Abstract

An antisymmetric tensor, the photon tensor, is defined for the description of the photon as a massless relativistic particle. The classical photon can then be visualized as an essentially two dimensional rotating object. The quantum mechanical description of a single photon is presented and the wave-particle duality for the photon is analysed. The intended relation between Schrödinger’s equation for the photon with Maxwell’s equations is discussed. This work is devoted to the attempt to understand the quantum of electromagnetic radiation, based on the assumption that the photons are the primary ontology and that the electromagnetic fields are macroscopic emergent properties of an ensemble of photons.

I. INTRODUCTION

On March 2005 we celebrated the one hundredth anniversary of the photon. At the beginning, the particle character of the photon was not recognized and it was considered to be a parcel of electromagnetic energy. Only after his twentieth anniversary and after the discovery of the Compton effect, the photon was accepted as a full fledged particle and he was given a name. Even though the concept of the photon is old, there are still many confusing matters that need clarification and there are opposing interpretations concerning the reality of the photon; indeed, near the end of his live, Einstein complained that after fifty years of conscious meditation he had not come any closer to the answer to the question What are light quanta?
In a previous work\textsuperscript{5}, the commutation relations of the operators corresponding to the electromagnetic fields were calculated from first principles, without reference to quantum field theory, and their highly singular character was discussed. The conclusion reached was that we should not consider the electromagnetic fields to be a primary ontology but that they are a macroscopic collective effect of a large, or undetermined, number of particles, the photons. One should however mention that the choice adopted, namely, that \textit{the photons are the primary ontology that must be treated quantum mechanically and that the electromagnetic fields are macroscopic manifestations of an ensemble of them}, is not a unique choice. There are indeed authors that consider the electromagnetic fields as the primary ontology and that the photons do not have an objective existence but are rather mathematical entities, “particle-like excitations”\textsuperscript{6}, corresponding to the normal mode decomposition of the fields. This alternative option can be adopted because, although the objective existence of the photons provide the most natural explanation of the photoelectric effect and of Compton scattering, these are not compelling evidence for their existence because these effects can also be explained by a semiclassical theory combining unquantized electromagnetic fields with quantum theory of matter\textsuperscript{7–9}. These semiclassical theories are however not capable of explaining some phenomena like the spontaneous emission of light by atoms or the correlations between distant photons in Clauser-Aspect test of Bell’s inequalities.

What is the meaning of trying to understand the photon? We know that the photon carries energy-momentum and has spin. Are these two things related? We can argue that they can not be related because spin has always the same value for all photons -one unit of $\hbar$- whereas the energy can vary continuously from zero to infinity. On the other side, spin is apparently the only property possessed by the photon and we don’t know what in the reality of the photon is the thing that carries energy. We expect that the energy of the photon resides in some internal property of it and is not purely kinematic because all photons move identically, with the same speed, but they can have different energies. What differentiates two photons of different energy? We will see that, as a consequence of \textit{special relativity}, not of quantum mechanics, the energy of a photon is given by some frequency. What is the thing that is changing in time with such a frequency? Is something within the photon rotating with that frequency? Wouldn’t it be natural to associate this rotation with the spin of the photon? All these questions and many other, indicate that we are still lacking a deep understanding of the quanta of electromagnetic radiation and although some progress has been achieved, Einstein’s question \textit{What are light quanta?} has not received a complete answer.

In this work we will present the quantum mechanical treatment of the photon in a “first
quantization” style, similar to a previous presentation\(^\text{10}\), but with the difference that here we do not rely on Maxwell’s equations as a starting point but instead we define the photon as a particle described with a “photon tensor”. We find this way more convenient because there is some misunderstanding relating the quantum mechanical state of a photon, solution of the photon’s Schrödinger equation, with the electromagnetic fields, solutions of Maxwell’s equations. We will show, indeed, that the intended derivation of Maxwell’s equation from Schrödinger equation is not justified. We will see that, in this interpretation, the photon is not the particle-like duality partner of the wave associated to the classical electromagnetic field. The particle-wave duality in the electromagnetic radiation corresponds to two different quantum states of the photon. On one side the space localized states \(\varphi_r\), eigenvectors of the position operator \(R\), and on the other side, states with sharp momentum and energy \(\phi_p\). In both cases we deal with single photon states and not with the electromagnetic field resulting from the combined effect of an undetermined number of photons. The confusion arises, probably, because the classical electromagnetic plane wave is closely related to a Bose-Einstein condensate of a large and indefinite number of photons in one single state \(\phi_p\).

Greek indices \(\mu, \nu, \sigma, \cdots\) assume values 0, 1, 2, 3 and the 0 is associated with the time coordinate; latin indices \(i, j, k, \cdots\) take values in 1, 2, 3; contravariant vectors will be occasionally decomposed in their time and space parts \(k^\mu = (k^0, k)\); the metric tensor is \(g^{\mu\nu} = g_{\mu\nu} = \text{Diag}(1, -1, -1, -1)\); \(\varepsilon_{\mu\nu\rho}\) and \(\varepsilon_{jkl}\) are the total antisymmetric tensor, \(\partial_k = \frac{\partial}{\partial x_k}\), \(\partial_t = \frac{\partial}{\partial t}\) and we adopt Einstein’s convention, stating that repeated indices indicate a summation.

