TURÁN TYPE INEQUALITIES FOR CLASSICAL AND GENERALIZED MITTAG-LEFFLER FUNCTIONS

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Abstract. In this paper some Turán type inequalities for classical and generalized Mittag-Leffler functions are considered. The method is based on proving monotonicity for special ratios of sections for series of such functions. Some applications are considered to Lazarević-type and Wilker-type inequalities for classical and generalized Mittag-Leffler functions.

Keywords: Mittag–Leffler functions; Turán type inequalities; Lazarević-type inequalities; Wilker-type inequalities.

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1. Introduction

We use a definition of Mittag–Leffler function by its series

\[ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n + \beta)} , \quad z \in \mathbb{C}, \ \alpha, \beta \in \mathbb{C}, \ \text{Re}(\alpha) > 0, \ \text{Re}(\beta) > 0. \]

This function was first introduced by G. Mittag–Leffler in 1903 for \( \alpha = 1 \) and by A. Wiman in 1905 for the general case (1). For the mathematical theory and properties of Mittag–Leffler functions cf. [7], [10], [48].

First applications of the function (1) by Mittag–Leffler and Wiman were in complex function theory (non–trivial examples of entire functions with fractional growth characteristics such as order and generalized summation methods). But its really important applications were found in 20–th century for fractional integral and differential equations. The most known result in this field is an explicit formula for the resolvent of Riemann–Liouville fractional integral proved by E. Hille and J. Tamarkin in 1930. On this and similar formulas many results are based still for solving fractional integral and differential equations. For numerous applications of the Mittag–Leffler function to fractional calculus cf. [37], [10], [13], [15], [34]. Due to many useful applications it was crowned by R. Gorenflo and F. Mainardi in [9] as a "Queen function of Fractional Calculus"! Besides fractional calculus the Mittag–Leffler function also plays an important role in various branches of applied mathematics and engineering sciences, such as chemistry, biology, statistics, thermodynamics, mechanics, quantum physics, informatics, signal processing and others.

There are further related generalizations of the Mittag–Leffler function, namely Wright and Fox functions. Wright functions are defined in the same way as (1) but with more gamma–functions both in numerator and denominator, sometimes these functions are called ”multi–indexed Mittag–Leffler functions”, cf. [17], [18], [42]. Fox function is defined by the Mellin transform, cf. [16], [23], [24], [42], [43], it has also important applications e.g. in fractional diffusion theory, cf. [19], [20], [8].

By means of the series representation Prabhakar in [36] studied the function

\[ E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{n!\Gamma(\alpha n + \beta)} , \quad z \in \mathbb{C}, \ \alpha, \beta, \gamma \in \mathbb{C}, \ \text{Re}(\alpha) > 0, \ \text{Re}(\beta) > 0, \ \text{Re}(\gamma) > 0. \]
where \((\gamma)_n\) is the Pochhammer symbol given by
\[
(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)}, \quad (\gamma)_0 = 1.
\]

In 2007 Shukla and Prajapati [43] studied another generalization with four parameters \(E^{\gamma,q}_{\alpha,\beta}(z)\) which is defined for \(z, \alpha, \beta, \gamma \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0\) and \(q \in (0,1) \cup \mathbb{N}\) as
\[
E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{n! (\alpha n + \beta)}.
\]

Note that
\[
E^{1,1}_{1,1}(z) = e^z, \quad E^{1,1}_{0,1}(z) = E_0(z), \quad E^{1,1}_{0,0}(z) = E^\gamma(z), \quad \text{and} \quad E^{\gamma,1}_{0,0}(z) = E^\gamma(z).
\]

These interesting generalizations are special cases of generalized Mittag–Leffler function, cf. [17]–18.

An important result which initiated a new field of researches on inequalities for special functions was proved by Paul Turán, it is:

\[
[P_n(x)]^2 - P_{n+1}(x)P_{n-1}(x) \geq 0,
\]

where \(-1 < x < 1, n \in \mathbb{N}\) and \(P_n(\cdot)\) stands for the classical Legendre polynomial. This inequality was published by Turán in 1950 in [43] but proved earlier in 1946 in a letter to Szegö. Since the publication of the above Turán inequality in 1948 by Szegö [47] many authors derived results of such type for classical orthogonal polynomials and different special functions. The Turán type inequalities now have an extensive literature and some of the results have been applied successfully to different problems in information theory, economic theory, biophysics, probability and statistics. For more details cf. [1], [2], [3], [22], [25], [36]. The results on Turán type inequalities are closely connected with log–convex and log–concave functions, cf. [11], [12], [13], [14].

Now Turán–type inequalities are proved for different classes of special functions: Kummer hypergeometric functions (cf. [3], [26], [27], [28], [33], [35], [40]), Gauss hypergeometric functions (cf. [13], [14], [26], [27], [28]), different types of Bessel functions (cf. [1], [24]), Dunkl kernel and \(q\)–Dunkl kernel (cf. [23]), \(q\)–Kummer hypergeometric functions (cf. [29], [30], [31]) and some others.

This paper is a continuation of some line of authors results. In 1990 one of the authors studied inequalities for sections of series for exponential function in [38]. Among other results in [38] a conjecture was proposed on monotonicity of ratios for Kummer hypergeometric function, cf. also [39]–[40]. This conjecture was proved recently by the authors in [26], [27], cf. also [28]–[29]. After that \(q\)–versions of these results were proved in [30]–[32].

The paper is organized as follows. In section 2 we collect some lemmas. In section 3 we give some Turán type inequalities for Mittag–Leffler functions. Moreover, we prove monotonicity of ratios for sections of series of Mittag–Leffler functions, the result is also closely connected with Turán-type inequalities. In section 4 we deduce new inequalities of Lazarević–type and Wilker–type for Mittag–Leffler functions. In sections 5 and 6 similar results are proved for generalized Mittag–Leffler functions [2]–[3], which demonstrate that the technics of this paper is applicable for these functions too. At the end of the paper two unsolved problems are included.

2. Useful lemmas

We need the following two useful lemmas proved in [4], [35].

**Lemma 1.** Let \((a_n)\) and \((b_n)\) \((n = 0, 1, 2,...)\) be real numbers, such that \(b_n > 0, n = 0, 1, 2,...\) and \(\left(\frac{a_{n+1} - \gamma a_n}{b_{n+1} + \gamma b_n}\right)_{n \geq 0}\) is increasing (decreasing), then \(\left(\frac{a_{n+1} - \gamma a_n}{b_{n+1} + \gamma b_n}\right)_{n \geq 0}\) is also increasing (decreasing).
Lemma 2. Let \((a_n)\) and \((b_n)\) \((n = 0, 1, 2, \ldots)\) be real numbers and let the power series \(A(x) = \sum_{n=0}^{\infty} a_n x^n\) and \(B(x) = \sum_{n=0}^{\infty} b_n x^n\) be convergent for \(|x| < r\). If \(b_n > 0\), \(n = 0, 1, 2, \ldots\) and the sequence \(\left(\frac{a_n}{b_n}\right)_{n \geq 0}\) is (strictly) increasing (decreasing), then the function \(A(x)/B(x)\) is also (strictly) increasing on \([0, r)\).

