Linear multi-photon storage based on dark modes with frequency tuning

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Abstract

We propose a quantum memory applicable to the optical regime based on a linear system. The system is in a symmetrical star configuration: one central mode is both connected to the input–output channel and uniformly coupled to a number of modes with tunable frequencies. We show that, as long as the number of these tunable modes is double an odd number, such a configuration is just flexible enough to perform the storage and on-demand recall of a number of individual photons. Tuning of the mode frequencies is feasible experimentally by means of adiabatic frequency tuning, and we show the system is scalable and robust against various type of imperfections. Moreover, the linearity of the system is compatible with the use of single-mode continuous-variable error correction code. Our results therefore provide a promising approach to the storage of many photons carrying protected quantum information.

1. Introduction

Quantum memories, which are integral to quantum communication and quantum information processing, are devices that can preserve quantum information for a period of time until an on-demand recall [1–8]. A wide range of quantum memories have been proposed, including related devices such as quantum registers [9–11], which have additional processing capabilities, as well as so-called quantum random access memories [12–16], which have quantum addressing capabilities. Particularly, quantum memory of light that preserves its multiple quantum states [17] attracts wide interest [18–23].

Compared to storing the quantum information carried by light as qubits or qudits of finite dimensions, linear storage of light has its advantages. For example, one potential way to improve the lifespan of quantum information in a quantum memory is to use error correction codes, which can protect encoded quantum information against various kind of noise. Recently, there have been great progress on continuous-variable error correction codes [24–26], among which are the single-mode bosonic codes [27–30]. They encode quantum information in multiple excitations of a single bosonic mode, which protects the information against linear noise on bosonic modes. To employ bosonic codes for quantum memory, storage of light with a linear system is a requirement. Though quantum mechanics is in general linear since Schrödinger equation is linear, linearity in quantum optics refers to the linearity of Heisenberg equation for bosonic annihilation operators. It can ensure that, apart from loss, bosonic excitations on the same mode would remain unseparated.

The main challenge to stopping light is a relation called delay-bandwidth product [18]: the faster a photon can enter a cavity, the shorter the photon shall stay in the cavity. One way around this challenge is via tunable coupling [31–33], which is available to superconduction systems. In the optical regime, it is instead easier to shift the frequency of optical modes [34]. Ever since the underlying physics was
experimentally observed [35], adiabatic frequency tuning of photon by tuning the refractive index [36] has been a rapidly maturing technique [37–40], which shows promising potential in photonic memory [41, 42]. In this work, we propose a linear storage device of multiple photons capable of individual on-demand recall based on adiabatic frequency tuning as well as dark modes [43–47]. We consider a number of tunable modes that are uniformly coupled to a central mode, which in turn is coupled with a single input–output channel. When the device is idle, all the tunable modes have the same frequency. Under such condition, by discrete Fourier transformation, these tunable modes can be redefined as a series of non-local modes, among which only one is bright, i.e., directly coupled to the central mode. All other non-local modes are isolated dark modes, which are obviously good candidate for quantum memories.

Dark modes are not straightforwardly accessible, and the approach of their effective manipulation remains an open discussion. Here, we assume that the only controllable parameters are the frequencies of the tunable modes. Within such constraint, we give explicit solutions to a series of control sequences, with which we show that a significant number of the aforementioned dark modes can be access indirectly but individually, enabling the storage and on-demand recall of multiple photons. The capability of accessing individual modes is important to the integrated implementation of devices such as quantum repeaters [48–50], which requires flexible quantum memory of sufficient capacity and has imminent application as the infrastructure of quantum internet [51, 52]. In general, for the device to work, the number of tunable modes has to be double an odd number, and the overall capacity of such a mass storage device is half the number of tunable modes.

The remaining of this paper is organized as follows: in section 2, we introduce the model of the multi-mode photonic storage device and characterize its physical constraints; in section 3, we give the arithmetic basis for the scalability of the device and outline its basic operations; in section 4, we show how the excitations on the dark modes can be safely shuffled among a set of non-local modes; in section 5, we explain and demonstrate the transfer of excitations between the input–output channel and one particular mode within the aforementioned set of modes; in section 6, we discuss and characterize the effect of various possible imperfections; and finally in section 7, we conclude our results.

2. Model

We begin with a collection of $n$ bosonic modes $a_j$, with tunable frequency $\omega_j, j = 0, 1, \ldots, n - 1$. As shown in figure 1(a), all of them are uniformly coupled to a single bosonic mode $c$ of frequency $\omega_c$, which is in turn connected to a single input–output waveguide. The Hamiltonian reads:

$$H = \sum_{j=0}^{n-1} \omega_j a_j^\dagger a_j + \sum_{j=0}^{n-1} \left( c^\dagger a_j + c a_j^\dagger \right) + H_B, \quad (1)$$

$$H_B = \omega_c c^\dagger c + \int_0^\infty d\nu \left\{ \nu B_0^\dagger B_0 + \sqrt{\frac{\kappa}{2\pi}} \left[ B_0^\dagger (c^\dagger c) + B_0 (c c^\dagger) \right] \right\}, \quad (2)$$

where $a_j, c, B_\nu$ are bosonic annihilation operators, and $g$ is the constant coupling coefficient between the uniformly coupled tunable modes $a_j$ and mode $c$. $H_B$ characterizes the energy of mode $c$ and the input–output channel. Channel mode $B_\nu$ is coupled to mode $c$ with a fixed coefficient $\kappa$ independent of frequency $\nu$.

We note that such a physical configuration was already realized experimentally in reference [46] as a copper box containing individually controllable yttrium iron garnet spheres. The same Hamiltonian and control is potentially available in other systems as well. In the optical regime, it has been shown that arbitrary coupling between cavities can be realized with a box of beam splitters [53]. Alternatively, one can arrange beam splitters into ring cavities [54–57]. Then, the frequency of each cavity can be controlled by either moving the mirrors of the cavity or tuning the refractive index of the medium within the cavity. Also, frequency tuning has also been realized in fiber cavities [58], where the interaction between fibers can be realized by evanescent coupling [59, 60]. Moreover, for more miniaturized application, in the highly connectable superconducting systems, a great advancement in tunable resonators [61] was recently made.

We consider $\omega_j \equiv \omega_r$ as the idle state for this device. Since the couplings of tunable modes $a_j$ are already the same, if all of them were to have identical frequency as well, they would be completely interchangeable in the system. It is quite intuitive that dark modes must emerge when there are interchangeable modes in a quantum system. Consider different scenarios where the same excitations were initialized on different ones of them. If full emission were to happen in each one of those scenarios, e.g., $\left| \psi_{\text{out}} \right> = e^{-iHt} \left| a_j^\dagger \right| \text{vac}$, then, with mathematically identical initial conditions, each of these scenarios would give identical predictions on
the final state in the environment, i.e., \( |\psi^\text{out}\rangle \equiv |\psi_k^\text{out}\rangle \), which would violate unitarity. Ergo, none of these modes can be allowed to have a full emission.

