Probing the early universe with a generalized action

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Abstract

Possibly, the most general action in the background of isotropic and homogeneous space-time has been considered to study the quantum evolution of the early universe. Hermiticity of the effective Hamiltonian operator in the presence of curvature squared terms suggests unitary time evolution of the quantum states, assuring conservation of probability. Oscillatory behaviour of the semi-classical wavefunction signals that the theory is classically allowed. Despite the presence of several coupling parameters, only one additional slow-roll condition is required to impose in order to study the inflation. Inflationary parameters lie well within the presently available Planck’s data. Since, the Gauss-Bonnet-dilatonic coupled term does not play any roll in the inflationary regime, it appears to play a significant roll in the late stage of cosmic evolution.

1 Introduction:

In quantum mechanics, ‘unitarity’ is a restriction on the allowed evolution of quantum systems that ensures the sum of probabilities of all possible outcomes of an event is always normalized to 1. So the operator which describes the evolution of a physical system in time, must be unitary. Likewise, the S-matrix, that describes how the physical system changes in scattering process, must also be a unitary operator implying optical theorem (It is a consequence of the conservation of probability. In wave scattering theory, it relates the forward scattering amplitude to the extinction cross-section of the scatterer). Therefore, the time evolution in quantum theory must be formulated as a unitary transformation generated by the Hamiltonian. Such formulation is possible in ‘Non-relativistic Quantum Theory’ where Hamiltonian is the total energy, and also in ‘Special Theory of Relativity’ where Hamiltonian is the time-component of the four-momenta. Nevertheless, problems appear in ‘General Theory of Relativity’ (GTR), since in GTR neither energy nor the momenta are local quantities, rather they are defined globally and only for suitable asymptotic behaviour. Further, the fact that gauge-invariant divergences make GTR non-renormalizable, is quite familiar by now. Non-renormalizable theories are acceptable as description of low energy physics, But these theories have intrinsic mass-scale at which the effective low energy theories break down. For GTR it is the Planck’s scale \((M_P = 10^{19} \text{ GeV})\). The Non-renormalizability of GTR indicates that one should opt for a new physics at Planck’s scale. Consequently, if Einstein-Hilbert action for GTR is modified in a manner such that principle candidates are the contracted quadratic products of the curvature tensor, then fourth derivative terms appear which lead to a suitable graviton propagator that behaves like \(k^{-4}\) for large momenta, and the resulting action

\[
A = \int \left[ \alpha R + \beta_1 R^2 + \beta_2 \left( \frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu} \right) \right] \sqrt{-g} d^4 x, \tag{1}
\]

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is renormalizable. Once renormalization is established for pure gravity, the inclusion of other renormalized fields through coupling, does not pose any further problem. However, it was also simultaneously realized that such fourth order theories lead to ghosts when expanded in the perturbative series about the linearized theory, since the linearized energy of the five massive spin-2 excitations is negative definite \([11]\). This poses major obstacles to their physical interpretation, since it destroys the ‘unitarity’, and thus appears to preclude the acceptability of the above action \([11]\) as a physical theory. On the contrary, there are counter arguments which indicate that the problem with unitarity appearing in perturbative expansion might be misleading. First of all, for \(N\) number of matter fields, the theory is unitary in \(1/N\) expansion, as \(N \to \infty\) \([2]\). Secondly, the standard non-perturbative ‘Osterwalder-Schrader’ construction (it satisfies the condition that correlation functions on Euclidean space-time have to be equivalent to the correlation functions of a Wightman Quantum Field Theory on Minkowski space-time, i.e. it assures that the Wick rotation is well defined isomorphism of ‘Quantum Field Theory’ on Minkowski and on Euclidean space-time) resulted in a Hilbert space with positive norm, which proves the unitarity of fourth order gravitational action \([11]\). Particularly, the action is found to be asymptotically free \([3, 4, 5]\). Asymptotic freedom is the property that causes interactions between particles to become arbitrarily weak at arbitrarily large energy scales, corresponding to arbitrarily small length scale. Thus asymptotic free theories are non-perturbatively renormalizable. Thirdly, it is pointed out that the presence of a massive spin-2 ghost in the bare propagator is inconclusive, since this excitation is unstable. It is shown that the physical \(S\) matrix between in and out states containing only transverse, massless gravitons and physical massless matter fields is gauge independent, and the contribution of all gauge-variant poles to its intermediate states must cancel. The physical \(S\) matrix should therefore, be unitary \([6, 7]\). Fourthly, up on quantization of the effective Hamiltonian corresponding to the conformal version of the above action \([11]\), no ghosts are found to the leading order, in a strong coupling expansion \([8]\). Finally, no ghosts are seen at the classical level, in the zero total energy theorem, even without Einstein-Hilbert version of the above action \([11]\) \([9]\).

In connection with the above discussion, it is well posed to study the role of the above action \([11]\) in the very early universe non-perturbatively, to get certain insights regarding the behaviour of our universe near Planck’s epoch. This requires Hamiltonian formulation of the action \([11]\), which is a non-trivial task due to the presence of higher order curvature invariant terms, and canonical quantization thereafter. Starting from cosmological principle i.e. treating the universe as isotropic and homogeneous a-priori (while observable anisotropy appears due to scalar and metric perturbations), one should consider Robertson-Walker metric described by,

\[
ds^2 = -N^2dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right],
\]

for which the term \( (\frac{1}{2}R^2 - R_{\mu\nu}R^{\mu\nu})\sqrt{-g} \, d^4x \) appearing in the action \([11]\) is a total derivative term, and thus do not contribute to the field equations. Further, a more general action should incorporate \( R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \) term. Nevertheless, the so-called Gauss-Bonnet combination, viz. \( G\sqrt{-g} \, d^4x = (R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})\sqrt{-g} \, d^4x \), which kills the ghost, is again topologically invariant in 4-dimensions. Nonetheless, both these terms may be incorporated in the action \([11]\), to obtain non-trivial contributions in the field equations, through suitable coupling terms, once a scalar field is taken into account. In that case, an action in its most general form may be expressed as,

\[
A = \int \left[ \alpha(\phi)R + \beta_1(\phi)R^2 + \beta_2(\phi) \left( R_{\mu\nu}^2 - \frac{1}{3}R^2 \right) + \gamma(\phi)G - \frac{1}{2}\phi,\mu\phi,\mu - V(\phi) \right] \, d^4x\sqrt{-g}.
\]

In the above, apart from the required functional dependence of \(\beta_2\) and \(\gamma\), we have also chosen functional dependence of \(\alpha\) and \(\beta_1\), for generality, and \(V(\phi)\) is an arbitrary potential. Particularly, \(\gamma(\phi)G\) is called the Gauss-Bonnet-dilatonic coupling term, which arises naturally as the leading order of the \(\alpha'\) expansion of heterotic superstring theory, where, \(\alpha'\) is the inverse string tension \([10]\) \([11]\) \([12]\) \([13]\). Further, the low energy limit of the string theory also gives rise to the dilatonic scalar field which is found to be coupled with various curvature invariant terms \([13]\) \([14]\). Therefore, the leading quadratic correction gives rise to Gauss-Bonnet term with a dilatonic coupling \([16]\). The primary reason for incorporating dilatonic coupled Gauss-Bonnet term in the action is due to the fact that, it does not play any role during Inflationary regime \([17]\). Thus, it might be deployed to play an important role at the late-stage evolution of the universe to exhibit accelerated expansion after a long Friedmann-like deceleration in the matter dominated era \([18]\) \([19]\). For a clarification, we remind that conformally invariant theory of gravity is specified by the conformal Weyl squared term, for which the Lagrangian density reads.
as \( \mathcal{L} = \alpha R - \frac{1}{4} \Gamma_{\alpha \beta \gamma} \Gamma^{\alpha \beta \gamma} \), \( m \) being the mass scale at which such correction becomes relevant. Nonetheless, it is worth to mention that Weyl square term vanishes in 4-dimensional isotropic metric. Now, while the expression for Weyl squared term is
\[
C^2 = \frac{1}{4} R^2 - 2 R_{\alpha \beta} R^{\alpha \beta} + R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta},
\]
therefore, \( C^2 = G - 2(\frac{1}{4} R^2 - R_{\alpha \beta} R^{\alpha \beta}) \). Since, in action \( (3) \) the two terms on the right hand side appear with two different coupling parameters, so it is even more general than incorporating Weyl squared term.

In the following section, we write the general field equations and their counterparts in the background of isotropic and homogeneous Robertson-Walker minisuperspace \( (2) \). Classical de-Sitter solutions are thereby explored. In section 3, we perform Canonical analysis as a mere prologue to canonical quantization scheme, followed by an appropriate semiclassical approximation. In section 4, inflation under slow-roll approximation is performed and the results are compared with recently released data sets from Planck’s collaborators \( (20, 21) \). Finally we conclude in section 5.