3. Turán type inequalities for Mittag–Leffler functions

Our first result is the next theorem.

Theorem 1. Let \(\alpha, \beta > 0\). Then the following assertions are true.

a. The function \(\beta \mapsto E_{\alpha, \beta}(z) = \Gamma(\beta) E_{\alpha, \beta}(z)\) is log–convex on \((0, \infty)\).

b. The following Turán type inequality

\[
\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n
\]

holds for all \(z \in (0, \infty)\).

In particular, the following inequality

\[
(e^z - 1)^2 \leq 2e^z(e^z - 1 - z)
\]

is valid for all \(z > 0\).

c. For \(n \in \mathbb{N}\) define the function \(E_{\alpha, \beta}^n(z)\) by

\[
E_{\alpha, \beta}^n(z) = E_{\alpha, \beta}(z) - \sum_{k=0}^{n} \frac{z^k}{\Gamma(\alpha k + \beta)} = \sum_{k=n+1}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.
\]

Then the following Turán type inequality

\[
E_{\alpha, \beta}^n(z) E_{\alpha, \beta}^{n+2}(z) \leq \left[ E_{\alpha, \beta}^{n+1}(z) \right]^2.
\]

is valid for all \(n \in \mathbb{N}\), and \(\alpha, \beta > 0\) and \(z > 0\).

Proof. a. For log–convexity of \(\beta \mapsto E_{\alpha, \beta}(z)\) we observe that it is enough to show the log–convexity of each individual term and to use the fact that the sum of log–convex functions is log–convex too. Thus, we just need to show that for each \(k \geq 0\) we have

\[
\frac{\partial^2}{\partial \beta^2} \log \left( \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha k)} \right) = \psi'(\beta) - \psi'(\beta + \alpha k) \geq 0,
\]

where \(\psi(x) = \Gamma'(x)/\Gamma(x)\) is the so–called digamma function. But \(\psi\) is known to be concave, and consequently the function \(\beta \mapsto \frac{\Gamma(\beta)}{\Gamma(\beta + k\alpha)}\) is log–convex on \((0, \infty)\).

b. Since the function \(\beta \mapsto E_{\alpha, \beta}(z)\) is log–convex, then for all \(\beta_1, \beta_2 > 0, z > 0\) and \(t \in [0, 1]\) we have

\[
E_{\alpha, t\beta_1 + (1-t)\beta_2}(z) \leq [E_{\alpha, \beta_1}(z)]^t [E_{\alpha, \beta_2}(z)]^{1-t}.
\]

Now choosing \(t = 1/2\), \(\beta_1 = \beta\), \(\beta_2 = \beta + 2\) we conclude that (5) holds. To prove the inequality (6) choose \(\alpha = \beta = 1\) and use a recurrence relation from (24), Theorem 5.1

\[
E_{\alpha, \beta}(z) = zE_{\alpha, \alpha + \beta}(z) + \frac{1}{\Gamma(\beta)}.
\]

c. Let \(n \in \mathbb{N}\), from the definition of the function \(E_{\alpha}(\alpha, \beta, z)\), we have

\[
E_{\alpha, \beta}^n(z) = E_{\alpha, \beta}^{n+1}(z) + \frac{z^{n+1}}{\Gamma(\beta + (n+1)\alpha)}
\]
holds for all $\alpha, \beta > 0$. It implies that

$$E_{\alpha, \beta}^{n+2}(z) = E_{\alpha, \beta}^{n+1}(z) - \frac{z^{n+2}}{\Gamma(\beta + (n + 2)\alpha)}.$$ 

Due to log-convexity of $\Gamma(x)$, the ratio $x \mapsto \frac{\Gamma(x+a)}{\Gamma(x)}$ is increasing on $(0, \infty)$ when $a > 0$. It implies the following inequality

$$\frac{\Gamma(x+a)}{\Gamma(x)} \leq \frac{\Gamma(x+a+b)}{\Gamma(x+b)}.$$ 

holds for all $a, b > 0$. For $n \geq 0$ and $k \geq n+3$, let $x = \beta + (n+1)\alpha$, $a = \alpha$, $\beta = \alpha(k-(n+2))$ in (8) we conclude that $A_k(\alpha, \beta) \leq 0$, and so the Turán type inequality (7) is proved.

**Corollary 1.** The following Turán type inequality

$$E_{\alpha, \beta+(n+1)\alpha}(z)E_{\alpha, \beta+(n+3)\alpha}(z) \leq [E_{\alpha, \beta+(n+2)\alpha}(z)]^2,$$

holds for all $n \in \mathbb{N}$, and $\alpha, \beta \geq 0$ and $z > 0$.

**Proof.** In [24] the following formula for Mittag–Leffler functions was proved

$$z^n E_{\alpha, \beta+n\alpha}(z) = E_{\alpha, \beta}(z) - \sum_{k=0}^{n-1} \frac{z^k}{\Gamma(\beta+k\alpha)}.$$ 

it holds for all $\alpha > 0$, $\beta > 0$ and $n \in \mathbb{N}$. Using (7) and (10) we conclude that (8) holds.

**Theorem 2.** Let $\alpha, \beta > 0$ and $n \in \mathbb{N}$. Then the function $h_n(\alpha, \beta, z)$ defined by

$$x \mapsto h_n(\alpha, \beta, z) = \frac{E_{\alpha, \beta}^n(z)E_{\alpha, \beta}^{n+2}(z)}{[E_{\alpha, \beta}^{n+1}(z)]^2}$$

is increasing on $(0, \infty)$. So the following Turán type inequality

$$\frac{\Gamma^2(\beta+(n+2)\alpha)}{\Gamma(\beta+(n+1)\alpha)\Gamma(\beta+(n+3)\alpha)}[E_{\alpha, \beta}^{n+1}(z)]^2 \leq E_{\alpha, \beta}^n(z)E_{\alpha, \beta}^{n+2}(z),$$

holds for all $\alpha, \beta > 0$, $n \in \mathbb{N}$ and $z > 0$. The constant in LHS of inequality (11) is sharp.
Proof. From the Cauchy series product we get
\[ h_n(\alpha, \beta, z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \frac{1}{\Gamma(\beta + (n+1+j)\alpha)} \Gamma(\beta + (n+3+k-j)\alpha) \right) z^{2n+2+k} \]
\[ = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{a_j(\alpha, \beta)}{\Gamma(\beta + (n+1+j)\alpha)} z^{2n+2+k} \]
\[ = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{b_j(\alpha, \beta)}{\Gamma(\beta + (n+1+j)\alpha)} z^{2n+2+k} . \]

Now we consider the sequence \((U_j)_{j \geq 0}\) defined by
\[ U_j = \frac{\Gamma(\beta + (n + 2 + j)\alpha)\Gamma(\beta + (n + 2 + k - j)\alpha)}{\Gamma(\beta + (n + 1 + j)\alpha)\Gamma(\beta + (n + 3 + k - j)\alpha)} . \]

