A UNIFIED GENERALIZATION OF REAL, GOULD–HOPPER, 1-D AND 2-D HOLOMORPHIC, AND POLYANALYTIC HERMITE POLYNOMIALS

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Abstract. We analyze a variety of basic properties of a new class of polynomials generalizing different classes of Hermite polynomials such as the real, Gould–Hopper, as well as the 1-d and 2-d holomorphic, ternary and polyanalytic complex Hermite polynomials. Mainly, we are interested in their operational representations, different generating functions, recurrence relationships, higher partial differential equations they obey. We also derive some special identities including addition formulas of Runge type, multiplication and Nielson formulas. Their relationship to the Gould–Hopper polynomials and the hypergeometric functions are also discussed.

1 Introduction

The Hermite polynomials

\[ H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \]

on the real line as well as their various generalizations are widely studied in the literature. They have found many applications in many branches of mathematics, physics and engineering (see e.g. [17, 27, 34, 19] and the references therein). Their tensor product \( H_m(x)H_n(y) \) is a specific 2d-generalization that form an orthogonal basis of \( L^2(\mathbb{R}^2; e^{-x^2-y^2} dx dy) \). The holomorphic Hermite polynomials \( H_n(z) \), obtained by replacing the real \( x \) by the complex variable \( z \), are another natural and interesting extension of \( H_n(x) \) to the complex plane \( \mathbb{C} \). The associated functions \( e^{-z^2/2} H_n(z) \) form an orthogonal basis of a like Bargmann-Fock space (see e.g. [36, 3]). For their analytic and combinatoric properties (see e.g. [34, 20]). The polyanalytic analog of \( H_m(z) \), for satisfying the generalized Cauchy–Riemann equation \( \partial^{n+1}_z f = 0 \), are defined by

\[ H_{n,m}(z, \bar{z}) := (-1)^{m+n} |z|^2 \partial_z^n \partial_{\bar{z}}^m \left( e^{-|z|^2} \right) \]

where \( n \wedge m = \min\{n, m\} \). Such polynomials, introduced by Itô [22] in the context of the complex Markov process, are basic tools in the nonlinear analysis of traveling wave tube amplifiers [2]. More specifically, they appear in the calculation of the effects of non-linearity on broadband radio frequencies in communication systems. For their properties and applications one can refer to [11, 12, 20, 10, 13]. Further classes of these generalizations are introduced and studied recently, such as the complex bivariate holomorphic Hermite polynomials \( H_{n,m}(z, \bar{z}) \) and the ternary Hermite polynomials \( H_{n,m}(z, w, x) \), see [14, 20] and (26), respectively. Another class of generalized Hermite
polynomials are those due to Gould and Hopper (\cite[Eq. (6.2), p. 58]{15})

\[ H_n^{(p)}(z|\gamma) = n! \sum_{k=0}^{\left[\frac{n}{p}\right]} \frac{\gamma^k}{k!} \frac{z^{n-pk}}{(n-pk)!}. \]  

(1.1)

They enter in the study of Novikov–Veselov equation \cite{16}. Moreover, it turns out that
\[ H_n^{(2p)}(x)((-1)^{p+1}t) \]
are the so-called heat polynomials which are solutions of the higher order heat equation \((-1)^{q+1} \frac{\partial^q}{\partial x^q} u(x, t) = \frac{\partial}{\partial t} u(x, t)\) (see \cite{16}). For further of their properties, we can refer to \cite{15} and the references therein.

In this paper, we are concerned with a special and unified generalization. More precisely, we deal with the polynomials

\[ H_n^{(p,q)}(z, w|\gamma) = n!m! \sum_{k=0}^{\left[\frac{n}{p}\right] \left[\frac{m}{q}\right]} \frac{\gamma^k z^{n-pk}}{k!} \frac{w^{m-qk}}{(n-pk)! (m-qk)!}. \]  

(1.2)

The polynomials \( H_n^{(p,q)}(z, w|\gamma) \) contain of the classes prescribed above. Moreover, they give rise to new classes of polynomials of Hermite type. The concrete study of \( H_n^{(p,q)}(z, w|\gamma) \) is presented here in a unified way and includes the following items

- Operational representations and connection to hypergeometric function (Section 2),
- Generating functions (Section 3),
- Addition formulas of Runge type (Section 4),
- Multiplication formulas (Section 5),
- Recurrence relationships (Section 6),
- Nielson formulas (Section 7),
- Higher order differential equation they obey (Section 8),
- Connection to Gould–Hopper polynomials (Section 9).

\section{The Hermite Polynomials \( H_n^{(p,q)}(z, w|\gamma) \)}

The polynomials \( H_n^{(p,q)}(z, w|\gamma) \) are defined by their explicit expression in (1.2) with the conventions that \( H_n^{(p,q)}(z, w|\gamma) = e^\gamma \) and \( \left[\frac{n}{p}\right] = +\infty \) when \( k = 0 \). Obviously, they generalize the real, Gould–Hopper, as well as the 1-d and 2-d holomorphic, ternary and polyanalytic complex polynomials. More precisely, we have

1. The sub-case of \( m = q = 0, p = 2, w = 1 \) and \( \gamma = -1 \) leads to the holomorphic Hermite polynomials. More exactly, we have \( H_n^{(2,0)}(2z, 1, -1) = H_n(z) \).

2. For \( m = q = 0 \) and \( w = 1 \), the polynomials \( H_n^{(p,0)}(z, 1|\gamma) \) reduces further to the Gould-Hopper polynomials \( H_n^{(p)}(z|\gamma) \).

3. We recover the 2d-holomorphic Hermite polynomials \( H_{n,m}(z, w) \) as well as their restriction to the non-analytic surface \( w = \bar{z} \), the polyanalytic Hermite polynomials \( H_{n,m}(z, \bar{z}) \), by taking \( p = q = 1 \) and \( \gamma = -1 \). Indeed, we have

\[ H_{n,m}^{(1,1)}(z, w| -1) = H_{n,m}(z, w), \]

\[ H_{n,m}^{(1,1)}(z, \bar{z}| -1) = H_{n,m}(z, \bar{z}). \]

4. The case of \( p = q = 1 \) leads to the ternary Hermite polynomials \( H_{n,m}(z, w|\gamma) \).

The monomials \( z^n w^m \) are recovered by considering special cases of the parameters \( \gamma \) and \( p, q \). For example, if \( \gamma = 0 \) or also \( n < p \) and \( m < q \), we get \( H_{n,m}^{(p,q)}(z, w|\gamma) = z^n w^m \).

Notice also that for \( n = p \) et \( m = q \), we have \( H_{p,p}^{(p,q)}(z, w|\gamma) = z^p w^q - \gamma p!q! \).
Remark 2.1. The following particular cases
\[ H_{n,m}^{(0,q)}(z,w|\gamma) = z^n H_m^{(q)}(w|\gamma) \quad \text{and} \quad H_{n,m}^{(p,0)}(z,w|\gamma) = w^m H_n^{(p)}(z|\gamma) \]
hold. The second identity follows also using the symmetry property
\[ H_{n,m}^{(p,q)}(z,w|\gamma) = H_{m,n}^{(q,p)}(w,z|\gamma) \]  
(2.1)
satisfied by the polynomials \( H_{n,m}^{(p,q)} \).

Remark 2.2. If \( p \mid n \) and \( q \mid m \), then
\[ H_{n,m}^{(p,q)}(0,0|\gamma) = n! \left[ \gamma \left[ \frac{n}{p} \right] \right] \left[ \gamma \left[ \frac{n}{q} \right] \right] \]
Otherwise (i.e., when \( p \nmid n \) or \( q \nmid m \)), we have
\[ H_{n,m}^{(p,q)}(0,0|\gamma) = 0 = H_{n,m}^{(p,q)}(z,0|\gamma). \]

We recover in particular the well-known identity for the real Hermite polynomials ([4, Theorem 5.4])
\[ H_{2n}(0) = (-1)^n \left( \frac{2n}{n!} \right); \quad H_{2n+1}(0) = 0. \]

An equivalent definition of (1.2) is given by the following operational formula.

**Theorem 2.3.** We have
\[ H_{n,m}^{(p,q)}(z,w|\gamma) = e^{\gamma \partial_z \partial_w} \{ z^n w^m \}. \]  
(2.2)

**Proof.** The result is clear for \( p = q = 0 \). The case of \( p = 0 \) and \( q \geq 1 \) immediately follows from the one corresponding to \( p \geq 1 \) and \( q = 0 \) by the symmetry property (2.1). For the last case \( (p \geq 1 \text{ and } q = 0) \), we use ([4, Eq. (6)]) to get
\[ e^{\gamma \partial_z} \{ z^n w^m \} = w^m e^{\gamma \partial_z} \{ z^n \} = w^m H_n^{(p)}(z|\gamma) = H_{n,m}^{(p,0)}(z,w|\gamma). \]

While for \( p \geq 1 \) and \( q \geq 1 \), we have
\[ e^{\gamma \partial_z \partial_w} \{ z^n w^m \} = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} (\partial_z \partial_w)^k \{ z^n w^m \} = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} \partial_z^k \{ z^n \} \partial_w^k \{ w^m \}. \]
Now, using the fact that \( \partial_z^k \{ z^n \} = \frac{n!}{(n-pk)!} z^{n-pk} \), when \( k \leq \left[ \frac{n}{p} \right] \) and vanishes otherwise, we obtain
\[ e^{\gamma \partial_z \partial_w} \{ z^n w^m \} = n! m! \sum_{k=0}^{\left[ \frac{n}{p} \right]} \frac{\gamma^k}{k!} \frac{z^{n-pk}}{(n-pk)!} \frac{w^{m-qk}}{(m-qk)!} = H_{n,m}^{(p,q)}(z,w|\gamma). \]
This completes the proof. \( \square \)

We conclude this section by giving the expression of \( H_{n,m}^{(p,q)}(z,w|\gamma) \) in terms of the hypergeometric function ([4, p. 203])
\[ _pF_q \left( \begin{array}{c} a_1, a_2, \ldots, a_r \\ c_1, c_2, \ldots, c_s \end{array} \mid x \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_r)_k}{(c_1)_k(c_2)_k \cdots (c_s)_k} \frac{x^k}{k!} \]
is given by the following.

