Magnetoresistance of a two-dimensional electron gas with spatially periodic lateral modulations: Exact consequences of Boltzmann’s equation

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On the basis of Boltzmann’s equation, and including anisotropic scattering in the collision operator, we investigate the effect of one-dimensional superlattices on two-dimensional electron systems. In addition to superlattices defined by static electric and magnetic fields, we consider mobility superlattices describing a spatially modulated density of scattering centers. We prove that magnetic and electric superlattices in \( x \)-direction affect only the resistivity component \( \rho_{xx} \) if the mobility is homogeneous, whereas a mobility lattice in \( x \)-direction in the absence of electric and magnetic modulations affects only \( \rho_{yy} \). Solving Boltzmann’s equation numerically, we calculate the positive magnetoresistance in weak magnetic fields and the Weiss oscillations in stronger fields within a unified approach.

I. INTRODUCTION

Two-dimensional electron systems (2DESs) in high-mobility Al\(_{1-x}\)Ga\(_x\)As-GaAs heterostructures with one-dimensional lateral superlattices show interesting magneto-transport properties at low temperatures (\( T \approx 4\)K or less) \([1,2]\). Experiments on typical samples with electron density \( n_{el} \approx 3 \cdot 10^{11} \) cm\(^{-2}\), mobility \( \mu \approx 10^6 \) cm\(^2\)/Vs, and a weak electric modulation \( E \) in \( x \)-direction of period \( a \approx 300 \) nm revealed the following features. If the homogeneous magnetic field \( B_0 \) applied perpendicular to the 2DES is very weak (\( B_0 \leq 0.03T \)), one finds a pronounced positive magnetoresistance in the component \( \rho_{xx} \). With increasing \( B_0 \), this saturates and is followed by the “Weiss oscillations” (\( 0.1T \leq B_0 \leq 1T \)), which at higher fields (typically \( B_0 \geq 0.6T \) at \( T = 4.2K \)) are superimposed by the Shubnikov-de Haas oscillations. The resistivity component \( \rho_{yy} \) does not exhibit such a positive magnetoresistance, but shows Weiss oscillations of opposite phase. No significant effect of the modulation on the Hall resistance has been observed. Experiments on samples with a stronger electric modulation \([3,4]\), and also the recent experiments on magnetically modulated systems \([5,6]\), led to similar results, although they usually did not provide the full information about the conductivity tensor. The Weiss oscillations were first understood within a quantum mechanical picture \([7,8]\). Solving Boltzmann’s equation numerically, we calculate the positive magnetoresistance and the Weiss oscillations of \( \rho_{xx} \) totally within classical dynamics, is still missing. The purpose of this paper is to present such a classical calculation.

We will extend Beenakker’s work \([6]\) by including other modulation sources in addition to a periodic electric field. We will consider a magnetic modulation and also a “mobility modulation”, i.e. a periodic position dependence of the collision operator \( C \) of Boltzmann’s equation, describing a spatial modulation of the density of scatterers. We will also allow for anisotropic scattering of the electrons by, e.g., impurity potentials of finite range, and thus go beyond the relaxation time approximation of Boltzmann’s equation. Within our classical approach we will prove that, in the absence of a mobility modulation, a one-dimensional superlattice defined by \( x \)-dependent electric and magnetic fields affects only \( \rho_{xx} \), even if anisotropic scattering is included. Allowing for a periodic mobility modulation in \( x \)-direction, we can obtain oscillations of \( \rho_{yy} \), however with the same phase as those of \( \rho_{xx} \) in the case of a pure electric modulation.

After giving some details about our model in Sect. II, we present the formal calculations revealing the structure of the magnetoresistance tensor in Sect. III and in Appendix A. In Sect. IV we present analytical results: For the regime of Weiss oscillations, we generalize Beenakker’s calculation to include magnetic and mobility modulations, and compare with simplified calculations based on the evaluation of the drift velocities of cyclotron...
orbits $[10,14,15]$. For the regime of channeled trajectories we concentrate on the collisionless limit, where we obtain an algebraic dependence of the conductance on the modulation strength. Therefore, in contrast to the regime of Weiss oscillations, a simple power expansion of the transport coefficients with respect to the modulation strength is not possible in this regime.

Numerical results based on a Fourier expansion of Boltzmann’s equation are presented in Sect. V. For sufficiently weak modulation strengths, these results cover the whole range of low-field magnetoresistance and Weiss oscillations, and they include the effect of anisotropic scattering as well as of electric, magnetic and mobility modulations. In a final Sect. VI we summarize the most important results.

II. THE MODEL

It is known from quantum mechanical calculations $[8,10,11]$ that the Weiss oscillations on typical high-mobility samples are observed in a peculiar temperature range. The temperature must be high enough, so that in the regime of Weiss oscillations the Shubnikov-de Hass oscillations are not resolved. On the other hand, it must be low enough, so that the transport coefficients are determined only by the electronic states near the Fermi energy $[8,10,11]$. To describe the relevant physics in this temperature window, we consider the 2DES in the $x$-$y$ plane as a degenerate Fermi gas, with Fermi energy $E_F = (m/2)v_F^2$, of (non-interacting) particles with effective mass $m$ and charge $-e$, obeying classical dynamics. The velocity $\mathbf{v} = \mathbf{v} = (\dot{x}, \dot{y}, 0)$ of an electron obeys Newton’s equation

$$m \ddot{\mathbf{r}} = -e (\mathbf{F} + (\mathbf{v} \times \mathbf{B})/c).$$  

To describe a modulated system with period $a$ in equilibrium, we write the electric field as $\mathbf{F}(x) = \nabla V(x)/e$ and the magnetic field as $\mathbf{B}(x) = (0, 0, B_0 + B_m(x))$, and assume that the periodic functions $V(x) = V(x + a)$ and $B_m(x) = B_m(x + a)$ have zero average values. The thermal equilibrium at constant temperature $T$ and electrochemical potential $\mu^*$ is described by the distribution function

$$f_{\text{eq}}(\mathbf{r}, \mathbf{v}) = f_0(E(\mathbf{r}, \mathbf{v}); T, \mu^*),$$  

where the Fermi function $f_0(E; T, \mu^*) = 1/[1 + \exp((E - \mu^*)/k_BT)]$ depends on the dynamical variables only through the conserved energy

$$E(\mathbf{r}, \mathbf{v}) = (m/2)v^2 + V(x).$$  

In more general situations, the distribution function $f(\mathbf{r}, \mathbf{v}, t)$, which yields electron density $n_{el}$ and current density $\mathbf{j}$ according to

$$n_{el}(\mathbf{r}, t) = \langle f \rangle, \quad \mathbf{j}(\mathbf{r}, t) = -e \mathbf{v} \langle f \rangle,$$

where $\langle f \rangle_v = [2m^2/(2\pi\hbar)^2] \int d^2v f$, is determined by Boltzmann’s equation

$$
\left( \frac{\partial}{\partial t} + \mathbf{i} \cdot \nabla + \mathbf{v} \cdot \nabla \right) f = C[f; \mathbf{r}, \mathbf{v}, t]$$  

and suitable boundary conditions. For any reasonable microscopic model of elastic and inelastic scattering $[9]$, the collision operator $C$ vanishes after averaging over the velocity, $\langle C \rangle_v = 0$. As a consequence, the $v$-integral over Eq. (2.5) yields the continuity equation,

$$-e \partial n_{el}/\partial t + \nabla \cdot \mathbf{j} = 0.$$  

If we want to choose a phenomenological model for $C$, we should be careful not to destroy this property.

In the following we consider the stationary, linear response of the modulated system to an external homogeneous electric field $\mathbf{E}^0$, which describes the average effect of a voltage applied across the sample. The result will be a non-zero current density $\mathbf{j}(x)$ and a correction $\delta n_{el}(x)$ to the equilibrium electron density. At low temperatures $[k_B T \ll \mu^* - V(x)]$, the equilibrium density follows from Eqs. (2.2) and (2.3) as $n_{el}(x) = D_0(E - V(x))$, where $D_0 = m/\pi\hbar^2$ is the density of states and $E_F = \mu^*(T \rightarrow 0)$ the Fermi energy of the homogeneous 2DES.

With the ansatz

$$f_{\text{stat}}(\mathbf{r}, \mathbf{v}) = f_0(E(\mathbf{r}, \mathbf{v}); T, \mu^*) - e \frac{\delta f_0}{\delta \mu^*}(\mathbf{r}, \mathbf{v})$$

for the stationary solution, and the assumption that the correction $\phi(\mathbf{r}, \mathbf{v})$ to the equilibrium distribution is linear in $\mathbf{E}^0$, we obtain the linearized Boltzmann equation

$$\mathcal{D}\phi(\mathbf{r}, \mathbf{v}) - C[\phi; \mathbf{r}, \mathbf{v}] = \mathbf{v} \cdot \mathbf{E}^0,$$

where $\mathcal{D} = \mathbf{v} \cdot \partial/\partial \mathbf{r} + \mathbf{v} \cdot \partial/\partial \mathbf{v}$, and $\mathbf{v}$ is determined by Eq. (2.7) with the force in the equilibrium state. For $T \rightarrow 0$, $\partial f_0/\partial \mu^* \rightarrow \delta(E(\mathbf{r}, \mathbf{v}) - E_F)$ and with Eq. (2.3), which implies for the magnitude of the velocity $v(x) = eF[1 - V(x)/E_F]^{1/2}$, the 2D $v$-integral in Eq. (2.3) reduces to an integral over the polar angle in the velocity space, e.g.

$$\int \frac{d\varphi}{2\pi} v(x) \mathbf{u}(\varphi) \phi(x, \varphi),$$

where $\mathbf{u}(\varphi) = (\cos \varphi, \sin \varphi)$, and the dependence of $\phi$ on $E_F$ is not explicitly indicated. The drift term of Eq. (2.7) reads in these variables

$$\mathcal{D} = v(x) \cos \varphi \frac{\partial}{\partial x} \left[ \omega_c + \omega_m(x) + \omega_e(x, \varphi) \right] \frac{\partial}{\partial \varphi},$$

with $\omega_c = eB_0/mc$ the cyclotron frequency due to the average magnetic field, and $\omega_m(x) = eB_m(x)/mc$ and $\omega_e(x, \varphi) = -\sin \varphi \partial v/dx$ are due to the magnetic and the electric modulation, respectively. The collision operator describes the change of the distribution function due to scattering events with a spatial extent which is short...
on the scale of the drift motion \( \mathbb{L} \). If there are different scattering mechanisms, we may write \( C[\phi; x, \varphi] = \sum_j C_j[\phi; x, \varphi] \), with
\[
C_j[\phi; x, \varphi] = \frac{1}{\tau_j(x)} \int \frac{d\varphi'}{2\pi} P_j(\varphi' - \varphi)[\phi(x, \varphi') - \phi(x, \varphi)],
\]
(2.10)
where we have expressed the renormalized differential cross sections in terms of relaxation times \( \tau_j(x) \) and dimensionless kernels \( P_j(\varphi) \). Here, the last term describes the decay of the distribution function due to scattering processes of the \( j \)-th type which start in phase space at \( (x, \varphi) \), whereas \( \phi(x', \varphi') \) describes the back-scattering into this state. Assuming that the distribution function changes little on the range of the scattering processes, we consider only \( x' = x \) in the back-scattering term. For scattering by impurities, the scattering rate is proportional to the impurity density, which may be a function of the position coordinate \( x \) in periodically modulated samples. In order to describe a periodic modulation of the electron mobility, we write \( 1/\tau_j(x) = 1/\bar{\tau}_j + \tau_j(x) \) and assume that the oscillating parts \( \tau_j(x) = \tau_j(x + a) \) of the scattering rates have zero average values. Further we assume that the kernels \( P_j(\varphi) \), with the normalization \( \int_0^{2\pi} d\varphi P_j(\varphi) = 2\pi \), have the symmetry \( P_j(-\varphi) = P_j(\varphi) \). This follows from the microscopic symmetry if the effect of the magnetic field during an individual scattering process is neglected.

