DEPTH THREE TOWERS OF RINGS AND GROUPS

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Abstract. Depth three and finite depth are notions known for subfactors via diagrams and Frobenius extensions of rings via centralizers in endomorphism towers. From the point of view of depth two ring extensions, we provide a clear definition of depth three for a tower of three rings $C \subseteq B \subseteq A$. If $A = \text{End}_B C$ and $B | C$ is a Frobenius extension, this captures the notion of depth three for a Frobenius extension. For example we provide an algebraic proof that if $B | C$ is depth three, then $A | C$ is depth two. If $A$, $B$ and $C$ correspond to a tower of subgroups $G > H > K$ via the group algebra over a fixed base ring, the depth three condition is the condition that subgroup $K$ has normal closure $K^G$ contained in $H$. For a depth three tower of rings, there is an interesting algebraic theory from the point of view of Galois correspondence theory for the ring $\text{End}_B A \otimes C$ with respect to the centralizers $A^B$ and $A^C$ involving Morita context bimodules, nondegenerate pairings and right coideal subrings.

1. Introduction

Depth $n$ is a notion from the classification of subfactors which describes where in the derived tower of centralizers, if at all, there occurs three successive algebras forming a basic construction $C \hookrightarrow B \hookrightarrow \text{End}_B C$. Depth two plays the most important role in finite depth classification theory [9]. This is partly because a finite depth subfactor embeds via its Jones tower into a depth two subfactor (see Theorem 2.3 for the depth three algebraic version). A subfactor $B \subseteq A$ is depth two then if the centralizers $V_A(B) \hookrightarrow V_{A_1}(B) \hookrightarrow V_{A_2}(B)$ is a basic construction, where $A \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow A_3$ is a Jones tower of iterated basic constructions. The subfactor $B \subseteq A$ is depth three if instead the centralizers $V_A(B) \hookrightarrow V_{A_2}(B) \hookrightarrow V_{A_3}(B)$ is a basic construction. The algebraic property of finite depth may be described most easily starting with a Frobenius extension $A \supseteq B$, where the definition guarantees the existence of a bimodule homomorphism $A \rightarrow B$ with dual bases for the finitely generated projective $B$-module $A$ [3].

A careful algebraic study of the depth two condition on subalgebra $B \subseteq A$ shows that it is most simply expressed as a type of central projectivity condition on the tensor-square $A \otimes_B A$ w.r.t. $A$ as natural $A$-$B$-bimodules and $B$-$A$-bimodules. There is a Galois theory connected to this viewpoint with Galois quantum groupoids, in the category of Hopf algebroids. Although a future viewpoint on depth two ring extension in this generality might be that it is better called a “normal extension,” depth two does presently suggest that it is part of a larger theory of depth 2, 3 and beyond for ring extensions. Indeed depth three does lend itself, after reformulation, to a notion for ring extensions (see the preprint version of [4]). However, in this
preprint we prefer to view depth three as a property most naturally associated to a tower of three algebras or rings $C \subseteq B \subseteq A$. This tower is right depth three (rD3) if $A \otimes B A$ is $A$-$C$-isomorphic to a direct summand of $A \oplus \cdots \oplus A$. The advantage of this definition over the one in [4] preprint version is that it is close to the depth two definition so that a substantial amount of depth two theory is available as we see in this paper. At the same time, it seems inevitable that depth three will play a role in any Galois correspondence theory involving depth two, simple algebras and purely algebraic hypotheses: see Theorem 2.5 the last section and compare with [9]. The connection with classical depth three subfactors may be seen as follows: if $C \subseteq B$ is a Frobenius extension with $A = \text{End} B C$, it follows that $A \cong B \otimes_C B$, that $A A \otimes_B A C$ reduces to $A B \otimes_C B \otimes_C B C$ and $A A_C$ to $A B \otimes_C B C$, the terms in which the depth three condition is expressed in [4] preprint version.

