The canonical structure of Wess-Zumino-Witten models

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ABSTRACT

The phase space of the Wess-Zumino-Witten model on a circle with target space a compact, connected, semisimple Lie group G is defined and the corresponding symplectic form is given. We present a careful derivation of the Poisson brackets of the Wess-Zumino-Witten model. We also study the canonical structure of the supersymmetric and the gauged Wess-Zumino-Witten models.
1. Introduction

The Wess-Zumino-Witten (WZW) models are two-dimensional sigma models with target space a group manifold and they constitute an important class of conformal field theories. In ref. [1] the left and right currents were used to study the quantisation of WZW models. The quantisation of the Poisson bracket (PB) algebra of the currents leads to two copies of a Kac-Moody algebra, one copy for each chiral sector. The Hilbert space of the theory is constructed from the representation theory of these Kac-Moody algebras.

Recently, in refs. [2-9] the quantisation of the WZW model was investigated in terms of the group elements of a loop group. The quantisation of the PB algebra of the WZW model in terms of these variables leads to a quantum group structure characteristic of these and other integrable two-dimensional models.

The calculation of the PBs of the group elements of the loop group is an interesting problem. One difficulty in calculating the PBs of the WZW models from their symplectic form arises from the fact that the phase space of the theory is an infinite dimensional space and the symplectic form is an operator. Thus the methods usually applied to finite dimensional systems are not directly applicable in this case. Several authors [7, 8, 9] attacked the problem of calculating the PBs of the WZW model using as a starting point the definition of the phase space of the theory as the space of solutions of the model [10], and as a symplectic form the one derived from the classical action of the system. In particular this approach was studied in detail by the authors of reference [8].

In this paper, we give a new derivation of the PBs for the WZW models. Our derivation follows from a careful study of the structure of the phase space of the WZW models, the use of sigma model methods and an application of methods in operator theory. The advantages of our derivation are that the PBs of the WZW model are constructed from first principles and in a geometric way. The Jacobi identities for these PBs are explicitly shown to be satisfied. These PBs are
consistent with the topology and geometry of the spacetime, which is taken to be a cylinder $S^1 \times \mathbb{R}$.

To achieve this, we introduce a model which is related to the WZW model but which has different left and right monodromies; we call this the LR model. The phase space $P_{LR}$ of the LR model is an enhanced version of the phase space $P$ of the WZW model. The PBs of the LR model are calculated and it is shown that this theory factorises into left and right moving sectors, i.e. the PB of any left sector variable with any right sector variable vanishes. We show how our results agree with those in the “chiral approach” of ref.[8]. The phase space of the WZW model is shown to be given by the phase space of the LR model with the addition of a first class constraint that enforces the equality of the left and right monodromies. We also study the canonical structure of the $(1, 1)$ supersymmetric and gauged WZW models.

In section two, we outline the WZW model, state its symplectic form and set up our notation. In section three, we derive the PBs of the LR model and describe the WZW model in terms of the phase space $P_{LR}$ of this model. In sections four and five, we examine the supersymmetric and gauged WZW models. Finally, in section six we present our conclusions and comment on the quantisation of the WZW model.

2. The WZW model

The WZW model is a sigma model with Wess-Zumino term whose target space is a group manifold $G$, where $G$ is a compact, connected, semisimple Lie group. The metric of the WZW model is a bi-invariant metric $h$ on $G$ and the Wess-Zumino term is a bi-invariant (closed) three form $H$. After an appropriate normalisation of the metric and Wess-Zumino terms the Lagrangian of the WZW model [1] is

$$L = -\frac{k}{16 \pi} \left( h_{ij} \eta^{\mu \nu} \partial_{\mu} \phi^i \partial_{\nu} \phi^j + b_{ij} \epsilon^{\mu \nu} \partial_{\mu} \phi^i \partial_{\nu} \phi^j \right),$$

where $\phi$ is a map from a cylinder $S^1 \times \mathbb{R}$ with co-ordinates $\{(x, t), 0 \leq x \leq l, -\infty < t < \infty \}$. 

