An explicit computation of the Hecke operator and the ghost conjecture

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Abstract
In this paper, we investigate the Hecke operator at $p = 5$ and show that the upper minors of the matrix have non zero corank and, interestingly, follow the same ghost pattern in the ghost conjecture of Bergdall and Pollack. Due to this facts, we conjecture that the slope of Hecke action in this case can be computed using an appropriate variant of ghost series. Assume this result, we achieve an upper bound for the slopes that is similar to the Gouvea's $(k - 1)/(p + 1)$ conjecture.

Keywords: Slope of $U_p$ operators, Overconvergent modular forms, Ghost conjecture, Gouvêa’s conjecture

1 Introduction
Let $p$ be a prime number, and $N$ be a positive integer coprime to $p$. For an integer $k \geq 3$, we use $S_k(\Gamma_1(pN))$ to denote the space of modular cuspforms of weight $k$, level $\Gamma_1(pN)$. A modular form $f$ in $S_k(\Gamma_1(pN))$ can be written as

$$f(\tau) = \sum_{n=1}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i \tau}.$$  

For each prime $l$, one can define a Hecke operator $T_l$ on $S_k(\Gamma_1(pN))$. When $l = p$, the Hecke operator is often denoted by $U_p$ instead. For an eigenform $f$, after normalization, one can deduce that

$$U_pf = a_pf.$$  

The $p$-adic valuation of $a_p$ is called the slope of the eigenform. The study of slopes plays an important role in understanding the geometry of the so-called eigencurve introduced by Coleman and Mazur using $p$-adic interpolation of overconvergent eigenforms [7]. The eigencurve has many applications in $p$-adic number theory, for example, Kisin’s proof of Fontaine-Mazur conjecture for $GL_2$ [11].

The first numerical data of the slopes was due to Gouvêa and Mazur in [9] using computer calculations. Buzzard and his co-authors computed in the case with small primes $p$ and small level [5,6,10]. Liu, Wan and Xiao found the slopes for the $U_p$-operator over the boundary of the weight space to be certain unions of arithmetic progressions [12]. Recently, Bergdall and Pollack proposed the ghost conjecture which predicts the slopes
over the entire weight space. In [2, 3], the authors defined a formal power series, called the \textit{ghost series}, and conjectured that the slopes of $U_p$ action are the same as the slopes of the Newton polygon of the ghost series.

We consider a variant of the ghost conjecture for overconvergent forms on definite quaternion algebras. The result can then be translated to modular forms using Jacquet–Langlands correspondence. The upshot is that this circumvented the difficulties posed by the geometry of a modular curve, traded with the arithmetic complication of a quaternion algebra, which turns out to be more accessible by our method.

One simple and nice computation in this direction is due to Jacobs in [10]. In his thesis, Jacobs studied the case when $p = 3$ with a particular level [10], and computed the slopes of the $U_p^\circ$-operation. The method is later refined by Wan–Xiao–Zhang [15].

In this paper, we investigate the Hecke operator $U_5$ and show that the $n \times n$ upper left minors of the matrix have nonzero corank, and interestingly, leads to the unimodal pattern in the ghost conjecture [2, 3]. This seems to give the ghost series of Bergdall and Pollack some theoretic explanation. We expect that the slopes of $U_5$-operator in this case can be computed using an appropriate variant of the \textit{ghost series}, defined in (8.1.1). Assuming this result, we achieve an upper bound for the slopes that is similar to the Gouëa’s $\frac{k-1}{p+1}$ conjecture.

The result can be generalized in my ongoing project with Ruochuan Liu, Liang Xiao and Bin Zhao to prove the ghost conjecture under a certain mild technical hypothesis.

2 Setup

2.1 The quaternion algebra

Our setup is a variant of [15]. In this paper, we investigate the case $p = 5$.

Let $D$ be a quaternion algebra over $\mathbb{Q}$. Explicitly, we set

$$D := \mathbb{Q}(i, j)/(i^2 + 1, j^2 + 1, ij + ji),$$

which ramifies exactly at 2 and $\infty$, and splits at all other primes $p$. In particular,

$$D \otimes_{\mathbb{Q}} \mathbb{Q}_5 \cong M_2(\mathbb{Q}_5).$$

We denote $\nu_5$ as the square root of $-1$ in $\mathbb{Q}_5$ that is congruent to 2 modulo 5. In 5-adic expansion,

$$\nu_5 = 2 + 5 + 2 \cdot 5^2 + \cdots .$$

We then fix an isomorphism between $D \otimes \mathbb{Q}_5$ and $M_2(\mathbb{Q}_5)$ so that

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j \leftrightarrow \begin{pmatrix} \nu_5 & 0 \\ 0 & -\nu_5 \end{pmatrix}.$$ 

The result in the paper is independent of the above isomorphism and the choice of the square root $\nu_5$. We set $k := ij$, whose image under the isomorphism is

$$k \leftrightarrow \begin{pmatrix} 0 & -\nu_5 \\ -\nu_5 & 0 \end{pmatrix}.$$ 

We also know that the unit group of $\mathcal{O}_D$ consists of 24 elements:

$$\mathcal{O}_D^\times = \{ \pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k) \}.$$ 

Their images under the above isomorphism are:
and the maximal open compact subgroup
\[ \hat{\text{GL}}_2(\mathbb{Z}_5) := D_5^\times(\mathbb{Z}_2) \times \prod_{\ell \neq 2,5} \text{GL}_2(\mathbb{Z}_\ell). \]

We have the following useful lemma, which explains the choice of \( \hat{\Gamma}_1(5) \).
Lemma 2.3 The following natural map is bijective.

\[ D^\times \times \tilde{\Gamma}_1(5) \longrightarrow D_f^\times \]

\( (\delta, u) \longmapsto \delta u. \)

Proof We follow the same argument in [15]. By [10, Lemma 1.22],

\[ D_f^\times = D^\times \cdot \tilde{\Gamma}_2(\mathbb{Z}_5). \]

Since \( D^\times \cap \tilde{\Gamma}_2(\mathbb{Z}_5) = O_D^\times \), taking into account of the duplication, we have

\[ D^\times \cap \tilde{\Gamma}_1(5) = O_D^\times \cap \tilde{\Gamma}_2(\mathbb{Z}_5) / \tilde{\Gamma}_1(5) = O_D^\times / \tilde{\Gamma}_1(5). \]

We have the isomorphism \( \tilde{\Gamma}_2(\mathbb{Z}_5) / \tilde{\Gamma}_1(5) \sim= GL_2(\mathbb{F}_5) / \mathbb{F}_5 \times \mathbb{F}_5 \), so it suffices to check that the image of \( O_D^\times \) forms a full set of coset representatives of \( GL_2(\mathbb{F}_5) / \mathbb{F}_5 \times \mathbb{F}_5 \). \( \Box \)

2.4 Overconvergent automorphic forms

In this section, we define the space of overconvergent automorphic forms for a definite quaternion algebra and describe the Hecke actions explicitly.

Fix an integer \( k \in \mathbb{Z} \). Note that any \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in the Iwahori subgroup \( Iw_{51} := \begin{pmatrix} \mathbb{Z}_5^\times & \mathbb{Z}_5 \\ 5\mathbb{Z}_5 & 1 + 5\mathbb{Z}_5 \end{pmatrix} \) of \( GL_2(\mathbb{Z}_5) \) has a right action on the Tate algebra \( \mathbb{Q}_5[z] \) over \( \mathbb{Q}_5 \) by

\[ (f || k\gamma)(z) := (cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right). \]

(2.4.1)

For any \( \delta \in D^\times \), we put \( (f || k\delta) := (f || k\gamma) \), where \( \gamma \) is the image of \( \delta \) in \( GL_2(\mathbb{Q}_5) \).

As in [15], we define the space of overconvergent automorphic forms of weight \( k \) and level \( \tilde{\Gamma}_1(5) \) by

\[ S_k^{(\Gamma)}(\tilde{\Gamma}_1(5)) := \left\{ \varphi : D^\times \times D_f^\times \rightarrow \mathbb{Q}_5[z] \mid \varphi(gu) = \varphi(g)|| k u_5, \text{ for any } g \in D_f^\times, u \in \tilde{\Gamma}_1(5) \right\}, \]

where \( u_5 \) is the 5-component of \( u \).

Using Lemma 2.3, we have the isomorphism:

\[ S_k^{(\Gamma)}(\tilde{\Gamma}_1(5)) \rightarrow \mathbb{Q}_5[z] \]

\[ \varphi \longmapsto \varphi(1). \]

(2.4.2)

Remark 2.5 The isomorphism gives a simpler description of the space of overconvergent automorphic form \( S_k^{(\Gamma)}(\tilde{\Gamma}_1(5)) \), which allows us to compute explicitly the matrix of the Hecke action.

2.6 \( U_5 \)-operator

The space \( S_k^{(\Gamma)}(\tilde{\Gamma}_1(5)) \) carries the actions of Hecke operators \( U_5 \), defined as follows.
We write
\[ \text{Iw}_{5,1}(\begin{smallmatrix} 5 & 0 \\ 0 & 1 \end{smallmatrix}) \text{Iw}_{5,1} = \prod_{i=0}^{4} \text{Iw}_{5,1} v_i, \quad \text{with } v_i = (\begin{smallmatrix} 5 \\ 5i \\ 1 \end{smallmatrix}). \]

Then the action of the operator \( U_S \) on \( S_{\hat{\Gamma}_1(5)} \) is defined to be
\[ U_S(\varphi) = \sum_{i=0}^{4} \varphi|_k v_i, \quad \text{with } (\varphi|_k v_i)(g) := \psi(\gamma v_i^{-1})|_k v_i. \quad (2.6.1) \]

3. **Explicit computation of the infinite matrix of \( U_S \)**

In terms of the explicit description of the space of overconvergent automorphic forms (2.4.2), the \( U_S \)-operator can be described by the following commutative diagram. For simplicity, we use \( U_S \) and \( U_5 \) interchangeably.

\[ \begin{array}{ccc}
S_5^{D^+}(\hat{\Gamma}_1(5)) & \xrightarrow{\varphi \mapsto \varphi(1)} & \mathbb{Q}_S[z] \\
\varphi \mapsto U_S(\varphi) & & U_5 \\
S_5^{D^+}(\hat{\Gamma}_1(5)) & \xrightarrow{\varphi \mapsto \varphi(1)} & \mathbb{Q}_S[z].
\end{array} \]

**Lemma 3.1** Let
\[ \begin{align*}
\delta_1 &= j - 2, \quad \delta_2 = \frac{1}{2}(1 + i - 3j - 3k), \\
\delta_3 &= \frac{1}{2}(1 + 3i - 3j + k), \\
\delta_4 &= \frac{1}{2}(1 - 3i - 3j - k), \quad \text{and } \delta_5 = \frac{1}{2}(1 - i - 3j + 3k).
\end{align*} \]

