Recent strategy to study fractal-order viscoelastic polymer materials using an ancient Chinese algorithm and He’s formulation

Alex Elías-Zúñiga\textsuperscript{1}, Luis M Palacios-Pineda\textsuperscript{2}, Daniel Olvera-Trejo\textsuperscript{2} and Oscar Martínez-Romero\textsuperscript{1}

Abstract
This paper introduces a novel methodology to determine the frequency-amplitude relationship of fractal-order viscoelastic polymer materials using the two-scale fractal dimension transform, the equivalent power-form representation of the conservative restoring forces, and a simple coordinate transformation to eliminate viscoelastic effects. Then, the ancient Chinese algorithm Ying Bu Zu Shu and He’s formulation are used for obtaining the frequency-amplitude relationship. Simulation results obtained from the derived expressions exhibit good agreement when compared to numerical integration solutions. This article elucidates how the molecular structure of polymer chains influences the relaxation oscillations as a function of the fractal parameter values.

Keywords
Two-scale fractal dimension transform, power-form equivalent transformation, ancient Chinese algorithm, He’s frequency formulation, fractal and damped polymer chains oscillator

Introduction
The material damping properties serve to modify the oscillations’ amplitude through energy dissipation. In this sense, polymers are used for energy absorption and vibration damping attenuation in several engineering devices since the damping forces are small compared to the elastic and inertia ones.\textsuperscript{1,2} The oscillation’s decay in free vibrations can be explained considering damping effects that prevent oscillations by energy dissipation due to the material molecule movement. Of course, factors such as viscoelastic material properties and glass-transition phenomena could influence damping effects. It is also known that when a rubber material vibrates, its molecules adopt new conformations releasing the vibration energy. Recently, Sarkheil\textsuperscript{3} identified that the mobility of polymer molecules is an indication of the molecular structure fractal behavior. Schiessel and Blumen\textsuperscript{4} found that dynamic processes in polymers such as mechanical relaxation dynamics and cross-linking at the sol-gel transition exhibit fractal behavior. Based on these findings, Elías-Zúñiga et al.\textsuperscript{5} investigated how the fractal molecular structure of polymer materials changes as the oscillation amplitude varies and how this polymer material behavior can be used to identify molecular structure defects and intermolecular cohesion.

It is evident that including damping effects in the mathematical models provides qualitative and quantitative material response behavior that could agree with experimental observations. Furthermore, products such as vibration dampers, bridge bearings, seismic absorbers, and building/engine mounts, to name a few, are designed considering a material’s high
damping value to mitigate undesirable vibration effects. As expected, the addition of damping effects in the governing equation of motion in conjunction with strongly nonlinear restoring forces increases the complexity in deriving approximate analytical solutions valid for the whole solution domain. Therefore, one needs to identify which solution technique must be used to get the desired solution. A good overview of some available techniques to get the approximate solution of strongly nonlinear and damped equations such as variational approaches, homotopy techniques, He’s frequency formulation, Taylor series, iteration methods, equivalent power-form approach, energy techniques, simple frequency formulations, and ancient Chinese mathematical methods are given in Refs. 7–34 and references cited therein.

The goal of this article is to introduce a straightforward methodology to solve the fractal and damped differential equation of motion that models the dynamic response of polymer materials using the two-scale fractal dimension transform, the equivalent power-form representation of the restoring force, and a coordinate transformation to eliminate the damping term. Then, the frequency-amplitude relationship is determined by using the Ying Bu Zu Shu or ancient Chinese algorithm of the Chinese method.39–42

Mathematical model

Before introducing the fractal mathematical model that describes the dynamics of a viscoelastic Langevin polymer chain first, we recall the simple definition of the two-scale dimension transform and its relevance in modeling multiscale phenomena that describe behavior at a given length or time scales, playing a critical role in bridging qualitative and quantitative methods for engineering problem analysis. In fact, the two-scale theory models each phenomenon with a large scale to characterize continuous media with the classic calculus, and on the smaller scale, to elucidate system molecular effects and to reflect discontinuous phenomena. The two-scale dimension transform has been used to study the N/MEMS in a fractal space (porous medium),43 the Fangzhu water collector from air,44–48 the vibration attenuation and vibration absorption of the porous concrete,49 the smart receptor system for exact printing of nano/micro devices modeled by a fractal Duffing equation,50 the rheological properties of SiC-based print paste using a fractal model for predicting the material viscosity,51 the fractal model of current generation in porous electrodes,52 the fractal Toda oscillator that describes the intensity fluctuation of Nd:Yag lasers,53 the nonlinear vibration of the carbon nanotube embedded in fractal medium,54 the shallow water waves traveling along an unsmooth boundary,55 to describe the flow of the shallow water of harbor considering an unsmooth boundary,56 and packaging cushioning systems57 among others. Furthermore, modeling polymer molecules’ dynamics using fractal derivatives and the two-scale dimension transform aids in identifying molecular structure defects, intermolecular cohesion, and the evolution of the polymer storage modulus as a function of frequency and of the fractal parameter.5

