Accelerating Power Methods for Higher-order Markov Chains

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Abstract

Higher-order Markov chains play a very important role in many fields, ranging from multilinear PageRank to financial modeling. In this paper, we propose three accelerated higher-order power methods for computing the limiting probability distribution of higher-order Markov chains, namely higher-order power method with momentum and higher-order quadratic extrapolation method. The convergence results are established, and numerical experiments are reported to show the efficiency of the proposed algorithms. In particular, the non-parametric quadratic extrapolation method is very competitive, and outperforms state-of-the-art competitors.

Keywords: Higher-order Markov chains, Limiting probability distribution vector, Transition probability tensor, Power method, Quadratic extrapolation, Momentum methods

1 Introduction

Markov chains are powerful tools to analyze and predict traffic flows, communications networks, genetic issues, and a variety of stochastic (probabilistic) processes over time, in which the probability of each event depends only on the state attained in the previous event. Considering a stochastic process \( \{X_t, t = 0, 1, 2, \ldots\} \) that takes on a finite set \( \{1, 2, \ldots, n\} \equiv \langle n \rangle \). An element in \( \langle n \rangle \) is called a state of the process. The definition of a Markov Chain can be given as follows.

Definition 1 Assume there exists a fixed probability \( p_{i,j} \) independent of time such that

\[
Prob(X_{t+1} = i|X_t = j, X_{t-1} = i_{t-1}, \ldots, X_0 = i_0) = Prob(X_{t+1} = i|X_t = j) = p_{i,j},
\]

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where \( i, j, i_0, i_1, \ldots, i_{t-1} \in \langle n \rangle \) and \( \{X_t\}(t = 0, 1, 2, \ldots) \) is a stochastic process. Then this is called a Markov chain process.

The probability \( p_{i,j} \) represents the probability that the process will make a transition to state \( i \) given that currently the process is state \( j \). Clearly one has

\[
p_{i,j} \geq 0, \quad \sum_{j=1}^{n} p_{i,j} = 1, \quad j = 1, \ldots, n.
\]

The matrix \( P = (p_{i,j}) \) is called the one-step transition probability matrix of the process. A vector \( \bar{x} \) is said to be a limiting or stationary probability distribution of a finite Markov chain having \( n \) states with

\[
\bar{x}_i \geq 0, \quad \forall i, \quad \sum_{i=1}^{n} \bar{x}_i = 1, \quad \text{and} \quad P\bar{x} = \bar{x}.
\]

In real world, there are many situations that one would like to employ higher-order Markov chain models as a mathematical tool to analyze data sequences, in which the probability of \( X_{t+1} = i \) not only depends on the adjacent time state \( X_t \) but also depends on more previous time states. The \((m - 1)^{th}\) order Markov chain model is given as follows.

**Definition 2** Assume there exists a fixed probability \( p_{i_1,i_2,\ldots,i_m} \) independent of time such that

\[
0 \leq p_{i_1,i_2,\ldots,i_m} = \text{Prob}(X_{t+1} = i_1 | X_t = i_2, \ldots, X_{t-m+2} = i_m) \leq 1,
\]

where \( i_1, \ldots, i_m \in \langle n \rangle \) and \( \sum_{i_1=1}^{n} p_{i_1,i_2,\ldots,i_m} = 1 \). Then this is called a \((m - 1)^{th}\) order Markov chain process.

It is clear that the \((m - 1)^{th}\) order Markov chain process will reduces to first-order Markov chain when \( m = 2 \). The probability \( p_{i_1,i_2,\ldots,i_m} \) represents that process make transition to the state \( i_1 \) given that currently the process is in the state \( i_2 \) and previously the process is in the states \( i_3, \ldots, i_m \). Tensor \( \mathcal{P} = (p_{i_1,i_2,\ldots,i_m}) \) is called transition probability tensor. A number of applications can be found in the literature, for example, chemistry\cite{7, 15}, physics\cite{1} and multilinear PageRank\cite{9}.

In \cite{17}, Li and Ng established the following approximated tensor model for the stationary probability distribution of Higher-order Markov chains:

\[
x = \mathcal{P}x^{m-1}, \quad x \geq 0, \quad \|x\|_1 = 1,
\]

where \( \mathcal{P}x^{m-1} \) is defined by:

\[
(\mathcal{P}x^{m-1})_i = \sum_{i_2,\ldots,i_m=1}^{n} p_{i_1i_2\ldots i_m}x_{i_2}\cdots x_{i_m}, \quad i = 1, 2, \ldots, n.
\]
and $x = (x_i)$ is called a stationary probability distribution vector of higher-order Markov chains. The stationary probability distribution vector is unique under suitable conditions [5, 8, 17, 10].

Later, many researchers employed the higher-order Markov chains model to explore some applications such as in random walk [3] and multilinear PageRank [9]. Gleich, Lim and Yu [9] first studied the following multilinear PageRank model:

$$x = \theta \hat{P} x^{m-1} + (1 - \theta)v,$$

(2)

where tensor $\hat{P}$ is a transition probability tensor, $v$ is transition probability vector, and $\theta \in (0, 1)$ is a damping parameter. We can rewrite the equation (2) as follows

$$x = \mathcal{P} x^{m-1}, \quad \|x\|_1 = 1, \quad \mathcal{P} = \theta \hat{P} + (1 - \theta) * \mathcal{V},$$

(3)

where $\mathcal{V} = (v_{i_1i_2...i_m})$ with $v_{i_1i_2...i_m} = v_{i_1}, \forall i_2, \ldots, i_m$. It is easy to see that the tensor $\mathcal{P}$ is also a transition probability tensor.

Recently, Li et al. [16] investigated the uniqueness of the fixed-point for the equation (3) and presented perturbation analysis for Multilinear PageRank model (2). In [9], several iterative algorithms (a fixed-point algorithm, a shifted fixed-point algorithm, a inner-outer iteration algorithm, an inverse iteration algorithm and a Newton algorithm) are proposed by Gleich et al. Furthermore, Meini and Poloni [20] proposed the Perron-based iteration and Cipolla et al. [4] presented some extrapolation methods for fixed-point multilinear PageRank computations.

As for solving the tensor equations (1), Li and Ng [17] extended the power method to compute the tensor equation (1). They given the convergence analysis of the proposed iterative algorithm. In [18], Liu et al. proposed several relaxation algorithms for solving equation (1). And a truncated power method is presented in [6] for sparse Markov chains. Power-type methods are very popular due to their simplicity and efficiency, especially for large-scale problems [6, 9]. However, as shown in [26], the convergence rate of the higher-order power method will be slow when the spectral gap is small. Moreover, they point out that there exists irreducible and aperiodic transition probability tensors where the Z-eigenvector type power iteration fails to converge.

A pair $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n \backslash \{0\}$ is called Z-eigenpair of the tensor $\mathcal{P}$ if

$$\mathcal{P} x^{m-1} = \lambda x, \quad \|x\| = 1.$$  

(4)

This definition was proposed by Qi [11] and Lim [19], independently. Here, $\| \cdot \|$ could be $l_1$-norm or $l_2$-norm. Moreover, for $x \in \mathbb{R}^n$ with $\|x\|_1 = 1$, $(x, \lambda)$ is the $Z_1$-eigenpair if and only if $(\frac{x}{\|x\|_1}, \frac{\lambda}{\|x\|_2})$ is the corresponding $Z_2$-eigenpair [5]. It is clear that the solution to the tensor equations (1), i.e. the stationary probability distribution vector $x$, is the $Z_1$-eigenvector of $\mathcal{P}$ while 1 is the largest $Z_1$-eigenvalue.

In this paper, we propose three algorithms for solving the stationary probability distribution of higher-order Markov chains by accelerating the higher-order power method. The main contributions of this paper are
• to present two novel higher-order power methods with momentum for Z-eigenvector computations of tensor;
• to propose a non-parametric higher-order quadratic extrapolation method to compute the stationary probability distribution of higher-order Markov chains;
• to establish the convergence theorems for the proposed algorithms;
• to use the proposed algorithms for some applications such as fixed-point multilinear PageRank computations.

