We formulate a symmetry principle on the basis of the duality of electric and magnetic fields and apply it to dispersion forces. Within the context of macroscopic quantum electrodynamics, we rigorously establish duality invariance for the free electromagnetic field in the presence of causal magnetoelectrics. Dispersion forces are given in terms of the Green tensor for the electromagnetic field and the atomic response functions. After discussing the behavior of the Green tensor under a duality transformation, we are able to show that Casimir forces on bodies in free space as well as local-field corrected Casimir–Polder and van der Waals forces are duality invariant.

Keywords: Dispersion forces; macroscopic quantum electrodynamics; interatomic potentials.

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1. Introduction

Dispersion forces such as the Casimir force\(^1\) on a body, the Casimir–Polder (CP) force\(^2\) between an atom and a body and the van der Waals (vdW) force\(^2\) between two atoms were originally conceived as effective electromagnetic forces between electrically neutral, but polarizable ground-state objects; they are typically attractive.\(^3\) Somewhat later, the investigations were extended to bodies and atoms with an additional magnetic response\(^4\), revealing that polarizable and magnetizable objects may repel each other. Forces between magnetoelectric systems have recently been subject to a renewed interest\(^5\) due to the availability of metamaterials with a controllable permittivity and permeability\(^6\).

As we will demonstrate in this article, investigations of this kind can be considerably simplified by exploiting the well-known duality of electric and magnetic fields.\(^7\) To that end, we first study the effect of duality transformations in the context of macroscopic quantum electrodynamics\(^8\) (Sec.\(2\)) and then use our results to prove that duality invariance is a valid symmetry of dispersion forces under very
2. Macroscopic quantum electrodynamics and duality

Duality is one of the inherent symmetries of the Maxwell equations. To see this, we group the electric and magnetic fields $\hat{E}(r, t)$, $\hat{B}(r, t)$ and excitations $\hat{D}(r, t)$, $\hat{H}(r, t)$ into dual pairs $(\hat{E}, Z_0 \hat{H})^T$ and $(Z_0 \hat{D}, \hat{B})^T$, where the vacuum impedance $Z_0 = \sqrt{\mu_0 / \varepsilon_0}$ has been introduced for dimensional reasons. In this dual-pair notation, the Maxwell equations in the absence of free charges and currents assume the compact form

$$\nabla \cdot \left( Z_0 \hat{D} \right) = 0,$$  

$$\nabla \times \left( \frac{\hat{E}}{Z_0} \right) + \frac{\partial}{\partial t} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} Z_0 \hat{D} \\ \hat{B} \end{array} \right) = 0.$$  

(1)

(2)

Grouping the polarization $\hat{P}(r, t)$ and the magnetization $\hat{M}(r, t)$ according to $(Z_0 \hat{P}, \mu_0 \hat{M})^T$, the relation between the fields and excitations reads

$$\left( \begin{array}{c} Z_0 \hat{D} \\ \hat{B} \end{array} \right) = \frac{1}{c} \left( \begin{array}{c} \hat{E} \\ Z_0 \hat{H} \end{array} \right) + \left( \begin{array}{c} 0 \\ Z_0 \hat{P} / \mu_0 \hat{M} \end{array} \right).$$  

(3)

It is now immediately obvious that Eqs. (1)–(3) are invariant with respect to a duality transformation

$$\left( \begin{array}{c} x \\ y \end{array} \right)^* = D(r, \theta) \left( \begin{array}{c} x \\ y \end{array} \right), \quad D(r, \theta) = \left( \begin{array}{cc} r \cos \theta & r \sin \theta \\ -r \sin \theta & r \cos \theta \end{array} \right) \in \mathbb{R}_+ \times \text{SO}(2),$$  

with $D(r, \theta)$ being the most general real matrix that commutes with the symplectic matrix in Eq. (2). This transformation may be viewed as a rotation in the space of dual pairs ($0 \leq \theta < 2\pi$) together with a rescaling of all fields ($r > 0$). In classical physics, the real-valued electromagnetic fields are often combined into complex Riemann–Silberstein vectors $x + iy$ in which case duality invariance manifests itself as a U(1) symmetry.