Considering the relevance that polymers have in engineering applications, there is a need for a better understanding of their network structure behavior that affects physical and mechanical properties; therefore, a mathematical model that takes into account fractal patterns that are formed in the preparation of polymers and their composites is highly desirable.58 In this article, we use a dynamics mathematical model that can be obtained from the fractal Lagrange equation

$$\frac{d}{dt}\alpha \left( \frac{\partial(T - V)}{\partial q_i} \right) - \frac{\partial(T - V)}{\partial q_i} = Q_i$$

(1)

that considers the Langevin polymer chain’s viscoelastic effects. In equation (1), the kinetic and Langevin potential energies $T$ and $V$ can be computed from the following expressions

$$T = \frac{1}{2} M \left( \frac{dq_i}{dt^\alpha} \right)^2$$

(2)

$$V = kNT \int_0^{\alpha_i} L^{-1}(q_i) dq_i$$

(3)

Here, $L^{-1}(x)$ is known as the inverse Langevin function defined as $x = L(\beta) = \coth(\beta) - 1/\beta$ with $\beta = L^{-1}(x)$.59,60 $q_i$ are the generalized coordinates, the overdot denotes derivative with respect to time, $Q_i$ are the generalized nonconservative forces, $M$ is the mass, and $\alpha$ is the fractal dimension.51–63 Thus, for a viscoelastic material, the governing equation of motion becomes
\[ \frac{d}{dt^{\alpha}} \left( \frac{d\lambda}{d\tau^{\alpha}} \right) + 2\nu \frac{d\lambda}{d\tau} + K_1 \mathcal{L}^{-1}(\lambda) = 0, \quad \lambda(0) = A, \quad \frac{d\lambda}{d\tau} = 0 \] (4)

where \( K_1 = kNT/(ML_1) \), \( 2\nu = v_0/(ML_1) \), \( \lambda \) is the chain stretch, \( L_1 \) is the undeformed chain length, and \( v_0 \) is the damping constant.

Recalling the fractal derivative definition with respect to time

\[ \frac{d\lambda}{d\tau^{\alpha}} \left( t_0 \right) = \Gamma(1 + \alpha) \lim_{t \to t_0} \frac{\lambda(t) - \lambda(t_0)}{(t - t_0)^{\alpha}} \] (5)

that agrees with the classical differential derivative when the two-scale fractal dimension takes the value of one. Thus, using the fractal dimension transform,\(^{64-66}\)

\[ \tau = t^{\alpha} \] (6)

Equation (4) can be written in the form

\[ \frac{d^2x}{d\tau^2} + 2\nu \omega_n \frac{dx}{d\tau} + K_1 \mathcal{L}^{-1}(\lambda) = 0, \quad \lambda(0) = A, \quad \frac{d\lambda}{d\tau} = 0 \] (7)

Next, the approximate representation form of \( \mathcal{L}^{-1}(\lambda) \) is introduced

\[ \mathcal{L}^{-1}(\lambda) = \frac{\lambda(30 - 26\lambda + 7\lambda^2)}{3(1 - a\lambda)(10 + a\lambda)} \] (8)

where \( \omega_n = \sqrt{K_1} \), \( a = 1/\lambda_0 \), and \( \lambda_0 \) describes the maximum possible chain extension\(^{61}\); therefore, equation (7) simplifies to

\[ \frac{d^2\lambda}{d\tau^2} + 2\nu \omega_n \frac{d\lambda}{d\tau} + K_1 \frac{\lambda(30 - 26\lambda + 7\lambda^2)}{3(1 - a\lambda)(10 + a\lambda)} = 0 \] (9)

To find the frequency-amplitude expression for equation (9) using the ancient Chinese algorithm Ying Bu Zu Shu, we first use the approach proposed in Refs.\(^{67-74}\) to find its power-form equivalent representation, and then, we introduce a coordinate transformation to eliminate the damping term.

