CLASSIFICATION OF ISOLATED SINGULARITIES OF POSITIVE SOLUTIONS FOR CHOQUARD EQUATIONS

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Abstract. In this paper we classify the isolated singularities of positive solutions to Choquard equation

$$-\Delta u + u = I_\alpha[u^p]u^q \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where $p > 0, q \geq 1, N \geq 3, \alpha \in (0, N)$ and $I_\alpha[u^p](x) = \int_{\mathbb{R}^N} \frac{u(y)^p}{|x-y|^{N-\alpha}} dy$. We show that any positive solution $u$ is a solution of

$$-\Delta u + u = I_\alpha[u^p]u^q + k\delta_0 \quad \text{in } \mathbb{R}^N$$

in the distributional sense for some $k \geq 0$, where $\delta_0$ is the Dirac mass at the origin. We prove the existence of singular solutions in the subcritical case:

$$p + q < \frac{N+\alpha}{N-2},$$

and prove that either the solution $u$ has removable singularity at the origin or satisfies

$$\lim_{|x| \to 0^+} u(x)|x|^N = C_N$$

which is a positive constant. In the supercritical case:

$$p + q \geq \frac{N+\alpha}{N-2} \quad \text{or} \quad p \geq \frac{N}{N-2} \quad \text{or} \quad q \geq \frac{N}{N-2}$$

we prove that $k = 0$.

1. Introduction

In this paper, we are concerned with the classification of all positive solutions to Choquard equation

$$-\Delta u + u = I_\alpha[u^p]u^q \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where $p > 0, q \geq 1, N \geq 3, \alpha \in (0, N)$ and

$$I_\alpha[u^p](x) = \int_{\mathbb{R}^N} \frac{u(y)^p}{|x-y|^{N-\alpha}} dy.$$
When \( N = 3, \alpha = p = 2 \) and \( q = 1 \), problem (1.1) was proposed by P. Choquard as an approximation to Hartree-Fock theory for a one component plasma, which has been explained in Lieb and Lieb-Simon’s papers [21, 22] respectively. It is also called Choquard-Pekar equation after a more early work of S. Paker for describing the quantum mechanics of a polaron at rest [32], or sometime the nonlinear Schrödinger-Newton equation in the context of self-gravitating matter [36]. The Choquard type equations also arise in the physics of multiple-particle systems, see [19]. Furthermore, the Choquard type equations appear to be a prototype of the nonlocal problems, which play a fundamental role in some Quantum-mechanical and non-linear optics, refer to [18, 31]. When \( \alpha \in (0, 2) \), the Riesz potential \( I_\alpha \) is related to the fractional Laplacian, which is a nonlocal operator, so the Choquard equation (1.1) could be divided into a system with the Laplacian in the linear part of the first equation and fractional Laplacian in the second one. For the related topics on the fractional equation we can refer for example to [7, 8, 10, 11].

The study of isolated singularities is initiated by Brezis and Lions in [5], where an useful tool to connect the singular solutions of elliptic equation in punctured domain and the solutions of corresponding elliptic equation in the distributional sense was built, by the study of

\[
\Delta u \leq au + f \quad \text{in} \quad \Omega \setminus \{0\}, \quad u > 0 \quad \text{in} \quad \Omega \setminus \{0\},
\]

\[
u \in L^1(\Omega), \quad \Delta u \in L^1(\Omega \setminus \{0\}),
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) containing the origin, the parameter \( a > 0 \) and function \( f \in L^1(\Omega) \). Later on, the classification of isolated singular problem

\[
-\Delta u = u^p \quad \text{in} \quad \Omega \setminus \{0\},
\]

\[
u > 0 \quad \text{in} \quad \Omega
\]

(1.2)

was performed by Lions in [23] for \( p \in (1, \frac{N}{N-2}) \), by Aviles in [1] for \( p = \frac{N}{N-2} \), by Gidas-Spruck in [16] for \( \frac{N}{N-2} < p < \frac{N+2}{N-2} \), by Caffarelli-Gidas-Spruck in [6] for \( p = \frac{N+2}{N-2} \). For the case that \( p > \frac{N+2}{N-2} \), the classification of isolated singularities for (1.2) is still open. In the particular case of \( p \in (1, \frac{N}{N-2}) \), any positive solution of (1.2) is a solution of

\[
-\Delta u = u^p + k\delta_0 \quad \text{in} \quad \Omega
\]

(1.3)

in the distributional sense for some \( k \geq 0 \). Furthermore, for suitable \( k \), problem (1.3) has at least two positive solutions including the minimal solution. More related topics could be referred to the references [2, 3, 4, 13, 14, 26, 35].

Our interest in this paper is to classify the singularities of positive classical solutions for Choquard equation (1.1). Here \( u \) is said to be a classical solution of (1.1) if \( u \in C^2(\mathbb{R}^N \setminus \{0\}) \), \( I_\alpha[u^q] \) is well-defined in \( \mathbb{R}^N \setminus \{0\} \) and \( u \) satisfies (1.1) pointwisely. The first result can be stated as follows.

**Theorem 1.1.** Assume that \( N \geq 3, \alpha \in (0, N), p > 0, q \geq 1 \) and \( u \) is a positive classical solution of (1.1) satisfying \( u \in L^p(\mathbb{R}^N) \).

Then \( I_\alpha[u^p]u^q \in L^1(\mathbb{R}^N) \) and there exists \( k \geq 0 \) such that \( u \) is a solution of

\[
-\Delta u + u = I_\alpha[u^p]u^q + k\delta_0 \quad \text{in} \quad \mathbb{R}^N
\]

(1.4)

in the distributional sense, that is the following identity holds,

\[
\int_{\mathbb{R}^N} [u(-\Delta \xi) + \xi] - I_\alpha[u^p]u^q \xi ] \, dx = k\xi(0), \quad \forall \xi \in C_c^\infty(\mathbb{R}^N),
\]

(1.5)

where \( C_c^\infty(\mathbb{R}^N) \) is the space of all the functions in \( C^\infty(\mathbb{R}^N) \) with compact support.
Furthermore, (i) when
\[ p + q \geq \frac{N + \alpha}{N - 2} \quad \text{or} \quad p \geq \frac{N}{N - 2} \quad \text{or} \quad q \geq \frac{N}{N - 2}, \] (1.6)
then \( k = 0 \).

(ii) When
\[ p + q < \frac{N + \alpha}{N - 2} \quad \text{and} \quad p < \frac{N}{N - 2} \quad \text{and} \quad q < \frac{N}{N - 2} \] (1.7)
and if \( k = 0 \), then \( u \) is a classical solution of
\[-\Delta u + u = I_\alpha[u^p]u^q \quad \text{in} \quad \mathbb{R}^N, \]
\[ \lim_{|x| \to +\infty} u(x) = 0; \] (1.8)
if \( k > 0 \), then \( u \) satisfies that
\[ \lim_{|x| \to 0^+} \frac{u(x)}{|x|^{N - 2}} = c_N k, \] (1.9)
where \( c_N \) is the normalized constant.

The solution of (1.1) in the distributional sense are sometimes called the very weak solution. We call also the pair exponent \((p, q)\) supercritical if (1.6) holds and \((p, q)\) is subcritical if (1.7) does. Theorem 1.1 shows that in the supercritical case, the singularities of positive solutions of (1.1) are not visible in the distribution sense by the Dirac mass. In the subcritical case the solutions of (1.1) may have the singularity as \(|x|^{2-N}\) or removable singularity at the origin. In the subcritical case and when \( k = 0 \), we improve the regularity of \( u \) by separating the factors \( I_\alpha[u^p], u^q \) of nonlinearity and using the bootstrap argument, however, the factors \( I_\alpha[u^p], u^q \) have different growth rates in \( L^t \) estimates and the key point is to balance them; while \( k > 0 \), in order to study (1.9), our strategy is to divide \( u \) into
\[ u \leq u_n + \sum_{i=1}^{n-1} \Gamma_i + k \Gamma_0, \]
where \( \Gamma_0 \) is the fundamental solution of \(-\Delta u + u = \delta_0 \) in \( \mathbb{R}^N \), \( \Gamma_i \) are generated by \( \Gamma_0 \) but with lower singularities, \( u_n \) is the remainder term. Our aim here is to find some \( n_0 \) such that \( u_{n_0} \) is bounded at the origin. The difficulty in this procedure is to control the singularity of \( \sum_{i=1}^{n-1} \Gamma_i \) and to improve the regularity of \( u_n \) at the same time. To this end, we develop the bootstrap argument, to reduce the singularity of \( \Gamma_n \) first until to be bounded and then to improve the regularity of \( u_n \) without the influence of singularities from \( \Gamma_n \). We mention that [18, 24, 28] show that the nonlinear Choquard equation admits variational solutions in the case of \( q = p - 1 \), which have no singularities at the origin. See a survey [29] and the references therein.

Our second aim of this paper is to decide whether (1.1) has singular solutions in the subcritical case. To this end, we shall search the weak solutions of (1.4), where the restriction \( \lim_{|x| \to +\infty} u(x) = 0 \) in (1.4) is viewed as
\[ \lim_{r \to +\infty} \text{esssup}_{x \in \mathbb{R}^N \setminus B_r(0)} u(x) = 0. \]
Now it is ready to state the existence and nonexistence of weak solutions of (1.4).

**Theorem 1.2.** Assume that \( N \geq 3 \), \( \alpha \in (0, N) \), \( p > 0 \), \( q \geq 1 \) satisfy (1.7) and denote
\[ k_q = \left( \frac{1}{c_1(p + q)} \right)^{\frac{1}{p+q-1}} \left( \frac{p + q - 1}{p + q} \right), \]
where \( c_1 \) is the constant from (1.11). Then there exists \( k^* \geq k_q \) such that...
(i) for $k \in (0, k^*)$, problem (1.4) admits a minimal positive weak solution $u_k$;
(ii) for $k > k^*$, problem (1.4) admits no positive weak solution.

Furthermore, if $k \leq k_q$, $u_k$ is a classical solution of (1.1) and satisfies (1.9).

