QUANTIFIER ELIMINATION IN C*-ALGEBRAS

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ABSTRACT. Conjecturally, the only C*-algebras that admit elimination of quantifiers in continuous logic are \( \mathbb{C}, \mathbb{C}^2, C(\text{Cantor space}) \) and \( M_2(\mathbb{C}) \). Focusing on the noncommutative case, we prove that \( M_2(\mathbb{C}) \) is the only exact noncommutative C*-algebra whose theory admits elimination of quantifiers and that every other example is purely infinite and simple.

INTRODUCTION

One of the key steps in using model theory in applications is to understand the definable objects in models of a particular theory. It is often the case that the objects which can be defined without the use of quantifiers have particularly natural descriptions, while definitions involving quantifiers are more difficult to analyze. Quantifier elimination, which is the property that every definable object can be defined without using quantifiers, is therefore a highly desirable feature for a theory to possess.

Quantifier elimination is a matter of the formal language used to study the structures of interest. It is easy to see that any theory can be extended to a theory with quantifier elimination in an expanded language by simply adding a new symbol for every object definable in the original one; the usefulness of quantifier elimination results therefore depends on using a natural language for the structures at hand. For this reason we consider C*-algebras as structures in the language for C*-algebras introduced in [FHS14]. This language contains symbols for the natural operations in a C*-algebra, and is sufficiently expressive that many natural classes of C*-algebras are either axiomatizable, or at least defined by the omission of certain types (many examples of this kind are given in [FHL+]). Nevertheless, this language is also sufficiently limited that quantifier-free formulas are quite simple, being continuous combinations of norms of *-polynomials with complex coefficients.

Conjecture 1. The only theories of unital C*-algebras admitting quantifier elimination are those of \( \mathbb{C}, \mathbb{C}^2, M_2(\mathbb{C}), \) and \( C(\text{Cantor space}) \).

Our main result is that a noncommutative counterexample to this conjecture, if it exists, is necessarily purely infinite, simple, nonexact, not embeddable into an ultrapower of Cuntz algebra \( \mathcal{O}_2 \) and not tensorially \( \mathcal{O}_2 \)-absorbing. We also prove that the theory of C*-algebras does not have model companion and give natural examples of C*-algebras whose theories are not \( \forall \exists \)-axiomatizable. In the appendix,
written with D.C. Amador, B. Hart, J. Kawach, and S. Kim, we show that a
commutative counterexample, if it exists, is of the form $C(X)$ for an indecomposable
continuum $X$.

Section 1 contains preliminaries and a test for quantifier elimination. In this section
we also completely answer the question of which finite-dimensional C*-algebras
have quantifier elimination. In Section 2 we prove our main results, implying in
particular that $M_2(\mathbb{C})$ is the only noncommutative exact C*-algebra whose
theory admits the elimination of quantifiers. In Section 3 we show that the theory of unital
C*-algebras does not have a model companion, and also obtain results related to
the $\forall \exists$-axiomatizability of some classes of C*-algebras.

Throughout the paper we consider only unital C*-algebras.

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1. Quantifier Elimination

In this section we recall the model-theoretic framework for studying C*-algebras,
as well as tests for quantifier elimination. The reader interested in a more complete
discussion of the model theory of C*-algebras can consult [FHS14], or the forth-
coming [FHL+]. For more on quantifier elimination in metric structures in general,
see [BYBH08, Section 13].

Definition 1.1. The formulas for C*-algebras are recursively defined as follows.
In each case, $\overline{v}$ denotes a finite tuple of variables (which will later be interpreted as
elements of a C*-algebra).

(1) If $P(\overline{v})$ is a $*$-polynomial with complex coefficients, then $\|P(\overline{v})\|$ is a for-
mula.

(2) If $\varphi_1(\overline{v}), \ldots, \varphi_n(\overline{v})$ are formulas and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then
$f(\varphi_1(\overline{v}), \ldots, \varphi_n(\overline{v}))$ is a formula.

(3) If $\varphi(\overline{v}, y)$ is a formula and $R \in \mathbb{R}^+$, then $\sup_{\|y\| \leq R} \varphi(\overline{v}, y)$ and $\inf_{\|y\| \leq R} \varphi(\overline{v}, y)$
are formulas.

We think of $\sup_{\|y\| \leq R}$ and $\inf_{\|y\| \leq R}$ as replacements for the first-order quantifiers
$\forall$ and $\exists$, respectively. A formula constructed using only clauses (1) and (2) of
the definition is therefore said to be quantifier-free.

The definition above is slightly different from the one in [FHS14]. In particular,
we have replaced their domains of quantification by requiring that our suprema and
infima range over closed balls of finite radius. Since the two versions are equivalent,
we have chosen an approach intended to minimize the technical complexity of the
definitions.

If $\varphi(\overline{v})$ is a formula, $A$ is a C*-algebra, and $\overline{a}$ is a tuple of elements of $A$ of
the same length as the tuple $\overline{v}$, then there is a natural way to evaluate $\varphi$ in $A$ with $\overline{v}$ replaced by $\overline{a}$; the result is a real number denoted $\varphi^A(\overline{a})$.

It is useful to collect all of the information about a tuple $\overline{v}$ which can be expressed
by formulas. Because of clause (2) of the definition of formulas, it is equivalent to
just consider which formulas take the value 0 when evaluated at $\overline{a}$.
Definition 1.2. Let $A$ be a C*-algebra, and $\overline{a} \in A^n$ be a tuple of elements from $A$. The type of $\overline{a}$ in $A$, denoted $\text{tp}^A(\overline{a})$, is defined to be the set of all formulas $\varphi(\overline{x})$ such that $\varphi^A(\overline{a}) = 0$. Similarly, the quantifier-free type of $\overline{a}$, denoted $\text{qftp}^A(\overline{a})$, is the set of all quantifier-free formulas $\varphi(\overline{x})$ such that $\varphi^A(\overline{a}) = 0$. If the algebra $A$ is clear from the context we omit it from the notation.

We note that it follows from the definition of formulas that if $\overline{a}$ and $\overline{b}$ are two tuples of finite length in a C*-algebra $A$, then $\text{qftp}(\overline{a}) = \text{qftp}(\overline{b})$ if and only if $\|\varphi(\overline{x})\| = \|\varphi(\overline{y})\|$ for all *-polynomials $\varphi$ with complex coefficients.

A formula without free variables is a sentence. A theory $T$ is a set of sentences, and a C*-algebra $A$ is a model of $T$ (written $A \models T$) if every sentence in $T$ takes the value 0 when interpreted in $A$. The theory of $A$, $\text{Th}(A)$, is the set of all sentences which take value 0 when interpreted in $A$.

