Applications of the Fictitious Compress Recovery Approach in Physical Geodesy

SHEN Wenbin  LI Jiancheng  LI Jin  NING Jinsheng  CHAO Dingbo

Abstract The fictitious compress recovery approach is introduced, which could be applied to the establishment of the Runge-Krarup theorem, the determination of the Bjerhammar’s fictitious gravity anomaly, the solution of the “downward continuation” problem of the gravity field, the confirmation of the convergence of the spherical harmonic expansion series of the Earth’s potential field, and the gravity field determination in three cases: gravitational potential case, gravitation case, and gravitational gradient case. Several tests using simulation experiments show that the fictitious compress recovery approach shows promise in physical geodesy applications.

Keywords fictitious compress recovery; physical geodesy; gravity field; determination

Introduction

Concerning the determination of the Earth’s external gravity field, Stokes’s method\cite{1} is simple but does not fit reality so well; Molodensky’s method\cite{2} and Bjerhammar’s method\cite{3} fit reality better, but they are complicated. In recent decades, a lot of methods for determining the Earth’s gravity field have been put forward. It is not the aim of this paper to judge the advantages and disadvantages of the various methods. However, it is well known that in physical geodesy the following two problems are still open, at least in theory\cite{4-10}: ① whether the spherical harmonic expansion series expressing the Earth’s potential field converges in the domain between the Earth’s surface and the surface of the Brillouin sphere\cite{11}, simply referred to as the convergence problem; ② how to determine the Earth’s external gravity field based only on the data (e.g., potential, gravitation or gravitational gradients) given on a simply closed surface (e.g., the surface corresponding to the satellite altitude, simply referred to as the “downward continuation” problem).

The method which has been put forward recently, the fictitious compress recovery approach\cite{12}, is simple, and based on this approach the above-mentioned two problems could be solved.

1 The fictitious compress recovery approach

Referring to [12,13], the basic idea of fictitious

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SHEN Wenbin, School of Geodesy and Geomatics, Wuhan University, 129 Luoyu Road, Wuhan 430079, China; The Key Laboratory of Geospace Environment and Geodesy, Wuhan University, 129 Luoyu Road, Wuhan 430079, China.

E-mail: wbshen@sgg.whu.edu.cn
compress recovery approach is that the “compress” and “recovery” procedures are successively executed (iterative approach) between the given boundary (the Earth’s surface or the satellite surface) and the surface of an inner sphere (or Bjerrum sphere) that lies inside the Earth, and finally one gets a fictitious harmonic series solution outside the inner sphere, which coincides with the Earth’s real field in the whole domain outside the Earth, no matter whether the given boundary is on the Earth’s surface, $\partial \Omega$, or on the satellite surface $\partial S$.[14].

Suppose there exists an unknown regular harmonic field $u(P)$ defined in the domain $\Omega$, the domain outside the Earth, and suppose the harmonic field $u(P)$ is uniquely determined by the given boundary value $u(P)|_{\partial \Omega}$ on the Earth’s surface $\partial \Omega$. The problem is to find the harmonic field $u(P)(P \in \Omega)$. The realization procedure could be stated as follows.

Set

$$\Theta^{(0)}(P) \equiv u(P), P \in \Omega$$

$$\Theta^{(n)}(P) = \Theta^{(n-1)}(P) - u^{(n)}(P), \quad P \in \Omega, n \geq 1$$

where $\Theta^{(n-1)}(P)(n \geq 1)$ is called the $(n-1)^{\text{th}}$-order residual (potential) field. Note that the $(n-1)^{\text{th}}$-order residual field (including the 0-order residual field) is only defined in the domain $\Omega$, and can be expressed on the Earth’s boundary as follows

$$\Theta_{\partial \Omega}^{(n)} \equiv u_{\partial \Omega}$$

$$\Theta_{\partial \Omega}^{(n)} = \Theta_{\partial \Omega}^{(n-1)} - u_{\partial \Omega}^{(n)}, \quad n \geq 1$$

