A New Perspective on Path Integral Quantum Mechanics in Curved Space-Time

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Reference: D. Singh and N. Mobed, arXiv:1008.1451 [gr-qc] (under review).
1 Introduction and Motivation

- In the search for a self-consistent quantum theory of gravity over the past *seventy years* or more, there now exist many *distinct avenues* available to pursue.

- A partial list of various *inequivalent approaches*:
  - *String Theory* (Witten, Polchinski, Horowitz, etc.),
  - *Loop Quantum Gravity* (Ashtekar, Smolin, Rovelli, etc.),
  - *Twistor Theory* (Penrose),
  - *Causal Set Theory* (Sorkin),
  - *Regge Calculus* (Williams),
  - *Causal Dynamical Triangulations* (Loll),
  - *Noncommutative Geometry* (Connes),
  - etc.
• Despite their differences in conceptual and computational details, all these approaches claim that quantum gravity is only relevant at the Planck length scale of $10^{-33}$ cm, some twenty orders of magnitude smaller than the effective radius of a proton.

• Given this dilemma, it is worthwhile to ask if there are more indirect and modest means to seek out quantum gravity, but strongly driven by a simple desire to acquire some readily identifiable predictions involving established theories of gravitation and quantum mechanics when put under extreme conditions.
• It is an undeniable fact that the **Feynman path integral approach to quantum mechanics (QM)** has made significant contributions towards the present-day understanding of theoretical physics:
  - subatomic particles,
  - condensed matter,
  - statistical mechanics, etc.

• Arguably, this is **not the simplest approach** for solving practical problems in QM.

• However, it provides a **naturally intuitive and physically insightful overview** about the relationship between **classical and quantum phenomena** within a **unified mathematical framework**.
Figure 1: A “sum-over-histories” view about QM, in which all possible classical paths combine to yield the extremized path of classical mechanics plus small quantum fluctuations.
Because of the conceptual overlap between classical and quantum mechanics via the path integral, this approach has obvious implications for better understanding QM in curved space-time.

While the complete lack of physical data as a motivating force to provide direction makes any approach to quantum gravity problematic, many useful directions have been explored—including path integral methods—to develop the mathematical machinery to make the effort worthwhile.
Relevant Questions About Path Integrals in Quantum Gravity:

To what extent does the **curved space-time manifold** have a **mathematically smooth structure** when applying the path integral approach?

Is it possible to effectively perform the sum-over-histories when the **intermediate classical paths deviate significantly from a classical geodesic**?

Is the presence of **space-time torsion necessarily required** to correctly define the path integral in curved space-time?
Relevant Questions About Path Integrals in Quantum Gravity:

Are the intermediate classical paths required to preserve local causality or should they be free to trace out causality-violating worldlines, including ones that imply propagation into the local past?

To what extent can a coarse-grained skeletonized form of the path integral in curved space-time be identified with a truly continuum form in the limit as the finitely chosen time step becomes infinitesimally small?
A New Perspective on Path Integral QM in Curved Space-Time:

- Instead of following the “standard approaches” on path integral QM established by DeWitt et al., as found in the literature, the approach taken here is fundamentally different, while simultaneously adhering to Feynman’s original approach as much as possible.

- It employs Fermi or Riemann normal co-ordinates, but does so within the context of how position states are identified within a locally curved background setting.

- A scalar particle propagator is derived that, while satisfying the expected form in flat space-time, reveals interesting new physical predictions that are not readily evident in the more standard approaches found within the literature.
2 A Representation of Position State Vectors in Normal Co-ordinate Frames

• Fermi or Riemann Normal Co-ordinates: \( x^\mu = (\tau, x(\tau)) \)

• Orthonormal Tetrad: \( \bar{e}^\mu_\nu = \delta^\mu_\nu + \tilde{R}^\mu_\nu, \quad \tilde{R}^\mu_\nu \ll \delta^\mu_\nu. \)

• For Fermi Normal Co-ordinates:
  \[
  F \tilde{R}^\mu_\nu = \left[ \frac{1}{2} F R^\mu_{lm0}(\tau) \delta^0_\nu + \frac{1}{6} F R^\mu_{lmk}(\tau) \delta^k_\nu \right] \delta x^l \delta x^m
  \]

