§0. Introduction.

There are two things that quickly become clear in surveying the work done on the subject of Calabi-Yau threefolds, by which we mean a threefold $X$ (with some specified class of possible singularities) with $K_X = 0$ and $h^1(O_X) = 0$. First, there are a huge number of such threefolds, even non-singular. Second, one reason there appears to be so many is this: suppose you degenerate a non-singular Calabi-Yau $X$ to a threefold $X'$ with canonical singularities. If $X'$ has a crepant desingularization $\tilde{X}$, then $\tilde{X}$ will not, in general, be in the same deformation family as $X$ or even be diffeomorphic to $X$. As a result, an innocent-enough Calabi-Yau threefold such as the quintic hypersurface in $\mathbb{P}^4$ can have hundreds of degenerations, if not more, with crepant resolutions, thus giving rise to huge numbers of other Calabi-Yaus.

Classification of these degenerations seems to be a hopeless problem. All we know, of course, is that there are only a finite number of families of such degenerate quintics. So, if we want to try to simplify the task of Calabi-Yau classification, it might help to concentrate on Calabi-Yau threefolds which do not arise as crepant resolutions of degenerations of other Calabi-Yau threefolds. This motivates the following definition:

**Definition.** A non-singular Calabi-Yau threefold $\tilde{X}$ is primitive if there is no birational contraction $\tilde{X} \to X$ with $X$ smoothable to a Calabi-Yau threefold which is not deformation equivalent to $\tilde{X}$.

(The last condition is included to rule out the sort of possibility that can occur, say, if one has contracted an elliptic scroll to a curve. See [48], Example 4.6 for such an example.)

While I will not study primitive Calabi-Yau threefolds directly in this paper (I defer this until [12]), this definition will still guide our inquiry. In particular, in order to understand which Calabi-Yaus are not primitive, we should understand the answer to the

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following question: given a Calabi-Yau threefold $\tilde{X}$, when can we find a birational contraction morphism $\tilde{X} \to X$ such that $X$ is smoothable? Now, any birational contraction will yield a threefold $X$ with at worst canonical singularities. So if we want to begin to understand the smoothability of such threefolds, we first need to understand if they have obstructed deformation theory or not. Thus we have

**Question.** Given a Calabi-Yau threefold $X$ with canonical singularities, is $\text{Def}(X)$ nonsingular? If not, can we get some reasonable dimension estimates for components of $\text{Def}(X)$?

As already shown in [11], if $X$ has canonical singularities, $\text{Def}(X)$ can indeed be singular. However, as we shall show in this paper, we can still control the dimension of components of $\text{Def}(X)$ if $X$ has canonical singularities. The principle we discover is that obstructions to deforming $X$ are essentially the obstructions to deforming a germ of the singularities of $X$. This gives a further generalization of the Bogomolov-Tian-Todorov unobstructedness theorem. This material is covered in §2, with preliminaries in §1.

Now, in the attempt to understand when Calabi-Yau threefolds with canonical singularities are smoothable, it will certainly be hopeless to try to understand all possible canonical singularities which can arise and then determine which ones are smoothable. Nevertheless, we still obtain strong results. We have

**Theorem 3.8.** Let $\tilde{X}$ be a non-singular Calabi-Yau threefold, and $\pi : \tilde{X} \to X$ be a birational contraction morphism, such that $X$ has isolated complete intersection singularities. Then there is a deformation of $X$ which smooths all singular points of $X$ except possibly the ordinary double points of $X$.

Results of Namikawa [28] then allow us to extend this result to the case that $\tilde{X}$ has terminal singularities.

In the case that the isolated singularity is not complete intersection, we have a much weaker statement. The hypothesis that $X$ is $\mathbb{Q}$-factorial is necessary to ensure that $X$ has enough infinitesimal deformations. Furthermore, since at the moment we do not have any real control over how bad the deformation space of a canonical singularity can be, we include a rather artificial hypothesis on the singularities we will consider (see Definition 4.2) called *good*. Hopefully some of these hypotheses can be removed at a future date. We show in §5 that some simple classes of singularities are good, and this is enough for initial applications of our results. We have

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Theorem 4.3. Let $\tilde{X}$ be a non-singular Calabi-Yau threefold and $\pi : \tilde{X} \to X$ a birational contraction, so that $X$ is $\mathbb{Q}$-factorial and for each $P \in \text{Sing}(X)$, the germ $(X, P)$ is good. Then $X$ is smoothable.

Recall that a birational projective contraction $\pi : \tilde{X} \to X$ is primitive if it cannot be factored in the projective category. One application of Theorem 4.3 given in §5 is

Theorem 5.8. Let $\pi : \tilde{X} \to X$ be a primitive contraction contracting a divisor $E$ to a point. Then $X$ is smoothable unless $E \cong \mathbb{P}^2$ or $F_1$.

Finally, a bit of history on these questions. The unobstructedness question has been answered positively for varying degrees of singularities: for non-singular Calabi-Yaus of any dimension by Bogomolov, Tian [44] and Todorov [46], with algebraic proofs given by Ran [32], Kawamata [18], and Deligne; for Calabi-Yaus with ordinary double points by Kawamata [18] and Tian [45]; for Calabi-Yaus with Kleinian singularities and orbikleinfold singularities by Ran [33,36]; and finally for Calabi-Yau threefolds with rational isolated complete intersection singularities by Namikawa [27]. Results on smoothability of singular Calabi-Yau threefolds were first obtained by Friedman [8] for Calabi-Yau threefolds with ordinary double points, and by [28] for Calabi-Yaus with arbitrary terminal singularities, as well as for a limited class of hypersurface singularities. The latter result is generalized here by Theorem 3.8. Most of the methods used in this paper are generalizations of ones applied by Namikawa in [27] and [28]. Right before submitting this paper, I received a new version of [28] which gives a proof of Corollary 3.10 in the hypersurface case.

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§1. Some Deformation Theory.

There is little that is new in this section, but I will be needing a number of minor variants on results primarily due to Ran and Kawamata. I give here complete proofs of the precise statements I will be needing. I begin by reviewing some facts of deformation theory.

(1.1) The general context from [40] is as follows. Let $k$ be a field, and $\Lambda$ a complete Noetherian local $k$-algebra with residue field $k$ and maximal ideal $m_\Lambda$. We denote by $C_\Lambda$ the category of Artin local $\Lambda$-algebras with residue field $k$ with local homomorphisms. We are interested in deformation functors $D : C_\Lambda \to \text{Ens}$ (here $\text{Ens}$ is the category of sets), with $D(k)$ consisting of one element.
We will always write an isomorphism on tangent spaces. Here Hom denotes local Λ-algebra homomorphisms. \( \alpha \) and let \( \text{pro-represented by } S \) (miniversal space). Thus there is a complete local Λ-algebra \( A \) with a morphism of functors Hom\((S, \cdot ) \to D \). This morphism is an isomorphism if \( D \) is pro-represented by \( S \). If \( S \) is a hull of \( D \), this morphism is only smooth and induces an isomorphism on tangent spaces. Here Hom denotes local Λ-algebra homomorphisms. We will always write \( S \cong R/J \), where \( R = \Lambda[[x_1, \ldots, x_r]] \) with maximal ideal \( m_R = m_\Lambda R + (x_1, \ldots, x_r) \), and \( J \subseteq m_\Lambda R + m_R^2 \) an ideal.

Following \[18\], set
\[
A_n = k[t]/(t^{n+1})
\]
\[
B_n = A_n \otimes_k A_1 = k[x, y]/(x^{n+1}, y^2)
\]
\[
C_n = B_{n-1} \times_{A_{n-1}} A_n = k[x, y]/(x^{n+1}, x^n y, y^2)
\]
and let \( \alpha_n : A_{n+1} \to A_n \), \( \beta_n : B_n \to A_n \), \( \gamma_n : B_n \to C_n \), and \( \xi_n : B_n \to B_{n-1} \) be the natural maps. Define \( \epsilon_n : A_{n+1} \to B_n \) by \( t \mapsto x + y \) and \( \epsilon'_n : A_n \to C_n \) by \( t \mapsto x + y \) also.

Now consider the case that \( \Lambda = k \). For \( X_n \in D(A_n) \) we define the first order tangent space of \( X_n \),
\[
T^1(X_n/A_n) = \{ Y_n \in D(B_n) | D(\beta_n)(Y_n) = X_n \}.
\]
If \( D \) is pro-representable, then \( T^1(X_n/A_n) \) has a natural \( A_n \)-module structure as follows. First, if \( \alpha, \beta \in T^1(X_n/A_n) \), we need to define \( \alpha + \beta \). We have \( \alpha \times \beta \in D(B_n) \times_{D(A_n)} D(B_n) = D(B_n \times_{A_n} B_n) \) by pro-representability. The ring \( B_n \times_{A_n} B_n \) is isomorphic to \( k[x, y, y']/(x^{n+1}, y^2, yy', y'^2) \), and there is a natural map \( B_n \times_{A_n} B_n \to B_n \) via \( x \mapsto x, y \mapsto y, y' \mapsto y \). Then \( \alpha + \beta \) is the image of \( \alpha \times \beta \) in \( D(B_n) \) under this map. It is easy to check that \( \alpha + \beta \in T^1(X_n/A_n) \).

Secondly, if \( a \in A_n \), we have the endomorphism \( a : B_n \to B_n \) given by \( x \mapsto x, y \mapsto ay \). Then if \( \alpha \in T^1(X_n/A_n) \), so is \( a\alpha = D(a)(\alpha) \).

In this paper, we will on occasion work with functors which are not necessarily pro-representable, but which do have a hull. \( D \) has a hull if and only if Schlessinger’s conditions \((H_1) - (H_3)\) of \[40\], Theorem 2.11 are satisfied. We recall these here. For any morphisms \( A' \to A, A'' \to A \) in \( \text{C}_\Lambda \), there is a natural functorial map
\[
\psi_{A', A'', A} : D(A' \times_A A'') \to D(A') \times_{D(A)} D(A'').
\]
A surjective map \( A' \to A \) in \( \text{C}_\Lambda \) is a small extension if its kernel is a non-zero principal ideal \( (t) \) such that \( m_{A'}(t) = 0 \). The conditions \((H_1) - (H_3)\) are:
(H₁) \( \psi_{A',A''} \) is a surjection whenever \( A'' \rightarrow A \) is a small extension.

(H₂) \( \psi_{A',A''} \) is a bijection when \( A = k \), \( A'' = A₁ \).

(H₃) \( \dim_k(T¹(X/k)) < \infty \).

(H₂) guarantees that \( T¹(X/k) \) has a \( k \)-vector space structure as in (1.4). If in addition

(H₄) \( \psi_{A',A''} \) is a bijection for any small extension \( A' \rightarrow A \)

is satisfied then \( D \) is pro-representable. (This is also a part of [40], Theorem 2.11)

For an arbitrary deformation functor \( D \), it may not be possible to define an \( A_n \)-module structure on \( T¹(X_n/A_n) \). However, we will be dealing with a class of deformation functors where this is possible. To do this, we need a condition we shall refer to as (H₅):

(H₅) If \( A' \rightarrow A \) and \( A'' \rightarrow A \) are surjections in \( Cₐ \), there exists a map

\[
\phi_{A',A'',A} : D(A') \times D(A) D(A'') \rightarrow D(A' \times A A'')
\]

with \( \psi_{A',A'',A} \circ \phi_{A',A'',A} \) the identity on \( D(A') \times D(A) D(A'') \). Furthermore, whenever there is a commutative diagram

\[
\begin{array}{ccc}
B & \rightarrow & A'' \\
\downarrow & & \downarrow \\
A' & \rightarrow & A
\end{array}
\]

inducing maps \( \alpha_1 : D(B) \rightarrow D(A' \times A A'') \) and \( \alpha_2 : D(B) \rightarrow D(A') \times D(A) D(A'') \), we have \( \phi_{A'',A',A} \circ \alpha_2 = \alpha_1 \).

If \( D \) has a hull and satisfies (H₅) in addition, then one can still define an \( A_n \)-module structure on \( T¹(X_n/A_n) \), by defining \( \alpha + \beta \) to be the image of \( \alpha \times \beta \) under the composed map

\[
D(B_n) \times D(A_n) D(B_n) \rightarrow D(B_n \times A_n B_n) \rightarrow D(B_n).
\]

(1.6) Let \( (X, O_X) \) be a ringed space with \( O_X \) a sheaf of \( k \)-algebras. For \( A \in Cₖ \), an infinitesimal deformation of \( X \) over \( A \) is a ringed space \( (X, O_{X,A}) \) such that \( O_{X,A} \) is a sheaf of flat \( A \)-algebras and \( O_{X,A} \otimes_A k = O_X \). \( (X, O_{X,A}) \) and \( (X, O'_{X,A}) \) are isomorphic deformations if there is an isomorphism \( O_{X,A} \rightarrow O'_{X,A} \) which is the identity upon tensoring with \( k \). We define \( D_X : Cₖ \rightarrow \text{Ens} \) by

\[
D_X(A) = \{ \text{Isomorphism classes of deformations of } X \text{ over } A \}.
\]

Then \( D_X \) satisfies (H₅). Indeed, given \( A' \rightarrow A \), \( A'' \rightarrow A \) surjective, \( (X, O_{X,A'}) \in D_X(A') \), \( (X, O_{X,A''}) \in D_X(A'') \) with \( O_{X,A'} \otimes_{A'} A = O_{X,A''} \otimes_{A''} A = O_{X,A} \), we can define

\[
\phi_{A',A'',A}((X, O_{X,A'}) \times (X, O_{X,A''})) = (X, O_{X,A'} \times_{O_{X,A}} O_{X,A''}).
\]
The last condition of \( (H_5) \) follows from the fact that

\[
\mathcal{O}_{X_B} \otimes_B A' \times_{\mathcal{O}_{X_B} \otimes_B A} \mathcal{O}_{X_B} \otimes_B A'' \cong \mathcal{O}_{X_B} \otimes_B (A' \times_A A'')
\]

for a flat \( B \)-algebra \( \mathcal{O}_{X_B} \) by [40], Corollary 3.6.

The methods of [40], (3.7) show also that \( (H_1) \) and \( (H_2) \) always hold for \( D_X \). Thus, if \( \dim_k T^1(X/k) < \infty \), \( D_X \) has a hull.

(1.7) If \( D \) is a deformation functor on \( \mathbf{C}_A \), we say that a \( k \)-vector space \( T^2 \) is an obstruction space for \( D \) if whenever we have a surjection \( \phi : A' \to A \) in \( \mathbf{C}_A \) with \( I = \ker \phi \) annihilated by the maximal ideal of \( A' \), we get a sequence

\[
D(A') \xrightarrow{D(\phi)} D(A) \xrightarrow{\delta} T^2 \otimes I,
\]

and this sequence is exact in the sense that if \( \alpha \in D(A) \), then \( \delta(\alpha) = 0 \) if and only if \( \alpha \) is in the image of \( D(\phi) \). Furthermore, the obstruction map should be functorial, so given in addition \( \phi' : B' \to B \) surjective, \( I' = \ker \phi' \) annihilated by the maximal ideal of \( B' \) and a commutative diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{\phi} & A \\
\downarrow{\beta'} & & \downarrow{\beta} \\
B' & \xrightarrow{\phi'} & B
\end{array}
\]

there is a commutative diagram

\[
\begin{array}{ccc}
D(A') & \xrightarrow{D(\phi)} & D(A) \\
\downarrow{D(\beta')} & & \downarrow{D(\beta)} \\
D(B') & \xrightarrow{D(\phi')} & D(B) \\
& & \downarrow{\delta'} \\
& & T^2 \otimes I'
\end{array}
\]

\[
\begin{array}{c}
\xrightarrow{1_{T^2} \otimes -}
\end{array}
\]

If \( D \) is prorepresentable by \( S \cong R/J \) as in (1.2), it is easy to describe the obstruction theory of \( D \). Let \( T^2 \) be the \( k \)-vector space \( (J/\mathfrak{m}_R J)^\vee \). The obstruction map \( \delta \) can be described as follows. Given \( f \in D(A) = \text{Hom}(R/J, A) \), let \( f(x_i) = \alpha_i \in A \). Choose any lifting of \( \alpha_i \) to \( \alpha'_i \in A' \); this defines a \( \Lambda \)-algebra homomorphism \( f' : R \to A' \). Now given an element \( \beta \in J \), we must have \( f'(\beta) \in I \), since \( f(\beta) = 0 \). Furthermore, if \( \beta \in \mathfrak{m}_R J \), then \( f'(\beta) = 0 \), since \( f' \) is a local homomorphism and \( I \) is annihilated by the maximal ideal of \( A' \). Thus \( f' \) induces a \( k \)-vector space map \( J/\mathfrak{m}_R J \to I \), i.e. an element of \( T^2 \otimes I \).

