LEARNING FROM STRATEGIC AGENTS: Accuracy, Improvement, and Causality

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ABSTRACT

In many predictive decision-making scenarios, such as credit scoring and academic testing, a decision-maker must construct a model (predicting some outcome) that accounts for agents’ incentives to “game” their features in order to receive better decisions. Whereas the strategic classification literature generally assumes that agents’ outcomes are not causally dependent on their features (and thus strategic behavior is a form of lying), we join concurrent work in modeling agents’ outcomes as a function of their changeable attributes. Our formulation is the first to incorporate a crucial phenomenon: when agents act to change observable features, they may as a side effect perturb hidden features that causally affect their true outcomes. We consider three distinct desiderata for a decision-maker’s model: accurately predicting agents’ post-gaming outcomes (accuracy), incentivizing agents to improve these outcomes (improvement), and, in the linear setting, estimating the visible coefficients of the true causal model (causal precision). As our main contribution, we provide the first algorithms for learning accuracy-optimizing, improvement-optimizing, and causal-precision-optimizing linear regression models directly from data, without prior knowledge of agents’ possible actions. These algorithms circumvent the hardness result of Miller et al. (2019) by allowing the decision maker to observe agents’ responses to a sequence of decision rules, in effect inducing agents to perform causal interventions for free.

1 Introduction

We want algorithmic transparency in decisions that affect us. Transparency lets us audit models for fairness and correctness, and allows us to understand what decision we might receive if our features were different. Why, then, are some models kept hidden from the view of those subject to their decisions? Beyond setting-specific concerns like intellectual property theft or training-data extraction, the canonical answer is that transparency would allow strategic individual agents to “game” the model. These individual agents will act to change their features to receive a better decision. The decision-maker, meanwhile, makes decisions based on its prediction of individuals’ true future outcomes. The entire field of strategic classification (Hardt et al., 2016) is predicated on the idea that these changes to individuals’ features are superficial, and will not affect an individuals’ true outcomes.

However, in reality, individuals’ actions really may change their future outcomes. Consider the case of U.S. college admissions processes admitting students partially based on their volunteering in high school. One view of this is that volunteering is now a purely cosmetic exercise: once upon a time, those teenagers who volunteered were more socially conscious and thus would prove to be better community members (the future outcome that the admissions decision aims to predict).
But now, a large fraction of students who wish to attend selective colleges know to volunteer as a means for improving their admissions prospects, which has undermined the utility of volunteering as a predictive signal. Yet it may be that volunteering itself makes high school students more socially conscious, and thus prospective applicants become better community members because they volunteered. In this scenario, each college wants not only to identify the best students, but to incentivize students volunteer in order to actually improve the distribution of outcomes among the students it will encounter. This same dynamic appears in many decision settings where the decision-maker has a meaningful stake in the future outcomes of its subject population, including credit scoring, academic testing, and insurance.

In this work, we propose a new methodology for designing models that must make predictive decisions about strategic agents who will, by acting to change their decisions, change their true outcomes as well. Specifically, we provide methods for optimizing the accuracy of the model, the improvement of outcomes in the agent population, and the causal precision of the recovered model parameters. It turns out that this is surprisingly challenging: in very recent work, Miller et al. (2019) explore the problem through the lens of causal graphs, and prove that finding an improvement-optimizing decision rule in additive noise models is as hard as causal inference.

The core innovation of our work is that while we may not know how agents will respond to a decision rule, they will naturally respond to any rule we provide. Thus, as we test different decision rules and observe strategic agents’ responses, we can improve our model over time. In the language of causality, the advantage of this setting is that individuals gaming the decision function are constantly launching their own interventions on their features. They thus generate counterfactuals which we can observe and leverage to implicitly learn attributes of the causal graph. This allows us to circumvent the hardness result and maximize outcome improvement! Contrary to conventional wisdom, this means decision-makers are incentivized not only to be transparent, but to actively inform individuals about the decision model. This would lead those individuals to provide informative counterfactuals, and generally change their outcomes in a way that benefits both them and the decision-maker.

In addition to Miller et al. (2019), concurrent work by Haghtalab et al. (2020) analyzes a setting similar to ours, in which agents take optimal linear actions to respond to either a linear regression mechanism or a thresholded linear classifier. They provide procedures for deriving the improvement-maximizing binary classifier when features are only partially visible. However, unlike this work, they do not consider the accuracy achieved by the decision-maker, and further assume that the agents can (with some cost) manipulate every feature-space direction independently. In our methodology, we assume that decision makers do not a priori know the actions available to agents, and that actions have multiple effects, some of which may affect hidden features that the decision-maker never observes.

It turns out that the omitted variable problem is a crucial challenge when learning to incentivize improvement-inducing decision-rules. Traditional omitted variable bias (Greene, 2003) occurs when causal features are omitted from a model, leading regressions to overweight the importance of correlated features that are included. When making predictions about strategic agents, this is an even greater challenge: in order to change a visible feature, an agent may take an action that simultaneously affects both a visible feature and an outcome-affecting hidden feature. This means that a naive model may predict a very different change in outcome than would be expected from a model trained only on the original visible data. All of our methods are designed to succeed even when actions perturb these omitted causal variables.

In this work, we provide the first algorithms for leveraging strategic agents’ responses in order to learn linear regression models that maximize improvement, accuracy, and causal precision. We focus on the setting where agents face a norm-squared cost to perform actions, and where the
true future outcome is generated by a noisy linear combination of the features. Our algorithms all instruct the decision maker to choose a sequence of decision models to incentivize agents to change their features in highly informative directions.

As much of the prior literature has focused on the case of binary classification, it’s worth meditating on why we focus on regression. Many decisions, such as loan terms or insurance premiums, are not binary “accept/reject”s but rather lie somewhere on a continuum based on a prediction of a real-valued outcome. Furthermore, some seemingly-binary decisions, like selecting the top 10 applicants to interview for a competitive job position, can instead be better viewed as real-valued decisions that are post-processed into a ranking.

1.1 Related Work

The literature on strategic classification (Hardt et al., 2016) has canonically analyzed gaming through the lens of an adversarial zero-sum game between a decision-maker, who wishes to determine agents’ true pre-determined outcome, and agents, which wish to trick the decision-maker into making a favorable prediction about them, regardless of what their true outcome may be. Other work has shown how to find an optimal mechanism in a linear regression setting (Chen et al., 2018). Yet other recent works (Khajehnejad et al., 2019; D’Amour et al., 2020) explore methods for finding good decision rules for fixed-outcome agents by dynamically retraining on data-outcome pairs.

