A NOTE ON AUSLANDER-REITEN QUIVER METHODS FOR ENDO-TRIVIAL MODULES

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Abstract. The aim of the present note is to use Auslander-Reiten quiver techniques, on the one hand to describe the structure of the group of endo-trivial modules for classes of groups with Klein-four Sylow subgroups, and on the other hand to discard the existence of simple endo-trivial modules for alternating groups, symmetric groups and groups of Lie type in their defining characteristic, providing us with alternative proofs of results obtained in [LMS13].

1. Introduction

In [Bes91, Thm. 2.6] C. Bessenrodt determined the position of endo-trivial modules in the stable Auslander-Reiten quiver $\Gamma_s(kG)$ of a finite group $G$. However, her result has scarcely been exploited in the classification problem of endo-trivial modules, apart in [Kaw93, CMT11b] to treat groups with semi-dihedral Sylow 2-subgroups, and more recently in [LMS13] where it is used to determine the position of simple endo-trivial modules on the Brauer tree.

The aim of the present note is to describe further applications of Bessenrodt’s result [Bes91, Thm. 2.6]. Firstly, in Section 3, we investigate the structure of the group of endo-trivial modules $T(G)$ for groups $G$ with Klein-four Sylow 2-subgroup. In this case it is clear that the torsion free rank of $T(G)$ is one (see e.g. [Car12]), yet the structure of the torsion subgroup $TT(G)$ of $T(G)$ has not been determined.

Secondly, motivated by a recent result of Robinson [Rob11] asserting that simple endo-trivial modules in $p$-rank > 1 are either induced from a strongly $p$-embedded subgroup or are endo-trivial simple for a quasi-simple normal subgroup, in [LMS13] we started a classification of simple endo-trivial modules for quasi-simple groups. The methods used are mainly character-theoretic or use the intrinsic combinatorics of the groups considered, but do not try to use the structure of the group $T(G)$ when it is known. For alternating groups, symmetric groups and groups of Lie type in their defining characteristic, combining Auslander-Reiten quiver methods with known results on the structure of $T(G)$, in Section 4, we obtain a unified criterion to discard the existence of simple endo-trivial modules. This provides us with links between results obtained in [LMS13] and previous work on endo-trivial modules in [CMN06, CMN09, CHM10].

2. Preliminaries

Throughout, unless otherwise specified, we let $p$ denote a prime number, $G$ a finite group such that $p$ divides $|G|$, $k = \overline{k}$ an algebraically closed field of characteristic $p$. Modules are all finitely generated left $kG$-modules and $\text{mod}(kG)$ denotes the category of all such modules. If $P \in \text{Syl}_p(G)$, we denote by $f$ the Green correspondence with respect to $(G, N_G(P), P)$. 

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Furthermore $\Gamma_s(kG)$ (respectively $\Gamma_s(B)$) denotes the stable Auslander-Reiten quiver of the group algebra $kG$ (of the block $B$ of $kG$) and $\Gamma_s(M)$ the component of the module $M \in \text{mod}(kG)$. For further results and standard terminology concerning the Auslander-Reiten theory, we refer to [Ben98, Erd90, Web82] and references therein.

### 2.1. Endo-trivial and V-endo-trivial modules

Let $V \in \text{mod}(kG)$ be a fixed module. As introduced by Okuyama [Oku91], a module $A \in \text{mod}(kG)$ is said to be $V$-projective if $A$ is a direct summand of $V \otimes_k B$ for some module $B \in \text{mod}(kG)$ (see also [CP96, Las12]). If $V$ is absolutely $p$-divisible, i.e. $p \mid \dim_k(U)$ for every direct summand $U$ of $V$, then a module $M \in \text{mod}(kG)$ is called $V$-endo-trivial (or relatively $V$-endo-trivial) if there is an isomorphism of $kG$-modules $\text{End}_k(M) \cong M^* \otimes M \cong k \oplus (V\text{-proj})$, where $k$ denotes the trivial $kG$-module and $V\text{-proj}$ some $V$-projective module. Here are some basic properties of $V$-endo-trivial modules (cf. [Las11]):

**Lemma-Definition 2.1.** Let $G$ be a finite group.

(a) If $M \in \text{mod}(kG)$ is $V$-endo-trivial, then $M \cong M_0 \oplus (V\text{-proj})$ where $M_0$ is indecomposable and $V$-endo-trivial.

