GLOBAL BOUNDEDNESS OF CLASSICAL SOLUTIONS TO A LOGISTIC CHEMOTAXIS SYSTEM WITH SINGULAR SENSITIVITY

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ABSTRACT. We consider a chemotaxis system with singular sensitivity and logistic-type source: $u_t = \Delta u - \chi \nabla \cdot (\chi \nabla v) + ru - \mu u^k$, $v_t = \epsilon \Delta v - v + u$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $\chi, r, \mu, \epsilon > 0$, $k > 1$ and $n \geq 2$. It is proved that the system possesses a globally bounded classical solution when $\epsilon + \chi < 1$. This shows that the diffusive coefficient $\epsilon$ of the chemical substance $v$ properly small benefits the global boundedness of solutions, without the restriction on the dampening exponent $k > 1$ in logistic source.

1. Introduction. In this paper, we consider the following chemotaxis system with singular sensitivity and logistic-type source

$$
\begin{cases}
  u_t = \Delta u - \chi \nabla \cdot (\chi \nabla v) + f(u), & x \in \Omega, \ t > 0, \\
  v_t = \epsilon \Delta v - v + u, & x \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
  (u(x, 0), v(x, 0)) = (u_0, v_0), & x \in \overline{\Omega},
\end{cases}
$$

(1)

where $\chi, \epsilon > 0$, $f(u) = ru - \mu u^k$ with $r, \mu > 0$ and $k > 1$. $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) is a bounded smooth domain. $\frac{\partial}{\partial \nu}$ denotes the derivation with respect to the outer normal of $\partial \Omega$, and for some $q > n$ the initial datum satisfy

$$
\begin{cases}
  u_0 \in C^0(\overline{\Omega}), \ u_0 > 0, \\
  v_0 \in W^{1,q}(\Omega), \ v_0 > 0.
\end{cases}
$$

(2)

Eq. (1) is an extended version of the well-known Keller-Segel system [10] (i.e., $f(u) = 0$), to describe the cells $u$ moving towards the concentration gradient of a chemical substance $v$ produced by the cells themselves. Here the involved singular chemotactic sensitivity function $\frac{\chi}{v}$ is derived by the Weber-Fechner laws. Recall the known results for $\epsilon = 1$ with $f(u) = 0$. All solutions are global in time when $n = 1$. 

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If $\chi < \frac{2}{n}$ with $n \geq 2$, there exists a globally bounded classical solution [6]. With larger chemotactic sensitivity coefficient

$$\chi < \begin{cases} \infty & \text{if } n = 2, \\ \sqrt{\frac{8}{n}} & \text{if } n = 3, \\ \frac{n}{n-2} & \text{if } n \geq 4. \end{cases}$$

the system possesses a global generalized solution [11]. See [1, 7, 12, 14] for more conclusions on chemotaxis models with singular sensitivity. Consider the case of $\epsilon = 1$ and $f(u) = ru - \mu u^k$. If $n, k = 2$, there exist global solutions [2], and globally bounded classical solutions [16] when

$$r > \begin{cases} \frac{\chi^2}{4}, & 0 < \chi \leq 2, \\ \chi - 1, & \chi > 2. \end{cases}$$

If $n, k \geq 2$, the global solvability of classical or generalized solutions has been discussed in [5, 17]. We refer to [8] on the related parabolic-elliptic case.

To study the global boundedness of solutions to the chemotaxis system with singular sensitivity, the main step is to establish the uniform lower bound estimate for chemical signal $v$. If $f(u) = 0$, this can be arrived thanks to the mass conservation for cell $u$. However, this a priori uniform estimate on $v$ will be false for the singular chemotaxis model with logistic-type source $ru - \mu u^k$, i.e., the singularity in the chemotactic sensitivity function $\chi^v$ may happen. By means of a weighted integral $\int_{\Omega} u^{-p} v^{-q} dx$, we can firstly obtain the desired uniformly lower bound estimate for $v$, and then study the global solvability of solutions to (1) for $\epsilon = 1$ [16, 17]. In this paper, with a transformation $w = uv^\frac{1}{1-\epsilon}$ inspired by [9, 18], we will obtain $\|w\|_{L^\infty(\Omega)}$ by Moser iteration, and hence the crucial estimate on $\|u\|_{L^q(\Omega)}$ for some $q > \frac{n}{2}$ via an iteration. This will be enough to ensure the global boundedness of solutions to system (1). Now, we state the main results.

**Theorem 1.1.** Let $n \geq 2$, $\chi, r, \mu > 0$ and $k > 1$. If $\epsilon \in (0, 1)$ and $\chi \in (0, 1 - \epsilon)$, the problem (1) admits a unique global bounded classical solution.

