UPPER BOUND OF THE LEAST QUADRATIC NONRESIDUES

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Abstract. Let $p \geq 3$ be a large prime and let $n(p) \geq 2$ denotes the least quadratic nonresidue modulo $p$. This note sharpens the standard upper bound of the least quadratic nonresidue from the unconditional upper bound $n(p) \ll p^{1/4+\epsilon}$ to the conjectured upper bound $n(p) \ll (\log p)^{1+\epsilon}$, where $\epsilon > 0$ is a small number, unconditionally. This improvement breaks the exponential upper bound barrier and proves the Elliot’s conjecture.

Contents

1. Introduction 1
2. Representations of the Characteristic Functions 2
3. Finite Fourier Transform and Summation Kernels 3
  3.1. Finite Fourier Transform 3
  3.2. Summation Kernels 4
  3.3. Harmonic Summation Kernels 6
4. Estimates Of Exponential Sums 7
  4.1. Incomplete Exponential Sums over Consecutive Index 8
  4.2. Incomplete Exponential Sums over Relatively Prime Index 9
  4.3. Equivalent Exponential Sums over Consecutive Index 10
  4.4. Equivalent Exponential Sums over Relatively Prime Index 11
5. Results for Gaussian Sums 13
6. Fibers and Multiplicities for Quadratic Residues 13
7. Evaluation of the Main Term 14
8. Estimate For The Error Term 15
9. Main Result 17
References 19

1. Introduction

Let $p \geq 3$ be a prime and consider the equation $x^2 - n \equiv 0 \mod p$. The integer $n \neq 0, 1$ is called a quadratic residue if the congruence has a solution $x = x_0$. Otherwise, $n$ is called a quadratic nonresidue. The Burgess upper bound of the least quadratic nonresidue claims that

$$n(p) \ll p^{1/4+\epsilon},$$

where $\epsilon > 0$ is a small number, see [2], and [3] for a survey and discussion. However, in general, the least quadratic nonresidue is significantly smaller. Conditioned on the generalized RH, the upper bound satisfies

$$n(p) \ll (\log p)^2,$$

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see [1], [23], et alii for fine details and explicit versions. The heuristic in [27] claims that
\[ n(p) \ll (\log p)(\log \log p). \quad (3) \]
Similar heuristic and extensive calculations discussed in [16, p. 841], lead to the following claim.

**Conjecture 1.1.** (Elliot 1969) For any \( \varepsilon > 0 \) and a large prime \( p \), the least quadratic nonresidue has the upper bound
\[ n(p) \ll (\log p)^{1+\varepsilon}. \]

This conjecture is a phenomenal improvement of the much older Vinogradov conjecture, which claims that \( n(p) \ll p^\varepsilon \). A closely related result due to Linnik in [24] proves that the number of primes \( p \in [x^\varepsilon, x] \) that fails the Vinogradov conjecture is finite, a recent proof appears in [6, Corollary 5].

On the other extreme, there is the lower bound
\[ n(p) \gg (\log p)(\log \log p), \quad (4) \]
see [25, Theorem 13.5].

On the statistical perspective, there is the probability that a random integer \( x < p \) is a quadratic nonresidue modulo \( p \) is \( P(x = n_p) = 1/2 \) and there is the expected value of the least quadratic nonresidue computed in [15]. The expected value is a small constant
\[ \frac{1}{\pi(x)} \sum_{p \leq x} n_p = (1 + o(1)) \sum_{n \geq 1} \frac{p_n}{2^n} = 3.6746439660113287789956763\ldots, \quad (5) \]
where \( p_n \) denotes the \( n \)th prime in increasing order. Furthermore, combining a result for Gauss quadratic sum and Weyl’s theorem, it is easy to verify that both the quadratic residues and quadratic nonresidues are equidistributed on the interval \([1, p - 1]\).

This short note proposes a resolution of the above conjecture.

**Theorem 1.1.** Let \( p \) be a large prime and let \( n(p) \) denotes the least quadratic nonresidue modulo \( p \). Then
\[ n(p) \ll (\log p)^{1+\varepsilon}, \quad (6) \]
where \( \varepsilon > 0 \) is a small number.

The analysis is completely different from the traditional literature on quadratic residues and quadratic nonresidues. This made possible by a new indicator function for quadratic nonresidues in finite fields introduced in Section 2. The foundational and supporting materials are covered in Section 2 to Section 8. The proof of Theorem 1.1 appears in Section 9.

### 2. Representations of the Characteristic Functions

For an odd prime \( p \) the quadratic symbol modulo \( p \) is defined by
\[ \left( \frac{n}{p} \right) = \begin{cases} 
1 & \text{if } n \text{ is a quadratic residues,} \\
-1 & \text{if } n \text{ is a quadratic nonresidues,} \\
0 & \text{if } n \text{ is divisible by } p,
\end{cases} \quad (7) \]

The classical characteristic functions of quadratic residues and quadratic nonresidues in the finite field \( \mathbb{F}_p \), which are defined in terms of the quadratic symbol, have the simple formulas described below.
Lemma 2.1. If $p \geq 2$ is a prime and $n \in \mathbb{F}_p$ is a nonzero element, then

(i) $\kappa_0(n) = \frac{1}{2} \left( 1 + \frac{n}{p} \right) = \begin{cases} 1 & \text{if } n \text{ is a quadratic residues,} \\ 0 & \text{if } n \text{ is a quadratic nonresidues,} \end{cases}$

(ii) $\kappa(n) = \frac{1}{2} \left( 1 - \frac{n}{p} \right) = \begin{cases} 1 & \text{if } n \text{ is a quadratic nonresidues,} \\ 0 & \text{if } n \text{ is a quadratic residues,} \end{cases}$

respectively.

A new representation of the characteristic function for quadratic nonresidues in the finite field $\mathbb{F}_p$ is introduced here.

Lemma 2.2. Let $p \geq 2$ be a prime and let $\tau$ be a primitive root mod $p$. If $n \in \mathbb{F}_p$ is a nonzero element, then

(i) $\kappa_0(n) = \sum_{0 \leq s < p/2} \frac{1}{p} \sum_{0 \leq t \leq p-1} e^{i2\pi \frac{(\tau^{2s+n})t}{p}} = \begin{cases} 1 & \text{if } n \text{ is a quadratic residues,} \\ 0 & \text{if } n \text{ is a quadratic nonresidues,} \end{cases}$

(ii) $\kappa(n) = \sum_{0 \leq s < p/2} \frac{1}{p} \sum_{0 \leq t \leq p-1} e^{i2\pi \frac{(\tau^{2s+1+n})t}{p}} = \begin{cases} 1 & \text{if } n \text{ is a quadratic nonresidues,} \\ 0 & \text{if } n \text{ is a quadratic residues,} \end{cases}$

respectively.

