Stochastic Representation of Quantum Interactions and Two-Level Systems

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Abstract

Stochastic representation for interaction of quantum systems is formulated which allows to replace some of them by equivalent but purely commutative random sources. The formalism is applied to two-level systems interacting with Gaussian thermal bath. Strong-coupling non-Markovian effects and besides long-living fluctuations in common susceptibility of two systems subjected to the same bath are considered.

1. Stochastic representation.

In quantum statistical physics and kinetics typically one deals with tasks about interaction between some “dynamical system” (DS, microscopical or with only a few degrees of freedom) and its macroscopic surroundings (“thermal bath”, or “thermostat”). Usually, an interaction Hamiltonian has bilinear structure (or can be reduced to that):

$$H_{\text{int}} = \sum_j B_j \ast D_j ,$$

where operators $D_j$ and $B_j$ act in different linear (Hilbert) spaces $D$ and $B$ (native spaces of DS and thermostat, respectively). In the Heisenberg picture, the operators $B_j(t) \equiv \exp(iH_b t)B_j\exp(-iH_b t)$ (with $H_b$ being Hamiltonian of autonomous thermostat) play like random perturbations with respect to DS. But, in opposite to classical Langevin forces, they are operator-valued non-commutative random sources and therefore withdraw the DS evolution into the immense direct product of the two spaces, $D \otimes B$.

Possibly, the task would become more simple, if $B_j(t)$ were replaced by equivalent commutative random sources, so that the DS evolution would remain in $D$. The equivalence means exact reproduction of results under interest, first of all, exact simulation of role of the thermostat temperature and and effects of DS self-action mediated by thermostat, in particular, dephasing and dissipation.

The variant of such a replacement was suggested in [1,5]. Let $H_d(t)$ be Hamiltonian of free DS. Introduce the notation

$$H(B, t) = H_d(t) + \sum_j B_j(t)D_j$$

This is joint Hamiltonian in the representation which is Heisenbergian with respect to the thermostat. Let some DS observables, $J_k$, are staying under actual observation (measurement). Then statistical operator of complete system “DS plus thermostat”, $R(t)$, undergoes the equation

$$\dot{R} = v(t)J \circ R + i \{ RH(B, t) - H(B, t)R \} ,$$

where $v(t)J \equiv \sum v_k(t)J_k$, with $v_k(t)$ being test functions of the observation and $\circ$ the symbol of symmetric product (Jordan product). At $v_k(t) = 0$, this is the usual von Neumann equation for joint density matrix. At $v_k(t) \neq 0$, the trace of $R(t)$ in $D \otimes B$ represents [1,5] the characteristic functional of the observables, $\Xi(t, v)$:

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\[ \text{Tr}_D \text{Tr}_B R = \text{Tr}_D \text{Tr}_B \exp \left[ \frac{1}{2} \int v(t') J(t') dt' \right] \times \exp \left[ \frac{1}{2} \int v(t') J(t') dt' \right] \]

Here \( R_{in} = R(-\infty) \), the presence of time argument in \( J_k(t) \) means that these operators are treated in the Heisenberg picture, left-hand (right-hand) oriented arrow indicates chronological (anti-chronological) time ordering, and the integrals are taken over the region \( t' < t \) (with \( t \) being present time moment). Let us replace the operators \( B(t) \) in Hamiltonian (2) by commutative (similar to \( C \)-numbers) variables \( \xi(t) = x(t) + iy(t)/2 \) or \( \eta(t) = x(t) - iy(t)/2 \) and, instead of Eq.3, consider the equation

\[ \dot{\rho}(t) = v(t) J \circ \rho(t) + L(t) \rho(t), \]

\[ L(t) \rho \equiv i \{ \rho H(\eta, t) - H(\xi, t) \rho \} = \sum_j y_j(t) D_j \circ \rho + i[\rho, H(x, t)], \]

treating all the \( \eta(t), \xi(t), x(t) \) and \( y(t) \) as random processes which reflect an influence by the thermostat. Assume that \( R_{in} \) has the factorized form: \( R_{in} = \rho(-\infty) \rho_b \), where \( \rho_b = \text{Tr}_D R(-\infty) \) is statistical operator of the thermostat. All of \( x(t) \) and \( y(t) \) are defined if their characteristic functional is defined. We make it as follows:

\[ \langle \exp \int [g(t)x(t) + f(t)y(t)] dt \rangle = \text{Tr}_B \exp \left\{ \frac{1}{2} \int g(t) B(t, f) dt \right\} \times \exp \left\{ \frac{1}{2} \int g(t) B(t, f) dt \right\} \rho_b, \]

(demonstrating formally equal rights of \( x(t) \) and \( y(t) \) on the right-hand side of Eq.7).

