Equilibria for a gyrostat in newtonian interaction with two rigid bodies

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Abstract

In this paper the non-canonical Hamiltonian dynamics of a gyrostat in the three body problem will be examined. By means of geometric-mechanics methods some relative equilibria of the dynamics of a gyrostat in Newtonian interaction with two rigid bodies will be studied. Taking advantage of the results obtained in previous papers, working on the reduced problem, the bifurcations of these relative equilibria will be studied. The instability of Eulerian relative equilibria if the gyrostat is close to a sphere is proven. Necessary and sufficient conditions will be provided for lineal stability of Lagrangian relative equilibria if the gyrostat is close to a sphere. The analysis is done in vectorial form avoiding the use of canonical variables and the tedious expressions associated with them. In this way, the classic results on equilibria of the three-body problem, many of them obtained by other authors who had used of more classic techniques, are generalized.

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1 Introduction

In the last few years a lot of papers about the problem of roto-translational motion of celestial bodies have appeared. These show a new interest in the study of configurations of relative equilibria by differential geometry methods instead of more classical ones.

In the present paper, references will be made to Wang et al. [9], concerning the problem of a rigid body in a central Newtonian field, and Maciejewski [8], concerning the problem of two rigid bodies in mutual Newtonian attraction. Let us remember that a gyrostat is a mechanical system $S$ composed of a rigid body $S'$ and other bodies $S''$ (deformable or rigid), connected to it in such a way that their relative motion with respect to its rigid
part do not change the distribution of mass of the total system $S$ (Leimanis, [2]). These papers have been generalized by Mondéjar and Viguera [4] to the case of two gyrostats in mutual Newtonian attraction.

With regards to the problem of three rigid bodies, references will have to be made to Vidiakin [8] and Duboshin [1] who proved the existence of Euler and Lagrange configurations of equilibria when the bodies possess symmetries. Later Zhuravlev and Petrutskii [10] reviewed the results until 1990.

Vera [5] and Vera and Viguera ([6],[7]) have studied the non-canonical Hamiltonian dynamics of $n + 1$ bodies in Newtonian attraction, in which $n$ of them are rigid bodies with a spherical distribution of masses or material points and the other one is a triaxial gyrostat. Working in the reduced problem, global considerations about the conditions for relative equilibria were made. Finally, in an approximated model of the dynamics, a study of some relative equilibria of a gyrostat in Newtonian interaction with three rigid bodies is carried out.

In this paper, the existence and stability of some periodic solutions of the dynamics of a gyrostat in Newtonian interaction with two rigid bodies will be investigated. In a first approach to the problem, the attitude dynamics of both rigid bodies are assumed to be the same as the dynamics of a rigid body in torque free motion. The two rigid bodies have revolution symmetry about the third axis of inertia. On the other hand, the Newtonian interaction of the gyrostat between both rigid bodies is simplified supposing the gyrostat a material particle. The Newtonian interaction between the two rigid bodies are the same of two material particles.

With these hypotheses, according to Vera and Viguera [7], working in the double reduced space of configuration of the problem, the equations of motion and those which determine the relative equilibria will be derived.

Two families of relative equilibria, Eulerian and Lagrangian will be investigated. The Eulerian relative equilibria are completely determined by a polynomial equation of degree nine. The bifurcations of these equilibria are made. The bifurcations of the Lagrangian relative equilibria are completely investigated and expressions for the Lagrangian relative equilibria when both solids are close to spheres are provided.

We should notice that the studied system, has potential interest both in astrodynamics (dealing with spacecrafts) as well as in the understanding of the evolution of planetary systems recently found (and more to appear), where some of the planets may be modeled like a rigid body rather than a rigid body. In fact, the equilibria reported might well be compared with the ones taken for the ‘parking areas’ of the space missions (GENESIS, SOHO, DARWIN, etc) around the Eulerian points of the Sun-Earth and the Earth-Moon systems. Some interesting numeric results have been calculated in the Appendix B.

The methods used in this work can be applied to similar problems. The study of the nonlinear stability of the relative equilibria obtained here is the logical continuation of this work.
2 Poisson dynamics

Let $S_0$ be a gyrostat of mass $m_0$, and $S_1, S_2$ two symmetrical rigid bodies of masses $m_1$ and $m_2$ respectively; $\mathcal{J} = \{\mathbf{O}, u_1, u_2, u_3\}$ an inertial reference frame; $\mathcal{J} = \{C_0, b_1, b_2, b_3\}$ a body frame fixed at the center of mass $C_0$ of $S_0$, (see Vera and Vigueras [7] for details).

The following notation is used

\[
M_2 = m_1 + m_2, \quad M_1 = m_1 + m_2 + m_0, \quad g_1 = \frac{m_1m_2}{M_2}, \quad g_2 = \frac{m_0M_2}{M_1}
\]

Where for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $\mathbf{u} \cdot \mathbf{v}$ is the dot product; $|\mathbf{u}|$ is the Euclidean norm of the vector $\mathbf{u}$; $\mathbf{u} \times \mathbf{v}$ is the cross product; $\mathbb{I}_{\mathbb{R}^3}$ is the identity matrix; and $\mathbf{0}$ is the zero matrix of order three. Consider $\mathbb{I}_i = \text{diag}(A_i, A_i, C_i)$ the diagonal tensor of inertia of the gyrostat, and $\mathbb{I}_i = \text{diag}(A_i, A_i, C_i)$ the diagonal tensors of inertia for the rigid bodies $S_i$, $i = 1, 2$. The generic expression $\mathbf{z} = (\mathbb{I}_1, \mathbb{I}_2, \mathbb{I}_0, \mathbf{\lambda}, \mathbf{p}_\mathbf{\lambda}, \mathbf{\mu}, \mathbf{p}_\mathbf{\mu}) \in \mathbb{R}^{21}$ is an vector of the twice reduced problem obtained by applying the symmetries of the system. $\mathbb{I}_0 = \mathbb{I}_0\mathbb{\Omega}_0 + \mathbb{I}_1$ is the total rotational angular momentum vector of the gyrostat in the body frame, which is attached to its rigid part and whose axes have the direction of the principal axes of inertia of $S_0$; the vector $\mathbf{I}_1 = (0, 0, l)$ is the constant gyrostatic momentum and $\mathbb{I}_i = \mathbb{I}_i\mathbb{\Omega}_i$ $(i = 1, 2)$ are the total rotational angular momentum vectors for the two rigid bodies. The elements $\mathbf{\lambda}, \mathbf{\mu}, \mathbf{p}_\mathbf{\lambda}$ and $\mathbf{p}_\mathbf{\mu}$ are respectively the barycentric coordinates and the linear momenta expressed in the body frame $\mathcal{J}$.

Following the results of Vera and Vigueras [7], according to the hypotheses formulated in the introduction of this paper, a good approximation to the potential of the system is expressed by the following expression

\[
\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2
\]

where

\[
\mathcal{V}_1 = - \left( \frac{G m_1 m_2}{|\mathbf{\lambda}|} + \frac{G m_1 m_0}{|\mathbf{\mu} - \frac{m_2}{M_2}\mathbf{\lambda}|} + \frac{G m_2 m_0}{|\mathbf{\mu} + \frac{m_1}{M_2}\mathbf{\lambda}|} \right)
\]

\[
\mathcal{V}_2 = - \frac{1}{2} \left( \frac{G m_0 \alpha_1}{|\mathbf{\mu} - \frac{m_2}{M_2}\mathbf{\lambda}|^3} + \frac{G m_0 \alpha_2}{|\mathbf{\mu} + \frac{m_1}{M_2}\mathbf{\lambda}|^3} - \frac{3G m_0 f_1}{|\mathbf{\mu} - \frac{m_2}{M_2}\mathbf{\lambda}|^5} - \frac{3G m_0 f_2}{|\mathbf{\mu} + \frac{m_1}{M_2}\mathbf{\lambda}|^5} \right)
\]

and

\[
\alpha_1 = 2A_1 + C_1, \quad \alpha_2 = 2A_2 + C_2
\]

\[
f_1(\mathbf{\lambda}, \mathbf{\mu}) = \mathbf{\mu} \cdot \mathbb{I}_1 \mathbf{\mu} - \frac{2m_2}{M_2} \mathbf{\lambda} \cdot \mathbb{I}_1 \mathbf{\mu} + \left(\frac{m_2}{M_2}\right)^2 \mathbf{\lambda} \cdot \mathbb{I}_1 \mathbf{\lambda}
\]

\[
f_2(\mathbf{\lambda}, \mathbf{\mu}) = \mathbf{\mu} \cdot \mathbb{I}_2 \mathbf{\mu} + \frac{2m_1}{M_2} \mathbf{\lambda} \cdot \mathbb{I}_2 \mathbf{\mu} + \left(\frac{m_1}{M_2}\right)^2 \mathbf{\lambda} \cdot \mathbb{I}_2 \mathbf{\lambda}
\]
The Hamiltonian function of the system adopts the form

$$\mathcal{H}(z) = \frac{|p_\lambda|^2}{2g_1} + \frac{|p_\mu|^2}{2g_2} + \frac{1}{2} \Pi_0 \Pi_0 - l_1 \cdot I_0 \Pi + \frac{1}{2} \Pi_1 \Pi_1 I_1 + \frac{1}{2} \Pi_2 \Pi_2 I_2 + V(\lambda, \mu)$$

Let \((M, \{, \}, \mathcal{H})\) with \(M = \mathbb{R}^{21}\) be the Poisson manifold, where \(\{, \}\) is the Poisson brackets defined by means of the Poisson tensor

$$B(z) = \begin{pmatrix}
\hat{\Pi}_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \hat{\Pi}_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \hat{\Pi}_0 & \hat{\lambda} & \hat{p}_\lambda & \hat{\mu} \\
0 & 0 & \hat{\lambda}_1 & 0 & I_{\mathbb{R}^3} & 0 \\
0 & 0 & \hat{p}_\lambda & -I_{\mathbb{R}^3} & 0 & 0 \\
0 & 0 & \hat{\mu} & 0 & 0 & I_{\mathbb{R}^3} \\
0 & 0 & \hat{p}_\mu & 0 & 0 & -I_{\mathbb{R}^3} \\
\end{pmatrix}$$

In \(B(z)\), \(\hat{v}\) is considered to be the image of the vector \(v \in \mathbb{R}^3\) by the standard isomorphism between the Lie Algebras \(\mathbb{R}^3\) and \(\mathfrak{so}(3)\), i.e.

$$\hat{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\
v_3 & 0 & -v_1 \\
-v_2 & v_1 & 0 \end{pmatrix}$$

The equations of the motion are given by the below expression

$$\frac{dz}{dt} = \{z, \mathcal{H}(z)\} = B(z) \nabla_z \mathcal{H}(z)$$

with \(\nabla_z \mathcal{V}\) being the gradient of \(\mathcal{V}\) with respect to an arbitrary vector \(z\).