In section 2 of this paper we note that right or left D3 ring towers are characterized in terms either of the tensor-square, $\mathbb{H}$-equivalent modules, quasibases or the endomorphism ring. In section 3 we show that a tower of subgroups $G > H > K$ of finite index with the condition that the normal closure $K^G < H$ ensures that the group algebras $F[G] \cong F[H] \cong F[K]$ are a depth three tower w.r.t. any base ring $F$. We propose that the converse is true if $G$ is a finite group and $F = \mathbb{C}$. In section 4 we study the right coideal subring $E = \text{End} B A_C$ as well as the bimodule and co-ring $P = (A \otimes_B A)^C$, which provide the quasibases for a right D3 tower $A \mid B \mid C$. We show that right depth three towers may be characterized by $P$ being finite projective as a left module over the centralizer $V = A^C$ and a pre-Galois isomorphism $A \otimes_B A \xrightarrow{\sim} A \otimes_Y P$.

2. Definition and first properties of depth three towers

Let $A$, $B$ and $C$ denote rings with identity element, and $C \to B$, $B \to A$ denote ring homomorphisms preserving the identities. We use ring extension notation $A \mid B \mid C$ for $C \to B \to A$ and call this a tower of rings: an important special case if of course $C \subseteq B \subseteq A$ of subrings $B$ in $A$ and $C$ in $B$. Of most importance to us are the induced bimodules such as $B A_C$ and $C A_B$. We may naturally also choose to work with algebras over commutative rings, and obtain almost identical results.

We denote the centralizer subgroup of a ring $A$ in an $A$-$A$-bimodule $M$ by $M^A = \{ m \in M \mid \forall a \in A, ma = am \}$. We also use the notation $V A(C) = A^C$ for the centralizer subring of $C$ in $A$. This should not be confused with our notation $K^G$ for the normal closure of a subgroup $K < G$. Notation like $\text{End} B C$ will denote the ring of endomorphisms of the module $B C$ under composition and addition (and not algebra homomorphisms fixing $C$ or the like).

**Definition 2.1.** A tower of rings $A \mid B \mid C$ is right depth three (rD3) if the tensor-square $A \otimes_B A$ is isomorphic as $A$-$C$-bimodules to a direct summand of a finite direct sum of $A$ with itself: in module-theoretic symbols, this becomes, for some positive integer $N$,

\[
\text{(1)} \quad A A \otimes_B A_C \oplus \ast \cong A A_C^N
\]

By switching to $C$-$A$-bimodules instead, we naturally define a left D3 tower of rings. The theory for these is entirely analogous to rD3 towers and is briefly considered at the end of this section.

Recall that over a ring $R$, two modules $M_R$ and $N_R$ are H-equivalent if $M_R \oplus \ast \cong N_R^n$ and $N_R \oplus \ast \cong M_R^m$ for some positive integers $n$ and $m$. In this case,
the endomorphism rings \( \text{End } M_R \) and \( \text{End } N_R \) are Morita equivalent with context bimodules \( \text{Hom} (M_R, N_R) \) and \( \text{Hom} (N_R, M_R) \).

**Lemma 2.2.** A tower \( A \mid B \mid C \) of rings is rD3 iff the natural \( A\text{-}C \)-bimodules \( A \otimes_B A \) and \( A \) are \( H \)-equivalent.

**Proof.** We note that for any tower of rings, \( A \oplus_* \cong A \otimes_B A \) as \( A\text{-}C \)-bimodules, since the epi \( \mu : A \otimes_B A \to A \) splits as an \( A\text{-}C \)-bimodule arrow.

Since for any tower of rings \( \text{End}_A AC \) is isomorphic to the centralizer \( V_A(C) = A^C \) (or anti-isomorphic according to convention), we see from the lemma that the notion of rD3 has something to do with classical depth three. Indeed, \( \Box \)

**Example 2.3.** If \( B \mid C \) is a Frobenius extension, with Frobenius system \( (E, x_1, y_1) \) satisfying for each \( a \in A \),

\[
\sum_i E(ax_i)y_i = a = \sum_i x_iE(y_ia)
\]

then \( B \otimes_C B \cong \text{End } B_C := A \) via \( x \otimes by \mapsto \lambda_x \circ E \circ \lambda_y \) for left multiplication \( \lambda_x \) by element \( x \in B \). Let \( B \to A \) be this mapping \( B \hookrightarrow \text{End } B_C \) given by \( b \mapsto \lambda_b \). It is then easy to show that \( A \otimes_B B \otimes_C B \cong A \otimes_B A \), so that condition (1) is equivalent to the condition for rD3 in preprint [4], which in turn slightly generalizes the condition in [3] for depth three free Frobenius extension. We should make note here that right or left depth three would be equivalent notions for Frobenius extensions, since \( \text{End } B_C \) and \( \text{End } C_B \) are anti-isomorphic for such.