When $q = p - 1$, V. Moroz and J. Van Schaftingen [27] have derived groundstates of (1.8) for $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ by variational method. Furthermore, this existence result is sharp in the sense that there is no nontrivial regular variational solution to (1.8) for $p \leq \frac{N+\alpha}{N}$ and $p \geq \frac{N+\alpha}{N-2}$. Usually, the derivation of the very weak solution is different by using nonvariational methods. The solution $u_k$ of (1.4) is derived by iterating procedure:

$$v_0 = kG[\delta_0], \quad v_n = G[v_{n-1}^p] + kG[\delta_0],$$

where $G$ is the Green’s operator defined by the Green kernel $G(x, y)$ of $-\Delta + id$ in $\mathbb{R}^N \times \mathbb{R}^N$. Here the main difficulty is to find a barrier function to control the sequence $\{v_n\}_n$. It is well-known that in the bounded domain $\Omega$ and $\gamma \in (1, \frac{N+\alpha}{N-2})$ the barrier function is constructed by the fact that

$$G_{\Omega}[G_{\Omega}^\gamma[\delta_0]] \leq c_2 G_{\Omega}[\delta_0] \quad \text{in} \quad \Omega \setminus \{0\}, \quad (1.10)$$

where $c_2 > 0$ and $G_{\Omega}[\cdot]$ is the Green’s operator defined by Green’s kernel of $-\Delta + id$ in $\Omega \times \Omega$. However, the estimate (1.10) is no longer valid for $G$. In order to find a barrier function when the domain is the whole space, our strategy here is to establish the following estimate

$$G[I_\alpha[\Phi_0^p] \Phi_0^q] \leq c_1 \Phi_0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \quad (1.11)$$

where $c_1 > 0$ and $\Phi_0$ satisfies that

$$-\Delta u + \frac{1}{4} u = \delta_0 \quad \text{in} \quad \mathbb{R}^N.$$

Recently, M. Ghergu and S.D. Taliaferro [15] have studied the behavior near the origin in $\mathbb{R}^n$ for the Choquard-Pekar type inequality

$$0 \leq -\Delta u \leq (|x|^{-\alpha} * u^\lambda) u^\sigma \quad \text{in} \quad B_2(0) \setminus \{0\}. \quad (1.12)$$

Here $u$ is assumed to be in $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^\lambda(\mathbb{R}^n)$ and $*$ is the convolution operator. In particular, they proved that for some suitable range of $\lambda, \sigma$ depending on $n$ and $\alpha$, the existence of pointwise bounds for nonnegative solutions of (1.12). We mention that the nonnegative solutions they considered are superharmonic functions, and the operator $-\Delta + id$ in our case make a great difference on the analysis of the singularities and the existence of singular solutions.

We emphasize that in this paper we consider the case where $q \geq 1$. When $q < 1$, [28, 27] show that the solutions of problem (1.1) may have polynomial decay at infinity, which makes the classification of singularities of (1.1) difficult and interesting. In fact, the polynomial can not guarantee that $I_\alpha[u^p]$ is well defined and then it may cause the nonexistence. The existence and nonexistence of isolated singular solution of (1.5) when $q \in (0, 1)$ is considered in [12].

The rest of this paper is organized as follows. In Section 2, we show the integrability of the solutions for equation (1.1) and the singularity of the functions generated by the fundamental solution of $-\Delta u + u = \delta_0$ in $\mathbb{R}^N$. Section 3 is devoted to the classification of the singularities of positive solutions for (1.1) and in Section 4, we search the weak solutions of (1.4) in the subcritical case.
2. Preliminary

We start the analysis from the integrability of the solutions to (1.1) near the origin. In what follows, we denote by $c_i$ a generic positive constant.

**Lemma 2.1.** Assume that $N \geq 3$, $\alpha \in (0, N)$, $p > 0$, $q > 0$ and $u$ is a positive classical solution of (1.1) such that $u \in L^p(\mathbb{R}^N)$. Then we have

$$I_\alpha[u^p]u^q \in L^1_{\text{loc}}(\mathbb{R}^N).$$  \hfill (2.1)

**Proof.** If $I_\alpha[u^p]u^q \not\in L^1_{\text{loc}}(\mathbb{R}^N)$, then

$$\lim_{r \to 0^+} \int_{B_1(0) \setminus B_r(0)} I_\alpha[u^p]u^q \, dx = +\infty$$

by the facts that $u \geq 0$ and $u \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$. Thus there exists a decreasing sequence $\{R_n\}_n \subset (0, 1)$ such that $\lim_{n \to \infty} R_n = 0$ and

$$\int_{B_1(0) \setminus B_{R_n}(0)} I_\alpha[u^p]u^q \, dx = n. \tag{2.2}$$

Let $w_n$ be the solution of

$$-\Delta w_n + w_n = \chi_{B_1(0) \setminus B_{R_n}(0)} I_\alpha[u^p]u^q \quad \text{in} \quad \mathbb{R}^N,$$

$$\lim_{|x| \to +\infty} w_n(x) = 0,$$

where $\chi_D$ denotes the standard characteristic function of a domain $D$. Let $\Gamma_0$ be the fundamental solution of $-\Delta + id$, then

$$\lim_{|x| \to 0^+} (u + \Gamma_0)(x) = +\infty \quad \text{and} \quad \lim_{|x| \to +\infty} (u + \Gamma_0)(x) = 0,$$

so it follows by Comparison Principle that for any $n \in \mathbb{N}$,

$$u + \Gamma_0 \geq w_n \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \tag{2.3}$$

Note that $G(x, y) \geq c_3$ for any $x, y \in B_1(0)$, then by (2.2), we have that

$$w_n(x) = G[\chi_{B_1(0) \setminus B_{R_n}(0)} I_\alpha[u^p]u^q](x) = \int_{B_1(0) \setminus B_{R_n}(0)} G(x, y) I_\alpha[u^p](y)u(y)^q \, dy$$

$$\geq c_3 \int_{B_1(0) \setminus B_{R_n}(0)} I_\alpha[u^p](y)u(y)^q \, dy = c_3 n$$

$$\to +\infty \quad \text{as} \quad n \to \infty, \quad \forall x \in B_1(0),$$

which, together with (2.3), implies that $u + \Gamma_0 \equiv +\infty$ in $B_1(0)$ and this is impossible. Therefore we have that $I_\alpha[u^p]u^q \in L^1_{\text{loc}}(\mathbb{R}^N)$.

The following asymptotic behavior of positive solutions to problem (1.1) plays an important role in the control of the integrability at infinity.

**Lemma 2.2.** Assume that $N \geq 3$, $\alpha \in (0, N)$, $p > 0$, $q \geq 1$ and $u$ is a positive classical solution of (1.1) such that $u \in L^p(\mathbb{R}^N)$. Then $I_\alpha[u^p] \in L^\infty(\mathbb{R}^N \setminus B_1(0))$, and for any $\theta \in (0, 1)$, there holds

$$\lim_{|x| \to +\infty} I_\alpha[u^p](x) = 0 \tag{2.4}$$

and for any $\theta \in (0, 1)$, there holds

$$\lim_{|x| \to +\infty} u(x)e^{\theta|x|} = 0. \tag{2.5}$$
Proof. For any \( x \in \mathbb{R}^N \setminus B_1(0) \), we have that
\[
I_\alpha[u^p](x) = \int_{\mathbb{R}^N} \frac{u(y)^p}{|x-y|^{N-\alpha}} dy
\]
\[
= \int_{\mathbb{R}^N \setminus B_\frac{1}{2}(x)} \frac{u(y)^p}{|x-y|^{N-\alpha}} dy + \int_{B_\frac{1}{2}(x)} \frac{u(y)^p}{|x-y|^{N-\alpha}} dy
\]
\[
\leq \left( \frac{1}{2} \right)^{\alpha-N} \|u\|_{L^p(\mathbb{R}^N)}^p + \|u\|_{L^\infty(B_\frac{1}{2}(x))} ^p \int_{B_\frac{1}{2}(x)} \frac{1}{|x-y|^{N-\alpha}} dy
\]
\[
\leq \left( \frac{1}{2} \right)^{\alpha-N} \|u\|_{L^p(\mathbb{R}^N)}^p + c_4 \|u\|_{L^\infty(\mathbb{R}^N \setminus B_\frac{1}{2}(0))}^p,
\]
thus \( I_\alpha[u^p] \in L^\infty(\mathbb{R}^N \setminus B_1(0)) \).

Similarly, for \( x \in \mathbb{R}^N \setminus B_2(0) \) and \( r \in (0, \frac{|x|}{2}) \) depending on \( |x| \), which will be chosen later, we have
\[
I_\alpha[u^p](x) \leq r^{\alpha-N} \|u\|_{L^p(\mathbb{R}^N)}^p + \|u\|_{L^\infty(B_r(x))}^p \int_{B_r(x)} \frac{1}{|x-y|^{N-\alpha}} dy
\]
\[
\leq r^{\alpha-N} \|u\|_{L^p(\mathbb{R}^N)}^p + r^{\alpha} \|u\|_{L^\infty(\mathbb{R}^N \setminus B_{|x|-r}(0))}^p
\]
\[
\leq r^{\alpha-N} \|u\|_{L^p(\mathbb{R}^N)}^p + r^{\alpha} \|u\|_{L^\infty(\mathbb{R}^N \setminus B_{|x|}(0))}^p.
\]

Since
\[
\lim_{|x| \to +\infty} \|u\|_{L^\infty(\mathbb{R}^N \setminus B_{|x|}(0))} = 0,
\]
then \( r := \min\{\|u\|_{L^\infty(\mathbb{R}^N \setminus B_{|x|}(0))}, \frac{|x|}{2}\} \to +\infty \) as \( |x| \to +\infty \), and thus
\[
\lim_{|x| \to +\infty} r^{\alpha-N} \|u\|_{L^p(\mathbb{R}^N)}^p = 0 \quad \text{and} \quad \lim_{|x| \to +\infty} r^{\alpha} \|u\|_{L^\infty(\mathbb{R}^N \setminus B_{|x|}(0))}^p = 0,
\]
which imply that (2.4) holds.

Now for any \( \theta' \in (0, 1) \), since \( q = 1 \), there exists \( r_1 > 2 \) such that
\[
I_\alpha[u^p](x) u(x)^{q-1} \leq 1 - \theta' \quad \text{for} \quad |x| \geq r_1
\]
and
\[
-\Delta e^{-\theta'|x|} + \theta' e^{-\theta'|x|} \geq 0, \quad x \in \mathbb{R}^N \setminus B_{r_1}(0).
\]
Then we have that
\[
-\Delta u + \theta' u \leq 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_{r_1}(0),
\]
\[
\lim_{|x| \to +\infty} u(x) = 0.
\]
It follows by Comparison Principle that
\[
u(x) \leq c_5 e^{-\theta'|x|} \quad \text{for} \quad |x| \geq r_1,
\]
which implies that (2.5) holds.

When \( q \geq 1 \), exponential decay of the solutions to equation (1.1) enables us to focus on the singularities at the origin. Precisely, from Lemma 2.1 and Lemma 2.2, we have the following conclusion.

