ON THE EXISTENCE OF SOLUTIONS CONNECTING 
IK SINGULARITIES AND IMPASSE POINTS 
IN FULLY NONLINEAR RLC CIRCUITS

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Abstract. Higher-dimensional nonlinear and perturbed systems of implicit 
ordinary differential equations are studied by means of methods of dynamical 
systems. Namely, the persistence of solutions are studied under nonautonomous 
perturbations connecting either impasse points with IK-singularities or two impasse points. Important parts of the paper are applications of the 
three to concrete perturbed fully nonlinear RLC circuits.

1. Introduction. Motivated by \cite{8, 16}, a fully nonlinear RLC circuit is described 
by the second order ordinary differential equation
\begin{equation}
    u + L(v) + R(v) = e(t), \quad v = C(u),
\end{equation}
where \( L, R \) are the nonlinear self-inductance and the ohmic resistance, respectively 
and \( C \) is the nonlinear capacitance. \( u \) is the potential/voltage, \( q = C(u) \) is the 
charge and \( v = \frac{dq}{dt} \) is the current intensity. We suppose that \( L, R, C \) and \( e \) are 
\( C^{\infty} \)-smooth.

We can write equation \eqref{eq:1.1} as a system in \( \mathbb{R}^2 \):
\begin{align}
    L'(v)v' &= -u - R(v) + e(t), \\
    C'(u)u' &= v.
\end{align}

Quasilinear implicit differential equations, such as \eqref{eq:1.1}, find applications in a large 
number of physical sciences and have been studied by several authors \cite{7, 9, 10, 11, 12, 13, 14}.

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In this paper, instead of (1.2) we consider the perturbed problem

\[ L'(v)v' = -u - \varepsilon R(v) + \varepsilon(t), \]

\[ C'(u)u' = v, \]  

that is we assume that the external e.m.f. and the ohmic resistance are small. We also assume that \( v_0, u_0 \in \mathbb{R} \setminus \{0\} \) exist such that \( L'(v_0) = 0, L''(v_0) \neq 0 \) and \( C'(u_0) = 0, C''(u_0) \neq 0 \). Setting \( \varepsilon = 0 \) we get the unperturbed system:

\[ L'(v)v' = -u, \]

\[ C'(u)u' = v, \]  

which has almost the same phase portrait as

\[ \begin{align*}
&\dot{v} = -C'(u)u, \\
&\dot{u} = L'(v)v
\end{align*} \]  

with \( t' = \frac{dt}{ds} \). Indeed, system (1.5) has the Hamiltonian:

\[ H(v, u) = \int_0^v zL'(z)dz + \int_0^u zC'(z)dz \]  

and it is easy to check that \( H(v, u) \) is constant along trajectories of equation (1.4).

Next, equation (1.5) has four equilibria: \( e_1 = (0, 0), e_2 = (v_0, 0), e_3 = (0, u_0) \) and \( e_4 = (v_0, u_0) \). The Jacobian matrix of system (1.5) at a point \((v, u)\) is

\[ J(v, u) = \begin{pmatrix} 0 & -C'(u) - uC''(u) \\
L'(v) + vL''(v) & 0 \end{pmatrix} \]

Thus:

\[ J(0, 0) = \begin{pmatrix} 0 & -C'(0) \\
L'(0) & 0 \end{pmatrix} \]

\[ J(v_0, 0) = \begin{pmatrix} 0 & -C'(0) \\
v_0L''(v_0) & 0 \end{pmatrix} \]

and

\[ J(0, u_0) = \begin{pmatrix} 0 & -u_0C''(u_0) \\
L'(0) & 0 \end{pmatrix} \]

\[ J(v_0, u_0) = \begin{pmatrix} 0 & -u_0C''(u_0) \\
v_0L''(v_0) & 0 \end{pmatrix} \]

Hence the characteristic polynomial at the fixed points are respectively:

\[ p_{(0, 0)}(\lambda) = \lambda^2 + C'(0)L'(0) \]

\[ p_{(v_0, 0)}(\lambda) = \lambda^2 + v_0C'(0)L''(v_0) \]

\[ p_{(0, u_0)}(\lambda) = \lambda^2 + u_0C''(u_0)L'(0) \]

\[ p_{(v_0, u_0)}(\lambda) = \lambda^2 + u_0v_0C''(u_0)L''(v_0). \]

We suppose \( L'(0) \neq 0 \) and \( C'(0) \neq 0 \). Thus the stability of the fixed points depend on the signs of \( u_0, v_0, C'(0), L'(0), C''(u_0), L''(u_0) \).

Let \( \omega(v, u) = C'(u)L'(v) \). Certainly \( \omega(e_i) = 0 \) for \( i = 2, 3, 4 \), while \( \omega'(e_2) \neq 0 \) and \( \omega'(e_3) \neq 0 \), but \( \omega'(e_4) = 0 \).

In this paper we suppose that equation (1.5) has a solution connecting the fixed point \( e_3 \) (at \( t = -\infty \)) with another point in the manifold \( \omega(v, u) = 0 \). We want to study persistence of this kind of solutions.

This situation arises, for example, if equation (1.5) has heteroclinic connection between the fixed points \( e_3, e_2 \) crossing the set \( S := \{(v, u) \mid \omega(v, u) = 0\} \), where we assume that 0 is a regular value of \( \omega \). In particular, if this happens, it must be:

\[ \mathcal{H}(e_2) = \int_0^{v_0} zL'(z)dz = \int_0^{u_0} zC'(z)dz = \mathcal{H}(e_3). \]
In Section 4 of this paper we will give a couple of examples of such a situation. So let a heteroclinic orbit \( \gamma(t) \) intersect the set \( S \) transversally at two points \( (v_1^*, u_1^*) \) and \( (v_2^*, u_2^*) \) with \( \gamma(t_j^*) = (v_j^*, u_j^*), \ j = 1, 2, t_1^* < t_2^* \). In Section 2 we will study the persistence of the branch \( \{ \gamma(t) \mid t \leq t_1^* \} \) to a solution of equation (1.3) tending to the fixed point \( e_3 \) as \( t \to -\infty \) (or \( +\infty \) possibly reversing time) and hitting \( S \) at finite time. In Section 3 we will prove a result concerning the middle part \( \{ \gamma(t) \mid t_1^* \leq t \leq t_2^* \} \).

We write (1.3) as the standard IODE equation:

\[
\begin{align*}
\omega(v,u)v' &= C'(u) (-u - \varepsilon R(v) + \varepsilon e(t)) \\
\omega(v,u)u' &= L'(v)v.
\end{align*}
\] (1.7)

It is easy to check that \( e_2 \) persists as equilibrium in (1.7) after perturbation, but \( e_3 \) does not. Next, since \( \omega(v,u) = 0 \) only on the lines \( u = u_0, \ v = v_0 \), the system is regular outside these lines. As in [1] it can be seen that the orbits of equations (1.7) outside the lines \( u = u_0 \) and \( v = v_0 \) are the same as those of equation

\[
\begin{align*}
\dot{v} &= C'(u) (-u - \varepsilon R(v) + \varepsilon e(t)), \\
\dot{u} &= L'(v)v.
\end{align*}
\] (1.8)

However, if a solution \((v(t), u(t))\) of equation (1.8) hits the line \( v = v_0 \) (resp. \( u = u_0 \)) at a point \((v_0, \bar{u})\), \( \bar{u} \neq 0, u_0 \), (resp. \((\bar{v}, u_0)\), \( \bar{v} \neq 0, v_0 \) at the time \( t = \bar{t} \) this solution cannot be extended to a regular solution of equation (1.7) since the right-hand-side of the first (resp. second) equation in (1.7) does not vanish, in general, at the point \((v_0, \bar{u})\) (resp. \((\bar{v}, u_0)\)). Indeed suppose we have a solution \((v(t), u(t))\) of equation (1.7) tending in finite time to a point such as \((\bar{v}, u_0)\) with \( \bar{v} \neq 0, v_0 \), as \( t \to \bar{t} \). From the second equation in (1.7) we get

\[
\lim_{t \to \bar{t}} u'(t) = \infty.
\]

Similarly any solution of (1.7) approaching, as \( t \to \bar{t} \), a point like \((v_0, \bar{u})\), \( \bar{u} \neq 0, u_0 \) has to satisfy:

\[
\lim_{t \to \bar{t}} v'(t) = \infty.
\]

As we said, in this paper we study the persistence of solutions of the unperturbed system

\[
\begin{align*}
\omega(v,u)v' &= -C'(u)u \\
\omega(v,u)u' &= L'(v)v
\end{align*}
\] (1.9)

connecting regular points in the manifold \( \omega(v,u) = 0 \) toward solutions of the perturbed equation (1.7). Furthermore, since the fixed point \( e_2 \) disappears it would also be interesting to consider persistence of solutions of equation (1.9) tending to \( e_2 \) as \( t \to \pm \infty \). We leave this last study to a forthcoming paper.

These problems are different from the ones studied in [1] where equation (1.5) has an hyperbolic equilibrium together with a homoclinic orbit \( \gamma(s) = (\gamma_v(s), \gamma_u(s)) \) to it and the fixed point persists after perturbation. In [1] it has been proved that a change of time scale (depending on the solution) exists so that \( \gamma(s) \) corresponds to a solution \( \Gamma(t) \) of equation (1.9) tending to the fixed point as \( t \to \pm \infty \), but \( \omega(\Gamma(t)) > 0 \) for any \( t \in (-T, T) \). The persistence of such orbits are studied in [1]. Similar problems are studied in [2] and [3].