Thus
\[ \frac{U_{j+1}}{U_j} = \frac{\Gamma(\beta + (n + 3 + j)\alpha)\Gamma(\beta + (n + 1 + j)\alpha)}{\Gamma(\beta + (n + 2 + j)\alpha)} . \]
\[ = \frac{\Gamma(\beta_1 + (n + 3)\alpha)\Gamma(\beta_1 + (n + 1)\alpha)}{\Gamma(\beta_2 + (n + 2)\alpha)} \frac{\Gamma(\beta_2 + (n + 1)\alpha)\Gamma(\beta_2 + (n + 3)\alpha)}{\Gamma(\beta_1 + (n + 2)\alpha)} , \]
where \(\beta_1 = \beta + j\alpha\) and \(\beta_2 = \beta + (k - j)\alpha\). And again using the Turán type inequality we conclude that \(\frac{U_{j+1}}{U_j} \geq 1\) and consequently the sequence \((U_j)_{j \geq 0}\) is increasing. So from lemma 3 we conclude that \(\frac{\sum_k a_j(\alpha, \beta)}{\sum_k b_j(\alpha, \beta)}\) is increasing. Therefore, the function \(x \mapsto h_n(\alpha, \beta, z)\) is also increasing on \((0, \infty)\) by lemma 2. Finally,
\[ \lim_{x \to 0} h_n(\alpha, \beta, z) = \frac{\Gamma^2(\beta + (n + 2)\alpha)}{\Gamma(\beta + (n + 1)\alpha)\Gamma(\beta + (n + 3)\alpha)} . \]
And it follows that the next constant equals \(\frac{\Gamma^2(\beta + (n + 2)\alpha)}{\Gamma(\beta + (n + 1)\alpha)\Gamma(\beta + (n + 3)\alpha)}\) is the best possible for which the inequality holds for all \(\alpha, \beta > 0, n \in \mathbb{N}\) and \(z > 0\). \(\square\)

Theorem 3. Let \(\alpha > 0, \beta_1, \beta_2 > 1\). If \(\beta_1 < \beta_2, (\beta_2 < \beta_1)\), then the function \(z \mapsto E_{\alpha, \beta_1}(z)/E_{\alpha, \beta_2}(z)\) is increasing (decreasing) on \((0, \infty)\). So, the following Turán type inequalities
\[ E_{\alpha, \beta_1}(z)E_{\alpha, \beta_1-1}(z) - E_{\alpha, \beta_1}(z)E_{\alpha, \beta_2-1}(z) + (\beta_2 - \beta_1)E_{\alpha, \beta_1}(z)E_{\alpha, \beta_2}(z) \geq 0, \]
holds for all \(\alpha, \beta_1, \beta_2 > 0\) such that \(\beta_2 > \beta_1\). In particular, the Turán type inequality
\[ E_{\alpha, \beta}(z)E_{\alpha, \beta+2}(z) - E_{\alpha, \beta+1}(z)^2 + (\beta + 1)E_{\alpha, \beta+1}(z)E_{\alpha, \beta+2}(z) \geq 0, \]
is valid for all \(\alpha, \beta, z > 0\).

Proof. By using the power-series representation of the Mittag-Leffler function \(E_{\alpha, \beta}(z)\), we have
\[ E_{\alpha, \beta_1}(z)/E_{\alpha, \beta_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta_1 + k\alpha)} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta_2 + k\alpha)} . \]
In view of Lemma 2 we need to study the monotonicity of sequence \((u_k)_{k \geq 0}\) defined by
\[ u_k = \frac{\Gamma(\beta_2 + k\alpha)}{\Gamma(\beta_1 + k\alpha)} , k \geq 0. \]

Thus
\[ \frac{u_{k+1}}{u_k} = \frac{\Gamma(\beta_2 + \alpha + k\alpha)\Gamma(\beta_1 + k\alpha)}{\Gamma(\beta_2 + k\alpha)\Gamma(\beta_1 + \alpha + k\alpha)} . \]
In the case \( \beta_1 > \beta_2 > 1 \), we let \( x = \beta_2 + k\alpha \), \( a = \alpha \) and \( b = \beta_1 - \beta_2 > 0 \) in \((8)\) we obtain that
\[
u_{k+1} / \nu_k = \frac{\Gamma(\beta_2 + \alpha + k\alpha)\Gamma(\beta_1 + \alpha + k\alpha)}{\Gamma(\beta_2 + \alpha + k\alpha)\Gamma(\beta_1 + \alpha + k\alpha)} \leq 1.
\]

Consequently, \( u_{k+1} \leq u_k \) for all \( k \geq 0 \) if and only if \( \beta_1 > \beta_2 \), and the function \( z \mapsto E_{\alpha, \beta_1}(z) / E_{\alpha, \beta_2}(z) \) is decreasing on \((0, \infty)\) if \( \beta_1 > \beta_2 \), by Lemma 2. In the case \( \beta_2 > \beta_1 \) we set \( x = \beta_1 + k\alpha \), \( a = \alpha \) and \( b = \beta_2 - \beta_1 > 0 \) in \((8)\) we conclude that \( u_{k+1} \geq u_k \) for all \( k \geq 0 \). So the function \( z \mapsto E_{\alpha, \beta_1}(z) / E_{\alpha, \beta_2}(z) \) is increasing on \((0, \infty)\) if \( \beta_2 > \beta_1 \), by means of Lemma 2. From the differentiation formula \((24)\), Theorem 5.1
\[
\frac{d}{dz}E_{\alpha, \beta}(z) = \frac{E_{\alpha, \beta-1}(z) - (\beta - 1)E_{\alpha, \beta}(z)}{\alpha z}
\]
we get for \( \beta_2 > \beta_1 \)
\[
\left[ \frac{E_{\alpha, \beta_1}(z)}{E_{\alpha, \beta_2}(z)} \right]' = \frac{E_{\alpha, \beta_1}(z)E_{\alpha, \beta_2}(z) - E_{\alpha, \beta_2}(z)E_{\alpha, \beta_1}(z) + (\beta_2 - \beta_1)E_{\alpha, \beta_1}(z)E_{\alpha, \beta_2}(z)}{\alpha z E_{\alpha, \beta_2}(z)} \geq 0,
\]
and with this the proof of the inequality \((26)\) is done. Finally, choosing \( \beta_1 = \beta + 1 \) and \( \beta_2 = \beta + 2 \) in the inequality \((26)\) we obtain \((27)\).

4. Applications: Lazarevič and Wilker–type inequalities for Mittag–Leffler functions

**Theorem 4.** Let \( \alpha, \beta_1, \beta_2 > 0 \) be such that \( \beta_1 \geq \beta_2 > 1 \). Then the following inequality
\[
\left[ E_{\alpha, \beta_1}(z) \right]^{\frac{\Gamma(\beta_1 - 1)}{\Gamma(\beta_2)}} \leq \left[ E_{\alpha, \beta_2}(z) \right]^{\frac{\Gamma(\beta_1 - 1)}{\Gamma(\beta_2)}}
\]
holds for all \( z \in \mathbb{R} \).