For a better picture of the dark modes, consider a mode transformation as follows:

\[
\vec{b} = \sqrt{n} F^{-1} \vec{a} \quad \Leftrightarrow \quad b_k \equiv \frac{1}{\sqrt{n}} \sum_j e^{2\pi i j n a_j},
\]

\[
\vec{a} = \frac{1}{\sqrt{n}} F \vec{b} \quad \Leftrightarrow \quad a_j = \frac{1}{\sqrt{n}} \sum_k e^{-2\pi i j n b_k},
\]

where \( \vec{a} \equiv \{a_j\} \), \( \vec{b} \equiv \{b_k\} \), satisfying \([b_k, b_{k'}^\dagger] = \delta_{kk'}\), are the annihilation operators of a collection of non-local modes. \( F \) and \( F^{-1} \) stands for discrete Fourier transformation and its inverse respectively. We therefore have:

\[
H_I = \sqrt{n} g \left( c^\dagger b_0 + c b_0^\dagger \right).
\]

As shown in figure 1(b), only the 0th non-local mode is directly coupled with mode \( c \). The other non-local modes are therefore potentially dark modes. We also have:

\[
H_0 = \sum_{kk'} h_{kk'} b_k^\dagger b_{k'},
\]

\[
h_{kk'} = \frac{1}{n} \sum_j \omega_j e^{2\pi i j (k-k') n} = h_{k-k'},
\]

where the indices \( k \) of non-local modes are always modulo \( n \). \( h_0 \) is the uniform frequency of all non-local modes, while \( h_x \), with \( x = 1, 2, \ldots, n-1 \), characterizes the coupling between mode \( b_k \) and mode \( b_{k+x} \) for any \( k \).

When \( \omega_j \equiv \omega_r \), we have \( h_x = \delta_{x0} \omega_r \), meaning all the non-local modes are decoupled from one another. Non-local modes other than \( h_0 \) are therefore isolated dark modes under a uniform frequency. And when the frequencies of each mode \( a_j \) are independently detuned, we have:

\[
\vec{h} = F^{-1} \vec{\omega},
\]

\[
\vec{\omega} = F \vec{h},
\]

where \( \vec{h} \equiv \{h_j\} \), \( \vec{\omega} \equiv \{\omega_j\} \). Apparently, we have some level of control over the couplings of the non-local modes, but not without limitations:

Firstly, shifts in the frequencies of the uniformly coupled modes are limited to real numbers, i.e. \( \omega_j \in \mathbb{R} \), for which the necessary and sufficient condition reads:

\[
h_{x-x} = h_x^r,
\]

where \( h_{x-x} \) and \( h_x^r \) are always a pair of coefficients that characterizes the interaction between a pair of non-local modes.

And secondly, as given by equation (6), the couplings between the non-local modes can only be invariant under cyclic permutations of the form \( b_k \rightarrow b_{k+1} \).
3. Compartmentalization

To realize a storage device for multiple photons, we would prefer to have individual access to each storage mode, which is clearly not possible for the given system. The challenges are as follows: when all the uniformly coupled modes are of the same frequency, only one non-local mode \( b_0 \) is indirectly connected to the input–output channel, which prohibits access to other modes; and when the frequencies are detuned, due to the constraint of permutation invariance on their couplings given by equation (6), each non-local mode is subject to couplings, which can potentially lead to loss of all photon in storage.

Despite the limitation, we find the given system is just flexible enough to realize a quantum memory device with a capacity of up to \( \frac{n}{3} \) bosonic modes available for on-demand access. We begin with how the excitations in storage can be preserved when the uniformly coupled modes are detuned, which demands some of the non-local modes to remain disconnected from the waveguide. Since it is not possible to prevent any non-local modes from coupling under detuning, we instead compartmentalize the coupling: couple the non-local modes in isolated groups as a result of the permutation invariance.

The simplest compartmentalization with \( n = 6 \) is shown in figure 2. When \( h_3 \) is the only non-zero \( h_x \) for \( x \neq 0 \), non-local modes become three isolated groups, as shown in figure 2(a). On the other hand, when \( h_2 = h_4 \) are the only non-zero \( h_x \) for \( x \neq 0 \), which limits couplings to the second nearest neighbors, non-local modes become two isolated groups, as shown in figure 2(b).

For clarity, we arrange \( b_4 \) in a \( 2 \times 3 \) formation. The arrangement is characterized by \( k = 3\alpha_1 + 2\alpha_2 \pmod{6} \), with \( \alpha_1 \in \{0, 1\} \) and \( \alpha_2 \in \{0, 1, 2\} \). \( \alpha_1, \alpha_2 \) are thereby considered as coordinates of mode \( h_x \), and they form an integer coordinate space, which is modular and finite. The two compartmentalization can then be illustrated as vertical and horizontal couplings, as shown in figures 2(c) and (d), respectively.

We then define two basic operations for such a device: the input–output operation and internal circle shift operations, which will be discussed in detail in the following two sections. Basically, the vertical couplings enables the input–output operation: a controlled transfer of bosonic excitations between any non-local modes in isolated groups as a result of the permutation invariance. The horizontal couplings, as shown in figures 2(c) and (d), enables the input–output operation: a controlled transfer of bosonic excitations between any non-local modes in isolated groups as a result of the permutation invariance.

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Similar compartmentalization can be shown for other number \( n \) as well. Generally, it requires \( n \) to be a product of multiple mutually primed numbers as follows:

\[
n = \prod_{l=1}^{s} q_l,
\]

where \( q_l \), with \( l = 1, \ldots, s \), are mutually prime with one another. Consider an integer \( k \) given by:

\[
k = \tilde{k} (\vec{\alpha}) \equiv \sum_{l=1}^{s} \frac{\alpha_l n}{q_l} \pmod{n},
\]

\[
\alpha_l \equiv \alpha_l \pmod{q_l},
\]

where \( \tilde{k}(\vec{\alpha}) \) is a function of integer vector \( \vec{\alpha} \equiv \{\alpha_l\} \). There are a total of \( n \) distinct \( \vec{\alpha} \) in a \( s \)-dimension integer coordinate space, and it can be shown that \( k \) and \( \vec{\alpha} \) have one-to-one correspondence (appendix A). This thereby allows us to arrange \( n \) non-local modes in a \( s \)-dimension formation with \( \vec{\alpha} \) being the integer coordinates of each non-local mode \( b_{\vec{\alpha}} \equiv b_k \) with \( k = \tilde{k}(\vec{\alpha}) \).

