LOOP-FUSION COHOMOLOGY AND TRANSGRESSION

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Abstract. ‘Loop-fusion cohomology’ is defined on the continuous loop space of a manifold in terms of Čech cochains satisfying two multiplicative conditions with respect to the fusion and figure-of-eight products on loops. The main result is that these cohomology groups, with coefficients in an abelian group, are isomorphic to those of the manifold and the transgression homomorphism factors through the isomorphism.

In this note we present a refined Čech cohomology of the continuous free loop space $L^cM$ of a manifold $M$ (or we could work throughout with the energy space instead). Compared to the standard theory, the cochains are limited by multiplicativity conditions under two products on loops, the fusion product (defined by Stolz and Teichner [ST]) and the figure-of-eight product (which appears implicitly in Barrett [Bar91] and explicitly in [KM13]). The main result of this paper is that the resulting ‘loop-fusion’ cohomology, $\check{H}^\bullet_{lf}(L^cM; A)$, recovers the cohomology of the manifold directly on the loop space.

Fusion and figure-of-eight configurations.

Theorem. For each $k \geq 1$ and abelian group $A$ there is an enhanced transgression isomorphism

$$T_{lf} : \check{H}^k(M; A) \xrightarrow{\cong} \check{H}^k_{lf}(L^cM; A)$$

forming a commutative diagram with the forgetful map, $f$, to ordinary cohomology and the standard transgression map $T$:

$$\begin{array}{ccc}
\check{H}^k(M; A) & \xrightarrow{T_{lf}} & \check{H}^k_{lf}(L^cM; A) \\
& \searrow_{T} & \downarrow_{f} \\
& & \check{H}^{k-1}(L^cM; A).
\end{array}$$

(1)

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For $A = \mathbb{Z}$ and $k = 2$ or $k = 3$ this result appears in [KM13]. There the cohomology classes are represented geometrically by functions and circle bundles over the loop space which satisfy the fusion property and are reparameterization equivariant; the figure-of-eight condition follows from these conditions.

The case $k = 2$ with integer coefficients is closely related to the problem of recovering a circle bundle on $M$ up to isomorphism from its holonomy as a function on $LM$, which has been considered by Teleman [Tel63], Barrett [Bar91] and Caetano-Picken [CP94]. In [Wal09], Waldorf considers principal bundles for general abelian groups and makes explicit use of the fusion product. The case $k = 3$ corresponds to an association between gerbes on $M$ and circle bundles on $LM$. Such a construction was first given by Brylinski [Bry93], and in [BM96], Brylinski and McLaughlin point out that the resulting bundle on the smooth loop space has an action by $\text{Diff}(S)$ and a multiplicativity with respect to the composition of loops based at the same point. In [Wal10], [Wal12] Waldorf identifies the fusion property for bundles on $LM$ given by the transgression of gerbes, and uses this to define an inverse functor.

The extension of such results to $k \geq 3$ to give an explicit transgression of geometric objects, such as higher gerbes, faces the usual obstacles associated with compatibility conditions. Here, the use of Čech cohomology allows for a short and unified treatment of the general case. In particular this shows that the two conditions included in the loop-fusion structure, without equivariance with respect to the variable on the circle or thin homotopy equivalence, suffice to capture the cohomology of $M$.

1. Spaces, covers and Čech cohomology

1.1. Base space. Let $M$ be a smooth manifold. In the subsequent discussion we fix a Riemann metric on $M$ and $\epsilon > 0$ smaller than the injectivity radius although refinement arguments show that none of the results depend on these choices. For each $m \in M$ let $U_m$ be the open geodesic ball of radius $\epsilon > 0$ centered at $m$ and consider the disjoint union of these balls as a cover of $M$:

$$U = \bigsqcup_{m \in M} U_m \rightarrow M.$$ 

This is a good cover: for $k \geq 1$, each of the $k$-fold intersections is empty or contractible. The disjoint union of these intersections is equivalent to the fiber product

$$U^{(k)} = U \times_M \cdots \times_M U \rightarrow M.$$

Remark 1. It is convenient to work with ‘maximal’ covers parameterized by the space itself. However it is possible throughout the discussion below to restrict to countable covers as is more conventional in Čech theory. Indeed, one can work here with the cover of $M$ by neighborhoods with centers at a countable dense subset. See the subsequent remark on paths and loops.

The collection $\{M^n : n \geq 1\}$ forms a simplicial space with the projections $\pi_i : M^n \rightarrow M^{n-1}$, $1 \leq i \leq n$, as face maps with the convention that $\pi_i$ omits the $i$th factor. Similarly $\{U^n : n \geq 1\}$ is a simplicial space, with face maps also denoted $\pi_i$; each $U^n \rightarrow M^n$ is also a good cover. Differentials deriving from this simplicial structure will be denoted by $\partial$. 
For each fixed \( n \) the successive fiber products \( \{(\mathcal{U}^{(k)})^n \equiv (\mathcal{U}^u)^{k} : k \geq 1\} \) also form a simplicial space with face maps \( \iota_j : (\mathcal{U}^{(k)})^{(k+1)} \rightarrow (\mathcal{U}^u)^{(k)} \) the inclusions of \((k+1)\)-fold intersections of the open sets into the \( k \)-fold intersections. This second simplicial space underlies the Čech cohomology of \( M^n \). Indeed, for an abelian group \( A \) the Čech cochains on \( M^n \) with respect to \( \mathcal{U}^u \) are the continuous maps