• For Riemann Normal Co-ordinates:
  \[
  R \tilde{R}^\mu_\nu = \frac{1}{6} R R^\mu_{\alpha\beta\nu}(\tau) \delta x^\alpha \delta x^\beta
  \]

• The \( \delta x^\mu \) denote space-time quantum fluctuations:
  \(|\delta x^\mu| \ll |x^\mu|\).
• Metric Tensor: \[ g_{\mu\nu} = \eta_{\alpha\beta} \bar{e}^\alpha \mu \bar{e}^\beta \nu \approx \eta_{\mu\nu} + 2 \tilde{R}_{(\mu\nu)} . \]

• Position Ket Vector: \[ |x^\mu\rangle = |(\tau, x)\rangle . \]

• Expression for the position ket vector defined in a local Lorentz frame:

\[
|X_{(G)}^\mu(\tau, x)\rangle = \bar{e}^\mu \nu x^\nu \rangle = U_{\text{Proj.}}(\tau, x) |x^\mu\rangle
\]

\[
U_{\text{Proj.}}(\tau, x) = 1 + \frac{i}{\hbar} \tilde{R}_{\beta\alpha} x^\alpha p^\beta ,
\]

an operator to project local space-time curvature onto a local tangent space, in terms of position and canonical momentum operators \( x^\alpha \) and \( p^\alpha \).
Figure 2: A projection operator $U_{\text{Proj.}}(\tau, \mathbf{x})$ transforms the position ket vector $|x^\mu\rangle$ into $|\hat{X}^{\mu}_{(G)}(\tau, \mathbf{x})\rangle = U_{\text{Proj.}}(\tau, \mathbf{x}) |x^\mu\rangle$ on a local tangent space defined at $\tau$. 
3 A New Proper Time Translation Operator in Curved Space-Time

- Consider the geometric description of Lie transport as applied to $|X^\mu(\tau, x)\rangle_G$.

- For infinitesimal proper time translation $\tau \rightarrow \tau + \Delta \tau$:

$$
|X'(\tau + \Delta \tau, x + \Delta x)\rangle \\
= |X(\tau + \Delta \tau, x + \Delta x)\rangle - \Delta \tau [(\mathcal{L}_\xi X)^\mu(\tau, x)] \\
= |X(\tau + \Delta \tau, x + \Delta x)\rangle \\
\quad - \frac{i}{\hbar} \Delta \tau (\mathcal{L}_\xi X)^\nu P^\nu |X^\mu(\tau, x)\rangle
$$
\[
\left| X_{(G)}^{\hat{\mu}} (\tau + \Delta \tau, x + \Delta x) \right\rangle = \left| \tilde{e}_{\nu}^{\hat{\mu}} (x^\alpha + \Delta x^\alpha) \right| (x^\nu + \Delta x^\nu) \right\rangle
\]

\[
(\mathcal{L}_\xi X)^{\hat{\mu}} = \xi^{\hat{\nu}} (\nabla_{\hat{\nu}} X^{\hat{\mu}}) - X^{\hat{\nu}} (\nabla_{\hat{\nu}} \xi^{\hat{\mu}})
\]

- Allow for \((\mathcal{L}_\xi X)^{\hat{\mu}} \to V^0 (\mathcal{L}_\xi X)^{\hat{\mu}}\) to demonstrate \textbf{path integral invariance under reparametrization} of \(\tau\).
- \(V^0\) is the time component of the four-vector \(V^\mu = \Delta x^\mu / \Delta \tau\), and serves as the \textbf{lapse function}.
- Therefore:

\[
\left| X_{(G)}^{\prime \mu} (\tau + \Delta \tau, x + \Delta x) \right\rangle = U_{\Delta \tau} (V^\alpha, \xi^\alpha) \left| X_{(G)}^{\mu} (\tau, x) \right\rangle
\]
A New Perspective on Path Integral Quantum Mechanics in Curved Space-Time