We then define \( \delta(f) \) to be this element. It is easy to see that \( \delta(f) \) does not depend on the choice of the lifting. Furthermore, it is clear that \( f' \) induces a map \( f' : R/J \to A' \) if and only if \( \delta(f) = 0 \). Note that if one choice of the \( \alpha'_i \) provides a lifting \( f' : R/J \to A' \), then any choice does, and the set of possible liftings is a principal homogeneous space over the vector space \( (\mathfrak{m}_R/(\mathfrak{m}_R + \mathfrak{m}_R^2))^\vee \otimes I) \).
We note also that \((J/\mathfrak{m}_RJ)^\vee\) is naturally isomorphic to \(T^1(S/A, k)\) by [25, 3.1.2], where here \(T^1\) is the first cotangent functor of Lichtenbaum and Schlessinger.

The following is a slight generalization of Ran’s and Kawamata’s statements of Ran’s \(T^1\)-lifting criterion. ([34] and [18]).

\textbf{Theorem 1.8.} (Ran). Let \(k\) be a field of characteristic 0 and \(D\) a deformation functor on \(C_k\) which has a hull and satisfies (H5). Suppose also that \(D\) has an obstruction space \(T^2\). Then for each \(X_n \in D(A_n), X_{n-1} = D(\alpha_{n-1})(X_n), \alpha = D(\epsilon_{n-1})(X_n) \in T^1(X_{n-1}/A_{n-1}),\) there exists \(X_{n+1} \in D(A_{n+1})\) with \(D(\alpha_n)(X_{n+1}) = X_n\) if and only if \(\alpha\) is in the image of the natural map

\[
T^1(X_n/A_n) \to T^1(X_{n-1}/A_{n-1}).
\]

Proof. This proof is simply a very minor modification of Kawamata’s proof of Ran’s \(T^1\)-lifting criterion in [18].

First note that \(D(\beta_{n-1})D(\epsilon_{n-1})(X_n) = D(\beta_{n-1}\epsilon_{n-1})(X_n) = D(\alpha_{n-1})(X_n) = X_{n-1},\) so that \(\alpha \in T^1(X_{n-1}/A_{n-1}).\) Now, as in [18] page 185, we have a commutative diagram with exact rows:

\[
\begin{array}{ccc}
D(A_{n+1}) & \xrightarrow{D(\alpha_n)} & D(A_n) \\
\downarrow{D(\epsilon_n)} & & \downarrow{D(\epsilon_n)} \\
D(B_n) & \xrightarrow{D(\gamma_n)} & D(C_n) \\
\end{array}
\xrightarrow{\delta_1} T^2 \otimes (t^{n+1}) \\
\downarrow{1_{T^2} \otimes \epsilon_n} \\
\xrightarrow{\delta_2} T^2 \otimes (x^n y)
\]

Note that \(1_{T^2} \otimes \epsilon_n\) is an isomorphism (here we need \(\text{char } k = 0\), since \(\epsilon_n(t^{n+1}) = (n+1)x^n y\) in \(B_n\)).

First we compute \(D(\epsilon'_n)(X_n).\) Let \(\epsilon'_n : D(A_n) \to D(B_{n-1}) \times_{D(A_{n-1})} D(A_n)\) be the map induced by \(D(\epsilon_{n-1}) : D(A_n) \to D(B_{n-1})\) and the identity map on \(D(A_n)\). By (H5), if we set \(\phi = \phi_{B_{n-1}, A_n,A_{n-1}}, D(\epsilon'_n) = \phi \circ \epsilon'_n,\) so that \(D(\epsilon'_n)(X_n) = \phi(\alpha \times X_n).\) Also, let \(\bar{\gamma}_n : D(B_n) \to D(B_{n-1}) \times_{D(A_{n-1})} D(A_n)\) be the map induced by \(D(\xi_n) : D(B_n) \to D(B_{n-1})\) and \(D(\beta_n) : D(B_n) \to D(A_n)\), so that by (H5), \(D(\gamma_n) = \phi \circ \bar{\gamma}_n.\) So \(D(\epsilon'_n)(X_n) = \phi(\alpha \times X_n)\) is in the image of \(D(\gamma_n)\) if and only if \(\alpha \times X_n\) is in the image of \(\bar{\gamma}_n\) restricted to \(T^1(X_n/A_n) \subseteq D(B_n),\) if and only if \(\alpha\) is in the image of \(T^1(X_n/A_n) \to T^1(X_{n-1}/A_{n-1}).\) But \(\phi(\alpha \times X_n) \in \text{im } D(\gamma_n)\) if and only if \(\delta_2(\phi(\alpha \times X_n)) = 0,\) but since \(1_{T^2} \otimes \epsilon_n\) is an isomorphism, this is true if and only if \(\delta_1(X_n) = 0,\) if and only if there exists \(X_{n+1} \in D(A_{n+1})\) as desired. \(

The following theorem is a slightly more specific version of Ran’s Theorem 1.1 of [35].

\textbf{Theorem 1.9.} Let \(k\) be a field of characteristic 0, \(D_1\) and \(D_2\) two deformation functors on \(C_k\) with a morphism of functors \(F : D_1 \to D_2.\) Suppose \(D_1\) is pro-representable by
a $k$-algebra $S \cong P/I$, $P = k[[x_1, \ldots, x_s]]$ as in (1.2), and suppose $D_2$ has a hull $\Lambda$ and satisfies $(H_3)$. Let $T^1_1$ and $T^2_2$ be $k$-vector spaces and $l : T^1_1 \to T^2_2$ a $k$-vector space map. Denote by $X$ the unique element of $D_1(k)$. Suppose, for all $n$ and for each $X_n \in D_1(A_n)$ inducing $X_{n-1} \in D_1(A_{n-1})$, there is a commutative diagram

\[
\begin{array}{ccc}
T^1_1(X_n/A_n) & \xrightarrow{F_2} & T^2_2(F(X_n)/A_n) \\
\downarrow{D_1(\xi_n)} & & \downarrow{D_2(\xi_n)} \\
T^1_1(X_{n-1}/A_{n-1}) & \xrightarrow{F_2} & T^2_2(F(X_{n-1})/A_{n-1}) \\
\downarrow{\delta_1} & & \downarrow{\delta_2} \\
T^2_2 & \xrightarrow{l} & T^2_2
\end{array}
\]

with exact columns and $l|_{\text{im}(\delta_1)} : \text{im}(\delta_1) \to T^2_2$ injective. Then $S \cong R/J$, with $R = \Lambda[[x_1, \ldots, x_r]]$, $r = \dim_k \ker(T^1_1(X/k) \xrightarrow{F_2} T^2_2(F(X)/k))$, $J \subseteq m_A R + m_R^2$ an ideal. In addition, there is an ideal $J' \subseteq J$ with $\text{Supp}(R/J) = \text{Supp}(R/J')$ and $m_R/(m_R^2 + J) \cong m_R/(m_R^2 + J')$ such that $J'$ is generated by $\dim_k \operatorname{coker}(T^1_1(X/k) \xrightarrow{F_2} T^2_2(F(X)/k))$ elements of $R$.

Proof. Since $\Lambda$ is a hull for $D_2$, there is an induced map $\Lambda \to S$, unique only up to the induced map $F_*$ on Zariski tangent spaces. Fix one such map. Now

$$
r = \dim_k \operatorname{coker}(\frac{m_A}{m_A^2} \xrightarrow{F_2^\vee} \frac{m_S}{m_S^2}),$$

and if we choose elements $\alpha_1, \ldots, \alpha_r \in m_S$ which along with $\operatorname{im} F_*^\vee$ generate $\frac{m_S}{m_S^2}$, we can define a map $R = \Lambda[[x_1, \ldots, x_r]] \to S$ by $x_i \mapsto \alpha_i$. This map is surjective, and if its kernel is $J$, $S \cong R/J$. Furthermore, $J \subseteq m_A R + m_R^2$. Let $D_0 : C_A \to \text{Ens}$ be the functor pro-represented by $S = R/J$. We will first prove

(1.10) If $V = \text{coker}(T^1_1(X/k) \to T^1_2(F(X)/k))$, then $V \subseteq (J/m_R J)^\vee$, and the obstruction map $\delta_0$ in

$$
D_0(A_{n+1}) \to D_0(A_n) \xrightarrow{\delta_0} (J/m_R J)^\vee \otimes (t^{n+1})
$$

always takes its values in $V \otimes (t^{n+1})$.

To prove this, first note the change of rings sequence for the functors $T^i$ of Lichtenbaum and Schlessinger ([25], pg. 235) for $k \to \Lambda \to S$ yields the exact sequence

$$
T^0(S/k, k) \to T^0(A/k, k) \to T^1(S/\Lambda, k) \to T^1(S/k, k),
$$

or equivalently

(1.11) 

$$
(m_S/m_S^2)^\vee \to (m_A/m_A^2)^\vee \to (J/m_R J)^\vee \xrightarrow{d} (I/m_P I)^\vee.
$$

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Thus \( \ker d = V \). Furthermore, the map \( d \) is compatible with the obstruction maps: in particular, the diagram

\[
\begin{array}{ccc}
D_0(A_{n+1}) & \rightarrow & D_1(A_{n+1}) \\
\downarrow & & \downarrow \\
D_0(A_n) & \rightarrow & D_1(A_n) \\
\delta_0 & & \delta_1 \\
(J/m_RJ) \hat{\otimes} (t^{n+1}) & \stackrel{d \otimes 1}{\rightarrow} & (I/m_PI) \hat{\otimes} (t^{n+1})
\end{array}
\]

is commutative for all \( n \).

Now fix a \( \Lambda \)-algebra structure on \( A_{n+1} \), inducing a \( \Lambda \)-algebra structure on \( A_n \) and \( A_{n-1} \). Since \( \Lambda \) is a hull for \( D_2 \), these \( \Lambda \)-algebra structures yield elements \( X'_i \in D_2(A_i) \), for \( i = n-1, n \) and \( n+1 \). Let \( X_n \in D_0(A_n) \subseteq D_1(A_n) \) be arbitrary, so that \( F(X_n) = X'_n \). Let \( \alpha_{n-1} \in T^1(X_{n-1}/A_{n-1}) \) be given by \( D_1(\epsilon_n)(X_n) \), and \( \alpha'_{n-1} \in T^1(X'_{n-1}/A_{n-1}) \) be given by \( D_2(\epsilon_n)(X'_n) = F(\alpha_{n-1}) \). By Theorem 1.8, \( \delta_2(\alpha'_{n-1}) = 0 \) since \( X'_{n+1} \) is a lifting of \( X'_n \). Since \( l|_{\text{im}(\delta_1)} \) is injective, \( \delta_1(\alpha_{n-1}) = 0 \). Again by Theorem 1.8, \( X_n \) lifts to \( X_{n+1} \in D_1(A_{n+1}) \), so \( (d \otimes 1) \circ \delta_0(X_n) = \delta_1(X_n) = 0 \), so \( \delta_0(X_n) \in \ker d \otimes 1 = V \otimes (t^{n+1}) \). Thus the image of \( \delta_0 \) is contained in \( V \otimes (t^{n+1}) \). This proves (1.10).

We now follow some of the ideas of Kawamata in [19] to construct \( J' \). Dualizing the sequence (1.11), we have

\[
J/m_RJ \xrightarrow{\phi} m_\Lambda/m^2_\Lambda F^*_S \rightarrow m_S/m^2_S.
\]

Let \( f_1, \ldots, f_t \in J \) be elements such that \( \phi(f_1), \ldots, \phi(f_t) \) generate \( \ker F^*_S \), with \( t = \dim_k \ker F^*_S = \dim_k V \). Set \( J' = (f_1, \ldots, f_t) \). There is a surjective map \( m_R/(m^2_R + J') \rightarrow m_R/(m^2_R + J) \) and the dimensions of these spaces are the same by construction, so we have equality.

The map \( J'/m_R J' \xrightarrow{i} J/m_R J \) induced by \( J' \subseteq J \) is an inclusion since \( \dim_k J'/m_R J' \leq t \) but \( \dim_k \text{im}(\phi \circ i) = t \). Thus dually the composition

\[
V \rightarrow (J/m_R J)^{\vee} \rightarrow (J'/m_R J')^{\vee}
\]

is an isomorphism. Let \( D'_0 : C_\Lambda \rightarrow \text{Ens} \) be the functor pro-represented by \( R/J' \). Clearly \( D_0 \) is a subfunctor of \( D'_0 \), and for each \( n \), we have a commutative diagram

\[
\begin{array}{ccc}
D_0(A_{n+1}) & \rightarrow & D_0(A_n) & \delta_0 & \rightarrow & V \otimes (t^{n+1}) \\
\downarrow & & \downarrow & & \downarrow & \cong \\
D'_0(A_{n+1}) & \rightarrow & D'_0(A_n) & \delta'_0 & \rightarrow & (J'/m_R J')^{\vee} \otimes (t^{n+1})
\end{array}
\]
This shows that an element of \( D_0(A_n) \) lifts to \( D_0(A_{n+1}) \) if and only if it lifts to \( D'_0(A_{n+1}) \). Since the choice of liftings is a principal homogeneous space over \( (m_R/(m_R \Lambda + m_R^2))^\vee \otimes (t^{n+1}) \) for both \( D_0 \) and \( D'_0 \), we see inductively that \( D_0(A_n) = D'_0(A_n) \) for any \( \Lambda \)-algebra structure on \( A_n \). Thus
\[
\text{Hom}_{\Lambda \text{-alg}}(R/J, k[[t]]) = \text{Hom}_{\Lambda \text{-alg}}(R/J', k[[t]])
\]
for any \( \Lambda \)-algebra structure on \( k[[t]] \), and so
\[
\text{Hom}_{k \text{-alg}}(R/J, k[[t]]) = \text{Hom}_{k \text{-alg}}(R/J', k[[t]])
\]
and we conclude that \( \text{Supp}(R/J) = \text{Supp}(R/J') \) as in the last paragraph of the proof of [19], Theorem 1. 

\section{Obstructions for Calabi-Yau threefolds with canonical singularities. (2.1)}

Let \( X \) be a Calabi-Yau threefold with canonical singularities over \( k = \mathbb{C} \), the complex numbers, i.e. a threefold with canonical singularities, \( K_X = 0 \), and \( h^1(O_X) = 0 \). Set \( Z = \text{Sing}(X) \). We want to study the deformation theory of \( X \). In particular, we will relate the deformation theory of \( X \) to the deformation theory of \( X \), the formal completion of \( X \) along \( Z \). To paraphrase Theorem 2.2 below, we will find that the obstructions to deforming \( X \) are contained in the obstructions to deforming \( X \), i.e. “obstructions to deforming \( X \) are local to the singularities of \( X \).”

To make this concept rigorous, let \( D \) be the functor of deformations of \( X \), and let \( D_{loc} \) be the functor of deformations of \( X \), as in (1.6). There is a natural morphism of functors \( F : D \to D_{loc} \) taking a deformation to its completion along \( Z \). Note that \( D \) is pro-representable since \( \text{Hom}(\Omega^1_X, O_X) = 0 \) by [17], Corollary 8.6.