**Theorem 2.4.** For every positive integers \( n, m, p, q \), and complex numbers \( z \) and \( w \), we have
\[ H_{n,m}^{(p,q)}(z,w|\gamma) = z^n w^m \, _pF_q \left( \begin{array}{c} -\frac{n}{p}, -\frac{1-n}{p}, \ldots, -\frac{p-1-n}{p}, -\frac{m}{q}, -\frac{1-m}{q}, \ldots, -\frac{q-1-m}{q} \\ \frac{1}{p}, \frac{1}{q} \end{array} \mid \frac{(-p)^p (-q)^q}{z^p w^q} \right). \]  
(2.3)
Proof. Starting from (1.2), we can rewrite the explicit expression of $H_{n,m}^{(p,q)}$ in the following form

$$H_{n,m}^{(p,q)}(z, w|\gamma) = n!m!z^n w^m \sum_{k=0}^{[^{p}/2]+[^{q}/2]} \frac{1}{(n-pk)!(m-qk)!} \left(\frac{\gamma}{z^p w^q}\right)^k.$$ 

The involved factorial $(n-pk)!$ can be expressed in terms of the Pochhammer symbol as

$$(n-pk)! = \frac{(-1)^{pk}n!}{(-n)_{pk}},$$

in virtue of $(n-pk)! = (1)_{n-pk}$, for $pk \leq n$, combined with the identity ([31, Eq. (21), p. 21])

$$\left(\frac{d}{da}\right)_n (a)_{n-pk} = \frac{(-1)^{pk}n!(1-a-n)_{pk}}{(-a)_{pk}}.$$ 

Therefore, the Gauss multiplication theorem ([31, Eq. (26), p. 23])

$$\left(\frac{d}{da}\right)_n (a)_{pk} = p^{pk} \prod_{j=1}^{p} \left(\frac{a+j-1}{p}\right)_k,$$

implies

$$(n-pk)! = \frac{(-1)^{pk}n!(1-a-n)_{pk}}{(-p)_{pk} \prod_{j=1}^{p} \left(\frac{j-1-n}{p}\right)_k}.$$ 

Thus, we obtain

$$H_{n,m}^{(p,q)}(z, w|\gamma) = z^n w^m \sum_{k=0}^{[^{p}/2]+[^{q}/2]} \prod_{i=1}^{p} \left(\frac{j-1-n}{p}\right)_k \prod_{i=1}^{q} \left(\frac{i-1-m}{q}\right)_k \left(\frac{(-1)^{p+q}p!q!\gamma}{z^p w^q}\right)^k.$$ 

This is exactly the desired result (2.8). \qed

**Corollary 2.5.** We have the following formulas

$$H_n(z) = 2^n z^n \, _2F_0 \left( -\frac{n}{2}, \frac{1-n}{2} \left| -\frac{1}{z^2} \right. \right)$$ (2.4)

$$H_{n,m}(z, \bar{z}) = z^n \, \bar{z}^m \, _2F_0 \left( -n, -m \left| -\frac{1}{|z|^2} \right. \right)$$ (2.5)

$$H_{n,m}(z, w) = z^n w^m \, _2F_0 \left( -n, -m \left| -\frac{1}{zw} \right. \right)$$ (2.6)

$$H_{n,m}(z, w|\gamma) = z^n w^m \, _2F_0 \left( -n, -m \left| -\frac{\gamma}{zw} \right. \right)$$ (2.7)

$$H_n^{(p)}(z|\gamma) = z^p \, _pF_0 \left( -\frac{n}{p}, \frac{1-n}{p}, \cdots, \frac{p-1-n}{p} \left| -\frac{(-p)^p\gamma}{z^p} \right. \right)$$ (2.8)

The previous classes of Hermite polynomials described in Corollary 2.5 can be expressed in terms of the confluent hypergeometric function $\, _1F_1$. To do so, we need to the following transformation formula.

**Lemma 2.6.** For every $z \in \mathbb{C}$ and $n, m \in \mathbb{N}$, we have the identity

$$z^{n+m} \, _2F_0 \left( -n, -m \left| -\frac{1}{z} \right. \right) = (-1)^{n+m} (n \lor m)! \, _1F_1 \left( -n \land m \left| -\frac{1}{n-m} \right. \frac{1}{\left( n-m \right)!} + 1 \left| z \right. \right)$$ (2.9)

with $n \land m = \min(n, m)$, $n \lor m = \max(n, m)$. 

Proof. By specializing the identity ([11 p. 504])

\[ 1F_1 \left( \begin{array}{c} a \\ b \end{array} \middle| z \right) = \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}2F_0 \left( \begin{array}{c} a, 1 + a - b \\ -1 \end{array} \middle| \frac{-1}{z} \right) + \frac{\Gamma(b)}{\Gamma(a)}z^a e^{z}2F_0 \left( \begin{array}{c} b - a, 1 - a \\ -1 \end{array} \middle| \frac{1}{z} \right) \]  

(2.10)

for \( a = -n, b = -m, \) such as \( n \leq m, \) and using the fact \( \frac{1}{\Gamma(-n)} = 0 \) (see [11 Theorem 2.11]), we get

\[ z^{m_2}F_0 \left( \begin{array}{c} -n, -m \\ -1 \end{array} \middle| \frac{-1}{z} \right) = \frac{(-1)^m m!}{(m-n)!}1F_1 \left( \begin{array}{c} -n \\ m-n+1 \end{array} \middle| z \right) \]

For \( m \leq n, \) we proceed in a similar as above to obtain

\[ z^{n_2}F_0 \left( \begin{array}{c} -n, -m \\ -1 \end{array} \middle| \frac{-1}{z} \right) = \frac{(-1)^n n!}{(n-m)!}1F_1 \left( \begin{array}{c} -m \\ n-m+1 \end{array} \middle| z \right). \]

This completes the proof for the desired identity. \( \square \)

Remark 2.7. Using the previous lemma we recover the following representations ([28 Eq. (1.2)]

\[ H_n(z) = \begin{cases} (-1)^\frac{n}{2} \frac{n!}{(\frac{1}{2})!}1F_1 \left( \begin{array}{c} -\frac{n}{2} \\ \frac{1}{2} \end{array} \middle| \frac{z^2}{\gamma} \right) & \text{(n even)} \\ (-1)^\frac{n-1}{2} \frac{n!}{(\frac{1}{2})!}2\gamma 1F_1 \left( \begin{array}{c} -\frac{n}{2} + \frac{1}{2} \\ \frac{1}{2} \end{array} \middle| z^2 \right) & \text{(n odd)} \end{cases} \]  

(2.11)

and (see [18 Eq. (2.3)]

\[ H_{n,m}(z, \gamma) = \frac{(-1)^{n+m}(n \vee m)!}{(|n-m|)!}e^{(n-m)\arg z}z^{|n-m|}1F_1 \left( \begin{array}{c} -n \wedge m \\ |n-m|+1 \end{array} \middle| |z|^2 \right). \]  

(2.12)

The next corollary gives the expression of \( H_{n,m}(z, w|\gamma) \) in terms of the confluent hypergeometric function.

Corollary 2.8. For any \( z, w \in \mathbb{C}, \) we have

\[ H_{n,m}(z, w|\gamma) = \frac{(n \vee m)!}{(|n-m|)!} 2F_1 \left( \begin{array}{c} -n \wedge m \\ |n-m|+1 \end{array} \middle| \frac{-zw}{\gamma} \right) \]  

(2.13)

Proof. Starting from

\[ H_{n,m}(z, w|\gamma) = \gamma^{n \wedge m} z^{-n \wedge m} w^{m-n \wedge m} 2F_0 \left( \begin{array}{c} -n, -m \\ - \end{array} \middle| \frac{\gamma}{zw} \right) \]

and then applying Lemma 2.9 we get the result desired. \( \square \)

3 Generating functions.

In this section we give different generating functions for the polynomials \( H_{n,m}^{(p,q)} \). We begin with the following partial generating functions.

