Abstract

It has proved convenient to define the effective lepton flavor mixing matrix $\tilde{U}$ and neutrino mass-squared differences $\tilde{\Delta}_{ji} \equiv \tilde{m}_j^2 - \tilde{m}_i^2$ (for $i, j = 1, 2, 3$) to describe the phenomena of neutrino mixing and flavor oscillations in a medium, but the prerequisite is to establish direct and transparent relations between these effective quantities and their fundamental counterparts in vacuum. With the help of two sets of sum rules for $\tilde{U}$ and $\tilde{\Delta}_{ji}$, we derive new and exact formulas for moduli of the nine elements of $\tilde{U}$ and the sides of its three Dirac unitarity triangles in the complex plane. The asymptotic behaviors of $|\tilde{U}_{\alpha i}|^2$ and $\tilde{\Delta}_{ji}$ (for $\alpha = e, \mu, \tau$ and $i, j = 1, 2, 3$) in very dense matter (namely, allowing the matter parameter $A = 2\sqrt{2} G_F N_e E$ to mathematically approach infinity) are analytically unraveled for the first time, and in this connection the confusion associated with the parameter redundancy of $\tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23}$ and $\tilde{\delta}$ in the standard parametrization of $\tilde{U}$ is clarified.

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1 Introduction

When a neutrino beam travels in a medium, its electron-flavor component undergoes some forward coherent scattering with the electrons in this medium via the weak charged-current interactions, leading to a nontrivial modification of the behaviors of neutrino oscillations [1, 2, 3]. Such matter effects have played very important roles in solving the long-standing solar neutrino problem and in explaining current atmospheric and long-baseline accelerator neutrino oscillation data [4], and they are even expected to have an appreciable impact on the sensitivity of a medium-baseline JUNO-like reactor antineutrino oscillation experiment [5, 6]. A lot of efforts have been made in the past decades to formulate matter effects on neutrino oscillations, and recently some interest has been shown in going beyond Freund’s analytical approximations [7] to reformulate probabilities of neutrino oscillations with weak or strong terrestrial matter contamination (see, e.g., Refs. [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]), or in describing matter effects on neutrino mixing and CP violation with the help of a language similar to the renormalization-group equations (see, e.g., Refs. [18, 19, 20, 21]).

With the help of two sets of sum rules for the effective neutrino mass-squared differences $\tilde{\Delta}_{ji} \equiv \tilde{m}_j^2 - \tilde{m}_i^2$ (for $i, j = 1, 2, 3$) and the effective Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton flavor matrix $\tilde{U}$ defined in matter, we are going to explore the properties of matter-corrected neutrino mixing and CP violation in the following two aspects.

- We derive new and exact formulas for moduli of the nine elements of $\tilde{U}$ and the sides of its three Dirac unitarity triangles in the complex plane. Different from the previous formulas of this kind [24, 25, 26, 27], our present results are more symmetric and independent of the uneasy terms $\tilde{m}_j^2 - m_i^2$ with $m_i$ and $\tilde{m}_j$ standing respectively for the genuine neutrino masses in vacuum and their effective counterparts in matter (for $i, j = 1, 2, 3$). This improvement makes sense because only $\tilde{\Delta}_{ji}$ are physical for neutrino oscillations in matter.

- We analytically unravel the asymptotic behaviors of $|\tilde{U}_{\alpha i}|^2$ and $\tilde{\Delta}_{ji}$ (for $\alpha = e, \mu, \tau$ and $i, j = 1, 2, 3$) in very dense matter (i.e., when the matter parameter $A = 2\sqrt{2} G_F N_e E$ is considerably large and even allowed to approach infinity). This is the first time that a full and analytical understanding of these matter-corrected quantities in the $A \to \infty$ limit has been achieved purely in terms of the fundamental quantities $|U_{\alpha i}|^2$ and $\Delta_{ji} \equiv m_j^2 - m_i^2$, although their asymptotic behaviors were partly observed in some previous numerical calculations (see, e.g., Refs. [20, 23]).

Of course, the sum rules that we have derived can also be used to calculate the effective Jarlskog invariant of CP violation $\tilde{\mathcal{J}}$ [29] in matter, from which it is straightforward to establish the Naumov relation between $\tilde{\mathcal{J}}$ and its counterpart $\mathcal{J}$ in vacuum [30].

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1 In Ref. [28] Blennow and Ohlsson have discussed an interesting scenario of the effective two-flavor neutrino mixing in the $A \to \infty$ limit and its validity to describe neutrino oscillations in a medium with a large but finite electron number density by taking the standard parametrization of the effective PMNS matrix $\tilde{U}$. Our formulas and main results in the present work are essentially different from theirs.
It is also worth stressing that our analytical results are parametrization-independent, and thus they can be used to clarify the confusion associated with the asymptotic results of \( \tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23} \) and \( \tilde{\delta} \) in the standard parametrization of \( \tilde{U} \). The point is that only one degree of freedom is needed to describe the effective PMNS matrix \( \tilde{U} \) in the \( A \to \infty \) limit, simply because \( |\tilde{U}_{ei}|^2 = 1 \) and \( |\tilde{U}_{ej}|^2 = |\tilde{U}_{ek}|^2 = 0 \) (for \( i \neq j \neq k = 1, 2, 3 \)) hold in this special case. So it is always possible to remove \( \tilde{\delta} \) from \( \tilde{U} \) if the \( A \to \infty \) limit is taken, and then we are left with a trivial flavor mixing angle (e.g., \( \tilde{\theta}_{13} = \pi/2 \)) and a nontrivial flavor mixing angle which is neither \( \tilde{\theta}_{12} \) nor \( \tilde{\theta}_{23} \). This kind of subtle parameter redundancy was not noticed in the previous papers (see, e.g., Refs. \([20, 21]\)), where specific but misleading values of \( \tilde{\delta}, \tilde{\theta}_{12} \) and \( \tilde{\theta}_{23} \) have been obtained in the \( A \to \infty \) limit.

### 2 Exact formulas

In the standard three-flavor scheme, the effective Hamiltonian responsible for a neutrino beam propagating in a medium can be expressed as

\[
H_m' = \frac{1}{2E} \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} V_{cc} + V_{nc} & 0 & 0 \\ 0 & V_{nc} & 0 \\ 0 & 0 & V_{nc} \end{pmatrix} \equiv \frac{1}{2E} \tilde{U} \begin{pmatrix} \tilde{m}_1^2 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 \\ 0 & 0 & \tilde{m}_3^2 \end{pmatrix} \tilde{U}^\dagger, \tag{1}
\]

where \( V_{cc} = \sqrt{2} G_F N_e \) and \( V_{nc} = -G_F N_n/\sqrt{2} \) are the so-called matter potential terms arising respectively from weak charged- and neutral-current interactions of neutrinos with electrons and neutrons in this medium \([1]\). When an antineutrino beam is concerned, the corresponding effective Hamiltonian in matter can directly be read off from Eq. (1) with the replacements \( U \to U^*, V_{cc} \to -V_{cc} \) and \( V_{nc} \to -V_{nc} \). Because neutrino (or antineutrino) oscillations depend only on the neutrino mass-squared differences, it is more convenient to rewrite Eq. (1) in the following way:

\[
H_m' = \frac{1}{2E} \left[ U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_{21} & 0 \\ 0 & 0 & \Delta_{31} \end{pmatrix} U^\dagger + \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \equiv \frac{1}{2E} \left[ \tilde{U} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{\Delta}_{21} & 0 \\ 0 & 0 & \tilde{\Delta}_{31} \end{pmatrix} \tilde{U}^\dagger + BI \right], \tag{2}
\]

where \( A = 2EV_{cc} \) and \( B = \tilde{m}_1^2 - m_1^2 - 2EV_{nc} \), and \( I \) denotes the identity matrix. Given the analytical expressions of \( \tilde{\Delta}_i^2 \) (for \( i = 1, 2, 3 \)) which have been derived in Refs. \([21, 31, 32]\), it is straightforward for us to obtain

\[
\tilde{\Delta}_{21} = \frac{2}{3} \sqrt{x^2 - 3y} \sqrt{3(1-z^2)} ,
\]

\[
\tilde{\Delta}_{31} = \frac{1}{3} \sqrt{x^2 - 3y} \left[ 3z + \sqrt{3 (1-z^2)} \right] ,
\]

\[
B = \frac{1}{3} x - \frac{1}{3} \sqrt{x^2 - 3y} \left[ z + \sqrt{3 (1-z^2)} \right] \tag{3}
\]
if three neutrinos have a normal mass ordering (NMO) with \( m_1 < m_2 < m_3 \) or \( \Delta_{31} > 0 \); or

\[
\tilde{\Delta}_{21} = \frac{1}{3} \sqrt{x^2 - 3y} \left[ 3z - \sqrt{3} (1 - z^2) \right],
\]
\[
\tilde{\Delta}_{31} = -\frac{2}{3} \sqrt{x^2 - 3y} \sqrt{3} (1 - z^2),
\]
\[
B = \frac{1}{3} x - \frac{1}{3} \sqrt{x^2 - 3y} \left[ z - \sqrt{3} (1 - z^2) \right],
\]

