On the postulation of lines and a fat line

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This talk is based on joint work (arXiv: 1706.02350) with
Thomas Bauer (Marburg),
Sandra Di Rocco (KTH Stockholm),
David Schmitz (Marburg)
and
Tomasz Szemberg (PU Cracow).
The slides are available at:
http://szpond.up.krakow.pl/MEGA2017.pdf
Definition

Let \( I \subset R \) be a homogeneous ideal in a polynomial ring \( R = \mathbb{K}[x_0, \ldots, x_N] \). The *Hilbert function* of \( I \) is

\[
HF_{R/I}(d) = \dim(R/I)_d.
\]
**Definition**

Let $I \subset R$ be a homogeneous ideal in a polynomial ring $R = \mathbb{K}[x_0, \ldots, x_N]$. The *Hilbert function* of $I$ is

$$HF_{R/I}(d) = \dim(R/I)_d.$$ 

**Remark**

*It is well-known that the Hilbert function becomes polynomial, i.e., there is a polynomial $HP_{R/I}(d)$ such that*

$$HF_{R/I}(d) = HP_{R/I}(d) \quad \text{for} \quad d \gg 0.$$
Remark

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$$HF_V(d) = HF_{R/I}(d)$$

and

$$HP_V(d) = HP_{R/I}(d).$$
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MEGA (Meta) Problem

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Problem

Determine Hilbert functions of subschemes in $\mathbb{P}^N(K)$. 

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_Determine Hilbert functions of subschemes in $\mathbb{P}^N(K)$._

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_This problem is much too hard in this generality and beyond reach._
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Problem

Determine Hilbert functions of subschemes in $\mathbb{P}^N(K)$.

Remark

This problem is much too hard in this generality and beyond reach.

The simplest Hilbert functions occur for subvarieties which impose independent (or predictable) conditions on forms of arbitrary degree.
Definition (Carlini, Catalisano, Geramita)

We say that a subscheme $V \subset \mathbb{P}^N(K)$ has a *bipolynomial Hilbert function* if

$$HF_V(d) = \min \{ HP_{\mathbb{P}^N}(d), \ HP_V(d) \}$$

for all $d$. 

Remark: There is $HP_{\mathbb{P}^N}(d) = (N + d) d$ for all $d$. 

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Example

Let $V$ be a finite union of $s$ general points in a projective space $\mathbb{P}^N(K)$. Then $V$ has a bipolynomial Hilbert function.
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Let $V$ be a finite union of $s$ general points in a projective space $\mathbb{P}^N(K)$. Then $V$ has a bipolynomial Hilbert function. More precisely we have

$$HF_V(d) = \min \left\{ \binom{N + d}{d}, s \right\}$$

for all $d$. 
Theorem (Alexander-Hirschowitz 1995)

Let $V$ be a general collection of $s$ double points in $\mathbb{P}^N(K)$ (over an algebraically closed field of characteristic zero). Then

$$HF_V(d) = \min \left\{ \left( \begin{array}{c} N + d \\ d \end{array} \right), \ s(N + 1) \right\}$$

except in the following cases

- $d = 2$, $2 \leq s \leq N$;
- $N = 2$, $d = 4$, $s = 5$;
- $N = 3$, $d = 4$, $s = 9$;
- $N = 4$, $d = 4$, $s = 14$;
- $N = 4$, $d = 3$, $s = 7$.
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- $N = 3, d = 4, s = 9$;
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Remark

The authors worked on this problem for over 10 years.

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The proof of Alexander and Hirschowitz is rather involved. It has been simplified by several authors including:

- Karen Chandler (Trans. Amer. Math. Soc. 353 (2001) and Compositio Math. 134 (2002));

Remark:
All proofs are based on some degeneration, i.e., if the claim holds for points in special position, then it holds for points in general position (provided both positions belong to a flat family).
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**Remark**

*Ciliberto and Miranda introduced a degeneration of the ambient space (replace $\mathbb{P}^2$ by some other scheme) combined with the degeneration of points.*
Passing from points to fat points, arbitrary $m$

Conjecture (SHGH, Segre-Harbourne-Gimigliano-Hirschowitz)

Let $V$ be a collection of $s$ general points of multiplicity $m$ in $\mathbb{P}^2(\mathbb{K})$. Then either

$$\text{HF}_V(d) = \min \left\{ \binom{d+2}{2}, s \binom{m+1}{2} \right\}$$

or the linear system

$$|\mathcal{O}_{\mathbb{P}^2}(d) \otimes \mathcal{I}_V^{(m)}|$$

contains a fat $(-1)$-curve in its base locus.
Theorem (Hartshorne-Hirschowitz 1982)

Let $V$ be a union of $s$ general lines in the projective space $\mathbb{P}^N(K)$, with $N \geq 3$. Then the Hilbert function of $V$ is bipolynomial.

