CORRECTION OF BDFK FOR FRACTIONAL FEYNMAN-KAC EQUATION WITH LÉVY FLIGHT

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Abstract. In this work, we present the correction formulas of the \( k \)-step BDF convolution quadrature at the starting \( k - 1 \) steps for the fractional Feynman-Kac equation with Lévy flight. The desired \( k \)-th order convergence rate can be achieved with nonsmooth data. Based on the idea of [Jin, Li, AND Zhou, SIAM J. Sci. Comput., 39 (2017), A3129–A3152], we provide a detailed convergence analysis for the correction BDF\( k \) scheme. The numerical experiments with spectral method are given to illustrate the effectiveness of the presented method. To the best of our knowledge, this is the first proof of the convergence analysis and numerical verified the space fractional evolution equation with correction BDF\( k \).

Key words. Fractional Feynman-Kac equation with Lévy flight, correction of BDF\( k \), fractional substantial derivative, error estimates.

1. Introduction. Functionals of Brownian motion have diverse applications in physics, mathematics, and other fields. The probability density function of Brownian functionals satisfies the Feynman-Kac formula, which is a Schrödinger equation in imaginary time. The functionals of non-Brownian motion, or anomalous diffusion, follow the general fractional Feynman-Kac equation \[\eqref{1.1}\], where the fractional substantial derivative is involved \[\eqref{1.2}\]. This paper focuses on providing the correction of \( k \)-step backward differential formulas (BDF\( k \)) for backward fractional Feynman-Kac equation with Lévy flight \[\eqref{1.2}\].

\[
C_s^\gamma D_t^\alpha G(t) + AG(t) = f(t) \quad \text{with} \quad A := (-\Delta)^{\alpha/2},
\]

where \( f \) is a given function and the initial condition \( G(0) = G_0 \) with the homogeneous Dirichlet boundary conditions. Here \( (-\Delta)^{\alpha/2} \) with \( \alpha \in (1, 2) \) is the fractional Laplacian, the definition of which is based on the spectral decomposition of the Dirichlet Laplacian \[\eqref{1.2}\], and the Caputo fractional substantial derivative with \( 0 < \gamma < 1 \) is defined by \[\eqref{1.2}\].

\[
C_s^\gamma D_t^\alpha G(t) = \frac{1}{\Gamma(1 - \gamma)} \int_0^t e^{-\sigma(t-s)} \left( \sigma + \frac{\partial}{\partial s} \right) G(s)ds
\]

with a constant \( \sigma > 0 \). It should be noted that \[\eqref{1.2}\] reduces to the Caputo fractional derivative if \( \sigma = 0 \).

High-order schemes for the time discretization of \[\eqref{1.1}\] with \( \sigma = 0 \) (Caputo fractional derivative) have been proposed by various authors. There are two predominant discretization techniques in time direction: L1-type approximation \[\eqref{1.1}\] and Lubich-Grünwald-Letnikov approximation \[\eqref{1.2}\]. For the first group, under the time regularity assumption on the solution, they developed the L1 schemes \[\eqref{1.1}\] for the Caputo fractional derivative and strictly proved the stability and convergence rate with \( O(\tau^{\alpha}) \). It is extended to the quadratic interpolation case \[\eqref{1.2}\] with a convergence rate \( O(\tau^{3-\alpha}) \). Recently, for a layer or blows up at \( t = 0 \), a sharp new discrete

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stability result has been considered in [29]. In the second group, using fractional linear multistep method and Fourier transform, error analysis of up to sixth order temporal accuracy for fractional ordinary differential equation has been discussed [21] with the starting quadrature weights schemes. A few years later, based on operational calculus with sectorial operator, nonsmooth data error estimates for fractional evolution equations have been studied in [8, 22] and developed in [16, 17] to restore $O(\tau^k)$. Under the time regularity assumption, high order finite difference method (BDF2) for the anomalous-diffusion equation has been studied in [18] by analyzing the properties of the coefficients. Application of Grenander-Szegö theorem, stability and convergence for time-fractional sub-diffusion equation have been provided in [13, 15] with weighted and shifted Grünwald operator.

In recent years, the numerical method for backward fractional Feynman-Kac equation (1.1) with $\alpha = 2$ were developed. For example, the time discretization of Caputo fractional substantial derivative was first provided in [4] with the starting quadrature weights schemes. Spectral methods for substantial fractional ordinary differential equations was presented in [14]. Under smooth assumption, numerical algorithms (finite difference and finite element) for (1.1) with $\alpha = 2$ are considered in [9]. Moreover, the second-convergence analysis are discussed in [6, 12]. In addition, the problem with nonsmooth solution is also discussed in [6]. Recently, the second-order error estimates are presented in [30] with nonsmooth initial data. However, it seems that there are no published works for more than three order accurate scheme for model (1.1) with Lévy flight. In this work, we provide a detailed convergence analysis of the correction BDF$k (k \leq 6)$ for (1.1) with nonsmooth data.

We first provide the solutions of fractional Feynman-Kac equation with Lévy flight.

**Solution representation for (1.1).** Let $\Omega$ be a bounded domain with a boundary $\partial\Omega$. If $\sigma = 0$, then (1.1) reduce to the following time-space Caputo-Riesz fractional diffusion equation [7]

\[
\begin{aligned}
C^\gamma \partial_t^\gamma G(x, t) + A G(x, t) &= f(x, t), \quad (x, t) \in \Omega \times (0, T] \\
G(x, 0) &= v(x), \quad x \in \Omega \\
G(x, t) &= 0, \quad (x, t) \in \partial\Omega \times (0, T],
\end{aligned}
\]

where $f$ is a given function and $A$ denotes the Laplacian $(-\Delta)^{\alpha/2}$. Based on the idea of [16, 27, 22] with the eigenpairs $\{(\lambda_j^{\alpha/2}, \varphi_j)\}_{j=1}^\infty$ of the operator $A$, it is easy to get

\[
G(x, t) = E(t)v + \int_0^t E(t - s)f(s)ds.
\]

Here the operators $E(t)$ and $\overline{E}(t)$ are, respectively, given by

\[
E(t) = \sum_{j=1}^\infty E_{\gamma, 1}(-\lambda_j^{\alpha/2} t^\gamma)(v, \varphi_j)\varphi_j(x)
\]

and

\[
\overline{E}(t)\chi = \sum_{j=1}^\infty t^{\gamma-1}E_{\gamma, \gamma}(-\lambda_j^{\alpha/2} t^\gamma)(v, \varphi_j)(\chi, \varphi_j)\varphi_j(x)
\]
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with the Mittag-Leffler function $E_{\gamma, \alpha/2}(z)$ \[29\] p. 17, i.e.,

$$E_{\gamma, \alpha/2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\gamma + \alpha/2)}, z \in C.$$ 

If $\sigma > 0$, using (1.3) and the following property \[4, 5\]

$$C_s D^\gamma_t e^{-\sigma t} G(t) = e^{-\sigma t} \left( C D^\gamma_t G(t) \right),$$

infer that

$$G(x, t) = e^{-\sigma t} E(t) v + e^{-\sigma t} \int_0^t E(t-s) e^{\sigma s} f(s) ds.$$ 

From the above solution representation of (1.1), we known that the smoothness of all the data of (1.1) do not imply the smoothness of the solution $G$. For example, if $G_0 \in L^2(\Omega)$ and $\gamma \in (0, 1)$ with $\alpha = 2$, the following estimate holds \[27, Theorem 2.1\]

$$\| C D^\gamma_t G(t) \|_{L^2(\Omega)} \leq c t^{-\gamma} \| G_0 \|_{L^2(\Omega)},$$

which reduces to a classical case $\| \partial_t G(t) \|_{L^2(\Omega)} \leq c t^{-1} \| G_0 \|_{L^2(\Omega)}$ if $\gamma = 1$ \[29\] Lemma 3.2]. This shows that $G$ has an initial layer at $t \to 0^+$ (i.e., unbounded near $t = 0$) \[29\]. Hence, the high-order convergence rates may not hold for nonsmooth data. Thus, the corrected algorithms are necessary in order to restore the desired convergence rate, even for smooth initial data. In this paper, based on the idea of \[17\], we present the correction of the $k$-step BDF convolution quadrature (CQ) at the starting $k-1$ steps for the backward fractional Feynman-Kac equation with Lévy flight \[14\]. The desired $k$th-order convergence rate can be achieved with nonsmooth data. To the best of our knowledge, this is the first proof of the convergence analysis and numerical verified the space fractional evolution equation with correction BDF$k$.

The paper is organized as follows. In the next Section, we provide the correction of the $k$-step BDF convolution quadrature at the starting $k-1$ steps for (1.1). In Section 3, based on operational calculus, the detailed convergence analysis of the correction BDF$k$ are provided. To show the effectiveness of the presented schemes, the results of numerical experiments are reported in Section 4.

