CHARACTERIZATIONS OF SOBOLEV SPACES ON SUBLEVEL SETS IN ABSTRACT WIENER SPACES

DAVIDE ADDONA, GIORGIO MENEGATTI, MICHELE MIRANDA JR

ABSTRACT. In this paper we consider an abstract Wiener space \((X, \gamma, H)\) and an open subset \(O \subseteq X\) which satisfies suitable assumptions. For every \(p \in (1, +\infty)\) we define the Sobolev space \(W^{1,p}_0(O, \gamma)\) as the closure of Lipschitz continuous functions which support with positive distance from \(\partial O\) with respect to the natural Sobolev norm, and we show that under the assumptions on \(O\) the space \(W^{1,p}_0(O, \gamma)\) can be characterized as the space of functions in \(W^{1,p}(O, \gamma)\) which have null trace at the boundary \(\partial O\), or, equivalently, as the space of functions defined on \(O\) whose trivial extension belongs to \(W^{1,p}(X, \gamma)\).

1. Introduction

In this paper we consider an abstract Wiener space \((X, \gamma, H)\), i.e., \(X\) is a real separable Banach space endowed with a centered non-degenerate Gaussian measure \(\gamma\) and \(H\) is the associated Cameron-Martin space, and a subset \(O \subseteq X\) with \(O = G^{-1}((-\infty, 0))\), where \(G\) is a function which satisfies suitable assumptions (see Hypotheses 3.1).

The topic of Sobolev spaces \(W^{k,p}(X, \gamma)\) in a Wiener space \(X\) is well established (see e.g. [5]), while Sobolev spaces in subsets of a Wiener space admit different definitions, and they have been treated for example in [7], [12] and [13].Following [7] (see also [3]), we consider \(W^{1,p}(O, \gamma)\) as the domain of closure \(V_H\) of the \(H\)-gradient operator on Lipschitz continuous functions.

In [7], the set \(O\) is the sublevel \(G^{-1}((-\infty, 0))\) of a function \(G\). Under some regularity assumptions on \(G\) it is possible to define a surface measure \(\rho\) (Hausdorff-Gauss infinite dimensional measure or Feyel-de La Pradelle measure, see e.g. [11]). Moreover, in [7] the authors show the existence of a bounded operator (trace operator) \(\text{Tr}\) from \(W^{1,p}(O, \gamma)\) to \(L^q(\partial O, \rho)\) with \(p > 1\) and \(q \in [1, p)\). Thanks to this operator, it is possible to introduce an integration-by-parts formula on \(O\) which generalize that on the whole space. Namely, for every \(\varphi \in W^{1,p}(O, \gamma)\) and every \(h \in H\), we have

\[
\int_O (\partial_h \varphi - \hat{\varphi}) \, d\gamma = \int_{\partial O} \text{Tr} \varphi \frac{\langle \nabla_H G; h \rangle_H}{\|\nabla_H G\|_H} \, d\rho,
\]

with \(\partial_h \varphi(x) = \langle \nabla_H \varphi(x), h \rangle_H\) and \(\hat{\varphi} = R_{\gamma}^{-1}(h)\), where \(R_{\gamma}\) is the covariance operator of \(\gamma\). Integration-by-parts formula as in [11] on domains have been also proved in [3][4] with different techniques. Strengthening the assumptions on \(G\), the trace operator can be extended as an operator from \(W^{1,p}(O, \gamma)\) onto \(L^p(\partial O, \rho)\) for every \(p > 1\). Finally, in [7] the authors prove that the subspace of \(f \in W^{1,p}(O, \gamma)\), consisting of functions with null trace on \(\partial O\), coincides with the subspace of \(f \in W^{1,p}(O, \gamma)\), whose elements are those functions whose trivial extension to the whole \(X\) belongs to \(W^{1,p}(X, \gamma)\).

In this paper we consider \(O = G^{-1}((-\infty, 0))\) and we define the space \(W^{1,p}_0(O, \gamma)\) as the closure, with respect to the \(W^{1,p}(O, \gamma)\)-norm, of Lipschitz continuous functions whose support has positive distance from \(O^c\). Eventually we prove that, under suitable conditions
on $G$, for every $p > 1$, $W^{1,p}_0(O, \gamma)$ is the space of functions in $W^{1,p}(O, \gamma)$ with null trace on $\partial O$ (Theorem 4.1).

Examples of spaces $O$ to which our results apply can be found in the Section 5; they include subgraphs of functions with some regularity, and subsets of $X$ in the particular case in which $X$ is the Wiener space which models the Brownian motion or in the case in which $X$ is the Wiener space which models the pinned Brownian motion.

We stress that these examples may not comprehend many regular sets like balls, neither if $X$ is a Hilbert space. This limitation is strictly related to our approach, and also appears in [7], in the case when the operator $\text{Tr}$ maps $W^{1,p}(O, \gamma)$ onto $L^p(\partial O, \rho)$ when $p > 1$. To the best of our knowledge, nowadays there is no other result about the definition of a trace operator from $W^{1,p}(O, \gamma)$ onto $L^p(\partial O, \rho)$ for more general subsets $O$, and also the case $p = 1$ is not reached. This is one of the main gaps with respect to the finite dimension, where the theory of traces for Sobolev functions is well-understood and complete. For the case $p = 1$, a possible alternative approach is to consider BV functions in open domains in Wiener spaces, which are studied and characterized in [2]. However, it is still not clear how to extend the theory of traces for BV functions in finite dimension to this setting.

Beside traces, another open question in infinite dimension is what domains $O$ allow the construction of extension operators for Sobolev functions. Again, the case when $O$ is an open ball is still an open problem, even when $X$ is a Hilbert space. A negative answer is given in [6], where the authors provide an example of open convex subset $O$ of a Hilbert space $X$ such that, for every $p > 1$, there exists a function $f \in W^{1,p}(O, \gamma)$ which does not admit a Sobolev extension to the whole $X$. On the contrary, an example of extension operator can be found in [8], where the authors show that if $O$ is an half-plane that it admits an extension operator, and explicitly provide such an extension.

Acknowledgements. G. M. wants to thank Michael Röckner for posing the problem which originated this work and for several important suggestions, and moreover for hosting him to Bielefeld University for a research period. D. A. claims that this research has financially been supported by the Programme “FIL-Quota Incentivante” of University of Parma and co-sponsored by Fondazione Cariparma.

2. Notations and preliminary results

In the following, for any $k, d \in \mathbb{N}$ we denote by $C^k_b(\mathbb{R}^d)$ the set of $k$-times differentiable functions on $\mathbb{R}^d$ with all derivatives uniformly bounded. $C^k_c(\mathbb{R}^d)$ is the set of bounded smooth functions on $\mathbb{R}^d$ which belongs to $C^k_b(\mathbb{R}^d)$ for every $k \in \mathbb{N}$. $C^\infty_c(\mathbb{R}^d)$ is the set of functions in $C^\infty_b(\mathbb{R}^d)$ with compact support.

For every real-valued function $f$ defined in a subset $A \subseteq X$, we denote by $\overline{f}$ its trivial extension on $X$, i.e., $\overline{f} = f$ on $A$ and $\overline{f} = 0$ on $A^c$.

For every $A \subseteq X$, we denote by $1_A$ the characteristic function of $A$, i.e., $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$.

Let $K$ be a real separable Hilbert space, and let $\mathcal{L}(K)$ be the space of linear bounded operators on $K$. We denote by $\mathcal{L}_2(K)$ the subspace of $\mathcal{L}(K)$ whose elements $L$ satisfy

$$\|L\|_{\mathcal{L}_2(K)}^2 := \sum_{n=1}^{\infty} \|Le_n\|_K^2 < +\infty,$$

where $\{e_n : n \in \mathbb{N}\}$ is any orthonormal basis of $K$. The elements of $\mathcal{L}_2(K)$ are called Hilbert-Schmidt operators, and the norm $\| \cdot \|_{\mathcal{L}_2(K)}$ is the Hilbert-Schmidt norm. The space
The space \( (L^2(K), \| \cdot \|_{L^2(K)}) \) is a separable Hilbert space if endowed with the inner product

\[
[L, M]_{L^2(K)} = \sum_{n=1}^{\infty} (L e_n, M e_n)_K, \quad L, M \in L^2(K),
\]

where \( \{e_n : n \in \mathbb{N} \} \) is any orthonormal basis of \( K \). Given a real separable Banach space \( X \), we denote by \( \mathscr{B}(X) \) the Borel subsets of \( X \).

We denote by \( \text{Lip}(X) \) the set of Lipschitz continuous functions from \( X \) onto \( \mathbb{R} \). For every open set \( O \subseteq X \) we denote by \( \text{Lip}(O) \) the set of Lipschitz continuous functions on \( O \), by \( \text{Lip}_b(O) \) the set of bounded Lipschitz continuous functions on \( O \), and by \( \text{Lip}_c(O) \) the set of Lipschitz continuous functions on \( O \) whose support has positive distance from \( O^c \).

We recall some definitions and properties of abstract Wiener spaces (see e.g. [5]). Let \( X \) be a separable Banach space, let \( X^* \) be its dual and let \( X^{**} \) be the dual of \( X^* \). We will suppose that \( \gamma \) is a centered non-degenerate Gaussian measure on \( X \).

We consider the embedding \( j : X^* \hookrightarrow L^2(X, \gamma) \), and we define the reproducing kernel \( K \) as the closure in \( L^2(X, \gamma) \) of \( j(X^*) \). It is a separable Hilbert space endowed with the \( L^2 \)-norm, and we introduce the covariance operator \( R_\gamma : X^* \to X^{**} \) defined as

\[
R_\gamma f(g) = \int_X f\left( \frac{g}{\|g\|} \right) d\gamma, \quad f \in X^*_\gamma, \ g \in X^*.
\]

\( R_\gamma \) is injective, and its range is contained in \( X \), by identifying \( X \) with its natural embedding in \( X^{**} \). We define the Cameron-Martin space \( H \) as \( R_\gamma(X^*_\gamma) \subseteq X; \) \( H \) inherits a structure of separable Hilbert space through \( R_\gamma \); we define \( \langle \cdot, \cdot \rangle_H \) as the inner product in \( H \) and \( \| \cdot \|_H \) as the associated norm. As a subspace of \( X, H \) is dense. If \( h \in H \), we define \( \hat{h} = R_\gamma^{-1}(h) \), so that \( \hat{h} \in X^*_\gamma \subseteq L^2(X, \gamma) \). The triple \( (X, H, \gamma) \) is called abstract Wiener space.

