Nonlinear Two-Time-Scale Stochastic Approximation: Convergence and Finite-Time Performance

Thinh T. Doan

Department of Electrical and Computer Engineering
Virginia Tech, USA

Abstract

Two-time-scale stochastic approximation, a generalized version of the popular stochastic approximation, has found broad applications in many areas including stochastic control, optimization, and machine learning. Despite of its popularity, theoretical guarantees of this method, especially its finite-time performance, are mostly achieved for the linear case while the results for the nonlinear counterpart are very sparse. Motivated by the classic control theory for singularly perturbed systems, we study in this paper the asymptotic convergence and finite-time analysis of the nonlinear two-time-scale stochastic approximation. Under some fairly standard assumptions, we provide a formula that characterizes the rate of convergence of the main iterates to the desired solutions. In particular, we show that the method achieves a convergence in expectation at a rate $O(1/k^{2/3})$, where $k$ is the number of iterations. The key idea in our analysis is to properly choose the two step sizes to characterize the coupling between the fast and slow-time-scale iterates.

1. Nonlinear two-time-scale stochastic approximation

Stochastic approximation (SA), introduced by Robbins and Monro (1951), is a simulation-based approach for finding the root (fixed point) of some unknown operator $F$. Specifically, this method seeks a point $x^*$ such that $F(x^*) = 0$ based on the noisy observations $F(x; \xi)$ of $F(x)$, where $\xi$ is some random variable. SA has found broad applications in many areas including statistics, stochastic optimization, machine learning, and reinforcement learning Bertsekas and Tsitsiklis (1999); Borkar (2008); Hastie et al. (2009); Sutton and Barto (2018); Lan (2020).

In this paper, we consider the so-called two-time-scale SA, a generalized variant of the classic SA, which is used to find the root of a system of two coupled nonlinear equations. Given two unknown operators $F: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $G: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, we seek to find $x^*$ and $y^*$ such that

$$\begin{cases} F(x^*, y^*) = 0 \\ G(x^*, y^*) = 0. \end{cases}$$

Since $F$ and $G$ are unknown, we assume that there is a stochastic oracle that outputs noisy values of $F(x, y)$ and $G(x, y)$ for a given pair $(x, y)$. In particular, for any given points $x$ and $y$ we have access to $F(x, y) + \xi$ and $G(x, y) + \psi$, where $\xi$ and $\psi$ are two random variables. Using this stochastic oracle, we study the two-time-scale nonlinear SA for solving problem (1), which iteratively updates the iterates $x_k$ and $y_k$, the estimates of $x^*$ and $y^*$, respectively, for any $k \geq 0$ as

$$\begin{align*}
x_{k+1} &= x_k - \alpha_k (F(x_k, y_k) + \xi_k) \\
y_{k+1} &= y_k - \beta_k (G(x_k, y_k) + \psi_k),
\end{align*}$$

© 2020 T.T. Doan.
where \( x_0 \) and \( y_0 \) are arbitrarily initialized in \( \mathbb{R}^d \). Here \( \alpha_k, \beta_k \) are two step sizes chosen such that \( \beta_k \ll \alpha_k \), i.e., the second iterate is updated using step sizes that are very small as compared to the ones used to update the first iterate. Thus, the update of \( x_k \) is referred to as the “fast-time scale” while the update of \( y_k \) is called the “slow-time scale”. In addition, the update of the fast iterate depends on the slow iterate and vice versa, that is, they are coupled to each other. To handle this coupling, the two step sizes have to be properly chosen to guarantee the convergence of the method. Indeed, an important problem in this area is to select the two step sizes so that the two iterates converge as fast as possible. Our main focus is, therefore, to derive the convergence rate of (2) in solving (1) under some proper choice of these two step sizes and to understand their impact on the performance of the nonlinear two-time-scale SA.

1.1. Main contributions
The focus of this paper is to derive the asymptotic convergence and finite-time performance of the nonlinear two-time-scale SA. In particular, we provide a formula that characterizes the rate of convergence of the main iterates to the desired solutions. Under some proper choice of step sizes \( \alpha_k \) and \( \beta_k \), we show that the method achieves a convergence in expectation at a rate \( O(1/k^{2/3}) \), where \( k \) is the number of iterations. Our key technique is motivated by the classic control theory for singularly perturbed systems, that is, we properly choose the two step sizes to characterize the coupling between the fast and slow-time-scale iterates. In addition, our convergence rate analysis also provides an insight for different choice of step sizes under different settings.

1.2. Motivating applications
Nonlinear two-time-scale SA, Eq. (2), has been found in numerous applications within engineering and sciences. One concrete example is to model the well-known stochastic gradient descent (SGD) with the Polyak-Rupert averaging for minimizing an objective function \( f \), i.e., we want to solve

\[
\min_{y \in \mathbb{R}^d} f(y)
\]

where we only have access to the noisy observations of the true gradients. In this case, the classic SGD iteratively update the iterates \( y_k \) as

\[
y_{k+1} = y_k - \beta_k (\nabla f(y_k) + \psi_k),
\]

where \( \psi_k \) is some zero-mean random variables. In order to improve the convergence of SGD, an additional averaging step is often used Polyak and Juditsky (1992); Ruppert (1988)

\[
x_{k+1} = \frac{1}{k+1} \sum_{t=0}^{k} y_k = x_k + \frac{1}{k+1} (y_k - x_k).
\]

Obviously, these two updates are special case of the nonlinear two-time-scale SA in (2). A more complicated version of two-time-scale SGD for optimizing the composite function \( f(g(x)) \) can also be viewed as a variant of (2) Wang et al. (2017); Zhang and Xiao (2019). In addition, two-time-scale methods have been used in distributed optimization to address the issues of communication constraints Doan et al. (2017, 2020) and in distributed control to handle clustered networks Romeres et al. (2013); Chow and Kokotovic (1985); Biyik and Arcak (2008); Boker et al. (2016);
Finally, two-time-scale SA has been used extensively to model reinforcement learning methods, for example, gradient temporal difference (TD) learning and actor-critic methods. As a specific application of (2) in reinforcement learning, we consider the gradient TD learning for solving the policy evaluation problem under nonlinear function approximations studied in Maei et al. (2009), which can be viewed as a variant of (2). In this problem, we want to estimate the cumulative rewards $V$ of a stationary policy using function approximations $V_y$, that is, our goal is to find $y$ so that $V_y$ is as close as possible to the true value $V$. Here, $V_y$ can be represented by a neural network where $y$ is the weight vector of the network. Let $\zeta$ be the environmental state, $\gamma$ be the discount factor, $\phi(\zeta) = \nabla V_y(\zeta)$ be the feature vector of state $\zeta$, and $r$ be the reward return by the environment. Given a sequence of samples $\{\zeta_k, r_k\}$, one version of GTD are given as

$$
x_{k+1} = x_k + \alpha_k(\delta_k - \phi(\zeta_k)^T x_k)\phi(\zeta_k)
$$

$$
y_{k+1} = y_k + \beta_k \left[ (\phi(\zeta_k) - \gamma\phi(\zeta_{k+1})) \phi(\zeta_k)^T x_k - h_k \right],
$$

where $\delta_k$ and $h_k$ are defined as

$$
\delta_k = r_k + \gamma V_{y_k}(\zeta_{k+1}) - V_{y_k}(\zeta_k) \text{ and } h_k = (\delta_k - \phi(\zeta_k)^T x_k)\nabla^2 V_{y_k}(\zeta_k)x_k,
$$

which is clearly a variant of (2) under some proper choice of $F$ and $G$. It has been observed that the GTD method is more stable and performs better compared to the single-time-scale counterpart (TD learning) under off-policy learning and nonlinear function approximations.

1.3. Related works

Given the broad applications of SA in many areas, its convergence properties have received much interests for years. In particular, the asymptotic convergence of SA, including its two-time-scale variant, can be established by using the (almost) Martingale convergence theorem when the noise are i.i.d or the ordinary differential equation (ODE) method for more general noise settings; see for example Borkar (2008); Benveniste et al. (2012); Bertsekas and Tsitsiklis (1999). Under the right conditions both of these methods show that the noise effects eventually average out and the SA iterate asymptotically converges to the desired solutions.

The convergence rate of the single-time-scale SA has been mostly studied in the context of SGD under the i.i.d noise model; see for example Bottou et al. (2018) and the references therein. Given the wide applications of SA in reinforcement learning, three are a significant interest in analyzing the finite-time analysis of SA under different conditions; see for example Bhandari et al. (2018); Karimi et al. (2019); Srikant and Ying (2019); Hu and Syed (2019) for linear SA and Chen et al. (2019) for nonlinear counterpart.

Unlike the single-time-scale SA, the convergence rates of the two-time-scale SA are less understood due to the complicated interactions between the two step sizes and the iterates. Specifically, the rates of the two-time-scale SA has been studied mostly for the linear settings, i.e, when $F$ and $G$ are linear functions w.r.t their variables; see for example in Konda and Tsitsiklis (2004); Dalal et al. (2018); Doan and Romberg (2019); Gupta et al. (2019); Doan (2020); Karimi et al. (2020). On the other hand, we are only aware of the work in Mokkadem and Pelletier (2006), which considers the finite-time analysis of the nonlinear two-time-scale SA in (2). In particular, under the stability condition (Assumption 1 in Mokkadem and Pelletier (2006), $\lim_{k\to\infty}(x_k, y_k) = (x^*, y^*)$) and when $F$
and $G$ can be locally approximated by linear functions in a neighborhood of $(x^*, y^*)$, a convergence rate of (2) in distribution is provided. They also show that the rates of the fast-time and slow-time scales are asymptotically decoupled under proper choice of step sizes, which agrees with the previous observations of two-time-scale SA; see for example Konda and Tsitsiklis (2004). In this paper, our focus is to study the finite-time analysis that characterizes the rates of (2) in mean square errors. We do this under different assumptions on the operators $F$ and $G$ as compared to the ones considered in Mokkadem and Pelletier (2006); for example, we do not require the stability condition. Our setting is motivated by the conditions considered in (Kokotović et al., 1999, Chapter 7), where the authors study the continuous-time and deterministic version of (2), i.e., $\xi_k = \psi_k = 0$.