Compress the $(n-1)^{\text{th}}$ -residual boundary value $\Theta_{\partial \Omega}^{(n-1)}$ on the surface $\partial K$ of an inner sphere $K$, that lies inside the Earth along the radial direction, searching for the solution of the following boundary value problem

$$\left\{ \begin{array}{l}
\Delta u^{(n)}(P) = 0, P \in K_i \\
u^{(n)}(P)|_{\partial K_i} = \Theta_{\partial \Omega}^{(n-1)} \\
\lim_{P \to \infty} u^{(n)}(P) = 0, n \geq 1
\end{array} \right.$$  

(3)

the solution is given by the Poisson integral

$$u^{(n)}(P) = \frac{\rho^2 - R^2}{4\pi R} \int_{\partial K} \frac{\Theta_{\partial \Omega}^{(n-1)}}{l^2} \mathrm{d}\sigma,$$  

(4)

$$P \in K_i, n \geq 1$$

which is a regular harmonic function in the domain $K_i$; the domain outside the inner sphere is $K$, with radius $R$, where $d\sigma$ is the surface integral element. When the above solution is constrained in the Earth’s external domain $\Omega$, it can be taken as the first approximation of the $(n-1)^{\text{th}}$-order residual field $\Theta^{(n-1)}(P)$, or $\sum_{n=1}^{\infty} u^{(n)}(P)$ can be taken as the $n^{\text{th}}$ approximation of the field $u(P)$.

Hence, one gets a series of the solution in the domain $K$:

$$u'(P) = \sum_{n=1}^{\infty} u^{(n)}(P), P \in K,$$  

(5)

which coincides with the real field $u(P)$ in $\Omega$,[15], and on the boundary $\partial K$ it holds that

$$u'(P)|_{\partial K} = \sum_{n=0}^{\infty} \Theta^{(n)}|_{\partial K},$$  

(6)

where $\Theta^{(n)}|_{\partial K}$ is determined by Eq.(2).

### 2 Applications in physical geodesy

#### 2.1 Runge-Krarup theorem

Based on Runge’s theorem[5], Krarup derived a result which could be briefly stated as follows[15]; any regular harmonic function $\phi$ defined in the domain outside the Earth can be always uniformly and infinitely approximated by a regular harmonic function defined in the domain outside an inner sphere that lies inside the Earth. This result is referred to as the Runge-Krarup theorem[5,16].

Now, suppose the function $V(P)$ (e.g., the gravitational potential field) is regular and harmonic in $\Omega$, which is uniquely determined by the boundary value $V_{\partial \Omega}$, then, based on the fictitious compress recovery approach, a fictitious regular harmonic function $V'(P)(P \in K_i)$ can be found out, which coincides exactly with the original field $V(P)$ in the domain $\Omega$. Hence, the Runge-Krarup theorem is derived. In fact, based on the fictitious compress recovery approach, “a regular harmonic function $\psi$ defined in the domain outside an inner sphere that lies inside the Earth” can be found, which not only uniformly and infinitely approximates the real field, but also coincides with the real field.
2.2 Bjerhammar’s fictitious gravity anomaly

With Bjerhammar’s method to determine the fictitious gravity anomaly \( \Delta g^* \), one needs to solve an integral equation. If one applies the fictitious compress recovery approach, the solution procedure is simple.

Not strictly speaking, 
\[
u(P) = r \Delta g(P), P \in \Omega\]
\[\text{(7)}\]
is regular and harmonic in the domain \( \Omega \), where \( \Delta g(P) \) is the difference between the gravity and the normal gravity on the Earth’s surface. Given the boundary value 
\[
u(P)_{|_{\partial \Omega}} = [r \Delta g(P)]_{|_{\partial \Omega}}\]
\[\text{(8)}\]
and then based on the fictitious compress recovery approach, the fictitious field \( u^*(P) \) could be easily determined:
\[
u^*(P) = \sum_{n=1}^{\infty} u^{(n)}(P), P \in \overline{K},\]
\[\text{(9)}\]
which coincides with the real field \( u(P) \) in the domain \( \Omega \). Taking into account Eq.(7), on the boundary of the inner sphere one gets
\[
u^*(P)_{|_{\partial \Omega}} = [r \Delta g(P)]_{|_{\partial \Omega}} = R_i [\Delta g(P)]_{|_{\partial \Omega}}\]
\[\text{(10)}\]
and consequently the fictitious gravity anomaly \( (\Delta g)^* \) required in Bjerhammar’s method is determined.