(4)

if three neutrinos have an inverted mass ordering (IMO) with \( m_3 < m_1 < m_2 \) or \( \Delta_{31} < 0 \), where \( x, y \) and \( z \) are given by

\[
x = \Delta_{21} + \Delta_{31} + A,
\]
\[
y = \Delta_{21} \Delta_{31} + A \left[ \Delta_{21} (1 - |U_{e2}|^2) + \Delta_{31} (1 - |U_{e3}|^2) \right],
\]
\[
z = \cos \left[ \frac{1}{3} \arccos \frac{2x^3 - 9xy + 27A\Delta_{21}\Delta_{31}|U_{e1}|^2}{2\sqrt{(x^2 - 3y)^3}} \right].
\]

(5)

Taking the trace of \( \mathcal{H}_m' \) in Eq. (2), we immediately arrive at

\[
B = \frac{1}{3} \left( \Delta_{21} + \Delta_{31} + A - \tilde{\Delta}_{21} - \tilde{\Delta}_{31} \right).
\]

(6)

Note again that Eqs. (2)—(6) are only valid for neutrino mixing and flavor oscillations in matter. When an antineutrino beam travelling in a medium is taken into account, one should make the replacements \( U \to U^* \) and \( A \to -A \) for Eqs. (2)—(6).

Eq. (2) allows us to obtain the following sum rules in an easy way:

\[
\sum_{i=1}^{3} \tilde{U}_{\alpha i} \tilde{U}^*_{\beta i} \tilde{\Delta}_{i1} = \sum_{i=1}^{3} U_{\alpha i} U^*_{\beta i} \Delta_{i1} + A\delta_{\alpha e} \delta_{\beta e} - B\delta_{\alpha \beta},
\]

(7)

where the Greek and Latin subscripts run over \((e, \mu, \tau)\) and \((1, 2, 3)\), respectively. On the other hand, a direct calculation of \( \mathcal{H}_m'^2 \) leads us to another set of sum rules:

\[
\sum_{i=1}^{3} \tilde{U}_{\alpha i} \tilde{U}^*_{\beta i} \tilde{\Delta}_{i1} (\tilde{\Delta}_{i1} + 2B) = \sum_{i=1}^{3} U_{\alpha i} U^*_{\beta i} \Delta_{i1} \left[ \Delta_{i1} + A (\delta_{e\alpha} + \delta_{e\beta}) \right] + A^2 \delta_{e\alpha} \delta_{e\beta} - B^2 \delta_{\alpha \beta}.
\]

(8)

Eqs. (7) and (8), together with the unitarity conditions of \( U \) and \( \tilde{U} \),

\[
\sum_{i=1}^{3} \tilde{U}_{\alpha i} \tilde{U}^*_{\beta i} = \sum_{i=1}^{3} U_{\alpha i} U^*_{\beta i} = \delta_{\alpha \beta},
\]

(9)

constitute a full set of linear equations of three unknown variables \( \tilde{U}_{\alpha 1} \tilde{U}^*_{\beta 1}, \tilde{U}_{\alpha 2} \tilde{U}^*_{\beta 2} \) and \( \tilde{U}_{\alpha 3} \tilde{U}^*_{\beta 3} \) for two given flavors \( \alpha \) and \( \beta \). One may therefore solve these equations and then express \( \tilde{U}_{\alpha i} \tilde{U}^*_{\beta i} \) in terms of \( U_{\alpha i} U^*_{\beta i}, \Delta_{ji}, \tilde{\Delta}_{ji}, A \) and \( B \).
2.1 Moduli of the matrix elements $\tilde{U}_{\alpha i}$

Taking $\alpha = \beta$, we obtain a full set of linear equations of $|\tilde{U}_{\alpha i}|^2$ from Eqs. (7)—(9) as follows:

$$
|\tilde{U}_{\alpha 1}|^2 + |\tilde{U}_{\alpha 2}|^2 + |\tilde{U}_{\alpha 3}|^2 = 1 ,
\tilde{\Delta}_{21}|\tilde{U}_{\alpha 2}|^2 + \tilde{\Delta}_{31}|\tilde{U}_{\alpha 3}|^2 = \xi ,
\tilde{\Delta}_{21}(\tilde{\Delta}_{21} + 2B)|\tilde{U}_{\alpha 2}|^2 + \tilde{\Delta}_{31}(\tilde{\Delta}_{31} + 2B)|\tilde{U}_{\alpha 3}|^2 = \zeta ,
$$

(10)

where

$$
\xi = \Delta_{21}|U_{\alpha 2}|^2 + \Delta_{31}|U_{\alpha 3}|^2 + A\delta_{\alpha} - B ,
\zeta = \Delta_{21}(\Delta_{21} + 2A\delta_{\alpha})|U_{\alpha 2}|^2 + \Delta_{31}(\Delta_{31} + 2A\delta_{\alpha})|U_{\alpha 3}|^2 + A^2\delta_{\alpha} - B^2 .
$$

(11)

The solutions of Eq. (10) turn out to be

$$
|\tilde{U}_{\alpha 1}|^2 = \frac{\zeta - 2\xi B - \xi \tilde{\Delta}_{21} - \xi \tilde{\Delta}_{31} + \tilde{\Delta}_{21}\tilde{\Delta}_{31}}{\Delta_{21}\Delta_{31}} ,
|\tilde{U}_{\alpha 2}|^2 = \frac{\xi \tilde{\Delta}_{31} + 2\xi B - \zeta}{\Delta_{21}\Delta_{32}} ,
|\tilde{U}_{\alpha 3}|^2 = \frac{\zeta - 2\xi B - \xi \tilde{\Delta}_{21}}{\Delta_{31}\Delta_{32}} ,
$$

(12)

where $\alpha = e, \mu, \tau$. With the help of Eqs. (6), (11) and (12), nine $|\tilde{U}_{\alpha i}|^2$ can be explicitly expressed as

$$
|\tilde{U}_{e 1}|^2 = \frac{1}{9} \left[ \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{31}} |U_{e 1}|^2 , \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{21} - \Delta_{32} - A}{\tilde{\Delta}_{21}} |U_{e 2}|^2 
+ \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{31}} |U_{e 3}|^2 \right] ,
$$

(13a)

$$
|\tilde{U}_{e 2}|^2 = \frac{1}{9} \left[ \frac{\tilde{\Delta}_{32} - \tilde{\Delta}_{21} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{32}} |U_{e 1}|^2 , \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{32} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{21}} |U_{e 1}|^2 
+ \frac{\tilde{\Delta}_{32} - \tilde{\Delta}_{21} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{32}} |U_{e 2}|^2 
+ \frac{\tilde{\Delta}_{32} - \tilde{\Delta}_{21} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{32}} |U_{e 3}|^2 \right] ,
$$

(13b)

$$
|\tilde{U}_{e 3}|^2 = \frac{1}{9} \left[ \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} - \Delta_{31} - \Delta_{32} + A}{\tilde{\Delta}_{31}} |U_{e 1}|^2 , \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{32}} |U_{e 1}|^2 
+ \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} - \Delta_{31} - \Delta_{32} + A}{\tilde{\Delta}_{31}} |U_{e 2}|^2 
+ \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{21} + \Delta_{31} + A}{\tilde{\Delta}_{31}} |U_{e 3}|^2 \right] ;
$$