More precisely we have $HF_V(d) = \min\{ (N+d) \cdot d, s \cdot (d+1) \}$. The hardest case is that of $N=3$. The proof in $\mathbb{P}^3$ is based on a careful specialization of some lines onto a smooth quadric accompanied by a careful collision of some pairs of lines in points on the quadric.
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Theorem (Carlini, Catalisano, Geramita 2013, printed 2016)

Let $V$ be a union of $s$ general lines and a general point of multiplicity $m$ in the projective space $\mathbb{P}^N(K)$, with $N \geq 4$. Then the Hilbert function of $V$ is bipolynomial.

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In the case $N = 3$ the picture is more complicated: The equality in (1) fails for $2 \leq s \leq m$ and $d = m$.
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$$\text{HF}_V(d) = \min \left\{ \binom{N + d}{d}, s(d + 1) + \binom{m + N - 1}{N} \right\}.$$  \hfill (1)
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The case $N = 3$ is solved by Aladpoosh-Ballico (Rend. Sem. Mat. Univ. Pol. Torino 2015) and Ballico (Mediterranean Journal of Mathematics (2016)).
Problem (Carlini, Catalisano, Geramita)

Identify Hilbert functions of subschemes in $\mathbb{P}^N$ consisting of the union of general lines and one fat linear subspace of arbitrary dimension.
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Motivation: Determine the dimension of (higher) secant varieties to Segre embeddings of products of projective spaces.
Definition (Ballico)

A "double" line $Y \subset \mathbb{P}^3$ is a connected divisor of type $(2, 0)$ on a smooth quadric surface.
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Theorem (Ballico 2012)
Let $V$ be a union of $s$ general lines and $t$ general "double" lines in $\mathbb{P}^3$. Then $V$ has a bipolynomial Hilbert function.
Definition

A double line $X \subset \mathbb{P}^3$ is a subscheme supported on a line $L$ whose structure is determined by the square of the saturated ideal $I_L$ defining $L$. 

Theorem (Aladpoosh 2016)

Let $V$ be a union of $s$ general lines and one general double line in $\mathbb{P}^N$, with $N \geq 3$. Then

$$HF_V(d) = \min \left\{ \left( N+d \right), s(d+1) + (Nd+1) \right\}$$

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One fat line in $\mathbb{P}^3$: the main Theorem of this talk

Theorem (Bauer, Di Rocco, Schmitz, Szemberg, Sz.)

Let $V$ be a union of $s$ general lines and one general line of multiplicity $m$ (i.e. defined by $l^m_i$) in $\mathbb{P}^3$. Then

$$\text{HF}_V(d) = \min \left\{ \binom{d+3}{d}, \ s(d+1) + \frac{1}{6}m(m+1)(3d+5-2m) \right\}$$

for all $d \geq 3\binom{m+1}{3}$.
Definition (Zig-zag)

A zig-zag of length $z$ is the limiting subscheme obtained by a collision of an ordered set of $z$ general lines $L_1, L_2, \ldots, L_z$ in such a way, that the line $L_1$ intersects $L_2$, the line $L_2$ intersects $L_1$ and $L_3$ and the intersection points are distinct, $L_3$ intersects $L_2$ and $L_4$ and the intersection points are again distinct, and so on, finally $L_{z-1}$ intersects $L_{z-2}$ and $L_z$ in two distinct points. The structure in the intersection points is the same as the structure of a sundial in the intersection point of its lines.
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A zig-zag of length \( z \) has thus \( (z - 1) \) singular points.

A sundial is a zig-zag of length 2.
Phase 1

The singular points of the zig-zag are specialized onto a general smooth quadric in $\mathbb{P}^3$. 

Phase 2

Every second line of the reduced zig-zag is specialized on a smooth quadric as a general line, all in the same ruling. This quadric is also exhibited as a base component of the studied linear system. Removing it from the system decreases the degree again by 2. The residue of the reduced zig-zag is a collection of disjoint (general) lines.
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How is a zig-zag applied in the proof

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Definition (Trace and residual scheme)

Let $Y$ be a smooth divisor in $\mathbb{P}^N$ and let $Z \subset \mathbb{P}^N$ be a closed subscheme. Then the subscheme $Z'' = \text{Tr}_Y(Z)$ defined in $Y$ by the ideal

$$I_{Z''}/Y = (I_Y + I_Z)/I_Y \subset O_Y$$

is the trace of $Z$ on $Y$.

