Aperiodic Sets of Prototiles Extracted From the Penrose Rhomb Tiling

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Abstract

We present aperiodic sets of prototiles whose shapes are based on the well-known Penrose rhomb tiling. Some decorated prototiles lead to an exact Penrose rhomb tiling without any matching rules. We also give an approximate solution to an aperiodic monotile that tessellates the plane (including five types of gaps) only in a nonperiodic way.

1 Introduction

The Penrose P2 (kite and dart) and P3 (rhomb) tilings are certainly the most popular examples of nonperiodic tilings. Both tilings are strongly related and generate the same mld-class\(^1\). For more details we refer the reader to [1][2][4]. In this article we will take a closer look at cutouts from the P3 tiling, which themselves form aperiodic sets of two prototiles (decorated or undecorated) and which have not yet been published or mentioned. The smallest structures of edge to edge connected rhombs we have found are given in Figure 1, whereby \(T_3\) is not a connected tile, and \(T_3\) represents a connected pair of \(T_1\) and \(T_2\). The aperiodic sets of \(T_1, T_2\) or \(T_2, T_3\) inevitably lead to an exact P3 tiling without matching rules. A slight modification of \(T_1\) and \(T_2\) will give us an aperiodic set of two undecorated tiles in the shape of a snake and a dog (Fig. 4).

\(^1\)Two tilings are called mld (mutually locally derivable), if one is obtained from the other in a unique way by local rules, and vice versa [1].
Tilings like P2 and P3 have a scaling self-similarity as a fractal. This is due to the substitution rules that allow each tile to be decomposed into smaller tiles of the same shape as those used in the tiling. Thus allow larger tiles to be composed of smaller ones. Penrose tilings are strongly related with Fibonacci numbers\(^2\). For instance, the ratio between the number of different prototiles which can tessellate a larger tile is always given by such numbers \([1][2]\). This Fibonacci ratio also applies to the prototiles in Figure 1. \(T_1\) consists of five thin and eight thick rhombs, \(T_2\) of eight thin and thirteen thick rhombs, and \(T_3\) of thirteen thin and 21 thick rhombs.

Figures 2 and 3 show the arrangements for \(T_1\), \(T_2\) respectively \(T_2\), \(T_3\) based on the superordinate kite and dart shape. The use of the substitution rules for the P2 tiling then lead to a nonperiodic tiling (Fig. 10). Please note that the right sided arrangements of Figures 2 and 3 are completely part of the left sided ones.

\(^2\)https://oeis.org/A000045
The dashed kite and dart shapes in Figure 3 can be completely decorated or filled by rhombs, including half rhombs with their diagonals on the dashed edges. The arrangements in Figures 2 and 3 can also be interpreted as modifications of the kite and dart shape, as mentioned in [2] on page 539, where the long and short sides of kite and dart are replaced by two J-curves.

2 Variants of the Prototiles

From $T_1$ and $T_2$ we can get two new undecorated prototiles, nicknamed Snake and Dog, by shifting two outside half thin rhombs to the corresponding gaps (Fig. 4).

Again the snake and dog tiles can be transformed into further and smoother shapes given by a hexagon and a pentagon, both concave and

\[^3\text{The black dots within the tiles should represent the animals’ eyes only.}\]
irregular. Figures 5 and 6 show this process from the left to the right by shifting three or four half rhombs as shown. $T_8$ and $T_9$, as well as Snake and Dog, are balanced tiles with the same acreage as $T_1$ and $T_2$. 

![Figures 5 and 6](image)

Figure 5: Building $T_8$ from the Snake.

![Figures 6](image)

Figure 6: Building $T_9$ from the Dog.

![Figure 7](image)

Figure 7: $T_8$ and $T_9$ with Robinson triangles.

As well as the Penrose tiles, $T_8$ and $T_9$ can be decomposed into Robinson triangles. $T_8$ can be decomposed into two golden gnomons and two golden triangles, $T_9$ into four golden gnomons and two golden triangles (Fig. 7).
The decorated snake and dog tiles (Fig. 5b and 6b) lead to a P3 tiling, whereas a tiling of the decorated $T_8$ and $T_9$ tiles will always contain small errors due to incomplete edges. These incorrect areas are related to the decoration in the top corner of $T_8$ and the lower left corner of $T_9$. Figure 8 shows an aperiodic set of two decorated prototiles (based on $T_8$ and $T_9$), which allow an exact P3 tiling again (Fig. 12). We leave it to the reader to locate the three areas with the rectified edge-errors.

Figure 8: Aperiodic set for a P3 tiling.