**Power-form equivalent representation**

Following the transformation approach introduced in Refs.\(^{67-74}\), equation (7) can be written in the following form

\[ \frac{d^2\lambda}{d\tau^2} + 2\nu \omega_n \frac{d\lambda}{d\tau} + a_1 \lambda + a_2 \text{sgn}(\lambda) |\lambda|^m = 0 \] (10)

where \( a_1, a_2, \) and \( m \) are parameters that can be determined using the weighted mean square error \( U \), which is defined as

\[ U = \int_0^\infty \left( \frac{K_1 \lambda(30 - 26\lambda + 7\lambda^2)}{3(1 - a\lambda)(10 + a\lambda)} - a_1 \lambda - a_2 \text{sgn}(\lambda) |\lambda|^m \right)^2 w(\lambda) d\lambda \] (11)

with

\[ \frac{\partial U}{\partial a_1} = 0, \quad \frac{\partial U}{\partial a_2} = 0 \] (12)

where the weighted function \( w(\lambda) \) is assumed to have the form \( w(\lambda) = \lambda^{N_1} \). Using equations (11) and (12), we get the following expressions for \( a_1 \) and \( a_2 \).
\[ a_1 = \frac{1}{(33a^2(-1 + m)^2)}K_1(11(-7a^2/m - 1)^2 - ((637 + 6a(39 + 5a))(3 + N_i)}
\]
\[ (2 + m + N_i)^3/(1 + N_i)a^2 + a(63 + 6a)(3 + N_i)(2 + m + N_i) + (2 + N_i)\sigma^2/70 + a(26 + 3a)(3 + N_i)(2 + 2m + N_i)\]
\[ F_1[1, 2 + m + N_i, 3 + m + N_i, a\sigma] + \Gamma[3 + N_i](2 + m + N_i)^2
\]
\[ \Gamma[1 + N_i(100(70 + a(26 + 3a)); F_1[1, 1 + N_i, 2 + N_i, a\sigma/10] + (7 - 26a + 30a^2)F_1[1, 1 + N_i, 2 + N_i, a\sigma])]
\]
\[ a_2 = \frac{1}{(33a^2(m - 1)^2)}K_1(2 + m + N_i)(1 + 2m + N_i)
\]
\[ \sigma^{-1-m}(11(3 + N_i)((637 + 6a(39 + 5a))(2 + N_i) - a(63 + 26a)(1 + N_i)\sigma) + a(26 + 3a)(1 + N_i)(2 + N_i)\sigma^2F_1[1, 2 + m + N_i,
\]
\[ 3 + m + N_i, a\sigma/10] + a^2(7 - 26a + 30a^2)(1 + N_i)(2 + N_i)\sigma^2
\]
\[ F_1[1, 2 + m + N_i, 3 + m + N_i, a\sigma] + \Gamma[4 + N_i](-100(70 + a(26 + 3a))F_1[1, 1 + N_i, 2 + N_i, a\sigma])
\]

where \( F_1[a, b; c; z] \) is the hypergeometric function \( F_1(a, b; c; z) \), and \( F_1[a, b, c, z] \) is the regularized hypergeometric function \( F_1[a, b, c, z] \). Substitution of equations (13) and (14) into equations (11) and minimization of the resulting expression, yield the value of \( \sigma, N_i, \) and \( m \).

**Frequency-amplitude formulation**

Before we start with the derivation of the frequency-amplitude formulation of equation (10) using the Chinese algorithm and He’s formulation, it is important to point out some of its advantages over traditional analytical methods. This method does not depend on small or large physical parameters. It has a high convergence rate, and its accuracy is higher when compared to Newton’s iteration method. This ancient algorithm has been also used to derive, by modifying Chun-Hui He’s algorithm, an efficient iterative algorithm that converges to the optimal solution with only a few number of iterations. Furthermore, it has excellent flexibility in choosing the functions needed to identify the trial solutions; it can be used to obtain the frequency-amplitude formulation of nonlinear oscillators without a linear term that makes it harder to find an approximate solution when using perturbation methods. This ancient Chinese algorithm can be also combined with other techniques such as the power-form equivalent representation, Padé approximants, Chebyshev polynomials, Fourier series, to name a few. One possible limitation of this algorithm might be related to the incapacity of getting the transient response of nonlinear forced oscillators. However, the extent of this limitation certainly requires further investigation.

