Improving some sequences convergent to Euler-Mascheroni constant

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Abstract

We obtain the following very fast sequences convergent to Euler-Mascheroni constant:

\[ \theta_n = H_n - \log \left( n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5} \right) \]

and

\[ \phi_n = H_n - \log \left( n + \frac{1}{2} + \frac{1}{24(n + \frac{1}{2})} - \frac{37}{5760(n + \frac{1}{2})^3} + \frac{10313}{2903040(n + \frac{1}{2})^5} \right) , \]

where \( H_n \) are the harmonic numbers defined by \( H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \).

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1. Introduction

Euler’s constant (or Euler Mascheroni constant) $\gamma$ was introduced by Leonhard Euler (1707-1783) in 1734 as the limit of the sequence

$$D_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n. \quad (1.1)$$

This constant is known to be the third most important mathematical constant, next to $\pi$ and $e$. It appears in a lot of places in mathematics such as number theory, analysis, theory of probability, special functions, and differential equations. The convergence of $D_n$ to $\gamma$ is very slowly since

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n} \quad (\text{R.M. Young [12]}),$$

which shows that it converges to $\gamma$ as $n^{-1}$. A faster convergent sequence to $\gamma$ were introduced by DeTemle in [10, 11]. He proved that the sequence $R_n$ defined by

$$R_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log \left(n + \frac{1}{2}\right). \quad (1.2)$$

converges to $\gamma$ with the speed like $n^{-2}$, since

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}. \quad (1.3)$$

Recently Chen [3] obtained sharp form of (1.3) as follows: For all $n \in \mathbb{N}$

$$\frac{1}{24(n+a)^2} < R_n - \gamma < \frac{1}{24(n+b)^2}$$

with the best possible constants

$$a = \frac{1}{\sqrt{24(1 - \gamma - \log(3/2))}} = 0.55106\ldots, \quad \text{and} \quad b = \frac{1}{2}.$$

In 1997, Negoi [9] introduced the sequence

$$T_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log \left(n + \frac{1}{2} + \frac{1}{24n}\right)$$

and showed that

$$\frac{1}{48(n+1)^3} < T_n - \gamma < \frac{1}{48n^3},$$
Improving some sequences convergent to Euler-Mascheroni constant

which shows that the approximation $T_n \approx \gamma$ has a significant superiority over the approximation $R_n \approx \gamma$. For other faster convergences of Euler-Mascheroni constant we refer to [1, 3, 5, 6, 7, 8]. To accelerate the sequence $(T_n)$, Chen and Mortici [2] established the following approximation formula

\[(1.4) \quad \gamma \approx H_n - \log \left( n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \ldots \right). \]

Our first aim here is to improve (1.4) and obtain bounds for $H_n$ in this form. Our second aim is to establish similar formulas to improve $(R_n)$. For this purpose we shall consider the following sequences:

\[(1.5) \quad A_n = H_n - \log \left( n + \frac{1}{2} + \frac{1}{24n} + \frac{\alpha}{n^2} + \frac{\beta}{n^3} + \frac{\delta}{n^4} + \frac{\epsilon}{n^5} \right), \]

and

\[B_n = H_n - \log \left( n + \frac{1}{2} + \frac{a}{n + \frac{1}{2}} + \frac{b}{(n + \frac{1}{2})^2} + \frac{c}{(n + \frac{1}{2})^3} + \frac{d}{(n + \frac{1}{2})^4} + \frac{e}{(n + \frac{1}{2})^5} \right),\]

where $\alpha, \beta, \delta, \epsilon$ and $a, b, c, d, e$ are real parameters. Precisely, we introduce the sequences $(\theta_n)$ and $(\phi_n)$ by

\[\theta_n = H_n - \log \left( n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5} \right), \]

and

\[\phi_n = H_n - \log \left( n + \frac{1}{2} + \frac{1}{24(n + \frac{1}{2})} - \frac{37}{5760(n + \frac{1}{2})^3} + \frac{10313}{2903040(n + \frac{1}{2})^5} \right), \]

both of which converge to $\gamma$ like $n^{-7}$, since

\[\lim_{n \to \infty} n^7(\theta_n - \gamma) = -\frac{2501}{1161216} \]

and

\[\lim_{n \to \infty} n^7\phi_n = -\frac{5509121}{1393459200}. \]

In order to prove our main results we need the following lemma, which is a strong tool to measure and improve the speed of convergence of some sequences having limit equal to zero.
Lemma 1.1. If \((s_n)\) is convergent to zero and there exists the limit
\[
\lim_{n \to \infty} n^k (s_n - s_{n+1}) = c \in \mathbb{R}
\]
then there exists the limit
\[
\lim_{n \to \infty} n^{k-1} s_n = \frac{c}{k-1}.
\]
see [4]. From this lemma it is clear that the speed of convergence of the sequence \((s_n)\) is as higher as the value of \(k\) satisfying (1.6) is as greater as.

