CORRESPONDENCE BETWEEN DIFFEOMORPHISM GROUPS AND SINGULAR FOLIATIONS

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Abstract. It is well-known that any isotopically connected diffeomorphism group $G$ of a manifold determines uniquely a singular foliation $\mathcal{F}_G$. A one-to-one correspondence between the class of singular foliations and a subclass of diffeomorphism groups is established. As an illustration of this correspondence it is shown that the commutator subgroup $[G, G]$ of an isotopically connected, factorizable and non-fixing $C^r$-diffeomorphism group $G$ is simple iff the foliation $\mathcal{F}_{[G, G]}$ defined by $[G, G]$ admits no proper minimal sets. In particular, the compactly supported $e$-component of the leaf preserving $C^\infty$-diffeomorphism group of a regular foliation $\mathcal{F}$ is simple iff $\mathcal{F}$ has no proper minimal sets.

1. Introduction

Throughout by a foliation we mean a singular foliation (Sussmann [17], Stefan [15]), and by a regular foliation we mean a foliation whose leaves have the same dimension. Introducing the notion of foliations, Sussmann and Stefan emphasized that they play a role of collections of "accessible" sets. Alternatively, they regarded foliations as integrable smooth distributions. Another point of view is to treat foliations as by-products of non-transitive geometric structures, c.f. [2], [20] and examples in [10]. In Molino's approach some types of singular foliations constitute collections of closures of leaves of certain regular foliations ([7], [21]). In this note we regard foliations as a special type of diffeomorphism groups.

Given a $C^\infty$ smooth paracompact boundaryless manifold $M$, $\text{Diff}^r(M)_0$ (resp. $\text{Diff}^r_c(M)_0$), where $1 \leq r \leq \infty$, is the subgroup of the group of all $C^r$ diffeomorphisms $\text{Diff}^r(M)$ on $M$ consisting of diffeomorphisms that can be joined to the identity through a $C^r$ isotopy (resp. compactly supported $C^r$ isotopy) on $M$. A diffeomorphism group $G \leq \text{Diff}^r(M)$, is called isotopically connected if any element $f$ of $G$ can be joined to $\text{id}_M$ through a $C^r$ isotopy.
in $G$. That is, there is a mapping $\mathbb{R} \times M \ni (t,x) \mapsto f_t(x) \in M$ of class $C^r$ with $f_t \in G$ for all $t$ and such that $f_0 = \text{id}$ and $f_1 = f$. It is well-known that any isotopically connected group $G \leq \text{Diff}^r(M)_0$ defines uniquely a foliation of class $C^r$, designated by $\mathcal{F}_G$ (see sect. 2).

Our first aim is to establish a correspondence between the class $\mathfrak{F}^r(M)$ of all $C^r$-foliations on $M$ and a subclass of diffeomorphism groups on $M$, and, by using it, to interpret some results and some open problems concerning non-transitive diffeomorphism groups. The second aim is to prove new results (Theorems 1.1 and 1.2) illustrating this correspondence.

A group $G \leq \text{Diff}^r(M)$ is called factorizable if for every open cover $U$ and every $g \in G$ there are $g_1, \ldots, g_r \in G$ with $g = g_1 \ldots g_r$ and such that $g_i \in G_{U_i}$, $i = 1, \ldots, r$, for some $U_1, \ldots, U_r \in U$. Here for $U \subset M$ and $G \leq \text{Diff}^r(M)$, $G_U$ stands for the identity component of the group of all diffeomorphisms from $G$ compactly supported in $U$. Next $G$ is said to be non-fixing if $G(x) \neq \{x\}$ for every $x \in X$.

**Theorem 1.1.** Assume that $G \leq \text{Diff}^r_c(M)_0$, $1 \leq r \leq \infty$, is isotopically connected, non-fixing and factorizable group of diffeomorphisms of smooth manifold $M$. Then the commutator group $[G,G]$ is simple if and only if the corresponding foliation $\mathcal{F}_{[G,G]}$ admits no proper (i.e. not equal to $M$) minimal set.

In early 1970’s Thurston and Mather proved that the group $\text{Diff}^r(M)_0$, where $1 \leq r \leq \infty$, $r \neq \dim(M) + 1$, is perfect and simple (see [13],[6], [1]). Next, similar results were proved for classical diffeomorphism groups of class $C^\infty$ ([1], [13]). For the significance of these simplicity theorems, see, e.g., [1], [13] and references therein.