2. Main Results

In this section, we present in details the main results of this paper, that is, we provide a finite-time analysis for the convergence rates of (2) in mean square errors. Under some certain conditions explained below, we show that the mean square errors converge to zero at a rate

$$
\mathbb{E} \left[ \|y_k - y^*\|^2 \right] + \left( \frac{\beta_k}{\alpha_k} \right) \mathbb{E} \left[ \|x_k - x^*\|^2 \right] \leq O \left( \frac{1}{(k + 1)^{2/3}} \right),
$$

where the choice of $\beta_k \ll \alpha_k$ is discussed in the next section. In addition, under the same choice of step sizes we obtain

$$
\lim_{k \to \infty} \|x_k - x^*\| = \lim_{k \to \infty} \|y_k - y^*\| = 0 \quad a.s.
$$

Before presenting the details of our results, we discuss the main idea behind our approach and assumptions, motivated by the one considered in Saberi and Khalil (1984); Kokotović et al. (1999). In particular, under some proper conditions of the noise and step sizes the convergence of (2) is reduced to study the stability of the following differential equations using constant step sizes $\alpha, \beta$

$$
\frac{dx}{dt} = -F(x(t), y(t))
$$

$$
\frac{dy}{dt} = -\frac{\beta}{\alpha}G(x(t), y(t)),
$$

where the ratio $\beta/\alpha$ represents the difference in time scale between these two updates. For simplicity, we assume that $(x^*, y^*) = (0, 0)$ is the unique equilibrium (3). Since $\beta/\alpha \ll 1$, the dynamic of $x(t)$ evolves much faster than the one of $y(t)$. Thus, one can consider $y(t) = y$ being fixed in $\dot{x}$ and study the stability of the following “fast” system

$$
\dot{x}(t) = -F(x(t), y).
$$

Given a fixed $y$ we assume that $x = H(y)$ is the unique solution of $F(x, y) = 0$ or unique equilibrium of $\dot{x}$. This can be guaranteed under some proper smoothness conditions on $F$. A natural condition to guarantee the stability of this system is the existence of a Lypaunov function $V_F(x, y)$ such that the following condition holds with some constant $\mu_x > 0$

$$
\frac{dV_F(x(t), y)}{dt} = -\frac{\partial V_F}{\partial x}F(x(t), y) \leq -\mu_x \|x(t) - H(y)\|^2 \quad \text{and} \quad V_F(x(t), y) = 0 \quad \text{iff} \quad x(t) = H(y).
$$
In this case, Lyapunov theorem says that \( \lim_{t \to \infty} x(t) = H(y) \) exponentially. However, since \( y_t \) is also changing the derivative of \( V_F \) is given as

\[
\frac{dV_F(x(t), y(t))}{dt} = -\frac{\partial V_F}{\partial x} F(x(t), y(t)) - \frac{\beta}{\alpha} \frac{\partial V_F}{\partial y} G(x(t), y(t)),
\]

which does not immediately yield the convergence of \( x(t) \) to \( H(y(t)) \) unless some additional requirements on the asymptotic convergence of \( y(t) \) is satisfied. On the other hand, we consider the slow dynamic \( \dot{y} \) as follows

\[
\dot{y}(t) = -\frac{\beta}{\alpha} G(H(y(t)), y(t)) - \frac{\beta}{\alpha} [G(x(t), y(t)) - G(H(y(t)), y(t))],
\]

where the “least” condition to guarantee the stability of this system is the existence of a Lyapunov function when \( x(t) = H(y(t)) \), i.e., we assume that there exists a Lyapunov function \( V_G(y) \) and a positive constant \( \mu_y \) such that

\[
\frac{dV_G(y(t))}{dt} = -\frac{\beta}{\alpha} \frac{\partial V_G}{\partial y} G(H(y(t)), y(t)) \leq -\mu_y \|y(t)\|^2,
\]

which implies the convergence of \( y(t) \) to 0. However, in our case \( x(t) \neq H(y(t)) \) which yields

\[
\frac{dV_G(y(t))}{dt} = -\frac{\beta}{\alpha} \frac{\partial V_G}{\partial y} G(H(y(t)), y(t)) - \frac{\beta}{\alpha} \frac{\partial V_G}{\partial y} [G(x(t), y(t)) - G(H(y(t)), y(t))]
\]

\[
\leq -\mu_y \|y(t)\|^2 + \frac{L_G \beta}{\alpha} \left\| \frac{\partial V_G}{\partial y} \right\| \|x(t) - H(y(t))\|,
\]

where we assume that \( G \) is Lipschitz continuous with constant \( L_G > 0 \). In this case, one cannot immediately declare the convergence of \( y(t) \) to 0, unless the asymptotic convergence of \( x(t) \) to \( H(y(t)) \) is guaranteed at a proper rate.

Based on the observations in Eqs. (4) and (5), to guarantee the stability of the system (3) one needs to combine the Lyapunov functions \( V_G \) and \( V_F \) to characterize the coupling asymptotic behavior of \( x(t) \) and \( y(t) \) as well as the convergence of individual variables. In this paper, we use the following Lyapunov function based on the time-coupling ratio \( \beta/\alpha \)

\[
V(x, y) = V_F(x, y) + \frac{\beta}{\alpha} V_G(x, y)
\]

Our settings and analysis in the sequel are established based on this important observation. Due to the impact of the noise, one cannot in general guarantee that the solutions of the stochastic systems will track the ones of (3). However, under some fairly standard assumptions we show that this goal can be achieved. In addition, we provide a finite-time bound to characterize the rates of this convergence.

2.1. Preliminaries and Assumptions

We start this section with the main assumptions, which are useful for our later analysis. Our first assumption is on the smoothness of \( F \) and \( G \), which basically guarantees the existence of the solutions of (3).
Assumption 1 Given any $x$ there exists an operator $H : \mathbb{R}^d \to \mathbb{R}^d$ such that $x = H(y)$ is the unique solution of

$$F(H(y), y) = 0,$$

where $H$ is assumed to be Lipschitz continuous with constant $L_H$, i.e.,

$$\|H(y_1) - H(y_2)\| \leq L_H \|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^d. \quad (6)$$

In addition, given a fixed $y$ $F(\cdot, y)$ is Lipschitz continuous with constant $L_F$, i.e.,

$$\|F(x_1, y) - F(x_2, y)\| \leq L_F \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^d. \quad (7)$$

Finally, we assume that $G(\cdot, \cdot)$ is Lipschitz continuous with constant $L_G$, i.e.,

$$\|G(x_1, y_1) - G(x_2, y_2)\| \leq L_G (\|x_1 - x_2\| + \|y_1 - y_2\|), \quad \forall x_1, x_2, y_1, y_2 \in \mathbb{R}^d. \quad (8)$$

Note that the Lipschitz continuity of $G$ on both variables is necessary since it guarantees the existence of the solutions of $\dot{y}$ when $x(t) = H(y(t))$ presented in the previous section. Our second assumption basically says that the operators $F$ and $G$ are strong monotone, which is used to guarantee the exponential convergence of the decoupled ODE systems (i.e., $y(t) = y$ in $\dot{x}$ and $x(t) = H(y(t))$ in $\dot{y}$). These assumptions are also considered under different variants in the context of both linear and nonlinear SA studied in Konda and Tsitsiklis (2004); Dalal et al. (2018); Doan and Romberg (2019); Gupta et al. (2019); Doan (2020); Karimi et al. (2020); Mokkadem and Pelletier (2006).

Assumption 2 $F$ is strongly monotone w.r.t $x$ when $y$ is fixed, i.e., there exists a constant $\mu > 0$

$$\langle x - z, F(x, y) - F(z, y) \rangle \geq \mu_F \|x - z\|^2. \quad (9)$$

Similarly, $G$ is assumed to be 1-point strongly monotone w.r.t the solution $y^*$, i.e., there exists a constant $\mu_G > 0$ such that

$$\langle y - y^*, G(H(y), y) \rangle \geq \mu_G \|y - y^*\|^2. \quad (10)$$

Finally, we consider the noise model being i.i.d, that is, $\xi_k$ and $\psi_k$ are Martingale difference.

Assumption 3 The random variables $\xi_k$ and $\psi_k$, for all $k \geq 0$, are independent of each other and across time, with zero mean and common variances given as follows

$$\mathbb{E}[\xi_k^T \xi_k] = \Gamma_{11}, \quad \mathbb{E}[\xi_k^T \psi_k] = \Gamma_{12} = \Gamma_{21}^T, \quad \mathbb{E}[\psi_k^T \psi_k] = \Gamma_{22}. \quad (11)$$

We denote by $\mathcal{Q}_k$ the filtration contains all the history generated by the algorithms upto time $k$, i.e.,

$$\mathcal{Q}_k = \{x_0, y_0, \xi_0, \psi_0, \xi_1, \psi_1, \ldots, \xi_{k-1}, \psi_{k-1}\}.$$

To the rest of this paper, we consider nonincreasing and nonnegative time-varying sequence of step sizes $\{\alpha_k, \beta_k\}$, e.g., $\alpha_k \leq \alpha_0$ and $\beta_k \leq \beta_0$. For convenience, we consider the following notation

$$L_1 = L_H L_G (1 + L_H \alpha_0) \quad \text{and} \quad L_2 = L_G^2 (1 + 2(L_H + 1)) \beta_0. \quad (12)$$
As observed in the previous section, the fast and slow-time-scale updates are coupled through the term \( x - H(y) \). Therefore, we introduce two residual variables \( \hat{x} \) and \( \hat{y} \) defined as
\[
\hat{x}_k = x_k - H(y_k) \\
\hat{y}_k = y_k - y^*.
\] (13)

Obviously, if \( \hat{y}_k \) and \( \hat{x}_k \) go to zero, \( (x_k, y_k) \to (x^*, y^*) \). Thus, to establish the convergence of \( (x_k, y_k) \) to \( (x^*, y^*) \) one can instead study the convergence of \( (\hat{x}_k, \hat{y}_k) \) to zero. To do that, we first study the relation of these two residual variables in the following two lemmas. For an ease of exposition, we delay the analysis of the results in this section to the Appendix.

