Hydrodynamic Interaction in Confined Geometries

Haim Diamant

School of Chemistry, Tel Aviv University, Tel Aviv 69978, Israel

This article gives an overview of recent theoretical and experimental findings concerning the hydrodynamic interaction between liquid-embedded particles in various confined geometries. A simple unifying description emerges, which accounts for the various findings based on the effect of confinement on conserved fields of the embedding liquid. It shows, in particular, that the hydrodynamic interaction under confinement remains long-ranged, decaying algebraically with inter-particle distance, except for the case of confinement in a rigid linear channel.

KEYWORDS: colloids, suspensions, microfluidics, response functions, pair diffusion, pair mobility, drag, effective viscosity, porous media

1. Introduction

Particles moving through a fluid affect each other’s velocity through the flow that their motions cause. These medium-induced, hydrodynamic interactions play a crucial role in the dynamics of all particulate liquids, such as colloid suspensions\(^1\) and polymer solutions,\(^2\) and have been thoroughly studied. In the past several years there has been significant progress in clarifying the effects of spatial confinement on hydrodynamic interactions. This research has been driven by new techniques for the fabrication of microfluidic channels\(^3\) and for the tracking and manipulation of individual particles.\(^4–6\) The findings highlight the dramatic effects that confinement on the scale of the size of particles has on their hydrodynamic interaction. It can change the sign of the interaction, its decay with distance, and its concentration dependence.\(^7–9\) Under conditions of driven flow it may also lead to a new type of density waves.\(^10,11\)

This article provides an overview of recent developments concerning the hydrodynamic interaction in confined geometries. Rather than summarizing various technical results, we attempt to present an intuitive unifying description, which derives from the effects of confining boundaries on the conserved fields of the fluid. To this end it is helpful to begin by recalling, in the following section, a few fundamentals concerning the hydrodynamic interaction in an unconfined liquid. We then proceed in §3 to show how these basic considerations are modified in three examples of confined, quasi-two-dimensional (q2D) systems. Section §4 addresses confinement in quasi-one-dimensional (q1D) channels, and in §5 we comment on hydrodynamic interactions in liquids embedded in three-dimensional (3D) solid matrices, such as gels and porous media. Finally, in §6 we discuss further implications and open issues.

\(^*\)E-mail: hdiamant@tau.ac.il
2. General Considerations

The dynamics of a fluid can be coarse-grained into continuous equations for its conserved fields.\textsuperscript{12} In an isotropic fluid these fields are the local densities of mass, momentum, and energy. If the heat conductivity is much larger than the kinematic viscosity $\nu$, thermal relaxation will be much faster than that of momentum, and the temperature can be assumed uniform. If the sound velocity $c$ is much larger than $\nu/r$, where $r$ is the length scale under consideration, sound (compressive) modes will be much faster than transverse-momentum (shear) ones, and the mass density $\rho$ can be assumed uniform. Finally, if the scales of length $a$ and velocity $v$ of the particles moving through the fluid are sufficiently small to yield a negligible Reynolds number, $va/\nu \ll 1$, the inertial terms in the (Navier-Stokes) equation for the momentum density can be neglected. Although this combination of conditions may seem restrictive, it actually holds in a broad range of circumstances relevant to particulate liquids.\textsuperscript{13} In such a case the steady-state flow satisfies the equations,

\begin{align}
&-\nabla p + \eta \nabla^2 \mathbf{v} + \mathbf{f} = 0, \hspace{1cm} (1) \\
&\nabla \cdot \mathbf{v} = 0, \hspace{1cm} (2)
\end{align}

where $\mathbf{v}(\mathbf{r})$ is the fluid velocity, $p(\mathbf{r})$ its pressure, $\mathbf{f}(\mathbf{r})$ the applied force density, and $\eta = \rho \nu$ the dynamic viscosity. Equations (1) and (2) reflect, respectively, the conservation of momentum and mass in an isothermal, incompressible liquid at zero Reynolds number.