(b) The relation $M \sim_V N \iff M_0 \cong N_0$ is an equivalence relation on the class of $V$-endo-trivial modules. We let $T_V(G)$ denote the set of resulting equivalence classes.

(c) $T_V(G)$, endowed with the law $[M] + [N] := [M \otimes_k N]$, is an abelian group called the group of $V$-endo-trivial modules of $G$. The zero element is the class $[k]$ and $-[M] = [M^*]$, the class of the dual module.

Setting $V := kG$ yields the ordinary notion of endo-trivial modules and $T_V(G) = T(G)$ is the group of endo-trivial modules of $G$. Notice that for any $V \in \text{mod}(kG)$ an endo-trivial can also be seen as a $V$-endo-trivial module and $T(G)$ identifies with a subgroup of $T_V(G)$. For further results and references on endo-trivial modules, we refer the reader to the survey articles [Thé07] and [Car12]. We recall that $T(G)$ is a finitely generated abelian group. In particular its torsion subgroup $TT(G)$ is finite and its torsion free-rank can be described explicitly (see [CMN09]).

An absolutely $p$-divisible $kG$-module of particular interest is the module $V(F_G) := \bigoplus_Q kG^Q$, where $Q$ runs over the set of proper $p$-subgroups of the Sylow $p$-subgroups of $G$. The generalised Dade group of $G$ defined in [Las13] is denoted $D(G)$ and identifies with a subgroup of $T_V(F_G(G))$.

If $G$ is a finite group, we denote by $X(G)$ the abelian group of all isomorphism classes of one-dimensional $kG$-modules, endowed with the tensor product $\otimes_k$. (Recall that $X(G) \cong (G^{ab})_{p'}$, the $p'$-part of the abelianisation of $G$.) Identifying $\chi \in X(G)$ with its class $[\chi] \in T(G)$, we obtain series of embeddings:

$$X(G) \subseteq T(G) \subseteq T_V(G) \quad \text{and} \quad X(G) \subseteq T(G) \subseteq D(G) \subseteq T_V(F_G(G)).$$

If $P \in \text{Syl}_p(G)$ and $N := N_G(P)$, we denote by $f^{-1}(X(N))$ the subgroup of $T_V(F_G(G))$ consisting of the classes of the $kG$-Green correspondents of the elements of $X(N)$. We have $f^{-1}(X(N)) \cong X(N)$, but we emphasise that $f^{-1}(X(N))$ is not contained in $T(G)$ in general, however $f^{-1}(X(N)) \cap T(G) \subseteq TT(G)$.

The aim of this note is to make use of the following result of C. Bessenrodt concerning the position of endo-trivial modules in $\Gamma_s(kG)$.

**Theorem 2.2** ([Bes91], Thm. 2.6). Let $G$ be a finite group and $P \in \text{Syl}_p(G)$. Let $M$ be an indecomposable endo-trivial $kG$-module. Let $\Gamma$ be the tree class of $\Gamma_s(M)$. Then:

(a) If $P$ is cyclic, then $\Gamma = A_n$ for some $n \in \mathbb{N}$. The endo-trivial modules in $\Gamma_s(M)$ are exactly the modules forming the two end $\Omega^2$-orbits.
(b) If $P \cong C_2 \times C_2$ and $N_G(P) = C_G(P)$, then $\Gamma = \tilde{A}_{1,2}$. All modules in $\Gamma_s(M)$ are endo-trivial.

(c) If $P$ is a dihedral 2-group and (b) does not hold, then $\Gamma = A_\infty^\infty$. All modules in $\Gamma_s(M)$ are endo-trivial.

(d) If $P$ is a semi-dihedral 2-group, then $\Gamma = D_\infty$. The endo-trivial modules in $\Gamma_s(M)$ are exactly the modules forming the two end $\Omega^2$-orbits.

(e) In all other cases, $\Gamma = A_\infty$, and the endo-trivial modules in $\Gamma_s(M)$ form the unique end $\Omega^2$-orbit.

