ON THE TWO-POINT CORRELATION FUNCTIONS FOR THE
$U_q[SU(2)]$ INVARIANT SPIN ONE-HALF HEISENBERG CHAIN
AT ROOTS OF UNITY

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Abstract

Using $U_q[SU(2)]$ tensor calculus we compute the two-point scalar operators (TPSO), their
averages on the ground-state give the two-point correlation functions. The TPSOs are identi-
fied as elements of the Temperley-Lieb algebra and a recurrence relation is given for them. We
have not tempted to derive the analytic expressions for the correlation functions in the general
case but got some partial results. For $q = e^{i\pi/3}$, all correlation functions are (trivially) zero,
for $q = e^{i\pi/4}$, they are related in the continuum to the correlation functions of left-handed
and right-handed Majorana fields in the half plane coupled by the boundary condition. In
the case $q = e^{i\pi/6}$, one gets the correlation functions of Mittag’s and Stephen’s parafermions
for the three-state Potts model. A diagrammatic approach to compute correlation functions
is also presented.
1 Introduction

This paper is a continuation of the effort to compute the correlation functions for quantum chains invariant under quantum group transformations with \( q \) being a root of unity. The simplest case which is the \( U_q[SU(1/1)] \) invariant chain was considered in Ref. [1]. Here we consider the \( U_q[SU(2)] \) invariant chain [2] described by the Hamiltonian

\[
H = - \sum_{j=1}^{L-1} e_j
\]  

where

\[
e_j = - \frac{1}{2} \left( \sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1} + \frac{q + q^{-1}}{2} (\sigma^z_j \sigma^z_{j+1} - 1) + \frac{q - q^{-1}}{2} (\sigma^z_j - \sigma^z_{j+1}) \right).
\]

In this expression, \( \sigma^x, \sigma^y, \) and \( \sigma^z \) are Pauli matrices and \( q \) is a complex parameter. We consider the antiferromagnetic case. The Hamiltonian (1.1) is the XXZ Heisenberg model with boundary terms. As is well known, if \( q \) is real, one is in the massive phase. For this case the correlation functions have been recently computed [3] taking first the limit \( L \to \infty \) and observing that the symmetry is larger than \( U_q[SU(2)] \) (the boundary terms disappear). This procedure does not apply when \( q \) is on the unit circle where one is in the massless phase. If however one takes periodic boundary conditions (the boundary terms again drop out and the \( U_q[SU(2)] \) symmetry is lost) the correlation functions can be computed as shown in Ref. [4]. We will insist in maintaining the \( U_q[SU(2)] \) symmetry, thus keeping the boundary terms. The interest in the calculation of the correlation functions in this case stems from the observation [5, 2] that the spectrum of the Hamiltonian (1.1) for \( q = e^{i\pi m+1} \) contains the spectrum of Hamiltonians corresponding to statistical models which in the continuum limits are described by a conformal field theory with a central charge

\[
c = 1 - \frac{6}{m(m+1)}.
\]  

In order to explain the problem, let us first consider the case \( q = 1 \) (\( SU(2) \) symmetry). We consider a two-point \( SU(2) \) scalar operator (TPSO)

\[
g_{l,m} = - \frac{1}{2} \bar{\sigma}_l \cdot \bar{\sigma}_m
\]  

and the two-point correlation function

\[
G(l, m, L) = \langle 0 | g_{l,m} | 0 \rangle = \langle g_{l,m} \rangle.
\]  

The reader should keep in mind that \( \bar{\sigma}_l \) and \( \bar{\sigma}_m \) are \( SU(2) \) tensor operators corresponding to the adjoint representation and their scalar product is thus an \( SU(2) \) invariant. The ground state is an \( SU(2) \) scalar (we always take an even number of sites). Because of the boundary terms, translation invariance is lost and the correlation function is not a function of \( (l - m) \) and \( L \) only. The continuum limit is obtained [3] taking \( L \to \infty \) with \( l \) and \( m \) fixed.

An important observation which is going to be extensively used in the text is that the \( e_j \) of Eq. (1.2) are the generators of the Temperley-Lieb algebra (with different quotients when \( q^p = \pm 1 \)) and that the elements

\[
c_{l,m} = \frac{1}{2} + g_{l,m}, \quad c_{l,l+1} = e_l
\]  

(1.6)
belong to that algebra. In fact, for \( l < m < n \) (and still with \( q = 1 \)) these elements satisfy the obvious recurrence relation

\[
c_{l,n} = c_{l,m} + c_{m,n} - c_{l,m} c_{m,n} - c_{m,n} c_{l,m}.
\]  

(1.7)

It is well known that, considered as elements of the Temperley-Lieb algebra, the \( SU(2) \) scalar states, of which the ground state is one, build an irreducible representation of the algebra. Eq. (1.7) thus allows to do the whole calculation of the correlation functions by purely algebraic means.

Obviously Eqs. (1.4,6,7) have to be extended for \( q \neq 1 \). As will be seen, the two-point scalar operator (TPSO) \( g_{l,m} \) generalizes in two ways and, unlike the \( q = 1 \) case, \( g_{l,m} \neq g_{m,l} \). One obvious question one would like to answer for \( q = e^{i\pi/3} \) when the continuum theory is conformal invariant, is what are the surface and bulk exponents one derives. The question is very relevant since, as we will see, unlike the \( \vec{\sigma}_l \) of the \( q = 1 \) case, the tensor operators are highly non-local objects and we would like to know first, whether one can define on the lattice the proper "local" operators directly for \( g_{l,m} \) and get their critical dimensions.

The paper is organized as follows. In Sec. 2 we remind the reader of the definition of tensor operators for the \( U_q[SU(2)] \) quantum group [7-11] and give the expression of the TPSOs which generalize Eq. (1.4) for \( q \neq 1 \). In Sec. 3 we make the contact with the Temperley-Lieb algebra (generalizing Eq. (1.6) and extending the recurrence relation (1.7) for \( q \neq 1 \)). At this point, using the Bethe-Ansatz calculations of Refs. [12-15] for the ground state wave function, one could start to compute the correlation functions, but we did not try to do it. We have preferred to do some exploratory work which could clarify what to expect from the results to come. Actually, in Appendix A we give a diagrammatic approach for the calculation of the correlation functions. This approach is entirely based on using irreducible representations of Temperley-Lieb algebras and their quotients. This approach can be useful for numerical calculations and to obtain some partial results. One can for example trivially show that for \( q = e^{i\pi/3} \), all the correlation functions are zero. Another important result for \( q = e^{i\pi/4} \) (which corresponds to the Ising model) is that the continuum limit \( (L \to \infty) \) has to be taken with care since one gets

\[
\langle g_{2l,2m} \rangle = \langle g_{2l+1,2m+1} \rangle = 0,
\]

(1.8)

while the other correlation functions are different from zero.

