Zeros and points of discontinuity of the commutator function of the free Dirac field

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"Evidently it can be asked how far the theory of Dirac goes in reconciling and fusing quantum conceptions and relativity conceptions, the former implying essentially discontinuity, the latter being imbued with continuity."

LOUIS DE BROGLIE
"The Revolution in Physics"

Abstract. The properties of the commutator function of the free electron Dirac field, zeros of which respond to the independent measurements of the electron charge density in two different points of the space-time, are discussed. The established properties of one-space dimensional commutator function \( D(x, t) \) in the discrete representation, show the dependency of the number and location of its zeros and points of discontinuity on the value of normalizing factor. In this connection, the following new property of the commutator function is found: the length of the time interval containing at least one zero of \( D(x, t) \) depends on the relation between the value of the normalizing factor and the Compton wavelength of the electron.

1. Introduction
In the relativistic theory of Dirac the (anti-)commutation relations determine the causal connections between events that happen at different times and places, and hence the relativistic properties of the theory as a whole.

In what follows, we shall call both commutation and anticommutation for brevity commutation.

Suppose we consider in the framework of Dirac’s theory a certain physically observable quantity, for example, the electron charge density. The measurements of the values of this observable at two different space-time points can be made without interference of one measurement with the other if the two operators that represent the observable at the two points commute with each other. In this case, the corresponding matrices can be diagonalized simultaneously and hence, exact results (eigenvalues) can be obtained from both measurements.

It is well known (see, for example [10, 11, 13]), that the charge density \( \rho(r', t) \) will commute with \( \rho(r'', t'') \) only when the commutator function \( D(r'' - r', t'' - t') \) is zero. Here the commutator function \( D(r, t) \) is defined by the expression (see [10, 11, 13])
\[ D(\vec{r}, t) = \sum_{k} L^{-3} e^{i\vec{k} \cdot \vec{r}} \frac{\sin \left( ct \sqrt{k^2 + k_0^2} \right)}{c \sqrt{k^2 + k_0^2}}, \quad (1) \]

where \( \vec{k} \) is the wave vector of the length \( |\vec{k}| = k = \frac{2\pi}{L} \sqrt{k_x^2 + k_y^2 + k_z^2} ; k_x, k_y, k_z — \text{any integers}, \)

\[ k_0 = \frac{2\pi mc}{\hbar}; \quad (2) \]

\( m \) is the mass of the electron, \( c \) is the speed of light, \( h \) is the Planck constant. In addition, \( L \) is the normalizing factor, representing the length of the edge of the cube, in which the eigenfunctions of the operators of the energy-impulse, subject to the periodicity conditions at the walls of this cube, are defined.

In early works on quantum field theory ([3, 4, 5, 6, 10] see also [1]), the field functions and together with them the commutator functions were introduced in accordance with the concept of the quantization in the form of sums over the states, defined by wave vectors, and only after that for the convenience of work with them, they are substituted by the corresponding integrals.

So, from the sum (1) it is obtained the integral which is called the Pauli-Jordan function (see [1])

\[ D_{PJ}(\vec{r}, t) = \frac{1}{(2\pi)^3 c} \int_{-\infty}^{+\infty} \frac{\sin \left( ct \sqrt{k^2 + k_0^2} \right)}{c \sqrt{k^2 + k_0^2}} d\vec{k}. \quad (3) \]

Later on, the commutator functions are considered merely in the form of the integrals, since this representation is relativistically-invariant and sufficiently simple for studying (see [1, 2, 9, 12, 14]). The space one-dimensional version of the function (3) can be represented in the following form [1, 4, 10, 11, 13]

\[ D_{PJ}(x, t) = \frac{1}{4\pi xc} \left( \delta \left( t - \frac{x}{c} \right) - \delta \left( t + \frac{x}{c} \right) \right) - \tilde{D}_{PJ}(x, t), \quad (4) \]

where

\[ \tilde{D}_{PJ}(x, t) = \begin{cases} \frac{k_0}{4\pi} J_1(k_0 \sqrt{c^2 t^2 - x^2}) \frac{1}{\sqrt{c^2 t^2 - x^2}}, & \text{if } ct > x \\ 0, & \text{if } -x < ct < x \\ -\frac{k_0}{4\pi} J_1(k_0 \sqrt{c^2 t^2 - x^2}) \frac{1}{\sqrt{c^2 t^2 - x^2}}, & \text{if } ct < -x. \end{cases} \quad (5) \]