We have
\[ U_5 = ||_{k} \delta_1 + ||_{k} \delta_2 + ||_{k} \delta_3 + ||_{k} \delta_4 + ||_{k} \delta_5. \]

In particular, the images of \( \delta'_s \) in \( \text{GL}_2(\mathbb{Q}_S) \) are given by
\[ \begin{align*}
\begin{bmatrix}
-2 + v_5 & 0 \\
0 & -(2 + v_5)
\end{bmatrix}, & \frac{1}{2} \begin{bmatrix}
1 - 3v_5 & 1 + 3v_5 \\
-1 + 3v_5 & 1 + 3v_5
\end{bmatrix}, & \frac{1}{2} \begin{bmatrix}
1 - 3v_5 & 3 - v_5 \\
-3(3 + v_5) & 1 + 3v_5
\end{bmatrix}, \\
\frac{1}{2} \begin{bmatrix}
1 - 3v_5 & -3 + v_5 \\
3 + v_5 & 1 + 3v_5
\end{bmatrix}, & \frac{1}{2} \begin{bmatrix}
1 - 3v_5 & -(1 + 3v_5) \\
1 - 3v_5 & 1 + 3v_5
\end{bmatrix},
\end{align*} \]

**Proof** We can compute \( U_5 \) explicitly by
\[ U_5(\varphi)(1) = \sum_{j=1}^{5} \psi(v_j^{-1})|_k v_j, \quad \text{for } v_j = (\begin{smallmatrix} 5 \\ 0 \\ 1 \end{smallmatrix}). \]

By Lemma 2.3, we can write each \( v_j^{-1} \) uniquely as \( \delta_j^{-1} u_j \) for \( \delta_j \in D^x \) and \( u_j \in \hat{\Gamma}_1(5) \). Then
\[ \psi(v_j^{-1})|_k v_j = \psi(\delta_j^{-1} u_j)|_k v_j = \psi(1)|_k (u_{j,5} v_j) = \psi(1)|_k \delta_{j,5}, \]
where \( u_{j,5} \) and \( \delta_{j,5} \) denote the 5-components of \( u_j \) and \( \delta_j \), respectively.

We see that
\[ \delta_{j,5} = u_{j,5} v_j \in D^x \cap \text{Iw}_{5,1} v_j \subseteq D^x \cap \text{Iw}_{5,1} (\begin{smallmatrix} 5 \\ 0 \\ 1 \end{smallmatrix}) \text{Iw}_{5,1} \subseteq D^x \cap U^*(\begin{smallmatrix} 5 \\ 0 \\ 1 \end{smallmatrix}), \]
where \( U^* \) is a subgroup of \( \text{GL}_2(\mathbb{Z}_S) \) such that the lower right entry belongs to \( 1 + 5\mathbb{Z}_S \). If we put \( \delta_j := \delta_j'(1 + 2j) \), then we have
\[ \delta_j' \in D^x \cap U^*(\begin{smallmatrix} 5 \\ 0 \\ 1 \end{smallmatrix})(1 + 2j)^{-1} = D^x \cap U^*(\begin{smallmatrix} 1 - 2v_5 \\ 0 \\ 0 \\ 1 + 2v_5 \end{smallmatrix}). \]
We know $D^* \cap \text{GL}_2(\mathbb{Z}_5) = \mathcal{O}_5^*$ and all $\delta'_j$ are distinct, so by taking modulo 5, and comparing to the list of $\mathcal{O}_5^*$, we have
\[
\delta'_j \in \left\{ j, -\frac{1}{2}(1 + i + j + k), -\frac{1}{2}(1 - i + j + k), -\frac{1}{2}(1 + i - j + k), \frac{1}{2}(-1 + i - j + k) \right\}.
\]
It is then clear that all $\delta'_j$'s are among the collections of the above right-multiplied by $1 + 2j$. The rest of the lemma is straightforward. \hfill \Box

4 Computation

In this section we compute the matrix for the operator $U_5$ explicitly and explore its upper left $n \times n$ principal minor.

**Theorem 4.1** Let $(P_{ij})_{i,j=0,1,...}$ denote the matrix for the operator $U_5$ on $\mathbb{Q}_5(z)$, defined by the commutative diagram in (3.0.1), with respect to the power basis $1, z, z^2, \ldots$. Then

- For $i = j$,
  \[
P_{ij} = \left( \frac{1 + 3v_5}{2} \right)^{k - 2} \left( \frac{1 - 3v_5}{2} \right)^i \left( \frac{4}{5} \sum_{n=0}^{\infty} (-1)^{i+n} \binom{j}{n} \binom{k - j - 2}{i} \right)
  \]
- For $i \neq j$ but $i \equiv j \pmod{4}$,
  \[
P_{ij} = 4 \left( \frac{1 + 3v_5}{2} \right)^{k - 2} \left( \frac{1 - 3v_5}{2} \right)^i \sum_{n=0}^{\min\{i,j\}} (-1)^{i+n} \binom{j}{n} \binom{k - j - 2}{i} \]
- otherwise $P_{ij} = 0$.

**Proof** The following argument is originally due to Jacobs [10]. For a matrix $P = (P_{ij})$, we define a generating series as the formal power series $H_P(x, y) := \sum P_{ij} x^i y^j$.

Note that we always treat $\delta \in D^*$ as its image in $\text{GL}_2(\mathbb{Q}_5)$. We have explicit expression of the generating series for the operator $U_5$ on $\mathbb{Q}_5(z)$ with respect to the power basis

\[
H_P(x, y) := \sum_{i,j \geq 0} P_{ij} x^i y^j = \sum_{s=1}^{5} \frac{(c_s x + d_s)^{k-1}}{c_s x + d_s - a_s xy - b_s y},
\]
where $\delta_s = \left( \begin{array}{cc} a_s & b_s \\ c_s & d_s \end{array} \right)$ are listed as in Lemma 3.1.

We include a proof here for the convenience of the readers. It is enough to compute the generating series of the operator $\|_k \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ acting on $\mathbb{Q}_5(z)$ with respect to the power basis. By definition,

\[
H(x, y) = \sum_{j \geq 0} y^j \left( ax + d \right)^{k-2} \left( \frac{ax + b}{cx + d} \right)^j = \frac{(cx + d)^{k-2}}{1 - y \cdot \frac{ax + b}{cx + d}} = \frac{(cx + d)^{k-1}}{cx + d - axy - by}.
\]

Note that
\[
H_{\delta_1}(x, y) = \frac{(-2 - v_5)^{k-1}}{-2 - v_5 + (2 - v_5)xy} = \frac{(-2 - v_5)^{k-2}}{1 - \frac{2 - v_5}{2 + v_5}xy} = (-2 - v_5)^{k-2} \sum_{i \geq 0} \left( \frac{2 - v_5}{2 + v_5} \right)^i x^i y^j.
\]
Let $\alpha = \frac{1 + 3v_5}{2}$, then

We observe that
\[
H_{\delta_1}(x, y) = H_{\delta_2}(v_5 x, -v_5 y), H_{\delta_2}(x, y) = H_{\delta_2}(-v_5 x, v_5 y), \text{ and } H_{\delta_3}(x, y) = H_{\delta_2}(-x, y).
\]
This combining with our computation of $H_{4.3}$ pose the space $U$

Remark 4.1 We note that $P_{ij} = 0$ unless $i \equiv j \pmod{4}$. Hence, we would like to decompose the space $S_k^{D,1}((\Gamma_1(5)))$ into 4 sub-spaces corresponding to the four sub-matrices $(P_{2i+a,2j+a})_{i=0,1,2,3}$. The decomposition comes from the study of the Hecke operator at $p = 2$.

4.2 $U_2$-operator

The quaternion algebra $D$ is ramified at $p = 2$, i.e. $D \otimes \mathbb{Q}_2$ is the central division algebra over $\mathbb{Q}_2$. Let $\pi_2$ be the element in $D \otimes \mathbb{Q}_2$ such that $\pi_2^2 = 2$. We define the Hecke operator $U_2$ to be the map

$$U_2 : S_k^{D,1}((\Gamma_1(5))) \rightarrow S_k^{D,1}((\Gamma_1(5))) \tag{4.2.1}$$

$$\varphi(g) \mapsto \varphi(g \pi_2).$$

We can write $U_2$ explicitly following the same recipe for $U_5$-operator:

$$S_k^{D,1}((\Gamma_1(5))) \xrightarrow{\varphi \mapsto \varphi(1)} Q_5[z]$$

$$\varphi \mapsto U_2(\varphi)$$

$$S_k^{D,1}((\Gamma_1(5))) \xrightarrow{\varphi \mapsto \varphi(1)} Q_5[z].$$

Lemma 4.3 The $U_2$ is given by the action $|k(1 + j)|^{-1}$ whose image in $GL_2(\mathbb{Q}_5)$ is

$$\begin{pmatrix}
-1 + 15 & 2 \\
0 & -1 - 15/2
\end{pmatrix}.$$
Explicitly, for any \( g \in \mathbb{Q}_5[z] \),
\[
\mathbb{U}_2(g(z)) = \left( \frac{-1 - v_5}{2} \right)^{k-2} g(-v_5z).
\]

**Proof** For each \( \varphi \in S_k^{D^+}(\tilde{\Gamma}_1(5)) \), we have

\[
\varphi(\pi_2, 1, 1, \ldots) = \varphi(-(1+j)^{-1}(\pi_2, 1, 1, \ldots))
\]
\[
= \varphi(-\pi_2 \left( \frac{1}{1+j}, \frac{1}{1+j}, \frac{1}{1+j} \right)_5, \ldots)
\]
\[
= \varphi(-1, -1, -\left( \frac{1}{1+j} \right)_5, \ldots)
\]
\[
= \varphi(1) \left| k -(1+j)^{-1} = \varphi(1) \left| k \left( \frac{-1}{2} \right)^{k-2} \begin{pmatrix} 0 & -1-v_5 \\ 2 & 0 \end{pmatrix} \right. \right).
\]

Here, the first equality holds because \( \varphi \) is \( D^\times \)-invariant, and the third equality holds because at all places \( \ell \neq 2, 5 \), \( (1+j)^{-1} \) belongs to \( \text{GL}_2(\mathbb{Z}_\ell) \) and \( -\pi_2 \left( \frac{1}{1+j} \right)_5 \) has norm 1, so belongs to \( D^\times(\mathbb{Z}_\ell) \), the maximal compact subgroup of \( (D \otimes \mathbb{Q}_\ell)^\times \). Hence by the commutativity of the diagram and the above chain of equalities, by replacing \( \varphi(1) \) with \( g(z) \), we have

\[
\mathbb{U}_2(g(z)) = g(z) \left| k \left( \frac{-1}{2} \right)^{k-2} \begin{pmatrix} 0 & -1-v_5 \\ 2 & 0 \end{pmatrix} \right. \right).\]

\[\square\]

The lemma gives us the decomposition:

\[
S_k^{D^+}(\tilde{\Gamma}_1(5)) = \bigoplus_{a=0, \ldots, 3} S_k^{D^+}(\tilde{\Gamma}_1(5)) \mathbb{U}_2 = \left( \frac{-1-v_5}{2} \right)^{k-2} \begin{pmatrix} 0 & -1-v_5 \\ 2 & 0 \end{pmatrix}.
\]

To ease the notation, we write

\[
S_{k,a}^{D^+}(\tilde{\Gamma}_1(5)) := S_k^{D^+}(\tilde{\Gamma}_1(5)) \mathbb{U}_2 = \left( \frac{-1-v_5}{2} \right)^{k-2} \begin{pmatrix} 0 & -1-v_5 \\ 2 & 0 \end{pmatrix}.
\]

### 5 SAGE computation of the \( n \times n \) minor

Explicitly, \( S_{k,a}^{D^+}(\tilde{\Gamma}_1(5)) \) corresponds under \( (2.4.2) \) to \( z^a \cdot \mathbb{Q}_5[z^4] \). The upper left minor matrices of the four subspaces have the following surprisingly ruled distribution. Let \( P_n(k, a) = (P_{4i+a,4j+a})_{0 \leq i,j \leq n-1} \) be the upper and left \( n \times n \) minor and choose the weight \( k = 4k_a + 2a - 2 \) for \( k_a \geq 1 \).

