Unit-log-symmetric models: Characterization, statistical properties and its use in analyzing internet access data

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Abstract

We present here a unit-log-symmetric model based on the bivariate log-symmetric distribution. It is a flexible family of distributions over the interval (0, 1). We then discuss its mathematical properties such as stochastic representation, symmetry, modality, moments, quantile function, entropy and maximum likelihood estimators, paying particular attention to the special cases of unit-log-normal, unit-log-Student-t and unit-log-Laplace distributions. Finally, some empirical results and practical illustrations are presented.

Keywords. Unit-log-symmetric distribution • Log-symmetric distribution • Bivariate model • Bounded distributions • MCMC.

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1 Introduction

Distribution theory has mostly focused on unbounded distributions. However, in the recent years, bounded distributions have also become a subject of interest. In many practical applications, such as in medicine, biology, economics and finance, bounded data arise naturally. Some of the most common models used in this case are the beta distribution and its generalizations,
Kumaraswamy and Topp-Leone distribution. In order to have including flexible models for real world problems, it is often desirable to have a whole family of distributions available. For this purpose, transformations have been developed in the statistical literature which construct a bounded distribution for any given distribution over the real line \((-\infty, \infty)\) or over \((0, \infty)\), respectively, such as logistic transformation.

In this work, we study a transformation that contains the latter as a special case (for \(\mu = 0\) and \(\sigma = 1\)). However, our transformation is derived from the bivariate log-symmetric distribution which is defined for the first time in this paper. We introduce the unit-log-symmetric (ULS) distribution in Section 2. We pay special attention to the cases of unit-log-normal, unit-log-Student-\(t\) and unit-log-Laplace distributions. In Section 3, mathematical properties of these models are studied. Stochastic representations of the ULS are given which are used to simplify the density considerably. The symmetry of the ULS density is established and then the above-mentioned special cases are explicitly formulated, and graphically illustrated. The modality is also discussed in detail. The unit-log-normal density may be unimodal or bimodal; the unit-log-Student-\(t\) density may have a bathtub shape or possess two minimum points; the unit-log-Laplace density may be unimodal or possess two minimum points. The quantile function is studied in Subsection 3.4 while the moments and the Shannon entropy in Subsections 3.5 and 3.6, respectively. Finally, the maximum likelihood (ML) estimators are studied in Subsection 3.7. In Section 4, the Monte Carlo simulation results are presented for demonstrating the performance of the estimators and for assessing the empirical distribution of the residuals. The paper ends with the application of the unit-log-normal, unit-log-Student-\(t\) and unit-log-Laplace distributions for modeling an internet access data. Specifically, the three models are fitted to the proportion of world population using Internet (Section 5). Finally, in Section 6, some concluding remarks are made.

## 2 Unit-Log-Symmetric Distribution

We say that a continuous random variable \(W\), with support \((0, 1)\), follows a unit-log-symmetric (ULS) distribution with parameter \(\sigma_\rho = \sigma \sqrt{2(1 - \rho)}\), denoted by \(W \sim \text{ULS}(\sigma_\rho, g_c)\), if its probability density function (PDF) is given by

\[
f_W(w; \sigma_\rho) = \frac{1}{w(1-w)\sigma^2\sqrt{1-\rho^2}Z_{g_c}} \int_0^\infty \frac{1}{t} g_c \left( \frac{\tilde{t}^2_w - 2\rho \tilde{t}_w \tilde{t} + \tilde{t}^2}{1-\rho^2} \right) dt, \quad 0 < w < 1,
\]

where \(t_w = \left( \frac{w}{1-w} \right) t\), \(\tilde{t}_w = \log \left( \frac{t_w}{\eta} \right)^{1/\sigma}\), \(\tilde{t} = \log \left( \frac{t}{\eta} \right)^{1/\sigma}\), \(\eta = \exp(\mu)\),

with \(\mu \in \mathbb{R}\), \(\sigma > 0\) and \(\rho \in (-1, 1)\). Moreover, \(Z_{g_c}\) is a positive constant defined as

\[
Z_{g_c} = \int_0^\infty \int_0^\infty \frac{1}{t_1t_2\sigma^2\sqrt{1-\rho^2}} g_c \left( \frac{\tilde{t}^2_1 - 2\rho \tilde{t}_1 \tilde{t}_2 + \tilde{t}^2_2}{1-\rho^2} \right) dt_1dt_2 = \pi \int_0^\infty g_c(u) du,
\]
with \( g_c \) being a scalar function, referred to as the density generator, which may or may not depend on extra parameters considered known (Fang et al., 1990).

**Remark 2.1.** We have defined the PDF in (2.1) as a uniparameter function for two reasons: first, to avoid identification problems (see Proposition 3.1) and second, because the shape of this PDF depends only on the parameter \( \sigma \) (see Subsection 3.3).

For simplicity in presentation, we focus our attention specifically on the density generators of unit-log-normal, unit-log-Student-\( t \) and unit-log-Laplace (Table 1), though other forms of density generators can be handled in an analogous manner.

**Tab. 1:** Normalization constants (\( Z_{g_c} \)) and density generators (\( g_c \)) for some ULS distributions.

| Distribution          | \( Z_{g_c} \) | \( g_c \)             | Parameter |
|-----------------------|----------------|------------------------|-----------|
| Unit-log-normal       | \( 2\pi \)    | \( \exp(-x/2) \)       | \(-\)      |
| Unit-log-Student-\( t \) | \( \frac{\Gamma(v/2)\nu\pi}{\Gamma((v+2)/2)} \) | \( (1 + \frac{z}{\nu})^{-(v+2)/2} \) | \( \nu > 0 \) |
| Unit-log-Laplace      | \( \pi \)     | \( K_0(\sqrt{2x}) \)  | \(-\)      |

Here, in Table 1, \( K_0(u) = \int_0^\infty t^{-1} \exp(-t - \frac{u^2}{4t}) \, dt/2, \ u > 0, \) is the Bessel function of the third kind (for more details on the main properties of \( K_0 \), one may refer to appendix of Kotz et al., 2001).

## 3 Some basic properties

We now discuss some mathematical properties of the unit-log-symmetric distribution stated in the last section. For this purpose, the following definition of bivariate log-symmetric distribution becomes essential.

