Suppressing charge-noise sensitivity in high-speed Ge hole spin-orbit qubits

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Abstract

Strong spin-orbit interactions make hole quantum dots central to the quest for electrical spin qubit manipulation enabling fast, low-power, scalable quantum computation. Yet it is important to establish to what extent spin-orbit coupling may expose the qubit to electrical noise, facilitating decoherence. Here we show that, unlike electron spin qubits, the hole spin-3/2 leads generically to sweet spots in the dephasing rate of gate-defined hole qubits as a function of the gate electric field. At these sweet spots, the dephasing rate vanishes to first order in the perpendicular electric field, the
EDSR dipole moment is maximized, and the relaxation rate can be drastically reduced by working at small magnetic fields. The existence of the sweet spots is traced to properties of the Rashba spin-orbit interaction unique to spin-3/2 systems. Our results suggest that the coherence of Ge hole spin qubits in quantum dots can be optimized at sweet spots where rapid electric control is also possible, characteristics that make hole spin qubits very favourable for scalable quantum computing.

**Introduction**

Quantum computing architectures require reliable initialization, robust single-qubit operations, long coherence times, and a clear pathway towards scaling up. Solid-state platforms are supported by the well developed solid-state device industry, with mature microfabrication and miniaturization technologies. Among solid-state platforms, semiconductor quantum dot (QD) spin qubits have been actively pursued, with an energetic recent focus on hole spins in diamond and zincblende nano-structures.

The primary motivation for this focus is the strong spin-orbit interaction of hole systems, which enables one to control qubits via electron dipole spin resonance (EDSR), making quantum computing faster, more power-efficient and easier to operate. This because electric fields are much easier to apply and localize than magnetic fields used in electron spin resonance. Only a global static magnetic field is required to split the qubit levels. In addition, the $p$-symmetry of the hole wave function causes the contact hyperfine interaction to vanish, and no complications involving valley degrees of freedom are present. Initial studies indicate that hole spins may possess sufficiently long coherence times for quantum computing. Meanwhile, much progress has been made in the initialization and readout of hole spin qubits.

The existential question that will determine the future of hole QD spin qubits is: Does the strong spin-orbit interaction that allows fast qubit operation also enhance undesired couplings to stray fields such as phonons and charge noise leading to intractable relaxation and
dephasing? In this paper, we demonstrate theoretically that this is emphatically not the case. Due to the spin-3/2 nature of holes that sets them entirely apart from electrons, dephasing can be essentially eliminated to at least the first order in the gate electric field at specific sweet spots in parameter space.\cite{15,16,37,45–50} At these sweet spots, electrical qubit rotations are at their most efficient, with the EDSR gate time reaching a minimum. At the same time, the relaxation rate due to phonons can be made as small as is desired by working at small magnetic fields, of the order of 0.1 T, which can allow $10^6 – 10^7$ operations in one relaxation time for an in-plane alternating field $E_{AC} \sim 10^5$ V/m. We argue that every gate-defined hole qubit has a sweet spot at a certain value of the gate electric field. Whereas our analysis is generically applicable to all gate-defined hole quantum dots, our focus in this paper is on Ge, which has witnessed enormous progress in the last few years. As a group-IV element, it has isotopes with zero nuclear spin and no piezoelectric phonons, while the bulk Dresselhaus spin-orbit interaction is absent.\cite{6,7,9,10,51} Holes in planar Ge quantum wells have a very large out of plane Landé g-factor, $g \approx 20$, enabling operation at very small magnetic fields, which would not impede coupling to a superconducting resonator. The low resistivity of Ge when contacting with metals makes couplings between other devices such as superconductors easier.\cite{9,52,53} In the past decade, spectacular results have been reported, for example, the EDSR detection techniques,\cite{18,51} structures of quantum confinement systems,\cite{7,54–56} the anisotropy of $g$-tensors,\cite{10,54} spin-orbit couplings and transport phenomena in two-dimensional hole systems.\cite{4,6,10,57,58} As we shall show below, strong cubic-symmetry terms enable EDSR with ultra-short gate times by inducing a special kind of Rashba interaction, yet can still be understood within a perturbative scheme.

Hole Quantum Dot

Our focus in this work is on single dots. A prototype device, including a neighboring dot, is shown in Figure 1. The Hamiltonian describing a single hole quantum dot has the general
Figure 1: A prototype double quantum dot in a two-dimensional hole gas. The red shaded circles represent two quantum dots confined by a set of gates. Our work is concerned with a single dot; two dots are shown to illustrate scaling up strategies, e.g. gates $B_2$ and $T_1$ control inter-dot tunnelling.

The form $H = H_{LK} + H_{BP} + H_Z + H_{ph} + H_{\text{conf}}$, where $H_{LK}$ represents the Luttinger-Kohn Hamiltonian, $H_Z$ is the Zeeman interaction between the hole and an external magnetic field, and $H_{ph}$ the hole-phonon interaction. $H_{\text{conf}}$ is the confinement potential including the vertical and lateral confinement. The vertical confinement is achieved by applying a gate electric field $F_z$ in the growth direction, leading to a term $eF_z z$ in the Hamiltonian; the lateral confinement is modelled as an in-plane parabolic potential well. The strain term $H_{BP}$ is represented by the Bir-Pikus Hamiltonian$^{18}$ due to the experimental implementation during the fabrication of the two-dimensional hole system. A typical configuration of holes in Ge is achieved by growing a thin strained Ge layer (usually about 10 nm to 20 nm) between SiGe layers such that, if the barrier between the two layers is high enough, a quantum well can be formed. In this paper, we consider Si$_x$Ge$_{1-x}$, where $x = 0.15$.$^{4,6,7,54}$

First we determine the effective $2 \times 2$ Hamiltonian for a qubit composed of HH states, we start from the bulk band structure of holes as derived by Luttinger and Kohn.$^{59}$ The spinor basis is formed by the eigenstates of $J_z$, $\{ |+3\rangle, |\frac{3}{2}\rangle, |\frac{1}{2}\rangle, |\frac{-1}{2}\rangle \}$. For a 2D hole
gas grown along \( \hat{z} \parallel (001) \), we write the Luttinger-Kohn Hamiltonian as:

\[
H_{\text{LK}}(k^2, k_z) = \begin{bmatrix}
P + Q & L & M \\
0 & P + Q & M^* & -L^* \\
L^* & M & P - Q & 0 \\
M^* & -L & 0 & P - Q
\end{bmatrix},
\]

where \( P = \frac{\hbar^2}{2m_0} (k^2 + k_z^2) \), \( Q = -\frac{\hbar^2}{2m_0} (2k_z^2 - k^2) \), \( L = -\sqrt{3} \frac{\hbar^2 \gamma_3}{2m_0} k_- k_z \), \( M = -\sqrt{3} \frac{\hbar^2 \gamma_2}{2m_0} (\gamma k_+^2 + \delta k_-^2) \), and \( m_0 \) is the free electron mass, \( \gamma_1, \gamma_2, \gamma_3 \) are Luttinger parameters which are determined by the band structure. The in-plane wave vector will be \( k^2 = k_x^2 + k_y^2 \), \( k_\pm = k_x \pm ik_y \), the wave vector in the growth direction will be \( \hat{k}_z = -i \partial / \partial z \). We have also used \( \bar{\gamma} = (\gamma_2 + \gamma_3) / 2 \) and \( \delta = (\gamma_3 - \gamma_2) / 2 \) to simplify the algebra. In Ge \( \delta / \bar{\gamma} < 0.15 \), hence \( \delta \) can be treated perturbatively, while bulk Dresselhaus terms are absent. Although interface inversion asymmetry terms with the same functional form may exist,\(^{109}\) at the strong gate fields considered here they will be overwhelmed by the Rashba interaction and are not discussed in detail. The diagonal terms of \( H_{\text{BP}} \) in the HH manifold are \( P_\varepsilon + Q_\varepsilon = -a_v (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \), while in the LH manifold they are \( P_\varepsilon - Q_\varepsilon = -(b_v / 2) (\varepsilon_{xx} + \varepsilon_{yy} - 2\varepsilon_{zz}) \), where \( a_v = -12 \text{ eV} \) and \( b_v = -2.3 \text{ eV} \) are deformation potential constants.\(^{18}\) In our chosen configuration \( \varepsilon_{xx} = \varepsilon_{yy} = -0.0063 \), the minus sign indicates that the germanium is compressed in \( xy \)-plane. In the \( \hat{z} \)-direction, the Ge layer will be stretched, and \( \varepsilon_{zz} = (2C_{12}/C_{11})\varepsilon_{xx} = 0.0044 \), with \( C_{12} = 44 \text{ GPa} \), \( C_{11} = 126 \text{ GPa} \) for Ge. The diagonal terms of the strain-relaxed barrier configuration will change the HH-LH energy splitting by a constant, which is approximately 50 meV.

The growth direction provides the spin quantization axis, with the heavy holes states (HHs) representing the \( \pm 3/2 \) angular momentum projection onto this axis, while the light holes states (LHs) represent \( \pm 1/2 \). In 2D hole systems, the HHs are the ground state.\(^{7,26,47,61,62}\) To define a quantum dot a series of gates are added on top of the 2D hole gas confinement, as in Figure [1]' and we ultimately seek an effective Hamiltonian describing the two lowest-lying HH states in a quantum dot. Since we expect the HH-LH splitting to be much larger
than the quantum dot confinement energy, we proceed with the standard assumptions of \( \mathbf{k} \cdot \mathbf{p} \) theory, retaining at first only terms containing \( k_z \), with \( k_x \) and \( k_y \) initially set to zero. This determines the approximate eigenstates \( \psi_{H,L}(z) \) corresponding to the growth-direction. These are described by two variational Bastard wave functions \( \psi_H \) and \( \psi_L \),

\[
\psi_{H,L}(z) = \sqrt{\frac{4\beta_{H,L} \left( \pi^2 + \beta_{H,L}^2 \right)}{(1 - e^{-2\beta_{H,L}}) d \pi^2}} \cos \left( \frac{\pi z}{d} \right) \exp \left[ -\beta_{H,L} \left( \frac{z}{d} + \frac{1}{2} \right) \right],
\]

where the dimensionless variational parameters \( \beta_{H,L} \) are sensitive to the gate electric field due to the term \( eF_z z \), and \( d \) is the width of the quantum well in the growth direction, which is an input parameter. The orthogonality of the HH and LH states is ensured by the spinors. This wave function is suitable for inversion layers as well as accumulation layers, although our focus will be primarily on the latter.\(^1\) We employ analogous wave functions for the first excited states, omitting the details.\(^6\)