We use SAGE to compute the co-rank of these minors. The result shows that for lots of \( k_a \), \( P_n(k, a) \) does not always have full rank as show in Tables 1, 2, 3 and 4. The blank places mean that the corresponding corank is zero.
Table 1 Corank of $P_n(4k_\bullet - 2, 0)$

| $n \times n$ | $k_\bullet = 4$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|--------------|-----------------|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|
| 2 x 2        | 1               | 1 | 1 | 1 | 1 |
| 3 x 3        | 1               | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 x 4        | 1               | 1 | 3 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 x 5        | 2               | 2 | 3 | 3 | 3 | 3 | 2 | 3 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 x 6        | 1               | 1 | 2 | 3 | 4 | 4 | 3 | 4 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 3 | 3 | 2 |
| 7 x 7        | 1               | 2 | 3 | 3 | 5 | 4 | 4 | 4 | 3 | 4 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 8 x 8        | 1               | 2 | 2 | 4 | 4 | 5 | 5 | 5 | 4 | 5 | 4 | 5 | 4 | 5 | 4 | 5 | 4 | 5 |
| 9 x 9        | 1               | 1 | 3 | 3 | 4 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 |
| 10 x 10      |                |   | 2 | 2 | 3 | 4 | 5 | 5 | 7 | 6 |   |   |   |   |   |   |   |   |   |

Table 2 Corank of $P_n(4k_\bullet, 1)$

| $n \times n$ | $k_\bullet = 3$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|--------------|-----------------|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| 2 x 2        | 1               | 1 | 1 | 1 | 1 | 1 |
| 3 x 3        | 1               | 2 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 x 4        | 1               | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 x 5        | 1               | 1 | 3 | 3 | 3 | 3 | 3 | 2 | 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 6 x 6        | 2               | 2 | 3 | 4 | 4 | 3 | 4 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 7 x 7        | 1               | 1 | 2 | 3 | 4 | 4 | 5 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 8 x 8        | 1               | 2 | 3 | 3 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 9 x 9        | 1               | 1 | 2 | 2 | 4 | 4 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 |
| 10 x 10      |                |   | 1 | 1 | 3 | 3 | 4 | 5 | 6 | 6 |   |   |   |   |   |   |   |   |   |   |

Table 3 Corank of $P_n(4k_\bullet + 2, 2)$

| $n \times n$ | $k_\bullet = 3$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|--------------|-----------------|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| 2 x 2        |                | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 x 3        |                | 1 | 2 | 3 | 2 | 3 | 2 | 2 | 2 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| 4 x 4        |                | 1 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 2 | 3 | 2 | 2 | 2 | 2 | 2 |
| 5 x 5        |                | 1 | 1 | 3 | 4 | 4 | 4 | 4 | 4 | 3 | 4 | 3 | 3 | 3 | 3 | 3 | 3 |
| 6 x 6        |                | 2 | 2 | 3 | 4 | 5 | 4 | 5 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 7 x 7        |                | 1 | 1 | 2 | 3 | 4 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 |
| 8 x 8        |                | 1 | 2 | 3 | 3 | 5 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 |
| 9 x 9        |                | 1 | 2 | 2 | 4 | 4 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 | 5 | 6 |
| 10 x 10      |                | 1 | 2 | 2 | 4 | 4 | 5 | 6 | 7 | 6 |   |   |   |   |   |   |   |   |   |

The corank of $P_n(k, a)$ shows the unimodal pattern that looks like the multiplicity of the ghost zeros [1]. For each fixed weight $k$, as $n$ increases, the corank $P_n(k, a)$ pattern is

1, 2, 3, \ldots, 3, 2, 1.

In the next chapter, we will state this pattern in a more precise form in term of the dimensions of the spaces of classical automorphic forms.
Table 4 Corank of $P_n(4k_\star + 4, 3)$

| $n \times n$ | $k_\star = 2$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|---------------|----------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 x 2         | 1              | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | | | | | |
| 3 x 3         | 2              | 2 | 2 | 2 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | | | | | |
| 4 x 4         | 1              | 1 | 2 | 3 | 3 | 2 | 3 | 2 | 2 | 2 | 1 | 2 | 1 | | | | |
| 5 x 5         | 1              | 2 | 3 | 3 | 4 | 3 | 3 | 3 | 3 | 2 | 3 | 2 | 2 | | | | |
| 6 x 6         | 1              | 2 | 2 | 4 | 4 | 4 | 4 | 3 | 4 | 3 | 3 | | | | | | |
| 7 x 7         | 1              | 1 | 3 | 3 | 4 | 5 | 5 | 4 | 5 | 4 | 4 | | | | | | |
| 8 x 8         | 2              | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 5 | 5 | 5 | | | | | | |
| 9 x 9         | 1              | 1 | 2 | 3 | 4 | 4 | 6 | 6 | 6 | | | | | | | | |
| 10 x 10       | 1              | 2 | 3 | 3 | 5 | 5 | 5 | | | | | | | | | | |

6 The classical automorphic forms

We recall spaces of classical automorphic forms and compute their dimensions:

$$S^0_k(\hat{GL}_2(\mathbb{Z}_5)) := \left\{ \psi : D^x \rightarrow \mathbb{Q}_5[z]^{\deg \leq k-2} \text{ s.t. } \psi(xu) = \psi(x)|_k u_5 \text{ for all } u \in \hat{GL}_2(\mathbb{Z}_5) \right\},$$

$$S^0_k(\hat{\Gamma}_1(5)) := \left\{ \psi : D^x \rightarrow \mathbb{Q}_5[z]^{\deg \leq k-2} \text{ s.t. } \psi(xu) = \psi(x)|_k u_5 \text{ for all } u \in \hat{\Gamma}_1(5) \right\}.$$  

Here we extend the action of $\hat{\Gamma}_1(5)$ defined in (2.4.1) to $\hat{GL}_2(\mathbb{Z}_5)$. The action $U_2$, defined in (4.2.1), stabilizes both $S^0_k(\hat{GL}_2(\mathbb{Z}_5))$ and $S^0_k(\hat{\Gamma}_1(5))$, and decomposes these space into 4 sub-spaces $S^{D}_{k,a}(\hat{GL}_2(\mathbb{Z}_5))$, $S^{D}_{k,a}(\hat{\Gamma}_1(5))$ with the dimensions $d_{unr}^{k,a}$ and $d_{Iw}^{k,a}$ respectively.

From now on, we only consider the pair $k$ and $a$ satisfies $k = 4k_\star + 2a - 2$. We can compute the dimensions explicitly.

**Theorem 6.1** For integers $k = 4k_\star + 2a - 2$, we have $d_{Iw}^{k,a} = k_\star$.

**Proof** The isomorphism in (2.4.2) induces the following commutative diagram

$$S^D_k(\hat{\Gamma}_1(5)) \cong S^D_{k,a}(\hat{\Gamma}_1(5)) \cong \mathbb{Q}_5[z]^{\deg \leq k-2} \rightarrow \mathbb{Q}_5[z].$$

Since the $U_2$-action is compatible with vertical maps being isomorphism, we have

$$S^D_{k,a}(\hat{\Gamma}_1(5)) \cong \mathbb{Q}_5[z]^d \oplus \mathbb{Q}_5[z]^{4d} \oplus \cdots = \bigoplus_{0 \leq i \leq k-2, i \equiv a(\mod 4)} \mathbb{Q}_5[z]^i.$$  

The dimension formula follows easily.  

It is more difficult to compute $d_{unr}^{k,a}$, which requires representation theory.

Let us consider the right representation $\text{Sym}^m \mathbb{Q}_5^{B2} \otimes \det^n$ of $\text{GL}_2(\mathbb{Q}_5)$ such that the action of $\text{GL}_2(\mathbb{Q}_5)$ is given by:

$$f \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) (z) := (\alpha \delta - \beta \gamma)^n (\gamma z + \delta)^m f \left( \frac{\alpha z + \beta}{\gamma z + \delta} \right).$$

We may identify $\text{Sym}^m \mathbb{Q}_5^{B2} \otimes \det^n$ with the vector space $\mathbb{Q}_5[z]^{\deg \leq m} \otimes \det^n$ by

$$\text{Sym}^m \mathbb{Q}_5^{B2} \otimes \det^n \rightarrow \mathbb{Q}_5[z]^{\deg \leq m} \otimes \det^n.$$
In particular, when \( m = k - 2 \) and \( n = 0 \), we get the \((k - 2)\)-th symmetric power \( \text{Sym}^{k-2} F^\oplus_5 \):

\[
\sum_{i=0}^{m} a_i X^i Y^{k-2-i} \quad \sum_{i=0}^{k-2} a_i z^i.
\]

We treat both \( \text{Sym}^{k-2} Z^2_5 \) and \( \text{Sym}^m F^\oplus_5 \) as \( O_5^\times \)-modules whose actions are provided by \( O_5^\times \) and its reduction in \( \text{GL}_2(F_5) \), respectively. From the proof of Lemma 2.3, we deduce that

\[
S_k^O(\text{GL}_2(Z_5)) \cong \text{Hom}_{\text{GL}_2(Z_5)}(O^\times_5 \setminus D^\times / \hat{\text{GL}}_2(Z_5), \text{Sym}^{k-2} Q^\oplus_5) \cong \left( \text{Sym}^{k-2} Q^\oplus_5 \right)^{O_5^\times}.
\]

Moreover, we see that

\[
\left( \text{Sym}^{k-2} Q^\oplus_5 \right)^{O_5^\times} \cong \left( \text{Sym}^{k-2} F^\oplus_5 \right)^{O_5^\times}.
\]

Since the order of \( O_5^\times \) is not divisible by 5, we have

\[
\left( \text{Sym}^{k-2} Q^\oplus_5 \right)^{O_5^\times} \equiv (\text{mod } 5) \cong \left( \text{Sym}^{k-2} F^\oplus_5 \right)^{O_5^\times}.
\]

Thus, we obtain

\[
\dim_{Q_5} \left( \text{Sym}^{k-2} Q^\oplus_5 \right)^{O_5^\times} = \text{rank}_{Z_5} \left( \text{Sym}^{k-2} Q^\oplus_5 \right)^{O_5^\times} = \text{rank}_{F_5} \left( \text{Sym}^{k-2} F^\oplus_5 \right)^{O_5^\times}.
\]

We use the following well-known fact from the cohomology theory of finite groups:

**Lemma 6.1** If \( G \) is a finite group and \( M \) is any \( G \)-module, then \( H^i(G, M) \) is annihilated by the order of the group \( G \) for any \( i \geq 1 \).