$$1$$
\( t < \infty \) to the group \( G \), \( H = \frac{3}{2} db \), \( \eta \) is a metric on \( S^1 \times \mathbb{R} \), \( \epsilon \) is a two form on \( S^1 \times \mathbb{R} \) (\( \epsilon^{01} = 1 \)), and \( i, j = 1, \cdots, \dim G \). The metric \( h \) and the form \( H \) can be expressed as

\[
h_{ij} = L_i^a L_j^b \delta_{ab} = R_i^a R_j^b \delta_{ab} \tag{2}
\]

and

\[
H_{ijk} = \frac{1}{2} L_i^a L_j^b L_k^c f_{abc} = -\frac{1}{2} R_i^a R_j^b R_k^c f_{abc} \tag{3}
\]

respectively. \( L (R) \) is the left (right) frame on the group \( G \), \( f^{a bc} \) are the structure constants of \( \text{Lie}_G \) and \( a, b, c = 1, \cdots, \dim \text{Lie}_G \) are Lie algebra indices.

The metric and three form \( H \) can be rewritten as

\[
h_{ij} = \text{tr}(g^{-1} \partial_i g \ g^{-1} \partial_j g),
H_{ijk} = - \text{tr}(g^{-1} \partial_i g \ g^{-1} \partial_j g \ g^{-1} \partial_k g). \tag{4}
\]

For simplicity, by \( g \) we mean the group element \( g \in G \) in some unitary representation \( \pi \) of \( G \). \( \text{tr} \) is normalised such that \( \text{tr}(t_a t_b) = -\delta_{ab} \) where \( t \) \( ([t_a, t_b] = i f_{ab}^c t_c) \) is the representation of \( \text{Lie}_G \) that corresponds to the representation \( \pi \) of \( G \), and \( g \in G \), \( g^{-1} \partial_i g = i L_i^a t_a \) and \( \partial_i g \ g^{-1} = -i R_i^a t_a \).

Due to the relative normalisation of the two terms in the Lagrangian (1), the equations of motion are

\[
\partial_-(\partial_+ g \ g^{-1}) = 0, \quad \partial_+(g^{-1} \partial_- g) = 0 \tag{5}
\]

where \( x^\pm = t \pm x \). The action (1) is invariant under the semilocal transformations \( g \rightarrow l(x^-) g \ r^{-1}(x^+) \) and the corresponding currents are

\[
J_-(x^-) = \frac{ik}{4\pi} g^{-1} \partial_- g, \quad J_+(x^+) = -\frac{ik}{4\pi} \partial_+ g \ g^{-1}. \tag{6}
\]

The general solution of the equations of motion (5) can be expressed in the
form
\[ g(x, t) = U(x^+) M^{2\pi t} V(x^-). \] (7)

where \( g, U, V \) are periodic functions of \( x \) and \( M \) is on a maximal torus of the group \( G \). This solution is invariant under two sets of transformations, one given by the action of the Weyl group, and the other being

\[ U \to U h, \quad V \to h^{-1} V, \quad M \to M, \] (8)

where \( h \) is any element of a maximal torus of \( G \).

The phase space \( P \) of the WZW model is the space of functions \( \{U, V, M\} \) up to the relations given by eqn. (8)*.

3. The Poisson Brackets

The symplectic form \( \omega \) of the WZW model is the space integral of the time component of the symplectic current \( S \) where \( S^\mu = \delta \phi^i \wedge \delta (\partial L/\partial \partial_{\mu} \phi^i) \). The symplectic form in the basis \( \{U, V, M\} \) is

\[
\omega = -\frac{k}{8\pi} \left[ \int_0^l dx \, \text{tr} \left( U^{-1} \delta U \, \partial_x (U^{-1} \delta U) + \frac{2i}{l} (U^{-1} \delta U)^2 A \right.ight.

\[ - \delta V V^{-1} \partial_x (\delta V V^{-1}) - \frac{2i}{l} (\delta V V^{-1})^2 A \]

\[ + \frac{2}{l} \text{tr} \left( (U^{-1} \delta U)_0 \delta MM^{-1} + \frac{2}{l} (\delta V V^{-1})_0 \delta MM^{-1} \right) \right]. \] (9)

where \( M = \exp i A \) and \( (U^{-1} \delta U)_0 \) \( ((\delta V V^{-1})_0 \) is the constant mode of \( U^{-1} \delta U \) \( (\delta V V^{-1}) \). \( \omega \) is closed, and independent of the time \( t \) and the origin \( x_0 \) of the interval \( [x_0, x_0 + l] \) (we take \( t = x_0 = 0 \)). \( \omega \) is degenerate along the directions of the transformations (8).