Nonlinear terms in (1.3) occur by considering the Kerr dielectric for nonlinear capacitance [8] or a nonlinear parallel-plate capacitor [5, p. 300], then a tunnel diode for nonlinear resistance [3, p. 52], and nonlinear toroidal inductor [5, p. 300].
We also refer to [6] for more examples of electronic nonlinearities like a gyrator circuit as a nonlinear inductor.

Finally, we interpret our study for (1.3) in the terminology of IODE [15] by rewriting system (1.3) as

\[ A(x)x' = h(x, t, \varepsilon) \]  

(1.10)

with

\[ x := \begin{pmatrix} v \\ u \end{pmatrix}, \quad A(x) := \begin{pmatrix} L'(v) & 0 \\ 0 & C'(u) \end{pmatrix}, \]

\[ h(x, t, \varepsilon) := \begin{pmatrix} -u - \varepsilon R(v) + \varepsilon e(t) \\ v \end{pmatrix}. \]

Since \( \det A(x) = \omega(x) \) and \( \omega(x^*) = 0, \, \omega'(x^*) \neq 0 \) for \( x^* \in \{e_2, e_3, \gamma(t_1^*), \gamma(t_2^*)\} \), the points \( x^* \) are noncritical 0-singularities of (1.10) (see [15, p. 163]). Moreover, for \( y^* \in \{\gamma(t_1^*), \gamma(t_2^*)\} \), we have \( h(y^*, t, 0) \notin RA(y^*) \) and \( (\det A'(y^*))w \neq 0 \) for any \( 0 \neq w \in NA(y^*) \) (see [15, (4.25),(4.26)]). So both 0-singularities \( y^* \) are impasse points for (1.4) (see [15, p. 163]). On the other hand, both \( e_2 \) and \( e_3 \) are IK—singularities of (1.4) (see [15, p. 168]). So, in Section 2 we study the persistence of a connection between an IK—singularity and an impasse point under the nonautonomous perturbation (1.3), while in Section 3 we study the persistence of a connection between two impasse points under the nonautonomous perturbation (1.3). We study in Sections 2 and 3 higher dimensional analogy of (1.7). We note that the above analysis for (1.10) cannot be applied to (1.7), since considering (1.7) in the form of (1.10), we get \( \dot{A}(x^*) = 0 \). Then alternative definitions are used, for instance, impasse points are determined by [15 (4.33)].

2. Connecting impasse points with IK-singularities. Motivated by Introduction, in this section we consider a singular equation as

\[ \omega(x)x' = F(x) + \varepsilon G(x, t, \varepsilon, \kappa), \quad x \in \mathbb{R}^n, \quad \kappa \in \mathbb{R}^m \]  

(2.1)

for \( \omega \in C^2(\mathbb{R}^n, \mathbb{R}), \, F \in C^2(\mathbb{R}^n, \mathbb{R}^n) \) and \( G \in C^2(\mathbb{R}^{n+m+2}, \mathbb{R}^n) \) with bounded derivatives. First we suppose

(C1) The unperturbed (2.1):

\[ \omega(x)x' = F(x) \]  

(2.2)

possesses noncritical singularities at \( x_0 \) and \( x_1 \), i.e. \( \omega(x_i) = 0 \) and \( \omega'(x_i) \neq 0 \), \( i = 1, 2 \).

(C2) The ODE:

\[ \dot{x} = F(x) \]  

(2.3)

has the hyperbolic equilibrium \( x_0, \) i.e. \( F(x_0) = 0 \) and the spectrum \( \sigma(DF(x_0)) \) has no eigenvalues on the imaginary axis, and a solution \( \gamma(s) \) on \( (-\infty, 1] \) such that \( \lim_{s \to -\infty} \gamma(s) = x_0 \), and \( \omega(\gamma(s)) \neq 0 \) for any \( s \in (-\infty, 1) \) with \( \gamma(1) = x_1 \).

Moreover \( G(x_0, t, \varepsilon, \kappa) = 0 \) for any \( t \in \mathbb{R}, \, \kappa \in \mathbb{R}^m \) and \( \varepsilon \) sufficiently small.

(C3) It results:

\[ \lim_{s \to -\infty} \frac{1}{s} \ln |\gamma(s) - x_0| = \mu_+ \]  

(2.4)

where \( \mu_+ \) is a simple positive eigenvalue of \( F'(x_0) \) with the corresponding eigenvector \( \gamma_- \) and all the other eigenvalues of \( F'(x_0) \) have real parts greater than \( \mu_+ \).

(C4) \( \langle \nabla \omega(x_0), \gamma_- \rangle \cdot \langle \nabla \omega(x_1), \gamma(1) \rangle < 0. \)
Suppose \( \omega(\gamma(s)) > 0 \) for any \( s \in (-\infty, 1) \). Since \( \lim_{s \to -\infty} \omega(\gamma(s)) = 0 \) and \( \omega(\gamma(1)) = 0 \), for any \( n \in \mathbb{N} \) there exists \( s_n < -n \) such that

\[
\left\langle \nabla \omega(\gamma(s_n)), \frac{\dot{\gamma}(s_n)}{\dot{\gamma}(s_n)} \right\rangle \geq 0 \geq (\nabla \omega(\gamma(1)), \dot{\gamma}(1)).
\]

However it is known (see [2 p. 1184]) that, under condition (C3), it holds \( \lim_{s \to -\infty} \frac{\dot{\gamma}(s)}{\dot{\gamma}(s)} = \gamma_- \). Thus \( \langle \nabla \omega(x_0), \gamma_- \rangle \geq 0 \geq \langle \nabla \omega(x_1), \dot{\gamma}(1) \rangle \) and hence condition (C4) reads:

\[
\langle \nabla \omega(x_0), \gamma_- \rangle > 0 \quad \text{and} \quad \langle \nabla \omega(x_1), \dot{\gamma}(1) \rangle < 0. \tag{2.5}
\]

These inequalities will be reversed if we assume that \( \omega(\gamma(s)) < 0 \), for any \( s \leq 0 \).

In the following we assume that \( \omega(\gamma(s)) > 0 \) for any \( s \in (-\infty, 1) \) and (2.5) instead of (C4), as we can always reduce to this situation possibly changing \( t \) with \( -t \) in equation (2.1).

First we consider equation (2.2) in the interval \((-T_*, 0]\), where \( T_* > 0 \) is given as follows. For \( s \leq 0 \), let

\[
t = \theta(s) := \int_0^s \omega(\gamma(\tau))d\tau,
\]

then \( \gamma(\theta^{-1}(t)) \) satisfies equation (2.2) for \( t \in (-T_*, 0]\), where

\[
T_* = \int_{-\infty}^0 \omega(\gamma(\tau))d\tau.
\]

In [2 p. 1164] it has been observed that \( T_* \) is finite.

Let \( k_- \) be the number of eigenvalues of \( F'(x_0) \) with negative real parts counted with multiplicities. Then \( F'(x_0) \) has \( n - k_- \) eigenvalues with positive real parts counted with multiplicities. According to arguments of [2 p. 1169], the linear system

\[
\dot{y} = \left[ F'(\gamma(s)) - \frac{F(\gamma(s))\omega'(\gamma(s))}{\omega(\gamma(s))} - \mu_+ I \right] y \tag{2.7}
\]

has an exponential dichotomy on \( \mathbb{R}_- \) with a projection \( P_- \) such that rank \( P_- = k_- + 1 \) or \( \dim \mathcal{N} P_- = n - k_- - 1 \).

We prove the following

**Lemma 2.1.** Assume conditions (C1)–(C4) hold and that \( \omega(\gamma(s)) > 0 \) for any \( s \leq 0 \). Then for any \( \kappa \in I \), for a compact subset \( I \subset \mathbb{R}^m \), \( \eta \in \mathcal{N} P_- \) and \( \varepsilon \) sufficiently small, equation (2.1) has a unique bounded solution \( \tilde{x}(t) = \tilde{x}(t, \eta, \varepsilon, \kappa) \) defined for \( t \in (-T_*, 0] \) and such that \( \omega(\tilde{x}(t)) > 0 \) along with \( (\mathbb{I} - P_-)(\tilde{x}(0) - \gamma(0)) = \eta \) and \( \lim_{t \to -T_*^+} \tilde{x}(t) = x_0 \). Moreover,

\[
\lim_{|\varepsilon| + |\eta| \to 0} \sup_{t \in (-T_*, 0]} |\tilde{x}(t) - \gamma(\theta^{-1}(t))| = 0 \tag{2.8}
\]

uniformly for \( \kappa \) in compact sets in \( \mathbb{R}^m \).