Definition 1.3. A theory $T$ has quantifier elimination if for every formula $\varphi(\overline{x})$ and every $\epsilon > 0$ there is a quantifier-free formula $\psi_\epsilon(\overline{x})$ such that whenever $A \models T$ and $\overline{a}$ is a tuple of the appropriate length and sorts from $A$, we have

$$|\varphi^A(\overline{a}) - \psi_\epsilon^A(\overline{a})| \leq \epsilon.$$

By a standard abuse of language, we say that a C*-algebra $A$ has quantifier elimination if $\text{Th}(A)$ does.

We record some general consequences of quantifier elimination for a C*-algebra. We will apply these results in the subsequent sections to show that various classes of C*-algebras do not admit quantifier elimination. The first of these results, Lemma 1.4, is straightforward but very useful as it gives an analytic description of a quantifier-free type of a tuple of commuting normal elements. The joint spectrum of commuting normal elements $a_1, \ldots, a_n$, $j\sigma(\overline{a})$, is the set of all $\lambda \in \mathbb{C}^n$ such that $\{\lambda_1 - a_1, \lambda_2 - a_2, \ldots, \lambda_n - a_n\}$ generates a proper ideal. Equivalently, if these elements belong to a commutative algebra $C(X)$ then

$$j\sigma(a_1, \ldots, a_n) = \{(a_1(x), \ldots, a_n(x)) : x \in X\}.$$

Lemma 1.4. In any C*-algebra, two finite tuples of commuting normal elements have the same quantifier-free type if and only if they have the same joint spectrum. Consequently, if the C*-algebra has quantifier elimination, then two finite tuples of commuting normal elements have the same type if and only if they have the same joint spectrum.

Proof. Let $\overline{a}$ and $\overline{b}$ be finite tuples of commuting normal elements. By (EV14) Prop. 5.23 the joint spectrum $j\sigma(\overline{a})$ is quantifier-free definable from $\overline{a}$, and hence if $\text{qftp}(\overline{a}) = \text{qftp}(\overline{b})$ then $j\sigma(\overline{a}) = j\sigma(\overline{b})$. Conversely, if $j\sigma(\overline{a}) = j\sigma(\overline{b})$, then $C^*(\overline{a}) \cong C^*(\overline{b})$ by an isomorphism sending $\overline{a}$ to $\overline{b}$, and so $\text{qftp}(\overline{a}) = \text{qftp}(\overline{b})$. \qed

The following easy generalization of Lemma 1.4 is also worth stating.

Lemma 1.5. For $n$-tuples $\overline{a}$ and $\overline{b}$ in C*-algebras $A$ and $B$, respectively, one has that $\text{qftp}(\overline{a}) = \text{qftp}(\overline{b})$ if and only if C*-algebras generated by $\overline{a}$ and $\overline{b}$ are isomorphic via an isomorphism that sends $\overline{a}$ to $\overline{b}$.

Proof. The quantifier-free type of $\overline{a}$ determines the values of $\|\varphi(\overline{a})\|$, where $\varphi$ ranges over all *-polynomials in $n$ variables with complex coefficients. This data exactly determines the structure of $C^*(\overline{a})$. \qed
By $A^U$ we denote an ultrapower of $A$ associated with an ultrafilter $\mathcal{U}$. All ultrafilters are assumed to be nonprincipal ultrafilters on $\mathbb{N}$. We shall need the following consequence of \[BYBHU08\,\text{Prop. 13.6}].

**Proposition 1.6.** The following are equivalent for every unital $C^*$-algebra $A$.

1. $A$ has quantifier elimination.
2. Whenever $B$ is elementarily equivalent to $A$ and separable, and $N$ is a finitely generated unital subalgebra of $B$, then every unital $^*$-homomorphism $\Phi: N \to B^U$ can be extended to a unital $^*$-homomorphism $\Psi: B \to B^U$.

**Proof.** Assume (1) holds and fix $B, N, \Phi$ as in (2). Let $\bar{a}$ be a tuple that generates $N$. By Lemma 1.5, the quantifier-free type of $\bar{a}$ determines $N$ up to the isomorphism. By (1), the quantifier-free type of $\bar{a}$ determines its type. Since $B$ is separable, one can enumerate a dense subset of its unit ball as $b_n$, for $n \in \mathbb{N}$. By the countable saturation of ultrapowers (see e.g., \[FHS14\]) one can recursively choose $c_n$, for $n \in \mathbb{N}$, in $B^U$ such that $\text{tp}_B(\bar{a}, b_1, \ldots, b_n) = \text{tp}_{B^U}(\Phi(\bar{a}), c_1, \ldots, c_n)$ for all $n$. Then the map $b_n \mapsto c_n$ is an isometry and its continuous extension to $B$ is $\Psi$ as required.

The converse implication is an easy consequence of \[BYBHU08\,\text{Prop. 13.6} and we shall not need it. $\square$

1.1. **Finite-dimensional $C^*$-algebras.** To conclude this section we treat the case of finite-dimensional $C^*$-algebras, proving that the only ones with quantifier elimination are $\mathbb{C}$, $\mathbb{C}^2$, and $M_2(\mathbb{C})$. When $A$ is finite-dimensional the closed balls of $A$ of finite radius are compact, and it follows that $A$ is (up to isomorphism) the only model of its theory (see \[BYBHU08\, p. 24]). We say that a projection $p$ is minimal if $pAp \cong \mathbb{C}$.

**Theorem 1.7.** For a finite-dimensional $C^*$-algebra the following are equivalent.

1. $A$ is isomorphic to a subalgebra of $M_2(\mathbb{C})$.
2. Every commutative subalgebra of $A$ is isomorphic to $\mathbb{C}$ or to $\mathbb{C}^2$.
3. $A$ is isomorphic to one of $\mathbb{C}, \mathbb{C}^2$, or $M_2(\mathbb{C})$.
4. $A$ has quantifier elimination.

**Proof.** The equivalence of (1), (2) and (3) is an easy consequence of the fact that every finite-dimensional $C^*$-algebra is isomorphic to a direct sum of full matrix algebras.

We prove that (4) implies (2). If (2) fails, then there are two projections $p$ and $q$ in $A$ which are both minimal, are orthogonal, and are such that $q \neq 1 - p$. If $A$ has quantifier elimination then every nontrivial projection has the same type as $p$, and in particular, is minimal. This contradicts the fact that $q + p$ is a nontrivial nonminimal projection.