The problem of finding the pair hydrodynamic interaction between particle 1 at the origin and particle 2 at $\mathbf{r}$ amounts to solving eqs. (1) and (2) with $\mathbf{f} = 0$, given that the particles translate with velocities $\mathbf{v}^1$ and $\mathbf{v}^2$, and subject to appropriate boundary conditions (e.g., no slip) at the surfaces of the particles and at the system boundaries. One can subsequently calculate the forces $\mathbf{F}^1$ and $\mathbf{F}^2$ acting on the particles, and from their linear dependence on the prescribed velocities establish a pair mobility tensor,

$$v_{ij}^\alpha = B_{ij}^{\alpha\beta}(\mathbf{r}) \mathbf{F}_j^\beta, \hspace{0.5cm} \alpha, \beta = 1, 2 \hspace{0.5cm} \alpha, \beta = x, y, z. \hspace{1cm} (3)$$

(Summation over repeated indices is implied throughout the article.) In particular, $B_{ij}^{21}(\mathbf{r})$ gives the velocity of a force-free particle 2 due to a force acting on particle 1, thus characterizing the pair coupling. If we define the $x$ axis along $\mathbf{r}$, the diagonal terms of $\mathbf{B}^{21}$ correspond to three coupling “polarizations”—a longitudinal coupling and two transverse ones,

$$B_{L1}(r) = B_{xx}^{21}(r\hat{x}), \hspace{1cm} B_{T1}(r) = B_{yy}^{21}(r\hat{x}), \hspace{0.5cm} B_{T2}(r) = B_{zz}^{21}(r\hat{x}). \hspace{1cm} (4)$$

For an unbounded, isotropic system these are also the eigenvalues of $\mathbf{B}^{21}$ (i.e., off-diagonal terms vanish), and the two transverse coefficients are equal. In the overdamped limit under consideration the Einstein relation safely holds. Thus, the coupling mobility coefficients are simply related to coupling diffusion coefficients, $D_{L,T}^{c} = k_B T B_{L,T}^{c}$, $k_B T$ being the thermal energy. The coupling diffusion coefficients, in turn, can be directly measured by tracking the correlated Brownian motion of particle pairs.
In the case of no-slip boundary conditions at the particle surfaces, and in the limit of large inter-particle distance $r$ compared to the particle size (radius) $a$, the procedure described above is significantly simplified. To leading order in $a/r$ the force distribution over the surface of particle 1 may be replaced by a force monopole, $f(r) = F^1 \delta(r)$, and the velocity of particle 2 may be assumed equal to the would-be velocity of the liquid at its position if it were absent, $v^2 = v(r)$. Thus, the coupling mobility is equal in this limit to the velocity Green’s function of eqs. (1) and (2) which, for an unbounded liquid, is given by the Oseen tensor,

$$B_{21}^{ij}(r) \simeq \frac{1}{8\pi \eta r} \left( \delta_{ij} + \frac{r_i r_j}{r^2} \right).$$

(5)

This leads, according to eq. (4), to

$$B_{cL}^{(r \gg a)} \simeq \frac{1}{4\pi \eta r},$$

$$B_{cT1}^{(r \gg a)} = B_{cT2}^{(r \gg a)} \simeq \frac{1}{8\pi \eta r}.$$  

(6)

Equations (5) and (6) are independent of the sizes and shapes of the particles. This universality is related to the fact that they can be obtained, up to a numerical prefactor, solely from conservation arguments. The force monopole associated with particle 1 introduces a momentum source in the liquid. To conserve the total momentum flux emanating from the source and passing through an envelope of radius $r$, the local flux must decay as $1/r^2$. That momentum flux is the liquid stress tensor, $\sigma \sim 1/r^2$, whose shear part is related to liquid velocity as $\sigma \sim \eta \nabla v$; hence, $v \sim 1/(\eta r)$. Momentum conservation in 3D, therefore, dictates the following form of the coupling mobility tensor: $B_{21}^{ij} \sim (\eta r)^{-1}(\delta_{ij} + Cr_i r_j / r^2)$. The constant $C$ is then forced by mass conservation (incompressibility), $\partial_i B_{21}^{ij} = 0$, to be $C = 1$.

This argument remains intact when the two test particles are surrounded by other particles, so long as the entire particulate liquid conserves momentum (i.e., remains translation-invariant). The only thing that can change at sufficiently large inter-particle distances is the prefactor in eqs. (5) and (6). The modified prefactor, depending on the volume fraction $\phi$ of particles, defines an effective viscosity, $\eta_{\text{eff}}(\phi)$. An explicit calculation for an unbounded suspension of hard spheres, to linear order in $\phi$, confirms this statement, yielding $\eta_{\text{eff}} \simeq \eta(1 + 5\phi/2)$, in agreement with Einstein’s classical result.