**Proof.** Taking the change of time given in (2.6) equation (2.1) reads

\[
\omega(z) \dot{z} = \omega(\gamma(s)) \left( F(z) + \varepsilon G(z, \theta(s), \varepsilon, \kappa) \right) \tag{2.9}
\]

with \( z(s) = x(\theta(s)) \). Note that \( \theta(0) = 0 \), \( \lim_{s \to -\infty} \theta(s) = -T_* \) and then \( z(0) = x(0) \). Then set

\[
\varphi(s) := e^{\mu s},
\]
and make in (2.9) the change of variables
\[ z(s) = \gamma(s) + \varphi(s)y(s) = x_0 + \varphi(s)(\eta(s) + y(s)) \]  
(2.10)
where \( \eta(s) \) is the bounded function \( \frac{\gamma(s)-x_0}{\varphi(s)} \). So we derive the equation
\[ \dot{y} = \frac{\omega(\gamma)}{\varphi\omega(\gamma + \varphi y)} F(\gamma + \varphi y) - \frac{F(\gamma)}{\varphi} - \mu + y \]
\[ + \varepsilon \frac{\omega(\gamma)}{\varphi\omega(\gamma + \varphi y)} G(\gamma + \varphi y, \theta(s), \varepsilon, \kappa). \]  
(2.11)
We solve equation (2.11) on \( C_b := C_b([\mathbb{R}_-, \mathbb{R}^n]), \mathbb{R}_- := (-\infty, 0] \), by following [2]. It is easy to check that the linearization of (2.11) at \( y = 0 \) and \( \varepsilon = 0 \) is equation (2.7) and we already noted that this equation has an exponential dichotomy on \( \mathbb{R}_- \) with projection of rank \( k_- + 1 \) or \( \dim \mathcal{N}P_- = n - k_- - 1 \).

Next we consider equation (2.1) in an interval \( 0 \leq t < \hat{T} \), where \( \hat{T} \) will be specified below, i.e., we study (2.1) near \( \gamma(\theta^{-1}(t)) \) for \( t \in [0, T^*] \), where
\[ T^* = \int_0^1 \omega(\gamma(\tau))d\tau. \]

Here we suppose that \( \gamma(s) \) is a solution of (2.3) defined in an open interval containing \((-\infty, 1]\), which is always possible to assume. We mimic the argument given in [15] Theorem 4.7 and prove the following
Lemma 2.2. Assume conditions (C1)–(C4) hold and that \( \omega(\gamma(s)) > 0 \) for any \( s \in [0,1] \). Then for any \( \kappa \in I \), for a compact subset \( I \subset \mathbb{R}^m \), and for any \( (\xi,\varepsilon) \) such that \( |\xi - \gamma(0)| + |\varepsilon| \) is sufficiently small, there exists a unique function \( x(t) = x(t, \xi, \varepsilon, \kappa) \) continuous in \( t \in [0,T] \), with \( T > 0 \), that satisfies equation (2.1) in \( [0,T) \) and it is such that \( x(0) = \xi, \omega(x(t)) > 0 \) and \( \omega(x(T)) = 0 \). Moreover,

\[
\lim_{|\xi-x_1|+|\varepsilon|\to 0} \sup_{t \in [0,T]} |x(t) - \gamma(\theta^{-1}(t))| = 0 \tag{2.15}
\]

uniformly for \( \kappa \) in compact sets in \( \mathbb{R}^m \). Furthermore, \( T \) is a \( C^2 \)-function of its arguments \( \xi, \varepsilon, \kappa \) with \( T \to T^* \) as \( |\xi-x_1|+|\varepsilon| \to 0 \) uniformly for \( \kappa \) in compact sets in \( \mathbb{R}^m \).

Proof. Let

\[
g(x,t,\varepsilon,\kappa) := F(x) + \varepsilon G(x,t,\varepsilon,\kappa)
\]

and \( w := \begin{pmatrix} x \\ \tau \end{pmatrix} \). For any \( \xi \in \mathbb{R}^n \) such that \( |\xi - \gamma(0)| \) is sufficiently small the Cauchy problem

\[
\begin{cases}
\dot{w} = 
\begin{pmatrix}
g(x,\tau,\varepsilon,\kappa) \\
\omega(x)
\end{pmatrix} \\
w(0) = 
\begin{pmatrix}
\xi \\
0
\end{pmatrix}
\end{cases}
\tag{2.16}
\]

has the unique solution \( \tilde{w}(s) = \begin{pmatrix} \tilde{x}(s) \\ \tilde{\tau}(s) \end{pmatrix} = \begin{pmatrix} \tilde{x}(s, \xi, \varepsilon, \kappa) \\ \tilde{\tau}(s, \xi, \varepsilon, \kappa) \end{pmatrix} \), with \( s \in [0,s^*] \) for some \( s^* \geq 0 \). Note that, for \( \varepsilon = 0, \xi = \gamma(0) \), and \( \kappa \) in any fixed compact set, equation (2.16) reduces to

\[
\begin{align*}
\dot{x} &= F(x) \\
\dot{\tau} &= \omega(x) \\
x(0) &= \gamma(0), \quad \tau(0) = 0
\end{align*}
\]

whose unique solution is \( (x,\tau) = (\gamma(s),\tau(s)) \), with

\[
\tau(s) := \int_0^s \omega(\gamma(\sigma))d\sigma.
\]

Because of smoothness with respect to parameters and initial conditions we can assume that \( \tilde{w}(s) \) is defined for \( 0 \leq s \leq 1 + \delta \), for some \( \delta > 0 \), provided \( |\varepsilon| \) and \( |\xi - \gamma(0)| \) are sufficiently small, uniformly with respect to \( \kappa \) in a compact set of \( \mathbb{R}^{m+1} \) (that is, we can assume \( s^* = 1 + \delta \)). Moreover:

\[
\sup_{0 \leq s \leq 1+\delta} |\tilde{x}(s, \xi, \varepsilon, \kappa) - \gamma(s)| \to 0 \tag{2.17}
\]

as \( (\xi,\varepsilon) \to (\gamma(0),0) \) uniformly for \( \kappa \) in a fixed compact set and the same holds for the derivatives of \( \tilde{x}(s, \xi, \varepsilon, \kappa) - \gamma(s) \) with respect to the parameters \( (\xi,\varepsilon,\kappa) \).

Next note that, using also (2.17):

\[
\begin{align*}
d\omega(\tilde{x}(t, \gamma(0),0,\kappa)) &= \\
\omega'(\tilde{x}(1, \gamma(0),0,\kappa))g(\tilde{x}(1, \gamma(0),0,\kappa), \tilde{\tau}(1, \gamma(0),0,\kappa),0,\kappa) \\
&= \omega'(\gamma(1))F(\gamma(1)) = \omega'(\gamma(1))\gamma(1) < 0.
\end{align*}
\]

Hence for \( |\xi - \gamma(0)| + |\varepsilon| \) sufficiently small, and uniformly with respect to \( \kappa \) in compact sets of \( \mathbb{R}^m \), we have:

\[
\frac{d\omega(\tilde{x}(s))}{dt} < 0
\]
for $1 - \delta < s < 1 + \delta$, possibly for a smaller $\delta > 0$. Next

\[
\lim_{(\xi,\varepsilon) \to (\gamma(0),0)} \omega(\hat{x}(1 - \delta)) = \omega(\gamma(1 - \delta)) > 0
\]

\[
\lim_{(\xi,\varepsilon) \to (\gamma(0),0)} \omega(\hat{x}(1 + \delta)) = \omega(\gamma(1 + \delta)) < 0.
\]

As a consequence there exists a unique $\bar{s} \in (1 - \delta, 1 + \delta)$ such that $\omega(\hat{x}(\bar{s})) = 0$ and $\omega(\hat{x}(s)) > 0$ for $1 - \delta \leq s < \bar{s}$. Then, from (2.17) it follows that $\omega(\hat{x}(s)) > 0$ for any $s \in [0, \bar{s})$, provided $|\xi - \gamma(0)| + |\varepsilon|$ is sufficiently small, and uniformly with respect to $\kappa$ in compact sets of $\mathbb{R}^m$. We assume that we made this choice and set

\[
t = \vartheta(s) := \int_0^s \omega(\hat{x}(\tau))d\tau
\]

and $x(t) = \hat{x}(\vartheta^{-1}(t))$, $\tau(t) := \tilde{\vartheta}(\vartheta^{-1}(t))$. Note that:

\[
\vartheta'(s) = \omega(\hat{x}(s)) > 0
\]

for $0 \leq s < \bar{s}$, and $\vartheta'(\bar{s}) = \omega(\hat{x}(\bar{s})) = 0$. Hence $\vartheta(s)$ is a strictly increasing function of $s \in [0, \bar{s}]$ taking values in $[0, \vartheta(\bar{s})]$. Note that $x(t)$ and $\tau(t)$ are $C^1$ functions in $[0, \vartheta(\bar{s})]$ that are continuous up to $\bar{T} = \vartheta(\bar{s})$. We get:

\[
\frac{d\tau}{dt} = \frac{d\hat{\tau}}{ds} \frac{1}{\vartheta'(\vartheta^{-1}(t))} = \frac{\omega(\hat{x}(\vartheta^{-1}(t)))}{\omega(\hat{x}(\vartheta^{-1}(t)))} = 1,
\]

\[
\tau(0) = \tilde{\vartheta}(0) = 0
\]

that is $\tau(t) = t$. Next, for $0 \leq t < \bar{T}$:

\[
\omega(x(t)) \frac{dx}{dt}(t) = \omega(x(t))g(\hat{x}(\vartheta^{-1}(t)), \tilde{\vartheta}(\vartheta^{-1}(t)), \varepsilon, \kappa) \frac{1}{\vartheta'(\vartheta^{-1}(t))}
\]

\[
= \frac{\omega(x(t))}{\omega(\hat{x}(\vartheta^{-1}(t)))}g(\hat{x}(\vartheta^{-1}(t)), \tilde{\vartheta}(\vartheta^{-1}(t)), \varepsilon, \kappa)
\]

\[
= g(x(t), \tau(t), \varepsilon, \kappa) = g(x(t), t, \varepsilon, \kappa),
\]

\[
x(\bar{T}) = \hat{x}(\bar{s}) \in S := \{\xi \in \mathbb{R}^m \mid \omega(\xi) = 0\}.
\]

Hence $x(t) = x(t, \xi, \varepsilon, \kappa)$ is a solution of equation (2.1) in $[0, \bar{T})$, continuous up to $t = \bar{T}$ and satisfying $\omega(x(\bar{T})) = 0$. Moreover:

\[
\sup_{0 \leq t < \bar{T}} |x(t, \xi, \varepsilon, \kappa) - \gamma(\vartheta^{-1}(t))| \to 0
\]

as $(\xi, \varepsilon) \to (\gamma(0),0)$ uniformly for $\kappa$ in a fixed compact set of $\mathbb{R}^m$ and the same holds for the derivatives of $x(t, \xi, \varepsilon, \kappa) - \gamma(\vartheta^{-1}(t))$ with respect to the parameters $(\xi, \varepsilon, \kappa)$. Note also that $\tau(t) = t$ is equivalent to $\tilde{\vartheta}(s) = \vartheta(s)$.