Proof. (ii) As the index $s \geq 0$ ranges over the odd integers up to $p-1$, the element $\tau^{2s+1} \in \mathbb{F}_p$ ranges over the quadratic nonresidues modulo $p$. Thus, the equation $\tau^{2s+1} - n = 0$ has a solution if and only if the fixed element $n \in \mathbb{F}_p$ is a quadratic nonresidue. In this case the inner sum in

$$\sum_{0 \leq s < p/2} \frac{1}{p} \sum_{0 \leq t \leq p-1} e^{i2\pi \frac{(\tau^{2s+1+n})t}{p}}$$

(8)

collapses to $\sum_{0 \leq t \leq p-1} 1 = p$. Otherwise, element is not a quadratic nonresidue. Thus, the equation $\tau^{2s+1} - n \neq 0$ has no solution and the inner sum in (8) collapses to $\sum_{0 \leq t \leq p-1} e^{i2\pi \frac{(\tau^{2s+1+n})t}{p}} = 0$. \hfill \blacksquare

3. Finite Fourier Transform and Summation Kernels

3.1. Finite Fourier Transform. Let $f : \mathbb{C} \to \mathbb{C}$ be a function, and let $q \in \mathbb{N}$ be a large integer.

Definition 3.1. The discrete Fourier transform of the function $f : \mathbb{N} \to \mathbb{C}$ and its inverse are defined by

$$\hat{f}(t) = \sum_{0 \leq s \leq q-1} e^{ist/q} \tag{9}$$

and

$$f(s) = \frac{1}{q} \sum_{0 \leq t \leq q-1} \hat{f}(m)e^{-i2\pi st/q}, \tag{10}$$

respectively.

The finite Fourier transform and its inverse are used here to derive a summation kernel function, which is almost identical to the Dirichlet kernel, in this application $q = p$ is a prime number.
Definition 3.2. Let $p$ be a prime, let $\omega = e^{2\pi i/p}$, and $\zeta = e^{2\pi i/p}$ be roots of unity. The finite summation kernel is defined by the finite Fourier transform identity

$$K(f(n)) = \frac{1}{p} \sum_{0 \leq t \leq p-1} \sum_{0 \leq s \leq p-1} \omega^{t(n-s)} f(s) = f(n). \quad (11)$$

This simple identity is very effective in computing upper bounds of some exponential sums

$$\sum_{n \leq x} f(n) = \sum_{n \leq x} K(f(n)), \quad (12)$$

where $x < p$.

3.2. Summation Kernels.

Lemma 3.1. Let $p \geq 2$ be a large primes, let $x < p - 1$ and let $\omega = e^{2\pi i/p}$ be a $p$th root of unity. If $t \in [1, p-1]$, then,

$$\left| \sum_{n \leq x} \omega^{tn} \right| \leq \frac{2p}{\pi t}. \quad (13)$$

Proof. Use the geometric series to compute this simple exponential sum as

$$\sum_{n \leq x} \omega^{tn} = \frac{\omega^t - \omega^{t(p-1)}}{1 - \omega^t} = \frac{\omega^t - \omega^{(t+1)\pi i/p}}{1 - \omega^t}.$$

Now, observe that $\omega = e^{2\pi i/p}$, the integers $t \in [1, p-1]$, and $d < p - 1$. This data implies that $\pi t / p \neq k\pi$ with $k \in \mathbb{Z}$, so the sine function $\sin(\pi t / p) \neq 0$ is well defined. Using standard manipulations, and $z/2 \leq \sin(z) < z$ for $0 < |z| < \pi/2$, the last expression becomes

$$\left| \frac{\omega^t - \omega^{(t+1)\pi i/p}}{1 - \omega^t} \right| \leq \frac{2}{\sin(\pi t / p)} \leq \frac{2p}{\pi t}. \quad (13)$$

Lemma 3.2. Let $p \geq 2$, let $x < p - 1$ a and let $\omega = e^{2\pi i/p}$ be a $p$th root of unity. Then,

$$\left| \sum_{\gcd(n, p-1)=1} \omega^{tn} \right| \ll \frac{2p^t \delta \log p}{\pi t},$$

where $\delta > 0$ is a small real number and $t \in [1, p-1]$. 

Proof. Set \( \omega = e^{2\pi i/p} \). Use the inclusion exclusion principle to rewrite the exponential sum as

\[
\sum_{\gcd(n,p-1)=1} \omega^{tn} = \sum_{d \mid p-1} \mu(d) \sum_{n \leq x \atop \gcd(n,p-1)=1} \omega^{tn}
\]

(14)

\[
= \sum_{d \mid p-1} \mu(d) \sum_{n \leq x \atop \gcd(n,p-1)=1} \omega^{tn}
\]

\[
= \sum_{d \mid p-1} \mu(d) \sum_{m \leq (p-1)/d} \omega^{dtm}
\]

\[
= \sum_{d \mid p-1} \mu(d) \left( \frac{1 - \omega^{dt(\frac{p-1}{d}+1)}}{1 - \omega^{dt}} - 1 \right)
\]

\[
= \sum_{d \mid p-1} \mu(d) \frac{\omega^{dt} - \omega^{dt(p-1)/d}}{1 - \omega^{dt}},
\]

the last 2 lines follows from a geometric summation. Taking absolute value yields

\[
\left| \sum_{d \mid p-1} \mu(d) \frac{\omega^{dt} - \omega^{dp/p/d}}{1 - \omega^{dt}} \right| \leq \sum_{d \mid p-1} \left| \mu(d) \right| \cdot \left| \frac{\omega^{dt} - \omega^{dp/p/d}}{1 - \omega^{dt}} \right|
\]

(15)

\[
\leq \sum_{d \mid p-1} \left| \frac{\omega^{dt} - \omega^{dp/p/d}}{1 - \omega^{dt}} \right|
\]

\[
\leq \sum_{d \mid p-1} \left| \frac{2}{\sin(\pi dt/p)} \right|.
\]

Now, observe that the integers \( t \in [1, p-1] \), and \( d \leq p-1 \). This data implies that \( \pi dt/p \neq k\pi \) with \( k \in \mathbb{Z} \). Accordingly, the sine function \( \sin(\pi dt/p) \neq 0 \) is well defined. In addition, For each \( d \mid p-1 \), the argument \( dt/p \leq t/p < 1 \). Thus, the sine function approximation \( z/2 \leq \sin(z) < z \) for \( 0 < |z| < \pi/2 \) over the subinterval \( [1, p/d) \) is feasible here. Under these conditions, the last expression becomes

\[
\sum_{d \mid p-1} \left| \frac{2}{\sin(\pi dt/p)} \right| \leq \sum_{d \mid p-1} \left| \frac{2}{\sin(\pi dt/p/d)} \right|
\]

(16)

\[
\leq \sum_{d \mid p-1} \left| \frac{2}{\sin(\pi t/p)} \right|
\]

\[
\leq \frac{2p}{\pi t} \sum_{d \mid p-1} 1
\]

\[
\leq \frac{2p^{1+\delta}}{\pi t},
\]

where \( \sum_{d \mid n} 1 \ll n^{\delta}, \delta > 0 \).
Additional information on the order of the divisor function has a nearly explicit upper bound of the form \( \sum_{d|n} 1 = n^{(\log 2 + o(1)) / \log \log n} \), this appears in [10, Proposition 7.12], [19, Theorem 315] and similar sources.