Separately, a budding literature (Ustun et al., 2019) has explored methods for providing algorithmic recourse: explaining to an individual how they may change their features to change the decision they receive. The philosophical basis for recourse is that it is inherently good to grant decision subjects agency over decisions that influence them. However, if we are to believe the strategic classification literature, agents’ changes to their features cannot change their true outcomes, meaning that a decision-maker’s accuracy can only ever worsen when agents can act to influence predictions made about them. While a decision-maker may be legally or socially compelled to provide recourse to decision subjects, they would never be incentivized to provide individuals agency over the decision they receive.

Crucially, if we instead understand an individual’s outcome to be a function of their features, we can see that a decision-maker can grant agency without losing accuracy, and (as we will explore) may even benefit from agency.

A few other works on adjacent problems merit mention. Kleinberg and Raghavan (2019) explore the feasibility of incentivizing individuals to make specific actions (without regard for the decision-maker’s accuracy), in a setting that is similar to ours except that they allow for a more general class of effort cost functions. They show that linear decision models generally suffice to incentivize any action profile incentivizable by a monotonic model. Liu et al. (2019) analyze the cross-group fairness of the equilibria resulting from agents of different groups responding differently to dynamically-updating binary classifiers.

Our work has several interesting connections to other research directions. Our problem can be understood as a type of adversarial machine learning defense (Chakraborty et al., 2018), where the agents are the adversaries, except that we are explicitly recognizing that some perturbations should actually change the outcome. The challenge of learning a model efficiently by selectively exploring regions of the data (in our case by incentivizing agents to game in certain directions) is studied in both the active learning (Cohn et al., 1996) and bandit learning literatures (Chu et al., 2011). More broadly, the challenge of aligning the incentives of the decision-maker and agents can be viewed as an instance of a principal-agent problem from economics. (For an excellent overview of these works’ connections to economics, see Kleinberg and Raghavan (2019).)
1.2 Contributions

In this work, we introduce a new setting for studying the performance of linear models that make continuous decisions about strategic agents. Our methodology incorporates the realities that agents’ actions causally affect their eventual outcomes, that a decision-maker can only observe a subset of agents’ features, and that agents’ actions are constrained to a subspace of the feature space. We require no prior knowledge of the agent feature distribution, or of the actions available to strategic agents, and require no knowledge of the true outcome function beyond that it is a noisy linear function of the feature vector.

We evaluate our algorithms with three criteria. The first, improvement, examines the utility received by the decision-maker. The second, accuracy, measures the accuracy of the learned regression. The third, causal precision, guarantees robustness to distributional changes.

In Section 3, we propose an algorithm for maximizing the expected future outcomes associated with each agent. This yields a general method for finding a decision rule that will incentivize agents to maximize nearly any measurable outcome, even without knowing how to achieve that outcome, so long as one has the time and incentivization power to get agents to explore. In Section 4, we propose an algorithm that can take any decision model and compute an adjustment that allows it to attain accuracy equivalent to the accuracy achievable without the presence of gaming. We show that this is optimal. That is, the optimal accuracy with gaming is equal to optimal accuracy without.

In Section 5, we show that in the case where all causal features are visible to the decision-maker, they can substantially improve their estimation of the true causal parameters governing the outcome. At a high-level, this is because by incentivizing agents to change their features in certain directions, we are able to increase the variance along dimensions of the feature space that had no variance in the non-gaming case. For example, if two features were perfectly correlated in the initial agent feature distribution, incentivizing agents to increase only one of these features will allow us to disambiguate between the causal effect of each of the features.

We conclude by discussing the implications and applicability of this methodology in the real world, and many promising directions for future work.

2 Problem Setup

2.1 Model and motivation

Let each individual agent be described by a set of features $x \in \mathbb{R}^d$, and let $P \in \Delta(\mathbb{R}^d)$ be the initial distribution of agents’ features, with covariance matrix $\Sigma$. Agents can choose an effort vector $a \in \mathbb{R}^m$ to change their features from $x$ to $x_g$, according to the following update rule: $x_g = x + Ma$, where the action matrix $M \in \mathbb{R}^{d \times m}$ is a matrix where entry $M_{i,j}$ corresponds to the change in the $i$th feature of $x$ as a result of spending one unit of effort along the $j$th direction of the effort space. Note that $M$ captures the setting where a single effort dimension perturbs multiple features simultaneously; for example, in the context of PhD progress evaluation, writing a paper may increase both the number of papers a student has published as well as their level of research skill, maybe. The paper-writing action might correspond to a column $M_1 = (1, 1)^T$, in which the two entries represent the action’s marginal impact on number of papers published and research skill, respectively. $M$ can also be low-rank, meaning some feature dimensions are not perturbable. Agents incur some cost $C(a)$ for each unit of effort. We will typically assume that agents incur

\footnote{We will assume that we have centered $P$, i.e. $\mathbb{E}_{x \sim P}[x] = 0$. We refer to its covariance matrix as $\Sigma = \mathbb{E}_{x \sim P}[xx^T]$.}
quadratic cost $C(a) = \frac{1}{2}||a||_2^2$. This corresponds to a setting with diminishing returns for effort invested in any particular action.\footnote{Note that we do not assume all actions have the same cost. We simply rescale the columns of $M$ to ensure that the units of effort in $a$ are interchangeable.}

Let $y$ be a random variable representing an agent’s true outcome, which we assume is decomposable into $y = f(x) + \eta$, where $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is deterministic and $\eta$ is a zero-mean independent sub-gaussian noise variable with variance $\sigma$. A decision-maker chooses their decisions based on a decision-making function $h(x) : \mathbb{R}^d \rightarrow \mathbb{R}$.

We assume that agents will respond optimally\footnote{One could alternatively consider a setting where some fraction of agents are not strategic. In this case our improvement and causal precision algorithms still apply with only minor adjustments.} to maximize the decision they receive from the model:

$$a_h(x) = \arg \max_{a \in \mathbb{R}^m} (h(x + Ma) - C(a)) \tag{1}$$

We define $P_h \in \mathcal{X}$ as the distribution of agents’ features resulting from drawing agents from distribution $P$ and then having them take action $a_h(x)$ to optimally game $h$.

The core question this work will explore is: how should a decision-maker choose the decision rule $h$? We assume the decision maker does not know the true outcome-yielding function $f$ nor the agent distribution $P$, nor do they know agents’ available actions $M$ (though the agents do, allowing them to maximize their internal utility). Further, the decision-maker can only measure and make decisions based on a subset of the agents’ features $Vx$, where $V$ is the projection from the original feature space to only the visible features. Specifically, $V$ is a diagonal matrix with $V_{ii} = 1$ if feature $i$ is visible to the decision maker, and $V_{ii} = 0$ when feature $i$ is not measured. In real life situations, there will often be hidden features that affect outcomes, but that a decision-maker cannot observe. Worse, these features may change as a side-effect of agents’ actions, and thus change the outcome, without any corresponding change in the visible features.

