Two-Dimensional Elliptic Determinantal Point Processes and Related Systems

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Abstract

We introduce new families of determinantal point processes (DPPs) on a complex plane $\mathbb{C}$, which are classified into seven types following the irreducible reduced affine root systems, $R_N = A_N, B_N, B_N^\vee, C_N, C_N^\vee, B_{CN}, D_N$, $N \in \mathbb{N}$. Their multivariate probability densities are totally elliptic functions with periods $(L, iW)$, $0 < L, W < \infty$, $i = \sqrt{-1}$. The construction is based on the orthogonality relations with respect to the double integrals over the fundamental domain, $[0, L) \times [0, iW)$, which are proved in this paper for the $R_N$-theta functions introduced by Rosengren and Schlosser. In the scaling limit $N \to \infty, L \to \infty$ with constant density $\rho = N/(LW)$ and constant $W$, we obtain four types of DPPs with an infinite number of points on $\mathbb{C}$, which have periodicity with period $iW$. In the further limit $W \to \infty$ with constant $\rho$, they are degenerated into three infinite-dimensional DPPs. One of them is uniform on $\mathbb{C}$ and equivalent with the Ginibre point process studied in random matrix theory, while other two systems are isotropic viewed from the origin, but non-uniform on $\mathbb{C}$. We show that the elliptic DPP of type $A_N$ is identified with the particle section, obtained by subtracting the background effect, of the two-dimensional exactly solvable model for one-component plasma studied by Forrester. Other two exactly solvable models of one-component plasma are constructed associated with the elliptic DPPs of types $C_N$ and $D_N$. Relationship to the Gaussian free field on a torus is discussed for these three exactly solvable plasma models.

1 Introduction

In a series of papers [9][11][12][13], we have studied elliptic extensions of determinantal point processes (DPPs) and determinantal processes (DPs), which are realized in the systems of noncolliding Brownian motions (BM) on a circle with radius $r > 0$ or in an interval $[0, \pi r]$ with appropriate boundary conditions at the edges $x = 0$ and $x = \pi r$. These stochastic processes are defined in a finite time duration $[0, t^\star]$, $0 < t^\star < \infty$, in which the particle configurations at the final time $t = t^\star$ are pinned at specified configurations. The basic idea of our elliptic extension is based on the fundamental fact that the Jacobi theta function $\vartheta_1(\xi; \tau)$ solves the following partial differential equation (PDE)

$$\frac{\partial^2 \vartheta_1(\xi; \tau)}{\partial \xi^2} = 4\pi i \frac{\partial \vartheta_1(\xi; \tau)}{\partial \tau},$$

(1.1)

where $i = \sqrt{-1}$. (Notations and formulas of the Jacobi theta functions used in this paper are shown in Appendix [A].) As functions of spatial and temporal coordinates $x$ and $t$, we have parameterized the two variables $\xi$ and $\tau$ as

$$\xi = \xi(x) = \frac{x}{2\pi r}, \quad \tau = \tau(t) = \frac{it}{2\pi r^2},$$

(1.2)
$x \in \mathbb{R}, t \in [0, \infty)$. Then the PDE (1.1) can be identified with the diffusion equation

$$
\left( \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) \vartheta_1(\xi(x); \tau(t)) = 0.
$$

With the positivity $\vartheta_1(\xi(x); \tau(t)) > 0$ for $x \in (0, 2\pi r), t > 0$, this fact suggests that $\vartheta_1(\xi(x); \tau(t))$ will be used to describe probability laws of the systems of BMs in the setting mentioned above. Regarding the modular parameter $\tau$ as an imaginary time and studying time-evolution of a system by continuously changing $\tau$ may provide new applications of elliptic functions and their related functions to stochastic analysis, but in our previous study, only a situation with $\xi \in [0, 1]$ has been considered and quasi-double-periodicity of $\vartheta_1(\xi; \tau)$ as a complex function of $\xi \in \mathbb{C}$ has not been used at all.

In the present paper, we assume $0 < L, W < \infty$ and consider another parameterization,

$$
\xi = \xi(x, y) = \frac{x + iy}{L}, \quad \tau = i\alpha \quad \text{with } \alpha = \frac{W}{L},
$$

(1.3)

for $(x, y) \in \mathbb{R}^2$. It is obvious by this parameterization of $\xi$, $\vartheta_1(\xi(x, y); i\alpha)$ is a harmonic function of $z = x + iy \in \mathbb{C}$. The nontrivial point of this parameterization (1.3) is found at $\tau$, which makes $\vartheta_1(\xi(x, y); i\alpha)$ have the following quasi-double-periodicity

$$
\begin{align*}
\vartheta_1(\xi(x + L, y); i\alpha) &= -\vartheta_1(\xi(x, y); i\alpha), \\
\vartheta_1(\xi(x, y + W); i\alpha) &= -e^{-2\pi i \xi(x, y) + \pi \alpha} \vartheta_1(\xi(x, y); i\alpha).
\end{align*}
$$

This fact suggests that, if we consider the absolute value $|\vartheta_1(\xi(x, y); i\alpha)|$ with an appropriate normalization, it will be used to describe probability laws of suitable random point processes defined on a complex plane $\mathbb{C}$ having a fundamental domain with periods $(L, iW)$, which is denoted as

$$
\Lambda_{(L, iW)} \equiv [0, L) \times [0, iW) \subset \mathbb{C}.
$$

In this paper. The positive parameter $\alpha$ defined in (1.3) gives an aspect ratio of the rectangular shape of this fundamental domain $\Lambda_{(L, iW)}$.

In order to obtain multivariate functions which can be used to describe probability laws of DP, the Macdonald denominator formulas were used in the previous papers [9] [11] [12] [13] and will be used in the present paper, which were obtained by Rosengren and Schlosser for the seven types of irreducible reduced affine root systems, $R_N = A_{N-1}$, $B_N$, $B_N^\vee$, $C_N$, $C_N^\vee$, $BC_N$, $D_N$, $N \in \mathbb{N}$ [13]. More precisely speaking, Rosengren and Schlosser introduced a series of $A_{N-1}$-theta functions of appropriate norm, $f_j^{A_{N-1}}(\xi; \tau), j \in \mathbb{N}$, and series of $R_N$-theta functions, $f_j^{R_N}(\xi; \tau), j \in \mathbb{N}$, for $R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$, and represented the seven types of Macdonald denominators, $W^{R_N}(\xi; \tau)$ with $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N$, using the determinants $\det_{1 \leq j, k \leq N} [f_j^{R_N}(\xi_k; \tau)]$.

A new aspect of the $R_N$-theta functions of Rosengren and Schlosser, $\{f_j^{R_N}(\xi, \tau)\}$, was reported in [13] such that, if we set

$$
M_j^{R_N}(x, t) = f_j^{R_N}(N^{R_N} \xi(x); N^{R_N} \tau(t))
$$

(1.4)

under the first parameterization (1.2), where $N^{R_N}$ is defined by (2.4) depending on $R_N, N \in \mathbb{N}$ below, the following biorthogonality relations are established:

$$
\int_0^{L_N} dx \overline{M_j^{R_N}(x, t_*)} M_k^{R_N}(x, t) = m_{jk}^{R_N}(t_*) \delta_{jk},
$$

for any $t \in (0, t_*)$, if $j, k \in \{1, 2, \ldots, N\}$, where $L^{A_{N-1}} = 2\pi r$ and $L^{R_N} = \pi r$ for $R_N = B_N, B_N^\vee, C_N, C_N^\vee, BC_N, D_N$, and $M_j^{R_N}(x, t_* - t)$ denotes the complex conjugate of $M_j^{R_N}(x, t_* - t)$. The functions $\{m_{jk}^{R_N}(t_*)\}_{j,k \in \{1, 2, \ldots, N\}}$ are explicitly given in Lemma 2.1 in [13]. In the present paper, we set

$$
M_j^{R_N}(x + iy) = f_j^{R_N}(N^{R_N} \xi(x + iy); i\alpha)
$$

(1.5)
under the second parameterization \([13]\), and we prove the orthogonality relations for the double integrals over the fundamental domain \(\Lambda_{(L, iW)}\),
\[
\int_0^L dx \int_0^W dy \exp \left( -\frac{2\pi N R_S}{LW} y^2 \right) M_j^{RN}(x + iy) M_k^{RN}(x + iy) = h_j^{RN} \delta_{jk},
\]
for \(j, k \in \{1, 2, \ldots, N\}\), where \(\{h_j^{RN}\}_{j \in \{1, 2, \ldots, N\}}\) are given in Proposition \([23]\) depending on \(R_N, N \in \mathbb{N}\).

Once such orthogonality relations are proved, it is rather easy to construct DPPs on a complex plane \(\mathbb{C}\). We can also consider the scaling limit \(N \to \infty, L \to \infty\) with constant particle density \(\rho = N/(LW)\) and constant \(W\). The obtained DPPs with an infinite number of points are of four types and they are defined on \(\mathbb{C}\) with a period \(iW\) in the direction of imaginary axis. We study the further limit \(W \to \infty\) with constant \(\rho\). In this limit, we have three types of infinite DPPs on \(\mathbb{C}\), one of which is identified with the Ginibre point process studied in random matrix theory as the eigenvalue ensemble of complex Gaussian random matrices \([6, 7, 20]\), while other two infinite DPPs on \(\mathbb{C}\) are new.

Appearance of the exponential weights \(\exp(-2\pi R_S N y^2/(LW))\) in the double integrals \([10]\) is essential, and we derive them by requiring that the probability measures of the DPPs on \(\mathbb{C}\) should be totally elliptic (in the sense of \([22]\), see Definition \([2.2]\) below) with respect to all \(N\) complex variables \((z_1, \ldots, z_N) = (x_1 + iy_1, \ldots, x_N + iy_N)\) which describe \(N\)-point configurations on \(\mathbb{C}\).

Forrester studied a particle system in \(\Lambda_{(L, iW)}\) with doubly periodic boundary conditions such that \(N\) mobile particles, each of which is charged \(+1\), are interacting via the pair potential given using logarithmic of \(\vartheta_1(\xi(x + iy); i\alpha)\) and these particles are confined to the domain \(\Lambda_{(L, iW)}\). He assumed that a uniform background with negative charge density \(-N/LW\) exists and the system is neutralized. Such a system consisting of positively charged \(N\) particles and negatively charged background is called a one-component plasma model \([8, 4, 5]\). In the present paper, we show that the elliptic DPP of type \(A_{N-1}\) is identified with the particle section of Forrester’s one-component plasma model, in which the background effect is subtracted. We also show that other two elliptic DPPs of types \(C_N\) and \(D_N\) are also realized as the particle sections of one-component plasma models, while these additional two systems are not perfectly neutralized. This consideration gives another derivation of the exponential weights \(\exp(-2\pi R_S N y^2/(LW))\) in \([10]\) at least for the three systems of types \(A_{N-1}, C_N\), and \(D_N\).

Forrester claimed the equivalence between exact-solvability and double-periodicity in his plasma model of type \(A_{N-1}\). This statement is extended to all seven types though the Macdonald denominator formulas of Rosengren and Schlosser \([18]\) and the orthogonality relations \([10]\), in which these two notions are mathematically restated as being determinantal and being totally elliptic, respectively.

Forrester discussed an interesting relationship of his one-component plasma model to the Gaussian free field (GFF) defined on a torus studied by Cardy \([2]\). We develop his argument to our two additional models of types \(C_N\) and \(D_N\). Correspondence to the modular invariance of the partition function discussed by Cardy for the GFF on a torus, we find the correction terms in large \(N\) expansion of the free energies of our new plasma models of types \(C_N\) and \(D_N\), which are invariant under the transformation \(\alpha = W/L \to 1/\alpha = L/W\) of the aspect ratio of \(\Lambda_{(L, iW)}\).

The paper is organized as follows. In Section \([2]\) we first introduce notations used in this paper associated with Appendix \([A]\) and we list up the Macdonald denominator formulas of Rosengren and Schlosser \([18]\) in our notations. Then we construct the seven types of probability weights, \(Q^{RN}(z)\) for \(R_N = A_{N-1}, B_N, B'_N, C_N, C'_N, BC_N, D_N, N \in \mathbb{N}\), which are all totally elliptic with periods \((L, iW)\) with respect to \(N\)-component complex variables \(z\) representing point configurations for the systems. The orthogonality relations \([10]\) are proved in Proposition \([2.5]\) for the \(R_N\)-theta functions of Rosengren and Schlosser with the setting of variables given by \([15]\). In Section \([3]\) the seven types of point processes on \(\mathbb{C}\), \((\Xi^{RN}, \mathbf{P}^{RN})\), \(R_N = A_{N-1}, B_N, B'_N, C_N, C'_N, BC_N, D_N\), are constructed by properly normalizing the totally elliptic weights \(Q^{RN}(z)\) to define the probability laws \(\mathbf{P}^{RN}\). Then we can prove that all of them are determinantal following the standard method in random matrix theory \([16, 5, 11, 10]\), and we give the correlation kernels using the orthogonal elliptic functions, \(\{M_j^{RN}(z)\}_{j=1}^{N}\) (Theorem \([3.2]\)). The scaling limits \(N \to \infty, L \to \infty\) with constant point-density \(\rho\) and \(W\) are calculated for the correlation kernels (Proposition \([3.3]\)), and the four types of DPPs with an infinite number of particles are derived on \(\mathbb{C}\) (Theorem \([5.1]\)). In the further limit \(W \to \infty\) with \(\rho = \text{const.}\),
these four systems are degenerated into three infinite DPPs, one of which is identified with the Ginibre point process [6, 7, 20] (Theorem 3.5). In Section 2 we first review the one-component plasma model studied by Forrester [11] and then introduce other two models, which are defined in $\Lambda_{(L, Aw)}$ with doubly periodic conditions. We show that the particle section of Forrester’s model is identified with the elliptic DPP of type $A_{N-1}$, $(\Xi^{A_{N-1}}, P^{A_{N-1}})$. We also prove that other two models realize $(\Xi^{C_{N}}, P^{C_{N}})$ and $(\Xi^{B_{N}}, P^{B_{N}})$ as their particle sections (Theorem 1.2). Relationship to the GFF defined on a torus studied by Cardy [2] is discussed for these three types of exactly solvable plasma models. Appendices B and C support this section. Section 2 is devoted to concluding remarks.

2 Preliminaries

2.1 Macdonald denominator formulas of Rosengren and Schlosser

Assume that $N \in \mathbb{N} \equiv \{1, 2, \ldots \}$. As extensions of the Weyl denominators for classical root systems, Rosengren and Schlosser studied the Macdonald denominators for the seven types of irreducible reduced affine root systems, $W^{R_{N}}(\xi), \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^{N}, R_{N} = A_{N-1}, B_{N}, B_{N}^{\vee}, C_{N}, C_{N}^{\vee}, B_{C}, D_{N}, N \in \mathbb{N}$ [18].