\[
\check{C}^k(M^n, A) \ni \alpha : (\mathcal{U}^u)^{(k+1)} \rightarrow A, \quad k \in \mathbb{N}
\]

with differential

\[
\delta : \check{C}^k(M^n; A) \rightarrow \check{C}^{k+1}(M^n; A),
\]

\[
\delta \alpha = \prod_{j=1}^{k+2} \iota_j^* f_j^{(-1)^j} : (\mathcal{U}^u)^{(k+2)} \rightarrow A.
\]

Note that these are unoriented Čech cochains, so that \( \alpha \) is not required to be odd with respect to permutations acting on the fiberwise factors of \( \mathcal{U}^{(k)} \rightarrow M \).

For a good cover such as \( \mathcal{U}^u \), the Čech cohomology is isomorphic to the sheaf cohomology of \( M^n \):\textsuperscript{[God73]}

\[
\check{H}^\bullet(M^n; A) := H^\bullet(\check{C}^\bullet(M^n; A), \delta) \cong H^\bullet(M^n; A),
\]

where \( A \) is the sheaf of continuous functions into \( A \).

**Lemma 1.1.** For each \( k \), the sequence

\[
0 \rightarrow \check{H}^k(M; A) \xrightarrow{\partial} \check{H}^k(M^2; A) \xrightarrow{\partial} \cdots
\]

\[
\partial : \check{H}^k(M^n; A) \ni \alpha \rightarrow \prod_{j=1}^{n+1} \pi_j^* \alpha^{(-1)^j} \in \check{H}^k(M^{n+1}; A)
\]

is exact.

**Proof.** By symmetry, \( \partial^2 = 0 \). Fix a point \( \bar{m} \in M \) and consider the inclusions

\[
i_n : M^n \hookrightarrow M^{n+1}, \quad (m_1, \ldots, m_n) \mapsto (\bar{m}, m_1, \ldots, m_n).
\]

Then

\[
\pi_j \circ i_n = \begin{cases} \text{Id} & j = 1 \\ i_{n-1} \circ \pi_{j-1} & j \geq 2 \end{cases}
\]

as maps from \( M^n \) to \( M^n \) and for \( \alpha \in \check{H}^k(M^n; A) \),

\[
\partial i_{n-1}^* \alpha = \prod_{j=1}^n \pi_j^* i_n^* \alpha^{(-1)^j} = \alpha (i_n^* \partial \alpha)
\]

Thus if \( \partial \alpha = 1 \) then \( \alpha = \partial i_{n-1}^* \alpha \). \( \square \)

**1.2. Path space.** Let \( \mathcal{I}M = \mathcal{C}([0,1]; M) \) be the free continuous path space of \( M \); it is a Banach manifold which fibers over \( M^2 \) by the endpoint map

\[
\varepsilon : \mathcal{I}M \rightarrow M^2, \quad \varepsilon(\gamma) = (\gamma(0), \gamma(1)).
\]

We make use of the join product

\[
j : \pi^*_2 \mathcal{I}M \times_{M^2} \pi^*_1 \mathcal{I}M \rightarrow \pi^*_2 \mathcal{I}M
\]

\[
(\gamma_1, \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2(t-1/2)) & 1/2 \leq t \leq 1, \end{cases}
\]
where \((\gamma_1, \gamma_2) \in \pi_3^X \mathcal{I}M \times \mathcal{M}^3 \pi_1^X \mathcal{I}M\) if and only if \(\gamma_1(1) = \gamma_2(0)\). Note that \(\mathcal{J}\) is a bijection and hence \(\pi_3^X \mathcal{I}M \times \mathcal{M}^3 \pi_1^X \mathcal{I}M\) can be identified with \(\mathcal{I}M\) fibering over \(\mathcal{M}^3\) by the map \(\gamma \mapsto (\gamma(0), \gamma(1/2), \gamma(1))\).

For \(\gamma \in \mathcal{I}M\), let \(\Gamma_\gamma = \{\gamma' \in \mathcal{I}M : \sup_t |\gamma(t) - \gamma'(t)| < \epsilon\}\) be the set of paths lying pointwise within the metric tube of radius \(\epsilon\) around \(\gamma\). Proceeding as above and setting

\[
\Gamma = \bigcup_{\gamma \in \mathcal{I}M} \Gamma_\gamma \rightarrow \mathcal{I}M, \quad \Gamma^{(k)} = \Gamma \times \mathcal{I}M \cdots \times \mathcal{I}M \Gamma,
\]

gives a good cover of \(\mathcal{I}M\), which factors through \(\mathcal{U}^2\), i.e. the diagram

\[
\begin{array}{ccc}
\Gamma^{(k)} & \xrightarrow{\varepsilon} & (\mathcal{U}^2)^{(k)} \\
\downarrow & & \downarrow \\
\mathcal{I}M & \xrightarrow{\varepsilon} & \mathcal{M}^2
\end{array}
\]

commutes for each \(k\). Furthermore, join lifts to a well-defined map

\[
j : \pi_3^X \Gamma^{(k)} \times_{(\mathcal{U}^2)^{(k)}} \pi_1^X \Gamma^{(k)} \rightarrow \pi_2^X \Gamma^{(k)}
\]

and there is a natural identification of \(\pi_3^X \Gamma^{(k)} \times_{(\mathcal{U}^2)^{(k)}} \pi_1^X \Gamma^{(k)}\) with \(\Gamma^{(k)}\).