Calculating \(\{z, \mathcal{H}(z)\}\), the below group of vectorial equations of the motion can be written as

$$\begin{align*}
\frac{d\Pi_0}{dt} &= \Pi_0 \times \Omega_0 + \lambda \times \nabla_\lambda \mathcal{V} + \mu \times \nabla_\mu \mathcal{V} \\
\frac{d\lambda}{dt} &= \frac{p_\lambda}{g_1} + \lambda \times \Omega_0, \quad \frac{dp_\lambda}{dt} = p_\lambda \times \Omega_0 - \nabla_\lambda \mathcal{V} \\
\frac{d\mu}{dt} &= \frac{p_\mu}{g_2} + \mu \times \Omega_0, \quad \frac{dp_\mu}{dt} = p_\mu \times \Omega_0 - \nabla_\mu \mathcal{V} \\
\frac{d\Pi_1}{dt} &= \Pi_1 \times \Omega_1, \quad \frac{d\Pi_2}{dt} = \Pi_2 \times \Omega_2
\end{align*}$$

(1)

Important elements of \(B(z)\) are the associated Casimir functions. The vector

$$L_0 = \Pi_0 + \lambda \times p_\lambda + \mu \times p_\mu$$
is a part of the total angular momentum $L$ given by

$$L = \Pi_2 + \Pi_1 + L_0$$

Then the below result can be concluded.

**Proposition 1** If $\varphi_i$, $(i = 0, 1, 2)$ are real smooth functions, then $\varphi_0\left(\frac{|L_0|^2}{2}\right)$, $\varphi_i\left(\frac{|\Pi_i|^2}{2}\right)$, $(i = 1, 2)$ are Casimir functions of the Poisson tensor $B(z)$. Furthermore, $\text{Ker} B(z) = \langle \nabla_z \varphi_0, \nabla_z \varphi_1, \nabla_z \varphi_2 \rangle$. We also have $\frac{\partial}{\partial t} = 0$, which means the total angular momentum vector remains constant. If $\Pi_0 = (\pi^1_0, \pi^2_0, \pi^3_0)$, then $\pi^3_0$ is an integral of the motion.

### 3 Relative Equilibria

If $z_e = (\Pi^e_2, \Pi^e_1, \Pi^e_0, \lambda^e, \mu^e, p^e_\lambda, p^e_\mu)$ is a generic relative equilibrium, the below vectorial equations are verified

$$\Pi^e_0 \times \Omega^e_0 + \lambda^e \times (\nabla_\lambda \mathcal{V})_e + \mu \times (\nabla_\mu \mathcal{V})_e = 0 \quad (2)$$

$$\frac{p^e_\lambda}{g_1} + \lambda^e \times \Omega^e_0 = 0, \quad \frac{p^e_\mu}{g_2} + \mu \times \Omega^e_0 = (\nabla_\lambda \mathcal{V})_e$$

$$\Pi^e_1 \times \Omega^e_1 = 0, \quad \Pi^e_2 \times \Omega^e_2 = 0 \quad (3)$$

where $(\nabla_\lambda \mathcal{V})_e$ and $(\nabla_\mu \mathcal{V})_e$ are the values of $\nabla_\lambda \mathcal{V}$ and $\nabla_\mu \mathcal{V}$ in $z_e$.

According to the relationships provided by Vera and Vigueras [7], the following results are obtained.

**Lemma 2** Whenever $z_e = (\Pi^e_2, \Pi^e_1, \Pi^e_0, \lambda^e, \mu^e, p^e_\lambda, p^e_\mu)$ is a relative equilibrium, the below relationships are verified

$$|\Omega^e_0|^2 \lambda^e|^2 - (\lambda^e \cdot \Omega^e_0)^2 \frac{1}{g_1} = (\lambda^e \cdot (\nabla_\lambda \mathcal{V})_e)$$

$$|\Omega^e_0|^2 \mu^e|^2 - (\mu^e \cdot \Omega^e_0)^2 \frac{1}{g_2} = (\mu^e \cdot (\nabla_\mu \mathcal{V})_e)$$

The previous two identities will be used to obtain necessary conditions for the existence of relative equilibria.

Certain relative equilibria will be studied assuming that vectors $\Omega^e_0, \lambda^e$ and $\mu^e$ satisfy special geometric properties.

**Definition 3** $z_e$ is said to be an Eulerian relative equilibrium when, $\lambda^e$ and $\mu^e$ are proportional and $\Omega_e$ is perpendicular to the straight line that is generate.
**Definition 4** \( z_e \) is said to be a Lagrangian relative equilibrium if \( \lambda^e \) and \( \mu^e \) are not proportional and \( \Omega_e \) is perpendicular to the plane that both of them generate.

From the above definitions, the following property is deduced.

**Proposition 5** In an Eulerian or Lagrangian relative equilibrium, momenta are not exercised over the gyrostat.

Next, necessary and sufficient conditions for the existence of Eulerian and Lagrangian relative equilibria will be obtained.

4 Eulerian relative equilibria

According to the relative position of the gyrostat \( S_0 \) with respect to \( S_1 \) and \( S_2 \), there are three possible equilibrium configurations: a) \( S_2 S_1 S_0 \), b) \( S_2 S_0 S_1 \) and c) \( S_0 S_2 S_1 \).

4.1 Necessary conditions of existence

The following lemma is a direct consequence of the geometry of the problem.

**Lemma 6** If \( z_e = (\Pi^e_2, \Pi^e_1, \Pi^e_0, \lambda^e, \mu^e, p^e_\lambda, p^e_\mu) \) is a relative equilibrium of Euler type, then for the configuration \( S_0 S_2 S_1 \)

\[
| \mu^e - \frac{m_2}{M_2} \lambda^e | = | \lambda^e | + | \mu^e + \frac{m_1}{M_2} \lambda^e |
\]

In a similar way, for the configuration \( S_2 S_0 S_1 \)

\[
| \lambda^e | = | \mu^e - \frac{m_2}{M_2} \lambda^e | + | \mu^e + \frac{m_1}{M_2} \lambda^e |
\]

Finally, for the configuration \( S_2 S_1 S_0 \)

\[
| \mu^e + \frac{m_1}{M_2} \lambda^e | = | \mu^e - \frac{m_2}{M_2} \lambda^e | + | \lambda^e |
\]

If \( z_e \) is an Eulerian relative equilibrium, then

\[
g_1 | \Omega^0_e |^2 | \lambda^e |^2 = \lambda^e \cdot (\nabla \lambda \nabla )^e
\]

\[
g_2 | \Omega^0_e |^2 | \mu^e |^2 = \mu^e \cdot (\nabla \mu \nabla )^e
\]

with

\[
\mu^e - \frac{m_2}{M_2} \lambda^e = \rho \lambda^e, \quad \mu^e + \frac{m_1}{M_2} \lambda^e = (1 + \rho) \lambda^e, \quad \mu^e = \frac{(1 + \rho)m_2 + \rho m_1}{M_2} \lambda^e
\]
where \( \rho \in (-\infty,-1) \) in the configuration a), \( \rho \in (-1,0) \) in the configuration b) and \( \rho \in (0, +\infty) \) in the configuration c). Moreover the following expressions are possible to be obtained
\[
(\nabla \chi) = h_1(\rho) \chi, \quad (\nabla \mu) = h_2(\rho) \chi
\]
with
\[
h_1(\rho) = \frac{Gm_1 m_2}{|\chi|^3} + \frac{Gm_0 m_1 \text{sgn}(1 + \rho)}{M_2 |\chi|^3} \left( \frac{m_2}{(1 + \rho)^2} + \frac{\beta_2}{(1 + \rho)^4 |\chi|^2} \right) -
\]
and
\[
h_2(\rho) = \frac{Gm_0 \text{sgn}(1 + \rho)}{|\chi|^3} \left( \frac{m_2}{(1 + \rho)^2} + \frac{\beta_2}{|\chi|^2 (1 + \rho)^4} \right) +
\]
where \( \beta_1 = 3(C_1 - A_1)/2, \beta_2 = 3(C_2 - A_2)/2 \) and \( \text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases} \).