Up to trivial factors they are written using the Jacobi theta functions as follows.

$$W^{A_{N-1}}(\xi; \tau) = \prod_{1 \leq j < k \leq N} \vartheta_{1}(\xi_j - \xi_k; \tau),$$

$$W^{B_{N}}(\xi; \tau) = \prod_{\ell=1}^{N} \vartheta_{1}(\xi_{\ell}; \tau) \prod_{1 \leq j < k \leq N} \left\{ \vartheta_{1}(\xi_j - \xi_k; \tau) \vartheta_{1}(\xi_j + \xi_k; \tau) \right\},$$

$$W^{B_{N}^{\vee}}(\xi; \tau) = \prod_{\ell=1}^{N} \vartheta_{1}(2\xi_{\ell}; 2\tau) \prod_{1 \leq j < k \leq N} \left\{ \vartheta_{1}(\xi_j - \xi_k; \tau) \vartheta_{1}(\xi_j + \xi_k; \tau) \right\},$$

$$W^{C_{N}}(\xi; \tau) = \prod_{\ell=1}^{N} \vartheta_{1}(2\xi_{\ell}; \tau) \prod_{1 \leq j < k \leq N} \left\{ \vartheta_{1}(\xi_j - \xi_k; \tau) \vartheta_{1}(\xi_j + \xi_k; \tau) \right\},$$

$$W^{C_{N}^{\vee}}(\xi; \tau) = \prod_{\ell=1}^{N} \left\{ \vartheta_{1}(\xi_{\ell}; \tau) \vartheta_{0}(2\xi_{\ell}; 2\tau) \right\} \prod_{1 \leq j < k \leq N} \left\{ \vartheta_{1}(\xi_j - \xi_k; \tau) \vartheta_{1}(\xi_j + \xi_k; \tau) \right\},$$

$$W^{B_{C}}(\xi; \tau) = \prod_{1 \leq j < k \leq N} \left\{ \vartheta_{1}(\xi_j - \xi_k; \tau) \vartheta_{1}(\xi_j + \xi_k; \tau) \right\},$$

$$W^{D_{N}}(\xi; \tau) = \prod_{1 \leq j < k \leq N} \left\{ \vartheta_{1}(\xi_j - \xi_k; \tau) \vartheta_{0}(\xi_j + \xi_k; \tau) \right\},$$

(2.1)

where $\tau \in \mathbb{H} \equiv \{z \in \mathbb{C}: \Re z > 0\}$. Rosengren and Schlosser introduced the notions of $A_{N-1}$-theta function of norm $t$ and $R_{N}$-theta function for $R_{N} = B_{N}, B_{N}^{\vee}, C_{N}, C_{N}^{\vee}, B_{C}, D_{N}$. They proved that, if $f^{A_{N-1}}_{j}, j = 1, 2, \ldots, N$ are $A_{N-1}$-theta function of norm $t$, then

$$\det_{1 \leq j, k \leq N} \left[ f^{A_{N-1}}_{j}(\xi_k; \tau) \right] = C^{A_{N-1}}(\tau) \vartheta_{1} \left( \sum_{\ell=1}^{N} \xi_{\ell} + t \right) W^{A_{N-1}}(\xi; \tau)$$

(2.2)

with $t = e^{2\pi i \tau}$, and if $f^{R_{N}}_{j}, j = 1, 2, \ldots, N$, are $R_{N}$-theta functions,

$$\det_{1 \leq j, k \leq N} \left[ f^{R_{N}}_{j}(\xi_k; \tau) \right] = C^{R_{N}}(\tau) W^{R_{N}}(\xi; \tau), \quad R_{N} = B_{N}, B_{N}^{\vee}, C_{N}, C_{N}^{\vee}, B_{C}, D_{N},$$

(2.3)

where $C^{R_{N}}(\tau)$ depend on $\tau$ and $N$ but not on $\xi$. The factors $C^{R_{N}}(\tau)$ are explicitly determined in Proposition 6.1 in [18] and the equalities (2.2) and (2.3) are called the Macdonald denominator formulas. See also [15, 23].
For \( N \in \mathbb{N} \), we set

\[
\mathcal{N}^{R_N} = \begin{cases} 
N, & R_N = A_{N-1}, \\
2N - 1, & R_N = B_N, \\
2N, & R_N = B_N', C_N' , \\
2(N + 1), & R_N = C_N, \\
2N + 1, & R_N = BC_N, \\
2(N - 1), & R_N = D_N.
\end{cases}
\] (2.4)

In the present paper, we use the \( A_{N-1} \)-theta functions \( \{ f_j^{A_{N-1}}(\xi; \tau) \}_{j \in \mathbb{N}} \) of norm \( t = e^{2\pi i R_N} \) with

\[
\tilde{\eta}_N = \begin{cases} 
N \alpha / 2, & \text{if } N \text{ is even}, \\
(1 + i N \alpha) / 2, & \text{if } N \text{ is odd},
\end{cases}
\]

and the \( R_N \)-theta functions, \( \{ f_j^{R_N}(\xi; \tau) \}_{j \in \mathbb{N}} \), for \( R_N = B_N, B_N', C_N, C_N', BC_N, D_N \), of Rosengren and Schlosser, and we define the following version of the seven finite sets of functions of a complex variable \( z \in \mathbb{C} \),

\[
M_j^{R_N}(z) = M_j^{R_N}(z; L, W) = f_j^{R_N}(\mathcal{N}^{R_N}(\xi(z)); i \alpha), \quad j = 1, 2, \ldots, N.
\] (2.5)

The explicit expressions of these functions are given by follows,

\[
M_j^{A_{N-1}}(z) = M_j^{A_{N-1}}(z; L, W) = e^{2\pi i j \xi^{A_{N-1}}(\xi(z))} \vartheta_2 \left( i j A_{N-1}(j) \alpha + \mathcal{N}^{A_{N-1}}(\xi(z)); i \mathcal{N}^{A_{N-1}} \alpha \right),
\]

\[
M_j^{R_N}(z) = M_j^{R_N}(z; L, W) = e^{2\pi i j \xi^{R_N}(\xi(z))} \vartheta_1 \left( i j R_N(j) \alpha + \mathcal{N}^{R_N}(\xi(z)); i \mathcal{N}^{R_N} \alpha \right) - e^{-2\pi i j \xi^{R_N}(\xi(z))} \vartheta_1 \left( i j R_N(j) \alpha - \mathcal{N}^{R_N}(\xi(z)); i \mathcal{N}^{R_N} \alpha \right),
\]

for \( R_N = B_N, B_N' \),

\[
M_j^{R_N}(z) = M_j^{R_N}(z; L, W) = e^{2\pi i j \xi^{R_N}(\xi(z))} \vartheta_2 \left( i j R_N(j) \alpha + \mathcal{N}^{R_N}(\xi(z)); i \mathcal{N}^{R_N} \alpha \right) - e^{-2\pi i j \xi^{R_N}(\xi(z))} \vartheta_2 \left( i j R_N(j) \alpha - \mathcal{N}^{R_N}(\xi(z)); i \mathcal{N}^{R_N} \alpha \right),
\]

for \( R_N = C_N, C_N', BC_N \),

\[
M_j^{D_N}(z) = M_j^{D_N}(z; L, W) = e^{2\pi i j \xi^{D_N}(\xi(z))} \vartheta_2 \left( i j D_N(j) \alpha + \mathcal{N}^{D_N}(\xi(z)); i \mathcal{N}^{D_N} \alpha \right) + e^{-2\pi i j \xi^{D_N}(\xi(z))} \vartheta_2 \left( i j D_N(j) \alpha - \mathcal{N}^{D_N}(\xi(z)); i \mathcal{N}^{D_N} \alpha \right),
\]

where

\[
J^{R_N}(j) = \begin{cases} 
\frac{j - 1}{2}, & R_N = A_{N-1}, C_N', \\
\frac{j - 1}{2}, & R_N = B_N, B_N', D_N, \\
j, & R_N = C_N, BC_N.
\end{cases}
\]

Let \( \eta(\tau) \) be the Dedekind modular function (see, for instance, Sec.23.15 in \[17\]),

\[
\eta(\tau) = e^{\pi i / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi n \tau i}).
\] (2.7)
In the present setting, the Macdonald denominator formulas (2.2) and (2.3) of Rosengren and Schlosser (Proposition 6.1 in [18]) are written as follows.

\[
\det_{1 \leq j, k \leq N} [M_j^{AN-1}(z_k)] =
\begin{cases}
  i^{-N(N+1)/2} a^{AN-1}(\alpha) \vartheta_0 \left( \sum_{j=1}^{N} \xi(z_j); i\alpha \right) W^{AN-1}(\xi(z); i\alpha), & \text{if } N \text{ is even,} \\
  i^{-(N-1)(N-2)/2} a^{AN-1}(\alpha) \vartheta_3 \left( \sum_{j=1}^{N} \xi(x_j); i\alpha \right) W^{AN-1}(\xi(x); i\alpha), & \text{if } N \text{ is odd,}
\end{cases}
\]

for \( N \in \mathbb{Z} \) and given by

\[
a^{AN-1}(\alpha) = e^{(2N+1)(2N+1)\pi \alpha/12 \eta(i\alpha)^{-(N-1)(N-2)/2}},
\]

\[
a^{BN}(\alpha) = 2e^{N(N-1)\pi \alpha/6 \eta(i\alpha)^{-N(N-1)}},
\]

\[
a^{B_{N}'}(\alpha) = 2e^{(N-1)(2N-1)\pi \alpha/12 \eta(i\alpha)^{-(N-1)^2 \eta(2i\alpha)^{-(N-1)}},
\]

\[
a^{CN}(\alpha) = e^{N(2N+1)\pi \alpha/12 \eta(i\alpha)^{-N(N-1)}},
\]

\[
a^{C_{N}'}(\alpha) = e^{(2N-1)(2N+1)\pi \alpha/24 \eta(i\alpha)^{-(N-1)^2 \eta(i\alpha/2)^{-(N-1)}},
\]

\[
a^{BC_{N}}(\alpha) = e^{N(N+1)\pi \alpha/6 \eta(i\alpha)^{-N(N-1)}} e^{(2i\alpha)^{-N}},
\]

\[
a^{D_{N}}(\alpha) = 4e^{N(2N+1)\pi \alpha/12 \eta(i\alpha)^{-N(N-2)}},
\]

(2.8)

where \( a^{RN}(\alpha) \in \mathbb{R} \) and given by

\[
C^{RN}(z) = C^{RN}(z; L, W)
\]

\[
= \begin{cases}
    \exp \left( \frac{-\pi N^{AN-1}}{LW} \sum_{j=1}^{N} y_j^2 \right) \vartheta_s(N) \left( \sum_{k=1}^{N} \xi(z_k); i\alpha \right), & \text{for } R_N = A_{N-1}, \\
    \exp \left( \frac{-\pi N^{RN}}{LW} \sum_{j=1}^{N} y_j^2 \right), & \text{for } R_N = B_{N}, B_{N}', C_{N}, C_{N}', B_{CN}, D_{N},
\end{cases}
\]

(2.10)

For \( z \in \mathbb{C} \), we write \( x = \Re z \) and \( y = \Im z \). Let \( s(N) = 0 \) if \( N \) is even, and \( s(N) = 3 \) if \( N \) is odd. For \( z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N \), define

\[
C^{RN}(z) = C^{RN}(z; L, W)
\]

\[
= \begin{cases}
    \exp \left( \frac{-\pi N^{AN-1}}{LW} \sum_{j=1}^{N} y_j^2 \right) \vartheta_s(N) \left( \sum_{k=1}^{N} \xi(z_k); i\alpha \right), & \text{for } R_N = A_{N-1}, \\
    \exp \left( \frac{-\pi N^{RN}}{LW} \sum_{j=1}^{N} y_j^2 \right), & \text{for } R_N = B_{N}, B_{N}', C_{N}, C_{N}', B_{CN}, D_{N},
\end{cases}
\]

(2.10)

For \( z = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N \), we consider the shift operators \( \sigma_m(w) \) with \( m = 1, 2, \ldots, N, w \in \mathbb{C} \) such that, for a function \( f \) of \( z \),

\[
\sigma_m(w)f(z) = f(z_1, z_2, z_{m-1}, z_m + w, z_{m+1}, \ldots, z_N).
\]

Lemma 2.1 For \( R_N = A_{N-1}, B_{N}, B_{N}', C_{N}, C_{N}', B_{CN}, D_{N} \), define

\[
q^{RN}(z) = C^{RN}(z) W^{RN}(\xi(z); i\alpha),
\]

where \( \xi(z) = (\xi(z_1), \ldots, \xi(z_N)) \). Then these multivariate functions \( q^{RN}(z) \) are quasi-double-periodic in the sense that, for \( m = 1, 2, \ldots, N \),

\[
q^{RN}(\sigma_m(L)z) = \text{sgn}_{(L)} q^{RN}(z),
\]

(2.11)

\[
q^{RN}(\sigma_m(iW)z) = \text{sgn}_{(iW)} e^{-2\pi N^{RN} \xi(z_m)} q^{RN}(z),
\]

(2.12)
where

\[
\begin{align*}
\text{sgn}^{R_N}_{(i)} &= \begin{cases} 
1, & \text{for } R_N = A_{N-1} \text{ with } N \text{ odd, } B_N^{\nu}, C_N, D_N, \\
-1, & \text{for } R_N = A_{N-1} \text{ with } N \text{ even, } B_N, C_N^{\nu}, B_C N, D_N,
\end{cases} \\
\text{sgn}^{R_N}_{(iW)} &= \begin{cases} 
1, & \text{for } R_N = A_{N-1}, C_N, C_N^{\nu}, B_C N, D_N, \\
-1, & \text{for } R_N = B_N, B_N^{\nu}.
\end{cases}
\end{align*}
\]

Proof \((i)\) First we consider the type \(A_{N-1}\). By \((A.3)\), it is easy to see that \(C^{A_{N-1}}(z)\) is invariant under the operation of \(\sigma_m(L)\)'s, and

\[\sigma_m(L)W^{A_{N-1}}(\xi(z); i\alpha) = (-1)^{N-1}W^{A_{N-1}}(\xi(z); i\alpha), \quad m = 1, 2, \ldots, N.\]

Thus we obtain \((2.11)\) with \((2.13)\) for \(R_N = A_{N-1}\). By \((A.4)\), we see that, for \(m = 1, 2, \ldots, N,\)

\[
\begin{align*}
\sigma_m(iW)\vartheta_{s(N)} \left(\sum_{k=1}^{N} \xi(z_k); i\alpha\right) &= (-1)^{N-1} \exp \left\{ -\pi i \left( 2 \sum_{\ell=1}^{N} \xi(z_{2\ell}) + i\alpha \right) \right\} \vartheta_{s(N)} \left(\sum_{k=1}^{N} \xi(z_k); i\alpha\right), \\
\sigma_m(iW)W^{A_{N-1}}(\xi(z); i\alpha) &= (-1)^{N-1} \exp \left\{ -\pi i \left( 2(N-1)\xi(z_m) - 2 \sum_{1 \leq \ell \leq N, \ell \neq m} \xi(z_{2\ell}) + (N-1)i\alpha \right) \right\} \\
&\times W^{A_{N-1}}(\xi(z); i\alpha),
\end{align*}
\]

and then

\[
\begin{align*}
\sigma_m(iW)\vartheta_{s(N)} \left(\sum_{k=1}^{N} \xi(z_k); i\alpha\right) W^{A_{N-1}}(\xi(z); i\alpha) &= e^{-\pi i N(2\xi(z_m) + i\alpha)} \vartheta_{s(N)} \left(\sum_{k=1}^{N} \xi(z_k); i\alpha\right) W^{A_{N-1}}(\xi(z); i\alpha) \\
&= e^{-2\pi i N(\xi(z_m) + 2\xi(y_m) + N\alpha)} \vartheta_{s(N)} \left(\sum_{k=1}^{N} \xi(z_k); i\alpha\right) W^{A_{N-1}}(\xi(z); i\alpha).
\end{align*}
\]

On the other hand, we have the equality

\[
\begin{align*}
\sigma_m(iW) &\exp \left( -\frac{\pi N}{LW} \sum_{j=1}^{N} y_j^2 \right) = \exp \left[ -\frac{\pi N}{LW} \sum_{1 \leq j \leq N, j \neq m} y_j^2 - \frac{\pi N}{LW} (3(z_m + iW))^2 \right] \\
&= e^{\pi i N(2\xi(y_m) + i\alpha)} \exp \left( -\frac{\pi N}{LW} \sum_{j=1}^{N} y_j^2 \right) = e^{-2\pi N\xi(y_m) - \pi N\alpha} \exp \left( -\frac{\pi N}{LW} \sum_{j=1}^{N} y_j^2 \right).
\end{align*}
\]

Hence we obtain \((2.12)\) with \((2.14)\) for \(R_N = A_{N-1}\).