As particle 1 moves through the liquid, it perturbs not only the liquid momentum density but also its mass density. (In the limit of an incompressible liquid this perturbation does not disappear but is accounted for by $p(r)$, which is determined from the incompressibility constraint.) To leading order in $a/r$ the mass perturbation may be replaced by a mass dipole (a source and a sink). A mass source would create a flow velocity proportional to $1/r^2$; hence, the mass dipole creates a flow that decays as $1/r^3$. This effect is manifest in the exact expression for the flow due to a single translating rigid sphere and, consequently, contributes to the Rotne–Prager mobility tensor, which is widely used in computer simulations of suspensions and polymer solutions. The resulting correction to the coupling mobility is smaller by an order of $(a/r)^2$ than the leading $1/r$ term and is negligible, therefore, in the limit $r \gg a$. The dominant momentum-source contribution leads to a flow field of monopolar shape and to strictly positive
coupling coefficients [eq. (6)].

These consequences of the conservation of transverse momentum and mass for the pair hydrodynamic interaction should be borne in mind as we turn in the following sections to the more complicated cases of confined liquids.

3. Quasi-Two-Dimensional Systems

In q2D systems one of the dimensions of the confined liquid (in the $\hat{z}$ direction, say) is much smaller than the other two. Assuming that this small width $w$ is not much larger than the particle size, we ignore particle motion in the $\hat{z}$ direction. Thus, particles move essentially in two dimensions, whereas the full dynamics of the system (e.g., momentum transport) remain three-dimensional.

Since the surface area of such a system scales with its volume, the type of contact between the confined liquid and the environment plays a crucial role and strongly affects the hydrodynamic interaction between embedded particles. In §3.1 and §3.2 we treat two useful limits for this contact—one in which transverse momentum is completely absorbed by the boundaries, and another in which it is fully conserved (corresponding, respectively, to no-slip and slip boundary conditions for the liquid velocity). Section 3.3 describes fluid membranes, which represent an interesting intermediate between those two limiting behaviors.

3.1 Confinement between two rigid surfaces

Hydrodynamic interactions in colloid suspensions confined between two parallel solid plates have been thoroughly investigated, both experimentally and theoretically, in the past several years.\textsuperscript{8,9,17–21} The plates, fixed in the lab frame, break the translational symmetry of the confined liquid; hence, liquid momentum is not conserved over distances larger than $w$. The loss of transverse momentum is usually taken into account by imposing no-slip boundary conditions, $v = 0$, at the confining surfaces, leading to a finite momentum flux, $\partial_r v \neq 0$, into the plates. Consequently, the component of the flow due to particle 1 which is analogous to eq. (5), i.e., arising from momentum conservation, is exponentially small in $r/w$.

Liquid mass, however, remains conserved. Hence, the mass-displacement term introduced by particle 1, as described in §2, should become the dominant contribution to the flow at $r > w$. Furthermore, for $r \gg w$ the flow becomes essentially two-dimensional, $v(r)$ lying in the $xy$ plane and having a symmetric (parabolic) profile in the $\hat{z}$ direction. Thus, at a large distance the flow due to the forced particle 1 looks as if it were due to a 2D mass dipole. Such a flow decays as $1/r^2$ (since that of a mass source in 2D decays as $1/r$ to preserve the flux through the perimeter of an envelope of radius $r$). Consequently, the coupling mobility tensor must be proportional to $-r^{-2}(\delta_{ij} + Cr_ir_j/r^2)$, where the minus sign stems from the direction of the mass dipole. (The pressure is higher in front of the particle and lower behind it.) Mass conservation, $\partial_i B_{ij}^{21} = 0$ ($i, j = x, y$), sets $C = -2$. In addition, on dimensional grounds, the dipole strength must be equal to $\alpha w/\eta$, where $\alpha(a/w)$ is a dimensionless prefactor depending on the confinement ratio $a/w$. The coupling tensor is, therefore,

$$B_{ij}^{21}(r) \approx -\alpha(a/w)\frac{w}{\eta r^2} \left( \delta_{ij} - 2\frac{r_ir_j}{r^2} \right),$$

(7)
leading, according to eq. (4), to
\[ B_{L,T}(r \gg w) \simeq \pm \alpha (a/w) \frac{w}{\eta r^2}, \] (8)
where the positive (negative) sign corresponds to the longitudinal (transverse) interaction.

