THE GLOBAL ATTRACTOR FOR THE WAVE EQUATION WITH NONLOCAL STRONG DAMPING

BIYUE CHEN
Department of Mathematics, Nanjing University
Nanjing, 210093, China

CHUNXIANG ZHAO
Department of Mathematics, Nanjing University
Nanjing, 210093, China
Institute of Applied System Analysis, Jiangsu University
Zhenjiang, 212013, China

CHENGKUI ZHONG*
Department of Mathematics, Nanjing University
Nanjing, 210093, China

Communicated by Chunyou Sun

Abstract. The paper is devoted to establishing the long-time behavior of solutions for the wave equation with nonlocal strong damping: \( u_{tt} - \Delta u - \|\nabla u_t\|^p \Delta u_t + f(u) = h(x) \). It proves the well-posedness by means of the monotone operator theory and the existence of a global attractor when the growth exponent of the nonlinearity \( f(u) \) is up to the subcritical and critical cases in natural energy space.

1. Introduction. In this paper, we study the long-time behavior of solutions to the following wave equation with nonlocal strong damping

\[
\begin{cases}
    u_{tt} - \Delta u - \|\nabla u_t\|^p \Delta u_t + f(u) = h(x), \\
    u|_{\partial \Omega} = 0, \\
    u(x, 0) = u_0(x), u_t(x, 0) = u_1(x),
\end{cases}
\tag{1.1}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial \Omega \), and the assumptions on \( f(u) \) and \( h \) will be specified later.

In the physical process, there have been extensive concerns on the well-posedness and the long-time behavior of solutions to the wave equations related to type (1.1)(cf.[1, 2, 11, 20, 21, 26, 15] and references therein). Global attractor is a basic concept to study the longtime dynamics of wave equations with various dissipations. For example, the weak damping \( u_t \)(e.g.[4, 16, 32]), the nonlinear \( g(u_t) \) (e.g.[9, 12, 13]). And with respect to the fractional one \( (-\Delta)^\theta u_t, 0 < \theta \leq 1 \), we can
see [33, 34] and references therein. In particular, when $\theta = 1$, the fractional one is the strong damping $-\Delta u_t$, Ghidaglia and Marzocchi [14] studied the asymptotic behaviour of solutions to strongly damped abstract nonlinear wave equations and after reviewing sufficient hypotheses for existence and uniqueness, uniform time estimates are given and a global attractor for the trajectories of the associated dynamical system is constructed. Pata and Zelik [29] investigated the existence of compact global attractors of optimal regularity for nonlinearities of critical and supercritical growth. See also [5, 24, 27, 35] and so on. Moreover, we note that the long-time dynamics of the evolution equations with nonlocal damping has been studied by many authors. One of them is the Kirchhoff type nonlocal damping

$$M \left( \int_\Omega |\nabla u|^2 \, dx \right) (-\Delta)^\theta u_t,$$

where $M(s) > 0$ is a nonlinear function and $0 \leq \theta \leq 1$ (e.g. [9, 17, 18, 22]). More recently, Jorge Silva and Narciso [19] studied the well-posedness, polynomial stability and non-exponential decay of a beam model

$$u_{tt} + \Delta^2 u - \kappa \Delta u - \gamma \left( \int_\Omega (|\Delta u|^2 + |u_t|^2) \, dx \right)^q \Delta u_t + f(u) = 0,$$

where the nonlocal energy damping $\left( \int_\Omega (|\Delta u|^2 + |u_t|^2) \, dx \right)^q \Delta u_t$ was first proposed by Balakrishnan and Taylor [6]. Moreover, in [36], the authors studied the long-time behavior of a class of extensible beams equation with the nonlocal weak damping

$$u_{tt} + \Delta^2 u - m(\|\nabla u\|^2) \Delta u + \|u_t\|^p u_t + f(u) = h, \quad \text{in } \Omega \times \mathbb{R}^+, \quad p > 0,$$

where the asymptotic smoothness of the semigroup is verified by the energy reconstruction method. And in [37], they also consider the asymptotic behaviour of the wave equation with the nonlocal weak damping $\|u_t\|^p u_t$ and nonlocal weak anti-damping $\int_\Omega K(x,y) u_t(y) \, dy$. It is worth mentioning that the nonlocal weak damping

$$\|u_t\|^p u_t, \quad p > 0,$$

was also first proposed.

Motivated by the literature mentioned above, we investigate the long-time dynamics of the model (1.1) with the following nonlocal strong damping

$$\|\nabla u_t\|^p (-\Delta u_t), \quad p > 0.$$  \hfill (1.2)

Similar to the nonlocal weak damping, the existence of a global attractor of problem (1.1) is dependent of $p$. To our best knowledge, so far there is few result on the long-time behaviour of wave equations with nonlocal strong damping in the form (1.2). Thus, it is worth for us to mention the for the well-posedness of problem (1.1). Due to the matter that the nonlocal coefficients $\|\nabla u_t\|^p$ is not weakly continuous in $L^2(\Omega)$, it is difficult for us to using Galerkin method to obtain the existence of the solution for Eq. (1.1). Therefore, we apply the monotone operator theory (thanks to [28, 30, 31]) to deal with the well-posedness in the manuscript. In addition, we deduce the existence of the global attractor of this system by the method of Condition (C) in [25] and asymptotic smoothness in [9].

Compared with nonlocal weak damping [38], [36] and [37], we found that the main differences and difficulties with these are mainly reflected in the following points.
(1) For well-posedness, due to the unboundedness of the damping in $L^2(\Omega)$, we can not employ proposition 1.12 in [9] (as in [38],[36],[37]) to verify the generalized solution of problem (1.1) is also a weak solution. Thus, we use the regularity $\|\nabla u_t(\cdot)\|^{p+2} \in L^1(0,T)$ to deal with the well-posedness of Eq.(1.1).

(2) When the nonlinearity $f(u)$ is up to the subcritical case, in the proof of the (C)-condition, we cannot obtain directly the boundedness of the damping $L^2(\Omega)$ norm to resolve the compactness of the dynamical system.

The remaining of the paper is organized as follows. We recall several notations, present the assumptions and obtain the well-posedness result in Section 2 and Section 3, respectively. Dissipativity is given in Section 4. Section 5 is dedicated to the proof of results on asymptotically smoothness and existence of global attractor for problem (1.1).

2. Preliminary results. Let us begin by introducing the notations that will be used throughout the remaining work.

2.1. Functions space. Initially, we recall the concept and recapitulate its basic properties, see [11] for more details.

We denote by $H = L^2(\Omega)$, with usual inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. We also consider the space $V = H^1_0(\Omega)$ with usual inner product $(\cdot,\cdot)_V$ and norm $\|\nabla\cdot\|$. When there is no possibility of confusion, we shall use the same notation $(\cdot,\cdot)$ to represent the duality pairing between any Banach space $W$ and its dual $W^\prime$.

Denoting $\lambda_1 > 0$ the first eigenvalue of the harmonic operator $-\Delta$ with Dirichlet boundary condition, then it holds

$$\|\nabla u\|^2 \geq \lambda_1 \|u\|^2. \quad (2.1)$$

Finally, we also consider the following phase space $\mathcal{H} = V \times H$ endowed with norm

$$\|(u,v)\|^2_{\mathcal{H}} = \|\nabla u\|^2 + \|v\|^2, (u,v) \in \mathcal{H},$$

where the analysis of the asymptotic behavior of solutions shall be done.

2.2. Assumptions. Now, we introduce the assumptions on the functions $f$ and $h$ as follows:

(H1) the function $h \in L^2(\Omega)$.

(H2) the function $f \in C^1(R)$ and $f(0) = 0$ is of the following polynomial growth condition: there exists a positive constant $C > 0$ such that

$$|f'(s)| \leq C(1 + |s|^q), \forall \ s \in R,$$

where $1 \leq q < \infty$ when $n \leq 2$, and $1 \leq q \leq \frac{2}{n-2}$ when $n \geq 3$.

Moreover, the following dissipativity condition holds:

$$\lim_{|s| \to +\infty} f'(s) > -\lambda_1; \quad (2.3)$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ with the Dirichlet boundary condition.

Setting $F(s) = \int_0^s f(\tau)d\tau$, the assumption (H2) implies that there exist positive constants $C$ and $0 < \lambda < \lambda_1$ such that

$$\int_\Omega F(u)dx \geq -\frac{\lambda}{2} \|u\|^2 - C, \quad (2.4)$$

$$(f(u),u) \geq \int_\Omega F(u)dx - \frac{\lambda}{2} \|u\|^2 - C. \quad (2.5)$$
Next, we list some definitions and lemmas which are used frequently in this paper.

**Definition 2.1.** The operator \( D : H^1_0(\Omega) \to H^{-1}(\Omega) \) is defined as follows
\[
(D(u), v) = \| \nabla u \|^p(\nabla u, \nabla v), \quad \text{for any } u, v \in H^1_0(\Omega).
\] (2.6)

**Definition 2.2.** Let \( W \) and hemicontinuous if for each \( u, v \in W \)
\[
\langle M(u) - M(v), u - v \rangle \geq 0, \quad \text{for any } u, v \in W,
\] (2.7)

and hemicontinuous if for each \( u, v \in W \) the real-valued function \( t \mapsto M(u + tv)(v) \) is continuous (clearly this last condition is true for the restriction of \( M \) to each line segment is continuous into \( W' \) with weak convergence.)

**Definition 2.3.** Let \( W \) be a Hillbert space, suppose \( A : D(A) \to W \) is an (unbounded) operator and \( D(A) \) is dense in \( W \). The operator \( A \) is accretive if
\[
(Ax, x)_W \geq 0, \quad \forall x \in D(A),
\] (2.8)

and it is a m-accretive operator if, in addition, \( I + A \) maps \( D(A) \) onto \( W \), i.e., \( R(I + A) = W \).