Now, from the identities
\[
\chi \cdot (\nabla \chi) = |\chi|^2 h_1(\rho)
\]
\[
\mu \cdot (\nabla \mu) = \frac{(1 + \rho)m_2 + \rho m_1}{M_2} |\chi|^2 h_2(\rho)
\]
these equations are deduced
\[
|\Omega^e_0|^2 = \frac{m_1 + m_2}{m_1 m_2} h_1(\rho)
\]
\[
|\Omega^e_0|^2 = \frac{m_0 + m_1 + m_2}{m_0 ((1 + \rho)m_2 + \rho m_1)} h_2(\rho)
\]
Then for an Eulerian relative equilibrium, \( \rho \) must be a real root of the below equation
\[
m_0(m_1 + m_2) ((1 + \rho)m_1 + \rho m_2) h_1(\rho) = m_1 m_2 (m_0 + m_1 + m_2) h_2(\rho)
\]
Proposition 7 If \( z_e = (\Pi^e_2, \Pi^e_1, \Pi^e_0, \chi^e, p^e_\chi, \mu^e, p^e_\mu) \) is an Eulerian relative equilibrium, the equation (8) has, at least, a real root in which the functions \( h_1(\rho) \) and \( h_2(\rho) \) are given by the expressions (4) and (5). The modulus of the angular velocity of the gyrostat is
\[
|\Omega^e_0|^2 = \frac{m_1 + m_2}{m_1 m_2} h_1(\rho)
\]
Remark 8 When \( |\chi_e| \) has a fixed value, if an Eulerian relative equilibrium exists, the equation (8) has real solutions. The number of real roots of the equation (8) will depend, obviously, on the parameters which exist in the system.
4.2 Sufficient conditions of existence

The below proposition indicates how to calculate solutions of eq. \([\mathbf{2}]\).

**Proposition 9** When \(|\lambda^e|\) has a fixed value, let \(\rho\) be a solution of the equation \([\mathbf{8}]\), where the functions \(h_1(\rho)\) and \(h_2(\rho)\) are given by the expressions \([\mathbf{4,5}]\). Then \(\mathbf{z}_e = (\Pi_2^e, \Pi_1^e, \lambda^e, \mu^e, p_{\lambda}^e, p_{\mu}^e)\) given by

\[
\lambda^e = (\lambda^e, 0, 0), \quad \mu^e = (\mu^e, 0, 0), \quad \Omega_0^e = (0, 0, \omega^e_0)
\]

\[
p_{\lambda}^e = (0, g_1\omega^e_0\lambda^e, 0), \quad p_{\mu}^e = (0, g_2\omega^e_0\mu^e, 0), \quad \Pi_0^e = (0, 0, C_0\omega^e_0 + l)
\]

where

\[
\mu^e = \frac{(1 + \rho)m_1 + \rho m_2}{M_2} \lambda^e, \quad (\omega^e_0)^2 = \frac{(m_1 + m_2)h_1(\rho)}{m_1 m_2}
\]

is an Eulerian relative equilibrium. The total angular momentum of the system is expressed by

\[
L = (0, 0, C_2\omega^e_2 + C_1\omega^e_1 + C_0\omega^e_0 + l + g_1\omega^e_0(\lambda^e)^2 + g_2\omega^e_0(\mu^e)^2)
\]

with \(l\) being the gyrostatic momentum. The vectors \(\Pi_2^e, \Pi_1^e\) verify the vectorial equations

\[
\Pi_1^e \times \Omega_1^e = 0, \quad \Pi_2^e \times \Omega_2^e = 0
\]

4.3 Eulerian relative equilibria when \(S_2\) and \(S_1\) are spherical rigid bodies

Consider the existence and number of solutions for Eulerian relative equilibria when \(S_2\) and \(S_1\) are spherical rigid bodies. In this case \(C_1 = A_1, C_2 = A_2\) and the equation \([\mathbf{8}]\) is equivalent to the below polynomial equation

\[
(m_1 + m_2)\rho^5 + (3m_2 + 2m_1)\rho^4 + (3m_2 + m_1 + m_0)(\text{sgn}(1 + \rho)\rho^3 + (m_2 - m_2\text{sgn}(1 + \rho) - m_1\text{sgn}(\rho) - 3m_0\text{sgn}(\rho))\rho^2
\]

\[-(3\text{sgn}(\rho)m_0 + 2\text{sgn}(\rho)m_1)\rho - (m_0\text{sgn}(\rho) + m_1\text{sgn}(\rho)) = 0 \tag{10}
\]

This equation has a unique real solution in the intervals \((-\infty, -1), (-1, 0)\) and \((0, +\infty)\). Therefore only one Eulerian relative equilibrium exists.

On the other hand

\[
|\Omega_0^e|^2 = \frac{G(m_1 + m_2)}{|\lambda^e|^3} \left(1 + \frac{m_0}{m_1 + m_2} \left(\frac{\text{sgn}(1 + \rho)}{(1 + \rho)^2} - \frac{\text{sgn}(\rho)}{\rho^2}\right)\right)
\]

where \(\rho\) is the only solution of the equation \([\mathbf{9}]\).

Proposition 5 gathers the results about Eulerian relative equilibria when \(S_2\) and \(S_1\) are spherical rigid bodies in the configurations a), b) and c).
Proposition 10

1. If $\rho$ is the unique positive root of the equation
$$
(m_1 + m_2)\rho^5 + (3m_2 + 2m_1)\rho^4 + (3m_2 + m_1)\rho^3 + \\
-(3m_0 + m_1)\rho^2 - (3m_0 + 2m_1)\rho - (m_0 + m_1) = 0
$$
with
$$
|\Omega_0^5|^2 = \frac{G(m_1 + m_2)}{|\lambda_e^5|} \left(1 + \frac{m_0}{(m_1 + m_2)} \left(\frac{1}{(1 + \rho)^2} - \frac{1}{\rho^2}\right)\right)
$$
then $z_e = (\Pi_2^e, \Pi_1^e, \Pi_0^e, \lambda_e^e, \mu_e^e, p_\lambda^e)$, given by
$$
\lambda_e^e = (\lambda_e^e, 0, 0), \quad \mu_e^e = (\mu_e^e, 0, 0), \quad \Omega_0^e = (0, 0, \omega_0^e)
$$
$$
p_\lambda^e = (0, g_1\omega_0^e\lambda_e^e, 0), \quad p_\mu^e = (0, g_2\omega_0^e\mu_e^e, 0), \quad \Pi_0^e = (0, 0, C_0\omega_0^e + l)
$$
is the unique solution of relative equilibrium of Euler type in the configuration $S_2S_1S_0$.

2. If $\rho \in (-1, 0)$ is the unique root of the equation
$$
(m_1 + m_2)\rho^5 + (3m_2 + 2m_1)\rho^4 + (3m_2 + m_1)\rho^3 + \\
+(3m_0 + 2m_1)\rho^2 + (3m_0 + 2m_1)\rho + (m_0 + m_1) = 0
$$
with
$$
|\Omega_0^5|^2 = \frac{G(m_1 + m_2)}{|\lambda_e^5|} \left(1 + \frac{m_0}{(m_1 + m_2)} \left(\frac{1}{(1 + \rho)^2} + \frac{1}{\rho^2}\right)\right)
$$
then $z_e = (\Pi_2^e, \Pi_1^e, \Pi_0^e, \lambda_e^e, \mu_e^e, p_\mu^e)$ given by
$$
\lambda_e^e = (\lambda_e^e, 0, 0), \quad \mu_e^e = (\mu_e^e, 0, 0), \quad \Omega_0^e = (0, 0, \omega_0^e)
$$
$$
p_\lambda^e = (0, g_1\omega_0^e\lambda_e^e, 0), \quad p_\mu^e = (0, g_2\omega_0^e\mu_e^e, 0), \quad \Pi_0^e = (0, 0, C_0\omega_0^e + l)
$$
is the unique solution of relative equilibrium of Euler type in the configuration $S_2S_0S_1$.

3. If $\rho \in (-\infty, -1)$ is the unique root of the equation
$$
(m_1 + m_2)\rho^5 + (3m_1 + 2m_2)\rho^4 + (2m_0 + 3m_2 + m_1)\rho^3 + \\
+(3m_0 + m_1)\rho^2 + (3m_0 + 2m_1)\rho + (m_0 + m_1) = 0
$$
where
$$
|\Omega_0^5|^2 = \frac{G(m_1 + m_2)}{|\lambda_e^5|} \left(1 + \frac{m_0}{(m_1 + m_2)} \left(\frac{1}{\rho^2} - \frac{1}{(1 + \rho)^2}\right)\right)
$$
then $z_e = (\Pi_2^e, \Pi_1^e, \Pi_0^e, \lambda_e^e, \mu_e^e, p_\mu^e)$ given by
$$
\lambda_e^e = (\lambda_e^e, 0, 0), \quad \mu_e^e = (\mu_e^e, 0, 0), \quad \Omega_0^e = (0, 0, \omega_0^e)
$$
$$
p_\lambda^e = (0, g_1\omega_0^e\lambda_e^e, 0), \quad p_\mu^e = (0, g_2\omega_0^e\mu_e^e, 0), \quad \Pi_0^e = (0, 0, C_0\omega_0^e + l)
$$
is the unique solution of relative equilibrium of Euler type in the configuration $S_0S_2S_1$. 
4.4 Eulerian relative equilibria when \( S_2 \) and \( S_1 \) are not spherical rigid bodies

In the present case, after carrying out the appropriate calculations, the equation (8) is reduced to the study of the positive real roots of the nine degree equation

\[
\beta_2 q(\rho) - m_1 m_2 a^2 \rho^2 (\rho + 1)^2 p(\rho) = 0
\]

where

\[
p(\rho) = (m_1 + m_2) \rho^5 + (3m_2 + 2m_1) \rho^4 + (3m_2 + m_1 + m_0 (\text{sgn}(1 + \rho) \\
- \text{sgn}(\rho))) \rho^3 + (m_2 - m_2 \text{sgn}(1 + \rho) - m_1 \text{sgn}(\rho) - 3m_0 \text{sgn}(\rho)) \rho^2 \\
- (3\text{sgn}(\rho) m_0 + 2\text{sgn}(\rho) m_1) \rho - (m_0 \text{sgn}(\rho) + m_1 \text{sgn}(\rho))
\]

and

\[
q(\rho) = m_0 (k \text{sgn}(\rho) m_2 - \text{sgn}(1 + \rho) m_1) \rho^5 + m_2 (\text{sgn}(1 + \rho) m_1 + 5 \text{sgn}(\rho) m_0 k + k m_1 \text{sgn}(\rho)) \rho^4 \\
+ 2k \text{sgn}(\rho) (2m_1 + 5m_0) \rho^3 + 2k \text{sgn}(\rho) (3m_1 + 5m_0) \rho^2 + k \text{sgn}(\rho) m_2 (4m_1 + 5m_0) \rho \\
+ k \text{sgn}(\rho) m_2 (m_0 + m_1)
\]

where

\[
\beta_1 = 3(C_1 - A_1)/2, \quad \beta_2 = 3(C_2 - A_2)/2
\]

with \( \beta_1 = k \beta_2 \), \( a = |\lambda_e| \) and \( k \in \mathbb{R} \).

