Distributed optimal control problems for a class of elliptic hemivariational inequalities with a parameter and its asymptotic behavior

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Dedicated to Professor Stanislaw Migórski on the occasion of his 60th birthday

Abstract.
In this paper, we study optimal control problems on the internal energy for a system governed by a class of elliptic boundary hemivariational inequalities with a parameter. The system has been originated by a steady-state heat conduction problem with non-monotone multivalued subdifferential boundary condition on a portion of the boundary of the domain described by the Clarke generalized gradient of a locally Lipschitz function. We prove an existence result for the optimal controls and we show an asymptotic result for the optimal controls and the system states, when the parameter, like a heat transfer coefficient, tends to infinity on a portion of the boundary.

Key words. Elliptic hemivariational inequality, optimal control problems, asymptotic behavior, Clarke generalized gradient, mixed elliptic problem, convergence.

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1 Introduction

We consider a bounded domain $\Omega$ in $\mathbb{R}^d$ whose regular boundary $\Gamma$ consists of the union of three disjoint portions $\Gamma_i$, $i = 1, 2, 3$ with $|\Gamma_i| > 0$, where $|\Gamma_i|$ denotes the $(d-1)$-dimensional Hausdorff measure of the portion $\Gamma_i$ on $\Gamma$. The outward normal vector on the boundary is denoted by $n$. We formulate the following steady-state heat conduction problem with mixed boundary conditions $[1, 2, 12, 13, 27, 28]$:

$$
-\Delta u = g \text{ in } \Omega, \quad u|_{\Gamma_1} = 0, \quad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q, \quad u|_{\Gamma_3} = b, \quad (1)
$$

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where \( u \) is the temperature in \( \Omega \), \( g \) is the internal energy in \( \Omega \), \( b \) is the temperature on \( \Gamma_3 \) and \( q \) is the heat flux on \( \Gamma_2 \), which satisfy the hypothesis: \( g \in H = L^2(\Omega) \), \( q \in Q = L^2(\Gamma_2) \) and \( b \in H^\frac{1}{2}(\Gamma_3) \).

Throughout the paper we use the following notation

\[
V = H^1(\Omega), \quad V_0 = \{ v \in V \mid v = 0 \text{ on } \Gamma_1 \},
\]

\[
K = \{ v \in V \mid v = 0 \text{ on } \Gamma_1, \ v = b \text{ on } \Gamma_3 \}, \quad K_0 = \{ v \in V \mid v = 0 \text{ on } \Gamma_1 \cup \Gamma_3 \},
\]

\[
a(u, v) = \int_\Omega \nabla u \nabla v \, dx, \quad L(v) = \int_\Omega g v \, dx - \int_{\Gamma_2} q \gamma(v) \, d\Gamma,
\]

where \( \gamma : V \to L^2(\Gamma) \) denotes the trace operator on \( \Gamma \). In what follows, we write \( u \) for the trace of a function \( u \in V \) on the boundary. In a standard way, we obtain the following variational formulation of (1):

\[
\text{find } u_\infty \in K \text{ such that } a(u_\infty, v) = L(v) \text{ for all } v \in K_0,
\]

(2)

The standard norms on \( V \) and \( V_0 \) are denoted by

\[
\|v\|_V = \left( \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega; \mathbb{R}^d)} \right)^{1/2} \text{ for } v \in V,
\]

\[
\|v\|_{V_0} = \|\nabla v\|_{L^2(\Omega; \mathbb{R}^d)} \text{ for } v \in V_0.
\]

It is well known by the Poincaré inequality, see \([4, 23, 27]\), that on \( V_0 \) the above two norms are equivalent. Note that the form \( a \) is bilinear, symmetric, continuous and coercive with constant \( m_a > 0 \), i.e.

\[
a(v, v) = \|v\|^2_{V_0} \geq m_a \|v\|^2_V \text{ for all } v \in V_0.
\]

(3)

We remark that, under additional hypotheses on the data \( g, q \) and \( b \), problem (1) can be considered as steady-state two-phase Stefan problem, see, for example, \([11, 25, 26, 28]\). We can particularly see it in \([11]\) (Example 1 in page 629, Example 2 in page 630, and Example 3 in page 631); in \([25]\) (Example (i) and (ii) in page 35, and Example (iii) in page 36), and in \([28]\) (Example 1 and Example 2 in page 180).

Now, in this paper, we consider the mixed nonlinear boundary value problem for an elliptic equation as follows:

\[
- \Delta u = g \text{ in } \Omega, \quad u|_{\Gamma_1} = 0, \quad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q, \quad -\frac{\partial u}{\partial n}|_{\Gamma_3} \in \alpha \partial j(u),
\]

(4)

which has been recently studied in \([9]\).

Here \( \alpha \) is a positive constant which can be considered as the heat transfer coefficient on the boundary while the function \( j : \Gamma_3 \times \mathbb{R} \to \mathbb{R} \), called a superpotential (nonconvex potential), is such that \( j(x, \cdot) \) locally Lipschitz for a.e. \( x \in \Gamma_3 \) and not necessary differentiable. Since in general \( j(x, \cdot) \) is nonconvex, so the multivalued condition on \( \Gamma_3 \) in problem (4) is described by a nonmonotone relation expressed by the generalized gradient of Clarke \([5]\). Such multivalued relation in problem (4) is
met in certain types of steady-state heat conduction problems (the behavior of a semipermeable membrane of finite thickness, a temperature control problems, etc.). Further, problem (4) can be considered as a prototype of several boundary semipermeability models, see [15, 19, 20, 30], which are motivated by problems arising in hydraulics, fluid flow problems through porous media, and electrostatics, where the solution represents the pressure and the electric potentials. Note that the analogous problems with maximal monotone multivalued boundary relations (that is the case when \( j(x, \cdot) \) is a convex function) were considered in [3, 17], see also references therein.

