QUANTUM CONTINUOUS $\mathfrak{gl}_\infty$: TENSOR PRODUCTS OF FOCK MODULES AND $W_n$ CHARACTERS

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Abstract. We construct a family of irreducible representations of the quantum continuous $\mathfrak{gl}_\infty$ whose characters coincide with the characters of representations in the minimal models of the $W_n$ algebras of $\mathfrak{gl}_n$ type. In particular, we obtain a simple combinatorial model for all representations of the $W_n$-algebras appearing in the minimal models in terms of $n$ interrelating partitions.

1. Introduction

In [FFJMM] we initiated a study of representations of the algebra which we denote by $\mathcal{E}$ and call quantum continuous $\mathfrak{gl}_\infty$. This algebra depends on two parameters $q_1, q_2$ and is closely related to the Ding-Iohara algebra introduced in [DI] and then considered in [FHHSY], [FT]. Conjecturally, the algebra $\mathcal{E}$ is isomorphic to the infinite spherical double affine Hecke algebra constructed in [SchV1], [SchV2].

We argued, that the representation theory of the algebra $\mathcal{E}$ has many features similar to the representation theory of $\mathfrak{gl}_\infty$. In particular, it has a family of vector representations. The algebra $\mathcal{E}$ is not a Hopf algebra. However there is “a comultiplication rule” which under some conditions, defines an action of $\mathcal{E}$ on a tensor product of $\mathcal{E}$-modules. We used the comultiplication rule and the vector representations to construct Fock modules by the standard semi-infinite construction (which can also be called an inductive limit construction or semi-infinite wedge representation). Similarly to the case of $\mathfrak{gl}_\infty$, the Fock modules have a natural basis labeled by partitions.

On the other hand, due to the quantum nature of the algebra $\mathcal{E}$, the representation theory of $\mathcal{E}$ is richer than that of $\mathfrak{gl}_\infty$. In [FFJMM], we used the semi-infinite construction to define another family of modules for the case when $q_1^{1-r}q_3^{k+1} = 1$ with $r \in \mathbb{Z}_{>1}$, $k \in \mathbb{Z}_{>0}$. These modules do not have $\mathfrak{gl}_\infty$ analogs. Their bases are given by the so-called $(k, r)$-admissible partitions $\lambda$ satisfying $\lambda_i - \lambda_{i+k} \geq r$.

In [FFJMM] it was shown that, when the parameters $q, t$ satisfy $q^{1-r}t^{k+1} = 1$, the Macdonald polynomials $P_\lambda(q, t)$ with $(k, r)$-admissible partitions $\lambda$ form a basis of the smallest ideal in the space of symmetric polynomials stable under the Macdonald operators, see also [K], [SV]. The $\mathcal{E}$-module we constructed is an inductive limit of these ideals in an appropriate sense.

In this paper we continue the study of representations of $\mathcal{E}$ started in [FFJMM]. We find that this algebra has a large class of modules which are tame in the following sense. The algebra $\mathcal{E}$ has a commutative subalgebra generated by $\psi_i^+, \psi_i^-$, where $i \in \mathbb{Z}_{>0}$, which serves as an analog of Cartan subalgebra. We call a module tame if this subalgebra acts diagonally with the simple joint spectrum.
In contrast to \cite{FFJMM}, to construct new modules, we do not use the semi-infinite constructions but use the Fock spaces as building blocks. Namely, we consider subquotients of the tensor product of several Fock spaces.

The Fock space depends on a continuous parameter \( u \) and has a basis labeled by partitions. Therefore the tensor product of \( n \) Fock modules depends on \( n \) complex parameters \( u_i \), where \( i = 1, \ldots, n \), and has a basis labeled by \( n \)-tuples of partitions. The subalgebra of \( \psi_1^\pm \) acts diagonally in this basis. Remarkably, all submodules and quotient modules we study also have a basis of \( n \)-tuples of partitions with some conditions. It means that we never have to deal with linear combinations of the basic vectors and all considerations are purely combinatorial.

For generic parameters \( q_1, q_2, u_i \), where \( i = 1, \ldots, n \), the tensor products of Fock modules are irreducible. We consider special values which we call resonances. In the case of the resonances, the \( E \) action on the tensor product of the Fock spaces is not defined. However we find a “subquotient” for which it is well-defined. This subquotient is an irreducible \( E \)-module with a basis labeled by \( n \) partitions \( \lambda^{(1)}, \ldots, \lambda^{(n)} \) with conditions

\[
\lambda^{(i)}_s \geq \lambda^{(i+1)}_{s+b_i} - a_i, \quad \text{where } i = 1, \ldots, n, \ s \in \mathbb{Z}_{\geq 0}.
\]

Here \( \lambda^{(n+1)} = \lambda^{(1)} \) and \( a_i, b_i \in \mathbb{Z}_{\geq 0} \) are the parameters of the module. The conditions with \( i = 1, \ldots, n-1 \) correspond to the submodules appearing under resonances \( u_i/u_{i+1} = q_1^{a_i+1} q_3^{b_i+1} \), where \( q_3 = (q_1 q_2)^{-1} \). The \( i = n \) condition corresponds to the quotient module appearing under the resonance \( q_1^{p_i} q_3^{p_i} = 1 \), where \( p_i = \sum_{i=1}^n (a_i + 1) \), \( p = \sum_{i=1}^n (b_i + 1) \).

We use a recursion to compute the graded character of the subquotient and find that it coincides with the character of the \( \mathcal{W}_n \)-module from the minimal \((p', p)\) theory labeled by the \( \mathfrak{sl}_n \) weights \( \eta = \sum_{i=1}^n a_i \omega_i \) and \( \xi = \sum_{i=1}^n b_i \omega_i \), see Theorem 4.5. The recursion in the case \( n = 2 \) is similar to the recursion in \cite{ABBBFV}.

For the information on \( \mathcal{W}_n \) and its representations we refer to \cite{FKW}. The \( \mathcal{W}_n \)-modules are not well understood apart from the Virasoro \( n = 2 \) case and we hope that such a relatively simple combinatorial description will shed new light on the subject. Moreover, there are many indications that the relation between \( E \) and \( \mathcal{W}_n \) algebras is much deeper than the one described in this paper.

The paper is constructed as follows. In Section 2 we recall the main definitions and constructions from \cite{FFJMM}. In Section 3 we construct the subquotients of the tensor products of Fock modules and prove that they are tame and irreducible. In Section 4 we describe the recursion for the set of \( n \)-tuples of partitions with conditions, give a solution of the recursion in a bosonic form and compare it to the characters of \( \mathcal{W}_n \) algebras. In Section 5 we show several isomorphisms of \( E \)-modules, in particular, the isomorphism of the subquotients in the case \( p = n + 1 \) to the \( E \)-modules with bases of \((k, r)\)-admissible partitions of \cite{FFJMM}.

2. Quantum continuous \( \mathfrak{gl}_\infty \)

In this section we recall the basic definitions about quantum continuous \( \mathfrak{gl}_\infty \) and its representations following \cite{FFJMM}.

2.1. Algebra. Let \( q_1, q_2, q_3 \in \mathbb{C} \) be complex parameters satisfying the relation \( q_1 q_2 q_3 = 1 \). In this paper we assume that neither of \( q_i \) is a root of unity. Let

\[
g(z, w) = (z - q_1 w)(z - q_2 w)(z - q_3 w).
\]
The quantum continuous $\mathfrak{gl}_\infty$ is an associative algebra $\mathcal{E}$ with generators $e_i$, $f_i$, where $i \in \mathbb{Z}$, $\psi^+_i$, $\psi^-_i$, where $i \in \mathbb{Z}_{\geq 0}$, and $(\psi^+_i)^{-1}$ satisfying the following defining relations:

\begin{align}
(2.1) \quad g(z, w)e(z)e(w) &= -g(w, z)e(w)e(z), \quad g(w, z)f(z)f(w) = -g(z, w)f(w)f(z), \\
(2.2) \quad g(z, w)\psi^+_i(z)e(w) &= -g(w, z)e(w)\psi^+_i(z), \quad g(w, z)\psi^+_i(z)f(w) = -g(z, w)f(w)\psi^+_i(z), \\
(2.3) \quad [e(z), f(w)] &= \frac{\delta(z/w)}{g(1, 1)}(\psi^+(z) - \psi^-(z)), \\
(2.4) \quad [\psi^+_i, \psi^+_{j}] &= 0, \quad [\psi^+_i, \psi^+_{j}] = 0, \\
(2.5) \quad \psi^+_i(\psi^+_i)^{-1} &= (\psi^+_i)^{-1}\psi^+_i = 1, \\
(2.6) \quad [e_0, [e_1, e_{-1}]] &= 0, \quad [f_0, [f_1, f_{-1}]] = 0.
\end{align}

Here $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ denotes the formal delta function and the generating series of the generators of $\mathcal{E}$ are given by

\begin{align}
e(z) &= \sum_{i \in \mathbb{Z}} e_i z^{-i}, \quad f(z) = \sum_{i \in \mathbb{Z}} f_i z^{-i}, \quad \psi^+_i(z) = \sum_{i \in \mathbb{Z}_{\geq 0}} \psi^+_i z^{-i}.
\end{align}

Note that the algebra $\mathcal{E}$ depends on the unordered set of parameters $\{q_1, q_2, q_3\}$, as all $q_i$ enter the relations symmetrically through the function $g(z, w)$.

The algebra generated by $\{\psi^+_i, \psi^-_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is commutative. We call an $\mathcal{E}$-module $V$ tame if the operators $\{\psi^+_i, \psi^-_i\}_{i \in \mathbb{Z}_{\geq 0}}$ act by diagonalizable operators with simple joint spectrum.

The elements $\psi^+_0 \in \mathcal{E}$ are central and invertible. For $x, y \in \mathbb{C}^\times$, we call an $\mathcal{E}$-module $V$ of level $(x, y)$ if $\psi^+_0$ acts by $x$ and $\psi^-_0$ by $y$.

Let $\phi^+_0(z) \in \mathcal{E}$ be a formal series in $z^\pm 1$. We call an $\mathcal{E}$-module $V$ highest weight module with highest vector $v \in V$ and highest weight $\phi^+_0(z)$ if $v$ generates $V$ and $f(z)v = 0$, $\psi^+_i(z)v = \phi^+_0(z)v$, $\psi^-(z)v = \phi^-_0(z)v$.

The comultiplication rule, see [FFJMM], [DI], is given by

\begin{align}
(2.7) \quad \Delta e(z) &= e(z) \otimes 1 + \psi^-(z) \otimes e(z), \\
(2.8) \quad \Delta f(z) &= f(z) \otimes \psi^+(z) + 1 \otimes f(z), \\
(2.9) \quad \Delta \psi^\pm(z) &= \psi^\pm(z) \otimes \psi^\pm(z).
\end{align}

These formulas do not define a comultiplication in the usual sense since the right hand sides are not the elements of $\mathcal{E} \otimes \mathcal{E}$, as they contain infinite sums. However, under some conditions the comultiplication rule does give an $\mathcal{E}$-module structure on $V_1 \otimes V_2$. We will discuss these conditions somewhere else. In [FFJMM] and in this paper we use the comultiplication rule as a way to obtain the formulas for the $\mathcal{E}$ action and then we check directly that the answer is consistent with the relations in the algebra $\mathcal{E}$.

Note that this comultiplication rule is compatible with permutations of $q_i$.