If \( X_n \) is a deformation of \( X \) over \( A_n \), we have \( T^1(X_n/A_n) \cong \text{Ext}^1_{O_X}(\Omega^1_{X_n/A_n}, O_{X_n}) \) and \( T^1_{loc}(X_n/A_n) \cong \text{Ext}^1_{O_{X_n}}(\Omega^1_{X_n/A_n} \otimes_{O_X} O_{X_n}, O_{X_n}) \). The former isomorphism is well-known. For the latter, if \( X_n/A_n \) is locally embedded in \( Y_n/A_n \) smooth with ideal sheaf \( I \), then the local \( T^1 \) sheaf of \( X_n/A_n \) is as usual given by

\[
\text{Hom}_{O_{X_n}}(\Omega^1_{Y_n/A_n}|_{X_n}, O_{X_n}) \to \text{Hom}_{O_{X_n}}(\mathbb{I}/\mathbb{I}^2, O_{X_n}) \to T^1 \to 0.
\]

Note that \( \Omega^1_{Y_n/A_n} \otimes_{O_X} O_{Y_n} \cong \Omega^1_{Y_n/A_n} \), the completion of \( \Omega^1_{Y_n/A_n} \) (see [14], Chapter 0, 20.7.14). Since \( O_{X_n} \) is complete,

\[
\text{Hom}_{O_{X_n}}(\Omega^1_{Y_n/A_n}|_{X_n}, O_{X_n}) = \text{Hom}_{O_{X_n}}(\Omega^1_{Y_n/A_n}|_{X_n}, O_{X_n})
\]

\[
= \text{Hom}_{O_{X_n}}(\Omega^1_{Y_n/A_n}|_{X_n} \otimes_{O_X} O_{X_n}, O_{X_n}).
\]
Thus from the exact sequence
\[ I/I^2 \otimes \mathcal{O}_{X_n} \mathcal{O}_{\hat{X}_n} \to \Omega^1_{Y_n/A_n} \vert_{X_n} \otimes \mathcal{O}_{X_n} \mathcal{O}_{\hat{X}_n} \to \Omega^1_{X_n/A_n} \otimes \mathcal{O}_{X_n} \mathcal{O}_{\hat{X}_n} \to 0 \]
we see that \( T^1 \cong \operatorname{Ext}^1_{\mathcal{O}_{\hat{X}_n}}(\Omega^1_{X_n/A_n} \otimes \mathcal{O}_{\hat{X}_n}, \mathcal{O}_{\hat{X}_n}) \). Local infinitesimal deformations of \( \hat{X}_n \) then patch together as usual to yield an element of \( \operatorname{Ext}^1_{\mathcal{O}_{\hat{X}_n}}(\Omega^1_{X_n/A_n} \otimes \mathcal{O}_{\hat{X}_n}, \mathcal{O}_{\hat{X}_n}) \).

Finally, let \( T^2 = \operatorname{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \) and \( T^2_{loc} = \operatorname{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \). Let \( l : T^2 \to T^2_{loc} \) be the map induced by the map \( \mathcal{O}_X \to \mathcal{O}_{\hat{X}} \).

The following theorem is our generalization of the Bogomolov-Tian-Todorov unobstructedness theorem.

**Theorem 2.2.** Let \( X \) be a Calabi-Yau threefold with isolated canonical singularities, and \( X_n/A_n \) a deformation of \( X \). There is a commutative diagram

\[
\begin{array}{ccc}
T^1(X_n/A_n) & \xrightarrow{F} & T^1_{loc}(X_n/A_n) \\
\downarrow & & \downarrow \\
T^1(X_{n-1}/A_{n-1}) & \xrightarrow{F} & T^1_{loc}(X_{n-1}/A_{n-1}) \\
\downarrow \delta & & \downarrow \delta_{loc} \\
T^2 & \xrightarrow{l} & T^2_{loc}
\end{array}
\]

where \( l_{|\operatorname{im}(\delta)} : \operatorname{im}(\delta) \to T^2_{loc} \) is injective.

We shall prove this result by generalising Namikawa’s argument in [27].

**Lemma 2.3.** Given the hypotheses of (2.1), if \( X_n \) is a deformation of \( X \) over \( A_n \), then there are natural isomorphisms

\[ \operatorname{Ext}^i_{\mathcal{O}_{X_n}}(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_n}) \cong \operatorname{Ext}^i_{\mathcal{O}_{X_m}}(\Omega^1_{X_m/A_m}, \mathcal{O}_{X_m}) \]

for \( m < n, i \leq 2 \) and \( X_m = X_n \otimes A_n A_m \).

Proof: The change of rings spectral sequence ([39], Theorem 11.65) tells us that

\[ \operatorname{Ext}^p_{\mathcal{O}_{X_m}}(\operatorname{Tor}^q_{\mathcal{O}_X}(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_n}), \mathcal{O}_{X_m}) \Rightarrow \operatorname{Ext}^i_{\mathcal{O}_{X_n}}(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_m}). \]

Now \( \Omega^1_{X_n/A_n} \) is a flat \( \mathcal{O}_{X_n} \)-module away from \( Z \subseteq X_n \), so \( \operatorname{Tor}^q_{\mathcal{O}_X}(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_n}) \) is supported on \( Z \) for \( q \geq 1 \). Since \( X_m \) is Cohen-Macaulay, \( \operatorname{Ext}^p_{\mathcal{O}_{X_m}}(\operatorname{Tor}^q_{\mathcal{O}_X}(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_n}), \mathcal{O}_{X_m}) = 0 \) for \( p \leq 1, q \geq 1 \). Thus

\[ \operatorname{Ext}^i_{\mathcal{O}_{X_m}}(\Omega^1_{X_m/A_m}, \mathcal{O}_{X_m}) = \operatorname{Ext}^i_{\mathcal{O}_{X_m}}(\Omega^1_{X_n/A_n} \otimes \mathcal{O}_{X_n} \mathcal{O}_{X_m}, \mathcal{O}_{X_m}) \]

for \( i \leq 2 \). The statement of the lemma for global Exts then follows from the local-global spectral sequence for Exts. •

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Lemma 2.4. There are isomorphisms

(a) \( \text{Ext}^1_{\mathcal{O}_{X_m}}(\Omega^1_{X_m/A_m}, \mathcal{O}_{X_m}) \cong T^1_{\text{loc}}(X_m/A_m) \) for \( m < n \).

(b) \( \text{Ext}^2_{\mathcal{O}_{X_m}}(\Omega^1_{X_m/A_m}, \mathcal{O}_{\hat{X}_m}) = T^2_{\text{loc}} \).

(c) \( (T^2_{\text{loc}})^\vee \cong H^1_Z(\Omega^1_X), (T^2)^\vee \cong H^1(\Omega^1_X) \), and the natural map \( H^1_Z(\Omega^1_X) \to H^1(\Omega^1_X) \) is dual to \( l \).

Proof: The change of rings spectral sequence yields

\[
\text{Ext}^p_{\mathcal{O}_{X_m}}(\text{Tor}^q_{\mathcal{O}_{X_n}}(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_m}), \mathcal{O}_{\hat{X}_m}) \Rightarrow \text{Ext}^i_{\mathcal{O}_{X_m}}(\Omega^1_{X_n/A_n}, \mathcal{O}_{\hat{X}_m}),
\]

so the same argument as in the proof of Lemma 2.3 shows that

\[
\text{Ext}^i_{\mathcal{O}_{X_m}}(\Omega^1_{X_n/A_n}, \mathcal{O}_{\hat{X}_m}) \cong \text{Ext}^i_{\mathcal{O}_{X_m}}(\Omega^1_{X_n/A_n}, \mathcal{O}_{\hat{X}_m})
\]

for \( i \leq 2 \). This proves (b) in particular. Furthermore, since \( \mathcal{O}_{\hat{X}_m} \) is a flat \( \mathcal{O}_{X_m} \)-module,

\[
\text{Ext}^i_{\mathcal{O}_{X_m}}(\Omega^1_{X_n/A_n}, \mathcal{O}_{\hat{X}_m}) \cong \text{Ext}^i_{\mathcal{O}_{X_m}}(\Omega^1_{X_n/A_n} \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{\hat{X}_m}, \mathcal{O}_{\hat{X}_m}),
\]

which is \( T^1_{\text{loc}}(X_m/A_m) \) by (2.1).

To prove (c), we use Alonso, Jeremías and Lipman’s generalization of local duality [1]. By [1], (0.3), (0.1), and (0.4.1),

\[
\mathbf{R}\text{Hom}_{\mathcal{O}_X}(\mathbf{R}\Gamma_Z \Omega^1_X, \mathcal{O}_X) \cong \mathbf{R}\text{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X).
\]

By the Grothendieck duality theorem of [15],

\[
\mathbf{R}\Gamma \mathbf{R}\text{Hom}_{\mathcal{O}_X}(\mathbf{R}\Gamma_Z \Omega^1_X, \mathcal{O}_X) \cong \text{Hom}_k(\mathbf{R}\Gamma \mathbf{R}\Gamma_Z \Omega^1_X, k[-3])
\]

\[= \text{Hom}_k(\mathbf{R}\Gamma_Z \Omega^1_X, k[-3]).\]

Putting these two isomorphisms together and taking cohomology of the two complexes, we have

\[
H^i(\text{Hom}_k(\mathbf{R}\Gamma_Z \Omega^1_X, k[-3])) \cong H^i(\mathbf{R}\text{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)).
\]

For \( i = 2 \), this yields the isomorphism

\[
(H^1_Z)(\Omega^1_X)^\vee \cong \text{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \cong T^2_{\text{loc}}.
\]

This isomorphism is compatible the map \( T^2 = \text{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \to \text{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = T^2_{\text{loc}} \) induced by the map \( \mathcal{O}_X \to \mathcal{O}_{\hat{X}} \) since [1], (0.3) also tells us that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathbf{R}\text{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) & \xrightarrow{\beta} & \mathbf{R}\text{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
\mathbf{R}\text{Hom}_{\mathcal{O}_X}(\mathbf{R}\Gamma_Z \Omega^1_X, \mathcal{O}_X) & \xrightarrow{\beta} & \mathbf{R}\text{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)
\end{array}
\]
where \( \alpha \) is induced by the natural map in the derived category \( \mathbf{R}\mathcal{H}_{Z}\Omega_{X}^{1} \to \Omega_{X}^{1} \) and \( \beta \) is induced by \( \mathcal{O}_{X} \to \mathcal{O}_{X} \).

We need a version of Namikawa’s Lemma 2.2 of [27]. Let \( U = X - Z \). Then \( X_{n}/A_{n} \) induces a deformation \( U_{n} \) of \( U \) over \( A_{n} \). We set \( \hat{\Omega}_{X_{n}/A_{n}}^{1} = j_{*}\Omega_{U_{n}/A_{n}}^{1} \) with \( j : U_{n} \to X_{n} \) the inclusion. We put \( \hat{\Omega}_{X}^{1} := \hat{\Omega}_{X/k}^{1} \). Observe that we have a natural map \( d\log : H^{1}(X, \mathcal{O}_{X}^{*}) \to H^{1}(X, \hat{\Omega}_{X}^{1}) \) which is the composition of \( H^{1}(X, \mathcal{O}_{X}^{*}) \) and \( H^{1}(X, \Omega_{X}^{1}) \to H^{1}(X, \hat{\Omega}_{X}^{1}) \). Also, if \( \pi : X \to \tilde{X} \) is a resolution of singularities of \( X \), then the map \( \Omega_{X_{n}} \to \hat{\Omega}_{X_{n}}^{1} \) factors through \( \Omega_{X}^{1} \to \pi_{*}\Omega_{X}^{1} \).

**Lemma 2.5.** The image of the map

\[
d\log : H^{1}(X, \mathcal{O}_{X}^{*}) \to H^{1}(X, \hat{\Omega}_{X}^{1})
\]

generates \( H^{1}(X, \hat{\Omega}_{X}^{1}) \) as a \( k \)-vector space.

Proof: The proof of [27, Lemma 2.2] actually shows in general that the image of \( H^{1}(X, \mathcal{O}_{X}^{*}) \to H^{1}(X, \pi_{*}\Omega_{X}^{1}) \) generates the latter as a vector space, using only the hypothesis that \( \tilde{X} \) has rational singularities, which is true of any canonical singularity. Now the kernel and cokernel of \( \pi_{*}\Omega_{X}^{1} \to \hat{\Omega}_{X}^{1} \) are supported on \( Z \). If we can show furthermore that \( \text{coker}(\pi_{*}\Omega_{X}^{1} \to \hat{\Omega}_{X}^{1}) \) is supported on a finite set of points, then in fact \( H^{1}(\pi_{*}\Omega_{X}^{1}) \to H^{1}(\hat{\Omega}_{X}^{1}) \) is surjective and the lemma follows.

By [37], except for a finite number of dissident points, \( X \) is analytically isomorphic in a neighborhood of a point of \( Z \) to \( \Delta \times S \), where \( \Delta \) is the germ of a curve and \( S \) is a germ of a du Val surface singularity, with resolution \( \pi' : \tilde{S} \to S \). Thus, in this neighborhood, \( \pi : \tilde{X} \to X \) looks like \( \pi : \Delta \times \tilde{S} \to \Delta \times S \). Let \( p_{1} \) and \( p_{2} \) be the projections of \( \Delta \times S \) onto \( \Delta \) and \( S \) respectively. The map \( \pi_{*}\Omega_{X}^{1} \to \hat{\Omega}_{X}^{1} \) is concentrated on the dissident points of \( \Delta \). •

**Lemma 2.6.** Let \( M_{n} \) be an \( A_{n} \)-module, with \( M_{i} = M_{n} \otimes_{A_{n}} A_{i} \) for \( i < n \). Let \( \phi : \text{Hom}_{A_{n}}(M_{n}, A_{n}) \to \text{Hom}_{A_{n-1}}(M_{n-1}, A_{n-1}) \) be given by \( f \mapsto f \otimes 1_{A_{n-1}} \), and \( \phi' : M_{n-1} \to M_{n} \) be given by multiplication by \( t \). Then there is a natural isomorphism \( \text{coker}(\phi) \cong (\text{ker}(\phi'))' \) as \( k \)-vector spaces.

Proof: Applying \( \text{Hom}_{A_{n}}(M_{n}, \cdot) \) to the exact sequence of \( A_{n} \)-modules

\[
0 \to k \to A_{n} \to A_{n-1} \to 0
\]

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yields
\[ \text{Hom}_A(M_n, A_n) \to \text{Hom}_A(M_n, A_{n-1}) \to \text{Ext}^1_{A_n}(M_n, k) \to 0. \]

But \( \text{Hom}_A(M_n, A_{n-1}) \cong \text{Hom}_{A_{n-1}}(M_{n-1}, A_{n-1}) \), so \( \text{coker} \phi \cong \text{Ext}^1_{A_n}(M_n, k) \).

Similarly, the sequence
\[ 0 \to A_{n-1} \overset{t}{\to} A_n \to k \to 0 \]
tensored with \( M_n \) yields
\[ 0 \to \text{Tor}^1_{A_n}(M_n, k) \to M_{n-1} \overset{\phi'}{\to} M_n. \]
Thus \( \text{ker} \phi' = \text{Tor}^1_{A_n}(M_n, k) \). Now
\[ \text{Hom}_k(\text{Tor}^1_{A_n}(M_n, k), k) \cong \text{Ext}^1_{A_n}(M_n, k), \]
as follows from the change of rings spectral sequence. \( \bullet \)

**Proof of the theorem:** To obtain the first column of the diagram, we apply \( \text{Hom}_{O_{X_n}}(\Omega^1_{X_n/A_n}, \cdot) \) to the exact sequence
\[ 0 \to O_X \to O_{X_n} \to O_{X_{n-1}} \to 0 \]
yielding
\[ \text{Ext}^1_{O_{X_n}}(\Omega^1_{X_n/A_n}, O_{X_n}) \to \text{Ext}^1_{O_{X_n}}(\Omega^1_{X_n/A_n}, O_{X_{n-1}}) \to \text{Ext}^2_{O_{X_n}}(\Omega^1_{X_n/A_n}, O_X), \]
from which we obtain the first column, by Lemma 2.3. The second column is obtained by applying \( \text{Hom}_{O_{X_n}}(\Omega^1_{X_n/A_n}, \cdot) \) to the sequence
\[ 0 \to O_{\tilde{X}} \to O_{\tilde{X}_n} \to O_{\tilde{X}_{n-1}} \to 0, \]
and applying Lemma 2.4. The maps between the columns induced by the maps \( O_{X_n} \to O_{\tilde{X}_n} \) yield a commutative diagram. For the first two rows, these maps coincide with the maps induced by \( F : D \to D_{loc} \).

We now only need to show that \( l|_{\text{im}(\delta)} : \text{im}(\delta) \to T^2_{loc} \) is injective for each such diagram. This is equivalent to \( l|_{\text{im}(\delta)}^\vee : (T^2_{loc})^\vee \to \text{im}(\delta)^\vee \) being surjective.

We have a diagram
\[
\begin{array}{cccc}
H^1_Z(\Omega^1_{X_n/A_n}) & \to & H^1(\Omega^1_{X_n/A_n}) & \to & H^1(\tilde{\Omega}^1_{X_n/A_n}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_Z(\Omega^1_X) & \to & H^1(\Omega^1_X) & \to & H^1(\tilde{\Omega}^1_X) \\
\downarrow & & \downarrow & & \downarrow \\
K_{loc} & \to & \tilde{K} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\]
Here the vertical maps between the first two rows are induced by the restriction map \( \Omega^1_{X_n/A_n} \to \Omega^1_X \), and \( K_{iloc} \) and \( K \) are just defined to make the columns exact. The map \( H^1_Z(\Omega^1_X) \to H^1(\Omega^1_X) \) is dual to \( t : T^2 \to T^2_{iloc} \), by Lemma 2.4 (c). I claim the rows of this diagram are exact.