In Sec. 4 we compute directly the correlation functions for \( q = e^{i\pi/4} \). This is done using the Ising representation of the Temperley-Lieb algebra and observing that the \( g_{l,m} \) can be written locally in terms of free fermions. The correlation functions one obtains are new. In Sec. 5, still for \( q = e^{i\pi/4} \), we compute the time-dependent correlation functions. The idea is quite simple. Some time ago, Symanzik [16] has considered a left-mover and a right-mover Majorana field in the half-plane (they are coupled through the boundary conditions) and computed their propagators. We perform a conformal transformation from the half-plane to the strip and discover that the equal-time correlation functions thus obtained are those derived in Sec. 4. In Sec. 6 we discuss the physical interpretation of the correlation functions for \( q = e^{i\pi/6} \) which corresponds to the three-states Potts model. One discovers that the "local" operators appearing here in the TPSOs are the parafermions of Mittag and Stephen [17]. The conclusions are presented in Sec. 7.
2 Two-point $U_q[SU(2)]$ invariant operators

Let us first again consider the $q = 1$ case. The TPSOs given by Eq. (1.3) can be written as follows

$$g_{l,m} = t_l^1 t_m^{-1} + t_l^{-1} t_m^1 - t_l^0 t_m^0,$$  \hspace{1cm} (2.1)

where

$$t_l^{\pm 1} = \mp \sigma_l^{\pm}, \hspace{1cm} t_l^0 = \frac{1}{\sqrt{2}} \sigma_l^z. \hspace{1cm} (\sigma_l^{\pm} = \frac{1}{2}(\sigma_l^x \pm i \sigma_l^y))$$  \hspace{1cm} (2.2)

The $t_l^p$ ($p = \pm 1, 0$) are irreducible tensor operators corresponding to the three-dimensional vector representation [18].

We now consider the case $q \neq 1$. First let us introduce some notations:

$$S_{i,j}^0 = \begin{cases} \frac{1}{2} \sum_{k=i}^{j} \sigma_k^z & i \leq j \\ 0 & i > j \end{cases}$$  \hspace{1cm} (2.6)

$$K_{i,j} = q^{S_{i,j}^0},$$  \hspace{1cm} (2.7)

$$S_{i,j}^{\pm} = \sum_{k=i}^{j} K_{k+1,j}^+ \sigma_k^+ K_{k-1,j}^- \hspace{1cm} (i < j)$$  \hspace{1cm} (2.8)

$$S^\pm = S_{1,L}^\pm, \hspace{1cm} K^\pm = K_{1,L}^\pm.$$  \hspace{1cm} (2.9)

$S^\pm$ and $K^\pm$ commute with the Hamiltonian $H$ (Eq. (1.1)) and are the generators of the $U_q[SU(2)]$ algebra:

$$K^+ K^- = K^- K^+ = 1$$  \hspace{1cm} (2.10)

$$K^+ S^\pm K^- = q^{\mp 1} S^\pm$$

$$[S^+, S^-] = \frac{K^+ - K^-}{q - q^{-1}}.$$  \hspace{1cm} (2.11)

Following Ref. [9] the tensor operators corresponding to the adjoint representation of $U_q[SU(2)]$ are defined through the relations

$$(ad S^\pm) t_j^{\pm 1} = 0$$

$$(ad S^\pm) t_j^0 = \sqrt{[2]_q} t_j^{\pm 1}$$

$$(ad S^\pm) t_j^{\mp 1} = \sqrt{[2]_q} t_j^0$$

$$(ad K^\pm) t_j^p = q^{\mp p} t_j^p, \hspace{1cm} (p = \pm 1, 0)$$
where
\[
(ad S^\pm) t_j^p = S^\pm t_j^p K^+ - q^{1/2} K^+ t_j^p S^\pm
\] (2.12)
and
\[
(ad K^\pm) t_j^p = K^\pm t_j^p K^\mp
\]
respectively
\[
t_j^{-1} = \left(K_{1,j-1}^{-1}\right)^2 \tau_j^-
\] (2.14)
\[
t_j^{+1} = \left(K_{1,j-1}^+\right)^2 \tau_j^+ + (q - q^{-1})q^{-1} \{ (q + q^{-1})^{1/2} K_{1,j-1}^+ S_{1,j-1}^- \tau_j^0
\] (2.15)
\[
t_j^0 = \tau_j^0 + (q - q^{-1}) (q + q^{-1})^{1/2} K_{1,j-1}^- S_{1,j-1}^+ \tau_j^-
\] (2.16)
\[
\tilde{t}_j^{-1} = \left(K_{1,j-1}^{-1}\right)^2 \tau_j^+
\] (2.17)
\[
\tilde{t}_j^{+1} = \left(K_{1,j-1}^+\right)^2 \tau_j^- - (q - q^{-1})q \{ (q + q^{-1})^{1/2} K_{1,j-1}^- S_{1,j-1}^+ \tau_j^0
\] (2.18)
\[
\tilde{t}_j^0 = \tau_j^0 - (q - q^{-1}) (q + q^{-1})^{1/2} K_{1,j-1}^+ S_{1,j-1}^- \tau_j^+
\] (2.19)
are the tensor operators for one site. As shown in Ref. [10], the tensor operators satisfy the following permutation properties among the lattice sites:
\[
t_i^p t_j^q = \sum_{u,v} \hat{R}_{u,v;p,r} t_i^u t_j^v \quad (i < j)
\] (2.22)
\[
\tilde{t}_i^p \tilde{t}_j^q = \sum_{u,v} \hat{R}_{u,v;p,r} \tilde{t}_i^u \tilde{t}_j^v \quad (i < j),
\] (2.23)
where the spin 1 \(\otimes\) spin 1 \(\hat{R}\) matrix has the following expression:
\[
\hat{R} = \sum \hat{R}_{u,v;p,r} E^{u,p} \otimes E^{v,r}
\]
\[
= (1 - q^{-2}) [2]_q (q E^{1,1} \otimes E^{0,0} + q E^{0,0} \otimes E^{-1,-1} + E^{0,1} \otimes E^{0,-1}
\] (2.24)
\[
+ q E^{1,0} \otimes E^{-1,0} + q(1 - q^{-2}) E^{1,1} \otimes E^{-1,-1})
\]
\[
+ g^2 (E^{1,1} \otimes E^{1,1} + E^{-1,-1} \otimes E^{-1,1})
\]
\[
+ g q (E^{1,0} \otimes E^{1,0} + E^{0,0} \otimes E^{-1,0} + E^{1,0} \otimes E^{0,1} + E^{-1,0} \otimes E^{0,-1} + E^{0,0} \otimes E^{0,0})
\].