We say that a continuous random vector \( (T_1, T_2) \), with \( T_1 \) and \( T_2 \) being identically distributed, follows a bivariate log-symmetric (BLS) distribution if its joint PDF is given by

\[
f_{T_1,T_2}(t_1,t_2;\theta) = \frac{1}{t_1 t_2 \sigma^2 \sqrt{1-\rho^2} Z_{g_c}} g_c \left( \frac{t_1^2 - 2\rho \bar{t}_1 \bar{t}_2 + \bar{t}_2^2}{1-\rho^2} \right), \quad t_1, t_2 > 0,
\]

where \( \bar{t}_i = \log \left[ \left( \frac{t_i}{\eta} \right)^{1/\sigma} \right], \ \eta = \exp(\mu), \ i = 1, 2, \)

with \( \theta = (\eta, \sigma, \rho) \) being the parameter vector, \( \mu \in \mathbb{R}, \ \sigma > 0, \ \rho \in (-1,1), \) and \( g_c \) and \( Z_{g_c} \) are as given in (2.1) and (2.2), respectively. Observe that \( Z_{g_c} \) in (2.2) is the normalization constant for \( f_{T_1,T_2} \) and that the joint PDF of \( (T_1, T_2) \) can be obtained as the PDF of an exponential transformation of an elliptically symmetric vector (Balakrishnan and Lai, 2009, Chapter 13, p. 591).
3.1 Stochastic representation

A simple observation shows that if \((T_1, T_2) \sim \text{BLS}(\theta, g_c)\), then

\[
P \left( \frac{T_1}{T_1 + T_2} \leq w \right) = P \left( \frac{T_1}{T_2} \leq \frac{w}{1-w} \right), \quad 0 < w < 1, \tag{3.1}\]

and so,

\[
f_{T_1/T_2}(w) = \frac{1}{(1-w)^2} f_{T_1/T_2} \left( \frac{w}{1-w} \right) = \frac{1}{(1-w)^2} \int_0^\infty t f_{T_1, T_2}(t, w; \theta) dt = f_W(w; \sigma _\rho ).\]

This means that if \((T_1, T_2) \sim \text{BLS}(\theta, g_c)\), the ULS random variable \(W\) admits the stochastic representation

\[
W = \frac{T_1}{T_1 + T_2}. \tag{3.2}\]

3.2 Characterizations

From the classic stochastic representation of a bivariate elliptical vector (Balakrishnan and Lai, 2009, Subsection 13.2.3, p. 593), it follows that \((T_1, T_2) \sim \text{BLS}(\theta, g_c)\) has the following representation:

\[
T_1 = \eta \exp(\sigma Z_1), \quad T_2 = \eta \exp(\sigma [\rho Z_1 + \sqrt{1-\rho^2} Z_2]), \tag{3.3}\]

where \(Z_1 = RU_1\) and \(Z_2 = R\sqrt{1-D^2} U_2\) with \(U_1, U_2, R\) and \(D\) being mutually independent, \(\rho \in (-1, 1)\), \(\eta = \exp(\mu)\), \(P(U_i = -1) = P(U_i = 1) = 1/2\), \(i = 1, 2\), and the variables \(D\) and \(R\) have PDFs \(f_D(d) = 2/(\pi \sqrt{1-d^2})\), \(d \in (0, 1)\), and \(f_R(r) = 2 g_c(r^2)/[\int_0^\infty g_c(u) du]\), \(r > 0\), respectively.

Therefore, \(T_1/T_2 = \exp(\sigma [(1-\rho)Z_1 - \sqrt{1-\rho^2} Z_2])\), and so we obtain

\[
P \left( \frac{T_1}{T_2} \leq \frac{w}{1-w} \right) = P(Z_\rho \leq A(w)), \tag{3.4}\]

where

\[
Z_\rho = \frac{1}{\sqrt{2(1-\rho)}} [ (1-\rho)Z_1 - \sqrt{1-\rho^2} Z_2] \tag{3.5}\]

and

\[
A(w) = \log \left[ \left( \frac{w}{1-w} \right)^{1/\sigma _\rho } \right]. \tag{3.5}\]

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Thus, from (3.1) and (3.2), the cumulative distribution function (CDF) of $W \sim \text{ULS}(\sigma_{\rho}, g_c)$, denoted by $F_W(w; \sigma_{\rho})$, can be expressed as

$$F_W(w; \sigma_{\rho}) = \mathbb{P}(Z_{\rho} \leq A(w)), \quad 0 < w < 1.$$  

(3.6)

From (3.6), the following result follows immediately.

**Proposition 3.1.** If the distribution of $Z_{\rho}$ depends (or not) only on the parameters of the density generator $g_c$, then the ULS distribution is identifiable.

The inverse function of $A$, denoted by $A^{-1}$, is given by

$$A^{-1}(w) = \frac{1}{1 + \exp(-\sigma_{\rho}w)}.$$  

(3.7)

So, using this notation, the following result follows immediately from (3.6).

**Proposition 3.2** (Another stochastic representation). Given the distribution of $Z_{\rho}$, we have $A^{-1}(Z_{\rho}) \sim \text{ULS}(\sigma_{\rho}, g_c)$. On the other hand, if $W \sim \text{ULS}(\sigma_{\rho}, g_c)$, then $A(W)$ and $Z_{\rho}$ have the same distribution.

Differentiating (3.6) with respect to $w$, the ULS PDF in (2.1) of $W$ is characterized as

$$f_W(w; \sigma_{\rho}) = \frac{1}{w(1-w)\sigma_{\rho}} f_{Z_{\rho}}(A(w)), \quad 0 < w < 1,$$  

(3.8)

where $f_{Z_{\rho}}$ denotes the PDF of $Z_{\rho}$.

This means that, the distribution of $W$ is completely determined by the distribution of $Z_{\rho}$.

**Proposition 3.3** (Symmetry). The ULS PDF in (2.1) is symmetric around $w_0 = 1/2$ provided that the PDF of $Z_{\rho}$ is symmetric around $z_0 = 0$. Furthermore, in this case, the median and the mean of a ULS distribution both occur at $w_0 = 1/2$.

**Proof.** A simple algebraic manipulation shows that

$$f_W\left(\frac{1}{2} - w; \sigma_{\rho}\right) = f_W\left(\frac{1}{2} + w; \sigma_{\rho}\right), \quad \forall 0 < w < 1,$$

provided that $f_{Z_{\rho}}(-z) = f_{Z_{\rho}}(z)$, $\forall z \in \mathbb{R}$. The required result then follows. $\square$

**Remark 3.4.** Since the PDFs of $Z_1$ and $Z_2$ are even functions (see Proposition 1; Item ii, of Saulo et al., 2022), and provided that $Z_1$ and $Z_2$ are independent, we have that $f_{Z_{\rho}}(-z) = f_{Z_{\rho}}(z)$, $\forall z \in \mathbb{R}$.
3.2.1 Unit-log-normal

It is well-known that the bivariate log-normal distribution admits a stochastic representation as in (3.3), where $Z_1$ and $Z_2$ are independent and identically distributed (i.i.d.) standard normal random variables (Corollary 1 of Saulo et al., 2022). Consequently, $Z_\rho$ in (3.4) is standard normally distributed. So, by using (3.6) and (3.8), we obtain

$$F_W(w; \sigma_\rho) = \Phi(A(w)), \quad f_W(w; \sigma_\rho) = \frac{1}{w(1-w)\sigma_\rho} \phi(A(w)), \quad 0 < w < 1,$$

(3.9)

where $\phi$ and $\Phi$ denote the PDF and CDF of a standard normal distribution, respectively.

3.2.2 Unit-log-Student-$t$

It is well-known that the bivariate log-Student-$t$ distribution has a stochastic representation as in (3.3), where $Z_1 = Z_1^* \sqrt{\nu/Q} \sim t_\nu$ and $Z_2 = Z_2^* \sqrt{\nu/Q} \sim t_\nu$ (Corollary 2 of Saulo et al., 2022). Here, $Q \sim \chi^2_\nu$ (chi-square with $\nu$ degrees of freedom) is independent by of $Z_1^*$ and $\rho Z_1^* + \sqrt{1-\rho^2} Z_2^*$, whereas $Z_1^*$ and $Z_2^*$ are i.i.d. standard normal random variables.