As a corollary, if \( p \nmid |G| \), then \(-G\) is exact on \( Z_p[G] \)-modules. In our case, the order of \( O_5^\times \) is 24 which is coprime to 5, thus \( H^i(O_5^\times, M) = 0 \) for all \( Z_5 \left[ O_5^\times \right] \)-module \( M \) and \( i \geq 1 \). Thus, we are left to find the Jordan-Hölder factors of \( \text{Sym}^{k-2} F^\oplus_5 \) as a representation of \( \text{GL}_2(F_5) \).

By Proposition 2.17 in [4], the Serre weights \( \left( \text{Sym}^m F^\oplus_5 \otimes \det^n \right) \) for \( m = 0, 1, \ldots, 4 \) and \( n = 0, \ldots, 3 \) exhaust all irreducible representations of \( \text{GL}_2(F_5) \) over \( F_5 \). We use \( \sigma_{m,n} \) to denote the Serre weight \( \left( \text{Sym}^m F^\oplus_5 \otimes \det^n \right) \).

**Lemma 6.2** \( \sigma_{m,n} \) is non-zero if and only if \( m = 0 \).

**Proof** We observe that all elements of \( O_5^\times \) mod 5 have determinant 1. Thus if \( m = 0 \) then \( \sigma_{0,n} \) is \( F_5 \)-isomorphism, so is non-zero. If \( m > 0 \), by checking the action of the element \( i \), we can see that \( \sigma_{m,n} \) is zero.

By Lemma 4.3, we know that \( U_2 \)-action on \( S_k^O(\text{GL}_2(Z_5)) \) is given by the action of \( \| k \cdot (1 + j)^{-1} \) where the image of \( (1 + j)^{-1} \) in \( \text{GL}_2(F_5) \) is \( \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \). Hence, checking the action of \( U_2 \) as in (4.3.1), we see that among the Serre weights \( \sigma_{0,n} \) \( n = 0, \ldots, 3 \), only \( \sigma_{0,3} \) contributes one to the dimension \( d^{\text{unr}}_{k,a} \). Let \( \text{Mult}_{\sigma_{m,n}}(\sigma_{k-2,0}) \) be the multiplicity of \( \sigma_{m,n} \) in the multiset of Jordan-Hölder factors of \( \sigma_{k-2,0} \). We have:

\[
d^{\text{unr}}_{k,a} = \text{Mult}_{\sigma_{0,3}}(\sigma_{k-2,0}). \quad (6.2.1)
\]
From now on, we use the notation
\[\delta_{a,b,c} := \begin{cases} 
1 & \text{if } a \equiv b \pmod{c}, \\
0 & \text{otherwise}.
\end{cases}\]

**Theorem 6.2** For integers \(k = 4k_* + 2a - 2\), we have \(d_{k,a}^{unr} = \lfloor \frac{k_* - a + 5}{6} \rfloor - \delta_{k,a+2,6}\).

**Proof** It is enough to prove the equality in the Grothendieck groups.

Let \(B := \begin{pmatrix} \mathbb{F}_5^\times & \mathbb{F}_5 \\ 0 & \mathbb{F}_5^\times \end{pmatrix}\) and \(G = \text{GL}_2(\mathbb{F}_5)\). We consider the following character \(\eta\) of \(B\):
\[\eta : B \to \mathbb{F}_5^\times, \quad \left( \begin{array}{cc} a & b \\ 0 & \delta \end{array} \right) \mapsto \alpha.\]

For \(k \in \mathbb{Z}\), we have an induced representation
\[\text{Ind}_B^G(\eta^k) := \{ f : G \to \mathbb{F} \mid f(bg) = \eta^k(b)f(g), \text{ for all } b \in B\},\]
which is equipped with the right action of \(G\) given by
\[f|_h(g) := f(gh^T), \quad \text{for } g, h \in G.\]

This is the transpose of the usual left action. We use the following two lemmas from [13, Lemma 4.9] and [14, Proposition 2.4], respectively.

**Lemma 6.3** For any integer \(1 \leq k \leq p-1\), we have an exact sequence of \(G\)-representations:
\[0 \to \sigma_{k,0} \to \text{Ind}_B^G(\eta^k) \to \sigma_{p-1-k,0} \to 0.\]

**Lemma 6.4** For any integer \(k \geq p+1\), we have an exact sequence of \(G\)-representations:
\[0 \to \sigma_{k-(p+1),1} \to \text{Ind}_B^G(\eta^k) \to \sigma_{0,0} \to 0.\]

Let \(p = 5\), for \(k \in \mathbb{Z}\), denote \(\tilde{k} = k \pmod{4}\). From the two exact sequences above, we have the following equality in \(\text{Groth}(\mathbb{F}_5[\text{GL}_2(\mathbb{F}_5)])\):
\[(\sigma_{k-2,0}) - (\sigma_{k-8,1}) = (\sigma_{4-k-2,0}) + (\sigma_{4-k-2,2}).\]  \hfill (6.4.1)

We consider the case \(k \equiv 2 \pmod{4}\). The other cases are very similar. The equality (6.4.1) becomes:
\[(\sigma_{k-2,0}) - (\sigma_{k-8,1}) = (\sigma_{0,0}) + (\sigma_{4,0}).\]

Replacing \(k-2\) by \(k-8\) and then twisting by the character \(\text{det}\), we have
\[(\sigma_{k-8,1}) - (\sigma_{k-14,2}) = (\sigma_{2,1}) + (\sigma_{2,3}).\]

Repeating this iteration, we obtain
\[(\sigma_{k-14,2}) - (\sigma_{k-20,3}) = (\sigma_{0,2}) + (\sigma_{4,2})\]
and
\[(\sigma_{k-20,1}) - (\sigma_{k-26,0}) = (\sigma_{2,3}) + (\sigma_{2,1}).\]

Adding up these equalities, on the left side we have \((\sigma_{k-2,0}) - (\sigma_{k-26,0})\) and on the right side the Serre weight \(\sigma_{0,2}\) for \(a = 0\) or \(2\) appears exactly once. Combining this with (6.2.1), we showed that \(d_{k+24,a}^{unr} = d_{k,a}^{unr} + 1\). For \(k \leq 26\), by counting the Serre weight \(\sigma_{0,a}\), we can compute the values of \(d_{k,a}^{unr}\) as in the table.

From this, we conclude that
\[d_{k,a}^{unr} = \left\lfloor \frac{k_* - a + 5}{6} \right\rfloor - \delta_{k,a+2,6}.\]
7 Main Theorem

Theorem 7.1 For an integer $k \equiv 2a - 2 \pmod{4}$ and an integer $n$ such that $d_{k,a}^{\text{unr}} \leq n \leq d_{k,a}^{\text{lw}} - d_{k,a}^{\text{unr}}$, the corank of $P_{n}(k,a)$ is at least

$$\min\{n - d_{k,a}^{\text{unr}}, d_{k,a}^{\text{lw}} - d_{k,a}^{\text{unr}} - n\}. \quad (7.0.2)$$

Proof Consider the maps $i : S_{k,a}^{\text{D}}(\hat{\Gamma}_{1}(5)) \rightarrow S_{k,a}^{\text{D}}(\hat{\Gamma}_{1}(5))$ and $\text{proj} : S_{k,a}^{\text{D}}(\hat{\Gamma}_{1}(5)) \rightarrow S_{k,a}^{\text{D}}(\hat{\Gamma}_{1}(5))$, given by

$$i(\phi)(x) = \phi(x(\begin{smallmatrix} 5 & 0 \\ 0 & 1 \end{smallmatrix})^{-1})(\begin{smallmatrix} 5 \\ 0 \end{smallmatrix}),$$

and

$$\text{proj}(\phi)(x) = \phi(x(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}) + \sum_{j=0}^{4} \phi(x(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})^{-1})(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}).$$

We note that the map $i$ is the analogue of the map $f(z) \mapsto f(pz)$ for modular form. For the formula for $\text{proj}$, the matrices are chosen from the following representative set of $\Gamma_{0}(5) \setminus \hat{\Gamma}_{1}(5)$

$$\{(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\}, \text{ for } j = 0, \ldots, 4, d = 1, \ldots, 4.$$  

As $S_{k,a}^{\text{D}}(\hat{\Gamma}_{1}(5))$ is invariant under the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$, the map $\text{proj}$ is well defined.