## 2 Action, field equations and classical solutions:

The field equation corresponding to the action \( (1) \) is found under the standard metric variation as \( (22) \),

\[
\alpha(\phi) G_{\mu \nu} = \Box \alpha(\phi) g_{\mu \nu} - \nabla_\mu \nabla_\nu \alpha(\phi) + 4 \left( \beta_1(\phi) - \frac{\beta_2(\phi)}{3} \right) R R_{\mu \nu} + 4 g_{\mu \nu} \Box \left[ \left( \beta_1(\phi) - \frac{\beta_2(\phi)}{3} \right) R \right] - 4 \nabla_\mu \nabla_\nu
\]

\[
\left[ \left( \beta_1(\phi) - \frac{\beta_2(\phi)}{3} \right) R \right] g_{\mu \nu} - g_{\mu \nu} \left( \beta_1(\phi) - \frac{\beta_2(\phi)}{3} \right) R^2 + 2 \beta_2(\phi) \left( g_{\mu \alpha} R_{\alpha \beta} R^{\alpha \beta} - 4 R_{\mu \alpha} R^{\alpha \beta} - 4 g_{\mu \alpha} \nabla_\alpha \nabla_\beta \beta_2(\phi) R^{\alpha \beta} \right)
\]

\[
+ 8 \nabla_\alpha \nabla_\beta \beta_2(\phi) R_{\mu \nu} - 4 \Box \beta_2(\phi) R_{\mu \nu} + 2 \gamma(\phi) H_{\mu \nu} + 8 (\gamma' \nabla_\rho \phi \nabla^\rho \phi - \gamma' \nabla_\rho \nabla^\rho \phi) P_{\mu \nu \rho \sigma} - T_{\mu \nu} = 0,
\]

where, \( G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \) and \( T_{\mu \nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu \nu} \nabla_\alpha \phi \nabla^\alpha \phi - g_{\mu \nu} V(\phi) \) are the Einstein tensor and the energy-momentum tensor, respectively. Further, \( H_{\mu \nu} = 2 \left( R_{\mu \rho \nu \sigma} - 2 R_{\mu \rho} R_{\nu \sigma} - 2 R_{\mu \rho \sigma \lambda} R^{\rho \lambda} - \frac{1}{4} g_{\mu \nu} (R^2 - 4 R_{\mu \nu} R^{\rho \sigma} - R_{\mu \rho \sigma \lambda} R_{\nu}^{\rho \lambda}) \right) \) and \( P_{\mu \nu \rho \sigma} = R_{\mu \rho \sigma} + 2 g_{\mu |\sigma} R_{\nu |\rho} + 2 g_{\mu |\rho} R_{\sigma |\nu} + R_{\mu |\rho \sigma |\nu} + R_{\mu |\rho \gamma} R_{\nu |\sigma \delta} \). Explicit form of equation \( (4) \) together with the \( \phi \) variation equation may be written as,

\[
\frac{1}{2} \left( \alpha G_{\mu \nu} + \Box \alpha g_{\mu \nu} - \alpha g_{\mu \nu} \right) + 2 \left( \beta_1(\phi) - \frac{\beta_2(\phi)}{3} \right) \left( R R_{\mu \nu} - \frac{1}{4} g_{\mu \nu} R^2 \right)
\]

\[
+ 2 \left[ \left[ \left( \beta_1(\phi) - \frac{\beta_2(\phi)}{3} \right) R \right] g_{\mu \nu} - \left[ \left( \beta_1(\phi) - \frac{\beta_2(\phi)}{3} \right) R \right] \right]_{\mu \nu}
\]

\[
+ \beta_2 g_{\mu \nu} R_{\alpha \beta} R^{\alpha \beta} - 4 \beta_2 R_{\mu \alpha} R^{\alpha \beta} - 2 g_{\mu \nu} \nabla_\alpha \nabla_\beta \beta_2 R^{\alpha \beta} + 4 \nabla_\alpha \nabla_\beta \beta_2 R_{\mu \nu} - 2 \Box \beta_2 R_{\mu \nu}
\]

\[
+ 2 \gamma \left[ R R_{\mu \nu} - 2 R_{\mu \rho} R^{\rho \sigma} - 2 R_{\mu \rho \sigma \lambda} R^{\rho \lambda} - \frac{1}{4} g_{\mu \nu} (R^2 - 4 R_{\mu \nu} R^{\rho \sigma} + R_{\mu \rho \sigma \lambda} R_{\nu}^{\rho \lambda}) \right]
\]

\[
+ 4 \left( \gamma^2 \phi \gamma' \phi - \gamma' \phi \phi \phi \right) \left[ R_{\mu \rho \sigma \lambda} + 2 g_{\mu |\sigma} R_{\rho |\lambda} + 2 g_{\mu |\rho \lambda} R_{\sigma |\lambda} + R_{\mu |\rho \lambda} g_{\sigma |\lambda} \right] = \frac{T_{\mu \nu}}{2},
\]

and,

\[
\Box \phi - \alpha' R - \beta_1 R^2 - \beta_2 \left( R^2 - \frac{1}{3} R^2 \right) - \gamma' \mathcal{G} - V' = 0,
\]

respectively, where prime denotes derivative with respect to \( \phi \). In the homogeneous and isotropic Robertson-Walker metric \( (2) \), the Ricci scalar reads as,

\[
R = \frac{6}{N^2} \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} + N^2 \frac{k}{a^2} - \frac{\ddot{a} N}{a N} \right).
\]

Not all the components of Einstein’s equations are independent. It therefore suffices to write the two independent components of Einstein’s field equations, viz., the \( (00) \) equation and the \( \dot{\phi} \) variation equation, under standard
gauge choice $N = 1$, in terms of the scale factor $a$ as,

$$
-\frac{6\alpha}{a^2}(\dot{a}^2 + k) - \frac{6\alpha'\dot{a}\dot{\phi}}{a} - 36\beta_1\left(\frac{2\dot{a}^2}{a^2} - \frac{a^2}{a^4} + \frac{2a^2\ddot{a}}{a^3} - \frac{3a^4}{a^4} - \frac{2ka^2}{a^4} + \frac{k^2}{a^4}\right) - 72\beta_1\dot{\phi}\left(\frac{\dot{a}^2}{a^2} + \frac{a^3}{a^3} + \frac{k\dot{a}}{a^3}\right) + 6\beta_2\phi\left(\frac{2a^3}{a^3} + \frac{3k\dot{a}}{a^3}\right) - 24\gamma\dot{\phi}\left(\frac{a^3}{a^3} + \frac{k\dot{a}}{a^3}\right) + \left(\frac{\phi^2}{2} + V\right) = 0
$$

(7)

and

$$
-6\alpha\left(a^2\ddot{a} + a\dot{a}^2 + ka\right) - 36\beta_1\left(\frac{a^2}{a^2} + \frac{2a^2\ddot{a}}{a} + \frac{\dot{a}^2}{a} + \frac{2ka^2}{a} + 2k\dot{a}\right) + 12\beta_2\left(a^2\ddot{a} + k\dot{a}\right) + 3a^2\dot{a}\phi - 24\gamma\left(\ddot{a}^2 + k\dot{a}\right) + 3\left(\dot{\phi} + V'\right) = 0
$$

(8)

We seek inflationary solution of the classical field equations (7), (8) in the following standard de-Sitter form,

$$
a = a_0e^{Ht}; \quad \phi = \phi_0e^{-Ht},
$$

(9)

where $H = \frac{\dot{a}}{a}$ denotes the expansion rate. The de-Sitter solution (9), restricts the forms of coupling parameters and the potential $V(\phi)$ (for $k = 0$), in view of the above classical Einstein field equations (7) and (8) as,

$$
\alpha = \alpha_0 + \frac{\alpha_1}{\phi} - \alpha_2\phi^2, \quad V = \frac{V_1}{\phi} + V_0, \quad \text{and} \quad 2\gamma' + 12\beta_1 = \beta_2,
$$

(10)

while the constants are restricted as,

$$
\alpha_0 = \frac{V_0}{6H^2}, \quad \alpha_1 = \frac{V_1}{12H^2}, \quad \alpha_2 = \frac{1}{12}.
$$

(11)

In the above, $V_0$ and $V_1$ are arbitrary constants. Thus the forms of the coupling parameter $\alpha(\phi)$ and the potential $V(\phi)$ are fixed once and forever, while $\gamma(\phi)$ or $\beta_1(\phi)$ and $\beta_2(\phi)$ remain as free parameters. Nevertheless, once any two of $\gamma(\phi)$, $\beta_1(\phi)$ and $\beta_2(\phi)$ are chosen in view of some physical/mathematical argument say, then the the equation (10) fixes the other. We shall require these solutions later.

3 Canonical formulation:

Canonical formulation of higher-order theories requires additional degrees of freedom. The action for the present case is chosen in such a manner that the field equations are of fourth order, and hence one additional degree of freedom is necessary. Ostrogradski’s technique [24] towards canonical formulation of higher order theories does not work for the singular Lagrangian (for which the determinant of the Hessian vanishes) under consideration, at least due to the presence of the Laplace function $N$, which essentially is a Lagrange multiplier. It is therefore required to follow Dirac’s algorithm of constrained analysis [24] [25]. In Dirac’s formalism, for treating higher-order theory of gravity, it is customary to assume $\delta h_{ij}|_{\partial V} = 0 = \delta K_{ij}|_{\partial V}$ at the boundary, where, $h_{ij}$ is the induced three metric, and $K_{ij}$ is the extrinsic curvature tensor. However, Modified Horowitz Formalism (MHF) [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] bypasses the constrained analysis. In MHF $\delta h_{ij}|_{\partial V} = 0 = \delta R|_{\partial V}$ at the boundary, and the action is required to be supplemented by appropriate boundary terms. There exists other techniques too, and the Hamiltonians obtained following different techniques are canonically equivalent [39]. Nevertheless, equivalence at the quantum level requires canonical transformation in the quantum domain, and since at the classical level canonical transformations are highly non-linear, so the different quantum descriptions so obtained are likely to be inequivalent [40]. In this connection it has recently been established [37] that MHF and the Dirac constraint analysis towards canonical formulation of higher-order theory of gravity lead to the same Hamiltonian, and therefore to identical quantum description, provided the action is first expressed in terms of $h_{ij}$, and Dirac algorithm is initiated only after taking care of the divergent terms appearing in the action. We shall here follow the MHF, while Dirac’s technique is exhibited in the appendix. As mentioned, in order to expatiate MHF, the action (3) should be supplemented by appropriate boundary terms, viz.,
\[ A = \int \left[ \alpha(\phi)R + \beta_1(\phi)R^2 + \beta_2(\phi) \left[ R_{\mu\nu}^2 - \frac{1}{3} R^2 \right] + \gamma(\phi)G - \frac{1}{2} \phi_{,\mu} \phi_{,\nu}^* - V(\phi) \right] \sqrt{-g} \, dt \, x \]

\[ + \alpha(\phi) R + \beta_1(\phi) R^2 + \beta_2(\phi) \Sigma_{R^2} + \gamma(\phi) \Sigma_{G} \]  

(12)