Let us next address the compatibility of the duality transformation with the constitutive relations, which may conveniently be formulated in terms of the Fourier components $\tilde{\mathbf{x}}$ of the fields, $\mathbf{x}(r, t) = \int_0^\infty d\omega \tilde{\mathbf{x}}(r, \omega, t) + \text{h.c.}$ For linear, local, isotropic, dispersing and absorbing media, the constitutive relations may be given as

$$\left( \begin{array}{c} Z_0 \tilde{\mathbf{D}} \\ \tilde{\mathbf{B}} \end{array} \right) = \frac{1}{c} \left( \begin{array}{c} \varepsilon & 0 \\ 0 & \mu \end{array} \right) \left( \begin{array}{c} \tilde{\mathbf{E}} \\ Z_0 \tilde{\mathbf{H}} \end{array} \right) + \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left( \begin{array}{c} \varepsilon \tilde{\mathbf{P}}_N \\ \mu \tilde{\mathbf{M}}_N \end{array} \right),$$  

(5)

where $\varepsilon = \varepsilon(r, \omega)$ and $\mu = \mu(r, \omega)$ denote the relative electric permittivity and magnetic permeability; and $\tilde{\mathbf{P}}_N$ and $\tilde{\mathbf{M}}_N$ are the noise polarization and magnetization which necessarily arise in the presence of absorbing media. Invariance of the
constitutive relations under the duality transformation requires that
\[
\begin{pmatrix}
\varepsilon^* & 0 \\
0 & \mu^*
\end{pmatrix} = \mathcal{D}(r, \theta) \begin{pmatrix}
\varepsilon & 0 \\
0 & \mu
\end{pmatrix} \mathcal{D}^{-1}(r, \theta) = \begin{pmatrix}
\varepsilon \cos^2 \theta + \mu \sin^2 \theta & (\mu - \varepsilon) \sin \theta \cos \theta \\
(\mu - \varepsilon) \sin \theta \cos \theta & \varepsilon \sin^2 \theta + \mu \cos^2 \theta
\end{pmatrix},
\]
(6)
\[
\begin{pmatrix}
\hat{P}_N \\
\hat{M}_N/c
\end{pmatrix}^* = \begin{pmatrix}
r \cos \theta & \mu r \sin \theta \\
-(1/\varepsilon) r \sin \theta & (\mu/\varepsilon) r \cos \theta
\end{pmatrix} \begin{pmatrix}
\hat{P}_N \\
\hat{M}_N/c
\end{pmatrix}.
\]
(7)
Condition (6) can be fulfilled in two ways: It holds if the relative impedance of the media is equal to unity, \(Z = \sqrt{\mu/\varepsilon} = 1\). In this case, which includes both free space and a perfect lens medium (\(\varepsilon = \mu = -1\)), the duality rotations form a continuous SO(2) symmetry of the electromagnetic field and one has \(\varepsilon^* = \mu^* = \varepsilon\) as well as
\[
\begin{pmatrix}
\hat{P}_N \\
\hat{M}_N/c
\end{pmatrix}^* = \begin{pmatrix}
r \cos \theta & \varepsilon r \sin \theta \\
-(1/\varepsilon) r \sin \theta & r \cos \theta
\end{pmatrix} \begin{pmatrix}
\hat{P}_N \\
\hat{M}_N/c
\end{pmatrix}.
\]
(8)
For media with a nontrivial impedance, Eq. (6) holds for \(\theta = n \pi/2\) with \(n \in \mathbb{Z}\) only. The presence of such media hence reduces the duality invariance from the full SO(2) group to a discrete \(\mathbb{Z}_4\) symmetry with the four distinct members
\[
\mathcal{D}_0 = \mathcal{I}, \quad \mathcal{D}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{D}_2 = -\mathcal{I}, \quad \mathcal{D}_3 = -\mathcal{D}_1,
\]
(9)
(\(\mathcal{I}\): unit matrix) where Eqs. (6) and (7) imply the transformations
\[
\begin{pmatrix}
\varepsilon \\
\mu
\end{pmatrix}^* = \begin{pmatrix}
\cos^2 \theta \sin^2 \theta \\
\sin^2 \theta \cos^2 \theta
\end{pmatrix} \begin{pmatrix}
\varepsilon \\
\mu
\end{pmatrix},
\]
(10)
\[
\begin{pmatrix}
\hat{P}_N \\
\hat{M}_N/c
\end{pmatrix}^* = \begin{pmatrix}
r \cos \theta & \mu r \sin \theta \\
-(1/\varepsilon) r \sin \theta & r \cos \theta
\end{pmatrix} \begin{pmatrix}
\hat{P}_N \\
\hat{M}_N/c
\end{pmatrix}.
\]
(11)
Duality is thus an exact symmetry of the Maxwell equations in the absence of free charges and currents. It must also be manifest in the underlying Hamiltonian\[11\]
\[
\hat{H}_F = \sum_{\lambda=e,m} \int d^3 r \int_0^\infty d\omega \hat{h}_\omega \hat{f}_\lambda(r, \omega) \cdot \hat{f}_\lambda(r, \omega),
\]
where the bosonic dynamical variables \(\hat{f}_e(r, \omega), \hat{f}_m(r, \omega)\) are associated with electric and magnetic medium–field excitations. Noting that the dynamical variables are related to the noise fields via
\[
\begin{pmatrix}
Z_0 \hat{P}_N \\
\mu_0 \hat{M}_N
\end{pmatrix} = \sqrt{\frac{\hbar \mu_0}{\pi}} \begin{pmatrix} 0 & \sqrt{\text{Im} \mu} \\ \sqrt{\text{Im} \mu} & 0 \end{pmatrix} \begin{pmatrix}
\hat{f}_e \\
\hat{f}_m
\end{pmatrix}
\]
(12)
and recalling Eqs. (5) and (11), they are seen to transform as
\[
\begin{pmatrix}
\hat{f}_e \\
\hat{f}_m
\end{pmatrix}^* = \begin{pmatrix}
r \cos \theta & -i(\mu/\varepsilon) r \sin \theta \\
-i(\mu/\varepsilon) r \sin \theta & r \cos \theta
\end{pmatrix} \begin{pmatrix}
\hat{f}_e \\
\hat{f}_m
\end{pmatrix}
\]
(13)
in both the continuous (\(\varepsilon = \mu\)) and discrete (\(\theta = n \pi/2\)) cases. It follows that a duality transformation leads to a rescaling of the Hamiltonian \(\hat{H}_F^* = \rho^2 \hat{H}_F\), such that the equations of motion remain invariant.
It is important to note that electromagnetic forces are not duality-invariant in general, even when acting on electrically neutral systems. For instance, the Lorentz force on a neutral magnetolectric body of volume $V$ can be written as:

$$\hat{F} = \int_{\partial V} dA \cdot \left\{ \varepsilon_0 \hat{E}(r) \hat{E}(r) + \frac{1}{\mu_0} \hat{B}(r) \hat{B}(r) - \frac{1}{2} \left[ \varepsilon_0 \hat{E}^2(r) + \frac{1}{\mu_0} \hat{B}^2(r) \right] I \right\}$$

$$- \varepsilon_0 \frac{d}{dt} \int_V d^3r \hat{E}(r) \times \dot{\hat{B}}(r)$$

(14)

where $I$ is the unit tensor; it is obviously not duality-invariant. Duality invariance would be realized for a stationary field acting on a body at rest (such that the total time derivative vanishes), provided that $\hat{D} \approx \varepsilon_0 \hat{E}$ and $\hat{H} \approx \hat{B}/\mu_0$ on the body surface. While this can never be true on an operator level due to the unavoidable presence of the noise fields $\hat{P}_N$ and $\hat{M}_N$, one would expect the Casimir force $\langle \hat{F} \rangle$ on a stationary body in free space to be invariant. The situation is very similar for the Lorentz force on a neutral atom (position $r_A$, polarization $\hat{P}_A$, magnetization $\hat{M}_A$) which can be written as:

$$\hat{F} = \nabla_A \int d^3r \left[ \hat{P}_A(r) \cdot \hat{E}(r) + \hat{M}_A(r) \cdot \hat{B}(r) + \hat{P}_A(r) \times \dot{r}_A \cdot \hat{B}(r) \right]$$