We first project the Luttinger Hamiltonian onto the wave functions for the growth-direction, which in our model comprise eight sub-bands: HH1, LH1, HH2, LH2, each with two spin projections. Carrying out a Schrieffer-Wolff transformation we obtain the effective 2 \( \times \) 2 spin-orbit coupling for a 2D hole gas,\(^6\) which, for a system with cubic symmetry, contains two terms with different rotational properties:

\[
H_{SO} = i\alpha_2 \left( k^3_+ \hat{\sigma}_- - k^3_- \hat{\sigma}_+ \right) + i\alpha_3 \left( k_+ k_- k_+ \hat{\sigma}_+ - k_- k_+ k_- \hat{\sigma}_- \right),
\]

where \( \hat{\sigma}_+ = (\hat{\sigma}_x + i\hat{\sigma}_y)/2, \hat{\sigma}_- = (\hat{\sigma}_x - i\hat{\sigma}_y)/2 \). In the absence of a magnetic field, the \( \alpha_2 \)-Rashba term winds around the Fermi surface three times, whereas the \( \alpha_3 \)-Rashba term winds only once. The latter term enables EDSR, as we show below. The two coefficients are evaluated

\(^1\)For inversion layers, the Bastard wave function will also be appropriate, because in experiments the electric field can be made large enough such that the hole gas sticks around the top of the quantum well.
\[
\alpha_2 = \frac{3}{2} \frac{\mu_2^2 \gamma_3}{E_H - E_L} \langle \psi_H \mid \psi_L \rangle \left[ \langle \psi_H \mid \hat{k}_z \mid \psi_L \rangle - \langle \psi_L \mid \hat{k}_z \mid \psi_H \rangle \right]
\]
\[
\alpha_3 = \frac{3}{2} \frac{\mu_2^2 \delta \gamma_3}{E_H - E_L} \langle \psi_H \mid \psi_L \rangle \left[ \langle \psi_H \mid \hat{k}_z \mid \psi_L \rangle - \langle \psi_L \mid \hat{k}_z \mid \psi_H \rangle \right],
\]

where \( E_H \) and \( E_L \), obtained by the variational method, are the energies of the lowest-lying heavy hole and light hole states, respectively, and are strong functions of the gate electric field. Next, in a perpendicular magnetic field, the in-plane wave functions are found from

\[
\left[ \frac{\hbar^2}{2m_p} \left( -i \nabla_{\parallel} + eA \right)^2 + \frac{1}{2} m_p \omega_0^2 (x^2 + y^2) \right] \phi = \varepsilon \phi,
\]

where \( m_p = m_0 / (\gamma_1 + \gamma_2) \) is the in-plane effective mass of the heavy holes, the subscript \( \parallel \) refers to the \( xy \)-plane. The vector potential is \( A = (B/2) (-y, x, 0) \), \( \omega_0 \) is the oscillator frequency, \( a_0 \) the QD radius which satisfy \( a_0^2 = \hbar / (m_p \omega_l) \), a magnetic field will narrow the QD radius. The ground and first excited states are:

\[
\phi_0 = \frac{1}{a_0 \sqrt{\pi}} \exp \left[ -\frac{(x^2 + y^2)}{2a_0^2} \right]
\]

\[
\phi_{\pm 1} = \frac{1}{a_0^2 \sqrt{\pi}} (x \pm iy) \exp \left[ -\frac{(x^2 + y^2)}{2a_0^2} \right],
\]

and the eigen-energies \( \varepsilon_{n_1,n_2} = \hbar(n_1 - n_2 + 1)\omega_l + \frac{1}{2} \hbar(n_2 - n_1)\omega_c \), where \( \omega_l = \sqrt{\omega_0^2 + \omega_c^2 / 4} \), \( \omega_c = eB/m_p \) is the cyclotron frequency. Finally, the hole-phonon interaction is:

\[
H_{i,j,s} = \sum_{\alpha,\beta=x,y,z} \frac{1}{2} \sqrt{\frac{\hbar}{2NV_c \rho \omega_s}} D^{i,j}_{\alpha,\beta} \left[ q_{\alpha} \hat{e}_{s,\beta} + q_{\beta} \hat{e}_{s,\alpha} \right] q \left( e^{-i \mathbf{q} \cdot \mathbf{r}} \hat{a}^\dagger_q + e^{i \mathbf{q} \cdot \mathbf{r}} \hat{a}_q \right),
\]

where \( q \) is the phonon wave vector, \( V_c \) is the unit cell volume, \( NV_c \) is the crystal volume, \( \hat{e}_s \) is the polarization directon vector. The density of the material is denoted by \( \rho \), \( D_{\alpha,\beta} \) represents the deformation potential matrix, and \( \hat{a}^\dagger \) and \( \hat{a} \) are the phonon creation and the annihilation operators. The details of the process of reducing the above to an effective QD
spin qubit Hamiltonian are presented in the Supporting Information.\cite{35,70,74}

**Results and Discussion**

![Spin qubit Hamiltonian](image)

**Figure 2:** (a) Qubit Zeeman splitting. When the gate electric field is turned off, the qubit Zeeman splitting will be $g_0 \mu_B B = 120 \mu$eV. As the gate electric field increases, the Rashba spin-orbit coupling will change the quantum dot energy levels, leading to a sweet spot. (b) A comparison of the magnitude of the $\alpha_2$- and $\alpha_3$- Rashba terms that lead to the change in the qubit Zeeman splitting. In all of these figures, we used the quantum well width $d = 15$ nm, and the dot radius $a_0 = 8$ nm. The out-of-plane magnetic field is $B = 0.1$ T, and the confinement energy $\hbar \omega_0 = 20$ meV.

We begin with the qubit Larmor frequency, which has been plotted in Figure 2a and 2b as a function of the gate electric field. The relationship between the spin-orbit coupling coefficients and the qubit Zeeman splitting is:

$$
\Delta \varepsilon_Z = \frac{8B m_p [\alpha_2^2 (6g_0 \mu_B m_p - 45e\hbar) + \alpha_3^2 (23e\hbar + 38g_0 \mu_B m_p)]}{9g_0^2 \hbar^4},
$$

where $\mu_B$ is the Bohr magneton, $g_0$ is the same as the bulk $g$-factor in Ge which is $6\kappa = 20.36$. We note the non-monotonic behaviour as a function of gate field, which is directly related to the behaviour of the two Rashba spin-orbit coupling terms $\alpha_2$ and $\alpha_3$, as shown in Figure 2), both of which contribute to the Zeeman energy. As is seen from the figures, the magnitude of the energy splitting is dominated by the $\alpha_2$-Rashba terms, and both the $\alpha_2$- and $\alpha_3$-terms have maxima at the same value of the gate field. Figure 3 shows the magnitude of

\[\text{[Footnote]}\]

\[\text{For the sake of simplicity, the expression quoted includes only contributions from the HH1 and LH1 sub-bands. The numerical evaluations include the HH2 and LH2 contributions as well.}\]

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the spin-orbit coupling coefficients. The magnitude of the $\alpha_2$-Rashba term for a specific quantum well width will be larger than the $\alpha_3$-term, which explains the relative magnitudes of the qubit Zeeman splitting in Figure 2b. The sweet spot in the qubit Zeeman splitting Figure 2 always coincides with the maximum in the spin-orbit coupling constants Figure 3. However, importantly, the location of the sweet spot is different for each qubit and can vary considerably depending on the width of the quantum well – it can be calculated or determined experimentally for each qubit.

Figure 3: Spin-orbit coupling coefficients, the quantum well width is $d = 15$ nm. Luttinger parameters for germanium are $\gamma_1 = 13.18$, $\gamma_2 = 4.24$, $\gamma_3 = 5.69$, $\bar{\gamma} = 4.97$, $\delta = -0.725$, the heavy hole effective mass is $m_{HH} = 0.213 m_0$ and the light hole effective mass is $m_{LH} = 0.046 m_0$, where $m_0 = 9.31 \times 10^{-31}$ kg is the free electron mass. Note that spin-orbit coupling coefficients do not depend on the dot size, but only the quantum well width $d$.

Physically, the behaviour of the qubit Zeeman splitting and Rashba coefficients is understood by recalling that the Rashba effect for the HH sub-bands is primarily driven by the off-diagonal matrix element $L$ in Eq. 1 connecting the HH and LH sub-bands. This term, which is $\propto k_z k_+$, increases with the top gate field. At small gate fields, therefore, the Rashba spin-orbit constants increase monotonically due to the increase in the $k_z$ overlap integral. This continues until a critical top gate field is reached at which the HH-LH splittings, determined by the matrix element $Q$, begin to increase faster than the off-diagonal matrix element $L$. This physics has been shown previously by Winkler and collaborators. Beyond this critical field the Rashba terms decrease, resulting in a relatively broad sweet spot, at which
the qubit is insensitive to background electric field fluctuations in the \( \hat{z} \)-direction and hence the dephasing rate vanishes to first order in the \( \hat{z} \)-electric field. As we shall show below, electric field fluctuations in the \( \hat{z} \)-direction are by far the most damaging to the qubit, and are the primary source of decoherence to be avoided. The breadth and smoothness of the extreme make the tuning of the electric field to reach the sweet spot easier, as will be quantified below. The sweet spot reflects the interplay of the quadrupole degree of freedom with the gate electric field unique to spin-3/2 systems.

Figure 4: (a) Relaxation time and EDSR Rabi time as a function of the gate electric field. (b) The allowable number of single qubit operations in one relaxation time. In all these plots, the quantum well width is \( d = 15 \text{ nm} \), \( a_0 = 8 \text{ nm} \), the external magnetic field is \( B = 0.1 \text{T} \). The cyclotron frequency is \( \omega_c = 3 \times 10^{11} \text{ Hz} \), the confinement frequency is \( \omega_l = 3.2 \times 10^{13} \text{ Hz} \), the density of Germanium \( \rho = 5.33 \times 10^3 \text{ kg/m}^3 \). The phonon propagation speed along the transverse direction is \( v_t = 3.57 \times 10^3 \text{ m/s} \), along the longitudinal direction it is \( v_l = 4.85 \times 10^3 \text{ m/s} \).

