Simulation of Fast Magnetic Reconnection using a Two-Fluid Model of Collisionless Pair Plasma without Anomalous Resistivity\[2

E. Alec Johnson (ejohnson@math.wisc.edu)[2
James A. Rossmanith (rossmani@math.wisc.edu)

Department of Mathematics
UW-Madison

Abstract

For the first time to our knowledge, we demonstrate fast magnetic reconnection near a magnetic null point in a fluid model of collisionless pair plasma without resorting to the contrivance of anomalous resistivity. In particular, we demonstrate that fast reconnection occurs in an anisotropic adiabatic two-fluid model of collisionless pair plasma with relaxation toward isotropy for a broad range of isotropization rates. For very rapid isotropization we see fast reconnection, but instabilities eventually arise that cause numerical error and cast doubt on the simulated behavior.

Overview

Motivating Problem. We have been working to develop algorithms that efficiently model fast magnetic reconnection in collisionless space plasmas. A plasma is a gas of charged particles and the accompanying magnetic field that it carries. The ability to simulate plasmas efficiently over a wide range of phenomena and scales is essential to understanding and predicting the behavior both of space plasmas and of industrial fusion plasmas.

The two basic types of plasma models are kinetic models and fluid models. Fluid models are computationally less expensive attempts to approximate kinetic models. Kinetic models represent particles or evolve the space-velocity distribution of particles. Fluid models evolve moment averages (e.g. density, momentum, or energy) of assumed (e.g. Maxwellian) velocity distributions. Fluid models give good accuracy for highly collisional plasmas. For rarefied space plasmas, however, particularly near magnetic null points, the regular velocity distributions assumed by fluid models often fail to hold, since the particles move essentially unconstrained.

The phenomenon for which fluid models of plasma have been most apt to fail is collisionless fast magnetic reconnection. It is precisely this phenomenon which has proved most critical to understanding and predicting the volatile dynamics of astrophysical plasmas, including solar storms and geomagnetic substorms in Earth’s magnetosphere. The critical physical role of fast reconnection and the failure of fluid models to capture it has prompted extensive studies using particle-based simulations of collisionless magnetic reconnection. Our objective is to study the ability of fluid models to match particle-based simulations of collisionless fast magnetic reconnection. We have

\[1\] Appeared in Proceedings of the 19th Annual Wisconsin Space Conference (2009).
\[2\] This research was supported by a Wisconsin Space Grant Consortium Graduate Fellowship for 2008-2009.
concentrated our effort specifically on fast magnetic reconnection in collisionless *pair* plasmas. A *pair plasma* is a plasma whose positively and negatively charged particles have the same charge-to-mass ratio. The physical example of a pair plasma is an electron-positron plasma, of interest to astrophysicists.

**Historical development.** The GEM magnetic reconnection challenge problem [3] identified Hall effects as critical to fast magnetic reconnection in electron-ion plasmas. Since Hall effects are absent for electron-positron (pair) plasmas, this prompted Bessho and Bhattacharjee [1, 2] to demonstrate via particle simulations that fast magnetic reconnection occurs even in collisionless pair plasma, which they attributed to pressure anisotropy.

The next challenge was to demonstrate fast reconnection in a fluid model of pair plasma. Assuming the ubiquitous presence of a strong background magnetic guide field (which constrains charged particles to move in tight spirals) allowed Chacón et al. [4] to develop an analytical fluid theory of fast reconnection in magnetized pair plasma.

For the case where there is a magnetic null point, Zenitani et al. [10] demonstrated fast reconnection in a two-fluid model of relativistic isotropic pair plasma. Their model assumes a spatially dependent anomalous resistivity, as has been used with resistive single-fluid MHD to simulate fast reconnection. By selecting a resistivity with an anomalously high value near the X-point, one can essentially prescribe the desired rate of reconnection (as determined from PIC simulations); the elusive goal is to find a simple and generic expression for anomalous resistivity that works for a broad range of problem conditions. We remark that resistive single-fluid MHD does not assume any particular ratio of mass-to-charge ratios between the two species and thus (with an appropriate choice of anomalous resistivity) could be used as a fluid model of pair plasma.