We have the key identity on the space $S_{k,a}^{\text{D}}(\hat{\Gamma}_{1}(5))$

$$i \circ \text{proj}(\phi)(x) = \sum_{j=0}^{4} \phi(x(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})^{-1})(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}))(\begin{smallmatrix} 5 \\ 0 \end{smallmatrix}),$$

$$+ \phi(x(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}))(\begin{smallmatrix} 5 \\ 0 \end{smallmatrix})^{-1})(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}))(\begin{smallmatrix} 5 \\ 0 \end{smallmatrix}),$$

$$= \sum_{j=0}^{4} \phi(x(\begin{smallmatrix} 5 & 0 \\ j & 1 \end{smallmatrix})^{-1})(\begin{smallmatrix} 5 \\ 0 \end{smallmatrix}))(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}))(\begin{smallmatrix} 5 \\ 0 \end{smallmatrix}) + \phi(x(\begin{smallmatrix} 0 & 1 \\ 5 & 0 \end{smallmatrix})^{-1})(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}))(\begin{smallmatrix} 5 \\ 0 \end{smallmatrix}),$$

$$= U_{5}(\varphi(z)) + \varphi(x(\begin{smallmatrix} 0 & 1 \\ 5 & 0 \end{smallmatrix})^{-1})(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})).$$

If we write this equality as matrices, it means that

$$(P_{k,a+4})_{0 \leq a \leq k} = \text{matrix for } i \circ \text{proj} - \text{matrix for } \varphi(\bullet) \mapsto \varphi(\bullet(\begin{smallmatrix} 0 & 1 \\ 5 & 0 \end{smallmatrix})^{-1})(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})). \quad (7.0.3)$$

Since the codomain of the map $\text{proj}$ has dimension $d_{k,a}^{\text{unr}}$, the matrix for $i \circ \text{proj}$ has rank not greater than $d_{k,a}^{\text{unr}}$. To see the matrix of the second operation we use the same argument in Lemma 3.1. We write $\left(\begin{smallmatrix} 0 & 1 \\ 5 & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 0 & -2v_{5} \\ -2v_{5} & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} -2 & 0 \\ -2v_{5} & -2v_{5} \end{smallmatrix}\right)$. Then

$$\varphi(x(\begin{smallmatrix} 0 & 1 \\ 5 & 0 \end{smallmatrix})^{-1})(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}) = \varphi(x)(\begin{smallmatrix} 0 & -2v_{5} \\ -2v_{5} & 0 \end{smallmatrix}).$$
In particular, this means that the matrix of the second operator of (7.0.3) on $S_k^D(\overline{\Gamma}_1(5))$ is the anti-diagonal matrix

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & (-2 + \nu_5)^a(-2 - \nu_5)^{k-2-a} \\
0 & 0 & \cdots & (-2 + \nu_5)^{a+4}(-2 - \nu_5)^{k-6-a} & 0 \\
& & \vdots & \vdots & \vdots \\
& & & & 0 & 0
\end{pmatrix}
$$

But if we only look at its upper left $n \times n$ submatrix, its rank is

$$\begin{cases}
0 & \text{if } n \leq d_{k,a}^{lw}/2, \\
2n - d_{k,a}^{lw} & \text{otherwise}.
\end{cases} \quad (7.0.4)
$$

Thus, the rank of $P_n(k, a)$ is at most the sum of (7.0.4) and $d_{k,a}^{unr}$. We deduce that the corank of $P_n(k, a)$ is at least (7.0.2).

\[\square\]

Remark 7.1 With the dimension formulas, we can compare the corank with the number (7.0.2). We observe that they appear to be always the same. But for the purpose of our project, we only need the inequality.

For example, when $a = 0$, $k_\bullet = 5, 6, \ldots, 14$ and $n = 3$, we have

| $k_\bullet$ | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Corank of $P_3(4k_\bullet - 2, 0)$ | 1   | 2   | 1   | 2   | 1   | 1   | 1   | 1   | 0   | 1   |

The numbers $\min\{3 - d_{k,0}^{unr}, d_{k,0}^{lw} - d_{k,0}^{unr} - 3\}$ in (7.0.2) are:

| $k_\bullet$ | $d_{k,0}^{unr}$ | $d_{k,0}^{lw}$ | $3 - d_{k,0}^{unr}$ | $d_{k,0}^{lw} - d_{k,0}^{unr} - 3$ | $\min\{3 - d_{k,0}^{unr}, d_{k,0}^{lw} - d_{k,0}^{unr} - 3\}$ |
|-------------|----------------|----------------|---------------------|-------------------------------|--------------------------------|
| 5           | 1              | 5              | 2                   | 1                             | 1                             |
| 6           | 1              | 6              | 2                   | 2                             | 2                             |
| 7           | 2              | 7              | 1                   | 2                             | 1                             |
| 8           | 1              | 8              | 2                   | 4                             | 2                             |
| 9           | 2              | 9              | 1                   | 4                             | 1                             |
| 10          | 2              | 10             | 1                   | 5                             | 1                             |
| 11          | 2              | 11             | 1                   | 6                             | 1                             |
| 12          | 2              | 12             | 1                   | 7                             | 1                             |
| 13          | 3              | 13             | 0                   | 7                             | 0                             |
| 14          | 2              | 14             | 1                   | 9                             | 1                             |

8 Ghost conjecture and its application

8.1 Ghost conjecture

In this section, we will state a version of the ghost conjecture and one useful theorem that can be used in the proof of the ghost conjecture.

As in [12], we can define the characteristic power series of $U_5$ on $S_k^D(\overline{\Gamma}_1(5))$ as

$$\text{Char}(U_5, k, a) = \det(I - tP(k, a)),$$

where $P(k, a) = (P_{i+a, j+a})_{i,j \geq 0}$ is the sub-matrix for the operator $U_5$ on the space $S_k^D(\overline{\Gamma}_1(5))$ with respect to the power basis. Our goal is to compute the 5-adic valuation of the $U_5$-eigenvalues, and slopes, by studying the Newton polygon of the power
series \(
\text{Char}(U_5, k, a)\). The ghost conjecture states that there is an explicitly defined power series that has the same Newton polygon.

Set \(w_k = \exp(5(k - 2)) - 1\). As in [12], there is a matrix \(M_a \in M_5(\mathbb{Z}_5[[w]])\) such that \(M_a|_{w} = w_k = P(k, a)\). We denote \(M_a\) by \(P(w, a)\). As a consequence, there is a power series \(\text{Char}(U_5, w, t) \in \mathbb{Z}_5[[w]][[t]]\) such that \(\text{Char}(U_5, w_k, a) = \text{Char}(U_5, k, a)\) for all weights \(k\).

Following [1], we define the ghost series for each \(a = 0, \ldots, 3\) to be the formal power series

\[
G^{(a)}(w, t) = 1 + \sum_{n=1}^{\infty} g_n^{(a)}(w)t^n \in \mathbb{Z}_5[[w]][[t]], \tag{8.1.1}
\]

where each coefficient \(g_n^{(a)}(w)\) is a product

\[
g_n^{(a)}(w) = \prod_{k=2a-2 \text{ (mod 4)}}^{k \geq 2} (w - w_k)^{m_n^{(a)}(w)k} \in \mathbb{Z}_5[w]
\]

with exponents \(m_n^{(a)}(w)\) given by the following recipe

\[
m_n^{(a)}(w) = \begin{cases} 
\min \{ n - d_{k,a}^{\text{unr}}, d_{k,a}^{\text{uw}} - d_{k,a}^{\text{unr}} - n \} & \text{if } d_{k,a}^{\text{unr}} < n < d_{k,a}^{\text{uw}} - d_{k,a}^{\text{unr}} \text{,} \\
0 & \text{otherwise.}
\end{cases}
\]

Note that the exponents \(m_n^{(a)}(w)\) has the same pattern as the corank of \(P_n(k, a)\). We expect that the ghost conjecture is correct in this case.

Conjecture 8.2 (Ghost conjecture) \(\text{Char}(U_5, w, a)\) and \(G^{(a)}(w, t)\) have the same Newton polygon for every \(w \in m_{C_5}\), where \(m_{C_5}\) represents the maximal ideal of the ring of integers of \(C_5\).

There are more technical difficulties to prove this conjecture. However, we believe the following theorem is one among many ingredients to deal with them.

Theorem 8.1 The determinant of the upper left \(n \times n\) minor \(P_n(w, a)\) is divisible by \(g_n^{(a)}(w)\).

Proof As noted above, each coefficients of the matrix is a power series in \(w\). Then we can find a power series \(f_{a,n}(w) \in \mathbb{Z}_5[[w]]\) such that \(f_{a,n}(w_k) = \text{det}(P_n(k, a))\). By the corank bound in (7.0.2), \(f_{a,n}(w)\) is divisible by \((w - w_k)^{m_n^{(a)}(w)}\), so is \(g_n^{(a)}(w)\). \(\square\)

8.3 Gouvêa conjecture

In this section we show that the first \(d_{k,a}^{\text{unr}}\)-slopes of the Newton polygon of the ghost series is less than \(\frac{k+4}{6}\), this is slightly larger than the Gouvêa’s \(\frac{k-1}{p+1}\) conjecture which in our case \(p = 5\). When no ambiguity rises, we shall suppress \(a\) from the notation.

First, in order to use them to compute the difference between the exponent \(m_n(k)\), we introduce some integers \(k_{\text{min}}(n), k_{\text{mid}}(n)\) and \(k_{\text{max}}(n)\) as follows.

Definition 8.4 For each \(n \in \mathbb{Z}_{\geq 0}\), we define:

\[
k_{\text{mid}}(n) := 2n + 1, \quad k_{\text{max}}(n) := 6n + a + 2, \quad k_{\text{min}}(n) := 6n - a + 4 + 5\delta_{n,a,5}, \quad \text{and} \quad k_{\text{min}}(n) := [k_{\text{min}}(n)/5].
\]
Lemma 8.5 For $k = 4k_* + 2a - 2$, fix $n \in \mathbb{Z}_{\geq 0}$.

1. $d^w_{k,a} \leq 2n$ if and only if $k_* < k_{\text{mid}}(n)$.
2. $d^\text{unr}_{k,a} < n$ if and only if $k_* \leq k_{\text{max}}(n-1)$ and $k_* \neq k_{\text{max}}(n-1) - 1$
3. $d^w_{k,a} - d^\text{unr}_{k,a} \leq n$ if and only if $k_* < k_{\text{min}}(n)$.

Proof Recall the dimension formulas in Theorems 6.1 and 6.2,

$$d^w_{k,a} = k_*$$ and $d^\text{unr}_{k,a} = \left\lfloor \frac{k_* - a + 5}{6} \right\rfloor - \delta_{k_*a+2,6}$.

(1) It is obvious.
(2) The condition on $k_*$ means $k_* \leq 6n + a - 4$ and $k_* \neq 6n + a - 5$.

If $k_* \equiv a + 2 \pmod{6}$, we have

$$d^\text{unr}_{k,a} = \left\lfloor \frac{k_* - a + 5}{6} \right\rfloor - 1 = \frac{k_* - a + 4}{6} - 1 < n.$$ 

Otherwise, $k_* \not\equiv a + 2 \pmod{6}$, then $d^\text{unr}_{k,a} = \left\lfloor \frac{k_* - a + 5}{6} \right\rfloor < n$.

Conversely, if $k_* = k_{\text{max}}(n-1) - 1 = 6n + a - 5$, then $n = \frac{k_* - a + 5}{6} = d^\text{unr}_{k,a}$, and if $k_* > k_{\text{max}}(n-1)$ then $k_* - a + 4 > 6n$, or $d^\text{unr}_{k,a} = \left\lfloor \frac{k_* - a + 5}{6} \right\rfloor - \delta_{k_*a+2,6} \geq n$.

(3) By the dimension formulas, $d^w_{k,a} - d^\text{unr}_{k,a} = k_* - \left\lfloor \frac{k_* - a + 5}{6} \right\rfloor + \delta_{k_*a+2,6}$. Hence, the inequality $d^w_{k,a} - d^\text{unr}_{k,a} \leq n$ is equivalent to

$$5k_* \leq 6n - a + 5 - 6\delta_{k_*a+2,6}.$$  \hspace{1cm} (8.5.1)

We consider two cases:

- Case 1: If $n \equiv a \pmod{5}$, we have $k_{\text{min}}(n) = \frac{6n-a+10}{5}$.
  