We say that a set of complex parameters $\{v_1, \ldots, v_n\}$ is generic if no monomial in $v_i$ is equal to 1: $\prod_{i=1}^n (\psi^+_i)^{j_i} = 1$ with $j_i \in \mathbb{Z}$ implies that $j_1 = \cdots = j_n = 0$.

Clearly, $q_1, q_2$ are generic if and only if $q_1, q_3$ are generic and $q_1, q_2$ are generic if and only if $q_2, q_3$ are generic. If a set of parameters is generic then any subset of that set is also generic. Throughout
the paper we will assume that neither of $q_i$ is a root of unity. That is we always assume that $q_1$ is
generic, $q_2$ is generic, $q_3$ is generic.

The algebra $\mathcal{E}$ is $\mathbb{Z}$-graded. The degrees of generators are given by
\begin{equation}
\deg e_i = 1, \quad \deg f_i = -1, \quad \deg \psi_i^\pm = 0.
\end{equation}

In fact $\mathcal{E}$ is $\mathbb{Z}^2$-graded, see [FFJMM], Lemma 2.4, but we do not need the second component of the
grading in this paper.

2.2. Vector representations. For $u \in \mathbb{C}^\times$, let $V(u)$ be a complex vector space spanned by basis
$|i\rangle_u$, where $i \in \mathbb{Z}$.

We also use the notation $u\langle i|$ with $i \in \mathbb{Z}$ for the dual basis. When it is clear we skip the subscript
$u$ and write simply $|i\rangle$, $\langle i|$.

Lemma 2.1. The formulas
\begin{align*}
(1 - q_1) e(z) |i\rangle & = \delta(q_1^{-1} u/z) |i + 1\rangle, \\
(q_1^{-1} - 1) f(z) |i\rangle & = \delta(q_1^{-1} u/z) |i - 1\rangle, \\
\psi^+(z) |i\rangle & = \frac{(1 - q_1^{-1} u/z)(1 - q_1^{-1} q_2 u/z)}{(1 - q_1^{-1} u/z)(1 - q_1^{-1} q_3 u/z)} |i\rangle, \\
\psi^-(z) |i\rangle & = \frac{(1 - q_1^{-1} q_3^{-1} u/z)(1 - q_1^{-1} q_2^{-1} u/z)}{(1 - q_1^{-1} z/u)(1 - q_1^{-1} q_3^{-1} z/u)} |i\rangle,
\end{align*}
define a structure of an irreducible tame $\mathcal{E}$-module on $V(u)$ of level $(1,1)$.

Proof. These formulas define an $\mathcal{E}$-module by [FFJMM], [FHJ]. It is easy to check that it is
tame and irreducible. $\square$

We call the $\mathcal{E}$-module $V(u)$ the vector representation. Note that $q_1$ plays a special role in the
definition of $V(u)$ while $q_2$ and $q_3$ participate symmetrically. Therefore there are two other vector
representations obtained from $V(u)$ by switching roles of $q_i$. We do not use these other two modules
in this paper.

Note that the vector representation is not a highest weight representation.

The comultiplication rule defines an $\mathcal{E}$-module structure on the tensor product of vector representations
$V(u_1) \otimes \cdots \otimes V(u_n)$ for generic $q_1, q_2, u_1, \ldots, u_n$. This is again an irreducible representation
of level $(1,1)$.

If these parameters are not generic, the comultiplication rule may fail to define the action of series
e(z) and $f(z)$. In general, some matrix coefficients are well-defined and some are not. Also, some
matrix coefficients which are generically non-zero become zero for special values of parameters.

The study of this phenomena is reduced to the case of $n = 2$, and the following simple lemma
about tensor products of two vector representations of $\mathcal{E}$ describes the situation.

Lemma 2.2. In $V(u) \otimes V(v)$, all matrix coefficients of operators $\psi^\pm(z)$, $e(z)$ and $f(z)$ are well-defined except possibly for
\begin{align*}
\langle i | \otimes | j \rangle e(z) |i\rangle \otimes |j - 1\rangle, \quad \langle i | \otimes | j \rangle f(z) |i + 1\rangle \otimes |j\rangle.
\end{align*}

Each of these matrix coefficients is undefined if and only if
\begin{align*}
v/u = q_1^{i-j} \quad \text{or} \quad v/u = q_1^{i-j+1},
\end{align*}
and is equal to zero if and only if

\[ v/u = q_1^{i-j}q_2^{-1} \quad \text{or} \quad v/u = q_1^{i-j}q_3^{-1}. \]

Moreover the matrix coefficients

\[ \langle i+1 \rangle \otimes \langle j \rangle \, e(z) \mid \langle i \rangle \otimes \langle j \rangle, \quad \langle i \rangle \otimes \langle j \rangle \, f(z) \mid \langle i \rangle \otimes \langle j+1 \rangle \]

are always non-zero.

**Proof.** The proof is straightforward. \( \square \)

### 2.3. Fock modules

A *partition* \( \lambda \) is the sequence \((\lambda_s)_{s \in \mathbb{Z}_{\geq 1}}\), such that \( \lambda_s \in \mathbb{Z} \) and \( \lambda_s \geq \lambda_{s+1} \) for all \( s \). Let \( \mathcal{P} \) be the set of all partitions. Let \( \mathcal{P}^+ \subset \mathcal{P} \) be the set of finite partitions: \( \lambda \in \mathcal{P} \) is in \( \mathcal{P}^+ \) if and only if only finitely many \( \lambda_s \) are non-zero.

For \( s \in \mathbb{Z}_{\geq 1} \), denote \( 1_s = (\delta_{sm})_{m \in \mathbb{Z}_{\geq 1}} \). For example, we have \( \lambda + 1_s = (\lambda_1, \ldots, \lambda_{s-1}, \lambda_s + 1, \lambda_{s+1}, \ldots) \).

For \( u \in \mathbb{C} \), let \( \mathcal{F}(u) \) be a complex vector space spanned by \( \mid \lambda \rangle_u \), where \( \lambda \in \mathcal{P}^+ \).

We use the notation \( u \langle \lambda \rangle \) with \( \lambda \in \mathcal{P}^+ \) for the dual basis. When it is clear we omit the subscript \( u \) and write simply \( \langle \lambda \rangle \) and \( | \lambda \rangle \).

For a partition \( \lambda \) we denote by \( \langle \lambda | \psi^\pm(z) \rangle_i | \lambda \rangle \) the eigenvalue of the series \( \psi^\pm(z) \) on the vector \( |\lambda_i - i + 1 \rangle_{uq_2^{-i+1}} \in V(uq_2^{-i+1}) \), i.e.

\[ \psi^\pm(z)|\lambda_i - i + 1 \rangle_{uq_2^{-i+1}} = \langle \lambda | \psi^\pm(z) \rangle_i |\lambda_i - i + 1 \rangle_{uq_2^{-i+1}}. \]

The subscript \( i \) in \( \psi^\pm(z)_i \) indicates the \( i \)-th component \( \lambda_i \) of the partition \( \lambda \), and at the same time the shifts

\[ \lambda_i \mapsto \lambda_i - i + 1, \quad u \mapsto uq_2^{-i+1} \]

in the vector \( |\lambda_i - i + 1 \rangle_{uq_2^{-i+1}} \).

Similarly, we introduce the matrix coefficients \( \langle \lambda + 1_i | e(z) \rangle_i |\lambda \rangle \) and \( \langle \lambda | f(z) \rangle_i |\lambda + 1_i \rangle \)

\[ \langle \lambda + 1_i | e(z) \rangle_i |\lambda \rangle = \frac{\delta(q_1^{\lambda_i}q_2^{-i}u/z)}{1 - q_1}, \quad \langle \lambda | f(z) \rangle_i |\lambda + 1_i \rangle = \frac{q_1 \delta(q_1^{\lambda_i}q_3^{-i}u/z)}{1 - q_1}. \]

In what follows we multiply these delta functions by rational functions \( \langle \lambda | \psi^\pm(z) \rangle_j | \lambda \rangle \). We mean that they are multiplied by the values of the rational functions at the support of the delta function: \( F(z)\delta(v/z) = F(v)\delta(v/z) \).

Let \( \emptyset \) be the empty partition, \( \emptyset_i = 0 \), \( i \in \mathbb{Z}_{\geq 1} \). Introduce the series \( \psi^\pm(z) \in \mathbb{C}[[z^\pm 1]] \) by

\[ \psi^+_\emptyset(z) = \frac{1 - q_2 z}{1 - z}, \quad \psi^-_\emptyset(z) = q_2 \frac{1 - q_3 z^{-1}}{1 - z^{-1}}. \]

Define the action of \( e(z) \) on \( \mathcal{F}(u) \) by

\[ \langle \lambda + 1_i | e(z) \rangle_i |\lambda \rangle = \langle \lambda + 1_i | e(z) \rangle_i |\lambda \rangle \prod_{j=1}^{i-1} \langle \lambda | \psi^+(z)_j |\lambda \rangle \]

(2.11)
and setting all other matrix coefficients to be zero. Define the action of \( f(z) \) on \( \mathcal{F}(u) \) by
\[
(\lambda|f(z)|\lambda + 1_i) = (\lambda|f(z)|\lambda + 1_i) \psi_0^+(uq_3^i/z) \prod_{j=i+1}^{\infty} \frac{(\lambda|\psi_0^+(z)|\lambda)}{(\emptyset|\psi_0^+(z)|\emptyset)}
\]
and setting all other matrix coefficients to be zero. Define the action of \( \psi^\pm(z) \) on \( \mathcal{F}(u) \) by
\[
(\lambda|\psi^\pm(z)|\lambda) = \psi_0^\pm(u/z) \prod_{j=1}^{\infty} \frac{(\lambda|\psi_j^\pm(z)|\lambda)}{(\emptyset|\psi_j^\pm(z)|\emptyset)}
\]
and setting all other matrix coefficients to be zero.

Note that although the formulas are written using infinite products, each product in fact is finite since \( \lambda_j = 0 = \emptyset_j \) for all but finitely many indices \( j \).

For example, explicitly we have
\[
\psi_+^+(z) |\lambda\rangle = \psi_\lambda(u/z) |\lambda\rangle, \quad \psi_\lambda(u/z) = \frac{1 - q_1^{\lambda_1-1}q_3^{-1}u/z}{1 - q_1^{\lambda_1}u/z} \prod_{i=1}^{\infty} \frac{(1 - q_1^{\lambda_i}q_3^{-1}u/z)(1 - q_1^{\lambda_{i+1}-1}q_3^{-1}u/z)}{(1 - q_1^{\lambda_i}q_3^{-1}u/z)(1 - q_1^{\lambda_{i+1}}q_3^{-1}u/z)}.
\]

**Lemma 2.3.** Formulas (2.11), (2.12), (2.13) define a structure of an irreducible tame \( \mathcal{E} \)-module on \( \mathcal{F}(u) \) of level \((1,q_2)\). It is a highest weight module with highest vector \(|\emptyset\rangle\) and highest weight \(\psi_0^\pm(u/z)\).

**Proof.** These formulas define an \( \mathcal{E} \)-module by [FFJMM], cf. also [FT]. It is easy to check that it is tame and irreducible, see also Theorem 3.4 below. The highest weight conditions are obvious. \(\square\)

We call \( \mathcal{F}(u) \) the Fock module. It is an analog of a Fock module for \( \mathfrak{g} l_\infty \). This module appeared in [FT] from geometric considerations.