Let $(M_i, \mathcal{F}_i)$, $i = 1, 2$, be foliated manifolds. A map $f : M_1 \to M_2$ is called foliation preserving if $f(L_x) = L_{f(x)}$ for any $x \in M_1$, where $L_x$ is the leaf meeting $x$. Next, if $(M_1, \mathcal{F}_1) = (M_2, \mathcal{F}_2)$ then $f$ is leaf preserving if $f(L_x) = L_x$ for all $x \in M_1$. Throughout $\text{Diff}^r(M, \mathcal{F})$ will stand for the group of all leaf preserving $C^r$-diffeomorphisms of a foliated manifold $(M, \mathcal{F})$. Define $\text{Diff}^r_c(M, \mathcal{F})_0$ and $\text{Diff}^r_c(M, \mathcal{F})_0$ analogously as above. Observe that a perfectness theorem for the compactly supported identity component $\text{Diff}^\infty_c(M, \mathcal{F})_0$, being a non-transitive counterpart of Thurston’s theorem, has been proved by the author in [9] and by Tsuboi in [19]. Next, the author in [10], following Mather [6], II, showed that $\text{Diff}^r_c(M, \mathcal{F})_0$ is perfect provided $1 \leq r \leq \dim \mathcal{F}$. Observe that, in general, the group $\text{Diff}^r_c(M, \mathcal{F})_0$ is not simple for obvious reasons.

**Theorem 1.2.** Let $(M, \mathcal{F})$ be a foliation on a $C^\infty$ smooth manifold $M$ with no leaves of dimension 0. Then the commutator subgroup

$$\mathcal{D} = [\text{Diff}^r_c(M, \mathcal{F})_0, \text{Diff}^r_c(M, \mathcal{F})_0]$$
is simple if and only if $F$ does not have any proper minimal set. In particular, if $F$ is regular, and $1 \leq r \leq \dim F$ or $r = \infty$, then $\text{Diff}_c(M, F)_0$ is simple if and only if $F$ has no proper minimal sets.

In the proof of Theorem 1.1 in sect. 3 some ideas from Ling [5] are in use.

2. Foliations correspond to a subclass of the class of diffeomorphism groups

Let $1 \leq r \leq \infty$ and let $L$ be a subset of a $C^r$-manifold $M$ endowed with a $C^r$-differentiable structure which makes it an immersed submanifold. Then $L$ is weakly imbedded if for any locally connected topological space $N$ and a continuous map $f : N \to M$ satisfying $f(N) \subset L$, the map $f : N \to L$ is continuous as well. It follows that in this case such a differentiable structure is unique. A foliation of class $C^r$ is a partition $F$ of $M$ into weakly imbedded submanifolds, called leaves, such that the following condition holds. If $x$ belongs to a $k$-dimensional leaf, then there is a local chart $(U, \varphi)$ with $\varphi(x) = 0$, and $\varphi(U) = V \times W$, where $V$ is open in $\mathbb{R}^k$, and $W$ is open in $\mathbb{R}^{n-k}$, such that if $L \in F$ then $\varphi(L \cap U) = V \times l$, where $l = \{w \in W|\varphi^{-1}(0, w) \in L\}$. A foliation is called regular if all leaves have the same dimension.

Sussmann ([17]) and Stefan ([15],[16]) regarded foliations as collections of accessible sets in the following sense.

**Definition 2.1.** A smooth mapping $\varphi$ of an open subset of $\mathbb{R} \times M$ into $M$ is said to be a $C^r$-arrow, $1 \leq r \leq \infty$, if

1. $\varphi(t, \cdot) = \varphi_t$ is a local $C^r$-diffeomorphism for each $t$, possibly with empty domain,
2. $\varphi_0 = \text{id}$ on its domain,
3. $\text{dom}(\varphi_t) \subset \text{dom}(\varphi_s)$ whenever $0 \leq s < t$.

Given an arbitrary set of arrows $A$, let $A^*$ be the totality of local diffeomorphisms $\psi$ such that $\psi = \varphi(t, \cdot)$ for some $\varphi \in A$, $t \in \mathbb{R}$. Next $\hat{A}$ denotes the set consisting of all local diffeomorphisms being finite compositions of elements from $A^*$ or $(A^*)^{-1} = \{\psi^{-1}|\psi \in A^*\}$, and of the identity. Then the orbits of $\hat{A}$ are called accessible sets of $A$.