As $[1/\sqrt{2(1-\rho)}][(1-\rho)Z_1^* - \sqrt{1-\rho^2}Z_2^*] \sim N(0,1)$, we have

$$Z_\rho = \frac{1}{\sqrt{2(1-\rho)}} [(1-\rho)Z_1^* - \sqrt{1-\rho^2}Z_2^*] \sqrt{Q/\nu} \sim t_\nu.$$  

(3.10)

By combining (3.6) and (3.8) with (3.10), we obtain

$$F_W(w; \sigma_\rho) = F_\nu(A(w)), \quad f_W(w; \sigma_\rho) = \frac{1}{w(1-w)\sigma_\rho} f_\nu(A(w)), \quad 0 < w < 1,$$

where $f_\nu$ and $F_\nu$ denote the PDF and CDF of a Student-$t$ distribution with $\nu$ degrees of freedom, respectively.

3.2.3 Unit-log-Laplace

By using the general algorithm for simulation of symmetric bivariate Laplace variables (Kotz et al., 2001, Subsection 5.1.4, p. 234), it follows that the bivariate log-Laplace distribution has a stochastic representation as in (3.3), where $Z_1 = Z_1^* \sqrt{B} \sim \text{Laplace}(0,1/\sqrt{2})$ and $Z_2 = Z_2^* \sqrt{B} \sim \text{Laplace}(0,1/\sqrt{2})$. Here, $B$ is a standard exponential variable independent by of $Z_1^*$ and $\rho Z_1^* + \sqrt{1-\rho^2} Z_2^*$, whereas $Z_1^*$ and $Z_2^*$ are i.i.d. standard normal random variables.

As $[1/\sqrt{2(1-\rho)}][(1-\rho)Z_1^* - \sqrt{1-\rho^2}Z_2^*] \sim N(0,1)$, we have

$$Z_\rho = \frac{1}{\sqrt{2(1-\rho)}} [(1-\rho)Z_1^* - \sqrt{1-\rho^2}Z_2^*] \sqrt{B} \sim \text{Laplace}(0,1/\sqrt{2}).$$

(3.11)
Then, by (3.6), (3.8) and (3.10) we obtain

\[ F_W(w; \sigma_\rho) = F_\ell(A(w)), \quad f_W(w; \sigma_\rho) = \frac{1}{w(1-w)\sigma_\rho} f_\ell(A(w)), \quad 0 < w < 1, \quad (3.12) \]

where \( F_\ell \) and \( f_\ell \) denote the CDF and PDF of Laplace distribution with location parameter 0 and scale parameter \( 1/\sqrt{2} \), respectively.

Figure 1 displays different shapes of the unit-log-symmetric PDFs for different choices of parameters. From this figure, we observe that the parameter \( \sigma_\rho \) controls the shape of the unit-log-normal, unit-log-Laplace and unit-log-Student-\( t \) densities. We also note that the PDFs of these three models are symmetric. Finally, from Figure 1(d), we see the clear effect the kurtosis parameter \( \nu \) has on the unit-log-Student-\( t \) density.

Fig. 1: Unit-log-symmetric PDFs for some choices of parameters.
3.3 Modality

Differentiating $A$ in (3.5) with respect to $w$, we get

$$A'(w) = \frac{1}{w(1-w)\sigma_{\rho}} \quad \text{and} \quad A''(w) = (2w - 1)[A'(w)]^2\sigma_{\rho}. \quad (3.13)$$

Suppose the derivative of $f_{W}(w;\sigma_{\rho})$ with respect to $w$ exists at every point in its domain. In the case there is a countable number of points on its domain where $f_{W}$ is not differentiable, by definition, these are considered to be critical points of $f_{W}$. Thus, from (3.8), we have $f_{W}(w;\sigma_{\rho}) = A'(w)f_{Z_{\rho}}(A(w))$ so that

$$f'_{W}(w;\sigma_{\rho}) = A''(w)f_{Z_{\rho}}(A(w)) + [A'(w)]^2 f'_{Z_{\rho}}(A(w)).$$

If $f'_{Z_{\rho}}(z) = -r(z)f_{Z_{\rho}}(z)$, for some real-valued function $r$, then

$$f'_{W}(w;\sigma_{\rho}) = \{A''(w) - [A'(w)]^2r(A(w))\}f_{Z_{\rho}}(A(w)) = \{(2w - 1)\sigma_{\rho} - r(A(w))\}A'(w)f_{W}(w;\sigma_{\rho}).$$

Hence,

$$f'_{W}(w;\sigma_{\rho}) = 0 \iff (2w - 1)\sigma_{\rho} = r(A(w)) \iff [2A^{-1}(A(w)) - 1]\sigma_{\rho} = r(A(w)).$$

Therefore, by using (3.7), a critical point $w$ of the ULS PDF (2.1) is such that

$$\left[\frac{2}{1 + \exp(-\sigma_{\rho}A(w))} - 1\right]\sigma_{\rho} = r(A(w)),$$

or, equivalently,

$$\tanh\left(\frac{\sigma_{\rho}^2}{2}y\right) = \frac{r(\sigma_{\rho}y)}{\sigma_{\rho}}, \quad \text{with} \quad y = A(w)/\sigma_{\rho}. \quad (3.14)$$

3.3.1 Unit-log-normal

In this case, from Subsubsection 3.2.1, we have $Z_{\rho} \sim N(0, 1)$, and then $r(z) = z$. So, from (3.14), the equation for the critical points of the ULS PDF is given by

$$\tanh\left(\frac{\sigma_{\rho}^2}{2}y\right) = y, \quad \text{with} \quad y = A(w)/\sigma_{\rho},$$

if and only if

$$\frac{\sigma_{\rho}^2}{2}y = \arctanh(y). \quad (3.15)$$

Observe that there always exists at least one solution for the above equation. The number of solutions of (3.15) depends on whether $\sigma_{\rho}$ is larger or smaller than $\sqrt{2}$. If $\sigma_{\rho} \leq \sqrt{2}$, then
(3.15) has a unique solution, given by \( y_0 = 0 \) (Figure 2), and if \( \sigma_\rho > \sqrt{2} \), then (3.15) has two additional non-trivial solutions (Figure 3), \( y_- = y_- (\sigma_\rho) < 0 \) and \( y_+ = y_+ (\sigma_\rho) > 0 \) (which depend on \( \sigma_\rho \)), with \( y_+ = -y_- \); this means that \( f_W(w; \sigma_\rho) \) has one critical point at \( w_0 = 1/2 \), or three critical points, \( w_- = A^{-1}(\sigma_\rho y_-), w_0 = 1/2 \) and \( w_+ = A^{-1}(\sigma_\rho y_+) \). As \( \lim_{w_+ \to 0^+} f_W(w; \sigma_\rho) = \lim_{w_- \to 1^-} f_W(w; \sigma_\rho) = 0 \), we then obtain the following result.