With modes arranged in an integer coordinate space, the permutation invariance required by equation (6) can be rewritten as:

\[
h_{\vec{\alpha}l'} \equiv h_{\vec{\alpha}l} = h_{\vec{\alpha}-l} = h_{\Delta \vec{\alpha}},
\]

\[
\Delta \vec{\alpha} \equiv \vec{\alpha} - \vec{\alpha}' \equiv [\Delta \alpha_1, \Delta \alpha_2, \ldots, \Delta \alpha_s].
\]

Couplings limited to the direction of each dimension in the integer coordinate space can therefore be realized independently: for any given \( q_l \), as long as \( h_0 = 0 \) for any \( x \neq \alpha_l \), \( \alpha_l \in \{0, 1, \ldots, q_l\} \), \( \Delta \alpha_l = 0 \) would be guaranteed for every \( l' \neq l \), which would in turn ensure that only coupling in the \( l \)th direction of the integer coordinates space is enabled. We thereby define compartmentalized operation in the \( l \)th direction with a Kronecker product as follows:
Figure 2. Illustration of the simplest compartmentalized couplings. (a) and (b) shows the permutation invariance of the coupling, while (c) and (d) shows the corresponding arithmetic interpretation. (a) and (c) are couplings for an input–output operation, while (b) and (d) can be engineered to realize an internal circle shift operation.

\[
\vec{h}(t) = \vec{h}_l(t) \otimes_{\text{Kron}} \vec{e}_{n/ql}, \quad (15)
\]

\[
\vec{e}_{n/ql} \equiv [1, 0, 0, \ldots, 0], \quad (16)
\]

where vector \(\vec{e}_{n/ql}\) has \(n/ql\) dimensions, and vector \(\vec{h}_l(t) \equiv \{h'_x(t)\}\) has \(ql\) dimensions. Note that the Kronecker product above simply insert 0s into vector \(\vec{h}_l(t)\) to create \(\vec{h}(t)\). We thereby have \(h_{n/ql} = h'_{n/ql}\) while guaranteeing \(h_x = 0\) for other \(x\). Under couplings of this form, only non-local modes on the same line as \(b_0\) in the \(l\)th direction are not isolated from the waveguide. Moreover, as long as \(h'_x\) satisfied \(h'_x - x = (h'_x)^*\), the equation (9) would be guaranteed.

To realize a similar input–output operation as shown in figure 2(c) in a general system, \(q_1 = 2\) in equation (10) is required. Other \(q_l\) then has to be relatively prime with 2, hence the minimum requirement for \(n\) is \(\Pi_{l=2}^{s} q_l = n/2\) being an odd number: \(n = 6, 10, 14, 18, 22, \ldots\). However, if the integer coordinate space had more dimensions, e.g. \(n = 30 = 2 \times 3 \times 5\), more different circle shift operations would be available, which could benefit moving photons within the device.

4. Circle shift operation

Non-local modes with \(\vec{\alpha} = [1, \alpha_2, \ldots, \alpha_s]\) make up a storage medium of \(s-1\) dimensions, in which \([1, 0, \ldots, 0]\) is ready for input or output under \(\vec{h} = \vec{h}_l \otimes_{\text{Kron}} \vec{e}_{n'/2}\). Apparently, one can access all of the storage space as long as one can rotate the storage medium in each of the remaining \(s-1\) directions, e.g., during an operation time denoted \(T\), an ideal single step forward in the \(l\)th direction is given by \(b_{\vec{\alpha} + \Delta \vec{\alpha}}(T) = b_{\vec{\alpha}}(0)\) with \(\Delta \vec{\alpha} = [0, 1, 0, \ldots, 0]\). In the following, we will show how to realize such an operation in the \(l\)th direction using \(\vec{h}(t) = \vec{h}_l(t) \otimes_{\text{Kron}} \vec{e}_{n/ql}\) with \(l \neq 1\).

To determine a protocol of circle shift operation on the given system, we take advantage of another system similar to \(H_0\) as follows:

\[
H'_l(t) = \sum_{j=0}^{n'-4} \omega'_{l,j}(t) \vec{a}_j \vec{a}'_j. \quad (17)
\]

With a similar treatment of mode transformation as equation (3), we also give:

\[
\tilde{b}_k \equiv \frac{1}{\sqrt{n'}} \sum_{j=0}^{n'-4} e^{2\pi i \frac{k}{n'} \vec{a}_j}, \quad (18)
\]
Figure 3. Numerical simulation of circle shift operation in an $n = 2 \times 3$ system with $T = 1$ ns, $q_l = 3$. Different curves correspond to different scenarios with different initial conditions: red dash line, blue solid line and black dot line are initialized with a single excitation on non-local mode $b_0, b_1, b_2$ respectively; whereas the seven boxes give the population of seven different modes over a $3T$ period of time: $\hat{n}_k(t) = \langle b_k^\dagger(t) b_k(t) \rangle$. The simulation is within the single-photon regime and we used $W_j = j (\text{mod } 3)$, $h_z = 0$, $g = 1$ ns$^{-1}$ and a constant $\omega(t) \equiv \omega$. The photon loss to the channel is characterized via a non-hermitian Hamiltonian using substitution $\omega \rightarrow \omega - i\kappa$ with $\kappa = 10$ ns$^{-1}$.

It is easy to show that, for a system in idle, i.e., $\omega^{(l)}_{m}(t) \equiv \omega_{l}$, we have $e^{-i\omega^{(l)}_{m} T} \tilde{b}_k(0) = \tilde{b}_k(T)$. In contrast, a circle shift operation is given as:

$$e^{-i\omega_{l} T} \tilde{b}_{k-m}(0) = \tilde{b}_k(T) \equiv e^{i\int_0^T \omega^{(l)}_{m}(t) \text{d}t} \tilde{b}_k e^{-i\int_0^T \omega^{(l)}_{m}(t) \text{d}t},$$

(19)

where $m$ is the number of steps of the circle shift operation. Substituted with equation (18), we find that it can be easily realized by local operations in the system of equation (17):

$$e^{-i\frac{2\pi}{n'} W_j} e^{-i\omega_{l} T} \tilde{a}_j = e^{i\int_0^T \omega^{(l)}_{m}(t) \text{d}t} \tilde{a}_j e^{-i\int_0^T \omega^{(l)}_{m}(t) \text{d}t} \tilde{a}_j,$$

(20)

which then straightforwardly gives:

$$\int_0^T \omega^{(l)}_{m}(t) \text{d}t = \frac{2\pi W_j}{n'} + \omega_{l} T,$$

(21)

$$W_j = jm \pmod{n'}.$$

(22)

Any $\tilde{\omega}^{(l)}_{n'} \equiv \{\omega^{(l)}_{m}(t)\}$ that satisfies the equations above can therefore realize an $m$-step circle shift operation during time $T$.

What we are after by considering the system of equation (17) is actually given by the inverse Fourier transformation of its operation under $n' = q_l$:

$$\tilde{b}_l(t) = \mathcal{F}^{-1} \tilde{\omega}^{(l)}_{q_l}(t),$$

(23)

which realizes an $m$-step circle shift in the $l$th dimension. Using equation (15) with the formula above, the protocol for circle shift operation with the full original system is therefore explicitly given as follows:

$$\tilde{\omega}(t) = \mathcal{F} \left\{ \left[ \mathcal{F}^{-1} \tilde{\omega}^{(l)}_{q_l}(t) \right] \otimes_{\text{Kron}} \tilde{a}_j \right\}.$$

(24)

We note that $\tilde{b}_l$ from equation (18) can be considered as corresponding to a line of non-local modes in the $l$th direction of a compartmentalized device, whereas $\tilde{a}_j$ has no direct correspondence with the original system.