**Remark 2.** As noted in Remark \(\mathcal{J}\) above, it is possible to work throughout with countable covers. One can restrict to neighborhoods centered on paths which are finite combinations of segments with rational end-points and which are affine geodesics between the chosen countable dense set in the manifold. The resulting cover has the crucial property of being closed under join, and the induced countable cover of loop space, considered below, is closed with respect to the two loop-fusion operations.

The finite-dimensionality of \(\mathcal{M}\) is not used, so the definition of the Čech cochain complex above carries over giving

\[
\tilde{C}^k(\mathcal{I}M; A) \ni f : \Gamma^{(k+1)} \rightarrow A, \quad \delta f = \prod_{j=1}^{k+2} \epsilon_j f^{(-1)} \in \tilde{C}^{k+1}(\mathcal{I}M; A)
\]

where we reuse the notation \(\epsilon_j : \Gamma^{(k+1)} \rightarrow \Gamma^{(k)}\) for the face maps of the simplicial space \(\{\Gamma^{(k)}; k \geq 1\}\), and observe that again \(H^k(\mathcal{I}M; A) \cong H^k(\mathcal{I}M; A)\) by the local contractibility of \(\mathcal{I}M\).

The identification of \(\pi_3^X \Gamma \times_{\mathcal{U}^3} \pi_1^X \Gamma\) with \(\Gamma\) and \(\mathcal{J}\) gives a second chain map on \(\tilde{C}^* \mathcal{I}M; A\) associated to the simplicial structure on \(\{M^n : n \geq 1\}\):

\[
\partial : \tilde{C}^k(\mathcal{I}M; A) \rightarrow \tilde{C}^k(\mathcal{I}M; A), \quad \partial f = \pi_3^X f^{-1} \pi_1^X f^{-1} j^*(\pi_3^X f) : \Gamma^{(k)} \rightarrow A.
\]

This does not lead to a complex, i.e. \(\partial^2\) is not trivial, since \(\mathcal{I}M\) is not itself a simplicial space over \(\{M^n : n \geq 1\}\) and the join operation is required to compare pullbacks.

The constant paths may be identified as an inclusion \(M \subset \mathcal{I}M\). Let

\[
\tilde{C}^k(\mathcal{I}M; A) = \{f \in \tilde{C}^k(\mathcal{I}M; A) : f|_M = 1\}
\]

denote the subcomplex of cochains which are trivial on them. Since the join map restricts to the trivial map on constant paths \(\tilde{\partial} : \tilde{C}^0(\mathcal{I}M; A) \rightarrow \tilde{C}^0(\mathcal{I}M; A)\).
Lemma 1.2. The subcomplex \((\tilde{C}_\ast(\mathcal{IM}; A), \delta)\) is acyclic.

Proof. The short exact sequence of chain complexes

\[ 0 \to \tilde{C}_0(\mathcal{IM}; A) \to \tilde{C}_\ast(\mathcal{IM}; A) \to \tilde{C}_\ast(M; A) \to 0 \]

induces a long exact sequence in cohomology, however \(\tilde{H}_\ast(\mathcal{IM}; A) \cong \tilde{H}_\ast(M; A)\) since there is a deformation retraction of \(\mathcal{IM}\) onto \(M\), from which it follows that \(\tilde{H}_\ast(\mathcal{IM}; A) = 0\). \(\square\)

1.3. Loop space. For \(l \geq 1\) we denote by \(\mathcal{I}^l M\) the fiber product

\[ \mathcal{I}^l M = \mathcal{IM} \times_M \cdots \times_M \mathcal{IM} \]

and observe that \(\mathcal{I}^2 M = \{(\gamma_1, \gamma_2) : \gamma_1(t) = \gamma_2(t), \ t = 0, 1\}\) may be identified with the Banach manifold of free continuous loops by fusion of paths:

\[ \psi : \mathcal{I}^2 M \to \mathcal{L}M = \mathcal{C}(S; M) , \quad \ell(t) = \psi(\gamma_1, \gamma_2)(t) = \begin{cases} \gamma_1(t) , & 0 \leq t \leq 1 \\ \gamma_2(-t) , & -1 \leq t \leq 0 \end{cases} \]

where \(S\) is parameterized as \([-1, 1]/\{-1\} \sim \{1\}\) for later convenience.