**Theorem 3.5.** Let \( W \sim \text{ULS}(\sigma_\rho, g_c) \), with \( g_c(x) = \exp(-x/2) \). Then, the following holds:

If \( \sigma_\rho \leq \sqrt{2} \), then the ULS PDF \( f_W(w; \sigma_\rho) \) is unimodal, with mode \( w_0 = 1/2 \). On the other hand, if \( \sigma_\rho > \sqrt{2} \), then the ULS PDF \( f_W(w; \sigma_\rho) \) is bimodal, with modes

\[
w_- = A^{-1}(\sigma_\rho y_-) = \frac{1}{1 + \exp(-\sigma_\rho^2 y_-)} \quad \text{and} \quad w_+ = A^{-1}(\sigma_\rho y_+) = \frac{1}{1 + \exp(-\sigma_\rho^2 y_+)},
\]

and minimum point \( w_0 = 1/2 \), so that \( 0 < w_- < w_0 < w_+ < 1 \). Moreover, the graph of \( f_W(w; \sigma_\rho) \) is symmetric around \( w_0 = 1/2 \) (Figure 1 a).

### 3.3.2 Unit-log-Student-\( t \)

In this case, from Subsubsection 3.2.2, we have \( Z_\rho \sim t_\nu \), and \( r(z) = (\nu + 1)z / (\nu + z^2) \). So, from (3.14), the equation for critical points of the ULS PDF is given by

\[
\tanh \left( \frac{\sigma_\rho^2}{2} y \right) = \frac{(\nu + 1)y}{\nu + \sigma_\rho^2 y^2}, \quad \text{with} \quad y = A(w)/\sigma_\rho, \\
\]

or, equivalently,

\[
\tanh(z) = \frac{2(\nu + 1)z}{\nu \sigma_\rho^2 + 4z^2}, \quad \text{with} \quad z = \sigma_\rho^2 y / 2. \tag{3.16}
\]

Note that \( w_0 = 1/2 \) is a critical point of the ULS PDF because \( z_0 = 0 \) satisfies (3.16).
Because \( \lim_{z \to \pm \infty} \tanh(z) = \pm 1 \), by analyzing the graphs of functions \( z \mapsto \tanh(z) \) and \( 2(\nu + 1)z/(\nu \sigma^2_\rho + 4z^2) \), we see that (3.16) has one solution, given by \( z_0 = 0 \), or three solutions, \( z_- < z_0 = 0 < z_+ = -z_- \), that depend on the choice of parameters \( \nu \) and \( \sigma_\rho \) (Figures 4 and 5). In other words, \( f_W(w; \sigma_\rho) \) has one critical point at \( w_0 = 1/2 \), or three critical points, \( 0 < w_- = A^{-1}(2z_-/\sigma_\rho) < w_0 = 1/2 < w_+ = A^{-1}(2z_+/\sigma_\rho) < 1 \). As \( \lim_{w \to 0^+} f_W(w; \sigma_\rho) = \lim_{w \to 1^-} f_W(w; \sigma_\rho) = \infty \) and \( w_0 = 1/2 \) is a critical point (and point of symmetry, see Proposition 3.3) of \( f_W(w; \sigma_\rho) \), we readily have the following result.

**Theorem 3.6.** If \( W \sim \text{ULS}(\sigma_\rho, g_c) \), with \( g_c(x) = (1+x/\nu)^{-(\nu+2)/2} \), then the ULS PDF \( f_W(w; \sigma_\rho) \) has a bathtub shape with minimum point \( w_0 = 1/2 \), or is decreasing-increasing-decreasing-increasing with minimum points

\[
w_- = A^{-1}(2z_-/\sigma_\rho) = \frac{1}{1 + \exp(-2z_-)} \quad \text{and} \quad w_+ = A^{-1}(2z_+/\sigma_\rho) = \frac{1}{1 + \exp(-2z_+)}
\]

and maximum point \( w_0 = 1/2 \). Moreover, the graph of the ULS PDF is symmetric with around \( w_0 = 1/2 \) (Figures 1 c, d).

### 3.3.3 Unit-log-Laplace

From Subsubsection 3.2.3, \( Z_\rho \sim \text{Laplace}(0, 1/\sqrt{2}) \), and \( r(z) = \sqrt{2}z/|z|, \ z \neq 0 \). The function \( f_W(w, \sigma_\rho) \) is not differentiable at \( w_0 = 1/2 \), and this point is a critical point with \( f_W(w_0, \sigma_\rho) = 4/\sigma_\rho \). So, Equation (3.14) for critical points (excluding \( w_0 \)) of the ULS PDF is given by

\[
\tanh\left(\frac{\sigma_\rho^2}{2} y\right) = \frac{\sqrt{2}}{\sigma_\rho} \frac{y}{|y|} \quad \text{with} \quad y = A(w)/\sigma_\rho,
\]

which implies that \( y_\pm = \pm 2 \arctanh(\sqrt{2}/\sigma_\rho)/\sigma_\rho^2 \). In other words, \( f_W(w; \sigma_\rho) \) has one critical point at \( w_0 = 1/2 \), or three critical points, \( 0 < w_- = A^{-1}(\sigma_\rho y_-) < w_0 = 1/2 < w_+ = A^{-1}(\sigma_\rho y_+) < 1 \).
Because
\[
\lim_{w \to 0^+} f_W(w, \sigma_\rho) = \lim_{w \to 1^-} f_W(w, \sigma_\rho) = \begin{cases} 
0, & \sigma_\rho < \sqrt{2}, \\
1/\sqrt{2}, & \sigma_\rho = \sqrt{2}, \\
\infty, & \sigma_\rho > \sqrt{2},
\end{cases}
\]
and \(w_0 = 1/2\) is a critical point of symmetry (Proposition 3.3) of \(f_W(w; \sigma_\rho)\), we readily obtain the following result.

**Theorem 3.7.** Let \(W \sim \text{ULS}(\sigma_\rho, g_c)\), with \(g_c(x) = K_0(\sqrt{2}x)\). Then, the following holds:

If \(\sigma_\rho \leq \sqrt{2}\) then the ULS PDF \(f_W(w; \sigma_\rho)\) is unimodal, with mode \(w_0 = 1/2\). On the other hand, if \(\sigma_\rho > \sqrt{2}\) then the ULS PDF \(f_W(w; \sigma_\rho)\) is decreasing-increasing-decreasing-increasing with minimum points
\[
w_- = A^{-1}(\sigma_\rho y_-) = \frac{1}{1 + \exp(2 \text{arctanh}(\sqrt{2}/\sigma_\rho))},
\]
\[
w_+ = A^{-1}(\sigma_\rho y_+) = \frac{1}{1 + \exp(-2 \text{arctanh}(\sqrt{2}/\sigma_\rho))},
\]
and maximum point \(w_0 = 1/2\), so that \(0 < w_- < w_0 < w_+ < 1\). Moreover, \(f_W(w; \sigma_\rho)\) is not differentiable and symmetric at \(w_0 = 1/2\) (Figure 1 b).