(ii) Next we consider other types \(R_N = B_N, B_N^{\nu}, C_N, C_N^{\nu}, B_C N, D_N\). It is easy to see that, for \(m = 1, 2, \ldots, N, C^{R_N}(z)\) and \(W^{D_N}(\xi(z); i\alpha)\) are invariant under \(\sigma_m(L), m = 1, 2, \ldots, N,\) and that

\[
\sigma_m(iW)C^{R_N}(z) = e^{\pi i A^{R_N}(2\xi(y_m) + i\alpha)} C^{R_N}(z), \quad R_N = B_N, B_N^{\nu}, C_N, C_N^{\nu}, B_C N, D_N,
\]

\[
\sigma_m(iW)W^{D_N}(\xi(z); i\alpha) = e^{-\pi i A^{D_N}(2\xi(z_m) + i\alpha)} W^{D_N}(\xi(z); i\alpha), \quad m = 1, 2, \ldots, N.
\]
Thus for $R_N = D_N$, (2.11) and (2.12) with (2.13) and (2.14) are proved. By (A.3) and (A.4), we can show that

$$\sigma_m(L) \vartheta_1(\xi(z_m); i\alpha) = -\vartheta_1(\xi(z_m); i\alpha),$$
$$\sigma_m(iW) \vartheta_1(\xi(z_m); i\alpha) = e^{-\pi i (2(z_m)+i\alpha)} \vartheta_1(\xi(z_m); i\alpha),$$
$$\sigma_m(L) \vartheta_1(2\xi(z_m); 2i\alpha) = \vartheta_1(2\xi(z_m); 2i\alpha),$$
$$\sigma_m(iW) \vartheta_1(2\xi(z_m); 2i\alpha) = e^{-\pi i (2(z_m)+i\alpha)} \vartheta_1(2\xi(z_m); 2i\alpha),$$
$$\sigma_m(L) \vartheta_1(2\xi(z_m); i\alpha) = \vartheta_1(2\xi(z_m); i\alpha),$$
$$\sigma_m(iW) \vartheta_1(2\xi(z_m); i\alpha) = e^{-\pi i (2(z_m)+i\alpha)} \vartheta_1(2\xi(z_m); i\alpha),$$
$$\sigma_m(L) \vartheta_1(\xi(z_m); i\alpha/2) = \vartheta_1(\xi(z_m); i\alpha/2),$$
$$\sigma_m(iW) \vartheta_1(\xi(z_m); i\alpha/2) = e^{-\pi i (2(z_m)+i\alpha)} \vartheta_1(\xi(z_m); i\alpha/2),$$
$$\sigma_m(L) \vartheta_1(\xi(z_m); i\alpha) \vartheta_0(2\xi(z_m); 2i\alpha) = \vartheta_1(\xi(z_m); i\alpha) \vartheta_0(2\xi(z_m); 2i\alpha),$$
$$\sigma_m(iW) \vartheta_1(\xi(z_m); i\alpha) \vartheta_0(2\xi(z_m); 2i\alpha) = e^{-\pi i (2(z_m)+i\alpha)} \vartheta_1(\xi(z_m); i\alpha) \vartheta_0(2\xi(z_m); 2i\alpha).$$

Since $N^D_N + 1 = N^{B_N}, N^D_N + 2 = N^{B_N}, N^D_N + 4 = N^{C_N}, N^D_N + 2 = N^{C_N}$, and $N^D_N + 3 = N^{BC_N}$, we obtain (2.11) and (2.12) with (2.13) and (2.14) for $R_N = B_N, B_N, C_N, C_N, BC_N, D_N$. \]

We write the complex conjugate of $z = x + iy \in \mathbb{C}$ as $\overline{z} = x - iy$ and $\overline{z} \equiv (\overline{z_1}, \ldots, \overline{z_N})$. By the definitions (2.1) and (2.10), and by parity (A.2) of the Jacobi theta functions,

$$W^{R_N}(\xi(z); i\alpha) = W^{R_N}(\xi(\overline{z}); i\alpha),$$
$$C^{A_N-1}(z) = \exp \left( -\frac{\pi N^{A_N-1}}{LW} \sum_{j=1}^{N} y_j^2 \right) \vartheta_{n(N)} \left( \sum_{k=1}^{N} \xi(\overline{z_k}); i\alpha \right),$$
$$C^{R_N}(z) = C^{R_N}(z) = \exp \left( -\frac{\pi N^{R_N}}{LW} \sum_{j=1}^{N} y_j^2 \right), \quad R_N = B_N, B_N, C_N, C_N, BC_N, D_N.$$

Now we define the real-valued functions as follows,

$$Q^{R_N}(z) \equiv |q^{R_N}(z)|^2 = q^{R_N}(z) q^{-R_N}(z)$$

$$= \begin{cases} \exp \left( -\frac{2\pi N^{A_N-1}}{LW} \sum_{j=1}^{N} y_j^2 \right) \left| \vartheta_{n(N)} \left( \sum_{k=1}^{N} \xi(z_k); i\alpha \right) \right|^2 \left| W^{A_N-1}(\xi(z); i\alpha) \right|^2, & \text{for } R_N = A_N-1, \\
\exp \left( -\frac{2\pi N^{R_N}}{LW} \sum_{j=1}^{N} y_j^2 \right) \left| W^{R_N}(\xi(z); i\alpha) \right|^2, & \text{for other } R_N. \end{cases} \quad (2.15)$$

Following (22), we define the following notion.

**Definition 2.2** For a pair of fixed positive values, $0 < L, W < \infty$, if the multivariate function $F$ of $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ satisfied the equalities

$$\sigma_m(L) F(z) = F(z),$$
$$\sigma_m(iW) F(z) = F(z),$$

for any $m = 1, 2, \ldots, N$, then the function $F(z)$ is said to be totally elliptic with periods $(L, iW)$. 

Then the following is immediately concluded from Lemma 2.1.

**Proposition 2.3** The multivariate functions $Q^{R_N}(z)$ defined by (2.15) are totally elliptic with periods $(L,iW)$.

### 2.3 Orthogonality

In Lemma 2.1 in [13], we proved the orthogonality for a version [13] of the seven series of $R_N$-theta functions of Rosengren and Schlosser [13]. By the similar argument, we can prove the following orthogonality for the present version of $\{M_j^{R_N}(z)\}_{j=1}^N$ defined by (2.24) for a complex variable $z = x + iy, x, y \in \mathbb{R}$, respect to the integral over $x \in [0,L)$. Here we note that the following relations hold for the complex conjugates of $\{M_j^{R_N}(z)\}_{j=1}^N, z \in \mathbb{C}$:

\[
\begin{align*}
M_j^{A_{N-1}}(z) &= M_j^{A_{N-1}}(-\overline{z}), \\
M_j^{R_N}(z) &= -M_j^{R_N}(-\overline{z}) = M^{R_N}(\overline{z}), \quad \text{for } R_N = B_N, B'_N, \\
M_j^{R_N}(z) &= M_j^{R_N}(-\overline{z}) = -M^{R_N}(\overline{z}), \quad \text{for } R_N = C_N, C'_N, BC_N, \\
M_j^{D_N}(z) &= M_j^{D_N}(-\overline{z}) = M^{D_N}(\overline{z}).
\end{align*}
\]

**Lemma 2.4** Let $z = x + iy, x, y \in \mathbb{R}$. For $R_N = A_{N-1}, B_N, B'_N, C_N, C'_N, BC_N, D_N$, if $j, k \in \{1,2,\ldots,N\}$,

\[
\int_0^L dx \, M_j^{R_N}(z)M_k^{R_N}(z) = m_j^{R_N}(y)\delta_{jk},
\]

where

\[
\begin{align*}
m_j^{A_{N-1}}(y) &= Le^{-4\pi J^A_{N-1}(j)y}\vartheta_2\left(2i\{J^A_{N-1}(j)\alpha + N^{A_{N-1}}\xi(y)\}; 2iN^{A_{N-1}}\alpha\right), \quad j \in \{1,2,\ldots,N\}, \\
m_j^{R_N}(y) &= L \left\{e^{4\pi J^R_N(j)y}\vartheta_2\left(2i\{J^R_N(j)\alpha - N^{R_N}\xi(y)\}; 2iN^{R_N}\alpha\right) + e^{-4\pi J^R_N(j)y}\vartheta_2\left(2i\{J^R_N(j)\alpha + N^{R_N}\xi(y)\}; 2iN^{R_N}\alpha\right)\right\}, \quad j \in \{1,2,\ldots,N\}, \\
&m_j^{R_N}(y) = \begin{cases} 4L\vartheta_2\left(2iN^{R_N}\xi(y); 2iN^{R_N}\alpha\right), & j = 1, \\
L \left\{e^{4\pi J^R_N(j)y}\vartheta_2\left(2i\{J^R_N(j)\alpha - N^{R_N}\xi(y)\}; 2iN^{R_N}\alpha\right) + e^{-4\pi J^R_N(j)y}\vartheta_2\left(2i\{J^R_N(j)\alpha + N^{R_N}\xi(y)\}; 2iN^{R_N}\alpha\right)\right\}, & j \in \{2,3,\ldots,N\}, \end{cases} \\
&m_j^{D_N}(y) = \begin{cases} 4L\vartheta_2\left(2iN^{D_N}\xi(y); 2iN^{D_N}\alpha\right), & j = 1, \\
2L \left\{e^{4\pi J^D_N(j)y}\vartheta_2\left(2i\{J^D_N(j)\alpha - N^{D_N}\xi(y)\}; 2iN^{D_N}\alpha\right) + e^{-4\pi J^D_N(j)y}\vartheta_2\left(2i\{J^D_N(j)\alpha + N^{D_N}\xi(y)\}; 2iN^{D_N}\alpha\right)\right\}, & j \in \{2,3,\ldots,N-1\}, \\
+e^{-4\pi J^D_N(N)y}\vartheta_2\left(2i\{J^D_N(N)\alpha + N^{D_N}\xi(y)\}; 2iN^{D_N}\alpha\right), & j = N. \end{cases}
\end{align*}
\]

In the present paper, we prove the following orthogonality for the version 2.25 of the $R_N$-theta functions of Rosengren and Schlosser [13] with respect to the double integrals over the fundamental domain $\Lambda_{(L,iW)}$. 


Proposition 2.5 Let $z = x + iy, x, y \in \mathbb{R}$. If $j,k \in \{1,2,\ldots,N\}$, then for $R_N = A_{N-1}, B_N, B'_N, C_N, C'_N, BC_N, D_N$, 
\[
\int_0^L dx \int_0^W dy \exp \left(-\frac{2\pi N\alpha}{LW}y^2\right) M_j^{R_N}(z) M_k^{R_N}(z) = h_j^{R_N} \delta_{jk}. \tag{2.17}
\]
where
\[
\begin{align*}
    h_j^{A_{N-1}} &= \frac{LW}{2N\alpha} 2\pi \alpha (j^{A_{N-1}}-1) / N, & j &\in \{1,2,\ldots,N\}, \\
    h_j^{R_N} &= \frac{2LW}{\sqrt{2N\alpha}} 2\pi \alpha (j^{R_N}) / N^{R_N}, & j &\in \{1,2,\ldots,N\}, \quad \text{for } R_N = C_N, C'_N, BC_N, \\
    h_j^{R_N} &= \left\{ \begin{array}{ll} \\
        \frac{4LW}{\sqrt{2N\alpha}} 2\pi \alpha (j^{R_N}) / N^{R_N}, & j = 1, \\
        \frac{2LW}{\sqrt{2N\alpha}} 2\pi \alpha (j^{R_N}) / N^{R_N}, & j \in \{2,3,\ldots,N\}, \quad \text{for } R_N = B_N, B'_N, \\
    \end{array} \right. \\
    h_j^{D_N} &= \left\{ \begin{array}{ll} \\
        \frac{4LW}{\sqrt{2N\alpha}} 2\pi \alpha (j^{D_N}) / N^{D_N}, & j \in \{1,2,\ldots,N-1\}, \\
        \frac{2LW}{\sqrt{2N\alpha}} 2\pi \alpha (j^{D_N}) / N^{D_N}, & j \in \{2,3,\ldots,N-1\}. \\
    \end{array} \right. \tag{2.18}
\end{align*}
\]

Proof By Lemma 2.4 the orthogonality (2.17) holds with
\[
h_j^{R_N} = \int_0^W dy e^{-2\pi N\alpha (\xi(y) + \frac{\xi(y)}{\alpha})} m_j^{R_N}(y), \quad j = 1,2,\ldots,N.
\]
If we apply Jacobi’s imaginary transformation (2.8), we obtain the equality
\[
\begin{align*}
    \vartheta_2^2 \left( 2i \left( J^{R_N}(j) \alpha \pm N^{R_N} \xi(y) \right); 2i N^{R_N} \alpha \right) &= \frac{1}{\sqrt{2N^{R_N} \alpha}} e^{2\pi \alpha (J^{R_N}(j)^2 / N^{R_N} \pm 4\pi J^{R_N}(j) \xi(y) \mp 2\pi N^{R_N} \xi(y)^2 / \alpha \vartheta_0 \left( J^{R_N}(j) / N^{R_N} \pm \frac{\xi(y)}{\alpha}; i \frac{j}{2N^{R_N} \alpha} \right).}
\end{align*}
\]
Hence
\[
\begin{align*}
    e^{-2\pi N^{R_N} \xi(y)^2 / \alpha} e^{\mp 4\pi J^{R_N}(j) \xi(y) \vartheta_2^2 \left( 2i \left( J^{R_N}(j) \alpha \pm N^{R_N} \xi(y) \right); 2i N^{R_N} \alpha \right)} &= \frac{1}{\sqrt{2N^{R_N} \alpha}} e^{2\pi \alpha (J^{R_N}(j)^2 / N^{R_N} \alpha)} \vartheta_0 \left( J^{R_N}(j) / N^{R_N} \pm \frac{\xi(y)}{\alpha}; i \frac{j}{2N^{R_N} \alpha} \right). \tag{2.19}
\end{align*}
\]
By the definition of $\vartheta_0$ given by (3.1),
\[
\int_0^W dy \vartheta_0 \left( J^{R_N}(j) / N^{R_N} \pm \frac{\xi(y)}{\alpha}; i \frac{j}{2N^{R_N} \alpha} \right) = \sum_{n \in \mathbb{Z}} (-1)^n e^{-n^2 \pi / (2N^{R_N} \alpha) + 2\pi J^{R_N}(j) / N^{R_N}} \int_0^W dy e^{\pm 2\pi i y / W}.
\]
Since $\int_0^W dy e^{\pm 2\pi i y / W} = W \delta_{\alpha 0}$, the above integral is equal to $W$. Then, from (2.16) in Lemma 2.4 (2.18) are derived. The proof is complete. 

