Note on the connectivity keeping spiders in $k$-connected graphs

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Abstract
W. Mader [J. Graph Theory 65 (2010), 61–69] conjectured that for any tree $T$ of order $m$, every $k$-connected graph $G$ with $\delta(G) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m − 1$ contains a tree $T' \cong T$ such that $G − V(T')$ remains $k$-connected. In 2010, Mader confirmed the conjecture for the $k$-connected graph if $T$ is a path; very recently, Liu et al. confirmed the conjecture if $k = 2, 3$. The conjecture is open for $k \geq 4$ till now. In this paper, we show that Mader’s conjecture is true for the $k+1$-connected graph if $T$ is a spider and $\Delta(G) = |G| − 1$.

Keywords: Connectivity; Spider; $k$-Connected Graph; Fragment

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1 Introduction
In this paper, all the graphs are finite, undirected simple. For graph-theoretical terminology and notations not defined here, we follow [1]. Some basic sym-

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bolts and definitions ones are needed to be introduced. The minimum degree and the connectivity number of a graph $G$ is denoted by $\delta(G)$ and $\kappa(G)$, respectively. For a subgraph $H \subseteq G$ and a subset $V' \subseteq V(G)$, write $V' \cap H$ for $V' \cap V(H)$. For any vertex $x \in G$, denote the set of neighbors of $x$ in $G$ by $N_G(x)$ and for a subgraph $H \subseteq G$, define $N_G(H) := \bigcup_{x \in H} N_G(x) - V(H)$.

For $H \subseteq G$, $G(H)$ stands for the subgraph induced by $H$ in $G$ and $G - H := G - V(H)$. For $H \subseteq G$, set $\delta_G(H) := \min_{x \in H} d_G(x)$. For any edge $xy \in E(G)$ and $H \subseteq G$, write $H \cup xy$ for the subgraph $(V(H) \cup \{x, y\}, E(H) \cup xy)$ of $G$.

Let $x, y$-path be a path $P$ from $x$ to $y$. For $u, v \in P$, let $P[u, v] = P[v, u]$ be the subpath of $P$ between $u$ and $v$, and $P[u, v)$ means $P[u, v] - v$. Denotes the complete graph on vertex set $S$ by $K(S)$.

Let $G$ be a graph. A separating set $S \subseteq G$ denotes a vertex cut-set, and the cardinality of a minimum separating set $S$ is denoted by $|S| = \kappa(G)$. If $S$ is a minimum separating set of the graph $G$, then the union components $F$ of $G - S$ with $G - S - F \neq \emptyset$ is called a fragment $F$ to $S$, and the complementary fragment $G - (S \cup V(F))$ is denoted by $\bar{F}$. If a fragment of $G$ does not contain any other fragments of $G$, then it is called an end of $G$. Clearly, every graph contains an end except for the complete graphs. For a fragment $F$ of $G$ to $S$, it definitely follows $S = N_G(F)$. For $S \subseteq G$, denote the complete graph induced by $S$ by $K(S)$ and the graph $G[S] := G \cup K(S)$.

In 1972, Chartrand, Kaigars and Lick [2] proved that every $k$-connected graph $G$ with $\delta(G) \geq \lceil \frac{3k}{2} \rceil$ has a vertex $x$ with $\kappa(G - x) \geq k$. Nearly forty years later, Fujita and Kawarabayashi [4] extended the result, that is, every $k$-connected graph $G$ with $\delta(G) \geq \lceil \frac{3k}{2} \rceil + 2$ has an edge $xy$ such that $G - \{x, y\}$ remains $k$-connected. Furthermore, Fujita and Kawarabayashi posed the following conjecture.

**Conjecture 1** (Fujita and Kawarabayashi [4]). For all positive integers $k$, $m$, there is a (least) non-negative integer $f_k(m)$ such that every $k$-connected graph $G$ with $\delta(G) \geq \left\lceil \frac{3k}{2} \right\rceil + f_k(m) - 1$ contains a connected subgraph $W$ of order $m$ such that $G - V(W)$ is still $k$-connected.
In 2010, Mader [10] proved Conjecture 1 is true if $f_k(m) = m$ and $W$ is a path. Mader [10] proposed the following conjecture. The author will show it by using the Mader’s method.