Rather than restricting our decision-maker to choose a model purely from observational data and then stick with it, we assume that they can update their new decision rule over time to incorporate new observations. Initially, the decision-maker observes some dataset of individuals who have not gamed, along with their outcomes $\{(Vx, y) \mid x \sim P, y \sim f(x) + \eta\}$. The decision-maker then chooses a decision rule $h_1$ and publicizes it to the agents. Next, fresh agents are drawn $x \sim P$, and game according to the current decision rule $h_1$ arriving at state $x_g$. The decision-maker only observes the agents’ features $Vx_g$ at decision time (not observing $Vx$) and then, after the decision has been made, the decision-maker observes their true outcome $y_g \sim f(x_g) + \eta$. The decision maker then uses these additional $(Vx_g, y_g)$ pairs, along with any previous information, to formulate a decision rule $h_2$, and the process repeats. Note that in practice, we generally view it as more difficult to switch to a new decision rule than to collect data about an additional agent, so we will often report algorithm complexity in rounds and samples-per-round separately.

We define two objectives which a decision-maker may want to optimize in choosing $h$:

**Objective 1.** Accuracy is how close the output of the model is to the true outcome. It is measured in terms of expected squared error:

$$\mathbb{E}_{x_g \sim P_h, y \sim f(x) + \eta} [(y - h(x))^2] \tag{2}$$

A decision-maker wants to optimize accuracy if they want the scores they assign to individuals to be as predictive of their true outcomes as possible. For example, insurers’ profitability is contingent on neither over- nor under-estimating client risk.
Objective 2. *Improvement* is the average outcome over the population:

$$\mathbb{E}_{x \sim P_h, \ y \sim f(x) + \eta} [y]$$

(3)

A decision-maker might care about maximizing improvement if it is in their interest for agents to achieve outcomes that are as positive as possible. For example, a teacher formulating a test for their students may care more about incentivizing the students to study the material than they care about accurately stratifying students based on their knowledge of the material.

Throughout this paper, we will be focusing on the case of linear decision-making functions:

$$h(x) = \omega^T V x + b,$$

where \( \omega \in \mathbb{R}^d \) and \( b \in \mathbb{R} \). We further assume that the outcome-producing function \( f \) is linear in both the visible and invisible features:

$$f(x) = \omega^* T x.$$  

Note that this setting, with the outcome function linear in the features and interventions in the form of linear perturbations, is equivalent to the well-studied additive noise model (Peters et al., 2014).

In this linear setting there is a natural third objective:

Objective 3. Let \( \omega^* \) be the true outcome-determining vector and let \( \omega \) be the weight vector of the decision-maker’s model. *Causal precision* measures how close the model coefficients are to the visible coefficients of the true linear model:

$$\| V(\omega - \omega^*) \|_2$$

(4)

To be clear, we are not using the word “causal” lightly: in the case where \( f \) is linear, \( \omega^* \) represents the true causal relationships between each visible feature and the outcome. A key advantage of this objective is that unlike the accuracy and improvement objectives, the causal precision of a model is invariant to the distribution of agents \( P \), their cost functions \( C \), and the action matrix \( M \).

One more definition: because the expression \( M M^T V \) is repeatedly used in our proofs, we let \( G = M M^T V \) as shorthand. (We will see that \( G \) maps \( \omega \) to the movement in feature vectors it incentivizes).

2.2 Illustrative example

Consider the following toy example of a decision-maker predicting creditworthiness of individuals via a linear model \( h(x) = (V \omega)^T x + b \). They attempt to approximate an individual’s true creditworthiness \( y \sim f(x) + \eta \), where \( f \) is a function of the following four features: (1) number of credit cards, (2) level of income, (3) number of dependents, and (4) level of conscientiousness. The first 3 features are visible to the decision-maker, while the last (conscientiousness) is not. Let \( \omega^* = (0, 1, 0, 1) \), that is, only changes in income and conscientiousness truly affect creditworthiness. Let the first action matrix column \( M_1 = (1, 0, 0, 0) \) be the act of opening a credit card, \( M_2 = (0, 1, 0, 0) \) be the act of increasing income one unit, and \( M_3 = (0, 0, 1, 1) \) be the act of gaining a dependent, which in this model, can only happen concurrently with becoming more conscientious. Our initial distribution \( x \) has covariance such that # credit cards and income are linearly correlated, and initially all agents have no dependents and zero conscientiousness (perhaps because the sample population is young and happens to be irresponsible).

Then the minimum \( L^2 \)-norm model that maximizes accuracy in the absence of gaming is \( V \omega_{acc-ng} = \left( \frac{1}{2}, \frac{1}{2}, 0, 0 \right) \). The unit-norm-bounded model that maximizes improvement is \( V \omega_{imp} = \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) \). The actions taken by agents in response to \( \omega_{imp} \) are shown in Figure 2.2. An accuracy-maximizing model with gaming is \( \omega_{acc} = (\alpha, 1 - \alpha, 0, 0) \) for any value \( \alpha \) of \( 1 > \alpha > 0 \), along with

Note that \( \alpha = 0 \) is the solution that maximizes accuracy even when some fraction of the population isn’t optimally gaming, though it forgoes any additional improvement due to \( M_3 \).
Figure 1: The four features of the illustrative example from Section 2.2 are shown in two 2D plots. When the improvement-maximizing decision rule \( \omega_{imp} \) is used as the decision function, agents are incentivized to increase their income level and number of dependents (and therefore conscientiousness), thereby improving their true credit-worthiness.

an appropriate bias term as in Theorem 2. Not only do all these criteria imply different models, but in fact none of them is proportional to the visible components of the causally precise model: \( V\omega^* = (0, 1, 0, 0) \). Clearly, there are non-trivial trade-offs in choosing a decision model.

3 Maximizing Improvement

In this section, we derive an algorithm for finding the decision function that incentivizes maximal improvement in agents’ true outcomes:

\[
\omega_{imp} = \arg \max_{\omega} \mathbb{E}_{x_g \sim P_{\omega}} \left[ \omega^T x_g \right]
\]

**Theorem 1.** Suppose the covariance matrix \( \Sigma \) has largest eigenvalue \( \lambda_{max} \), and suppose the outcome noise \( \eta \) is 1-subgaussian. Then Algorithm 4 learns an approximation of \( \omega_{imp} \) with squared \( \ell_2 \) error at most \( \epsilon \) in \( \text{rank}(V) + 1 \) rounds with \( O(e^{-1}\lambda_{max}\text{rank}(V) + 1) \) samples in each round. Furthermore, Algorithm 1 is non-adaptive. It succeeds and achieves the desired error with high probability.