Another litmus test for a correct notion of depth three is that depth two extensions should be depth three in a certain sense. Recall that a ring extension \( A \mid B \) is right depth two (rD2) if the tensor-square \( A \otimes_B A \) is \( A\text{-}B \)-bimodule isomorphic to \( N \) copies of \( A \) in a direct sum with itself:

\[
A \otimes_B A \cong A^\oplus \cong A^{N_A}
\]

Since the notions pass from ring extension to tower of rings, there are several cases to look at.

**Proposition 2.4.** Suppose \( A \mid B \mid C \) is a tower of rings. We note:

1. If \( B = C \) and \( B \to C \) is the identity mapping, then \( A \mid B \mid C \) is rD3 \( \iff \) \( A \mid B \) is rD2.
2. If \( A \mid B \) is rD2, then \( A \mid B \mid C \) is rD3 w.r.t. any ring extension \( B \mid C \).
3. If \( A \mid C \) is rD2 and \( B \mid C \) is a separable extension, then \( A \mid B \mid C \) is rD3.
4. If \( B \mid C \) is left D2, and \( A = \text{End } B_C \), then \( A \mid B \mid C \) is left D3.
5. If \( C \) is the trivial subring, any ring extension \( A \mid B \), where \( BA \) is finite projective, together with \( C \) is rD3.

**Proof.** The proof follows from comparing eqs. (1) and (3), noting that \( A \otimes_B A \oplus_* \cong A \otimes_C A \) as natural \( A\text{-}A \)-bimodules if \( B \mid C \) is a separable extension (and having a separability element \( e = e^1 \otimes_C e^2 \in (B \otimes_C B)^B \) satisfying \( e^1 e^2 = 1 \)), and finally from [5] that \( B \mid C \) left D2 extension \( \Rightarrow A \mid B \) is left D2 extension if \( A = \text{End } B_C \). The last statement follows from tensoring \( BA \oplus_* \cong B^n \) by \( A \otimes_C B \).

The next theorem is a converse and algebraic simplification of a key fact in subfactor Galois theory (the \( n = 3 \) case): a depth three subfactor \( N \subseteq M \) yields a depth two subfactor \( N \subseteq M_1 \), w.r.t. its basic construction \( M_1 \cong M \otimes_N M \). In
preparation, let us call a ring extension $B\mid C$ rD3 if the endomorphism ring tower $A\mid B\mid C$ is rD3, where $A = \text{End}_B C$ and $A\mid B$ has underlying map $\lambda : B \to \text{End}_B C$, the left regular mapping given by $\lambda(x)(b) = xb$ for all $x, b \in B$.

**Theorem 2.5.** Suppose $B\mid C$ is a Frobenius extension and $A = \text{End}_B C$. If $B\mid C$ is rD3, then the composite extension $A\mid C$ is D2.

**Proof.** There is a well-known bimodule isomorphism for a Frobenius extension $B\mid C$, between its endomorphism ring and its tensor-square, $BA_B \cong B \otimes_C B$. Tensoring by mappings just described. We compute:

From the condition (1), there are obviously two.

Thus $A\mid C$ is right D2. Since it is a Frobenius extension as well, it is also left depth two. \hfill \square

We introduce quasi-bases for right depth three towers.

**Theorem 2.6.** A tower $A\mid B\mid C$ is right depth three iff there are $N$ elements each of $\gamma_i \in \text{End}_B AC$ and of $u_i \in (A \otimes_B A)^C$ satisfying (for each $x, y \in A$)

\begin{equation}
  x \otimes_B y = \sum_{i=1}^N x\gamma_i(y)u_i
\end{equation}

**Proof.** From the condition (1), there are obviously $N$ maps each of

\begin{equation}
  f_i \in \text{Hom}(AA_C, AA \otimes_B AC), \quad g_i \in \text{Hom}(AA \otimes_B AC, AA_C)
\end{equation}

such that $\sum_{i=1}^N f_i \circ g_i = \text{id}_{A \otimes_B A}$. First, we note that for any tower of rings, not necessarily rD3,

\begin{equation}
  \text{Hom}(AA_C, AA \otimes_B AC) \cong (A \otimes_B A)^C
\end{equation}

via $f \mapsto f(1_A)$. The inverse is given by $p \mapsto ap$ where $p = p^1 \otimes_B p^2 \in (A \otimes_B A)^C$ using a Sweedler-type notation that suppresses a possible summation over simple tensors.