3 Elliptic Determinantal Point Processes on a Complex Plane

3.1 DPPs with a finite number of points

Combination of the Macdonald denominator formulas (2.8) with (2.9) and the orthogonality (Proposition 2.5) for the present version (2.5) of the $R_N$-theta functions of Rosengren and Schlosser [18], the following multiple integral formulas are derived for the totally elliptic weight functions $Q^{R_N}(z)$ defined by (2.15).
Lemma 3.1 For \( R_N = A_{N-1}, B_{N}, B^\vee_{N}, C_{N}, C^\vee_{N}, BC_{N}, D_{N} \),

\[
\frac{1}{N!} \int_{\mathcal{A}(L,W)^N} dz \, Q^R_N(z) = \frac{1}{N!} \prod_{j=1}^{N} \int_{0}^{L} dx_j \int_{0}^{W} dy_j \, Q^R_N(z) = Z^R_N, \tag{3.1}
\]

where

\[
Z^R_N = Z^R_N(L,W) = 2^{\eta N} \frac{(LW)^N}{(N^R_N \alpha)^N/2} (\eta(\alpha))^\kappa N \, g^R_N(\alpha), \tag{3.2}
\]

with

\[
\delta^R_N = \begin{cases} 
-N/2, & R_N = A_{N-1}, \\
(N-2)/2, & R_N = B_{N}, B^\vee_{N}, \\
N/2, & R_N = C_{N}, C^\vee_{N}, BC_{N}, \\
(N-4)/2, & R_N = D_{N}, 
\end{cases}
\]

\[
\kappa^R_N = \begin{cases} 
(N-1)(N-2), & R_N = A_{N-1}, \\
2N(N-1), & R_N = B_{N}, C_{N}, \\
2(N-1)(N+1), & R_N = B^\vee_{N}, \\
(N-1)(2N-1), & R_N = C^\vee_{N}, \\
2N(N+1), & R_N = BC_{N}, \\
2N(N-2), & R_N = D_{N}, 
\end{cases}
\]

\[
g^R_N(\alpha) = \begin{cases} 
1, & R_N = A_{N-1}, B_{N}, C_{N}, D_{N}, \\
\left(\frac{\eta(\alpha)}{\eta(\alpha)^2}\right)^{2(N-1)}, & R_N = B^\vee_{N}, \\
\left(\frac{\eta(\alpha)^2}{\eta(\alpha)}\right)^{N-1}, & R_N = C^\vee_{N}, \\
\left(\frac{\eta(\alpha)}{\eta(\alpha)^2}\right)^{2N}, & R_N = BC_{N}. 
\end{cases} \tag{3.3}
\]

Proof By the Macdonald denominator formulas (2.8) with (2.9),

\[
Q^R_N(z) = \frac{1}{a^R_N(\alpha)^2} \exp \left( -\frac{2\pi N^R_N}{LW} \sum_{j=1}^{N} y_j^2 \right) \det_{1 \leq j, k \leq N} M^R_j(z) \det_{1 \leq \ell, m \leq N} M^R_\ell(z_m) = \frac{1}{a^R_N(\alpha)^2} \det_{1 \leq j, k \leq N} \left[ e^{-\pi N^R_N y_j^2/(LW)} M^R_j(z_k) \right] \det_{1 \leq \ell, m \leq N} \left[ e^{-\pi N^R_N y_m^2/(LW)} M^R_\ell(z_m) \right].
\]

Use of the Heine identity (see, for instance, Eq.(C.4) in [13]) gives

\[
\frac{1}{N!} \int_{\mathcal{A}(L,W)^N} dz \, \det_{1 \leq j, k \leq N} \left[ e^{-\pi N^R_N y_j^2/(LW)} M^R_j(z_k) \right] \det_{1 \leq \ell, m \leq N} \left[ e^{-\pi N^R_N y_m^2/(LW)} M^R_\ell(z_m) \right] = \det_{1 \leq j, k \leq N} \left[ \int_{0}^{L} dx \int_{0}^{W} dy \, e^{-2\pi N^R_N y^2/(LW)} M^R_j(z) M^R_\ell(z) \right].
\]

Then Lemma 2.5 implies that this multiple integral is given by \( \prod_{n=1}^{N} h^R_N \) and hence

\[
\frac{1}{N!} \int_{\mathcal{A}(L,W)^N} dz \, Q^R_N(z) = \frac{1}{a^R_N(\alpha)^2} \prod_{n=1}^{N} h^R_N.
\]
From the expressions (2.18) for $h_j^{R_N}$, $j \in \{1, 2, \ldots, N\}$, the formulas (3.1) with (3.2) and (3.3) are obtained.

Now we define seven types of point processes on $\mathbb{C}$ as statistical ensembles of nonnegative integer-valued Radon measures,

$$\Xi^{R_N}(\cdot) = \sum_{j=1}^{N} \delta_{Z_j^{R_N}}(\cdot),$$

for $R_N = A_{N-1}, B_N, B_N', C_N, C_N', BC_N, D_N$, provided that the number of particles in $\Lambda_{(L,iW)}$ is $N$ and the distributions of points $Z^{R_N} = \{Z_j^{R_N}\}_{j=1}^{N}$ on $\mathbb{C}$ are governed by the probability measures

$$P^{R_N}(Z^{R_N} \in dz) = P^{R_N}(z)dz = \frac{Q^{R_N}(z)}{Z^{R_N}}dz,$$  \hspace{1cm} (3.4)

where

$$\int_{\Lambda_{(L,iW)}} dz P^{R_N}(z) = 1.$$  \hspace{1cm}

The functions, $P^{R_N}(z)$, $R_N = A_{N-1}, B_N, B_N', C_N, C_N', BC_N, D_N$, are called probability density functions. By this construction, the probability density functions are symmetric and totally elliptic with periods $(L, iW)$ with respect to $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$.

Given the determinantal expressions (2.15) and (3.4) with (2.8) for the probability measures $P^{R_N}$ associated with the orthogonal functions (Proposition 2.8), we can readily prove the following by the standard method in random matrix theory [16, 5, 1, 10]. (See, for instance, Appendix C in [13].)

**Theorem 3.2** The seven types of point processes on $\mathbb{C}$, $(\Xi^{R_N}, P^{R_N})$, $R_N = A_{N-1}, B_N, B_N', C_N, C_N', BC_N, D_N$, are determinantal with the correlation kernels,

$$K^{R_N}(z, z') = e^{-\pi N^{R_N} [y^2 + (y')^2]/(LW)} \sum_{n=1}^{N} h_n^{R_N} M_n^{R_N}(z)M_n^{R_N}(z'), \hspace{1cm} z, z' \in \mathbb{C}. \hspace{1cm} (3.5)$$

These point processes are doubly periodic with periods $(L, iW)$.

**Remark 1** The correlation kernels are quasi doubly periodic as shown below,

$$K^{R_N}(z + L, z') = K^{R_N}(z, z' + L)$$

$$= \begin{cases} (-1)^{N^{A_{N-1}}} K^{R_N}(z, z'), & R_N = A_{N-1}, \\ -K^{R_N}(z, z'), & R_N = B_N, C_N', BC_N, \\ K^{R_N}(z, z'), & R_N = B'_N, C_N, D_N. \end{cases}$$

$$K^{R_N}(z + iW, z') = e^{2\pi i N^{A_{N-1}} z/L} K^{R_N}(z, z'), \hspace{1cm} R_N = A_{N-1}, C_N, C_N', BC_N, D_N,$$

$$-e^{2\pi i N^{B_N} z/L} K^{R_N}(z, z'), \hspace{1cm} R_N = B_N, B'_N, C_N,$$

$$K^{R_N}(z, z' + iW) = e^{2\pi i N^{A_{N-1}} z'/L} K^{R_N}(z, z'), \hspace{1cm} R_N = A_{N-1}, C_N, C_N', BC_N, D_N,$$

$$-e^{2\pi i N^{B_N} z'/L} K^{R_N}(z, z'), \hspace{1cm} R_N = B_N, B'_N.$$

Since Theorem 3.2 states that the present point processes are determinantal, for any $1 \leq N' \leq N$, $N'$-point correlation function at configuration $z_{N'} = (z_1, \ldots, z_{N'}) \in \mathbb{C}^{N'}$ is given by

$$\rho^{R_N}(z_{N'}) = \det_{1 \leq j, k \leq N'} [K^{R_N}(z_j, z_k)].$$  \hspace{1cm} (3.6)

Although $K^{R_N}(z, z')$ are quasi doubly periodic, by basic property of determinant, the correlation functions (3.6) are all totally elliptic with periods $(L, iW)$ with respect to $z_{N'} = (z_1, \ldots, z_{N'})$. This is a matter of
course, since the probability density functions $p^{R_N}(z)$ are symmetric and totally elliptic with respect to $z = (z_1, \ldots, z_N)$, and the correlation functions are defined by

$$
\rho^{R_N}(z_N') \equiv \frac{1}{(N-N')!} \int_{\Lambda_{L,W}^{N-N'}} \prod_{j=N}^N dz_j p^{R_N}(z_1, \ldots, z_N', z_{N+1}, \ldots, z_N), \quad z_{N'} \in \mathbb{C}^N.
$$

3.2 DPPs with an infinite number of points

Now we consider the double limit $N \to \infty, L \to \infty$. We fix the density of points,

$$
\rho = \frac{N}{LW}, \quad (3.7)
$$

and here we assume that the value of $W$ is also fixed. This implies $\alpha = W/L = \rho W^2/N \to 0$ in the limit $N \to \infty$. Then we obtain the following.

**Proposition 3.3** The following scaling limits are obtained for correlation kernels.

(i) For $R_N = A_{N-1}$,

$$
K_{W,\rho}^A(z, z') \equiv \lim_{N \to \infty, L \to \infty, N/L = \rho W} K_{A}^{N-1}(z, z')
$$

\[= \sqrt{2} \rho e^{-\pi \rho (y^2 + (y')^2)} \int_0^{\sqrt{N}W} d\lambda e^{-2\pi \rho (\lambda^2 + 2\pi i \sqrt{z} \sqrt{-\lambda})} \lambda \times \vartheta_2(\sqrt{W}(i\lambda + \sqrt{\rho z}); i\rho W^2) \vartheta_1(\sqrt{W}(i\lambda - 2\sqrt{\rho z}); i\rho W^2), \quad (3.8)
\]

(ii) For $R_N = B_{N}, B_{N}'$

$$
K_{W,\rho}^B(z, z') \equiv \lim_{N \to \infty, L \to \infty, N/L = \rho W} K^{R_N}(z, z')
$$

\[= -\rho e^{-2\pi \rho (y^2 + (y')^2)} \times \left\{ \int_{\sqrt{N}W}^{\sqrt{\rho W}} d\lambda e^{-\pi \lambda^2 + 2\pi i \sqrt{z} \sqrt{-\lambda}} \vartheta_1(\sqrt{\rho W}(i\lambda + 2\sqrt{\rho z}); 2i\rho W^2) \vartheta_1(\sqrt{\rho W}(i\lambda - 2\sqrt{\rho z}); 2i\rho W^2) \right\}, \quad (3.9)
\]

(iii) For $R_N = C_{N}, C_{N}', BC_{N}$

$$
K_{W,\rho}^C(z, z') \equiv \lim_{N \to \infty, L \to \infty, N/L = \rho W} K^{R_N}(z, z')
$$

\[= \rho e^{-2\pi \rho (y^2 + (y')^2)} \times \left\{ \int_{\sqrt{N}W}^{\sqrt{\rho W}} d\lambda e^{-\pi \lambda^2 + 2\pi i \sqrt{z} \sqrt{-\lambda}} \vartheta_2(\sqrt{\rho W}(i\lambda + 2\sqrt{\rho z}); 2i\rho W^2) \vartheta_2(\sqrt{\rho W}(i\lambda - 2\sqrt{\rho z}); 2i\rho W^2) \right\}, \quad (3.10)
\]
In this converges into the following integral in $N$:

$$K^D_{W,\rho}(z, z') \equiv \lim_{N \to \infty, L \to \infty} K^{D_N}(z, z')$$

$$= \rho e^{-2\pi \rho (y^2 + (y')^2)}$$

$$\times \left[ \int_{-\sqrt{W}}^{\sqrt{W}} d\lambda e^{-\pi \lambda^2 + 2\pi \rho \sqrt{\rho - \lambda^2}} \vartheta_2(\sqrt{W}(i\lambda + 2\sqrt{\rho} z); 2i\rho W^2) \vartheta_2(\sqrt{W}(i\lambda - 2\sqrt{\rho} z); 2i\rho W^2) \right. + \left. \int_{-\sqrt{W}}^{\sqrt{W}} d\lambda e^{-\pi \lambda^2 + 2\pi \rho \sqrt{\rho - \lambda^2}} \vartheta_2(\sqrt{W}(i\lambda + 2\sqrt{\rho} z); 2i\rho W^2) \vartheta_2(\sqrt{W}(i\lambda + 2\sqrt{\rho} z); 2i\rho W^2) \right], \quad (3.11)$$

$z, z' \in \mathbb{C}.$

**Proof** We fix the values of $\rho$ and $W$ and change the value of $N$. We have $\alpha = \rho W^2/N$ and $L = N/(\rho W)$. First we consider the case (i). $K^{A_N-1}(z, z')$ is written as

$$K^{A_N-1}(z, z') = \sqrt{2} \rho^{3/2} W e^{-\pi \rho (y^2 + (y')^2)} \frac{1}{N} \sum_{n=1}^{N} e^{-2\pi \rho W^2(n-1/2)^2/N^2} e^{2\pi i \rho W(z - \overline{z})(n-1)/N}$$

$$\times \vartheta_2(i\rho W^2(n - 1/2)/N + \rho W z; i\rho W^2) \vartheta_2(i\rho W^2(n - 1/2)/N - \rho W \overline{z}; i\rho W^2).$$

This converges into the following integral in $N \to \infty$,

$$\sqrt{2} \rho^{3/2} W e^{-\pi \rho (y^2 + (y')^2)}$$

$$\times \int_{0}^{1} du e^{-2\pi W^2 u^2 + 2\pi i W u (z - \overline{z})} \vartheta_2(i\rho W^2 u + \rho W z; i\rho W^2) \vartheta_2(i\rho W^2 u - \rho W \overline{z}; i\rho W^2).$$

We change the integral valuable as $u \to \lambda = \sqrt{\rho W} u$, and then $\lambda$ is obtained.

Next we consider the case (ii). Using $\rho$ and $W$, $K^{R_N}(z, z')$ is written as

$$K^{R_N}(z, z') = \sqrt{\rho W^2 \frac{N^{R_N}}{2N}} \rho e^{-2\pi \rho (y^2 + (y')^2)} N^{R_N}/(2N)$$

$$\times \frac{1}{N} \sum_{n=1}^{N} e^{-2\pi \rho J_{R_N}(n)^2/(N N^{R_N})} \left[ e^{2\pi i \rho W z J_{R_N}(n)/N} \vartheta_2 \left( i\rho W^2 \frac{J_{R_N}(n)}{N} + 2\rho W z \frac{N^{R_N}}{2N} \right) \right. - e^{-2\pi i \rho W z J_{R_N}(n)/N} \vartheta_2 \left( i\rho W^2 \frac{J_{R_N}(n)}{N} - 2\rho W z \frac{N^{R_N}}{2N} \right) \right]$$

$$\times \left[ e^{-2\pi i \rho W \overline{z} J_{R_N}(n)/N} \vartheta_2 \left( i\rho W^2 \frac{J_{R_N}(n)}{N} - 2\rho W \overline{z} \frac{N^{R_N}}{2N} \right) \right. - e^{2\pi i \rho W \overline{z} J_{R_N}(n)/N} \vartheta_2 \left( i\rho W^2 \frac{J_{R_N}(n)}{N} + 2\rho W \overline{z} \frac{N^{R_N}}{2N} \right) \right].$$

Since $N^{R_N}/(2N) \to 1$ as $N \to \infty$, this converges to the following integral in $N \to \infty$,

$$\rho^{3/2} W e^{-\pi \rho (y^2 + (y')^2)}$$

$$\times \int_{0}^{1} du e^{-\pi \rho W^2 u^2} \left[ e^{2\pi i \rho W z u} \vartheta_2(i\rho W^2 u + 2\rho W z; 2i\rho W^2) - e^{-2\pi i \rho W z u} \vartheta_2(i\rho W^2 u - 2\rho W z; 2i\rho W^2) \right]$$

$$\times \left[ e^{-2\pi i \rho W \overline{z} u} \vartheta_2(i\rho W^2 u - 2\rho W \overline{z}; 2i\rho W^2) - e^{2\pi i \rho W \overline{z} u} \vartheta_2(i\rho W^2 u + 2\rho W \overline{z}; 2i\rho W^2) \right].$$
If we change the integral variable as \( u = \lambda = \sqrt{\rho W} u \) and use the fact that \( \vartheta_{2}(-v; \tau) = \vartheta_{2}(v; \tau) \), (3.10) is obtained.