In figure 3, we simulate the circle operation as figure 2(d) within the single-photon regime. For simplicity, photon loss to the input–output channel is characterized by non-Hermitian Hamiltonian [62].
As can be seen from the figure, the initial single excitation on modes $b_1$ is safely shuffled among $b_1$, $b_2$, $b_3$, whereas initial excitations on $b_0$ and $b_2$ are leaked to mode $c$ and lost to the channel.

5. Input–output operation

5.1. Dynamics

$b_0$, with coordinate $s$ being $[0, 0, \ldots, 0]$, is always connected to the waveguide. We need coupling between $b_0$ and only one other non-local mode $b_\nu$, enabling transfer of quantum information from the waveguide to $b_\nu$ and vice versa. Recall $h_{n-x} = h_n^*$, the only option is therefore with $y = \frac{x}{2}$ and:

$$h_n(t) = h_n^*(t) = \delta_{y0} h_y(t) + \delta_{x0} h_0(t).$$  \hspace{1cm} (25)$$

The couplings enabled by such a control sequence is not limited to that between $b_0$ and $b_\nu$. Though $[1, \alpha_2, \ldots, \alpha_s]$ are not all coupled to $[0, 0, \ldots, 0]$, each one of them are coupled to their counterparts among $[0, \alpha_2, \ldots, \alpha_s]$ with exactly the same coupling coefficient $h_j(t)$. With any given $h(t)$, the Hamiltonian of the uniformly coupled modes reads:

$$H_{0}(t) = \sum_{k=0}^{y-1} H_{k}(t),$$  \hspace{1cm} (26)$$

$$H_{k}(t) \equiv h_0(t) \left( b_k^\dagger b_k + b_{y+k}^\dagger b_{y+k} \right) + h_j(t) \left( b_j^\dagger b_{j+k} + b_{j+k}^\dagger b_{j+k} \right).$$  \hspace{1cm} (27)$$

In other words, during the input–output operation, all non-local modes are affected. However, apart from $[0, 0, \ldots, 0]$ and $[1, 0, \ldots, 0]$, the whole process can still preserve the excitations stored on the non-local modes as long as they return to where they were by the end of the operation, which can be easily guaranteed by:

$$\int_0^T h_j(t) dt = 2\pi X,$$  \hspace{1cm} (28)$$

where $X$ can be any arbitrary integer. This requirement can in turn be ensured by introducing arbitrary ‘movement’ of $h_j(t)$ before an input operation or after an output operation.

The first and the second term of equation (27) commute. For $k \neq 0$, it is then easy to show under equation (28) that their Heisenberg operators satisfy:

$$b_k(T) = e^{-i\int_0^T h_0(t) dt} b_k(0).$$  \hspace{1cm} (29)$$

With all other pairs of non-local modes taken care of, the Hamiltonian involved with the input–output process is thereby limited to:

$$H_{0} = H_{0} + H_{1} + H_{C}$$

$$= h_0(t) \left( b_0^\dagger b_0 + b_\nu^\dagger b_\nu \right) + \omega c^\dagger c + h_j(t) \left( b_j^\dagger b_j + b_{j+k}^\dagger b_{j+k} \right) + \sqrt{\kappa} \left( c^\dagger b_0 + cb_0^\dagger \right)$$

$$+ \int_{-\infty}^{\infty} \left\{ \nu B_{\nu}^2 \right\} \left\{ \frac{\kappa}{2\pi} \left[ b_\nu^\dagger c + B_{\nu} c \right] \right\} d\nu,$$

where a rotating frame of high frequency is assumed and the lower bound of the frequency integral is ignored. For clarity, we apply physically trivial mode substitutions $b_\nu \rightarrow -i\hbar_0, c \rightarrow -c, B_{\nu} \rightarrow iB_{\nu}$ to the Hamiltonian above, which in turn gives the following dynamics of Heisenberg operators [63]:

$$\frac{\partial}{\partial t} b_j(t) = -i\hbar_0(t) b_j(t) - h_j(t) b_0(t),$$  \hspace{1cm} (31)$$

$$\frac{\partial}{\partial t} b_0(t) = -i\hbar_0(t) b_0(t) + h_j(t) b_j(t) - \sqrt{\kappa} c(t),$$  \hspace{1cm} (32)$$

$$\frac{\partial}{\partial t} c(t) = -i\omega c(t) + \sqrt{\kappa} b_0(t) - \frac{\sqrt{\kappa}}{2} c(t),$$  \hspace{1cm} (33)$$

$$C(t) \equiv B_{\nu}(t) + \frac{\sqrt{\kappa}}{2} c(t) = B_{\nu}(t) - \frac{\sqrt{\kappa}}{2} c(t),$$  \hspace{1cm} (34)$$

...
where $B_i(t)$ and $B_o(t)$ are time-domain bosonic operators of the input and output field, which are given as Fourier transformations of Heisenberg operators $B_r(\tau)$ from before and after the interaction, respectively:

$$B_i(t) = \lim_{\tau \to +\infty} \int_{-\infty}^{+\infty} \frac{d\nu}{\sqrt{2\pi}} B_r(\tau) e^{-i\nu(t-\tau)},$$

$$B_o(t) = \lim_{\tau \to +\infty} \int_{-\infty}^{+\infty} \frac{d\nu}{\sqrt{2\pi}} B_r(\tau) e^{-i\nu(t-\tau)}.$$

### 5.2. Analysis

To better illustrate the limitations on the input–output control, we consider mirror systems that also obey equations (31)–(33). Generally, a mirror system can be given as follows:

\begin{align*}
B'_i(t) &= B_i(-t), & \omega' &= -\omega, & g' &= -g, & C'(t) &= C(-t), & h'_o(t) &= -h_o(t), \\
B'_r(t) &= b_o(-t), & c' &= c(-t), & b'_o(t) &= b_o(-t), & h'_o(t) &= -h_o(t), & h'_r(t) &= h_r(-t),
\end{align*}

where all the mirror system counterparts of all operators and numbers are denoted with the prime symbol.

We note that a negative frequency $\omega'$ can be considered valid in a rotating frame. Alternatively, within $h_0(t) = h'_0(t) = \omega_c = \omega'_c = 0$, which can also be valid in a rotating frame, there’s a simpler mirror system:

\begin{align*}
B'_i(t) &= b_i(-t), & c'(t) &= c(-t), & C'(t) &= -C(-t), & h'_o(t) &= h_o(-t), \\
B'_r(t) &= -b_o(-t), & g' &= g, & h'_r(t) &= h_r(-t).
\end{align*}

In both mirror systems, we have $C'(t) = -C(-t)$ and $c'(t) = c(-t)$, equation (34) thereby gives:

$$B'_i(t) = -B_o(-t), \quad B'_r(t) = -B_i(-t).$$

These formulas directly connect the perfect absorption of the system to the full emission of the system, which allow us to examine the input field $B_i(t)$ under a given $h_i(t), h_o(t)$ via a straightforward calculation of emission $B'_r(t)$ of either mirror system. A full emission of the mirror system therefore produces the photon wave function that can be perfectly absorbed by the original system.