\[ U_{\Delta \tau}(V^\alpha, \xi^\alpha) = 1 - \frac{i}{\hbar} V^0 \Delta \tau \left\{ \left[ \delta^0_\alpha - F(\tilde{R})_\alpha\beta \frac{V^\beta}{V^0} + (\mathcal{L}_\xi X)_\alpha \right] \right. \]

\[ + \frac{i}{\hbar} \eta_{\alpha\beta} \tilde{R}_{\lambda\sigma} x^\sigma \frac{V^\beta}{V^0} P^\lambda \right\} P^\alpha, \]

\[ F(\tilde{R})_{\alpha\beta} = \left( \tilde{R}_{\alpha\beta,\mu} + \frac{1}{2!} \tilde{R}_{\alpha\sigma,\beta\mu} x^\sigma \right) \Delta x^\mu \]

\[ + \left( \frac{1}{2!} \tilde{R}_{\alpha\beta,\mu\nu} + \frac{1}{3!} \tilde{R}_{\alpha\sigma,\beta\mu\nu} x^\sigma \right) \Delta x^\mu \Delta x^\nu \]

\[ + O ((\Delta x)^3). \]

- Note that \( U_{\Delta \tau}(V^\alpha, \xi^\alpha) \) is not unitary:

\[ U_{\Delta \tau}^{-1}(V^\alpha, \xi^\alpha) = U_{-\Delta \tau}(V^\alpha, \xi^\alpha) \neq U_{\Delta \tau}^{\dagger}(V^\alpha, \xi^\alpha) \]
4 Configuration Space Path Integral in Curved Space-Time

- Consider determining a **scalar particle propagator** in terms of an initial position ket vector: \( \left| X^{(G)}_{(i)}(\tau_i, x_i) \right\rangle \)
and final position ket vector:

\[
\left| X^{(G)}_{(f)}(\tau_f, x_f) \right\rangle = U_{(\tau_f - \tau_i)}^{-1}(V^\alpha, \xi^\alpha) \left| X^{(G)}_{(f)}(\tau_i, x_i) \right\rangle,
\]

\[
\tau_f - \tau_i = N \Delta \tau
\]
• **Scalar Particle Propagator:**

\[
\langle X'_{(f)}^{(G)}(\tau_f, x_f) \mid X_{(i)}^{(G)}(\tau_i, x_i) \rangle = \langle X_{(f)}^{(G)}(\tau_i, x_i) \mid U_{N \Delta \tau}^{-1 \dagger} \mid X_{(i)}^{(G)}(\tau_i, x_i) \rangle \\
= \langle X_{(f)}^{(G)}(\tau_i, x_i) \mid \left( \prod_{n=1}^{N} 1_{(n)} U_{\Delta \tau}^{-1 \dagger} \right) 1_{(0)} \mid X_{(i)}^{(G)}(\tau_i, x_i) \rangle,
\]

\[
1_{(n)} = \int_{-\infty}^{\infty} d^3 X_{(n)} \langle X_{(n)}(\tau_n, x_n) \rangle \langle X_{(n)}(\tau_n, x_n) \rangle
\]
\[
\left\langle X_{(0)} \left| X_{(i)}^{(G)} (\tau_i, x_i) \right.\right\rangle = \int_{-\infty}^{\infty} \frac{d^3 P_{(0)}}{(2\pi \hbar)^3} \left[ 1 + \frac{i}{\hbar} \tilde{R}_{ij} (\tau_i, x_i) X_j^{(i)} P_i^{(0)} \right] \\
\times \exp \left[ \frac{i}{\hbar} (X_{(0)} - X_{(i)}) \cdot P_{(0)} \right] \\
\approx \delta^3 \left( X_{(0)} - \left[ X_{(i)} - \tilde{R}_{ij} (\tau_i, x_i) X_j^{(i)} \hat{x}^i \right] \right),
\]

\[
\left\langle X_{(f)}^{(G)} (\tau_i, x_i) \left| X_{(N)} \right.\right\rangle \approx \delta^3 \left( X_{(N)} - \left[ X_{(f)} - \tilde{R}_{ij} (\tau_i, x_i) X_j^{(f)} \hat{x}^i \right] \right).
\]

