Hofer’s diameter and Lagrangian intersections

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The purpose of the present note is to show that the group of Hamiltonian diffeomorphisms of the 2-sphere has infinite diameter with respect to Hofer’s metric. For surfaces of higher genus this fact follows from the energy-capacity inequality in the universal cover (see Lalonde-McDuff [LM], and a recent preprint by M. Schwarz [S] which establishes infiniteness of Hofer’s diameter for arbitrary aspherical symplectic manifolds). Our approach is based on a reduction of the statement to certain Lagrangian intersection problem, which in the case of $S^2$ can be easily solved by existing methods. The same approach works in some higher dimensional situations (see discussion in section 6 below).

1. Let $(M,\Omega)$ be a closed symplectic manifold. Denote by $F$ the space of all smooth functions $F: M \times S^1 \to \mathbb{R}$ which satisfy a normalization condition $\int_M F(x,t)\Omega^n = 0$ for every $t \in S^1$. We write $\text{Ham}(M,\Omega)$ for the group of Hamiltonian diffeomorphisms of $M$. Given a transformation $\phi \in \text{Ham}(M,\Omega)$, its Hofer’s norm $\rho(\text{id},\phi)$ is defined as

$$\inf \int_0^1 \max_x |F(x,t)| dt,$$

where the infimum is taken over all Hamiltonians $F \in \mathcal{F}$ which generate $\phi$ (see [H]).

Consider the 2-sphere $S^2$ endowed with a symplectic form, and denote by $L \subset S^2$ an equator (that is a simple closed curve which divides the sphere into two parts of equal areas).

Theorem 1.A. Let $\phi$ be a Hamiltonian diffeomorphism of $S^2$ generated by a Hamiltonian function $F \in \mathcal{F}$. Assume that for some positive number $c$ holds $F(x,t) \geq c$ for all $x \in L$ and $t \in S^1$. Then $\rho(\text{id},\phi) \geq c$.

1In [H] and [LM] the Hofer’s norm is defined as $\inf \int_0^1 \max_x F(x,t) - \min_x F(x,t) dt$. This norm is equivalent to the one we consider, and the results 1.A, 1.B and 6.A below remain valid for it as well.
Corollary 1.B. Hofer’s diameter of Ham($S^2$) is infinite.

Indeed, choose an autonomous normalized Hamiltonian $F$ on $S^2$ which equals on the equator to an arbitrarily large constant. Theorem 1.A implies that the corresponding Hamiltonian diffeomorphism lies on an arbitrarily large distance from the identity.

The proof of 1.A is divided into several steps.

2. Define a norm $||F|| = \max_{x,t} |F(x,t)|$ on $\mathcal{F}$. First we note that the Hofer’s distance can be expressed via this norm as follows.

Lemma 2.A. For every $\phi \in \text{Ham}(M)$

$$\rho(\text{id}, \phi) = \inf ||F||,$$

where the infimum is taken over all Hamiltonians from $\mathcal{F}$ which generate $\phi$.

In the terminology of [P2] this means that the “coarse” Hofer’s norm coincides with the usual one. The proof is based on a suitable choice of the time reparametrizations. We present this elementary argument at the end of the note.

3. Denote by $\mathcal{H} \subset \mathcal{F}$ the subset of all Hamiltonians from $\mathcal{F}$ which generate the identity map (or in other words which correspond to loops of Hamiltonian diffeomorphisms).

Lemma 3.A. Assume that a Hamiltonian diffeomorphism $\phi$ is generated by some $F \in \mathcal{F}$. Then

$$\rho(\text{id}, \phi) = \inf_{H \in \mathcal{H}} ||F - H||.$$

Proof: Let $g = \{g_t\}, t \in [0; 1]$ be any other path of Hamiltonian diffeomorphisms joining the identity with $\phi$. Then $g_t = h_t \circ f_t$ for some loop $h = \{h_t\}$. Write $G$ and $H$ for the normalized Hamiltonians of the paths $g$ and $h$ respectively. Then

$$G(x,t) = H(x,t) + F(h_t^{-1}x, t).$$

Assume that $G \in \mathcal{F}$. This is equivalent to the fact that $H \in \mathcal{H}$. Set $H'(x,t) = -H(h_t x, t)$. Note that $H'$ generates a loop $\{h_t^{-1}\}$, and thus belongs to $\mathcal{H}$. On the other hand the expression for $G$ above implies that $||G|| = ||F - H'||$. The required statement follows immediately from Lemma 2.A. \qed

Assume now that $M = S^2$, and $L$ is an equator.