**Proof.** For convenience, we let \( r = \text{rank}(V) \). First we note that it is straightforward to compute the action that each agent will take. The agent maximizes \( \omega^T V (x + Ma) - \frac{1}{2} \|a\|^2 \) over \( a \in \mathbb{R}^m \). Note that \( \nabla_a (\omega^T V Ma - \frac{1}{2} \|a\|^2) = M^T V \omega - a \). Thus,

\[
\arg \max_a \omega^T V (x + Ma) - \frac{1}{2} \|a\|^2 = M^T V \omega
\]

\[
\arg \max_a \omega^T V Ma - \frac{1}{2} \|a\|^2
\]

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5Note that the bias term \( b \) doesn’t affect improvement, so we ignore it in this section.

6We will require, for the related algorithms, that \( \|\omega\|_2 \leq 1 \) because without a bound on \( \omega \), the decision-maker could provide arbitrarily large incentives to agents, who would then take actions of arbitrary magnitude.
That is, every agent chooses \( x_g = x + MM^T V \omega = x + G \omega \). This means that if the decision-maker publishes \( \omega \), the resulting improvement value is \( \omega^*^T (x + G \omega) \). Hence, the optimal value of \( \omega \) for the decision-maker to choose is
\[
\frac{G^T \omega^*}{\|G^T \omega^*\|_2}
\]

Imagine for a moment that there were no outcome noise. We could directly make linear measurements of perfect quality by choosing \( \omega \)'s that form a basis. We do the same thing in Algorithm [1] just in a robust fashion. We now describe the algorithm.

In the first round we set \( \omega = 0 \) in order to obtain an estimate for \( E[\omega^*^T x] \). Since \( \omega \) is a unit vector, the variance of \( \omega^*^T x \) is at most \( \lambda_{\text{max}} \) plus a constant (from the 1–subgaussian noise). This means that \( O(\epsilon^{-1} r \lambda_{\text{max}}) \) samples suffice for the empirical estimator of \( E[\omega^*^T x] \) to have no more than \( \frac{\epsilon}{2^r} \) error with failure probability \( \Omega(\frac{1}{2^r}) \). We call the output of this estimator \( \hat{\mu} \) and let \( \hat{\mu}_r \) be the \( r \)-dimensional vector with \( \hat{\mu} \) in every coordinate.

Now we choose \( \omega_1, \ldots, \omega_r \) that form an orthonormal basis of the image of the diagonal matrix \( V \). For each \( \omega \) we observe the reward \( \omega^*^T (x + G \omega) + \eta \), subtract out \( \hat{\mu}_r \), and plug it into the empirical mean estimator. For each \( \omega_i \), let \( \hat{\omega}^* \) be the resulting coefficient. After \( O(r \lambda_{\text{max}}) \) samples, each coefficient has at most \( \frac{\epsilon}{2^r} \) error with failure probability at most \( \frac{1}{2^r} \). Since we have computed \( r + 1 \) estimators, each one with failure probability at most \( \frac{1}{2^r} \), a union bound gives us a total failure probability that is sub-constant.

We can now bound the total squared \( \ell_2 \) error between said coefficients and \( \omega_{\text{imp}} \) in the \( \omega_1 \ldots \omega_r \) basis (noting that the choice of basis does not affect the magnitude of the error). We can break up the error into two components using the triangle inequality: the error due to \( \hat{\mu}_r \) and the error in the subsequent rounds. Each coordinate of \( \hat{\mu}_r \) has magnitude at most \( \frac{\epsilon}{2^r} \), so the total magnitude of the error in \( \hat{\mu}_r \) is at most \( \frac{\epsilon}{2^r} \). The same argument applies for the error in the coordinate estimates, leading to a total squared \( \ell_2 \) error of at most \( \epsilon \).

We note that if our estimate is outside the unit ball, re-normalizing to within the unit ball only decreases the algorithm’s error.

This algorithm has several desirable characteristics. First of all, the decision-maker who implements the algorithm does not need to have any knowledge of \( M \) or even of the number of hidden features \( d - r \). Secondly, the algorithm is non-adaptive, in that the published decision function in each round does not depend on the results of previous rounds. Hence, the algorithm can easily be parallelized, for example, by simultaneously applying \( r \) separate decision functions to \( r \) separate segments of a population. Thirdly, the algorithm is able to incentivize outcome-improving movement in agents’ hidden features by assigning weight to visible features that are related in effort space. Finally, [Miller et al., 2019] proved that optimizing improvement is as hard as performing a non-trivial causal inference task. Our algorithm overcomes this barrier by incentivizing agents to perform the necessary causal interventions.

4 Maximizing Accuracy

In this section, we tackle the task of finding a decision making function \( h(x) = \omega^T x + b \) that has optimal accuracy. Specifically, we want to minimize expected squared error when predicting the true outcomes of agents who have gamed with respect to \( h \):

\[
\omega_{\text{acc}} = \arg\min_{\omega \in \mathbb{R}^d} \mathbb{E}_{x_g \sim p_g} \left[ \left( \omega^T V x_g + b - (\omega^* T x_g + \eta) \right)^2 \right]
\]
Algorithm 1 Improvement Maximization

Input: scalar parameters $\lambda_{\text{max}}, \epsilon$, matrix $V$

Let $n = O(\epsilon^{-1}\lambda_{\text{max}}r)$
Let $r = \text{rank}(V)$

Sample $x_1 \ldots x_n$ with $\omega = 0$.
Let $\hat{\mu} = \frac{1}{n} \sum x_i$

for $i = 1$ to $r$
    Sample $x_1 \ldots x_n$ with $\omega = e_i$
    Let $\hat{\omega}_i = \frac{1}{n} \sum x_i$
end for

if $||\hat{\omega}|| > 1$
    Let $\hat{\omega} = \frac{\hat{\omega}}{||\hat{\omega}||}$
end if

Return $\hat{\omega}$

We begin by noting a useful decomposition of accuracy in the linear setting:

**Lemma 1.** Suppose that $\mathbb{E}[x] = 0$. Let $h(x) = \omega^T x + b$ and let $a$ be the action taken by agents in response to $h$. Then the expected squared error of $h$ on the gamed distribution can be decomposed as the sum of the following two positive terms:

1. The expected squared error of $\omega$ on the original distribution
2. The squared error of $h$ in predicting the expected impact (on agents’ outcomes) of $a$.

