Comment on ‘Solution of the Dirac equation for the
Woods-Saxon potential with spin and pseudospin
symmetry’ [J. Y. Guo and Z-Q. Sheng, Phys. Lett.
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Abstract

The bound-state method employed by Guo and Sheng (loc. cit.) is shown inadequate
since only one of their solutions remains compatible, in the spin-symmetric low-mass
regime, with the physical boundary conditions. We clarify the problem and construct
a new, correct solution in the pseudospin-symmetric regime.

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Recent letter [1] paid attention to the radial Dirac $s$–wave bound-state problem considered in the two different dynamical regimes. This problem has been reduced to the solution of the respective Schrödinger-like bound-state equations

$$-\frac{d^2}{dr^2} F(r) + (M + E - C) [M - E + \Sigma(r)] F(r) = 0, \quad F(0) = F(\infty) = 0, \quad (1)$$

(cf. eqs. (5) or (9) in [1]) and

$$-\frac{d^2}{dr^2} G(r) + (M - E + C) [M + E - \Delta(r)] G(r) = 0, \quad G(0) = G(\infty) = 0, \quad (2)$$

(cf. eqs. (6) or (29) in [1]) where $M$ denotes the mass, $E$ is the bound-state energy, $C$ is a parameter and the functions $\Sigma(r)$ and $\Delta(r)$ represent certain phenomenological external potentials which have been chosen in the elementary Woods-Saxon form,

$$\Sigma(r) = -\frac{\Sigma_0}{1 + \exp\left(\frac{r-R}{a}\right)}, \quad \Delta(r) = -\frac{\Delta_0}{1 + \exp\left(\frac{r-R}{a}\right)}. \quad (3)$$

In the non-relativistic context, the solution of a very similar differential equation has been described in fair detail in ref. [2]. Unfortunately, the existence of certain specific physical features of the relativistic eqs. (1) and (2) forced one of us to imagine that the constructions of ref. [1] should be re-examined.

In a preparatory step the authors of ref. [1] replaced the combinations of constants appearing in eqs. (1), (2) and (3) by the life-simplifying abbreviations

$$\mu^2_{(i)} = (M - E)(M + E - C)a^2, \quad \mu^2_{(ii)} = (M + E)(M - E + C)a^2,$$

$$\nu^2_{(i)} = (M + E - C)\Sigma_0 a^2, \quad \nu^2_{(ii)} = -(M - E + C)\Delta_0 a^2.$$

This converts both eqs. (1) and (2) to the same eigenvalue problem,

$$-a^2 \frac{d^2}{dr^2} y(r) + \mu^2 y(r) - \frac{\nu^2}{1 + \exp[(r-R)/a]} y(r) = 0, \quad y(0) = y(\infty) = 0. \quad (4)$$

The elementary change of variables

$$r \to x = 1/(1 + \exp[(r-R)/a])$$

reduces eq. (4) to the Gauss’s hypergeometric differential equation (compare eq. (64.3) in [2] with the same formulae preceding eqs. (9) and (29) in [1]). One arrives at the
exact wave functions sampled, say, by the unnumbered equation preceding eq. (64.10) in the non-relativistic construction \[2\]. Its relativistic analogue

\[
\psi(x) = N x^\mu \left[ (1 - x)^\delta f(\mu, \delta) \, _2F_1(\mu + \delta, \mu + \delta + 1, 1 + 2\delta; 1 - x) \\
+ (1 - x)^{-\delta} f(\mu, -\delta) \, _2F_1(\mu - \delta, \mu - \delta + 1, 1 - 2\delta; 1 - x) \right]
\]

contains a certain ratio \( f(\mu, \delta) \) of products of pairs of \( \Gamma \)–functions given in full detail in eq. (13) of \[1\]. In this notation, the boundary condition in the origin becomes reduced to the single and exact transcendental equation

\[
\psi \left( \frac{1}{1 + e^{-R/a}} \right) = 0.
\]

Usually \[2\], the purely numerical exact quantization condition \(6\) is being replaced by its approximate asymptotic simplification

\[
\psi \left( 1 - e^{-R/a} \right) \cdot \left[ 1 + \mathcal{O} \left( e^{-R/a} \right) \right] = 0, \quad R/a \gg 1.
\]

The authors of ref. \[1\] did the same [cf. their four main quantization conditions (17), (22), (32) and (37)]. In this sense, the slightly misleading first line of their abstract (saying that their “...Dirac equation is solved exactly ...”) should rather read “...Dirac equation is solved approximatively ...”.