The rest of this paper is organized as follows. Firstly, some preliminary knowledge and existing methods are presented in Section 2. In Section 3, we propose three provable power-type methods for calculating the limiting probability distribution vector of higher-order Markov chains. In Section 4, the convergence theorems for the proposed methods are established. Numerical experiments are given and analyzed in Section 5. The last section is the conclusions.

Before end of this section, we would like to describe notations and show some preliminary knowledge on tensors. For the details of basic definitions and properties of tensors, we refer to reader to [12] and the references therein.

Let $\mathbb{R}$ be the real field. An $m$th-order $n$-dimensional real tensor $\mathcal{P}$ consists of $n^m$ entries in real numbers: $\mathcal{P} = (a_{i_1i_2\ldots i_m})$, $p_{i_1i_2\ldots i_m} \in \mathbb{R}$, for any $i_1, i_2, \ldots, i_m \in [n]$, where $[n] = \{1, 2, \ldots, n\}$. $\mathcal{P}$ is called non-negative (or, respectively, positive) if $p_{i_1i_2\ldots i_m} \geq 0$ (or, respectively, $p_{i_1i_2\ldots i_m} > 0$). Given two vectors $x, y \in \mathbb{R}^n$, we define

$$\mathcal{P}(x^{m-1} - y^{m-1}) \equiv \mathcal{P}x^{m-1} - \mathcal{P}y^{m-1}.$$

Let $x_+ = \max(x, 0)$ and $\text{proj}(x) = \frac{x_+}{\|x_+\|}$. It is easy to get that $\text{proj}(x)$ is a transition probability vector. We also show the definition of irreducible tensors as follows.

**Definition 3** An $m$-order $n$-dimensional tensor $\mathcal{P}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$P_{i_1i_2\ldots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.$$

If $\mathcal{P}$ is not reducible, then we call $\mathcal{P}$ irreducible.

## 2 The existing methods

To some extent, the limit probability distribution problem (1) can be transformed into an optimization problem:

$$\min f(x) = \min_{\|x\|_1=1} \frac{1}{2} x^T x - \frac{1}{m} x^T \mathcal{P}x^{m-2}x,$$
where $\mathcal{P}_{x^{m-2}}$ is a matrix with its component as

$$(\mathcal{P}_{x^{m-2}})_{i_1i_2} = \sum_{i_3, \ldots, i_m=1}^{n} p_{i_1i_2i_3\ldots i_m} x_{i_3} \cdots x_{i_m} \text{ for all } i_1, i_2 = 1, \ldots, n.$$ 

Let $\nabla f(x) = x - \mathcal{P}_{x^{m-1}}$, and minimizing the above problems via gradient descent scheme

$$x_k = x_{k-1} - \alpha \nabla f(x_{k-1}) = x_{k-1} - \alpha (x_{k-1} - \mathcal{P}_{x_{k-1}^{m-1}}),$$

setting $\alpha = 1$, then we can get the following higher-order power method (HOPM) for solving the tensor equation [1].

**Algorithm 1: higher-order power method (HOPM) [17]**

1. Given a transition probability tensor $\mathcal{P}$, maximum $k_{\text{max}}$, termination tolerance $\epsilon$ and an initial point $x_0$;
2. Initialize $k = 1$.
3. $x_k = \mathcal{P}_{x_{k-1}^{m-1}}$;
4. $\delta = \|x_k - x_{k-1}\|$;
5. $k = k + 1$;
6. until $\delta < \epsilon$.

**Remark 1.** The main computational cost of the algorithm depends on the cost of performing tensor operation. Assume that there are $O(N)$ nonzero entries (sparse data) in tensor $\mathcal{P}$, the cost of this tensor calculation are of $O(N)$ arithmetic operations. Under some suitable conditions, they established the linear convergence of the above algorithm.

In [14], Kolda and Mayo presented an adaptive, monotonically convergent, shifted power method for computing tensor $Z$-eigenpairs, called GEAP method.

**Algorithm 2: Z-Eigenpair Adapative Power Method (GEAP Method) [14]**

Given a transition probability tensor $\mathcal{P}$, maximum $k_{\text{max}}$, termination tolerance $\epsilon$ and an initial point $x_0$; Let $\tau > 0$ is the tolerance on being positive definite.

Initialize $k = 1$.
1. Precompute $\mathcal{P}_{x_{k-1}^{m-2}}, \mathcal{P}_{x_{k-1}^{m-1}}$
2. $H_{k-1} \leftarrow H(x_{k-1}) = m(m-1)\mathcal{P}_{x_{k-1}^{m-2}}$
3. $\alpha_k \leftarrow \max\{0, (\tau - \lambda_{\text{min}}(H_{k-1}))/m\}$
4. $\hat{x}_k \leftarrow A_{x_{k-1}^{m-1}} + \alpha_k x_{k-1}, x_k = \text{proj}(\hat{x}_k)$.
5. $\delta = \|x_k - x_{k-1}\|$;
6. $k = k + 1$;
7. until $\delta < \epsilon$. 

5
In [18], Liu et al. proposed several relaxation methods for computing tensor equation (1). In particular, by using relaxation technique to the higher-order power method, they developed a novel algorithm as follows.

Algorithm 3: Relaxation higher-order power method, RHOPM (Alg.2 in [18])
1. Given a transition probability tensor $P, \gamma > 0$, termination tolerance $\epsilon$ and an initial point $x_0$;
2. Initialize $k = 1$.
3. $y_k = P x_{k-1}^{m-1}$;
4. $\hat{x}_k = \gamma y_k + (1 - \gamma)x_{k-1}$, $x_k = \text{proj}(\hat{x}_k)$
5. $\delta = \|x_k - x_{k-1}\|$
6. $k = k + 1$;
7. until $\delta < \epsilon$.

3 Accelerating Power methods

In this section, we will propose three accelerated higher-order power methods, including two higher-order power methods with momentum term and quadratic extrapolation higher-order power method, respectively.

3.1 Higher-order power methods with momentum

In [23, 21], some accelerated first-order methods are proposed by adding momentum terms to classic gradient method, called heavy-ball method and Nesterov’s accelerated gradient method (NAG), respectively. Recently, Xu et al. [27] proposed a power method with momentum for principal component analysis. Along this line, motivated by the efficiency of momentum methods, we propose the following two algorithms for solving the tensor equations (1) by respectively adding two different momentum terms to higher-order power method, referred to as the HOPMM-I and HOPPM-II.

HOPMM-I
1. Given a transition probability tensor $P$, maximum $k_{\text{max}}, \beta > 0$, termination tolerance $\epsilon$ and an initial point $x_0$;
2. Initialize $k = 1$.
3. repeat
4. $x_k = P x_{k-1}^{m-1}$;
5. periodically;
6. $\hat{x}_k = x_k + \beta(x_{k-1} - x_{k-2})$, $x_k = \text{proj}(\hat{x}_k)$;
7. If $\|x_k - x_{k-1}\| < \epsilon$, break and output $x_k$;
8. \( k = k + 1 \), back to step 4.

**Remark 2.** The \( \beta(x_{k-1} - x_{k-2}) \) is called momentum term. By choosing a suitable parameter \( \beta \), the HOPMM-I will performs better than higher-order power method. In particular, if \( \beta = 0 \), the HOPMM-I will reduces to higher-order power method that proposed by Li et al.\[17\]. Compared with the RHOPM, our proposed method uses three iterative points to generate next iterative point. In the HOPMM-I, we will execute the momentum extrapolation at every 3 steps.

Furthermore, we can also add a “heavy-ball” momentum term to higher-order power method and obtain the following algorithm.

**HOPMM-II**

1: Given a transition probability tensor \( P \), maximum \( k_{\text{max}} \), \( \eta > 0 \), termination tolerance \( \epsilon \) and an initial point \( x_0 \);
2: Initialize \( k = 1 \).
3. repeat
4: \( x_k = P x_{k-1}^{m-1} \);
5. periodically;
6: \( \hat{x}_k = x_k + \eta(x_k - x_{k-1}) \), \( x_k = \text{proj}(\hat{x}_k) \);
7: If \( \|x_k - x_{k-1}\|_1 < \epsilon \), break and output \( x_k \);
8: \( k = k + 1 \), back to step 4.