Proposition 3.1. For every complex numbers \( u, v, z \) and \( w, \) we have

\[ \sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n}{n!} = H_{m}^{(q)}(w|u|\gamma)e^{zu} \]  

(3.1)
and
\[
\sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z,w|\gamma) \frac{u^m v^m}{m!} = H_n^{(p)}(z|v^q\gamma)e^{uv}.
\] (3.2)

**Proof.** We only need to establish (3.1), since (3.2) can be obtained from using (2.1). The proof of (3.1) lies on the operational realization of \( H_{n,m}^{(p,q)}(z,w|\gamma) \) provided by Theorem 2.3. Indeed, we have

\[
\sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(z,w|\gamma) \frac{u^n}{n!} = e^{\gamma \partial_u^p \partial_v^q} \left\{ \sum_{n=0}^{\infty} \frac{(zu)^n}{n!} w^m \right\} = e^{\gamma \partial_u^p \partial_v^q} \{e^{uw}w^m\}.
\]

Next, direct computation of \( e^{\gamma \partial_u^p \partial_v^q} \{e^{uw}w^m\} \) gives rise to

\[
e^{\gamma \partial_u^p \partial_v^q} \{e^{uw}w^m\} = \sum_{n=0}^{\infty} \frac{u^n}{n!} H_{n,m}^{(p,q)}(z,w|\gamma) \frac{v^m}{m!} = e^{\gamma \partial_u^p \partial_v^q} \{e^{uw}w^m\} = e^{\gamma \partial_u^p \partial_v^q} \{w^m\}e^{uv} = H_m^{(q)}(w|u^p\gamma)e^{zu}.
\]

This completes the proof of (3.1). \(
\)

**Remark 3.2.** For \( p = q = 1 \), we recover the partial generating functions for the complex Hermite polynomials ([12, Proposition 3.4]):

\[
\sum_{n=0}^{\infty} \frac{u^n}{n!} H_{n,m}(z,\bar{z}) = (z - u)^m e^{uz}
\]

and

\[
\sum_{m=0}^{\infty} \frac{v^m}{m!} H_{n,m}(z,\bar{z}) = (z - v)^n e^{\bar{z}v}
\]

The next generating function is a consequence of the previous one and gives the closed expression of

\[
R_{\gamma}^{p,q}(z,w|u,v) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z,w|\gamma) \frac{u^n v^m}{n!m!}.
\] (3.3)

**Proposition 3.3.** For every \( u, v, z, w, \in \mathbb{C} \), we have

\[
R_{\gamma}^{p,q}(z,w|u,v) = e^{zw+uv+\gamma uv^q}.
\] (3.4)

**Proof.** Direct application of (3.1) infers

\[
R_{\gamma}^{p,q}(z,w|u,v) = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} \frac{H_{n,m}^{(p,q)}(z,w|\gamma) u^n}{n!} \right) \frac{v^m}{m!} = \sum_{m=0}^{\infty} H_m^{(q)}(w|u^p\gamma) \frac{v^m}{m!} e^{zu} = e^{wv+\gamma uv^q} e^{zu}.
\]

The last equality is obtained making use of the generating function of Gould–Hopper polynomials ([8, p. 72])

\[
\sum_{m=0}^{\infty} H_m^{(q)}(w|\gamma) \frac{v^m}{m!} = e^{wv+\gamma vq}.
\]

**Remark 3.4.** The last generating function (3.3) englobes the known ones for the Hermite polynomials described in the introductory section. A direct proof of (3.3) can be given using the explicit expression of \( H_{n,m}^{(p,q)} \) (see Appendix 9).
Proposition 3.5. For any \( a, b \in \mathbb{C} \), we have
\[
a^n b^m H_{n,m}^{(p,q)}(z, w|\gamma) = H_{n,m}^{(p,q)}(az, bw|\gamma a^p b^q).
\]

Proof. The result (3.5) follows from
\[
\sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma)\frac{u^n v^m}{n! m!} = \sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(az, bw|\gamma a^p b^q)\frac{u^n v^m}{n! m!},
\]
which can be obtained making use of the generating function (3.3) and the observation that
\[
e^{auz+bvw+\gamma(au+bv)} = e^{auz+bvw+(\gamma a^p b^q)\omega e^{\eta}}.
\]

As immediate consequence

Corollary 3.6. We have
\[
\lim_{t \to 0} t^{n+m} H_{n,m}^{(p,q)}\left(\frac{z}{t}, \frac{w}{t} \big| \gamma\right) = z^n w^m.
\]

Proof. By replacing \( a \) and \( b \) by \( \frac{1}{t} \) in (3.5), we get
\[
H_{n,m}^{(p,q)}\left(\frac{z}{t}, \frac{w}{t} \big| \gamma\right) = \left(\frac{1}{t}\right)^n H_{n,m}^{(p,q)}(z, w|t^{n+m} \gamma).
\]
Therefore, we have
\[
\lim_{t \to 0} t^{n+m} H_{n,m}^{(p,q)}\left(\frac{z}{t}, \frac{w}{t} \big| \gamma\right) = H_{n,m}^{(p,q)}(z, w|0)
\]
with \( H_{n,m}^{(p,q)}(z, w|0) = z^n w^m \).

Remark 3.7. For particular values of \( p \) and \( q \) (giving rise to the holomorphic Hermite polynomials \( H_n \) and \( H_{n,m} \)), we recover from (3.6) the limits (36, 14)
\[
\lim_{t \to 0} \left(\frac{1}{2}\right)^n H_n\left(\frac{z}{t}\right) = z^n \quad \text{and} \quad \lim_{t \to 0} t^{m+n} H_{n,m}\left(\frac{z}{t}, \frac{w}{t}\right) = z^m w^n.
\]

The last generating function can be extended to a general setting and concerns
\[
G_{\gamma}^{p,q}(z, w|u, v) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n)_j(m)_k H_{n,m}^{(p,q)}(z, w|\gamma) \frac{v^n w^m}{n! m!},
\]
where \((x)_n = x(x+1)\cdots(x+n-1)\) denotes the Pochhammer symbol. We define also the falling factorial to be \((x)^m = x(x-1)\cdots(x-m+1)\). The closed formula of \( G_{\gamma}^{p,q}(z, w|u, v) \) involves the polynomials
\[
P_k^{p}(z) = \sum_{j=0}^{k-1} (n)^{(k-j)} \binom{k}{j} z^j.
\]

Theorem 3.8. For any \( j, k, p, q \in \mathbb{N} \) and \( z, w, u, v, \gamma \in \mathbb{C} \), we have:
\[
G_{\gamma}^{p,q}(z, w|u, v) = uvzwe^{zuw+vwp\eta}(z^{j-1} + w^{k-1} + P_{j-1}^{q}(uz) + P_{k-1}^{q}(vw)).
\]
Proof. Using the definition of \( H_{n,m}^{p,q} \), we get
\[
G_{\gamma}^{p,q}(z,w|u,v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (n)_j(m)_k \frac{[\frac{n}{p}] [\frac{m}{q}]}{[\frac{n}{p}][\frac{m}{q}]} \gamma^k z^{n-pk} w^{m-qk} u^n v^m
= \sum_{k=0}^{\infty} \left( \frac{\gamma^k P_{j}^{k}}{k!} \right) \sum_{n=0}^{\infty} (n)_j(uz)^n n! \sum_{m=0}^{\infty} (m)_k(vw)^m m!.
\]
Hence, making use of the generating function ([29, Theorem 2.1]),
\[
\sum_{i=0}^{\infty} (n)_j \frac{z^n}{n!} = z e^z \left( z^{-1} + P_{j-1}^{j}(z) \right),
\]
we arrive at
\[
G_{\gamma}^{p,q}(z,w|u,v) = e^{\gamma u^p v^q} u z e^{uz} \left( (uz)^{j-1} + P_{j-1}^{j}(uz) \right) v w e^{vw} \left( (vw)^{k-1} + P_{k-1}^{k}(vw) \right).
\]
This ends the proof. \(\square\)

**Remark 3.9.** For the special case \( j = k = 0 \), we recover the generating function ([3.3]).

We conclude this section by proving another generalized generating function for the polynomials \( H_{n,m}^{p,q} \), thanks to Theorem 2.4. It gives the closed formula of the quantity
\[
S_{\gamma,a,b}^{p,q}(z,w|u,v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (a)_n(b)_m H_{n,m}^{p,q}(z,w|\gamma) \frac{u^n v^m}{n! m!}.
\]

**Theorem 3.10.** For any \( n, m, p, q \in \mathbb{N} \) and \( z, w \in \mathbb{C} \), we have:
\[
S_{\gamma,a,b}^{p,q}(z,w|u,v) = (1 - uz)^{-a} (1 - vw)^{-b} \sum_{k=0}^{\infty} \frac{[\frac{n}{p}] [\frac{m}{q}]}{[\frac{n}{p}][\frac{m}{q}]} \gamma^k z^{n-pk} w^{m-qk} u^n v^m \sum_{k=0}^{\infty} (m)_k(vw)^m m!.
\]

**Proof.** By means of ([1.2]), we get
\[
S_{\gamma,a,b}^{p,q}(z,w|u,v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (a)_n(b)_m u^n v^m \sum_{k=0}^{\infty} \frac{[\frac{n}{p}] [\frac{m}{q}]}{[\frac{n}{p}][\frac{m}{q}]} \gamma^k z^{n-pk} w^{m-qk} u^n v^m \sum_{k=0}^{\infty} (m)_k(vw)^m m!.
\]
The identity ([31, Eq. (5), p. 101])
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n-pk),
\]
as well as the formula \((a)_{m+n} = (a)_m (a+m)_n\) (see [31, Eq. (20), p. 22]) yield
\[
S_{\gamma,a,b}^{p,q}(z,w|u,v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} (a)_{n+pk}(b)_{m+qk} u^{nk} v^{qk} (uz)^n (vw)^m n! m!.
\]
Now, using ([2]) and the fact
\[
\sum_{n=0}^{\infty} (a + pk)_n \frac{(uz)^n}{n!} = (1 - uz)^{-a},
\]
we get
\[ S_{\gamma,a,b}^{p,q}(z, w|u, v) = (1 - uz)^{-a}(1 - vw)^{-b} \sum_{k=0}^{\infty} \prod_{j=1}^{p} \left( \frac{a + j - 1}{p} \right) \prod_{k=1}^{q} \left( \frac{b + i - 1}{q} \right) \times \frac{1}{k!} \left( \frac{p^p q^q \gamma uv}{(1 - uz)^p(1 - vw)^q} \right)^k. \]