**Our work.** Rather than resort to an anomalous resistivity, we seek generic two-fluid moment closures that give reconnection behavior in agreement with particle-based simulations. We have studied reconnection in five-moment (isotropic) and ten-moment (anisotropic) adiabatic fluid models of collisionless pair plasma with varying rates of relaxation toward isotropy. We implemented conservative shock-capturing Discontinuous Galerkin two-fluid five-moment and ten-moment plasma models, following Hakim et al. [5, 6].

We initially adopted the modified GEM settings of [1, 2], but found that for pair plasmas the large aspect ratio of the reconnection region gives rise to a secondary instability, namely, the unpredictable formation of magnetic islands, making it difficult to obtain demonstrably converged results, as seen in our paper, [8]. Essentially, for the pair plasma case of the GEM problem, in contrast to the electron-proton case, the tearing instability wants to produce smaller magnetic islands; since the GEM problem *is* the formation of one big magnetic island, we can avoid the instability by reducing the size of the domain. Pair plasma involves no need to resolve scale separation between species, so arguably a smaller domain is acceptable. Therefore, to eliminate the secondary instability and to reduce computational expense we multiplied the dimensions of the domain by one half.

We also chose to focus on the case where both species have the same temperature. In this case there is complete symmetry between the two species, and the number of equations needed is halved. In
the case of zero guide field the GEM problem is symmetric about both the horizontal and vertical axes. We enforced all these symmetries, reducing computational expense by a factor of eight.

We simulated the GEM problem using the collisionless adiabatic ten-moment pair plasma model supplemented with a globally prescribed rate of pressure tensor isotropization. We have neglected all diffusivities and relaxation terms except isotropization. We varied the rate of pressure isotropization and studied the resulting variation in the rate of reconnection and the contributions of the terms in Ohm’s law.

Model

Generic physical equations for the ten-moment two-fluid model are:

- conservation of mass for each species:
  \[ \partial_t \rho_s + \nabla \cdot (\rho_s u_s) = 0, \]

- conservation of momentum for each species:
  \[ \partial_t (\rho_s u_s) + \nabla \cdot (\rho_s u_s \otimes u_s + P_s) = \frac{q_s}{m_s} \rho_s (E + u_s \times B) + R_s, \]

- evolution of the pressure tensor for each species:
  \[ \partial_t P_s + \nabla \cdot (u_s P_s) + 2 \text{Sym} (P_s \cdot \nabla u_s) + \nabla \cdot Q_s = 2 \text{Sym} \left( \frac{q_s}{m_s} P_s \times B \right) + R_s, \]

- Maxwell’s equations for evolution of electromagnetic field:
  \[ \partial_t B + \nabla \times E = 0, \]
  \[ \partial_t E - c^2 \nabla \times B = -J / \varepsilon, \]

- and Maxwell’s divergence constraints:
  \[ \nabla \cdot B = 0, \]
  \[ \nabla \cdot E = \sigma / \varepsilon. \]

In these equations, Sym denotes the symmetric part of the argument tensor (i.e. the average over all permutations of subscripts), \( \vee \) denotes symmetric outer product (i.e. the symmetric part of the tensor product), and \( i \) and \( e \) are positive (“ion”) and negative (“electron”) species indices; for species \( s \in \{i, e\} \), \( q_s = \pm e \) is particle charge, \( m_s \) is particle mass, \( n_s \) is particle number density, \( \rho_s = m_s n_s \) is mass density, \( \sigma_s = q_s n_s \) is charge density, \( u_5 \rho_s \) is momentum, \( J_s = u_s \sigma_s \) is current.

\(^3\)For conservation and shock-capturing purposes we actually evolve the energy tensor \( E_s := P_s + \rho_s u_s u_s \) rather than directly evolving the pressure tensor.
density, and $P_s$ is a pressure tensor; $B$ is magnetic field, $E$ is electric field, $c$ is the speed of light, $\varepsilon$ is vacuum permittivity, $J = J_i + J_e$ is net current density, and $\sigma = \sigma_i + \sigma_e$ is net charge density. To close the system, constitutive relations must be supplied for the generalized heat fluxes $Q_s$, the interspecies drag force on the ions $R_i = -R_e$, and $R_s$, the production of generalized thermal energy due to collisions. We nondimensionalize these equations, choosing the timescale to be the gyroperiod of a typical particle of mass 1, and choosing the typical velocity to be a typical Alfvén speed. The nondimensionalized equations retain the form of the dimensional equations above, with the simplifications that $e = 1$, $m_i + m_e = 1$, and $\frac{1}{\varepsilon} = c^2$.

