Higgs Mass Bound in the Minimal
Standard Model

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Abstract
A brief review of the role of the Higgs mechanism and the ensuing Higgs particle in the Minimal Standard Model is given. Then the property of triviality of the scalar sector in the Minimal Standard Model and the upper bound on the Higgs mass that follows is discussed. It is emphasized that the bound is obtained by limiting cutoff effects on physical processes. Actions that allow a parameterization and tuning of the leading cutoff effects are studied both analytically, in the large $N$ limit of the generalization of the $O(4)$ symmetry of the scalar sector to $O(N)$, and numerically for the physical case $N = 4$. Combining those results we show that the Minimal Standard Model will describe physics to an accuracy of a few percent up to energies of the order 2 to 4 times the Higgs mass, $M_H$, only if $M_H \leq 710 \pm 60$ GeV. This bound is the result of a systematic search in the space of dimension six operators and is expected to hold in the continuum.

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1. Introduction

The elementary particles and their interactions are described in a highly economical and successful way in the Minimal Standard Model. The symmetry among the elementary fermions is described by the internal symmetry group

\[ G = SU(3)_{\text{color}} \times SU(2)_L \times U(1)_Y \]

and we have three known families of such fermions. Indeed, precision measurements of the width of the Z vector boson at LEP have yielded for the number of massless neutrinos (and thus of families, since we have one massless left-handed neutrino per family) \( N_\nu = 3.04 \pm 0.04 \).\textsuperscript{[1]}

The interactions among the elementary fermions are introduced by making the global symmetry group \( G \) into a local symmetry group with the help of \( 8 + 3 + 1 = 12 \) massless vector bosons (8 gluons for the \( SU(3)_{\text{color}} \) factor, 3 vector bosons for \( SU(2)_L \) and 1 for the \( U(1) \) hypercharge). The resulting theory is perturbatively renormalizable, which means in particular that it is applicable for energies ranging over many orders of magnitude and that it has a lot of predictive power.

However, so far we do not describe nature appropriately. We know that explicitly realized is only the subgroup \( SU(3)_{\text{color}} \times U(1)_{\text{em}} \) of \( G \), where here the \( U(1) \) factor describes electromagnetism. Furthermore the theory does not allow for masses. In the case of fermions, mass terms are forbidden by the chiral nature of the symmetry group \( G \), and in the case of the vector bosons, mass terms are forbidden by the gauge symmetry. But in nature 14 out of the 15 fermions per family are massive, as are the weak vector bosons \( W^\pm \) and \( Z \). In the case of the three vector bosons we don’t even have enough degrees of freedom, since massless vector boson have two (transverse) polarizations, while massive ones need in addition a state with longitudinal polarization.

In the Minimal Standard Model we correct these shortcomings by adding an elementary complex scalar field, transforming as a doublet under \( SU(2)_L \). Group theory determines that the scalar self-interactions have an enhanced symmetry, \( SU(2)_L \times SU(2)_{\text{custodial}} \simeq O(4) \). This symmetry is then arranged to be broken spontaneously to \( O(3) \) by giving the scalar field a non-vanishing vacuum expectation value, \( F \). (This notation is chosen from the analogy to the current algebra describing the soft pions of QCD, in which the analog of the vacuum expectation value is the pion decay constant, \( f_\pi \)). This spontaneous symmetry breaking turns out to do all the tricks we need. It breaks the symmetry group from \( G \) to the subgroup \( SU(3)_{\text{color}} \times U(1)_{\text{em}} \), explicitly realized. The three Goldstone bosons of the breaking \( O(4) \to O(3) \) provide the longitudinal degrees of freedom of the \( W^\pm \) and \( Z \), making those massive, and finally the spontaneous symmetry breaking gives masses to the fermions via gauge invariant Yukawa couplings. And all these good things happen while the theory remains perturbatively renormalizable and maintains its predictive power. After the spontaneous symmetry breaking, one out of the 4 scalar degrees of freedom that we added is left, the so far elusive Higgs boson.