**Conjecture 2 (Mader [10]).** For every positive integer $k$ and finite tree $T$ of order $m$, every $k$-connected finite graph $G$ with minimum degree $\delta(G) \geq \left\lceil \frac{3k}{2} \right\rceil + |T| - 1$ contains a subgraph $T' \cong T$ such that $G - V(T')$ remains $k$-connected.

In the case that $G$ is a graph with small connectivity, in 2009 Diwan and Tholiya [3] confirmed the conjecture for $k = 1$; Tian et al. [12] and Tian et al. [13] studied some special trees, such as stars, double stars and path-star and so on, for 2-connected graphs; Hasunuma [5] proved some results on 2-connected graphs with girth conditions; Hasunuma, Ono [6] and Lu, Zhang [9] showed that it holds for 2-connected graphs if $T$ is a tree with diameter condition; Very recently, Hong and Liu [8] showed that it holds for 2-connected graphs if $T$ is a caterpillar or a spider. Furthermore, Hong et al. [7] confirmed the conjecture for $k = 2, 3$. In the case that $G$ is a graph with general connectivity, Mader [10] proved the conjecture is true if $T$ is a path; Mader [11] showed that the conjecture is true for $k$-connected graphs if $\delta \geq 2(k - 1 + m)^2 + m - 1$. In this paper, the authors will confirm Mader’s conjecture for the spider tree using Mader’s structural theorem in Section 2; in Section 3 we conclude the paper. In order to show our theorems, we need some structural lemmas to prove our results.

**Lemma 1 (10).** Let $G$ be a graph with $\kappa(G) = k$ and let $F$ be a fragment of $G$ to $S$. Then it follows that if $F$ is an end of $G$ with $F \geq 2$, then $G[S] - V(F)$ is $(k + 1)$-connected.

**Lemma 2 (10).** Let $G$ be a $k$-connected graph and let $S$ be a separating set of $G$ with $|S| = k$. Then the following holds.

(a) For every fragment $F$ of $G$ to $S$, $G[S] - V(F)$ is $k$-connected.

(b) Assume $\delta(G) \geq \left\lceil \frac{3k}{2} \right\rceil + m - 1$ and let $F$ be a fragment of $G$ to $S$. If $W \subseteq G - (S \cup V(F))$ has order at most $m$ and $k(G[S] - V(F \cup W)) \geq k$
holds, then also $\kappa(G - V(W)) \geq k$ holds.

(c) Assume $(G, C) \in \mathcal{F}_k(m)$ and let $F$ be a fragment of $G$ to $S$ with $C \subseteq G(F \cup S)$. If $W \subseteq G - (S \cup V(F))$ has order at most $m$ and $\kappa(G[S] - V(F \cup W)) \geq k$ holds, then also $\kappa(G - V(W)) \geq k$ holds.

Lemma 3 ([10]). For all $(G, C) \in \mathcal{F}_k^+(m)$ and $p_0 \in G - V(C)$, there is a path $P \subseteq G - V(C)$ of length $m - 1$ starting from $p_0$, such that $\kappa(G - V(P)) \geq k$ holds.

Theorem 1 ([3]). Let $G$ be a connected graph with minimum degree $d \geq 1$. Then for any tree $T$ of order $d$, $G$ contains a subtree $T'$ isomorphic to $T$ such that $G - T'$ is connected.

| $\kappa(G)$ | $\delta$ | $T$ | Authors and References |
|------------|---------|-----|------------------------|
| 1          | $\delta \geq m$ | $T$ is any tree | Diwan and Tholiya [3]; |
| 2 or 3     | $\delta \geq m + 2$ or $m + 3$ | $T$ is any tree | Hong and Liu [7]; |
| $k$        | $\delta \geq \left\lceil \frac{3k}{2} \right\rceil + m - 1$ | $T$ is a path | Mader [10]; |
| $k$        | $2(k - 1 + m)^2 + m - 1$ | $T$ is any tree | Mader [11]; |

