Monodromy Matrix in the PP-Wave Limit

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Abstract: We construct the monodromy matrix for a class of gauged WZNW models in the plane wave limit and discuss various properties of such systems.
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1. Introduction

It is recognized that two dimensional field theories have played an important role in describing a variety of physical systems [1, 2, 3]. Furthermore, some of these models possess the interesting property of integrability which corresponds to exactly solvable models. The nonlinear $\sigma$-models in $1+1$ dimensions have been studied in great detail from different perspectives. Such models are endowed with a rich symmetry structure. The integrability properties of $\sigma$-models in flat space have attracted considerable attention in the past. In the context of Einstein's gravity and supergravity theories, when a four dimensional action is dimensionally reduced to two space-time dimensions, the resulting action, in general, describes a $\sigma$-model on a coset, $G/H$, where $G$ turns out to be a noncompact group in many cases and $H$ is its maximal compact subgroup. In order to study integrability properties of such models, one of the approaches involves constructing the monodromy matrix associated with the system under consideration [4, 5, 6, 7]. In the case of $\sigma$-models in flat space, it is customary to introduce a constant spectral parameter and then construct a suitable current which is curvature free. This single condition fulfills the requirement of integrability as well as yields the field equations. On the other hand, when we consider a $\sigma$-model in a curved background, the spectral parameter assumes space-time dependence to satisfy consistency requirements in order that the curvaturelessness of the appropriate currents are maintained.

In the framework of string theory, we encounter systems with a large number of isometries. Some of these systems are described by a two dimensional effective action which may be derived from the dimensional reduction of a ten dimensional string effective action [8, 9, 10, 11, 12]. For example, a four dimensional stringy spherically symmetric black hole can be described by a two dimensional effective action and such an action is endowed with a rich duality symmetry group. In addition to the duality symmetry, the theory, in several cases, possesses additional symmetries special to two dimensional models.

In the past, the symmetry contents of theories, when the fields depend only on one of the light-cone (LC) variables, have been explored. One starts from a two dimensional action whose integrability properties are best described when the equations of motion and integrability conditions (more appropriately curvaturelessness of the currents with a spectral parameter) are expressed in terms of the LC variables. One expects an enhancement of symmetries when we have a theory described by only one coordinate. Julia was first to propose that such symmetry enhancements might occur in dimensionally reduced theory where fields depend on a single variable [13]. It is worthwhile to note that symmetry properties of supergravity theories/M-theory have been studied in the vicinity of space-like singularities. Near the singularity, the dynamical evolution of fields at each spatial point is asymptotically governed by a set of second order differential equations in time. In other words, the spatial points
decouple. An elegant description can be given in terms of a nonlinear $\sigma$-model in one dimension and the Kac-Moody algebra makes an appearance [14]. Thus, it is worthwhile to consider an exactly solvable model in order to study some of these features and we study the WZWN model at hand in the Penrose limit.

The AdS/CFT duality in string theories [15] is a very powerful result which exhibits the intimate connection between string theory and supersymmetric Yang-Mills theory. When one considers the pp-wave limit of string theory in $AdS_5 \times S_5$ background in the presence of RR flux, the exact spectrum of the theory can be deduced. Therefore, such a theory may be identified as an integrable system [16, 17, 18]. Furthermore, in this limit, the BMN conjecture [19] has been used as a powerful tool to explore more interesting connections between the string states and the states of the $N = 4$ supersymmetric Yang-Mills theory.

The integrability of dimensionally reduced gravity and supergravity to two dimensions have been studied extensively. As mentioned earlier, the first step in this direction is to introduce the spectral parameter and construct a set of currents which satisfy curvaturelessness condition. In the context of string theory, the graviton and the shifted dilaton appear together in the two dimensional string effective action in many problems of interest. The construction of the monodromy matrix and its transformation properties under T-duality has been investigated in the recent past [20]

The purpose of this article is to construct the monodromy matrix for a WZWN model and examine its behavior in the pp-wave limit. The WZWN models are exactly solvable theories. In the past, we have taken advantage of this attribute of WZWN models to verify our proposal regarding duality properties of the monodromy matrix. We find that the monodromy matrix has interesting behavior in the pp-wave limit.