**Lemma 2.4.** ([31]) If \( A \) is m-accretive and \( B \) is accretive and Lipschitz, then \( A + B \) is m-accretive.

3. **Well-posedness.** The existence and uniqueness of solution to problem (1.1) is presented below.

**Definition 3.1.** ([9]) A function \( u \in C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \) possessing the properties \( u(0) = u_0 \) and \( u_0(0) = u_1 \) is said to be

- (S) a strong solution to problem (1.1) on the interval \([0, T]\), iff
  - \( u \in W^{1,1}(a, b; H^1_0(\Omega)) \) and \( u_t \in W^{1,1}(a, b; L^2(\Omega)) \) for any \( 0 < a < b < T \);
  - \( -\Delta u(t) + \| \nabla u(t) \|^p(-\Delta u_1)(t) \in L^2(\Omega) \) for almost all \( t \in [0, T] \);
  - the equation (1.1) is satisfied in \( L^2(\Omega) \) for almost all \( t \in [0, T] \);
- (G) a generalized solution to problem (1.1) on the interval \([0, T]\), iff there exists a sequence of strong solution \( \{ u_n(t) \} \) to the problem (1.1) with initial data \( (u_0; u_1) \) instead of \( (u_0; u_1) \) such that
\[
\lim_{n \to \infty} \max_{t \in [0, T]} \left\{ \| u(t) - u_n(t) \| + \| \nabla u(t) - u_n(t) \| \right\} = 0.
\] (3.1)

Let the hypotheses \((H_1)\) and \((H_2)\) hold, the well-posedness of problem (1.1) is given by the following results.

**Theorem 3.2.** Under the assumptions \((H_1)\) and \((H_2)\), for any \( T > 0 \), the following statements hold.

(i) For every \( (u_0, u_1) \in H^1_0(\Omega) \times H^1_0(\Omega) \) such that \(-\Delta u_0 - \| \nabla u_1 \|^p \Delta u_1 \in L^2(\Omega)\), there exists a unique strong solution \( (u, u_t) \in H^1_0(\Omega) \times H^1_0(\Omega) \) to the problem (1.1) on the interval \([0, T]\) such that
\[
(u_t, u_{tt}) \in L^\infty(0, T; \mathcal{H}), \quad u_t \in C_r([0, T]; H^1_0(\Omega)), \quad u_{tt} \in C_r([0, T]; L^2(\Omega)),
\] (3.2)

and
\[
-\Delta u(t) + \| \nabla u(t) \|^p(-\Delta u_1)(t) \in C_r([0, T]; L^2(\Omega)),
\] (3.3)
Proof of Theorem 3.2. Denote operator $D$ a self-adjoint m-accretive operator and holds for every $\psi$

(ii) For any initial data $(u_0, u_1) \in \mathcal{H} = H^1_0(\Omega) \times L^2(\Omega)$, there exists a unique 
                        generalized solution $(u, u_t) \in \mathcal{H}$ such that 
                        
                        $$(u, u_t) \in C([0, T]; \mathcal{H}), \ \forall \ T \geq 0. \quad (3.4)$$

(iii) Furthermore, both the strong solution and generalized solution $(u, u_t)$ satisfy the energy relation
                        
                        $$E(t) + \int_0^t \|\nabla u_t(s)\|^{p+2} ds = E(0), \quad (3.5)$$
                        
                        where
                        $E(t) = E(u(t), u_t(t)) = \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|\nabla u(t)\|^2 + \int_\Omega F(u) dx - (h, u), \quad (3.6)$
                        
                        and denote
                        $E_0(t) = E_0(u(t), u_t(t)) = \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|\nabla u_t(t)\|^2. \quad (3.7)$
                        
                        Accordingly, we can explore the regularity,
                        $u_t \in L^{p+2}(0, T; H^1_0(\Omega)), \ \forall \ T > 0. \quad (3.8)$
                        
                        Moreover, for every generalized solution, the relation
                        $$(u_t(t), \psi) = (u_1, \psi) - \int_0^t (-\Delta u(\tau), \psi) - (Du_t(\tau), \psi) - (f(u), \psi) + (h, \psi) d\tau \quad (3.9)$$
                        holds for every $\psi \in H^1_0(\Omega)$.

Proof of Theorem 3.2. Denote operator $A = -\Delta$, it is clear that $A$ is a positive self-adjoint m-accretive operator and $\mathcal{D}(A^\frac{1}{2}) = H^1_0(\Omega)$ in Dirichlet condition. In addition, there exist a constant $C_p$ such that

$$(Du_t - Du_t, u_t - v_t) = (\|\nabla u_t\|^p\nabla u_t - \|\nabla u_t\|^p\nabla v_t, \nabla u_t - v_t) \geq C_p \|\nabla (u_t - v_t)\|^{p+2}, \quad (3.10)$$

for any $u_t, v_t \in H^1_0(\Omega)$, which indicates the damping operator $D : H^1_0(\Omega) \to H^{-1}(\Omega)$ possess strong monotonicity. The hemicontinuity of $D$ can follow from Lebesgue convergence theorem.

Let us define the operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ where $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega)$, and

$$A = \begin{pmatrix} 0 & -I \\ A & D \end{pmatrix} \quad (3.11)$$

with $\mathcal{D}(A) = \{(x, y) \in H^1_0(\Omega) \times H^1_0(\Omega) : Ax + Dy \in L^2(\Omega)\}$. Hence, the original evolution problem (1.1) is equivalent to the equation

$$\begin{cases} \frac{d}{dt} U(t) + AU(t) = \begin{pmatrix} 0 \\ h - f(u) \end{pmatrix}, \\ U(0) = U_0 = (u_0, u_1), \end{cases} \quad (3.12)$$

where $U(t) = (u(t), u_t(t))$.

Step 1. We show the operator $A$ defined in (3.11) is m-accretive.
For any elements \( u = (u_1, u_2), w = (w_1, w_2) \in \mathcal{D}(A) \), let \( \xi = (\xi_1, \xi_2) \in A(u) \) and \( \eta = (\eta_1, \eta_2) \in A(w) \). Thus, we can get \( \xi_1 = -u_2, \xi_2 = Au_1 + Dw_2 \) and \( \eta_1 = -w_2, \eta_2 = Aw_1 + Dw_2 \). Furthermore, we obtain that

\[
(A(u) - A(w), u - w)_{\mathcal{H}} = -(A(u_2 - w_2), u_1 - w_1) + (A(u_1 - w_1) + (Du_2 - Dw_2), u_2 - w_2) \geq 0, \tag{3.13}
\]

thus, \( A \) is a accretive operator.

And the next, we need to show that \( R(I + A) = \mathcal{H} \). For every \( f_0 \in H^1_0(\Omega) \) and \( f_1 \in L^2(\Omega) \) find \( (x, y) \in \mathcal{D}(A) \) such that

\[
f_0 = -y + x,
\]

\[
f_1 = Ax + Dy + y. \tag{3.14}
\]

Eliminating \( x \) yields,

\[
Ay + Dy + y = f_1 - Af_0 \in [D(A^{\frac{1}{2}})]' = H^{-1}(\Omega). \tag{3.15}
\]

If we denote \( v = A^{\frac{1}{2}}y \), then we obtain the relation

\[
v + Sv = A^{-\frac{1}{2}}(f_1 - Af_0) \in L^2(\Omega), \tag{3.16}
\]

where \( Sv = A^{-\frac{1}{2}}D(A^{-\frac{1}{2}}v) + A^{-1}v \). It is clear that the operator \( A^{-\frac{1}{2}}DA^{-\frac{1}{2}} \) is m-accretive in \( L^2(\Omega) \), in addition, the operator \( A^{-1} \) is Lipschitz and accretive in \( L^2(\Omega) \). Therefore, by Lemma 2.4, the operator \( S \) is m-accretive in \( L^2(\Omega) \), i.e., \( R(I + S) = L^2(\Omega) \), which candidates that there exists a \( v \in L^2(\Omega) \) to solve (3.14), consequently, \( y = A^{-\frac{1}{2}}v \in H^1_0(\Omega) \).

From these, we obtain \( x \in H^1_0(\Omega) \) in the equation (3.14), it is obvious that \( (x, y) \in \mathcal{D}(A) \). Therefore, the proof of m-accretive property of \( A \) has been completed.

**Step 2.** Setting the operator

\[
B(U) = \begin{pmatrix} 0 \\ h - f(u) \end{pmatrix}, \tag{3.17}
\]

which is locally Lipschitz on \( \mathcal{H} \) under the assumption (\( H_2 \)).

Thus, using [32, Theorems 4.1 and 4.1A, Chapter 4] and applying the rather standard application of the monotone operator theory with locally Lipschitz perturbations, we obtain the follow conclusions.

For any \( U_0 \in \mathcal{D}(A) \), there exists \( t_{\max} \leq +\infty \) such that (3.12) has \( s \) unique strong solution \( U \) on the interval \([0, t_{\max}]\).

Assume only \( U_0 \in \overline{\mathcal{D}(A)} = \mathcal{H} \), we get a unique generalized solution \( U \in C([0, t_{\max}], \mathcal{H}) \) to (3.12).

In both cases we have

\[
\lim_{t \to t_{\max}} \|U(t)\|_{\mathcal{H}} = +\infty, \text{ as } t_{\max} < +\infty. \tag{3.18}
\]

If \( t_{\max} < +\infty \), let \( u(t) \) be a strong solution on some interval \([0, t_{\max}]\). Multiplying (1.1) by \( u_t \) in \( L^2(\Omega) \), we obtain

\[
E(t) + \int_0^t \|\nabla u_t(s)\|^{p+2} ds = E(0), \tag{3.19}
\]

which holds for \( t \in [0, t_{\max}] \). By the equation (3.18), we deduce \( \lim_{t \to t_{\max}} E(t) = +\infty \). Hence, there exists a contradiction, and then \( t_{\max} = +\infty \). From these, it is valid to acquire the global existence of strong and generalized solution.
Furthermore, for any $t > 0$, the energy relation (3.19) is holding for the strong solution of problem (1.1).