2.3 The “downward continuation”

Suppose the boundary value \( V(P)_{|_{\partial \Omega}} \) (or gravitation \( V_i(P)_{|_{\partial \Omega}} \) or gravitational gradients \( V_{ij}(P)_{|_{\partial \Omega}} \)) on the satellite surface is given. It is noted that the boundary value \( V_{|_{\partial \Omega}} \) on the satellite surface might be determined by using the well known energy integral approach\[17,18\]. To determine the real field \( V(P) \) in the domain \( \Omega - S \), the domain between the Earth’s physical surface \( \partial \Omega \) and the surface \( \partial S \), the “downward continuation” problem occurs, which has not been solved satisfactorily by using conventional methods due to the “ill-posed” problem and/or divergence problem\[4-9\].

The fictitious compress recovery approach can be applied in determining the Earth’s external field \( V(P) (P \in \Omega) \), provided that a boundary value \( V(P)_{|_{\partial \Omega}} \) on a “simply closed surface” \( \partial S \) (it means that there exists a one-to-one continuous map \( f \) between \( \partial S \) and \( \partial K \)) is given, where the closed surface \( \partial K \) completely encloses the whole Earth. Based on the boundary value \( V_{|_{\partial \Omega}} \) and the fictitious compress recovery approach, a fictitious field \( V^*(P) (P \in \overline{K}) \) can be determined, which coincides with the real field \( V(P) \) in the domain outside the surface \( \partial S \). It has been further proven that the determined fictitious field \( V^*(P) (P \in \overline{K}) \) coincides with the real field \( V(P) \) in the whole domain outside the Earth\[13,14\]. Consequently, the “downward continuation” problem is solved. This approach is referred to as the “fictitious downward continuation”\[19-21\].

2.4 The convergence of the spherical harmonic expansion series

Given continuous or discrete boundary values on \( \partial \Omega \) or \( \partial S \), in practical applications, the Earth’s global gravity field is determined based on the spherical harmonic expansion series. However, it cannot be guaranteed that the considered series is convergent in the domain between the Earth’s surface and the surface of the Brillouin sphere. Although the truncation technique is applied, the theoretical problem still exits: if the series is divergent, we should not take more and more terms; but if a more precise result is required, we need more and more terms.

Now we apply the fictitious compress recovery approach. Based on the given boundary value \( V_{|_{\partial \Omega}} \) (or \( V_{ij|_{\partial \Omega}} \)), a fictitious field \( V^*(P) \) is determined, which is regular and harmonic in \( \overline{K} \), and coincides with the real field \( V(P) \) in \( \Omega \). Since \( V^*(P) (P \in \overline{K}) \) is regular and harmonic, it could be expressed as a uniformly convergent spherical harmonic series in the domain \( \overline{K} \)[22]. Further, since it holds that
\[
V(P) = V^*(P), P \in \overline{\Omega}\]
\[\text{(11)}\]
the following conclusion is achieved: the Earth’s potential field \( V(P) \) could be expressed as a uniformly convergent spherical harmonic expansion series in the whole domain outside the Earth. Consequently, based on the fictitious compress recovery approach, the convergence problem is solved.

2.5 Gravity field determination in three cases

For any regular harmonic function \( u(P) \) in \( \Omega \), if the boundary value \( u(P)_{|_{\partial S}} \) is given, and where
the boundary $\partial S$ is a simply closed surface that encloses the whole Earth, then the real field $u(P)(P \in \Omega)$ can be determined based on the fictitious compress recovery approach.

Suppose that on the boundary $\partial S$ the gravitational potential $V(P)$ is given \[18\], or the gravitation $V_i(P) (i = 1, 2, 3)$ is given, or the gravitational gradients $V_{ij}(P) (i = 1, 2, 3)$ are given. Concerning the potential $V(P)$, the gravitation components $V_i(P)$, and the gravitational gradients $V_{ij}(P)$, all of them are regular and harmonic in $\Omega$ (in a rectangular coordinate system). Hence, applying the fictitious compress recovery approach, the real potential field $V(P)(P \in \overline{\Omega})$ , or the real gravitation field $V_i(P)(P \in \overline{\Omega})$, or the real gravitational gradient field $V_{ij}(P)(P \in \overline{\Omega})$, could be determined based only on the given boundary values $V_{\partial S}$, or $V_i|_{\partial S}$, or $V_{ij}|_{\partial S}$.

