EXPONENTIAL MIXING FOR THE FRACTIONAL MAGNETO-HYDRODYNAMIC EQUATIONS WITH DEGENERATE STOCHASTIC FORCING

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Abstract. We establish the existence, uniqueness and exponential attraction properties of an invariant measure for the MHD equations with degenerate stochastic forcing acting only in the magnetic equation. The central challenge is to establish time asymptotic smoothing properties of the associated Markovian semigroup corresponding to this system. Towards this aim we take full advantage of the characteristics of the advective structure to discover a novel H"{o}rmander-type condition which only allows for several noises in the magnetic direction.

1. Introduction. The dynamics of the velocity and the magnetic field in electrically conducting fluids and basic physics conservation laws can be described by the Magneto-Hydrodynamic (MHD) equations (c.f. [3]). The existence, uniqueness, regularity and stability of the MHD equations have been extensively studied in many papers, see [5, 6, 17, 21].

Recently there has been mounting interest in the generalized fractional MHD equations,
\begin{align*}
\partial_t u + [u \cdot \nabla u + \mu (-\Delta)^\alpha u] &= [-\nabla p + b \cdot \nabla b], \\
\partial_t b + [u \cdot \nabla b + \nu (-\Delta)^\beta b] &= b \cdot \nabla u, \\
u(0, x) &= u_0, b(0, x) = b_0,
\end{align*}
(1.1)
where $u(t, x), b(t, x) \in \mathbb{R}^2$. Wu [24] proved that equations (1.1) have a unique weak solution when $\mu > 0, \nu > 0, \alpha \geq \frac{1}{2} + \frac{d}{4}, \beta \geq \frac{1}{2} + d$, and $u_0, b_0 \in L^2$ and equations (1.1) have a classical solution when $\mu > 0, \nu > 0, \alpha \geq \frac{1}{2} + \frac{d}{4}, \beta \geq \frac{1}{2} + \frac{d}{4}$, and $u_0, b_0$ are sufficiently smooth. For the 2D incompressible MHD equations with horizontal

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dissipation and horizontal magnetic diffusion, Cao, Dipendra and Wu [4] proved that equations possesses a global regular solution if $\varepsilon, \delta > 0$ and $u_0, b_0$ are sufficiently smooth.

Meanwhile, for the MHD equations driven by non-degenerate stochastic forcing terms both in the velocity and in the magnetic field, the existence and uniqueness of invariant measure was obtained via coupling method in [2]. Huang and Shen [12] proved the well-posedness and the existence of a random attractor for the stochastic 2D incompressible fractional MHD equations driven by Gaussian multiplicative noise. For the stochastic fractional MHD equations with degenerate multiplicative noise on the Torus $\mathbb{T}^2$, Shen, Huang and Zeng [23] proved the existence and uniqueness of the invariant measure for the associated transition semigroup. The noise in [23] is degenerate in the sense that it drives the system only in the first finite Fourier modes.

In this paper, we consider the following MHD equations driven by degenerate additive noise on two-dimensional torus $\mathbb{T}^2 = [-\pi, \pi]^2$,

\[
\begin{aligned}
du + \left[u \cdot \nabla u + (-\Delta)^{\alpha} u\right] dt &= \left[\nabla p + b \cdot \nabla b\right] dt, \\
db + \left[u \cdot \nabla b + (-\Delta)^{\beta} b\right] dt &= b \cdot \nabla ud t + \sum_{k = (k_1, k_2) \in \mathbb{Z}_0} \left(\frac{k_2}{|k|} - \frac{k_1}{|k|}\right)^T \alpha_k^0 \cos(k \cdot x) dW_{k,0}^0 \\
+ \sum_{k = (k_1, k_2) \in \mathbb{Z}_0} \left(-\frac{k_2}{|k|} \frac{k_1}{|k|}\right)^T \alpha_k^1 \sin(k \cdot x) dW_{k,1}^1, \\
\n\end{aligned}
\tag{1.2}
\]

with periodic boundary value conditions

\[ u_i(x, t) = u_i(x + 2\pi j, t), \quad b_i(x, t) = b_i(x + 2\pi j, t), \quad i = 1, 2, \]
where $\alpha > 1, \beta > 1, t \geq 0, j \in \mathbb{Z},$ $u = (u_1, u_2)$ and $b = (b_1, b_2)$ denote the velocity field and magnetic field respectively, $p$ is a scalar pressure, $Z_0$ is a subset of $\mathbb{Z}^2 \setminus \{0, 0\}$, $(\mathbb{W}^{k, m})_{k \in Z_0, m \in (0, 1)}$ is a $2|Z_0|$-dimensional Brownian motion defined relative to a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $\{\alpha_k^m\}_{k \in Z_0, m \in (0, 1)}$ are non-zero constants. Throughout this paper, we assume that $d := 2|Z_0| < \infty$.

For $n \geq 0$, define

\[ Z_0 := \{k \mid k \in Z_0 \text{ or } -k \in Z_0\}, \]
\[ Z_n := \{k + \ell \mid k \in Z_{n-1}, \ell \in Z_0, (k, \ell^\perp) \neq 0, |k| \neq |\ell|\}, \]
where $\ell^\perp = (-\ell_2, \ell_1)$ for any $\ell = (\ell_1, \ell_2)$ and $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^2$.

**Hypothesis 1.1**

\[ \bigcup_{k=0}^\infty Z_{2k} = \mathbb{Z}^2 \setminus \{0, 0\}, \quad \bigcup_{k=0}^\infty Z_{2k+1} = \mathbb{Z}^2 \setminus \{0, 0\}. \]

Hypothesis 1.1 may not seem intuitive, however it falls into the most interesting case of degenerate noise—"hypoellipticity" setting, and includes many interesting examples, see Remark 1 and Example 2.1. Now we state our main result as below.

**Theorem 1.1.** Assume that Hypothesis 1.1 holds, then the associated Markov semigroup corresponding to (1.2) possesses a unique, exponentially mixing invariant measure. Furthermore, a law of large numbers together with a central limit theorem is established under the current circumstances.
We remark that Theorem 1.1 is a simplified version of Theorem 2.1 and refer readers to Section 2.2 for more details.

Nowadays ergodicity research on infinite-dimensional systems driven by degenerate stochastic forcing has attracted considerable attention ([1, 7, 8, 9, 10, 11, 14, 15, 18, 20]), not only because this poses many interesting mathematical challenges, but also provides rigorous justification for the explicit or implicit statistical measurement assumptions invoked in a physical environment. It is exciting that recently there have been remarkable breakthroughs (c.f. [8, 9, 11]), initiating the development a theory of "hypoellipticity" for degenerated forced infinite-dimensional stochastic systems. However, the whole theory is far from mature and remains in an involved formation. The reason for this is unlike in the case of finite-dimensional systems, the invertibility of Malliavin matrix is hard to prove, not to mention characterise its range. Experts have thus devised a tactful strategy to take full advantage of the structure of turbulent systems. Roughly speaking, infinite-dimensional as these systems are, their unstable directions are confined to be finitely many, and it is reasonable that one just focus on proving the Malliavin matrix to possess small eigenvalues on some spanning cones.

The technical difficulties of this method lie in how to generate successively larger finite dimensional spaces through the interaction between the nonlinear and stochastic terms and how to exert delicate spectral analysis on these spaces. To be more specific, one digging into the technical details will find that the proof virtually relies heavily on the results of progressive computation of Lie brackets using constant vector fields and nonlinear terms, by virtue of which the whole involved arguing process will be decomposed in an inductive manner and most importantly an appropriate Hörmander condition will thus be determined. It is worth emphasising that finding out such an ideal collection of Lie brackets to accomplish the task is case-by-case, there is no general recipe for all. For instance, Navier-Stokes equations and Boussinesq equations are treated quite differently and therefore lead to different Hörmander condition (c.f. [8, 9, 11]).

The main contribution of the manuscript is, we successfully devised a special pattern of Lie bracket computations suitable for the fractional MHD equation, and thus propose a novel Hörmander condition. Apart from [8, 9], the considered fractional MHD equations are of original formation instead of vorticity formation. Furthermore, Due to the special form of stochastic fractional Magneto-Hydrodynamic equations (1.2), we exert a series of Lie bracket computation strategically to exploit the distinctive structure of nonlinear advective terms. Roughly speaking, we activate the noise term within the magnetic equation to spread to the velocity equation through advection, then perturb it again with stochastic forcing to generate new directions in the \( b \) component of the phase space. On the flip side, new \( v \) directions can be generated similarly except being stochastically driven once. This procedure can be repeated iteratively so as to span the whole phase space as long as Hypothesis 1.1 is satisfied (c.f. Section 4 for more details). Attentive readers may find that the whole deductive process and derived Hörmander condition distinguish from that within [8, 9, 11].

We would also like to add that the degenerate noise in [22, 23] is in a fairly simple manner and belongs to the so-called "essentially elliptic" setting. More precisely, although driven modes are assumed to be finite, they are forced to be one by one and required to be sufficiently many, while in this paper we adopt a hypoellipticity setting, which allows for a limit number of directions to be driven on and off. We
will further exemplify this essential difference with Example 2.1, which also exhibits a distinct picture compared with [8, 9, 11]. All in all, our analysis gets the utmost out of existing techniques in the recent works but yields something peculiar and we believe it will enrich ergodic research upon systems of SPDEs.

This article is organized as follows: In Section 2 we introduce general definitions and formulate our main result (Theorem 2.1). Section 3 is devoted to some moment estimates which will be used frequently. In Section 4 we illustrate progressive computations of Lie brackets in detail. Then in Section 5 we focus on proving the spectral properties of Malliavin matrix (Theorem 5.1) and give a gradient estimate of the Markov semigroup (Proposition 3). Finally, we provide a proof of Theorem 2.1 in Section 6.

2. Preliminaries.

2.1. Mathematical setting. In this section we introduce a functional setting for the equations (1.2). Then we describe specifically the stochastic forcing, and thus formulate (1.2) as an abstract stochastic evolution equation. Finally, we introduce some basic elements of the Malliavin calculus centering on the Malliavin matrix.

The higher order Sobolev spaces are denoted by

\[ H^s_t := \left\{ u = (u_1, u_2) \in (W^{s,2}(\mathbb{T}^2))^2 : \nabla \cdot u = 0, \int_{\mathbb{T}^2} u_1(x)dx = \int_{\mathbb{T}^2} u_2(x)dx = 0 \right\} \]

for any \( s \geq 0 \), where \( W^{s,2}(\mathbb{T}^2) \) is classical Sobolev-Slobodeckii space, and \( H^s_1 \) is equipped with the norm

\[ \|u\|_{H^s_1}^2 := \|u_1\|^2_{W^{s,2}} + \|u_2\|^2_{W^{s,2}}, \]

here \( u = (u_1, u_2) \). Let \( H^s_2 = H^s_1 \times H^s_2 \) and \( H^s_2 \) is equipped with the same norm with \( H^s_1 \). For any \( U = (u, b) \in H^s \), the norm of \( U \) on the space \( H^s \) is given by

\[ \|U\|^2_{H^s} = \|u\|^2_{H^s_1} + \|b\|^2_{H^s_2}. \]

We also denote \( H^{-s} := (H^s)^* \) the dual space to \( H^s \). Specially,

\[ H_1 := H^0_1 = \left\{ u = (u_1, u_2) \in (L^2(\mathbb{T}^2))^2 : \nabla \cdot u = 0, \int_{\mathbb{T}^2} u_1(x)dx = \int_{\mathbb{T}^2} u_2(x)dx = 0 \right\}. \]

The norm on the space \( H_1 \) is given by

\[ \|u\|^2_2 = \|(u_1, u_2)\|^2_2 := \|u_1\|^2_{L^2} + \|u_2\|^2_{L^2}. \]

Likewise, let \( H := H^0 \). By a slight abuse of notation, \( \langle \cdot, \cdot \rangle \) may denote the inner product on Hilbert space \( H \) or \( H_1 \). Let \( \Pi \) be the projection operator from \( (L^2(\mathbb{T}^2))^2 \) to the space \( H_1 \).

For any \( u \in H^1_1 \), let \( \Lambda^\alpha u = (-\Delta)^{\alpha/2} u \). For any \( b \in H^2_2 \), let \( \Lambda^\beta b = (-\Delta)^{\beta/2} b \).

For any \( m, n \in \mathbb{R} \), we denote by

\[ H^{m,n} = \left\{ w = (u, b) \mid u \in H^m_1, b \in H^n_2 \right\}, \]

endowed with the norm \( \|w\|^2_{H^{m,n}} = \|u\|^2_{H^m_1} + \|b\|^2_{H^n_2} \). We also denote \( H^{-m,-n} := (H^{m,n})^* \) the dual space to \( H^{m,n} \).

Next, we need to construct the stochastic forcing based on an orthogonal basis of \( H \), therefore for \( k = (k_1, k_2) \), denote

\[ e_k^0 = \left( \frac{k_2}{|k|}, -\frac{k_1}{|k|} \right)^T \cos(k \cdot x), \quad e_k^1 = \left( -\frac{k_2}{|k|}, \frac{k_1}{|k|} \right)^T \sin(k \cdot x). \]
It is commonsense that \( \{e^m_k\}_{k \in \mathbb{Z}^2 \setminus \{0,0\}, m \in \{0,1\}} \) forms an orthogonal basis of \( H_1 \) exactly.