### 3.4 Quantile function

From (3.6), the \(p\)-quantile function \(Q_W(p)\) of \(W \sim \text{ULS}(\sigma_\rho, g_c)\), for \(p \in (0, 1)\), satisfies
\[
p = F_W(Q_W(p); \sigma_\rho) = \mathbb{P}(Z_\rho \leq A(Q_W(p))),
\]
where \(Z_\rho\) is as in (3.4). Hence,
\[
Q_{Z_\rho}(p) = A(Q_W(p)) \iff Q_W(p) = A^{-1}(Q_{Z_\rho}(p)) \overset{(3.7)}{=} \frac{1}{1 + \exp(-\sigma_\rho Q_{Z_\rho}(p))}.
\]

As the quantiles of Gaussian, Student-\(t\), and Laplace variables are well-known, by using the results of Subsubsections 3.2.1, 3.2.2, and 3.2.3 on the distribution of \(Z_\rho\), we have the quantiles as presented in Table 2.

**Tab. 2:** Density generators (\(g_c\)) and \(p\)-quantiles of some ULS distributions.

| Distribution      | \(g_c\)                          | \(Q_W(p)\)                        |
|-------------------|----------------------------------|----------------------------------|
| Unit-log-normal   | \(\exp(-x/2)\)                   | \([1 + \exp(-\sigma_\rho \Phi^{-1}(p))]^{-1}\) |
| Unit-log-Student-\(t\) | \((1 + \frac{x}{\nu})^{-(\nu+2)/2}\) | \([1 + \exp(-\sigma_\rho F_{\nu}^{-1}(p))]^{-1}\) |
| Unit-log-Laplace  | \(K_0(\sqrt{2}x)\)              | \([1 + (2p)^{-\sigma_\rho/\sqrt{2}}]^{-1}1_{(0, 1/2]}(p) + [1 + (2 - 2p)\sigma_\rho/\sqrt{2}]^{-1}1_{(1/2, \infty]}(p)\) |

Here, \(\Phi^{-1}\) and \(F_{\nu}^{-1}\) denote the quantiles of the standard normal and Student-\(t\) distributions, respectively.
3.5 Moments

For $W \sim \text{ULS}(\sigma_\rho, g_c)$, since $W = T_1/(T_1 + T_2)$ with $(T_1, T_2) \sim \text{BLS}(\theta, g_c)$, it is clear that $0 \leq \mathbb{E}(W^r) \leq 1$, $r > 0$. Consequently, all positive moments of $W$ always exist.

In general, the moments of $W$ admit the following representation:

$$\mathbb{E}(W^r) = \mathbb{E}([A^{-1}(Z_\rho)]^r) \overset{(3.7)}{=} \mathbb{E}\left[\frac{1}{(1 + \exp(-\sigma_\rho Z_\rho))^r}\right], \quad r \in \mathbb{R},$$

where $Z_\rho$ is as given in (3.4).

Upon using binomial expansion, the negative integer moments of $W$ can be written as

$$\mathbb{E}(W^{-n}) = \mathbb{E}[(1 + \exp(-\sigma_\rho Z_\rho))^n] = \sum_{k=0}^{n} \binom{n}{k} M_{Z_\rho}(-k\sigma_\rho), \quad n \in \mathbb{N},$$

where $M_{Z_\rho}(t)$ is the moment generating function (MGF) of $Z_\rho$.

As the MGFs of Gaussian and Laplace variables are available in some simple explicit forms, by using the results of Subsubsections 3.2.1 and 3.2.3 on the distribution of $Z_\rho$, we have the negative moments corresponding to unit-log-normal and unit-log-Laplace (Table 3). On the other hand, since $Z_\rho \sim t_\nu$ as $g(x) = (1 + x/\nu)^{-(\nu+2)/2}$ (Subsubsection 3.2.2), it is clear that the negative moments corresponding to unit-log-Student-$t$ do not exist.

| Distribution       | $g_c$                 | $\mathbb{E}(W^{-n})$                           | Restriction               |
|--------------------|-----------------------|------------------------------------------------|---------------------------|
| Unit-log-normal    | $\exp(-x/2)$          | $\sum_{k=0}^{n} \binom{n}{k} \exp(\frac{1}{2}k^2\sigma_\rho^2)$ | -                         |
| Unit-log-Student-$t$| $(1 + \frac{x}{\nu})^{-(\nu+2)/2}$ | $\sum_{k=0}^{n} \binom{n}{k} (1 - \frac{1}{2}k^2\sigma_\rho^2)^{-1}$ | $n < \sqrt{2}/\sigma_\rho$, $\sigma_\rho \leq \sqrt{2}/2$ |
| Unit-log-Laplace   | $K_0(\sqrt{2x})$      | $\sum_{k=0}^{n} \binom{n}{k} (1 - \frac{1}{2}k^2\sigma_\rho^2)^{-1}$ | -                         |

3.6 Shannon entropy

The entropy of a random variable can be interpreted as the average level of uncertainty inherent in the possible outcomes of the variable. Following Shannon and Weaver (1949), the entropy $H$ of a random variable $W \sim \text{ULS}(\sigma_\rho, g_c)$, which takes values in the interval $(0, 1)$ and is distributed according to $f_W(\cdot, \sigma_\rho)$ in (3.8), is given by

$$H(W) = -\mathbb{E}[\log f_W(W, \sigma_\rho)].$$

Upon using (3.8) and the representation in (3.3), we observe that

$$H(W) = 2\mu + \log(\sigma_\rho) + \sigma[(1 + \rho)\mathbb{E}(Z_1) + \sqrt{1 - \rho^2}\mathbb{E}(Z_2)] + H(Z_\rho),$$

12
where $Z_1$ and $Z_2$ are as given in Subsection 3.2 and $Z_\rho$ is as in (3.4). Because $Z_1$ and $Z_2$ are identically distributed (Proposition 1; Item ii, of Saulo et al., 2022), the above identity can be expressed as

$$H(W) = 2\mu + \log(\sigma_\rho) + \sigma(1 + \rho + \sqrt{1 - \rho^2})E(Z_1) + H(Z_\rho).$$

It is known that the entropies of Gaussian, Student-$t$, and Laplace variables all exist. Then, by using the results of Subsubsections 3.2.1, 3.2.2, and 3.2.3 on the distribution of $Z_\rho$, we obtain the entropies for the three models as presented in 4.

**Tab. 4:** Density generators ($g_c$) and entropies for some ULS distributions.

| Distribution       | $g_c$                               | $H(W)$                                                                 |
|-------------------|------------------------------------|----------------------------------------------------------------------|
| Unit-log-normal   | $\exp(-x/2)$                       | $2\mu + \log(\sigma_\rho) + \frac{1}{2} \log(2\pi) + 1$            |
| Unit-log-Student-t| $(1 + \frac{x}{\nu})^{-(\nu+2)/2}$ | $2\mu + \log(\sigma_\rho) + \frac{\nu+1}{2} \left[ \Psi \left( \frac{\nu+1}{2} \right) - \Psi \left( \frac{\nu}{2} \right) \right] + \log \left[ \sqrt{\nu} B \left( \frac{\nu}{2}, \frac{1}{2} \right) \right]$ |
| Unit-log-Laplace  | $K_0(\sqrt{2x})$                   | $2\mu + \log(\sigma_\rho) + \log(\sqrt{2e})$                      |

Here, $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function and $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is the complete beta function, with $\Gamma(t) = \int_0^\infty x^{t-1} \exp(-x) \, dx, \ t > 0,$ being the gamma function.