Note that this lemma is true regardless of the cost function $C$.

**Proof.** Suppose $h = \omega^T x + b$. Then:

$$
\mathbb{E}_{x \sim P} \left[ \left( \omega^T V (x + Ma) + b - \omega^* T (x + Ma) - \eta \right)^2 \right]
$$

$$
= \mathbb{E} \left[ \left( (\omega^T Vx - \omega^* T x - \eta) + (\omega^T VMa + b - \omega^* T Ma) \right)^2 \right]
$$

$$
= \mathbb{E} \left[ (\omega^T Vx - \omega^* T x - \eta)^2 \right] + (\omega^T VMa + b - \omega^* T Ma)^2
$$

where the last line follows because $\mathbb{E}[x], \mathbb{E}[\eta] = 0$ and $a$ and $M$ do not depend on $x$. \hfill \Box

**Corollary 1.** The expected squared error of $\omega, b$ under gaming is always greater than or equal to the expected squared error of $\omega$ without gaming.

The above is an immediate consequence of Lemma 1 and is an unfortunate reality: transparency (which leads to gaming) can never improve linear regression accuracy, though it can lead to improvement in agent outcomes, and, as we discuss in Section 5, the presence of gaming may allow the decision-maker to design a function whose accuracy is more robust to shifts in $P$ or $M$. Perhaps surprisingly, we now show that transparency doesn’t worsen accuracy either:
**Theorem 2.** Suppose $p$ is a procedure that learns an $\omega_0$. Then there exists a procedure that learns a bias term $b \in \mathbb{R}$ such that $\omega_0, b$ achieves accuracy with gaming that is equivalent to the accuracy achieved by $\omega_0$ without gaming. In particular, $b$ can be learned with additive error $\leq \epsilon$ with $O(\frac{1}{\epsilon} \lambda_{\text{max}} + 1)$ samples in a single round. $\lambda_{\text{max}}$ is defined as in Theorem 1.

The above theorem essentially states that transparency and the gaming that comes with it have no inherent accuracy cost in the setting we propose. By slightly modifying a decision rule, we can recover the accuracy that would have been present in a setting with no transparency. Furthermore, the number of samples required to learn the modified decision rule is quite small.

**Proof.** We choose a value of $b$ to maximize the accuracy. That is:

$$b = \langle \omega^*, G\omega_0 \rangle - \langle V\omega_0, G\omega_0 \rangle$$

We discuss the procedure to estimate the value of $b$ in a later part of the proof.

We first evaluate the prediction of $\omega_0, b$ on $x_g$:

$$\langle V\omega_0, x_g \rangle = \langle V\omega_0, x \rangle + \langle V\omega_0, G\omega_0 \rangle + b$$

Which allows us to show that the accuracy of $\omega_0, b$ under gaming is equal to the accuracy of $\omega_0$ without gaming:

$$\text{accuracy with gaming} = \mathbb{E}[\langle (\omega_0, x_g) + b - \langle \omega^*, x_g \rangle \rangle^2$$

$$= \mathbb{E}(\langle V\omega_0, x \rangle + \langle \omega^*, G\omega_0 \rangle - \langle \omega^*, x \rangle - \langle \omega^*, G\omega_0 \rangle)^2$$

$$= \mathbb{E}(\langle V\omega_0, x \rangle - \langle \omega^*, x \rangle)^2$$

$$= \text{accuracy without gaming}$$

Now we show how to efficiently learn $b$. Recall from the proof of Theorem 1 that when proposing an $\omega$, for every sample, the decision maker receives $\omega^T(x + G\omega) + \eta = \omega^T x + \omega^T G\omega + \eta$. When $\omega = \omega_0$, subtracting the estimate for $\omega^T x$ obtained in Theorem 1 gives a random variable is exactly $b$ in expectation.

The same bounds computed in Theorem 1 apply. That is, we can estimate $b$ with at most additive error $\epsilon$ with $O(\frac{1}{\epsilon} \lambda_{\text{max}} + 1)$ samples.

By combining Corollary 1 and Theorem 2, we immediately obtain the following result:

**Corollary 2.** When the decision-maker uses our optimal algorithms, gaming, and thus transparency, do not affect accuracy. In particular, suppose $\omega$ achieves the optimal accuracy in the non-gaming setting. Then in the presence of gaming $\omega$ along with an efficiently learnable $b$ as defined in Theorem 2 constitute an accuracy-optimal decision function in the presence of gaming.

### 5 Maximizing Causal Precision

Finally, we provide an algorithm for improving the estimate of the true causal parameters of the outcome, specifically in the case where the features are fully visible ($V = I$). Estimating the causal parameters is desirable both because they deepen our understanding of the outcome-generating phenomenon. More relevantly, when used as a decision rule in the fully-visible case, the causally-precise define a decision rule that ensures maximal accuracy and improvement even when the distribution of agents $P$ or their actions $M$ shifts.
Theorem 3. (Informal) Given $V = I$ (all dimensions are visible) and $\Sigma + \lambda MM^T$ is full rank for some $\lambda$ (that is, there exist actions that will allow change in the full feature space), we can recover $\omega^*$ to arbitrary precision. We do so by computing an $\omega$ that results in more informative samples, and then gathering samples under that $\omega$. The procedure requires $\tilde{O}(d)$ rounds. See Appendix A for details.

The algorithm that achieves this result is actually simple to sketch. It consists of the following steps:

1. Estimate the covariance of the initial agent feature distribution before strategic behavior $\Sigma$ by initially not disclosing any decision rule to the agents, and observing their features.

2. Estimate parameters of the Gramian of the action matrix $G = MM^T$ by incentivizing agents to vary each feature sequentially.

3. Use this information to learn the decision function $\omega$ which will yield the most informative samples in identifying $\omega^*$, via a convex optimization.

4. Use the new, more informative samples in order to run OLS to compute an estimate of the causally precise regression parameters $\omega^*$.

At its core, this can be understood as running OLS after acquiring a better dataset via the smartest choice of $\omega$ (which is again, surprisingly, unique!). Whereas the convergence of OLS without gaming would be controlled by the minimum eigenvalue of the second moment, convergence of our method is governed by the minimum eigenvalue of following matrix:

$$
E[(x + G\omega)(x + G\omega)^T] = \Sigma + 2\mu\omega^T G^T + G\omega\omega^T G^T
$$

Our method learns a value of $\omega$ that results in the above matrix having a larger minimum eigenvalue, potentially resolving issues of rank and improving the convergence rate.

The proof and complete algorithm description is left to Appendix A.