**2.4. The module** \( G^{k,r}_a \). Assume \( q_1^{-1}q_3^{k+1} = 1 \), where \( r, k + 1 \in \mathbb{Z}_{\geq 2} \). More precisely, we mean that \( q_1^x q_3^y = 1 \) if and only if \( x = (1-r)\kappa, y = (k+1)\kappa \) for some \( \kappa \in \mathbb{Z} \).

Fix a sequence of non-negative integers \( a = (a_1, \ldots, a_k) \) satisfying \( \sum_{i=1}^{k} a_i = r \), and set \( c_j = \sum_{i=1}^{j} a_i \). Define the vacuum partition \( \Lambda^0 \in \mathcal{P} \) by
\[
\Lambda^0_{vk+i+1} = -\nu r - c_i, \quad \text{where} \quad \nu \in \mathbb{Z}_{\geq 0}, \ i = 0, \ldots, k - 1.
\]

We define the sets of \((k,r)\)-admissible partitions
\[
\mathcal{P}^{k,r}_a = \{ \Lambda \in \mathcal{P} | \Lambda_j - \Lambda_{j+k} \geq r, \ \text{for} \ j \in \mathbb{Z}_{\geq 2}; \ \Lambda_i = \Lambda_i^0 \ \text{for all sufficiently large} \ i \}.
\]

We recall the semi-infinite construction of \( \mathcal{E} \)-module given in [FFJMM]. Let \( W^{k,r}_a(u) \) be the space spanned by \(|\Lambda\rangle\), where \( \Lambda \in \mathcal{P}^{k,r}_a \).

**Remark 2.4.** Note that our notation is different from the one used in [FFJMM]. The module \( W^{k,r}_c(u) \) in [FFJMM] is denoted by \( W^{k,r}_a(u) \) here, where \( c_j = \sum_{i=1}^{j} a_i \). The formulas for the action are shorter in terms of \( c_i \), on the other hand \( a_i \) are natural parameters for many related objects appearing in this paper.
Define series \( \varphi_0^\pm(z) \in \mathbb{C}[[z^{\pm 1}]] \) by
\[
\varphi_0^+(z) = \frac{1 - q_3 z}{1 - z}, \quad \varphi_0^-(z) = \frac{1 - q_3^{-1} z^{-1}}{1 - z^{-1}}.
\]

Define the action of \( e(z) \) on \( W_{a^r}^k(u) \) by
\[
(2.15) \quad \langle \Lambda + 1_i | e(z) | \Lambda \rangle = \langle \Lambda + 1_i | e(z) | \Lambda \rangle \prod_{j=1}^{i-1} \langle \Lambda | \psi_-(z) | j \rangle | \Lambda \rangle
\]
and setting all other matrix coefficients to be zero. Define the action of \( f(z) \) on \( W_{a^r}^k(u) \) by
\[
(2.16) \quad \langle \Lambda | f(z) | \Lambda + 1_i \rangle = \langle \Lambda | f(z) | \Lambda + 1_i \rangle \prod_{j=i+1}^{\infty} \langle \Lambda | \psi_+(z) | j \rangle | \Lambda \rangle \prod_{j=0}^{k-1} \varphi_0^+(u q_1^{a_j+1} q_3^j z)\)
and setting all other matrix coefficients to be zero. Define the action of \( \psi(z) \) on \( W_{a^r}^k(u) \) by
\[
(2.17) \quad \langle \Lambda | \psi(z) | \Lambda \rangle = \prod_{i=0}^{k-1} \varphi_0^+(u q_1^{-a_i} q_3^i z) \prod_{i \geq 1} \langle \Lambda | \psi(z) | i \rangle | \Lambda \rangle \prod_{i \geq 1} \langle \Lambda | \psi(z) | i \rangle | \Lambda \rangle
\]
and setting all other matrix coefficients to be zero.

**Lemma 2.5.** Suppose \( q_1^{1-r} q_3^{k+1} = 1 \) with \( r, k + 1 \in \mathbb{Z}_{\geq 2} \). Then formulas \( (2.15), (2.16), (2.17) \) define a structure of an irreducible tame \( E \)-module on \( W_{a^r}^k(u) \) of level \( (1, q_3^k) \). It is a highest weight module with highest vector \( | \Lambda^0 \rangle \) and highest weight \( \prod_{i=0}^{k-1} \varphi_0^+(u q_1^{-a_i} q_3^i z) \).

**Proof.** These formulas define an \( E \)-module by \([E]_a, [M]_M\). It is easy to check that it is tame and irreducible. (It also follows from Theorem 5.3 of this paper.) The highest weight conditions are obvious. \( \square \)

Let \( q_1^{p'} q_3^p = 1 \) and \( p = k + 1, p' = k + r \), where as above \( r, k + 1 \in \mathbb{Z}_{\geq 2} \). This is equivalent to \( q_1^{1-r} q_3^{k+1} = 1 \). Again, by that we mean that \( q_1^{p'} q_3^p = 1 \) is and only if \( x = (k + r) \kappa, y = (k + 1) \kappa \) for some \( \kappa \in \mathbb{Z} \).

Let \( G_{a^r}^{k} \) be the space spanned by \( | \lambda \rangle \), where \( \lambda \in \mathcal{P}_{a^r}^{k} \). Define the action of operators \( e(z), f(z), \psi(z) \) on \( G_{a^r}^{k} \) by formulas \( (2.15), (2.16), (2.17) \), where \( q_2 \) is replaced with \( q_3 \), and \( q_3 \) is replaced with \( q_2 \).

**Lemma 2.6.** Suppose \( q_1^{p'} q_3^p = 1 \) and \( p = k + 1, p' = k + r \), where \( k + 1, r \in \mathbb{Z}_{\geq 2} \). Then formulas \( (2.15), (2.16), (2.17) \), where \( q_2 \) is replaced with \( q_3 \), and \( q_3 \) is replaced with \( q_2 \), define a structure of an irreducible tame \( E \)-module on \( G_{a^r}^{k} \) of level \( (1, q_3^k) \). It is a highest weight module with highest vector \( | \Lambda^0 \rangle \) and highest weight \( \prod_{i=0}^{k-1} \psi_0^+(u q_1^{-a_i} q_3^i z) \).

**Proof.** The lemma follows from Lemma 2.5 and the symmetry of the algebra \( E \) with respect to permutations of parameters \( q_1, q_2, q_3 \). \( \square \)

3. Construction of \( E \)-modules

In this section we construct \( E \)-modules \( M_{a^r}^{p', q}(u) \) as subquotients of tensor products of Fock modules.
3.1. Generic tensor products. Consider a tensor product of $n$ Fock modules $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ with $n \geq 2$. In this section we assume that $q_1, q_2, u_1, \ldots, u_n$ are generic.

A basis of $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ is given by $|\lambda^{(i)}\rangle_{u_1} \otimes \cdots \otimes |\lambda^{(n)}\rangle_{u_n}$, where $\lambda^{(i)} \in \mathcal{P}$ for $i = 1, \ldots, n$.

We use the following notation. We write basic vectors in $\mathcal{F}(u_i)$ with upper index $i$ and skip the index $u_i$: we write simply $|\lambda^{(i)}\rangle$ instead of $|\lambda^{(i)}\rangle_{u_i}$. Moreover, we use the notation $|\lambda^{(1)}, \ldots, \lambda^{(n)}\rangle$ and $\langle \lambda^{(1)}, \ldots, \lambda^{(n)}|$ for $|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(n)}\rangle$ and $\langle \lambda^{(1)}| \otimes \cdots \otimes \langle \lambda^{(n)}|$. Sometimes we use the bold font notation $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ and then we denote $1_s^{(i)}$ in the $i$-th place by $1_s^i$:

$$\lambda + 1_s^i = (\lambda^{(1)}, \ldots, \lambda^{(i)} + 1, \ldots, \lambda^{(n)}).$$

Define a $\mathbb{Z}$ grading on $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ by (cf. (2.10)):

$$\text{deg}(|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(n)}\rangle) = \sum_{i=1}^{n} |\lambda^{(i)}| = \sum_{i=1}^{n} \sum_{s=1}^{\infty} \lambda^{(i)}_s.$$

**Lemma 3.1.** Assume that $q_1, q_2, u_1, \ldots, u_n$ are generic. The comultiplication rule defines on $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ a structure of an irreducible graded tame $\mathcal{E}$-module of level $(1, q_1^n)$.

The module $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ is a highest weight module with highest vector $|0^{(1)}\rangle \otimes \cdots \otimes |0^{(n)}\rangle$ and highest weight $\prod_{i=1}^{n} \psi_0^\pm(u_i/z)$.

**Proof.** To check that the comultiplication rule gives well-defined formulas, it is sufficient to consider the case $n = 2$. In the case $n = 2$, the well-definedness is obvious due to Lemma 2.2.

These formulas give a well-defined action of $\mathcal{E}$, that is that the relations in $\mathcal{E}$ are respected. Indeed, the check is reduced to the case of a tensor product of vector representation, which is done in Lemma 2.5 in [FFJMM].

Let us prove the simplicity of the spectrum of $\psi(z)$. Recall the eigenvalue $\psi_\lambda(u/z)$ of $\psi^+(z)$ on $|\lambda\rangle$ in $\mathcal{F}(u)$, see (2.14). Assume that

$$\prod_{i=1}^{n} \psi_{\lambda^{(i)}}(u_i/z) = \prod_{i=1}^{n} \psi_{\mu^{(i)}}(u_i/z).$$

We need to show that this implies $\lambda^{(i)} = \mu^{(i)}$ for $i = 1, \ldots, n$.

Note that $\psi_{\lambda^{(i)}}(u_i/z)$ has a pole at $z = q_1^{\lambda^{(i)}} u_i$. Since $q_1, q_2, u_1, \ldots, u_n$ are generic this pole can be canceled only by the poles $z = q_1^{\mu^{(i)}} u_i$ or $z = q_1^{\mu^{(i)}-1} u_i$. The latter is impossible because in such a case $\lambda^{(i)} = \mu^{(i)} - 1$, and the pole $z = q_1^{\mu^{(i)}} u_i$ is not canceled. Therefore we obtain $\lambda^{(i)} = \mu^{(i)}$ for $i = 1, \ldots, n$. Cancel the terms with $\lambda^{(1)}_i$ and $\mu^{(1)}_i$, and replace $u_i$ with $u_i/q_2$. Then the same argument gives $\lambda^{(i)}_2 = \mu^{(i)}_2$ for $i = 1, \ldots, n$. Repeating the process we obtain that $\lambda^{(i)} = \mu^{(i)}$ for $i = 1, \ldots, n$.

Since the representation is tame, to show it is irreducible, it is sufficient to check that the matrix coefficients $\langle \lambda + 1_s^{(i)}| e(z)|\lambda \rangle$ and $\langle \lambda| f(z) |\lambda + 1_s^{(i)}\rangle$ are non-zero for all $i, s$. This is reduced to $n = 2$ case where it follows from Lemma 2.2. \[\square\]

3.2. Resonance in $u_i/u_{i+1}, q_1, q_3$. We turn to special values of parameters. In doing so we always keep in mind that the matrix coefficients of the considered modules are rational functions of parameters, sometimes multiplied by the delta functions. When we go to special values of parameters
we just take limits of these rational functions. In particular, we first cancel factors at generic values of parameters as much as possible and then simply substitute the special values.