  Hence, $k_* < k_{\text{min}}(n)$ is equivalent to

$$5k_* < 6n - a + 10.$$ \hspace{1cm} (8.5.2)

If $k_* \not\equiv a + 2 \pmod{6}$, the inequality (8.5.1) would become $5k_* \leq 6n - a + 5$ which is equivalent to (8.5.2) since $6n - a + 5$ and $6n - a + 10$ are two consecutive multiples of 5. If $k_* \equiv a + 2 \mod{6}$, by considering modulo 6, we see that the inequality (8.5.2) becomes $5k_* \leq 6n - a + 4$. However, since $n \equiv a \mod{5}$, we have $5k_* \leq 6n - a - 2$, equivalent to (8.5.1) since in this case neither $6n - a - 1$ nor $6n - a - 2$ is a multiple of 5.

- Case 2: If $n \not\equiv a \pmod{5}$, we have $k_{\text{min}}(n) = \left\lceil \frac{6n-a+4}{5} \right\rceil$. Hence, $k_* < k_{\text{min}}(n)$ is equivalent to the inequality

$$5k_* < 6n - a + 4.$$ \hspace{1cm} (8.5.3)

If $k_* \not\equiv a + 2 \mod{6}$, the inequality (8.5.1) becomes $5k_* \leq 6n - a + 5$. Since $n \not\equiv a \mod{5}$, and $k_* \not\equiv a + 2 \mod{6}$, $5k_*$ could not be $6n - a + 5$ or $6n - a + 4$. Thus, (8.5.1) is equivalent to (8.5.3). If $k_* \equiv a + 2 \mod{6}$, then by consider modulo 6, we see that the inequality (8.5.3) becomes $5k_* \leq 6n - a - 2$, equivalent to (8.5.1).

Hence, let $k_{\text{min}}(n) = 6n - a + 5\delta_{k,a,5}$, then we have $d^w_{k,a} - d^\text{unr}_{k,a} \leq n$ if and only if $k_* < \left\lceil \frac{k_{\text{min}}(n)}{5} \right\rceil$. \hfill $\square$
Lemma 8.6 For $k = 4k_* + 2a - 2$, we have
\[
m_{n+1}(k) - m_n(k) = \begin{cases} 
-1 & \text{if } k_{\min}(n) \leq k_* < k_{\text{mid}}(n), \\
0 & \text{if } k_* = k_{\text{mid}}(n) \text{ or } k_{\text{max}}(n) - 1, \\
1 & \text{if } k_{\text{mid}}(n) < k_* \leq k_{\text{max}}(n) \text{ and } k_* \neq k_{\text{max}}(n) - 1.
\end{cases}
\]

Proof For a fixed integer $n$, we consider $k$ such that $d_{k,a}^{\text{unr}} < n < d_{k,a}^{\text{lw}} - d_{k,a}^{\text{unr}}$, and have
\[
m_n(k) = \min \{n - d_{k,a}^{\text{unr}}, d_{k,a}^{\text{lw}} - d_{k,a}^{\text{unr}} - n\} = \begin{cases} 
n - d_{k,a}^{\text{unr}} & \text{if } n \leq \frac{d_{k,a}^{\text{lw}}}{2}, \\
d_{k,a}^{\text{lw}} - d_{k,a}^{\text{unr}} - n & \text{otherwise.}
\end{cases}
\]

From Lemma 8.5, we list the value of $m_{n+1}(k)$ and $m_n(k)$ for various $k$’s in the following table

| $k_*$ | $m_{n+1}(k)$ | $m_n(k)$ |
|-------|--------------|----------|
| $k_{\min}(n) \leq k_* < k_{\text{mid}}(n)$ | $d_{k,a}^{\text{lw}} - d_{k,a}^{\text{unr}} - (n + 1)$ | $d_{k,a}^{\text{lw}} - d_{k,a}^{\text{unr}} - n$ |
| $k_* = k_{\text{mid}}(n)$ | $d_{k,a}^{\text{lw}} - d_{k,a}^{\text{unr}} - (n + 1)$ | $n - d_{k,a}^{\text{unr}}$ |
| $k_{\text{mid}}(n) < k_* \leq k_{\text{max}}(n) - 1$ | $n + 1 - d_{k,a}^{\text{unr}}$ | $n - d_{k,a}^{\text{unr}}$ |
| $k_* = k_{\text{max}}(n) - 1$ | 0 | 0 |
| $k_{\text{max}}(n) - 1 < k_* \leq k_{\text{max}}(n), k_* \neq k_{\text{max}}(n) - 1$ | 1 | 0 |

The first case and the last case follows directly from the above lemma. We explain the other two cases.

When $k_* = k_{\text{mid}}(n) = 2n + 1$, by the lemma 8.5, $m_{n+1}(k) = d_{k,a}^{\text{lw}} - d_{k,a}^{\text{unr}} - (n + 1)$ and $m_n(k) = n - d_{k,a}^{\text{unr}}$. Hence, $m_{n+1}(k) - m_n(k) = d_{k,a}^{\text{lw}} - 2n - 1 = k_* - 2n - 1 = 0$.

When $k_* = k_{\text{max}}(n) - 1$, we have $k_* = 6n + a + 1$. By the formula in Theorem 6.2, we have $d_{k,a}^{\text{unr}} = n + 1$ so $m_{n+1} = 0$, and since $d_{k,a}^{\text{lw}} > n$, $m_n = 0$.

We also need the following useful lemmas.

For a positive integer $m$, we denote $\text{Dig}(m)$ as the sum of all digits in the $p$-adic expansion of $m$. For a real number $\alpha$, the number $\lfloor \alpha \rfloor$ is the smallest integer that is not smaller than $\alpha$.

Lemma 8.7 The sums of valuations of consecutive integers in $(m_1, m_2]$ with $m_2 > m_1 > 0$ is
\[
\sum_{m_1 < i \leq m_2} v_p(i) = \frac{(m_2 - \text{Dig}(m_2)) - (m_1 - \text{Dig}(m_1))}{p - 1}.
\]

Proof We use the well-known formula $v_p(n!) = \frac{n - \text{Dig}(n)}{p - 1}$ to conclude that
\[
\sum_{m_1 < i \leq m_2} v_p(i) = v_p(m_2!) - v_p(m_1!) = \frac{m_2 - \text{Dig}(m_2)}{p - 1} - \frac{m_1 - \text{Dig}(m_1)}{p - 1}.
\]

Lemma 8.8 Let $n, m$ be positive integers such that $pn > m$, then
\[
p\left\lfloor \frac{m}{p} \right\rfloor + \text{Dig} \left( n - \left\lfloor \frac{m}{p} \right\rfloor \right) = m + \text{Dig}(pn - m).
\]
Proof  Write \( m = p\lfloor \frac{m}{p} \rfloor - r \) where \( r = 0, \ldots, p - 1 \). We have

\[
\text{Dig}\left(n - \left\lfloor \frac{m}{p} \right\rfloor \right) = \text{Dig}\left(pn - p\left\lfloor \frac{m}{p} \right\rfloor \right) = \text{Dig}(pn - m - r).
\]

Since \( pn - m - r \) is divisible by \( p \), the number \( pn - m \) ends with digit \( r \) in its \( p \)-based expression. Hence,

\[
\text{Dig}(pn - m - r) = \text{Dig}(pn - m) - r.
\]

\[ \Box \]

Lemma 8.9  For all \( m, C \in \mathbb{Z}_+ \)

\[
\text{Dig}((p + 1)C) \leq \text{Dig}(C) + \text{Dig}(C + m) + m(p - 2).
\]

Proof  We prove by induction on the number of the digit of \( C \) written in \( p \)-base. If \( C \) has only one digit, we have

\[
\text{Dig}(C) + \text{Dig}(C + m) + m(p - 2) \geq \text{Dig}(C) + p - 1 \geq 2\text{Dig}(C) = 2C = \text{Dig}\left((p + 1)C \right).
\]

Assuming that the inequality is true for any \( C \) with at most \( n - 1 \) digits, we prove for the case \( C \) has \( n \) digits.

If \( m \) has at least \( r \) digits then \( m(p - 2) \geq p^{n-1}(p - 2) \geq (n-1)p(p-2) \geq (n+1)(p-1) \). The last inequality holds for \( n \geq 2 \), and \( p \geq 5 \). We observe that either \((p+1)C\) has \( n + 1 \) digits or it has \( n+2 \) digits and start with 10, so in both case the sum of its digit is at most \((n+1)(p-1)\). The lemma follows. Now we assume \( m \) has \( r \leq n - 1 \) digits and consider 2 cases.

Case 1: If the first \( n - r \) digits of \( C \) are all equal \( p - 1 \), we write \( C = p^r D + E \) where \( 0 \leq E < p^r \). If \( E + m < p^r \), then \( \text{Dig}(C + m) = \text{Dig}(D) + \text{Dig}(E + m) \). By the inductive hypothesis on \( E \), we deduce

\[
\text{Dig}(C) + \text{Dig}(C + m) + m(p - 2) = 2\text{Dig}(D) + \text{Dig}(E) + \text{Dig}(E + m) + m(p - 2) \\
\geq \text{Dig}((p + 1)D) + \text{Dig}((p + 1)E) \geq \text{Dig}((p + 1)C).
\]

If \( E + m \geq p^r \), we can compute \( \text{Dig}(C + m) = \text{Dig}(E + m) \), \( \text{Dig}(C) = (n-r)(p-1) + \text{Dig}(E) \) and \( \text{Dig}((p + 1)C) \leq (n-r)(p-1) + \text{Dig}((p + 1)E) \). The lemma follows from the inductive hypothesis on \( E \).

Case 2: \( C \) has a digit other than \( p - 1 \) in the first \( n - r \) digit, let say it happen at \( p^r \). We write \( C = p^{r-1} D + E \), where \( 0 \leq E < p^{r-1} \). In this case either \( E + m \geq p^r \) or \( E + m < p^r \), we have \( \text{Dig}(C + m) = \text{Dig}(D) + \text{Dig}(E + m) \). This is the same situation as in Case 1. \( \Box \)

Now we are ready to find an upper bound for the slope.