We now prove that $M_2(\mathbb{C})$ has quantifier elimination, using Proposition 1.6. Every ultrapower of $M_2(\mathbb{C})$ is isomorphic to $M_2(\mathbb{C})$. If $M$ and $N$ are isomorphic unital subalgebras of $M_2(\mathbb{C})$, then by the equivalence of (2) and (3) and easy computation the isomorphism of $M$ and $N$ is implemented by a unitary in $M_2(\mathbb{C})$. Therefore the isomorphism extends to an automorphism of $M_2(\mathbb{C})$, and this completes the proof.

We omit the proofs that $\mathbb{C}$ and $\mathbb{C}^2$ have quantifier elimination, which are similar but easier (see also Lemma 1.4). $\square$

The following is also worth recording.
Proposition 1.8. The only $C^*$-algebras that have a minimal projection and admit elimination of quantifiers are $\mathbb{C}$, $\mathbb{C}^2$ and $M_2(\mathbb{C})$.

Proof. Assume $A$ has elimination of quantifiers and a minimal projection $p$. If $A$ is not $\mathbb{C}$ then $p$ is nontrivial and by Lemma 1.4 every other nontrivial projection in $A$ has the same quantifier-free type as $p$. In particular, using quantifier elimination, $1 - p$ has the same type as $p$, and hence is also minimal. It follows that $A \cong \mathbb{C}^2$ or $A \cong M_2(\mathbb{C})$.

2. Noncommutative $C^*$-algebras

For the remainder of the paper all $C^*$-algebras are assumed to be infinite-dimensional, noncommutative, and unital, unless mentioned otherwise. Our first result shows that quantifier elimination has strong consequences for the structure of projections:

Proposition 2.1. Let $A$ be unital, noncommutative, and infinite-dimensional. If $A$ has quantifier elimination then $A$ is purely infinite and simple.

The proof of Proposition 2.1 requires some definitions and lemmas.

Lemma 2.2. Assume $A$ is a $C^*$-algebra (not necessarily unital) with no minimal projections. Then it contains a positive element with spectrum equal to $[0, 1]$.

Proof. We may assume $A$ is separable. Let $(X, d)$ be a locally compact metric space such that $C_0(X)$ is isomorphic to a masa of $A$. By the continuous functional calculus we need to find $f \in C_0(X)$ whose range is a nontrivial interval. Since $A$ has no minimal projections, $X$ has no isolated points and is therefore uncountable.

Let us first consider the case when $X$ has an uncountable connected component $Y$. Choose a point $y \in Y$ and $r > 0$ small enough to have $\sup_{z \in Y} d(z, y) \geq r$ and that $\{x \in X : d(x, y) < r\}$ is relatively compact. Define $g : [0, \infty) \to [0, 1]$ by $g(t) = \frac{2t}{r}$ if $t \leq r/2$, $g(t) = \frac{2t}{r} + 2$ if $r/2 < t \leq r$, and $g(t) = 0$ elsewhere. Then $f : X \to [0, 1]$ defined by $f(x) = g(d(x, y))$ is in $C_0(X)$ and its range is equal to $[0, 1]$.

If there is no such $Y$ then every connected component of $X$ consists of a single point and therefore $X$ is zero-dimensional. Being locally compact and with no isolated points, $X$ has a clopen subset homeomorphic to the Cantor set. Since the Cantor set maps continuously onto $[0, 1]$, we can find $f$ as required.

Definition 2.3 (Cuntz78). For positive elements $a$ and $b$ in a $C^*$-algebra $A$ we write $a \preceq b$, and say that $a$ is Cuntz-subequivalent to $b$, if there is a sequence $\{z_n\}_{n \in \mathbb{N}}$ such that

$$\lim_n \|z_n b z_n^* - a\| = 0.$$ 

Lemma 2.4. For every $x$ in every $C^*$-algebra one has $x^* x \preceq xx^*$. Moreover, for every $n$ there exists $z_n$ with $\|z_n\| \leq n$ such that $\|x^* x - z_n^* xx^* z_n\| \leq 1/n$.

Proof. For $n \in \mathbb{N}$ let $f_n : [0, 1] \to [0, n]$ be defined by $f_n(t) = t^{-1/2}$ if $t \geq 1/n^2$ and $f_n(t) = n$ if $t < 1/n$. Let $z_n = x f_n(x^* x)$. Clearly $\|z_n\| \leq n$. Also we have the following computation, which takes place in the commutative algebra $C^*(x^* x)$:

$$z_n^* xx^* z_n = f_n(x^* x)(x^* x)^2 f_n(x^* x) = g_n(x^* x),$$

where $g_n(t) = t$ if $t \geq 1/n^2$ and $g_n(t) = t(1 - t/n)$ if $t < 1/n^2$. Since $|t - g_n(t)| < 1/n$ we have that $\|x^* x - z_n^* xx^* z_n\| < 1/n$, as required.
Let us temporarily write \( a \sim b \) if

1. \( a \) and \( b \) are positive
2. for every \( n \) there is \( z_n \) such that \( \|b - z_n^* a z_n\| \leq 1/n \) and \( \|z_n\| \leq n \), and
3. for every \( n \) there is \( y_n \) such that \( \|a - y_n^* b y_n\| \leq 1/n \) and \( \|y_n\| \leq n \).

This is an equivalence relation, and the type of the pair \( a, b \) is capable of detecting \( a \sim b \). The following technical lemma is needed in the proof of Proposition 2.1.

**Lemma 2.5.** If \( A \) is noncommutative, infinite-dimensional, and has quantifier elimination, then there exist \( a, b \in A \) such that \( a \) and \( b \) are orthogonal positive elements each with spectrum \([0, 1]\); moreover, for any such pair, \( a \sim b \).

**Proof.** Since \( A \) is noncommutative, by [Bla06 II.6.4.14], there is \( x \) such that \( \|x\| = 1 \) and \( x^2 = 0 \). Then \( xx^* \) and \( x^*x \) are orthogonal positive elements of norm 1. Moreover, the spectra of \( a = xx^* \) and \( b = x^*x \) both contain 0 and are therefore equal.

Let us prove that we may assume \( \sigma(a) = [0, 1] \). If \( \sigma(a) \neq [0, 1] \) then by continuous functional calculus we can find a nonzero projection \( p \in C^*(a) \). Since by Proposition [Bla06 II.3.8] \( A \) has no minimal projections, the algebra \( pAp \) is infinite-dimensional and by Lemma 2.3 we can find positive \( a_1 \in pAp \) such that \( \sigma(a_1) = [0, 1] \). Let \( x_1 = pa_1 \). Note that \( x_1 = xa_1 = a_1x \), hence \( (x_1)^2 = 0 \). Also, if we let \( a_2 = x_1^* x_1 \) then \( a_2 = a_1^2 \) and hence \( \sigma(a_2) = [0, 1] \). Therefore by replacing \( x \) with \( x_1 \) and re-evaluating \( a \) and \( b \) we may assume \( \sigma(a) = [0, 1] \). In particular, \( A \) contains a unital copy of \( C([0, 1]) \).