For the generalized solution $u(t)$ to problem (1.1), there exists a sequence of strong solution $\{u_n(t)\}$ to the problem (1.1) with initial data $(u_0; u_1)$ instead of $(u_0; u_1)$ and

\[
\lim_{n \to +\infty} \max_{t \in [0, T]} \left\{ \| u_t(t) - u_{nt}(t) \| + \| \nabla (u(t) - u_n(t)) \| \right\} = 0, \tag{3.20}
\]

which holds for $t > 0$, the energy relation (3.19) is holding for the strong solution of problem (1.1).

Multiplying with $u_{nt} - u_{nt}$ and integrating from 0 to $T$, we obtain

\[
\frac{1}{2} \| w_i^{n,m}(T) \|^2 + \frac{1}{2} \| \nabla w_i^{n,m}(T) \|^2 \\
+ \int_0^T \int_\Omega (\| \nabla u_{nt}(t) \|^2 - \| \nabla u_{nt}(t) \|^2 p(-\Delta u_{nt}(t))) (u_{nt} - u_{nt}) dx dt \\
= \frac{1}{2} \| w_i^{n,m}(0) \|^2 + \frac{1}{2} \| \nabla w_i^{n,m}(0) \|^2 + \int_0^T \int_\Omega (f(u_m - f(u_n)) (u_{nt} - u_{nt}) dx dt,
\]

where $w^{n,m} = u_n - u_m$ and $w_i^{n,m} = u_{nt} - u_{nt}$.

Making use of (3.20), it is obviously to achieve

\[
\lim_{n,m \to +\infty} \frac{1}{2} \| w_i^{n,m}(t) \|^2 + \frac{1}{2} \| \nabla w_i^{n,m}(t) \|^2 = 0, \tag{3.23}
\]

which holds for $t \in [0, T]$.

Applying Young’s inequality and growth condition (2.2), we can infer that

\[
\left| \int_\Omega (f(u_n) - f(u_m)) (u_{nt} - u_{nt}) dx \right| \\
\leq C \int_\Omega \left( 1 + \| u_n \|^2 + \| u_n \|^2 \right) \| u_n - u_m \| u_{nt} - u_{nt} dx \\
\leq C \left( 1 + \| u_n \|^{2(q+1)} + \| u_n \|^{2(q+1)} \right) \| u_n - u_m \|^{2(q+1)} \| u_{nt} - u_{nt} \| \\
\leq C \left( 1 + \| \nabla u_n \|^q + \| \nabla u_n \|^q \right) \| \nabla u_n - \nabla u_m \| \| u_{nt} - u_{nt} \|.
\]

Using Lebesgue convergence theorem, formulas (3.23) and (3.24), we obtain

\[
\lim_{n,m \to +\infty} \int_0^T \int_\Omega (f(u_n - f(u_m)) (u_{nt} - u_{nt}) dx dt = 0. \tag{3.25}
\]

In addition, we have

\[
\int_0^T \int_\Omega (\| \nabla u_{nt}(t) \|^p (-\Delta u_{nt}(t)) - \| \nabla u_{nt}(t) \|^p (-\Delta u_{nt}(t))) (u_{nt} - u_{nt}) dx dt \\
= \int_0^T \int_\Omega (\| \nabla u_{nt}(t) \|^p \nabla u_{nt} - \| \nabla u_{nt}(t) \|^p \nabla u_{nt}) (\nabla u_{nt} - \nabla u_{nt}) dx dt \\
\geq C_p \int_0^T \| \nabla (u_{nt}(t) - u_{nt}(t)) \|^{p+2} \geq 0.
\]

However, for any $t > 0$, the energy relation (3.19) is holding for the strong solution of problem (1.1).
Applying the formula (3.23), (3.25) and (3.26) in (3.22), we conclude that
\[
\lim_{n,m \to +\infty} \int_0^T \int_{\Omega} (\|\nabla u_{n,t}\|^p(-\Delta u_{n,t}) - \|\nabla u_{m,t}\|^p(-\Delta u_{m,t})) (u_{n,t} - u_{m,t}) \, dx \, dt = 0.
\] (3.27)

From these, we infer
\[
u_n \to u, \text{ in } C([0,T], H^1_0(\Omega));
\] (3.28)
\[
u_{n,t} \to \psi, \text{ in } C([0,T], L^2(\Omega));
\] (3.29)
\[
u_{n,t} \to \psi, \text{ in } L^{p+2}(0,T; H^1_0(\Omega)).
\] (3.30)

Since every strong solution \( u_n(t) \) satisfies
\[
\mathcal{E}(u_n(t), \psi) + \int_0^t \|\nabla \psi(s)\|^{p+2} \, ds = \mathcal{E}(u_n(0), \psi),
\] (3.31)

furthermore, applying with (3.20) and the continuity of \( F \), we have
\[
\lim_{n \to +\infty} \mathcal{E}(u_n(t), \psi) = \mathcal{E}(u(t), \psi), \quad \text{for } t \in [0,T].
\] (3.32)

Taking \( n \to +\infty \) in (3.31) and applying with (3.30), (3.32), we obtain every generalized solution of the problem (1.1) also satisfies
\[
\mathcal{E}(u(t), \psi) + \int_0^t \|\nabla \psi(s)\|^{p+2} \, ds = \mathcal{E}(u(0), \psi), \quad \text{for } t \in [0,T].
\] (3.33)

which also indicates that every generalized solution \( u(t) \) satisfies the energy relation equality (3.5).

Furthermore, the generalized solution also satisfies
\[
(u_t(t), \psi) = (u_1, \psi) - \int_0^t (-\Delta u(\tau), \psi) - (Du_t, \psi) - (f(u), \psi) + (h, \psi) \, d\tau,
\] (3.34)

for every \( \psi \in H^1_0(\Omega) \), which infers the every generalized solution of (1.1) is also a weak solution, that is the required result (3.9).

Ultimately, from the energy identity (3.5), we can explore the further regularity
\[
u_t \in L^{p+2}(0,T; H^1_0(\Omega)), \quad \forall \ T > 0,
\] (3.35)

which holds for both the strong solution and the generalized solution, that is the required result (3.8). Thus, we complete the proof of well-posedness.

**Remark 1.** According to Komura-kato proposition, it is obvious that the semigroup \( \{S(t) : t \geq 0\} \) for strong solution of (1.1) is continuous on \( R^+ \) and \( D(A) \).

By taking the (uniformly) continuous extension of each \( S_t \), we obtain a family of functions \( S(t) : D(A) \to \overline{D(A)} \) which satisfies
- \( S(t_1 + t_2) = S(t_1) \circ S(t_2) \);
- \( S(\cdot)u_0 \) is continuous on \( R^+ \) for each \( u_0 \in \overline{D(A)} \);
- \( S(t)(\cdot) \) is continuous in \( \overline{D(A)} \) for any \( t \geq 0 \).

Which gives the semigroup \( \{S(t) : t \geq 0\} \) for generalized solution of (1.1) is also continuous on \( R^+ \) and \( \overline{D(A)} \).
4. Dissipativity. In this section, we concerned with the dissipation property for the dynamical system \( \{ H; S(t) \} \) corresponding to (1.1).

**Theorem 4.1.** Let assumptions (H₁) and (H₂) be in force. Then the system \( \{ H; S(t) \} \) generated by (1.1) in the space \( H \) is dissipative, i.e. there exists \( \rho_0 > 0 \) possessing the property: for any bounded set \( B \subseteq H \), there exists \( t_0 = t(B) \) such that

\[
\| S(t)y \|_H = \| (u(t), u(t)) \|_H \leq \rho_0, \tag{4.1}
\]

for any \( y \in B \) and \( t \geq t_0 \). Correspondingly, the set

\[
B_0 = \{ (u, v) \in H; \| (u, v) \|_H \leq \rho_0 \}
\]

is a bounded absorbing set for the system \( \{ H; S(t) \} \).

**Proof.** Multiplying (1.1) by \( u_t \) and integrating over \( \Omega \), we obtain

\[
\frac{d}{dt} \left( \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| \nabla u \|^2 + \int_{\Omega} F(u) \, dx - \int_{\Omega} h(x) u \, dx \right) + \| \nabla u_t \|^2 = 0. \tag{4.2}
\]

Rewriting (4.2) as

\[
\frac{d}{dt} E(t) + \| \nabla u_t \|^2 = 0. \tag{4.3}
\]

From (2.4) and combining with (2.1), we attain that

\[
\int_{\Omega} F(u) \, dx \geq -\frac{\lambda}{2\lambda_1} \| \nabla u \|^2 - C. \tag{4.4}
\]

By Hölder inequality, Young’s inequality and (2.1), we arrive that

\[
\left| \int_{\Omega} h \, u \, dx \right| \leq \frac{1}{4\lambda_1} \| h \|^2 + \kappa \lambda_1 \| u \|^2 \leq \frac{1}{4\lambda_1} \| h \|^2 + \kappa \| \nabla u \|^2. \tag{4.5}
\]

Choosing \( \kappa = \frac{1}{2}(\frac{1}{2} - \frac{\lambda}{2\lambda_1}) \), we get

\[
E(t) \geq c_0 E_0(t) - C_0, \tag{4.6}
\]

where \( 0 < c_0 < 1 \) and \( C_0 > 0 \).