Lemma 3.B. For every $H \in \mathcal{H}$ there exists a point $(x,t) \in L \times S^1$ such that $H(x,t) = 0$.

Proof of Theorem 1.A: In view of 3.B the inequality $||F - H|| \geq c$ holds for all $H \in \mathcal{H}$. Thus 3.A implies that $\rho(\text{id}, \phi) \geq c$. This completes the proof. \qed
4. Proof of 3.B:

In order to prove Lemma 3.B recall the Lagrangian suspension construction (which was used for study of Hofer’s geometry in [P1]). Let \( L \subset M \) be a Lagrangian submanifold, and \( h = \{ h_t \} \) be a loop of Hamiltonian diffeomorphisms. Then the embedding

\[
L \times S^1 \to (M \times T^*S^1, \Omega \oplus dr \wedge dt)
\]

which takes \((x, t)\) to \((h_t x, -H(h_t x, t), t)\) is Lagrangian. Denote its image by \( N(L, h) \). Note that homotopic loops \( h \) lead to Hamiltonian isotopic Lagrangian submanifolds \( N(L, h) \) (this follows from a result due to Weinstein [W]).

Return now to the case when \( M = S^2 \) and \( L \) is an equator. We claim that for every loop \( h \) the corresponding suspension \( N(L, h) \) is Hamiltonian isotopic to the "standard" Lagrangian torus \( N_0 = L \times \{ r = 0 \} \). Indeed, it is known that \( \pi_1(\text{Ham}(S^2)) = \mathbb{Z}_2 \). If \( h \) is homotopic to the constant loop \( \text{id} \) then the claim follows from the fact that \( N_0 = N(L, \text{id}) \). If \( h \) is not contractible then it is homotopic to a path \( f \) which is given by the rotation around the axis orthogonal to the equator. The corresponding Hamiltonian vanishes on \( L \), and again we see that \( N_0 = N(L, f) \). This completes the proof of the claim.

Our next claim is that the Lagrangian submanifold \( N_0 \) has the Lagrangian intersection property. Namely, if \( N \) is an arbitrary Lagrangian submanifold which is Hamiltonian isotopic to \( N_0 \) then \( N \) intersects \( N_0 \). This fact could be considered as a stabilization of the (obvious) Lagrangian intersection property of the equator in \( S^2 \). The proof is given in the next section.

Now we are ready to finish off the proof of the Lemma. Assume that \( N = N(L, h) \) where \( h \) is a loop of Hamiltonian diffeomorphisms of \( S^2 \). We see that \( N(L, h) \) intersects \( N_0 \) and thus there exist \( x \in L \) and \( t \in S^1 \) such that \( h_t x \in L \) and \( H(h_t x, t) = 0 \). This completes the proof of Lemma 3.B. \( \square \)

5. Here we prove the Lagrangian intersection property which was used in the previous section. The proof is based on the Floer homology theory for monotone Lagrangian submanifolds developed by Y.-G. Oh (see [O2, section 8] and references therein). Let \( (M, \Omega) \) be a tame symplectic manifold, and \( L \subset M \) be a closed monotone Lagrangian submanifold whose minimal Maslov number is at least 2. Let \( L' \) be the image of \( L \) under a generic Hamiltonian isotopy such that \( L \) and \( L' \) intersect transversally. Choose a generic \( \Omega \)-compatible almost complex structure \( J \) on \( M \). It is proved by Oh that in this case the Floer homology \( HF(L, L') \) over \( \mathbb{Z}_2 \) is well defined and (up to an isomorphism) does not depend on the generic choices of \( L' \), the Hamiltonian isotopy and \( J \). If \( HF(L, L') \neq 0 \) then \( L \) has the Lagrangian intersection property, in other words \( L \cap L' \) is non-empty. In particular, this is the case when \( L \) is the equator of \( S^2 \), or more generally when \( L = RP^n \subset CP^n \) (see [O1]).

