Nonrigidity of a class of two dimensional surfaces with positive curvature and planar points

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Introduction

The problem considered here deals with the bendings of an orientable, embedded surface $S$ in $\mathbb{R}^3$. We assume that $S$ has a vanishing first homology group, that $S$ is a $C^\infty$ compact surface with boundary, that it has positive curvature except at finitely many planar points in $S$. The main result states that for any $k \in \mathbb{Z}^+$, $S$ has nontrivial infinitesimal bendings of class $C^k$. That is, there is a $C^k$ function $U : S \to \mathbb{R}^3$ such that the first fundamental form of the deformation surface $S_\sigma = \{p + \sigma U(p), \ p \in S\}$ satisfies $dS^2_\sigma = dS^2 + O(\sigma^2)$ as $\sigma \to 0$, where $\sigma$ is a real parameter. Furthermore, $S_\sigma$ is not obtained from $S$ through a rigid motion of $\mathbb{R}^3$. A consequence of this result is the nonrigidity of $S$ in the following sense. Any given $\epsilon$-neighborhood of $S$ (for the $C^k$ topology) contains isometric surfaces that are not congruent.

The study of bendings of surfaces in $\mathbb{R}^3$ has a rich history and many physical applications. In particular, it is used in the theory of elastic shells. We refer to the survey article of Sabitov (14) and the references therein. The results of this paper are also related to those contained in the following papers [2], [3], [4], [5], [6], [8], [10], [12], [19], [15].

Our approach is through the study of the associated (complex) field of asymptotic directions on $S$. We prove that such a vector field generates an integrable structure on $S$. We reformulate the equations for the bending field $U$ in terms of a Bers-Vekua type equation (with singularities). Then use recent results about the solvability of such equations to construct the bending fields.
1 Integrability of the field of asymptotic directions

For the surfaces considered here, we show that the field of asymptotic directions on $S$ has a global first integral.

Let $S \subset \mathbb{R}^3$ be an orientable $C^\infty$ surface with a $C^\infty$ boundary. We assume that $H_1(S) = 0$. The surface $S$ is diffeomorphic to a relatively compact domain $\Omega \in \mathbb{R}^2$ with a $C^\infty$ boundary. Hence,

$$\overline{S} = \{R(s,t) \in \mathbb{R}^3; (s,t) \in \overline{\Omega}\},$$

where the position vector $R: \overline{\Omega} \rightarrow \mathbb{R}^3$ is a $C^\infty$ parametrization of $\overline{S}$. Let $E, F, G$ and $e, f, g$ be the coefficients of the first and second fundamental forms of $S$. Thus,

$$E = R_s \cdot R_s, \quad F = R_s \cdot R_t, \quad G = R_t \cdot R_t,$$

$$e = R_{ss} \cdot N, \quad f = R_{st} \cdot N, \quad g = R_{tt} \cdot N,$$

where $N = \frac{R_s \times R_t}{|R_s \times R_t|}$ is the unit normal of $S$. The Gaussian curvature of $S$ is $K = \frac{eg - f^2}{EG - F^2}$. We assume that $\overline{S}$ has positive curvature except at a finite number of planar points in $S$. That is, there exist $p_1 = (s_1, t_1) \in \Omega, \cdots, p_l(s_l, t_l) \in \Omega$ such that

$$K(s,t) > 0, \quad \forall (s,t) \in \overline{\Omega}\backslash\{p_1, \cdots, p_l\}. \tag{1.2}$$

The (complex) asymptotic directions on $S$ are given by the quadratic equation

$$\lambda^2 + 2f\lambda + eg = 0 \tag{1.3}$$

Thus $\lambda = -f + i\sqrt{eg - f^2} \in \mathbb{R} + i\mathbb{R}^+$ except at the planar points $p_1, \cdots, p_l$ where $\lambda = 0$.

Consider the structure on $\overline{\Omega}$ generated by the $\mathbb{C}$-valued vector field

$$L = g(s,t) \frac{\partial}{\partial s} + \lambda(s,t) \frac{\partial}{\partial t}. \tag{1.4}$$

This structure is elliptic on $\overline{\Omega}\backslash\{p_1, \cdots, p_l\}$. That is, $L$ and $\overline{T}$ are independent outside the planar points. The next proposition shows that $L$ has a global first integral on $\overline{S}$.