### II. THE CLASSICAL PHOTON

Let us postulate the existence of a massless physical system, called photon, transporting energy \(E\), momentum \(P\) and intrinsic angular momentum (spin) \(S\). The relativistic kinematics of a massless particle implies that its speed must be \(c\) (Einstein’s constant) and that the energy, momentum and spin are related by

\[
E = c|P| \quad (1)
\]

and

\[
S \times P = 0 \ . \quad (2)
\]

The constraint imposed by the second relation, that for reasons to be clarified later can be called transversality constraint, follows from the fact that the intrinsic angular momentum
of an extended object moving at the speed $c$ can not have any component perpendicular to the direction of propagation. This can be easily understood if we consider a sphere moving with speed $v$ less than $c$ in a direction $\mathbf{k}$ and rotating around an axis along $\mathbf{s}$, perpendicular to the direction of propagation. Two points on the periphery “above” and “below” the plane of $\mathbf{k}$ and $\mathbf{s}$ will have different velocities resulting from the relativistic combination of the velocity $v$ of the center with the tangential velocity $\pm v_T$ due to the rotation. Now, if the center moves with speed $c$, the two points on the periphery will also have the speed $c$ and no rotation is possible. Therefore an extended massless object moving with speed $c$ can only rotate along an axis collinear with the direction of propagation as required by Eq.2. This heuristic argument would not be valid for a point like particle without spatial extension; however the quantum nature of particles suggest that no particle is exactly point like: massive particles can not be localized beyond their Compton wave length and a photon moving in a well defined direction must be extended in a direction perpendicular to the direction of movement as imposed by Heisenberg’s uncertainty principle.

The massless character of the photon suggests that its energy must be proportional to some frequency; that is, something in the photon must be changing periodically in time. The energy-frequency relation is therefore a consequence of special relativity. Although this was known for long, it has not received much attention. In order to see this, let us consider a photon propagating in the $x^1$ direction with four momentum (with $c = 1$) $p^\mu = (E, E, 0, 0)$. We can now perform a Lorentz transformation with speed $\beta$ (Lorentz factor $\gamma = 1/\sqrt{1 - \beta^2}$) in the direction of the $x^1$ coordinate axis. Doing this we see that the energy $E$ of the photon decreases to $E'$ in the “primed” reference frame given by

$$E' = E \gamma (1 - \beta) = E \sqrt{\frac{1 - \beta}{1 + \beta}}. \quad (3)$$

However this is precisely the transformation property of a frequency (Doppler shift) and we can therefore expect that the energy of a photon depends on some frequency. What makes this most remarkable, is that we have found that the energy of the photon depends on its frequency using only classical relativistic (that is, non-quantum) arguments in contrast with usual saga that presents the energy of a photon $h\nu$ as an essential quantum mechanical fact. The energy-frequency relation was discovered by Einstein in his seminal photoelectric effect paper in 1905, and is usually publicized as one of the first quantum postulates. However, after the consolidation of the photon concept as a particle this relation becomes a consequence of the massless character of any relativistic particle and, by the way, the same should hold for gluons and gravitons.

In order to build our model of a photon we must decide what is the thing or quality
that carries energy and momentum. For an electron, we would think on a electrically charged massive particle carrying one half unit of angular momentum and possibly other qualities such as the leptonic charge and so on. We must then postulate the existence of some “elements of physical reality”, as Einstein could have called it, in terms of which, the energy and spin of the photon should be expressed. Einstein’s relativity requires that this elements of physical reality should be formalized by a mathematical entity having well defined properties under a Lorentz transformation relating different observers, that is, it should be a scalar, vector or tensor of appropriate rank. Clearly, a scalar, being an invariant, can not represent the properties of a photon and neither can we describe the photon with a vector (different from $p^\mu$). We can then consider an antisymmetric, or symmetric or general second rank tensor with 6, 10, 16 components respectively. We have no profound reason to decide, except that if we choose the simplest case, an antisymmetric tensor $f^{\mu\nu}$ in order to describe such an element of physical reality, then we can accommodate all known physical properties of the photons. Let us choose the name “Photon Tensor” in order to denote this antisymmetric tensor that, therefore, contains six independent real numbers. It is very convenient to describe these six real numbers in terms of two three-vectors $e$ and $b$. However we should warn that, in rigour, the quantities $e$ and $b$ are not the space components of a four-vector because they do not transform as such in a general Lorentz transformation. They should be considered as a convenient notation for the six nonvanishing components of the photon tensor $f^{\mu\nu}$, according to the assignment given by

$$f^{\mu\nu} = \begin{pmatrix}
  0 & e_1 & e_2 & e_3 \\
  -e_1 & 0 & b_3 & -b_2 \\
  -e_2 & -b_3 & 0 & b_1 \\
  -e_3 & b_2 & -b_1 & 0
\end{pmatrix}.$$  
(4)

The notation is anyway justified because $e$ and $b$ transform as three-vectors under the subgroup of Lorentz transformations corresponding to space rotations and translations.

Associated to any second rank antisymmetric tensor there is another second rank anti-symmetric tensor, the dual tensor, defined by

$$f^{*\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\sigma\rho} f_{\sigma\rho} = \begin{pmatrix}
  0 & -b_1 & -b_2 & -b_3 \\
  b_1 & 0 & e_3 & -e_2 \\
  b_2 & -e_3 & 0 & e_1 \\
  b_3 & e_2 & -e_1 & 0
\end{pmatrix}.$$  
(5)

Notice that the dual photon tensor is obtained from the photon tensor performing a dual transformation defined by $e \rightarrow -b$ and $b \rightarrow e$. The description of the photon by means
of the photon tensor is equivalent to its description with the dual photon tensor and we postulate that the photon is invariant under the dual transformation.