For EDSR, an in-plane oscillating electric field represented in the Hamiltonian by \( eE_{AC}(t)x \) drives spin-conserving transitions between the QD ground state \( \phi_{0\uparrow,\downarrow} \) and the first excited state \( \phi_{\pm 1\uparrow,\downarrow} \). Spin flips come from the spin-orbit interaction. In a single-hole dot the Rashba term \( \propto \alpha_2 \), which has a winding number of three, only couples the QD ground state to the third excited state and does not give rise to EDSR. On the other hand, the Rashba
term $\propto \alpha_3$ gives rise to spin-flip transitions between the ground state $\phi_{0\uparrow,\downarrow}$ and the first excited state $\phi_{\pm 1\downarrow,\uparrow}$. The combined action of the electric field term and the $\alpha_3$-Rashba term is a second-order process resulting in a spin-flip in the ground state, namely EDSR. For a multiply occupied hole dot the excited state structure may be more complex but the argument above remains valid because the $\alpha_2$ and $\alpha_3$ Rashba terms couple the ground state to different excited states. The EDSR Rabi time describes the time taken to accomplish an operation. The EDSR Rabi frequency, expanded to first order in the magnetic field $B$, reads (details in the Supporting Information)

$$f_R = a_0^2 \alpha_3 B e E_{AC} g_2 \mu_B m_p^2 / \hbar^3.$$  

The in-plane electric field $E_{AC}$ is set to be $10^5 \text{V/m}$. The EDSR Rabi frequency can be tuned by changing the gate electric field and with it the Rashba spin-orbit coupling constant. Note, however, that, because the two Rashba terms directly determine the correction to the $g$-factor, clearly the Rashba interaction and the $g$-factor cannot be tuned independently at present.

Next, we discuss qubit relaxation. Hyperfine interactions and phonon-hole interactions are two major factors affecting the relaxation time, hence the quality of the qubit. However, the $p$-type symmetry of the valence band excludes the contact hyperfine interaction. There is no bulk inversion asymmetry in group IV elements; this leads to no Dresselhaus spin-orbit coupling. However, there is still the Rashba spin-orbit coupling due to the structure inversion asymmetry, which couples the heavy-hole states to the light-hole states. Neither the spin nor the orbit angular momentum will be a good quantum number, as the admixture of the spin-down and the spin-up states will modify the wave functions Eqs. 7 and 8. We emphasize that, whereas EDSR comes only from the $\alpha_3$-Rashba term, the qubit relaxation is caused by both the $\alpha_2$- and the $\alpha_3$-Rashba terms.

Using Fermi’s golden rule, we can evaluate the relaxation time of the QD, as we can see in Figure 4. For completeness, we also consider two-phonon relaxation processes, which include virtual emission and absorption of a phonon between two heavy hole states, since in the first-order relaxation calculation there is no direct matrix element between the two heavy-hole states (see Supporting Information for a detailed explanation). However, the two-phonon
process calculation returns a negligible relaxation rate, which will not contribute significantly to the relaxation time. The relaxation rate will depend on the external magnetic field as $(1/T_1) \propto B^7$ for the $\alpha_3$-Rashba term and $(1/T_1) \propto B^9$ for the $\alpha_2$-Rashba term. This is shown in Figure 4a.

We also plot the ratio between the relaxation time and the EDSR time, demonstrating that the system allows for a large number of operations. The allowable number of single qubit operations is calculated by evaluating the ratio of the relaxation time and the EDSR time. The in-plane electric field we used is $E_{AC} = 10^5$ V/m. In Figure 4, we plot the relaxation time, EDSR Rabi time, comparison of the magnitude of the relaxation time and EDSR time and an estimation of red the allowable number of single qubit operations as the function of the gate electric field at a magnetic field $B = 0.1$ T which is parallel to the growth direction. The relaxation time calculations mainly consider the hole-phonon interactions and the details can be found in the supplementary information. Both the relaxation time and the EDSR time will depend on the spin-orbit coupling coefficients, therefore, their extrema coincide.

From Figure 4a, we can see that the Ge hole quantum dot has a long relaxation time at dilution refrigerator temperatures. It is also useful to study the relaxation time at slightly higher temperatures, e.g. 4 K, at which both phonon absorption and emission must be taken into account. The phonon occupation number is given by the Bose-Einstein distribution $N = (e^{\hbar \omega/(k_B T)} - 1)^{-1}$, where $N$ is the occupation number, $\omega = qv$, $q$ is the phonon wave vector and $v$ is the phonon propagation velocity, $T$ is the temperature, $k_B$ is the Boltzmann constant. More details can be found in the Supporting Information, where a plot of the temperature dependence of the relaxation rate is presented in Figure 8. For $T = 4$ K, the relaxation time is 56 ms, suggesting that the qubit can easily be operated at this temperature.

Finally, we focus on dephasing, for which the main mechanism is provided by fluctuating electrical fields such as charge noise. We focus on random telegraph noise (RTN) due to charge defects, noting that a similar discussion can be presented for $1/f$ noise, which is
typically caused by an incoherent superposition of RTN sources. For this reason, we expect the trends for the two types of noise to be similar, while reliable numbers for $1/f$ noise must await experimental determination of the noise spectral density $S(\omega)$ for hole qubits. To begin with, we estimate the dephasing time $T_2^*$, which is expected to be primarily determined by fluctuations in the Larmor frequency of the qubit stemming from fluctuations in the spin-orbit coupling constants $\alpha_2$ and $\alpha_3$ induced by charge noise. The electric potential induced at the qubit by a defect located at $r_D$, which may give rise to RTN, can be modelled as a quasi-2D screened Coulomb potential:

$$U_{\text{scr}} = \frac{e^2}{2\epsilon_0\epsilon_r} \int_0^{2k_F} \frac{e^{-i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}_D)}}{q + q_TF} \frac{d^3q}{(2\pi)^3},$$

(11)

where $\epsilon_0$ is the vacuum permeability, $\epsilon_r$ is the relative permeability for Ge, $q_TF$ is the Thomas-Fermi wave vector, and $k_F$ is the Fermi wave vector.\(^{45}\) In a dilution refrigerator, the high energy modes of the Coulomb potential is negligible, therefore the $q > 2k_F$ part is ignored. Another source of dephasing is dipole defects due to the asymmetry in bond polarities.

$$U_{\text{dip}} (\mathbf{R}_D) = \frac{p \cdot \mathbf{R}_D}{4\pi\epsilon_0\epsilon_r R_D^3},$$

(12)

where $\mathbf{R}_D$ is the distance between the dot and the unscreened charge dipole. $p$ is the dipole moment of the charge $p = e\ell$, the size of the dipole is about 1Å.

As a worst-case estimate of the dephasing time, we use the motional narrowing result,\(^{45,46}\) the dephasing time $T_2^{-1} = (\delta \omega)^2 \tau / 2$, where $\delta \omega$ is the change in qubit Larmor frequency due to the fluctuator, and we consider $\tau = 10^3 t_{\text{Rabi}}$, where $t_{\text{Rabi}}$ is the single-qubit operation time (the inverse of the EDSR frequency), which can be found from Figure 4b. Because of the weak coupling between the spin degree of freedom and external reservoirs, slower fluctuators can be eliminated via pulse sequences and the spin echo techniques.\(^{78}\) We consider a defect located 20 nm away from the quantum dot in the plane of the dot as a worst-case scenario. Here we used $r_D = 20$ nm since regions inside this range will be depleted by the top gate,
and the defects will not be active; the dipole defect is right under the gate, and we assume $R_D = 20\,\text{nm}$, in $\hat{z}$ direction. The sweet spot is at $F = 1.3 \times 10^7\,\text{V/m}$. To estimate the pure dephasing time at the sweet spot due to such a defect, we first note that the in-plane electric field will not contribute to dephasing. An in-plane electric field enters the QD Hamiltonian as $E_\parallel \cdot r_\parallel$. This in-plane electric field term does not couple states with different spin orientations. When we consider the qubit Zeeman splittings, the corrections to the effective quantum dot levels due to the in-plane electric field will read the same for $H_{1,1}$ and $H_{2,2}$ up to the second-order, therefore, fluctuations in qubit Zeeman splitting $H_{1,1} - H_{2,2}$ will not depend on the in-plane electric field, a detailed calculation can be found in the Supporting Information. However, higher-order terms in the expansion of the electrostatic potential of the defect will lead to dephasing, and these are responsible for dephasing at the sweet spot itself. To determine their effect, we write the ground state energy as $E_0 + E_z + v_0$ where $E_0$ is the lateral confinement energy, $E_z$ is the Zeeman energy, and $v_0$ is the energy correction due to the defect.

We would like to estimate the approximate qubit window of operation around the sweet spot. Away from the sweet spot, due to the fluctuating electric potential of the defect, the energy levels of the quantum dot will gain a correction, and there will be fluctuations in $\alpha_2$ and $\alpha_3$ due to the $\hat{z}$-electric field of the defect, given by $F_z = \frac{dU}{dz}$. This fluctuation $F_z$ will affect the Bastard wave functions, changing the HH-LH energy splitting, and the spin-orbit coupling constants according to Eqs. 4 and 5. A plot of variational parameters as a function of the gate electric field is given in the Supporting Information. Diagrammatically, we can read off the gradient of the variational parameters $\beta_H$ or $\beta_L$ as a function of the gate electric field; and estimate the fluctuations in $\beta$ as: $\delta\beta_H = \frac{\partial\beta}{\partial F} \delta F_z$, $\delta\beta_H = \frac{\partial\beta}{\partial F} \delta F_z$. Consequently, the spin-orbit coupling constants will also gain a correction via the fluctuations in the variational parameters. To evaluate the change of spin-orbit coupling constants, we have: $\delta\alpha = \frac{\partial\alpha}{\partial\beta_H} \delta\beta_H + \frac{\partial\alpha}{\partial\beta_L} \delta\beta_L$, where $\alpha \in \{\alpha_2, \alpha_3\}$. With these assumptions, the dephasing time is plotted as a function of the gate electric field in Figure 5a. At the sweet spot, the
The dephasing time due to the out-of-plane fluctuations is calculated to the second-order, since the first-order fluctuation vanishes, the in-plane fluctuations will dominate the dephasing. Away from the sweet spot, the motional narrowing result is much smaller than the quasi-static limit result. This is because the first-order variation of the qubit Zeeman splitting will weaken the correlation time, while the quasi-static limit does not consider any correlations. However, as the gate electric field approaches to the sweet spot, the variation of qubit Zeeman splitting is getting smaller; at the sweet spot, compared with the quasi-static limit result, longer correlation time will lead to a larger dephasing time. We also determine the pure dephasing time in the quasi-static limit, where the switching time is the longest time scale in the system. This is essentially given by $T_2 = 2\pi/(\delta\omega)$, and is plotted in Figure 5b.

![Figure 5](image.png)

Figure 5: (a) Dephasing time in the motional narrowing regime. (b) Dephasing time in the quasi-static limit. In both plots, the quantum well width is $d = 15$ nm and the size of the quantum dot is $a_0 = 8$ nm. We considered both the in-plane fluctuations due to the screened potential and the out-of-plane fluctuations due to the screened potential and dipole defects.

