Graded cluster algebras

Jan E. Grabowski†
Department of Mathematics and Statistics, Lancaster University,
Lancaster, LA1 4YF, United Kingdom
24th September 2013

Abstract

In recent work with S. Launois, we introduced a framework for $\mathbb{Z}$-gradings on cluster algebras (and their quantum analogues) that are compatible with mutation. To do this, one chooses the degrees of the cluster variables in an initial seed subject to a compatibility with the initial exchange matrix, and then one extends this to all cluster variables by mutation.

In this note, we classify gradings on cluster algebras of finite type $A$, $D$ and $E$, in the coefficient-free setting and also for give some examples with coefficients. We find a close connection to properties of the associated cluster category and see that gradings on cluster algebras yield tropical frieze patterns on the Auslander–Reiten quiver of the cluster category.

Contents

1 Introduction 2
2 Preliminaries 3
3 Graded seeds and graded cluster algebras 4
4 Gradings in type $A$ 6
5 Gradings in type $D$ without coefficients 12
6 Gradings in type $E$ without coefficients 15

†Email: j.grabowski@lancaster.ac.uk Website: http://www.maths.lancs.ac.uk/~grabowsj/
1 Introduction

When studying any class of rings or algebras, the existence of gradings often has a big impact on what can be said about the members of the class. Typically, establishing the existence of a grading is difficult and a full classification of gradings even more so. The potential for this problem to be difficult is perhaps best seen by interpreting it in the language of algebraic geometry and homogeneous coordinate rings.

In the few years since their inception, cluster algebras have been found in numerous places and have been shown to be responsible for a plethora of combinatorial patterns. Many algebras of interest have been shown to admit cluster algebra structures, these mostly being (homogeneous) coordinate rings of varieties arising in Lie theory, such as flag varieties. However until very recently gradings on cluster algebras have not been considered in any detail. One might have observed that a certain homogeneous coordinate ring was a cluster algebra and that the cluster variables behaved nicely with respect to the natural grading but relationships of this sort between the two structures were not typically explored in depth.

In recent work with Stéphane Launois ([9]), in which we prove that the quantum versions of homogeneous coordinate rings of Grassmannians are quantum cluster algebras, we introduced the notion of a $\mathbb{Z}$-grading for a quantum cluster algebra and noted that the definition also applies to the classical commutative case. The same notion in a different form, namely toric actions on (commutative) cluster algebras, also appeared in work of Gekhtman, Shapiro and Vainshtein ([8]), in a more restricted setting. We also note that some other types of grading associated to cluster algebras have been used previously, most notably a natural $\mathbb{Z}^n$-grading in the original work of Fomin and Zelevinsky ([7]).

In our work with Launois, we used the grading on quantum cluster algebras to give a construction analogous to cocycle twisting and hence to prove our general result. So far as we are aware, this remains the only cluster algebra result that depends crucially on using $\mathbb{Z}$-gradings in this way. That this notion is more significant in the noncommutative version of cluster algebra theory should not be a surprise, however.

This note has a different aim to our previous work. We wish to promote the use of gradings in cluster algebra theory and to show that there are interesting questions and especially combinatorial phenomena associated with gradings. To do this, we consider the usual starting cases: cluster algebras without coefficients of types $A$, $D$ and $E$.

Two highlights of the resulting analysis are firstly the close link between the gradings on the cluster algebra and the representation theory encoded in the associated cluster category, and secondly the emergence of a combinatorial pattern called a tropical frieze on the Auslander–Reiten quiver of the cluster category arising from the degrees of the cluster variables. For us, this illustrates how deeply integrated into the theory of cluster algebras gradings are, even though they have mostly not been noticed to date.

The structure of this note is as follows. We begin with a brief exposition of the definition of a grading for a cluster algebra and some associated lemmas (Section 3), adapted from those in [9] for the simpler case at hand of classical cluster algebras. We then classify gradings for coefficient-free cluster algebras of types $A$ (Section 4.1), $D$ (Section 5) and $E$ (Section 6), with a brief interlude in Section 4.2 on some examples of gradings for cluster algebras of type $A$ with coefficients.
Acknowledgements

The author would particularly like to thank Stéphane Launois for many helpful discussions throughout the collaboration that this work originated in.

2 Preliminaries

The construction of a cluster algebra of geometric type from an initial seed \((x, B)\), due to Fomin and Zelevinsky ([7]), is now well-known. Here \(x\) is a transcendence base for a certain field of fractions of a polynomial ring and \(B\) is a skew-symmetric integer matrix; often \(B\) is replaced by the quiver \(Q = Q(B)\) it defines in the natural way. We refer the reader who is unfamiliar with this construction to the survey of Keller ([12]) and the recent book of Gekhtman, Shapiro and Vainshtein ([8]) for an introduction to the topic and summaries of the main related results in this area.