Consider the tensor product of \( n \) Fock modules \( \mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n) \) with \( n \geq 2 \) and let

\[
(3.1) \quad u_i = u_{i+1} q_1^{a_i+1} q_3^{b_i+1}, \quad \text{where } a_i, b_i \in \mathbb{Z}_{\geq 0} \text{ and } i = 1, \ldots, n - 1.
\]

Let \( \mathbf{a} = (a_1, \ldots, a_{n-1}), \mathbf{b} = (b_1, \ldots, b_{n-1}), u_i = u \) and let

\[
\mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) = \text{span}\{ |\lambda^{(1)} \rangle, \ldots, |\lambda^{(n)} \rangle | \lambda_s^{(i+1)} \geq \lambda_{s+b_i}^{(i+1)} - a_i, \text{ where } s \in \mathbb{Z}_{\geq 1}, i = 1, \ldots, n - 1 \}.
\]

Note that if \( a_i \) was negative for some \( i \) then the space \( \mathcal{M}_{\mathbf{a}, \mathbf{b}} \) would be trivial.

The following lemma shows that the definition of \( \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \) is in fact a superposition of \( n = 2 \) conditions. Note that for \( 1 \leq i < j \leq n \) we have

\[
|a|_i = u_j q_1^{a_i+1} q_3^{b_i+1}, \quad a_{ij} = \sum_{l=i}^{j-1} (a_l + 1) - 1, \quad b_{ij} = \sum_{l=i}^{j-1} (b_l + 1) - 1.
\]

Lemma 3.2. We have \( |\lambda^{(1)} \rangle, \ldots, |\lambda^{(n)} \rangle \in \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \) if and only if for all \( i, j, 1 \leq i < j \leq n, \)

\[
|\lambda^{(i)} \rangle \otimes |\lambda^{(j)} \rangle \in \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u).
\]

Proof. The lemma is straightforward.

We have an obvious inclusion of vector spaces \( \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \rightarrow \mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n) \). In particular, the space \( \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \) inherits the \( \mathbb{Z} \)-grading.

We define the action of operators \( \psi^\pm(z), e(z), f(z) \) on \( \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \) using the action of \( \mathcal{E} \) on the tensor product \( \mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n) \). Namely, let the matrix coefficients of operators \( \psi^\pm(z), e(z), f(z) \) acting on \( \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \) in the basis \( |\lambda^{(1)} \rangle, \ldots, |\lambda^{(n)} \rangle \) be the same as the corresponding matrix coefficients for the tensor action.

Proposition 3.3. Assume that \( q_1, q_2, u \) are generic. Then the action of operators \( \psi^\pm(z), e(z), f(z) \) in \( \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \) is well-defined and gives a structure of a graded \( \mathcal{E} \)-module.

Proof. Consider the case \( n = 2 \). Let \( a_1 = a, b_1 = b \).

It is sufficient to perform the following checks.

(i) If \( |\lambda, \mu \rangle \in \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \), then the matrix coefficients \( \langle \lambda, \mu | e(z) |\lambda, \mu - 1_s \rangle, \langle \lambda, \mu | f(z) |\lambda + 1_s, \mu \rangle \) are well defined.

(ii) If \( |\lambda, \mu \rangle \in \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \), then

\[
|\lambda, \mu + 1_s \rangle \not\in \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \Rightarrow \langle \lambda, \mu + 1_s | e(z) |\lambda, \mu \rangle = 0,
\]

\[
|\lambda - 1_s, \mu \rangle \not\in \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \Rightarrow \langle \lambda - 1_s, \mu | f(z) |\lambda, \mu \rangle = 0.
\]

All the checks are straightforward using Lemma 2.2.

For example, let \( |\lambda \rangle \otimes |\mu \rangle, |\lambda + 1_s \rangle \otimes |\mu \rangle \in \mathcal{M}_{\mathbf{a}, \mathbf{b}}(u) \) and consider \( \langle \lambda | \otimes |\mu | f(z) |\lambda + 1_s \rangle \otimes |\mu \rangle \). By Lemma 2.2 the poles of this matrix coefficient happen if

\[
u q_2^{1-s}/(q_2 v_2^{-1}) = q_1^{\mu-l-\lambda_s+s} \quad \text{or} \quad q_1^{1-s}/(v_2 q_3^{1-l}) = q_1^{\mu-l-\lambda_s+s-1}.
\]

Which means \( q_1^{\mu-\lambda_s-a-1} = q_3^{-l+b+1} \) or \( q_1^{\mu-\lambda_s-a-2} = q_3^{-l+b+1} \). Equivalently, \( l = s + b + 1 \), and \( \lambda_s = \mu+s+b+1-a-1 \) or \( \lambda_s = \mu+s+b+1-a-2 \). This is impossible, because

\[
\lambda_s \geq \mu_s+b-a = \mu_s+b+1-a-1 > \mu_s+b+1-a-2.
\]
Similarly, let $|\lambda| \otimes |\mu| \in M_{a,b}(u)$ and $|\lambda| \otimes (|\mu| + 1_s) \notin M_{a,b}(u)$. Then we have $s - b \geq 1$ and $\lambda - b = \mu - a$. It follows that the coefficient $\langle \lambda \otimes (|\mu| + 1_s) | e(z) | \lambda \otimes |\mu| \rangle$ vanishes by Lemma 2.2 since $vq_2^{1-s}/(uq_2^{1-(s-b)}) = q_1^{b-a}q_2 = q_1^{\lambda_s - s + b - b - 1}q_3^{-1}$.

We omit further details.

Since all the necessary checks reduce to the case of $n = 2$ due to Lemma 3.2 the general case Proposition 3.3 follows.

**Theorem 3.4.** Assume that $q_1, q_2, u$ are generic. Then $M_{a,b}(u)$ is an irreducible, tame, highest weight $E$-module with highest weight $\prod_{i=1}^n \psi_{\theta_i}^+(u_i/z)$.

**Proof.** First, let us show the module is tame. Assume that for some $|\lambda^{(1)}, \ldots, \lambda^{(n)}|, |\mu^{(1)}, \ldots, \mu^{(n)}| \in M_{a,b}(u)$, we have

$$\prod_{i=1}^n \psi_{\lambda^{(i)}}(u_i/z) = \prod_{i=1}^n \psi_{\mu^{(i)}}(u_i/z).$$

Recall that $\psi_{\lambda^{(i)}}(u_i/z)$ has a pole at $z = q_1^{\lambda^{(i)}} u_i$, see (2.14). We show that for $i = 1, \ldots, n$, $\lambda^{(i)} = \mu^{(i)}$ by showing that the pole $z = q_1^{\lambda^{(i)}} u_i$ in the left hand side is canceled by the pole $z = q_1^{\mu^{(i)}} u_i$ in the right hand side.

Suppose the pole $z = q_1^{\lambda^{(i)}} u_i$ is canceled by zeroes of $\psi_{\lambda^{(j)}}(u_j/z)$. There are two possible cases:

1. $q_1^{\lambda^{(i)}} u_i = q_1^{\lambda^{(j)}} q_3^s u_j$, where $s \in \mathbb{Z}_{\geq 1}$ and $q_1^{\lambda^{(i)}} u_i = q_1^{\lambda^{(j)} - 1} q_3^{s-1} u_j$, where $s \in \mathbb{Z}_{\geq 0}$.

2. Suppose $q_1^{\lambda^{(i)}} u_i = q_1^{\lambda^{(j)}} q_3^s u_j$, where $s \in \mathbb{Z}_{\geq 1}$. Since $s \geq 1$, and (3.1) holds, it implies that $j > i$ and

$$\lambda^{(i)} = \lambda^{(j)} - \sum_{l=i}^{j-1} (a_l + 1), \quad s = \sum_{l=i}^{j-1} (b_l + 1).$$

But then we have

$$\lambda^{(i)} \geq \lambda^{(j)} - \sum_{l=i}^{j-1} a_l > \lambda^{(j)} - \sum_{l=i}^{j-1} (b_l + 1) - \sum_{l=i}^{j-1} (a_l + 1) = \lambda^{(i)},$$

which is a contradiction.

Suppose $q_1^{\lambda^{(i)}} u_i = q_1^{\lambda^{(j)} - 1} q_3^{s-1} u_j$, where $s \in \mathbb{Z}_{\geq 0}$. By a similar argument we show that it is possible only if $s = 0, j = i - 1, b_{i-1} = 0$ and $\lambda^{(i)} = \lambda^{(i-1)} - 1$.

Now suppose the pole $z = q_1^{\lambda^{(i)}} u_i$ is canceled by poles of $\psi_{\mu^{(j)}}(u_j/z)$. We again have two cases. For example, suppose $q_1^{\lambda^{(i)}} u_i = q_1^{\mu^{(j)} - 1} q_3^s u_j$ and $(i, j) \neq (j, s + 1)$. Then we necessarily have $i < j$ and

$$\lambda^{(i)} = \mu^{(j)} - \sum_{l=i}^{j-1} (a_l + 1), \quad s = \sum_{l=i}^{j-1} (b_l + 1).$$

It implies that

$$\lambda^{(i)} \geq \lambda^{(j)} \sum_{l=i}^{j-1} b_l - \sum_{l=i}^{j-1} a_l > \mu^{(j)} - \sum_{l=i}^{j-1} (b_l + 1) - \sum_{l=i}^{j-1} (a_l + 1) = \lambda^{(i)}.$$
Similarly, we obtain $\mu^{(i)}_1 > \lambda^{(i)}_1$ in the other case of $q^{(i)}_1 u_i = q^{(i)}_1 u_i - q^{s-1}_3 u_j$.

We claim $\lambda^{(1)}_1 = \mu^{(1)}_1$. Indeed, for $i = 1$ the cancellation of the poles with zeroes is impossible and as we saw, the cancellation of poles $z = q^{(1)}_1 u_1$ and $z = q^{(1)}_1 u_1$ with other poles imply both $\mu^{(1)}_1 > \lambda^{(1)}_1$ and $\lambda^{(1)}_1 > \mu^{(1)}_1$, which is a contradiction.

Next we claim that $\lambda^{(2)}_1 = \mu^{(2)}_1$. Indeed, since the terms with $\lambda^{(1)}_1$ and $\mu^{(1)}_1$ cancel each other, the cancellation of the poles $z = q^{(2)}_1 u_1$ and $z = q^{(2)}_1 u_1$ with zeroes is again impossible and cancellation with other poles leads to a contradiction.

Repeating, we obtain $\lambda^{(i)}_1 = \mu^{(i)}_1$ for $i = 1, \ldots, n$.

Cancel the corresponding factors and replace $u_i$ with $u_i q_3^{-1}$. Then, a similar argument gives $\lambda^{(i)}_2 = \mu^{(i)}_2$, where $i = 1, \ldots, n$. Repeating the argument, we prove that the module $M_{a,b}(u)$ is tame.

Now, to prove that $M_{a,b}(u)$ is irreducible, it is sufficient to show that if vectors $|\lambda\rangle$, $|\lambda + 1^{(i)}_s\rangle$ are both in $M_{a,b}(u)$, then $|\lambda + 1^{(i)}_s e(z)|\lambda\rangle$ and $|\lambda f(z)|\lambda + 1^{(i)}_s\rangle$ are non-zero. It is similar to that of Proposition 3.3. We omit further details.

The character of $M_{a,b}(u)$ is given in Theorem 4.6.

