Derivation of Dirac, Klein-Gordon, Schrödinger, diffusion and quantum heat transport equations from a universal quantum wave equation

A. I. Arbab

Department of Physics, Faculty of Science, University of Khartoum - P.O. Box 321, Khartoum 11115, Sudan

received 3 October 2010; accepted in final form 28 October 2010 published online 16 November 2010

PACS 03.65.-w – Quantum mechanics
PACS 31.30.J- – Relativistic and quantum electrodynamic (QED) effects in atoms, molecules, and ions
PACS 03.65.Ge – Solutions of wave equations: bound states

Abstract – A universal quantum wave equation that yields Dirac, Klein-Gordon, Schrödinger and quantum heat equations is derived. These equations are related by complex transformation of space, time and mass. The new symmetry exhibited by these equations is investigated. The universal quantum equation yields the Dirac equation in two ways: firstly by replacing the particle mass $m_0$ by $im_0$, and secondly by changing space and time coordinates by $it$ and $i\vec{r}$, respectively.

Introduction. – Adler and others have developed the quaternionic quantum mechanics [1]. However, we have recently adopted a different view and came up with a generalized Klein-Gordon equation [2]. Moreover, the Dirac equation can be obtained from this equation by either replacing the time and space coordinates by their imaginary ones, or by replacing the mass of the particle $m_0$ by the imaginary mass $im_0$. Moreover, this equation exhibits the invariance of the transformation of space-time and mass by their imaginary ones. The universal quantum wave equation we have derived here is the relativistic quantum heat transport equation of the equation proposed recently by Kozlowski and Marciak-Kozlowska [3]. It is also invariant under the combined space-time and mass imaginary transformations, viz., $t \rightarrow it$, $\vec{r} \rightarrow i\vec{r}$, and $m_0 \rightarrow -im_0$ [4]. The Dirac, Klein-Gordon and Schrödinger equations are found to be invariant under this complex transformations. t’Hooft and Nobbenhuis have recently argued that the invariance under space-time complex transformation results in a vacuum state with strictly vanishing vacuum energy [5]. However, the authors in ref. [5] have left the mass of the particle unchanged (massless), a fact that affected the integrity and the invariance of the other physical quantity under this transformation. Inasmuch as energy and mass are intimately related, so that a transformation of the former would transform the latter. Allowing these transformations will retain the whole physics of the system. This property has been recently investigated [4].

Quaternionic quantum mechanics: Dirac-like equation. – Consider a particle described by the quaternion wave function $\Psi = (\frac{1}{i} \psi_0, \vec{\psi})$. This is equivalent to the spinor representation of ordinary quantum mechanics that we have recently proposed [2]. The evolution of this quaternion wave function is defined by the three equations [2]

$$\vec{\nabla} \cdot \vec{\psi} - \frac{1}{c^2} \frac{\partial \psi_0}{\partial t} - \frac{m_0}{\hbar} \psi_0 = 0,$$  \hspace{1cm} (1)

$$\vec{\nabla} \psi_0 - \frac{\partial \vec{\psi}}{\partial t} - \frac{m_0 c^2}{\hbar} \vec{\psi} = 0,$$  \hspace{1cm} (2)

and

$$\vec{\nabla} \times \vec{\psi} = 0.$$  \hspace{1cm} (3)

Equations (1)–(3) yield the two wave equations

$$\frac{1}{c^2} \frac{\partial^2 \vec{\psi}}{\partial t^2} - \nabla^2 \vec{\psi} + 2 \left( \frac{m_0}{\hbar} \right) \frac{\partial \vec{\psi}}{\partial t} + \left( \frac{m_0 c}{\hbar} \right)^2 \vec{\psi} = 0,$$  \hspace{1cm} (4)

and

$$\frac{1}{c^2} \frac{\partial^2 \psi_0}{\partial t^2} - \nabla^2 \psi_0 + 2 \left( \frac{m_0}{\hbar} \right) \frac{\partial \psi_0}{\partial t} + \left( \frac{m_0 c}{\hbar} \right)^2 \psi_0 = 0.$$  \hspace{1cm} (5)

