FUNCTORIAL COMPACTIFICATION OF LINEAR SPACES

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Abstract. We define compactifications of vector spaces which are functorial with respect to certain linear maps. These “many-body” compactifications are manifolds with corners, and the linear maps lift to b-maps in the sense of Melrose. We derive a simple criterion under which the lifted maps are in fact b-fibrations, and identify how these restrict to boundary hypersurfaces. This theory is an application of a general result on the iterated blow-up of cleanly intersecting submanifolds which extends related results in the literature.

1. Introduction

One approach to the inherent challenge of solving an analytical problem on a non-compact space is to look for the “right compactification” and seek a solution there. To facilitate the analysis, the compactification should reside in a good category, say a smooth manifold with boundary or with corners, and the new points in the compactification should have some geometrical or analytical significance. For example, the (possibly singular) limits of a solution at the boundary faces of a compact manifold with corners typically encode important asymptotic regimes of original problem on the non-compact interior. While examples of this approach are far too numerous to list completely, we highlight [MM87, MM90, Mel93, Mel94, GZ03, Vas10] as a small sample of particular historical importance. In general, the question of which compactification is the “right” one is highly problem-specific and not always immediately obvious, even when the original space is something simple, like a vector space.

In this paper we present a class of compactifications of vector spaces with the key property that they admit smooth extensions of a given set of linear maps, or in other words are functorial. The starting point is a category, Lin, whose objects are finite dimensional real vector spaces $X$ equipped with a linear system—finite sets, $S_X$, of subspaces including $\{0\}$ and $X$ which are closed under intersection—and whose morphisms are the admissible linear maps $f : X \to Y$, which are required to satisfy $f^{-1}(V) \in S_X$ for every $V \in S_Y$. The many body compactification, $\hat{X}$, of a vector space $X \in \text{Lin}$ is the manifold with corners obtained from the radial compactification, $\overline{X}$, by the iterated blow-up of each boundary $\partial V$ of a subspace $V \in S_X$, in order from smallest to largest. This compactification is known in the literature going back to the work of Vasy on scattering theory in many body systems [Vas01]; however, that the association $X \mapsto \hat{X}$ is actually a functor from the category Lin to the category of manifolds with corners and b-maps is a new observation leading to novel applications which are discussed below.

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In fact, more can be said: if an admissible map $f : X \to Y$ is a so-called admissible quotient, then its functorially associated morphism $\hat{f}$ is actually a b-fibration, the analogue for corners of a fibration in the usual category of manifolds without corners. In this case we obtain a detailed description of the restriction of $\hat{f}$ to the boundary hypersurfaces of $\hat{X}$. Referring to §3 precise definitions, our results may be summarized as follows.

Theorem (Theorem 4.1, 5.1, and 5.3).

(i) The association $X \mapsto \hat{X}$ is a functor from Lin to the category of manifolds with corners, so every admissible map $f : X \to Y$ extends to a unique b-map $\hat{f} : \hat{X} \to \hat{Y}$. If $f$ is an admissible quotient, meaning $f(S_X) = S_Y$, then $\hat{f}$ is a b-fibration.
The boundary hypersurfaces of the compactifications $\tilde{X}$, which are indexed by $V \in S_X \setminus \{0\}$, are diffeomorphic to products $B_V \times \tilde{X}/V$, where $\tilde{X}/V$ is the many body compactification of the quotient $X/V$, and $B_V$ denotes the space obtained from the sphere $\partial V$ by the blow-up of each subsphere $\partial W$ where $W \in S_X$ and $W \subset V$.

(iii) In the case that $f : X \rightarrow Y$ is an admissible quotient, the restriction of the $b$-fibration $f$ to each boundary hypersurface $B_V \times \tilde{X}/V$ of $\tilde{X}$ is identified with the product $\partial \tilde{f}_V \times \int \tilde{f}_V$, where $\partial \tilde{f}_V : B_V \rightarrow B_{f(V)}$ is induced by the extension of the map $f|_V : V \rightarrow f(V)$ and $\int \tilde{f}_V : \tilde{X}/V \rightarrow Y/f(V)$ is the extension of the quotient map $f/f_V : X/V \rightarrow Y/f(V)$.

This theory leads to a particularly simple solution to a problem posed in [MS08] by Melrose and Singer, namely, to find compactifications of the products $X^n$, $n \in \mathbb{N}$ which lift all of the projections $X^n \rightarrow X^m$ for $m < n$ as well as the difference maps $X^n \rightarrow X$, $(x_1, \ldots, x_n) \mapsto x_i - x_j$. (See §6.)

In addition, the many body compactifications of vector spaces serve as key examples of a structure of interest in the manifolds with corners community known as a fibered corners structure [AM11, ALMP12, DLR15, CDR16] (see Remark 5.2 for details). Further equipping the vector spaces $X$ with Euclidean metrics, the compactifications $\tilde{X}$ become examples of quasi fibered boundary (QFB) manifolds, a class of spaces which has been of recent interest in Calabi-Yau and hyperKähler geometry [CDR16], as it generalizes the QALE manifolds of Joyce [Joy00]. While of course not topologically interesting, the $\tilde{X}$ have the advantage of a wealth of easily constructed $b$-maps (and in particular $b$-fibrations) coming from the underlying linear maps, making them important test cases for developing analytical results on QFB spaces.

In particular, they will play an important role in a forthcoming work of the author along with K. Fritzsch and M. Singer on a QFB compactification of the hyperKähler moduli space of SU(2) monopoles, where in addition to serving as simplified models for the compactified moduli space, the functoriality of many body spaces is an essential part of the analytical machinery used to obtain the compactification.

The functoriality of the many body compactifications is a consequence of our other main result (of independent interest) concerning the iterated blow-up of submanifolds in a manifold with corners. If $\mathcal{P}$ is a finite set of $p$-submanifolds of a manifold with corners $M$ (meaning submanifolds positioned nicely with respect to the boundary faces of $M$; see §3 for a precise definition), we say $\mathcal{P}$ is closed under clean intersection if each pair $P_i, P_j \in \mathcal{P}$ intersects cleanly and its intersection $P_i \cap P_j$ is also an element of $\mathcal{P}$. Among the many possible total orders on $\mathcal{P}$, a size order is any one which extends the partial order by inclusion, so that $P_i \cap P_j$ must precede both $P_i$ and $P_j$. More generally, an intersection order is any total order in which either $P_i$ or $P_j$ is allowed to precede $P_i \cap P_j$, but not both.