2 Main results

We define the set $\mathcal{F}_k(m)$ containing all pairs $(G, C)$ satisfying the following conditions:

- $G$ is a $k$-connected graph with $|G| \geq k + 1$;
- $C \subseteq G$ is a complete subgraph with $|C| = k$ and $\delta_G(G - V(C)) \geq \left\lceil \frac{3k}{2} \right\rceil + m - 1$;
- we denote by $\mathcal{F}_k^+(m)$ all the pairs $(G, C) \in \mathcal{F}_k(m)$ with $\kappa(G) \geq k + 1$. 

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The spiders are considered and defined now. For a tree, if there is at most one vertex of degree at least 3, then this tree is called a spider (Specially, a path is also a spider; but we a star is not a spider). Each leg of a spider is a path from the vertex adjacent to the root \(x_0\) to a vertex of degree 1; if there are \(z\) legs, then denote the spider by \(T_{m}^{t_1, t_2, \ldots, t_z}\), where \(|T_{m}^{t_1, t_2, \ldots, t_z}| = m\) and \(t_i\) denotes the order of the \(i\)th leg with \(t_1 + t_2 + \cdots + t_z + 1 = m\). If there are \(t\) legs of order one, then we abbreviate \(T_{m}^{1, 1, \ldots, 1, m-t-1}\) as \(T_{m}^{t, m-t-1}\).

**Lemma 4.** Let \(t \geq 0\) be an integer. For any \((G, C) \in F_k^+(m)\) and any \(s_0 \in G - V(C)\) with \(d(s_0) = |G| - 1\), there is a spider \(T_{m}^{t, m-t-1} \subseteq G - V(C)\) of order \(m\) rooted at \(s_0\) such that

\[
\kappa(G - V(T_{m}^{t, m-t-1})) \geq k.
\]

**Proof.** We perform an induction on the order \(n\) of the graph \(G\) for the lemma. Clearly, the order of the graph \(G\) must no less than \([\frac{3k}{2}] + m\) since \(\delta(G - C) \geq [\frac{3k}{2}] + m - 1\). Then it holds for \((G, C) \in F_k^+(m)\) if \(G\) is a complete graph with order at least \([\frac{3k}{2}] + m\). So we just need to consider the case that \(G\) is not complete and \(|G| \geq [\frac{3k}{2}] + m + 1\). Now assume that \(G\) is a graph with smallest order and with \(\Delta(G) = |G| - 1\) such that \((G, C) \in F_k^+(m)\) is a counterexample to Lemma 4 for \(k\), and some \(C \subseteq G\), and \(m\).

Subject to above assumption, we find out on the order \(m\) of tree \(T_{m}^{t, m-t-1}\) satisfying above assumption. From Lemma 3 there exists a path \(P \subseteq G - V(C)\) of length \(m - 1\) starting from \(s_0\) such that \(\kappa(G - V(P)) \geq k\) holds. Let \(P = \{s_0p_1\} \cup P[p_1, p]\). Since \(|N_G(u) \cap (G - P)| \geq [\frac{3k}{2}] + m - 1 - (|P| - 1) \geq k\) for any vertex \(u \in V(P[p_1, p])\), then \(\kappa(G - s_0p_1) \geq k\), where \(s_0p_1 \in E(G)\) is a subpath of \(P\) and also a spider \(T_{2}^{1,0}\). Hence, Lemma 4 holds when \(m = 1, 2\). Now suppose that a maximal spider \(T_{t+j+1}^{t, j}\) with root \(s_0\) and legs \(s_1, s_2, \ldots, s_t\) satisfies the following conditions.