**Exactness of the first two rows:** For any \( n \), we have the exact sequence

\[
0 \to H^0_Z(\Omega^1_{X_n/A_n}) \to \Omega^1_{X_n/A_n} \to \hat{\Omega}^1_{X_n/A_n} \to H^1_Z(\Omega^1_{X_n/A_n}) \to 0.
\]

Now \( H^0_Z(\Omega^1_{X_n/A_n}) \) is supported on the set of points of \( X \) which are not locally complete intersection, and so has finite support. Thus \( H^1_Z(\Omega^1_{X_n/A_n}) = 0 \) and \( H^1(\hat{\Omega}^1_{X_n/A_n}) = H^0_Z(\Omega^1_{X_n/A_n}) \). From this follows exactness of the first two rows. The last row is then clear.

**Exactness of the last row:** As in [27], there is a diagram

\[
\begin{array}{ccc}
H^1(O^*_X) & \longrightarrow & H^1(\hat{\Omega}^1_X) \\
\downarrow \alpha & & \downarrow \\
H^1(O^*_X) & \longrightarrow & H^1(\hat{\Omega}^1_X)
\end{array}
\]

with \( \alpha \) surjective since \( H^2(O_X) = 0 \), and by Lemma 2.5, the image of \( H^1(O^*_X) \) in \( H^1(\hat{\Omega}^1_X) \) generates the latter as a \( k \)-vector space. Thus the composed map \( H^1(O^*_X) \otimes Z k \to H^1(\hat{\Omega}^1_X) \) is surjective, and so the map \( H^1(\hat{\Omega}^1_X) \to H^1(\hat{\Omega}^1_X) \) is surjective. Now a simple diagram chase shows that the last row is exact.

This diagram shows that the composed map \( H^1_Z(\Omega^1_X) \to K \) is surjective. Now we also have the sequence

\[
0 \longrightarrow T \longrightarrow \Omega^1_{X_{n-1}/A_{n-1}} \longrightarrow \Omega^1_{X_n/A_n} \longrightarrow \Omega^1_X \longrightarrow 0
\]

obtained by tensoring

\[
0 \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow k \longrightarrow 0
\]

by \( \Omega^1_{X_n/A_n} \). \( T \) is a sheaf supported on \( Z \). This yields an exact sequence

\[
H^1(\Omega^1_{X_n/A_n}) \longrightarrow H^1(\Omega^1_X) \longrightarrow H^2(\Omega^1_{X_{n-1}/A_{n-1}}) \longrightarrow H^2(\Omega^1_{X_n/A_n})
\]

showing that \( K = \ker(H^2(\Omega^1_{X_{n-1}/A_{n-1}}) \to H^2(\Omega^1_{X_n/A_n})) \). On the other hand, there is a natural map

\[
H^2(\Omega^1_{X_n/A_n}) \otimes A_n A_{n-1} \to H^2(\Omega^1_{X_{n-1}/A_{n-1}}),
\]
(see [16, III Prop. 12.5]). This is in fact an isomorphism: we have from
\[ A_n \xrightarrow{t^n} A_n \longrightarrow A_{n-1} \longrightarrow 0 \]
the diagram
\[
\begin{array}{ccc}
H^2(\Omega^1_{X_n/A_n}) & \xrightarrow{t^n} & H^2(\Omega^1_{X_n/A_n}) \\
\downarrow \cong & & \downarrow \cong \\
H^2(\Omega^1_{X_n/A_n}) & \xrightarrow{t^n} & H^2(\Omega^1_{X_n/A_n})
\end{array}
\]
with the exactness of the bottom row because \( H^3(\Omega^1_X) = \text{Hom}(\Omega^1_X, \mathcal{O}_X) \cong 0 \). Thus \( \phi \) is an isomorphism. So

\[
K = \ker(H^2(\Omega^1_{X_n/A_n}) \otimes_{A_n} A_{n-1} \xrightarrow{t} H^2(\Omega^1_{X_n/A_n})) \\
\cong \text{coker}(\text{Hom}_{A_n}(H^2(\Omega^1_{X_n/A_n}), A_n) \rightarrow \text{Hom}_{A_{n-1}}(H^2(\Omega^1_{X_{n-1}/A_{n-1}}, A_{n-1})) \cong \text{coker}(\text{Ext}^1_{\mathcal{O}_{X_n}}(\Omega^1_{X_n/A_n}, \mathcal{O}_{X_n}) \rightarrow \text{Ext}^1_{\mathcal{O}_{X_{n-1}}}(\Omega^1_{X_{n-1}/A_{n-1}}, \mathcal{O}_{X_{n-1}})) \cong (\text{im } \delta)^\vee.
\]

by Lemma 2.6 and Serre duality,

\[
\cong \text{coker}(T^1(X_n/A_n) \rightarrow T^1(X_{n-1}/A_{n-1})) \cong (\text{im } \delta)^\vee.
\]

So the map \( H^1(\Omega^1_X) \rightarrow K \rightarrow 0 \) is dual to \( 0 \rightarrow \text{im } \delta \rightarrow T^2 \). Thus \( (T^2_{\text{loc}})^\vee \rightarrow \text{im}(\delta)^\vee \) is the surjection \( H^1_Z(\Omega^1_X) \rightarrow K \). This is the desired surjectivity. ●

Remark 2.7. We cannot apply Theorem 2.2 immediately to the situation of Theorem 1.9 without first knowing that \( D_{\text{loc}} \) has a hull. By (1.6), this is the case if and only if \( T^1_{\text{loc}}(X/k) \) is finite dimensional. This is of course the case if \( X \) has isolated singularities, in which case, we can just as well consider the complex germ \((X, Z)\) instead of the formal scheme \( \hat{X} \). However, \( T^1_{\text{loc}} \) need not be finite dimensional if \( X \) has non-isolated singularities. Nevertheless, even in this case, Theorem 2.2 can be useful. In any event, Theorem 2.2 tells us that

\[
T^1(X_n/A_n) \rightarrow T^1(X_{n-1}/A_{n-1}) \rightarrow T^2_{\text{loc}}
\]
is exact. If \( T^2_{\text{loc}} = 0 \), then \( D \) is unobstructed by the \( T^1 \)-lifting criterion. If \( T^2_{\text{loc}} \neq 0 \), then we obtain dimension estimates for \( \text{Def}(X) \) using methods similar to [19]. See [12] for an application in the non-isolated case. Note in particular that if \( X \) has isolated complete intersection singularities, then \( T^2_{\text{loc}} = 0 \), reproducing the unobstructedness result of [27] in a rather more inefficient way.
Example 2.8. We give here a simple example of a Calabi-Yau with obstructed deformation theory. Let \( Y \subseteq P^9 \) be a cone over a non-singular del Pezzo surface in \( P^8 \) isomorphic to \( P^2 \) blown up in one point. By [2], a hull of the deformation functor of the singular point of \( Y \) is \( \Lambda = k[t]/(t^2) \). In fact, in [2], (9.2), Altmann gives explicit equations for a deformation of the affine cone in \( C^9 \) over the del Pezzo surface in \( P^8 \), and these equations are easily projectivized to yield a non-trivial but obstructed infinitesimal deformation of \( Y, \mathcal{Y}/\text{Spec } \Lambda \).

Let \( X \) be a double cover of \( Y \) branched over the intersection of \( Y \) with a general quartic hypersurface in \( P^9 \). It is easy to see that \( K_X = 0, X \) has two singular points analytically isomorphic to the singular point of \( Y \), and the deformation \( \mathcal{Y}/\text{Spec } \Lambda \) lifts to a deformation \( \mathcal{X}/\text{Spec } \Lambda \), which is obstructed. In fact, \( \text{Def}(X) \) is non-reduced.

See [11] for an example in any dimension of obstructed deformations for a Calabi-Yau with non-isolated singularities. Such an example is much more subtle, and requires a more global analysis.

§3. Calabi-Yaus with complete intersection singularities.

(3.1) We will first consider, very generally, the situation that \( (X,0) \) is the germ of an isolated rational complex threefold singularity, and that \( \pi: (\tilde{X}, E) \to (X,0) \) is a resolution of singularities. We have a natural map of germs of analytic spaces \( \text{Def}(\tilde{X}) \to \text{Def}(X) \) by [23], Proposition 11.4 (by [47], Theorem 1.4 (c) for the map on the level of deformation functors,) since \( H^1(\mathcal{O}_X) = 0 \). We denote by \( \mathcal{O}_{X,0} \) the local ring of \( X \) at the origin with maximal ideal \( m \), and we denote by \( T^1 \) the tangent space of \( \text{Def}(X) \).

Lemma 3.2. The tangent space to \( \text{Def}(\tilde{X}) \) is \( H^0(R^1\pi_*\mathcal{T}_\tilde{X}) \), of \( \text{Def}(X) \) is \( T^1 = H^2_0(\mathcal{T}_X) \), where \( \mathcal{T}_X = \text{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \), and there is an exact sequence of \( \mathcal{O}_{X,0} \)-modules

\[
H^0(R^1\pi_*\mathcal{T}_\tilde{X}) \to H^2_0(\mathcal{T}_X) \to T^1 \to 0
\]

with \( T^1 = \ker(H^2_0(\mathcal{T}_X) \to H^0(R^2\pi_*\mathcal{T}_\tilde{X})) \), \( Z = \text{Sing}(X) \) and \( E \) the exceptional locus of \( \pi \).

The map \( H^0(R^1\pi_*\mathcal{T}_\tilde{X}) \to H^2_0(\mathcal{T}_X) \) is the differential of the map \( \text{Def}(\tilde{X}) \to \text{Def}(X) \).

Proof: Since \( X \) is a germ, \( H^1(\pi_*\mathcal{T}_\tilde{X}) = H^2(\pi_*\mathcal{T}_\tilde{X}) = 0 \). Thus the tangent space to \( \text{Def}(\tilde{X}) \) is \( H^1(\mathcal{T}_\tilde{X}) \cong H^0(R^1\pi_*\mathcal{T}_\tilde{X}) \) by the Leray spectral sequence. Similarly, \( H^2(\mathcal{T}_X) = H^0(R^2\pi_*\mathcal{T}_\tilde{X}) \). Also, the tangent space to \( \text{Def}(X) \) is \( H^1(X - \{0\}, \mathcal{T}_X) = H^1(\tilde{X} - E, \mathcal{T}_\tilde{X}) = H^2_0(\mathcal{T}_X) \), by Theorem 2 of [41]. The map \( H^1(\tilde{X}, \mathcal{T}_\tilde{X}) \to H^1(\tilde{X} - E, \mathcal{T}_\tilde{X}) \) is the differential of \( \text{Def}(\tilde{X}) \to \text{Def}(X) \). Hence the exact sequence

\[
H^1(\tilde{X}, \mathcal{T}_\tilde{X}) \to H^1(\tilde{X} - E, \mathcal{T}_\tilde{X}) \to H^2_0(\tilde{X}, \mathcal{T}_\tilde{X}) \to H^2(\tilde{X}, \mathcal{T}_\tilde{X})
\]
is identical to
\[ H^0(R^1\pi_*T_X) \to H^2_Z(T_X) \to H^2_{\tilde{E}}(T_{\tilde{X}}) \to H^0(R^2\pi_*T_{\tilde{X}}) \]
which yields the desired sequence. Elements of \( \mathcal{O}_{X,0} \) pull back to elements of \( \mathcal{O}_{\tilde{X}} \), which then act on \( T_{\tilde{X}} \), and so \( H^1(\tilde{X}, T_{\tilde{X}}) \) and \( H^1(\tilde{X} - E, T_{\tilde{X}}) \) are naturally \( \mathcal{O}_{X,0} \)-modules.

(3.3) In our situation we will be interested in the case that \((X, 0)\) is an isolated rational Gorenstein point, and that \(\tilde{X} \to X\) is a crepant resolution. Recall from [37] that there is an invariant \( k \) associated with a rational Gorenstein point as follows:

\begin{itemize}
  \item \( k = 0 \) if \((X, 0)\) is a cDV point, so is terminal.
  \item \( k = 1 \) if \((X, 0)\) is a hypersurface singularity locally of the form \( x^2 + y^3 + f(y, z, t) = 0 \) where \( f = yf_1(z, t) + f_2(z, t) \) (respectively \( f_1 \) (respectively \( f_2 \)) is a sum of monomials \( z^a t^b \) of degree \( a + b \geq 4 \) (respectively \( \geq 6 \)).
  \item \( k = 2 \) if \((X, 0)\) is a hypersurface singularity locally of the form \( x^2 + f(y, z, t) = 0 \) where \( f \) is a sum of monomials of degree \( \geq 4 \).
  \item \( k \geq 3 \) if \( \text{mult}_0 X = k \) and \( \text{emb.dim.}(X, 0) = k + 1 \). The exceptional divisor of the blow-up of \( 0 \in X \) is a del Pezzo surface of degree \( k \).
\end{itemize}

So in particular, for \( k \leq 3 \), \((X, 0)\) is a hypersurface singularity, and for \( k = 4 \), \((X, 0)\) is a complete intersection (Gorenstein in codimension 2 implies complete intersection) of two equations whose leading terms are quadratic and define a del Pezzo surface in \( \mathbb{P}^4 \). However \((X, 0)\) is never a complete intersection if \( k > 4 \).

**Proposition 3.4.** Let \( \tilde{X} \to X \) be a crepant resolution of an isolated rational Gorenstein threefold singularity \((X, 0)\). Then \( \text{Def}(\tilde{X}) \) is non-singular.

Proof. The Hodge theory of \( \tilde{X} \) is well-behaved above the middle dimension from [30]: in particular the spectral sequence

\[ H^q(\Omega^p_{\tilde{X}}) \Rightarrow H^n(\tilde{X}, \mathbb{C}) \]

degenerates at the \( E_1 \) level for \( p + q > 3 \). Thus as in the proof of 5.5 of [4], if \( \tilde{X}_n/A_n \) is a deformation of \( \tilde{X} \) over \( A_n \), then \( H^2(\Omega^2_{\tilde{X}_n/A_n}) \) is a locally free \( A_n \)-module of rank \( \dim H^2(\Omega^2_{\tilde{X}}) \). Then, as a case of Ran’s “\( T^2 \)-injecting” criterion [35], we see we have an exact sequence

\[ H^1(\Omega^2_{\tilde{X}_n/A_n}) \to H^1(\Omega^2_{\tilde{X}_{n-1}/A_{n-1}}) \to H^2(\Omega^2_{\tilde{X}}) \to H^2(\Omega^2_{\tilde{X}_n/A_n}) \to H^2(\Omega^2_{\tilde{X}_{n-1}/A_{n-1}}) \]

with \( \phi \) injective, so that \( T^1(\tilde{X}_n/A_n) \to T^1(\tilde{X}_{n-1}/A_{n-1}) \) is always surjective. Thus, by the \( T^1 \)-lifting criterion, (Theorem 1.8) \( \text{Def}(\tilde{X}) \) is smooth. \( \blacksquare \)
(3.5) Suppose furthermore that $X$ is a complete intersection singularity with a crepant resolution $\pi : \tilde{X} \rightarrow X$, with embedding dimension $e$ given by $f_1 = \cdots = f_n = 0$ with $f_i \in \mathbb{C}\{x_1, \ldots, x_e\}$. So $\mathcal{O}_{X,0} = \mathbb{C}\{x_1, \ldots, x_e\}/(f_1, \ldots, f_n)$. (Of course, given that $X$ is a threefold canonical singularity, either $n = 1, e = 4$ or $n = 2, e = 5$.) Then by [43], pg. 634,

$$T^1 \cong \mathcal{O}_{X,0}^n/J$$

where $J$ is the submodule of $\mathcal{O}_{X,0}^n$ generated by $(\partial f_1/\partial x_i, \ldots, \partial f_n/\partial x_i)$, $1 \leq i \leq e$. The obvious $\mathcal{O}_{X,0}$-module structure on $\mathcal{O}_{X,0}^n/J$ coincides with the $\mathcal{O}_{X,0}$-module structure on $T^1$ from Lemma 3.2. If we choose elements $(g_{11}, \ldots, g_{1n}), \ldots, (g_{m1}, \ldots, g_{mn}) \in \mathfrak{m}\mathcal{O}_{X,0}^n$ which along with $(1,0,\ldots,0), \ldots, (0,\ldots,0,1)$ form a basis for $T^1$ after reducing modulo $J$, then a miniversal family over the germ $Def(X) = (T^1,0)$ about the origin of $T^1$ is given by

$$f_1 + a_1 + b_1 g_{11} + \cdots + b_m g_{m1} = 0$$

$$\vdots$$

$$f_n + a_n + b_1 g_{1n} + \cdots + b_m g_{mn} = 0$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m$ are coordinates on $(T^1,0)$ given by our choice of basis. This defines a miniversal deformation $F : (X,0) \rightarrow (T^1,0)$ of $(X,0)$. $F$ has a discriminant locus $D \subseteq (T^1,0)$, over which the fibres of $F$ are singular. From Lemma 3.2, we have a quotient $T'$ of $T^1$ which is an $\mathcal{O}_{X,0}$-module.