For $x \in M$ let $A(x)$, $\hat{A}(x)$ be the vector subspaces of $T_xM$ generated by

$$\{\dot{\varphi}(t, y)|\varphi \in A, \varphi_t(y) = x\}, \quad \{d_y\psi(v)|\psi \in \hat{A}, \psi(y) = x, v \in A(y)\},$$

respectively. Then we have ([15])

**Theorem 2.2.** Let $A$ be an arbitrary set of $C^r$-arrows on $M$. Then

1. every accessible set of $A$ admits a (unique) $C^r$-differentiable structure of a connected weakly imbedded submanifold of $M$;
2. the collection of accessible sets defines a foliation $F$; and
(3) \( \mathcal{D}(F) := \{ A(x) \} \) is the tangent distribution of \( F \).

Let \( G \leq \text{Diff}^r(M) \) be an isotopically connected group of diffeomorphisms. Let \( \mathcal{A}_G \) be the totality of restrictions of isotopies \( \mathbb{R} \times M \ni (t,x) \mapsto f_t(x) \in M \) in \( G \) to open subsets of \( \mathbb{R} \times M \). Then by \( F_G \) we denote the foliation defined by the set of arrows \( \mathcal{A}_G \). Observe that \( \tilde{A}_G = A_G \) and, consequently, \( \tilde{A}_G(x) = A_G(x) \).

**Remark 2.3.** (1) Of course, any subgroup \( G \leq \text{Diff}^r(M) \) determines uniquely a foliation. Namely, \( G \) defines uniquely its maximal subgroup \( G_0 \) which is isotopically connected.

(2) Denote by \( G_c \) the subgroup of all compactly supported elements of \( G \). Then \( G_c \) need not be isotopically connected even if \( G \) is so. In fact, let \( G = \text{Diff}^r(\mathbb{R}^n)_0 \), \( 1 \leq r \leq \infty \). Then every \( f \in G_c \) is isotopic to the identity but the isotopy need not be in \( G_c \). That is, \( G_c \) is not isotopically connected. Observe that the exception is the \( C^0 \) case: due to Alexander’s trick for \( r = 0 \) (see, e.g., [3], p.70) \( G_c \) is isotopically connected.

Likewise, let \( C = \mathbb{R} \times S^1 \) be the annulus and let \( G = \text{Diff}^r(C)_0 \). Then there is the twisting number epimorphism \( T : G_c \to \mathbb{Z} \). It is easily seen that \( f \in G_c \) is joined to id by a compactly supported isotopy iff \( T(f) = 0 \). Consequently, \( G_c \) is not isotopically connected.

Denote by \( \mathcal{G}^r(M) \) (resp. \( \mathcal{G}^r_c(M) \)), \( 1 \leq r \leq \infty \), the totality of isotopically connected (resp. compactly supported, through compactly supported isotopies) groups of \( C^r \) diffeomorphisms of \( M \). Next the symbol \( \mathcal{F}^r(M) \) will stand for the totality of foliations of class \( C^r \) on \( M \). Then each \( G \in \mathcal{G}^r(M) \) determines uniquely a foliation from \( \mathcal{F}^r(M) \), denoted by \( F_G \). That is, we have the mapping \( \beta_M : \mathcal{G}^r(M) \ni G \mapsto F_G \in \mathcal{F}^r(M) \). Conversely, to any foliation \( F \in \mathcal{F}^r(M) \) we assign \( G_F := \text{Diff}^r_c(M,F)_0 \) and we get the mapping \( \alpha_M : \mathcal{F}^r(M) \ni F \mapsto G_F \in \mathcal{G}^r_c(M) \). The following is obvious.

**Proposition 2.4.** One has \( \beta_M \circ \alpha_M = \text{id}_{\mathcal{G}^r(M)} \). In particular

\[
\alpha_M : \mathcal{F}^r(M) \ni F \mapsto G_F \in \mathcal{G}^r_c(M)
\]

is an injection identifying the class of \( C^r \)-foliations with a subclass of \( C^r \)-diffeomorphism groups.

Observe that usually \( (\alpha_M \circ \beta_M)(G) \in \mathcal{G}^r_c(M) \) is not a subgroup of \( G \) even if \( G \in \mathcal{G}^r_c(M) \). For instance, take the group of Hamiltonian diffeomorphisms of a Poisson manifold, see [20]. See also examples in [11].