Identical results to eqs. (7) and (8) can be obtained by considering the exact solution for the flow due to a point force in this geometry,\textsuperscript{22} applying a lubrication approximation to eqs. (1) and (2),\textsuperscript{23} or treating the particles as suspended in a 2D Brinkman fluid\textsuperscript{24}—i.e., a fluid satisfying eqs. (1) and (2) in 2D with an additional friction term \( \sim -(\eta/w^2)v \) on the left-hand side of eq. (1). This merely highlights the generality of the results, which arise from conserved liquid mass in 2D and unconserved momentum. For example, it is evident from the aforementioned arguments that allowing for finite slip at the confining surfaces will not qualitatively change eqs. (7) and (8)—the fact that only part of the transverse momentum imparted to the plates is lost does not change the basic behavior of the q2D suspension as momentum-leaking and mass-conserving. Nor will these asymptotic results for \( r \gg w \) change if we include the effect of particle motion in the third (\( z \)) dimension, as both the momentum monopole and mass dipole created by such a motion will result in a flow which is exponentially small in \( r/w \).\textsuperscript{22} Consequently, even in cases of weak confinement, \( w \gg a \), where there may be many layers of particles between the two surfaces, the crossover to the 3D hydrodynamic coupling of eq. (6) will occur only at sufficiently small distances, \( a \ll r \ll w \). (In such cases the prefactor \( \alpha \) will depend on particle concentration.)

Thus, confinement between two rigid surfaces qualitatively changes the pair hydrodynamic interaction. It strongly suppresses the momentum-monopole contribution (from a long-ranged \( 1/r \) effect to an exponential decay), while amplifying the mass-dipole one (from a 3D \( 1/r^3 \) effect to a 2D \( 1/r^2 \) one). The amplification of the mass term, in fact, makes the hydrodynamic interaction in this geometry decay more slowly than the one near a single rigid surface (which decays only as \( 1/r^3 \)).\textsuperscript{25,26} The confinement also changes the sign of the transverse interaction from positive [eq. (6)] to negative [eq. (8)], which is a consequence of the dipolar shape of eq. (7). Most interestingly, since the mass dipole induced by particle 1, \( \alpha w/\eta \), is unaffected by the presence of surrounding particles (so long as the suspension of particles is sufficiently confined and/or dilute so as not to have a correlation length smaller than \( w \)),\textsuperscript{14} the hydrodynamic interaction is independent of particle concentration. Thus, unlike the unconfined case of \( \text{§2} \), the effective viscosity, as defined by the prefactors of eq. (7), is not modified. This statement has been verified by an explicit calculation to first order in the particle area fraction \( \phi \), yielding a vanishing concentration correction to the interaction at large distances.\textsuperscript{9} The leading correction at high area fractions is a short-ranged effect reflecting the equilibrium structure (pair correlation function) of the concentrated suspension.\textsuperscript{9}

All the aforementioned predictions arising from eq. (8)—the \( 1/r^2 \) decay, the opposite-sign couplings, the concentration-independence of the large-distance interaction—have been confirmed to high accuracy in video-microscopy experiments.\textsuperscript{8,9,17} In addition, the prefactor \( \alpha \) is found to have a moderate dependence on the confinement ratio. In the limit \( a/w \to 0 \) (yet continuing to assume that the particles lie at the mid-plane between the two plates), it is analytically found as \( \alpha = 3/(32\pi) \simeq 0.030 \).\textsuperscript{22} The measured value for \( a/w \simeq 0.45 \) (quite close to the upper bound of 1/2) is \( \alpha \simeq 0.019 \).
The dipolar hydrodynamic interactions of eq. (8) have further fundamental consequences when the inversion symmetry in the \( xy \) plane is broken as well—e.g., under driven flow in a microfluidic channel, where the particles (or droplets) exhibit a new type of density waves with an unusual dispersion relation,\(^{10}\) or when the confining plates are moved relative to one another.\(^{21}\)

### 3.2 Free-standing liquid films

The situation is drastically different when the q2D system is bounded by vacuum or gas, as in a soap film. In this case both momentum and mass are conserved, and the main effect of confinement is to make the flow at \( r \gg w \) essentially two-dimensional.\(^{27,28}\) The momentum flux emanating from a momentum monopole in 2D must decay as \( 1/r \) and, therefore, the flow velocity due to the forced particle falls off logarithmically with distance. This necessitates a cutoff length, \( \kappa^{-1} \), which may arise from the lateral system size, liquid inertia, or viscosity of the outer medium, depending on the particular system. (In micron-scale free-standing films it is the system size which usually determines \( \kappa \).\(^{29}\)) The mass-dipole effect is of order \((w/r)^2\) smaller and can be neglected. The logarithmic decay slightly complicates the general procedure used in §2 and §3.1 to deduce \( B^{21} \).