* More precisely, the phase space of the WZW model is the space of orbits of the group action (8) on the bundle space of a fibre bundle with base space a maximal torus of the group \( G \) which parametrises the monodromy and fibre \( LG \times LG \) where \( LG \) is the loop group of \( G \).
To continue, we introduce a new model called the LR model. The phase space \( P_{LR} \) of this model is the space of functions \( \{U, V, A_L, A_R\} \), with symplectic form \( \Omega = \omega_L + \omega_R \) where

\[
\omega_L(U, A_L) = -\frac{k}{8\pi} \left[ \int_0^l dx \left\{ \text{tr}(U^{-1}\delta U \partial_x(U^{-1}\delta U)) + \frac{2i}{l}(U^{-1}\delta U)^2 A_L \right\} \right.
\]
\[
+ \frac{2}{l} \text{tr}(U^{-1}\delta U)_0 \delta M_L M^{-1}_L \right] \tag{10}
\]

and \( \omega_R(V, A_R) := -\omega_L(V^{-1}, A_R) \).

The forms \( \omega_L, \omega_R \) are closed provided the monodromies \( M_L, M_R \) are restricted to be in a maximal torus of \( G \) [7], i.e \( A_L, A_R \) are in a Cartan subalgebra of \( \text{LieG} \). The symplectic forms \( \Omega, \omega_L, \omega_R \) are not degenerate. The transformations (8) do not leave these symplectic forms invariant. In particular \( \Omega \) is not invariant because the monodromy of the left sector is different from that of the right sector. The Hamiltonian of the LR model is the one given by the sum of the Sugawara Hamiltonians of the left and right chiral sectors.

It is clear that the phase space \( P_{LR} \) and the symplectic form \( \Omega \) of the LR model factorise, i.e. \( P_{LR} = P_L \times P_R \) and \( \Omega = \omega_L \oplus \omega_R \). In the following we study the inverse of \( \omega_L \) only since the methods used for this inverse can be applied straightforwardly to calculate the inverse of \( \omega_R \) and hence the PBs of the theory.

The symplectic form \( \omega_L \) can be expressed in a representation independent way. Indeed, using \( U^{-1}\delta U = \sum_a t_a L^a_i \delta X^i \), we can rewrite it as

\[
\omega_L = -\frac{k}{8\pi} \left[ \int_0^l dx \left\{ \text{tr}(L^a_i \delta X^i (\delta_{ab} \partial_x - \frac{1}{l} A^c_L f_{cab} (L^b_j \delta X^j)) \right\} \right.
\]
\[
+ \frac{2}{l} \delta_{ab} (L^a_i \delta X^i)_0 \delta A^b_L \right]. \tag{11}
\]

To find the PBs of the left sector of the theory, we study the inverse of the
The operator $D$ acts on periodic functions with period $l$, i.e. it is an operator on a circle of circumference $l$. The construction of the inverse of $D$ is considerably simplified by observing that $A_L$ is a vector in the Cartan subalgebra $\mathfrak{h}$ of $\text{Lie}G$.

In a Cartan-Weyl basis for $\text{Lie}G$, the operator $D$ can be rewritten as follows

\begin{equation}
D_{rs} = \frac{d}{dx}\delta_{rs}, \quad r, s = 1, \ldots, d = \text{dim}\, \mathfrak{h}, \quad (13)
\end{equation}

\begin{equation}
D_{\alpha-\alpha} = \frac{d}{dx}\delta_{\alpha-\alpha} + i \frac{l}{l} a^\alpha(A_L)_r \quad (14)
\end{equation}

and

\begin{equation}
D_{-\alpha\alpha} = \frac{d}{dx}\delta_{-\alpha\alpha} - i \frac{l}{l} a^\alpha(A_L)_r \quad (15)
\end{equation}

where $\{\alpha\}$ is the set of positive roots of $\text{Lie}G$. To invert $D$ we invert each component separately. $D_{rs}$ is an elliptic anti-Hermitian operator with a space of zero modes of dimension equal to the rank of the group $G$ (there is a natural inner product for the operator $D$ on the circle). Moreover the operator $D_{rs}$ is invariant under $O(d)$ rotations. This operator does not have a unique inverse. The kernel $K_{rs}(x,y)$ of the inverse operator* of $D_{rs}$ is specified up to a constant matrix. The antisymmetry of the PBs requires that the kernel $K_{rs}(x,y)$ be antisymmetric, i.e