In our closure we assume that $R_s = 0$, and to provide for isotropization we let

$$R_s = \frac{1}{\tau_s} \left( \frac{1}{3} (\text{tr} P_s) I - P_s \right),$$

where $\tau_s$ is the isotropization period of species $s$, $\text{tr}$ denotes tensor trace, and $I$ is the identity tensor.

In our present work we assume that $Q_s = 0$. This assumption might not be satisfactory, though: the particle simulations of Hesse et al. [7] showed, at least for the case of guide-field electron-proton reconnection, that generalized heat flux contributions to the evolution of the pressure tensor are necessary to obtain an appropriate approximation for the pressure nongyrotropy near the X-point. We therefore plan to investigate C. David Levermore’s closure [9],

$$Q_s = \frac{9}{5} (v_0 - v_1) \nabla \cdot \nabla \left( \nabla \cdot \Theta_s^{-1} \right) + 3v_1 \left( \nabla \cdot \Theta_s^{-1} \right),$$

where $\Theta_s := P_s/\rho_s$ and $v_0 \simeq v_1$ is proportional to collision frequency. We expect to determine whether we can get fast reconnection without isotropization by using such a non-vanishing generalized heat flux.

**Ohm’s law.** Combining the momentum equations gives net current balance. Assuming quasineutrality (zero net charge) and solving for electric field gives Ohm’s law for the electric field:

$$E = \frac{m_i + m_e}{\rho} (-R_s) \quad \text{(resistive term)}$$

$$+ B \times u \quad \text{(ideal term)}$$

$$+ \frac{\dot{m}_i - \dot{m}_e}{\rho} J \times B \quad \text{(Hall term)}$$

$$+ \frac{1}{\rho} \nabla \cdot (\dot{m}_e P_i - \dot{m}_i P_e) \quad \text{(pressure term)}$$

$$+ \frac{\dot{m}_i \dot{m}_e}{\rho} \left( \partial_t J + \nabla \cdot (uJ + J\dot{u} + \frac{\dot{m}_e - \dot{m}_i}{\rho} J J) \right) \quad \text{(inertial term)},$$

where $\dot{m}_i := \frac{m_i}{\varepsilon}$ and $\dot{m}_e := \frac{m_e}{\varepsilon}$ and the resistive term is usually assumed to be of the form $\eta \cdot J$, i.e., a linear function of current.
GEM magnetic reconnection challenge problem

The GEM magnetic reconnection challenge problem studies the evolution of 2-dimensional plasma in a rectangular box aligned with the coordinate axes and centered at the origin. The top and bottom of the box are conducting walls and periodic symmetry in the x-axis defines the width of the box. The plasma is initially in near-equilibrium. The upper half of the box is occupied by strong magnetic field lines pointing to the right and the lower half is occupied by strong magnetic field lines pointing to the left, separated by a thin, potentially volatile transition layer along the x-axis. Some studies (e.g. [4]) add a constant out-of-plane component to the magnetic field, called a “guide field”. We do not have a guide field (nor did the original GEM problem or the studies we are trying to replicate).

Domain. The computational domain is the rectangular domain $[-L_x/2, L_x/2] \times [-L_y/2, L_y/2]$. The problem is symmetric under 180 degree rotation around the origin, and in the case of zero guide field is also symmetric under reflection across either the horizontal or vertical axis. In the original GEM problem, $L_x = 8\pi$ and $L_y = 4\pi$. We halved the dimensions, so that $L_x = 4\pi$ and $L_y = 2\pi$.

Boundary conditions. The domain is periodic along the x-axis. The boundaries parallel to the x-axis are thermally insulating conducting wall boundaries. A conducting wall boundary is a solid wall boundary (with slip boundary conditions in the case of ideal plasma) for the fluid variables, and the electric field at the boundary has no component parallel to the boundary. We also assume that magnetic field does not penetrate the boundary.

Model Parameters. We set the speed of light to 10 (rather than 20 as in [1]) and set the mass of each species to 0.5 (rather than the GEM values of 1 for ions and 1/25 for electrons).