(i) In the first, spin-symmetric Dirac’s dynamical regime the authors of ref. \[1\] had to separate the solution of the ordinary differential Dirac bound-state problem \[1\] in two subcases. Incidentally, the first subcase [to be numbered or indexed as (i.a) in what follows] provides the only case where the result of ref. \[1\] is correct. We shall show that their construction fails to incorporate the boundary conditions properly in all the other cases.

(i.a) Assuming that \( \mu_{(i)}^2 = \mu^2 > 0 \) and returning to eq. (9) of \[1\] in the first subcase one deduces that \( \nu_{(i)}^2 = \nu^2 > 0 \) and requires that the potential is sufficiently attractive, i.e., that \( \nu^2 - \mu^2 \equiv \gamma^2 > 0 \). In such a case, strictly speaking, the extraction of the exact spectrum from its implicit definition \(6\) represents a straightforward, albeit purely numerical task. Unpleasant due to the well known slowness of the convergence of the infinite hypergeometric series

\[
_2F_1(a, b, c; z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{2c(c+1)} z^2 + \ldots
\]
in \(\psi(x)\). Fortunately, in eq. (5) the value of the argument \(z\) remains exponentially small whenever \(R/a \gg 1\). This means that in eq. (7) re-written in its fully explicit form

\[
\left[ e^{-R/a} \left( 1 + e^{-R/a} \right) \right]^{2\delta} f(\mu, \delta) \cdot {}_{2}F_{1} \left( \mu + \delta, \mu + \delta + 1, 1 + 2\delta; e^{-R/a} \left( 1 + e^{-R/a} \right) \right) = -f(\mu, -\delta) \cdot {}_{2}F_{1} \left( \mu - \delta, \mu - \delta + 1, 1 - 2\delta; e^{-R/a} \left( 1 + e^{-R/a} \right) \right)
\]

the approximation \(\text{re}F_1(a, b, c; z) \rightarrow 1\) will lead to very good precision. The authors of ref. [1] paralleled the non-relativistic exercise of ref. [2] and obtained the definitions (20) and (21) of the two components \(F_n(r)\) and \(G_n(r)\) of the related Dirac’s bound-state wave functions. Their parallel implicit formula (19) for energies \(E_n\)

\[
\arg \Gamma(2i\gamma) - \arg \Gamma(\mu + i\gamma) - \arctan(\gamma/\mu) + \gamma R/a = (n + 1/2)\pi
\]

was simply based on the approximation (7).

Let us note here that the next-order truncation \(\text{re}F_1(a, b, c; z) \rightarrow 1 + abz/c\) clearly and manifestly confirms the expectations as it leads to an elementary explicit incorporation of the next-order corrections via transition from eq. (8) to another closed formula,

\[
\arg \Gamma(2i\gamma) - \arg \Gamma(\mu + i\gamma) - \arctan(\gamma/\mu) + \gamma R/a = \left( n + \frac{1}{2} \right)\pi.
\]

This improvement illustrates, persuasively enough, the exponential smallness of the higher-order corrections under the relativistic kinematics.

(i.b) In an alternative range of parameters one admits that both the values of \(\mu\) and \(\nu\) may be imaginary, i.e., \(\mu^2(i) = \mu^2 \leq 0\) and \(\nu^2(i) = \nu^2 \leq 0\) while \(\mu^2 - \nu^2 \equiv \delta^2 \geq 0\) (cf. [1]). It is fairly elementary to notice that at the very large \(r \gg 1\) we only have to deal with the significantly simplified constant-coefficient form of the original radial Dirac equation (1),

\[
- a^2 \frac{d^2}{dr^2} F(r) + \mu^2(i) F(r) = 0, \quad r \gg 1.
\]

This equation obviously possesses just the non-localized, asymptotically oscillatory solutions (remember that \(\mu\) is now purely imaginary). As a consequence, all the
“bound-state” solutions obtained in [1] remain non-localized and correspond in fact to the scattering states. In the other words, the implicit formula (24) of [1] for the energies \( E_n \) is incorrect while the respective definitions (25) and (26) of the related Dirac’s “bound-state” wave functions remain unnormalizable and, hence, irrelevant.