Here \( J_1(z) \) is a well-known Bessel function of the first kind [15], \( \delta(z) \) is the Dirac delta-function:

\[ \delta(z) = 0, \quad z \neq 0, \quad \int_{-\infty}^{+\infty} \delta(z) dz = 1. \]

Thus, the Pauli-Jordan function has the following important property: it vanishes exterior to the light cone, for the space one-dimensional version, for \(-x < ct < x\). In physical understanding it is connected with a more general postulate that no signals can be transmitted with velocities greater than that of light, and therefore the measurements, for example, of the electron charge density, at two space points with a space-like distance can never disturb each other.

The commutator function in the discrete representation hasn’t aforementioned properties.

The question arises: whether the passage from the discrete representation to the continuous one has any deficiencies? Whether all properties of the commutator function in the discrete representation are reflected in its integral analogue?
The aim of the present paper is to show that the properties of the commutator function in the discrete representation are different from the properties of the commutator function in the integral representation, and that in the problems of quantum theory, in which the quantum effects are essential, the use of the commutator function in the discrete representation can help to get more precise results.

We remark that in accordance with (4), (5) the Pauli-Jordan function has infinitely many zeros (together with the Bessel function $J_1(k_0\sqrt{c^2t^2 - x^2})$) interior of the domain $ct = |x|$. Now we consider the commutator function in the discrete representation. For simplicity, we shall study the space one-dimensional version of the function (1)

$$D(x,t) = L^{-1} \sum_{k=\infty}^\infty e^{i\frac{2\pi}{L}xk}\frac{\sin\left(c\sqrt{\left(\frac{2\pi}{L}\right)^2 k_x^2 + k_0^2}\right)}{c\sqrt{\left(\frac{2\pi}{L}\right)^2 k_x^2 + k_0^2}}.$$ (6)

In what follows, for the real $x$, the function fractional part of the number $x$, $y = \{x\}$, is defined by the equality

$$y = \{x\} = x - \lfloor x \rfloor,$$

where $\lfloor x \rfloor$ is the integral part of $x$, that is an integer such that $x - 1 < \lfloor x \rfloor \leq x$.

2. The points of discontinuity of the function $D(x,t)$

Rewrite (6) in the form

$$D(x,t) = L^{-1} \left( \frac{\sin \left(\frac{ctk_0}{k_0}\right)}{k_0} + 2 \sum_{k=1}^{\infty} \cos \left(\frac{2\pi}{L}k\right) \frac{\sin\left(\frac{ct\sqrt{\left(\frac{2\pi}{L}\right)^2 k_x^2 + k_0^2}}{c\sqrt{\left(\frac{2\pi}{L}\right)^2 k_x^2 + k_0^2}}\right)}{k}\right).$$ (7)

We remark that (7) is a conditionally convergent series. The function $D(x,t)$ is an odd function in $t$ and an even function in $x$. Besides, $D(x+L,t) = D(x,t)$ and $D(x,0) = 0$. Further we shall consider $D(x,t)$ for

$$t > 0, \quad 0 \leq x \leq L.$$ (8)

From (7) we obtain (see for details [8]):

$$D(x,t) = (2\pi c)^{-1} \left( k_1^{-1} \sin \left(\frac{2\pi}{L}xk_1\right) + 2f(x,t) + 2g(x,t) \right),$$ (9)

where

$$k_1 = k_0 \frac{L}{2\pi}.$$ (10)