**Theorem** (Theorem 3.2, Corollary 3.5).

(i) If $\mathcal{P}$ is closed under clean intersection, then the iterated blow-up

$$[M; \mathcal{P}] := [M; P_1, \ldots, P_N]$$

is well-defined up to diffeomorphism, where $P_1, \ldots, P_N$ is any intersection order of the elements in $\mathcal{P}$; in particular the iterated blow-ups taken in two different intersection orders are canonically diffeomorphic.

(ii) If $\mathcal{Q} \subset \mathcal{P}$ is an intersection closed subset, then the blow-down $[M; \mathcal{P}] \rightarrow M$ factors through a unique $b$-map $[M; \mathcal{P}] \rightarrow [M; \mathcal{Q}]$.

Part (i) generalizes a result proved in [Vas01] for size orders, under an additional assumption that each pair $\{P_i, P_j\}$ forms a so-called normal family, while parts (i) and (ii) were proved in [MS08] in the special case that $\mathcal{P}$ is a set of boundary faces.

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An outline of the paper is as follows. In §2, we define the category of linear systems and admissible maps. §3 contains background information on manifolds with corners and blow-up, and concludes with a proof of the second Theorem. In §4 we prove the functoriality of the many body compactification—part (i) of the first Theorem above—and in §5 we characterize the boundary hypersurfaces of the many body compactification, proving parts (ii) and (iii). Finally, as an application, we show in §6 how appropriate many body compactifications of the products $X^n$ furnish an alternative to the scattering products of [MS08].

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2. Linear Systems and Many Body Compactification

Definition 2.1. A linear system in a vector space $X$ is a finite set $S_X$ of subspaces of $X$ such that

(i) $\{0\}$ and $X$ are in $S_X$, and
(ii) whenever $W$ and $V$ are in $S_X$, then $W \cap V$ is in $S_X$.

Given another vector space $Y$ with linear system $S_Y$, we say a linear map $f : X \rightarrow Y$ is admissible if $f^{-1}(S_Y) \subset S_X$, that is, if $f^{-1}(W) \in S_X$ for every $W \in S_Y$. In particular, $\ker f$ is required to be in $S_X$. The collection of finite dimensional vector spaces with linear systems and admissible maps form a category $\text{Lin}$.

If $K$ is a subspace of $X$ (not necessarily in $S_X$), then $S_X/K := \{V/(V \cap K) : V \in S_X\}$ is a linear system in the quotient $X/K$. In general, the quotient map $\pi : X \rightarrow X/K$ need not be admissible. In fact, admissibility of $\pi$ is equivalent to the condition that $V + K$ is in $S_X$ for every $V$ in $S_X$ (in particular, for $V = \{0\}$). By an admissible quotient map, we will understand that the target $X/K$ is equipped with the linear system $S_X/K$ (as opposed to some subsystem, with respect to which the quotient would still be admissible).

Lemma 2.2. The following are equivalent:

(i) $f : X \rightarrow Y$ is admissible and $f(S_X) := \{f(V) : V \in S_X\} = S_Y$.
(ii) $Y \cong X/\ker f$, and $f$ is an admissible quotient map.

Proof of Lemma 2.2. The condition $f(S_X) = S_Y$ implies in particular that $Y = f(X)$, so $f$ is surjective and we may identify $Y$ with $X/K$, where $K = \ker f$. Under this identification, $f(V) \cong V/(V \cap K)$, so the condition $f(S_X) = S_Y$ becomes the statement that $S_Y \cong S_X/K$. $\square$

Definition 2.3. The many body compactification of a linear system $(X, S_X)$ is the manifold with corners

$\hat{X} = [\mathcal{X}; \partial S_X], \quad \partial S_X = \{\partial V : V \in S_X\}$

obtained by iteratively blowing up the boundaries of the subspaces inside the radial compactification $\bar{X}$ of $X$. The blow-up is performed in some size order, meaning any total order on $S_X$ extending the partial order defined by inclusion of subspaces. We prove in Corollary 3.4 that $\hat{X}$ is well-defined, following a digression through the theory of manifolds with corners, referring the reader to [Mel93, Mel] for a comprehensive account.

3. On Manifolds with Corners

Recall that the radial compactification of a vector space $X$ is the manifold with boundary $\overline{X}$ obtained by adjoining a sphere of dimension $\dim(X) - 1$ at infinity, with the smooth structure induced by taking $1/r$ as a boundary defining function for any choice of Euclidean norm $r$. Equivalently,
The following local coordinate characterization is convenient: given a coordinate chart in which
preimage in $\mathbb{R}$

$x$ with $\hat{x}$ where $R$

The boundary hypersurface

$M$

boundary face

we use the term boundary face if the codimension is positive. In particular, boundary hypersurfaces
are boundary faces of codimension one, and we require as part of the definition of a manifold with
corners that these are embedded—that is, upon taking the closure of a set of points of codimension
one, no self-intersections occur.

A $p$-submanifold $P \subset M$ is a submanifold of some face $F$ of $M$, with $F$ taken as small as possible,
which is required to intersect any boundary hypersurface of $M$ transversally in $F$; in particular
$P$ is covered by “product-type” coordinate charts valued in $\mathbb{R}^{n-l} \times \mathbb{R}^l$ for various $l$, in which it is
locally defined by the vanishing of codim$(P)$ of the coordinates. It follows from the definition that
if $P$ and $Q$ are $p$-submanifolds of $M$ and $P \subseteq Q$, then $P$ is also a $p$-submanifold of $Q$ (there is
some confusion about this point in the literature). The (radial) blow-up of $P$ in $M$, denoted by
$[M;P]$, is the space $(M \setminus P) \cup S_+P$ where $S_+P$ is the inward pointing spherical normal bundle
(that is, the set of unit vectors in $NP$ with respect to some inner product which integrate to
flows of $M$) with smooth structure induced by polar coordinates normal to $P$. This space admits
a blow-down map $\beta : [M;P] \rightarrow M$, which is the smooth surjection defined by the identity on
$M \setminus P$ and the bundle projection on $S_+P$; this is an example of a $b$-map, as defined in §4 below.
The boundary hypersurface $\beta^{-1}(P) \subset [M;P]$ is referred to as the front face of the blow-up. The
following local coordinate characterization is convenient: given a coordinate chart in which $P$ has
the form $\{(x_1,\ldots,x_n) : x_1 = \cdots = x_k = 0\}$ where the $x_i$ are each valued in either $\mathbb{R}_+$ or $\mathbb{R}$, the
preimage in $[M;P]$ is covered by charts with “projective coordinates”

