Abstract

In this work we study the Lie group analysis of the equation $u_t + g(u) u_x + \lambda h(u) = 0$ which is one of the generalization of the classical invicid Burgers equations ($\lambda = 0$). Seven inequivalent classes of this generalized equation were classified and many exact and transformed solutions were obtained for each class.

1 Introduction

Damping can be regarded as an effect that tends to stabilize or reduce other effects in a given system such as friction in a mechanical system. Damping devices are used in almost all the systems that we know, thus the importance of having an equation that models appropriate damping systems cannot be overstated. In this paper, we shall use Lie symmetry to perform analysis on a generalized invicid Burgers’ equation with damping in the form

$$u_t + g(u) u_x + \lambda h(u) = 0$$

(1)

where $\lambda \neq 0$ is a constant and $g(u)$ and $h(u)$ are smooth functions of $u = u(x,t)$ in the domain of definition. Numerous applicable cases of equation (1) are available in the literature, for instance, researchers have used it to model the Gunn effect in semi conductors, rotating thin liquid films, chloride concentration in the kidney and flow of petroleum in underground reservoirs, see [7] for all the references. If $h(u) = O(u^{\alpha})$, $\alpha > 0$, $0 < u << 1$, Murray (see [6]) shows that a finite initial disturbance zero outside a finite range in $x$ decays under certain conditions and for each condition, the asymptotic speed of propagation of the discontinuity was given together with its role in the decay process. Also for $\lambda = 1$, $g = u$ and $h = u(u - 1)$, equation (1) becomes the limiting case of the Burgers’-Fishers equation

$$u_t + uu_x + u(u - 1) = \frac{\delta}{2} u_{xx}$$
which has been used to model many physical phenomenon in Mathematical Biology and Ge-
netics.

Symmetry analysis based on local transformation group is one of the most powerful and prolific methods used for solving nonlinear partial differential equations (PDEs). Its application to the study of PDEs was laid down by Sophus Lie, a Norwegian Mathematician, in the later half of the nineteenth century. A symmetry of a system of differential equations is a transformation that maps any solution to another solution of the system. Such transformations are groups that depend on continuous parameters and consist of either point transformations, acting on the systems’ space of independent and dependent variables, or, more generally, contact transformations, acting on the space of independent and dependent variables as well as on all first derivatives of the dependent variables. Lie showed that the algebra of all vector fields (infinitesimal generators) that leave a given system of PDEs invariant could be found by solving over-determined (or under-determined) auxiliary system of linear homogeneous PDEs, known as the determining equations of the group. The calculations involved in obtaining these determining equations and their possible solutions are cumbersome, hence researchers have written codes for implementing Lie’s algorithms in various packages for symbolic computations like Mathematica, Maple, etc. Though these softwares help in circumventing the tedious tasks encountered in obtaining similarity solutions, they fall short of providing any tangible results when the underlying PDE has an arbitrary function like that in (1). The key feature of a Lie group, which makes it very useful, is the parametric representation of smooth functions on a continuous open interval in $\epsilon$; the group parameter. This ensures that the mapping is differentiable and invertible and that the mapping functions can be expanded in a Taylor series about any value of $\epsilon$. For detail ramifications on the subject, see the monographs [2]-[5].

This work is organized as follows. In section two, we carry out the group analysis of equation (1) and obtain seven inequivalent classes where both $g(u)$ and $h(u)$ are restricted to order of powers of $u$. In section three, we present theorems on transformed solutions for these classes and show how they can be used to generate non-trivial many-parameters invariant solutions of equation (1) from the trivial solution $u(x,t) = 0$. Section four deals with invariants solutions,while conclusion is presented in section five.

2 Group Classification

The generalized inviscid Burgers’ equation with damping (1) is said to be invariant under the first order prolonged symmetry operator $X^{(1)} = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t}$ if and only if

$$X^{(1)} [u_t + g(u)u_x + \lambda h(u)] |_{(1)} = 0$$

(2)

where

$$\eta^x = \eta_x + (\eta_u - \xi_x)u_x - \tau xu_t - \xi u^2 u_x - \tau u_x u_t,$$

$$\eta^t = \eta_t - \xi u_x + (\eta_u - \tau_t)u_t - \xi u_x u_t - \tau u^2 t.$$

Expanding equation (2) leads to under-estimated system of partial differential equations in $\xi(x,t,u)$, $\tau(x,t,u)$, and $\phi(x,t,u)$ which are the coefficients of symmetry generator to be
determined. Due to this predicament, group classifications of equation \([2]\) is very difficult, if not impossible, to achieve. To circumvent this, we shall assume \(\xi_u = \tau_u = 0\) which, when applied in the expansion of equation \([2]\), results in the following equations

\[
\begin{align*}
\lambda h' (u) \phi + \lambda h (u) \tau_t + \lambda g (u) h (u) \tau_x - \lambda h (u) \phi_u + \phi_t + g (u) \phi_x &= 0, \\
g' (u) \phi - \xi_t - g (u) \xi_x + g (u) \tau_t + g^2 (u) \tau_x &= 0.
\end{align*}
\]

Differentiating equation \([4]\) twice and re-arranging, we get

\[
\phi = \frac{1}{g'(u)} \left( g^2 (u) A (x, t) + g (u) B (x, t) + C (x, t) \right)
\]

subject to the consistency conditions

\[
\begin{align*}
\tau_x &= -A (x, t), \\
\xi_x - \tau_t &= B (x, t), \\
\xi_t &= C (x, t),
\end{align*}
\]

where \(A (x, t), B (x, t),\) and \(C (x, t)\) are arbitrary functions of \(x\) and \(t\). Using equations \([5]\) – \([8]\) into \([3]\) leave us with a very complex equation which is absolutely difficult to solve if \(g\) and \(h\) are left as arbitrary functions of \(u (x, t)\). But much can be achieved if both \(g\) and \(h\) are considered as functions of order of \(u\) which lead us to the following results.

**Theorem 1** Let \(g (u) = u^k, h (u) = u^m\) where \(k\) and \(m\) are real numbers such that \(k \neq m\) and both not equal to zero. Then we have the following exclusive (inequivalent) cases for \(k\).