2. Main results

Let \(A_n\) be the sequence defined by (1.5). We are interested to find the values of \(\alpha, \beta, \delta\) and \(\epsilon\) which provide the fastest sequence \(A_n\). First, let us write
\[
A_n - A_{n+1} = -\frac{1}{n+1} - \log \left( n + \frac{1}{2} + \frac{1}{24n} + \frac{\alpha}{n^2} + \frac{\beta}{n^3} \right) + \log \left( n + 1 + \frac{1}{2} + \frac{1}{24(n+1)} + \frac{\alpha}{(n+1)^2} + \frac{\delta}{(n+1)^3} \right).
\]

We are concentrated to compute a limit of the form (1.6). In order to do this we use a computer software to obtain the following power series representation in \(\frac{1}{n}\):
\[
A_n - A_{n+1} = \left( -\frac{1}{16} - 3\alpha \right) \frac{1}{n^4} + \left( \frac{263}{1440} + 8\alpha - 4\beta \right) \frac{1}{n^5} + \left( -\frac{139}{8} \right) \frac{1}{n^6} + \left( \frac{3685}{6048} + \frac{229\alpha}{8} + 3\alpha^2 - \frac{115\beta}{4} + 18\delta - 6\epsilon \right) \frac{1}{n^7} + \left( -\frac{8663}{9216} - \frac{27517\alpha}{576} - 14\alpha^2 + \frac{1379\beta}{24} + 7\alpha\beta - \frac{1127\delta}{24} + \frac{49\epsilon}{2} \right) \frac{1}{n^8} + O(n^{-9}).
\]

By Lemma 1.1 faster convergences are obtained by imposing the conditions that the first four coefficients vanish. Now this results in
Improving some sequences convergent to Euler-Mascheroni constant

\[
\begin{align*}
-\frac{1}{16} - 3\alpha &= 0, \\
\frac{263}{1440} + 8\alpha - 4\beta &= 0, \\
\frac{139}{384} - \frac{385\alpha}{24} + \frac{25\beta}{2} - 5\delta &= 0, \\
\frac{3685}{6048} + \frac{229\alpha}{8} + 3\alpha^2 - \frac{115\beta}{4} + 18\delta - 6\epsilon &= 0, \\
-\frac{8663}{9216} - \frac{27517\alpha}{576} - 14\alpha^2 + \frac{1379\beta}{24} + 7\alpha\beta - \frac{1127\delta}{24} + \frac{49\epsilon}{2} &= 0,
\end{align*}
\]

namely,

\begin{align*}
\alpha &= -\frac{1}{48}, \quad \beta = \frac{23}{5760}, \quad \delta = \frac{17}{3840}, \quad \text{and} \quad \epsilon = -\frac{10099}{2903040}. 
\end{align*}

These solutions correspond to the following sequence

\[
\theta_n = H_n - \log \left(n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5}\right),
\]

(2.2)

By replacing the solutions (2.1) above

\[
\theta_n - \theta_{n+1} = -\frac{2501}{165888n^8} + O(n^{-9}).
\]

Now we can state the following

**Theorem 2.1.** Let \((\theta_n)\) be the sequence defined by (2.2). Then

\[
\lim_{n \to \infty} n^8(\theta_n - \theta_{n+1}) = -\frac{2501}{165888} \quad \text{and} \quad \lim_{n \to \infty} n^7(\theta_n - \gamma) = -\frac{2501}{1161206},
\]

namely, the speed of convergences of the sequence \((\theta_n)\) is like \(n^{-7}\)

Let \((B_n)\) be as defined by (1). Then we have

\[
B_n - B_{n+1} = -\frac{1}{n+1} - \log \left(n + \frac{1}{2} + \frac{a}{(n+\frac{1}{2})^2} + \frac{b}{(n+\frac{1}{2})^3} + \frac{c}{(n+\frac{1}{2})^4} + \frac{d}{(n+\frac{1}{2})^5} \right) + \frac{e}{(n+\frac{1}{2})^5} + \log \left(n + \frac{1}{2} + \frac{a}{(n+\frac{1}{2})^2} + \frac{b}{(n+\frac{1}{2})^3} + \frac{c}{(n+\frac{1}{2})^4} + \frac{d}{(n+\frac{1}{2})^5} + \frac{e}{(n+\frac{1}{2})^5}\right).
\]