**Remark 2.5.** Note that we can also define \( \alpha'_M : \mathcal{F}^r(M) \ni F \mapsto G'_F \in \mathcal{G}^r(M) \), where \( G'_F := \text{Diff}^r(M,F) \in \mathcal{G}^r(M) \), and we get another identification of the class of \( C^r \)-foliations with a subclass of \( C^r \)-diffeomorphism groups. However we prefer \( \alpha_M \) to \( \alpha'_M \) because of Proposition 2.11 below.
For $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}^r(M)$ we say that $\mathcal{F}_1$ is a subfoliation of $\mathcal{F}_2$ if each leaf of $\mathcal{F}_1$ is contained in a leaf of $\mathcal{F}_2$. We then write $\mathcal{F}_1 \prec \mathcal{F}_2$. By a flag structure we mean a finite sequence $\mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k$ of foliations of $M$. Next, by the intersection of $\mathcal{F}_1, \mathcal{F}_2$ we mean the partition $\mathcal{F}_1 \cap \mathcal{F}_2 := \{L_1 \cap L_2 : \exists i \in \mathcal{F}_1, \mathcal{F}_2 \}$ of $M$. Clearly, if $\mathcal{F}_1 \cap \mathcal{F}_2$ is a foliation then $\mathcal{F}_1 \cap \mathcal{F}_2 \prec \mathcal{F}_i, i = 1, 2$.

It is a rare phenomenon that $\mathcal{F}_1 \cap \mathcal{F}_2$ would be a regular foliation, provided $\mathcal{F}_1, \mathcal{F}_2$ are regular. In the category of (singular) foliations it may happen more often.

**Proposition 2.6.** (1) If the distribution $\mathcal{D}(\mathcal{F}_1 \cap \mathcal{F}_2)$ is of class $C^r$ ([15]) then $\mathcal{F}_1 \cap \mathcal{F}_2$ is a foliation.

(2) If $G_1, G_2 \in \mathfrak{G}^r(M)$ have the intersection $G = G_1 \cap G_2$ isotopically connected then $\mathcal{F}_G = \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2}$.

(3) For $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}^r(M)$, if the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is a foliation then there is $G \in \mathfrak{G}^r(M)$ such that $G \leq G_{\mathcal{F}_1} \cap G_{\mathcal{F}_2}$ and $\mathcal{F}_G = \mathcal{F}_1 \cap \mathcal{F}_2$.

(4) For $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}^r(M)$, if $G_{\mathcal{F}_1} \cap G_{\mathcal{F}_2}$ is isotopically connected then the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ is a foliation.

**Proof.** (1) In fact, the distribution of $\mathcal{F}_1 \cap \mathcal{F}_2$ is then integrable.

(2) Denote by $\mathcal{I}G$ the set of all isotopies in $G$. Clearly, $\mathcal{I}(G_1 \cap G_2) = \mathcal{I}G_1 \cap \mathcal{I}G_2$ for arbitrary $G_1, G_2 \in \mathfrak{G}^r(M)$. For $x \in M$, set $\mathcal{I}G(x) := \{y \in M : (\exists f \in \mathcal{I}G)(\exists t \in I) y = f_t(x)\}$. By definition, $L_x = \mathcal{I}G(x)$, where $L_x \in \mathcal{F}_G$ is a leaf meeting $x$. Therefore, since $G_1, G_2, G$ are isotopically connected we have $L_x = \mathcal{I}G(x) = \mathcal{I}G_1(x) \cap \mathcal{I}G_2(x) = L^1_x \cap L^2_x$, where $L^i_x \in \mathcal{F}_{G_i}, i = 1, 2$.

(3) Set $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ and $G = \mathcal{F}_G$. Use Prop. 2.4.

(4) In view of Prop. 2.4 we have $\mathcal{F}_{G_{\mathcal{F}_0}} = \mathcal{F}_0$ for all $\mathcal{F}_0 \in \mathfrak{F}^r(M)$. Put $G = G_{\mathcal{F}_1} \cap G_{\mathcal{F}_2}$. Therefore, in view of (2), $\mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}_{G_{\mathcal{F}_1} \cap G_{\mathcal{F}_2}} = \mathcal{F}_G$ is a foliation.