As it was shown in [1], if statistics of commutative random sources in Eqs.5 and 6 is determined by Eqs.7 and 8 then the sources exactly imitate real quantum thermostat. Concretely,

\[ \langle \rho(t) \rangle = \text{Tr}_B R(t), \]

\[ \text{Tr}_D \langle \rho(t) \rangle = \Xi(t, v), \]

where the angle brackets \( \langle ... \rangle \) denote statistical averaging with respect to \( x(t) \) and \( y(t) \) (or to \( \eta(t) \) and \( \xi(t) \)) in accordance with (7-8).

2. Properties of the random sources.

The pay for the desired replacement is the doubling of random sources: instead of each real observable (Hermitian operator) \( B_j(t) \) we obtain the pair of either real variables \( (x_j(t) \text{ and } y_j(t)) \)
or mutually conjugated complex variables \((\eta_j\text{ and } \xi_j = \eta_j^*)\). From Eqs.5-6 it is seen that \(x_j(t)\) play direct role of random forces (potentials), while \(y_j(t)\) stay in positions of test functions which correspond to observing DS by thermostat (in analogy with \(v_k(t)\) in Eq.3). In absence of \(y_j(t)\), under random pump introduced by \(x_j(t)\), both the energy and entropy of DS would grow as much as possible. However, like in any measurement process, influence of thermostat to the “stochastic Liouville operator” (6) destroys unitarity of evolution of the statistical operator \(\rho(t)\) and decreases phase volume (entropy) of DS. At the same time \(y_j(t)\) are responsible for the energy outflow back to thermostat, i.e. for dissipation, and hence for non-uniform (thermal) DS energy distribution over states of DS. It should be emphasized that according to (9-10) on the average the unitarity is ensured.

Due to Eqs.7-8,

\[
\left\langle \prod_j x(t_j) \prod_m y(t'_m) \right\rangle = \left[ \prod_m \delta \left( t'_m - t_j \right) \right] \left\langle \prod_j B(t_j, f) \right\rangle_{f=0}.
\]

Here from it is clear that \(\langle y(t) \rangle = 0\), \(\langle y(t) y(t') \rangle = 0\), and all the higher self-correlators of \(y(t)\) also are equal to zero. But the cross-correlators between \(y(t)\) and \(x(t)\) differ from zero representing the response of thermostat to its perturbation by DS. Thus, \(y(t)\) are not \(C\)-numbers in literal sense. But the singularity of statistical properties of \(y(t)\) only facilitates factual calculations. Importantly, \(y(t)\) can possess correlations with more late values \(x(t' > t)\) only, in correspondence with the causality principle.

In case of Gaussian equilibrium (“bosonic”) thermostat, all the functions (11) can be expressed through pair correlators

\[
\langle x_j(\tau)x_m(0) \rangle = \int_0^\infty \cos(\omega\tau)S_{jm}(\omega)\frac{d\omega}{\pi}, \quad (12)
\]

\[
\langle x_j(\tau)y_m(0) \rangle = 2\vartheta(\tau) \int_0^\infty \sin(\omega\tau) \times \times \tanh \left( \frac{\omega}{2T} \right) S_{jm}(\omega)\frac{d\omega}{\pi}
\]

Here \(T\) is temperature of thermostat, \(\vartheta(\tau)\) is Heavyside function, and \(S_{jm}(\omega)\) is a non-negatively defined spectral matrix. The formulas (12) show that neglecting the \(y(t)\)-component of random sources in (5-6) in essence would be equivalent to infinity of thermostat temperature.

In Eqs.9-10, only the average value of the “stochastic density matrix of DS”, \(\rho(t)\), does appear. Naturally, its higher statistical moments

\[
\langle \rho(t) \otimes \ldots \otimes \rho(t) \rangle
\]

describe several copies of DS interacting with the same thermostat. The undistinguishness of the copies require to concretize their quantum statistics (second quantization rule). In [1] we considered how the case of Fermi statistics reduces to analysing the moments (13), thus replacing second quantization in a “many-particle problem” by equivalent “second randomization” in one-particle problem. In this approach, an additional direct (e.g. Coulombian) interaction between samples of DS can be included as interaction through the second (Gaussian) thermostat, i.e. additional pairs of stochastic sources \(x(t)\), \(y(t)\).

