On the classification of the laminar-turbulent transition process using the methods of nonlinear dynamics: general analysis and the future

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Abstract The definition of a turbulent flow is still not mathematically defined. Different methods exist that provide different definitions resulting in different approaches and classifications of the laminar-turbulent transition as well as the fully developed turbulent flows. The paper includes the analysis of different stages of turbulence development, specific methods that are used in each stage and possible methods of unification of these methods based on the nonlinear-dynamical approach. Results are demonstrated for some characteristic fluid dynamics problems. Future highlights on the path to possible unification are discussed.

Introduction

This paper tries to generalize the approach to laminar-turbulent transition (LTT) process in the wake of the nonlinear dynamical approach. The definition of the laminar-turbulent transition is a difficult one. There's no mathematical definition of turbulence, so the definition is physical in most cases and varies depending on the problem being considered. Note, that the cited literature serves as examples rather than the literature overview in detail.

For example, the typical definition of laminar-turbulent transition for physical problems are related to the statistical properties of the developed turbulence. The transitional process is understood differently. Research groups that are mainly focused on the physical problems in engineering (e.g. flow over the blunt bodies, overflow over weirs, channels etc) can understand the process of transition as the process of the turbulence development in the streamwise flow direction for the fixed flow parameters. In this case the definition of the laminar-turbulent process is given in terms of boundary layer separation and further development of the flow with hydrodynamic variables being governed by random process with certain properties. The Reynolds number is assumed to be fixed for such problems. For example, the definition in [1] and other papers by Rozhdestvenskii and co-authors define turbulence as flows subjected to pulsations of hydrodynamical variables and otherwise called laminar. There are two states which a flow can take. In means that the the intermittent part of the process is neglected, but such definition seems true from the engineering point of view.

Other research groups that are mainly focused on the mechanics of the flows use the definition of the laminar-turbulent transition in terms of the Reynolds number variation (or other Π-theorem complexes). In this case the first problem is the stability analysis of the main flow that can be treated analytically [3] or numerically [4]. The most important parameter obtained in the problem is the
critical Reynolds number at which the flow looses stability (either through linear or non-linear process). For example the combination of analytical reduction of equations and numerical methods allowed the authors in [4] to obtain critical Reynolds numbers for complex wall shapes. In this case the laminar-turbulent transition is understood as the stability loss of the main flow. The second group of definitions for the laminar-turbulent transition are related to the teams which are dealing with near-wall turbulent flows like Couette, Taylor-Couette and Poiseuille flows. Such flows are known to be stable for infinitesimal perturbations either for the whole interval of Reynolds numbers or for a much wider period, then the actual instability takes place. In these cases the transitional process is related to the nonlinear amplification of finite most dangerous perturbations. For example in [5,6], for the Poiseuille pipe flow, the process of laminar-turbulent transition can be characterized as the process of the development of coherent turbulent structures that are self-sustained (starting from some Reynolds number), which later evolve into the developed turbulence as the Reynolds number increases. The developed turbulence is understood in the statistical sense. In papers [6,7] the authors analyze the Poiseuille flow in the transitional regime on the phase space separatrix (between base flow and unstable turbulent streak) and show the structure of this transitional solution. Physically, authors show that the sustaining process is due to the lift-up effect of perturbations being transported from walls to the center of the pipe. The further process of the turbulence development is described from statistical and spectral points of view, e.g. [8].

The definition of laminar-turbulent transition is different for flows which are triggered by buoyancy or shear layer instability, such as Rayleigh-Taylor, Kelvin-Helmholtz, Richtmayer-Meshkov instabilities and others. In this case it is difficult to select unique parameter dependence (Atwood number and Raynolds number) since the flow without external energy deposition cannot reach quasistationary state, i.e. the Reynolds averaging for the fixed parameter value is not applicable. In this case one usually measures evolution of statistical properties (e.g. shear layer width or mixing length scales) as functions of time. These flows are known to posses the property of self-similarity, namely, for some time, grater than the threshold, the flow may forget its initial conditions and enter a self-similar growth phase.