Before we start with the physical analysis of the photon tensor it is important to clarify that, in spite of the formal similarity, the photon tensor \( f^{\mu \nu} \) is not a field like the electromagnetic field tensor \( F^{\mu \nu}(t, \mathbf{r}) \). The components of \( f^{\mu \nu} \) are not functions of space-time and therefore derivatives like \( \partial_\mu f^{\mu \nu} \) are meaningless. There are, indeed, many formal differences between the photon tensor and the electromagnetic tensor. It is therefore wrong to think about \( f^{\mu \nu} \) as being “the electromagnetic field of one photon” because, in this interpretation, the electromagnetic field is an emergent property of an ensemble of photons; the idea of the electromagnetic field of one photon is as meaningless as the idea of the density or temperature of one molecule.

The three scalars associated with the photon tensor are

\[
\begin{align*}
\mathbf{f}_\mu &= 0, \\
f^{\mu \nu} f_{\mu \nu} &= 2 \left( e^2 - b^2 \right), \\
f^{\mu \nu} f^{\nu \star} &= -4 \mathbf{e} \cdot \mathbf{b},
\end{align*}
\]

where \( e = |\mathbf{e}| \) and \( b = |\mathbf{b}| \). The first scalar vanishes trivially and we will later see that the other two scalar also vanish.

Since every proper orthochronous Lorentz transformation is equivalent to a standard Lorentz transformation preceded and followed by an appropriate space rotation and translation, we only need to consider the transformation of the photon tensor under a boost in the \( x^1 \) coordinate axis. Doing this we obtain the photon tensor in the “primed” reference frame that is moving with speed \( \beta \) (Lorentz factor \( \gamma = 1/\sqrt{1-\beta^2} \)) along the \( x^1 \) axis with respect to the unprimed frame:

\[
\begin{pmatrix}
0 & e_1 & \gamma(e_2 - \beta b_3) & \gamma(e_3 + \beta b_2) \\
-e_1 & 0 & \gamma(b_3 - \beta e_2) & -\gamma(b_2 + \beta e_3) \\
-\gamma(e_2 - \beta b_3) & -\gamma(b_3 - \beta e_2) & 0 & b_1 \\
-\gamma(e_3 + \beta b_2) & \gamma(b_2 + \beta e_3) & -b_1 & 0
\end{pmatrix},
\]

where we can read the components of \( \mathbf{e}' \) and \( \mathbf{b}' \) in the primed reference frame. We can notice here that, indeed, \( \mathbf{e} \) and \( \mathbf{b} \) do not transform as the space components of a four-vector.

The photon tensor is not invariant under space rotations. This means that the photon has some intrinsic orientation or directionality, manifest in the tree-vectors \( \mathbf{e} \) and \( \mathbf{b} \). On the other side, the photon propagates in space along a direction defined by a unit vector \( \mathbf{k} \). These two directional properties of the photon, the intrinsic orientation of \( f^{\mu \nu} \) and the direction
of propagation $\mathbf{k}$, are not independent but are coupled as consequence of special relativity. Because of the Lorentz contraction for an object moving with speed $c$, all lengths in the direction of propagation must collapse to zero; therefore we can think about the photon as being an *essentially* two dimensional object with all its elements of physical reality in a plane orthogonal to the direction of propagation, that is, we must have $\mathbf{k} \cdot \mathbf{e} = 0$ and $\mathbf{k} \cdot \mathbf{b} = 0$. However, these two conditions must be expressed in a way valid in all reference frames, that is, in a covariant way under Lorentz transformations. For this we notice that, due to the massless character of the photon, the unit vector $\mathbf{k}$ must be the space part of a propagation null four-vector, tangent to the photon world line, given by $k^\mu = (1, \mathbf{k})$. The coupling of the intrinsic and external orientations of the photon becomes then

$$ k_\mu f^{\mu \nu} = (\mathbf{k} \cdot \mathbf{e}, \mathbf{e} + \mathbf{k} \times \mathbf{b}) = 0 ,$$

and the dual transformation of this relation that must also be true, \( k_\mu f^{* \mu \nu} = (-\mathbf{k} \cdot \mathbf{b}, -\mathbf{b} + \mathbf{k} \times \mathbf{e}) = 0 \).

Therefore we have

$$\begin{align*}
\mathbf{k} \cdot \mathbf{e} &= 0 , \quad \mathbf{k} \times \mathbf{e} = \mathbf{b} \\
\mathbf{k} \cdot \mathbf{b} &= 0 , \quad \mathbf{k} \times \mathbf{b} = -\mathbf{e} .
\end{align*}$$

From these (redundant) equations it follows that in every Lorentz frame, $\mathbf{e}$ and $\mathbf{b}$ lay in a plane orthogonal to the direction of propagation $\mathbf{k}$, that they are orthogonal and with equal modulus, and $(\mathbf{e}, \mathbf{b}, \mathbf{k})$ build a right handed set of orthogonal vectors. Therefore we have

$$\mathbf{e} \cdot \mathbf{b} = 0 , \quad e = b .$$

This result, valid in all reference frames, imply that the two scalar in Eqs.(7,8) vanish, as mentioned before. Notice that there is an arbitrary orientation of the orthogonal pair $(\mathbf{e}, \mathbf{b})$ in their plane; therefore there is a reference frame, with the $x^1$ axis along the direction of propagation, where the photon tensor of Eq.(4) assumes the simplest form with all components vanishing except for $e_2$ and $b_3$ that are equal.