2. The General Construction

In this section, we briefly review the general procedure for construction of the monodromy matrix for two dimensional string effective actions. We shall follow the notations and conventions of our earlier works [20]. The essential point is that the action describes a $\sigma$-model coupled to gravity in two dimensions. In the conformal gauge ($g_{\alpha\beta} = \eta_{\alpha\beta} e^{\bar{\phi}}$), the action for such a sigma model (derived from dimensional reduction of the effective string theory to two dimensions) takes the following form [21, 22]

$$S_\sigma = \frac{1}{8} \int dx^0 dx^1 e^{-\bar{\phi}} \eta^{\alpha\beta} \text{Tr} \left( \partial_\alpha M^{-1} \partial_\beta M \right), \quad (2.1)$$

where $\alpha, \beta = 0, 1$, $\bar{\phi}$ the shifted dilaton and the matrix $M$ belongs to the T-duality group and we identify it to be $O(d, d)$, $d$ being the number of isometries when di-
imensional reduction is implemented. The matrix $M$ is

$$
M = \begin{pmatrix}
G^{-1} & -G^{-1}B \\
BG^{-1} & G - BG^{-1}B
\end{pmatrix},
$$

(2.2)

where $G_{ij}$ and $B_{ij}$ represent respectively the metric in the internal space and the moduli coming from dimensional reduction of the NS-NS two form potential in higher space-time dimensions. The moduli $G$ and $B$ parameterize the coset $O(d, d)/O(d) \times O(d)$. Under the global $O(d, d)$ transformation,

$$
M \rightarrow \Omega^T M \Omega, \quad \Omega \in O(d, d),
$$

(2.3)

while the shifted dilaton remains unchanged. The matrix $M$ can be written in the factorized form

$$
M = V V^T, \quad V \in \frac{O(d, d)}{O(d) \times O(d)},
$$

(2.4)

where $V$ has the triangular form

$$
V = \begin{pmatrix}
E^{-1} & 0 \\
BE^{-1} & E^T
\end{pmatrix},
$$

(2.5)

with $(E^T E)_{ij} = G_{ij}$. The matrix $V$ parameterizing the coset $\frac{O(d, d)}{O(d) \times O(d)}$ transforms nontrivially under global $O(d, d)$ as well as a local $O(d) \times O(d)$ as

$$
V \rightarrow \Omega^T V h(x), \quad \Omega \in O(d, d), \quad h(x) \in O(d, d) \times O(d, d),
$$

(2.6)

and consequently

$$
M = V V^T \rightarrow \Omega^T V V^T \Omega = \Omega^T M \Omega.
$$

(2.7)

Next, we construct the current $V^{-1} \partial_\alpha V$ which belongs to the Lie algebra of $O(d, d)$ and it can be decomposed as

$$
V^{-1} \partial_\alpha V = P_\alpha + Q_\alpha.
$$

(2.8)

Here, $Q_\alpha$ belongs to the Lie algebra of the maximally compact subgroup $O(d) \times O(d)$ and $P_\alpha$ belongs to the complement. Furthermore, it follows from the symmetric space automorphism property of the coset $\frac{O(d, d)}{O(d) \times O(d)}$ that $P_\alpha^T = P_\alpha, Q_\alpha^T = -Q_\alpha$; therefore,

$$
P_\alpha = \frac{1}{2} \left( V^{-1} \partial_\alpha V + (V^{-1} \partial_\alpha V)^T \right),
$$

$$
Q_\alpha = \frac{1}{2} \left( V^{-1} \partial_\alpha V - (V^{-1} \partial_\alpha V)^T \right).
$$

(2.9)

It is now straightforward to show that

$$
\text{Tr} \left( \partial_\alpha M^{-1} \partial_\beta M \right) = -4 \text{Tr} \left( P_\alpha P_\beta \right).
$$

(2.10)
Furthermore, the currents in (2.8) are invariant, under global $O(d,d)$ rotations whereas under a local $O(d) \times O(d)$ transformation, $V \rightarrow V h(x)$,

$$P_\alpha \rightarrow h^{-1}(x) P_\alpha h(x), \quad Q_\alpha \rightarrow h^{-1}(x)Q_\alpha h(x) + h^{-1}(x)\partial_\alpha h(x). \quad (2.11)$$

Thus, $Q_\alpha$ transforms like a gauge field under local $O(d) \times O(d)$ transformations, while $P_\alpha$ transforms in the adjoint representation under a gauge transformation. It is clear, therefore, that (2.10) is invariant under the global $O(d,d)$ as well as the local $O(d) \times O(d)$ transformations. Consequently, the action in (2.1) is also invariant under the local $O(d) \times O(d)$ transformations.