**Remark 3.** HOPMM-II is obtained by adding the “heavy ball” momentum term \( \eta(x_k - x_{k-1}) \) to higher-order power method. In the HOPMM-II, we will execute the momentum extrapolation at every 2 steps.

How to choose the parameters \( \beta, \eta \) is crucial for the performance of HOPMM-I and HOPMM-II. However, it is difficult to select the parameter \( \beta, \eta \) so far. Thus we further propose a free-parameter quadratic extrapolation method for solving the tensor equation \( (\Pi) \) in the following subsection.

### 3.2 Higher-order quadratic extrapolation method

In this subsection, we extend the quadratic extrapolation method in \[13\] for solving the tensor equation \( (\Pi) \), referred to as the QEHOPM. For the classic quadratic extrapolation method, Sidia in \[25\] has proved that this method is faster than power method. We develop the Quadratic Extrapolation Higher Order Power Method (QEHOPM) as follows.

Letting \( B = P \bar{x}^{m-2} \), where \( \bar{x} \) is a solution of equation \( (\Pi) \), it is clear that \( B \) is also a Markov Matrix, and \( \bar{x} \) is the principal eigenvector of \( B \). Assume that the matrix \( B = P \bar{x}^{m-2} \) has only 3 eigenvectors. Then, the iterate \( x_{k-3} \) can be expressed as a linear combination of these 3 eigenvectors. Of course, \( B \) has more than 3 eigenvectors, and
$x_{k-3}$ can only be approximated as a linear combination of the first three eigenvectors. Therefore, the $\hat{x}$ that we compute in QEHOPM algorithm is only an estimate for the true $\bar{x}$.

Similar to the quadratic extrapolation method in [13], we assume that $B$ has only three eigenvectors, and approximating $x_{k-3}$ as a linear combination of these three eigenvector. We then define the successive iterates $x_{k-2} = Bx_{k-3}, x_{k-1} = Bx_{k-2}, x_k = Bx_{k-1}$. From the analysis in Theorem 3, the QEHOPM will converge to the fixed-point $\bar{x}$. So, in practice, we can use $P_{x_{m-2}}$ to approximate $P_{x_{m-2}}$ in $k_{th}$ iteration.

Since we assume $B$ has 3 eigenvectors, the characteristic polynomial $p_B(\lambda)$ is given by

$$p_B(\lambda) = \gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2 + \gamma_3\lambda^3$$

Moreover, since $\lambda = 1$ is the leading eigenvalue,

$$p_B(\lambda) = (\lambda - 1)(\beta_0 + \beta_1\lambda + \beta_2\lambda^2) = (\lambda - 1)q_B(\lambda),$$

where $\beta_0 = \gamma_1 + \gamma_2 + \gamma_3$, $\beta_1 = \gamma_2 + \gamma_3$, $\beta_2 = \gamma_3$.

By the Cayley-Hamilton Theorem, for any vector $z \in \mathbb{R}^n$, we have

$$p_B(B)z = (\gamma_0I + \gamma_1B + \gamma_2B^2 + \gamma_3B^3)z = 0.$$  

Letting $z = x_{k-3}$, we obtain that

$$\gamma_0x_{k-3} + \gamma_1x_{k-2} + \gamma_2x_{k-1} + \gamma_3x_k = 0.$$  

Since $p_B(1) = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 = 0$, we have $\gamma_0 = -(\gamma_1 + \gamma_2 + \gamma_3)$. Letting $y_k = x_k - x_{k-3}$, $y_{k-1} = x_{k-1} - x_{k-3}$, $y_{k-2} = x_{k-2} - x_{k-3}$, combing with (7), we have

$$\gamma_1y_{k-2} + \gamma_2y_{k-1} + \gamma_3y_k = 0.$$  

Fixing $\gamma_3 = 1$, and then

$$\gamma_1y_{k-2} + \gamma_2y_{k-1} = -y_k.$$  

By Least-Square method and QR factorization, we can compute approximatively the above overdetermined system.

Again, by the Cayley-Hamilton Theorem, for any vector $z \in \mathbb{R}^n$, follows from (6), $q_B(B)z$ is the eigenvector of $B$ corresponding to eigenvalue 1 (the principal eigenvector). Letting $z = x_{k-2}$, we have

$$q_B(B)x_{k-2} = (\beta_0I + \beta_1B + \beta_2B^2)x_{k-2} = \beta_0x_{k-2} + \beta_1x_{k-1} + \beta_2x_k.$$  

Scaling the above equation by $1/(\beta_0 + \beta_1 + \beta_2)$, we have

$$\bar{x} \approx \frac{\beta_0}{\beta_0 + \beta_1 + \beta_2}x_{k-2} + \frac{\beta_1}{\beta_0 + \beta_1 + \beta_2}x_{k-1} + \frac{\beta_2}{\beta_0 + \beta_1 + \beta_2}x_k$$

$$= x_k + \frac{\beta_0}{\beta_0 + \beta_1 + \beta_2}(x_{k-2} - x_{k-1}) + \frac{\beta_0 + \beta_1}{\beta_0 + \beta_1 + \beta_2}(x_{k-1} - x_k).$$
So, according to (10), we know that QEHOPM algorithm is the power method with 2 momentum terms. In particular, all of the parameters could be calculated in closed form.

Now, the QEHOPM algorithm is shown as follows.

**QEHOPM**
1. Given a transition probability tensor $\mathcal{P}$, maximum $k_{\text{max}}$, $\beta > 0$, termination tolerance $\epsilon$ and an initial point $x_0$;
2. Initialize $k = 1$.
3. repeat
   4. $x_k = \mathcal{P}x_{k-1}^m$;
   5. $\delta = \|x_k - x_{k-1}\|$;
   6. periodically,
   7. $\hat{x}_k = \text{Quadratic Extrapolation}(x_{k-3}, \ldots, x_k)$, $x_k = \text{proj}(\hat{x}_k)$;
   8. $k = k + 1$;
9. until $\delta < \epsilon$

The quadratic extrapolation algorithm is defined as follows.

**Quadratic Extrapolation**
function $\hat{x} = \text{Quadratic Extrapolation}(x_{k-3}, \ldots, x_k)$
for $j = k - 2 : k$ do
$y_j = x_j - x_{k-3}$;
end
$Y = (y_{k-2}, y_{k-1}); \gamma_3 = 1$;
$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = -Y^+y_k$;
$\beta_0 = \gamma_1 + \gamma_2 + \gamma_3$;
$\beta_1 = \gamma_2 + \gamma_3$;
$\beta_2 = \gamma_3$;
$\hat{x} = \frac{\beta_0}{\beta_0 + \beta_1 + \beta_2}x_{k-2} + \frac{\beta_1}{\beta_0 + \beta_1 + \beta_2}x_{k-1} + \frac{\beta_2}{\beta_0 + \beta_1 + \beta_2}x_k$;
\}

Using the following Gram-Schmidt to solve $\gamma_1$ and $\gamma_2$.

**Gram-Schmidt**
1. Compute the reduced $QR$ factorization $Y = QR$ using 2 steps of Gram-Schmidt.
2. Compute the vector $-Q^Ty_k$.
3. Solve the upper triangular system:
$R \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = -Q^Ty_k$;
for $R\left(\frac{\gamma_1}{\gamma_2}\right)$ using back substitution.

In this paper, we will apply quadratic extrapolation at every 4 steps.

4 Convergence analysis for the proposed methods

In this section, we present the convergence analysis of the proposed algorithms. Before giving these Theorems, some lemmas that established by Li et al. in [17, 18] are shown as follows.

**Lemma 1** If $P$ is a non-negative transition probability tensor of order $m$ and dimension $n$, then there exists a nonzero non-negative vector $\bar{x}$ satisfies (1). In particular, if $P$ is irreducible, then $\bar{x}$ must be positive.