The cases (ii) and (iv) are proved by the similar calculation. The proof is complete.

The limit correlation kernels have the following quasi periodicity with period \( iW \),

\[ K_{R, \rho}^{R}(z + iW, z') = \begin{cases} e^{-2\pi i W z} K_{R, \rho}^{R}(z, z'), & R = A, C, D, \\ -e^{-2\pi i W z} K_{R, \rho}^{R}(z, z'), & R = B, \end{cases} \]

\[ K_{R, \rho}^{R}(iW, z' + iW) = \begin{cases} e^{2\pi i W z} K_{R, \rho}^{R}(z, z'), & R = A, C, D, \\ -e^{-2\pi i W z} K_{R, \rho}^{R}(z, z'), & R = B. \end{cases} \]

We see that the limit kernels, \( K_{R, \rho}^{R}(z, z') \), \( R = A, B, C, D \), are continuous functions of \( (z, z') \in \mathbb{C}^2 \) and

\[ \sup_{z, z' \in \mathcal{D}} |K_{R, \rho}^{R}(z, z')| < \infty, \quad \forall N \in \mathbb{N}, \]

for any compact domain \( \mathcal{D} \subset \mathbb{C} \). Then we can obtain the convergence of generating functions of correlation functions, which are expressed by Fredholm determinants [14, 10], as \( N \to \infty \). Generating functions of correlation functions can be identified with the Laplace transforms of probability densities of point processes. Hence this implies the convergence of probability laws in the sense of finite dimensional distributions weakly in the vague topology [14, 10]. We write the probability laws for the infinite dimensional DPPs associated with the limit kernels \( K_{R, \rho}^{R}, R = A, B, C, D \) given in Proposition 3.3 as \( P_{R, \rho}^{R} \), \( R = A, B, C, D \), respectively.

The density profile of DPP is given by \( \rho_{W, \rho}^{R}(z) = K_{W, \rho}^{R}(z, z), z \in \mathbb{C} \). It is easy to verify that \( K_{W, \rho}^{C}(0, 0) = K_{W, \rho}^{C}(0, 0) = 0 \) for (3.9) and (3.10).

**Theorem 3.4** In the scaling limit \( N \to \infty, L \to \infty \) with constant density of points (3.7) and constant \( W \), the seven types of elliptic DPPs on \( \mathbb{C}, (\Xi^R, P_{W, \rho}^{R}), R_{N} = A_{N-1}, B_{N}, B_{N}', C_{N}, C_{N}', B_{N}^{C}, D_{N}, \) converge in the sense of distribution to the four types of infinite dimensional point processes as follows,

\[ (\Xi^{A_{N-1}}, P^{A_{N-1}}) \implies (\Xi^{A}, P_{W, \rho}^{A}), \]

\[ (\Xi^{B_{N}}, P_{B_{N}}) \implies (\Xi^{B}, P_{W, \rho}^{B}), \]

\[ (\Xi^{C_{N}}, P_{C_{N}}) \implies (\Xi^{C}, P_{W, \rho}^{C}), \]

\[ (\Xi^{D_{N}}, P_{D_{N}}) \implies (\Xi^{D}, P_{W, \rho}^{D}). \]

The four types of infinite DPPs on \( \mathbb{C}, (\Xi^R, P_{W, \rho}^{R}), R = A, B, C, D \), have the periodicity with period \( iW \). In particular, the densities at the points \( \{nW\} \in \mathbb{Z} \) on the imaginary axis are fixed to be zero in \( P_{W, \rho}^{B} \) and \( P_{W, \rho}^{C} \);

\[ \rho_{W, \rho}^{B}(iW) = \rho_{W, \rho}^{C}(iW) = 0, \quad n \in \mathbb{Z}. \]

Now we consider the further limit

\[ W \to \infty \]

with constant \( \rho \).

In this limit, \( \Im(i\rho W^2) = \rho W^2 \to \infty \), and application of (A.5) to (3.8) gives

\[ K_{R, \rho}^{A}(z, z') \simeq 4\sqrt{2} \rho e^{-\rho(y^2+(y')^2)} \int_{0}^{\sqrt{\rho W}} d\lambda e^{-2\pi \lambda^2 + 2\pi \sqrt{\rho(z-z')} \lambda - \pi \rho W^2/2} \]

\[ \times \cos[\pi \sqrt{\rho W}(i\lambda + \sqrt{\rho z})] \cos[\pi \sqrt{\rho W}(i\lambda - \sqrt{\rho z'})] \quad \text{as } W \to \infty. \]
We can rewrite this as
\[
K_{W,\rho}^A(z, z') \simeq \sqrt{2}pe^{-\pi\rho(y^2+(y')^2)} \left[ e^{-\pi\rho(z-z')^2/2} e^{-2\pi\rho^2 d\eta} \right.
\]
\[
+2e^{-\pi\rho W^2/2-\pi\rho(z-z')^2/2} \cos(\pi\rho W(z+z')) \int_{\sqrt{\rho(y+y')}}^{\sqrt{\rho(y+y')}} e^{-2\pi\rho^2 d\eta} \right]
\]
as \( W \to \infty \). Then we obtain the limit
\[
\lim_{W \to \infty} K_{W,\rho}^A(z, z') = pe^{-\pi\rho(y^2+(y')^2)} e^{-\pi\rho(z-z')^2/2}
\]
\[
eq \frac{e^{2\pi i\rho y'}}{e^{2\pi i\rho y}} K_{\text{Ginibre},\rho}^A(z, z'),
\]
which is the correlation kernel for the *Ginibre ensemble* studied in random matrix theory [6, 17, 20] with uniform density
\[
\rho_{\text{Ginibre},\rho}^A(z, z) = K_{\text{Ginibre},\rho}^A(z, z) = \rho, \quad z \in \mathbb{C}.
\]
Similarly, we can obtain the following limit kernels,
\[
\lim_{W \to \infty} K_{W,\rho}^B(z, z') = \lim_{W \to \infty} K_{W,\rho}^C(z, z')
\]
\[
= pe^{-2\rho(y^2+(y')^2)} \left[ e^{-\pi\rho(z-z')^2} - e^{-\pi\rho(z+z')^2} \right]
\]
\[
= \frac{e^{2\pi i\rho y'}}{e^{2\pi i\rho y}} K_{\text{Ginibre},\rho}^B(z, z'),
\]
\[
\lim_{W \to \infty} K_{W,\rho}^D(z, z') = pe^{-2\rho(y^2+(y')^2)} \left[ e^{-\pi\rho(z-z')^2} + e^{-\pi\rho(z+z')^2} \right]
\]
\[
= \frac{e^{2\pi i\rho y'}}{e^{2\pi i\rho y}} K_{\text{Ginibre},\rho}^D(z, z'),
\]
where
\[
K_{\text{Ginibre},\rho}^C(z, z') \equiv 2pe^{-\pi\rho(|z|^2+|z'|^2)} \sinh(2\pi\rho z\overline{z}),
\]
\[
K_{\text{Ginibre},\rho}^D(z, z') \equiv 2pe^{-\pi\rho(|z|^2+|z'|^2)} \cosh(2\pi\rho z\overline{z}), \quad z \in \mathbb{C}.
\]
We write the probability laws of the DPPs governed by \( \mathcal{K}_{\text{Ginibre},\rho}^R(z, z') \) as \( \mathcal{P}_{\text{Ginibre},\rho}^R \) for \( R = A, C, \) and \( D \).

**Theorem 3.5** In the limit \( W \to \infty \) with constant density of points \( \rho \), the four types of DPPs on \( \mathbb{C} \), \((\Xi^R, \mathcal{P}_{W,\rho}^R), \) \( R = A, B, C, D \) converge in the sense of distribution to the three types of infinite dimensional point processes as follows,
\[
(\Xi^A, \mathcal{P}_{W,\rho}^A) \Rightarrow (\Xi^A, \mathcal{P}_{\text{Ginibre},\rho}^A),
\]
\[
(\Xi^B, \mathcal{P}_{W,\rho}^B) \Rightarrow (\Xi^C, \mathcal{P}_{\text{Ginibre},\rho}^C),
\]
\[
(\Xi^C, \mathcal{P}_{W,\rho}^C) \Rightarrow (\Xi^D, \mathcal{P}_{\text{Ginibre},\rho}^D).
\]
From (3.13), we see that the density profiles are not uniform on $C$ in $P_{Ginibre,\rho}^{R}$ for $R = C$ and $D$:

$$\mathcal{K}_{Ginibre,\rho}^{C}(z, z) = \rho_{Ginibre,\rho}^{C}(z) = \rho_{Ginibre,\rho}(|z|) = \rho[1 - e^{-4\pi\rho|z|^2}],$$

$$\mathcal{K}_{Ginibre,\rho}^{D}(z, z) = \rho_{Ginibre,\rho}^{D}(z) = \rho_{Ginibre,\rho}(|z|) = \rho[1 + e^{-4\pi\rho|z|^2}],$$

in which

$$\min_{z \in C} \rho_{Ginibre,\rho}^{C}(z) = \rho_{Ginibre,\rho}^{C}(0) = 0,$$

$$\max_{z \in C} \rho_{Ginibre,\rho}^{D}(z) = \rho_{Ginibre,\rho}^{D}(0) = 2,$$

and

$$\lim_{|z| \to \infty} \rho_{Ginibre,\rho}^{R}(z) = \rho, \quad R = C, D.$$

**Remark 2** Combination of Theorem 3.4 and Theorem 3.5 implies that in the bulk scaling limit,

$$N \to \infty, \quad L \to \infty, \quad W \to \infty \quad \text{with constant} \quad \rho = \frac{N}{LW},$$

the seven types of elliptic DPPs, $P_{R}^{R_{N}}, P_{C}^{R_{N}}, P_{D}^{R_{N}}, R_{N} = A_{N-1}, B_{N}, B_{N}', C_{N}, C_{N}', B_{C}, B_{D}, D_{N},$ converge to the three types of Ginibre-like DPPs, $P_{R}^{R_{Ginibre,\rho}}, R = A, C, D.$

### 4 Realization as One-Component Plasma Systems and Relationship to Gaussian Free Field

#### 4.1 General setting of one-component plasma models

We consider the following two types of two-point potential functions in the rectangular domain $\Lambda_{(L, iw)} \subset \mathbb{C}$;

$$\Phi_{0}^{-}(z, z') = \Phi_{0}^{-}(z, z'; i\alpha) \equiv -\log(|\vartheta_{1}(\xi(z) - \xi(z'); i\alpha)|), \quad (4.1)$$

$$\Phi_{0}^{+}(z, z') = \Phi_{0}^{+}(z, z'; i\alpha) \equiv -\log(|\vartheta_{1}(\xi(z) + \xi(z'); i\alpha)|\vartheta_{1}(\xi(z) - \xi(z'); i\alpha)|)$$

$$= -\left\{ \log(|\vartheta_{1}(\xi(z) + \xi(z'); i\alpha)|) + \log(|\vartheta_{1}(\xi(z) - \xi(z'); i\alpha)|) \right\}, \quad (4.2)$$

$z, z' \in \Lambda_{(L, iw)}$.

**Remark 3** Note that $\Phi_{0}^{\pm}(z, z')$ is different from the two-point potential functions in the system with the ant metallic boundary condition found in Section 15.9 in [3], since for $\xi = \Re \xi + i\Im \xi, \overline{\xi} = \Re \xi - i\Im \xi \neq -\xi = -\Re \xi - i\Im \xi$, in general.

These two-point potential functions are related with the absolute values of the complex-valued Macdonald denominators of types $A_{N-1}$ and $D_{N}$ as

$$|W_{A_{N-1}}^{\pm}(\xi(z); i\alpha)| = \exp \left( - \sum_{1 \leq j < k \leq N} \Phi_{0}^{\pm}(z_{k}, z_{j}; i\alpha) \right),$$

$$|W_{D_{N}}^{\pm}(\xi(z); i\alpha)| = \exp \left( - \sum_{1 \leq j < k \leq N} \Phi_{0}^{\pm}(z_{k}, z_{j}; i\alpha) \right), \quad z = (z_{1}, z_{2}, \ldots, z_{N}) \in \Lambda_{(L, iw)}^{N}.$$
Since \( \vartheta_1(\xi; i\alpha) \simeq \xi \partial \vartheta_1(\xi; i\alpha)/\partial \xi \big|_{\xi=0} \) as \( \xi \to 0 \), the relation (4.7) gives

\[
\Phi_0^-(z, z') \simeq -\log \left( \frac{\eta(i\alpha)^2 2\pi}{L} |z - z'| \right),
\]

\[
\Phi_0^+(z, z') \simeq -\log \left( \frac{\eta(i\alpha)^2 2\pi}{L} |z - z'| \right) - \frac{1}{2} \{ \log(|\vartheta_1(2\xi(z); i\alpha)|) + \log(|\vartheta_1(2\xi(z'); i\alpha)|) \},
\]

as \( |z - z'| \to 0 \). Hence, if we define

\[
\Phi^-(z, z') = \Phi_0^-(z, z') + 3 \log(\eta(i\alpha)) + \log(2\pi/L),
\]

\[
\Phi^+(z, z') = \Phi_0^+(z, z') + 3 \log(\eta(i\alpha)) + \log(2\pi/L) + \frac{1}{2} \{ \log(|\vartheta_1(2\xi(z); i\alpha)|) + \log(|\vartheta_1(2\xi(z'); i\alpha)|) \},
\]

(4.3)

(4.4)

then they have the common asymptotic form,

\[
\Phi^-(z, z') \simeq -\log |z - z'|, \quad \Phi^+(z, z') \simeq -\log |z - z'|, \quad \text{as} \quad |z - z'| \to 0,
\]

which solves the two-dimensional Poisson equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x + iy, x' + iy') = -2\pi \delta(x - x') \delta(y - y')
\]

with free boundary condition.

We consider two types of systems of \( N \) particles each of which has charge +1. The \( N \) particles are mobile and interacting via the potential (4.3) or (4.4), but they are confined to the rectangle domain \( \Lambda_{(L,W)} \subset \mathbb{C} \). The former system was studied by Forrester [4], but the latter is new. The total energies of the particle-particle interaction in particle configurations \( z \) are given by

\[
E^p_{pp}(z) = \sum_{1 \leq j, k \leq N} \Phi^-(z_k, z_j)
\]

\[
= -\log \left( \prod_{1 \leq j < k \leq N} |\vartheta_1(\xi(z_k) - \xi(z_j); i\alpha)| \right) + \frac{3}{2} N(N - 1) \log(\eta(i\alpha)) + \frac{1}{2} N(N - 1) \log(2\pi/L),
\]

\[
E^p_{pp}(z) = \sum_{1 \leq j, k \leq N} \Phi^+(z_k, z_j)
\]

\[
= -\log \left( \prod_{1 \leq j < k \leq N} |\vartheta_1(\xi(z_k) - \xi(z_j); i\alpha)||\vartheta_1(\xi(z_k) + \xi(z_j); i\alpha)| \right)
\]

\[
+ \frac{3}{2} N(N - 1) \log(\eta(i\alpha)) + \frac{1}{2} N(N - 1) \log(2\pi/L) + \frac{N - 1}{2} \log \left( \prod_{j=1}^{N} |\vartheta_1(2\xi(z_j); i\alpha)| \right),
\]

respectively.