This completes the proof of the required result. \( \square \)

**Remark 3.11.** For the special case \( p = 2 \) and \( m = q = 0 \), we recover the following generating function for the real Hermite polynomials \([30]\) and the polyanalytic Hermite polynomials \([12]\).

\[ \sum_{n=0}^{\infty} \frac{(a)_n}{n!} H_n(x) = (1 - 2xt)^{-a/2} F_0 \left( \frac{a}{2}, \frac{a+1}{2}, \frac{-4t^2}{(1-2xt)^2} \right) \]

and for the case \( p = q = 1 \) and \( w = \bar{z} \), we find the corresponding of the polyanalytic Hermite polynomials (\([20] \text{ Eq. (4.5)}\))

\[ \sum_{m,j=0}^{\infty} \frac{(a)m(b)_n}{m!n!} u^m v^n H_{m,n}(z, \bar{z}) = (1 - uz)^{-a}(1 - v\bar{z})^{-b} F_0 \left( a, b, \frac{uv}{(1-uz)(1-v\bar{z})} \right). \]

### 4 Runge Formulas

In this section, we establish a generalized Runge type formula with respect to the three variables \( z, w \) and \( \gamma \). It gives rise to special generalization of the well-known ones for the real Hermite polynomials \([30]\) and the polyanalytic Hermite polynomials \([12]\).

**Theorem 4.1.** For any \( z, z', w, w', \gamma, \gamma' \in \mathbb{C} \), we have:

\[ H_{n,m}^{(p,q)}(z + z', w + w' | \gamma + \gamma') = n!m! \sum_{k=0}^{n} \sum_{j=0}^{m} \frac{H_{k,j}^{(p,q)}(z, w | \gamma) H_{n-k,m-j}^{(p,q)}(z', w' | \gamma')}{(n-k)!(m-j)!}. \]

**Proof.** We perform

\[ T_{\gamma,a,b}^{p,q}(z, z', w, w'|u, v) := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z + z', w + w' | \gamma + \gamma') \frac{u^m v^n}{n!m!}. \]

Then, using the generating function \((3.3)\) as well as

\[ e^{(z+z')u+(w+w')v+\gamma+\gamma'w^pv^q} = e^{zu+vw+\gamma+\gamma'}w^pv^q, \]

we obtain

\[ T_{\gamma,a,b}^{p,q}(z, z', w, w'|u, v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w | \gamma) \frac{u^m v^n}{n!m!} \left( \sum_{n'=0}^{\infty} \sum_{m'=0}^{\infty} H_{n',m'}^{(p,q)}(z', w' | \gamma') \frac{u^{n'} v^{m'}}{n!'m'!} \right). \]

Now, replacing \( n \) by \( n + n' \) and \( m \) by \( m + m' \), and applying the formula (\([31] \text{ Eq. (1)}, \text{ p. 100}\))

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(n, k) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(n - k, k), \]

we get

\[ T_{\gamma,a,b}^{p,q}(z, z', w, w'|u, v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{j=0}^{m} \frac{H_{k,j}^{(p,q)}(z, w | \gamma) H_{n-k,m-j}^{(p,q)}(z', w' | \gamma')}{(n-k)!(m-j)!} \right) u^m v^n. \]

Finally, the result follows by identification. \( \square \)
Remark 4.2. The special case of $\gamma = -\gamma'$ in (4.1) leads to the following identity
\[
\sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \binom{m}{j} H_{k,j}^{(p,q)} \left( \frac{z}{2}, \frac{w}{2} \right) H_{n-k,m-j}^{(p,q)} \left( \frac{z}{2}, \frac{w}{2} \right) - \gamma = z^n w^m.
\]

Remark 4.3. The addition formula (4.1) for the special case $z = z' = \frac{z}{2}$, $w = w' = \frac{w}{2}$ and $\gamma = \gamma' = \frac{\gamma}{2}$, combined with (3.5) gives rise to the identity
\[
H_{n,m}^{(p,q)}(z, w|\gamma) = n!m! \sum_{k=0}^{n} H_{k,j}^{(p,q)}(z, w|2^{p+q-1}\gamma) \frac{H_{n-k,m-j}^{(p,q)}(z, w|2^{p+q-1}\gamma)}{(n-k)!(m-j)!}.
\]
As immediate consequence of Theorem 4.1, we assert the following.

Corollary 4.4. For any $z, z', w, w', \gamma, \gamma' \in \mathbb{C}$, we have:
\[
H_{n,m}^{(p,q)} \left( \frac{z + z'}{2\sqrt{2}}, \frac{w + w'}{2\sqrt{2}} \right) = n!m!2^{-\left(\frac{p}{2} + \frac{q}{2}\right)} \sum_{k=0}^{n} \sum_{j=0}^{m} \frac{H_{k,j}^{(p,q)}(z, w|\gamma) H_{n-k,m-j}^{(p,q)}(z', w'|\gamma)}{k!j! (n-k)!(m-j)!}.
\]

Proof. By taking $\gamma = \gamma'$ in Theorem 4.1 and next using the fact that
\[
H_{n,m}^{(p,q)}(z, w|2\gamma) = 2^{-\frac{1}{2}(\frac{p}{2} + \frac{q}{2})} H_{n,m}^{(p,q)}(z, w|2\gamma),
\]
we get (4.4). \qed

A direct proof of Corollary 4.4 is given in the Appendix 9.

Remark 4.5. For $p = q = 1$ and $w = \bar{z}$, we recover the Runge formula for the polyanalytic Hermite polynomials (12, p.9), Eq 3.22
\[
H_{n,m} \left( \frac{z + \bar{z}}{\sqrt{2}}, \frac{z + \bar{w}}{\sqrt{2}} \right) = 2^{-\frac{n+m}{2}} m! \sum_{j=0}^{n} \sum_{k=0}^{m} H_{j,k}(z, \bar{z}) \frac{H_{n-j,m-k}(w, \bar{w})}{j!k! (n-j)!(m-k)!}.
\]

For $p = 2$ and $m = q = 0$, we find the Runge formula for the real Hermite polynomials [30]
\[
H_n \left( \frac{x + y}{\sqrt{2}} \right) = 2^{-\frac{n}{2}} \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(y) H_k(x).
\]

5 Multiplication Formulas

We begin with the following multiplication formula needed to prove a recursion relation with respect to parameters $p$ and $q$.

Proposition 5.1. We have the identity
\[
H_{n,m}^{(p,q)}(z, w|c\gamma) = n!m! \sum_{k=0}^{\left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor} \binom{c-1}{k} \frac{H_{n-pk,m-qk}^{(p,q)}(z, w|\gamma)}{(n-pk)!(m-qk)!}.
\]

Proof. From
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|c\gamma) \frac{u^n v^m}{n!m!} = e^{uz+vw+c\gamma uv} = e^{uz+vw+c\gamma uv} e^{(c-1)uv},
\]
we get
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|c\gamma) \frac{u^n v^m}{n!m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{n!m!} \left( \sum_{k=0}^{\infty} \frac{(c-1)^k \gamma^k}{k!} \right)
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c-1)^k \gamma^k}{k!} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^{np+mp} v^{mk+mk}}{n!m!}.
\]
Thus, using Eq. (5), p. 101

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B(k, n + pk) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n),
\]

we obtain

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{p,q}(z, w|c\gamma) \frac{u^nv^m}{n!m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n!m! \sum_{k=0}^{\infty} \frac{(c - a^pb^q)^k}{k!} \frac{\gamma^k}{(n - pk)!(m - qk)!} H_{n,pk,m-qk}(z, w|\gamma) u^n v^m.
\]

Thus, (5.1) readily follows by identification of power series in \(u\) and \(v\). □

Another multiplication formula is the following.

**Proposition 5.2.** For every \(a, b, c \in \mathbb{C}\), we have

\[
H_{n,m}^{p,q}(az, bw|c\gamma) = n!m! a^n b^m \sum_{k=0}^{\infty} \frac{(c - a^pb^{-q} - 1)^k}{k!} \frac{\gamma^k}{(n - pk)!(m - qk)!} H_{n-pk,m-qk}^{p,q}(z, w|\gamma).
\]

**Proof.** By applying (3.5), we get

\[
H_{n,m}^{p,q}(az, bw|c\gamma) = a^n b^m H_{n,m}^{p,q}(z, w|ca^{-p}b^{-q}\gamma).
\]

By (5.1), we obtain

\[
H_{n,m}^{p,q}(az, bw|c\gamma) = n!m! a^n b^m \sum_{k=0}^{\infty} \frac{(c - a^pb^{-q} - 1)^k}{k!} \frac{\gamma^k}{(n - pk)!(m - qk)!} H_{n-pk,m-qk}^{p,q}(z, w|\gamma)
\]

This infers (5.3). □

**Remark 5.3.** Accordingly, the following multiplication formula for the Gould–Hopper polynomials

\[
H_{n}^{p}(az|c\gamma) = n! \sum_{k=0}^{\infty} \frac{(c - a^p)^k}{k!} \frac{\gamma^k}{(n - pk)!} H_{n-pk}^{p}(z|\gamma)
\]

is new. Notice also that for \(p = 2, q = 0, w = 1\) and \(\gamma = -1\), we recover the multiplication formula for real Hermite polynomials \(H_{n}\) (13 Eq. (4.6.33))

\[
H_{n}(cx) = n! \sum_{k=0}^{\infty} \frac{(c^2 - 1)^k}{k!} \frac{\gamma^k (c^n - 2k)!}{(n - 2k)!} H_{n-2k}(x),
\]

while for \(p = q = 1, w = \bar{z}\) and \(\gamma = -1\), we find the multiplication formula for polyanalytic polynomials \(H_{n,m}\) (20 Eq. (4.13))

\[
H_{m,n}(cz, \bar{cz}) = m!n! \sum_{k=0}^{m+n} \frac{(c - 1)^k}{k!} \frac{c^{n-k} \gamma^k}{(m - k)!(n - k)!} H_{m-k, n-k}(z, \bar{z}).
\]
6 Recurrence formulas

In this section, we investigate the derivative relations and the multiplication formulas in order to obtain the recursion relations that correspond to the four parameters \( n, m, p, q \). We give first the derivative relations.