In the above, the supplementary boundary terms \( \Sigma_R = 2 \oint_{\partial \Omega} k \sqrt{|h|} d^3x \), \( \Sigma_{R^2} = \Sigma_{R^2}^1 + \Sigma_{R^2}^2 = 4 \oint_{\partial \Omega} \left( 3 + (4 - 3R) \right) K \sqrt{|h|} d^3x \), and \( \Sigma_{G} = 4 \oint_{\partial \Omega} \left( 2G_{ij} K^{ij} + \frac{K}{2} \right) \sqrt{|h|} d^3x \) are the Gibbons-Hawking-York term and its subsequent modifications \( \boxed{11 \, 12 \, 13} \), while \( K = K^2 - 3K K^{ij} K_{ij} + 2K^{ij} K_k k^k \), \( K \) being the trace of extrinsic curvature tensor \( K_{ij} \). Let us briefly elucidate our programme. First, we express the action (12) in terms of the basic variable \( h_{ij} = a^2 \delta_{ij} = z \delta_{ij} \), where, \( a^2 = z \), and remove the divergent terms under integrate by parts. Next, we shall introduce an auxiliary variable, remove divergent terms again, and finally translate the auxiliary variable to the other basic variable, viz. \( k_{ij} = -\frac{b_{ij}}{2N} = -\frac{a^2}{N} \delta_{ij} = \frac{z}{N} \delta_{ij} \), to obtain the phase-space structure. So we first write the action (12) in terms of \( h_{ij} = a^2 \delta_{ij} = z \delta_{ij} \), using the form of the Ricci scalar (6) as,

\[ A = \int \left[ 3\alpha(\phi) \sqrt{2} \left( \frac{\dot{z}}{N} - \frac{\dot{N} z}{N^2} + 2kN \right) + \frac{9\beta_1(\phi)}{\sqrt{2}} \left( \frac{z^2}{N^3} + \frac{\dot{z} \dot{N} z}{N^4} + \frac{\dot{z}^2 N^2}{N^5} - \frac{4kN}{N^2} + \frac{4k^2}{N} + 4k^2 N \right) \right. \]

\[ - \beta_2(\phi) \left( \frac{3\dot{z} z}{N^2} - \frac{3 \dot{N} z}{N^3} - \frac{3 \dot{z}^2}{2N^3} + \frac{6k \dot{z} z}{N^2 \sqrt{2}} - \frac{6kN}{N^2 \sqrt{2}} + \frac{3k^2 \dot{z}^2}{N^2} \right) \]

\[ + \frac{3\gamma(\phi)}{\sqrt{2}} \left( \frac{\dot{z}^2}{N^2} - \frac{\dot{z}^2 N + 4k - \frac{2k^2}{N} - \frac{4kN}{N} - \frac{4k^2}{N} \right) \left] \right. \left( \frac{1}{2N} \dot{\phi}^2 - N(\phi) \right) \right] \]

\[ + \alpha(\phi) \Sigma_R + \beta_1(\phi) \Sigma_{R^2} + \beta_2(\phi) \Sigma_{R^2} - V(\phi) \]

(13)

In the above, \( \Sigma_R = -\frac{3\sqrt{z}}{N^2} + \Sigma_{R^2}^1 + \Sigma_{R^2}^2 = -\frac{4\sqrt{z}}{N^2} \left( \frac{\dot{z}}{N} - \frac{\dot{N} z}{N^2} \right) \), \( \Sigma_{R^2}^1 = \frac{36k}{N^2} + 12k \) supplementary surface terms known as Gibbons-Hawking-York (GHY) and its subsequent modified versions in the isotropic and homogeneous Robertson-Walker metric (12). It is important to mention that unlike GTR, here the lapse function appears in the action with its time derivative, behaving like a true variable. Despite such uncanny situation, one can still bypass Dirac’s algorithm. First, under integrating the above action (13) by parts, the counter terms \( \Sigma_R, \Sigma_{R^2}, \Sigma_{R^2}^1, \Sigma_{R^2}^2 \) and \( \Sigma_{G} \) get cancelled and the above action (13) reads as,

\[ A = \int \left[ \left( \frac{-3\alpha \dot{z}}{N} - \frac{3 \dot{z} z}{2N \sqrt{2}} + 6k N \alpha \sqrt{2} \right) + \frac{9\beta_1}{\sqrt{2}} \left( \frac{\dot{z}^2}{N^3} + \frac{2 \dot{N} \dot{z} z}{N^4} + \frac{\dot{z}^2 N^2}{N^5} + \frac{2 k \dot{z}^2}{N^3} + \frac{4 k^2 N}{N} \right) \right. \]

\[ - \frac{36k}{N^2} \sqrt{2} \left( \frac{\dot{z}^2}{2N^3 \sqrt{2}} + \frac{6 k \dot{z}}{N \sqrt{2}} \right) - \gamma' \dot{\phi} \left( \frac{\dot{z}^2}{N^2} + 12k \right) + \frac{3}{2N} \dot{\phi}^2 - N(\phi) \right] \left( \frac{1}{2N} \dot{\phi}^2 - N(\phi) \right) \]

(14)

At this stage we introduce an auxiliary variable,

\[ Q = \frac{\partial A}{\partial \dot{z}} = \frac{18\beta_1}{N^3 \sqrt{2}} \left( \frac{\dot{z}}{N} - \frac{\dot{N} z}{N} \right), \]

(15)

judiciously into the action (14), so that it takes the following form,

\[ A = \int \left[ \left( \frac{-3\alpha \dot{z}}{N} - \frac{3 \dot{z} z}{2N \sqrt{2}} + 6k N \alpha \sqrt{2} \right) + \left( Q \frac{\dot{z}}{N} - \frac{N^3 \sqrt{2} Q^2}{36 \beta_1} \right) + \frac{9\beta_1}{\sqrt{2}} \left( \frac{2 k \dot{z}^2}{N^3} + \frac{4 k^2 N}{N} \right) \right. \]

\[ - \frac{36k}{N^2} \sqrt{2} \left( \frac{\dot{z}^2}{2N^3 \sqrt{2}} + \frac{6 k \dot{z}}{N \sqrt{2}} \right) - \gamma' \dot{\phi} \left( \frac{\dot{z}^2}{N^2} + 12k \right) + \frac{3}{2N} \dot{\phi}^2 - N(\phi) \right] \left( \frac{1}{2N} \dot{\phi}^2 - N(\phi) \right) \]

(16)

Now, integrating by parts yet again, the last of the surface terms gets cancelled with the total derivative term and
the action (16) can finally be expressed as,

\[
A = \int \left[ \left( -\frac{3\alpha^2}{N} \frac{\dot{z}}{\sqrt{z}} + \frac{3\alpha^2}{2N} \right) + 6kN\alpha \sqrt{z}\right) - \left( \frac{\dot{N}z}{N} + \frac{NHQ}{N} + \frac{9N}{z^2} \left( \frac{2k^2}{N} + 4N^2 \right) - \frac{36k\beta_1\dot{z}}{N} + \frac{z^3}{2N^3} \right) + \frac{6k\dot{z}}{N} \right) - \frac{3\gamma^2}{N} \left( \frac{2k^2}{N} + 4N^2 \right) \right) \left( \dot{z}^2 + 12N \right) + z^2 \left( \frac{\dot{\phi}^2}{2N} - NV \right) \right] \, dt.
\]

Therefore, the canonical momenta are

\[
p_Q = -\dot{z},
\]
\[
p_z = \frac{3\alpha^2}{N} \frac{\dot{z}}{\sqrt{z}} - \frac{3\alpha^2}{2Nz} - \dot{Q} - \frac{NQ}{N} + \frac{36k\beta_1}{Nz} + \frac{36k\beta_1}{2N^3z} + \frac{6k\dot{z}}{N} \right) - \frac{3\gamma^2}{N} \left( \frac{2k^2}{N} + 4N^2 \right),
\]
\[
p_\phi = \frac{3\alpha^2}{N} \frac{\dot{z}}{\sqrt{z}} - \frac{36k\beta_1}{Nz} + \frac{36k\beta_1}{Nz^3} + \frac{1}{N} \left( \frac{3\dot{z}^2}{N^3} + 12k \right) + \frac{\dot{z}^2}{N},
\]
\[
p_N = -\frac{Q}{N}.
\]

The action (20) still contains time derivative of the Lapse function \(N\) and not all the momenta are invertible, implying degeneracy of the Lagrangian. One can bypass Dirac’s constraint analysis as claimed earlier up on finding the following relation in view of the definition of momenta [18],

\[
p_Qp_z = \frac{3\alpha^2}{N} \frac{\dot{z}}{\sqrt{z}} + \frac{3\alpha^2}{2Nz} + \dot{Q} + \frac{NQ}{N} + \frac{36k\beta_1}{Nz} + \frac{36k\beta_1}{2N^3z} + \frac{6k\dot{z}}{N} \right) - \frac{3\gamma^2}{N} \left( \frac{2k^2}{N} + 4N^2 \right),
\]

and using the above relation and the definitions of momenta [18], to obtain the phase space structure of the Hamiltonian constraint equation as,

\[
H_c = -p_Qp_z + \frac{N^2Q}{2N^3} - \frac{N^2p_\phi}{2N^3} - \frac{3\alpha^2p_Qp_\phi}{Nz} + \frac{36k\beta_1p_Qp_\phi}{Nz^3} + \frac{6k\dot{z}}{N} - \frac{3\gamma^2}{N} \left( \frac{2k^2}{N} + 4N^2 \right) + \frac{\dot{z}^2}{N},
\]

\[
\right)\left( \dot{z}^2 + 12N \right) + \frac{\dot{\phi}^2}{2N} - NV = 0.
\]

The problem with the above Hamiltonian is that, firstly it does not exhibit diffeomorphic invariance \(H = NH\), and second, the momenta \(p_Q\) appears up to sixth degree. This uncanny situation is improved considerably, as soon as the auxiliary variable \((Q)\) is replaced by the basic variable. In fact, under the following canonical transformations \(Q = \frac{z^2}{N}\) and \(p_Q = -Nx\), the phase-space structure of the Hamiltonian can be expressed in terms of basic variables as,

\[
H_c = N \left[ xp_z + \frac{\sqrt{2}p_\phi}{36\beta_1} + \frac{z^3}{N} + \frac{3\alpha^2p_xp_\phi}{z} + \frac{36k\beta_1p_xp_\phi}{z^3} - \frac{6kx}{\sqrt{z}} + \frac{6kx}{N} \right] + \frac{\dot{\phi}^2}{2N} - NV = NH = 0.
\]
In the process diffeomorphic invariance is established, and the momenta appear only with second degree. It is now also possible to express the action (10) in the canonical form with respect to the basic variables as,

\[ A = \int \left( \dot{z}_x + \dot{\phi}_x + \dot{\phi}_\phi - N\mathcal{H} \right) dt dx = \int \left( h_{ij}\pi^{ij} + K_{ij}\Pi^{ij} + \dot{\phi}_p - N\mathcal{H} \right) dt dx, \]  

where \(\pi^{ij}\) and \(\Pi^{ij}\) are momenta canonically conjugate to \(h_{ij}\) and \(K_{ij}\) respectively.