- Assume a **Hamiltonian** of the form: \( H(P) = \sqrt{m^2 + P \cdot P} \).
- **Reduced Compton Wavelength**: \( \lambda = \hbar/m \).
It is possible to integrate out the intermediate momentum states exactly to first-order in $\tilde{R}_{\mu\nu}$, by using the integral form of the modified Bessel function:

$$K_{\pm \nu}(\mu\beta) = \frac{\beta-\nu}{2} e^{-i\nu\pi/2} \int_0^\infty dN \, N^{\nu-1} \exp \left[ \frac{i\mu}{2} \left( N - \frac{\beta^2}{N} \right) \right],$$

for $\nu = 1/2$, with $\text{Im}(\mu) > 0$ and $\text{Im}(\mu\beta^2) < 0$.

Assume that $\text{Im}(\Delta\tau) \lesssim 0$ and identify:

$$\mu = -\frac{V_0 \Delta\tau}{\chi},$$

$$\beta = -\frac{i}{m} \left[ 1 - G(\tilde{R})_{0\alpha} \frac{V^\alpha}{V_0} + (\mathcal{L} \xi \mathbf{X})_0 + \frac{i}{\hbar} \left( \tilde{R}_{0\alpha} x^\alpha \frac{V_j}{V_0} - \tilde{R}_{j\alpha} x^\alpha \right) P^j \right] \times \sqrt{P \cdot P + m^2}.$$

$$G(\tilde{R})_{\mu\nu} = \mathcal{F}(\tilde{R})_{\mu\nu} + \tilde{R}_{\mu\nu} + \tilde{R}_{\mu\alpha,\nu} x^\alpha + (\tilde{R}^\alpha_{\alpha}) \eta_{\mu\nu}.$$
It is relatively straightforward to demonstrate that when
\[ V^0 \Delta \tau \to d\tau, \quad (V^0 \Delta \tau)^{-1} \to \delta(0), \quad \text{and} \]
\[ V^\mu / V^0 \to \dot{x}^\mu (\tau) = (1, \dot{x}(\tau)), \] the integration measure for the skeletonized path integral becomes

\[
\lim_{N \to \infty} \left( \frac{1}{2\pi \lambda i V^0 \Delta \tau} \right)^{3N/2} \prod_{n=0}^{N} \int_{-\infty}^{\infty} d^3 X_{(n)} \to \int \mathcal{D}[X(\tau)],
\]

and the configuration space propagator in curved space-time becomes

\[
\langle X'_{(f)}^{(G)}(\tau_f, x_f) \mid X^{(G)}_{(i)}(\tau_i, x_i) \rangle = \int \mathcal{D}[X(\tau)] \exp \left[ \frac{i}{\hbar} \int_{\tau_i}^{\tau_f} d\tau L^{(G)}_{(Re,0)} \right]
\]
\[\times \exp \left[ \frac{i}{\hbar} \int_{\tau_i}^{\tau_f} d\tau \left( L^{(G)}_{(Re,1)} + i \left[ L^{(G)}_{(Im,0)} + \lambda \delta(0) L^{(G)}_{(Im,1)} \right] \right) \right].\]
• Real Contributions to the Free-Particle Lagrangian in Curved Space-Time:

\[
\begin{align*}
L^{(G)}_{(\text{Re},0)} &= -m \left[ 1 - \frac{1}{2} \left\{ 2 G(\tilde{R})_{(00)} + 4 G(\tilde{R})_{(0j)} \dot{x}^j \right. \right. \\
&\quad + \left[ \eta_{ij} + 2 G(\tilde{R})_{(ij)} \right] \ddot{x}^i \ddot{x}^j \left. \right\} \right] \\
&\approx -m \left[ -g^{(\text{eff.})}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right]^{1/2}, \quad g^{(\text{eff.})}_{\mu\nu} = \eta_{\mu\nu} + 2 G(\tilde{R})_{(\mu\nu)},
\end{align*}
\]

\[
\begin{align*}
L^{(G)}_{(\text{Re},1)} &= -m \left\{ (L_{\xi X})_{\mu} + 2 \left[ (L_{\xi X})_0 \delta^0_\mu - G(\tilde{R})_{0\mu} \right] \eta_{ij} \dot{x}^i \dot{x}^j \right\} \\
&\quad \times \dot{x}^\mu.
\end{align*}
\]
• Imaginary Contributions to the Free-Particle Lagrangian in Curved Space-Time:

\[
\begin{align*}
L^{(G)}_{(\text{Im,0})} &= m \left\{ \lambda \left[ (\mathcal{L}_\xi X)^\alpha,_{\alpha} - \mathcal{H}(\tilde{R})_\alpha \dot{x}^\alpha \right] \\
&\quad + \frac{1}{2\lambda} \tilde{R}_{k\alpha} x^\alpha \dot{x}^k \left( 1 - 3 \eta_{ij} \dot{x}^i \dot{x}^j \right) \\
&\quad - \frac{1}{\lambda} \tilde{R}_{0\alpha} x^\alpha \left( 1 - 3 \eta_{ij} \dot{x}^i \dot{x}^j \right) \left( 1 - \eta_{kl} \dot{x}^k \dot{x}^l \right) \right\}, \\
L^{(G)}_{(\text{Im,1})} &= m \left\{ \frac{3}{2} \left[ (\mathcal{L}_\xi X)_0 - \mathcal{G}(\tilde{R})_{00} \right] + \frac{5}{2} \left[ (\mathcal{L}_\xi X)_j - \mathcal{G}(\tilde{R})_{j0} \right] \dot{x}^j \\
&\quad - \frac{5}{4} \left[ \eta_{ij} + 2 \mathcal{G}(\tilde{R})_{(ij)} \right] \dot{x}^i \dot{x}^j \\
&\quad + \left[ (\mathcal{L}_\xi X)_0 - \mathcal{G}(\tilde{R})_{0\mu} \dot{x}^\mu \right] \eta_{ij} \dot{x}^i \dot{x}^j \right\}, \\
\mathcal{H}(\tilde{R})_\mu &= \mathcal{F}(\tilde{R})^\alpha_{\mu,\alpha} + \tilde{R}^\alpha_{\alpha,\mu}. 
\end{align*}
\]
5 Physical Consequences for the Configuration Space Path Integral

- Already at a purely formal level, the configuration space path integral correctly yields the free-particle propagator in the limit as space-time curvature vanishes, while simultaneously revealing some very interesting predictions.
• While, all the curvature-dependent terms that correspond to the conservation of probability also satisfy the weak equivalence principle, at least to leading order in $\bar{\lambda}$, all the probability violating contributions due to curvature result in a direct coupling of $\bar{\lambda}$ with the gravitational background.

• This is a quantum violation of the weak equivalence principle at the Compton wavelength scale.

• This detail also indicates a breakdown of time reversal symmetry in the scalar propagator under the interchange of $\tau_i \leftrightarrow \tau_f$, providing a potentially satisfactory explanation as to why there exists a preference for time to propagate in the forward direction only.
Regulation of the Path Integral in Cartesian Co-ordinates:

- Explicit evaluation of the **scalar particle propagator in skeletonized form** is a straightforward exercise involving multiple Gaussian integrations with respect to $\prod_{n=1}^{N-1} d^3 X(n)$.

- However, it is necessary to then **regularize the propagator** in order to remove all of its **singular contributions** in the limit as $V^0 \to 0$. 
• Normally, this involves describing the skeletonized propagator according to the ansatz

\[
\left\langle X^{(G)}_{(f)}(\tau_f, x_f) \right| X^{(G)}_{(i)}(\tau_i, x_i) \right\rangle_{\text{reg.}} \equiv \left( \frac{1}{2\pi \lambda i V^0 (\tau_f - \tau_i)} \right)^{3/2} \\
\times \exp \left[ \frac{i}{2} \frac{m}{\hbar} \left( \eta_{\mu \nu} \Delta X^\mu_{(i \rightarrow f)} \Delta X^\nu_{(i \rightarrow f)} \right) \right] \sum_{k=0}^\infty a_k(x^\mu_i, x^\mu_f) (i V^0)^k,
\]

where \( \Delta X^\mu_{(i \rightarrow f)} = X^\mu_{(f)} - X^\mu_{(i)} \) and \( a_k(x^\mu_i, x^\mu_f) \) are the curvature-dependent Seeley-DeWitt coefficients, whose values are determined from solving the heat kernel equation.