**Lemma 1** Suppose that Assumptions 1–3 hold. Let \( \{x_k, y_k\} \) be generated by (2). Then, for any constant \( \eta_x > 0 \) we have
\[
\mathbb{E} \left[ \|\hat{x}_{k+1}\|^2 \mid Q_k \right] \leq (1 - 2\mu_F \alpha_k) \|\hat{x}_k\|^2 + \Gamma_{22} \beta_k^2 + \frac{L_1 \Gamma_{22} \beta_k^2}{\eta_x \alpha_k} \\
+ 2L_{11}^2 L_2^2 (L_H + 1)^2 \beta_k^2 \|\hat{y}_k\|^2 + L_{11}^2 (2L_{11}^2 + 1) \alpha_k \|\hat{x}_k\|^2 \\
+ \frac{L_1 (L_H + 1)^2 \beta_k^2}{\eta_x \alpha_k} \|\hat{y}_k\|^2 + 2L_1 (\beta_k + \eta_x \alpha_k) \|\hat{x}_k\|^2.
\] (14)

**Lemma 2** Suppose that Assumptions 1–3 hold. Let \( \{x_k, y_k\} \) be generated by (2). Then we have
\[
\mathbb{E} \left[ \|\hat{y}_{k+1}\|^2 \mid Q_k \right] \leq (1 - \mu_G \beta_k) \|\hat{y}_k\|^2 + \Gamma_{22} \beta_k^2 + \frac{L_2^2 \beta_k^2}{\mu_G} \|\hat{x}_k\|^2 \\
+ L_2^2 (1 + 2L_{11}^2 \beta_k) \left( \|\hat{x}_k\|^2 + \|\hat{y}_k\|^2 \right).
\] (15)

Next, we show that \( \mathbb{E} \left[ \|\hat{x}_k\|^2 + \|\hat{y}_k\|^2 \right] \) is bounded in the following lemma.

**Lemma 3** Suppose that Assumptions 1–3 hold. Let \( \{x_k, y_k\} \) be generated by (2). In addition, consider \( 1/2 < a < b \leq 1, 2b - a > 1 \), and let \( \alpha_k, \beta_k, \) and \( \eta_x \) be chosen as
\[
\frac{\beta_0}{\alpha_0} \leq \min \left\{ \frac{2\mu_F \mu_G}{3(2L_1 \mu_G + L_2^2)}, \frac{\mu_G \eta_x}{2L_1 (L_H + 1)^2} \right\}, \quad \eta_x \leq \frac{\mu_F}{3L_1}
\] (16)
\[
\alpha_k = \frac{\alpha_0}{(k + 2)^a}, \quad \beta_k = \frac{\beta_0}{(k + 2)^b},
\]
where \( L_1, L_2 \) are defined in (12) and \( \eta_x \) are given in Lemma 1. We denote by \( C_1 \) and \( C \) as
\[
C_1 = \exp \left\{ (1 + 4L_2^2 (L_H + 1)^4) \left( \frac{\alpha_0^2 + 1}{2\alpha_0 - 1} \right) \right\}
\]
\[
C_2 = \frac{2\Gamma_{22} \beta_0^2 (2b - 1) + \Gamma_{11} (\alpha_0^2 (2a - 1) + 1) + L_1 \Gamma_{22} \beta_0^2 (2b - a - 1) + \alpha_0}{2b - 1} + \frac{\Gamma_{11} (\alpha_0^2 (2a - 1) + 1) + L_1 \Gamma_{22} \beta_0^2 (2b - a - 1) + \alpha_0}{\eta_x \alpha_k \beta_k}.
\] (17)

Let \( \mu = \min \{\mu_F, \mu_G\} \). Then we have for any \( k \geq 0 \)
\[
\mathbb{E} \left[ \|\hat{x}_{k+1}\|^2 + \|\hat{y}_{k+1}\|^2 \mid Q_k \right] \leq (1 - \mu \beta_k + 4L_2^2 (L_H + 1)^4 \alpha_k^2) \left[ \|\hat{x}_k\|^2 + \|\hat{y}_k\|^2 \right] \\
+ 2\beta_k^2 \Gamma_{22} + \alpha_k^2 \Gamma_{11} + \frac{L_1 \beta_k^2}{\eta_x \alpha_k} \Gamma_{22}.
\] (18)
In addition, we obtain
\[ \mathbb{E} [\|\hat{x}_k\|^2 + \|\hat{y}_k\|^2] \leq C = C_1 (\mathbb{E} [\|\hat{x}_0\|^2 + \|\hat{y}_0\|^2] + C_2) \exp \left\{ - \frac{(1 + 4L_2^2(L_H + 1)^4)}{2a-1} \right\}. \] (19)

Finally, in our analysis we utilize the well-known almost supermartingale convergence result to establish the asymptotic convergence of our iterates Robbins and Siegmund (1971).

**Lemma 4 (Robbins and Siegmund (1971))** Let \( \{w_k\}, \{v_k\}, \{\sigma_k\}, \text{and} \{\delta_k\} \) be non-negative sequences of random variables and satisfy
\[ \mathbb{E} \left[ w_{k+1} \mid \mathcal{F}_k \right] \leq (1 + \sigma_k) w_k - v_k + \delta_k \] (20)
\[ \sum_{k=0}^{\infty} \sigma_k < \infty \ a.s., \quad \sum_{k=0}^{\infty} \delta_k < \infty \ a.s., \] (21)
where \( \mathcal{F}_k = \{w(0), \ldots, w_k\} \), the history of \( w \) up to time \( k \). Then \( \{w_k\} \) converges a.s., and \( \sum_{k=0}^{\infty} v_k < \infty \ a.s.

### 2.2. Main Analysis

In this section, we present the analysis for our main result stated in Section 2. The main idea of our analysis is to combine the results in Lemmas 1–3 and utilize the following Lyapunov equation which characterizes the coupling between the two iterates
\[ V(\hat{x}_k, \hat{y}_k) = \|\hat{y}_k\|^2 + \eta \frac{\beta_k}{\alpha_k} \|\hat{x}_k\|^2 = \|y_k - y^*\|^2 + \eta \frac{\beta_k}{\alpha_k} \|x_k - H(y_k)\|^2. \] (22)

**Theorem 1** Suppose that Assumptions 1–3 hold. Let \( \{x_k, y_k\} \) be generated by (2). In addition, consider \( 1/2 < a < b \leq 1, \ 2b - a > 1 \), and let \( \alpha_k, \beta_k, \text{and} \eta_k \) be chosen as
\[ \frac{\alpha_0}{\alpha} \leq \min \left\{ \frac{2\mu F \mu G}{3(2L_1 \mu G + L_2^2)}, \frac{\mu G \eta_k}{2L_1 (L_H + 1)^2}, \frac{2\mu F}{3(\mu G + 2L_1)} \right\} \] (23)
\[ \alpha_k = \frac{\alpha_0}{(k + 1)^a}, \quad \beta_k = \frac{\beta_0}{(k + 1)^b}, \quad \eta_k \leq \frac{\mu F}{3L_1}, \quad \eta = \frac{3L_2^2}{2\mu G \mu F} \]
where \( L_1, L_2 \) are defined in (12) and \( \eta_k \) are given in Lemma 1. Then we have
\[ \lim_{k \to \infty} \hat{x}_k = \lim_{k \to \infty} \hat{y}_k = 0 \ a.s. \] (24)

In addition, consider the constant \( C \) defined in (19) and \( L = \max\{L_H, L_G, L_F\} \). Let \( a = 2/3 \), \( b = 1 \), and \( \beta_0 = 1/\mu_G \). Then we have
\[ \mathbb{E} [V(\hat{x}_{k+1}, \hat{y}_{k+1})] \leq \mathbb{E} \left[ V(\hat{x}_0, \hat{y}_0) \right] + \left( \frac{3CL_2^2(1 + 2L_2^2)}{2\mu_G \mu F} + \frac{(2\mu_F + 3L_2) \Gamma_{22}}{2\mu_G \mu F} \right) \frac{1 + \ln(k + 1)}{k + 2} \]
\[ + \left( \frac{3L_2 \alpha_0 \Gamma_{11}}{2\mu_G \mu F} + \frac{3C L_2 L_1 \Gamma_{22}}{2\mu_G \mu F \eta_k \alpha_0} \right) \frac{1}{(k + 2)^{2/3}} \]
\[ + \left( \frac{3CL_2 (L + 1)^2}{2\mu_G \mu F \eta_k \alpha_0} + \frac{4CL_2 L^4 (L + 1)^2 \alpha_0}{\mu_G \mu F} \right) \frac{1}{(k + 2)^{2/3}}. \] (25)
Proof

1. We start our analysis with the proof of the asymptotic convergence of \( \hat{x}_k \) and \( \hat{y}_k \) to zero by utilizing Lemma 4. Indeed, let \( z_k = ||\hat{x}_k||^2 + ||\hat{y}_k||^2 \). Since the step sizes \( \alpha_k, \beta_k \) satisfy (23), they also satisfy (16). Thus, by (18) we have

\[
\mathbb{E}[z_{k+1} | Q_k] \leq \left( 1 - \mu \beta_k + 4L_G^2(L_H + 1)^4 \alpha_k^2 \right) z_k + 2\beta_k^2 \Gamma_2 + \alpha_k^2 \Gamma_{11} + \frac{L_1 \beta_k^2 \Gamma_2}{\eta_x \alpha_k}
= (1 - \mu \beta_k + a_k)z_k + b_k.
\]