3. Two-level system in Gaussian thermostat.

Let us apply the above formalism to two-level system (TLS) interacting with Gaussian thermal bath. We can put on

\[
J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (14)
\]

and \(H_d(t) = u(t)J/2\). Here \(u(t)\) is energy difference between two states, possibly depending on time because of external perturbation of TLS. The operator \(D\) performs contact with thermostat which forces random switchings of TLS.
state. The operator $J$ corresponds to observation of TLS state, for instance, spin orientation, or alternate velocity of quantum particle in the simplest model of Brownian motion. We use this model to illustrate role of the $y(t)$-component of random source, i.e. “dissipative” term $y(t)\rho(t)$ in Eqs.5-6.

First, let both states of free TLS have equal energies: $u(t) \equiv 0$. Then the Eqs.5-6 can be integrated up to rather visual expressions. In particular, for the trace

$$\theta(t,v) \equiv \text{Tr}_D \rho(t) = \rho_{11} + \rho_{22},$$

one can derive the recurrent equation

$$\theta(t,v) = Y(t, -\infty) +$$

$$+ \int Y(t,t_1)u(t_1)X(t_1,t_2)v(t_2)\theta(t_2,v)dt_1 dt_2,$$

$$X(t_1,t_2) \equiv \cos \left[ 2 \int_{t_2}^{t_1} x(t') dt' \right],$$

$$Y(t_1,t_2) \equiv \cosh \left[ \int_{t_2}^{t_1} y(t') dt' \right],$$

where $t > t_1 > t_2$, and $\Xi(t,v) = \langle \theta(t,v) \rangle$.

Consider the correlation function of spin fluctuations (or velocity fluctuations if speak about Brownian particle, or so on), $K(\tau) = \langle J(\tau)J(0) \rangle$, and the diffusivity $\Delta = \int^\infty_0 K(\tau)d\tau$. First of all the case is curious when the noise of thermostat is “white”, i.e. $S(\omega) = \text{const}$ in (12). To be precise, it should be underlined that both the correlators (12) can not simultaneously turn into $\delta$-functions, and in this sense quantum noise never can be white. After expanding Eq.15 into series, averaging it with taking into account Eqs.12 and then differentiating by the test function, one obtains

$$K(\tau) = e^{-2S} \cos \left\{ \frac{2S}{\pi} \int_0^\tau \ln \tanh \left( \frac{\pi T \tau'}{2} \right) d\tau' \right\}$$

(16)

Under increase of the order of magnitude of the dimensionless noise intensity, $S/T$, initially monotonous relaxation changes to more and more oscillating relaxation. From the formal point of view, this is just the effect of the $y(t)$-component, i.e. inverse influence by DS onto thermostat (indeed, in absence of $y(t)$ the correlation would be purely exponential independently on $S/T$).

However, practically the opposite approximation can occur more appropriate:

$$S(\omega) = S/[1 + (\tau_0 \omega)^2], \tau_0 T >> 1,$$

(17)

when sufficiently large correlation time of the thermostat noise (characteristic time of thermostat response), $\tau_0$, excludes high frequencies. Then it is convenient to introduce the coupling energy $\epsilon$ by means of $S(0) = 2\epsilon^2\tau_0$. Calculating the correlation function under assumptions (17), we find again that at weak coupling ($\epsilon/T < 1$) monotonous (nearly exponential) relaxation takes place, while at $\epsilon/T > 1$ (strong coupling)

$$K(\tau) \approx \exp(-2\epsilon^2\tau^2) \cos(\epsilon^2\tau/T),$$

(18)

that is again the oscillations arise and then become multiplied.

Clearly, oscillations of relaxation lead to an excess decrease of the diffusivity $\Delta$ (spectral power density of $J(t)$ at zero frequency) as compared with its value $\Delta_0$ which would realize in absence of $y(t)$. In the case of Lorenz noise with $T\tau_0 > 1$, the diffusivity can become extremely small resulting in spatial localization of Brownian particle.

The non-monotonous relaxation means that the TLS response to periodic external perturbation can acquire “resonant” character having maximum at some non-zero frequency of the perturbation (or, if the frequency is fixed, maximum at some non-zero level of thermostat noise). Such a kind of phenomena is known as “stochastic resonance” (see, for example, the review [2]).
We would like to emphasize also that oscillatory relaxation represents non-Marcovian effect which could not be adequately considered by the theory of Marcovian quantum dynamical semi-groups, i.e. time-local kinetic equations (this theory takes beginning from the classical work [3]). In terms of this theory, we considered “stochastic dilation” of a semi-group which is as much non-Marcovian as strongly the correlator \( \langle x(\tau)y(0) \rangle \) determining oscillations in (16) and (18) differs from \( \delta \)-function.