Here \(E^{u,p}\) is the 3 \(\times\) 3 matrix whose sole non-vanishing matrix element is on the \(u\) th row and the \(p\) th column, and this element is equal to one. The matrix \(\hat{R}\) has been calculated from
From Eqs. (2.22-2.26) we derive the following important relation: 

\[
\text{complex 3-space it coincides with the } \hat{R} \text{ matrix of the quantum group } SO(3) \text{ [20]. Relations like Eqs. (2.22), (2.23) are well known in conformal field theory (see Ref. [21] and references therein).}
\]

Using \( U_q[SU(2)] \) Clebsch-Gordan coefficients we now define two TPSOs:

\[
g_{t,m}^+ = q^{-2} (q^{-1} t_{t+1}^- t_{m}^- + q t_{t+1}^- t_{m}^- - t_{t}^0 t_{m}^0) \tag{2.25}
\]

\[
g_{t,m}^- = q^{+2} (q^{-1} t_{t+1}^- t_{m}^- + q t_{t+1}^- t_{m}^- - t_{t}^0 t_{m}^0) \tag{2.26}
\]

From Eqs. (2.23, 2.26) we derive the following important relation:

\[
g_{t,m}^+ = q^{+4} g_{m,t}^\pm, \quad (l < m) \tag{2.27}
\]

We now give the explicit expressions of the TPSOs. Using Eqs. (2.14-2.21) we obtain for \( l < m \)

\[
g_{t,m}^+ = -\left[ \sigma_t^+ (K_{t+1, m-1})^2 \sigma_m^- + \sigma_t^- (K_{t+1, m-1})^2 \sigma_m^+ \right. \tag{2.28}
\]

\[
+ \frac{q + q^{-1}}{4} (\sigma_t^+ \sigma_m^-) - \frac{1}{4} \left( \frac{q - q^{-1}}{q + q^{-1}} \right) \left( \frac{q - q^{-1}}{2} \right) (\sigma_t^- \sigma_m^+ S_{t+1, m-1}^- K_{t+1, m-1}^- \sigma_m^-)
\]

\[
- \frac{q^{-1/2} (q - q^{-1})^2}{2} (S_{t+1, m-1}^+ K_{t+1, m-1}^- \sigma_m^-)
\]

\[
+ \frac{q^{-1/2} (q^2 - q^{-2})}{2} (\sigma_t^+ S_{t+1, m-1}^+ K_{t+1, m-1}^- \sigma_m^-)
\]

\[
- \frac{q^{1/2} (q - q^{-1})^2}{2} (\sigma_t^- S_{t+1, m-1}^+ K_{t+1, m-1}^- \sigma_m^-)
\]

\[
- \frac{q^{1/2} (q^2 - q^{-2})}{2} (\sigma_t^- S_{t+1, m-1}^+ K_{t+1, m-1}^- \sigma_m^-)
\]

\[
= (q - q^{-1})^2 (\sigma_t^- (S_{t+1, m-1}^+)^2 \sigma_m^-)
\]

and

\[
g_{t,m}^- = -\left[ \sigma_t^+ (K_{t+1, m-1})^2 \sigma_m^- + \sigma_t^- (K_{t+1, m-1})^2 \sigma_m^+ \right. \tag{2.29}
\]

\[
+ \frac{q + q^{-1}}{4} (\sigma_t^+ \sigma_m^-) - \frac{1}{4} \left( \frac{q - q^{-1}}{q + q^{-1}} \right) \left( \frac{q - q^{-1}}{2} \right) (\sigma_t^- \sigma_m^+ S_{t+1, m-1}^- K_{t+1, m-1}^- \sigma_m^-)
\]

\[
- \frac{q^{-1/2} (q - q^{-1})^2}{2} (K_{t+1, m-1}^- S_{t+1, m-1}^+ \sigma_m^+)
\]

\[
+ \frac{q^{-1/2} (q^2 - q^{-2})}{2} (\sigma_t^+ K_{t+1, m-1}^- S_{t+1, m-1}^+ \sigma_m^+)
\]

\[
- \frac{q^{1/2} (q - q^{-1})^2}{2} (\sigma_t^- K_{t+1, m-1}^- S_{t+1, m-1}^+ \sigma_m^-)
\]

\[
- \frac{q^{1/2} (q^2 - q^{-2})}{2} (\sigma_t^- K_{t+1, m-1}^- S_{t+1, m-1}^+ \sigma_m^-)
\]

\[
= (q - q^{-1})^2 (\sigma_t^+ (S_{t+1, m-1}^-)^2 \sigma_m^+)
\]
Notice that
\[ g_{l+1}^e = g_{l-1}^e = e_l - (q + q^{-1})^{-1}. \]

The \( g_{l,m} \) given by Eqs. (2.28-2.29) generalize Eq. (1.4). The generalization of Eqs. (1.6) and (1.7) is going to be presented in the next section.

## 3 Identification of the TPSOs as elements of the Temperley Lieb algebra. Recurrence relations.

The Temperley-Lieb algebra \[22\] is defined by the generators \( e_j \) \((j = 1, \ldots, L - 1)\) satisfying the conditions
\[ e_j^2 = (q + q^{-1}) e_j = x e_j \]  \quad (3.1)
\[ e_j e_{j+1} e_j = e_j \]  \quad (3.2)
\[ [e_i, e_j] = 0. \quad (j \neq i \pm 1) \]  \quad (3.3)

As already mentioned earlier, the \( e_j \) of Eq. (1.2) verify the conditions (3.1-3.3). We now generalize Eq. (1.6) defining
\[ c_{l,m}^\pm = g_{l,m}^\pm + x^{-1} \]  \quad \((l < m)\)  \quad (3.4)

where the TPSOs are defined in (2.28-2.29). Here \( x = q + q^{-1} \). The \( c_{l,m}^\pm \) are elements of the Temperley-Lieb algebra. The proof is simple. We first notice that
\[ c_{j,j+1}^\pm = e_j. \]  \quad (3.5)

Moreover, we can show that for \( l < m < n \) one has the following recurrence relation which generalizes Eq. (1.7):
\[ c_{l,n}^\pm = c_{l,m}^\pm + c_{m,n}^\pm - q^\pm c_{l,m}^\pm c_{m,n}^\pm - q^\mp c_{m,n}^\pm c_{l,m}^\pm \]  \quad (3.6)

and this proves our claim. Let us also notice that the \( c_{l,m}^\pm \) satisfy the relations
\[ (c_{l,m}^\pm)^2 = x c_{l,m}^\pm \]  \quad (3.7)
and for \( l < m < n \) with \( \mu, \nu = \pm 1 \)
\[ c_{l,m}^\mu c_{m,n}^\nu c_{l,m}^\mu = c_{l,m}^\mu \]  \quad (3.8)
\[ c_{m,n}^\mu c_{l,m}^\nu c_{m,n}^\mu = c_{m,n}^\mu \]  \quad (3.9)

and for \( i < j < l < m \)
\[ [c_{i,j}^\mu, c_{i,m}^\nu] = [c_{i,m}^\mu, c_{j,l}^\nu] = 0 \]  \quad (3.10)

which resemble the relations (3.1-3.3) for the generators.