Under the above notation, the weak formulation of the elliptic problem (4) becomes the following elliptic boundary hemivariational inequality [9]:

\[
\text{find } u \in V_0 \text{ such that } a(u, v) + \alpha \int_{\Gamma_3} j^0(u; v) d\Gamma \geq L(v) \text{ for all } v \in V_0. \tag{5}
\]

Here and in what follows we often omit the variable \( x \) and we simply write \( j(r) \) instead of \( j(x, r) \). The stationary heat conduction models with nonmonotone multivalued subdifferential interior and boundary semipermeability relations cannot be described by convex potentials. They use locally Lipschitz potentials and their weak formulations lead to hemivariational inequalities, see [19, Chapter 5.5.3] and [20].

We mention that theory of hemivariational and variational inequalities has been proposed in the 1980s by Panagiotopoulos, see [19, 21, 22], as variational formulations of important classes of inequality problems in mechanics. In the last few years, new kinds of variational, hemivariational, and variational-hemivariational inequalities have been investigated, see recent monographs [4, 18, 24], and the theory has emerged today as a new and interesting branch of applied mathematics.

We consider the distributed optimal control problem of the type studied in [10, 14, 29] given by:

\[
\text{find } g^* \in H \text{ such that } J(g^*) = \min_{g \in H} J(g) \tag{6}
\]

with

\[
J(g) = \frac{1}{2} \| u_g - z_d \|_H^2 + \frac{M}{2} \| g \|_H^2 \tag{7}
\]

where \( u_g \) is the unique solution to the variational equality (2), \( z_d \in H \) given and \( M \) a positive constant.

The goal of this paper is to formulate, for each \( \alpha > 0 \), the following new distributed optimal control problem

\[
\text{find } g_{\alpha}^* \in H \text{ such that } J_{\alpha}(g_{\alpha}^*) = \min_{g \in H} J_{\alpha}(g) \tag{8}
\]

with

\[
J_{\alpha}(g) = \frac{1}{2} \| u_{\alpha g} - z_d \|_H^2 + \frac{M}{2} \| g \|_H^2 \tag{9}
\]

where \( u_{\alpha g} \) is a solution to the hemivariational inequality (5), \( z_d \in H \) given and \( M \) a positive constant, and to study the convergent to problem (8) when the parameter \( \alpha \) goes to infinity.
The paper is structured as follows. In Section 2 we establish preliminaries concepts of the hemivariational inequalities theory, which are necessary for the development of the following sections. In Section 3, for each $\alpha > 0$, we obtain an existence result of solution to the optimal control problem (8). Finally, in Section 4, we prove the strong convergence of a sequence of optimal controls of the problems (8) to the unique optimal control of the problem (6), when the parameter $\alpha$ goes to infinity. Moreover, we obtain the strong convergence of the system states related to the problems (8) to the system state related to the problem (6), when $\alpha$ goes to infinity. These results generalize for a locally Lipschitz function $j$, under the hypothesis $H(j)$ and ($H_1$), the classical results obtained in [10] for a quadratic superpotential $j$.

2 Preliminaries

In this section we recall standard notation and preliminary concepts, which are necessary for the development of this paper.

Let $(X, \| \cdot \|_X)$ be a reflexive Banach space, $X^*$ be its dual, and $\langle \cdot, \cdot \rangle$ denote the duality between $X^*$ and $X$. For a real valued function defined on $X$, we have the following definitions [5, Section 2.1] and [6, 18].

Definition 1. A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exist $U_x$ a neighborhood of $x$ and a constant $L_x > 0$ such that

$$|\varphi(y) - \varphi(z)| \leq L_x \|y - z\|_X \quad \text{for all } y, z \in U_x.$$ 

For such a function the generalized (Clarke) directional derivative of $j$ at the point $x \in X$ in the direction $v \in X$ is defined by

$$\varphi^0(x; v) = \limsup_{y \to x, \lambda \to 0^+} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$ 

The generalized gradient (subdifferential) of $\varphi$ at $x$ is a subset of the dual space $X^*$ given by

$$\partial \varphi(x) = \{ \zeta \in X^* \mid \varphi^0(x; v) \geq \langle \zeta, v \rangle \quad \text{for all } v \in X \}.$$ 

We consider the following hypothesis.

$H(j)$: $j : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ is such that

(a) $j(\cdot, r)$ is measurable for all $r \in \mathbb{R}$,

(b) $j(x, \cdot)$ is locally Lipschitz for a.e. $x \in \Gamma_3$,

(c) there exist $c_0, c_1 \geq 0$ such that $|\partial j(x, r)| \leq c_0 + c_1|r|$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$,

(d) $j^0(x, r; b - r) \leq 0$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$ with a constant $b \in \mathbb{R}$.