3.3. Harmonic Summation Kernels. The harmonic summation kernels naturally arise in the partial sums of Fourier series and in the studies of convergences of continuous functions.

**Definition 3.3.** The Dirichlet kernel is defined by

\[
D_x(z) = \sum_{-x \leq n \leq x} e^{2\pi nz} = \frac{\sin((2x + 1)z)}{\sin(z)},
\]

where \( x \in \mathbb{N} \) is an integer and \( z \in \mathbb{R} - \pi \mathbb{Z} \) is a real number.

**Definition 3.4.** The Fejer kernel is defined by

\[
F_x(z) = \sum_{0 \leq n \leq x} \sum_{-n \leq k \leq n} e^{2\pi k z} = \frac{1}{2} \frac{\sin((x + 1)z)^2}{\sin(z)^2},
\]

where \( x \in \mathbb{N} \) is an integer and \( z \in \mathbb{R} - \pi \mathbb{Z} \) is a real number.

These formulas are well known, see [21] and similar references. For \( z \neq k\pi \), the harmonic summation kernels have the upper bounds \( |K_x(z)| = |D_x(z)| \ll |x| \), and \( |K_x(z)| = |F_x(z)| \ll |x|^2 \).

An important property is the that a proper choice of the parameter \( x \geq 1 \) can shifts the sporadic large value of the reciprocal sine function \( 1/\sin z \) to \( K_x(z) \), and the term \( 1/\sin(2x + 1)z \) remains bounded. This principle will be applied to the lacunary sequence \( \{p_n : n \geq 1\} \), which maximize the reciprocal sine function \( 1/\sin z \), to obtain an effective upper bound of the function \( 1/\sin z \).

The Dirichlet kernel in **Definition 3.3** is a well defined continued function of two variables \( x, z \in \mathbb{R} \). Hence, for fixed \( z \), it has an analytic continuation to all the real numbers \( x \in \mathbb{R} \).

A measure theoretical result for the maximal magnitude of the Dirichlet kernel, Fejer and other related exponential sums is stated here.

**Theorem 3.1.** ([14, Theorem 1]) Let \( N \geq 2 \) be a large integer and let \( \mathcal{U} \subset [-N, N] \) be a large subset of cardinality \( \# \mathcal{U} = N + 1 \). If \( T = \{s \in \mathbb{C} : |s| = 1\} \), then the subset of real numbers \( \mathcal{A} = \{\alpha \in \mathbb{T}\} \) such that the exponential sum

\[
\sum_{u \in \mathcal{U}} e^{2\pi \alpha u} \geq \frac{2\sqrt{2}}{\pi} (N + 1)
\]

has a large magnitude has cardinality

\[
\# \mathcal{A} = \frac{2\theta}{N + 1} + O \left( \frac{1}{N^3} \right),
\]

where \( \theta \in (0, 1) \). Furthermore, the largest measure \( \mathcal{A} \) is attained if and only if \( \mathcal{U} \) is an arithmetic progression.

This is related to the cosine problem, which asks for the maximal of the finite sum

\[
\sum_{u \in \mathcal{U}} \cos \alpha u,
\]

(19)
and the Littlewood conjecture, which states that
\[ \sum_{u \in \mathbb{Z}/N\mathbb{Z}} e^{i2\pi au} \gg \log N. \] (20)

The lower bound
\[ \sum_{u \in \mathbb{Z}/N\mathbb{Z}} e^{i2\pi au} \gg \frac{\log N}{(\log \log N)^2}, \] (21)

which very close to the claim is achieved in [28]. Now, observe that (21) leads to the norm lower bound
\[ \left| \sum_{1 \leq u \leq N} e^{i2\pi au} \right|^2 = \sum_{1 \leq u \leq N} e^{i2\pi au} \cdot \sum_{1 \leq v \leq N} e^{-i2\pi av} + \sum_{1 \leq u, v \leq N, u \neq v} e^{i2\pi a(u-v)} \gg N + N \frac{\log N}{(\log \log N)^2}. \] (22)

For irrational \( \alpha \neq 0 \), this implies that exponential sums such as
\[ \sum_{1 \leq u \leq N} e^{i2\pi au} \gg \sqrt{N \log N} \] (23)

and the Dirichlet kernel (17) have large lower bound.

4. Estimates Of Exponential Sums

Exponential sums indexed by the powers of elements of nontrivial orders have applications in mathematics and cryptography. These applications have propelled the development of these exponential sums.

**Theorem 4.1.** ([13, Lemma 4]) For integers \( a, k, N \in \mathbb{N} \), assume that \( \gcd(a, N) = c \), and that \( \gcd(k, t) = d \).

(i) If the element \( \theta \in \mathbb{Z}_N \) is of multiplicative order \( t \geq t_0 \), then
\[ \max_{1 \leq x \leq t} \left| \sum_{1 \leq x \leq t} e^{i2\pi a x^{t}/N} \right| < cd^{1/2} N^{1/2}. \] (24)

(ii) If \( H \subset \mathbb{Z}/N\mathbb{Z} \) is a subset of cardinality \( \#H \geq N^\delta \), \( \delta > 0 \), then
\[ \max_{\gcd(a,\varphi(N))=1} \left| \sum_{x \in H} e^{i2\pi a x^{t}/N} \right| < N^{1-\delta}. \] (25)

Various upper bounds of exponential sums over subsets of elements in finite rings \((\mathbb{Z}/N\mathbb{Z})^X\) can be used to prove the next result. These estimates are useful in the proof of Lemma 4.3. The reader should consult the literature, such as [7], [5], [4, Theorem 2.1], [20], and references within the cited papers.
4.1. **Incomplete Exponential Sums over Consecutive Index.** The simple finite Fourier transform identity

\[ \sum_{n \leq x} f(n) = \sum_{n \leq x} F(f(n)), \quad (26) \]

see Definition 3.2, is very effective in computing upper bounds of some exponential sums. An improved version of Theorem 4.1 and a few other applications are illustrated here.