The laminar-turbulent transition in natural convection, convection in binary mixtures, Rayleigh-Benard convection [9], are the classic problems. These problems are usually studied in terms of the Rayleigh number dependence on the instability. The dynamical system approach was the first to be extensively used in these flows since the famous Lorenz system was derived as the 0-th order Galerkin approximation of the Rayleigh-Benard convection [10]. These problems are known to form complex patterns as the Rayleigh number increases [11].

Another well known property is that the turbulence flow reaches the self-similar limit w.r.t. the Reynolds number as soon as it’s value passes over a certain point (problem dependent). For this well developed turbulence only statistical methods exist that are based either on the Kolmogorov-like models [12] or the topological models which are being developed by V.I. Arnold and co-authors [13], where the hydrodynamic flow is considered as the Hamiltonian group structure. In this case the well known KAM (Kolmogorov–Arnold–Moser)-theory [14] can be applied to the problem at question. However this approach is only developing.

Another numerical approach which can be applied to the study of the developed turbulence is the modal analysis (It is closely related to the dynamical system approach). This approach includes the following methods: the proper orthogonal dsecomposition (POD) method [15], Dynamic Mode Decomposition (DMD) method and Koopman analysis [16] method. The former determines the optimal set of modes to represent numerically or experimentally obtained flow data based on the $L_2$ norm. The DMD method captures dynamic modes with associated growth rates and frequencies which is basically a linear approximation to the nonlinear dynamics. The Koopman method is the infinite dimensional generalization of the DMD, introducing linear infinite dimensional operator. These methods can be used as a substantial addition to the nonlinear dynamical system analysis.

There are enormous various definitions of laminar-turbulence transition process. The author wishes to stress that all of the considered problems above can be recast into the definition in terms of the
nonlinear dynamical system approach if the right methods are applied and the conjecture of the inertial manifold existence [17,18] for the 3D Navier-Stokes equations is assumed. For further reading on the mathematical questions regarding 3D Navier-Stokes equations, regularities and smooth solution existence we refer reader to the review in [19]. We also state that we refer to turbulence only in connection with the 3D spatial domains under consideration and call solutions, obtained in 2D spatial domains as chaotic. We wish to stress that these methods, being modern and relatively new, can be used as addition to the nonlinear dynamical system analysis.

Here we would like to encompass the mentioned above definitions and approaches and outline the general laminar turbulent transition process by four stages:

- Stability loss of the main solution. In this case the basic laminar flow regime switches to the other (possibly laminar) flow regimes. Examples of such stability loss with smooth regime change (supercritical bifurcation) are the classical Rayleigh-Benard convection and the 2D Kolmogorov flow problems. However, the stability loss can be triggered also by the subcritical bifurcation. In this case the flow regime changes suddenly and abruptly. The classical theory of hydrodynamic stability studies these flows analysing energy and linear stability as well as nonlinear transition process. Examples of such flows are the 3D Kolmogorov flow, Couette flow and Poissele flow problems. From the dynamical system point of view the process of subcritical bifurcation and multistability is well understood as will be demonstrated later.

- Multiple bifurcations of the secondary flows. In this stage the flow exhibits multiple regime changes which can be characterised by low order dynamics (invariant cycles and tori in the phase space). Such flow regimes can be characterised by a complex behaviour in the physical space despite the fact that corresponding phase space trajectories belong to the low dimensional attractor.

- Crisis of particular flow regimes. This stage is characterised by a sudden growth of the solution dimension in the phase space. An analysis of this situation for the Rayleigh-Benard convection is provided in paper [20], where the Lameray diagram is constructed for the crisis on the invariant 2D torus in the phase space. In this stage noise intermittency is frequent, since the phase flow trajectories become intangled on the attractor and small (but finite) amplitude perturbations can trigger jumps between different singular solutions. This stage can also contain global bifurcations which include homoclinic and heteroclinic bifurcations.