Although we have used a simple parabolic model for the in-plane QD confinement, our conclusions are very general. Firstly, the dephasing sweet spot will be present for potentials of arbitrary complexity (for example hut wire geometries), since it is due to the fundamental interplay between the HH and LH that gives rise to the Rashba spin-orbit coupling in the HH manifold. Secondly, we have examined the possibility that the insensitivity of the $g$-factor to in-plane electric fields is an artefact of the model. We have tested three deviations from parabolicity and found that none of them exposes the qubit to dephasing by fluctuating in-plane electric fields. This implies (i) that the dot does not have to be perfectly parabolic allowing for some flexibility in the gate structure; (ii) that in-plane electric field fluctuations
generally have a negligible effect on the \(g\)-factor, while out-of-plane electric field fluctuations cause fluctuations in the Rashba spin-orbit coupling and affect the \(g\)-factor, therefore it is most important to avoid the effect of the out-of-plane field; and (iii) that dephasing at the sweet spot itself comes about primarily from higher-order terms in the electrical potential, i.e. electrical quadrupole and higher. We expect our results to hold qualitatively in Si as well, where the spin-orbit interaction is weaker than in Ge, while \(\delta\) is larger. However, the large \(\delta\) and frequent failure of the Schrieffer-Wolff approximation in Si calls for a fully numerical treatment.\(^{64}\) In GaAs, the hyperfine interaction is a source of dephasing that cannot be tuned away by the gate, albeit much weaker for holes than for electrons.

Experimentally, the configuration we describe requires a double-gated device with separate plunger gates and barrier gates allowing the number density and the gate electric-field (and spin-orbit coupling) to be controlled independently.\(^{54}\) The numerical estimates above suggest that, in general, a smooth and broad sweet spot will enable the Ge hole qubit to work insensitively to the charge noise inside a large range of gate electric field accessible to experiment. Exchange-based two-qubit gates are expected to be possible for hole QDs, and their speed depends on the values of exchange obtained, which is expected to be tunable by gates. Moreover, it is likely to simplify the coupling between the two qubits since the valley degree of freedom is absent in hole systems. However, two-qubit gates in the setup discussed here is not well optimized for long distance coupling, which leads to the two-qubit gate time is of the order of microseconds for dipole-dipole interactions and hundreds of microseconds for circuit QED, limited by the Ge Luttinger parameters. They can be sped up by using strain to enhance the spin-orbit interaction, but we defer the discussion to a future publication.
Summary

We have demonstrated that electrostatically defined hole quantum dot spin qubits naturally exhibit a sweet spot at which sensitivity to charge noise is minimized while the speed of electrical operation is maximized. The location of the sweet spot can be determined from the width of the quantum well and the strain tensors applied. Relaxation times are long even at 4 K, while dephasing is determined by higher-order terms in the expansion of the electrostatic potential due to charge defects, but are expected to allow for a large window of operation around the sweet spot. Our results provide a theoretical guideline for achieving fast, highly coherent, low-power electrically operated spin qubits experimentally. Future studies must consider in-plane magnetic fields, which interact much more weakly with HH spins and are considerably more complicated to treat theoretically.

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References

(1) Loss, D.; DiVincenzo, D. P. Quantum computation with quantum dots. *Phys. Rev. A* 1998, 57, 120–126.

(2) Fischer, J.; Trif, M.; Coish, W.; Loss, D. Spin interactions, relaxation and decoherence in quantum dots. *Solid State Communications* 2009, 149, 1443 – 1450, Fundamental Phenomena and Applications of Quantum Dots.
(3) Chesi, S.; Wang, X. J.; Coish, W. A. Controlling hole spins in quantum dots and wells. *The European Physical Journal Plus* **2014**, *129*, 86.

(4) Lodari, M.; Tosato, A.; Sabbagh, D.; Schubert, M. A.; Capellini, G.; Sammak, A.; Veldhorst, M.; Scappucci, G. Light effective hole mass in undoped Ge/SiGe quantum wells. *Phys. Rev. B* **2019**, *100*, 041304.

(5) Liles, S. D.; Li, R.; Yang, C. H.; Hudson, F. E.; Veldhorst, M.; Dzurak, A. S.; Hamilton, A. R. Spin and orbital structure of the first six holes in a silicon metal-oxide-semiconductor quantum dot. *Nature Communications* **2018**, *9*, 3255.

(6) Watzinger, H.; Kukučka, J.; Vukušić, L.; Gao, F.; Wang, T.; Schäffler, F.; Zhang, J.-J.; Katsaros, G. A germanium hole spin qubit. *Nature Communications* **2018**, *9*, 3902.

(7) Watzinger, H.; Kloeffel, C.; Vukui, L.; Rossell, M. D.; Sessi, V.; Kukuka, J.; Kirchschlager, R.; Lausecker, E.; Truhlar, A.; Glaser, M.; Rastelli, A.; Führer, A.; Loss, D.; Katsaros, G. Heavy-Hole States in Germanium Hut Wires. *Nano Letters* **2016**, *16*, 6879–6885, PMID: 27656760.

(8) Vukui, L.; Kukuka, J.; Watzinger, H.; Milem, J. M.; Schaffler, F.; Katsaros, G. Single-Shot Readout of Hole Spins in Ge. *Nano Letters* **2018**, *18*, 7141–7145, PMID: 30359041.

(9) Ares, N.; Katsaros, G.; Golovach, V. N.; Zhang, J. J.; Prager, A.; Glazman, L. I.; Schmidt, O. G.; De Franceschi, S. SiGe quantum dots for fast hole spin Rabi oscillations. *Applied Physics Letters* **2013**, *103*, 263113.

(10) Mizokuchi, R.; Maurand, R.; Vigneau, F.; Myronov, M.; De Franceschi, S. Ballistic One-Dimensional Holes with Strong g-Factor Anisotropy in Germanium. *Nano Letters* **2018**, *18*, 4861–4865, PMID: 29995419.

(11) Bohuslavskiyi, H.; Kotekar-Patil, D.; Maurand, R.; Corna, A.; Barraud, S.; Bourdet, L.; Hutin, L.; Niquet, Y.-M.; Jehl, X.; De Franceschi, S.; Vinet, M.; Sanquer, M. Pauli
blockade in a few-hole PMOS double quantum dot limited by spin-orbit interaction. *Applied Physics Letters* 2016, 109, 193101.

(12) Zwanenburg, F. A.; van Rijmenam, C. E. W. M.; Fang, Y.; Lieber, C. M.; Kouwenhoven, L. P. Spin States of the First Four Holes in a Silicon Nanowire Quantum Dot. *Nano Letters* 2009, 9, 1071–1079, PMID: 19226128.

(13) Spruijtenburg, P. C.; Ridderbos, J.; Mueller, F.; Leenstra, A. W.; Brauns, M.; Aarnink, A. A. I.; van der Wiel, W. G.; Zwanenburg, F. A. Single-hole tunneling through a two-dimensional hole gas in intrinsic silicon. *Applied Physics Letters* 2013, 102, 192105.

(14) Brauns, M.; Ridderbos, J.; Li, A.; Bakkers, E. P. A. M.; van der Wiel, W. G.; Zwanenburg, F. A. Anisotropic Pauli spin blockade in hole quantum dots. *Phys. Rev. B* 2016, 94, 041411.

(15) Salfi, J.; Mol, J. A.; Culcer, D.; Rogge, S. Charge-Insensitive Single-Atom Spin-Orbit Qubit in Silicon. *Phys. Rev. Lett.* 2016, 116, 246801.

(16) Salfi, J.; Tong, M.; Rogge, S.; Culcer, D. Quantum computing with acceptor spins in silicon. *Nanotechnology* 2016, 27, 244001.

(17) Hung, J.-T.; Marcellina, E.; Wang, B.; Hamilton, A. R.; Culcer, D. Spin blockade in hole quantum dots: Tuning exchange electrically and probing Zeeman interactions. *Phys. Rev. B* 2017, 95, 195316.

(18) Terrazos, L. A.; Marcellina, E.; Coppersmith, S. N.; Friesen, M.; Hamilton, A. R.; Hu, X.; Koiller, B.; Saraiva, A. L.; Culcer, D.; Capaz, R. B. Qubits Based on Hole Quantum Dots in Strained Ge. 2018,

(19) van der Heijden, J.; Kobayashi, T.; House, M. G.; Salfi, J.; Barraud, S.; Lavieville, R.;
Simmons, M. Y.; Rogge, S. Readout and control of the spin-orbit states of two coupled acceptor atoms in a silicon transistor. *Science Advances* **2018**, *4*.

(20) Chekhovich, E. A.; Makhonin, M. N.; Tartakovskii, A. I.; Yacoby, A.; Bluhm, H.; Nowack, K. C.; Vandersypen, L. M. K. Nuclear spin effects in semiconductor quantum dots. *Nature Materials* **2013**, *12*, 494 EP –.

(21) Kyrychenko, F. V.; Kossut, J. Diluted magnetic semiconductor quantum dots: An extreme sensitivity of the hole Zeeman splitting on the aspect ratio of the confining potential. *Phys. Rev. B* **2004**, *70*, 205317.

(22) Nadj-Perge, S.; Frolov, S. M.; Bakkers, E. P. A. M.; Kouwenhoven, L. P. Spin-orbit qubit in a semiconductor nanowire. *Nature* **2010**, *468*, 1084 EP –.

(23) Golovach, V. N.; Khaetskii, A.; Loss, D. Spin relaxation at the singlet-triplet crossing in a quantum dot. *Phys. Rev. B* **2008**, *77*, 045328.

(24) Rossi, A.; Ferrus, T.; Lin, W.; Kodera, T.; Williams, D. A.; Oda, S. Detection of variable tunneling rates in silicon quantum dots. *Applied Physics Letters* **2011**, *98*, 133506.

(25) Yoneda, J.; Takeda, K.; Otsuka, T.; Nakajima, T.; Delbecq, M. R.; Allison, G.; Honda, T.; Kodera, T.; Oda, S.; Hoshi, Y.; Usami, N.; Itoh, K. M.; Tarucha, S. A quantum-dot spin qubit with coherence limited by charge noise and fidelity higher than 99.9%. *Nature Nanotechnology* **2018**, *13*, 102–106.

(26) Bulaev, D. V.; Loss, D. Electric Dipole Spin Resonance for Heavy Holes in Quantum Dots. *Phys. Rev. Lett.* **2007**, *98*, 097202.

(27) Golovach, V. N.; Borhani, M.; Loss, D. Electric-dipole-induced spin resonance in quantum dots. *Phys. Rev. B* **2006**, *74*, 165319.
(28) Coish, W. A.; Golovach, V. N.; Egues, J. C.; Loss, D. Measurement, control, and decay of quantum-dot spins. *physica status solidi (b)* 2006, 243, 3658–3672.

(29) Szumniak, P.; Bednarek, S.; Pawlowski, J.; Partoens, B. All-electrical control of quantum gates for single heavy-hole spin qubits. *Phys. Rev. B* 2013, 87, 195307.