4. Stochastic linear response of TLS to non-stochastic perturbation.

Because of the symmetry (degeneration) of states of TLS, the trace (15) contains even powers of \( v(t) \) only. If \( u(t) \neq 0 \) then this symmetry breaks down. Define integral operators \( \hat{I} \), \( \hat{C} \) and \( \hat{S} \) by the formulas

\[
\hat{I} f(t) = \int_{-\infty}^{t} f(t') dt',
\]

\[
\hat{C}, \hat{S} f(t) = \int_{-\infty}^{t} \cos, \sin \left\{ \int_{t'}^{t} u(t'') dt'' \right\} f(t') dt',
\]

At \( u(t) \neq 0 \) instead of (15) the following equation can be derived:

\[
\theta = 1 + \hat{I} y \hat{C} y \theta + \hat{I} (v + 2 y \hat{S} x) [1 + 4 \hat{I} x \hat{C} x]^{-1} \hat{I} (v + 2 x \hat{S} y) \theta.
\]

Now, in the expansion of \( \theta \) into power series of \( v(t) \),

\[
\theta(t,v) = \theta_0(t) + \int_{-\infty}^{t} \bar{J}(t_\alpha)v(t_\alpha)dt_\alpha +
\]

\[
+ \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} \bar{K}(t_1,t_2)v(t_1)v(t_2)dt_1 dt_2 + \ldots,
\]

the first order term is present (as well as other odd terms). For simplicity, consider this contribution in the “hot thermostat limit”: \( S/T << 1 \), \( T\tau_0 >> 1 \), when in accordance with (12) any multiplier \( y(t) \) contributes excess order of smallness. Besides, assume the perturbation \( u(t) \) is infinitely weak. Then, saving lowest powers of \( y(t) \) and \( u(t) \) only, we can write

\[
\bar{J}(t_\alpha) = J_+(t_\alpha) + J_-(t_\alpha),
\]

\[
J_+ = \int_{-\infty}^{t_\alpha} \sin \left\{ 2 \int_{t_1}^{t_\alpha} x(t') dt' \right\} \times
\]

\[
\times u(t_1) \int_{-\infty}^{t_1} y(t_2) dt_2 dt_1,
\]

\[
J_- = \int_{t_\alpha}^{t} \sin \left\{ 2 \int_{t_1}^{t_\alpha} x(t') dt' \right\} u(t_1) \int_{t_1}^{t} y(t_2) dt_2 dt_1
\]

Of course, here the factual observation time \( t_\alpha \) is more early than formal running time \( t \). Under this condition, \( t \) drops out from any result of statistical averaging, therefore it may be convenient to set \( t = \infty \).

Obviously, \( \bar{J}(t_\alpha) \) disappears if we neglect \( y(t) \)-component of the bath noise. Consequently, this component (in company with \( x(t) \)) is responsible for non-zero response of DS to external perturbations. The integral operators which act onto \( u(t) \) in the expressions \( J_+ \) and \( J_- \) are nothing but random linear response functions (random susceptibility, or mobility, or so on), in other words, these operators represent multiplicative (parametric) noise. What is hidden beyond it is the randomness of quantum probabilities of quantum transitions, i.e. dependence of the probabilities on thermostat noise [4].

Formulas (20) clearly highlight that the response functions in their non-averaged form has no decay and can connect past and future events regardless of how far they are distanced in time.

5. Two TLS in shared thermostat.

Consider two identical TLS contacting with one and the same thermostat. Suppose that the systems are formally distinguishable and that they give additive contributions to the variable under observation. Then characteristic functional of the summary observable is presented by
\[ \Xi(t, v) = \langle \theta^2(t, v) \rangle, \quad (21) \]

instead of (10). For its correlation function, to be designated as \( K_2 \), in the hot thermostat limit the Eq.(21) yields

\[ K_2(t_1, t_2) = 2 \{ \langle \tilde{K}(t_1, t_2) \rangle + \langle J_+(t_1) J_-(t_2) \rangle + \langle J_+(t_1) J_+(t_2) \rangle - 2 \langle J_+(t_1) \rangle \langle J_+(t_2) \rangle \}, \quad (22) \]

where \( t_1 > t_2 \) is mentioned.