2. Correction of BDF$k$. Let $t_n = n\tau, n = 0, 1, \ldots, N$ with $\tau = \frac{T}{N}$ the uniform time steplength, and let $G^n$ denote the approximation of $G(t)$ and $f^n = f(t_n)$. The convolution quadrature generated by BDF$k$, $k = 1, 2, \ldots, 6$, approximates the Riemann-Liouville fractional substantial derivative $\mathcal{D}^\gamma_t$ by \[4\]

$$\mathcal{D}^\gamma_t \varphi^n := \frac{1}{\tau^\gamma} \sum_{j=0}^{n} q_j \varphi^{n-j}$$

with $\varphi^n = \varphi(t_n)$. Here the weights $q_j = e^{-\sigma j \tau} b_j$ and $b_j$ are the coefficients in the series expansion

$$\delta^\gamma_t (\xi) = \frac{1}{\tau^\gamma} \sum_{j=0}^{\infty} b_j \xi^j \quad \text{with} \quad \delta^\gamma_t (\xi) := \frac{1}{\tau} \sum_{j=1}^{k} \frac{1}{j} (1 - \xi)^j \quad \text{and} \quad \delta (\xi) := \delta_1 (\xi).$$
Then the standard BDF$_k$ for (1.1) is as following

\begin{equation}
\mathcal{D}_\tau \left( G^n - e^{-\sigma t_n} G(0) \right) + A G^n = f(t_n).
\end{equation}

To obtain the $k$-order accuracy with nonsmooth data, we correct the standard BDF$_k$ (2.3) at the starting $k-1$ steps by

\begin{equation}
\mathcal{D}_\tau \left( G^n - e^{-\sigma t_n} G(0) \right) + A G^n = - a^{(k)}_n e^{-\sigma n \tau} AG^0 + b^{(k)}_n f(0)
\end{equation}

\begin{equation}
+ \sum_{l=1}^{k-2} d^{(k)}_{l,n} \tau^l \partial_t^l f(0) + f(t_n) \quad 1 \leq n \leq k-1;
\end{equation}

\begin{equation}
\mathcal{D}_\tau \left( G^n - e^{-\sigma t_n} G(0) \right) + A G^n = f(t_n) \quad k \leq n \leq N.
\end{equation}

Here the correction coefficients $a^{(k)}_n$ and $b^{(k)}_n$ are given in Table 2.1 and $d^{(k)}_{l,n}$ is given in Table 2.2. We noted that the correction coefficients share similarities with the fractional Caputo equations [17].

Table 2.1: The coefficients $a^{(k)}_n$ and $b^{(k)}_n$.

| Order of BDF | $a^{(k)}_{1,n}$ | $a^{(k)}_{2,n}$ | $a^{(k)}_{3,n}$ | $a^{(k)}_{4,n}$ | $a^{(k)}_{5,n}$ | $b^{(k)}_{1,n}$ | $b^{(k)}_{2,n}$ | $b^{(k)}_{3,n}$ | $b^{(k)}_{4,n}$ | $b^{(k)}_{5,n}$ |
|--------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $k = 2$      | $\frac{1}{2}$  | $\frac{1}{2}$  | $\frac{1}{2}$  | $\frac{1}{2}$  | $\frac{1}{2}$  | $\frac{1}{2}$  | $\frac{1}{2}$  | $\frac{1}{2}$  | $\frac{1}{2}$  | $\frac{1}{2}$  |
| $k = 3$      | $\frac{3}{4}$  | $\frac{3}{4}$  | $\frac{3}{4}$  | $\frac{3}{4}$  | $\frac{3}{4}$  | $\frac{3}{4}$  | $\frac{3}{4}$  | $\frac{3}{4}$  | $\frac{3}{4}$  | $\frac{3}{4}$  |
| $k = 4$      | $\frac{5}{6}$  | $\frac{5}{6}$  | $\frac{5}{6}$  | $\frac{5}{6}$  | $\frac{5}{6}$  | $\frac{5}{6}$  | $\frac{5}{6}$  | $\frac{5}{6}$  | $\frac{5}{6}$  | $\frac{5}{6}$  |
| $k = 5$      | $\frac{11}{10}$ | $\frac{11}{10}$ | $\frac{11}{10}$ | $\frac{11}{10}$ | $\frac{11}{10}$ | $\frac{11}{10}$ | $\frac{11}{10}$ | $\frac{11}{10}$ | $\frac{11}{10}$ | $\frac{11}{10}$ |
| $k = 6$      | $\frac{28}{25}$ | $\frac{28}{25}$ | $\frac{28}{25}$ | $\frac{28}{25}$ | $\frac{28}{25}$ | $\frac{28}{25}$ | $\frac{28}{25}$ | $\frac{28}{25}$ | $\frac{28}{25}$ | $\frac{28}{25}$ |

Table 2.2: The coefficients $d^{(k)}_{l,n}$.

| Order of BDF | $l$ | $i$ | $d^{(k)}_{i,l,1}$ | $d^{(k)}_{i,l,2}$ | $d^{(k)}_{i,l,3}$ | $d^{(k)}_{i,l,4}$ | $d^{(k)}_{i,l,5}$ |
|--------------|-----|-----|-------------------|-------------------|-------------------|-------------------|-------------------|
| $k = 3$      | 1   | 1   | $\frac{1}{2}$    | 0                 |                   |                   |                   |
| $k = 4$      | 1   | 1   | $\frac{1}{8}$    | $\frac{1}{2}$    | 0                 |                   |                   |
|              | 2   |     | 0                 | 0                 | 0                 |                   |                   |
| $k = 5$      | 1   | 1   | $\frac{3}{20}$   | $\frac{3}{20}$   | $\frac{3}{20}$   | 0                 |                   |
|              | 2   |     | $\frac{3}{20}$   | $\frac{3}{20}$   | 0                 | 0                 |                   |
|              | 3   |     | $\frac{3}{20}$   | $\frac{3}{20}$   | $\frac{3}{20}$   | 0                 | 0                 |
| $k = 6$      | 1   | 1   | $\frac{7}{20}$   | $\frac{7}{20}$   | $\frac{7}{20}$   | $\frac{7}{20}$   | 0                 |
|              | 2   |     | $\frac{1}{8}$    | $\frac{1}{8}$    | $\frac{1}{8}$    | $\frac{1}{8}$    | 0                 |
|              | 3   |     | $\frac{1}{8}$    | $\frac{1}{8}$    | $\frac{1}{8}$    | $\frac{1}{8}$    | 0                 |
|              | 4   |     | 0                 | 0                 | 0                 | 0                 | 0                 |

2.1. Solution representation with CQ for (1.1). First we split right-hand side $f$ into

\begin{equation}
f(t) = f(0) + \sum_{l=1}^{k-2} \frac{t^l}{l!} \partial_t^l f(0) + R_k.
\end{equation}
Here

\[(2.6) \quad R_k = f(t) - f(0) - \sum_{l=1}^{k-2} \frac{t^l}{l!} \partial^l_t f(0) = \frac{t^{k-1}}{(k-1)!} \partial^{k-1}_t f(0) + \frac{t^{k-1}}{(k-1)!} \ast \partial^k_t f,\]

and the symbol \(\ast\) denotes Laplace convolution. Let \(W(t) := G(t) - e^{-\sigma t} G(0)\) with \(W(0) = 0\). Then we can rewrite (1.1) as

\[(2.7) \quad sD^\gamma_t W(t) + AW(t) = -Ae^{-\sigma t} G(0) + f(0) + \sum_{l=1}^{k-2} \frac{1}{z^{l+1}} \partial^l_t f(0) + R_k\]

with \(\frac{D^\gamma_t G(t)}{G(t)} = sD^\gamma_t [G(t) - e^{-\sigma t} G(0)] = sD^\gamma_t W(t)\) in [4].

Applying Laplace transform in [4] to both sides of the above equation, we have

\[(\sigma + z)^\gamma \hat{W}(z) + A\hat{W}(z) = -A(\sigma + z)^{-1} G(0) + z^{-1} f(0) + \sum_{l=1}^{k-2} \frac{1}{z^{l+1}} \partial^l_t f(0) + \hat{R}_k(z),\]

where \(\hat{W}\) and \(\hat{R}_k\) denote the Laplace transform, i.e., \(\hat{u}(z) = \int_0^\infty e^{-zt} u(t) dt\). Then

\[\hat{W}(z) = ((\sigma + z)^\gamma + A)^{-1} \left[ -A(\sigma + z)^{-1} G(0) + z^{-1} f(0) + \sum_{l=1}^{k-2} \frac{1}{z^{l+1}} \partial^l_t f(0) + \hat{R}_k(z) \right].\]

By the inverse Laplace transform, we can obtain the solution \(W(t)\) as following

\[(2.8) \quad W(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zt} K(\sigma + z) \left( -AG(0) + \frac{\sigma + z}{z} f(0) \right) dz + \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zt} (\sigma + z) K(\sigma + z) \left( \sum_{l=1}^{k-2} \frac{1}{z^{l+1}} \partial^l_t f(0) + \hat{R}_k(z) \right) dz\]

with

\[(2.9) \quad K(\sigma + z) = ((\sigma + z)^\gamma + A)^{-1} (\sigma + z)^{-1}.\]

Here the \(\Gamma_{\theta,\kappa}\) is defined by [10]

\[(2.10) \quad \Gamma_{\theta,\kappa} = \{ z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta \} \cup \{ z \in \mathbb{C} : z = re^{\pm i\theta}, \kappa \leq r < \infty \}.\]

It should be noted that we have the following resolvent bound [8]

\[(2.11) \quad ||(z^\gamma + A)^{-1}|| \leq c|z|^{-\gamma} \quad \text{and} \quad ||K(z)|| \leq c|z|^{-1-\gamma}\]

with a positive constant \(c\).

