Estimation of the $r$-th derivative of a density function by the tilted kernel estimator

Abstract

We consider the problem of estimating the $s$-th derivative of a density function $f$ by the tilted Kernel estimator introduced in [Hall and Doosti (2012)]. Then we further show this estimator achieves the same convergence rate, in probability, the wavelet estimators achieved as shown in [Hall and Patil (1995)]. That is, the convergence rate of $O_p\left( n^{-\frac{r-s}{2r+1}} \right)$.

1 Tilted Kernel estimators for derivative

1.1 Settings and assumptions

Suppose we have an i.i.d sample of random variables $X_1, X_2, \ldots, X_n$. Hall and Doosti (2012) introduced the tilted Kernel estimator $\hat{f}_n(x)$ as

$$\hat{f}_n(x) = \frac{1}{h} \sum_{i=1}^{n} p_{i,n} K \left( \frac{x - X_i}{h} \right),$$

where $\sum_{i=1}^{n} p_{i,n} = 1$, $K$ is bounded, symmetric and compactly supported. Note that the conventional Kernel estimator is a special case of when $p_{i,n} = \frac{1}{n}$ for all $i = 1, 2, \ldots, n$.

Observe that $p_n = \{p_{1,n}, p_{2,n}, \ldots, p_{n,n}\}$ forms a probability measure on $n$ points for $n \in \mathbb{N}$, let us agree to call the family $\{p_n : n \in \mathbb{N}\}$ the family of probability measures associated to $\hat{f}_n$.

Suppose we know the density function $f$ is $r$ times differentiable, it would be intuitive to just take the $s$-th derivative of $\hat{f}_n$ to be the estimator of $f^{(s)}$, for $s \leq r$, namely,

$$\hat{f}_n^{(s)}(x) = \frac{1}{h} \sum_{i=1}^{n} p_{i,n} \frac{\partial^s}{\partial x^s} K \left( \frac{x - X_i}{h} \right).$$
1.2 Selection of the associated probability measures

In Hall and Doosti (2012), the authors selected the associated probability measures to be, for \( n \in \mathbb{N} \),

\[
p_{i,n} = \frac{1 + h_n^2 g(X_i)}{n(1 + \Delta)}, \quad \text{where} \quad \Delta_n = \frac{h_n^2}{n} \sum_{i=1}^{n} g(X_i), \quad i = 1, 2, \ldots, n, \tag{1.3}
\]

with \( h_n \) the bandwidth and \( g \) defined as follow,

\[
g = \frac{1}{2} f'' + h^2 e_2 f + \ldots + h^{2r-2} e_{2r-2} \frac{f}{f}. \tag{1.4}
\]

The functions \( e_2, e_4, \ldots, e_{2r-2} \) are obtained recursively as in Hall and Doosti (2012). Consequently, we can express the expectation of \( \hat{f}_n \) as,

\[
E \left( \hat{f}_n(x) \right) = \psi(x) = f(x) + \frac{h^{r-s}}{(r-1)!} f^{(r)}(\xi) \int u^r K(u) du.
\]

1.3 Convergence speed

Suppose we are estimation the \( s \)-th derivative of a density. It is shown in Hall and Patil (1995) that the convergence rate, in probability, of \( n^{-\frac{(r-s)}{2s+1}} \) can be achieved by the wavelet estimator. We show that, in Theorem 2.1, the same convergence speed can be achieved by the tilted kernel estimator with an appropriate associated probability measures. That is, we show that

\[
\hat{f}_n^{(s)}(x) - f^{(s)}(x) = O_p \left( n^{-\frac{r-s}{2s+1}} \right), \tag{1.5}
\]

with the appropriate choice of the bandwidth \( h \).

2 Technical results

**Theorem 2.1.** Suppose \( f \) is a \( r \)-th time differentiable density function where \( r \) is even. Assume \( f^{(2)}, f^{(4)}, \ldots, f^{(r)} \) and \( f^{(2)}/f, f^{(4)}/f, \ldots, f^{(r)}/f \) exist and are bounded on \( \mathbb{R} \). Then there exists \( h_n > 0, n \in \mathbb{N} \) and a set of probability measures \( \{\mu_n : n \in \mathbb{N}\} \) associated to \( \hat{f}_n \) such that for all \( x \) and \( s \leq r \),

\[
\hat{f}_n^{(s)}(x) - f^{(s)}(x) = O_p \left( n^{-\frac{r-s}{2s+1}} \right) \tag{2.1}
\]

where
Proof. Let $\tilde{p}_{i,n} = n^{-1} (1 + h^2 g(X_i))$, the non-standardised version of $p_{i,n}$ as in (1.3) and

$$\tilde{f}_n(x) = \frac{1}{h} \sum_{i=1}^{n} \tilde{p}_{i,n} K \left( \frac{x - X_i}{h} \right). \quad (2.2)$$