It is interesting to note that the trigonometric functions are eigenfunctions of the collision operator: for \( \psi_m(\varphi) = \exp(\im \varphi m) \) one has
\[
C[\psi_m; x, \varphi] = -\frac{1}{\tau_{tr}(m)}(x) \psi_m(\varphi),
\]
(2.11)
with the eigenvalues
\[
1/\tau_{tr}(m)(x) = \sum_j (1 - \gamma_j(m))/\tau_j(x),
\]
(2.12)
where \( \gamma_j(m) = (2\pi)^{-1} \int_0^{2\pi} d\varphi P_j(\varphi) \cos(m\varphi) \). The same holds for the real part and the imaginary part of \( \psi_m \) separately. We will demonstrate in the following that several transport properties do not depend on all the details of the collision operator defined by Eq. (2.10), but only on the transport time \( \tau_{tr}(x) = \tau_{tr}(1)(x) \), which we write in the form \( 1/\tau_{tr}(x) = 1/\bar{\tau}_{tr} + \tau_{tr}(x) \). For instance, the Drude conductivity tensor of the homogeneous 2DES, with \( \tau_{tr}(x) = 0 \), depends only on \( \bar{\tau}_{tr} \). Thus, the transport scattering rates due to the different scattering mechanisms simply add.

The special case \( P_j(\varphi) \equiv 1 \) describes isotropic scattering, and is identical with the relaxation time approximation, \( C_j[\phi] = -\phi/\tau_j \), where the relaxation is towards the local equilibrium distribution \( \phi_{loc}(x) = \int_0^{2\pi} d\varphi \phi(x, \varphi)/2\pi \). This is the correct form of the relaxation time approximation for inhomogeneous systems

\[\text{III. STRUCTURE OF THE RESISTIVITY TENSOR}\]

In this section we prove from Boltzmann’s equation that a combined electric and magnetic modulation in \( x \)-direction affects only the component \( \rho_{xx} \) of the effective resistivity tensor, if there is no position dependence of the scattering rates, \( \tau_j(x) \equiv 0 \). All other components then remain those of the unmodulated system. If, on the other hand, there is only a modulation of the scattering rates while \( V(x) = 0 \) and \( B_m(x) \equiv 0 \), only \( \rho_{yy} \) will be affected. To obtain (within the classical Boltzmann equation approach) a modulation effect on more than one component of the resistivity tensor, one needs a mobility modulation in addition to an electric and/or a magnetic modulation. We will make explicit use of the equation of continuity to derive these results, which generalize an early observation by Beenakker \[10] to more general types of modulations and scattering mechanisms. Beenakker solved the problem for a purely electric modulation in the relaxation time approximation. To appreciate the important role of the continuity equation in this context, it is instructive to consider first the simple local limit.

We want to emphasize that we assume translational invariance in the \( y \)-direction throughout this work, so that, together with the modulations, current densities and electric field components may depend only on \( x \). Boundary effects are neglected. The experimental situation of a Hall bar with current flow in \( x \)-direction will be simulated by assuming that the average current density in \( y \)-direction vanishes. The experimental boundary conditions of exactly vanishing \( j_y(x) \) on the sample boundaries in \( y \)-direction would destroy the translational invariance in \( y \)-direction and, thus, the simplification resulting from the merely unidirectional modulation. They would require a fully numerical treatment already in the local limit \[11] which may become necessary under certain circumstances, but will not be discussed in the present work.

\[\text{A. Local limit}\]

In the local limit we describe the linear relation between current density \( \mathbb{J}(x) \) and driving electric field \( \mathbb{F}(x) \) by
\[
\hat{\rho}^{loc}(x) \mathbb{J}(x) = \mathbb{F}(x),
\]
(3.1)
with a local resistivity tensor \( \hat{\rho}^{loc}(x) \). It is obtained from the Drude resistivity tensor \( \hat{\rho}^D \) of the homogeneous system, with components \( \rho_{xx}^D = \rho_{yy}^D = \rho_0 \) and
\[\text{\[17\ LS\]. Neglecting } \phi_{loc} \text{ would describe relaxation towards the total equilibrium distribution [cf. Eq. (2.6)], and would violate explicitly the equation of continuity. As we will discuss below, this can lead to qualitatively wrong results for the transport coefficients.}\]
\( \rho_{xy}^D = -\rho_{yx}^D = \omega_c \tau_{tr}, \rho_0, \) where \( \rho_0 = m/(e^2 \tau_{tr} n_{el}) \), by replacing the spatially constant electron density \( n_{el}, \) cyclotron frequency \( \omega_c, \) and transport time \( \tau_{tr} \) by their spatially modulated counterparts \( n_{el}(x) = \bar{n}_{el}(1 - V(x)/E_F), \) \( \omega(x) = \omega_c + \omega_m(x), \) and \( \tau_{tr}(x) = \bar{\tau}_{tr}/[1 + \bar{\tau}_{tr} r_{tr}(x)], \) respectively. In thermal equilibrium, Eq. (3.1) applies with a positive magnetoresistance (\( \propto \)).

The stationary response to a homogeneous external electric field \( \mathbf{E}^0 \) is accompanied by a change \( \delta n_{el}(x) \) of the electron density, so that \( \mathbf{F}(x) = \mathbf{E}^0 + \nabla \delta n_{el}(x)/(e \mathbf{D}) \) has to be inserted into Eq. (3.1). Since \( \delta n_{el}(x) \) must have the periodicity of the electric modulation, we find \( \langle \mathbf{F}(x) \rangle = \mathbf{E}^0 \), where \( \langle \ldots \rangle \) denotes the average over a period of the modulation.

In the second, even simpler calculation, we set \( j_x = 0 \) and Maxwell’s equation \( \nabla \times \mathbf{F} = 0 \) imply that \( j_x \) and \( F_y \) must be independent of \( x \). With this result, we can define an effective resistivity tensor \( \tilde{\rho} \) by

\[ \tilde{\rho}(j(x)) = \mathbf{E}^0, \] (3.2)

and calculate it by considering two experimentally relevant situations.

First, we assume that the average current density in \( y \)-direction is zero. Then, solving the \( y \)-component of Eq. (3.1) for \( j_y(x) \), we obtain

\[ \tilde{\rho}_{yx} = \frac{F_y}{j_y} = \frac{\langle \rho_{yx}^{loc} / \rho_{yy}^{loc} \rangle}{\langle 1 / \rho_{yy}^{loc} \rangle}. \] (3.3)

Inserting \( j_y(x) \) into the \( x \)-component of Eq. (3.1), one obtains

\[ \tilde{\rho}_{xx} = \frac{\langle F_x \rangle}{j_x} = \left( \rho_{xx}^{loc} + \langle \rho_{yx} - \rho_{yx}^{loc} \rangle \rho_{xx}^{loc} / \rho_{yy}^{loc} \right). \] (3.4)

In the second, even simpler calculation, we set \( j_x = 0 \) and obtain

\[ \tilde{\rho}_{yy} = \frac{F_y}{j_y} = \frac{1}{\langle 1 / \rho_{yy}^{loc} \rangle}, \] (3.5)

and

\[ \tilde{\rho}_{xy} = \frac{\langle F_x \rangle}{j_y} = \frac{\langle \rho_{yx}^{loc} / \rho_{yy}^{loc} \rangle}{\langle 1 / \rho_{yy}^{loc} \rangle}. \] (3.6)

Since \( \rho_{yx}^{loc} = -\rho_{yx}^{loc} \), one finds from Eqs. (3.3) and (3.6) \( \tilde{\rho}_{yx} = -\tilde{\rho}_{xy} \). If there is no mobility modulation, \( r(x) \equiv 0 \), one has \( \langle 1 / \rho_{yy}^{loc} \rangle = \sigma_0 / 1 = \rho_0 \) the Drude conductivity of the homogeneous 2DES. Since \( \langle \omega(x) \rangle = \omega_c \tau_{tr} \), Eqs. (3.3), (3.4), and (3.5) reduce to the corresponding components of the Drude resistivity for the homogeneous system. Only \( \tilde{\rho}_{xx} = \langle \rho_{xx}^{loc} \rangle + \tau_{tr}^2 \langle (\omega(x)^2 \rho_{xx}^{loc}(x) - \omega_c^2) \sigma_0 \rangle \) is different from the Drude value \( 1/\rho_0 \), and shows a positive magnetoresistance (\( \propto \omega_m^2 \)). Note that this result differs from the simple average \( \langle \rho_{xx}^{loc} \rangle \), which does not yield any magnetoresistance. If only the mobility is modulated, one has \( \langle 1 / \rho_{yy}^{loc} \rangle = \sigma_0 \langle \tau_{tr}(x) \rangle / \tau_{tr} \) and only \( \tilde{\rho}_{yy} \) deviates from the corresponding Drude value for the unmodulated system. If all types of modulation are present, all components of the effective resistivity tensor deviate from the Drude values. For weak modulations, one obtains up to second order in the modulation amplitudes

\[ \tilde{\rho}_{xx} - \frac{\rho_{xx}^D}{\rho_0} = \frac{\langle \left( \frac{V}{E_F} + \omega_c \tau_{tr} V/E_F + \omega_m \tau_{tr} \right)^2 \rangle}{\langle \left( \omega_c \tau_{tr} V/E_F \right)^2 \rangle}, \] (3.7)

\[ \tilde{\rho}_{yy} - \frac{\rho_{yy}^D}{\rho_0} = \frac{\langle \left( \omega_c \tau_{tr} V/E_F + \omega_m \tau_{tr} \right)^2 \rangle}{\langle \left( \omega_c \tau_{tr} V/E_F \right)^2 \rangle}, \] (3.8)

\[ \tilde{\rho}_{yy} - \frac{\rho_{yy}^D}{\rho_0} = \frac{\langle \left( \tau_{tr} r_{tr} \right)^2 \rangle}{\langle \tau_{tr} r_{tr} V/E_F \rangle}. \] (3.9)

We want to emphasize that, in order to obtain these results, it was important to satisfy the equation of continuity. If instead we would have taken naively the spatial average of the local conductivity tensor, the resulting effective resistivity tensor would not have the correct structure. A pure electric modulation would have no effect at all. A pure magnetic modulation would affect \( \tilde{\rho}_{xx} \) and \( \tilde{\rho}_{yy} \) in the same manner. Although the leading order correction to the \( xx \)-component would agree with Eq. (3.7), the corresponding correction to the \( yy \)-component would disagree with the correct result (3.9).

### B. Nonlocal calculation

The local limit applies to systems in which the electronic mean free path is much shorter than the modulation period. In order to exhibit the Weiss oscillations, the system must be in the opposite limit, with a mean free path considerably larger than the modulation period. For such systems, we consider the current density (2.8), resulting from the solution of Boltzmann’s equation (2.7), and define the effective resistivity tensor by Eq. (3.1). We want to emphasize that the spatial average affects only \( j_y(x) \), whereas \( j_x \) must be independent of \( x \) by virtue of the continuity equation.