Let \( Z \subset T^*S^1 \) be the zero section, and let \( Z' \subset T^*S^1 \) be the graph of the function \( r = \cos 2\pi t \). It is easy to see that Floer homology \( HF(L, L') \) is stable
in the following sense:

\[(5.A) \quad HF(L \times Z, L' \times Z') = HF(L, L') \otimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2).\]

Let us explain this formula. First of all notice that the left hand side is well defined since \(L \times S^1\) is again monotone with the same minimal Maslov number. Further,

\[C(L \times Z, L' \times Z') = C(L, L') \otimes C(Z, Z'),\]

and

\[\partial(x \otimes y) = \partial(x) \otimes y + x \otimes \partial(y).\]

Here \((C(\ldots), \partial)\) stands for the Floer complex, where \(M \times T^*S^1\) is endowed with the split almost complex structure, and the Hamiltonian isotopy between \(L \times Z\) and \(L' \times Z'\) splits as well. The expression for the differentials above reflects the fact that every discrete Floer’s gradient trajectory is the product of a discrete gradient trajectory in one factor and a constant trajectory in the other factor. Clearly \(C(Z, Z') = \mathbb{Z}_2 \oplus \mathbb{Z}_2\), and the corresponding differential vanishes. Thus we get formula 5.A.

Applying 5.A to the case when \(L\) is the equator of \(S^2\) and taking into account that \(HF(L, L')\) is non-trivial we obtain the required intersection property for \(L \times S^1\).

6. A generalization and discussion. Here we discuss some applications of our method to higher-dimensional manifolds. Let \((M, \Omega)\) be a closed symplectic manifold which satisfies the following two conditions:

(i) The fundamental group of \(\text{Ham}(M, \Omega)\) is finite (or, more generally, the Hofer’s length spectrum of \(\text{Ham}(M, \Omega)\) is bounded away from infinity).

(ii) There exists a monotone closed Lagrangian submanifold \(L \subset M\) whose minimal Maslov number is at least 2, such that \(HF(L, L') \neq 0\).

**Theorem 6.A.** The Hofer’s diameter of \(\text{Ham}(M, \Omega)\) is infinite.

Note that this theorem applies to \(S^2 \times S^2\) endowed with the split symplectic form such that the areas of the factors are equal, as well as to \(\mathbb{CP}^2\). Indeed, take \(L\) as the product of equators in the first case, and as \(\mathbb{RP}^2\) in the second case, and note that the corresponding fundamental groups are equal to \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) and \(\mathbb{Z}_3\) respectively.

**Proof of 6.A:** Let \(F \in \mathcal{F}\) be a normalized autonomous Hamiltonian which equals to some positive constant \(c\) on \(L\). Denote by \(f = \{f_t\}, \ t \in [0; 1]\) the corresponding Hamiltonian isotopy which generates a Hamiltonian diffeomorphism \(\phi = f_1\). Set \(D = \text{Ham}(M, \Omega)\) and let \((\tilde{D}, \tilde{\rho})\) be the universal cover of \(D\) endowed with the lift of the Hofer’s metric. Here we take the identity element of \(D\) as the base point, and consider smooth paths and smooth homotopies only.
Consider the lift $\psi$ of $\phi$ associated to the path $f$. Arguing exactly as in the proof of Theorem 1.A we get that $\tilde{\rho}(id, \psi) \geq c$. Indeed, let $h$ be any contractible loop of Hamiltonian diffeomorphisms. The condition (ii) combined with 5.A guarantees that the Lagrangian suspension $N(L, h)$ intersects $L \times Z$ in $M \times T^*S^1$. Then Lemmas 2.A and 3.A complete the job.

Denote by $u$ the upper bound for the length spectrum of $G$. In other words, every closed loop on $G$ can be homotoped to a loop whose Hofer’s length does not exceed $u$. The existence of such $u$ follows from the condition (i) above. Since the Hofer’s length structure is biinvariant, it follows that for an arbitrary lift $\theta$ of $\phi$ holds $\tilde{\rho}(\psi, \theta) \leq u$. Thus the triangle inequality for $\tilde{\rho}$ implies that $\tilde{\rho}(id, \theta) \geq c - u$, and hence $\rho(id, \phi) \geq c - u$. Taking $c$ arbitrary large we get the statement of the theorem.