The set \(\{\mathcal{I}^l M : l \geq 1\}\) forms another simplicial space, with face maps given by the fiber projections \(\partial_j : \mathcal{I}^l M \to \mathcal{I}^{l-1} M, 1 \leq j \leq l\), and \(\{\Gamma^l : l \geq 1\}\) forms a good cover, where

\[ \Gamma^l = \Gamma \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_2 \Gamma \to \mathcal{I}^l M , \]

is lifted from the path space with \(k\)-fold overlaps

\[ (\Gamma^k)^l \equiv (\Gamma^l)^k = \Gamma^l \times_{\mathcal{I}^l M} \cdots \times_{\mathcal{I}^l M} \Gamma^l . \]

For clarity of notation, we denote this cover of loop space by

\[ \Lambda = \Gamma^{[2]} \to \mathcal{LM} . \]

We will denote differentials derived from this simplicial space or its cover by \(d\).

Passing to \(\mathcal{I}^l M\) in (2) gives rise to a map

\[ j^l : \pi^3_\ast \mathcal{I}^l M \times_M \cdots \times_M \mathcal{I}^l M \to \pi^\ast_\mathcal{L} \mathcal{I}^l M \]

and its local version

\[ j^l : \pi^3_\ast (\Gamma^l)^k \times (\mathcal{U}_2)^k \pi^1_\ast (\Gamma^l)^k \to \pi^\ast_\mathcal{L} (\Gamma^l)^k . \]

In the case \(l = 2\), we call this the **figure-of-eight product** on loops as in [KM13].

The product of two loops \(\ell_1 = \psi(\gamma_{11}, \gamma_{12})\) and \(\ell_2 = \psi(\gamma_{21}, \gamma_{22})\) such that \(\ell_1(1) = \ell_2(0)\) is the loop \(\ell_3 = \psi(j(\gamma_{11}, \gamma_{21}), j(\gamma_{12}, \gamma_{22}))\). The domain in (3) with \(l = 2\) may be identified with the subspace of **figure-of-eight loops** in \(M\):

\[ \mathcal{L}_8 M = \{\ell \in \mathcal{LM} : \ell(1/2) = \ell(-1/2)\} \to M^3 \]

This Banach manifold fibers over \(M^3\) and has a good cover given by the domain in (4) with \(l = 2\) and \(k = 1\). Unlike the case \(l = 1\), \(\mathcal{L}_8 M\) cannot be identified with the full loop space nor is \(j^l\) invertible.

There is a more fundamental product on loop space, considered already in [ST], associated to \(\mathcal{I}^3 M\). If \((\gamma_1, \gamma_2, \gamma_3) \in \mathcal{I}^3 M\), then \(\ell_3 = \psi(\gamma_1, \gamma_3)\) is the fusion product of \(\ell_1 = \psi(\gamma_1, \gamma_2)\) and \(\ell_2 = \psi(\gamma_2, \gamma_3)\).

Within the Čech cochain complex \((\check{C}_\ast(\mathcal{LM}; A), \delta)\) for loop space:

\[ \check{C}^k(\mathcal{LM}; A) \ni f : \Lambda^{k+1} \to A , \quad \delta f = \prod_{j=1}^{k+2} \ell^j(f^{(-1)})^j \in \check{C}^{k+1}(\mathcal{LM}; A) . \]
consider the subcomplex of fusion cochains
\[ \tilde{C}_{\text{fus}}(\mathcal{M}; A) = \{ f \in \tilde{C}^k(\mathcal{M}; A) : df = 1 \} \]
\[ df = \varrho_1^* f^{-1} \varrho_2 f \varrho_1^* f^{-1} \in \tilde{C}^k(\mathcal{I}^{[3]} M; A). \]
Note that \( d^2 : \tilde{C}^k(\mathcal{I}^{[0]}; A) \to \tilde{C}^k(\mathcal{I}^{[0]}; A) \) vanishes and \( \delta d = d\delta \) so this is indeed a subcomplex.

The subspace \( \mathcal{L}_8 M \subset \mathcal{L} M \) is closed under fusion so \( \tilde{C}_{\text{fus}}^*(\mathcal{L}_8 M; A) \) is well-defined, and imposing a condition over the figure-of-eight product leads to the loop-fusion subcomplex
\[ \tilde{C}_{\mathcal{I}_8}^k(\mathcal{M}; A) = \{ f \in \tilde{C}_{\text{fus}}^k(\mathcal{M}; A) : \bar{\partial} f = \delta g \text{ for } g \in \tilde{C}_{\text{fus}}^{k-1}(\mathcal{L}_8 M; A) \}, \]
\[ \bar{\partial} f = \pi_3^* f^{-1} \pi_1^* f^{-1} (j^2)^* (\pi_2^* f) \in \tilde{C}_{\text{fus}}^k(\mathcal{L}_8 M; A). \]
Thus, this complex consists of those fusion cochains which are multiplicative with respect to the figure-of-eight product up to a fusion boundary. The image of \( \bar{\partial} \) on these chains lies in the space of fusion Čech cochains on the space of figure-of-eight loops; \( \bar{\partial}^2 \) is not sensibly defined without more constraints. That \( \mathcal{L}_{8} \) is a subcomplex follows from the fact that \( \delta \bar{\partial} = \bar{\partial} \delta \). It is also the case that \( \bar{\partial} \partial = \partial \bar{\partial} \) on suitably defined spaces, in particular as maps from \( \tilde{C}^k(\mathcal{I} M; A) \) to \( \tilde{C}^{k+1}(\mathcal{I} M; A) \) and from \( \tilde{C}_*^k(\mathcal{M}; A) \) to \( \tilde{C}_{\mathcal{I}_8}^k(\mathcal{M}; A) \).