(13c)
and

\[ |\tilde{U}_\mu|^2 = \frac{1}{9} \left[ \frac{\tilde{\Delta}_{21} - \tilde{\Delta}_{32} + \Delta_{21} + \Delta_{31} + A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{21} + \Delta_{31} + A}{\tilde{\Delta}_{31}} |U_\mu|^2 + \frac{\tilde{\Delta}_{21} - \tilde{\Delta}_{32} + \Delta_{32} - \Delta_{21} + A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{32} - \Delta_{21} + A}{\tilde{\Delta}_{31}} |U_\mu|^2 + \frac{\tilde{\Delta}_{21} - \tilde{\Delta}_{32} - \Delta_{31} - \Delta_{32} + A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} - \Delta_{31} - \Delta_{32} + A}{\tilde{\Delta}_{31}} |U_\mu|^2 \right], \quad (14a) \]

\[ |\tilde{U}_\mu|^2 = \frac{1}{9} \left[ \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} - \Delta_{21} - \Delta_{31} + A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{21} + \Delta_{31} + A}{\tilde{\Delta}_{31}} |U_\mu|^2 + \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{21} - \Delta_{32} - A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{21} - \Delta_{32} + A}{\tilde{\Delta}_{31}} |U_\mu|^2 + \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} - \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{31}} |U_\mu|^2 \right], \quad (14b) \]

\[ |\tilde{U}_\mu|^2 = \frac{1}{9} \left[ \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} - \Delta_{21} - \Delta_{31} + A}{\tilde{\Delta}_{31}} \cdot \frac{\tilde{\Delta}_{32} - \tilde{\Delta}_{21} - \Delta_{21} - \Delta_{31} + A}{\tilde{\Delta}_{32}} |U_\mu|^2 + \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{21} - \Delta_{32} - A}{\tilde{\Delta}_{31}} \cdot \frac{\tilde{\Delta}_{32} - \tilde{\Delta}_{21} + \Delta_{21} - \Delta_{32} + A}{\tilde{\Delta}_{32}} |U_\mu|^2 + \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{31}} \cdot \frac{\tilde{\Delta}_{32} - \tilde{\Delta}_{21} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{32}} |U_\mu|^2 \right]; \quad (14c) \]

as well as

\[ |\tilde{U}_\tau| = \frac{1}{9} \left[ \frac{\tilde{\Delta}_{21} - \tilde{\Delta}_{32} + \Delta_{21} + \Delta_{31} + A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{21} + \Delta_{31} + A}{\tilde{\Delta}_{31}} |U_\tau|^2 + \frac{\tilde{\Delta}_{21} - \tilde{\Delta}_{32} + \Delta_{32} - \Delta_{21} + A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{32} - \Delta_{21} + A}{\tilde{\Delta}_{31}} |U_\tau|^2 + \frac{\tilde{\Delta}_{21} - \tilde{\Delta}_{32} - \Delta_{31} - \Delta_{32} + A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} - \Delta_{31} - \Delta_{32} + A}{\tilde{\Delta}_{31}} |U_\tau|^2 \right], \quad (15a) \]

\[ |\tilde{U}_\tau| = \frac{1}{9} \left[ \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} - \Delta_{21} - \Delta_{31} + A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{21} + \Delta_{31} + A}{\tilde{\Delta}_{31}} |U_\tau|^2 + \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{21} - \Delta_{32} - A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{21} - \Delta_{32} + A}{\tilde{\Delta}_{31}} |U_\tau|^2 + \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{21}} \cdot \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} - \Delta_{31} + \Delta_{32} - A}{\tilde{\Delta}_{31}} |U_\tau|^2 \right], \quad (15b) \]
\[ |\tilde{U}_{\tau 3}|^2 = \frac{1}{9} \left[ \frac{\Delta_{21} + \Delta_{31} - \Delta_{21} - \Delta_{31} - A}{\Delta_{31}} \cdot \frac{\Delta_{32} - \Delta_{21} - \Delta_{31} - A}{\Delta_{32}} \right] |U_{\tau 1}|^2 \\
+ \frac{\Delta_{21} + \Delta_{31} + \Delta_{21} - \Delta_{32} - A}{\Delta_{31}} \cdot \frac{\Delta_{32} - \Delta_{21} - \Delta_{32} + \Delta_{21} - A}{\Delta_{32}} |U_{\tau 2}|^2 \\
+ \frac{\Delta_{21} + \Delta_{31} + \Delta_{31} + \Delta_{32} - A}{\Delta_{31}} \cdot \frac{\Delta_{32} - \Delta_{21} + \Delta_{31} + \Delta_{32} - A}{\Delta_{32}} |U_{\tau 3}|^2 \right] . \quad (15c)\]

Since the expressions of \( \tilde{\Delta}_{ji} \) have been given in Eq. (3) for the NMO case and in Eq. (4) for the IMO case, it is straightforward to calculate \( |\tilde{U}_{\alpha i}|^2 \) by taking a specific value of the matter parameter \( A \) and inputting the experimental values of two neutrino mass-squared differences and four flavor mixing parameters in vacuum.

From a phenomenological point of view, we emphasize that the analytical results of \( |\tilde{U}_{\alpha i}|^2 \) obtained above are more advantageous than the previous ones obtained in Refs. [26, 27] in two aspects. First, the present results are more symmetric and transparent in reflecting the relations between \( |\tilde{U}_{\alpha i}|^2 \) and \( |U_{\alpha i}|^2 \). Second, the present expressions of \( |\tilde{U}_{\alpha i}|^2 \) are free from the uneasy terms \( \tilde{m}_j^2 - m_i^2 \) in which the effective neutrino masses \( \tilde{m}_j^2 \) do not have a definite physical meaning. In fact, only \( \tilde{\Delta}_{ji} \) are physical for neutrino oscillations in matter.

Once again, one should keep in mind that the above results are only valid for a neutrino beam travelling in matter. It is necessary to make the replacements \( U \to U^* \) and \( A \to -A \) when an antineutrino beam is taken into account.

### 2.2 Sides of the Dirac unitarity triangles

As in vacuum, the orthogonality conditions of \( \tilde{U} \) given in Eq. (9) can define three distinct unitarity triangles in the complex plane — the so-called effective Dirac unitarity triangles in matter [11, 33]

\[
\begin{align*}
\tilde{\Delta}_e : \ & \tilde{U}_{\mu 1} \tilde{U}_{\tau 1}^* + \tilde{U}_{\mu 2} \tilde{U}_{\tau 2}^* + \tilde{U}_{\mu 3} \tilde{U}_{\tau 3}^* = 0 , \\
\tilde{\Delta}_\mu : \ & \tilde{U}_{\tau 1} \tilde{U}_{e 1}^* + \tilde{U}_{\tau 2} \tilde{U}_{e 2}^* + \tilde{U}_{\tau 3} \tilde{U}_{e 3}^* = 0 , \\
\tilde{\Delta}_\tau : \ & \tilde{U}_{e 1} \tilde{U}_{\mu 1}^* + \tilde{U}_{e 2} \tilde{U}_{\mu 2}^* + \tilde{U}_{e 3} \tilde{U}_{\mu 3}^* = 0 ,
\end{align*}
\]

which are insensitive to a redefinition of the phases of three neutrino fields and thus have nothing to do with the Majorana phases of the PMNS matrix \( U \). Each of these three triangle is named after the flavor index that does not show up in its three sides. The areas of \( \tilde{\Delta}_e \), \( \tilde{\Delta}_\mu \) and \( \tilde{\Delta}_\tau \) are all equal to half of the magnitude of the effective Jarlskog invariant of CP violation in matter, denoted by \( \tilde{J} \). The latter, together with its fundamental counterpart \( J \) in vacuum [29], is defined as

\[
\begin{align*}
\text{Im}(U_{\alpha i} U_{\beta j} U_{\alpha j}^* U_{\beta i}^*) = J \sum_{\gamma} \varepsilon_{\alpha \beta \gamma} \sum_{k} \varepsilon_{ijk} , \\
\text{Im}(\tilde{U}_{\alpha i} \tilde{U}_{\beta j} \tilde{U}_{\alpha j}^* \tilde{U}_{\beta i}^*) = \tilde{J} \sum_{\gamma} \varepsilon_{\alpha \beta \gamma} \sum_{k} \varepsilon_{ijk} .
\end{align*}
\]

\begin{align} \text{Im}(U_{\alpha i} U_{\beta j} U_{\alpha j}^* U_{\beta i}^*) &= J \sum_{\gamma} \varepsilon_{\alpha \beta \gamma} \sum_{k} \varepsilon_{ijk} , \\
\text{Im}(\tilde{U}_{\alpha i} \tilde{U}_{\beta j} \tilde{U}_{\alpha j}^* \tilde{U}_{\beta i}^*) &= \tilde{J} \sum_{\gamma} \varepsilon_{\alpha \beta \gamma} \sum_{k} \varepsilon_{ijk} . \end{align}
where \( \varepsilon_{\alpha\beta\gamma} \) and \( \varepsilon_{ijk} \) are the three-dimension Levi-Civita symbols.