**Lemma 2** Suppose $P$ is a non-negative transition probability tensor of order $m$ and dimension $n$. If $\delta_m > \frac{m-2}{m-1}$, the $\delta_m$ is given as follows

$$
\delta_m := \min_{S \subset \langle n \rangle} \left\{ \min_{i_2, \ldots, i_m \in \langle n \rangle} \sum_{i \in S'} p_{i, i_2, \ldots, i_m} + \min_{i_2, \ldots, i_m \in \langle n \rangle} \sum_{i \notin S} p_{i, i_2, \ldots, i_m} \right\}.
$$

(11)

where $\langle n \rangle = \{1, 2, \ldots, n\}$, $S$ is a subset of $\langle n \rangle$ and $S'$ be its complementary set in $\{1, 2, \ldots, n\}$, i.e., $S' = \{1, 2, \ldots, n\} \setminus S$. then the nonzero non-negative vector $\bar{x}$ in Lemma 1 is unique.

Lemma 1 and Lemma 2 give the existence and uniqueness conditions of the solution for equation (1), respectively.

**Lemma 3** Suppose $P$ is a non-negative transition probability tensor of order $m$ and dimension $n$ and $x, y \in \mathbb{R}^n$ are transition probability vectors. Then we have

$$
\|P(x^{m-1} - y^{m-1})\|_1 \leq \eta_m \|x - y\|_1,
$$

(12)

where $\eta_m = (1 - \delta_m)(m - 1)$.

The proof of Lemma 3 can be found in the Lemma 2 of [18].

**Lemma 4** Let $\hat{x}, y \in \mathbb{R}^n$ and $\|\hat{x}\|_1 = 1, \|y\|_1 = 1$. If $x = \text{proj}(\hat{x})$, then $\|\hat{x} - y\|_1 \geq \|x - y\|_1$.

The proof can be obtained by Lemma 3 of [18].

Based on these above Lemmas, we establish the following convergence Theorems for HOPMM-I, HOPMM-II and QEHOPM, respectively.
4.1 Convergence analysis for HOPMM-I and HOPMM-II

**Theorem 1** Let $\mathcal{P}$ be a non-negative transition probability tensor of order $m$ and dimension $n$ with $\delta_m > \frac{m-2}{m-1}$ and $\bar{x}$ is a solution of equation (11). Then, if $\beta < 1 - \eta_m$ the iterative sequence $\{x_k\}$ generated by HOPMM-I exists a convergent subsequence $\{x_{k_n}\}$ that converges to the solution $\bar{x}$ for any initial transition probability vector $x_0$, i.e.,

$$\lim_{n \to \infty} x_{k_n} = \bar{x}. \quad (13)$$

**Proof.** According to the Lemma 2 and condition $\delta_m > \frac{m-2}{m-1}$, we get that equation (11) has a unique solution $\bar{x}$. From the HOPMM-I, it is easy to get that $x_k \geq 0$ for all $k$.

Let $\hat{e}_k = \hat{x}_k - \bar{x}$ and $e_k = x_k - \bar{x}$. By Algorithm 1, we can obtain

$$\hat{e}_k = x_k + \beta (x_{k-1} - x_{k-2}) - \bar{x}, \quad (14)$$

where $\beta > 0$.

By substituting the $\bar{x} = \mathcal{P} \bar{x}^{m-1}$ and $x_k = \mathcal{P} x_k^{m-1}$ into (14), we have

$$\hat{e}_k = \mathcal{P}(x_{k-1} - \bar{x}^{m-1}) + \beta (x_{k-1} - \bar{x}) + \beta (\bar{x} - x_{k-2})$$

$$= \mathcal{P}(x_{k-1} - \bar{x}^{m-1}) + \beta e_{k-1} - \beta e_{k-2}. \quad (15)$$

By Lemma 2, we have

$$\|\mathcal{P}(x_{k-1} - \bar{x}^{m-1})\|_1 \leq \|e_{k-1}\|_1. \quad (16)$$

and

$$\|e_{k-1}\|_1 = \|\mathcal{P} x_{k-2}^{m-1} - \mathcal{P} \bar{x}^{m-1}\|_1 \leq \eta_m \|e_{k-2}\|_1 \quad (17)$$

Then, by (16) and (17), we have

$$\|\hat{e}_k\|_1 \leq (\eta_m + \beta) \|e_{k-1}\|_1 + \beta \|e_{k-2}\|_1$$

$$\leq (\eta_m + \beta) \eta_m \|e_{k-2}\|_1 + \beta \|e_{k-2}\|_1 \quad (18)$$

By Lemma 1, we can obtain

$$\|e_k\|_1 \leq \|\hat{e}_k\|_1 \leq [(\eta_m + \beta) \eta_m + \beta] \|e_{k-2}\|_1. \quad (19)$$

If $\beta < 1 - \eta_m$, it is easy to get $(\eta_m + \beta) \eta_m + \beta < 1$, which proves that the iterative sequence $\{x_k\}$ generated by HOPMM-I exists a convergent subsequence $\{x_{k_n}\}$.

**Theorem 2** Let $\mathcal{P}$ be a non-negative transition probability tensor of order $m$ and dimension $n$ with $\delta_m > \frac{m-2}{m-1}$ and $\bar{x}$ is a solution of equation (11). Then, if $\eta < \frac{1 - \delta_m}{1 + \delta_m}$, the iterative sequence $\{x_k\}$ generated by HOPMM-II converges to the solution $\bar{x}$ for any initial transition probability vector $x_0$. Furthermore, we have the following error bound

$$\|x_k - \bar{x}\|_1 \leq \epsilon_k \|x_0 - \bar{x}\|_1. \quad (20)$$

where $\epsilon_k = (\eta + 1) \eta_m + \eta$.
Proof. According to the Theorem 1, we get that equation (1) has a unique solution \( \bar{x} \).

From the HOPMM-II, it is obvious that \( x_k \geq 0 \) for all \( k \).

Let \( \hat{e}_k = \hat{x}_k - \bar{x} \) and \( e_k = x_k - \bar{x} \). By Algorithm 2, we obtain
\[
\hat{e}_k = x_k + \eta (x_k - x_{k-1}) - \bar{x},
\]
where \( \eta > 0 \).

By substituting the \( \bar{x} = P \bar{x}^{m-1} \) and \( x_k = P x_k^{m-1} \) into (21), we have
\[
\hat{e}_k = P (x_k^{m-1} - \bar{x}^{m-1}) + \eta P (x_{k-1}^{m-1} - \bar{x}^{m-1}) - \eta e_{k-1}
\]
(22)

By Lemma 3, we have
\[
\|P (x_k^{m-1} - \bar{x}^{m-1})\|_1 \leq \eta_m \|e_{k-1}\|_1.
\]
(23)

Then, by (22) and (23), we have
\[
\|\hat{e}_k\| \leq [(1 + \eta) \eta_m + \eta] \|e_{k-1}\|_1
\]
(24)

By Lemma 3, we get
\[
\|e_k\| \leq \|\hat{e}_k\| \leq [(1 + \eta) \eta_m + \eta] \|e_{k-1}\|_1.
\]
(25)

It follows from (25) that the error bound (20) holds. It is obvious to get \( 0 < \epsilon_\eta < 1 \)
if \( 0 < \eta < \frac{1 - \eta_m}{1 + \eta_m} \), which proves the convergence theorem. This completes the proof of the theorem. \( \square \)

### 4.2 Convergence analysis for QEHOPM

Now, we establish the convergence Theorem for QEHOPM.

**Theorem 3** Assume \( P \) is a non-negative transition probability tensor of order \( m \) and dimension \( n \) with \( \delta_m > \frac{m-2}{m-1} \) and \( \bar{x} \) is a solution of equation (1). Then, the iterative sequence \( \{x_k\} \) generated by QEHOPM has a convergent subsequence \( \{x_{k_n}\} \) that converges to the solution \( \bar{x} \) for any initial transition probability vector \( x_0 \), i.e.,
\[
\lim_{n \to \infty} x_{k_n} = \bar{x}.
\]
(26)

**Proof.** According to the Lemma 2 and condition \( \delta_m > \frac{m-2}{m-1} \), we get that equation (1) has a unique solution \( \bar{x} \). From the QEHOPM, it is easy to get that \( x_k \geq 0 \) for all \( k \).