(30) Budich, J. C.; Rothe, D. G.; Hankiewicz, E. M.; Trauzettel, B. All-electric qubit control in heavy hole quantum dots via non-Abelian geometric phases. *Phys. Rev. B* 2012, 85, 205425.

(31) Salis, G.; Kato, Y.; Ensslin, K.; Driscoll, D. C.; Gossard, A. C.; Awschalom, D. D. Electrical control of spin coherence in semiconductor nanostructures. *Nature* 2001, 414, 619–622.

(32) Pribiag, V. S.; Nadj-Perge, S.; Frolov, S. M.; van den Berg, J. W. G.; van Weperen, I.; Plissard, S. R.; Bakkers, E. P. A. M.; Kouwenhoven, L. P. Electrical control of single hole spins in nanowire quantum dots. *Nature Nanotechnology* 2013, 8, 170 EP –.

(33) Khaetskii, A. V. Spin relaxation in semiconductor mesoscopic systems. *Physica E: Low-dimensional Systems and Nanostructures* 2001, 10, 27 – 31, Proceedings of the First International Conference on the Physics and Applications of Spin-Related Phenomena in Semiconductors.

(34) Gvozdić, D. M.; Ekenberg, U. Superiority of p-type spin transistors. *Physica Scripta* 2006, T126, 21–26.

(35) Woods, L. M.; Reinecke, T. L.; Kotlyar, R. Hole spin relaxation in quantum dots. *Phys. Rev. B* 2004, 69, 125330.

(36) Hsieh, C.-Y.; Cheriton, R.; Korkusinski, M.; Hawrylak, P. Valence holes as Luttinger spinor based qubits in quantum dots. *Phys. Rev. B* 2009, 80, 235320.
(37) Kloeffel, C.; Trif, M.; Loss, D. Strong spin-orbit interaction and helical hole states in Ge/Si nanowires. *Phys. Rev. B* **2011**, *84*, 195314.

(38) Szumniak, P.; Bednarek, S.; Partoens, B.; Peeters, F. M. Spin-Orbit-Mediated Manipulation of Heavy-Hole Spin Qubits in Gated Semiconductor Nanodevices. *Phys. Rev. Lett.* **2012**, *109*, 107201.

(39) Khaetskii, A. V.; Nazarov, Y. V. Spin relaxation in semiconductor quantum dots. *Phys. Rev. B* **2000**, *61*, 12639–12642.

(40) Khaetskii, A. V.; Loss, D.; Glazman, L. Electron Spin Decoherence in Quantum Dots due to Interaction with Nuclei. *Phys. Rev. Lett.* **2002**, *88*, 186802.

(41) Fischer, J.; Loss, D. Hybridization and Spin Decoherence in Heavy-Hole Quantum Dots. *Phys. Rev. Lett.* **2010**, *105*, 266603.

(42) Kobayashi, T.; Salfi, J.; van der Heijden, J.; Chua, C.; House, M. G.; Culcer, D.; Hutchison, W. D.; Johnson, B. C.; McCallum, J. C.; Riemann, H.; Abrosimov, N. V.; Becker, P.; Pohl, H. J.; Simmons, M. Y.; Rogge, S. Engineering long spin coherence times of spin-orbit systems. 2018.

(43) Yamahata, G.; Kodera, T.; Churchill, H. O. H.; Uchida, K.; Marcus, C. M.; Oda, S. Magnetic field dependence of Pauli spin blockade: A window into the sources of spin relaxation in silicon quantum dots. *Phys. Rev. B* **2012**, *86*, 115322.

(44) Hendrickx, N. W.; Franke, D. P.; Sammak, A.; Scappucci, G.; Veldhorst, M. Fast two-qubit logic with holes in germanium. 2019.

(45) Bermeister, A.; Keith, D.; Culcer, D. Charge noise, spin-orbit coupling, and dephasing of single-spin qubits. *Applied Physics Letters* **2014**, *105*, 192102.

(46) Culcer, D.; Hu, X.; Das Sarma, S. Dephasing of Si spin qubits due to charge noise. *Applied Physics Letters* **2009**, *95*, 073102.
(47) Fischer, J.; Coish, W. A.; Bulaev, D. V.; Loss, D. Spin decoherence of a heavy hole coupled to nuclear spins in a quantum dot. *Phys. Rev. B* **2008**, *78*, 155329.

(48) Csontos, D.; Brusheim, P.; Zülicke, U.; Xu, H. Q. Spin-$3/2$ physics of semiconductor hole nanowires: Valence-band mixing and tunable interplay between bulk-material and orbital bound-state spin splittings. *Phys. Rev. B* **2009**, *79*, 155323.

(49) Meier, L.; Salis, G.; Shorubalko, I.; Gini, E.; Schön, S.; Ensslin, K. Measurement of Rashba and Dresselhaus spin–orbit magnetic fields. *Nature Physics* **2007**, *3*, 650 EP –.

(50) Governale, M.; Zülicke, U. Spin accumulation in quantum wires with strong Rashba spin-orbit coupling. *Phys. Rev. B* **2002**, *66*, 073311.

(51) Li, S.-X.; Li, Y.; Gao, F.; Xu, G.; Li, H.-O.; Cao, G.; Xiao, M.; Wang, T.; Zhang, J.-J.; Guo, G.-P. Measuring hole spin states of single quantum dot in germanium hut wire. *Applied Physics Letters* **2017**, *110*, 133105.

(52) Li, Y.; Li, S.-X.; Gao, F.; Li, H.-O.; Xu, G.; Wang, K.; Liu, D.; Cao, G.; Xiao, M.; Wang, T.; Zhang, J.-J.; Guo, G.-C.; Guo, G.-P. Coupling a Germanium Hut Wire Hole Quantum Dot to a Superconducting Microwave Resonator. *Nano Letters* **2018**, *18*, 2091–2097, PMID: 29468882.

(53) Kloeffel, C.; Trif, M.; Stano, P.; Loss, D. Circuit QED with hole-spin qubits in Ge/Si nanowire quantum dots. *Phys. Rev. B* **2013**, *88*, 241405.

(54) Sammak, A.; Sabbagh, D.; Hendrickx, N. W.; Lodari, M.; Paquelet Wuetz, B.; Tosato, A.; Yeoh, L.; Bollani, M.; Virgilio, M.; Schubert, M. A.; Zaumseil, P.; Capellini, G.; Veldhorst, M.; Scappucci, G. Shallow and Undoped Germanium Quantum Wells: A Playground for Spin and Hybrid Quantum Technology. *Advanced Functional Materials* **2019**, *29*, 1807613.
(55) Hardy, W. J.; Harris, C. T.; Su, Y.-H.; Chuang, Y.; Moussa, J.; Maurer, L. N.; Li, J.-Y.; Lu, T.-M.; Luhman, D. R. Single and double hole quantum dots in strained Ge/SiGe quantum wells. *Nanotechnology* **2019**, *30*, 215202.

(56) Takeda, K.; Obata, T.; Fukuoka, Y.; Akhtar, W. M.; Kamioka, J.; Kodera, T.; Oda, S.; Tarucha, S. Characterization and suppression of low-frequency noise in Si/SiGe quantum point contacts and quantum dots. *Applied Physics Letters* **2013**, *102*, 123113.

(57) Katsaros, G.; Golovach, V. N.; Spathis, P.; Ares, N.; Stoffel, M.; Fournel, F.; Schmidt, O. G.; Glazman, L. I.; De Franceschi, S. Observation of Spin-Selective Tunneling in SiGe Nanocrystals. *Phys. Rev. Lett.* **2011**, *107*, 246601.

(58) Rol, M.; Battistel, F.; Malinowski, F.; Bultink, C.; Tarasinski, B.; Vollmer, R.; Haider, N.; Muthusubramanian, N.; Bruno, A.; Terhal, B.; et al., Fast, High-Fidelity Conditional-Phase Gate Exploiting Leakage Interference in Weakly Anharmonic Superconducting Qubits. *Physical Review Letters* **2019**, *123*.

(59) Luttinger, J. M.; Kohn, W. Motion of Electrons and Holes in Perturbed Periodic Fields. *Phys. Rev.* **1955**, *97*, 869–883.

(60) Durnev, M. V.; Glazov, M. M.; Ivchenko, E. L. Spin-orbit splitting of valence subbands in semiconductor nanostructures. *Phys. Rev. B* **2014**, *89*, 075430.

(61) Marcellina, E.; Srinivasan, A.; Miserev, D. S.; Croxall, A. F.; Ritchie, D. A.; Farrer, I.; Sushkov, O. P.; Culcer, D.; Hamilton, A. R. Electrical Control of the Zeeman Spin Splitting in Two-Dimensional Hole Systems. *Phys. Rev. Lett.* **2018**, *121*, 077701.

(62) Liu, H.; Marcellina, E.; Hamilton, A. R.; Culcer, D. Strong Spin-Orbit Contribution to the Hall Coefficient of Two-Dimensional Hole Systems. *Phys. Rev. Lett.* **2018**, *121*, 087701.
(63) Bastard, G.; Mendez, E. E.; Chang, L. L.; Esaki, L. Variational calculations on a quantum well in an electric field. *Phys. Rev. B* **1983**, *28*, 3241–3245.

(64) Marcellina, E.; Hamilton, A. R.; Winkler, R.; Culcer, D. Spin-orbit interactions in inversion-asymmetric two-dimensional hole systems: A variational analysis. *Phys. Rev. B* **2017**, *95*, 075305.

(65) Ahn, D.; Chuang, S. L. Variational calculations of subbands in a quantum well with uniform electric field: GramSchmidt orthogonalization approach. *Applied Physics Letters* **1986**, *49*, 1450–1452.

(66) Bravyi, S.; DiVincenzo, D. P.; Loss, D. SchriefferWolff transformation for quantum many-body systems. *Annals of Physics* **2011**, *326*, 2793 – 2826.

(67) Golovach, V. N.; Khaetskii, A.; Loss, D. Phonon-Induced Decay of the Electron Spin in Quantum Dots. *Phys. Rev. Lett.* **2004**, *93*, 016601.

(68) Trif, M.; Simon, P.; Loss, D. Relaxation of Hole Spins in Quantum Dots via Two-Phonon Processes. *Phys. Rev. Lett.* **2009**, *103*, 106601.

(69) Bulaev, D. V.; Loss, D. Spin Relaxation and Decoherence of Holes in Quantum Dots. *Phys. Rev. Lett.* **2005**, *95*, 076805.

(70) Maier, F.; Kloeffel, C.; Loss, D. Tunable g factor and phonon-mediated hole spin relaxation in Ge/Si nanowire quantum dots. *Phys. Rev. B* **2013**, *87*, 161305.