Theorem 8.2  For an integer \( k_0 = 2a - 2 + 4k_0^s \), the first \( d_{k_0,a}^{\text{unr}} \)-slopes of the Newton polygon of \( G^t(w_{k_0}, t) \) are at most \( \frac{k_0+4}{6} \). More precisely, they are less than or equal to

\[
4(d_{k_0,a}^{\text{unr}} - 1 + \delta_{k_0,a+2,6}) + a + 1.
\]

(8.9.1)

Proof  We first check that the number in (8.9.1) is less than \( \frac{k_0+4}{6} \).
From Theorem 6.2, we have:

\[ 4(d_{k_0}^{unr} - 1 + \delta_{k_0, a+2, 6}) + a + 1 = 4 \left( \frac{k_0 - a - 1}{6} \right) + a + 1 \leq \frac{4k_0 + 2a + 2}{6} = \frac{k_0 + 4}{6}. \]

Now, we prove that the first \( d_{k_0}^{unr} \)-slopes of NP(\( G(w_{k_0}) \)) are all less than or equal to (8.9.1). It suffices to show that for \( i = 1, \ldots, d_{k_0}^{unr} \), we have

\[ v_5(g_{d_{k_0}^{unr}}(w_{k_0})) - v_5(g_{d_{k_0}^{unr}-i}(w_{k_0})) \leq \left( 4 \left( d_{k_0}^{unr} - 1 + \delta_{k_0, a+2, 6} \right) + a + 1 \right). \]

We write

\[
\begin{align*}
A_n &= k_0 - k_{\text{max}}(n) - 1 = k_0 - 6n - a - 3, \\
B_n &= 5k_0 - 6n - 6 + a, \\
C_n &= k_0 - k_{\text{mid}}(n) = k_0 - 2n - 2.
\end{align*}
\]

First let us consider the case \( k_0 \equiv a + 1 \) (mod 6). In this case, \( k_0 > k_{\text{max}}(n) \) for all integers \( n = 0, 1, \ldots, d_{k_0}^{unr} - 1 \).

**Lemma 8.10** For integers \( n = 0, 1, \ldots, d_{k_0}^{unr} - 1 \) and \( k_0 \equiv a + 1 \) mod 6, we have

\[ v_5(g_{n+1}(w_{k_0})) - v_5(g_n(w_{k_0})) \leq 4n + a + \frac{1}{2} \left( \text{Dig}(A_n) + \text{Dig}(B_n) - \text{Dig}(C_n) - \text{Dig}(C_n + 1) \right) \]

**Proof** We have

\[
\begin{align*}
v_5(g_{n+1}(w_{k_0})) - v_5(g_n(w_{k_0})) &= \sum_{k_0 \neq k_{\text{max}}(n) - 1} \left( v_5(k_0) + 1 \right) - \sum_{k_{\text{mid}}(n) < k_0 < k_{\text{max}}(n)} \left( v_5(k_0) + 1 \right) \\
&\leq 5(k_{\text{max}}(n) - k_{\text{mid}}(n)) - 4 + \text{Dig}(k_0 - k_{\text{max}}(n) - 1) - \text{Dig}(k_0 - k_{\text{mid}}(n) - 1) \\
&\leq \frac{5(k_{\text{max}}(n) - k_{\text{mid}}(n)) + \text{Dig}(k_0 - k_{\text{max}}(n)) - \text{Dig}(k_0 - k_{\text{mid}}(n))}{4}.
\end{align*}
\]

Note that in the inequality, we add an extra term \( v_5(k_0 - k_{\text{max}}(n)) + 1 \) in order to use the following Lemma 8.7. We will subtract it when it is necessary later. Using Lemmas 8.8 and 8.5, one can deduce that

\[ 5k_{\text{min}}(n) + \text{Dig}(k_0 - k_{\text{min}}(n)) = \tilde{k}_{\text{min}}(n) + \text{Dig}(5k_0 - \tilde{k}_{\text{min}}(n)). \]

Hence, the expression is equal to

\[
\begin{align*}
&\frac{5k_{\text{max}}(n) + \tilde{k}_{\text{min}}(n) - 10k_{\text{mid}}(n) - 4}{4} \\
&= \frac{\text{Dig}(k_0 - k_{\text{max}}(n) - 1) + \text{Dig}(5k_0 - \tilde{k}_{\text{min}}(n))}{4} \\
&\quad - \frac{\text{Dig}(k_0 - k_{\text{mid}}(n) - 1) + \text{Dig}(k_0 - k_{\text{mid}}(n))}{4}.
\end{align*}
\]

From Lemma 8.5, we see that the first term is equal to

\[
\frac{5(6n + a + 2) + (6n + 4 - a + \delta_{n,a,5}) - 10(2n + 1) - 4}{4} = 4n + a + \frac{\delta_{n,a,5}}{4}. \quad (8.10.1)
\]

Now we simplify the rest of the expression. We have

\[ \text{Dig}(5k_0 - \tilde{k}_{\text{min}}(n)) = \text{Dig}(5k_0 - (6n + 4 - a + \delta_{n,a,5})) . \]
If \( n \equiv a \mod 5 \) then it is equal to \( \text{Dig}(5k_0 - 6n - 9 + a) \). We observe that \( 5k_0 - 6n - 4 + a \) end with digit 1 when written in 5-base. Hence,

\[
\text{Dig}(5k_0 - 6n - 9 + a) = \text{Dig}(5k_0 - (6n + 4 - a) - 2) - 3.
\]

If \( n \not\equiv a \mod 5 \), we have

\[
\text{Dig}(5k_0 - (6n + 4 - a + \delta_{n,a,5})) = \text{Dig}(5k_0 - 6n - 4 + a) \\
\leq \text{Dig}(5k_0 - 6n - 6 + a) + 2.
\]

Thus,

\[
\text{Dig}(5k_0 - k_{\min}(n)) \leq \text{Dig}(5k_0 - 6n - 6 + a) + 2 - \delta_{n,a,5}.
\]

The lemma follows from the formulas of \( k_{\min}(n), k_{\text{mid}}(n) \) and \( k_{\max}(n) \) in Lemma 8.5.

Thus, we have:

\[
v_p\left(\frac{\G_{k_0}^{\text{lurr}}(w_{k_0})}{\G_{k_0}^{\text{lurr}}(w_{k_0})}\right) - v_p\left(\frac{\G_{k_0}^{\text{lurr}}(w_{k_0})}{\G_{k_0}^{\text{lurr}}(w_{k_0})}\right) - i \cdot \left(4d_{k_0,a}^{\text{lurr}} - 1 + a + 1\right) \leq i \left(\frac{3}{2} - 2i\right)
\]

\[
+ \sum_{1 \leq j \leq i} \text{Dig}\left(A_{d_{k_0}^{anr} - j}\right) + \text{Dig}\left(B_{d_{k_0}^{anr} - j}\right) - \text{Dig}\left(C_{d_{k_0}^{anr} - j}\right) - \text{Dig}\left(C_{d_{k_0}^{anr} - 1}\right) = 4i^2 + 2i.
\]

**Lemma 8.11** For integers \( i = 1, 2, \ldots, d_{k_0}^{\text{lurr}} - 1 \) and \( k_0 \not\equiv a + 1, a + 2 \mod 6 \), we have

\[
\sum_{1 \leq j \leq i} \text{Dig}\left(A_{d_{k_0}^{anr} - j}\right) + \text{Dig}\left(B_{d_{k_0}^{anr} - j}\right) - \text{Dig}\left(C_{d_{k_0}^{anr} - j}\right) - \text{Dig}\left(C_{d_{k_0}^{anr} - 1}\right) \leq 4i^2 + 2i.
\]

**Proof** For \( j = 1, 2, \ldots, d_{k_0}^{\text{lurr}} - 1 \), we observe that \( B_{d_{k_0}^{anr} - j} = 6C_{d_{k_0}^{anr}} - A_{d_{k_0}^{anr} - j} + 3 \).

We write \( A_{d_{k_0}^{anr} - j}, B_{d_{k_0}^{anr} - j}, C_{d_{k_0}^{anr} - j} \) in terms of \( A = A_{d_{k_0}^{anr} - 1}, \) and \( C = C_{d_{k_0}^{anr} - 1} \).

Note that \( A_{d_{k_0}^{anr} - 1} = k_0 - 6(d_{k_0}^{\text{lurr}} - 1) - a - 3 = k_0 - a - 1 - \left\lfloor \frac{k_0 - 6 - a - 1}{\delta_{n,a,5}} \right\rfloor - 1 \in \{0, 1, 2, 3\} \).

\[
A_{d_{k_0}^{anr} - j} = A + 6(j - 1) \quad \text{and} \quad C_{d_{k_0}^{anr} - j} = C + 2(j - 1).
\]

Hence, \( B_{d_{k_0}^{anr} - j} = 6(C + j - 1) + 3 - A \).

Then using the inequality \( \text{Dig}(X + Y) \leq \text{Dig}(X) + \text{Dig}(Y) \) and \( \text{Dig}(6X) \leq 2\text{Dig}(X) \leq 2X \), we conclude that:

\[
\text{Dig}(A_{d_{k_0}^{anr} - j}) \leq \text{Dig}(A) + 2(j - 1) = A + 2(j - 1),
\]

\[
\text{Dig}(B_{d_{k_0}^{anr} - j}) \leq \text{Dig}(6(C + j - 1)) + \text{Dig}(3 - A) \leq \text{Dig}(6(C + j - 1)) + 3 - A.
\]

Adding these inequalities:

\[
\sum_{1 \leq j \leq i} (\text{Dig}A_{d_{k_0}^{anr} - j} + \text{Dig}B_{d_{k_0}^{anr} - j}) \leq i^2 + 2i + \sum_{1 \leq j \leq i} \text{Dig}(6(C + j - 1)).
\]

Applying the Lemma 8.9 for \( C, C + 1, \ldots, C + i - 1, m = i \) and \( p = 5 \), then summing up the inequalities, we deduce

\[
\sum_{1 \leq j \leq i} \text{Dig}(6(C + j)) \leq \sum_{1 \leq j \leq 2i} \text{Dig}((C + j - 1) + 3i^2).
\]
Hence,
\[
\sum_{1 \leq j \leq i} \text{Dig} \left( A_{d_{k_0}^{\text{unr}} - j} \right) + \text{Dig} \left( B_{d_{k_0}^{\text{unr}} - j} \right) - 2 \text{Dig} \left( C_{d_{k_0}^{\text{unr}} - j} \right) \leq 4i^2 + 2i.
\]

Thus, the inequality (8.10.2) becomes
\[
v_5(g_{d_{k_0}^{\text{unr}}(w_{k_0})}) - v_p(g_{d_{k_0}^{\text{unr}} - 1}(w_{k_0})) - i \cdot (4(d_{k_0}^{\text{unr}} - 1) + a) \leq i(2 - i).
\]
We only need to check for the case \( i = 1 \).

For \( i = 1 \), using the notation in Lemma 8.11,
\[
v_5(g_{d_{k_0}^{\text{unr}}(w_{k_0})}) - v_5(g_{d_{k_0}^{\text{unr}} - 1}(w_{k_0})) - (4(d_{k_0}^{\text{unr}} - 1) + a + 1)
\leq -\frac{1}{2} + \text{Dig}(A + 2) - \text{Dig}(C + 1).
\]
- If \( A = 0, 1, \) or \( 2 \), then we have
  \[
  \text{Dig}(6C + 3 - A) \leq \text{Dig}(5C) + \text{Dig}(C + 1) + \text{Dig}(2 - A)
  = 2 - A + \text{Dig}(C) + \text{Dig}(C + 1).
  \]
- If \( A = 3 \), recall that in 8.1.1, we put the extra term
  \[
v_5 \left( k_0 - k_{\text{max}(a)} + 1 \right) = 4(v_5(A + 2) = 1.
  \]
So in both cases, \( v_5(g_{d_{k_0}^{\text{unr}}(w_{k_0})}) - v_5(g_{d_{k_0}^{\text{unr}} - 1}(w_{k_0})) - (4(d_{k_0}^{\text{unr}} - 1) + a + 1) \leq 0 \). This proves Theorem 8.2 when \( k_0 \equiv a + 1, a + 2 \mod 6 \).