The other hom-group above also has a simplification. We note that for any tower,

\begin{equation}
  \text{Hom}(AA \otimes_B AC, AA_C) \cong \text{End}_{BA_C}
\end{equation}

via $F \mapsto F(1_A \otimes_B -)$. Given $\alpha \in \text{End}_B AC$, we define an inverse sending $\alpha$ to the homomorphism $x \otimes_B y \mapsto x\alpha(y)$.

Let $f_i$ correspond to $u_i \in (A \otimes_B A)^C$ and $g_i$ correspond to $\gamma_i \in \text{End}_B AC$ via the mappings just described. We compute:

\begin{equation}
  x \otimes_B y = \sum_i f_i(g_i(x \otimes y)) = \sum_i f_i(x\gamma_i(y)) = \sum_i x\gamma_i(y)u_i,
\end{equation}

which establishes the rD3 quasibases equation in the theorem, given an rD3 tower.

For the converse, suppose we have $u_i \in (A \otimes_B A)^C$ and $\gamma_i \in \text{End}_B AC$ satisfying the equation in the theorem. Then map $\pi : A^N \to A \otimes_B A$ by

\begin{equation}
  \pi : (a_1, \ldots, a_N) \mapsto \sum_i a_i u_i,
\end{equation}
an $A$-$C$-bimodule epimorphism split by the mapping $\sigma : A \otimes_B A \to A^N$ given by

$$\sigma(x \otimes_B y) := (x \gamma_1(y), \ldots, x \gamma_N(y)).$$

It follows from the equation above that $\pi \circ \sigma = \text{id}_{A \otimes_B A}$. \hfill \Box

2.1. **Left D3 towers and quasibases.** A tower of rings $A \mid B \mid C$ is left D3 if the tensor-square $A \otimes_B A$ is an $C$-$A$-bimodule direct summand of $A^N$ for some $N$. If $B = C$, this recovers the definition of a left depth two extension $A \mid B$. There is a left version of all results in this paper: we note that $A \mid B \mid C$ is a right D3 tower if and only if $A^{\text{op}} \mid B^{\text{op}} \mid C^{\text{op}}$ is a left D3 tower (cf. [5]).

The next theorem refers to notation established in the example above.

**Theorem 2.7.** Suppose $B \mid C$ is a Frobenius extension with $A = \text{End}_B C$. Then $A \mid B \mid C$ is right depth three if and only if $A \mid B \mid C$ is left depth three.

**Proof.** It is well-known that also $A \mid B$ is a Frobenius extension. Then $A \otimes_B A \cong \text{End}_B A$ as natural $A$-$A$-bimodules. Also $A \otimes_B A \cong \text{End}_B A$ by a similar mapping utilizing the Frobenius homomorphism in one direction, and the dual bases in the other. As a result, the left and right endomorphism rings are anti-isomorphic.

Now note the following characterization of left D3 with proof almost identical with that of [7, Prop. 3.8]: If $A \mid B \mid C$ is a tower where $A_B$ is finite projective, then $A \mid B \mid C$ is left D3 $\iff \text{End}_A B \oplus * \cong A^N$ as natural $A$-$C$-bimodules. The proof involves noting that $\text{End}_A B \cong \text{Hom}(A \otimes_B A, A_A)$ via

$$f \mapsto (a \otimes a' \mapsto f(a)a').$$

Similarly, if $A \mid B \mid C$ is a tower where $B_A$ is finite projective, then $A \mid B \mid C$ is right D3 if and only if $\text{End}_B A \oplus * \cong A^N$ as natural $C$-$A$-bimodules.

Of course a Frobenius extension satisfies both finite projectivity conditions. The anti-isomorphism of the left and right endomorphism rings twists the $C$-$A$-structure to an $A$-$C$-structure, thereby demonstrating the equivalence of left and right D3 conditions on $A \otimes_B A$ relative to $A \cong \text{End}_A A$. \hfill \Box

In a fairly obvious reversal to opposite ring structures in the proof of theorem 2.6, we see that a tower $A \mid B \mid C$ is left D3 iff there are $N$ elements $\beta_j \in \text{End}_C A_B$ and $N$ elements $t_j \in (A \otimes_B A)^C$ such that for all $x, y \in A$, we have

$$(8) \quad x \otimes_B y = \sum_{j=1}^N t_j \beta_j(x)y.$$

We record the characterization of left D3, noted above in the proof, for towers satisfying a finite projectivity condition.