**Case 1.** For \(k \neq \pm (m - 1), k \neq \pm \frac{m - 1}{2}, k \neq \frac{m - 1}{3}\) and \(m \neq 1\), equation \([7]\) admits three dimensional symmetry algebra spanned by the closed vector fields

\[
M_1 = \partial_x, \quad M_2 = \partial_t, \quad M_3 = \frac{k - m + 1}{k} x \partial_x + \frac{1 - m}{k} t \partial_t + \frac{1}{k} u \partial_u.
\]

**Case 2.** For \(k = m - 1 (m \neq 1)\), equation \([7]\) admits eight parameters group of projective transformations spanned by the closed vector fields \(M_1, M_2,\)

\[
\begin{align*}
M_3 &= t \partial_x + \lambda (m - 1) t^2 \partial_t + \left( \frac{u^2 - m}{m - 1} - 2 \lambda t \right) \partial_u, \\
M_4 &= \left[ t \partial_x + \left( \frac{u^2 - m}{m - 1} - \lambda t \right) \partial_u \right] e^{-\lambda (m - 1) x}, \\
M_5 &= \left( \frac{-1}{\lambda (m - 1)} \partial_t + \frac{u}{m - 1} \partial_u \right) e^{\lambda (m - 1) x}, \\
M_6 &= -t \partial_t + \frac{u}{m - 1} \partial_u, \\
M_7 &= \left( \frac{-1}{\lambda (m - 1)} \partial_x + \frac{u}{m - 1} \partial_u \right) e^{-\lambda (m - 1) x}, \\
M_8 &= (\partial_x + \lambda (m - 1) t \partial_t - \lambda^2 (m - 1) tu^m \partial_u) e^{\lambda (m - 1) x}.
\end{align*}
\]

**Case 3.** For \(k = 1 - m (m \neq 1)\), equation \([7]\) admits eight parameters group of projective
transformations spanned by the vector fields $M_1, M_2,$

\[ M_3 = \left(-x^2 + \frac{\lambda^2 (1 - m)^2}{4} t^4\right) \partial_x - \left(x t + \frac{\lambda (1 - m)}{2} t^3\right) \partial_t \]
\[ + \frac{1}{1 - m} \left[tu^{2-m} + \left(\frac{3}{2} \lambda (1 - m) t^2 - x\right) u + \lambda^2 (1 - m)^2 t^4 u^m\right] \partial_u, \]

\[ M_4 = \left(3\lambda (1 - m) xt - \frac{\lambda^2 (1 - m)^2}{2} t^3\right) \partial_x + \left(\frac{3}{2} \lambda (1 - m) t^2 - x\right) \partial_t \]
\[ + \frac{1}{1 - m} \left[u^{2-m} + \left(3\lambda (1 - m) x - \frac{3}{2} \lambda^2 (1 - m)^2 t^2\right) u^m\right] \partial_u, \]

\[ M_5 = \left(\frac{\lambda (1 - m)}{2} t^3 - xt\right) \partial_x - t^2 \partial_t + \frac{1}{1 - m} \left[tu + \left(\frac{3}{2} \lambda (1 - m) t^2 - x\right) u^m\right] \partial_u, \]

\[ M_6 = t \partial_x + \frac{u^m}{1 - m} \partial_u, \quad M_7 = \left(x + \frac{\lambda (1 - m)}{2} t^2\right) \partial_x + \frac{1}{1 - m} (u + \lambda (1 - m) tu^m) \partial_u, \]

\[ M_8 = \left(x - \frac{\lambda (1 - m)}{2} t^2\right) \partial_x + t \partial_t - \lambda tu^m \partial_u. \]

**Case 4.** For $k = \frac{m - 1}{2} (m \neq 1),$ equation (1) admits eight parameters group of projective transformations spanned by the vector field $M_1, M_2,$

\[ M_3 = \left(x t - \frac{\lambda (m - 1)}{4} x^3\right) \partial_x + \left(t^2 + \frac{\lambda^2 (m - 1)^2}{16} x^4\right) \partial_t \]
\[ + \frac{2}{m - 1} \left[\frac{\lambda^2 (m - 1)^2}{4} x^3 u^{m+1} - \left(t + \frac{3}{4} \lambda (m - 1) x^2\right) u + t u^{\frac{3-m}{2}}\right] \partial_u, \]

\[ M_4 = \left(t - \frac{3}{4} \lambda (m - 1) x^2\right) \partial_x - \frac{\lambda^2 (m - 1)^2}{4} x^3 \partial_t \]
\[ + \frac{2}{m - 1} \left[\frac{3}{4} \lambda^2 (m - 1)^2 x^2 u^{m+1} - \frac{3}{2} \lambda (m - 1) xu + u^{\frac{3-m}{2}}\right] \partial_u, \]

\[ M_5 = -x^2 \partial_x - \left(x t - \frac{\lambda}{4} (m - 1) x^3\right) \partial_t + \frac{2}{m - 1} \left[\left(t + \frac{3}{4} \lambda (m - 1) x^2\right) u^{m+1} - xu\right] \partial_u, \]

\[ M_6 = x \partial_x + \left(t + \frac{\lambda}{4} (m - 1) x^2\right) \partial_t - \lambda xu^{m+1} \partial_u, \]

\[ M_7 = x \partial_x + \frac{\lambda}{2} (m - 1) x^2 \partial_t + \frac{2}{m - 1} \left(u - \lambda (m - 1) xu^{m+1}\right) \partial_u, \quad M_8 = -x \partial_x + \frac{2}{m - 1} u^{m+1} \partial_u. \]

**Case 5.** For $k = \frac{1-m}{2} (m \neq 1),$ equation (1) admits three parameters group of projective transformations spanned by the closed vector field

\[ M_1, M_2, M_3 = \frac{3}{2} x \partial_x + t \partial_t + \frac{1}{1 - m} u \partial_u. \]
Case 6. For \( k = \frac{m-1}{3} (m \neq 1) \), equation (1) admits three parameters group of projective transformations spanned by the closed vector field

\[
M_1, \ M_2, \ M_3 = -2x \partial_x - 3t \partial_t + \frac{3}{m-1} u \partial_u.
\]

Case 7. For \( m = 1 (k \neq 0) \), equation (1) admits eight parameters group of projective transformations spanned by the closed vector field \( M_1, M_2, \ldots, M_8 \).

\[
M_3 = \lambda k x^2 \partial_x - x \partial_t + \frac{1}{k} (u^{k+1} + 2\lambda k x u) \partial_u, \quad M_4 = \left[-x \partial_t + \frac{1}{k} (u^{k+1} + \lambda k x u) \partial_u\right] e^{\lambda k t},
\]

\[
M_5 = \frac{1}{k} \left(\frac{-1}{\lambda} \partial_x + u^{1-k} \partial_u\right) e^{-\lambda k t}, \quad M_6 = x \partial_x + \frac{u}{k} \partial_u,
\]

\[
M_7 = \frac{1}{k} \left(\frac{-1}{\lambda} \partial_t + u \partial_u\right) e^{\lambda k t}, \quad M_8 = \lambda k \left(-x \partial_x + \frac{1}{\lambda k} \partial_t + \lambda x u^{1-k} \partial_u\right) e^{-\lambda k t}.
\]