\begin{equation}
K_{rs}(x,y) = -K_{sr}(y,x). \quad (16)
\end{equation}

The kernels that obey this condition are

\begin{equation}
K_{rs}(x,y) = -\left[(x - y)\text{mod}\, l - \frac{l}{2}\right]\delta_{rs} + C_{rs}; \quad C_{rs} = -C_{sr} \quad (17)
\end{equation}

where $C = C(A_L)$ is a constant matrix and $0 \leq (x - y)\text{mod}\, l < l$. The Jacobi

\* The kernel $K(x,y)$ of the inverse operator of an operator $D$ on a space $M$ obeys the condition

\[ \frac{1}{\text{Vol}(M)} \int_M dy K(x,y)\phi_\kappa(y) = \frac{\phi_\kappa(x)}{\kappa} \] where $\kappa$ are the eigenvalues and $\phi_\kappa$ are the eigenfunctions of the operator $D$ ($\kappa \neq 0$).
identities of the PBs impose additional restrictions on the matrix $C$, in fact it is found that $C$ is a closed two-form on a maximal torus of the group $G$.

There is a unique inverse of the operator $D_{rs}$ which is invariant under $O(d)$ rotations. This inverse operator has kernel $K_{rs}$ given in eqn. (17) with $C = 0$. The zero modes of the integral operator given by the kernel $K_{rs}$ ($C = 0$) are the constant functions on the circle, i.e. they are the same as the zero modes of the operator $D_{rs}$. Therefore $K_{rs}$ ($C = 0$) is the appropriate kernel which inverts the symplectic form outside the zero modes of $D_{rs}$. Finally, the zero modes of $D_{rs}$ are “conjugate” to the monodromy $A_L$ in the symplectic form $\omega_L$ (see the last term of the symplectic form $\omega_L$), giving a non-degenerate symplectic form and consistent PBs$^\dagger$.

Next we turn our attention to the operator $D_{\alpha - \alpha}$. If $\alpha^r(A_L)_r \neq 2\pi n$, $n$ an integer, the operator $D_{\alpha - \alpha}$ has no zero modes and it admits a unique inverse. The kernel of the inverse is

$$K_{\alpha - \alpha}(x, y) = \frac{-i\frac{l}{2}}{\sin(\frac{\alpha A_L}{2})} \exp \left[ -\frac{i\alpha A_L}{l} ((x - y) \mod l - \frac{l}{2}) \right].$$

(18)

Similarly the kernel of the inverse of the operator $D_{-\alpha \alpha}$ is

$$K_{-\alpha \alpha}(x, y) = \frac{i\frac{l}{2}}{\sin(\frac{\alpha A_L}{2})} \exp \left[ \frac{i\alpha A_L}{l} ((x - y) \mod l - \frac{l}{2}) \right].$$

(19)

The PBs expressed in the basis given by the maps $X$ are thus

$$\{X^i(x), X^j(y)\}_{PB} = \frac{1}{\beta} L^i_a(X(x)) K^{ab}(x, y) L^j_b(X(y))$$

$$= \frac{1}{\beta} \left[ L^i_r(X(x)) \{-(x - y) \mod l - \frac{l}{2}\} \delta^r s L^j_s(X(y)) \right.$$

$$+ \sum_{\alpha} \frac{i\frac{l}{2}}{\sin(\frac{\alpha A_L}{2})} L^i_a(X(x)) \exp \left\{ \frac{i\alpha A_L}{l} ((x - y) \mod l - \frac{l}{2}) \right\} L^j_{-\alpha}(X(y)) \left. \right]$$

(20)

$^\dagger$ Note that step functions are not the kernels of the inverse of $D_{rs}$ on the circle. One reason is that they have an even number of discontinuous points when they are considered on the circle.
\[ \{ X^i(x), (A_L)_r \}_{PB} = -\frac{l}{\beta} L^i_r \] 

(21)

and

\[ \{(A_L)_r, (A_L)_s \}_{PB} = 0, \] 

(22)

where \( \beta = -\frac{k}{8\pi} \) and \( \sum_\alpha \) is the sum over all roots.