Note that the existence results for elliptic hemivariational inequalities can be found in several contributions, see [4, 16, 17, 18, 19]. In [11, Theorem 4], the hypothesis $H(j)(d)$ is considered in order to obtain existence of a solution to problem (5). Moreover, under this condition the authors have studied the asymptotic behavior when $\alpha \to \infty$ (see [11, Theorem 7]).
We note that, if the hypothesis $H(j)(d)$ is replaced by the relaxed monotonicity condition (see [9, Remark 10] for details)
\[(e) \quad J^0(x, r; s - r) + J^0(x, s; r - s) \leq m_j |r - s|^2\]
for all $r, s \in \mathbb{R}$, a.e. $x \in \Gamma_3$ with $m_j \geq 0$, and the following smallness condition
\[(f) \quad m_a > \alpha m_j \|\gamma\|^2\]
is assumed, then problem (3) is uniquely solvable, see [17, Lemma 20] for the proof. However, this smallness condition is not suitable in the study to problem (5) since for a sufficiently large value of $\alpha$, it is not satisfied. Finally, in [9] we can find several examples of locally Lipschitz (nondifferentiable and nonconvex) functions which satisfies the above hypotheses.

3 Optimal control problems

We know, by [10], that there exists a unique optimal pair $(g^*, u_{g^*}) \in H \times V_0$ of the distributed optimal control problem (6). Now, we pass to a result on existence of solution to the optimal control problem (8) in which the system is governed by the hemivariational inequality (5).

THEOREM 2. For each $\alpha > 0$, if $H(j)(a) - (d)$ holds, then the distributed optimal control problems (5) has a solution.

Proof. By definition, for each $\alpha > 0$, the functional $J_\alpha$ is bounded from below. Next, taking into account that the hemivariational inequality (5) has solution (see [9, Theorem 4]), for each $\alpha > 0$ and each $g \in H$, we denote by $T_\alpha(g)$ the set of solutions of (5) and we have that
\[m = \inf \{J_\alpha(g), g \in H, u_{g^*} \in T_\alpha(g)\} \geq 0. \tag{10}\]
Let $g_n \in H$ be a minimizing sequence to (10) such that
\[m \leq J_\alpha(g_n) \leq m + \frac{1}{n}. \tag{11}\]
Taking into account that the functional $J_\alpha$ satisfies
\[\lim_{\|g\|_H \to +\infty} J_\alpha(g) = +\infty\]
we obtain that there exists $C_1 > 0$ such that
\[\|g_n\|_H \leq C_1. \tag{12}\]
Moreover, we can prove that there exists $C_2 > 0$ such that
\[\|u_{g^*_n}\|_{V_0} \leq C_2. \tag{13}\]
In fact, let $u_\infty \in K$ be the solution to problem (2). We have

$$a(u_{\alpha gn}, u_\infty - u_{\alpha gn}) + \alpha \int_{\Gamma_3} j^0(u_{\alpha gn}; u_\infty - u_{\alpha gn}) d\Gamma \geq \int_\Omega g_n(u_\infty - u_{\alpha gn}) dx$$

$$- \int_{\Gamma_2} q(u_\infty - u_{\alpha gn}) d\Gamma.$$ 

Hence

$$a(u_\infty - u_{\alpha gn}, u_\infty - u_{\alpha gn}) \leq a(u_\infty, u_\infty - u_{\alpha gn}) + \alpha \int_{\Gamma_3} j^0(u_{\alpha gn}; b - u_{\alpha gn}) d\Gamma$$

$$+ \int_\Omega g_n(u_{\alpha gn} - u_\infty) dx - \int_{\Gamma_2} q(u_\infty - u_{\alpha gn}) d\Gamma.$$ 

From hypothesis $H(j)(d)$, since the form $a$ is bounded (with positive constant $M_a$), we get

$$\|u_\infty - u_{\alpha gn}\|^2_{V_0} \leq a(u_\infty, u_\infty - u_{\alpha gn}) + \int_\Omega g_n(u_{\alpha gn} - u_\infty) dx - \int_{\Gamma_2} q(u_\infty - u_{\alpha gn}) d\Gamma$$

$$\leq M_a \|u_\infty\|_V \|u_\infty - u_{\alpha gn}\|_V + (\|g_n\|_H + \|q\|_Q\|\gamma\|) \|u_\infty - u_{\alpha gn}\|_V$$

$$\leq (M_a \|u_\infty\|_V + C_1 + C_3 \|q\|_Q\|\gamma\|) \|u_\infty - u_{\alpha gn}\|_{V_0}$$

where $\|\gamma\|$ denote the norm of trace operator and $C_3$ is a positive constant due to the equivalence of norms. Subsequently, we obtain (13). Therefore, there exist $f \in H$ and $\eta_\alpha \in V_0$ such that

$$u_{\alpha gn} \rightharpoonup \eta_\alpha \quad \text{in} \quad V_0 \quad \text{weakly} \quad \text{and} \quad g_n \rightarrow f \quad \text{in} \quad H \quad \text{weakly}.$$ 

