Solitons in the one-dimensional forest fire model

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Fires in the one-dimensional Bak-Chen-Tang forest fire model propagate as solitons, resembling shocks in Burgers turbulence. The branching of solitons, creating new fires, is balanced by the pairwise annihilation of oppositely moving solitons. Two distinct, diverging length scales appear in the limit where the growth rate of trees, \(p\), vanishes. The width of the solitons, \(w\), diverges as a power law, \(1/p\), while the average distance between solitons diverges much faster as \(d \sim \exp(\pi^2/12p)\).

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The Bak-Chen-Tang (BCT) forest fire model \([1]\) is a simplified model of turbulent phenomena. Propagating fires dissipate or burn trees at an average rate determined by the tree growth probability, \(p\), representing power fed into the system. The fires are self-sustaining, with no spontaneous ignition.

The fires exhibit non-trivial spatio-temporal correlations in the slow driving limit, where the growth rate of trees, \(p\), vanishes \([2]\). In two and three dimensions, a “scale-dependent” critical behavior emerges \([2]\), which is not adequately understood at present. The fractal dimension of fires depends on the length scale of observation, up to the correlation length, \(\xi\), where the fires become space filling. This correlation length diverges with the inverse growth rate \(1/p\) to a power \(2/d\) for \(d = 2, 3\) \([3]\). The behavior in the one dimensional (\(d = 1\)) BCT model, where the fires are constrained by dimensionality to have a much simpler structure, has not previously been investigated.

Here we show that in one dimension, as \(p \to 0\), fires self-organize into a state characterized by propagating, branching and annihilating solitons. These are localized objects, compared to their scale of separation, moving with constant velocity. Outside of the solitons only solid forest exists. As an advancing fire moves into solid forest, each advancing fire is trailed by a cloud of smaller forests separated by holes, with possibly a few fires. Each fire advancing into solid forest together with its trail of finite forests and fires constitutes a soliton. New solitons are created by a process where forward moving solitons emit opposite moving ones.

The width of the solitons, \(w\), diverges as \(1/p\). Within the solitons, the finite forests, which are connected segments of trees, have a distribution of sizes, \(P(s) \sim 1/s^2\) up to the cut-off scale \(1/p\) imposed by the width of the solitons. The other statistical properties are completely determined by the branching rate for solitons, which can also be calculated analytically. This branching process creates new solitons, and must be balanced by the collision rate of oppositely moving solitons, which annihilates both solitons. We find that the density of fires remains finite but decreases extremely fast, \(n \sim \exp(-\pi^2/12p)\), in the limit \(p \to 0\). Thus, even in one dimension, the fires are self-sustaining with no spontaneous ignition in the slow driving limit. Although the solitons are diverging in width, their width vanishes compared to their separation, and they are well-defined in the limit \(p \to 0\).

All of this behavior is entirely different from the behavior of the Drossel-Schwabl (DS) forest fire model \([1]\), which self-organizes into a critical state with a power law distribution of forest fires. (The BCT model does not have a power-law distribution of forest fires.) In the DS model, fires are not self-sustaining but are injected at a small rate, \(f\). The DS model is critical in the limit \(f/p \to 0\) where a separation in time scales between burning of entire forests and the growth of trees occurs. Since the forests burn down instantly, and the fires are injected rather than self-sustaining, this limit gives a completely different physical picture than the propagating fires in the strict \(f = 0\) problem, corresponding to the original BCT model. Solitons, obviously, cannot appear in the DS model.

The BCT forest fire model is defined as follows. Each site on a \(d\)--dimensional lattice of linear extent \(L\), can be in one of three states: occupied by a fire, occupied by a tree, or empty. During a time step, all sites which contain fire burn down, leaving an empty site, and ignite neighboring trees on the lattice. Then all empty sites are independently occupied with trees with probability \(p\). This two step process is repeated indefinitely. The model can be studied with either open or periodic boundary conditions, and the results discussed below are found in either case. After a transient period, the system enters a statistically stationary state with a complex distribution of fires, and forests, which are connected clusters of trees. The fire is self-sustaining and fed by the tree growth process, for \(p > 1/\ln L\) (for \(d = 1\)). Otherwise the fire dies out completely.
FIG. 1. Solitons in the one-dimensional forest fire model for $p = 0.15$. The red squares are fires, the blue squares are empty sites. The white sites are trees. This figure shows the essential dynamical processes that enable an estimate of the contribution of various terms including collisions between solitons, followed by complete re-growth of the surrounding forest, and emission of solitons through back-propagating fires.

In order to get a clear picture, it is useful to view the space-time dynamics of the burning process. The numerical simulations were initiated with a random distribution of trees and fires. The spatio-temporal behavior in the steady-state is shown in Figure 1 for $p = 0.15$. The fires propagate either left or right, moving into solid forests, with unit velocity. Each such fire leaves a trail of forests, empty sites, and possibly some fires in their wake. Each moving trail, led by a fire advancing with unit velocity into a fully connected forest, constitutes a soliton. After some time, of order $1/p$, the small forests trailing in the wake of the advancing fire have grown to form a completely connected forest again, with no trace of the passing fire. Thus the width of the soliton scales as $1/p$. For small $p$, such as that shown in Figure 1, the solitons are well separated from each other and collisions are rare.