Let $\mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k$ be a flag structure on $M$ and let $x \in M$. If $x \in L_i \in \mathcal{F}_i$ we write $p_i(x) = \dim L_i, \bar{p}_i(x) = p_i(x) - p_{i-1}(x)$ $(i = 2, \ldots, k)$ and $q_i(x) = m - p_i(x)$.

**Definition 2.7.** A chart $(U, \varphi)$ of $M$ with $\varphi(0) = x$ is called a distinguished chart at $x$ with respect to $\mathcal{F}_1 \prec \cdots \prec \mathcal{F}_k$ if $U = V_1 \times \cdots \times V_k \times W$ such that $V_1 \subset \mathbb{R}^{p_1(x)}, \ldots, V_k \subset \mathbb{R}^{p_k(x)} (i \geq 2)$ and $W \subset \mathbb{R}^{q_k(x)}$ are open balls and for any $L_i \in \mathcal{F}_i$ we have

$$\varphi(U) \cap L_i = \varphi(V_1 \times \cdots \times V_i \times l_i),$$

where $l_i = \{w \in V_{i+1} \times \cdots \times V_k \times W : \varphi(0, w) \in L_i\}$ for $i = 1, \ldots, k$.

Observe that actually the above $\varphi$ is an inverse chart; following [16] we call it a chart for simplicity. Notice as well that in the above definition one need not assume that $\mathcal{F}_i$ is a foliation but only that it is a partition by weakly imbedded submanifolds; that $\mathcal{F}_i$ is a foliation follows then by definition.
Theorem 2.8. Let $G_1 \leq \ldots \leq G_k \leq \text{Diff}^r(M)$ be an increasing sequence of diffeomorphism groups of $M$. Then $\mathcal{F}_{G_1} \prec \cdots \prec \mathcal{F}_{G_k}$ admits a distinguished chart at any $x \in M$.

In fact, it is a straightforward consequence of Theorem 2 in [11].

Corollary 2.9. Let $G_1 \leq \ldots \leq G_k \leq \text{Diff}^r(M)$ and let $(L, \sigma)$ be a leaf of $\mathcal{F}_{G_k}$. Then all $G_i$ preserve $L$, and $\mathcal{F}_{G_i|L} \prec \cdots \prec \mathcal{F}_{G_{k-1}|L}$ is a flag structure on $L$. Moreover, a distinguished chart at $x$ for $\mathcal{F}_{G_i|L} \prec \cdots \prec \mathcal{F}_{G_{k-1}|L}$ is the restriction to $L$ of a distinguished chart at $x$ for $\mathcal{F}_{G_i} \prec \cdots \prec \mathcal{F}_{G_k}$.

The following property of paracompact spaces is well-known.

Lemma 2.10. If $X$ is a paracompact space and $\mathcal{U}$ is an open cover of $X$, then there exists an open cover $\mathcal{V}$ starwise finer than $\mathcal{U}$, that is for all $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $\text{star}^V(V) \subset U$. Here $\text{star}^V(V) := \bigcup\{V' \in \mathcal{V} : V' \cap V \neq \emptyset\}$. In particular, for all $V_1, V_2 \in \mathcal{V}$ with $V_1 \cap V_2 \neq \emptyset$ there is $U \in \mathcal{U}$ such that $V_1 \cup V_2 \subset U$.

Proposition 2.11. If $\mathcal{F} \in \mathcal{F}^c(M)$ then $G_{\mathcal{F}} = \alpha(\mathcal{F})$ is factorizable.

Proof. Let $\mathcal{X}_c(M, \mathcal{F})$ be the Lie algebra of all compactly supported vector fields on $M$ tangent to $\mathcal{F}$. Then there is a one-to-one correspondence between isotopies $f_t$ in $G_{\mathcal{F}}$ and smooth paths $X_t$ in $\mathcal{X}_c(M, \mathcal{F})$ given by the equation

$$\frac{df_t}{dt} = X_t \circ f_t \quad \text{with} \quad f_0 = \text{id}.$$ 

Let $f = (f_t) \in IG_{\mathcal{F}}$ and let $X_t$ be the corresponding family in $\mathcal{X}_c(M, \mathcal{F})$. By considering $f_{(p/m)t}^{-1} f_{(p-1/m)t}$, $p = 1, \ldots, m$, instead of $f_t$ we may assume that $f_t$ is close to the identity.