While the Minimal Standard Model, briefly outlined above, has had many spectacular successes, relatively little is known experimentally about the Higgs sector. One exception is the value of the vacuum expectation value \( F \). From its relation to
the $W$ boson mass and the latter’s to the four-fermion coupling $G_F$ one can easily deduce that $F = 246 \text{ GeV}$. Furthermore, recent experiments at LEP led to a lower bound on the Higgs mass of about 60 GeV [2].

In the remainder of this seminar I will describe what we know about the mass of this Higgs particle on purely theoretical grounds. In particular I will present the arguments leading to an upper bound on the mass of the Higgs particle, the so called triviality bound. And finally I’ll describe a non-perturbative computation of the bound, that we found to be $710 \pm 60 \text{ GeV}$.

2. Perturbative indications of triviality

As indicated in the introduction we are interested in an upper bound on the Higgs mass. It turns out, as we will see below, that the Higgs mass increases with increasing scalar self coupling. At the energy scale of the upper bound of Higgs mass, all gauge interactions are relatively weak and can be treated perturbatively. The same holds true for the Yukawa interactions, including the top if it is not heavier than about 200 GeV, as is favored by experiment [1]. Therefore we concentrate here on the scalar sector of the Minimal Standard Model alone. It is described by an $O(4)$ invariant scalar field theory with potential

$$V(\vec{\phi}) = \frac{1}{2} \mu_0^2 \vec{\phi}^2 + \frac{g_0}{4!} (\vec{\phi}^2)^2$$

To have spontaneous symmetry breaking we assume $\mu_0^2 < 0$. As usual the theory so far described is ill defined. We need to introduce a cutoff $\Lambda$ to regulate and then renormalize the theory. A two-loop perturbative computation together with application of the renormalization group leads to the relation between the cutoff, $\Lambda$, the physical Higgs mass, $M_H$, and the renormalized coupling, $g_R = 3 M_H^2 / F^2$

$$\frac{M_H}{\Lambda} = C \left( \frac{g_R}{4 \pi^2} \right)^{13/24} \exp \left\{ -\frac{4 \pi^2}{g_R} \right\} \left[ 1 + O(g_R) \right]$$

Usually, at the end of the calculation one would like to remove the cutoff by taking the limit $\Lambda \to \infty$. However, from eq. (3) we see that the limit $\Lambda \to \infty$ implies $g_R \to 0$, i.e. we are left with a non-interacting, trivial theory. But we need an interacting scalar sector for the Higgs mechanism to work. Therefore we need to keep the cutoff finite: the Minimal Standard Model has to be viewed as an effective theory that describes physics at energies below the cutoff scale.

Since we have to keep a finite cutoff, we may ask what happens if we try to make the (renormalized) scalar self-interactions stronger. Eq. (3) tells us that as $g_R$ increases, so does the ratio $M_H / \Lambda$. But since the Higgs mass is one of the physical quantities that the standard model is supposed to describe, we certainly need $M_H / \Lambda < 1$. Hence we have arrived at an upper bound on the Higgs mass. Since it comes from the triviality of the scalar sector, this bound is referred to as the triviality bound.

In the next three sections I will make the definition of the bound, namely the meaning of the “$\,<\,$” in $M_H / \Lambda < 1$ more precise and give the results of a numerical computation of the bound.
3. Cutoff effects and generalized actions

The triviality of the scalar sector of the Minimal Standard Model, and therefore the need to retain a finite cutoff $\Lambda$, are by now very well established \cite{3, 4, 5, 6}. As a consequence, all observable predictions have a weak cutoff dependence, of order $1/\Lambda^2$. I will later on show explicit examples of such cutoff effects, computed in the large $N$ limit of the generalization of the $O(4)$ symmetric scalar sector to $O(N)$. This generalization is useful, because we can solve the model analytically in the large $N$ limit, and therefore e.g. compute cutoff effects. The cutoff effects become larger when the ratio $M_H/\Lambda$, and hence $g_R$, increases. Therefore, by limiting the cutoff effect on some physical observable – we shall use the square of the invariant scattering amplitude of Goldstone bosons at 90° in the center of mass frame – we obtain a more precise definition of the upper bound on the Higgs mass.