Taking \( \alpha \neq \beta \), Eqs. (7)–(9) can now be simplified to the following set of linear equations of three variables \( \tilde{U}_{\alpha i} \tilde{U}_{\beta i}^* \) (for \( i = 1, 2, 3 \)):

\[
\begin{align*}
\tilde{U}_{\alpha 1} \tilde{U}_{\beta 1}^* + \tilde{U}_{\alpha 2} \tilde{U}_{\beta 2}^* + \tilde{U}_{\alpha 3} \tilde{U}_{\beta 3}^* &= 0, \\
\tilde{\Delta}_{21} \tilde{U}_{\alpha 2} \tilde{U}_{\beta 2}^* + \tilde{\Delta}_{31} \tilde{U}_{\alpha 3} \tilde{U}_{\beta 3}^* &= \xi', \\
\tilde{\Delta}_{21} (\tilde{\Delta}_{21} + 2B) \tilde{U}_{\alpha 2} \tilde{U}_{\beta 2}^* + \tilde{\Delta}_{31} (\tilde{\Delta}_{31} + 2B) \tilde{U}_{\alpha 3} \tilde{U}_{\beta 3}^* &= \zeta',
\end{align*}
\]

where

\[
\begin{align*}
\xi' &= \Delta_{21} U_{\alpha 2} U_{\beta 2}^* + \Delta_{31} U_{\alpha 3} U_{\beta 3}^*, \\
\zeta' &= \Delta_{21} \left( \Delta_{21} + A(\delta_{\epsilon \alpha} + \delta_{\epsilon \beta}) \right) U_{\alpha 2} U_{\beta 2}^* + \Delta_{31} \left( \Delta_{31} + A(\delta_{\epsilon \alpha} + \delta_{\epsilon \beta}) \right) U_{\alpha 3} U_{\beta 3}^*. 
\end{align*}
\]

Solving Eq. (18) in a straightforward way, we are left with the solutions

\[
\begin{align*}
\tilde{U}_{\alpha 1} \tilde{U}_{\beta 1}^* &= \frac{\zeta' - 2\xi' B - \xi' \tilde{\Delta}_{21} - \xi' \tilde{\Delta}_{31}}{\tilde{\Delta}_{21} \tilde{\Delta}_{31}}, \\
\tilde{U}_{\alpha 2} \tilde{U}_{\beta 2}^* &= \frac{\xi' \tilde{\Delta}_{31} + 2\xi' B - \xi'}{\tilde{\Delta}_{21} \tilde{\Delta}_{32}}, \\
\tilde{U}_{\alpha 3} \tilde{U}_{\beta 3}^* &= \frac{\zeta' - 2\xi' B - \xi' \tilde{\Delta}_{21}}{\tilde{\Delta}_{31} \tilde{\Delta}_{32}}.
\end{align*}
\]

After Eq. (6) is taken into account, the explicit expressions of nine \( \tilde{U}_{\alpha i} \tilde{U}_{\beta i}^* \) can be obtained from Eqs. (19) and (20). Namely,

\[
\begin{align*}
\tilde{U}_{\mu 1} \tilde{U}_{\tau 1}^* &= \frac{1}{3} \left[ \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{21} - \Delta_{32} + 2A}{\Delta_{21}} \cdot \frac{\Delta_{31}}{\Delta_{31}} U_{\mu 1} U_{\tau 1}^* \\
&\quad + \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} - \Delta_{21} - \Delta_{31} + 2A}{\Delta_{21}} \cdot \frac{\Delta_{32}}{\Delta_{31}} U_{\mu 2} U_{\tau 2}^* \right], \\
\tilde{U}_{\mu 2} \tilde{U}_{\tau 2}^* &= \frac{1}{3} \left[ \frac{\tilde{\Delta}_{32} - \tilde{\Delta}_{21} + \Delta_{31} + \Delta_{32} + 2A}{\Delta_{32}} \cdot \frac{\Delta_{21}}{\Delta_{21}} U_{\mu 2} U_{\tau 2}^* \\
&\quad + \frac{\tilde{\Delta}_{32} - \tilde{\Delta}_{21} + \Delta_{21} - \Delta_{32} + 2A}{\Delta_{32}} \cdot \frac{\Delta_{31}}{\Delta_{21}} U_{\mu 3} U_{\tau 3}^* \right], \\
\tilde{U}_{\mu 3} \tilde{U}_{\tau 3}^* &= \frac{1}{3} \left[ \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} + \Delta_{31} + \Delta_{21} - 2A}{\Delta_{31}} \cdot \frac{\Delta_{32}}{\Delta_{31}} U_{\mu 3} U_{\tau 3}^* \\
&\quad - \frac{\tilde{\Delta}_{31} + \tilde{\Delta}_{32} - \Delta_{31} - \Delta_{32} - 2A}{\Delta_{31}} \cdot \frac{\Delta_{21}}{\Delta_{32}} U_{\mu 1} U_{\tau 1}^* \right],
\end{align*}
\]

and

\[
\begin{align*}
\tilde{U}_{\tau 1} \tilde{U}_{e 1}^* &= \frac{1}{3} \left[ \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} + \Delta_{21} - \Delta_{32} - A}{\Delta_{21}} \cdot \frac{\Delta_{31}}{\Delta_{31}} U_{\tau 1} U_{e 1}^* \\
&\quad + \frac{\tilde{\Delta}_{21} + \tilde{\Delta}_{31} - \Delta_{21} - \Delta_{31} - A}{\Delta_{21}} \cdot \frac{\Delta_{32}}{\Delta_{31}} U_{\tau 2} U_{e 2}^* \right].
\end{align*}
\]
\( \tilde{U}_{r_2} \tilde{U}_{e_2}^* = \frac{1}{3} \left[ \frac{\Delta_{32} - \Delta_{21} + \Delta_{31} + \Delta_{22} - A}{\Delta_{32}} \frac{\Delta_{21}}{\Delta_{31}} U_{r_2} U_{e_2}^* \right. \)

\[ + \left. \frac{\Delta_{32} - \Delta_{21} + \Delta_{21} - \Delta_{32} - A}{\Delta_{32}} \frac{\Delta_{31}}{\Delta_{31}} U_{r_3} U_{e_3}^* \right], \quad (22b) \]

\( \tilde{U}_{r_3} \tilde{U}_{e_3}^* = \frac{1}{3} \left[ \frac{\Delta_{31} + \Delta_{32} + \Delta_{31} + \Delta_{32} + A}{\Delta_{31}} \frac{\Delta_{21}}{\Delta_{32}} U_{r_3} U_{e_3}^* \right. \)

\[ - \left. \frac{\Delta_{31} + \Delta_{32} - \Delta_{31} - \Delta_{32} + A}{\Delta_{31}} \frac{\Delta_{21}}{\Delta_{32}} U_{r_1} U_{e_1}^* \right], \quad (22c) \]

as well as

\( \tilde{U}_{e_1} \tilde{U}_{\mu_1}^* = \frac{1}{3} \left[ \frac{\Delta_{21} + \Delta_{31} + \Delta_{21} - A}{\Delta_{21}} \frac{\Delta_{31}}{\Delta_{21}} U_{e_1} U_{\mu_1}^* \right. \)

\[ + \left. \frac{\Delta_{21} + \Delta_{31} - \Delta_{21} - \Delta_{31} - A}{\Delta_{21}} \frac{\Delta_{31}}{\Delta_{21}} U_{e_2} U_{\mu_2}^* \right], \quad (23a) \]

\( \tilde{U}_{e_2} \tilde{U}_{\mu_2}^* = \frac{1}{3} \left[ \frac{\Delta_{31} + \Delta_{32} + \Delta_{21} + \Delta_{21} - A}{\Delta_{32}} \frac{\Delta_{21}}{\Delta_{32}} U_{e_2} U_{\mu_2}^* \right. \)

\[ + \left. \frac{\Delta_{32} - \Delta_{21} + \Delta_{21} - \Delta_{32} - A}{\Delta_{32}} \frac{\Delta_{31}}{\Delta_{32}} U_{e_3} U_{\mu_3}^* \right], \quad (23b) \]

\( \tilde{U}_{e_3} \tilde{U}_{\mu_3}^* = \frac{1}{3} \left[ \frac{\Delta_{31} + \Delta_{32} + \Delta_{31} + \Delta_{21} + A}{\Delta_{31}} \frac{\Delta_{21}}{\Delta_{31}} U_{e_3} U_{\mu_3}^* \right. \)

\[ - \left. \frac{\Delta_{31} + \Delta_{32} - \Delta_{31} - \Delta_{32} + A}{\Delta_{31}} \frac{\Delta_{21}}{\Delta_{32}} U_{e_1} U_{\mu_1}^* \right], \quad (23c) \]

where \( \tilde{\Delta}_{ji} \) have been given in Eq. (3) for the NMO case and in Eq. (4) for the IMO case. It is obvious that the shapes of three Dirac unitarity triangles in vacuum will be deformed by matter effects, and this implies the change of \( \tilde{J} \) as compared with \( J \).