**Proposition 2.1.** Under the assumptions of Lemma 2.2, we have that
\[
I_\alpha[u^p]u^q \in L^1(\mathbb{R}^N) \quad \text{and} \quad u \in L^1(\mathbb{R}^N).
\]
Furthermore, if \( u \in L^t(B_1(0)) \) for \( t \in [1, +\infty) \), then \( u \in L^t(\mathbb{R}^N) \).
Proof. From Lemma 2.1, we know that $I_\alpha [u^p] \in L^1_{\text{loc}}(\mathbb{R}^N)$ and by Lemma 2.2, we have that $I_\alpha [u^p] u^q \in L^1(\mathbb{R}^N)$. Since $u$ is a positive solution, then

$$u \geq c_6 \quad \text{in} \quad B_2(0) \setminus B_{\frac{1}{2}}(0)$$

and for $x \in B_1(0) \setminus \{0\}$,

$$I_\alpha [u^p](x) \geq \int_{B_2(0) \setminus B_{\frac{1}{2}}(0)} \frac{u(x)^p}{|x-y|^{N-\alpha}} dy \geq c_7. \quad (2.6)$$

Then

$$\int_{B_1(0)} u(x) dx \leq |B_1(0)|^{1-\frac{1}{q}} \left( \int_{B_1(0)} u(x)^q dx \right)^{\frac{1}{q}} \leq |B_1(0)|^{1-\frac{1}{q}} \left( \frac{1}{c_7} \int_{B_1(0)} I_\alpha [u^p] u^q dx \right)^{\frac{1}{q}} < +\infty,$$

that is, $u \in L^1_{\text{loc}}(\mathbb{R}^N)$. By Lemma 2.2, it implies that $u \in L^1(\mathbb{R}^N)$.

If $u \in L^t(B_1(0))$ for $t \in [1, +\infty]$, it infers by Lemma 2.2 that $u \in L^t(\mathbb{R}^N)$. \hfill \Box

To tackle the singularity estimate (1.9), we establish the following lemma.

Lemma 2.3. Assume that $\alpha, \tau \in (0, N)$ and $\nu_\tau : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}_+$ satisfies

$$V_\tau(x) \leq c_8 |x|^{-\tau} \quad \text{for} \quad x \in B_2(0).$$

If $V_\tau \in L^\infty(\mathbb{R}^N \setminus B_1(0))$, then for $x \in B_{\frac{1}{2}}(0) \setminus \{0\}$,

$$\mathbb{G}[V_\tau](x) \leq \begin{cases} \ c_9 |x|^{-\tau+2} & \text{if} \quad \tau > 2, \\
- c_9 \log |x| & \text{if} \quad \tau = 2, \\
c_9 & \text{if} \quad \tau < 2. \end{cases} \quad (2.7)$$

If $V_\tau \in L^1(\mathbb{R}^N)$, then for $x \in B_{\frac{1}{2}}(0) \setminus \{0\}$,

$$I_\alpha [V_\tau](x) \leq \begin{cases} \ c_9 |x|^{-\tau+\alpha} & \text{if} \quad \tau > \alpha, \\
- c_9 \log |x| & \text{if} \quad \tau = \alpha, \\
c_9 & \text{if} \quad \tau < \alpha. \end{cases} \quad (2.8)$$

Proof. Since the Green’s function $G(x, \cdot)$ decays exponentially, then for $x \in B_{\frac{1}{2}}(0) \setminus \{0\}$, we have that

$$\begin{align*}
\mathbb{G}[V_\tau](x) & \leq c_{10} \int_{B_1(0)} \frac{1}{|x-y|^{N-\tau}} \frac{1}{|y|^\tau} dy + c_{10} \|V_\tau\|_{L^\infty(\mathbb{R}^N \setminus B_1(0))} \\
& = c_{10} |x|^{2-\tau} \int_{B_{\frac{1}{2}}(0)} \frac{1}{|y|^{N-\tau}} \frac{1}{|y|^\tau} dy + c_{10} \|V_\tau\|_{L^\infty(\mathbb{R}^N \setminus B_1(0))} \\
& \leq c_{11} |x|^{2-\tau} \int_{B_{\frac{1}{2}}(0)} \frac{1}{1+|y|^{N-\tau+\alpha}} dy + c_{10} \|V_\tau\|_{L^\infty(\mathbb{R}^N \setminus B_1(0))} \\
& \leq \begin{cases} \ c_9 |x|^{-\tau+2} & \text{if} \quad \tau > 2, \\
- c_9 \log |x| & \text{if} \quad \tau = 2, \\
c_9 & \text{if} \quad \tau < 2, \end{cases}
\end{align*}$$

where $e_x = \frac{x}{|x|}$. This implies (2.7).
Corollary 2.1. Let $\alpha \in (0, N)$, $p \in (0, \frac{N}{N-2})$, $q \in (1, \frac{N}{N-2})$ and $V_\tau(x) \leq c_8|x|^{-\tau}$ for $x \in B_2(0)$ with $\tau \in (0, N-2]$.

If $V_\tau \in L^p(\mathbb{R}^N \setminus B_1(0)) \cap L^\infty(\mathbb{R}^N \setminus B_1(0))$ and $(p\tau - \alpha)t < N$, then for $x \in B_{1/2}(0) \setminus \{0\}$,

$$G[(I_\alpha[V_\tau^p])^t](x) \leq \begin{cases} 
c_{13}|x|^{t[(\alpha-p\tau)+2]} & \text{if } \tau > \frac{1}{p}(\alpha + \frac{2}{q}), \\
-c_{13} \log |x| & \text{if } \tau = \frac{1}{p}(\alpha + \frac{2}{q}), \\
c_{13} & \text{if } \tau < \frac{1}{p}(\alpha + \frac{2}{q}).
\end{cases}$$

(2.9)

If $V_\tau \in C(\mathbb{R}^N \setminus \{0\}) \cap L^\infty(\mathbb{R}^N \setminus B_1(0))$ and $\tau q t < N$, then for $x \in B_{1/2}(0) \setminus \{0\}$,

$$G[(V_\tau^p)^t](x) \leq \begin{cases} 
c_{13}|x|^{-\tau q t+2} & \text{if } \tau > \frac{2}{qt}, \\
-c_{13} \log |x| & \text{if } \tau = \frac{2}{qt}, \\
c_{13} & \text{if } \tau < \frac{2}{qt}.
\end{cases}$$

(2.10)

Proof. For $y \in \mathbb{R}^N \setminus B_1(0)$, we have that

$$I_\alpha[V_\tau^p](y) \leq c_8^p \int_{B_1(0)} \frac{1}{|y-z|^{N-\alpha}} \frac{1}{|z|^{p\tau}} dz + c_{10} \|V_\tau\|_{L^p(\mathbb{R}^N \setminus B_1(0))}^p \|V_\tau\|_{L^1(\mathbb{R}^N \setminus B_1(0))}^p$$

\[ \leq c_8^p \|V_\tau\|_{L^\infty(\mathbb{R}^N \setminus B_1(0))}^p \|V_\tau\|_{L^1(\mathbb{R}^N \setminus B_1(0))}^p, \]

that is, $I_\alpha[V_\tau^p] \in L^\infty(\mathbb{R}^N \setminus B_1(0))$. Now we apply Lemma 2.3 to obtain (2.9). It is similar to obtain (2.10). \qed

Lemma 2.4. Assume that $\alpha \in (0, N)$, $p > 0$, $q \geq 1$ and $u \in L^1(\mathbb{R}^N)$ is a positive weak solution of (1.4) with $k \geq 0$. Then

$$u \geq k \Gamma_0 \quad \text{a.e. in} \quad \mathbb{R}^N,$$

where $\Gamma_0$ is the fundamental solution of $-\Delta + id$. 

\[ \]
Proof. Let $w = u - k\Gamma_0$, then $w$ is a weak solution of
\[
-\Delta w + w = I_\alpha[w^p]u^q \quad \text{in} \quad \mathbb{R}^N,
\]

in the distribution sense, that is
\[
\int_{\mathbb{R}^N} w(-\Delta \xi + \xi) dx = \int_{\mathbb{R}^N} I_\alpha[w^p]u^q \xi dx, \quad \forall \xi \in C_c^\infty(\mathbb{R}^N).
\]

For any $n \in \mathbb{N}$, denote
\[
\xi_n(x) = G[\text{sign}(w_-)](x)\eta_0(\frac{x}{n}), \quad \forall x \in \mathbb{R}^N,
\]
where $t_- = \min\{t, 0\}$ and $\eta_0 : \mathbb{R}^N \to [0, 1]$ is a $C^\infty$-function with the support in $B_2(0)$ and satisfying $\eta_0 = 1$ in $B_1(0)$, then $\xi_n \in C_c^\infty(\mathbb{R}^N)$ for any $n \in \mathbb{N}$. Thus, we have that
\[
\int_{\mathbb{R}^N} w (-\Delta \xi_n + \xi_n) dx = \int_{\mathbb{R}^N} w(x)\text{sign}(w_-)(x)\eta_0(\frac{x}{n}) dx + \int_{\mathbb{R}^N} w(x)\nabla G[\text{sign}(w_-)](x) \cdot \nabla \eta_0(\frac{x}{n}) dx + \int_{\mathbb{R}^N} w(x)G[\text{sign}(w_-)](x)(-\Delta)\eta_0(\frac{x}{n}) dx,
\]
where
\[
\int_{\mathbb{R}^N} w(x)\text{sign}(w_-)(x)\eta_0(\frac{x}{n}) dx = -\int_{\mathbb{R}^N} w_-(x)\eta_0(\frac{x}{n}) dx \geq \int_{B_n(0)} |w_-(x)| dx,
\]
\[
\int_{\mathbb{R}^N} w(x)\nabla G[\text{sign}(w_-)](x) \cdot \nabla \eta_0(\frac{x}{n}) dx \leq \frac{c_1}{n} \int_{B_2n(0)\setminus B_n(0)} |w(x)| dx,
\]
and
\[
\int_{\mathbb{R}^N} w(x)G[\text{sign}(w_-)](x)(-\Delta)\eta_0(\frac{x}{n}) dx \leq \frac{c_1}{n^2} \int_{B_2n(0)\setminus B_n(0)} |w(x)| dx.
\]
Since $I_\alpha[w^p]u^q \in L^1(\mathbb{R}^N)$ and $\xi_n$ is non-positive in $\mathbb{R}^N$, we have
\[
\int_{\mathbb{R}^N} I_\alpha[w^p]u^q \xi_n dx \leq 0,
\]
which implies that
\[
\int_{B_n(0)} |w_-(x)| dx \leq \int_{\mathbb{R}^N} w(x)G[\text{sign}(w_-)](x)(-\Delta)\eta_0(\frac{x}{n}) dx + \int_{\mathbb{R}^N} w(x)G[\text{sign}(w_-)](x)(-\Delta)\eta_0(\frac{x}{n}) dx.
\]
Therefore by taking $n \to \infty$, we obtain that
\[
w_- = 0 \quad \text{a.e. in} \quad \mathbb{R}^N,
\]
that is, $u - k\Gamma_0 \geq 0$ a.e. in $\mathbb{R}^N$. 

3. Classification of singularities

In this section, we classify the singularities of positive solutions to equation (1.1).