Let \( \hat{e}_k = \hat{x}_k - x \) and \( e_k = x_k - x \). By Algorithm 2, we can obtain
\[
\hat{e}_k = \alpha_1 x_k + \alpha_2 x_{k-1} + \alpha_3 x_{k-2} - \bar{x}
\]
(27)

where \( \alpha_1 = \frac{\beta_2}{\beta_0 + \beta_1 + \beta_2}, \alpha_2 = \frac{\beta_1}{\beta_0 + \beta_1 + \beta_2}, \alpha_3 = \frac{\beta_0}{\beta_0 + \beta_1 + \beta_2} \).
By substituting the $x_k = \mathcal{P}^{m-1}_{x_{k-1}}, \bar{x} = \mathcal{P}^{m-1}$ into (7), we have
\[
\hat{e}_k = \alpha_1 \mathcal{P}^{m-1}_{x_{k-1}} + \alpha_1 \mathcal{P}^{m-1}_{\bar{x}} - \alpha_1 \mathcal{P}^{m-1} + \alpha_2 x_{k-1} + \alpha_3 x_{k-2} - \bar{x}
\]
\[
= \alpha_1 \mathcal{P}^{m-1}_{x_{k-1}} + (\alpha_1 - 1) \bar{x} + \alpha_2 x_{k-1} + (1 - \alpha_1 - \alpha_2) x_{k-2}
\]
\[
= \alpha_1 \mathcal{P}^{m-1}_{x_{k-1}} + (1 - \alpha_1)(x_{k-2} - \bar{x}) + \alpha_2 (x_{k-1} - \bar{x} + \bar{x} - x_{k-2})
\]
\[
= \alpha_1 \mathcal{P}^{m-1}_{x_{k-1}} + (1 - \alpha_1 - \alpha_2) e_{k-2} + \alpha_2 e_{k-1}
\]
(28)

By Lemma 3, we have
\[
\|\mathcal{P}^{m-1}_{x_{k-1}} - \bar{x}\|_1 \leq \eta_m \|e_{k-1}\|_1.
\]
(29)

and
\[
\|e_{k-1}\|_1 = \|\mathcal{P}^{m-1}_{x_{k-2}} - \mathcal{P}^{m-1}_{\bar{x}}\|_1 \leq \eta_m \|e_{k-2}\|_1
\]
(30)

Then, by (28), (29) and (30), we have
\[
\|\hat{e}_k\|_1 \leq (\alpha_1 \eta_m + \alpha_2) \|e_{k-1}\|_1 + \alpha_3 \|e_{k-2}\|_1
\]
\[
= [(\alpha_1 \eta_m + \alpha_2) \eta_m + \alpha_3] \|e_{k-2}\|_1.
\]
(31)

By Lemma 4, we can obtain
\[
\|e_k\|_1 \leq \|\hat{e}_k\|_1 \leq [(\alpha_1 \eta_m + \alpha_2) \eta_m + \alpha_3] \|e_{k-2}\|_1.
\]
(32)

It is easy to get that $0 \leq \eta_m < 1$ when $\delta_m > \frac{m-2}{m-1}$. Then, we have
\[
(\alpha_1 \eta_m + \alpha_2) \eta_m + \alpha_3 < 1.
\]
(33)

Now, by (32) and (33), we can get that sequence $\{e_k\}$ has a convergent subsequence $\{e_{k_n}\}$ that will converges to zero vector, which proves that the iterative sequence $\{x_k\}$ has a convergent subsequence $\{x_{k_n}\}$ that converges to the solution $\bar{x}$. This completes the proof of the theorem.

5 Numerical experiments

In this section, a number of numerical experiments are presented to verify the efficiency and superiority of our methods, compared with the original higher-order power method (HOPM) [15], the relaxation higher-order power method ((RHOPM)) [15], the shifted power method (S) [9] and the inner-outer iteration method (IO) [9]. Three measure indexes are reported, including the number of iterations (denoted IT), the CPU time in seconds (denoted by CPU) and the relative residual (denoted by RR) defined by $\|\mathcal{P}^{m-1}_{x_{k}} - x_k\|_1$.

In the numerical experiments, all initial points are chosen to be $x_0 = \text{ones}(n, 1)/n$, all algorithms are performed with Tensor Toolbox 2.6 [2] in MATLAB R2010a and are terminated when the condition $\|x_{k+1} - x_k\| < 10^{-10}$ is satisfied. The maximum iterative number is set to 1000. The curve of the norm of relative residual vector versus the number of iteration step is plotted. The selection of parameter in RHOPM are the same to that of in [15].
5.1 Numerical results for higher-order Markov chains

In this subsection, we use the proposed methods (i.e., QEHOPM, HOPMM-I and HOPMM-II), HOPM, RHOPM and GEAP for solving the limiting probability distribution vector of four transition probability tensors (which were contained in the appendix).

Figure 1: The curve of the norm of relative residual vector versus the number of iteration step for (i)-(iv).
The numerical results are reported in Table 1. As we can see, from the Table 1, the number of the iteration steps in QEHOPM and HOPMM-I/HOPMM-II are less than that of HOPM, RHOPM and GEAP. Furthermore, our methods (i.e., QEHOPM and HOPMM) spend less time than HOPM, RHOPM and GEAP. Specially, QEHOPM performs the best among all methods.

Figure 1 plotted the norm of relative residual vector versus the number of iteration step for the above examples. Compared with HOPM, RHOPM need more iterations while HOPMM-I/HOPMM-II is much faster. Noticed that QEHOPM is the best one. Especially for (i) and (ii), only one quadratic extrapolation can reach to the solution.

5.2 Numerical results for multilinear PageRank

In this subsection, we display the numerical results when the tested algorithms are applied for solving the multilinear PageRank. We use the benchmark set of 29 stochastic
tensors constructed by Gleich et al. For the sake of fairness, we rewrite the codes of IO method (the inner-outer iteration in [9]) and S method (the shifted fixed-point iteration in [9]) by using the function `ttv` of the package Tensor Toolbox 2.6. The vector \( v \) is set to \( \frac{1}{n}e \), where \( e = \text{ones}(n, 1) \), and the damping parameter \( \theta \) is set to 0.70, 0.85, 0.90, 0.95, and 0.99, respectively. We list the numerical results in Tables 2-6 for different \( \theta \), where ‘’ means that the corresponding algorithm is not available.

Table 2: The numerical results for multilinear PageRank with \( \theta = 0.7 \).