The loop-fusion cohomology of \( \mathcal{L} M \) is then defined to be
\[ \check{H}_*^k(\mathcal{L} M; A) = H_*^k(\tilde{C}_{\mathcal{I}_8}^*(\mathcal{M}; A), \delta) \to \check{H}_*^k(\mathcal{L} M; A) \]
with its homomorphism, \( f \), to ordinary Čech cohomology induced by the inclusion of \( \tilde{C}_*^*(\mathcal{L} M; A) \) in \( \tilde{C}_*^*(\mathcal{M}; A) \).

2. Transgression and Regression

We proceed to the proof of the Theorem above.

2.1. Transgression. We first construct the map \( T_{\mathcal{I}_8} \). Let \( \alpha \in \tilde{C}^k(M; A) \) be a co-cycle for \( k \geq 1 \), and consider
\[ \varepsilon^* \partial \alpha \in \tilde{C}_0^k(\mathcal{I} M; A), \quad \partial \alpha = \pi_1^* \alpha^{-1} \pi_2^* \alpha \in \tilde{C}^k(M^2; A). \]
Since \( \delta \varepsilon^* \partial \alpha = \varepsilon^* \partial \delta \alpha = 1 \) and \( \tilde{C}_*^k(\mathcal{I} M; A) \) is exact by Lemma 1.2, it follows that \( \varepsilon^* \partial \alpha = \delta \beta \) for some \( \beta \in \tilde{C}_0^{k-1}(\mathcal{I} M; A) \); set
\[ \omega = d \beta = \varrho_1^* \beta^{-1} \varrho_2 \beta \in \tilde{C}^{k-1}(\mathcal{L} M; A). \]
Then \( \varepsilon \circ \varrho_1 = \varepsilon \circ \varrho_2 \) implies
\[ \delta \omega = d \delta \beta = \varrho_1^* (\varepsilon^* \partial \alpha)^{-1} \varrho_2^* (\varepsilon^* \partial \alpha) = 1. \]
Moreover \( d^2 = 1 \)
\[ d \omega = d^2 \beta = 1 \implies \omega \in \tilde{C}_{\text{fus}}^{k-1}(\mathcal{L} M; A). \]
Finally, \( \omega \) is fusion-figure-of-eight since \( \bar{\partial} \omega = d \bar{\partial} \beta \) and \( \bar{\partial} \bar{\partial} \), which lies in \( \tilde{C}_0^k(\mathcal{I} M; A) \) by Lemma 1.2, is a boundary. Indeed, for any path \( \gamma = j(\gamma_1, \gamma_2) \),
\[ \delta \bar{\partial} \beta(\gamma) = \bar{\partial} \varepsilon^* \partial \alpha(\gamma) = \varepsilon^* \partial \alpha^{-1}(\gamma_1) \varepsilon^* \partial \alpha^{-1}(\gamma_2) \varepsilon^* \partial \alpha(\gamma) \]
\[ = \alpha(\gamma_1(0)) \alpha^{-1}(\gamma_1(1)) \alpha(\gamma_2(0)) \alpha^{-1}(\gamma_2(1)) \alpha^{-1}(\gamma(0)) \alpha(\gamma(1)) = 1. \]
Thus $\bar{\partial}\beta$ is a cocycle and as $\check{C}_k^*(\mathcal{M}; A)$ is acyclic there exists $\eta \in \check{C}_0^{k-2}(\mathcal{M}; A)$ such that $\bar{\partial}\beta = \partial \eta$. It follows that
\[
\bar{\partial}\omega = d\bar{\partial}\beta = d\partial \eta = \partial d\eta, \quad d(d\eta) = 1 \implies \omega \in \check{C}_0^{k-1}(\mathcal{L}; A).
\]

Consider next the effect of the choices made. If $\beta' \in \check{C}_0^{k-1}(\mathcal{M}; A)$ is another cochain such that $\delta \beta' = \varepsilon^*\partial \alpha$, then $\delta (\beta' \beta^{-1}) = 1$ implies that $\beta' = \beta \delta \nu$ for some $\nu \in \check{C}_0^{k-2}(\mathcal{M}; A)$, which alters $\omega$ by the boundary term $\partial \nu$. Similarly if $\alpha' = \alpha \delta \mu$ is another representative for $[\alpha] \in \check{H}^k(M; A)$, it follows that $\omega' = \omega \delta \sigma$, where $\sigma$ is the result of the same construction applied to $\mu$. Thus the transgression map
\[
T_{\mathfrak{I}} : \check{H}^k(M; A) \rightarrow \check{H}_\mathfrak{I}^{k-1}(\mathcal{L}; A), \quad T_{\mathfrak{I}}[\alpha] = [\omega]^{-1}
\]
is well-defined.