We fix an orthonormal basis \( \{h_i : i \in \mathbb{N}\} \) of \( H \) such that \( h_i \in R_\gamma(X^*) \) for every \( i \in \mathbb{N} \). We have that \( \{\hat{h}_i : i \in \mathbb{N}\} \) is an orthonormal basis of \( X^*_\gamma \subseteq L^2(X, \gamma) \), and for every \( f \in L^2(X, \gamma) \) we get

\[
\sum_{i=1}^{\infty} |\langle f, \hat{h}_i \rangle_{L^2(X, \gamma)}|^2 \leq \|f\|_{L^2(X, \gamma)}^2.
\]

For every \( n \in \mathbb{N} \) we define the projection \( \pi_n : X \to \text{span}\{h_1, \ldots, h_n\} \subseteq H \) as

\[
\pi_n(x) = \sum_{i=1}^{n} \hat{h}_i(x) h_i, \quad x \in X.
\]

We denote by \( L^p(X, \gamma; H) \) the space of (equivalence classes of) Bochner integrable functions \( f : X \to H \) such that

\[
\|f\|_{L^p(X, \gamma; H)} := \left( \int_X \|f(x)\|_H^p \ d\gamma \right)^{1/p} < \infty.
\]

\( L^p(X, \gamma; H) \) is a Banach space endowed with the norm \( \| \cdot \|_{L^p(X, \gamma; H)} \) (see e.g. [9]).

Let \( n \in \mathbb{N} \) and let \( F \) be a \( n \)-dimensional subspace of \( R_\gamma(X^*) \subseteq H \). If \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( F \), we define the projection \( \pi_F \) of \( X \) on \( F \) as the bounded linear function

\[
\pi_F(x) = \sum_{i=1}^{n} \hat{e}_i(x) e_i
\]

for every \( x \in X. \) \( \pi_F \) is uniquely defined, independently from the choice of the basis.

We denote by \( \gamma_F \) the image measure \( \gamma \circ \pi_F^{-1} \) on \( F \), i.e.,

\[
\gamma_F(A) = \gamma(\pi_F^{-1}(A))
\]
for every $A$ Borel set in $F$. It follows that $\gamma_F$ is a non-degenerate centered Gaussian measure, and there exist a Banach space $X_{F^\perp}$ and a non-degenerate centered Gaussian measure $\gamma_{F^\perp}$ such that we have an isometry between $F \times X_{F^\perp}$ and $X$, and $\gamma = \gamma_F \otimes \gamma_{F^\perp}$. This is said factorization of $\gamma$ with respect to $F$.

We define the space of bounded infinitely many times differentiable cylindrical functions $\mathcal{F}C^\infty_b(X)$ as the set of functions $f : X \to \mathbb{R}$ such that

$$f(x) = g(l_1(x), \ldots, l_n(x)), \quad x \in X,$$

where $l_1, \ldots, l_n \in X^*$ are bounded linear functions on $X$ and $g \in C^\infty_b(\mathbb{R}^n)$ for some $n \in \mathbb{N}$. We recall that $\mathcal{F}C^\infty_b(X)$ is dense in $L^p(X, \gamma)$ for every $p \in [1, +\infty)$. $\mathcal{F}C^\infty_b(X; H)$ denotes the set of functions $f : X \to H$ with finite dimensional range such that, for every $l \in H$, we have $x \mapsto \langle l, f(x) \rangle_H \in \mathcal{F}C^\infty_b(X)$. In particular, $\mathcal{F}C^\infty_b(X; H)$ is spanned by functions $\phi h$ with $\phi \in \mathcal{F}C^\infty_b(X)$ and $h \in H$. It is easy to prove that $\mathcal{F}C^\infty_b(X; H)$ is dense in $L^p(X, \gamma; H)$.

For every smooth function $f : X \to \mathbb{R}$, every $h \in H$ and every $x \in X$, we define the partial derivative $\partial_h f(x)$ of $f$ at $x$ along $h$ as

$$\partial_h f(x) := \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon} \quad (2.2)$$

and the partial logarithmic derivative $\partial^*_h f(x)$ of $f$ at $x$ along $h$ as

$$\partial^*_h f(x) := \partial_h f(x) - f(x) \hat{h}(x).$$

We say that $f$ is $H$-differentiable in $x \in X$ if there exists $\nabla_H f(x) \in H$ such that

$$\partial_h f = \langle \nabla_H f, h \rangle_H, \quad h \in H.$$

If $f \in \mathcal{F}C^\infty_b(X)$ then it is everywhere $H$-differentiable, $\nabla_H f = R_D f$, where $D f$ is the Fréchet derivative of $f$, and $\nabla_H f \in L^\infty(X, \gamma; H)$. Further, the operator $\nabla_H$ is well defined for any Lipschitz continuous function $f$ and $\nabla_H f \in L^\infty(X, \gamma; H)$ (see e.g. [4, Theo. 5.11.2]).

For every $p \in [1, +\infty)$, $\nabla_H : \mathcal{F}C^\infty_b(X) \to L^p(X, \gamma; H)$ is a closable operator in $L^p(X, \gamma)$. We still denote its closure as $\nabla_H$ and we define the Sobolev space $W^{1,p}(X, \gamma)$ as the domain of this closure (see [4, Sec. 5.2]). Moreover, if $f \in W^{1,p}(X, \gamma)$, then $\nabla_H f \in L^p(X, \gamma; H)$ and for every $h \in H \setminus \{0\}$ we set $\partial_h f = \langle \nabla_H f, h \rangle_H$.

Let $f : X \to H$. We say that $f$ is $H$-differentiable at $x \in X$ if there exists a Hilbert-Schmidt operator $D_H f(x)$ on $H$ such that

$$D_H f(x) h = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h) - f(x)}{\varepsilon}, \quad x \in X, \; h \in H.$$

For every $p \in [1, +\infty)$, $D_H : \mathcal{F}C^\infty_b(X; H) \to L^p(X, \gamma; \mathcal{L}_2(H))$ is a closable operator on $L^p(X, \gamma; H)$. We still denote by $D_H$ its closure and we define $W^{1,p}(X, \gamma; H)$ as the domain of this closure (see [4, Sec. 5.2]).

Let $f : X \to \mathbb{R}$ be such that $\nabla_H$ is defined at each point $x \in X$. We say that $f$ is twice $H$-differentiable at $x \in X$ if $\nabla_H f$ is $H$-differentiable at $x$. We set $\partial^2_H f(x) := D_H(\nabla_H f)(x)$, and we recall that the operator $D^2_H f(x) : H \times H \to \mathbb{R}$ is a Hilbert-Schmidt operator on $H$. $D^2_H f(x)$ is said $H$-second derivative of $f$ at $x$.

The operator $(\nabla_H, D^2_H) : \mathcal{F}C^\infty_b(X) \to L^p(X, \gamma; H) \times L^p(X, \gamma; \mathcal{L}_2(H))$ is a closable operator on $L^p(X, \gamma)$ for every $p \in [1, +\infty)$. We denote by $W^{2,p}(X, \gamma)$ the domain of the closure of the operator $(\nabla_H, D^2_H)$ (see [4, Sec. 5.2]).
We recall the concept of $H$-divergence (see [5, Sec. 5.8]). For every $f \in \mathcal{F}C^n(X;\mathbb{R})$ we define the $H$-divergence $\text{div}_\gamma$ with respect to $\gamma$ as

$$\text{div}_\gamma f = \sum_{i=1}^{\infty} (\partial_{h_i} f_i - \hat{h}_i f_i) = \sum_{i=1}^{\infty} \partial_{h_i} f_i,$$

where $\{h_i : i \in \mathbb{N}\}$ is an orthonormal basis of $H$ and $f_i = \langle f, h_i \rangle_H$ for every $i \in \mathbb{N}$. The definition of $\text{div}_\gamma$ does not depend on the choice of the basis of $H$. Further, if $f : X \to H$ is everywhere $H$-differentiable with $D_H f$ uniformly bounded, then $\text{div}_\gamma f$ is defined everywhere (through formula (2.3)).

Let $f \in W^{1,2}(X;\mathbb{R})$. For every $n \in \mathbb{N}$ we define $f_n(x) = \pi_n \circ f(x)$ for every $x \in X$. It follows that the divergence $\text{div}_\gamma f_n$ is defined $\gamma$-a.e. in $X$, it belongs to $L^2(X,\gamma)$ and it converges in $L^2(X,\gamma)$ to a function $g \in L^2(X,\gamma)$ which we denote by $\text{div}_\gamma f$ (see [5, Theo. 5.8.3]). Moreover, the operator $\text{div}_\gamma$ is the adjoint of $-\nabla_H$ in $L^2(X,\gamma)$ in the sense that, if $f \in W^{1,2}(X;\mathbb{R})$ then

$$\int_X \langle f, \nabla_H g \rangle_H \, d\gamma = -\int_X \text{div}_\gamma f g \, d\gamma$$

for every $g \in W^{1,2}(X,\gamma)$.

A function $f : X \to \mathbb{R}$ is said to be $H$-Lipschitz continuous if there exists a positive constant $c$ such that for $\gamma$-a.e. $x \in X$ we have

$$|f(x + h) - f(x)| \leq c \|h\|_H, \quad h \in H.$$

The constant $c$ is called the $H$-Lipschitz constant of $f$, and we denote by $\text{Lip}_H(X)$ the space of $H$-Lipschitz continuous functions.