**Theorem 4.2.** ([29], [26]) Let \( p \geq 2 \) be a large prime, and let \( \tau \in \mathbb{F}_p \) be an element of large multiplicative order \( \text{ord}_p(\tau) \mid p - 1 \). Then, for any \( b \in [1, p - 1] \), and \( x \leq p - 1 \),

\[ \sum_{n \leq x} e^{i2\pi b\tau^n/p} \ll p^{1/2} \log p. \quad (27) \]

**Proof.** Let \( p \) be a prime, and let \( f(n) = e^{i2\pi b\tau^n/p} \), where \( \tau \) is a primitive root modulo \( p \). Applying the finite summation kernel in (26) yields

\[ \sum_{n \leq x} e^{i2\pi b\tau^n/p} = \sum_{n \leq x} \frac{1}{p} \sum_{0 \leq t \leq p - 1, 1 \leq s \leq p - 1} \omega^{t(n-s)} e^{i2\pi bs^t/p}. \quad (28) \]

The term \( t = 0 \) contributes \( -x/p \), and rearranging it yield

\[ \sum_{n \leq x} e^{i2\pi b\tau^n/p} = \frac{1}{p} \sum_{n \leq x} \sum_{1 \leq t \leq p - 1, 1 \leq s \leq p - 1} \omega^{t(n-s)} e^{i2\pi bs^t/p} - \frac{x}{p} \quad (29) \]

\[ = \frac{1}{p} \sum_{1 \leq t \leq p - 1} \left( \sum_{1 \leq s \leq p - 1} \omega^{-ts} e^{i2\pi bs^t/p} \right) \left( \sum_{n \leq x} \omega^{tn} \right) - \frac{x}{p}. \]

Taking absolute value, and applying Lemma 3.1 and Lemma 5.1, yield

\[ \left| \sum_{n \leq x} e^{i2\pi b\tau^n/p} \right| \leq \frac{1}{p} \sum_{1 \leq t \leq p - 1} \left| \sum_{0 \leq s \leq p - 1} \omega^{-ts} e^{i2\pi bs^t/p} \right| \cdot \left| \sum_{n \leq x} \omega^{tn} \right| + \frac{x}{p} \]

\[ \ll \frac{1}{p} \sum_{1 \leq t \leq p - 1} \left( 2p^{1/2} \log p \right) \cdot \left( \frac{2p}{\pi t} \right) + \frac{x}{p} \quad (30) \]

\[ \ll p^{1/2} \log^2 p. \]

The third line in (30) uses the estimate

\[ \sum_{1 \leq t \leq p - 1} \frac{1}{t} \ll \log p \ll \log p. \quad (31) \]

This appears to be the best possible upper bound. The above proof generalizes the sum of resolvents method used in [26]. Here, it is reformulated as a finite Fourier transform method, which is applicable to a wide range of functions. A similar upper bound for composite moduli \( p = m \) is also proved, [op. cit., equation (2.29)].
4.2. Incomplete Exponential Sums over Relatively Prime Index.

**Theorem 4.3.** ([29]) Let $p \geq 2$ be a large prime, and let $\tau \in \mathbb{F}_p$ be an element of large multiplicative order $p - 1 = \text{ord}_p(\tau)$. If $x \leq p - 1$ and $s \in [1, p - 1]$, then
\[ \sum_{n \leq x} e^{i2\pi s\tau^n/p} \ll p^{1/2} \log p. \] (32)

This appears to be the best possible upper bound. A similar upper bound for composite moduli $p = m$ is also proved, [op. cit., equation (2.29)]. A simpler proof and generalization of this exponential is is provided in [26].

**Lemma 4.1.** Let $p \geq 2$ be a large prime, let $x \leq p$ and let $\tau$ be a primitive root modulo $p$. If $s \in [1, p - 1]$, then,
\[ \sum_{\gcd(n, p - 1) = 1, n \leq x} e^{i2\pi s\tau^n/p} \ll p^{1/2 + \delta}, \] (33)
where $\delta > 0$ is a small real number.

**Proof.** Use the inclusion exclusion principle to rewrite the exponential sum as
\[ \sum_{n \leq x, \gcd(n, p - 1) = 1} e^{i2\pi s\tau^n/p} = \sum_{n \leq p - 1} e^{i2\pi s\tau^n/p} \sum_{\gcd(n, p - 1) = 1} \mu(d) \] (34)
\[ = \sum_{d|p-1} \mu(d) \sum_{n \leq p - 1, d|n} e^{i2\pi s\tau^n/p} \]
\[ = \sum_{d|p-1} \mu(d) \sum_{m \leq (p-1)/d} e^{i2\pi s\tau^{dm}/p}. \]

Taking absolute value, and invoking Theorem 4.3 yield
\[ \left| \sum_{n \leq x, \gcd(n, p - 1) = 1} e^{i2\pi s\tau^n/p} \right| \leq \sum_{d|p-1} |\mu(d)| \left| \sum_{m \leq (p-1)/d} e^{i2\pi s\tau^{dm}/p} \right| \]
\[ \ll \sum_{d|p-1} |\mu(d)| \left( \left( \frac{p - 1}{d} \right)^{1/2} \log p \right) \] (35)
\[ \ll (p - 1)^{1/2} \log(p - 1) \sum_{d|p-1} \frac{|\mu(d)|}{d^{1/2}} \]
\[ \ll (p - 1)^{1/2} \log(p - 1) \cdot p^\delta \]
\[ \ll p^{1/2 + \delta}. \]

The last inequality follows from
\[ \sum_{d|p-1} \frac{|\mu(d)|}{d^{1/2}} \leq \sum_{d|p-1} 1 \ll p^\delta \] (36)
for any arbitrary small number $\delta > 0$, and any sufficiently large prime $p \geq 2$. □
A different approach to this result appears in [11, Theorem 6], and related results are given in [7], [12], [18], and [17, Theorem 1]. The upper bound given in Theorem 4.2 seems to be optimum. A different proof, which has a weaker upper bound, appears in [11, Theorem 6], and related results are given in [7], [12], [17], and [17, Theorem 1].

### 4.3. Equivalent Exponential Sums over Consecutive Index.

For any fixed $0 \neq b \in \mathbb{F}_p$, the map $\tau^n \mapsto b\tau^n$ is one-to-one in $\mathbb{F}_p$. Consequently, the subsets

$$\{\tau^n : n \leq x\} \quad \text{and} \quad \{b\tau^n : n \leq x\} \subset \mathbb{F}_p$$

have the same cardinalities. As a direct consequence the exponential sums

$$\sum_{n \leq x} e^{i2\pi b\tau^n/p} \quad \text{and} \quad \sum_{n \leq x} e^{i2\pi\tau^n/p},$$

have the same upper bound up to an error term up to an error term. The result below expresses the first exponential sum in (38) as a sum of simpler exponential sum and an error term.

**Lemma 4.2.** Let $p \geq 2$ be a large primes. If $\tau$ be a primitive root modulo $p$, then,

$$\sum_{n \leq x} e^{i2\pi b\tau^n/p} = \sum_{n \leq x} e^{i2\pi\tau^n/p} + O(p^{1/2}\log^2 p),$$

for any $b \in [1, p - 1]$.