6 Conclusion

In this work, we have introduced a model and techniques for analyzing decision-making about strategic agents. We provide algorithms for leveraging this strategic behavior to maximize improvement, accuracy, and causal precision.

Let us dwell on several real-world considerations that should inform the utility of these algorithms. First, while these algorithms eventually yield more desirable decision-making functions, they substantially reduce the decision-maker’s accuracy in the short term while exploration is occurring, and this tradeoff should inform their use. In general, these procedures make the most sense in scenarios with a fairly small period of delay between decision and outcome (e.g. predicting short-term creditworthiness rather than long-term professional success), as at each new round the decision-maker must wait this length of time to receive the first samples gamed according to the new rule. That said, our algorithms are either non-adaptive procedures, or have very limited adaptivity. This allows them to be parallelized in a straightforward fashion by using different decision rules for different agents.

It is also important that these methods only be applied to agents whose actions do not change their state in irreversible and possible detrimental ways. In particular, in the case of repeated decision-making (as in a credit scoring setting), an agent may make changes to respond to a temporary decision rule, only to realize their features leave them in a worse position when the
decision rule changes as part of the type of algorithm we’ve described. Only if agents have no future expectation of the consistency of the decision rule, or if they receive a decision only once (as in college admissions), can we be certain that the exploration induced by the decision rule is not exploitative. (After all, agents will only incur action cost if they actually benefit from the decision they’ll receive.)

As we have mentioned, our model has several notable social implications. First, in many settings, our results show that decision-makers are incentivized not only to be fully transparent, but to be actively informative. Sharing details about workings of their algorithm can potentially maximize both the decision-maker’s and agents’ utilities. More radically, agents may actually be incentivized to join together to construct a decision-maker if one does not exist. The agents themselves may wish to know \( \omega_{imp} \), and the way to do that is to aggregate their data, and have the agent-led decision-maker reward different agents for gaming in different directions, in order to more quickly identify the true causal parameters, as in Section 5.

This method can work to incentivize agents to act toward any desired outcome. This generality means it can be used for good, but also for ill. One can imagine an authoritarian state trying to figure out how to incentivize actions that will causally increase “loyalty to the state”. Our best hope is that such a decision rule is difficult to learn, or that it is not well-modeled by a linear function.

Our work opens up numerous avenues for future work. First, one can explore algorithms that work under a more general set of action cost models, or where different agents have different action costs, or where the actions available to agents are state-dependent. One can explore methods for minimizing the regret of decision-makers during the exploration phase of our algorithms. One could also explore the dynamics of decision rules when agents have persistent states across multiple decisions, like web content creators gaming the incentives of repeatedly-changing recommendation systems. One could also explore improving the efficiency of the algorithms we proposed. Finally, one could extend these results to the setting where both the decision rule, and the true outcome-producing function, are non-linear.

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A Causal Precision

In this section we describe how we recover $\hat{\omega}_{opt}$ in $L^2$-distance when there exists an $\omega$ such that $\Sigma + G\omega$ is full rank. Before we proceed we make a couple of observations. When there is no way to make the above matrix full rank, we cannot hope to recover the optimal $\hat{\omega}_{opt}$. If there is no natural variation in e.g. the last two features, and furthermore no agent can act along those features, it is not possible to disentangle their potential effects on the outcome. This also suggests that the causal precision is a more substantive demand for the decision maker than the standard linear regression setting. To discover this additional information, the decision maker can incentivize the agents to take actions that help the decision-maker recover the true outcome-governing parameters.

This motivates the algorithm we present in this section. It operates in two stages. First, it recovers the information necessary in order to identify the decision rule which will provide the most informative agent samples after those agents have gamed. Second, it collects data while incentivizing this action. Finally, it computes an estimate of $\hat{\omega}_{opt}$ using the collected data. We present the complete procedure in Algorithm 2.

Algorithm 2 Recovering the Causal Model

1: Let $k_1 = \lambda_{max}(G^T G)$ and $k_2 = ||\Sigma||^2$
2: Let $\kappa_{min} = \lambda_{min}(\Sigma)$
3: Choose an $\epsilon > 0$
4: Let $n_1 = O(\max(\frac{d\kappa_1}{\kappa_{min}}, \frac{d^2\kappa_2}{\kappa_{min}}))$
5: Collect samples $x_1, \ldots, x_{n_1}$
6: Let $\hat{\mu} = \frac{1}{n_1} \sum x_i$
7: Let $\hat{\Sigma} = \frac{1}{n_1} \sum x_i x_i^T$
8: Let $n_2 = O(\max(d^2||\hat{\mu}||^2 \text{tr}(\Sigma), d^3||G||^2 \text{tr}(\Sigma)))$
9: for $i = 1 \ldots d$
10: $\omega = e_i$
11: Sample $x_1, \ldots, x_{n_2}$ and subtract $\hat{\mu}$ from each one.
12: Let $\hat{G}_i = \frac{1}{n_2} \sum_{j=1}^{n_2} x_j$
13: end for
14: Let $\hat{\omega}_{opt} = \arg \min \hat{\Sigma} + 2\mu\omega^T \hat{G}^T + \hat{G}\omega\omega^T \hat{G}^T$
15: Let $n_3 = O(\frac{d}{\kappa_{min}})$
16: Sample $x_1, \ldots, x_{n_3}$ with $\omega = \hat{\omega}_{opt}$
17: Return the output of OLS on $x_1, \ldots, x_{n_3}$

The procedure in Algorithm 2 can be summarized as follows:

1. Estimate the first and second moments of the distribution of agents’ features.
2. Estimate the Gramian of the action matrix $G$.
3. Compute the most informative choice of $\omega$.
4. Collect samples under the most informative $\omega$ and then return the output of OLS.

Before we proceed to the proof of correctness of Algorithm 2 let us build some intuition for why this procedure of choosing a single $\omega$ and collecting samples under said $\omega$ makes sense. As
we show later, the convergence of OLS for linear regression can be controlled by the minimum eigenvalue of the second moment matrix of the samples. Our algorithm finds the value of \( \omega \) that, after agents game, maximizes this minimum eigenvalue in expectation. It turns out the minimum eigenvalue of the expected second moment matrix of post-gaming samples is convex with respect to the choice of \( \omega \). The convexity of the objective suggests that a priori, when choosing \( \omega \)s to obtain informative samples, the optimal strategy is choose a single specific \( \omega \).

The main difficulty in the rest of the algorithm is achieving the necessary precision in the estimation to be able to set up the above optimization problem to identify such an \( \omega \).

**Theorem 3.** When \( V = I \), the output of Algorithm 2 run with parameter \( \epsilon \) satisfies \( ||\omega - \omega^*|| \leq \epsilon \) with probability greater than \( \frac{2}{3} \).