**Proposition 2.8.** Suppose $A \mid B \mid C$ is a tower of rings where $A_B$ is finite projective. Then this tower is left D3 if and only if the natural $A$-$C$-bimodules satisfy for some $N$,

$$(9) \quad \text{End}_A B \oplus * \cong A^N.$$
3. Depth three for towers of groups

Fix a base ring $F$. Groups give rise to rings via $G \mapsto F[G]$, the functor associating the group algebra $F[G]$ to a group $G$. Therefore we can pull back the notion of depth 2 or 3 for ring extensions or towers to the category of groups (so long as reference is made to the base ring).

In the paper \cite{2}, a depth two subgroup w.r.t. the complex numbers is shown to be equivalent to the notion of normal subgroup for finite groups. This consists of two results. The easier result is that over any base ring, a normal subgroup of finite index is depth two by exhibiting left or right D2 quasibases via coset representatives and projection onto cosets. This proof suggests that the converse hold as well. The second result is a converse for complex finite dimensional D2 group algebras where normality of the subgroup is established using character theory and Mackey’s subgroup theorem.

In this section, we will similarly do the first step in showing what group-theoretic notion corresponds to depth three tower of rings. Let $G > H > K$ be a tower of groups, where $G$ is a finite group, $H$ is a subgroup, and $K$ is a subgroup of $H$. Let $A = F[G]$, $B = F[H]$ and $C = F[K]$. Then $A \mid B \mid C$ is a tower of rings, and we may ask what group-theoretic notion on $G > H > K$ will guarantee, with fewest possible hypotheses, that $A \mid B \mid C$ is rD3.

**Theorem 3.1.** The tower of groups algebras $A \mid B \mid C$ is D3 if the corresponding tower of groups $G > H > K$ satisfies

$$K^G < H$$

where $K^G$ denotes the normal closure of $K$ in $G$.

**Proof.** Let $\{g_1, \ldots, g_N\}$ be double coset representatives such that $G = \bigsqcup_{i=1}^{N} H g_i K$. Define $\gamma_i(g) = 0$ if $g \not\in Hg_i K$ and $\gamma_i(g) = g$ if $g \in Hg_i K$. Of course, $\gamma_i \in \text{End}_{BAC}$ for $i = 1, \ldots, N$.

Since $K^G \subseteq H$, we have $gK \subseteq Hg$ for each $g \in G$. Hence for each $k \in K$, $g_jk = hg_j$ for some $h \in H$. It follows that

$$g_j^{-1} \otimes \_B g_j k = g_j^{-1} h \otimes \_B g_j = kg_j^{-1} \otimes \_B g_j.$$

Given $g \in G$, we have $g = hg_jk$ for some $j = 1, \ldots, N$, $h \in H$, and $k \in K$. Then we compute:

$$1 \otimes \_B g = 1 \otimes \_B hg_j k = hg_j g_j^{-1} \otimes \_B g_j k = hg_j kg_j^{-1} \otimes \_B g_j$$

so $1 \otimes \_B g = \sum_i \gamma_i(g) g_i^{-1} \otimes \_B g_i$ where $g_i^{-1} \otimes \_B g_i \in (A \otimes \_B A)^C$. By theorem then, $A \mid B \mid C$ is an rD3 tower.

The proof that the tower of group algebras is left D3 is entirely symmetrical via the inverse mapping. \hfill \Box

The theorem is also valid for infinite groups where the index $[G : H]$ is finite, since $HgK = Hg$ for each $g \in G$.

Notice how the equivalent notions of depth two and normality for finite groups over $\mathbb{C}$ yields the proposition \cite{2} for groups. Suppose we have a tower of groups $G > H > K$ where $K^G \subseteq H$. If $K = H$, then $H$ is normal (D2) in $G$. If $K = \{e\}$, then it is rD3 together with any subgroup $H < G$. If $H \triangleleft G$ is a normal subgroup, then necessarily $K^G \subseteq H$. If $K \triangleleft G$, then $K^G = K < H$ and the tower is D3.
Question: Can the character-theoretic proof in [2] be adapted to prove that a D3 tower $C \triangleright C \triangleright C$ are bimodules with respect to the two rings familiar from depth two theory, $A|B|C$ denoting a right depth three tower of rings, $P := (A \otimes_B A)^C$, $Q := (A \otimes_C A)^B$, which are bimodules with respect to the two rings familiar from depth two theory, $T := (A \otimes_B A)^B$, $U := (A \otimes_C A)^C$

Note that $P$ and $Q$ are isomorphic to two $A$-$A$-bimodule Hom-groups:

(11) $P \cong \text{Hom} (A \otimes C A, A \otimes B A), \quad Q \cong \text{Hom} (A \otimes B A, A \otimes C A)$.