(26)

Note that by (23) we have
\[
\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} b_k = \infty \quad \text{and} \quad \max \left\{ \sum_{k=0}^{\infty} a_k^2, \sum_{k=0}^{\infty} b_k^2 \right\} < \infty.
\]

Thus, we apply the results in Lemma 4 to (26), where \( w_k = z_k, v_k = \mu G \beta_k z_k, \sigma_k = a_k, \) and \( \delta_k = b_k \) to have
\[
\left\{ \begin{align*}
& z_k \text{ converges} \\
& \sum_{k=0}^{\infty} \mu G \beta_k z_k < \infty.
\end{align*} \right.
\]

Using the definition of \( z_k \) in (22) we obtain \( ||\hat{y}_k||^2 + ||\hat{x}_k||^2 \) converges a.s. In addition, since \( \sum_{k=0}^{\infty} \beta_k = \infty \) a.s., we have \( z_k \) converges to zero a.s. Thus, we obtain
\[
\lim_{k \to \infty} \hat{x}_k = \lim_{k \to \infty} \hat{y}_k = 0 \quad \text{a.s.}
\]

2. Next, we derive the rate in (25). Since \( \alpha_k, \beta_k \) satisfy (23) we have \( \beta_{k+1}/\alpha_{k+1} \) is a nonincreasing sequence. Thus, multiplying both sides of (14) by \( \eta \beta_{k+1}/\alpha_{k+1} \) for some \( \eta > 0 \), we obtain

\[
\frac{\eta \beta_{k+1}}{\alpha_{k+1}} \mathbb{E}[||\hat{x}_{k+1}||^2 | Q_k] \leq \frac{\eta \beta_k}{\alpha_k} \mathbb{E}[||\hat{x}_{k+1}||^2 | Q_k]
\]

\[
\leq \left( \frac{\eta \beta_k}{\alpha_k} - 2 \mu F \eta \beta_k \right) ||\hat{x}_k||^2 + \Gamma_2 \beta_k^2 \alpha_k + \Gamma_{11} \alpha_k \beta_k + \frac{L_1 \beta_k^2 \Gamma_2}{\eta_x \alpha_k^2}
+ 2L_2^2 L_G^2 (L_H + 1)^2 \eta \beta_k \alpha_k \beta_k \eta \beta_k ||\hat{x}_k||^2 + L_2^2 (2L_G^2 + 1) \eta \beta_k \alpha_k \beta_k \eta \beta_k ||\hat{x}_k||^2
+ \frac{L_1 \beta_k^2 \Gamma_2}{\eta_x \alpha_k^2} \eta \beta_k ||\hat{x}_k||^2 + 2L_1 \eta \left( \frac{\beta_k^2 \alpha_k}{\eta_x} + \eta \beta_k \alpha_k \beta_k \right) ||\hat{x}_k||^2
= (1 - \mu G \beta_k) \frac{\eta \beta_k}{\alpha_k} ||\hat{x}_k||^2 + \Gamma_2 \frac{\beta_k^2}{\alpha_k} + \Gamma_{11} \alpha_k \beta_k + \frac{L_1 \beta_k^2 \Gamma_2}{\eta_x \alpha_k^2}
+ 2L_2^2 L_G^2 (L_H + 1)^2 \eta \beta_k \alpha_k \beta_k \eta \beta_k ||\hat{x}_k||^2 + L_2^2 (2L_G^2 + 1) \eta \beta_k \alpha_k \beta_k ||\hat{x}_k||^2
+ \frac{L_1 \beta_k^2 \Gamma_2}{\eta_x \alpha_k^2} \eta \beta_k ||\hat{x}_k||^2 + 2L_1 \eta \left( \frac{\beta_k^2 \alpha_k}{\eta_x} + \eta \beta_k \alpha_k \beta_k \right) ||\hat{x}_k||^2
- \left( 2 \mu F \eta \beta_k - \frac{\mu G \eta \beta_k^2}{\alpha_k} \right) ||\hat{x}_k||^2,
\]

9
which when adding to (15) gives
\[
\frac{\eta \beta_{k+1}}{\alpha_{k+1}} \mathbb{E} \left[ \|\hat{x}_{k+1}\|^2 \mid Q_k \right] + \mathbb{E} \left[ \|\hat{y}_{k+1}\|^2 \mid Q_k \right] \\
\leq (1 - \mu_G \beta_k) \left( \frac{\eta \beta_k}{\alpha_k} \|\hat{x}_k\|^2 + \|\hat{y}_k\|^2 \right) + \frac{\eta \beta_k}{\alpha_k} \left( \Gamma_{22} \beta_k^2 + \Gamma_{11} \alpha_k^2 + \frac{L_1 \Gamma_{22} \beta_k^2}{\eta_x} \right) + \Gamma_{22} \beta_k^2 \\
+ 2L_H^2 L_G^2 (L_H + 1)^2 \eta \frac{\beta_k^3}{\alpha_k} \|\hat{y}_k\|^2 + L_H^2 (2L_G + 1) \eta \beta_k \alpha_k \|\hat{x}_k\|^2 \\
+ \frac{L_1 (L_H + 1)^2 \eta \beta_k^3}{\eta_x \alpha_k^2} \|\hat{y}_k\|^2 + L_G^2 (1 + 2L_H^2) \beta_k^2 (\|\hat{x}_k\|^2 + \|\hat{y}_k\|^2) \\
- \left( 2 \mu_F \eta \beta_k - \frac{\mu_G \beta_k^2}{\alpha_k} - 2L_1 \eta \left( \frac{\beta_k^2}{\alpha_k} + \eta_x \beta_k \right) - \frac{L_2^2}{\mu_G} \beta_k \right) \|\hat{x}_k\|^2.
\]
(27)

Recall that $\eta = 3L_2/(2\mu_F)$ and since $\alpha_k, \beta_k$ and $\eta_x$ satisfy (23) the last term on the right-hand side of (27) is negative, i.e.,
\[
2 \mu_F \eta \beta_k - \frac{\mu_G \eta \beta_k^2}{\alpha_k} - 2L_1 \eta \left( \frac{\beta_k^2}{\alpha_k} + \eta_x \beta_k \right) - \frac{L_2^2}{\mu_G} \beta_k \\
= \left( \frac{2 \mu_F \eta}{3} - \frac{L_2^2}{\mu_G} \right) \beta_k + \left( \frac{2 \mu_F}{3} - (\mu_G + 2L_1) \frac{\beta_k}{\alpha_k} \right) \eta \beta_k + \left( \frac{2 \mu_F}{3} - 2L_1 \eta_x \right) \eta \beta_k \\
\geq 0 + \left( \frac{2 \mu_F}{3} - (\mu_G + 2L_1) \frac{\beta_0}{\alpha_0} \right) \eta \beta_k \geq 0,
\]
which when applying to (27) and using the definition of $V$ in (22) gives
\[
\mathbb{E} \left[ V(\hat{x}_{k+1}, \hat{y}_{k+1}) \mid Q_k \right] \leq (1 - \mu_G \beta_k) \mathbb{E} \left[ V(\hat{x}_k, \hat{y}_k) \right] \\
+ \eta \left( \Gamma_{22} \frac{\beta_k^3}{\alpha_k} + \Gamma_{11} \beta_k \alpha_k + \frac{L_1 \Gamma_{22} \beta_k^2}{\eta_x} \right) + \Gamma_{22} \beta_k^2 \\
+ 2L_H^2 L_G^2 (L_H + 1)^2 \eta \frac{\beta_k^3}{\alpha_k} \mathbb{E} \left[ \|\hat{y}_k\|^2 \right] + L_H^2 (2L_G + 1) \eta \beta_k \alpha_k \mathbb{E} \left[ \|\hat{x}_k\|^2 \right] \\
+ \frac{L_1 (L_H + 1)^2 \eta \beta_k^3}{\eta_x \alpha_k^2} \mathbb{E} \left[ \|\hat{y}_k\|^2 \right] + L_G^2 (1 + 2L_H^2) \beta_k^2 \mathbb{E} \left[ \|\hat{x}_k\|^2 + \|\hat{y}_k\|^2 \right],
\]
which when taking the expectation on both sides, using (19), and since $\beta_k \leq \alpha_k$ yields
\[
\mathbb{E} \left[ V(\hat{x}_{k+1}, \hat{y}_{k+1}) \right] \leq (1 - \mu_G \beta_k) \mathbb{E} \left[ V(\hat{x}_k, \hat{y}_k) \right] \\
+ \eta \left( \Gamma_{22} \frac{\beta_k^3}{\alpha_k} + \Gamma_{11} \beta_k \alpha_k + \frac{L_1 \Gamma_{22} \beta_k^2}{\eta_x} \right) + \Gamma_{22} \beta_k^2 \\
+ 2L_H^2 L_G^2 (L_H + 1)^2 \eta \beta_k \alpha_k \mathbb{E} \left[ \|\hat{y}_k\|^2 \right] + L_H^2 (2L_G + 1) \eta \beta_k \alpha_k \mathbb{E} \left[ \|\hat{x}_k\|^2 \right] \\
+ \frac{L_1 (L_H + 1)^2 \eta \beta_k^3}{\eta_x \alpha_k^2} \mathbb{E} \left[ \|\hat{y}_k\|^2 \right] + L_G^2 (1 + 2L_H^2) \beta_k^2 \mathbb{E} \left[ \|\hat{x}_k\|^2 + \|\hat{y}_k\|^2 \right]
\[ \begin{align*}
&\leq (1 - \mu_G \beta_k) \mathbb{E}[V(\hat{x}_k, \hat{y}_k)] + \frac{(2\mu_F + 3L_2)\Gamma_{22}}{2\mu_F} \beta_k^2 + \frac{3L_2 \Gamma_{11}}{2\mu_F} \beta_k \alpha_k + \\
&\quad \frac{3L_2 L_1 \Gamma_{22}}{2\mu_F \eta_x} \frac{3\beta_k^3}{2} + \frac{3CL_2 L_1 (L + 1)^2 \beta_k^3}{2\mu_F \eta_x \alpha_k^2} + \\
&\quad \frac{4CL_2 L^4 (L + 1)^2}{\mu_F} \beta_k \alpha_k + \frac{3CL_2 L^2 (1 + 2L^2)}{2\mu_F} \beta_k^2,
\end{align*} \]