The need for a finite cutoff in the Minimal Standard Model means, that it is only an effective theory, applicable for energies below the cutoff. There will be some, as yet unknown, embedding theory. But we assume that for “small” energies – energies smaller than a few times the Higgs mass – the scalar sector is representable by an effective action (for a more detailed discussion see e.g. \cite{7} and references therein)

$$L_{\text{eff}} = L_{\text{ren}} + \frac{1}{\Lambda^2} \sum A c_A O_A \, , \quad \dim O_A \leq 6$$

with $L_{\text{ren}}$ the usual renormalized $\phi^4$ Lagrangian. $O_A$ are operators with the correct symmetry properties and dimension less than or equal to 6, and the coefficients $c_A$ depend on the embedding theory. Since we don’t know this theory, we just parameterize our ignorance with these $c_A$’s, i.e., we consider reasonable bare cutoff models with enough free parameters to reproduce the effective action, eq. (4). This allows us to tune the cutoff effects. Eliminating redundant operators, which leave the S-matrix unchanged, we end up with two “measurable” $c_A$’s. The Higgs mass bound is now obtained as the maximal value $M_H$ can take, when varying the parameters $c_A$ while maintaining our requirement of limiting the cutoff effects by some prescribed value, typically a few percent. However, since we do not know the embedding theory, we will in this maximization avoid excessive fine tuning that might eliminate leading cutoff effects, of order $1/\Lambda^2$.

The most straightforward implementation is to start with an action of the form eq. (4) on the level of bare fields and parameters. It turns out, however, that, as our intuition would tell us, the maximal renormalized coupling $g_R$, and hence the maximal Higgs mass, is obtained at maximal bare $\phi^4$ self coupling, i.e., at $g_0 \to \infty$ (see e.g. \cite{4, 8}). But in this limit the model becomes nonlinear, with the field having a fixed length, and all dimension six operators become trivial. In a non-linear theory it are terms with four derivatives that allow us to tune the cutoff effects of order $1/\Lambda^2$. Maintaining $O(N)$ invariance, there are three different terms with four derivatives, and we are led to consider actions of the form

$$S = \int_x \left[ \frac{1}{2} \tilde{\phi}(-\partial^2 + 2b_0 \partial^4)\tilde{\phi} - \frac{b_1}{2N}(\partial_{\mu}\tilde{\phi}\cdot\partial_{\mu}\tilde{\phi})^2 - \frac{b_2}{2N}(\partial_{\mu}\tilde{\phi}\cdot\partial_{\nu}\tilde{\phi} - \frac{1}{4}\delta_{\mu,\nu}\partial_{\sigma}\tilde{\phi}\cdot\partial_{\sigma}\tilde{\phi})^2 \right]$$

(5)
with $\phi^2 = N\beta$ fixed. Up to terms with more derivatives, the parameter $b_0$ can be eliminated with a field redefinition

$$\tilde{\phi} \rightarrow \frac{\tilde{\phi} + b_0 \partial^2 \tilde{\phi}}{\sqrt{\tilde{\phi}^2 + b_0^2 (\partial^2 \tilde{\phi})^2 + 2 b_0 \tilde{\phi} \partial^2 \tilde{\phi}}} \sqrt{N\beta}. \quad (6)$$

This leaves two free parameters to tune the cutoff effects, exactly the number of measurable coefficients $c_A$ in (4). Therefore (3) should give a good parameterization to obtain the Higgs mass bound.

4. Solution at large $N$

To understand the effect of the four-derivative couplings in the action eq. (3) we studied these models first in the solvable large $N$ limit [8]. We considered different regularizations: a class of Pauli-Villars regularizations obtained by replacing the term $\tilde{\phi}(-\partial^2)\tilde{\phi}$ by $\tilde{\phi}(-\partial^2(1 + (-\partial^2/\Lambda^2)^n)\tilde{\phi}$ with $n \geq 3$ and $b_0$ set to zero, and some transcriptions of action (3) on a lattice, i.e. lattice actions such that eq. (3) appears in their expansion in slowly varying fields.