We think that in both types of systems there are uniform backgrounds which are negatively charged and the negative charge densities are given by \(-N^-/LW\). Here \( N^- \) denotes a total number of negative charges constructing the background, whose value is not equal to \( N \) in general and will be determined later. The particle-background potential energies are given by

\[
E^p_{pb}(z) = \sum_{j=1}^{N} V^p(z_j),
\]

with

\[
V^p(z) = -\frac{N^-}{LW} \int_{0}^{L} dx' \int_{0}^{W} dy' \Phi^+(z, z'), \quad z = -, \pm.
\]

(4.5)
The background-background potential energies are then given by

\[ E_{bb}^\sharp = -\frac{1}{2} N W \int_0^L dx \int_0^W dy \, V(z), \quad \sharp = -, \pm. \]

The probability weights for particle configuration \( z \) at the inverse temperature \( \beta > 0 \) are then given by the Boltzmann factors corresponding to the total energies,

\[ Q^\sharp_{\text{plasma}}(z) = Q^\sharp_{\text{plasma}}(z; N, N, \beta) \equiv \exp \left[ -\beta (E^\sharp_{pp}(z) + E^\sharp_{pb}(z) + E^\sharp_{bb}) \right], \quad \sharp = -, \pm, \quad z \in \Lambda(L, LW)^N. \]

These systems consisting of positively charged particles embedded in negatively charged background are called one-component plasma models [8, 4, 5].

### 4.2 Calculation of potential energies

Now we evaluate \( V(z) \) explicitly. By the definition (4.5) with (4.1), (4.2), (4.3), and (4.4),

\[ V^\pm(z) = \frac{N}{LW} \left( \Re I^\pm(z) + \Re I^\mp(z) - \frac{1}{2} \Re I^0 \right) - 3 N^\pm \log(\eta(i\alpha)) - N^\pm \log(2\pi/L) - \frac{1}{2} N^\pm \log(|\vartheta_1(2\xi(z); i\alpha)|), \]

where

\[ I^\pm(z) \equiv \int_0^L dx' \int_0^W dy' \log(\vartheta_1(\xi(z) \pm \xi(z'); i\alpha)), \]

\[ I^0 \equiv \int_0^L dx' \int_0^W dy' \log(\vartheta_1(2\xi(z'); i\alpha)). \]

As shown in Appendix B, we can evaluate that

\[ I^-(z) = LW \log(\eta(i\alpha)) + \pi \left( y - \frac{W}{2} \right)^2 + \frac{\pi W^2}{12} - \pi i(2xy - Wx - 2Ly + LW), \]  

\[ I^+(z) = LW \log(\eta(i\alpha)) + \pi y^2 + \pi W y + \frac{\pi W^2}{3} - \pi i(2xy + Wx), \]

\[ I^0 = LW \log(\eta(i\alpha)) + \frac{13\pi}{12} W^2 - \pi i LW. \]

Hence we have

\[ V^-(z) = -2 N^- \log(\eta(i\alpha)) - N^- \log(2\pi/L) + \frac{\pi N^-}{LW} \left( y - \frac{W}{2} \right)^2 + \frac{\pi N^- W}{12L}, \]

\[ V^\pm(z) = -\frac{3}{2} N^- \log(\eta(i\alpha)) - N^- \log(2\pi/L) + \frac{2\pi N^-}{LW} y^2 + \frac{\pi N^- W}{8L} - \frac{1}{2} N^- \log(|\vartheta_1(2\xi(z); i\alpha)|), \]
and then we obtain

$$E_{pb}^-(z) = -2NN^{-} \log(\eta(\text{i}a)) - NN^{-} \log(2\pi/L) + \frac{\pi N^{-}}{LW} \sum_{j=1}^{N} (y_j - \frac{W}{2})^2 + \frac{\pi NN^{-} W}{12L},$$

$$E_{pb}^+(z) = -\frac{3}{2} NN^{-} \log(\eta(\text{i}a)) - NN^{-} \log(2\pi/L) + \frac{2\pi N^{-}}{LW} \sum_{j=1}^{N} y_j^2 + \frac{\pi NN^{-} W}{8L}$$

- \frac{1}{2} NN^{-} \sum_{j=1}^{N} \log(|\vartheta_1(2\xi(z_j); \text{i}a)|),$$

and

$$E_{bb}^- = (N^{-})^2 \log(\eta(\text{i}a)) + \frac{1}{2} (N^{-})^2 \log(2\pi/L) - \frac{\pi (N^{-})^2 W}{12L},$$

$$E_{bb}^+ = (N^{-})^2 \log(\eta(\text{i}a)) + \frac{1}{2} (N^{-})^2 \log(2\pi/L) - \frac{\pi (N^{-})^2 W}{8L}.$$

The above gives the following results,

$$E_{pp}^- + E_{pb}^- + E_{bb}^- = -\log \left( \prod_{1 \leq j < k \leq N} |\vartheta_1(\xi(z_k) - \xi(z_j); \text{i}a)| \right) + \frac{\pi N^{-}}{LW} \sum_{j=1}^{N} \left( y_j - \frac{W}{2} \right)^2$$

$$+ \frac{1}{2} \{3N(N-1) - 4NN^{-} + 2(N^{-})^2 \} \log(\eta(\text{i}a))$$

$$+ \frac{1}{2} \{N(N-1) - 2NN^{-} + (N^{-})^2 \} \log(2\pi/L)$$

$$+ N^{-} (N^--N^{-}) \frac{\pi W}{12L},$$

(4.10)

and

$$E_{pp}^+ + E_{pb}^+ + E_{bb}^+ = -\log \left( \prod_{1 \leq j < k \leq N} |\vartheta_1(\xi(z_k) - \xi(z_j); \text{i}a)| |\vartheta_1(\xi(z_k) + \xi(z_j); \text{i}a)| \right)$$

$$+ \frac{1}{2} \{(N-1) - N^- \} \log \left( \prod_{j=1}^{N} |\vartheta_1(2\xi(z_j); \text{i}a)| \right) + \frac{2\pi N^{-}}{LW} \sum_{j=1}^{N} y_j^2$$

$$+ \frac{1}{2} \{3N(N-1) - 3NN^{-} + 2(N^{-})^2 \} \log(\eta(\text{i}a))$$

$$+ \frac{1}{2} \{N(N-1) - 2NN^{-} + (N^{-})^2 \} \log(2\pi/L)$$

$$+ N^{-} (N^--N^{-}) \frac{\pi W}{8L}.$$  

(4.11)

### 4.3 Solvability conditions

First we review the argument given by Forrester [4] for the first system with the two-point potential function $\Phi^-$. In (4.10), put

$$N^- = N = N^{\Lambda N-1}.$$
This neutralizes the system and the last term in the RHS of (4.10) vanishes. The probability weight (4.6) becomes
\[
Q_{\text{plasma}}^{-}(z; N, N^{A_{N-1}}, \beta) = (2\pi/L)^{\beta N/2} \eta(\iota \alpha)^{-\beta N(N-3)/2} 
\times \exp \left[ -\frac{\beta \pi N}{LW} \sum_{j=1}^{N} \left( y_j - \frac{W}{2} \right)^2 \right]
\prod_{1 \leq j < k \leq N} |\vartheta_2(\xi(z_j) - \xi(z_k); \iota \alpha)|^\beta.
\]

Then we set
\[
\beta = 2,
\]
and perform the following transformation of the weight from \(Q_{\text{plasma}}^{-}\) to \(\tilde{Q}_{\text{plasma}}^{-}\),
\[
\tilde{Q}_{\text{plasma}}^{-}(z; N, N^{A_{N-1}}, \beta = 2) = \left| \vartheta_{\tilde{s}(N)} \left( \sum_{k=1}^{N} \xi(z_k - (L + iW)/2); \iota \alpha \right) \right|^2 Q_{\text{plasma}}^{-}(z; N, N^{A_{N-1}}, \beta = 2), \tag{4.12}
\]
where \(\tilde{s}(N) = 0\), if \(N\) is even, and \(\tilde{s}(N) = 1\), if \(N\) is odd.

We can prove the following identities \[4.1\]. (The proof is given in Appendix \[3\].)

**Lemma 4.1** Let \(s(N) = 0, \tilde{s}(N) = 0\), if \(N\) is even, and \(s(N) = 3, \tilde{s}(N) = 1\), if \(N\) is odd. Then the following equalities hold,
\[
\exp \left[ -\frac{2\pi N}{LW} \sum_{j=1}^{N} \left( y_j - \frac{W}{2} \right)^2 \right] \left| \vartheta_{\tilde{s}(N)} \left( \sum_{k=1}^{N} \xi(z_k - (L + iW)/2); \iota \alpha \right) \right|^2 = \exp \left[ -\frac{2\pi N}{LW} \sum_{j=1}^{N} y_j^2 \right] \left| \vartheta_{s(N)} \left( \sum_{k=1}^{N} \xi(z_k); \iota \alpha \right) \right|^2. \tag{4.13}
\]

Then we arrive at the following equality,
\[
\tilde{Q}_{\text{plasma}}^{-}(z; N, N^{A_{N-1}}, \beta = 2) = c^{A_{N-1}}(L, \alpha)Q^{A_{N-1}}(z), \tag{4.14}
\]
with
\[
c^{A_{N-1}}(L, \alpha) = (2\pi/L)^{N} \eta(\iota \alpha)^{-N(N-3)}, \tag{4.15}
\]
where \(Q^{A_{N-1}}(z)\) is given by (2.15) for \(R_N = A_{N-1}\). As stated in Proposition \[2.3\] the probability weight \(Q^{A_{N-1}}(z)\) is totally elliptic with respect to the \(N\)-particle configuration \(z \in \mathbb{C}^N\), and hence so is \(Q_{\text{plasma}}(z; N, N^{A_{N-1}}, \beta = 2)\) by the proportionality (1.14). Moreover, as shown in Section \[3\] this probability weight can be well normalized and the point process \(\Xi^{A_{N-1}}\) is proved to be determinantal governed by the correlation kernel \(K^{A_{N-1}}\) given by (3.5) with \(\{M_n^{A_{N-1}}(z)\}_{n=1}^{N}\) defined in the first line of (2.6). Although Forrester did not give the correlation kernel for this DPP explicitly, he claimed the fact that if and only if we perform the transformation (4.12) in the probability weight at \(\beta = 2\), the one-component plasma model becomes exactly solvable in the sense that all correlation functions are explicitly given by determinants generated by \(K^{A_{N-1}}\). The origin of this key transformation (4.12) is found in the Macdonald denominator formula (2.5) for \(R_N = A_{N-1}\) given by Rosengren and Schlosser [15]. (Note that Forrester [4, 5] proved the equivalent equalities to the Macdonald denominator formula for \(R_N = A_{N-1}\) independently of [15]. See Remark 6 in [13].)

Now we consider the second system with the two-point potential function \(\Phi^\pm\). We will report two cases which give us the exactly solvable plasma models.
Type $D_N$

In (4.11), first we set

$$N^* = N - 1 = \frac{N^{D_N}}{2}.$$  

In this setting, the second term in the RHS of (4.11) vanishes. Since neutralization in not achieved, $N > N^*$, the last term in the RHS of (4.11) remains positive. However, if we set $\beta = 2$ again, we obtain the following equality,

$$Q_{\text{plasma}}^+(z; N, N^{D_N}/2, \beta = 2) = c^{D_N}(L, \alpha) \exp \left( -\frac{2\pi N^{D_N}}{LW} \sum_{j=1}^{N} y_j^2 \right) |W^{D_N}(\xi(z); i\alpha)|^2$$

$$= c^{D_N}(L, \alpha) Q^{D_N}(z)$$  \hfill (4.16)

with

$$c^{D_N}(L, \alpha) = (2\pi/L)^{N-1} \eta(i\alpha)^{-2(N-1)^2} e^{-\pi(N-1)\alpha/L}.$$  \hfill (4.17)

Type $C_N$

Next we set

$$N^- = N + 1 = \frac{N^{C_N}}{2}$$

in (4.11). The system is charged $-1$, and if we set $\beta = 2$, the following equality is established,

$$Q_{\text{plasma}}^+(z; N, N^{C_N}/2, \beta = 2) = c^{C_N}(L, \alpha) \exp \left( -\frac{2\pi N^{C_N}}{LW} \sum_{j=1}^{N} y_j^2 \right) |W^{C_N}(\xi(z); i\alpha)|^2$$

$$= c^{C_N}(L, \alpha) Q^{D_N}(z)$$  \hfill (4.18)

with

$$c^{C_N}(L, \alpha) = (2\pi/L)^{N-1} \eta(i\alpha)^{-2(N^2-N+1)} e^{-\pi(N+1)\alpha/L}.$$  \hfill (4.19)

The above results are summarized as follows.

**Theorem 4.2** In the following three cases, the one-component plasma model in $\Lambda_{(L, W)} \subset \mathbb{C}$ becomes exactly solvable in the sense that the particle configuration $z \in \Lambda_{(L, W)}$ is given by DPP.

1. The case such that the two-point potential is $\Phi^-$ given by (4.3), $N^* = N^{A_{N-1}}$, $\beta = 2$, and the transform (4.12) is performed. The correlation kernel is given by $K^{A_{N-1}}$. The system is neutral.

2. The case such that the two-point potential is $\Phi^+$ given by (4.4), $N^* = N^{C_N}/2$, and $\beta = 2$. The correlation kernel is given by $K^{C_N}$. The system is negatively charged by unit, $-1$.

3. The case such that the two-point potential is $\Phi^+$ given by (4.4), $N^* = N^{D_N}/2$, and $\beta = 2$. The correlation kernel is given by $K^{D_N}$. The system is positively charged by unit, $+1$.  