**Proposition 6.1.** Expressions of partial derivatives of \( H_{n,m}^{(p,q)}(z, w|\gamma) \) are given by:

\[
\partial_z H_{n,m}^{(p,q)}(z, w|\gamma) = nH_{(n-1,m)}^{(p,q)}(z, w|\gamma). \tag{6.1}
\]

\[
\partial_w H_{n,m}^{(p,q)}(z, w|\gamma) = mH_{(n,m-1)}^{(p,q)}(z, w|\gamma). \tag{6.2}
\]

\[
\partial_{\gamma,j} H_{n,m}^{(p,q)}(z, w|\gamma) = \partial_z \partial_w H_{n,m}^{(p,q)}(z, w|\gamma). \tag{6.3}
\]

**Proof.** We use the operational formula (6.7) combined with the fact that \( e^{\gamma \partial_z \partial_w} \) and \( \partial_z \) commute to get (6.1). Indeed,

\[
\partial_z H_{n,m}^{(p,q)}(z, w|\gamma) = e^{\gamma \partial_z \partial_w} \partial_z \{ z^n w^m \} = ne^{\gamma \partial_z \partial_w} \{ z^{n-1} w^m \}.
\]

We obtain (6.2) by the symmetry 2.1. The third partial derivative (6.3) is obtained by noticing that

\[
\partial_{\gamma,j} e^{\gamma \partial_z \partial_w} = \partial_z \partial_w e^{\gamma \partial_z \partial_w}.
\]

**Remark 6.2.** By mathematical induction, one can establish the following formula

\[
\partial_{\gamma,j} \partial_z^k H_{n,m}^{(p,q)}(z, w|\gamma) = n! \frac{m!}{(n-j)! (m-k)!} H_{(n-j,m-k)}^{(p,q)}(z, w|\gamma) \tag{6.4}
\]

when \( j \leq n \) and \( k \leq m \), and vanishes otherwise. Accordingly, we deduce

\[
\partial_{\gamma,j} H_{n,m}^{(p,q)}(z, w|\gamma) = \frac{n!}{(n-pj)! (m-kq)!} \frac{m!}{(m-kq)!} H_{(n-pj,m-kq)}^{(p,q)}(z, w|\gamma) \tag{6.5}
\]

if \( j \leq \lfloor \frac{n}{p} \rfloor \) and \( k \leq \lfloor \frac{m}{q} \rfloor \), and vanishes otherwise.

Thanks to the previous proposition, we can assert the following.

**Proposition 6.3.** For any \( n, m, p, q \in \mathbb{N} \), we have

\[
z^n w^m = n! m! \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor} \left( -\gamma \right)^k \frac{H_{n-pk,m-qk}^{(p,q)}(z, w|\gamma)}{k! (n-pk)! (m-qk)!}. \tag{6.6}
\]

**Proof.** By writing down the Taylor series of the polynomials \( H_{n,m}^{(p,q)} \),

\[
H_{n,m}^{(p,q)}(z, w|\gamma + h) = \sum_{k=0}^{n} \frac{h^k}{k!} \partial_{\gamma,j} H_{n,m}^{(p,q)}(z, w|\gamma),
\]

and next using (6.5), we get

\[
H_{n,m}^{(p,q)}(z, w|\gamma + h) = n! m! \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor} \frac{h^k}{k!} \frac{H_{n-pk,m-qk}^{(p,q)}(z, w|\gamma)}{(n-pk)! (m-qk)!}.
\]

Finally, by taking \( h = -\gamma \) and using the fact that \( H_{n,m}^{(p,q)}(z, w|0) = z^n w^m \), we find

\[
z^n w^m = n! m! \sum_{k=0}^{\left\lfloor \frac{n}{p} \right\rfloor} \left( -\gamma \right)^k \frac{H_{n-pk,m-qk}^{(p,q)}(z, w|\gamma)}{k! (n-pk)! (m-qk)!}.
\]

\[\square\]
Corollary 6.4. The following operational formula holds true,
\[ e^{-\gamma \partial_x^p \partial_y^q} \{ H_{n,m}^{(p,q)}(z, w|\gamma) \} = z^n w^m. \] (6.7)

Proof. Direct computation yields
\[
e^{-\gamma \partial_x^p \partial_y^q} \{ H_{n,m}^{(p,q)}(z, w|\gamma) \} = \sum_{k=0}^{\infty} \frac{(\gamma)^k}{k!} \partial_x^p \partial_y^q H_n^{(p,q)}(z, w|\gamma)
= n!m! \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \frac{H_{n-pk,m-qk}^{(p,q)}(z, w|\gamma)}{(n-pk)!(m-qk)!} = z^n w^m.
\]

The first recursion relation in this section is the following.

Proposition 6.5. The polynomials \(H_{n,m}^{(p,q)}\) obey the recursion relations
\[
H_{n+1,m}^{(p,q)}(z, w|\gamma) = z H_{n,m}^{(p,q)}(z, w|\gamma) + \gamma p! q! \binom{n}{p-1} \binom{m}{q} H_{n+1-m,p}^{(p,q)}(z, w|\gamma).
\] (6.8)

and
\[
H_{n+1,m}^{(p,q)}(z, w|\gamma) = (z + p\gamma \partial_x^{p-1} \partial_y^q) H_{n,m}^{(p,q)}(z, w|\gamma).
\] (6.9)

Proof. Using the fact that \(\partial_x e^{u z + w v + \gamma p q v^q} = (z + p\gamma u^{p-1} v^q) e^{u z + w v + \gamma p q v^q}\), we obtain
\[
(u^{p-1} v^q) e^{u z + w v + \gamma p q v^q} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^{n+p-1} v^{m+q}}{n! m!}.
\]

Now, replacing \(n\) by \(n + p - 1\) and \(m\) by \(m + q\), we get
\[
(u^{p-1} v^q) e^{u z + w v + \gamma p q v^q} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n+1-p,m-q}^{(p,q)}(z, w|\gamma) \frac{u^n v^m}{(n+1-p)!(m-q)!}.
\]

Thus
\[
\partial_x e^{u z + w v + \gamma p q v^q} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (z H_{n,m}^{(p,q)}(z, w|\gamma) + \gamma p! q! \binom{n}{p-1} \binom{m}{q} H_{n+1-m,p}^{(p,q)}(z, w|\gamma)) \frac{u^n v^m}{n! m!}.
\]

On the other hand, because of
\[
H_{n+1,m}^{(p,q)}(z, w|\gamma) = \partial_u^{n+1} \partial_v^m e^{u z + w v + \gamma p q v^q}|_{u=v=0} = \partial_u^n \partial_v^m (\partial_u e^{u z + w v + \gamma p q v^q}|_{u=v=0}),
\]

we have
\[
H_{n+1,m}^{(p,q)}(z, w|\gamma) = z H_{n,m}^{(p,q)}(z, w|\gamma) + \gamma p! q! \binom{n}{p-1} \binom{m}{q} H_{n+1-m,p}^{(p,q)}(z, w|\gamma).
\]

This completes our check of (6.8). For the proof of (6.9), notice first that
\[
\partial_x^{p-1} \partial_y^q H_{n,m}^{(p,q)}(z, w|\gamma) = \frac{n! m!}{(n-p+1)!(m-q)!} H_{n+1-p,m-q}^{(p,q)}
= (p-1)! q! \binom{n}{p-1} \binom{m}{q} H_{n+1-p,m-q}^{(p,q)}.
\]

Therefore,
\[
H^{(p,q)}_{n+1,m}(z, w|\gamma) = zH^{(p,q)}_{n,m}(z, w|\gamma) + \gamma p!q! \binom{n'}{p-1} \binom{m'}{q} H^{(p,q)}_{n'+1-p,m'-q}(z, w|\gamma)
\]
\[
= (z + p\gamma \partial_z^{p-1} \partial_w^q) H^{(p,q)}_{n,m}(z, w|\gamma).
\]

\[
\square
\]

Remark 6.6. By the symmetry identity [2.1], we obtain the recursion relations
\[
H^{(p,q)}_{n,m+1}(z, w|\gamma) = wH^{(p,q)}_{n,m}(z, w|\gamma) + \gamma p!q! \binom{n}{p} \binom{m}{q-1} H^{(p,q)}_{n,m}(z, w|\gamma) \quad (6.10)
\]
and
\[
H^{(p,q)}_{n,m+1}(z, w|\gamma) = (w + q\gamma \partial_z^{p-1} \partial_w^{q-1}) H^{(p,q)}_{n,m}(z, w|\gamma). \quad (6.11)
\]

In virtue of the previous recursion formulas, the polynomials \(H^{(p,q)}_{n,m}\) can be rewritten, according to the values of \(p\) and \(q\), in terms of some creating differential operators with the monomials \(z^n\) and \(w^m\) as generators. More precisely, we assert the following.