In what follows, we focus our efforts on finding the frequency-amplitude expression for equation (10). One can see that the determination of the frequency-amplitude expression for equation (10) using He’s formulation and the ancient Chinese algorithm is not a straightforward process because of the presence of the damping term. Therefore, to avoid potential complications in the derivation of the frequency-amplitude relationship using this ancient algorithm, the following coordinate transformation

\[ \lambda = e^{-t\omega_n}y(t) \]

is introduced to eliminate the damping term in equation (10). This coordinate transformation yields the following equation of motion

\[ \frac{d^2\lambda}{dt^2} + \left(a_1 - \omega_n^2\right)\lambda + \omega_n^2e^{-t\omega_n}\left(m+1\right)sgn(y)|y|^{m} = 0 \]

Thus, the frequency-amplitude expression of equation (16) will be determined using He’s formulation based on the ancient Chinese algorithm Ying Bu Zu Shu originated from an ancient Chinese mathematics monograph—The Nine Chapters on the Mathematical Art (see Refs. and references cited therein). In this formulation, it is first assumed that the approximate trial solutions \( x_1(t) \) and \( x_2(t) \) of equation (16) are given as
where $A_1$ and $A_2$ need to be defined. Then, the substitution of equation (17) into equation (16) yields the trial residual functions

$$R_i = A_i \cos(\omega_i \tau) \left( a_1 \omega_i^2 - v^2 \omega_i^2 \right) + a_2 \exp(- (m + 1) v \tau \omega_i) A_i^m \cos(\omega_i \tau)^m$$

where $i = 1, 2$ (no sum). Next, the average trail residuals $R_i$ defined by the following equation

$$\bar{R}_i = \frac{4}{T_i} \int_0^{T_i/4} R_i d \tau$$

are determined with $T_i = 2\pi/\omega_i$. To simplify the integration of equation (19), we use Fourier series to write the term $\cos(S \omega_i)^{m}$ in equivalent form as

$$\cos(S \omega_i)^{m} = b_{1m} \cos(S \omega_i) + b_{3m} \cos(3S \omega_i)$$

with

$$b_{1m} = \frac{2 \Gamma(1 + m/2)}{\pi^{1/2} \Gamma(3 + m/2)}, \quad b_{3m} = \frac{(m - 1) \Gamma(1 + m/2)}{\pi^{1/2} \Gamma(5 + m/2)}$$

where $\Gamma[...]$ is the Gamma function. Thus, substitution of equation (18) into equation (19), yields

$$\bar{R}_i = \frac{4}{T_i} \left( A_i a_1 / \omega_i - A_i \omega_i - A_i \frac{v^2 \omega_i^2}{\omega_i} + \left( A_i^m b_{1m} a_2 e^{-1/4(1 + m) T \omega_i} (\omega_i + (1 + m) \nu (e^{1/4(1 + m) T \omega_i} - 1)) / (\omega_i^2 + (1 + m)^2 v^2 \omega_i^2) + \left( A_i^m b_{3m} a_2 e^{-1/4(1 + m) T \omega_i} (-3 \omega_i + (1 + m)^2 v^2 \omega_i^2) \right) / (9 \omega_i^2 + (1 + m)^2 v^2 \omega_i^2) \right)$$

By using He’s formula

$$\omega_{AE}^2 = \frac{\omega_{2i}^2 - \omega_{2i}^2 \bar{R}_2}{R_2 - R_1}$$

the approximate frequency-amplitude expression for equation (9) is found. Here, it is assumed that $\omega_1 = 1$, $\omega_2 = 2$, $A_1 = A_2 = A$ as discussed in Ref.\textsuperscript{39-42}

The following section will address the applicability of equation (23) to obtain the fractal frequency-amplitude response curves of equation (9).