Let $f : \Omega \to X$ be a $H$-Lipschitz continuous function with $H$-Lipschitz constant $c > 0$. Then, $f$ is Gâteaux differentiable $\gamma$-a.e. in $X$, it is $H$-differentiable and it belongs to $W^{1,p}(X,\gamma)$. Moreover, $\nabla_H f$ is defined $\gamma$-a.e. in $X$, and

$$\|\nabla_H f\|_H \leq c, \quad \gamma\text{-a.e. in } X$$

(see e.g. [5, Theorem 5.11.2]).

2.1. The Hausdorff-Gauss spherical measure. In the above setting, by following [11], it is possible to define a Borel measure $\mathcal{H}^{m-1}$ on $X$ which replace the $(d-1)$-Hausdorff measure in $\mathbb{R}^d$ in abstract Wiener spaces (hence, it can be seen as an area measure for $(\infty - 1)$-hypersurfaces). The measure $\mathcal{H}^{m-1}$ is called the Hausdorff-Gauss measure or Feyel-de la Pradelle measure and we denote it by $\rho$. Let us briefly show the construction of $\rho$.

Let $F \subseteq R_f(X^*)$ be an $m$-dimensional subspace of $H$. We identify $F$ with $\mathbb{R}^m$ by choosing an orthonormal basis of $F$ in $H$, and by identifying it with the canonical basis of $\mathbb{R}^m$. For every $m \in \mathbb{N}$, $\mathcal{H}^{m-1}$ denotes the spherical $(m-1)$-dimensional Hausdorff measure on space $F$, and for every $y \in X_F$ and every $B \in \mathcal{B}(X)$, we denote by $B_y$ the section

$$B_y = \{z \in F : y + z \in B\}$$

and the function

$$G_m(y) = (2\pi)^{-\frac{m}{2}} e^{-\frac{|y|^2}{2}}.$$

The spherical $(\infty - 1)$-dimensional Hausdorff-Gauss measure in $X$ with respect to $F$ is

$$\mathcal{H}^{m-1}_F(B) = \int_{X_F} \int_{B_y} G_m(z) d\mathcal{H}^{m-1}(z) d\gamma^F(y), \quad B \subseteq X.$$
$\mathcal{F}^{-1}$ is a $\sigma$-additive Borel measure on $X$, and for every Borel set $B \in \mathcal{B}(X)$, the map $y \mapsto \int_B G_m \, d\mathcal{F}^{-1}$ is $\gamma'$-measurable in $F^{-1}$. Since the measures $\mathcal{F}^{-1}$ are monotone with respect to $F$, we set

$$\rho(B) = \sup_{F \subseteq R_d(X^*)} \mathcal{F}^{-1}(B)$$

for every $B \in \mathcal{B}(X)$, where the supremum is meant as a supremum in a direct set. It turns out that $\rho$ is a Borel measure.

### 2.2. Definition of the Sobolev spaces $W^{1,p}(O,\gamma)$ and $W^{1,p}_0(O,\gamma)$

Let $O \subseteq X$ be an open set. We denote by $\mathcal{F}C^0_b(O)$ the set of the restrictions to $O$ of elements of $\mathcal{F}C^0_b(X)$. The next Lemma is proved, for instance, in [3] Lem. 2.1.

**Lemma 2.1.** For every $p \in [1, +\infty)$, the operator $\nabla_H : \mathcal{F}C^0_b(O) \to L^p(O, \gamma, H)$ is closable in $L^p(O, \gamma, H)$. The same is true if we use Lip$(O)$ instead of $\mathcal{F}C^0_b(O)$, and the domains of the closures coincide. We still denote by $\nabla_H$ the closure of $\nabla_H$.

**Proof.** The proof is the same of [3] Lemma 2.1 for both the space functions. The closures coincide because $\mathcal{F}C^0_b(O) \subseteq $ Lip$(O)$, and every Lipschitz continuous function can be extended to $X$ by the McShane extension, and then approximated in $L^p$ by $\mathcal{F}C^0_b(X)$ functions.

Actually, the proof in [3] uses spaces $\mathcal{F}C^1_b(O)$ and Lip$_b(O)$, respectively, but the arguments are the same. From Lemma 2.1 we introduce the following spaces.

**Definition 2.1.** We denote by $W^{1,p}(O,\gamma)$ the domain of the closure of $\nabla_H$ in $L^p(O,\gamma)$. If endowed with the norm

$$\|f\|_{W^{1,p}(O,\gamma)} := \left( \|f\|_{L^p(O,\gamma)}^p + \|\nabla_H f\|_{L^p(O,\gamma, H)}^p \right)^{1/p}, \quad f \in W^{1,p}(O,\gamma),$$

the space $W^{1,p}(O,\gamma)$ is a Banach space. If $p = 2$ then $W^{1,2}(O,\gamma)$ is a Hilbert space with inner product

$$\langle f, g \rangle_{W^{1,2}(O,\gamma)} = \langle f, g \rangle_{L^2(O,\gamma)} + \langle \nabla_H f, \nabla_H g \rangle_{L^2(O,\gamma, H)}, \quad f, g \in W^{1,2}(O,\gamma).$$

We now define the Sobolev spaces $W^{1,p}_0(O,\gamma)$.

**Definition 2.2.** For every $p \in [1, +\infty)$, we denote by $W^{1,p}_0(O,\gamma)$ the closure of Lip$_c(O)$ in $W^{1,p}(O,\gamma)$.

We want to prove that $W^{1,p}_0(O)$ actually coincides with the closure of different subspaces of $W^{1,p}(O,\gamma)$. To this aim, we introduce the following spaces of functions.

**Definition 2.3.** The space Lip$_c,H(O)$ is the space of functions $f : O \to \mathbb{R}$, with support contained in an open set $A$ with positive distance from $O^c$, such that there exists a positive constant $\ell$ such that for $\gamma$-a.e. $x \in O$ we have

$$|f(x+h) - f(x)| \leq \ell \|h\|_H, \quad \forall h \in H, \; x+h \in O. \quad (2.4)$$

**Definition 2.4.** With $\mathcal{H}^1(X)$ we denote the set of all continuous functions $f$ (not necessarily bounded) which are $H$-differentiable on $X$ and such that $\nabla_H f : X \to H$ is bounded and continuous.

$\mathcal{H}^1_0(O)$ is the subset of $\mathcal{H}^1(X)$ of functions $f$ whose support has positive distance from $O^c$. 

The following result shows that \( \mathcal{H}_0^1(O) \) is not empty. To prove this fact, we introduce the Ornstein-Uhlenbeck semigroup \( (T_t)_{t \geq 0} \) on \( X \), characterized by the Mehler formula

\[
T_t f(x) = \int_X f \left( e^{-t} x + \sqrt{1 - e^{-2t}} y \right) \gamma(dy), \quad f \in C_b(X), \ x \in X, \ t \geq 0,
\]

which extends to a bounded strongly continuous semigroup on \( L^p(X, \gamma) \) for every \( p \in [1, +\infty) \), which we again denote by \( (T_t)_{t \geq 0} \). We recall that for every \( f \in C_b(X) \), we have \( T_t f \) is \( \mathcal{H} \)-differentiable and

\[
\langle \nabla H T_t f(x), h \rangle \mathcal{H} = \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int_X f \left( e^{-t} x + \sqrt{1 - e^{-2t}} y \right) \hat{h}(y) \gamma(dy), \quad x \in X, \ t \geq 0.
\]

**Lemma 2.2.** \( \mathcal{H}_0^1(O) \) is not empty.

**Proof.** Let us fix a bounded closed set \( B \subseteq X \) and \( \varepsilon > 0 \). From [14 Lemma 2.5] and its proof, we infer that there exists a Lipschitz continuous function \( f \) with Lipschitz constant \( 2\varepsilon^{-1}, \ t > 0 \) and a smooth function \( \Phi \in C_c^\infty(\mathbb{R}) \) with \( 0 \leq \Phi \leq 1, \ \Phi = 0 \) on \((-\infty, 1/3)\) and \( \Phi = 1 \) on \((2/3, +\infty)\), such that the function \( F_{B,\varepsilon} = \Phi(T_t f) \) equals 1 on \( B \) and \( F_{B,\varepsilon} = 0 \) on \( X \setminus \{ x \in X : d(x, B) > \varepsilon \} \).

Hence, for every bounded open set \( A \subseteq O \) with positive distance \( d \) from \( O^c \), with the choice \( B = \overline{A} \) and \( \varepsilon < d \), the function \( F = F_{B,\varepsilon} \) has \( \nabla H F \) everywhere defined and bounded by [5 Theorem 5.11.2]. \( F \) belongs to \( \mathcal{H}_0^1(O) \), providing that we prove that \( \nabla H F \) is continuous. To this aim, for every \( x, y \in X \) we have

\[
\| \nabla H F(x) - \nabla H F(y) \|_{\mathcal{H}}^2 \\
\leq \frac{e^{-2t} \| \Phi \|_{C_c^\infty(\mathbb{R})}^2}{1 - e^{-2t}} \sum_{n=1}^{\infty} \left( \int_X \left( f(e^{-t} x + \sqrt{1 - e^{-2t}} z) - f(e^{-t} y + \sqrt{1 - e^{-2t}} z) \right) \hat{h}_n(z) \gamma(dz) \right)^2 \\
\leq \frac{e^{-2t} \| \Phi \|_{C_c^\infty(\mathbb{R})}^2}{1 - e^{-2t}} \| f(e^{-t} x + \sqrt{1 - e^{-2t}} z) - f(e^{-t} y + \sqrt{1 - e^{-2t}} z) \|_{L^2(X, \gamma)}^2 \\
\leq \frac{4e^{-4t} \| \Phi \|_{C_c^\infty(\mathbb{R})}^2}{1 - e^{-2t}} \| x - y \|_{\mathcal{H}}^2
\]

where the second inequality is a consequence of [24], and this gives the continuity of \( \nabla H F \). Further, \( F \) is Lipschitz continuous due to the Lipschitz continuity of \( f \), the definition of \( T_t \) and the smoothness of \( \Phi \). \( \square \)

From the definition of \( \mathcal{H}_0^1(O) \), it follows that \( \mathcal{H}_0^1(O) \subseteq \text{Lip}_{c,H}(O) \). Further, for every \( f \in \text{Lip}_{c,H}(O) \), its trivial extension \( \tilde{f} \) belongs to \( W^{1,p}(X, \gamma) \) for every \( p \in (1, +\infty) \), and so \( \text{Lip}_{c,H}(O) \subseteq W^{1,p}(O, \gamma) \) for every \( p \in (1, +\infty) \) (see [5 Theorem 5.11.2]).

