Abstract

In this paper we investigate the construction of state models for link invariants using representations of the braid group obtained from various gauge choices for a solution of the trigonometric Yang–Baxter equation. Our results show that it is possible to obtain invariants of regular isotopy (as defined by Kauffman) which may not be ambient isotopic. We illustrate our results with explicit computations using solutions of the trigonometric Yang–Baxter equation associated with the one-parameter family of minimal typical representations of the quantum superalgebra $U_q[gl(2|1)]$. We have implemented Mathematica code to evaluate the invariants for all prime knots up to 10 crossings.

1 Introduction

The close connection between solutions of the quantum Yang–Baxter equation (QYBE) and the evaluation of invariants for oriented knots and links through representations of the braid group is well known [9]. The algebraic properties of quantum algebras and superalgebras provide systematic means to construct solutions of the QYBE which can in turn be used to explicitly compute these invariants. The seminal example of this method is the use of the six vertex solution of the QYBE in the evaluation of the Jones polynomial invariant [7].

The spectral parameter dependent solutions of the trigonometric Yang–Baxter equation (TYBE) arise as evaluation (or loop) representations of affine quantum algebras and superalgebras. Braid group representations are obtained from these solutions by taking the limit as the spectral parameter approaches infinity. We refer
to these limiting cases as *quantum R matrices* which satisfy the QYBE. As demonstrated by Bracken et al. \[1\], gauge equivalent solutions of the TYBE are obtained by considering different gradations of the underlying affine (super)algebraic structure. The significant point is that the explicit braid group representation obtained in the infinite spectral parameter limit depends on the choice of the gradation. This feature has already been observed in \[1\]. This means that given a single solution of the TYBE, there are many possibilities to in turn obtain a solution of the QYBE.

Our aim in this work is to investigate the construction of link invariants for the case of solutions of the QYBE obtained through non-standard choices of gradation (i.e., not the *homogeneous* gradation) for the affine (super)algebraic structure which underlies the TYBE solutions. We find that in general we are only able to define invariants of *regular isotopy*, which is to say that the results are invariant under the second and third Reidemeister moves. However, we will show that in our examples we can relate the regular isotopy invariants to well known invariants of *ambient isotopy*, viz invariant under all three Reidemeister moves. (Our terminology for regular and ambient isotopy is adopted from \[6\].)

Our specific calculations are performed using the solution of the TYBE associated with the one-parameter family of minimal typical representations of the quantum superalgebra $U_q[\mathfrak{gl}(2|1)]$. For the choice of the homogeneous gradation for the untwisted affine extension $U_q[\mathfrak{gl}(2|1)(1)]$, one obtains the Links–Gould invariants recently investigated in detail in \[2, 5\]. However, for different choices of the gradation the process yields other invariants which can be related back to both the Jones and Alexander–Conway polynomial invariants.

We begin the paper with a description of gauge equivalent solutions of the TYBE without appealing to the gradation structure of quantum affine (super)algebras. For the sake of simplicity, we present these solutions in terms of a simple basis transformation satisfying some particular constraints.

## 2 Gauge equivalent solutions of the TYBE

For an arbitrary vector space $V$, let $R(x) \in \text{End}(V \otimes V)$ satisfy the TYBE on the tensor product space $V \otimes V \otimes V$:

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x).$$

(1)

where $R_{12}(x) \triangleq R(x) \otimes I$, etc, and $x$ and $y$ are arbitrary complex parameters. Consider an invertible matrix $A(x) \in \text{End}(V)$, with the properties:

$$\begin{align*}
A(x)A(y) & = A(xy) \\
[R(x), A_1(y)A_2(y)] & = 0,
\end{align*}$$

(2)

where $A_1(x) \triangleq A(x) \otimes I$ and $A_2(x) \triangleq I \otimes A(x)$. We immediately deduce that:

$$\begin{align*}
A(1) & = I, \\
A(x)A(y) & = A(y)A(x), \\
A^{-1}(x) & = A^{-1}(x),
\end{align*}$$

where throughout the paper, we liberally write $X$ for $X^{-1}$, for various $X$. 