**Proof.** From part (a.) of the theorem 1 the function \( \beta \mapsto \log E_{\alpha, \beta}(z) \) is convex and hence it follows that \( \beta \mapsto \log[ E_{\alpha, \beta_1+a}(z) ] - \log[ E_{\alpha, \beta_2}(z) ] \) is increasing for each \( a > 0 \). Thus, choosing \( a = 1 \) we obtain that indeed the function \( \beta \mapsto E_{\alpha, \beta_1}(z) / E_{\alpha, \beta_2}(z) \) is increasing on \((0, \infty)\). Now, providing that \( \beta_1 \geq \beta_2 > 1 \) let define the function \( \Phi : \mathbb{R} \rightarrow \mathbb{R} \) by
\[
\Phi(x) = \frac{\Gamma(\beta_2)\Gamma(\beta_1 - 1)}{\Gamma(\beta_2 - 1)\Gamma(\beta_1)} \log[ E_{\alpha, \beta_1}(z) ] - \log[ E_{\alpha, \beta_2}(z) ].
\]
From the differentiation formula \((14)\) we get
\[
\Phi'(x) = \frac{1}{\alpha z} \left[ \frac{\Gamma(\beta_2)\Gamma(\beta_1 - 1)}{\Gamma(\beta_2 - 1)\Gamma(\beta_1)} \frac{E_{\alpha, \beta_1-1}(z)}{E_{\alpha, \beta_1}(z)} - \frac{E_{\alpha, \beta_2}(z) - (\beta_2 - 1)}{E_{\alpha, \beta_2}(z)} + \frac{\Gamma(\beta_2)\Gamma(\beta_1 - 1)}{\Gamma(\beta_2 - 1)\Gamma(\beta_1)}(\beta_1 - 1) \right]
\]
\[
= \frac{\Gamma(\beta_2)}{\alpha z \Gamma(\beta_2 - 1)} \left[ \frac{E_{\alpha, \beta_1-1}(z)}{E_{\alpha, \beta_1}(z)} - \frac{E_{\alpha, \beta_2-1}(z)}{E_{\alpha, \beta_2}(z)} \right].
\]
Since the function \( \beta \mapsto E_{\alpha, \beta+1}(z) / E_{\alpha, \beta}(z) \) is increasing on \((0, \infty)\) we derive for all \( \beta_1 \geq \beta_2 > 1 \) that
\[
\frac{E_{\alpha, \beta_1-1}(z)}{E_{\alpha, \beta_1}(z)} \leq \frac{E_{\alpha, \beta_2-1}(z)}{E_{\alpha, \beta_2}(z)}.
\]
From this we conclude that the function \( z \mapsto \Phi(z) \) is decreasing on \([0, \infty)\) and increasing on \((-\infty, 0]\). Consequently \( \Phi(z) \leq \Phi(0) = 0 \) for all \( z \in \mathbb{R} \). So the proof of the theorem \((4)\) is complete.
Clearly, if \( \beta > \frac{1}{2} \), then \( E(\alpha, \beta) \) is log-convex on \( (0, \infty) \).

**Remark** 2. For \( \beta_1 = \beta + 1, \beta_2 = \beta \) in (16) we obtain
\[
E_{\alpha, \beta}((\alpha, \beta)) \leq \left[ E_{\alpha, \beta}(z) \right]^{\beta_1 - \beta_2}, \quad z \in \mathbb{R}.
\]
If \( \beta = \frac{3}{2} \) we derive the Lazarević-type inequality \([21]\) for the Mittag-Leffler function
\[
E_{\alpha, \beta}(z) \leq \left[ E_{\alpha, \beta}(z) \right]^3, \quad z \in \mathbb{R}.
\]

**Corollary 2.** Let \( \alpha, \beta_1, \beta_2 > 0 \) be such that \( \beta_1 \geq \beta_2 > 1 \). Then the following inequality
\[
[ E_{\alpha, \beta_2}(z) ]^{\beta_1 - \beta_2} + \frac{E_{\alpha, \beta_2}(z)}{E_{\alpha, \beta_1}(z)} \geq 2,
\]
holds for all \( z \in \mathbb{R} \).

**Proof.** By using the inequality (19) and the arithmetic–geometric mean inequality we conclude that
\[
\frac{1}{2} \left( [ E_{\alpha, \beta_2}(z) ]^{\beta_1 - \beta_2} + \frac{E_{\alpha, \beta_2}(z)}{E_{\alpha, \beta_1}(z)} \right) \geq \sqrt{\frac{[ E_{\alpha, \beta_2}(z) ]^{\beta_1 - \beta_2}}{E_{\alpha, \beta_1}(z)}} \geq 1.
\]

Note that with use of generalizations of AGM–inequality we may refine (18), on generalizations of the AGM–inequality cf. [3], [33] and related Cauchy–Bunyakovsky inequality cf. [41].

**Remark** 2. For \( \beta_1 = \beta + 1, \beta_2 = \beta \) in (18) we obtain
\[
[ E_{\alpha, \beta}(z) ]^{\beta_1 - \beta_2} + \frac{E_{\alpha, \beta}(z)}{E_{\alpha, \beta_1}(z)} \geq 2, \quad z \in \mathbb{R}.
\]

In case \( \beta = \frac{3}{2} \) the next Wilker-type inequality \([19]\) for the Mittag-Leffler function follows
\[
[ E_{\alpha, \beta}(z) ]^{2} + \frac{E_{\alpha, \beta}(z)}{E_{\alpha, \beta}(z)} \geq 2, \quad z \in \mathbb{R}.
\]

5. **Túrán type inequalities for the Generalized Mittag–Leffler functions**

**Theorem 5.** Let \( \alpha > 0, \beta > 0, \gamma > 0, \) and \( q \in (0, 1) \cup \mathbb{N} \). Then the following assertions are true:

a. The function \( \gamma \mapsto E_{\alpha, \beta}^{\gamma, q}(z) \) is increasing on \( (0, \infty) \).

b. The function \( \beta \mapsto E_{\alpha, \beta}^{\gamma, q}(z) = \Gamma(\beta)E_{\alpha, \beta}^{\gamma, q}(z) \) is log-convex on \( (0, \infty) \).

c. Let \( z > 0 \). Then following Túrán type inequalities
\[
\left( E_{\alpha, \beta+1}^{\gamma, q}(z) \right)^2 \leq E_{\alpha, \beta}^{\gamma, q}(z)E_{\alpha, \beta+2}^{\gamma, q}(z),
\]
holds for all \( \alpha > 0, \beta > 0, \gamma > 0, \) and \( q \in (0, 1) \cup \mathbb{N} \).

