On Fixed Point Sets and Lefschetz Modules for Sporadic Simple Groups

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• $G$ is a finite group and $p$ a prime dividing its order
• $Q$ a nontrivial $p$-subgroup of $G$
• $Q$ is $p$-radical if $Q = O_p(N_G(Q))$
• $Q$ is $p$-centric if $Z(Q) \in Syl_p(C_G(Q))$
• $G$ has characteristic $p$ if $C_G(O_p(G)) \leq O_p(G)$
• $G$ has local characteristic $p$ if all $p$-local subgroups of $G$ have characteristic $p$
• $G$ has parabolic characteristic $p$ if all $p$-local subgroups which contain a Sylow $p$-subgroup of $G$ have characteristic $p$
Collection $\mathcal{C}$ family of subgroups of $G$
  closed under $G$-conjugation
  partially ordered by inclusion

Subgroup complex $|\mathcal{C}| = \Delta(\mathcal{C})$
  simplices: $\sigma = (Q_0 < Q_1 < \ldots < Q_n), \ Q_i \in \mathcal{C}$
  isotropy group of $\sigma$: $G_\sigma = \cap_{i=0}^n N_G(Q_i)$
  fixed point set of $Q$: $|\mathcal{C}|^Q = \Delta(\mathcal{C})^Q$

Standard collections
  Brown $\mathcal{I}_p(G)$ $p$-subgroups
  Quillen $\mathcal{A}_p(G)$ elementary abelian $p$-subgroups
  Bouc $\mathcal{B}_p(G)$ $p$-radical subgroups
  $\mathcal{B}_{p\text{cen}}(G)$ $p$-centric and $p$-radical subgroups

Equivariant homotopy equivalences: $\mathcal{A}_p(G) \subseteq \mathcal{I}_p(G) \supseteq \mathcal{B}_p(G)$
**Terminology and Notation: Lefschetz Modules**

- $k$: field of characteristic $p$
- $\Delta$: subgroup complex
- $\Delta/G$: the orbit complex of $\Delta$

The reduced Lefschetz module

- alternating sum of chain groups
  \[ \tilde{L}_G(\Delta; k) := \sum_{i=-1}^{\lfloor \frac{\text{dim} \Delta}{p} \rfloor} (-1)^i C_i(\Delta; k) \]
- element of Green ring of $kG$
  \[ \tilde{L}_G(\Delta; k) = \sum_{\sigma \in \Delta/G} (-1)^{\text{dim} \sigma} \text{Ind}_{G\sigma}^G k - k \]

- for a Lie group in defining characteristic $\tilde{L}_G(|\mathcal{I}_p(G)|; k)$ is equal to the Steinberg module
- $\tilde{L}_G(|\mathcal{I}_p(G)|; k)$ is virtual projective module
- Thévenaz (1987): $\tilde{L}_G(\Delta; k)$ is $\mathcal{K}$-relatively projective
  \( \mathcal{K} \) is a collection of $p$-subgroups
  $\Delta^Q$ is contractible for every $p$-subgroup $Q \notin \mathcal{K}$
if $\Delta^Q$ is contractible for $Q$ any subgroup of order $p$ then $\tilde{L}_G(\Delta; \mathbb{Z}_p)$ is virtual projective module and $\hat{H}^n(G; M)_p = \sum_{\sigma \in \Delta/G} (-1)^{|\sigma|} \hat{H}^n(G_\sigma; M)_p$

- Webb, 1987

- sporadic geometries with projective reduced Lefschetz modules
  - Ryba, Smith and Yoshiara, 1990

- relate projectivity of the reduced Lefschetz module for sporadic geometries to the $p$-local structure of the group
  - Smith and Yoshiara, 1997

- $\tilde{L}(|B_p^{\text{cen}}|; k)$ is projective relative to the collection of $p$-subgroups which are $p$-radical but not $p$-centric
  - Sawabe, 2005

- connections between 2-local geometries and standard complexes for the 26 sporadic simple groups
  - Benson and Smith, 2008
A 2-Local Geometry for $Co_3$

$G$ - Conway’s third sporadic simple group $Co_3$
$\Delta$ - standard 2-local geometry with vertex stabilizers given below:

$$G_p = 2.Sp_6(2)$$
$$G_L = 2^{2+6}3.(S_3 \times S_3)$$
$$G_M = 2^4.L_4(2)$$

**Theorem [MO]**

The 2-local geometry $\Delta$ for $Co_3$ is equivariant homotopy equivalent to the complex of distinguished 2-radical subgroups $|\hat{B}_2(Co_3)|$; 2-radical subgroups containing 2-central involutions in their centers.
An element of order $p$ in $G$ is $p$-central if it lies in the center of a Sylow $p$-subgroup of $G$.