Denote
\[
\psi^0_k(x) := (e^0_k, 0)^T \in H_1 \times H_2, \quad \psi^1_k(x) = (e^1_k, 0)^T \in H_1 \times H_2,
\]
and
\[
\sigma^0_k(x) := (0, e^0_k)^T \in H_1 \times H_2, \quad \sigma^1_k(x) = (0, e^1_k)^T \in H_1 \times H_2. \tag{2.1}
\]

Let \( \{\psi^m_k\}_{k \in \mathbb{Z}^2, m \in \{0,1\}} \) be the standard basis of \( \mathbb{R}^{2|\mathbb{Z}_0|} \). We define a linear map \( Q_b : \mathbb{R}^{2|\mathbb{Z}_0|} \to H \) such that
\[
Q_b e^m_k := \alpha^m_k \psi^m_k.
\]
Denote the Hilbert-Schmidt norm of \( Q_b \) by
\[
\mathcal{E}_0 := \|Q_b Q_b\| = \sum_{k \in \mathbb{Z}_0, m \in \{0,1\}} (\alpha^m_k)^2.
\]

We consider stochastic forcing of the form
\[
Q_b dW = \sum_{k \in \mathbb{Z}_0, m \in \{0,1\}} \alpha^m_k \sigma^m_k dW^{k,m}. \tag{2.2}
\]

For \( U = (u, b)^T \) and \( \tilde{U} = (\tilde{u}, \tilde{b})^T \), denote \( A^{\alpha,\beta} U = ((-\Delta)^{\alpha} u, (-\Delta)^{\beta} b)^T \), and
\[
B(U, \tilde{U}) = \begin{pmatrix}
\Pi [u \cdot \nabla \tilde{u} - b \cdot \nabla \tilde{b}]
\Pi [u \cdot \nabla \tilde{b} - b \cdot \nabla \tilde{u}]
\end{pmatrix},
\]
where
\[
B(U) = B(U, U), \quad F(U) = -A^{\alpha,\beta} U - B(U, U).
\]

With these preliminaries in hand, the equations (1.2) may be written as an abstract stochastic evolution equation on \( H \)
\[
dU + (A^{\alpha,\beta} U + B(U, U)) dt = Q_b dW, \quad U_0 = (u_0, b_0), \tag{2.3}
\]
or in a more compact formulation
\[
dU = F(U) dt + Q_b dW. \tag{2.4}
\]

We say that \( U = U(t, U_0) \) is a solution of (2.3) if it is \( \mathcal{F}_t \)-adapted, \( U \in C([0, \infty); H) \cap L^2_{loc}([0, \infty); H^1) \) a.s. and \( U \) satisfies (2.3) in the mild sense, that is,
\[
U_t = e^{-A^{\alpha,\beta} t} U_0 - \int_0^t e^{-A^{\alpha,\beta}(t-s)} B(U_s, U_s) ds + \int_0^t e^{-A^{\alpha,\beta}(t-s)} G dW_s.
\]

The well-posedness can be established similarly as in [12]. Hence we let \( U = U(t, U_0) \) be the unique solution of (2.3) with initial value \( U_0 \). For any \( \xi = (\xi_1, \xi_2) \in H, \ t \geq s \geq 0 \), the Jacobian \( J_{s,t} \xi \) is actually the unique solution of
\[
\begin{cases}
\partial_t J_{s,t} \xi + A^{\alpha,\beta} J_{s,t} \xi + B(U_t, J_{s,t} \xi) + B(J_{s,t} \xi, U_t) = 0, \\
J_{s,s} \xi = \xi. \tag{2.5}
\end{cases}
\]

In the interest of brevity, set \( J_t \xi := J_0,t \xi \). Let \( J^{(2)}_{s,t} : H \to \mathcal{L}(H, \mathcal{L}(H)) \) be the second derivative of \( U \) with respect to an initial value \( U_0 \). Observe that for fixed \( U_0 \in H \) and any \( \xi, \xi' \in H \) the function \( \varrho = \varrho_t := J^{(2)}_{s,t} (\xi, \xi') \) is the solution of
\[
\partial_t \varrho_t + A^{\alpha,\beta} \varrho_t + \nabla B(U_t) \varrho_t + \nabla B(J_t \xi) J_{t,t} \xi = 0, \quad \varrho_s = 0, \tag{2.6}
\]
where \( \nabla B(\theta) \partial = B(\theta, \partial) + B(\partial, \theta) \).
Let \( d = 2|Z_0| \). The Malliavin derivative \( D : L^2(\Omega, H) \to L^2(\Omega; L^2(0, T; \mathbb{R}^d) \times H) \) satisfies, for each \( v \in L^2(0, T; \mathbb{R}^d) \)

\[
\langle DU, v \rangle_{L^2(0, T; \mathbb{R}^d)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(U(T, U_0, W + \varepsilon \int_0^T v_s ds) - U(T, U_0, W)\right).
\]

One may infer from Duhamel’s formula that (c.f. [11]) for \( v \in L^2(\Omega; L^2(0, T; \mathbb{R}^d)) \),

\[
\langle DU, v \rangle_{L^2(0, T; \mathbb{R}^d)} = \int_0^T J_{s, T} Q_0 v_s ds.
\]

We define the random operator \( A_{s, t} : L^2(s, t; \mathbb{R}^d) \to H \) by

\[
A_{s, t} v := \int_s^t J_{r, t} Q_0 v_r dr.
\]

Direct computation shows that \( A_{s, t} v \) satisfies the following equation

\[
\begin{cases}
\partial_t A_{s, t} v + A^{\alpha, \beta} A_{s, t} v + B(U_t, A_{s, t} v) + B(A_{s, t} v, U_t) = Q_0 v_t, \\
A_{s, s} v = 0.
\end{cases}
\]

For any \( s < t \), let \( A^*_{s, t} : H \to L^2(s, t; \mathbb{R}^d) \) be the adjoint of \( A_{s, t} \), then

\[
(A^*_{s, t} \xi)(r) = Q^*_0 K_{r, t} \xi, \text{ for any } \xi \in H, r \in [s, t],
\]

where \( Q^*_0 : H \to \mathbb{R}^d \) is the adjoint of \( Q_0 \), and for \( s < t \), \( K_{s, t} \xi \) is the solution of the following “backward” system

\[
\partial_s \rho^* = A^{\alpha, \beta} \rho^* + (\nabla B(U))^* \rho^* = -B(U)' \rho^*, \quad \rho^*_t = \xi. \quad (2.7)
\]

It is time to define the Malliavin matrix as

\[
M_{s, t} := A_{s, t} A^*_{s, t} : H \to H.
\]

Observe that \( \rho_t := J_{0, t} \xi - A_{0, t} v \) satisfies

\[
\begin{cases}
\partial_t \rho_t + A^{\alpha, \beta} \rho_t + B(U_t, \rho_t) + B(\rho_t, U_t) = -Q_0 v_t, \\
\rho_0 = \xi.
\end{cases}
\]

This equation enables us to translate the ergodicity issue into a control problem. Actually in conjunction with the Malliavin integration by parts formula, one can obtain the estimate on \( \nabla P_t \Phi \) through spectral analysis on the Malliavin matrix \( M \) (c.f. Section 5).

### 2.2. Main Theorem

Before stating the main theorem of the manuscript, let us recall some basic notations with regard to the associated Markovian semigroup. It is necessary to introduce new functional spaces first.

Denote by \( M_b(H) \) and \( C_b(H) \) respectively, the spaces of bounded measurable and bounded continuous real valued functions on \( H \) equipped with the supremum norm. We also define

\[
O_\eta := \left\{ \Phi \in C^1(H) : \|\Phi\|_\eta < \infty \right\},
\]

where \( \|\Phi\|_\eta := \sup_{U_0 \in H} \left( \exp\left(-\eta \|U_0\|\right) \|\Phi(U_0)\| + \|\nabla \Phi(U_0)\| \right) \),

for any \( \eta > 0 \), which is the special admissible functional space for Theorem 2.1.

The transition function associated to \((2.3)\) is given by

\[
P_t(U_0, E) = P(U(t, U_0) \in E) \text{ for any } U_0 \in H, E \in \mathcal{B}(H), t \geq 0,
\]
where $\mathcal{B}(H)$ is the collection of Borel sets on $H$, $U(t,U_0)$ is the solution of (2.3) with initial value $U_0$. We also define the Markov semigroup $\{P_t\}_{t \geq 0}$ with $P_t : \mathcal{M}_b(H) \to \mathcal{M}_b(H)$ associated to (1.2) by

$$P_t \Phi(U_0) := \mathbb{E}\Phi(U(t,U_0)) = \int_H \Phi(U) P_t(U_0, dU) \text{ for any } \Phi \in \mathcal{M}_b(H), t \geq 0. \quad (2.8)$$

Now we will give our main results in this article.

**Theorem 2.1.** Assume Hypothesis 1.1 holds, then there exists an unique invariant measure $\mu_*$ associated to (1.2) and for each $t \geq 0$ the map $P_t$ is ergodic relative to $\mu_*$. Moreover, there exists a constant $\eta^*$ such that $\mu_*$ satisfies for each $\eta \in (0, \eta^*)$

(i) (Mixing) There is $\gamma = \gamma(\eta) > 0$ and $C = C(\eta)$ such that

$$\left| \mathbb{E}\Phi(U(t,U_0)) - \int_H \Phi(U) d\mu(U) \right| \leq C \exp(-\gamma t + \eta||U_0||) \|\Phi\|_\eta \quad (2.9)$$

holds for any $\Phi \in \mathcal{O}_\eta$, $U_0 \in H$ and any $t \geq 0$.

(ii) (Weak law of large numbers) For any $\Phi \in \mathcal{O}_\eta$ and any $U_0 \in H$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi(U(t,U_0)) dt = \int_H \Phi(\hat{U}) d\mu_*(\hat{U}) =: m_\Phi, \text{ in probability.} \quad (2.10)$$

(iii) (Central limit theorem) For any $\Phi \in \mathcal{O}_\eta$, every $U_0 \in H$ and $\xi \in \mathbb{R}$

$$\lim_{T \to \infty} \mathbb{P} \left( \frac{1}{\sqrt{T}} \int_0^T \left( \Phi(U(t,U_0)) - m_\Phi \right) dt < \xi \right) = \mathcal{N}(\xi), \quad (2.11)$$

where $\mathcal{N}$ is the distribution function of a normal random variable with zero mean and variance equal to

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left( \int_0^T \left( \Phi(U(t,U_0)) - m_\Phi \right) dt \right)^2.$$

**Remark 1.** Interestingly, under Hypothesis 1.1 it is possible that the noise allows to be so degenerate that only four modes in the magnetic direction are actually driven. The next example allows one to get a primary idea into this phenomenon, and meanwhile to notice the specificity of Hypothesis 1.1 in comparison to [8, 9, 10].

**Example 2.1** If $Z_0 = \{(0,1),(1,1),(1,0),(1,2)\}$, then Hypothesis 1.1 holds.

**Proof.** For $n \geq 0$, define

$$\hat{Z}_0 = \{(0,1),(1,1),-(0,1),-(1,1)\},$$

$$\hat{Z}_n := \{k + \ell \mid k \in \hat{Z}_{n-1}, \ell \in \hat{Z}_0, (k,\ell) \neq 0, |k| \neq |\ell|\},$$

then it is not difficult to check that

$$\hat{Z}_1 = \{-1,0\}, \{1,0\}, \{1,2\}, \text{ and } \cup_{n=0}^{\infty} \hat{Z}_n = \mathbb{Z}^2 \setminus \{(0,0)\}.$$.

By (1.3),

$$Z_0 = \hat{Z}_0 \cup \hat{Z}_1, \quad Z_1 \supseteq \hat{Z}_1 \cup \hat{Z}_2, \quad Z_2 \supseteq \hat{Z}_2 \cup \hat{Z}_3, \quad \cdots \quad Z_k \supseteq \hat{Z}_k \cup \hat{Z}_{k+1}, \quad \cdots$$

Therefore, one sees that

$$\cup_{k=0}^{\infty} Z_{2k} \supseteq \cup_{n=0}^{\infty} \hat{Z}_n = \mathbb{Z}^2 \setminus \{(0,0)\}, \quad \cup_{k=0}^{\infty} Z_{2k+1} \supseteq \cup_{n=0}^{\infty} \hat{Z}_n = \mathbb{Z}^2 \setminus \{(0,0)\},$$

which yields the desired result. \qed
3. Moment estimates on $U_t, J_{s,t}^*, J_{s,t}^{(2)}, \mathcal{K}_{s,t}^*, \mathcal{M}_{s,t}$. In this section we provide moment bounds with respect to the unique solution $U$ and its linearizations. They may seem familiar for readers who are familiar with research works with regard to ergodicity on the stochastic Navier-Stokes equations and so on. Hence some proofs are sketched or omitted if they do not distinguish from existing methods. However for the fractional MHD equations (1.2), we have to impose $\alpha > 1, \beta > 1$ to compensate for the complicated advective operator $B$. This is accomplished through delicate interpolation and weighting. Lemma 3.1 and Lemma 3.2 give one a glimpse of this strategy.