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Now set $\mathcal{R}(x) = A_1(x)R(x)A_1(\pi)$. It is an algebra exercise to prove:

$$
\mathcal{R}_{12}(x)\mathcal{R}_{13}(xy)\mathcal{R}_{23}(y) = A_1(x)\mathcal{R}_{12}(x)A_1(\pi)A_1(xy)\mathcal{R}_{13}(xy)A_1(\pi y)A_2(y)\mathcal{R}_{23}(y)A_2(\pi y)A_2(\pi y).
$$

Confirming this involves manipulating expressions in a total of 6 variables, viz:

$$
A_{12}A_{13}A_1\mathcal{R}_{12}(x)A_1(\pi)A_1(xy)\mathcal{R}_{13}(xy)A_1(\pi y)A_2(y)\mathcal{R}_{23}(y)A_2(\pi y)A_2(\pi y).
$$

Applying the above to a particular example, the following trigonometric $R$ matrix with gauge parameters $x_1$, $x_2$, $x_3$, $x_4$, $x_5$, $x_6$, $x_7$, and $x_8$ is discussed in [1]. For our purposes, we need not describe this in detail, and simply refer to $R(x)$ and $\mathcal{R}(x)$ as gauge equivalent. The essential point is that the limit of $\mathcal{R}(x)$ as $x \to \infty$ yields different quantum R matrices depending on the choice of $\mathcal{A}(x)$.

3 Trigonometric R matrix with gauge parameters

Applying the above to a particular example, the following trigonometric $R$ matrix $\mathcal{R}^{r,s}(u)$ with gauge parameters $r$ and $s$ arises from the one-parameter family of minimal typical representations of $U_q[gl(2|1)]$. (Here, we have replaced variable $x$ with $u$, defined by $x \equiv q^u$.) The operator $\mathcal{R}^{r,s}(u)$ has 36 nonzero components, and is scaled such that its first component is 1. It satisfies the TYBE in the additive form:

$$
\tilde{\mathcal{R}}_{12}(u)\tilde{\mathcal{R}}_{23}(u+v)\tilde{\mathcal{R}}_{12}(v) = \tilde{\mathcal{R}}_{23}(v)\tilde{\mathcal{R}}_{12}(u+v)\tilde{\mathcal{R}}_{23}(u).
$$

Conferring this involves manipulating expressions in a total of 6 variables, viz: the representation variables $q$ and $\alpha$, the spectral variables $u$ and $v$, and the gauge parameters $r$ and $s$. Explicitly, $\mathcal{R}^{r,s}(u)$ is:

$$
\left\{e_{11}^{11}\right\}, \quad \frac{\alpha + u}{\alpha - u} \left\{\begin{array}{c} e_{12}^{22} \\ e_{33}^{33} \end{array}\right\}, \quad \frac{\alpha + u}{\alpha - u} \left\{\begin{array}{c} 1 + \alpha + u \\ 1 + \alpha - u \end{array}\right\} \left\{e_{44}^{44}\right\},
$$

$$
\frac{\alpha - u}{\alpha - u} \left\{\begin{array}{c} r^u q^u e_{12}^{12}, r^v q^v e_{31}^{31} \\ r^u q^u e_{21}^{21}, r^v q^v e_{31}^{31} \end{array}\right\}, \quad \frac{\alpha - u}{\alpha - u} \left\{\begin{array}{c} 1 + \alpha \\ 1 + \alpha + u \end{array}\right\} \left\{\begin{array}{c} r^u q^u e_{12}^{12}, r^v q^v e_{31}^{31} \\ r^u q^u e_{21}^{21}, r^v q^v e_{31}^{31} \end{array}\right\},
$$

$$
\frac{1}{\Delta^2} \left\{\begin{array}{c} u \\ 1 + \alpha - u \end{array}\right\} \left\{\begin{array}{c} e_{12}^{12}, e_{31}^{31} \\ e_{21}^{21}, e_{31}^{31} \end{array}\right\}, \quad \frac{1}{\Delta^2} \left\{\begin{array}{c} 1 - u \\ 1 + \alpha - u \end{array}\right\} \left\{\begin{array}{c} e_{12}^{12}, e_{31}^{31} \\ e_{21}^{21}, e_{31}^{31} \end{array}\right\},
$$