(i) \(2 \leq |T_{t+j+1}^{t, j}| = t + j + 1 < m\);

(ii) \(\kappa(G - T_{t+j+1}^{t, j}) \geq k\).
Note that $s_1, s_2, \ldots, s_t \in V(G)$ and $s_0 s_i \in E(G)$, $1 \leq i \leq t$, and $T^{t,j}_{t+j+1} - \{s_i \mid 1 \leq i \leq t\}$ is a path of order $j + 1$, say $P = s_0 p_1 \cup P[p_1, p_j] := p_1 p_2 \cdots p_j$. Simply, we set $H = G - T^{t,j}_{t+j+1}$.

Claim 1. $\kappa(H) = k$.

Proof. Assume, to the contrary, that $\kappa(H) > k$. Since

$$|N_G(x) \cap (H - C)| \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1 - k - (m - 2) = \left\lfloor \frac{k}{2} \right\rfloor + 1$$

for $x \in \{s_0, p_j\}$, it follows that there exists a vertex $s \in H - C$ such that $xs \in E(G)$. Note that $T^{t,j}_{t+j+1} - xs$ is a spider rooted at $s_0$ of order $t+j+2 \leq m$, which contradicts the choice of $T^{t,j}_{t+j+1}$. Complete the proof of Claim 1. \hfill \square

Since $H$ is not a complete graph, it follows that $|H| \geq k+2$. An end $E$ is contained in $H$ with $E \cap C = \emptyset$. Set $S := N_H(E)$. Then $|S| = k$. Furthermore, let $\bar{E} = H - S - E$.

Claim 2. $|E| \geq 2$.

Proof. Assume, to the contrary, that $|E| = 1$. It satisfies $k = d_H(z) \geq \left\lceil \frac{3k}{2} \right\rceil + m - 1 - |T^{t,j}_{t+j+1}| \geq \left\lceil \frac{3k}{2} \right\rceil$ for each $z \in E$, which means $k = 1$ and $|T^{t,j}_{t+j+1}| = m - 1$. We get $V(T^{t,m-1-t-2}_{m-1}) \subseteq N_G(z)$ because of $\delta(G) \geq \left\lceil \frac{3k}{2} \right\rceil + m - 1$. Then $|T^{t,m-1-t-1}_{m-1}| := |T^{t,m-1-t-2}_{m-1} \cup zx| = m$ for $x \in \{s_0, p_{m-t-2}\}$ and $G - T^{t,m-1-t-1}_{m-1} = H - z$ is 1-connected. And also, $|T^{t+1,m-t-2}_{m-1}| := |T^{t+1,m-t-2}_{m-1} \cup zx| = m$ and $G - T^{t+1,m-t-2}_{m-1} = H - z$ is 1-connected, which contradicts that $T^{t+1,j}_{t+j+1}$ is a maximal spider. Complete the proof of Claim 2. \hfill \square

From Claim 2, we have $|E| \geq 2$. Then the graph $H[S] - \bar{E}$ is $(k+1)$-connected from Lemma 1. From above assumption, we know $\kappa(G) > k = \kappa(H)$, thus $N_G(T^{t,j}_{t+j+1}) \cap E \neq \emptyset$. Let $y$ be one of the farthest vertices to $s_0$ on $T^{t,j}_{t+j+1}$ with $N_G(y) \cap \bar{E} = \emptyset$. Otherwise, $S$ is also a separating set of $G$, which contradicts $\kappa(G) > k$. Suppose that $q$ is one vertex in $N_G(y) \cap \bar{E}$. We distinguish the following two cases to show this lemma. We will construct two larger spiders $T^{t+1,j+1}_{t+j+2}$ and $T^{t+1,j+2}_{t+j+2}$ such that $G - T^{t,j+1}_{t+j+2}$ and $G - T^{t+1,j}_{t+j+2}$ remains $k$-connected, respectively.
Claim 3. There is a larger spider $T_{t+j+1}^{t+j}$ such that $G - T_{t+j+2}^{t+j+1}$ remains $k$-connected.