(71) Woods, L. M.; Reinecke, T. L.; Lyanda-Geller, Y. Spin relaxation in quantum dots. *Phys. Rev. B* **2002**, *66*, 161318.

(72) Climente, J. I.; Segarra, C.; Planelles, J. Spin–orbit-induced hole spin relaxation in InAs and GaAs quantum dots. *New Journal of Physics* **2013**, *15*, 093009.

(73) Kornich, V.; Kloeffel, C.; Loss, D. Phonon-mediated decay of singlet-triplet qubits in double quantum dots. *Phys. Rev. B* **2014**, *89*, 085410.
(74) Ares, N.; Golovach, V. N.; Katsaros, G.; Stoffel, M.; Fournel, F.; Glazman, L. I.; Schmidt, O. G.; De Franceschi, S. Nature of Tunable Hole $g$ Factors in Quantum Dots. Phys. Rev. Lett. 2013, 110, 046602.

(75) Winkler, R. Rashba spin splitting in two-dimensional electron and hole systems. Phys. Rev. B 2000, 62, 4245–4248.

(76) Winkler, R.; Papadakis, S. J.; De Poortere, E. P.; Shayegan, M. Highly Anisotropic $g$-Factor of Two-Dimensional Hole Systems. Phys. Rev. Lett. 2000, 85, 4574–4577.

(77) Habib, B.; Tutuc, S. S. M., E.vand Melinte; Wasserman, D.; Lyon, S. A.; Winkler, R. Negative differential Rashba effect in two-dimensional hole systems. Applied Physics Letters 2004, 85, 3151–3153.

(78) Press, D.; De Greve, K.; McMahon, P. L.; Ladd, T. D.; Friess, B.; Schneider, C.; Kamp, M.; Höfling, S.; Forchel, A.; Yamamoto, Y. Ultrafast optical spin echo in a single quantum dot. Nature Photonics 2010, 4, 367.
Supporting Information Available

The following files are available free of charge.

Strain terms, Bir-Pikus Hamiltonian

The Bir-Pikus Hamiltonian reads:

\[
H_{\text{strain}} = \begin{bmatrix}
P_\varepsilon + Q_\varepsilon & 0 & L_\varepsilon & M_\varepsilon \\
0 & P_\varepsilon + Q_\varepsilon & M^*_\varepsilon & -L^*_\varepsilon \\
L^*_\varepsilon & M_\varepsilon & P_\varepsilon - Q_\varepsilon & 0 \\
M^*_\varepsilon & -L_\varepsilon & 0 & P_\varepsilon - Q_\varepsilon
\end{bmatrix},
\]

(13)

where \( P_\varepsilon = -a_v(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \), \( Q_\varepsilon = -(b_v/2)(\varepsilon_{xx} + \varepsilon_{yy} - 2\varepsilon_{zz}) \), \( L_\varepsilon = (\sqrt{3}/2)b_v(\varepsilon_{xx} - \varepsilon_{yy} - id\varepsilon_{xy}) \), \( M_\varepsilon = d(\varepsilon_{xx} - i\varepsilon_{yz}) \). where \( a_v, b_v, d \) are deformation potential constants, and \( \varepsilon_{i,j} \) are components of the strain tensor. In our case, we have \( \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0 \), therefore, we only have diagonal matrix elements.

To calculate the magnitude of the strain, we use the Vegard’s law:

\[
a_{\text{Si}_{x-Ge_{1-x}}} = xa_{\text{Si}} + (1 - x)a_{\text{Ge}_{1-x}}.
\]

(14)

Where \( a \) is the lattice constant, for Si, \( a_{\text{Si}} = 0.543 \text{ nm} \), for Ge, \( a_{\text{Ge}} = 0.566 \text{ nm} \). To match the Ge layer on the top of the \( \text{Si}_{0.15}\text{Ge}_{0.85} \), the lattice constant for Ge layer should be \( a_{\text{Si}_{0.15}\text{Ge}_{0.85}} \). Therefore, we can find the compressive strain for the Ge in the \( xy \)-plane \( \varepsilon_{xx} = \varepsilon_{yy} = -0.0063 \). Due to the compression in the \( xy \)-plane, the Ge layer will expand in the \( z \)-direction, and the tensile strain due to the expansion can be found using Poisson’s ratio, and the compressive strain in the \( xy \)-plane will be \( \varepsilon_{zz} = (-2C_{12}/C_{11})\varepsilon_{xx} = 0.0044 \).
Energy splitting between heavy holes and light holes

In this section, we first plot the variational parameters $\beta_H$ and $\beta_L$ used in Eq. 2, then we plot the energy splitting between the heavy-hole state (HH) and the light-hole state (LH). The variational parameters are evaluated by minimizing the expectation value of the Hamiltonian in the states $\psi_H$ and $\psi_L$:

$$E_{h,l}(\beta_{h,l}) = \langle \psi_{h,l} | \hat{H} | \psi_{h,l} \rangle = \langle \psi_{h,l} | -\frac{h^2}{2m_{h,l}} \frac{\partial^2}{\partial z^2} + eFz | \psi_{h,l} \rangle,$$  \hspace{1cm} (15)

where the subscript can be either $h$ for the heavy-hole state or $l$ for the light-hole state. The energy then reads:

$$E_{h,h}(\beta_{h,l}) = eFd \left( \frac{1}{2\beta_{h,l}} + \frac{\beta_{h,l}}{\pi^2 + \beta_{h,l}^2} \right) + \frac{\hbar^2 \pi^2 + \beta_{h,l}^2}{2d^2 m_{h,l}} - \frac{1}{2} eFd \coth(\beta_{h,l}),$$  \hspace{1cm} (16)

where $F$ is the applied gate electric field, $d$ is the quantum well width, the effective masses $m_h$ and $m_l$ are obtained from fitting the band diagram, which can be described by Luttinger parameters. For germanium, we have $\gamma_1 = 13.18$, $\gamma_2 = 4.24$, $\gamma_3 = 5.69$, which give the following effective masses:

$$m_h = \frac{m_0}{\gamma_1 - 2\gamma_2} = 0.213 m_0 \quad m_l = \frac{m_0}{\gamma_1 + \gamma_2} = 0.046 m_0,$$  \hspace{1cm} (17)

where $m_0$ is the bare electron mass. By minimising Eqs. 15 and 16, we obtain the variational parameters $\beta_h$ and $\beta_l$ (Figure 6a), as well as the HH-LH energy splitting (Figure 6b), as a function of the gate electric field.
Spin-orbit coupling coefficients

In this section, we first derive the spin-orbit coupling coefficients. To this end we first project the Luttinger Hamiltonian onto the zero-node HH and LH states $\psi_{0HH1}^0, \psi_{0HH1}^0, \psi_{0LH1}^0, \psi_{0LH1}^0$, where the superscript denote zero-node wave-functions.

$$H_{LK} = \begin{pmatrix} P + Q & 0 & L & M \\ 0 & P + Q & M^{\dagger} & -L^{\dagger} \\ L^{\dagger} & M & P - Q & 0 \\ M^{\dagger} & -L & 0 & P - Q \end{pmatrix}. \quad (18)$$

Where $P = \frac{\hbar^2 \gamma_1}{2m_0} (k^2 + k_z^2), Q = -\frac{\hbar^2 \gamma_2}{2m_0} (2k_z^2 - k^2), L = -\sqrt{3}\frac{\hbar^2 \gamma_3}{2m_0} k_z k_x, M = -\sqrt{3}\hbar^2 (\tau k_x^2 + \delta k_z^2)$, the in-plane wave vector will be $k^2 = k_x^2 + k_y^2, k_{\pm} = k_x \pm ik_y$, the wave vector in the growth direction will be $\hat{k}_z = -i\partial/\partial z$. Now we include the contribution from one-node HH and LH states $\psi_{H1}^1, \psi_{H1}^1, \psi_{L1}^1, \psi_{L1}^1$, where the superscript $1$ denotes one-node wave-functions. To find the one-node HH and LH wave-functions, we use the Gram-Schmidt process, first we define a set of orthogonal basis to span the excited states

$$u_n(z) = \sin \left[ \frac{n\pi}{d} \left( z + \frac{L}{2} \right) \right] \times \exp \left[ -\beta^n \left( \frac{z}{d} + \frac{1}{2} \right) \right], \quad |z| < \frac{L}{2}, \quad (19)$$
where $\beta^n$ denotes the variational parameter for the corresponding energy levels, in our calculations, we only consider $\beta^1_k$ and $\beta^1_l$, which are the one-node HH and LH states variational parameters. Variational parameters are solved by minimizing the expectation of the Hamiltonian under $\psi_H^1, \psi_H^0$. Then the one-node HH and LH states can be constructed as $\psi^1 = u_2 - \langle u_2 | \phi^1 \rangle \phi^1$. We project the Luttinger Hamiltonian onto basis $\psi^0_{HH1}, \psi^0_{HH1}, \psi^0_{LH1}, \psi^0_{LH1}, \psi^1_{HH1}, \psi^1_{HH1}, \psi^1_{LH1}, \psi^1_{LH1}$:

$$H_{LK} = \begin{pmatrix}
P + Q & 0 & L & M & P + Q & 0 & L & M \\
0 & P + Q & M^\dagger & -L^\dagger & 0 & P + Q & M^\dagger & -L^\dagger \\
L^\dagger & M & P - Q & 0 & L^\dagger & M & P - Q & 0 \\
M^\dagger & -L & 0 & P - Q & M^\dagger & -L & 0 & P - Q \\
P + Q & 0 & L & M & P + Q & 0 & L & M \\
0 & P + Q & M^\dagger & -L^\dagger & 0 & P + Q & M^\dagger & -L^\dagger \\
L^\dagger & M & P - Q & 0 & L^\dagger & M & P - Q & 0 \\
M^\dagger & -L & 0 & P - Q & M^\dagger & -L & 0 & P - Q \\
\end{pmatrix}. \quad (20)$$

To obtain an effective Hamiltonian between the two zero-node HH states, we apply the Schrieffer-Wolff transformation on the 8-band Luttinger Hamiltonian. The off-diagonal element $H_{1,2}$ will now read:

$$H_{1,2} = H_{1,2} = \frac{1}{2} \sum_{k=1}^{8} H_{1,k} H_{k,2} \left[ \frac{1}{E_1 - E_k} - \frac{1}{E_k - E_2} \right].$$

$$= \frac{3}{2} \frac{\mu_0^2 \gamma_3}{E_1 - E_3} \left[ \langle \psi_H | k_z | \psi_L \rangle \langle \psi_H | \psi_L \rangle - \langle \psi_L | k_z | \psi_H \rangle \langle \psi_H | \psi_L \rangle \right] k_-^3$$

$$+ \frac{3}{2} \frac{\mu_0^2 \delta \gamma_3}{E_1 - E_3} \left[ \langle \psi_H | k_z | \psi_L \rangle \langle \psi_H | \psi_L \rangle - \langle \psi_L | k_z | \psi_H \rangle \langle \psi_H | \psi_L \rangle \right] k_- k_+^2. \quad (21)$$

Now we can write down the expression for $\alpha_3$-Rashba term and $\alpha_2$-Rashba term:

$$\alpha_2 = \frac{3}{2} \frac{\mu_0^2 \gamma_3}{E_1 - E_3} \left[ \langle \psi_H | k_z | \psi_L \rangle \langle \psi_H | \psi_L \rangle - \text{h.c.} \right], \quad \alpha_3 = \frac{3}{2} \frac{\mu_0^2 \delta \gamma_3}{E_1 - E_3} \left[ \langle \psi_H | k_z | \psi_L \rangle \langle \psi_H | \psi_L \rangle - \text{h.c.} \right]. \quad (22)$$
We plot the spin-orbit coupling coefficients with contributions from the zero-node states and the one-node states in Figure 7. We should note that including the one-node wave-functions will not change the sweet spot for a give quantum well width because the energy splitting between the one-node state and the zero-node state is very large.