- Statistical turbulence. This stage is the least studied in the terms of dynamical system approach. The flow regimes are fully turbulent with usually observed intermittency between different chaotic regimes. It was noticed, however, that there's a direct correlation between the number of unstable cycles being found for a particular Reynolds number and the turbulence energy cascade [21]. For example, a more chaotic regime corresponds to a larger dimension of the unstable manifold w.r.t. the periodic trajectories in the phase flow.

We assume that all these stages can be analized and understood from the dynamical system point of view.

The paper contains short overview of the results, obtained and still being processed.

General approach

Assume that the infinite dimensional system is reduced to the finite dimensional under the assumption of the approximate inertial manifold existence [17,18,22]. In this case the following procedure is suggested:

- Obtain the necessary number of degrees of freedom to encompass the maximum attractor, e.g. by using the method in [23]. If the conjecture of the existence of the inertial manifold is assumed, then there exists an approximate inertial manifold and then the maximum attractor [24] is contained in it.

- Select the range of Reynolds numbers and estimate the influence of the high frequency harmonics on the finite dimensional system.

- Perform numerical simulation or other procedures on the discrete finite set of coefficients while controlling the amplitude of high frequency harmonics of the obtained solutions.
As the result one obtains the numerical method that approximates inertial manifold and, thus, contains the maximum attractor.

The analysis includes construction of bifurcation diagrams and schemes as well as analysis of low dimensional attractors evolution for near turbulent flow regimes.

In order to perform the above-mentioned analysis one must perform the following procedure: find all stationary, periodic and quasi-periodic (up to maximum possible dimension) solutions, determine their type and stability, describe their evolution by locating bifurcation points and describing type up until one reaches turbulent regimes, where regular attractors can no longer be traced. At these regimes one must continue the solutions form laminar to turbulent regimes. Or, if these solutions contain turning bifurcations, one must deflate the unstable regular attractors for the turbulent regime. The allows one to determine the influence of regular low dimensional attractors on the turbulent flow as well as describe the transitional process that undergoes from regular attractors to turbulence.

Many books and papers exist that are related to the problems of numerical bifurcation analysis [25,26]. However, applying these methods to the large scale systems is a difficult task which requires the change of many approaches common for the low-dimensional bifurcation analysis [27]. Review of some applicable methods of bifurcation analysis for the Navier-Stokes systems is give in the review paper [28]. Here we describe the methods which are developed and applied by the author for this problem, the review of the papers related to these problems are given in author's cited papers. First, the solutions are to be found for a particular value of the bifurcation parameter. It is done by using the deflation method [29] which is adopted for large scale problem [30] and multiple GPU architecture. This method allows one to find distinct, even disconnected, solutions. Continuation of the found solutions is performed using the pseudo arc-length continuation process which is explained in [21,30]. It allows one to follow the bifurcation branch in the parameter space, thus constructing the curve of the topologically equivalent phase protraits. The bifurcation diagram is constructed by continuing each deflated solution in the parameter space using pseudo arc-length continuation. The problem of bifurcation point detection and their classification on stationary and periodic solutions is treated by the solution of the leading eigenvalue problems for the linear operator of the original system of equations. More details on the particular methods used in authors research are available in [31, 32].

One can observe that in order to solve all these problems one faces with two most common but most difficult problems in numerical algebra: solving linear systems and finding leading eigenvalues of large poorly conditioned linear operators which are sometimes not available in explicit matrix form. Unfortunately there is no unique recipe in general case and each problem being considered has it's own approach to the solution of these problems which are beyond the scope of this paper. Please, see [31-34] for more information.

All these methods allow one to construct the bifurcation diagrams and schemes for particular problems. Some problems were investigated and the results are presented bellow. The analysis of the developed turbulent flows from the nonlinear dynamical point of view is difficult and contains no particular applied mathematical methods. The main basis for such analysis is the influence of unstable regular attractors on the flow. This analysis is performed in terms of phase-flow recurrence near unstable stationary and periodic solutions. This approach is now being developed and is currently under analysis. Some results are presented bellow.

Results
Governing equations and problem formulation for all considered problems can be written in the following general explicit form.