The relations (3.6)-(3.10) have first been conjectured on the basis of a detailed computer investigation of the explicit expressions (2.28), (2.29) for the \( g_{l,m}^\pm \). This induced us to study the tensor product of tensor operators from a more fundamental point of view \[10\], which, in particular, led us to the following factorized expressions for the operators \( c_{l,m}^\pm \). Let \( R \) be the
well-known spin $\frac{1}{2} \otimes \text{spin } \frac{1}{2}$ \(R\) matrix (not to be confused with the matrix in Eq. (2.24)) and let us define the operators

\[ b_j = \hat{R}_{j,j+1} \quad (j = 1, 2, \ldots, L - 1) \tag{3.11} \]

Assuming that \(\hat{R}\) is normalized appropriately, we have

\[ b_j = q - e_j, \quad b_j^{-1} = q^{-1} - e_j, \tag{3.12} \]

and then the \(c_{l,m}^\pm\) are given by

\[ c_{l,m}^\pm = b_{m-1}^{\mp 1} b_{m-2}^{\mp 1} \cdots b_{l+1}^{\mp 1} e_l b_{l+1}^{\mp 1} \cdots b_{m-2}^{\mp 1} b_{m-1}^{\mp 1} \tag{3.13} \]

if \(l < m\). The aforementioned statements and relations now follow immediately. Obviously, the relation (3.13) has a wider range of applications than the special case of the XXZ chain considered here. For more details we refer the reader to a forthcoming publication [11].

A direct application of the recurrence relations (3.6) to a diagrammatic approach for the calculation of the correlation functions is given in Appendix A. Other applications are shown in Secs. 4 and 6.

## 4 Correlation functions for \(q = e^{i\pi/4}\)

In this section we specialize to the case \(q = e^{i\pi/4} \quad (x = \sqrt{2})\). The quotient of the Temperley-Lieb algebra for this case is given in Appendix A but we will not need it here since we are going to do the calculations in a different way. We take the Hamiltonian (1.1) with an even number of sites \(L = 2N\) such that the ground state is an \(U_q[SU(2)]\) scalar. For \(x = \sqrt{2}\), the Temperley-Lieb algebra has the following well-known representation in terms of Pauli matrices

\[ e_{2j} = c_{2j,2j+1}^\pm = \frac{1}{\sqrt{2}} (1 + \sigma_j^x \sigma_{j+1}^x) \tag{4.1} \]

\[ e_{2j-1} = c_{2j-1,2j}^\pm = \frac{1}{\sqrt{2}} (1 + \sigma_j^z) \tag{4.2} \]

which gives the Ising Hamiltonian with free boundary conditions:

\[ H = - \sum_{j=1}^{2N-1} e_j = - \frac{1}{\sqrt{2}} \left( \sum_{j=1}^{N-1} \sigma_j^x \sigma_{j+1}^x + \sum_{j=1}^{N} \sigma_j^z \right) - \frac{2N-1}{\sqrt{2}}. \tag{4.3} \]

We use the recurrence relations (3.6) to get

\[ g_{2j,2k}^\pm = \pm \frac{i}{\sqrt{2}} \sigma_j^y \sigma_k^y \tag{4.4} \]

\[ g_{2j-1,2k-1}^\pm = \pm \frac{i}{\sqrt{2}} \sigma_j^y \sigma_k^y \tag{4.5} \]

\[ g_{2j,2k-1}^\pm = - \frac{i}{\sqrt{2}} \sigma_j^x \sigma_k^x \tag{4.6} \]

\[ g_{2j-1,2k}^\pm = - \frac{i}{\sqrt{2}} \sigma_j^x \sigma_k^y \tag{4.7} \]
where
\[ \tau_j^x = \left( \prod_{i=1}^{j-1} \sigma_i^x \right) \sigma_j^x, \quad \tau_j^y = \left( \prod_{i=1}^{j-1} \sigma_i^y \right) \sigma_j^y \] (4.8)
are fermionic operators:
\[ \{ \tau_i^x, \tau_j^x \} = 2 \delta_{i,j}, \quad \{ \tau_i^y, \tau_j^y \} = 2 \delta_{i,j} \] (4.9)
\[ \{ \tau_i^x, \tau_j^y \} = 0. \] (4.10)

Notice that the TPSOs are hermitean operators and that
\[ g_{i,m} = -g_{m,l} \] (4.11)
in agreement with Eq. (2.27).

The two point correlation functions can now be computed using standard techniques [23]. One obtains
\[ \langle g_{2j,2k}^\pm \rangle = \langle g_{2j-1,2k-1}^\pm \rangle = 0 \] (4.12)
in agreement with the results of Appendix A. One also obtains
\[ \langle g_{2j,2k-1}^\pm \rangle = \frac{2 \sqrt{2}}{2L+1} \sum_{n=0}^{L-1} \sin\left(\frac{2n+1}{2L+1}j\right) \cos\left(\frac{2n+1}{2L+1}(k-\frac{1}{2})\right) \] (4.13)
\[ \langle g_{2j-1,2k}^\pm \rangle = \frac{2 \sqrt{2}}{2L+1} \sum_{n=0}^{L-1} \sin\left(\frac{2n+1}{2L+1}k\right) \cos\left(\frac{2n+1}{2L+1}(j-\frac{1}{2})\right). \] (4.14)
In the continuum limit \( L \to \infty \) these correlation functions reduce to
\[ \langle g_{2j,2k-1}^\pm \rangle = \frac{1}{\pi \sqrt{2}} \left( \frac{1}{R} + \frac{1}{S} \right) \] (4.15)
\[ \langle g_{2j-1,2k}^\pm \rangle = \frac{1}{\pi \sqrt{2}} \left( \frac{1}{R} - \frac{1}{S} \right), \] (4.16)
where
\[ R = k - j, \quad S = k + j. \] (4.17)

We can now use [3] the standard definition of critical exponents: The correlation function is of the form
\[ G(R,S) = \frac{F(\rho)}{R^{2x}}, \quad (\rho = \frac{S^2}{R^2}) \] (4.18)
where \( x \) is the bulk critical exponent. From Eqs. (4.15) and (4.16) we get \( x = \frac{1}{2} \) for both correlation functions. If
\[ F(\rho) = F(1 + \alpha); \quad \lim_{\alpha \to 0} F(\rho) = \alpha^{x_s - x} \] (4.19)
one gets \( x_s = \frac{1}{2} \) for one correlation function and \( x_s = \frac{3}{2} \) for the other.