Proof. Suppose that $y \in \{p_1, p_2, \ldots, p_j, s_0\}$. Let $P = P[p_j, y]$. Consider the graph $G - (T_{t+j+1}^{t+j+1} - \bar{P}) := H \cup \bar{P}$. Since $|N_G(x) \cap H| \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1 - (t + j) \geq \left\lfloor \frac{3k}{2} \right\rfloor + 1 \geq k$ for any $x \in V(\bar{P})$, it follows that

$$\kappa(G - (T_{t+j+1}^{t+j+1} - \bar{P})) = \kappa(H \cup \bar{P}) \geq k.$$  

As $y$ is the farthest vertex to $s_0$ on $T_{t+j+1}$, we have $N_G(\bar{P}) \cap E = \emptyset$. Naturally, $S$ is also a minimum separating set of $H \cup \bar{P}$, and $E$ is an end of $H \cup \bar{P}$. From Lemma 1, $(H \cup \bar{P})[S](E \cup S) = H[S] - \bar{E}$ is $(k + 1)$-connected. Furthermore, it follows that $C \subseteq H \cup \bar{P} - E$ and

$$\delta_{(H \cup \bar{P})}(H \cup \bar{P} - C) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1 - (t + j + 1 - |\bar{P}|) \geq \left\lfloor \frac{3k}{2} \right\rfloor + |\bar{P}|.$$  

Hence, $(H \cup \bar{P}, C) \in \mathcal{F}_k(|\bar{P}| + 1)$ and $(H \cup \bar{P}[S](E \cup S), K(S)) \in \mathcal{F}^+_k(|\bar{P}| + 1)$. From Lemma 3, there exists a path $Q \subseteq E$ of order $|\bar{P}| + 1$ starting from $q$ such that $(H \cup \bar{P})[S](E \cup S) - Q$ is $k$-connected. Now for $(H \cup \bar{P}, C) \in \mathcal{F}_k(|\bar{P}| + 1)$ by Lemma 2(c), then $\kappa((H \cup \bar{P})[S](E \cup S) - Q) \geq k$. The spider $T_{t+j+2}^{t+j+1} := (T_{t+j+1}^{t+j+1} \setminus \bar{P}) \cup yq \cup Q$ rooted at $s_0$ has order $|T_{t+j+1}^{t+j+1}| + 1 \leq m$ and $G - V(T_{t+j+2}^{t+j+1}) = H - Q$ is $k$-connected, a contradiction.

Suppose that $y \in \{s_1, s_2, \ldots, s_t\}$. Let $\bar{P} = P[p_j, p_t]$ and then there is a spider $T_{t+j+2}^{t+j+1} := (T_{t+j+1}^{t+j+1} - \bar{P}) \cup yp \cup Q$ with order $t + j + 2 \leq m$ according to the same way as above case. □

Claim 4. There is a larger spider $T_{t+j+2}^{t+j+1}$ such that $G - T_{t+j+2}^{t+j+1}$ remains $k$-connected.

Proof. Since $d_G(s_0) = |G| - 1$, then $N(s_0) \cap E \neq \emptyset$. Let $N(s_0) \cap E = \{q\}$. Furthermore, from Lemma 3, there exists a vertex $q \subseteq E$ such that $H[S](E \cup S) - q$ is $k$-connected. Hence, $G - V(T_{t+j+2}^{t+j+1}) = H - q$ is $k$-connected, a contradiction. □

By Claim 3 and Claim 4 we completed the proof. □
**Theorem 2.** Every $k + 1$-connected graph $G$ with $\delta(G) \geq \left\lceil \frac{3k}{2} \rightceil + m - 1$ and $\Delta(G) = |G| - 1$ for positive integers $k, m, t$, contains a spider $T^{t;m-t-1}_m$ such that

$$\kappa(G - V(T^{t;m-t-1}_m)) \geq k.$$ 

**Proof.** Clearly, the complete graph with order at least $\left\lceil \frac{3k}{2} \rightceil + m$ holds. Suppose that $G$ is $k + 1$-connected with $\delta(G) \geq \left\lceil \frac{3k}{2} \rightceil + m - 1$ and not complete.