where we use \( L = \max\{L_H, L_G, L_F\} \) and \( \eta = 3L_2/(2\mu_F) \) in the second inequality. Next, using \( \beta_k = 1/\mu_G(k + 1) \) in the preceding relation we obtain

\[ \mathbb{E}[V(\hat{x}_{k+1}, \hat{y}_{k+1})] \leq \mathbb{E}[V(\hat{x}_k, \hat{y}_k)] + \frac{(2\mu_F + 3L_2)\Gamma_{22}}{2\mu_F} \beta_k^2 + \frac{3L_2 \Gamma_{11}}{2\mu_F} \beta_k \alpha_k + \\
\frac{3L_2 L_1 \Gamma_{22}}{2\mu_F \eta_x} \frac{3\beta_k^3}{2} + \frac{3CL_2 L_1 (L + 1)^2 \beta_k^3}{2\mu_F \eta_x \alpha_k^2} + \\
\frac{4CL_2 L^4 (L + 1)^2}{\mu_F} \beta_k \alpha_k + \frac{3CL_2 L^2 (1 + 2L^2)}{2\mu_F} \beta_k^2,
\]

which by iteratively updating over \( k \) yields

\[ \mathbb{E}[V(\hat{x}_{k+1}, \hat{y}_{k+1})] \leq \frac{\mathbb{E}[V(\hat{x}_0, \hat{y}_0)]}{k + 2} + \sum_{t=0}^{k} \left( \frac{(2\mu_F + 3L_2)\Gamma_{22}}{2\mu_F} \beta_t^2 + \frac{3L_2 \Gamma_{11}}{2\mu_F} \beta_t \alpha_t \right) \prod_{\ell=t+1}^{k} \frac{\ell + 1}{\ell + 2} + \\
\sum_{t=0}^{k} \left( \frac{3L_2 L_1 \Gamma_{22}}{2\mu_F \eta_x} \frac{3\beta_t^3}{2} + \frac{3CL_2 L_1 (L + 1)^2 \beta_t^3}{2\mu_F \eta_x \alpha_t^2} \right) \prod_{\ell=t+1}^{k} \frac{\ell + 1}{\ell + 2} + \\
\sum_{t=0}^{k} \left( \frac{4CL_2 L^4 (L + 1)^2}{\mu_F} \beta_t \alpha_t + \frac{3CL_2 L^2 (1 + 2L^2)}{2\mu_F} \beta_t^2 \right) \prod_{\ell=t+1}^{k} \frac{\ell + 1}{\ell + 2} = \\
\frac{\mathbb{E}[V(\hat{x}_0, \hat{y}_0)]}{k + 2} + \sum_{t=0}^{k} \left( \frac{(2\mu_F + 3L_2)\Gamma_{22}}{2\mu_F} \beta_t^2 + \frac{3L_2 \Gamma_{11}}{2\mu_F} \beta_t \alpha_t \right) \frac{t + 2}{k + 2} + \\
\sum_{t=0}^{k} \left( \frac{3L_2 L_1 \Gamma_{22}}{2\mu_F \eta_x} \frac{3\beta_t^3}{2} + \frac{3CL_2 L_1 (L + 1)^2 \beta_t^3}{2\mu_F \eta_x \alpha_t^2} \right) \frac{t + 2}{k + 2} + \\
\sum_{t=0}^{k} \left( \frac{4CL_2 L^4 (L + 1)^2}{\mu_F} \beta_t \alpha_t + \frac{3CL_2 L^2 (1 + 2L^2)}{2\mu_F} \beta_t^2 \right) \frac{t + 2}{k + 2}. \tag{28} \]
Using the integral test and \( \alpha_k = \alpha_0/(k + 1)^{2/3} \) we have
\[
\sum_{t=0}^{k} \beta^2_t(t + 2) \leq \frac{1}{\mu^2_G} \left( 1 + \int_0^k \frac{1}{(t + 1)^{2/3}} \, dt \right) \leq \frac{1 + \ln(k + 1)}{\mu^2_G}
\]
\[
\sum_{t=0}^{k} \alpha_t \beta_t(t + 2) \leq \frac{\alpha_0}{\mu_G} \left( 1 + \int_0^k \frac{1}{(t + 1)^{2/3}} \right) \leq \frac{\alpha_0(k + 1)^{1/3}}{\mu_G}
\]
\[
\sum_{t=0}^{k} \beta_t^3(t + 2) \leq \frac{1}{\mu^3_G \alpha^2_0} \left( 1 + \int_0^k \frac{1}{(k + 1)^{2/3}} \right) \leq \frac{(k + 1)^{1/3}}{\mu^3_G \alpha^2_0}.
\]
Substituting these inequalities into (28) immediately gives (25).

**Remark 1** As observed from Eq. (28), the two terms decide the rate of the two-time-scale SA are \( \beta_k \alpha_k \) and \( \beta_k^3/\alpha_k^2 \), which also characterize the coupling between the two iterates. Our choice of \( a \) and \( b \) in the theorem is to balance these two terms, i.e., we want to achieve
\[
\alpha_k \beta_k = \frac{\beta_k^3}{\alpha_k^2} \Rightarrow \alpha_k^3 = \beta_k^2 \Rightarrow b = 1, \ a = \frac{2}{3}.
\]
However, one can choose different values of \( a \) and \( b \) for different purposes. For example, if we choose \( b \) smaller, e.g., \( b = 3/4 \), then one can quickly remove the impacts of the initial conditions on the performance of the method. However, this will cause a slow convergence of the variance term.

**Appendix A. Proofs of Lemmas 1–3**

**A.1. Proof of Lemma 1**

Proof We consider
\[
\dot{x}_{k+1} = x_{k+1} - H(y_{k+1}) = x_k - \alpha_k F(x_k, y_k) - \alpha_k \xi_k - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k)
\]
\[
= x_k - H(y_k) - \alpha_k F(x_k, y_k) - \alpha_k \xi_k + H(y_k) - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k),
\]
which implies that
\[
\|
\dot{x}_{k+1}
\|^2 = \|x_k - H(y_k) - \alpha_k F(x_k, y_k) - \alpha_k \xi_k + H(y_k) - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k)\|^2
\]
\[
= \|x_k - H(y_k) - \alpha_k F(x_k, y_k)\|^2 + \|H(y_k) - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k) - \alpha_k \xi_k\|^2
\]
\[
+ 2(x_k - H(y_k) - \alpha_k F(x_k, y_k))^T (H(y_k) - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k))
\]
\[
- 2\alpha_k (x_k - H(y_k) - \alpha_k F(x_k, y_k))^T \alpha_k \xi_k.
\]