In order to relate the energy of the photon with the components of $f^{\mu \nu}$, let us consider that the energy is invariant under space rotations; therefore it must depend on rotationally invariant quantities like $e = |\mathbf{e}|$ or $b = |\mathbf{b}|$. Furthermore we expect the energy to be invariant under the duality transformation. However, since $e = b$, we only have to consider the dependence of the energy from $e$ in a way that should have the appropriate transformation
property given by Eq.(3). We must therefore calculate the transformation property of $e$ for a photon moving along $x^1$ under a standard Lorentz transformation. In this case we have $e_1 = b_1 = 0$ and considering the components of the transformed photon tensor given in Eq.(9) we have

$$e' = \sqrt{\gamma^2(e_2 - \beta b_3)^2 + \gamma^2(e_3 + \beta b_2)^2} = \gamma\sqrt{e^2 + \beta^2 b^2 - 2\beta(e_2 b_3 - e_3 b_2)} .$$

(15)

Notice that the term in parenthesis is the component of $e \times b$ along the $x^1$ axis; therefore using Eq.(14) we get

$$e' = e\gamma\sqrt{1 + \beta^2 - 2\beta} = e\gamma(1 - \beta) .$$

(16)

Comparing this with Eq.(3) we see that the energy of the photon $E$, and the modulus of the photon tensor $e = b$, have identical transformation property, that is, they both transform as a frequency. Then, the energy can be given by an homogeneous function of $e$ of first degree, that is, $E$ must be proportional to $e$, and we introduce the symbol $\omega = e = b$ in order to denote the frequency. The proportionality constant between the energy and the frequency must have units of action or of angular momentum and therefore we set it equal to Planck constant. We have then

$$E = \hbar\omega .$$

(17)

Both vectors $e$ and $b$ have equal modulus, $\omega$, and we can express them in terms of a set of orthogonal unit vectors $\mathbf{\hat{e}}$ and $\mathbf{\hat{b}}$ as $e = \omega\mathbf{\hat{e}}$ and $b = \omega\mathbf{\hat{b}}$. Also the photon tensor can have the frequency $\omega$ factored out and be given in terms of unit vectors. We can now fix the orientation of the unit vectors $\mathbf{\hat{e}}$ and $\mathbf{\hat{b}}$ in their plane by requiring that they rotate with the frequency $\omega$. For this rotation we have two choices corresponding to a clockwise or counterclockwise rotation. Considering the propagation of the photon along the direction $k$, we see that in these two choices the tip of the unit vector $\mathbf{\hat{e}}$ will make a right handed or a left handed helix. These correspond to a positive or a negative helicity photon. The rotation and propagation of the photon allows us to introduce a useful length scale for the photon corresponding to the step of the helix given by

$$\lambda = \frac{2\pi c}{\omega} .$$

(18)

Notice that we do not call this length scale “wave length” because, so far, for the classical -that is, non quantum- photon we don’t have any “wave”. It is natural to associate this rotation of the vectors $\mathbf{\hat{e}}$ and $\mathbf{\hat{b}}$ with the spin of the photon; however we must remind that spin can only take one value, $\pm \hbar$, whereas the frequency (the energy) can take any positive
value. This is no real problem because we can think of a rotating energy distribution whose associated angular momentum is constant. In the Appendix we show several examples of these mechanical models. However we should not take too seriously these mechanical models of the photon. All that we want to show is that there is no contradiction between a constant (i.e. for all frequencies) value of angular momentum of a rotating system and an energy linearly dependent on the frequency of rotation, therefore we are allowed to think that spin and energy are due to the rotation of the element of physical reality of the photon in the plane perpendicular to the direction of propagation.

The behaviour of the photon tensor under the proper and orthochronous Lorentz transformations proved to be very useful in order to determine its classical features. For completeness we will give now all the transformation properties of the photon tensor. One of the reasons for characterizing the photon tensor $f^{\mu\nu}$ in terms of the two three-vectors $e$ and $b$ is that these have simpler properties under space and time inversion and also under charge conjugation and duality transformation. From the definition of the photon tensor, it follows that under a space inversion transformation $P$, $e$ changes sign as a vector and $b$ remains invariant as a pseudovector. Let us now consider time inversion, that is better characterized as inversion in the direction of movement because we don’t have the “time” variable on our photon tensor. The direction of movement, $k$, is given by $e \times b$; therefore under the time inversion transformation $T$, either $e$ or $b$ must change sign. Since $T$ must be different from $P$ we choose that $b$ changes sign whereas $e$ remains invariant. In order to fix the transformation properties of $e$ and $b$ under the charge conjugation transformation $C$, we require that $PTC$ should be the identity; therefore both $e$ and $b$ must change sign under charge conjugation. We have then the transformation properties, including the duality transformation $D$, given by

\[
P: \begin{cases} e \rightarrow -e \\ b \rightarrow b \end{cases}, \quad T: \begin{cases} e \rightarrow e \\ b \rightarrow -b \end{cases}, \quad C: \begin{cases} e \rightarrow -e \\ b \rightarrow -b \end{cases}, \quad D: \begin{cases} e \rightarrow -b \\ b \rightarrow e \end{cases}.
\]  

(19)

Let us summarize all that we have learnt about the classical photon: in any reference frame, we can visualize a positive or negative helicity photon of energy $E$ and spin $\hbar$ propagating with speed $c$ in a direction given by a unit vector $k$ as a unit vector $\hat{e}$ rotating clockwise or counterclockwise in a plane orthogonal to $k$ with frequency $\omega = E/\hbar$. In the same plane we have another unit vector $\hat{b} = k \times \hat{e}$ and with the vectors $e = \omega \hat{e}$ and $b = \omega \hat{b}$ we can build the photon tensor $f^{\mu\nu}$ whose Lorentz transformations provide the description of the photon in other reference frames.
III. QUANTUM MECHANICAL TREATMENT OF ONE PHOTON

For the quantum mechanical treatment of the photon we must first define a Hilbert space whose elements are possible states of one photon and where the photon observables act as hermitian operators. There are clear empirical facts, for instance, the emission of one photon in atomic transitions \( p \to s \), that indicate that the photon has spin one. Therefore the description of spin states of the photon requires a three dimensional Hilbert space \( \mathcal{H}^S \). On the other side, the kinematic description of the photon, that is, its movement in physical space, requires an infinite dimensional Hilbert space \( \mathcal{H}^K \). The states of a photon are then elements of the Hilbert space

\[
\mathcal{H} = \mathcal{H}^S \otimes \mathcal{H}^K.
\]

Photon observables are represented in this space by hermitian operators of the form \( 1 \otimes R \) for position, \( 1 \otimes P \) for momentum or \( S \otimes 1 \) for spin. Having clarified this, we will use the simpler notation \( R, P \) and \( S \) to denote the operators.