Now, for all $g_n \in H$, we have

$$a(u_{\alpha gn}, v) + \alpha \int_{\Gamma_3} j^0(u_{\alpha gn}; v) d\Gamma \geq \int_\Omega g_n v dx - \int_{\Gamma_2} q v d\Gamma \quad \text{for all} \quad v \in V_0$$

and taking the upper limit, we obtain

$$a(\eta_\alpha, v) + \alpha \limsup_{n \to +\infty} \int_{\Gamma_3} j^0(u_{\alpha gn}; v) d\Gamma \geq \int_\Omega f v dx - \int_{\Gamma_2} q v d\Gamma \quad \text{for all} \quad v \in V_0. \quad (14)$$

By the compactness of the trace operator from $V$ into $L^2(\Gamma_3)$, we have $u_{\alpha gn}|_{\Gamma_3} \rightharpoonup \eta_\alpha|_{\Gamma_3}$ in $L^2(\Gamma_3)$, as $n \to +\infty$, and at least for a subsequence, $u_{\alpha gn}(x) \rightharpoonup \eta_\alpha(x)$ for a.e. $x \in \Gamma_3$ and $|u_{\alpha gn}(x)| \leq h_\alpha(x)$ a.e. $x \in \Gamma_3$, where $h_\alpha \in L^2(\Gamma_3)$. Since the function $\mathbb{R} \times \mathbb{R} \ni (r, s) \mapsto j^0(x, r; s) \in \mathbb{R}$ a.e. on $\Gamma_3$ is upper semicontinuous, see [3], we obtain

$$\limsup_{n \to +\infty} j^0(x, u_{\alpha gn}(x); v(x)) \leq j^0(x, \eta_\alpha(x); v(x)) \quad \text{a.e.} \quad x \in \Gamma_3.$$

Next, from $H(j)(c)$, we deduce the estimate

$$|j^0(x, u_{\alpha gn}(x); v(x))| \leq (c_0 + c_1|u_{\alpha gn}(x)|)|v(x)| \leq k_\alpha(x) \quad \text{a.e.} \quad x \in \Gamma_3$$
where $k_\alpha \in L^1(\Gamma_3)$, $k_\alpha(x) = (c_0 + c_1 h_\alpha(x))|v(x)|$ and we apply the dominated convergence theorem, see [6] to get

$$
\limsup_{n \to +\infty} \int_{\Gamma_3} j^0(u_{\alpha g_n}; v) \, d\Gamma \leq \int_{\Gamma_3} \limsup_{n \to +\infty} j^0(u_{\alpha g_n}; v) \, d\Gamma \leq \int_{\Gamma_3} j^0(\eta; v) \, d\Gamma.
$$

Using the latter in (14), we obtain

$$
a(\eta, v) + \alpha \int_{\Gamma_3} j^0(\eta; v) \, d\Gamma \geq \int_\Omega f v \, dx - \int_{\Gamma_2} q v \, d\Gamma \quad \text{for all} \quad v \in V_0
$$

that is, $\eta \in V_0$ is a solution to the hemivariational inequality (5). Next, we have proved that

$$
\eta_\alpha = u_{\alpha f}
$$

where $u_{\alpha f}$ is a solution of the hemivariational inequality (5) for data $f \in H$ and $q \in Q$. Finally, from (11) and the weak lower semicontinuity of $J_\alpha$, we have

$$
m \geq \liminf_{n \to +\infty} J_\alpha(g_n)
$$

$$
\geq \frac{1}{2} \liminf_{n \to +\infty} \|u_{\alpha g_n} - z_d\|_H^2 + \frac{M}{2} \liminf_{n \to +\infty} \|g_n\|_H^2
$$

$$
\geq \frac{1}{2} \|u_{\alpha f} - z_d\|_H^2 + \frac{M}{2} \|f\|_H^2 = J_\alpha(f),
$$

and therefore, $(f, u_{\alpha f})$ is an optimal pair to optimal control problem (8).

\[\square\]

4 Asymptotic behavior of the optimal controls

In this section we investigate the asymptotic behavior of the optimal solutions to problem (8) when $\alpha \to \infty$. To this end, we need the following additional hypothesis on the superpotential $j$.

(H1): if $j^0(x, r; b - r) = 0$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_3$, then $r = b$.

Theorem 3. Assume $H(j)$ and (H1). If $(g_\alpha, u_{\alpha g_n})$ is a optimal solution to problem (8) and $(g^*, u_{\alpha g^*})$ is the unique solution to problem (6), then $g_\alpha \to g^*$ in $H$ strongly and $u_{\alpha g_n} \to u_{\alpha g^*}$ in $V$ strongly, when $\alpha \to \infty$.

Proof. We will make the prove in three steps.

**Step 1.** Since $(g_\alpha, u_{\alpha g_n})$ is a optimal solution to problem (8), we have the following inequality

$$
\frac{1}{2} \|u_{\alpha g_n} - z_d\|_H^2 + \frac{M}{2} \|g_n\|_H^2 \leq \frac{1}{2} \|u_{\alpha g} - z_d\|_H^2 + \frac{M}{2} \|g\|_H^2, \quad \forall g \in H
$$

and taking $g = 0$, we obtain that there exists a positive constant $C_1$ such that

$$
\frac{1}{2} \|u_{\alpha g_n} - z_d\|_H^2 + \frac{M}{2} \|g_n\|_H^2 \leq \frac{1}{2} \|u_{\alpha 0} - z_d\|_H^2 \leq C_1
$$