3 Experiments

3.1 Simulation test 1 \[20\]

With a $10^5 \times 10^6$ grid and discrete values obtained by the EGM96 model, the (disturbing) potential values $V_{ak}$ (648 point values ranging from $-83.676$ m$^2$·s$^{-2}$ to $+64.869$ m$^2$·s$^{-2}$) on the boundary $\partial K_1$ of a smaller sphere $K_1$ with radius $R_1 = 6338$ km are given \[20\]. Then the potential values $V_{ak}$ on the “satellite surface” $\partial K_2$ are calculated by using the Poisson integral. Suppose we know only the boundary values $V_{ak}$, the problem is to determine the real field in the domain outside $K_1$. Based on the fictitious compress recovery approach and the given values $V_{ak}$, the fictitious field $V'(P)$ is determined. By comparison, it is found that the determined fictitious field $V'(P)$ coincides with the real values $V_{ak}$ on the boundary $\partial K_1$ at least under the accuracy (RMS) level 0.1 cm, and the largest difference $\Delta V = V - V'$ on the boundary $\partial K_1$ is 0.4 cm. Hence, the preliminary experiment shows that both the fictitious compress recovery approach and the “fictitious downward continuation” might be valid and reliable \[20\].

3.2 Simulation test 2 \[21\]

Choose two spherical boundaries $\partial K_1$ and $\partial K_2$ of two spheres $K_1$ and $K_2$ with radii $R_1 = 55$ km and $R_2 = 80.35$ km, respectively, where $K_1$ is taken as the inner sphere, and choose two mass points $Gm_1 = Gm_2 = 100$ m$^3$·s$^{-2}$, which are located outside and inside the smaller sphere, respectively \[21\].

In the experimental test, using the discrete approach, dividing the spherical surface into $10^5 \times 10^6$ grids, consequently there are 64800 point values on the boundary $\partial K_2$ : $V'_{iak} (i = 1, 2, \cdots, 64800)$. Then based on the fictitious compress recovery approach, the fictitious boundary values $V'_{iak} (i = 1, 2, \cdots, 64800)$ are determined (which are stored for future use), based on which the fictitious field $V'(P)(P \in \overline{K_1})$ can be determined, i.e., the value $V'(P)$ at any point in $\overline{K_1}$ could be determined. 11 test points were chosen, which are listed in Table 1. The results are listed in Table 2: the second column lists the real values $V(P_j)$ at test points $P_j (j = 1, 2, \cdots, 11)$, the third column lists the “fictitious values” corresponding to the test points $P_j$, with the iterative procedure times $N = 30$, where the “fictitious values” were calculated based on the Poisson integral by using the “fictitious boundary values” $V'_{iak}$, which were determined by the fictitious compress recovery approach, and the last column lists the differences between the real values and the “fictitious values” at the test points $P_j$.

The first point mass $m_1 = m$ and second point mass $m_2 = m$ are located at $(r, \phi, \lambda) = (30.0, 90, 90)$ and $(r, \phi, \lambda) = (63.0, 90, 90)$, respectively. The fictitious compress recovery approach predicts that all the fictitious values $V'(P_j) (j = 1, 2, \cdots, 11)$ on the test points should be coinciding with the corresponding real values $V(P_j) (j = 1, 2, \cdots, 11)$ except for the test point $P_1 (60.5, 90, 90)$, because it is located at the position directly under the point mass $m_2$ that is located at point $(63.0, 90, 90)$, and in this case, one cannot construct a “simply closed surface” $\partial S$ which encloses point masses $m_1$ and $m_2$ but does not enclose the first test point $P_1$ so that there exists a one-to-one continuous map between $\partial \Omega$ and $\partial K_1$. Hence, the experimental test supports the “fictitious downward continuation”.

Since the potential obeys the supposition principle, the above experimental results have important meaning: for a body of mass (e.g., the Earth), the fictitious
compress recovery approach, as well as the “fictitious downward continuation”, is valid and reliable.