5 Time-dependent correlation functions for \( q = e^{i \pi/4} \)

Our aim is to compute (see Eqs. (1.4)-(1.7))
\[ \langle \tau_j^y(t_1) \tau_j^y(t_2) \rangle = \langle 0 | \tau_j^y(t_1, \theta_1) \tau_j^y(t_2, \theta_2) | 0 \rangle \] (5.1)
\[ \langle \tau_j^x(t_1) \tau_k^x(t_2) \rangle = \langle 0 | \tau_j^x(t_1, \theta_1) \tau_k^x(t_2, \theta_2) | 0 \rangle \]
\[ \langle \tau_j^y(t_1) \tau_k^x(t_2) \rangle = \langle 0 | \tau_j^y(t_1, \theta_1) \tau_k^x(t_2, \theta_2) | 0 \rangle \]
\[ \langle \tau_j^x(t_1) \tau_k^y(t_2) \rangle = \langle 0 | \tau_j^x(t_1, \theta_1) \tau_k^y(t_2, \theta_2) | 0 \rangle \]
directly in the continuum limit. Here $\theta_1 = ja$, $\theta_2 = ka$ and $a$ is the lattice spacing. We first consider \[24\] the Majorana fields $\psi_1(r)$ and $\psi_2(r)$

$$\tau^x_j = \psi_1(r) + \psi_2(r), \quad \tau^y_j = \psi_1(r) - \psi_2(r)$$

with the properties:

$$\{\psi_\alpha(r_1), \psi_\beta(r_2)\} = \delta_{\alpha,\beta}\delta_{r_1,r_2}, \quad \psi_\alpha^\dagger = \psi_\alpha. \quad (\alpha, \beta = 1, 2)$$

The continuum version of the Hamiltonian \[4.3\] is \[24\]

$$H = \frac{i}{2}(\psi_1(r)\frac{\partial}{\partial r}\psi_1(r) - \psi_2(r)\frac{\partial}{\partial r}\psi_2(r)).$$

With

$$\psi_\alpha(r, t) = e^{Ht}\psi_\alpha(r)e^{-Ht}$$

we get the equations of motion

$$\partial_z \psi_1 = 0, \quad \partial_\bar{z} \psi_2 = 0,$$

where $z = r + it$ and $\bar{z} = r - it$. The correlation functions in the plane are well known:

$$\langle \psi_1(z_1) \psi_1(z_2) \rangle = \frac{1}{2\pi(z_1 - \bar{z}_2)}$$

$$\langle \psi_2(z_1) \psi_2(z_2) \rangle = \frac{1}{2\pi(z_1 - \bar{z}_2)}.$$

For our purpose these are not the proper correlation functions. What we need are the correlation functions in the half plane. They have been calculated by Symanzik \[16\]. The boundary condition for $\psi_1(r, t)$ and $\psi_2(r, t)$ is

$$\tau^x(r, t = 0^+) = \psi_1(r, 0^+) + \psi_2(r, 0^+) = 0.$$ 

This boundary condition couples $\psi_1$ with $\psi_2$ (the other boundary condition $\tau^y(r, t = 0^+) = 0$ will not be useful here).

We define the $2 \times 2$ matrix

$$G_{\alpha,\beta} = \langle \psi_\alpha \psi_\beta \rangle \quad (\alpha, \beta = 1, 2)$$

and the propagators computed in Ref. \[16\] are:

$$G(z_1, \bar{z}_1, z_2, \bar{z}_2) = \frac{1}{2\pi} \begin{pmatrix} \frac{1}{z_1 - \bar{z}_2} & \frac{1}{\bar{z}_1 - \bar{z}_2} \\ \frac{1}{\bar{z}_1 - \bar{z}_2} & \frac{1}{\bar{z}_1 - \bar{z}_2} \end{pmatrix}.$$

Next we make a conformal transformation which takes the half-plane into a strip of width $N$ \[3\]

$$z = e^{\frac{w}{N}} = e^{\frac{w}{N}(t + i\theta)}$$

and get

$$G^{\text{strip}}(t_1, \theta_1, t_2, \theta_2) = \frac{1}{2\pi} \begin{pmatrix} \frac{1}{w_1 - \bar{w}_2} & \frac{1}{\bar{w}_1 - \bar{w}_2} \\ \frac{1}{\bar{w}_1 - \bar{w}_2} & \frac{1}{\bar{w}_1 - \bar{w}_2} \end{pmatrix}.$$
We denote
\[ \Delta t = t_2 - t_1 , \quad R = \theta_2 - \theta_1 , \quad S = \theta_1 + \theta_2 \]
and obtain for the correlation functions (5.1) the following expressions:
\[ \langle \tau^x(t_1, \theta_1) \tau^x(t_2, \theta_2) \rangle = \frac{1}{\pi} \left( \frac{1}{\Delta t + R^2/\Delta t} + \frac{1}{\Delta t + S^2/\Delta t} \right) \]
\[ \langle \tau^y(t_1, \theta_1) \tau^y(t_2, \theta_2) \rangle = \frac{1}{\pi} \left( \frac{1}{\Delta t + R^2/\Delta t} - \frac{1}{\Delta t + S^2/\Delta t} \right) \]
\[ \langle i \tau^y(t_1, \theta_1) \tau^x(t_2, \theta_2) \rangle = - \frac{1}{\pi} \left( \frac{1}{R + \Delta t^2/R} + \frac{1}{S + \Delta t^2/S} \right) \]
\[ \langle i \tau^x(t_1, \theta_1) \tau^y(t_2, \theta_2) \rangle = - \frac{1}{\pi} \left( \frac{1}{R + \Delta t^2/R} - \frac{1}{S + \Delta t^2/S} \right) \]
(5.14)

We now get convinced that we have taken the proper boundary condition (5.8) since taking \( \Delta t = 0 \), we recover the results of Sec. 4 (see Eqs. (4.12-4.16)). The second boundary condition of Ref. [16] \( \tau^y(r, t = 0^+) = 0 \) is useful if one changes the overall sign of the Hamiltonian (5.4).

6 Physical interpretation of the correlation functions for \( q = e^{i\pi/6} \)

We saw in the last two sections that for \( q = e^{i\pi/4} \) \( (g_{l,m} = -g_{m,l}^\pm) \), the correlation functions were given by fermionic propagators in the strip. In the present case, we have
\[ g_{l,m}^\pm = \omega^\pm_1 g_{m,l} . \quad (l < m, \ \omega = e^{2i\pi/3}) \]
(6.1)

This suggests that the "local" operators in the subspace where the irreducible representation of the Temperley-Lieb algebra acts are the parafermions of Mittag and Stephen [17] occurring in the three-state Potts model.