Problem 1
Let the piecewise Lipschitz domain \( \Omega \in \mathbb{R}^3 \) with boundary \( \partial \Omega \) is given as well as vector-function \( g \), number \( T \) and initial conditions \( \rho_0, u_0, p_0 \). For the given range of parameters \( \mu_1 \leq \mu \leq \mu_2 \) find the velocity vector-function \( u: \Omega \times (0,T] \to \mathbb{R}^3 \) and scalar functions of density \( \rho: \Omega \times (0,T] \to \mathbb{R} \), pressure \( p: \Omega \times (0,T] \to \mathbb{R} \) and energy \( E: \Omega \times (0,T] \to \mathbb{R} \), that satisfy the Navier-Stokes equations:
Perform numerical bifurcation analysis of these solutions using chaotic dynamic methods. Construct bifurcation diagrams and schemes until the solution reaches chaotic regimes. Analyse the influence of unstable solutions at turbulent regimes.

This problem encompasses both compressible and incompressible flows. For the latter case the density is assumed constant hence the first equation is reduced to $\nabla \cdot u = 0$, energy conservation equation becomes an identity and the pressure becomes a gauge that insures incompressibility. In order to bring the system (1) to the abstract form one uses the divergence free projection operator which is constructed as $\text{id} \cdot \nabla \cdot u$. Then the pressure is eliminated from the system. This projection can be done in several steps, see [20,35] for examples. Let us present some new results for the considered problems. The overview of the previous results can be found in [36].

$\text{• 3D Kolmogorov flow problem, multistability of stationary solutions.}$ The domain is defined as a 3D stretched torus $\Omega = [-\pi/\alpha, \pi/\alpha] \times [-\pi, \pi] \times [-\pi, \pi]$, where $\alpha \in R^+$ is a stretching factor. The flow is sustained by the forcing term $g = (\sin(y), \cos(z), 0, 0)^T$.

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} \left[ \rho u_j \right] = 0
\]
\[
\frac{\partial}{\partial t} (\rho u_i) + \frac{\partial}{\partial x_j} \left[ \rho u_i u_j + p \delta_{ij} - 2 \mu S_{ij} \right] = \rho g_i, \quad \{i, j, k\} = 1, 2, 3;
\]
\[
\frac{\partial}{\partial t} (\rho E) + \frac{\partial}{\partial x_j} \left[ \rho u_j E + u_j p - 2 \mu u_i S_{ij} \right] = \rho u_k g_k,
\]
\[
\rho E = \frac{1}{2} \rho u^2 + \rho e, \quad \rho = (\gamma - 1)(\rho E - 1/2 \rho u^2).
\]
Figure 2. Square modulus of the velocity vector of the stable stationary main solution at $\lambda = 20.2$
Square modulus of the velocity vector of one of the stationary alternative stable solutions at $\lambda = 20$.

The bifurcation diagram of disconnected solutions in shown for the Reynolds number in the interval [1;22] for $\alpha = 1$ in Figure 1. It is known that at this stretching factor value the transitional process undergoes subcritical cascade of bifurcations [21]. This bifurcation diagram demonstrates an example of multistable behaviour: there are five stationary stable solutions for the Reynolds number in the parameter interval [16.7;20.67] - the main analytical solution and four solutions that are continued from four saddle-node bifurcations at $\lambda = 13.2475$, see Figure 1. These solutions are disconnected from the main analytical solution branch, at least at the considered parameter interval, and cannot be obtained without the deflation process.

Example of the main solution at $\lambda = 20$ is presented in Figure 2 (left) and example of the alternative stable solution for the same parameter value is presented in Figure 3 (right). If one starts simulation of a time dependent problem at the Reynolds number 20 from some perturbed initial conditions, one can end up converging either to the main solution or to one of the alternative solutions depending on the perturbation magnitude. This example demonstrates multistable behaviour of the Navier-Stokes attractor i.e. there are regions where multiple stable solutions co-exist i.e. the observing attractor is the maximum attractor. Hence, it is unlikely that a global attractor exists for the 3D Navier-Stokes equations.