If \( c \) and \( d \) are positive orthogonal elements with \( \sigma(c) = \sigma(d) = [0, 1] \) we have that \( j\sigma(a, b) = j\sigma(c, d) \) hence by quantifier elimination \( tp^A(c, d) = tp^A(a, b) \). In particular \( \inf_{\|y\| \leq 1} \|y - y^*\| = 0 \), and so by Lemma 2.4 we also have \( c \sim d \).

We now have all of the ingredients necessary to prove Proposition 2.1.

**Proof of Proposition 2.1.** We are assuming that \( A \) is unital, noncommutative, infinite-dimensional and has quantifier elimination. To show that \( A \) is simple and purely infinite it suffices, by [Ror02 Proposition 1.4.1 (i)], to check that for every two positive elements \( a \) and \( b \) we have \( b \not\prec a \). Before doing so we will need two preliminary claims.

**Claim 2.5.1.** Suppose that \( f, g \in A \) are positive elements each with spectrum \([0, 1]\), and \( fg = gf \). Then \( f \sim g \).

**Proof of Claim 2.5.1.** Choose elements \( a, b \) and \( c \) in \( A \) such that \( ab = 0 \) and \( bc = cb = c \), each with spectrum equal to \([0, 1]\) - such elements can be found in \( C([0, 1]) \) and by Lemma 2.3 \( A \) contains a unital copy of \( C([0, 1]) \). Then again by Lemma 2.3 we have that \( c \sim a \sim b \) and therefore \( c \sim b \). Since the spectra of \( f \) and \( g \) are both \([0, 1]\) and \( fg = gf = g \) we have \( qftp(f, g) = qftp(b, c) \). By quantifier elimination and the fact that \( b \sim c \) we conclude that \( f \sim g \).

**Claim 2.5.2.** Every positive element \( f \) with spectrum \([0, 1]\) satisfies \( 1 \not\prec f \).

**Proof of Claim 2.5.2.** Choose \( a, b, c \in A \), each with spectrum \([0, 1]\), such that \( ab = ac = bc = 0 \), and let \( d = 1 - a \). Then \( \sigma(d) = [0, 1] \) and \( db = bd = b \), hence \( d \sim b \) by Claim 2.5.1. Also \( a \sim c \). In fact, we will only need to know that \( d \not\prec b \) and \( a \not\prec c \). Since \( bc = 0 \), by [Cun78 Proposition 1.1] we have that \( a + d \not\prec b + c \). But \( a + d = 1 \)
and \( \sigma(b + c) = [0, 1] \). In particular, \( b + c \) is a positive element with spectrum \([0, 1]\) such that \( 1 \precsim b + c \). By quantifier elimination it follows that every positive \( f \) with spectrum \([0, 1]\) has \( 1 \precsim f \). \( \square \)

Let \( a, b \) be positive; we will show that \( b \precsim a \). For every positive \( b \) with \( \|b\| \leq 1 \) we have \( b \leq 1 \) and therefore \( b \precsim 1 \). Then Claim \( \ref{claim:positive-contractions} \) implies that we have \( b \precsim a \) for every two positive contractions \( a \) and \( b \) such that \( \sigma(a) = [0, 1] \).

Now consider the case when \( \sigma(a) \neq [0, 1] \) but \( \|a\| = 1 \). Let \( a_1 = (a - 1/2)_+ \), the positive part of the self-adjoint \( a - 1/2 \). By Lemma \( \ref{lemma:positive-contractions} \) we can find \( a_2 \in a_1 A a_1 \) such that \( \sigma(a_2) = [0, 1] \). Then \( a_2 \leq 2a \). Fix \( \epsilon > 0 \). Since \( 1 \precsim a_2 \) there is \( x \) such that \( xa_2x^* = 1 - \epsilon \) then we have

\[
2xax^* \geq xa_2x^* \geq 1 - \epsilon
\]

and therefore, by continuous functional calculus, one can find \( z \) such that \( zaz^* = 1 - 2\epsilon \). Since \( \epsilon \) was arbitrary, by \cite[Proposition 2.4]{Ror92} we get \( 1 \precsim a \) and hence \( b \precsim a \). This completes the proof. \( \square \)

2.1. \( \mathcal{O}_2 \) and quantifier elimination. Any C*-algebra generated by \( n \) isometries with orthogonal ranges with sum 1 is isomorphic to the Cuntz algebra \( \mathcal{O}_n \) \cite{Cun77}. Hence \( \mathcal{O}_2 \) is the universal algebra defined by relations \( s^*s = t^*t = 1 \) and \( ss^* + tt^* = 1 \). This algebra plays a pivotal role in Elliott’s classification program (see \cite{Ror10}). Notably, \( \mathcal{O}_2 \) has some properties implied by quantifier elimination; for example, every unital embedding of \( \mathcal{O}_2 \) into itself, or into any other model of its theory, is elementary (see e.g., \cite{GSL}, Lemma \( \ref{lemma:elementary} \) below). It also satisfies the conclusion of Lemma \( \ref{lemma:quartic} \).

**Proposition 2.6.** Suppose that \( \overline{a} \) and \( \overline{b} \) are finite tuples of commuting normal elements in \( \mathcal{O}_2 \) (of the same length) with \( j\sigma(\overline{a}) = j\sigma(\overline{b}) \). Then \( \text{tp}(\overline{a}) = \text{tp}(\overline{b}) \).

**Proof.** Let \( \overline{a} = (a_1, \ldots, a_k) \) and \( \overline{b} = (b_1, \ldots, b_k) \). Without loss of generality assume that each \( a_i \) and each \( b_i \) has norm \( \leq 1 \). It is well-known that \( \mathcal{O}_2 \) is purely infinite, has real rank zero and has trivial \( K_1 \) group.