### 2.2. Discrete solution representation with CQ for (2.4)

In this subsection, we provide the discrete solution of (2.4).
Lemma 2.1. Let \( f \in C^{k-1}([0, T]; L^2(\Omega)) \) and \( \int_0^t (t-s)^{\gamma-1} ||\partial_s^\gamma f(s)||_{L^2(\Omega)} ds < \infty \). Let discrete solution \( W^n = G^n - e^{-\sigma n\tau}G(0) \) with \( W^0 = 0 \). Then

\[
W^n = \frac{1}{2\pi i} \int_{\Gamma_{\hat{\theta}, \hat{\kappa}}} e^{t_n z} K \left( \delta_x (e^{-\sigma z}) \right) \left[ -\mu_1 (e^{-\sigma z}) AG^0 + \frac{\delta_x (e^{-\sigma z})}{\delta_x (e^{-\sigma z})} \mu_2 (e^{-\sigma z}) f(0) \right] dz
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_{\hat{\theta}, \hat{\kappa}}} e^{t_n z} K \left( \delta_x (e^{-\sigma z}) \right) \delta_x (e^{-\sigma z}) \left( -\mu_1 (e^{-\sigma z}) AG^0 + \frac{\delta_x (e^{-\sigma z})}{\delta_x (e^{-\sigma z})} \mu_2 (e^{-\sigma z}) f(0) \right) dz
\]

\[
\times \sum_{l=1}^{k-2} \left( \frac{\gamma_l (e^{-\sigma z})}{l!} + \sum_{j=1}^{k-1} d_{l,n}^{(k)} e^{-\sigma j} \right) z^{l+1} \partial_{l} f(0) dz
\]

\[
+ \frac{1}{2\pi i} \int_{\Gamma_{\hat{\theta}, \hat{\kappa}}} e^{t_n z} K \left( \delta_x (e^{-\sigma z}) \right) \delta_x (e^{-\sigma z}) \left( -\mu_1 (e^{-\sigma z}) AG^0 + \frac{\delta_x (e^{-\sigma z})}{\delta_x (e^{-\sigma z})} \mu_2 (e^{-\sigma z}) f(0) \right) dz.
\]

Here the contour is

\[
\Gamma_{\hat{\theta}, \hat{\kappa}} = \{ z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta \} \cup \left\{ z \in \mathbb{C} : z = re^{\pm i\theta}, \kappa \leq r < \frac{\pi}{\tau \sin(\theta)} \right\},
\]

and \( \tilde{R}_k (e^{-\sigma z}) = \sum_{n=1}^{\infty} e^{-z \sigma n} R_k(t_n) \), and

\[
\mu_1 (\xi) = \delta (\xi) \left( \frac{\xi}{1 - \xi} + \sum_{j=1}^{k-1} a_j^{(k)} \xi^j \right),
\]

(2.13)

\[
\mu_2 (\xi) = \delta (\xi) \left( \frac{\xi}{1 - \xi} + \sum_{j=1}^{k-1} b_j^{(k)} \xi^j \right) \text{ and } \gamma_l (\xi) = \sum_{n=1}^{\infty} n^l \xi^n.
\]

Proof. Let \( W^n = G^n - e^{-\sigma n\tau}G(0) \). From (2.1), it holds

\[
\mathcal{D}_x W^n + AW^n = - (1 + a_n^{(k)}) e^{-\sigma n\tau} AG^0 + (1 + a_n^{(k)}) f(0)
\]

\[
+ \sum_{l=1}^{k-2} \left( \frac{l!}{l} + \sigma_{l,n}^{(k)} \right) \sigma_{l} f(0) + R_k(t_n) \quad 1 \leq n \leq k - 1;
\]

(2.14)

\[
\mathcal{D}_x W^n + AW^n = - e^{-\sigma n\tau} AG^0 + f(0) + \sum_{l=1}^{k-2} \frac{l!}{l} \sigma_{l} f(0) + R_k(t_n) \quad k \leq n \leq N.
\]
Multiplying (2.14) by $\xi^n$ and summing over $n$, we have

\[
(2.15) \quad \sum_{n=1}^{\infty} \xi^n (\mathcal{D}_\tau W^n + AW^n) = - \left( \sum_{n=1}^{\infty} \xi^n e^{-\sigma n \tau} + \sum_{j=1}^{k-1} \xi^j a_j^{(k)} e^{-\sigma j \tau} \right) AG^0 \\
+ \left( \sum_{n=1}^{\infty} \xi^n + \sum_{j=1}^{k-1} \xi^j b_j^{(k)} \right) f(0) + \sum_{l=1}^{k-2} \left( \sum_{n=1}^{\infty} \xi^n \frac{l!}{l!} + \sum_{j=1}^{k-1} \xi^j d_{l,n}^{(k)} \right) \partial_l f(0) + \tilde{R}_k(\xi) \\
= - \left( \frac{\xi e^{-\sigma \tau}}{1 - \xi e^{-\sigma \tau}} + \sum_{j=1}^{k-1} \xi^j a_j^{(k)} e^{-\sigma j \tau} \right) AG^0 + \left( \frac{\xi}{1 - \xi} + \sum_{j=1}^{k-1} \xi^j b_j^{(k)} \right) f(0) \\
+ \sum_{l=1}^{k-2} \left( \frac{\gamma_l(\xi)}{l!} + \sum_{j=1}^{k-1} \xi^j d_{l,n}^{(k)} \right) \tau^l \partial_l f(0) + \tilde{R}_k(\xi),
\]

where $\tilde{R}_k(\xi) = \sum_{n=1}^{\infty} \xi^n R_k(t_n)$ and $\gamma_l(\xi) = \sum_{n=1}^{\infty} n^l \xi^n$. Here we use the property $\sum_{n=1}^{\infty} \xi^n = \frac{1}{1 - \xi}$. Since

\[
\sum_{n=1}^{\infty} \xi^n \mathcal{D}_\tau W^n = \sum_{n=1}^{\infty} \xi^n \frac{1}{\tau} \sum_{j=1}^{n} q_{n-j} W^j = \sum_{j=1}^{\infty} \frac{1}{\tau} \sum_{n=1}^{\infty} \xi^n q_n \sum_{j=1}^{\infty} \xi^j W^j = \delta_\tau^j (e^{-\sigma \tau} \xi) \tilde{W}(\xi)
\]

with $W^0 = 0$ and $\sum_{n=1}^{\infty} \xi^n AW^n = \tilde{W}(\xi)$. We obtain

\[
(2.17) \quad \tilde{W}(\xi) = K \left[ \delta_\tau (e^{-\sigma \tau} \xi) \right] \delta_\tau (e^{-\sigma \tau} \xi) \left[ - \left( \frac{e^{-\sigma \tau} \xi}{1 - e^{-\sigma \tau} \xi} + \sum_{j=1}^{k-1} a_j^{(k)} e^{-\sigma j \tau} \xi^j \right) AG^0 \\
+ \left( \frac{\xi}{1 - \xi} + \sum_{j=1}^{k-1} b_j^{(k)} \xi^j \right) f(0) + \sum_{l=1}^{k-2} \left( \frac{\gamma_l(\xi)}{l!} + \sum_{j=1}^{k-1} d_{l,n}^{(k)} \xi^j \right) \tau^l \partial_l f(0) + \tilde{R}_k(\xi) \right] \\
= K \left[ \delta_\tau (e^{-\sigma \tau} \xi) \right] \left[ - \tau^{-1} \mu_1 (e^{-\sigma \tau} \xi) AG^0 + \tau^{-1} \frac{\delta_\tau (e^{-\sigma \tau} \xi)}{\delta_\tau (\xi)} \mu_2 (\xi) f(0) \\
+ \delta_\tau (e^{-\sigma \tau} \xi) \sum_{l=1}^{k-2} \left( \frac{\gamma_l(\xi)}{l!} + \sum_{j=1}^{k-1} d_{l,n}^{(k)} \xi^j \right) \tau^l \partial_l f(0) + \delta_\tau (e^{-\sigma \tau} \xi) \tilde{R}_k(\xi) \right],
\]

where $K$ is given by (2.14) and

\[
\mu_1 (e^{-\sigma \tau} \xi) = \delta (e^{-\sigma \tau} \xi) \left( \frac{e^{-\sigma \tau} \xi}{1 - e^{-\sigma \tau} \xi} + \sum_{j=1}^{k-1} a_j^{(k)} e^{-\sigma j \tau} \xi^j \right),
\]

(2.18) \[
\mu_2 (\xi) = \delta (\xi) \left( \frac{\xi}{1 - \xi} + \sum_{j=1}^{k-1} b_j^{(k)} \xi^j \right).
\]
According to Cauchy’s integral formula, and the change of variables $\xi = e^{-z\tau}$, and Cauchy’s theorem of complex analysis, we have

\[
W^n = \frac{1}{2\pi i} \int_{|\xi| = \rho} \xi^{-n-1} \tilde{W}(\xi) d\xi = \frac{1}{2\pi i} \int_{\Gamma^{\tau}} e^{nz\tau} \tilde{W}(e^{-z\tau}) dz
\]

(2.19)

\[
= \frac{1}{2\pi i} \int_{\Gamma^{\tau,\kappa}} e^{nz\tau} \tilde{W}(e^{-z\tau}) dz
\]

with the Bromwich contours [26]

\[
\Gamma^{\tau} = \{ z = \kappa + iy : y \in \mathbb{R} \text{ and } |y| \leq \pi/\tau \}.
\]

The proof is completed. □

3. Convergence analysis. In this section, based on the idea of [17], we provided the detailed convergence analysis of the correction BDF$k$ for [14] with Lévy flight.