These PBs satisfy the Jacobi identities. The only non-trivial Jacobi identity is the vanishing of the expression

\[ \{ \{ X^i(x), X^j(y) \}_{PB}, X^k(z) \}_{PB} + \text{cyclic}. \] 

(23)

To verify this Jacobi identity, we substitute eqn. (20) into the above expression, giving

\[ \frac{1}{\beta^2} \left[ \partial_x K_{ab}(x, y) + \partial_y K_{ca}(z, x) + \partial_a K_{bc}(y, z) \right. \]

\[ - f_b^{\ \ cd} K_{ad}(x, y) K_{cc}(y, z) - f_a^{\ \ cd} K_{cd}(z, x) K_{eb}(x, y) \]

\[ - f_c^{\ \ ed} K_{bd}(y, z) K_{ca}(z, x) \right] L^a_i(x) L^b_j(y) L^c_k(z), \]

(24)

where \( \partial_a = \frac{\partial}{\partial x_a} \). This can be shown to be identically zero by substituting the \( K \) of eqns. (18) - (20) into (24) and by restricting \( C \) to be a closed two-form, i.e. \( \partial_\nu C_{\nu \alpha \beta \gamma} = 0 \).

Having calculated the PBs in the \((X, A_L)\) basis, we can calculate the PBs of any other functionals of \( X \) and \( A_L \). For example, the PBs of the variables \( U \) are

\[ \{ U(x) \otimes U(y) \}_{PB} = -\frac{1}{\beta} U(x) t_a \otimes U(y) t_b K^{ab}(x, y). \]

(25)

It is also possible to calculate the Poisson brackets of the variables \( U \) or \( V \) that lie in different representations of the group \( G \). To compare with the results of the authors of ref. [8], we note that they use the variable \( u(x) = U(x) M^x \), where \( x \in \mathbb{R} \). The PBs of the \( u \) variables can then be checked to be those given in this reference.

‡ We would like to thank Meifang Chu for pointing this out to us.
Because the symplectic form $\Omega$ factorises, the PB of $X$ with $Y$ is zero, where $V = V(Y)$. This is consistent with the Jacobi identities. As a consequence of this, the PB of the current $J_-(Y)$ with the current $J_+(X)$ is zero and the theory has two commuting Kac-Moody algebras, one for each chiral sector.

To describe the WZW model we introduce a constraint $Q = A_L - A_R$ on the phase space $(P_{LR}, \Omega)$ of the LR model. This constraint does not generate other constraints and it is first class. Moreover, under the PBs of the LR model the constraint $Q$ generates the transformations of eqn. (8).

The introduction of further gauge fixing conditions was suggested in ref. [8] as another way to construct the inverse of the symplectic form of the WZW model. However, it is not clear to us that the PBs presented in this approach satisfy the Jacobi identities.

4. THE SUPERSYMMETRIC WZW MODEL

The Lagrangian of the (1,1) supersymmetric WZW model is

$$L = \frac{k}{4\pi} (h_{ij} - b_{ij}) D_+ \phi^j D_- \phi^i,$$

(26)

where $h, b$ are defined as in the bosonic sigma model, $\phi$ is a map from a (1,1) superspace with co-ordinates $\{z\} = \{x, t, \theta^+, \theta^-\}$ to a group manifold $G$ and $D_-, D_+$ are superspace derivatives with $D_-^2 = i\partial_-, D_+^2 = i\partial_+$ and $\{D_+, D_-\} = 0$. The action is invariant under the semilocal transformations $g(z) \rightarrow l(z^-)g(z)r^{-1}(z^+)$ and the corresponding currents are

$$J_-(z^-) = \frac{k}{4\pi} g^{-1} D_- g, \quad J_+(z^+) = -\frac{k}{4\pi} D_+ g g^{-1},$$

(27)

where $\{z^+\} = \{x^+, \theta^+\}$ and $\{z^-\} = \{x^-, \theta^-\}$.

The equations of motion of the supersymmetric WZW model can be solved as in the case of the bosonic WZW model. The general solution can be expressed in
the form
\[ g(x, t, \theta^+, \theta^-) = U(z^+)M^2tV(z^-), \tag{28} \]
where \( g, U, V \) are periodic in the \( x \) coordinate and the monodromy \( M \) is on a maximal torus of the group \( G \). The solution (28) of the supersymmetric WZW model is invariant under the analogue of the right transformations given in eqn (8), for the bosonic model.