We first give the Temperley-Lieb generators for the \( q = e^{i\pi/6} \) quotient:
\[ e_{2j} = \frac{1}{\sqrt{3}} (1 + \Gamma_j \Gamma_{j+1}^\dagger + \Gamma_j^\dagger \Gamma_{j+1}) \]
\[ e_{2j-1} = \frac{1}{\sqrt{3}} (1 + \sigma_j + \sigma_j^\dagger) , \]
(6.2)

where
\[ \sigma \Gamma = \omega \Gamma \sigma , \quad \sigma^3 = \Gamma^3 = 1 \]
(6.3)

with the representation:
\[ \sigma = \begin{pmatrix} 1 & \omega \\ \omega & \omega^2 \end{pmatrix} , \quad \Gamma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} . \]
(6.4)

Using Eqs (6.1) and (6.2) one obtains
\[ H = - \sum_{j=1}^{2N-1} e_j = - \frac{1}{\sqrt{3}} \left( \sum_{j=1}^{N-1} (\Gamma_j \Gamma_{j+1}^\dagger + \Gamma_j^\dagger \Gamma_{j+1}) + \sum_{j=1}^{N} (\sigma_j + \sigma_j^\dagger) \right) - \frac{2N - 1}{\sqrt{3}} . \]
(6.5)
We now define the parafermionic operators \[\Pi_j = \left(\prod_{i=1}^{j-1} \sigma_i\right) \Gamma_j \] (6.6)

\[Q_j = \Gamma_j^\dagger \left(\prod_{i=1}^{j} \sigma_i^\dagger\right)\]

with \[\Pi_j^2 = \Pi_j^\dagger, \quad Q_j^2 = Q_j^\dagger\] (6.7)

and (for \(l < m\))

\[Q_m Q_l = \omega Q_l^\dagger Q_m, \quad Q_m \Pi_l = \omega^2 \Pi_l Q_m, \quad \Pi_m Q_l = \omega^2 Q_l \Pi_m, \quad \Pi_m \Pi_l = \omega \Pi_l \Pi_m. \]

Beside \(Q_l\) and \(\Pi_l\) it is also useful to consider their complex conjugates \(Q_l^\ast\) and \(\Pi_l^\ast\). In terms of parafermions the ferromagnetic three-state Potts Hamiltonian is:

\[H = -\left[\sum_{j=1}^{N-1} (Q_j \Pi_{j+1} + \Pi_{j+1}^\dagger Q_j^\dagger) + \sum_{j=1}^{N} (\Pi_j Q_j + Q_j^\dagger \Pi_j^\dagger) + \frac{2N + 1}{\sqrt{3}}\right]\] (6.9)

Like in the Ising case, we take \(L = 2N\), where \(L\) is the number of sites in the XXZ chain. We use again the recurrence relations (3.6) and notice that we can express the \(g_{l,m}^\pm\) in terms of parafermions. With \(l < m\), we have:

\[g_{2l,2m}^\pm = \frac{1}{\sqrt{3}} (\omega Q_l Q_m^\dagger + \omega^2 Q_m Q_l^\dagger) \] (6.10)

\[g_{2l,2m-1}^\pm = \frac{1}{\sqrt{3}} (Q_l \Pi_m + \Pi_m^\dagger Q_l^\dagger) \]

\[g_{2l-1,2m}^\pm = \frac{1}{\sqrt{3}} (Q_l^\dagger Q_m + Q_m^\dagger \Pi_l^\dagger) \]

\[g_{2l-1,2m-1}^\pm = \frac{1}{\sqrt{3}} (\omega Q_l \Pi_m^\dagger + \omega^2 Q_m \Pi_l^\dagger) \]

Notice that the \(g_{l,m}^\pm\) are hermitean. We also obtain

\[g_{l,m}^- = (g_{l,m}^+)\ast. \] (6.11)

Since the Hamiltonian and the ground state are invariant under conjugation and so is the ground state, the correlation functions one obtains from \(g_{l,m}^+\) and \(g_{l,m}^-\) are the same

\[\langle g_{l,m}^+ \rangle = \langle g_{l,m}^- \rangle \] (6.12)

as one can check from the examples computed in Appendix A (see Eqs. (A.19)-(A.21)).
7 Conclusions

We think that we have clarified some properties of the $U_q[SU(2)]$ invariant two-point correlation functions for $q^p = \pm 1$. From the examples we considered we have seen that although the objects one has to compute (see Eqs. (2.28) and (2.29)) look hopeless, by taking proper sequences of sites (see Eqs. (4.12)-(4.16)) one can discover new local operators with a clear physical meaning (see Eqs. (4.4)-(4.7) and (6.10)) in terms of which the correlation functions are expressed in a transparent way. The key points are the recurrence and the symmetry relations (3.6) and (2.27) which help formulating the whole problem as a purely algebraic one. The same picture should be valid for the correlation functions for other quantum chains where the underlying algebra is not anymore the Temperley-Lieb one (see Eq. (3.13) which is more general). As we have often done in the text, we would like refer the reader to the Appendix A where a diagrammatic approach to the calculation of correlation functions is given. The full range of applications of this method is not yet known to us. As the reader might have guessed, this paper will have an obvious sequel.

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A A diagrammatic calculation of the correlation functions

Since at this point we have not yet been able to compute the two-point correlation functions $g_{i,m}^{\pm}$ for $q^p = \pm 1$ except in the case $p = 4$, we would like to suggest a diagrammatic approach [25] to the problem which is complementary to the usual Bethe-Ansatz method. This different approach cannot only bring a different insight to the problem of computing correlation functions but, as will be seen at the end of this appendix, give unexpected results.

We start by giving a diagrammatic description to the generators $c_{j,j+1}$ of the Temperley-Lieb algebra (see Eq. (1.7)). All the rules can be understood taking a chain with four sites. The three generators $c_{1,2}$, $c_{2,3}$ and $c_{3,4}$ are represented as:

$$
\begin{align*}
  c_{1,2} &= \begin{array}{c}
  1' \ 2' \ 3' \ 4' \\
  1 \ 2 \ 3 \ 4
  \end{array}, \\
  c_{2,3} &= \begin{array}{c}
  1' \ 2' \ 3' \ 4' \\
  1 \ 2 \ 3 \ 4
  \end{array}, \\
  c_{3,4} &= \begin{array}{c}
  1' \ 2' \ 3' \ 4' \\
  1 \ 2 \ 3 \ 4
  \end{array}.
\end{align*}
$$

(A.1)

