Accurate Community Detection in the Stochastic Block Model via Spectral Algorithms

Se-Young Yun  
MSR-Inria, 23 Avenue d’Italie, 75013 Paris, France  
SEYOUNG.YUN@INRIA.FR

Alexandre Proutiere  
KTH, The Royal Institute of Technology, EE School / ACL, Osqudasv. 10, Stockholm 100-44, Sweden  
ALEPRO@KTH.SE

Abstract

We consider the problem of community detection in the Stochastic Block Model with a finite number $K$ of communities of sizes linearly growing with the network size $n$. This model consists in a random graph such that each pair of vertices is connected independently with probability $p$ within communities and $q$ across communities. One observes a realization of this random graph, and the objective is to reconstruct the communities from this observation. We show that under spectral algorithms, the number of misclassified vertices does not exceed $s$ with high probability as $n$ grows large, whenever $pn = \omega(1)$, $s = o(n)$ and

$$
\liminf_{n \to \infty} \frac{n(\alpha_1 p + \alpha_2 q - (\alpha_1 + \alpha_2)p^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} q^{\frac{\alpha_2}{\alpha_1 + \alpha_2}})}{\log(\frac{n}{s})} > 1, \quad (1)
$$

where $\alpha_1$ and $\alpha_2$ denote the (fixed) proportions of vertices in the two smallest communities. In view of recent work by Abbe et al. (2014) and Mossel et al. (2014), this establishes that the proposed spectral algorithms are able to exactly recover communities whenever this is at all possible in the case of networks with two communities with equal sizes. We conjecture that condition (1) is actually necessary to obtain less than $s$ misclassified vertices asymptotically, which would establish the optimality of spectral method in more general scenarios.

1. Introduction

Extracting structures or communities in networks is a central task in many disciplines including social sciences, biology, computer science, statistics, and physics. The Stochastic Block Model (SBM) was introduced a few decades ago as a performance benchmark to study the problem of community detection in random graphs, and it has, since then, attracted a lot attention. In this paper, we provide new results on the performance of spectral algorithms for detecting communities in the SBM. We consider a network consisting of a set $V$ of $n$ nodes. $V$ admits a hidden partition of $K$ non-overlapping subsets or communities $V_1, \ldots, V_K$ ($V = \bigcup_{k=1}^{K} V_k$). The size of community $V_k$ is $\alpha_k n$ for some $\alpha_k > 0$. Without loss of generality, let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_K$. We assume that when the network size $n$ grows large, the number of communities $K$ and their relative sizes are kept fixed. The communities have to be reconstructed from an observed realization of a random graph constructed as follows. Each pair of vertices is connected independently with probability $p$ within communities and $q$ across communities, where $p$ and $q$ may depend on the network size $n$. We assume that there exists $\epsilon > 0$ such that $\frac{p}{q} \geq 1 + \epsilon$ uniformly in $n$. We further restrict our attention to the sparse case such that $p = o(1/\log^* n)$ and the case where $pn = \omega(1)$, which is a necessary
condition for asymptotically accurate community detection i.e., for the existence of algorithms that yield a vanishing proportion of misclassified vertices.

We show that under certain spectral algorithms, the number of misclassified vertices does not exceed $s$ with high probability as $n$ grows large, whenever $s = o(n)$ and

$$
\lim \inf_{n \to \infty} \frac{n(\alpha_1 p + \alpha_2 q - (\alpha_1 + \alpha_2) p^{\frac{\alpha_1}{\alpha_1+\alpha_2}} q^{\frac{\alpha_2}{\alpha_1+\alpha_2}})}{\log(n/s)} > 1. \tag{1}
$$

This result extends recent work about exact community reconstruction in the binary symmetric SBM (i.e., in the specific case of two communities of equal sizes). Indeed, by choosing $s < 1$ in (1), we get a condition under which spectral algorithms exactly recover the structure of any asymmetric networks with an arbitrary (but finite) number of communities. However our results is not limited to exact reconstruction, as we may choose any $s = o(n)$, e.g., $s = \sqrt{n}$.