Finally, from (2.17) it follows:

\[
\vartheta(s) - \theta(s) = \int_0^s (\omega(x(\sigma, \xi, \varepsilon, \kappa)) - \omega(\gamma(\sigma))) d\sigma \to 0
\]

as $(\xi, \varepsilon) \to (\gamma(0),0)$ uniformly with respect to $\kappa$ in compact sets of $\mathbb{R}^m$. The proof is finished.

Thus, if in Lemma 2.2 we take $\xi = \hat{x}(0, \eta, \varepsilon, \kappa)$, $\hat{x}(t, \eta, \varepsilon, \kappa)$ being the solution whose existence is stated in Lemma 2.1 we obtain the following result.
Theorem 2.3. Assume conditions (C1)-(C4) hold and that \( \omega(\gamma(s)) > 0 \) for any \( s \leq 1 \). Then given a compact subset \( K \subset \mathbb{R}^m \) there exist \( \eta_0 > 0 \) and \( \varepsilon_0 > 0 \) such that for any \( (\eta, \varepsilon, \kappa) \in N \mathbb{P}_- \times \mathbb{R} \times K \) satisfying \( |\eta| \leq \eta_0, |\varepsilon| \leq \varepsilon_0 \), there exists a unique continuous function \( x(t, \eta, \kappa, \varepsilon) \) defined for \( t \in [-T_*, \bar{T}] \), with \( \bar{T} > 0 \) and satisfying equation \((2.1)\) and \( \omega(x(t, \eta, \kappa, \varepsilon)) > 0 \) for any \( t \in (-T_*, \bar{T}) \) as well as \( \omega(x(-T_*, \eta, \kappa, \varepsilon)) = \omega(x(\bar{T}, \eta, \kappa, \varepsilon)) = 0 \). Moreover, \[
\lim_{t \to -T_*} |x(t, \eta, \kappa, \varepsilon) - x_0| = 0
\]
and
\[
\sup_{-T_* < t < \bar{T}} |x(t, \eta, \kappa, \varepsilon) - \gamma(\theta^{-1}(t))] \to 0 \tag{2.19}
\]
as \( |\varepsilon| + |\eta| \to 0 \) uniformly with respect to \( \kappa \) in compact sets of \( \mathbb{R}^m \). Furthermore, \( \bar{T} \) is a \( C^2 \)-function of its arguments \( \eta, \varepsilon, \kappa \) with \( \bar{T} \to \bar{T}^* \) as \( |\eta| + |\varepsilon| \to 0 \) uniformly for \( \kappa \) in compact sets in \( \mathbb{R}^m \).

Note that \((2.19)\) follows from \((2.8)\) and \((2.15)\).

Remark 2.4. If \( \omega(\gamma(s)) < 0 \), for \( s < 1 \), the proofs of Lemma \(2.1, 2.2\) and Theorem \(2.3\) still go through. The only difference is that the functions \( \theta(s) \) and \( \vartheta(s) \) are in this case decreasing. Thus, the appropriate intervals must be reversed. Consequently, instead of \((2.19)\) we have:
\[
\sup_{t \in (\bar{T}, -T_*)} |x(t, \eta, \kappa, \varepsilon) - \gamma(\theta^{-1}(t))] \to 0
\]
as \( |\varepsilon| + |\eta| \to 0 \).

3. Connecting two impasse points. In this section we modify the assumptions (C1)-(C4) to handle the case where a solution of equation \((2.2)\) exists connecting in finite time impasse points on the singularity set \( \omega(x) = 0 \). So we assume the following conditions hold

(H1) The unperturbed \((2.2)\) possesses noncritical singularities at \( x_{-1} \) and \( x_1 \), i.e. \( \omega(x_{-1}) = \omega(x_1) = 0 \) and \( \omega'(x_{-1}) \neq 0 \neq \omega'(x_1) \)

(H2) The ODE \((2.3)\) has a solution \( \gamma(s) \) on \([s_{-1}, s_1]\) such that \( \gamma(s_i) = x_i, i = -1, 1 \) and \( \omega(\gamma(s)) \neq 0 \) for any \( s \in (s_{-1}, s_1) \). Without loss of generality, we may, and will, assume \( \omega(\gamma(s)) > 0 \) for any \( s \in (s_{-1}, s_1) \).

(H3) \( \langle \nabla \omega(x_{-1}), \dot{\gamma}(s_{-1}) \rangle > 0 \) and \( \langle \nabla \omega(x_1), \dot{\gamma}(s_1) \rangle < 0 \).

Arguing as in the proof of Theorem \(2.2\) we prove the following

Theorem 3.1. Assume conditions (H1)-(H3) hold. Then for any \( (\xi, \varepsilon) \) such that \( |\xi - x_1| + |\varepsilon| \) is sufficiently small and \( \omega(\xi) = 0 \), there exists a unique continuous function \( x(t) = x(t, \xi, \varepsilon, \kappa) \) on the interval \([\bar{t}, s_1^*]\), for some \( \bar{t} < s_1^* \), that satisfies equation \((2.1)\) as well as \( \omega(x(t)) > 0 \) for \( t \in (\bar{t}, s_1^*) \), \( x(s_1^*) = \xi \) and \( \omega(x(\bar{t})) = 0 \). Moreover,
\[
\lim_{|\xi - x_1| + |\varepsilon| \to 0} \sup_{t \in [\bar{t}, s_1^*]} |x(t) - \gamma(\bar{\theta}^{-1}(t))] = 0
\]
uniformly for \( \kappa \) in compact sets of \( \mathbb{R}^m \) for
\[
\bar{\theta}(s) := s_1^* - \int_{s}^{s_1^*} \omega(\gamma) \, d\tau. \tag{3.1}
\]
Furthermore, \( \bar{t} \) is a \( C^2 \)-function of the parameters \( \xi, \varepsilon, \kappa \) with the property \( \bar{t} \to s_1^* - \int_{s_{-1}}^{s_1^*} \omega(\gamma) \, d\tau \) as \( |\xi - x_1| + |\varepsilon| \to 0 \) uniformly for \( \kappa \) in compact sets in \( \mathbb{R}^m \).
Theorem 3.2. Assume conditions (H1)–(H3) hold. Then for any $s \in S = \{ x \in \mathbb{R}^n \mid \omega(x) = \mathbf{0}\}$, such that $|\xi - \gamma(s^*_1)|$ is sufficiently small, a solution $\hat{x}(s)$ of
\begin{equation}
\begin{cases}
\dot{x} = F(x) + \varepsilon G(x, \tau, \varepsilon, \kappa) \\
\dot{\tau} = \omega(x) \\
x(s^*_1) = \xi, \quad \tau(s^*_1) = s^*_1
\end{cases}
\end{equation}
exists on $[s^*_1 - \delta, s^*_1]$ for a small $\delta > 0$ such that $\sup_{s^*_1 - \delta \leq s \leq s^*_1} |\hat{x}(s) - \gamma(s)| \to 0$, as $(\varepsilon, \xi) \to (0, \gamma(s^*_1))$ and the same holds concerning $|\frac{d}{ds} [\hat{x}(s) - \gamma(s)]|$. Note we extend also $\gamma(s)$ on $[s^*_1 - \delta, s^*_1]$. Then, from (H3) it follows that for any $\varepsilon$ sufficiently small $\omega'(\xi)\hat{x}(s^*_1) < 0$.

Moreover, by the implicit function theorem from (H2) and (H3), there is a unique $\bar{s}$ near $s^*_1$ such that $\omega(\hat{x}(\bar{s})) = 0$, $\omega'(\hat{x}(\bar{s}))\hat{x}(\bar{s}) > 0$ and $\omega(\hat{x}(s)) > 0$ for $s \in (\bar{s}, s^*_1)$. Moreover, $\bar{s}$ is a $C^2$-function of the parameters $\xi, \varepsilon, \kappa$. Then taking
$$t = \vartheta(s) := s^*_1 - \int_{s}^{s^*_1} \omega(\hat{x}(\tau))d\tau, \quad s \in [\bar{s}, s^*_1],$$
and setting $x(t) := \hat{x}(\vartheta^{-1}(t))$, $\tau(t) := \tilde{\tau}(\vartheta^{-1}(t))$ we see, as in the proof of Theorem 2.2 that
$$\frac{d\tau}{dt} = 1, \quad \tau(s^*_1) = \hat{\tau}(s^*_1) = s^*_1.$$
As a consequence $\tau(t) = t$. Hence we see that $x(t)$ is a continuous function on $[\bar{t}, s^*_1]$, $\bar{t} := \vartheta(\bar{s})$ solving equation (2.1) in $(\bar{t}, s^*_1)$ and satisfying $\lim_{t \to s^*_1^-} x(t) = \xi$ as well as:
$$\sup_{\bar{t} \leq t \leq s^*_1} |x(t) - \gamma(\vartheta^{-1}(t))| \to 0 \text{ as } (\varepsilon, \xi) \to (0, \gamma(s^*_1)).$$
Since $\bar{s} \to s^*_1$ as $|\varepsilon| + |\xi - \gamma(s^*_1)| \to 0$ we see that $\bar{t} \to s^*_1 - \int_{s^*_1}^{s^*_1} \omega(\gamma(\tau))d\tau$ as $|\xi - x_1| + |\varepsilon| \to 0$. The proof is finished.