**Numerical results**

To illustrate the applicability of our proposed approach to obtain the frequency-amplitude response curve of the fractal and damped polymer chain oscillator, let us consider the case for which the parameter values are $\alpha = 0.5$, $\nu = 0.1$; $K_1 = 1$; and $\alpha = 0.9$, 1, and 1.1 with $y(0) = A = 0.5$ and $\dot{y}(0) = 0$. Figure 1 shows the amplitude-time response curves plotted using the values of $a_1 = 1.0032$, $a_2 = -0.3113$, $\sigma = 1$, $N_1 = -2.9$, $m = 1.75$, and $\omega_{AE} = 0.9265$. Small discrepancies are observed in Figure 1 between the numerical integration solution (solid lines) of equation (9) and the approximate solution (dashed lines) obtained from

$$\lambda(\tau) = A e^{-\nu \tau / \omega_i} \cos\left(\omega_{AE} \tau^{1/\alpha}\right)$$

As a second example, let us consider the system parameter values of $\alpha = 0.85$; $\nu = 0.1$; $K_1 = 0.25$; and $\alpha = 0.9$, 1, and 1.1 with $y(0) = A = 0.9$ and $\dot{y}(0) = 0$. Figure 2 shows that our proposed solution (dashed lines) given by equation (24) follows the numerical integration solution of equation (9) well. Furthermore, our derived solution is able to capture the variation of the wave length propagation as well as the decaying system oscillations as a function of the fractal exponent values. In this case, the parameter values of $m$, $N_1$, and $\sigma$ used to plot the amplitude-time curves shown in Figure 2 were 2.5, $-2.9$, and 0.5, respectively, with $a_1 = 0.2495$, $a_2 = 0.0097$, and $\omega_{AE} = 0.5025$. 

$$\lambda(\tau) = A e^{-\nu \tau / \omega_i} \cos\left(\omega_{AE} \tau^{1/\alpha}\right)$$
The proposed methodology capture fractal phenomena that occur at the polymer molecular level. In fact, the computed amplitude-time curves indicate that the chain’s mobility is less localized when $\alpha < 1$ since the polymer molecular structure exhibits fluid-like behavior with steady flow viscosity and low vibration frequency.\textsuperscript{83,84} Although there is evidence that in polymer fluids (PFGs), high energy-dissipation property can be precisely tailored at desired frequencies,\textsuperscript{85} one needs to bear in mind that the localized strain accumulation process, that occurs at lower frequencies, could lead to the crack initiation, followed by its propagation, and failure of the polymer part. In other words, the polymer material macroscopic behavior strongly depends on the material fractal structure.\textsuperscript{86}

When considering $\alpha > 1$, the polymer material exhibits solid-like behavior due to geometrical changes in the chemical network during gelation because the dynamic properties depend strongly on the material composition and structure.\textsuperscript{87} Therefore, understanding the fractal nature of the material molecular structure can help to develop a polymer material with the desired dynamic response that fulfills engineering specifications.

**Conclusions**

This article elucidates how our proposed solution methodology can be extended to study the relaxation oscillations of a fractal viscoelastic polymer chain oscillator. We have used this methodology to obtain the frequency-amplitude relationship...
of a mathematical model that describes the dynamics of fractal viscoelastic polymer Langevin chains at the material molecular level. The accuracy of the derived solution is confirmed by plotting the amplitude-time curves that indicate less localized chains mobility when $\alpha < 1$ because the polymer exhibits fluid-like behavior. For fractal values bigger than one, the material exhibits solid-like behavior which induces the largest decay in the oscillations amplitude of the polymer chain oscillator.

The derived approximate frequency-amplitude relationship sheds new light on understanding how the fractal parameter value correlates with relaxation oscillations of polymer chains. Therefore, understanding the fractal nature of the material molecular structure can help to have polymer materials with the desired dynamic response.

Declaration of conflicting interests
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: Tecnológico de Monterrey funded this research through the Research Group of Nanotechnology for Devices Design, and by the Consejo Nacional de Ciencia y Tecnología de México (Conacyt), Project Numbers 242269, 255837, and 296176.

ORCID iDs
Alex Elías-Zúñiga https://orcid.org/0000-0002-5661-2802
Luis M Palacios-Pineda https://orcid.org/0000-0001-5297-2950
Daniel Olvera Trejo https://orcid.org/0000-0002-4385-6269