Figure 7: (a,b) Comparison of the Rashba SOC coefficients calculated with the first excited states vs. only the zero-node states. In (a) we plot the $\alpha_3$-Rashba term, In (b) we plot the $\alpha_2$-Rashba term.

EDSR Rabi time

In this section, we derive the EDSR Rabi time. The in-plane wave-functions are the Fock-Darwin wave-functions

$$\phi_0 = \frac{1}{a_0\sqrt{\pi}} \exp \left[ -\frac{(x^2 + y^2)}{2a_0^2} \right], \quad \phi_{\pm 1} = \frac{1}{a_0\sqrt{\pi}} (x \pm iy) \exp \left[ -\frac{(x^2 + y^2)}{2a_0^2} \right]. \quad (23)$$

The EDSR Hamiltonian include two terms: the spin-orbit coupling constants and the in-plane electric field. If we project the EDSR Hamiltonian onto the Fock-Darwin states, the only relevant matrix elements are

\[
H_{1,6}^{\text{EDSR}} = 2i\alpha_3 \langle \phi_0 | k_+ k_- | \phi_{-1} \rangle = 4\alpha_3 \frac{a_0^3}{a_0^3} + 2\alpha_3 Be \frac{a_0^3}{a_0\hbar},
\]

\[
H_{2,3}^{\text{EDSR}} = -2i\alpha_3 \langle \phi_0 | k_- k_+ | \phi_{+1} \rangle = -\frac{4\alpha_3}{a_0^3} \frac{a_0^3}{a_0\hbar} + 2\alpha_3 Be \frac{a_0^3}{a_0\hbar},
\]

and

\[
H_{1,3}^{\text{EDSR}} = H_{24}^{\text{EDSR}} = H_{1,5}^{\text{EDSR}} = H_{26}^{\text{EDSR}} = \frac{1}{2}a_0 e E_{AC}. \quad (25)
\]
Where $a_0$ is the size of the quantum dot and $\alpha_3$ is the spin-orbit coupling constants defined in section 2. Considering all contributions, perform the Schrieffer-Wolff transformation, we can write down our effective quantum dot Hamiltonian $H_{1,2}$, therefore we can find its complex conjugate to find $H_{2,1}$. The effective $2 \times 2$ EDSR Hamiltonian can be written as

$$H_{\text{EDSR}} = \frac{1}{2}g_0\mu_B B\sigma_z + \hbar\Omega_{\text{EDSR}}\sigma_x.$$  \hfill (26)

If we expand the $\Omega_{\text{EDSR}}$ onto first order and convert it to frequency, we have

$$f_{\text{EDSR}} = \frac{a_0^2\alpha_3 BeE_{\text{AC}}g_0\mu_Bm_p^2}{\pi\hbar^5},$$ \hfill (27)

where $E_{\text{AC}} = 10^3$ V/m is the in-plane electric field, $g_0$ is the bulk $g$-factor for germanium, $m_p = m_0/(\gamma_1 + \gamma_2) = 0.057 m_0$ is the in-plane effective mass.

**Relaxation time**

In this section, we evaluate the relaxation time. We first calculate the first-order contribution to the relaxation time at a dilution refrigerator temperature $T = 100$ mK. Then, we discussed the relaxation due to the second-order processes. Finally, we calculate the relaxation time when $T = 4$ K, and a plot is given to illustrate the relation between the relaxation rate and the temperature.

To calculate the first-order contributions to the relaxation time, we need to find the transition matrix elements describing the emission of a phonon. We use the Fermi’s golden
rule to find the transition rate. The hole-phonon interaction Hamiltonian can be written as

$$H_{H-P} = \begin{bmatrix}
P_p + Q_p & 0 & L_p & -M_p \\
0 & P_p + Q_p & M_p^\dagger & L_p^\dagger \\
L_p^\dagger & M_p & P_p - Q_p & 0 \\
-M_p^\dagger & L_p & 0 & P_p - Q_p
\end{bmatrix}.$$  \hspace{0.5cm} (28)

We consider the three polarization directions:

$$\hat{e}_l = \begin{bmatrix}
\cos(\theta) \sin(\phi) \\
\sin(\theta) \sin(\phi) \\
\cos(\phi)
\end{bmatrix}, \quad \hat{e}_t = \begin{bmatrix}
\sin(\theta) \\
\cos(\theta) \\
0
\end{bmatrix}, \quad \hat{e}_w = \begin{bmatrix}
\cos(\theta) \cos(\phi) \\
\sin(\theta) \cos(\phi) \\
-\sin(\phi)
\end{bmatrix}.$$  \hspace{0.5cm} (29)

For each polarization direction, the phonon matrix elements will read differently. For example, in the longitudinal direction, we have:

$$P_p^l = ia \sqrt{\frac{\hbar}{N V_c \rho \omega_l}} \frac{q}{\sqrt{q}} \left( a_{q_s}^\dagger + a_{q_s} \right) \exp \left( i q_s \cdot \mathbf{r} \right)$$

$$Q_p^l = \frac{ib}{2} \sqrt{\frac{\hbar}{N V_c \rho \omega_l}} \frac{1}{\sqrt{q}} \left( q - \frac{3q_z^2}{q} \right) \left( a_{q_s}^\dagger + a_{q_s} \right) \exp \left( i q_s \cdot \mathbf{r} \right)$$

$$L_p^l = -i \sqrt{\frac{\hbar}{N V_c \rho \omega_l}} \frac{1}{\sqrt{q}} \left( \frac{\sqrt{3}}{2} \frac{q_z^2 - q_y^2}{q} - \frac{i d q_x q_y}{q} \right) \left( a_{q_s}^\dagger + a_{q_s} \right) \exp \left( i q_s \cdot \mathbf{r} \right)$$

$$M_p^l = id \sqrt{\frac{\hbar}{N V_c \rho \omega_l}} \frac{1}{\sqrt{q}} \frac{q_z (q_x - i q_y)}{q} \left( a_{q_s}^\dagger + a_{q_s} \right) \exp \left( i q_s \cdot \mathbf{r} \right).$$  \hspace{0.5cm} (30)

where $a, b, d$ are the deformation constants, $N$ is the number of the unit cells, $V_c$ is the unit crystal volume, $\rho$ is the density of germanium. We can then project the phonon Hamiltonian onto our in-plane quantum dot Hamiltonian. To calculate the relaxation rate we use the Fermi’s golden rule:

$$W = \frac{2\pi}{\hbar} \int q \|H_{12}\|^2 \delta \left( E_i - E_f - \hbar \omega \right) \frac{L^3}{8\pi^3 q^2} \sin \theta dq d\theta d\phi.$$  \hspace{0.5cm} (31)
The delta function indicates that the relaxation is completed by emitting a phonon, \( \delta (g_0 \mu_B B - \hbar q v) = \delta (\hbar v (q - g_0 \mu_B B / \hbar v)) = \frac{1}{\hbar v} \delta (q - g_0 \mu_B B / \hbar v) \). To find the effective transition matrix elements, we use the Schrieffer-Wolff transformation and substitute the effective matrix element into the transition rate:

\[
W_L^L = \frac{2\pi}{\hbar} \int \frac{1}{4} A^2 q \left[ \frac{a + b}{2} (1 - 3 \cos^2 \theta) \right]^2 \frac{1}{4} \alpha_0^2 \theta^2 \sin^2 \theta e^{-\frac{1}{2} \alpha_0^2 \theta^2 \sin^2 \theta} \times \left\{ \left( -\frac{4 \alpha_3}{a_0^3} + \frac{2 \alpha_3 B \epsilon}{a_0 h} \right) \left( \frac{1}{E_1 - E_3} - \frac{1}{E_3 - E_2} \right) + \left( \frac{4 \alpha_3}{a_0^3} + \frac{2 \alpha_3 B \epsilon}{a_0 h} \right) \left( \frac{1}{E_1 - E_6} - \frac{1}{E_6 - E_2} \right) \right\}^2 \frac{1}{\hbar v} \delta \left( q - \frac{g_0 \mu_B B}{\hbar v} \right) \frac{L^3}{8\pi^3} q^2 \sin \theta dq d\theta d\phi,
\]

where \( E_1, E_2, E_3, E_6 \) are the eigen-energies of the unperturbed in-plane QD Hamiltonian, i.e., the Fock-Darwin Hamiltonian. We then repeat this process for other two polarization directions and the \( \alpha_2 \)-Rashba terms.