By formal, but exact manipulations of the Boltzmann equation we show in Appendix A that the tensor inverse of \( \tilde{\rho} \) can be written as

\[ \tilde{\delta} = \delta^D + \rho_0 \delta^D \hat{\Gamma} \delta^D \] (3.10)

where \( \delta^D \) is the Drude conductivity tensor of the corresponding homogeneous system, and \( \hat{\Gamma} \) has the components

\[ \Gamma_{xx} = -\langle \tau_{tr} r_{tr} V/E_F \rangle - G_{xx}, \quad \Gamma_{xy} = -G_{xy} \]
\[ \Gamma_{yy} = +\langle \tau_{tr} r_{tr} V/E_F \rangle + G_{yy}, \quad \Gamma_{yx} = +G_{yx}, \] (3.11)

with
To obtain the conductivity correct to second order, we replace the square bracket in Eq. (3.15) by unity.

and \( \chi^\nu \) is the solution of the modified Boltzmann equation

\[ [D - C] \chi^\nu = g^\nu \]  \tag{3.14} \]

Then, by tensor inversion, \( \tilde{\sigma}^{-1} = \tilde{\rho} = \rho^D + \Delta \tilde{\rho} \), we obtain from Eq. (3.10)

\[ \Delta \tilde{\rho}/\rho_0 = - \hat{\Gamma} \left[ 1 + \rho_0 \sigma^D \hat{\Gamma} \right]^{-1} \cdot \]  \tag{3.15} \]

Explicit matrix inversion and multiplication finally leads to the relatively simple result

\[ \frac{\Delta \tilde{\rho}}{\rho_0} = - \left[ \hat{\Gamma} + \frac{\det \hat{\Gamma}}{1 + \omega_0^2 \hat{\Gamma}_{xx}} \rho_0 \right] \frac{1}{D}, \]  \tag{3.16} \]

with \( \det \hat{\Gamma} = \Gamma_{xx} \Gamma_{yy} - \Gamma_{xy} \Gamma_{yx} \) and

\[ D = 1 + \Gamma_{xx} + \Gamma_{yy} + \omega_0 \hat{\Gamma}_{xx} (\Gamma_{xy} - \Gamma_{yx}) + \det \hat{\Gamma} \left[ 1 + \omega_0^2 \hat{\Gamma}_{xx} \right] \]  \tag{3.17} \]

If \( \Gamma_{xy} \) or \( \Gamma_{yy} \) is the only nonzero matrix element of \( \hat{\Gamma} \), Eq. (3.16) reduces to (A.10) or (A.15), respectively, i.e., either only \( \Delta \rho_{xx} \) or only \( \Delta \rho_{yy} \) is nonzero, and we obtain the same tensor structure as in the local limit. As shown in Appendix A, these results can be derived without the explicit formal result (3.16).

We want to emphasize that only the global transport time \( \hat{\tau}_x \), defined below Eq. (2.12), enters these considerations about the structure of the effective resistivity tensor, not any further details of the scattering mechanisms. The latter enter, of course, the solutions of Eq. (3.14) and determine, according to (3.12), the numerical values of the tensor components of \( \hat{\Gamma} \).

\[ G_{\mu\nu} = \frac{2\pi^2}{v_F^2} \left[ \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} g^\mu \chi^\nu \right] \]  \tag{3.12} \]

Here

\[ g_x = \frac{v_F^2}{2} \frac{dV}{dx} + \omega_m v_y, \quad g_y = -r_{tr} v_y, \]  \tag{3.13} \]

and \( \chi^\nu \) is the solution of the modified Boltzmann equation

The latter enter, of course, the solutions of Eq. (3.14) and determine, according to (3.12), the numerical values of the tensor components of \( \hat{\Gamma} \).

IV. ANALYTIC RESULTS

A. Weiss oscillations

If the modulation is weak, in the sense that the forces due to the electric and the magnetic modulations are weak as compared to the Lorentz force due to the average magnetic field \( B_0 \), a perturbation analysis with respect to the amplitudes of the modulation is possible. To obtain the conductivity correct to second order, we need, according to Eq. (3.12), \( \chi^\nu \) to first order, and we can replace the square bracket in Eq. (3.15) by unity. Since the right hand side of Eq. (3.14) is already of first order, we should neglect modulation effects on \( C \) and on \( D \), i.e. replace in Eq. (2.4) \( v(x) \) with \( v_F \) and omit \( \omega_m(x) \) and \( \omega_r(x, \varphi) \). In this approximation the different Fourier coefficients of the expansions \( V(x) = \sum_{q \neq 0} V_q \exp(iq x) \), and similar for \( \omega_m(x) \) and \( r_{tr}(x) \) with coefficients \( \omega_q \) and \( r_q \), respectively, do not mix in Eq. (3.14) and lead to uncoupled integro-differential equations of the variable \( \varphi \). We now follow Beenakker\(^{10}\) and consider only isotropic scattering, i.e. we take \( \gamma_0^{(m)} = 1 \) and \( \gamma_0^{(m)} = 0 \) for \( m \neq 0 \) in Eqs. (2.11), and define the relaxation time \( \tau = \hat{\tau} \) by the spatial average \( 1/\tau = \langle 1/\tau(x) \rangle \). Then we can solve these equations analytically and obtain

\[ G_{xx} = \frac{1}{2} \sum_{q \neq 0} \left\{ |\lambda q V_q/E_F|^2 Q_{Q e}^{ee} \right. \]

\[ - 4 \lambda q \tau \omega_q V_{-q}/E_F Q_{Q m}^{em} + 2 \omega_q^2 Q_{Q m}^{mm} \bigg\}, \]  \tag{4.1} \]

\[ G_{xy} = \sum_{q \neq 0} \tau r_{tr-q}(\lambda q V_q/E_F)Q_{Q e}^{em} - 2 \tau r_q Q_{Q m}^{mm}, \]  \tag{4.2} \]

\[ G_{yy} = - G_{xy}, \quad G_{yx} = 2 \sum_{q \neq 0} |\tau r_q|^2 Q_{Q m}^{mm}, \]  \tag{4.3} \]

with \( Q_{Q e}^{ee} = S_q^{(0,0)}/N_q, \quad Q_{Q e}^{em} = S_q^{(0,1)}/N_q, \) and \( Q_{Q m}^{mm} = S_q^{(1,1)} + (S_q^{(0,1)})^2/N_q, \) where \( N_q = 1 - S_q^{(0,0)} \) and

\[ S_q^{(\mu,\nu)} = \sum_{m=-\infty}^{\infty} \frac{J_m^{(\mu)}(Rq)J_m^{(\nu)}(Rq)}{1 + (\mu \omega_0 \tau)^2}, \]  \tag{4.4} \]

where \( \lambda = v_F \tau \) is the mean free path, \( R = v_F/\omega_c \) is the cyclotron radius, and \( J_m^{(0)}(z) = J_m(z) \) and \( J_m^{(1)}(z) = J'_m(z) \) are the Bessel functions and their derivatives. For \( \omega_q \equiv 0, \quad r_q \equiv 0, \) and \( V_\varphi = \delta_{q,2\pi/a} V_{rms}/\sqrt{2} \) this reduces to Beenakker’s result (10). The term \( S_q^{(0,0)} \) in the denominator \( N_q \) arises from the back-scattering term in the collision operator, which describes relaxation towards the local equilibrium. It is thus a consequence of the equation of continuity.

1. Clean limit, \( \omega_c \tau \gg 1 \)

In the limit \( \omega_c \tau \gg 1 \), only the term with \( m = 0 \) survives in the sum of Eq. (4.1), and in Eq. (A.11), the term \( \Gamma_{xx} \) may be neglected, so that \( \Delta \rho_{xx}/\rho_0 \approx G_{xx} \). Then, rearranging terms with \( q + \nu \), the curly bracket in Eq. (4.1) can be replaced by \( |\lambda q S_q/E_F|^2 \) with \( S_q = V_q J_0(Rq) + (k_F/q) \sqrt{\omega_q} J_1(Rq) \),

\[ \Delta \rho_{xx}/\rho_0 \approx \frac{1}{2} \sum_{q \neq 0} \frac{|\lambda q S_q/E_F|^2}{1 - J_0^2(Rq)}. \]  \tag{4.5} \]

This result is, apart from the denominator \( 1 - J_0^2(Rq) \), in agreement with a recent simplified calculation (13) based on an evaluation of Chambers’ formula for the conductivity in terms of the drift velocity of cyclotron orbits.
That calculation corresponds to a naive relaxation time approximation in Boltzmann’s equation with relaxation towards the total instead of the local equilibrium distribution function, and thus violates the equation of continuity for the modulated system. Indeed, the nontrivial denominator in the present calculation results from the back-scattering term in Eq. (2.10).

In Fig. 1 we demonstrate the range of validity of several approximations to the results (4.1) - (4.4) for the special case of a simply harmonic electrical modulation $V(x) = V_0 \cos qx$. Figure 1(a) shows, for several values of the mean free path $\lambda$, the quantity $S_0^{(0,0)}$ of Eq. (4.4), which determines the denominator of the magnetoresistance corrections. Figures 1(b) and (c) show results for the magnetoresistivity, calculated from Eq. (4.1), with different normalizations which are suitable to discuss the local limit, (b), and the limit of infinite $\lambda$, (c), respectively. The curves are plotted versus the inverse cyclotron radius, which is proportional to the average magnetic field $B_0$. In the limit $\lambda \to \infty$, $S_0^{(0,0)} = J_0^2(Rq) \approx (2/\pi Rq) \cos^2(\pi Rq - \pi/4)$ decreases with decreasing $B_0$, with an infinite number of zeroes and with values at the maxima, which approach zero linearly with $B_0$. The same then holds for the magnetoresistivity curve obtained from Eq. (4.5) and for the thin line in Fig. 1(c), which corresponds to the simplified result of Refs. [14,15] and is calculated from Eq. (4.3) by replacing the denominator by 1. For $1/Rq < 0.1$ the denominator of Eq. (4.5) is close to unity $(S_0^{(0,0)} < 0.1)$, and the thin line approximates the result of Eq. (4.3) quite well. At higher magnetic fields, $1/Rq > 0.2$, the result of Eq. (4.3) agrees well with the magnetoresistivity curve obtained for $\lambda_0 = 30$ in Fig. 1(c). For a sample with a given value of the mean free path, Eq. (4.5) does not hold uniformly for all values of the average magnetic field. With decreasing $B_0$ values, $\lambda_0/Rq = \omega_c \tau$ becomes smaller and more and more terms contribute to the sum in Eq. (4.4). As a consequence, the minima of the Weiss oscillations of $S_0^{(0,0)}$ and $\Delta \rho_{xx}$ are risen to positive values, the amplitudes of the oscillations are increasingly suppressed with decreasing $B_0$ values, and, in the limit $B_0 \to 0$, a finite positive value is approached with zero slope.

Whereas $S_0^{(0,0)}$ in the limit of large $B_0$ ($Rq \to 0$) approaches 1 for all values of $\lambda$ [since $J_0(0) = 1$ and $J_m(0) = 0$ for $m \neq 0$], it shows an overall increase with decreasing $\lambda$ for all values of $Rq$, and approaches 1 uniformly in $B_0$ for $\lambda_0 \to 0$ [since $\sum_{q} \int J_n^2(z) = 1$]. Thus, the denominator $N_0$ in Eqs. (4.1) - (4.3) becomes most important in the local limit $\lambda_0 \to 0$ and in the “quasi-local” limit $Rq < 1$, where the cyclotron radius $R$ becomes smaller than the period $a = 2\pi/q$ of the modulation. Figure 1(b) shows the magnetoresistivity data for $\lambda_0 = 9$ and $\lambda_0 = 4$ in a representation which is more convenient for the discussion of the local limit $\lambda_0 \to 0$. For $\lambda_0 = 1$, the corresponding result approximates already the local result of Eq. (4.4), $\Delta \rho_{xx}/\rho_0 = (1/2)(V_0/E_F)^2[1 + (\lambda/R)^2]$, which is indicated by the dash-dotted line.