We set some notation for later use. For \(k\) a mutable index, set

\[ b^+_k = -e_k + \sum_{b_{ik}>0} b_{ik} e_i \quad \text{and} \quad b^-_k = -e_k - \sum_{b_{ik}<0} b_{ik} e_i \]

where the vector \(e_i \in \mathbb{C}^r\) (\(r\) being the number of rows of \(B\)) is the \(i\)th standard basis vector. Note that the \(k\)th row of \(B\) may be recovered as \(B_k = b^+_k - b^-_k\).

Then given a cluster \(x = (X_1, \ldots, X_r)\) and exchange matrix \(B\), the exchange relation for mutation in the direction \(k\) is given by

\[ X'_k = M(b^+_k) + M(b^-_k) \]

with

\[ M(a_1, \ldots, a_r) \overset{\text{def}}{=} X_1^{a_1} \cdots X_r^{a_r} \]

By construction, the integers \(a_i\) are all non-negative except for \(a_k = -1\).

Later we will briefly discuss cluster algebras with coefficients (also called frozen variables). That is, we designate some of the elements of the initial cluster to be mutable (i.e. we are allowed to mutate these) and the remainder to be non-mutable. We will also talk about the corresponding indices for the variables as being mutable or not; in [3] the former are referred to as exchangeable indices. The rank of the cluster algebra is the number of mutable variables in a cluster; we will refer to the total number of variables, mutable and not, as the cardinality of the cluster.

Note that we will adopt the convention that \(B\) will be a square matrix—in the literature it is more common to let \(B\) have as column indices just the mutable indices ([8] adopts the transpose convention, of having the rows indexed by the mutable indices). At some points, we will need the submatrix \(B_{\text{mut}}\) of \(B\) given by taking only the columns of \(B\) with mutable indices, or the transpose of this. The matrix \(B_{\text{mut}}\) is often referred to as the extended exchange matrix and its submatrix \(B^\text{mut}_{\text{mut}}\) with row set also the mutable indices is what is usually called the principal part of \(B\). Our square matrix \(B\) is simply the “skew-symmetric extension” of \(B_{\text{mut}}\), i.e. completing \(B_{\text{mut}}\) to a square matrix in the unique way so that \(B\) is skew-symmetric and so that if \(i\) and \(j\) are non-mutable indices then \(b_{ij} = 0\). (The latter choice accords with the convention that the exchange quiver has no arrows between frozen vertices.)
3 Graded seeds and graded cluster algebras

Berenstein and Zelevinsky ([3, Definition 6.5]) have given a definition of graded quantum seeds, which give rise to module gradings but not algebra gradings. In our previous work and in what follows, we are concerned with algebra gradings specifically and our definition is inspired by that of Berenstein and Zelevinsky but not equivalent to it.

Similarly, \( \mathbb{Z}^n \)-gradings have been considered previously, starting with the foundational work of Fomin and Zelevinsky ([7]). Here \( n \) is precisely the rank of the cluster algebra. We will also encounter \( \mathbb{Z}^r \)-gradings for \( r > 1 \) in due course but begin with the simplest case of \( \mathbb{Z} \)-gradings.

The definition below and the subsequent lemmas and proofs have appeared in [9] (for quantum cluster algebras) but we include them here for completeness.

**Definition 3.1.** A graded seed is a triple \((x, B, G)\) such that

(a) \( (x = (X_1, \ldots, X_r), B) \) is a seed of cardinality \( r \) and

(b) \( G \in \mathbb{Z}^r \) is an integer (column) vector such that for all mutable indices \( k \), the \( k \)th row of \( B \), \( B_k \), satisfies \( B_k G = 0 \).

We will set \( \deg_G(X_i) = G_i \) for all \( X_i \) belonging to the cluster \( x \). Then the second condition of the definition is equivalent to asking that every exchange relation (as encoded by the rows \( B_k \)) is homogeneous with respect to this degree, in the sense that the two monomials in the variables \( x \{X_k\} \) determining \( X'_k \) are of the same homogeneous degree. From the quiver perspective, this asks that the sum of the degrees of the variables with arrows to a given mutable vertex is equal to the sum of the degrees of the variables at the end of arrows leaving that vertex.

In contrast to the definition of Berenstein and Zelevinsky, the above can be extended to an algebra grading on the field of rational functions in the initial variables, associated to \((x, B)\), simply by setting \( \deg_G(X_i^{-1}) = -\deg_G(X_i) \) and extending \( \deg_G \) additively to all (Laurent) monomials.