$$
\frac{1}{\Delta^2} \left\{\begin{array}{c} u^2 \\ 1 + \alpha - u \end{array}\right\} \left\{\begin{array}{c} e_{23}^{23} \\ e_{23}^{23} \end{array}\right\}, \quad \frac{1}{\Delta^2} \left\{\begin{array}{c} u^2 \\ 1 + \alpha - u \end{array}\right\} \left\{\begin{array}{c} e_{23}^{23} \\ e_{23}^{23} \end{array}\right\},
$$

$$
\frac{1}{\Delta^2} \left\{\begin{array}{c} 1 + \alpha + \frac{1}{2} |u| \\ 1 + \alpha - u \end{array}\right\} \left\{\begin{array}{c} q^u f(q) e_{12}^{12}, q^v f(q) e_{21}^{21} \\ q^u f(q) e_{21}^{21}, q^v f(q) e_{12}^{12} \end{array}\right\}, \quad \frac{1}{\Delta^2} \left\{\begin{array}{c} 1 + \alpha + \frac{1}{2} |u| \\ 1 + \alpha - u \end{array}\right\} \left\{\begin{array}{c} q^u f(q) e_{12}^{12}, q^v f(q) e_{21}^{21} \\ q^u f(q) e_{21}^{21}, q^v f(q) e_{12}^{12} \end{array}\right\},
$$

where $f(q) \equiv -2q + q^{2\alpha}(q - \tilde{q}) + q^{1+2\alpha} + \tilde{q}^{1+2\alpha}$ and we have defined $\Delta \equiv q - \tilde{q}$, as well as writing $|x|$ for $(q^x - \tilde{q}^x)/(q - \tilde{q})$. 

3
This solution of the TYBE originates in the following trigonometric R matrix
\( \hat{R}(u) \equiv R^{r=1,s=1}(u) \), which may be regarded as a gauge-free version of \( R^{r,s}(u) \).

\[
\begin{align*}
&\{ e_{11}^1 \}, \quad [\alpha + u] \left\{ \frac{e_{23}^2}{e_{33}^2} \right\}, \quad [\alpha + u][1 + \alpha + u] \left\{ \frac{e_{44}^1}{e_{33}^1} \right\}, \\
&\left[ \alpha \right] \left\{ \varpi^e e_{12}^2, \varpi^e e_{13}^3 \right\}, \quad [\alpha][1 + \alpha] \left\{ \varpi^e e_{14}^4 \right\}, \\
&\frac{1}{\Delta^2[\alpha - u][1 + \alpha - u]} \left\{ f(q)e_{24}^3, f(q)e_{34}^3 \right\}, \quad \frac{1 + \alpha}[\alpha + u] \left\{ \varpi^e e_{24}^3, \varpi^e e_{34}^3 \right\}, \\
&\frac{-[u]}{[\alpha - u][1 + \alpha - u]} \left\{ e_{12}^2, e_{13}^3 \right\}, \quad \frac{[1 - u][u]}{[\alpha - u][1 + \alpha - u]} \left\{ e_{14}^4 \right\}, \\
&\frac{-[u]}{[\alpha - u][1 + \alpha - u]} \left\{ e_{32}^2, e_{23}^3 \right\}, \quad \frac{[u][\alpha + u]}{[\alpha - u][1 + \alpha - u]} \left\{ e_{24}^4, e_{34}^4 \right\}, \\
&\frac{[\alpha]^{1/2}[1 + \alpha]^{1/2}[u]}{[\alpha - u][1 + \alpha - u]} \left\{ -\varpi^{-1} \right\} \begin{cases} e_{32}^2, & -q^{-1} \end{cases}, \quad -\varpi^{-1} \begin{cases} e_{32}^3, & +q \end{cases}, \quad -\varpi^{-1} \begin{cases} e_{32}^3, & +q \end{cases}, \quad -\varpi^{-1} \begin{cases} e_{32}^2, & -q^{-1} \end{cases}.
\end{align*}
\]

The above solution of (3) is obtained from the representation theory of the quantum superalgebra \( U_q[gl(2|1)] \) and the Baxterization procedure (e.g. see §6). It is related to \( R(u) \) through a transposition:

\[
\hat{R}(u)_{ac}^{bd} = R(u)_{bd}^{ca}.
\]

From it we may construct a quantum R matrix \( \hat{R} \equiv \lim_{u \to \infty} \hat{R}(u) \), and build a link invariant \( \hat{R}^{r,s}(u) \).