Here

$$g(x,t) = \sum_{k=1}^{\infty} \cos \left(\frac{2\pi}{L}k\right) \left( \frac{\sin\left(\frac{2\pi}{L}\sqrt{k_x^2 + k_1^2}\right)}{\sqrt{k_x^2 + k_1^2}} - \frac{\sin \left(\frac{2\pi}{L}k\right)}{k}\right).$$ (11)

is a continuous function in $x$ and $t$ in the domain $|x| \leq X, |t| \leq \tau$, where $X$ and $\tau$ are arbitrary numbers. In addition, the function

$$f(x,t) = \sum_{k=1}^{\infty} k^{-1} \cos \left(\frac{2\pi}{L}xk\right) \sin \left(\frac{2\pi}{L}ck\right),$$
is representable in the form

\[ f(x, t) = \frac{\pi}{2} \left( \varrho_0 \left( \frac{x + ct}{L} \right) - \varrho_0 \left( \frac{x - ct}{L} \right) \right), \quad (12) \]

where

\[ \varrho_0(y) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin (2\pi yk)}{k} \]

is the Fourier series for the Euler \( \varrho \)-function, \( \varrho(y) = \frac{1}{2} - \{y\} \) (see, for example, [7]). The function \( \varrho_0(y) \) is a piecewise smooth function with the points of discontinuity of the first kind at the integer points

\[ y = n; \quad n = 0; \pm 1; \pm 2; \ldots; \]

and also

\[ \varrho_0(y) = \begin{cases} \varrho(y) = \frac{1}{2} - \{y\}, & \text{if } \{y\} \neq 0 \\ 0, & \text{if } \{y\} = 0. \end{cases} \quad (13) \]

It is easy to see from (9) –(13) that the commutator function \( D(x, t) \) is a piecewise smooth function and can be represented in the form

\[ D(x, t) = (2\pi c)^{-1} \left( k_1^{-1} \sin \left( \frac{2\pi ct}{L} k_1 \right) + \pi \varrho_0 \left( \frac{x + ct}{L} \right) - \pi \varrho_0 \left( \frac{x - ct}{L} \right) + 2g(x, t) \right), \]

where \( g(x, t) \) is the continuous function (11), and the points of discontinuity of \( D(x, t) \) coincide with the points of discontinuity of \( f(x, t) \). We remark that the points of discontinuity of the functions \( \varrho_0 \left( \frac{x + ct}{L} \right) \) and \( \varrho_0 \left( \frac{x - ct}{L} \right) \) are correspondingly the points

\[ x + ct = Ln; \quad n = 0; \pm 1; \pm 2; \ldots; \]
\[ x - ct = Ln; \quad n = 0; \pm 1; \pm 2; \ldots. \]

Since \( g(x, t) \) is the continuous function of \( x \) and \( t \), and \( g(x, 0) = 0 \), then for certain values of \( x \) and \( t \), for example, for small values of \( t \) with any \( x \), the behavior of the function \( D(x, t) \) is determined by the behavior of \( f(x, t) \). According to (12)–(13), this function has the following properties in the domain of interest (8):

I. \( f \left( x + L, t \right) = f(x, t) \); \( f \left( x, t + \frac{L}{c} \right) = f(x, t) \).

II. for the fixed \( x = x_1; 0 \leq x_1 < L; \)

1) for \( x_1 = 0, \)

\[ \frac{2}{\pi} f(0, t) = \frac{2}{\pi} f(t) = \begin{cases} 0, & \text{if } t = 0 \\ 1 - 2\frac{t}{L}, & \text{if } 0 < t < \frac{L}{c} \end{cases} \]

2) for \( 0 < x_1 < \frac{L}{2}, \)

\[ \frac{2}{\pi} f(x_1, t) = \frac{2}{\pi} f(t) = \begin{cases} -2\frac{t}{L}, & \text{if } 0 \leq t < \frac{x_1}{c} \\ \frac{1}{2} - 2\frac{t}{L}, & \text{if } t = \frac{x_1}{c} \\ 1 - 2\frac{t}{L}, & \text{if } \frac{x_1}{c} < t < \frac{L - x_1}{c} \\ -\frac{1}{2} + 2\frac{t}{L}, & \text{if } t = \frac{L - x_1}{c} \\ 2 - 2\frac{t}{L}, & \text{if } \frac{L - x_1}{c} < t \leq \frac{L}{c} \end{cases} \quad (14) \]
3) for \( x_1 = \frac{L}{2} \),
\[
\frac{2}{\pi} f(x_1,t) = \frac{2}{\pi} f(t) = \begin{cases} 
-2\frac{c}{L}, & \text{if } 0 \leq t < \frac{L}{2c} \\
0, & \text{if } t = \frac{L}{2c} \\
2 - 2\frac{c}{L}, & \text{if } \frac{L}{2c} < t \leq \frac{L}{\pi} 
\end{cases}
\]

The case, when \( \frac{L}{2} < x_1 < L \), is obtained from (14), if we replace \( x_1 \) by \( L - x_1 \).