Let us introduce a one parameter family of matrices $\hat{V}(x,t)$ $t$ being the spectral parameter (we denote time coordinate as $x^0$), such that $\hat{V}(x,t = 0) = V(x)$ and

$$\hat{V}^{-1}\partial_\alpha \hat{V} = Q_\alpha + \frac{1 + t^2}{1 - t^2} P_\alpha + \frac{2t}{1 - t^2} \epsilon_{\alpha\beta} P^\beta. \quad (2.12)$$

In the case of a sigma model in the flat space, the spectral parameter turns out to be a constant. However, when we consider a sigma model in curved space-time, it is necessary for the spectral parameter to be a local function satisfying the first order differential equation

$$\partial_\alpha e^{-\tilde{\phi}} = -\frac{1}{2} \epsilon_{\alpha\beta} \partial^\beta \left( e^{-\tilde{\phi}} \left( t + \frac{1}{t} \right) \right), \quad (2.13)$$

in order to fulfill consistency conditions arising from integrability properties. The solution for the shifted dilaton can be written as a sum

$$\rho(x) = e^{-\tilde{\phi}(x)} = \rho_+(x^+) + \rho_-(x^-), \quad (2.14)$$

in terms of which the solution to Eq. (2.13) is expressed as

$$t(x) = \frac{\sqrt{\omega + \rho_+} - \sqrt{\omega - \rho_-}}{\sqrt{\omega + \rho_+} + \sqrt{\omega - \rho_-}}, \quad (2.15)$$

where $\omega$ is the constant of integration, which can be thought of as a global spectral parameter. It is straightforward to check that the zero curvature condition following from (2.12) leads to the integrability of the current in (2.8) as well as the dynamical equation for the nonlinear sigma model derived from (2.1).

Several comments are in order at this stage which follow from the foregoing discussions. First, the one parameter family of connections (currents) does not determine the potential $\hat{V}(x,t)$ uniquely, namely, $\hat{V}$ and $S(\omega)\hat{V}$, where $S(\omega)$ is a constant matrix, yield the same one parameter family of connections. Second, in the presence of the spectral parameter, the symmetric space automorphism can be generalized as

$$\eta^\infty(\hat{V}(x,t)) = \eta \left( \hat{V} \left( x, \frac{1}{t} \right) \right) = \left( \hat{V}^{-1} \left( x, \frac{1}{t} \right) \right)^T. \quad (2.16)$$
Thus the family of matrices, $\hat{V}(x,t)$, satisfy

$$
\left(\hat{V}^{-1}(x,\frac{1}{t}) \partial_\alpha \hat{V}(x,\frac{1}{t})\right)^T = -\hat{V}^{-1}(x,t)\partial_\alpha \hat{V}(x,t).
$$

(2.17)

We are now in a position to define the monodromy matrix as

$$
\mathcal{M} = \hat{V}(x,t)\hat{V}^T(x,\frac{1}{t}).
$$

(2.18)

It follows, from Eq. (2.17), that

$$
\partial_\alpha \mathcal{M} = 0.
$$

(2.19)

In other words, $\mathcal{M} = \mathcal{M}(\omega)$ is independent of space-time coordinates. The monodromy matrix for a system under investigation encodes integrability properties such as the conserved charges associated with the theory.