2.2. Groups with Klein-four Sylow 2-subgroups. Assume $G$ is a finite group with a Klein-four Sylow 2-subgroup $P$ and $\text{char}(k) = 2$. Set $N := N_G(P)$ and $\bar{N} := N/O_2(N)$. Then one of the following holds (see [Gor80, Thm 7.1]):

(i) $|N_G(P) : C_G(P)| = 1$ and involutions fuse. In fact by Burnside transfer theorem and its converse this happens if and only if $G$ is 2-nilpotent, that is $\bar{N} \cong C_2 \times C_2$; or

(ii) $|N_G(P) : C_G(P)| = 3$ and there are three conjugacy classes of involutions. In this case $\bar{N} \cong A_4$.

In case (i), the Auslander-Reiten component containing the trivial module is $\Gamma_s(k) \cong Z\tilde{A}_{1,2}$, and in case (ii), $\Gamma_s(k) \cong Z\tilde{A}_3$ (with tree class $A_\infty^\infty$). See Theorem 2.2 and [Ben98, §4.17].

**Theorem 2.3** ([Las11] Thm. 6.0.4 and [Las12] Cor. 8.1.3). Let $G$ be a finite group with a Klein-four Sylow 2-subgroup $P$. Let $V$ be an absolutely 2-divisible $kG$-module. Then the following holds:

(a) If $P \trianglelefteq G$, then there is a group isomorphism $T_V(G) \rightarrow T(G) : [M] \mapsto [M_0]$ where $M \cong M_0 \oplus (V - \text{proj})$ with $M_0$ the unique indecomposable and $V$-endo-trivial summand of $M$.

(b) The group $T_V(G)$ identifies via restriction with a subgroup of $T(N) \cong T_V(F_G)(G)$. Furthermore $D(G) \cong T_V(F_G)(G) = f^{-1}(X(N)) \oplus \langle \Omega(k) \rangle$, where $f^{-1}(X(N)) \cong X(N)$.

**Remark 2.4.** (a) Dade showed that $T(C_2 \times C_2) = \langle \Omega(k) \rangle \cong Z$, where $\Omega$ denotes the Heller operator. Moreover $T(A_4) = X(A_4) \oplus \langle \Omega(k) \rangle \cong Z/3Z \oplus Z$, where $X(A_4) \cong Z/3Z$ consists of the three one-dimensional $kA_4$-module, that we denote by $k, k_\omega, k_\zeta$. (See [The07, Car12] and references therein.)

(b) In general the restriction map $\text{Res}^G_V : T(G) \rightarrow T(N) : [M] \mapsto [\text{Res}^G_V(M)]$ is an injective group morphism (see [CMN06, Prop. 2.6]) and $T(G) = TT(G) \oplus \langle \Omega(k) \rangle$ where $TT(G) = \ker(\text{Res}^G_V)$ and $\langle \Omega(k) \rangle \cong Z$. Note that $f^{-1}(X(N))$ is the kernel of the restriction map $R^G_V : T_V(F_G)(G) \rightarrow T(P) : [M] \mapsto [\text{Res}^G_V(M)]$, however, a priori $f^{-1}(X(N))$ and $TT(G)$ need not be equal.

3. Endo-trivial modules for groups with Klein-four Sylow subgroups

Throughout this section, we let $G$ be a finite group with a Klein-four Sylow 2-subgroup $P$, $\text{char}(k) = 2$, and set $N := N_G(P)$ and $\bar{N} := N/O_2(N)$.

In [Las12], we conjectured that the group of endo-trivial modules $T(G)$ should always be isomorphic to the group $T_V(G)$ for any absolutely 2-divisible $kG$-module $V$. First we give an example to show that this conjecture is false in general and second we exhibit typical situations where it does hold.
3.1. The 2-nilpotent case. First assume $G$ is a 2-nilpotent group, that is $G = N \rtimes P$ for some $N \trianglelefteq G$ and satisfies (i) of \S 2.2. It was conjectured in [CMT11a] and proved in [NR10] that

$$T(G) \cong X(G) \oplus T(P) \cong X(G) \oplus \mathbb{Z},$$

where the torsion free-part of $T(G)$ is generated by the class of the first syzygy $\Omega(k)$. Under these assumptions, it may happen that $T(G) \ncong T_{V(F_G)}(G)$. Here is an example:

**Example 3.1.** Let $P$ be a Klein four-group with generators $u$ and $v$. Let $p$ be an odd prime and $X_{p^3}$ denote the extraspecial group of order $p^3$ and exponent $p$, given by the presentation

$$X_{p^3} = \langle a, b, z \mid a^3 = b^3 = z^3 = 1, [a, b] = z, [a, z] = 1 = [b, z] \rangle.$$  

Then build $G := X_{p^3} \rtimes P$ to be the semi-direct product where $P$ acts on $X_{p^3}$ as follows: $u$ acts trivially and $u \cdot a := a^{-1}, u \cdot b := b^{-1}, u \cdot z := z$. We have $N_G(P) = C_G(P) = Z \times P$ where $Z := \langle z \rangle \cong C_p$ is the center of $X_{p^3}$.