Proof. Using \( g = u^k, h = u^m \) and equation (5) in (3) and simplifying we get

\[
\frac{1}{k} A_x u^{2k+1} + \lambda \left[\frac{1}{k} (m-k-1) A + \tau_x\right] u^{k+m} + \lambda \left[\frac{1}{k} (m-1) B + \tau_t\right] u^m
\]

\[
+ \frac{\lambda}{k} (m+k-1) C u^{m-k} + \frac{1}{k} (A_t + B_x) u^{k+1} + \frac{1}{k} (B_t + C_x) u + \frac{1}{k} C_t u^{1-k} = 0.
\]

(9)

Case 1. Since all the coefficient in (9) are independent of \( u \), we have \( \frac{1}{k} (m-k-1) A + \tau_x = 0 \) which implies \( A = 0 \) using equation (6), thus \( \tau_x = 0 \) and \( B_x = 0 \). Also \( C = 0 \) implies \( \xi_t = 0 \) and \( B_t = 0 \) and so \( B = \alpha_3 \), a constant. Therefore, the required infinitesimal transformations are

\[
\xi = \frac{k-m+1}{k} \alpha_3 x + \alpha_1, \quad \tau = \frac{1-m}{k} \alpha_3 t + \alpha_2, \quad \phi = \frac{1}{k} \alpha_3 u
\]

(10)

which proves the result.

Case 2. Under this case, equation (2) becomes

\[
\frac{1}{m-1} (A_x + \lambda (m-1) \tau_x) u^{2m-1} + \frac{1}{m-1} [\lambda (m-1) (B + \tau_t) + A_t + B_x] u^m
\]

\[
+ \frac{1}{m-1} (2\lambda (m-1) C + B_t + C_x) u + \frac{1}{m-1} C_t u^{2-m} = 0,
\]

which holds if and only if

\[
C (x,t) = C (x), \quad A_x - \lambda (m-1) A = 0, \quad \lambda (m-1) \xi_x + A_t + B_x = 0, \quad 2\lambda (m-1) C + B_t + C_x = 0.
\]
From (8), (12) and (13), we have

\[
B = - (C'(x) + 2\lambda (m - 1) C) t + a(x),
A = \frac{1}{2} [C''(x) + \lambda (m - 1) C'(x)] t^2 - [\lambda (m - 1) b'(x) + a'(x)] t + d(x),
\]

where \(a(x), b(x)\) and \(d(x)\) are arbitrary functions. Consistency criterion for \(A\) in equations (11) results into

\[
C''(x) - \lambda^2 (m - 1)^2 C'(x) = 0,
\lambda^2 (m - 1)^2 b'(x) - \lambda (m - 1) b''(x) + \lambda (m - 1) a'(x) - a''(x) = 0,
d'(x) - \lambda (m - 1) d(x) = 0.
\]

Using (14) with (6) and (7), and solving the resulting equations that follow, we get the infinitesimal transformations to be

\[
\xi = \left(\alpha_1 + \alpha_2 e^{-\lambda(m-1)x}\right) t - \frac{\alpha_5}{\lambda(m-1)} e^{-\lambda(m-1)x} + \alpha_7 e^{\lambda(m-1)x} + \alpha_6,
\tau = \lambda(m-1) \alpha_1 t^2 - \frac{\alpha_3}{\lambda(m-1)} e^{\lambda(m-1)x} + (-\alpha_4 + \lambda(m-1) \alpha_7 e^{\lambda(m-1)x}) t + \alpha_8,
\phi = \frac{1}{m-1} \left[ \left(\alpha_3 - \lambda^2 (m-1)^2 \alpha_7 t\right) u_m e^{\lambda(m-1)x} + \left(\alpha_1 + \alpha_2 e^{-\lambda(m-1)x}\right) u^{2-m} \right].
\]

The Lie algebras are obtained from these infinitesimals. The closure of these algebras is shown in the following commutator table where \(a = \lambda(m-1)\) and \(V = aM_6 - 2M_1\).

|   | \(M_1\) | \(M_2\) | \(M_3\) | \(M_4\) | \(M_5\) | \(M_6\) | \(M_7\) | \(M_8\) |
|---|--------|--------|--------|--------|--------|--------|--------|--------|
| 1 | 0      | 0      | 0      | -aM_4 | aM_5   | 0      | -aM_7 | aM_8   |
| 2 | 0      | 0      | M_1 - 2aM_6 | -aM_7 | 0      | -M_2  | 0      | -a^2M_5 |
| 3 | 0      | -M_1 + 2aM_6 | 0      | 0      | M_5 - aM_1 | M_3 | M_4 | aM_3 |
| 4 | aM_5  | aM_7   | 0      | 0      | -M_5 | -M_5 | aM_3 | M_8 - aM_8 |
| 5 | 0      | M_2    | -M_3  | -M_4 | M_5  | 0      | 0      | V      |
| 6 | aM_7  | 0      | -M_4  | 0      | M_8  | 0      | 0      | -V     |
| 7 | 0      | aM_8   | 0      | -aM_3 | 0      | 0      | -V     |
| 8 | 0      | 0      | -aM_3 | 0      | -M_3 | 0      | 0      | 0      |

**Case 7.** We have from (9),

\[
A = A(t),
A'(t) - 2\lambda k A(t) + B_x = 0, \quad \lambda k \tau_t + B_t + C_x = 0, \quad \lambda k C + C_t = 0.
\]
Equations (6), (15) and (16) respectively imply
\[ \tau = -A(t)x + p(t), \]
\[ B = (2\lambda kA(t) - A'(t))x + q(t), \]
\[ C = \frac{1}{2}(A''(t) - \lambda kA'(t))x^2 - (\lambda kp'(t) + q'(t))x + r(t), \]
where \( p(t), q(t) \) and \( r(t) \) are arbitrary functions. Hence equation (17) gives
\[ A(t) = \alpha_1 + \alpha_2 e^{\lambda kt} + \alpha e^{-\lambda kt}, \]
\[ \lambda k (p''(t) + \lambda kp'(t)) + q''(t) + \lambda kq'(t) = 0, \]
\[ r(t) = \alpha_3 e^{-\lambda kt}. \]