We conjecture that the condition (1) is necessary for the existence of algorithms yielding less than $s$ misclassified vertices. The conjecture is true in the case of exact reconstruction ($s < 1$) for the binary symmetric SBM. Please refer to the next section for a more detailed description on the related work.

2. Previous Results

Exact Detection. Asymptotically exact community reconstruction in the SBM has been recently addressed in Abbe et al. (2014), Mossel et al. (2014), and Hajek et al. (2014). These papers only consider the binary symmetric SBM. They establish a necessary and sufficient condition for asymptotically exact reconstruction that coincides with (1) when applied to two communities of equal sizes ($\alpha_1 = \alpha_2 = 1/2$) and $s < 1$. For example, when $p = \frac{a \log(n)}{n}$ and $q = \frac{b \log(n)}{n}$ for $a > b$, (1) becomes equivalent to $\frac{a+b}{2} - \sqrt{ab} > 1$. The three aforementioned papers further provide optimal algorithms, i.e., algorithms exactly recovering the network structure when this is possible. Note that in Abbe et al. (2014), and Hajek et al. (2014), the proposed algorithms are based on SDP, and can be computationally expensive. In contrast, we prove that simple spectral algorithms are optimal.

Asymptotically Accurate Detection. Necessary and sufficient conditions for asymptotically accurate detection (i.e., the proportion of misclassified vertices vanishes when $n$ grows large) in the SBM has been derived in Yun and Proutiere (2014). This condition is $n(p - q)^2/(p + q) = \omega(1)$. In the present paper, we provide results that fill the gap between exact detection and asymptotically accurate detection.

Detectability. In the sparse regime where $p, q = o(1)$, and for the binary symmetric SBM, the main focus recently has been on identifying the phase transition threshold (a condition on $p$ and $q$) for detectability: It was conjectured in Decelle et al. (2011) that if $n(p - q) < \sqrt{2n(p + q)}$ (i.e., under the threshold), no algorithm can perform better than a simple random assignment of vertices to communities, and above the threshold, communities can partially be recovered. The conjecture was recently proved in Mossel et al. (2012) (necessary condition), and Massoulié (2013) (sufficient condition).

A more exhaustive list of papers related to the SBM can be found in Yun and Proutiere (2014).
Algorithm 1 Spectral Partition

Input: Observation matrix $A$.

1. **Trimming.** Construct $A_{\Gamma} = (A_{vw})_{v, w \in \Gamma}$ where $\Gamma = \{v : \sum_{w \in V} A_{vw} \leq 5K \frac{\sum_{(v, w) \in E} A_{vw}}{n}\}$.

2. **Spectral Decomposition.** Run Algorithm 2 with input $A_{\Gamma}, \frac{\sum_{(v, w) \in E} A_{vw}}{n^2}$, and output $(S_k)_{k=1, \ldots, K}$.

3. **Improvement.**
   
   $S_{k}^{(0)} \leftarrow S_k$, for all $k$
   
   for $i = 1$ to $\log n$ do
   
   $S_k^{(i)} \leftarrow \emptyset$, for all $k$
   
   for $v \in V$ do
   
   Find $k^* = \arg \max_k \{\sum_{w \in S_k^{(i-1)}} A_{vw} / |S_k^{(i-1)}|\}$ (tie broken uniformly at random)
   
   $S_{k^*}^{(i)} \leftarrow S_{k^*}^{(i)} \cup \{v\}$
   
   end for
   
   end for

   $\hat{V}_k \leftarrow S_k^{(i)}$, for all $k$

Output: $(\hat{V}_k)_{k=1, \ldots, K}$.

Algorithm 2 Spectral decomposition

Input: $A_{\Gamma}, \frac{\sum_{(v, w) \in E} A_{vw}}{n^2}$.