A few comments about our plan to prove Theorem 3.8 are in order here. The basic strategy is to show that certain tangent vectors in $T^1$ always correspond to smoothing directions. Lemma 3.7 will show these tangent vectors will be the tangent vectors which do not land in $\mathfrak{m}T'$ under the projection $T^1 \rightarrow T'$. Lemma 3.6 helps us identify tangent vectors for which this is not the case.

**Lemma 3.6.** In the situation of (3.5), suppose $(X,0)$ is not an ordinary double point (analytically isomorphic to $x_1^2 + \cdots + x_2^2 = 0$). Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,0}$. If $x \in T'$ is annihilated by $\mathfrak{m}$, then $x \in \mathfrak{m}T'$.

Proof: We split this into the hypersurface case and the codimension two complete intersection case, the former being simpler than the latter. First suppose $(X,0)$ is a hypersurface singularity, so that $T' \cong \mathcal{O}_{X,0}/I$ for some ideal $I$ containing the jacobian ideal $J$. If some element of $T'$ not in $\mathfrak{m}T'$ were annihilated by $\mathfrak{m}$, we would then have a non-zero element of $T'/\mathfrak{m}T'$ killed by $\mathfrak{m}/\mathfrak{m}^2$ under the surjective multiplication map $T'/\mathfrak{m}T' \times \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}T'/\mathfrak{m}^2T'$. But $\dim_{\mathbb{C}} T'/\mathfrak{m}T' = 1$, so this is only possible if $\mathfrak{m}T' = 0$ and $\dim_{\mathbb{C}} T' = 1$. Thus we just need to show that $\dim_{\mathbb{C}} T' > 1$. 

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By Proposition 3.4, $Def(\tilde{X})$ is smooth. On the other hand, the fibre of the miniversal space of $X$ over a general point of $D \subseteq Def(X)$ has one ODP. I claim that in fact the image of $Def(\tilde{X})$ in $Def(X)$ cannot contain any points corresponding to deformations of $X$ to a non-singular germ or a germ with one ODP, and thus the codimension of the image of $Def(\tilde{X})$ in $Def(X)$ is at least 2, from which we conclude $\dim T' \geq 2$. To show this claim, if the invariant $k$ of $(X, 0)$ is at least one, then $(\tilde{X}, 0)$ contains some exceptional divisors, and thus by [26], Lemma 3.1, any deformation of $(\tilde{X}, 0)$ does also. Since any deformation of $(\tilde{X}, 0)$ also has trivial canonical bundle, these divisors blow down to yield a singularity which is not an ODP. If $k = 0$, $\tilde{X} \to X$ is a small resolution, and the claim follows from [29], Lemma (1.8).

In the case that $(X, 0)$ is a codimension two complete intersection, let $\tilde{X} = \pi_2X_1 \to X$ be a factorization of $\pi$, with $\pi_2$ the blowing up of $X$ at 0. By the argument of [47],(1.5), we have $\Ext^1(\Omega^1_{X_1}, \mathcal{O}_{X_1}) \cong \Ext^1(\pi_1^*\Omega^1_{X_1}, \mathcal{O}_X)$, $\Ext^1(\Omega^1_X, \mathcal{O}_X) \cong \Ext^1(\pi^*\Omega^1_X, \mathcal{O}_X)$, and the natural maps $\pi^*\Omega^1_X \to \pi^*\Omega_{X_1} \to \Omega^1_{X_1}$ induce the maps on tangent spaces

$$\Ext^1(\Omega^1_{X_1}, \mathcal{O}_{X_1}) \to \Ext^1(\pi_1^*\Omega^1_{X_1}, \mathcal{O}_X) \to \Ext^1(\pi^*\Omega^1_X, \mathcal{O}_X).$$

Thus if

$$T'' = \text{coker}(\Ext^1(\Omega^1_{X_1}, \mathcal{O}_{X_1}) \to \Ext^1(\Omega^1_X, \mathcal{O}_X)),$$

there is a surjection $T' \to T'' \to 0$ and it is enough to show that $T'/mT' \cong T''/mT''$ and any element not in $mT''$ is not annihilated by $m$. To do this, we consider $X \subseteq Y = (\mathbb{C}^5, 0)$ and $Y_1 \to Y$ the blowing-up of the origin, $X_1 \subseteq Y_1$ the proper transform of $X$ in $Y_1$. We will first show that every infinitesimal deformation of $X_1$ is a deformation of $X_1$ inside of $Y_1$. We have an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{Y_1}|_{X_1} \to \Omega^1_{X_1} \to 0$$

with $\mathcal{I}$ the ideal sheaf of $X_1 \subseteq Y_1$, which yields

$$\Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{X_1}) \to \Ext^1(\Omega^1_{X_1}, \mathcal{O}_{X_1}) \to \Ext^1(\Omega^1_{Y_1}|_{X_1}, \mathcal{O}_{X_1}).$$

Since $\Omega^1_{Y_1}|_{X_1}$ is locally free, $\Ext^1(\Omega^1_{Y_1}|_{X_1}, \mathcal{O}_{X_1}) = H^1(\mathcal{T}_{Y_1}|_{X_1})$, and we have an exact sequence

$$H^1(\mathcal{T}_{Y_1}) \to H^1(\mathcal{T}_{Y_1}|_{X_1}) \to H^2(\mathcal{T}_{Y_1} \otimes \mathcal{I}).$$

Let $F$ be the exceptional $\mathbb{P}^4$ of the blowing-up $Y_1 \to Y$. Then using

$$H^1(\mathcal{T}_{Y_1}) = \lim \leftarrow H^1(\mathcal{T}_{Y_1} \otimes \mathcal{O}_{Y_1}/\mathcal{I}^n),$$

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the exact sequences
\[ 0 \to \mathcal{T}_F \to \mathcal{T}_{Y_1}|_F \to \mathcal{O}_F(-1) \to 0 \]
and
\[ 0 \to \mathcal{O}_F(n-1) = \mathcal{T}_F^n/\mathcal{T}_F^n \to \mathcal{O}_{Y_1}/\mathcal{I}_F^n \to \mathcal{O}_{Y_1}/\mathcal{T}_F^n \to 0, \]
we see that \( H^1(\mathcal{T}_{Y_1}) = 0. \) To show that \( H^2(\mathcal{T}_Y \otimes \mathcal{I}) = 0, \) we use the Koszul resolution of \( \mathcal{I} \)
\[ 0 \to \mathcal{L}_1 \to \mathcal{L}_2 \oplus \mathcal{L}_3 \to \mathcal{I} \to 0 \]
where \( \mathcal{L}_1, \mathcal{L}_2 \) and \( \mathcal{L}_3 \) are line bundles, with \( \mathcal{L}_1|_F \cong \mathcal{O}_F(-4), \mathcal{L}_2|_F \cong \mathcal{O}_F \mathcal{L}_2|_F \cong \mathcal{O}_F(-2). \)
We then see as before that \( H^2((\mathcal{L}_2 \oplus \mathcal{L}_3) \otimes \mathcal{T}_{Y_1}) = 0 \) and \( H^3(\mathcal{L}_1 \otimes \mathcal{T}_{Y_1}) = 0. \) Thus \( H^2(\mathcal{T}_{Y_1} \otimes \mathcal{I}) = 0. \) To conclude, we have shown that \( \text{Ext}^1(\Omega^1_{Y_1}|_{X_1}, \mathcal{O}_{X_1}) = 0, \) and so any infinitesimal deformation of \( X_1 \) comes from an infinitesimal deformation of \( X_1 \) in \( Y_1. \)

Now a choice of \( g = (g_1, g_2) \in \mathcal{O}^2_{X,0} \) gives a deformation \( \bar{X}/A_1 \) of \( X \) by the equations
\[ f_1 + tg_1 = 0 \]
\[ f_2 + tg_2 = 0. \]
Note that if \( \bar{X}/A_1 \) is normally flat along \( 0 \times \text{Spec} A_1 \) then \( (g_1, g_2) \in \mathfrak{m}^2 \mathcal{O}^2_{X,0}. \) Indeed, since \( f_1 = f_2 = 0 \) defines a rational Gorenstein singularity, the leading terms of \( f_1 \) and \( f_2 \) must be quadratic, and thus if \( (g_1, g_2) \notin \mathfrak{m}^2 \mathcal{O}^2_{X,0}, \) the blow-up of \( \bar{X}/A_1 \) at \( 0 \times \text{Spec} A_1 \) would not yield an exceptional divisor flat over \( A_1. \)

If we write \( T'' = \mathcal{O}^2_{X,0}/I, \) with \( I \supseteq J, \) then \( (g_1, g_2) \in I \) if \( (g_1, g_2) \) yields a deformation \( \bar{X}/A_1 \) which lifts to a deformation \( \bar{X}_1/A_1 \) of \( X_1. \) Since \( \bar{X}_1/A_1 \) is equivalent to a deformation \( \bar{X}'_1/A_1 \subseteq Y_1 \times \text{Spec} A_1, \) we can blow down this deformation to obtain a deformation \( \bar{X}'/A_1 \) equivalent to \( \bar{X}/A_1, \) which in particular is normally flat along \( 0 \times \text{Spec} A_1. \) Since two deformations of \( X \) given by \( (g_1, g_2) \) and \( (g'_1, g'_2) \) are equivalent if they differ by an element \( (h_1, h_2) \in J, \) we must have \( (g_1 + h_1, g_2 + h_2) \in \mathfrak{m}^2 \mathcal{O}^2_{X,0} \) for some \( (h_1, h_2) \in J. \)

Now \( J \subseteq \mathfrak{m} \mathcal{O}^2_{X,0}, \) so if \( (g_1, g_2) \notin \mathfrak{m} \mathcal{O}^2_{X,0}, \) we cannot have \( (g_1, g_2) \in I. \) Thus \( \dim \mathfrak{t} T''/\mathfrak{m} T'' = 2, \) and since \( \dim \mathfrak{t} T'/\mathfrak{m} T' \leq 2, \) we must have the surjection \( T'/\mathfrak{m} T' \to T''/\mathfrak{m} T'' \) being an isomorphism.

If \( (g_1, g_2) \notin \mathfrak{m} \mathcal{O}^2_{X,0}, \) but \( x_i(g_1, g_2) \in I \) for all \( i, \) then in particular for each \( i, \) there exists \( (h_1, h_2) \in J \) such that \( x_i(g_1, g_2) + (h_1, h_2) \in \mathfrak{m}^2 \mathcal{O}^2_{X,0} \), from which one easily sees that the leading (quadratic) terms of \( f_1 \) and \( f_2 \) have proportional partial derivatives, and hence must be proportional themselves, ruling out the possibility of having a del Pezzo surface as an exceptional divisor. Thus \( (g_1, g_2) \) is not annihilated by all elements of \( \mathfrak{m} \) in \( T''. \)
Lemma 3.7. Let $F' : (X', 0) \to (S', 0)$ be a flat deformation of $(X, 0)$, with $S'$ non-singular at $0 \in S'$ and the tangent space at $0 \in S'$ the vector space $T$. Assume furthermore that $X$ is not an ordinary double point. Since $F : (X, 0) \to (T^1, 0)$ given in (3.5) is a miniversal family for $(X, 0)$, there is a (non-unique) map $S' \to (T^1, 0)$ inducing a unique map $T \to T^1$. Composing this map with $T^1 \to T'$, we obtain a map $T \to T'$. If $\text{im}(T \to T') \not\subseteq mT'$, then a general fibre of $X' \to S'$ is non-singular.

Proof. Following [43, pg. 645], let $D \subseteq (T^1, 0)$ be the discriminant locus of $F$, $C \subseteq (X, 0)$ the critical locus. From [43], $F|_C : C \to D$ is the normalization of $D$. If every fibre of $F'$ is singular, the induced map $S' \to (T^1, 0)$ factors through $D$. Since $S'$ is non-singular, in particular this map factors through $C \to D$, and thus the image of $T$ in $T^1$ is contained in the image of the tangent space $T_{x,0}$ of $(X, 0)$ at 0 in $T^1$ via $F_*$. Using the explicit equations for $(X, 0)$ and basis for $T^1$ given in (3.5), we see that the image of $T_{x,0}$ in $T^1$ is a vector space $W \subseteq T^1$ given by $a_1 = \cdots = a_n = 0$. Again from the explicit description of $T^1$ in (3.5), $W = mT^1$, so its image in $T'$ is a subspace $V \subseteq mT'$. Thus, we see that if $F'$ only has singular fibres, the image of $T$ is contained in $mT'$. 

Theorem 3.8. Let $\tilde{X}$ be a non-singular Calabi-Yau threefold, and $\pi : \tilde{X} \to X$ be a birational contraction morphism such that $X$ has isolated, canonical, complete intersection singularities. Then there is a deformation of $X$ which smooths all singular points of $X$ except possibly the ordinary double points of $X$. In particular, if $X$ has no ordinary double points, then $X$ is smoothable.

Proof. Let $P \in Z = \text{Sing}(X)$ be a singular point of $X$ which is not an ordinary double point. We will show that there is a deformation of $X$ which smooths $P$. We have a diagram

$$
\begin{array}{cccc}
H^1(\tilde{X}, T_{\tilde{X}}) & \to & H^1(U, T_{\tilde{X}}) & \to & H^2_{\pi^{-1}(Z)}(\tilde{X}, T_{\tilde{X}}) & \to & H^2(\tilde{X}, T_{\tilde{X}}) \\
H^1(X, T_X) & \to & H^1(U, T_X) & \to & H^2_Z(X, T_X) & \to & H^2(X, T_X) \\
\end{array}
$$

(3.9)

$\text{Def}(X)$ is smooth by [27] or Theorem 2.1. Thus to show that there is a deformation of $X$ which smooths $P \in X$, it is enough to show by Lemma 3.7 that the image of the composed map

$$H^1(U, T_{\tilde{X}}) \to H^2_{\pi^{-1}(Z)}(\tilde{X}, T_{\tilde{X}}) \to H^2_E(\tilde{X}, T_{\tilde{X}})$$

with $E = \pi^{-1}(P)$ contains an element not in $m_PT_P \subseteq T_P \subseteq H^2_E(\tilde{X}, T_{\tilde{X}})$. Here $m_P$ is the maximal ideal of $O_{X,P}$, and $T_P$ is the subspace of $H^2_E(\tilde{X}, T_{\tilde{X}})$ given by Lemma 3.2.
applied to the germ \((X, P)\). To show this, it is enough to show that there is an element of \(\ker(H^2_E(\tilde{X}, \mathcal{T}_\tilde{X}) \to H^2(\tilde{X}, \mathcal{T}_\tilde{X}))\) not in \(m_P T'_p\).

To see this, first consider the dual map

\[
H^1(\Omega^1_{\tilde{X}}) \xrightarrow{\phi^\vee} (R^1\pi_*\Omega^1_X)_P.
\]

Since \(H^1(\Omega^1_{\tilde{X}}) \cong \text{Pic}\, \tilde{X} \otimes \mathbb{C}\), this map factors through \(\text{Pic}(\tilde{X}, E) \otimes \mathbb{Z} \otimes \mathbb{C}\), where \((\tilde{X}, E)\) denotes the germ of \(\tilde{X}\) at \(E\). Now let \(E = \bigcup E_i\), with \(E_i\) irreducible.

**Claim:** The map \(\text{Pic}(\tilde{X}, E) \otimes \mathbb{Z} \otimes \mathbb{C} \to \bigoplus H^1(\Omega^1_{E_i})\) is injective.

**Proof:** Let \((\tilde{X}', E') \to (\tilde{X}, E)\) be a composition of blow-ups of non-singular subvarieties so that \((\tilde{X}', E') \to (X, P)\) is a resolution with \(E' = \bigcup E'_i\) simple normal crossings. Then we have a diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Pic}(\tilde{X}', E') \otimes \mathbb{Z} \otimes \mathbb{C} \\
& & \uparrow \\
& & \bigoplus H^1(\Omega^1_{E'_i}) \\
& & \uparrow \\
& & \text{Pic}(\tilde{X}, E) \otimes \mathbb{Z} \otimes \mathbb{C} \\
& & \uparrow \\
& & \bigoplus H^1(\Omega^1_{E_i}) \\
& & \uparrow \\
& & 0
\end{array}
\]

The first column is exact since \(\tilde{X}' \to \tilde{X}\) is a series of blow-ups. The map \(\bigoplus H^1(\Omega^1_{E_i}) \to \bigoplus H^1(\Omega^1_{E'_i})\) is defined via the induced maps \(E'_i \to E_j\) for all \(i, j\). Suppose the first row were exact. Then the claimed map must be injective.