III. for the fixed \( t = t_1 \); 0 < \( t_1 < \frac{L}{2c} \);

1) for \( 0 < t_1 < \frac{L}{2c} \),
\[
\frac{2}{\pi} f(x,t_1) = \frac{2}{\pi} f(x) = \begin{cases} 
1 - 2\frac{c}{L}, & \text{if } 0 \leq x < ct_1 \\
\frac{1}{2} - 2\frac{c}{L}, & \text{if } x = ct_1 \\
-2\frac{c}{L}, & \text{if } ct_1 < x < L - ct_1 \\
\frac{1}{2} - 2\frac{c}{L}, & \text{if } x = L - ct_1 \\
1 - 2\frac{c}{L}, & \text{if } L - ct_1 < x \leq L 
\end{cases}
\]

2) for \( t_1 = \frac{L}{2c} \), for \( 0 \leq x \leq L \),
\[
\frac{2}{\pi} f(x,t_1) = \frac{2}{\pi} f(x) = 0.
\]

3) for \( \frac{L}{2c} < t_1 < \frac{L}{c} \),
\[
\frac{2}{\pi} f(x,t_1) = \frac{2}{\pi} f(x) = \begin{cases} 
1 - 2\frac{c}{L}, & \text{if } 0 \leq x < L - ct_1 \\
\frac{3}{2} - 2\frac{c}{L}, & \text{if } x = L - ct_1 \\
2 - 2\frac{c}{L}, & \text{if } L - ct_1 < x < ct_1 \\
\frac{3}{2} - 2\frac{c}{L}, & \text{if } x = ct_1 \\
1 - 2\frac{c}{L}, & \text{if } ct_1 < x \leq L 
\end{cases}
\]

From the foregoing properties of the function \( f(x,t) \), the following statements are valid on its discontinuities, and therefore, on the discontinuities of the function \( D(x,t) \):

**Proposition 2.1** For any fixed \( x = x_1 \), 0 \( \leq x_1 < L \), the function \( D(x_1,t) \) has at most one point of discontinuity

— on the segment \( \frac{L}{2c} n \leq t \leq \frac{L}{2c} (n+1); \quad n = 0, 1, 2, \ldots \); if \( x_1 \neq \frac{L}{2} \);

— on the segment \( \frac{L}{2} n \leq t \leq \frac{L}{2c} (n+1); \quad n = 0, 1, 2, \ldots \); if \( x_1 = \frac{L}{2} \).

On every segment \( 0 \leq t \leq \frac{L}{c} n; \quad n = 1, 2, 3, \ldots \); the function \( D(x_1,t) \) has at most \( 2n \) points of discontinuity, if \( x_1 \neq \frac{L}{2} \); and at most \( n \) points of discontinuity, if \( x_1 = \frac{L}{2} \).

**Proposition 2.2** For any fixed \( t = t_1 \), 0 \( < t_1 < \frac{L}{2c} \), the function \( D(x,t_1) \)

— has at most one point of discontinuity on the segment \( \frac{L}{2} n \leq x \leq \frac{L}{2c} (n+1); \quad n = 0, 1, 2, \ldots \); if \( t_1 \neq \frac{L}{2c} \);

— is continuous, if \( t_1 = \frac{L}{2c} \).

On every segment \( 0 \leq x \leq L n; \quad n = 1, 2, 3, \ldots \); the function \( D(x,t_1) \) has at most \( 2n \) points of discontinuity, if \( t_1 \neq \frac{L}{2c} \); and \( D(x,t_1) \) is continuous, if \( t_1 = \frac{L}{2c} \).