For every \( \epsilon > 0 \) and normal element \( a \in \mathcal{O}_2 \) we can find projections \( p_1, \ldots, p_n \), and \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) with \( \frac{1}{n} < \epsilon \leq \frac{1}{n^2+1} \) and \( \|a - \sum_{i=1}^n \lambda_i p_i\| < 2\epsilon \). More generally, since \( a_1, \ldots, a_k \) commute we can find \( M \in \mathbb{N} \), \( \lambda_{i,j} \in \mathbb{C} \) such that \( \|a_i - \sum_{j=1}^M \lambda_{i,j} p_j\| \) where \( p_i p_l = 0 \) for \( i \neq l \leq M \) and \( \sum_{j=1}^M p_j = 1 \). Note that \( M \) depends only on the norms of the \( a_i \)'s and on \( k \). For the same reason, since \( j\sigma(\overline{a}) = j\sigma(\overline{b}) \), we can pick projections \( q_j \), for \( j \leq M \), such that \( \|b_i - \sum_{j=1}^M \lambda_{i,j} q_j\| < \epsilon \) with \( q_j q_l = 0 \) and \( \sum_{j=1}^M q_j = 1 \). Moreover we can arrange that \( q_j = 0 \) if and only if \( p_j = 0 \).

The two \( M \)-tuples \( (p_1, \ldots, p_M) \) and \( (q_1, \ldots, q_M) \) are unitarily equivalent in \( \mathcal{O}_2 \), so there is an automorphism of \( \mathcal{O}_2 \) sending \( \overline{a} \) to \( \overline{b} \) (up to \( 2\epsilon \)). This implies that \( \overline{a} \) and \( \overline{b} \) have the same type. \( \square \)

By the Weyl–von Neumann theorem (see e.g., \cite{Dav96}) an analogous statement holds for tuples of commuting self-adjoint elements in the Calkin algebra. It is, however, not true for normal elements: Because of the Fredholm index obstruction not all unitaries with full spectrum have the same type. To see this, let \( s \) be the unilateral shift in \( \mathcal{B}(H) \) and \( \pi \) the usual quotient map. Then both \( \pi(s) \) and \( \pi(s^2) \) are unitaries with full spectrum, but \( \pi(s) \) does not have a square root. As pointed
out in the introduction to [PW07], this failure of model completeness (see [3] is one of the reasons why it was difficult to construct an outer automorphism of the Calkin algebra.

We shall need $C^*_t(F_2)$, the reduced C*-algebra of $F_2$, the free group on 2 generators. This algebra is exact (see [Kir93, p. 453, 1., 1-3], or [BO08, Proposition 5.1.8]) and therefore embeds into $O_2$ (see [KP00]).

**Theorem 2.7.** If $A$ is a separable, infinite-dimensional, noncommutative C*-algebra such that $A \otimes O_2 \cong A$ then $A$ does not have quantifier elimination.

We shall need the following well-known fact (see e.g., [PHrT15, Lemma 2.8]).

**Lemma 2.8.** If $A$ is unital and $A \otimes O_2 \cong A$ then any two embeddings of $O_2$ into the ultrapower $A^U$ are unitarily conjugate.

**Proof.** This is a consequence of the fact that $O_2$ has an *approximately inner flip*, i.e., that the automorphism of $O_2 \otimes O_2$ which interchanges $a \otimes b$ and $b \otimes a$ is *approximately inner* ([TW07]). (An automorphism is approximately inner if it is a point-norm limit of inner automorphisms.) Let $\Phi_j : O_2 \to A^U$ be unital *-homomorphisms for $j = 1, 2$. Let us first assume that the ranges of $\Phi_1$ and $\Phi_2$ commute. In this case, by Proposition 2.6, the type of the unitary $u$ such that $\Phi_j = \text{Ad } u \circ \Phi_j$ is consistent and by the countable saturation of the ultrapower the conclusion follows.

Now consider the case when the ranges of $\Phi_j$ do not necessarily commute. Since $A \otimes O_2 \cong A$ we can find a third *-homomorphism $\Psi : O_2 \to A^U$ whose range commutes with ranges of $\Phi_1$ and $\Phi_2$. Then by the above argument one sees that the range of $\Psi$ is unitarily conjugate to the range of $\Phi_j$ for $j = 1, 2$, and by composing inner automorphisms the conclusion follows.

A formula $\varphi$ is weakly stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every C*-algebra $A$ and every $a \in A$, $\varphi(a) < \delta$ implies that the distance from $a$ to the zero-set of $\varphi$ in $A$ is $< \epsilon$. In the language of logic of metric structures, being zero-set of a weakly stable formula is equivalent to being *definable* (as defined in [BYBHU08]). See [CCF14, Lemma 2.1] for details. By [BYBHU08], every formula involving quantification over a definable set is equivalent to a standard formula.

**Proof of Theorem 2.7.** We shall use Proposition 1.6 with $B = A$ and $N = C^*_t(F_2)$.

Let $U$ be a nonprincipal ultrafilter on $N$. Since $A \otimes O_2 \cong A$ and $C^*_t(F_2)$ embeds in $O_2$, we take an embedding $\iota_1 : C^*_t(F_2) \to O_2$ and extend it to an embedding $\Phi : C^*_t(F_2) \to A$.

Now we construct a second embedding $\Psi : C^*_t(F_2) \to A^U$. Each $M_n(\mathbb{C})$ embeds into $O_2$ and therefore into $A$. We therefore have an embedding of the ultraproduct $M = \prod_U M_n(\mathbb{C})$ inside $O_2^U \subseteq A^U$, denoted by $\iota_2 : M \to A^U$. By [HT03, Thm. B], there are unitary representations $\pi_n : F_2 \to M_n(\mathbb{C})$ such that for every $m \in \mathbb{N}$, all $h_1, \ldots, h_m$ in $F_2$ and all $c_1, \ldots, c_m$ in $\mathbb{C}$ one has (with $\lambda$ denoting the left-regular representation of $F_2$ on $\ell_2(F_2)$):

$$\lim_{n \to \infty} \| \sum_{j=1}^m c_j \pi_n(h_j) \| = \| \sum_{j=1}^m c_j \lambda(h_j) \|.$$  

Therefore for every nonprincipal ultrafilter $U$ on $\mathbb{N}$ the map

$$\sum_{j=1}^m c_j h_j \mapsto (\sum_{j=1}^m c_j \pi_n(h_j))/U$$

is an isometry from a dense subset of $C^*_t(F_2)$ into $M$. The continuous extension $\varphi : C^*_t(F_2) \to M$ of this isometry is therefore a unital *-homomorphism and we can take $\Psi = \iota_2 \circ \varphi$. 
We claim that \( \Phi \) cannot be extended to an embedding \( \Phi' \) of \( \mathcal{O}_2 \) into \( A^U \). Otherwise, by Lemma 2.8 we have a unitary \( u \in A^U \) conjugating the diagonal copy of \( \mathcal{O}_2 \) in \( A^U \) with \( \Phi'(\mathcal{O}_2) \). Since the defining relation for a being a unitary, \( \|xx^* - 1\| + \|x^*x - 1\| \), is weakly stable, \( u \) can be lifted to a unitary \( w \) in \( \ell_\infty(A) \). In particular, the composition of the diagonal *-homomorphism of \( \mathcal{O}_2 \) into \( \ell_\infty(A) \) and the quotient map, when restricted to \( C^*_\tau(F_2) \), gives a *-homomorphism \( \alpha: C^*_\tau(F_2) \to \ell_\infty(A) \) that lifts \( \Psi \).