**Proposition 3.1.** Assume that \( N \geq 3, \alpha \in (0, N), \ p > 0, \ q \geq 1 \) and \( u \) is a positive classical solution of (1.1) satisfying \( u \in L^p(\mathbb{R}^N) \). Then \( u \) is a weak solution of (1.4) for some \( k \geq 0 \). Furthermore, if \((p, q)\) satisfies (1.6), then \( k = 0 \).

**Proof.** From Proposition 2.1, we know that \( I_\alpha[u^p]u^q \in L^1(\mathbb{R}^N) \) and \( u \in L^1(\mathbb{R}^N) \). Define the operator \( L \) by the following

\[
L(\xi) := \int_{\mathbb{R}^N} [u(-\Delta \xi + \xi) - I_\alpha[u^p]u^q \xi] \, dx, \quad \forall \xi \in C_0^\infty(\mathbb{R}^N). \tag{3.1}
\]

First we claim that for any \( \xi \in C_0^\infty(\mathbb{R}^N) \) with the support in \( \mathbb{R}^N \setminus \{0\} \),

\[
L(\xi) = 0.
\]

In fact, since \( \xi \in C_0^\infty(\mathbb{R}^N) \) has the support in \( \mathbb{R}^N \setminus \{0\} \), then there exists \( r \in (0, 1) \) such that \( \xi = 0 \) in \( B_r(0) \cup (\mathbb{R}^N \setminus B_\frac{1}{r}(0)) \) and then

\[
L(\xi) = \int_{B_{\frac{1}{r}}(0) \setminus B_r(0)} [u(-\Delta \xi + \xi) - I_\alpha[u^p]u^q \xi] \, dx \\
= \int_{B_{\frac{1}{r}}(0) \setminus B_r(0)} (-\Delta u + u - I_\alpha[u^p]u^q) \xi \, dx \\
= 0.
\]

From Theorem 1.1 in [23], it implies that

\[
L = k\delta_0 \quad \text{for some} \ k \geq 0, \tag{3.2}
\]

that is,

\[
L(\xi) = \int_{\mathbb{R}^N} [u(-\Delta \xi + \xi) - I_\alpha[u^p]u^q \xi] \, dx = k\xi(0), \quad \forall \xi \in C_0^\infty(\mathbb{R}^N). \tag{3.3}
\]

Then \( u \) is a weak solution of (1.4) for some \( k \geq 0 \).

Next we prove that \( k = 0 \) if \((p, q)\) satisfies (1.6). By contradiction, if \( k > 0 \), then Lemma 2.4 implies that

\[
u \geq k\Gamma_0 \quad \text{in} \ B_1(0) \setminus \{0\}.
\]

It is well known that

\[
\Gamma_0(x) \geq c_{15}|x|^{2-N}, \quad \forall x \in B_1(0) \setminus \{0\}.
\]

**Case I:** \( p + q \geq \frac{N+\alpha}{N-2} \). For \( x \in B_1(0) \setminus \{0\} \), we have that

\[
I_\alpha[u^p]u^q(x) \geq k^{p+q}I_\alpha[T^p_0]\Gamma_0^q(x)
\]

\[
> c_{15}k^{p+q} \int_{B_1(0)} \frac{|y|^{(2-N)p}}{|x-y|^{N-\alpha}} \, dy |x|^{(2-N)q}
\]

\[
= c_{15}k^{p+q} \int_{B_{\frac{1}{|x|}}(0)} \frac{|z|^{(2-N)p}}{|e_x - z|^{N-\alpha}} \, dz |x|^{(2-N)(p+q)+\alpha}
\]

\[
\geq c_{15}k^{p+q} \int_{B_1(0)} \frac{|z|^{(2-N)p}}{|e_x - z|^{N-\alpha}} \, dz |x|^{(2-N)(p+q)+\alpha},
\]

where \( e_x = \frac{x}{|x|} \). But in **Case I**, the function \(|x|^{(2-N)(p+q)+\alpha}\) does not belong to \( L^1_{loc}(\mathbb{R}^N) \).

This contradicts Lemma 2.1 and we have that \( k = 0 \).
Case II: \( p \geq \frac{N}{N-2} \). We note that
\[
\Gamma_0^q(x) \geq c_{15}^q |x|^{q(2-N)} \geq c_{15}^q |x|^{-N} \quad \text{for} \quad 0 < |x| < 1,
\]
then \( I_\alpha[\Gamma_0^q] \equiv +\infty \) in \( B_1(0) \setminus \{0\} \), and then for \( x \in B_1(0) \setminus \{0\} \), we have that
\[
I_\alpha[u^p](x) \geq k^{p+q} I_\alpha[\Gamma_0^p](x) = +\infty,
\]
which is impossible. Thus \( k = 0 \).

Case III: \( q \geq \frac{N}{N-2} \). We note that \( \Gamma_0^q(x) \geq c_{15}^q |x|^{q(2-N)} \) for \( 0 < |x| < 1 \), where \( q(2-N) \leq -N \). It follows from (2.6) that \( I_\alpha[u^p] \geq c_7 \) in \( B_1(0) \setminus \{0\} \), then
\[
I_\alpha[u^p]u^q(x) \geq c_7 k^q \Gamma_0^q(x) \geq c_7 k^q c_{15}^q |x|^{q(2-N)} \geq c_7 c_{15}^q k^q |x|^{-N} \quad \text{for} \quad 0 < |x| < 1,
\]
which contradicts Lemma 2.1. Therefore we have that \( k = 0 \). \( \square \)

Now we focus on the subcritical case.

**Proposition 3.2.** Assume that \( N \geq 3, \alpha \in (0, N), (1.7) \) holds for \( p > 0, q \geq 1 \) and \( u \) is a positive classical solution of (1.1) satisfying \( u \in L^p(\mathbb{R}^N) \). Assume more that \( u \) is a weak solution of (1.4) with \( k = 0 \). Then \( u \) is a classical solution of (1.8).

**Proof.** Since \( I_\alpha[u^p]u^q \in L^1(\mathbb{R}^N) \) and \( k = 0 \), we have that
\[-\Delta u + u = I_\alpha[u^p]u^q \quad \text{in the distribution sense}
\]
and then \( u \in L^1(\mathbb{R}^N) \) with \( t < \frac{N}{N-2} \).

Case 1: \( p < \frac{\alpha}{N-2} \). It follows from Proposition 5.2 that \( I_\alpha[u^p] \in L_\infty^{\infty}(\mathbb{R}^N) \). Then applying the standard bootstrap argument, we have that \( u \in L^\infty(\mathbb{R}^N) \) and then \( u \) is a classical solution of (1.8).

Case 2: \( p = \frac{\alpha}{N-2} \). Again by Proposition 5.2, we see that \( I_\alpha[u^p] \in L_\infty^{\infty}(\mathbb{R}^N) \) for any \( t > 1 \). By Hölder’s inequality, we have that
\[
\int_{B_1(0)} (I_\alpha[u^p]u^q)^s \, dx \leq \left[ \int_{B_1(0)} (I_\alpha[u^p])^s \, dx \right]^\frac{t}{s} \left[ \int_{B_1(0)} u^{qs+1} \, dx \right]^{1-\frac{t}{s}} \tag{3.4}
\]
for \( s, t > 1 \) satisfying
\[
\begin{cases}
st < +\infty, \\
\frac{s}{t-1} < \frac{1}{q} \frac{N}{N-2}.
\end{cases}
\]

Since \( q < \frac{N}{N-2} \), we choose \( t \) big enough, then
\[I_\alpha[u^p]u^\theta \in L^s(\mathbb{R}^N)\]
for any \( s \in (1, \frac{1}{q} \frac{N}{N-2}) \). Then by Proposition 5.1, \( u \in L^{N_s} (\mathbb{R}^N) \) and \( \frac{1}{p} \frac{N-s}{N-2s} > \frac{N}{\alpha} \), thus \( I_\alpha[u^p] \in L^{\infty}(\mathbb{R}^N) \) and by standard elliptic regularity theory, \( u \) is a classical solution of (1.8).

Case 3: \( p > \frac{\alpha}{N-2} \). We have that \( I_\alpha[u^p] \in L^{\frac{N_s}{p} \frac{N}{N-2}}(\mathbb{R}^N) \) for any \( \theta < \frac{N}{p} \frac{N}{N-2} \). By Hölder’s inequality, (3.4) holds for \( s \geq 1, t > 1 \) satisfying
\[
\begin{cases}
st < \frac{N}{p} \frac{N}{N-2} = \frac{N}{p(N-2) - \alpha}, \\
\frac{s}{t-1} < \frac{1}{q} \frac{N}{N-2}.
\end{cases}
\tag{3.5}
\]
When $s = 1$, (3.5) reduces to
\[
\frac{N}{N - q(N - 2)} < t < \frac{N}{p(N - 2) - \alpha}. \tag{3.6}
\]
Clearly the existence of $t$ satisfying (3.6) is guaranteed by (1.7). Now choose
\[
t = t_1 := \frac{(p + q)(N - 2) - \alpha}{p(N - 2) - \alpha} \tag{3.7}
\]
such that
\[
\frac{1}{t_1} \frac{N}{p(N - 2) - \alpha} = \frac{t_1 - 1}{t_1} \frac{N}{q(N - 2)}. \tag{3.8}
\]
holds, then (3.5) becomes to
\[
s < \frac{N}{(p + q)(N - 2) - \alpha}
\]
and
\[
I_\alpha[u^p]u^q \in L^s(\mathbb{R}^N) \quad \text{for any } s < \frac{N}{(p + q)(N - 2) - \alpha}.
\]
If \(\frac{N}{(p+q)(N-2)-\alpha} > \frac{N}{2}\), by Proposition 5.1, it implies that \(u \in L^\infty(\mathbb{R}^N)\), then \(u\) is a classical solution of (1.8).