In order to study the number of real roots of the polynomial (11) the rational function will be studied

\[
\beta_2 = R(\rho) = \frac{m_1 m_2 a^2 \rho^2 (\rho + 1)^2 p(\rho)}{q(\rho)}
\]

In practical applications \( m_0 \) is very small, then up to the first order in \( m_0 \)

\[
\beta_2 = R_1(\rho) = \frac{a^2 \rho^2 (\rho + 1)^2 p_1(\rho)}{q_1(\rho)} + o(m_0)
\]

where

\[
p_1(\rho) = \rho^5 + (2 + \mu) \rho^4 + (1 + 2\mu) \rho^3 + (\mu - \mu \text{sgn}(1 + \rho) \\
- (1 - \mu) \text{sgn}(\rho)) \rho^2 - 2 \text{sgn}(\rho) (1 - \mu) \rho - (1 - \mu) \text{sgn}(\rho)
\]

and

\[
q_1(\rho) = (\text{sgn}(1 + \rho) + \text{sgn}(\rho) k) \rho^4 + 4 \text{sgn}(\rho) k \rho^3 + 6 \text{sgn}(\rho) k \rho^2 + 4 \text{sgn}(\rho) k \rho + \text{sgn}(\rho) k
\]

where \( \mu = \frac{m_1}{m_2 + m_1} \).
The polynomial $q_1$ has no roots in $(0, +\infty)$ and $(-1, 0)$ if $k > 0$ and $k < 0$, respectively. On the other hand, $q_1$ has only one root, $\rho_1$, in $(0, +\infty)$ and $(-1, 0)$ if $k < 0$ and $k > 0$, respectively. $\rho_0$ will be denoted as the only root of $p_1$. The implicit curve $Res(k, \mu) = 0$, where $Res$ is the resultant of the polynomials $p_1$ and $q_1$, is used to study the graph of $R_1$. When $\mu_0$ has a fixed value, the only $k_0$ which verifies $Res(k_0, \mu_0) = 0$, exists according to the Implicit Function Theorem. Consider the only root, $\tilde{\rho}_1 = \rho_0(\mu_0)$, of $q_1$ for $k_0$ and $\mu_0$.

The expressions

$$ (\rho_{\text{max}}, \xi_1(k) = R_1(\rho_{\text{max}})), \quad (\rho_{\text{min}}, \xi_2(k) = R_1(\rho_{\text{min}})) $$

are the local maximum and minimum of the function $R_1$.

In the configuration $S_2S_1S_0$, for any value of $\mu$ fixed

$$ \lim_{k \rightarrow +\infty} \xi_2(k) = 0, \quad \lim_{k \rightarrow 0^+} \xi_2(k) = -\infty $$

if $k > 0$. For $k < 0$

$$ \lim_{k \rightarrow +\infty} \xi_1(k) = 0, \quad \lim_{k \rightarrow 0^+} \xi_2(k) = +\infty $$

if $\rho_1 > \tilde{\rho}_1$. If $\rho_1 \leq \tilde{\rho}_1$, $R_1$ is strictly increasing.

For the configuration $S_2S_0S_1$ if $\tilde{\rho}_1 \leq \rho_1$ and $k > 0$, the function $R_1$ has just a minimum, which verifies

$$ \lim_{k \rightarrow 0^+} \xi_2(k) = \xi_0 $$

If $\rho_1 < \tilde{\rho}_1$, then

$$ \lim_{k \rightarrow +\infty} \xi_2(k) = 0 $$

If $k < 0$, then $R_1$ verifies that

$$ \lim_{k \rightarrow 0^-} \xi_2(k) = \xi_0, \quad \lim_{k \rightarrow 0^-} \xi_1(k) = +\infty $$

$$ \lim_{k \rightarrow -\infty} \xi_2(k) = 0, \quad \lim_{k \rightarrow -\infty} \xi_1(k) = 0 $$

The results for the configuration $S_0S_2S_1$ will be deduced from the configuration $S_2S_1S_0$.

According to these statements, the following proposition can be stated.

**Proposition 11** In the configuration $S_2S_1S_0$ the following results are verified.

a) For $k > 0$ then:

1. If $\beta_2 < R_1(\rho_{\text{min}})$, then Eulerian relative equilibria do not exist.

2. If $\beta_2 = R_1(\rho_{\text{min}})$, a unique 2-parametric family of Eulerian relative equilibria exist.

3. If $R_1(\rho_{\text{min}}) < \beta_2 < 0$, two 2-parametric families of Eulerian relative equilibria exist.
4. If $\beta_2 > 0$ a unique 2-parametric family of Eulerian relative equilibria exists.

b) For $k_0 < k < 0$ and $\beta_2 > 0$, then:

1. If $\beta_2 \in (\xi_1(k), \xi_2(k))$, Eulerian relative equilibria do not exist.

2. If $\beta_2 = \xi_1(k)$ or $\beta_2 = \xi_2(k)$, then a unique 2-parametric family of Eulerian relative equilibria exists.

3. If $\beta_2 > \xi_2(k)$ two 2-parametric families of Eulerian relative equilibria exists.

4. If $0 < \beta_2 < \xi_1(k)$ two 2-parametric families of Eulerian relative equilibria exists.

c) For $k_0 < k < 0$ and $\beta_2 < 0$, then:

1. A unique 2-parametric family of Eulerian relative equilibria exist.

d) For $k < k_0$ and $\beta_2 > 0$, then:

1. Two 2-parametric families of Eulerian relative equilibria exists.

e) For $k < k_0$ and $\beta_2 < 0$, then:

1. A unique 2-parametric family of Eulerian relative equilibria exist.

Similarly we obtain the following result.

**Proposition 12** In the configuration $S_2S_0S_1$ the following results are verified.

a) For $k > 0$ then:

1. If $\beta_2 < R_1(\rho_{\min})$, then a unique 2-parametric family of Eulerian relative equilibria exists.

2. If $\beta_2 = R_1(\rho_{\min})$, a two 2-parametric family of Eulerian relative equilibria exist.

3. If $R_1(\rho_{\min}) < \beta_2 < 0$, three 2-parametric families of Eulerian relative equilibria exist.

4. If $\beta_2 > 0$ a unique 2-parametric family of Eulerian relative equilibria exists.

b) For $k < 0$ and $\beta_2 > 0$, then:

1. If $\beta_2 \in (\xi_1(k), \xi_2(k))$, Eulerian relative equilibria do not exist.
2. If \( \beta_2 = \xi_1(k) \) or \( \beta_2 = \xi_2(k) \) a unique 2-parametric family of Eulerian relative equilibria exist.

3. If \( \beta_2 > \xi_2(k) \), two 2-parametric families of Eulerian relative equilibria exist.

4. If \( 0 < \beta_2 < \xi_1(k) \), two 2-parametric families of Eulerian relative equilibria exist.

c) For \( k < 0 \) and \( \beta_2 < 0 \), then:

1. If \( R_1(\rho_{\min}) < \beta_2 < 0 \), then two 2-parametric families of Eulerian relative equilibria exist.

2. If \( R_1(\rho_{\min}) = \beta_2 \), then a unique 2-parametric family of Eulerian relative equilibria exists.

3. If \( \beta_2 < R_1(\rho_{\min}) \), then Eulerian relative equilibria do not exist.

4. If \( 0 < \beta_2 < R_2(\rho_{\max}) \), then two 2-parametric families of Eulerian relative equilibria exist.

5. If \( \beta_2 = R_2(\rho_{\max}) \), then a unique 2-parametric family of Eulerian relative equilibria exists.

6. If \( \beta_2 > R_2(\rho_{\max}) \), then Eulerian relative equilibria do not exist.

5 Stability of Eulerian relative equilibria

The tangent flow of the equations (11) in an Eulerian relative equilibrium \( z_e \) is expressed by

\[
\frac{d\delta z}{dt} = \Upsilon(z_e)\delta z
\]

with \( \delta z = z - z_e \) and \( \Upsilon(z_e) \) being the Jacobian matrix of (11) in \( z_e \).