We do not discuss the first term on the righthand side of Eq.(22). What is for the next terms, they describe “excess noise” of summary observable which is proportional to squared perturbation \( u^2 \) and comes just from the fluctuations in random linear responses (similar in both TLS and determined by Eqs.20). If \( t_1 - t_2 >> \tau_c \), where \( \tau_c \) is correlation time of the fast component of the \( J(t) \) fluctuations (that is decay time of equilibrium correlation function \( K(\tau) \) at \( u(t) = 0 \)), then a contribution from the cross-correlator of \( J_+ \) and \( J_- \) does survive only, and further it does not decay at all. A rough formal estimate of these long-living fluctuations by order of magnitude gives

\[ | \langle J_+(\infty) J_-(\infty) \rangle | \sim (\tau_c^2/\tau_0^2) \langle \bar{J} \rangle^2 \]

Here the ratio \( \tau_c/\tau_0 \) crucially depends on the parameter \( \epsilon \tau_0 \) and can be much greater than unit (at \( \epsilon \tau_0 << 1 \) ) as well as much smaller than unit (at \( \epsilon \tau_0 >> 1 \) ).

Interestingly, the expression \( J_-(t_0) \) in (20) corresponds to “reaction of the past on the future” because it includes future values of all the variables only with respect to the observation time. With no doubts, any correlator in which time argument of \( J_- \) is most late should turn into zero, otherwise it would have no physical sense. However, non-zero correlations between \( J_- \) and more late \( J_+ \) does not contradict to causality principle.

The non-decaying tails of such correlations in two-TLS system are quite similar to properties of fluctuations of quantum transport probabilities in many-electron system considered in [4]. Their physical origin is the coherence (unitarity and time reversibility) of joint evolution of DS and thermostat: the coherence can be interrupted by external observation acts but conserves between them. What is important, unlike [4], in the present example formal analysis of non-decaying correlations is not restricted by any time frames.

6. Appendix. Generalized stochastic representation.

The text above almost coincides with the paper in Russian [5] (to be translated into English in JETP Letters). Since the proof of the basic relations (9),(10) is omitted in [5] (see [1] for it), below we compensate this by proving generalized stochastic representation which includes the above one as a particular case.

Let the joint evolution (Liouville) operator, \( L \), of combined system “DS plus thermostat” has the bilinear form as follows:

\[ L = L_D + L_B + \sum_j \Lambda_{Dj} \Lambda_{Bj}, \quad (23) \]

where \( L_D \) and \( L_B \) are evolution (Liouville) operators of autonomous DS and thermostat, respectively, and \( \Lambda_{Dj} \) and \( \Lambda_{Bj} \) are some superoperators acting in \( D \) and \( B \), respectively (to be precise, they act in the spaces of linear operators defined in \( D \) and \( B \) ). For instance, the superoperators \( \Lambda_{Dj} \) and \( \Lambda_{Bj} \) can be Liouville operators (commutators) or multiplication operators (Jordan products). For shortness, assume that the terms \( \sum v_k(t) J_k \) corresponding to external watching for DS are included into \( L_D \).

Notice, that, from one hand, if \( \alpha(t) \) is any random process with characteristic functional

\[ \Psi\{f\} = \exp \left[ \int f(t) \alpha(t) dt \right] \quad (24) \]

(angle brackets mean averaging over \( \alpha \)’s probability distribution), then the average of any functional \( \Phi\{\alpha\} \) can be formally represented by

\[ \langle \Phi\{\alpha\} \rangle = \Psi \left\{ \frac{\delta}{\delta \alpha} \right\} \Phi\{\alpha\} \bigg|_{\alpha=0} \quad (25) \]
Similar relation holds also for a set of random processes \( \alpha_j(t) \).

From another hand, since all of \( L_D \) and \( \Lambda_{Dj} \) commute with all of \( L_B \) and \( \Lambda_{Bj} \), we can write:

\[
\hat{\exp}\left\{ \int \left[ L_D + L_B + \sum \Lambda_{Dj} \Lambda_{Bj} \right] dt \right\} = \hat{\exp}\left\{ \int \left[ L_B + \sum \Lambda_{Bj} \frac{\delta}{\delta \alpha_j(t)} \right] dt \right\} \times
\]

\[
\times \hat{\exp}\left\{ \int \left[ L_D + \sum \alpha_j(t) \Lambda_{Dj} \right] dt \right\}_{\alpha(t)=0}
\]

Here mutual chronological ordering of the two chronological exponents in their product is taken in mind. This is merely the consequence from chronological ordering and besides from the formal identity

\[
\exp(O_1O_2) = \exp\left( O_1 \frac{\partial}{\partial \alpha} \right) \exp(\alpha O_2) \bigg|_{\alpha=0},
\]

where \( O_1 \) and \( O_2 \) are arbitrary mutually commuting objects: \( [O_1, O_2] = 0 \).