Table 1  A list of the coordinates at test points

| Point No. | Radius /km | Co-latitude /° | Longitude /° |
|-----------|------------|----------------|--------------|
| 1         | 60.5       | 90.0           | 90.0         |
| 2         | 67.0       | 80.0           | 60.0         |
| 3         | 100.0      | 20.0           | 30.0         |
| 4         | 100.0      | 50.0           | 90.0         |
| 5         | 100.0      | 120.0          | 150.0        |
| 6         | 100.0      | 160.0          | 70.0         |
| 7         | 90.0       | 60.0           | 80.0         |
| 8         | 70.0       | 160.0          | 30.0         |
| 9         | 70.0       | 80.0           | 30.0         |
| 10        | 72.0       | 120.0          | 80.0         |
| 11        | 120.0      | 30.0           | 80.0         |

Table 2  Experimental results at 11 test points/(m²·s⁻²)

| P. No. | $V(P_j)$ | $V'(P_j)_{VS}$ | $\Delta V$ |
|--------|----------|----------------|------------|
| 1      | 68.147 498 3 | 20.362 421 0 | 49.785 077 276 |
| 2      | 4.291 468 1  | 4.237 077 1  | 0.054 390 962 |
| 3      | 2.302 151 2  | 2.302 461 7  | -0.000 310 552 |
| 4      | 2.708 610 8  | 2.708 806 0  | -0.000 195 131 |
| 5      | 1.934 462 7  | 1.933 830 6  | 0.000 632 143  |
| 6      | 1.784 740 3  | 1.784 804 5  | -0.000 646 234 |
| 7      | 3.290 160 7  | 3.302 812 4  | -0.012 651 715 |
| 8      | 2.186 571 3  | 2.189 414 3  | -0.002 842 985 |
| 9      | 2.900 040 0  | 2.903 915 1  | -0.003 875 168 |
| 10     | 3.761 099 0  | 3.761 877 9  | -0.000 778 967 |
| 11     | 2.007 407 7  | 2.006 954 8  | 0.000 452 841  |

3.3 Simulation test 3[23]

Choose a gravitational potential model[23], a 4-sphere anomaly model: three small spheres $O_i$ (i =1, 2, 3) are located inside a large sphere $O_L$. Based on this model, the real potential field outside the large sphere (which simulates the surface of the “Earth”) is known.

It is supposed that only the boundary value $V_{vs}$ on a satellite surface $\partial S$ is known, calculated from the given model, and the aim is to determine the real field $V(P)$ in the domain outside the “Earth”, based only on the given boundary value $V_{vs}$, which, in practice, is supposed to be obtained based on a polar satellite. The satellite surface is supposed to be a rotation-ellipsoidal surface, with its geometric centre coinciding with the coordinate origin, the major-axis $a = 6371 + 250$ km, and the eccentricity $e = 0.01$. The radius of the inner sphere is taken as 6 000 km. The simulation calculations are executed based on a $1° \times 1°$ grid approach.

With the given (discrete) boundary values $V_{vs}$, the fictitious distribution on the surface of the inner sphere is determined based on the fictitious compress recovery approach (Cf. Sec.2), and consequently the fictitious field $V'$ in the domain outside the inner sphere is determined. In theory, the fictitious field coincides with the real field in the domain outside the “Earth”.

Based on the calculations[23], the following conclusions can be drawn: based on the boundary value and the fictitious compress recovery approach, after 15 iterative procedures, one gets a fictitious (regular harmonic) field which coincides with the real field on the surface of the “Earth” under the accuracy (RMS) level 0.1 mm (note that 0.001 22ms⁻² corresponds to the height 0.1 mm). And based on the extreme value principle[22], one can conclude that the fictitious field coincides with the real field in the whole domain outside the “Earth” at least under the accuracy (RMS) level 0.1 mm.

4 Conclusions

Concerning the boundary value problem, the fictitious compress recovery approach could be applied not only in physical geodesy, but also in any field related to the determination of a regular harmonic field based on the given boundary value. For instance, applying the fictitious compress recovery approach, the stable magnetic field generated by the Earth could be determined, provided that the boundary value on the Earth’s surface or a satellite surface is given.

Also, the corresponding fictitious compress recovery approach for solving the internal problem might be formulated. However, geodesists have little interest in this problem.

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