By the above, $T(G) \cong X(G) \oplus \mathbb{Z}$ and embeds as a subgroup of $T(N_G(P)) \cong X(N_G(P)) \oplus \mathbb{Z}$ via restriction. Now

$$X(G) = (G^{ab})_{2^r} = \left[\left(\left(X_{p^3}^{ab}\right)_{2^r}\right) \times P^{ab}\right]_{2^r} = (X_{p^3}/Z)_P = \langle \pi, \overline{a}, \overline{b} \rangle_{p=1}$$

whereas $X(N_G(P)) = (N_G(P)^{ab})_{2^r} = Z_P = Z \cong C_p$. (The subscript $P$ denotes taking the coinvariants with respect to the action of $P$.) It ensues that $T(G) \ncong T(N_G(P)) \cong T_{V(F_G)}(G)$.

3.2. Case 2: Involutions fuse. We now assume that $G$ satisfies (ii) of \S 2.2.

**Proposition 3.2.** Let $G$ be a finite group with a Klein-four Sylow 2-subgroup $P$ such that $|N_G(P) : C_G(P)| = 3$. Then $T(\mathfrak{A}_4) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}$ embeds as a subgroup of $T(G)$ via inflation from $\bar{N} \cong \mathfrak{A}_4$ and Green correspondence.

**Proof.** First $\bar{N} \cong \mathfrak{A}_4$ by assumption that $|N_G(P) : C_G(P)| = 3$ (see \S 2.2 and by Remark 2.4(a), $T(\mathfrak{A}_4) = X(\mathfrak{A}_4) \oplus ([\Omega(k)]) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}$. Both the inflation map $\text{Inf}_N^\mathbb{N} : T(\bar{N}) \to T(N) : [M] \mapsto [\text{Inf}_N^\mathbb{N}(M)]$ and $\text{Res}_N^\mathbb{N} : T(G) \to T(N)$ are injective group morphisms. In fact, if $M$ is an indecomposable endo-trivial $kG$-module, then $\text{Res}_N^\mathbb{N}([M]) = [Gr(M)]$ the class of the $kN$-Green correspondent of $M$. Thus in view of Remark 2.4(b) we only need to show that $f^{-1}(\text{Inf}_N^\mathbb{N}(X(\mathfrak{A}_4))) \cong \mathbb{Z}/3\mathbb{Z}$ is a subgroup of $T(G)$.

Now the three one-dimensional $k\mathfrak{A}_4$-modules $k, k_a, k_b$ all belong to the same component $\Theta(k) \cong \mathbb{Z}A_5$ of $\Gamma_s(k\mathfrak{A}_4)$ (see e.g. [Ben98, \S 4.1]). By Webb’s Theorem [Webb82, Thm. D], $\Theta(k) \cong \Gamma_s(kG)$ the component of the trivial module in $\Gamma_s(kG)$ via inflation from $\bar{N} \to N$ followed
by Green correspondence. Moreover by Theorem 2.2(c) all modules in $\Gamma_{s}(k)$ are endo-trivial, so that the $kG$-modules $k$, $f^{-1}(\text{Inf}_{N}^{G}(k_{w}))$ and $f^{-1}(\text{Inf}_{N}^{G}(k_{w}))$ are endo-trivial, as required. \qed

**Corollary 3.3.** Assume $G$ is a finite group with a self-centralising Klein-four Sylow $2$-subgroup $P$ such that $|N_{G}(P) : C_{G}(P)| = 3$, then $T_{V}(G) \cong T(G) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}$ for any absolutely $p$-divisible module $V \in \text{mod}(kG)$.