3.1. A few technical Lemmas. We first introduce a few technical lemmas.

Lemma 3.1. Let $\delta(\xi)$ be given by (2.2) for $1 \leq k \leq 6$. Then

\[
\delta(e^{-y}) = y - \frac{1}{k+1} y^{k+1} + O(y^{k+2}).
\]

Proof. According to (2.2) and the Taylor series expansion $e^{-y} = \sum_{n=0}^{\infty} \frac{(-y)^n}{n!} = \sum_{n=0}^{k+1} \frac{(-y)^n}{n!} + O(y^{k+2})$, we have

\[
\delta(e^{-y}) = 1 - e^{-y} = y - \frac{1}{2} y^2 + O(y^3), \quad k = 1;
\]

\[
\delta(e^{-y}) = \frac{3}{2} \left( 1 - \frac{4}{3} e^{-y} + \frac{1}{3} e^{-2y} \right) = y - \frac{1}{3} y^3 + O(y^4), \quad k = 2;
\]

\[
\delta(e^{-y}) = \frac{11}{6} \left( 1 - \frac{18}{11} e^{-y} + \frac{9}{11} e^{-2y} - \frac{2}{11} e^{-3y} \right) = y - \frac{1}{4} y^4 + O(y^5), \quad k = 3;
\]

\[
\delta(e^{-y}) = \frac{25}{12} \left( 1 - \frac{48}{25} e^{-y} + \frac{36}{25} e^{-2y} - \frac{16}{25} e^{-3y} + \frac{3}{25} e^{-4y} \right) = y - \frac{1}{5} y^5 + O(y^6), \quad k = 4;
\]

\[
\delta(e^{-y}) = \frac{137}{60} \left( 1 - \frac{300}{137} e^{-y} + \frac{300}{137} e^{-2y} - \frac{200}{137} e^{-3y} + \frac{75}{137} e^{-4y} - \frac{12}{137} e^{-5y} \right)
\]

\[= y - \frac{1}{6} y^6 + O(y^7), \quad k = 5;
\]

\[
\delta(e^{-y}) = \frac{147}{60} \left( 1 - \frac{360}{147} e^{-y} + \frac{450}{147} e^{-2y} - \frac{400}{147} e^{-3y} + \frac{225}{147} e^{-4y} - \frac{72}{147} e^{-5y} + \frac{10}{147} e^{-6y} \right)
\]

\[= y - \frac{1}{7} y^7 + O(y^8), \quad k = 6.
\]

The proof is completed. □

Lemma 3.2. Let $\delta_{\tau}(\xi)$ be given by (2.2) for $1 \leq k \leq 6$. Then there exist the positive constants $c_1, c_2$ and $c$ such that

\[
c_1 |z| \leq |\delta_{\tau}(e^{-z\tau})| \leq c_2 |z| \quad \text{and} \quad |\delta_{\tau}^2(e^{-z\tau}) - z^\gamma| \leq c \tau^k |z|^{k+\gamma} \quad \forall z \in \Gamma^{\tau,\kappa}.
\]
Proof. From Lemma 3.1 we have $c_1 |z| \leq |\delta_\tau (e^{-z\tau})| \leq c_2 |z|$. On the other hand, according to (39) of [17] and Lemma 3.1 we have

$$\left| \delta_\tau^2 (e^{-z\tau}) - z^\gamma \right| = \gamma \left| \int_{z}^{\delta_\tau (e^{-z\tau})} \xi^{\gamma-1} d\xi \right| \leq \max_{\xi} |\xi|^{\gamma-1} \left| \delta_\tau (e^{-z\tau}) - z \right| \leq c\tau^{k+1}|z|^{k+\gamma}.$$  

The proof is completed. \[ \square \]

Lemma 3.3. Let $\delta_\tau$ and $K$ be given by (2.22) and (2.24), respectively. Then there exist a positive constants $c$ such that

$$\left\| K(\delta_\tau (e^{-(\sigma+z)\tau})) - K(\sigma + z) \right\| \leq c\tau^{k+1}|\sigma + z|^{k-1-\gamma} \leq c\tau^{k+1}|z|^{k-1-\gamma},$$

where $\kappa$ in (2.10) is large enough compared with $\sigma$.

Proof. Using the triangle inequality and (2.11), we have

$$\left\| K(\delta_\tau (e^{-(\sigma+z)\tau})) - K(\sigma + z) \right\| = \left\| (\delta_\tau (e^{-(\sigma+z)\tau}))^{-1}(\delta_\tau^2 (e^{-(\sigma+z)\tau}) + A)^{-1} - (\sigma + z)^{-1}((\sigma + z)^\gamma + A)^{-1} \right\|$$

$$\leq \left\| (\delta_\tau (e^{-(\sigma+z)\tau}))^{-1} - (\sigma + z)^{-1} \right\| \left\| (\delta_\tau^2 (e^{-(\sigma+z)\tau}) + A)^{-1} \right\|$$

$$\leq c\tau^{k+1}|\sigma + z|^{k-1-\gamma}.$$  

Here we use [17]

$$\left\| (\delta_\tau^2 (e^{-(\sigma+z)\tau}) + A)^{-1} \right\| \leq c|\sigma + z|^{-\gamma}$$

and

$$\left\| (\delta_\tau (e^{-(\sigma+z)\tau}) + A)^{-1} - ((\sigma + z)^\gamma + A)^{-1} \right\|$$

$$= \left( (\sigma + z)^\gamma - \delta_\tau^2 (e^{-(\sigma+z)\tau}) \right) \left( \delta_\tau^2 (e^{-(\sigma+z)\tau}) + A \right)^{-1} ((\sigma + z)^\gamma + A)^{-1}.$$  

The proof is completed. \[ \square \]

Lemma 3.4. Let $\delta_\tau$ be given by (2.24) with $1 \leq k \leq 6$ and $\mu_1(\xi), \mu_2(\xi), \gamma_l(\xi)$ be given by (2.13). Let $\gamma_j^{(k)}, \delta_j^{(k)}$ and $\delta_j^{(k)}$ be given in Table 2.1 and Table 2.2. Then

$$\left| \delta_\tau (e^{-(\sigma+z)\tau}) - \sigma + z \right| \leq c\tau^{k+1}|\sigma + z|^{k+1} \leq c\tau^{k+1}|z|^{k+1},$$

$$\mu_1(e^{-(\sigma+z)\tau}) - 1 \leq c\tau^{k+1}|\sigma + z|^{k+1} \leq c\tau^{k+1}|z|^{k+1},$$

$$\left| \mu_2(e^{-(\sigma+z)\tau}) - 1 \right| \leq c\tau^{k+1}|\sigma + z|^{k+1} \leq c\tau^{k+1}|z|^{k+1},$$

$$\left| \left( \frac{\gamma_l(e^{-(\sigma+z)\tau})}{l!} + \sum_{j=1}^{k-1} \delta_j^{(k)} e^{-z\tau} \right) e^{-(\sigma+z)\tau} - \frac{1}{z^{l+1}} \right| \leq c\tau^{k+1}|\sigma + z|^{k-1-\gamma} \leq c\tau^{k+1}|z|^{k-1-\gamma},$$

where $\kappa$ in (2.10) is large enough compared with $\sigma$.

Proof. From Lemma 5.1 infer that

$$\left| \delta_\tau (e^{-(\sigma+z)\tau}) - \sigma + z \right| \leq c\tau^{k+1}|\sigma + z|^{k+1} \leq c\tau^{k+1}|z|^{k+1}.$$
The others inequality similar arguments can be performed as in [17], we omit it here.

**Lemma 3.5.** Let $\delta_r(\xi)$ be given by (2.2) with $1 \leq k \leq 6$. Then

$$
\left| \frac{\delta_r(e^{-(\sigma+z)\tau})}{\delta_r(e^{-\tau})} - \frac{\sigma + z}{z} \right| \leq c \tau^k \left( |\sigma + z|^{k+1} |z|^{-1} + |\sigma + z| |z|^{k-1} \right) \leq c \tau^k |z|^k,
$$

where $\kappa$ in (2.10) is large enough compared with $\sigma$.