By the following procedure, \( \hat{R}^{r,s}(u) \) may be obtained from \( \hat{R}(u) \). Let us define:

\[
\hat{R}(u)_{ac}^{bd} = \hat{A}(u)_{ac}^{bd} R(u)_{bc}^{cd} A(-u)_{db}^{cf},
\]

where \( \hat{A}(u) \) is a \( 4 \times 4 \) matrix satisfying the properties (3) which ensure that \( \hat{R}(u) \) also satisfies (3). Specifically, we will use the following diagonal \( A(u) \equiv A^{r,s}(u) \) containing gauge parameters \( r \) and \( s \):

\[
A^{r,s}(u) \triangleq \text{diag} \{ 1, r^u, s^u, r^u s^u \},
\]

thus we will write \( \hat{R}^{r,s}(u) \) rather than \( \hat{R}(u) \). We define \( \hat{R}^{r,s}(u) \) by again transposing:

\[
\hat{R}^{r,s}(u)_{ac}^{bd} = \hat{R}^{r,s}(u)_{bd}^{ca}.
\]

This \( \hat{R}^{r,s}(u) \) satisfies (3), and indeed is the object which we first introduced.

To investigate the possibility of constructing link invariants, we will select appropriate gauge choices \( r = r_i(q), s = s_i(q) \) and take the spectral limit \( u \to \infty \) of \( \hat{R}^{r,s_i}(u) \) to yield a quantum R matrix \( \hat{R}^i \):

\[
\hat{R}^i \triangleq \lim_{u \to \infty} \hat{R}^{r_i,s_i}(u),
\]

which satisfies the (non-parametric) QYBE:

\[
\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23},
\]

which is graphically depictable in terms of braids. (In the above, \( i \) is an index to number the gauge choice, and the functions \( r_i \) and \( s_i \) may also contain variable \( q \).)
Substituting (3) and (4) into (5) allows us to write (no sums on \( b \) and \( c \)):
\[
\hat{R}^{r,s}(u)_{cd}^\alpha = A^{r,s}(u)_c^\alpha \cdot \hat{R}(u)_{db}^\alpha \cdot A^{r,s}(-u)_b^\alpha,
\]
hence we have for grading choice \( i \) a quantum R matrix \( R^i \) with components:
\[
(\hat{R}^i)_{bd}^{\alpha c} = \lim_{u \to \infty} A^{r,s}(u)^{\alpha c}_c \cdot \hat{R}(u)_{db}^{\alpha c} \cdot A^{r,s}(-u)_b^{\alpha c}.
\]

The gauge choices that we shall make are depicted in Table 1.

| Case (i) | \( r_i, s_i \) | \( A^{r_i,s_i}(u) \) |
|----------|----------------|-----------------|
| 1        | \( r_1 = 1, s_1 = 1 \) | \( I \) |
| 2        | \( r_2 = 1, s_2 = q \) | \( \text{diag} \{ 1, 1, q^u, q^u \} \) |
| 3        | \( r_3 = q, s_3 = q \) | \( \text{diag} \{ 1, q^u, q^u, q^{2u} \} \) |
| 4        | \( r_4 s_4 = q^2, s_4 > r_4 > 1 \) | \( \text{diag} \{ 1, q^u, q^u, q^{2u} \} \) |

Table 1: Gauge choices.

From Case 1, we recover the ungauged situation.

4 Quantum R matrices

In the spectral limit \( u \to \infty \), our trigonometric R matrix \( \hat{R}^{r,s}(u) \) becomes a quantum R matrix \( \hat{R}^i \) in variables \( q \) and \( \alpha \), where \( i \) is the index of the gauge choice (4). Here, we present the \( \hat{R}^i \), in terms of ‘internal’ variables, \( p \triangleq q^{u+\frac{1}{2}} \) and \( Q \triangleq q^{2u} \), which simplify computations.

