Fermion localization on a split brane

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Abstract

In this work we analyze the localization of fermions on a brane embedded in five-dimensional, warped and non-warped, space-time. In both cases we use the same non-linear theoretical model with a non-polynomial potential featuring a self-interacting scalar field whose minimum energy solution is a soliton (a kink) which can be continuously deformed into a two-kink. Thus a single brane splits into two branes. The behavior of spin 1/2 fermions wavefunctions on the split brane depends on the coupling of fermions to the scalar field and on the geometry of the space-time.

1 Introduction

The idea that our Universe can be realized inside a domain wall embedded in a (4,1)-dimensional world [1] has provided many creative ways to solve the hierarchy of interactions problem [2]-[3] and the cosmological constant problem [4] in worlds with large extra dimension, without resorting to the compactification of the extra dimension. It has also been shown [5] that the effective gravitational potential between two particles recovers the Newtonian behavior, since one has localization of gravitons on a thin brane in five-dimensional space-time with a warped geometry and the cosmological constant is related to the brane tension. Localization of matter (spin-zero, spin-1/2 and spin-3/2) in the Randall-Sundrum framework was shown to be possible, under certain conditions over the brane tension [6]. The localization of spin 1/2 fermions on thin branes is due to a soliton, via the mechanism created by Jackiw and Rebbi [7] to demonstrate the phenomenon of fermion charge fractionization. By its turn, the domain wall which we would live in, according to the scenario proposed by Rubakov and Shaposhnikov [1], is the topological defect provided by the very same soliton.

The Rubakov-Shaposhnikov scenario [1] has been extended to five-dimensional warped space-time by means of a self-interacting scalar field, or a set of (self-)interacting scalar fields
also coupled to gravity \cite{8}. In those nonlinear models, also inspired on previous studies on the stabilization of gravity fluctuations on domain walls in supergravity theories \cite{9}, the thick branes solutions are minimum energy configurations (in many cases they are Bogomol’nyi-Prasad-Sommerfield (BPS) solutions) that separates the space in two patches with a peculiar warp-factor whose asymptotic behavior is an anti-de Sitter (AdS) space.

In some models the minimum energy configurations are wide solitons \cite{10}-\cite{12}, also called double-wall. Double-wall configurations also appear as solutions, degenerate in energy \cite{13}, of a model with two interacting scalar fields \cite{14}-\cite{15}, and they have been called degenerate Bloch branes. The model used in \cite{14} also comprises critical Bloch branes \cite{13}, which are analogues of the extreme domain walls described in \cite{9}. Such a variety of solutions \cite{16} is due to an arbitrary constant of integration of the equations of motion. The continuous deformation from a single brane (domain wall) to a double-brane, up to an extreme one, by varying the constant of integration, is similar to the phenomenon of brane splitting discussed in \cite{17}, in analogy to first-order phase transitions in condensed matter systems. Such a transition is usually approached by using a sixth degree polynomial potential, as was done in \cite{17}, which is characterized by the increase of a disordered phase (a wet phase in surface physics) or an increase of the thickness of the domain wall (brane). As the temperature of the system approaches a certain limit, one has the appearance of two interfaces between the disordered phase and the ordered ones, that is, the formation of a double-wall. That phenomenon has been called brane splitting \cite{17}. As the temperature of the system goes towards a critical one, a complete domination of the disordered phase (complete wetting) might happen, which we interpret, in brane worlds scenario, as the formation of an extreme brane. Such a phase transition can also be described by effective non-polynomial potentials \cite{18} constructed from the model with two scalar fields \cite{14}, \cite{16}, \cite{13} with the advantage of having a model whose minimum energy solutions are the BPS ones.

The localization of fermions on a double-wall (double-brane) in warped space-time has been studied in \cite{19} and in \cite{20}. It has been shown that double-wall can localize massless fermions. Notwithstanding, a close inspection on the behavior of the zero-mode eigenfunction reveals that it is peaked just in the region between the branes and has tails inside the branes, where the possibility of detecting massless fermions is very small. It can be noticed that this behavior is due to the Yukawa coupling, \(\phi\bar{\Psi}\Psi\), of fermions to the soliton. We think that the peak of the zero-mode eigenfunction should follows the trend of the brane, that is, it should also split. In this work we show that this behavior can be achieved by means of a convenient coupling of the fermion to the scalar field which is reminiscent from \(N = 1\) supersymmetry (SUSY), namely \(W_{\phi\phi}\bar{\Psi}\Psi\), where \(W_{\phi\phi}\) is the 2nd derivative of the superpotential, of the field theory model, with respect to the field taken at the BPS configuration.