$$+ \frac{d}{dt} \int d^3r \hat{P}_A(r) \times \dot{\hat{B}}(r)$$

(15)

when neglecting diamagnetic interactions. Noting that $\hat{P}_A$ and $\hat{M}_A$ transform under duality like $\hat{P}$ and $\hat{M}$, one sees that duality invariance can only hold for an atom at rest (so that the velocity-dependent terms do not contribute) prepared in an incoherent superposition of energy eigenstates and subject to a stationary field (so that the time-derivative does not contribute), provided that $\hat{D} \approx \varepsilon_0 \hat{E}$ and $\hat{H} \approx \hat{B}/\mu_0$ hold within the volume occupied by the atom. Again, this is never true on an operator level, but one may expect CP and vdW forces on atoms to be duality invariant. We will confirm the conjectured duality invariance of dispersion forces in Sec. 3. In the following, we concentrate our attention on the particular transformation $r = 1, \theta = \pi/2$ which is a generator of the discrete duality group $Z_4$.

### 3. Duality invariance of dispersion forces

As a preparation for studying the duality invariance of dispersion forces, let us first establish a few general expressions for these forces in terms of the relevant response functions. By taking the ground-state expectation value of the Lorentz force (14), one easily finds that the Casimir force on a stationary homogeneous magnetolectric body is given by:

$$\hat{F} = \frac{\hbar}{\pi} \int_V d^3r \int_0^\infty d\omega \left( \frac{\omega^2}{c^2} \nabla \cdot \text{Im}G^{(1)}(r, r, \omega) \right)$$

$$+ \text{Tr} \left\{ I \times \left[ \nabla \times \nabla \times - \frac{\omega^2}{c^2} \right] \text{Im}G^{(1)}(r, r, \omega) \times \nabla \right\}$$

(16)
where $G^{(1)}$ is the scattering part of the Green tensor $G$ of the electromagnetic field,
\[
\left[ \nabla \times \frac{1}{\mu(r,\omega)} \nabla \times -\frac{\omega^2}{c^2} \varepsilon(r,\omega) \right] G(r, r', \omega) = \delta(r - r').
\] (17)

Alternatively, the Casimir force may be given as a surface integral
\[
F_\lambda = \frac{\hbar}{\pi} \int_0^\infty d\xi \int_{\partial V} dA \left[ G^{(1)}_{\lambda\lambda}(r, r, i\xi) - \frac{1}{2} \text{Tr} G^{(1)}_{\lambda\lambda}(r, r, i\xi) \right] \quad (\lambda = e, m)
\] (18)

with $G_{ee}(r, r', \omega) = (i\omega/c)G(r, r', \omega)/(i\omega/c)$, $G_{mm}(r, r', \omega) = \nabla \times G(r, r', \omega) \times \nabla'$. The CP force on a single atom can be derived from the Casimir force in its volume-integral form by considering the force on a dilute gas of atoms [number density $\eta(r)$, polarizability $\alpha(\omega)$, magnetizability $\beta(\omega)$] occupying an otherwise empty volume $V$. Within leading order in $\eta$, the contributions of the atoms to the permittivity and the inverse permeability ($\kappa = \mu^{-1}$) is given by the linearized Clausius–Mosotti relations $\Delta \varepsilon(r, \omega) = \eta(\omega)\alpha(\omega)/\varepsilon_0$ and $\Delta \kappa(r, \omega) = -\eta(\omega)\beta(\omega)\mu_0$. Using a linear Born expansion, one finds that the resulting change of the Green tensor reads
\[
\Delta G(r, r', \omega) = \int d^3s \frac{\eta(s)}{\varepsilon_0} \left\{ \frac{\omega^2}{c^2} \alpha(\omega) G(r, s, \omega) \cdot G(s, r', \omega) - \beta(\omega) \left[ G(r, s, \omega) \times \nabla_s \cdot \nabla_s \times G(s, r', \omega) \right] \right\}.
\] (19)