Equations (4) and (5) have been obtained using the commutator brackets proposed recently by Arbab and

\[a^\text{E-mail: aiarbab@uofk.edu}\]
The electric current and charge densities, \( \mathbf{J} \), so that the vector and scalar fields, \( \mathbf{E} \) and \( \mathbf{B} \), and inductance (\( L \)) transform as \( q \rightarrow q, \ V \rightarrow -i V, \ I \rightarrow -i I, \ C \rightarrow i C, \ L \rightarrow i L. \)

Moreover, owing to UCT, the permittivity, permeability, Planck's constant and Newton's constant transform as
\[
\varepsilon_0 \rightarrow \varepsilon_0, \quad \mu_0 \rightarrow \mu_0, \quad G \rightarrow -G, \quad h \rightarrow h. \tag{15}
\]

We, therefore, believe that the UCT is the symmetry of nature. Thus, the physical world must be invariant under the UCT. Recently, t'Hooft and Nobbenhuis have considered a complex invariance that results in a vacuum state with strictly vanishing vacuum energy \([5]\). They have argued that the vacuum state is the only unique state that is invariant under this transformation and all other states break this symmetry.

In the context of gravitation, the invariance of Einstein field equations under UCT requires that \([4,8]\)
\[
R_{\mu\nu} \rightarrow - R_{\mu\nu}, \quad \rho_m \rightarrow \rho_m, \quad G \rightarrow -G, \quad \tag{16}
\]
where \( \rho_m \) is the matter density. The force transformation in eq. (11) implies that in the complex world forces are inverted. However, since gravity is always attractive in the real world, a negative \( G \) guarantees that the gravitational force between two masses in the complex world would still be attractive.

The quantum heat transport equation. – More recently, Kozłowski and Marcik-Kozłowska have derived a new quantum heat transport equation \([3]\). According to their equation the temperature, \( T \), satisfies the wave equation
\[
\frac{1}{v^2} \frac{\partial^2 T}{\partial T^2} - \nabla^2 T + \left( \frac{m}{\hbar} \right) \frac{\partial T}{\partial t} + \frac{2Vm}{\hbar^2} T = 0, \tag{17}
\]
where \( m \) is the mass of the heat carriers and \( V \) is the nonthermal potential. They found that for an undistorted thermal wave, \( i.e. \), a wave which preserves the shape in the field of the potential \( V \), the relation
\[
V \tau = \frac{\hbar}{8}, \quad \tau = \frac{\hbar}{2mv^2}. \quad \tag{18}
\]
holds. If we compare eq. (17) with eq. (4), we obtain
\[
m = 2m_0, \quad V = \frac{1}{4} m_0 c^2. \quad \tag{19}
\]
Hence, our equation, eq. (4), represents the quantum relativistic version of eq. (17). Equation (4) can also represent the equation of an undistorted particle (wave) that is attenuated as it propagates in space-time due to its mass (inertia).

Ordinary quantum mechanics: Dirac equation. – Dirac’s equation can be written in the form \([9]\)
\[
\frac{1}{c} \frac{\partial \psi}{\partial t} + \vec{\alpha} \cdot \vec{\nabla} \psi + \frac{im_0 c}{\hbar} \beta \psi = 0, \tag{20}
\]
where \( \beta = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \alpha = \left( \begin{array}{cc} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{array} \right), \quad \alpha^2 = \beta^2 = 1 \) and \( \vec{\sigma} \) are the Pauli matrices.
Now, we define
\[ \psi = \bar{\alpha} \psi, \quad \psi_0 = -c \psi, \] (21)
and let
\[ m_0 \rightarrow i m_0 \beta. \] (22)

Equation (22) implies that
\[ m_0 \rightarrow \pm i m_0. \] (23)

Equation (23) is compatible with eq. (8), the UCT. Substituting eqs. (21) and (22) into eq. (1) yields the Dirac equation. Moreover, substituting eq. (22) into eq. (5) yields
\[ \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + 2 \left( \frac{m_0 i}{h} \right) \frac{\partial \bar{\psi}}{\partial t} - \left( \frac{m_0 c}{h} \right)^2 \psi = 0. \] (24)

Using eq. (20), eq. (24) becomes
\[ \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + 2 \left( \frac{m_0 i}{h} \right) \beta c \bar{\alpha} \cdot \nabla \psi + \left( \frac{m_0 c}{h} \right)^2 \psi = 0. \] (25)

This equation represents a new form of the Dirac equation which has not been known before.