**Proof.** According to Lemma 3.2 and 3.4, we have

$$
\left| \frac{\delta_r(e^{-(\sigma+z)\tau})}{\delta_r(e^{-\tau})} - \frac{\sigma + z}{z} \right| \leq \left| \frac{\delta_r(e^{-(\sigma+z)\tau}) - (\sigma + z)}{\delta_r(e^{-\tau})} \right| + \left| \frac{(\sigma + z) (z - \delta_r(e^{-\tau}))}{z \delta_r(e^{-\tau})} \right| 
$$

$$
\leq c \tau^k |\sigma + z|^{k+1} + c \tau^k |\sigma + z| |z|^k 
$$

$$
\leq c \tau^k \left( |\sigma + z|^{k+1} |z|^{-1} + |\sigma + z| |z|^{k-1} \right) \leq c \tau^k |z|^k.
$$

The proof is completed.

**Lemma 3.6.** Let $\gamma_l(\xi)$ be given by (2.13). Then

$$
\left| \frac{\gamma_l(e^{-\tau})}{l!} e^{l+1} \right| \leq c \tau^{l+1} \quad l = 1, 2, \ldots, 5, \quad z \in \Gamma_{\delta, \kappa}^r.
$$

**Proof.** From (2.14), we have

$$
\gamma_1(e^{-\tau}) = \frac{e^{-\tau}}{(1 - e^{-\tau})^2}, \quad \gamma_2(e^{-\tau}) = \frac{e^{-\tau} + e^{-2\tau}}{(1 - e^{-\tau})^3},
$$

$$
\gamma_3(e^{-\tau}) = \frac{e^{-\tau} + 4e^{-2\tau} + e^{-3\tau}}{(1 - e^{-\tau})^4},
$$

$$
\gamma_4(e^{-\tau}) = \frac{e^{-\tau} + 11e^{-2\tau} + 11e^{-3\tau} + e^{-4\tau}}{(1 - e^{-\tau})^5},
$$

$$
\gamma_5(e^{-\tau}) = \frac{e^{-\tau} + 26e^{-2\tau} + 66e^{-3\tau} + 26e^{-4\tau} + e^{-5\tau}}{(1 - e^{-\tau})^6}.
$$

Using the Taylor series expansion, we have

$$
(3.1) \quad \left| (1 - e^{-\tau})^{l+1} \right| \geq c |z|^{2l+2}.
$$

For simplicity, we denote $\gamma_l(e^{-\tau}) = \frac{\psi_l(e^{-\tau})}{\rho_l(e^{-\tau})}$ with $\rho_l(e^{-\tau}) = (1 - e^{-\tau})^{l+1}$, it yields

$$
\left| \left( \frac{\psi_l(e^{-\tau})}{\rho_l(e^{-\tau})} \right) z^{l+1} - \frac{l \rho_l(e^{-\tau})}{(l+1)!} \right| 
$$

$$
= \left| \tau^{l+2} z^{2l+3} \sum_{n=0}^{\infty} \frac{\sum_{j=1}^{l} p_i j (-j z \tau)^n}{(n + l + 1)!} - l \frac{\sum_{j=1}^{l+1} c_{i,j} (-j z \tau)^n}{(n + 2l + 2)!} \right| 
$$

$$
\leq c \tau^{l+2} |z|^{2l+3} \leq c \tau^{l+1} |z|^{2l+2} \quad \text{if} \quad l = 2, 4,
$$
and

\[
\left| (\psi(e^{-z\tau})) z^{l+1} - \sum_{n=0}^{\infty} \left( \sum_{j=1}^{l} p_{l,j} \frac{(-j z \tau)^n}{(n+1)!} - l! \frac{\sum_{j=1}^{l+1} c_{l,j} (-j z \tau)^n}{(n+2l+2)!} \right) \right| \\
\leq c t^{l+1} |z|^{2l+2} \quad \text{if} \quad l = 1, 3, 5.
\]

Here the coefficients \( p_{l,j} \) and \( c_{l,j} \) are, respectively, given in Table 3.1 and Table 3.2.

| \( l \) | \( p_{1,1} \) | \( p_{1,2} \) | \( p_{1,3} \) | \( p_{1,4} \) | \( p_{1,5} \) |
|------|------|------|------|------|------|
| 1    | 1    |      |      |      |      |
| 2    | 1    | 16   |      |      |      |
| 3    | 1    | 64   | 8    |      |      |
| 4    | 1    | 704  | 8019 | 4096 |      |
| 5    | 1    | 1664 | 48114| 106496| 15625 |

Table 3.2: The coefficients \( c_{l,j} \).

| \( l \) | \( c_{1,1} \) | \( c_{1,2} \) | \( c_{1,3} \) | \( c_{1,4} \) | \( c_{1,5} \) | \( c_{1,6} \) |
|------|------|------|------|------|------|------|
| 1    | -2   | 16   |      |      |      |      |
| 2    | 3    | -384 | 2187 |      |      |      |
| 3    | -4   | 1536 | -26244| 65536|      |      |
| 4    | 5    | -20480| 1771470 | -209715201| 48828125|      |
| 5    | -6   | 61440 | -10628820 | 251658240 | -1464843750| 2176782336|

The proof is completed.

3.2. Error analysis. We now given the error analysis of correction BDFk \((2.4)\) for \((1.1)\).

**Lemma 3.7.** Let \( G(t) \) and \( G^n \) be the solutions of \((1.1)\) and \((2.4)\), respectively. If \( G_0 = 0 \) and \( f(t) = \frac{t^{k-1}}{(k-1)!} \) with \( g = \partial_t^{k-1} f(0) \), then

\[
\| G^n - G(t_n) \| \leq c t^n \int_0^{t_n} (t_n - s)^{\gamma-1} \| g \| ds.
\]

**Proof.** From \((2.8)\) and \((2.12)\), we have

\[
G(t_n) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zt_n} ((\sigma + z)^\gamma + A)^{-1} \frac{1}{z^k} gdz
\]

and

\[
G^n = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zt_n} (\delta^\gamma (e^{-z\tau} + A)^{-1} \frac{1}{(k-1)!} \tau^k gdz.
\]
Then

\[ G(t_n) - G^n = I + II. \]

Here

\[
I = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zt_n} \left( ((\sigma + z)\gamma + A)^{-1} \frac{1}{z^k} - (\delta_\gamma^2(e^{-(\sigma+z)\gamma} + A)^{-1}\frac{\gamma^{k-1}(e^{-z\gamma}}{(k-1)!}) \right) g(z) dz
\]

and

\[
II = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa} \setminus \Gamma_{\theta,\kappa}^r} e^{zt_n} ((\sigma + z)\gamma + A)^{-1} z^{-k} g(z) dz
\]

with

\[
\Gamma_{\theta,\kappa} \setminus \Gamma_{\theta,\kappa}^r = \left\{ z \in \mathbb{C} : z = re^{\pm i\theta}, -\frac{\pi}{r \sin(\theta)} \leq r < \infty \right\}.
\]

Using Lemma 3.6, it yields

\[
\|I\| \leq c t^{-k} \|g\| \left( \int_{\Gamma_{\theta,\kappa}} e^{zt_n} \cos \theta r^{-\gamma} dr + \int_{-\theta}^{0} e^{\kappa t_n \cos \psi} r^{1-\gamma} d\psi \right)
\]

\[
= c t^{-k} \|g\| \left( \int_{t_n}^{t_n \frac{1}{\cos \theta}} e^{\kappa t_n \cos \psi} r^{1-\gamma} d\psi + \int_{-\theta}^{0} e^{\kappa t_n \cos \psi} d\psi \right)
\]

\[
\leq c t^{-k} \|g\| \left( t_n^{-1} + \kappa^{1-\gamma} \int_{-\theta}^{0} e^{\kappa T} d\psi \right) \leq c t^{-k} \|g\| (t_n^{-1} + \kappa^{1-\gamma} e^{\kappa T})
\]

\[
\leq c t^{-k} \|g\| (t_n^{\gamma-1} + (\kappa T)^{1-\gamma} e^{\kappa T} t_n^{\gamma-1}) \leq c t^{-k} \Gamma_{\theta,\kappa}^r \|g\|
\]

and

\[
\|II\| \leq c \|g\| \int_{\Gamma_{\theta,\kappa} \setminus \Gamma_{\theta,\kappa}^r} e^{zt_n} \cos \theta r^{-\gamma} dr
\]

\[
\leq c t^{-k} \|g\| \int_{\Gamma_{\theta,\kappa} \setminus \Gamma_{\theta,\kappa}^r} e^{zt_n} \cos \theta r^{-\gamma} dr \leq c t^{-k} \Gamma_{\theta,\kappa}^r \|g\|.
\]

According to the triangle inequality, the desired result are obtained. □

**Lemma 3.8.** Let \( G(t) \) and \( G^n \) be the solutions of (1.1) and (2.4), respectively. If \( G_0 = 0 \) and \( f(t) = \frac{t^{k-1}}{(k-1)!} \ast g(t) \), then

\[
\|G^n - G(t_n)\| \leq c t^{-k} \int_{0}^{t_n} (t_n - s)^{\gamma-1} \|g(s)\| ds.
\]

**Proof.** From (2.3), we have

\[
G(t_n) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zt_n} ((\sigma + z)\gamma + A)^{-1} g(z) dz
\]

\[
= \left( \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zt_n} ((\sigma + z)\gamma + A)^{-1} dz \ast f \right) (t_n)
\]

\[
= (\mathcal{E} \ast f)(t_n) = \left( \mathcal{E} \ast \left( \frac{t^{k-1}}{(k-1)!} \ast g(t) \right) \right) (t_n) = \left( \mathcal{E} \ast \left( \frac{t^{k-1}}{(k-1)!} \ast g(t) \right) \right) (t_n)
\]
with \( \mathcal{E}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,n}} e^{zt} ((\sigma + z)^{\gamma} + A)^{-1} dz \).