Lemma 3.1. For any $U_0 \in H$, let $U_t = U(t, U_0)$ be the unique solution of (2.3) with initial value $U_0$. Then there exists $\eta^* > 0$ such that

1. for any $\eta \in (0, \eta^*]$ and some $C = C(\eta, \rho, \kappa) > 0$, there holds
   $$\mathbb{E}\left[\exp\left\{ \eta \|U_t\|^2 + \frac{\eta}{2} e^{-t/2} \int_0^t \|\Lambda^\alpha u_s\|^2 ds + \frac{\eta}{2} e^{-t/2} \int_0^t \|\Lambda^\beta b_s\|^2 ds \right\} \right] \leq C \exp\{\eta \|U_0\|^2 e^{-t}\}.$$  
2. (2) for some $C > 0$ and any $\eta \in (0, \eta^*]$, $\mathbb{E}\left\{ \eta \|U_t\|^2 - \eta \|U_0\|^2 + \eta \int_0^t \|\Lambda^\alpha u_s\|^2 ds + \eta \int_0^t \|\Lambda^\beta b_s\|^2 ds - \eta \mathcal{E}_0 t \right\} \leq C$. (3.1)
3. For any $\eta > 0$ and $\eta \in (0, \eta^*]$, $\mathbb{E}\left\{ \eta \sum_{k=0}^N \|U_k\|^2 \right\} \leq \exp (\rho \eta \|U_0\|^2 + \kappa N)$,

where $\rho, \kappa > 0$ are positive constants independent of $N$ and $U_0$.
4. For any $s \geq 0, p \geq 2$, and $\eta \in (0, \eta^*]$, there exists $C = C(\eta, s, T, p)$ such that
   $$\mathbb{E}\left( \sup_{t \in [T/2, T]} \|U_t\|^p_{\mathcal{H}_t} \right) \leq C \exp (\eta \|U_0\|^2),$$

and
   $$\mathbb{E}\left( \|U\|^p_{C^{1/4}(T/2, T), H^s} \right) \leq C \exp (\eta \|U_0\|^2).$$

Proof. 1. By Ito’s formula, for $\eta > 0$,
   $$\eta \|U_t\|^2 - \eta \|U_0\|^2 + 2\eta \int_0^t \|\Lambda^\alpha u_s\|^2 ds + 2\eta \int_0^t \|\Lambda^\beta b_s\|^2 ds = \eta \mathcal{E}_0 t + 2\eta \int_0^t \langle b_s, Q_b dW_s \rangle.$$  
Set $\bar{Z}_t := \eta \|\Lambda^\alpha u_s\|^2 + \eta \|\Lambda^\beta b_s\|^2$, then $\eta \mathcal{E}_0 - 2\eta \|\Lambda^\alpha u_s\|^2 - 2\eta \|\Lambda^\beta b_s\|^2 \leq \eta \mathcal{E}_0 - 2\bar{Z}_t, \quad 4\eta^2 \|\langle b, Q_b \rangle\|^2 \leq 4\eta \mathcal{E}_0 \bar{Z}_t.$

Applying [10, lemma 5.1] with $\bar{U}_t := \eta \|U_t\|^2$, one arrives that there exists $\eta^* > 0$, such that for any $\eta \in (0, \eta^*]$
   $$\mathbb{E}\left[\exp\left\{ \eta \|U_t\|^2 + \frac{1}{2} e^{-t/2} \int_0^t \eta \|\Lambda^\alpha u_s\|^2 ds + \frac{1}{2} e^{-t/2} \int_0^t \eta \|\Lambda^\beta b_s\|^2 ds \right\} \right] \leq C(\eta, \mathcal{E}_0) \exp\{\eta \|U_0\|^2 e^{-t}\}.$$

2. Ito’s formula yields that
   $$\|U_t\|^2 - \|U_0\|^2 + 2\int_0^t \|\Lambda^\alpha u_s\|^2 ds + 2\int_0^t \|\Lambda^\beta b_s\|^2 ds = \mathcal{E}_0 t + 2\int_0^t \langle b_s, Q_b dW_s \rangle,$$
then for any \( \eta > 0 \),
\[
\eta \| U_t \|^2 - \eta \| U_0 \|^2 + 2\eta \int_0^t \| \Lambda^\alpha u_s \|^2 ds + \eta \int_0^t \| \Lambda^\beta b_s \|^2 ds - \eta \xi_0 t
\]
\[
\leq 2\eta \int_0^t \langle b_s, Q_s dW_s \rangle - \eta \int_0^t \| \Lambda^\beta b_s \|^2 ds.
\]
If \( \eta \leq \frac{1}{4\| \xi_0 \|^2} \), the following inequality holds from the exponential martingale argument for some absolute constant \( C \),
\[
\mathbb{E} \exp \left\{ \eta \| U_t \|^2 - \eta \| U_0 \|^2 + \eta \int_0^t \| \Lambda^\alpha u_s \|^2 ds + \eta \int_0^t \| \Lambda^\beta b_s \|^2 ds - \eta \xi_0 t \right\} \leq C.
\]

3. The proof of (3) follows similarly as in [9, proof of Lemma 4.10].
4. The proof of (4) follows similarly as in [16, Proposition 2.4.12] and the fact \( \| W^{k,t} \|_{C_{1/4},2,T} \) has finite \( p \)th moment for any \( p \geq 1 \). \( \square \)

The next lemmata include necessary estimates on linearizations of (2.3). Referring back to (2.5), (2.7) and (2.6), one finds that \( \mathcal{J}_{s,t} \) is the Jacobian operator with its adjoint \( \mathcal{K}_{s,t} \) and derivative \( \mathcal{J}_{s,t}^{(2)} \). At first glance the following bounds are closely related to those on the Malliavin derivative \( \mathcal{M}_{s,t} \). The technically oriented readers may jump to Section 5 for further details.

**Lemma 3.2.** For \( \xi \in H \), assume \( \mathcal{J}_{s,t} \xi = (\mathcal{J}_{s,t}^1 \xi, \mathcal{J}_{s,t}^2 \xi) \in H_1 \times H_2 \). For each \( \eta > 0 \) and \( 0 < s < t \), we have the following estimate
\[
\| \mathcal{J}_{s,t} \xi \|^2 + \int_s^t \left[ \| \Lambda^\alpha J^1_{s,r} \xi \|^2 + \| \Lambda^\beta J^2_{s,r} \xi \|^2 \right] dr \leq C \exp \left( \eta \int_s^t \| U_s \|_{H^1}^2 ds + C(\eta)(t - s) \right) \| \xi \|^2,
\]
where \( C(\eta) \) is independent of \( s, t \). Moreover, for each \( r \leq t, p \geq 1 \) and any \( \eta > 0 \), there exists \( C = C(\eta, t - r, p) \) such that
\[
\mathbb{E} \sup_{s < t \leq r, T} \| \mathcal{J}_{s,t} \xi \|^p \leq C \exp (\eta \| U_0 \|^2) \| \xi \|^p,
\]
\[
\mathbb{E} \sup_{s < t \leq r, T} \| \mathcal{K}_{s,t} \xi \|^p \leq C \exp (\eta \| U_0 \|^2) \| \xi \|^p,
\]
\[
\mathbb{E} \sup_{s < t \leq r, T} \| \mathcal{J}_{s,t}^{(2)}(\xi, \xi') \|^p \leq C \exp (\eta \| U_0 \|^2) \| \xi \|^p \| \xi' \|^p.
\]

**Proof.** Recalling (2.5), for \( \alpha' \in (1, \alpha), \beta' \in (0, \beta) \) and any \( \eta \in (0, 1) \), we deduce from the interpolation inequality and Young inequality that
\[
\text{d}\| \mathcal{J}_{s,t} \xi \|^2
\]
\[
= -2(A^{\alpha,\beta} J_{s,t} \xi, J_{s,t} \xi) dt - 2(B(U_t, J_{s,t} \xi, J_{s,t} \xi) dt - 2(B(J_{s,t} \xi, U_t, J_{s,t} \xi) dt
\]
\[
= -2\| \Lambda^\alpha J_{s,t}^1 \xi \|^2 - 2\| \Lambda^\beta J_{s,t}^2 \xi \|^2 + C\| U_t \|_{H^1} \cdot \left[ \| \Lambda^\alpha J_{s,t}^1 \xi \| + \| \Lambda^\beta J_{s,t}^2 \xi \| \right] \cdot \| J_{s,t} \xi \|
\]
\[
\leq -2\| \Lambda^\alpha J_{s,t}^1 \xi \|^2 - 2\| \Lambda^\beta J_{s,t}^2 \xi \|^2 + C(\eta) \cdot \left[ \| \Lambda^\alpha J_{s,t}^1 \xi \|^2 + \| \Lambda^\beta J_{s,t}^2 \xi \|^2 \right]
\]
\[
+ \eta\| U_t \|_{H^1} \cdot \| J_{s,t} \xi \|^2
\]
\[
\leq -\| \Lambda^\alpha J_{s,t}^1 \xi \|^2 - \| \Lambda^\beta J_{s,t}^2 \xi \|^2 + \eta\| U_t \|_{H^1} \cdot \| J_{s,t} \xi \|^2 + C(\eta)\| J_{s,t} \xi \|^2,
\]
Lemma 3.3. which leads to (3.2).

(3.3) and (3.4) follows from (3.2) and Lemma 3.1.

For fixed \( U_0 \in H \) and any \( \xi, \xi' \in H \), it follows from (2.5) that the second derivative \( \varrho_t := J^\alpha(\xi, \xi') = (\varrho^1_t, \varrho^2_t) \in H_1 \times H_2 \) satisfies
\[
\varrho_t = 0.
\]

Then
\[
\varrho_t = 0.
\]

Therefore again by Young inequality and Interpolation inequality, for any \( \alpha' \in (1, \alpha), \beta' \in (0, \beta) \) and \( \eta > 0 \),
\[
\varrho_t \leq C(\eta) \left( \| \varrho_t \|^2 + 2 \| \Lambda^{\alpha'} \varrho_t \|^2 + 2 \| \Lambda^{\beta'} \varrho_t \|^2 \right)
\]

Setting \( \eta \) small enough, one reaches from Gronwall’s inequality that
\[
\varrho_t \leq C(\eta) \left( \| \varrho_t \|^2 + 2 \| \Lambda^{\alpha'} \varrho_t \|^2 + 2 \| \Lambda^{\beta'} \varrho_t \|^2 \right)
\]

Now, Lemma 3.1 in combination with (3.2)-(3.4) leads to (3.5).
then this lemma follows from Lemma 3.2.

For any $N \geq 1$, define

$$H_N := \text{span}\{\sigma_k^0, \sigma_k^1, \psi_k^0, \psi_k^1 : 0 < |k| \leq N\},$$

along with the associated projection operators

$$P_N : H \rightarrow H_N \text{ the orthogonal projection onto } H_N, \quad Q_N := I - P_N.$$

The following three lemmas are particularly useful in translating the bounds on the Malliavin matrix into gradient estimates on the Markov semigroup (c.f. Proposition 3). Since their proofs adopt similar approach as above in combination with a straightforward modification of existing methods (c.f. [8, 9]), they are omitted to save space.

**Lemma 3.4.** For every $p \geq 1, T > 0, \delta, \gamma > 0$ there exists $N_* = N_*(p, T, \delta, \gamma)$, such that for any $N \geq N_*$ one has

$$\mathbb{E}\|Q_N J_{0,T}\|_{L^p(H,H)}^p \leq \gamma \exp(\delta \|U_0\|^2), \quad \mathbb{E}\|J_{0,T} Q_N\|_{L^p(H,H)}^p \leq \gamma \exp(\delta \|U_0\|^2).$$

Here, $\mathcal{L}(X,Y)$ denotes the operator norm of the linear map between the given Hilbert spaces $X$ and $Y$.

**Lemma 3.5.** For $0 < s < t$,

$$\|A_{s,t}\|_{\mathcal{L}(L^2([s,t], \mathbb{R}^d), H)} \leq C \left( \int_s^t \|J_{r,t}\|_{L^2(H,H)}^2 dr \right)^{1/2}$$

holds for a constant $C$ independent of $s, t$. Moreover, for any $\kappa > 0$

$$\|A_{s,t}^* (M_{s,t} + \kappa I)^{-1/2}\|_{\mathcal{L}(H,L^2([s,t], \mathbb{R}^d))} \leq 1,$$

$$\|(M_{s,t} + \kappa I)^{-1/2} A_{s,t}\|_{\mathcal{L}(L^2([s,t], \mathbb{R}^d), H)} \leq 1,$$

$$\|(M_{s,t} + \kappa I)^{-1/2}\|_{\mathcal{L}(H,H)} \leq \kappa^{-1/2}.$$

Recall that $\mathcal{D}$ is the Malliavin derivative. We adopt the notions

$$\mathcal{D}_s F := (DF)(s), \quad s \in [0, T], \quad \mathcal{D}^j F := (DF)^j, \quad j = 1, \ldots, d.$$

Then observe that for $\tau \leq t$

$$\mathcal{D}_s J_{s,t} \xi = \begin{cases} J_{s,t}^{(2)}(Q_0 e_j, J_{s,\tau} \xi) & \text{if } s \leq \tau, \\ J_{s,t}^{(2)}(J_{s,\tau} Q_0 e_j, \xi) & \text{if } s > \tau. \end{cases}$$

**Lemma 3.6.** For any $\eta > 0, \xi \in H$ and $p \geq 1$ we have the bounds

$$\mathbb{E}\|\mathcal{D}_s J_{s,t} \xi\|^p \leq C \exp(\eta \|U_0\|^2) \|\xi\|^p,$$

$$\mathbb{E}\|\mathcal{D}_s^j A_{s,t} \xi\|^p_{L^p([s,t], \mathbb{R}^d)} \leq C \exp(\eta \|U_0\|^2),$$

$$\mathbb{E}\|\mathcal{D}_s^j A_{s,t}^* \xi\|^p_{L^p(H, L^p([s,t], \mathbb{R}^d))} \leq C \exp(\eta \|U_0\|^2),$$

where $C = C(\eta, p)$. 

\[ \square \]
4. Details of Lie bracket computations. For any Fréchet differentiable $E_1, E_2 : H \to H$,
\[ [E_1, E_2](u) := \nabla E_2(u) E_1(u) - \nabla E_1(u) E_2(u). \]

$[E_1, E_2]$ is referred to as the Lie bracket of two "vector fields" $E_1, E_2$.