### 3.7 Maximum likelihood estimation

Let $\{W_i; i = 1, \ldots, n\}$ be a univariate random sample of size $n$ from the ULS($\sigma_\rho, g_c$) distribution with PDF as in (3.8), and let $w_i$ be the sample observations of $W_i$. Then, the log-likelihood function for $\sigma_\rho$ is given by

$$\ell(\sigma_\rho) = -n \log(\sigma_\rho) - \sum_{i=1}^n \log[w_i(1 - w_i)] + \sum_{i=1}^n \log f_{Z_\rho}(A(w_i)),$$

where the random variable $Z_\rho$ is as defined in (3.4) and $A$ is as in (3.5). In the case when a supremum $\tilde{\sigma}_\rho$ exists, it must be such that

$$\left. \frac{\partial \ell(\sigma_\rho)}{\partial \sigma_\rho} \right|_{\sigma_\rho = \tilde{\sigma}_\rho} = 0 \quad \text{and} \quad \left. \frac{\partial^2 \ell(\sigma_\rho)}{\partial \sigma^2_\rho} \right|_{\sigma_\rho = \tilde{\sigma}_\rho} < 0,$$

(3.19)

with

$$\frac{\partial \ell(\sigma_\rho)}{\partial \sigma_\rho} = -\frac{n}{\sigma_\rho} - \frac{1}{\sigma^2_\rho} \sum_{i=1}^n \log \left( \frac{w_i}{1 - w_i} \right) G(A(w_i)),$$

$$\frac{\partial^2 \ell(\sigma_\rho)}{\partial \sigma^2_\rho} = -\frac{1}{\sigma_\rho} \left[ 2 \frac{\partial \ell(\sigma_\rho)}{\partial \sigma_\rho} + \frac{n}{\sigma_\rho} \right] + \frac{1}{\sigma^3_\rho} \sum_{i=1}^n \log^2 \left( \frac{w_i}{1 - w_i} \right) G'(A(w_i)).$$

(3.20)
We shall now adopt the notation
\[ G(x) = f'_{Z_\rho}(x) / f_{Z_\rho}(x). \] (3.21)
Observe now that the likelihood equation \[ \partial \ell(\sigma_\rho) / \partial \sigma_\rho \bigg|_{\sigma_\rho = \hat{\sigma}_\rho} = 0 \] in (3.19) can be written as
\[ -\frac{1}{n} \sum_{i=1}^{n} \log \left( \frac{w_i}{1 - w_i} \right) G(A(w_i)) \bigg|_{\sigma_\rho = \hat{\sigma}_\rho} = \hat{\sigma}_\rho. \] (3.22)
Any nontrivial root \( \hat{\sigma}_\rho \) of the above equation is the ML estimate of \( \sigma_\rho \) in the loose sense. When the parameter value provides the absolute maximum of the log-likelihood function, it is called an ML estimate in the strict sense.

### 3.7.1 Unit-log-normal

From Subsubsection 3.2.1, \( Z_\rho \sim N(0, 1) \). Then, \( G(x) \) in (3.21) and its derivative are given by \( G(x) = -x \) and \( G'(x) = -1 \). Thus, by using (3.22), the ML estimate of \( \sigma^2_\rho \) in the loose sense is given by
\[ \hat{\sigma}^2_\rho = \frac{1}{n} \sum_{i=1}^{n} \log^2 \left( \frac{w_i}{1 - w_i} \right). \]
From (3.20), it is then easy to verify that \( \partial^2 \ell(\sigma_\rho) / \partial \sigma^2_\rho \bigg|_{\sigma_\rho = \hat{\sigma}_\rho} < 0 \), and so \( \hat{\sigma}^2_\rho \) is an ML estimate in the strict sense.

Let \( \hat{\sigma}^2_\rho \) be the corresponding ML estimator of \( \sigma^2_\rho \). By using Proposition 3.2, we have \( A(W) = Z_\rho \sim N(0, 1) \), for \( W \sim ULS(\sigma_\rho, g_c) \). Then (for \( k = 1, 2, \ldots \)),
\[ E[(\hat{\sigma}^2_\rho)^k] = E\left[ \log^k \left( \frac{W}{1 - W} \right) \right] = \sigma^2_\rho E[A^k(W)] = \sigma^2_\rho E[Z^{2k}_\rho] = \sigma^2_\rho \frac{(2k)!}{2^k k!}. \] (3.23)

**Proposition 3.8.** Let \( W_1, \ldots, W_n \) be independent and identically distributed random variables with the unit-log-normal distribution in (3.9), where \( \sigma_\rho \) is unknown. Then, the ML estimator \( \hat{\sigma}^2_\rho \) of \( \sigma^2_\rho \) is
1. unbiased;
2. consistent;
3. asymptotically normal; that is, \( \sqrt{n}(\hat{\sigma}^2_\rho - \sigma^2_\rho) \) converges in distribution to a normal distribution with mean zero and variance \( 2\sigma^2_\rho \);
4. efficient.

**Proof.** By taking \( n = 1 \) in (3.23), the unbiasedness of \( \hat{\sigma}^2_\rho \) follows.

By the Strong Law of Large Numbers and the Central Limit Theorem, it follows that \( \hat{\sigma}^2_\rho \) is consistent and asymptotically normal.

Finally, since the variance of \( \hat{\sigma}^2_\rho \) coincides with the Cramér-Rao lower bound \( [n I(\sigma_\rho)]^{-1} \), where \( I(\sigma_\rho) = -E[\partial^2 \log f_W(w; \sigma_\rho) / \partial (\sigma^2_\rho)^2] = 1 / (2\sigma^2_\rho) \) is the Fisher information in one observation from \( f_W(w; \sigma_\rho) \), the efficiency of \( \hat{\sigma}^2_\rho \) follows readily. \( \square \)
3.7.2 Unit-log-Student-t

From Subsubsection 3.2.2, \( Z_\rho \sim t_\nu \). Then, \( G(x) \) in (3.21) and its derivative are given by \( G(x) = -(\nu + 1)x/(\nu + x^2) \) and \( G'(x) = -(\nu + 1)(\nu - x^2)/(\nu + x^2)^2 \). Thus, by using (3.22), the ML estimate of \( \sigma_\rho^2 \) in the loose sense satisfies

\[
\frac{\nu + 1}{n} \sum_{i=1}^{n} \log^2 \left( \frac{w_i}{1-w_i} \right) = 1. \tag{3.24}
\]

**Proposition 3.9.** Any ML estimate of \( \sigma_\rho^2 \) in the loose sense is an ML estimate in the strict sense.

**Proof.** Suppose \( \hat{\sigma}_\rho^2 \) is an ML estimate of \( \sigma_\rho^2 \) in the loose sense. From (3.20), it follows that \( \hat{\sigma}_\rho^2 \) is an ML estimate in the strict sense whenever

\[
\frac{\nu + 1}{n} \sum_{i=1}^{n} \log^2 \left( \frac{w_i}{1-w_i} \right) \frac{\nu \hat{\sigma}_\rho^2 - \log^2 \left( \frac{w_i}{1-w_i} \right)}{\nu \hat{\sigma}_\rho^2 + \log^2 \left( \frac{w_i}{1-w_i} \right)} > -1.
\]