The tensor action of $\mathcal{E}$ on the space $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ for generic $u_i$ does not have a limit to the case (3.1) in the basis $|\lambda^{(1)}, \ldots, \lambda^{(n)}\rangle$. This limit exists only on $M_{a,b}(u)$. However, we think of $M_{a,b}(u)$ as “a submodule of $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$”.

Remark 3.5. Note that in the case of (3.1), the action of operators $\psi^\pm_{s,i}$ on $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ is well-defined, however the joint spectrum of $\psi^\pm_{s,i}$ is not simple. For example, consider the case $n = 2$, $a_1 = b_1 = 0$. Thus we consider $\mathcal{F}(u) \otimes \mathcal{F}(u q_2)$. Then the vectors $|0\rangle_u \otimes |(2, 2)\rangle_{u q_2}$ and $|(1)\rangle_u \otimes |(2, 1)\rangle_{u q_2}$ have the same $\psi^\pm_{s,i}$ eigenvalues.

3.3. Resonance in $q_1, q_3$. Consider the tensor product of $n$ Fock modules $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ with $n \geq 2$.

Assume (3.1) and let $p', p$ be some integers such that

$$a_n = p' - 1 - \sum_{i=1}^{n-1} (a_i + 1), \quad b_n = p - 1 - \sum_{i=1}^{n-1} (b_i + 1)$$

belong to $\mathbb{Z}_{\geq 0}$. Assume further that

$$q_1^{p'} q_3^p = 1, \quad p \neq p'.$$

More precisely, by equality (3.2) we mean that $q_1^{p'} q_3^p = 1$ if and only if $x = p' \kappa$, $y = p \kappa$ for some $\kappa \in \mathbb{Z}$.

We use a cyclic modulo $n$ convention for indices and suffixes: $u_{n+1} = u_1, \lambda^{(0)} = \lambda^{(n)}$, etc. Let

$$M_{a,b}^{p', p}(u) = \text{span}\{|\lambda^{(1)}, \ldots, \lambda^{(n)}\}, \quad \lambda^{(i)}_s \geq \lambda^{(i+1)}_{s+b_i} - a_i, \quad \text{where } s \in \mathbb{Z}_{\geq 1}, \quad i = 1, \ldots, n\}.$$

The following lemma shows that the definition of $M_{a,b}^{p', p}(u)$ is in fact a superposition of $n = 2$ conditions.
Lemma 3.6. We have $|\lambda^{(1)}, \ldots, \lambda^{(n)}| \in M_{a,b}^{\nu, \nu}(u)$ if and only if for all $i, j$, $1 \leq i < j \leq n$, $|\lambda^{(i)} \otimes |\lambda^{(j)}| \in M_{a_{ij}, b_{ij}}^{\nu, \nu}(u_i)$.

Proof. The lemma is straightforward. □

We have an obvious surjective map of linear spaces: $M_{a,b}(u) \rightarrow M_{a,b}^{\nu, \nu}(u)$, sending $|\lambda^{(1)}, \ldots, \lambda^{(n)}|$ to either zero or to $|\lambda^{(1)}, \ldots, \lambda^{(n)}|$. In particular, the space $M_{a,b}^{\nu, \nu}(u)$ inherits the $\mathbb{Z}$-grading.

We define the action of operator $\psi^{\pm}(z), e(z), f(z)$ on $M_{a,b}^{\nu, \nu}(u)$ as the factorized action of $E$ on $M_{a,b}(u)$. Namely, let the matrix coefficients of operators $\psi^{\pm}(z), e(z), f(z)$ in the basis $|\lambda^{(1)}, \ldots, \lambda^{(n)}|$ be the same as the corresponding matrix coefficients in $M_{a,b}(u)$.

Proposition 3.7. The action of operators $\psi^{\pm}(z), e(z), f(z)$ in $M_{a,b}^{\nu, \nu}(u)$ is well-defined and gives a structure of a graded $E$-module.

Proof. Consider the case $n = 2$ and set $a_1 = a, b_1 = b$.

It is sufficient to perform the following checks.

(i) If $|\lambda, \mu| \in M_{a,b}^{\nu, \nu}(u)$, then the matrix coefficients $\langle \lambda, \mu + 1_s | e(z) | \lambda, \mu \rangle$, $\langle \lambda - 1_s, \mu | f(z) | \lambda, \mu \rangle$ are well defined.

(ii) If $|\lambda, \mu| \in M_{a,b}^{\nu, \nu}(u)$, then

$|\lambda, \mu - 1_s| \notin M_{a,b}^{\nu, \nu}(u) \Rightarrow \langle \lambda, \mu | e(z) | \lambda, \mu - 1_s \rangle = 0$

$|\lambda + 1_s, \mu| \notin M_{a,b}^{\nu, \nu}(u) \Rightarrow \langle \lambda, \mu | f(z) | \lambda + 1_s, \mu \rangle = 0$.

All the checks are straightforward using Lemma 2.2.

For example, consider $\langle \lambda | \otimes | \mu + 1_s | e(z) | \lambda \otimes | \mu \rangle$.

By Lemma 2.2 the poles of this matrix coefficient happen if $vq_1^{l-b}(uvq_2^{l-1}) = q_1^{\lambda - b_1 - b_2 + 2}$ or $uq_2^{l-b}(uvq_2^{l-1}) = q_1^{\lambda - b_1 - b_2 + 2}$. Which means $q_1^{\lambda - b_1 - b_2 + 2} = 0$ or $q_1^{\lambda - b_1 - b_2 + 2} = 0$.

Equivalently, due to (3.2),

$l = s - b - 1 + \gamma(b_1 + b_2 + 2), \quad \lambda_l - \mu_s + a_1 + 1 = \gamma(a_1 + a_2 + 2)$

or

$l = s - b - 1 + \gamma(b_1 + b_2 + 2), \quad \lambda_l - \mu_s + a_1 = \gamma(a_1 + a_2 + 2)$

for some $\gamma \in \mathbb{Z}$.

Therefore $\lambda_s = \mu + (b_1 + b_2 + 2) + \gamma(a_1 + a_2 + 2) - a_1 - 1$ or $\lambda_s = \mu + (b_1 + b_2 + 2) + \gamma(a_1 + a_2 + 2)$.

Let, first, $\gamma < 0$. Then

$\lambda_l - \mu_l - (b_1 + b_2 + 2) - b_1 + 1 \geq \lambda_l - \mu_l - (b_1 + b_2 + 2) + \gamma(a_1 + a_2 + 2)$

and therefore the poles of the matrix coefficient do not occur.

Let $\gamma > 0$. Then

$\lambda_l - \mu_l - (b_1 + b_2 + 2) - b_1 + 1 \leq \lambda_l - (\gamma - 1) - (b_1 + b_2 + 2) + \gamma(a_1 + a_2 + 2)$

and again, such a pole is impossible.

Let now $\gamma = 0$. That is $s = l + b_1 + 1$ and $\lambda_l - \mu_l - b_1 + 1 = -a_1$ or $\lambda_l - \mu_l - b_1 + 1 = -a_1 - 1$.

Since $\lambda_l - \mu_l - b_1 + 1 \geq \lambda_l - \mu_l - b_1 \geq -a_1$, the second case is impossible and in the first case we have
Since \( a > 0 \), otherwise in the case \( l = 1 \) the index of \( \mu_{s-1} = \mu_{l+b_1} \) is non-positive and the coefficient does not have to be zero.

We omit further details.

The general case of Proposition 3.7 reduces to the case of \( n = 2 \) by Lemma 3.6.

\[ \text{Theorem 3.8. Assume in addition that } p > n. \text{ Then the } E \text{-module } M_{a,b}^p(u) \text{ is an irreducible, tame, highest weight } E \text{-module with highest weight } \prod_{i=1}^n \psi_i^+ \left( u_i/z \right). \]

\[ \text{Proof. The proof is similar to the proof of Theorem 3.4. Assume that for some } (\lambda^{(1)}, \ldots, \lambda^{(n)}) \text{, } (\mu^{(1)}, \ldots, \mu^{(n)}) \in M_{a,b}^p(u), \text{ we have } \]

\[ \prod_{i=1}^n \psi_{\lambda^{(i)}} \left( u_i/z \right) = \prod_{i=1}^n \psi_{\mu^{(i)}} \left( u_i/z \right). \]

We then show that this implies \( \lambda^{(i)} = \mu^{(i)} \).

For example, let us check that the pole \( z = q^{\lambda^{(i)}} \) is not canceled by the zero \( z = q_1^{\lambda^{(j)}} q_3^{a_j} u_j \) of \( \psi_{\lambda^{(j)}} \left( u_j/z \right) \), that is \( q_1^{\lambda^{(i)}} u_i = q_1^{\lambda^{(j)}} q_3^{a_j} u_j \).

It is easy to see that \( i = j \) is impossible.

Consider the case \( j > i \). Then for some \( \kappa \in \mathbb{Z} \) we have

\[ \lambda_1^{(i)} = \lambda_s^{(j)} - \sum_{l=i}^{j-1} (a_l + 1) - \kappa p', \quad s = \sum_{l=i}^{j-1} (b_l + 1) + \kappa p. \]

Since \( s \geq 1 \), we obtain that \( \kappa \geq 0 \). This is impossible since

\[ \lambda_1^{(i)} \geq \lambda_1^{(i)} + \kappa p' \geq \lambda_{1+\kappa}^{(j)} - \sum_{l=i}^{j-1} (a_l + 1) - \kappa p' > \lambda_{s}^{(j)} - \sum_{l=i}^{j-1} (a_l + 1) - \kappa p'. \]

Here we used that \( a_l, b_l \geq 0 \).

In the case \( j < i \), we have

\[ \lambda_1^{(i)} = \lambda_s^{(j)} + \sum_{l=j}^{i-1} (a_l + 1) - \kappa p', \quad s = - \sum_{l=j}^{i-1} (b_l + 1) + \kappa p. \]

Since \( s \geq 1 \), we obtain that \( \kappa \geq 1 \). This is impossible since

\[ \lambda_1^{(i)} \geq \lambda_{1+\kappa}^{(i-p)} - (\kappa - 1) p' \geq \lambda_{1+\kappa}^{(j)} - \sum_{l=j}^{i-1} (a_l + 1) - \kappa p' \geq \lambda_s^{(j)} - \sum_{l=i}^{j-1} (a_l + 1) - \kappa p'. \]

The case \( q_1^{\lambda^{(i)}} u_i = q_1^{\lambda^{(j)}} q_3^{a_j} u_j \) is again possible only if \( j = i - 1, s = 0 \) and \( b_{i-1} = 0 \).

Note that since \( p > n \), there exists \( i \), such that \( b_{i-1} \neq 0 \). In such a case the pole \( z = q^{\lambda^{(i)}} u_i \) is not canceled with a zero.

We omit further details.
Note that if some of $b_i$ were negative then the theorem would not hold.

The character of $\mathcal{M}_{a,b}^{\nu,p}(u)$ is given in Theorem 4.5.

The tensor action of $\mathcal{E}$ on the space $\mathcal{M}_{a,b}(u)$ for generic $q_1, q_2$ does not have a limit to the case (3.2) in the basis $|\lambda^{(1)}, \ldots, \lambda^{(n)}\rangle$. However, we think of $\mathcal{M}_{a,b}^{\nu,p}(u)$ as of “a quotient module of $\mathcal{M}_{a,b}(u)$” and even as of “a subquotient of $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$”.