**Proof.** Let $V \in \text{mod}(kG)$ be absolutely $p$-divisible. By Theorem 2.3,

$$T(G) \leq T_{V}(G) \leq T_{V}(\mathcal{F}_{G})(G) \cong X(N_{G}(P)) \oplus \mathbb{Z}. $$

Since $P$ is self-centralising and $|N_{G}(P) : C_{G}(P)| = 3$, we have $N_{G}(P) \cong A_{4}$, so that $X(N_{G}(P)) \cong ((\mathfrak{A}_{4})^{ab})^{2} \cong \mathbb{Z}/3\mathbb{Z}$. Thus the claim follows from Proposition 3.2. \qed

Note that a finite group satisfying the assumptions of 3.3 is such that $G/O_{2'}(G) \cong L_{2}(q)$, $q \equiv \pm 3 \pmod{8}$. The groups $L_{2}(q)$ with $q \equiv \pm 3 \pmod{8}$ and $q > 3$ are in fact the only simple groups with Klein-four Sylow $2$-subgroups. (See e.g. [Gor80, §15, Thm. 2.1].)

**Corollary 3.4.** Let $G := L_{2}(q) \times A$ with $q \equiv \pm 3 \pmod{8}$ and $A$ an arbitrary $2'$-group. Then $T_{V}(G) \cong T_{V}(\mathcal{F}_{G})(G)$ for any absolutely $p$-divisible $V \in \text{mod}(kG)$. In particular $T(G) \cong f^{-1}(X(N_{G}(P))) \oplus \mathbb{Z}$.

In other words, in the above situation the $kG$-Green correspondents of the one-dimensional $kN_{G}(P)$-modules are all endo-trivial.

**Proof.** By Proposition 2.3(b) and Remark 2.4, it suffices to show that $TT(G) = f^{-1}(X(N))$. Let $P \in \text{Syl}_{2}(G)$. Then $N \cong \mathfrak{A}_{4} \times A$ so that $X(N) \cong (N^{ab})^{2} \cong (\mathfrak{A}_{4})^{ab} \times A^{ab}$. By Proposition 3.2 and its proof, $f^{-1}((\mathfrak{A}_{4})^{ab})$ is a subgroup of $TT(G)$, while $f^{-1}(A^{ab}) = X(G)$ is certainly also a subgroup of $TT(G)$. Indeed $X(G) \cong (G^{ab})^{2} = A^{ab}$. The claim follows. \qed

**Remark 3.5.** In general if $G$ is non-nilpotent with Klein-four Sylow $2$-subgroups one can say the following. First, as endo-trivial $k[C_{2} \times C_{2}]$-modules are non-periodic, so are endo-trivial $kG$-modules. Now a block of $kG$ containing an endo-trivial module must be of full defect and thus Morita equivalent to either $k\mathfrak{A}_{4}$ or the $B_{0}(k\mathfrak{A}_{5})$. These blocks have exactly one non-periodic $AR$-component isomorphic to $\mathbb{Z}\mathfrak{A}_{5}$ with tree class $A_{\infty}^{\mathfrak{A}_{5}}$ (see [Erd90, §V.4]). Moreover by Theorem 2.2, if such a block contains an indecomposable endo-trivial module $M$, then all modules in $\Gamma_{s}(M) \cong \mathbb{Z}\mathfrak{A}_{5}$ are endo-trivial and $\Gamma_{s}(M)$ contains exactly three elements of $TT(G)$. Therefore $|TT(G)|$ is bounded above by three times the number of blocks of full defect of $G$.

### 3.3. Endo-trivial modules for $L_{2}(q)$, $q \equiv 3 \pmod{8}$ via character theory

Throughout this subsection let $G := L_{2}(q)$ with $q \equiv 3 \pmod{8}$ and as before $N := N_{G}(P) \cong \mathfrak{A}_{4}$. It is known from Erdmann’s work [Erd77] that the $kG$-Green correspondents of the three one-dimensional $kN$-modules are simple. Thus in this case we have an explicit description of $TT(G) \cong \mathbb{Z}/3\mathbb{Z}$ (see Corollary 3.3). We show with the following proposition that character theory provides us with a simpler method to prove that the latter simple modules are torsion endo-trivial.