Clearly, also \( \text{Lip}_{c}(O) \subseteq \text{Lip}_{c,H}(O) \), and so

\[
W_0^{1,p}(O, \gamma) \subseteq \text{Lip}_{c,H}(O)^{W^{1,p}(O, \gamma)}, \quad \mathcal{H}_0^1(O)^{W^{1,p}(O, \gamma)} \subseteq \text{Lip}_{c,H}(O)^{W^{1,p}(O, \gamma)}. \]

We prove that the above inclusions are indeed equalities.

**Lemma 2.3.** We have

\[
W_0^{1,p}(O, \gamma) = \mathcal{H}_0^1(O)^{W^{1,p}(O, \gamma)} = \text{Lip}_{c,H}(O)^{W^{1,p}(O, \gamma)},
\]

The closure of \( \mathcal{H}_0^1(O) \) in \( W^{1,p}(O, \gamma) \) coincides with \( W_0^{1,p}(O, \gamma) \) for every \( p \in (1, +\infty) \).
Proof. Let us fix $p \in (1, +\infty)$. To prove the statement, we show that $\mathcal{H}^1_0(O) \subseteq W^{1,p}(O, \gamma)$ and that $\text{Lip}_p(O) \subseteq \mathcal{H}^1_0(O)$. Without loss of generality, we assume that the support of the considered functions is bounded.

Let $g \in \mathcal{H}^1_0(O)$. Its trivial extension $\overline{g}$ belongs to $W^{1,p}(X, \gamma)$ and so there exists a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathcal{F} \mathcal{C}^0(X)$ which converges to $\overline{g}$ in $W^{1,p}(X, \gamma)$ as $n \to +\infty$. Let $A \subseteq O$ be a bounded open set with positive distance from $O^c$ such that $\text{supp}(g) \subseteq A$, and let $F$ be the function defined in Lemma 2.2. $F$ is Lipschitz continuous: indeed, for every $x, y \in X$ we have

$$|F(x) - F(y)| \leq \|\Phi\|_{L_\infty} |T_f(x) - T_f(y)|$$

$$\leq \|\Phi\|_{L_\infty} \int_X |f(e^{-t}x + \sqrt{1 - e^{2t}}z) - f(e^{-t}y + \sqrt{1 - e^{2t}}z)| \gamma(dz)$$

$$\leq \frac{2}{\epsilon} e^{-t}\|\Phi\|_{L_\infty} |x - y|,$$

where we have used the fact that $f$ is a $\frac{2}{\epsilon}$-Lipschitz continuous function. Then, the sequence $(F g_n)_{n\in\mathbb{N}} \subseteq \text{Lip}_p(O)$ and it converges to $g$ in $W^{1,p}(O, \gamma)$.

This implies that $\mathcal{H}^1_0(O) \subseteq W^{1,p}(O, \gamma)$.

Let $g \in \text{Lip}_p(O)$. Its trivial extension $\overline{g}$ belongs to $W^{1,p}(X, \gamma)$. Hence, there exists a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathcal{F} \mathcal{C}^0(X)$ such that $g_n \to \overline{g}$ in $W^{1,p}(X, \gamma)$ as $n \to +\infty$. Let $A \subseteq O$ be a bounded open set with positive distance from $O^c$ such that $\text{supp}(g) \subseteq A$, and let $F$ be the function defined in Lemma 2.2. The sequence $(F g_n)_{n\in\mathbb{N}}$ converges to $g$ in $W^{1,p}(O, \gamma)$, with $F \in \mathcal{H}^1_0(O)$ and $g_n \in \mathcal{F} \mathcal{C}^0(X)$, which give $F g_n \in \mathcal{H}^1_0(O)$ for every $n \in \mathbb{N}$.

This gives $\text{Lip}_p(O) \subseteq \mathcal{H}^1_0(O)$. \hfill \Box

By the operator theory, there exists a unique unbounded operator $L_O$, with dense domain in $W^{1,2}_0(O, \gamma)$, such that, for every $f \in D(L_O)$ and $g \in W^{1,2}_0(O, \gamma)$, we get

$$\int_O L_O f \cdot g \, d\gamma = - \int_O \langle \nabla_H f, \nabla_H g \rangle_H \, d\gamma.$$  

Definition 2.5. The operator $L_O : D(L_O) \to L^2(O, \gamma)$ is called Ornstein-Uhlenbeck operator on $O$ with homogeneous Dirichlet boundary conditions.

When $O = X$, we denote $L_X$ by $L$, and it is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $(T_t)_{t \geq 0}$.

3. Traces in regular sets

In the following, for every $\delta > 0$ we denote by $I_\delta$ the real interval $(-\delta, \delta) \subseteq \mathbb{R}$. Inspired by [7] Hypothesis 3.1, we state our assumptions on $O$.

Hypotheses 3.1. Let $G : X \to \mathbb{R}$ and $\delta > 0$ satisfy:

(i) $G$ is a continuous function which belongs to $\text{Lip}_H(X)$;
(ii) $G \in W^{2,p}(X, \gamma)$ for some $p > 1$ and $\text{esssup}_X \|D_H^2 G\|_{L_2(H)} < +\infty$;
(iii) $\|\nabla_H G\|_{L_1}^2 \in L^\infty(X)$;
(iv) $LG \in L^\infty(G^{-1}(I_\delta)).$

Hereafter, we set $O := G^{-1}((-\infty, 0))$ and we assume that $O$ and $\partial O$ are not the empty set.

Remark 3.1. Let us comment the above assumptions.
i) $O$ is an open set and $\partial O = G^{-1}(\{0\})$. Hence, $\gamma(O) > 0$ since every open set has positive measure by an immediate consequence of [5, Prop. 2.4.10].

ii) From Hypothesis [7, ii) we infer that $G \in W^{2,q}(X, \gamma)$ for all $q > 1$, and so Hypothesis 3.1 is fulfilled.

iii) By the points (ii) and (iii) of the Hypotheses [3,1] it follows that

$$\nabla H G \|_{H} \in W^{1,2}(X; H).$$

By adding the point (iv) we have also that $\text{div}_{\gamma} \left( \frac{\nabla H G}{\| \nabla H G \|_{H}} \right) \in L^{\infty}(G^{-1}(I_{\delta}))$, since

$$\text{div}_{\gamma} \left( \frac{\nabla H G}{\| \nabla H G \|_{H}} \right) = \frac{L G}{\| \nabla H G \|_{H}} - \frac{\langle D_{G}^{2} G(\nabla H G), \nabla H G \rangle}{\| \nabla H G \|_{H}^{3}}.$$

iv) The Hypothesis (iii) is very restrictive, for example it is not satisfied by $X$ is a Hilbert space, which would allow to consider balls.

v) For $-\delta < \varepsilon < \delta$ the assumption remains true if we replace $G$ with $G + \varepsilon$ or with $G + \varepsilon$, with the value $\delta$ replaced by $\delta = \delta - |\varepsilon|$.

vi) From Hypotheses [7, it follows that $L G$, and so $\text{div}_{\gamma}(\nabla H G / \| \nabla H G \|_{H})$, belongs to $L^{p}(X, \gamma)$ for every $p \in (1, +\infty)$.

In the sequel, we will need the Sobolev regularity of the modulus of elements of $W^{1,p}(X)$, which is proved in the following lemma.

**Lemma 3.1.** Let $u \in W^{1,p}(X, \gamma)$ with $p > 1$. Then, for every $q \in (1, p)$, the function $|u|^q$ belongs to $W^{1,p/q}(X, \gamma)$, and $\nabla H |u|^q = q \text{sgn}(u) |u|^{q-1} \nabla H u$.

**Proof.** The classical method consists in introducing the function $\eta_{n}(\xi) := (\xi^{2} + 1/4)^{q/2}$ and approximating $|u|^q$ by means of the sequence $(\eta_{n} \circ u_{n}) \subseteq \mathcal{F}_{b}^{\infty}(X)$ in $W^{1,p/q}(O, \gamma)$, where $(u_{n}) \subseteq \mathcal{F}_{b}^{\infty}(X)$ is a sequence which converges to $u$ in $W^{1,p}(O, \gamma)$. However, we provide a different proof.

At first, we notice that, for every $q > 1$ and every $a, b \in \mathbb{R}$, we have

$$|\text{sgn}(a) |a|^q - \text{sgn}(b) |b|^q| \leq q |a - b| \left| |a|^{q-1} + |b|^{q-1} \right|. \quad (3.1)$$

Let $(u_{n})_{n \in \mathbb{N}} \subseteq \mathcal{F}_{b}^{\infty}(X)$ be a sequence which converges to $u$ in $W^{1,p}(X, \gamma)$. Without loss of generality, we may suppose that $(u_{n})_{n \in \mathbb{N}}$ pointwise converges to $u$. We split the proof into three steps.