### 3.1 Canonical quantization:

The quantum counterpart of the Hamiltonian (21) under standard canonical quantization reads as,

\[ \frac{i\hbar}{\sqrt{z}} \frac{\partial \Psi}{\partial z} = \left[ -\frac{\hbar^2}{36\beta_1 x} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) - \frac{\hbar^2}{2x^2z^2} \frac{\partial^2}{\partial \phi^2} + \frac{36k^2\beta_1\beta_2}{z^4} \left( \frac{x^3}{2z^2} + \frac{12k}{z^2} \right) + \frac{3\alpha\beta_1^2}{z^2} \left( 3\frac{x^3}{2z^2} + \frac{6k}{z^2} \right) + \frac{3\alpha\beta_2^2}{z^2} \left( 2\frac{x^3}{2z^2} + \frac{6k}{z^2} \right) - \frac{\alpha\beta_1\gamma}{z^2} \left( \frac{x^3}{2z^2} + \frac{6k}{z^2} \right) \right] \Psi, \]

where, \(n\) is the operator ordering index which removes some but not all of the operator ordering ambiguities appearing between \(\dot{x}\) and \(\dot{p}_x\). Now, in order to remove additional ambiguities in connection with the pairs \(\dot{\phi}_\phi, \dot{\phi}_\gamma\) and \(\dot{\phi}_\phi, \dot{\phi}_\gamma\), we need to know the functional dependence of the coupling parameters. While, \(\alpha(\phi)\) is already known, a linear relation exists amongst \(\beta_1, \beta_2\) and \(\gamma\) vide (10), in view of the classical de-Sitter solution. For the sake of simplicity, we choose,

\[ \beta_1 = \beta_{01}\phi; \quad \beta_2 = \beta_{02}\phi; \quad \gamma = \gamma_0\phi, \quad \Rightarrow \quad \beta_{02} = 2\gamma_0 + 12\beta_{01}, \]

particularly to avoid further complications arising out of operator ordering between \(\{\dot{\phi}_\phi\}, \{\dot{\phi}_\gamma\}\) and \(\{\dot{\phi}_\phi\}\) etc. Thus, performing Weyl symmetric ordering carefully, equation (23) takes the following form,

\[ \frac{i\hbar}{\sqrt{z}} \frac{\partial \Psi}{\partial z} = \left[ -\frac{\hbar^2}{36\beta_1 x} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) - \frac{\hbar^2}{2x^2z^2} \frac{\partial^2}{\partial \phi^2} + \frac{3i\hbar\alpha_1}{z^2} \left( \frac{1}{\phi^2} \frac{\partial}{\partial \phi} - \frac{1}{\phi^3} \right) + \frac{3i\hbar\alpha_2}{z^2} \left( \frac{2\phi}{\partial \phi} + 1 \right) \right. \]

\[ + \frac{6i\hbar\alpha_1 x^2}{z^2} \left( \frac{\partial}{\partial \phi} \right) + \frac{9x}{2z} \left( \alpha_1 \frac{\phi^3}{\partial \phi^4} + \frac{4\alpha_1\alpha_2}{\phi^2} + 4\alpha_2\phi^2 \right) + \frac{18\beta_0^2 x^3}{z^3} + \frac{18\beta_0 x^3}{z^3} \left( \frac{\alpha_1}{\phi^2} + 2\alpha_2 \phi \right) \]

\[ + \left. \frac{3x}{2z} \left( \alpha_0 + \frac{\alpha_1}{\phi} - \alpha_2 \phi^2 \right) + \frac{z}{x} \left( \frac{V_1}{\phi} + V_0 \right) \right] \Psi. \]

Now, under a change of variable, the above modified Wheeler-de-Witt equation, takes the look of Schrödinger equation, viz.,

\[ i\hbar \frac{\partial \Psi}{\partial \sigma} = \left[ -\frac{\hbar^2}{54\beta_0 x} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) - \frac{\hbar^2}{3x^2 \sigma^2} \frac{\partial^2}{\partial \phi^2} + \frac{2i\hbar}{3\sigma} \left( \frac{3\alpha_1}{\phi^2} + 6\alpha_2 \phi + \frac{6\beta_0 x^2}{\sigma^2} \right) \frac{\partial}{\partial \phi} + \frac{2i\hbar}{\sigma} \left( \alpha_2 - \frac{\alpha_1}{\phi^3} \right) + V_\epsilon \right] \Psi \]

\[ = H_\psi \Psi. \]

In the above Schrödinger-like equation, the effective potential \(V_\epsilon\) is given by,
$$V_c = \frac{3x}{\sigma^3} \left( \alpha_1^2 + \frac{4\alpha_1\alpha_2}{\phi} + 4\alpha_2^2\phi^2 \right) + \frac{12\beta_0 x^5}{\sigma^3} + \frac{12\beta_0 x^3}{\sigma^2} \left( \frac{\alpha_1}{\phi^2} + 2\alpha_2\phi \right) + \frac{x}{\sigma^3} \left( \alpha_0 + \frac{\alpha_1}{\phi} - \alpha_2\phi^2 \right) + \frac{2\phi^2}{3x} \left( V_1 + V_0 \right).$$

and, $\sigma = z^2 = a^3$ plays the role of internal time parameter. It is quite important to mention that, since time itself acts as a dynamical variable in the theory of gravity, the Hamiltonian appears as a constraint, and upon quantization for any state $\Psi$, $\hat{H}\Psi = 0$, time disappears. Thus GTR confronts with standard probabilistic interpretation, and one is not supposed to ask what happened earlier, since time collapses. However, despite the fact that the proper volume of the universe itself is a dynamical variable, it acts as an internal time parameter in the quantum description of higher-order theory, and standard quantum mechanical probabilistic interpretation holds, as we explore in the following subsection.

### 3.2 Hermiticity of $\hat{H}_e$ and probabilistic interpretation:

We initiated our discussion in connection with unitarity, which is the fundamental requirement of a viable quantum theory. In fact, unitarity and consistency are synonym. Now, hermiticity of a time independent Hamiltonian leads to unitary time evolution, which assures conservation of probability. Further, in quantum scattering theory, hermiticity is necessary both for reciprocity and unitarity. Thus, the requirement of ‘hermiticity’ is necessary and sufficient condition for the unitary time evolution. In the following we demonstrate that the effective Hamiltonian so obtained, is a hermitian operator. Splitting the effective Hamiltonian $\hat{H}_e$ obtained in (26) we express it as,

$$\hat{H}_e = \hat{H}_1 + \hat{H}_2 + \hat{H}_3 + V_c, \tag{27}$$

where,

$$\hat{H}_1 = -\frac{\hbar^2}{54\beta_0 \phi x} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right), \tag{29}$$

$$\hat{H}_2 = -\frac{\hbar^2}{3x \sigma^3} \frac{\partial^2}{\partial \phi^2}, \tag{30}$$

$$\hat{H}_3 = \frac{2i\hbar}{\sigma} \left( \frac{\alpha_1}{\phi} + 2\alpha_2\phi + \frac{2\beta_0 x^2}{\sigma^2} \right) \frac{\partial}{\partial \phi} + \frac{2i\hbar}{\sigma} \left( \alpha_2 - \frac{\alpha_1}{\phi^3} \right) \tag{31}$$

$$\hat{V}_c = V_c. \tag{32}$$

Now, let us consider the first term,

$$\int (\hat{H}_1 \Psi)^* \Psi dx = -\frac{\hbar^2}{54\beta_0 \phi} \int \left( \frac{1}{x} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{n}{x^2} \frac{\partial \Psi^*}{\partial x} \right) \Psi dx = -\frac{\hbar^2}{54\beta_0 \phi} \int \left( \frac{\Psi \frac{\partial^2 \Psi^*}{\partial x^2} + n \Psi \frac{\partial \Psi^*}{\partial x}}{x^2} \right) dx. \tag{33}$$

Under integration by parts twice and dropping the first term due to fall-of condition, we obtain,

$$\int (\hat{H}_1 \Psi)^* \Psi dx = -\frac{\hbar^2}{54\beta_0 \phi} \int \Psi^* \left[ \frac{1}{x} \frac{\partial^2 \Psi}{\partial x^2} - \frac{n + 2}{x^2} \frac{\partial \Psi}{\partial x} + \frac{2(n + 1)}{x^3} \Psi \right] dx. \tag{34}$$

Up on choosing the operator ordering index $n = -1$, \[34\] turns out to be,

$$\int (\hat{H}_1 \Psi)^* \Psi dx = -\frac{\hbar^2}{54\beta_0 \phi} \int \Psi^* \left[ \frac{1}{x} \frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{x^2} \frac{\partial \Psi}{\partial x} \right] dx = \int \Psi^* \hat{H}_1 \Psi dx. \tag{35}$$
Thus, \( \hat{H}_1 \) is hermitian, for a particular choice of operator ordering parameter \( n = -1 \). Now since it is trivial to prove that \( \hat{H}_2 \) is hermitian, let us consider the third term, viz. \( \hat{H}_3 \),

\[
\int (\hat{H}_3 \Psi)^* \Psi d\phi = -\frac{2i\hbar}{\sigma} \int \left( \frac{\alpha_1}{\phi^2} + 2\alpha_2\phi + \frac{2\beta_01x^2}{\sigma^2} \right) \frac{\partial \Psi^*}{\partial \phi} \Psi d\phi - \frac{2i\hbar}{\sigma} \int \left( \alpha_2 - \frac{\alpha_1}{\phi^3} \right) \Psi^* \Psi d\phi.
\] (36)