Initial conditions. The initial conditions are a perturbed Harris sheet equilibrium. The unperturbed equilibrium is given by

$$B(y) = B_0 \tanh(y/\lambda) e_x, \quad p(y) = \frac{B_0^2}{2n_0} n(y),$$

$$n(y) = n_i(y) = n_e(y) = n_0 (1/5 + \text{sech}^2(y/\lambda)), \quad p_e(y) = \frac{T_e}{T_i + T_e} p(y), \quad p_i(y) = \frac{T_i}{T_i + T_e} p(y).$$

On top of this the magnetic field is perturbed by

$$\delta B = -e_z \times \nabla(\psi), \quad \text{where} \quad \psi(x,y) = \psi_0 \cos(2\pi x/L_x) \cos(\pi y/L_y).$$

In the GEM problem the initial condition constants are

$$T_i/T_e = 5, \quad \lambda = 0.5, \quad B_0 = 1, \quad n_0 = 1, \quad \psi_0 = B_0/10;$$

we reset the initial temperature ratio to 1 to get symmetry between the species, and we set $\psi_0$ to $r_s^2 B_0/10$, where $r_s = 0.5$ is our domain rescaling factor, so that in the vicinity of the X-point our initial conditions agree (up to first-order Taylor expansion) with the initial conditions of the GEM problem.
**Properties of the GEM problem**

**Reconnected flux.** We define magnetic reconnection to be the loss of magnetic flux through the vertical axis into the first quadrant. Using Faraday’s law, $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$, one can show that the rate of reconnection is minus the value of the out-of-plane component of the electric field at the origin (i.e. the X-point) \[8\].

**Ohm’s law at the origin.** Since the electric field at the origin is the rate of reconnection, we are lead to study the terms of Ohm’s law for the electric field at the origin. Since the problem is symmetric under 180 degree rotational symmetry about the origin, only the out-of-plane component of vectors is nonzero at the origin. In Ohm’s law only the out-of-plane components of the resistive, pressure, and inertial terms survive.

As a proxy for Ohm’s law\[4\], we select a species and write the momentum equation solved for the electric field:

$$d_t (\text{reconnected flux}) = \mathbf{E}_3 \bigg|_{\text{origin}} = \left[ \frac{-\mathbf{R}_i}{e n_i} + \frac{\nabla \cdot \mathbf{P}_i}{e n_i} + \frac{m_i}{e} \partial_t \mathbf{u}_i \right]_{\text{origin}}^3.$$

In a perfectly collisionless, gyrotropic plasma, the resistive term and pressure divergence vanish, and \textit{reconnected flux should exactly track with species velocity} (a proxy for the current) at the origin.

**Results**

We simulated the GEM magnetic reconnection challenge problem for a pair plasma, varying the mesh resolution and varying the rate of isotropization from zero to instantaneous. We plotted the contribution of proxy Ohm’s law terms to the reconnected flux. We find that for a broad intermediate range of isotropization rates reconnection is fast and that the pressure term makes the dominant contribution to reconnected flux.

When isotropization is very slow or absent, there is oscillatory exchange between the inertial term and the pressure term at roughly a typical gyrofrequency, and reconnection proceeds at a slow to moderate rate (our simulations for the case of no isotropization are not sufficiently resolved). For a broad intermediate range of isotropization, there is little oscillation and, in agreement with PIC simulations (see \[1\] \[2\]), the pressure divergence dominates and provides for faster reconnection. As the rate of isotropization becomes very fast, however, the pressure divergence is forced to vanish and the inertial term is the only remaining term in the equation that can provide for reconnected flux. This forces the current at the origin to ramp up in track with reconnected flux. The system seems unable to sustain this ramp-up in current, however, and numerical instability kicks in, as evidenced by the sudden appearance of strong, very rapid oscillatory exchange (less evident in the accumulation integrals shown) between the inertial term and the residual. The numerical

\[4\] Ohm’s law involves the approximating assumption of quasineutrality, whereas the momentum equation holds exactly and the inertial term reduces to a simpler form at the origin.
residual displaces the inertial term, allowing the the current to peak and then decay while the re-connected flux maintains the same smooth, rapid ascent seen for intermediate isotropization rates, as if unconcerned whether pressure, inertia, numerical resistivity – or even anomalous resistivity, as our cursory investigations suggest – provides for its determined course. We can conclude that fast reconnection at least commences in an isotropic pair plasma model, and we conjecture that the numerical instability we see corresponds to some physical (perhaps streaming?) instability that provides for an effective anomalous resistivity (whose functional form we have not analyzed). As expected, the five-moment simulations show agreement with instantaneous relaxation of the ten-moment system to isotropy.