We next analyze each term on the right-hand side of (29). First, using \( F(H(y_k), y_k) = 0 \) we have
\[
\|x_k - H(y_k) - \alpha_k F(x_k, y_k)\|^2
\]
\[
= \|x_k - H(y_k)\|^2 - 2\alpha_k (x_k - H(y_k))^T F(x_k, y_k) + \|\alpha_k F(x_k, y_k)\|^2
\]
\[
= \|
\dot{x}_k
\|^2 - 2\alpha_k (x_k - H(y_k))^T (F(x_k, y_k) - F(H(y_k), y_k)) + \alpha_k^2 \|F(x_k, y_k) - F(H(y_k), y_k)\|^2
\]
\[
\leq \|
\dot{x}_k
\|^2 - 2\mu F \alpha_k \|x_k - H(y_k)\|^2 + \mu^2 H \alpha_k^2 \|x_k - H(y_k)\|^2
\]
\[
= \left( 1 - 2\mu F \alpha_k + \mu^2 H \alpha_k^2 \right) \|
\dot{x}_k
\|^2,
\]
(30)
where in the first inequality we use the strong monotone and Lipschitz continuity of $F$, i.e., Eqs. (9) and (7), respectively. Next, we take the conditional expectation of the second term w.r.t $Q_k$ and using Assumption 3 to have
\[
\mathbb{E} \left[ \|H(y_k) - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k) - \alpha_k \xi_k \|^2 \mid Q_k \right] \\
= \mathbb{E} \left[ \|H(y_k) - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k)\|^2 \mid Q_k \right] + \alpha_k^2 \mathbb{E} \left[ \|\xi_k\|^2 \mid Q_k \right] \\
\leq L_H^2 \mathbb{E} \left[ \|\beta_k G(x_k, y_k) - \beta_k \psi_k\|^2 \mid Q_k \right] + \alpha_k^2 \mathbb{E} \left[ \|\xi_k\|^2 \mid Q_k \right] \\
= L_H^2 \beta_k^2 \mathbb{E} \left[ \|G(x_k, y_k)\|^2 \right] + \alpha_k^2 \mathbb{E} \left[ \|\xi_k\|^2 \mid Q_k \right] \\
= L_H^2 \beta_k^2 \mathbb{E} \left[ \|G(x_k, y_k)\|^2 \right] + \alpha_k^2 \mathbb{E} \left[ \|\xi_k\|^2 \mid Q_k \right] \\
\leq 2L_H^2 \beta_k^2 \mathbb{E} \left[ \|\hat{x}_k\|^2 \right] + 2L_H^2 \beta_k^2 (L_H + 1)^2 \mathbb{E} \left[ \|\hat{y}_k\|^2 \right] + \alpha_k^2 \mathbb{E} \left[ \|\xi_k\|^2 \mid Q_k \right],
\]
where the last inequality we use $G(H(y^*), y^*) = 0$ and the Lipschitz continuity of $G$ and $H$ to obtain
\[
\|G(x_k, y_k)\|^2 \leq (\|G(x_k, y_k) - G(H(y_k), y_k)\|^2 + \|G(H(y_k), y_k) - G(H(y^*), y^*)\|^2) \\
\leq (L_G \|\hat{x}_k\| + L_G (\|H(y_k) - H(y^*)\| + \|y_k - y^*\|))^2 \\
\leq 2L_G^2 \|\hat{x}_k\|^2 + 2L_G^2 (L_H + 1)^2 \|\hat{y}_k\|^2.
\]
Finally, we analyze the third term by using (30) and the preceding relation
\[
(x_k - H(y_k) - \alpha_k F(x_k, y_k))^T (H(y_k) - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k)) \\
\leq \|x_k - H(y_k) - \alpha_k F(x_k, y_k)\| \|H(y_k) - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k)\| \\
\leq \sqrt{1 - 2\mu_F \alpha_k + L_H^2 \alpha_k^2} \|\hat{x}_k\| \|L_H \beta_k G(x_k, y_k) + \psi_k\| \\
\leq L_H (1 + L_H \alpha_k) \beta_k \|\hat{x}_k\| (\|G(x_k, y_k)\| + \|\psi_k\|),
\]
where the last inequality we use the fact that $\sqrt{1 + a^2} \leq 1 + a$ for all $a \geq 0$. Applying the inequality (32) and the notation in (12) to the preceding relation yields
\[
(x_k - H(y_k) - \alpha_k F(x_k, y_k))^T (H(y_k) - H(y_k + \beta_k G(x_k, y_k) - \beta_k \psi_k)) \\
\leq L_H (1 + L_H \alpha_k) \beta_k \|\hat{x}_k\| (L_G \|\hat{x}_k\| + L_G (L_H + 1) \|\hat{y}_k\| + \|\psi_k\|) \\
= L_H L_G (1 + L_H \alpha_k) \beta_k \|\hat{x}_k\|^2 + (L_H + 1) \|\hat{x}_k\| \|\hat{y}_k\| + L_H L_G (1 + L_H \alpha_k) \beta_k \|\hat{x}_k\| \|\psi_k\| \\
\leq L_1 \beta_k \left( \|\hat{x}_k\|^2 + \frac{\eta_x \alpha_k}{2 \beta_k} \|\hat{x}_k\|^2 + \frac{(L_H + 1)^2 \beta_k}{2 \eta_x \alpha_k} \|\hat{y}_k\|^2 \right) + L_1 \beta_k \|\hat{x}_k\| \|\psi_k\| \\
\leq L_1 \beta_k \|\hat{x}_k\|^2 + L_1 \eta_x \alpha_k \|\hat{x}_k\|^2 + \frac{L_1 (L_H + 1)^2 \beta_k}{2 \eta_x \alpha_k} \|\hat{y}_k\|^2 + L_1 \beta_k \|\hat{x}_k\| \|\psi_k\|^2,
\]
where the second and last inequalities are due to the following two relations for $\eta_x > 0$, respectively,
\[
(L_H + 1) \|\hat{x}_k\| \|\hat{y}_k\| \leq \frac{\eta_x \alpha_k}{2 \beta_k} \|\hat{x}_k\|^2 + \frac{(L_H + 1)^2 \beta_k}{2 \eta_x \alpha_k} \|\hat{y}_k\|^2,
\]
\[
2 \beta_k \|\hat{x}_k\| \|\psi_k\| \leq \eta_x \alpha_k \|\hat{x}_k\|^2 + \frac{\beta_k^2}{\eta_x \alpha_k} \|\psi_k\|^2.
\]
Taking the conditional expectation on both sides of (29) w.r.t \( F_k \) and using (30)–(33) and Assumption 3 we obtain (14), i.e.,

\[
\mathbb{E} [\|\hat{x}_{k+1}\|^2 | Q_k] = (1 - 2\mu_F \alpha_k) \|\hat{x}_k\|^2 + L_H^2 (L_H + 1)^2 \beta_k^2 \|\hat{y}_k\|^2 + \beta_k^2 \Gamma_{22} + \alpha_k^2 \Gamma_{11} + \frac{L_1(L_H + 1)^2 \beta_k^2}{\eta_x \alpha_k} \|\hat{y}_k\|^2 \\
+ 2L_1 \beta_k \|\hat{x}_k\|^2 + 2L_1 \alpha_k \|\hat{x}_k\|^2 + \frac{L_1 \beta_k^2}{\eta_x \alpha_k} \mathbb{E} [\|\psi_k\|^2 | Q_k] \\
\leq (1 - 2\mu_F \alpha_k) \|\hat{x}_k\|^2 + \beta_k^2 \Gamma_{22} + \alpha_k^2 \Gamma_{11} + \frac{L_1 \beta_k^2}{\eta_x \alpha_k} \|\hat{y}_k\|^2 \\
+ 2L_2 \beta_k \|\hat{x}_k\|^2 + 2L_2 \alpha_k \|\hat{x}_k\|^2 + \frac{L_2 (L_H + 1)^2 \beta_k^2}{\eta_x \alpha_k} \|\hat{y}_k\|^2 \\
+ \frac{L_1(L_H + 1)^2 \beta_k^2}{\eta_x \alpha_k} \|\hat{y}_k\|^2 + 2L_1 (\beta_k + \eta_x \alpha_k) \|\hat{x}_k\|^2,
\]

where the last inequality we use the fact that \( \beta_k \leq \alpha_k \).

\[\blacksquare\]

### A.2. Proof of Lemma 2

**Proof** Using (2) we consider

\[
\hat{y}_{k+1} = y_{k+1} - y^* = y_k - y^* - \beta_k G(x_k, y_k) - \beta_k \psi_k \\
= y_k - y^* - \beta_k G(H(y_k), y_k) + \beta_k (G(H(y_k), y_k) - G(x_k, y_k)) - \beta_k \psi_k,
\]

which implies that

\[
\|\hat{y}_{k+1}\|^2 = \|y_k - y^* - \beta_k G(H(y_k), y_k) + \beta_k (G(H(y_k), y_k) - G(x_k, y_k)) - \beta_k \psi_k\|^2 \\
= \|y_k - y^* - \beta_k G(H(y_k), y_k)\|^2 + \|\beta_k (G(H(y_k), y_k) - G(x_k, y_k)) - \beta_k \psi_k\|^2 \\
+ 2 \beta_k (y_k - y^* - \beta_k G(H(y_k), y_k))^T (G(H(y_k), y_k) - G(x_k, y_k) - \psi_k). \tag{34}
\]

We next analyze each term on the right-hand side of (34). First, using \( G(H(y^*), y^*) = 0 \), (10), (6) and (8) we consider

\[
\|y_k - y^* - \beta_k G(H(y_k), y_k)\|^2 \\
= \|y_k - y^*\|^2 - 2\beta_k (y_k - y^*)^T G(H(y_k), y_k) + \beta_k^2 \|G(H(y_k), y_k)\|^2 \\
\overset{(10)}{\leq} \|\hat{y}_k\|^2 - 2\mu_G \beta_k \|y_k - y^*\|^2 + \beta_k^2 \|G(H(y_k), y_k) - G(H(y^*), y^*)\|^2 \\
= (1 - 2\mu_G \beta_k) \|\hat{y}_k\|^2 + \beta_k^2 \|G(H(y_k), y_k) - G(H(y_k), y_k) + G(H(y_k), y^*) - G(H(y^*), y^*)\|^2 \\
\leq (1 - 2\mu_G \beta_k) \|\hat{y}_k\|^2 + 2\beta_k^2 \|G(H(y_k), y_k) - G(H(y_k), y^*)\|^2 \\
+ 2\beta_k^2 \|G(H(y_k), y^*) - G(H(y^*), y^*)\|^2 \\
\leq (1 - 2\mu_G \beta_k) \|\hat{y}_k\|^2 + 2L_H \beta_k^2 \|H(y_k) - H(y^*)\|^2 + \|y_k - y^*\|^2 \\
\leq (1 - 2\mu_G \beta_k + 2(L_H^2 + 1) \beta_k^2) \|\hat{y}_k\|^2. \tag{35}
\]
Next, taking the conditional expectation of the second term w.r.t $Q_k$ and using Assumption 3 and (8) we have

$$
E \left[ \| \beta_k (G(H(y_k), y_k) - G(x_k, y_k)) - \beta_k \psi_k \|^2 \mid Q_k \right]
$$

$$
= \beta_k^2 \| G(H(y_k), y_k) - G(x_k, y_k) \|^2 + \beta_k^2 E \left[ \| \psi_k \|^2 \mid Q_k \right]
$$

$$
\leq L_G^2 \beta_k^2 E \left[ \| \hat{x}_k \|^2 \right] + \beta_k^2 \Gamma_{22}.
$$