We remark that from (4), (5) the Pauli-Jordan function has one point of discontinuity for any fixed \( x = x_1 \) (when \( t = \frac{x_1}{c} \)) and one point of discontinuity for any fixed \( t = t_1 \) (when \( x = ct_1 \).
3. Zeros of the function $D(x,t)$ in time variable

The following theorem on the number of the changes of sign of the function $D(x,t)$ in $t$ takes place.

**Theorem 3.1** Let $\lambda_c$ be the Compton wavelength of the electron. Then for any $x, x \geq 0$, on any interval $(t_a, t_b)$, $t_b > t_a \geq 0$, the function $D(x,t)$ changes sign at least $j$ times, where

$$j \geq 2 \left( \frac{(t_b - t_a)c}{\lambda_c} - \frac{L^2}{\lambda_c^2} - \frac{3}{2} \right).$$

For the proof we introduce the notations

$$\alpha_k = \frac{c}{L} \sqrt{k^2 + k_1^2}; \quad \beta_k = \frac{k}{L}; \quad k = 0, 1, 2, \ldots.$$

Then $D(x,t)$ takes the form:

$$D(x,t) = L^{-1} \left( \frac{\sin (2\pi \alpha_0 t)}{2\pi \alpha_0} + 2 \sum_{k=1}^{\infty} \cos (2\pi \beta_k x) \frac{\sin (2\pi \alpha_k t)}{2\pi \alpha_k} \right). \tag{15}$$

Let $N$ be an arbitrary natural number, and

$$\tilde{D}_{2N}(x,t) = \frac{(-1)^N}{L} \left( \frac{\sin (2\pi \alpha_0 t)}{(2\pi \alpha_0)^{2N+1}} + 2 \sum_{k=1}^{\infty} \cos (2\pi \beta_k x) \frac{\sin (2\pi \alpha_k t)}{(2\pi \alpha_k)^{2N+1}} \right).$$

It is obvious that

$$\frac{d^{2N}}{dt^{2N}} \tilde{D}_{2N}(x,t) = D(x,t).$$

To find a lower bound of the number of zeros of the function $\tilde{D}_{2N}(x,t)$ on the interval $t_a < t < t_b$, we represent $\tilde{D}_{2N}(x,t)$ in the following form

$$\tilde{D}_{2N}(x,t) = \frac{(-1)^N}{L} (2\pi \alpha_0)^{-2N-1} \left( \sin (2\pi \alpha_0 t) + \tilde{F}(t) \right), \tag{16}$$

where

$$\tilde{F}(t) = 2\alpha_0^{2N+1} \sum_{k=1}^{\infty} \alpha_k^{-2N-1} \cos (2\pi \beta_k x) \sin (2\pi \alpha_k t).$$

It is not difficult to prove (see for details [8]) that if

$$N = \max([k_1^2] + 1; 5), \tag{17}$$

then

$$\left| \tilde{F}(t) \right| < \frac{124}{125} < 1.$$

From here and from (16) it follows that the function $\tilde{D}_{2N}(x,t)$ has on the interval $(t_a, t_b)$ no fewer zeros than the function $y = \sin (2\pi \alpha_0 t)$, which has at least

$$2\alpha_0(t_b - t_a) - 1$$

zeros. Consequently, $D(x,t)$ changes sign on the interval $(t_a, t_b)$ at least $j_1$ times, where

$$j_1 \geq 2\alpha_0(t_b - t_a) - 2N - 1. \tag{18}$$
Taking into account that the Compton wavelength of the electron is \( \lambda_c = h/(mc) \), and hence from (2), (10)

\[
k_1 = \frac{L}{\lambda_c},
\]

the statement of the theorem follows from (17), (18).

We have the following important corollaries from the theorem.

**Corollary 3.2** On any interval \((t_a, t_b)\) of its continuity, the function \(D(x, t)\) has at least

\[
2 \left( \frac{(t_b - t_a)c}{\lambda_c} - \frac{L^2}{\lambda_c^2} - \frac{3}{2} \right)
\]

zeros.