We let $S_{+}, S_{-}$ denote the two dual simple $kG$-modules with dimension $(q-1)/2$. They belong to the principal block and are liftable to simple $CG$-modules affording the Deligne-Lusztig induced characters $R_{+}(\theta_{0})$ and $R_{-}(\theta_{0})$, respectively, say. Here we use, and refer the reader to, Bonnafé’s notation for the ordinary characters of $\text{SL}_{2}(q)$, see [Bon11, Part II and Chap. 9].

**Proposition 3.6.** Let $G := L_{2}(q)$ with $q \equiv 3 \pmod{8}$, then the two simple $kG$-modules $S_{+}, S_{-}$ of dimension $(q-1)/2$ are endo-trivial and their classes in $T(G)$ lie in $TT(G)$. 
Proof. One easily computes from the character table of $L_2(q)$ that
\[ R'_+(\theta_0) \otimes R'_-(\theta_0) = 1_G + \sum R(\alpha), \]
where $\alpha$ runs over $\{\alpha \in \text{Irr}(T) \setminus \{1_T\} \mid \alpha(-1) = 1\}$ ($T$ is the split torus of $\text{SL}_2(q)$, consisting of diagonal matrices, and is isomorphic to the group $\mu_{q-1}$ of $q-1$-th roots of unity). As each $R(\alpha)$ is a defect zero character, reduction modulo $p$ yields $(S_-)^* \otimes_k S_- \cong S_+ \otimes_k S_- \cong k \oplus (\text{proj})$. Hence both $S_+$ and $S_-$ are endo-trivial. Moreover
\[ R'_+ (\theta_0) \otimes R'_-(\theta_0) = R'_-(\theta_0) + \sum R'(\theta), \]
where $\theta$ runs over $\{\theta \in \text{Irr}(T') \setminus \{1_{T'}, \theta_0\} \mid \theta(-1) = 1, R'(\theta) \text{ belongs to a block of defect } 1\}$ (here $T' \cong \mu_{q+1}$ is a non-split torus of $\text{SL}_2(q)$). Therefore reduction modulo $p$ yields $(S_+)^{\otimes 2} \cong S_+ \oplus F$, where $F$ is a module whose indecomposable summands lie in blocks of defect one. Since $S_+$ is endo-trivial, $F$ is in fact projective and $2[S_+] = [(S_+)^{\otimes 2}] = [S_-] \in T(G)$. Finally, as $(S_+)^* \cong S_-$, it follows that $[S_+],[S_-] \in TT'(G)$.

Remark 3.7. Similar computations show that the two simple modules of dimension $(q-1)/2$ for $L_2(q)$, $q \equiv 7 \pmod{8}$ are endo-trivial, although they are not torsion elements of $T(G)$ in this case. This completes a left-open case in the character-theoretic study of endo-trivial modules for some quasi-simple groups. However we emphasise that the character-theoretic approach of [LMS13] seems more straightforward and powerful than the more involved module theoretic methods presented below.

4. SIMPLE ENDO-TRIVIAL MODULES VIA THE AUFLANDER-REITEN QUIVER

Combining results on the position of simple modules in the stable Auslander-Reiten quiver [KMU00, KMU01] and results on the structure of the group of endo-trivial modules described in [CMN06, CMN09, CHM10], one obtains the following criteria to discard the existence of simple endo-trivial modules for some quasi-simple groups. However we emphasise that the character-theoretic approach of [LMS13] seems more straightforward and powerful than the more involved module theoretic methods presented below.

Proposition 4.1. Let $G$ be a finite group and $B_0$ be the principal block of $kG$. Assume $B_0$ has wild representation type and $G$ satisfies the following conditions:

(i) each AR-component of type $ZA_\infty$ of $\Gamma_\mathfrak{s}(B_0)$ contains at most one simple module;

(ii) $T(G) \cong \mathbb{Z}$.

Then $G$ does not have non-trivial simple endo-trivial modules.

