A SIMPLIFIED DEFINITION OF LOGICALLY DISTRIBUTIVE CATEGORIES

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Abstract. In [Ben13] the notion of logically distributive category has been introduced to provide a sound and complete semantics to multi-sorted first-order logical theories based on intuitionistic logic. In this note, it will be shown that the definition of logically distributive category can be simplified by dropping two requirements, the one saying that product must distribute over sum, and the one saying that product must distribute over existential quantification.

The purpose of this note is to show how one can drop two conditions in the definition of logically distributive categories. We remind from [Ben13] the original definition:

**Definition 0.1** (Logically distributive category). Fixed a λ-signature Σ = (S, F, R, Ax), a category C together with a map M: ΛTypes(Σ) → Obj C is said to be logically distributive if it satisfies the following seven conditions:

1. C has finite products;
2. C has finite co-products;
3. C has exponentiation;
4. C is distributive, i.e., for every A, B, C ∈ Obj C, the arrow Δ = [1_A × i_1, 1_A × i_2]: (A × B) + (A × C) → A × (B + C) has an inverse, where [_, _] is the co-universal arrow of the (A × B) + (A × C) co-product, _ × _ is the product arrow, see [GoI06], 1_A is the identity arrow on A, and i_1: B → B + C, i_2: C → B + C are the canonical injections of the B + C co-product.
5. All the subcategories (1) have terminal objects, and all the subcategories C_{(1: x:A)} have initial objects;
6. The M map is such that
   a) M(0) = 0, the initial object of C;
7. For every s ∈ S, A ∈ ΛTypes(Σ), and x ∈ V_s, let Σ_A(x: s): LTerms(Σ)(s) → C be the functor from the discrete category LTerms(Σ)(s) = {t: s | t: s ∈ LTerms(Σ)} defined by t: s → M(A)(t(x)). Also, for every s ∈ S, A ∈ ΛTypes(Σ), and x ∈ V_s, let C_{(∀ x:A)} be the subcategory of C whose objects are the vertices of the cones on Σ_A(x: s) such that they are of the form MB for some B ∈ ΛTypes(Σ) and x: s /∈ FV(B).
8. For every s ∈ S, A ∈ ΛTypes(Σ), and x ∈ V_s, let C_{(∃ x:A)} be the subcategory of C whose objects are the vertices of the co-cones on Σ_A(x: s) such that they are of the form MB for some B ∈ ΛTypes(Σ) and x: s /∈ FV(B). Moreover, the arrows of C_{(∃ x:A)}, apart identities, are the arrows in the category of objects of C_{(∀ x:A)} as domain and M(∀ x:A) as co-domain. Finally, for every s ∈ S, A ∈ ΛTypes(Σ), and x ∈ V_s, let C_{(∃ x:A)} be the subcategory of C whose objects are the vertices of the co-cones on Σ_A(x: s) such that they are of the form MB for some B ∈ ΛTypes(Σ) and x: s /∈ FV(B).

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(b) $M(1) = 1$, the terminal object of $C$;
(c) $M(A \times B) = MA \times MB$, the binary product in $C$;
(d) $M(A + B) = MA + MB$, the binary co-product in $C$;
(e) $M(A \rightarrow B) = MB^A$, the exponential object in $C$;
(f) $M(\forall x : s. A)$ is the terminal object in the subcategory $C_{\forall x : s. A}$;
(g) $M(\exists x : s. A)$ is the initial object in the subcategory $C_{\exists x : s. A}$.

(7) For every $x \in V$, $A, B \in \lambda$Types($\Sigma$) with $x : s \not\in \text{FV}(A)$, $MA \times M(\exists x : s. B)$ is an object of $C_{\exists x : s. A \times B}$ since, if $\{M(\exists x : s.B), \{[\delta]_t : t \in \text{Terms}(\Sigma)\}\}$ is a co-cone over $\Sigma_B(x : s)$, and there is one by condition (5), then $\{MA \times M(\exists x : s.B), \{[\delta_A \times \delta_I]_t : t \in \text{Terms}(\Sigma)\}\}$ is a co-cone over $\Sigma_{A \times B}(x : s)$. Thus, there is a unique arrow $!: M(\exists x : s. A \times B) \rightarrow MA \times M(\exists x : s. B)$ in $C_{\exists x : s. A \times B}$. Our last condition requires that the arrow $!$ has an inverse.

Actually, since conditions (1) and (3) amount to say that $C$ is Cartesian closed\(^1\), the product functors $A \times -$ preserve co-limits, because it has the exponential functor $(-)^A$ as right adjoint — a standard check that can be found in any textbook, see, e.g., [Bor94].