For all massive particles, the kinematic and the spin degrees of freedom are decoupled and any combination of spin states with kinematic states are acceptable. For the photon, however, we have seen in Eq.(2) that spin must be collinear with momentum, therefore states corresponding to null projection of spin in the direction of momentum must be excluded. This means that spin states are limited to a two dimensional subspace of \( \mathcal{H}^S \) spanned by two states \( \{ \chi_+, \chi_- \} \) corresponding to helicities \( \pm 1 \). That is,

\[
(k \cdot S) \chi_\pm = \pm \hbar \chi_\pm,
\]

where \( k \) is a unit vector (not an operator) in the direction of \( P \). This two dimensional subspace is not fixed in \( \mathcal{H}^S \) because it is coupled to the direction of momentum in physical space and this requires a special treatment of spin adequate to this situation. It is usual to choose the \( z \) axis as the quantization axis for angular momentum making the operator \( S_z \) diagonal in such a representation. For the case of a photon where spin is collinear with momentum such a choice is not convenient. We will therefore adopt a representation for the spin operators more adequate for the implementation of the constraint \( S \times P = 0 \) with arbitrary direction of the momentum. In this representation, the three spin operators \( S_k \), \( k = 1, 2, 3 \), satisfying the commutation relations \( [S_j, S_k] = i\hbar \varepsilon_{jkl} S_l \), are given by the matrices

\[
S_x = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, 
S_y = \hbar \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, 
S_z = \hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};
\]

(22)
that is, with the matrix elements given by

\[(S_j)_{kl} = -i\hbar\varepsilon_{jkl} . \quad (23)\]

Applying the useful identity \(\varepsilon_{jkl}\varepsilon_{jrs} = \delta_{kr}\delta_{ls} - \delta_{ks}\delta_{lr}\) one can check that the commutation relations are satisfied. Now we can write Eq.(21) in matrix form and find the eigenvectors

\[
\chi_{\pm} = \frac{1}{2\sqrt{1-k_xk_y-k_yk_z-k_zk_x}} \begin{pmatrix}
1-k_x(k_x+k_y+k_z) \pm i(k_y-k_z) \\
1-k_y(k_x+k_y+k_z) \pm i(k_z-k_x) \\
1-k_z(k_x+k_y+k_z) \pm i(k_x-k_y)
\end{pmatrix}, \quad \chi_0 = \begin{pmatrix}
k_x \\
k_y \\
k_z
\end{pmatrix},
\]

where we have included the eigenvector \(\chi_0\) incompatible with the transversality constraint given in Eq.(2).

The two dimensional subspace of \(H^S\) containing the spin states is orthogonal (and therefore the name \textit{transversality} given to the constraint) to the Hilbert space element \(\chi_0\) that has, \textit{numerically}, the same components as the vector \(k\) in physical space. It is important not to confuse these two objects: \(\chi_0\) belongs to the Hilbert space of states whereas \(k\) belongs to physical space. This confusion of the three dimensional Hilbert space of states with the three dimensional physical space is the source of an erroneous identification of Maxwell’s equations with Schrödinger’s equation. Let us look in more details at this interesting but, alas, erroneous argument.

\section*{IV. ON SCHRODINGER AND MAXWELL EQUATIONS}

Using the mathematical tools developed for quantum mechanics, it has been shown that Maxwell’s equations can be written as an evolution equation for a spinor\textsuperscript{12,10}. However it should be clear that this spinorial equation remains a \textit{classical} equation for the \textit{classical} electromagnetic field disguised as a quantum mechanical equation and we should resist the temptation of an interpretation of this equation as some “Schrödinger’s equation” for some quantum system like the photon. In this section we will see that the attempted derivation of Maxwell’s equations from the photon’s Schrödinger equation is erroneous.

Let us consider a photon state \(\{\Psi_j\} \in H^S \otimes H^K\) where the index \(j = 1,2,3\) denotes the components of the state in the three dimensional space \(H^S\). Its time evolution will be determined by a hamiltonian operator such that it leaves invariant a photon state \(\chi_{\pm} \otimes \phi_p\), corresponding to the momentum eigenvalue \(p\) in the direction \(k = p/|p|\), with helicity \(\pm 1\) and energy \(E = c|p|\). One candidate to achieve this, is the operator.
\[ S \cdot P = k \cdot S \otimes |P| , \] (25)

where the operator \(|P|\) is such that \(|P|\phi_p = |p|\phi_p\). Indeed we have

\[ S \cdot P \chi^k \otimes \phi_p = \pm \frac{\hbar}{c} \chi^k \otimes \phi_p . \] (26)