Equations (7), (8), (18) and their stability criteria lead to
\[ \alpha = 0, \]
\[ p(t) = \alpha_6 + \alpha_7 e^{-\lambda kt} - \frac{\alpha_3}{\lambda k} e^{\lambda kt} \]
\[ q(t) = \alpha_4 + \alpha_5 e^{\lambda kt} \]
\[ B(x,t) = (2\lambda k\alpha_1 + \lambda k\alpha_2 e^{\lambda kt})x + \alpha_4 + \alpha_5 e^{\lambda kt} \]
\[ C(x,t) = \alpha_3 e^{-\lambda kt} + (\lambda k)^2 \alpha_7 e^{-\lambda kt}x \]

Thus, the coefficients of the symmetry generator are
\[ \xi = \lambda k\alpha_1 x^2 + (\alpha_4 - \lambda k\alpha_7 e^{-\lambda kt})x - \frac{\alpha_3}{\lambda k} e^{\lambda kt} + \alpha_8, \]
\[ \tau = -\left(\alpha_1 + \alpha_2 e^{\lambda kt}\right)x - \frac{\alpha_5}{\lambda k} e^{\lambda kt} + \alpha_6 + \alpha_7 e^{-\lambda kt}, \]
\[ \phi = \frac{1}{k} \left[ (\alpha_1 + \alpha_2 e^{\lambda kt}) u^{k+1} + (\lambda k (2\alpha_1 + \alpha_2 e^{\lambda kt})x + \alpha_4 + \alpha_5 e^{\lambda kt}) u \right]. \]

The Lie algebras are obtained from the infinitesimals above and their closure is shown in the following commutator table, \( N = M_2 - 2\lambda k M_6. \)

|     | \( M_1 \) | \( M_2 \) | \( M_3 \) | \( M_4 \) | \( M_5 \) | \( M_6 \) | \( M_7 \) | \( M_8 \) |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( M_1 \) | 0         | 0         | \(-N\)    | \(\lambda k M_7\) | 0         | \( M_1 \) | 0         | \(\lambda^2 k^2 M_5\) |
| \( M_2 \) | 0         | 0         | 0         | \(\lambda k M_4\) | \(-\lambda k M_5\) | 0         | \(\lambda k M_7\) | \(-\lambda k M_8\) |
| \( M_3 \) | \( N \)   | 0         | 0         | 0         | \(-M_8\) | \(-M_3\) | \(-M_4\) | 0         |
| \( M_4 \) | \(-\lambda k M_7\) | \(-\lambda k M_4\) | 0         | 0         | \(-M_8\) | \(-M_3\) | \(-M_4\) | 0         |
| \( M_5 \) | 0         | \(\lambda k M_5\) | \( M_8 \) | \(\frac{4M_2}{M_k}\) + \( M_6 \) | 0         | \( M_5 \) | \(\frac{M_1}{M_k}\) | 0         |
| \( M_6 \) | \(-M_1\) | 0         | \( M_3 \) | \( M_4 \) | \(-M_5\) | 0         | 0         | 0         |
| \( M_7 \) | 0         | \(-\lambda k M_5\) | \( M_4 \) | 0         | \(-\frac{M_1}{M_k}\) | 0         | 0         | \(2M_2 - \lambda k M_6\) |
| \( M_8 \) | \(-\lambda^2 k^2 M_5\) | \(\lambda k M_5\) | 0         | \(\lambda k M_3\) | 0         | \(-2M_2 + \lambda k M_6\) | 0         | 0         |

Cases 5 and 6 can be obtained by substituting respective values of \( k \) in (10) while the remaining cases can be proved in a similar way as did in case two.
3 Group Transformations

Each of the infinitesimal generators $M_i$ in theorem □ above can be used to obtain corresponding one-parameter ($\varepsilon$) Lie group of point transformations $G_i$ through exponentiation. To achieve this, we solved the following system of differential equations

$$\frac{d\tilde{x}_k(\varepsilon)}{d\varepsilon} = \xi_k(\tilde{x}_k(\varepsilon), \tilde{u}(\varepsilon)), $$

$$\frac{d\tilde{u}(\varepsilon)}{d\varepsilon} = \phi(\tilde{x}_k(\varepsilon), \tilde{u}(\varepsilon)),$$

subject to the initial conditions

$$(\tilde{x}_k, \tilde{u})|_{\varepsilon=0} = (x_k, u)$$

for all $X_i : i = 1, 2, ..., 8$ and obtained the following (in addition to $G_1 : (x, t, u) \mapsto (x + \varepsilon, t, u)$ and $G_2 : (x, t, u) \mapsto (x + t + \varepsilon, u)$):

Case 1.

$G_3 : (x, t, u) \mapsto \left(e^{\frac{k-m+1}{k}} x, e^{\frac{1-m}{k}} t, e^{\frac{1}{k}} u\right).$

Case 2 ($a = \lambda (m-1)$).

$G_3 : (x, t, u) \mapsto \left(x - \ln \sqrt{1-a\varepsilon t}, \frac{t}{1-a\varepsilon}, \frac{m-1}{2} \sqrt{\varepsilon (1-a\varepsilon t) + (1-a\varepsilon t)^2 u^{m-1}} \right),$

$G_4 : (x, t, u) \mapsto \left(\ln \sqrt{a\varepsilon t + e^{ax}}, t, \frac{m-1}{2} \sqrt{e^{ax}} \varepsilon u^{m-1} \right),$

$G_5 : (x, t, u) \mapsto \left(x, t - \frac{e^{ax}}{a}, \frac{1-m}{a} e^{ax} + u^{1-m} \right),$

$G_6 : (x, t, u) \mapsto \left(x, e^{-\varepsilon t}, e^{\frac{1}{m-1}} u \right),$

$G_7 : (x, t, u) \mapsto \left(\ln \sqrt{-a\varepsilon + e^{ax}}, t, e^{\lambda x} \sqrt{-a\varepsilon + e^{ax}} \right),$

$G_8 : (x, t, u) \mapsto \left(-\ln \sqrt{-a\varepsilon + e^{ax}} + e^{ax}, t, \frac{1}{1-a\varepsilon e^{ax}}, \frac{1-m}{a\varepsilon e^{ax}} \sqrt{e^{ax}} u^{1-m} \right).$

Case 3 ($a = \lambda (1-m)$).

$G_6 : (x, t, u) \mapsto \left(x + \varepsilon t, t, \frac{1-m}{a\varepsilon} \sqrt{\varepsilon + u^{1-m}} \right),$

$G_7 : (x, t, u) \mapsto \left(\left(x + \frac{a t^2}{2} \right) e^{\varepsilon} - \frac{a t^2}{2}, t, e^{ax} \left(u^{1-m} + 2at - 2at \right) \right),$

$G_8 : (x, t, u) \mapsto \left(\left[x + \frac{a t^2}{2} (1 - e^{\varepsilon}) \right] e^{\varepsilon}, t, \frac{1}{1-a\varepsilon e^{ax}} u^{1-m} \right).$

Case 4 ($a = \frac{\lambda(m-1)}{2}$).