This section is technical, however, reveals some intrinsic thoughts of the manuscript. As a matter of fact, we present that for any $N \in \mathbb{N}$, how finite dimensional subspaces $H_N$ of $H$ can be generated through the iterations of Lie brackets. It is worth mentioning that these computations are motivated by the celebrated Hörmander condition for the Kolmogorov-Fokker-Planck equations associated to (2.3). The following is split into two parts. Firstly, we describe how the velocity direction $u$ is covered.

4.1. Covering velocity direction. For $u, \tilde{u} \in H_1 = H_2$, denote $b(u, \tilde{u}) := u \cdot \nabla \tilde{u}$. For any $\ell, k \in \mathbb{Z}^2, m, m' \in \{0, 1\}$ and $U = (u, b) \in H_1 \times H_2$, we introduce
\[
Y_k^m(U) := [F(U), \sigma_k^m] = A^{\alpha, \beta} \sigma_k^m + B(\sigma_k^m, U) + B(U, \sigma_k^m),
\]
\[
J_{k, \ell}^{m, m'}(U) := -[Y_k^m(U), \sigma_\ell^{m'}] = B(\sigma_k^m, \sigma_\ell^{m'}) + B(\sigma_\ell^{m'}, \sigma_k^m)
\]
\[
= \begin{pmatrix}
-\Pi b(e_k^m, e_\ell^{m'}) - \Pi b(e_\ell^{m'}, e_k^m) \\
0
\end{pmatrix} := \begin{pmatrix}
-\Pi J_{k, \ell}^{m, m'}
0
\end{pmatrix}. \tag{4.1}
\]

In fact, $Y_k^m(U)$ and $J_{k, \ell}^{m, m'}(U)$ are devised elaborately by calculation to guarantee that the following two lemmas hold.

Lemma 4.1. For $k, \ell \in \mathbb{Z}^2_+$,
\[
b(e_k^1, e_\ell^1) = \frac{\langle k, \ell^\perp \rangle}{|k||\ell|} \sin(k \cdot x) \cos(\ell \cdot x)(\ell_2, -\ell_1)^T,
\]
\[
b(e\ell^1, e_k^1) = \frac{\ell \cdot k^\perp}{|k||\ell|} \sin(\ell \cdot x) \cos(k \cdot x)(k_2, -k_1)^T,
\]
\[
b(e_k^0, e_\ell^0) = \frac{\langle k, \ell^\perp \rangle}{|k||\ell|} \sin(k \cdot x) \sin(\ell \cdot x)(-\ell_2, \ell_1)^T,
\]
\[
b(e\ell^0, e_k^0) = \frac{\ell \cdot k^\perp}{|k||\ell|} \sin(\ell \cdot x) \sin(k \cdot x)(-k_2, k_1)^T,
\]

and
\[
b(e_k^0, e_\ell^1) = \frac{\langle k, \ell^\perp \rangle}{|k||\ell|} \cos(k \cdot x) \cos(\ell \cdot x)(\ell_2, -\ell_1)^T,
\]
\[
b(e\ell^0, e_k^1) = \frac{\ell \cdot k^\perp}{|k||\ell|} \cos(\ell \cdot x) \cos(k \cdot x)(k_2, -k_1)^T,
\]
\[
b(e_k^0, e_\ell^0) = \frac{\langle k, \ell^\perp \rangle}{|k||\ell|} \cos(k \cdot x) \sin(\ell \cdot x)(-\ell_2, \ell_1)^T,
\]
\[
b(e\ell^0, e_k^0) = \frac{\ell \cdot k^\perp}{|k||\ell|} \cos(\ell \cdot x) \sin(k \cdot x)(-k_2, k_1)^T.
\]

Lemma 4.2. Let $a = \frac{\langle k, \ell^\perp \rangle}{|k||\ell|}$, then for any $k, \ell \in \mathbb{Z}^2_+$,
\[
J_{k, \ell}^{0, 1} = b(e_k^0, e_\ell^1) + b(e\ell^0, e_k^0)
\]
\[
= a \cos((k + \ell) x) (\ell_2 - k_2, -\ell_1 + k_1)^T + a \cos((k - \ell) x) (\ell_2 + k_2, -\ell_1 - k_1)^T,
\]
\[
J_{k, \ell}^{1, 0} = b(e_k^1, e_\ell^0) + b(e\ell^1, e_k^0)
\]
\[
= a \cos((k + \ell) x) (-\ell_2, \ell_1)^T + a \cos((k - \ell) x) (\ell_2, -\ell_1)^T,
\]
\[
J_{k, \ell}^{0, 0} = b(e_k^0, e_\ell^0) + b(e\ell^0, e_k^0)
\]
\[
= a \cos((k + \ell) x) (-\ell_2, \ell_1)^T + a \cos((k - \ell) x) (\ell_2, -\ell_1)^T.
\]
\[
\mathcal{J}_{\ell,k}^{0,1} = b(e_{\ell}^0, e_k^1) + b(e_{\ell}^1, e_k^0)
\]
\[
= -a \cos((k + \ell)x)(k_2 - \ell_2, -k_1 + \ell_1)^T - a \cos((\ell - k)x)(\ell_2 + k_2, -\ell_1 - k_1)^T,
\]
and furthermore
\[
\mathcal{J}_{\ell,k}^{0,1} + \mathcal{J}_{\ell,k}^{0,1} = 2a \cos((k + \ell)x)(\ell_2 - k_2, -\ell_1 + k_1)^T,
\]
\[
\Pi[\mathcal{J}_{\ell,k}^{0,1} + \mathcal{J}_{\ell,k}^{0,1}] = ac \frac{1}{|k + \ell|} \cdot (|\ell|^2 - |k|^2)e_{k+\ell}^0,
\]
\[
\Pi[\mathcal{J}_{\ell,k}^{0,1} - \mathcal{J}_{\ell,k}^{0,1}] = ac -|\ell|^2 + |k|^2 \cdot e_{k-\ell}^0,
\]
\[
\Pi[\mathcal{J}_{\ell,k}^{1,1} + \mathcal{J}_{\ell,k}^{0,0}] = ac \frac{|\ell|^2 - |k|^2}{|k - \ell|} \cdot e_{k-\ell}^1,
\]
\[
\Pi[\mathcal{J}_{\ell,k}^{1,1} - \mathcal{J}_{\ell,k}^{0,0}] = ac \frac{|\ell|^2 - |k|^2}{|k + \ell|} \cdot e_{k+\ell}^1,
\]
where \(c\) is an absolutely non-zero constant independent of \(k, \ell\) and may change from line to line.

**Proof.** Since all of the above can be proved in a similar way by direct calculating, we only give the proof of (4.3).

It is from (4.2) that
\[
\Pi[\mathcal{J}_{\ell,k}^{0,1} + \mathcal{J}_{\ell,k}^{0,1}] = (2a \cos((k + \ell)x)(\ell_2 - k_2, -\ell_1 + k_1)^T, e_{k+\ell}^0)e_{k+\ell}^0
\]
\[
= ac \frac{1}{|k + \ell|} \cdot (|\ell|^2 - |k|^2)e_{k+\ell}^0,
\]
where \(\langle \cdot, \cdot \rangle\) denotes the inner product on \(H_1\), \(c\) is a non-zero constant. \(\square\)

By lemma 4.2 and (4.1), we can generate suitable directions in the \(u\) component.

**Lemma 4.3.** Let \(a = \frac{(k,\ell)}{|k||\ell|}\), then for some absolutely non-zero constant \(c\) which is independent of \(k, \ell\), the following inequalities hold.

\[
J_{\ell,k}^{0,1} + J_{\ell,k}^{0,1} = ac \frac{1}{|k + \ell|} \cdot (|\ell|^2 - |k|^2)\psi_{k+\ell}^0, \quad J_{\ell,k}^{0,1} - J_{\ell,k}^{0,1} = ac \frac{-|\ell|^2 + |k|^2}{|k - \ell|} \cdot \psi_{k-\ell}^0,
\]
\[
J_{\ell,k}^{1,1} + J_{\ell,k}^{0,0} = ac \frac{|\ell|^2 - |k|^2}{|k - \ell|} \cdot \psi_{k-\ell}^1, \quad J_{\ell,k}^{1,1} - J_{\ell,k}^{0,0} = ac \frac{|\ell|^2 - |k|^2}{|k + \ell|} \cdot \psi_{k+\ell}^1.
\]

### 4.2. Covering magnetic direction.

Likewise, we will need the following notations for the \(b\) direction, which are also obtained through the iteration of Lie brackets computation.

\[
\mathcal{Y}^{m}_{k}(U) := [F(U), \psi_{k}^{m}] = A^{\alpha,\beta}\psi_{k}^{m} + B(\psi_{k}^{m}, U) + B(U, \psi_{k}^{m})
\]
\[
\mathcal{Z}^{m,m'}_{k,\ell} := -\mathcal{Y}^{m}_{k}(U), \sigma_{m}^{m'}) = B(\psi_{k}^{m}, \sigma_{m'}^{m'}) + B(\sigma_{m'}^{m'}, \psi_{k}^{m})
\]

\[
= \begin{pmatrix}
0 \\
\Pi[b(e_{\ell}^{m}, e_{k}^{m'}) - b(e_{\ell}^{m'}, e_{k}^{m})]
\end{pmatrix} = \begin{pmatrix}
0 \\
\Pi[Z_{k,\ell}^{m,m'}]
\end{pmatrix},
\]

where \(Z_{k,\ell}^{m,m'} := b(e_{\ell}^{m}, e_{k}^{m'}) - b(e_{\ell}^{m'}, e_{k}^{m})\).

The following lemma is the counterpart of Lemma 4.2.
Lemma 4.4. Denote $a = \frac{(k, \ell^1)}{|k||\ell|}$, then for some absolutely non-zero constant $c$ which is independent of $k, \ell$ (it may change from line to line), the following equalities hold.

\[
\Pi Z_{k, \ell}^0 + \Pi Z_{\ell, k}^0 = ac|k - \ell|e_{k-\ell}^0, \quad \Pi Z_{k, \ell}^0 - \Pi Z_{\ell, k}^0 = ac|k + \ell|e_{k+\ell}^0,
\]

\[
\Pi Z_{k, \ell}^1 + \Pi Z_{\ell, k}^1 = ac|k - \ell|e_{k+\ell}^1, \quad \Pi Z_{k, \ell}^1 - \Pi Z_{\ell, k}^1 = ac|k + \ell|e_{k+\ell}^1.
\]

Proof. By the definition of $Z_{k, \ell}^{m, n}$, we get

\[
Z_{k, \ell}^0 + Z_{\ell, k}^0 = (b(e_k^0, e_\ell^1) - b(e_k^1, e_\ell^0)) + (b(e_k^0, e_\ell^1 - b(e_\ell^1, c_k^0))). \tag{4.5}
\]

By Lemma 4.1, it holds that

\[
(b(e_k^0, e_\ell^1) - b(e_k^1, e_\ell^0)) = \frac{(k, \ell^1)}{|k||\ell|} \cos(k \cdot x) \cos(\ell \cdot x)(\ell_2, -\ell_1)^T - \frac{(k, \ell^1)}{|k||\ell|} \sin(k \cdot x) \sin(\ell \cdot x)(-\ell_2, \ell_1)^T
\]

\[
= (k, \ell^1) \frac{|k||\ell|}{|k||\ell|}(\ell_2, -\ell_1)^T \cos(k - \ell)x. \tag{4.6}
\]

With a similar way, one sees that

\[
(b(e_k^0, e_\ell^1) - b(e_k^1, e_\ell^0)) = \frac{(k, \ell^1)}{|k||\ell|} (-k_2, k_1)^T \cos(k - \ell)x. \tag{4.7}
\]

Combining (4.7) with (4.6), we obtain

\[
Z_{k, \ell}^0 + Z_{\ell, k}^0 = a(\ell_2 - k_2, -\ell_1 + k_1)^T \cos(k - \ell)x.
\]

Therefore,

\[
\Pi Z_{k, \ell}^0 + \Pi Z_{\ell, k}^0 = (a \cos((k - \ell)x)(\ell_2 - k_2, -\ell_1 + k_1)^T, e_{k-\ell}^0)
\]

\[
= a \left( \cos((k - \ell)x)(\ell_2 - k_2, -\ell_1 + k_1)^T, \cos ((k - \ell)x) \frac{k_2 - \ell_2}{|k - \ell|}, \frac{-k_1 + \ell_1}{|k - \ell|} \right)^T e_{k-\ell}^0
\]

\[
= ac |k_2 - \ell_2|^2 + |k_1 - \ell_1|^2 e_{k-\ell}^0 = ac|k - \ell|e_{k-\ell}^0,
\]

where

\[
c = -\int_{-\pi, \pi} \cos^2((k_1 - \ell_1)x_1 + (k_2 - \ell_2)x_2) dx_1 dx_2
\]

\[
= -\int_{-\pi, \pi} \frac{1 + \cos 2((k_1 - \ell_1)x_1 + (k_2 - \ell_2)x_2)}{2} dx_1 dx_2 = -\frac{1}{2}(2\pi)^2.
\]

The proof of other equalities are similar. \qed

Likewise, by Lemma 4.4 and (4.4) we can generate suitable directions in the $b$ component.