$\hat{A} \leftarrow K$-rank approximation of $A_{\Gamma}$

for $i = 1$ to $\log n$ do

$Q_{i,v} \leftarrow \{w \in \Gamma : \|\hat{A}_w - \hat{A}_v\|^2 \leq \frac{i \sum_{(v, w) \in E} A_{vw}}{100n^2}\}$

$T_{i,0} \leftarrow \emptyset$

for $k = 1$ to $K$ do

$v_k^* \leftarrow \arg \max_v |Q_{i,v} \setminus \bigcup_{l=1}^{k-1} T_{i,l}|$

$T_{i,k} \leftarrow Q_{i,v_k^*} \setminus \bigcup_{l=1}^{k-1} T_{i,l}$ and $\xi_{i,k} \leftarrow \sum_{v \in T_{i,k}} \hat{A}_v / |T_{i,k}|$

end for

for $v \in \Gamma \setminus (\bigcup_{k=1}^K T_{i,k})$ do

$k^* \leftarrow \arg \min_k \|\hat{A}_v - \xi_{i,k}\|$ and $T_{i,k^*} \leftarrow T_{i,k^*} \cup \{v\}$

end for

$r_i \leftarrow \sum_{k=1}^K \sum_{v \in T_{i,k}} \|\hat{A}_v - \xi_{i,k}\|^2$

end for

$i^* \leftarrow \arg \min_i r_i$

$S_k \leftarrow T_{i^*,k}$, for all $k$

Output: $(S_k)_{k=1, \ldots, K}$.

3. Spectral Algorithms and Their Performance

The proposed algorithm, referred to as Spectral Partition, is the same as that in Yun and Proutiere (2014), and is simple modifications of algorithms initially presented in Coja-Oghlan (2010). In this paper, we present a more precise analysis of its performance than that of Yun and Proutiere (2014).

Let $A$ denote the observed random adjacency matrix. The algorithm consists in three steps.
1. **Trimming.** We first trim the adjacency matrix $A$, i.e., we keep the entries corresponding to a set $\Gamma$ of vertices whose degrees are not too large. More precisely, $\Gamma = \{ v : \sum_{w \in V} A_{vw} \leq 10 \sum_{(v,w) \in E} A_{vw} \}$. The resulting trimmed observation matrix is denoted by $A_\Gamma$.

2. **Spectral decomposition.** We then extract the communities from the spectral analysis of $A_\Gamma$.

3. **Improvement.** Finally, we further improve the estimated communities. After the spectral decomposition step, the identified communities $(S_k)_{k=1}^2$ are good approximations of the true communities. The improvement is obtained by sequentially considering each vertex and by moving it to the community with which it has the largest number of edges.

The pseudo-code of the algorithm is presented in Algorithms 1 and 2. The next theorem provides performance guarantees for the Spectral Partition algorithm.

**Theorem 1** Assume that for $n$ large enough:

$$n(\alpha_1 p + \alpha_2 q - (\alpha_1 + \alpha_2)p^{\alpha_1/x_1} q^{\alpha_2/x_2}) - \frac{np}{\log np} \geq \log\left(\frac{n}{s}\right).$$

Then under the Spectral Partition algorithm, the number of misclassified vertices is less than $s$ with high probability.

By assumption, we have $\frac{p}{q} \geq 1 + \epsilon$, and $pn = \omega(1)$. We may deduce that:

$$n(\alpha_1 p + \alpha_2 q - (\alpha_1 + \alpha_2)p^{\alpha_1/x_1} q^{\alpha_2/x_2}) = \omega\left(\frac{np}{\log np}\right).$$

This can be proven using extensions of the weighted Arithmetic-Mean Geometric-Mean inequality. From Theorem 1, we deduce that: if

$$\liminf_{n \to \infty} \frac{n(\alpha_1 p + \alpha_2 q - (\alpha_1 + \alpha_2)p^{\alpha_1/x_1} q^{\alpha_2/x_2})}{\log\left(\frac{n}{s}\right)} > 1,$$

then the Spectral Partition algorithm yields less than $s$ misclassified vertices with high probability.

We conclude this paper by exemplifying the condition (1). Consider the binary symmetric SBM, with $p = \frac{a \log(n)}{n}$ and $q = \frac{b \log(n)}{n}$ for some $a > b$.