- 3D Kolmogorov flow, formation of singular limited cycles. The same problem for the $\alpha = 1/2$. This is a stretched domain with a rich low dimensional dynamics that was analysed in [21,34], however these results were not published due to the paper size limitations. It was possible to trace the formation of the singular attractors with fractal dimension. One of the simplest singular attractors at Raynolds number 8.67 was considered. This attractor was selected on the basis of least number of excited degrees of freedom. The limited cycle corresponded to the flow regime in question is provided in Figure 4 and physical flow at different intervals on this cycle are presented in Figure 3.

First we find the basis of attraction of the current attractor. We select an arbitrary section and put the section hyperplane ($N - 1$ dimensional, where $N$ - number of degrees of freedom (DOF), represented by Fourier coefficients) transversal to the trajectory with respect to the right hand side vector of the system. We run computation until 100 intersection points are obtained. Intersection is performed using bisections, tolerance is set to $1 \cdot 10^{-12}$. Next, we solve the dimension reduction problem by using a linear regression and obtain all points in the hyperplane as the linear function of
two selected vectors. Then, the initial rectangle is defined (see 5, top, left) in coordinates of these two selected vectors. We execute our calculations again using this $N-1$ dimensional rectangle set $B$ as the initial condition for the system. Then we find all trajectories intersections with the hyperplane and analyse all return maps.

Figure 3. Iso-surfaces of the velocity vector magnitude at different time intervals for the Reynolds number 8.67 on a singular periodic orbit for scratched domain $\alpha = 1/2$. 3

Figure 4. Singular cycle projection at the Reynolds number 8.67 and Poincare section. 4
First observe, that our set $B$ is not optimal. The first mapping $P(B)$ just turns the domain, scratches it in one direction (collinear with vector $(0.95)$ on the set $B$) and squishes in another direction (collinear with vector $(0,4)$ on the set $B$) and bends it. The final set $B^1 := P(B)$ is now located on the attractor and aligned with the phase flow. Such action is very close to an 1D discrete mapping.

We fix three point $(1,2,3)$ on the $B^1$ (designated with circles in Figure 5) and observe the action of the mapping in the neighbourhood of these points for $P^2(B)$ and $P^3(B)$:

\begin{align}
1 & \mapsto 2, \\
2 & \mapsto 3, \\
3 & \mapsto 1.
\end{align}

This mapping is common to the cycle period 3, but in this case we are considering a singular cycle. We can trace the formation sequence of $B^{n-1} \mapsto B^n$ by the numbers which are used to designate the initial set. First, the domain is stretched in the direction, aligned with the phase flow. This can be seen by the analysis of numbers in the upper left corner (sequence $0,5,10,\ldots$). The stretching is performed in the direction $1 \mapsto 2$ and $3 \mapsto 1$. Then, the folding is executed such, that all points in the neighbourhood of the point 2 (the upper right corner) are mapped to the neighbourhood of the point 3. Formally, we cannot state that such construction is the Smale horseshoe, however the action of this Poincare map resembles the horseshoe action of stretching, squishing and bending. Also, this low dimensional behaviour resembles unimodal mapping on a chaotic attractor in a discrete dynamic system, based on a cycle period 3. The latter implies chaos by the Sharkovsky theorem. Thus an existence of the chaotic low dimensional dynamics is clear in this example. This is a good illustration of the second stage of the LTT, where secondary solutions undergo cascades of bifurcations. Thus, the flow in Figure 3 is, basically, a generalization of the horseshoe map on the Navier-Stokes equations.

- **3D Rayleigh-Benard convection, crisis of the quasi-periodic solution.** The results are partly provided in [20]. The Oberbeck-Boussinesq approximation is used to describe the flow. The domain is $[0,4] \times [0,4] \times [0,1]$ with temperature values and zero velocity values are prescribed on boundaries which are orthogonal to Z axis, the gravity vector is co-linear with the Z axis. Other boundaries are periodic. The third stage of the LTT is observed for Rayleigh number round 8351 – 8383 and a unit Prandtl number for a particular non-symmetric solution branch which is formed after the symmetry break of the main solution. The corresponding physical flow and phase space projections are presented in Figures 6, 7 and 8.
Figure 6. Temperature distribution for Rayleigh number 8300 (left) and 8400 (right) that corresponds to the invariant torus (left) and to the singular torus (right) in the phase space.