Under integration by parts and dropping the integrated out terms due to fall-off condition, we obtain,

\[
\int (\hat{H}_3 \Psi)^* \Psi d\phi = \frac{2i\hbar}{\sigma} \int \Psi^* \left( \frac{\alpha_1}{\phi^2} + 2\alpha_2\phi + \frac{2\beta_01x^2}{\sigma^2} \right) \frac{\partial \Psi}{\partial \phi} d\phi + \frac{2i\hbar}{\sigma} \int \left( \alpha_2 - \frac{\alpha_1}{\phi^3} \right) \Psi^* \Psi d\phi = \int \Psi^* \hat{H}_3 \Psi d\phi,
\] (37)

indicating \( \hat{H}_3 \) is hermitian too. Thus, the effective Hamiltonian \( \hat{H}_e \) turns out to be a hermitian operator. The hermiticity of \( \hat{H}_e \) now allows one to write the continuity equation in its standard form as,

\[
\frac{\partial \rho}{\partial \sigma} + \nabla \cdot \mathbf{J} = 0,
\] (38)

in the following manner. This requires to find \( \frac{\partial \rho}{\partial \sigma} \), where, \( \rho = \Psi^* \Psi \), is the probability density and \( \mathbf{J} \) is the current density. A little algebra leads to the following equation,

\[
\frac{\partial \rho}{\partial \sigma} = -\frac{\partial}{\partial x} \left[ \frac{i\hbar}{54\beta_01\phi x} (\Psi \Psi^* - \Psi^* \Psi) \right] - \frac{\partial}{\partial \phi} \left[ \frac{i\hbar}{3\sigma^2} (\Psi^* \Psi - \Psi \Psi^*) - \frac{2}{\sigma} \left( \frac{\alpha_1}{\phi^2} + 2\alpha_2\phi + \frac{2\beta_01x^2}{\sigma^2} \right) \Psi^* \Psi \right]
\] (39)

Clearly, the continuity equation can be written, only under the choice \( n = -1 \) as,

\[
\frac{\partial \rho}{\partial \sigma} + \frac{\partial \mathbf{J}_x}{\partial x} + \frac{\partial \mathbf{J}_\phi}{\partial \phi} = 0,
\] (40)

where the current density \( \mathbf{J} = (\mathbf{J}_x, \mathbf{J}_\phi, 0) \), where,

\[
\mathbf{J}_x = \frac{i\hbar}{54\beta_01\phi x} (\Psi \Psi^* - \Psi^* \Psi)
\] (41)

\[
\mathbf{J}_\phi = \frac{i\hbar}{3\sigma^2} (\Psi \Psi^* - \Psi^* \Psi) - \frac{2}{\sigma} \left( \frac{\alpha_1}{\phi^2} + 2\alpha_2\phi + \frac{2\beta_01x^2}{\sigma^2} \right) \Psi^* \Psi.
\] (42)

Here, as already mentioned, the variable \( \sigma \) plays the role of internal time parameter.

### 3.3 Semiclassical approximation:

Unitarity only proves the viability of a quantum equation in quantum domain. A quantum equation can only play an effective role in the physical world, if it admits an appropriate semiclassical approximation. Semiclassical approximation is essentially a method of finding an approximate wavefunction associated with a quantum equation. If the integrand in the exponent of the semiclassical wavefunction is imaginary, then the behaviour of the approximate wave function is oscillatory, and falls within the classical allowed region. Otherwise it is classically forbidden. Of-course, a quantum theory is justified, only when semiclassical approximation works, i.e. admits classical limit. Consequently, when the classical limit is admissible, most of the important physics are inherent in the classical action. A quantum theory therefore, may only be accepted as viable, if it admits and also found to be well-behaved under, an appropriate semiclassical approximation. To further justify the quantum equation \([24]\), in this context, we therefore need to study its behaviour under certain appropriate semiclassical limit in the standard WKB approximation. For this purpose it is much easier to handle the equation \([23]\), when it is expressed in the following form,
\[
\left[ -\frac{\hbar^2}{36\beta_0x} \left( \frac{\partial^2}{\partial x^2} + \frac{n}{x} \frac{\partial}{\partial x} \right) - \frac{\hbar^2}{2xz^2} \frac{\partial^2}{\partial z^2} - i\hbar \frac{\partial}{\partial z} + \frac{3i\hbar}{z} \left( \frac{\alpha_1}{\phi^2} + 2\alpha_2 + \frac{2\beta_0x^2}{z^2} \right) \frac{\partial}{\partial \phi} - \frac{3i\hbar}{z} \left( \frac{\alpha_1}{\phi^2} - \alpha_2 \right) + V \right] \Psi = 0,
\]

where
\[
V = \left[ \frac{3x}{2\sqrt{z}}(\alpha_0 - \alpha_2^2) + \frac{9x}{2\sqrt{z}} \left( \frac{\alpha_1^2}{\phi} + 4\alpha_1\alpha_2 + 4\alpha_2^2 \right) + \frac{18\beta_0x^5}{z^3} + \frac{18\beta_0x^3}{z^2} \left( \frac{\alpha_1}{\phi^2} + 2\alpha_2 \right) + \frac{\sqrt{z}}{x} \left( \frac{\alpha_1}{\phi^2} \right) \right] V_1 + V_0
\]

Equation (43) may be treated as time independent Schrödinger equation with three variables \((x, z, \phi)\), and therefore as usual, let us seek the solution of equation (43) as,

\[
\psi = \psi_0 e^{\frac{i}{\hbar}S(x,z,\phi)},
\]

and expand \(S\) in power series of \(\hbar\) as,

\[
S = S_0(x, z, \phi) + \hbar S_1(x, z, \phi) + \hbar^2 S_2(x, z, \phi) + ... .
\]

Now inserting the expressions (46) and (44) and their appropriate derivatives in equation (43) and equating the coefficients of different powers of \(\hbar\) to zero, one obtains the following set of equations (upto second order),

\[
\frac{\sqrt{z}}{36\beta_0x}S_{0,0}^2 + \frac{S_{0,0}^2}{2xz^2} + S_{0,z} - \frac{1}{z} \left( \frac{3\alpha_1}{\phi^2} + 6\alpha_2 + \frac{6\beta_0x^2}{z^2} \right) S_{0,\phi} + V(x, z, \phi) = 0,
\]

\[
-\frac{i\sqrt{z}}{36\beta_0x}S_{0,xx} - \frac{in\sqrt{z}}{36\beta_0x^2}S_{0,xx} - \frac{iS_{0,\phi\phi}}{2xz^2} + S_{1,zz} + \frac{\sqrt{z}S_{0,xx}S_{1,xx}}{18\beta_0x} + \frac{S_{0,\phi}S_{1,\phi}}{xz^2} - \frac{3i}{z} \left( \frac{\alpha_1}{\phi^3} - \alpha_2 \right) - \frac{1}{z} \left( \frac{3\alpha_1}{\phi^2} + 6\alpha_2 + \frac{6\beta_0x^2}{z^2} \right) S_{1,\phi} = 0,
\]

\[
-\frac{i\sqrt{z}S_{1,xx}}{36\beta_0x} - \frac{i\sqrt{z}S_{1,xx}}{36\beta_0x^2} + \frac{\sqrt{z}}{36\beta_0x} \left( S_{1,xx}^2 + 2S_{0,xx}S_{2,xx} + \frac{1}{2xz^3} \left( S_{1,\phi\phi}^2 + 2S_{0,\phi\phi}S_{2,\phi} \right) - \frac{i}{2xz^3} + S_{2,\phi} \right) - \frac{1}{z} \left( \frac{3\alpha_1}{\phi^2} + 6\alpha_2 + \frac{6\beta_0x^2}{z^2} \right) S_{2,\phi} = 0
\]

which are to be solved successively to find \(S_0(x, z, \phi)\), \(S_1(x, z, \phi)\) and \(S_2(x, z, \phi)\) and so on. Now identifying \(S_{0,xx}\) as \(p_x\); \(S_{0,zz}\) as \(p_z\) and \(S_{0,\phi}\) as \(p_\phi\) one can recover the classical Hamiltonian constraint equation \(H_c = 0\), given in equation (21) from equation (47). This identifies equation (47) as the Hamilton-Jacobi equation. Thus, \(S_0(x, z)\) can now be expressed as,

\[
S_0 = \int p_x dz + \int p_z dx + \int p_\phi d\phi
\]

apart from a constant of integration which may be absorbed in \(\psi_0\). The integrals in the above expression can be evaluated using the classical solution for \(k = 0\) presented in equation (49), the definition of \(p_x\) and \(p_\phi\) given in equation (13) and the definition \(p_x = Q\). Further, recalling the expression for \(Q\) given in (13), remembering the relation, \(x = \hat{z}\), where, \(z = \alpha^2\). Further, we choose \(n = -1\), since probability interpretation holds only for such value of \(n\). Using solution (49), \(\alpha(\phi), \beta_1(\phi), \beta_2(\phi), \) and \(\gamma(\phi)\), \(x(= \hat{z})\) and hence the expressions of \(p_x, p_z, p_\phi\)
can be expressed in term of \( x \), \( z \) and \( \phi \) as,

\[
\alpha' = -\left( \frac{\alpha_1}{\phi^4} + 2\alpha_2\phi \right) \tag{51a}
\]

\[
x = 2Hz \tag{51b}
\]

\[
p_x = 72\beta_0H^2a_0\phi_0 = \text{constant} \tag{51c}
\]

\[
p_z = -6\alpha_0H\sqrt{z} - \frac{9\alpha_1H}{a_0\phi_0}z - 6\beta_0H^3a_0\phi_0 + 12\gamma_0H^3a_0\phi_0 = -6\alpha_0H\sqrt{z} - \frac{9\alpha_1H}{a_0\phi_0}z - 72\beta_0H^3a_0\phi_0 \tag{51d}
\]

\[
p_\phi = \frac{6\alpha_1H_0a_0^3\phi_0^3}{\phi^5} + \frac{4\beta_0H^3a_0^3\phi_0^3}{\phi^3} - \frac{8\gamma_0H^3a_0^3\phi_0^3}{\phi^3} = \frac{6\alpha_1H_0a_0^3\phi_0^3}{\phi^5} + \frac{48\beta_0H^3a_0^3\phi_0^3}{\phi^5} \tag{51e}
\]