\( \hat{R}^1 \) has 26 nonzero components:
\[
\{ e_{11}^1 \}, \quad -p^2 Q \left\{ e_{23}^{22} e_{33}^{22} \right\}, \quad p^4 \left\{ e_{44}^{44} \right\}, \\
-p Q (p Q - Q) \left\{ e_{21}^{21} e_{31}^{21} \right\}, \quad p^3 Q (p Q - Q) \left\{ e_{23}^{12} e_{32}^{12} \right\}, \\
p^2 (p Q - Q) (p Q - Q) \left\{ e_{31}^{41} \right\}, \quad p^2 (Q^2 - Q^2) \left\{ e_{12}^{32} \right\}, \\
p Q \left\{ e_{12}^{12} e_{31}^{12} e_{13}^{13} \right\}, \quad p^2 Q \left\{ e_{22}^{12} e_{32}^{12} e_{42}^{12} e_{43}^{12} \right\}, \quad p^2 Q \left\{ e_{13}^{14} e_{31}^{14} e_{34}^{14} e_{44}^{14} \right\}, \quad p^2 Q \left\{ e_{23}^{32} e_{33}^{32} \right\}, \\
p^2 (p Q - Q) \left\{ e_{12}^{21} e_{22}^{21} e_{32}^{21} \right\} \cdot Q \left\{ e_{22}^{12} e_{32}^{12} \right\}.
\]

\( \hat{R}^2 \) has 20 nonzero components:
\[
\{ e_{11}^1 \}, \quad -p^2 Q \left\{ e_{23}^{22} e_{33}^{22} \right\}, \quad p^4 \left\{ e_{44}^{44} \right\}, \\
-p Q (p Q - Q) \left\{ e_{21}^{21} e_{31}^{21} \right\}, \quad p^2 Q (p Q - Q) \left\{ e_{31}^{41} \right\}, \\
p Q \left\{ e_{12}^{12} e_{31}^{12} e_{13}^{13} \right\}, \quad p^2 Q \left\{ e_{22}^{12} e_{32}^{12} e_{42}^{12} e_{43}^{12} \right\}, \quad p^2 Q \left\{ e_{13}^{14} e_{31}^{14} e_{34}^{14} e_{44}^{14} \right\}, \quad p^2 Q \left\{ e_{23}^{32} e_{33}^{32} \right\}, \\
-p^2 Q (p Q - Q) \left\{ e_{12}^{21} e_{22}^{21} e_{32}^{21} \right\} \cdot Q \left\{ e_{22}^{12} e_{32}^{12} \right\}.
\]
\( R^3 \) has 17 nonzero components:

\[
\begin{align*}
\{ e_{11}^1 \}, & \quad -p^2 Q \{ e_{22}^{11} \}, & \quad p^4 \{ e_{44}^1 \}, \\
p^2 (Q^2 - Q^2) \{ e_{32}^{11} \}, & \\
p^2 (Q^2 + Q^2) \{ e_{22}^{11}, e_{44}^1 \}, & \\
p^2 (Q^2 + Q^2) \{ e_{22}^{11}, e_{44}^1 \}, & \\
p^2 (Q^2 + Q^2) \{ e_{22}^{11}, e_{44}^1 \}, & \\
p^2 (Q^2 + Q^2) \{ e_{22}^{11}, e_{44}^1 \}.
\end{align*}
\]

\( R^4 \) has 16 nonzero components:

\[
\begin{align*}
\{ e_{11}^1 \}, & \quad -p^2 Q \{ e_{22}^{11} \}, & \quad p^4 \{ e_{44}^1 \}, \\
p^2 Q \{ e_{22}^{11}, e_{44}^1 \}, & \\
p^2 Q \{ e_{22}^{11}, e_{44}^1 \}, & \\
p^2 Q \{ e_{22}^{11}, e_{44}^1 \}, & \\
p^2 Q \{ e_{22}^{11}, e_{44}^1 \}.
\end{align*}
\]

Each \( R^i \) has a distinct set of eigenvalues, with some overlap. Immediately, the construction of \( R(u) \) reminds us that the three distinct (diagonal) components \((R^i)_{jj}\) must be (gauge-independent) eigenvalues. In the spectral limit \( u \to \infty \), these become 1, \(-q^{2a}\) (twice), and \(q^{4a+2}\) respectively. No matter what the gauge, we will have a minimum of these 3 eigenvalues; in fact \( R^1 \) contains only these 3. Overall, we expect a maximum of 15 distinct eigenvalues; these are described in Table 2.