The colon ideal $I_{Z'} = (I_Z : I_Y) \subset O_{\mathbb{P}^N}$ defines $Z' = \text{Res}_Y(Z)$, the residual scheme of $Z$ with respect to $Y$. 

The residual sequence

$$0 \to I_{Z'}(-Y) \to I_Z \to I_{Z''}/Y \to 0$$
Definition (Trace and residual scheme)

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The trace and residual schemes

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**Residual sequence**

$$0 \longrightarrow I_{Z'}(-Y) \longrightarrow I_Z \longrightarrow I_{Z''}/Y \longrightarrow 0$$

We apply this sequence with $Y$ a smooth quadric in $\mathbb{P}^3$ and all terms twisted by $\mathcal{O}_{\mathbb{P}^3}(d)$. 
Lemma

Let $Y \subset \mathbb{P}^N$ be a divisor of degree $e$ and let $d \geq e$ be an integer. Let $Z \subset \mathbb{P}^N$ be a closed subscheme. Then

$$h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{I}_Z) \leq h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d - e) \otimes \mathcal{I}_{\text{Res}_Y(Z)}) +$$

$$+ h^0(Y, \mathcal{O}_Y(d) \otimes \mathcal{I}_{\text{Tr}_Y(Z)/Y}).$$
Lemma

Let \( Y \subset \mathbb{P}^N \) be a divisor of degree \( e \) and let \( d \geq e \) be an integer. Let \( Z \subset \mathbb{P}^N \) be a closed subscheme. Then

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\begin{align*}
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    & + h^0(Y, \mathcal{O}_Y(d) \otimes \mathcal{I}_{\text{Tr}_Y(Z)/Y}).
\end{align*}
\]

We call the space \( H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d - e) \otimes \mathcal{I}_{\text{Res}_Y(Z)}) \) the residual linear system of \( H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{I}_Z) \) with respect to \( Y \) and \( H^0(Y, \mathcal{O}_Y(d) \otimes \mathcal{I}_{\text{Tr}_Y(Z)/Y}) \) the trace linear system of \( H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{I}_Z) \) on \( Y \).
Outline of the proof

The proof of the Main Theorem consists of

- specializing lines and collision points of lines onto a smooth quadric;

Remark
We have developed a computer software to make correct guesses on the bound on $d$; test various reduction steps.
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The proof of the Main Theorem consists of

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- book-keeping!

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Remark

*We have developed a computer software to*
- *make correct guesses on the bound on d;*
- *test various reduction steps.*
And that’s it.

thank you!
Remark

The bound on $d$ given in the Main Theorem is due to the fact, that for $d$ big enough certain invariants of the ideal of $Z$ can be described by an explicit function. This makes the induction possible.
The Hilbert function in $\mathbb{P}^3$ is bipolynomial.

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Remark

We have checked by computer hundreds of cases and have found no irregularities in the Hilbert function.
Example

Already in $\mathbb{P}^4$ some special cases come up. The easiest one is 1 double line $L$ and 2 ordinary lines $L_1$ and $L_2$. The union of two hypersurfaces generated by $L$ and $L_i$ vanishes double along $L$ and once along each $L_i$, whereas it is unexpected from the naive dimension count.
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Remark

We expect however, that with a similar bound as in $\mathbb{P}^3$, the values of the Hilbert function in $\mathbb{P}^N$ can be computed by the bipolynomial formula.
Very rough outline of induction procedure

| For          | a sequence of length | yields                        |
|--------------|----------------------|-------------------------------|
| $B(k,0,m)$   | 1                    | $B(k-1,1,m-1)$               |
| $B(k,1,m)$   | 2                    | $B(k-1,0,m-1)$               |
| $B(k,2,m)$   | 1                    | $B(k,0,m-1)$                 |
| $I(k,0,m)$   | 2                    | $I(k-2,2,m-2)$               |
| $I(k,1,m)$   | 1                    | $I(k-1,2,m-1)$               |
| $I(k,2,3\ell)$ | $3\ell - 1$ | $B(k-2\ell+1,1,1)$         |
| $I(k,2,3\ell+1)$ | $3\ell + 1$ | $B(k-2\ell,0,0)$         |
| $I(k,2,3\ell+2)$ | $3\ell + 1$ | $B(k-2\ell,0,1)$         |