From the above expression of \( p_x \), it is apparent that \( x \) is a cyclic coordinate. Hence the integrals in \( 50 \) are evaluated as,

\[
\int p_x dx = 72\beta_0H^2a_0\phi_0x; \tag{52a}
\]

\[
\int p_z dz = -4\alpha_0Hz^2 - \frac{9\alpha_1H}{2a_0\phi_0}z^2 - 72\beta_0H^3a_0\phi_0z; \tag{52b}
\]

\[
\int p_\phi d\phi = \frac{3\alpha_1H_0a_0^3\phi_0^3}{2\phi^4} - \frac{24\beta_0H^3a_0^3\phi_0^3}{\phi^2}. \tag{52c}
\]

Therefore, explicit form of \( S_0 \) in terms of \( z \) is found as,

\[
S_0 = -4\alpha_0Hz^\frac{3}{2} - \frac{6\alpha_1H}{a_0\phi_0}z^2 + 48\beta_0H^3a_0\phi_0z. \tag{53}
\]

For consistency, one can trivially check that the expression for \( S_0 \) so obtained, satisfies equation \( 17 \) identically. In fact it should, because, equation \( 17 \) coincides with Hamiltonian constraint equation \( 21 \) for \( k = 0 \), and therefore is the Hamilton-Jacobi equation, as already mentioned. Moreover, one can also compute the zeroth order on-shell action \( 16 \). For example, using the relation of \( V_0, V_1, \alpha_2 \) and \( \gamma_0 \) and classical solution \( 9 \) one may express all the variables in terms of \( t \) and substitute in the action \( 16 \) to obtain,

\[
A = A_{cl} = \int \left[ -12\alpha_0H^2a_0^3\phi_0^3\phi e^{4Ht} - \frac{24\alpha_1H^2a_0^3}{\phi_0} e^{4Ht} + 96\beta_0H^4a_0^3\phi_0 e^{2Ht} \right] dt. \tag{54}
\]

Integrating we have,

\[
A = A_{cl} = -4\alpha_0H_0^3 e^{3Ht} - \frac{6\alpha_1H_0^3}{\phi_0} e^{4Ht} + 48\beta_0H^3a_0^3\phi_0 e^{2Ht}, \tag{55}
\]

which is the same as we obtained in \( 53 \), and at this end, the wave function is

\[
\Psi = \psi_0 e^{\int \left[ -4\alpha_0Hz^\frac{3}{2} - \frac{6\alpha_1H}{a_0\phi_0}z^2 + 48\beta_0H^3a_0\phi_0z \right].} \tag{56}
\]

### 3.4 First order approximation

Now for \( n = -1 \), equation \( 18 \) may be expressed as,

\[
-\frac{\sqrt{z}}{36\beta_0Ht} \left( iS_{0,xx} - 2S_{0,x}S_{1,x} - \frac{i}{x} S_{0,x} \right) - \frac{1}{2xz^2} \left( iS_{0,\phi} - 2S_{0,x}S_{1,\phi} \right) - \frac{3\alpha_1}{\phi^2z} + \frac{6\alpha_2\phi}{z} + \frac{6\beta_0H^3x^2}{z^3} \right) S_{1,\phi} \tag{57}
\]

\[-\frac{3i}{z} \left( \frac{\alpha_1}{\phi^3} - \alpha_2 \right) + S_{1,z} = 0,
\]

Using the expression of \( S_0 \) from \( 53 \), we can find \( S_{1,z} \) from the above equation as,

\[
S_{1,z} = \left[ C_1 \sqrt{z} + C_2 \sqrt{z} + C_3 \frac{z}{x} + C_4 \right] \left[ D_1 \sqrt{z} + D_2 z + D_3 z^\frac{3}{2} + D_4 \right]. \tag{58}
\]
where, \( C_1 = \left( \frac{\alpha_0}{a_0 a_0 a_0} + \frac{72\beta_0 H^2}{a_0 a_0} \right) \), \( C_2 = -\frac{27\alpha_1}{a_0 a_0}, \ C_3 = \frac{\gamma}{12}, \ C_4 = \frac{-12\alpha_2}{a_0 a_0}, \)
\( D_1 = \left( -\frac{\alpha_4}{24a_0 a_0 H a_0} + \frac{144\beta_0 H^2}{a_0 a_0} \right), \ D_2 = -\left( \frac{12\alpha_2}{a_0 a_0} + \frac{\alpha_1}{12\beta_0 H^2 a_0 a_0} \right), \ D_3 = -\frac{18\alpha_3}{a_0 a_0}, \) and \( D_4 = \frac{7}{5}. \)

On integration the form of \( S_1 \) is found as,

\[
S_1 = iF(z).
\]  

(59)

Therefore the wave function up to first-order approximation reads as,

\[
\Psi = \psi_0 e^{\int \left[-4\alpha_0 H^2 \frac{2}{a_0 a_0} - \frac{6\alpha_1 H^2}{a_0 a_0^2} + 48\beta_0 H^3 a_0 a_0 z\right]},
\]  

(60)

where,

\[
\psi_0 = \psi_0 e^{-F(z)}.
\]  

(61)

which only tells upon the pre-factor keeping the exponent part unaltered. We have therefore exhibited a technique to find the semiclassical wavefunction, on-shell. One can proceed further to find higher order approximations. Nevertheless, it is clear that higher order approximations too, in no way would affect the form of the semiclassical wavefunction, which has been found to be oscillatory around the classical inflationary solution. Since, the semiclassical wavefunction exhibits oscillatory behaviour and therefore strongly peaked around classical de-Sitter solution \(^9\). Thus we prove that the quantum counterpart of the action \( \{12\} \) produces a reasonably viable theory.

### 4 Inflation under Slow Roll Approximation

Inflation is a quantum phenomena and it must have occurred just very close to Planck’s era. In the previous sub-section we have mentioned that if a quantum theory admits a viable semiclassical approximation, then most of the important physics may be extracted from the classical action itself. Having proved the viability of the action \( \{12\} \) in the quantum domain, we now proceed to test inflation with currently released data sets in this regard \( \{20, 21\} \). For this purpose, let us rearrange the \( \{9\} \) and the \( \phi \) variation equations of Einstein, viz., \( \{7\} \) and \( \{8\} \) respectively as,

\[
-6\alpha H^2 - 6\alpha' \phi H - 36\beta_1 H^4 \left[ 4 \left( 1 + \frac{\dot{H}}{H^2} \right) + \frac{4}{H^2} \left( 1 + \frac{\dot{H}}{H^2} \right) + 2 \left( \frac{\dot{H}}{H^2} - \frac{2 \dot{H}^2}{H^2} \right) - \left( 1 + \frac{\dot{H}}{H^2} \right)^2 - 3 \right]  
\]

\[\]

\[
-72\beta_1' \phi H^3 \left[ 1 + \frac{\dot{H}}{H^2} \right] + 1 + 12\beta_2' \phi H^3 - 24\gamma' \phi H^3 + \frac{\dot{\phi}^2}{2} + V = 0
\]

\[
\dot{\phi} + 3H\phi = -V' + 6H' \left[ 1 + \frac{\dot{H}}{H^2} \right] + 36\beta_1 H^4 \left[ \left( 1 + \frac{\dot{H}}{H^2} \right)^2 + 2 \left( 1 + \frac{\dot{H}}{H^2} \right) + 1 \right]
\]

\[\]

\[
-12\beta_2 H^4 \left( 1 + \frac{\dot{H}}{H^2} \right) + 24\gamma' H^4 \left( 1 + \frac{\dot{H}}{H^2} \right).
\]

(62)

where, \( H = \frac{\ddot{a}}{a} \) denotes the expansion rate. Remember, we have already presented inflationary solutions of the classical field equations \( \{7\} \) and \( \{8\} \) in standard de-Sitter form in \( \{9\} \), which restricts the potential and the coupling parameters through conditions appearing in \( \{10\} \) and \( \{11\} \). Now, the standard slow-roll conditions in the minimally coupled theory read as, \( \dot{\phi}^2 \ll V \) and \( |\dot{\phi}| \ll 3H\phi \). However, due to the presence of coupling, which imposes additional degrees of freedom through coupling parameters, it is required to improvise the conditions by taking into account some additional conditions \( \{44\} \), viz. \( 4|\dot{\phi}|H \ll 1 \) and \( |\dot{\phi}| \ll |\dot{\phi}|H \). However, instead of standard slow roll parameters, we introduce a combined hierarchy of Hubble and coupling flow parameters in the following manner, which appears to be much suitable \( \{17\} \) \( \{33\} \) \( \{34\} \) \( \{45\} \) \( \{46\} \) \( \{47\} \) \( \{48\} \). Firstly, the background evolution of the
theory under consideration is described by a set of horizon flow functions (the behaviour of Hubble distance during inflation) starting from,

\[ \epsilon_0 = \frac{dH}{dN}, \text{ where } dH = H^{-1}, \]  

(64)

where, \( dH = H^{-1} \) is the Hubble distance, also called the horizon in our chosen units. We use suffix \( i \) to denote the era at which inflation was initiated. Now hierarchy of functions is defined in a systematic way as,

\[ \epsilon_{l+1} = \frac{d\ln|\epsilon_i|}{dN}, \quad l \geq 0. \]  

(65)

In view of the definition \( \mathcal{N} = \ln \frac{d}{\alpha} \), implying \( \mathcal{N} = H \), one can compute \( \epsilon_1 = \frac{d\ln dH}{dN} \), which is the logarithmic change of Hubble distance per e-fold expansion \( \mathcal{N} \), which is the first slow-roll parameter, \( \epsilon_1 = \dot{H} = -\frac{H}{H} \). This signals that the Hubble parameter almost remains constant during inflation. The above hierarchy allows one to compute \( \epsilon_2 = \frac{d\ln \epsilon_1}{dN} = \frac{1}{H} \dot{\epsilon}_1 \), which implies \( \epsilon_1 \epsilon_2 = d_1 \dot{d}_1 = -\frac{1}{H} \left( \frac{H}{H} - 2 \frac{H^2}{H^2} \right) \). In the same manner higher slow-roll parameters may be computed. Equation (65) essentially defines a flow in space with cosmic time being the evolution parameter, which is described by the equation of motion,

\[ \epsilon_0 \dot{\epsilon}_1 - \frac{1}{dH} \epsilon_1 \epsilon_{l+1} = 0, \quad l \geq 0. \]  