Since each \( M_n(\mathbb{C}) \) is an injective von Neumann algebra, by [Bro04] Proposition IV.2.1.4, there are coordinatewise conditional expectations \( \theta_n: A \to M_n(\mathbb{C}) \); taking \( \theta \) to be the conditional expectation induced by the expectations \( \theta_n \), we have that \( \theta: \ell_\infty(A) \to \prod M_n(\mathbb{C}) \) is a completely positive contraction, hence

\[
\theta \circ \alpha: C^*_\tau(F_2) \to \prod M_n(\mathbb{C})
\]

is a completely positive contractive lifting for \( \varphi \). We have therefore constructed an injective unital *-homomorphism of \( C^*_\tau(F_2) \) into \( M \) with a completely positive contractive lifting. C*-algebras with this property are said to be quasi-diagonal (see [Bro04]). However, by a result of Rosenberg ([BO08 Corollary 7.1.18]) quasi-diagonality of \( C^*_\tau(F_2) \) implies amenability of the nonamenable group \( \mathbb{F}_2 \). This contradiction concludes the proof.

**Lemma 2.9.** Let \( A \) be a unital infinite-dimensional noncommutative C*-algebra with quantifier elimination. Then \( \mathcal{O}_2 \) embeds unitally in \( A \).

**Proof.** By Proposition 2.1 we know that \( A \) is purely infinite. Hence there is \( s \in A \) such that \( s^*s = 1 \) and \( p = ss^* < 1 \). Then \( q = 1 - p \) has the same type as \( p \) and therefore we have \( \inf_x \|xx^* - q\| + \|x^*x - 1\| = 0 \). By the weak stability of Murray–von Neumann equivalence we conclude that there exists \( t \in A \) such that \( t^*t = 1 \) and \( tt^* = q \). Hence \( C^*(s,t) \) is isomorphic to \( \mathcal{O}_2 \).

**Theorem 2.10.** Let \( A \) be an infinite-dimensional, unital noncommutative C*-algebra which embeds into an ultrapower of \( \mathcal{O}_2 \). Then \( A \) does not have quantifier elimination.

**Proof.** The proof uses the sandwich argument of [GHS13 Proposition 3.2]. By taking an elementary submodel, without loss of generality we may assume \( A \) is separable. Since, by Proposition 2.1, \( A \) is infinite, by Lemma 2.8 there is a unital embedding \( \mathcal{O}_2 \) in \( A \). Since \( \mathcal{O}_2 \) is isomorphic to each of its corners \( p\mathcal{O}_2p \), \( A \) embeds into \( \mathcal{O}_2^U \) unitally. The composition of these two embeddings is by Lemma 2.8 conjugate to the diagonal embedding and therefore elementary. Taking ultrapower of the diagram

\[
\mathcal{O}_2 \to A \to \mathcal{O}_2^U
\]

we obtain elementary embeddings

\[
\mathcal{O}_2^U \to A^U \to (\mathcal{O}_2^U)^U.
\]

Iterating the construction, we obtain a sequence of embeddings \( B_0 \to A_0 \to B_1 \to A_1 \to \ldots \) such \( B_i \equiv \mathcal{O}_2 \), \( A_i \equiv A \) and embeddings \( B_i \to B_{i+1} \) are elementary for all \( i \). The inductive limit of this chain is therefore elementarily equivalent to \( \mathcal{O}_2 \) and (by the quantifier elimination assumed for \( A \)) to \( A \). Since by Theorem 2.7 \( \mathcal{O}_2 \) does not have quantifier elimination, the conclusion follows.

Exact algebras embed into \( \mathcal{O}_2 \) ([KP00]) so we immediately obtain:
Corollary 2.11. The only noncommutative exact $C^*$-algebra with quantifier elimination is $M_2(\mathbb{C})$.

A problem raised by the third author asks whether every separable $C^*$-algebra embeds into an ultrapower of $O_2$. Theorem 2.10 therefore implies:

Corollary 2.12. If the Kirchberg’s Embedding Problem has a positive solution then $M_2(\mathbb{C})$ is the only noncommutative $C^*$-algebra with quantifier elimination. □

3. Model completeness and model companions

A theory is said to be model complete if every embedding between models of the theory is elementary, in the sense of preserving the values of all formulas. It is easy to see that quantifier elimination implies model completeness, while the converse is false. For example, the theory of every finite-dimensional $C^*$-algebra is model complete. Model completeness is a useful tool in applications of model theory to algebra; for example, the fact that the (discrete) theory of algebraically closed fields is model complete is the key ingredient in a model-theoretic proof of Hilbert’s Nullstellensatz (see [Mar02 Theorem 3.2.11]).

A theory $T^*$ is said to be a model companion of theory $T$ if: (i) every model of $T$ is a submodel of a model of $T^*$ and vice versa, and (ii) $T^*$ is model complete. It is well-known that a theory can have at most one model companion. For example, the model companion of the theory of fields is the theory of algebraically closed fields. In [GS14 Proposition 5.10] it was proved that, assuming Kirchberg’s Embedding Problem has a positive solution, the theory of $C^*$-algebras does not have a model companion. Isaac Goldbring has observed that Theorem 2.10, together with the methods of [GS14], allows us to remove the dependence on Kirchberg’s Embedding Problem.

Theorem 3.1. The theory of unital $C^*$-algebras does not have a model companion.

Proof. Assume otherwise and let $T^*$ be the model companion of the theory of unital $C^*$-algebras.

If $A$ is a separable model of $T^*$, then there is a model $B$ of $T^*$ such that $A \otimes O_2$ is a submodel of $B$. The theory $T^*$ is model complete, so $A$ is an elementary submodel of $B$, and hence the theory of $A$ implies that $O_2$ embeds into the relative commutant of the ultrapower of $A$. By standard methods this implies that $A \otimes O_2 \cong A$. Thus Theorem 2.10 implies that $T^*$ does not have quantifier elimination.