Particularly, in the second mirror system, the only physical change to the system is a flip of $h_i(t)$. In other words, a device that operates within $h_0(t) \equiv \omega_c$ can always perfectly absorb the time reverse of its own output field with a time flipped operation of $h_i(t)$. In figure 4(a), such a system emits a single photon with a particular wave function to the channel. In figure 4(b), within the same rotating frame, another photon with the time reversion of this wave function is fully absorbed using a time reversion of the same control sequence.
To address the controlled absorption of arbitrary input line shape, we also consider a mirrored emission process. For simplicity, we assume an initial state $|y\rangle = b^\dagger_i |\text{vac}\rangle$ within the single-photon regime and ignore the input field. Under $h_0(t) \equiv \omega_c = 0$, equations (31)–(34) gives:

$$\hat{\phi}(t) = \sqrt{\mu g} b_i(t) y(t) - n g^2 \phi'(t) - \frac{\kappa}{2} \hat{\phi}(t),$$

where the probability amplitude of the photon on mode $b_i$ is denoted as $y(t) = \langle \text{vac}|b_i H(t,0)|y\rangle = \langle \text{vac}|b_i(t)b_i^\dagger|\text{vac}\rangle$, where $H(t,0)$ is the evolution operator. $\phi(t) = \langle \text{vac}|b_i(t)b_i^\dagger|\text{vac}\rangle$ is the wave function of the emitted photon, and they satisfy $y(0) = 1$, $\phi(0) = \phi(0) = 0$. Considering that $\hat{\phi}(t)$ and $\phi(t)$ are both determined by the past value of $\hat{\phi}(t)$, the second order derivative $\phi'(t)$ can be considered as directly controlled by $h_i(t)$ as long as $\phi(t) \neq 0$. Two conclusions thereby follow:

First, as long as $h_i(t)\phi'(t)$ and $\phi(t)$ are finite, $\hat{\phi}(t)$ has to be continuous or $\hat{\phi}(t)$ would be infinite. Therefore, the first order derivative of the single photon wave function $\phi(t)$ must be continuous for either full emissions or perfect absorptions.

Second, even if $h_i(t)$ can take arbitrarily large value, it could still loss control of $\hat{\phi}(t)$ if $\phi(t) = 0$ occurs before the emission process is completed. Therefore, not all desired wave function can be achieved with this device.

However, a high efficiency absorption may still be achieved, as long as the wave function given by the control sequence closely approximates the wave function of the input photon. This can be quantified in terms of the mirror emission process as well. Suppose a desired emission wave function is $\phi(t)$, and it is still possible to obtain a wave function closely approximated by $h_i(t)$ as long as $\phi(t) \neq 0$. Two conclusions thereby follow:

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6. Effect of imperfections

In principle, the proposed device is scalable: by adding more tunable cavities, more photons can be stored in this integrated device. However, in experiment and application, noise and defects in fabrication are unavoidable. In the following, we analyze the impact of several expected defects, and show that they do not increase with the size of the system.

6.1. Coupling deviation in idle

We assume that, out of all the parameters of the system, the coupling coefficients between modes $a_i$ and mode $c$ can no longer be adjusted after the manufacture of this theoretical device. Such deviations are therefore likely costly to reduce, and they affect equation (1) as follows:

$$H_i \rightarrow \sqrt{n g} \left( c^\dagger d + \bar{d} c \right),$$

$$d \equiv \sum_{j=0}^{n-1} \frac{g_j}{\sqrt{n g}} a_j,$$

where coupling coefficients $g_j$ are no longer perfectly the same. Also, we assume $[d, \bar{d}] = 1$, which entails $ng^2 = \sum g_j^2$. $\bar{g}$ is therefore the root square mean of $g_j$. It can then be shown that:

$$\left| \left\langle \left[ d, b^\dagger_i \right] \right\rangle \right|^2 = \left| \left\langle \text{vac}|db^\dagger_i|\text{vac}\rangle \right\rangle^2 = \left[ \sum_{j=0}^{n-1} g_j^2 \right] \sum_{j=0}^{n-1} g_j^2 > 1 - S^2_n / \bar{g}^2,$$

$$S^2_n \equiv \frac{1}{n-1} \left\{ \sum_{j=0}^{n-1} g_j^2 - \left[ \sum_{j=0}^{n-1} g_j \right]^2 \right\},$$

where commutator $[d, b^\dagger_i]$ is always the product of a complex number and the identity operator. The mean of the commutator is therefore independent of quantum state, and with regard to the vacuum state, it equals the inner product of single excitation states on bosonic mode $d$ and $b_i$. We thereby use commutators
to evaluate the overlap between bosonic modes. Also, \( S_g^2 \) is the sample variance of \( g_j \), which quantifies the deviation of \( g_j \) from uniformity.

When the tunable cavities are no longer identical, the general argument of unitarity from section 2 no longer applies. However, as we will show in the following, for this particular system, the presence of dark modes can still be guaranteed, which still prevents a full emission in long-time limit.

Given an arbitrary bosonic mode \( \hat{d}_n \) therewithin, an arbitrary linear bosonic system with \( n \) modes denote as \( H_n \) can always be transformed into a chain of bosonic modes [64] with only nearest neighbor interactions. With \( H_n \) transformed into a chain that begins with mode \( \hat{d}_d \), if the interaction between two neighboring modes were 0, all modes on the other side would be isolated from mode \( \hat{d}_d \), mode \( c \) and the channel. It can be shown that the modes and couplings in such a chain can be determined iteratively by the following formulas:

\[
H_{n-1} = H_n - D_{n,r_{n-1}} \hat{d}_n - D_{n,n-1} \hat{d}_{n-1} - D_{n-1,n} \hat{d}_{n-1} \hat{d}_n,
\]

\[
D_{n,n-1} \hat{d}_{n-1} = [\hat{d}_n, H_n] - D_n d_n,
\]

\[
D_{mn} = \left[ [\hat{d}_{an}, H_n] \hat{d}_m \right],
\]

where interaction between neighbors is characterized by \( D_{n,n-1} = D_{n-1,n} \) and \( [\hat{d}_n, H_{n-1}] = 0 \).

For our device, which has a mode-degenerate \( H_n = \omega \sum_{j=0}^n a_j^\dagger a_j \) in its idle state, the arbitrary bosonic mode \( \hat{d}_n \equiv \hat{d} \) satisfies \( [\hat{d}_n, H_n] = \omega \hat{d}_n \), equations (46) and (47) then give:

\[
D_{n,n-1} \hat{d}_{n-1} = 0,
\]

which means the chain breaks after mode \( d \), and hence all the modes after \( d \) are dark modes. This result applies as long as the device is idle. Excitations that are allowed to dissipate is thereby restricted to only one mode \( d \) directly coupled to mode \( c \), albeit mode \( d \) may deviate slightly from \( b_0 \).

In the following, we quantify the effect of the deviation of mode \( d \) from \( b_0 \). For simplicity, we still consider an excitation on mode \( b_k \) as the standard of a photon in perfect storage, which maintains the same definition as the ideal case in equation (4).