$\[\square\]$
because \( \{u_{\alpha 0}\} \) is convergent when \( \alpha \to \infty \), see \cite{9} Theorem 7. Therefore, there exist positive constants \( C_2 \) and \( C_3 \), independent of \( \alpha \), such that

\[
\|g_\alpha\|_H \leq C_2 \quad \text{and} \quad \|u_{\alpha 0}\|_H \leq C_3. \tag{15}
\]

Now, we choose \( v = u_{\infty \alpha} - u_{\alpha 0} \in V_0 \) as a test function in the elliptic boundary hemivariational inequality \cite{5} to obtain

\[
a(u_{\alpha 0}, u_{\infty \alpha} - u_{\alpha 0}) + \alpha \int_{\Gamma_3} j^0(u_{\alpha 0}; u_{\infty \alpha} - u_{\alpha 0}) \, d\Gamma \geq L(u_{\infty \alpha} - u_{\alpha 0}).
\]

From the equality

\[
a(u_{\alpha 0}, u_{\infty \alpha} - u_{\alpha 0}) = -a(u_{\infty \alpha} - u_{\alpha 0}, u_{\infty \alpha} - u_{\alpha 0}) + a(u_{\infty \alpha}, u_{\infty \alpha} - u_{\alpha 0}),
\]

we get

\[
a(u_{\infty \alpha} - u_{\alpha 0}, u_{\infty \alpha} - u_{\alpha 0}) - \alpha \int_{\Gamma_3} j^0(u_{\alpha 0}; u_{\infty \alpha} - u_{\alpha 0}) \, d\Gamma \leq a(u_{\infty \alpha}, u_{\infty \alpha} - u_{\alpha 0}) - L(u_{\infty \alpha} - u_{\alpha 0}). \tag{16}
\]

Taking into account that \( j^0(x, u_{\alpha 0}; u_{\infty \alpha} - u_{\alpha 0}) = j^0(x, u_{\alpha 0}; b - u_{\alpha 0}) \) on \( \Gamma_3 \), and by \( H(j)(d) \), we have \( j^0(x, u_{\alpha 0}; u_{\infty \alpha} - u_{\alpha 0}) \leq 0 \) on \( \Gamma_3 \). Hence

\[
a(u_{\infty \alpha} - u_{\alpha 0}, u_{\infty \alpha} - u_{\alpha 0}) \leq a(u_{\infty \alpha}, u_{\infty \alpha} - u_{\alpha 0}) - L(u_{\infty \alpha} - u_{\alpha 0}).
\]

By the boundedness and coerciveness of \( a \), we infer

\[
m_a \|u_{\infty \alpha} - u_{\alpha 0}\|_V^2 \leq (M_a \|u_{\infty \alpha}\|_V + \|L\|_{V^*}) \|u_{\infty \alpha} - u_{\alpha 0}\|_V
\]

with \( M_a > 0 \), and subsequently

\[
\|u_{\alpha 0}\|_V \leq \|u_{\infty \alpha} - u_{\alpha 0}\|_V + \|u_{\infty \alpha}\|_V
\]

\[
\quad \leq \frac{1}{m_a} (M_a \|u_{\infty \alpha}\|_V + \|L\|_{V^*}) + \|u_{\infty \alpha}\|_V
\]

\[
=: C_4,
\]

where \( C_4 > 0 \) is a constant independent of \( \alpha \). Hence, since \( a(u_{\infty \alpha} - u_{\alpha 0}, u_{\infty \alpha} - u_{\alpha 0}) \geq 0 \), from \( \text{(16)} \), we have

\[
-\alpha \int_{\Gamma_3} j^0(u_{\alpha 0}; u_{\infty \alpha} - u_{\alpha 0}) \, d\Gamma \leq (M_a \|u_{\infty \alpha}\|_V + \|L\|_{V^*}) \|u_{\infty \alpha} - u_{\alpha 0}\|_V
\]

\[
\quad \leq \frac{1}{m_a} (M_a \|u_{\infty \alpha}\|_V + \|L\|_{V^*})^2
\]

\[
=: C_5,
\]

where \( C_5 > 0 \) is independent of \( \alpha \). Thus

\[
-\int_{\Gamma_3} j^0(u_{\alpha 0}; u_{\infty \alpha} - u_{\alpha 0}) \, d\Gamma \leq \frac{C_5}{\alpha}. \tag{18}
\]
It follows from (17) that \( \{u_{a\alpha}\} \) remains in a bounded subset of \( V \). Thus, there exists \( \eta \in V \) such that, by passing to a subsequence if necessary, we have

\[
u_{a\alpha} \rightarrow \eta \quad \text{weakly in } V, \quad \alpha \rightarrow \infty. \tag{19}\]

Moreover, from (15) we have that there exists \( h \in H \) such that

\[
g_{\alpha} \rightarrow h \quad \text{weakly in } H, \quad \alpha \rightarrow \infty. \tag{20}\]

**Step 2.** Next, we will show that \( h = g^* \) and \( \eta = u_{\infty g^*} \). We observe that \( \eta \in V_0 \) because \( \{u_{a\alpha}\} \subset V_0 \) and \( V_0 \) is sequentially weakly closed in \( V \). Let \( w \in K \) and \( v = w - u_{a\alpha} \in V_0 \). From (21), we have

\[
L(w - u_{a\alpha}) \leq a(u_{a\alpha}, w - u_{a\alpha}) + \alpha \int_{\Gamma_3} j^0(u_{a\alpha}; w - u_{a\alpha}) d\Gamma.
\]