Let $\mathcal{U}$ be an open cover of $M$. We choose a family of open sets, $(V_j)_{j=1}^s$, which is starwise finer than $\mathcal{U}$, and satisfies $\text{supp}(f_t) \subset V_1 \cup \cdots \cup V_s$ for each $t$. Let $(\lambda_j)_{j=1}^s$ be a partition of unity subordinate to $(V_j)$, and let $Y_t^j = \lambda_j X_t$. We set

$$X_t^j = Y_t^1 + \cdots + Y_t^j, \quad j = 1, \ldots, s,$$

and $X_t^0 = 0$. Each of the smooth families $X_t^j$ integrates to an isotopy $g_t^j$ with support in $V_1 \cup \cdots \cup V_j$. We get the fragmentation

$$f_t = g_t^s \circ \cdots \circ g_t^1,$$

where $f_t^j = g_t^j \circ (g_t^{j-1})^{-1}$, with the required inclusions

$$\text{supp}(f_t^j) = \text{supp}(g_t^j \circ (g_t^{j-1})^{-1}) \subset \text{star}(V_j) \subset U_{i(j)}$$

which hold if $f_t$ is sufficiently small. Thus the group of isotopies of $G_{\mathcal{F}}$ is factorizable. Consequently, $G_{\mathcal{F}}$ itself is factorizable. \qed
Remark 2.12. The identification $\alpha_M$ enables us to consider several new properties of foliations from $\mathcal{F}^r(M)$. For instance, one can say that a foliation $\mathcal{F}$ is perfect if so is the corresponding diffeomorphisms group $G_{\mathcal{F}} = \alpha_M(\mathcal{F})$. As we mentioned before it is known that $G_{\mathcal{F}} = \text{Diff}_{\mathcal{F}}^r(M)$ is perfect provided $\mathcal{F}$ is regular and $1 \leq r \leq \text{dim} \mathcal{F}$ or $r = \infty$ ([9], [19], [10]). It is not known whether $G_{\mathcal{F}}$ is perfect for singular foliations and a possible proof seems to be very difficult. In turn, a possible perfectness of $G_{\mathcal{F}} = \text{Diff}_{\mathcal{F}}^r(M)$ with $r$ large is closely related to the simplicity of $\text{Diff}_{\mathcal{F}}^{n+1}(M)$, see [3].

Likewise, one can consider uniformly perfect or bounded foliations by using the corresponding notions for groups, see [14] and references therein.

Finally consider the following important feature of subclasses of the class $\mathcal{F}^r(M)$, depending also on $M$ and $r$. A subclass $\mathcal{K}$ of $\mathcal{F}^r(M)$ is called faithful if the following holds: For all $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{K}$ and for any group isomorphism $\Phi : \alpha_M(\mathcal{F}_1) \cong \alpha_M(\mathcal{F}_2)$ there is a $C^r$ foliated diffeomorphism $\varphi : (M, \mathcal{F}_1) \cong (M, \mathcal{F}_2)$ such that $\forall f \in \alpha_M(\mathcal{F}_1)$, $\Phi(f) = \varphi \circ f \circ \varphi^{-1}$. From reconstruction results of Rybicki [12] and Rubin [8] it is known that the class of regular foliations of class $C^\infty$, $\mathcal{F}_\text{reg}^\infty(M)$, is faithful.

3. Proof of Theorem 1.1 and 1.2

Proof of Theorem 1.1. First observe that the fact that a foliation $\mathcal{F}$ has no proper minimal set is equivalent to the statement that all leaves of $\mathcal{F}$ are dense.

$(\Rightarrow)$ Assume that $\emptyset \neq L \subset M$ is a proper closed saturated subset of $M$. Choose $x \in M \setminus L$. We prove the following statement:

$\text{(*)} \text{ there are a ball } U \subset M \setminus L \text{ with } x \in U \text{ and } g \in [G_U, G_U] \text{ such that } g(x) \neq x.$

We are done in view of $(\ast)$ by setting $H := \{g \in [G, G] : g|_L = \text{id}_L\}$. To prove $(\ast)$, choose balls $U$ and $V$ in $M$ such that $x \in V \subset \overline{V} \subset U$. Take $f \in G$ such that $f(x) \neq x$. In light of the assumption, for $\mathcal{U} = \{U, \overline{V}\}$ we may write $g = g_r \ldots g_1$, where all $g_i$ are supported in elements of $\mathcal{U}$. Let $s := \min\{i \in \{1, \ldots, r\} : \text{ supp}(g_i) \subset U \text{ and } g_i(x) \neq x\}$. Then $g_s \in G_U$ satisfies $g_s(x) \neq x$.