Proposition 6.7. For any \(n, m, p, q = 0, 1, 2, \ldots\), we have
\[
H^{(p,q)}_{n,m}(z, w|\gamma) = \begin{cases} e^\gamma z^n w^m, & \text{if } p = q = 0, \\ (z + p\gamma \partial_z^{p-1} \partial_w^q)^n \{w^m\}, & \text{if } p \geq 1 \text{ and } q = 0, \\ (z + p\gamma \partial_z^{p-1} \partial_w^q)^n \{w^m\}, & \text{if } p \geq 1 \text{ and } q \geq 1. \end{cases} \quad (6.12)
\]

Proof. The first identity in (6.12) is obvious keeping in mind the convention that \([\frac{1}{x}] = +\infty\) when \(k = 0\). While the second one, i.e., when \(q = 0\) and \(p \geq 1\), can be derived making use of ([9 Eq (6), p. 18])
\[
H^{(p)}_{n}(z|\gamma) = (z + p\gamma \partial_z^{p-1})^n \{1\}.
\]
Indeed, we have
\[
H^{(p,0)}_{n,m}(z, w|\gamma) = w^m H^{(p)}_{n}(z|\gamma) = (z + p\gamma \partial_z^{p-1})^n \{w^m\}.
\]
The proof of the last identity, corresponding to \(p \geq 1\) and \(q \geq 1\), can be handled by induction on \(n\). Indeed, we have
\[
H^{(p,q)}_{n,m}(z, w|\gamma) = (z + p\gamma \partial_z^{p-1} \partial_w^q) H^{(p,q)}_{n-1,m}(z, w|\gamma),
\]
and therefore,
\[
H^{(p,q)}_{n,m}(z, w|\gamma) = (z + p\gamma \partial_z^{p-1} \partial_w^q)^n H^{(p,q)}_{0,m}(z, w|\gamma),
\]
where \(H^{(p,q)}_{0,m}(z, w|\gamma) = w^m\). Then, we have
\[
H^{(p,q)}_{n,m}(z, w|\gamma) = (z + p\gamma \partial_z^{p-1} \partial_w^q)^n \{w^m\}.
\]
\[
\square
\]

Remark 6.8. The symmetry identity for the second recursion formula, corresponding to \(p = 0\) and \(q \geq 1\), is given by \(H^{(p,q)}_{n,m}(z, w|\gamma) = (w + q\gamma \partial_w^{q-1})^n \{z^n\}\), while the symmetry identity for the third one reads \(H^{(p,q)}_{n,m}(z, w|\gamma) = (w + q\gamma \partial_z^{p-1} \partial_w^{q-1})^n \{z^n\}\).

An immediate consequence is the following operational rule.

Corollary 6.9. For any \(n, m, p, q \in \mathbb{C}\), we have
\[
H^{(p,q)}_{n,m}(z, w|\gamma) = (z + p\gamma \partial_z^{p-1} \partial_w^q)^n (w + q\gamma \partial_z^{p} \partial_w^{q-1})^m(1). \quad (6.13)
\]
Proof. Notice first that we have
\[ H_{n,m}^{(p,q)}(z, w|\gamma) = (z + p\gamma \partial_z^{p-1} \partial_w^q)^n H_{0,m}^{(p,q)}(z, w|\gamma) = (z + p\gamma \partial_z^{p-1} \partial_w^q)^n (w + q\gamma \partial_z^{p-1} \partial_w^q)^m H_{0,0}^{(p,q)}(z, w|\gamma). \]
Therefore, the result follows since \( H_{0,0}^{(p,q)}(z, w|\gamma) = 1 \). □

**Proposition 6.11.** We have
\[ H_{n,m}^{(p+1,q)}(z, w|\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \sum_{j=0}^{\lfloor \frac{m}{q} \rfloor} \binom{j}{k} \gamma^k (-1)^{k-j} \frac{H_{n-j-pk,m-k}^{(p,q)}(z, w|\gamma)}{(n-j-pk)!(m-qk)!}. \] (6.14)

**Proof.** For the proof, we make appeal of \( R_{\gamma}^{p,q}(z, w|u, v) \) in (3.3). Hence, using the generating function (3.3), we get
\[ R_{\gamma}^{p+1,q}(z, w|u, v) = e^{az+uw+\gamma pv^q} = e^{az+uw+(\gamma u)wv^q} = R_{\gamma}^{p+1,q}(z, w|u, v). \]
Now, in view of (5.3), we obtain
\[ R_{\gamma}^{p+1,q}(z, w|u, v) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\lfloor \frac{m}{q} \rfloor} \gamma^k (-1)^{k-j} \frac{H_{n-j-pk,m-k}^{(p,q)}(z, w|\gamma)}{(n-j-pk)!(m-qk)!} \sum_{j=0}^{\lfloor \frac{n}{p} \rfloor} \binom{j}{k} \gamma^k (-1)^{k-j} \frac{H_{n-j-pk,m-k}^{(p,q)}(z, w|\gamma)}{(n-j-pk)!(m-qk)!} \frac{n!m!}{n!m!} \]
By identification, we get (6.14). □

The previous recursion relation can be realized by the following formulae:

**Proposition 6.11.** We have
\[ H_{n,m}^{(p+1,q)}(z, w|\gamma) = e^{\gamma (\partial_z - 1) \partial_z \partial_w^q} H_{n,m}^{(p,q)}(z, w|\gamma). \] (6.15)

**Proof.** Starting from
\[ H_{n,m}^{(p+1,q)}(z, w|\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \sum_{j=0}^{\lfloor \frac{m}{q} \rfloor} \binom{j}{k} \gamma^k (-1)^{k-j} \frac{H_{n-j-pk,m-k}^{(p,q)}(z, w|\gamma)}{(n-j-pk)!(m-qk)!}, \]
and using (3.3), we obtain
\[ H_{n,m}^{(p+1,q)}(z, w|\gamma) = e^{\gamma (\partial_z - 1) \partial_z \partial_w^q} H_{n,m}^{(p,q)}(z, w|\gamma) \]

Subsequently, by symmetry of \( H_{n,m}^{(p,q)} \), we have
\[ H_{n,m}^{(p+1,q)}(z, w|\gamma) = n!m! \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \sum_{j=0}^{\lfloor \frac{m}{q} \rfloor} \binom{j}{k} \gamma^k (-1)^{k-j} \frac{H_{n-j-pk,m-j-qk}^{(p,q)}(z, w|\gamma)}{(n-j-pk)!(m-j-qk)!} \]
\[ H_{n,m}^{(p,q+1)}(z, w|\gamma) = e^{\gamma (\partial_w - 1) \partial_z \partial_w^q} H_{n,m}^{(p,q)}(z, w|\gamma). \]
Moreover, we can prove the following.

**Corollary 6.12.** We have

\[
H_{n,m}^{(p+1,q+1)}(z,w|\gamma) = e^{\gamma(\partial_{z}+\partial_{w}-2)\partial_{z}^{p}\partial_{w}^{q}} H_{n,m}^{(p,q)}(z,w|\gamma). \tag{6.16}
\]

**Proof.** The result follows since

\[
H_{n,m}^{(p+1,q+1)}(z,w|\gamma) = e^{\gamma(\partial_{z}-1)\partial_{z}^{p}\partial_{w}^{q}} H_{n,m}^{(p,q+1)}(z,w|\gamma)
= e^{\gamma(\partial_{w}-1)\partial_{w}^{p}\partial_{Z}^{q}} H_{n,m}^{(p,q)}(z,w|\gamma)
\]

and keeping in mind that \((\partial_{z} - 1)\partial_{Z}^{p}\partial_{w}^{q}\) and \((\partial_{w} - 1)\partial_{w}^{p}\partial_{Z}^{q}\) commute. \(\square\)

**Remark 6.13.** As a particular case, we have a new recursion relation for the Gould–Hopper polynomials, to wit

\[
H_{n}^{(p+1)}(z|\gamma) = n! \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \sum_{j=0}^{k} \binom{j}{k} \gamma^{k} (-1)^{k-j} H_{n-k,p}^{(p,q)} \frac{H_{n-j-pk}^{(p,q)}}{(n-j-pk)!}, \tag{6.17}
\]

and which can be re-expressed as follows

\[
H_{n}^{(p+1)}(z|\gamma) = e^{\gamma(\partial_{z}-1)\partial_{z}^{p}} H_{n}^{(p)}(z|\gamma). \tag{6.18}
\]

### 7 Nielsen identities

In this section, we prove some summation formulas of Nielsen type for the polynomials \(H_{n,m}^{(p,q)}\). As immediate consequence we derive others addition formulas for these polynomials.