$G_6 : (x, t, u) \mapsto \left(e^{\varepsilon} x, t + \frac{a}{2} \left(e^{\varepsilon} - 1 \right) x^2 \right) e^{\varepsilon} \left[a \left(e^{\varepsilon} - 1 \right) x + u^{1-m} \frac{2}{1-m} \right],$

$G_7 : (x, t, u) \mapsto \left(e^{\varepsilon} x + \frac{a}{2} \left(e^{2\varepsilon} - 1 \right) x^2, \left[a \left(e^{2\varepsilon} - 1 \right) x + u^{1-m} \frac{2}{1-m} \right] e^{-\varepsilon} \right),$

$G_8 : (x, t, u) \mapsto \left(e^{\varepsilon} x, t, \left(-\varepsilon + u^{1-m} \frac{2}{1-m} \right) \right).$
Case 5.
\[
G_3: (x, t, u) \mapsto \left( e^{\frac{3}{2}\varepsilon} x, e^{\varepsilon} t, e^{\frac{3}{2}m^{\varepsilon}} u \right).
\]

Case 6.
\[
G_3: (x, t, u) \mapsto \left( e^{-2\varepsilon} x, e^{-3\varepsilon} t, e^{-\frac{3}{2}m^{\varepsilon}} u \right).
\]

Case 7.
\[
G_3: (x, t, u) \mapsto \left( \frac{x}{1-\lambda e kx}, t - \frac{\ln(1-\lambda e kx)}{\lambda k}, \frac{u}{\sqrt{1-\lambda e kx - \varepsilon u}} \right),
\]
\[
G_4: (x, t, u) \mapsto \left( x, -\frac{\ln(\lambda e kx + e^{-\lambda k t})}{\lambda k}, \frac{k}{1-\varepsilon e^{\lambda k t} u} \right),
\]
\[
G_5: (x, t, u) \mapsto \left( x - \frac{\varepsilon}{\lambda k} e^{-\lambda k t}, t, \frac{\sqrt{e^{-\lambda k t} + u}}{e}\right),
\]
\[
G_6: (x, t, u) \mapsto \left( xe^{\varepsilon}, t, e^{\varepsilon} u \right),
\]
\[
G_7: (x, t, u) \mapsto \left( x, -\frac{\ln(\frac{\varepsilon}{k} + e^{-\lambda k t})}{\lambda k}, \frac{\sqrt{1 - \varepsilon e^{\lambda k t} u}}{u} \right),
\]
\[
G_8: (x, t, u) \mapsto \left( \frac{x}{1+\lambda e k - \lambda k t}, \frac{\ln(\lambda e k + e^{\lambda k t})}{\lambda k}, \frac{k}{\sqrt{\lambda e k^2 e^{-\lambda k t} + \varepsilon u}} \right).
\]

From all the above calculated one parameter Lie groups of transformations, the following theorems on transformed solutions hold true.

**Theorem 2** If \( u = \varphi (x, t) \) is an invariant solution of \( \square \) obtained through the generators \( M_1 \) and \( M_2 \), then so are the following functions:
\[
G_1: \varphi (x, t) = \varphi (x - \varepsilon, t),
\]
\[
G_2: \varphi (x, t) = \varphi (x, t - \varepsilon).
\]

**Theorem 3** If \( u = \varphi (x, t) \) is an invariant solution of \( \square \) obtained through the remaining generator of case one, then so is the following function:
\[
G_3: \varphi (x, t) = e^{-\frac{\varepsilon}{k}} \varphi \left( e^{\frac{k-1}{k} \varepsilon} x, e^{\frac{1-m}{k} \varepsilon} t \right).
\]

**Theorem 4** If \( u = \varphi (x, t) \) is an invariant solution of \( \square \) obtained through the generators of case two, then so are the following functions:
\[
G_3: \varphi (x, t) = m^{-1} \left[ (x - \ln \sqrt{1 - \varepsilon t} + \frac{t}{1 - \varepsilon t}) \right]^{m-1} - \varepsilon e^{-ax},
\]
\[
G_4: \varphi (x, t) = m^{-1} \left[ 1 + a \varepsilon e^{-ax} \right]^{m-1} \left[ \varphi \left( \ln \sqrt{a \varepsilon t + e^{ax}}, t \right) \right]^{m-1} - \varepsilon e^{-ax},
\]
\[
G_5: \varphi (x, t) = 1^{-m} \left[ \varphi \left( x, t - \frac{\varepsilon}{a} \right) \right]^{1-m} - \varepsilon e^{ax},
\]
\[
G_6: \varphi (x, t) = e^{-\frac{\varepsilon}{m-1}} \varphi (x, e^{-\varepsilon} t),
\]
\[
G_7: \varphi (x, t) = e^{-ax} \left[ \varepsilon e^{-ax} - a e^{ax} \right] \left( \varphi \left( \ln \sqrt{a \varepsilon e^{ax}}, t \right) \right),
\]
\[
G_8: \varphi (x, t) = m^{-1} \left[ \varphi \left( \ln \sqrt{-a \varepsilon + e^{-ax}} + \frac{t}{1 - a e^{ax}} \right) \right]^{m-1} - \frac{a^2 e^{2\varepsilon} t}{a e^{ax} - \varepsilon e^{-ax}}.
\]
Theorem 5 If \( u = \varphi(x,t) \) is an invariant solution of (1) obtained through the generators of case three, then so are the following functions:

\[
G_6.\varphi(x,t) = \frac{1}{1-m} \left[ \varphi(e^{t}+x,t) \right]^{1-m} - e,
\]

\[
G_7.\varphi(x,t) = \frac{1}{1-m} \left[ \varphi \left( \left( x + \frac{a}{2}t^2 \right) e^x, t \right) \right]^{1-m} + 2at e - 2at,
\]

\[
G_8.\varphi(x,t) = \frac{1}{1-m} \left[ \varphi \left( \left( x + \frac{a}{2}t^2 (1-e^x) \right) e^x, t \right) \right]^{1-m} - at (1-e^x).
\]

Theorem 6 If \( u = \varphi(x,t) \) is an invariant solution (1) obtained through the generators of case four, then so are the following functions:

\[
G_6.\varphi(x,t) = \left[ \varphi \left( xe^x, \left( t + \frac{a}{2}x^2 (e^x - 1) \right) e^x \right) \right]^{1-m} - ax (e^x - 1)\frac{2}{1-m},
\]

\[
G_7.\varphi(x,t) = \left[ e^{x} \left[ \varphi \left( xe^x, t + \frac{a}{2}x^2 (e^{2x} - 1) \right) \right]^{1-m} - ax (e^{2x} - 1)\right]^{2-m},
\]

\[
G_8.\varphi(x,t) = \left[ \varphi \left( xe^{-x}, t \right) \right]^{1-m} + \epsilon\frac{2}{1-m}.
\]