Recall that $T$ and $U$ have multiplications given by

$$tt' = t'^1 \otimes_B t'^2, \quad uu' = u'^1 \otimes_C u'^2,$$

where $1_T = 1_A \otimes 1_A$ and a similar expression for $1_U$. Namely, the bimodule $TP_U$ is given by

(12) $TP_U : t \cdot p \cdot u = u^1 t^1 \otimes_B t^2 p^2 u^2$.

The bimodule $UQ_T$ is given by

(13) $UQ_T : u \cdot q \cdot t = t^1 q^1 u^1 \otimes_C u^2 q^2 t^2$.

We have the following result, also mentioned in passing in [3] with several additional hypotheses.

**Proposition 4.1.** The bimodules $P$ and $Q$ over the rings $T$ and $U$ form a Morita context with associative multiplications

(14) $P \otimes_U Q \rightarrow T, \quad p \otimes q \mapsto pq = q^pp^q$,

(15) $Q \otimes_T P \rightarrow U, \quad q \otimes p \mapsto qp = p^q q^p$.

If $B|C$ is an $H$-separable extension, then $T$ and $U$ are Morita equivalent rings via this context.

**Proof.** The equations $p(qp') = (pq)p'$ and $q(pq') = (qp)q'$ for $p, p' \in P$ and $q, q' \in Q$ follow from the four equations directly above.

Note that $T \cong \text{End}_A A \otimes_B A A$, $U \cong \text{End}_A A \otimes_C A A$ as rings. We now claim that the hypotheses on $A|B$, $A|C$ and $B|C$ imply that the $A$-$A$-bimodules $A \otimes B A$ and $A \otimes C A$ are H-equivalent. Then the endomorphism rings above are Morita equivalent via context bimodules given by eqs. (11), which proves the proposition.

Since $B|C$ is H-separable, it is in particular separable, and the canonical $A$-$A$-epi $A \otimes C A \rightarrow A \otimes B A$ splits via an application of a separability element. Thus, $A \otimes B A \oplus * \cong A \otimes C A$. Also, $B \otimes C B \oplus * \cong B^N$ as $B$-$B$-bimodules for some positive integer $N$. Therefore, $A \otimes C A \oplus * \cong A \otimes B A^N$ as $A$-$A$-bimodules by an application of the functor $A \otimes_B ? \otimes_B A$. Hence, $A \otimes B A$ and $A \otimes C A$ are H-equivalent $A^r$-modules (i.e., $A$-$A$-bimodules). $\square$
We denote the centralizer subrings $A^C$ and $A^B$ of $A$ by
\[
R := V_A(B) \subseteq V_A(C) := V
\]
We have generalized anchor mappings [6],
\[
R \otimes_T P \to V, \quad r \otimes p \mapsto p^1 r p^2
\]
\[
V \otimes_U Q \to R, \quad v \otimes q \mapsto q^1 v q^2
\]
**Proposition 4.2.** The two generalized anchor mappings are bijective if $B \mid C$ is $H$-separable.

**Proof.** Denote $r \cdot p := p^1 r p^2$ and $v \cdot q := q^1 v q^2$. From the previous proposition, there are elements $p_i \in P$ and $q_i \in Q$ such that $\sum_i p_i q_i = 1_T$; in addition, $p'_j \in P$ and $q'_j \in Q$ such that $1_U = \sum_j q'_j p'_j$. Let $v \in V$, then
\[
v = v \cdot 1_U = \sum_j v \cdot (q'_j p'_j) = \sum_j (v \cdot q'_j) \cdot p'_j
\]
and a similar computation starting with $r = r \cdot 1_T$ shows that the two generalized anchor mappings are surjective.