References
1. Crandall SH. The role of damping in vibration theory. J Sound Vib 1970; 11(1): 3–18.
2. Liu N, Song G, Yi J, et al. Damping analysis of polyurethane/polyacrylate interpenetrating polymer network composites filled with graphite particles. Polym Compos 2013; 34(2): 288–292.
3. Sarkheil H and Rahbari S. Fractal geometry analysis of chemical structure of natural starch modification as a green biopolymeric product. Arab J Chem 2019; 12: 2430–2438.
4. Schiessel H and Blumen A. Fractal aspects in polymer science. Fractals 1995; 3: 483–490.
5. Elías-Zúñiga A, Martínez-Romero O, Olvera-Trejo D, et al. Fractal equation of motion of a non-Gaussian polymer chain: investigating its dynamic fractal response using an ancient Chinese algorithm. J Math Chem 2021; 60: 461–473. DOI: 10.1007/s10910-021-01310-x.
6. Geethamma V G, Asaletha R, Kalarikkal N, et al. Vibration and sound damping in polymers. Resonance 2014; 19: 821–833.
7. He JH. Some asymptotic methods for strongly nonlinear equations. Int J Mod Phys B 2006; 20(1): 1141–1199.
8. He CH. An introduction an Ancient Chinese algorithm and its modification. Int J Numer Methods Heat Fluid Flow 2016; 26(8): 2486–2491.
9. He CH. A simple analytical approach to a non-linear equation arising in porous catalyst. Int J Numer Methods Heat Fluid Flow 2017; 27(4): 861–866.
10. Sedighi HM and Daneshmand F. Nonlinear transversely vibrating beams by the homotopy perturbation method with an auxiliary term. J Appl Comput Mech 2015; 1(1): 1–9.
11. Anjum N and He J-H. Two modifications of the homotopy perturbation method for nonlinear oscillators. J Appl Comput Mech 2020; 6(SI): 1420–1425.
12. Amer TS, Galal AA, and Elnaggar S. The vibrational motion of a dynamical system using homotopy perturbation technique. Appl Math 2020; 11(11): 1081–1099.
13. He J-H, Amer TS, Elnaggar S, et al. Periodic property and instability of a rotating pendulum system. Axioms 2021; 10: 191.
14. He J-H, Yang Q, He CH, et al. A simple frequency formulation for the tangent oscillator. Axioms 2021; 10: 320.
15. He J-H, El-Dib YO, and Mady AA. Homotopy perturbation method for the fractal Toda oscillator. Fractal and Fract 2021; 5(3): 93.
16. Nadeem M and He J-H. The homotopy perturbation method for fractional differential equations: part 2, two-scale transform. Int J Numer Methods Heat Fluid Flow 2021; 11(3): 3490–3504.
17. Sedighi HM. Size-dependent dynamic pull-in instability of vibrating electrically actuated microbeams based on the strain gradient elasticity theory. Acta Astronaut 2014; 95: 111–123.
18. He CH, Liu C, He JH, et al. Passive atmospheric water harvesting utilizing an ancient Chinese ink slab. *Facta Univ Ser Mech Eng* 2021; 19(2): 229–239.
19. Wang K. He’s frequency formulation for fractal nonlinear oscillator arising in a microgravity space. *Numer Methods Partial Differ Equ* 2021; 37(2): 1374–1384.
20. Qie N, Houa WF, and He J-H. The fastest insight into the large amplitude vibration of a string. *Rep Mech Eng* 2020; 2(1): 1–5.
21. He J-H. Iteration perturbation method for strongly nonlinear oscillators. *J Vib Control* 2001; 7(5): 631–642.
22. Sedighi HM and Daneshmand F. Static and dynamic pull-in instability of multi-walled carbon nanotube probes by He’s iteration perturbation method. *J Mech Sci Technol* 2014; 28(9): 3459–3469.
23. He J-H, Kong HY, Chen RX, et al. Variational iteration method for Bratu-like equation arising in electrospinning. *Carbohydr Polym* 2014; 105: 229–230.
24. He J-H. A fractal variational theory for one-dimensional compressible flow in a microgravity space. *Fractals* 2020; 28(02): 2050024.
25. Wang KL. Fractal variational theory for Chaplygin-He gas in a microgravity condition. *Comput Methods Appl Mech Eng* 2020; 6(31): 1606–1612.