**Proof.** For $b \neq 1$, the exponential sum has the representation

$$\sum_{n \leq x} e^{i2\pi b\tau^n/p} = \frac{1}{p} \sum_{1 \leq t \leq p-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi bs/p} \right) \left( \sum_{n \leq x} \omega^{tn} \right) - \frac{\varphi(p)}{p},$$

confer equation (29) for more details. And, for $b = 1$,

$$\sum_{n \leq x} e^{i2\pi\tau^n/p} = \frac{1}{p} \sum_{1 \leq t \leq p-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi s/p} \right) \left( \sum_{n \leq x} \omega^{tn} \right) - \frac{\varphi(p)}{p}.$$

Differencing (40) and (41) produces

$$S = \sum_{n \leq x} e^{i2\pi b\tau^n/p} - \sum_{n \leq x} e^{i2\pi\tau^n/p}$$

$$= \frac{1}{p} \sum_{0 \leq t \leq p-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi bs/p} - \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi s/p} \right) \left( \sum_{n \leq x} \omega^{tn} \right).$$

By Lemma 3.1, the summation kernel is bounded by

$$\left| \sum_{n \leq x} \omega^{tn} \right| \leq \frac{2p}{\pi t},$$

and by Lemma 5.1, the difference of two Gauss sums is bounded by

$$G = \left| \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi bs/p} - \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi s/p} \right|$$

$$\leq \left| \sum_{1 \leq s \leq p-1} \chi(s)\psi_b(s) \right| + \left| \sum_{1 \leq s \leq p-1} \chi(s)\psi_1(s) \right|$$

$$\leq 4p^{1/2}\log p,$$
where \( \chi(s) = e^{ist/p} \), and \( \psi_b(s) = e^{i2\pi \tau^s/p} \). Taking absolute value in (42) and replacing (43), and (44), return

\[
|S| = \left| \sum_{n \leq p-1 \atop \gcd(n,p-1)=1} e^{i2\pi \tau^n/p} - \sum_{n \leq p-1 \atop \gcd(n,p-1)=1} e^{i2\pi \tau^n/p} \right| \\
\leq \frac{1}{p} \sum_{0 \leq t \leq p-1} \left( 4p^{1/2} \log p \right) \cdot \left( \frac{2p}{t} \right) \\
\leq 8p^{1/2} (\log p) (\log \log p) \\
\leq 8p^{1/2} \log^2 p. 
\]

### 4.4. Equivalent Exponential Sums over Relatively Prime Index.

For any fixed primitive root \( \tau \) and \( 0 \neq b \in \mathbb{F}_p \), the maps \( n \rightarrow \tau^n \) and \( n \rightarrow b\tau^n \) are one-to-one in \( \mathbb{F}_p \). Consequently, the subsets

\[
\{ \tau^n : n \leq x \} \text{ and } \{ b\tau^n : n \leq x \} \subset \mathbb{F}_p
\]

have the same cardinalities. As a direct consequence the exponential sums

\[
\sum_{n \leq x \atop \gcd(n,p-1)=1} e^{i2\pi b\tau^n/p} \quad \text{and} \quad \sum_{n \leq x \atop \gcd(n,p-1)=1} e^{i2\pi \tau^n/p},
\]

have the same upper bound up to an error term up to an error term. An asymptotic formula is provided in Lemma 4.3. The proof is based on finite Fourier transform version of the Lagrange resolvent

\[
(\omega^t, \zeta^s) = \zeta^s + \omega^{-t} \zeta^{st} + \omega^{-2t} \zeta^{s2t} + \cdots + \omega^{-(p-1)t} \zeta^{st^{p-1}},
\]

where \( \omega = e^{i2\pi/p}, \zeta = e^{i2\pi/p}, \) and \( 0 \neq s, t \in \mathbb{F}_p \).

The result below expresses the first exponential sum in (47) as a sum of simpler exponential sum and an error term.

**Lemma 4.3.** Let \( p \geq 2 \) be a large primes. If \( \tau \) be a primitive root modulo \( p \), then,

\[
\sum_{n \leq x \atop \gcd(n,p-1)=1} e^{i2\pi b\tau^n/p} = \sum_{n \leq x \atop \gcd(n,p-1)=1} e^{i2\pi \tau^n/p} + O\left(p^{1/2+\delta}\right),
\]

for any \( b \in [1, p-1] \) and \( \delta > 0 \) is a small real number.

**Proof.** For \( b \neq 1 \), the exponential sum has the representation

\[
\sum_{n \leq x \atop \gcd(n,p-1)=1} e^{i2\pi b\tau^n/p} = \frac{1}{p} \sum_{1 \leq t \leq p-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi \tau^n/p} \right) \left( \sum_{n \leq x \atop \gcd(n,p-1)=1} \omega^{tn} \right) - \frac{\varphi(p)}{p},
\]

where \( \chi(s) = e^{i2\pi st/p} \), and \( \psi_b(s) = e^{i2\pi \tau^s/p} \).
confer equation (29) for more details. And, for \( b = 1 \),
\[
\sum_{\substack{n \leq x \\ \gcd(n, p-1) = 1}} e^{i2\pi \frac{bn}{p}} = \frac{1}{p} \sum_{1 \leq t \leq p-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi \frac{bs}{p}} \right) \left( \sum_{\substack{n \leq x \\ \gcd(n, p-1) = 1}} \omega^{tn} \right) - \frac{\varphi(p)}{p}. \tag{51}
\]

Differencing (50) and (51) produces
\[
S = \sum_{\substack{n \leq x \\ \gcd(n, p-1) = 1}} e^{i2\pi b \frac{n}{p}} - \sum_{\substack{n \leq x \\ \gcd(n, p-1) = 1}} e^{i2\pi \frac{n}{p}}
\]
\[
= \frac{1}{p} \sum_{0 \leq t \leq p-1} \left( \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi \frac{bs}{p}} - \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi \frac{s}{p}} \right) \left( \sum_{\substack{n \leq x \\ \gcd(n, p-1) = 1}} \omega^{tn} \right).
\]

By Lemma 3.2, the summation kernel is bounded by
\[
\left| \sum_{\substack{n \leq x \\ \gcd(n, p-1) = 1}} \omega^{tn} \right| \ll \frac{2p^{1+\delta} \log p}{\pi t}, \tag{53}
\]
and by Lemma 5.1, the difference of two Gauss sums is bounded by
\[
G = \left| \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi \frac{bs}{p}} - \sum_{1 \leq s \leq p-1} \omega^{-ts} e^{i2\pi \frac{s}{p}} \right| \tag{54}
\]
\[
\leq \left| \sum_{1 \leq s \leq p-1} \chi(s) \psi_b(s) \right| + \left| \sum_{1 \leq s \leq p-1} \chi(s) \psi_1(s) \right|
\]
\[
\leq 4p^{1/2} \log p,
\]
where \( \chi(s) = e^{i\pi st/p} \), and \( \psi_b(s) = e^{i2\pi brs/p} \). Taking absolute value in (52) and replacing (53), and (54), return
\[
|S| = \left| \sum_{\substack{n \leq x \\ \gcd(n, p-1) = 1}} e^{i2\pi b \frac{n}{p}} - \sum_{\substack{n \leq x \\ \gcd(n, p-1) = 1}} e^{i2\pi \frac{n}{p}} \right|
\]
\[
\ll \frac{1}{p} \sum_{0 \leq t \leq p-1} \left( 4p^{1/2} \log p \right) \cdot \left( \frac{2p^{1+\delta} \log p}{\pi t} \right)
\]
\[
\ll 8p^{1/2+\delta}(\log p)(\log p)(\log p)
\]
\[
\ll p^{1/2+\delta},
\]
where \((\log p)^3\) is absorbed by the term \( p^\delta \). \(\blacksquare\)
5. Results for Gaussian Sums

Some elementary exponential sums estimates are provided in this section.