Within the single-photon regime, we formalize a complete photon loss on an arbitrary mode \( a_j \) using operator \( U(a_j) \equiv 1 - a_j^\dagger a_j \), which satisfies:

\[
\langle \text{vac}| a_j U(a_k) a_j^\dagger |\text{vac} \rangle = 1 - \delta_{kj}.
\]

By using \( d = \sum \sum |d, b^\dagger_m \rangle b_m \) and \( \langle \text{vac}| b^\dagger = 0 \), and by reducing several commutators to complex numbers, we get:

\[
\langle \text{vac}| b_k U(d) = \langle \text{vac}| \left( 1 - |f_k|^2 \right) b_k - \langle \text{vac}| \sum m \neq k f_m^* b_m,
\]

where complex number \( f_m \equiv \langle b_m, d^\dagger \rangle \). In the right side of the equation above, the first term characterizes photon loss, and the second term characterizes crosstalk. Using \( \sum m = 0 \sum |f_m|^2 = 1 \) and equation (43), we find the inner product of the second term with itself satisfies \( \sum m \neq k |f_m|^2 < |f_k|^2 < S_{g}^2 / 2 \). Apparently, as the device remains idle for a long time, the photon loss and crosstalk eventually introduced by the deviation of couplings is limited by \( S_{g}^2 / 2 \).

6.2. Coupling deviation during input–output

During the input–output operation, equations (8) and (25) give:

\[
\omega_j(t) = \sum x e^{-2i \xi_j} h_x(t) = h_0(t) + e^{-i \pi} h_j(t),
\]

in which \( e^{-i \pi} = \pm 1 \), depending on whether \( j \) is even or odd. Substituting the formula above into the definition of \( H_0 \) in equation (1), \( H_0(t) \) thereby is a sum of two mode-degenerate systems with respective frequencies \( h_0(t) \pm h_j(t) \).

We then repeat the same analysis of dark modes as a chain of bosonic modes. For a sum of two mode-degenerate systems, which we will denote with subscripts \( A \) and \( B \), the total Hamiltonian \( H_n \) and an arbitrary beginning of the chain \( d_n \) can be written as follows:

\[
H_n = H_{An} + H_{Bn},
\]

\[
d_n = \alpha d_{An} + \beta d_{Bn},
\]
where bosonic modes $d_{An}$ and $d_{Bn}$ satisfy:

\[
[d_{An}, H_{An}] = E_A d_{An},
\]

\[
[d_{Bn}, H_{Bn}] = E_B d_{Bn},
\]

\[
[d_{An}, H_{Bn}] = [d_{Bn}, H_{An}] = 0,
\]

where $E_A$ and $E_B$ are the photon energies in the two mode-degenerate subsystems, respectively. Note that, at moment $t$ during an input–output operation of our device, $E_A = h_0(t) + h_y(t)$ and $E_B = h_0(t) - h_y(t)$. For each such moment, equations (46) and (47) give:

\[
D_{n, n-1} d_{n-1} = \alpha \beta (E_A - E_B) (\beta^* d_{An} - \alpha^* b_{Bn}).
\]

Subsequently, equation (45) gives:

\[
[d_{n-1}, H_{n-1}] = [d_{n-1}, H_n] = D_{n-1, n} d_{n},
\]

which make it easy to show with equations (46) and (47) that:

\[
D_{n-1, n-2} d_{n-2} = 0.
\]

This shows that for a sum of two mode-degenerate systems, a chain starting at arbitrary bosonic mode $d_n$ ends after $d_{n-1}$, which is applicable to our device throughout the input–output operation: mode $c$ is coupled to mode $d_n = d$, which in turn is coupled with another mode $d_{n-1}$, after which the chain must end regardless of $g_j$.

The chain of mode $c$, $d_n$ and $d_{n-1}$ can be considered as a replacement of the chain of $c$, $b_0$ and $b_1$ from equation (30). Note that equation (57) shows $d_{n-1}$ is independent of the energy difference, which guarantees a consistent $d_{n-1}$ throughout an input–output process with a time-dependent energy difference. Therefore, the same structure of equation (30) remains in the presence of deviation in $g_j$, with $d_n \equiv d$ replacing $b_0$ and $d_{n-1}$ replacing $b_1$. In the mean time, the coupling coefficient $h_x(t)$ between $b_0$ and $b_1$ in equation (30) is replaced by coupling coefficient $2 \alpha \beta h_y(t)$ between $d_n$ and $d_{n-1}$ given in equation (57).

With a proper $2 \alpha \beta h_y(t)$ and a compatible wave function of photon, a full transfer between the channel and mode $d_{n-1}$ can still be realized despite the deviation in $g_j$. However, mode $d_{n-1}$ differs from mode $b_1$, which is where the photon is supposed to be stored. This contributes to photon loss and crosstalk. Nevertheless, we note that equation (57) gives a linear relation between $d_{n-1}$ and mode $d$. According to equation (43), the deviation of $d$ from $b_0$ is limited by $S_j^2 / \tilde{g}^2$, which therefore guarantees a similar limit to the deviation of $d_{n-1}$ from $b_1$ as well as the subsequent photon loss and crosstalk.

6.3. Photon loss

We characterize the photon loss of physical modes with Lindbladian superoperator as follows [65]:

\[
\mathcal{L} (\rho) = \sum_j \gamma_j \left[ a_j \rho a_j^\dagger - \frac{1}{2} \left( a_j^\dagger a_j \rho + \rho a_j^\dagger a_j \right) \right]
\]

\[
= \sum_{k k'} \Gamma_{k k'} \left[ b_k \rho b_k^\dagger - \frac{1}{2} \left( b_k^\dagger b_k \rho + \rho b_k^\dagger b_k \right) \right],
\]

\[
\Gamma_{k k'} = \Gamma_{k' k} = \frac{1}{n} \sum_j \gamma_j \mathrm{e}^{-2 \pi i \frac{\delta_{k k'}}{\tilde{g}}},
\]

where $\rho$ is the density matrix of the full system. For $k = k'$, it is apparent that $\Gamma_0 = \gamma \equiv \frac{4}{n} \sum_j \gamma_j$, hence the photon loss of each non-local mode does not increase with device capacity.