- Exact reconstruction: with $s < 1$, (1) is equivalent to $\frac{a+b}{2} - \sqrt{ab} > 1$, which also constitutes a necessary condition for exact reconstruction. Theorem 1 then states that Spectral Partition is optimal for exact reconstruction, i.e., it extracts the communities exactly whenever this is at all possible.

- Accurate reconstruction: choose $s = n^x$ for some $x \in (0, 1)$. Then (1) is equivalent to

$$\frac{a+b}{2} - \sqrt{ab} > 1 - x.$$  

**References**

E. Abbe, A. Bandeira, and G. Hall. Exact recovery in the stochastic block model. *arXiv preprint arXiv:1405.3267*, 2014.
A. Coja-Oghlan. Graph partitioning via adaptive spectral techniques. *Combinatorics, Probability & Computing*, 19(2):227–284, 2010.

A. Decelle, F. Krzakala, C. Moore, and L. Zdeborová. Inference and phase transitions in the detection of modules in sparse networks. *Phys. Rev. Lett.*, 107, Aug 2011.

U. Feige and E. Ofek. Spectral techniques applied to sparse random graphs. *Random Structures & Algorithms*, 27(2):251–275, 2005.

B. Hajek, Y. Wu, and J. Xu. Achieving exact cluster recovery threshold via semidefinite programming. 2014.

L. Massoulié. Community detection thresholds and the weak ramanujan property. *CoRR*, abs/1311.3085, 2013.

E. Mossel, J. Neeman, and A. Sly. Stochastic block models and reconstruction. *arXiv preprint arXiv:1202.1499*, 2012.

E. Mossel, J. Neeman, and A. Sly. Consistency thresholds for binary symmetric block models. *arXiv preprint arXiv:1407.1591*, 2014.

S. Yun and A. Proutiere. Community detection via random and adaptive sampling. In *COLT*, 2014.

**Appendix A. Proof of Theorem 1**

**A.1. Preliminaries**

In what follows, we use the standard matrix norm $\|A\| = \sup_{\|x\|_2 = 1} \|Ax\|_2$. We define $X_\Gamma = A_\Gamma - \mathbb{E}[A_\Gamma]$, where $A_\Gamma$ is the adjacency matrix obtained after trimming (Step 1 in Algorithm 1). We also denote by $e(v, S) = \sum_{w \in S} A_{vw}$ the total number of edges in the observed graph including node $v$ and a node from $S$.

We first provide key intermediate results.

**Lemma 2 (Lemma 8.5 of Coja-Oghlan (2010))** With high probability, $\|X_\Gamma\| = O(\sqrt{np})$.

The proof of Lemma 2 relies on arguments used in the spectral analysis of random graphs, see Feige and Ofek (2005). The next lemma provides a bound on the number of misclassified nodes after spectral decomposition applied to the trimmed matrix $A_\Gamma$, see Algorithm 2.

**Lemma 3 (Lemma 15 of Yun and Proutiere (2014))** Assume that $|V \setminus \Gamma| = O(1/p)$ and $\|X_\Gamma\| = O(\sqrt{np})$. Let $(S_k)_{1 \leq k \leq K}$ denotes the output of Algorithm 2. With high probability, there exists a permutation $\sigma$ of $\{1, \ldots, K\}$ such that:

$$|\bigcup_{k=1}^{K} (V_{\sigma(k)} \setminus S_k) \cap \Gamma| = O(1/p).$$
Observe that using Chernoff bound, we have \(|V \setminus \Gamma| = O(1/p)|.

**Corollary 4** Assume that \(np = \omega(1)|.

The output \((S_k)_{1 \leq k \leq K}\) of Algorithm 2 satisfies: with high probability, there exists a permutation \(\sigma\) of \(\{1, \ldots, K\}\) such that \(\frac{1}{n} \left| \bigcup_{k=1}^{K} V_k \setminus S_k \right| = O\left(\frac{1}{np}\right)|.