**Proof.** a. Let us write
\[
E_{\alpha, \beta}^{\gamma, q}(z) = \sum_{n=0}^{\infty} a_{n}^{\alpha, \beta}(\alpha, \beta, \gamma)z^n, \quad a_{n}^{\alpha, \beta}(\alpha, \beta, \gamma) = \frac{(\gamma)^{n}q}{n!\Gamma(\alpha n + \beta)}, \quad n \geq 0.
\]
Clearly if \( \gamma_1 \geq \gamma_2 > 0 \), then \( (\gamma_1)^{n}q \geq (\gamma_2)^{n}q \), and consequently \( a_{n}^{\alpha, \beta}(\alpha, \beta, \gamma_1) \geq a_{n}^{\alpha, \beta}(\alpha, \beta, \gamma_2) \). Thus the function \( \gamma \mapsto E_{\alpha, \beta}^{\gamma, q}(z) \) is increasing \( (0, \infty) \).

b. For log-convexity property of the function \( \beta \mapsto E_{\alpha, \beta}^{\gamma, q}(z) \), we observe that it is enough to show the log-convexity of each individual term and to use the fact that sums of log-convex functions.

Thus, we just need to show that for each \( k \geq 0 \) we have
\[
\frac{\partial^2}{\partial \beta^2} \log \left[ \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha k)} \right] = \psi'(\beta) - \psi'(\beta + \alpha k) \geq 0,
\]
where \( \psi(x) = \Gamma'(x)/\Gamma(x) \) is the so-called digamma function. But \( \psi \) is known to be concave, and consequently the function \( \beta \mapsto \Gamma(\beta) \) is log-convex on \((0, \infty)\).

**c.** Since the function \( \beta \mapsto \mathbb{E}^{\gamma,q}_{\alpha,\beta}(z) \) is log-convex, then for all \( \beta_1, \beta_2 > 0, \ z > 0 \) and \( t \in [0, 1] \) we have

\[
\mathbb{E}^{\gamma,q}_{\alpha,\beta_1+(1-t)\beta_2}(z) \leq \left[ \mathbb{E}^{\gamma,q}_{\alpha,\beta_1}(z) \right]^t \left[ \mathbb{E}^{\gamma,q}_{\alpha,\beta_2}(z) \right]^{1-t}.
\]

Now choosing \( t = 1/2, \ \beta_1 = \beta, \ \beta_2 = \beta + 2 \), we conclude that (21) holds.

**Theorem 6.** Let \( \alpha > 0, 0 < \gamma \leq 1 \), and \( \beta > x^* - 1 \), where \( x^* \) is the abscissa of the minimum of the \( \Gamma \) function. Then the following Turán type inequality

\[
E^{\gamma_1,2}_{\alpha,\beta}(z)E^{\gamma_1,2}_{\alpha,\beta+2}(z) \geq \frac{1}{1(\beta + 1)1(\beta + 2)(1 - z)^2},
\]

holds for all \( z \in [0, 1] \).

**Proof.** Let \( \alpha > 0, \ \beta > 0 \) and \( \gamma > 0 \). The Cauchy product reveals that

\[
E^{\gamma_1,2}_{\alpha,\beta}(z)E^{\gamma_1,2}_{\alpha,\beta+2}(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(\gamma)_{k}(\gamma)_{n-k}}{k!(n-k)!\Gamma(ak+\beta)\Gamma(a(n-k)+\beta+2)} \right) z^{2n},
\]

and

\[
(E^{\gamma_1,2}_{\alpha,\beta+1}(z))^2 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(\gamma)_{k}(\gamma)_{n-k}}{k!(n-k)!\Gamma(ak+\beta+1)\Gamma(a(n-k)+\beta+1)} \right) z^{2n}.
\]

Thus,

\[
E^{\gamma_1,2}_{\alpha,\beta}(z)E^{\gamma_1,2}_{\alpha,\beta+2}(z) - (E^{\gamma_1,2}_{\alpha,\beta+1}(z))^2 = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(\gamma)_{k}(\gamma)_{n-k}[a(2k-n)-1]}{k!(n-k)!\Gamma(ak+\beta+1)\Gamma(a(n-k)+\beta+2)} \right) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} T^{\gamma_1,2}_{n,k}(\alpha, \beta) z^n - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\gamma)_{k}(\gamma)_{n-k}z^n}{k!(n-k)!\Gamma(ak+\beta+1)\Gamma(a(n-k)+\beta+2)}.
\]

where \( T^{\gamma_1,2}_{n,k}(\alpha, \beta) = \frac{(\gamma)_{k}(\gamma)_{n-k}[a(2k-n)]}{k!(n-k)!\Gamma(ak+\beta+1)\Gamma(a(n-k)+\beta+2)} \). If \( n \) is even, then

\[
\sum_{k=0}^{n} T^{\gamma_1,2}_{n,k}(\alpha, \beta) = \sum_{k=0}^{n/2-1} T^{\gamma_1,2}_{n,k}(\alpha, \beta) + \sum_{k=n/2+1}^{n} T^{\gamma_1,2}_{n,k}(\alpha, \beta) + T^{\gamma_1,2}_{n,n/2}(\alpha, \beta)
\]

\[
= \sum_{k=0}^{n/2-1} (T^{\gamma_1,2}_{n,k}(\alpha, \beta) + T^{\gamma_1,2}_{n,n-k}(\alpha, \beta))
\]

\[
= \sum_{k=0}^{[n-1]/2} (T^{\gamma_1,2}_{n,k}(\alpha, \beta) + T^{\gamma_1,2}_{n,n-k}(\alpha, \beta)),
\]

where \( [\cdot] \) denotes the greatest integer function. Similarly, if \( n \) is odd, then

\[
\sum_{k=0}^{n} T^{\gamma_1,2}_{n,k}(\alpha, \beta) = \sum_{k=0}^{[(n-1)/2]} (T^{\gamma_1,2}_{n,k}(\alpha, \beta) + T^{\gamma_1,2}_{n,n-k}(\alpha, \beta)).
\]
Simplifying, we find that

\[
T_{\alpha,\beta}^{\gamma,1}(\alpha, \beta) + T_{n,\alpha,\beta}^{\gamma,1}(\alpha, \beta) = \frac{(\gamma)_k(\gamma)_{n-k}}{k!(n-k)!\Gamma(\alpha k + \beta + 1)\Gamma(\alpha(n-k) + \beta + 1)} \left[ \frac{\alpha(2k-n)}{\alpha(n-k) + \beta + 1} + \frac{\alpha(n-2k)}{\alpha k + \beta + 1} \right] \\
= \frac{\alpha^2(\gamma)_k(\gamma)_{n-k}(n-2k)^2}{k!(n-k)!\Gamma(\alpha k + \beta + 2)\Gamma(\alpha(n-k) + \beta + 2)}.
\]

Since the function \( \gamma \mapsto (\gamma)_k \) is increasing on \((0, 1)\), and since \( \gamma \in (0, 1) \), we conclude that \((\gamma)_k \leq (1)_k = k!\), and consequently \((\gamma)_k(\gamma)_{n-k} \leq 1\). In addition, the function \( x \mapsto \Gamma(x) \) is increasing on \([x^*, \infty)\) where \( x^* \approx 1.461632144... \) is the abscissa of the minimum of the Gamma function, this implies that \( \frac{1}{\Gamma(\alpha k + \beta + 1)\Gamma(\alpha(n-k) + \beta + 2)} \leq \frac{1}{\Gamma(\beta + 1)\Gamma(\beta + 2)} \).