If \(\frac{N}{(p+q)(N-2)-\alpha} \leq \frac{N}{2}\), fix some \(s_1 \in (1, \frac{N}{(p+q)(N-2)-\alpha})\), then \(u^p \in L^\theta(\mathbb{R}^N)\) with \(\theta \leq \frac{1}{p} \frac{Ns_1}{N-2s_1}\) and it follows by Proposition 5.2 that \(I_\alpha[u^p] \in L^\infty_{loc}(\mathbb{R}^N)\) for any \(\theta \leq \frac{1}{p} \frac{Ns_1}{N-2s_1}\). Now (3.4) holds for \(s, t > 1\) satisfying
\[
\begin{cases}
st & \leq \frac{Ns_1}{p(N - 2s_1) - \alpha s_1}, \\
\frac{s}{t} - 1 & \leq \frac{Ns_1}{qN - 2s_1}.
\end{cases} \tag{3.9}
\]
Take \(s = s_1\), then (3.9) reduces to
\[
\frac{N}{N - q(N - 2s_1)} \leq t \leq \frac{N}{p(N - 2s_1) - \alpha s_1}. \tag{3.10}
\]
Choose \(t = t_2 := \frac{(p+q)(N-2s_1)-\alpha s_1}{p(N-2s_1)-\alpha s_1}\) and then
\[
\frac{1}{t_2} \frac{N s_1}{p(N - 2s_1) - \alpha s_1} = \frac{t_2 - 1}{qt_2} \frac{N s_1}{N - 2s_1}.
\]
Condition (3.9) becomes to
\[
s \leq \frac{1}{t_2} \frac{N s_1}{p(N - 2s_1) - \alpha s_1}. \tag{3.11}
\]
Choose \(s = s_2 := \frac{Ns_1}{t_2 (p(N-2s_1)-\alpha s_1)} = \frac{Ns_1}{(p+q)(N-2s_1)-\alpha s_1}\), then \(I_\alpha[u^p]u^q \in L^{s_2}(\mathbb{R}^N)\) and
\[
\frac{N}{(p + q)(N - 2) - \alpha s_1}.
\]
If \(s_2 > \frac{N}{2}\), we are done. If not, step by step, assume that \(u \in L^{s_{n-1}}(\mathbb{R}^N)\) with \(s_{n-1} < \frac{N}{2}\), then we can find \(s > s_{n-1}\) such that \(I_\alpha[u^p]u^q \in L^s(\mathbb{R}^N)\) and (3.4) holds for \(s, t > 1\) satisfying
\[
\begin{cases}
st & \leq \frac{Ns_{n-1}}{p(N - 2s_{n-1}) - \alpha s_{n-1}}, \\
\frac{s}{t} - 1 & \leq \frac{Ns_{n-1}}{qN - 2s_{n-1}}.
\end{cases} \tag{3.12}
\]
Choose \( t_n := \frac{(p+q)(N-2s_n-1)-\alpha s_n-1}{p(N-2s_n-1)-\alpha s_n-1} \) and
\[
s_n = \frac{N}{(p+q)(N-2s_n-1)-\alpha s_n-1}^{s_n-1}.
\]
Observing that \( s_n > 1 \) and \( \{s_n\}_n \) is increasing with respect to \( n \) satisfying
\[
s_n \geq \frac{N}{(p+q)(N-2) - \alpha} s_n^{-1} \geq \left( \frac{N}{(p+q)(N-2) - \alpha} \right)^{n-1} s_1 \to +\infty \text{ as } n \to +\infty.
\]
Then there exists \( n_0 \) such that \( s_{n_0-1} \leq \frac{N}{2} \) and \( s_{n_0} > \frac{N}{2} \), thus we have that \( \mathcal{G}[I_\alpha[u^p]u^q] \in L^\infty(\mathbb{R}^N) \) and the rest of the proof is standard to obtain that \( u \) is a classical solution of (1.8).

Next we consider the subcritical case with \( k > 0 \). We have the following

**Proposition 3.3.** Assume that \( \alpha \in (0, N) \), (1.7) holds for \( p > 0 \), \( q \geq 1 \) and \( u \) is a positive classical solution of (1.1) satisfying \( u \in L^p(\mathbb{R}^N) \). Assume more that \( u \) is a weak solution of (1.4) with \( k > 0 \).

Then
\[
\lim_{|x| \to 0^+} u(x)|x|^{N-2} = c_N k.
\]

**Proof.** Observe that
\[
\lim_{|x| \to 0^+} \Gamma_0(x)|x|^{N-2} = c_N
\]
and
\[
u = \mathcal{G}[I_\alpha[u^p]u^q] + k\Gamma_0,
\]
then \( u^p \in L^t(\mathbb{R}^N) \) with \( t < \frac{N}{p-\frac{\alpha}{N-2}} \).

**Case 1:** \( p < \frac{\alpha}{N-2} \). We see that \( \frac{N}{p-\frac{\alpha}{N-2}} > \frac{N}{\alpha} \), then it follows from Proposition 5.2 that \( I_\alpha[u^p] \in L^\infty(\mathbb{R}^N) \).

In this case, (3.14) could be reduced to
\[
u \leq c_{16}[u^q] + k\Gamma_0,
\]
then it follows by [23, Theorem 1.1] that (3.13) holds.

**Case 2:** \( p = \frac{\alpha}{N-2} \). We observe that \( I_\alpha[u^p] \in L^t(\mathbb{R}^N) \) for any \( t > 1 \). For any \( s < \frac{1}{q'} \), there exists \( \tilde{t} > 1 \) such that
\[
s \frac{\tilde{t}}{t-1} < \frac{N}{q N - 2}
\]
holds. Then by using again (3.4) with \( t = \tilde{t} \), we get \( I_\alpha[u^p]u^q \in L^s(\mathbb{R}^N) \) for any \( s < \frac{1}{q N - 2} \).
Let \( u_1 := \mathcal{G}[I_\alpha[u^p]u^q] \), then \( u = u_1 + k\Gamma_0 \). By Young’s inequality and the fact that \( (a+b)^r \leq 2^r(a^r + b^r) \) for \( a, b, r > 0 \), we have that
\[
u_1 = \mathcal{G}[I_\alpha[(u_1 + k\Gamma_0)^p]^q(u_1 + k\Gamma_0)^q]
\]
\[
\leq c_{17} \left( \mathcal{G}[I_\alpha[u^p]u^q] + k\mathcal{G}[I_\alpha[\Gamma_0^p]u^q] + k^q \mathcal{G}[I_\alpha[u_1]^{\Gamma_0^q}] + \mathcal{G}[I_\alpha[\Gamma_0^p]^{\Gamma_0^q}] \right)
\]
\[
\leq c_{18} \mathcal{G} \left[ I_\alpha[u_1]^p \Gamma_0^q \right] + c_{18} \mathcal{G} \left[ u_1^{\frac{q}{t-1}} \right] + \Gamma_1,
\]
where
\[
\Gamma_1 = c_{18} \mathcal{G} \left[ I_\alpha[\Gamma_0^p]^{\Gamma_0^q} \right] \leq c_{19} + c_{18} \mathcal{G} \left[ \Gamma_0^{\frac{q}{t-1}} \right]
\]
by the fact that $I_\alpha[I^p_0](x) \leq -c_0 \log |x|$ for $0 < |x| < \frac{t_1}{2}$ and $\mathbb{G} \left[ I_\alpha[I^p_0] \right] \in L^\infty(\mathbb{R}^N)$. Since $I_\alpha[u^p_1] \in L^\theta(\mathbb{R}^N)$ for any $\theta > 1$, we obtain that $\mathbb{G} \left[ I_\alpha[u^p_1] \right] \in L^\infty(\mathbb{R}^N)$. Therefore (3.14) deduces into

$$u \leq c_{18} \mathbb{G}[u^{q_1 \frac{t_1}{t}}_1] + c_{18} \mathbb{G} \left[ I_\alpha[u^p_1] \right]_L \infty(\mathbb{R}^N) + \Gamma_1 + k \Gamma_0 \quad \text{in} \quad B_1(0) \setminus \{0\}. \quad (3.16)$$

Then we repeat the procedure in Case 1 since $q_1 \frac{t}{t_1} < \frac{N}{N-2}$.

Case 3: $p > \frac{\alpha}{N-2}$. We take again $t_1 > 1$ given by (3.7) such that (3.8) holds. Since $I_\alpha[u^p] \in L^{s_1}(\mathbb{R}^N)$ and $u^q \in L^{q_1 \frac{t}{t_1}}(\mathbb{R}^N)$ for $s < \frac{N}{t_1 p(N-2)-\alpha}$, we obtain that:

if $\frac{N}{(p+q)(N-2)-\alpha} > \frac{N}{2}$, we have $u_1 \in L^\infty(\mathbb{R}^N)$ and we are done;

if not, re-denote $u_1 = \mathbb{G}[I_\alpha[u^p]u^q] \in L^{\frac{Nq}{N-2\alpha}}(\mathbb{R}^N)$ for $\theta \in (1, \frac{N}{(p+q)(N-2)-\alpha})$ if $(p+q)(N-2) - 2 - \alpha > 0$, or for $\theta \in (1, \infty)$ if $(p+q)(N-2) - 2 - \alpha = 0$. By Young’s inequality, we have that

$$u_1 = \mathbb{G}[I_\alpha[(u_1 + k \Gamma_0)^p](u_1 + k \Gamma_0)^q]$$

$$\leq c_{19} \left( \mathbb{G}[I_\alpha[u^p_1]u^q_1] + k \mathbb{G}[I_\alpha[I^p_0]u^q_1] + k^q \mathbb{G}[I_\alpha[u^p_1] \Gamma^p_0] + \mathbb{G}[I_\alpha[I^p_0] \Gamma^q_0] \right)$$

$$\leq c_{20} \mathbb{G} \left[ I_\alpha[u^p_1] \Gamma^{q_1 \frac{t_1}{t}}_1 + u^{q_1 \frac{t_1}{t}}_1 \right] + \Gamma_1,$$

where

$$\Gamma_1 = c_{20} \mathbb{G} \left[ I_\alpha[I^p_0] \Gamma^{q_1 \frac{t_1}{t}}_1 + \Gamma^{q_1 \frac{t_1}{t}}_0 \right].$$

Let $T_0 := 2 - N < 0$. We notice that if $(p+q)(N-2) - 2 - \alpha > 0$, we have

$$\frac{N}{(p+q)(N-2)-\alpha} = \frac{N}{(p+q)(N-2)-\alpha} > \frac{N}{2}$$

and

$$\Gamma_1(x) \leq c_{21} |x|^{T_1} \quad \text{for} \quad 0 < |x| < 1,$$

where

$$T_1 := 2 + \frac{q t}{t_1 - 1} T_0 = 2 + \alpha - (p+q)(N-2) > 0$$

and thus $u_1 \in L^{\frac{N}{N-2}}(\mathbb{R}^N)$.

If $(p+q)(N-2) - 2 - \alpha = 0$, 

$$\Gamma_1(x) \leq -c_{21} \ln |x| \quad \text{for} \quad 0 < |x| < \frac{1}{2},$$

and it is obvious that $u_1 \in L^{\frac{N}{N-2}}(\mathbb{R}^N)$ and

$$I_\alpha[u^p_1] \Gamma^{q_1 \frac{t_1}{t}}_1 \in L^\infty_{loc}(\mathbb{R}^N) \quad \text{and} \quad u^{q_1 \frac{t_1}{t}}_1 \in L^{\frac{t_1 - 1}{t_1} \frac{N}{N-2}}(\mathbb{R}^N).$$

Letting

$$s_1 = \frac{N}{(p+q)(N-2)-\alpha} \quad \text{and} \quad u_2 = c_{20} \mathbb{G} \left[ I_\alpha[u^p_1] \Gamma^{q_1 \frac{t_1}{t}}_1 + u^{q_1 \frac{t_1}{t}}_1 \right],$$

we have that $\frac{N}{N-2} < s_1 < \frac{N}{2}$ and $u_1 \leq u_2 + \Gamma_1$, where $u_2 \in L^{\frac{N}{N-2}}(\mathbb{R}^N)$.