Multiplying (1.1) by \( \alpha u \) and integrating over \( \Omega \), we have that

\[
\frac{d}{dt} \alpha(u_t, u) - \alpha \| u_t \|^2 + \alpha \| \nabla u \|^2 + \alpha \int_{\Omega} f(u) \, dx - \alpha \int_{\Omega} h(x) u \, dx + \alpha \int_{\Omega} \| \nabla u_t \|^2 \| \nabla u_t \| \, dx = 0. \tag{4.7}
\]

Combining (4.2) with (4.7), we get that

\[
\frac{d}{dt} \left( E(t) + \alpha(u_t, u) \right) + \| \nabla u_t \|^2 - \| u_t \|^2 + \alpha \| \nabla u \|^2 + \alpha \int_{\Omega} f(u) \, dx - \alpha \int_{\Omega} h(x) u \, dx \tag{4.8}
\]

\[
- \alpha \int_{\Omega} h(x) u \, dx + \alpha \int_{\Omega} \| \nabla u_t \|^2 \| \nabla u_t \| \, dx = 0.
\]

Denote \( V(t) = E(t) + \alpha(u_t, u) \), then by Hölder inequality and Young’s inequality, we have

\[
| \alpha(u_t, u) | \leq \alpha \left( \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| u \|^2 \right) \leq \frac{\alpha}{2} \| u_t \|^2 + \frac{\alpha}{2\lambda_1} \| \nabla u \|^2. \tag{4.9}
\]
Combining (4.6) with (4.9), there exists $0 < \alpha_0 < 1$ such that

$$V(t) \geq c_1 E_0(t) - C_1,$$

(4.10)

holds for any $0 < \alpha < \alpha_0$, where $0 < c_1 < 1$ and $C_1$ are positive constants. Therefore, it is clear that

$$\frac{d}{dt} V(t) + \|\nabla u_t\|^{p+2} - \alpha \|u_t\|^2 + \alpha \|\nabla u\|^2 + \alpha \int_{\Omega} f(u) \, dx$$

$$- \alpha \int_{\Omega} h(x) \, dx + \alpha \int_{\Omega} \|\nabla u_t\|^{p}\nabla u_t \cdot \nabla u \, dx = 0.$$  

(4.11)

And then, we rewrite the formula (4.11) as follows

$$\frac{d}{dt} V(t) + \alpha V(t) = \Gamma,$$  

(4.12)

where

$$\Gamma = -\|\nabla u_t\|^{p+2} + \alpha \|u_t\|^2 - \alpha \|\nabla u\|^2 - \alpha \int_{\Omega} f(u) \, dx$$

$$- \alpha \int_{\Omega} \|\nabla u_t\|^{p}\nabla u_t \cdot \nabla u \, dx + \frac{\alpha}{2} \|u_t\|^2$$

$$+ \frac{\alpha}{2} \|\nabla u\|^2 + \alpha \int_{\Omega} F(u) \, dx + \alpha^2 (u_t, u).$$

Using Hölder inequality and (2.1), we have

$$|\alpha^2 (u_t, u)| \leq \alpha^2 \left( \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u\|^2 \right) \leq \frac{\alpha^2}{2} \|u_t\|^2 + \frac{\alpha^2}{2\lambda_1} \|\nabla u\|^2.$$  

(4.14)

Based on (2.1), (2.4) and (2.5), the following estimate is valid,

$$\alpha \int_{\Omega} F(u) \, dx - \alpha \int_{\Omega} f(u) \, dx \leq \frac{\alpha\lambda}{2} \|u\|^2 + C\alpha \leq \frac{\alpha\lambda}{2\lambda_1} \|\nabla u\|^2 + C\alpha.$$  

(4.15)

By Cauchy inequality, Young’s inequality, we infer that

$$| - \alpha \int_{\Omega} \|\nabla u_t\|^{p}\nabla u_t \cdot \nabla u \, dx| \leq \alpha \|\nabla u_t\| p \left( \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 \right)$$

$$\leq \frac{\alpha}{2} \|\nabla u_t\|^{p+2} + \frac{\alpha}{2} \|\nabla u_t\|^{p}\|\nabla u\|^2)$$

$$\leq \frac{\alpha}{2} \|\nabla u_t\|^{p+2} + \frac{\alpha}{2} (\delta + C\delta \|\nabla u_t\|^{p+2}) \|\nabla u\|^2$$

$$\leq \frac{\alpha}{2} \|\nabla u_t\|^{p+2} + \frac{\alpha\delta}{2} \|\nabla u\|^2 + \frac{\alpha C\delta}{2} \|\nabla u_t\|^{p+2} \|\nabla u\|^2$$

$$\leq \frac{\alpha}{2} \|\nabla u_t\|^{p+2} + \frac{\alpha C\delta}{2} \|\nabla u_t\|^{p+2} E_0(t) + \frac{\alpha\delta}{2} \|\nabla u\|^2.$$  

(4.16)

Hence,

$$| - \alpha \int_{\Omega} \|\nabla u_t\|^{p}\nabla u_t \cdot \nabla u \, dx| \leq \frac{\alpha}{2} \|\nabla u_t\|^{p+2} + \frac{\alpha C\delta}{2} \|\nabla u_t\|^{p+2} E_0(t) + \frac{\alpha\delta}{2} \|\nabla u\|^2.$$  

(4.17)

Applying the energy relation (3.5) and (4.6), there exists a positive constant $C_B$ depending on $B$ such that

$$E_0(t) \leq C(1 + E(t)) \leq C(1 + E(0)) \leq C_B.$$  

(4.18)

Therefore, we attain

$$| - \alpha \int_{\Omega} \|\nabla u_t\|^{p}\nabla u_t \cdot \nabla u \, dx| \leq \frac{\alpha}{2} \|\nabla u_t\|^{p+2} + \frac{\alpha C\delta C_B}{2} \|\nabla u_t\|^{p+2} + \frac{\alpha\delta}{2} \|\nabla u\|^2.$$  

(4.19)
As a consequence of (4.14)-(4.19)

\[
\frac{d}{dt}V(t) + \alpha V(t)
\leq \frac{3\alpha}{2} \|u_t\|^2 - \frac{\alpha}{2} \|\nabla u\|^2 + \frac{\alpha \lambda}{2\lambda_1} \|\nabla u\|^2 + C_0 \alpha + \frac{\alpha^2}{2} \|u_t\|^2 + \frac{\alpha^2}{2\lambda_1} \|\nabla u\|^2
\]

\[
- \|\nabla u_t\|^{p+2} + \frac{\alpha}{2} \|\nabla u_t\|^{p+2} + \frac{\alpha C_3 C_B}{2} \|\nabla u_t\|^{p+2} + \frac{\alpha \delta}{2} \|\nabla u\|^2
\]

\leq \left( \frac{3\alpha}{2} + \frac{\alpha^2}{2} \right) \|u_t\|^2 - \left( \frac{\alpha}{2} - \frac{\alpha \lambda}{2\lambda_1} - \frac{\alpha^2}{2\lambda_1} - \frac{\alpha \delta}{2} \right) \|\nabla u\|^2 + C\alpha
\]

\[
- (1 - \frac{\alpha}{2} - \frac{\alpha C_3 C_B}{2}) \|\nabla u_t\|^{p+2}.
\]

By Poincare’s inequality and Young’s inequality, we infer that there exist constants \(c_2, c_3 > 0\), such that

\[
\|u_t\|^2 \leq \frac{1}{\lambda_1} \|\nabla u_t\|^2 \leq \frac{c_2}{\lambda_1} \|\nabla u_t\|^{p+2},
\]

hence

\[
\left( \frac{3\alpha}{2} + \frac{\alpha^2}{2} \right) \|u_t\|^2 \leq \frac{c_3}{\lambda_1} \left( \frac{3\alpha}{2} + \frac{\alpha^2}{2} \right) \|\nabla u_t\|^{p+2} + \frac{c_2}{\lambda_1} \left( \frac{3\alpha}{2} + \frac{\alpha^2}{2} \right).
\]

Substituting these estimates (4.21)-(4.22) into (4.20), it follows that

\[
\frac{d}{dt}V(t) + \alpha V(t) \leq - (1 - \frac{\alpha}{2} - \frac{\alpha C_3 C_B}{2} - \frac{c_3}{\lambda_1} \left( \frac{3\alpha}{2} + \frac{\alpha^2}{2} \right)) \|\nabla u_t\|^{p+2}
\]

\[
- \left( \frac{\alpha}{2} - \frac{\alpha \lambda}{2\lambda_1} - \frac{\alpha^2}{2\lambda_1} - \frac{\alpha \delta}{2} \right) \|\nabla u\|^2
\]

\[
+ \frac{c_2}{\lambda_1} \left( \frac{3\alpha}{2} + \frac{\alpha^2}{2} \right) + C\alpha.
\]

Choosing \(\alpha\) small enough, such that

\[
\frac{\alpha}{2} - \frac{\alpha \lambda}{2\lambda_1} - \frac{\alpha^2}{2\lambda_1} - \frac{\alpha \delta}{2} > 0;
\]

\[
1 - \frac{\alpha}{2} - \frac{\alpha C_3 C_B}{2} - \frac{c_3}{\lambda_1} \left( \frac{3\alpha}{2} + \frac{\alpha^2}{2} \right) > 0;
\]

which implies

\[
\frac{d}{dt}V(t) + \alpha V(t) \leq \frac{c_2}{\lambda_1} \left( \frac{3\alpha}{2} + \frac{\alpha^2}{2} \right) + C\alpha
\]

\[
\leq C_3 \alpha,
\]

where \(C_3 = \frac{2\alpha}{\lambda_1} + C\) is independent of \(B\) and \(\alpha\). Applying Gronwall’s Lemma, we have

\[
V(t) \leq V(0)e^{-\alpha t} + \frac{C_3 \alpha}{\alpha} \left( 1 - e^{-\alpha t} \right)
\]

\[
\leq V(0)e^{-\alpha t} + C_3(1 - e^{-\alpha t}).
\]

Therefore, there exists a time \(t_0 = \frac{1}{\alpha} \ln \frac{V(0)}{C_3}\), such that \(V(t) \leq 2C_3, \forall t \geq t_0\). Using energy inequality (4.10),

\[
\|(u,u_t)\|_H \leq \frac{2C_3 + C_1}{C_1} = \rho_0.
\]
This concludes the proof of the dissipativity. \hfill \square

5. The existence of the global attractor. It is well known that if we want to prove the existence of global attractors, the key point is to obtain the compactness of the semigroups in some sense.