**Proposition 1.1** Let $S$ be surface given by (1.1) whose curvature $K$ satisfies (1.2). Then there exists an injective function

$$Z: \overline{\Omega} \rightarrow \mathbb{C}$$

such that
1. \( Z \) is \( C^\infty \) on \( \overline{\Omega}\setminus\{p_1, \ldots, p_l\} \);

2. \( LZ = 0 \) on \( \overline{\Omega}\setminus\{p_1, \ldots, p_l\} \); and

3. For every \( j = 1, \ldots, l \), there exists \( \mu_j > 0 \) and polar coordinates \((r, \theta)\) centered at \( p_j \) such that in neighborhood of \( p_j \) we have

\[
Z(r, \theta) = Z(0,0) + r^{\mu_j} e^{i\theta} + O(r^{2\mu_j}) \tag{1.5}
\]

**Proof.** Since \( \mathcal{L} \) is \( C^\infty \) and elliptic on \( \overline{\Omega}\setminus\{p_1, \ldots, p_l\} \), then it follows from the uniformization of complex structures on planar domains (see\(^{[17]}\)) that there exists a \( C^\infty \) diffeomorphism

\[
Z : \overline{\Omega}\setminus\{p_1, \ldots, p_l\} \rightarrow Z(\overline{\Omega}\setminus\{p_1, \ldots, p_l\}) \subset \mathbb{C}
\]

such that \( LZ = 0 \). It remains to show that \( Z \) has the form (1.5) in a neighborhood of a planar point.

Let \( p_j \) be a planar point of \( S \). We can assume that \( S \) is given in a neighborhood of \( p_j \) as the graph of a function \( z = z(x,y) \) with \( p_j = (0,0) \), \( z(0,0) = 0 \), and \( z_x(0,0) = z_y(0,0) = 0 \). The assumption on the curvature implies that

\[
z(x,y) = z_m(x,y) + o(\sqrt{x^2 + y^2})
\]

where \( z_m(x,y) \) is a homogeneous polynomial of degree \( m > 2 \), satisfying \( z_{xx}z_{yy} - z_{xy}^2 > 0 \) for \((x,y) \neq (0,0)\). We can also assume that \( z(x,y) > 0 \) for \((x,y) \neq (0,0)\). The complex structure generated by the asymptotic directions is given by the vector field

\[
\mathcal{L} = z_{yy} \frac{\partial}{\partial x} + \left( -z_{xy} + i\sqrt{z_{xx}z_{yy} - z_{xy}^2} \right) \frac{\partial}{\partial y}.
\]

With respect to the polar coordinates \( x = \rho \cos \phi, y = \rho \sin \phi \), we get

\[
z = \rho^m P(\phi) + \rho^{m+1} A(\rho, \phi),
\]

where \( P(\phi) \) is a trigonometric polynomial of degree \( m \) satisfying \( P(\phi) > 0 \) and (curvature)

\[
m^2 P(\phi)^2 + m P(\phi) P''(\phi) - (m - 1) P'(\phi)^2 > 0 \quad \forall \phi \in \mathbb{R}.
\]

With respect to the coordinates \((\rho, \phi)\), the vector field \( \mathcal{L} \) becomes

\[
\mathcal{L} = m(m - 1) \rho^{m-2}(P(\phi) + O(\rho)) L_0
\]

with

\[
L_0 = \frac{\partial}{\partial \phi} + \rho (M(\phi) + iN(\phi) + O(\rho)) \frac{\partial}{\partial \rho}
\]
and
\[ M = \frac{P'}{mP} \quad \text{and} \quad N = \frac{1}{m} \sqrt{\frac{m^2 P^2 + mPP'' - (m - 1)P^2}{(m - 1)P^2}}. \]

We know (see [7]) that such a vector field $L_0$ is integrable in a neighborhood of the circle $\rho = 0$. Moreover, we can find coordinates $(r, \theta)$ in which $L_0$ is $C^1$-conjugate to the model vector field
\[ T = \mu_j \frac{\partial}{\partial \theta} - ir \frac{\partial}{\partial r}, \]
where $\mu_j > 0$ is given by
\[ \frac{1}{\mu_j} = \frac{1}{2\pi} \int_0^{2\pi} (N(\phi) - iM(\phi))d\phi = \frac{1}{2\pi} \int_0^{2\pi} N(\phi)d\phi. \]
The function $u_j(r, \theta) = r^{\mu_j}e^{i\theta}$ is a first integral of $T$ in $r > 0$.