**Theorem 7.1.** For any \(n, m, n', m' \in \mathbb{N}\) and \(z, w, z', w' \in \mathbb{C}\), we have

\[
H_{n,m+n',m'}^{(p,q)}(z,w|\gamma) = \sum_{i=0}^{n} \sum_{j=0}^{n'} \binom{n}{i} \binom{n'}{j} (z-z')^{i+j} H_{n-i-k,m}^{(p,q)}(z',w|\gamma) \tag{7.1}
\]

and

\[
H_{n,m+m',m'}^{(p,q)}(z,w|\gamma) = \sum_{k=0}^{m} \sum_{l=0}^{m'} \binom{m}{k} \binom{m'}{l} (w-w')^{k+l} H_{n,m+k-l}^{(p,q)}(z,w'|\gamma). \tag{7.2}
\]

**Proof.** We need to prove only the first identity, thanks to the symmetry identity satisfied by \(H_{n,m}^{(p,q)}(z,w|\gamma)\). Thus, according to \([\text{3.1}])

\[\sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(z,w|\gamma) \frac{u^{n}}{n!} = H_{m}^{(q)}(w|u^{p}w) e^{zu}\]

for every fixed \(m\). By applying \([\text{PII Eq. (1), p. 100})

\[\sum_{n=0}^{\infty} A(n) \frac{(x+y)^{n}}{n!} = \sum_{n,m=0}^{\infty} A(n+m) \frac{x^{n} y^{m}}{n! m!}, \tag{7.3}
\]

it follows

\[H_{m}^{(q)}(w|(u+t)^{p}w)e^{z(u+t)} = \sum_{n=0}^{\infty} H_{n,m}^{(p,q)}(z,w|\gamma) \frac{(u+t)^{n}}{n!} = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m}^{(p,q)}(z,w|\gamma) \frac{u^{n} t^{n'}}{n! n'!}. \tag{7.4}
\]

On the other hand, by replacing \(z\) by \(z'\), we obtain

\[H_{m}^{(q)}(w|(u+t)^{p}w) = e^{-z'(u+t)} N_{\gamma}^{p,q,m}(z',w|u,t), \tag{7.5}
\]
where we have set
\[ N_{\gamma}^{p,q,m}(z', w|u, t) := \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m'}^{(p,q)}(z', w|\gamma) \frac{u^n t^{n'}}{n! n'!}. \] (7.6)

By equating the right hand-side of (7.4) and (7.6), we find
\[ N_{\gamma}^{p,q,m}(z', w|u, t) = e^{(z-z')(u+t)} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m'}^{(p,q)}(z', w|\gamma) \frac{u^n t^{n'}}{n! n'!} \]
\[ = \sum_{k=0}^{\infty} (z - z')^k (u + t)^k \left( \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m'}^{(p,q)}(z', w|\gamma) \frac{u^n t^{n'}}{n! n'!} \right) \]
\[ = \sum_{k=0}^{\infty} \sum_{j=0}^{k} (z - z')^{k+j} \frac{u^k t^j}{k! j!} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m'}^{(p,q)}(z', w|\gamma) \frac{u^n t^{n'}}{n! n'!}. \]

The last equality follows using (7.3). Now, the substitution of \( n \) by \( n - k \) and \( n' \) by \( n' - j \) as well as the use of (4.2) lead to
\[ N_{\gamma}^{p,q,m}(z', w|u, t) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} (z - z')^{k+j} \frac{u^k t^j}{k! j!} \left( \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m'}^{(p,q)}(z', w|\gamma) \frac{u^n t^{n'}}{n! n'!} \right) \]
\[ = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} n! n'! \sum_{k=0}^{n} \sum_{j=0}^{n'} (z - z')^{k+j} \frac{u^k t^j}{k! j!} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} H_{n+n',m'}^{(p,q)}(z', w|\gamma) \frac{u^n t^{n'}}{n! n'!}. \]

This proves (7.1). \( \square \)

A generalization of Theorem 7.1 is the following

**Proposition 7.2.** For any \( n, m, n', m' \in \mathbb{N} \) and \( z, w, z', w' \in \mathbb{C} \), we have
\[ H_{n+n',m+m'}^{(p,q)}(z, w|\gamma) = \sum_{i+j=0}^{i=n, j=n'} \sum_{k+l=0}^{k=m, l=m'} \binom{n}{i} \binom{n'}{j} \binom{m}{k} \binom{m'}{l} (z - z')^{i+j} (w - w')^{k+l} \] (7.7)
\[ \times H_{n+n'-i-k, m+m'-k-l}^{(p,q)}(z', w'|\gamma). \] (7.8)

**Proof.** From (7.1) applied to \( H_{n+n',m+m'}^{(p,q)}(z, w|\gamma) \), we get
\[ H_{n+n',m+m'}^{(p,q)}(z, w|\gamma) = \sum_{i=0}^{n'} \sum_{j=0}^{n'} \binom{n}{i} \binom{n'}{j} (z - z')^{i+j} H_{n+n'-i-k, m+m'}^{(p,q)}(z', w'|\gamma). \]

Now, by applying (7.2) to \( H_{n+n'-i-k, m+m'}^{(p,q)}(z', w'|\gamma) \), we arrive at
\[ H_{n+n',m+m'}^{(p,q)}(z, w|\gamma) = \sum_{i=0}^{n'} \sum_{j=0}^{n'} \binom{n}{i} \binom{n'}{j} (z - z')^{i+j} \sum_{k=0}^{m} \sum_{l=0}^{m'} \binom{m}{k} \binom{m'}{l} (w - w')^{k+l} \]
\[ \times H_{n+n'-i-k, m+m'-k-l}^{(p,q)}(z', w'|\gamma) \]
\[ = \sum_{i=0}^{n'} \sum_{j=0}^{n'} \sum_{k=0}^{m} \sum_{l=0}^{m'} \binom{n}{i} \binom{n'}{j} \binom{m}{k} \binom{m'}{l} (z - z')^{i+j} (w - w')^{k+l} \]
\[ \times H_{n+n'-i-k, m+m'-k-l}^{(p,q)}(z', w'|\gamma). \] \( \square \)
Remark 7.3. Notice that by taking \( m = q = 0 \) in Proposition 7.2, we get the formula for the Gould-Hopper polynomials [32]:

\[
H^p_{n+n'}(z, w|\gamma) = \sum_{i=0}^{n} \binom{n'}{i} \binom{n}{j} (z - z')^{i+j} H^p_{n+n'-i-k}(z'|\gamma)
\]

Remark 7.4. By taking \( z' = \frac{z}{2} \) and \( w' = \frac{w}{2} \) in Proposition 7.2, it follows

\[
H^{(p,q)}_{n+n',m+m}(z, w|\gamma) = \sum_{i=0}^{n} \sum_{j=0}^{n'} \binom{n}{i} \binom{n'}{j} (z+w)^{i+j} H^{(p,q)}_{n-i,m-j}(z', w')|\gamma|\gamma = 0.
\]

As immediate consequence, we obtain the following addition formula with respect to the variables \( z \) and \( w \).

Corollary 7.5. We have

\[
H^{(p,q)}_{n,m}(z + z', w + w'|\gamma) = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} z^i w^j H^{(p,q)}_{n-i,m-j}(z', w'|\gamma).
\]

Proof. This readily follows from (7.7) by specifying \( n' = m' = 0 \), and replacing \( z \) by \( z - z' \) and \( w \) by \( w - w' \).

Corollary 7.6. We have

\[
H^{(p,q)}_{n,m}(z, w|\gamma) = 2^{n+m} \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} z^i w^j H^{(p,q)}_{n-i,m-j}(z, w|2^{p+q-1}||\gamma).
\]

Proof. It suffices to replace \( z \) and \( z' \) by \( \frac{z}{2} \), and \( w \) and \( w' \) by \( \frac{w}{2} \) in (7.9), and next applying (3.5).

8 Different higher order partial differential equations

The obtained recurrence formulas, in the previous section, show that the polynomials \( H^{(p,q)}_{n,m} \) satisfy certain partial differential equations, generalizing, somehow, those obtained for the real and complex Hermite type polynomials considered in this paper. Indeed, from (6.3) we have

\[
(\partial_z - \partial^p_z \partial^q_w) H^{(p,q)}_{n,m}(z, w|\gamma) = 0.
\]

This mean that the polynomials \( H^{(p,q)}_{n,m} \) are solutions for the heat \( (p, q) \)-differential equation

\[
\partial_z u = \partial^p_z \partial^q_w u.
\]

Another partial differential equation satisfied by \( H^{(p,q)}_{n,m} \) follows making use of (6.1) and the recursion relation (6.8). Namely, we have

\[
(z + \gamma \partial_z^{-1}\partial^q_w) \partial_z H^{(p,q)}_{n,m}(z, w|\gamma) = n(z + \gamma \partial_z^{-1}\partial^q_w) H^{(p,q)}_{n-1,m}(z, w|\gamma) = nH^{(p,q)}_{n,m}(z, w|\gamma).
\]

By the symmetry property (2.1), we also have

\[
(w + \gamma \partial^p_z \partial^q_w^{-1}) \partial_w H^{(p,q)}_{n,m}(z, w|\gamma) = mH^{(p,q)}_{n,m}(z, w|\gamma).
\]

This means that the polynomials \( H^{(p,q)}_{n,m} \) are commune eigenfunctions of the partial differential operators

\[
\Delta^{p,q}_z := z \partial_z + \gamma \partial^p_z \partial^q_w \quad \text{and} \quad \Delta^{p,q}_w := w \partial_w + \gamma \partial^p_z \partial^q_w.
\]
and therefore they are solutions of the following system
\[
\begin{aligned}
\Delta^{p,q}_{\gamma} u &= nu, \\
\Delta^{p}_{\gamma} u &= mu.
\end{aligned}
\] (8.5)

Subsequently and since the operator \(\partial_z\) and \((w + \gamma \partial_z \partial_{\tilde{w}}^{-1})\) commute, the polynomials \(H_{n,m}^{(p,q)}\) obey to the following higher order partial differential equation
\[
(z + \gamma \partial_z \partial_{\tilde{w}}^{-1})(w + \gamma \partial_z \partial_{\tilde{w}}^{-1})\partial_z \partial_{\tilde{w}} u = nm u.
\] (8.6)

The last equation means also that the polynomials \(H_{n,m}^{(p,q)}\) are eigenfunction of the operator \((z + \gamma \partial_z \partial_{\tilde{w}}^{-1})(w + \gamma \partial_z \partial_{\tilde{w}}^{-1})\partial_z \partial_{\tilde{w}},\) corresponding to the eigenvalue \(nm\).