Since

$$\left( \tilde{f}_n(x) - f(x) \right)^2 \leq 2 \left[ \left( \tilde{f}_n(x) - \hat{f}_n(x) \right)^2 + \left( \hat{f}_n(x) - f(x) \right)^2 \right],$$

the result can be shown by showing

$$\left( \hat{f}_n^{(s)}(x) - \tilde{f}_n^{(s)}(x) \right)^2 = O_p \left( n^{-\frac{2(r-s+1)}{2r+1}} \right) \quad (2.3)$$

and

$$\left( \tilde{f}_n^{(s)}(x) - f^{(s)}(x) \right)^2 = O_p \left( n^{-\frac{2(r-s)}{2r+1}} \right) \quad (2.4)$$

for some $h = h_n > 0$ and probability measures $\{\mu_n : n \in \mathbb{N}\}$.

To show (2.3), first let $\Delta_n = n^{-1} \sum_{i=1}^{n} h^2 g(X_i)$ and observe that by choosing the associated measure as in (1.3), we have

$$\left( \hat{f}_n^{(s)}(x) - \tilde{f}_n^{(s)}(x) \right)^2 = \left( 1 - \frac{1}{1 + \Delta_n} \right)^2 \left[ \frac{1}{nh} \sum_{i=1}^{n} \left( 1 + h^2 g(X_i) \right) \frac{\partial}{\partial x} K \left( \frac{x - X_i}{h} \right) \right]^2 \quad (2.5)$$

Since $g$ is, by definition and assumption, bounded on the real line, $\Delta_n \leq 1/2$ almost surely for sufficiently large $n$. Therefore the first term on the right hand side of (2.5) can be bounded, almost surely, by $4\Delta^2_n$ and hence

$$E \left[ \left( 1 - \frac{1}{1 + \Delta_n} \right)^2 \right] \leq 4E\Delta^2_n = O \left( n^{-1} \right),$$

where the last equality follows from the Law of Large Numbers applied to the triangular array $\{h^2_i g(X_j) : i, j \in \mathbb{N}\}$. Thus,

$$\left( 1 - \frac{1}{1 + \Delta_n} \right)^2 = O_p \left( n^{-1} \right) = O_p \left( n^{-\frac{2(r-s+1)}{2r+1}} \right)$$

Again, since $g, K^{(r)}$ are bounded, the expectation of the second term of (2.5) can be bounded as follow,

$$E \left[ \frac{1}{nh} \sum_{i=1}^{n} \left( 1 + h^2 g(X_i) \right) \frac{\partial}{\partial x} K \left( \frac{x - X_i}{h} \right) \right]^2 \leq C_1 \left[ \text{Var} \hat{f}_n^{(s)}(x) - \left( E\hat{f}_n^{(s)}(x) \right)^2 \right].$$

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Thus we have shown \((2.3)\).

To show \((2.4)\), let \(\psi(x) = E\left(\tilde{f}_n^{(s)}(x) - f^{(s)}(x)\right)\), note that

\[
E\left(\tilde{f}_n^{(s)}(x) - f^{(s)}(x)\right)^2
= \psi(x)^2 + \frac{1}{nh^2} \ Var \left[ (1 + h^2 g(X_i)) \frac{\partial}{\partial x} K \left(\frac{x - X_i}{h}\right) \right]
\leq \psi(x)^2 + \frac{C_1^2}{nh^2} Var \left[ \frac{\partial}{\partial x} K \left(\frac{x - X_i}{h}\right) \right].
\]

Hence it suffices to show that the result hold for \(\psi\). To see this, we observe that as

\[
\frac{\partial^s}{\partial x^s} K \left(\frac{x - X_i}{h}\right) = h^{-s} K^{(s)} \left(\frac{x - X_i}{h}\right).
\]

Therefore

\[
\psi(x) = f(x) - f^{(s)}(x) + \frac{h^{r-s}}{(r-1)!} f^{(r)}(\xi) \int u^r K(u) du,
\]

for some \(\xi \in [x, x + hu]\). Thus, by choosing \(h_n = n^{-\frac{1}{2r+1}}\), we have

\[
E\left(\tilde{f}_n^{(s)}(x) - f^{(s)}(x)\right)^2 = O\left(n^{-\frac{2(r-s)}{2r+1}}\right),
\]

hence proving \((2.4)\). \(\Box\)

References

P. Hall and H. Doosti. Making a nonparametric density estimator more attractive, as well as more accurate, by tilting. *pre-print*, 2012.

Peter Hall and Prakash Patil. Formulae for mean integrated squared error of nonlinear wavelet-based density estimators. *The Annals of Statistics*, 23 (3):pp. 905–928, 1995.