$$ (\tilde{x}, x' / \tilde{x}), \quad \tilde{x} = \pm x_i \geq 0, \; i \in \{1,\ldots,k\}, \quad x' = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \quad (1) $$

where $\tilde{x}$ is a $\mathbb{R}_+$ coordinate ranging over $\pm x_1,\ldots, \pm x_k$ (we take only the + sign if the original
coordinate was $\mathbb{R}_+$-valued, and both signs if it was $\mathbb{R}$-valued), and $x'$ denotes the tuple $(x_1,\ldots,x_n)$
with $x_i$ removed; in such a chart $\tilde{x}$ is a boundary defining coordinate for the front face.

If $S \subset M$ is another $p$-submanifold, the lift (or proper transform) of $S$ in $[M;P]$ is defined to be
the set $\beta^{-1}(S)$ if $S \subset P$, and the closure of $\beta^{-1}(S \setminus P)$ in $[M;P]$ otherwise. Committing a minor
abuse of notation, we will continue to use the same letter to denote both a $p$-submanifold of $M$ and
its lift to $[M;P]$. Finally, we use the abbreviated notation $[M;P;S]$ to denote the iterated blow-up
$[[M;P];S]$, given by first blowing up $P$ in $M$ and then the lift of $S$ in $[M;P]$, provided that this
lift is a $p$-submanifold in $[M;P]$.

In the setting of Definition 2.3 above, we may identify the closure of each $V \in \mathcal{S}_X$ in $\overline{X}$ with its
radial compactification $\overline{V}$; these along with their boundaries $\partial \overline{V}$ are clearly $p$-submanifolds in $\overline{X}$.
That $\overline{X}$ is well-defined is a direct consequence of the following general results about commuting
blow-ups in manifolds with corners.

Recall that a set $P = \{P_1,\ldots,P_\ell\}$ of $p$-submanifolds of $M$ is a normal family if it admits simultaneous
product-type coordinates; in other words, $\bigcup_i P_i$ is covered by coordinate charts $\{\{x_1,\ldots,x_n\}\}$
in which every $P_j$ has the form $\{\{x_1,\ldots,x_n\} : x_i = 0, i \in I_j\}$ for some $I_j \subset \{1,\ldots,n\}$.

**Proposition 3.1** ([Mel], Prop. 5.8.1 and Prop. 5.8.2). Let $M$ be a manifold with corners, with
$p$-submanifolds $P,Q \subset M$. If one of the three following conditions hold:

(i) one of the submanifolds is included in the other, say $P \subset Q$, or
(ii) $P$ and $Q$ are disjoint in $M$, or
(iii) \( P \) and \( Q \) are normally transverse, meaning \( P \) and \( Q \) constitute a normal family and intersect transversally in \( M \), then there is a natural diffeomorphism \([M; P, Q] \cong [M; Q, P]\).

Though this result is well-known, we include a proof below since the arguments we employ form the basis for other results later on.

**Proof of Proposition 3.1.** Observe that in all three cases \( \{ P, Q \} \) is a normal family. Indeed, by induction any nested sequence \( P_1 \subset P_2 \subset \cdots \subset P_\ell \) of p-submanifolds forms a normal family, since we may assume that we have simultaneous product coordinates for \( \{ P_1, \ldots, P_\ell \} \) and use the fact that \( P_k \) is a p-submanifold of \( P_{k+1} \) to extend these to simultaneous local product coordinates for \( \{ P_1, \ldots, P_{k+1} \} \).

If \( P \) and \( Q \) are disjoint, then \([M; P, Q] \cong [M; Q, P]\) obviously holds. Suppose next that \( P \subset Q \). This means that locally along \( P \) there exist coordinates \( (x, y, z) \), where \( P = \{ y = 0, z = 0 \} \) and \( Q = \{ z = 0 \} \). Here \( x, y \) and \( z \) are tuples (e.g. \( z = (z_1, \ldots, z_k) \)) each component of which is valued either in \( \mathbb{R} \) or \( \mathbb{R}_+ \). The preimage of such a coordinate chart in \([M; Q]\) is covered by charts having coordinates of the form \((x, y, \hat{z}, z'/\hat{z})\), where in notation as above \( \hat{z} \in \mathbb{R}_+ \), runs through \( \pm z_i, i = 1, \ldots, k \), and \( z' \) denotes the tuple \( z \) with \( z_i \) removed. The lift of \( P \) in any such chart is the set \( \{ y = 0, \hat{z} = 0 \} \). On the blow-up \([M; Q, P]\), we then have coordinates \((x, y/\hat{z}, \hat{z}, z'/\hat{z})\) and \((x, \hat{y}, y'/\hat{y}, \hat{z}/\hat{y}, z'/\hat{z})\), using similar notation.

In the other direction, \([M; P]\) has coordinates of the form \((x, \hat{y}, y'/\hat{y}, z/\hat{y})\), in which \( Q \) lifts as \( \{ z/\hat{y} = 0 \} \), and of the form \((x, y/\hat{z}, \hat{z}, z'/\hat{z})\), which \( Q \) does not meet. Further passing to \([M; P, Q]\), we have a cover by coordinates of the form \((x, \hat{y}, y'/\hat{y}, \hat{z}/\hat{y}, (z'/\hat{y})/(\hat{z}/\hat{y}) = z'/\hat{z})\). Clearly we can identify these coordinate charts in \([M; P, Q]\) with the respective ones from \([M; Q, P]\), which patch together into a diffeomorphism.

Finally suppose \( P \) and \( Q \) are normally transverse; this amounts to the local existence of coordinates \((x, y, z)\) as above in which now \( P \) is the set \( \{ z = 0 \} \) and \( Q \) is the set \( \{ y = 0 \} \). The blow-up \([M; P]\) admits coordinate charts of the form \((x, y, \hat{z}, \hat{z}, z'/\hat{z})\) with \( Q \) lifting as \( \{ y = 0 \} \), so that \([M; P, Q]\) has charts of the form \((x, \hat{y}, y'/\hat{y}, \hat{z}/\hat{y}, z'/\hat{z})\). This is clearly symmetric upon interchanging the roles of \( P \) and \( Q \), so we can again patch together a diffeomorphism \([M; P, Q] \cong [M; Q, P]\). \( \square \)

**Theorem 3.2.** Let \( M \) be a manifold with corners, and \( \mathcal{P} \) a collection of p-submanifolds which is closed under clean intersection (meaning the submanifolds intersect cleanly pairwise, and each such non-empty intersection is an element of \( \mathcal{P} \)).