Finally, taking the conditional expectation of the third term w.r.t $Q_k$ and using Assumption 3 and (35) we obtain for any $\eta_y > 0$

$$
2\beta_k E \left[ (y_k - y^* - \beta_k G(H(y_k), y_k))^T (G(H(y_k), y_k) - G(x_k, y_k) - \psi_k) \mid Q_k \right]
$$

$$
= 2\beta_k \| y_k - y^* - \beta_k G(H(y_k), y_k) \|^2 \| G(H(y_k), y_k) - G(x_k, y_k) \|
$$

$$
\leq 2\beta_k \sqrt{1 - 2\mu_G \beta_k + 2(L_H^2 + 1)L_G^2 \beta_k^2 \| \hat{y}_k \| L_G \| \hat{x}_k \|}
$$

$$
\leq 2L_G \sqrt{1 + 2(L_H + 1)L_G \beta_k \| \hat{y}_k \| \| \hat{x}_k \|} \leq 2L_2 \beta k \| \hat{y}_k \| \| \hat{x}_k \|
$$

$$
\leq \frac{L_2^2}{\mu_G} \beta_k \| \hat{x}_k \|^2 + \mu_G \beta_k \| \hat{y}_k \|^2,
$$

where the last inequality we use the relation $2ab \leq \eta a^2 + (1/\eta)b^2$ for any $a, b$ and $\eta > 0$. Thus, taking the conditional expectation of (29) w.r.t $Q_k$ and using (35)–(37) we obtain (15), i.e.,

$$
E \left[ \| \hat{y}_{k+1} \|^2 \mid Q_k \right] \leq (1 - 2\mu_G \beta_k + 2(L_H^2 + 1)L_G^2 \beta_k^2) \| \hat{y}_k \|^2 + L_G^2 \beta_k^2 \| \hat{x}_k \|^2 + \beta_k^2 \Gamma_{22}
$$

$$
+ \frac{L_2^2}{\mu_G} \beta_k \| \hat{x}_k \|^2 + \mu_G \beta_k \| \hat{y}_k \|^2
$$

$$
\leq (1 - \mu_G \beta_k) \| \hat{y}_k \|^2 + \beta_k^2 \Gamma_{22} + \frac{L_2^2}{\mu_G} \beta_k \| \hat{x}_k \|^2
$$

$$
+ L_G^2 \sqrt{1 + 2(L_H^2 + 1)L_G \beta_k \| \hat{x}_k \| + \| \hat{y}_k \|^2},
$$

where the last inequality is due to $\beta_k \leq \beta_0 \leq \mu_G / L_2^2$. 

\[\square\]
A.3. Proof of Lemma 3

Proof For convenience let \( z_k = \| \hat{x}_k \|^2 + \| \hat{y}_k \|^2 \). Adding (14) to (15) yields

\[
\mathbb{E} [ z_{k+1} | Q_k ] \leq (1 - 2 \mu_F \alpha_k) \mathbb{E} [ \| \hat{x}_k \|^2 ] + \beta_k^2 \Gamma_{22} + \alpha_k^2 \Gamma_{11} + \frac{L_1 \beta_k^2}{\eta_x \alpha_k} \Gamma_{22} + 2L_H^2 \mathbb{E} [ \| \hat{y}_k \|^2 ] + 2L_1 \beta_k \mathbb{E} [ \| \hat{x}_k \|^2 ] + 2L_1 (\beta_k + \eta_x \alpha_k) \mathbb{E} [ \| \hat{x}_k \|^2 ] + 2L_1 \beta_k \mathbb{E} [ \| \hat{y}_k \|^2 ] \\
+ L_1 (L_H + 1)^2 \beta_k \mathbb{E} [ \| \hat{x}_k \|^2 ] + 2L_1 (\beta_k + \eta_x \alpha_k) \mathbb{E} [ \| \hat{x}_k \|^2 ] + 2L_1 \beta_k \mathbb{E} [ \| \hat{y}_k \|^2 ] \\
+ (1 - \mu_G \beta_k) \mathbb{E} [ \| \hat{y}_k \|^2 ] + \beta_k^2 \Gamma_{22} + \frac{L_2^2 \beta_k}{\mu_G} \mathbb{E} [ \| \hat{x}_k \|^2 ] + \frac{L_1 (L_H + 1)^2 \beta_k}{\alpha_k} \mathbb{E} [ \| \hat{y}_k \|^2 ] \\
+ L_2 (1 + 2L_H^2) \beta_k E \left[ \| \hat{x}_k \|^2 \right] + \mathbb{E} \left[ \| \hat{y}_k \|^2 \right]
\]

\[
\leq z_k + 2 \beta_k^2 \Gamma_{22} + \alpha_k^2 \Gamma_{11} + \frac{L_1 \beta_k^2}{\eta_x \alpha_k} \Gamma_{22} + 3L_H \alpha_k \zeta_k + L_2 \beta_k (1 + 2L_H^2) \beta_k z_k \\
- 2 \mu_F \alpha_k \mathbb{E} [ \| \hat{x}_k \|^2 ] - \mu_G \beta_k \mathbb{E} [ \| \hat{y}_k \|^2 ] + \frac{L_1 (L_H + 1)^2 \beta_k}{\alpha_k} \mathbb{E} [ \| \hat{y}_k \|^2 ] \\
+ 2L_1 (\beta_k + \eta_x \alpha_k) \mathbb{E} [ \| \hat{x}_k \|^2 ] + \frac{L_2^2 \beta_k}{\mu_G} \mathbb{E} [ \| \hat{x}_k \|^2 ] .
\]

(38)

Using (16) we obtain

\[
\mu_F - L_1 \frac{\beta_k}{\alpha_k} - L_1 \eta_x - \frac{L_2^2 \beta_k}{2 \mu_G \alpha_k} = \mu_F - \left( L_1 + \frac{L_2^2}{2 \mu_G} \right) \frac{\beta_0}{\alpha_0 (k + 1)^{b-a}} - L_1 \eta_x \\
\geq \mu_F - \left( L_1 + \frac{L_2^2}{2 \mu_G} \right) \frac{\beta_0}{\alpha_0} - L_1 \eta_x \geq \mu_F - \frac{\mu_F}{3} - \frac{\mu_F}{3} = \frac{\mu_F}{3}.
\]

In addition, using (16) one more time we have

\[
- \mu_G + \frac{L_1 (L_H + 1)^2 \beta_k}{\eta_x \alpha_k} \leq - \mu_G + \frac{L_1 (L_H + 1)^2 \beta_0}{\eta_x \alpha_0} \leq - \frac{\mu_G}{2}.
\]

Thus, applying the previous two relations into (38) and using \( \mu = \min \{ \mu_F, \mu_G \} \) yield (18), i.e.,

\[
\mathbb{E} [ z_{k+1} | Q_k ] \leq z_k - \frac{2 \mu_F}{3} \alpha_k \| \hat{x}_k \|^2 - \frac{\mu_G}{2} \beta_k \| \hat{y}_k \|^2 + 3L_H^2 \mathbb{E} [ \| \hat{x}_k \|^2 ] + 2L_2 \beta_k \zeta_k \\
+ \frac{L_1 \beta_k^2}{\eta_x \alpha_k} \Gamma_{22} + 2L_1 \beta_k \mathbb{E} [ \| \hat{x}_k \|^2 ] + \frac{L_2 \beta_k^2}{\mu_G} \mathbb{E} [ \| \hat{x}_k \|^2 ] + \frac{L_1 \beta_k^2}{\eta_x \alpha_k} \Gamma_{22} \\
\leq (1 - \mu \beta_k + 4L_2 \alpha_k \zeta_k + 2 \beta_k \Gamma_{22} + \alpha_k \Gamma_{11} + \frac{L_1 \beta_k^2}{\eta_x \alpha_k} \Gamma_{22}.
\]

(39)

Next, since \( \alpha_k = \alpha_0 / (k + 1)^a \) for \( a \in (1/2, 1) \) using the integral test we have

\[
\sum_{t=0}^{\infty} \frac{\alpha_0}{(k + 1)^{2a}} \leq \alpha_0^2 + \int_0^\infty \frac{1}{(t + 1)^{2a}} dt \leq \alpha_0^2 + \frac{1}{2a - 1}.
\]
which by using $1 + x \leq \exp(x)$ for any $x \geq 0$ yields
\[
\prod_{t=k}^{\infty} (1 + 4L_G^2(L_H + 1)^4\alpha_0^2) \leq \exp \left\{ (1 + 4L_G^2(L_H + 1)^4) \sum_{t=k}^{\infty} \alpha_k^2 \right\}
\]
\[
\leq \exp \left\{ (1 + 4L_G^2(L_H + 1)^4) \sum_{t=0}^{\infty} \frac{\alpha_0^2}{(k + 1)^{2a}} \right\}
\]
\[
\leq \exp \left\{ (1 + 4L_G^2(L_H + 1)^4) \left( \alpha_0^2 + \frac{1}{2a - 1} \right) \right\} \triangleq C_1.
\]

Let $w_k$ be defined as
\[
w_k = \prod_{t=k}^{\infty} (1 + 4L_G^2(L_H + 1)^4\alpha_0^2) \mathbb{E}[z_k]
\]
Taking the expectation both sides of (39) and then multiplying by a finite number
\[
\prod_{t=k+1}^{\infty} (1 + 4L_G^2(L_H + 1)^4\alpha_0^2),
\]
and using (40) we obtain
\[
w_{k+1} \leq w_k + C_1 \left( 2\beta_0^2\Gamma_{22} + \alpha_k^2\Gamma_{11} + \frac{L_1\Gamma_{22}}{\eta_x} \frac{\beta_k^2}{\alpha_k} \right),
\]
which when summing up both sides over $k = 0, \ldots, K$ for some $K > 0$ yields
\[
w_{K+1} \leq w_0 + C_1 \left( 2\Gamma_{22} \sum_{k=0}^{K} \beta_k^2 + \Gamma_{11} \sum_{k=0}^{K} \alpha_k^2 + \frac{L_1\Gamma_{22}}{\eta_x} \sum_{k=0}^{K} \frac{\beta_k^2}{\alpha_k} \right)
\]
\[
\leq C_1 z_0 + C_1 \left[ \frac{2\Gamma_{22}(\beta_0^2(2b - 1) + 1)}{2b - 1} + \frac{\Gamma_{11}(\alpha_0^2(2a - 1) + 1)}{2a - 1} + \frac{L_1\Gamma_{22}}{\eta_x} \frac{\beta_0^2(2b - a - 1) + \alpha_0}{\alpha_0(2b - a - 1)} \right]
\]
\[
\triangleq C_1(z_0 + C_2),
\]
where the last inequality we use the integral test to have
\[
\sum_{k=0}^{K} \frac{\beta_k^2}{(k + 2)^{2b}} \leq \beta_0^2 + \frac{1}{2b - 1}, \quad \sum_{k=0}^{K} \frac{\alpha_k^2}{(k + 2)^{2a}} \leq \alpha_0^2 + \frac{1}{2a - 1}
\]
\[
\sum_{k=0}^{K} \frac{\beta_k^2}{\alpha_0(k + 2)^{2b - a}} \leq \frac{\beta_0^2}{\alpha_0^2} + \frac{1}{2b - a - 1}
\]
where we use $2b - a > 1$.