The result of our investigations is that at $N = \infty$, after $b_0$ has been eliminated, $b_2$ has no effect and that the bound depends monotonically on $b_1$, increasing with decreasing $b_1$. Overall stability of the homogeneous broken phase restricts the range of $b_1$ and thus we find a finite optimal value for $b_1$. The physical picture that emerges is that among the nonlinear actions the bound is further increased by reducing as much as possible the attraction between low momentum pions in the $I = J = 0$ channel.

The rule in the above paragraph does not lead to an exactly universal bound. Different bare actions that give the same effective parameter $b_1$ can give somewhat different bounds because the dependence of physical observables on the bare action is highly nonlinear. For example, at the optimal $b_1$ value, Pauli–Villars regularizations lead to bounds higher by about 100 GeV than some lattice regularizations. This difference between the lattice and Pauli–Villars can be traced to the way the free massless inverse Euclidean propagator departs from the $O(p^2)$ behavior at low momenta. For Pauli–Villars it bends upwards to enforce the needed suppression of higher modes in the functional integral, while on the lattice it typically bends downwards to reflect the eventual compactification of momentum space.

When considering lattice regularizations, because we desire to preserve Lorentz invariance to order $1/\Lambda^2$, we use the $F_4$ lattice. The $F_4$ lattice can be thought of as embedded in a hypercubic lattice from which odd sites (i.e. sites whose integer coordinates add up to an odd sum) have been removed. The $F_4$ lattice turns out to have a larger symmetry group than the hypercubic lattice, which forbids Lorentz invariance breaking terms at order $1/\Lambda^2$.

On the basis of the above observations, we went through three stages of investigation. The first stage was to investigate the naïve nearest–neighbor model. This should be viewed as the generic lattice case where no special effort to increase the bound is made. The next stage is to write down the simplest action that has a tunable parameter $b_1$. We should emphasize that on the $F_4$ lattice, unlike on the
Figure 1: Large $N$ prediction of the Higgs mass $M_H = M_H/F \times 246 \text{ GeV}$ in physical units vs. the Higgs mass $m_H = aM_H$ in lattice units for the three actions on the $F_4$ lattice.

In the large $N$ limit, to simplify the calculation somewhat, we actually considered slight variants of actions $S_2$ and $S_3$ (see [8]), that have the same expansion for slowly varying fields and are expected to give the same physics. We indicate this by denoting the modified actions with a $'$. In each case, at constant $\beta_2$, $\beta_0$ is varied.

The coupling $\beta_2$ here plays the role of $b_1$ in eq. (5).

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$$S_1 = -2\beta_0 \sum_{<x,x'>} \Phi(x) \cdot \Phi(x')$$

$$S_2 = S_1 - \frac{\beta_2}{8} \sum_x \sum_{l,l',x \neq \emptyset, l \cap x \neq \emptyset, l' \cap x' \neq \emptyset} \left[ \left( \Phi(x) \cdot \Phi(x') \right) \left( \Phi(x) \cdot \Phi(x'') \right) \right]$$

$$S_3 = -2(2\beta_0 + \beta_2) \sum_{<x,x'>} \Phi(x) \cdot \Phi(x') + (\beta_0 + \beta_2) \sum_{<x,x'>} \Phi(x) \cdot \Phi(x')$$

$$-\frac{\beta_2}{8} \sum_x \sum_{l,l',x \neq \emptyset, l \cap x \neq \emptyset, l' \cap x' \neq \emptyset} \left[ \left( \Phi(x) \cdot \Phi(x') \right) \left( \Phi(x) \cdot \Phi(x'') \right) \right].$$

The coupling $\beta_2$ here plays the role of $b_1$ in eq. (5).
tracing out a line in parameter space approaching a critical point from the broken phase. This line can also be parameterized by $M_H/\Lambda$ (on the lattice $\Lambda = a^{-1}$) or $g_R$. For actions $S'_2$ and $S'_3$, $\beta_2$ is chosen so that on this line the bound on $M_H$ is expected to be largest. A simulation produces a graph showing $M_H/F$ as a function of $M_H/\Lambda$ along this line. The y-axis is turned into an axis for $M_H$ by $M_H = M_H/F \times 246 \text{ GeV}$. The large $N$ predictions for these graphs are shown in Figure 1.