For $k \neq k'$, with $\bar{\Gamma} \equiv \{ \Gamma_j \}, \bar{\gamma} \equiv \{ \gamma_j \}$, we have $\bar{\Gamma} = \frac{4}{n} \bar{F} \bar{\gamma}$. Similar to equation (3), it is guaranteed that $|\bar{\Gamma}|^2 = \frac{4}{n} |\bar{\gamma}|^2$. This straightforwardly leads to:

\[
\sum_{x=1}^{n-1} |\bar{\Gamma}_x|^2 = \frac{n-1}{n} S_j^2,
\]

\[
S_j^2 \equiv \frac{1}{n-1} \sum_{j=0}^{n-1} (\gamma_j - \bar{\gamma})^2,
\]
Figure 5. The probability evolution of a single photon within the $n = 6$ device. 30 iterations are drawn in superimposed semi-transparent blue lines under random noise parameters. The device is undergoing a series of arbitrary operations that can be discerned from the figure. Ideal $\omega_j(t)$ used are given by equations (21)–(24): under $m = 1$ and $\omega_r = 0$, for $0 < \tau < 1$, $n' = 2$ and $T = 0.5$ ns are used; for $1 < \tau < 4$, $n' = 3$ and $T = 1$ ns are used, where $\tau \equiv t n^{-1} (\text{mod} 4)$. In each iteration, the couplings are random numbers in the range of $2$ ns$^{-1}$ < $g_j$ < $4$ ns$^{-1}$. Also, the energies of the tunable modes are off and they are dissipating, which we characterize by $\omega_j(t) \rightarrow \omega_j(t) + \Delta \omega_j - i \Delta \kappa_j$, where $0 < \Delta \omega_j < 0.02$ ns$^{-1}$, $0 < \Delta \kappa_j < 0.02$ ns$^{-1}$. All random numbers are given by an even distribution of probabilities.

where $S^2_\gamma$ is the sample variance of $\gamma_j$, which can characterize the crosstalk of the device. We focus on any one of the non-local mode $b_K$. Items from equation (60) with $k \neq k'$ that satisfy either $k = K$ or $k' = K$ add up to:

$$L_K(\rho) = \Gamma' \left[ b_K \rho b_K^\dagger - \frac{1}{2} \left( b_K^\dagger b_K^\dagger \rho + \rho b_K b_K^\dagger \right) \right] + \text{h.c.},$$

$$\bar{b}_K \equiv \sum_{x=1}^{n-1} \frac{\Gamma_x}{\Gamma'_x} b_{K+x},$$

$$|\Gamma'|^2 = \sum_{x=1}^{n-1} |\Gamma_x|^2 = \frac{n-1}{n} S^2_\gamma < S^2_\gamma,$$

where bosonic mode $\bar{b}_K$, a superposition of other non-local modes $b_{K+x}$ with $x \neq 0$, is fully responsible for the crosstalk noise on $b_K$. Apparently, for each non-local mode $b_K$, the magnitude of crosstalk noise from photon loss is limited by the sample variance of photon loss $S^2_\gamma$. Statistically, $S^2_\gamma$ is largely determined by the variance that characterizes the fabrication process of mode $a_j$, which is not expected to change as the same process is repeated. Therefore, for each non-local mode, the crosstalk noise is not expected to increase with the capacity of the device either.

6.4. Deviation in energies and numerical demonstration

Another possible source of error is the deviation of frequencies in the tunable modes. Random frequencies would lead to random couplings among non-local modes, which in-turn leads to both crosstalk and loss. In figure 5, we simulate the operation of the device under multiple kinds of imperfection. Under the combined effect of deviations in mode frequencies, couplings and dissipations, a modest level of loss and crosstalk for a single photon in storage are observed as the device undergoes a series of operations.

7. Discussion and conclusion

In this paper, we propose a scalable design of a linear optical device that can store and recall multiple individual photons. The device is in a star configuration: multiple single-mode cavities for storage are
uniformly coupled to one central single-mode cavity that is also connected to the input--output channel. The dynamics of the storage cavities are understood via a Fourier transformation that gives a collection of non-local modes. When the device is idle, only one of the non-local modes is actually coupled to the central cavity, while the others are dark modes.

We show that by applying specific adiabatic frequency tuning on the storage cavities, couplings of the dark modes can be controlled in a compartmentalized manner: couplings are limited to groups of dark modes. In this manner, we realize the input--output operation between one specific dark mode and the channel, as well as the means to move excitations safely among the dark modes. Overall, the number of safely and independently accessible dark modes, i.e., the storage capacity, is half the number of storage cavities.

As a basic requirement, the number of the cavities for storage has to be the product of 2 and an odd number. However, it is preferable that the number is a product of many mutually prime numbers. More mutually prime numbers means more versions of compartmentalized operations, which can potentially reduce the time to address each bosonic mode in storage.

Moreover, certain effect of imperfection are analyzed, including violation of the uniform couplings and photon loss in each of the cavities for storage. The dark modes are robust against deviation in the couplings, but the presence of the deviation do introduce limited loss and crosstalk to each operation. Photon loss can also introduce crosstalk when it is happening with different dissipation rates on different storage cavities, and the effect of such crosstalk on each non-local mode is shown to be smaller than the sample variance of the different dissipation rates. These effects are not expected to increase with the capacity of the system, which indicates that the design is scalable.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Relation between \(k\) and \(\vec{\alpha}\)

Here we show that \(k\) and \(\vec{\alpha}\) from equation (11) are mutually uniquely determined. Consider \(k\) and \(k'\) as follows:

\[
k = \sum_i \alpha_i \frac{n}{q_i} \pmod{n},
\]

\[
k' = \sum_i \alpha'_i \frac{n}{q_i} \pmod{n}.
\]

We begin with the assumption:

\[
k - k' = 0 \pmod{n},
\]

which gives:

\[
\sum_i (\alpha_i - \alpha'_i) \frac{n}{q_i} + Xn = 0,
\]

where, as a substitute of modulo, \(X\) is an arbitrary integer. The equation above can be easily rewritten as:

\[
\sum_{i \neq m} (\alpha_i - \alpha'_i) \frac{n}{q_i q_m} + X \frac{n}{q_m} q_m = - (\alpha_m - \alpha'_m) \frac{n}{q_m} q_m,
\]

where an arbitrary \(q_m\) among \(q_i\) is singled out. Apparently, given equation (10), every term in the left-hand side is an integer multiple of \(q_m\). In the right-hand side, since \(n/q_m\) is supposed to be mutually prime with \(q_m\), it thereby follows that \(\alpha_m - \alpha'_m\) has to be an integer multiple of \(q_m\). In other words:
\( \alpha_m' - \alpha_m = 0 \pmod{q_m} \). \hfill (A.6)

The assumption of equation (A.3) is thereby shown to guarantee equation (A.6) for any given \( m \). This in turn shows that two different \( \alpha \) cannot be associated with the same \( k \). There are a total of \( n \) unique \( \alpha \), which have to give \( n \) different \( k \). Considering that there are only a total of \( n \) distinct \( k \) and \( \alpha \) thereby have one-to-one correspondence.