We also need to be able to mutate our grading in a sensible fashion and it is clear what we ought to do. Let \((x', B')\) be the seed given by mutation of \((x, B)\) in the direction \( k \). We set \( G'_i = G_i \) for \( i \neq k \) (i.e. the degrees of variables we are not mutating at this point remain the same). Then the homogeneity of the exchange relation \( X'_k = M(b^+_k) + M(b^-_k) \) implies that we should set

\[
G'_k = \deg_G'(X'_k) = \deg_G(M(b^+_k)) = \deg_G(M(b^-_k)).
\]

As discussed in [2] and [3], the mutation operations can also be expressed in terms of row and column operations, or more concisely as corresponding matrix multiplications. To this end, we recall the definition of a matrix \( E \) (denoted \( E_+ \) in [3]) that encodes mutation of a seed with exchange matrix \( B \) in the direction \( k \) as follows:

\[
E_{rs} = \begin{cases} 
\delta_{rs} & \text{if } s \neq k; \\
-1 & \text{if } r = s = k; \\
\max(0, -b_{rk}) & \text{if } r \neq s = k.
\end{cases}
\]

Then \( B' = EBE^T \). Our mutation of \( G \) can be written in terms of \( E \) similarly.

**Lemma 3.2.** \( G' = E^TG \).

**Proof:** This is straightforward to check directly. \( \square \)
We may also re-express this in terms of the vectors \( b_k^\pm \) defined above:
\[
G'_i = \begin{cases} 
G_i & \text{if } i \neq k \\
b_k^- \cdot G & \text{if } i = k 
\end{cases}
\]

Since \( B_k = b_k^+ - b_k^- \) and \( G \) is a grading, so \( B_k G = 0 \), we have \( b_k^+ \cdot G = b_k^- \cdot G \) so that we may use \( b_k^+ \) instead of \( b_k^- \) in calculating \( G' \).

We also need to know that this mutation operation does indeed produce another graded seed.

**Lemma 3.3.** For each mutable index \( j \), \( (B')_j G' = 0 \).

**Proof:** As noted in [3, Proposition 3.4], \( E^2 = 1 \) and so
\[
(B')_j G' = (EBE^T)_j (E^T G) = (EB(E^T)^2 G)_j = (EBG)_j = E_j (BG) = 0
\]
since \( (BG)_j = B_j G = 0 \). Here, \( (\ )_j \) refers to the \( j \)th row for matrices and column vectors as appropriate.

That is, \( (\mathbf{x}', B', G') \) is again a graded seed. Furthermore, this mutation is involutive (cf. [3, Proposition 4.10]). Then we see that repeated mutation propagates a grading on an initial seed to every cluster variable and hence to the associated cluster algebra, as every exchange relation is homogeneous.

**Corollary 3.4.** The cluster algebra \( A(\mathbf{x}, B, G) \) associated to an initial graded seed \( (\mathbf{x}, B, G) \) is a \( \mathbb{Z} \)-graded algebra.

We note in particular that this construction—by definition—says that every cluster variable of a graded cluster algebra is homogeneous for this grading. This is especially important for cluster algebra structures on known algebras, such as homogeneous coordinate rings, where one wishes to establish if a cluster algebra grading is related to one already known. We refer to Example 4.1 below as an illustration of this.

We may calculate this homogeneous degree of a cluster variable directly if we know its expression as a Laurent polynomial: knowing the grading on the initial cluster \( \mathbf{x} = (X_1, \ldots, X_r) \), the value of the grading is simply given by calculating the degrees of the numerator and denominator, both of which are homogeneous polynomials. (The homogeneity of the denominator is clear and the numerator is in fact homogeneous, being the product of the cluster variable itself and the denominator, both of which are homogeneous.)

**Remark 3.5.** Gekhtman, Shapiro and Vainshtein have also noted the above definition and lemmas, in the dual language and in a slightly more restricted setting ([8, Section 5.2]). In the case that the underlying field \( F \) is \( \mathbb{R} \) or \( \mathbb{C} \), they discuss a toric action of \( F^* \) determined by a choice of integer weight. Their Lemma 5.3 notes that a local toric action extends to a global one on the associated cluster algebra precisely when condition [b] of Definition 3.1 holds.

For comparison, our definition in [9] is more general, encompassing quantum cluster algebras, and the above Definition 3.1 is phrased in a way that permits more general ground fields, but clearly the essence of the notion is the same. The toric action setting suggests a different set of associated questions and we recommend Chapter 5 of [8] to readers interested in the more geometrical aspects. We will concentrate on algebraic and combinatorial ones here.
Remark 3.6. It is clear that all of the above is insensitive to replacing $G$ with $-G$, i.e. reversing the sign of every degree. Indeed for each graded cluster algebra $A(x, B, G)$ we have an isomorphic graded cluster algebra $A^- = A(x, B, -G)$ (where “isomorphic” here means as cluster algebras, not just as algebras).

Remark 3.7. From the definition of a grading, we see that the existence of a grading is a linear algebra problem: if $B_{\text{mut}}$ has rank equal to the number of mutable indices\(^1\), the only solution is the zero grading $0$. Clearly the zero grading assigns degree 0 to every cluster variable.

Classification of gradings for a particular $B$ is also a linear algebra problem, of finding a basis for the kernel in the case that the rank is not maximal.