With the help of Eqs. (21)—(23), one may calculate \( \tilde{J} \) by using any two sides of the Dirac unitarity triangle \( \tilde{\Delta}_\alpha \) (for \( \alpha = e, \mu, \tau \)). For example,

\[ \tilde{J} = \text{Im} \left[ \left( \tilde{U}_{\mu_2} \tilde{U}_{\tau_2}^* \right) \left( \tilde{U}_{\mu_3} \tilde{U}_{\tau_3}^* \right)^* \right] \equiv \frac{\Delta_{21} \Delta_{31}}{\Delta_{21} \Delta_{31} \Delta_{32}} \text{Im} \left[ \left( \Delta_{31} - \Delta_{21} \right) \left( \Delta_{31} - \Delta_{21} \right) U_{\mu_2} U_{\tau_3} U_{\mu_3} U_{\tau_3} \right. \]

\[ + \left. \left( \Delta_{31} - \Delta_{31} \right) \left( \Delta_{21} - \Delta_{21} \right) U_{\mu_2} U_{\tau_3} U_{\mu_3} U_{\tau_3} \right] = \Delta_{21} \frac{\Delta_{31} \Delta_{32}}{\Delta_{21} \Delta_{31} \Delta_{32}} \mathcal{J}, \quad (24) \]

where \( \mathcal{J} = \text{Im} \left( U_{\mu_2} U_{\tau_3} U_{\mu_3} U_{\tau_3} \right) \) has been used. One can see that Eq. (24) is just the well-known Naumov relation between \( \tilde{J} \) and \( J \) [30].
The above parametrization-independent expressions of $|\tilde{U}_{\alpha i}|^2$ and $\tilde{U}_{\alpha i}^\dagger \tilde{U}_{\beta i}$ can easily be used to derive the effective neutrino mixing angles ($\tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23}$) and the effective CP-violating phase ($\tilde{\delta}$) in the standard parametrization of $\tilde{U}$,

$$
\tilde{U} = \begin{pmatrix}
\tilde{c}_{12}\tilde{c}_{13} & \tilde{s}_{12}\tilde{c}_{13} & \tilde{s}_{13} e^{-i\tilde{\delta}} \\
-\tilde{s}_{12}\tilde{c}_{23} - \tilde{c}_{12}\tilde{s}_{13}\tilde{s}_{23} e^{i\tilde{\delta}} & \tilde{c}_{12}\tilde{c}_{23} - \tilde{s}_{12}\tilde{s}_{13}\tilde{s}_{23} e^{i\tilde{\delta}} & \tilde{c}_{13}\tilde{c}_{23}
\end{pmatrix},
$$

(25)

in which $\tilde{c}_{ij} \equiv \cos \tilde{\theta}_{ij}$ and $\tilde{s}_{ij} \equiv \sin \tilde{\theta}_{ij}$ with $\tilde{\theta}_{ij}$ lying in the first quadrant (for $ij = 12, 13, 23$), and $\tilde{\delta}$ is allowed to vary between 0 and $2\pi$. For instance,

$$
\tan \tilde{\theta}_{12} = \frac{|\tilde{U}_{e2}|}{|\tilde{U}_{e1}|}, \quad \sin \tilde{\theta}_{13} = |\tilde{U}_{e3}|, \quad \tan \tilde{\theta}_{23} = \frac{|\tilde{U}_{\mu 3}|}{|\tilde{U}_{\tau 3}|},
$$

(26)

where the moduli of the effective PMNS matrix elements have been given in Eqs. (13)—(15), and then $\sin \tilde{\delta}$ can be obtained from the Toshev relation $\sin 2\tilde{\theta}_{23} \sin \tilde{\delta} = \sin 2\theta_{23} \sin \delta$.

### 3 Asymptotic behaviors

Now we apply the exact formulas of $|\tilde{U}_{\alpha i}|^2$ to an extreme case, in which the matter density is considerably large or equivalent to $A \to \infty$, to examine the asymptotic behaviors of $|\tilde{U}_{\alpha i}|^2$. Although the behaviors of $|\tilde{U}_{\alpha i}|^2$ changing with the matter parameter $A$ have been numerically illustrated in the literature (see, e.g., Ref. [20]), a comprehensive and analytical understanding of their asymptotic properties in the $A \to \infty$ limit has been lacking. On the other hand, it has been shown that in the standard parametrization of $\tilde{U}$ the effective CP-violating phase $\tilde{\delta}$ approaches a finite value even if $\tilde{\theta}_{13} \to \pi/2$ in the $A \to \infty$ limit (see, e.g., Refs. [20, 21]). This result is confusing because $\tilde{\delta}$ can be rotated away when $|\tilde{U}_{e1}|^2 = |\tilde{U}_{e2}|^2 = |\tilde{U}_{\mu 3}|^2 = |\tilde{U}_{\tau 3}|^2 = 0$ holds as a result of $\cos \tilde{\theta}_{13} \to 0$. We are going to clarify this parameter redundancy by demonstrating that $\tilde{U}$ only contains one degree of freedom when $A$ approaches infinity.

#### 3.1 Case A: (NMO, $\nu$)

Let us first consider the case of a neutrino beam ($\nu$) propagating in matter with a normal mass ordering (NMO). When taking $A \to \infty$, we can simplify Eq. (3) and arrive at

$$
\tilde{\Delta}_{21} = \sqrt{p^2 - 4q},
$$

$$
\tilde{\Delta}_{31} = \Delta_{21} + \Delta_{31} + A - \frac{1}{2} \left( 3p - \sqrt{p^2 - 4q} \right),
$$

$$
B = \frac{1}{2} \left( p - \sqrt{p^2 - 4q} \right),
$$

(27)

where $p = \Delta_{21} (1 - |U_{e2}|^2) + \Delta_{31} (1 - |U_{e3}|^2)$ and $q = \Delta_{21} \Delta_{31} |U_{e1}|^2$. In a good approximation, we find that $\tilde{\Delta}_{21} \approx \Delta_{31} (1 - |U_{e3}|^2) - \Delta_{21} |U_{e1}|^2$ is finite and $\tilde{\Delta}_{31} \approx \Delta_{32} \approx A$ approaches infinity in the $A \to \infty$ limit.
With the help of Eq. (27), one may use Eqs. (13)—(15) to calculate the nine elements of $\tilde{U}$ in the $A \to \infty$ limit. The results are

$$|	ilde{U}_{e_1}|^2 = |	ilde{U}_{e_2}|^2 = |	ilde{U}_{\mu_3}|^2 = |	ilde{U}_{\tau_3}|^2 = 0, \quad |	ilde{U}_{e_3}|^2 = 1,$$

$$|	ilde{U}_{\mu_1}|^2 = |	ilde{U}_{\tau_2}|^2 = \frac{1}{2} + \frac{\Delta_{21} (|U_{\tau_2}|^2 - |U_{\mu_2}|^2) + \Delta_{31} (|U_{\tau_3}|^2 - |U_{\mu_3}|^2)}{2\sqrt{p^2 - 4q}},$$

$$|	ilde{U}_{\mu_2}|^2 = |	ilde{U}_{\tau_1}|^2 = \frac{1}{2} - \frac{\Delta_{21} (|U_{\tau_2}|^2 - |U_{\mu_2}|^2) + \Delta_{31} (|U_{\tau_3}|^2 - |U_{\mu_3}|^2)}{2\sqrt{p^2 - 4q}}. \quad (28)$$

To be more intuitive and instructive, let us take $\alpha \equiv \Delta_{21}/\Delta_{31}$ and $|U_{e_3}|^2$ as two small expansion parameters to simplify Eq. (28), because both of them are of $\mathcal{O}(10^{-2})$. Then we arrive at

$$|	ilde{U}_{\mu_1}|^2 = |	ilde{U}_{\tau_2}|^2 \simeq 1 - |U_{\mu_3}|^2 (1 + |U_{e_3}|^2) + \alpha (|U_{\mu_3}|^2 - |U_{\mu_3}|^2 |U_{e_1}|^2 - |U_{\tau_1}|^2),$$

$$|	ilde{U}_{\mu_2}|^2 = |	ilde{U}_{\tau_1}|^2 \simeq |U_{\mu_3}|^2 (1 + |U_{e_3}|^2) - \alpha (|U_{\mu_3}|^2 - |U_{\mu_3}|^2 |U_{e_1}|^2 - |U_{\tau_1}|^2). \quad (29)$$

It becomes clear that the asymptotic form of $\tilde{U}$ for $A \to \infty$ contains only a single degree of freedom and thus can be parametrized as

$$\tilde{U} \big|_{A \to \infty} = \begin{pmatrix} 0 & 0 & 1 \\ \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \end{pmatrix}, \quad (30)$$

where

$$\tan \theta = \frac{\sqrt{p^2 - 4q} - \Delta_{21} (|U_{\tau_2}|^2 - |U_{\mu_2}|^2) - \Delta_{31} (|U_{\tau_3}|^2 - |U_{\mu_3}|^2)}{\sqrt{p^2 - 4q} + \Delta_{21} (|U_{\tau_2}|^2 - |U_{\mu_2}|^2) + \Delta_{31} (|U_{\tau_3}|^2 - |U_{\mu_3}|^2)} \simeq \frac{|U_{\mu_3}|^2 (1 + |U_{e_3}|^2 - |U_{\mu_3}|^2)}{(1 - |U_{\mu_3}|^2)^2} \frac{|U_{\tau_1}|^2 - |U_{e_2}|^2 |U_{\mu_3}|^2}{(1 - |U_{\mu_3}|^2)^2}. \quad (31)$$

It is easy to see that matter effects do preserve the $\mu$-$\tau$ symmetry (i.e., $|\tilde{U}_{\mu_i}| = |\tilde{U}_{\tau_i}|$ will hold as a consequence of $|\tilde{U}_{\mu_i}| = |\tilde{U}_{\tau_i}|$ for $i = 1, 2, 3$) even in very dense matter (i.e., $A \to \infty$).