| Case (i) | Eigenvalues of \( R^i \) | # |
|---|---|---|
| 1 | \(-q^{2a}\), \(q^{4a+2}\) | 3 |
| 2 | \(-q^{2a}\), \(q^{4a+2}\), \(\pm q^{2a}\), \(\pm q^{3a+1}\) | 7 |
| 3 | \(-q^{2a}\), \(q^{4a+2}\), \(\pm q^{2a}\), \(\pm q^{3a+1}\), \(q^{2a}\), \(q^{2a+2}\), \(p^4\), \(p^2 Q\) | 9 |
| 4 | \(-q^{2a}\), \(q^{4a+2}\), \(\pm q^{2a}\), \(\pm q^{3a+1}\), \(q^{2a}\), \(q^{2a+1}\), \(\pm q^{2a}\), \(\pm q^{2a+1}\) | 10 |

Table 2: Eigenvalues of quantum R matrices for various gauges.

### 5 Ambient isotopy link invariants

A state model for evaluation of link invariants of ambient isotropy based on a quantum R matrix \( R \), is defined by two parameters: a representation of the braid generator \( \sigma \), and a representation of the ‘left handle’ \( C \). In our case, we seek a scaling factor \( \kappa \) such that \( \sigma^{\pm 1} = \kappa^{\pm 1} R^{\pm 1} \), and a 4 × 4 matrix \( C \) such that (Einstein summation convention):

\[
C_{d}^{c} \cdot (\sigma^{\pm 1})_{ca} = \delta_{bc}^{a}, \tag{9}
\]

is satisfied.
This ensures that the value of the link invariant over a single loop of writhe is unity, viz that the first Reidemeister move is satisfied. This is depicted in Figure 1. In fact, this only determines \( \kappa \) up to a factor of \(+1\), but we shall ignore one case.

As \( \hat{R} \) necessarily satisfies the QYBE, abstract tensors built from \( \sigma \) are invariant under the second and third Reidemeister moves, hence we may construct representations of arbitrary braids from \( \sigma \). As all links may be represented by braids combined with left handles, together these are sufficient parameters.

Our state model then yields a polynomial invariant of oriented of (1, 1) tangles. Evaluation of such a model for arbitrary links is described in our previous work \([2, 5]\). Here, we use the same principles and Mathematica code – only the state model parameters are changed.

**Case 1**

Suitable \( C \) that satisfy the QYBE are:

\[
\kappa \cdot \bar{\sigma}^2 Q^2 A = \kappa \cdot \bar{Q}^2 A, \quad \text{where} \quad A = \text{diag}\left\{+Q^2, -Q^2, -\bar{Q}^2, +\bar{Q}^2\right\},
\]

hence \( \kappa = \bar{\sigma}^2 Q^2 \) suffices to yield: \( C = p^2 A \). The associated braid generator \( \sigma \) is:

\[
\begin{align*}
\bar{\sigma}^2 Q^2 & \left\{ e_{11} \right\}, \quad -1 \left\{ e_{22} \right\}, \quad p^2 Q^2 \left\{ e_{44} \right\}, \\
-\bar{p}Q(p\bar{Q} - pQ) & \left\{ e_{21} \right\}, \quad pQ(p\bar{Q} - pQ) \left\{ e_{42} \right\}, \\
Q^2(p\bar{Q} - pQ)(pQ - \bar{p}Q) & \left\{ e_{41} \right\}, \quad Q^2(Q^2 - \bar{Q}^2) \left\{ e_{32} \right\}, \\
\bar{p}Q & \left\{ e_{121}, e_{13} \right\}, \quad pQ \left\{ e_{242}, e_{23} \right\}, \quad \left\{ e_{34} \right\}, \quad -Q^2 \left\{ e_{43} \right\}, \\
Q^2(p\bar{Q} - pQ) & \left\{ e_{23} \right\}, \quad Q \left\{ e_{31} \right\}, \quad \left\{ e_{32} \right\}.
\end{align*}
\]

These choices of course lead to the Links–Gould invariant \( LG \) of \([2, 5]\).