The coupling tensor is bound to be proportional to \([ (C_1 + C_2 \ln \kappa r) \delta_{ij} + (1 + C_3 \ln \kappa r) r_i r_j/r^2 ] (i,j = x,y)\). Mass conservation in 2D, \( \partial_i B^{21}_{ij} = 0 \) for any \( r \), sets \( C_3 = 0 \) and \( C_2 = -1 \). The value of the last constant, \( C_1 \), depends on the boundary condition imposed at the cutoff perimeter \( r = \kappa^{-1} \). For example, imposing a vanishing radial velocity at the edge, \( B^{21}_{ij} r_j \big|_{r = \kappa^{-1}} = 0 \), leads to \( C_1 = -1 \). If the strength of the 3D momentum monopole, associated with the forced particle 1, is taken as unity (i.e., a unit point force), then the strength of the resulting 2D monopole is equal to \( w \). This additional requirement sets the prefactor of \( B^{21} \) to be \((4\pi\eta w)^{-1}\). Thus, in summary, we find

\[
B^{21}_{ij}(r) \simeq \frac{1}{4\pi\eta w} \left[ - (1 + \ln(\kappa r)) \delta_{ij} + \frac{r_i r_j}{r^2} \right],
\]

leading, according to eq. (4), to

\[
B^L_{21}(r \gg w) \simeq -\frac{1}{4\pi\eta w} \ln(\kappa r),
\]

\[
B^T_{21}(r \gg w) \simeq -\frac{1}{4\pi\eta w} [1 + \ln(\kappa r)].
\]

(The vanishing of the longitudinal interaction vs. the finite value of the transverse one at the perimeter \( r = \kappa^{-1} \) stem from the specific boundary conditions imposed above.) The prefactor in eqs. (9) and (10) defines a two-dimensional film viscosity, \( \eta_m = \eta w \). As in the unconfined case of §2, and unlike the two-plate confinement of §§3.1, eqs. (9) and (10) are independent of the particle size and shape. This is because they stem directly from the unit force introduced by particle 1 and not from the effective mass dipole associated with it.

The ultra-long-ranged hydrodynamic interaction described by eq. (10) has recently been observed in soap films containing colloid particles.\(^{29,30}\) The crossover between this 2D behavior and the 3D one for \( r < w \) has been demonstrated as well.\(^{30}\)

Since hydrodynamic correlations in the film are carried over large distances by transverse momentum,
we expect the 2D viscosity, entering the prefactors of eqs. (9) and (10), to be modified by the presence of surrounding particles, similar to the unconfined case. Embedding rigid cylindrical inclusions of height \( w \) and radius \( a \) in the film makes the problem purely 2D. An explicit calculation of the modified prefactors, to linear order in the particle area fraction \( \phi \), leads to

\[
\eta_{\text{m,eff}} = \eta_m (1 + 2 \phi),
\]

in agreement with the known effective viscosity of such a 2D suspension of hard disks.\(^{32}\)

### 3.3 Membranes

Fluid membranes provide a particularly important example of hydrodynamic interactions in confined geometry. Membranes form the envelopes of all living cells and are also used to make vesicles (liposomes) for various applications. The membrane is a self-assembled bilayer of amphiphilic molecules (e.g., lipids), which are free to move in the lateral directions, thus forming a q2D liquid. Biomembranes contain also a high concentration of membrane-embedded proteins, whose motion is confined to the membrane surface as well. Since the membrane viscosity \( \eta \) is much higher than the viscosity \( \eta_f \) of the surrounding aqueous medium, yet not infinitely so (typically by a factor of \( \sim 10^3 \)), this q2D system presents an interesting intermediate between the two systems studied in §3.1 and §3.2. From the conservation arguments that underlie our discussion it is clear that one should distinguish between two cases—one in which the membrane is freely suspended in the solution, and another in which it is immobilized.

In the first case the system is translationally invariant and, hence, conserves momentum. The hydrodynamics of such a membrane was first studied by Saffman and Delbrück\(^{33, 34}\) and later, using a different approach, by Levine and MacKintosh.\(^{35}\) The large viscosity contrast introduces a length scale,

\[
\kappa^{-1} = \eta w / (2 \eta_f),
\]

which is much larger than the molecular thickness of the membrane and protein size, \( \kappa^{-1} \gg w \sim a \). The Saffman-Delbrück length \( \kappa^{-1} \), which is typically of micron scale, sets the distance beyond which momentum is transported primarily through the surrounding liquid rather than through the membrane.