The hamiltonian chosen is then

\[ H = \frac{c}{\hbar} S \cdot P , \] (27)

however this operator is not positive, as can be seen in Eq.26, and therefore the minus sign should be correctly interpreted as a photon with positive energy but negative helicity. With this, the time evolution of a general photon state will be given by

\[ i\hbar \partial_t \Psi_j = (H)_{jk} \Psi_k , \] (28)

where we have explicitly written the hamiltonian as a 3 \times 3 matrix in \(\mathcal{H}^S\) (whose components are operators in \(\mathcal{H}^K\)). That is,

\[ i\hbar \partial_t \Psi_j = \frac{c}{\hbar} (S \cdot P)_{jk} \Psi_k . \] (29)

We can now specialize this equation for a particular representation of the Hilbert space. For the spin part we choose the representation where spin is given by Eq.(23) and for the kinematic part \(\mathcal{H}^K\) we choose the space of square integrable functions of \(r\), \(L_2\) where the momentum operator is given by the derivative operator \(P = -i\hbar \nabla\). In this way we obtain what we may call *Schrödinger’s equation for the photon*

\[ \frac{i}{c} \partial_t \Psi_j(t, r) = \varepsilon_{jk} \partial_r \Psi_k(t, r) . \] (30)

Notice that in this equation the Planck constant \(\hbar\) does not appear although it is essentially a quantum mechanical equation. This can be interpreted as an indication that this equation does not have a classical limit \(\hbar \to 0\). This equation is correct but we will now see how, doing some illegal moves, we can get from it Maxwell’s equations. This intended derivation contains an important conceptual and mathematical error consisting in identifying the three dimensional Hilbert space of states with the three dimensional physical space and to equate \(\Psi_j\) with the components of the electromagnetic vector fields \(E\) and \(B\) according to

\[ \Psi_j(t, r) = E_j(t, r) + iB_j(t, r) \] (31)

Replacing above and separating the real and imaginary part, we obtain the first two of Maxwell’s equations
\[
-\varepsilon_{jlk} \partial_l E_k = \frac{1}{c} \partial_t B_j ,
\]
\[
\varepsilon_{jlk} \partial_l B_k = \frac{1}{c} \partial_t E_j ,
\]
\[
\partial_k E_k = 0 ,
\]
\[
\partial_k B_k = 0 .
\]

The last two equations are obtained repeating the same error: the photon state must be orthogonal to the state \( \chi_0 \) of \( \mathcal{H}^S \) given in Eq.(24), that is, \( k_k \Psi_k = 0 \). Now replacing the real numbers \( k_k \) by the operators \( \mathcal{P}_k \) in the \( L_2 \) representation, another error, we get \( \partial_k \Psi_k = 0 \) that with the misidentification of Eq.(31) leads to the last two Maxwell’s equations.

Certainly, the manipulations shown can not be considered to be a derivation of Maxwell’s equations from the photon Schrödinger’s equation. The intended identification in Eq.(31) of the state function of a quantum system with a field having physical existence, that is carrying energy an momentum in physical space, was considered in the early days of quantum mechanics by Schrödinger himself; however he had to abandon such an interpretation mainly because of the failure of the interpretation for a system of many particles. Indeed, the state function for \( N \) particles depends on \( 3N \) coordinates and therefore it can not be considered to be a field in three dimensional physical space. It is today well known that the quantum mechanical state can not be interpreted as an objectively existing field carrying energy and momentum as the electromagnetic field does. Indeed, if the quantum state were a field carrying energy and momentum, then well established quantum effects like nonlocality, entanglement, teleportation and others would imply unacceptable violations of relativity.

Another useful representation of Eq.(29) results from the observation that the equation will take its simplest form in the Hilbert space where the momentum operator has the simplest expression and this is, of course, the space of square integrable functions of \( p \) where the momentum operator amounts simply to a multiplication by \( p \). Taking, as before, for \( \mathcal{H}^S \) the representation where spin is given by Eq.(23) and \( L_2 \) for the kinematic part \( \mathcal{H}^K \), we obtain the photon Schrödinger’s equation in momentum representation
\[
\frac{\hbar}{c} \partial_t \Psi_j(t, p) = -\varepsilon_{jlk} p_l \Psi_k(t, p) .
\]

In spite of the formal similarity, the right hand side of this equation should not be written as a “vector product” \( p \times \Psi \) because the two “vectors” belong to different spaces: \( p \), the momentum eigenvalue, is in physical three dimensional space and \( \Psi \) is in the three dimensional Hilbert space \( \mathcal{H}^S \). In Eqs.(36) and (30) we have used the same letter \( \Psi \) to denote different functions (actually related by Fourier transformation) but this should cause no confusion.
We can now investigate the stationary state solutions to Eq.(29), that is, solutions corresponding to a fixed value of the energy $E$ of the form

$$\Psi_j = \exp(-\frac{i}{\hbar}Et)\Phi_{j,E}$$  \hspace{1cm} (37)

where $\Phi_{j,E}$ is the solution of the time independent equation

$$(\mathbf{S} \cdot \mathbf{P})_{jk}\Phi_{k,E} = \frac{E}{c}\hbar \Phi_{j,E}.$$  \hspace{1cm} (38)

Again, this equation will take its simplest form in the momentum representation. With respect to the spin part, the equation above is essentially the same as Eq.(21) and therefore the spin part of the solution is given by the helicity states given in Eq.(24). The kinematic part of the solution must relate the momentum eigenvalue $\mathbf{p}$ with the energy $E$. Therefore we have

$$\Phi_{j,E}(\mathbf{p}) = \chi_\pm \otimes \delta(|\mathbf{p}| - \frac{E}{c}),$$  \hspace{1cm} (39)