Combining this with Eq. (16) one can show that the Casimir force on the atomic cloud can be written as
\[
F = -\int_V d^3r \eta(r) \nabla U(r) \text{ where } U(r_A) = U_e(r_A) + U_m(r_A)
\]
with
\[
U_\lambda(r_A) = \frac{\hbar}{2\pi \varepsilon_0} \int_0^\infty d\xi \alpha_\lambda(i\xi) \text{Tr} G^{(1)}_{\lambda\lambda}(r_A, r_A, i\xi) \quad (\lambda = e, m)
\] (20)

($\alpha_e = \alpha$, $\alpha_m = \beta/c^2$) is the CP potential sought. The vDW potential between two atoms $A$ and $B$ can be obtained in an analogous way by introducing a second dilute cloud of atoms and applying Eq. (19) to the CP potential. This results in
\[
U(r_A, r_B) = \int_V d^3r \eta(r) U(r_A, r)
\]
with
\[
U_{\lambda\lambda}(r_A, r_B) = \frac{\hbar}{2\pi \varepsilon_0} \int_0^\infty d\xi \alpha_\lambda^A(i\xi) \alpha_\lambda^B(i\xi) \text{Tr} \left[ G_{\lambda\lambda}(r_A, r_B, i\xi) \cdot G_{\lambda\lambda}(r_B, r_A, i\xi) \right] \quad (\lambda, \lambda' = e, m)
\]
(21)

with $\mathbf{G}_{ee}(r, r', \omega) = (i\omega/c)\mathbf{G}(r, r', \omega) \times \nabla'$, $\mathbf{G}_{mm}(r, r', \omega) = \nabla \times \mathbf{G}(r, r', \omega)(i\omega/c)$.

Dispersion forces and potentials can thus be given in terms of the response functions of the electromagnetic field and the atoms, so their behavior under a duality transformation can be determined from that of $G$, $\alpha$ and $\beta$. By virtue of the linearized Clausius–Mosotti relations, the known transformation properties $\varepsilon^* = \mu$, $\lambda^* = \lambda$, the contributions of the atoms to $\varepsilon$ and $\mu$ are linear in the density $\eta(r)$ and $\mu_0/c$ instead of $\varepsilon_0$ and $\mu_0$. The vDW potential reads
\[
U_{\lambda\lambda}(r_A, r_B) = \frac{\hbar}{2\pi \varepsilon_0} \int_0^\infty d\xi \alpha_\lambda^A(i\xi) \alpha_\lambda^B(i\xi) \text{Tr} \left[ G_{\lambda\lambda}(r_A, r_B, i\xi) \cdot G_{\lambda\lambda}(r_B, r_A, i\xi) \right] \quad (\lambda, \lambda' = e, m)
\]
(21)
\( \mu^* = \varepsilon \) imply that \( \alpha^* = c^2 \beta, \beta^* = \alpha/c^2 \). As shown in Appendix A, the Green tensor transforms according to

\[
\begin{align*}
\mathbf{G}^{ee}_{ee}(\mathbf{r}, \mathbf{r}', \omega) &= \mu^{-1}(\mathbf{r}, \omega)\mathbf{G}_{mm}(\mathbf{r}, \mathbf{r}', \omega)\mu^{-1}(\mathbf{r}', \omega) + \mu^{-1}(\mathbf{r}, \omega)\delta(\mathbf{r} - \mathbf{r}'), \quad (22) \\
\mathbf{G}^{mm}_{mm}(\mathbf{r}, \mathbf{r}', \omega) &= \varepsilon(\mathbf{r}, \omega)\mathbf{G}_{ee}(\mathbf{r}, \mathbf{r}', \omega)\varepsilon(\mathbf{r}', \omega) - \varepsilon(\mathbf{r}, \omega)\delta(\mathbf{r} - \mathbf{r}'), \quad (23) \\
\mathbf{G}^{em}_{em}(\mathbf{r}, \mathbf{r}', \omega) &= -\mu^{-1}(\mathbf{r}, \omega)\mathbf{G}_{me}(\mathbf{r}, \mathbf{r}', \omega)\varepsilon(\mathbf{r}', \omega), \quad (24) \\
\mathbf{G}^{me}_{me}(\mathbf{r}, \mathbf{r}', \omega) &= -\varepsilon(\mathbf{r}, \omega)\mathbf{G}_{em}(\mathbf{r}, \mathbf{r}', \omega)\mu^{-1}(\mathbf{r}', \omega). \quad (25)
\end{align*}
\]