Now we prove that the function $Z$ which is defined in $\Omega \setminus \{p_1, \cdots, p_l\}$ extends to $p_j$ with the desired form given by (1.5). Let $O_j$ be a disc centered at $p_j$ where $L$ is conjugate to a multiple of $T$ in the $(r, \theta)$ coordinates. Since $u_j$ and $Z$ are both first integrals of $L$ in the punctured disc $O_j \setminus p_j$, then there exists a holomorphic function $h_j$ defined on the image $u_j(O_j \setminus p_j)$ such that $Z(r, t) = h_j(u_j(r, t))$. Since both $Z$ and $u_j$ are homeomorphisms onto their images, then $h_j$ is one to one in a neighborhood of $u_j(p_j) = 0 \in \mathbb{C}$ and since $h_j$ is bounded, then
\[ h_j(\zeta) = C_0 + C_1\zeta + O(\zeta^2) \quad \text{for } \zeta \text{ close to } 0 \in \mathbb{C} \]
with $C_1 \neq 0$. This means that after a linear change of the coordinates $(r, \theta)$ (to remove the constant $C_1$), the function $Z$ has the form (1.5) \[ \Box \]

2 Equations of the bending fields in terms of $L$

Let $S$ be a surface given by (1.1). An infinitesimal bending of class $C^k$ of $S$ is a deformation surface $S_\sigma \subset \mathbb{R}^3$, with $\sigma \in \mathbb{R}$ a parameter, given by the position vector
\[ R_\sigma(s, t) = R(s, t) + \sigma U(s, t), \quad (2.1) \]
whose first fundamental form satisfies
\[ dR^2_\sigma = dR^2 + O(\sigma^2) \quad \text{as } \sigma \to 0. \]
This means that the bending field $U : \overline{\Omega} \to \mathbb{R}^3$ is of class $C^k$ and satisfies
\[ dR \cdot dU = 0. \quad (2.2) \]
The trivial bendings of $S$ are those induced by the rigid motions of $\mathbb{R}^3$. They are given by $U(s,t) = A \times R(s,t) + B$, where $A$ and $B$ are constants in $\mathbb{R}^3$, and where $\times$ denotes the vector product in $\mathbb{R}^3$.

Let $L$ be the field of asymptotic directions defined by (1.4). For each function $U : \Omega \rightarrow \mathbb{R}^3$, we associate the $C$-valued function $w$ defined by

$$w(s,t) = LR(s,t) \cdot U(s,t) = g(s,t)u(s,t) + \lambda(s,t)v(s,t),$$

where $u = R_s \cdot U$ and $v = R_t \cdot U$. The following theorem proved in [9] will be used in the next section.

**Theorem 2.1** [9] If $U : \Omega \rightarrow \mathbb{R}^3$ satisfies (2.2), then the function $w$ given by (2.3) satisfies the equation

$$CLw = Aw + Bw,$$

where

$$A = (LR \times LR) \cdot (L^2 R \times LR),$$

$$B = (LR \times LR) \cdot (L^2 R \times LR),$$

$$C = (LR \times LR) \cdot (LR \times LR).$$

**Remark 2.1** If $w$ solves equation (2.4). The function $w' = aw$, where $a$ is a nonvanishing function solves the same equation with the vector field $L$ replaced by $L' = aL$.

## 3 Main Results

**Theorem 3.1** Let $S$ be a surface given by (1.1) and such that its curvature $K$ satisfies (1.2). Then for every $k \in \mathbb{Z}^+$, the surface $S$ has a nontrivial infinitesimal bending $U : \Omega \rightarrow \mathbb{R}^3$ of class $C^k$.