Where $\mathbf{k} = \mathbf{p}/|\mathbf{p}|$. As said before, the momentum representation is the most adequate one in order to describe the quantum state of the photon because in this case the transversality constraint, that is, the condition that the spin and momentum must be collinear, is most evident. In order to obtain the position representation of the photon energy eigenstates we must take the Fourier transform of the equation above that does not have a simple expression. For this reason, in the position representation it is more convenient to describe the photon in momentum eigenstate rather than in energy eigenstates. These states are $\chi_{\mp k_0} \otimes \phi_{p_0}$ with $\mathbf{k_0} = \mathbf{p_0}/|\mathbf{p_0}|$ corresponding to a photon with definite momentum $\mathbf{p_0}$ and helicity $\pm 1$. These states are also eigenstates of the hamiltonian with degenerate energy eigenvalue $E = c|\mathbf{p_0}|$. The kinematic part $\phi_{p_0}$ in the momentum representation is given by

$$\phi_{p_0}(\mathbf{p}) = \delta(\mathbf{p} - \mathbf{p_0}),$$  \hspace{1cm} (40)

and in position representation they are

$$\phi_{p_0}(\mathbf{r}) = \frac{1}{(\sqrt{2\pi\hbar})^3} \exp(i\frac{\mathbf{p_0} \cdot \mathbf{r}}{\hbar}),$$  \hspace{1cm} (41)

showing that one photon behaves as a wave in these states. Notice that this is the first appearance of a wave related to the photon. We will later see the relation of this wave with the macroscopic electromagnetic waves.
The idea of complementarity, introduced by Bohr in quantum mechanics, is a generalization of the observed wave-particle duality in the behaviour of matter. According to complementarity, every description or observation of a physical system involves a set of observables that necessarily excludes other observables. The simultaneous and exact treatment of all observables of the system is not possible. As as metaphor for complementarity one can think that the observation of a human face must be necessarily done from only one particular point of view. Every perspective excludes other possible perspectives. A philosophically realist physicist postulates the objective existence of physical reality that may be in different quantum states corresponding to all possible complementary descriptions or observations, in the same way that the tree dimensional real existence of a human face generates all possible two dimensional perspectives. The marvellous invention of Picasso, that combines mutually exclusive perspectives of a human face in one single image is, in this sense, a representation of reality more faithful than a conventional photograph showing only one perspective.

As it happens with any particle, the photon also exhibits the wave-particle duality in different experimental situations. It behaves like a particle in the photoelectric effect and in Compton scattering and it behaves like a wave in diffraction experiments. The photon exhibits particle like behaviour when it is in a localized quantum state \( \varphi_r \), eigenvector of the position operator \( \textbf{R} \) and it will behave as a wave when it is in state \( \phi_p \) eigenstate of the momentum operator \( \textbf{P} \). Both dual states, or any other, are equally valid; however the transversality constraint requires a well defined momentum and therefore it introduces a preference for momentum eigenstates. In order to impose the transversality constraint in an arbitrary state we must expand it on the basis \( \{ \phi_p \} \) of \( \mathcal{H}^K \) and every component of it will determine the two dimensional subspace of \( \mathcal{H}^S \) that contains the spin state. An interesting remark is that in such a state, spin and momentum are entangled because every momentum eigenvalue is associated with a spin superposition. An arbitrary state for one photon is then

\[
\Psi = \int d^3 \textbf{p} \ C(\textbf{p}) \left( \alpha(\textbf{p}) \chi^\text{P}_+ + \beta(\textbf{p}) \chi^\text{P}_- \right) \otimes \phi_p .
\]

where \( C(\textbf{p}) \), \( \alpha(\textbf{p}) \) and \( \beta(\textbf{p}) \) are the coefficients that determine the state, and \( \chi^\text{P}_\pm \) are the helicity states, given by Eq.(24) with \( k = \textbf{p}/|\textbf{p}| \).

It is necessary to emphasize that the wave-particle duality for one photon, as for any other single material particle, corresponds to different quantum states associated to noncommuting observables of position and momentum, because there is some confusion proposing that the photon is the dual, particle-like, partner of the wave-like electromagnetic field. The
electromagnetic field is not a quantum state for a particle exhibiting wave like behaviour. The electromagnetic field is a macroscopic observable of a system composed by a large number, or more precisely, an undetermined number of photons (we will see however one special case where all these photons are Bose-Einstein condensed in a single photon state). The explicit construction of the electromagnetic fields in terms of an ensemble of photons will be treated in another work.\textsuperscript{13}

VI. CONCLUSIONS

In this work devoted to the attempt to understand the quantum of electromagnetic radiation, we have described the single free photon, first as a classical relativistic particle that requires for its description an antisymmetric tensor, the photon tensor. In this description we developed an image of the photon as an essentially two dimensional object with its elements of physical reality on a plane orthogonal to the direction of propagation. As a consequence of special relativity, not of quantum mechanics, the energy of the photon is related to a frequency that we associate to a clockwise or counterclockwise rotation in the plane of the photon. This rotation is simultaneously related to the spin of the photon. It is shown that this image of the photon is compatible with the empirical fact that the energy depends linearly on the frequency of rotation and the spin is constant independent of the frequency.

The quantum mechanical description of a free photon was given in similar lines as the description of any particle, but with the complication that spin and momentum are coupled, or entangled by the transversality constraint arising from the massless character of the photon. This condition implies a preference for momentum eigenstates compared to localized states. It was shown that the quantum states of a photon can not be associated to the electromagnetic fields as could be suggested by a mistaken derivation of Maxwell’s equations from the photon Schrödinger’s equation.