**Step 1.** Here we prove that $|u_{n}|^q$ converges to $|u|^q$ in $L^{p/q}(X, \gamma)$. Since $q > 1$, from (3.1) it follows that

$$\| |u_{n}|^q - |u|^q \|_{L^{p/q}(X, \gamma)} \leq q \left( |u_{n}| - |u| \right) \left( |u_{n}|^{q-1} + |u|^{q-1} \right) \| |u_{n}|^q - |u|^q \|_{L^{p/(q-1)}(X, \gamma)}, \quad (3.2)$$

where we have used the Hölder inequality with $q$ and $q/(q-1)$. The first factor in the right-hand side of (3.2) converges to 0 as $n \to +\infty$, while the second is bounded, uniformly with respect to $n \in \mathbb{N}$, since

$$\| |u_{n}|^{q-1} + |u|^{q-1} \|_{L^{p/(q-1)}(X, \gamma)} \leq \| |u_{n}|^{q-1} \|_{L^{p/(q-1)}(X, \gamma)} + \| |u|^{q-1} \|_{L^{p/(q-1)}(X, \gamma)},$$

which is uniformly bounded with respect to $n \in \mathbb{N}$.
Step 2. We want to show that $\text{sgn}(u_n) |u_n|^{q-1}$ converges to $\text{sgn}(u) |u|^{q-1}$ in $L^{p/(q-1)}(X, \gamma)$. Let us suppose $q \geq 2$, hence $q - 1 \geq 1$, and from (3.1) we infer that

\[
\int_X |\text{sgn}(u_n) |u_n|^{q-1} - \text{sgn}(u) |u|^{q-1}|^{p/(q-1)} d\gamma \\
\leq (q-1)^{p/(q-1)} \int_X |u_n - u|^{p/(q-1)} |u_n|^{q-2} + |u|^{q-2}|^{p/(q-1)} d\gamma \\
\leq (q-1)^{p/(q-1)} \left( \int_X |u_n - u|^p d\gamma \right)^{1/p} \\
\times \left( \int_X |u_n|^{q-2} + |u|^{q-2} \right)^{(q-2)/(q-1)}^{p/(q-1)} \\
=: (q-1)^{p/(q-1)} A_n B_n.
\]

$A_n$ converges to 0 as $n \to +\infty$, and

\[
B_n^{(q-1)/p} \leq \left( \int_X |u_n|^{q-2} + |u|^{q-2} \right)^{p/(q-1)} \leq \left( \int_X |u_n|^{q-2} \right)^{p/(q-1)} + \left( \int_X |u|^{q-2} \right)^{p/(q-1)} \\
\leq \|u_n\|_{L^{p/(q-2)}(X, \gamma)}^{q-2} + \|u\|_{L^{p/(q-2)}(X, \gamma)}^{q-2},
\]

which is uniformly bounded with respect to $n \in \mathbb{N}$.

Instead, if $q \in [1, 2)$,

\[
\|\text{sgn}(u_n) |u_n|^{q-1} - \text{sgn}(u) |u|^{q-1}\|_{L^{p/(q-1)}(X, \gamma)} \\
\leq \|\text{sgn}(u_n) |u_n|^{q-1} - \text{sgn}(u_n) |u|^{q-1}\|_{L^{p/(q-1)}(X, \gamma)} \\
+ \|\text{sgn}(u_n) |u|^{q-1} - \text{sgn}(u) |u|^{q-1}\|_{L^{p/(q-1)}(X, \gamma)} \\
\leq \|u_n|^{q-1} - |u|^{q-1}\|_{L^{p/(q-1)}(X, \gamma)} \\
+ \left( \int_X \left( |\text{sgn}(u_n) - \text{sgn}(u)\right) |u|^{q-1} \right)^{p/(q-1)} d\gamma) \\
=: P_n + Q_n.
\]

Since $t \mapsto t^{q-1}$ is concave, we get

\[
P_n \leq \left( \int_X |u_n| - |u|^{q-1} \right)^{p/(q-1)} = \|u_n - u\|_{L^{p}(X, \gamma)}^{q-1}
\]

which converges to 0 as $n \to +\infty$, while, if we set $U := \{x \in X | u(x) \neq 0\}$, then

\[
Q_n = \int_U (\text{sgn}(u_n) - \text{sgn}(u)) |u|^p d\gamma,
\]

and it converges to 0 by the dominated convergence, since $(u_n)_{n \in \mathbb{N}}$ pointwise converges to $u$.

Step 3. By approximation, it is possible to show that $\nabla_H|u_n|^q = q \text{sgn}(u_n) |u_n|^{q-1} \nabla_H u_n$ for every $n \in \mathbb{N}$. In this last step we prove that $\nabla_H|u_n|^q$ converges to $q \text{sgn}(u) |u|^{q-1} \nabla_H u$ in $L^{p/q}(X, \gamma, H)$ as $n \to +\infty$. We have

\[
\|q \text{sgn}(u_n) |u_n|^{q-1} \nabla_H u_n - q \text{sgn}(u) |u|^{q-1} \nabla_H u\|_{L^{p/q}(X, \gamma)} \\
\leq \|q \text{sgn}(u_n) |u_n|^{q-1} \nabla_H u_n - q \text{sgn}(u) |u|^{q-1} \nabla_H u\|_{L^{p/q}(X, \gamma, H)} \\
+ \|q \text{sgn}(u) |u|^{q-1} \nabla_H u_n - q \text{sgn}(u) |u|^{q-1} \nabla_H u\|_{L^{p/q}(X, \gamma, H)} =: R_n + S_n.
\]
As far as \( R_n \) is concerned, by applying the Hölder inequality with \( q \) and \( q/(q-1) \) we get
\[
R_n \leq q \left\| \text{sgn}(u_n) |u_n|^{q-1} - \text{sgn}(u_n) |u_n|^{(q-1)/q} \right\|_{L^p((q-1)/q) \Gamma \Omega(\gamma)} \cdot \left\| \nabla_H u_n \right\|_{L^p(\Gamma \Omega(\gamma))}^{1/2}
\]
The first factor converges to 0 as \( n \to +\infty \) from Step 2 and the second one is uniformly bounded with respect to \( n \in \mathbb{N} \), while
\[
S_n \leq q \left\| u \right\|_{L^p((q-1)/q) \Gamma \Omega(\gamma)} \cdot \left\| \nabla_H u_n - \nabla_H u \right\|_{L^p(\Gamma \Omega(\gamma))}^{1/2}
\]
and the last term converges to 0 as \( n \to +\infty \), where again we have used the Hölder inequality with \( q \) and \( a/(a-1) \).

The proof is concluded.

From [7], under Hypotheses 3.1 (i)-(iii), for every \( t \in (-\delta, \delta) \) and every \( q < p \) it is defined the trace operator
\[
\text{Tr}_t : W^{1,p}(G^{-1}((-\infty,t)), \gamma) \to L^q(G^{-1}({\{t\}}), \rho).
\]
If \( f \in W^{1,p}(G^{-1}((-\infty,t)), \gamma) \) is the restriction of a continuous function on \( X \), then
\[
\text{Tr}_t f = f|_{G^{-1}({\{t\})}}.
\]

The following three results are proved in [7].

**Lemma 3.2.** [7] Cor. 3.2 Assume Hypotheses 3.1 (i)-(iii), let \( \delta_0 > 0 \) and \( O_{\delta_0} := G^{-1}(I_{\delta_0}) \). Then, for every \( f \in \text{Lip}(X) \subseteq L^1(O_{\delta_0}, \gamma) \), the function
\[
q_f(\xi) := \int_{G^{-1}(\{\xi\})} \frac{f}{\|\nabla_H G\|_H} \ d\rho, \quad -\delta_0 < \xi < \delta_0,
\]
belongs to \( L^1(I_{\delta_0}, \mathcal{L}^1) \) (with \( \mathcal{L}^1 \) being the 1-dimensional Lebesgue measure). Moreover, \( q_f \) is a density of the measure \( f \gamma \circ G^{-1} \) with respect to \( \mathcal{L}^1 \), and
\[
\| q_f \|_{L^1(-\delta_0, \delta_0)} \leq \| f \|_{L^1(O_{\delta_0}, \gamma)}.
\]

**Lemma 3.3.** [7] Prop. 4.10] Under Hypotheses 3.1 for every \( p > 1 \), every \( t \in I_\delta \) and every \( f \in W^{1,p}(G^{-1}((-\infty,t)), \gamma), \text{Tr}_t f \equiv 0 \) if and only if the trivial extension \( \overline{f} \) of \( f \) out of \( G^{-1}((-\infty,t)) \) belongs \( W^{1,p}(X, \gamma) \).

**Lemma 3.4.** [7] Prop. 4.1] Under Hypotheses 3.1 for every \( p > 1 \), every \( q \in [1,p) \) and every \( t \in I_\delta \), if \( f \in W^{1,p}(X, \gamma) \) then
\[
\int_{G^{-1}({\{t\}})} |\text{Tr}_t f|^q \ d\rho = q \int_{G^{-1}((-\infty,t))} |f|^{q-2} f \frac{\langle \nabla_H f, \nabla_H G \rangle_H}{\|\nabla_H G\|_H} \ d\gamma + q \int_{G^{-1}((-\infty,t))} \text{div}_\gamma \left( \frac{\nabla_H G}{\|\nabla_H G\|_H} \right) |f|^q \ d\gamma
\]
\[
= q \int_{G^{-1}((t,+\infty))} |f|^{q-2} f \frac{\langle \nabla_H f, \nabla_H G \rangle_H}{\|\nabla_H G\|_H} \ d\gamma + q \int_{G^{-1}((t,+\infty))} \text{div}_\gamma \left( \frac{\nabla_H G}{\|\nabla_H G\|_H} \right) |f|^q \ d\gamma. \tag{3.3}
\]
Remark 3.2. An easy consequence of Lemma 3.2 is that $\gamma(G^{-1}(\{t\})) = 0$ for every $t \in I$. Further, if we take $f = 1$ in (3.3), we infer that

$$
\rho(G^{-1}(\{t\})) = \int_{G^{-1}(\{t\})} d\rho
\leq \left\| \text{div}_\gamma \left( \frac{\nabla H \varphi}{\left\| \nabla H \varphi \right\|_H} \right) \right\|_{L^1(O,T)} + \left\| \text{div}_\gamma \left( \frac{\nabla H \varphi}{\left\| \nabla H \varphi \right\|_H} \right) \right\|_{L^\infty(G^{-1}(I))} < +\infty.
$$

(3.4)

4. Equivalent definitions of $W^{1,p}_0(O,\gamma)$

We set $A_r := G^{-1}(r, \delta)$, and we prove the following two intermediate results.