Using the generating function \( \bar{f}(\xi) = \sum_{n=0}^{\infty} f(t_n) \xi^n \) and

\[
G(\xi) = (\delta_{\tau}^{\gamma} (e^{-\sigma \tau} \xi) + A)^{-1} \bar{f}(\xi) := \bar{\mathcal{E}}(\delta_{\tau}(\xi)) \bar{f}(\xi)
\]

in (2.17), we obtain

\[
G^n = \sum_{j=0}^{n} \mathcal{E}_\tau^n f(t_j) \quad \text{with} \quad \bar{\mathcal{E}}(\delta_{\tau}(\xi)) = \sum_{n=0}^{\infty} \mathcal{E}_\tau^n \xi^n.
\]

From Cauchy's integral formula and taking the change of variables \( \xi = e^{-z\tau} \), we have the following integral representation:

\[
\mathcal{E}_\tau^n = \frac{\tau}{2\pi i} \int_{\Gamma_{\theta,n}} e^{z\tau} \left( \delta_{\tau}^{\gamma} \left( e^{-(\sigma+z)\tau} \right) + A \right)^{-1} dz.
\]

Using Lemma 3.2, it means that

\[
\mathcal{E}_\tau^n = \sum_{j=0}^{n} \mathcal{E}_\tau^{n-j} f(t_j) = G^n.
\]

Moreover, we have

\[
(\mathcal{E}_\tau^{k}(\xi)) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \mathcal{E}_\tau^{n-j} t_j^{k-1} \xi^n = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \mathcal{E}_\tau^{n-j} t_j^{k-1} \xi^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \mathcal{E}_\tau^{n} t_j^{k-1} \xi^{n+j} = \sum_{n=0}^{\infty} \mathcal{E}_\tau^{n} \xi^n \sum_{j=0}^{\infty} t_j^{k-1} \xi^j
\]

\[
= \mathcal{E}(\delta_{\tau}(\xi)) t^{k-1} \sum_{j=0}^{\infty} j^{k-1} \xi^j = \mathcal{E}(\delta_{\tau}(\xi)) t^{k-1} \gamma_{k-1}(\xi).
\]

From Lemma 3.7, we have the following estimate

\[
\left\| \left( \mathcal{E}_\tau - \mathcal{E} \right) \ast \frac{t^{k-1}}{(k-1)!} \right\| (t_n) \leq c \tau^{k} t_n^{\gamma_{k-1}}.
\]

Next, we prove the following inequality [3.3] for \( t > 0 \)

\[
\left\| \left( \mathcal{E}_\tau - \mathcal{E} \right) \ast \frac{t^{k-1}}{(k-1)!} \right\| (t) \leq c t^{k} t^{\gamma_{k-1}}, \quad \forall t \in (t_{n-1}, t_n).
\]
Using the Taylor series expansion of $\mathcal{E}(t)$ at $t = t_n$, we get

\[
\left(\mathcal{E} \ast \frac{t^{k-1}}{(k-1)!}\right)(t) = \left(\mathcal{E} \ast \frac{t^{k-1}}{(k-1)!}\right)(t_n) + (t-t_n) \left(\mathcal{E} \ast \frac{t^{k-2}}{(k-2)!}\right)(t_n) + \cdots + \frac{(t-t_n)^{k-2}}{(k-2)!} (\mathcal{E} \ast t)(t_n) + \frac{(t-t_n)^{k-1}}{(k-1)!} (\mathcal{E} \ast 1)(t_n) + \frac{1}{(k-1)!} \int_{t_n}^{t} (t-s)^{k-1} \mathcal{E}(s) ds.
\]

This above expansion also holds for \(\left(\mathcal{E}_\tau \ast \frac{t^{k-1}}{(k-1)!}\right)(t)\). Then we have

\[
\left\|\left(\mathcal{E}_\tau - \mathcal{E}\right) \ast \frac{t^k}{k!}\right\|(t_n) \leq c t^{k+1} t^\gamma - 1.
\]

Using (2.11), we have

\[
\left\|\mathcal{E}(t)\right\| \leq c \left(\int_{-\infty}^{\infty} e^{t \cos \theta} r^{-\gamma} dr + \int_{-\theta}^{\theta} e^{t \cos \phi} r^{1-\gamma} d\phi\right) \leq c t^{\gamma-1}
\]

and

\[
\left\|\int_{t_n}^{t} (s-t)^{\gamma-1} ds\right\| \leq c t^{\gamma-1}.
\]

Similarly, from (3.2), we deduce

\[
\left\|\int_{t_n}^{t} (s-t)^{\gamma-1} \mathcal{E}_\tau(s) ds\right\| \leq c t^{\gamma-1} \left\|\mathcal{E}^\gamma\right\| \leq c t^{\gamma-1}.
\]

Then we can obtain (3.20) by $t_n^{\gamma-1} \leq t^{\gamma-1}$ for $t \in (t_{n-1}, t_n)$ and $\gamma \in (0, 1)$. The proof is completed. \[ \square \]

**Theorem 3.9.** Let $f \in C^{k-1}(0, T; L^2(\Omega))$ and $\int_{0}^{t} (s-t)^{-\gamma-1} \left\|\partial_s^l f(s)\right\|_{L^2(\Omega)} ds < \infty$. Let $G(t_n)$ and $G^n$ be the solutions of (1.1) and (2.4) at the point $t_n$, respectively. Let $\varepsilon^n = G^n - G(t_n)$ with $\varepsilon^0 = 0$. Then

\[
\left\|\varepsilon^n\right\| = \left\|G^n - G(t_n)\right\| \leq c t^k \left(\varepsilon_n^{\gamma-k} \left\|A G^0\right\| + t_n^{\gamma-k} \left\|f(0)\right\| + \sum_{l=1}^{k-1} t_n^{\gamma+l-k} \left\|\partial_s^l f(0)\right\| \right) + \int_{0}^{t_n} (t_n-s)^{\gamma-1} \left\|\partial_s^k f(s)\right\| ds.
\]

**Proof.** Using

\[
G^n - G(t_n) = W^n + e^{-\sigma_n t} G(0) - (W(t_n) + e^{-\sigma_n t} G(0)) = W^n - W(t_n)
\]

and subtracting (2.8) from (2.12), we have

\[
G^n - G(t_n) = I_1 + I_2 + \sum_{l=1}^{k-2} I_{l,3} + I_4 - I_5.
\]