The diagrams should always be followed from the top to the bottom. The word $c_{1,2}c_{2,3}$ can be obtained connecting the lower part of $c_{1,2}$ with the upper part of $c_{2,3}$:

$$
\begin{align*}
  c_{1,2}c_{2,3} &= \begin{array}{c}
  1' \ 2' \ 3' \ 4' \\
  1 \ 2 \ 3 \ 4
  \end{array} = \begin{array}{c}
  1' \ 2' \ 3' \ 4' \\
  1 \ 2 \ 3 \ 4
  \end{array}.
\end{align*}
$$

(A.2)
Notice that through multiple applications of $c_{j,j+1}$ one never generates intersecting lines. If we now multiply $c_{1,2}c_{2,3}$ with $c_{1,2}$ (put $c_{1,2}$ to the bottom of (A.2)) we obtain

$$
\begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cup & \cup & \cup & \cup \\
1' & 2' & 3' & 4'
\end{array}
\end{array}
= c_{1,2} c_{2,3} c_{1,2} = c_{1,2} = \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cap & \cap & \cap & \cap \\
1' & 2' & 3' & 4'
\end{array}
\end{array}
$$

(A.3)

in agreement with Eq. (4.3). A new rule appears if one multiplies $c_{1,2}$ with itself:

$$
\begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cup & \cup & \cup & \cup \\
1' & 2' & 3' & 4'
\end{array}
\end{array}
= x \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cap & \cap & \cap & \cap \\
1' & 2' & 3' & 4'
\end{array}
\end{array}
= x c_{1,2} = (q + q^{-1}) c_{1,2}.
$$

(A.4)

Thus a closed blob is taken away from the diagram and is substituted by the number $x = q + q^{-1}$. The diagrammatic calculation described by Eq. (A.4) gives the defining identity (4.3). One can now use the recurrence relation (4.8) to get the diagrammatic expression of the $c_{i,m}^{\pm}$. For example

$$c_{2,4}^{+} = c_{2,3} + c_{3,4} - q c_{2,3} c_{3,4} - q^{-1} c_{3,4} c_{2,3}
$$

has the following diagram

(A.5)

\begin{equation}
\begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cap & \cap & \cap & \cap \\
1' & 2' & 3' & 4'
\end{array}
\end{array}
\end{equation}

(A.6)

We want to do the calculation of the correlation functions purely algebraically. The ground-state of the system is a $U_q[SU(2)]$ scalar. It has been shown [25] that the linear combination of words corresponding to the scalars are of a special type. The lower part is disconnected from the upper part and the lower part is the same for all the words (it can be taken in an arbitrary way). For example there are two scalars appearing in the four sites problem and an eigenword $w$ can be written as

$$w = a_1 \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cap & \cap & \cap & \cap \\
1' & 2' & 3' & 4'
\end{array}
\end{array}
+ a_2 \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cap & \cap & \cap & \cap \\
1' & 2' & 3' & 4'
\end{array}
\end{array}$$

(A.7)

where the constants $a_1$ and $a_2$ have to be determined from the eigenvalue problem (each diagram can be understood as a basis ket vector). Similarly, for the six sites problem (there are five $U_q[SU(2)]$ scalars in this case) the ground-state has the following representation:

$$w = a_1 \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cap & \cap & \cap & \cap \\
1' & 2' & 3' & 4'
\end{array}
\end{array}
+ a_2 \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cap & \cap & \cap & \cap \\
1' & 2' & 3' & 4'
\end{array}
\end{array}
+ a_3 \begin{array}{c}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cap & \cap & \cap & \cap \\
1' & 2' & 3' & 4'
\end{array}
\end{array}$$

(A.8)
In order to find the eigenwords of $H$ given by Eq. (1.1), we solve the equation

$$H w = \lambda w.$$  \hfill (A.9)

This is easily done for the four-site case. Using Eqs. (A.1) and (A.7), one obtains:

$$c_{1,2} w = (a_1 x + a_2),$$
$$c_{2,3} w = (a_1 + a_2 x),$$
$$c_{3,4} w = (a_1 x + a_2)$$  \hfill (A.10)

and instead of Eq. (A.9) we get

$$(2a_1 x + 2a_2 + \lambda a_1) + (a_1 + a_2 x + \lambda a_2) = 0$$  \hfill (A.11)

from which we obtain

$$\lambda^{(I)} = -\frac{3x + \sqrt{x^2 + 8}}{2}, \quad a^{(I)}_1 = \frac{a^{(I)}_2}{2} (x + \sqrt{x^2 + 8})$$  \hfill (A.12)

$$\lambda^{(II)} = -\frac{3x - \sqrt{x^2 + 8}}{2}, \quad a^{(II)}_1 = \frac{a^{(II)}_2}{2} (x - \sqrt{x^2 + 8}).$$

Obviously $\lambda^{(I)}$ corresponds to the ground-state energy. One can check that

$$w^T = a_1 + a_2, \quad w^T H = \lambda w^T.$$  \hfill (A.13)

where $a_1$ and $a_2$ correspond to the two solutions given by Eq. (A.12), are left eigenwords:

$$w^T H = \lambda w^T.$$  \hfill (A.14)

One can also check that the normalization condition

$$w^T w = 1$$  \hfill (A.15)
gives
\[(a_1^2 + a_2^2)x^2 + 2x a_1 a_2 = 1 \tag{A.16}\]
and that one has
\[w^{T(I)} w^{(II)} = w^{T(II)} w^{(I)} = 0. \tag{A.17}\]

We now compute correlation functions taking averages on the word \(w\) given by (A.7) without specifying whether it is the eigenword corresponding to the ground-state or not:
\[w^T c_{l,m} w = \langle c_{l,m} \rangle. \tag{A.18}\]

This calculation can be done using the diagrammatic approach, one obtains:
\[
\begin{align*}
\langle c_{1,2}^\pm \rangle &= \langle c_{3,4}^\pm \rangle = x (a_1 x + a_2)^2 \\
\langle c_{2,3}^\pm \rangle &= \langle c_{1,4}^\pm \rangle = x (a_1 + x a_2)^2 \\
\langle c_{1,3}^\pm \rangle &= \langle c_{2,4}^\pm \rangle = x [a_1^2 + a_2^2 + a_1 a_2 (3x - x^3)] .
\end{align*}
\tag{A.19-21}\]

As the reader has already noticed, the advantage of the diagrammatic approach is that one avoids to work with matrices altogether and that one does not have to use \(q\)-Young symmetrizers \([26]\) in order to construct the \(U_q[SU(2)]\) singlet states.