**Proof of Theorem 1:** Let \(H\) be the largest set of vertices \(v \in V\) satisfying:

(H1) When \(v \in V_k\), \(e(v, V_k) - e(v, V_j) \geq \frac{p}{\log^4 np}\) for all \(j \neq k\).

(H2) \(e(v, V) \leq 10np\)

(H3) \(e(v, V \setminus H) \leq 2 \log^2 np\).

The proof proceeds as follows. We first show that \(|V \setminus H| \leq s\) with high probability. To this aim, we control the number of vertices satisfying (H1), (H2), and (H3), see Lemma 5, Claim 1 and Lemma 6, respectively. The result is summarised in Lemma 6. Next Lemma 7 establishes that there is no misclassified vertices in \(H\) with high probability, which concludes the proof.

**Lemma 5** For \(v \in V_k\), and for all \(j \neq k\),

\[
P\left\{ \frac{e(v, V_k)}{|V_k|} - \frac{e(v, V_j)}{|V_j|} \leq \frac{p}{\log^4 np} \right\} \leq \exp\left(-n(\alpha_1 p + \alpha_2 q - (\alpha_1 + \alpha_2)p^{\alpha_1 + \alpha_2}q^{\alpha_1 + \alpha_2}) + \frac{np}{2 \log np}\right).
\]

From Lemma 5, with high probability, the number of vertices that do not satisfy (H1) is less than \(s/3\) when \(n(\alpha_1 p + \alpha_2 q - (\alpha_1 + \alpha_2)p^{\alpha_1 + \alpha_2}q^{\alpha_1 + \alpha_2}) - \log(n/s) - \frac{np}{2 \log np} = \omega(1)|,

 ściance

\[
\frac{1}{s/3} \leq 3n \exp\left(-n(\alpha_1 p + \alpha_2 q - (\alpha_1 + \alpha_2)p^{\alpha_1 + \alpha_2}q^{\alpha_1 + \alpha_2}) + \frac{np}{2 \log np}\right) = o(1).
\]

**Claim 1.** From Chernoff bound, we can easily show that \(v\) does not satisfy (H2) with probability at most \(\exp(-5np)|\) and thus, with high probability, the number of vertices that do not satisfy (H2) is less than \(\frac{s}{10}\), since

\[
\frac{1}{s/10} \leq \frac{10n}{s} \exp(-np) = o(1).
\]

In Lemma 6, we conclude that \(|V \setminus H| \leq s\) after showing the number of vertices that do not satisfy (H3) is less that \(\frac{s}{2}\) with high probability.

**Lemma 6** When \(n(\alpha_1 p + \alpha_2 q - (\alpha_1 + \alpha_2)p^{\alpha_1 + \alpha_2}q^{\alpha_1 + \alpha_2}) - \log(n/s) - \frac{np}{2 \log np} = \omega(1)|,

 with high probability.
Lemma 7 shows that when initial (after Algorithm 2) number of misclassified vertices is \(O(1/p)\),

\[
\frac{\text{# misclassified vertices in } H \text{ at } i + 1\text{-th iteration}}{\text{# misclassified vertices in } H \text{ at } i\text{-th iteration}} \leq e^{-2}.
\]

Since the initial number of misclassified vertices is negligible compared to \(n\) by Lemma 4, after \(\log n\) iterations, there is no misclassified vertex in \(H\).

**Lemma 7** If \(|\bigcup_{k=1}^{K} (S_k^{(i)} \setminus V_k) \cap H| + |V \setminus H| = O(1/p)\),

\[
\frac{|\bigcup_{k=1}^{K} (S_k^{(i+1)} \setminus V_k) \cap H|}{|\bigcup_{k=1}^{K} (S_k^{(i)} \setminus V_k) \cap H|} \leq \frac{1}{\sqrt{np}}.
\]

**A.2. Proof of Lemma 5**

From Chernoff bound, we know that for all \(1 \leq t \leq K\),

\[
P\{e(v, V_t) \geq \alpha_t np \log np\} = o(\exp(-np)).
\]