Let $\mathcal{C}_p(G)$ be a collection of $p$-subgroups of $G$.

**Definition**

The distinguished collection $\hat{\mathcal{C}}_p(G)$ is the collection of subgroups in $\mathcal{C}_p(G)$ which contain $p$-central elements in their centers.
Proposition [MO]

The inclusion \( \widehat{A}_p(G) \hookrightarrow \widehat{F}_p(G) \) is a \( G \)-homotopy equivalence.
Proposition [MO]

The inclusion $\hat{\mathcal{A}}_p(G) \hookrightarrow \hat{\mathcal{P}}_p(G)$ is a $G$-homotopy equivalence.

A poset $C$ is conically contractible if there is a poset map $f : C \to C$ and an element $x_0 \in C$ such that $x \leq f(x) \geq x_0$ for all $x \in C$.

Theorem [Thévenaz and Webb, 1991]:

Let $C \subseteq D$. Assume that for all $y \in D$ the subposet $C_{\leq y} = \{x \in C | x \leq y\}$ is $G_y$-contractible. Then the inclusion is a $G$-homotopy equivalence.
A Homotopy Equivalence

Proposition [MO]

The inclusion $\hat{A}_p(G) \hookrightarrow \hat{\mathcal{F}}_p(G)$ is a $G$-homotopy equivalence.

A poset $\mathcal{C}$ is conically contractible if there is a poset map $f : \mathcal{C} \to \mathcal{C}$ and an element $x_0 \in \mathcal{C}$ such that $x \leq f(x) \geq x_0$ for all $x \in \mathcal{C}$.

Theorem [Thévenaz and Webb, 1991]:
Let $\mathcal{C} \subseteq \mathcal{D}$. Assume that for all $y \in \mathcal{D}$ the subposet $\mathcal{C}_{\leq y} = \{ x \in \mathcal{C} | x \leq y \}$ is $G_y$-contractible. Then the inclusion is a $G$-homotopy equivalence.

Proof.

Let $P \in \hat{\mathcal{F}}_p(G)$ and let $Q \in \hat{\mathcal{A}}_p(G) \leq P$. $\hat{P}$ is the subgroup generated by the $p$-central elements in $Z(P)$. The subposet $\hat{\mathcal{A}}_p(G) \leq P$ is contractible via the double inequality:

$$Q \leq \hat{P} \cdot Q \geq \hat{P}$$

The poset map $Q \to \hat{P} \cdot Q$ is $N_G(P)$-equivariant.
If $G$ has parabolic characteristic $p$, then $\hat{B}_p(G) \leftrightarrow \hat{P}_p(G)$ is a $G$-homotopy equivalence.
If $G$ has parabolic characteristic $p$, then $\hat{B}_p(G) \hookrightarrow \hat{P}_p(G)$ is a $G$-homotopy equivalence.

Webb’s alternating sum formula holds for $\hat{B}_p(G)$.

- $H^*(G; \hat{L}_G(\hat{B}_p; k)) = 0$
If $G$ has parabolic characteristic $p$, then $\hat{\mathcal{B}}_p(G) \hookrightarrow \hat{\mathcal{I}}_p(G)$ is a $G$-homotopy equivalence.

Webb’s alternating sum formula holds for $\hat{\mathcal{B}}_p(G)$:

- $H^\ast(G; \hat{L}_G(\lvert \hat{\mathcal{B}}_p \rvert; k)) = 0$
- $\mathcal{B}_p^{\text{cen}} \subseteq \hat{\mathcal{B}}_p \subseteq \mathcal{B}_p$

If $G$ has parabolic characteristic $p$ then $\hat{\mathcal{B}}_p = \mathcal{B}_p^{\text{cen}}$.
If $G$ has parabolic characteristic $p$, then $\hat{B}_p(G) \hookrightarrow \hat{P}_p(G)$ is a $G$-homotopy equivalence.

Webb’s alternating sum formula holds for $\hat{B}_p(G)$
- $H^*(G; L_G(|\hat{B}_p|; k)) = 0$

$\mathcal{B}_p^{\text{cen}} \subseteq \hat{B}_p \subseteq \mathcal{B}_p$
- if $G$ has parabolic characteristic $p$ then $\hat{B}_p = \mathcal{B}_p^{\text{cen}}$

$|\hat{B}_p(G)|$ is homotopy equivalent to the standard 2-local geometry for all but two ($Fi_{23}$ and $O’N$) sporadic simple groups.
If $G$ has parabolic characteristic $p$, then $\hat{\mathcal{B}}_p(G) \hookrightarrow \hat{\mathcal{I}}_p(G)$ is a $G$-homotopy equivalence.