Let us now return to our results given by Eqs. (A.19)-(A.21) and make a few observations. We first take \(x = 1\) (\(q = e^{i\pi/3}\)). Using the normalization condition (A.16) one discovers that
\[\langle c_{l,m}^\pm \rangle = 1 \tag{A.22}\]
which implies (see Eq. (4.6)) that
\[\langle g_{l,m}^\pm \rangle = 0 . \tag{A.23}\]

This result can be understood in the following way. For \(x = 1\), the Temperley-Lieb algebra has the quotient \([27]\):
\[e_j = 1. \tag{A.24}\]
This makes (using Eq. (3.6)) all the \(c_{l,m}^\pm = 1\) and \(g_{l,m}^\pm = 0\) in the general case (not only for four sites).

Let us next rewrite \(\langle g_{1,3}^\pm \rangle\) using the normalization condition (A.16), we obtain:
\[\langle g_{1,3}^\pm \rangle = a_1 a_2 (x^2 - 1) (2 - x^2) . \tag{A.25}\]

One notices that \(\langle g_{1,3}^\pm \rangle = \langle g_{2,4}^\pm \rangle\) vanishes not only for \(x = 1\) but also for \(x = \sqrt{2} \) (\(q = e^{i\pi/4}\)). This result is not new for the average on the ground state (energy \(\lambda^{(I)}\) in Eq. (A.12)) as was shown by explicit calculation in Eq. (4.12), but it is new for the excited state (energy \(\lambda^{(II)}\) in Eq. (A.12)). Using the diagrammatic approach, one can show that for any number of sites and \(x = \sqrt{2}\) one has
\[\langle g_{2j,2k}^\pm \rangle = \langle g_{2j-1,2k-1}^\pm \rangle = 0 \tag{A.26}\]
for all singlet states. We first sketch the proof through two examples. For \(x = \sqrt{2}\), the Temperley-Lieb algebra has the quotient \([27]\):
\[\sqrt{2} (c_{k,k+1} + c_{k+1,k+2}) - c_{k,k+1} c_{k+1,k+2} - c_{k+1,k+2} c_{k,k+1} = 1 . \tag{A.27}\]
Other quotients corresponding to different values of \( x \) are also described in Ref. [27]. Using Eq. (A.27) we get for example

\[
g_{1,3}^\pm = \mp \frac{i}{\sqrt{2}} c_{1,2} c_{2,3} \pm \frac{i}{\sqrt{2}} c_{2,3} c_{1,2}.
\] (A.28)

Thus \( g_{1,3}^\pm \) has the diagram

\[
g_{1,3}^\pm = \mp \frac{i}{\sqrt{2}} \begin{array}{cc}
\text{G}\text{G}\text{G}
\end{array} \pm \frac{i}{\sqrt{2}} \begin{array}{cc}
\text{G}\text{G}\text{G}
\end{array} .
\] (A.29)

Notice that \( (g_{1,3}^\pm)^T = g_{1,3}^\pm \). The transposition means (see also Eq. (A.14)) taking the reflected (up-down) diagrams. Since \( w \) and \( w^T \) in the calculation of averages (see Eq. (A.18)) also correspond to up-down reflected diagrams, the contributions of the two diagrams in (A.29) cancel. One can go one step further. Quite generally, the recurrence relation (3.6) is equivalent to

\[
g_{l,m}^\pm = -q^\pm_1 g_{l,m}^\pm g_{m,n}^\pm - q^\mp_1 g_{m,n}^\pm g_{l,m}^\pm
\] (A.30)

for \( l < m < n \). In particular, we have

\[
g_{1,5}^\pm = -q^\pm_1 g_{1,3}^\pm g_{3,5}^\pm - q^\mp_1 g_{3,5}^\pm g_{1,3}^\pm
\] (A.31)

The diagram corresponding to \( g_{1,3}^\pm g_{3,5}^\pm \) is

\[
-\frac{1}{2} \left( \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} + \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} \right) + \frac{1}{2} \left( \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} + \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} \right) .
\] (A.32)

Explicit calculation of \( w^T g_{1,3}^\pm g_{3,5}^\pm \) shows that the contributions of the two terms in Eq. (A.32) cancel, and the same is true for \( g_{3,5}^\pm g_{1,3}^\pm \).

In order to give a general proof we start from the preceding example and re-express \( g_{1,3}^\pm g_{3,5}^\pm \) using the Ising quotient relation (A.27) in the form

\[
\begin{array}{ccc}
\text{G}\text{G}\text{G}
\end{array} = \sqrt{2} \left( \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} - \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} \right) - \left( \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} + \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} \right)
\] (A.33)

to eliminate diagrams with three descending lines. We can get, for example,

\[
g_{1,3}^\pm g_{3,5}^\pm = \frac{1}{2} \left[ \sqrt{2} \left( \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} - \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} \right) + \left( \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} - \begin{array}{c}
\text{G}\text{G}\text{G}
\end{array} \right) \right]
\] (A.34)
so that

\[ g_{1,3} g_{3,5} = -(g_{1,3} g_{3,5})^T. \]  

(A.35)

Since \((g_{1,3} g_{3,5})^T = (g_{3,5}^T g_{1,3})^T\) and, for example, \((g_{1,3})^T = -g_{1,3}\) (see Eq. (A.29)), we conclude that

\[ g_{1,3} g_{3,5} + g_{3,5} g_{1,3}^T = 0. \]  

(A.36)

Let us next show that

\[ (g_{l,l+2k})^T = -g_{l,l+2k} \]  

(A.37)

in the Ising case. In fact, from the recurrence relation (A.30) we note that \(g_{l,n}^\pm\) is effectively linear in \(g_{l+2}^\pm\). Consequently, using Eq. (A.36) and the fact that \([g_{i,j}, g_{k,l}] = 0\) for \(i < j < k < l\), we obtain

\[ g_{l-2}^\pm g_{l,n}^\pm + g_{l,n}^\pm g_{l-2}^\pm = 0. \]  

(A.38)

Eq. (A.37) now follows by induction on \(k\), the case \(k = 1\) being settled by Eq. (A.29).

Finally, we are ready to prove Eq. (A.26). Consider any element \(w\) of the Temperley-Lieb algebra of the form \(w = Ye_1 e_3 \ldots e_{2N-1}\), where \(Y\) is an arbitrary element of the algebra (thus \(w\) is an element of the left ideal generated by \(e_1 e_3 \ldots e_{2N-1}\)). If \(X\) is one more element of the Temperley-Lieb algebra, we have

\[ w^T X w = \chi(X) e_1 e_3 \ldots e_{2N-1}, \]  

(A.39)

where \(\chi(X)\) is just a complex number (to see this, draw the corresponding diagram). Since obviously \((e_1 e_3 \ldots e_{2N-1})^T = e_1 e_3 \ldots e_{2N-1}\), this implies that

\[ w^T X w = (w^T X w)^T = w^T X^T w. \]  

(A.40)

Consequently, this expression vanishes whenever \(X^T = -X\), in particular, for \(X = g_{l,l+2k}^\pm\).

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