The exactness of the first row follows from an argument of Namikawa in a preprint version of [27]; since this argument did not appear in the final version of the paper, we sketch it here.

First, by [37], the \(E'_i\) are all rational or ruled surfaces, since they are exceptional divisors in the resolution of a rational Gorenstein point. Let \(\tau^p_{E'} \subseteq \Omega^p_{E'}\) be the torsion subsheaf. Then there is a spectral sequence

\[
E_1^{p,q} = H^q(E', \Omega^p_{E'}/\tau^p_{E'}) \Rightarrow H^{p+q}(E', \mathbb{C})
\]

which degenerates at the \(E_1\) term, by [7], Proposition 1.5. Note that \(H^{p+q}(E', \mathbb{C}) \cong H^{p+q}((\tilde{X}', E'), \mathbb{C})\), the cohomology of the germ \((\tilde{X}', E')\). Let

\[
E'_{[p]} = \coprod_{i_0 < \cdots < i_p} E'_{i_0} \cap \cdots \cap E'_{i_p}.
\]

There is an exact sequence

\[
0 \to \Omega^p_{E'}/\tau^p_{E'} \to \Omega^p_{E'_{[0]}} \to \Omega^p_{E'_{[1]}} \to \cdots
\]
for each $p$, again by [7], Proposition 1.5. First consider this sequence for $p = 0$. Since $H^2(O_{E'}) = 0$, as $(X, P)$ is a rational singularity, we find $H^1(O_{E'_{[0]}}) \to H^1(O_{E'_{[1]}})$ is surjective, and thus $H^0(\Omega^1_{E'_{[0]}}) \to H^0(\Omega^1_{E'_{[1]}})$ is surjective.

By the same sequence for $p = 1$, we find $H^1(\Omega^1_{E'_{[0]}}/\tau_{E'}) \to H^1(\Omega^1_{E'_{[1]}})$ is injective. With $p = 2$, we obtain $H^0(\Omega^2_{E'_{[0]}}/\tau_{E'}) = H^0(\Omega^2_{E'_{[0]}}) = 0$ since all components of $E'$ are ruled. Thus our spectral sequence yields $H^2(E', C) \cong H^1(\Omega^1_{E'_{[0]}}/\tau_{E'})$ and so the map $H^2(E', C) \to H^1(\Omega^1_{E'_{[0]}})$ is injective. Since $\text{Pic}(\tilde{X}', E') \otimes C \cong H^2((\tilde{X}', E'), C) \cong H^2(E', C)$, the result follows. $\bullet$

Thus the composed map

$$\text{Pic}(\tilde{X}, E) \otimes C \to (R^1\pi_*\Omega^1_X)_P \to \bigoplus_i H^1(\Omega^1_X|_{E_i})$$

must also be injective, as is then the composed map

$$\text{coim}(\phi^\vee) \to (R^1\pi_*\Omega^1_X)_P \to \bigoplus_i H^1(\Omega^1_X|_{E_i}).$$

Dually, we get that the composed map

$$\bigoplus_i \text{Ext}^2_{O_X}(O_{E_i}, T_{\tilde{X}}) \to H^2(E)(T_{\tilde{X}}) \to \text{im}(\phi)$$

is surjective. Now clearly $\text{Ext}^2_{O_X}(O_{E_i}, T_{\tilde{X}})$ is annihilated by the maximal ideal $m_P \subseteq O_{X, P}$ since $O_{E_i}$ is. Let $W \subseteq \bigoplus \text{Ext}^2_{O_X}(O_{E_i}, T_{\tilde{X}})$ be a subspace mapping isomorphically via $\phi$ to $\text{im}(\phi)$. Identifying $W$ with its image in $H^2(E)(T_{\tilde{X}})$, we must then have $H^2(E)(T_{\tilde{X}}) = W + \text{ker}(\phi)$. $W$ is annihilated by $m_P$, and so $W' \cap T_P' \subseteq m_P T_P'$ by Lemma 3.6; thus $\text{ker}(\phi)$ contains some elements not in $m_P T_P$. $\bullet$

Combining this with Namikawa’s results in [28], we obtain

**Corollary 3.10.** Let $\tilde{X}$ be a factorial Calabi-Yau threefold with terminal singularities, and suppose $\pi : \tilde{X} \to X$ is a birational contraction morphism such that $X$ has isolated, canonical, complete intersection singularities. Then there is a deformation of $X$ to a variety with at worst ordinary double points.

Proof: By [28], there is a small deformation of $\tilde{X}$ to a non-singular Calabi-Yau $\tilde{X}'$. If $H$ is an ample Cartier divisor on $X$, $\pi^*H$ is a nef and big divisor on $\tilde{X}$, and by [27], Theorem C, this divisor deforms to a nef and big divisor on a general smoothing of $\tilde{X}$. Thus the morphism $\pi$ deforms to a morphism $\pi' : \tilde{X}' \to X'$, with $X'$ a deformation of $X$. 24
§4. Calabi-Yaus with non-complete intersection singularities.

If \( X \) is a Calabi-Yau with non-complete intersection isolated canonical singularities, then there is no statement as complete as Theorem 3.8. There are three difficulties. First, not every isolated singularity is smoothable. Second, as seen in Example 2.8, the deformation theory of the Calabi-Yau can be quite bad. Third, as Example 4.1 shows, even if the singularity is smoothable, we may not even have any infinitesimal deformations of \( X \) which yield a non-trivial infinitesimal deformation of the singularity.

**Example 4.1.** Let \( S \) be a non-singular del Pezzo surface of degree \( \geq 5 \), and consider an elliptic fibration \( f : \tilde{X} \to S \) defined by the Weierstrass equation \( y^2 = x^3 + ax + b \), for general \( a \in H^0(\omega_S^{-4}) \), \( b \in H^0(\omega_S^{-6}) \). \( \tilde{X} \) will be a non-singular Calabi-Yau threefold, and \( f \) has a section \( \sigma \), the section at infinity obtained after compactifying the affine equation given. Identifying \( S \) with \( \sigma(S) \), it is possible to find a contraction \( \pi : \tilde{X} \to X \) contracting \( S \). As in (3.9), we have the exact sequence

\[
H^1(\tilde{X}, T_{\tilde{X}}) \to H^1(U, T_{\tilde{X}}) \to H^2(S, T_{\tilde{X}}) \to H^2(\tilde{X}, T_{\tilde{X}})
\]

where \( U = \tilde{X} - S \). Dualizing the last map we obtain the map \( H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \to H^0(R^1\pi_*\Omega^1_{\tilde{X}}) \). Using \( \text{deg} S \geq 5 \), a straightforward calculation shows that \( H^0(R^1\pi_*\Omega^1_{\tilde{X}}) \cong H^1(S, \Omega^1_S) \), and \( H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \to H^1(S, \Omega^1_S) \) is the restriction map \( \sigma^* \) and is thus surjective, since \( \sigma^*f^* \) is the identity on \( H^1(S, \Omega^1_S) \). Thus \( H^1(\tilde{X}, T_{\tilde{X}}) \to H^1(U, T_{\tilde{X}}) \) is surjective, and so there are no deformations of \( X \) which don’t come from deformations of \( \tilde{X} \). Since the exceptional locus \( S \) deforms in any deformation of \( \tilde{X} \), \( X \) is not smoothable.

It is clear from this example that one issue is controlling the map \( T^1 \to T^1_{\text{loc}} \), where \( T^1 \) is the tangent space to \( \text{Def}(X) \) and \( T^1_{\text{loc}} \) is the tangent space to \( \text{Def}(X, Z) \), \( Z = \text{Sing}(X) \). An assumption which will give us as much control over this map as possible is that \( X \) is \( \mathbb{Q} \)-factorial. Nevertheless, this does not guarantee surjectivity of \( T^1 \to T^1_{\text{loc}} \). Hence I make here some further assumptions which I feel are quite artificial and which hopefully can be removed in the future, once more is known about the deformation theory of rational Gorenstein threefold singularities.

We make the following rather ad hoc definition:

**Definition 4.2.** An isolated non-complete intersection rational Gorenstein point \((X, P)\) is good when

1. \( X' \to X \) is the blow-up of \( X \) at \( P \) with exceptional divisor \( E \), then \( E \) is irreducible and \( X' \) has only isolated singularities.
(2) \(\text{Def}(X')\) is non-singular, and the natural map \(\text{Def}(X') \to \text{Def}(X)\) is an immersion. (3) There is a smoothing component of \(\text{Def}(X)\) containing the image of \(\text{Def}(X') \to \text{Def}(X)\).

This is a particularly strong set of assumptions. In §5, we will prove certain singularities are good. Here, we just want to make explicit the assumptions we need.

**Theorem 4.3.** Let \(\tilde{X}\) be a non-singular Calabi-Yau threefold and \(\pi: \tilde{X} \to X\) a birational contraction, so that \(X\) is \(\mathbb{Q}\)-factorial and for each \(P \in \text{Sing}(X)\), \((X, P)\) is good. Then \(X\) is smoothable.

First we need

**Lemma 4.4.** If \(X\) is a compact, \(\mathbb{Q}\)-factorial algebraic variety with rational singularities, \(\pi: \tilde{X} \to X\) a resolution of singularities with irreducible \(E_1, \ldots, E_n \subseteq \tilde{X}\) the \(\pi\)-exceptional divisors, and \(H^2(\mathcal{O}_\tilde{X}) = 0\), then

\[
\text{im} \left[ H^1(\Omega^1_{\tilde{X}}) \xrightarrow{p_1} H^0(R^1\pi_*\Omega^1_X) \right] = \text{im} \left[ \bigoplus CE_i \xrightarrow{p_2} H^0(R^1\pi_*\Omega^1_X) \right]
\]

with the map \(p_2\) the composition of \(\bigoplus CE_i \to H^1(\Omega^1_X)\) and \(p_1\).

Proof: [23], 12.1.6 gives a similar statement for rational cohomology, i.e. \(X\) \(\mathbb{Q}\)-factorial implies that

\[
\text{im} \left[ H^2(\tilde{X}, \mathbb{Q}) \xrightarrow{p_1} H^0(X, R^2\pi_* \mathbb{Q}) \right] = \text{im} \left[ \bigoplus \mathbb{Q}E_i \xrightarrow{p_2} H^0(X, R^2\pi_* \mathbb{Q}) \right].
\]

We can clearly replace \(\mathbb{Q}\) by \(\mathbb{C}\), and since \(H^2(\mathcal{O}_\tilde{X}) = 0\), \(H^2(\tilde{X}, \mathbb{C}) = H^1(\tilde{X}, \Omega^1_{\tilde{X}})\). Now \(R^i\pi_* \mathcal{O}_\tilde{X} = 0\) for \(i > 0\) since \(X\) has rational singularities. Thus by the spectral sequence

\[
R^q\pi_* \Omega^p_{\tilde{X}} \Rightarrow R^n\pi_* \mathbb{C},
\]

there is a natural map

\[
H^0(X, R^2\pi_* \mathbb{C}) \to H^0(X, R^1\pi_* \Omega^1_X),
\]

yielding a commutative diagram

\[
\begin{array}{ccc}
H^2(\tilde{X}, \mathbb{C}) & \xrightarrow{\cong} & H^1(\tilde{X}, \Omega^1_{\tilde{X}}) \\
\downarrow & & \downarrow \quad p_1 \\
H^0(X, R^2\pi_* \mathbb{C}) & \longrightarrow & H^0(X, R^1\pi_* \Omega^1_X)
\end{array}
\]

This gives the desired result. 

The reason for condition (1) in the definition of good singularity is the following lemma, which shows that we can “smooth in one step.”

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Lemma 4.5. Let $\mathcal{X} \to \Delta$ be a one-parameter deformation of a good singularity $(X, P)$. This induces a map $\Delta \to \text{Def}(X)$. Then either $\text{im}(\Delta \to \text{Def}(X)) \subseteq \text{im}(\text{Def}(X') \to \text{Def}(X))$ or $\mathcal{X}_t$ has only hypersurface singularities for general $t \in \Delta$.

Proof: Suppose that for general $t \in \Delta$, $\mathcal{X}_t$ has worse than hypersurface singularities. $\mathcal{X}$ must be singular at each point where $\mathcal{X}_t$ has worse than hypersurface singularities, so we can assume there is a curve $C \subseteq \mathcal{X}$ dominating $\Delta$ along which $\mathcal{X}$ is singular. By making a suitable base change, we can assume $C$ is the image of a section $\Delta \to \mathcal{X}$. It is enough to show that $\mathcal{X}$ is normally flat along $C$; this will permit us to blow up $\mathcal{X}$ along $C$ to obtain a deformation $\mathcal{X}' \to \Delta$ of $X'$. To show that $\mathcal{X}$ is normally flat along $C$, it is enough to show that the multiplicity of $\mathcal{X}_t$ at $C_t$ is constant for $t \in \Delta$. Indeed, $\mathcal{X}$ is normally flat along $C$ if the Hilbert-Samuel function of the point $C_t \in \mathcal{X}_t$ is constant for $t \in \Delta$. For a rational Gorenstein non-cDV point, the exceptional locus upon blowing-up is a del Pezzo surface, and the Hilbert-Samuel function for a family of del Pezzo surfaces of the same degree is constant. To show that the multiplicity of $\mathcal{X}$ along $C$ is constant, it is enough to show the same thing for a general hyperplane section of $\mathcal{X}$ containing $C$.

To this end, take a general hyperplane section of $\mathcal{X}$ containing $C$; this yields a deformation $\mathcal{S} \to \Delta$ of surface singularities, with $\mathcal{S}_0$ a general hyperplane section of $\mathcal{X}_0 = X$. $\mathcal{S}_0$ is then a Gorenstein elliptic singularity, and the blowing up of $\mathcal{S}_0$ at $P$ resolves the singularity, since $(X, P)$ is good. Let $\tilde{\mathcal{S}} \to \Delta$ be a minimal model of $\mathcal{S} \to \Delta$. By [24], Theorem 5.3, either $\tilde{\mathcal{S}}_0$ is a minimal resolution of $\mathcal{S}_0$ or $\mathcal{S}$ has only canonical singularities. In the latter case, $\mathcal{S}_t$ must have at worst hypersurface (du Val) singularities for general $t \in \Delta$, and thus $\mathcal{X}$ has at worst hypersurface singularities along $C$, contradicting the assumption. Thus $\mathcal{S} \to \Delta$ is a deformation of the minimal resolution $\tilde{\mathcal{S}}_0$ of $\mathcal{S}_0$. Since $\mathcal{S}$ is singular along $C$, the (irreducible) exceptional curve of $\tilde{\mathcal{S}}_0 \to \mathcal{S}_0$ must deform to an exceptional curve in $\tilde{\mathcal{S}}_t$ for $t \in \Delta$, thus showing that $\mathcal{S}$ has constant multiplicity along $C$. ●

Proof of Theorem 4.3: Diagram (3.9) is still valid in this setting. Let $Z = \text{Sing}(X)$ and let $E = \pi^{-1}(Z)$ be the exceptional locus. First, we claim that

$$\ker(H^2(E(T_{\mathcal{X}})) \to H^0(R^2\pi_*T_{\mathcal{X}})) = \ker(H^2(E(T_{\mathcal{X}})) \to H^2(T_{\mathcal{X}})).$$

That the first space contains the second is clear, since the first map factors through the second. Dualizing this statement, we need to prove that

$$\text{coker}(H^1(E(\Omega^1_{\mathcal{X}})) \to \text{coker}(H^1(\Omega^1_{\mathcal{X}}) \to H^0(R^1\pi_*\Omega^1_{\mathcal{X}}))).$$

The first surjects on the second, i.e. $\text{im}(p_1) \subseteq \text{im}(p_2)$. To show equality, we need to show
im(p_1) \supseteq \text{im}(p_2). \text{ Let } \{E_i\} \text{ be the irreducible components of } E. \text{ We have the diagram}

\[
\begin{array}{cccc}
\bigoplus \mathbb{C}E_i & \to & \text{Pic} \tilde{X} \otimes_{\mathbb{Z}} \mathbb{C} & \to & \text{Pic}U \otimes_{\mathbb{Z}} \mathbb{C} & \to & 0 \\
\downarrow & & \downarrow d \log X & & \downarrow d \log U & & \\
0 & \to & H^1_E(\Omega^1_X) & \to & H^1(\Omega^1_X) & \to & H^1(U, \Omega^1_X) \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & & & & & \\
\end{array}
\]

Here, \( H^0(U, \Omega^1_X) = H^0(U, \Omega^1_X) = 0 \) by [42]. This shows that \( \text{im}(\bigoplus \mathbb{C}E_i \to H^1(\Omega^1_X)) \subseteq \text{im}(H^1_E(\Omega^1_X) \to H^1(\Omega^1_X)) \), and thus by Lemma 4.4 and the hypothesis that \( X \) is \( \mathbb{Q} \)-factorial, \( \text{im}(p_2) \subseteq \text{im}(p_1) \).