3. A Wess-Zumino-Witten Model

We are interested in investigating attributes of the monodromy matrix in the pp-wave limit. We focus our attention on a specific WZWN model due to Sfetsos and Tseytlin [23]. The pp-wave limit exists for this model. The model describes a gauged WZWN model based on $(E_2^c \otimes U(1))/U(1)$, where $E_2^c$ represents the two dimensional Euclidean group with a central extension. Following the notation of [23], the action is given by

$$
S = I_0(g_1) + \frac{1}{\pi} \int d^2z \left[ \frac{1}{2} \partial \phi \bar{\partial} \phi + i A \left( \sqrt{q_0} \partial \phi + \bar{\partial} x_1 + \cos u \partial x_2 \right)
- i \bar{A} \left( \sqrt{q_0} \partial \phi + \bar{\partial} x_2 + \cos u \partial x + \bar{A} A (1 + \cos u + 2q_0) \right) \right],
$$

(3.1)

where we have identified $g_1 = E_2^c, g_2 = U(1)$ and $I_0(g_1)$ represents the action for the WZWN model for the group $E_2^c$. The group elements are

$$
g = \text{diag} (g_1, g_2),
g_1 = e^{ix_1 P_1} e^{iuJ} e^{ix_2 P_2} e^{iF} \in E_2^c, 
g_2 = e^{i\phi P_0} \in U(1).
$$

(3.2)

Here, $P_i, J, i = 1, 2$ represent the generators of the two dimensional Euclidean group, $F$ the central extension generator and $P_0$ is the generator for $U(1)$. Note that $q_0$ appearing in (3.1) is a constant parameter which parameterizes the embedding of $U(1)$ into $E_2^c \otimes U(1)$.

The action (3.1) is invariant under the infinitesimal gauge transformations

$$
\delta x_1 = \delta x_2 = \alpha(x), \quad \delta \phi = 2 \sqrt{q_0} \alpha(x), \quad \delta A = i \partial \alpha(x), \quad \delta \bar{A} = -i \bar{\partial} \alpha(x),
$$

(3.3)
Furthermore, $u$ and $v$ remain inert under the gauge transformation. Thus we use this property to fix the gauge $\phi = 0$. The metric, the anti-symmetric tensor fields as well as the shifted dilaton can be identified from (3.1) (after some re-scaling) to be

\begin{equation}
    ds^2 = 2dvd\!u + \frac{\cos^2 u}{q_0 + \cos^2 u} d\!x_1^2 + \frac{\sin^2 u}{q_0 + \cos^2 u} d\!x_2^2, \tag{3.4}
\end{equation}

\begin{equation}
    B_{12} = -B_{21} = b = \sqrt{q_0(1 + q_0)} q_0 + \cos^2 u, \tag{3.5}
\end{equation}

\begin{equation}
    \bar{\phi} = -\ln \sin^2 2u, \tag{3.6}
\end{equation}

where we have chosen a specific value for the constant (in the expression for the dilaton) for later convenience. We note that the anti-symmetric tensor field $b = 0$ when $q_0 = 0$. We recall, from our earlier experience, that it might be convenient to start from the following set of background configurations

\begin{equation}
    ds^2 = 2dvd\!u + d\!x_1^2 + \tan^2 u d\!x_2^2, \tag{3.7}
\end{equation}

\begin{equation}
    b_0 = 0, \tag{3.8}
\end{equation}

\begin{equation}
    \bar{\phi}_0 = -\ln \sin^2 2u, \tag{3.9}
\end{equation}

and then implement an appropriate T-duality transformation to obtain the backgrounds associated with the action (3.1).

Let us note from the structure of the backgrounds in (3.1) as well as from the definition of the $M$-matrix

\begin{equation}
    M = VV^T = \begin{pmatrix}
        G^{-1} & -G^{-1}B \\
        BG^{-1} & G - BG^{-1}B
    \end{pmatrix}, \tag{3.10}
\end{equation}

that we can identify

\begin{equation}
    G = \begin{pmatrix}
        G_{11} & 0 \\
        0 & G_{22}
    \end{pmatrix}, \tag{3.11}
\end{equation}

where

\begin{equation}
    G_{11} = \frac{\cos^2 u}{q_0 + \cos^2 u}, \quad G_{22} = \frac{\sin^2 u}{q_0 + \cos^2 u}, \tag{3.12}
\end{equation}

and the anti-symmetric field is given by (3.5). For $q_0 = 0$, on the other hand, we have (see (3.7), (3.8) and (3.10))

\begin{equation}
    M_0 = \begin{pmatrix}
        G_0^{-1} & 0 \\
        0 & G_0
    \end{pmatrix}, \tag{3.13}
\end{equation}