Since \( w = b \) on \( \Gamma_3 \), by \( H(j)(d) \), we have

\[
\alpha \int_{\Gamma_3} j^0(u_{a\alpha}; w - u_{a\alpha}) d\Gamma = \alpha \int_{\Gamma_3} j^0(u_{a\alpha}; b - u_{a\alpha}) d\Gamma \leq 0
\]

which implies

\[
L(w - u_{a\alpha}) \leq a(u_{a\alpha}, w - u_{a\alpha}). \tag{21}
\]

Next, we use the weak lower semicontinuity of the functional \( V \ni v \mapsto a(v, v) \in \mathbb{R} \) and from (21), we deduce

\[
\eta \in V_0 \quad \text{satisfies} \quad L(w - \eta) \leq a(\eta, w - \eta) \quad \text{for all } w \in K. \tag{22}
\]

Subsequently, we will show that \( \eta \in K \). In fact, from (19), by the compactness of the trace operator, we have \( u_{a\alpha}|_{\Gamma_3} \rightarrow \eta|_{\Gamma_3} \) in \( L^2(\Gamma_3) \), as \( \alpha \rightarrow \infty \). Passing to a subsequence if necessary, we may suppose that \( u_{a\alpha}(x) \rightarrow \eta(x) \) for a.e. \( x \in \Gamma_3 \) and there exists \( f \in L^2(\Gamma_3) \) such that \( |u_{a\alpha}(x)| \leq f(x) \) a.e. \( x \in \Gamma_3 \). Using the upper semicontinuity of the function \( \mathbb{R} \times \mathbb{R} \ni (r, s) \mapsto j^0(x, r; s) \in \mathbb{R} \) for a.e. \( x \in \Gamma_3 \), see [9] Proposition 3 (iii), we get

\[
\limsup_{\alpha \rightarrow \infty} j^0(x, u_{a\alpha}(x); b - u_{a\alpha}(x)) \leq j^0(x, \eta(x); b - \eta(x)) \quad \text{a.e. } x \in \Gamma_3.
\]

Next, taking into account the estimate

\[
|j^0(x, u_{a\alpha}(x); b - u_{a\alpha}(x))| \leq (c_0 + c_1|u_{a\alpha}(x)|) |b - u_{a\alpha}(x)| \leq k(x) \quad \text{a.e. } x \in \Gamma_3
\]

with \( k \in L^1(\Gamma_3) \) given by \( k(x) = (c_0 + c_1 f(x))(|b| + f(x)) \), by the dominated convergence theorem, see [6], we obtain

\[
\limsup_{\alpha \rightarrow \infty} \int_{\Gamma_3} j^0(u_{a\alpha}; b - u_{a\alpha}) d\Gamma \leq \int_{\Gamma_3} j^0(\eta; b - \eta) d\Gamma.
\]
Consequently, from $H(j)(d)$ and (18), we have

$$0 \leq - \int_{\Gamma_3} j^0(\eta; b - \eta) \, d\Gamma \leq \liminf_{\alpha \to \infty} \left( - \int_{\Gamma_3} j^0(u_{\alpha g_a}; b - u_{\alpha g_a}) \, d\Gamma \right) \leq 0$$

which gives $\int_{\Gamma_3} j^0(\eta; b - \eta) \, d\Gamma = 0$. Again by $H(j)(d)$, we get $j^0(x, \eta; b - \eta) = 0$ a.e. $x \in \Gamma_3$. Using $(H_1)$, we have $\eta(x) = b$ for a.e. $x \in \Gamma_3$, which together with (22) implies

$$\eta \in K \text{ satisfies } L(w - \eta) \leq a(\eta, w - \eta) \text{ for all } w \in K.$$ 

Next, we will prove that $\eta = u_{\infty h}$. To this end, let $v := w - \eta \in K_0$ with arbitrary $w \in K$. Hence, $L(v) \leq a(\eta, v)$ for all $v \in K_0$. Recalling that $v \in K_0$ implies $-v \in K_0$, we obtain $a(\eta, v) \leq L(v)$ for all $v \in K_0$. Hence, we conclude that

$$\eta \in K \text{ satisfies } a(\eta, v) = L(v) \text{ for all } v \in K_0,$$

i.e., $\eta \in K$ is a solution to problem (2). By the uniqueness of solution to problem (2), we have $\eta = u_{\infty h}$ and hence $u_{\alpha g_a} \rightharpoonup u_{\infty h}$ weakly in $V$, as $\alpha \to \infty$.