(i) The iterative blow-up \([M; \mathcal{P}] = [\cdots [M; P_1]; P_2]; \cdots; P_N]\) is well-defined where \( P_1 < P_2 < \cdots < P_N \) is any choice of size order, meaning a total order on \( \mathcal{P} \) extending the partial order by inclusion.

(ii) If \( Q \subset \mathcal{P} \) is an intersection closed subset of \( \mathcal{P} \), then the blow-down \([M; \mathcal{P}] \to M\) factors through a unique b-map \([M; \mathcal{P}] \to [M; Q]\).

**Proof.** In part (i), given a choice of total order on \( \mathcal{P} \), we must first justify why the iterated blow-up is defined; more precisely we must show that upon blowing up some \( Q \), those \( P \) such that \( Q < P \) lift again to p-submanifolds. If \( Q \) and \( P \) are disjoint in \( M \) to begin with then this is obvious. If \( Q \) and \( P \) meet but \( Q \not\subset P \), then by the clean intersection property \( Q \) and \( P \) lift to disjoint p-submanifolds in the blow-up of \( Q \cap P \), which must precede both in the given order. Finally, if \( Q \subset P \) in \( M \), then this inclusion relation persists to their lifts under blow-up of those elements preceding \( Q \) in the size order, and then the fact that \( P \) lifts to a p-submanifold upon blowing up \( Q \) was observed in the local coordinate computation in the proof of Proposition 3.1 above.

To see that \([M; \mathcal{P}] \) is well-defined independent of the choice of size order, note that any size order may be obtained from any other one by a sequence of size orders in which pairs of adjacent elements (necessarily incomparable in the original partial order) are swapped. Thus we consider a
While \( Q \)-submanifold in the blow-up of the boundary point \( P \) defined, since upon blowing up some \( Q \)-submanifolds. The simplest example of this situation occurs in the half space \( M = \mathbb{R}^2 \times \mathbb{R}_+ \), with \( Q = \{ x = y = 0 \} \) the z-axis and \( P = \{ x = z, y = 0 \} \), with \( P \) a “diagonal” line meeting \( Q \) cleanly at the boundary point \( P \cap Q = (0, 0, 0) \). Nevertheless, it so happens that the lift of such \( P \) eventually becomes p-submanifolds after further blow-ups.

**Corollary 3.5.** Theorem 3.2 holds with “size order” replaced by “intersection order”.

Remark 3.3. In [Vas01], (c.f. Lemmas 2.7 and 2.8), Vasy proves a result similar to part (i) under the additional hypothesis that the p-submanifolds are pairwise normal, i.e., each pair \( \{ P_i, P_j \} \) is a normal family. In the proof above, we only use normality for nested sequences (which is automatic), so this additional hypothesis can be removed.

**Corollary 3.4.** The many body compactification \( \hat{X} \) of a linear system \((X, S_X)\) is well-defined.
Proof. It suffices to show that the iterated blow-up of $\mathcal{P}$ in an intersection order is well-defined and diffeomorphic to the blow-up of $\mathcal{P}$ in some size order. We do this by induction on the size of $\mathcal{P}$, the case $|\mathcal{P}| = 1$ being trivial. (Also, the case $|\mathcal{P}| = 2$ is covered by Proposition 3.1.) Thus, given an intersection ordered set $\mathcal{P} = \{P_1 < \cdots < P_n\}$, we assume by induction that $[M; P_1, \ldots, P_{n-1}]$ is well-defined, and without any loss of generality we may assume that $\mathcal{Q} = \{P_1 < \cdots < P_{n-1}\}$ is in a size order. Then that $[M; P_1, \ldots, P_{n-1}, P_n] = [M; \mathcal{Q}, P_n]$ is well-defined and diffeomorphic to a blow-up of $\mathcal{P}$ in $M$ in a size order was shown in the proof of part (ii) of Theorem 3.2 above. \[ \square \]

Remark 3.6. This generalizes Proposition 3.5 and Corollary 3.8 in [MS08], where the authors prove Corollary 3.5 in the case that $\mathcal{P}$ and $\mathcal{Q}$ are collections of boundary faces of $M$. In fact it is not quite a full generalization of their results, since Melrose and Singer relax the condition that the sets be closed under intersection, requiring only that they be closed under non-transversal intersection, meaning that $P \cap Q$ is only required to be in $\mathcal{P}$ if $P$ and $Q$ are not (normally) transverse. (Note that transverse boundary faces are automatically normally transverse.) The proofs of Theorems 3.2 and Corollary 3.5 would go through under this weaker hypothesis thanks to Proposition 3.1.(iii), provided the lifts of a normally transverse pair $P$ and $Q$ remain normally transverse upon blowing up the elements preceding them in any intersection order. While this is automatic for boundary faces, it is not clear (to the author at least) that it holds for pairwise cleanly intersecting $p$-submanifolds without further hypotheses.

The intersection order condition in Corollary 3.5 is sharp in the sense that $[M; P, Q, P \cap Q]$, even if well-defined, is generally not diffeomorphic to $[M; P \cap Q, P, Q]$. This is evident in simple examples, such as that of two distinct lines meeting at a point in $\mathbb{R}^3$.