Now we consider the second-order contribution to the relaxation rate, which includes virtual emission and absorption between the heavy-hole state and the light-hole state. We first start from the time-dependent perturbation

\[
c_n^{(2)}(t) = \left( \frac{i}{\hbar} \right)^2 \sum_m \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' e^{i\omega_m t'} V_{nm}(t') e^{i\omega_m t''} V_{mi}(t''),
\]

where \( V_{n,m} \) are the phonon matrix elements in Eq. [30] but we should also consider the time-dependence:

\[
V = C(q)e^{-i\omega t} + C^\dagger(q)e^{i\omega t},
\]

where \( C(q) \) are the time-independent parts. We then substitute the occupation numbers and phonon-hole interactions back to the transition rate, integrating over the momentum space. For example, we consider the longitudinal polarization and denote two heavy-hole energy levels by \( E_1 \) and \( E_2 \), as well as two virtual energy levels \( E_3 \) and \( E_4 \). One possible secondary process can be caused by a heavy state in \( E_1 \) undergoing virtual emissions and adsorptions which then goes back to its original energy level \( E_1 \). The transition probability
for this process can be written as

\[
\int_{q'} C^\dagger (q') \tilde{C} (q') \frac{\delta (\omega_{21} - \omega - \omega')}{(\omega_{31} + \omega')^2} \omega^2 \sin \theta' d\phi' d\theta' d\omega'
= \int_{q'} A^2 C^\dagger C \langle \phi_1 | e^{-i\mathbf{q}\cdot\mathbf{r}} | \phi_3 \rangle \langle n' | \hat{a}^\dagger | n \rangle \frac{\delta (\omega_{21} - \omega + \omega')}{(\omega_{31} + \omega')^2} \omega^2 \sin \theta' d\phi' d\theta' d\omega'.
\]

(35)

Considering the longitudinal direction Eq. 35 becomes

\[
C^\dagger \tilde{C} = \left( \frac{3}{4} b^2 q \sin^4 \theta \cos^2 2\phi + \frac{d^2}{4} q \sin^4 \theta \sin^2 2\phi \right).
\]

(36)

Repeating the calculations for other polarizations, we evaluate the second-order contribution to the relaxation time. As expected, the second order contribution to the relaxation time is small.

To test the quality of the qubit, we also calculate the contribution of the first-order process to the relaxation rate at \( T = 4 \, \text{K} \). When the temperature is getting larger, both the absorption and emission process are getting very important. The logic of calculating the relaxation time at higher temperature is similar to the calculations at \( T = 100 \, \text{mK} \). We have to evaluate the occupation numbers for both absorption and emission process.

\[
\langle n' | \hat{a}^\dagger | n \rangle = \sqrt{N_q + 1} = \sqrt{\exp \left( \frac{\hbar \omega}{K_b T} \right) - 1}, \quad \langle n' | \hat{a}^\dagger | n \rangle = \sqrt{N_q} = \sqrt{\frac{1}{\exp \left( \frac{\hbar \omega}{K_b T} \right) - 1}}.
\]

(37)

As an example, we consider the longitudinal polarization, the projection of the phonon-hole Hamiltonian onto the ground QD state and the first excited QD state should read

\[
H_{1,3} = i \sqrt{\frac{\hbar}{NV \epsilon \rho}} \sqrt{\frac{q}{\bar{q}}} \left[ a + \frac{b}{2} (1 - 3 \cos^2 \theta) \right] \exp \left( \frac{\hbar \omega}{K_b T} \right) \frac{1}{\exp \left( \frac{\hbar \omega}{K_b T} \right) - 1} e^{i\mathbf{q}\cdot\mathbf{r}} + \exp \left( \frac{\hbar \omega}{K_b T} \right) \frac{1}{\exp \left( \frac{\hbar \omega}{K_b T} \right) - 1} e^{i\mathbf{q}\cdot\mathbf{r}} \phi_{1,3}.
\]

(38)

Similarly, we can obtain the matrix elements for other directions. We then substitute the
new matrix elements into the Fermi’s golden rule. We can then evaluate the new relaxation
time. Here we present a plot of the temperature dependence of the relaxation rate Figure 8.
The relaxation rate increases with temperature, because a higher temperature will increase
the possibility of adsorptions of a phonon which weaken the coherence of the system.

![Figure 8: Temperature dependence of the relaxation time when $d = 15$ nm. In this plot, we used $a_0 = 8$ nm, $B = 0.1$ T.](image)

## In-plane electric field

In this section, we demonstrate why the in-plane electric cannot have contribution to the qubit Zeeman splitting. First, we should note that the qubit Zeeman splitting is the difference between two effective QD energy levels. Therefore, if the in-plane electric field has the same contribution to the effective QD energy levels, the qubit Zeeman splitting will not be changed.

Now we consider an in-plane electric field (for example, the in-plane electric field $E_\parallel = E_{AC}$ in the EDSR Hamiltonian) $E_\parallel$ leading to a term $eE_\parallel \cdot r_\parallel$, the subscript is used to denote the in-plane field. The projections of this term onto the quantum dot levels (consider both $\alpha_2$-Rashba term and $\alpha_3$-Rashba term, there are 20 quantum dot levels) are:

$$\langle \phi | qE_\parallel \cdot r_\parallel | \phi \rangle$$  (39)

where $\phi$ will be Fock-Darwin states. Applying the Schrieffer-Wolff transformation on the
$20 \times 20$ QD Hamiltonian, we obtain the corrections on the diagonal elements due to the in-plane electric field:

$$H_{1,1} = \frac{H_{1,3}H_{3,1}}{E_1 - E_3} + \frac{H_{1,5}H_{5,1}}{E_1 - E_5} + \frac{H_{1,6}H_{6,1}}{E_1 - E_6}, \quad H_{2,2} = \frac{H_{2,3}H_{3,2}}{E_2 - E_3} + \frac{H_{2,4}H_{4,2}}{E_2 - E_4} + \frac{H_{2,6}H_{6,2}}{E_2 - E_6}. \quad (40)$$

where

$$H_{1,3} = H_{2,4} = H_{1,5} = H_{2,6} = \frac{1}{2}eE_ACa_0. \quad (41)$$

and

$$E_1 - E_3 = E_2 - E_4 \quad E_1 - E_5 = E_2 - E_6. \quad (42)$$

Therefore, when we evaluate the qubit Zeeman splitting, the corrections due to the in-plane electric field will be cancelled out. So, the in-plane electric field does not contribute to the fluctuations in the qubit Zeeman splitting, therefore, it leads to no dephasing. However, the dephasing can be from higher-order terms of the in-plane electric field.

**Distortion of the parabolic confinement**

In experiments, it is hard to establish a perfect parabolic confinement described in Eq. To describe the distortion due to the parabolic confinements, we study the following three models as new perturbation:

$$V_1' = \lambda|x|^3, \quad V_2' = \frac{1}{2}(\delta\omega)^2x^2, \quad V_3' = \xi|x|. \quad (43)$$

For each of the distortion model, we set the perturbation parameters $\lambda, \delta, \xi$ to satisfy $2V'/(m_\pi\omega_0^2a_0^2) = 0.1$. If we consider regions close to the quantum dot, the energy correction due to the distortion $\langle \phi|V'|\phi \rangle$ will be small and we can treat it as off-diagonal terms and use the Schrieffer-Wolff transformation to evaluate the correction to the quantum dot energy levels. In this regime, the corrections to the first two effective quantum dot levels are
the same, therefore will be no change in the qubit Zeeman splitting.

However, if we consider a larger region, the energy correction due to the distortion $\langle \phi | V' | \phi \rangle$ will be comparable to the confinement energy $\hbar \omega_0$, i.e, the new quantum dot energy levels will read $E = E_0 + E_z + \langle \phi_i | V' | \phi_i \rangle$. For example, when the quantum well width is $d = 15 \text{ nm}$, $F = 1.3 \times 10^7 \text{ V/m}$. The first model (cubic term) will change the qubit Zeeman splitting by 1.5%, and the second model (quadratic term) will change the qubit Zeeman splitting by 0.46% and the third model (linear term) will change the qubit Zeeman splitting by 0.24%.

**A different quantum well width**

In this section, we discussed the effect on the sweet spot due to the quantum well width. In our early calculations, we used the quantum well width $d = 15 \text{ nm}$. Here we produce some plots for the quantum well width $d = 9 \text{ nm}$, but the strain terms will the same as before.

In Figure 9a, due to a smaller quantum well width, the variational parameters will be smaller. As we decrease the quantum well width, the energy splitting Figure 9b is getting larger.

![Figure 9: (a) Variational parameters as a function of the gate electric field. (b) Energy splitting as a function of the gate electric field. In both plots, we use the width of the quantum well $d = 9 \text{ nm}$, the size of the quantum dot $a_0 = 7 \text{ nm}$.](image)

As in Figure 10, the decrease of the quantum well width will shift the sweet spot to a higher gate electric field, which suggests that for each different quantum well width the sweet
spot, the qubit Larmor frequency will be changed significantly.

Figure 10: (a) Qubit Zeeman splitting, when the gate electric field is turned off, the qubit Zeeman splitting will be $g_0\mu_B B = 120 \mu eV$. As the gate electric field increase, the Rashba spin-orbit coupling will change the quantum dot energy levels, leading to a sweet spot. (b) A comparison of the magnitude of $\alpha_2$- and $\alpha_3$- Rashba terms that leads to the change in the qubit Zeeman splitting. In both plots we used the width of the quantum well $d = 9 \text{ nm}$, the size of the quantum dot $a_0 = 7 \text{ nm}$. We can notice that the sweet spot move to a higher gate field compared with the configuration when $d = 15 \text{ nm}$. Now the sweet spot is at about $45 \text{ MV/m}$, and the confinement energy $\hbar \omega_0 = 27 \text{ meV}$.

The change in the qubit Zeeman splitting as a function of the gate electric field follows the change in the Rashba spin-orbit coupling coefficients with the gate electric field as shown in Figure 11.

Figure 11: Spin-orbit coupling coefficients, the width of the quantum well $d = 9 \text{ nm}$, the size of the quantum dot $a_0 = 7 \text{ nm}$.

We also report the change in relaxation time and EDSR Rabi time and the allowable number of single qubit operations in Figure 12. As we can see that the relaxation time is getting larger if we decrease the size of our quantum dot. However, the EDSR Rabi term is getting larger due to smaller spin-orbit couplings and smaller dot size.
Figure 12: (a) A comparison of the relaxation time and the EDSR Rabi time, the sweet spot appears at the same gate electric field. (b) The allowable number of single qubit operations in unit relaxation time. In all these plots, we use the width of the quantum well $d = 9\,\text{nm}$, the size of the quantum dot $a_0 = 7\,\text{nm}$, $B = 0.1\,\text{T}$, $\omega_c = 3.05 \times 10^{11}\,\text{Hz}$, $\omega_l = 4.1 \times 10^{13}\,\text{Hz}$, $\rho = 5.33 \times 10^{3}\,\text{kg/m}^3$, The phonon propagation speed along the transverse direction is $v_t = 3.57 \times 10^{3}\,\text{m/s}$, the phonon propagation speed along the longitudinal direction is $v_l = 4.85 \times 10^{3}\,\text{m/s}$.

For the dephasing time of the system, we expect a larger dephasing time for a smaller confinement width (given that that dot size does not vary to much). This is because the relative fluctuations in $\alpha_2$- and $\alpha_3$-Rashba coefficients will be getting smaller.

Figure 13: (a) Dephasing time in motional narrowing regime. (b) Dephasing time in quasi-static limit. In all these plots, we use the width of the quantum well $d = 9\,\text{nm}$, the size of the quantum dot $a_0 = 7\,\text{nm}$. 