What we have then shown is that if \( T^1 \cong H^1(U, T_X) \) is the tangent space to \( \text{Def}(X) \), \( T^1_{\text{loc}} \) the tangent space to \( \text{Def}(X, Z) \), and \( T' \subseteq H^2_E(T_{\tilde{X}}) \) the quotient space of \( T^1_{\text{loc}} \) defined in Lemma 3.2, then the composed map

\[
(4.6) \quad T^1 \to T^1_{\text{loc}} \to T'
\]

is surjective.

Adopting the notation of \S\S 1 and 2, let \( S \) be a complete local ring which pro-represents the deformation functor of \( X \), and let \( \Lambda, \Lambda' \) and \( \tilde{\Lambda} \) be hulls of the deformation functors of \( (X, Z) \), \( (X, E') \) and \( (\tilde{X}, E) \) respectively. Here \( X' \) is the blow-up of \( X \) at \( Z \), with exceptional locus \( E' \). By Theorems 2.2 and 1.9, if we put \( r = \text{dim}_k V_0 \) and \( s = \text{dim}_k V_1 \) with \( V_0 = \ker(T^1 \to T^1_{\text{loc}}) \) and \( V_1 = \text{coker}(T^1 \to T^1_{\text{loc}}) \), then we have \( S \cong R/J \) with \( R = \Lambda[[x_1, \ldots, x_r]] \). Furthermore, there is an ideal \( J' \subseteq J \) with \( \text{Supp}(R/J) = \text{Supp}(R/J') \), \( \mathfrak{m}_R/(\mathfrak{m}_R^2 + J) \cong \mathfrak{m}_R/(\mathfrak{m}_R^2 + J') \) and \( J' \) is generated by \( s \) elements. Let \( p : \text{Spec } R \to \text{Spec } \Lambda \) be the projection.

Let \( S \subseteq \text{Spec } R \) be the pull-back via \( p \) of a smoothing component given by item (3) of Definition 4.2 in \( \text{Spec } \Lambda \). Let \( B \subseteq \text{Spec } R \) be the subscheme of \( \text{Spec } R \) defined by \( J' \). Furthermore, let \( \tilde{D} = \text{Spec } \tilde{\Lambda} \times_{\text{Spec } \Lambda} \text{Spec } R \), and let \( D' \) be the pull-back under \( p \) of the image of \( \text{Spec } \Lambda' \to \text{Spec } \Lambda \). By (2) of Definition 4.2, \( D' \) is a non-singular subscheme of \( \text{Spec } R \), and \( \text{im}(\tilde{\pi} : \tilde{D} \to \text{Spec } R) \subseteq D' \). We denote by \( T_{D', 0} \) and \( T_{B, 0} \) the Zariski tangent spaces of \( D' \) and \( B \) respectively, at the closed point \( 0 \in \text{Spec } R \). These are all contained in \( T = T_{\text{Spec } R, 0} = T^1_{\text{loc}} \oplus V_0 \). Let \( T_{\tilde{D}, 0} \) be the tangent space to \( \tilde{D} \) at \( 0 \in \tilde{D} \). Then \( \tilde{\pi}_*(T_{\tilde{D}, 0}) \subseteq T_{D', 0} \), and \( \tilde{\pi}_*(T_{\tilde{D}, 0}) \) is the kernel of the composed map \( T \to T^1_{\text{loc}} \to T' \).

Note that since \( \mathfrak{m}_R/(\mathfrak{m}_R^2 + J) \cong \mathfrak{m}_R/(\mathfrak{m}_R^2 + J') \), \( T_{B, 0} \cong T^1 \). Thus \( \dim(T_{B, 0} + T_{D', 0}) \geq \dim(\tilde{\pi}_*(T_{\tilde{D}, 0}) + \text{dim } T') \), since \( \tilde{\pi}_*(T_{\tilde{D}, 0}) \subseteq T_{D', 0} \) and (4.6) gives a surjection \( T^1 \to T' \) with
kernel containing $\tilde{\pi}_*(T_{D,0}) \cap T^1$. Also, $s = \dim T - \dim T_{B,0} = \dim \tilde{\pi}_*(T_{D,0}) + \dim T' - \dim T_{B,0}$. Then

$$
\dim \mathcal{B} \cap \mathcal{D}' \leq \dim T_{B,0} \cap T_{D',0}
$$

$$
= \dim T_{B,0} + \dim T_{D',0} - \dim(T_{B,0} + T_{D',0})
$$

$$
\leq \dim T_{B,0} + \dim T_{D',0} - \dim \tilde{\pi}_*(T_{D,0}) - \dim T'
$$

$$
= \dim T_{D',0} - s
$$

$$
= \dim \mathcal{D}' - s.
$$

On the other hand,

$$
\dim \mathcal{B} \cap \mathcal{S} \geq \dim \mathcal{S} - \# \text{ of equations generating } J'
$$

$$
= \dim \mathcal{S} - s
$$

$$
\geq \dim \mathcal{D}' + 1 - s
$$

$$
> \dim \mathcal{B} \cap \mathcal{D}'.
$$

Thus $\mathcal{B} \cap \mathcal{S} \not\subseteq \mathcal{B} \cap \mathcal{D}'$.

This yields a small deformation $Y$ of $X$ corresponding to a point of $Def(X)$ not in $\mathcal{D}'$. By Lemma 4.5, $Y$ must have at worst hypersurface singularities. By [23], 12.1.11, $Y$ is $\mathbb{Q}$-factorial. By Corollary 3.10, we can deform $Y$ to something with only ordinary double points which is still $\mathbb{Q}$-factorial, and hence can be smoothed entirely by [8].

Remark 4.7. More generally, if every singularity of $X$ is smoothable and, in the notation of (2.1), $T^1 \to T_{loc}^1$ is surjective, then it is clear that $X$ is smoothable. This will hold, for example, if the map $H^0(R^1\pi_*\mathcal{T}_{\tilde{X}}) \to H^2_Z(\mathcal{T}_X)$ of Lemma 3.2 is zero and $X$ is $\mathbb{Q}$-factorial. This follows as in the proof of Theorem 4.3.

§5. Primitive Contractions.

Recall from [48] that if $\tilde{X}$ is a non-singular Calabi-Yau, then $\pi : \tilde{X} \to X$ is a primitive contraction if $\pi$ cannot be factored in the algebraic category. The goal of this section is to apply the results of the previous sections to the case that $\pi$ is a primitive contraction. We will find strong restrictions on the possible primitive contractions of $\tilde{X}$ if we assume that $\tilde{X}$ is primitive in the sense of the introduction. Recall from [48] the following classification of primitive contractions:

Type I: $\pi$ contracts a union of curves.

Type II: $\pi$ contracts a divisor to a point.

Type III: $\pi$ contracts a divisor to a curve.

We treat the first two cases in this section. The type III case will be treated in [12]. Type I contractions have already been treated by Namikawa:
**Theorem 5.1.** Suppose $\pi : \tilde{X} \to X$ is a primitive type I contraction. Then $X$ is smoothable unless $\pi$ is the contraction of a single $\mathbb{P}^1$ to an ordinary double point.

Proof: If $X$ has only ordinary double points, and $C_1, \ldots, C_n$ are the exceptional curves of $\pi$, then the cohomology classes $[C_1], \ldots, [C_n] \in H_2(\tilde{X}, \mathbb{Z})$ coincide, since $\pi$ is primitive, and thus unless $n = 1$, there is a non-trivial linear dependence relation on $[C_1], \ldots, [C_n]$. Thus by [8], $X$ is smoothable. If $X$ does not have only ordinary double points, then let $Z \to X$ be a (non-projective) small resolution of the ordinary double points of $X$. Then if $C_1, \ldots, C_n \subseteq Z$ are the exceptional curves, $[C_1], \ldots, [C_n] = 0$ in $H_2(Z, \mathbb{Z})$, and so by [29], Theorem 2.5, $X$ is smoothable. ●

For type II contractions, we will need a more refined classification.

**Theorem 5.2.** Let $\pi : \tilde{X} \to X$ be a primitive type II contraction. Then if $X$ has a non-hypersurface singularity, the exceptional divisor $E$ of $\pi$ is either

(i) a normal rational del Pezzo surface of degree $\leq 9$ or

(ii) a nonnormal del Pezzo surface of degree 7.

Proof: By [37], Theorem 2.11, $\pi$ is the blowing-up of $X$ at $P$, the singular point of $X$. The exceptional surface $E$ is a generalized del Pezzo surface ($\omega_E$ is ample) of degree $k$, where $k$ is Reid’s invariant (see §3). Since $\pi$ is primitive, $E$ is integral.

We then have the following possibilities for $E$:

(i) $E$ is a normal, rational del Pezzo surface, in which case $\deg E \leq 9$.

(ii) $E$ is a nonnormal del Pezzo surface, as classified in [38]. The possibilities are:

(a) Let $F_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$ be the rational scroll, with $\text{Pic} F_a$ generated by $C_0$, the negative section, and $f$, the class of a fibre. Embed $F_a$ in $\mathbb{P}^{a+5}$ via $|C_0 + (a + 2)f|$. $E$ is the projection of $F_a$ into $\mathbb{P}^{a+4}$ from a point in a plane spanned by the conic $C_0$. This projection maps $C_0$ two-to-one to a line $l$, and makes no other identifications.

(b) Embed $F_a \subseteq \mathbb{P}^{a+3}$ via $|C_0 + (a + 1)f|$. $E$ is the projection of $F_a$ into $\mathbb{P}^{a+2}$ from a point in the plane spanned by the line $C_0$ and one fibre $f$. This projection identifies $C_0$ and $f$.

(c) Take $E$ to be a cone over a rational nodal or cuspidal curve of degree $d$ spanning $\mathbb{P}^{d-1}$. For $d > 3$, the vertex of this cone will not be a hypersurface singularity, and hence such a del Pezzo surface cannot be contained in a non-singular Calabi-Yau threefold. Thus this case does not occur.
(iii) $E$ is a cone over an elliptic normal curve, in which case this does not occur just as in the cones in case (ii) (c).

Thus we need to deal with cases (ii), (a) and (b). Suppose $D \subseteq X$ is such a del Pezzo surface, $X$ a non-singular Calabi-Yau. Let $\tilde{D} \to D$ be the normalization; $\tilde{D}$ is a scroll. Let $i : \tilde{D} \to X$ be the induced map. Let $S \subseteq \tilde{D}$ be the subscheme defined by the zeroth Fitting ideal of $\Omega^1_{\tilde{D}/X}$. This is the subscheme defined by the condition that $i^* \Omega^1_X \xrightarrow{di} \Omega^1_{\tilde{D}}$ drops rank, and so is defined by the $2 \times 2$ minors of the map $di$. (See [21], III A.) The degree of $S$ is the number of pinch points of $D$. Now in case (ii) (a), $S$ consists precisely of the two pinch points corresponding to the ramification points of $C_0 \to I$, and $S$ is a length two scheme. In case (ii) (b), there is just one similar such point, but it is not an ordinary pinch point. To analyze this point, we can consider $A^2 \subseteq \tilde{D}$ with coordinates $u$ and $v$ in such a way that $C_0$ and $f$ coincide with $u = 0$ and $v = 0$ respectively. The map $\tilde{D} \to D$ then identifies the $u$ and $v$ axes. Following the recipe of [38], 2.1, $\tilde{D} \to D$ then locally looks like

$$(u, v) \in A^2 \mapsto (u + v, uv, uv^2) \in A^3.$$ 

The Jacobian of this map is

$$
\begin{pmatrix}
1 & v & u^2 \\
1 & u & 2uv
\end{pmatrix}
$$

and the ideal of $2 \times 2$ minors is

$$(u - v, 2uv - v^2, uv^2) = (u - v, uv),$$

which defines a scheme of length two at the origin. Thus in either case, $\deg S = 2$.

Alternatively, we can compute $\deg S$ by [21], (III, 8),

$$\deg S = c_2(i^* T_X - T_{\tilde{D}}).$$

Now the Chern polynomial in $t$ of $T_{\tilde{D}}$ is

$$c_t(T_{\tilde{D}}) = 1 - K_{\tilde{D}}t + 4t^2$$

and of $i^* T_X$

$$c_t(i^* T_X) = 1 + c_2(X).D t^2.$$ 

We compute $c_2(X).D$ using Riemann-Roch:

$$\chi(O_X(D)) = \frac{1}{6} D^3 + \frac{1}{12} c_2(X).D.$$ 

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Now $H^i(\mathcal{O}_X(D)) = H^{3-i}(\mathcal{O}_X(-D))$ by Serre duality, and the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

shows that $H^i(\mathcal{O}_X(D)) = 0$ for $i > 0$. ($H^i(\mathcal{O}_X) = 0$ for $i = 1, 2$ and $H^1(\mathcal{O}_D) = 0$ by [38, 4.10].) Also, $H^0(\mathcal{O}_X(D)) = 1$, so $\chi(\mathcal{O}_X(D)) = 1$. Thus

$$c_2(X).D = 12 - 2D^3.$$

We then find that

$$c_t(i^*\mathcal{T}_X - \mathcal{T}_D) = c_t(i^*\mathcal{T}_X)/c_t(\mathcal{T}_D) = 1 + K_D t + (c_2(X).D + 4)t^2$$

so $\deg S = 16 - 2D^3$. Thus we must have $2 = 16 - 2D^3$, or $D^3 = 7$. Thus $D$ must be a del Pezzo surface of degree 7. •

Remark 5.3. While the above theorem was stated for the non-hypersurface singularity case, in the hypersurface singularity case we can still rule out the non-normal case as above, but cones over elliptic curves can, and do, occur.

Proposition 5.4. Suppose $(X, 0)$ is an isolated rational Gorenstein threefold point with $k = \text{mult}_0X \geq 5$, such that if $\hat{X} \to X$ is the blowing-up of $X$ at 0, then $\hat{X}$ is non-singular and the exceptional divisor $E$ is non-singular. Then $(X, 0)$ is analytically isomorphic to a cone over $E$.

Proof. Let $\mathcal{I}_E$ be the ideal sheaf of $E$ in $\hat{X}$. If $H^1(\mathcal{T}_E \otimes (\mathcal{I}_E/\mathcal{I}_E^2)^n) = H^1((\mathcal{I}_E/\mathcal{I}_E^2)^n) = 0$ for all $n \geq 1$, then [9], Corollary to Satz 7, tells us that $(\hat{X}, E)$ is analytically isomorphic to an open neighborhood of $E$ embedded in the normal bundle of $E$ in $\hat{X}$ as the zero section. This will then give the theorem.

$E$ is a del Pezzo surface of degree between 5 and 9, and $\mathcal{I}_E/\mathcal{I}_E^2 = \omega_E^{-1}$. For a del Pezzo surface, $H^1(\omega_E^{-n}) = 0$ for all $n \geq 1$. If we show that $H^1(\mathcal{T}_E) = 0$, then if $H$ is a hyperplane section of $E$, the exact sequence

$$0 \to \mathcal{T}_E \otimes \omega_E^{-n+1} \to \mathcal{T}_E \otimes \omega_E^{-n} \to (\mathcal{T}_E \otimes \omega_E^{-n})|_H \to 0$$

shows us that $H^1(\mathcal{T}_E \otimes \omega_E^{-n}) = 0$ for all $n \geq 1$. Now $E$ is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2$ blown up in $9 - k$ points. In the first two cases, $H^1(\mathcal{T}_E) = 0$ is immediate. In the other cases, $H^1(\mathcal{T}_E)$ is the tangent space to $Def(E)$, and the moduli space of non-singular del Pezzo surfaces of degree $\geq 5$ consists of a single point (reduced since $H^2(\mathcal{T}_E) \cong H^0(\Omega_{E}^1 \otimes \omega_E)^\vee = 0$) by [5], VII, 2. Thus $H^1(\mathcal{T}_E) = 0$. •
We review here the deformation theory of the singularity \((X, 0)\) which is the cone over a non-singular del Pezzo surface \(E\) of degree \(k\), \(5 \leq k \leq 9\). The cases \(6 \leq k \leq 9\) follow from [2].