However it will in general be difficult to find the degrees of every cluster variable, especially in infinite types. In finite types, one can reasonably expect to solve this problem and we will do so in the coefficient-free case, making use of cluster categories and associated techniques.

Remark 3.8. For some cluster algebra problems, the presence or absence of coefficients does not play a large part and the phenomena seen are determined by the cluster type. This is not the case for gradings, though. As the examples below will illustrate, adding or removing coefficients can radically change the gradings that can exist. This is to be expected: adding coefficients increases the number of rows of the associated exchange matrix and this can impact on the rank and hence the solutions $G$ that are the grading vectors.

In the remainder of this work, we examine the possible gradings on cluster algebras of type $A$, $D$ and $E$ with no coefficients, as well as some particular examples with coefficients.

4 Gradings in type $A$

4.1 The coefficient-free case

We begin by considering a cluster algebra of type $A_n$ with no coefficients. Since the rank of an exchange matrix is unchanged by mutation ([8, Lemma 3.23]), we may consider any of the exchange matrices for this cluster algebra when establishing the existence or otherwise of a grading. Every orientation of the Dynkin diagram $A_n$ occurs as an exchange quiver (this is true for all cluster algebras of Dynkin type [8, Theorem 3.29]) so let us without loss of generality consider the quiver

$$
1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow \cdots \leftarrow n
$$

Examining the corresponding exchange matrix $B_{\text{mut}}$, we find that $B_{\text{mut}}$ has rank $n$ if $n$ is even and $B_{\text{mut}}$ has rank $n - 1$ if $n$ is odd. This gives our first result: if $n$ is even, the cluster algebra $A$ of type $A_n$ (with no coefficients) admits no non-zero grading.

We note however that if we were to add a coefficient corresponding to an additional vertex 0 to the quiver $A_{2m}$ to give the ice quiver (this is true for all cluster algebras of Dynkin type [8, Theorem 3.29]) so let us without loss of generality consider the quiver

$$
0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow \cdots \leftarrow n
$$

we would still have a cluster algebra $A'$ of type $A_{2m}$ but $A'$ would then admit a non-zero grading. We refer also to Example 4.2\(^1\).

\(^1\)That is, if the (row) rank of the matrix $B_{\text{mut}}$ equals the rank of the cluster algebra—an unfortunate coincidence of terminology.
Figure 1: The Auslander–Reiten quiver for the cluster category of type $A_5$. Each quiver representation is given in terms of its Loewy series, writing just “$i$” for the simple module $S_i$. (Note that the left- and right-hand edges are to be identified.)

From now on, we consider the case of $n = 2m + 1$ odd. The rank of the exchange matrix in this case is $2m$, so that the nullity is 1. The nullspace is spanned by the vector $G = (1, 0, 1, \ldots, 1, 0, 1)$ with value 1 in odd-numbered positions and 0 otherwise. One also sees this from the quiver: every vertex has the same sum of the degrees at incoming and outgoing vertices, with the convention that empty sums are zero (for sources and sinks).

Integer multiples of this vector are also gradings but without loss of generality we may consider $G$ as above, as multiplying the grading vector by $\lambda$ simply has the effect of multiplying every degree by $\lambda$. We will be concerned with questions such as how many cluster variables are there of each degree and the answer to this is unchanged by scaling by $\lambda$.

Our goal is now to understand this grading completely, by identifying which cluster variables have which degree. We will see that the degree of any given cluster variable may be calculated directly from knowledge of the pre-image of the cluster variable in the cluster category of type $A$ provided by the Caldero–Chapoton map.

We will not recall here details of the construction of the cluster category $\mathcal{C}_Q$ associated to a quiver $Q$ but refer to the excellent survey of Keller ([12]). As an example, the Auslander–Reiten quiver of the cluster category of type $A_5$ is as in Figure 1. In that figure, $V[1]$ refers to the image of $V$ under the suspension functor (often also denoted $\Sigma$).

The Caldero–Chapoton map is then the map from the cluster category to $\mathbb{Q}(X_1, \ldots, X_n)$ taking an (isomorphism class of an) indecomposable representation $V$ having dimension vector $d = [d_1, \ldots, d_n]$ to the element

$$X_V = \frac{1}{\prod_{i=1}^n X_{d_i}} \sum_{\underline{e} \subseteq \underline{d}} \chi(\text{Gr}_{\underline{e}}(V)) \prod_{i=1}^n X_{i}^{\sum_{j \rightarrow i} e_j + \sum_{i \rightarrow j} d_j - e_j}$$