26. Wang KL, Yao SW, Liu YP, et al. A fractal variational principle for the telegraph equation with fractal derivatives. *Fractals* 2020; 28(04): 2050058.
27. Wang KL. A new fractal model for the soliton motion in a microgravity space. *Int J Numer Methods Heat Fluid Flow* 2021; 31(1): 442–451.
28. Goharian M and Koochi A. Nonlinear oscillations of CNT nano-resonator based on nonlocal elasticity: the energy balance method. *Rep Mech Eng* 2021; 2(1): 41–50.
29. Elias-Zúñiga A and Martínez-Romero O. Energy method to obtain approximate solutions of strongly nonlinear oscillators. *Math Probl Engr* 2013; 2013: 620591.
30. He J-H, Hou WF, Qie N, et al. Hamiltonian-frequency-amplitude formulation for nonlinear oscillators. *Facta Univ Ser Mech Eng* 2021; 19(2): 199–208.
31. He J-H and Garcia A. The simplest amplitude-period formula for non-conservative oscillators. *Rep Mech Eng* 2021; 2(1): 143–148.
32. He J-H and El-Dib YO. The enhanced homotopy perturbation method for axial vibration of strings. *Facta Univ Ser Mech Eng* 2021; 2: 25033. DOI: 10.22190/FUME210125033H.
33. He CH, Shen Y, Ji FY, et al. Taylor series solution for fractal Bratu-type equation arising in electrospinning process. *Fractals* 2020; 28(01): 2050011.
34. Elias-Zúñiga A, Martínez-Romero O, Trejo DO, et al. An efficient ancient Chinese algorithm to investigate the dynamics response of a fractal microgravity forced oscillator. *Fractals* 2021; 29(06): 2150144.
35. Ain QT and He J-H. On two-scale dimension and its applications. *Therm Sci* 2019; 4: 1707–1712.
36. Wang KL, Wang KJ, and He CH. Physical insight of local fractional calculus and its application to fractional Kdv-burgers-Kuramoto equation. *Fractals* 2019; 27: 1950122.
37. He J-H and Ji FY. Two-scale mathematics and fractional calculus for thermodynamics. *Therm Sci* 2019; 23(4): 2131–2133.
38. He J-H and Ain QT. New promises and future challenges of fractal calculus: from two-scale thermodynamics to fractal variational principle. *Therm Sci* 2020; 24: 659–681.
39. He J-H. Ancient Chinese algorithm: the Ying Buzu Shu (method of surplus and deficiency) vs. Newton iteration method. *Appl Math Mech* 2002; 23: 1407–1412.
40. He J-H. An improved amplitude-frequency formulation for nonlinear oscillators. *Int J Nonlinear Sci Numer Simul* 2008; 9: 211–212.
41. He J-H. Amplitude-frequency relationship for conservative nonlinear oscillators with odd nonlinearities. *Int J Appl Comput Math* 2017; 3: 1557–1560.
42. Elias-Zúñiga A, Palacios-Pineda LM, Jiménez-Cedeño IH, et al. Enhanced He’s frequency-amplitude formulation for nonlinear oscillators. *Results Phys* 2020; 19: 103626.
43. Tian D, Ain QT, Anjum N, et al. Fractal N/Mems: from pull-in instability to pull-in stability. *Fractals* 2021; 29(2): 2150030.
44. Wang KL. Effect of Fangzhuz’s nanoscale surface morphology on water collection. *Math Meth Appl Sci* 2020; 1–10. DOI: 10.1002/mmma.6569.
45. He CH, He J-H, and Sedighi HM. Fangzhuz(方诸): an ancient Chinese nanotechnology for water collection from air: history, mathematical insight, promises and challenges. *Math Meth Appl Sci* 2020; 1–10. DOI: 10.1002/mmma.6384.
46. He J-H and El-Dib YO. Homotopy perturbation method for Fangzhuz oscillator. *J Math Chem* 2020; 58: 2245–2253.
47. Akgül A and Ahmad H. Reproducing kernel method for Fangzhuz’s oscillator for water collection from air. *Math Methods Appl Sci* 2020; 1–10. DOI: 10.1002/mmma.6853.
48. Elias-Zúñiga A, Palacios-Pineda LM, Martínez-Romero O, et al. Dynamics response of the forced Fangzhuz fractal device for water collection from air. *Fractals* 2021; 29(7): 2150186.
49. He CH, Liu C, He J-H, et al. Low frequency property of a fractal vibration model for a concrete beam. Fractals 2021; 29(05): 2150117.