The proof of Theorem 3 relies on several lemmas. First we bound the \( L^2 \) error of OLS as a function of the empirical second moment matrix in Lemma 2. Note that the usual bound for the convergence of OLS is distribution dependent. That is, the expected error is small.

**Lemma 2.** Assume \( V = I \). Consider samples \( x_1, \ldots, x_n \) and \( y_i = \hat{\omega}_{\text{opt}}^T x_i + \eta_i \). Let \( \omega \) be the output of OLS \((x_i, y_i)\). Then

\[
E_\eta \left[ ||\omega - \hat{\omega}_{\text{opt}}||^2 \right] \leq \frac{d}{n\kappa_{\text{min}}}
\]

The above proof is elementary and a slight modification of the standard textbook proof (see for example, (Liang, 2016)).

The proof also requires that the optimization to choose the optimal \( \omega \) is convex.

**Lemma 3.** The minimum eigenvalue of the following matrix is convex with respect to \( \omega \) for any values of \( x, G \).

\[
\sum_i (x_i + \hat{G}\omega)(x_i + \hat{G}\omega)^T
\]

Furthermore, when the following conditions are true, then the minimum eigenvalue of the above is within a constant factor of the optimal value.

\[
E[(x + G\omega)(x + \hat{G}\omega)^T].
\]

- \( ||\hat{\Sigma} - \Sigma||^2 \leq \epsilon \)
- \( ||\hat{\mu} - \hat{\mu}||^2 \leq \frac{\lambda_{\text{max}}(G^TG)\epsilon}{d} \)
- \( ||\hat{G} - G||^2 \leq \frac{\epsilon}{d||G||^2} \)
- \( ||\hat{G} - G||^2 \leq \frac{\epsilon}{d^2||G||^2} \)

Finally, the above holds true even for an \( \omega \) with distance at most \( O\left(\frac{1}{\text{poly}(d)}\right) \) from the optimum.

Finally, we use a minor lemma for recover of a random vector via the empirical mean estimator. Note that we treat the matrix \( G \) as a vector.

**Lemma 4.** Assume \( V = I \). Let \( g_1, \ldots, g_n \) be drawn from the distribution \( G_i + \xi \) and \( \hat{G} \) be the empirical mean estimator computed from said \( g_i \)'s. Let \( \Sigma \) be the expected second moment matrix of the \( \xi \)'s. Then

\[
E_\xi ||G - \hat{G}||^2 \leq \frac{d^2\text{tr}(\Sigma)}{n}
\]

We proceed with the proof of Theorem 3 below.
Proof. The first step of the algorithm is for recovering an estimate of \( \Sigma \) and \( \mu \). Note that \( n_1 \) samples suffice to recover \( \hat{\Sigma} \) and \( \hat{\mu} \) such that:

- \[ ||\hat{\Sigma} - \Sigma||^2 \leq \epsilon \]
- \[ ||\mu - \hat{\mu}||^2 \leq \frac{\lambda_{\text{max}}(G^TG)\epsilon}{d} \]

The for loop recovers an estimate of \( G \). Via Lemma 4, the samples suffice to ensure that the following two conditions hold:

- \[ ||\hat{G} - G||^2 \leq \frac{\epsilon}{d||\mu||^2} \]
- \[ ||\hat{G} - G||^2 \leq \frac{\epsilon}{d^2||G||^2} \]

Then the algorithm computes an estimate of the optimal \( \omega \). Via Lemma 5, we have that the optimum guarantees the minimum eigenvalue of an approximate solution will be within a constant factor of the optimum.

This \( \omega \) guarantees that \( n_3 \) samples suffice to ensure the recovery of \( \omega^* \) within squared \( L^2 \)-distance of \( O(\epsilon) \) in expectation.

Finally the expectations can be used with a Markov inequality to ensure the algorithm succeeds with (arbitrarily high) constant probability.

Now we prove the lemmas. We begin with Lemma 2. This proof is a slight modification of the textbook proof for the convergence of OLS.

Proof. In this section we derive a bound on the convergence of the least squares estimator when a fixed design matrix \( X \) is used. Note this is exactly the case we encounter, since the choice of \( \omega \) lets us affect the entries of the design matrix. This is a standard, textbook result and not a main contribution of the paper.

In order to state the result more formally we have to introduce some notation. The goal of the procedure is to recover \( \hat{\omega}_{\text{opt}} \), when given tuples \((x_i, \hat{\omega}_{\text{opt}}x_i + \eta)\) where \( \eta \) is 1-subgaussian. We aim to characterize \( ||\omega - \hat{\omega}_{\text{opt}}|| \) where \( \omega \) is obtained from ordinary least squares. Let \( X \) be the vector with the \( x_i \)'s in its columns. Let \( \kappa_{\text{min}} \) be the minimum eigenvalue of \( \frac{1}{n}X^TX \) (the second moment matrix).

Below all expectations are taken only over the random noise. We assume the second moment matrix is full rank.

\[
\mathbb{E}[||\omega - \hat{\omega}_{\text{opt}}||^2] \leq \mathbb{E}\left[ \frac{1}{n\kappa_{\text{min}}} (\omega - \hat{\omega}_{\text{opt}})X^TX(\omega - \hat{\omega}_{\text{opt}}) \right] \\
= \mathbb{E}\left[ \frac{1}{n\kappa_{\text{min}}} ||X(\omega - \hat{\omega}_{\text{opt}})||^2 \right] \\
= \frac{1}{n\kappa_{\text{min}}} \mathbb{E}[||X(X^TX)^{-1}X\hat{\omega}_{\text{opt}} + \eta - X\hat{\omega}_{\text{opt}}||^2] \\
= \frac{1}{n\kappa_{\text{min}}} \mathbb{E}[||X(X^TX)^{-1}X\eta||^2] \\
\leq \frac{d}{n\kappa_{\text{min}}}
\]

This motivates our procedure to learn the causal model. We do so in a fashion that attempts to maximize \( \kappa_{\text{min}} \). Note that it is the minimum eigenvalue that determines the convergence rate. This
is due to the fact that little variation along a dimension makes it hard to disentangle the features’ effect on the outcome via $\hat{\omega}_{opt}$ from the constant-variance noise $\eta$.

Lemma 3 is somewhat more involved. It is proven in three parts. The first is that the optimization problem is convex. The second is that approximate recovery of $S$, $\mu$, and $G$ suffice for approximately minimizing the original expression. The third is that an approximate solution suffices.

**Proof.** In this section we describe how to choose the value of $\omega$ that maximizes the value of $\kappa_{min}$ for the samples we obtain.