By a similar proof we can also show the following

Theorem 3.2. Assume conditions (H1)–(H3) hold. Then for any $(\xi, \varepsilon)$ such that $|\xi - x_1| + |\varepsilon|$ is sufficiently small and $\omega(\xi) = 0$, there exists a unique continuous function $x(t) = x(t, \xi, \varepsilon, \kappa)$ on the interval $[s^*_1, \hat{t}]$, for some $\hat{t} > s^*_1$, that satisfies equation (2.1) in $(s^*_1, \hat{t})$ with $x(s^*_1) = \xi$ and such that $\omega(x(t)) > 0$, for $t \in (s^*_1, \hat{t})$ and $\omega(x(t)) = 0$. Moreover,
$$\lim_{|\xi - x_1| + |\varepsilon| \to 0} \sup_{t \in [s^*_1, \hat{t}]} |x(t) - \gamma(\tilde{\vartheta}^{-1}(t))| = 0$$
uniformly for $\kappa$ in compact sets of $\mathbb{R}^m$ for $\tilde{\vartheta}(s) := s^*_1 + \int_{s^*_1}^{s^*_1} \omega(\gamma(\tau))d\tau$. Furthermore, $\tilde{t}$ is a $C^2$-function of parameters $\xi, \varepsilon, \kappa$ such that $\tilde{t} \to s^*_1 + \int_{s^*_1}^{s^*_1} \omega(\gamma(\tau))d\tau$ as $|\xi - x_1| + |\varepsilon| \to 0 \to 0$ uniformly for $\kappa$ in compact sets in $\mathbb{R}^m$.

Remark 3.3. i) Of course the sign of $\omega(\gamma(s))$ in the interior of the interval is not crucial. Similar statements hold when $\omega(\gamma(s)) < 0$ for $s \in (s^*_1, s^*_2)$. However in this case the functions $\vartheta(s), \tilde{\vartheta}(s)$ and $\hat{\vartheta}(s)$ are decreasing and then time $t$ has to be reversed as in Remark 2.4.

ii) Suppose that $\gamma(s)$ is a solution of equation (2.3) defined in the interval $(\bar{t}, T]$, $T > 1$, such that $\omega(\gamma(1)) = \omega(\gamma(T)) = 0$, $\omega(\gamma(s)) > 0$ for $s < 1,$
where \( \omega(\gamma(s)) < 0 \) for \( 1 < s < T \). Then we can apply both Theorems 2.3 and 3.1 (this last with time reversed since \( \omega(\gamma(s)) < 0 \) for \( 1 < s < T \)) and prove the existence of \( \rho > 0 \) such that for any \( \eta \in \mathcal{N}P_- \) with \( |\eta| < \rho \) and \( \xi \) such that \( \omega(\xi) = 0 \) and \( |\xi - \gamma(1)| < \rho \), there are two solutions of equation (2.1), \( x(t, \eta, \kappa, \varepsilon) \) and \( x(t, \xi, \varepsilon, \kappa) \), the first defined for \( t \in (-T_*, T) \) and satisfying

\[
\sup_{t \in (-T_*, T)} |x(t, \kappa, \varepsilon, \eta) - \gamma(\theta(t))| \rightarrow 0
\]

as \( |\varepsilon| + |\eta| \rightarrow 0 \) and the second one defined for \( t \in (1, \tilde{t}) \) and satisfying

\[
\sup_{t \in (1, \tilde{t})} |x(t, \xi, \varepsilon, \kappa) - \gamma(\tilde{\theta}(t))| \rightarrow 0
\]

as \( |\varepsilon| + |\xi - \gamma(1)| \rightarrow 0 \), where \( \theta(s) \) and \( \tilde{\theta}(s) \) are as in (2.6) and (3.1), respectively.

4. Examples. This section is devoted to the construction of two examples of applications of our results. These examples fall in the class of systems (1.7) with \( \omega(v, u) = L'(v)C''(u) \). In applying Theorem 2.3 we consider the singularities \( x_0 = (0, u_0) \) and \( x_1 = (v_0, u_0) \) (for some \( u_* \)) and \( x_{-1} = (v_0, u_1) \) (for some \( v_* \)) when we apply Theorem 3.2. It is then clear that (C1) and (H1) hold provided

\[
L'(0) \neq 0, L'(v_*) \neq 0, L''(v_0) \neq 0, L''(u_0) \neq 0.
\]

Next \( x_0 \) is a hyperbolic equilibrium of equation (1.5) if and only if

\[
u_0C''(u_0)L'(0) < 0.
\]

If this condition holds then, with reference to Section 2, we have \( \mathcal{N}P_- = \{0\} \).

Finally

\[
\nabla \omega(0, u_0) = \begin{pmatrix} 0 \\ C''(u_0)L'(0) \end{pmatrix},
\]

\[
\nabla \omega(v_0, u_*) = \begin{pmatrix} C'(u_*)L''(v_0) \\ 0 \end{pmatrix},
\]

\[
\nabla \omega(v_*, u_0) = \begin{pmatrix} 0 \\ C''(u_0)L'(v_*) \end{pmatrix}.
\]

4.1. First example. We take \( C(u) = h(u) \), \( L(v) = h(v) \), where \( h(z) \) is a \( C^3 \) strictly convex function having two simple zeros \( z = 0 \) and \( z = k_* \) (then \( h'(0) \neq 0 \neq h'(k_*) \)). Then \( h(v, u) = h'(v)h'(u) \). Since \( h(z) \) is convex we see that \( h'(0) > 0 \Rightarrow k_* < 0 \) and \( h'(0) < 0 \Rightarrow k_* > 0 \). As a consequence

\[
h'(0)z_0 < 0
\]

for \( z_0 \in (0, k_*] \) with \( h'(z_0) = 0 \). Hence condition 4.2 is satisfied. To apply our above results, we need to find a solution of the unperturbed equation

\[
\dot{v} = -h'(u)u,
\]

\[
\dot{u} = h'(v)u.
\]

asymptotic, as \( t \rightarrow -\infty \), to \( x_0 = (0, z_0) \). Note that (1.5) has the Hamiltonian:

\[
H(v) + H(u) \quad \text{where} \quad H(x) := \int_0^x sh'(s) ds.
\]

Then we have a heteroclinic connection between \((0, z_0), (z_0, 0)\) in the manifold:

\[
H(v) + H(u) = H(z_0).
\]
As a concrete example, we take \( h(z) = z + z^2 \). We have \( k^* = -1 \) and \( z_0 = -\frac{1}{2} \). Since
\[
H(x) := \int_0^x s h'(s) \, ds = \int_0^x s(1 + 2s) \, ds = \frac{x^2}{2} + \frac{2}{3} x^3,
\]
the unperturbed system
\[
\dot{v} = -u(1 + 2u), \\
\dot{u} = v(1 + 2v)
\] (4.5)
has the Hamiltonian \( H(v,u) = \frac{v^2}{2} + \frac{2}{3} v^3 + \frac{u^2}{2} + \frac{2}{3} u^3 \). The heteroclinic connection between the two saddles satisfies the equation:
\[
\frac{v^2}{2} + \frac{2}{3} v^3 + \frac{u^2}{2} + \frac{2}{3} u^3 - \frac{1}{24} = 0
\]

But:
\[
\frac{v^2}{2} + \frac{2}{3} v^3 + \frac{u^2}{2} + \frac{2}{3} u^3 - \frac{1}{24} = \frac{1}{24} (2u + 2v + 1)(8u^2 - 8uv + 8v^2 + 2u + 2v - 1)
\]
Thus we have three heteroclinic orbits connecting the saddles \((-1/2,0)\) and \((0,-1/2)\), one lying in the line \( \ell := \{2u + 2v + 1 = 0\} \) and the other two in the ellipse \( E = \{8u^2 - 8uv + 2u + 8v^2 + 2v - 1 = 0\} \).