(2.3)
Using again a computer software we get

\[
B_n - B_{n+1} = \left( \frac{1}{12} - 2a \right) \frac{1}{n^3} + \left( -\frac{1}{4} + 6a - 3b \right) \frac{1}{n^4}
+ \left( \frac{41}{80} - 13a + 2a^2 + 12b - 4c \right) \frac{1}{n^5}
+ \left( -\frac{43}{48} + 25a - 10a^2 - \frac{65b}{2} + 5ab + 20c - 5d \right) \frac{1}{n^6}
+ \left( \frac{645}{448} - \frac{363a}{8} + \frac{65a^2}{2} - 2a^3 + 75b - 30ab + 3b^2 - 65c + 6ac + 30d - 6e \right) \frac{1}{n^7}
\left( -\frac{141}{64} + \frac{637a}{8} - \frac{175a^2}{2} + 14a^3 - \frac{2541b}{16} + \frac{455ab}{4} - 7a^2b - 21b^2 + 175c
- 42ac + 7bc - \frac{455d}{4} + 7ad + 42e \right) \frac{1}{n^8} + O(n^{-9}).
\]

(2.4)

According to Lemma 1.1 we can see that the fastest sequence \( \phi_n \) is obtained in the case when as many of the first coefficients of (2.3) is cancelled. As we have five parameters \( a, b, c, d, e \), they produce the best result if and only if

\[
\frac{1}{12} - 2a = 0,
-\frac{1}{4} + 6a - 3b = 0,
\frac{41}{80} - 13a + 2a^2 + 12b - 4c = 0,
-\frac{43}{48} + 25a - 10a^2 - \frac{65b}{2} + 5ab + 20c - 5d = 0,
\frac{645}{448} - \frac{363a}{8} + \frac{65a^2}{2} - 2a^3 + 75b - 30ab + 3b^2 - 65c + 6ac + 30d - 6e = 0,
\frac{141}{64} + \frac{637a}{8} - \frac{175a^2}{2} + 14a^3 - \frac{2541b}{16} + \frac{455ab}{4} - 7a^2b - 21b^2 + 175c
- 42ac + 7bc - \frac{455d}{4} + 7ad + 42e = 0.
\]
From these we obtain the following solutions:

\( a = \frac{1}{24}, \ b = 0, \ c = -\frac{37}{5760}, \ d = 0, \ e = \frac{10313}{2903040} \)

and these solutions correspond to the following sequence

\[
\phi_n = H_n - \log \left( n + \frac{1}{2} + \frac{1}{24(n + \frac{1}{2})} - \frac{37}{5760(n + \frac{1}{2})^3} + \frac{10313}{2903040(n + \frac{1}{2})^5} \right).
\]

By replacing the solutions given in (2.5) in (2.4) we get

\[
\phi_n - \phi_{n+1} = -\frac{5509121}{174182400}n^8 + O(n^{-9}).
\]

These can be summarized as follow.

**Theorem 2.2.** Let \((\phi_n)\) be the sequence given by (2.6). Then it holds that

\[
\lim_{n \to \infty} n^8(\phi_n - \phi_{n+1}) = -\frac{5509121}{174182400} \quad \text{and} \quad \lim_{n \to \infty} n^7\phi_n = -\frac{5509121}{1393459200},
\]

that is, the speed of convergence of \((\phi_n)\) is like \(n^{-7}\).

**Theorem 2.3.** Let the sequences \((\theta_n)\) and \((\phi_n)\) be as defined (2.2) and (2.6). Then, both \((\theta_n)\) and \((\phi_n)\) are strictly decreasing for \(n \geq 2\) and all natural numbers \(n\), respectively.