4. Many body compactification as a functor

Returning to our original setting, we now show that the many body compactification is a functor from Lin to the category $\mathbf{MwC}$ of manifolds with corners. While there are various choices of morphisms between manifolds with corners (in addition to the conventions of [Mel93] observed here, compare for instance [Joy12] or [Joy16], Definition 2.1), we take the morphisms in $\mathbf{MwC}$ to be the $b$-maps\footnote{These are called “smooth maps” in [Joy16]. Joyce uses the term “weakly smooth” for what we call smooth here.} \( g : M \to N \), which are by definition those smooth maps (i.e., \( g^* (C^\infty (N)) \subset C^\infty (M) \)) such that for each boundary defining function \( \rho_H \) of a boundary hypersurface \( H \subset N \), the pullback \( g^* (\rho_H) \) either vanishes identically (implying that \( g(M) \subset H \)) or has the form

\[
g^* (\rho_H) = a \prod_{H'} \rho_{H'}^{e_{H'H}}, \quad e_{H'H'} \in \mathbb{N}_0, \quad a > 0 \in C^\infty (M).
\]  

Here the index \( H' \) ranges over boundary hypersurfaces of \( M \), and \( \rho_{H'} \) is a boundary defining function for \( H' \). In this note all $b$-maps are interior, meaning that (3) always holds. Examples include the blow-down maps \( \beta : [M; P] \to M \). Of particular importance are the $b$-fibrations, which are fibrations in the usual sense over the interiors and restrict again to $b$-fibrations over each boundary face of the domain to some boundary face of the range. They are defined to be $b$-maps whose natural differential (c.f. [Mel93]) is surjective pointwise, and for which at most one exponent \( e_{H'H'} \) is nonzero for each \( H' \) in (3); equivalently each boundary hypersurface of \( M \) is mapped surjectively either onto some boundary hypersurface of \( N \) or onto \( N \) itself.

Theorem 4.1. Every admissible map \( f : X \to Y \) extends to a unique $b$-map \( \hat{f} : \hat{X} \to \hat{Y} \). Moreover, if \( f \) is an admissible quotient, then \( \hat{f} \) is a $b$-fibration.

Proof. We consider first the case that \( f : X \to X/K \) is an admissible quotient, with \( \mathcal{S}_X = \{0\}, K, X \), so we must show that \( \hat{X} = \left[X; \partial K\right] \to X/K \) is a $b$-fibration. For this we choose
a complement $X = W \oplus K$ and write $x = (x_1, x_2) \in W \oplus K \cong \mathbb{R}^{n-k} \oplus \mathbb{R}^k$ in “product radial”
coordinates

$$x = (x_1, x_2) = R\omega = R(r\xi_1, s\xi_2),$$

$$R = |x|, \quad \omega = \frac{x}{R}, \quad r = \frac{x_1}{R}, \quad s = \frac{x_2}{R}, \quad \xi_1 = \frac{x_1}{Rr} \in S^{n-k-1}, \quad \xi_2 = \frac{x_2}{Rs} \in S^{k-1}.$$ 

As with standard polar coordinates if $R$ or $r$ or $s$ vanishes than the spherical variables are under-
determined; it is more accurate to view the coordinates as a map

$$\mathbb{R}_+ \times [0, 1] \times S^{n-k-1} \times S^{k-1} = \{(R, r, \xi_1, \xi_2)\} \mapsto (Rr\xi_1, R(\sqrt{1-r^2})\xi_2) \in X$$

which is a diffeomorphism away from the zero sets of $R$, $r$ or $s = \sqrt{1-r^2}$. In any case, coordinates on the radial compactification of $X$ are given by $(\rho, \omega) = (\rho, (r, \xi_1), (s, \xi_2))$, where

$$\rho = 1/R,$$ 

and coordinates on $X/K$ are given by $(\sigma, \xi_1)$ where $\sigma = 1/Rr$. The submanifold $\partial K$ is given by $\{\rho = r = 0\}$, and its blow-up in $X$ is parameterized near the corner by coordinates $(\sigma, (r, \xi_1), (\sqrt{1-r^2}, \xi_2))$, where again $\sigma = \rho/r = 1/Rr$, and $\{r = 0\}$ is no longer singular. The projection map $X \rightarrow X/K; (x_1, x_2) \mapsto x_1$ extends by continuity to the map $[X; \partial K] \rightarrow X/K$, $(\sigma, r, \xi_1, \xi_2) \mapsto (\sigma, \xi_1)$, which is manifestly a b-fibration.

Returning to the general case of an admissible quotient, let us assume inductively that we have a b-fibration $[X; \partial S'] \rightarrow [X/K; \partial (S'/K)]$ for an intersection closed subset $S' \subset S_X$ of the linear system in $X$, the base case $S' = \{0\}, K, X$ having been shown above. Let $W \in S_X \setminus S'$ be a minimal element, meaning there is no $V \in S_X \setminus S'$ with $V \subset W$. In particular $W \cap V \subset S'$ for all $V \in S'$. There are three possibilities:

1. $W$ is contained in $K$. In this case the image of $W$ in $X/K$ is the trivial subspace, and does not induce an additional blow-up in the target $[X/K; \partial (S'/K)]$. In the domain, the composition of the blow-down $[X; \partial S', \partial W] \rightarrow [X; \partial S']$ with the b-fibration to $[X/K; \partial (S'/K)]$ is again a b-fibration, since the front face associated to $\partial W$ maps into the interior of the target.

2. $W$ intersects $K$ transversally. In this case the image of $W$ in $X/K$ is the whole space, and again does not induce an additional blow-up in the target. In the blow-up $[X; \partial S', \partial W]$, the front face maps to the original radial boundary of $[X/K; \partial (S'/K)]$, which is of codimension one, so this is again a b-fibration.

3. If neither of the above holds, then $W$ descends to the proper nontrivial subspace $W/(W \cap K)$ in $X/K$. If we haven’t yet blown up (the lift of) $\partial W/(W \cap K)$ in the target, then we may blow this up along with its preimage in the domain, which is (the lift of) $\partial W + K$; by admissibility $W + K$ is an element of $S_X$. The old b-fibration lifts to these blow-ups, and is again a b-fibration since the new front face of the b-domain is mapped onto the new front face of the target which has codimension one. In so doing we may assume that $W + K \in S'$. Then if $W \neq W + K$, the composition of the blow-down $[X; \partial S', \partial W] \rightarrow [X; \partial S']$ with the b-fibration to $[X/K; \partial (S'/K)]$ is a b-fibration since the front face maps onto the hypersurface associated to $\partial W/(W \cap K)$.

In any case, by Theorem 3.2, the space $[X; \partial S', \partial W]$ is diffeomorphic to a size order blow-up $[X; \partial S'']$ where $S'' = S' \cup \{W\}$, and we may then replace $S'$ by $S''$ to complete the induction. This completes the proof that admissible quotients lift to b-fibrations.