The symplectic current of the theory is given by
\[ S^\pm = \delta \phi^i \wedge \delta (\partial L/\partial D^\pm \phi^i) \] and the symplectic form is
\[ \omega = -\frac{i}{2} \int_0^l dx (D^+S^+ + D^-S^-)|_{\theta^\pm = 0}. \tag{29} \]

This symplectic form can be factorised into a left and a right symplectic form by introducing different monodromies for the \( U \) and \( V \) sectors. The left form \( \omega_L \) written in a periodic basis and in components is
\[ \omega_L = -\frac{k}{8\pi} \left[ \int_0^l dx \left\{ tr(U^{-1}\delta U \partial_x(U^{-1}\delta U)) + \frac{2i}{l}(U^{-1}\delta U)^2 A_L \right\} - i\delta_{ab}\lambda^a_+\delta\lambda^b_+ + \frac{2}{l} tr(U^{-1}\delta U) \delta M_L M^{-1}_L \right]\tag{30} \]
where \( U, A_L \) are defined as in the bosonic case and \( -i\lambda^a_+ t_a = (D_+UU^{-1})|_{\theta^+ = 0} \) \( (i\lambda^a_- t_a = (V^{-1}D_-V)|_{\theta^- = 0} \). \( U \) is periodic.

It is evident that \( \omega_L \) factorises into bosonic and fermionic sectors and that these two sectors can be inverted separately. The PBs of the bosonic sector are the same as the PBs of the bosonic WZW model and they are given in eqns (20) - (22). The remaining PBs are
\[ \{ \lambda^a_+(x), \lambda^b_+(y) \}_{PB} = \frac{i}{\beta} \delta^{ab} \delta(x, y), \tag{31} \]
\[ \{ X^i(x), \lambda^a_+(y) \}_{PB} = \{ A_L, \lambda^a_+(x) \}_{PB} = 0. \]

These PBs satisfy the Jacobi identities. Moreover, the Poisson bracket algebra
of the currents (27) is the supersymmetric Kac-Moody algebra.

Finally, we can introduce antiperiodic boundary conditions for the fermions \( \lambda_+ \) and/or \( \lambda_- \) instead of the periodic ones, due to the existence of two spin structures on \( S^1 \). In all these cases, the PBs of the theory remain the same as in eqns (20) - (22) and (31).

5. The gauged WZW model

Another class of interesting conformal field theories is the gauged WZW models. We consider the case where one gauges a subgroup \( H \) of the diagonal subgroup of the \( G \times G \) rigid symmetry group of the WZW model with target space the group \( G \). The action of \( H \) on \( G \) is \( g \rightarrow hgh^{-1} \), with \( g \in G \) and \( h \in H \). The Lagrangian of the corresponding gauged WZW model can be written

\[
L = -\frac{k}{16\pi} (h_{ij} \eta^{\mu\nu} \nabla_\mu \phi^i \nabla_\nu \phi^j + b_{ij} \epsilon^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j - \epsilon^{\mu\nu} a^A_\mu \partial_\nu \phi^i w_{iA} + c_{AB} \epsilon^{\mu\nu} a^A_\mu a^B_\nu),
\]

(32)

where \( \phi \) is a section of an \( H \)-bundle over \( S^1 \times \mathbb{R} \), \( \nabla \) is the covariant derivative of the connection \( a (\nabla_\mu \phi^i = \partial_\mu \phi^i + a^A_\mu \xi_A) \), \( \xi_A, A = 1, \cdots, \text{dim} H \), are the vector fields generated by the group action of \( H \) on \( G \), \( w \) is given by \( \xi_A^i H_{ijk} = 3 \partial_j w_{kA} \) and \( c_{AB} = \xi_A^i w_{iB} \) (c.f. ref.[11]).