In the “quasi-local” high magnetic field limit, $Rq \ll 1$, the denominator becomes 1 $- J_0^2(Rq) \approx (Rq)^2/2 \ll 1$, so that, for $V(x) = V_0 \cos qx$, we obtain a quadratic magnetoresistance,

$$\Delta \rho_{xx}/\rho_0 \approx (\omega_c \tau)^2(V_0/E_F)^2/2, \quad (Rq \ll 1), \quad (4.6)$$

whereas the calculation of Ref. [15] yields a saturation for large $\omega_c$, $\Delta \rho_{xx}/\rho_0 \approx (\lambda q)^2(V_0/E_F)^2/4$, which becomes small $\propto B_0^{-2}$. We want to emphasize that, in this “quasi-local” limit, the $B_0$ dependence of the magnetoresistance obtained from Eq. (4.3) agrees with that of the local approximation (2.3).

Figure 2 shows the predictions of Eqs. (4.1) - (4.4) for the case when two types of modulations are simultaneously present. In Fig. 2(a) an electric and a magnetic cosine modulation of comparable effective strengths are considered. The dash-dotted line refers to a pure electric and the solid line to a pure magnetic modulation, whereas the dashed lines refer to a combination of both. For fixed amplitudes of the electric and the magnetic cosine modulations, the resulting magnetoresistivity curve depends strongly on the phase difference between both. The long-dashed line is Fig. 2(a) is obtained for in-phase modulations, and the short-dashed line for a phase shift of a half period. A systematic investigation of the interference effects occurring due to the superposition of an electric and a magnetic modulation has been given within the simplified treatment of Ref. [15]. Since for $\lambda_0 > 20$ this treatment is qualitatively correct in the regime of Weiss oscillations, we will not repeat this discussion here.

Figure 2(b) shows the components of the magnetoresistivity tensor for a combination of electric and mobility modulations. We assume here in-phase cosine modulations for the electrostatic potential energy and the scattering rate of the electrons, and comment on opposite phases below. According to Eqs. (4.3) - (4.4), $\Delta \rho_{xx}$ is determined by the quantity $Q_{ee}^\tau$, $\Delta \rho_{yy}$ by $Q_{mm}^\tau$, and

$$\Delta \rho_{xy} = -\Delta \rho_{yx} \approx Q_{em}^c, \quad (4.7)$$

all of which contain different combinations of Bessel functions and their derivatives. For $\lambda q > 20$, in the regime of Weiss oscillations $Q_{ee}^\tau$ is proportional to the square of $J_0(Rq)$ and $Q_{mm}^\tau$, which also determines the results of a pure magnetic modulation, is proportional to the square of the derivative $J_0'(Rq)$. This leads to antiphase oscillations of $Q_{ee}^\tau$ and $Q_{mm}^\tau$ as depicted in Fig. 2(a) by the dash-dotted and the solid lines, respectively. However, according to Eqs. (4.1),

$$\Delta \rho_{xx}/\rho_0 \approx (\omega_m^2 \tau^2)/2, \quad (Rq \ll 1),$$

whereas Ref. [15] yields $\Delta \rho_{xx}/\rho_0 \approx (\lambda q)^2(\omega_m^2/\omega_c^2)/4$, which becomes small $\propto B_0^{-2}$. We want to emphasize that, in this “quasi-local” limit, the $B_0$ dependence of the magnetoresistance obtained from Eq. (4.3) agrees with that of the local approximation (2.3).
explain the early experiments \[1\], as mentioned in the Introduction. In the situation considered in Fig. 2, \(Q^m\) is approximately given by the derivative of \(Q^e\) with respect to \(Rq\). Thus, the mobility modulation cannot explain the anti-phase oscillations of \(\Delta \rho_{yy}\) observed in the early experiments \[1\], as mentioned in the Introduction. In the situation considered in Fig. 2, \(Q^m\) is approximately given by the derivative of \(Q^e\) with respect to \(Rq\). Thus, the mobility modulation cannot explain the anti-phase oscillations of \(\Delta \rho_{yy}\) observed in the early experiments \[1\], as mentioned in the Introduction.

We do not show a figure analog to Fig. 3 (b) for a superposition of a magnetic and a mobility modulation, since in this case all components of the magnetoresistivity tensor are determined by \(Q^m\). As a consequence, \(\Delta \rho_{yy}\) would show the same appearance as \(\Delta \rho_{xx}\), i.e. as the solid line in Fig 2 (a), and \(\Delta \rho_{yy}\) would appear as \(-\Delta \rho_{xx}\), i.e. with the opposite phase. In contrast to the case of electric and mobility modulation, in the case of magnetic and mobility modulation there is no constant offset in the diagonal components of the resistivity tensor, and \(\Delta \rho_{xy}\) does not assume negative values.

2. Damping of Weiss oscillations

The value of \(S_q^{(0,0)}\) for weak magnetic fields, \(Rq \geq 2\), depends strongly on \(\lambda q\), and Weiss oscillations are washed out for \(1/Rq < 0.5/\lambda q\), i.e., for \(\omega c \tau < 0.5\). If the mean free path is too small, \(\lambda q \leq 2\) (here we have in mind that the period of the modulations, and thus \(q\) is fixed), no oscillations survive, since too many Bessel functions contribute to the sum in Eq. (4.4) \((\omega c \tau < 1)\) in the \(1/Rq\) regime where oscillations can be expected.

In this regime the calculation of Ref. \[15\] is not applicable, since it assumes \(\lambda \gg R\).

We can also re-obtain the local limit \(\lambda q \to 0\) from Eqs. (4.1) - (4.3), if we expand the Bessel functions in Eq (4.4) for small arguments, \(Rq \to 0\), and keep only the leading orders. This means that we consider the limit of large modulation period, but make no assumption about the magnitude of \(\omega c \tau\). In this limit, only the terms with \(m = 0\) and \(m = \pm 1\) contribute to Eq. (4.4), and we exactly recover Eqs. (4.7) - (4.9), with \(\tau\) instead of \(\tau_r\). To obtain this result correctly, it is essential to calculate the denominator (which vanishes in this limit) consistently up to the order \((Rq)^2\), which includes contributions from the Bessel functions \(J_0\) and \(J_1\). This demonstrates nicely that the denominator in Eqs. (4.1) - (4.3) indeed is important in order to obtain the correct local-limit results, Eqs. (4.7) - (4.9), which were derived by exploiting explicitly the equation of continuity.

B. Low-field magnetoresistance

If the average magnetic field \(B_0\) becomes too small, the analytic solution of Eq. (4.14) for small modulation amplitudes is no longer applicable. It was obtained under the assumption, that for the calculation of \(\chi_\nu\) the modulation contributions \(\omega_m(x)\) and \(\omega_c(x, \varphi)\) to the drift operator (2.9) could be neglected as compared to the average cyclotron frequency \(\omega_c\). Physically this means that the modulations modify the cyclotron motion in the average magnetic field only weakly and lead essentially to a drift of the cyclotron orbits. If however, for a given modulation, \(B_0\) becomes smaller than a critical \(B_{\text{crit}}\) (or \(B_{\text{crit}}\)), this is no longer true and other types of orbits ("channeled orbits") occur which are confined in the \(x\)-direction to a single period of the modulation superlattice. The effect of these orbits, which lead to a positive magnetoresistance at small \(B_0\), is not included in Eqs. (4.1) - (4.3).

1. Classification of orbits

To characterize the channeled orbits and to elucidate their importance for the magnetoresistance, we proceed as follows. We note that, for a one-dimensional modulation in \(x\)-direction, there exists a second conserved quantity in addition to the energy, Eq. (2.3). Integrating the \(y\)-component of Newton’s equation (2.1), we can write

\[
v_y = \frac{e}{mc} \left[ x B_0 + \int_0^x dx’ B_m(x’) \right] = \frac{p_y}{m},
\]

where \(p_y\) is a constant of motion, which has the meaning of a canonical momentum. Solving for \(v_y\) and inserting this into Eq. (2.3), we obtain for the projection of the orbit on the \(x\)-direction

\[
(m/2)v_x^2 + \tilde{V}(x) = E_F,
\]

with the effective one-dimensional potential

\[
\tilde{V}(x) = V(x) + \frac{m}{2} \omega_c^2 \left[ x - x_0 + \int_0^x dx’ \frac{B_m(x’)}{B_0} \right]^2.
\]

(4.10)

Here we assume \(B_0 \neq 0\), and the constant of motion is written as \(x_0 = -cp_y/eB_0\), which in the absence of modulations is the \(x\) coordinate of the center of the cyclotron orbit. For each value of \(x_0\) one obtains a different location of the effective potential \(\tilde{V}(x)\) on the \(x\)-axis, and thus a different orbit located between turning points, at which \(v_x = 0\). If \(x\) is a turning point, \(\tilde{V}(x) = E_F\), Eq. (4.10) yields two possible values for the constant \(x_0\):

\[
x_0^\pm(x) = x + \int_0^x dx’ \frac{B_m(x’)}{B_0} \pm R \sqrt{1 - \frac{V(x)}{E_F}}.
\]

(4.11)
Since \( \tilde{V}(x) \leq E_F \) holds for any orbit passing through a
given position \( x \), only orbits with \( x_0 \) values in the interval
\( x_0^- (x) \leq x_0 \leq x_0^+ (x) \) can pass through \( x \). (We specify
orbits only up to an arbitrary shift in \( y \) direction.) On
the other hand, a given value of \( x_0 \) determines an or-
bit between turning points \( x_1 \) satisfying \( x_0^- (x_1) = x_0 \) and
their right turning point \( x_\tau \) satisfying \( x_0^- (x_\tau) = x_0 \). For
sufficiently smooth modulation (cf. Appendix B) and
large \( B_0 \) this is the only type of orbits, as we demon-
strate in the left panels of Fig. 3 for an electric cosine
modulation. In the limit of vanishing modulation these
“drifting orbits” reduce to cyclotron orbits, i.e., to cir-
cles, and \( x_\tau - x_1 \to 2R \). The effect of the modulation in
\( x \)-direction is to deform the circular cyclotron orbits and
to superimpose on the cyclotron motion a drift velocity
in \( y \)-direction. The drift velocity depends on the position
of the orbit in the modulation field and may vanish for
highly symmetric positions. In the left panel of Fig. 3
one orbit is nearly symmetric with respect to the poten-
tial maximum at \( x = 0 \), another one nearly symmetric
about the minimum at \( x = 1.5a \). The small distance
between neighboring loops of the orbits (three loops are
shown) indicates a low drift velocity.

Orbits of the second type, which we call “channeled or-
bits”, occur if the curves \( x_0^+ (x) \) have local extrema, i.e.,
for sufficiently strong modulation or weak magnetic field
\( B_0 \). For “channeled orbits” we require that their turn-
ing points both satisfy either \( x_0^- (x_1) = x_0 (x_\tau) = x_0 \) or
\( x_0^- (x_1) = x_0^- (x_\tau) = x_0 \), i.e. in plots like those in the up-
er panels of Fig. 3, both turning points are located either
on \( x_0^+ (x) \) or on \( x_0^- (x) \). These channeled orbits (see mid-
dle and right panels of Fig. 3) are confined to a single pe-
riod \( a \) of the modulation and have no counterparts in the
unmodulated system. For an electric modulation they oc-
cur near potential minima. For a magnetic modulation
they occur where the total magnetic field \( B_0 + B_m (x) \nchanges sign 21.