2.2. Regression. Next we define a map which is shown below to be the inverse of $T_{\mathfrak{I}}$. Suppose $\omega \in \check{C}_0^{k-1}(\mathcal{L}; A)$ is a cocycle, so
\[
\delta \omega = 1, \quad d\omega = 1, \quad \bar{\partial}\omega = \delta \nu, \quad d\nu = 1.
\]
Then $\omega$ gives descent data for the trivial principal $A$-bundle
\[
\Gamma^{(k)} \times A \rightarrow \Gamma^{(k)}.
\]
over $(\mathcal{U}^{(k)})$. That is, multiplication by $\omega$ determines a relation on the fibers, with the content of $d\omega = 1$ being that this is an equivalence relation so inducing a well-defined principal $A$-bundle $P_k \rightarrow (\mathcal{U}^{(k)})$
\[
(P_k)_{(m,m')} = \left\{ (\gamma, a) \in \Gamma^{(k)} \times A : \varepsilon(\gamma) = (m, m') \right\} / \sim \omega
\]
\[
(\gamma, a) \sim \omega (\gamma', a') \iff a = \omega(\gamma, \gamma')a'.
\]
The condition $\delta \omega = 1$ implies that $P_k$ is a simplicial bundle (see [BM99, MS03]), i.e. the bundle over $(\mathcal{U}^{(k)})$ consisting of the alternating tensor products of the pullbacks of $P_k$ by the maps $\iota_j : (\mathcal{U}^{(k)}) \rightarrow (\mathcal{U}^{(k)})$ is canonically trivial:
\[
\delta P_k = \bigotimes_j \iota_j P_k = (\mathcal{U}^{(k)}) \times A \rightarrow (\mathcal{U}^{(k+1)}).
\]
Similarly, $\nu$ determines a principal $A$-bundle
\[
R_{k-1} = \Gamma^{(k-1)} \times A / \sim \nu \rightarrow (\mathcal{U}^{(k-1)}),
\]
and by functoriality of descent there is a canonical isomorphism
\[
\partial P_k \cong \delta R_{k-1} \rightarrow (\mathcal{U}^{(k)}) \quad \partial P_k = \pi_1^* P_k^{-1} \otimes \pi_2^* P_k \otimes \pi_3^* P_k^{-1}.
\]
The components of $(\mathcal{U}^{(k)})$ and $(\mathcal{U}^{(k-1)})$ are contractible so there exist sections
\[
s : (\mathcal{U}^{(k)}) \rightarrow P_k, \quad r : (\mathcal{U}^{(k-1)}) \rightarrow R_{k-1}.
\]
These pull back to give sections $\delta s$ of $\delta P_k$ and $\delta r$ of $\delta R_{k-1}$ and as $\delta P_k$ is canonically trivial $\delta s$ gives rise to a cocycle
\[
\kappa = \delta s \in \check{C}_k^k(M^2; A), \quad \delta \kappa = \delta \delta s = 1
\]
where $\delta^2 s$ coincides with the canonical trivialization of $\delta^2 P$ for any section $s$. Another choice of section $s'$ alters $\kappa$ by a term $\delta \gamma$, where $\gamma \in \check{C}_1^{k-1}(M^2; A)$ is fixed by $s' = s \gamma$. Thus $[\kappa] \in \check{H}^k(M^2; A)$ is determined by $\omega$. Similarly, another choice $\omega'$ such that $\omega' = \omega \delta \mu$, $d\mu = 1$ leads to a bundle $P'_k$ and a canonical isomorphism
\[
P'_k \cong P_k \otimes \delta Q_{k-1}, \quad \text{where } Q_{k-1} \text{ is formed by descent using } \mu. \text{ If } \kappa = \delta s \text{ and } \kappa' = \delta s' \text{ then }
\]
for respective sections \( s \) and \( s' \) of \( P_k \) and \( P'_k \), if \( q \) is any section of \( Q_{k-1} \), and 
\( s' = (s \otimes \delta q) \circ \nu \) for some \( \nu \in \check{C}^{k-1}(M^2; A) \), then \( \kappa' = \kappa \delta^2 q \delta \nu = \kappa \delta \nu \). Thus the map 
from \( \check{H}_{il}^{-1}(\mathcal{L}M; A) \) to \( \check{H}^k(M^2; A) \) is well-defined.

Finally, we may compare \( \partial s \) and \( \partial \tau \) as sections of \( (10) \), namely \( \partial s = \delta^2 \tau \) and 
\( \tau \in \check{C}^{k-1}(M^3; A) \), from which it follows that 
\[
\partial \kappa = \delta(\partial s) = \delta^2 \tau \delta \tau = \delta \tau \in \check{C}^k(M^3; A).
\]

Thus \( \partial[\kappa] = 1 \in \check{H}^k(M^3; A) \) and so by Lemma \( \ref{lem:1} \) \( [\kappa] = \partial[\alpha] \) for a unique class 
\( [\alpha] \in \check{H}^k(M; A) \). It follows that the regression map is well-defined by 
\[
(11) \quad R : \check{H}_{il}^{-1}(\mathcal{L}M; A) \to \check{H}^k(M; A), \quad R[\omega] = [\alpha]^{-1}.
\]

**Proposition 2.1.** The maps \( 8 \) and \( 11 \) are inverses.

**Proof.** To see that \( T_{il}R = \text{Id} \) fix a cocycle \( \omega \in \check{C}^{k-1}_{il}(\mathcal{L}M; A) \) and let \( \alpha \in \check{C}^k(M; A) \)
represent \( R[\omega]^{-1} \), so that \( \partial \alpha = \kappa \delta \nu \) for some \( \nu \in \check{C}^{k-1}(M^2; A) \), where \( \kappa = \delta \tau \in \check{C}^k(M^2; A) \) for a choice of section \( s \) of the bundle \( P_k \). Replacing \( s \) by \( s \nu^{-1} \) if necessary, we may assume that \( \partial \alpha = \kappa = \delta \tau \).