In general, we have the corestriction of the inclusion $T \subseteq A \otimes_B A$,
\[
\tau T \hookrightarrow \tau P
\]
which is split as a left $T$-module monic by $p \mapsto e^1 pe^2$ in case there is a separability element $e = e^1 \otimes_C e^2 \in B \otimes_C B$. Similarly,
\[
u Q \hookrightarrow \nu U
\]
is a split monic in case $B \mid C$ is separable. Of course, if $B \mid C$ is $H$-separable, we note from proposition [4, 5, 6] and Morita theory that $P$ and $Q$ are projective generators on both sides.

It follows from faithful flatness that the anchor mappings are also injective. □

Note that $P$ is a $V$-$V$-bimodules (via the commuting homomorphism and antihomomorphism $V \to U \leftarrow V)$:
\[
\nu P_V : \quad v \cdot p \cdot v' = v p^1 \otimes_B p^2 v'
\]
Note too that $E = \text{End}_{B A_C}$ is an $R$-$V$-bimodule via
\[
\text{RE}_V : \quad r \cdot \alpha \cdot v = r \alpha(-) v
\]
Note the subring and over-ring
\[
\text{End}_{B A_B} \subseteq E \subseteq \text{End}_{C A_C}
\]
which are the total algebras of the left $R$- and $V$-bialgebroids in depth two theory [4, 5, 6].

**Lemma 4.3.** The modules $\nu P$ and $E_V$ are finitely generated projective. In case $A \mid C$ is left $D2$, the subring $E$ is a right coideal subring of the left $V$-bialgebroid $\text{End}_{C A_C}$.
Proposition 4.5. There is a mapping has the inverse $\alpha$ via $\gamma_i(p^2)u_i$ where $u_i \in P$ and $p \mapsto \gamma_i(p^2)$ in Hom $(\mathcal{V}P, \mathcal{V}V)$, thus dual bases for a finite projective module. The second claim follows similarly from

$$\alpha = \sum \gamma_i(-)u_i^1\alpha(u^2)$$

where $\gamma_i \in E$ and $\alpha \mapsto u^1\alpha(u^2)$ are mappings in Hom $(E_V, V_V)$.

Now suppose $\beta_j \in S := \text{End}_{C}A_C$ and $t_j \in (A \otimes C)C$ are left D2 quasibases of $A \otimes C$. Recall that the coproduct $\Delta : S \rightarrow S \otimes V S$ given by ($\beta \in S$)

$$\Delta(\beta) = \sum \beta(-t_1^j)t_2^j \otimes \beta_j$$

makes $\alpha$ a left $V$-bialgebroid [4]. Of course this restricts and corestricts to $\alpha \in E$ as follows: $\Delta(\alpha) \in E \otimes V S$. Hence, $E$ is a right coideal subring of $S$.

In fact, if $A \mid B$ is also D2, and $S = \text{End}_B A_B$, then $E$ is similarly shown to be an $S$-$S$-bicomodule ring For we recall the coaction $E \rightarrow S \otimes_R E$ given by

$$\alpha(-1) \otimes_R \alpha(0) = \sum \tilde{\gamma}_i \otimes \tilde{u}^1_\gamma \alpha(\tilde{u}^2_i)$$

where $\tilde{\gamma}_i \in S$ and $\tilde{u}_i \in (A \otimes B)B$ are right D2 quasibases of $A \mid B$ (restriction of [5 eq. (19)]).

Twice above we made use of a $V$-bilinear pairing $P \otimes E \rightarrow V$ given by

$$(p, \alpha) := p^1 \alpha(p^2), \ (p \in P = (A \otimes B)C, \ \alpha \in E = \text{End}_B A_C)$$

Lemma 4.4. The pairing above is nondegenerate. It induces $E_V \cong \text{Hom}(\mathcal{V}P, \mathcal{V}V)$ via $\alpha \mapsto \langle -, \alpha \rangle$.

Proof. The mapping has the inverse $F : \mathcal{V}P \rightarrow \mathcal{V}E, \ {\mathcal{V}P \rightarrow \mathcal{V}E}$, where $\gamma_i \in E, u_i \in P$ are rD3 quasibases for $A \mid B \mid C$. Indeed, $\sum \gamma_i(p)F(u_i) = F(\sum p^1 \gamma_i(p^2)u_i) = F(p)$ for each $p \in P$ since $F$ is left $V$-linear, and for each $\alpha \in E$, we note that $\sum \gamma_i(-)u_i, \ \alpha = \alpha$.