Assume that $\kappa(G) \geq k + 1$. We again find a maximal spider $T^{t_j}_{t+j+1}$ with root $s_0$ and legs $s_1, s_2, \ldots, s_t$ satisfying the conditions: (i) $2 \leq |T^{t_j}_{t+j+1}| = t + j + 1 < m$; (ii) $\kappa(G - T^{t_j}_{t+j+1}) \geq k$. Then in the following proof we prove exactly in the same way and symbols as in the proof of Lemma 4. Naturally, we can show

$$\kappa(H) = k \text{ and } |E| \geq 2.$$ 

Since $H := G - T^{t_j}_{t+j+1}$ is not a complete graph, it follows that $|H| \geq k + 2$. An end $E$ is contained in $H$ with $E \cap C = \emptyset$. Set $S := N_H(E)$. Then $|S| = k$. Furthermore, let $\bar{E} = H - S - E$.

Then the graph $H[S] - \bar{E}$ is $(k + 1)$-connected from Lemma 4. From above assumption, we know $\kappa(G) > k = \kappa(H)$, thus $N_G(T^{t_j}_{t+j+1}) \cap E \neq \emptyset$. Let $y$ be one of farthest vertices to $s_0$ on $T^{t_j}_{t+j+1}$ with $N_G(y) \cap E \neq \emptyset$. Set $q \in N_G(y) \cap E$.

**Claim 1.** There is a larger spider $T^{t_{j+1}}_{t+j+2}$ such that $G - T^{t_{j+1}}_{t+j+2}$ remains $k$-connected.

Suppose that $y \in \{p_1, p_2, \ldots, p_j, s_0\}$. Let $\bar{P} = P[p_j, y]$. Consider the graph $G - (T^{t_j}_{t+j+1} - \bar{P}) := H \cup \bar{P}$. Since $|N_G(x) \cap H| \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1 - (t + j) \geq \left\lfloor \frac{3k}{2} \right\rfloor + 1 \geq k$ for any $x \in V(\bar{P})$, it follows that

$$\kappa(G - (T^{t_j}_{t+j+1} - \bar{P})) = \kappa(H \cup \bar{P})) \geq k.$$ 

As $y$ is the farthest vertex to $s_0$ on $T^{t_j}_{t+j+1}$, we have $N_G(\bar{P}) \cap E = \emptyset$. Naturally, $S$ is also a minimum separating set of $H \cup \bar{P}$, and $E$ is an end of $H \cup \bar{P}$. From Lemma 4 $(H \cup \bar{P})[S](E \cup S) = H[S] - \bar{E}$ is $(k + 1)$-connected. Furthermore,
it follows that \( C \subseteq H \cup \bar{P} - E \) and
\[
\delta_{(H \cup \bar{P})}(H \cup \bar{P} - C) \geq \left\lfloor \frac{3k}{2} \right\rfloor + m - 1 - (t + j + 1 - |\bar{P}|) \geq \left\lfloor \frac{3k}{2} \right\rfloor + |\bar{P}|.
\]

Hence, \((H \cup \bar{P}, C) \in F_k(\bar{P} + 1)\) and \((H \cup \bar{P}[S](E \cup S), K(S)) \in F_k^+(\bar{P} + 1)\).

From Lemma 3, there exists a path \( Q \subseteq E \) of order \(|\bar{P}| + 1\) starting from \( q \) such that \((H \cup \bar{P})[S](E \cup S) - Q \) is \( k \)-connected. Now for \((H \cup \bar{P}, C) \in F_k(\bar{P} + 1)\) by Lemma 2(c), then \( \kappa((H \cup \bar{P})[S](E \cup S) - Q) \geq k \). The spider \( T_{t+j+1}^{t+j+1} := (T_{t+j+1}^{t+j} \setminus \bar{P}) \cup yq \cup Q \) rooted at \( s_0 \) has order \(|T_{t+j+1}^{t+j}| + 1 \leq m\) and \( G - V(T_{t+j+1}^{t+j+1}) = H - Q \) is \( k \)-connected, a contradiction.

We can again prove it holds for \( y \in \{s_1, s_2, \ldots, s_t\} \) by the above way, where \( \bar{P} = \bar{P}[j, y] \). Complete the proof of Lemma 2.