Theorem 7 If \( u = \varphi(x,t) \) is an invariant solution of (1) obtained through the respective generator of case five and six, then so are the following respective function:

\[
G_3.\varphi(x,t) = e^{-\frac{\epsilon}{1-m}} \varphi \left( e^\frac{3\epsilon}{2}x, e^\frac{3\epsilon}{2}t \right).
\]

\[
G_3.\varphi(x,t) = e^{-\frac{3\epsilon}{m-\epsilon}} \varphi \left( e^{-2\epsilon}x, e^{-3\epsilon}t \right).
\]

Theorem 8 If \( u = \varphi(x,t) \) is an invariant solution of (1) obtained through the generators of case seven, then so are the following functions:

\[
G_3.\varphi(x,t) = \sqrt{\frac{(1-\lambda \epsilon k)}{1+\lambda \epsilon k}} \left[ \varphi \left( \frac{x}{1-\lambda \epsilon k}, t - \ln \sqrt{1-\lambda \epsilon kx} \right) \right],
\]

\[
G_4.\varphi(x,t) = \frac{\varphi(x, -\ln \lambda k e + \lambda \epsilon k)}{\sqrt{1+\epsilon \lambda k \left( \lambda k x + \varphi \left( x, -\ln \lambda k \sqrt{\lambda \epsilon k + e^{-\lambda \epsilon k t}} \right) \right)^k}},
\]

\[
G_5.\varphi(x,t) = \sqrt{\frac{\varphi \left( x - \frac{\epsilon}{\lambda k} e^{-\lambda \epsilon k t}, t \right)}{\sqrt{\lambda \epsilon k + e^{-\lambda \epsilon k t}}}},
\]

\[
G_6.\varphi(x,t) = e^{-\frac{\epsilon}{\lambda k}} \varphi(x, e^\epsilon t),
\]

\[
G_7.\varphi(x,t) = \sqrt{\frac{1}{1+\epsilon \lambda k t}} \varphi \left( x, -\ln \lambda k - \epsilon + e^{-\lambda \epsilon k t} \right),
\]

\[
G_8.\varphi(x,t) = \sqrt{\frac{1}{1+\lambda \epsilon k e^{-\lambda \epsilon k t}}} \varphi \left( x, \ln \lambda k \sqrt{k + \lambda \epsilon k} + e^{-\lambda \epsilon k t} \right)^k - \frac{\lambda \epsilon k e^{-\lambda \epsilon k t}}{1+\lambda \epsilon k e^{-\lambda \epsilon k t}}.
\]

Example 9 This example highlights the importance of the above theorems. We start with the trivial solution of equation (1), that is \( u(x,t) = 0 \). By applying the transformations of theorem 8 on this solution we found that \( G_3, G_4, G_6, \) and \( G_7 \) all give no new solution but
follows:

\[ u = \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x} \]

as the new non-trivial one parameter invariant solutions of \( G \). We next pick one of these solutions and apply the rest of the transformations on it so as to generate more new solutions. Suppose we pick \( u(x,t) = \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x} \) which is invariant under \( G_5 \), the actions of the rest of the \( G_i \) are as follows:

\[ G_7 : G_5 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}, \]

\[ G_8 : G_7 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}, \]

\[ G_4 : G_8 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}, \]

\[ G_3 : G_4 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}, \]

\[ G_6 : G_3 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}. \]

Hence we see that by applying the transformations of theorem 8 to \( u(x,t) = 0 \), we obtained a non-trivial five-parameters solution

\[ u(x,t) = e^{-\frac{\lambda k e x}{1 + \lambda k e x}} \]

that keeps equation \( G \) invariant. If we have chosen \( u = \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x} \) which is invariant under \( G_8 \), then the actions of the rest of the \( G_i \) will be as follows:

\[ G_7 : G_8 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}, \]

\[ G_5 : G_7 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}, \]

\[ G_4 : G_5 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}, \]

\[ G_3 : G_4 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}, \]

\[ G_6 : G_3 \rightarrow \frac{\sqrt{1 - \lambda k e x}}{1 + \lambda k e x}. \]

Therefore,

\[ u(x,t) = \frac{e^{-\frac{\lambda k e x}{1 + \lambda k e x}} (1 - \lambda k e x) [e_2 (1 - \lambda k e x) + (1 - \lambda k e x) e^{2 \xi x}] - \lambda^3 k^2 e_2 e^{2 \xi x} x^2 + (1 - \lambda k e x) e^{\lambda k t}}{\lambda (1 - \lambda k e x) [k e - \lambda k e x] e^{\lambda k t}}. \]

is the new six-parameters invariant solution of equation \( G \).

- Transformations \( G_1 \) and \( G_2 \) were not used because they only generate space and time translations respectively.
- The order of applying the transformations is immaterial.
• $G_i : G_j$ means the action of $G_i$ on the solution of $G_j$.

**Theorem 10** If $g(u) = h(u) = u^k (k \neq 0)$, then all the above results hold as follows:
1. For $k \neq \pm 1$, $k \neq \pm \frac{1}{2}$, and $k \neq \frac{1}{3}$, equation (1) admits three parameters group of projective transformations spanned by the vector fields of case 1.
2. For $k = 1$, equation (1) admits eight parameters group of projective transformations spanned by the vector fields of case 7.
3. For $k = \frac{1}{2}$, equation (1) admits eight parameters group of projective transformations spanned by the vector fields of case 3.
4. For $k = -1$, equation (1) admits eight parameters group of projective transformations spanned by the vector fields of case 4.
5. For $k = \frac{1}{3}$, equation (1) admits three parameters group of projective transformations spanned by the vector fields of case 5.
6. For $k = -\frac{1}{2}$, equation (1) admits three parameters group of projective transformations spanned by the vector field of case 6.

4 Reductions and Exact Invariant Solutions

The reason behind going through all the tedious process of finding symmetry generators of any given differential equation is mainly to use them for obtaining symmetry reductions and possibly, exact solutions of the underlying differential equation. In this section, we shall use the symmetry generators of each of the classified cases in the previous section to derive corresponding reduced equations and subsequently, obtain exact invariant solutions of equation (1) whenever feasible.