The characteristic polynomial \( \Upsilon(z_e) \) is expressed as follows

\[
P(\lambda) = \lambda^3(\lambda^2 + \Phi_0^2)(\lambda^2 + \Phi_1^2)(\lambda^4 + m\lambda^2 + n)(\lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s) \quad (14)
\]

with \( \Phi_i^2 = \frac{(C_i - A_i)\omega_i^c + l}{A_i} \). The coefficients present in the above polynomial are functions of the parameters of the problem and \( \rho \), where \( \rho \) is taken as the root of the equation (8).
5.1 S2 and S1 are spherical rigid bodies

The characteristic polynomial (14) of \( \Phi(z_e) \) simplifies to
\[
P(\lambda) = \lambda^5(\lambda^2 + \Phi_0^2)(\lambda^2 + \Phi_1^2)(\lambda^2 + (\omega_0^e)^2)^2(\lambda^2 + p)(\lambda^4 + q\lambda^2 + r)
\]
with coefficients shown in Appendix B.

If \( p \geq 0, q \geq 0, r \geq 0, q^2 - 4r \geq 0 \), then \( z_e \) is spectrally stable. These conditions are not verified since \( r < 0 \).

**Proposition 13** If \( z_e \) is the only relative equilibrium in the configuration \( S_0S_2S_1 \) of the zero order approximate dynamics then it is unstable.

5.2 S2 and S1 are close to a sphere

The case in which \( S_i \) \((i = 1, 2) \) are close to a sphere will now be analyzed. In this case \( C_i - A_i \approx 0 \), and this is the reason why by applying the Implicit Function Theorem \( z_e \) is unstable.

If \( C_i - A_i \) is not close to zero, the coefficients of the polynomial (14) have very complicated expressions. Numeric calculations prove that linear stable Eulerian relative equilibria exist for certain values of the parameters \( C_i - A_i \) \((i = 1, 2) \) (see Vera and Vigueras [7] for details). These results are also applicable to configurations \( S_2S_0S_1 \) and \( S_2S_1S_0 \).

6 Lagrangian relative equilibria

6.1 Necessary conditions of existence

If \( z_e = (\Pi_2^e, \Pi_1^e, \Pi_0^e, \lambda_e^e, \mu_e^e, p_e^e, p_\mu^e) \) is a Lagrangian relative equilibrium, the below identities are verified
\[
\begin{align*}
\lambda^e \times (\nabla_\lambda V)_e &= 0, \\
g_1 \mid \Omega_0^e \mid^2 (\lambda^e \times \mu^e) &= (\nabla_\lambda V)_e \times \mu^e \\
\mu^e \times (\nabla_\mu V)_e &= 0, \\
g_2 \mid \Omega_0^e \mid^2 (\lambda^e \times \mu^e) &= \lambda^e \times (\nabla_\mu V)_e
\end{align*}
\]

After some calculations, it is concluded
\[
\begin{align*}
(A_{12})_e(\lambda^e \times \mu^e) &= 0, \\
g_1 \mid \Omega_0^e \mid^2 (\lambda^e \times \mu^e) &= (A_{11})_e(\lambda^e \times \mu^e) \\
(A_{21})_e(\lambda^e \times \mu^e) &= 0, \\
g_2 \mid \Omega_0^e \mid^2 (\lambda^e \times \mu^e) &= (A_{22})_e(\lambda^e \times \mu^e)
\end{align*}
\]

The expressions \( (A_{ij})_e \) are certain values of the potential function in the Lagrangian relative equilibrium (see Appendix A, for details).

Concluding, the following relationships are verified
\[
(A_{12})_e = 0, \quad \mid \Omega_0^e \mid^2 = \frac{(A_{11})_e}{g_1} = \frac{(A_{22})_e}{g_2}
\]
and the below proposition can be obtained.
Proposition 14 Let \( z_e = (\Pi^e_2, \Pi^e_1, \Pi^e_0, \lambda^e, \mathbf{p}^e_\lambda, \mu^e, \mathbf{p}^e_\mu) \) be a Lagrangian relative equilibrium. Then it is seen that

\[
g_2(\tilde{A}_{11})_e = g_1(\tilde{A}_{22})_e, \quad (\tilde{A}_{12})_e = 0, \quad |\Omega^e_0|^2 = \frac{(A_{11})_e}{g_1}
\]

If \( z_e = (\Pi^e_2, \Pi^e_1, \Pi^e_0, \lambda^e, \mathbf{p}^e_\lambda, \mu^e, \mathbf{p}^e_\mu) \) is a Lagrangian relative equilibrium, then using the expressions of \((A_{ij})_e\) it is proven

\[
m_1 \lambda^e = m_1 \mu^e - \frac{m_2}{M_2} \lambda^e, \quad m_2 \lambda^e = m_2 \mu^e + \frac{m_1}{M_2} \lambda^e
\]

and

\[
|\Omega^e_0|^2 = \frac{G(m_0 + m_1 + m_2)}{Z^3}
\]

From the previous expressions, the following result is deduced.

Proposition 15 If \( z_e = (\Pi^e_2, \Pi^e_1, \Pi^e_0, \lambda^e, \mathbf{p}^e_\lambda, \mu^e, \mathbf{p}^e_\mu) \) is a Lagrangian relative equilibrium, then by denoting \( |\lambda^e| = Z, \quad |\mathbf{p}^e_\lambda - \frac{m_2}{M_2} \lambda^e| = X, \quad |\mathbf{p}^e_\mu + \frac{m_1}{M_2} \lambda^e| = Y \), the system of equations

\[
\begin{align*}
X^5 - Z^3X^2 - \beta_1 Z^3 &= 0 \\
Y^5 - Z^3Y^2 - \beta_2 Z^3 &= 0
\end{align*}
\]

has positive real solutions.

Remark 16 \( Z, \beta_1 \) and \( \beta_2 \) will be the parameters that exert a strong influence on the study of the number of the different configurations of a Lagrangian relative equilibrium.

6.2 Sufficient conditions of existence

If \( Z \) has a fixed value and \( X, Y \) verify the below system of equations

\[
\begin{align*}
X^5 - Z^3X^2 - \beta_1 Z^3 &= 0 \\
Y^5 - Z^3Y^2 - \beta_2 Z^3 &= 0
\end{align*}
\]

, with respect to an appropriate reference system, the coordinates of Lagrangian relative equilibria are completely determined. If \( X = Y \neq Z \) is a solution to the previous system, then the gyrostat \( S_0 \) and the rigid bodies \( S_i \) \((i = 1, 2)\) form an isosceles triangle when \( \beta_1 = \beta_2 \). On the other hand, if \( \beta_1 \neq \beta_2 \) then \( X \neq Y \neq Z \) and \( S_i \) \((i = 0, 1, 2)\) form a scalene triangle.

The following proposition, whose demonstration is based on verifying the equations of the equilibria, shows how Lagrangian relative equilibria are when \( S_0, S_1, S_2 \) form an isosceles triangle. Following the same line of reasoning, Lagrangian relative equilibria could be described when \( S_0, S_1, S_2 \) form a scalene triangle.
Proposition 17  With respect to an appropriate reference system, \( z_e = (\Pi_2^e, \Pi_1^e, \Pi_0^e, \lambda^e, \mu^e, p_\lambda^e, p_\mu^e) \) expressed by

\[
\lambda^e = (x_1, y_1, 0), \quad p_\lambda^e = g_1 \omega_0^e (-y_1, x_1, 0), \quad \mu^e = (x_2, y_2, 0) \tag{17}
\]

with

\[
x_1 = Z, \quad y_1 = 0, \quad x_2 = \frac{Z(m_2 - m_1)}{2(m_1 + m_2)}, \quad y_2 = \pm \sqrt{\frac{4X^2 - Z^2}{2}}, \quad (\omega_0^e)^2 = \frac{G(m_0 + m_1 + m_2)}{Z^3}
\]

are an isosceles Lagrangian relative equilibrium. The total angular momentum vector of the system is given by

\[
L = (0, 0, C_2 \omega_2^e + C_1 \omega_1^e + C_0 \omega_0^e + l + \omega_0^e \sum_{i=1}^2 g_i(x_i^2 + y_i^2))
\]

Proposition 18  With respect to an appropriate reference system, \( z_e = (\Pi_2^e, \Pi_1^e, \Pi_0^e, \lambda^e, \mu^e, p_\lambda^e, p_\mu^e) \) given by

\[
\lambda^e = (x_1, y_1, 0), \quad p_\lambda^e = g_1 \omega_0^e (-y_1, x_1, 0), \quad \mu^e = (x_2, y_2, 0) \tag{18}
\]

with

\[
x_1 = Z, \quad y_1 = 0, \quad x_2 = \frac{m_2(X^2 + Z^2 - Y^2) - m_1(Y^2 + Z^2 - X^2)}{2(m_1 + m_2)Z}, \quad (\omega_0^e)^2 = \frac{G(m_0 + m_1 + m_2)}{Z^3}, \quad y_2 = \pm \sqrt{\frac{(Z + X + Y)(Z + X - Y)(Z + Y - X)(X + Y - Z)}{2Z}}
\]

are a scalene Lagrangian relative equilibrium. The total angular momentum vector of the system is expressed as follows

\[
L = (0, 0, C_2 \omega_2^e + C_1 \omega_1^e + C_0 \omega_0^e + l + \omega_0^e \sum_{i=1}^2 g_i(x_i^2 + y_i^2))
\]

Next the Lagrangian relative equilibria will be studied when \( S_2 \) and \( S_1 \) are spherical rigid bodies.
6.3 Lagrange relative equilibria when $S_2$ and $S_1$ are spherical rigid bodies

If $S_2$ and $S_1$ are spherical rigid bodies, then $C_1 = A_1$, $C_2 = A_2$. The equations (15) are

\[
\begin{align*}
X^3 &= Z^3 \\
Y^3 &= Z^3
\end{align*}
\]

being easily deduced from them that $X = Y = Z$, i.e. $S_0$, $S_1$ and $S_2$ form an equilateral triangle. It is also obtained

\[|\Omega^e_0|^2 = \frac{G(m_0 + m_1 + m_2)}{Z^3}\]

On the other hand, one parametrization of the former equilibria $z_e = (\Pi^e_2, \Pi^e_1, \Pi^e_0, X^e, p^e_\lambda, \mu^e, p^e_\mu)$ is given by the relationship (17) being

\[
x_1 = Z, \quad y_1 = 0, \quad x_2 = \frac{Z(m_2 - m_1)}{2(m_1 + m_2)}, \quad y_2 = y_2 = \pm \frac{\sqrt{3}Z}{2}, \quad (\omega^e_0)^2 = \frac{G(m_0 + m_1 + m_2)}{Z^3}
\]

This parametrization of the relative equilibria will be useful to study the stability of the same ones.