For a general admissible map $f : X \rightarrow Y$, we make a series of reductions. By admissibility, $f^{-1}(S_Y)$ is an intersection closed linear subsystem of $S_X$. Then $[X; \partial S_X]$ admits a b-map to $[X; \partial f^{-1}(S_Y)]$ by Theorem 3.2, so we can suppose from now on that $S_X = f^{-1}(S_Y)$. We may factor $f$ as the quotient $X \rightarrow X/K$ and an injection $X/K \hookrightarrow Y$, where $K = \ker f$. Since every element of $f^{-1}(S_Y)$ contains $K := \ker f$, the map $X \rightarrow X/K$ is an admissible quotient, which extends to a b-map as shown above, so it remains to consider the case that $X$ is a subspace of
elements, and for each pair \( V \) and \( S \) many-body compactification with system \( S \). Moreover, if \( W \supset X \), then the lift of \( X \) to the blow-up of \( \partial W \) in \( Y \) is diffeomorphic again to \( X \). It follows iteratively then that \( X \subset Y \) lifts to a p-submanifold of \( Y \) which is diffeomorphic to \( \tilde{X} = [X; \partial S_Y \cap X] \).

5. Boundary faces

A linear system \( S_X \) is a set which is partially ordered by inclusion, has minimal and maximal elements, and for each pair \( V \), \( W \) of elements has a unique infimum \( V \cap W \). It is notationally convenient at this point to use \( S_X \) as an abstract partially ordered indexing set, and from now on we will use Greek letters \( \lambda, \mu \in S_X \) for elements, with the order and infimum denoted by \( \lambda \leq \mu \) and \( \lambda \wedge \mu \), respectively. We denote the minimal element by 0 and sometimes denote the maximal element by 1. We write \( X_\lambda \) instead of \( \lambda \) when we wish to emphasize the actual subspaces of \( X \), thus \( X_0 = \{0\} \), \( X_1 = X \), and \( X_{\lambda \wedge \mu} = X_\lambda \cap X_\mu \).

**Theorem 5.1** (c.f. [Vas01]). There is a bijective correspondence between boundary hypersurfaces of \( \tilde{X} \) and \( S_X \setminus \{0\} \), under which \( \lambda \in S_X \setminus \{0\} \) corresponds to a hypersurface \( N_\lambda \) diffeomorphic to the product

\[
N_\lambda \cong B_\lambda \times F_\lambda, \quad B_\lambda = [\partial X_\lambda; \{\partial X_\mu: \mu < \lambda\}], \quad F_\lambda = \overline{X/X_\lambda}.
\]

Moreover \( N_{\lambda_1} \cap \cdots \cap N_{\lambda_k} \neq \emptyset \) if and only if \( \{\lambda_1 < \cdots < \lambda_k\} \subset S_X \) is a totally ordered subset. In this case

\[
N_{\lambda_1} \cap \cdots \cap N_{\lambda_k} \cong B_{\lambda_1} \times B_{\lambda_1 \wedge \lambda_2} \times \cdots \times B_{\lambda_{k-1} \wedge \lambda_k} \times F_{\lambda_k},
\]

\[
B_{\nu, \mu} = [\partial X_\mu/X_\nu; \{\partial X_\kappa/(X_\kappa \cap X_\nu) : \kappa < \mu\}].
\]

In particular there is a maximal \( S_X \) boundary hypersurface \( N_1 = B_1 \) with respect to \((S_X, \leq)\) which in the many-body compactification with system \( S_F_X \cong \{\mu \in S_X : \mu \leq \lambda\} \) and maximal element \( 1 = \lambda \). The other factor \( B_\lambda \), which may be identified with the free region of \( \tilde{X}_\lambda \), is not a many-body compactification, but it is a manifold with corners whose boundary hypersurfaces are indexed by the ordered set \( \{\mu \in S_X : \lambda > \mu\} \).

It is convenient to set \( N_0 = X \), which is not of course a boundary hypersurface, but is consistent with (4), since \( B_0 \) is a point (being the radial compactification of \( \{0\} \)), and \( F_0 = \overline{X}/\{0\} = \tilde{X} \).

**Proof of Theorem 5.1.** The definition of \( \tilde{X} \) makes it clear that there is indeed a boundary hypersurface \( N_\lambda \) for each element \( \lambda \) of \( S_X \setminus \{0\} \). To see the structure of \( N_\lambda \), consider its origin as the submanifold \( \partial X_\lambda \) inside \( X \). We first blow-up in \( X \) all those submanifolds \( \partial X_\mu \) such that \( \mu < \lambda \) (note that if \( \mu \) precedes \( \lambda \) in the size order but not in the original partial order on \( S_X \) then \( \partial X_\mu \) and \( \partial X_\lambda \) do not meet), after which the lift of \( \partial X_\lambda \) is diffeomorphic to the space \( B_\lambda = [\partial X_\lambda; \{\partial X_\mu: \mu < \lambda\}] \).

We then blow-up this lift \( B_\lambda \) of \( \partial X_\lambda \) itself, introducing as a front face the inward pointing spherical normal bundle of \( B_\lambda \) in \( X \). However, the inward spherical normal bundle of \( \partial X_\lambda \) in \( X \) is equivalent (essentially by definition) to the radial compactification of the normal bundle to \( X_\lambda \) in \( X \) and this remains true even after passing to \( B_\lambda \) by blow-up. Since the spaces are linear, this bundle is simply the product \( B_\lambda \times X/X_\lambda \).

Finally, we proceed to blow-up the lifts of those \( \partial X_\mu \) such that \( \mu > \lambda \), which meet \( B_\lambda \times X/X_\lambda \) in the submanifolds \( B_\lambda \times \partial X/X_\mu \) sitting inside the boundary face \( B_\lambda \times \partial X/X_\mu \), from which the first claim follows.
The second claim follows from the fact that for any pair of subspaces \( X_\lambda \) and \( X_\mu \) such that \( X_\lambda \not\subset X_\mu \) and \( X_\mu \not\subset X_\lambda \), the lifts of \( \partial X_\lambda \) and \( \partial X_\mu \) are made disjoint by the blow-up of \( \partial X_{\lambda \wedge \mu} \).