Consequently, we get

\[
\frac{(\gamma)_k(\gamma)_{n-k}}{k!(n-k)!\Gamma(\alpha k + \beta + 1)\Gamma(\alpha(n-k) + \beta + 2)} \leq \frac{1}{\Gamma(\beta + 1)\Gamma(\beta + 2)}.
\]

Combining (23), (24) and (25) we obtain (22).

**Theorem 7.** Let \( \alpha > 0 \), \( \beta_1, \beta_2 > 0 \), \( \gamma > 0 \), and \( q \in (0, 1) \cup \mathbb{N} \). If \( \beta_1 < \beta_2 \) (resp. \( \beta_2 < \beta_1 \)), then the function \( z \mapsto E_{\alpha,\beta_1}^{\gamma,q}(z)/E_{\alpha,\beta_2}^{\gamma,q}(z) \) is increasing (resp. decreasing) on \((0, \infty)\). Therefore, the following Turán type inequalities

\[
E_{\alpha,\beta_1}^{\gamma,q}(z)E_{\alpha,\beta_2}^{\gamma,q}(z)E_{\alpha,\beta_1+1}^{\gamma,q}(z)E_{\alpha,\beta_2+1}^{\gamma,q}(z) \\
= (\beta_2 - \beta_1)E_{\alpha,\beta_1}^{\gamma,q}(z)E_{\alpha,\beta_2}^{\gamma,q}(z)E_{\alpha,\beta_1+1}^{\gamma,q}(z)E_{\alpha,\beta_2+1}^{\gamma,q}(z) \geq 0,
\]

holds for all \( \alpha, \beta_1, \beta_2 > 0 \) such that \( \beta_2 > \beta_1 \). In particular, the Turán type inequality

\[
E_{\alpha,\beta_1}^{\gamma,q}(z)E_{\alpha,\beta_2}^{\gamma,q}(z) - (E_{\alpha,\beta_1}^{\gamma,q}(z))^2 + (\beta_1 + 1)(E_{\alpha,\beta_1+1}^{\gamma,q}(z))^2 \geq 0,
\]

is valid for all \( \alpha, \beta, z > 0 \).

**Proof.** From the power-series representation of the Mittag-Leffler function \( E_{\alpha,\beta}^{\gamma,q}(z) \), we have

\[
E_{\alpha,\beta_1}(z)/E_{\alpha,\beta_2}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\beta_1 + ka)} \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\beta_2 + ka)}.
\]

In view of Lemma 2, to prove the monotonicity properties of the function \( E_{\alpha,\beta_1}^{\gamma,q}(z)/E_{\alpha,\beta_2}^{\gamma,q}(z) \) it is sufficient to prove the monotonicity of the sequence \( u_k = \{\Gamma(\beta_2 + ka)/\Gamma(\beta_1 + ka)\}_k \). In [10], the authors proved that the sequences \( u_k \) is increasing if and only if \( \beta_1 < \beta_2 \) and \( u_k \) is decreasing if and only if \( \beta_2 < \beta_1 \). Consequently, if \( \beta_1 < \beta_2 \), (resp. \( \beta_2 < \beta_1 \)), then the function \( z \mapsto E_{\alpha,\beta_1}^{\gamma,q}(z)/E_{\alpha,\beta_2}^{\gamma,q}(z) \) is increasing (resp. decreasing) on \((0, \infty)\). By again using the differentiation formula \([44], \text{Theorem 2.1}\]

\[
\frac{d}{dz} E_{\alpha,\beta_1}^{\gamma,q}(z) = \frac{E_{\alpha,\beta_1+1}^{\gamma,q}(z) - (\beta - 1)E_{\alpha,\beta_1}^{\gamma,q}(z)}{\alpha z}
\]

we obtain for \( \beta_2 > \beta_1 \)

\[
\left[ E_{\alpha,\beta_1}^{\gamma,q}(z)/E_{\alpha,\beta_2}^{\gamma,q}(z) \right]' = \frac{E_{\alpha,\beta_1+1}^{\gamma,q}(z) - E_{\alpha,\beta_2}^{\gamma,q}(z)E_{\alpha,\beta_1+1}^{\gamma,q}(z) + (\beta_2 - \beta_1)E_{\alpha,\beta_1}^{\gamma,q}(z)E_{\alpha,\beta_2}^{\gamma,q}(z)}{\alpha z[E_{\alpha,\beta_2}^{\gamma,q}(z)]^2} \geq 0.
\]

So, the inequality (26) is proved. Now, choosing \( \beta_1 = \beta + 1 \) and \( \beta_2 = \beta + 2 \) in the inequality (26) we obtain (27). The proof of Theorem 7 is now completed.

□
Theorem 8. Let $n \in \mathbb{N}, \alpha > 0, \beta > 0$, and $\gamma > 0$. We define the function $E^{\gamma, n}_{\alpha, \beta}(z)$ on $(0, \infty)$ by
\[
E^{\gamma,1,n}_{\alpha, \beta}(z) = E^{\gamma,1}_{\alpha, \beta}(z) - \sum_{k=0}^{n} \frac{(\gamma)_{k} z^{k}}{k! \Gamma(ak + \beta)}.
\]

Then the following Turán type inequality
\[
E^{\gamma,1,n}_{\alpha, \beta}(z)E^{\gamma,1,n+2}_{\alpha, \beta}(z) \leq (E^{\gamma,1,n+1}_{\alpha, \beta}(z))^{2},
\]
is valid.

Proof. From the definition of the function $E^{\gamma,1,n}_{\alpha, \beta}(z)$ we get
\[
E^{\gamma,1,n}_{\alpha, \beta}(z) = E^{\gamma,1,n+1}_{\alpha, \beta}(z) + \frac{(\gamma)_{n+1} z^{n+1}}{(n+1)! \Gamma(\alpha(n+1)+\beta)}
\]
and
\[
E^{\gamma,1,n+2}_{\alpha, \beta}(z) = E^{\gamma,1,n+1}_{\alpha, \beta}(z) - \frac{(\gamma)_{n+2} z^{n+2}}{(n+2)! \Gamma(\alpha(n+2)+\beta)}.
\]

