A model of complex spins (corresponding to a non-minimal model in the language of CFT) coupled to the binary branched polymer sector of quantum gravity is considered. We show that this leads to new behaviour.

1. Introduction

Branched polymers play an important role in the discretized version of bosonic non-critical strings. Both numerical simulations \[1\] and semi-rigorous results \([2], [3], [4]\) lead us to believe they dominate the theory for values of the central charge \(c\) greater than 1. These configurations are characterized by a critical exponent \(\gamma_{str} = \frac{1}{2}\). It is possible that coupling non-minimal models to branched polymers might lead to new behaviour \([5]\). We investigate this possibility. We also consider the interpolation between a fixed configuration and the fully fluctuating case.

2. Binary Trees

The properties of the ensemble of tree graphs are well known \([6]\). In this paper we will consider trees made of cubic vertices (a subset of the full branched polymer ensemble) but modify the graphs slightly so that all the external lines except the root are attached to another line. Letting \(T_N\) be the number of graphs with \(N\) external vertices (not counting the root) we have

\[
T_N = \sum_{k=1}^{N-1} T_{N-k} T_k , \quad T_1 = 1
\]

or

\[
T(z) = \sum_{N=1}^{\infty} T_N z^N = \frac{1}{2} \left(1 - \sqrt{1 - 4z} \right)
\]

The exponent \(\gamma_{str}\) for the ensemble of graphs with one marked point (the root in this case) is defined so that the generating function \(G\) for the number of graphs of a given size has leading non-analytic behaviour

\[
\frac{\partial G}{\partial z} = (z_{cr} - z)^{-\gamma_{str}}
\]

so for the tree ensemble \(\gamma_{str} = \frac{1}{2}\), the same value as the full branched polymer case.

3. Matter coupled to binary trees

We can extend the model by coupling matter to the trees by placing an Ising spin \(\sigma_i = \pm 1\) at each of the vertices. We obtain the recurrence relation (see fig. 1)

\[
Z_N = \sum_{q=1}^{N-1} \sum_{abcd} (1 + t\sigma_1 a)(1 + t\sigma_1 c)(1 + tbd) Z_{N-q}(a, \sigma_2, b) Z_q(c, d, \sigma_3)
\]

In the absence of a magnetic field the dependence of \(Z_N\) on the external configuration \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) must take the form

\[
Z_N(\sigma) = A_N + \sigma_1 \sigma_2 B_N + \sigma_1 \sigma_3 B_N + \sigma_2 \sigma_3 C_N
\]
Inserting (5) into the partition function (4) and equating coefficients depending on the same combination of spins on both LHS and RHS, we find a system of coupled equations for the \(A_N, B_N, C_N\) in terms of \(A_{N-1}, A_{N-2}\), etc. Defining the grand-canonical partition functions

\[
A(z, t) = \sum_{N=1}^{\infty} z^N A_N(t)
\]

and similarly for \(B(z, t), C(z, t)\), we obtain the system of equations:

\[
A = z + A^2 + t^3 B^2
\]

\[
B = z + tAB + t^2 BC
\]

\[
C = z + tC^2 + t^2 B^2
\]

Singularities in any of the functions \(A, B, C\) (or in any of their derivatives) will signal a critical point in the full partition function (4). It can be shown that one has \(\gamma_{\text{str}} = \frac{1}{2}\) for \(t = [0,1]\). The critical structure of the theory is not changed by the addition of Ising spins. This result is expected because the Ising spins never have a diverging correlation length in less than two dimensions and so cannot affect the global properties of the geometry. Again we recover the behaviour observed when we couple the Ising model to the full branched polymer ensemble.