50. Zuo Y. A gecko-like fractal receptor of a three-dimensional printing technology: a fractal oscillator. J Math Chem 2021; 59: 735–744.

51. Zuo Y. A fractal rheological model and experimental verification. Therm Sci 2021; 25(3B): 2405–2409.

52. Elias-Zúñiga A, Palacios-Pineda LM, Jiménez-Cedeño IH, et al. A fractal model for current generation in porous electrodes. J Electroanal Chem 2021; 880(1): 114883.

53. Elias-Zúñiga A, Palacios-Pineda LM, Jiménez-Cedeño IH, et al. Equivalent power-form representation of the fractal Toda oscillator. Fractals 2021; 29(1): 21500341.

54. Wang KJ. Research on the nonlinear vibration of carbon nanotube embedded in fractal medium. Fractals 2021; 30: 22500165, DOI: 10.1142/S0218348X22500165.

55. Wang KJ, Li J, Liu JH, et al. Solitary waves of the fractal regularized long wave equation travelling along an unsmooth boundary. Fractals 2021; 30: 22500086, DOI: 10.1142/S0218348X22500086.

56. Wang KJ, Wang GD, and Zhu HW. A new perspective on the study of the fractal coupled Boussinesq-Burger equation in shallow water. Fractals 2021; 29(5): 2150122.

57. Palacios-Pineda LM, Elias-Zúñiga A, Martínez-Romero O, et al. Investigation of the fractal response of a nonlinear packaging system. Fractals 2022; 30(1): 2250007. DOI: 10.1142/S0218348X22500074.

58. Duan Q, An J, Mao H, et al. Review about the application of fractal theory in the research of packaging materials. Materials 2021; 14: 860. DOI: 10.3390/ma14040860.

59. Kuhn W and Grün F. Beziehungen zwischen elastischen Konstanten und Dehnungsdoppelbrechung hochelastischer Stoffe. Colloid Polym Sci 1942; 101: 248–271.

60. Elias-Zúñiga A. A non-Gaussian network model for rubber elasticity. Polymer 2006; 47: 907–914.

61. Jedynak R. New facts concerning the approximation of the inverse Langevin function. J Non-Newtonian Fluid Mech 2017; 249: 8–25.

62. Rickaby SR and Scott NH. A comparison of limited-stretch models of rubber. Int J Nonlin Mech 2015; 68: 71–86.

63. Sheikholeslami SA and Aghdam MM. A novel rational Padé approximation of the inverse Langevin function. Appl Math Comput 2018; 8(4): 649.

64. Rickaby SR and Scott NH. A comparison of limited-stretch models of rubber. Int J Nonlin Mech 2015; 68: 71–86.

65. Rickaby SR and Scott NH. A comparison of limited-stretch models of rubber. Int J Nonlin Mech 2015; 68: 71–86.
78. Zúñiga AE and Beatty MF. Forced vibrations of a body supported by viscohyperelastic shear mountings. *J Eng Math* 2001; 40(4): 333–353.

79. Wang KJ. Generalized variational principle and periodic wave solution to the modified equal width-Burgers equation in nonlinear dispersion media. *Phys Lett A* 2021; 419: 127723.

80. Wang KJ. Periodic solution of the time-space fractional complex nonlinear Fokas-Lenells equation by an ancient Chinese algorithm. *Optik* 2021; 243: 167461.

81. Wang KJ and Zhang PL. Investigation of the periodic solution of the time-space fractional Sasa-Satsuma equation arising in the monomode optical fibers. *EPL* 2021, In press, DOI: 10.1209/0295-5075/ac2a62.

82. Kovacic I. Forced vibrations of oscillators with a purely nonlinear power-form restoring force. *J Sound Vib* 2011; 330: 4313–4327.

83. Schiessel H and Blumen A. Fractal aspects in polymer science. *Fractals* 1995; 3: 483–490.

84. Puente-Córdoba JG, Reyes-Melo ME, Palacios-Pineda LM, et al. Fabrication and characterization of isotropic and anisotropic magnetorheological elastomers, based on silicone rubber and carbonyl iron microparticles. *Polymers* 2018; 10: 1343. DOI: 10.3390/polym10121343.

85. Huang J, Xu Y, Qi S, et al. Ultra-high energy-dissipation elastomers by precisely tailoring the relaxation of confined polymer fluids. *Nat Commun* 2021; 12: 3610, DOI: 10.1038/s41467-021-23984-2.

86. Emri I, Zupančič B, Gergesova M, et al. Importance of viscoelastic characteristics in determining functionality of time-dependent materials. In: *Dyna, año 79*. Medellín, octubre: Edición Especial; 2012, pp. 97–104. ISSN 0012-7353.

87. Scanlan JC and Winter HH. Composition dependence of the viscoelasticity of end-linked poly(dimethylsiloxane) at the gel point. *Macromolecules* 1991; 24(1): 47–54.