**Figure 1.** The heteroclinic orbits of equation (4.5)

Since the right-hand-side of equation (4.5) at the point \((-\frac{1}{4}, -\frac{1}{4})\) is given by
\[
\frac{1}{8} \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]
the heteroclinic orbit in \( \ell \) connects \( e_2 = (-\frac{1}{2},0) \) with \( e_3 = (0,-\frac{1}{2}) \). On the other hand the intersection of \( E \) with the line \( v = u \) is given by the points \( \left(\frac{\sqrt{3} - 1}{4}, \frac{\sqrt{3} - 1}{4}\right) \).
If, instead, we take \( t = ( - \frac{\sqrt{3} + 1}{4}, - \frac{\sqrt{3} + 1}{4} ) \). The vector field at these points is:
\[
\frac{1}{4} \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]
and hence the (two) heteroclinic connections lying on \( \mathcal{E} \) go from \( e_3 = (0, -\frac{1}{2}) \) to \( e_2 = (-\frac{1}{2}, 0) \). However, the first of these two orbits does not intersect the lines \( u = -\frac{1}{2} \) and \( v = -\frac{1}{2} \) while the second does two times at the points
\[
\left( -\frac{1}{2}, -\frac{3}{4} \right) \quad \text{and} \quad \left( -\frac{3}{4}, -\frac{1}{2} \right).
\]
As a consequence:
\[
u^* = v^* = -\frac{3}{4}
\]
and \( L'(v^*) \neq 0 \neq C'(u^*) \). Hence conditions \([4.1], [4.2]\) are satisfied.

We want to find the explicit expression of this heteroclinic connection. This solution lie in the ellipse \( \mathcal{E} = \{ 8u^2 - 8uv + 2u + 8v^2 + 2v - 1 \} \). Let
\[
v = \frac{1}{2}(x + y - 1) \quad \Rightarrow \quad x = 2v - 2u \\
u = \frac{1}{2}(y - x - 1) \quad \Rightarrow \quad y = 2v + 2u + 1
\]
Then the equation for \( \mathcal{E} \) in the new coordinates is \( 3x^2 + y^2 - 3 = 0 \). Since \( \dot{x} = 2(\dot{v} - \dot{u}) = -2(v(2v + 1) + u(2u + 1)) = \frac{1}{2}(1 - x^2 - y^2) \), we get the equation for \( x \) in \( \mathcal{E} \):
\[
\dot{x} = x^2 - 1
\]
to which we should add the initial condition. Starting from \( x(0) = 0 \) we get the solution:
\[
x(s) = \frac{1 - e^{2s}}{1 + e^{2s}} = \tanh s
\]
Since the solution we look for satisfies \( 2v + 2u + 1 < 0 \), we get:
\[
y(s) = -\sqrt{3(1 - x^2)} = -\frac{\sqrt{3}}{\cosh s} = -\frac{2\sqrt{3}e^s}{e^{2s} + 1}
\]
and then
\[
v_{h,0}(s) = \frac{e^s(e^s + \sqrt{3})}{2(e^{2s} + 1)}, \quad u_{h,0}(s) = -\frac{\sqrt{3}e^s + 1}{2(e^{2s} + 1)}.
\]
If, instead, we take \( y(s) = \sqrt{3(1 - x^2)} = \frac{2\sqrt{3}e^s}{e^{2s} + 1} \), we obtain the solution
\[
v_{h,1}(s) = \frac{e^s(\sqrt{3} - e^s)}{2(e^{2s} + 1)}, \quad u_{h,1}(s) = \frac{\sqrt{3}e^s - 1}{2(e^{2s} + 1)}.
\]
Note that
\[
\lim_{s \to -\infty} v_{h,i}(s) = \lim_{s \to -\infty} u_{h,i}(s) = 0 \\
\lim_{s \to \infty} v_{h,i}(s) = \lim_{s \to \infty} u_{h,i}(s) = -\frac{1}{2}
\]
for \( i = 0, 1 \). For completeness we also determine the heteroclinic connection in the line \( \ell \). Since \( 2v(2v) + 2u(s) = 1 \), we get \( \dot{v}(s) + \dot{u}(s) = 0 \) that is \( u(1 + 2u) = v(1 + 2v) \). Hence equation \([4.5]\) on the line \( \ell \) is equivalent to:
\[
\dot{v} = -v(2v + 1), \quad 2v + 2u + 1 = 0
\]
So we get
\[
v(s) = \frac{3}{2} \frac{1}{e^s + 3}, \quad u(s) = -\frac{1}{2} \frac{e^s}{e^s + 3}
\]
It is easy to check that, indeed, \((v(s), u(s))\) satisfies equation (4.5) and the following hold:

\[
- \frac{1}{2} < u(s), \quad v(s) < 0 \quad \forall t \in \mathbb{R} \\
\lim_{s \to -\infty} v(s) = \lim_{s \to -\infty} u(s) = 0 \\
\lim_{s \to -\infty} v(s) = \lim_{s \to -\infty} u(s) = -\frac{1}{2}
\]

The results of this paper apply to the heteroclinic connection

\[
\gamma_h(s) = (v_h, u_h(s))
\]

since \((v_{h,1}(s), u_{h,1}(s))\) and \((v(s), u(s))\) do not intersect the lines \(u = -\frac{1}{2}, v = -\frac{1}{2}\).

However, since \(v_{h,0}(s) = -\frac{1}{2}\) for \(s = s^*_1 := -\frac{1}{2}\ln 3\) and \(u_{h,0}(s) = -\frac{1}{2}\), for \(s = s^*_1 = \frac{1}{2}\ln 3\), we need to consider

\[
\gamma(s) := \gamma_h(s + s^*_1 - 1) = (v_0(s), u_0(s)) = \left( -\frac{e^s(e^s + 3e)}{2(e^{2s} + 3e^2)}, -\frac{3e(e^s + e)}{2(e^{2s} + 3e^2)} \right)
\]

instead of \(\gamma_h(s)\). Note that \(u_{h,0}(s^*_1) = v_{h,0}(s^*_1) = -\frac{3}{4}\).

It is easy to check that in the interval \((-\infty, 1]\), \(\gamma(s)\) satisfies assumptions (C1)-(C4) with \(x_0 = (0, u_0) = (0, -\frac{1}{2})\) and \(x_1 = (v_0, u^*) = (-\frac{1}{2}, -\frac{3}{4})\) and \(\omega(\gamma(s)) = (1 + 2v_{h,0}(s + s^*_1 - 1)) (1 + 2u_{h,0}(s + s^*_1 - 1)) < 0\) for any \(s < 1\). Indeed, for \(s < s^*_1 - 1\) we have \(-\frac{1}{2} < v_{h,0}(s) < 0, -\frac{3}{4} < u_{h,0}(s) < -\frac{1}{2}\). Next it is easy to check that

\[
\lim_{s \to -\infty} \frac{1}{s} \ln |\gamma(s) - x_0| = \lim_{s \to -\infty} \frac{1}{s} \ln |\gamma_h(s) - x_0| = 1.
\]

So far we proved that (C1)-(C3) are satisfied. As for (C4) we observe that

\[
\gamma(1) = \gamma_h(s^*_1) = \left( -\frac{3}{8}, 0 \right)
\]

and it is easy to see that

\[
\gamma_- = \left( \frac{-1}{-1} \right)
\]

so that \(\langle \nabla \omega(x_1), \gamma(1) \rangle = \frac{3}{4} > 0\) and \(\langle \nabla \omega(x_0), \gamma_- \rangle = -2 < 0\).

With reference to equation (2.6) we have, for \(s \leq 1\)

\[
\theta(s) = \int_0^s \omega(\gamma(\tau))d\tau = \int_{s^*_1}^{s + s^*_1 - 1} (1 + 2v_{h,0}(\tau))(1 + 2u_{h,0}(\tau))d\tau \\
= \int_{s^*_1}^{s + s^*_1 - 1} e^\tau (-\sqrt{3}e^{2\tau} + 4e^\tau - \sqrt{3}) d\tau \\
= \int_{s^*_1}^{s + s^*_1 - 1} \frac{e^\tau}{(e^{2\tau} + 1)^2} - \sqrt{3}\sigma^2 + 4\sigma - \sqrt{3} d\sigma \\
= \left[ -\sqrt{3}\arctan\sigma - \frac{2}{\sigma^2 + 1} \right]_{s^*_1}^{s^*_1 + \frac{\pi}{\sqrt{3}}}.
\]

Thus

\[
T_\ast = -\theta(-\infty) = \left[ \sqrt{3}\arctan\sigma + \frac{2}{\sigma^2 + 1} \right]_{\frac{-\pi}{\sqrt{3}}}^0 \\
= \frac{2}{3e^2 + 1} - \sqrt{3}\arctan\frac{1}{e\sqrt{3}} = -0.28
\]
Proposition 4.1. Consider equation (1.3) with \( C(u) = u + u^2 \), \( L(v) = v + v^2 \) and \( R(v) \), \( e(t) \) are \( C^3 \) functions. Then there exists \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in \mathbb{R} \), with \( |\varepsilon| < \varepsilon_0 \), equation (1.3) has a unique solution \((v(t), u(t))\), with \( t \in (\bar{T}, -T_*) \) and satisfying \( \lim_{t \to -T_*} (v(t), u(t)) = (0, -\frac{1}{2}) \) along with

\[
\sup_{t \in (\bar{T}, -T_*)} |v(t) - v_0(\theta^{-1}(t))| + |u(t) - u_0(\theta^{-1}(t))| \to 0, \quad \bar{T} \to T^*
\]

as \( |\varepsilon| \to 0 \).