**References**

[1] Choi K S, Deng H, Laurat J and Kimble H J 2008 *Nature* 452 67–71
[2] Héhet G, Longdell J J, Alexander A L, Lam P K and Sellars M J 2008 *Phys. Rev. Lett.* 100 023601
[3] Lvovsky A I, Sanders B C and Tittel W 2009 *Nat. Photon.* 3 706–14
[4] Hedges M P, Longdell J J, Li Y and Sellars M J 2010 *Nature* 465 1052–6
[5] Hosseini M, Campbell G, Sparks B M, Lam P K and Buchler B C 2011 *Nat. Phys.* 7 794–8
[6] Zhou Z-Q, Lin W-B, Yang M, Li C-F and Guo G-C 2012 *Phys. Rev. Lett.* 108 190505
[7] Gündoan M, Ledingham P M, Almasi A, Cristiani M and de Riedmatten H 2012 *Phys. Rev. Lett.* 108 190504
[8] Reagor M et al 2016 *Phys. Rev. B* 94 014506
[9] Dutt M V G, Childress L, Tang K, Togan E, Maze J, Jelezko F, Zibrov A S, Hemmer P R and Lukin M D 2007 *Science* 316 1312–6
[10] Zhang J, Hegde S S and Suter D 2018 *Phys. Rev. A* 98 043802
[11] Nguyen C T et al 2019 *Phys. Rev. B* 100 165428
[12] Giovannetti V, Lloyd S and Maccone L 2008 *Phys. Rev. A* 78 052310
[13] Giovannetti V, Lloyd S and Maccone L 2008 *Phys. Rev. Lett.* 100 160501
[14] Hong F Y, Xiang Y, Zhu Z Y, Jiang L N and Wu L N 2012 *Phys. Rev. A* 86 010306
[15] Bang J, Dutta A, Lee S W and Kim J 2019 *Phys. Rev. A* 99 012326
[16] Hann C T, Zou C-L, Zhang Y, Chu Y, Schoelkopf R J, Girvin S M and Jiang L 2019 *Phys. Rev. Lett.* 123 250501
[17] Hosseini M, Sparks B M, Campbell G, Lam P K and Buchler B C 2011 *Nat. Commun.* 2 174
[18] Yanik M F and Fan S 2007 *Phys. Rev.* 6 372–4
[19] Yoshikawa J I, Makino K, Kurata S, van Loock P and Furusawa A 2013 *Phys. Rev. X* 3 041028
[20] Cho Y-W et al 2016 *Optica* 3 100
[21] Merklein M, Stiller B, Vu K, Madden S J and Eggleton B J 2017 *Nat. Commun.* 8 574
[22] Everett J L, Campbell G T, Cho Y-W, Vernaz-Gris P, Higginbottom D B, Pinel O, Robins N P, Lam P K and Buchler B C 2017 *Nat. Phys.* 13 68–73
[23] Li W, Islam P and Windpassinger P 2020 *Phys. Rev. Lett.* 125 150501
[24] Braunstein S L and van Loock P 2005 *Rev. Mod. Phys.* 77 513–77
[25] Niu M Y, Chuang I L and Shapiro J H 2018 *Phys. Rev. A* 97 032323
[26] Hillmann T, Quijandría F, Johansson G, Ferraro A, Gasparinetti S and Ferrini G 2020 *Phys. Rev. Lett.* 125 160501
[27] Michael M H, Silveri M, Brierley R T, Albert V V, Salmilehto J, Jiang L and Girvin S M 2016 *Phys. Rev. X* 6 031006
[28] Albert V V et al 2018 *Phys. Rev. A* 97 032346
[29] Lami L, Plenio M B, Giovannetti V and Holevo A S 2020 *Phys. Rev. Lett.* 125 110504
[30] Noh K and Chamberland C 2020 *Phys. Rev. A* 101 012316
[31] Yin Y et al 2012 *Phys. Rev. Lett.* 110 107001
[32] Wenner J et al 2014 *Phys. Rev. Lett.* 112 211505
[33] Sete E A, Milana E and Kotrolov A N 2015 *Phys. Rev. B* 91 144509
[34] Fan L, Zou C-L, Poot M, Cheng R, Guo X, Han X and Tang H X 2016 *Nat. Photon.* 10 766–70
[35] Reed E J, Soljačić M and Joannopoulos J D 2003 *Phys. Rev. Lett.* 90 203904
[36] Notomi M and Mitsugi S 2006 *Phys. Rev. A* 73 051805
[37] Preble S, Cao L, Elshaari A, Aboketaf A and Adams D 2012 *Appl. Phys. Lett.* 101 171110
[38] Kabakova I V, Yu Z, Halliwel D, Fonjallaz P Y, Tarasenko O, de Sterke C M and Margulis W 2012 *J. Opt. Soc. Am. B* 29 155
[39] Khurgin J B et al 2020 *Optica* 7 226
[40] Minet Y, Reis L, Szabados J, Werner C S, Zappe H and Breunig J 2020 *Opt. Express* 28 2939
[41] Tanabe T, Notomi M, Taniyama H and Kuramochi E 2009 *Phys. Rev. Lett.* 102 043907
[42] Miller J M L, Ansari A, Heinz D B, Chen Y, Fader I B, Shin D D, Villanueva L G and Kenny T W 2018 *Appl. Phys. Rev.* 5 041307
[43] Dong C, Fiore V, Kuzyl M C and Wang H 2012 *Science* 338 1609–13
[44] Wang Y-D and Clerk A A 2012 *New J. Phys.* 14 105010
[45] Panaro S, Nazir A, Liberale C, Wang H, De Angelis F, Proietti Zaccaria R, Di Fabrizio E and Toma A 2014 *Plasmon. Met.* *Nanostuctures Their Opt. Prop.* 9163
[46] Zhang X, Zou C-L, Zhu N, Marquardt F, Jiang L and Tang H X 2015 *Nat. Commun.* 6 8914
[47] Yu Z, Xi X, Ma J, Tsang H K, Zou C-L and Sun X 2019 *Optica* 6 1342
[48] Briegel H-J, Dür W, Cirac J I and Zoller P 1998 *Phys. Rev. Lett.* 81 5932–5
[49] Sangouard N, Simon C, de Riedmatten H and Gisin N 2011 *Rev. Mod. Phys.* 83 33–80
[50] Li Z-D et al 2019 *Nat. Photon.* 13 644–8
[51] Kimble H J 2008 *Nature* 453 1023–30
[52] Chen Y-A et al 2021 *Nature* 589 214–9
[53] Kyoseva E, Beige A and Kwek L C 2012 *New J. Phys.* 14 023023
[54] Bunkowski A, Burmeister O, Danzmann K and Schnabel R 2005 *Opt. Lett.* 30 1183
[55] Paris M G A 2007 J. Phys. B: At. Mol. Opt. Phys. 40 F63–8
[56] Bao X-H et al 2012 Nat. Phys. 8 517–21
[57] Nunn J et al 2017 Phys. Rev. A 96 012338
[58] Srinivasan K and Painter O 2007 Appl. Phys. Lett. 90 031114
[59] Lepert G, Trupke M, Hartmann M J, Plenio M B and Hinds E A 2011 New J. Phys. 13 113002
[60] Anetsberger G, Arcizet O, Unterreithmeier Q P, Riviè re R, Schliesser A, Weig E M, Kotthaus J P and Kippenberg T J 2009 Nat. Phys. 5 909–14
[61] Mahashabde S, Otto E, Montemurro D, de Graaf S, Kubatkin S and Danilov A 2020 Phys. Rev. Appl. 14 044040
[62] Dalibard J, Castin Y and Mølmer K 1992 Phys. Rev. Lett. 68 580–3
[63] Zhang J, Liu Y X, Wu R B, Jacobs K and Nori F 2013 Phys. Rev. A 87 032117
[64] Vasile R, Galve F and Zambrini R 2014 Phys. Rev. A 89 022109
[65] Gardiner C and Zoller P 2004 Quantum Noise (Berlin: Springer)