where $\chi(\text{Gr}_{\underline{e}}(V))$ is the Euler characteristic of the quiver Grassmannian of submodules of $V$ of dimension vector $\underline{e}$ (and $\chi$ is taken with respect to étale cohomology). Here, “$\sum_{i \rightarrow j}$” means that the sum is taken over all arrows of this form in the exchange quiver $Q$.
The Caldero–Chapoton map is a bijection between the isoclasses of indecomposables in the cluster category and the cluster variables, when we identify the initial cluster variable \(X_i\) with the indecomposable \(P_i\). We may now simplify the formula above for the particular case at hand, knowing that the arrows in the quiver \(Q\) are all of the form \(i \to i - 1\): we have

\[
X_V = \frac{1}{\prod_{i=1}^{n} X_i^{d_i}} \sum_{0 \leq e \leq d} \chi(\text{Gr}_e(V)) \prod_{i=1}^{n} X_i^{e_{i-1}+d_i+1-e_{i+1}}
\]

taking \(e_k\) and \(d_k\) to be equal to zero if \(k < 1\) or \(k > n\).

Recall that we have set \(n = 2m + 1\). We would like to calculate the degree of the numerator in particular. As noted above, the numerator is a homogeneous polynomial; we can see this explicitly by seeing that for fixed \(e\), the term \(\prod_{i=1}^{n} X_i^{e_i-1+d_i+1-e_{i+1}}\) has degree \(\sum_{j=1}^{m+1} e_{2j-2} + d_{2j} - e_{2j} = \sum_{j=1}^{m} d_{2j}\), independent of \(e\). This will be the degree of the numerator provided that the numerator is not equal to 1, or equivalently that some \(\chi(\text{Gr}_e(V))\) is non-zero. This is the case, as for either \(e = 0\) or \(e = d\), the quiver Grassmannian is a point so that the Euler characteristic is 1.

It follows that the degree of \(X_V\) is equal to the sum of \(-\sum_{j=1}^{m+1} d_{2j-1}\) (from the denominator) and \(\sum_{j=1}^{m} d_{2j}\) (from the numerator). That is, the degree of \(X_V\) is \(\sum_{i=1}^{n} (-1)^i d_i\).

From this formula and knowledge of the structure of the representations in the cluster category, we may deduce that

- the degree of any cluster variable is \(-1\), 0 or 1.
- the number of cluster variables of degree 0 is equal to the number of even-dimensional representations, namely \(m(m + 2)\).
- the number of cluster variables of degree 1 is equal to the number of odd-dimensional \(kA_{2m+1}\) representations having socle isomorphic to \(S_{2l}\) for some \(l\), plus \(m + 1\) (for the initial cluster). This number is \((m + 1)(m + 2)/2\).
- the number of cluster variables of degree \(-1\) is equal to the number of odd-dimensional \(kA_{2m+1}\) representations having socle isomorphic to \(S_{2l+1}\) for some \(l\). This number is also \((m + 1)(m + 2)/2\).

These counts accord with the total number of cluster variables being the number of almost positive roots in a root system of type \(A_{2m+1}\), which is \((2m + 1)(m + 2)\).

In Figure 2 we replace the modules by the degrees of their corresponding cluster variables in the Auslander–Reiten quiver of type \(A_5\): one sees striking combinatorial patterns emerging. These patterns are mostly explained by the parity of the dimension of the corresponding module but one additional pattern can be seen in the meshes.

Recall that a mesh in an Auslander–Reiten quiver is a square such as those seen in the figures, coming from an Auslander–Reiten sequence (also called an almost split sequence). The modules at the left and right corners form the left and right terms of the sequence and the middle term is the direct sum of the modules at the top and bottom.

Since dimension vectors are additive with respect to these sequences, it follows easily that the grading given by the alternating sum above is also. Hence the degrees have the property that around any mesh, the sum of the left and right vertex degrees is equal to the sum of the top and bottom vertex degrees. (In fact these are always equations of the form “1 + (−1) = 0 + 0” in type \(A_{2m+1}\).)
This type of pattern is similar to—but different from—the notion of a frieze pattern, as introduced by Conway and Coxeter ([5], [4]). For a frieze pattern of integers in the plane, the principal requirement is the unimodular rule: that is, in a diamond of adjacent integers

\[ \begin{array}{c}
  a & d \\
  b & c
\end{array} \]

we have \( ad - bc = 1 \).

Our pattern does not have this property: the corresponding determinants are either +1 or −1. However the sum rule on meshes referred to in the previous paragraph can be considered as a tropical version of the unimodular rule. Following Fock–Goncharov ([6]) and Propp ([13]), we see that the tropical version of the unimodular rule should be \( a + d = \max(b + c, 0) \). We do indeed have a tropical frieze, therefore; the “max” is redundant as our sums \( b + c \) are all zero in any case.

We refer the reader to the work of Guo ([10]) for a detailed study of tropical friezes associated to Dynkin type cluster categories. We note that the frieze given here is not explicitly described in Guo’s work but that her main theorem tells us that it is determined by certain categorical data (a choice of cluster-tilting object and an element of an associated Grothendieck group). Guo’s work constructs tropical friezes on more cluster categories than we consider in this paper but at the expense of one needing to work in a more categorical setting.