4. Characters

All modules considered in Section 3 are graded modules with finite-dimensional graded components. Therefore we have well-defined formal characters which we study in this section.

4.1. Finitized characters and recursion. Recall that we have constructed a family of $\mathcal{E}$-modules $\mathcal{M}_{a,b}^{\nu,p}$. Here $p, p'$ are positive integers satisfying $p, p' \geq n$, $p' \neq p$, and $a = (a_1, \ldots, a_{n-1})$, $b = (b_1, \ldots, b_{n-1}) \in \mathbb{Z}^{n-1}$ are such that there exist $a_n, b_n \in \mathbb{Z}_{\geq 0}$ satisfying

$$\sum_{i=1}^{n} (a_i + 1) = p', \quad \sum_{i=1}^{n} (b_i + 1) = p.$$ 

We shall always assume that $a_n, b_n$ are determined from $a, b$ as above. Throughout this section, $b$ will be fixed. We shall also assume that $p' > n$.

The module $\mathcal{M}_{a,b}^{\nu,p}$ has a basis labeled by the set of $n$-tuples of partitions

(4.1) $P_{a,b}^{\nu,p} := \{ (\lambda^{(1)}, \ldots, \lambda^{(n)}) \mid \lambda^{(j)} \in \mathbb{P}^+, \lambda^{(i)}_j \geq (i+1) - a_i, \text{ where } i = 1, \ldots, n, \ j \in \mathbb{Z}_{>0} \}$,

where $\lambda^{(i)} = (\lambda^{(i)}_j)_{j \geq 0}$ and $\lambda^{(n+1)} = \lambda^{(1)}$. In this section, we study their characters

(4.2) $\chi_{a,b}^{\nu,p} := \sum_{(\lambda^{(1)}, \ldots, \lambda^{(n)}) \in P_{a,b}^{\nu,p}} q^{n \sum_{i=1}^{n} \sum_{j=1}^{\infty} \lambda^{(i)}_j}.$

Our goal is to show that they coincide with the characters of modules from the $\mathcal{W}_n$-minimal series of $\mathfrak{s}\mathfrak{l}_n$-type, up to an overall factor corresponding to the presence of an extra Heisenberg algebra (see Theorem 4.5 below).

As a technical tool for studying (4.2), let us introduce a finitized version of the characters. For $N \in \mathbb{Z}^n$, define the subset

$$P_{a,b}^{\nu,p}[N] := \{ (\lambda^{(1)}, \ldots, \lambda^{(n)}) \in P_{a,b}^{\nu,p} \mid \lambda^{(i)}_{N_i+1} = 0, \text{ where } i = 1, \ldots, n \},$$

and its character

$$\chi_{a,b}^{\nu,p}[N] := \sum_{(\lambda^{(1)}, \ldots, \lambda^{(n)}) \in P_{a,b}^{\nu,p}[N]} q^{n \sum_{i=1}^{n} \sum_{j=1}^{\infty} \lambda^{(i)}_j}.$$

We set also

(4.3) $\chi_{a,b}^{\nu,p}[N] = 0$ if $N_i < 0$ for some $i$.

Clearly we have

(4.4) $\chi_{a,b}^{\nu,p}[0] = 1.$
In the following, we extend the suffix $i$ for $a_i$ by $a_{i+n} = a_i$. A similar convention will be used for $b_i, N_i$.

**Proposition 4.1.** The finitized characters $\chi_{a,b}^{p}[N]$ satisfy the following recursion relations for each $i = 1, \ldots, n$:

\begin{align}
(4.5) \quad &\chi_{a,b}^{p}[N] = \chi_{a,b}^{p}[N - 1] + q N_i \chi_{a - 1 - 1, b}^{p}[N] & \text{if } N_{i+1} - N_i \leq b_i \text{ and } a_{i-1} \geq 1, \\
(4.6) \quad &\chi_{a,b}^{p}[N] = \chi_{a,b}^{p}[N - i] & \text{if } N_i - N_{i-1} = b_{i-1} + 1 \text{ and } a_{i-1} = 0.
\end{align}

In the right hand side of (4.5), $a - 1_{i-1} + 1_i$ means $a + 1_i$ for $i = 1$ and $a - 1_{n-1}$ for $i = n$.

**Proof.** We fix $i$, and assume first that $N_i > 0$. Then the set $P_{a,b}^{p}[N]$ is partitioned into a disjoint union of subsets $P' \sqcup P''$, where

$$P' = \{ \lambda \in P_{a,b}^{p}[N] \mid \lambda^{(i)}_{N_i} = 0 \}, \quad P'' = \{ \lambda \in P_{a,b}^{p}[N] \mid \lambda^{(i)}_{N_i} > 0 \}.$$ 

By the definition, $P'$ coincides with $P_{a,b}^{p}[N - 1]$. Suppose $N_{i+1} - N_i \leq b_i$ and $a_{i-1} \geq 1$. For $\lambda \in P''$, the conditions involving $\lambda^{(i)}$ read

$$\lambda^{(i)}_j \geq \lambda^{(i)}_{j+b_i} + 1 = (a_{i-1} - 1),$$
$$\lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+b_i} - a_i.$$

Since $N_i + 1 + b_i > N_{i+1}$ and $a_i \geq 0$, the second condition is void if $j > N_i$. Hence it can be replaced by

$$\lambda^{(i)}_j - \theta(j \leq N_i) \geq \lambda^{(i+1)}_{j+b_i} - (a_i + 1),$$

where $\theta(P) = 1$ if the statement $P$ is true and $\theta(P) = 0$ otherwise. This gives rise to a bijection $P'' \to P_{a-1,b+1}^{p}[N]$ sending $\lambda$ to $\bar{\lambda}$, with $\lambda^{(i)}_{\bar{\lambda}} = \lambda^{(i)}_j - \delta_{j,i}\theta(j \leq N_i)$. The recursion (4.5) follows from this. When $N_i = 0$, the same consideration applies to show that $P_{a,b}^{p}[N]$ is in bijective correspondence with $P_{a-1,b+1}^{p}[N]$.

Next suppose $N_i - N_{i-1} = b_{i-1} + 1$ and $a_{i-1} = 0$ (in this case necessarily $N_i > 0$). The condition $\lambda^{(i-1)}_{N_i-1} = \lambda^{(i)}_{N_i} - a_i$ implies $\lambda^{(i)}_{N_i} = 0$, so that $P'' = \emptyset$. Hence (4.6) holds true.

In general, the recursions (4.5), (4.6) are not enough to determine the characters $\chi_{a,b}^{p}[N]$ completely. Nevertheless they are in certain region of the parameters $a, b, N$ as the following proposition shows.

**Proposition 4.2.** The set

$$\{ \chi_{a,b}^{p}[N] \mid N_i, a_i \in \mathbb{Z}_{\geq 0}, N_{i+1} - N_i \leq b_i + 1, \text{ where } i = 1, \ldots, n \}$$

is uniquely determined by the recursion relations (4.5), (4.6), along with the initial condition (4.4) and the boundary condition (4.3).
Proof. The proof is by induction on \( d = \sum_{i=1}^{n} N_i \). When \( d = 0 \), there is nothing to show. Suppose \( d > 0 \), and assume that the assertion is true for \( \sum_{i=1}^{n} N_i < d \). We divide into two cases,

Case a): \( N_{i+1} - N_i \leq b_i \) for all \( i = 1, \ldots, n \).

Case b): \( N_{i+1} - N_i = b_i + 1 \) for some \( i \).

Consider Case a). Since \( p' - n > 0 \), there is an \( i \) such that \( a_{i-1} > 0 \). Applying (4.5) successively for \( i, i + 1, \ldots \), we obtain

\[
\chi_{a,b}^{p'}[N] = \chi_{a,b}^{p'}[N - 1] + q N_i \chi_{a - 1_{i+1},b}^{p'}[N - 1] + q N_i + N_{i+1} \chi_{a - 1_{i+1} + 1_{i+1},b}^{p'}[N] = \ldots = \sum_{j=1}^{i+n-1} q^{N_i + \ldots + N_{j-1}} \chi_{a,1_{j-1} + 1_{j-1},b}^{p'}[N - 1] + q^{N_i + \ldots + N_n} \chi_{a,b}^{p'}[N].
\]

Since \( N_1 + \ldots + N_n = d \), \( \chi_{a,b}^{p'}[N] \) is determined in terms of those with \( \sum_{i=1}^{n} N_i < d \).

Next consider Case b). Since \( \sum_{i=1}^{n} (b_i + 1) > 0 \), we cannot have the equality \( N_{i+1} - N_i = b_i + 1 \) for all \( i \). Choose an \( i \) such that \( N_i - N_{i-1} = b_i - 1 + 1 \) and \( N_i + 1 - N_{i-1} \leq b_i \). If \( a_{i-1} = 0 \), then (4.6) implies \( \chi_{a,b}^{p'}[N] = \chi_{a,b}^{p'}[N - 1] \) and we are done. Otherwise (4.5) is applicable. Repeating it \( a_{i-1} \) times we obtain

\[
\chi_{a,b}^{p'}[N] = \sum_{j=1}^{a_{i-1}} q^{(j-1)N_i} \chi_{a - 1_{j-1} + 1_{j-1},b}^{p'}[N - j1] + q^{a_{i-1}N_i} \chi_{a - 1_{i-1} + 1_{i-1},b}^{p'}[N].
\]

The last term reduces to the case \( a_{i-1} = 0 \) already discussed above. \( \square \)

4.2. Bosonic formulas and comparison to \( \mathcal{W}_n \) characters. Our next task is to relate \( \chi_{a,b}^{p'} \) to the characters from the \( \mathcal{W}_n \)-minimal series. Let us prepare some notation concerning the affine Lie algebra \( \hat{\mathfrak{so}}_3 \). Denote the simple roots by \( \alpha_0, \ldots, \alpha_{n-1} \) and the fundamental weights by \( \omega_0, \ldots, \omega_{n-1} \). We set \( \rho = \sum_{i=0}^{n-1} \omega_i \). Let \( W = S_n \rtimes Q \) be the affine Weyl group of type \( A^{(1)}_{n-1} \), where \( Q = \oplus_{i=0}^{n-1} \mathbb{Z} \omega_i \) denotes the classical root lattice. Let further \( L = \oplus_{i=0}^{n-1} \mathbb{Z} \omega_i \) be the weight lattice and \( \mathcal{L}_l = \{ \sum_{i=0}^{n-1} c_i \omega_i \mid c_0, \ldots, c_{n-1} \in \mathbb{Z}_{\geq 0}, \sum_{i=0}^{n-1} c_i = l \} \) be the set of dominant integral weights of level \( l \).