Consider the transgression of \( \alpha \). This involves a choice of \( \beta \in \check{C}^{k-1}(\mathcal{L}M; A) \) such 
that \( \delta \beta = \varepsilon^* \partial \alpha = \varepsilon^* \kappa \) but there is a natural choice available. Namely, the section 
\( s \) of \( P_k \) lifts canonically to a section of the trivial \( A \)-bundle over \( \Gamma^{(k)} \), from which \( P_k \) is descended, so defining a cochain 
\[
\bar{s} \in \check{C}^{k-1}_0(\mathcal{L}M; A), \quad \bar{s}(\gamma) = a \iff s(\varepsilon(\gamma)) = [\gamma, a] \in P_k.
\]

That \( \bar{s} \) is trivial on constant paths is a consequence of the fact that fusion condition implies that the descent data \( \omega \) for \( P_k \) is trivial on constant loops. Since \( \delta P_k \) is trivially descended from the trivial bundle over \( \Gamma^{(k+1)} \),
\[
\delta \bar{s} = (\delta \bar{s}) = \varepsilon^* \delta s = \varepsilon^* \kappa
\]
and hence \( \beta = \bar{s} \in \check{C}^{k-1}_0(\mathcal{L}M; A) \) is an element such that \( \delta \beta = \varepsilon^* \kappa \). It then follows that 
\( d \beta = \theta^1 \bar{s} - \theta^2 \bar{s} = \omega \in \check{C}^{k-1}_{il}(\mathcal{L}M; A) \) since 
\[
\bar{s}(\gamma) \bar{s}(\gamma')^{-1} = a a'^{-1}, \quad \text{such that}
\]
\[
s(\varepsilon(\gamma)) = s(\varepsilon(\gamma')) = [\gamma, a] = [\gamma', a'] \iff a = \omega(\gamma, \gamma') a'.
\]

In the other direction, fix a cocycle \( \alpha \in \check{C}^k(M^2; A) \) and let \( \omega \in \check{C}^{k-1}_{il}(\mathcal{L}M; A) \)
represent \( T[\alpha]^{-1} \), given by \( \omega = d \beta \) where \( \delta \beta = \varepsilon^* \partial \alpha \in \check{C}^{k}_0(\mathcal{L}M; A) \). The regression of 
\( \omega \) involves a choice, of section of the bundle \( P_k \), but here too there is a natural one which recovers \( \partial \alpha \in \check{C}^k(M^2; A) \). Indeed, since \( \omega = \theta^1 \beta - \theta^2 \beta, \) the equivalence relation defining \( P_k \) takes the particular form 
\[
P_k \ni [\gamma, a] = [\gamma', a'] \iff a = \beta(\gamma) \beta(\gamma')^{-1} a'
\]
and an appropriate section of \( P_k \) is defined by 
\[
s(m, m') = [(\gamma, \beta(\gamma))] = [(\gamma', \beta(\gamma'))]
\]
since this equivalence class is independent of the particular \( \gamma \in \varepsilon^{-1}(m, m') \). With 
\( s \) so defined, it follows that \( \delta s \in \check{C}^k(M; A) \) is given by 
\[
\delta s(m, m') = [(\gamma, \delta \beta(\gamma))] = [(\gamma, \varepsilon^* \partial \alpha(\gamma))] = \delta \alpha(m, m'). \quad \square
\]
2.3. Compatibility. The commutativity of the diagram (11) asserts that the ‘enhanced transgression’ map constructed above is compatible with transgression in the usual sense. The latter corresponds to pullback of cohomology under the evaluation map followed by projection onto the second factor under the decomposition for the product:

\[
\text{(12) } \text{ev}^* : H^k(M; A) \to H^k(S \times LM; A) = H^k(LM; A) \oplus H^{k-1}(LM; A) \to H^{k-1}(LM; A).
\]

To realize this in Čech cohomology, fix a small parameter \(\delta > 0\) and consider the open cover indexed by points

\[
S_{t,\ell} = \{(t', \ell') \in S \times LM : \ell' \in \Lambda_\ell, \ t' \in (t - \delta, t + \delta), \ \ell'(t') \in U_{i(t)} \},
\]

\[
S_{t,\ell} \to \Lambda_\ell, \ S_{t,\ell} \to I_{t,\ell} \subset S, \ ev : S_{t,\ell} \ni (t', \ell') \mapsto \ell'(t') \in U_{i(t)}.
\]

The interval \((t - \delta, t + \delta) \subset S\) is to be interpreted as the ‘short’ signed interval on \(S\). This is a good cover, with respect to which we consider the Čech complex on \(S \times LM\). The evaluation map \(ev : S \times LM \to M\) and projections \(S \times LM \to LM\) and \(S \times LM \to S\) lift to maps of the covers \(S \to \mathcal{U}, \ S \to \Lambda\), and \(S \to \mathcal{V}\), respectively, where \(\mathcal{V}\) is the cover of \(S\) by intervals of length \(2\delta\) around each point.