When fires moving in opposite directions collide, they annihilate each other, leaving clusters of small forests. After a time interval of order $1/p$, the small forests have all healed and joined the large surrounding forest, and the remnants of both solitons have completely disappeared.

Now and then, solitons emit solitons propagating in the opposite direction. This is the fundamental process for the self-organizing dynamics. Consider a state with a density $n$ of random placed solitons, each with a randomly chosen direction. In the stationary state, the rate of soliton death due to collisions must balance the rate of new solitons emitted as back-fires. The expected lifetime of a soliton is equal to the average distance between solitons, $d = 1/n$. Assume that each soliton emits back-propagating solitons at a rate $r$. Stationarity requires $r = 1/d = n$.

In order for a backward-moving fire to survive, and emerge as a soliton, the new fire must survive a number of time steps, by using the new, growing forests inside the “parent” soliton as stepping stones, before it can escape from its parent soliton and propagate in a fully-connected forest. Since the system is one-dimensional, one might suspect that this probability will vanish, for any $p < 1$, since sooner or later the newly created fire would always meet an empty site inside its parent soliton. Actually, for any finite $p$ this is not the case.

At the first step, where the initial fire moves forward one unit into a fully connected forest, there is a probability $p$ that this fire will branch backwards. In this case, there will be two fires at the same time at neighboring
sites. This means that a site that was on fire in the previous time step, \( t - 1 \), is regrown with a tree and set again on fire at time \( t \). Thus \( p \) is the basic branching probability for new fires to be created.

We will consider the process where the newly emitted fire only moves backward (see Fig. 1), opposite from the original fire, with no forward motion, or wandering. This is the dominant process for \( p \to 0 \). What is the probability that such a newly created fire will survive one additional time step? It is \( (1 - (1 - p)^3) \), since it is now three time steps since the forward-moving, parent fire passed the advancing position of the new, daughter fire. At the \( m' \)th time step for the new fire, the conditional probability of surviving one more time step is \( 1 - (1 - p)^{2m+1} \). Thus, the probability for surviving \( m \) steps is

\[
P_{\text{surv}}(m) = p \prod_{m'=1}^{m} (1 - (1 - p)^{2m'+1}) .
\]

Taking the limit \( m \to \infty \) and \( p \to 0 \) gives

\[
\lim_{p \to 0} \ln P_{\text{surv}} = \frac{1}{2p} \int_{0}^{\infty} \ln(1 - e^{-u})du = \frac{-\pi^2}{12p} .
\]

This is the rate of emitting back-propagating solitons. From the previous argument \( P_{\text{surv}} = n \). This result has been compared with numerical simulations as shown in Fig. 2. Numerical simulation results for \( 0.1 < p < 0.3 \), and soliton distances ranging from 20 to 10000, yield an excellent fit to the form \( n = \exp(-C/p) \), with \( C = 0.834 \), within 2\% of the exact value \( \pi^2/12 \).

Note that there are two length scales, each diverging with \( 1/p \), but in very different ways. One is the width of solitons, \( w \sim 1/p \). The other is the much larger distance between solitons, \( d = \exp(\pi^2/12p) \). Because of the exponentially growing distance between solitons, it is numerically feasible to simulate the model only for \( p > 0.1 \). However, the length scales are already extremely well separated for these values, establishing the soliton picture.

Also for any finite \( p \) there is always a finite probability that a soliton will annihilate itself by encountering a rare hole in the nominally fully connected forest. However the rate of this annihilation process for a soliton is \( e^{-apd} \), where \( a \) is a constant of order unity. This is derived by considering that such a rare hole left over was created by a soliton approximately \( d \) time steps in the past. This rate vanishes, in the limit \( p \to 0 \), compared to the rate \( 1/d \) for the pair-wise annihilation described above.

The distribution of the forest sizes \textit{within} the comparatively narrow solitons has a structure determined entirely by a cascade process in which isolated clusters of trees form bigger and bigger forests. This is caused by tree growth filling in the empty sites between neighboring forests. Forests of size \( s \) are both created and destroyed by this same process. We obtain analytically a power-law distribution of forests within solitons, \( P(s) \sim 1/s^2 \), up to a cut-off, which is the size of the soliton (of order \( 1/p \)).

In higher dimensions, the propagating fires interact with each other at a distance. This interaction is mediated by their effect on the forest density and tree-tree correlations. The resulting spatio-temporal structure of dissipation is much more complicated. Thus it is not clear if the soliton description will remain valid or useful for the higher dimensional BCT models, or other systems where the interactions between dissipating structures are long-range. Nevertheless, the simple results found here for the one-dimensional BCT forest fire model suggests that solitons, or shocks, may govern the dynamics of other self-organized critical models of turbulent, intermittent phenomena. In fact, we have observed similar behavior.

*FIG. 2.* A fit of numerical measurements of the density of fires, \( n \), to the form \( n = \exp(-C/p) \), giving \( C = 0.834 \).
in an interface depinning model \[12\]. In this case the sites on fire correspond to the sites which are moving, and the limit \( p \to 0 \) corresponds to the limit where the driving force approaches the depinning threshold from the moving phase.

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