**Remark 3.1** It should be mentioned that without the assumption that $K > 0$ up to the boundary $\partial S$, the surface could be rigid under infinitesimal bendings. Indeed, let $T^2$ be a standard torus in $\mathbb{R}^3$, it is known (see [1] or [3]) that if $S$ consists of the portion of $T^2$ with positive curvature, then $S$ is rigid under infinitesimal bendings. Here the curvature vanishes on $\partial S$.

Before we proceed with the proof, we give a consequence of Theorem 3.1.

**Theorem 3.2** Let $S$ be as in Theorem 3.1. Then for every $\epsilon > 0$ and for every $k \in \mathbb{Z}^+$, there exist surfaces $\Sigma^+$ and $\Sigma^-$ of class $C^k$ in the $\epsilon$-neighborhood of $S$ (for the $C^k$-topology) such that $\Sigma^+$ and $\Sigma^-$ are isometric but not congruent.
Proof. Let \( U : \overline{\Omega} \rightarrow \mathbb{R}^3 \) be a nontrivial infinitesimal bending of \( S \) of class \( C^k \). Consider the surfaces \( \Sigma_{\sigma} \) and \( \Sigma_{-\sigma} \) defined the position vectors

\[
R_{\pm\sigma}(s, t) = R(s, t) \pm \sigma U(s, t).
\]

Since \( dR \cdot dU = 0 \), then \( dR^2_{\pm\sigma} = dR^2 + \sigma^2 dU^2 \). Hence \( \Sigma_{\sigma} \) and \( \Sigma_{-\sigma} \) are isometric. Furthermore, since \( U \) is nontrivial, then \( \Sigma_{\sigma} \) and \( \Sigma_{-\sigma} \) are not congruent (see [16]). For a given \( \epsilon > 0 \), the surfaces \( \Sigma_{\pm\sigma} \) are contained in the \( \epsilon \)-neighborhood of \( S \) if \( \sigma \) is small enough.

Proof of Theorem 3.1. First we construct non trivial solutions \( w \) of equation (2.4) and then deduce the infinitesimal bending fields \( U \). For this, we use the first integral \( Z \) of \( L \) to transform equation (2.4) into a Bers-Vekua type equation with singularities. Let \( Z_1 = Z(p_1), \ldots, Z_l = Z(p_l) \) be the images of the planar points by the function \( Z \). The pushforward of equation (2.4) via \( Z \) gives rise to an equation of the form

\[
\frac{\partial W}{\partial Z} = \frac{A(Z)}{\prod_{j=1}^l (Z - Z_j)} W + \frac{B(Z)}{\prod_{j=1}^l (Z - Z_j)} \overline{W},
\]

where \( w(s, t) = W(Z(s, t)) \) and \( A, B \in C^\infty(Z(\overline{\Omega}\setminus\{p_1, \ldots, p_l\})) \cap L^\infty(Z(\overline{\Omega})) \).

The local study of the solutions of such equations near a singularity is considered in [11], [18], [20]. To construct a global solution of (3.1) with the desired properties, we proceed as follows. We seek a solution \( W \) in the form

\[
W(Z) = H(Z)W_1(Z),
\]

where \( M \) is a (large) positive integer to be chosen. In order for \( W = HW_1 \) to solve (3.1), the function \( W_1 \) needs to solve the modified equation

\[
\frac{\partial W_1}{\partial Z} = \frac{A(Z)}{\prod_{j=1}^l (Z - Z_j)} W_1 + \frac{B(Z)}{\prod_{j=1}^l (Z - Z_j)} \overline{H(Z)} \overline{W_1}. \tag{3.2}
\]

Since \( A \) and \( B\overline{H}/H \) are bounded functions on \( Z(\overline{\Omega}) \), then a result of [18] gives a continuous solution \( W_1 \) of (3.2) on \( Z(\overline{\Omega}) \). Furthermore, such a solutions is \( C^\infty \) on \( Z(\overline{\Omega}\setminus\{Z_1, \ldots, Z_l\}) \) since the equation is elliptic and the coefficients are \( C^\infty \) outside the \( Z_j \)'s.