**Corollary 3.3** On any interval \((t_a, t_b)\) such that \(t_a \geq 0, t_b \leq \frac{L}{c} n\), the function \(D(x, t)\) has at least

\[
2 \left( \frac{(t_b - t_a)c}{\lambda_c} - \frac{L^2}{\lambda_c^2} - n - \frac{3}{2} \right)
\]

zeros.

**Corollary 3.4** For any natural number \(n\) such that

\[
L^2 + \left( n + \frac{3}{2} \right) \frac{\lambda_c^2}{c} < nL\lambda_c,
\]

the function \(D(x, t)\) has at least one zero on the interval

\[
\frac{L^2}{c\lambda_c} + \left( n + \frac{3}{2} \right) \frac{\lambda_c}{c} < t_b - t_a < n\frac{L}{c}.
\]

The proofs of the Corollaries 3.2–3.4 follow immediately from the Theorem 3.1 and the Proposition 2.1.

**4. Zeros of the function \(D(x, t)\) in space variable**

The problem of existence of zeros of \(D(x, t)\) in \(x\) is more complicated. We shall follow the arguments of the Section 3 with certain modification. We denote by

\[ J = J(x_a, x_b) \]

the number of the changes of sign of the function \(D(x, t)\) represented by (15) on the interval \(x_a < x < x_b\). If \(D(x, t)\) is continuous on the interval \((x_a, x_b)\), then \(J\) is the number of zeros of \(D(x, t)\) on \((x_a, x_b)\). Let \(N\) be an arbitrary natural number, and

\[
\tilde{D}_{2N}(x, t) = \frac{(x - x_a)^{2N}}{(2N)!} \sum_{k=0}^{\infty} \frac{\cos \left( \frac{2\pi \beta_k x}{L} \right) \sin \left( \frac{2\pi \alpha_0 t}{2N} \right)}{2 \pi \alpha_0} + (-1)^N \sum_{k=1}^{\infty} \frac{\cos \left( \frac{2\pi \beta_k x}{L} \right) \sin \left( \frac{2\pi \alpha_k t}{2N} \right)}{2 \pi \alpha_k}. \tag{20}
\]

It is obvious that

\[
\frac{1}{L} \frac{d^{2N}}{dx^{2N}} \tilde{D}_{2N}(x, t) = D(x, t).
\]

Besides, if \(J_{2N}\) is equal to the number of zeros of the function \(\tilde{D}_{2N}(x, t)\) on \((x_a, x_b)\), then
\[ J \geq J_{2N} - 2N. \]

Let \( n \geq 2 \), \( n \) be an integer. Isolating in the sum over \( k \) from (20) the summand with \( k = n \) and assuming that \( \sin (2\pi \alpha_n t) \neq 0 \), we rewrite \( \widetilde{D}_{2N}(x,t) \) in the form

\[
\widetilde{D}_{2N}(x,t) = (-1)^N \frac{2}{(2\pi \beta_n)^{2N}} \frac{\sin (2\pi \alpha_n t)}{2\pi \alpha_n} (\cos (2\pi \beta_n x) + R_1(x,t) + R_2(x,t)),
\]

where

\[
R_1(x,t) = \frac{(-1)^N}{2} \frac{2\pi \alpha_n}{\sin (2\pi \alpha_n t)} \frac{2N}{(2N)!} \frac{\sin (2\pi \alpha_0 t)}{2\pi \alpha_0} \alpha_n \sin (2\pi \alpha_k t) \cos (2\pi \beta_k x) \alpha_k \sin (2\pi \alpha_n t) \cos (2\pi \beta_k x)
\]

\[
+ \sum_{k=1}^{n-1} \left( \frac{\beta_n}{\beta_k} \right)^{2N} \frac{\alpha_n \sin (2\pi \alpha_k t) \alpha_k \sin (2\pi \alpha_n t)}{\cos (2\pi \beta_k x) \sin (2\pi \alpha_n t) \cos (2\pi \beta_k x)}
\]

\[
+ \sum_{k=n+1}^{2n} \left( \frac{\beta_n}{\beta_k} \right)^{2N} \frac{\alpha_n \sin (2\pi \alpha_k t) \alpha_k \sin (2\pi \alpha_n t)}{\cos (2\pi \beta_k x) \sin (2\pi \alpha_n t) \cos (2\pi \beta_k x)}
\]

\[
R_2(x,t) = \sum_{k=2n+1}^{\infty} \left( \frac{\beta_n}{\beta_k} \right)^{2N} \frac{\alpha_n \sin (2\pi \alpha_k t) \alpha_k \sin (2\pi \alpha_n t)}{\cos (2\pi \beta_k x) \sin (2\pi \alpha_n t) \cos (2\pi \beta_k x)}
\]