By using the partial fraction expansion

\[
\frac{x - y}{(x + y)^2} = \frac{2x}{(x + y)^2} - \frac{1}{x + y},
\]

the above inequality is written as

\[
\frac{2\nu(\nu + 1)\hat{\sigma}_\rho^2}{n} \sum_{i=1}^{n} \left[ \frac{\nu \hat{\sigma}_\rho^2 - \log^2 \left( \frac{w_i}{1-w_i} \right)}{\nu \hat{\sigma}_\rho^2 + \log^2 \left( \frac{w_i}{1-w_i} \right)} \right] - \frac{\nu + 1}{n} \sum_{i=1}^{n} \frac{\log^2 \left( \frac{w_i}{1-w_i} \right)}{\nu \hat{\sigma}_\rho^2 + \log^2 \left( \frac{w_i}{1-w_i} \right)} > -1.
\]

As \( \hat{\sigma}_\rho^2 \) is an ML estimate in the loose sense, it satisfies (3.24), and consequently, the above inequality is true. This completes the proof of the result. \( \square \)

3.7.3 Unit-log-Laplace

In this case, note that \( f_W(w; \sigma_\rho) = \exp[\theta T(w) - \psi(\theta)]h(w), W \sim ULS(\sigma_\rho, g_c) \), with \( \theta = 1/\sigma_\rho \),

\[
T(w) = -\sqrt{2} \left| \log \left( \frac{w}{1-w} \right) \right|, \quad \psi(\theta) = -\log(\theta) \quad \text{and} \quad h(w) = \frac{1}{\sqrt{2}w(1-w)}.
\]

Then, \( f_W \) belongs to the one-parameter exponential family with \( \Theta = \{\theta : \theta > 0\} \) parameter space.
Proposition 3.10. 1. For \( \theta \in \Theta^o \), the interior of \( \Theta \), all moments of \( T(W) \) exist, and \( \psi(\theta) \) is infinitely differentiable at any such \( \theta \). Furthermore,
\[
\mathbb{E} \left( \sqrt{2} \left| \log \left( \frac{W}{1 - W} \right) \right| \right) = \sigma_\rho \quad \text{and} \quad \text{Var} \left( \sqrt{2} \left| \log \left( \frac{W}{1 - W} \right) \right| \right) = \sigma_\rho^2,
\]

2. Given an i.i.d. sample of size \( n \) from \( f_W(w; \sigma_\rho) \),
\[
- \sqrt{2} \sum_{i=1}^{n} \left| \log \left( \frac{W}{1 - W} \right) \right|
\]
is minimally sufficient;

3. The Fisher information function exists, is finite at all \( \theta \in \Theta^o \) and equals \( I(\theta) = [I(\sigma_\rho)]^{-1} = \sigma_\rho^2 \).

Proof. The proof follows by direct application of Proposition 16.1 of DasGupta (2008), and is therefore omitted for the sake of conciseness.

Proposition 3.11. Let \( \theta = \theta_0 \in \Theta^o \) be the true value. Then, for all large \( n \), with probability 1, a unique ML estimator of \( \sigma_\rho \) exists, and is consistent and asymptotically normal.

Proof. As \( \psi''(\theta) = \sigma_\rho^2 > 0 \), by Theorem 16.1 of DasGupta (2008), a unique ML estimator of \( \theta = 1/\sigma_\rho \) exists, and is consistent and asymptotically normal. Hence, by the invariance property of the ML estimator, by the continuous application theorem for convergence in probability and by delta method, the proof of the result gets completed.

Furthermore, in the unit-log-Laplace case, the likelihood function for \( \sigma_\rho \) is given by
\[
L(\sigma_\rho) = \prod_{i=1}^{n} \frac{1}{w_i(1 - w_i)\sigma_\rho} f_\ell(A(w_i)),
\]
where \( A \) is as given in (3.5) and \( f_\ell \) is the PDF of Laplace distribution with location parameter 0 and scale parameter \( 1/\sqrt{2} \). Because \( f_\ell(x) \) is not differentiable at \( x = 0 \) we cannot use the log-likelihood function differentiation method as in the last two cases. In what follows, we maximize \( L(\sigma_\rho) \).

Indeed, an algebraic manipulation yields
\[
L(\sigma_\rho) = \frac{1}{2^{n/2}} \left[ \prod_{i=1}^{n} \frac{1}{w_i(1 - w_i)} \right] \exp \left( -\frac{\sqrt{2}}{\sigma_\rho} \sum_{i=1}^{n} \log \left( \frac{w_i}{1 - w_i} \right) \right) \frac{\sigma_\rho^n}{\sigma_\rho^n}.
\]
As the function \( \sigma_\rho \mapsto \exp(-t/\sigma_\rho)/\sigma_\rho^n, t > 0 \), reaches a maximum at \( \sigma_\rho = t/n \), the right-hand expression in the above equation is at most
\[
\frac{\exp(-n)}{2^n} \left[ \prod_{i=1}^{n} \frac{1}{w_i(1 - w_i)} \right] \left/ \left[ \frac{1}{n} \sum_{i=1}^{n} \log \left( \frac{w_i}{1 - w_i} \right) \right] \right.^n.
\]
So, the ML estimate of $\sigma_\rho^2$ is

$$\hat{\sigma}_\rho^2 = \frac{\sqrt{2}}{n} \sum_{i=1}^{n} \left| \log \left( \frac{w_i}{1 - w_i} \right) \right|.$$ 

This corroborates the uniqueness of the ML estimator evidenced in Proposition 3.11. Moreover, from Item 1 of Proposition 3.10, the ML estimator of $\sigma_\rho^2$ is unbiased and consistent.

4 Simulation results

In this section, we present the results of Monte Carlo simulation studies for unit-log-normal model (the results of the unit-log-Student-$t$ and unit-log-Laplace are somewhat similar and so are omitted here due to space limitations), considering different scenarios of parameters and sample sizes. The first part of the study presents the evaluation of the ML estimates, while the second part evaluates the empirical distribution of the generalized Cox-Snell (GCS) and randomized quantile (RQ) residuals, which are given, respectively, by

$$\hat{r}_{GCS}^i = - \log [1 - \hat{F}_W(w_i; \hat{\sigma}_\rho)] \quad \text{and} \quad \hat{r}_{RQ}^i = \Phi^{-1}(\hat{F}_W(w_i; \hat{\sigma}_\rho)), \quad i = 1, \ldots, n,$$

where $\hat{F}_W$ is the fitted unit-log-normal or unit-log-Student-$t$ CDF, and $\hat{\sigma}_\rho$ is the ML estimate of $\sigma_\rho$. When the models are correctly specified, the GCS is asymptotically standard exponentially distributed, while the RQ is asymptotically standard normally distributed. The distributions of these residuals are important as they will be used to assess the goodness-of-fit of the considered model.