Similarly, noting that \( \omega(\gamma_h(s)) > 0 \) for \( s \in (s_{-1}, s_1^*) \), we can apply Theorem 3.2 to \( \gamma_h(s) \), with \( s \in (s_{-1}^*, s_1^*) \). Computing:

\[
\hat{\theta}(s) = s_{-1}^* + \int_{s_{-1}^*}^{s} \omega(\gamma_h(\tau))d\tau = \frac{\pi}{2\sqrt{3}} - \sqrt{3}\arctan(e^s) + \frac{3e^{2s} - 1}{2(e^{2s} + 1)} - \frac{1}{2}\ln 3
\]

\[
\hat{\theta}(s) = s_1^* + \int_{s_1^*}^{s} \omega(\gamma_h(\tau))d\tau = \frac{\pi}{2\sqrt{3}} - \sqrt{3}\arctan(e^s) + \frac{e^{2s} - 3}{2(1 + e^{2s})} + \frac{1}{2}\ln 3
\]

\[
\hat{t}^* = \hat{\theta}(s_1^*) = 1 - \frac{\pi}{2\sqrt{3}} - \frac{1}{2}\ln 3 = -0.46
\]

\[
\hat{t}^* = \hat{\theta}(s_{-1}^*) = -1 + \frac{\pi}{2\sqrt{3}} + \frac{1}{2}\ln 3 = 0.46
\]

we obtain the following

Proposition 4.2. Consider equation (1.3) with \( C(u) = u + u^2 \), \( L(v) = v + v^2 \) and \( R(v) \), \( e(t) \) are \( C^3 \) functions. Then there exist \( \varepsilon_0 > 0 \), \( \rho_0 > 0 \) such that, for any \( (v_0, u_0) \) such that \( |v_0 + \frac{1}{2}| < \rho_0 \), \( |u_0 + \frac{1}{2}| < \rho_0 \) and any \( \varepsilon \in \mathbb{R} \) with \( |\varepsilon| < \varepsilon_0 \), there exist positive numbers \( t \to \hat{t}^* \), \( \hat{t} \to t^* \), as \( \varepsilon \to 0 \), and unique solutions \((\hat{v}(t), \hat{u}(t))\), \((\tilde{v}(t), \tilde{u}(t))\) of (1.3), defined respectively in the intervals \((\hat{t}, s_1^*)\) and \((s_{-1}^*, \tilde{t})\) satisfying

\[
\omega(\tilde{v}(t), \tilde{u}(t)) > 0 \quad \text{for } t \in (\hat{t}, s_1^*)
\]

\[
\lim_{t \to \hat{t}} \omega(\tilde{v}(t), \tilde{u}(t)) = 0, \quad \lim_{t \to s_{-1}^*} (\tilde{v}(t), \tilde{u}(t)) = (v_0, -\frac{1}{2})
\]

\[
\sup_{t \in (\hat{t}, s_1^*)} |\tilde{v}(t) - v_{h,0}(\theta^{-1}(t))| + |\tilde{u}(t) - u_{h,0}(\theta^{-1}(t))| \to 0 \quad \text{as } \varepsilon \to 0
\]

and

\[
\omega(\hat{v}(t), \hat{u}(t)) > 0 \quad \text{for } t \in (s_{-1}^*, \tilde{t})
\]

\[
\lim_{t \to \tilde{t}} \omega(\hat{v}(t), \hat{u}(t)) = 0, \quad \lim_{t \to s_{-1}^*} (\hat{v}(t), \hat{u}(t)) = (-\frac{1}{2}, u_0)
\]

\[
\sup_{t \in (s_{-1}^*, \tilde{t})} |\hat{v}(t) - v_{h,0}(\theta^{-1}(t))| + |\hat{u}(t) - u_{h,0}(\theta^{-1}(t))| \to 0 \quad \text{as } \varepsilon \to 0
\]

4.2. Second example. In this second example, we take \( C(u) = h(u) \), \( L(v) = 2v - h(v) \) for some \( C^3 \) convex function \( h(z) \) having two simple zeros \( z = 0 \) and \( z = \bar{z} < 0 \) and \( 0 < h'(0) < 2 \). Hence there exists \( u_0 \in (\bar{z}, 0) \) such that \( h'(u_0) = 0 \). We also assume the existence of \( v_0 > 0 \) such that \( h'(v_0) = 2 \). Since \( C'(u) = h'(u) \), \( L'(v) = 2 - h'(v) \) we get \( C'(u_0) = 0 \) and \( L'(v_0) = 0 \). It is easy to check that \((0, 0), \ldots\)
(v₀, u₀) are centre and (0, u₀), (v₀, 0) are saddle. The unperturbed equation (1.5) reads:
\[ \begin{align*}
\dot{v} &= -h'(u)u, \\
\dot{u} &= (2 - h'(v))v
\end{align*} \quad (4.6) \]
and has the Hamiltonian:
\[ H(v, u) := v^2 + \int_u^0 \text{sh}'(s)ds. \]

Note that:
\[ \begin{align*}
H(0, u_0) &= \int_0^{u_0} \text{sh}'(s)ds = -\int_{u_0}^0 \text{sh}'(s)ds \\
H(v_0, 0) &= v_0^2 + \int_0^{v_0} \text{sh}'(s)ds = v_0^2 - \int_0^{v_0} \text{sh}'(s)ds.
\end{align*} \]

So a heteroclinic connection between (0, u₀) and (v₀, 0) exists if and only if
\[ v_0^2 - \int_0^{v_0} \text{sh}'(s)ds + \int_0^{u_0} \text{sh}'(s)ds = 0. \]

For definiteness take again \( h(z) = z + z^2 \) so that \( C(u) = u + u^2 \) and \( L(v) = v - v^2 \). Then \( L'(v)C'(u) = (1 + 2u)(1 - 2v) \), \( u_0 = -\frac{1}{2} \), \( v_0 = \frac{1}{2} \). The unperturbed system
\[ \begin{align*}
\dot{v} &= -u(1 + 2u), \\
\dot{u} &= v(1 - 2v)
\end{align*} \quad (4.7) \]
has the Hamiltonian \( H(v, u) = \frac{1}{2} (v^2 + u^2) - \frac{1}{3} (v^3 - u^3) \). Since
\[ H \left( \frac{1}{2}, 0 \right) = H \left( 0, -\frac{1}{2} \right) = \frac{1}{24}, \]
the heteroclinic connection between the two saddles satisfies the equation:
\[ \frac{v^2}{2} - \frac{2}{3}v^3 + \frac{u^2}{2} + \frac{2}{3}u^3 - \frac{1}{24} = 0 \]

But:
\[ \frac{v^2}{2} - \frac{2}{3}v^3 + \frac{u^2}{2} + \frac{2}{3}u^3 - \frac{1}{24} = \frac{1}{24} (2u - 2v + 1)(8u^2 + 8uv + 8v^2 + 2u - 2v - 1) \]
thus there are three heteroclinic connections between \( e_2 = \left( \frac{1}{2}, 0 \right) \) and \( e_3 = \left( 0, -\frac{1}{2} \right) \), one in the invariant line \( 2v - 2u = 1 \) and the other two in the ellipsis \( 8u^2 + 8uv + 8v^2 + 2u - 2v - 1 \).

Arguing as in the previous example, we see that the orbit in the line \( 2v - 2u = 1 \) is:
\[ \begin{align*}
v(s) &= \frac{1}{2} e^{\frac{s}{2}}, \\
u(s) &= -\frac{1}{2} e^{\frac{s}{2}} + 1
\end{align*} \]
and the two orbits lying in the ellipsis \( 8u^2 + 8uv + 8v^2 + 2u - 2v - 1 = 0 \) are:
\[ \begin{align*}
v_{h, \pm}(s) &= \frac{1 \pm e^{s\sqrt{3}}}{2(1 + e^{2s})}, \\
u_{h, \pm}(s) &= -\frac{e^{s}(e^{s} \pm \sqrt{3})}{2(e^{2s} + 1)}
\end{align*} \]
but only \( \gamma_h(s) := (v_{h, +}(s), u_{h, +}(s)) \) intersects the set \( S = \{ \omega(v, u) = 0 \} \), precisely at the points:
\[ \left( \frac{1}{2}, -\frac{3}{4} \right) \quad \text{and} \quad \left( \frac{3}{4}, -\frac{1}{2} \right) \]
corresponding to $s = \frac{1}{2} \ln 3$ and $s = -\frac{1}{2} \ln 3$, respectively. Note that, in this case \( \gamma_h(s) \rightarrow e_3 \) as $s \rightarrow \infty$, so, in order to apply Theorems 2.3 and 3.1 we change $s$ with $-s$ in (4.5), that is, instead of (4.3) we consider the equivalent equation:

\[
(2v - 1)v' = u + \varepsilon R(v) - \varepsilon e(t)
\]

\[
(1 + 2u)u' = v
\]

and the associated equation

\[
\dot{v} = u(1 + 2u)
\]

\[
\dot{u} = v(2v - 1)
\]

has the solution $\gamma_h(-s)$ tending to $e_3$ as $s \rightarrow -\infty$ and intersecting $S$ at the time $s_{-1}^* = -\frac{1}{2} \ln 3$ (at the point \( \left( \frac{1}{2}, -\frac{1}{2} \right) \)) and $s_1^* = \frac{1}{2} \ln 3$ (at the point \( \left( \frac{1}{2}, -\frac{1}{2} \right) \)). Then $\omega(v, u) = (2v - 1)(1 + 2u)$ and it is easy to check that

\[
\omega(\gamma_h(-s)) > 0 \quad \text{for} \quad s < s_{-1}^*
\]

\[
\omega(\gamma_h(-s)) < 0 \quad \text{for} \quad s_{-1}^* < s < s_1^*
\]

As in the first example, we apply Theorem 2.3 to the function

\[
\gamma(s) := \gamma_h(1 - s_{-1}^* - s) = (v_0(s), u_0(s))
\]

\[
= \left( e^s(e^s + 3e) - 3e(e^s + e) \right) \left( 2(e^{2s} + 3e^2) - 2(e^{2s} + 3e^2) \right), \quad s \leq 1.
\]