|       | HOPM IT | HOPM CPU | QEHOPM IT | QEHOPM CPU | RHOPM IT | RHOPM CPU | IO IT | IO CPU | S IT | S CPU |
|-------|---------|----------|-----------|------------|----------|----------|-------|--------|------|-------|
| \( R_3,1 \) | 52 0.1271 | 9 0.0183 | 28 0.0468 | 69 0.4481 | 37 0.0645 |
| \( R_3,2 \) | 15 0.0949 | 9 0.0167 | 14 0.0183 | 60 0.3419 | 36 0.5075 |
| \( R_3,3 \) | 15 0.1078 | 9 0.0179 | 14 0.0202 | 60 0.3822 | 35 0.0877 |
| \( R_3,4 \) | 43 0.1189 | 9 0.0101 | 18 0.0551 | 48 0.3955 | 20 0.0429 |
| \( R_3,5 \) | 86 0.1704 | 17 0.0242 | 96 0.6651 | 134 0.2289 |
| \( R_4,1 \) | 77 0.1317 | 37 0.0438 | 62 0.3955 | 20 0.0429 |
| \( R_4,2 \) | 77 0.1780 | 34 0.0450 | 64 0.3419 | 30 0.0575 |
| \( R_4,3 \) | 50 0.1300 | 37 0.0678 | 51 0.0777 | 114 0.9529 | 79 0.1481 |
| \( R_4,4 \) | 77 0.1521 | 59 0.0563 | 65 0.0846 | 118 1.1107 | 80 0.1689 |
| \( R_4,5 \) | 53 0.1254 | 34 0.0386 | 40 0.0648 | 115 0.9562 | 81 0.1530 |
| \( R_4,6 \) | 64 0.1417 | 33 0.0352 | 51 0.0776 | 117 0.8601 | 93 0.1785 |
| \( R_4,7 \) | 51 0.1279 | 44 0.0450 | 53 0.0799 | 109 0.9250 | 75 0.1347 |
| \( R_4,8 \) | 68 0.1421 | 56 0.0570 | 61 0.0890 | 112 0.8877 | 94 0.1534 |
| \( R_4,9 \) | 52 0.1359 | 27 0.0406 | 45 0.0628 | 110 0.9428 | 81 0.1365 |
| \( R_4,10 \) | 90 0.2322 | 67 0.0904 | 57 0.0742 | 111 0.9534 | 80 0.1535 |
| \( R_4,11 \) | 54 0.1327 | 38 0.0454 | 49 0.0715 | 114 0.8166 | 82 0.1408 |
| \( R_4,12 \) | 38 0.1215 | 31 0.0400 | 39 0.0628 | 86 0.8176 | 61 0.1156 |
| \( R_4,13 \) | 58 0.1333 | 49 0.0488 | 64 0.0800 | 125 1.0120 | 86 0.1468 |
| \( R_4,14 \) | 56 0.1175 | 20 0.0269 | 41 0.0641 | 111 0.8955 | 82 0.1403 |
| \( R_4,15 \) | 70 0.1494 | 37 0.0422 | 59 0.0788 | 122 0.9808 | 101 0.1638 |
| \( R_4,16 \) | 56 0.1342 | 49 0.0496 | 62 0.0799 | 118 0.8728 | 85 0.1481 |
| \( R_4,17 \) | 53 0.1209 | 23 0.0351 | 56 0.0709 | 114 0.8236 | 78 0.1273 |
| \( R_4,18 \) | 52 0.1323 | 29 0.0316 | 55 0.0719 | 116 0.8247 | 78 0.1386 |
| \( R_4,19 \) | 49 0.1277 | 40 0.0436 | 54 0.0722 | 103 0.8040 | 75 0.1312 |
| \( R_6,1 \) | 40 0.1124 | 26 0.0266 | 24 0.0442 | 100 0.6884 | 65 0.1045 |
| \( R_6,2 \) | 49 0.1234 | 25 0.0290 | 44 0.0773 | 110 0.7558 | 64 0.1175 |
| \( R_6,3 \) | 35 0.1091 | 22 0.0304 | 40 0.0667 | 89 0.6202 | 57 0.0928 |
| \( R_6,4 \) | 40 0.1180 | 21 0.0289 | 25 0.0404 | 104 0.7710 | 66 0.1096 |
| \( R_6,5 \) | 33 0.1033 | 19 0.0270 | 26 0.0455 | 87 0.4741 | 55 0.0946 |
Table 3: The numerical results for multilinear PageRank with $\theta = 0.85.$

|       | HOPM | QEHOPM | RHOPM | IO | S  |
|-------|------|--------|-------|----|----|
|       | IT   | CPU    | IT    | CPU| IT | CPU |
| $R_{3,1}$ | 130  | 0.2141 | 13    | 0.0193 | 34  | 0.0368 | 60  | 0.5886 | 44  | 0.0783 |
| $R_{3,2}$ | 21   | 0.0843 | 13    | 0.0220 | 29  | 0.0334 | 55  | 0.3816 | 37  | 0.0708 |
| $R_{3,3}$ | 21   | 0.0822 | 13    | 0.0218 | 17  | 0.0220 | 55  | 0.3814 | 37  | 0.0681 |
| $R_{3,4}$ | 70   | 0.1359 | 13    | 0.0203 | 19  | 0.0205 | 39  | 0.3800 | 21  | 0.0414 |
| $R_{3,5}$ | 103  | 0.1617 | 29    | 0.0350 | 115 | 0.1022 | 160 | 1.9054 | 158 | 0.2571 |
| $R_{4,1}$ | 134  | 0.1857 | 102   | 0.0978 | 146 | 0.1287 | 166 | 1.9235 | 191 | 0.3081 |
| $R_{4,2}$ | 261  | 0.2957 | 33    | 0.0356 | 318 | 0.2866 | 158 | 1.8214 | 296 | 0.4762 |
| $R_{4,3}$ | 101  | 0.1550 | 70    | 0.0702 | 146 | 0.1287 | 166 | 1.9235 | 191 | 0.3081 |
| $R_{4,4}$ | 262  | 0.3049 | 172   | 0.1639 | 181 | 0.1648 | 167 | 1.9003 | 232 | 0.3767 |
| $R_{4,5}$ | 162  | 0.2413 | 41    | 0.0514 | 144 | 0.1648 | 160 | 2.2219 | 220 | 0.4797 |
| $R_{4,6}$ | 225  | 0.2751 | 29    | 0.0356 | 225 | 0.1976 | 155 | 1.6964 | 269 | 0.4301 |
| $R_{4,7}$ | 120  | 0.1703 | 110   | 0.1044 | 132 | 0.1159 | 141 | 1.6796 | 170 | 0.2723 |
| $R_{4,8}$ | 300  | 0.3332 | 57    | 0.0572 | 279 | 0.2490 | 136 | 1.5656 | 296 | 0.4816 |
| $R_{4,9}$ | 144  | 0.1996 | 49    | 0.0537 | 132 | 0.1158 | 144 | 1.6666 | 196 | 0.3150 |
| $R_{4,10}$ | 364  | 0.3767 | 113   | 0.1092 | 130 | 0.1242 | 154 | 1.8034 | 172 | 0.2779 |
| $R_{4,11}$ | 133  | 0.2074 | 64    | 0.0726 | 115 | 0.1015 | 174 | 1.6932 | 195 | 0.3086 |
| $R_{4,12}$ | 88   | 0.1495 | 43    | 0.0451 | 80  | 0.0728 | 140 | 1.5340 | 137 | 0.2395 |
| $R_{4,13}$ | 151  | 0.2028 | 122   | 0.1161 | 162 | 0.1428 | 177 | 2.0442 | 215 | 0.3434 |
| $R_{4,14}$ | 134  | 0.1915 | 33    | 0.0363 | 117 | 0.1041 | 135 | 1.5947 | 181 | 0.2915 |
| $R_{4,15}$ | 238  | 0.2808 | 33    | 0.0374 | 238 | 0.2088 | 154 | 1.7142 | 276 | 0.4491 |
| $R_{4,16}$ | 149  | 0.1969 | 101   | 0.0976 | 138 | 0.1203 | 167 | 1.9701 | 209 | 0.3348 |
| $R_{4,17}$ | 208  | 0.2491 | 35    | 0.0377 | 136 | 0.1209 | 177 | 1.9880 | 186 | 0.3062 |
| $R_{4,18}$ | 141  | 0.1897 | 49    | 0.0518 | 156 | 0.1419 | 209 | 2.3378 | 215 | 0.3558 |
| $R_{4,19}$ | 120  | 0.1733 | 92    | 0.0904 | 113 | 0.1022 | 125 | 1.4163 | 163 | 0.3173 |
| $R_{5,1}$ | 106  | 0.1659 | 30    | 0.0347 | 60  | 0.0542 | 162 | 1.5742 | 163 | 0.2793 |
| $R_{5,2}$ | 98   | 0.1679 | 46    | 0.0511 | 88  | 0.0773 | 130 | 1.3974 | 123 | 0.2517 |
| $R_{5,3}$ | 67   | 0.1325 | 28    | 0.0289 | 74  | 0.0667 | 109 | 1.0417 | 103 | 0.1839 |
| $R_{5,4}$ | 85   | 0.1482 | 26    | 0.0293 | 52  | 0.0504 | 129 | 1.3442 | 131 | 0.2406 |
| $R_{5,5}$ | 66   | 0.1288 | 26    | 0.0299 | 47  | 0.0455 | 113 | 0.9191 | 106 | 0.1970 |
Table 4: The numerical results for multilinear PageRank with $\theta = 0.9$.