Now

$$J_\alpha(g_\alpha) \leq J_\alpha(f), \quad \forall f \in H$$

next

$$J(h) = \frac{1}{2} ||u_{\infty h} - z_d||_H^2 + \frac{M}{2} ||h||_H^2 - \frac{1}{2} ||\eta - z_d||_H^2 + \frac{M}{2} ||h||_H^2$$

$$\leq \liminf_{\alpha \to \infty} J_\alpha(g_\alpha) \leq \liminf_{\alpha \to \infty} J_\alpha(f)$$

$$= \lim_{\alpha \to \infty} J_\alpha(f) = J(f), \quad \forall f \in H$$

and from the uniqueness of the optimal control problem (3), see (11), we obtain that

$$h = g^*,$$

therefore $u_{\infty h} = u_{\infty g^*}$. Next, we have that, when $\alpha \to \infty$

$$g_\alpha \rightharpoonup g^* \text{ weakly in } H \text{ and } u_{\alpha g_a} \rightharpoonup u_{\infty g^*} \text{ weakly in } V.$$

**Step 3.** Now, we prove the strong convergence $u_{\alpha g_a} \to u_{\infty g^*}$ in $V$, as $\alpha \to \infty$. Choosing $v = u_{\infty g^*} - u_{\alpha g_a} \in V_0$ in problem (5), we obtain

$$a(u_{\alpha g_a}, u_{\infty g^*} - u_{\alpha g_a}) + \alpha \int_{\Gamma_3} j^0(u_{\alpha g_a}; u_{\infty g^*} - u_{\alpha g_a}) \, d\Gamma \geq L(u_{\infty g^*} - u_{\alpha g_a}).$$

Hence

$$a(u_{\infty g^*} - u_{\alpha g_a}, u_{\infty g^*} - u_{\alpha g_a}) \leq a(u_{\infty g^*}, u_{\infty g^*} - u_{\alpha g_a}) + L(u_{\alpha g_a} - u_{\infty g^*})$$

$$+ \alpha \int_{\Gamma_3} j^0(u_{\alpha g_a}; u_{\infty g^*} - u_{\alpha g_a}) \, d\Gamma.$$
Since \( u_{\infty g^*} = b \) on \( \Gamma_3 \), by \( H(j)(d) \) and the coerciveness of the form \( a \), we have
\[
m_a \| u_{\infty g^*} - u_{\alpha g^*} \|_V^2 \leq a(u_{\infty g^*}, u_{\infty g^*} - u_{\alpha g^*}) + L(u_{\alpha g^*} - u_{\infty g^*}).
\]
Employing the weak continuity of \( a(u_{\infty g^*}, \cdot) \), the compactness of the trace operator and taking into account that \( u_{\alpha g^*} \to u_{\infty g^*} \) strongly in \( H \), we conclude that \( u_{\alpha g^*} \to u_{\infty g^*} \) strongly in \( V \), as \( \alpha \to \infty \).

Finally, we prove the strong convergence of \( g_\alpha \) to \( g^* \) in \( H \), when \( \alpha \to \infty \). In fact, from \( u_{\alpha g^*} \to u_{\infty g^*} \) strongly in \( H \), we deduce
\[
\lim_{\alpha \to \infty} \frac{1}{2} \| u_{\alpha g^*} - z_d \|_H^2 = \frac{1}{2} \| u_{\infty g^*} - z_d \|_H^2 \quad (23)
\]
and as \( g_\alpha \rightharpoonup g^* \) weakly in \( H \), then
\[
\| g^* \|_H^2 \leq \liminf_{\alpha \to \infty} \| g_\alpha \|_H^2. \quad (24)
\]

Next, from (23) and (24), we obtain
\[
\frac{1}{2} \| u_{\infty g^*} - z_d \|_H^2 + \frac{M}{2} \| g^* \|_H^2 \leq \liminf_{\alpha \to \infty} \left( \frac{1}{2} \| u_{\alpha g^*} - z_d \|_H^2 + \frac{M}{2} \| g_\alpha \|_H^2 \right),
\]
that is
\[
J(g^*) \leq \liminf_{\alpha \to \infty} J_\alpha(g_\alpha).
\]
On the other hand, from the definition of \( g_\alpha \), we have
\[
J_\alpha(g_\alpha) \leq J_\alpha(g^*)
\]
then, taking into account that \( u_{\alpha g^*} \to u_{\infty g^*} \) strongly in \( H \), see [9, Theorem 7], we obtain
\[
\limsup_{\alpha \to \infty} J_\alpha(g_\alpha) \leq \limsup_{\alpha \to \infty} J_\alpha(g^*) = J(g^*)
\]
and therefore
\[
\lim_{\alpha \to \infty} J_\alpha(g_\alpha) = J(g^*)
\]
or equivalently
\[
\lim_{\alpha \to \infty} \left( \frac{1}{2} \| u_{\alpha g^*} - z_d \|_H^2 + \frac{M}{2} \| g_\alpha \|_H^2 \right) = \frac{1}{2} \| u_{\infty g^*} - z_d \|_H^2 + \frac{M}{2} \| g^* \|_H^2. \quad (25)
\]
Now, from (23) and (25), when \( \alpha \to \infty \), we have
\[
\| g_\alpha \|_H^2 \to \| g^* \|_H^2
\]
and as \( g_\alpha \rightharpoonup g^* \) weakly in \( H \), we deduce that \( g_\alpha \to g^* \) strongly in \( H \). This completes the proof.

We remark that we can find examples of several locally Lipschitz functions \( j \) which satisfies the hypothesis \( H(j) \) and \( (H_1) \) in [9].
5 Conclusions

We have studied a parameter optimal control problems for systems governed by elliptic boundary hemivariational inequalities with a non-monotone multivalued subdifferential boundary condition on a portion of the boundary of the domain which is described by the Clarke generalized gradient of a locally Lipschitz function. We prove an existence result for the optimal controls and we show an asymptotic result for the optimal controls and the system states, when the parameter (the heat transfer coefficient on a portion of the boundary) tends to infinity. These results generalize for a locally Lipschitz function $j$, under the hypothesis $H(j)$ and $(H_1)$, the classical results obtained in [10] for a quadratic superpotential $j$.