If, on the other hand, \( \lambda < \mu \), then \( N_\mu \) and \( N_\lambda \) meet precisely in the boundary hypersurface of \( N_\lambda = B_\lambda \times F_\lambda \) introduced by the blow up of \( B_\lambda \times \partial X_\mu \) inside \( B_\lambda \times \partial X_\lambda \), which corresponds in the second factor to the face \( N_{\lambda, \mu} = B_{\lambda, \mu} \times F_\lambda \) of \( F_\lambda \), giving \( N_\mu \cap N_\lambda = B_\lambda \times B_{\lambda, \mu} \times F_\lambda \). The general case follows by induction. 

Remark 5.2. The structure (4) is a primary example of what is known variously in the manifolds with corners literature as a resolution structure [AM11], an iterated fibration structure [ALMP12], or (the term we use here) a fibered corners structure [CDR16]. In general, this means a manifold with corners \( M \) whose boundary hypersurfaces \( N_\lambda \) are indexed by a partially ordered set and are equipped with fibrations \( \phi_\lambda : N_\lambda \rightarrow B_\lambda \) with typical fiber \( F_\lambda \), such that:

(i) Each \( F_\lambda \) and \( B_\lambda \) are also manifolds with corners.

(ii) \( N_\lambda \cap \cdots \cap N_{\lambda_N} \neq \emptyset \) if and only if \( \lambda_1 < \cdots < \lambda_N \) is a totally ordered chain.

(iii) If \( \lambda < \mu \), then \( \phi_\lambda|_{N_\lambda \cap N_\mu} : N_\lambda \cap N_\mu \rightarrow B_\lambda \) is a fibration whose typical fiber is a boundary hypersurface \( \partial \mu F_\lambda \) of \( F_\lambda \), while \( \phi_\mu|_{N_\lambda \cap N_\mu} \) is a restriction of \( \phi_\mu \) over a boundary hypersurface \( \partial \lambda B_\mu := \phi_\mu(N_\lambda \cap N_\mu) \) of \( B_\mu \); moreover there is a fibration \( \phi_{\lambda, \mu} : \partial \lambda B_\mu \rightarrow B_\lambda \) such that \( \phi_{\lambda, \mu} \circ \phi_\mu = \phi_\lambda \).

(iv) Every boundary hypersurface of \( F_\lambda \) is of the form \( \partial \mu F_\lambda \) for some \( \mu > \lambda \) and likewise every boundary hypersurface of \( B_\mu \) is of the form \( \partial \lambda B_\mu \) for some \( \lambda < \mu \). In particular it follows that each \( F_\lambda \) and \( B_\lambda \) has a fibered corners structure induced by the maps \( \phi_\lambda \) and \( \phi_{\mu, \lambda} \), respectively.

(The fibered corners structure on a many body compactification \( \bar{X} \) is suggested by (4), with \( F_\lambda \) the fiber and \( B_\lambda \) the base, which is the correct interpretation from the point of view of QFB metrics (see below). However, since the fibrations are products in this case, \( \bar{X} \) also admits an inequivalent fibered corners structure with the indexing set, as well as the roles of the \( F_\lambda \) and \( B_\lambda \), reversed!)

In fact, by [ALMP12], a manifold \( M \) with fibered corners is equivalent to the resolution of a smoothly stratified space \( \bar{M} = M/\sim \), obtained by collapsing the fibers of each boundary hypersurface, i.e., taking the quotient by the equivalence relation where \( p \sim q \) if \( \phi_\lambda(p) = \phi_\lambda(q) \) for some \( \lambda \). Conversely, a smoothly stratified space may be defined intrinsically as a stratified space \( S \) with control data in the sense of Mather—in particular, the strata admit tubular neighborhoods in \( S \) which are assumed to be locally trivial cone bundles (see [ALMP12] for a detailed definition)—and then the resolution by iterative radial blowup of the strata of \( S \) yields a manifold with fibered corners.

In our case, the smoothly stratified space in question is simply the original radial compactification \( X \), with strata consisting of the interior of \( \overline{X} \lambda \) along with the boundaries \( \partial \overline{X}_\lambda \setminus \{ \partial \overline{X}_\mu : \mu < \lambda \} \) of the subspaces in the system \( S_X \).

In addition to this combinatorial topological structure, there is a natural geometric structure on \( \bar{X} \) induced by any Euclidean metric on \( X \). Such structure may characterized equivalently in terms of the vector fields which are bounded with respect to such a metric, and in turn these vector fields admit a metric-independent description. In the general setting of a manifold \( M \) with fibered corners, a quasi-fibered boundary (QFB) structure (see [CDR16]) is defined by a Lie subalgebra \( \mathcal{V}_{\text{QFB}}(M) \subset \mathcal{V}_b(M) \) of vector fields (here \( \mathcal{V}_b(M) \) is the algebra of vector fields tangent to all boundary faces of \( M \)), defined as those vector fields \( V \) such that

(i) \( V \) is tangent to the fibers \( F_\lambda \) at each boundary hypersurface \( N_\lambda \), and

(ii) \( V(\rho) \in \rho^2 C^\infty(M) \) where \( \rho = \prod \rho_\lambda \) is a choice of total boundary defining function.

A QFB metric may be then be defined as a Riemannian metric on the interior such that the pointwise norm of each \( V \in \mathcal{V}_{\text{QFB}}(M) \) extends smoothly up to the boundary of \( M \). In the special
case that the boundary fibrations are trivial for maximal $\lambda$ (so $N_\lambda \cong B_\lambda$ with $\phi_\lambda \cong \text{Id}$), a QFB structure is known as a quasi-asymptotically conic (QAC) structure (c.f. [DM14, CDR16]).

That the lift of a Euclidean metric on $X$ furnishes a QAC metric on $\hat{X}$ is trivial to verify: indeed, on the radial compactification $\overline{X}$, the inverse radial function $\rho = r^{-1}$ furnishes a canonical boundary defining function, which then lifts to a total boundary defining function on $\hat{X}$. Moreover, the vector fields which are bounded with respect to the Euclidean metric are precisely those $V$ on $\overline{X}$ such that $V\rho \in \rho^2 C^\infty(\overline{X})$ (these are the scattering vector fields in the sense of Melrose [Mel94]), and these are easily seen to lift to be tangent to the boundary fibrations on $\hat{X}$.

From now on we consider the b-fibration $\hat{f} : \hat{X} \to \hat{Y}$ associated to an admissible quotient $f : X \to Y$. As $\hat{f}$ is a b-fibration, the smallest face of $\hat{Y}$ containing the image $f(N_\lambda)$ of each hypersurface $N_\lambda$ is either a hypersurface $M_{\mu}$ of $\hat{Y}$ or $\hat{Y}$ itself. We recall that the restriction of a b-fibration to an arbitrary boundary face of the domain is again a b-fibration.