with

\begin{equation}
    G_0 = \begin{pmatrix}
        1 & 0 \\
        0 & \tan^2 u
    \end{pmatrix}. \tag{3.14}
\end{equation}
It is straightforward to show from these that the two matrices, $M$ and $M_0$, are related through
\[ M = \Omega^T M_0 \Omega, \tag{3.15} \]
where the rotation matrix $\Omega$ has the explicit form
\[
\Omega = \begin{pmatrix}
\sqrt{1+q_0} & 0 & 0 & -\sqrt{q_0} \\
0 & \sqrt{1+q_0} & \sqrt{q_0} & 0 \\
0 & \sqrt{q_0} & \sqrt{1+q_0} & 0 \\
-\sqrt{q_0} & 0 & 0 & \sqrt{1+q_0}
\end{pmatrix}. \tag{3.16}
\]

4. The Monodromy matrix

The construction of the monodromy matrix for this model can be carried out in a manner parallel to the case of the Nappi-Witten model [24]. For $q_0 = 0$, we can write (see (2.5) and (3.14))
\[
V^{(B=0)} = \begin{pmatrix} (E^{(B=0)})^{-1} & 0 \\ 0 & E^{(B=0)} \end{pmatrix}, \quad E^{(B=0)} = \begin{pmatrix} 1 & 0 \\ 0 & \tan u \end{pmatrix}, \tag{4.1}
\]
so that we can determine (see (2.9) and we will use light-cone variables where we identify $u = x^+, v = x^-$)
\[
P^{(B=0)}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\csc 2x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\csc 2x \end{pmatrix},
\]
\[
P^{(B=0)}_- = 0 = Q^{(B=0)}_\pm. \tag{4.2}
\]
Therefore, the one parameter family of potentials $\hat{V}^{(B=0)}(x, t)$ satisfy (since $B = 0$)
\[
(\hat{V}^{(B=0)})^{-1} \partial_{\pm} \hat{V}^{(B=0)} = \frac{1 \pm t}{1 \mp t} P^{\pm}, \tag{4.3}
\]
where $t$ denotes the space-time dependent spectral parameter.

From the form of $P_+$ in (4.2), we expect $\hat{V}^{(B=0)}$ in (4.3) to be diagonal. We note that, with the identification of the shifted dilaton in (3.6) (or (3.9)), we obtain the solution
\[
\rho = e^{-\phi} = \rho_+(x^+) + \rho(x^-), \quad \rho_+ = \frac{1}{2} \left( 1 - 2 \cos^2 2x^+ \right), \quad \rho_- = \frac{1}{2}. \tag{4.4}
\]
In this case, the spectral parameter does not depend on $x^-$ and satisfies the equation (see (2.13))
\[
\partial_{\pm} t = -\frac{4t(1-t)}{(1+t)} \frac{\cos 2x^+}{\sin 2x^+}. \tag{4.5}
\]
Following our earlier method [20], we can now determine the diagonal form of the matrix $\hat{V}$ to be

$$\hat{V}^{(B=0)}(x,t) = \text{diag} \left( \hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{V}_4 \right)$$

$$= \text{diag} \left( 1, \frac{(1-t)\tan x^+}{t + \tan^2 x^+}, \frac{t + \tan^2 x^+}{1-t\tan x^+} \right). \quad (4.6)$$

We note that, for the case at hand, we have $t_2 = 1$ and $t_4 = -\tan^2 x^+$ corresponding to $\omega_2 = \frac{1}{2} = -\omega_4$ respectively. It is easy to check that when $t = 0$, the matrix $\hat{V}^{(B=0)}$ indeed reduces to $V^{(B=0)}$ in (4.3). The monodromy matrix can be easily constructed now and has the form

$$\mathcal{M}^{(B=0)} = \hat{V}^{(B=0)}(x,t) \left( \hat{V}^{(B=0)} \right)^T \left( x, \frac{1}{t} \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-2\omega}{1+2\omega} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1+2\omega}{1-2\omega} \end{pmatrix}. \quad (4.7)$$

The poles of the monodromy matrix at $\omega_2 = \frac{1}{2} = -\omega_4$ are now manifest. However, the other two trivial diagonal elements of the monodromy matrix would seem to suggest that we are missing two other poles. A careful analysis shows that the location of the other two poles has moved to infinity. Thus we recognize that for $\omega_1 = -\omega_3 \to \infty$, we obtain the following relations

$$\frac{\omega_1 - \omega}{\omega_1 + \omega} = 1 = \frac{\omega_1 + \omega}{\omega_1 - \omega}. \quad (4.8)$$

In other words, in the pp-wave limit, the monodromy matrix for our model has even a simpler structure than that of a generic WZWN model.