At first, we will review the abstract results of the global attractors.

Definition 5.1. ([9]) Let \((X, S_t)\) be a dynamical system.

- A closed set \(B \subset X\) is said to be absorbing for \((X, S_t)\) iff for any bounded set \(D \subset X\) there exists \(t_0(D)\) such that \(S_t D \subset B\) for all \(t \geq t_0(D)\).
- \((X, S_t)\) is said to be (ultimately) dissipative iff it possesses a bounded absorbing set \(B\). If \(X\) is an Banach space, then a value \(R > 0\) is said to be a radius of dissipativity of \((X, S_t)\) iff for any \(x \in B\), \(\|x\|_X \leq R\).
- \((X, S_t)\) is said to be asymptotically smooth iff for any bounded set \(D\) such that \(S_t D \subset D\) for \(t > 0\) there exists a compact set \(K\) in the closure \(\overline{D}\) of \(D\), such that
  \[
  \lim_{t \to +\infty} d_X \{S_t D | A\} = 0, \tag{5.1}
  \]
  where \(d_X \{S_t D | A\} = \sup_{x \in A} \text{dist}_X(S_t x, B)\) is the Hausdorff semidistance.

Definition 5.2. ([9]) A bounded closed set \(A \subset X\) is said to be a global attractor of the dynamical system \((X, S_t)\) iff

(i) \(A\) is an invariant set, i.e. \(S_t A = A\) for \(t \geq 0\), and
(ii) \(A\) is uniformly attracting, i.e. for all bounded set \(D \subset X\)

\[
\lim_{t \to +\infty} d_X \{S_t D | A\} = 0. \tag{5.2}
\]

Definition 5.3. ([25])(\(\omega\)-limit compact) For any bounded set \(B\) of the Banach space \(X\), and for any \(\varepsilon > 0\), there exist \(T > 0\) and a finite dimensional subspace \(X_1\) of \(X\), such that

\[
\kappa(\bigcup_{t \geq T} S(t)B) \leq \varepsilon, \tag{5.3}
\]

where \(\kappa\) denotes the Kuratowski noncompactness measure, the definition as follows;

\[
\kappa(B) = \inf \{\delta > 0 : B\text{ has a finite cover of diameter }< \delta\}.
\]

Theorem 5.4. ([25]) Let \(\{S(t)\}_{t \geq 0}\) be a continuous semigroup in the complete metric space \(X\). Then \(\{S(t)\}_{t \geq 0}\) possesses a compact global attractor \(A\) if and only if

(i) \(\{S(t)\}_{t \geq 0}\) has a bounded absorbing set \(B \subset X\);
(ii) \(\{S(t)\}_{t \geq 0}\) is \(\omega\)-limit compact.

5.1. When the nonlinearity \(f(u)\) is up to the subcritical case. On the basis of the above, we have the following theorem for determining the existence of global attractor.

Ulteriorly, in the specific application, Ma, Wang and Zhong give a more convenient verification method in [25], that is, the following (C) condition.

Definition 5.5. ([25]) (Condition (C)) For any bounded set \(B\) of the Banach space \(X\), and for any \(\varepsilon > 0\), there exist \(T(B) > 0\) and a finite dimensional subspace \(X_1\) of \(X\), such that \(\|PS(t)B\|\) is bounded and

\[
\|(I - P)S(t)B\| < \varepsilon, \quad \forall t \geq T(B) \tag{5.4}
\]

where \(P : X \to X_1\) is a bounded projector.
Theorem 5.6. ([25]) Let $\{S(t)\}_{t \geq 0}$ be a continuous semigroup in the complete metric space $X$, and meet the following conditions,

(i) $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set $B \subset X$,
(ii) $\{S(t)\}_{t \geq 0}$ satisfies the condition (C). Then $\{S(t)\}_{t \geq 0}$ possesses a compact global attractor $\mathcal{A}$.

Furthermore, when $X$ is a uniformly convex Banach space (especially, $X$ is a Hilbert space), condition (i) and (ii) are also the necessary condition for the existence of global attractor.

Therefore, in order to prove $\{S(t) : t \geq 0\}$ is $\omega$-limit compact, we need to show that the exterior estimate is small enough. Assuming that $\{w_j\}$ is the complete orthonormal basis of $L^2(\Omega)$, such that

$$
\begin{cases}
-\Delta w_j = \lambda_j w_j, w_j|_{\partial \Omega} = 0, \ j = 1, 2, \\
(w_i, w_j) = \delta_{i,j}, i \neq j; \ (w_i, w_i) = 0 = \eta, \ i = j,
\end{cases}
$$

(5.5)

where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \to +\infty$, as $j \to +\infty$.

Let $V_m = \text{span}\{w_1, \ldots, w_m\}$ is the subspace of $H^1_0(\Omega)$ and the orthogonal projector $P_m : H^1_0(\Omega) \to V_m$. Hence, each $u \in H^1_0(\Omega)$ can be decomposed as
$$
u = P_m u + (I - P_m) u = u^1 + u^2, \ u^1 \in V_m, \ u^2 \in V_m^1.
$$

Consequently, the following theorem provides the estimates for $(u^2, u^2)$.

Theorem 5.7. Assume that the assumptions (H1) and (H2) hold. For any $\varepsilon > 0$ and positive invariant bounded subset $B$, there exist $t_0 > 0$, $m_0 > 0$, such that if $t \geq t_0$, $m \geq m_0$, we have
$$
\|\langle u^2(t), u^2(t) \rangle\|^2_H \leq \varepsilon, \text{ for any } (u^1, u^1) \in B.
$$

(5.7)

Proof. Since $B$ be a positively invariant bounded subset in $\mathcal{H}$, then for any $(u_0, u_1) \in B$, by dissipative of the system, it holds that
$$
\|(u(t), u(t))\|_H \leq \rho_0,
$$

(5.8)

for all $t \geq t(\rho_0)$. With the energy relation (3.5) and (4.6), there exists a positive constant $M$ (depends on $E_0(0)$) such that
$$
\int_0^t \|\nabla u(t)\|^{p+2} ds \leq E(0) - c_0 E_0(t) + C_0 \leq M,
$$

(5.9)

holds for any $t \geq t(\rho_0)$.

Choosing $0 < \eta \leq \min\{\frac{1}{2}, \frac{\lambda_1}{2}, \frac{\lambda_2}{3}\}$ and multiplying $u^2_t, \eta u^2$ respectively to the equation (1.1) in $L^2(\Omega)$, we get
$$
\frac{d}{dt} \left( \frac{1}{2} \|u^2_t\|^2 + \frac{1}{2} \|\nabla u^2_t\|^2 \right) + \|\nabla u_t\|^p \|\nabla u^2_t\|^2 + (f(u), u^2_t) = (h, u^2_t),
$$

(5.10)

and
$$
\frac{d}{dt} \eta(u^2, u^2_t) - \eta \|u_t\|^2 + \eta \|\nabla u^2_t\|^2 + \eta \|\nabla u_t\|^p (\nabla u^2_t, \nabla u^2_t) + \eta (f(u), u^2_t) = \eta (h, u^2).
$$

(5.11)

Denote
$$
W_2(t) = \frac{1}{2} \|u^2_t\|^2 + \frac{1}{2} \|\nabla u^2_t\|^2 + \eta (u^2, u^2_t),
$$

(5.12)
hence,
\[
\frac{d}{dt}W_2(t) + \eta W_2(t) \\
= -\|\nabla u_t\|^p \|\nabla u_t^2\|^2 - (f(u), u_t^2 + \eta u^2) + (h, u_t^2 + \eta u^2) \\
+ \frac{3\eta}{2} \|u_t^2\|^2 - \frac{\eta}{2} \|\nabla u_t^2\|^2 - \eta \|\nabla u_t\|^p (\nabla u_t^2, \nabla u^2) + \eta^2 (u^2, u_t^2).
\] (5.13)

By assumptions (H1)-(H2) and the compact embedding \(H^1_0(\Omega) \hookrightarrow L^{2(q+1)}(\Omega)\) (where \(q < \frac{2}{m-2}\)), for above \(\varepsilon > 0\), there exists \(m_i \in N^+\) (i=1,2,3) and \(t_j \geq 0\) (j=2,3,4) such that
\[
\|u^2(t)\| < \varepsilon, \ \forall \ t \geq t_2, \ m \geq m_1; \quad (5.14)
\]
\[
\| (I - P_m)f(u(t)) \| < \eta \varepsilon^2, \ \forall \ t > t_3, m \geq m_2; \quad (5.15)
\]
\[
\| (I - P_m)h \| < \eta \varepsilon^2, \ \forall \ t > t_4, m \geq m_3. \quad (5.16)
\]

Using Hölder inequality, it follows that
\[
\|u_t\|^p \|\nabla u_t^2\|^2 \leq \eta \|\nabla u_t\|^p \|\nabla u_t^2\| \leq \eta \|\nabla u_t\|^p \left(\frac{1}{2} \|\nabla u_t^2\|^2 + \frac{1}{2} \|\nabla u_t\|^2 \right).
\] (5.17)

By the Young's inequality with \(\varepsilon\), we obtain
\[
\|u_t^2\|^2 \leq \frac{1}{\lambda_1} \|\nabla u_t^2\|^2 \\
\leq \frac{1}{\lambda_1} (\varepsilon + \varepsilon^{-\frac{p}{2}} \|\nabla u_t^2\|^{p+2}) \quad (5.18)
\]

Applying the dissipation, it is obvious that there exists a bounded positive constant \(C(\eta)\) (about \(\eta, \rho_0\)) such that
\[
\|u_t^2(t) + \eta u^2(t)\| \leq 2(1 + \eta)\rho_0^\frac{4}{p} = C(\eta), \ \text{as} \ t > t(\rho_0). \quad (5.19)
\]