Both studies consider simulated data generated from each of the unit-log-normal and unit-log-Student-$t$ models according to the following stochastic relation:

$$W = A^{-1}(F_\nu^{-1}(U)) \overset{(3.7)}{=} \frac{1}{1 + \exp(-\sigma_\rho F_\nu^{-1}(U))},$$

where $F_\nu$ is the unit-log-normal or unit-log-Student-$t$ CDF and $U \sim U[0,1]$. The Monte Carlo simulation experiments were performed using the R environment; see http://www.r-project.org.

4.1 Maximum likelihood estimation

The simulation scenario used the following settings: sample size $n \in \{50, 100, 150, 250, 600\}$ and true parameter $\sigma_\rho \in \{0.15, 0.25, 0.50, 1.0, 1.5\}$, with 500 Monte Carlo replications for each sample size. To study the performance of the ML estimators, we computed the bias and root mean square error (RMSE), defined by

$$\text{Bias}(\hat{\theta}) = \frac{1}{M} \sum_{i=1}^{M} \hat{\theta}^{(i)} - \theta, \quad \text{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (\hat{\theta}^{(i)} - \theta)^2},$$
where $\theta$ and $\hat{\theta}^{(i)}$ are the true parameter value and its respective $i$-th ML estimate, and $M$ is the number of Monte Carlo replications.

The ML estimation results are shown in Figure 6 wherein the empirical bias and RMSE are both presented. It can be seen that the simulations produced the expected outcomes, namely, that the bias and RMSE both approach zero as $n$ increases.

![Graph showing empirical bias and RMSE for different values of $\sigma_\rho$.](image)

**Fig. 6:** Monte Carlo simulation results for the unit-log-normal model.

### 4.2 Empirical distribution of residuals

In this subsection, we evaluate the performance of GCS and RQ residuals. We present the empirical means of the following descriptive statistics: mean, median, standard deviation (Sd), coefficient of skewness and coefficient of kurtosis, whose values are expected to be 1, 0.69, 1, 2 and 6, respectively, for the GCS residuals, and 0, 0, 1, 0 and 0, respectively, for the RQ residuals. From Figures 7 and 8, we note that the considered residuals conform well with their reference distributions, and we can therefore use the RQ and GCS residuals to verify well the fit of the proposed models.

### 5 Illustration with internet accesss data

In this section, a real-world data set, corresponding to the share of the population using internet, is analyzed. Figure 9 shows the share of the population that is accessing internet for 201 countries of the world in 2015. Here, the internet can be accessed through a computer, mobile phone, games machine, digital TV, etc.; see [https://ourworldindata.org/internet](https://ourworldindata.org/internet). From Figure 9, we observe high rates of people online in richer countries and much lower rates in the developing world. For example, three-quarters (74.55%) of people in the US were online, while in India only 14.9% used the internet.

In terms of stochastic representation, the share of the population using internet, represented by the random variable $W$, can be expressed as

$$W = \frac{T_1}{T_1 + T_2} \quad (5.1)$$
Fig. 7: Monte Carlo simulation results of the RQ residuals for the unit-log-normal model.

Fig. 8: Monte Carlo simulation results of the GCS residuals for the unit-log-normal model.
where $T_1 \in \mathbb{R}^+$ and $T_2 \in \mathbb{R}^+$ are two random variables following log-symmetric distributions representing the number of people with internet access and the number of people without internet access, respectively; in other words, the sum $T_1 + T_2$ represents the total population while the ratio in (5.1) has support in the unit interval $(0,1)$ with PDF in (2.1).

Table 5 provides descriptive statistics for the share of the population using the internet, including the mean, median, standard deviation (SD), coefficient of variation (CV), skewness (CS), (excess) kurtosis (CK), and minimum and maximum values. From this table, we note that the mean is almost equal to the median. In addition, the CV (dispersion around the mean) is less than 100%, indicating a low dispersion of data around the mean. Finally, we observe that the data show skewness near zero and low degree of kurtosis.

**Tab. 5:** Summary statistics for the internet data.

| Mean  | Median | SD      | CV   | CS    | CK    | minimum | maximum | size |
|-------|--------|---------|------|-------|-------|---------|---------|------|
| 0.480 | 0.497  | 0.289   | 60.276% | 0.008 | -1.316 | 0.011   | 0.983   | 201  |

Table 6 presents the ML estimates and SEs for the unit-log-normal, unit-log-Student-$t$ and unit-log-Laplace model parameters. This table also presents the log-likelihood value, and Akaike (AIC) and Bayesian information (BIC) criteria. For comparative purpose, the results of the beta model (Ferrari and Cribari-Neto, 2004) are provided as well. The results in Table 6 reveal that
the proposed unit-log-normal and unit-log-Student-\(t\) models provide better adjustments than the beta model based on the values of log-likelihood, AIC and BIC. Furthermore, the unit-log-normal model provides the best fit to these data.

**Tab. 6:** ML estimates (with SE in parentheses) and model selection measures for fit to the internet data.

|               | Unit-log-normal | Unit-log-Student-\(t\) | Unit-log-Laplace | Beta    |
|---------------|-----------------|------------------------|------------------|---------|
| \(\sigma\) (or \(\mu\)) | 1.6645 (0.0830)  | 1.5390 (0.0860)        | 1.9794 (0.2341)  | -0.7386 (0.0397) |
| \(\nu\) (or \(\phi\)) | 10              |                        |                  | 2.1860 (0.1849)  |
| Log-lik.      | 3.2534          | 1.2758                 | -21.9354         | 1.1450   |
| AIC           | -4.5068         | -0.5516                | 45.8709          | 1.7098   |
| BIC           | -1.2035         | 2.7517                 | 49.1742          | 8.3164   |

Figure 10 shows the QQ plots with simulated envelope of the GCS and RQ residuals for the unit-log-normal, unit-log-Student-\(t\), unit-log-Laplace and beta models considered in Table 6. We observe that the unit-log-normal model provides better fit than the unit-log-Student-\(t\), unit-log-Laplace and beta models. Figure 11 displays the histogram and superimposed fitted PDFs, and fitted CDFs (empirical CDF in gray).

6 Concluding remarks

In the present paper, we have introduced a family of flexible models over the interval \((0, 1)\). By suitably defining the density generator, we can transform any distribution over the real line into a bounded distribution over the interval \((0, 1)\). In this way, we have transformed the normal, Student-\(t\) and Laplace distributions into unit-log-normal, unit-log-Student-\(t\) and unit-log-Laplace distributions, respectively. These three cases have been studied at great length. However, many other unit-log-symmetric distributions, such as the unit-log Kotz type, unit-log Pearson Type VII, unit-log hyperbolic, unit-log slash, unit-log-logistic and unit-log-power exponential distributions can all be studied similarly. These may be useful alternatives to “conventional” models like beta generalizations, Kumaraswamy and the Topp-Leone distributions.

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Fig. 10: QQ plot and its envelope for the GCS (top) and RQ (bottom) residuals in the indicated model for the internet data.

Fig. 11: Histogram of the internet data with fitted PDFs (left) and fitted CDFs (right).

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