\(k = 9\): \((X, 0)\) is rigid. (This also follows from [41].)

\(k = 8\): There are two cases. If \(E \cong \mathbf{P}^1 \times \mathbf{P}^1\), then \((X, 0)\) can be smoothed by taking a hyperplane section of a cone over \(\mathbf{P}^3\) embedded via the 2-uple embedding. If \(E \cong F_1\), there is no smoothing. In both cases, \(\dim_k T^1 = 1\).

\(k = 7\): \((X, 0)\) can be smoothed by taking a hyperplane section of a cone over \(\mathbf{P}^3\) blown up at a point, embedded in \(\mathbf{P}^8\) by projecting \(\mathbf{P}^3\) embedded in \(\mathbf{P}^9\) via the 2-uple embedding from a point on the \(\mathbf{P}^3\). Here \(\dim_k T^1 = 2\).

\(k = 6\): There are two distinct smoothings, one coming from taking a hyperplane section of a cone over \(\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \subseteq \mathbf{P}^7\), the other from taking two hyperplane sections of a cone over \(\mathbf{P}^2 \times \mathbf{P}^2 \subseteq \mathbf{P}^8\). Here \(\dim_k T^1 = 3\).

\(k = 5\): Any codimension 3 Gorenstein subscheme of the spectrum of a regular local ring is a Pfaffian subscheme [3], and any Pfaffian subscheme is smoothable by [22]. Here \(\dim_k T^1 = 4\).

We need one more technical lemma:

**Lemma 5.6.** Let \(E\) be a del Pezzo surface which is either rational and normal of any degree or else is non-normal of degree 7 of type (ii) (a) or (b) as given in the proof of Theorem 5.2. Then

(i) \(H^2(T_E \otimes \omega_{E}^{-n}) = 0\) for all \(n \geq 0\).

(ii) \(H^2(\Omega_E^1 \otimes \omega_{E}^{-n}) = 0\) for all \(n \geq 0\).

(iii) \(E\) is smoothable.

Proof: (i) By Serre duality, \(H^2(T_E \otimes \omega_{E}^{-n})^\vee \cong H^0(\Omega_E^{1 \vee} \otimes \omega_E^{n+1})\). If \(E\) is a normal del Pezzo surface and thus has only quotient singularities, then \(H^0(\Omega_E^{1 \vee}) = 0\) by Hodge theory. Since \(\omega_{E}^{-1}\) is effective, \(H^0(\Omega_E^{1 \vee} \otimes \omega_E^{n+1}) = 0\) for all \(n \geq 0\) also.

If \(E\) is non-normal, let \(n : \tilde{E} \to E\) be the normalization, so that \(\tilde{E}\) is a scroll. Then the map \(n^*\Omega_E^1 \to \Omega_{\tilde{E}}^1\) yields a map \((n^*\Omega_E^1)^{\vee} \to \Omega_{\tilde{E}}^{1 \vee} \cong \Omega_{\tilde{E}}^1\), and since \((n^*\Omega_E^1)^{\vee} \cong n^*\Omega_{\tilde{E}}^{1 \vee}\), the adjoint map is \(\Omega_{\tilde{E}}^{1 \vee} \to n_*\Omega_{\tilde{E}}^1\). Since \(\Omega_{\tilde{E}}^{1 \vee}\) is torsion-free, this map is injective, so \(H^0(\Omega_{\tilde{E}}^{1 \vee}) \subseteq H^0(\Omega_{\tilde{E}}^1) = 0\). Thus the result still follows in this case.

(ii) If \(E\) is normal, then again \(H^2(\Omega_E^1) \cong H^2(\Omega_{\tilde{E}}^{1 \vee}) = 0\) by Hodge theory, and so (ii) follows. Now suppose \(E\) is not normal, with \(n : \tilde{E} \to E\) the normalization. We need to
consider type (ii) (a) and (b) separately. If \( E \) is of type (a), \( n \) maps the section \( C_0 \) two-to-one to the singular curve \( l \). Let \( \iota : C_0 \to C_0 \) be the induced involution on \( C_0 \). \( \iota \) induces an involution \( \iota^* : n_* \Omega^1_{C_0} \to n_* \Omega^1_{C_0} \). Let \( F \subseteq n_* \Omega^1_{C_0} \) be the sheaf of anti-invariants of this involution. \( F \) is the image of the map \( \delta' : n_* \Omega^1_{C_0} \to n_* \Omega^1_{C_0} \) given by \( \delta'(\alpha) = \alpha - \iota^*(\alpha) \).

Since \( n_* \Omega^1_{C_0} = n_* \mathcal{O}_{C_0}(-2) = \mathcal{O}_l(-2) \oplus \mathcal{O}_l(-1) \), in fact \( F \cong \mathcal{O}_l(-1) \). Let \( \tau^1_E \subseteq \Omega^1_E \) be the torsion subsheaf of \( \Omega^1_E \). We then have a complex

\[
0 \to \tau^1_E \to \Omega^1_E \to n_* \Omega^1_E \to F \to 0
\]

where \( \delta \) is the (surjective) composition of the surjective restriction map \( n_* \Omega^1_E \to n_* \Omega^1_{C_0} \) and \( \delta' \). This sequence is exact at \( \Omega^1_E \) since \( n_* \Omega^1_E \) is torsion-free. It is exact at \( n_* \Omega^1_E \) where \( E \) has normal crossings by [7], (1.5), and thus this complex splits into exact sequences

\[
0 \to K \to n_* \Omega^1_E \to F \to 0
\]

and

\[
0 \to \Omega^1_E / \tau^1_E \to K \to \tau' \to 0
\]

with \( \tau' \) supported on points, from which we conclude that \( H^2(\Omega^1_E) = 0 \) as desired, and (ii) follows in this case.

If \( E \) is of case (ii) (b), then we can follow a similar procedure. The normalization map \( n \) maps \( C_0 \) and \( f \) to the singular line \( l \). Let \( \iota : C_0 \coprod f \to C_0 \coprod f \) be the induced involution interchanging these two lines. This induces \( \iota^* : n_* \Omega^1_{C_0} \coprod f \to n_* \Omega^1_{C_0} \coprod f \); let \( F \cong \Omega^1_l \) be the anti-invariant part. Again, \( F \) is the image of \( \delta' : n_* \Omega^1_{C_0} \coprod f \to n_* \Omega^1_{C_0} \coprod f \) given by \( \delta'(\alpha) = \alpha - \iota^*(\alpha) \). We still have the complex (5.7), and we finish the argument as before, noting now that the map \( H^1(\Omega^1_E) \to H^1(F) \) is surjective.

(iii) A nodal del Pezzo surface is easily seen to be smoothable: the local-global Ext sequence yields

\[
\text{Ext}^1(\Omega^1_E, \mathcal{O}_E) \to H^0(\text{Ext}^1(\Omega^1_E, \mathcal{O}_E)) \to H^2(\mathcal{T}_E) = 0,
\]

and since \( \text{Def}(E) \) is smooth, any small deformation of a neighborhood of the rational double points of \( E \) is realised by a global deformation of \( E \).

If \( E \) is not normal, then \( E \) is degree 7 of type (ii) (a) or (b). First suppose \( E \subseteq \mathbb{P}^7 \) is of type (ii) (a). Then there is a map \( p : E \to l \) taking a point \( x \in E \) to the point of \( l \subseteq E \) which is contained in the same ruling of \( E \) as \( x \). Each fibre is a singular conic: a union of two \( \mathbb{P}^1 \)'s or in two cases a doubled line. Each conic spans a plane, and thus
$E$ is contained in a three dimensional scroll. Abstractly, this scroll can be described as the image of the $\mathbb{P}^2$-bundle $P(p_*\mathcal{O}_E(1))$ in $\mathbb{P}^7$. Note that $p_*\mathcal{O}_E(1)$ is generated by global sections since $\mathcal{O}_E(1)$ is. Since $h^0(p_*\mathcal{O}_E(1)) = h^0(\mathcal{O}_E(1)) = 8$, $p_*\mathcal{O}_E(1)$ must be a rank 3 vector bundle on $l$ isomorphic to $\mathcal{E}(a,b,c) = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ with $a,b,c \geq 0$ and $a + b + c + 3 = 8$. If $t = c_1(\mathcal{O}_P(\mathcal{E}(a,b,c))(1))$ and $f$ is the class of a fibre of $P(\mathcal{E}(a,b,c))$, then $\text{Pic}P(\mathcal{E}(a,b,c)) = \mathbb{Z}t \oplus \mathbb{Z}f$. In order for $E \subseteq P(\mathcal{E}(a,b,c))$ to be a conic bundle of degree 7, it must have class $2t - 3f$. The linear system $|t|$ induces the map of $P(\mathcal{E}(a,b,c))$ into $\mathbb{P}^7$, and if $|t|$ were not ample (i.e. one of $a$, $b$, or $c$ were zero), it is easy to see that this would imply two different fibres of $p$ were not disjoint. Thus $(a,b,c) = (1,2,2)$ or $(1,1,3)$. In the latter case, the linear system $|2t - 3f|$ has a fixed component given by $t - 3f$, and so $E \subseteq P(\mathcal{E}(1,2,2))$.

Now a non-singular del Pezzo surface $E'$ of degree 7 also has a conic bundle structure, so an identical argument also shows that $E' \subseteq P(\mathcal{E}(1,2,2))$ and is in the same linear system $|2t - 3f|$. Thus $E$ is smoothable.

If $E$ is of type (ii) (b), it is easy to see that it is a degenerate case of (ii) (a), and so this case is also smoothable. One can consult [49] for an explicit description of this degeneration. •

**Theorem 5.8.** Let $\pi : \tilde{X} \to X$ be a primitive type II contraction with exceptional divisor $E$. Then $X$ is smoothable unless

1. $E \cong \mathbb{P}^2$
2. $E \cong F_1$.

Proof. We want to apply Theorem 4.3, so we need to verify the hypotheses of the theorem. First, since $\pi$ corresponds to the contraction of an extremal ray corresponding to $K_{\tilde{X}} + E$, $X$ is $\mathbb{Q}$-factorial by [20], Proposition 5.1.6.

From now on we will assume that $k \geq 5$. If $k \leq 4$, then $X$ has only complete intersection singularities and we can apply Theorem 3.8. We have to show that the singularity of $X$ is good.

Let $P \in X$ be the singular point, with $(\tilde{X}, E) \to (X, P)$ the resolution of the germ $(X, P)$. We first consider deformations of the inclusion map $i : E \to (\tilde{X}, E)$. If we denote by $T^1_i$ the tangent space to the deformation space of the triple $((\tilde{X}, E), E, i)$, there is an exact sequence by [31]

$$\text{Hom}(\Omega^1_E, \mathcal{O}_E) \oplus H^0(T_{\tilde{X}}) \to H^0(T_{\tilde{X}}|_E) \to T^1_i \xrightarrow{(d_1,d_2)} \text{Ext}^1(\Omega^1_E, \mathcal{O}_E) \oplus H^1(T_{\tilde{X}}) \to H^1(T_{\tilde{X}}|_E).$$
also, this map is an isomorphism.) Similarly, \( d \) which is the case since \( \text{Def} \) and \( \text{H} \) surjective, which is true if \( \text{E} \) is smoothable unless \( \text{E} \) is non-singular but \( E \) may not be.) The composed maps \( d_1 : T_i^1 \rightarrow \text{Ext}^1(\Omega_E^1, \mathcal{O}_E) \) and \( d_2 : T_i^1 \rightarrow H^1(\mathcal{T}_X) \) are the differentials of the maps \( \text{Def}((\tilde{X}, E), E, i) \rightarrow \text{Def}(E) \) and \( \text{Def}((\tilde{X}, E), E, i) \rightarrow \text{Def}(\tilde{X}, E) \) respectively.

Claim 1: \( d_1 \) and \( d_2 \) are surjective.

Proof: Note that \( d_2 \) is surjective if the map \( \text{Ext}^1(\Omega_E^1, \mathcal{O}_E) \rightarrow H^1(\mathcal{T}_X|_E) \) is surjective, which is the case since \( H^1((\mathcal{I}_E/\mathcal{I}_E^2)^{\vee}) = H^1(\omega_E) = 0 \). (In fact, since \( H^0((\mathcal{I}_E/\mathcal{I}_E^2)^{\vee}) = 0 \) also, this map is an isomorphism.) Similarly, \( d_1 \) is surjective if \( H^1(\mathcal{T}_X) \rightarrow H^1(\mathcal{T}_X|_E) \) is surjective, which is true if \( H^2(\mathcal{T}_X(-E)) = \lim_{\rightarrow} H^2(\mathcal{T}_X(-E) \otimes \mathcal{O}_X/\mathcal{I}_E^n) = 0 \). From

\[
0 \rightarrow \mathcal{T}_E(-E) \rightarrow \mathcal{T}_X(-E)|_E \rightarrow \mathcal{O}_E \rightarrow 0
\]

we see that \( H^2(\mathcal{T}_X(-E)|_E) = 0 \) if \( H^2(\mathcal{T}_E(-E)) = H^2(\mathcal{T}_E \otimes \omega_E^{-1}) = 0 \). This is the case by Lemma 5.6 (i). Tensoring the exact sequence

\[
0 \rightarrow \mathcal{T}_E^n/\mathcal{T}_E^{n+1} = \omega_E^{-n} \rightarrow \mathcal{O}_X/\mathcal{T}_E^n \rightarrow \mathcal{O}_X/\mathcal{T}_E^n \rightarrow 0
\]

with \( \mathcal{T}_X(-E) \) shows that \( H^2(\mathcal{T}_X(-E) \otimes \mathcal{O}_X/\mathcal{I}_E^n) = 0 \) for all \( n \), so \( d_1 \) is surjective. ●

Claim 2: \( \text{Def}((\tilde{X}, E), E, i) \) is unobstructed.

Proof: As noted in the proof of Claim 1, the map \( \text{Ext}^1(\Omega_E^1, \mathcal{O}_E) \rightarrow H^1(\mathcal{T}_X|_E) \) is an isomorphism. Also the map \( \text{Hom}(\Omega_E^1, \mathcal{O}_E) \rightarrow H^0(\mathcal{T}_X|_E) \) is a surjection. This shows that \( T_i^1 \cong H^1(\mathcal{T}_X) \), and it then follows from Proposition 3.4 as in the proof of [34], Theorem 2.1, that \( \text{Def}((\tilde{X}, E), E, i) \) is unobstructed. ●

Thus, since \( d_2 \) is surjective by Claim 1, any deformation of \( (\tilde{X}, E) \) is of the form \((\tilde{X}', E')\) with \( E' \) a deformation of \( E \). Since \( d_1 \) is surjective, for any small deformation of \( E \) to \( E' \), there is a deformation \((\tilde{X}', E')\) of \((\tilde{X}, E)\). Thus in particular, if \( E \) is smoothable, the general deformation of \((\tilde{X}, E)\) to \((\tilde{X}', E')\) yields \( E' \) smooth. By Lemma 5.6 (iii), any of the possible del Pezzo surfaces under consideration are smoothable.

To summarize, \((\tilde{X}, E)\) can be deformed to \((\tilde{X}', E')\) with \( E' \) non-singular. So \((X, P)\) can be deformed to \((X', P')\) with \((\tilde{X}', E') \rightarrow (X', P')\) the blow-up, and by Theorem 5.4, \((X', P')\) is analytically isomorphic to a cone over a del Pezzo surface. By (5.5), \((X', P')\) is smoothable unless \( E' \cong P^2 \) or \( F_1 \). In this latter case, as it is easy to see that the only normal del Pezzo surfaces of degree 8 or 9 are \( P^2, F_1, P^1 \times P^1 \) or a quadric cone appropriately embedded, we must also have \( E \cong P^2 \) or \( F_1 \).

If \( E \) is not \( P^2 \) or \( F_1 \), then \((X', P')\) is smoothable. Thus the general point in the image of \( \text{Def}(\tilde{X}, E) \rightarrow \text{Def}(X, P) \) is smoothable, so this image is contained in a smoothing component. The only remaining thing to check is that \( \text{Def}(\tilde{X}, E) \rightarrow \text{Def}(X, P) \) is an
immersion. Thus we need to show that the differential of this map is injective. This
differential is given by Lemma 3.2 to be the map $H^0(R^1\pi_* T_\tilde{X}) \rightarrow H^2_Z(T_X)$, and the kernel
of this map is $H^1_E(T_\tilde{X}) \cong H^0(R^2\pi_* \Omega^1_\tilde{X})^\vee$, which is easily seen to be zero using similar
methods as above via Lemma 5.6 (ii).

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