Using (2),

\[
P(e(v, V_k) - e(v, V_j) < -p \log^4 np)
\]

\[
= P(-p \log np \leq e(v, V_k) - e(v, V_j) < -p \log^4 np) + P(e(v, V_k) - e(v, V_j) \leq -p \log np)
\]

\[
\leq P(-p \log np \leq e(v, V_k) - e(v, V_j) < -p \log^4 np) + o(\exp(-np))
\]

\[
\leq np \log np P(e(v, V_k) - \frac{\alpha_k}{\alpha_j} e(v, V_j) = \lceil \frac{np}{\log^4 np} \rceil) + o(\exp(-np))
\]

\[
\leq \exp \left( \left( \alpha_k + \alpha_j \right) np q^{-\frac{\alpha_k}{\alpha_j}} q^{-\frac{\alpha_j}{\alpha_k}} - \alpha_k np - \alpha_j nq + \frac{np}{2 \log np} \right).
\]

We conclude the proof by proving (3) and (4).

**Proof of (3):** Since \(P\{e(v, V_k) - \lceil \alpha_k/\alpha_j e(v, V_j) \rceil = x\} \leq P\{e(v, V_k) - \lceil \alpha_k/\alpha_j e(v, V_j) \rceil = \lceil \frac{np}{\log^4 np} \rceil\}\) for \(-\lceil \alpha_k np \log np \rceil \leq x \leq \lceil \frac{\alpha_k np}{\log^4 np} \rceil\),

\[
P{-p \log np \leq e(v, V_k) - e(v, V_j) < -p \log^2 np}
\]

\[
\leq \sum_{x = -\lceil \alpha_k np \log np \rceil}^{\lceil \frac{\alpha_k np}{\log^4 np} \rceil} P\{e(v, V_k) - \lceil \frac{\alpha_k}{\alpha_j} e(v, V_j) \rceil = x\}
\]

\[
\leq np \log np P\{e(v, V_k) - \lceil \frac{\alpha_k}{\alpha_j} e(v, V_j) \rceil = \lceil \frac{np}{\log^4 np} \rceil\}.
\]
Proof of (4): Let \( x^* = \lfloor \frac{np}{\log^2 np} \rfloor \).

\[
P\{e(v, V_k) - \lfloor \frac{\alpha_k}{\alpha_j} e(v, V_j) \rfloor = x^*\} - P\{e(v, V_k) > 10np\}
\]

\[
\leq P\{e(v, V_k) - \lfloor \frac{\alpha_k}{\alpha_j} e(v, V_j) \rfloor = x^*, e(v, V_k) \leq 10np\}
\]

\[
\leq \sum_{i=0}^{10np} \left( \frac{\alpha_k n}{i + x^*} \right) \left( \frac{\alpha_j n}{i} \right) \left( \frac{p}{1-p} \right)^{i+x^*} \left( \frac{q}{1-q} \right)^{\left\lfloor \frac{\alpha_j}{\alpha_k} i + \ell \right\rfloor} (1-p)^{\alpha_k n (1-q)^{\alpha_j n}}
\]

\[
\leq \left( 10np + 1 \right) \left( \left\lfloor \frac{\alpha_j}{\alpha_k} \right\rfloor + 1 \right) \exp \left( \left( \alpha_k + \alpha_j \right) np \frac{\alpha_k + \alpha_j}{\alpha_k} q^{\frac{\alpha_j}{\alpha_k}} \alpha_k np - \alpha_k np - \alpha_j nq \frac{np}{4 \log np} \right),
\]

where \((a)\) stems from the inequality \(\lfloor n/k \rfloor \leq (ne/k)^k\). Since \(P\{e(v, V_k) > 10np\} \leq o(\exp(-np))\) from Chernoff bound, (5) implies (4).

A.3. Proof of Lemma 6

Let \( Z_1 \) denote the set of vertices that do not satisfy at least one of (H1) and (H2). From Lemma 5 and Chernoff bound, \(|Z_1| < \frac{n}{2}\) with high probability.