It is simple to check that $\gamma(s)$ verifies assumptions (C1)--(C4) and that $\gamma_h(-s)$ satisfies (H1)--H(3) for $s_{-1}^* < s < s_1^*$, apart from the sign of $\omega(\gamma_h(-s))$ that is negative in this interval. We get:

\[
\theta(s) = \int_0^s \omega(\gamma(\tau))d\tau = \int_{1-s-s_{-1}^*}^{1-s_{-1}^*} \omega(\gamma_h(\tau))d\tau =
\]
\[
\int_{e^{1-s^*}}^{e^s} \frac{e^\tau (\sqrt{3} e^{2\tau} - 4e^\tau + \sqrt{3})}{(e^{2\tau} + 1)^2} d\tau = \\
\int_{e^{1-s^*}}^{e^s} \frac{\sqrt{3} \sigma^2 - 4\sigma + \sqrt{3}}{(\sigma^2 + 1)^2} d\sigma = \int_{e^{1-s^*}}^{e^s} \left( \frac{\sqrt{3}}{\sigma^2 + 1} - \frac{4\sigma}{(\sigma^2 + 1)^2} \right) d\sigma \\
= \frac{\sqrt{3}}{\sigma} \left[ \arctan(e\sqrt{3}) - \arctan(e^{-s^*}\sqrt{3}) \right] - 2 \frac{e^2(1 - e^{2s})}{3e^2 + 1} + \frac{2}{3e^2 + 1}
\]

and
\[
T_* = -\theta(-\infty) = \sqrt{3} \left[ \frac{\pi}{3} - \arctan(e\sqrt{3}) \right] - \frac{2}{3e^2 + 1} \approx 0.28 \\
T^* = \theta(1) = \sqrt{3} \arctan \left( \frac{3(1 - e^2)}{2(3e^2 + 1)} \right) \approx 0.13
\]

Then, from Theorem 2.3 we obtain the following

**Proposition 4.3.** Consider equation (1.3) with \( C(u) = u + u^2 \), \( L(v) = v - v^2 \) and \( R(v), e(t) \) are \( C^3 \) functions. Then there exist \( \varepsilon_0 > 0 \) and a positive number \( \bar{T} \to T^* \) as \( \varepsilon \to 0 \), such that for any \( \varepsilon \in \mathbb{R} \), with \( |\varepsilon| < \varepsilon_0 \), (1.3) has a unique solution \( (v(t), u(t)) \), defined for \( t \in (-T_*, \bar{T}) \) and satisfying
\[
\lim_{t \to T^*} |v(t)| + |2u(t) + 1| = 0 \\
\sup_{t \in (-T_*, \bar{T})} |v(t) - v_0(\sigma^{-1}(t)) + |u(t) - u_0(\sigma^{-1}(t))| \to 0
\]
as \( \varepsilon \to 0 \).

Next, we apply Theorems 3.1–3.2 and Remark 3.3 to equation (1.3), with \( \gamma(s) = \gamma_h(-s) \), \( s^*_1 \leq s \leq s^*_1 \). To this end, we set
\[
\tilde{\theta}(s) = s^*_1 + \int_{s^*_1}^s \omega(\gamma_h(-\tau)) d\tau = -\sqrt{3} \arctan(e^{-s}) + \frac{1 - 3e^{2s}}{2(1 + e^{2s})} \\
+ \frac{\pi}{\sqrt{3}} - \frac{1}{2} \ln 3 \\
\tilde{\theta}(s) = s^*_1 + \int_{s^*_1}^s \omega(\gamma_h(-\tau)) d\tau = -\sqrt{3} \arctan(e^{-s}) + \frac{3 - e^{2s}}{2(1 + e^{2s})} \\
+ \frac{\pi}{2\sqrt{3}} + \frac{1}{2} \ln 3 \\
\tilde{\tau} = \tilde{\theta}(s^*_1) = -1 - \frac{1}{2} \ln 3 + \frac{\pi}{2\sqrt{3}} \approx -0.64 \\
\tilde{\tau} = \tilde{\theta}(s^*_1) = 1 + \frac{1}{2} \ln 3 - \frac{\pi}{2\sqrt{3}} \approx 0.64
\]

Then from Theorems 3.1 and 3.2 we get:

**Proposition 4.4.** Consider equation (1.3) with \( C(u) = u + u^2 \), \( L(v) = v - v^2 \) and \( R(v), e(t) \) are \( C^3 \) functions. Then there exist \( \varepsilon_0 > 0 \), \( \rho_0 > 0 \) such that, for any \( (v_0, u_0) \) such that \( |v_0 - \frac{3}{4}| < \rho_0 \), \( |u_0 + \frac{3}{4}| < \rho_0 \) and any \( \varepsilon \in \mathbb{R} \) with \( |\varepsilon| < \varepsilon_0 \), there exist positive numbers \( \tilde{t} \to \tilde{t}^* \), \( \tilde{t} \to \tilde{t}^* \), as \( \varepsilon \to 0 \), and unique solutions \( (\tilde{v}(t), \tilde{u}(t)) \),
$(\tilde{v}(t), \tilde{u}(t))$ of $[1,3]$, defined respectively in the intervals $(s_1^*, \tilde{t})$ and $(\tilde{t}, s_{-1}^*)$ satisfying

$$\omega(\tilde{v}(t), \tilde{u}(t)) < 0 \quad \text{for} \quad t \in (s_1^*, \tilde{t})$$

$$\lim_{t \to \tilde{t}} \omega(\tilde{v}(t), \tilde{u}(t)) = 0, \quad \lim_{t \to s_1^*} (\tilde{v}(t), \tilde{u}(t)) = (v_0, -\frac{1}{2})$$

$$\sup_{t \in (s_1^*, \tilde{t})} |\tilde{v}(t) - v_{h,+}(-\tilde{\theta}^{-1}(t))| + |\tilde{u}(t) - u_{h,+}(-\tilde{\theta}^{-1}(t))| \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and

$$\omega(\tilde{v}(t), \tilde{u}(t)) < 0 \quad \text{for} \quad t \in (\tilde{t}, s_{-1}^*)$$

$$\lim_{t \to \tilde{t}} \omega(\tilde{v}(t), \tilde{u}(t)) = 0, \quad \lim_{t \to s_{-1}^*} (\tilde{v}(t), \tilde{u}(t)) = (\frac{1}{2}, u_0)$$

$$\sup_{t \in (\tilde{t}, s_{-1}^*)} |\tilde{v}(t) - v_{h,+}(-\tilde{\theta}^{-1}(t))| + |\tilde{u}(t) - u_{h,+}(-\tilde{\theta}^{-1}(t))| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

REFERENCES

[1] F. Battelli and M. Fečkan, Nonlinear RLC circuits and implicit ODEs, *Differential Integral Equations*, 27 (2014), 671–690.

[2] F. Battelli and M. Fečkan, Melnikov theory for nonlinear implicit ODEs, *J. Differential Equations*, 256 (2014), 1157–1190.

[3] F. Battelli and M. Fečkan, Melnikov theory for weakly coupled nonlinear RLC circuits, *Bound. Value Probl.*, 2014:101 (2014), 27pp.

[4] F. Battelli and M. Fečkan, On the existence of solutions connecting singularities in nonlinear RLC circuits, *Nonlinear Anal.*, 116 (2015), 26–36.

[5] L. O. Chua, Ch. A. Desoer and E. S. Kuh, *Linear and Nonlinear Circuits*, McGraw-Hill, New York, 1987.

[6] E. Gluskin, A nonlinear resistor and nonlinear inductor using a nonlinear capacitor, *J. Franklin Inst.*, 336 (1999), 1035–1047.

[7] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations, Analysis and Numerical Solution*, European Math. Soc. 2006.

[8] N. Lazarides, M. Eleftheriou and G. P. Tsironis, Discrete breathers in nonlinear magnetic metamaterials, *Phys. Rev. Lett.*, 97 (2006), 157406.

[9] M. Medved’, Normal forms of implicit and observed implicit differential equations, *Riv. Mat. Pura ed Appl.*, 10 (1991), 95–107.

[10] M. Medved’, Qualitative properties of generalized vector fields, *Riv. Mat. Pura ed Appl.*, 15 (1994), 7–31.

[11] P. J. Rabier and W. C. Rheinboldt, A general existence and uniqueness theorem for implicit differential algebraic equations, *Differential Integral Equations*, 4 (1991), 563–582.

[12] P. J. Rabier and W. C. Rheinboldt, A geometric treatment of implicit differential-algebraic equations *J. Differential Equations*, 109 (1994), 110–146.

[13] P. J. Rabier and W. C. Rheinboldt, On impasse points of quasilinear differential algebraic equations, *J. Math. Anal. Appl.*, 181 (1994), 429–454.

[14] P. J. Rabier and W. C. Rheinboldt, On the computation of impasse points of quasilinear differential algebraic equations, *Math. Comp.*, 62 (1994), 133–154.

[15] Rinza, *Differential-Algebraic Systems, Analytical Aspects and Circuit Applications*, World Scien. Publ. Co. Pte. Ltd.; 2008.

[16] G. P. Veldes, J. Cuevas, P. G. Kevrekidis and D. J. Frantzeskakis, Quasidiscrete microwave solitons in a split-ring-resonator-based left-handed coplanar waveguide, *Phys. Rev. E*, 83 (2011), 046608.

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