By Young’s inequality, we have that

$$u_2 \leq c_{22} \mathbb{G} \left[ I_\alpha[u^p_2] + u^{q_1 \frac{t_1}{t}}_2 \right] + \Gamma_2,$$
where
\[ \Gamma_2 = c_{22} \mathfrak{G} \left[ I_{\alpha} [\Gamma_1]^{t_1} + \Gamma_1^{\frac{q t_1}{1 - t_1}} \right]. \]

We notice that
\[ 2 + (\alpha + p T_1) t_1 > 2 + \frac{q t_1}{t_1 - 1} T_1 > T_1 \]
and
\[ \Gamma_2(x) \leq c_{23} |x|^{T_2} \quad \text{for} \quad 0 < |x| < 1, \]
where
\[ T_2 := 2 + \frac{q t_1}{t_1 - 1} T_1 > T_1. \]

Note that
\[ I_{\alpha} [u_2^{p}]^{t_1} \in L_{\text{loc}}^{\frac{1}{t_1} p(N - 2 s_1) - \alpha s_1} (\mathbb{R}^N) \quad \text{and} \quad u_2^{\frac{q t_1}{1 - t_1}} \in L_{\text{loc}}^{\frac{q t_1}{t_1 - 1}} s_1 (\mathbb{R}^N), \]
where
\[
\frac{1}{t_1} p(N - 2 s_1) - \alpha s_1 > 1 \quad \text{and} \quad \frac{t_1 - 1}{q t_1} s_1 > 1.
\]

Then we have that
\[ u_2 \in L^{\frac{N}{(p + q)(N - 2) - \alpha}} (\mathbb{R}^N) \quad \text{for} \quad \theta \in \left( 1, \frac{N}{(p + q)(N - 2) - \alpha} \right). \]

Inductively, we assume that
\[ u_{n-1} \leq c_{n-1} \mathfrak{G} \left[ I_{\alpha} [u_{n-1}^{p}]^{t_1} + u_{n-1}^{\frac{q t_1}{1 - t_1}} \right] + \Gamma_{n-1} \]
for some suitable constant \( c_{n-1} \). Denote
\[ u_n := c_{n-1} \mathfrak{G} \left[ I_{\alpha} [u_{n-1}^{p}]^{t_1} + u_{n-1}^{\frac{q t_1}{1 - t_1}} \right], \]
then \( u_{n-1} \leq u_n + \Gamma_{n-1} \) and
\[ u_n \leq c_n \mathfrak{G} \left[ I_{\alpha} [u_n^{p}]^{t_1} + u_n^{\frac{q t_1}{1 - t_1}} \right] + \Gamma_n, \quad \text{(3.18)} \]
where
\[ \Gamma_n = c_{24} \mathfrak{G} \left[ I_{\alpha} [\Gamma_{n-1}]^{t_1} + \Gamma_{n-1}^{\frac{q t_1}{1 - t_1}} \right]. \]

We notice that
\[ I_{\alpha} [u_{n-1}^{p}]^{t_1} \in L_{\text{loc}}^{\frac{1}{t_1} p(N - 2 s_1) - \alpha s_1} (\mathbb{R}^N) \quad \text{and} \quad u_{n-1}^{\frac{q t_1}{1 - t_1}} \in L_{\text{loc}}^{\frac{q t_1}{t_1 - 1}} s_1 (\mathbb{R}^N), \]
where \( t_1, s_1 \) satisfy (3.17). Then we get again
\[ u_n \in L^{\frac{N}{(p + q)(N - 2) - \alpha}} (\mathbb{R}^N) \quad \text{for} \quad \theta \in \left( 1, \frac{N}{(p + q)(N - 2) - \alpha} \right). \]

Furthermore, we have that for \( 0 < |x| < \frac{1}{2} \),
\[ \Gamma_n(x) \leq \begin{cases} 
  c_{25} |x|^{T_n} & \text{if} \ T_n < 0, \\
  -c_{25} \ln |x| & \text{if} \ T_n = 0, \\
  c_{25} & \text{if} \ T_n > 0,
\end{cases} \]
where
\[ T_n := 2 + \frac{q t_1}{t_1 - 1} T_{n-1}. \]
Since \( \frac{qt_1}{t_1-1} > 1 \) and \( T_1 - T_0 > 0 \), then
\[
T_n - T_{n-1} = \frac{qt_1}{t_1-1}(T_{n-1} - T_{n-2}) = \left(\frac{qt_1}{t_1-1}\right)^{n-1} (T_1 - T_0) \to +\infty \quad \text{as} \quad n \to \infty.
\]

Then there exists \( n_0 \geq 1 \) such that
\[
T_{n_0} > 0 \quad \text{and} \quad T_{n_0-1} \leq 0.
\]

Thus, \( \Gamma_{n_0} \in L^\infty(\mathbb{R}^N) \) and
\[
u \leq u_{n_0} + \sum_{i=1}^{n_0-1} \Gamma_i + k\Gamma_0.
\]

Finally, our aim is to prove \( u_{n_0} \in L^\infty(\mathbb{R}^N) \). Observing that (3.18) holds for \( n = n_0 \) and \( \Gamma_{n_0} \in L^\infty(\mathbb{R}^N) \), that is,
\[
u_{n_0} \leq c_{26} \left( e \left[ I_\alpha[u_{n_0}]_t^{t_1} + \frac{qt_1}{t_1-1} \right] + 1 \right),
\]
where \( u_{n_0} \in L^{\frac{N\alpha}{N-s_1}}(\mathbb{R}^N) \). Then
\[
I_\alpha[u_{n_0}]_t^{t_1} \in L_{\text{loc}}^{\frac{1}{p(N-2s_1)-\alpha s_1}}(\mathbb{R}^N) \quad \text{and} \quad \frac{qt_1}{t_1-1} u_{n_0} \in L^{\frac{t_1-1}{qt_1-1} \frac{N\alpha}{N-s_1}}(\mathbb{R}^N).
\]
We see that, by the definition of \( t_1, s_1 \),
\[
\frac{1}{t_1} \frac{N s_1}{p(N-2s_1)-\alpha s_1} - \frac{t_1-1}{qt_1} \frac{N s_1}{N-2s_1} = \frac{1}{t_1} \frac{N s_1}{p(N-2s_1)-\alpha s_1} \frac{\alpha N(s_1-1)}{(p+q)(N-2) - \alpha N - 2s_1} > 0
\]
and
\[
s_2 := \frac{t_1-1}{qt_1} \frac{N s_1}{N-2s_1} = \frac{N-2}{(p+q)(N-2) - \alpha N - 2s_1} s_1 > s_1
\]
by the fact that \( p + q < \frac{N+\alpha}{N-2} \). Therefore by Proposition 5.1 we obtain that
\[
u_{n_0} \in L^{\frac{N\alpha}{N-s_2}}(\mathbb{R}^N).
\]

Inductively, assume that
\[
u_{n_0} \in L^{\frac{N\alpha}{N-s_n-1}}(\mathbb{R}^N)
\]
for \( s_{n-1} \in (1, \frac{N}{2}) \). then we have that
\[
I_\alpha[u_{n_0}]_t^{t_1} \in \begin{cases}
L_{\text{loc}}^{\frac{1}{p(N-2s_{n-1})-\alpha s_{n-1}}} (\mathbb{R}^N) & \text{if} \quad p(N-2s_{n-1}) - \alpha s_{n-1} > 0, \\
L_{\text{loc}}^{t_1}(\mathbb{R}^N) & \text{for any} \quad t > 1 \quad \text{if} \quad p(N-2s_{n-1}) - \alpha s_{n-1} = 0, \\
L_{\text{loc}}^{\infty}(\mathbb{R}^N) & \text{if} \quad p(N-2s_{n-1}) - \alpha s_{n-1} < 0
\end{cases}
\]
and
\[
\frac{qt_1}{t_1-1} u_{n_0} \in L^{\frac{t_1-1}{qt_1-1} \frac{N\alpha}{N-s_{n-1}}}(\mathbb{R}^N).
\]
For $p(N - 2s_{n-1}) - \alpha s_{n-1} > 0$, we see that
\[
\frac{1}{t_1} \frac{Ns_{n-1}}{p(N - 2s_{n-1}) - \alpha s_{n-1}} - \frac{t_1 - 1}{qt_1} \frac{Ns_{n-1}}{N - 2s_{n-1}} = \frac{1}{t_1 p(N - 2) - \alpha} \left[ \alpha (s_{n-1} - 1) \right] > 0
\]
and
\[
s_n := \frac{t_1 - 1}{qt_1} \frac{Ns_{n-1}}{N - 2s_{n-1}} = \frac{N - 2}{p + q} (N - 2) - \alpha N s_{n-1} > s_{n-1}
\]
due to the facts that $p + q < \frac{N + \alpha}{N - 2}$ and $s_{n-1} > 1$, then we obtain that
\[
\begin{cases}
u_{n_0} \in L^{\frac{Ns_n}{N-2s_n}}(\mathbb{R}^N) & \text{if } s_n < \frac{N}{2}, \\
u_{n_0} \in L^t(\mathbb{R}^N) & \text{for any } t > 1 \text{ if } s_n = \frac{N}{2}, \\
u_{n_0} \in L^{\infty}(\mathbb{R}^N) & \text{if } s_n > \frac{N}{2}.
\end{cases}
\]
For $s_n > \frac{N}{2}$, we are done, for $s_n = \frac{N}{2}$, we may repeat the above process again to have $u_{n_0} \in L^{\infty}(\mathbb{R}^N)$, and then we are done. For $s_n < \frac{N}{2}$, we have that
\[
s_n \geq \left( \frac{t_1 - 1}{qt_1} \frac{N}{N - 2s_n} \right)^{n-1} \rightarrow +\infty \text{ as } n \rightarrow +\infty.
\]
Thus, there exists $n_1$ such that $s_{n_1} \geq \frac{N}{2}$ and then
\[
u_{n_0} \in L^{\infty}(\mathbb{R}^N).
\]
Therefore,
\[
k_0 \leq u \leq u_{n_0} + \sum_{i=1}^{n_0-1} \Gamma_i + k_0. \quad (3.19)
\]
Note that for $i = 1, 2, \cdots, n_0 - 1$,
\[
\Gamma_i(x) \leq c_{2\gamma} |x|^{T_i},
\]
where $T_i = 2 - N$. As a consequence, we obtain the conclusion. \hfill \Box

**Proof of Theorem 1.1.** From Proposition 3.1, we obtain that $I_\alpha[w^p]u^q \in L^1(\mathbb{R}^N)$ and $u$ is a weak solution of (1.4) for some $k \geq 0$. Furthermore, if $(p, q)$ is supercritical, we have that $k = 0$. For the subcritical case, we derive that $u$ is a classical solution of (1.8) if $k = 0$ from Proposition 3.2, and (1.9) holds by Proposition 3.3 if $k > 0$. \hfill \Box