On the other hand, for universally axiomatizable theories (such as the theory of unital $C^*$-algebras), quantifier elimination for the model companion is equivalent to the original theory having amalgamation (the proof is the same as the one in the discrete setting, for which see [CK90 Prop. 3.5.19]). Unital $C^*$-algebras have amalgamation (i.e., free products—see [Blu06 IL.8.3.5]). Unital $C^*$-algebras are only $\forall \exists$-axiomatizable (see definition below) in the language introduced above, but they are universally axiomatizable in an expanded language that has predicates for all $\ast$-polynomials (see [FHS14 p. 485]). However, the difference between the languages and axiomatizations of $C^*$-algebras is only definitional and it does not rise to the level of concern for elementary substructures. This completes the proof. □

A regularity property of a theory than is even weaker than being model complete is that of being $\forall \exists$-axiomatizable. A sentence $\varphi$ is $\forall \exists$ if it is of the form

$$\sup_{\overline{x}} \inf_{\overline{y}} \varphi(\overline{x}, \overline{y})$$
where \( \varphi(\bar{x}, \bar{y}) \) is quantifier-free, and a theory is \( \forall \exists \)-axiomatizable if it has a set \( \forall \exists \) axioms. If a theory \( T \) is model complete then it is preserved by taking inductive limits of its models. By the standard preservation theorem \( T \) is \( \forall \exists \)-axiomatizable (see e.g., [FHL+]).

The Cuntz algebra \( \mathcal{O}_2 \) belongs to the important class of strongly self-absorbing C*-algebras. A C*-algebra \( D \) is strongly self-absorbing (s.s.a.) if \( D \cong D \otimes D \) and the \(*\)-homomorphism of \( D \) into \( D \otimes D \) that sends \( d \) to \( d \otimes 1 \) is approximately unitarily equivalent to an isomorphism between \( D \) and \( D \otimes D \) ([TW07]). S.s.a. C*-algebras play an important role in the classification program of C*-algebras and exhibit interesting model-theoretic properties (see [Far14, §2.2 and §4.5] and [FHrT15]).

**Theorem 3.2.** Assume \( A \) has the same universal theory as an s.s.a. algebra \( D \). If the theory of \( A \) is model complete (or even just \( \forall \exists \)-axiomatizable), then \( A \) is elementarily equivalent to \( D \).

A standard use of saturation of ultrapowers shows that the hypothesis of Theorem 3.2 is equivalent to asserting that \( D \) embeds into an ultrapower of \( A \) and \( A \) embeds into an ultrapower of \( D \) (see e.g., [GS14]).

We shall need the following folklore fact.

**Lemma 3.3.** If \( D \) is s.s.a. then all unital \(*\)-homomorphisms of \( D \) into its ultrapower are elementary.

**Proof.** The proof of Lemma 2.8 gives that every unital \(*\)-homomorphism of \( D \) into \( D^U \) is unitarily conjugate to the diagonal embedding. Loš’s theorem implies that the diagonal embedding is elementary and the conclusion follows.

**Proof of Theorem 3.2.** Since \( A \) has the same universal theory as \( D \), \( D \) embeds into an ultrapower of \( A \) and \( A \) embeds into an ultrapower of \( D \). We therefore have a chain

\[
D \to A^U \to (D^U)^U
\]

such that the embedding of \( D \) into \((D^U)^U\) is elementary. The sandwich argument as in the proof of Theorem 2.10 produces two intertwined chains of C*-algebras, one of which is an elementary chain of ultrapowers of \( D \) and the other consists of algebras elementarily equivalent to \( A \). The inductive limit is elementarily equivalent to \( D \) (by the elementarity) and to \( A \) (by the well-known fact that \( \forall \exists \)-theories are preserved under direct limits), and the conclusion follows.

**Corollary 3.4.** If \( n \geq 3 \) then the theory of \( M_{n-1}(\mathcal{O}_n) \) is not \( \forall \exists \)-axiomatizable.

**Proof.** All of these algebras are nuclear and nonisomorphic and \( \mathcal{O}_2 \) is the only separable nuclear model of its theory (this is a consequence of Kirchberg’s theorem that \( A \otimes \mathcal{O}_2 \cong \mathcal{O}_2 \) for all separable, nuclear, unital simple C*-algebras \( A \); see [GS14] or [FHL+]). Therefore \( \mathcal{O}_2 \) is not elementarily equivalent to \( M_{n-1}(\mathcal{O}_n) \) for any \( n \geq 3 \).

By Theorem 3.2 it suffices to show that \( \mathcal{O}_2 \) unitaly embeds into \( A := M_{n-1}(\mathcal{O}_n) \) and \( A \) unitaly embeds into \( \mathcal{O}_2 \). The latter follows from \( A \) being nuclear and [KP00]. The former assertion is a straightforward \( K \)-theoretic argument reproduced here for the reader’s convenience. For details see e.g., [Rør02]. The \( K_0 \) group of \( \mathcal{O}_n \) is equal to \( \mathbb{Z}/(n-1)\mathbb{Z} \), with the \( K_0 \)-class of \( 1_{\mathcal{O}_n} \) being the generator of \( K_0(\mathcal{O}_n) \). Therefore the \( K_0 \) class of \( 1_{M_{n-1}(\mathcal{O}_n)} \) is 0, and since Kirchberg algebras—and \( \mathcal{O}_n \) in particular—have cancellation, this implies that \( \mathcal{O}_2 \) embeds unitaly in \( M_{n-1}(\mathcal{O}_n) \).
It is shown in [GS14, Prop. 5.7] that a positive solution to the Kirchberg’s Embedding Problem implies that $\text{Th}(O_2)$ is not model complete. We should also remark that in the case of II$_1$ factors the only strongly self-absorbing algebra is the hyperfinite II$_1$ factor $R$, and its theory is shown (relying on [Bro11]) not to be model-complete in [GHS13].

In positive results related to the topics we have been considering, Goldbring and Lupini [GL15] recently showed that the noncommutative Gurarij operator space and the noncommutative Gurarij operator system both have quantifier elimination (in the appropriate languages of operator systems and spaces, respectively).

Having shown that many natural examples of C*-algebras do not have quantifier elimination, we may ask whether they have quantifier reduction, that is, whether it can be shown that every formula is equivalent to one with a fixed number of alternations of quantifiers. For example, in the discrete setting Sela [Sel06] showed that in the theory of nonabelian free groups every formula is equivalent to a boolean combination of $\forall\exists$ formulas.

Question 3.5. Do any natural examples of C*-algebras admit quantifier reduction?