At distances \( a \ll r \ll \kappa^{-1} \) the membrane behaves much like the soap film of §3.2, conserving both momentum and mass in 2D. Over such intermediate distances, therefore, the coupling mobility tensor of a protein pair should be [cf. eq. (9)]

\[
B_{ij}^{21}(r) \simeq \frac{1}{4\pi \eta_m} \left[ - (1 + \ln(\kappa' r)) \delta_{ij} + \frac{r_i r_j}{r^2} \right],
\]

where \( \eta_m = \eta w \) is the membrane 2D viscosity, and \( \kappa' \sim \kappa \) up to a numerical constant. This leads to [cf. eq. (10)]

\[
B_{ij}^c(a \ll r \ll \kappa^{-1}) \simeq -\frac{1}{4\pi \eta_m} \ln(\kappa' r),
\]

\[
B_{ij}^c(a \ll r \ll \kappa^{-1}) \simeq \frac{1}{4\pi \eta_m} \left[ 1 + \ln(\kappa' r) \right].
\]

At sufficiently large distances, \( r > \kappa^{-1} \), momentum is transported through the surrounding liquid as well and is conserved in 3D rather than within the q2D membrane. As in §3.2 this dictates the following form for the coupling mobility tensor: \( B_{ij}^{21} \sim (\eta_f r)^{-1} (C \delta_{ij} + r_i r_j / r^2) \). Yet, unlike the unconfined case of
the membrane (lipid) mass is conserved in 2D, \( \partial_i B_{ij}^{21} = 0 \) \((i, j = x, y)\), which sets \( C = 0 \). We thus have

\[
B_{ij}^{21}(r) \simeq \frac{\epsilon}{\eta_f} \frac{r_i r_j}{r^3},
\]

where \( \epsilon \) is a dimensionless prefactor. This leads, according to eq. (1), to

\[
B_{ij}^c(r \gg \kappa^{-1}) \simeq \frac{\epsilon}{\eta_f r},
\]

and a transverse interaction \( B_T^c \) which is of order \((\kappa r)^{-1}\) smaller. Equations (13) and (14) are independent of any membrane property, reflecting the fact that the hydrodynamic interaction over such long distances is mediated solely by the surrounding medium. In particular, the interaction in this regime will not be modified by the presence of other membrane inclusions, i.e., it is independent of protein concentration.

A more detailed calculation\(^{31,35}\) yields the crossover between these two distance regimes, as well as the values for the numerical factors left unspecified above: \( \kappa' = (e^{\gamma_E - 1/2} / 2) \kappa \) (\( \gamma_E \) being the Euler constant), and \( \epsilon = 1/(4\pi) \). An explicit calculation of the effect of rigid cylindrical inclusions on the interaction\(^{31}\) shows that, to linear order in the area fraction \( \phi \) of inclusions, one can replace \( \eta_m \) with \( \eta_{m,\text{eff}} = \eta_m (1 + 2\phi) \), as in §3.2, except that this substitution should be made also in the Saffman-Delbrück length, \( \kappa^{-1} = \eta_{m,\text{eff}} / (2\eta_f) \rightarrow \eta_{m,\text{eff}} / (2\eta_f) \).

In the second relevant case, where the membrane is immobilized (e.g., supported on solid substrates) the system is not translationally invariant and, hence, does not conserve momentum. Another length scale, \( \lambda \), should be considered, beyond which membrane momentum is lost to the solid. The Saffman-Delbrück length being of micron scale, we have in general \( \lambda \ll \kappa^{-1} \). Since the membrane conserves mass in 2D, this scenario is equivalent to the one considered in §3.1, with \( \lambda \) replacing \( w \), and the particles can be treated as embedded in a 2D Brinkman fluid.\(^{36–38}\)

The dependencies summarized in eqs. (11)–(14) were observed in the dynamics of domains\(^{39,40}\) and colloids\(^{41}\) embedded in monolayers of amphiphilic molecules at the water–air interface. (In such systems, where one of the bounding fluids has a vanishingly small viscosity, one should replace \( \eta_f \) in the aforementioned results with \( \eta_f / 2 \).) Two-particle tracking experiments for membrane-embedded proteins, to our best knowledge, have not been performed yet. Nor are we aware of similar experiments involving immobilized membranes.