**Claim 2.** There is a larger spider \( T_{t+j+2}^{t+j+2} \) such that \( G - T_{t+j+2}^{t+j+2} \) remains \( k \)-connected.

**Proof.** Since \( d_G(s_0) = |G| - 1 \), it follows that \( N(s_0) \cap E \neq \emptyset \). Let \( N(s_0) \cap E = \{q\} \). Furthermore, from Lemma 3 there exists a vertex \( q \subseteq E \) such that \( H[S](E \cup S) - q \) is \( k \)-connected. Hence, \( G - V(T_{t+j+2}^{t+j+2}) = H - q \) is \( k \)-connected, a contradiction. \( \square \)

By Claims 1 and 2 we completed the proof. \( \square \)

### 3 Concluding remark

As we all know, the higher the connectivity is, the more complex the structure of the graph is. In order to complete the proof of Mader’s conjecture, we need to prove more structure theorems. According to our research experience, it needs a totally new method if one would to confirm thoroughly Mader’s conjecture for the highly connected graph \( G \). Naturally, maybe one could replace the minimum degree condition with some others, such as degree sum condition. In addition, one could also confirm it for some special \( k \)-connected graphs \( G \).
Maybe one could prove it for more special trees utilizing our results. For the tree $T^{t;i,m-t-1-i}_m$, it means $T^{t;i,m-t-1-i}_m$ has $t$ legs of length one, one leg of length $i$, and one leg of length $m - t - i - 1$.

**Problem 1.** Every $k + 1$-connected graph $G$ with $\delta(G) \geq \lceil \frac{3k}{2} \rceil + m - 1$ and $\Delta(G) = |G| + 1$ for positive integers $k, m, t$, contains a spider $T^{t;i,m-t-1-i}_m$ such that

$$\kappa(G - V(T^{t;i,m-t-1-i}_m)) \geq k.$$  

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**Declaration of interests**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**References**

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics 244, Springer, Berlin, 2008.

[2] G. Chartrand, A. Kaigars, and D.R. Lick, Critically $n$-connected graphs, *Proc. Amer. Math. Soc.* 32 (1972), 63–68.

[3] A.A. Diwan and N.P. Tholiya, Non-separating trees in connected graphs, *Discrete Math.* 309 (2009), 5235–5237.

[4] S. Fujita and K. Kawarabayashi, Connectivity keeping edges in graphs with large minimum degree, *J. Combin. Theory, Ser. B* 98 (2008), 805–811.

[5] T. Hasunuma, Connectivity keeping trees in 2-connected graphs with girth conditions, *Combin. Algor.* (2020), 316–329.
[6] T. Hasunuma and K. Ono, Connectivity keeping trees in 2-connected graphs, *J. Graph Theory* 94 (2020), 20–29.

[7] Y. Hong and Q. Liu, Mader’s conjecture for graphs with small connectivity, *J. Graph Theory* 101(3) 2022, 379–388.

[8] Y. Hong, Q. Liu, C. Lu, and Q. Ye, Connectivity keeping caterpillars and spiders in 2-connected graphs, *Discrete Math.* 344(3) (2021), 112236.

[9] C. Lu and P. Zhang, Connectivity keeping trees in 2-connected graphs, *Discrete Math.* 343(2) (2020), 1–4.

[10] W. Mader, Connectivity keeping paths in $k$-connected graphs, *J. Graph Theory* 65 (2010), 61–69.

[11] W. Mader, Connectivity keeping trees in $k$-connected graphs, *J. Graph Theory* 69 (2012), 324–329.

[12] Y. Tian, H. Lai, L. Xu, and J. Meng, Nonseparating trees in 2-connected graphs and oriented trees in strongly connected digraphs, *Discrete Math.* 342 (2019), 344–351.

[13] Y. Tian, J. Meng, and L. Xu, Connectivity keeping stars or double-stars in 2-connected graphs, *Discrete Math.* 341 (2018), 1120–1124.