Assume that \( t = t_\nu \) is such that for every \( x \) from \((x_a, x_b)\) the inequality

\[ |R_1(x,t_\nu) + R_2(x,t_\nu)| < 1 \quad (21) \]

is valid. Then the number of zeros of the function \( \widetilde{D}_{2N}(x,t_\nu) \) coincides on \((x_a, x_b)\) with the number of zeros of the function \( \cos (2\pi \beta_n x) \), that is

\[ J_{2N} \geq \frac{2n}{L} (x_a - x_b) - 1. \]

In [8] it is proved that in order for the value of \( R_1(x,t) \) to be small, we need that there should exist such \( t = t_\nu \) that the value \( |\sin (2\pi \alpha_n t_\nu)| \) would be sufficiently great, for example,

\[ |\sin (2\pi \alpha_n t_\nu)| \geq \frac{1}{A}, \quad (22) \]

where \( A \) is a constant, \( A > 1 \), and at the same time all the values \( |\sin (2\pi \alpha_k t_\nu)|; \ k = 0, 1, \ldots, n - 1, n + 1, \ldots, 2n; \) would be sufficiently small, for example,

\[ |\sin (2\pi \alpha_k t_\nu)| < \frac{1}{B_k}, \quad (23) \]

where \( P \) is a constant, \( P \geq 1 \), and

\[ \rho_0 = \frac{(2\pi \beta_n (x - x_a))^2 \alpha_n}{(2N)! \alpha_0}; \]

\[ \rho_k = 2 \left( \frac{\beta_n}{\beta_k} \right)^{2N} \frac{\alpha_n}{\alpha_k}; \ k = 1, 2, \ldots, 2n. \]

If the conditions (22), (23) are satisfied, then
\[ |R_1(x, t_\nu)| \leq \frac{1}{P} \left( 1 - \frac{1}{2n+1} - \frac{1}{2^{2n+1}} \right). \]

At the same time, the following bound for \( R_2(x, t) \) is proved in [8]:

\[ |R_2(x, t)| \leq 2A \frac{N + n}{2N - 1} \left( \frac{n}{2n + 1} \right)^{2N}. \]

Selecting the suitable parameters \( P, A, n, N \), it is possible to refine the estimates for the values \( R_1(x, t), R_2(x, t) \), and at the same time to make more precise the estimate of the amount of zeros of \( D(x, t) \) in one or another interval \((x_a, x_b)\).

Thus we have the following statement on the condition of the existence of at least one zero of the function \( D(x, t) \) on the interval, the length of which doesn’t exceed \( L \).

**Proposition 4.1** If the parameters \( A, P, n, N \) such that the bound (21) is realized and besides, \( N + \frac{3}{2} < n \), then on the interval \( x_a < x < x_b, \ x_a \geq 0, \ x_b \leq L \), such that

\[ x_a - x_b > L \frac{2N + 3}{2n}, \tag{24} \]

the function \( D(x, t) \) has at least one zero.

The main problem is, whether it is possible to find \( t = t_\nu \), such that for all \( x \) in \((x_a, x_b)\) the conditions (22),(23), were satisfied, which for a suitable choice of the parameters, would have provided the estimate (21). The answer gives the following

**Lemma 4.2** Let \( n \geq 1 \); let real numbers \( \alpha_0, \alpha_1, \ldots, \alpha_{2n} \) be linearly independent over the field of rational numbers \( Q \). Let \( 1 < A \leq 2; \ B_k \geq 2 \) for \( k = 0, 1, 2, \ldots, n - 1, n + 1, \ldots, 2n; \ B_n = 1 \).