**Lemma 8.12** For integers \( i = 1, 2, \ldots, d_{k_0}^{\text{unr}} - 1 \) and \( k_0 \equiv a + 2 \mod 6 \), we have
\[
\sum_{1 \leq j \leq i} \text{Dig} \left( A_{d_{k_0}^{\text{unr}} - j} \right) + \text{Dig} \left( B_{d_{k_0}^{\text{unr}} - j} \right) - \text{Dig} \left( C_{d_{k_0}^{\text{unr}} - j} \right) - \text{Dig} \left( C_{d_{k_0}^{\text{unr}} - j} + 1 \right) \leq 4i^2 + 4i.
\]

**Proof** In this case we have
\[
\text{Dig} \left( A_{d_{k_0}^{\text{unr}} - j} \right) = \text{Dig} \left( 5 + 6(j - 1) \right) \leq \text{Dig}(5) + \text{Dig}(6(j - 1)) \leq 2j - 1.
\]
Let \( C = C_{d_{k_0}^{\text{unr}}} \), then we can write \( C_{d_{k_0}^{\text{unr}} - j} = C + 2j \). Thus,
\[
\text{Dig} \left( B_{d_{k_0}^{\text{unr}} - j} \right) = \text{Dig} \left( 6C_{d_{k_0}^{\text{unr}} - j} - A_{d_{k_0}^{\text{unr}} - j} + 3 \right) \leq \text{Dig}(6(C + j)) + 4.
\]
We have
\[
\text{Dig} \left( A_{d_{k_0}^{\text{unr}} - j} \right) + \text{Dig} \left( B_{d_{k_0}^{\text{unr}} - j} \right) \leq 2j + 3 + \text{Dig}(6(C + j)).
\]
Following the same argument in Lemma 8.11, we find the upper bound is \( 4i^2 + 4i \). \( \Box \)

Thus, we have
\[
v_5(g_{d_{k_0}^{\text{unr}}(w_{k_0})}) - v_5(g_{d_{k_0}^{\text{unr}} - 1}(w_{k_0})) - i \cdot \left( 4(d_{k_0}^{\text{unr}} - 1 + \delta_{k_0,a + 2,6}) + a + 1 \right)
\leq i \left( \frac{5}{2} - 2i \right) + i^2 + i < 0.
\]
Theorem 8.2 is proved for the case \( k_0 \equiv a + 2 \mod 6 \).
We are left to prove the case \( k_0 \equiv a + 1 \mod 6 \). We following the same argument with some modification.
**Lemma 8.13** For integers \( n = 0, 1, \ldots, d_{k_0}^{unr} - 2 \), we have
\[
v_5(g_{n+1}(w_{k_0})) - v_5(g_n(w_{k_0})) \leq 4n + a + \frac{1}{2} \left( \text{Diag}(A_n) + \text{Diag}(B_n) - \text{Diag}(C_n) - \text{Diag}(C_n + 1) \right).
\]

For \( n = d_{k_0}^{unr} - 1 \) and \( k_0 \equiv a + 1 \mod 6 \), we have
\[
v_5(g_{n+1}(w_{k_0})) - v_5(g_n(w_{k_0})) \leq 4n + a + \frac{1}{4} \left( \text{Diag}(B_n) - \text{Diag}(C_n) - \text{Diag}(C_n + 1) \right).
\]

**Proof** Lemma 8.10 still work for \( n \leq d_{k_0}^{unr} - 2 \), as for such \( n, k_0 > k_{\max(n)} \).

When \( n = d_{k_0}^{unr} - 1 \), \( k_0 = k_{\max(n)} - 1 \). In this case we can compute the difference \( v_5(g_{n+1}(w_{k_0})) - v_5(g_n(w_{k_0})) \) as follows:
\[
1 + \sum_{k_{\min(n)} < k < k_{\max(n)} - 1} (v_5(k_0 - k) + 1) - \sum_{k_{\min(n)} < k < k_{\max(n)} - 1} (v_5(k_0 - k) + 1)
\]
\[
= 5(k_{\max(n)} - k_{\min(n)}) - 6 - \text{Diag}(k_0 - k_{\min}(n) - 1) - \frac{5(k_{\mid(n)} - k_{\min(n)}) + \text{Diag}(k_0 - k_{\min}(n)) - \text{Diag}(k_0 - k_{\min}(n))}{4}.
\]

The lemma follows the same argument in Lemma 8.10. \( \square \)

Thus, we have:
\[
v_p(g_{k_0}^{unr}(w_{k_0})) - v_p(g_{k_0}^{unr-1}(w_{k_0})) - i \cdot (4(d_{k_0}^{unr} - 1) + a + 1) \leq i \left( \frac{3}{2} - 2i \right) - \frac{1}{2}
\]
\[
+ \sum_{2 \leq j \leq i} \frac{\text{Diag}(A_{d_{k_0}^{unr} - j})}{4} + \sum_{1 \leq j \leq i} \frac{\text{Diag}(B_{d_{k_0}^{unr} - j}) - \text{Diag}(C_{d_{k_0}^{unr} - j}) - \text{Diag}(C_{d_{k_0}^{unr} - j + 1})}{4}.
\]

**Lemma 8.14** For integers \( i = 1, 2, \ldots, d_{k_0}^{unr} - 1 \) and \( k_0 \equiv a + 1 \mod 6 \), we have
\[
\sum_{2 \leq j \leq i} \text{Diag}(A_{d_{k_0}^{unr} - j}) + \sum_{1 \leq j \leq i} \text{Diag}(B_{d_{k_0}^{unr} - j}) - \text{Diag}(C_{d_{k_0}^{unr} - j}) - \text{Diag}(C_{d_{k_0}^{unr} - j + 1}) \leq 4i^2 + 4i - 2.
\]

**Proof** We recall that for \( j = 1, 2, \ldots, d_{k_0}^{unr} - 1 \),
\[
B_{d_{k_0}^{unr} - j} = 6C_{d_{k_0}^{unr} - j} - A_{d_{k_0}^{unr} - j} + 3.
\]

When \( k_0 \equiv a + 1 \mod 6 \), we have \( A_{d_{k_0}^{unr} - 2} = 4 \), so \( A_{d_{k_0}^{unr} - j} = 4 + 6(j - 2) \). Hence, for \( j \geq 2 \)
\[
\text{Diag}(A_{d_{k_0}^{unr} - j}) \leq \text{Diag}(4) + 2(j - 2) = 2j.
\]

Let \( C = C_{d_{k_0}^{unr} - 1} \), we have \( C_{d_{k_0}^{unr} - j} = C + 2(j - 1) \) and \( B_{d_{k_0}^{unr} - j} = 6(C + j - 1) + 5 \).

Thus, for \( j \geq 1 \)
\[
\text{Diag}(B_{d_{k_0}^{unr} - j}) \leq \text{Diag}(6(C + j - 1)) + 1.
\]

Adding these inequalities:
\[
\sum_{2 \leq j \leq i} \text{Diag}(A_{d_{k_0}^{unr} - j}) + \sum_{1 \leq j \leq i} \text{Diag}(B_{d_{k_0}^{unr} - j}) \leq i^2 + 2i - 2 + \sum_{1 \leq j \leq i} \text{Diag}(6(C + j - 1)).
\]

Following the same argument in Lemma 8.11, we find the upper bound is \( 4i^2 + 4i - 2 \). \( \square \)
Hence, we have
\[ v_5(g_{d,0}^{\text{unr}}(w_{k_0})) - v_p(g_{d,0}^{\text{unr}}(w_{k_0})) - i \cdot (4d_{k_0,a}^{\text{unr}} - 1) + a \leq i(2 - i) - 1 \leq 0. \]

We complete the proof of Theorem 8.2. □

9 SAGE code

The readers can find the SAGE code used to compute the $U_5$ matrices and the co-rank of the principal minors at https://github.com/NhaXTruong/Hecke-operator-U5.git.

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Data availability

All data generated or analyzed during this study are included in this published article.

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References

1. Bergdall, J., Pollack, R.: Arithmetic properties of Fredholm series for $p$-adic modular forms. Proc. Lond. Math. Soc. 113(4), 419–444 (2016)
2. Bergdall, J., Pollack, R.: Slopes of modular forms and the ghost conjecture. IMRN 4, 1125–1144 (2019)
3. Bergdall, J., Pollack, R.: Slopes of modular forms and the ghost conjecture II. Trans. AMS 372(1), 357–388 (2019)
4. Breuil, C.: Notes for his lecture at Columbia university, available on his webpage. https://www.imo.universite-paris-saclay.fr/~breuil/PUBLICATIONS/New-York.pdf
5. Buzzard, K., Calegari, F.: A counterexample to the Gouvêa–Mazur conjecture. C. R. Math. 338, 751–753 (2004)
6. Buzzard, K., Kilford, L.: The 2-adic eigencurve at the boundary of weight space. Compos. Math. 141(3), 605–619 (2005)
7. Coleman, R., Mazur, B.: The Eigencurve, in Galois Representations in Arithmetic Algebraic Geometry (Durham, 1996). London Math. Soc. Lecture Note Series, vol. 254, pp. 1–113. Cambridge Univ. Press, Cambridge (1998)
8. Gouvêa, F.: Where the slopes are. J. Ramanujan Math. Soc. 16, 75–99 (2001)
9. Gouvêa, F., Mazur, B.: Families of modular eigenforms. Math. Comp. 66, 793–805 (1992)
10. Jacobs, D.: Slopes of Compact Hecke Operators, Thesis. University of London, Imperial College, London (2004)
11. Kisin, M.: The Fontaine–Mazur conjecture for $GL_2$. J. Am. Math. Soc. 22(3), 641–690 (2009)
12. Liu, R., Wan, D., Xiao, L.: Eigencurve over the boundary of the weight space. Duke Math. J. 166, 1739–1787 (2017)
13. Paskunas, V.: Coefficient systems and supersingular representations of $GL_2(F)$, Mem. Soc. Math. Fr. 99, 84 (2004)
14. Reduzzi, D.: Reduction mod $p$ of cuspidal representations of $GL_2(\mathbb{F}_p)$ and symmetric powers. J. Algebra 324(12), 3507–3531 (2010)
15. Wan, D., Xiao, L., Zhang, J.: Slope of eigencurves over boundary disks. Math. Ann. 369, 487–537 (2017)

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