6.4 Lagrangian relative equilibria when $S_2$ and $S_1$ are not spherical rigid bodies

The number of real positive solutions of the below system will be now studied

\[
\begin{align*}
X^5 - Z^3X^2 - \beta_1 Z^3 &= 0 \\
Y^5 - Z^3Y^2 - \beta_2 Z^3 &= 0
\end{align*}
\]

(19)

where $Z$, $\beta_1$ and $\beta_2$ are parameters. With this aim in view, let us proceed with the study of the number of the positive real roots of the polynomial

\[p(X) = X^5 - Z^3X^2 - \beta Z^3\]

according to the values of the parameters $Z$ and $\beta$.

If $\beta \geq 0$, then this polynomial can only have a positive real root by applying the rule of the signs of Descartes.

If $\beta < 0$, then we can have two positive real roots, just one positive real root or none. The discriminant of the polynomial, denoted by $\text{discrim}(p, X)$, is expressed by

\[\text{discrim}(p, X) = \beta Z^{12}(3125\beta^3 + 108Z^6)\]
Then if \( \text{discrim}(p, X) < 0 \) the polynomial \( p \) has two real roots, if \( \text{discrim}(p, x) = 0 \) it has a positive double root, whereas if \( \text{discrim}(p, x) > 0 \) it has no positive root.

The discriminant is zero when the following relationship is verified

\[
\beta = -\frac{3\sqrt{20}}{25}Z^2
\]

From the previous results a detailed study can be carried out of the bifurcations of the Lagrangian relative equilibria when \( \beta_1 = \beta_2 \).

**Proposition 19** Let \( z_e = (\Pi_2^e, \Pi_1^e, \Pi_0^e, X^e, p^e_\lambda, \mu^e, p^e_\mu) \) be a Lagrangian relative equilibrium and \( \beta_1 = \beta_2 \), then

1. If \( \beta_1 > 0 \), a unique 2-parametric family exists, making \( S_0, S_1 \) and \( S_2 \) an isosceles triangle.

2. If \( \beta_1 < 0 \) (gyrostat prolate), then:

   a1) If \(-\frac{7Z^2}{32} < \beta_1 < 0\), there are two types of relative equilibria:
   - One 2-parametric family of relative equilibria, making \( S_0, S_1 \) and \( S_2 \) an isosceles triangle with \( X = Y \neq Z \).
   - Two 2-parametric families of relative equilibria, making \( S_0, S_1 \) and \( S_2 \) a scalene triangle with \( X \neq Y \neq Z \).

   a2) If \(-\frac{3\sqrt{20}}{25}Z^2 < \beta_1 < -\frac{7Z^2}{32}\), there are two types of relative equilibria:
   - Two 2-parametric families of relative equilibria, making \( S_0, S_1 \) and \( S_2 \) an isosceles triangle with \( X = Y \neq Z \).
   - Four 2-parametric families of relative equilibria, making \( S_0, S_1 \) and \( S_2 \) a scalene triangle with \( X \neq Y \neq Z \).

   b) If \( \beta_1 = -\frac{3\sqrt{20}}{25}Z^2 \), a unique 2-parametric family exists making \( S_0, S_1 \) and \( S_2 \) an isosceles triangle, with \( X = Y \neq Z \).

   c) If \( \beta_1 < -\frac{3\sqrt{20}}{25}Z^2 \), relative equilibria do not exist.

On the other hand, we have that \( z_e = (\Pi_2^e, \Pi_1^e, \Pi_0^e, X^e, p^e_\lambda, \mu^e, p^e_\mu) \), with equations (17) given by

\[
x_1 = Z, \quad y_1 = 0, \quad x_2 = \frac{Z(m_2 - m_1)}{2(m_1 + m_2)}, \quad y_2 = \pm \frac{\sqrt{4X^2 - Z^2}}{2}, \quad (\omega_0^e)^2 = \frac{G(m_0 + m_1 + m_2)}{Z^3}
\]

describes a parametrization of the Lagrangian relative equilibria when \( S_0, S_1 \) and \( S_2 \) form an isosceles triangle. Similar results are obtained when \( S_0, S_1 \) and \( S_2 \) form a scalene triangle.
Proposition 20 Let $\mathbf{z}_e = (\Pi^e_2, \Pi^e_1, \Pi^e_0, X^e, p^e_\lambda, \mu^e, p^e_\mu)$ be a Lagrangian relative equilibrium and $\beta_1 \neq \beta_2$, then

1. If $\beta_1, \beta_2 > 0$ a unique 3-parametric family exists, making $S_0$, $S_1$ and $S_2$ a scalene triangle.

2. If $\beta_1 < 0$ and $\beta_2 > 0$ then
   
   a1) If $-\frac{7Z^2}{32} < \beta_1 < 0$, there are two 3-parametric families of relative equilibria, making $S_0$, $S_1$ and $S_2$ a scalene triangle with $X \neq Y \neq Z$.

   a2) If $-\frac{3\sqrt{20}}{25}Z^2 < \beta_1 < -\frac{7Z^2}{32}$, there are four 3-parametric families of relative equilibria, making $S_0$, $S_1$ and $S_2$ a scalene triangle with $X \neq Y \neq Z$.

   b) If $\beta_1 = -\frac{3\sqrt{20}}{25}Z^2$ a unique 3-parametric family exists, making $S_0$, $S_1$ and $S_2$ a scalene triangle with $X = Y \neq Z$.

   c) If $\beta_1 < -\frac{3\sqrt{20}}{25}Z^2$, relative equilibria do not exist.

3. The case $\beta_2 < 0$ and $\beta_1 > 0$ is similar to the previous one.

4. If $\beta_1, \beta_2 < 0$ then:
   
   b1) If $-\frac{7Z^2}{32} < \beta_1 < 0$ and $-\frac{7Z^2}{32} < \beta_2 < 0$, there are four 3-parametric families of relative equilibria, making $S_0$, $S_1$ and $S_2$ a scalene triangle with $X \neq Y \neq Z$.

   b2) If $-\frac{7Z^2}{32} < \beta_1 < 0$ and $-\frac{3\sqrt{20}}{25}Z^2 < \beta_2 < -\frac{7Z^2}{32}$, there are six 3-parametric families of relative equilibria, making $S_0$, $S_1$ and $S_2$ a scalene triangle with $X \neq Y \neq Z$.

   b3) If $-\frac{3\sqrt{20}}{25}Z^2 < \beta_1 < -\frac{7Z^2}{32}$ and $-\frac{3\sqrt{20}}{25}Z^2 < \beta_2 < -\frac{7Z^2}{32}$, there are eight 3-parametric families of relative equilibria, making $S_0$, $S_1$ and $S_2$ a scalene triangle with $X \neq Y \neq Z$.

   b4) If $\beta_1 = -\frac{3\sqrt{20}}{25}Z^2$ and $-\frac{3\sqrt{20}}{25}Z^2 < \beta_2 < -\frac{7Z^2}{32}$, there are two 3-parametric families of relative equilibria, making $S_0$, $S_1$ and $S_2$ a scalene triangle with $X \neq Y \neq Z$.

   b5) If $\beta_1 = -\frac{3\sqrt{20}}{25}Z^2$ and $\beta_2 < -\frac{3\sqrt{20}}{25}Z^2$ or $\beta_2 = -\frac{3\sqrt{20}}{25}Z^2$ and $\beta_1 < -\frac{3\sqrt{20}}{25}Z^2$, relative equilibria do not exist.
6.5 Lagrangian relative equilibria when $S_2$ and $S_1$ are close to spherical rigid bodies

If $\beta_1, \beta_2 \approx 0$, the solutions of (19) up to the second order in the parameters $\beta_1$ and $\beta_2$ are

$$X = Z + \frac{\beta_1}{3Z} - \frac{\beta_1^2}{3Z^3} + o(\beta_1^2)$$

$$Y = Z + \frac{\beta_2}{3Z} - \frac{\beta_2^2}{3Z^3} + o(\beta_2^2)$$

Using the previous relationships, the expressions of $\mu^e = (x_2, y_2, 0)$ are obtained with

$$x_2 = \frac{(m_2 - m_1)Z}{2(m_2 + m_1)} - \frac{\beta_1}{3Z} + \frac{\beta_2}{3Z} + \frac{5\beta_1^2}{18Z^3} - \frac{5\beta_2^2}{18Z^3} + o(\beta_1^2 + \beta_2^2)$$

$$y_2 = \frac{\sqrt{3}Z}{2} + \frac{\sqrt{3}}{9} (\beta_1 + \beta_2) - \left( \frac{23\sqrt{3}\beta_1^2}{162Z^3} - \frac{4\sqrt{3}\beta_1\beta_2}{81Z^3} + \frac{23\sqrt{3}\beta_2^2}{162Z^3} \right) + o(\beta_1^2 + \beta_2^2)$$

and $\lambda^e = (Z, 0, 0)$.