### 9 Connection to Gould-Hopper Polynomials

The main aim here is to express the polynomials \(H_{n,m}^{(p,q)}\) in terms of the Gould-Hopper polynomials \(H_n^{(p)}\). We begin by expressing \(H_n^{(p)}\) in function of \(H_{n,m}^{(p,q)}\). To this end, we establish the following.

**Proposition 9.1.** For any \(n, p, q \in \mathbb{N}\), we have
\[
H_n^{p+q}(z + w|\gamma) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k,k}^{(p,q)}(z, w|\gamma)
\] (9.1)

**Proof.** Taking \(v = u\) in the generating function (3.3) infers
\[
\sum_{n,k=0}^{\infty} H_{n,k}^{(p,q)}(z, w|\gamma) \frac{u^{n+k}}{n!k!} = e^{(z+w)u + \gamma u^{p+q}}.
\]

The substitution of \(n\) by \(n - k\) entails
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} H_{n-k,k}^{(p,q)}(z, w|\gamma) \frac{u^{n}}{(n-k)!k!} = e^{(z+w)u + \gamma u^{p+q}}.
\]

But, since
\[
\sum_{n=0}^{\infty} H_{n}^{(p)}(z|\gamma) \frac{u^{n}}{n!} = e^{u z + \gamma u^p},
\]
we get
\[
\sum_{n=0}^{\infty} \sum_{k} H_{n,k}^{(p,q)}(z, w|\gamma) \frac{u^{n}}{(n-k)!k!} = \sum_{n=0}^{\infty} H_{n}^{p+q}(z|\gamma) \frac{u^{n}}{n!}.
\]

Finally, it follows
\[
H_n^{p+q}(z + w|\gamma) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k,k}^{(p,q)}(z, w|\gamma).
\]

\[\square\]

Accordingly, the expression of \(H_n^{(p)}\) in terms of \(H_{n,m}^{(p,q)}\) asserted by the following.

**Corollary 9.2.** For any \(n, p, q \in \mathbb{N}\), we have:
\[
H_n^{(p)}(z|\gamma) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k,k}^{(p-q,q)}(z - w, w|\gamma)
\] (9.2)

**Remark 9.3.** The case of \(p = q = 1, \gamma = -1\) and \(m = 0\) gives rise to a new expression of the holomrphic Hermite polynomials \(H_n(z)\) in terms of the polyanalytic Hermite polynomials,
\[
H_n(z) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k,k}(2i\Im(z), \bar{z}).
\]
Conversely, the next theorem is main result of this section.

**Theorem 9.4.** We have

\[
H^{(p,q)}_{n,m}(z, w|\gamma) = n! m! \sum_{k=j=0}^{[\frac{n}{2}]} \sum_{l=0}^{[\frac{m}{2}]} l! (l-k)! \sum_{i=0}^{[\frac{m-q}{2}]} \frac{(2)^{-l-i}(-\gamma)^{k+j}}{(n-p(l+k))! (n-q(i+j))!} H_n^{(p)}(z|\gamma) \frac{H_q^q(w|\gamma)}{(m-q(i+j))!}.
\]

*Proof.* Using the generating function (3.3) and the fact that it follows

\[
\frac{\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H^{(p,q)}_{n,m}(z, w|\gamma) u^n v^m}{n! m!} = \left( \sum_{n=0}^{\infty} H_n^{(p)} \left( \frac{z}{2} \right) \frac{u^n}{n!} \right) \left( \sum_{m=0}^{\infty} H_m^{(q)} \left( \frac{w}{2} \right) \frac{v^m}{m!} \right).
\]

The identity (5.4) infers

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H^{(p,q)}_{n,m}(z, w|\gamma) u^n v^m = \left( \sum_{n=0}^{\infty} H_n^{(p)} \left( \frac{z}{2} \right) \frac{u^n}{n!} \right) \left( \sum_{m=0}^{\infty} H_m^{(q)} \left( \frac{w}{2} \right) \frac{v^m}{m!} \right).
\]

Replacing \( n + ip \) by \( n \) and \( m + iq \) by \( m \), we get

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H^{(p,q)}_{n,m}(z, w|\gamma) u^n v^m = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n! m! \sum_{k=0}^{[\frac{n}{2}]} \sum_{j=0}^{[\frac{m}{2}]} \sum_{l=0}^{[\frac{m-q}{2}]} \frac{(2)^{-l-i}(-\gamma)^{k+j}}{(n-p(l+k))! (n-q(i+j))!} H_n^{(p)}(z|\gamma) \frac{H_q^q(w|\gamma)}{(m-q(i+j))!} u^n v^m.
\]

This completes the proof of (9.3). \( \square \)

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Appendix: A direct proof of (3.3)
More specifically, if $p$ and $q$ are non-zero positive integers, we have:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^m v^n}{n!m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\binom{\frac{p}{q}}{\frac{m}{q}}}{n!m!} \sum_{k=0}^{\infty} \frac{(z u)^{n-k}}{(n-k)!} \frac{(w v)^{m-k}}{(m-k)!} u^m v^n \frac{k!}{(n-k)!} \frac{(m-k)!}{(m-q)!} \frac{n!}{m!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\binom{\frac{p}{q}}{\frac{m}{q}}}{n!m!} \sum_{k=0}^{\infty} \frac{(z u)^{n-k}}{(n-k)!} \frac{(w v)^{m-k}}{(m-k)!} u^m v^n \frac{k!}{(n-k)!} \frac{(m-k)!}{(m-q)!} \frac{n!}{m!}$$

We suppose that $\left[\frac{p}{q}\right] \land \left[\frac{m}{q}\right] = \left[\frac{2p}{q}\right]$ and use (5.2) we get:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^m v^n}{n!m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\binom{\frac{p}{q}}{\frac{m}{q}}}{n!m!} \sum_{k=0}^{\infty} \frac{(z u)^{n-k}}{(n-k)!} \frac{(w v)^{m-k}}{(m-k)!} u^m v^n \frac{k!}{(n-k)!} \frac{(m-k)!}{(m-q)!} \frac{n!}{m!}$$

In this expression, we replace $m$ by $m - kq$, then we get:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(p,q)}(z, w|\gamma) \frac{u^m v^n}{n!m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\binom{\frac{p}{q}}{\frac{m}{q}}}{n!m!} \sum_{k=0}^{\infty} \frac{(z u)^{n-k}}{(n-k)!} \frac{(w v)^{m-k}}{(m-k)!} u^m v^n \frac{k!}{(n-k)!} \frac{(m-k)!}{(m-q)!} \frac{n!}{m!}$$

In case of $p = 0$, we have:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(0,q)}(z, w|\gamma) \frac{u^m v^n}{n!m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{n,m}^{(0,q)}(w|\gamma) \frac{v^m}{m!} \frac{u^n}{n!}$$

In case of $q = 0$, we use (2.1) and the result of the case of $p = 0$.

**Appendix: Proof of Corollary 4.4**

Notice first that we have:

$$e^{\frac{z' + z'}{2\sqrt{2}} + \frac{w' + w'}{2\sqrt{2}} + \gamma u v^p} = e^{\frac{z}{\sqrt{2}} u + \frac{w}{\sqrt{2}} v + \frac{z'}{\sqrt{2}} u v^p} e^{\frac{z'}{\sqrt{2}} u + \frac{w'}{\sqrt{2}} v + \frac{z}{\sqrt{2}} u v^p}.$$  

Using

$$H_{n,m}^{(p,q)}(z, w|\gamma) = \partial_u^p \partial_v^m e^{z'u + w'v + \gamma u v^p} |_{u=v=0},$$

we see that

$$H_{n,m}^{(p,q)}\left(\frac{z + z'}{2\sqrt{2}}, \frac{w + w'}{2\sqrt{2}} \right| \gamma\right) = \partial_u^p \partial_v^m \frac{z + z'}{2\sqrt{2}} + \frac{w + w'}{2\sqrt{2}} + \gamma u v^p |_{u=v=0}$$

is equal to

$$\sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \binom{m}{j} \partial_u^k \partial_v^j \frac{z + z'}{2\sqrt{2}} + \frac{w + w'}{2\sqrt{2}} + \gamma u v^p |_{u=v=0}$$

Therefore, we obtain

$$H_{n,m}^{(p,q)}\left(\frac{z + z'}{2\sqrt{2}}, \frac{w + w'}{2\sqrt{2}} \right| \gamma\right) = n!m! \sum_{k=0}^{n} \sum_{j=0}^{m} \frac{H_{k,j}^{(p,q)}(\frac{z + z'}{\sqrt{2}} \frac{w + w'}{\sqrt{2}} \frac{\gamma}{2}) H_{n-k,m-j}^{(p,q)}(\frac{z + z'}{\sqrt{2}} \frac{w + w'}{\sqrt{2}} \frac{\gamma}{2})}{k!j!(n-k)!(m-j)!}.$$
Now, by applying (3.5), we get
\[ H_{k,j}^{(p,q)} \left( \frac{2 \sqrt{2} z'}{2}, \frac{2 \sqrt{2} w'}{2} \right) = \left( \frac{2 \sqrt{2}}{2} \right)^k \left( \frac{2 \sqrt{2}}{2} \right)^j H_{k,j}^{(p,q)}(z, w) . \]
This proves (4.4).

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