|       | HOPM  | QEHOPM | RHOPM | IO  | S     |
|-------|-------|---------|-------|-----|-------|
|       | IT CPU| IT CPU  | IT CPU| IT CPU| IT CPU |
| $R_{4,1}$ | 245 0.2159 | 13 0.0105 | 36 0.0268 | 58 0.5878 | 46 0.0849 |
| $R_{4,2}$ | 906 0.7776 | 30 0.0274 | 667 0.5866 | 181 1.9300 | 643 0.9784 |
| $R_{4,3}$ | 157 0.1373 | 64 0.0494 | 174 0.1737 | 197 1.9505 | 239 0.3715 |
| $R_{4,4}$ | 660 0.5677 | 53 0.0452 | 432 0.2646 | 213 2.4346 | 481 0.7326 |
| $R_{4,5}$ | 537 0.4451 | 53 0.0438 | 512 0.4648 | 203 2.4789 | 535 0.8161 |
| $R_{4,6}$ | 873 0.6930 | 34 0.0297 | 621 0.5976 | 182 2.0236 | 629 1.0454 |
| $R_{4,7}$ | 243 0.2177 | 74 0.0593 | 226 0.1159 | 182 1.9439 | 315 0.5120 |
| $R_{4,8}$ | 108 0.0876 | 42 0.0353 | 337 0.2705 | 173 1.9783 | 368 0.5618 |
| $R_{4,9}$ | 233 0.2030 | 108 0.0876 | 235 0.1642 | 202 2.3554 | 302 0.5203 |
| $R_{4,11}$ | 222 0.1938 | 92 0.0881 | 126 0.1011 | 202 2.2082 | 230 0.3806 |
| $R_{4,11}$ | 314 0.2661 | 69 0.0554 | 328 0.2208 | 226 2.8959 | 406 0.6183 |
| $R_{4,12}$ | 272 0.1175 | 20 0.0269 | 271 0.2241 | 161 0.8955 | 82 0.1403 |
| $R_{4,13}$ | 799 0.1494 | 37 0.0422 | 599 0.5088 | 178 0.9808 | 101 0.1638 |
| $R_{4,14}$ | 299 0.1342 | 49 0.0496 | 285 0.1403 | 216 0.8728 | 85 0.1481 |
| $R_{4,15}$ | 23 0.0351 | 23 0.0351 | 258 0.1209 | 287 0.8236 | 78 0.1273 |
| $R_{4,16}$ | 154 0.1323 | 29 0.0316 | 140 0.1009 | 161 0.8247 | 78 0.1386 |
| $R_{4,17}$ | 218 0.1277 | 40 0.0436 | 213 0.1022 | 141 0.8040 | 75 0.1312 |
| $R_{4,18}$ | 232 0.1124 | 26 0.0266 | 123 0.0942 | 300 0.6884 | 65 0.1045 |
| $R_{4,19}$ | 118 0.1234 | 25 0.0290 | 118 0.0773 | 143 0.7558 | 64 0.1175 |
| $R_{4,20}$ | 95 0.1091 | 22 0.0304 | 106 0.0667 | 133 0.6202 | 57 0.0928 |
| $R_{4,21}$ | 152 0.1180 | 21 0.0289 | 96 0.0504 | 164 0.7710 | 66 0.1096 |
| $R_{4,22}$ | 98 0.1033 | 19 0.0270 | 64 0.0455 | 142 0.4741 | 55 0.0946 |
Table 5: The numerical results for multilinear PageRank with $\theta = 0.95$.

|     | HOPM IT | HOPM CPU | QEHOPM IT | QEHOPM CPU | RHOPM IT | RHOPM CPU | IO IT | IO CPU | S IT | S CPU |
|-----|--------|----------|-----------|------------|---------|-----------|------|-------|-----|-------|
| $R_{3,1}$ | - | - | 21 0.0254 | 45 0.0454 | 56 0.4543 | - | - |
| $R_{3,2}$ | 28 | 0.1009 | 13 0.0222 | 17 0.0255 | 55 0.4514 | 46 0.0902 |
| $R_{3,3}$ | 28 | 0.1029 | 13 0.0224 | 17 0.0262 | 55 0.4522 | 46 0.0900 |
| $R_{3,4}$ | 114 0.1775 | 13 0.0174 | 19 0.0288 | 34 0.3778 | 21 0.0140 |
| $R_{3,5}$ | 43 0.1168 | 37 0.0458 | 48 0.1198 | 69 0.7245 | 69 0.1312 |
| $R_{4,1}$ | - | - | 21 0.0230 | - | - | 331 4.4715 | - | - |
| $R_{4,2}$ | - | - | 22 0.0241 | - | - | 220 2.6415 | - | - |
| $R_{4,3}$ | 410 0.4276 | 49 0.0531 | 454 0.4973 | 358 4.1896 | 586 0.9188 |
| $R_{4,4}$ | - | - | 25 0.0262 | - | - | 323 4.1730 | - | - |
| $R_{4,5}$ | - | - | 105 0.0891 | - | - | 282 3.5991 | - | - |
| $R_{4,6}$ | - | - | 41 0.0378 | - | - | 224 2.6912 | - | - |
| $R_{4,7}$ | - | - | 37 0.0350 | - | - | 263 3.2073 | - | - |
| $R_{4,8}$ | - | - | 42 0.0411 | - | - | 174 2.2292 | - | - |
| $R_{4,9}$ | - | - | 42 0.0391 | - | - | 246 2.9428 | - | - |
| $R_{4,10}$ | - | - | 32 0.0315 | - | - | 326 4.1776 | - | - |
| $R_{4,11}$ | 945 0.8263 | 137 0.1178 | 873 0.7070 | 568 5.7697 | - | - |
| $R_{4,12}$ | 209 0.2597 | 30 0.0319 | 174 0.1493 | 202 2.5081 | 295 0.4853 |
| $R_{4,13}$ | - | - | 43 0.0402 | - | - | 340 4.1508 | - | - |
| $R_{4,14}$ | - | - | 38 0.0356 | - | - | 217 2.6353 | - | - |
| $R_{4,15}$ | - | - | 42 0.0397 | - | - | 221 3.9453 | - | - |
| $R_{4,16}$ | - | - | 41 0.0396 | - | - | 326 0.8728 | - | - |
| $R_{4,17}$ | - | - | 934 0.7409 | - | - | - | - | - |
| $R_{4,18}$ | 867 0.7706 | 41 0.0443 | 812 0.7020 | 280 3.5042 | 855 1.1386 |
| $R_{4,19}$ | 866 0.7500 | 62 0.0577 | 626 0.6308 | 163 1.9576 | 632 1.0034 |
| $R_{6,1}$ | 257 0.2888 | 65 0.0603 | 185 0.1803 | 240 3.0334 | 367 0.5613 |
| $R_{6,2}$ | 528 0.4939 | 90 0.0814 | 543 0.5102 | 285 3.5599 | 638 0.9902 |
| $R_{6,3}$ | 172 0.2253 | 53 0.0521 | 159 0.2301 | 204 2.0102 | 299 0.4097 |
| $R_{6,4}$ | 426 0.4216 | 90 0.0792 | 405 0.3529 | 248 2.8871 | 535 0.8490 |
| $R_{6,5}$ | 185 0.2308 | 65 0.0594 | 116 0.1308 | 226 2.3169 | 284 0.4334 |
Table 6: The numerical results for multilinear PageRank with $\theta = 0.99$.