The function \( W(Z) = H(Z)W_1(Z) \) is therefore a solution of (3.1). It is \( C^\infty \) on \( Z(\overline{\Omega}\setminus\{Z_1, \ldots, Z_l\}) \) and vanishes to order \( M \) at each point \( Z_j \). Consequently, the function \( w(s, t) = W(Z(s, t)) \) is \( C^\infty \) on \( \overline{\Omega}\setminus\{p_1, \ldots, p_l\} \) and vanishes to order \( M\mu_j \) at each planar point (\( \mu_j \) is the positive number appearing in Proposition 1.1).
Now we recover the bending field $U$ from the solution $w$ of (2.4) and the relation $w = LR \cdot U$. Set $w = gu + \lambda v$, where $\lambda$ is the asymptotic direction given in (1.3). The functions $u$ and $v$ are uniquely determined by

$$v = \frac{w - \bar{w}}{2i\sqrt{eg - f^2}} \quad \text{and} \quad u = \frac{w + \bar{w} + 2fv}{2g},$$

provided that the function $w$ vanishes to a high order at the planar points (order of vanishing of $W$ at $p_j$ larger than that of the curvature). These functions are $C^\infty$ outside the planar points. At each planar point $p_j$, it $m_j$ is the order of vanishing of $K$, then the functions $v$ and $u$ vanish to order $M \mu_j - m_j$. It follows from $LR \cdot U = w$ that $Rs \cdot U = u$ and $Rt \cdot U = v$. The condition $dR \cdot dU = 0$ implies that

$$R_{ss} \cdot U = u_s, \quad Rt \cdot U = v_t, \quad \text{and} \quad 2R_{st} \cdot U = u_t + v_s. \quad (3.3)$$

In terms of the components $(x, y, z)$ of $R$ and $(\xi, \eta, \zeta)$ of $U$, we have

$$\begin{cases} x_s \xi + y_s \eta + z_s \zeta &= u \\ x_t \xi + y_t \eta + z_t \zeta &= v \\ x_{ss} \xi + y_{ss} \eta + z_{ss} \zeta &= u_s \\ x_{tt} \xi + y_{tt} \eta + z_{tt} \zeta &= v_t \\ 2x_{st} \xi + 2y_{st} \eta + 2z_{st} \zeta &= u_t + v_s \end{cases} \quad (3.4)$$

(equation (2.4) guarantees the compatibility of this system). Note that at each point $p \in \Omega$ where $K > 0$, the functions $\xi$, $\eta$, and $\zeta$ are uniquely determined by $u$, $v$, and $u_s$ (or $v_t$). Indeed, at such a point the determinant of the first three equations of (3.4) is

$$R_{ss} \cdot (Rs \times Rt) = |Rs \times Rt| \neq 0.$$

With our choice that $w$ (and so $u$ and $v$) vanishing to an order larger than that of the curvature at each planar point, the functions $\xi$, $\eta$, and $\zeta$ are also uniquely determined to be 0 at each planar point. To see why, assume that at $p_j$, we have $x_s y_t - x_t y_s \neq 0$, then after solving the first two equations for $\xi$ and $\eta$ in terms of $\zeta$, $u$, and $v$, the third equation becomes

$$R_{ss} \cdot (Rs \times Rt) \zeta = \begin{vmatrix} x_s & y_s & u \\ x_t & y_t & v \\ x_{ss} & y_{ss} & u_s \end{vmatrix} . \quad (3.5)$$

Since the zeros $p_j$ of $R_{ss} \cdot (Rs \times Rt)$ is isolated and since $u$ and $v$ vanish to a high order at $p_j$, the function $\zeta$ is well defined by (3.5). Consequently, for any given $k \in \mathbb{Z}^+$, a nonzero solution $w$ of (2.4) which vanishes at high orders ($M$ large), gives rise to a unique field of infinitesimal bending $U$ of $S$, so that it is $C^\infty$ on $\overline{\Omega \setminus \{p_1, \cdots, p_l\}}$ and vanishes to an order $k$ at each
p_j. Such a field is therefore of class $C^k$ at each planar point. It remains to verify that $U$ is not trivial. If such a field were trivial ($U = A \times R + B$), then the vanishing of $dU = A \times dR$ at $p_j$ together with $dR \cdot dU = 0$ gives $A = 0$ and so $U = B = 0$ since $U = 0$ at $p_j$. This would give $w \equiv 0$ which is a contradiction □

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