At this point it is appropriate to clarify the confusing results obtained before for $\tilde{\theta}_{12}$, $\tilde{\theta}_{23}$ and $\tilde{\delta}$ in the $A \to \infty$ limit [21, 20]. Given $\tilde{\theta}_{13} \to \pi/2$ in this case, Eq. (25) becomes

$$\tilde{U} \big|_{A \to \infty} = \begin{pmatrix} 0 & 0 & e^{-i\tilde{\delta}} \\ -\tilde{s}_{12} \tilde{c}_{23} - \tilde{c}_{12} \tilde{s}_{23} e^{i\delta} & \tilde{c}_{12} \tilde{c}_{23} - \tilde{s}_{12} \tilde{s}_{23} e^{i\delta} & 0 \\ \tilde{s}_{12} \tilde{s}_{23} + \tilde{c}_{12} \tilde{c}_{23} e^{i\delta} & -\tilde{c}_{12} \tilde{s}_{23} - \tilde{s}_{12} \tilde{c}_{23} e^{i\delta} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\tilde{\delta}} & 0 & 0 \\ 0 & e^{i\tilde{\delta}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ X e^{i\phi} & Y e^{-i\phi} & 0 \\ -Y e^{i\phi} & X e^{-i\phi} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & e^{i(\phi + \delta)} \\ 0 & e^{i(\tilde{\delta} - \phi)} & 0 \\ 0 & 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ X & Y & 0 \\ -Y & X & 0 \end{pmatrix} \begin{pmatrix} e^{i(\phi + \delta)} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (32)$$
where \( X \equiv |\tilde{c}_{12}\tilde{s}_{23} + \tilde{s}_{12}\tilde{c}_{23}e^{-i\theta}|, Y = |\tilde{s}_{12}\tilde{s}_{23} - \tilde{c}_{12}\tilde{c}_{23}e^{-i\theta}|, \varphi \equiv \pi + \arg(\tilde{c}_{12}\tilde{s}_{23} + \tilde{s}_{12}\tilde{c}_{23}e^{-i\theta}) \) and \( \phi \equiv \pi + \arg(\tilde{s}_{12}\tilde{s}_{23} - \tilde{c}_{12}\tilde{c}_{23}e^{i\theta}) \). Then Eq. (32) is equivalent to Eq. (30) for the following two reasons: first, the two diagonal phase matrices in Eq. (32) can be absorbed by redefining the phases of the charged-lepton and neutrino fields \(^2\) second, \( X^2 + Y^2 = 1 \) holds, and thus one may always take \( X = \cos \theta \) and \( Y = \sin \theta \) with \( \theta \) being in the first quadrant.

The above discussion implies that the individual values of \( \tilde{\theta}_{12} \) and \( \tilde{\theta}_{23} \) in the \( A \to \infty \) limit do not make much sense, and in particular the finite value of \( \tilde{\delta} \) in this case is misleading. The latter observation is also supported by the fact \( \tilde{\mathcal{J}} \to 0 \) for \( A \to \infty \), as guaranteed by the Naumov relation in Eq. (24). A question turns out to be why \( \tilde{\mathcal{J}} \) and \( |\tilde{U}_{\alpha i}|^2 \) have the well-defined asymptotic behaviors in very dense matter, but the parameters \( \tilde{\theta}_{12}, \tilde{\theta}_{13}, \tilde{\theta}_{23} \) and \( \tilde{\delta} \) may not have. The answer to this question is very simple: a specific parametrization of \( \tilde{U} \) is always basis-dependent and hence its parameters are not guaranteed to be fully physical in the extreme case in which a redefinition of the basis becomes available to remove the possible parameter redundancy. In contrast, \( \tilde{\mathcal{J}} \) and \( |\tilde{U}_{\alpha i}|^2 \) do not suffer from this kind of subtlety because they are rephasing-invariant or basis-independent.

### 3.2 Case B: (IMO, \( \nu \))

Now we turn to the case of a neutrino beam (\( \nu \)) propagating in matter with an inverted mass ordering (IMO). In the \( A \to \infty \) limit, Eq. (4) is reduced to

\[
\tilde{\Delta}_{21} = \Delta_{21} + \Delta_{31} + A - \frac{1}{2} \left(3p + \sqrt{p^2 - 4q}\right), \\
\tilde{\Delta}_{31} = -\sqrt{p^2 - 4q}, \\
B = \frac{1}{2} \left(p + \sqrt{p^2 - 4q}\right),
\]

where \( p \) and \( q \) have already been defined below Eq. (27). In this case we find that \( \tilde{\Delta}_{31} \simeq \Delta_{31} (1 - |U_{e3}|^2) - \Delta_{21} |U_{e1}|^2 \) is finite and \( \tilde{\Delta}_{21} \simeq -\tilde{\Delta}_{32} \simeq A \) approaches infinity when \( A \to \infty \) is taken for very dense matter.

With the help of Eq. (33) and Eqs. (13)—(15), we calculate the nine elements of \( \tilde{U} \) in the \( A \to \infty \) limit and get

\[
|\tilde{U}_{e1}|^2 = |\tilde{U}_{e3}|^2 = |\tilde{U}_{\mu 2}|^2 = |\tilde{U}_{\tau 2}|^2 = 0, \quad |\tilde{U}_{e2}|^2 = 1, \\
|\tilde{U}_{\mu 1}|^2 = |\tilde{U}_{\tau 3}|^2 = \frac{\Delta_{21} (|U_{\tau 2}|^2 - |U_{\mu 2}|^2) + \Delta_{31} (|U_{\tau 3}|^2 - |U_{\mu 3}|^2)}{2\sqrt{p^2 - 4q}}, \\
|\tilde{U}_{\mu 3}|^2 = |\tilde{U}_{\tau 1}|^2 = \frac{\Delta_{21} (|U_{\tau 2}|^2 - |U_{\mu 2}|^2) + \Delta_{31} (|U_{\tau 3}|^2 - |U_{\mu 3}|^2)}{2\sqrt{p^2 - 4q}}.
\]

Just as Eq. (29), the expressions of \( |\tilde{U}_{\mu 1}|^2, |\tilde{U}_{\mu 3}|^2, |\tilde{U}_{\tau 1}|^2 \) and \( |\tilde{U}_{\tau 3}|^2 \) in Eq. (34) can be

\(^2\)Since neutrino oscillations are completely insensitive to the Majorana phases of three massive neutrinos no matter whether matter effects are involved or not, this rephasing treatment of the neutrino fields is definitely allowed in this connection.
expanded in terms of $\alpha$ and $|U_{e3}|^2$. As a result,

\[
|\tilde{U}_\mu|^2 = |\tilde{U}_{\tau 3}|^2 \simeq 1 - |U_{\mu 3}|^2 (1 + |U_{e3}|^2) + \alpha (|U_{\mu 3}|^2 - |U_{\mu 3}|^2 |U_{e1}|^2 - |U_{\tau 1}|^2) ,
\]
\[
|\tilde{U}_{\mu 3}|^2 = |\tilde{U}_{\tau 1}|^2 \simeq |U_{\mu 3}|^2 (1 + |U_{e3}|^2) - \alpha (|U_{\mu 3}|^2 - |U_{\mu 3}|^2 |U_{e1}|^2 - |U_{\tau 1}|^2) .
\]

(35)

In this case the asymptotic form of $\tilde{U}$ also contains only a single degree of freedom and thus can be rewritten as

\[
\tilde{U} \bigg|_{A \to \infty} = \begin{pmatrix} 0 & 1 & 0 \\ \cos \theta & 0 & \sin \theta \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} ,
\]

(36)

where

\[
\tan \theta = \frac{\sqrt{p^2 - 4q} + \Delta_{21} (|U_{r 2}|^2 - |U_{\mu 2}|^2) + \Delta_{31} (|U_{r 3}|^2 - |U_{\mu 3}|^2)}{\sqrt{p^2 - 4q} - \Delta_{21} (|U_{r 2}|^2 - |U_{\mu 2}|^2) - \Delta_{31} (|U_{r 3}|^2 - |U_{\mu 3}|^2)} \simeq \frac{|U_{\mu 3}|^2 (1 + |U_{e3}|^2 - |U_{\mu 3}|^2)}{(1 - |U_{\mu 3}|^2)^2} + \alpha \frac{|U_{r 1}|^2 - |U_{e 2}|^2 |U_{\mu 3}|^2}{(1 - |U_{\mu 3}|^2)^2} .
\]

(37)

3.3 Case C: (NMO, $\bar{\nu}$)

When it comes to the case of an antineutrino beam ($\bar{\nu}$) propagating in matter with a normal mass ordering (NMO), we simplify Eq. (3) in the $A \to \infty$ limit as follows:

\[
\tilde{\Delta}_{21} = - \Delta_{21} - \Delta_{31} + A + \frac{1}{2} \left( 3p - \sqrt{p^2 - 4q} \right) ,
\]
\[
\tilde{\Delta}_{31} = - \Delta_{21} - \Delta_{31} + A + \frac{1}{2} \left( 3p + \sqrt{p^2 - 4q} \right) ,
\]
\[
B = |U_{e 2}|^2 \tilde{\Delta}_{21} + |U_{e 3}|^2 \tilde{\Delta}_{31} - A ,
\]

(38)

with $p$ and $q$ having been defined below Eq. (27). In this case we find that $\tilde{\Delta}_{32} = \sqrt{p^2 - 4q} \simeq \Delta_{31} (1 - |U_{e 3}|^2) - \Delta_{21} |U_{e 1}|^2$ is finite and $\tilde{\Delta}_{21} \simeq \tilde{\Delta}_{31} \simeq A$ approaches infinity.