We have concentrated on the cluster category, for obvious reasons. However we observe that the combinatorics described above can be extended to the bounded derived category \( \mathcal{D}_Q \) of \( kQ \)-modules. By work of Happel ([11]), the full subcategory of indecomposables of \( \mathcal{D}_Q \) is equivalent to the mesh category of the repetition quiver \( ZQ \) (see [12, Section 5] for detailed definitions). The Auslander–Reiten quiver of \( \mathcal{D}_Q \) then takes the form of an infinite strip and in the case of type \( A \), we may describe it as the quiver given by taking the triangle corresponding to \( kA_n \)-modules repeated and reflected, as in Figure 3 for type \( A_5 \).
Figure 3: Part of the Auslander–Reiten quiver for the bounded derived category of type $A_5$. (The morphisms going between the shifts of the $kA_5$-module category are indicated by dashed lines, to highlight the repetitive structure.)

Figure 4: Part of the Auslander–Reiten quiver for the bounded derived category of type $A_5$, with degrees replacing modules.

Each repeated triangle corresponds to an application of the suspension (or shift) functor. The cluster category is constructed from the derived category by taking a certain quotient, which in the case at hand yields the Möbius strip of Figure 1.

We notice that if $n$ is odd, when we have the non-zero grading, the degree pattern above extends to give a tropical frieze pattern on the derived category. The shift functor reverses the parity of the degree, so that the degree of a module $V[s]$ in $\mathcal{D}_{A_{2m+1}}$ is $\sum_{i=1}^{2m+1} (-1)^{i+s}d_i$. This is illustrated for $n = 5$ in Figure 1.

We recall that we have no non-zero grading in the case of $n$ even. One could certainly use the formula $\sum_{i=1}^{n} (-1)^{i}d_i$ to assign an integer to $kQ$-modules for $n$ even but this does not extend to the cluster category: simply continuing the combinatorial pattern from $kA_4$-modules to the cluster category of type $A_4$ would see the isomorphic objects 1 and 2[1] being given degrees $-1$ and 0 respectively.
4.2 With coefficients

As previously remarked, the addition of coefficients has a significant impact on the possible gradings on a cluster algebra. We saw one example previously, extending $A_{2m}$ by adding one coefficient, and we now give two more.

**Example 4.1 (Grassmannians).** The homogeneous coordinate ring of a Grassmannian $\text{Gr}(k, n)$ of $k$-dimensional subspaces of $n$-space is well-known to have the structure of a cluster algebra; this is due to Scott ([14]). As noted in [8, Section 5.2] or [9], this cluster algebra structure admits a grading in the above sense that is seen to be equal to the standard grading on the homogeneous coordinate ring. The generating Plücker coordinates have degree one and every other cluster variable is a homogeneous polynomial in these.

For $k = 2$ in particular, we obtain a family of cluster algebras of type $A$, with coefficients. In that case, every cluster variable is a Plücker coordinate and so is of degree one—markedly different to the coefficient-free situation above.

For all $k > 2$, there are clusters consisting only of Plücker coordinates but there are also cluster variables of higher degree. For example, $\mathcal{O}(\text{Gr}(3, 8))$ (of type $E_8$) has 56 cluster variables of degree one, 56 of degree two and 24 of degree three.

It follows from the identification of these cluster algebras with homogeneous coordinate rings that these particular gradings are positive, i.e. have no cluster variables in negative degrees, again in contrast to the coefficient-free case.

**Example 4.2 (Principal coefficients).** Given a cluster algebra without coefficients and its $r \times r$ initial exchange matrix $B$, one may construct a related cluster algebra with so-called principal coefficients. For each initial variable $X_i$, one adds a corresponding coefficient $Y_i$ and the exchange matrix $B$ is given by forming the block matrix

\[
\begin{pmatrix}
  B & I_r \\
  -I_r & 0
\end{pmatrix}
\]

where $I_r$ is the identity matrix. In the quiver picture, one adds an arrow from each original vertex $i$ to a new vertex $i'$.

For example, if we consider extending the cluster algebra without coefficients of type $A_2$ by adding principal coefficients, the initial exchange quiver becomes

```
1 ----2
\   \   \\
1' 2'
```

This quiver admits non-zero gradings, in contrast to the coefficient-free case, where the cluster algebra of type $A_2$ admits no non-zero grading. The space of solutions to $B_k G = 0$ for mutable indices $k$ is 2-dimensional, with basis

\[
\{ G = (1, 0, 0, -1), \ H = (0, 1, 1, 0) \}.
\]

\[\text{Recall that we are using the convention that exchange matrices are square.}\]
These are represented by the following diagrams, placing degrees at the relevant vertices:

Note that the frozen vertices $1'$ and $2'$ are exempted from the condition that the sums of the degrees at arrows entering and at arrows leaving the vertex are equal—we only require this at mutable vertices.