Acknowledgements
I thank the Wisconsin Space Grant Consortium for their support. In addition to my advisor, James Rossmanith, I also thank Nick Murphy, Ellen Zweibel, and Ping Zhu for helpful conversations.

References

[1] N. Bessho and A. Bhattacharjee. Collisionless reconnection in an electron-positron plasma. *Phys. Rev. Letters*, 95:245001, December 2005.

[2] N. Bessho and A. Bhattacharjee. Fast collisionless reconnection in electron-positron plasmas. *Physics of Plasmas*, 14:056503, 2007.

[3] J. Birn, J.F. Drake, M.A. Shay, B.N. Rogers, R.E. Denton, M. Hesse, M. Kuznetsova, Z.W. Ma, A. Bhattacharjee, A. Otto, and P.L. Pritchett. Geospace environmental modeling (GEM) magnetic reconnection challenge. *Journal of Geophysical Research – Space Physics*, 106:3715–3719, 2001.

[4] L. Chacón, Andrei N. Simakov, V.S. Lukin, and A. Zocco. Fast reconnection in nonrelativistic 2D electron-positron plasmas. *Phys. Rev. Letters*, 101:025003, July 2008.

[5] A. Hakim, J. Loverich, and U. Shumlak. A high-resolution wave propagation scheme for ideal two-fluid plasma equations. *J. Comp. Phys.*, 219:418–442, 2006.

[6] A.H. Hakim. Extended MHD modelling with the ten-moment equations. *J. Fusion Energy*, 27(1–2):36–43, June 2007.

[7] M. Hesse, M. Kuznetsova, and J. Birn. The role of electron heat flux in guide-field magnetic reconnection. *Physics of Plasmas*, 11(12):5387–5397, 2004.

[8] E.A. Johnson and J.A. Rossmanith. Collisionless magnetic reconnection in a five-moment two-fluid electron-positron plasma. In *Proceedings of Symposia in Applied Mathematics*, volume 67.2, 2009.

[9] C. David Levermore. Kinetic theory, Gaussian moment closures, and fluid approximations. Presented at IPAM KT2009 Culminating Retreat, Lake Arrowhead, California, June 2009.

[10] S. Zenitani, M. Hesse, and A. Klimas. Two-fluid magnetohydrodynamic simulations of relativistic magnetic reconnection. *The Astrophysical Journal*, 696:1385–1401, May 2009.
Figure 1: Increase in reconnection rate as isotropization increases. For very slow or no isotropization the simulations we report are not sufficiently resolved. For instantaneous relaxation the positron bulk velocity ceases to track with reconnected flux and the residual becomes unacceptably large after $t = 15$, indicating that we are no longer solving the momentum equation faithfully.
Figure 2: Convergence comparisons for coarse (left) and fine (right) meshes. For intermediate isotropization rates a coarser mesh is satisfactory, but for no isotropization and for fast isotropization a finer mesh is required. For fast and instantaneous isotropization, even for a very fine mesh, the residual in “Ohm’s law” (the momentum equation) becomes unacceptably large, overwhelming and displacing the inertial and pressure terms. The five-moment simulations and the instantaneously relaxed ten-moment simulations agree well where convergence is demonstrated.
Figure 3: Examples of agreement of reconnected flux with minus the accumulation integral of the out-of-plane electric field at the origin for fine and coarse mesh. (For the fine mesh the two plots are exactly superimposed and so are indistinguishable.) Flux across the vertical axis represents the portion of field lines that have not reconnected. We obtained similar excellent agreement in all our simulations.

Figure 4: Snapshots of magnetic field lines. The reconnected flux is proportional to (the change in) the number of field lines through the horizontal axis of symmetry, and the unreconnected flux is proportional to the number of field lines through the vertical axis of symmetry.