These laws immediately show that the discrete global duality transformation \( \varepsilon \leftrightarrow \mu, \alpha \leftrightarrow \beta/c^2 \) leaves ground-state dispersion forces on stationary objects (and associated potentials) invariant, where the individual electric and magnetic components \( (18), (20) \) and \( (21) \) transform into one another according to \( U^* \) and \( \leftrightarrow \). When applying a duality transformation, the factors \( \varepsilon \) and \( \mu^{-1} \) appearing in Eqs. \( (22) - (25) \), this only holds for forces on atoms and bodies which are situated in free space.

In order to extend duality invariance to atoms which are embedded in a medium, local-field corrections need to be taken into account. Using the real-cavity model, one can show that local field effects give rise to correction factors \( \lambda \)

\[
c_e(\omega) = \left[ \frac{3\varepsilon(\omega)}{2\varepsilon(\omega) + 1} \right]^2, \quad c_m(\omega) = \left[ \frac{3}{2\mu(\omega) + 1} \right]^2,
\]

so the potentials \( (20) \) and \( (21) \) generalize to

\[
\begin{align*}
U_{\lambda}(\mathbf{r}_A) &= \frac{\hbar}{2\pi \varepsilon_0} \int_0^\infty d\xi \, c_\lambda(\mathbf{i}\xi) \alpha_\lambda(\mathbf{i}\xi) \text{Tr} \mathbf{G}^{(1)}_{\lambda\lambda}(\mathbf{r}_A, \mathbf{r}_A, \mathbf{i}\xi) \quad (\lambda = e, m) \quad (27) \\
U_{\lambda\lambda'}(\mathbf{r}_A, \mathbf{r}_B) &= -\frac{\hbar}{2\pi \varepsilon_0} \int_0^\infty d\xi \, c^{A}_\lambda(\mathbf{i}\xi) c^{B}_{\lambda'}(\mathbf{i}\xi) \alpha^{A}_\lambda(\mathbf{i}\xi) \alpha^{B}_{\lambda'}(\mathbf{i}\xi) \\
&\quad \times \text{Tr} \left[ \mathbf{G}^{(1)}_{\lambda\lambda'}(\mathbf{r}_A, \mathbf{r}_B, \mathbf{i}\xi) \cdot \mathbf{G}^{(1)}_{\lambda\lambda}(\mathbf{r}_B, \mathbf{r}_A, \mathbf{i}\xi) \right], \quad (\lambda, \lambda' = e, m). \quad (28)
\end{align*}
\]

When applying a duality transformation, the factors \( \varepsilon \) and \( \mu^{-1} \) arising from the transformation of the Green tensor \( (22) - (25) \) combine with those contained in the local-field correction factors \( (20) \) in such a way that the corrected potentials \( (21) \) and \( (28) \) transform into one another according to \( U_e \leftrightarrow U_m, U_{ee} \leftrightarrow U_{mm} \) and \( U_{em} \leftrightarrow U_{me} \). When including local-field corrections, the total ground-state dispersion potentials are hence also duality invariant for embedded atoms.

4. Summary
We have studied the behavior of electromagnetic fields, response functions and dispersion forces under duality transformations. In the presence of media with a non-trivial impedance, the SO(2) symmetry of duality rotations reduces to a discrete \( \mathbb{Z}_4 \) symmetry. The duality transformations induced by the generator of this group are displayed in Tab. 1. Note that the transformation needs to be applied four times to the fields (first block) in order to return to the original state. On the contrary,
Table 1. Effect of the duality transformation with $r = 1$, $\theta = \pi/2$ on fields, response functions and dispersion forces.