• For this skeletonized propagator, it is unnecessary to compute the Seeley-DeWitt coefficients, since the propagator prior to the Gaussian integrations can easily be put into power series form, such that all inverse powers of \( V^0 \) are identified by inspection alone and subsequently removed by hand.
\[
\left\langle X'_{(f)}(\tau_f, \mathbf{x}_f) \right| X_{(i)}^{(G)}(\tau_i, \mathbf{x}_i) \right\rangle_{\text{(reg.)}} = \\
\lim_{N \to \infty} \left( \frac{1}{2\pi \lambda} \right)^{3/2} \frac{1}{\sqrt{V^0(\tau_f - \tau_i)}} \exp \left[ \frac{i}{\hbar} \frac{\eta_{\mu\nu} \Delta X^\mu_{(i\to f)} \Delta X^\nu_{(i\to f)}}{V^0(\tau_f - \tau_i)} \right] \times \\
\sum_{k=0}^{\infty} \sum_{n=1}^{N} \frac{1}{k!} \left\{ \begin{array}{c} C^{(k,0)}_{(n)}(x^\mu) + \sum_{l=1}^{4} \frac{k!}{(k + l)!} \left( -\frac{i}{2N} \right)^l C^{(k+l,l)}_{(n)}(x^\mu) \\
+ \exp \left[ \frac{n}{N} \Delta X^\alpha_{(i\to f)} \frac{\partial}{\partial x^\alpha} \right] \left[ C^{(k,0)}_{(n)}(x^\mu) + \sum_{l=1}^{2} \frac{k!}{(k + l)!} \left( -\frac{i}{2N} \right)^l C^{(k+l,l)}_{(n)}(x^\mu) \right] \right. \right. \\
\times \left. \left. \left( \frac{-i V^0 \Delta \tau}{2\lambda} \right)^k \exp \left[ -\frac{i}{\hbar} \frac{\tilde{R}_{ij}(\tau_i, \mathbf{x}_i) \Delta X^i_{(i\to f)} \Delta X^j_{(i\to f)}}{V^0(\tau_f - \tau_i)} \right] \right. \right. \\
N \Delta \tau = \tau_f - \tau_i \equiv 1.
\]

- The overall phase can be identified as a gravitational analogue of the Aharonov-Bohm effect and Berry’s phase.
To leading-order in curvature, the coefficients $C^{(k, l)}_{(n)}(x^\mu)$ and $C^{(k, l)}_{(\mathcal{L}, n)}(x^\mu)$—the latter of which are proportional to $(\mathcal{L}_\xi X)^\mu_\tau(x_1, x_i)$—are generally complex-valued.

For $l = 0$:

$$C^{(k, 0)}_{(n)}(x^\mu) = 1 - \left(2k - \frac{3}{2}\right) \mathcal{G}(\tilde{R})_{00} + 2i k \left(\lambda \mathcal{H}(\tilde{R})_0 + \frac{1}{\lambda} \tilde{R}_{00} \tau_{(n)} \right)$$

$$+ \frac{1}{\lambda} \tilde{R}_{0j} \left[ X^j_{(i)} + \frac{n}{N} \Delta X^j_{(i \rightarrow f)} \right],$$

$$C^{(k, 0)}_{(\mathcal{L}, n)}(x^\mu) = \left(2k - \frac{3}{2}\right) (\mathcal{L}_\xi X)_0(\tau_1, x_i) - 2i k \lambda (\mathcal{L}_\xi X)^\alpha_\alpha(\tau_1, x_i).$$
6 Conclusion

- This scalar particle propagator in a locally curved space-time background, following a fundamentally different approach than what currently exists in the literature.

- It reveals what appear to be very significant physical predictions with potentially broad implications concerning quantum mechanical interactions in a non-trivial gravitational field.

- It is worthwhile to consider further developments of this approach when applied to:
  - non-zero integer and half-integer spin particles,
  - many-body particles,
  - quantum field description while approaching a continuum limit.