We remark here that under UCT, eq. (5) yields the equation (24)
\[ \frac{1}{c^2} \frac{\partial^2 \bar{\psi}}{\partial t^2} - \nabla^2 \bar{\psi} + 2 \left( \frac{m_0 i}{h} \right) \beta c \alpha \cdot \nabla \psi + \left( \frac{m_0 c}{h} \right)^2 \bar{\psi} = 0. \] (26)
and
\[ \partial t \frac{\partial}{\partial t} \bar{\psi} \pm i m_0 c^2 \frac{\partial}{\partial t} \beta \bar{\psi} = 0. \] (27)

Equations (1) and (2) are invariant under the UCT. It is also invariant under the following transformations:
\[ \bar{\psi} \rightarrow \bar{\psi} + \frac{i m_0 c}{h} \beta \bar{\alpha}, \] (28)
and
\[ \bar{\psi} \rightarrow \bar{\psi} \pm i m_0 c^2 \frac{\partial}{\partial t} \beta \bar{\alpha}. \] (29)

Equations (28) and (29) can be obtained from eq. (4) by replacing the mass \( m_0 \) by \( \pm i m_0 \), respectively. They can be compared with the Klein-Gordon equation of spin-0 particles, i.e.,
\[ \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + \left( \frac{m_0 c}{h} \right)^2 \psi = 0. \] (30)

If the dissipation term in eq. (4) can be neglected, the resulting equation is the Klein-Gordon equation, eq. (30). Writing eq. (4) as
\[ \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = 0, \] (31)
where
\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + m_0 c^2 \] (32)
reveals that the resulting equation represents an equation of zero mass. Hence, eq. (4) tells us that the particle moves as if it was massless and then moves as a free particle as governed by Klein-Gordon equation. Thus, the particle is in a continuous state of creation and annihilation. One can estimate the characteristic time during which it moves as a massless particle as \( t_c = \frac{m_0}{m_{net}} \). For a pion \( (\pi^\pm) \), one finds \( t_c = 4.69 \times 10^{-19} \) s. With attosecond spectroscopy, one can detect this motion. Therefore, a pion moves as a massless particle before this time. Generally, heavier particles free themselves after a shorter time. Such a time can be measured experimentally from the motion of spin-0 particles and compared with the theoretical findings.

The cause of this initial resistance is the inertia of the particle that the wave describes. However, in de Broglie theory the motion of the particle is not determined. But de Broglie associates a wave with a particle motion. Can we say that the wave has an inertial property?

It is interesting to notice that the Dirac equation is invariant under UCT. It is also invariant under the following transformations:
\[ \bar{\psi} \rightarrow \bar{\psi} + \frac{i m_0 c}{h} \beta \bar{\alpha}, \] (34)
and
\[ \bar{\psi} \rightarrow \bar{\psi} \pm i m_0 c^2 \frac{\partial}{\partial t} \beta \bar{\alpha}. \] (35)

Equations (1) and (2) are invariant under the above space and time transformations, provided that, \( \psi_0 = c \bar{\alpha} \cdot \bar{\psi} \). It is interesting to remark that eqs. (24) and (25) are derived by using the commutator brackets developed recently by Arbab and Yassein [6]. Equation (24) can be split into positive- and negative-energy equations, respectively, as
\[ \frac{1}{c^2} \frac{\partial^2 \chi}{\partial t^2} - \nabla^2 \chi + 2 \left( \frac{m_0 i}{h} \right) \frac{\partial \bar{\chi}}{\partial t} - \left( \frac{m_0 c}{h} \right)^2 \chi = 0, \] (36)
and
\[ \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi - 2 \left( \frac{m_0 i}{h} \right) \frac{\partial \varphi}{\partial t} - \left( \frac{m_0 c}{h} \right)^2 \varphi = 0. \] (37)

Equations (36) and (37) can be obtained from eq. (4) by replacing the mass \( m_0 \) by \( \pm i m_0 \), respectively. They can be compared with the Klein-Gordon equation of spin-0 particles, i.e.,
\[ \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + \left( \frac{m_0 c}{h} \right)^2 \psi = 0. \] (38)