(66)

One can also check that (66) yields all the results obtained from the hierarchy defined in (65), using the definition (64). As already mentioned, additional degree of freedom appearing due to the function \( \alpha(\phi) \) is required to introduce yet another hierarchy of flow parameters as,

\[ \delta_1 = 4\alpha H \ll 1, \quad \delta_{i+1} = \frac{d\ln|\delta_i|}{dN}, \quad \text{with}, \quad i \geq 1, \]  

(67)

Clearly, for \( i = 1, \delta_2 = \frac{d\ln|\delta_1|}{dN} = \frac{1}{\delta_1} \frac{\dot{\delta}_1}{N}, \) and \( \delta_1 \delta_2 = \frac{4}{H} \left( \dot{\alpha}H + \alpha \ddot{H} \right), \) and so on. The slow-roll conditions therefore read as \( |\epsilon_i| \ll 1 \) and \( |\delta_i| \ll 1 \), which are analogous to the standard slow-roll approximation. In view of the slow-roll parameters, the above equations (62) and (68) may therefore be expressed as,

\[ -6\alpha H^2 - \frac{3}{2} \left( 1 + \delta_1 \right) + \frac{3}{2} - 36\beta_1 H^4 \left[ 3(1 - \epsilon_1)^2 - 2(1 + \epsilon_1 \epsilon_2) - 1 \right] - 72\beta'_1 \dot{\phi} H^3 \left[ (1 - \epsilon_1) + 1 \right] + 12\beta''_1 \dot{\phi} H^3 
- 24\gamma' \dot{\phi} H^3 + \left( \frac{\phi^2}{2} + V \right) = 0, \]  

(68)

and

\[ \ddot{\phi} + 3H \dot{\phi} = -V' + 6\alpha H^2 \left[ 3 - (1 + \epsilon_1) \right] + 36\beta_1 H^4 \left[ (1 - \epsilon_1)^2 + 2(1 - \epsilon_1) + 1 \right] - 12\beta'_1 H^4 (1 - \epsilon_1) + 24\gamma' H^4 (1 - \epsilon_1), \]  

(69)

respectively. Under slow-roll approximation, Eqs. (68) and (69) may finally be approximated to,

\[ H^2 \simeq \frac{V}{6\alpha}, \]  

(70)

and

\[ H \dot{\phi} \simeq -\frac{1}{3} V' + 4\alpha' H^2, \]  

(71)
Now, combining the above equations (70) and (71) we get, \( \dot{\phi} = \frac{V'}{3H} \). Therefore, the number of e-folds at which the present Hubble scale equals the Hubble scale during inflation, may be computed as usual in view of the following relation:

\[
N(\phi) \simeq \int_{t_i}^{t_f} \frac{H dt}{\dot{\phi}} = \int_{\phi_i}^{\phi_f} \frac{H d\phi}{\dot{\phi}} \simeq \int_{\phi_i}^{\phi_f} \left( \frac{3H^2}{V} \right) d\phi, \tag{72}
\]

where, \( \phi_i \) and \( \phi_f \) denote the values of the scalar field at the beginning \( (t_i) \) and the end \( (t_f) \) of inflation. Now, the number of e-foldings \( (72) \) reads as,

\[
N(\phi) = \int_{\phi_i}^{\phi_f} \frac{1}{4\alpha_1} \phi^2 d\phi = \frac{1}{12\alpha_1}(\phi_f^3 - \phi_i^3). \tag{73}
\]

The slow-roll parameters can now be written as:

\[
\epsilon = \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2 = \frac{M_P^2}{2\dot{\phi}^2 \left( 1 + \frac{\alpha_0 \phi}{2\alpha_1} \right)^2}, \tag{74}
\]

\[
\eta = \frac{M_P^2}{V} \frac{V''}{\dot{\phi}^2} = \frac{2M_P^2}{\dot{\phi}^2 \left( 1 + \frac{\alpha_0 \phi}{2\alpha_1} \right)}. \tag{75}
\]

Further, the expressions for the scalar to tensor ratio and the spectral index of scalar perturbation are given by \( r = 16\epsilon \) and \( n_s = 1 - 6\epsilon + 2\eta \) respectively. It is important to note that presence of other coupling parameters viz. \( \beta_1, \beta_2 \) and \( \gamma \) do not require additional slow roll conditions. Now, in view of all these expressions we compute the inflationary parameters and present them for different values of the parameter \( \alpha_1 \) and \( \alpha_0 \) in table 1 and table 2 respectively. We also present respective \( n_s \) versus \( r \) plots in figure 1 and figure 2.

Table 1 depicts that under the variation of \( \alpha_1 \) within the range \( 9M_P^3 \leq \alpha_1 \leq 10M_P^3 \), the spectral index of scalar perturbation and the scalar to tensor ratio lie within the range \( 0.96 \leq n_s \leq 0.97 \) and \( 0.385 \leq r \leq 0.0536 \) respectively, which show excellent agreement with the recently released data [20][21]. The number of e-folding varies within the range \( 41 < N \leq 45 \), which is sufficient to solve the horizon and flatness problems. For better perception, we present the spectral index of scalar perturbation versus the scalar to tensor ratio plot in figure 1. On the other hand, table 2 depicts that under the variation of \( \alpha_0 \) within the range \( -1.86M_P^3 \leq \alpha_0 \leq -2.0M_P^3 \), the spectral index of scalar perturbation and the scalar to tensor ratio again lie within the range \( 0.96 \leq n_s \leq 0.97 \) and \( 0.385 \leq r \leq 0.0536 \) respectively, which are exactly the same as depicted in table 1, and hence show excellent agreement with the recently released data [20][21]. The number of e-foldings remains fixed at \( N \approx 45 \), which is again sufficient to solve the horizon and flatness problems. Here again, for the sake of visualization we present the the spectral index of scalar perturbation versus the scalar to tensor ratio plot in figure 2.

| \( \alpha_1 \) in \( \text{M}_P^3 \) | \( \phi_f \) in \( \text{M}_P \) | \( n_s \) | \( r \) | \( N \) |
|---|---|---|---|---|
| 9.1 | 0.7727 | 0.9702 | 0.0367 | 45 |
| 9.2 | 0.7718 | 0.9690 | 0.0385 | 45 |
| 9.3 | 0.7710 | 0.9681 | 0.0403 | 44 |
| 9.4 | 0.7702 | 0.9670 | 0.0434 | 44 |
| 9.5 | 0.7694 | 0.9658 | 0.0444 | 43 |
| 9.6 | 0.7686 | 0.9645 | 0.0465 | 43 |
| 9.7 | 0.7678 | 0.9632 | 0.0488 | 42 |
| 9.8 | 0.7671 | 0.9619 | 0.0512 | 42 |
| 9.9 | 0.7664 | 0.9605 | 0.0538 | 41 |
| 10.0 | 0.7663 | 0.9590 | 0.0565 | 41 |

Table 1: Data set for the inflationary parameters taking \( \phi_i = 17\text{M}_P \) and \( \alpha_0 = -2.0\text{M}_P^3 \), and varying \( \alpha_1 \).

Figure 1: This plot depicts almost smooth variation within experimental limit of \( n_s \) with \( r \), varying \( \alpha_1 \).
5 Concluding remarks

In the present manuscript, perhaps the most general action upto curvature squared term, in the minisuperspace model guided by cosmological principle, has been considered to open a possible window through which a glimpse of the very early universe might enable us to acquire an intuitive picture with certain insights. The fact that the action admits vacuum de-Sitter solution assures the viability of the action as a first check. Interestingly, the solutions so obtained admit identical forms as in [36] without $R_{\mu\nu}^2$ and Gauss-Bonnet-dilatonic coupled terms. It may be noticed that the de-Sitter solution restricts the potential in such a manner that $|V'(\phi)\gamma| \ll 0$ for a wide range of the value of the scalar field $\phi$. Therefore, and one does not require additional assumption, such as $\phi \approx 0$, as in the case of minimally coupled theory in the background of GTR, which is procured under appropriate initial/boundary condition on the wavefunction. It is also interesting to note that the form of the potential so obtain, tends to become flat for large value of the scalar field $\phi$, admitting slow roll. Phase-space structure of the action has been presented executing MHF, and its canonical quantization is performed. The effective Hamiltonian operator is hermitian which is the necessary and sufficient condition for unitarity. Thus, despite the presence of $R_{\mu\nu}^2$ term, the action is non-perturbatively well behaved. Although as usual, time ceases to exist, nevertheless, an internal parameter (the proper volume) plays its role, which allows to establish the standard probabilistic interpretation. Based on the only primitive prediction that the universe is approximately classical when it is large, the semiclassical approximation (on-shell) has been performed. The approximate wave-function so obtained, exhibits oscillatory behaviour about classical inflationary solution, i.e. it is peaked about the de-Sitter solutions so obtained admit identical forms as in [36] without $R_{\mu\nu}^2$ and varying $\alpha_0$. The initial value of the scalar field $\phi_i \sim 17M_P$ that drives the inflation in the present model on the contrary, implies it must have occurred earlier, in the Trans-Planck era. However, it ends in the post-Planck era.

In the presence of dilatonic coupling, a combined hierarchy of Hubble and GaussBonnet flow parameters was required to introduce, since additional condition apart from the standard slow roll condition was necessary [33, 34, 45, 46, 47, 48]. Even, in the case of a non-minimally coupled scalar-tensor theory of gravity in the presence of scalar curvature squared term with constant coupling (without Gauss-Bonnet term) again a combined hierarchy of Hubble and non-minimal flow parameters was required [36]. It therefore appears that additional conditions are required corresponding to the number of coupling parameters present in the theory. However, although in the present case, we have a pair of on-shell independent functional coupling parameters ($\alpha(\phi)$ and $\gamma(\phi)$, say, since the rest are related through equation (14)), we do not need any further condition. In fact, a combined hierarchy of Hubble and non-minimal flow parameters has been found to be enough to evaluate inflationary parameters. Particularly, the hierarchy of the GaussBonnet flow parameter is not required any more. The inflationary parameters obtained have excellent agreement with the latest released Planck’s data [20, 21], as depicted in figure-1 and figure-2.