Substituting these estimates, combining (5.14)-(5.19) with (5.13), it follows that
\[
\frac{d}{dt}W_2(t) + \eta W_2(t) \\
\leq -\|\nabla u_t\|^p \|\nabla u_t^2\|^2 + \frac{3\eta}{2} \|u_t^2\|^2 - \frac{\eta}{2} \|\nabla u_t\|^2 - \eta \|\nabla u_t\|^p (\nabla u_t^2, \nabla u^2) \\
- \| (I - P_m)f(u) \| \|u_t^2\|^2 + \| (I - P_m)h \| \|u_t^2\|^2 + C\eta^2 \varepsilon \\
\leq \frac{\eta}{4} \|\nabla u_t\|^p \|\nabla u_t^2\|^2 - \left(2 - \frac{3\eta}{2\lambda_1 \varepsilon^{\frac{p}{2}}} \right) \|\nabla u_t\|^p \|\nabla u_t^2\|^2 - \frac{\eta}{4} \|\nabla u_t\|^2 \\
+ \frac{3\eta}{2\lambda_1} + C\eta^2 \varepsilon + C(\eta) \eta \varepsilon \\
\leq \frac{\eta}{4} \|\nabla u_t\|^p \|\nabla u_t^2\|^2 + C^*(\eta) \eta \varepsilon,
\]
where \(C^*(\eta) = \frac{3}{2\lambda_1} + C\eta + C(\eta)\). To sum up, we get
\[
\frac{d}{dt}W_2(t) + \eta W_2(t) \leq \frac{\eta}{4} \|\nabla u_t\|^p \|\nabla u_t^2\|^2 + C^*(\eta) \eta \varepsilon, \ \text{t} \geq t_1, m \geq m_0. \quad (5.21)
\]
where \( t_1 = \max\{t_2, t_3, t_4, t(\rho_0)\} \), \( m_0 = \max\{m_1, m_2, m_3\} \in \mathbb{N}^+ \). By Hölder inequality,
\[
|\eta(u_t^2, u^2)| \leq \frac{\eta}{2} \|u_t^2\|^2 + \frac{\eta}{2\lambda_1} \|\nabla u^2\|^2. \tag{5.22}
\]
From this and the range of \( \eta \), it is easy to deduce that
\[
W_2(t) \geq \frac{1}{2}(1 - \eta)\|u_t^2\|^2 + \frac{1}{2}(1 - \frac{\eta}{\lambda_1})\|\nabla u^2\|^2 \tag{5.23}
\]
which indicates
\[
\frac{d}{dt}W_2(t) + \eta W_2(t) \leq \eta\|\nabla u_t\|^p + 2W_2(t) + C^*(\eta)\eta \varepsilon, \ t \geq t_1, m \geq m_0. \tag{5.24}
\]
Thus, applying Gronwall’s inequality, we obtain
\[
W_2(t) \leq W_2(t_1)e^{-\eta(t-t_1) + \eta \int_{t_1}^t \|\nabla u_t(\tau)\|^p + 2d\tau}
+ C^*(\eta)\varepsilon(1 - e^{-\eta(t-t_1)})e^{\eta \int_{t_1}^t \|\nabla u_t(\tau)\|^p + 2d\tau}
\leq \rho_0 M_1 e^{-\eta(t-t_1)} + C^*(\eta)\varepsilon M_1,
\tag{5.25}
\]
where \( M_1 = e^{\frac{\eta}{2}M} \), \( t \geq t_1 \) and \( m \geq m_0 \). Choosing \( t_0 = t_1 + \frac{1}{\eta} \ln \frac{\rho_0 M_1}{\varepsilon} \), we have
\[
W_2(t) \leq (1 + C^*(\eta)M_1)\varepsilon, \ t \geq t_0, \ m \geq m_0. \tag{5.26}
\]
Then there exists a positive constant \( \tilde{C} \) such that
\[
\|(u_t^2, u^2)\|^2_t \leq \tilde{C}\varepsilon, \ \forall \ t \geq t_0, \ m \geq m_0, \tag{5.27}
\]
which completes the proof. □

Combining Theorem 4.1, Theorem 5.7 with Theorem 5.6, we can attain the existence of global attractor for the system \((\mathcal{H}, S(t))\) generated by (1.1), which given by the following theorem.

**Theorem 5.8.** Let assumptions \((H_1)\) and \((H_2)\) be in force, then the system \((\mathcal{H}, S(t))\) generated by the equation (1.1) in the space \( \mathcal{H} \) possesses a compact global attractor \( \mathcal{A} \).

### 5.2. When the nonlinearity \( f(u) \) is up to the critical case.

In this section, we will research the existence of global attractor when the nonlinearity term \( f(u) \) is up to critical case. Below, we list some theorems and proposition which need to be applied into solving this issue.

**Theorem 5.9.** ([9]) Let \( X \) be a dissipative dynamical system in a complete metric space \( X \). Then \((X, S_t)\) possesses a compact global attractor \( A \) if and only \((X, S_t)\) is asymptotically smooth.

**Theorem 5.10.** ([9]) Let \((X, S_t)\) be a dynamical system on a complete metric space \( X \) endowed with a metric \( d \). Assume that for any bounded positively invariant set \( B \) in \( X \) and for any \( \varepsilon > 0 \), there exists \( T = T(\varepsilon, B) \) such that
\[
d(S_{T} y_1, S_{T} y_2) \leq \varepsilon + \Psi_{\varepsilon, B, T}(y_1, y_2), \ y_i \in B, \tag{5.28}
\]
where \( \Psi_{\varepsilon, B, T}(y_1, y_2) \) is a function defined on \( B \times B \) such that
\[
\lim_{n \to +\infty} \lim_{m \to +\infty} \Psi_{\varepsilon, B, T}(y_n, y_m) = 0 \tag{5.29}
\]
for every sequence $\{y_n\}$ from $B$. Then, $(X,S_t)$ is an asymptotically smooth dynamical system.

According to Theorem 4.1, it can be indicated that the dynamical system is dissipative with a bounded absorbing set. Therefore, we need to verify the asymptotically smooth of the dynamical system $(H,S(t))$ generated by problem (1.1). In order to obtain this conclusion, we use the following notations.

**Lemma 5.11.** Under assumptions $(H_1)$ and $(H_2)$, there exists $T_0 > 0$ and a constant $C > 0$ independent of $T$ such that for any pair $w$ and $v$ of strong solutions to (1.1), we have the following relation:

$$TE_z(T) + \int_0^T E_z(t)dt$$

$$\leq C \left\{ \int_0^T \|z(t)\|dt + \int_0^T (\|\nabla v(t)\|^p \Delta v_t - \|\nabla w(t)\|^p \Delta w_t, z_t)dt \right\}$$

$$\leq C \left\{ \int_0^T (\|\nabla v(t)\|^p \Delta v_t - \|\nabla w(t)\|^p \Delta w_t, z_t)dt \right\}$$

for every $T > T_0$, where $z(t) = w(t) - v(t)$, $(w_0, w_1), (v_0, v_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$, and we use the following notations:

$$E_z(t) = \frac{1}{2}(\|z(t)\|^2 + \|\nabla z(t)\|^2),$$

and

$$\Psi_T(w, v) = \left| \int_0^T (f(w) - f(v), z)dt \right| + \left| \int_0^T \int_t^T (f(w) - f(v), z_t)d\tau dt \right|$$

$$+ \left| \int_0^T (f(w) - f(v), z_t)dt \right|.$$

**Proof.** Due to $z(t) = w(t) - v(t)$, it is clear that the variable $z$ satisfies the following equation

$$z_{tt} - \Delta z + \|\nabla v(t)\|^p \Delta v_t - \|\nabla w(t)\|^p \Delta w_t + f(w) - f(v) = 0. \quad (5.33)$$

Multiplying (5.33) by $z_t(t)$ and integrating over $\Omega \times [t,T]$, we obtain

$$E_z(T) + \int_t^T (\|\nabla v(t)\|^p \Delta v_t - \|\nabla w(t)\|^p \Delta w_t, z_t)dt$$

$$= E_z(t) - \int_t^T (f(w) - f(v), z_t)d\tau. \quad (5.34)$$

Multiplying (5.33) by $z$ and integrating over $\Omega \times [0,T]$, we have

$$2 \int_0^T E_z(t)dt - 2 \int_0^T \|z(t)\|^2dt + \int_0^T (z, z_t)dt$$

$$= \int_0^T (\|\nabla v(t)\|^p \Delta v_t - \|\nabla w(t)\|^p \Delta w_t, z)dt - \int_0^T (f(w) - f(v), z)dt. \quad (5.35)$$

Applying Hölder’s inequality and (2.1), we get

$$|(z(t), z_t)| \leq \frac{1}{2} (\|z(t)\|^2 + \|z_t(t)\|^2) \leq CE_z(t), \quad (5.36)$$
and bring this estimate into (5.35), we obtain
\[
2 \int_0^T E_z(t) dt \leq C_0 (E_z(T) + E_z(0)) + 2 \int_0^T \| z_t(t) \|^2 dt \\
+ \int_0^T \left( \| \nabla v_t(t) \|^p \Delta v_t - \| \nabla w_t(t) \|^p \Delta w_t, z_t \right) dt - \int_0^T (f(w) - f(v), z_t) dt.
\]
(5.37)

Furthermore, integrating (5.34) from 0 to \( T \) and using the monotonicity of the damping, it can be derived that
\[
TE_z(T) \leq \int_0^T E_z(t) dt - \int_0^T \int_t^T (f(w) - f(v), z_t) d\tau dt.
\]
(5.38)

Setting \( t = 0 \) in (5.34) and combining with (5.37), (5.38), it can be obtained that
\[
TE_z(T) + \int_0^T E_z(t) dt \\
\leq C \left\{ \int_0^T \| z_t(t) \|^2 dt + \int_0^T \left( \| \nabla v_t(t) \|^p \Delta v_t - \| \nabla w_t(t) \|^p \Delta w_t, z_t \right) dt \\
+ \left| \int_0^T \left( \| \nabla v_t(t) \|^p \Delta v_t - \| \nabla w_t(t) \|^p \Delta w_t, z_t \right) dt \right| \right\}.
\]
(5.39)

Thus, the formula (5.30) is established, which completes the proof. \( \square \)

According to Theorem 4.1, let \( B_0 \) be a bounded absorbing set, by the definition, there exists \( t_0 \geq 0 \) such that \( S(t) B_0 \subset B_0 \) for all the \( t \geq t_0 \). Let \( B = \bigcup_{t \geq t_0} S(t) B_0 \).