Figure 7. Sections of the 2D invariant torus by 2D planes in the phase subspace for Rayleigh number 8381.
Figure 8. Crisis on the 2D invariant torus (one 2D section shown). Attractor dimension (dim) 2 for Ra 8351, dim 2 for Ra 8377, dim 2.05 for Ra 8381, dim 2.1 for Ra 8382, dim 2.24 for 8383.

The fractal dimension values provided in the Figure 8 are estimated by the fractal box tool. The evolution of the crisis is due to the dimension increase of the invariant torus which is formed topologically by the tensor product of two limited cycles, where one of them is regular and the other is singular. The singular cycle undergoes the phase lock followed by the formation of the Smale's horseshoe attractor on the invariant torus. After that the trajectories around the second periodic orbit become chaotic. This leads to the abrupt attractor dimension increase and formation of the chaotic flow in the phase space. Resulting change in the temperature distribution can be noticed in the provided figures. This change happens in a narrow parameter window exactly due to the observed crisis.

• Unstable periodic orbits embedded into the developed turbulence. The analysis of the forth stage of the LTT have being, basically, never discussed through the dynamical system approach, except by the topological methods, see the review [37]. We also only start our research in this area. The main difficulty is that the standard and more or less tested methods of nonlinear dynamics don't work for fully chaotic solutions. The first steps in this field is taken by Chandler and Kerswell [38], where the behaviour of recurrent solutions embedded in a chaotic two-dimensional Kolmogorov flow is studied. The recurrent flow represents a sustained closed cycle of dynamical processes which underpins the turbulent solution. In such way the authors try to link the unstable manifold dimension estimated through the analysis of unstable periodic orbits (UPO) with the developed turbulence behaviour. It is done by computing UPO with the Krylov-Newton-Raphson's method while the chaotic 2D turbulence is calculated using the DNS solver. The Newton's method is based on the estimation of the near recurrent flows by checking the approximate invariance of the applied symmetry group on the flow. If such solutions are found then those are subject to converge attempts by the extended Newton-Raphson's method where extension of the original vector is done by the parameters of the period under iterations. For the 2D Kolmogorov flow and Reynolds number around 100 it was shown that about 65 recurrent flows were extracted using this technique. The developed turbulence from the DNS data is
analysed using the estimation of the probability distribution function (PDF) which is depicted in the same coordinates as the converged recurrent periodic cycles. Presented results indicate, that the probable phase trajectory areas in the PDF sense are mostly concentrated around the found recurrent flows. Thus the authors assume that these recurrent flows are the essential contribution and (maybe) the drivers of the 2D turbulence. The continuation of this research was performed in [39] by Lucas and Kerswell for different aspect ratios for the 2D Kolmogorov flow. The same conclusion is drawn in the latter work as well.

However the extension of this approach to the 3D flow became a challenge. The results are summerised by Lucas and Kerswell in [40] for the 3D Kolmogorov flow. It was noticed, that starting from around Reynolds ($R$) number 20, the UPOs are difficult to trace and continuation from the UPOs for lower Reynolds number could not be carried out due to the trajectory retraction (saddle-point bifurcation). It was reported that around 39 UPOs, 19 traveling waves (solutions in the form $u(x,t) = U(x-c t)$, $c$ is a constant velocity) and 6 unstable stationary solutions were found. Some UPOs were analysed using the decomposition technics to determine the self sustaining process of the turbulence flow. However these UPOs are certainly not all UPOs that exist in the problem for $R = 20$. Extraction of UPOs for the increased $R$ was unsuccessful. Authors state that the chaotic attractor dimension increases rapidly for $R > 20$. For $R = 20$ the chaotic flow appears driven by the UPOs self sustaining process.