Each value \( x_0 \) of the constant of motion determines
uniquely a single drifting orbit (of course, modulo a shift
in \( y \)-direction). Channeled orbits may occur with the
same \( x_0 \) value, however only in intervals of the \( x \) axis
which are not entered by that drifting orbit. At a given
position \( x \), each allowed \( x_0 \) value characterizes uniquely a
single orbit through \( x \). In the limit \( B_0 = 0 \), drifting orbits
degenerate into wavy trajectories with periodic velocity
\( \mathbf{v}(x + a) = \mathbf{v}(x) \) and nonzero mean velocity \( \langle v_x \rangle \n- direction. The mean value in \( y \)-direction is \( \langle v_y \rangle = p_y/m \n(note that \( x_0/R = -p_y/mE_F \), and for pure electric mod-
ulation \( v_y = p_y/m \) is constant). In this limit two trajec-
tories with \( \langle v_x \rangle > 0 \) and \( \langle v_x \rangle < 0 \) occur with the same
\( \langle v_y \rangle \), as indicated by the dashed lines in the lower right
panel of Fig. 3.

If \( x_0 \) coincides with the value of a local maximum of
\( x_0^+ (x) \) or a local minimum of \( x_0^- (x) \), the orbits near the
corresponding turning point \( x_1 \) become critical in the
sense that they do not reach this turning point, but ap-
proach in the \( x - y \) plane the straight line \( x = x_1 \) asymp-
totically.

2. Effect of channeled orbits

As has been emphasized by Beton et al. [4], the channe-
led orbits are responsible for the strong positive magne-
toresistance of modulated systems in weak perpendicular
magnetic fields. In Appendix 3 we calculate the fraction
\( f_{co}(B_0, V_0) \) of the phase space covered by channeled or-
bits, as a function of the (average) magnetic field \( B_0 \) and
the modulation strengths \( V_0 \), for two electrostatic mod-
ulation models. For the smooth cosine modulation with
a fixed \( V_0 \), channeled orbits exist only at sufficiently low
\( B_0 < B_{crit} (V_0) \), whereas for a discontinuous step modu-
lation skipping orbits survive at arbitrarily high magnetic
fields. For both models the fraction of channeled orbits
at zero magnetic field, \( f_{co}(0, V_0) \), increases at low mod-
ulation strengths proportional to \( |V_0/E_F|^{3/2} \). We also
sketch, for \( B_0 = 0 \) and in the nonlocal limit of large
mean free path, a calculation in the spirit of Ref. 13
which shows that the resistivity correction increases as
\( |V_0/E_F|^{3/2} \) with the modulation amplitude. This indi-
cates that, for \( B_0 = 0 \) and weak modulation, no expa-
nion of the transport coefficients in a power series of \( V_0 \npossible. The same is true at finite magnetic field if
channeled orbits exist. Indeed, for the step model which
allows for channeled orbits at arbitrary strong \( B_0 \), the
sum over the Fourier coefficients in the second-order for-
ma 1,4 for the magnetoconductivity diverges for all
values of \( B_0 \), indicating that such an expansion does not
exist 4.

To include the effect of channeled orbits on the magne-
toresistance properly, we have solved Boltzmann’s equa-
tion numerically. The results show in the limit of long
mean free path and zero magnetic field the same
\( |V_0/E_F|^{3/2} \) behavior as obtained analytically in the sim-
plified approach, and will be presented in the next sec-

V. NUMERICAL RESULTS

A. Some technical details

In order to cover the whole magnetic field range, we have
solved Eq. (2.11) numerically, using Fourier expan-
sions for the dependences on both \( x \) and \( \omega \). According
to Eq. (2.11), the collision operator, Eq. (2.10), is di-
gonal in the Fourier representation. The drift term,
Eq. (2.9), becomes simple for strictly harmonic modu-
lations, provided that the electric modulation is weak
enough and \( v(x) \) can be approximated by \( v_F \). Then the drift term couples only neighboring Fourier coefficients, and Eq. (8.1) leads to an infinite set of linear equations for the Fourier coefficients with a sparsely occupied matrix. Truncating the infinite set by considering only the lowest \( N_x \) and \( N_\varphi \) Fourier coefficients for the \( x- \) and the \( \varphi- \)dependence, respectively, we could exploit the sparse nature of the system matrix and handle approximations of the infinite set with large dimensions \( N_x \cdot N_\varphi \). The number of Fourier coefficients needed to obtain converged results increases with increasing modulation strengths and with increasing mean free path. The convergence in the regime of Weiss oscillations is much better than in the low-\( B_0 \) regime, where the channeled orbits are important. For the pure electric cosine modulation with \( V_0/E_F = 0.02 \) we needed \( N_x = 13 \) and \( N_\varphi = 550 \) to achieve convergence in the low-magnetic-field regime \( 5 \cdot 10^{-5} < 1/qR < 0.02 \), whereas \( N_x = 4 \) and \( N_\varphi = 450 \) was sufficient for \( 1/qR > 0.5 \). The reason for the poor convergence at low \( B_0 \) is that the distribution function calculated from Eq. (3.14) develops very sharp and rapidly changing structures near the angles \( \varphi = \pi/2 \) and \( 3\pi/2 \), originating from the turning points of the channeled orbits. Owing to these numerical problems, our Fourier expansion approach is reliable only for weak modulations (\( V_0/E_F \leq 0.2 \)).

**B. Relaxation time approximation**

A typical low-magnetic-field result for the pure electric modulation \( V(x) = V_0 \cos qx \) with \( V_0/E_F = 0.02 \) is shown in Fig. 4. The solid line is the result of our numerical calculation. The dotted line shows the result of the analytical approximation (4.1). The dash-dotted line indicates the contribution of the channeled orbits and is obtained as follows. For weak modulation and \( \omega_\varphi \tau \gg 1 \), the correction to the Drude resistivity tensor can be written as \( \Delta \rho_{xx}/\rho_0 \approx \omega_\varphi^2 \tau^2 \Delta \sigma_{yy}/\sigma_0 \). The contribution of channeled orbits may be estimated by their mean square drift velocity \( \Delta \sigma_{yy}/\sigma_0 = \langle v_y^2 \rangle / \langle v_F^2 \rangle / 2 \), cf. Eq. (3.2). Approximating \( v_y^2 \approx v_F^2 \), this becomes \( \approx 2 f_{cc} (B_0, V_0) \), i.e., twice the fraction of channeled orbits at the Fermi energy (see appendix E). The dash-dotted line in Fig. 4 represents \( \Delta \rho_{xx}/\rho_0 \approx 2 \omega_\varphi^2 \tau^2 f_{cc} (B_0, V_0) \) with an additional vertical shift to match our numerical result at \( B_0 = 0 \). The vertical straight line indicates the critical magnetic field \( B_{\text{crit}}^{\text{Brton}} = q_0 V_0 / e v_F \) at which the channeled orbits die out. Our numerical result exhibits a maximum at the magnetic field value \( B_p \approx 0.8 B_{\text{crit}}^{\text{Brton}} \). This is in very good agreement with recent experiments [24], although the numerical factor 0.8 should depend on the specific form of the potential modulation. The corresponding results for a pure magnetic modulation look qualitatively very similar and are not shown.

The inset of Fig. 4 shows our numerical results for strong purely magnetic modulations of the form \( B_m(x) = B_m^0 \cos qx \). The modulation strength is given in units of \( 1/qR_m = a e B_m^0 / (2 \pi m e v_F) \), \( \Delta \rho_{xx}/\rho_0 \) in units of \( \omega_m^2 \tau^2 = \lambda^2 / R_m^2 \). With increasing modulation strength (different curves in the inset of Fig. 4), the extent of the positive magnetoresistance regime at low \( B_0 \) values increases and \( \Delta \rho_{xx} \) reaches a maximum at \( B_p \approx 0.7 B_m^0 \) (note that \( B_{\text{crit}} = B_m^0 \), see appendix E). The magnetoresistivity at \( B_p \) increases with increasing modulation strength, and the low-field Weiss oscillations are successively suppressed (the fluctuations in the curve for \( 1/qR_m = 0.34 \) are due to numerical errors, which become larger with increasing modulation strength). Again the dependence of the magnetoresistance on the modulation strength is very similar for the electric cosine modulation and will not be shown explicitly. For large mean free path and \( V_0/E_F > 0.2 \), fluctuations of the calculated curves indicate convergence problems. We found, however, that the relation \( B_p \approx 0.8 B_{\text{crit}}^{\text{Brton}} \), i.e., a linear increase of \( B_p \) with the modulation amplitude \( V_0 \), is reasonably well satisfied up to \( V_0/E_F \approx 0.5 \), in agreement with Ref. 1 and with a recent modification of Středa’s semiclassical approach [35]. The peak values of the magnetoresistivity can be well fitted by

\[
\Delta \rho_{xx}(B_p)/\rho_0 = 7.01 (V_0/E_F)^2 + 54.7 (V_0/E_F)^3. \tag{5.1}
\]

Whereas in the regime of Weiss oscillations (\( B_0 > 3 B_{\text{crit}} \)), \( \Delta \rho_{xx} \) is proportional to \( (V_0/E_F)^2 \) and the analytic approximation (1.1) is valid, the peak value at \( B_p \) is dominated by the third order term in Eq. (5.1) for \( V_0/E_F > 0.15 \). For weak modulation (\( V_0/E_F < 0.15 \)), \( \lambda q \gg 1 \) and \( B_0 = 0 \), our numerical results are consistent with the analytically calculated algebraic dependence \( \Delta \rho_{xx}(0)/\rho_0 \propto (V_0/E_F)^{3/2} \) (see Appendix E). This indicates that, at low magnetic fields where channeled orbits exist, an expansion of the magnetoresistance in powers of the modulation strength is not possible. To check this, we expanded the Boltzmann equation (8.1) in powers of \( V_0/E_F \) and solved it up to the order \( (V_0/E_F)^3 \). In the regime around \( B_p \), the higher order contributions lead to high-frequency oscillations, but we found no indication of convergence or positive magnetoresistance.

**C. Anisotropic scattering**

Experiments in the regime of Weiss oscillations give evidence that the mobility of the 2DES is dominated by scattering of the electrons by remote ionized impurities, which should lead to predominant forward scattering. To investigate the effect of anisotropic scattering, we simulate the angular dependence of the differential scattering cross section in the collision operator, Eq. (2.14), by the simple model

\[
P_p(\varphi) = c + (1 - c) \frac{(pt)^2}{(2p)!} \left( \frac{2 \cos \varphi}{2} \right)^{2p}, \tag{5.2}
\]
where \( c \) controls a constant background \((0 \leq c \leq 1)\), and \( p \) the sharpness of the forward scattering peak. This model is convenient, since it has simple Fourier coefficients, \( \gamma_0 = 1 \), \( \gamma_m = (1 - c)(p^2)/(p + m)((p + m)/(p - m)) \) for \( |m| \leq p \), and \( \gamma_m = 0 \) for \( |m| > p \) [cf. Eq. (2.1)]. According to Eq. (2.12), the ratio of transport time and relaxation time \( \tau \) increases then with increasing \( p \) as \( \tau_{ep}/\tau = (p + 1)/(cp + 1) \).

Keeping \( \tau \) fixed, we have calculated magnetoresistivity curves for several values of the parameters \( c \) and \( p \). We found as a general trend that, with increasing \( \tau_{ep}/\tau \), for both electric and magnetic modulations the values of \( \Delta \rho_{xx}/\rho \) [with \( \rho_{0} = m/(e^2\hbar c \tau_{ep}) \)] increase slightly faster than proportional to \( \tau_{ep}/\tau \). On an absolute scale this means that the values of \( \Delta \rho_{xx}/\rho \) change little near well developed minima but increase strongly at the peaks, resulting in a strong increase of the amplitudes of the Weiss oscillations. In Fig. 4 we have, for increasing \( \tau_{ep}/\tau \), reduced the relaxation time \( \tau \) so that the maximum value of \( \Delta \rho_{xx}/\rho \) near \( 1/qR = 0.26 \) remains unchanged. In this plot higher-index Weiss oscillations are increasingly damped with increasing importance of small-angle scattering. Comparison with typical experiments shows that small-angle scattering is indeed important. Apart from the quantum mechanical Shubnikov-de Haas oscillations at large \( B_0 \), we could nicely fit experimental curves in the whole region \( 1/qR \leq 0.8 \) with typical values of \( \tau_{ep}/\tau \geq 2 \).