### 4. Quasi-One-Dimensional Systems

Following the line of argument of §3, a liquid confined in a linear channel with rigid walls does not conserve momentum beyond a distance comparable to the channel width \( w \). The flow at a distance \( x \) from the momentum monopole due to the forced particle 1, therefore, is exponentially small in \( x/w \). However, unlike the q2D channel, the flow due to the mass dipole is also short-ranged. This is because at \( x \gg w \) the flow velocity becomes essentially one-dimensional, pointing in the \( \hat{x} \) direction. Since a mass source in 1D creates a flow which is uniform in \( x \), a 1D mass dipole, as well as all higher moments, create no flow at all. Thus, q1D confinement in a rigid channel leads to exponential screening of the hydrodynamic interaction beyond \( x > w \). This has been confirmed experimentally.\(^{7}\) The effect of the other particles in the q1D suspension on this screened interaction has been accounted for as well.\(^{42}\)
As in §3.1, introducing partial slip at the channel boundaries should not change this qualitative result. Nevertheless, if the particles are made bigger so as to fit the channel cross-section, one expects the flows to become plug-like, and the range of coupling between particles in an incompressible liquid to tend to infinity. Indications for such a divergence were observed in the dynamics of droplets driven in narrow microfluidic channels.\textsuperscript{11}

5. Gels and Porous Media

A commonly encountered confinement is that of a liquid pervading a solid matrix, such as a porous medium or a polymer network. It is often (wrongly) assumed that the hydrodynamic interaction between particles embedded in such a system is screened beyond a distance comparable to the correlation length $\xi$ of the matrix. The liquid, whose translational symmetry is broken by the stationary matrix, does not conserve momentum, and the flow at a distance $r$ away from the momentum monopole due to particle 1 will be exponentially small in $r/\xi$. Liquid mass, however, is conserved. Thus, at a large distance the flow due to the forced particle 1 looks as if it were due to a 3D mass dipole.\textsuperscript{14} Such a flow decays as $1/r^3$. Consequently, the coupling mobility tensor must be proportional to $-r^{-3}(\delta_{ij} + Cr_i r_j/r^2)$, where the minus sign is implied by the mass dipole direction. Mass conservation in 3D, $\partial_i B_{ij}^2 = 0$ ($i,j = x,y,z$), sets $C = -3$. Additionally, on dimensional grounds, the dipole strength must be equal to $\gamma \xi^2/\eta$, where $\gamma (a/\xi)$ is a dimensionless prefactor depending on the confinement ratio $a/\xi$. The coupling tensor is, therefore,

$$B_{ij}^2(r) \simeq -\gamma (a/\xi) \xi^2 \eta r^{-3} \left( \delta_{ij} - 3 \frac{r_i r_j}{r^2} \right), \quad \text{(15)}$$

leading, according to eq. (11), to

$$B_{L}^c(r \gg \xi) \simeq 2\gamma (a/\xi) \xi^2 \eta r^{-3},$$

$$B_{T1}^c(r \gg \xi) = B_{T2}^c(r \gg \xi) \simeq -\gamma (a/\xi) \xi^2 \eta r^{-3}. \quad \text{(16)}$$

Identical results to eqs. (15) and (16) can be obtained by considering the particles as embedded in a 3D Brinkman fluid\textsuperscript{24}—i.e., a fluid that satisfies eqs. (1) and (2) in 3D with an additional friction term $\sim -(\eta/\xi^2)\mathbf{v}$ on the left-hand side of eq. (1).\textsuperscript{43}

These results for confinement in a solid matrix are similar to those of §3.1 and so are their consequences. (Indeed, the two-plate geometry may be viewed as a particular example of a porous medium.) Confinement qualitatively changes the hydrodynamic interaction by strongly suppressing the momentum-monopole contribution (from a long-ranged $1/r$ effect to an exponential decay), while keeping the mass-dipole one intact. The resulting $1/r^3$ interaction decays faster with distance than the unconfined one but is still long-ranged. The confinement also changes the sign of the transverse interaction from positive [eq. (6)] to negative [eq. (16)], which is a consequence of the dipolar shape of eq. (15). As in §3.1 we reach the surprising conclusion that, so long as the suspension of particles is sufficiently confined and dilute so as not to have a correlation length smaller than $\xi$, the hydrodynamic interaction between particles embedded in the matrix is independent of particle concentration. This is because the mass dipole induced by particle
1, $\gamma \xi^2/\eta$, will be unaffected by the presence of surrounding particles. In this case the prefactors in eqs. (15) and (16) are independent of concentration, and the effective viscosity of the suspension, as defined by those prefactors, is $\eta_{\text{eff}}(\phi) = \eta$.