Next we prove the following intermediate claim: there is no subset \( S \subset V \) such that \( e(S, S) \geq s \log^2 np \) and \(|S| = s\) with high probability. For any subset \( S \subset V \) such that \(|S| = s\), by Markov inequality,

\[
P\{e(S, S) \geq s \log^2 np\} \leq \inf_{t \geq 0} \frac{E[\exp(e(S, S)t)]}{st \log^2 np}
\]

\[
\leq \inf_{t \geq 0} \frac{\prod_{i=1}^{s/2} (1 + p \exp(t))}{st \log^2 np}
\]

\[
\leq \inf_{t \geq 0} \exp \left( \frac{s^2 p}{2} \exp(t) - st \log^2 np \right)
\]

\[
\leq \exp \left( -nps \left( \log np - \frac{s}{2n} \exp \left( \frac{np}{\log np} \right) \right) \right)
\]

\[
\leq \exp \left( -\frac{nps \log np}{2} \right),
\]

where, in the last two inequalities, we have set \( t = \frac{np}{\log np} \) and used the fact that: \( \frac{n}{s} \geq \exp \left( \frac{np}{\log np} \right) \), which comes from the assumptions made in the theorem. Since the number of subsets \( S \subset V \) with
size $s$ is $\binom{n}{s} \leq \left(\frac{en}{s}\right)^s$, from (6), we deduce:

$$
\mathbb{E}[\{|S : e(S, S) \geq s \log^2 np \text{ and } |S| = s\}] \leq \left(\frac{en}{s}\right)^s \exp\left(-\frac{nps \log np}{2}\right)
= \exp\left(-s\left(\frac{np \log np}{2} - \log\frac{en}{s}\right)\right)
\leq \exp\left(-\frac{nps \log np}{4}\right).
$$

Therefore, by Markov inequality, we can conclude that there is no $S \subseteq V$ such that $e(S, S) \geq s \log^2 np$ and $|S| = s$ with high probability.

To conclude the proof of the lemma, we build the following sequence of sets. Let $\{Z(i) \subseteq V\}_{1 \leq i \leq i^*}$ be generated as follows:

- $Z(0) = Z_1$.
- For $i \geq 1$, $Z(i) = Z(i-1) \cup \{v_i\}$ if there exists $v_i \in V$ such that $e(v_i, Z(i-1)) \geq 2 \log^2 np$ and $v_i \notin Z(i-1)$ and if there does not exist, the sequence ends.

The sequence ends after the construction of $Z(i^*)$. By construction, every $v \in V \setminus Z(i^*)$ satisfies the conditions (H1), (H2), and (H3). Since $H$ is the largest set of vertices satisfying (H1), (H2), and (H3), $|H| \geq |V \setminus Z(i^*)|$.

The proof is hence completed if we show that $|Z(i^*)| < s$. Let $t^* = s - |Z_1|$. If $i^* \geq t^*$, $|Z(t^*)| = s$ and since $|Z_1| \leq \frac{s}{2}$,

$$
e(Z(t^*), Z(t^*)) \geq \sum_{i=1}^{t^*} e(v_i, Z(i-1)) \geq 2t^* \log^2 np \geq s \log^2 np,
$$

However, from the previous claim, we know that with high probability, all $S \subseteq V$ such that $|S| = s$ have to satisfy $e(S, S) \leq s \log^2 np$. Therefore, with high probability, $i^* < t^*$ and

$$|Z(i^*)| = i^* + |Z_1| < t^* + |Z_1| = s.$$

A.4. Proof of Lemma 7

We use the notation: $\mu(v, S) = \mathbb{E}[e(v, S)]$. Let $\mathcal{E}^{(i)}_{jk} = (S_j^{(i)} \cap V_k) \cap H$ and $\mathcal{E}^{(i)} = \bigcup_{j,k : j \neq k} \mathcal{E}^{(i)}_{jk}$. At each improvement step, vertices move to a community with more connections to it. Thus,

$$
\sum_{j,k : j \neq k} \sum_{v \in \mathcal{E}^{(i+1)}_{jk}} \frac{e(v, S_j^{(i)})}{|S_j^{(i)}|} - \frac{e(v, S_k^{(i)})}{|S_k^{(i)}|} \geq 0.
$$