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4.4 Relationship to Gaussian free field on a torus

We define the partition functions for the present three types of exactly solvable one-component plasma models as

\[
Z_{\text{plasma}}^{R_N} = \frac{1}{N!} \int_{N(L, W)^N} dz \tilde{Q}_{\text{plasma}}(z; N, N^-, \beta = 2),
\]

with

\[
\tilde{Q}_{\text{plasma}}(\cdot; \cdot, N^-, \cdot) = \begin{cases} 
\tilde{Q}_{\text{plasma}}^- (\cdot; \cdot, N^{A_{N-1}}, \cdot) & \text{for } R_N = A_{N-1}, \\
\tilde{Q}_{\text{plasma}}^C (\cdot; \cdot, N^{R_N/2}, \cdot) & \text{for } R_N = C_N, D_N.
\end{cases}
\]

By the equalities (4.14), (4.16), and (4.18) with (4.15), (4.17) and (4.19), we have the equalities

\[
Z_{\text{plasma}}^{R_N} = c^{R_N} (L, \alpha) Z_N^{\text{plasma}}, \quad R_N = A_{N-1}, C_N, D_N,
\]

where \(Z_N^{\text{plasma}}\) are given in Lemma 3.1. Hence we obtain the following exact formulas for the solvable plasma models,

\[
\begin{align*}
Z_{\text{plasma}}^{A_{N-1}} &= \left(2\pi^2 \frac{LW}{N} \right)^{N/2} \eta^2 (\alpha)^2, \\
Z_{\text{plasma}}^{C_N} &= \left(2\pi^2 \frac{LW}{N + 1} \right)^{N/2} \frac{L}{2\pi} e^{\pi(N+1)\alpha/\eta (\alpha)} - 2, \\
Z_{\text{plasma}}^{D_N} &= \left(2\pi^2 \frac{LW}{N - 1} \right)^{N/2} \frac{L}{2\pi} e^{-\pi(N-1)\alpha/\eta (\alpha)} - 2.
\end{align*}
\]

The free energy per particle multiplied by the inverse temperature \(\beta\) is defined by

\[
F_{\text{plasma}}^{R_N} (\beta) = \beta f_{\text{plasma}}^{R_N} \equiv -\frac{1}{N} \log Z_{\text{plasma}}^{R_N}, \quad R_N = A_{N-1}, C_N, D_N.
\]

The exact formulas (4.20) give the following,

\[
\begin{align*}
F_{\text{plasma}}^{A_{N-1}} (\beta = 2) &= F_0^A + \frac{1}{N} F_1^A, \\
F_{\text{plasma}}^{C_N} (\beta = 2) &= F_0^C - \frac{\log N}{2N} + \frac{1}{N} F_1^C + O(N^{-2}), \\
F_{\text{plasma}}^{D_N} (\beta = 2) &= F_0^D - \frac{\log N}{2N} + \frac{1}{N} F_1^D + O(N^{-2})
\end{align*}
\]

with

\[
F_0^\varepsilon = \begin{cases} 
\frac{1}{2} \log \left( \frac{\rho}{2\pi^2} \right), & \varepsilon = A, \\
\frac{1}{2} \log \left( \frac{\rho}{4\pi^2} \right) - \frac{\pi}{4} \alpha, & \varepsilon = C, \\
\frac{1}{2} \log \left( \frac{\rho}{4\pi^2} \right) + \frac{\pi}{4} \alpha, & \varepsilon = D,
\end{cases}
\]

and

\[
F_1^\varepsilon = \begin{cases} 
- \log(\eta (\alpha)^2), & \varepsilon = A, \\
\log(\sqrt{\alpha} \eta (\alpha)^2) + \frac{1}{2} \log(2^3 \pi^2) - \left( \frac{\pi}{4} \alpha + \frac{1}{2} \right), & \varepsilon = C, \\
\log(\sqrt{\alpha} \eta (\alpha)^2) + \frac{1}{2} \log(2^6 \pi^2) - \left( \frac{\pi}{4} \alpha + \frac{1}{2} \right), & \varepsilon = D,
\end{cases}
\]

(4.21)
where \( \rho = N/(LW) \).

As pointed out by Forrester \[^4\] , it was shown by Cardy \[^2\] that, if we ignore the zero mode, the partition function of the Gaussian free field (GFF) on a torus with the modular parameter \( \tau \in \mathbb{H} \) is given by

\[
Z_{\text{GFF}}^{(k \neq 0)}(\tau) = \frac{1}{\eta(\tau)\eta(-\tau)}.
\]

while if we treat the zero mode with appropriate regularization, it is determined in the form with a factor proportional to the inverse square root of \( 3\tau \),

\[
Z_{\text{GFF}}(\tau) = \frac{1}{\sqrt{3\tau\eta(\tau)\eta(-\tau)}}.
\]

Forrester clarified the following equality with a negative sign \[^8,4\]

\[
F_1^A = -F_{\text{GFF}}^{(k \neq 0)}(i\alpha) \quad \text{with} \quad F_{\text{GFF}}^{(k \neq 0)}(i\alpha) \equiv -\log Z_{\text{GFF}}^{(k \neq 0)}(i\alpha) = \log(\eta(i\alpha)^2).
\]

On the other hand, here we state that \( F_1^C \) and \( F_1^D \) include the term

\[
F_{\text{GFF}}(i\alpha) \equiv -\log Z_{\text{GFF}}(i\alpha) = \log(\sqrt{\alpha}\eta(i\alpha)^2)
\]

without change of sign. Dedekind’s \( \eta \) function satisfies the functional equation,

\[
\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau) \quad \text{[17].}
\]

This implies the equality

\[
\sqrt{\alpha}\eta(i\alpha)^2 = \frac{1}{\sqrt{\alpha}}\eta\left(\frac{i}{\alpha}\right)^2,
\]

which means that the term \( \log(\sqrt{\alpha}\eta(i\alpha)^2) \) is invariant under the transformation,

\[
\alpha = \frac{W}{L} \quad \rightarrow \quad \frac{1}{\alpha} = \frac{L}{W}.
\]

### 5 Concluding Remarks

In the present paper, we proposed a new parameterization \[^25\] of the seven families of \( R_N \)-theta functions given by Rosengren and Schlosser \[^18\] associated with the seven types of irreducible reduced affine root systems, \( W^{R_N}(\mathfrak{g}) \), \( \mathfrak{g} = (\xi_1, \ldots, \xi_N) \subset \mathbb{C}^N \). Forrester clarified the following equality with a negative sign \[^8,4\],

\[
F_1^A = -F_{\text{GFF}}^{(k \neq 0)}(i\alpha) \quad \text{with} \quad F_{\text{GFF}}^{(k \neq 0)}(i\alpha) \equiv -\log Z_{\text{GFF}}^{(k \neq 0)}(i\alpha) = \log(\eta(i\alpha)^2).
\]

For the elliptic DPPs, (\( \Xi^{R_N} \), \( P^{R_N} \)) on a complex plane \( \mathbb{C} \). The connection to one-component plasma models was able to be discussed based on these orthogonality relations, but the results are limited to the three types \( A_{N-1}, C_N, D_N \). Since the partition functions of our elliptic DPPs, (\( \Xi^{R_N} \), \( P^{R_N} \)), are explicitly evaluated for all seven types in Lemma \[^5-1\] other plasma models desired whose particle sections realize (\( \Xi^{R_N} \), \( P^{R_N} \)) for \( R_N = A_{N-1}, C_N, D_N \). Generalization of the present results to two-component plasma models \[^3,8,4\] (see also Section 2.2 in \[^21\]) will be a challenging future problem.

In Theorem \[^3,4\] the DPPs with an infinite number of points are obtained. They are defined on \( \mathbb{C} \) having periodicity with period \( iW \). Appearance of such stripe structures, \( \mathbb{R} \times [inW, i(n+1)W) \), \( n \in \mathbb{Z} \), on \( \mathbb{C} \) is due to the scaling limit \( N \to \infty, L \to \infty \) with constant density \( \rho = N/(LW) \) and constant \( W \). In the further limit \( W \to \infty \) with constant \( \rho \), three types of infinite-dimensional DPPs are obtained. One of them is identified with the Ginibre point process, which is uniform on \( \mathbb{C} \) and realized as the eigenvalue distribution of complex Gaussian random matrices \[^6,7,20\]. Other two point processes are isotropic viewed from the origin, but non-uniform in \( \mathbb{C} \). Random matrix ensembles which give these point processes as eigenvalue distribution should be clarified.

We reported the relationship between the elliptic DPPs and the GFF on a torus following the argument given by Forrester \[^4\] . It is not yet known whether the relationship found in the large-\( N \) expansions of...

\( \beta = N/(LW) \).

\[
\alpha = \frac{W}{L} \quad \rightarrow \quad \frac{1}{\alpha} = \frac{L}{W}.
\]
the free energies means more direct connections between some limit systems of the present elliptic DPPs and random fields related to GFF \cite{19} in the level of probability laws and geometrical structures. In the systems of types C and D, the difference from type A of signs for the $\eta(i\alpha)$-terms found in \cite{21} suggests the boundary condition of the corresponding random field may be different from the Dirichlet boundary condition. The regularized elliptic determinantants should be invariant under the transformation \cite{22}, and hence they will be invariant under the modular group $SL(2, \mathbb{Z})$ of transformations of the form, $i\alpha \to (i\alpha + b)/(i\alpha + d)$, $a,b,c,d \in \mathbb{Z}, ad - bc = 1$.

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### A The Jacobi Theta Functions

Let

$$ z = e^{\pi i \tau}, \quad q = e^{\pi i \tau}, $$

where $v, \tau \in \mathbb{C}$ and $\Im \tau > 0$. The Jacobi theta functions are defined as follows \cite{23} \cite{17},

$$
\vartheta_0(v; \tau) = -ie^{\pi i(v+\tau/4)}\vartheta_1 \left( v + \frac{\tau}{2}; \tau \right) = \sum_{n \in \mathbb{Z}} (-1)^n q^n z^{2n} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{\pi i n^2 \tau} \cos(2n\pi v),
$$

$$
\vartheta_1(v; \tau) = i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n-1/2)^2} z^{2n-1} = 2 \sum_{n=1}^{\infty} (-1)^n \tau^{n-1/2} \sin \{ (2n-1)\pi v \},
$$

$$
\vartheta_2(v; \tau) = \vartheta_1 \left( v + \frac{1}{2}; \tau \right) = \sum_{n \in \mathbb{Z}} q^{(n-1/2)^2} z^{2n-1} = 2 \sum_{n=1}^{\infty} \tau^{n-1/2} \cos \{ (2n-1)\pi v \},
$$

$$
\vartheta_3(v; \tau) = e^{\pi i(v+\tau/4)}\vartheta_1 \left( v + \frac{1 + \tau}{2}; \tau \right) = \sum_{n \in \mathbb{Z}} q^{n^2} z^{2n} = 1 + 2 \sum_{n=1}^{\infty} \tau^{n} \cos(2n\pi v).
$$  \hspace{1cm} (A.1)

(Note that the present functions $\vartheta_{\mu}(v; \tau), \mu = 1, 2, 3$ are denoted by $\vartheta_{\mu}(v, q)$, and $\vartheta_0(v; \tau)$ by $\vartheta_4(v, q)$ in \cite{21}. For $\Im \tau > 0$, $\vartheta_{\mu}(v; \tau), \mu = 0, 1, 2, 3$ are holomorphic for $|v| < \infty$. The parity with respect to $v$ is given by

$$
\vartheta_1(-v; \tau) = -\vartheta_1(v; \tau), \quad \vartheta_{\mu}(-v; \tau) = \vartheta_{\mu}(v; \tau), \quad \mu = 0, 2, 3,
$$  \hspace{1cm} (A.2)

and they have the quasi-double-periodicity:

$$
\vartheta_{\mu}(v + 1; \tau) = \begin{cases} 
\vartheta_{\mu}(v; \tau), & \mu = 0, 3, \\
-\vartheta_{\mu}(v; \tau), & \mu = 1, 2,
\end{cases}
$$  \hspace{1cm} (A.3)

$$
\vartheta_{\mu}(v + \tau; \tau) = \begin{cases} 
-e^{-\pi i (2v+\tau)} \vartheta_{\mu}(v; \tau), & \mu = 0, 1, \\
e^{-\pi i (2v+\tau)} \vartheta_{\mu}(v; \tau), & \mu = 2, 3.
\end{cases}
$$  \hspace{1cm} (A.4)

By the definition \cite{21}, when $\tau = i\alpha$ with $\alpha > 0$,

$$
\vartheta_1(0; \tau) = \vartheta_1(1; \tau) = 0, \quad \vartheta_1(x; \tau) > 0, \quad x \in (0, 1),
$$

$$
\vartheta_2(-1/2; \tau) = \vartheta_2(1/2; \tau) = 0, \quad \vartheta_2(x; \tau) > 0, \quad x \in (-1/2, 1/2),
$$

$$
\vartheta_3(x; \tau) > 0, \quad x \in \mathbb{R}.
$$
We see the asymptotics
\[
\vartheta_0(v; \tau) \simeq 1, \quad \vartheta_1(v; \tau) \simeq 2 e^{\pi i/4} \sin(\pi v), \quad \vartheta_2(v; \tau) \simeq 2 e^{\pi i/4} \cos(\pi v), \quad \vartheta_3(v; \tau) \simeq 1,
\]
in \( \Im \tau \to +\infty \) (i.e., \( q = e^{\pi i} \to 0 \)). \quad \text{(A.5)}

The Jacobi theta function \( \vartheta_1 \) defined by (A.1) has the following infinite-product expressions,
\[
\vartheta_1(v; \tau) = -i q^{1/4} \prod_{j=1}^{\infty} \left( 1 - q^{2j} z^2 \right) \left( 1 - q^{2j-2} z^2 \right) \left( 1 - q^{2j} \right).
\]
\[
= 2 q^{1/4} \sin(\pi v) \prod_{j=1}^{\infty} \left\{ (1 - 2q^{2j} \cos(2\pi v) + q^{4j})(1 - q^{2j}) \right\}, \quad \text{(A.6)}
\]

The following is also known,
\[
\frac{\partial \vartheta_1(v; \tau)}{\partial v} \bigg|_{v=0} = 2\pi \eta(\tau)^3, \quad \text{(A.7)}
\]

where \( \eta(\tau) \) is the Dedekind modular function \cite{24}. The following functional equalities are known as Jacobi’s imaginary transformations \cite{24 17},
\[
\vartheta_0(v; \tau) = e^{\pi i/4} \tau^{-1/2} e^{-\pi i v^2/\tau} \vartheta_2 \left( \frac{v}{\tau}; -\frac{1}{\tau} \right),
\]
\[
\vartheta_1(v; \tau) = e^{3\pi i/4} \tau^{-1/2} e^{-\pi i v^2/\tau} \vartheta_1 \left( \frac{v}{\tau}; -\frac{1}{\tau} \right),
\]
\[
\vartheta_2(v; \tau) = e^{\pi i/4} \tau^{-1/2} e^{-\pi i v^2/\tau} \vartheta_0 \left( \frac{v}{\tau}; -\frac{1}{\tau} \right),
\]
\[
\vartheta_3(v; \tau) = e^{\pi i/4} \tau^{-1/2} e^{-\pi i v^2/\tau} \vartheta_3 \left( \frac{v}{\tau}; -\frac{1}{\tau} \right). \quad \text{(A.8)}
\]

B Evaluation of Integrals

B.1 Integral \( I^-(z) \)

Consider the integral
\[
I^-(z) = \int_0^L dx' \int_0^W dy' \log(\vartheta_1(\xi(z) - \xi(z'); i\alpha))
\]
\[
= \int_0^L dx' \int_0^W dy' \log \left\{ \vartheta_1 \left( \frac{x-x'}{L} + i \frac{y-y'}{L}; i\alpha \right) \right\}.
\]

By the product formula (A.6),
\[
I^-(z) = \sum_{j=1}^{5} I_j^-
\]
where
\[
I_5 = \int_0^L dx' \int_0^W dy' \left( -\frac{\pi \alpha}{4} \right) = -\frac{\pi \alpha}{4} LW = -\frac{\pi}{4} W^2,
\]
\[
I_2 = \int_0^L dx' \int_0^W dy' \log \left[ 2 \sin \left( \frac{\pi}{L} \{ (x - x') + i(y - y') \} \right) \right],
\]
\[
I_3 = \sum_{n=1}^\infty \int_0^L dx' \int_0^W dy' \log(1 - q^{2n} e^{2\pi i(x-x')/L - 2\pi(y-y')/L}),
\]
\[
I_4 = \sum_{n=1}^\infty \int_0^L dx' \int_0^W dy' \log(1 - q^{2n} e^{-2\pi i(x-x')/L + 2\pi(y-y')/L}),
\]
\[
I_5 = \int_0^L dx' \int_0^W dy' \sum_{n=1}^\infty \log(1 - q^{2n}) = LW \sum_{n=1}^\infty \log(1 - q^{2n}),
\]

where \( q = e^{-\pi \alpha} \). By the definition of Dedekind’s function \(2.7\), we readily see that
\[
I_5^- = LW \left( \frac{\pi \alpha}{12} + \log(\eta(\alpha)) \right) = \frac{\pi}{12} W^2 + LW \log(\eta(\alpha)).
\]

Note that
\[
q^{2n} e^{\pm 2\pi i(x-x')/L + 2\pi(y-y')/L} = \exp \left[ \frac{2\pi}{L} (nW \pm (y - y')) \right] e^{\pm 2\pi i(x-x')/L}.
\]

Since \( 0 \leq y, y' \leq W \), \( |y - y'| \leq W \) and thus for \( n \geq 1 \), \( nW \pm (y - y') \geq 0 \). Then we have the expansion for \( I_3^- \) as
\[
I_3^- = -\sum_{n=1}^\infty \sum_{k=1}^\infty \frac{q^{2nk}}{k} \int_0^L dx' e^{2\pi i(x-x')k/L} \int_0^W dy' e^{-2\pi(y-y')k/L}.
\]

For \( k \geq 1 \), \( \int_0^L dx' e^{2\pi i(x-x')k/L} = 0 \), and hence \( I_3^- = 0 \). Similarly, \( I_4^- = 0 \).