Let us take as our initial cluster $(X_1, X_2, X_3, X_4)$ (writing $X_3$ for $X_{1'}$ etc., for clarity). Then the cluster variables and their degrees for the two gradings $G$ and $H$ above—and their sum $G + H$—are as follows.

|        | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_3 + X_3$ | $X_1 X_4 + 1$ | $X_2 X_3 + X_3 X_4$ |
|--------|------|------|------|------|-------------|---------------|------------------|
| $G$    | 1    | 0    | 0    | 0    | 0           | 0             | 0                |
| $H$    | 0    | 1    | 1    | 0    | 1           | -1            | 0                |
| $G + H$| 1    | 1    | 1    | -1   | 0           | -1            | -1               |

While we do again have the property of variables being concentrated in degrees $-1$, 0 and 1 for both gradings, neither is balanced, by which we will mean that the number of variables in degree $-1$ is equal to the number in degree 1, as we had previously for type $A_{2m+1}$ without coefficients. There does exist a balanced grading however, namely $G + H$.

5 Gradings in type $D$ without coefficients

We now turn our attention to type $D$, taking as our initial quiver

We make this choice without loss of generality for the same reasons as discussed previously, but for the convenience of having the Auslander–Reiten quiver for the case $n = 4$ to hand ([I, VII.5.15]) which we will use as an illustration.

The corresponding exchange matrix $B_{\text{mut}}$ has rank $n - 1$ if $n$ is odd and rank $n - 2$ if $n$ is even, so we have non-zero gradings in all cases.

5.1 Odd $n$

For odd $n$, where $B_{\text{mut}}$ has nullity 1, we find that the nullspace is spanned by $G = (0, 0, \ldots, 0, 1, -1)$. Carrying out a similar analysis using the images under Caldero–Chapoton map, we find that for this grading the numerator has degree zero. This is because $X_{n-1}$ and $X_n$ will appear to the same power, $d_{n-2} - e_{n-2}$, for any $e$ and these are the only variables that contribute to the degree.

The degree of the monomial in the denominator is clearly $d_{n-1} - d_n$ so that we conclude that the degree of $X_{1'}$ is $d_n - d_{n-1}$. 

We may use the fact that both the cluster variables and the indecomposable representations in the cluster category are in bijection with the almost positive roots of the root system of type $D_n$. In particular, the latter bijection, coming from Gabriel’s theorem, is enacted by taking dimension vectors of modules. Thus it suffices to examine the positive roots in order to calculate the numbers of modules (and hence cluster variables) of a given degree. The negative simple roots correspond to the initial cluster and we know the degrees here too.

Using an explicit description of the positive roots of type $D_n$ (written in terms of the simple roots), we find that

- the degree of any cluster variable is $-1, 0$ or $1$.
- the number of cluster variables of degree $0$ is $n(n - 2)$.
- the number of cluster variables of degree $1$ is $n$.
- the number of cluster variables of degree $-1$ is also $n$.

One sees that this gives the correct total of $n^2$ variables. We remark that there are striking similarities with the type $A$ case, even though the grading a priori appears to be very different. In particular, the variables are all in degree $-1, 0$ or $1$ and the number of those of degree $-1$ and those of degree $1$ are equal, so this grading is again balanced.

Also as before, we will have found a frieze pattern on the cluster category of type $D_{2m+1}$ in this way. We will not illustrate this case now but shall do so as part of the even $n$ case that follows.

5.2 Even $n$

When $n$ is even, we have nullity 2 and basis for the nullspace

$$\{(1, 0, 1, 0, \ldots, 1, 0, -1, 0), (0, 0, \ldots, 0, 1, -1)\}$$

Note that the first of these is very reminiscent of the type $A_{2m+1}$ case and the second is exactly the grading vector that occurs in the $D_{2m+1}$ case just analysed.

The analysis of the previous subsection carries over verbatim to fully describe the grading $(0, 0, \ldots, 0, 1, -1)$ for even $n$ also: none of the considerations there are affected by the parity of $n$.

For the grading vector $(1, 0, 1, 0, \ldots, 1, 0, -1, 0)$, the analysis is similar to type $A_{2m+1}$ and we find that the degree of $X_V$ (for $V$ of dimension vector $d$ as before) is given by $(\sum_{i=1}^{n-3}(-1)^i d_i) + d_{n-1}$.

Again we may examine the positive roots of type $D_{2m}$ to calculate the number of variables of each degree and we find that

- the degree of any cluster variable is $-1, 0$ or $1$.
- the number of cluster variables of degree $0$ is $2m^2$
- the number of cluster variables of degree $1$ is $m^2$
- the number of cluster variables of degree $-1$ is also $m^2$
Figure 5: The Auslander–Reiten quiver for the cluster category of type $D_4$. Here we use the alternative quiver representation convention of writing the dimension of the vector space at each vertex of the quiver. (Note that the left- and right-hand edges are to be identified, in the sense that the arrows at the right edge should be regarded as pointing to the representation on the far left.)