**D. Mobility modulation**

Finally, we have considered, within the relaxation time approximation, a mixed electric modulation \( V(x) = V_0 \cos qx \) and mobility modulation \( 1/\tau(x) = \bar{\tau}_x + \tau_m \cos qx \). We choose an in-phase modulations, \( V_0 \tau_m > 0 \). We found that, for \( V_0/EF \ll 1 \) and \( \tau_0 q \gg 1 \), the effect of the mobility modulation on \( \rho_{xx} \) is rather small. This is understandable in the regime of Weiss oscillations, since, according to Fig. 4, \( \Delta \rho_{xx}/\rho_0 \) becomes independent of \( \lambda_B \) for \( \lambda_B \geq 25 \). It is also expected for \( B_0 = 0 \) in the local limit, where \( \Delta \rho_{xx}/\rho_0 \) becomes proportional to \( (V_0/EF) \cdot \tau_r \). (For the corresponding step modulations one has \( \Delta \rho_{xx}/\rho_0 = [1 + (V_0/EF)\tau_r]/[1 - (V_0/EF)^2] \).

In Fig. 4 we show the results for pure electric and pure mobility modulations as solid lines. Whereas the electric modulation enhances \( \rho_{xx} \), the mobility modulation reduces \( \rho_{yy} \). The Weiss oscillations of both resistivity components are in phase, as we have seen already from the analytical solution Eq. (4.2,4.3). For the mobility modulation, there is no positive magnetoresistance at small \( B_0 \) values, since there are no channeled orbits in this case.

Results for the case of mixed modulations are shown as broken lines. For the diagonal resistivities, the changes compared to the pure cases are essentially a positive offset of \( \rho_{xx} \) and a negative offset of \( \rho_{yy} \), as expected from Eqs. (4.11) and (4.16). The enhancement of \( \rho_{xx} \) at small \( B_0 \) values \((R > R_{cr})\) is plausible, since now in the regime of channeled trajectories the mobility is increased. Therefore, the importance of these trajectories and thus the value of the magnetoresistivity is enhanced.

For the mixed case, we obtain also a correction to the Hall resistivity. To accommodate this correction in the same figure, we have subtracted the Drude value, which increases linearly with \( B_0 \). In the regime of Weiss oscillations, \( \Delta \rho_{xy}/\rho \) is roughly proportional to the slope of \( \rho_{xx}/\rho_0 \) as function of \( B_0 \) \( \propto 1/qR \).

**VI. SUMMARY**

Using the linearized stationary Boltzmann equation and including anisotropic scattering in the collision operator, we have investigated the effect of one-dimensional periodic modulations on the resistivity tensor of a two-dimensional electron system in a perpendicular magnetic field \( B_0 \). Even in the relaxation time approximation (for isotropic scattering) we have carefully considered the back-scattering term of the collision operator, which ensures relaxation of the distribution function towards its local equilibrium value in the stationary state, and not to the total equilibrium distribution. This is of principle importance for the modulated systems, since it ensures that the approximation does not violate the equation of continuity, and it has qualitative as well as quantitative consequences for the resistivity tensor.

The qualitative consequences concern the form of the resistivity tensor. In the absence of mobility modulations, electric and magnetic modulations in \( x \)-direction affect only \( \rho_{xx} \), whereas the other components of the resistivity tensor remain those of the unmodulated system. This result, which is not obvious in the presence of a magnetic field, has first been obtained by Beenakker [10] for a pure electric modulation in the relaxation time approximation. In Appendix A we show that this result also holds for mixed electric and magnetic modulations and for anisotropic scattering. Moreover, we show that, in the absence of electric and magnetic modulations, a mobility modulation in \( x \)-direction affects only \( \rho_{yy} \). In Sect. III A we have derived the same tensor structure in the local limit. This more transparent derivation elucidates the important role played by the continuity equation.

Simpler calculations based on an evaluation of the familiar velocity-velocity-correlation formula for the conductivity tensor are not in agreement with these exact results on the structure of the resistivity tensor [11,14,15], although they may yield reasonable approximate results under certain conditions.

For weak modulations, isotropic scattering, and sufficiently high magnetic field \( B_0 \), we followed an approximation scheme proposed by Beenakker [10] and obtained analytic results for the magnetoresistivity tensor (Sect. III A). In the clean limit (mean free path much larger than modulation period) these results reduce to those
of the simpler calculations, apart from a characteristic denominator, which can be traced back to result from the back-scattering term in the collision operator. The absence of this denominator in the simplified treatment indicates that this treatment does not correctly describe the relaxation towards the local equilibrium, and thus violates the equation of continuity. Quantitatively, the deviation of the denominator from unity is not large if the cyclotron radius $R$ (of an electron at the Fermi energy) is larger than the period $a$ of the modulation. Therefore, in the regime of Weiss oscillations the simplified treatment gives reasonable results. However, in the quasi-local regime of high magnetic fields ($R < a$), the denominator becomes small and the simplified treatment becomes wrong. In this regime the correct treatment of the equation of continuity is quantitatively important, see Sect. [V A I].

The analytic approximation, which implicitly assumes weakly perturbed cyclotron motion, breaks down at low magnetic fields, where “channeled orbits” become important [1]. At $B_0 = 0$ and for $\lambda q \gg 1$, these should lead to an interesting algebraic dependence of the resistivity on the modulation strength, as we demonstrate in Appendix B. To include the effect of the channeled orbits, we have solved the linearized Boltzmann equation numerically. The numerical results in the relaxation time approximation (isotropic scattering) give a qualitatively correct description of the experimental result for $\rho_{xx}$ [1], which can be traced back to result from a numerical calculation considering predominant forward scattering (Sect. [V A I]).

In an attempt to find a classical explanation of the anti-phase Weiss oscillations observed in $\rho_{yy}$ [1], we have also investigated a mobility modulation. We obtained, however, only weak in-phase oscillations, both analytically and numerically. Thus, this attempt of a classical explanation of the $\rho_{yy}$ oscillations fails, whereas the quantum treatment provides a convincing explanation of these oscillations as a density-of-states effect [1].

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### APPENDIX A: MOMENTS OF BOLTZMANN’S EQUATION

To investigate the structure of the tensor $\hat{\rho}$, we multiply the linearized Eq. (2.7) with $e^{i\omega t} u(x) \varphi$ and average with respect to both $\varphi$ and $x$ over a full period. Integrating by parts eliminates the derivatives of the distribution function $\phi(x, \varphi)$, see Eq. (2.9), and the collision operator is evaluated using the real and imaginary part of

$$
\int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} e^{i\varphi} C[\phi; x, \varphi] = -\frac{1}{\tau_{tr}(x)} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} e^{i\varphi} \phi(x, \varphi),
$$

(A1)

where only the effective transport scattering rate $1/\tau_{tr}(x)$ enters [cf. Eq. (2.11)]. Multiplying with $\hat{\tau}_{tr}$ and rearranging terms we can write the result as

$$
\hat{\rho}^D \langle \delta j(x) \rangle = E^0 + \Delta,
$$

(A2)

where $\hat{\rho}^D$ is the Drude resistivity tensor of the homogeneous system, and [with $V'(x) = dV/dx$]

$$
\Delta_x = -\langle [V'(x)/E_F] \delta^{loc}(x) + \rho_0 \hat{\tau}_{tr} \omega_m(x) j_y(x) \rangle, \quad \Delta_y = -\rho_0 \tau_{tr} \langle j_x(x) j_y(x) \rangle .
$$

(A3)

Here we have explicitly used that, according to the equation of continuity, $j_x(x) = \langle j_x(x) \rangle$ is spatially constant, and that the spatial average values of $\omega_m(x)$, $V(x)$, and $\tau_{tr}(x)$ vanish.

For the unmodulated system $\Delta = 0$, and Eq. (A2) reduces to the well known Drude form of Ohm’s law with homogeneous current density $\langle \delta j(x) \rangle = \langle j(x) \rangle = j^0$. For the modulated system with the same $E^0$, we may write $\langle \delta j(x) \rangle = j^0 + \langle \delta j(x) \rangle$, so that

$$
\hat{\rho}^D \langle \delta j(x) \rangle = \Delta .
$$

(A5)

To calculate these corrections, it is convenient to write the solution of Eq. (2.7) in the form

$$
\phi(x, \varphi) = \phi^0(\varphi) v(x)/v_F + \rho_0 \hat{\tau}_{tr} \chi(x, \varphi),
$$

(A6)

where $\phi^0(\varphi) = \rho_0 \tau_{tr} v_F u(\varphi) \cdot j^0$ is the corresponding solution for the unmodulated system. This yields

$$
\langle \delta j(x) \rangle = \langle j^0 v^2(x)/v_F^2 + \delta j(x) \rangle ,
$$

(A7)

with

$$
\delta j(x) = \frac{2}{v_F^2} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} v(x) u(\varphi) \chi(x, \varphi) ,
$$

(A8)

and $\chi(x, \varphi)$ satisfies Eq. (2.3) with a modified inhomogeneity which is linear in the modulation quantities $\omega_m(x)$, $V(x)$, and $\tau_{tr}(x)$,

$$
[D - C] \chi = (-v_F e/dx + \omega_m v_y - \tau_{tr} v_x) j^0_x - (\omega_m v_x + \tau_{tr} v_y) j^0_y .
$$

(A9)
where \((v_x, v_y) = v(x)(\cos \varphi, \sin \varphi)\). We may write \(\chi = \chi_1 + \chi_2\), where \(\chi_2\) solves

\[
[D - C] \chi_2 = -v(x) \cos \varphi [r_{tr} j_x^0 + \omega_m j_y^0].
\]  

(A10)

The solution is independent of \(\varphi\), and satisfies

\[
d\chi_2/dx = -[r_{tr} j_x^0 + \omega_m j_y^0].
\]  

(A11)

It does not contribute to the current density, Eq. (A8), but it contributes to the local equilibrium distribution, and thus contributes to the inhomogeneity of Eq. (A3).

\[
\langle dV/dx \rangle_{\phi^{loc}} = \rho_0 \tilde{\tau}_{tr} \left( \frac{dV}{dx} \right)_{\phi^{loc}},
\]

\[
\chi
\]

From Eqs. (A7) to (A12) we obtain

\[
\Delta_x = -\rho_0 \tilde{\tau}_{tr} \left( \frac{V}{E_F} \right) j_x^0 + \frac{2}{v_F^2} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \left( \frac{v_F^2}{E_F} \frac{dV}{dx} + \omega_m v_y \right) \chi_1,
\]

\[
\Delta_y = -\rho_0 \tilde{\tau}_{tr} \left( -r_{tr} \frac{V}{E_F} j_y^0 + \frac{2}{v_F^2} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} r_{tr} v_y \chi_1 \right). \quad \text{(A14)}
\]

Finally, we write \(\chi_1 = \chi_{xx} j_x^0 + \chi_{yy} j_y^0\), where, according to Eqs. (A10) and (A11), \(\chi_{xx}\) (for \(\nu = x, y\)) satisfies Eq. (3.14) with \(g_\nu\) given by Eq. (3.13). We obtain

\[
\Delta = \rho_0 \hat{\Gamma} j^0,
\]  

(A15)

where \(\hat{\Gamma}\) is defined by Eq. (3.11).