Now we consider the integrand of \( I_2^- \),
\[
\log \left[ 2 \sin \left( \frac{\pi}{L} \{ (x - x') + i(y - y') \} \right) \right] = -\frac{\pi i}{2} + \log \left( e^{\pi i(x-x')/L} e^{-\pi i(y-y')/L} - e^{-\pi i(x-x')/L} e^{\pi i(y-y')/L} \right).
\]

When \( y \geq y' \), this is written as
\[
-\frac{\pi i}{2} + \log \left[ (e^{-\pi i(x-x')/L} e^{\pi i(y-y')/L})(1 - e^{2\pi i(x-x')/L} e^{-2\pi(y-y')/L}) \right]
= -\frac{\pi i}{2} - \frac{\pi i}{L}(x + iy) + \frac{\pi i}{L}(x' + iy') - \sum_{k=1}^\infty \frac{1}{k} e^{2\pi i(x-x')k/L} e^{-2\pi(y-y')k/L},
\]
and, when \( y < y' \), the above integrand is written as
\[
-\frac{\pi i}{2} + \log \left[ e^{\pi i(x-x')/L} e^{-\pi i(y-y')/L}(1 - e^{-2\pi i(x-x')/L} e^{2\pi(y-y')/L}) \right]
= -\frac{\pi i}{2} + \frac{\pi i}{L}(x + iy) - \frac{\pi i}{L}(x' + iy') - \sum_{k=1}^\infty \frac{1}{k} e^{-2\pi i(x-x')k/L} e^{2\pi(y-y')k/L}.
\]

Since \( \int_0^Ldx' e^{\pm 2\pi i(x-x')/L} = 0, k \geq 1 \), for given \( (x, y) \in [0, L] \times [0, W] \), we have
\[
I_2^- = I_2^-< + I_2^->.
\]
with

\[ I_2^{-}\epsilon \equiv \int_0^L dx' \int_0^y dy' \left\{ \frac{\pi i}{2} - \frac{\pi i}{L} (x + iy) + \frac{\pi i}{L} (x' + iy') \right\} \]

\[ = -\frac{\pi i}{2} L y - \frac{\pi i}{L} (x + iy) L y + \frac{\pi i}{L} \left( \frac{L^2}{2} y + i \frac{y^2}{2} \right), \]

\[ I_2^{+}\epsilon \equiv \int_0^L dx' \int_y^W dy' \left\{ -\frac{\pi i}{2} + \frac{\pi i}{L} (x + iy) - \frac{\pi i}{L} (x' + iy') \right\} \]

\[ = -\frac{\pi i}{2} L (W - x) + \frac{\pi i}{L} (x + iy) L (W - y) - \frac{\pi i}{L} \left( \frac{L^2}{2} (W - y) + i L \left( \frac{W^2}{2} - \frac{y^2}{2} \right) \right), \]

and hence

\[ I_2^0 = \pi \left( y - \frac{W}{2} \right)^2 + \frac{\pi}{4} W^2 - \pi i (2xy - Wx - 2L + LW). \]

Combining the above results, we obtain (4.7).

B.2 Integral \( I^+(z) \)

Consider the integral

\[ I^+(z) = \int_0^L dx' \int_0^W dy' \log(\vartheta_1(\xi(z) + \xi(z') + i\alpha)) \]

\[ = \int_0^L dx' \int_0^W dy' \log \left\{ \vartheta_1 \left( \frac{x + x'}{L} + i \frac{y + y'}{L} + i\alpha \right) \right\}. \]

By the quasi-double-periodicity (A.4), the integrand is equal to

\[ \pi i - 2 \frac{\pi i}{L} (x + x') + 2 \frac{\pi}{L} y + \frac{\pi}{4} W + \log \vartheta_1 \left( \frac{x + x'}{L} + i \frac{y + y'}{L} + i\alpha \right). \]

By the product formula (A.6),

\[ I^+(z) = \sum_{j=0}^5 I_j^+ \]

with

\[ I_0^+ = \int_0^L dx' \int_0^W dy' \left\{ \frac{\pi i}{2} - \frac{\pi i}{L} (x + x') + \frac{2\pi}{L} (y + y') - \frac{\pi W}{L} \right\} \]

\[ = 2\pi y W - 2\pi W x, \]

\[ I_1^+ = \int_0^L dx' \int_0^W dy' \left( -\frac{\pi \alpha}{4} \right) = -\frac{\pi \alpha}{4} LW = -\frac{\pi}{4} W^2, \]

\[ I_2^+ = \int_0^L dx' \int_0^W dy' \log \left[ 2 \sin \left( \frac{\pi}{L} \left( (x + x') + i(y + y') - W \right) \right) \right], \]

\[ I_3^+ = \sum_{n=1}^{\infty} \int_0^L dx' \int_0^W dy' \log \left( 1 - q^{2n} e^{2\pi i(x+x')/L - 2\pi(y+y'-W)/L} \right), \]

\[ I_4^+ = \sum_{n=1}^{\infty} \int_0^L dx' \int_0^W dy' \log \left( 1 - q^{2n} e^{-2\pi(x+x')/L + 2\pi(y+y'-W)/L} \right), \]

\[ I_5^+ = \int_0^L dx' \int_0^W dy' \sum_{n=1}^{\infty} \log \left( 1 - q^{2n} \right) = \frac{\pi}{12} W^2 + LW \log(\eta(i\alpha)). \]
Note that
\[ q^{2n} e^{\pm 2\pi i(x+x')/L \mp 2\pi i(y+y'-W)/L} = \exp \left( -\frac{2\pi}{L} \{ nW \pm (y+y'-W) \} \right) e^{\pm 2\pi i(x+x')/L}. \]
Since \( 0 \leq y, y' \leq W, |y + y' - W| \leq W \) and thus for \( n \geq 1, nW \pm (y+y'-W) \geq 0 \). Then we have the expansion for \( I_3^+ \) as
\[ I_3^+ = -\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} q^{2nk}/k \int_0^L dx' e^{2\pi i(x+x')k/L} \int_0^W dy' e^{-2\pi i(y+y'-W)k/L}. \]
For \( k \geq 1, \int_0^L dx' e^{2\pi i(x+x')k/L} = 0 \), and hence \( I_3^+ = 0 \). Similarly, \( I_3^- = 0 \).

Now we consider the integrand of \( I_2^+ \),
\[ \log \left[ 2 \sin \left( \frac{\pi}{L} \{ (x + x') + i(y + y') \} \right) \right] = -\frac{\pi i}{2} + \log \left( e^{\pi i(x+x')/L} e^{-\pi (y+y'-W)/L} - e^{-\pi (x+x')/L} e^\pi (y+y'-W)/L \right). \]
When \( y + y' - W \geq 0 \), this can be expanded as
\[ -\frac{\pi i}{2} + \pi i \frac{y + y' - W}{L} - \frac{\pi W}{L} - \sum_{k=1}^{\infty} \frac{1}{k} e^{2\pi i(x+x')k/L} e^{-2\pi i(y+y'-W)k/L}, \]
and, when \( y + y' - W < 0 \), this can be expanded as
\[ -\frac{\pi i}{2} + \pi i \frac{y + y' - W}{L} + \frac{\pi W}{L} - \sum_{k=1}^{\infty} \frac{1}{k} e^{-2\pi i(x+x')k/L} e^{2\pi i(y+y'-W)k/L}. \]
Since \( \int_0^L dx' e^{\pm 2\pi i(x+x')/L} = 0 \), for given \((x, y) \in [0, L] \times [0, W]\), we have
\[ I_2^+ = I_2^+ > + I_2^+ < \]
with
\[ I_2^+ > = \int_0^L dx' \int_{-y+W}^W dy' \left\{ \frac{\pi i}{2} - \frac{\pi i}{L} (x + iy) - \frac{\pi i}{L} (x' + iy') - \frac{\pi W}{L} \right\} \]
\[ = \frac{\pi}{2} y^2 - \pi i xy, \]
\[ I_2^+ < = \int_0^L dx' \int_{-y+W}^W dy' \left\{ \frac{\pi i}{2} + \frac{\pi i}{L} (x + iy) + \frac{\pi i}{L} (x' + iy') + \frac{\pi W}{L} \right\} \]
\[ = \frac{\pi}{2} y^2 - \pi Wy + \frac{\pi}{2} W^2 - \pi i x (y - W), \]
and hence
\[ I_2^+ = \pi y^2 - \pi Wy + \frac{\pi}{2} W^2 - \pi i (2xy - Wx). \]
Combining the above results, we obtain (4.8).

**B.3 Integral I**
Consider the integral
\[ I = \int_0^L dx' \int_{0}^{W} dy' \log(\varphi_1(2\xi(z'); \imath)) \]
\[ = \int_0^L dx' \int_{0}^{W} dy' \log \left\{ \varphi_1 \left( \frac{2x'}{L} + \imath \frac{2y'}{L} \right) \right\}. \]

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By the quasi-double-periodicity (A.4), the integrand is equal to
\[ \pi i - \frac{4\pi i}{L}(x' + y'i) - \frac{\pi W}{L} + \log \vartheta_1 \left( \frac{2x'}{L} + \frac{2y'}{L} - W; i\alpha \right). \]

By the similar argument to those given in the previous two subsection, we can show that
\[ I^0(z) = (\pi W^2 - \pi i LW) - \frac{\pi}{4} W^2 + I_2^0 + \left( \frac{\pi}{12} W^2 + LW \log(\eta(i\alpha)) \right), \]

with
\[ I_2^0 = \int_{0}^{L} dx' \int_{0}^{W} dy' \log \left[ \frac{\pi}{L} \{(2x' + i(2y' - W))\} \right], \]
\[ = \frac{\pi^2}{2} W^2. \]

Then we obtain (4.9).

C Proof of Lemma 4.1

First we assume \( N \) is even and put \( N = 2n, n \in \mathbb{N} \). In this case
\[ \vartheta_0 \left( \sum_{k=1}^{N} \xi(z_k - (L + iW)/2); i\alpha \right) = \vartheta_0 \left( \sum_{k=1}^{N} \xi(z_k) - N(1 + i\alpha)/2; i\alpha \right) \]
\[ = \vartheta_0 \left( -\sum_{k=1}^{N} \xi(z_k) + (1 + i\alpha)n; i\alpha \right), \]
where we used (A.2). By the quasi-double-periodicity (A.3) and (A.4) of \( \vartheta_0(v; \tau) \), we have the equality
\[ \vartheta_0(v + (1 + \tau); \tau) = -e^{-\pi i (2v + \tau)} \vartheta_0(v; \tau). \]

Using this equality \( n \) times, we obtain
\[ \vartheta_0 \left( -\sum_{k=1}^{N} \xi(z_k) + (1 + i\alpha)n; i\alpha \right) \]
\[ = (-1)^n \exp \left[ -\pi i \left\{ -2n \sum_{j=1}^{N} \xi(z_j) + 2(1 + i\alpha) \sum_{j=1}^{n-1} j + i\alpha n \right\} \right] \vartheta_0 \left( \sum_{k=1}^{N} \xi(z_k); i\alpha \right) \]
\[ = (-1)^n \exp \left[ \pi i \left\{ \frac{N}{L} \sum_{j=1}^{N} x_j - (n - 1)n \right\} \right] \exp \left[ -\frac{N\pi}{L} \sum_{j=1}^{N} y_j + \frac{\pi n^2}{4} \right] \vartheta_0 \left( \sum_{k=1}^{N} \xi(z_k); i\alpha \right). \]

Since
\[ -\frac{N\pi}{L} \sum_{j=1}^{N} y_j + \frac{\pi n^2}{4} = -\frac{\pi N}{LW} \left\{ \sum_{j=1}^{N} y_j^2 - \sum_{j=1}^{N} \left( y_j - \frac{W}{2} \right)^2 \right\}, \]
(C.1)

(4.13) is obtained for even \( N \).

Next we assume \( N \) is odd and put \( N = 2n + 1, n \in \mathbb{N} \). In this case
\[ \vartheta_1 \left( \sum_{k=1}^{N} \xi(z_k - (L + iW)/2); i\alpha \right) = \vartheta_1 \left( -\sum_{k=1}^{N} \xi(z_k) + n(1 + i\alpha) + \frac{1}{2}(1 + i\alpha); i\alpha \right). \]
By the definition (A.1),
\[ \vartheta_1(v + (1 + \tau)/2; \tau) = e^{-\pi i (v+\tau)/4} \vartheta_3(v; \tau), \]
and by (A.3) and (A.4),
\[ \vartheta_3(\tau + (1 + \tau); \tau) = e^{-\pi i (2v+\tau)} \vartheta_3(v; \tau). \]

Thus we have
\[ \vartheta_1 \left( -\sum_{k=1}^{N} \xi(z_k) + n(1 + i\alpha) + \frac{1}{2}(1 + i\alpha); i\alpha \right) \]
\[ = \exp \left[ -\pi \left\{ -\sum_{j=1}^{N} \xi(z_j) + n(1 + i\alpha) + \frac{i\alpha}{4} \right\} \right] \vartheta_3 \left( -\sum_{k=1}^{N} \xi(z_k) + n(1 + i\alpha); i\alpha \right) \]
\[ = \exp \left[ -\pi \left\{ -\sum_{j=1}^{N} \xi(z_j) + n(1 + i\alpha) + \frac{i\alpha}{4} \right\} \right] \exp \left[ -\pi i \left\{ -2n \sum_{j=1}^{N} \xi(z_j) + 2(1 + i\alpha) \sum_{j=1}^{n-1} j + i\alpha n \right\} \right] \]
\[ \times \vartheta_3 \left( -\sum_{k=1}^{N} \xi(z_k) + n(1 + i\alpha); i\alpha \right) \]
\[ = \exp \left[ N\pi i \sum_{j=1}^{N} \xi(z_j) - \pi n^2 + \frac{\pi \alpha N^2}{4} \right] \vartheta_3 \left( \sum_{k=1}^{N} \xi(z_k); i\alpha \right). \]

This is written as
\[ \exp \left[ \pi i \left\{ \frac{N}{L} \sum_{j=1}^{N} x_j - n^2 \right\} \right] \exp \left[ -\frac{N\pi}{L} \sum_{j=1}^{N} y_j + \frac{\pi \alpha N^2}{4} \right] \vartheta_3 \left( \sum_{k=1}^{N} \xi(z_k); i\alpha \right). \]

Then, through (C.1), (4.13) is obtained for odd \( N \). The proof is complete. ■

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