Suppose that initial statistical operator \( R_{in} \) factorizes, \( R_{in} = \rho(-\infty) \rho_b \) where \( \rho \) stands for DS density matrix. Then, comparing formulas (25) and (26), we conclude, firstly, that from the point of view of DS its joint evolution together with thermostat is equivalent to its individual evolution but influenced by the set of random sources \( \alpha_j(t) \) and governed by stochastic evolution operator as follows:

\[
\dot{\rho} = \left\{ v(t)J \circ \rho + L_D + \sum \alpha_j(t) \Lambda_{Dj} \right\} \rho
\]

(27)

(we extracted external DS observation back from \( L_D \)). Secondly, in analogy with (24) and (25), characteristical functional of these sources is determined by

\[
\left\langle \exp\left[ \int \sum f_j(t) \alpha_j(t) dt \right] \right\rangle = \Tr_B \hat{\exp}\left\{ \int \left[ L_B + \sum f_j(t) \Lambda_{Bj} \right] dt \right\} \rho_b
\]

\[
= \Tr_B \hat{\exp}\left\{ \int \left[ L_B + \sum f_j(t) \Lambda_{Bj} \right] dt \right\} \rho_b
\]

Formulas (27) and (28) give the stochastic representation of the joint evolution. In fact, it is obtained by nothing but a variant of the Stratonovich transformation. Dependently on concrete contents of operators \( \Lambda_{Dj} \) and \( \Lambda_{Bj} \) the random sources \( \alpha_j(t) \) can behave either similar to usual classical random processes (like \( x(t) \) above) or in a singular manner (like \( y(t) \) above) or in some mixed fashion. Of course, in general all the sources are mutually correlated.

In particular, the above considered case of the bilinear interaction Hamiltonian (1) can be reduced to the representation (27)-(28) or derived from it if take into account the identity

\[
[R, DB] = [R, D] \circ B + [R, B] \circ D
\]

(29)

which is valid for any pair of mutually commuting \( D \) and \( B \), at \( [D, B] = 0 \). Hence, every pair \( D_j \) and \( B_j \) from (1) produces two terms in (23). Clearly, in the corresponding pair of \( \alpha \)'s one conjugated with the first term on the right-hand side of (29) plays the role of \( x(t) \) (see Sec.1), while another (conjugated with second term) plays the role of \( y(t) \).

It is useful to demonstrate a simple case which does not reduce to bilinear interaction (1). Concretely, let the joint evolution operator \( L \) is defined by

\[
LR = (L_D + L_B)R + iD' \circ [R, B] + iB' \circ [R, D],
\]

(30)

where \( D' \) differs from \( D \), \( B' \) differs from \( B \), and \( L_{D,B}R = i[R, H_{d,b}] \). Then, instead of the representation expressed by Eqs.5-8, we arrive to the representation

\[
\dot{\rho} = v(t)J \circ \rho + y(t)D' \circ \rho + i[\rho, H_d + x(t)D],
\]

(31)

\[
\left\langle \exp\left[ g(t)x(t) + f(t)y(t) \right] dt \right\rangle = \Tr_B \hat{\exp}\left\{ \frac{1}{2} \int g(t)B'(t,f) dt \right\} \times
\]

(32)

(33)
\[ \times \exp \left\{ \frac{1}{2} \int g(t)B'(t, f)dt \right\} \rho_b , \]

\[ B'(t, f) = U^+(t, f)B'U(t, f) , \quad (33) \]

\[ \dot{U}(t, f) = -i\{H_b + f(t)B\}U(t, f) \]

Now the two random sources involve not two but four operators \( D, D', B \) and \( B' \). Nevertheless, again it is possible to separate analysis of thermostat, under given classical perturbation, and analysis of DS under influence by commutative stochastic sources with given statistics (and again statistical properties of a number \( N \) of DS copies interacting with the same thermostat can be obtained from the moments \( \langle (\text{Tr}_B \rho)^N \rangle \), if the copies are distinguishable and undergo Boltzmann statistics, or as it was considered in [1], if the copies undergo Fermian second quantization statistics). Concrete application of this formal example will be done elsewhere.

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