Thus
\[
E^{\gamma,1,n}_{\alpha, \beta}(z)E^{\gamma,1,n+2}_{\alpha, \beta}(z) - (E^{\gamma,1,n+1}_{\alpha, \beta}(z))^{2} =
\]
\[
= E^{\gamma,1,n+1}_{\alpha, \beta}(z) \left( \frac{(\gamma)_{n+1} z^{n+1}}{(n+1)! \Gamma(\alpha(n+1)+\beta)} - \frac{(\gamma)_{n+2} z^{n+2}}{(n+2)! \Gamma(\alpha(n+2)+\beta)} \right) - \frac{(\gamma)_{n+1}(\gamma)_{n+2} z^{2n+3}}{(n+1)! (n+2)! \Gamma(\alpha(n+1)+\beta) \Gamma(\alpha(n+2)+\beta)}
\]
\[
= \frac{(\gamma)_{n+1}}{(n+1)! \Gamma(\alpha(n+1)+\beta)} \sum_{k=n+3}^{\infty} \frac{(\gamma)_{k} z^{k+n+1}}{k! \Gamma(ak+\beta)} - \frac{(\gamma)_{n+2}}{(n+2)! \Gamma(\alpha(n+2)+\beta)} \sum_{k=n+2}^{\infty} \frac{(\gamma)_{k} z^{k+n+2}}{k! \Gamma(ak+\beta)}
\]
\[
= \sum_{k=n+3}^{\infty} \left[ \frac{(\gamma)_{n+k+1}}{(n+k+1)! \Gamma(\alpha(n+k)+\beta) \Gamma(\alpha(n+2)+\beta)} \frac{(\gamma)_{n+1}}{(n+1)! \Gamma(\alpha(n+1)+\beta) \Gamma(\alpha(n+2)+\beta)} \right] z^{k+n+1}
\]
\[
= \sum_{k=n+3}^{\infty} \frac{A^{\gamma,1}_{k}(\alpha, \beta)}{k!} z^{k+n+1},
\]
where
\[
A^{\gamma,1}_{k}(\alpha, \beta) = (n+1)(\gamma)_{n+1}(\gamma)_{k} \Gamma(\alpha(n+2)+\beta) \Gamma(\alpha(n+2)+\beta) - (k-1)(\gamma)_{n+2}(\gamma)_{k-1} \Gamma(\alpha(n+1)+\beta) \Gamma(k-1) \Gamma(\alpha(k-1)+\beta).
\]

On the other hand, we have
\[
A^{\gamma,1}_{k}(\alpha, \beta) = (n+1)(\gamma)_{n+1}(\gamma)_{k} \left[ \Gamma(\alpha(n+2)+\beta) \Gamma(\alpha(n+2)+\beta) - \frac{(k-1)(\gamma)_{n+2}(\gamma)_{k-1}}{(n+2)! \Gamma(\alpha(n+1)+\beta) \Gamma(\alpha(n+2)+\beta)} \right]
\]
\[
= (n+1)(\gamma)_{n+1}(\gamma)_{k} \left[ \Gamma(\alpha(n+2)+\beta) \Gamma(\alpha(n+2)+\beta) - \frac{\Gamma(\gamma+n+2) \Gamma(\gamma+k-1)}{(\gamma+n+1) \Gamma(\gamma+k)} \Gamma(\alpha(n+1)+\beta) \Gamma(\alpha(n+2)+\beta) \right]
\]
\[
\leq (n+1)(\gamma)_{n+1}(\gamma)_{k} \left[ \Gamma(\alpha(n+2)+\beta) \Gamma(\alpha(n+2)+\beta) - \Gamma(\alpha(n+1)+\beta) \Gamma(\alpha(n+2)+\beta) \right].
\]

Taking into account the inequality [10, p. 4]
\[
\frac{\Gamma(\alpha(k+2)+\beta)}{\Gamma(\alpha(k+1)+\beta)} \leq \frac{\Gamma(\alpha(k+2)+\beta)}{\Gamma(\alpha(k-1)+\beta)},
\]
which holds for all $\alpha, \beta > 0$ and $k \geq n + 3$, and using [30], clearly we have $A^{\gamma,1}_{k}(\alpha, \beta) \leq 0$. This in turn implies that the inequality [29] hold.
Theorem 9. Let $n \in \mathbb{N}, \alpha > 0, \beta > 0,$ and $\gamma > 0$. We define the function $H_{\alpha,\beta}^{\gamma,1,n}$ on $(0, \infty)$, by
\[
H_{\alpha,\beta}^{\gamma,1,n}(z) = \frac{E_{\alpha,\beta}^{\gamma,1,n}(z)E_{\alpha,\beta}^{\gamma,1,n+2}(z)}{\left( E_{\alpha,\beta}^{\gamma,1,n+1}(z) \right)^2}, \quad z > 0.
\]
Then the function $x \mapsto H_{\alpha,\beta}^{\gamma,1,n}(z)$ is increasing on $(0, \infty)$. So, the following Turán type inequality
\[
\frac{(n + 1)(\gamma + n + 2)}{(n + 2)(\gamma + n + 1)} \frac{\Gamma^2(\beta + \alpha(n + 2))}{\Gamma(\beta + \alpha(n + 1))\Gamma(\beta + \alpha(n + 3))} \left( E_{\alpha,\beta}^{\gamma,1,n+1}(z) \right)^2 \leq E_{\alpha,\beta}^{\gamma,1,n}(z)E_{\alpha,\beta}^{\gamma,1,n+2}(z),
\]
valid for all $n \in \mathbb{N}, \alpha > 0, \beta > 0,$ and $\gamma > 0$. The constant in left hand side of inequality (33) is sharp.

Proof. By again using the Cauchy product we gave
\[
H_{\alpha,\beta}^{\gamma,1,n}(z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \frac{(\gamma)_{k-n+1}(\gamma)_{k-j-n+3}}{(\beta+\alpha(n+j+1))\Gamma(\beta+\alpha(k-j-n+3))} \right) z^{2n+2k+4},
\]
We define the sequence $U_j(\alpha, \beta, \gamma)$, by
\[
V_j(\alpha, \beta, \gamma) = \frac{(\gamma)_{j-n+1}(\gamma)_{k-j-n+3}(j + n + 2)!}{(\gamma)_{j-n+2}(\gamma)_{k-j-n+2}(j + n + 1)!} U_j(\alpha, \beta),
\]
where the sequence $U_j(\alpha, \beta)$ is defined by [16]
\[
U_j(\alpha, \beta) = \frac{\Gamma(\beta + (n + 2 + j)\alpha)}{\Gamma(\beta + (n + 1 + j)\alpha)} \frac{\Gamma(\beta + (n + 3 + j - k)\alpha)}{\Gamma(\beta + (n + 3 + j - k)\alpha)}.
\]
In [16], the authors proved that the sequence $U_j(\alpha, \beta)$ is increasing for all $j = 0, 1, ...$ Thus
\[
\frac{V_{j+1}(\alpha, \beta, \gamma)}{V_j(\alpha, \beta, \gamma)} = K_{k,j}(\alpha, \beta, \gamma) \frac{U_{j+1}(\alpha, \beta)}{U_j(\alpha, \beta)} \geq K_{k,j}(\alpha, \beta, \gamma),
\]
with
\[
K_{k,j}(\alpha, \beta, \gamma) = \frac{(\gamma + k - j + n + 1)(j + n + 2)(j + n + 3)(\gamma + j + n + 1)}{(\gamma + j + n + 2)(k - j + n + 1)(\gamma + k - j + n + 2)}. \]
By using the inequality
\[
\frac{(\gamma + j + n + 1)(j + n + 2)}{(\gamma + j + n + 2)(j + n + 1)} \geq 1,
\]
we have
\[
K_{k,j}(\alpha, \beta, \gamma) = \left[ \frac{(\gamma + k - j + n + 1)(k - j + n + 3)(\gamma + j + n + 1)}{(\gamma + j + n + 2)(k - j + n + 1)(\gamma + k - j + n + 2)} \right] \geq 1.
\]
So, the sequence $V_j(\alpha, \beta, \gamma)$ is increasing for $j = 0, 1, ...$, and $\alpha, \beta, \gamma > 0$. This implies that the ratios
\[
\sum_{j=0}^{k} \frac{(\gamma)_{j-n+1}(\gamma)_{k-j-n+3}}{(\beta+\alpha(n+j+1))\Gamma(\beta+\alpha(k-j-n+3))}, \quad \sum_{j=0}^{k} \frac{(\gamma)_{j-n+2}(\gamma)_{k-j-n+2}}{(\beta+\alpha(n+j+2))\Gamma(\beta+\alpha(k-j-n+2))}
\]
is increasing, by Lemma 1. So the function $x \mapsto H_{\alpha,\beta}^{\gamma,1,n}(z)$ is increasing by Lemma 2. Finally, it is easy to see that
\[
\lim_{x \to 0} H_{\alpha,\beta}^{\gamma,1,n}(z) = \frac{(n + 1)(\gamma + n + 2)}{(n + 2)(\gamma + n + 1)} \frac{\Gamma^2(\beta + \alpha(n + 2))}{\Gamma(\beta + \alpha(n + 1))\Gamma(\beta + \alpha(n + 3))}.
\]
Corollary 3. Let \( \alpha, \beta, \gamma > 0 \) and \( n \in \mathbb{N} \). If \( \beta_1 \geq \beta_2 \), then the Wilker type inequality for generalized Mittag–Leffler functions