For Cases 2–4, there are problems in finding \( \kappa \), as the process yields no consistent choices of \( \kappa \) without reducing the variables \( p \) and \( Q \). (Furthermore, in each case, relaxing the condition that \( C \) be diagonal immediately yields the conclusion that \( C \) must be diagonal if it does exist.)
Case 2

Appropriate $C$ are:

$$\pi \cdot \mathbf{p}^2 Q \cdot \text{diag} \left\{ +pQ, -pQ, -\overline{pQ}, +\overline{pQ} \right\},$$

$$\kappa \cdot p Q \cdot \text{diag} \left\{ +pQ, -pQ, -p \overline{Q}, +p \overline{Q} \right\}.$$  

The only solution to this system is found by setting $p = \pm 1$, whence $\kappa = Q$. In this case, we have:

$$C = \text{diag} \left\{ +Q, -Q, -\overline{Q}, +\overline{Q} \right\}$$

and the following one-variable braid generator $\sigma$ (note the imaginary components):

$$Q \left\{ e_1 \right\}, \quad -\overline{Q} \left\{ e_{23}^{23}, e_{33}^{33} \right\}, \quad Q \left\{ e_4^{44} \right\},$$

$$(Q - \overline{Q}) \left\{ e_{31}^{21}, e_{31}^{13} \right\}, \quad \pm \left\{ e_{12}^{12}, e_{11}^{11}, e_{13}^{13} \right\}, \quad \mp \left\{ e_{24}^{24}, e_{34}^{34} \right\}, \quad Q \left\{ e_{14}^{14}, e_{44}^{44} \right\}, \quad -Q \left\{ e_{32}^{23}, e_{33}^{33} \right\},$$

$$-i(Q - \overline{Q}) \left\{ e_{41}^{41}, e_{41}^{14} \right\}.$$  

After the scaling, the 7 distinct eigenvalues of $\overline{R^2}$ coalesce to 4 of $\sigma$; they are $\{q^2, -\overline{q}^2 \pm 1\} \equiv \{Q, -\overline{Q}, \pm 1\}$. Inspection of the eigenvectors shows that the $\sigma$ is not diagonalizable, as it has only 14 (out of 16) distinct eigenvectors.

The resulting invariant is nontrivial. It turns out to be the Alexander–Conway invariant in variable $q$. (This is an experimental observation only, but has been verified for all prime knots of up to 10 crossings.)

Case 3

Appropriate $C$ are:

$$\pi \cdot \mathbf{p}^2 Q^2 \cdot \text{diag} \left\{ +p^2 Q^2, -Q^4, -\overline{Q}^4, +\overline{p}^2 Q^2 \right\}$$

and

$$\kappa \cdot p^2 Q \cdot \text{diag} \left\{ +p^2 Q^2, -Q^4, -\overline{Q}^4, +p^2 Q^2 \right\}.$$  

The only solutions to this system leave us with an integer invariant (e.g. $p = Q = 1$ and $\kappa = 1$). In any case, setting $Q = 1$ means that the R matrix degenerates to being a special case of Case 4, so we ignore these solutions.

Case 4

Appropriate $C$ are:

$$\pi \cdot \mathbf{p}^2 \cdot \text{diag} \left\{ p^2, -Q^2, -Q^2, +p^2 \right\}$$

and

$$\kappa \cdot p^2 \cdot \text{diag} \left\{ p^2, -Q^2, -Q^2, +p^2 \right\}.$$  

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Again, the only solutions to this system leave us with an integer invariant. For example, for \( p = Q = \pm 1 \) and \( \kappa = 1 \), we obtain \( C = \text{diag} \{ +1, -1, -1, +1 \} \), and the braid generator:

\[
\begin{pmatrix} e_{11}^{11} \\ e_{22}^{11} \\ e_{33}^{11} \\ e_{44}^{11} \end{pmatrix}, \quad \begin{pmatrix} e_{22}^{22} \\ e_{33}^{22} \\ e_{44}^{22} \end{pmatrix}, \quad \begin{pmatrix} e_{44}^{44} \end{pmatrix}, \quad \begin{pmatrix} e_{12}^{12}, e_{13}^{13}, e_{14}^{14} \\ e_{21}^{21}, e_{23}^{23}, e_{24}^{24} \\ e_{31}^{31}, e_{32}^{32}, e_{34}^{34} \end{pmatrix}, \quad \begin{pmatrix} e_{12}^{23}, e_{13}^{23}, e_{14}^{23} \\ e_{21}^{24}, e_{23}^{24}, e_{24}^{24} \end{pmatrix}, \quad \begin{pmatrix} e_{31}^{24}, e_{32}^{24}, e_{34}^{24} \end{pmatrix}, \quad \begin{pmatrix} e_{41}^{23}, e_{42}^{23}, e_{43}^{23} \end{pmatrix}, \quad \begin{pmatrix} e_{41}^{41}, e_{42}^{41}, e_{43}^{41} \end{pmatrix}.
\]

The 10 distinct eigenvalues of \( \hat{R}^4 \) coalesce to 2 of \( \sigma \); they are just \( \{ \pm 1 \} \), and \( \sigma \) is of course diagonalizable.