Webb’s alternating sum formula holds for $\hat{\mathcal{B}}_p(G)$:

1. $H^*(G; \hat{L}_G(|\hat{\mathcal{B}}_p|; k)) = 0$

2. $\mathcal{B}^\text{cen}_p \subseteq \hat{\mathcal{B}}_p \subseteq \mathcal{B}_p$

   - if $G$ has parabolic characteristic $p$ then $\hat{\mathcal{B}}_p = \mathcal{B}^\text{cen}_p$

3. $|\hat{\mathcal{B}}_p(G)|$ is homotopy equivalent to the standard 2-local geometry for all but two ($\text{Fi}_{23}$ and $O'N$) sporadic simple groups.

4. $\hat{\mathcal{B}}_p(G)$ preserves the geometric interpretation of the points of the geometry in cases where $\mathcal{B}^\text{cen}_p$ does not
   - in $\text{Co}_3$, the 2-central involutions (the points of the geometry) are 2-radical but not 2-centric.
Proposition 1 [MO]

Let $G$ be a finite group of parabolic characteristic $p$. Let $z$ be a $p$-central element in $G$ and let $Z = \langle z \rangle$. Then the fixed point set $|\hat{B}_p(G)|^Z$ is $N_G(Z)$-contractible.
Proposition 1 [MO]
Let $G$ be a finite group of parabolic characteristic $p$. Let $z$ be a $p$-central element in $G$ and let $Z = \langle z \rangle$. Then the fixed point set $|\hat{B}_p(G)|^Z$ is $N_G(Z)$-contractible.

Proposition 2 [MO]
Let $G$ be a finite group of parabolic characteristic $p$. Let $t$ be a noncentral element of order $p$ and let $T = \langle t \rangle$. Assume that $O_p(C_G(t))$ contains a $p$-central element. Then the fixed point set $|\hat{B}_p(G)|^T$ is $N_G(T)$-contractible.
Fixed Point Sets

Theorem 3 [MO]

Assume $G$ is a finite group of parabolic characteristic $p$. Let $T = \langle t \rangle$ with $t$ an element of order $p$ of noncentral type in $G$. Let $C = C_G(t)$. Suppose that the following hypotheses hold:

- $O_p(C)$ does not contain any $p$-central elements;
- The quotient group $\overline{C} = C/O_p(C)$ has parabolic characteristic $p$.

Then there is an $N_G(T)$-equivariant homotopy equivalence

$$|\widehat{B}_p(G)|^T \simeq |\widehat{B}_p(\overline{C})|$$
The proof requires a combination of equivariant homotopy equivalences:

\[
|\hat{\mathcal{B}}_p(G)|^T \simeq |\hat{\mathcal{P}}_p(G)|^T \simeq |\hat{\mathcal{P}}_p(G)_{\leq C}^T| \simeq |\hat{\mathcal{P}}_p(G)_{\leq C}^T| \\
\simeq |\hat{\mathcal{P}}_p(G)_{\leq C}^{\leq C}_O| \simeq |\hat{\mathcal{P}}_p(G)_{\leq C}^{\leq C}_O| \simeq |\mathcal{S}| \simeq |\hat{\mathcal{P}}(\overline{C})| \simeq |\hat{\mathcal{B}}_p(\overline{C})|
\]

Some of the notations used:

- \(\tilde{S}_p(G) = \{ p\)-subgroups of \(G\) which contain \(p\)-central elements\},
- \(C_{\leq H}^P = \{ Q \in C \mid P < Q \leq H \}\),
- \(O_C = O_p(C)\) and \(C = C_G(t)\),
- \(\mathcal{S} = \{ P \in \hat{\mathcal{P}}_p(G)_{\leq C}^{\leq C}_O \mid Z(P) \cap Z(S) \neq 1, \text{ for } S_T \text{ and } S \text{ such that } P \leq S_T \leq S\}\),
- \(S_T \in \text{Syl}_p(C)\) which extends to \(S \in \text{Syl}_p(G)\).
A 2-Local Geometry for $\text{Fi}_{22}$