Given Eq. (38) and Eqs. (13)—(15), the elements of $\tilde{U}$ in the $A \to \infty$ limit read as

\[
|\tilde{U}_{\mu 2}|^2 = |\tilde{U}_{\mu 3}|^2 = |\tilde{U}_{\tau 2}|^2 = |\tilde{U}_{\tau 1}|^2 = 0 ,
\]
\[
|\tilde{U}_{e 2}|^2 = 1 ,
\]
\[
|\tilde{U}_{e 3}|^2 = |\tilde{U}_{\mu 1}|^2 = |\tilde{U}_{\tau 3}|^2 = 0 ,
\]
\[
|\tilde{U}_{e 1}|^2 = 1 ,
\]

(39)

Expanding $|\tilde{U}_{\mu 2}|^2$, $|\tilde{U}_{\mu 3}|^2$, $|\tilde{U}_{\tau 2}|^2$ and $|\tilde{U}_{\tau 3}|^2$ in terms of $\alpha$ and $|U_{e 3}|^2$, we obtain

\[
|\tilde{U}_{\mu 2}|^2 = |\tilde{U}_{\tau 3}|^2 \simeq 1 - |U_{\mu 3}|^2 (1 + |U_{e 3}|^2) + \alpha (|U_{\mu 3}|^2 - |U_{\mu 3}|^2 |U_{e 1}|^2 - |U_{\tau 1}|^2) ,
\]
\[
|\tilde{U}_{\mu 3}|^2 = |\tilde{U}_{\tau 2}|^2 \simeq |U_{\mu 3}|^2 (1 + |U_{e 3}|^2) - \alpha (|U_{\mu 3}|^2 - |U_{\mu 3}|^2 |U_{e 1}|^2 - |U_{\tau 1}|^2) .
\]

(40)
In this case the asymptotic form of $\mathcal{U}$ can be parameterized with only a single degree of freedom as follows:

$$
\mathcal{U} \bigg|_{A \to \infty} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix},
$$

(41)

where

$$
\tan \theta = \frac{\sqrt{p^2 - 4q} - \Delta_{21} (|U_{\tau 2}|^2 - |U_{\mu 2}|^2) - \Delta_{31} (|U_{\tau 3}|^2 - |U_{\mu 3}|^2)}{\sqrt{p^2 - 4q} + \Delta_{21} (|U_{\tau 2}|^2 - |U_{\mu 2}|^2) + \Delta_{31} (|U_{\tau 3}|^2 - |U_{\mu 3}|^2)}
$$

$$
\cong \frac{|U_{\mu 3}|^2 (1 + |U_{e 3}|^2 - |U_{\mu 3}|^2)}{\left(1 - |U_{\mu 3}|^2\right)^2} + \alpha \frac{|U_{\tau 1}|^2 - |U_{e 2}|^2 |U_{\mu 3}|^2}{\left(1 - |U_{\mu 3}|^2\right)^2}.
$$

(42)

### 3.4 Case D: (IMO, $\tau$)

Similarly, in the case of an anti-neutrino beam ($\tau$) propagating in matter with an inverted mass ordering (IMO), we simplify Eq. (4) in the $A \to \infty$ limit and arrive at

$$
\Delta_{21} = \sqrt{p^2 - 4q} ,
$$

$$
\Delta_{31} = \Delta_{21} + \Delta_{31} - A - \frac{1}{2} \left(3p - \sqrt{p^2 - 4q}\right) ,
$$

$$
B = \frac{1}{2} \left(p - \sqrt{p^2 - 4q}\right) ,
$$

(43)

where $p$ and $q$ have been given below Eq. (27). We are therefore left with finite $\Delta_{21} \simeq -\Delta_{31} (1 - |U_{e 3}|^2) + \Delta_{21} |U_{e 1}|^2$ and infinite $\Delta_{31} \simeq \Delta_{32} \simeq -A$. The expressions of $|\mathcal{U}_{\alpha 1}|^2$ in the $A \to \infty$ limit turn out to be

$$
|\mathcal{U}_{e 1}|^2 = |\mathcal{U}_{e 2}|^2 = |\mathcal{U}_{\mu 3}|^2 = |\mathcal{U}_{\tau 3}|^2 = 0 ,
$$

$$
|\mathcal{U}_{e 3}|^2 = 1 ,
$$

$$
|\mathcal{U}_{\mu 1}|^2 = |\mathcal{U}_{\mu 2}|^2 = \frac{\Delta_{21} (|U_{\tau 2}|^2 - |U_{\mu 2}|^2) + \Delta_{31} (|U_{\tau 3}|^2 - |U_{\mu 3}|^2)}{2\sqrt{p^2 - 4q}},
$$

$$
|\mathcal{U}_{\tau 1}|^2 = \frac{\Delta_{21} (|U_{\tau 2}|^2 - |U_{\mu 2}|^2) + \Delta_{31} (|U_{\tau 3}|^2 - |U_{\mu 3}|^2)}{2\sqrt{p^2 - 4q}}.
$$

(44)

In a good approximation, we find

$$
|\mathcal{U}_{\mu 1}|^2 = |\mathcal{U}_{\tau 2}|^2 \simeq |U_{\mu 3}|^2 (1 + |U_{e 3}|^2) - \alpha (|U_{\mu 3}|^2 - |U_{\mu 3}|^2 |U_{e 1}|^2 - |U_{\tau 1}|^2) ,
$$

$$
|\mathcal{U}_{\mu 2}|^2 = |\mathcal{U}_{\tau 1}|^2 \simeq 1 - |U_{\mu 3}|^2 (1 + |U_{e 3}|^2) + \alpha (|U_{\mu 3}|^2 - |U_{\mu 3}|^2 |U_{e 1}|^2 - |U_{\tau 1}|^2) .
$$

(45)

In this case the asymptotic form of $\mathcal{U}$ contains only a single degree of freedom and can be parametrized as

$$
\mathcal{U} \bigg|_{A \to \infty} = \begin{pmatrix}
0 & 0 & 1 \\
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0
\end{pmatrix} ,
$$

(46)
where

\[
\tan \theta = \frac{\sqrt{p^2 - 4q} - \Delta_{21} (|U_{\tau 2}|^2 - |U_{\mu 2}|^2) - \Delta_{31} (|U_{\tau 3}|^2 - |U_{\mu 3}|^2)}{\sqrt{p^2 - 4q} + \Delta_{21} (|U_{\tau 2}|^2 - |U_{\mu 2}|^2) + \Delta_{31} (|U_{\tau 3}|^2 - |U_{\mu 3}|^2)}
\]

\[
\simeq \frac{|U_{\tau 3}|^2}{|U_{\mu 3}|^2} - \frac{\alpha (|U_{\tau 1}|^2 - |U_{\mu 2}|^2 |U_{\mu 3}|^2)}{|U_{\mu 3}|^4}.
\]  

(47)

Comparing between cases A and B (or between C and D), we immediately find the interesting relations

\[
|\tilde{U}_{\mu 1}|^2 \bigg|_{A \to \infty}^{(NMO, \nu)} = |\tilde{U}_{\mu 2}|^2 \bigg|_{A \to \infty}^{(NMO, \nu)},
\]

\[
|\tilde{U}_{\mu 1}|^2 \bigg|_{A \to \infty}^{(IMO, \nu)} = |\tilde{U}_{\mu 2}|^2 \bigg|_{A \to \infty}^{(IMO, \nu)},
\]

(48)

which are equivalent to \(\tan \theta_{A \to \infty}^{(NMO, \nu)} = \tan \theta_{A \to \infty}^{(IMO, \nu)}\) and \(\tan \theta_{A \to \infty}^{(IMO, \nu)} = \cot \theta_{A \to \infty}^{(IMO, \nu)}\).