Lemma 4.5. Denote $a = \frac{(k, \ell^1)}{|k||\ell|}$, then for some absolutely non-zero constant $c$ which is independent of $k, \ell$ (it may change from line to line), the following equalities hold.

\[
Z_{k, \ell}^0 + Z_{\ell, k}^0 = ac|k - \ell|\sigma_{k-\ell}^0, \quad Z_{k, \ell}^0 - Z_{\ell, k}^0 = ac|k + \ell|\sigma_{k+\ell}^0,
\]

\[
Z_{k, \ell}^1 + Z_{\ell, k}^1 = ac|k - \ell|\sigma_{k-\ell}^1, \quad Z_{k, \ell}^1 - Z_{\ell, k}^1 = ac|k + \ell|\sigma_{k+\ell}^1.
\]
In conclusion, we give an illustration in Figure 1 how the new directions generated from the existing directions via the iterations of the chain of bracket computations. The construction is interesting that in the upper half part $\psi$’s are generated by $\sigma$’s, while in the lower half part $\sigma$’s are generated by $\psi$’s. This antisymmetric relationship is originated from the advective structure in $B$.

\[
\sigma^m_k, k \in \mathbb{Z}_{2n} \rightarrow Y^m_k(U) \rightarrow \sigma^m_{\ell} \rightarrow Y^m_k(U) \rightarrow \psi^m_k, k \in \mathbb{Z}_{2n+1}
\]

**Figure 1.** An illustration of how the new directions generated from the existing directions via the iterations of the chain of bracket computations. In this figure, $m, m' \in \{0, 1\}$, $\ell \in \mathbb{Z}_0$. Solid arrows mean that the new function is generated from a Lie bracket, with the type of bracket indicated above the arrow. Dashed arrows with green color signify that the new element is generated as a linear combination of elements from the previous position. The dotted arrows with red color shows that the process is iterative. The doubled arrow with yellow color ($\Rightarrow$) shows that $k \pm \ell$ is an element belongs to $\mathbb{Z}_{2n+1}$ or $\mathbb{Z}_{2n+2}$ actually.

### 5. Spectral properties of $\mathcal{M}$.

For any $\alpha > 0$, $N \in \mathbb{N}$, we define

\[
S_{\alpha,N} := \{ \phi \in H : \| P_N \phi \|^2 \geq \alpha \| \phi \|^2 \}.
\]

The aim of this section is to prove the following theorem, which gives information on the probability of eigenvectors with sizable projections in the unstable directions to have small eigenvalues. Broadly speaking, this provides us the invertibility of the Malliavin matrix on the space spanned by the unstable directions. Since it is finite dimensional under current circumstances, one can thus formulate a control problem through the Malliavin integration by parts formula to obtain the gradient estimate on the Markov semigroup, which is extremely useful in establishing ergodicity (c.f. Proposition 3).

**Theorem 5.1.** For any $N \geq 1, \alpha \in (0,1]$ and $\eta > 0$, there exists a positive constant $\varepsilon^* = \varepsilon^*(\alpha, \eta, N, T) > 0$, such that, for any $n \geq 0$, and $\varepsilon \in (0, \varepsilon^*]$, there exists a measurable set $\Omega_\varepsilon = \Omega_\varepsilon(\alpha, N, T) \subseteq \Omega$ satisfying

\[
\mathbb{P}(\Omega_\varepsilon^c) \leq r(\varepsilon) \exp(\eta \| U_0 \|^2),
\]

where $r = r(\alpha, \eta, N, T) : (0, \varepsilon^*) \rightarrow (0, \infty)$ is a non-negative, decreasing function with $\lim_{\varepsilon \rightarrow 0} r(\varepsilon) = 0$, and on the set $\Omega_\varepsilon$,

\[
\inf_{\phi \in S_{\alpha,N}} \frac{\langle \mathcal{M}_0 \phi, \phi \rangle}{\| \phi \|^2} \geq \varepsilon.
\]

In order to prove this theorem, we will first introduce a series of quadratic forms $Q_N$ and their lower bounds, next in Subsection 5.1 we introduce or recall some notational conventions and technical tools which will be used frequently. Then we
estimate upper bounds on $Q_N$ in Subsection 5.2. Finally in Subsection 5.3, we complete the proof of Theorem 5.1. To start with, denote

$$
\langle Q_N \phi, \phi \rangle := \sum_{n=0}^{N} \sum_{k \in \mathbb{Z}_{2n}, m \in \{0,1\}} |\langle \phi, \sigma_n^m \rangle|^2 + \sum_{n=0}^{N} \sum_{k \in \mathbb{Z}_{2n+1}, m \in \{0,1\}} |\langle \phi, \psi_n^m \rangle|^2.
$$

Lower bounds on these Quadratic forms are fairly simple since we are merely focusing on $\phi \in S_{\alpha,N}$.

**Proposition 1.** Fix any integer $N \in \mathbb{N}$ and $\alpha \in (0,1]$, 

$$
\langle Q_N \phi, \phi \rangle \geq \frac{\alpha}{2} \| \phi \|^2
$$

holds for every $\phi \in S_{\alpha,N}$.

**Proof.** Its proof is trivial. \qed

### 5.1. Preliminaries.

Denote by $\tilde{U} = U - Q_b W$, then

$$
\begin{align*}
\begin{cases}
\partial_t \tilde{U} = F(U) = F(U + Q_b W), \\
\tilde{U}_0 = U_0,
\end{cases}
\end{align*}
$$

and by expanding $U = \tilde{U} + Q_b W$ we find

$$
Y_k^m(U) = Y_k^m(\tilde{U}) - \sum_{\ell \in \mathbb{Z}_0, m' \in \{0,1\}} \alpha_{\ell}^{m'} |Y_k^m(U), \sigma_{\ell}^{m'}| W^{\ell, m'},
$$

and

$$
Y_k^m(U) = Y_k^m(\tilde{U}) - \sum_{\ell \in \mathbb{Z}_0, m' \in \{0,1\}} \alpha_{\ell}^{m'} |Y_k^m(U), \sigma_{\ell}^{m'}| W^{\ell, m'}.
$$

We introduce for $\alpha \in [0,1], \phi \in H$

$$
N_\alpha(\phi) := \max_{\ell \in \mathbb{Z}_0, m' \in \{0,1\}} \left\{ \| \langle K_{T, \phi}, Y_k^m(\tilde{U}) \rangle \|_{C^\alpha}, \| |\langle K_{T, \phi}, Y_k^m(U), \sigma_{\ell}^{m'} | W^{\ell, m'} | \|_{C^\alpha} \right\},
$$

and

$$
M_\alpha(\phi) := \max_{\ell \in \mathbb{Z}_0, m' \in \{0,1\}} \left\{ \| \langle K_{T, \phi}, Y_k^m(\tilde{U}) \rangle \|_{C^\alpha}, \| |\langle K_{T, \phi}, Y_k^m(U), \psi_{\ell}^{m'} | W^{\ell, m'} | \|_{C^\alpha} \right\},
$$

where for any function $g : [T/2, T] \to \mathbb{R}$, $\| g \|_{C^\alpha}$ is defined by

$$
\| g \|_{C^\alpha} := \sup_{t_1 \neq t_2 \in [T/2, T]} \frac{|g(t_1) - g(t_2)|}{|t_1 - t_2|^\alpha},
$$

and for $\alpha = 0$, $\| g \|_{C^\alpha}$ is defined by

$$
\| g \|_{C^\alpha} := \sup_{t \in [T/2, T]} |g(t)|.
$$

**Lemma 5.2.** For any $p \geq 1, \eta > 0$,

$$
E \left[ \sup_{\phi \in H, \| \phi \|=1} N_0(\phi)^p \right] \leq C(\eta, k, p) \exp (\eta \| U_0 \|^2),
$$

$$
E \left[ \sup_{\phi \in H, \| \phi \|=1} N_1(\phi)^p \right] \leq C(\eta, k, p) \exp (\eta \| U_0 \|^2),
$$

$$
E \left[ \sup_{\phi \in H, \| \phi \|=1} M_0(\phi)^p \right] \leq C(\eta, k, p) \exp (\eta \| U_0 \|^2),
$$

$$
E \left[ \sup_{\phi \in H, \| \phi \|=1} M_1(\phi)^p \right] \leq C(\eta, k, p) \exp (\eta \| U_0 \|^2).
$$

Proof. (5.6) and (5.8) follow directly from Lemma 3.1 and Lemma 3.2. By the expressions of $Y_k^m(U), F(U), Z_{k,t}^{m,m'}$ and Lemma 3.1, there exist $C = C(k,p,\eta)$ and $q = q(p,k)$ such that
\[
\mathbb{E} \sup_{t \in [T/2,T]} ||Y_k^m(\bar{U}), F(U)||^{2p} + \mathbb{E} \sup_{t \in [T/2,T]} ||[F(U), Z_{k,t}^{m,m'}]||^{2p} \leq C \mathbb{E}\left[1 + ||U||_{H^s}^q\right] \leq C \exp(\eta ||U_0||^2/2).
\]
(5.10)

This along with Lemma 3.2 yields
\[
\mathbb{E}\left[||\langle K_{t,T}\phi, Y_k^m(\bar{U}) \rangle||_{C^1}^p\right] \leq C \mathbb{E}\left[||\partial_t \langle K_{t,T}\phi, Y_k^m(\bar{U}) \rangle||^p\right] \leq C\mathbb{E}\left[||\langle K_{t,T}\phi, Y_k^m(\bar{U}) \rangle||^p\right] \leq C \exp(\eta ||U_0||^2/2) \left(\mathbb{E} \sup_{t \in [T/2,T]} ||Y_k^m(\bar{U}), F(U)||^{2p}\right)^{1/2}
\]
(5.11)

Combining (5.10) with Lemma 3.2, one arrives that
\[
\mathbb{E}\left[||\langle K_{t,T}\phi, Y_k^m(U), \sigma_k^m \rangle||_{C^1}^p\right] = \mathbb{E}\left[||\langle K_{t,T}\phi, Z_{k,t}^{m,m'} \rangle||_{C^1}^p\right] \leq C \mathbb{E}\left[||\partial_t \langle K_{t,T}\phi, Z_{k,t}^{m,m'} \rangle||^p\right] \leq C \mathbb{E}\left[||\langle K_{t,T}\phi, [F(U), Z_{k,t}^{m,m'}] \rangle||^p\right] \leq C \exp(\eta ||U_0||^2/2) \left(\mathbb{E} \sup_{t \in [T/2,T]} ||[F(U), Z_{k,t}^{m,m'}]||^{2p}\right)^{1/2}
\]
(5.12)

Immediately (5.7) follows from (5.11) and (5.12). The proof of (5.9) is similar to that of (5.7). \qed

We finish this subsection with citing two technical tools.

**Lemma 5.3** (Földes et al. [8]). Fix $T > 0$, $\alpha \in (0,1]$ and an index set $\mathcal{I}$. Consider a collection of random functions $g_\phi$ taking values in $C^{1,\alpha}([T/2,T])$ and indexed by $\phi \in \mathcal{I}$. Define, for each $\varepsilon > 0$,
\[
\Lambda_{\varepsilon,\alpha} := \bigcup_{\phi \in \mathcal{I}} \Lambda_{\varepsilon,\alpha}^\phi, \text{ where } \Lambda_{\varepsilon,\alpha}^\phi := \left\{ \sup_{t \in [T/2,T]} |g_\phi(t)| \leq \varepsilon \text{ and } \sup_{t \in [T/2,T]} |g_\phi'(t)| > \varepsilon \right\}.
\]
Then, there is $\varepsilon_0 = \varepsilon_0(\alpha,T)$ such that for each $\varepsilon \in (0,\varepsilon_0)$
\[
\mathbb{P}(\Lambda_{\varepsilon,\alpha}) \leq C\varepsilon \mathbb{E}\left(\sup_{\phi \in \mathcal{I}} ||g_\phi||_{C^{1,\alpha}([T/2,T])}^{2/\alpha}\right).
\]

Given any multi-index $\alpha := (\alpha_1, \cdots, \alpha_d) \in N^d$, recall the standard notation $W^\alpha := W_{\alpha_1}^{1} \cdots W_{\alpha_d}^{d}$. Consider the collection $\mathcal{B}_M$ of $M$th degree of ‘Wiener polynomials’ of the form
\[
F = A_0 + \sum_{|\alpha| \leq M} A_\alpha W^\alpha,
\]

**Theorem 5.4** (Hairer-Mattingly [11]). Fix $M,T > 0$. Consider the collection $\mathcal{B}_M$ of $M$th degree of ‘Wiener polynomials’ of the form
\[
F = A_0 + \sum_{|\alpha| \leq M} A_\alpha W^\alpha,
\]
where for each multi-index \( \alpha \), with \( |\alpha| \leq M \), \( A_\alpha : \Omega \times [0, T] \to \mathbb{R} \) is an arbitrary stochastic process. Then for all \( \varepsilon \in (0, 1) \) and \( \beta > 0 \), there exists a measurable set \( \Omega_{\varepsilon,M,\beta} \) with \( \mathbb{P}(\Omega_{\varepsilon,M,\beta}) \leq C \varepsilon \), such that on \( \Omega_{\varepsilon,M,\beta} \) and for every \( F \in \mathcal{B}_M \)

\[
\sup_{t \in [0,T]} |F(t)| < \varepsilon^\beta \Rightarrow \begin{cases} 
\text{either} & \sup_{\alpha \leq M} \sup_{t \in [0,T]} |A_\alpha(t)| \leq \varepsilon^{(\beta-1)M}, \\
\text{or} & \sup_{\alpha \leq M} \sup_{s \neq t \in [0,T]} |A_\alpha(t) - A_\alpha(s)| \geq \varepsilon^{-(\beta-1)(M+1)}. \end{cases}
\]

5.2. Quadratic forms: upper bounds. The purpose of this subsection is to give a proof of the following proposition.

**Proposition 2.** Fix \( T > 0 \), for any \( N \geq 1 \), \( \alpha \in \{0, 1\} \) and \( \eta > 0 \), there are positive constant \( q_1 = q_1(\alpha, N, T, \eta) \), \( q_2 = q_2(\alpha, N, T, \eta) \) such that the following holds. There exists a positive constant \( \varepsilon^* = \varepsilon^*(\alpha, N, T, \eta) > 0 \), such that, for any \( \varepsilon \in (0, \varepsilon^*] \), there exists a measurable set \( \Omega^*_\varepsilon = \Omega^*_{\varepsilon}(\alpha, N, T, \eta) \subseteq \Omega \) and positive constants \( C_1 = C_1(\alpha, N, T, \eta) \), \( C_2 = C_2(\alpha, N, T, \eta) \) such that

\[
\mathbb{P}(\Omega^*_\varepsilon) \leq C_1 \varepsilon^{q_1} \exp(\eta \|U_0\|^2),
\]

and on the set \( \Omega^*_\varepsilon \) one has,

\[
\langle M_{0,T} \phi, \phi \rangle \leq \varepsilon \|\phi\|^2 \Rightarrow \langle Q_N(U) \phi, \phi \rangle \leq C_2 \varepsilon^{q_2} \|\phi\|^2
\]

which is valid for any \( \phi \in \mathcal{S}_{\alpha,N} \).