It is clear that \( B \) is a closed bounded forward invariant set for this system. Since for any bounded set \( B \), we have \( S(t) B \subset B_0 \) for all the \( t \geq t(B) \), we obtain that \( S(t) B \subset B \) for all \( t \geq t_0 + t(B) \). Hence, \( B \) is also an absorbing set for this system \( \mathcal{H}; S(t) \).

**Theorem 5.12.** Let assumptions \((H_1), (H_2)\) hold, then the system \((\mathcal{H}, S(t))\) generated by (1.1) in the space is asymptotically smooth.

**Proof.** Let \( w(t) \) and \( v(t) \) be two generalized solutions to (1.1) corresponding to two different initial data in invariant set \( B \):
\[
(w(t), w_t(t)) \equiv S(t)y_0, \quad (v(t), v_t(t)) \equiv S(t)y_1, \quad y_0, \ y_1 \in B.
\]
(5.40)

Since the generalized solution of the problem (1.1) should be approximated by a sequence strong solutions for (1.1), we may as well assume that \( w(t) \) and \( v(t) \) are two strong solutions with \( y_0, \ y_1 \in \mathcal{H} \cap (H^1_0(\Omega) \times H^1_0(\Omega)) \).

Since \( B \) is bounded forward invariant set, from energy equality (3.5) and the continuity of \( f \), we could derive
\[
\int_0^T (\| \nabla w_t(t) \|^p (-\Delta w_t), w_t) dt + \int_0^T (\| \nabla v_t(t) \|^p (-\Delta v_t), v_t) dt \leq C_B.
\]
(5.41)

Setting \( z(t) = w(t) - v(t) \), thus \( z \) satisfies the equation
\[
z_{tt} - \Delta z + \| \nabla v_t(t) \|^p \Delta v_t - \| \nabla w_t(t) \|^p \Delta w_t + f(w) - f(v) = 0.
\]
(5.42)
Let us start with the following consequence of Lemma 5.11. Using Cauchy’s inequality and, Young’s inequality, we get
\[
\left| \left( \| \nabla w_t(t) \|^p (-\Delta w_t), z \right) \right| 
\leq \left| \left( \| \nabla w_t(t) \|^p \nabla w_t, \nabla z \right) \right| 
\leq \| \nabla w_t(t) \|^p (C_\delta \| \nabla w_t(t) \|^2 + \delta \| \nabla z \|^2)
\leq C_\delta \| \nabla w_t(t) \|^p + \delta (\varepsilon + C_\varepsilon \| \nabla w_t(t) \|^p) \| \nabla z \|^2
\leq (C_\delta + \delta C_\varepsilon \| \nabla z \|^2) \| \nabla w_t(t) \|^p + \delta \| \nabla z \|^2
\leq (C_\delta + \delta C_\varepsilon \| \nabla z \|^2) \| \nabla w_t(t) \|^p + \varepsilon \delta C_B
= C^\delta_B \| \nabla w_t(t) \|^p + \varepsilon \delta C_B,
\]
where \( C^\delta_B := C_\delta + \delta C_\varepsilon C_B \). Rescaling \( \varepsilon \delta C_B := \varepsilon \), therefore,
\[
\left| \left( \| \nabla v_t(t) \|^p \Delta v_t - \| \nabla w_t(t) \|^p \Delta w_t, z \right) \right| 
\leq \left| \left( \| \nabla w_t(t) \|^p \nabla w_t, \nabla z \right) \right| + \left| \left( \| \nabla v_t(t) \|^p \nabla v_t, \nabla z \right) \right|
\leq C^\delta_B \| \nabla w_t(t) \|^p + \| \nabla v_t(t) \|^p + 2\varepsilon,
\]
combining with (5.41), which indicates clearly that
\[
\left| \int_0^T \left( \| \nabla v_t(t) \|^p \Delta v_t - \| \nabla w_t(t) \|^p \Delta w_t, z_t \right) dt \right| \leq 2\varepsilon T + C^\delta_B C_B. \tag{5.45}
\]
And with Young’s inequality, it is obvious that for any \( \eta > 0 \) there exists \( C_\eta \) satisfies
\[
\int_0^T \| z_t(t) \|^2 dt \leq \int_0^T (\| w_t(t) \|^2 + \| v_t(t) \|^2) dt
\leq \int_0^T 2\eta + C_\eta \| \nabla w_t(t) \|^p + C_\eta \| \nabla v_t(t) \|^p dt
\leq 2\eta T + C^\eta_B. \tag{5.46}
\]
Let \( t = 0 \) in (3.34) and since \( E_z(0) \leq C_B \), we obtain
\[
\left| \int_0^T \left( \| \nabla v_t(t) \|^p \Delta v_t - \| \nabla w_t(t) \|^p \Delta w_t, z_t \right) dt \right|
\leq \left| E_z(0) - \int_0^T (f(w) - f(v), z_t) dt \right|
\leq C_B + \left| \int_0^T (f(w) - f(v), z_t) dt \right|
\leq C_B + \Psi_T(w, v),
\]
where \( \Psi_T(w, v) \) is same as the notation in (5.32).
Therefore, combining (5.45), (5.46) and (5.47) together, from Lemma 5.11, we have the estimates
\[
TE_z(T) \leq C \left\{ 2\eta T + C^\eta_B + 2\varepsilon T + C^\delta_B C_B + C_B + 2\Psi_T(w, v) \right\}
\leq 2(\varepsilon \delta C_B + \eta) CT + C_B, \tag{5.48}
\]
for any positive $\delta, \eta$, where $C_{B, \delta, \eta} := C(C_{B}^{\eta} + C_{B}^{\delta}C_{B} + C_{B})$. Therefore, substituting $2C(\eta + \varepsilon) := \eta$ into (5.48), we get that
\[
E_{z}(T) \leq \eta + \frac{C_{B, \delta, \eta}}{T}(1 + \Psi_{T}(w, v)).
\] (5.49)
Furthermore, let us show that
\[
\lim_{n \to +\infty} \lim_{m \to +\infty} \Psi_{T}(w_{n}, w_{m}) = 0,
\] (5.50)
where $(w_{n}(t), w_{n,t}(t)) = S(t)y_{n}^{0}$ with initial data $\{y_{n}^{0}\}$ from $B$.
From the above observation, without loss of generality (or by passing to subsequences), we assume that
\[
w_{n} \to w* -\text{weakly in } L^{\infty}(0, T; H_{0}^{1}(\Omega)),
\] (5.51)
\[
w_{n,t} \to w_{t} -\text{weakly in } L^{\infty}(0, T; L^{2}(\Omega)),
\] (5.52)
\[
w_{n} \to w \text{ strongly in } L^{2}(0, T; L^{2}(\Omega)),
\] (5.53)
and
\[
w_{n}(0) \to w(0) \text{ and } w_{n}(T) \to w(T) \text{ in } L^{q+2}(\Omega),
\] (5.54)
here we have used the compact embedding $H_{0}^{1}(\Omega) \hookrightarrow L^{q+2}(\Omega)$ (where $q \leq \frac{2n}{n-2}$).
Using the similar method in (3.24), we have
\[
\left| \int_{0}^{T} (f(w_{n}(t)) - f(w_{m}(t)), w_{n}(t) - w_{m}(t))dt \right|
\leq \int_{0}^{T} \left| \int_{\Omega} (f(w_{n}(t)) - f(w_{m}(t))(w_{n}(t) - w_{m}(t)) \right| dt
\leq \int_{0}^{T} \|f(w_{n}(t)) - f(w_{m}(t))\| \|w_{n}(t) - w_{m}(t)\| dt
\leq C(B) \int_{0}^{T} \|w_{n}(t) - w_{m}(t)\|^{2} dt,
\] (5.55)
and then combining with (5.53),
\[
\lim_{n \to +\infty} \lim_{m \to +\infty} \left| \int_{0}^{T} (f(w_{n}(t)) - f(w_{m}(t)), w_{n}(t) - w_{m}(t))dt \right| = 0.
\] (5.56)
Next, according to decomposing items, we obtain
\[
\int_{0}^{T} (f(w_{n}(t)) - f(w_{m}(t)), w_{n,t}(t) - w_{m,t}(t))dt
= \int_{\Omega} F(w_{n}(T))dx - \int_{\Omega} F(w_{n}(0))dx + \int_{\Omega} F(w_{m}(T))dx
- \int_{\Omega} F(w_{m}(0))dx - \int_{0}^{T} (f(w_{n}(t)), w_{n,t}(t))dt - \int_{0}^{T} (f(w_{m}(t)), w_{m,t}(t))dt.
\] (5.57)
By (5.51), (5.52) and (5.54), taking first

\[
\lim_{n \to +\infty} \lim_{m \to +\infty} \int_0^T (f(w_n(t)) - f(w_m(t)), w_{n,t}(t) - w_{m,t}(t))dt = 2 \left( \int_\Omega F(w(T))dx - \int_\Omega F(w(0))dx \right)
\]

- \lim_{n \to +\infty} \lim_{m \to +\infty} \int_0^T (f(w_n(t)), w_{m,t}(t))dt

- \lim_{n \to +\infty} \lim_{m \to +\infty} \int_0^T (f(w_m(t)), w_{n,t}(t))dt.