Lemma 4.1. Let Hypotheses 3.1 be satisfied. Then, for every $q \in [1, +\infty)$, there exists $C > 0$ such that for every $t \in (-\delta, 0)$ and $f \in W^{1,q}(X, \gamma)$ we have

$$
\| \text{Tr}_t f \|_{L^q(G^{-1}(t), \rho)} \leq C \| f \|_{L^q(A_r, \gamma)} \left\| \nabla H \varphi \right\|_{L^p(A_r, H)} + \| f \|_{L^q(A_r, \gamma)}^q.
$$

(4.1)

Proof. By density, it is enough to consider Lipschitz continuous functions $f$. Arguing as in the proof of [12] Prop. 4.1, we introduce a function $\theta \in C^\infty_0(\mathbb{R})$ such that $\theta = 1$ in $(-\delta, 0)$, $\theta = 0$ in $[\delta, +\infty)$ and $\theta(x) \in [0, 1]$ for every $x \in \mathbb{R}$. We define the function $\psi := f \cdot (\theta \circ G)$. The function $\psi$ belongs to $W^{1,q}(X, \gamma)$ for every $s \in [1, +\infty)$ (because $f$ is Lipschitz continuous and $G \in \mathcal{H}^1(X)$) with $\nabla H \psi = (\theta \circ G) \nabla H f + f (\theta' \circ G) \nabla H G$. From its definition, $\psi = 0$ on $G^{-1}(\langle \delta, +\infty \rangle)$, $|\psi| \leq |f|$ on $X$, and

$$
\| \psi \|_{L^q(G^{-1}(t, +\infty), \gamma)} = \| \psi \|_{L^q(A_r, \gamma)}, \quad \| \nabla H \psi \|_{L^q(G^{-1}(t, +\infty), \gamma)} = \| \nabla H \psi \|_{L^q(A_r, \gamma)}.
$$

Finally, $\psi_{G^{-1}(t)} \equiv f$. Hence, we can apply Lemma 3.3 which gives

$$
\int_{G^{-1}(\{t\})} |\text{Tr}_t f|^q d\rho = \int_{G^{-1}(\{t\})} |\text{Tr}_t \psi|^q d\rho
= q \int_{A_r} \| \psi \|^{q-2} \psi (\nabla H \psi, \nabla H G) \frac{d\gamma}{\| \nabla H G \|_H} + \int_{A_r} \text{div}_\gamma \left( \frac{\nabla H \psi}{\| \nabla H \psi \|_H} \right) |\psi|^q \frac{d\gamma}{\| \nabla H G \|_H}
\leq q \int_{A_r} |\psi|^{q-1} \| \nabla H \psi \| \frac{d\gamma}{\| \nabla H G \|_H} + \int_{A_r} \text{div}_\gamma \left( \frac{\nabla H \psi}{\| \nabla H G \|_H} \right) |\psi|^q \frac{d\gamma}{\| \nabla H G \|_H}.
$$

Recalling that $\text{div}_\gamma(\nabla H G / \| \nabla H G \|_H) \in L^\infty(A_r, \gamma)$ (see Remark 3.1 (iii)), we infer that

$$
\int_{G^{-1}(\{t\})} |\text{Tr}_t f|^q d\rho
\leq q \left( \| f \|_{L^q(A_r, \gamma)} \left\| \nabla H \psi \right\|_{L^p(A_r, \gamma)} + \left\| \text{div}_\gamma \left( \frac{\nabla H G}{\| \nabla H G \|_H} \right) \right\|_{L^\infty(A_r, \gamma)} \left\| \psi \right\|_{L^q(A_r, \gamma)}^q \right)
\leq C_1 \| f \|_{L^q(A_r, \gamma)} \left\| \nabla H f \right\|_{L^p(A_r, \gamma)} + \| f \|_{L^q(A_r, \gamma)}^q + C_2 \| f \|_{L^q(A_r, \gamma)}^q
\leq (C_1 + C_2) \left( \| f \|_{L^q(A_r, \gamma)} + \left\| \nabla H f \right\|_{L^p(A_r, \gamma)} \right),
$$

where $C_1 = q(1 + \| \theta' \|_{L^\infty(X, H)})$ and $C_2 = \left\| \text{div}_\gamma \left( \frac{\nabla H G}{\| \nabla H G \|_H} \right) \right\|_{L^\infty(A_r, \gamma)}$. The proof is now complete. \qed
Lemma 4.2. Let Hypotheses 3 and 4 be satisfied. Then, for every $p > 1$ there exists $C_0 > 0$ and $\delta_0 \in (0, \delta]$ such that, for every $f \in W^{1,p}(O, \gamma)$ with $\text{Tr}_0 f \equiv 0$ on $G^{-1}(\{0\})$ and every $t \in (-\delta_0, 0)$, we have

$$\|f\|_{L^p(G^{-1}(t, 0), \gamma)} \leq 2C_1 |t|\|\nabla_Hf\|_{L^p(G^{-1}(t, 0), \gamma_H)}.$$  

Proof. By density, it is enough to consider Lipschitz continuous functions $f$.

Let us assume that $\|f\|_{L^p(G^{-1}(t, 0), \gamma)} \neq 0$. From Lemma 3.3, the trivial extension $\tilde{f}$ of $f$ belongs to $W^{1,p}(X, \gamma)$. From Lemma 3.1 it follows that $\tilde{f}^q \in L^p/(X, \gamma)$ for every $q \in (1, p)$, and by applying Lemma 4.1 to the function $\tilde{f}^q$, for every $s \in (-\delta, 0)$ we get

$$\|\text{Tr}_s |f|^{q/p}_{L^p/(G^{-1}(s), \rho)} \leq \|\text{Tr}_s \tilde{f}^q\|^{q/p}_{L^p/(G^{-1}(s), \rho)} \leq C^0 |t|\|\nabla_Hf\|_{L^p(G^{-1}(s), \gamma_H)} + \|\text{div}_\gamma \left( \frac{\nabla_Hf}{\|\nabla_Hf\|_{H}} \right)\|_{L^p(G^{-1}(s), \gamma_H)},$$

where $C = \frac{p^q}{p} (1 + \|\theta\|_{\infty}) \|\nabla_Hf\|_{L^p(-\delta, \gamma_H)}$. From Lemma 3.2 for every $s \in (-\delta, 0)$ we have

$$\int_{G^{-1}(t, 0)} |f|^q d\gamma = \int_{G^{-1}(t, 0)} \int_{G^{-1}(\xi)} \text{Tr}_s |f|^q d\rho d\xi \leq |t| \sup_{s \in (-\delta, 0)} \|\text{Tr}_s |f|^{q/p}_{L^p/(G^{-1}(s), \rho)}\|_{L^p(G^{-1}(t, 0), \gamma_H)} \leq C' |t| \sup_{s \in (-\delta, 0)} \|\text{Tr}_s |f|^{q/p}_{L^p/(G^{-1}(s), \rho)}\|_{L^p(G^{-1}(t, 0), \gamma_H)},$$

where

$$C' := \|\nabla_Hf\|_{H} \sup_{s \in (-\delta, 0)} (1 \wedge \sup_{s \in (-\delta, 0)} \rho(G^{-1}(\{s\}))^{(p-1)/p}),$$

and we have used the fact that $\sup_{s \in (-\delta, 0)} \rho(G^{-1}(\{s\})) < +\infty$ (see estimate 4.24). By replacing estimate 4.21 in (4.3), it follows that

$$\|f\|_{L^p(G^{-1}(t, 0), \gamma)} \leq C_1 |t| \sup_{s \in (-\delta, 0)} \|\text{Tr}_s |f|^{q/p}_{L^p/(G^{-1}(s), \rho)}\|_{L^p(G^{-1}(t, 0), \gamma_H)} \leq C_1 |t| \|\text{Tr}_s |f|^{q/p}_{L^p/(G^{-1}(s), \rho)}\|_{L^p(G^{-1}(t, 0), \gamma_H)} + \|\text{Tr}_s |f|^{q/p}_{L^p/(G^{-1}(s), \rho)}\|_{L^p(G^{-1}(t, 0), \gamma_H)},$$

(4.4)
with \( C_1 := C \bar{C} \). Letting \( q \to p \) in the left-hand side of (4.4) we infer that
\[
\|f\|_{L^p(G^{-1}(I,0),\gamma)}^p \leq C_1 \|r\| \|f\|_{L^p(G^{-1}(I,0),\gamma)}^{p-1} \|\nabla_H f\|_{L^p(G^{-1}(I,0),\gamma,H)} + \|f\|_{L^p(G^{-1}(I,0),\gamma)}^p.
\]
(4.5)

We set \( \delta_1 := \delta \wedge \frac{1}{2} C_1^{-1} \). Then, for every \( t \in (-\delta_1,0) \) we get
\[
\|f\|_{L^p(G^{-1}(I,0),\gamma)}^p \leq C_1 |r|\|f\|_{L^p(G^{-1}(I,0),\gamma)}^{p-1} \|\nabla_H f\|_{L^p(G^{-1}(I,0),\gamma,H)} + \frac{1}{2} \|f\|_{L^p(G^{-1}(I,0),\gamma)}^p,
\]
which gives
\[
\|f\|_{L^p(G^{-1}(I,0),\gamma)}^p \leq 2C_1 |r|\|f\|_{L^p(G^{-1}(I,0),\gamma)}^{p-1} \|\nabla_H f\|_{L^p(G^{-1}(I,0),\gamma,H)}.
\]
(4.6)

By dividing both the sides of (4.6) by \( \|f\|_{L^p(G^{-1}(I,0),\gamma)}^{p-1} \) (which we assumed different from 0), we infer that
\[
\|f\|_{L^p(G^{-1}(I,0),\gamma)} \leq 2C_1 |r\| \|\nabla_H f\|_{L^p(G^{-1}(I,0),\gamma,H)},
\]
and the thesis is proved.