**Case 1.** The invariant surface condition of the generator $M_3 = \frac{k-m+1}{k} x \partial_x + \frac{1-m}{k} t \partial_t + \frac{1}{k} u \partial_u$ is

$$
\frac{k-m+1}{k} \frac{\partial u}{\partial x} + \frac{1-m}{k} \frac{\partial u}{\partial t} = \frac{1}{k} u
$$

with corresponding characteristics equations

$$
\frac{k}{k-m+1} \frac{dx}{x} = \frac{k}{1-m} \frac{dt}{t} = \frac{du}{u}.
$$

Solving these equations give two invariants $\psi = x^{m/k+1} t^{-1/k}$ and $\rho = ut^{-1/k}$. Since $\psi$ is independent of $u$, then $\rho = F(\psi)$, $F$ arbitrary. So the invariants become

$$
\psi = x^{m/k+1} t^{-1/k}, \quad u = t^{-1/k} F(\psi).
$$

By chain rule, we have

$$
\begin{align*}
u_t &= \frac{1}{1-m} t^{\frac{m}{1-m}} (F - \psi F') , \\
u_x &= \frac{1}{k-m+1} x^{\frac{m-k}{k-m+1}} F''
\end{align*}
$$
which changes (1) to the reduced equation
\[
F - \psi F' + \frac{1 - m}{k - m + 1} \psi^{m-k} F^k F' + \lambda (1 - m) F^m = 0.
\]

This reduced equation is very difficult to solve for arbitrary \(k\) and \(m\), but much can be obtained when appropriate choices of \(k\) and \(m\) are made. For example, if \(m = k = 2\), then the above equation becomes
\[
F - \psi F' - F^2 F' - \lambda F^2 = 0.
\]

After re-arrangement and simple quadrature, we obtain
\[
F = \frac{-(\lambda \psi + A) \pm \sqrt{(\lambda \psi + A)^2 + 4\psi}}{2}
\]
where \(A\) is a constant of integration. Hence the required invariant solution of equation (1) with \(m = k = 2\) is
\[
u(x, t) = \frac{-(\lambda xt + A) \pm \sqrt{(\lambda xt + A)^2 + 4xt}}{2t}.
\]

Case 2. The characteristics equations of \(M_3\) are
\[
\frac{dx}{t} = \frac{dt}{\lambda(m-1) t^2} = \frac{(m-1) du}{u^2 - 2\lambda(m-1) tu}
\]

having invariants \(\psi = \frac{\lambda^{(m-1)x}}{t}\) and \(u = \left(\frac{1}{\lambda(m-1)t} + \frac{F(\psi)}{t^2}\right)^{\frac{1}{m-1}}\). Substitution of \(u\) and its respective derivatives in (1) yield the reduced equation \(\psi F' + F = 0\) having solution \(F = \frac{A}{\psi}\). Thus the required invariant solution is
\[
u = \left(\frac{1}{\lambda(m-1)t} + \frac{Ae^{-\lambda(m-1)x}}{t}\right)^{\frac{1}{m-1}}.
\]

The operators \(M_4\) and \(M_6\) also produced same result as \(M_3\) with the constant of integration \(A\) replaced with \(\frac{A}{\lambda(1-m)}\) and \(\frac{e^{\lambda A(m-1)}}{\lambda(m-1)}\) respectively. Reduction with \(M_5\) or \(M_8\) yield the invariant solution \((A + \lambda(m-1)t)^{\frac{1}{m-1}}\) while \(u = Ae^{-\lambda x}\) is obtained as a solution of (1) by using \(M_7\).

Case 3. Two invariants of \(M_6\) are \(\psi = t\) and \(u = \left(\frac{x}{t} + F\right)^{\frac{1}{1-m}}\). These lead to the reduced equation \(F' + \frac{F}{\psi} + \lambda (1 - m) = 0\) having solution \(F = \left(\frac{4}{\psi} - \frac{\lambda(1-m)}{2} \psi\right)^\frac{1}{1-m}\). Thus the solution of (1) with \(k = 1 - m\) is
\[
u = \left(\frac{A + x}{t} - \frac{\lambda (1 - m)}{2} t\right)^{\frac{1}{1-m}}.
\]

Solving the characteristic equations of \(M_7\) give the invariants \(\psi = t\) and \(u = [(2x + at^2) F - at]^{\frac{1}{1-m}}\)
with the second one obtained by solving the Bernoulli equation
\[
\frac{du}{dx} - \frac{\lambda u}{a (2x + at^2)} = \frac{\lambda au^n}{a (2x + at^2)}
\]
where \(a = \lambda (1 - m)\). The reduced equation obtained from these invariants is \(F' + 2F^2 = 0\) leading to the solution \(F = \frac{1}{2\psi - A}\). Hence
\[
u = \left(\frac{2x + \lambda (1 - m) (A - t)}{2t - A}\right)^{\frac{1}{1-m}}.
\] (19)

Reduction with \(M_8\) produced same result as that in (19) with \(A = 0\). \(M_8\) also give additional solution \(u = (A - \lambda (1 - m) t)^{\frac{1}{1-m}}\).

We next use a linear combination of \(M_7\) and \(M_8\) to obtain a solution of (1). The characteristic equations of \(M_7 + M_8\) are
\[
\frac{dx}{2x} = \frac{dt}{t} = (1 - m) \frac{du}{u}
\]
with invariants \(\psi = \frac{\sqrt{x}}{t}, u = (t F)^{\frac{1}{1-m}}\). The reduced equation obtained is
\[
(F - 2\psi^2) F' + 2\psi (F - a) = 0
\] (20)

where \(a = \lambda (m - 1)\). To solve this innocent looking differential equation, we let
\[
F = \frac{a + 2\psi^2 (z (\psi) - 1)}{z (\psi)}.
\] (21)

Thus
\[
F' = \frac{(2\psi^2 - a) z' + 4\psi z (z - 1)}{z^2},
\]
\[
a - F = \frac{(a - 2\psi^2) (z - 1)}{z},
\] \[F - 2\psi^2 = \frac{a - 2\psi^2}{z}.
\] (22)

Substituting (21) and (22) into (20) lead to the following separable first order differential equation
\[
\frac{dz}{z (z - 1) (z - 2)} = \frac{2\psi d\psi}{a - 2\psi^2}
\]

having solution
\[
\frac{z (z - 2)}{(z - 1)^2 (2\psi^2 - a)} = A
\]

Without any lost of generality, we let \(A = 1\). Hence the solution of the reduced equation (20) is
\[
F^2 - 2 (a + 1) F + 2\psi^2 + a (a + 1) = 0
\]
which is a quadratic equation in $F$ and so

$$F = (a + 1) \pm \sqrt{a + 1 - 2\psi^2}. $$

Therefore

$$u = \left[ \lambda (m - 1) + 1 \right] t \pm \sqrt{\lambda (m - 1) + 1} t^2 - 2x \right] \frac{t}{m}$$

is the required invariant solution of (11) when $M_7 + M_8$ is used as reducing generator.