Without substitution rules the undecorated $T_8$, $T_9$, and $T_{11}$ tiles also allow periodic tilings (Fig. 9). A tessellation of the plane only with $T_8$ or $T_{10}$ tiles is not possible.

Figures 11 and 12 show larger portions with the snake and dog tiles and the undecorated $T_{10}$, $T_{11}$ set. Note the special feature in the last one, where the $T_{11}$ tiles never touch each other.

Figure 9: Matching rules for periodic tilings with $T_8$, $T_9$ and $T_{11}$. 
The substitution rules for all aperiodic sets in this article are the same as for the Penrose kite and dart, due to their same mld-class. Figure 10 shows these rules using $T_8$ and $T_9$ as an example.

Figure 10: Substitution rules for $T_8$ and $T_9$.

Figure 11: Snake and Dog tiling.
3 Aperiodic Monotiles

One of the most exciting open problems in plane geometry is the existence of an aperiodic monotile. It asks about a single connected prototile that by itself forms a strongly aperiodic set. Such a tile can tessellate the Euclidean plane only in a nonperiodic way without matching rules. The smallest aperiodic sets known to date consist of two prototiles, like the Penrose tiles. Note that the P2 and P3 sets must be modified to get strongly aperiodic sets without matching rules. Examples can be seen in [2] on page 539 (kite, dart) and page 544 (rhomb).4

The currently best approximations to an aperiodic monotile were given by Petra Gummelt in 1996, and Joshua Socolar and Joan Taylor in 2010.

4Further examples: https://en.wikipedia.org/wiki/Penrose_tiling#Rhombus_tiling_(P3)
Gummelt constructed a decorated decagonal tile and showed that when two kinds of overlaps between pairs of tiles are allowed, these tiles can cover the plane only nonperiodically [3]. Socolar and Taylor presented an undecorated, but not connected aperiodic monotile that is based on a regular hexagon [5].

Figure 13 shows an approximation to an aperiodic monotile from the Penrose tiles, a shape that represents both kite and dart as well as possible. Please note that it is not possible to create a truly aperiodic monotile with a 5-fold rotational symmetry from the Penrose prototiles because they do not have the same acreage. But other shapes as given by $T_{12}$ are possible. It is also possible to modify edges (e.g. by identically shaped bumps and notches) to enforce the tiling rules.

$T_{12}$, an irregular concave dodecagon, is based on a connected pair of $T_{10}, T_{11}$ tiles (Fig. 13a), and can be subdivided into a rhombus $(a, b, c, p)$, two congruent kites $(c, d, r, p)$ and $(g, h, i, r)$, two non-congruent trapezoids $(d, e, f, r)$ and $(i, j, k, r)$, which we will denote by $T_{13}$ and $T_{14}$, and an irregular quadrilateral $(l, n, q, r)$ (Fig. 13b).

Several properties and common features of the Penrose tilings involve the golden ratio $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ [2]. The edge lengths in Figure 13b, in relation to the unit length edges of the decorated rhombs, are $|fg| = |ij| = |pq| = \varphi$, $|ab| = |bc| = |cd| = |gh| = |hi| = |lo| = |op| = |pa| = |pc| = |qr| = |fr| = \varphi + 1$, $|dr| = |gr| = |hj| = |ir| = |lr| = |pr| = 2\varphi + 1$, and $|de| = |kl| = |lm| = 2$. 

Figure 13: The approximated aperiodic monotile $T_{12}$. 
A $T_{12}$ tiling contains always five types of gaps. An irregular triangle $(l, m, n)$, an irregular quadrilateral $(n, o, p, q)$, and three types of rotors (concave pentadecagons) that can be subdivided into $T_{13}$ and $T_{14}$ tiles (Fig. 13b and 15). Note that a connected pair of $T_{13}$ and $T_{14}$ could build a rhombus with edge lengths of $2\varphi + 1$. The decorated $T_{12}$ tiles (Fig. 13a) also allow a P3 tiling with the mentioned gaps. The substitution rules for $T_{12}$ are given in Figure 16.

![Figure 14: A $T_{12}$ tiling with gaps.](image)

The possibility of an exact P3 tiling by the decorated tile sets mentioned in this article can be proved with the seven possible vertex figures in a P2 tiling. Details are given in [2] on page 561 or at Wikipedia.\(^5\)

\(^5\)https://en.wikipedia.org/wiki/Penrose_tiling#Kite_and_dart_tiling_(P2)
Figure 15: The three types of rotor gaps in a $T_{12}$ tiling.

Figure 16: Substitution rules for $T_{12}$.

References

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