To our best knowledge, the aforementioned effects have never been experimentally observed. They should be readily testable using two-particle tracking, e.g., in a polymer gel. For eqs. (15) and (16) to hold in such an experiment, the particles should be sufficiently small, $a < \xi$, and the frequency sufficiently low, so as to avoid viscoelastic effects.

6. Discussion

The scenarios addressed in the preceding sections demonstrate how the hydrodynamic interaction between particles embedded in a confined fluid is strongly and differently affected by the confinement, depending on the specific geometry. On the one hand, confinement may suppress the transverse-momentum contribution to the interaction, which is the dominant one in unconfined fluids. This occurs when solid boundaries break the translational symmetry of the confined fluid. On the other hand, the mass-displacement contribution, which is negligible in unconfined fluids, may be either amplified, due to a reduction of the effective dimensionality of the flow, or left intact. Additionally, even severe confinement may have a negligible effect when the response of the outer medium, rather than that of the confined fluid, is dominant (as is the case for the large-distance interaction in membranes). As a result of these opposing trends, and somewhat against one’s naive expectation, in most cases confinement does not lead to overall suppression, or screening, of the hydrodynamic interaction at distances larger than the confinement width $w$. For instance, the hydrodynamic interactions in fluid membranes, a two-plate geometry, and porous matrices decay only algebraically, as $1/r$, $1/r^2$, and $1/r^3$, respectively.

Several of the theoretical results presented here have been convincingly confirmed in experiment (e.g., the ones pertaining to the q2D geometries of §3.1 and §3.2). Yet, other predictions are still to be tested—primarily those related to gels and liquid-filled porous media (§5). The dipolar shape of the coupling in this case (as well as in the two-plate geometry) yields a vanishing effect upon angular averaging. Consequently, the long-ranged interaction will not be observed in conventional scattering measurements. Two-particle tracking, nonetheless, should readily reveal it.

The agreement between theory and available experimental results suggests that the combination of assumptions put forth in §2 on which the entire analysis has relied, is valid under common experimental conditions. It would be beneficial, nevertheless, to examine the consequences of relaxing some of these assumptions under different conditions. The following key issues are left open for future study. (i) As has been mentioned above, introducing partial liquid slip at the rigid boundaries should not lead to a qualitative change in the results. However, considering finite rigidity of the outer medium will allow momentum exchange with it, thus affecting the hydrodynamic interaction at large distances. (ii) The limit of zero Reynolds number has allowed us to ignore the time dependence of the hydrodynamic interaction. It has been already shown that the switch of dominance between momentum and mass transport in certain confined geometries should lead to a drastically faster buildup of the interaction. Including temporal
dependence is obviously crucial also when considering viscoelastic effects. (iii) The assumption of liquid incompressibility implies that sound propagates infinitely fast through the system. Confinement by rigid surfaces is known to strongly affect the sound modes of the confined liquid, making them diffusive.\textsuperscript{28,44} The resulting diffusivity, $c^2 w^2/\nu$, is still very large ($\sim 1$ m$^2$/s for water in a micron-scale channel), yet the qualitatively different sound propagation may have an interesting effect on the short-time hydrodynamic coupling. (iv) Breaking inversion symmetry by driven flow has already been shown, both experimentally and theoretically, to lead to novel dynamics of confined particles.\textsuperscript{10,11} These new findings clearly call for further investigation—e.g., of confined sedimentation.

This article has shown how key results concerning the hydrodynamic interaction in varied confined geometries can be simply and accurately derived from the effect of confinement on the conserved fields of the fluid. Owing to their generality, such arguments may be found helpful in other scenarios. They can provide asymptotes against which more detailed theories or simulations are to be tested. Furthermore, analogous approaches can be applied to confinement effects on other medium-induced interactions. For example, when particles are confined between two plates of high dielectric permittivity, their static (London) dispersion interaction is suppressed due to the concentration of field lines in the plates, whereas the dynamic (Casimir-Polder) one is enhanced due to the reduced effective dimensionality for radiation.\textsuperscript{45}

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