We now consider a generalised Ising model in which the spins take the values 1, \(e^{\pm \pi i/3}\). By allowing the partition function to include complex weights, we obtain a richer phase structure than that of the 2-state case. The partition function is given by

\[
Z = \sum_{\text{trees}} \sum_{\{S_i\}} \prod_{\text{links}} (1 + \mu S_i S_j^\dagger + \nu S_i^\dagger S_j)
\]

Again we construct the most general form for the partition function:

\[
Z_N = A_N + S_1 S_2 B_N + S_1 S_3 \bar{B}_N + S_1^\dagger S_2 D_N
+ S_1^\dagger S_3 \bar{D}_N + S_2 S_3^\dagger C_N + S_3 S_2^\dagger E_N
+ S_1 S_2 S_3 F_N + S_1^\dagger S_2^\dagger S_3^\dagger G_N
\]

The system of equations which emerges from this is rather more complicated than the obtained from the simple Ising (7)-(9). But it still falls under the same generic category and can be solved for different values of the coupling constants \(\mu\) and \(\nu\); the results are neatly summarized in the phase diagram of the model, fig. 2.

We see that, unlike the case of the Ising, the non-minimal \(Z_3\) model leads to the possibility of critical exponents other than \(\gamma_{\text{str}} = \frac{1}{2}\).
a given \( N \geq 3 \) there are precisely two of these; however, \( N = 1 \) and \( N = 2 \) are special because there are only two graphs in the whole ensemble. All the non-ladder graphs we will call “trees”; so in fact there are no trees for \( N < 4 \). We will begin by considering the case of pure gravity.

Letting \( L_N \) be the number of ladders with \( N \) external vertices we have

\[
L_1 = L_2 = 1 \\
L_{N \geq 3} = 2
\]

so that

\[
L(z) = \frac{2z}{1-z} - z - z^2 = z \left( \frac{1 + z^2}{1-z} \right) \tag{12}
\]

and so the exponent \( \gamma_{str} \) takes the value 2 for a pure ladder ensemble. The number of trees \( T_N \) satisfies

\[
T_1 = T_2 = T_3 = 0 \\
T_N = q \left( \sum_{k=1}^{N-1} (T_{N-k} + L_{N-k})(T_k + L_k) \right) - \left( \sum_{k=1}^{N-1} L_{N-1} - \delta_{N3} \right) \tag{13}
\]

The factor \( q \) enables us to assign a different relative weight in the ensemble to trees and ladders: a typical tree with \( N \) external vertices gets a factor of \( q^{N-1} \). For the generating function we find

\[
T(z) = q \left\{ (T(z) + L(z))^2 - zL(z) - z^3 \right\} \tag{14}
\]

which is easily solved to yield the generating function \( G = T(z) + L(z) \) for the modified ensemble:

\[
G(z) = \frac{1}{2q} \left( 1 - \sqrt{1 - 4q ((1 - qz)L - qz^3)} \right) \tag{15}
\]

For \( q = 1 \) this just becomes the usual tree generating function with \( \gamma_{str} = \frac{1}{2} \) whereas at \( q = 0 \) it is equal to \( L(z) \). However for any positive non-zero value of \( q \) we find that \( \gamma_{str} = \frac{1}{2} \). This is easily seen by considering the behaviour of the argument of the square root as \( z \) is increased from zero; as \( z \) increases \( L(z) \) increases but before it diverges the argument of the square root must vanish (because it goes to \( -\infty \) if \( L(z) \) goes to \( \infty \)). Thus only at \( q = 0 \) exactly do we manage to “freeze out” the general trees and get a system which contains only ladders. This behaviour is very reminiscent of the \( R^2 \) model discussed in [5]; this remains in the ordinary gravity phase for all finite \( R^2 \) coupling.

A similar phenomenon continues to occur when we couple matter to the ensemble. For \( q = 0 \) we have an exceptional regime, where the exponent \( \gamma_{str} \) takes values different from those of the dynamical phase, which sets in for any positive, non-zero value of \( q \).

4. Conclusions

We have shown how the introduction of complex matter can change the value of \( \gamma_{str} \) for a branched polymer ensemble very much as the introduction of ordinary matter can change it in two dimensional quantum gravity. In this models we can also solve the problem of interpolation between a “gravity” regime and a “crystalline” regime; we have found that crystalline behaviour is only obtained when geometry fluctuations are completely forbidden.

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