Fix $\lambda \in S_X$ and let $\mu = f(\lambda) \in S_Y$. Note then that

$$f_\lambda := f|_{X_\lambda} : X_\lambda \to Y_\mu, \quad \text{and} \quad f/f_\lambda : X/X_\lambda \to Y/Y_\mu$$

are admissible linear maps, the former of which sends the maximal element of $S_{X_\lambda}$ to the maximal element in $S_{Y_\mu}$ and the latter of which is an admissible quotient.

**Theorem 5.3.** Let $f : X \to Y$ be an admissible quotient, let $N_\lambda$ be a boundary hypersurface of $\hat{X}$, and set $\mu = f(\lambda) \in S_Y$. Then the restriction of $\hat{f}$ to $N_\lambda$ is a b-fibration onto $M_\mu$ which is diffeomorphic to the product map

$$\hat{f}_\lambda \times \frac{f}{f_\lambda} : B_\lambda \times F_\lambda \to B_\mu \times F_\mu$$

where $B_\lambda \subset \hat{X}_\lambda$ and $B_\mu \subset \hat{Y}_\mu$ are the free regions of $\hat{X}_\lambda$ and $\hat{Y}_\mu$, respectively, and $F_\lambda = X/X_\lambda$ and $F_\mu = Y/Y_\mu$ as above.

Note that the statement applies as well in the case that $f(\lambda) = 0$ (i.e., $X_\lambda \in \ker f$), in which case $N_\lambda$ maps onto $M_0 = \hat{Y}$ itself, via the map

$$0 \times \hat{f}/f_\lambda : B_\lambda \times X/X_\lambda \to \{0\} \times \hat{Y}.$$

**Proof of Theorem 5.3.** By uniqueness of the continuous extensions of $f$, $f_\lambda$ and $f/f_\lambda$ to the compactifications of their respective domains, it suffices to show that $\hat{f}$ and $\hat{f}_\lambda \times \frac{f}{f_\lambda}$ agree on the interior of $N_\lambda$. As noted in the proof of Theorem 5.1 above, this interior may be identified with the normal bundle of (an open dense subset of) $\partial X_\lambda$, which is just the product $\partial X_\lambda \times X/X_\lambda$. On the other hand, as a linear map $f$ may be identified with its normal differential along $X_\lambda$, which may be in turn identified with the product map $f_\lambda \times f/f_\lambda$, it suffices to show that $\hat{f}_\lambda \times \frac{f}{f_\lambda}$ from $X \cong X_\lambda \times X/X_\lambda$ to $Y \cong Y_\mu \times Y/Y_\mu$.

The extension of this by continuity over $\partial X_\lambda \times X/X_\lambda$ agrees by definition with $\hat{f} \times f/f_\lambda$, and therefore with $\hat{f} \times f/f_\lambda$ on the interior of its domain.

6. An application

In [MS08], Melrose and Singer consider the problem of compactifying the products $X^n$ of a vector space $X$ as manifolds with corners $X^n_{sc}$ in such a way that

(i) $X^n_{sc} = \overline{X}$, the radial compactification,

(ii) the action of the permutation group $\Sigma_n$ lifts to $X^n_{sc}$,

(iii) the various projections

$$\pi_I : X^n \ni (u_1, \ldots, u_n) \mapsto (u_{i_1}, u_{i_2}, \ldots, u_{i_k}) \in X^k, \quad I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\} \quad (5)$$

lift to b-fibrations $X^n_{sc} \to X^k_{sc}$, and
(iv) the difference maps
\[
\delta_{ij} : X^n \ni (u_1, \ldots, u_n) \mapsto u_i - u_j \in X, \quad i \neq j
\]
lift to b-fibrations \(X^n_{sc} \to \overline{X}\).

In fact they work in the setting of a general compact manifold with boundary \(M\) in place of \(\overline{X}\), so generalizing to higher \(n\) the scattering spaces \(M^2_{sc}\) and \(M^3_{sc}\) introduced in [Mel94] to support kernels of pseudodifferential operators and their compositions. (Note that (iv) does not make sense at this level of generality.) In order to work in this general setting, Melrose and Singer must start with the manifolds with corners \(M^n = (\overline{X})^n\) and develop quite a few delicate and technical results about commutativity of blow-up of various families of submanifolds in order to obtain spaces satisfying the required properties.

On the other hand, provided one is willing to stick to the original setting of vector spaces, the comparatively simpler theory developed here furnishes an immediate solution. Indeed, within the product \(X^n\) consider two families of subspaces: the axes \(\{(u_1, \ldots, u_n) : u_i = 0 \text{ for } i \in J\}\) and the diagonals \(\{(u_1, \ldots, u_n) : u_i = u_j \text{ for } i, j \in J\}\), where here \(J\) runs over all subsets of \(\{1, \ldots, n\}\). The following is immediate.

**Theorem 6.1.** Let \(X\) be a vector space, and for \(n \in \mathbb{N}\), equip \(X^n\) with the linear system generated by all axes and diagonals. Then the permutations \(\Sigma_n \ni \sigma : X^n \to X^n\), the projections (5), and the difference maps (6) are all admissible quotients, hence lift to respective b-fibrations
\[
\hat{\sigma} : \hat{X}^n \to \hat{X}^n, \quad \hat{\pi}_J : \hat{X}^n \to \hat{X}^k, \quad \text{and} \quad \hat{\delta}_{ij} : \hat{X}^n \to \overline{X}.
\]

**Remark 6.2.** The main reason why the “many-body space” solution to the above compactification problem is so much simpler than the “scattering products” solution is the availability in the linear setting of the radial compactification \(\overline{X^n}\) of the product as an alternative to the product \((\overline{X})^n\) of the radial compactifications. If \(M\) is a manifold with boundary, it is possible to define the analogue of the radial compactification of the products \((M^n)^n\), though unless \(\partial M\) is a sphere these will be singular stratified spaces, and it is far from clear that an analogue of Theorem 3.2 holds in such a category.

Finally, we note here that the spaces \(\hat{X}^n\) and \(X^n_{sc}\) are not diffeomorphic if \(n \geq 3\), the verification of which we leave as an exercise to the interested reader.

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