**Remark 1.** Recall from [16, 17] that the problem (1) with $\epsilon = 1$ admits a globally bounded classical solution provided the chemotactic sensitivity coefficient $\chi$ suitably small with respect to $r > 0$, and $k \geq 2$ for $n = 2$ or $k > \frac{n+2}{n+4}$ for $n \geq 3$. Theorem 1.1 says that the system (1) possesses a globally bounded classical solution provided $\chi + \epsilon \in (0, 1)$, without the restriction on the dampening exponent $k > 1$ in the logistic source $ru - \mu u^k$.

**Remark 2.** By [16, 17] it was known that the uniformly lower bound estimate for $v$ is the main step to establish the global boundedness of the solution to system (1) with $\epsilon = 1$. Differently, without considering this crucial estimate ($v$ may tend to 0), it is proved in Theorem 1.1 that the system admits a globally bounded classical solution via the transformation $uw^{\frac{1}{1-\epsilon}}$ with $\epsilon \in (0, 1 - \chi)$. In other words, the diffusive coefficient $\epsilon \in (0, 1)$ of the chemical signal $v$ properly small benefits the global boundedness of classical solution to system (1).

2. **Preliminaries.** We firstly introduce the local existence of classical solutions to (1) without proof in detail, which can be obtained by the standard contraction argument like that in [16, Lemma 2.1].
Lemma 2.1. With the initial data (2), there exists $T_{\text{max}} \in (0, +\infty)$ and a unique pair $(u, v)$ of functions
\[
\begin{cases}
u \in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})), \\
v \in C^0(\Omega \times [0, T_{\text{max}})) \cap C^{2,1}(\Omega \times (0, T_{\text{max}})) \cap L^\infty(0, T_{\text{max}}); W^{1,q}(\Omega),
\end{cases}
\]
satisfying (1) in the classical sense with $u, v > 0$ in $\Omega \times (0, T)$. Moreover, either $T_{\text{max}} = \infty$, or $\lim_{t \to T_{\text{max}}} \left(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,q}(\Omega)}\right) = \infty$.

Next, we give some a priori estimates of the local classical solution $(u, v)$ to system (1). For convenience, denote $T := T_{\text{max}}$.

Lemma 2.2. It holds for $k > 1$ that
\[
\iint_\Omega ud\mu \leq m_0, \quad t \in (0, T)
\]
with $m_0 = \max\left\{\int_\Omega u_0 d\mu, |\Omega(\frac{1}{\mu})^\frac{1}{p-1}\right\}$.

Proof. Integrate the first equation in (1) over $\Omega$ to get
\[
\frac{d}{dt} \iint_\Omega ud\mu = r \iint_\Omega ud\mu - \mu \iint_\Omega u^k d\mu \leq r \iint_\Omega ud\mu - \frac{\mu}{|\Omega|^{k-1}} \left(\iint_\Omega ud\mu\right)^k, \quad t \in (0, T)
\]
by the Hölder inequality, which yields (4) by the Bernoulli inequality [4, Lemma 1.2.4].

According to estimates of the homogeneous Neumann heat semigroups $\{e^{t\Delta}\}_{t > 0}$ in [15, Lemma 1.3 (i)], we have the following lemma.

Lemma 2.3. Assume for $q \geq 1$ that
\[
\|u\|_{L^q(\Omega)} \leq C_1, \quad t \in (0, T)
\]
with some $C_1 > 0$. Then for $r \in \left(1, \frac{na}{n-2q}\right)$ if $q \in \left[1, \frac{n}{2}\right]$, or $r = \infty$ if $q > \frac{n}{2}$ there exists $C_2 > 0$ such that
\[
\|v\|_{L^r(\Omega)} \leq C_2, \quad t \in (0, T).
\]

Proof. Denote $\bar{u} = \frac{1}{|\Omega|} \int_\Omega u d\mu$, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in $\Omega$ under Neumann boundary conditions. Applying [15, Lemma 1.3 (i)] to the second equation in (1), then for $r \in \left(1, \frac{na}{n-2q}\right)$ if $q \in \left[1, \frac{n}{2}\right]$ we have from (4) and (5) that
\[
\|v\|_{L^r(\Omega)} \leq \|v_0\|_{L^r(\Omega)} + \int_0^t \|e^{(t-s)(\epsilon\Delta-1)}u\|_{L^r(\Omega)} ds
\]
\[
\leq \|v_0\|_{L^r(\Omega)} + \int_0^t \|e^{(t-s)(\epsilon\Delta-1)}(u - \bar{u})\|_{L^r(\Omega)} ds + \int_0^t \|e^{(t-s)(\epsilon\Delta-1)}\bar{u}\|_{L^r(\Omega)} ds
\]
\[
\leq \|v_0\|_{L^r(\Omega)} + K_1 \int_0^t \left(1 + [e(t-s)]^{-\frac{n}{2}k}\right)e^{-(\lambda_1 + \frac{n}{2})[e(t-s)]\|u\|_{L^q(\Omega)} ds + m_0|\Omega|^{\frac{1}{q}-1}
\]
\[
\leq \|v_0\|_{L^r(\Omega)} + \frac{K_1 C_1}{\epsilon} \int_0^{\infty} (1 + \alpha)^{-\frac{n}{2}k} e^{-\lambda_1\alpha} d\alpha + m_0\|\Omega\|^{\frac{1}{q}-1} =: C_2
\]
with $K_1 > 0$ for $t \in (0, T)$. Similar calculation to the case of $q > \frac{n}{2}$, we derive the estimate (6) for $r = \infty$ with $C_2 := \|v_0\|_{L^\infty(\Omega)} + \frac{K_1 C_1}{\epsilon} \int_0^{\infty} (1 + \alpha)^{-\frac{n}{2}k} e^{-\lambda_1\alpha} d\alpha + m_0\|\Omega\|^{\frac{1}{q}-1}$.