The equations of motion of the theory are

\[
\nabla_- (\nabla_+ gg^{-1}) + iF_{+ -}(a) = 0 \\
(\nabla_+ gg^{-1})|_{\text{Lie } H} = 0, \quad (g^{-1} \nabla_- g)|_{\text{Lie } H} = 0,
\]

(33)

where \( \nabla_\pm g = \partial_\pm g + ia_\pm - i g a_\pm \) and \( F(a) \) is the curvature of the connection \( a \). Equations (33) imply that \( a \) is a flat connection. The general solution of the first equation of (33) can then be written in the form

\[
g(x, t) = m^* U(x^+) M^2 V(x^-) m^* \]

(34)

where \( g, U \) and \( V \) are periodic in the co-ordinate \( x \) and \( M (M = \exp iA) \) is the monodromy familiar from the WZW model case. \( m \) is the holonomy of the flat
connection \(a\). We have used the \(H\) gauge invariance to put \(m\) on a maximal torus of the group \(H\).

The symplectic form of the gauged WZW model is given by

\[
\omega(g, a) = -\frac{k}{8\pi} \int_0^l dx \, tr \left[ \delta gg^{-1} \nabla_-(\delta gg^{-1}) + g^{-1}\delta g \nabla_+(g^{-1}\delta g) + 2i\delta gg^{-1}\delta a_+ - 2ig^{-1}\delta g \delta a_+ \right].
\]

Substituting the solution (34) into this expression for \(\omega(g, a)\), we get the same symplectic form as in the case of the (ungauged) WZW model. In particular this \(\omega\) does not depend on the connection \(a\). However, we have not imposed the last two equations of motion of eqn. (33) on \(U\) and \(V\). This turns out to be a rather involved problem which we will study in a future publication. Instead, we will examine a special case of the gauged WZW model, that with gauge group \(H = G\). This is a topological model with a finite number of degrees of freedom [12] and the phase space of the theory is a finite dimensional manifold.

The space of solutions of the equations of motion (33) of the topological model on a cylinder is given by the space of flat connections \(a\) of the group \(G\) times the space of constant elements \(g\) that lie on a maximal torus \(T\) of the group \(G\). Thus the phase space of the topological model is \(T \times T\). The symplectic form (eqn. (35)) is

\[
\omega = -\frac{k}{4\pi} (\delta a \wedge \delta b)
\]

where \(g = \exp it_r b^r, r = 1, \cdots, \text{rank}G\). The quantisation of quantum mechanical systems with phase space structure similar to that of the topological model has been studied in ref. [13] using geometric quantisation methods.
6. Concluding Remarks

The topology and geometry of the spacetime is important in determining the form of the PBs of any field theory and in particular the PBs of the WZW model. A change of geometry and topology of the spacetime has a dramatic effect on the form of the PBs. For example, consider the WZW model on $\mathbb{R}^2$ with the flat Lorenzian metric. In this case the model has no monodromy and the operator that we have to invert is $D_{ab} = \frac{d}{dx}\delta_{ab}$. There are several difficulties in defining this theory properly, one of them being that the symplectic form $\omega$ is not well defined for all $C^\infty$-functions $u^{-1}\delta u$ and $\delta vv^{-1}$ on $\mathbb{R}$ and a careful treatment of the associated analysis should be undertaken. However, if we proceed formally, the operator $D$ has zero modes and therefore does not have a unique inverse. There is a family of inverses with kernels

$$K_{ab} = \delta_{ab} \epsilon(x, y) + C_{ab}$$

where $\epsilon(x, y) = \frac{1}{2}, x > y, \epsilon(x, y) = -\frac{1}{2}, y > x$ and $C$ is an undetermined constant antisymmetric matrix. The Jacobi identities impose an additional restriction on $C$ which is that $C$ should satisfy the modified Yang-Baxter equation. The modified Yang-Baxter equation is a Nijenhuis tensor condition for the endomorphism $C$ of the Lie algebra $\text{Lie}G$. Solutions of this equation were discussed in ref.[14].

The phase space of a WZW model is an infinite dimensional space with non-trivial topology. In general, there is no unique way to quantise phase spaces with non-trivial topological and global structure [15]. One reason for this is that there is no “natural” choice of a set of variables with respect to which we can quantise the system. Different choices of variables may lead to inequivalent quantisations. One set of variables that has already been used to quantise the WZW models is the set of currents $\{J_+, J_-\}$ [1]. However there may be other choices of variables which may give new insight into the quantum structure of the model. For example, if this theory is quantised in terms of the currents the monodromy does not play any role in the quantum theory of the WZW model.
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