From these results we can show that the magnetoresistivity tensor reveals a very simple structure in the two important special cases, where there is either no modulation of the scattering rate, or where only the scattering rate is modulated. Let us first assume that we have an electrical and a magnetic modulation, but no mobility modulation, \(r(x) \equiv 0\). Then \(\Gamma_{xx} = -\Sigma_{xx}\) and all other tensor components of \(\hat{\Gamma}\) vanish, so that \(\Delta_x = \rho_0 \Gamma_{xx} j_x^0\), while \(\Delta_y = 0\). Thus, according to Eq. (A3), \(\langle \delta j_y \rangle = \omega_c \tilde{\tau}_r \langle \delta j_x \rangle\) and \(\langle \delta j_x \rangle = \Gamma_{xx} j_x^0\), with \(\Gamma_{xx} = \Gamma_{xx}/[1 + (\omega_c \tilde{\tau}_r)^2]\). If we now consider the effective conductivity tensor \(\hat{\sigma} = \tilde{\sigma}^{-1}\) of the modulated system, \(\hat{\sigma}^0 + \langle \hat{\delta} \hat{j} \rangle = \sigma_x \vec{E}^0\), we obtain for the components of \(\hat{\sigma}\) in terms of those of the Drude \(\hat{\rho}^D = [\hat{\rho}^D]^{-1}\): \(\sigma_{\mu \nu} = \sigma_{\mu \nu}^D(1 + \Gamma_{xx})\) for \((\mu \nu) = (xx), (xy)\), and \((yx)\), whereas \(\sigma_{yy} = \sigma_{yy}^D[1 - (\omega_c \tilde{\tau}_r)^2]\). Tensor inversion shows that \(\hat{\rho}\) differs from the Drude resistivity tensor \(\rho^D\) of the unmodulated system only in the \(xx\) component,

\[
\Delta \rho_{xx}/\rho_0 = \rho_{xx}/\rho_0 - 1 = -\Gamma_{xx}/[1 + \tilde{\Gamma}_{xx}], \quad \text{(A16)}
\]

while all other components are not modified by the modulation. This result, first pointed out by Beenakker [10] for purely electrostatic modulation within the relaxation time approach, is thus shown to hold also in the presence of a magnetic modulation and anisotropic scattering.

The other simple case is that without electric and magnetic modulation, \(V(x) \equiv 0, \omega_r(x) \equiv 0\), but with a finite mobility modulation \(r_{tr}(x)\). In this case \(\Delta_x = 0\) and \(\Delta_y = \rho_0 \Gamma_{yy} j_y^0\). The same procedure yields now a resistivity tensor, which differs from that of the unmodulated system only in the \(yy\) component,

\[
\Delta \rho_{yy}/\rho_0 = -\Gamma_{yy}/[1 + \Gamma_{yy}/(\omega_c \tilde{\tau}_r)^2]]. \quad \text{(A17)}
\]

In the general case, with mobility modulation and electric or magnetic modulation, we obtain from Eqs. (A12), (A12), and (A13) the effective conductivity tensor given by Eq. (3.11).

**APPENDIX B: CHANNELED ORBITS**

1. **Critical field \(B_{crit}\)**

For a smooth modulation, the condition for the existence of channeled orbits is apparently, that the functions \(x_0(x)\) and \(x_0^+(x)\) have local extrema, i.e., that the equation

\[
1 + \frac{B_m(x)}{B_0} \pm \frac{R}{2} \sqrt{\frac{V'(x)/E_F}{1 - V(x)/E_F}} = 0
\]

(A1)

has a solution. For a pure magnetic modulation of the form \(B_m(x) = B_m^0 \cos qx\), we obviously find \(B_{crit} = B_m^0\). For \(B_0 < B_{crit}\) the total magnetic field \(B(x) = B_0 + B_m(x)\) changes sign, and channeled orbits exist close to lines with \(B(x) = 0\).

For a pure electric modulation of the form \(V(x) = -V_0 \cos qx\), Eq. (A1) is equivalent with a quadratic equation for \(\cos qx\), which has a solution only if \(1 - (qR)^2 + [(qR)^2 V_0/2E_F]^2 > 0\). This yields for the critical cyclotron radius

\[
- \frac{2qR_{crit}}{V_0} = V_0 \left[ \frac{2}{1 - \sqrt{1 - V_0/E_F}} \right].
\]

(B2)

For \(V_0 \ll E_F\), we recover the result \(B_{crit} \approx B_{crit}^{Beton}(V_0) = cqV_0/eF\), which has previously been obtained by Beton et al. [8] from the balance of the average Lorentz force and the maximum electrical force \(qV_0\) due to the modulation. For stronger modulation, the \(B_{crit}\) obtained from Eq. (A1) increases faster than linearly with \(V_0\) and reaches the value \(B_{crit} = \sqrt{2} B_{crit}^{Beton}\) at \(V_0 = E_F\). Then all trajectories are confined to a single period of the potential. For \(B_0 > B_{crit}\), i.e., for \(2R < a\sqrt{2}\pi\), all trajectories are “drifting” orbits, with left turning points on \(x_0^-(x)\) and right turning points on \(x_0^+(x)\), whereas for \(B_0 < B_{crit}\) there exist also “channeled” trajectories having all their turning points either on \(x_0^+(x)\) or on \(x_0^-(x)\).
For the step modulation \( V(x) = -V_0 \text{sign}(\cos qx) \), \( x_0^*(x) \) and \( x_0^-(x) \) are piecewise linear functions with discontinuities at the walls of the potential wells, so that Eq. (B1) cannot be used to determine \( B_{\text{crit}} \). Two types of channeled orbits may exist near a local minimum of \( x_0^-(x) \) (maximum of \( x_0^+(x) \)): traversing trajectories with turning points on opposite walls of a potential well, and skipping orbits which are reflected only by one of the walls. Traversing trajectories exist for sufficiently weak magnetic fields, \( B_0 < B_{\text{crit}}^{\text{step}}(V_0) = \frac{mcF}{eF_{\text{crit}}} \) with

\[
a = 2F_{\text{crit}}^{\text{step}} = \sqrt{1 + V_0/E_F} - \sqrt{1 - V_0/E_F}.
\]  

(B3)

Skipping orbits survive at arbitrarily high magnetic fields. Physically, this is the result of the fact that the singular electric field at a potential discontinuity never becomes negligible as compared to the Lorentz force due to \( B_0 \). For small modulation amplitude we find \( B_{\text{crit}}^{\text{step}}(V_0) = (2/\pi)B_{\text{Barton}}(V_0) \). This indicates that the linear dependence of the critical magnetic field on the (weak) modulation strength holds independently of the detailed shape of the potential modulation, which may, however, affect the numerical prefactor.

A characteristic feature of the channeled orbits is that, even at \( B_0 \neq 0 \), their velocity in \( y \)-direction, \( v_y = \omega_c (x - x_0) \), does not change sign.

2. Number of channeled orbits

We now calculate the fraction of electrons occupying channeled orbits. The electron density \( n_{\text{el}}(x; E_F) = D_0/[E_F - V(x)] \theta(E_F - V(x)) \), i.e., the area density of occupied states at \( x \) with energy below \( E_F \), is proportional to the area \( \pi v^2(x) \) of the circle with radius \( v(x) = v_F [1 - V(x)/E_F] \) in \( v \)-space. The areal density of states at \( x \) with energy at \( E \) is given by the local density of states \( D(x; E) = (2\pi)^2 \rho_{\text{el}}(x/E) dE = D_0 \). Both, \( n_{\text{el}}(x; E) \) and \( D(x; E) \) are independent of the magnetic field \( B_0 \), and so are their spatial average values \( \bar{n}_{\text{el}}(E) \) and \( D(E) \). An electron trajectory through \( x \), at \( B_0 = 0 \), is a channeled orbit, if the energy available for the motion in the \( x \)-direction is smaller than the height of the potential barrier, \( E_F - (m/2)v^2 > V_{\text{max}} \), where \( V_{\text{max}} \) is the maximum value of \( V(x) \). With \( v = v(x) \sin \varphi \), this means \( \sin^2 \varphi > \sin^2 \varphi_0(x; E_F) \), where

\[
\varphi_0(x; E_F) = \arcsin \sqrt{\frac{E_F - V_{\text{max}}}{E_F - V(x)}}.
\]  

(B4)

Thus, the contribution of channeled orbits to the local density of states is \( D_{\text{co}}(x; E) = D_0 [1 - (2/\pi)\varphi_0(x; E)] \), and the density of electrons in channeled orbits (with energy below \( E_F \)) is \( n_{\text{co}}(x; E_F) = D_0 [E_F - V(x)] \theta(E_F - V(x)) \) if \( E_F < V_{\text{max}} \), and

\[
n_{\text{co}}(x; E_F) = 2D_0 \left[ \sqrt{E_F - V_{\text{max}}} \sqrt{V_{\text{max}} - V(x)} \right] + (E_F - V(x)) \arcsin \sqrt{\frac{V_{\text{max}} - V(x)}{E_F - V(x)}}.
\]  

(B5)

if \( E_F > V_{\text{max}} \). For fixed average electron density \( \bar{n}_{\text{el}} \), the Fermi energy is independent of the modulation strength, if \( V_0 < E_F = \bar{n}_{\text{el}}/D_0 \). For \( V_0 > E_F \) and \( B_0 = 0 \), all trajectories become channeled orbits. For the step potential \( V(x) = -V_0 \text{sign}(\cos qx) \), channeled orbits exist only in the potential wells, and Eq. (B3) yields the area density of channeled orbits in the well with \( V_{\text{max}} = V_0 \) and \( V(x) = -V_0 \). For \( V_0 > E_F > V_{\text{max}} \) the requirement of fixed average density \( \bar{n}_{\text{el}} \) yields a constant value of \( E_F \) for which the situation is identical to that at \( V_F = E_F \), with vanishing electron density in the barrier regions. Since the local density of states is \( D_0 \), we normalize, for arbitrary \( V_0 \), the fraction of channeled orbits for the step modulation on the total density of states in the wells, \( f_{\text{co}}^{\text{step}}(B_0 = 0, V_0) = D_{\text{co}}^\text{step}(x; E)/D_0 \) (averaging also over the barrier regions, where no channeled orbits exist, would reduce \( f_{\text{co}}^{\text{step}}(B_0, V_0) \) by a factor of 2). For \( B_0 = 0 \) we obtain

\[
f_{\text{co}}^{\text{step}}(0, V_0) = \frac{2}{\pi} \arcsin \sqrt{\frac{2V_0}{E_F + V_0},}
\]  

(B6)

which for small \( \epsilon = V_0/E_F \) has the expansion

\[
f_{\text{co}}^{\text{step}} = \frac{2\sqrt{2\epsilon}}{\pi} - \frac{(\sqrt{2\epsilon})^3}{6\pi} + O(\epsilon^{5/2}).
\]  

(B7)

This algebraic dependence on the modulation strength \( V_0 \) is not an artifact of the step model. For the cosine modulation \( V(x) = -V_0 \cos qx \) one obtains, with \( f_{\text{co}}^{\text{cos}}(B_0, V_0) = D_{\text{co}}^{\text{cos}}(x; E)/D_0 = (D_{\text{co}}^{\text{cos}}(x, E_F))/D_0 \),

\[
f_{\text{co}}^{\text{cos}}(0, V_0) = \frac{2}{\pi} \int_0^a \frac{dx}{a} \arcsin \sqrt{\frac{V_0 - V(x)}{E_F - V(x)}},
\]  

(B8)

with the expansion

\[
f_{\text{co}}^{\text{cos}} = \frac{4\sqrt{2\epsilon}}{\pi^2} + \frac{(\sqrt{2\epsilon})^3}{9\pi^2} + O(\epsilon^{5/2})
\]  

(B9)

for small amplitudes.