Taking account of the best-fit values of the six neutrino oscillation parameters (i.e., \(\Delta_{21} = 7.39 \times 10^{-5} \text{ eV}^2\), \(\Delta_{31} = 2.525 \times 10^{-3} \text{ eV}^2\), \(\theta_{12} = 33.82^\circ\), \(\theta_{13} = 8.61^\circ\), \(\theta_{23} = 49.7^\circ\) and \(\delta = 217^\circ\) for the NMO case; or \(\Delta_{21} = 7.39 \times 10^{-5} \text{ eV}^2\), \(\Delta_{31} = -2.438 \times 10^{-3} \text{ eV}^2\), \(\theta_{12} = 33.82^\circ\), \(\theta_{13} = 8.65^\circ\), \(\theta_{23} = 49.7^\circ\) and \(\delta = 280^\circ\) for the IMO case) \([35, 36]\), we have

\[
\begin{pmatrix}
|U_{e 1}|^2 & |U_{e 2}|^2 & |U_{e 3}|^2 \\
|U_{\mu 1}|^2 & |U_{\mu 2}|^2 & |U_{\mu 3}|^2 \\
|U_{\tau 1}|^2 & |U_{\tau 2}|^2 & |U_{\tau 3}|^2
\end{pmatrix}
\simeq
\begin{pmatrix}
0.675 & 0.303 & 0.022 \\
0.084 & 0.347 & 0.569 \\
0.241 & 0.350 & 0.409
\end{pmatrix}
\]  

(49)

for the NMO case; or

\[
\begin{pmatrix}
|U_{e 1}|^2 & |U_{e 2}|^2 & |U_{e 3}|^2 \\
|U_{\mu 1}|^2 & |U_{\mu 2}|^2 & |U_{\mu 3}|^2 \\
|U_{\tau 1}|^2 & |U_{\tau 2}|^2 & |U_{\tau 3}|^2
\end{pmatrix}
\simeq
\begin{pmatrix}
0.674 & 0.303 & 0.023 \\
0.151 & 0.281 & 0.568 \\
0.175 & 0.416 & 0.409
\end{pmatrix}
\]  

(50)

for the IMO case. Starting from the best-fit values of \(|U_{\alpha i}|^2\) at \(A = 0\) as given in Eq. (49) and Eq. (50), each of the nine effective quantities \(|\tilde{U}_{\alpha i}|^2\) evolves with the matter parameter \(A\) in a way shown in Figs. 1 and 2, where case A (NMO, \(\nu\)), case B (IMO, \(\nu\)), case C (NMO, \(\bar{\nu}\)) and case D (IMO, \(\bar{\nu}\)) have all been taken into account. One can see that \(|\tilde{U}_{\alpha i}|^2 \simeq |U_{\alpha i}|^2\) is a good approximation when \(A\) is small enough (i.e., \(A \lesssim 10^{-6} \text{ eV}^2\)). If the matter parameter \(A\) lies in the range \(10^{-6} \text{ eV}^2 \lesssim A \lesssim 10^{-2} \text{ eV}^2\), matter effects turn out to be significant and can make important corrections to the genuine lepton flavor mixing matrix \(U\) in vacuum. When taking the value of \(A\) larger than \(10^{-2} \text{ eV}^2\), we find that the effective PMNS matrix \(\tilde{U}\) asymptotically approaches a constant matrix in the \(A \to \infty\) limit. To be more explicit, we obtain

\[
\left(\begin{pmatrix}
|\tilde{U}_{\alpha i}|^2 
\end{pmatrix} \right)_{A \to \infty}^{(NMO, \nu)} = \begin{pmatrix}
0 & 0 & 1 \\
0.417 & 0.583 & 0 \\
0.583 & 0.417 & 0
\end{pmatrix},
\]

(51a)
Figure 1: The evolution of \(|\tilde{U}_{\alpha i}|^2\) (for \(\alpha = e, \mu, \tau\) and \(i = 1, 2, 3\)) with the matter effect parameter \(A\) in the normal neutrino mass ordering case, where the best-fit values of six neutrino oscillation parameters have been input [35].

\[
\begin{pmatrix}
|\tilde{U}_{\alpha i}|^2 & |\tilde{U}_{\beta j}|^2 & |\tilde{U}_{\gamma k}|^2
\end{pmatrix}_{A \to \infty} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0.417 & 0.583 \\
0 & 0.583 & 0.417
\end{pmatrix}, \quad (51b)
\]

\[
\begin{pmatrix}
|\tilde{U}_{\alpha i}|^2 & |\tilde{U}_{\beta j}|^2 & |\tilde{U}_{\gamma k}|^2
\end{pmatrix}_{A \to \infty} = \begin{pmatrix}
0.418 & 0 & 0.582 \\
0.582 & 0 & 0.418 \\
0 & 0 & 1
\end{pmatrix}, \quad (51c)
\]

\[
\begin{pmatrix}
|\tilde{U}_{\alpha i}|^2 & |\tilde{U}_{\beta j}|^2 & |\tilde{U}_{\gamma k}|^2
\end{pmatrix}_{A \to \infty} = \begin{pmatrix}
0.582 & 0.418 & 0 \\
0.418 & 0.582 & 0
\end{pmatrix}. \quad (51d)
\]

If the 3\(\sigma\) ranges of six neutrino oscillation parameters [35] are taken into account, we get

\[
|\tilde{U}_{\mu 1}|^2_{A \to \infty}^{\text{(NMO, } \nu)} = |\tilde{U}_{\mu 2}|^2_{A \to \infty}^{\text{(NMO, } \nu)} = 0.373 \to 0.574,
\]

\[
|\tilde{U}_{\mu 1}|^2_{A \to \infty}^{\text{(IMO, } \nu)} = |\tilde{U}_{\mu 2}|^2_{A \to \infty}^{\text{(IMO, } \nu)} = 0.375 \to 0.569. \quad (52)
\]
Figure 2: The evolution of $|\tilde{U}_{\alpha i}|^2$ (for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$) with the matter effect parameter $A$ in the inverted neutrino mass ordering case, where the best-fit values of six neutrino oscillation parameters have been input [35].

4 Summary

How to transparently describe matter effects on the behaviors of neutrino oscillations has been an interesting and important topic in neutrino phenomenology, and among a number of useful approaches making use of the effective PMNS matrix $\tilde{U}$ and the effective neutrino mass-squared differences $\tilde{\Delta}_{ji}$ has proved convenient to discuss neutrino mixing in a medium and formulate the matter-corrected probabilities of flavor oscillations. Then it makes sense to explore possible asymptotic behaviors of these effective quantities in the regime where the matter density is sufficiently large, a case which is mathematically equivalent to assuming the matter parameter $A = 2\sqrt{2} \cdot G_F\cdot N_e\cdot E$ to approach infinity.

In this paper we have established some direct and concise relations between $(\tilde{U}_{\alpha i}, \tilde{\Delta}_{ji})$ in matter and their fundamental counterparts $(U_{\alpha i}, \Delta_{ji})$ in vacuum (for $\alpha = e, \mu, \tau$ and $i, j = 1, 2, 3$) with the help of two sets of sum rules for them. These sum rules allow us to derive new and exact formulas for both nine $|\tilde{U}_{\alpha i}|^2$ and nine $\tilde{U}_{\alpha i}\tilde{U}_{\beta j}^*$ in a parametrization-
independent way, by which we have analytically unraveled the asymptotic behaviors of $|\tilde{U}_{\alpha i}|^2$ and $\tilde{\Delta}_{ji}$ in very dense matter for the first time. We have also clarified the confusion associated with the parameter redundancy of $\tilde{\theta}_{12}$, $\tilde{\theta}_{13}$, $\tilde{\theta}_{23}$ and $\tilde{\delta}$ in the standard parametrization of $\tilde{U}$ in the $A \to \infty$ limit. We conclude that $\tilde{U}$ contains only a single degree of freedom in this extreme case, with no CP violation in neutrino oscillations.

Finally it is worth mentioning that our approach can easily be extended to the (3+1) neutrino mixing scheme in matter, in which three active neutrinos are mixed with one light sterile neutrino species denoted as $\nu_s$ (flavor) or $\nu_4$ (mass). The corresponding sum rules for $\tilde{U}$ and $\tilde{\Delta}_{ji}$ will help derive the exact formulas for sixteen $|\tilde{U}_{\alpha i}|^2$ and sixteen $\tilde{U}_{\alpha i}\tilde{U}_{\beta i}^*$ (for $\alpha, \beta = e, \mu, \tau, s$ and $i = 1, 2, 3, 4$), as previously done [37]. In this case the matter potential term proportional to $V_{\text{nc}}$ must be taken into account.

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