Observe, that the methodology, suggested in the cited above papers relay on the extraction of UPOs from the DNS data. Not all UPOs can be extracted this way (as authors stated) and this process is computationally extensive. On the contrary, we state here that the dynamical system analysis of developed turbulence should start form the consideration of stationary unstable solutions and then UPOs. The analysis of this kind is imposible without the deflation methods [29,30]. Such methods, potentially, can extract unstable fixed point and periodic solutions directly from the system of equations without any DNS calculations needed. This research is currently underway.

Discussion

In this report we demonstrated that the dynamical system approach can be applied to study turbulence. Four stages of the LTT are introduced which describe the turbulence on each stage traceable with the underlined methodology. The general approach is closely linked to the numerical methods being the primary object of the research. These numerical method relay on the conjecture of the inertial manifold existence and are still questionable from the mathematical point of view. In order to reduce possible errors due to discretization we utilize higher order discretization methods and apply the Kolmogorov inertial scale estimation to choose the number of degrees of freedom for the discrete reduced Navier-Stokes systems. Resulting discrete systems contain around $10^6 - 10^8$ DOF, depending on the problem and estimated maximum Reynolds number under consideration. Only parallel implementation of the numerical methods guarantees successful application of the methodology in reasonable time span. The basic methods which use these large discrete systems of equations are the Newton's method for deflation and continuation and an eigensolver used to find dimension of unstable manifold and classify local bifurcations.

We demonstrated that using the described methods one can analyse three stages of the LTT. Two commonly known ways of stability loss through supercritical and subcritical bifurcations can be easily traced via continuation process of stationary (or periodic) solutions. It is demonstrated in the first example in section 3 that using deflation methods one can find multistable solutions as well. This, basically, explains the general idea behind such flows, where the main solution is linear stable while the flow may abruptly change to another solution (Poiseuille and Couette flows), see bifurcation diagram in Figure 1. If one is more interested in physics of this phenomenon, one can draw basins of attraction for each stable solution in the multistable region and predict the stability loss due to multistability and, hence, deriving optimal perturbations. However, it can be rather difficult.
Preliminary analysis by the author suggests that for some problem such basis of attraction may be fractal [41].

The other example demonstrates that the low Reynolds flows with supercritical smooth transitions demonstrate low dimensional dynamics. It is a well known phenomenon, however here we demonstrate that the formation of singular attractors is going through the heteroclinic global bifurcation forming Smale’s horseshoe attractor locally on the underlying UPO. Further chaotic regimes are formed by hyperbolic attractors. A good example of a crisis is provided next. It is shown that a tensor product of a stable and unstable periodic orbits results in an abrupt flow regime change going from quasi-periodic laminar flow to the chaotic flow with irregular flow structures. Such flow change happens in a relatively small variation of the parameter value, thus the name -- crisis. This happens due to the fast growth of the attractor fractal dimension in the phase space. This dimension growth is the results of two crisis phenomena - phase locking and heteroclinic bifurcation of one of the periodic orbits that form an invariant torus. It is demonstrated in the Poincare section in Figure 8.

The last stage of the LTT is poorly studied. We cited pioneer results of Chandler, Lucas and Kerswell that relate turbulence with the UPOs which form a self sustaining process. However we assume that the deflation methodology is better suited for this analysis. We are currently developing appropriate numerical methods that can allow one to find unstable stationary solutions and UPOs using deflation with no regard to the DNS data. We believe that along with the DMD methods, deflation can bring more light to the problem of turbulence for arbitrary, not only fully periodic, boundary conditions. This is the main focus of our current study.

Up to now we were able to trace LTT (from the first to the third stage) in the following problems: 3D Kolmogorov flow, Rayleigh-Benard flow with 4×4×1 domain with periodic boundaries and 1×1×1 boundary with zero Dirichlet boundaries, Backwards facing step flow with wall boundaries (except inflow and outflow), compressible flows with Rayleigh-Taylor and Kelvin-Helmholtz instabilities triggered by gradients of density and velocity, accordingly. All bifurcation scenarios are described in the recent paper [42].

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