An explanation of such a failure of the method returns us to the transition between formulae (11) and (12) in [1] where the latter formula has been erroneously declared to be the only solution compatible with the correct asymptotic bound-state boundary conditions. In fact, such an argument fails completely when \( \mu \) becomes imaginary.

(ii) Let us now move to the second, pseudospin-symmetric regime where the authors of ref. [1] claim the existence of the bound-state solutions of the Dirac bound-state problem (2) in the two separate subcases again [let us mark them as (ii.a) and (ii.b) in what follows]. We arrived at a different conclusion.

(ii.a) In eq. (2) let us follow ref. [1] and assume, firstly, that \( \mu_{(ii)}^2 = \mu^2 < 0 \) while \( \nu_{(ii)}^2 = \nu^2 > 0 \) and \( \nu^2 - \mu^2 \equiv \gamma^2 > 0 \). This leads to the problem

\[
- a^2 \frac{d^2}{dr^2} G(r) + \mu_{(ii)}^2 G(r) = 0, \quad r \gg 1 \tag{11}
\]

which parallels eq. (10). Hence, our conclusions remain the same. The implicit formula (34) for the bound-state energies \( E_n \) in [1] is incorrect while the related wave functions (35) and (36) are not normalizable and represent merely a random sample of the scattering states.

(ii.b) In the last subcase, equation (2) with the alternative choice of the real \( \mu \) (with positive square \( \mu_{(ii)}^2 = \mu^2 \geq 0 \)) is to be combined with the purely imaginary \( \nu \) such that \( \nu_{(ii)}^2 = \nu^2 \leq 0 \) while \( \mu^2 - \nu^2 \equiv \delta^2 \geq 0 \). Once we try to re-analyze such a model in present setting, we simply imagine that we deal with the Schrödinger-like equation

\[
\left( - \frac{d^2}{dr^2} + \kappa^2 [M + E - \Delta(r)] \right) G_n(r) = 0 \tag{12}
\]

where \( \kappa^2 = M - E + C > 0 \) and \( M + E > 0 \) in the original notation. There are no bound states for the positive \( \Delta_0 > 0 \), of course. This observation is not incompatible with the observation made in ref. [1] where the wave functions (39) and (40) with the most elementary energy formula (38) represented just the single state lying at the very boundary \( \mu = 0 \) of the domain, anyhow. Nevertheless, a return to the latter
\( \mu = 0 \) solution of eq. (2) “on the boundary” reveals that it does not need to exist at all. For example, we may verify that the insertion of the parameters \( \mu = \nu = 0 \) in the general solution (say, of the form similar to eq. (11) in [1]) leads to the exact wave function which is a constant. This constant must vanish due to the boundary conditions.

In our final remark we have to emphasize that the situation in the pseudospin-symmetric regime is by far not so hopeless as it seems to be. As long as our potential functions are always asymptotically vanishing, our last equation (12) will possess (a finite number of) bound states whenever the effective potential well proves, paradoxically, sufficiently strongly repulsive, \(-\Delta(0) = +\Delta_0 \leq \Delta_0^{\text{critical}} < 0\).

The existence of the latter family of bound states was not considered in ref. [1] at all. We repeat that in a search for a new relativistic Woods-Saxon model, one has to opt for a “paradoxical” choice of the repulsive barrier with \( \Delta_0 < 0 \) (i.e., with \( \Delta(r) > 0 \) at all \( r \)). This choice, obviously, represents the only eligible pseudospin-symmetric model where we would be allowed to construct bound states. The details of such a construction are entirely straightforward and may be left to the readers.

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