(3.4)
Here
\begin{align*}
I_1 &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{t_n z} \left[ K \left( \delta_T (e^{-(\sigma + z)\tau}) \right) \mu_1 (e^{-(\sigma + z)\tau}) - K(\sigma + z) \right] AG^0 dz; \\
I_2 &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{t_n z} \left[ K \left( \delta_T (e^{-(\sigma + z)\tau}) \right) \frac{\delta_T (e^{-(\sigma + z)\tau})}{\delta_T (e^{-z\tau})} \mu_2 (e^{-z\tau}) - K(\sigma + z) \frac{\sigma + z}{z} \right] f(0) dz; \\
I_{1,3} &= -\frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{t_n z} \left[ K \left( \delta_T (e^{-(\sigma + z)\tau}) \right) \delta_T (e^{-(\sigma + z)\tau}) \gamma_{l+1} + \sum_{j=1}^{k-1} d_{l,j} e^{-z_j \tau} \right] f(0) dz; \\
I_4 &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{t_n z} \frac{z}{z} K \left( \frac{\sigma + z}{z} \right) \tau R_k (e^{-z\tau}) dz \\
&\quad - \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{z \tau} K(\sigma + z)(\sigma + z) R_k dz; \\
I_5 &= \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa} \setminus \Gamma_{\theta,\kappa}} e^{t_n z} K(\sigma + z) \left( -AG(0) + \frac{\sigma + z}{z} f(0) + \mu_2 (e^{-z\tau}) \sum_{l=1}^{k-2} \int_{z_l+1}^{z} \partial_l f(0) \right) dz.
\end{align*}
According to Lemma 3.4 and Lemma 3.3 we estimate the first term $I_1$ as following
\begin{align*}
\| I_1 \| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{t_n z} \left[ K \left( \delta_T (e^{-(\sigma + z)\tau}) \right) \mu_1 (e^{-(\sigma + z)\tau}) - K(\sigma + z) \right] AG^0 \right\| dz \\
&\leq c\tau^k \left\| AG^0 \right\| \left( \int_{\kappa} e^{t_n \cos \theta} t^{k-1-\gamma} \int_{-\theta}^\theta e^{t_n \cos \psi} \kappa^{-1-\gamma} d\psi \right) \\
&\leq c\tau^k \left\| AG^0 \right\| t_n^{-k} \left( \int_{\kappa} e^{t_n \cos \theta} t^{k-1-\gamma} \int_{-\theta}^\theta e^{t_n \cos \psi} \kappa^{-1-\gamma} d\psi \right) \\
&\leq c\tau^k \left\| AG^0 \right\| (t_n^{\gamma-k} + \kappa^{\gamma-k} e^T \kappa) \\
&\leq c\tau^k \left\| AG^0 \right\| (t_n^{\gamma-k} + (T\kappa)^{\gamma-k} e^T \kappa t_n^{\gamma-k}) \\
&\leq c\tau^k \left\| AG^0 \right\| t_n^{\gamma-k}.
\end{align*}
From Lemma 3.4, Lemma 3.5 and Lemma 3.6 we estimate the second term $I_2$ as following in a similar way to $I_1$, i.e.,
\begin{align*}
\| I_2 \| &\leq c\tau^k \left\| f(0) \right\| t_n^{\gamma-k}.
\end{align*}
By Lemma 3.4 and Lemma 3.6 we estimate the third term $I_{1,3}$
\begin{align*}
\| I_{1,3} \| &\leq c\tau^k \left\| \partial_l f(0) \right\| \left( \int_{\kappa} e^{t_n \cos \theta} t^{k-1-\gamma} \int_{-\theta}^\theta e^{t_n \cos \psi} \kappa^{-1-\gamma} d\psi \right) \\
&\leq c\tau^k \left\| \partial_l f(0) \right\| t_n^{\gamma-l-k}, \quad l = 1, 2, \ldots, k - 2.
\end{align*}
Direct calculation $I_5$ as following
\[
\|I_5\| \leq c \int_{\Gamma_{\theta,n} \setminus \Gamma_{\theta,n}} |e^{zt_n}| |\sigma + z|^{-1-\gamma} \|AG(0)\|dz
\]
\[
+ c \int_{\Gamma_{\theta,n} \setminus \Gamma_{\theta,n}} |e^{zt_n}| |\sigma + z|^{-1-\gamma} \left| \frac{\sigma + z}{z} \right| \|f(0)\|dz
\]
\[
+ c \int_{\Gamma_{\theta,n} \setminus \Gamma_{\theta,n}} |e^{zt_n}| |\sigma + z|^{-1-\gamma} \sum_{\ell = 1}^{k-2} \frac{1}{|z|^\ell} \|\delta_\ell f(0)\|dz
\]
\[
\leq c \|AG(0)\| \int_{\frac{\pi}{\sin \theta}}^{\infty} e^{t_n \cos \theta r^{-1-\gamma}} dr + c \|f(0)\| \int_{\frac{\pi}{\sin \theta}}^{\infty} e^{t_n \cos \theta r^{-1-\gamma}} dr
\]
\[
+ c \sum_{\ell = 1}^{k-2} \|\delta_\ell f(0)\| \int_{\frac{\pi}{\sin \theta}}^{\infty} e^{t_n \cos \theta r^{-1-\gamma}} dr
\]
\[
\leq c t_n^{\gamma - k} (t_n^{\gamma - k} \|AG(0)\| + t_n^{\gamma - k} \|f(0)\| + \sum_{\ell = 1}^{k-2} t_n^{\gamma + l - k} \|\delta_\ell f(0)\|)
\]
for the last inequation, we use
\[
\int_{\frac{\pi}{\sin \theta}}^{\infty} e^{t_n \cos \theta r^{-1-\gamma}} dr \leq c t_n^{\gamma - k} \int_{\frac{\pi}{\sin \theta}}^{\infty} e^{t_n \cos \theta s^{-1-\gamma}} ds \leq c t_n^{\gamma - k},
\]
for the above inequality, we using $1 \leq \left( \frac{\sin \theta}{\pi} \right)^k r^k$, since $r \geq \frac{\pi}{\sin \theta}$.

Next we estimate $I_4$, from (2.6) $R_k = \frac{t_n^{k-1}}{(k-1)!} \partial_t^{k-1} f(0) + \frac{t_n^{k-1}}{(k-1)!} \partial_t^{k-1} f(t) = R_k^1 + R_k^2$, so $I_4 = I_4^1 + I_4^2$, we have
\[
I_4^1 = \frac{1}{2\pi i} \int_{\Gamma_{\theta,n}} e^{zt_n} \left( \delta_\tau (e^{-z \tau}) + A \right)^{-1} \frac{\gamma_{k-1} e^{-z \tau}}{(k-1)!} \partial_{\tau}^{k-1} f(0) dz
\]
\[
- \frac{1}{2\pi i} \int_{\Gamma_{\theta,n}} e^{zt_n} K(\sigma + z)(\sigma + z) R_k^1 e^{-z \tau} dz
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma_{\theta,n}} e^{zt_n} \left( \delta_\tau (e^{-z \tau}) + A \right)^{-1} \frac{\gamma_{k-1} e^{-z \tau}}{(k-1)!} \partial_{\tau}^{k-1} f(0) dz
\]
\[
- \frac{1}{2\pi i} \int_{\Gamma_{\theta,n}} e^{zt_n} ((\sigma + z)^\gamma + A)^{-1} \frac{\partial_{\tau}^{k-1} f(0)}{z^k} dz
\]
\[
- \frac{1}{2\pi i} \int_{\Gamma_{\theta,n}} e^{zt_n} ((\sigma + z)^\gamma + A)^{-1} \frac{\partial_{\tau}^{k-1} f(0)}{z^k} dz.
\]
From Lemma 3.7, it yields
\[
\|I_4^1\| \leq c t_n^{\gamma - 1} \|\partial_{\tau}^{k-1} f(0)\|.
\]
Similarly, using Lemma 3.8 with $g(t) = \partial_t^{k} f(t)$, we get
\[
\|I_4^2\| \leq c t_n^{\gamma - 1} \|\partial_{\tau}^{k} f(0)\| ds.
\]
The proof is completed.
4. Numerical results. We now numerically verify the above theoretical results including convergence orders of correction BDFk scheme (2.4) for (1.1) in one spatial dimension. In space direction, it is discretized with the spectral collocation method with the Chebyshev-Gauss-Lobatto points \[28\] in the interval \(\Omega = (-1, 1)\). Since the convergence rate of the spatial discretization is well understood, we focus on the time direction convergence order. Let us consider the following three examples:

(a) \(G_0 = \sqrt{1 - x^2}\) and \(f(x, t) = 0\).
(b) \(G_0 = 0\) and \(f(x, t) = (t + 1)^5 \left(1 + \chi_{(0,1)}(x)\right)\).
(c) \(G_0 = \sqrt{1 - x^2}\) and \(f(x, t) = \cos(t) \left(1 + \chi_{(0,1)}(x)\right)\).

Since the analytic solutions is unknown, the order of the convergence of the numerical results are computed by the following formula

\[
\text{Convergence Rate} = \frac{\ln \left(\frac{||G_N^{N/2} - G_N^N||_\infty}{||G_N - G_{2N}^N||_\infty}\right)}{\ln 2}.
\]

Table 4.1: The maximum errors and convergent order of correction BDFk scheme (2.4) for example (a) with \(\sigma = 0.5\) and \(T = 1\).

| \((\alpha, \gamma)\) | \(N\) | \(k\) | 40           | 80           | 160          | 320          | Rate |
|----------------|-----|-----|-------------|-------------|-------------|-------------|------|
| \((1.7, 0.3)\) | 2   |     | 8.7495e-06 | 2.1453e-06 | 5.3116e-07 | 1.3215e-07 | ≈2.0163|
|                | 3   |     | 5.0931e-07 | 6.0766e-08 | 7.4234e-09 | 9.1742e-10 | ≈3.0389|
|                | 4   |     | 4.2355e-08 | 2.4277e-09 | 1.4544e-10 | 8.9015e-12 | ≈4.0721|
|                | 5   |     | 6.9984e-09 | 1.2741e-10 | 3.7184e-12 | 1.1269e-13 | ≈5.3075|
|                | 6   |     | 7.8500e-07 | 4.8068e-09 | 1.5732e-13 | 2.7200e-15 | ≥6.0000|
| \((1.3, 0.7)\) | 2   |     | 2.7568e-05 | 6.8210e-06 | 1.6966e-06 | 4.2309e-07 | ≈2.0086|
|                | 3   |     | 1.0888e-06 | 1.3273e-07 | 1.6383e-08 | 2.0350e-09 | ≈3.0212|
|                | 4   |     | 6.4066e-08 | 3.8130e-09 | 2.3168e-10 | 1.4275e-11 | ≈4.0486|
|                | 5   |     | 6.0435e-09 | 1.6306e-10 | 4.8147e-12 | 1.2657e-13 | ≈5.1811|
|                | 6   |     | 2.9492e-05 | 3.3112e-08 | 2.3309e-13 | 5.1903e-14 | ≥6.0000|

Tables 4.4 and 4.5 show that the stand BDFk scheme in (2.3) just achieves the first-order convergence and the corrected BDFk scheme in (2.4) preserves the high-order convergence rate with the nonsmooth data.