**Lemma 5.1.** (Gauss sums) Let \( p \geq 2 \) be a prime, let \( \chi(t) = e^{it\pi/p} \) and \( \psi(t) = e^{it\pi/p} \) be a pair of characters. Then, the Gaussian sum has the upper bound

\[
\left| \sum_{1 \leq t \leq p-1} \chi(t)\psi(t) \right| \leq 2p^{1/2} \log p.
\]

**Proof.** The proof for \( \left| \chi(t)\psi(t) \right| \leq p^{1/2} \log p \) appears in [26]. Hence, the difference

\[
\left| \sum_{1 \leq t \leq p-1} \chi(t)\psi(t) \right| \leq 2p^{1/2} \log p.
\]

**Lemma 5.2.** Let \( p \geq 2 \) be a prime. If \( \omega = e^{i\pi/p}, \zeta = e^{i\pi/p}, \) and \( 0 \neq s, t \in F_p, \) then, the difference of two Lagrange resolvents has the upper bound

\[
\left| (\omega^s, \zeta^{sr^p}) - (\omega^t, \zeta^{rt^p}) \right| \leq 2p^{1/2} \log p. \quad (56)
\]

**Proof.** The proof for \( (\omega^s, \zeta^{sr^p}) \leq p^{1/2} \log p \) appears in [26]. Hence, the difference

\[
\left| (\omega^s, \zeta^{sr^p}) - (\omega^t, \zeta^{rt^p}) \right| \leq \left| (\omega^s, \zeta^{sr^p}) \right| + \left| (\omega^t, \zeta^{rt^p}) \right| \leq 2p^{1/2} \log p. \quad (57)
\]

**Lemma 5.3.** Let \( p \geq 2 \) be a prime and let \( \tau \in F_p \) be a primitive root. If \( t \neq 0, \) then

\[
\sum_{0 \leq s < p/2} e^{i2\pi s t + \pi i} = \frac{w}{2} \left( \frac{(\tau t)^{-1}}{p} \right) p^{1/2},
\]

where \( w \neq 1 \) is a root of unity.

**Proof.** Rewrite the finite sum in term of the quadratic symbol \( (a \mid p) \) in the form

\[
\sum_{0 \leq s < p/2} e^{i2\pi s t + \pi i} = \frac{1}{2} \sum_{0 \leq a < p} \left( 1 + \left( \frac{a}{p} \right) \right) e^{i2\pi a t/p} \quad (58)
\]

\[
= \frac{1}{2} \sum_{0 \leq a < p} \left( \frac{a}{p} \right) e^{i2\pi a t/p} \quad (59)
\]

\[
= \frac{1}{2} \left( \frac{(\tau t)^{-1}}{p} \right) \sum_{0 \leq a < p} \left( \frac{a}{p} \right) e^{i2\pi a/p} \quad (60)
\]

\[
= \frac{w}{2} \left( \frac{(\tau t)^{-1}}{p} \right) p^{1/2},
\]

where \( w \in C \) is a root of unity. \( \square \)

6. Fibers and Multiplicities for Quadratic Residues

The multiplicities of the fibers occurring in the estimate of the error term are computed in this section.

**Lemma 6.1.** Let \( p \) be a prime, let \( x = (\log p)^{1+\varepsilon} \) and let \( \tau \in F_p \) be a primitive root in the finite field \( F_p. \) Define the maps

\[
\alpha(s, n) \equiv (\tau^{2s+1} - n) \pmod{p} \quad \text{and} \quad \beta(u, v) \equiv uv \pmod{p}. \quad (59)
\]

Then, the fibers \( \alpha^{-1}(m) \) and \( \beta^{-1}(m) \) of an element \( 0 \neq m \in F_p \) have the cardinalities

\[
\# \alpha^{-1}(m) \leq x - 1 \quad \text{and} \quad \# \beta^{-1}(m) = x
\]

respectively.
Proof. Let \( \mathcal{S} = \{ s < p^{1-\varepsilon} \} \). Given a fixed \( n \in [2, x] \), the map
\[
\alpha : \mathcal{S} \times [2, x] \longrightarrow \mathbb{F}_p \quad \text{defined by} \quad \alpha(s, n) \equiv (\tau^{2s+1} - n) \mod p,
\]
is one-to-one. This follows from the fact that the map \( s \rightarrow \tau^s \mod p \) is a permutation the nonzero elements of the finite field \( \mathbb{F}_p \), and the map \( (s, n) \rightarrow (\tau^{2s+1} - n) \mod p \) is a shifted permutation of the subset of quadratic nonresidues
\[
\mathcal{N} = \{ \tau^{2s+1} : s < p^{1-\varepsilon} \} \subset \mathbb{F}_p,
\]
see [22, Chapter 7] for more extensive details on the theory of permutation functions of finite fields. Thus, as \((s, n) \in \mathcal{S} \times [2, x]\) varies, each value \( m = \alpha(s, n) \) is repeated at most \( x - 1 \) times. Moreover, the premises no quadratic nonresidues \( n \leq x = (\log p)^{1+\varepsilon} \) implies that \( m = \alpha(s, n) \neq 0 \). This verifies that the cardinality of the fiber is
\[
\#\alpha^{-1}(m) = \#\{(s, n) \in \mathcal{S} \times [2, x] : m \equiv (\tau^{2s+1} - n) \mod p \}
\leq x - 1.
\]
Similarly, given a fixed \( u \in [1, x] \), the map
\[
\beta : [1, x] \times [1, p - 1] \longrightarrow \mathbb{F}_p \quad \text{defined by} \quad \beta(u, v) \equiv uv \mod p,
\]
is one-to-one. Here the map \( v \rightarrow uv \mod p \) permutes the elements of the finite field \( \mathbb{F}_p \). Thus, as \((u, v) \in [1, x] \times [1, p - 1]\) varies, each value \( m = \beta(u, v) \) is repeated exactly \( x \) times. This verifies that the cardinality of the fiber is
\[
\#\beta^{-1}(m) = \#\{(u, v) \in [1, x] \times [1, p - 1] : m \equiv uv \mod p \} = x.
\]
Now each value \( m = \alpha(s, n) \neq 0 \) (of multiplicity up to \((x - 1)\) in \( \alpha^{-1}(m) \)), is matched to \( m = \alpha(s, n) = \beta(u, v) \) for some \((u, v)\), (of multiplicity exactly \( x \) in \( \beta^{-1}(m) \)). Comparing (63) and (65) prove that \( \#\alpha^{-1}(m) \leq \#\beta^{-1}(m) \). ■

7. Evaluation of the Main Term

An asymptotic formula for the main term \( M(x) \) is evaluated in this section.