Notice that having constructed the monodromy matrix for the case $b = 0$ (or $q_0 = 0$), and analyzed its properties, we are in a position to generalize the results to the case for nontrivial values of the anti-symmetric tensor field. We note that

$$V^{(B)}(x) = \begin{pmatrix} (E^{(B)})^{-1} & 0 \\ 0 & E^{(B)} \end{pmatrix} = \Omega^T V^{(B=0)}(x) h(x), \quad (4.9)$$

where $\Omega$ represents the global rotation defined in (3.16 and $h(x) \in H$ which is the maximal compact subgroup of the T-duality group.

$$E^{(B)} = \frac{1}{\sqrt{q_0 + \cos^2 x^+}} \begin{pmatrix} \cos x^+ & 0 \\ 0 & \sin x^+ \end{pmatrix}, \quad (4.10)$$
and

\[
    h(x) = \begin{pmatrix}
    \cos \theta & 0 & 0 & \sin \theta \\
    0 & \cos \theta - \sin \theta & 0 & 0 \\
    0 & \sin \theta & \cos \theta & 0 \\
    -\sin \theta & 0 & 0 & \cos \theta
    \end{pmatrix},
\]

where the angular variable \( \theta \) is defined by

\[
    \sin^2 \theta = \frac{q_0 \tan^2 x^+}{q_0 + \cos^2 x^+}.
\]

It follows now directly from the above relations that

\[
    P^{(B)}_\pm = h^{-1}(x)P^{(B=0)}_\pm h(x) = \frac{2}{q_0 + \cos^2 x^+} \begin{pmatrix}
    q_0 \tan x^+ & 0 & 0 & -\sqrt{q_0(q_0 + 1)} \\
    0 & -\frac{(q_0 + 1)}{\tan x^+} & \sqrt{q_0(q_0 + 1)} & 0 \\
    0 & -\sqrt{q_0(q_0 + 1)} & -q_0 \tan x^+ & 0 \\
    \sqrt{q_0(q_0 + 1)} & 0 & 0 & \frac{(q_0 + 1)}{\tan x^+}
    \end{pmatrix},
\]

\[
    P^{(B)}_- = h^{-1}(x)P^{(B=0)}_- h(x) = 0,
\]

\[
    Q^{(B)}_\pm = h^{-1}(x)\partial_+ h(x) + h^{-1}(x)Q^{(B=0)}_\pm h(x)
    \]

\[
    = \frac{2\sqrt{q_0(q_0 + 1)}}{q_0 + \cos^2 x^+} \begin{pmatrix}
    0 & 0 & 0 & 1 \\
    0 & 0 & -1 & 0 \\
    0 & 1 & 0 & 0 \\
    -1 & 0 & 0 & 0
    \end{pmatrix},
\]

\[
    Q^{(B)}_- = h^{-1}(x)\partial_- h(x) + h^{-1}(x)Q^{(B=0)}_- h(x) = 0.
\]

We have computed \( P^{(B)}_\pm \) and \( Q^{(B)}_\pm \) from the ST model directly and verified that \( P^{(B)}_\pm \) and \( Q^{(B)}_\pm \) obtained through the duality transformations from the \( B = 0 \) model agree with these direct computations.