### 4. Existence

In this section, we give the proof of Theorem 1.2. To this end, denote by $\Phi_0$ the solution of
\[
-\Delta u + \frac{1}{4} u = \delta_0 \quad \text{in } \mathbb{R}^N, \\
\lim_{|x| \rightarrow +\infty} u(x) = 0.
\]
By constructing suitable super and sub solution, we derive that
\[
\lim_{|x| \rightarrow 0^+} \frac{\Gamma_0(x)}{\Phi_0(x)} = 1, \quad \lim_{|x| \rightarrow +\infty} \frac{\Gamma_0(x)}{\Phi_0(x)} = 0 \quad (4.1)
\]
and
\[
\Gamma_0 \leq \Phi_0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (4.2)
\]
Proposition 4.1. Assume that $p > 0$, $q \geq 1$ satisfy (1.7), then there exists $c_{28} > 0$ such that

$$G[I_{\alpha} [\Phi_0^p] [\Phi_0^q]] \leq c_{28} \Phi_0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (4.3)$$

Proof. Observe that $G[I_{\alpha} [\Phi_0^p] [\Phi_0^q]]$ is in $C^2(\mathbb{R}^N \setminus \{0\})$ and has the singularity $|x|^{(2-N)(p+q)+\alpha+2}$ near the origin, which is weaker than $\Phi_0^0$ by the fact that

$$\lim_{|x| \to 0^+} \Phi_0(x)|x|^{N-2} = c_N.$$ 

Thus we only need to consider the asymptotic behavior of $G[I_{\alpha} [\Phi_0^p] [\Phi_0^q]]$ at infinity. Since

$$\lim_{|x| \to +\infty} \Phi_0(x)|x|^\frac{N-1}{2} e^{\frac{1}{2}|x|} = e^\frac{1}{2}$$

and $\Phi_0$ is radially symmetric and decreasing with respect to $|x|$, we have that for $|x| > 2$,

$$I_\alpha[\Phi_0^p](x) = \int_{B_1(0)} \frac{\Phi_0^p(y)}{|x - y|^{N-\alpha}} dy + \int_{\mathbb{R}^N \setminus B_1(0)} \frac{\Phi_0^p(y)}{|x - y|^{N-\alpha}} dy$$

$$\leq c_{29} \|\Phi_0\|_{L^p(\mathbb{R}^N)} |x|^{\alpha-N} + \Phi_0^p \left( \frac{|x|}{2} \right) \int_{\mathbb{R}^N} \frac{\Phi_0^p(y)}{|x - y|^{N-\alpha}} dy$$

$$\leq c_{30} \|\Phi_0\|_{L^p(\mathbb{R}^N)} |x|^{\alpha-N} + c_{33} \Phi_0^p \left( \frac{|x|}{2} \right)$$

thus, there exists $r > 2$ such that

$$\Phi_0 < 1, \quad I_\alpha[\Phi_0^p] \leq \frac{1}{4} \quad \text{in } \mathbb{R}^N \setminus B_r(0).$$

Moreover, we observe that for $|x| \geq r$,

$$-\Delta \Phi_0 + \Phi_0 = \frac{3}{4} \Phi_0 \geq I_\alpha[\Phi_0^p] \Phi_0^q$$

and

$$G[I_{\alpha} [\Phi_0^p] [\Phi_0^q]] \leq c_{32} \Phi_0 \quad \text{on } \partial B_r(0),$$

then it implies by Comparison Principle that

$$G[I_{\alpha} [\Phi_0^p] [\Phi_0^q]] \leq c_{32} \Phi_0 \quad \text{in } \mathbb{R}^N \setminus B_r(0).$$

This ends the proof. \(\square\)

Proof of Theorem 1.2. Let $v_0 := k \Gamma_0 > 0$. We define the sequence $\{v_n\}_n$ by the iteration

$$v_n = G[I_{\alpha} [v_{n-1}^p]] v_{n-1}^q + k \Gamma_0, \quad n \geq 1.$$ 

Observe that

$$v_1 = G[I_{\alpha} [v_0^p]] v_0^q + k \Gamma_0 > v_0$$

and assume that

$$v_{n-1} \geq v_{n-2} \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

for $n \geq 2$, then we have

$$v_n = G[I_{\alpha} [v_{n-1}^p]] v_{n-1}^q + k \Gamma_0 \geq G[I_{\alpha} [v_{n-2}^p]] v_{n-2}^q + k \Gamma_0 = v_{n-1}.$$ 

Thus, the sequence $\{v_n\}_n$ is increasing with respect to $n$. Moreover, we have that

$$\int_{\mathbb{R}^N} v_n (-\Delta \xi + \xi) dx = \int_{\mathbb{R}^N} I_{\alpha} [v_{n-1}^p] v_{n-1}^q \xi dx + k \xi(0), \quad \forall \xi \in C^\infty_c(\mathbb{R}^N). \quad (4.4)$$
We next build an upper bound for the sequence \( \{v_n\}_n \). For \( t > 0 \), denote by
\[
w_t := tk^{p+q}G[I_\alpha[u^p_0]^q] + k\Phi_0 \leq (c_{28}tk^{p+q} + k)\Phi_0,
\]
where \( c_{28} > 0 \) is from Proposition 4.1, then
\[
G[I_\alpha[u^p_t]^q] + k\Phi_0 \leq (c_{28}tk^{p+q} + k)^{p+q}G[I_\alpha[\Phi^p_0]^q] + k\Phi_0 \leq w_t,
\]
if
\[
(c_{28}tk^{p+q} + k)^{p+q} \leq tk^{p+q},
\]
that is,
\[
(c_{28}tk^{p+q-1} + 1)^{p+q} \leq t. \quad (4.5)
\]
Note that the convex function \( f_k(t) = (c_{28}tk^{p+q-1} + 1)^{p+q} \) can intersect the line \( g(t) = t \), if
\[
c_{28}k^{p+q-1} \leq \frac{1}{p+q} \left( \frac{p+q-1}{p+q} \right)^{p+q-1}.
\]
Let \( k_q = \left( \frac{1}{c_{28}(p+q)} \right)^{\frac{p+q-1}{p+q}} \), then if \( k \leq k_q \), it always holds that \( f_k(t_q) \leq t_q \) for \( t_q = \left( \frac{p+q}{p+q-1} \right)^{p+q} \). Hence we have \( w_{t_q} > v_0 \) and
\[
v_1 = G[I_\alpha[v^p_0]^q] + k\Gamma_0 \leq t_qk^{p+q}G[I_\alpha[\Phi^p_0]^q] + k\Phi_0 = w_{t_q}.
\]
Inductively, we obtain
\[
v_n \leq w_{t_q} \quad (4.7)
\]
for all \( n \in \mathbb{N} \). Therefore the sequence \( \{v_n\}_n \) converges to some function \( u_k \). By (4.4), \( u_k \) is a weak solution of (1.4) and satisfies (1.9).

For \( k \leq k_q \), we have that \( u_k \leq w_{t_q} \) in \( \mathbb{R}^N \setminus \{0\} \), so \( u_k \in \text{L}^p(\mathbb{R}^N) \) and \( I_\alpha[u^p_k]^q \) is bounded uniformly locally in \( \mathbb{R}^N \setminus \{0\} \), then \( u_k \) is a classical solution of (1.1).

We claim that \( u_k \) is the minimal solution of (1.1), that is, for any positive solution \( u \) of (1.4), we always have \( u_k \leq u \). Indeed, there holds
\[
u = G[I_\alpha[u^p]^q] + k\Gamma_0 \geq v_0,
\]
and then
\[
u = G[I_\alpha[u^p]^q] + k\Gamma_0 \geq G[I_\alpha[v^p_0]^q] + k\Gamma_0 = v_1.
\]
We may show inductively that
\[
u \geq v_n
\]
for all \( n \in \mathbb{N} \). The claim follows.

Similarly, if problem (1.4) has a positive solution \( u \) for some \( k_1 > 0 \), then (1.4) admits a minimal solution \( u_k \) for all \( k \in (0, k_1] \). As a result, the mapping \( k \mapsto u_k \) is increasing. So we may define
\[
k^* = \sup\{k > 0 : \text{ (1.4) has minimal positive solution for } k\},
\]
then \( k^* \geq k_q \).

We next prove that \( k^* < +\infty \).

Let \( \eta_0 \) be a radially symmetric \( C^\infty_c \)-function such that \( \eta_0 = 0 \) in \( \mathbb{R}^N \setminus B_2(0) \) and \( \eta_0 = 1 \) in \( B_1(0) \). For \( \epsilon \in (0, 1) \), denote
\[
\xi_\epsilon(x) = \eta_0(\epsilon x)G[\eta_0](x), \quad x \in \mathbb{R}^N.
\]
By the direct computation, we have that
\[
-\Delta \xi_\epsilon(x) + \xi_\epsilon(x) = \epsilon^2(-\Delta)\eta_0(\epsilon x)G[\eta_0](x) + 2\epsilon \nabla \eta_0(\epsilon x) \cdot \nabla G[\eta_0](x) + \eta_0(\epsilon x)\eta_0(x).
\]
Choosing \( \epsilon > 0 \) small, we deduce that
\[
\int_{\mathbb{R}^N} u_k(-\Delta \xi + \xi) \, dx = \int_{\mathbb{R}^N} u_k(x)[-\epsilon^2 \Delta \eta_0(\epsilon x)G[\eta_0](x) + 2\epsilon \nabla \eta_0(\epsilon x) \cdot \nabla G[\eta_0](x) \\
+ \eta_0(\epsilon x)\eta_0(x)] \, dx
\leq \int_{B_2(0)} u_k(x) \, dx + c_{33}(\epsilon + \epsilon^2) \int_{B_2(0) \setminus B_{\frac{1}{2}}(0)} u_k(x) \, dx
\leq 2 \int_{B_2(0)} u_k(x) \, dx,
\]
where we have used \( \text{ess}\sup_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} u_k(x) \to 0 \) as \( \epsilon \to 0 \).

Since
\[
u_k \geq k\Gamma_0 \geq c_{34} k \quad \text{in} \quad B_1(0)
\]
and
\[
I_\alpha[u_k^p] \geq c_{35} k^p \quad \text{in} \quad B_1(0)
\]
for some \( c_{40} > 0 \) independent of \( k \), then
\[
2 \int_{B_2(0)} u_k(x) \, dx \geq \int_{\mathbb{R}^N} u_k[(-\Delta) \xi + \xi] \, dx = \int_{\mathbb{R}^N} I_\alpha[u_k^p]u_k^q \xi \, dx
\geq c_{36} k^{p+q-1} \int_{B_2(0)} u_k(x) \, dx,
\]
where \( p + q > 1 \). Thus,
\[
k \leq c_{37},
\]
so does \( k^* \) which ends the proof. \( \square \)

**Remark 4.1.** Concerning the existence of weak solutions of (1.4) for \( 0 < k < k^* \) in the subcritical case, we may consider the stability of the minimal solution \( u_k \) and then construct the second solution by using Mountain Pass Theorem [34, Theorem 6.1].