The results for the remaining cases shall be presented in tabular form, $A$ is a constant wherever it appears.

**Case 4** ($b = \lambda (m - 1)$).

| Subalgebra | Invariants | Reduced Equation | Solution |
|------------|------------|------------------|----------|
| $M_6$      | $\psi = \frac{t}{x} - \frac{b}{4} x$ $u = \left( F + \frac{b}{2} x \right)^{-\frac{2}{m}}\sqrt{m}$ | $(F - \psi) F'$ | $u = \left( \frac{t}{x} + \frac{b}{4} x \right)^{-\frac{2}{m-1}}$ |
| $M_7$      | $\psi = \frac{b}{4} x^2 - t$ $u = \left( F + \frac{b}{2} x \right)^{-\frac{2}{m}}\sqrt{m}$ | $F' + 1 = 0$ | $u = \left( A + \frac{b}{4} x \right)^{-\frac{2}{m-1}}$ |
| $M_6 - \frac{1}{2} M_7$ | $\psi = \frac{a^2}{t}, u = \left( \frac{F}{t} \right)^{-\frac{2}{m}}\sqrt{m}$ | $(2\psi F - \psi^2) F'^2 - \frac{b}{2} F'^3 - F'^2 = 0$ | - |

**Case 5 & 6.**

| Case | Subalgebra | Invariants | Reduced Equation |
|------|------------|------------|------------------|
| 5    | $M_3$      | $\psi = \frac{b}{\sqrt{c}}, u = \left( \sqrt{t} F \right)^{-\frac{2}{m}}\sqrt{m}$ | $(2F_x^2 - 3\psi F) F'' + 3F^2 + 3\lambda (1 - m) = 0$ |
| 6    | $M_3$      | $\psi = \frac{\sqrt{c}}{\sqrt{t}}, u = \left( \frac{F}{\sqrt{t}} \right)^{-\frac{2}{m}}\sqrt{m}$ | $(3F - 2\psi^2) F' F'' - \psi F + 2\lambda (m - 1) \psi F^4 = 0$ |

**Case 7.**

| Subalgebra | Invariants | Reduced Equation | Solution |
|------------|------------|------------------|----------|
| $M_3$      | $\psi = xe^{\lambda kt}$, $u = \left( \frac{F}{\psi} - \frac{1}{\lambda kt} \right)^{-\frac{1}{k}}$ | $\psi F' - F = 0$ | $u = \left( \frac{Ae^{\lambda kt}}{x} - \frac{1}{\lambda k} \right)^{-\frac{1}{k}}$ |
| $M_4$      | $\psi = k, u = \left( \frac{\lambda ke^{\lambda kt} F}{1 - e^{-\lambda kt}} \right)^{\frac{1}{k}}$ | $F' = 0$ | $u = \left( \frac{\lambda ke^{\lambda kt}}{1 - e^{-\lambda kt}} \right)^{\frac{1}{k}}$ |
| $M_5$      | $\psi = t, u = \left( F - \lambda k \right)^{\frac{1}{k}}$ | $F' = 0$ | $u = \left( A - \lambda k \right)^{\frac{1}{k}}$ |
| $M_6$      | $\psi = t, u = \left( F \right)^{\frac{1}{k}}$ | $F'' + F^2 + \lambda k F = 0$ | $u = \left( \frac{Bxe^{-\lambda kt}}{1 - Be^{-\lambda kt}} \right)^{\frac{1}{k}}, B = \lambda k e^{\lambda kt}$ |
| $M_7$      | $\psi = x, u = \left( e^{-\lambda kt} F \right)^{\frac{1}{k}}$ | $F' = 0$ | $u = \left( A e^{-\lambda kt} \right)^{\frac{1}{k}}$ |
| $M_8$      | $\psi = xe^{\lambda kt}$, $u = \left( F - \lambda k \right)^{\frac{1}{k}}$ | $F' = 0$ | $u = \left( A - \lambda k \right)^{\frac{1}{k}}$ |

Finally, we present results obtained through the generators $M_1 = \partial_x$ and $M_2 = \partial_t$ which are known to have travelling waves solution. The invariant of $c\partial_x + \partial_t$ is $x - ct$ for any arbitrary constant $c \neq 0$. Since $\partial_u$ is missing, we can only proceed by letting $u = F(\psi)$ where $\psi = x - ct$, hence $u_t = -cF'$ and $u_x = F'$. The obtained results are as presented in
the following table.

| Case | Reduced Equation | Solution |
|------|------------------|----------|
| 1    | \((F^k - c) F' + \lambda F^m = 0\) | \(u^{1-m} = \frac{u^k\ c}{(1-m)} = A - (x - ct)\) |
| 2    | \((F^{m-1} - c) F' + \lambda F^m = 0\) | \(u^{1-m} = \frac{c}{e^{-\lambda(1-m)(x-ct)}}\) |
| 3    | \((F^{m-1} - c) F' + \lambda F^m = 0\) | \(u^{1-m} = c \pm \sqrt{c^2 - 2\lambda(1-m)(x - ct)}\) |
| 4    | \((F^{m-1} - c) F' + \lambda F^m = 0\) | \(u^{m-1} = \frac{1 \pm \sqrt{1-c(1-m)[A-\lambda(x-ct)]}}{c}\) |
| 5    | \((F^{m-1} - c) F' + \lambda F^m = 0\) | \(2u^{\frac{m(1-m)}{2}} - 3cu^{1-m} = 3(1-m)[A - \lambda(x - ct)]\) |
| 6    | \((F^{m-1} - c) F' + \lambda F^m = 0\) | \(3u^{\frac{m(1-m)}{2}} - 2cu^{1-m} = 2(1-m)[A - \lambda(x - ct)]\) |
| 7    | \((F^k - c) F' + \lambda F = 0\) | \(u^{-ck}e^{uk} = Ae^{-\lambda k(x-ct)}\) |

**Remark 1** All the invariant solutions obtained in this section can be subjected to the transformations of the previous section to generate new solutions.

## 5 Conclusion

In this paper, we have used symmetry analysis to perform classifications and subsequently, exhibit many invariant solutions for the damped inviscid Burger’s equation (1). Though various researchers have used this equation to model important physical phenomena, I have search the literature but did not see any work that deal with its analysis using Lie method as presented in this paper, hence this is the first among the series of work dedicated to the analysis of equation (1).
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