Lemma 7.1. Let \( \varepsilon > 0 \) be a small real number. If \( p \geq 2 \) is a large prime and \( x = (\log p)^{1+\varepsilon} \), then
\[
\sum_{2 \leq n \leq x} 1 \sum_{0 \leq s < p/2} \frac{1}{p} = \frac{x}{2} + O(1).
\]

Proof. A routine calculation returns
\[
M(x) = \sum_{2 \leq n \leq x} 1 \sum_{0 \leq s < p/2} \frac{1}{p}
= (x - O(1)) \cdot \frac{1}{p} \left( \frac{p}{2} + 1 \right)
= \frac{x}{2} + O(1).
\]
■
8. Estimate For The Error Term

A nontrivial upper bound for the error term \( E(x) \) is computed in this section. The error term is partitioned as \( E(x) = E_0(x) + E_1(x) \). The upper bound of the first term \( E_0(x) \) for \( n \leq p/x \) is derived using a geometric series/sine function techniques, and the upper bound of the second term \( E_1(x) \) for \( p/x \leq n \leq p/2 \) is derived using exponential sums techniques.

**Lemma 8.1.** Let \( \varepsilon > 0 \) be a small real number. Suppose \( p \geq 2 \) is a large prime and \( n \leq x = (\log p)^{1+\varepsilon} \). If there is no quadratic nonresidue \( n \leq x = (\log p)^{1+\varepsilon} \), then

\[
\sum_{2 \leq n \leq x} \sum_{0 \leq s < p/2} \sum_{1 \leq t \leq p-1} \frac{1}{p} e^{i2\pi \frac{(t^2+1-n)l}{p}} = O \left( (\log p)(\log \log p) \right).
\]

**Proof.** The product of a point \((u, v) \in [1, x] \times [1, p/x]\) satisfies \( uv < p \). This leads to the partition \([1, p/x] \cup [p/x, p/2]\) of the index \( n \), which is suitable for the sine approximation \( uv/p \ll \sin(\pi uv/p) \ll uv/p \) for \( |uv/p| < 1 \) on the first subinterval \([1, p/x]\), see (72) for more details. Thus, consider the partition of the triple finite sum

\[
E(x) = \sum_{2 \leq n \leq x} \sum_{s < p/2} \sum_{1 \leq t \leq p-1} \frac{1}{p} e^{i2\pi \frac{(t^2+1-n)l}{p}}
\]

\[
= \sum_{2 \leq n \leq x} \sum_{s < p/x} \sum_{1 \leq t \leq p-1} \frac{1}{p} e^{i2\pi \frac{(t^2+1-n)l}{p}} + \sum_{2 \leq n \leq x} \sum_{p/x \leq s < p/2} \sum_{1 \leq t \leq p-1} \frac{1}{p} e^{i2\pi \frac{(t^2+1-n)l}{p}}
\]

\[
= E_0(x) + E_1(x).
\]

Summing yields

\[
E(x) = E_0(x) + E_1(x) \ll (\log x)(\log p) + \frac{x(\log p)^2}{p^{1/2}} \ll (\log x)(\log p).
\]

This completes the estimate of the error term.

**Lemma 8.2.** Let \( p \geq 2 \) be a large primes. If \( \tau \) be a primitive root modulo \( p \), then

\[
E_0(x) = \sum_{2 \leq n \leq x} \sum_{s < p/\tau} \sum_{1 \leq t \leq p-1} \frac{1}{p} e^{i2\pi \frac{(t^2+1-n)l}{p}} = O((\log x)(\log p)).
\]

**Proof.** To apply the geometric series/sine function techniques, the subsum \( E_0(x) \) is partition as follows.

\[
E_0(x) = \sum_{2 \leq n \leq x} \sum_{s < p/\tau} \sum_{1 \leq t \leq p-1} \frac{1}{p} e^{i2\pi \frac{(t^2+1-n)l}{p}}
\]

\[
= \sum_{2 \leq n \leq x} \sum_{s < p/\tau} \sum_{1 \leq t < p/2} \frac{1}{p} e^{i2\pi \frac{(t^2+1-n)l}{p}} + \sum_{p/2 \leq t \leq p-1} \frac{1}{p} e^{i2\pi \frac{(t^2+1-n)l}{p}}
\]

\[
= E_{0,0}(x) + E_{0,1}(x).
\]
Now, a geometric series summation of the inner finite sum in the first term yields

\[ E_{0,0}(x) = \sum_{2 \leq n \leq x} \frac{1}{p} \sum_{s < p/x, 1 \leq t < p/2} e^{i2\pi \left(\frac{2t+1-n}{p}u\right)} \]

(71)

\[ = \frac{1}{p} \sum_{2 \leq n \leq x} \sum_{s < p/x} e^{i2\pi \left(\frac{2t+1-n}{p}(\frac{t}{s}+1)\right) - 1} \]

\[ \leq \frac{1}{p} \sum_{2 \leq n \leq x} \sum_{s < p/x} \left| \sin \pi \left(\frac{\tau s + 1 - n}{p}\right) \right|^2 \]

\[ \leq \frac{1}{p} \sum_{2 \leq n \leq x} \sum_{s < p/x} \left| \sin \pi \left(\frac{\tau s + 1 - n}{p}\right) \right|, \]

see [8, Chapter 23] for similar geometric series calculation and estimation. The last line in (71) follows from the hypothesis that \( u \) is not a primitive root. Specifically, \( 0 \neq \tau^s - n \in \mathbb{F}_p \) for any \( n \geq 1 \) and any \( n \leq x = (\log p)^{1+\varepsilon} \). Utilizing Lemma 6.1, the first term has the upper bound

\[ E_{0,0}(x) = \frac{1}{p} \sum_{2 \leq n \leq x} \sum_{s < p/x, 1 \leq \nu < p} \frac{1}{\sin \pi(\tau s + 1 - n)/p} \]

(72)

\[ \ll \frac{2}{p} \sum_{1 \leq u \leq x, 1 \leq v < p} \frac{1}{\sin \pi uv/p} \]

\[ \ll \frac{2}{p} \sum_{1 \leq u \leq x, 1 \leq v < p} \frac{p}{\pi uv} \]

\[ \approx \sum_{1 \leq u \leq x} \frac{1}{u} \sum_{1 \leq v < p} \frac{1}{v} \]

\[ \ll (\log x)(\log p), \]

where \( uv < p \) and \( |\sin \pi uv/p| \neq 0 \) since \( p \nmid uv \). Similarly, the second term has the upper bound

\[ E_{0,1}(x) = \sum_{2 \leq n \leq x} \frac{1}{p} \sum_{s < p/x, p/2 \leq t < p-1} e^{i2\pi \left(\frac{2t+1-n}{p}t\right)} \]