The characters of the irreducible modules from the \( \mathcal{W}_n \)-minimal series of \( \mathfrak{so}_3 \)-type are parametrized by a pair of dominant integral weights \( (\eta, \xi) \in \mathcal{L}_{p'}^+ \times \mathcal{L}_{p-n}^+ \). Explicitly they are given by the alternating series [FKW],

\[
\chi_{w,\xi}^{p'} = \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2} \left| \frac{w \cdot \xi}{p} - \frac{\xi}{p} \right|^2 + \left( \frac{w \cdot \xi}{p} \xi, p' (\xi + \rho) - p (\eta + \rho) \right)} = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} q^{\frac{1}{2} \left( \sigma (\alpha,\alpha) \right) + \left( \rho', \sigma (\xi + \rho) - p (\eta + \rho), \alpha \right) + \left( \xi + \rho - \sigma (\xi + \rho), \eta + \rho \right)}.
\]

Here \( w \ast \xi = w (\xi + \rho) - \rho = \sigma (\xi + \rho) - \rho + p \alpha \), where \( w = (\sigma, \alpha) \), and \( \ell(w) \) denotes the length function.
We need also their finitization. For \( \mathbf{N} \in \mathbb{Z}_{\geq 0}^n \) and \( \eta, \xi \in L \), define
\[
\chi^{\rho', p'}_{\eta, \xi}[\mathbf{N}] = \sum_{w \in W} (-1)^{\ell(w)} q^{|w \cdot \xi - \xi|} \left( w \cdot \xi - \xi, \rho + p(\eta + \rho) \right) \\
\times (q)|\mathbf{N}| \prod_{i=1}^n \frac{1}{(q)_{N_i - (w \cdot \xi - \xi, \omega_i - \omega_{i-1})}}.
\]
Here \( (q)_m = \prod_{i=1}^m (1 - q^i) \) for \( m \in \mathbb{Z}_{\geq 0} \), \( |\mathbf{N}| = \sum_{i=1}^n N_i \). We set also
\[
\frac{1}{(q)_m} = 0 \quad \text{if} \ m < 0.
\]
We retain the modulo \( n \) convention for the indices, such as \( \omega_n = \omega_0 \).

**Proposition 4.3.**

(i) For all \( \xi, \eta \in L \) and \( i = 1, \ldots, n \), we have
\[
\chi^{\rho', p'}_{\eta, \xi}[\mathbf{N}] = q^{N_i} \chi^{\rho', p'}_{\eta - \omega_i + \omega_{i-1}, \xi}[N_i] + (1 - q^{|N_i|}) \chi^{\rho', p'}_{\eta, \xi}[\mathbf{N} - \mathbf{1}_i].
\]

(ii) If \( N_{i+1} = N_i + (\xi + \rho, \alpha_i) \) and \( (\eta + \rho, \alpha_i) = 0 \) for \( i = 1, \ldots, n \), then
\[
\chi^{\rho', p'}_{\eta, \xi}[\mathbf{N}] = 0.
\]

(iii) If \( \xi \in L^+_{p-\eta} \), then
\[
\chi^{\rho', p'}_{\eta, \xi}[\mathbf{0}] = 1.
\]

**Proof.** The relation (i) can be verified directly, term by term. To see (ii), let \( \sigma_i \) be the simple reflection with respect to the root \( \alpha_i \). The assumption can be written as
\[
N_i - (w \cdot \xi - \xi, \omega_i - \omega_{i-1}) = N_{i+1} - ((\sigma_i w) \cdot \xi - \xi, \omega_{i+1} - \omega_i),
\]
\[
\sigma_i \cdot \eta = \eta.
\]
Under these circumstances, the terms with \( w \) and \( \sigma_i w \) cancel out in the sum pairwise.

Finally, under the assumption of (iii) and \( \mathbf{N} = \mathbf{0} \), only the term with \( w = \text{id} \) survives. \( \Box \)

**Proposition 4.4.** For all \( \mathbf{N}, a, b \) such that \( N_i, a_i, b_i \geq 0 \) and \( N_{i+1} - N_i \leq b_i + 1 \) for \( i = 1, \ldots, n \), we have the equality
\[
\chi^{\rho', p'}_{a, b}[\mathbf{N}] = \frac{1}{(q)|\mathbf{N}|} \chi^{\rho', p'}_{\eta, \xi}[\mathbf{N}],
\]
\[
\eta = \sum_{i=1}^n a_i \omega_i, \quad \xi = \sum_{i=1}^n b_i \omega_i.
\]
We recall that \( a_n = p' - n - \sum_{i=1}^{n-1} a_i, \ b_n = p - n - \sum_{i=1}^{n-1} b_i \).

**Proof.** This follows from Proposition 4.3 and Proposition 4.2. \( \Box \)

Letting \( N_i \rightarrow \infty \), we arrive at the following result.
Theorem 4.5. The character of the module $M_{a,b}^{p',p}$ is given by

$$\chi_{a,b}^{p',p} = \frac{1}{(q)_{\infty}} \chi_{\eta,\xi}^{p',p},$$

where

$$\eta = \sum_{i=1}^{n} a_{i} \omega_{i}, \quad \xi = \sum_{i=1}^{n} b_{i} \omega_{i}.$$ 

4.3. Characters of $M_{a,b}$. The module $M_{a,b}$ has a basis labeled by the set of $n$-tuples of partitions

$$P_{a,b} := \{(\lambda^{(1)}, \ldots, \lambda^{(n)}) | \lambda^{(i)} \in \mathcal{P}, \lambda^{(i)}_{j} \geq \lambda^{(i+1)}_{j+b_{i}} - a_{i}, \text{ where } i = 1, \ldots, n-1, \ j \in \mathbb{Z}_{>0}\}.$$ 

Define their characters

$$\chi_{a,b} := \sum_{(\lambda^{(1)}, \ldots, \lambda^{(n)}) \in P_{a,b}} q^{\sum_{i=1}^{n} \sum_{j=1}^{+} \lambda^{(i)}_{j}}.$$ 

Theorem 4.6. We have

$$\chi_{a,b} = \frac{1}{(q)_{\infty}} \sum_{w \in S_{n}} (-1)^{\ell(w)} q^{(\xi + \rho - w(\xi + \rho) + \rho)_{\infty}}.$$ 

Proof. Clearly, the set $P_{a,b}$ is the limit of the set $P_{a,b}^{p',p}$ as $p', p \to \infty$. The theorem then follows from Theorem 4.5.

5. Isomorphisms of representations

In this section we establish several isomorphisms between representations of $\mathcal{E}$ discussed in this paper. All these isomorphisms preserve the basis described in terms of partitions. This is no wonder since the modules are tame. In general, we expect that any two highest weight $\mathcal{E}$-modules with the same highest weight are isomorphic. We check this statement here in several cases. At the moment our proofs are strictly computational.

5.1. Permutations of factors in the tensor products of Fock spaces. In this section we assume that $q_1, q_2, u_1, \ldots, u_n$ are generic.

Theorem 5.1. Let $\sigma \in S_{n}$ and let $q_1, q_2, u_1, \ldots, u_n$ be generic. There exist non-zero constants $b_{\lambda}$, where $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \in \mathbb{P}^{n}$, such that the map

$$\iota: \mathcal{F}(u_{1}) \otimes \cdots \otimes \mathcal{F}(u_{n}) \to \mathcal{F}(u_{\sigma(1)}) \otimes \cdots \otimes \mathcal{F}(u_{\sigma(n)}),$$

$$|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(n)}\rangle \mapsto b_{\lambda} |\lambda^{(\sigma(1))}\rangle \otimes \cdots \otimes |\lambda^{(\sigma(n))}\rangle$$

is an isomorphism of $\mathcal{E}$-modules.

Proof. It is sufficient to prove the theorem in the case $n = 2$.

Let $n = 2$ and $\sigma = (12)$. It is necessary and sufficient to show that there exist coefficients $b_{\lambda}$ satisfying the conditions

$$b_{\lambda + 1_{s}^{(i)}}(\lambda + 1_{s}^{(i)}|e(z)|\lambda) = b_{\lambda}^{(i)}(\lambda^{(i)} + 1_{s}^{(i)}|e(z)|\lambda^{(i)}),$$

$$b_{\lambda}(\lambda|f(z)|\lambda + 1_{s}^{(i)}) = b_{\lambda + 1_{s}^{(i)}}(\lambda^{(i)}|f(z)|\lambda^{(i)} + 1_{s}^{(i)}).$$

Here $i = 1, 2$ and if $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ then $\lambda^{(i)} = (\lambda^{(2)}, \lambda^{(1)}).$
In order that these equations for \( b_\lambda \) to be consistent the following conditions are necessary and sufficient.

\[
\frac{\langle \lambda | f(w) | \lambda + 1^{(i)} \rangle}{\langle \lambda' | f(w) | \lambda' + 1^{(3-i)} \rangle} = \frac{\langle \lambda' + 1^{(3-i)} | e(z) | \lambda' \rangle}{\langle \lambda + 1^{(i)} | e(z) | \lambda \rangle},
\]

\[
\frac{\langle \lambda + 1^{(i)} + 1^{(j)} | e(z) | \lambda + 1^{(i)} \rangle}{\langle \lambda' + 1^{(3-i)} + 1^{(3-j)} | e(z) | \lambda' + 1^{(3-i)} \rangle} \cdot \frac{\langle \lambda + 1^{(i)} | e(w) | \lambda \rangle}{\langle \lambda' + 1^{(3-i)} | e(w) | \lambda' \rangle} = \frac{\langle \lambda + 1^{(j)} | e(z) | \lambda + 1^{(j)} \rangle}{\langle \lambda' + 1^{(3-j)} + 1^{(3-j)} | e(z) | \lambda' + 1^{(3-j)} \rangle} \cdot \frac{\langle \lambda + 1^{(j)} | e(w) | \lambda \rangle}{\langle \lambda' + 1^{(3-j)} | e(w) | \lambda' \rangle}.
\]

The precise meaning of such equations is as follows. Suppose that \( \delta_i(z) = c_i \sum_{n \in \mathbb{Z}} (a/z)^n \), where \( i = 1, 2 \), are delta functions with the same support multiplied by non-zero constants \( c_i \). Then by the ratio \( \delta_1(z) / \delta_2(z) \) we mean the ratio \( c_1 / c_2 \). For example, we have

\[
\frac{\langle \lambda + 1^{(i)} | e(z) | \lambda \rangle}{\langle \lambda | f(z) | \lambda + 1^{(i)} \rangle} = q_1^{-1}.
\]

Equations (5.1), (5.2) are checked by a straightforward computation.

5.2. The \( Z_n \) symmetry of \( M^p_{a,b} \). In this section we assume that the parameters \( q_1, q_2, u_1, \ldots, u_n \) satisfy (3.1) and (3.2). There exist non-zero constants \( c_\lambda \), where \( \lambda \in P_{a,b}^p \), such that the map

\[
u: M^p_{(a_1, \ldots, a_{n-1}, (b_1, \ldots, b_{n-1})} \rightarrow M^p_{(a_2, \ldots, a_n), (b_2, \ldots, b_n)},
\]

\[
|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(n-1)}\rangle \otimes |\lambda^{(n)}\rangle \mapsto c_\lambda |\lambda^{(2)}\rangle \otimes \cdots \otimes |\lambda^{(n)}\rangle \otimes |\lambda^{(1)}\rangle
\]

is an isomorphism of \( E \)-modules.

Proof. Clearly the set theoretic map

\[
\nu: P_{(a_1, \ldots, a_{n-1}, (b_1, \ldots, b_{n-1})} \rightarrow P_{(a_2, \ldots, a_n), (b_2, \ldots, b_n)},
\]

\[
(\lambda^{(1)}, \ldots, \lambda^{(n-1)}, \lambda^{(n)}) \mapsto (\lambda^{(2)}, \ldots, \lambda^{(n)}, \lambda^{(1)})
\]

is a bijection.

Then the equations on the constants \( c_\lambda \) are the same as in Theorem 5.1 with \( \sigma = (1, 2, \ldots, n) \). Therefore, the theorem follows.