It is interesting to know that Dirac, Klein-Gordon, Schrödinger, diffusion etc.
in eq. (4) yields the Dirac positive-energy equation, \textit{i.e.}, eq. (28). This implies that eq. (4) is relativistically invariant. Thus, eq. (4) is equivalent to Dirac equation in imaginary space and time. The coordinates transformation in eq. (37) is equivalent to the rotation of the space and time coordinates by an angle of 90 degrees. It is also equivalent, in the relativity theory, to replacing the squared interval $ds^2$ by $-ds^2$.

**Schrödinger equation.** – Now let us apply the transformation in eq. (37) to eq. (1) to obtain

$$\vec{\nabla} \cdot \vec{\psi} - \frac{1}{c^2} \frac{\partial \psi_0}{\partial t} - \frac{m_0 i}{\hbar} \psi_0 = 0.$$  \hspace{1cm} (38)

Equation (3) can be satisfied by choosing

$$\vec{\psi} = a \vec{\nabla} \psi_0, \quad a = \text{const.}$$  \hspace{1cm} (39)

Substituting eq. (39) into eq. (38) yields

$$i \hbar \frac{\partial \psi_0}{\partial t} = -\frac{\hbar^2}{2m_0} \nabla^2 \psi_0 + m_0 c^2 \psi_0,$$  \hspace{1cm} (40)

where $a = \frac{i \hbar}{2m_0 c^2}$. Equation (40) is the Schrödinger equation for a particle with potential equal to the rest mass energy of the particle, \textit{i.e.}, $V = m_0 c^2$. It can also be written as

$$H \psi_0 = E \psi_0, \quad \text{where} \quad E = E_K + m_0 c^2,$$  \hspace{1cm} (41)

where $E_K = \frac{p^2}{2m_0}$ is the nonrelativistic kinetic energy of the particle. Equation (40) is thus the nonrelativistic approximation of the relativistic equation

$$H \psi = E \psi,$$  \hspace{1cm} (42)

where

$$E = \sqrt{p^2 c^2 + m_0^2 c^4}.$$  \hspace{1cm} (43)

Notice that unlike the Klein-Gordon equation which yields the Schrödinger equation in the nonrelativistic limit, the Dirac equation yields the Weyl equation instead. However, we have found here that the universal quantum wave equation yields the Schrödinger equation. Alternatively, we can say that eq. (40) represents the motion of a free particle. Such an equation should replace the ordinary Schrödinger equation for a free particle which does not involve the rest mass energy term.

**Concluding remarks.** – We have derived a new wave equation that retains Schrödinger, Dirac, Klein-Gordon, diffusion and quantum heat transport equations for certain transformations and approximations. This equation is relativistically invariant. We have found that the UCT is a true law of nature where all equations of motion must satisfy. The full physical meaning of the UCT will be our future endeavor.

***

I would like to thank H. M. Widatallah for enlightening and constructive comments.

**REFERENCES**

[1] Adler S. L., *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford University Press, New York) 1995; Finkelstein D., Josef M. J., Schiminovich S. and Speiser D., *J. Math. Phys.*, 3 (1962) 207.

[2] Arbab A. I., *The quaternionic quantum mechanics*, arXiv:1003.0075 (2010).

[3] Kozlowski M. and Marcia-Kozlawska J., arXiv:q-bio/0501031v2 (2005).

[4] Arbab A. I. and Widatallah H. M., *EPL*, 92 (2010) 23002.

[5] 't Hooft G. and Nobbenhuis S., *Class Quantum Grav.*, 23 (2006) 3819.

[6] Arbab A. I. and Yassein F. A., *A new formulation of quantum mechanics*, to be published in J. Mod. Phys. (2010).

[7] Coulson C. A., *Waves* (Logman Group Ltd.) 1977, p. 17.

[8] Weinberg S., *Gravitation and Cosmology* (John Wiley & Sons, Inc., New York) 1972.

[9] Bjorken J. D. and Drell S. D., *Relativistic Quantum Mechanics* (McGraw-Hill, New York) 1964.