The next result (based on [7 Prop. 4.10]) is the infinite-dimensional version of a well-known theorem (see e.g. [10 Thm. 5.5.2]). We recall that the space \( W_0^{1,p}(O,\gamma) \) was defined in Definition 2.2, and characterized as the closure of \( \mathcal{H}_0^1(O) \) in \( W^{1,p}(O,\gamma) \) in Lemma 2.3.

**Theorem 4.1.** Let \( O = G^{-1}(\mathbb{R}) \) with \( G \) satisfying Hypotheses 3.1 and let \( f \in W^{1,p}(O,\gamma) \) for some \( p \in (1, +\infty) \), then, the following are equivalent:

i) \( f \in W_0^{1,p}(O,\gamma) \);

ii) \( \mathcal{T}_0 f \equiv 0 \);

iii) the trivial extension \( \overline{f} \) of \( f \) belongs to \( W^{1,p}(\mathbb{R},\gamma) \).

**Proof.** The points ii) and iii) are equivalent by Lemma 2.3.

By definition, if \( f \in W_0^{1,p}(O,\gamma) \), then it is the limit of a sequence \( (f_n) \subseteq \text{Lip}_p(O) \); clearly \( f_n \) can be extended as 0 out of \( O \), and the sequence of trivial extensions \( (\overline{f}_n) \) converges to the trivial extension \( \overline{f} \in W^{1,p}(\mathbb{R},\gamma) \) of \( f \) as \( n \to +\infty \). This gives i) \( \Rightarrow \) iii).

Now we prove iii) \( \Rightarrow \) i). Let us set
\[
O_m := G^{-1}\left(-\frac{2}{m}, \frac{2}{m}\right), \quad m \in \mathbb{N}.
\]
(O_m) is a sequence of open decreasing sets such that
\[
\bigcap_{m \in \mathbb{N}} O_m = G^{-1}(0).
\]
(4.7)

Let \( \eta \in C^\infty(\mathbb{R}) \) be such that \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) in \((-\infty,-1] \), \( \eta = 0 \) in \([-1/2, +\infty) \), and \( \eta' \leq 0 \) everywhere. For every \( m \in \mathbb{N} \), we define \( \chi_m \) as
\[
\chi_m(x) = \begin{cases} 
\eta(mG(x)+1), & \text{if } x \in O, \\
0, & \text{if } x \notin O.
\end{cases}
\]

It follows that \( \chi_m \in \text{Lip}_{p,H}(O) \) for every \( m \in \mathbb{N} \), that \( \chi_m = 1 \) on \( G^{-1}\left((-\infty,-\frac{2}{m}]\right) \) and \( \chi_m = 0 \) on \( G^{-1}\left((-\frac{1}{m}, +\infty)\right) \). Further,
\[
\nabla_H \chi_m(x) = (m\eta'(mG(x)+1))\nabla_G G(x), \quad \gamma\text{-a.e. } x \in O,
\]
(4.8)
\[ \nabla_H \chi_m \rho_{O_m} = 0 \quad \text{and} \quad \| \nabla_H \chi_m \|_{L^\infty(X,H)} \leq Cm, \tag{4.9} \]
for some positive constant \( C \) independent of \( m \). Let \( f \in W^{1,p}(O, \gamma) \) be such that \( \text{Tr}_0 f = 0 \). We have
\[
\int_O |f|^p \| \nabla_H \chi_m \|_H^p \, d\gamma \leq C^p m^p \int_{O_m} |f|^p \, d\gamma,
\]
and from Lemma 4.2 it follows that
\[
\int_O |f|^p \| \nabla_H \chi_m \|_H^p \, d\gamma \leq C_0 \| f \|_{W^{1,p}(O_m, \gamma)}, \tag{4.10}
\]
for some positive constant \( C_0 \) independent of \( m \) and \( \varphi \).

Now we prove that \( f \in W^{1,p}_0(O, \gamma) \), by finding a sequence \((f_m) \subseteq \text{Lip}_{c,H}(O)\) which converges to \( f \) in \( W^{1,p}(O, \gamma) \) as \( m \to +\infty \). We know that there exists a sequence \((g_n) \subseteq \mathcal{F} C_0^\infty(X)\) such that \( g_n \to f \) in \( W^{1,p}(X, \gamma) \) as \( n \to +\infty \). We fix \( n \in \mathbb{N} \). Then,
\[
\int_O |g_n \chi_m - f|^p \, d\gamma \leq 2^{p-1} \left( \int_O |g_n - f|^p \, d\gamma + \int_O |f|^p |\chi_m - 1|^p \, d\gamma \right),
\]
and the right-hand side converges to \( 0 \) as \( m \to \infty \), because \( \chi_m \to \chi_O \) in \( L^p(X, \gamma) \) as \( m \to +\infty \) for every \( r \in (1, +\infty) \) by the dominated convergence theorem, and \( g_n \to f \) in \( L^p(O, \gamma) \) as \( n \to +\infty \). Further, for every \( n, m \in \mathbb{N} \) we have
\[
\int_O \| \nabla_H (g_n \chi_m) - \nabla_H f \|_H^p \, d\gamma
\leq 4^{p-1} \left( \int_O |g_n - f|^p \| \nabla_H \chi_m \|_H^p \, d\gamma + \int_O \chi_m \nabla_H g_n - \nabla_H \chi_m - \chi_m \nabla_H f \|_H^p \, d\gamma \right)
\leq 4^{p-1} \left( m^p \| g_n - f \|_{L^p(O, \gamma)} + \| \nabla_H g_n - \nabla_H f \|_{L^p(O, \gamma)} \right) + \int_{O_m} \| \nabla_H f \|_H^p \, d\gamma
\leq 4^{p-1} \left( m^p \| g_n - f \|_{L^p(O, \gamma)} + \| \nabla_H g_n - \nabla_H f \|_{L^p(O, \gamma)} + (1 + C_0) \| f \|_{W^{1,p}(O_m, \gamma)} \right),
\]
where in the last inequality we exploit 4.10. By recalling that \( g_n \to f \) in \( W^{1,p}(X, \gamma) \) for \( n \to +\infty \), it follows that, for every fixed \( m \in \mathbb{N} \), there exists \( n_m \geq m \) such that \( \| g_{n_m} - f \|_{W^{1,p}(X, \gamma)} \leq m^{-p-1} \), which gives
\[
\int_O \| \nabla_H (g_{n_m} \chi_m) - \nabla_H f \|_H^p \, d\gamma \leq 4^{p-1} (m^{-p-1} + (1 + C_0) \| f \|_{W^{1,p}(O_m, \gamma)}).
\tag{4.11}
\]
We recall that \( \gamma(G^{-1} \{0\}) = 0 \) by Remark 3.2. Hence, from 4.7 the last addend in the right-hand side of 4.11 converges to \( 0 \) as \( m \to \infty \). We now define \( f_m = g_{n_m} \chi_m \) for every \( m \in \mathbb{N} \). Then, \((f_m)\) converges to \( f \) in \( W^{1,p}(O, \gamma) \) as \( m \to \infty \), and it is not hard to show that \( f_m \in \text{Lip}_{c,H}(O) \) for every \( m \in \mathbb{N} \). \( \square \)
5. Examples

5.1. Region below graphics. As above, we consider a basis \( \{ h_i \}_{i \in \mathbb{N}} \) in \( R_T(X^*) \). In the following, we will denote \( \pi_i \) with \( \pi \). We will define a function \( G \) such that \( O = G^{-1}((\infty, 0)) \) is the region below the graph of a smooth function.

Let \( \Phi \) be a real-valued function on \( X \) such that \( \partial \pi_i(\Phi) \equiv 0 \). Hence, for every \( x \) we have \( \Phi(x) = \Phi(x - \pi_i(x)) \). We set
\[
G(x) = \hat{h}_1(x) - \Phi(x), \quad x \in X.
\]
In this case, \( O \) is just the region below the graph of \( \Phi \). We assume that \( \Phi \) is continuous and also satisfies the following conditions:

1. \( \Phi \in \text{Lip}_H(X) \);
2. \( \Phi \in W^{2,p}(X, \gamma) \) for some \( p > 1 \) and \( \| D^2_H\Phi \|_{L^2(H)} \) is essentially bounded;
3. \( \Phi + L\Phi \in L^\infty(X, \gamma) \).

Under these assumptions, it follows that
\[
\nabla_H G(x) = h_1 - \nabla_H \Phi(x - \pi_i(x)),
\]
\[
D^2_H G(x) = -D^2_H \Phi(x - \pi_i(x)),
\]
\[
L G(x) = -\hat{h}_1(x) - L\Phi(x - \pi_i(x)),
\]
\( \gamma \)-a.e. \( x \in X \), and \( G \) satisfies Hypotheses [3.1]. Indeed, we have
\[
L G(x) = -\hat{h}_1(x) - L\Phi(x - \pi_i(x)) + \Phi(x - \pi_i(x)) - \Phi(x - \pi_i(x))
\]
\[= (-\hat{h}(x) + \Phi(x - \pi_i(x))) - (\Phi(x - \pi_i(x)) + L\Phi(x - \pi_i(x)))
\]
\[= G(x) - (\Phi(x - \pi_i(x)) + L\Phi(x - \pi_i(x))),(\gamma \text{-a.e. } x \in X \text{, and both the addends are bounded on } G^{-1}(I_0)). \]
If we take for instance \( \Phi = c \in \mathbb{R} \), we get that open half-planes satisfy our assumptions.

5.2. Brownian motion and pinned Brownian motion. For the following examples we refer to [3, Section 5].

We recall (see [5, Section 2.3]) that a Brownian motion starting from 0 can be modelled by a Wiener space \( (X, \gamma_W) \) where \( X = L^2(0, 1) \) (with Lebesgue measure), and \( \gamma_W \) concentrates on the set of the elements of \( L^2(0, 1) \) which have a continuous representative \( f \) such that \( f(0) = 0 \). The Cameron-Martin space \( H \) is the set of the elements of \( L^2(0, 1) \) which have an absolutely continuous representative \( f \) such that \( f'(0) \in L^2(0, 1) \) and \( f(0) = 0 \). Finally, for every \( f_1, f_2 \in H \) the inner product in \( H \) is defined as \( (f_1, f_2)_H = \int_0^1 f_1'(s)f_2'(s)ds \). In the following, for every \( h \in H \), we will identify \( h \) with its absolutely continuous representative.