Roughly speaking, this theorem suggests that the quadratic forms \( Q_N \) are bound to have small eigenvalues on \( \mathcal{S}_{\alpha,N} \) with large probability once the Malliavin matrix \( M_{0,T} \) possesses a small eigenvalue.

Motivated by Section 4, we will adopt an iterative and inductive strategy to prove Proposition 2. To make this more precise, notice that

\[
\langle M_{0,T} \phi, \phi \rangle = \sum_{\ell \in \mathbb{Z}_0, m \in \{0, 1\}} (\alpha^m_{\ell})^2 \int_0^T \langle \sigma^m_{\ell,K_{r,T}} \phi \rangle^2 dr.
\]

Therefore we start from that \( \langle M_{0,T} \phi, \phi \rangle \) is small to deduce that \( \langle \sigma^m_{\ell,K_{r,T}} \phi \rangle \) are small, which is the content of Lemma 5.3. Then by Lie brackets computation as suggested by Figure 1, we estimate progressively that \( \langle Y^m_{\ell}(U), K_{r,T} \phi \rangle \), \( \langle [Y^m_{\ell}(U), \sigma^m_{\ell}], K_{r,T} \phi \rangle \) and \( \langle \psi^m_{\ell,K_{r,T}} \phi \rangle \) are all small, which are the contents of Lemma 5.4, Lemma 5.5 and Lemma 5.6 respectively. We also need to integrate all these results, since they only hold on different large sets, which is the content of Lemma 5.10. Likewise, in the other direction we start from \( \langle \psi^m_{\ell,K_{r,T}} \phi \rangle \) are small to estimate progressively that \( \langle Y^m_{\ell}(U), K_{r,T} \phi \rangle \), \( \langle [Y^m_{\ell}(U), \sigma^m_{\ell}], K_{r,T} \phi \rangle \) and \( \langle \psi^m_{\ell,K_{r,T}} \phi \rangle \) are all small on some large sets, which are the contents of Lemma 5.7, Lemma 5.8 and Lemma 5.9 respectively. Lemma 5.11 serves to integrate all these results. The whole process is iterative and inductive so that we can tackle with successively larger finite dimensional subspace. To be specific, we refer the readers to Figure 2 for an illustration of the arguing structure in this subsection that lead to the proof of Proposition 2.

**Lemma 5.5.** For any \( 0 < \varepsilon < \varepsilon_0(T) \) and every \( \eta > 0 \), there exists a set \( \Omega_{\varepsilon,\mathcal{M}} \) and \( C = C(\eta, T) \) with

\[
\mathbb{P}(\Omega_{\varepsilon,\mathcal{M}}) \leq C \exp\{\eta \|U_0\|^2\} \varepsilon
\]
such that on the set $\Omega_{\epsilon,M}$

$$\langle \mathcal{M}_0, T \phi, \phi \rangle \leq \epsilon \|\phi\|^2 \Rightarrow \sup_{t \in [T/2, T]} \langle \mathcal{K}_{t,T} \phi, \sigma_{\ell}^{m'} \rangle \leq \epsilon^{1/8} \|\phi\|^2$$

(5.13)

for each $\ell \in \mathbb{Z}_0, m', m' \in \{0, 1\}$ and $\phi \in H$.

**Proof.** Notice that

$$\langle \mathcal{M}_0, T \phi, \phi \rangle = \sum_{\ell \in \mathbb{Z}_0, m', m' \in \{0, 1\}} (\alpha_{\ell}^{m'})^2 \int_0^T \langle \sigma_{\ell}^{m'}, \mathcal{K}_{t,T} \phi \rangle^2 \mathrm{d}t.$$

Define the function $g_\phi(\cdot) : [T/2, T] \to \mathbb{R}^+$ as

$$g_\phi(t) := \sum_{\ell \in \mathbb{Z}_0, m', m' \in \{0, 1\}} (\alpha_{\ell}^{m'})^2 \int_0^t \langle \sigma_{\ell}^{m'}, \mathcal{K}_{t,T} \phi \rangle^2 \mathrm{d}t,$$

then

$$g_\phi'(t) = \sum_{\ell \in \mathbb{Z}_0, m', m' \in \{0, 1\}} (\alpha_{\ell}^{m'})^2 \langle \sigma_{\ell}^{m'}, \mathcal{K}_{t,T} \phi \rangle^2,$$

$$g_\phi''(t) = 2 \sum_{\ell \in \mathbb{Z}_0, m', m' \in \{0, 1\}} (\alpha_{\ell}^{m'})^2 \langle \sigma_{\ell}^{m'}, \mathcal{K}_{t,T} \phi \rangle \langle \sigma_{\ell}^{m'}, \partial_t \mathcal{K}_{t,T} \phi \rangle.$$

Let

$$\Omega_{\epsilon,M} = \bigcap_{\phi \in H, \|\phi\|=1} \left\{ \sup_{t \in [T/2, T]} |g_\phi(t)| \geq \epsilon \text{ or } \sup_{t \in [T/2, T]} |g_\phi'(t)| \leq \epsilon^{1/4} \right\}.$$ 

Noticing the definition of $\mathcal{Z}_0$ and

$$\langle \mathcal{K}_{t,T} \phi, \sigma_{\ell}^{m'} \rangle = \langle \mathcal{K}_{t,T} \phi, \sigma_{-\ell}^{m'} \rangle,$$
then on $\Omega_{\varepsilon,M}$, (5.13) holds. By Lemma 5.3, Lemma 3.3 and (3.4), we have

$$P(\Omega_{\varepsilon,M}^c) \leq P \left( \bigcup_{\phi \in H, \|\phi\| = 1} \left\{ \sup_{t \in [T/2,T]} |g_\phi(t)| \leq \varepsilon \text{ and } \sup_{t \in [T/2,T]} |g'_\phi(t)| \geq \varepsilon^{1/4} \right\} \right) \leq C\varepsilon \sum_{t \in Z_0, m' \in \{0,1\}} (\alpha_{\ell}^{m'})^4 E \left[ \sup_{\phi \in H, \|\phi\| = 1} \sup_{t \in [T/2,T]} |(\sigma_{\ell}^{m'}, \mathcal{K}_{t,T}\phi) (\sigma_{\ell}^{m'}, \partial_t \mathcal{K}_{t,T}\phi)|^2 \right] \leq C\varepsilon \exp\{\eta \|U_0\|^2\}.$$

\section*{Lemma 5.6}

Fix a certain $k \in \mathbb{Z}^2, m \in \{0,1\}$. For any $0 < \varepsilon < \varepsilon_0(T)$ and $\eta > 0$, there exists a set $\Omega_{\varepsilon,k}^{1,m}$ and $C = C(k,\eta,T)$ with

$$P((\Omega_{\varepsilon,k}^{1,m})^c) \leq C \exp\{\eta \|U_0\|^2\} \varepsilon,$$

such that on the set $\Omega_{\varepsilon,k}^{1,m}$, it holds that

$$\sup_{t \in [T/2,T]} |(\mathcal{K}_{t,T}\phi, \sigma_{\ell}^{m})| \leq \varepsilon \|\phi\| \Rightarrow \sup_{t \in [T/2,T]} |(\mathcal{K}_{t,T}\phi, Y_k^{m}(U))| \leq \varepsilon^{1/10} \|\phi\|.$$

\section*{Proof}

Define $g_\phi(t) := (\mathcal{K}_{t,T}\phi, \sigma_{\ell}^{m}), \forall t \in [0,T]$ and observe by (2.7) that

$$g_\phi(t) = (\mathcal{K}_{t,T}\phi, [F(U), \sigma_{\ell}^{m}]) = (\mathcal{K}_{t,T}\phi, Y_k^{m}(U)).$$

Let $\alpha = \frac{1}{4}$, and define

$$\Omega_{\varepsilon,k}^{1,m} = \{ \phi \in H, \|\phi\| = 1 \} \left\{ \sup_{t \in [T/2,T]} |g_\phi(t)| \geq \varepsilon \text{ or } \sup_{t \in [T/2,T]} |g'_\phi(t)| \leq \varepsilon^{\alpha/2(1+\alpha)} \right\}.$$

Then on $\Omega_{\varepsilon,k}^{1,m}$, (5.14) holds. By Lemma 5.3 we have

$$P((\Omega_{\varepsilon,k}^{1,m})^c) \leq P \left( \bigcup_{\phi \in H, \|\phi\| = 1} \left\{ \sup_{t \in [T/2,T]} |g_\phi(t)| \leq \varepsilon \text{ and } \sup_{t \in [T/2,T]} |g'_\phi(t)| \geq \varepsilon^{\alpha/2(1+\alpha)} \right\} \right) \leq C\varepsilon \sup_{\phi \in H, \|\phi\| = 1} \|g'_\phi\|_2^{2/\alpha}.$$

Since

$$g'_\phi(t) = (\mathcal{K}_{t,T}\phi, Y_k^{m}(U)) = (\mathcal{K}_{t,T}\phi, Y_k^{m}(\bar{U})) + \sum_{t \in Z_0, m' \in \{0,1\}} \alpha_{\ell}^{m'}(\mathcal{K}_{t,T}\phi, [Y_k^{m}(U), \sigma_{\ell}^{m'}]) W_{t,m'},$$

there follows

$$\|g'_\phi\|_{C^1[T/2,T]} \leq C \sup_{t \in [T/2,T]} |\partial_t (\mathcal{K}_{t,T}\phi, Y_k^{m}(\bar{U}))| + C \sum_{t \in Z_0, m' \in \{0,1\}} \sup_{t \in [T/2,T]} |(\mathcal{K}_{t,T}\phi, [Y_k^{m}(U), \sigma_{\ell}^{m'}])| \cdot |W_{t,m'}(U)| + C \sum_{t \in Z_0, m' \in \{0,1\}} |(\mathcal{K}_{t,T}\phi, [Y_k^{m}(U), \sigma_{\ell}^{m'}])| C^1[T/2,T] \sup_{t \in [T/2,T]} |W_{t,m'}(U)|.$$
Therefore, by Lemma 5.2 one gets
\[ \mathbb{E} \left[ \sup_{\phi \in H, \|\phi\| = 1} \|g_\phi\|^{2/\alpha}_{\mathfrak{C}^\infty[T/2, T]} \right] \leq C(\eta, \alpha) \exp \left( \eta \|U_0\|^2 \right). \]

**Lemma 5.7.** Fix a certain \( k \in \mathbb{Z}_2^s \). For any \( 0 < \varepsilon < \varepsilon_0(T) \) and \( \eta > 0 \), there exists a set \( \Omega^{2,m}_{\varepsilon,k} \) and \( C = C(\eta, T, k) \) with
\[ \mathbb{P}(\Omega^{2,m}_{\varepsilon,k}) \leq C \exp\{\eta\|U_0\|^2\} \varepsilon^{1/9}, \]
such that on the set \( \Omega^{2,m}_{\varepsilon,k} \), it holds
\[ \sup_{t \in [T/2, T]} |\langle K_{t,T}^\phi, Y_k^m(U) \rangle| \leq \varepsilon \|\phi\| \]
implies
\[ \sup_{t \in \mathbb{Z}, m \in \{0, 1\}} \sup_{t \in [T/2, T]} |\alpha_{t}^m| \cdot |\langle K_{t,T}^\phi, [Y_k^m(U), \sigma_{t}^m] \rangle| \leq \varepsilon^{1/3} \|\phi\|. \]

**Proof.** By expanding one finds
\[ \langle K_{t,T}^\phi, Y_k^m(U) \rangle = \langle K_{t,T}^\phi, Y_k^m(U) \rangle - \sum_{t \in \mathbb{Z}, m \in \{0, 1\}} \alpha_{t}^m \langle K_{t,T}^\phi, [Y_k^m(U), \sigma_{t}^m] \rangle W_{t,T}^m. \]

For \( \alpha \in \{0, 1\}, \phi \in H \), we recall that
\[ N_\alpha(\phi) = \max_{t \in \mathbb{Z}, m \in \{0, 1\}} \left\{ \|\langle K_{t,T}^\phi, Y_k^m(U) \rangle\|_{C^\infty}, |\alpha_{t}^m| \right\} \|\langle K_{t,T}^\phi, [Y_k^m(U), \sigma_{t}^m] \rangle\|_{C^\infty}. \]

Then by Theorem 5.4, there exists a set \( \Omega^\#_{\varepsilon} \) such that
\[ \mathbb{P}(\Omega^\#_{\varepsilon}) \leq C \varepsilon, \]
and on \( \Omega^\#_{\varepsilon} \)
\[ \sup_{t \in [T/2, T]} |\langle K_{t,T}^\phi, Y_k^m(U) \rangle| \leq \varepsilon \|\phi\| \Rightarrow \left\{ \begin{array}{l} \text{either } N_0(\phi) \leq \varepsilon^{1/3}, \\
\text{or } N_1(\phi) \geq \varepsilon^{-1/9}. \end{array} \right. \]

Let
\[ \Omega^{2,m}_{\varepsilon,k} := \Omega^\#_{\varepsilon} \cap \bigcap_{\phi \in H, \|\phi\| = 1} \{ N_1(\phi) < \varepsilon^{-1/9} \}. \]

Then this lemma follows from Lemma 5.2, (1.3) and the fact
\[ |\langle K_{t,T}^\phi, [Y_k^m(U), \sigma_{t}^m] \rangle| = |\langle K_{t,T}^\phi, [Y_k^m(U), \sigma_{t}^m] \rangle|. \]

**Lemma 5.8.** For any \( n \in \mathbb{N} \) there exists a constant \( C_n \) such that for any \( k \in \mathbb{Z}_{2n} \),
\[ \sup_{t \in \mathbb{Z}, m \in \{0, 1\}} \sup_{t \in [T/2, T]} |\langle K_{t,T}^\phi, [Y_k^m(U), \sigma_{t}^m] \rangle| \leq C_n \|\phi\| \]
implies
\[ \sup_{t \in \mathbb{Z}, t \in \{k, k^{-1}\}} \sup_{t \in [T/2, T]} |\langle K_{t,T}^\phi, [\psi_k^m, \psi_{k+1}^m] \rangle| \leq C_n \varepsilon \|\phi\| \]
with probability one.