As a result of (5.51) and \( f(w_n) \in L^\infty(0, T; L^2(\Omega)) \), we can have

\[
f(w_n) \to f(w(t)) \ast -\text{weakly in } L^\infty(0, T; L^2(\Omega)),
\]

therefore, by (5.52) and (5.59), it can be achieved that

\[
\lim_{n \to +\infty} \lim_{m \to +\infty} \int_0^T (f(w_n(t)), w_{m,t}(t))dt = \lim_{n \to +\infty} \int_0^T (f(w(t)), w_t(t))dt = \int_\Omega F(w(T))dx - \int_\Omega F(w(0))dx.
\]

Adding (5.59), (5.60) into (5.58), we deduce

\[
\lim_{n \to +\infty} \lim_{m \to +\infty} \int_0^T (f(w_n(t)) - f(w_m(t)), w_{n,t}(t) - w_{m,t}(t))dt = 0.
\]

Meanwhile, it is clear that

\[
\int_t^T (f(w_n(t)) - f(w_m(t)), w_{n,t}(t) - w_{m,t}(t))dt
= \int_\Omega F(w_n(T))dx - \int_\Omega F(w_n(t))dx + \int_\Omega F(w_m(T))dx - \int_\Omega F(w_m(t))dx - \int_t^T (f(w_n(t)), w_{m,t}(t))dt - \int_t^T (f(w_m(t)), w_{n,t}(t))dt,
\]

correspondingly, \( \left| \int_t^T (f(w_n(t)) - f(w_m(t)), w_{n,t}(t) - w_{m,t}(t))dt \right| \) is bounded for every fixed \( T > 0 \). Thus, applying Lebesgue dominated convergence theorem, we have

\[
\lim_{n \to +\infty} \lim_{m \to +\infty} \int_0^T \int_t^T (f(w_n(\tau)) - f(w_\tau(\tau)), w_{n,t}(\tau) - w_{m,t}(\tau))d\tau dt
= \int_0^T \int_t^T \lim_{n \to +\infty} \lim_{m \to +\infty} (f(w_n(\tau)) - f(w_\tau(\tau)), w_{n,t}(\tau) - w_{m,t}(\tau))d\tau dt = 0.
\]

Thus, we obtain

\[
\lim_{n \to +\infty} \lim_{m \to +\infty} \Psi_T(w_n, w_m, T) = 0
\]

Applying Proposition 5.10, the dynamical system \((\mathcal{H}, S(t))\) is asymptotically smooth, which completes the proof.
Theorem 5.13. Let assumptions (H₁) and (H₂) hold, then the system \((\mathcal{H}, S(t))\) generated by the equation (1.1) in the space \(\mathcal{H}\) possesses a compact global attractor \(A\).

Proof. Combining Theorem 4.1 and Theorem 5.12, according to Theorem 5.9, we can attain the existence of global attractor for the system \((\mathcal{H}, S(t))\). \qed

REFERENCES

[1] G. Andrews and J. M. Ball, Asymptotic behavior and changes in phase in one-dimensional nonlinear viscoelasticity, J. Differential Equations, 44 (1982), 306–341.
[2] D. D. Ang and A. P. N. Dinh, Strong solutions of a quasi-linear wave equation with nonlinear damping term, SIAM J. Math. Anal., 19 (1988), 337–347.
[3] F. Abou, I. Ben Hassen and A. Harra, Compactness of trajectories to some nonlinear second order evolution equations and applications, J. Math. Pures Appl., 100 (2013), 295–326.
[4] J. M. Ball, Global attractors for damped semilinear wave equations, Discrete Continuous Dynam. Systems, 10 (2004), 31–52.
[5] V. Belleri and V. Pata, Attractors for semilinear strongly damped wave equations on \(\mathbb{R}^3\), Discrete Continuous Dynam. Systems, 7 (2001), 719–735.
[6] A. V. Balakrishnan and L. W. Taylor, Distributed Parameter Nonlinear Damping Models for Flight Structures, Proceedings Damping 89, Flight Dynamics Lab and Air Force Wright Aeronautical Labs, WPAFB, 1989.
[7] F. Chen, B. Guo and P. Wang, Long time behavior of strongly damped nonlinear wave equations, J. Differential Equations, 147 (1998), 231–241.
[8] I. D. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, University Lectures in Contemporary Mathematics, AKTA, Kharkiv, 1999.
[9] I. Chueshov and I. Lasiecka, Long-Time Behavior of Second Order Evolution Equations with Nonlinear Damping, Memoirs of AMS, 2008.
[10] I. Chueshov and S. Kolbasin, Long-time dynamics in plate models with strong nonlinear damping, Commun. Pure Appl. Anal., 11 (2012), 659–674.
[11] J. Clements, On the existence and uniqueness of solutions of the equation \(u_{tt} - \alpha \Delta u_{tt} + \beta \Delta x_{1} = f\), Canad. Math. Bull., 18 (1975), 181–187.
[12] E. Feireisl, Attractors for wave equations with nonlinear dissipation and critical exponent, C. R. Acad. Sci. Paris Sér. I Math, 315 (1992), 551–555.
[13] E. Feireisl, Finite-dimensional asymptotic behavior of some semilinear damped hyperbolic problems, J. Dynam. Differential Equations, 6 (1994), 23–35.
[14] J. M. Ghidaglia and A. Marzocchi, Long-time behaviour of strongly damped wave equations, global attractors and their dimension, SIAM J. Math. Anal., 22 (1991), 879–895.
[15] S. Gatti, V. Pata and S. Zelik, A Gronwall-type lemma with parameter and dissipative estimates for PDEs, Nonlinear Analysis: Theory, Methods Applications Volume., 70 (2009), 2337–2343.
[16] J. K. Hale, Asymptotic Behavior of Dissipative Systems, AMS, Providence, RI, 1988.
[17] M. A. Jorge Silva and V. Narciso, Long-time behavior for a plate equation with nonlocal weak damping, Differential Integral Equations, 27 (2014), 931–948.
[18] M. A. Jorge Silva and V. Narciso, Long-time dynamics for a class of extensible beams with nonlinear nonlocal damping, Evol. Equ. Control Theory, 6 (2017), 437–470.
[19] M. A. Jorge Silva, V. Narciso and A. Vicente, On a beam model related to flight structures with nonlocal energy damping, Discrete Continuous Dynam. Systems - B., 24 (2019), 3281–3298.
[20] S. Kawashima and Y. Shibata, Global existence and exponential stability of small solutions to nonlinear viscoelasticity, Comm. Math. Phys., 148 (1992), 189–208.
[21] T. Kobayashi, H. Pecher and Y. Shibata, On a global in time existence theorem of smooth solutions to a nonlinear wave equation with viscosity, Math. Ann., 296 (1993), 215–234.
[22] V. Kalantarov and S. Zelik, Finite-dimensional attractors for the quasi-linear strongly-damped wave equation, J. Differential Equations., 247 (2009), 1120–1155.
[23] H. Lange and G. P. Menzala, Rates of decay of a nonlinear beam equation, Differential Integral Equations., 10 (1997), 1075–1092.
[24] F. J. Meng, M. H. Yang and C. K. Zhong, Attractors for wave equations with nonlinear damping on timedependent space, *Discrete Continuous Dyn. Systems.*, **21** (2016), 205–225.

[25] Q. F. Ma, S. H. Wang and C. K. Zhong, Necessary and sufficient conditions for the existence of global attractors for semigroups and applications, *Indiana Univ. Math. J.*, **51** (2002), 1541–1559.

[26] M. Nakao, Energy decay for the quasi-linear wave equation with viscosity, *Math. Z.*, **219** (1995), 289–299.

[27] V. Pata and M. Squassina, On the strongly damped wave equation, *Comm. Math. Phys.*, **253** (2005), 511–533.

[28] I. Perai, *Multiplicity of Solutions for the p-Laplacian*, Second School of Nonlinear Functional Analysis and Applications to Differential Equations 21 April-9 May, 1997.

[29] V. Pata and S. Zelik, Smooth attractor for strongly damped wave equation, *Nonlinearity*, **19** (2006), 1495–1506.

[30] J. Simon, Compact sets in the space $L_p(0,T;B)$, *Ann. Mat. Pure Appl.*, **146** (1987), 65–96.

[31] R. E. Showalter, *Monotone Operator in Banach Spaces and Nonlinear Partial Differential Equations*, Mathematical Surveys and Monographs, 49. AMS, Providence, RI, 1997.

[32] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd edition, Applied Mathematical Sciences, 68, SpringerVerlag, New York, 1997.

[33] Z. J. Yang, P. Y. Ding and L. Li, Longtime dynamics of the Kirchhoff equations with fractional damping and supercritical nonlinearity, *J. Math. Anal. Appl.*, **442** (2016), 485–510.

[34] Z. J. Yang, Z. M. Liu and P. P. Niu, Exponential attractor for the wave equation with structural damping and supercritical exponent, *Commun. Contemp. Math.*, **18** (2016), 1550055, 13 pp.

[35] Z. J. Yang and Z. M. Liu, Global attractor of the quasi-linear wave equation with strong damping, *J. Math. Anal. Appl.*, **458** (2018), 1292–1306.

[36] C. X. Zhao, C. Y. Zhao and C. K. Zhong, The global attractor for a class of extensible beams with nonlocal weak damping, *Discrete Continuous Dyn. Systems-B*, **25** (2020), 935–955.

[37] C. Y. Zhao, C. X. Zhao and C. K. Zhong, Asymptotic behaviour of the wave equation with nonlocal weak damping and anti-damping, *J. Math. Anal. Appl.*, **490** (2020), 124186, 10 pp.

[38] C. Zhao, S. Ma and C. Zhong, Long-time behavior for a class of extensible beams with nonlocal weak damping and critical nonlinearity, *J. Math. Phys.*, **61** (2020), 032701, 15 pp.

Received December 2019; 1st revision July 2020; final revision October 2020.

E-mail address: bychenmath@163.com
E-mail address: zhaocxnju@163.com
E-mail address: ckzhong@nju.edu.cn