Figure 6: The Auslander–Reiten quiver for the cluster category of type $D_4$, with degrees corresponding to the initial grading vector $(0, 0, 1, -1)$. This accords with $4m^2 = n^2$ variables in total. Note that the distribution of degrees in this case certainly differs from the previous one. (However, by a numerical coincidence this is not visible in the $D_4$ examples.) Still, it is again the case that the variables are all in degrees $-1$, 0 and 1 and the grading is balanced, with the numbers in degrees $-1$ and 1 equal.

Of course, with a nullspace of dimension 2, there are many choices of grading vector given by taking integer linear combinations of the two basis vectors and these will give different distributions of degrees. However these are ultimately reducible to knowing the two formulæ above.

A more natural way to consider the situation where the nullity is $l > 1$ is to view this as the existence of a $\mathbb{Z}^l$-grading. So in type $D_{2m+1}$ we have a bi-grading, assigning to each variable the bi-degree $\left(\left(\sum_{i=1}^{n-3}(-1)^id_i\right) + d_{n-1}, d_n - d_{n-1}\right)$. A different choice of basis leads to a different bi-grading but we may regard these as equivalent.

We illustrate the above in Figures 5, 6, 7 and 8.
Figure 7: The Auslander–Reiten quiver for the cluster category of type $D_4$, with degrees corresponding to the initial grading vector $(1, 0, -1, 0)$.

Figure 8: The Auslander–Reiten quiver for the cluster category of type $D_4$, with bi-degrees corresponding to the initial grading vectors $(1, 0, -1, 0)$ and $(0, 0, 1, -1)$ (in that order).

6 Gradings in type $E$ without coefficients

One easily checks that the exchange matrices of type $E_6$ and $E_8$ have maximal rank, so that the coefficient-free cluster algebras of these types admit no non-zero gradings.

However, the exchange matrix $B_{\text{mut}}$ of type $E_7$ has rank 6, so we do have a grading in this case. For the quiver

```
  2
 /   \
1 --- 3 -- 4 -- 5 -- 6 -- 7
```

we have the grading

```
  1
 /   \
0 --- 0 --- 0 --- 1 --- 0 --- 1
```

By computer-aided calculation of the cluster variables in this case, we find that this grading has

- 15 cluster variables in degree 1,
- 40 in degree 0 and
- 15 in degree $-1$. 

15
We again notice the balanced nature of this grading, with the equality of the numbers of variables of degree 1 and $-1$. Since this cluster algebra is the only finite-type cluster algebra of type $E$ to admit a non-zero grading, we can only speculate at this point as to whether this is part of a larger pattern. Examination of a type $E_9$ cluster algebra would be the natural continuation but this is an infinite-type cluster algebra and so the correct generalisation of “balanced” would need to be identified. We expect it will be necessary to use more categorical methods such as in Section 4.1 to investigate this case.

References

[1] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006. Techniques of representation theory. MR 2197389 (2006j:16020)

[2] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. 126 (2005), no. 1, 1–52.

[3] Arkady Berenstein and Andrei Zelevinsky, *Quantum cluster algebras*, Adv. Math. 195 (2005), no. 2, 405–455.

[4] J. H. Conway and H. S. M. Coxeter, *Triangulated polygons and frieze patterns*, Math. Gaz. 57 (1973), no. 400, 87–94. MR 0461269 (57 #1254)

[5] H. S. M. Coxeter, *Frieze patterns*, Acta Arith. 18 (1971), 297–310. MR 0286771 (44 #3980)

[6] Vladimir V. Fock and Alexander B. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 6, 865–930. MR 2567745 (2011f:53202)

[7] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529 (electronic).

[8] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein, *Cluster algebras and Poisson geometry*, Mathematical Surveys and Monographs, vol. 167, American Mathematical Society, Providence, RI, 2010. MR 2683456 (2011k:13037)

[9] Jan E. Grabowski and Stéphane Launois, *Graded quantum cluster algebras and an application to quantum Grassmannians*, preprint, 2013.

[10] Lingyan Guo, *On tropical friezes associated with Dynkin diagrams*, preprint, 2012.

[11] Dieter Happel, *On the derived category of a finite-dimensional algebra*, Comment. Math. Helv. 62 (1987), no. 3, 339–389. MR 910167 (89c:16029)

[12] Bernhard Keller, *Cluster algebras, quiver representations and triangulated categories*, Triangulated categories, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge Univ. Press, Cambridge, 2010, pp. 76–160. MR 2681708 (2011h:13033)

[13] James Propp, *The combinatorics of frieze patterns and Markoff numbers*, preprint, 2005.

[14] Joshua S. Scott, *Grassmannians and cluster algebras*, Proc. London Math. Soc. (3) 92 (2006), no. 2, 345–380.