4. Appendix with Diego Caudillo Amador, Bradd Hart, Jamal Kawach, and Se-jin Kim

We give some evidence to the conjecture that the only theories of commutative C*-algebras that admit elimination of quantifiers are $\mathbb{C}$, $\mathbb{C}^2$ and $C(2^\mathbb{N})$, where $2^\mathbb{N}$ denotes the Cantor space. In [EVL14] it was proved that the latter algebra has quantifier elimination. Since $2^\mathbb{N}$ is (up to homeomorphism) the unique zero-dimensional, compact metrizable space with no isolated points, $C(2^\mathbb{N})$ is the unique separable model of its theory. Therefore, if $X$ is any compact zero-dimensional space with no isolated points then $C(X)$ is elementarily equivalent to $C(2^\mathbb{N})$.

Lemma 4.1. Let $C(X)$ be an infinite-dimensional commutative C*-algebra that admits elimination of quantifiers. Then either $X$ is connected, or $C(X)$ is elementarily equivalent to $C(2^\mathbb{N})$.

Proof. By Łośenheim-Skolem we may assume that $X$ is metrizable, and we assume also that $X$ is not connected. We first show that $X$ continuously surjects onto the Cantor space $2^\mathbb{N}$. Let $A \subseteq X$ be a nontrivial clopen set, and let $p$ be the characteristic function of $A$. Then $p$ is a nontrivial projection in $C(X)$. Assume for a contradiction that $A$ is connected. Then $p$ is a minimal projection, so by quantifier elimination all nontrivial projections in $C(X)$ are minimal; in particular, $X \setminus A$ is also connected. Let $f, g \in C(X)$ be such that $f[A] = [0, 1]$, $f[X \setminus A] = \{0\}$, $g[A] = [0, 1/2]$, and $g[X \setminus A] = [1/2, 1]$. Note that $f$ and $g$ are self-adjoints with the same spectrum, so by Lemma 1.4 they have the same type. However, there is a nontrivial projection $q$ (namely $p$) such that $fq = f$, while for any projection $q$ we have $\|gq - g\| \geq 1/2$, a contradiction.

We have shown that no nontrivial clopen subset of $X$ is connected; by repeatedly splitting each clopen set we obtain a binary tree, and hence a continuous surjection of $X$ onto $2^\mathbb{N}$. On the other hand, $X$ is a compact metrizable space, so $2^\mathbb{N}$ continuously surjects onto $X$. Now we use the sandwich method previously used in Theorems 2.10 and 3.2 above. We form a chain:

$$C(X) \hookrightarrow C(2^\mathbb{N}) \hookrightarrow C(X) \hookrightarrow C(2^\mathbb{N}) \hookrightarrow C(X) \hookrightarrow \cdots$$
By assumption $C(X)$ has quantifier elimination, and it is shown in [EV14] that $C(2^N)$ has quantifier elimination as well. Therefore both $C(X)$ and $C(2^N)$ are elementarily equivalent to the limit of the chain, and hence also to each other. □

It follows from Lemma 4.1 and Theorem 4.7 that the only theories of unital commutative real rank zero C*-algebras with quantifier elimination are the theories of $C, C^2$, and $C(2^N)$. We now turn to the other side of the dichotomy in Lemma 4.1 and consider the case where $X$ is connected. Recall that a connected compact Hausdorff space (i.e., a continuum) is said indecomposable if it is not the union of two of its proper subcontinua. This property is equivalent (see e.g., [Kur68, §48, Theorem 2]) to not having a connected non dense open set.

Theorem 4.2. Assume $X$ is a continuum such that $C(X)$ has elimination of quantifiers. Then $X$ is indecomposable.

Proof. A peak function is $f \in C(X)$ such that $\sigma(f) = [0,1]$ and the set $\{x : f(x) > 4/5\}$ is connected. A volcano function is $f \in C(X)$ such that $\sigma(f) = [0,1]$ and $f = g + h$ for some $g$ and $h$ that satisfy $\sigma(g) = \sigma(h) = [0,1]$ and $gh = 0$. Proposition 2.2 implies that every every C*-algebra with no minimal projectors. Then Proposition 2.2 implies that every every C*-algebra with no minimal projectors contains a volcano function $f_1$. We shall construct a peak function $f_2$ and show that $f_1$ and $f_2$ have different types. By Lemma 4.3 this will conclude the proof.

We will show that $f_1$ and $f_2$ do not have the same type. Consider the formula

$$\varphi(x) = \inf_{y,z} \max(||x - yy^* - zz^*||, |1 - \|y\||, |1 - \|z\||, \|yy^*zz^*\||).$$

Taking $y = g^{1/2}$ and $z = h^{1/2}$ we see that $\varphi(f_1) = 0$.

Now we construct a peak function using our assumptions on $X$. Let $U$ be a connected open subset which is not dense in $X$, and fix $z \in X$ such that $\text{dist}(z,U) > r > 0$ for some $r$. With $F = X \setminus U$, the function $h_0(x) = \text{dist}(x,F)$ has the closure of $U$ as its support. We normalize and let $h = \|h_0\|^{-1}h_0$. Function $g(x) = r^{-1}\max(0, r - d(x, z))$ satisfies $\sigma(g) = [0,1]$ and its support is disjoint from $U$. Therefore $f_2 = \frac{1}{5}h + \frac{1}{5}(1 - g)$ is a peak function.

Assume $\varphi(f_2) < 1/10$. Then there are $a = yy^*$ and $b = zz^*$ such that

$$\max(||f_2 - a - b||, |1 - \|a\||, |1 - \|b\||, ||ab||) < 1/10.$$  

Then there are $s$ and $t$ in $X$ such that $a(s) > 9/10$ and $b(t) > 9/10$. Since $|f_2(x) - a(x) - b(x)| < 1/10$ for all $x \in X$, we have $f_2(s) > 4/5$ and $f_2(t) > 4/5$ and $a(s) < 1/5$ and $b(s) < 1/5$. Since $\{x \in X : f_2(x) > 4/5\}$ is connected, there is $x \in X$ such that $a(x) = b(x)$ and $f_2(x) > 4/5$. Therefore $a(x) = b(x) > 7/20$ and $a(x)b(x) > 1/10$. This violates our assumptions and completes the proof. □

Theorem 4.2 precludes $X$ such that $C(X)$ admits elimination of quantifiers from being a finite-dimensional manifold or CW-complex. It remains open whether there are any connected infinite spaces such that $C(X)$ has quantifier elimination. The most natural candidate for an example of such a space is the pseudo-arc, a hereditarily indecomposable (i.e., every subcontinuum of $X$ is indecomposable) 1-dimensional subspace of $\mathbb{R}^2$ (see e.g., [Lew91]).

Question 4.3. Does $C(\text{pseudo-arc})$ admit quantifier elimination?

We were able to prove that $\beta(0,1) \setminus [0,1)$, the Stone-Čech remainder of the half line, is an example of indecomposable continuum without quantifier elimination. We omit the proof, which is long and far from elegant.
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