Since $|\mathcal{E}^{(i)}| = O(1/p)$ and $e(v, V) \leq 10np$ when $v \in H$,

$$
0 \leq \sum_{j,k : j \neq k} \sum_{v \in \mathcal{E}^{(i+1)}_{jk}} \frac{e(v, S_j^{(i)})}{|S_j^{(i)}|} - \frac{e(v, S_k^{(i)})}{|S_k^{(i)}|} \leq \sum_{j,k : j \neq k} \sum_{v \in \mathcal{E}^{(i+1)}_{jk}} \frac{e(v, S_j^{(i)})}{|V_j|} - \frac{e(v, S_k^{(i)})}{|V_k|} + \frac{\log np}{n} |\mathcal{E}^{(i+1)}|.
$$
With the above inequality and (H1), we can bound $\frac{\|E^{(i+1)}\|}{\|E^{(i)}\|}$ as follows:

$$-\frac{\log np}{n} |E^{(i+1)}| \leq \sum_{j,k:j \neq k} \sum_{v \in E^{(i+1)}} \frac{e(v, S_j)}{|V_j|} - \frac{e(v, S_k)}{|V_k|}$$

$$\leq \sum_{j,k:j \neq k} \sum_{v \in E^{(i+1)}} \frac{e(v, V_j)}{|V_j|} - \frac{e(v, V_k)}{|V_k|} + \sum_{v \in E^{(i+1)}} \frac{e(v, E^{(i)} \cup H)}{\alpha_1 n}$$

$$(a) \leq - |E^{(i+1)}| \frac{p}{\log^4 np} + \sum_{v \in E^{(i+1)}} \frac{e(v, E^{(i)})}{\alpha_1 n} + \sum_{v \in E^{(i+1)}} \frac{e(v, H)}{\alpha_1 n}$$

$$= - |E^{(i+1)}| \frac{p}{\log^4 np} + \sum_{v \in E^{(i+1)}} \frac{\mu(v, E^{(i)})}{\alpha_1 n} + \sum_{v \in E^{(i+1)}} \frac{(e(v, E^{(i)}) - \mu(v, E^{(i)}))}{\alpha_1 n} + \sum_{v \in E^{(i+1)}} \frac{e(v, H)}{\alpha_1 n}$$

$$(b) \leq - |E^{(i+1)}| \frac{p}{\log^4 np} + \sum_{v \in E^{(i+1)}} \frac{p|E^{(i)}||E^{(i+1)}|}{\alpha_1 n} + \sum_{v \in E^{(i+1)}} \frac{\sqrt{|E^{(i)}||E^{(i+1)}|}}{\alpha_1 n} + \sum_{v \in E^{(i+1)}} \frac{e(v, H)}{\alpha_1 n}$$

$$(c) \leq - |E^{(i+1)}| \frac{p}{\log^4 np} + \sum_{v \in E^{(i+1)}} \frac{p|E^{(i)}||E^{(i+1)}|}{\alpha_1 n} + \sum_{v \in E^{(i+1)}} \frac{\sqrt{|E^{(i)}||E^{(i+1)}|} np \log np}{\alpha_1 n} + \frac{2|E^{(i+1)}| \log^2 np}{\alpha_1 n},$$

where $(a)$ stems from (H1), $(b)$ stems from the fact that $\sum_{v \in E^{(i+1)}} (e(v, E^{(i)}) - \mu(v, E^{(i)})) = 1^T_{E^{(i)}} \cdot X_{\Gamma} \cdot 1_{E^{(i+1)}}$ where $1_S$ indicates the vector $v$-th value is 1 if $v \in S$ and 0 otherwise, and $(c)$ stems from (H3). Since $|E^{(i)}| = O(1/p)$, we conclude that

$$\frac{|E^{(i)}|}{|E^{(i+1)}|} \leq \frac{1}{\sqrt{np}}.$$