Proof. The condition $T(G) \cong \mathbb{Z}$ implies that an indecomposable endo-trivial $kG$-module is isomorphic to a syzygy module $\Omega^s(k)$ for some $n \in \mathbb{Z}$, and thus lies in $B_0$. Moreover $\Gamma_\mathfrak{s}(B_0)$ has only two connected components containing endo-trivial modules: $\Gamma_\mathfrak{s}(k)$ and $\Gamma_\mathfrak{s}(\Omega(k))$, both isomorphic to $ZA_\infty$ as $B_0$ is a wild block by 2.2). By assumption (i), the trivial module $k$ is the unique simple module in $\Gamma_\mathfrak{s}(k)$. If $\Gamma_\mathfrak{s}(\Omega(k))$ contains a simple endo-trivial module $S$, then $S = \Omega^{2n+1}(k)$ for some $n \in \mathbb{Z}$. Therefore $S^* = \Omega^{-2n-1}(k)$ is also simple and lies in $\Gamma_\mathfrak{s}(\Omega(k))$, which contradicts assumption (i). The claim follows. □
Theorem 4.2. Let $G$ be a finite group of one of the following types:

(a) $G$ is a perfect group of Lie type defined over a field of characteristic $p$ and $G$ is not of type $A_1(p)$ ($p > 2$), $^2A_2(p)$, $A_2(p)$, $B_2(p)$ ($p \geq 5$), $G_2(p)$ ($p \geq 7$), $^2B_2(2^{3+4}t^2)$ ($a \geq 0$) or $^2G_2(3^{3+4}t^2)$ ($a \geq 0$);

(b) $p = 2$ and $G = S_n$ is a symmetric group such that $n \geq 6$;

(c) $G = A_n$ is an alternating group such that $n \geq 8$ if $p = 2$, or such that $3p \leq n < p^2$ or $p^2 + p \leq n$ if $p \geq 3$.

Then $G$ does not have non-trivial simple endo-trivial modules.

Proof. The claim is a direct consequence of Proposition 4.1. Indeed, by [CMN06] and [CMN09, CHM10] any group of type (a), (b) or (c) is such that $T(G) \cong \mathbb{Z}$. Moreover by [KMU00, KU01] and [Uno00, Thm. 6] these groups also satisfy condition (i) of Proposition 4.1. □

4.1. Bounds for the number of simple endo-trivial modules. For the remainder of the section, assume $G$ is a finite group such that $kG$ has wild representation type. Let $P \in \text{Syl}_p(G)$, and $N := N_G(P)$. We write $se(G)$ for the number of isomorphism classes of simple endo-trivial $kG$-modules. In this situation, by Theorem 2.2, endo-trivial module lie at the end of $ZA_\infty$ components of $\Gamma_s(kG)$.

Lemma 4.3. Let $T$ be a self-dual indecomposable $kG$-module lying on an AR-component of $\Gamma_s(kG)$ of type $ZA_\infty$ and assume that both $\Gamma_s(T)$ and $\Gamma_s(\Omega(T))$ contain at most one simple module. If the module $\Omega^n(T)$ is simple, then $n = 0$.

Proof. Assume $S := \Omega^n(T)$ is simple. Then $S^* \cong \Omega^{-n}(T^*) \cong \Omega^{-n}(S) \cong \Omega^{-2n}(S)$, so that $S$ and $S^*$ belong to the same AR-component. As $ZA_\infty$ components contain at most one simple module, this forces $S$ to be self-dual, which in turn forces $S \cong T$, i.e. $n = 0$. Indeed if $M \in \text{mod}(kG)$, then there is at most one self-dual module in the set $\{\Omega^n(M) \mid n \in \mathbb{N}\}$. □

The next property is essentially due to [KMU00], though it is not written in the same terms.

Lemma 4.4. Assume $kG$ has wild representation type. Then the indecomposable representatives of the classes in $f^{-1}(X(N_G(P)))$ lie in pairwise distinct AR-components of $\Gamma_s(kG)$.

Proof. Applying [KMU00, Lem. 1.6] to $P$ and $N_G(P)$, we have that one-dimensional $kN_G(P)$-module all lie in pairwise distinct AR-component of $\Gamma_s(kN_G(P))$ (at the end as they are endo-trivial). Then, as the Green correspondence $f$ gives a graph monomorphism (see [KMU00, Lem. 1.1(iv)]), the $kG$-Green correspondents of the modules in $X(N_G(P))$ also lie in pairwise distinct AR-components of $\Gamma_s(kG)$. □

Remark 4.5. Since $kG$ has wild representation type, $T(P)$ is torsion-free and $TT(G)$ identifies via restriction with a subgroup of $X(N)$ (see [The07, Car12]).