To do so, we examine the expectation of the second moment matrix and make several observations. Let $\Sigma$ denote the expected second moment matrix of $x$ (i.e. $\mathbb{E}[xx^T]$). We have:

$$\mathbb{E}[(x + G\omega)(x + G\omega)^T] = \Sigma + 2\mu\omega^T G^T + G\omega\omega^T G^T$$

1. The minimum eigenvalue of the above expression is concave with respect to $\omega$. This follows due to the following: $x + G\omega$ is a linear operator, the minimum eigenvalue of a Gramian matrix $X^T X$ is concave with respect to $X$, and the expectation of a concave function is concave (Boyd and Vandenberghe 2004).

2. Since the agent attempts to maximize their motion in the $\omega$ direction, we want to ensure that we move them toward toward the direction that maximizes the minimum eigenvalue of $\mathbb{E}[(x + G\omega)(x + G\omega)^T]$.

However, we do not operate with exact knowledge of $G$, etc. It turns out that even approximately solving this optimization problem with estimates for $G$, $\Sigma$, $\mu$ suffices for our purposes, as long as the $\omega$ we obtain from our optimization (using the estimates) results in a high value for the minimum eigenvalue of $\mathbb{E}[(x + G\omega)(x + G\omega)^T]$. Let $\hat{\omega}$ be the maximizing argument for the estimated optimization problem and let $\omega$ be the maximizing argument for the original optimization problem. Let $\hat{Q}$ be the true maximized second moment matrix including gaming, and $\hat{\Sigma}$ be the maximizing second moment matrix with gaming resulting from replacing the true $\Sigma$, $\mu$, $G$ with the estimates. In formal terms, we need to show the minimum eigenvalue of the following is large:

$$\mathbb{E}[(x + \hat{G}\hat{\omega})(x + \hat{G}\hat{\omega})^T].$$

We note that when $y^T \hat{Q}y$ is within $\epsilon$ of $y^T Qy$ for all $y$ in the unit ball, the minimum eigenvalues may differ by at most $\epsilon$.

$$||y^T \hat{Q}y - y^T Qy||^2 = ||y^T (\hat{Q} - Q)y||^2 \leq \lambda_{max}(\hat{Q} - Q)(y) ||y||^2 \leq ||\hat{Q} - Q||^2$$

And now we bound the norm of $||\hat{\Sigma} - \Sigma||^2$ assuming the following:

1. $||\hat{\Sigma} - \Sigma||^2 \leq \epsilon$
2. $||\mu - \hat{\mu}||^2 \leq \frac{\lambda_{max}(G^T G)\epsilon}{d}$
3. $||\hat{G} - G||^2 \leq \frac{\epsilon}{d||\mu||^2}$
4. $||\hat{G} - G||^2 \leq \frac{\epsilon}{d||G||^2}$
We work out the bound below.

\[ ||\hat{Q} - Q||^2 = ||\Sigma + 2\mu\hat{\omega}_{opt}^T G T + G\omega_{opt}\hat{\omega}_{opt}^T G T - (\hat{\Sigma} + 2\hat{\mu}\omega^T \hat{G} T + \hat{G}\omega_{opt}\hat{\omega}_{opt}^T \hat{G} T)||^2 \]
\[ \leq ||\Sigma - \hat{\Sigma}||^2 + 2||\mu\omega_{opt}^T G T - \hat{\mu}\hat{\omega}_{opt}^T \hat{G} T||^2 + \ldots \]
\[ \leq e + 2||\mu\omega_{opt}^T G T + \hat{\mu}\omega_{opt}^T G T - \hat{\mu}\omega_{opt}^T \hat{G} T||^2 + \ldots \]
\[ \leq e + d||\mu - \hat{\mu}||^2||\hat{\omega}_{opt}^T G||^2 + ||\hat{G} - G||^2||\hat{\mu}\omega^T||^2 + \ldots \]
\[ \leq e + d||\mu - \hat{\mu}||^2||\hat{\omega}_{opt}^T G||^2 + ||\hat{G} - G||^2||\hat{\mu}\omega^T||^2 + \ldots \]
\[ \leq e + d||\mu - \hat{\mu}||^2 + \ldots \]
\[ \leq 3e + ||\hat{G}\omega_{opt}\hat{\omega}_{opt}^T \hat{G} T - G\omega_{opt}\hat{\omega}_{opt}^T G T||^2 \]
\[ \leq 3e + ||\hat{G}\omega_{opt}\hat{\omega}_{opt}^T \hat{G} T - \hat{G}\omega_{opt}\hat{\omega}_{opt}^T \hat{G} T + \hat{G}\omega_{opt}\hat{\omega}_{opt}^T G - G\omega_{opt}\hat{\omega}_{opt}^T G T||^2 \]
\[ \leq 3e + ||(\hat{G} - G)\omega_{opt}\hat{\omega}_{opt}^T \hat{G} T||^2 + ||(\hat{G} - G)\hat{\omega}_{opt}\hat{\omega}_{opt}^T \hat{G} T||^2 + ||(\hat{G} - G)\omega_{opt}\hat{\omega}_{opt}^T G T||^2 \]
\[ \leq 4e + d^2||\hat{G} - G||^2 + ||(\hat{G} - G)\hat{\omega}_{opt}\hat{\omega}_{opt}^T \hat{G} T||^2 \]
\[ \leq 5e + d^2||\hat{G} - G||^2 \]
\[ \leq 6e \]

This means if we find an \( \epsilon \)–approximate solution to the system with the estimated values, we obtain a \( \kappa_{min} \) within \( 6\epsilon \) of the optimal.

Finally, we present the proof of Lemma \[3\].

**Proof.** Recall that when the decision-maker fixes \( \omega \), it receives samples of the form \( x + G\omega \). We note this can be used to recover the matrix \( G \). In particular, we show how \( d \) rounds, each with \( O\left(\frac{\text{tr}(\Sigma)}{\epsilon}\right) \) samples, suffices to recover the matrix to squared frobenius norm \( \epsilon \). Recall the procedure we propose simply chooses \( \omega = \epsilon_1, ... \epsilon_d \), one-hot coordinate vectors in each round. We first bound the error in \( \hat{G} \), coordinate-wise: \( \mathbb{E}[||\hat{G}_{ij} - G_{ij}||^2] \leq \frac{\mathbb{E}[\xi^2]}{n} \). A union bound across coordinates shows that \( O\left(\frac{\text{tr}(\Sigma)}{\epsilon^2}\right) \) samples suffice to recover \( G \) within squared Frobenius norm \( \epsilon \).