5. **Appendix**

It is well-known that the Green kernel \( G(x, y) \) of \( -\Delta + id \) in \( \mathbb{R}^N \times \mathbb{R}^N \) is \( \Gamma_0(x - y) \), which has exponential decay at infinity. We recall that \( G[\cdot] \) the Green operator defined as
\[
G[f](x) = \int_{\mathbb{R}^N} G(x, y)f(y) \, dy.
\]

**Proposition 5.1.** [30, Lemma A.3] Assume that \( h \in L^s(\mathbb{R}^N) \) with \( s \geq 1 \), then
(i)
\[
\| G[h] \|_{L^\infty(\mathbb{R}^N)} \leq c_{38} \| h \|_{L^s(\mathbb{R}^N)} \quad \text{if} \quad \frac{1}{s} < \frac{2}{N};
\]
(ii)
\[
\| G[h] \|_{L^r(\mathbb{R}^N)} \leq c_{38} \| h \|_{L^s(\mathbb{R}^N)} \quad \text{if} \quad \frac{1}{s} \leq \frac{1}{r} + \frac{2}{N} \quad \text{and} \quad s > 1;
\]
(iii)
\[
\| G[h] \|_{L^r(\mathbb{R}^N)} \leq c_{38} \| h \|_{L^1(\mathbb{R}^N)} \quad \text{if} \quad \frac{1}{r} < \frac{1}{r} + \frac{2}{N}.
\]

Recall that
\[
I_\alpha[h](x) = \int_{\mathbb{R}^N} \frac{h(y)}{|x - y|^{N-\alpha}} \, dy \quad \text{for} \quad h \in L^1(\mathbb{R}^N).
Proposition 5.2. Suppose that $\alpha \in (0, N)$, $\Omega \subset B_{R/2}(0)$ for some $R > 0$ and $h \in L^s(B_R(0)) \cap L^1(\mathbb{R}^N)$ for some $s \geq 1$. Then

$$
\|I_\alpha[h]\|_{L^\infty(\Omega)} \leq c_{39} \left( \|h\|_{L^s(B_R(0))} + \|h\|_{L^1(\mathbb{R}^N)} \right) \quad \text{if } \frac{1}{s} < \frac{\alpha}{N},
$$

(5.4)

$$
\|I_\alpha[h]\|_{L^r(\Omega)} \leq c_{39} \left( \|h\|_{L^s(B_R(0))} + \|h\|_{L^1(\mathbb{R}^N)} \right) \quad \text{if } \frac{1}{s} \leq \frac{1}{r} + \frac{\alpha}{N} \quad \text{and} \quad s > 1
$$

(5.5)

and

$$
\|I_\alpha[h]\|_{L^r(\Omega)} \leq c_{39} \|h\|_{L^1(\mathbb{R}^N)} \quad \text{if } 1 < \frac{1}{r} + \frac{\alpha}{N}.
$$

(5.6)

Proof. Observe that $|x - y| > \frac{R}{2}$ for $x \in \Omega$ and $y \in \mathbb{R}^N \setminus B_R(0)$ and then for $x \in \Omega$,

$$
|I_\alpha[h](x)| \leq \int_{B_R(0)} \frac{|h(y)|}{|x - y|^{N-\alpha}} dy + \int_{\mathbb{R}^N \setminus B_R(0)} \frac{|h(y)|}{|x - y|^{N-\alpha}} dy
$$

$$
\leq \int_{B_R(0)} \frac{|h(y)|}{|x - y|^{N-\alpha}} dy + \left( \frac{R}{2} \right)^{\alpha - N} \int_{B_R(0)} \frac{1}{|x - y|^{N-\alpha}} dy,
$$

Without loss of generality, we can assume $h \geq 0$ and in the following, we only need to consider

$$
J_\alpha[h](x) := \int_{B_R(0)} \frac{h(y)}{|x - y|^{N-\alpha}} dy, \quad x \in \Omega.
$$

First we prove (5.4). By Hölder’s inequality,

$$
J_\alpha[h](x) \leq \left( \int_{B_R(0)} \frac{1}{|x - y|^{(N-\alpha)s'}} dy \right)^{\frac{1}{s'}} \left( \int_{B_R(0)} |h(y)|^s dy \right)^{\frac{1}{s}}
$$

$$
\leq c_{40} \|h\|_{L^s(B_R(0))} \int_{B_R(0)} \frac{1}{|x - y|^{(N-\alpha)s'}} dy,
$$

where $s' = \frac{s}{s-1}$. Since $\frac{1}{s} < \frac{2}{N}$ and $(N - \alpha)s' < N$, we have

$$
\int_{B_R(0)} \frac{1}{|x - y|^{(N-\alpha)s'}} dy = c_{41} R^{N-1-(N-\alpha)s'} \leq c_{42} R^{N-(N-\alpha)s'},
$$

then (5.4) holds.

Next, we prove (5.5) for $r \leq s$ and (5.6) for $r = 1$. There holds

$$
\left[ \int_{B_R(0)} J_\alpha[h]^r(x) dx \right]^{\frac{1}{r}} \leq \left\{ \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} \frac{h(y) \chi_{B_R(0)}(x) \chi_{B_R(0)}(y)}{|x - y|^{N-\alpha}} dy \right]^s dx \right\}^{\frac{1}{s}}
$$

$$
= \left\{ \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} \frac{h(x - y) \chi_{B_R(0)}(x) \chi_{B_R(0)}(y)}{|y|^{N-\alpha}} dy \right]^r dx \right\}^{\frac{1}{r}}.
$$

By Minkowski’s inequality, we have that

$$
\left[ \int_{B_R(0)} J_\alpha[h]^r(x) dx \right]^{\frac{1}{r}} \leq \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} \frac{h^r(x - y) \chi_{B_R(0)}(x) \chi_{B_R(0)}(y)}{|y|^{(N-\alpha)r}} dy \right]^{\frac{1}{r}} dx
$$

$$
\leq \int_{B_R(0)} \left[ \int_{\mathbb{R}^N} \frac{h^r(x - y) \chi_{B_R(0)}(x) \chi_{B_R(0)}(y)}{|y|^{(N-\alpha)r}} dx \right]^{\frac{1}{r}} \frac{1}{|y|^{N-\alpha}} dy
$$

$$
\leq \|h\|_{L^s(B_R(0))} \int_{B_R(0)} \frac{1}{|y|^{N-\alpha}} dy
$$

$$
\leq c_{43} \|h\|_{L^s(B_R(0))}.
$$
Finally, we prove (5.5) in the case $r > s \geq 1$ and $\frac{1}{s} \leq \frac{1}{r} + \frac{q}{N}$, and (5.6) for $r > 1$ and $1 < \frac{1}{r} + \frac{q}{N}$. We claim that if $r > s$ and $\frac{1}{r} = \frac{1}{s} - \frac{q}{N}$, the mapping $h \rightarrow J_\alpha(h)$ is of weak-type $(s, r^*)$ in the sense that

$$|\{x \in \Omega : |J_\alpha[h](x)| > t\}| \leq \left(A_{s,r^*} \frac{\|h\|_{L^s(B_R(0))}}{t} \right)^{r^*}, \quad h \in L^s(B_R(0)), \quad \forall t > 0, \quad (5.7)$$

where $A_{s,r^*}$ is a positive constant. Defining

$$J_0(x, y) = \begin{cases} \frac{|x - y|^{\alpha - N}}{\nu} & \text{if } |x - y| \leq \nu, \\ 0 & \text{if } |x - y| > \nu \end{cases}$$

for $\nu > 0$ and $J_\infty(x, y) = |x - y|^{\alpha - N} - J_0(x, y)$. Then we have that

$$|\{x \in \Omega : J_\alpha[h](x) > 2t\}| \leq |\{x \in \Omega : J_0[h](x) > t\}| + |\{x \in \Omega : J_\infty[h](x) > t\}|,$$

where $J_0[h] = \int_{B_R(0)} J_0(x, y) h(y) dy$ and $J_\infty[h] = \int_{B_R(0)} J_\infty(x, y) h(y) dy$. By Minkowski’s inequality, we obtain that

$$|\{x \in B_R(0) : |J_0[h](x)| > t\}| \leq \frac{\|J_0(h)\|_{L^s(B_R(0))}^s}{t^s} \leq \frac{\|\int_{B_R(0)} \chi_{B_r}(x - y)|x - y|^{\alpha - N} h(y)|dy\|_{L^s(B_R(0))}}{t^s} \leq \frac{\left[\int_{B_R(0)} \int_{B_R(0)} |h(x - y)|^s dx \right]^\frac{1}{s} |y|^{\alpha - N}\chi_{B_r}(y)dy^s}{t^s} \leq \frac{\|h\|_{L^s(B_R(0))}^s \int_{B_r} |x|^{-N+\alpha} dx}{t^s} = c_{44} \frac{\|h\|_{L^s(B_R(0))}^{s\frac{1}{T}}}{t^s}.$$

On the other hand,

$$\|J_\infty[h]\|_{L^\infty(B_R(0))} \leq \|\int_{B_R(0)} \chi_{B_r}(x - y)|x - y|^{\alpha - N} h(y)dy\|_{L^\infty(B_R(0))} \leq \left(\int_{B_R(0)} |h(y)|^s dy\right)^\frac{1}{s} \left(\int_{\Omega \setminus B_r(y)} |x - y|^{\alpha - N} dy\right)^\frac{1}{s} \|L^\infty(B_R(0))\| \leq c_{45} \|h\|_{L^s(B_R(0))}^{s\frac{1}{T}} \frac{1}{t^s},$$

where $s' = \frac{s}{s-1}$ if $s > 1$, and if $s = 1$, $s' = \infty$. Choosing $\nu = \left(\frac{t}{c_{45} \|h\|_{L^s(B_R(0))}}\right)^{\alpha - \frac{N}{2}}$, we obtain

$$\|J_\infty[h]\|_{L^\infty(B_R(0))} \leq t,$$

which means that

$$|\{x \in \Omega : |J_\infty[h](x)| > t\}| = 0.$$

With this choice of $\nu$, we have that

$$|\{x \in \Omega : |J_\alpha[h]| > 2t\}| \leq c_{46} \frac{\|h\|_{L^s(B_R(0))}^{s\frac{1}{T}}}{t^s} \leq c_{47} \left(\frac{\|h\|_{L^s(B_R(0))}}{t}\right)^{r^*}.$$

The claim for $r > s$ follows from the Marcinkiewicz interpolation theorem. \qed

Acknowledgements: The research of the first author is supported by NSFC (11401270). The research of the second author is partially supported by NSFC (11271133 and 11431005) and and Shanghai Key Laboratory of PMMP.
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