(a) If a self-dual (simple) $kG$-module $S$ is endo-trivial, then obviously $[S]$ is an element of order 1 or 2 in $T(G)$. Hence $se(G)$ is bounded above by the number of elements of order at most 2 in $TT(G)$. Therefore a larger bound, but easier to compute, is given by $|X(N)| = |N : [N, N]|_{p'}$.

(b) As a consequence, in characteristic 2, there is no self-dual (simple) endo-trivial module.

Now we let $l_s$ denote the number of elements of order 1 or 2 of $TT(G)$ and $l_r := |TT(G)| - l_s$.

Corollary 4.6. Assume $kG$ has wild representation type and $G$ satisfies the following conditions:

(i) each AR-component of type $ZA_\infty$ of $\Gamma_s(kG)$ contains at most one simple module;

(ii) the torsion-free rank of $T(G)$ is one.

Then $se(G) \leq l_s + 2l_r$. 
Proof. As $kG$ has wild representation type, under assumption (ii), $T(G) = TT(G) \oplus \mathbb{Z}$ where $\mathbb{Z} = \langle \Omega(k) \rangle$ and $TT(G) \leq f^{-1}(X(N)),$ (see [Car12]). Therefore any endo-trivial $kG$-module has the form $\Omega^n(T)$ with $[T] \in TT(G), \ n \in \mathbb{Z}$.

As noticed above the number of self-dual simple endo-trivial modules is bounded above by the number $l_s$. By Theorem 2.2 and Lemma 4.3, if $S$ is a self-dual endo-trivial, then there is no other simple module in $\Gamma_s(S) \cup \Gamma_s(\Omega(S))$. Let $T$ be a non self-dual indecomposable such that $[T] \in TT(G)$. Then by assumption (ii) there is at most one simple endo-trivial module in $\Gamma_s(T)$ and one in $\Gamma_s(\Omega(T))$. The claim follows.

\[ \square \]

Remark 4.7. Similarly to Remark 4.5(a), a larger bound than that of Corollary 4.6, but easier to compute, is obtained by replacing $TT(G)$ with $X(N)$.

Example 4.8. (a) If $G$ is one of the groups of Lie type $^2A_2(p)$ ($p+1 \not\equiv 0 \pmod{3}$), $^2B_2(2^{a+ \frac{1}{2}})$ ($a \geq 0$) or $^2G_2(3^{a+ \frac{1}{2}})$ ($a \geq 0$) in their defining characteristic, then we have $\text{se}(G) \leq l_s + l_r$. Indeed in these cases $T(G) \cong X(N) \oplus \mathbb{Z}$ by [CMN06]. Then using the fact that if $S$ is a simple $kG$-module, then $\Omega^n(S), \ n \in \mathbb{Z}$ is not simple unless $n = 0$ by [KMU00, Cor. 3.2], the bound obtained in the proof of Corollary 4.6 is in fact $l_s + l_r$.

(b) In order to complete partially the results on $\text{se}(G)$ obtained in Theorem 4.2(c) for alternating groups via quiver methods, let $k$ be a field of characteristic $p \geq 3$ and let $G := \mathfrak{A}_n$ be an alternating group. Then it follows from Corollary 4.6 and [Uno00, Thm. 6(b)] that:

$$\text{se}(G) \leq \begin{cases} 6 & \text{if } p = 3, \ n \in \{6, 7\}, \\ 4 & \text{if } p > 3, \ n \in \{2p, 2p + 1\}, \\ 2 & \text{if } 2p + 2 \leq n < 3p. \end{cases}$$

Moreover, in the 2nd and the 3rd case, the simple endo-trivial modules have to be self-dual by Lemma 4.3. Indeed by [CMN09, Thm. B], we have $T(G) \cong \mathbb{Z} \oplus TT(G)$ with $TT(G) \cong \mathbb{Z}/4\mathbb{Z}$, $TT(G) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $TT(G) \cong \mathbb{Z}/2\mathbb{Z}$, respectively to the three cases above.

In both cases we refer the reader to [LMS13] for the explicit determination of the simple endo-trivial $kG$-modules.

In conclusion, it would be interesting to have an answer to the following question:

Question 4.9. If $kG$ has wild representation type and $S$ is a simple endo-trivial $kG$-module (hence non-periodic), is $\Omega^n(S)$ simple if and only if $n = 0$? In particular, is $\Omega^n(k)$ simple if and only if $n = 0$?

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Auslander-Reiten quiver methods for endo-trivial modules

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