The one parameter family of potentials can also be constructed in a straightforward manner and take the form

\[
    \dot{V}^{(B)}(x, t) = \Omega^T\dot{V}^{(B=0)} h(x) = \begin{pmatrix}
    a_1 & 0 & 0 & a_2 \\
    0 & a_3 & a_4 & 0 \\
    0 & a_5 & a_6 & 0 \\
    a_7 & 0 & 0 & a_8
    \end{pmatrix},
\]

\[
\]

\[\text{– 10 –}\]
with
\[
\begin{align*}
a_1 &= \sqrt{1 + q_0} \cos \theta + \sqrt{q_0} \hat{V}_4 \sin \theta, \\
a_2 &= \sqrt{1 + q_0} \sin \theta - \sqrt{q_0} \hat{V}_4 \cos \theta, \\
a_3 &= \sqrt{q_0} \sin \theta + \sqrt{1 + q_0} \hat{V}_2 \cos \theta, \\
a_4 &= \sqrt{q_0} \cos \theta - \sqrt{1 + q_0} \hat{V}_2 \sin \theta, \\
a_5 &= \sqrt{1 + q_0} \sin \theta + \sqrt{q_0} \hat{V}_4 \cos \theta, \\
a_6 &= \sqrt{1 + q_0} \cos \theta - \sqrt{q_0} \hat{V}_4 \sin \theta, \\
a_7 &= -\sqrt{q_0} \cos \theta + \sqrt{1 + q_0} \hat{V}_4 \sin \theta, \\
a_8 &= -\sqrt{q_0} \sin \theta + \sqrt{1 + q_0} \hat{V}_4 \cos \theta,
\end{align*}
\] (4.15)
(4.16)
(4.17)
(4.18)
(4.19)
(4.20)
(4.21)
(4.22)

with \( \hat{V}_i \) and \( \theta \) defined respectively in (4.3) and (4.12). In turn, this gives us the monodromy matrix for the general case of the form
\[
\mathcal{M}^{(B)} = \hat{V}^{(B)}(x, t) \left( \hat{V}^{(B)} \right)^T (x, \frac{1}{\omega})
\]
(4.23)

where
\[
\begin{align*}
b_1 &= 1 + \frac{2q_0}{1 - 2\omega}, \\
b_2 &= -\sqrt{q_0(1 + q_0)} - \sqrt{q_0} \frac{1 + 2\omega}{1 - 2\omega}, \\
b_3 &= -1 + \frac{2(1 + q_0)}{1 + 2\omega}, \\
b_4 &= \sqrt{q_0} + \sqrt{q_0(1 + q_0)} \frac{1 - 2\omega}{1 + 2\omega}, \\
b_5 &= 1 + \frac{2q_0}{1 + 2\omega}, \\
b_6 &= -1 + \frac{2(1 + q_0)}{1 - 2\omega}.
\end{align*}
\] (4.24)
(4.25)
(4.26)
(4.27)
(4.28)
(4.29)

There are several interesting features to be noted from the form of the general monodromy matrix. First, it is straightforward to check that
\[
\mathcal{M}^{(B)} = \Omega^T \mathcal{M}^{(B=0)} \Omega
\] (4.30)
and that it reduces to the monodromy matrix (4.7) for the simpler background i.e. for vanishing anti-symmetric field when \( q_0 = 0 \).

The monodromy matrix for the Nappi-Witten model can also be obtained in the pp-wave limit adapting the same procedure. We may recall that in the case of the Nappi-Witten model

\[
\begin{align*}
\text{d}s^2 & = -\text{d}\tau^2 + \text{d}x^2 + \frac{1}{1 - \cos 2\tau \cos 2x} \\
& \quad \times 4 (\cos^2 \tau \cos^2 x \text{d}y^2 + \sin^2 \tau \sin^2 x \text{d}z^2), \\
\phi & = -\frac{1}{2} \ln (1 - \cos 2\tau \cos 2x), \\
B_{12} & = -B_{21} = b = \frac{\cos 2\tau - \cos 2x}{1 - \cos 2\tau \cos 2x}. \quad (4.31)
\end{align*}
\]

Here \( \tau \) represents time (to avoid confusion with the spectral parameter) and we have set an arbitrary constant parameter of the model to zero for conveniences.