|     | HOPM IT | HOPM CPU | QEHOPM IT | QEHOPM CPU | RHOPM IT | RHOPM CPU | IO IT | IO CPU | S IT | S CPU |
|-----|---------|----------|-----------|------------|----------|-----------|------|-------|------|-------|
| $R_{3,1}$ | - | - | 13 | 0.0185 | 47 | 0.0386 | 54 | 0.6360 | 52 | 0.0860 |
| $R_{3,2}$ | 34 | 0.1222 | 13 | 0.0249 | 14 | 0.0306 | 60 | 0.5520 | 56 | 0.0899 |
| $R_{3,3}$ | 34 | 0.1017 | 13 | 0.0222 | 14 | 0.0326 | 60 | 0.5866 | 56 | 0.1036 |
| $R_{3,4}$ | 148 | 0.2185 | 17 | 0.0248 | 21 | 0.0369 | 32 | 0.4296 | 21 | 0.0447 |
| $R_{3,5}$ | 27 | 0.1040 | 19 | 0.0252 | 31 | 0.0835 | 50 | 0.5139 | 47 | 0.0887 |
| $R_{4,1}$ | - | - | 70 | 0.0438 | - | - | 667 | 6.1107 | - | - |
| $R_{4,2}$ | - | - | 30 | 0.0294 | - | - | 283 | 2.0479 | - | - |
| $R_{4,3}$ | - | - | 32 | 0.0297 | - | - | - | - | - | - |
| $R_{4,4}$ | - | - | 29 | 0.0299 | - | - | 584 | 4.5101 | - | - |
| $R_{4,5}$ | - | - | 26 | 0.0272 | - | - | 386 | 3.5180 | - | - |
| $R_{4,6}$ | - | - | 22 | 0.0243 | - | - | 276 | 2.5554 | - | - |
| $R_{4,7}$ | - | - | 25 | 0.0259 | - | - | 427 | 4.9250 | - | - |
| $R_{4,8}$ | - | - | 30 | 0.0295 | - | - | 191 | 1.8877 | - | - |
| $R_{4,9}$ | - | - | 42 | 0.0399 | - | - | 391 | 5.5578 | - | - |
| $R_{4,10}$ | - | - | 63 | 0.0553 | - | - | 660 | 6.1657 | - | - |
| $R_{4,11}$ | - | - | 62 | 0.0678 | - | - | - | - | - | - |
| $R_{4,12}$ | - | - | 172 | 0.1404 | - | - | - | - | - | - |
| $R_{4,13}$ | - | - | 113 | 0.1121 | - | - | 623 | 5.8067 | - | - |
| $R_{4,14}$ | - | - | 34 | 0.0348 | - | - | 334 | 3.3455 | - | - |
| $R_{4,15}$ | - | - | 34 | 0.0332 | - | - | 283 | 2.9808 | - | - |
| $R_{4,16}$ | - | - | 29 | 0.0310 | - | - | 638 | 6.8728 | - | - |
| $R_{4,17}$ | - | - | 121 | 0.1012 | 510 | 0.1209 | - | - | 642 | 0.9793 |
| $R_{4,18}$ | - | - | 229 | 0.1963 | - | - | 529 | 5.5947 | - | - |
| $R_{4,19}$ | - | - | 42 | 0.0390 | - | - | 185 | 1.3840 | - | - |
| $R_{6,1}$ | - | - | 41 | 0.0310 | - | - | 902 | 9.9988 | - | - |
| $R_{6,2}$ | - | - | 76 | 0.0599 | - | - | 765 | 8.1938 | - | - |
| $R_{6,3}$ | - | - | 320 | 0.2306 | - | - | - | - | - | - |
| $R_{6,4}$ | - | - | 74 | 0.0557 | - | - | 420 | 5.3745 | - | - |
| $R_{6,5}$ | - | - | 404 | 0.3316 | - | - | - | - | - | - |

As we can see from Tables 2-6, the proposed QEHOPM method is faster than HOPM, RHOPM, IO and S methods. The QEHOPM method spends less iterations or CPU time than that of HOPM, RHOPM, IO and S methods. In particular, QEHOPM method has the most reliable convergence, which can solve successfully all of the benchmark set of 29 problems even when $\theta = 0.99$. In a word, from the above results, we conclude that QEHOPM algorithm is very effective and competitive.
6 Conclusion

In this paper, we have proposed three accelerated higher-order power method for higher-order Markov chains and multilinear PageRank, referred to as the HOPMM-I/HOPMM-II and QEHOPM, respectively. In particular, the QEHOPM method is non-parametric. We established the convergence results for the proposed algorithms. Numerical experiments are carried out to illustrate that HOPMM-I/HOPMM-II outperform the higher-order power method and the QEHOPM is the best one.

Appendix

Test Examples: The first three tensors come from DNA sequence data in the works of Raftery et al. [24]. On the other hand, their orders $m$ are 3 and their numbers of states $n$ are 3 or 4 by considering three categories($\{A/G,C,T\}$). By using the Matlab multi-dimensional array notation, the transition probability tensors are given as follows.

\[
\begin{align*}
\text{i)} P(\cdot,\cdot,1) &= \begin{pmatrix}
0.6000 & 0.4083 & 0.4935 \\
0.2000 & 0.2568 & 0.2426 \\
0.2000 & 0.3349 & 0.2639 \\
\end{pmatrix}, \\
P(\cdot,\cdot,2) &= \begin{pmatrix}
0.5217 & 0.3300 & 0.4152 \\
0.2232 & 0.2800 & 0.2658 \\
0.2551 & 0.3900 & 0.3190 \\
\end{pmatrix}, \\
P(\cdot,\cdot,3) &= \begin{pmatrix}
0.5565 & 0.3648 & 0.4500 \\
0.2174 & 0.2742 & 0.2600 \\
0.2261 & 0.3610 & 0.2900 \\
\end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
\text{ii)} P(\cdot,\cdot,1) &= \begin{pmatrix}
0.5200 & 0.2986 & 0.4462 \\
0.2700 & 0.3930 & 0.3192 \\
0.2100 & 0.3084 & 0.2346 \\
\end{pmatrix}, \\
P(\cdot,\cdot,2) &= \begin{pmatrix}
0.6514 & 0.4300 & 0.5776 \\
0.1970 & 0.3200 & 0.2462 \\
0.1516 & 0.2500 & 0.1762 \\
\end{pmatrix}, \\
P(\cdot,\cdot,3) &= \begin{pmatrix}
0.5638 & 0.3424 & 0.4900 \\
0.2408 & 0.3638 & 0.2900 \\
0.1954 & 0.2938 & 0.2200 \\
\end{pmatrix}.
\end{align*}
\]

\[
\begin{align*}
\text{iii)} \\
P(\cdot,\cdot,1) &= \begin{pmatrix}
0.2091 & 0.2834 & 0.2194 & 0.1830 \\
0.3371 & 0.3997 & 0.3219 & 0.3377 \\
0.3265 & 0.0560 & 0.3119 & 0.2961 \\
0.1723 & 0.2608 & 0.1468 & 0.1832 \\
\end{pmatrix}, \\
P(\cdot,\cdot,2) &= \begin{pmatrix}
0.1952 & 0.2695 & 0.2055 & 0.1690 \\
0.3336 & 0.3962 & 0.3184 & 0.3342 \\
0.2954 & 0.0249 & 0.2808 & 0.2650 \\
0.1758 & 0.3094 & 0.1953 & 0.2318 \\
\end{pmatrix}.
\end{align*}
\]
By considering three categories ($\{A,C,T,G\}$), we construct a transition probability tensor of order 4 and dimension 3 for the DNA sequence in [22]:

$$P(:,:,3) = \begin{pmatrix}
0.3145 & 0.3887 & 0.3248 & 0.2883 \\
0.0603 & 0.1203 & 0.0451 & 0.0609 \\
0.2293 & 0.3628 & 0.2487 & 0.2852 \\
0.2293 & 0.3628 & 0.2487 & 0.2852
\end{pmatrix}.$$  

$$P(:,:,4) = \begin{pmatrix}
0.1685 & 0.2429 & 0.1789 & 0.1425 \\
0.3553 & 0.4180 & 0.3402 & 0.3559 \\
0.3189 & 0.0484 & 0.3043 & 0.2885 \\
0.1571 & 0.2907 & 0.1766 & 0.2131
\end{pmatrix}.$$  

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