Last but not the least important finding is, slow roll approximation with the additional condition on non-minimal flow parameter, leads to the same approximate classical equations as in our earlier work [36]. This means, that Gauss-Bonnet term does not play any role in exhibiting inflation. This consequently might allow the Gauss-Bonnet term to play an important role at the late-stage of cosmic evolution [18, 19].
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A Canonical formulation by Dirac’s constraint analysis:

The aim of the appendix is to show that if one initiates Dirac formalism of constraint analysis only after taking care of the divergent terms appearing in the action, then it leads to identical Hamiltonian \([21]\) as obtained following ‘Modified Horowitz’ Formalism’. We therefore integrate the appropriate terms appearing in the action \([13]\) by parts, to express the point Lagrangian in view of \([14]\), in the following form,

\[
L = \left[ \left( -3\alpha' \phi \sqrt{2} - \frac{3kN}{2N\sqrt{z}} + \frac{6kN\alpha \sqrt{z}}{2N}\right) \right] + \frac{9\beta_1}{\sqrt{z}} \left( \frac{x^2}{N} - \frac{2kNz^2}{z} + \frac{2kN^2}{N} + 4k^2N \right) \\
- \frac{36\beta' kx \dot{\phi}}{\sqrt{z}} + \frac{\beta_2 x^2}{2N\sqrt{z} + 6kz \dot{\phi}} - \frac{\gamma \dot{x} \dot{\phi}}{\sqrt{z}} \left( \frac{x^2}{z} + 12k \right) + 2k \frac{\dot{\phi}^2}{2N} - VN \right] \tag{76}
\]

Now to initiate Dirac formalism, we substitute \(\dot{z} = N x\), i.e.; \(\ddot{z} = N \dot{x} + \dot{N} x\), so that the point Lagrangian may be expressed in the following form,

\[
L = \left[ \left( -3\alpha' \phi \sqrt{2} - \frac{3kN^2}{2\sqrt{z}} + 6kN\alpha \sqrt{z} \right) \right] + \frac{9\beta_1}{\sqrt{z}} \left( \frac{x^2}{N} + \frac{2kNz^2}{z} + 4k^2N \right) - \frac{36\beta' kx \dot{\phi}}{\sqrt{z}} \\
+ \frac{\beta_2 x^2}{2N\sqrt{z}} + 6kx \dot{\phi} - \frac{\gamma \dot{x} \dot{\phi}}{\sqrt{z}} \left( \frac{x^2}{z} + 12k \right) + 2k \frac{\dot{\phi}^2}{2N} - VN \right] \tag{77}
\]

where the expression \(\frac{\dot{z}}{N} - x\) is treated as a constraint and therefore introduced through the Lagrangian multiplier \(u\) in the above point Lagrangian. The canonical momenta are,

\[
p_x = \frac{18\beta_1 \dot{x}}{N \sqrt{z}}; \quad p_z = \frac{u}{N}; \quad p_{\phi} = 3kN \sqrt{2} x - 3\beta_1 \sqrt{2} x^2 + \frac{36\beta' kx \dot{\phi}}{N \sqrt{z}} - \frac{\gamma \dot{x}}{\sqrt{z}} \left( \frac{x^2}{z} + 12k \right) + 2k \frac{\dot{\phi}^2}{N \sqrt{z}} - \frac{\dot{\phi}^2}{N} \tag{78}
\]

Therefore, the primary constraint Hamiltonian reads as,

\[
H_{p_1} = N \left[ \frac{\sqrt{2} p_z^2}{\beta_1 \sqrt{2} x^2} + \frac{p_{\phi}^2}{\beta_1 \sqrt{2} x^2} + \frac{3kN \sqrt{2} x \dot{\phi}}{\beta_1 \sqrt{2} x^2} + \frac{6k \beta' kx \dot{\phi}}{\beta_1 \sqrt{2} x^2} + \frac{3k \beta' \sqrt{2} x \dot{\phi}}{\beta_1 \sqrt{2} x^2} + \frac{\gamma \dot{x}}{\sqrt{z}} \left( \frac{x^2}{z} + 12k \right) + 2k \frac{\dot{\phi}^2}{2N \sqrt{z}} - \frac{9\alpha \sqrt{2} x^2}{2N \sqrt{z}} \right] \\
+ 3k \left( \frac{x^2}{2N \sqrt{z} - 2k \sqrt{z}} \right) + \frac{10k \alpha' \beta_1^2 x^2}{\beta_1 \sqrt{2} x^2} - \frac{3k \beta' \beta_2 x \dot{\phi}}{\beta_1 \sqrt{2} x^2} + \frac{\gamma \dot{x}}{\sqrt{z}} \left( \frac{x^2}{z} + 12k \right) + 2k \frac{\dot{\phi}^2}{2N \sqrt{z}} - \frac{64k^2 \beta_1^2 x^2}{\beta_1 \sqrt{2} x^2} \tag{79}
\]

Now introducing the constraints \(\phi_1 = N \dot{p}_z - u = 0\) and \(\phi_2 = p_u = 0\) through the Lagrange multipliers \(u_1\) and
Note that the Poisson brackets \( \{ z, p_z \} = \{ \phi, p_\phi \} = \{ u, p_u \} = 1 \), hold. Now constraints should remain preserved in time, which are exhibited through the following Poisson brackets

\[ \dot{\phi}_1 = \{ \phi_1, H_{p_1} \} = -u_2 - N \frac{\partial H_{p_1}}{\partial z} = 0 \Rightarrow u_2 = -N \frac{\partial H_{p_1}}{\partial z}; \quad \dot{\phi}_2 = \{ \phi_2, H_{p_1} \} \approx 0 \Rightarrow u_1 = x. \]  

Therefore the primary Hamiltonian is modified to

\[ H_{p_2} = N \left[ x p_x + \sqrt{\frac{z}{2 \sqrt{z}}} p^2 - \frac{3 \alpha' x p_\phi}{z} - \frac{36 k^2 \beta_1' x p_\phi}{z^2} - \frac{\beta_2 p_\phi}{z^2} \left( \frac{x^3}{2 z^2 \sqrt{z}} + \frac{6 k x}{z^2 \sqrt{z}} \right) + \frac{\gamma' x p_\phi}{z^2} \left( \frac{x^2}{z} + 12 k \right) + \frac{9 \alpha'^2 x^2}{2 \sqrt{z}} \ight] \]

\[ + 3 \alpha \left( \frac{x^2}{2 \sqrt{z}} - 2 k \sqrt{z} \right) + \frac{108 k \alpha' \beta_1' x^2}{z^2 \sqrt{z}} - \frac{3 \alpha' \beta_2^2 x}{z^2} \left( \frac{x^3}{2 z^2 \sqrt{z}} + \frac{6 k x}{z^2 \sqrt{z}} \right) + \frac{3 \alpha' \gamma' x^2}{z^2} \left( \frac{x^2}{z} + 12 k \right) + \frac{648 k^2 \beta_1^2 x^2}{z^2 \sqrt{z}} \]

\[ - \frac{18 k \beta_1^2 x}{z^2} \left( \frac{x^3}{2 z^2 \sqrt{z}} + \frac{6 k x}{z^2 \sqrt{z}} \right) \left( \frac{x^2}{z} + 12 k \right) + \frac{\gamma' x^2}{z^2} \left( \frac{x^2}{z} + 12 k \right)^2 + z \frac{1}{2} \frac{\partial H_{p_1}}{\partial z} \]

As the constraint should remain preserved in time in the sense of Dirac, so

\[ \dot{\phi}_1 = \{ \phi_1, H_{p_2} \} = -N \left[ \frac{\partial H_{p_2}}{\partial z} - N p_u \frac{\partial^2 H_{p_2}}{\partial z^2} \right] + N \frac{\partial H_{p_1}}{\partial z} \approx 0 \Rightarrow p_u = 0. \]  

Finally the phase-space structure of the Hamiltonian, being free from constraints reads as,

\[ H = N \left[ x p_x + \sqrt{\frac{z}{2 \sqrt{z}}} p^2 - \frac{3 \alpha' x p_\phi}{z} - \frac{36 k^2 \beta_1' x p_\phi}{z^2} - \frac{\beta_2 p_\phi}{z^2} \left( \frac{x^3}{2 z^2 \sqrt{z}} + \frac{6 k x}{z^2 \sqrt{z}} \right) + \frac{\gamma' x p_\phi}{z^2} \left( \frac{x^2}{z} + 12 k \right) + \frac{9 \alpha'^2 x^2}{2 \sqrt{z}} \ight] \]

\[ + 3 \alpha \left( \frac{x^2}{2 \sqrt{z}} - 2 k \sqrt{z} \right) + \frac{108 k \alpha' \beta_1' x^2}{z^2 \sqrt{z}} - \frac{3 \alpha' \beta_2^2 x}{z^2} \left( \frac{x^3}{2 z^2 \sqrt{z}} + \frac{6 k x}{z^2 \sqrt{z}} \right) + \frac{3 \alpha' \gamma' x^2}{z^2} \left( \frac{x^2}{z} + 12 k \right) + \frac{648 k^2 \beta_1^2 x^2}{z^2 \sqrt{z}} \]

\[ - \frac{18 k \beta_1^2 x}{z^2} \left( \frac{x^3}{2 z^2 \sqrt{z}} + \frac{6 k x}{z^2 \sqrt{z}} \right) \left( \frac{x^2}{z} + 12 k \right) + \frac{\gamma' x^2}{z^2} \left( \frac{x^2}{z} + 12 k \right)^2 + z \frac{1}{2} \frac{\partial H_{p_1}}{\partial z} \]

\[ = N H, \]  

which is exactly same Hamiltonian as obtained in (24).