Let us define the light-cone coordinates

\[
\begin{align*}
x^+ & = \frac{\tau + x}{\sqrt{2}}, \\
x^- & = \frac{\tau - x}{\sqrt{2}},
\end{align*}
\]

and take the pp-wave limit \( v \to 0 \). Note from (4.31) that the anti-symmetric tensor field vanishes in this limit and we have

\[
\begin{align*}
\text{d}s^2 & = -2\text{d}x^+ \text{d}x^- + \tan^{-2} x^+ \text{d}y^2 + \tan^2 x^- \text{d}z^2, \\
\bar{\phi} & = -\ln \sin^2 \sqrt{2}x^+ , \quad (4.33)
\end{align*}
\]

where \( \bar{\phi} \) represents the shifted dilaton as before. In this case, the one parameter family of connections can be obtained. For the specific identifications \( \omega_1 = \omega_2 = \frac{1}{2} \) we have two independent solutions (depending on the choice of the sign of the square root) and we get

\[
\hat{V}(x, t) = \text{diag} \left( \hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{V}_4 \right), \quad (4.34)
\]

where

\[
\begin{align*}
\hat{V}_1 & = -\frac{t - 1}{t \tan^2 \frac{x^+}{\sqrt{2}} + 1} = \hat{V}_3^{-1}, \\
\hat{V}_2 & = -\frac{(t - 1) \tan^2 \frac{x^+}{\sqrt{2}}}{t - \tan^2 \frac{x^+}{\sqrt{2}}} = \hat{V}_4^{-1}. \quad (4.35)
\end{align*}
\]
We note that when \( t = 0 \), this reduces to \( V \)-matrix for this theory. The monodromy matrix is now given by

\[
M^{(NW)} = \hat{V}(x, t) \left( \hat{\nu} \right)^T \left( x, \frac{1}{t} \right)
\]

\[
= \begin{pmatrix}
\frac{1-2\omega}{1+2\omega} & 0 & 0 & 0 \\
0 & \frac{1-2\omega}{1+2\omega} & 0 & 0 \\
0 & 0 & \frac{1+2\omega}{1-2\omega} & 0 \\
0 & 0 & 0 & \frac{1+2\omega}{1-2\omega}
\end{pmatrix}.
\] (4.36)

Therefore, we conclude that in the pp-wave limit the monodromy matrix in the Nappi-Witten model has degenerate poles.

5. Summary and Discussion

In this paper we have constructed the monodromy matrix for a one dimensional \( \sigma \)-model arising in the pp-wave limit of the gauged WZWN \( \sigma \)-model \[23\]. In order to study the integrability properties of the effective action, we introduced the spectral parameter, \( t \), following the standard procedure. It has been argued that \( t \) is space-time dependent from consistency requirements and the integration constant, \( \omega \), may also be identified as another global spectral parameter. Furthermore, as we have shown, it is possible to trade \( \omega \) for \( t \) in constructing the monodromy matrix. Let us recall that with the introduction of \( t \) (alternatively \( \omega \)), we are led to envisage a ‘vielbein’, \( \hat{E}(x, t) \), which depends on the continuous parameter \( t \) besides space-time coordinates \( x \). When we set \( t = 0 \), we get back \( E(x) \) which is utilized to define the action and in turn the action is invariant under the global symmetry group \( G \) and the local group \( H \). In defining \( \hat{E} \), we have introduced an infinite family of matrices parameterized by a continuous spectral parameter. Indeed, the appearance of the infinite-dimensional symmetries of Kac-Moody type, arising in such two dimensional integrable models, can be intuitively understood from this perspective. It has been argued \[9, 13\] that this symmetry can be even larger when the theory is reduced to one dimension. In the present case, however, the algebra of charges can be worked out in a manner completely parallel to that in \[4\] and we find that it coincides with the algebra in \[4\]. We do not find any further enhancement of the algebra beyond what is already known under similar situation. In several known cases, the resulting symmetry is of hyperbolic type \[25\]. For instance, the hyperbolic algebra \( E_{10} \) \[14, 24, 27, 28\] has been argued to be the symmetry of the eleven-dimensional supergravity dimensionally reduced to one dimension \[13, 23\]. Furthermore, when four-dimensional supergravity is reduced to one dimension, the resulting symmetry has been shown to be the hyperbolic extension of \( A_1^1 \) algebra \[30\]. However, these hyperbolic algebras do not generally allow supersymmetric extensions. Therefore,
such an enlarged symmetry should already contain both bosonic and fermionic generators. From this perspective, it would be interesting to study the supersymmetric generalization of the $\sigma$-model considered in this paper. We speculate that, in such a case, one should find a hyperbolic extension of an algebra reminiscent of $A_1^1$. We refer the reader to the review article [6] for a detailed discussion of these issues and for further references.

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