So, by condition (2), one can deduce the existence of the arrow required in condition (4). Actually, it is more convenient to have a direct construction of the arrows making the interpretations of $A \times (B + C)$ and $(A \times B) + (A \times C)$ equivalent. This can be found in [Ben14].

By this result, condition (4) is redundant.

In [Ben14], the natural bijection $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C^B)$ has been explicitly constructed. Specifically, $\text{tr}: \text{Hom}(A \times B, C) \rightarrow \text{Hom}(A, C^B)$, which maps an arrow to its exponential transpose, is the isomorphism in one direction, and $\theta$ works in the opposite direction, where $\theta$ derives from a co-universal construction.

Turning to condition (7), we want to show that it is redundant, too. This amount to construct an inverse for the unique arrow

$$\alpha: M(\exists x : s. A \times B) \rightarrow MA \times M(\exists x : s. B)$$

with $x : s \not\in \text{FV}(A)$.

The corresponding diagram in $\langle C, M \rangle$ is

$$\begin{array}{ccc}
M(\exists x : s. A \times B) & \xrightarrow{\alpha} & MA \times M(\exists x : s. B) \\
\downarrow{\exp_1_{t:s}} & & \downarrow{\exp_1_{t:s}} \\
\{MA \times M(B[t/x])\}_{t:s} & \xrightarrow{\{\delta_I\}_t_{(t/s)}} & \{MA \times M(B[t/x])\}_{t:s}
\end{array}$$

so that we have two co-cones and $\alpha$ makes the diagram to commute, i.e., it is an arrow in the category of co-cones. Also, we know that $M(\exists x : s. A \times B)$ is the initial object in the sub-category $C_{\exists x : s. A \times B}$, i.e., a co-limit.

We may think to the co-cone on the right of the diagram (1) as the one produced by applying the product functor $(A \times -)$ to the following co-cone

$$\begin{array}{c}
M(\exists x : s. B) \\
\downarrow{\exp_1_{t:s}} \\
\{M(B[t/x])\}_{t:s}
\end{array}$$

\(^1\)Some authors define Cartesian closed categories as those having all finite limits and exponentiation, and reserve the adjective "weakly" for the categories having just products and exponentiation. Since the result holds for both definition, we avoid the adjective.
Now, consider any co-cone of the form

\[
\begin{array}{c}
\text{MC} \\
\{1_{\text{MC}} \times \text{ev}\}_{1:s} \\
\{\text{MA} \times M(B[t/s])\}_{1:s}
\end{array}
\]

where \( x:s \not\in \text{FV}_C \), i.e., any co-cone whose vertex lies in \( C_{\exists x:s.A \times B} \).

By exponentiation, we can construct

\[
\begin{array}{c}
\{\text{MA} \times M(B[t/s])\}_{1:s} \\
\{1_{\text{MA}} \times \text{tr}p_{1}\}_{1:s} \\
\{\text{ev}\}_{1:s}
\end{array}
\]

Since \( MC^{\text{MA}} = M(A \supset C) \), \( x:s \not\in \text{FV}(A \supset C) \), thus

\[
\begin{array}{c}
\text{MA} \times MC^{\text{MA}} \\
\{\text{ev}\}_{1:s}
\end{array}
\]

is a co-cone whose vertex is in \( C_{\exists x:s.B} \).

So, there is a co-universal arrow \( \gamma \) as in the diagram

\[
\begin{array}{c}
\text{MC} \\
\{\text{ev}\}_{1:s}
\end{array}
\]

Recalling that \( \theta \) is the inverse of \( \text{tr} \), as previously remarked, and applying the product functor, we get

\[
\begin{array}{c}
\text{MC} \\
\{\text{ev}\}_{1:s}
\end{array}
\]

So, \( \text{MA} \times M(\exists x:s.B) \) is initial in \( C_{\exists x:s.A \times B} \) and thus isomorphic to \( M(\exists x:s.A \times B) \) via the \( \alpha \) arrow of diagram (1) in one direction, and the \( \theta(\gamma) \) arrow above in the opposite.

REFERENCES

[Ben13] Marco Benini, *Intuitionistic first-order logic: Categorical semantics via the Curry-Howard isomorphism*, Tech. report, arXiv.org, 2013, http://arxiv.org/abs/1307.0188.

[Ben14] Cartesiann closed categories are distributive, Tech. report, arXiv.org, 2014, http://arxiv.org/abs/1406.0961.

[Bor94] Francis Borceux, *Handbook of categorical algebra 1: Basic category theory*, Encyclopedia of Mathematics and its Applications, no. 50, Cambridge University Press, Cambridge, UK, 1994, ISBN: 978-0-521-06119-3.

[Gol06] Robert Goldblatt, *Topoi: The categorical analysis of logic*, Dover, Mineola, NY, USA, 2006, ISBN: 0-486-45026-0.