| Dual partners | Transformation |
|---------------|---------------|
| $E, H$:       | $E^* = Z_0 H$, $H^* = -E/Z_0$ |
| $D, B$:       | $D^* = B Z_0$, $B^* = -Z_0 D$ |
| $P, M$:       | $P^* = M/c$, $M^* = -c P$ |
| $P_A, M_A$:   | $P_A^* = M_A/c$, $M_A^* = -c P_A$ |
| $P_N, M_N$:   | $P_N^* = \mu M_N/c$, $M_N^* = -c P_N/\epsilon$ |
| $\hat{f}_e, \hat{f}_m$: | $\hat{f}_e^* = -i(\mu/|\mu|) \hat{f}_m$, $\hat{f}_m^* = -i(|\epsilon|/\epsilon) \hat{f}_e$ |
| $\varepsilon, \mu$: | $\varepsilon^* = \mu$, $\mu^* = \varepsilon$ |
| $\alpha, \beta$: | $\alpha^* = c^2 \beta$, $\beta^* = \alpha/c^2$ |
| $G_{ee}, G_{mm}$: | $G_{ee}^* = (1/\mu) G_{mm}(1/\mu) + (1/\mu) \delta$, $G_{mm}^* = \varepsilon G_{ee} - \varepsilon \delta$ |
| $G_{em}, G_{me}$: | $G_{em}^* = -(1/\mu) G_{me} \varepsilon$, $G_{me}^* = -\varepsilon G_{em}(1/\mu)$ |

The transformation is self-inverse when acting on the response functions of bodies, atoms and the electromagnetic field (second block). We have shown that the electric and magnetic components of dispersion forces on neutral bodies and atoms, which are at rest and situated in free space, depend on these response functions in such a way that they inherit a similar transformation behavior (third block). As a consequence, the total dispersion forces are duality invariant. As demonstrated, duality invariance can be extended to atoms embedded in media provided that local-field corrections are taken into account.

The duality invariance established in our work provides an important consistency check for investigations of dispersion forces. It can further serve as calculational tool: Once the force on an object in a particular magnetolectric environment is known, an expression for the force in the dual arrangement can be generated by simply making the replacements $\alpha \leftrightarrow \beta/c^2$, $\varepsilon \leftrightarrow \mu$.

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Appendix A. Duality transformation of the Green tensor

To derive the transformed Green tensor $G^*$, which is a solution to the Eq. (17) with $\varepsilon^* = \mu$ and $\mu^* = \varepsilon$ instead of $\varepsilon$ and $\mu$, we first note that the Maxwell equations (1)
and \( \mathbf{D}(r, \omega) \) together with the constitutive relations \( \mathbf{G} \) are uniquely solved by

\[
\vec{E}(r, \omega) = \frac{1}{\varepsilon_0} \int d^3r' \mathbf{G}_{ee}(r, r', \omega) \cdot \vec{P}_N(r', \omega) - Z_0 \int d^3r' \mathbf{G}_{em}(r, r', \omega) \cdot \vec{M}_N(r', \omega),
\]

(A.1)

\[
\vec{B}(r, \omega) = -Z_0 \int d^3r' \mathbf{G}_{me}(r, r', \omega) \cdot \vec{P}_N(r', \omega) - \mu_0 \int d^3r' \mathbf{G}_{mm}(r, r', \omega) \cdot \vec{M}_N(r', \omega),
\]

(A.2)

\[
\vec{D}(r, \omega) = -\varepsilon(r, \omega) \int d^3r' \mathbf{G}_{em}(r, r', \omega) \cdot \vec{M}_N(r', \omega) - \int d^3r' \left[ \varepsilon(r, \omega) \mathbf{G}_{ee}(r, r', \omega) - \delta(r-r') \right] \cdot \vec{P}_N(r', \omega),
\]

(A.3)

\[
\vec{H}(r, \omega) = -\frac{c}{\mu(r, \omega)} \int d^3r' \mathbf{G}_{me}(r, r', \omega) \cdot \vec{P}_N(r', \omega) - \int d^3r' \left[ \frac{\mathbf{G}_{mm}(r, r', \omega)}{\mu(r, \omega)} + \delta(r-r') \right] \cdot \vec{M}_N(r', \omega).
\]

(A.4)

Applying the duality transformation to Eqs. (A.1) and (A.2) with the aid of the transformation laws established in Sec. 2 and using Eqs. (A.3) and (A.4), the unknown quantities \( \mathbf{G}_{\lambda \lambda'} \) on the rhs of the transformed equations can be related to the untransformed ones \( \mathbf{G}_{\lambda \lambda'} \) appearing on the lhs, and one obtains Eqs. (22)–(25).

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