**Proof.** It directly follows from Lemma 4.3 and (4.1).

**Lemma 5.9.** Fix some \( k \in \mathbb{Z}_2^s \), \( m \in \{0, 1\} \). For any \( 0 < \varepsilon < \varepsilon_0(T) \) and \( \eta > 0 \), there exists a set \( \Omega^{3,m}_{\varepsilon,k} \) and \( C = C(\eta, k, T) \) with
\[ \mathbb{P}(\Omega^{3,m}_{\varepsilon,k}) \leq C \exp\{\eta\|U_0\|^2\} \varepsilon, \]
such that on the set $\Omega_{\epsilon,k}^{3,m}$, for each $m \in \{0, 1\}$, it holds

$$\sup_{t \in [T/2,T]} |(\mathcal{K}_{t,T}\phi, \psi_k^m)| \leq \epsilon \|\phi\| \Rightarrow \sup_{t \in [T/2,T]} |(\mathcal{K}_{t,T}\phi, [F(U), \psi_k^m])| \leq \epsilon^{1/10} \|\phi\|. \quad (5.15)$$

**Proof.** Define $g_\phi(t) := \langle \mathcal{K}_{t,T}\phi, \psi_k^m \rangle$ and observe by (2.7) that

$$g_\phi'(t) = \langle \mathcal{K}_{t,T}\phi, [F(U), \psi_k^m] \rangle.$$

Let $\alpha = \frac{1}{4}$, and define

$$\Omega_{\epsilon,k}^{3,m} = \bigcap_{\phi \in H, \|\phi\| = 1} \left\{ \sup_{t \in [T/2,T]} |g_\phi(t)| \geq \epsilon \text{ or } \sup_{t \in [T/2,T]} |g_\phi'(t)| \leq \epsilon^{\alpha/2(1+\alpha)} \right\}.$$

Then on $\Omega_{\epsilon,k}^{3,m}$, (5.15) holds. By Lemma 5.3 we have

$$\mathbb{P}\left( (\Omega_{\epsilon,k}^{3,m})^c \right) \leq C \mathbb{E}\left[ \sup_{\phi \in H, \|\phi\| = 1} \|g_\phi'^{2/\alpha}_{\epsilon}\|_{C^0_{\epsilon}}\right].$$

Since,

$$g_\phi'(t) = \langle \mathcal{K}_{t,T}\phi, \mathcal{Y}_k^m(U) \rangle = \langle \mathcal{K}_{t,T}\phi, \mathcal{Y}_k^m(U) \rangle - \sum_{t \in \mathbb{Z}_0, m' \in \{0, 1\}} \alpha_{m'} \langle \mathcal{K}_{t,T}\phi, [\mathcal{Y}_k^m(U), \sigma_{t,m'}] \rangle W_{t,m'},$$

there follows

$$\|g_\phi'^{2/\alpha}_{\epsilon}\|_{C^0_{\epsilon}} \leq C \sup_{t \in [T/2,T]} |\mathcal{K}_{t,T}\phi, \mathcal{Y}_k^m(U)| + C \sum_{t \in \mathbb{Z}_0, m' \in \{0, 1\}} \sup_{t \in [T/2,T]} |\langle \mathcal{K}_{t,T}\phi, [\mathcal{Y}_k^m(U), \sigma_{t,m'}] \rangle \cdot |W_{t,m'}|_{C^0_{\epsilon}}| + C \sum_{t \in \mathbb{Z}_0, m' \in \{0, 1\}} |\langle \mathcal{K}_{t,T}\phi, [\mathcal{Y}_k^m(U), \sigma_{t,m'}] \rangle|_{C^0_{\epsilon}} \cdot \sup_{t \in [T/2,T]} |W_{t,m'}(t)|.$$

By Lemma 5.2 one gets

$$\mathbb{E}\left[ \sup_{\phi \in H, \|\phi\| = 1} \|g_\phi'^{2/\alpha}_{\epsilon}\|_{C^0_{\epsilon}} \right] \leq C(\eta, \alpha) \exp(\eta \|U_0\|)^2).$$

\[\square\]

**Lemma 5.10.** Fix some $k \in \mathbb{Z}_0^2$, $m \in \{0, 1\}$. For any $0 < \epsilon < \epsilon_0(T)$ and $\eta > 0$, there exists a set $\Omega_{\epsilon,k}^{4,m}$ and $C = C(k, \eta, T)$ with

$$\mathbb{P}(\Omega_{\epsilon,k}^{4,m})^c \leq C \exp\{\eta \|U_0\|^2\} \epsilon,$$

such that on the set $\Omega_{\epsilon,k}^{4,m}$, it holds

$$\sup_{t \in [T/2,T]} |\langle \mathcal{K}_{t,T}\phi, \mathcal{Y}_k^m(U) \rangle| \leq \epsilon \|\phi\|$$

$$\Rightarrow \sup_{t \in \mathbb{Z}_0, m' \in \{0, 1\}} \sup_{t \in [T/2,T]} \|\alpha_{m'} \cdot |\langle \mathcal{K}_{t,T}\phi, [\mathcal{Y}_k^m(U), \sigma_{t,m'}] \rangle| \leq \epsilon^{1/3} \|\phi\|.$$
Proof. By expanding one finds
\[ \langle K_t, T \phi, Y_k^m(U) \rangle = \langle K_t, T \phi, Y_k^m(U) \rangle - \sum_{\ell \in Z, m' \in \{0,1\}} \alpha^{m'}_\ell \langle K_t, T \phi, [Y_k^m(U), \sigma^{m'}_{\ell}] \rangle W^{\ell, m'} . \]

Recall that, for \( \alpha \in \{0,1\}, \phi \in H \)
\[ \mathcal{M}_\alpha(\phi) := \max_{\ell \in Z, m' \in \{0,1\}} \left\{ \| \langle K_t, T \phi, Y_k^m(U) \rangle \|_{C^{\alpha}}, |\alpha^{m'}_\ell| \cdot \| \langle K_t, T \phi, [Y_k^m(U), \sigma^{m'}_{\ell}] \rangle \|_{C^{\alpha}} \right\} . \]

Then by Theorem 5.4, there exists a set \( \Omega^\#_\varepsilon \) such that
\[ \mathbb{P}(\langle Omega^\#_\varepsilon \rangle) \leq C\varepsilon, \]
and on \( \Omega^\#_\varepsilon \)
\[ \sup_{t \in [T/2, T]} \| \langle K_t, T \phi, Y_k^m(U) \rangle \| \leq \varepsilon \| \phi \| \rightarrow \begin{cases} \text{either } \mathcal{M}_0(\phi) \leq \varepsilon^{1/3}, \\ \text{or } \mathcal{M}_1(\phi) \geq \varepsilon^{-1/9}. \end{cases} \]

Let
\[ \Omega^{4,m}_{\varepsilon,k} = \Omega^\#_\varepsilon \cap \cap_{\phi \in H, \| \phi \| = 1} \{ \mathcal{M}_1(\phi) < \varepsilon^{-1/9} \} . \]

Then this lemma follows from Lemma 5.2, (1.3) and the fact
\[ \langle K_t, T \phi, [Y_k^m(U), \psi^{m'}_{\ell}] \rangle = \langle K_t, T \phi, [Y_k^m(U), \psi^{m'}_{\ell}] \rangle . \]

\[ \Box \]

Lemma 5.11. For any \( n \in \mathbb{N} \) there exists a constant \( C = C(n) \) such that for any \( k \in \mathbb{Z}_{2n+1} \)
\[ \sup_{t \in [0, \mu]} \sup_{t \in [T/2, T]} \| \langle K_t, T \phi, Y_k^m(U) \rangle \sigma^{m'}_{\ell} \| \leq \varepsilon \| \phi \| \]
implies
\[ \sup_{t \in [0, \mu]} \sup_{t \in [T/2, T]} \sup_{t \in [T/2, T]} \| \langle K_t, T \phi, \sigma^{m'}_{k + \ell} \rangle \| \leq C\varepsilon \| \phi \| 
\]
with probability one.

Proof. It directly follows from (4.4) and Lemma 4.5. \[ \Box \]

Lemma 5.12. For any \( n \in \mathbb{N} \), and \( q_{2n}, C_{2n} > 0 \), there exist \( p_{2n+1}, q_{2n+1}, C_{2n+1} > 0 \), a set \( \Omega^\varepsilon_{2n} \) and a constant \( C = C(n, \eta, T) \) with
\[ \mathbb{P}(\Omega^c_{2n}) \leq C \exp\{n\|U_0\|^2\} \varepsilon^{q_{2n+1}}, \]
such that on the set \( \Omega^\varepsilon_{2n} \), it holds
\[ \sum_{k \in \mathbb{Z}_{2n}, m \in \{0,1\}} \sup_{t \in [T/2, T]} \| \langle K_t, T \phi, \sigma_k^m \rangle \| \leq C_{2n} \varepsilon^{q_{2n}} \| \phi \| \]
\[ \Rightarrow \sum_{k \in \mathbb{Z}_{2n+1}, m \in \{0,1\}} \sup_{t \in [T/2, T]} \| \langle K_t, T \phi, \psi_k^m \rangle \| \leq C_{2n+1} \varepsilon^{q_{2n+1}} \| \phi \| . \]

Proof. For any \( k \in \mathbb{Z}_{2n}, m \in \{0,1\} \), by Lemma 5.6, there exist \( p'_{2n}, C'_{2n+1}, q'_{2n+1} \) and a set \( \Omega^1_{\varepsilon,k} \) such that on \( \Omega^1_{\varepsilon,k} \)
\[ \sup_{t \in [T/2, T]} \| \langle K_t, T \phi, \sigma_k^m \rangle \| \leq C_{2n} \varepsilon^{q_{2n}} \| \phi \| \Rightarrow \sup_{t \in [T/2, T]} \| \langle K_t, T \phi, Y_k^m(U) \rangle \| \leq C_{2n+1} \varepsilon^{q_{2n+1}} \| \phi \| , \]
and
\[ \mathbb{P}(\langle \Omega^1_{\varepsilon,k} \rangle) \leq C \exp\{n\|U_0\|^2\} \varepsilon^{p'_2} . \]
Next by Lemma 5.7, there exist \( p_{2n}, C_{2n+1}, q_{2n+1} \) and a set \( \Omega_{\varepsilon,k}^{2,m} \) such that on \( \Omega_{\varepsilon,k}^{2,m} \),

\[
\sup_{t \in [T/2, T]} |\langle \mathcal{K}_t \phi, Y_k^m(U) \rangle| \leq C_{2n+1}^\prime \varepsilon_{q_{2n+1}} \| \phi \|
\]

\[
\Rightarrow \sup_{t \in Z_{p,m} \times \{0,1\}} \sup_{t \in [T/2, T]} |\langle \mathcal{K}_t \phi, [Y_k^m(U), \sigma_t^m] \rangle| \leq C_{2n+1}^\prime \varepsilon_{q_{2n+1}} \| \phi \|
\]

and

\[
\mathbb{P}(\Omega_{\varepsilon,K}^{2,m}^c) \leq C \exp\{\eta \|U_0\|^2\} \varepsilon_{p_{2n}}.
\]

Set

\[
\Omega_{\varepsilon,2n} = \bigcap_{k \in Z_{2n}, m \in \{0,1\}} \left[ \Omega_{\varepsilon,k}^{2,m} \cap \Omega_{\varepsilon,k}^{2,m} \right],
\]

then this lemma follows from (4.1), Lemma 5.8 and Lemma 4.3.