- $G = \text{Fi}_{22}$ has parabolic characteristic 2
- $G$ has three conjugacy classes of involutions:
  - $C_{\text{Fi}_{22}}(2A) = 2.U_6(2)$
  - $C_{\text{Fi}_{22}}(2B) = (2 \times 2^{1+8} : U_4(2)) : 2$, are 2-central
  - $C_{\text{Fi}_{22}}(2C) = 2^{5+8} : (S_3 \times 3^2 : 4)$
- $\Delta$ is the standard 2-local geometry for $G$, it is $G$-homotopy equivalent to $\hat{B}_2(G)$ and has vertex stabilizers:

\[
\begin{align*}
H_1 &= (2 \times 2^{1+8} : U_4(2)) : 2 \\
H_5 &= 2^{5+8} : (S_3 \times A_6) \\
H_6 &= 2^6 : \text{Sp}_6(2) \\
H_{10} &= 2^{10} : M_{22}
\end{align*}
\]
A 2-Local Geometry for $Fi_{22}$

**Proposition 4 [MO]**

Let $\Delta$ be the 2-local geometry for the Fischer group $Fi_{22}$.

a. The fixed point sets $\Delta^{2B}$ and $\Delta^{2C}$ are contractible.

b. The fixed point set $\Delta^{2A}$ is equivariantly homotopy equivalent to the building for the Lie group $U_6(2)$.

c. There is precisely one nonprojective summand of the reduced Lefschetz module, it has vertex $\langle 2A \rangle$ and lies in a block with the same group as defect group.

d. As an element of the Green ring:

$$\tilde{L}_{Fi_{22}}(\Delta) = -P_{Fi_{22}}(\varphi_{12}) - P_{Fi_{22}}(\varphi_{13}) - 6\varphi_{15} - 12P_{Fi_{22}}(\varphi_{16}) - \varphi_{16}.$$
Proof of the Proposition 4

Theorem [Robinson]: \textit{The number of indecomposable summands of } \widetilde{L}_G(\Delta) \textit{ with vertex } Q \textit{ is equal to the number of indecomposable summands of } \widetilde{L}_{NG}(Q)(\Delta^Q) \textit{ with the same vertex } Q. \textit{no vertex of an indecomposable summand of } \widetilde{L}_G(\Delta) \textit{ contains an involution of type } 2B \textit{ or } 2C \textit{ (Smith theory) thus } \widetilde{L}_{NG}(Q)(\Delta^Q) = 0. \textit{There is one nonprojective summand, it has vertex } \langle 2A \rangle \textit{ then } \Delta Q \textit{ is contractible}.
Theorem [Robinson]: *The number of indecomposable summands of* \( \tilde{L}_G(\Delta) \)*
*with vertex* \( Q \) *is equal to the number of indecomposable summands of* \( \tilde{L}_{NG}(Q)(\Delta^Q) \)*
*with the same vertex* \( Q \).

- the involutions \( 2B \) are central, Proposition 1 implies \( \Delta^{2B} \) is contractible
- \( O_2(C_G(2C)) \) contains 2-central elements, Proposition 2 implies that \( \Delta^{2C} \) is contractible
- \( \Delta^Q \) is mod-2 acyclic for any 2-group \( Q \) containing an involution of type \( 2B \) or \( 2C \) (Smith theory) thus \( \tilde{L}_{NG}(Q)(\Delta^Q) = 0 \)
Theorem [Robinson]: *The number of indecomposable summands of* $\tilde{L}_G(\Delta)$ *with vertex* $Q$ *is equal to the number of indecomposable summands of* $\tilde{L}_{NG}(Q)(\Delta^Q)$ *with the same vertex* $Q$.

- the involutions $2B$ are central, Proposition 1 implies $\Delta^{2B}$ is contractible
- $O_2(C_G(2C))$ contains 2-central elements, Proposition 2 implies that $\Delta^{2C}$ is contractible
- $\Delta^Q$ is mod-2 acyclic for any 2-group $Q$ containing an involution of type $2B$ or $2C$ (Smith theory) thus $\tilde{L}_{NG}(Q)(\Delta^Q) = 0$
- no vertex of an indecomposable summand of $\tilde{L}_G(\Delta)$ contains an involution of type $2B$ or $2C$
Proof of the Proposition 4

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- \( \Delta^Q \) is mod-2 acyclic for any \( 2 \)-group \( Q \) containing an involution of type \( 2B \) or \( 2C \) (Smith theory) thus \( \tilde{L}_{NG}(Q)(\Delta^Q) = 0 \)
- no vertex of an indecomposable summand of \( \tilde{L}_G(\Delta) \) contains an involution of type \( 2B \) or \( 2C \)
- \( C_G(2A)/O_2(C_G(2A)) = U_6(2) \), Theorem 3 implies that \( \Delta^{2A} \) is homotopy equivalent to the building for \( U_6(2) \)
- \( \Delta^Q \) is contractible for any \( Q > \langle 2A \rangle \)
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- \( \Delta^Q \) is contractible for any \( Q > \langle 2A \rangle \)
- there is one nonprojective summand, it has vertex \( \langle 2A \rangle \)