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References

[1] Azzam A. - Kreyszig E., On solutions of elliptic equations satisfying mixed boundary conditions, SIAM J. Math. Anal. 13 (1982), 254-262.

[2] Bacuta C. - Bramble J.H. - Pasciak J.E., Using finite element tools in proving shift theorems for elliptic boundary value problems, Numer. Linear Algebra Appl., 10 (2003), 33-64.

[3] Barbu V., Boundary control problems with non linear state equation, SIAM J. Control Optim., 20 (1982), 125-143.

[4] Carl S. - Le V.K. - Motreanu D., Nonsmooth Variational Problems and Their Inequalities, Springer, New York (2007)

[5] Clarke F.H., Optimization and Nonsmooth Analysis, Wiley, Interscience, New York (1983).

[6] Denkowski Z. - Migorski S. - Papageorgiou N.S., An Introduction to Nonlinear Analysis: Theory, Kluwer Academic/Plenum, Boston (2003).

[7] Duvaut G. - Lions J.L., Les Inéquations en Mécanique et en Physique, Dunod, Paris (1972).

[8] Garguichevich G.G. - Tarzia D.A., The steady-state two-phase Stefan problem with an internal energy and some related problems, Atti Sem. Mat. Fis. Univ. Modena, 39 (1991), 615-634.
[9] Gariboldi C. M. - Migorski S. - Ochal A. - Tarzia D.A., Existence, comparison, and convergence results for a class of elliptic hemivariational inequalities, Appl. Math. Optim., (2021), DOI: 10.1007/s00245-021-09800-9.

[10] Gariboldi C. M. - Tarzia D. A., Convergence of distributed optimal controls on the internal energy in mixed elliptic problems when the heat transfer coefficient goes to infinity, Appl. Math. Optim., 47 (2003), 213-230.

[11] Gariboldi C. M. - Tarzia D.A., Distributed optimal control problems for a class of elliptic hemivariational inequalities with a parameter and its asymptotic behavior, Commun. Nonlinear Sci. Numer. Simul., 104 No.106027 (2021), 1-9.

[12] Grisvard P., Elliptic Problems in Nonsmooth Domains, Pitman, London, (1985).

[13] Lanzani L. - Capagna L. - Brown R.M., The mixed problem in $L^p$ for some two-dimensional Lipschitz domain, Math. Ann., 342 (2008), 91-124.

[14] Lions J.L., Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles, Dunod, Paris, 1968.

[15] Migorski S. - Ochal A., Boundary hemivariational inequality of parabolic type, Nonlinear Analysis, 57 (2004), 579-596.

[16] Migorski S. - Ochal A., A unified approach to dynamic contact problems in viscoelasticity, J. Elasticity, 83 (2006), 247-275.

[17] Migorski S. - Ochal A. - Sofonea M., A class of variational-hemivariational inequalities in reflexive Banach spaces, J. Elasticity, 127 (2017), 151-178.

[18] Migorski S. - Ochal A. - Sofonea M., Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems, Springer, New York (2013).

[19] Naniewicz Z. - Panagiotopoulos P.D., Mathematical Theory of Hemivariational Inequalities and Applications, Marcel Dekker, Inc., New York, (1995).

[20] Panagiotopoulos P.D., Nonconvex problems of semipermeable media and related topics, Z. Angew. Math. Mech., 65 (1985), 29-36.

[21] Panagiotopoulos P.D., Inequality Problems in Mechanics and Applications, Birkhäuser, Boston (1985).

[22] Panagiotopoulos P.D., Hemivariational Inequalities, Applications in Mechanics and Engineering, Springer, Berlin (1993).

[23] Rodrigues J.F., Obstacle Problems in Mathematical Physics, North-Holland, Amsterdam (1987).
[24] Sofonea M. - Migorski S., *Variational-Hemivariational Inequalities with Applications*, CRC Press, Boca Raton (2018).

[25] Tabacman E.D. - Tarzia D.A., *Sufficient and/or necessary condition for the heat transfer coefficient on $\Gamma_1$ and the heat flux on $\Gamma_2$ to obtain a steady-state two-phase Stefan problem*, J. Differential Equations, 77 (1989), 16-37.

[26] Tarzia D.A., *Sur le problème de Stefan à deux phases*, C. R. Acad. Sci. Paris Ser. A, 288 (1979), 941-944.

[27] Tarzia D.A., *Una familia de problemas que converge hacia el caso estacionario del problema de Stefan a dos fases*, Mathematicae Notae, 27 (1979/80), 157-165.

[28] Tarzia D.A., *An inequality for the constant heat flux to obtain a steady-state two-phase Stefan problem*, Eng. Anal., 5 (4) (1988), 177-181.

[29] Trölstzsch F., *Optimal control of partial differential equations. Theory, methods and applications*, American Math. Soc., Providence, 2010.

[30] Zeng B. - Liu Z. - Migorski S., *On convergence of solutions to variational-hemivariational inequalities*, Z. angew. Math. Phys., 69 (87) (2018), 1-20.