We define an orthonormal basis \( \{ e_n \}_{n \in \mathbb{N}} \) of \( L^2(0, 1) \) as
\[
e_n(s) = \sqrt{2} \sin\left(\frac{s}{\sqrt{\lambda_n}}\right) = \sqrt{2} \sin\left(\frac{2n+1}{2}\pi s\right)
\]
where
\[
\lambda_n = \frac{1}{\pi^2 (n + \frac{1}{2})^2}.
\]
For every \( n \in \mathbb{N} \) we set \( h_n = \sqrt{\lambda_n} e_n \). It follows that \( \{ h_n \}_{n \in \mathbb{N}} \) is an orthonormal basis of \( H \).

We consider a function \( g \in C^2(\mathbb{R}) \), with bounded first and second order derivative, such that there exists \( C > 0 \) such that
\[
|g''(\xi) - g''(\eta)| \leq C|\xi - \eta|(|\xi| + |\eta|), \quad (5.1)
\]
for every $\xi, \eta \in \mathbb{R}$. Further, we assume that there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that $|g'(\xi)| \geq a$ (hence $g'(\xi) \neq 0$) for every $\xi \in \mathbb{R}$ and

$$\alpha_1 g(\xi) + \beta_1 \leq \xi g'(\xi) \leq \alpha_2 g(\xi) + \beta_2$$  \hspace{1cm} (5.2)

for every $\xi \in \mathbb{R}$.

The above assumptions are satisfied, for instance, by the function $g = p/q$, where $q$ is a positive polynomial of degree $m \in \mathbb{N}$ and $p$ polynomial of degree $m+1$ such that $p'(\xi) \neq 0$ for every $\xi \in \mathbb{R}$.

**Proposition 5.1.** Let us assume that $g \in C^2(\mathbb{R})$ satisfies (5.1) and (5.2), and let $r$ belong to the range of $g$. We define the function

$$G(x) := \int_0^1 g(x(s)) \, ds - r$$

for every $x \in X = L^2(0,1)$. Then, $G$ satisfies Hypotheses 3.1.

**Proof.** It is not hard to show that $G$ is $H$-differentiable. For every $h \in H$ and every $x \in X$ we have

$$\langle \nabla_H G(x), h \rangle_H = \int_0^1 g'(x(s)) h(s) \, ds$$

and

$$\|\nabla_H G(x)\|_H \leq \sqrt{\int_0^1 |g'(x(s))|^2 \, ds} \leq \|g'\|_\infty.$$  

Moreover, for every $x, y \in X$,

$$\|\nabla_H G(x) - \nabla_H G(y)\|_H^2 \leq \int_0^1 |g'(x(s)) - g'(y(s))|^2 \, ds \leq \int_0^1 \|g''\|_\infty^2 |x(s) - y(s)|^2 \, ds$$

$$\leq \|g''\|_\infty^2 \|x - y\|_X^2,$$

from which it follows that $G \in \mathcal{H}^1(X)$. Further, $D^2_H G$ is everywhere defined and, for every $h, k \in H$ and every $x \in X$ we get

$$\langle \langle D^2_H G(x) \rangle (h), k \rangle_H = \int_0^1 g''(x(s)) h(s) k(s) \, ds.$$  

Hence,

$$\|D^2_H G(x)\|_{\mathcal{F}_2^2(H)} = \sum_{n=1}^\infty \|D^2_H G(x) h_n\|_H^2 \leq \sum_{n=1}^\infty \lambda_n \|g''\|_\infty^2 < +\infty,$$

for every $x \in X$. Let us consider the function $\tilde{h} \in H$ defined by $\tilde{h}(s) = s$ for every $s \in [0,1]$. It follows that $\tilde{h} > 0$ and $\|\tilde{h}\|_H = 1$. Further, since $g'$ has constant sign (from $|g'| \geq a$), we have

$$|\langle \nabla_H G(x), \tilde{h} \rangle_H| = \int_0^1 |g'(x(s))| \tilde{h}(s) \, ds \geq a \int_0^1 \tilde{h}(s) \, ds = \frac{a}{2},$$

for every $x \in X$, which implies that

$$\|\nabla_H G\|_H^{-1} \leq \frac{2}{a}.$$  \hspace{1cm} (5.3)
If we consider the sequence \( \{h_k = \sqrt{n_k} e_k\}_{k \in \mathbb{N}} \), then the series \( \sum_{k=1}^{\infty} h_k^2 \) uniformly converges to a function \( f \in C([0, 1]) \). Moreover,

\[
LG(x) = \sum_{i=1}^{\infty} \langle D^2_H G(x)(h_i), h_i \rangle_H - \sum_{i=1}^{\infty} \langle \nabla_H G(x), h_i \rangle_H \hat{h}_i(x) \\
= \sum_{i=1}^{\infty} \int_{0}^{1} g''(x(s)) h_i^2(s) ds - \int_{0}^{1} g'(x(s)) x(s) ds \\
= \int_{0}^{1} g''(x(s)) f(s) ds - \int_{0}^{1} g'(x(s)) x(s) ds.
\]

The first addend in the last right-hand side of the above chain of equality is bounded because \( g'' \in C_b(\mathbb{R}) \) and \( f \in C([0, 1]) \). Further, from (5.2) we infer that

\[
\int_{0}^{1} g'(x(s)) x(s) ds \geq \int_{0}^{1} (\alpha_1 g(x(s)) + \beta_1) ds = \alpha_1 G(x) - \alpha_1 r + \beta_1
\]

and

\[
\int_{0}^{1} g'(x(s)) x(s) ds \leq \int_{0}^{1} (\alpha_2 g(x(s)) + \beta_2) ds = \alpha_2 G(x) - \alpha_2 r + \beta_2.
\]

Therefore, \( LG \) is bounded in \( G^{-1}((-\delta, \delta)) \) for every \( \delta > 0 \).

Finally, since \( r \) belongs to range of \( g \) it follows that that \( G^{-1}(\{0\}) \neq \emptyset \), we conclude that \( G \) fulfills Hypotheses 5.1.

An analogous example can be provided for pinned Wiener space, which models Brownian bridge with starting point at 0 and subject to the condition that in 1 the arriving point is 0. \((X, \mathfrak{F}_W)\) where \( X = L^2(0, 1) \), the Cameron-Martin space is \( H = W_{0,1}(0, 1) \). We recall that \( \{e_n\}_{n \in \mathbb{N}} \) with \( e_n = \sqrt{\frac{2}{n}} \sin(n\pi \cdot) \) for every \( n \in \mathbb{N} \) is an orthonormal basis of \( X \), and \( \{h_n\}_{n \in \mathbb{N}} \), where \( h_n = \sqrt{\frac{2}{n}} \sin(n\pi \cdot) \) for every \( n \in \mathbb{N} \) is an orthonormal basis of \( H \).

If we consider a function \( g \in C^2(\mathbb{R}) \) which enjoys (5.1) and (5.2), arguing as in the proof of Proposition 5.1, we gain the following result.

**Proposition 5.2.** Given \( r \) in the range of \( g \), we define

\[
G(x) = \int_{0}^{1} g(x(s)) ds - r
\]

for every \( x \in X = L^2(0, 1) \). Then, the function \( G \) satisfies Hypotheses 5.1.

**References**

[1] D. Addona, G. Cappa, S. Ferrari, *Domains of elliptic operators on sets in Wiener space*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 23 (2020), 2050004, 42 pp.

[2] D. Addona, G. Menegatti, M. Miranda Jr, *BV functions on open domains: the Wiener case and a Fomin differentiable case*, Commun. Pure Appl. Anal. 19 (2020), 2679-2711.

[3] D. Addona, G. Menegatti, M. Miranda Jr, *On integration by parts formula on open convex sets in Wiener spaces*, J. Evol. Equ., 21 (2021), 1917-1944.

[4] S. Bonaccorsi, L. Tubaro, M. Zanella, *Surface measures and integration by parts formula on levels sets induced by functionals of the Brownian motion in \( \mathbb{R}^m \)*, NoDEA Nonlinear Differential Equations Appl., 27 (2020), 22 pp.

[5] V. I. Bogachev, *Gaussian Measures*, Mathematical Surveys and Monographs, American Mathematical Society, 1998.

[6] V. I. Bogachev, A. Y. Pilipenko, A. V. Shaposhnikov, *Sobolev functions on infinite-dimensional domains*, J. Math. Anal. Appl. 419 (2014), 1023-1044.

[7] P. Celada, A. Lunardi, *Traces of Sobolev functions on regular surfaces in infinite dimensions*, J. Funct. Anal., 266 (2014), 1948-1987.
[8] G. Da Prato, A. Lunardi, *Maximal $L^2$ regularity regularity for Dirichlet problems in Hilbert spaces*, J. Math. Pures Appl., 99 (2013), 741-765.

[9] J. Diestel, J. J. Uhl, *Vector measures*, Mathematical Surveys and Monographs, 15, American Mathematical Society, 1977.

[10] L. Evans, *Partial Differential Equations*. American Mathematical Society, 1998.

[11] D. Feyel, *Hausdorff-Gauss Measures*, in: Stochastic Analysis and Related Topics, VII., Progr. in Probab. 98, Birkhäuser, 2001, 59-76.

[12] M. Hino, *Dirichlet spaces on $H$-convex sets in Wiener space*, Bull. Sci. Math., 135 (2011) 667-683; Erratum: Bull. Sci. Math., 137 (2013) 688-689.

[13] M. Hino, *On Dirichlet spaces over convex sets in infinite dimensions*, Finite and Infinite Dimensional Analysis in Honor of Leonard Gross, Contemp. Math., 317 (2003), 143-156.

[14] H. Sugita, *Positive generalized Wiener functions and potential theory over abstract Wiener spaces*, Osaka J. Math., 25 (1988), 665-696.