For the step modulation, also the \( B_0 \)-dependence of \( f_{\text{co}}^{\text{cos}}(B_0, V_0) \) can be calculated analytically. We suppress the lengthy result for \( 0 < B_0 < B_{\text{crit}}^{\text{step}} \) [cf. Eq. (B3)] and give only the result for \( B_0 > B_{\text{crit}}^{\text{step}} \).

\[
f_{\text{co}}^{\text{step}} = \frac{4R}{a} \left[ \sqrt{2\epsilon - \sqrt{1 - \epsilon}} \arcsin \sqrt{\frac{2\epsilon}{1 + \epsilon}} \right].
\]  

(B10)

For the cosine model at \( 0 < B_0 < B_{\text{crit}}^{\text{cos}} \), we can calculate the positions and the values of the local extrema of \( x_0^*(x) \) [and similar for \( x_0^-(x) \)] analytically, and also \( D_{\text{co}}^{\text{cos}}(x; E) \). Contributions to the average of \( D_{\text{co}}^{\text{cos}}(x; E) \) come from the \( x \)-interval between the location \( x_{\text{max}} \) of the local maximum and the value \( x_{\text{upper}} \) satisfying
and reaches at \(x_{\text{upper}}(x_{\text{max}})\). The value \(x_{\text{upper}}\) and the integral of \(D_{xx}^\text{cos}(x; E)\) between \(x_{\text{max}}\) and \(x_{\text{upper}}\) are calculated numerically. The results are plotted in Fig. 2(a) for both the cosine and the step model. Figure 2(a) shows the magnetic field dependence and (b) shows the dependence on the modulation strength of \(f_{\text{co}}(B_0, V_0)\), for \(V_0 \leq E_F\) in all cases. For the cosine model at a given value of the magnetic field \(B_0\), channeled orbits exist if \(V_0/E_F > (2/qR)(1 - (\pi/2R)^2)^{1/2}\). Their number increases with increasing \(V_0\). For \(V_0 \to E_F\), \(f_{\text{co}}(B_0, V_0)\) approaches, with vertical slope, a finite value, which decreases with increasing \(B_0\). No channeled orbits survive if \(B_0\) becomes so large that \(2/qR \geq \sqrt{2}\). For the step model, the given \(B_0\)-value sets a critical modulation strength \(V_{\text{crit}}\) given by \(V_{\text{crit}}/E_F = (\pi/qR)(1 - (\pi/2R)^2)^{1/2}\). For \(V_0 < V_{\text{crit}}\), only a single orbit exists, whereas for \(V_0 > V_{\text{crit}}\) also other orbits exist, which can be calculated numerically. For \(V_0 > V_{\text{crit}}\) the curves of the curvature of the modulating potential \(V(x)\) vanish on average, and the trajectories extended in \(x\)-direction, one has \(\bar{v}_x(\varphi, x_0) = a/T(\varphi, x_0)\), where

\[
\frac{\sigma_{xx}}{\sigma_0} = \left(1 + \epsilon \right) \Phi \left( \frac{2 \epsilon}{1 + \epsilon} \right) ,
\]

where \(\epsilon = V_0/E_F\),

\[
\Phi(\eta) = \eta^{3/2} \int_0^1 dt \left[ t^2 K(t) \sqrt{t - \eta} \right]^{-1}
\]

and \(K(t)\) is the complete elliptic integral of the first kind \(K\). Apparently \(\sigma_{xx}^\text{cos} \to 0\) for \(V_0 \to E_F\). For \(\epsilon \to 0\) one obtains a power expansion in \(\sqrt{\epsilon}\), the coefficients of which can be calculated numerically. To leading order in \(\epsilon\) one obtains

\[
\sigma_{xx}^\text{cos}/\sigma_0 \to \frac{0.69454}{\rho_0} \left( \frac{V_0}{E_F} \right)^{3/2} + O(\|V_0/E_F\|^{5/2}) .
\]

For the step model the corresponding results can be calculated analytically. The exact result for the conductivity is

\[
\frac{\sigma_{xx}^\text{step}}{\sigma_0} = 1 - \frac{3 - \epsilon}{\pi} \sqrt{\frac{1 - \epsilon}{2 \pi}} + \frac{3 - 2 \epsilon - 5 \epsilon^2}{2 \pi \epsilon} \arcsin \sqrt{\frac{2 \epsilon}{1 + \epsilon}} ,
\]

with the leading term of the resistance correction

\[
\frac{\Delta \rho_{xx}^\text{step}}{\rho_0} \approx 16 \left( \frac{2V_0}{15 \pi} \right) \left( \frac{V_0}{E_F} \right)^{3/2} + O(\|V_0/E_F\|^{5/2}).
\]
of the squared modulation potentials equal, i.e., replace $V_0$ in the step potential by $V_0/\sqrt{2}$, the right hand side of Eq. (B15) becomes $0.571 \left(\frac{V_0}{E_F}\right)^{3/2}$, where the prefactor is comparable with that in Eq. (B14).

The $B_0 = 0$ conductivities for pure magnetic modulations can be calculated similarly. For the modulation $B_m(x) = B_0 \cos qx$ one obtains

$$\frac{\sigma_{xx}^{\cos}}{\sigma_0} = (1 + \alpha)^2 \Phi \left(\frac{4\alpha}{(1 + \alpha)^2}\right),$$  \hspace{1cm} (B19)

where $\Phi(\eta)$ is given by Eq. (B15), and $\alpha = 1/qR_m = \omega_m/qv_F$. For $\alpha \to 0$ this yields a power expansion in $\sqrt{\alpha}$ similar to Eq. (B16).

$$\frac{\Delta \rho_{xx}^{\cos}}{\rho_0} = 1.9645 (qR_m)^{-3/2} + O((qR_m)^{-2}),$$  \hspace{1cm} (B20)

but now the correction term is larger. In this collisionless limit, the conductivity vanishes if all trajectories become bound in $x$-direction. For the cosine modulation this happens at $1/qR_m = 1$. For the magnetic step modulation, $B_m(x) = B_0 \text{sign}(\cos qx)$ the corresponding condition is $a/4R_m = 1$, which means that a strip of constant $B_m(x)$ just can accommodate a cyclotron orbit with diameter $2R_m$. As a function of $a/4R_m$, the conductivity for the magnetic step modulation shows a similar behavior as that for the magnetic cosine modulation as a function of $1/qR_m$.

It can be shown analytically that, for $B_0 = 0$, $\sigma_{yy} = \sigma_0$ is not changed by the modulation. It is, however, interesting to calculate the fraction $\Delta \sigma_{yy}^{\cos} / \sigma_0$ contributed by the channeled orbits to $\sigma_{yy}$. This is easily shown to equal the fraction contributed to the density,

$$\frac{\Delta \sigma_{yy}^{\cos}}{\sigma_0} = \int_0^a \frac{dx}{a} \frac{n_{\text{ch}}(x ; E_F)}{n_{\text{el}}}.$$  \hspace{1cm} (B21)
FIG. 1. (a) $S^{\beta(0.0)}_0 \propto 1/qR \propto B_0$ for different values of the mean free path $\lambda$, given in terms of the modulation period $a = 2\pi/q$; (b) and (c): selected magnetoresistance results for pure electric modulation $V(x) = V_0 \cos qx$ to order $V_0^2$, calculated from Eq. (1.1). In (b) $\Delta \rho_{xx}$ is plotted in units of $\rho_1 = (\rho_0/2)(V_0/E_F)^2$, and the dash-dotted line represents the local limit for $\lambda q = 1.0$. In (c) $\Delta \rho_{xx}$ is plotted in units of $\rho_2 = \rho_0(V_0/E_F)^2(\lambda q/2)^2$, and the thin line represents the approximation of Ref. [15].

FIG. 2. Magnetoresistivity vs. $1/qR \propto B_0$ for (a) mixed electric and magnetic, and (b) mixed electric and mobility modulations, calculated from Eqs. (4.1)-(4.4) for cosine modulations $V(x) = V_0 \cos qx$, $\omega_m(x) = \omega_0 \cos qx$, and $r(x) = r_0 \cos qx$, of comparable effective amplitudes $\alpha_V \equiv \lambda qV_0/2E_F$, $\alpha_B \equiv \tau_0 \omega_0$ and $\alpha_\mu \equiv \tau_0$, respectively, with $\lambda q = 30$. $\gamma$ is a small, but otherwise arbitrary, dimensionless constant.

FIG. 3. Typical orbits (lower panel) in the electric superlattice potential $V(x) = 0.4 E_F \cos qx$, with $q = 2\pi/a$, for three values of the magnetic field $B_0$ corresponding to $\beta = 1/qR = 0.25$, 0.1, and 0.001. The corresponding values of the constant of motion $x_0$ are indicated in the upper panels by horizontal bars ending on the curves $x_0^\pm(x)$ defining the turning points. Channeled orbits exist for $\beta > \beta_{\text{crit}} \approx 0.2$.

FIG. 4. Magnetoresistivity versus $1/qR \propto B_0$ for a weak electric cosine modulation with $V_0/E_F=0.02$ and $\lambda q=27.7$. Solid line: numerical result, dotted line: analytical approximation (4.1), dash-dotted: contribution of channeled orbits (see text). Inset: the same for magnetic cosine modulations with $1/qR_m = 0.34, 0.15, and 0.05$ (from top to bottom at arrow, for further explanations see text) and $\lambda q=27.7$.

FIG. 5. Magnetoresistivity vs. $1/qR \propto B_0$ for anisotropic scattering, modelled by Eq. (5.2), and an electric cosine modulation with $V_0/E_F = 0.02$; $\tau_0 = 27.7/\nu_F$.

FIG. 6. Magnetoresistivity vs. $1/qR$ for cosine modulation of the electric potential and the mobility, with $\tau_0 = 27.7$. Solid lines: $\rho_{xx}/\rho_0$ for pure electric modulation with $V_0/E_F = 0.2$ and $\rho_{yy}/\rho_0$ for pure mobility modulation with $r_m = 1/\tau$; broken lines: mixed modulation.

FIG. 7. Fraction of channeled orbits, as function of magnetic field $B$ (a) and modulation strength $V_0$ (b), for an electric cosine modulation $V(x) = -V_0 \cos qx$ and a step modulation $V(x) = -V_0 \text{sign}(\cos qx)$; (a): $B$-dependence for fixed $V_0/E_F = 0.01$ (solid lines) and 0.95 (dash-dotted lines), (b): $V_0$-dependence for fixed $1/qR = 0.0, 0.05, 0.2$, and 0.4 (from top to bottom), with solid lines for the cosine and dash-dotted lines for the step modulation.
R. Menne, Figure 1
\[ \frac{\Delta \rho_{\mu \nu}}{(\gamma^2 \rho_0)} = \begin{cases} \gamma, & \alpha_\mu = \gamma, \\ 0, & \alpha_\mu = 0, \\ -\gamma, & \alpha_\mu = -\gamma \end{cases} \]

\( \alpha_\nu = \gamma, \alpha_B = 0 \)

\( \alpha_\nu = 0, \alpha_B = \gamma \)

\( \alpha_\nu = \gamma, \alpha_B = \gamma \)

\( \alpha_\nu = \gamma, \alpha_B = -\gamma \)

R. Menne, Figure 2
R. Menne, Figure 3
$\Delta \rho_{xx} / \rho_0$ [%]

$1/qR_{\text{crit}} = 0.5 V_0/E_F$

$\Delta \rho_{xx} / (\rho_0 \omega_m^2 \tau)$

R. Menne, Figure 4
\[ \frac{\rho_{\mu\nu}}{\rho_0} \]

\[ \frac{\rho_{xx}}{\rho_0} \]

\[ \frac{\rho_{yy}}{\rho_0} \]

\[ \frac{\Delta\rho_{xy}}{\rho_0} \]
$f_{co}(B_0, V_0) / f_{co}(0, V_0)$

$B_0 / B_{crit}$

$V_0/E_F = 0.01$

$V_0/E_F = 0.95$

R. Menne, Figure 7