To obtain a well defined bound on the Higgs mass from such graphs, as we already explained, we need to compute the cutoff effects on some physical quantity. We show the cutoff effect in the square of the invariant $\pi-\pi$ scattering amplitude at $90^\circ$ at several center of mass energies in Figure 2.

Figure 2: Leading order cutoff effects in the invariant $\pi-\pi$ scattering amplitude at $90^\circ$ at center of mass energy $W = 2M_H$ vs. the Higgs mass $m_H = aM_H$ in lattice units for the three actions. The values of $M_H$ in GeV determined from $M_H = M_H/F \times \text{GeV}$ are put on the three horizontal lines at $\delta_{|A|^2} = 0.005, 0.01, 0.02$.

If one considers only the magnitude of cutoff effects as a function of $M_H/\Lambda$, one might conclude that the bound obtained with action $S_1$ would be larger than the bound obtained with $S'_2$. This conclusion proves to be wrong when the mass in physical units is considered. The values of $M_H$ in GeV, determined from $M_H = M_H/F \times 246 \text{ GeV}$, are put on three horizontal lines in Figure 2 at $\delta_{|A|^2} = 0.005, 0.01, 0.02$. At large $N$ the bound increases when going from $S_1$ to $S'_2$ and then to $S'_3$ by a little
over 10% at each step. For example, for $\delta |A|^2 = 0.01$ we get bounds on $M_H$ of 680, 764, 863 GeV for $S_1$, $S'_2$ and $S'_3$ respectively.

5. The physical case $N = 4$.

For the physical case, $N = 4$, we do not have at our disposal non-perturbative analytical methods of computation. We therefore resort to numerical simulations of the models in eq. (7). The result of these simulations are show in Figure 3 which shows $M_H = 246 \sqrt{g_R/3}$ GeV as a function of $aM_H$ for all three actions. One clearly sees the progressive increase of $M_H$ from action $S_1$ to $S_2$ and then $S_3$, just as in the large $N$ limit.

![Figure 3: The Higgs mass $M_H = M_H/F \times 246$ GeV in physical units vs. the Higgs mass $m_H = aM_H$ in lattice units from the numerical simulations. The diamonds correspond to action $S_1$, the squares to action $S_2$ and the crosses to action $S_3$.](image)

To obtain a bound on the Higgs mass we also need estimates of the cutoff effects. We do not know how to compute cutoff effects in numerical simulations. Therefore we take the estimates from the large $N$ calculation, which should be accurate enough for our purpose here (observe that the cutoff effects are relatively insensitive to the Higgs mass in lattice units, $m_H = aM_H$). A glance at Figure 2 shows that in all
cases the cutoff effects on the pion–pion scattering are below a few percent even at the maximal $M_H$ of each curve. Thus we can take the largest of these maxima as our bound. The ordering of the points and their relative positions are in agreement with Figure 1, while the differences in overall scale, reflecting the difference between $N = \infty$ and $N = 4$, come out compatible with $1/N$ corrections, as expected [8].

We conclude that the Minimal Standard Model will describe physics to an accuracy of a few percent up to energies of the order 2 to 4 times the Higgs mass, $M_H$, only if $M_H \leq 710 \pm 60$ GeV. The error quoted accounts for the statistical errors, shown in Figure 3, as well as the systematic uncertainty associated with the remaining regularization dependence, e.g. a not completely optimal choice of $b_1$ in (5) and the possible small dependence on $b_2$ for $N \neq \infty$. Since this bound is the result of a systematic search in the space of dimension six operators, we expect it to hold in the continuum. A Higgs particle of mass 710 GeV is expected to have a width between 180 GeV (the perturbative estimate) and 280 GeV (the large $N$ non-perturbative estimate). Thus, if the Higgs mass bound turns out to be saturated in nature, the Higgs would be quite strongly interacting.

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