The resulting invariant is integer, and in fact trivial. This has been confirmed by the application of the ‘Matveev \( \Delta - \nabla \) test’ \[8\]. This test applies a little theorem which shows that if an invariant cannot distinguish the braids \( \sigma_1 \sigma_2 \sigma_1 \) and \( \sigma_2 \sigma_1 \sigma_2 \), then it cannot distinguish any knot from the unknot, since repeated interchanges of these braids are sufficient to untangle any knot.

In fact, applying this test to \( \hat{R}^4 \) shows that, regardless of \( C \) or any special choice of the variables \( p \) and \( Q \), any associated invariant will be trivial. (Applying the test to Cases 2 and 3 shows that we may have a nontrivial invariant.)

6 Regular isotopy link invariants

For Cases 2 and 3, in general, we fail to build a link invariant of ambient isotopy as we cannot satisfy \( \hat{R}^4 \). We now build invariants associated with these cases that are only of regular isotopy.

Case 2

The choice \( \kappa = \overline{p}^2 Q \) and \( C = \overline{p} \cdot \text{diag} \{ +Q, -Q, -\overline{Q}, +\overline{Q} \} \), yields the symmetric results:

\[
(\text{tr} \otimes I) (C \otimes I) \sigma^{\pm 1} = \text{diag} \{ \overline{p}, \overline{p}, p, p \}^{\pm 1}.
\]

Applying our link invariant evaluation engine to these parameters, we obtain an open tangle invariant which is a diagonal \( 4 \times 4 \) matrix. For a knot \( K \) (presented as the closure \( \hat{\beta} \) of a braid \( \beta \)), we obtain the corresponding \text{regularly isotopic} \( (1, 1) \)-tangle invariant of the following form:

\[
\text{diag} \{ \overline{p}^w \Delta_K(Q^2 \overline{p}^2), p^w \Delta_K(Q^2 p^2) \},
\]

(twice)

\[
(\text{twice})
\]

where \( w \) is the writhe of \( \beta \), and \( \Delta_K \) is the Alexander–Conway invariant of \( K \). Again, this result is an experimental observation only, checked to be valid for all prime knots of up to 10 crossings. Setting \( p = 1 \) recovers our previous observation that the only possible invariant of ambient isotopy is \( \Delta_K \).
Case 3

The choices \( \kappa = p^2 \) and \( C = \text{diag} \{ 1, Q^2, Q^2, 1 \} \), yield the symmetric results:

\[
(\text{tr} \otimes I) \ (C \otimes I) \ \sigma^{\pm 1} = \text{diag} \left\{ +p^2, -Q^4, -Q^4, +p^2 \right\}^{\pm 1}.
\]

Again, we obtain an open-tangle invariant which is a diagonal \( 4 \times 4 \) matrix:

\[
\text{diag}\{ p^{2w}, (-Q^4)^w V_K(Q^4), p^{2w} \},
\]

twice

where \( V_K \) is the Jones polynomial of the link \( K \). Again, we emphasise that this is an experimental observation, known to be valid for all prime knots of up to 10 crossings.

7 Conclusions

Our calculations show that it is possible to access several invariants from a single solution of the TYBE. Specifically, we have shown that it is possible to recover both the Jones and the Alexander–Conway invariants from the solution of the TYBE which was originally employed to define the Links–Gould invariant \( LG \).

Repeating this process for other solutions may well uncover hitherto unknown invariants! Particularly, new solutions to the TYBE recently reported in [3, 4] warrant investigation in this context.

Acknowledgements

Jon Links is grateful to the Australian Research Council for financial support and the cherry blossoms of “Hanami” in Kyoto, April 2000 for inspiration.

David De Wit’s research at Kyoto University is funded by a Postdoctoral Fellowship for Foreign Researchers (# P99703), provided by the Japan Society for the Promotion of Science. Dōmo arigatō gozaimashita!

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