7 Linear stability of the Lagrangian relative equilibria

In the equilibrium $z_e$ the tangent flow of the equations (1) are

$$\frac{d\delta z}{dt} = \mathcal{U}(z_e)\delta z$$

where $\delta z = z - z_e$ and $\mathcal{U}(z_e)$ is the Jacobian matrix of (1) in $z_e$.

7.1 $S_2$ and $S_1$ are spherical rigid bodies

The characteristic polynomial of $\mathcal{U}(z_e)$ has the bellow expression

$$P(\lambda) = \lambda^5(\lambda^2 + \Phi_0^2)(\lambda^2 + \Phi_1^2)(\lambda^2 + \Phi_2^2)(\lambda^2 + \omega_e^2)^3(\lambda^4 + \omega_e^2\lambda^2 + q)$$

where

$$\omega_e^2 = \frac{G(m_0 + m_1 + m_2)}{Z^3}, \quad q = \frac{27G^2(m_1m_0 + m_2m_0 + m_1m_2)}{4Z^6}$$

and $\Phi_i^2 = \frac{(C_i - A_i)\omega_e + l}{A_i}$.

The minimum polynomial of $\mathcal{U}(z_e)$ has this expression

$$Q(\lambda) = \lambda^2(\lambda^2 + \Phi_0^2)(\lambda^2 + \Phi_1^2)(\lambda^2 + \Phi_2^2)(\lambda^2 + \omega_e^2)(\lambda^4 + \omega_e^2\lambda^2 + q)$$

Then the bellow results are verified.
Proposition 21 \( z_e \) is spectral stable if

\[
(m_0 + m_2 + m_1)^2 \geq 27(m_1m_0 + m_2m_0 + m_1m_2)
\] (20)

If

\[
(m_0 + m_2 + m_1)^2 < 27(m_1m_0 + m_2m_0 + m_1m_2)
\]

then \( z_e \) is unstable

As the minimum polynomial of \( \mathfrak{U}(z_e) \) has the \( \lambda = 0 \) as double root, the matrix \( \mathfrak{U}(z_e) \) is not diagonalizable, being the following proposition verified.

Proposition 22 The linear system

\[
\frac{d\delta z}{dt} = \mathfrak{U}(z_e)\delta z
\]

is unstable.

7.2 \( S_2 \) and \( S_1 \) are not spherical rigid bodies

Similar results show that the characteristic polynomial has this expression

\[
P(\lambda) = \lambda^3(\lambda^2 + \Phi_0^2)(\lambda^2 + \Phi_1^2)(\lambda^2 + \Phi_2^2)(\lambda^2 + m)(\lambda^2 + n)(\lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s)
\]

The minimum polynomial of \( \mathfrak{U}(z_e) \) is expressed by

\[
Q(\lambda) = \lambda(\lambda^2 + \Phi_0^2)(\lambda^2 + \Phi_1^2)(\lambda^2 + \Phi_2^2)(\lambda^2 + m)(\lambda^2 + n)(\lambda^8 + p\lambda^6 + q\lambda^4 + r\lambda^2 + s)
\]

The coefficients of the characteristic polynomial will be expressed in function of the parameters of the problem, i.e. the masses and the coefficients \( \beta_i \) \( (i = 1, 2) \). We conclude the following result by employing the Sturm Theorem.

We conclude that the Lagrangian relative equilibria are spectral stable (lineally stable) if the following conditions are verified

\[
p^2q^2 - 3rp^3 - 6p^2s - 4q^3 + 14pqr + 16qs - 18r^2 \geq 0 \ (> 0)
p^2qr - 48sr - 9sp^3 + 32pqs - 4q^2r + 3pr^2 \geq 0 \ (> 0)
r, s \geq 0 \ (> 0), \ 3p^2 - 8q \geq 0 \ (> 0), \ pr - 16s \geq 0 \ (> 0)
m, n \geq 0 \ (> 0), \ \text{discrim}(h) \geq 0 \ (> 0)
\]

If \( S_i \) \( (i = 1, 2) \) are arbitrary rigid bodies, the former conditions are very complicated expressions in the parameters of the problem and can only be studied by means of numerical analysis.
If \( S_i (i = 1, 2) \) are close to a sphere, the coefficients of \( P \), up to the first order in the parameters \( \beta_1 \) and \( \beta_2 \) are expressed as follows

\[
\begin{align*}
 m &= \frac{G(m_0 + m_1 + m_2)}{Z^3} + o(\beta_1) + o(\beta_2), \quad n = \frac{G(m_0 + m_1 + m_2)}{Z^3} + o(\beta_1) + o(\beta_2) \\
r &= \frac{27G^3(m_0 + m_1 + m_2)(m_1m_0 + m_2m_0 + m_1m_2)}{4Z^9} + o(\beta_1) + o(\beta_2) \\
q &= \frac{G^2(4m_0^2 + 4m_1^2 + 4m_2^2 + 35m_0m_1 + 35m_0m_2 + 35m_1m_2)}{4Z^6} + o(\beta_1) + o(\beta_2) \\
p &= \frac{2G(m_0 + m_1 + m_2)}{Z^3} + o(\beta_1) + o(\beta_2), \quad s = o(\beta_1) + o(\beta_2)
\end{align*}
\]

If the function

\[
s = \frac{81G^4m_0(m_0 + m_1 + m_2)^2(\beta_1m_1 + \beta_2m)}{4} + o(\beta_1^2 + \beta_2^2)
\]

is positive and

\[
(m_0 + m_2 + m_1)^2 > 27(m_1m_0 + m_2m_0 + m_1m_2)
\]

\( z_e \) is linearly stable. Then if \( S_i (i = 1, 2) \) is close to a sphere and

\[
m_1(C_1 - A_1) + m_2(C_2 - A_2) > 0
\]

\( z_e \) is linearly stable when \([20]\) is verified. If \( m_1(C_1 - A_1) + m_2(C_2 - A_2) = 0 \), second order terms of the coefficients in \( \beta_1 \) and \( \beta_2 \) are necessary for the study of the problem.

### 8 Conclusions and future works

In this paper it has been investigated some important periodic solutions of the dynamics of a gyrostat in Newtonian interaction with two symmetric rigid bodies. With the hypotheses formulated in the introduction of this paper, working in the double reduced space of configuration of the problem, both the equations of motion and the those which determine the relative equilibria have been derived.

Two families of relative equilibria, Eulerian and Lagrangian are studied. The Eulerian relative equilibria have been completely determinated by a polynomial equation of degree nine. The bifurcations of these equilibria have been carried out when \( m_0 \) is very small. The bifurcations of the Lagrangian relative equilibria have been completely investigated and expressions for the Lagrangian relative equilibria, when both solids are close to a sphere, are provided. These expressions are useful for the posterior study of the stability of the same ones.

The instability of Eulerian relative has been proven and necessary and sufficient conditions for lineal stability of Lagrangian relative equilibria are provided.
Different results, which had been obtained by means of classic methods in previous works, have been generalized in a different way, and other results, not previously considered, have been studied. Some interesting numeric results have been detailed in Appendix B.

Numerous problems are open, and among them the study of the "inclined" relative equilibria, in which \( \Omega_0^0 \) form an angle \( \alpha \neq 0, \pi/2 \) with the vector \( \mathbf{X} \times \mathbf{\mu}^e \), is considered.

The methods used in this work are susceptible of being used in similar problems. The nonlinear stability of the relative equilibria obtained here is the logical continuation of this work.

9 The \( (A_{ij})_e, \ i, j = 1, 2 \).

The below expressions of the potential \( \mathbf{V} \) are obtained.

\[
(\nabla_\mathbf{X}\mathbf{V})_e = \frac{G m_1 m_2 \mathbf{X}^e}{|\mathbf{X}^e|^3} - \frac{G m_0 m_2}{M_2} \sum_{i=0}^{1} \frac{\alpha_i^1 (\mathbf{\mu}^e - \frac{m_i}{M_2} \mathbf{X}^e)}{|\mathbf{\mu}^e - \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}} + \frac{G m_0 m_1}{M_2} \sum_{i=0}^{1} \frac{\alpha_i^2 (\mathbf{\mu}^e + \frac{m_i}{M_2} \mathbf{X}^e)}{|\mathbf{\mu}^e + \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}}
\]

\[
(\nabla_\mathbf{\mu}\mathbf{V})_e = G m_0 \left( \sum_{i=0}^{1} \frac{\alpha_i^1 (\mathbf{\mu}^e - \frac{m_i}{M_2} \mathbf{X}^e)}{|\mathbf{\mu}^e - \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}} + \sum_{i=0}^{k} \frac{\alpha_i^2 (\mathbf{\mu}^e + \frac{m_i}{M_2} \mathbf{X}^e)}{|\mathbf{\mu}^e + \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}} \right)
\]

where \( \alpha_0^1 = \alpha_2^2 = m_0, \ \alpha_1^1 = \beta_1 = 3/2(C_1 - A_1) \) and \( \alpha_1^2 = \beta_2 = 3/2(C_2 - A_2) \). The following identities are also verified

\[
(\nabla_\mathbf{X}\mathbf{V})_e = (A_{11})_e \mathbf{X}^e + (A_{12})_e \mathbf{\mu}^e, \quad (\nabla_\mathbf{\mu}\mathbf{V})_e = (A_{21})_e \mathbf{X}^e + (A_{22})_e \mathbf{\mu}^e
\]

where

\[
(A_{11})_e = \frac{G m_1 m_2}{|\mathbf{X}^e|^3} + \frac{G m_0 m_2}{M_2^2} \left( \sum_{i=0}^{1} \frac{\alpha_i^1}{|\mathbf{\mu}^e - \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}} \right) - \frac{G m_0 m_1}{M_2^2} \left( \sum_{i=0}^{1} \frac{\alpha_i^2}{|\mathbf{\mu}^e + \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}} \right)
\]

\[
(A_{12})_e = \frac{G m_0 m_1}{M_2} \left( \sum_{i=0}^{1} \frac{\alpha_i^1}{|\mathbf{\mu}^e - \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}} \right) - \frac{G m_0 m_2}{M_2} \left( \sum_{i=0}^{1} \frac{\alpha_i^2}{|\mathbf{\mu}^e + \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}} \right)
\]

\[
(A_{21})_e = (A_{22})_e
\]

\[
(A_{22})_e = G m_0 \left( \sum_{i=0}^{1} \frac{\alpha_i^1}{|\mathbf{\mu}^e - \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}} + \sum_{i=0}^{1} \frac{\alpha_i^2}{|\mathbf{\mu}^e + \frac{m_i}{M_2} \mathbf{X}^e|^{2i+3}} \right)
\]
10 Some numerical results