This completes the proof. 

\qed
3. Proof of Theorem 1.1. Let \( m := \frac{1}{1-\epsilon} \) with \( \epsilon \in (0, 1) \). Moreover, denote
\[ w := uw^{-m}. \] (7)

Then the system (1) becomes
\[
\begin{align*}
  w_t &= \Delta w + (1 + \epsilon)m \frac{\nabla w \cdot \nabla v}{v^2} + cm(m-1)w \frac{\nabla v^2}{v^2} + (m + r)w - mw^2v^{m-1} - m\mu w^kv^{m(k-1)}, & x \in \Omega, & t > 0, \\
  v_t &= \epsilon \Delta v - v + uw^m, & x \in \Omega, & t > 0, \\
  \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, & t > 0, \\
  (w(x, 0), v(x, 0)) &= (u_0v_0^{-m}, v_0), & x \in \bar{\Omega}.
\end{align*}
\]
(8)

Notice that \((w, v)\) is also a local classical solution to (8) with \( w, v > 0 \) on \( \bar{\Omega} \times (0, T) \).

Now, we establish the estimate on \( \|w\|_{L^\infty(\Omega)} \).

Lemma 3.1. Let \( n \geq 2 \), \( \chi, r, \mu > 0 \) and \( k > 1 \). If \( \epsilon \in (0, 1) \) and \( \chi \in (0, 1-\epsilon) \), then there exists \( M_1 > 0 \) such that
\[
\|w\|_{L^\infty(\Omega)} \leq M_1, \quad t \in (0, T). \] (9)

Proof. For any \( p > 1 \) and \( \epsilon_0 > 0 \), multiply the first equation of (8) by \( w^{p-1} \) and integrate over \( \Omega \) to have
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega w^p \, dx
= -(p-1) \int_\Omega w^{p-2} |\nabla w|^2 \, dx + (1 + \epsilon)m \int_\Omega \frac{w^{p-1}}{v} \nabla w \cdot \nabla v \, dx
+ cm(m-1) \int_\Omega w^{p-2} \frac{|\nabla v|^2}{v^2} \, dx + (m + r) \int_\Omega w^{p-1} \, dx
- m \int_\Omega w^{p+1} v^{m-1} \, dx - \mu \int_\Omega w^{p+k-1} v^{m(k-1)} \, dx
\leq -(p-1 - \epsilon_0) \int_\Omega w^{p-2} |\nabla w|^2 \, dx + \left[ cm(m-1) + \frac{(1 + \epsilon)^2 m^2}{4 \epsilon_0} \right] \int_\Omega w^{p-2} \frac{|\nabla v|^2}{v^2} \, dx
+ (m + r) \int_\Omega w^p \, dx - m \int_\Omega w^{p+1} v^{m-1} \, dx - \mu \int_\Omega w^{p+k-1} v^{m(k-1)} \, dx, \] (10)

by Young’s inequality with \( \epsilon_0 > 0 \) for \( t \in (0, T) \). Notice that \( m = \frac{1}{1-\epsilon} < 1 \) for \( \epsilon \in (0, 1) \) with \( \chi \in (0, 1-\epsilon) \). Again by Young’s inequality, we know
\[
\int_\Omega w^p \, dx = \int_\Omega \left( w^{p+1} v^{m-1} \right)^{\frac{m}{m+p(k-1)+p(1-m)+k-1}} \left( w^{p+k-1} v^{m(k-1)} \right)^{\frac{p(1-m)}{m+p(k-1)+p(1-m)+k-1}} \leq \epsilon_1 \int_\Omega w^{p+1} v^{m-1} \, dx + \epsilon_2 \int_\Omega w^{p+k-1} v^{m(k-1)} \, dx + C(\epsilon_1, \epsilon_2) \|\Omega\|, \] (11)