Now take an open $W$ such that $x \in W \subset U$ and $g_s(x) \notin W$. Choose $f \in G_W$ with $f(x) \neq x$ by an argument similar to the above. It follows that $f(g_s(x)) = g_s(x) \neq g_s(f(x))$ and, therefore, $[f, g_s](x) \neq x$. Thus $g = [f, g_s]$ satisfies the claim.

$(\Leftarrow)$ First observe the following commutator formulae for all $f, g, h \in G$

\begin{equation}
[f, g, h] = f[g, h]f^{-1}[f, h], \quad [f, gh] = [f, g][f, h]g^{-1}.
\end{equation}
Next, in view of a theorem of Ling [5] we have that \([G, G]\) is a perfect group, that is
\[ (3.2) \quad [G, G] = [[G, G], [G, G]]. \]
Suppose that \(H\) is a nontrivial normal subgroup of \([G, G]\). Let \(x \in M\) satisfy \(h(x) \neq x\) for some \(h \in H\). Fix a ball \(U_0\) such that \(h(U_0) \cap U_0 = \emptyset\). By the definition of \(\mathcal{F}_{[G, G]}\) and the assumption that each leaf \(L \in \mathcal{F}_{[G, G]}\) is dense, for every \(y \in M\) there are a ball \(U_y\) with \(y \in U_y\) and \(f_y \in [G, G]\) such that \(f_y(U_y) \subset U\). Let \(\mathcal{U} = \{U_y\}_{y \in M}\).

Due to Lemma 2.10 we can find an open cover \(\mathcal{V}\) starwise finer than \(\mathcal{U}\). We denote \(\mathcal{U}^G = \{g(U) | U \in \mathcal{U}, g \in [G, G]\}\) and
\[ \mathcal{F}^G = \prod_{U \in \mathcal{U}^G} [G_U, G_U]. \]
By assumption \(G\) is factorizable with respect to \(\mathcal{V}\). First we show that \([G, G] \subset \mathcal{F}^G\), i.e. that any \([g_1, g_2] \in [G, G]\) can be expressed as a product elements of \(\mathcal{F}^G\) of the form \([h_1, h_2]\), where \(h_1, h_2 \in G_U\) for some \(U \in \mathcal{U}^G\). In view of (3.1) and (3.2) we may assume that \(g_1, g_2 \in [G, G]\). Now the relation \([G, G] \subset \mathcal{F}^G\) is an immediate consequence of (3.1) and the fact that \(\mathcal{V}\) is starwise finer than \(\mathcal{U}\).

Next we have to show that \(\mathcal{F}^G \subset H\). It suffices to check that for every \(f, g \in G_U\) with \(U \in \mathcal{U}\) the bracket \([f, g]\) belongs to \(H\). This implies that for every \(f, g \in G_U\) with \(U \in \mathcal{U}^G\) one has \([f, g] \in H\), since \(H\) is a normal subgroup in \([G, G]\).

We have fixed \(h \in H\) and \(U_0\) such that \(h(U_0) \cap U_0 = \emptyset\). If \(U \in \mathcal{U}\) and \(f, g \in G_U\), take \(k \in [G, G]\) such that \(k(U) \subset U_0\), and put \(\bar{f} = kfk^{-1}, \bar{g} = kgk^{-1}\). It follows that \([h, f], \bar{g}] = \text{id}\). Therefore, \([\bar{f}, \bar{g}] = [(h, f], \bar{g}] \in H\), and we have also that \([f, g] \in H\). Thus we have \(\mathcal{F}^G \subset H\) and, consequently, \([G, G] \leq H\), as required. \(\square\)

\textit{Proof of Theorem 1.2.} By assumption and Prop. 2.11 \(\text{Diff}^r(M, \mathcal{F})_0\) is factorizable and non-fixing. Since \(\text{Diff}^r(M, \mathcal{F})_0\) is isotopically connected, the first assertion follows from Theorem 1.1. The second assertion is a consequence of theorems stating that \(\text{Diff}^r(M, \mathcal{F})_0\) is perfect ([9] and [19] for \(r = \infty\), and [6] and [10] for \(1 \leq r \leq \dim \mathcal{F}\)). \(\square\)

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