In order to obtain the values of $C_i - A_i, (i = 1, 2)$, the below relationships will be utilized

$$C_1 - A_1 = (1 - \mu) \left( \frac{e_{S_1}}{Z} \right)^2 J_{s_1}^{S_1}$$

$$C_2 - A_2 = \mu \left( \frac{e_{S_2}}{Z} \right)^2 J_{s_2}^{S_2}$$

where $e_{S_i}$ and $p_{S_i}$ represent the equatorial and polar radius of $S_i (i = 1, 2)$, $J_{s_i}^{S_i} = \frac{2}{5} \varepsilon_i$, respectively, and $\varepsilon_i = \frac{e_{S_i} - p_{S_i}}{e_{S_i}}$. $S_1, S_2$ are considered to be homogeneous ellipsoids. The distances are measured in kilometers.

| $S_2S_1S_0 (m_0 \to 0)$ | Without oblat. | Oblat. of $S_2$ | Oblat. of $S_2$ and $S_1$ |
|------------------------|----------------|-----------------|---------------------------|
| Earth-Moon-$S_0$       | 448879.206     | 448879.221      | 448879.251                |
| Mars-Phobos-$S_0$      | 9414.945       | 9414.958        | 9414.958                  |
| $S_0S_2S_1 (m_0 \to 0)$| Without oblat. | Oblat. of $S_2$ | Oblat. of $S_2$ and $S_1$ |
| $S_0$-Earth-Moon       | 381679.691     | 381679.763      | 381679.763                |
| $S_0$-Mars-Phobos      | 9310.642       | 9310.666        | 9310.668                  |
| $S_2S_0S_1 (m_0 \to 0)$| Without oblat. | Oblat. of $S_2$ | Oblat. of $S_2$ and $S_1$ |
| Earth-$S_0$-Moon       | 326409.744     | 326409.780      | 326409.751                |
| Mars-$S_0$-Phobos      | 9339.156       | 9339.196        | 9339.196                  |
11 Coefficients of the characteristic polynomial in Eulerian relative equilibria \( S_0 S_2 S_1 \)

The coefficients of the characteristic polynomial \([14]\) are

\[
\omega_e^2 = \frac{G ((m_2 + m_1) \rho^4 + (2 m_1 + 2 m_2) \rho^3 + (m_2 + m_1) \rho^2 - 2 m_0 \rho - m_0)}{\lambda_e^2 (1 + \rho)^2 \rho^2}
\]

\[
p = \frac{G((m_2 + 4 m_0 + m_1) \rho^3 + (3 m_2 + 6 m_0) \rho^2 + (4 m_0 + 3 m_2) \rho + m_0 + m_2)}{(1 + \rho)^3 \rho^3 \lambda_e^3}
\]

\[
q = \frac{G((-2 m_1 \rho^4 m_2 + (-2 m_0 m_1 + m_1^2 + m_2^2 - 2 m_1 m_2 - 2 m_0 m_2) \rho^3 + (3 m_2^2 + m_1 m_2 - 6 m_0 m_1) \rho^2 + (-m_1 m_2 + 3 m_2^2 + 2 m_0 m_2 - 4 m_0 m_1) \rho + m_2^2 - m_0 m_1 + m_0 m_2 - m_1 m_2))}{(1 + \rho)^3 \rho^3 \lambda_e^3}
\]

\[
r = \frac{G^2 (a_1 \rho^4 + a_2 \rho^4 + a_3 \rho^2 + a_4 \rho + a_5)}{(1 + \rho)^8 \rho^8 \lambda_e^9}
\]

11.1 Coefficients \( a_i \) \( (i = 1, \ldots, 5) \)

\[
a_1 = -42 m_1 m_1 - 48 m_1 m_0 - 147 m_0 m_1 m_0 - 129 m_0 m_1 m_0 - 336 m_0 m_1 m_0 - 129 m_0 m_1 m_0 - 207 m_0 m_1 m_0 - 782 m_0 m_1 m_0 - 673 m_0 m_1 m_0 - 81 m_0 m_1 m_0 - 150 m_0 m_1 m_0 - 869 m_0 m_1 m_0 - 1325 m_0 m_1 m_0 - 378 m_0 m_1 m_0 - 64 m_0 m_1 m_0 - 513 m_0 m_1 m_0 - 1270 m_0 m_1 m_0 - 702 m_0 m_1 m_0 - 14 m_0 m_1 m_0 - 165 m_0 m_1 m_0 - 610 m_0 m_1 m_0 - 648 m_0 m_1 m_0 - 24 m_0 m_1 m_0 - 119 m_0 m_1 m_0 - 297 m_0 m_1 m_0 + 2 m_0 m_1 m_0 - 54 m_0 m_1 m_0
\]
\[ a_2 = -60m_2^5m_1 - 54m_2^5m_0 - 243m_2^6m_1^2 - 474m_2^6m_1m_0 - 173m_2^6m_0^2 \\
-399m_2^5m_1^3 - 1345m_2^5m_1^2m_0 - 999m_2^5m_1m_0^2 - 135m_2^5m_0^3 - 329m_2^4m_1^4 \\
-1846m_2^4m_1^3m_0 - 2223m_2^4m_1^2m_0^2 - 648m_2^4m_1m_0^3 - 138m_2^4m_0^4 \\
-1364m_2^3m_1^5m_0 - 2506m_2^3m_1^4m_0^2 - 1242m_2^3m_1^3m_0^3 - 24m_2^3m_1^2m_0^4 \\
-536m_2^2m_1^5m_0^2 - 1530m_2^2m_1^4m_0^3 - 1188m_2^2m_1^3m_0^4 - 90m_2^2m_1^2m_0^5 \\
-477m_2m_1^5m_0^4 - 567m_2m_1^4m_0^5 - 56m_1^6m_0^2 - 108m_1^5m_0^3 \\
\]

\[ a_3 = -42m_2^5m_1 - 36m_2^5m_0 - 183m_2^6m_1^2 - 342m_2^6m_1m_0 - 93m_2^6m_0^2 \\
-349m_2^5m_1^3 - 1097m_2^5m_1^2m_0 - 630m_2^5m_1m_0^2 - 81m_2^5m_0^3 - 358m_2^4m_1^4 \\
-1776m_2^4m_1^3m_0 - 166m_2^4m_1^2m_0^2 - 405m_2^4m_1m_0^3 - 189m_2^4m_0^4 \\
-1614m_2^3m_1^5m_0 - 2256m_2^3m_1^4m_0^2 - 810m_2^3m_1^3m_0^3 - 31m_2^3m_1^2m_0^4 \\
-827m_2^2m_1^5m_0^2 - 1683m_2^2m_1^4m_0^3 - 810m_2^2m_1^3m_0^4 - 6m_2m_1^7 \\
-228m_2m_1^6m_0 - 666m_2m_1^5m_0^2 - 405m_2m_1^4m_0^3 - 30m_1^6m_0 \\
-81m_1^5m_0^3 - 111m_1^6m_0^2 \\
\]

\[ a_4 = -12m_2^5m_1 - 12m_2^5m_0 - 56m_2^6m_1^2 - 114m_2^6m_1m_0 - 24m_2^6m_0^2 \\
-130m_2^5m_1^3 - 387m_2^5m_1^2m_0 - 162m_2^5m_1m_0^2 - 179m_2^5m_0^3 \\
-687m_2^4m_1^4m_0 - 432m_2^4m_1^3m_0^2 - 140m_2^4m_1^2m_0^3 - 693m_2^4m_1m_0^4 \\
-588m_2^3m_1^5m_0 - 52m_2^3m_1^4m_0^2 - 387m_2^3m_1^3m_0^3 - 432m_2^3m_1^2m_0^4 - 6m_2m_1^7 \\
-108m_2m_1^6m_0 - 162m_2m_1^5m_0^2 - 12m_1^7m_0 - 24m_1^6m_0^2 \\
\]

\[ a_5 = -(m_0 + m_2)(18m_0m_2^6 + 12m_1m_2^6 + 94m_2m_0m_1 + 36m_2^2m_0^2 \\
+81m_2m_0m_1^2 + 168m_2m_0m_1^2 + 42m_2^2m_0^3 + 128m_2^2m_0m_0^3 \\
+27m_2m_0^4 + 15m_2^2m_1^5 + 31m_2^2m_0m_1^4 + 126m_2m_0^2m_1^3 + 18m_2^2m_0^2m_1^2 \\
+54m_2m_0^4m_1^2 + 12m_2m_0^4m_1^2 + 5m_2m_1^4 + 7m_1^6m_0 + 9m_0^5m_1^5 \\
+144m_2^3m_0^5m_1^2) \\
\]

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