with \( C(\epsilon_1, \epsilon_2) = \frac{m(1-m)(k-1)^2}{p(1-m)(k-1)^2} \). Let \( \epsilon_0 = \frac{m(1+\epsilon)^2}{4\epsilon(1-m)} \), \( \epsilon_1 = \frac{m}{m+r+1} \) and \( \epsilon_2 = \frac{p}{m+r+1} \). Then for \( p \geq 1 + \epsilon_0 \) it holds from (10) and (11) that
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega w^p \, dx + \frac{1}{p} \int_\Omega w^p \, dx \leq (m + r + 1) \int_\Omega w^p \, dx - m \int_\Omega w^{p+1} v^{m-1} \, dx
- \mu \int_\Omega w^{p+k-1} v^{m(k-1)} \, dx
\leq (m + r + 1) \|\Omega\| C(\epsilon_1, \epsilon_2) =: C_3 \] (12)
for \( t \in (0, T) \). Thus from an ODE comparison argument with (12) we obtain
\[
\int_{\Omega} w^p dx \leq \max \left\{ \int_{\Omega} w_0^p dx, pC_3 \right\}, \ t \in (0, T).
\]
This proves (9) with some \( M_1 > 0 \) via a Moser iteration procedure in [3].

Based on Lemma 3.1, we will derive the estimate on \( \|u\|_{L^q(\Omega)} \) for some \( q > \frac{n}{2} \).

**Lemma 3.2.** Under the conditions in lemma 3.1, there exists \( q > \frac{n}{2} \) with \( M_2 > 0 \) such that
\[
\int_{\Omega} u^q dx \leq M_2, \ t \in (0, T).
\]  
(13)

**Proof.** If \( 1 < q < p \), we have from Young's inequality that
\[
\int_{\Omega} u^q dx = \int_{\Omega} w^q v^{mq} dx \leq \frac{q}{p} \int_{\Omega} w^p dx + \frac{p-q}{p} \int_{\Omega} v^{pq \frac{m}{p-1}} dx.
\]  
(14)

According to Lemma 2.3 with (4), it is shown for \( r \in (1, \frac{n}{n-2}) \) that
\[
\|v\|_{L^r(\Omega)} \leq C_4, \ t \in (0, T).
\]  
(15)

with some \( C_4 > 0 \).

Let \( q_1 := m^{-1} \) and \( p > \frac{n}{2m} \). Then \( 1 < q_1 < p \) and \( \frac{pm}{p-q_1} < \frac{n}{n-2} \). Combining (9), (14) and (15), we obtain
\[
\int_{\Omega} u^{q_1} dx \leq \frac{q_1}{p} \int_{\Omega} w^p dx + \frac{p-q_1}{p} \int_{\Omega} v^{pq_1 \frac{m}{p-1}} dx \leq C_5, \ t \in (0, T)
\]  
(16)

with some \( C_5 > 0 \).

Without loss of generality, assume that \( n > 2 \) and \( 1 < q_1 \leq \frac{n}{2} \). Based on Lemma 2.3 with (16), we get for \( r \in (1, \frac{nq_1}{n-2q_1}) \) that
\[
\|v\|_{L^r(\Omega)} \leq C_6, \ t \in (0, T)
\]  
(17)

with some \( C_6 > 0 \). Now let \( q_2 := m^{-2} \) and \( p > \max\{q_2, \frac{n}{2m}\} \). A simple calculation shows that \( \frac{pm}{p-q_2} < \frac{n}{n-2q_2} \). Hence, it is shown from (14) with (9) and (17) that
\[
\int_{\Omega} u^{q_2} dx \leq \frac{q_2}{p} \int_{\Omega} w^p dx + \frac{p-q_2}{p} \int_{\Omega} v^{pq_2 \frac{m}{p-1}} dx \leq C_7, \ t \in (0, T)
\]  
(18)

with \( C_7 > 0 \). Since \( q_l = m^{-l} \to \infty \) as \( l \to \infty \), we can realize \( q_l > \frac{n}{2} \) after finite steps.

**Proof of Theorem 1.1.** Invoke Lemmas 3.2 and 2.3 to know
\[
\|v\|_{L^\infty(\Omega)} \leq C_8, \ t \in (0, T)
\]  
(19)

with some \( C_8 > 0 \). By Lemma 3.1 with(19), we obtain
\[
\|u\|_{L^\infty(\Omega)} \leq \|w_0^m\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} \|v^m\|_{L^\infty(\Omega)} \leq \tilde{M}, \ t \in (0, T)
\]
with some \( \tilde{M} > 0 \). This together with Lemma 2.1 indicates \( T = T_{\max} = \infty \), i.e.,
the classical solutions are globally bounded.

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