**Proof.** We set \(\theta_n - \theta_{n+1} = f(n)\), where

\[
f(x) = -\frac{1}{x+1} - \log \left( x + \frac{1}{2} + \frac{1}{48x^2} + \frac{23}{384x^4} + \frac{17}{3840x^6} - \frac{10099}{2903040x^8} \right)
\]

\[
+ \log \left( x + \frac{3}{2} + \frac{1}{48(x+1)^2} + \frac{23}{5760(x+1)^4} + \frac{17}{3840(x+1)^6} - \frac{10099}{2903040(x+1)^8} \right).
\]

Differentiation gives

\[
f'(x) = \frac{p(x)}{q(x)},
\]

where

\[
p(x) = -223661795575 - 1556403370554x - 4175585115408x^2
\]

\[-4951284518880x^3 - 1613300443776x^4 + 1495234411776x^5
\]

\[+1016470425600x^6
\]

and
By expanding \( p(x) \) and \( q(x) \) as a power series of \( x - 2 \) we get

\[
p(x) = 27439716165461 + 185481302397702(x - 2) + 290969125220676(x - 2)^2 + 204586956497952(x - 2)^3 + 74327269209984(x - 2)^4 + 13692879518976(x - 2)^5 + 1016470425600(x - 2)^6,
\]

and

\[
q(x) = 10423677493515991770 + 62668051134141291321(x - 2) + \ldots + 328678008422400(x - 2)^14 + 8427641241600(x - 2)^15,
\]

which is a polynomial with all positive coefficients. It follows \( f'(x) > 0 \) for \( x \geq 2 \), so that \( f \) is strictly increasing in \((2, \infty)\) with \( \lim_{x \to \infty} f(x) = 0 \). It results that \( f(x) < 0 \) for \( x \geq 2 \), namely \( \theta_n \) is strictly increasing for \( n \geq 2 \). This completes the first part of Theorem 2.3. To prove the second part of the theorem we denote

\[
g(x) = -\frac{1}{x+1} - \log \left( x + \frac{1}{2} + \frac{1}{24(x+\frac{3}{2})} - \frac{37}{5760(x+\frac{3}{2})^3} + \frac{10313}{290304(x+\frac{3}{2})^5} \right)
\]

By differentiation we get

\[
g'(x) = \frac{t(x)}{s(x)},
\]

where

\[
t(x) = 9678358492223 + 57880272188784x + 144357200961720x^2 + 192184418280960x^3 + 144005296337280x^4 + 57575515060224x^5 + 9595919176704x^6,
\]

and

\[
s(x) = 5912418259515 + 110278703811038x + 996749749920191x^2 + \ldots + 539369039462400x^{15} + 33710564966400x^{16},
\]

which is a polynomial with all positive coefficients. Since both \( t(x) \) and \( s(x) \) are positive for \( x \geq 1 \), \( g \) is strictly increasing with \( \lim_{x \to \infty} g(x) = 0 \), consequently, the sequence \((\phi_n)\) is strictly increasing for \( n = 1, 2, 3, \ldots \). This completes the proof of Theorem 2.3. \( \Box \)
As a direct consequence of the fact that $\theta$ is strictly increasing for $n = 2, 3, 4, \ldots$ we have $\theta_2 \leq \theta_n < \lim_{n \to \infty} \theta_n = \gamma$ for all $n \geq 2$. As $\theta_2 = \frac{3}{2} - \log\left(\frac{58804553}{23224320}\right)$, we have

**Corollary 2.4.** Let $n \geq 2$ be an integer. Then we have

$$
\alpha + \log\left( n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5}\right) \\
\leq H_n < \beta + \log\left( n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} + \frac{17}{3840n^4} - \frac{10099}{2903040n^5}\right),
$$

where $\alpha = \frac{3}{2} - \log\left(\frac{58804553}{23224320}\right) = 0.5709807216\ldots$ and $\beta = \gamma = 0.5772156\ldots$ are the best possible.

Similarly from monotonic increase of the sequence $(\phi_n)$ with $\lim_{n \to \infty} \phi_n = \gamma$ and $\phi_1 = 1 - \log\left(\frac{6729631}{4408992}\right) = 0.57712577887\ldots$, we get

$$
\alpha^* + \log\left( n + \frac{1}{2} + \frac{1}{24(n+\frac{1}{2})} - \frac{37}{5760(n+\frac{1}{2})^3} + \frac{10313}{2903040(n+\frac{1}{2})^5}\right) \\
\leq H_n \leq \beta^* + \log\left( n + \frac{1}{2} + \frac{1}{24(n+\frac{1}{2})} - \frac{37}{5760(n+\frac{1}{2})^3} + \frac{10313}{2903040(n+\frac{1}{2})^5}\right),
$$

where $\alpha^* = 1 - \log\left(\frac{6729631}{4408992}\right) = 0.57712577887\ldots$ and $\beta^* = \gamma = 0.5772156\ldots$ are the best possible constants.

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