The first factor in the product (12) corresponds to pullback to \(LM\) under the evaluation map at any fixed point on the circle. Consequently, to consider the projection to the second factor of (12) we modify the pullback \(ev^* \alpha \in \check{C}^k(S \times LM; A)\) to

\[
\alpha' = (ev_0^*)^{-1} ev^* \alpha \in \check{C}^k(S \times LM; A)
\]

instead, where \(ev_0 : S \times LM \ni (t, \ell) \mapsto \ell(0) \in M\) factors through the projection to \(LM\). The the class of (14) projects to zero in \(\check{H}^k(LM; A)\) and has the same projection as \(ev^* \alpha\) to \(\check{H}^{k-1}(LM; A)\).

To compute the latter, consider the space \([-1, 1] \times LM\) which map to \(S \times LM\) by the identification of the endpoints. This has a good cover \(\mathcal{T} = \bigsqcup_{T_{t,\ell}} T_{t,\ell}\) where \(T_{t,\ell}\) is defined as in (13) except that the interval is restricted to \([-1, 1]\). The map to \(S \times LM\) then lifts to a continuous map of the covers. The image of (14) lies in the subcomplex \(\check{C}^k_0([-1, 1] \times LM; A)\) of chains which are trivial at \(\{0\} \times LM\). This subcomplex is acyclic as in the proof of Lemma (12) since \([-1, 1] \times LM\) retracts onto \(\{0\} \times LM\). Thus

\[
\alpha' = \delta \sigma, \quad \sigma \in \check{C}^{k-1}([-1, 1] \times LM; A)
\]

and the transgression class is represented by the difference

\[
(\sigma|_{[-1] \times LM}) (\sigma^{-1}|_{[-1] \times LM}) \in \check{C}^{k-1}(LM; A).
\]

That this is a cocycle follows from the fact that its Čech differential is the difference of \(\alpha'\) at 1 and \(-1\) which is trivial since \(\alpha'\) is pulled back from the circle.

On the other hand, the initial portion of the enhanced transgression construction in (21) may be modified as follows. Consider the pullback

\[
\tilde{\varepsilon}^* \partial \alpha \in \check{C}^k([0, 1] \times IM; A),
\]

\[
\tilde{\varepsilon} : [0, 1] \times IM \to M^2, \quad \tilde{\varepsilon}(t, \gamma) = (\gamma(0), \gamma(t)).
\]

As before this lies in an exact subcomplex, so \(\tilde{\varepsilon}^* \partial \alpha = \delta \tilde{\beta}\) where \(\tilde{\beta} \in \check{C}^{k-1}([0, 1] \times IM; A)\), the restriction \(\beta = \tilde{\beta}|_{\{1\} \times IM}\) to a cochain on \(IM\) reduces to the earlier
construction and \( \tilde{\beta}_{|0\times L}\) is trivial. Then the product

\[
\sigma = \varsigma_1^* \tilde{\beta} \varsigma_2^* \tilde{\beta} \in \tilde{C}_{k-1}([-1, 1] \times \mathcal{Z}^2 M; A),
\]

\[
\varsigma_i : [-1, 1] \times \mathcal{Z}^2 M \rightarrow [0, 1] \times L M,
\]

\[
\varsigma_1(t, (\gamma_1, \gamma_2)) = (\max(0, t), \gamma_1), \quad \varsigma_2(t, (\gamma_1, \gamma_2)) = (\min(0, t), \gamma_2)
\]
is a cochain on \([-1, 1] \times L M\) with differential equal to \(\alpha'\). Indeed,

\[
\delta_{\sigma}(t, \ell) = (\varsigma_1^* \delta \tilde{\beta} \varsigma_2^* \delta \tilde{\beta})(t, (\gamma_1, \gamma_2)) = \begin{cases} 
\alpha(\gamma_1(t))\alpha^{-1}(\gamma_1(0)), & 0 \leq t \leq 1 \\
\alpha(\gamma_2(-t))\alpha^{-1}(\gamma_2(0)), & -1 \leq t \leq 0 \\
\alpha(\ell(t))\alpha^{-1}(\ell(0)), & \end{cases}
\]

where \(\ell = \psi(\gamma_1, \gamma_2)\). Finally, observe that the transgression class \((15)\) is represented by the ‘enhanced transgression’ class \(d\beta^{-1}\):

\[
(\sigma_{|[-1] \times L}\mathcal{M}) (\sigma^{-1}_{|[-1] \times L_M}) (\gamma_1, \gamma_2) = \tilde{\beta}(1, \gamma_1) \beta^{-1}(1, \gamma_2) = d\beta^{-1}(\gamma_1, \gamma_2).
\]

This completes the proof of the Theorem.

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