Proposition 4.5. There is a $V$-coring structure on $P$ left dual to the ring structure on $E$.

Proof. We note that

$$p \otimes_V P \cong (A \otimes B A \otimes B A)^C$$

via $p \otimes p' \mapsto p^1 \otimes p^2 p'^1 \otimes p'^2$ with inverse

$$p = p^1 \otimes p^2 \otimes p^3 \mapsto \sum (p^1 \otimes B p^2 \gamma_i(p^3)) \otimes_V u_i.$$

Via this identification, define a $V$-linear coproduct $\Delta : P \rightarrow P \otimes_V P$ by

$$\Delta(p) = p^1 \otimes_B 1_A \otimes_B p^2.$$ 

Alternatively, using Sweedler notation and rD3 quasibases

$$p_{(1)} \otimes_V p_{(2)} = \sum (p^1 \otimes_B \gamma_i(p^2)) \otimes_V u_i.$$
Define a $V$-linear counit $\varepsilon : P \to V$ by $\varepsilon(p) = p^1p^2$. The counitial equations follow readily [11].

Recall from Sweedler [10] that the $V$-coring $(P, V, \Delta, \varepsilon)$ has left dual ring $^*P := \hom_V(P, V)$ given by Sweedler notation by

\begin{equation}
(f * g)(p) = f(p_1)g(p_2)
\end{equation}

with $1 = \varepsilon$. Let $\alpha, \beta \in E$. If $f = \langle -, \alpha \rangle$ and $g = \langle -, \beta \rangle$, we compute $f * g = \langle -, \alpha \circ \beta \rangle$ below, which verifies the claim:

\[ f(p_1)g(p_2)) = \sum_i (p^1 \otimes_B \gamma_i(p^2)(u_i, \beta), \alpha) = (p^1 \otimes \beta(p^2), \alpha) = \langle p, \alpha \circ \beta \rangle. \]

\[ \Box \]

In addition, we note that $P$ is $V$-coring with grouplike element

\begin{equation}
g_P := 1_A \otimes_B 1_A
\end{equation}

since $\Delta(g_P) = 1 \otimes 1 \otimes 1 = g_P \otimes V g_P$ and $\varepsilon(g_P) = 1$.

There is a pre-Galois structure on $A$ given by the right $P$-comodule structure $\delta : A \to A \otimes V P$, $\delta(a) = a_0 \otimes V a_{(1)}$ defined by

\begin{equation}
\delta(a) := \sum_i \gamma_i(a) \otimes_V u_i.
\end{equation}

The pre-Galois isomorphism $\beta : A \otimes_B A \xrightarrow{\sim} A \otimes_V P$ given by

\begin{equation}
\beta(a \otimes a') = a a'_{(0)} \otimes V a'_{(1)}
\end{equation}

is utilized below in another characterization of right depth three towers.

**Theorem 4.6.** A tower of rings $A \mid B \mid C$ is right depth three if and only if $V P$ is finite projective and $A \otimes_V P \cong A \otimes_B A$ as natural $A$-$C$-bimodules.

**Proof:** ($\Rightarrow$) If $V P \oplus * \cong V N$ and $A \otimes_V P \cong A \otimes_B A$, then tensoring by $A \otimes_V -$ , we obtain $A \otimes_B A \oplus * \cong A N$ as natural $A$-$C$-bimodules, the rD3 defining condition on a tower.

($\Leftarrow$) In proposition we see that $V P$ is f.g. projective. Map $A \otimes_V P \to A \otimes_B A$ by $a \otimes p \mapsto a_1 \otimes_B p^2$, clearly an $A$-$C$-bimodule homomorphism. The inverse is the “pre-Galois” isomorphism,

\begin{equation}
\beta : A \otimes_B A \to A \otimes_V P, \quad \beta(a \otimes_B a') = \sum_i a \gamma_i(a') \otimes_V u_i
\end{equation}

since $\sum_i a^i \gamma_i(p^2) \otimes_V u_i = a \otimes_V p$ and $\sum_i a \gamma_i(a') u_i = a \otimes a'$ for $a, a' \in A, p \in P$. \[ \Box \]

It is an intriguing to continue this study, also for depth $n$ towers, and study the possibility of an algebraic version of the Galois theory for subfactors in Nikshych and Vainerman [9].

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