**Lemma 5.13.** For any \( n \in \mathbb{N} \), and \( q_{2n+1}, C_{2n+1} > 0 \), there exist \( p_{2n+2}, q_{2n+2}, C_{2n+2} > 0 \), a set \( \Omega_{\varepsilon,2n+1} \) and a constant \( C = C(n, \eta, T) \) with

\[
\mathbb{P}(\Omega_{\varepsilon,2n+1}^c) \leq C \exp\{\eta \|U_0\|^2\} \varepsilon_{p_{2n+2}},
\]

such that on the set \( \Omega_{\varepsilon,2n+1} \), it holds

\[
\sum_{k \in Z_{2n+1}, m \in \{0,1\}} \sup_{t \in [T/2, T]} |\langle \mathcal{K}_t \phi, \psi_k^m \rangle| \leq C_{2n+1} \varepsilon_{q_{2n+1}} \| \phi \|
\]

\[
\Rightarrow \sum_{k \in Z_{2n+2}, m \in \{0,1\}} \sup_{t \in [T/2, T]} |\langle \mathcal{K}_t \phi, \sigma_k^m \rangle| \leq C_{2n+2} \varepsilon_{q_{2n+2}} \| \phi \|
\]

**Proof.** By Lemma 5.9, for any \( m \in \{0,1\}, k \in Z_{2n+1}, \varepsilon > 0 \), there exist set \( \Omega_{\varepsilon,k}^{3,m} \) and \( p_{2n+2}, q_{2n+2}, C_{2n+2} > 0 \) such that

\[
\mathbb{P}(\Omega_{\varepsilon,k}^{3,m}^c) \leq C_{2n+2} \exp\{\eta \|U_0\|^2\} \varepsilon_{p_{2n+2}},
\]

such that on the set \( \Omega_{\varepsilon,k}^{3,m} \), it holds

\[
\sup_{t \in [T/2, T]} |\langle \mathcal{K}_t \phi, \psi_k^m \rangle| \leq \varepsilon_\phi \|
\]

\[
\Rightarrow \sup_{t \in [T/2, T]} |\langle \mathcal{K}_t \phi, Y_k^m(U) \rangle| \leq \varepsilon_{q_{2n+2}} \| \phi \|
\]

Next by Lemma 5.10, for any \( m \in \{0,1\}, k \in Z_{2n+1}, \varepsilon > 0 \), there exist set \( \Omega_{\varepsilon,k}^{4,m} \) and \( p_{2n+2}, q_{2n+2}, C_{2n+2} > 0 \) such that

\[
\mathbb{P}(\Omega_{\varepsilon,k}^{4,m}^c) \leq C_{2n+2} \exp\{\eta \|U_0\|^2\} \varepsilon_{p_{2n+2}},
\]

and on the set \( \Omega_{\varepsilon,k}^{4,m} \), it holds

\[
\sup_{t \in [T/2, T]} |\langle \mathcal{K}_t \phi, Y_k^m(U) \rangle| \leq \varepsilon_{q_{2n+2}} \| \phi \|
\]

\[
\Rightarrow \sup_{t \in Z_{p,m} \times \{0,1\}} \sup_{t \in [T/2, T]} |\langle \mathcal{K}_t \phi, [Y_k^m(U), \sigma_t^m] \rangle| \leq \varepsilon_{q_{2n+2}} \| \phi \|
\]

Set

\[
\Omega_{\varepsilon,2n+1} = \bigcap_{k \in Z_{2n+1}, m \in \{0,1\}} \left[ \Omega_{\varepsilon,k}^{3,m} \cap \Omega_{\varepsilon,k}^{4,m} \right],
\]

then this lemma follows from (4.4), Lemma 5.11 and Lemma 4.5. \(\square\)
Proof of Proposition 2. First, we recall the definition of $\Omega_{\varepsilon,M}$ from Lemma 5.5 and let $C_0 = 1, q_0 = \frac{1}{2}$. Then for any $n \in \mathbb{N}$, just after constants $C_{2n}, q_{2n}$ are fixed, we set $p_{2n+1}, q_{2n+1}, C_{2n+1}, \Omega_{\varepsilon,2n}$ by Lemma 5.12 and $p_{2n+2}, q_{2n+2}, C_{2n+2}, \Omega_{\varepsilon,2n+1}$ by Lemma 5.13. Recursively, for any $n \in \mathbb{N}, \Omega_{\varepsilon,n}, C_n, p_n, q_n$ are well chosen.

Let

$$\Omega_*^\varepsilon = \Omega_{\varepsilon,M} \cap \bigcap_{n=0}^{\infty} \Omega_{\varepsilon,n}.$$ 

Integrating Lemma 5.12 and Lemma 5.13 with Lemma 5.5, we have for some positive constants $p^*_N, q^*_N, C = C(\eta, T, N)$ that

$$\mathbb{P}((\Omega_*^\varepsilon)^c) \leq C\varepsilon^{\eta N} \exp(\eta\|U_0\|^2),$$

and on the set $\Omega_*^\varepsilon$

$$(\mathcal{M}_{0,T} \phi, \phi) \leq \varepsilon\|\phi\|^2 ~ \Rightarrow ~ (Q_N \phi, \phi) \leq C\varepsilon^{q^*_N} \|\phi\|^2,$$

which is valid for any $\phi \in \mathcal{S}_{\alpha,N}$. The proof is finished.

5.3. Proof of Theorem 5.1. Now we are in a position to prove Theorem 5.1.

Proof. Set $\Omega_\varepsilon = \Omega_*^\varepsilon$, which is given by Proposition 2. Let $\varepsilon^*$ be a constant such that for any $\varepsilon \in (0, \varepsilon^*]

$$\frac{\alpha}{2} > C_2 \varepsilon^{q_2}.$$  \hspace{1cm} (5.16)

Again $C_2, q_2$ are constants given by Proposition 2.

First, by Proposition 2, (5.1) holds.

Next on the set $\Omega_\varepsilon = \Omega_*^\varepsilon$, for any $\phi \in \mathcal{S}_{\alpha,N}$ satisfying

$$\langle \mathcal{M}_{0,T} \phi, \phi \rangle < \varepsilon \|\phi\|^2,$$

Proposition 1 and Proposition 2 imply

$$\frac{\alpha}{2} \|\phi\|^2 \leq \langle Q_N \phi, \phi \rangle \leq C_2 \varepsilon^{q_2} \|\phi\|^2,$$

which contradicts with (5.16). Therefore, (5.2) holds on the set $\Omega_\varepsilon$.

Once Proposition 2 is established, one can translate spectral bounds on the Malliavin matrix $\mathcal{M}$ to the estimate on $\nabla P_t \Phi$. This constitutes the main content of the next proposition, and since the Malliavin matrix $\mathcal{M}$ only prove to be nondegenerate on finite dimensional cones, gradient estimates are bound to be in an asymptotic form, which leads Hairer [9] to introduce the celebrated concept of asymptotic strong Feller.

Proposition 3. For some $\gamma_0 > 0$ and every $\eta > 0, U_0 \in H$, the Markov semigroup \{P_t\}_{t \geq 0} defined by (2.8) satisfies the following estimate

$$\|\nabla P_t \Phi(U_0)\| \leq C \exp(\eta\|U_0\|^2) \left( \sqrt{P_t(\|\Phi\|^2)(U_0)} + e^{-\gamma_0 t} \sqrt{P_t(\|\nabla \Phi\|^2)(U_0)} \right)$$

for every $t \geq 0$ and $\Phi \in C_b(H)$, where $C = C(\eta, \gamma_0)$ is independent of $t$ and $\Phi$.

Proof. Ever since [9] the proof of this type of gradient inequality have been attached great importance to and improved all along. Now the method to prove it is more or less standard. Broadly speaking, supplied with moment estimates of $U, J_{s,t} \xi, K_{s,t} \xi, J^\perp_{s,t}(\xi, \xi')$ listed in Section 2, one need to formulate a control problem through the Malliavin integration by parts formula, then do some decay estimates adopting an iterative construction with the aid of Lemma 3.4, Lemma 3.5, Lemma 3.6. We refer the readers to [8, 9, 10, 11] and omit the details.
6. Proof of Theorem 2.1. Our strategy in this section is to apply [10, Theorem 3.4] and [13, Theorem 2.1], separately, to draw the conclusion of mixing rates and central limit theorem. Since it is very straightforward and similar to the proof of [8, Theorem 2.3], we will sketch our arguments. Before carrying them out, we need to introduce a type of 1-Wasserstein distance. Referring to Lemma 3.1 (1) to fix some \( \eta^* > 0 \), then for any \( \eta \in (0, \eta^*], r \in (0, 1] \), define the metric \( \rho_r \) on \( H \) by

\[
\rho_r(U_1, U_2) := \inf_{\gamma} \int_0^1 \exp(\eta r \|\gamma(t)\|) \|\gamma'(t)\| dt,
\]

where the infimum runs over all paths \( \gamma \) such that \( \gamma(0) = U_1 \) and \( \gamma(1) = U_2 \). For brevity of notation, we set \( \rho := \rho_1 \).

**Proof of Theorem 2.1.** (a) Ito’s formula yields that

\[
\|U_t\|^2 - \|U_0\|^2 + 2 \int_0^t \|\mathcal{A}u_s\|^2 ds + 2 \int_0^t \|\mathcal{A}^2 b_s\|^2 ds = \mathcal{E}_0 t + 2 \int_0^t \langle b_s, \mathcal{Q}_b dW_s \rangle,
\]

then it follows that

\[
\frac{1}{T} \mathbb{E} \int_0^T \|U_t\|_{H^1} dt \leq \frac{\|U_0\|^2}{T} + \mathcal{E}_0.
\]

By the classical Krylov-Bogoliubov averaging method, one arrives at that there exists an invariant measure for the semigroup \( P_t \).

(b) Finding a strong type of Lyapunov structure: Let \( \kappa = \frac{3}{2}, r_0 = \frac{1}{3}, \) and \( \eta' = \frac{1}{4} \cdot \frac{7}{2} \cdot e^{-1/2} \), by Lemma 3.2 we have

\[
\|J_t \xi\| \leq C \exp(\eta' \int_0^t \|U_s\|_{H^1} ds).
\]

Then by Lemma 3.1, for \( r \in [r_0, 2\kappa] \) and \( t \in [0, 1] \) we get

\[
\mathbb{E} \left[ \exp(\eta \|U_t\|^2)(1 + \|J_t \xi\|) \right] \leq C \mathbb{E} \left[ \exp(\eta \|U_t\|^2 + \eta' \int_0^t \|U_s\|_{H^1}^2) \right]
\]

\[
\leq C \mathbb{E} \left[ \exp(\eta \|U_t\|^2 + \frac{\eta' r}{2} e^{-1/2} \int_0^t \|U_s\|_{H^1}^2) \right]
\]

\[
\leq C \exp\{\eta r \|U_0\|^2 e^{-1/2}\}.
\]

Therefore, [9, Assumption 4] is verified with \( \kappa = \frac{3}{2}, \eta \in (0, \frac{1}{2} \eta^*], r_0 = \frac{1}{4} \).

\[
V_s(x) = \exp\left(\frac{3}{4} qx^2\right), \forall x \in \mathbb{R}, \quad V^*(x) = \exp\left(\frac{17}{16} qx^2\right), \forall x \in \mathbb{R},
\]

\[
V(x) = \exp(\eta x^2), \forall x \in \mathbb{R}, \quad \xi(t) = e^{-\frac{t}{2}}, \ t \in [0, 1],
\]

which suggests the desirable Lyapunov structure.

(c) Gradient inequality on the Markov semigroup: This is just reemphasizing. By Proposition 3, for some \( \gamma_0 > 0 \) and every \( \eta > 0, U_0 \in H \), the Markov semigroup \( \{P_t\}_{t \geq 0} \) defined by (2.8) satisfies the following estimate

\[
\|\nabla P_t \Phi(U_0)\| \leq C \exp(\eta \|U_0\|^2) \left( P_t(\Phi^2)(U_0) + e^{-\gamma_0 t} P_t(\|\nabla \Phi\|^2)(U_0) \right)
\]

for every \( t \geq 0 \) and \( \Phi \in C_b(H) \), where \( C = C(\eta, \gamma_0) \) is independent of \( t \) and \( \Phi \).

(d) What we need now is to establish a relatively weak form of irreducibility, i.e., for any \( \theta, \varepsilon > 0, r \in (0, 1) \), there exists \( T^* = T^*(\theta, \varepsilon, r) \) such that for any \( T > T^* \),

\[
\inf_{\|U_1\|, \|U_2\|} \sup_{t \in [C(\theta, \varepsilon r_1), C(\theta, \varepsilon r_2)]} \Gamma((U', U''), H \times H : \rho_r(U', U'') < \varepsilon) > 0,
\]

where

\[
\rho_r(U_1, U_2) := \inf_{\gamma} \int_0^1 \exp(\eta r \|\gamma(t)\|) \|\gamma'(t)\| dt.
\]
\[ \delta_U \] is the Dirac measure concentrated on \( U \) and \( \mathcal{C}(\mu_1, \mu_2) \) denotes the set of all coupling measures \( \pi \) on \( H \times H \) such that \( \pi(A \times H) = \mu_1(A) \) and \( \pi(H \times A) = \mu_2(A) \) for every Borel set \( A \subset H \).

In fact, this can be deduced immediately by irreducibility in general sense, which is, for any \( \rho, \varepsilon > 0 \) there exists \( T_* = T_*(\rho, \varepsilon) \geq 0 \) such that
\[
\inf_{\|U_0\| \leq \rho} P_T(U_0, \{U \in H, \|U\| \leq \varepsilon\}) > 0, \quad (6.3)
\]
for any \( T > T_* \). Utilizing the dissipativity of deterministic system and properties of Gaussian distribution, this can be proved following the classical arguments (c.f. [19]).

It is common sense that (a)-(d) implicates there exits a unique invariant measure \( \mu_* \) for \( P_t \).

For every \( \Phi \in \mathcal{O}_\eta \), one can show that
\[
\int \Phi(U) d\mu_*(U) \leq C\|\Phi\|_\eta,
\]
for some constant \( C \) dependent on \( \eta \). From this, Applying [10, Theorem 3.4] yields that the unique invariant measure \( \mu_* \) is exponentially mixing.

(e) To apply [13, Theorem 2.1], the following inequality is critical
\[
\int [\rho(0, U)]^3 P_t(U_0, dU) \leq C \exp (\eta^* \|U_0\|^2),
\]
where the constant \( C \) is independent of \( U_0 \) and \( t \geq 0 \). With the definition of \( \rho \), this can be easily established from Lemma 3.1. The proof is finished.

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