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Table 4.2: The maximum errors and convergent order of correction BDF$k$ scheme \[2.4\] for example (b) with $\sigma = 0.5$ and $T = 1$.

| $(\alpha, \gamma)$ | $N$  | 40  | 80  | 160 | 320 | Rate  \\
|---------------------|------|-----|-----|-----|-----|--------|
| (1.7, 0.3)          | 2    | 1.2062e-03 | 3.0857e-04 | 7.8041e-05 | 1.9624e-05 | $\approx$ 1.9806 |
|                     | 3    | 5.5613e-05 | 7.1582e-06 | 9.0795e-07 | 1.1432e-07 | $\approx$ 2.9754 |
|                     | 4    | 2.1564e-06 | 1.3904e-07 | 8.8267e-09 | 5.5061e-10 | $\approx$ 3.9738 |
|                     | 5    | 6.4535e-08 | 2.1478e-09 | 6.8484e-11 | 2.1938e-12 | $\approx$ 4.9481 |
|                     | 6    | 7.8075e-07 | 2.3185e-09 | 5.7554e-13 | 4.2633e-14 | $\geq$ 6.0000 |
| (1.3, 0.7)          | 2    | 9.0605e-03 | 2.3200e-03 | 5.8706e-04 | 1.4766e-04 | $\approx$ 1.9798 |
|                     | 3    | 4.3027e-04 | 5.6219e-05 | 7.1349e-06 | 8.9864e-07 | $\approx$ 2.9744 |
|                     | 4    | 1.7567e-05 | 1.1341e-06 | 7.2043e-08 | 4.5394e-09 | $\approx$ 3.9727 |
|                     | 5    | 6.0435e-09 | 1.6306e-10 | 4.8147e-12 | 1.2657e-13 | $\approx$ 5.1811 |
|                     | 6    | 4.6825e-05 | 5.7562e-08 | 5.6595e-12 | 1.1688e-12 | $\geq$ 6.0000 |

Table 4.3: The maximum errors and convergent order of correction BDF$k$ scheme \[2.4\] for example (c) with $\sigma = 0.5$ and $T = 1$.

| $(\alpha, \gamma)$ | $N$  | 40  | 80  | 160 | 320 | Rate  \\
|---------------------|------|-----|-----|-----|-----|--------|
| (1.7, 0.3)          | 2    | 8.5901e-06 | 2.1399e-06 | 5.3403e-07 | 1.3339e-07 | $\approx$ 2.0030 |
|                     | 3    | 1.0845e-07 | 1.3026e-08 | 1.5976e-09 | 1.9785e-10 | $\approx$ 3.0328 |
|                     | 4    | 8.3418e-09 | 4.6450e-10 | 2.7389e-11 | 1.6622e-12 | $\approx$ 4.0977 |
|                     | 5    | 2.2068e-09 | 3.2239e-11 | 9.4552e-13 | 2.9421e-14 | $\approx$ 5.3982 |
|                     | 6    | 2.9255e-07 | 2.5148e-09 | 4.6241e-14 | 1.4988e-15 | $\geq$ 6.0000 |
| (1.3, 0.7)          | 2    | 2.8858e-05 | 7.4063e-06 | 1.8749e-06 | 4.7159e-07 | $\approx$ 2.0030 |
|                     | 3    | 2.5394e-06 | 3.0437e-07 | 3.7255e-08 | 4.6082e-09 | $\approx$ 3.0354 |
|                     | 4    | 1.8942e-07 | 1.1115e-08 | 6.7379e-10 | 4.1483e-11 | $\approx$ 4.0522 |
|                     | 5    | 1.4510e-08 | 4.3197e-10 | 2.6757e-12 | 1.3672e-13 | $\approx$ 5.0899 |
|                     | 6    | 1.6651e-05 | 2.4673e-08 | 4.4675e-13 | 1.4988e-14 | $\geq$ 6.0000 |

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Table 4.4: The maximum errors and convergent order of stand BDF$k$ scheme for example (a) with $\sigma = 0.5$ and $T = 1$.

| $(\alpha, \gamma)$ | $N$ | 40  | 80  | 160 | 320 | Rate |
|------------------|-----|-----|-----|-----|-----|------|
| 2                | 2   | 1.9994e-04 | 9.9876e-05 | 4.9910e-05 | 2.4948e-05 | $\approx 1.0009$ |
| 3                | 2   | 2.0105e-04 | 1.0014e-04 | 4.9974e-05 | 2.4963e-05 | $\approx 1.0032$ |
| 4                | 2   | 2.0106e-04 | 1.0013e-04 | 4.9973e-05 | 2.4963e-05 | $\approx 1.0031$ |
| 5                | 2   | 2.0101e-04 | 1.0013e-04 | 4.9973e-05 | 2.4963e-05 | $\approx 1.0031$ |
| 6                | 2   | 2.0586e-04 | 1.0014e-04 | 4.9972e-05 | 2.4963e-05 | $\approx 1.0146$ |

$(1.7, 0.3)$

| 2                | 2   | 8.1394e-04 | 4.0421e-04 | 2.0145e-04 | 1.0057e-04 | $\approx 1.0055$ |
| 3                | 2   | 8.0891e-04 | 4.0313e-04 | 2.0121e-04 | 1.0051e-04 | $\approx 1.0033$ |
| 4                | 2   | 8.0950e-04 | 4.0319e-04 | 2.0121e-04 | 1.0051e-04 | $\approx 1.0033$ |
| 5                | 2   | 8.0946e-04 | 4.0318e-04 | 2.0121e-04 | 1.0051e-04 | $\approx 1.0033$ |
| 6                | 2   | 8.1383e-04 | 4.0318e-04 | 2.0121e-04 | 1.0051e-04 | $\approx 1.0058$ |

$(1.3, 0.7)$

| 2                | 2   | 9.4869e-05 | 4.8909e-05 | 2.4823e-05 | 1.2503e-05 | $\approx 0.9745$ |
| 3                | 2   | 1.0120e-04 | 5.0491e-05 | 2.5218e-05 | 1.2602e-05 | $\approx 1.0018$ |
| 4                | 2   | 1.0120e-04 | 5.0492e-05 | 2.5218e-05 | 1.2602e-05 | $\approx 1.0018$ |
| 5                | 2   | 1.0120e-04 | 5.0492e-05 | 2.5218e-05 | 1.2602e-05 | $\approx 1.0018$ |
| 6                | 2   | 1.0363e-04 | 5.0496e-05 | 2.5218e-05 | 1.2602e-05 | $\approx 1.0153$ |

Table 4.5: The maximum errors and convergent order of stand BDF$k$ scheme for example (c) with $\sigma = 0.5$ and $T = 1$.

| $(\alpha, \gamma)$ | $N$ | 40  | 80  | 160 | 320 | Rate |
|------------------|-----|-----|-----|-----|-----|------|
| 2                | 2   | 9.4869e-05 | 4.8909e-05 | 2.4823e-05 | 1.2503e-05 | $\approx 0.9745$ |
| 3                | 2   | 1.0120e-04 | 5.0491e-05 | 2.5218e-05 | 1.2602e-05 | $\approx 1.0018$ |
| 4                | 2   | 1.0120e-04 | 5.0492e-05 | 2.5218e-05 | 1.2602e-05 | $\approx 1.0018$ |
| 5                | 2   | 1.0120e-04 | 5.0492e-05 | 2.5218e-05 | 1.2602e-05 | $\approx 1.0018$ |
| 6                | 2   | 1.0363e-04 | 5.0496e-05 | 2.5218e-05 | 1.2602e-05 | $\approx 1.0133$ |

$(1.7, 0.3)$

| 2                | 2   | 3.4862e-04 | 1.5918e-04 | 7.5788e-05 | 3.6941e-05 | $\approx 1.0794$ |
| 3                | 2   | 2.9348e-04 | 1.4535e-04 | 7.2325e-05 | 3.6075e-05 | $\approx 1.0081$ |
| 4                | 2   | 2.9348e-04 | 1.4535e-04 | 7.2325e-05 | 3.6075e-05 | $\approx 1.0081$ |
| 5                | 2   | 2.9350e-04 | 1.4535e-04 | 7.2325e-05 | 3.6075e-05 | $\approx 1.0081$ |
| 6                | 2   | 3.0291e-04 | 1.4536e-04 | 7.2325e-05 | 3.6075e-05 | $\approx 1.0233$ |

$(1.3, 0.7)$

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