Finally, in these attempts to understand the quantum of electromagnetic radiation, the basic assumption was made considering that the photons are the primary ontology, the building blocks, of electromagnetism and that the electromagnetic fields are a macroscopic construct emergent as a collective effect of an ensemble of photons. The details of the construction of the electromagnetic fields with a collection of photons will be the subject of another work.\textsuperscript{13}
In this section we build a toy model for a photon. For this, we show that it is possible to have a rotating energy distribution in a plane, such that the total energy $E$ is proportional to the frequency of rotation $\omega_0$ and the associated angular momentum $S$ is constant, i.e. independent of $\omega_0$. Furthermore $E = \omega_0 S$.

Let $\varepsilon(r)$ be the energy distribution in a plane, where $r$ is the radial coordinate. The total energy is then

$$ E = \int_0^\infty \! dr \, 2\pi r \, \varepsilon(r) \, . $$

(43)

The energy distribution is rotating with an angular frequency $\omega_0$ at the center $r = 0$. The rotation is not rigid and at any distance $r$ we have a local angular velocity $\omega(r)$ and a tangential velocity $v(r)$ related by $v(r) = r\omega(r)$. We must determine the local tangential and angular velocity in a way consistent with relativity, that is, for increasing $r$ the velocity should not exceed Einstein’s constant $c$. For this, let us consider two points at $r$ and $r + dr$. The velocity $v(r + dr)$ is given by the relativistic combination of the velocity $v(r)$ and the radial increase in velocity $dr \, \omega(r)$. That is,

$$ v(r + dr) = \frac{v(r) + dr \, \omega(r)}{1 + v(r) \, dr \, \omega(r)/c^2} \, . $$

(44)

That is, in terms of angular velocity alone,

$$ \omega(r + dr) = \frac{\omega(r)}{1 + r \, \omega^2(r) \, dr/c^2} \, . $$

(45)

With this we build the derivative

$$ \frac{d\omega(r)}{dr} = -\frac{\omega^3(r)}{c^2} \, r \, , $$

(46)

and integrating we get

$$ -c^2 \int_\omega_0^{\omega(r)} \! d\omega \, \omega^{-3} = \int_0^r \! dr \, r \, . $$

(47)

That is,

$$ \omega(r) = \frac{\omega_0}{\sqrt{\left(\frac{\omega_0 r}{c}\right)^2 + 1}} \, , $$

(48)

and the velocity is

$$ v(r) = r \, \omega(r) = \frac{\omega_0 r}{\sqrt{\left(\frac{\omega_0 r}{c}\right)^2 + 1}} \, . $$

(49)
With increasing $r$, the angular velocity decreases and the tangential velocity increases approaching the constant $c$.

In order to determine the angular momentum associated to the rotating energy density we consider that an element of area at position $r$ will have an effective mass $\varepsilon(r)/c^2$ and a tangential velocity $v(r)$ and therefore we have momentum density $\pi(r) = \frac{\varepsilon(r)}{c^2} \gamma(r) v(r)$ where $\gamma(r)$ is the Lorentz factor corresponding to the velocity $v(r)$. After some simple algebra we get

$$\pi(r) = \frac{\varepsilon(r)}{c^2} \omega_0 r . \quad (50)$$

Now, multiplying the momentum density by the radial distance and integrating we get the total angular momentum

$$S = \int_0^\infty dr \pi(r) r = 2\pi \int_0^\infty dr \varepsilon(r) \frac{\varepsilon(r)}{c^2} \omega_0 r . \quad (51)$$

Now, considering Eqs.(43,51) we can find a frequency depending energy distribution $\varepsilon(r)$ such that $S = \hbar$ and $E = \omega_0 \hbar$. There are many energy distributions that fulfill these conditions providing several toy models for the photon. The simplest example is a rotating disk of radius $R = \frac{\lambda}{\sqrt{2\pi}}$ with constant energy distribution

$$\varepsilon(r) = \begin{cases} \frac{\hbar \omega_0^3}{2\pi c^2}, & \text{for } r \leq \frac{\lambda}{\sqrt{2\pi}} \\ 0, & \text{for } r > \frac{\lambda}{\sqrt{2\pi}} \end{cases} . \quad (52)$$

In another example the energy distribution is not constant and has support in an annular region with internal radius $R_1$ and external radius $R_2$:

$$\varepsilon(r) = \begin{cases} 0, & \text{for } r < R_1 \\ K, & \text{for } R_1 \leq r \leq R_2 \\ 0, & \text{for } r > R_2 \end{cases} . \quad (53)$$

where the $K, R_1, R_2$ are given in terms of a parameter $k$ by

$$R_1 = \frac{\lambda}{2\pi} \left( \sqrt{k^2 + 1} - k \right) , \quad R_2 - R_1 = \frac{\lambda}{\pi} k , \quad K = \frac{\hbar c}{4\pi k} . \quad (54)$$

When $k \to 0$ the annular region becomes a ring of radius $\lambda/(2\pi)$. In these two examples the total surface where the energy doesn’t vanish, that is, the photon total cross section, depends on the photon energy as $\sigma_T \propto E^{-2}$.

Another last example for a toy photon is provided by an energy distribution along a rotating segment or string. In this case the expressions for energy and spin are
\[ E = \int_{-\infty}^{\infty} dx \varepsilon(x) , \quad (55) \]

\[ S = \int_{-\infty}^{\infty} dx x^2 \frac{\varepsilon(x)}{c^2} \omega_0 , \quad (56) \]

and an energy distribution

\[ \varepsilon(x) = \begin{cases} \frac{\hbar \omega_0^2}{2\sqrt{3}c}, & \text{for } |x| \leq \frac{\sqrt{3}}{2\pi} \lambda \\ 0, & \text{otherwise} \end{cases} , \quad (57) \]

will have \( E = \hbar \omega_0 \) and \( S = \hbar \).

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