\[
E_{\alpha, \beta_2}^\gamma(z) \leq \frac{E_{\alpha, \beta_1}^\gamma(z)}{E_{\alpha, \beta_1}^\gamma(1)},
\]

holds for all \( z \in \mathbb{R} \). In particular, the following inequality

\[
E_{\alpha, \beta_2}^\gamma(z) \leq \frac{E_{\alpha, \beta_1}^\gamma(z)}{E_{\alpha, \beta_1}^\gamma(2)},
\]

holds for all \( z \in \mathbb{R}, \alpha, \gamma > 0 \) and \( n \in \mathbb{N} \).

Proof. Since the function \( \beta \mapsto E_{\alpha, \beta}^\gamma(z) \) is log-convex on \((0, \infty)\), thus the function \( \beta \mapsto \frac{E_{\alpha, \beta}^\gamma(z)}{E_{\alpha, \beta}^\gamma(1)} \) is increasing on \((0, \infty)\). For \( \beta_1 \geq \beta_2 > 0 \), we define the function \( \Phi \) by

\[
\Phi(z) = \frac{\beta_2}{\beta_1} \log \left( E_{\alpha, \beta_1}^\gamma(z) \right) - \log \left( E_{\alpha, \beta_2}^\gamma(z) \right).
\]

From the differentiation formula \([44], \text{Theorem 2.1}\)

\[
E_{\alpha, \beta}^\gamma(z) = \beta E_{\alpha, \beta_1}^\gamma(z) + \alpha z \frac{d}{dz} E_{\alpha, \beta_1}^\gamma(z),
\]

we obtain that

\[
\Phi'(z) = \frac{\beta_2}{\alpha \beta_1} \frac{E_{\alpha, \beta_1}^\gamma(z) - \beta_1 E_{\alpha, \beta_1}^\gamma(1)}{E_{\alpha, \beta_1}^\gamma(z)} - \frac{\beta_2}{\alpha \beta_1} \frac{E_{\alpha, \beta_2}^\gamma(z) - \beta_2 E_{\alpha, \beta_2}^\gamma(1)}{E_{\alpha, \beta_2}^\gamma(z)}
\]

Thus, for all \( \beta_1 \geq \beta_2 > 0 \), we conclude that the function \( z \mapsto \Phi(z) \) is decreasing on \([0, \infty)\), and decreasing on \((-\infty, 0)\). Therefore, for all \( z \in \mathbb{R} \), we have \( \Phi(z) \leq \Phi(0) = 0 \). Now, Choosing \( \beta_1 = 3/2 \) and \( \beta_1 = 1/2 \) in \([44]\) we obtain \([45]\). This completes the proof of Theorem 10. \(\blacksquare\)

Corollary 3. Let \( \alpha, \beta, \gamma > 0 \) and \( n \in \mathbb{N} \). If \( \beta_1 \geq \beta_2 \), then the Wilker type inequality for generalized Mittag–Leffler functions

\[
\left[ E_{\alpha, \beta_2}^\gamma(z) \right]^\frac{\beta_1-\beta_2}{\beta_2} + \frac{E_{\alpha, \beta_1}^\gamma(z)}{E_{\alpha, \beta_1}^\gamma(1)} \geq 2,
\]

holds for all \( z \in \mathbb{R} \). In particular, the following inequality

\[
\left[ E_{\alpha, \beta_2}^\gamma(z) \right]^2 + \frac{E_{\alpha, \beta_1}^\gamma(z)}{E_{\alpha, \beta_1}^\gamma(2)} \geq 2,
\]

is valid for all \( z \in \mathbb{R}, \alpha, \gamma > 0 \) and \( n \in \mathbb{N} \).
Proof. From the inequality (34), we get
\begin{equation}
E_{\alpha,\beta_1+1}^{\gamma,q}(z) \leq E_{\alpha,\beta_2+1}^{\gamma,q}(z) \left[ E_{\alpha,\beta_2+1}^{\gamma,q}(z) \right]^{\frac{\beta_1 - \beta_2}{\beta_2}},
\end{equation}
combining this inequality and the arithmetic–geometric mean inequality, we conclude that (36) holds. Finally, let $\beta_1 = \frac{3}{2}$ and $\beta_2 = \frac{1}{2}$, we get (37).

7. Open Problems
Motivated by Theorems 8 and 9 we pose the following problems.

Problem 1. Motivated by the inequality (29) in Theorem 8 we pose the following problem: find the generalization of the inequality (29) in the following inequality
\begin{equation}
E_{\alpha,\beta}^{\gamma,q,n}(z)E_{\alpha,\beta}^{\gamma,q,n+2}(z) \leq (E_{\alpha,\beta}^{\gamma,q,n+1}(z))^2,
\end{equation}
where $\alpha, \beta, \gamma > 0, q \in (0,1) \cup \mathbb{N}$ and $z > 0$.

Problem 2. For $z \in (0,\infty)$, find the monotonicity of the function
\begin{equation}
H_{\alpha,\beta}^{\gamma,q,n}(z) = \frac{E_{\alpha,\beta}^{\gamma,q,n}(z)E_{\alpha,\beta}^{\gamma,q,n+2}(z)}{(E_{\alpha,\beta}^{\gamma,q,n+1}(z))^2}
\end{equation}
for all $n \in \mathbb{N}, \alpha > 0, \beta > 0, \gamma > 0$ and $q \in (0,1) \cup \mathbb{N}$.

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