Then on any interval \([T, 2T]) \), \( T \geq 1 \), there exists a set \( E \subset [T, 2T) \), the measure of which \( \mu(E) \) is defined by the relation

\[ \mu(E) = VT + o(T), \]

where

\[ V = (B_0B_1 \ldots B_{2n})^{-1} (2\pi)^{-2n-1} \sqrt{\frac{2}{\pi}} \sqrt{1 - \frac{1}{A}}, \]

and such that for any \( t \in E \) the inequalities

\[ |\sin (2\pi \alpha_0 t)| \geq \frac{1}{A}, \]

\[ |\sin (2\pi \alpha_k t)| \leq \frac{1}{B_k}, \ k = 0, 1, 2, \ldots, n - 1, n + 1, \ldots, 2n, \]

are fulfilled.

The proof of the lemma (see for details [8]) is based on application of the Kronecker theorem (see [7]) in the following statement.

Let \( s \geq 1 \); let \( \alpha_0, \alpha_1, \ldots, \alpha_s \) be real numbers which are linearly independent over \( Q \); let \( \theta_0, \theta_1, \ldots, \theta_s \) be arbitrary real numbers; let \( b_0, b_1, \ldots, b_s; \ \Delta_0, \Delta_1, \ldots, \Delta_s \) be arbitrary nonnegative numbers such that

\[ b_0 + \Delta_0 < 1, \ b_1 + \Delta_1 < 1, \ldots, \ b_s + \Delta_s < 1. \]
Let $\gamma$ be a domain of $s + 1$-dimensional Euclidean space $(x_0, x_1, \ldots, x_s)$ of the form $b_0 < x_0 \leq b_0 + \Delta_0, \ b_1 < x_1 \leq b_1 + \Delta_1, \ldots, \ b_s < x_s \leq b_s + \Delta_s$. Then on the interval $[0, T)$, $T \geq 1$, there exists a set $E$ of the numbers $t$, $E \subset [0, T)$, the measure of which $\mu(E)$ is equal $\mu(E) = VT + o(T)$, where $V = \Delta_0 \Delta_1 \ldots \Delta_s$, and such that for any $t \in E$ the relation $\left(\{\alpha_0 t + \theta_0\}, \{\alpha_1 t + \theta_1\}, \ldots, \{\alpha_s t + \theta_s\}\right) \in \gamma$ is satisfied.

In the considered problem the satisfaction of all conditions of the Kronecker theorem provides the proof of the lemma.

Thus, for any fixed $x = x_1$ the commutator function $D(x, t)$ has at least one zero on the interval $(t_a, t_b)$ which is determined by (19). From the other side, there are infinitely many intervals $(T_a, T_b)$ such that for certain $t_1 \in (T_a, T_b)$ the commutator function in the discrete representation has at least one zero on the interval $(x_a, x_b)$ which is determined by (24).

5. Conclusion
The amount of points of discontinuity and zeros and their location in one or another time or space interval for the commutator function in the discrete representation $D(x, t)$ and its integral analogue, the Pauli-Jordan function $D_{PJ}(x, t)$, are not the same.

The function $D(x, t)$ detects the following interesting inherent property: the number and location of its zeros in $x$ and in $t$ depend on the normalizing factor $L$, moreover the length of the time interval containing at least one zero of $D(x, t)$ depends on the relation between the value of the normalizing factor and the Compton wavelength of the electron.

A particular role in the study of zeros on space intervals plays the Kronecker theorem.

The described method of the studying $D(x, t)$ is applicable also to $D(\bar{r}, t)$, taking into account the effects, connected with the summing the multiple series.

At the same time, for both functions, $D(x, t)$ and $D_{PJ}(x, t)$, the same result on the zeros in the domain $-x > ct > x$ is obtained, and it is possible to generalize it also to three space dimensional case. Thus, the Dirac theory establishes theoretical possibility of the exact measurements of the electron charge density interior of the light cone $|ct| = \bar{r}$.

Acknowledgments. I thank A A Karatsuba for helpful discussions of the Kronecker theorem and the ways of its application.

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