Thus, diving both sides of (42) by $\prod_{t=K+1}^{\infty} (1 + 4L_G^2(L_H + 1)^4\alpha_0^2)$ gives (19), i.e.,
\[
w_{K+1} \leq \frac{C_1(z_0 + C_2)}{\prod_{t=K+1}^{\infty} (1 + 4L_G^2(L_H + 1)^4\alpha_0^2)}
\]
\[
\leq \frac{C_1(z_0 + C_2)}{C_1(z_0 + C_2)} \exp \left\{ -\frac{(1 + 4L_G^2(L_H + 1)^4)}{2a - 1} \right\} \triangleq C,
\]
where the second inequality is due to the relation $1 + x \geq \exp(-x)$ for any $x \geq 0$ and the lower bound of the integral test

$$
\prod_{t=K+1}^{\infty} \left( 1 + 4L_C^2(L_H + 1)^4 \alpha_t^2 \right) \geq \exp \left\{ - \left( 1 + 4L_C^2(L_H + 1)^4 \right) \sum_{t=K+1}^{\infty} \alpha_t^2 \right\} \\
\geq \exp \left\{ - \frac{\left( 1 + 4L_C^2(L_H + 1)^4 \right)}{2\alpha - 1} \right\}.
$$
References

Albert Benveniste, Michel Méritier, and Pierre Priouret. *Adaptive algorithms and stochastic approximations*, volume 22. Springer Science & Business Media, 2012.

Dimitri Bertsekas and John Tsitsiklis. *Neuro-Dynamic Programming*. Athena Scientific, Belmont, MA, 2nd edition, 1999.

J. Bhandari, D. Russo, and R. Singal. A finite time analysis of temporal difference learning with linear function approximation. In COLT, 2018.

E. Biyik and M. Arcak. Area aggregation and time-scale modeling for sparse nonlinear networks. *Systems and Control Letters*, 57(2):142–149, 2008.

A. M. Boker, C. Yuan, F. Wu, and A. Chakrabortty. Aggregate control of clustered networks with inter-cluster time delays. In *Proc. of 2016 American Control Conference*, pages 5340–5345, 2016.

Vivek S. Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press, 2008.

L. Bottou, F. Curtis, and J. Nocedal. Optimization methods for large-scale machine learning. *SIAM Review*, 60(2):223–311, 2018.

Z. Chen, S. Zhang, T. T. Doan, S. T. Maguluri, and J-P. Clarke. Performance of Q-learning with Linear Function Approximation: Stability and Finite-Time Analysis. Available at: [https://arxiv.org/abs/1905.11425](https://arxiv.org/abs/1905.11425), 2019.

J. Chow and P. Kokotovic. Time scale modeling of sparse dynamic networks. *IEEE Transactions on Automatic Control*, 30(8):714–722, 1985.

G. Dalal, G. Thoppe, B. Szörényi, and S. Mannor. Finite sample analysis of two-timescale stochastic approximation with applications to reinforcement learning. In COLT, 2018.

T. T. Doan. Finite-time analysis and restarting scheme for linear two-time-scale stochastic approximation. Available at: [https://arxiv.org/abs/1912.10583](https://arxiv.org/abs/1912.10583), 2020.

T. T. Doan and J. Romberg. Linear two-time-scale stochastic approximation a finite-time analysis. In *2019 57th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 399–406, 2019.

T. T. Doan, C. L. Beck, and R. Srikant. On the convergence rate of distributed gradient methods for finite-sum optimization under communication delays. *Proceedings ACM Meas. Anal. Comput. Syst.*, 1(2):37:1–37:27, 2017.

T. T. Doan, S. T. Maguluri, and J. Romberg. Convergence rates of distributed gradient methods under random quantization: A stochastic approximation approach. *IEEE on Transactions on Automatic Control*, 2020.
H. Gupta, R. Srikant, and L. Ying. Finite-time performance bounds and adaptive learning rate selection for two time-scale reinforcement learning. In *Advances in Neural Information Processing Systems*, 2019.

Trevor Hastie, Robert Tibshirani, and Jerome Friedman. *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*. Springer, 2009.

M. Hong, H-T Wai, Z. Wang, and Z. Yang. A two-timescale framework for bilevel optimization: Complexity analysis and application to actor-critic. Available at: https://arxiv.org/abs/2007.05170, 2020.

B. Hu and U. Syed. Characterizing the exact behaviors of temporal difference learning algorithms using markov jump linear system theory. In *Advances in Neural Information Processing Systems* 32. 2019.

B. Karimi, B. Miasojedow, E. Moulines, and H-T Wai. Non-asymptotic analysis of biased stochastic approximation scheme. In *Conference on Learning Theory, COLT 2019, 25-28 June 2019, Phoenix, AZ, USA*, pages 1944–1974, 2019.

B. Karimi, B. Miasojedow, E. Moulines, and H-T Wai. Finite time analysis of linear two-timescale stochastic approximation with Markovian noise. In *Conference on Learning Theory*, 2020.

Petar Kokotović, Hassan K. Khalil, and John O’Reilly. *Singular Perturbation Methods in Control: Analysis and Design*. Society for Industrial and Applied Mathematics, 1999.

V. R. Konda and J. N. Tsitsiklis. On actor-critic algorithms. *SIAM J. Control Optim.*, 42(4), 2003.

V. R. Konda and J. N. Tsitsiklis. Convergence rate of linear two-time-scale stochastic approximation. *The Annals of Applied Probability*, 14(2):796–819, 2004.

Guanghui Lan. *Lectures on Optimization Methods for Machine Learning*. Springer-Nature, 2020.

H. R. Maei, C. Szepesvári, S. Bhatnagar, D. Precup, D. Silver, and R. S. Sutton. Convergent temporal-difference learning with arbitrary smooth function approximation. In *Proceedings of the 22nd International Conference on Neural Information Processing Systems*, page 1204–1212, 2009.

A. Mokkadem and M. Pelletier. Convergence rate and averaging of nonlinear two-time-scale stochastic approximation algorithms. *The Annals of Applied Probability*, 16(3):1671–1702, 2006.

T. V. Pham, T. T. Doan, and D. H. Nguyen. Distributed two-time-scale methods over clustered networks. Available at: https://arxiv.org/abs/2010.00355, 2020.

B. T. Polyak and A. B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM Journal on Control and Optimization*, 30(4):838–855, 1992.

H. Robbins and S. Monro. A stochastic approximation method. *The Annals of Mathematical Statistics*, 22(3):400–407, 1951.
H. Robbins and D. Siegmund. A convergence theorem for nonnegative almost supermartingales and some applications. *Optimization Methods in Statistics, Academic Press, New York*, pages 233–257, 1971.

D. Romeres, F. Dörfler, and F. Bullo. Novel results on slow coherency in consensus and power networks. In *Proc. of 2013 European Control Conference*, pages 742–747, 2013.

D. Ruppert. Efficient estimations from a slowly convergent robbins-monro process. *Technical Report 781, School of Operations Research and Industrial Engineering, Cornell Univ.*, 02 1988.

A. Saberi and H. Khalil. Quadratic-type lyapunov functions for singularly perturbed systems. *IEEE Transactions on Automatic Control*, 29(6):542–550, 1984.

R. Srikant and L. Ying. Finite-time error bounds for linear stochastic approximation and TD learning. In *COLT*, 2019.

R. Sutton, H. R. Maei, D. Precup, S. Bhatnagar, D. Silver, C. Szepesvári, and E. Wiewiora. Fast gradient-descent methods for temporal-difference learning with linear function approximation. In *Proceedings of the 26th International Conference On Machine Learning, ICML*, volume 382, 01 2009a.

R. Sutton, H. R. Maei, and C. Szepesvári. A convergent o(n) temporal-difference algorithm for off-policy learning with linear function approximation. In *Advances in Neural Information Processing Systems 21*, 2009b.

Richard S. Sutton and Andrew G. Barto. *Reinforcement Learning: An Introduction*. MIT Press, Cambridge, MA, 2nd edition, 2018.

M. Wang, E. X. Fang, and H. Liu. Stochastic compositional gradient descent: algorithms for minimizing compositions of expected-value functions. *Mathematical Programming*, 161(1), Jan 2017.

Y. Wu, W. Zhang, P. Xu, and Q. Gu. A finite time analysis of two time-scale actor critic methods. Available at: [https://arxiv.org/abs/2005.01350](https://arxiv.org/abs/2005.01350), 2020.

T. Xu, Z. Wang, and Y. Liang. Improving sample complexity bounds for actor-critic algorithms. Available at: [https://arxiv.org/abs/2010.00335](https://arxiv.org/abs/2010.00335), 2020.

J. Zhang and L. Xiao. A stochastic composite gradient method with incremental variance reduction. In *Advances in Neural Information Processing Systems 32*, pages 9078–9088. 2019.