5.3. The modules \( M_{a,0}^{n+r,n+1} \) and \( G_{a}^{n,r} \). In this section we assume the following resonance condition

\[
q_1^{n+r} q_3^{n+1} = 1.
\]
We consider the special case of $M^{p', p}_{a, b}$ where

\[ p' = n + r, \ p = n + 1, \ r = \sum_{i=1}^{n} a_i, \]

\[ a = (a_1, \ldots, a_{n-1}), \ b = 0 = (0, \ldots, 0). \]

We abbreviate the representation $M^{n+k, n+1}_{a, 0}$ to $M_a$, where $\bar{a} = (a_1, \ldots, a_n)$, and the set of $n$-tuple partitions $P^{n+r, n+1}_{a, 0}$ to $P_a$:

\[ P_a = \{ \lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \mid \lambda^{(i)} \in \mathbb{P}^+, \ \lambda^{(i)}_{j(i)} \geq \lambda^{(i+1)}_{j(i)} - a_i, \ \text{where} \ i = 1, \ldots, n, \ j \in \mathbb{Z}_{\geq 0} \}. \]

Similarly, we abbreviate the representation $G^{n, r}_{a, 0}$ to $G_a$ and the set $\mathcal{P}^{n, r}_{a, 0}$ to $\mathcal{P}_a$.

Our aim is to prove that two representations $M_{\bar{a}}$ and $G_{\bar{a}}$ are isomorphic:

\[ (5.4) \quad \iota : M_{\bar{a}} \xrightarrow{\sim} G_{\bar{a}}. \]

Recall that the vector space $G_{\bar{a}}$ has a basis labeled by the set $\mathcal{P}_a$ of $(n, r)$-admissible partitions. Let $\iota$ be a bijection given by

\[ \iota : P_a \to \mathcal{P}_a, \quad \lambda \mapsto \Lambda, \]

\[ \Lambda_{ns+i} = \lambda_{s+1}^{(i)} + \lambda_0^{(r)} \text{ for } i = 1, \ldots, n, \ s \in \mathbb{Z}_{\geq 0}. \]

**Theorem 5.3.** There exist non-zero constants $c_\Lambda, \ \Lambda \in \mathcal{P}_a$, such that the linear map

\[ (5.5) \quad \iota : M_a \to G_a, \]

\[ |\lambda\rangle \mapsto c_\Lambda |\Lambda\rangle, \quad \Lambda = \iota(\lambda), \]

is an isomorphism of graded $\mathcal{E}$-modules.

**Proof.** The proof is similar to the proof of Theorem 5.1. However, the checks are slightly more involved and we give some details.

Note that the vectors $e(z)|\Lambda\rangle$ is a finite linear combination of the vectors $|\Lambda + 1_j\rangle$, where $\Lambda + 1_j = (\Lambda_1, \Lambda_2, \ldots, \Lambda_{j-1}, \Lambda_j + 1, \Lambda_{j+1}, \ldots)$, and $f(z)|\Lambda\rangle$ is a finite linear combination of the vectors $|\Lambda - 1_j\rangle$.

Therefore, it is necessary and sufficient to show that there exist coefficients $c_\Lambda$ satisfying the conditions

\[ c_{\Lambda + 1_{s+i}}(\lambda + 1_{s+1})e(z)|\lambda\rangle = c_{\Lambda}(\lambda + 1_{s+i})e(z)|\lambda\rangle, \]

\[ c_\Lambda(\lambda)|f(z)|\lambda\rangle + 1_{s+1}\rangle = c_{\Lambda + 1_{s+i}}((\lambda)|f(z)|\lambda\rangle + 1_{n+i}). \]

Here $(\lambda + 1_{s+1})e(z)|\lambda\rangle$ and $(\lambda)|f(z)|\lambda\rangle + 1_{s+1}\rangle$ denote the matrix coefficients of $e(z)$ and $f(z)$ respectively in the module $M_{\bar{a}}$, and $(\lambda + 1_{n+i})e(z)|\lambda\rangle$ and $(\lambda)|f(z)|\lambda\rangle + 1_{n+i})$ are those in the module $G_{\bar{a}}$. 
In order that these equations for $c_\Lambda$ to be consistent the following conditions are necessary and sufficient.

\begin{equation}
\langle \lambda | f(w) | \lambda + 1^{(i)}_{s+1} \rangle = \langle \Lambda + 1_{ns+i-1} | e(z) | \Lambda \rangle
\end{equation}

\begin{equation}
\langle \lambda + 1^{(i)}_{s+1} + 1^{(j)}_{t+1} | e(z) | \lambda + 1^{(i)}_{t+1} \rangle = \langle \Lambda + 1_{n+1} | e(w) | \Lambda \rangle
\end{equation}

\begin{equation}
\langle \lambda + 1^{(i)}_{s+1} + 1^{(j)}_{t+1} | e(z) | \lambda + 1^{(i)}_{t+1} \rangle = \langle \Lambda + 1_{n+1} | e(w) | \Lambda \rangle
\end{equation}

and the condition, similar to (5.8) for $f(z)$. Here we follow the conventions for the ratios of delta functions with the same support as in (5.1), (5.2).

We rewrite the action of $E$ on $G_\Theta$ in such a way that the matrix coefficients of $e(z)$ and $f(z)$ acting on $G_\Theta$ are given in terms of certain quantities which are used in the expressions for the matrix coefficients of $e(z)$ and $f(z)$ acting on $M_\Theta$. This makes the proof of these equation shorter.

The vector in $G_\Theta$ which corresponds to $\Lambda \in \mathcal{P}_\Theta$ is given by a semi-infinite tensor product:

$$| \Lambda \rangle = | \Lambda_1 \rangle_u \otimes | \Lambda_2 - 1 \rangle_{u q_3^{-1}} \otimes \cdots \otimes | \Lambda_j - j + 1 \rangle_{q_3^{j-1} u} \otimes \cdots$$

Let $\langle \Lambda | \psi^\pm(z)_{ns+i} | \Lambda \rangle$ be the matrix coefficient of $\psi^\pm(z)$ in the module $V(q_3^{-ns-i+1} u)$:

$$\psi^\pm(z) | \Lambda_{ns+i} - ns - i + 1 \rangle_{q_3^{ns-i+1} u} = \langle \Lambda | \psi^\pm(z)_{ns+i} | \Lambda \rangle | \Lambda_{ns+i} - ns - i + 1 \rangle_{q_3^{ns-i+1} u}.$$

Recall that we also write $\langle \lambda^{(i)} | \psi^\pm(z)_{s+1} | \lambda^{(i)} \rangle$ for the matrix coefficient of $\psi^\pm(z)$ in the module $V(q_2^{-s} u)$:

$$\psi^\pm(z) | \lambda^{(i)}_{s+1} - s \rangle_{q_2^{-s} u} = \langle \lambda^{(i)} | \psi^\pm(z)_{s+1} | \lambda^{(i)} \rangle | \lambda^{(i)}_{s+1} - s \rangle_{q_2^{-s} u},$$

where

\begin{equation}
u_i = q_1^{-c_i-1} (a_j+1) \frac{q_3^{-i+1} u}{q_3^{-n-1} u} = q_1^{-c_i-1} q_2^{-j-1} u.
\end{equation}

Note that we consider these quantities as rational functions in $z$, not as series in $z^{-1}$ therefore these is no distinction between $\psi^\pm(z)$.

The following is the basic equality for the proof of the isomorphism:

$$q_1^{\lambda^{(i)}_{s+1}} q_3^{s} u_i = q_1^\Lambda q_2^{n_{s+i-1} u}.$$

From this follows

$$\langle \lambda^{(i)} | \psi^\pm(z)_{s+1} | \lambda^{(i)} \rangle = \langle \Lambda | \psi^\pm(z)_{ns+i} | \Lambda \rangle.$$
Using these equalities one can write the formula for the actions of $e(z), f(z)$ in $G_\mathfrak{a}$ in the form

$$
\langle \Lambda + 1_{n+i} | e(z) | \Lambda \rangle = \frac{\delta(q_1^{\lambda(i)} q_3^{i} u_i / z)}{1 - q_1} \prod_{j=1}^{i-1} \left( \frac{1 - q_2 u_j / z}{1 - u_j / z} \frac{1 - q_3^{s+1} u_j / z}{1 - q_2 q_3^{s+1} u_j / z} \prod_{m=1}^{s} \frac{\langle \lambda(j) | \psi^-(z)_m | \lambda(j) \rangle}{\langle \theta(j) | \psi^-(z)_m | \theta(j) \rangle} \right) \times \prod_{j=i}^{n} \left( \frac{1 - q_2 q_3^{s} u_j / z}{1 - q_3^{s+1} u_j / z} \prod_{m=s+2}^{\infty} \frac{\langle \lambda(j) | \psi^+(z)_m | \lambda(j) \rangle}{\langle \theta(j) | \psi^+(z)_m | \theta(j) \rangle} \right).
$$

We also have the following formulas for the matrix coefficients in $M_\mathfrak{a}$.

$$
\langle \lambda | f(z) | \lambda + 1^{(i)}_{s+1} \rangle = \frac{\delta(q_1^{\lambda(i)} q_3^{i} u_i / z)}{1 - q_1} \prod_{j=1}^{i-1} \left( \frac{1 - q_2 u_j / z}{1 - u_j / z} \frac{1 - q_3^{s+1} u_j / z}{1 - q_2 q_3^{s+1} u_j / z} \prod_{m=1}^{s} \frac{\langle \lambda(i) | \psi^-(z)_m | \lambda(i) \rangle}{\langle \theta(i) | \psi^-(z)_m | \theta(i) \rangle} \right) \times \prod_{j=i}^{n} \left( \frac{1 - q_2 q_3^{s} u_j / z}{1 - q_3^{s+1} u_j / z} \prod_{m=s+2}^{\infty} \frac{\langle \lambda(i) | \psi^+(z)_m | \lambda(i) \rangle}{\langle \theta(i) | \psi^+(z)_m | \theta(i) \rangle} \right).
$$

Here $\theta(i)$ is used for the trivial partition $\lambda(i) = (0, 0, \ldots)$.

It is straightforward to check (5.7) and (5.8) by using these formulas.

\[ \square \]

As a corollary we have a bosonic formula for the character of set of the $(k, r)$-admissible partitions.

Let

$$
\chi_{\mathfrak{a}}^{k,r} = \sum_{\Lambda \in \mathfrak{X}^{k,r}} q^{\Lambda - \Lambda^0} = \sum_{\Lambda \in \mathfrak{X}^{k,r}} q^{\sum_{j=1}^{p} (\Lambda_j - \Lambda^0_j)}
$$

be the character of set of the $(k, r)$-admissible partitions.

**Corollary 5.4.** The character of the set of the $(k, r)$-admissible partitions $\mathfrak{X}^{k,r}_\mathfrak{a}$ coincides with the character of the set $P^{p,p'}_\mathfrak{a}$ with $p' = k + r$, $p = k + 1$, and, in particular, we have the bosonic formula

$$
\chi_{\mathfrak{a}}^{k,r} = \frac{1}{(q)_\infty} \chi_{\mathfrak{a},0}^{k+r,k+1},
$$

where $\chi_{\mathfrak{a},0}^{k+r,k+1}$ is given by (4.7).
There are other known formulas of the sets of \((k,r)\)-admissible partitions: for bosonic formulas see \[FJLMM\], for fermionic formulas in the case \((p = 3)\), see \[FJMMT1, FJMMT2\].

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