(73)

\[ = \frac{1}{p} \sum_{2 \leq n \leq x} \sum_{s < p/x} \frac{1 - e^{i2\pi \left(\frac{s^2-n}{p}(\frac{t}{s}+1)\right)}/p}{1 - e^{i2\pi \left(\frac{s^2-n}{p}\right)}} \]

\[ \leq \frac{1}{p} \sum_{2 \leq n \leq x} \sum_{s < p/x} \left| \sin \pi \left(\frac{\tau^s - n}{p}\right) \right|/p \]

\[ \ll (\log x)(\log p). \]

This is computed in the way as done in (71) to (72), mutatis mutandis. Hence,

\[ E_0(x) = E_{0,0}(x) + E_{0,1}(x) \ll (\log x)(\log p). \]

(74)
Lemma 8.3. Let $p \geq 2$ be a large primes. If $\tau$ be a primitive root modulo $p$, then,

$$E_1(x) = \sum_{2 \leq n \leq x} \frac{1}{p} \sum_{p/x \leq s < p/2} \sum_{1 \leq t \leq p-1} e^{i2\pi \left(\frac{s \tau + 1 - n}{p}\right)} = O\left(\frac{(\log p)^2}{p^{1/2}}\right), \quad (75)$$

for any $a \in [1, p - 1]$ and $\delta > 0$ is a small real number.

Proof. Rearrange the second subsum and apply Lemma 4.3 to the inner sum to obtain this:

$$E_1(x) = \frac{1}{p} \sum_{2 \leq n \leq x} \sum_{1 \leq t \leq p-1} e^{-i2\pi n t/p} \sum_{p/x \leq s < p/2} e^{i2\pi \left(\frac{s \tau + 1 - n}{p}\right)} \quad (76)$$

$$= \frac{1}{p} \sum_{2 \leq n \leq x} \sum_{1 \leq t \leq p-1} e^{-i2\pi n t/p} \left( \sum_{p/x \leq s < p/2} e^{i2\pi \left(\frac{s \tau + 1}{p}\right)} + O(p^{1/2}(\log p)^2) \right).$$

The application of Lemma 4.3 to the inner exponential sum on the second line of (76) removes the dependence on the variable $s \neq 0$ in exchange for a simpler exponential sum and an error term, which are independent of the variable $s$. Now, use the exact evaluation $\sum_{1 \leq t \leq p-1} e^{-i2\pi n t/p} = -1$, take absolute value and apply the triangle inequality:

$$E_1(x) \ll \frac{1}{p} \sum_{2 \leq n \leq x} \left| \sum_{1 \leq t \leq p-1} e^{-i2\pi n t/p} \right| \left| e^{i2\pi \left(\frac{s \tau + 1}{p}\right)} \sum_{p/x \leq s < p/2} e^{i2\pi \left(\frac{s \tau + 1}{p}\right)} + p^{1/2}(\log p)^2 \right| \quad (77)$$

$$\ll \frac{1}{p} \sum_{2 \leq n \leq x} \left| \sum_{1 \leq t \leq p-1} e^{-i2\pi n t/p} \right| \left( \left| e^{i2\pi \left(\frac{s \tau + 1}{p}\right)} \sum_{p/x \leq s < p/2} e^{i2\pi \left(\frac{s \tau + 1}{p}\right)} \right| + |p^{1/2}(\log p)^2| \right)$$

$$\ll \frac{1}{p} \sum_{2 \leq n \leq x} \left| -1 \right| \left| p^{1/2}(\log p)^2 \right|$$

$$\ll \frac{x(\log p)^2}{p^{1/2}}. \quad (77)$$

The third line in (77) follows from Lemma 4.1.

9. Main Result

Define the quadratic nonresidue counting function by

$$N_p(x) = \sum_{n \leq x} \begin{cases} 1 & \left(\frac{n}{p}\right) = -1 \end{cases}. \quad (78)$$

An asymptotic formula for this function is computed below.

Proof of Theorem 1.1. Let $p > 2$ be a large prime number, let $x = (\log p)^{1+\varepsilon}$, where $\varepsilon > 0$ is a small number. Suppose the least quadratic nonresidue $n > x$ and consider the sum of the characteristic function over the short interval $[2, x]$, that is,

$$N_p(x) = \sum_{2 \leq n \leq x} \chi(n) = 0. \quad (79)$$
Replacing the characteristic function, Lemma 2.2, and expanding the nonexistence equation (79) yield

$$N_p(x) = \sum_{2 \leq n \leq x} \chi(n)$$

(80)

$$= \sum_{2 \leq n \leq x} 1 \sum_{0 \leq s < p/2} \frac{1}{p} \sum_{0 \leq t \leq p-1} e^{i2\pi \frac{(2s+1-n)t}{p}}$$

$$= \sum_{2 \leq n \leq x} 1 \sum_{0 \leq s < p/2} \frac{1}{p} + \sum_{2 \leq n \leq x} 1 \sum_{0 \leq s < p/2} \frac{1}{p} \sum_{1 \leq t \leq p-1} e^{i2\pi \frac{(2s+1-n)t}{p}}$$

$$= M(x) + E(x).$$

The main term $M(x)$ is determined by $t = 0$, which reduces to the exponential to $e^{i2\pi st/p} = 1$, it is evaluated in Lemma 7.1. The error term $E(x)$ is determined by $t \neq 0$, which reduces to the exponential to $e^{i2\pi st/p} \neq 1$, it is estimated in Lemma 8.1.

Substituting these values yield

$$N_p(x) = \sum_{2 \leq n \leq x} \chi(n)$$

(81)

$$= M(x) + E(x)$$

$$= \left[ \frac{1}{2} (\log p)^{1+\varepsilon} + O(1) \right] + [O ((\log p)(\log \log p))]$$

$$= \frac{1}{2} (\log p)^{1+\varepsilon} + O ((\log p)(\log \log p)).$$

Consequently, the main term in (81) dominates the error term:

$$N_p(x) = \sum_{2 \leq n \leq x} \chi(n)$$

(82)

$$\gg (\log p)^{1+\varepsilon} \left(1 + O \left( \frac{\log \log p}{(\log p)^{\varepsilon}} \right) \right)$$

$$> 0$$

as $p \to \infty$. Clearly, this contradicts the hypothesis (79) for all sufficiently large prime numbers $p \geq p_0$. Therefore, there exists a quadratic nonresidue $n \leq x = (\log p)^{1+\varepsilon}$. ■
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