Note that Theorem 1.A is a particular case of Theorem 6.A, however the proof of 1.A is in a sense more explicit than the one we just completed. Namely it uses a more detailed information about the Hamiltonians generating representatives of all homotopy classes of loops in Ham($S^2$). One can try to generalize this way of arguing to some symplectic manifolds which violate condition (i) above. For instance let $(M, \Omega)$ be the blow up of $\mathbb{CP}^2$ at one point endowed with the monotone symplectic structure. It is proved recently in [AM] that the fundamental group of Ham($M, \Omega$) equals to $\mathbb{Z}$ and is generated by an explicitly known $S^1$-action. Moreover the length spectrum is unbounded (see [P2]).

Denote by $H$ the normalized Hamiltonian of this loop (see [P2] for the precise expression). It is easy to see that the zero level $\{H = 0\}$ contains a Lagrangian torus $L$ which turns out to be non-monotone. It would be interesting to check whether in spite of that non-monotonicity $L$ has the stable Lagrangian intersection property. If it really does, then one concludes that the Hofer’s diameter in this case is infinite as well.

In higher dimensions we face a difficulty. Namely, for applying our method we need an information about the fundamental group of Ham($M, \Omega$) which is not yet available. On the other hand for some manifolds like $\mathbb{CP}^n$ or the product of $n$ spheres of equal areas one can use our approach in order to show that there exist arbitrarily long Hamiltonian paths which cannot be shortened with fixed endpoints at least in some homotopy classes.

Note finally that our arguments show that if there exists a subset $L$ in $M$ with the open complement such that every function $H \in \mathcal{H}$ vanishes at some point of $L \times S^1$ then the Hofer’s diameter is infinite. We took $L$ to be a Lagrangian submanifold in order to apply the existing Lagrangian intersection techniques, however it would be quite natural to work with more general (say, open) subsets. Here is a question which sounds as a first step towards such a generalization.

Take $L$ to be the complement of a tiny ball in $M$, and define $N$ as the product of $L$ with the zero section in $T^*S^1$. Is it true that $N$ always intersects its image under an arbitrary Hamiltonian isotopy of $M \times T^*S^1$? Obviously this is the case for split Hamiltonian isotopies. Can one use the folding construction due
to Lalonde and McDuff [LM] for general Hamiltonian isotopies?

7. Proof of 2.A: For \(\phi \in \text{Ham}(M,\Omega)\) set \(r(\text{id},\phi) = \inf \|F\|\) where \(F\) runs over all Hamiltonians \(F \in \mathcal{F}\) which generate \(\phi\). Clearly \(r(\text{id},\phi) \geq \rho(\text{id},\phi)\). Our task is to prove the converse inequality. Fix a positive number \(\epsilon\). Choose a path \(f = \{f_t\}, t \in [0;1]\) of Hamiltonian diffeomorphisms generated by some \(F \in \mathcal{F}\) such that \(f_0 = \text{id}, f_1 = \phi\) and such that \(\int_0^1 m(t) \leq \rho(\text{id},\phi) + \epsilon\) where \(m(t) = \max_x |F(x,t)|\). We can always assume that \(m(t)\) is strictly positive (geometrically this means that \(f\) is a regular curve). Denote by \(\mathcal{C}\) the space of all \(C^1\)-smooth orientation preserving diffeomorphisms of \(S^1\) which fix 0. Note that for \(a \in \mathcal{C}\) the path \(f_a = \{f_{a(t)}\}\) is generated by the normalized Hamiltonian function \(F_a(x,t) = a'(t)F(x,a(t))\), where \(a'\) denotes the derivative with respect to \(t\). Take now \(a(t)\) as the inverse of \(\int_0^t m(s)ds\). Note that

\[
||F_a|| = \max_t a'(t)m(a(t)) = \max_t (m(t)/b'(t)) = \int_0^1 m(t).
\]

We conclude that \(||F_a|| \leq \rho(\text{id},\phi) + \epsilon\). Approximating \(a\) in the \(C^1\)-topology by a smooth diffeomorphism from \(\mathcal{C}\) we see that one can find a smooth normalized Hamiltonian, say \(\tilde{F}\), which generates \(\phi\) and such that \(||\tilde{F}|| \leq \rho(\text{id},\phi) + 2\epsilon\). Since this can be done for an arbitrary \(\epsilon\) we conclude that \(r(\text{id},\phi) \leq \rho(\text{id},\phi)\). This completes the proof. \(\blacksquare\)

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