We develop the calculations by using one of the models obtained in \cite{18}, but they could also be done by using any other nonlinear model which admits two-kink solutions for the scalar field. We also analyze the localization of massless and massive fermions in the Rubakov-Shaposhnikov framework \cite{1}, that is, in a non-warped space-time. In this latter case one clearly sees that the coupling \(\phi\bar{\Psi}\Psi\) provides an effective single well potential where the fermions would be trapped in, but that well is just in the region between the branes, while the coupling \(W_{\phi\phi}\bar{\Psi}\Psi\) provides a double-well potential with wells in the cores of the branes, as it should be. In the case of flat space-time, the Numerov method is used to obtain the massive localized modes, with emphasis on the SUSY inspired coupling with which there
is room to prepare a chiral mixed state of quasi-degenerate massless and tiny mass fermion states. We also show that such a mixed state can tunnel between the two branes with time of tunneling inversely proportional to the tiny mass which, in its turn, decreases as the distance between the walls increases. We would like to warn the reader that the brane splitting is not considered here as a dynamic process; in fact, the distance between the walls is one of the parameters of the nonlinear potential that could be seen as dependent on the temperature and the tunneling of fermions is analyzed at a fixed distance between the walls, that is, at a given temperature close to the critical temperature for the formation of an extreme wall.

In the next section we show the main features of the nonlinear model coupled to gravity in 5-dimensional space-time [18]. In the third section we are concerned to the localization of massless fermions on a split brane in warped space-time. In the fourth section we deal with the localization and tunneling of massive fermions in flat 5-dimensional space-time. A few remarks on fermion localization on double-walls are left to the last section.

2 The model, the double-brane and the warp factor

The model we are going to deal with includes a self-interacting scalar field coupled minimally to gravity with one extra space dimension, denoted by $r$. The action that leads to Euler-Lagrange for the scalar field and Einstein equations is given by

$$S = \int d^4 x dy \sqrt{|g|} \left( -\frac{1}{4} R + \frac{1}{2} g_{ab} \partial^a \phi \partial^b \phi - V(\phi) \right),$$

where $g \equiv \text{Det}(g_{ab})$ and the metric is

$$ds^2 = g_{ab}dx^a dx^b = e^{2A(r)} \eta_{\mu \nu} dx^\mu dx^\nu - dr^2, \quad a, b = 0, \ldots, 4,$$

where $\eta_{\mu \nu}$ is the Minkowski metric and $e^{2A(r)}$ is the warp factor, which is supposed to depend only on the extra dimension $r$. The Greek indices run from 0 to 3.

We consider that the potential $V(\phi)$ can be written as

$$V(\phi) = \frac{1}{2} \left( \frac{dW(\phi)}{d\phi} \right)^2 - \frac{4}{3} (W(\phi))^2,$$

In this case the BPS solution of the following first-order differential equations

$$\frac{d\phi}{dr} = \pm \frac{dW(\phi)}{d\phi} \quad \text{and} \quad \frac{dA}{dr} = \mp \frac{2}{3} W(\phi)$$

are also solutions of the second-order differential equations of motion in the static limit.

By taking $W(\phi)$, which we call the superpotential, given by

$$W(\phi) = 2 \mu \left[ \phi \left( \frac{\phi^2}{3} + f^2 + \frac{b}{2} \sqrt{\phi^2 + f^2} \right) + \frac{bf^2}{2} \sinh^{-1} \left( \frac{\phi}{f} \right) \right],$$

with $f = \sqrt{b^2 - a^2}$ and $b < -a < 0$, we have

$$\frac{dW(\phi)}{d\phi} = W_\phi(\phi) = 2 \mu (\phi^2 + f^2 + b \sqrt{\phi^2 + f^2}),$$

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and the solutions of the first-order differential equations (4) can be found straightforwardly. The classical configuration for the scalar field is given by

$$\phi = \pm a \frac{\sinh(2\mu ar)}{\cosh(2\mu ar) - b/f},$$

(7)

where the (lower)upper sign stands for (anti-)kink configuration. The reference [18] has more details on the soliton profile, as well as on the behavior of the warp factor in this model. As has been noted in the second paper of reference [16], the solution (7) can be conveniently written as

$$\phi = \pm a \left[ \tanh(\mu a(r + L)) + \tanh(\mu a(r - L)) \right],$$

(8)

where we have identified $b = -\cosh(2L)/\sqrt{\cosh(2L)^2 - 1}$. The above expression can be seen as a merging of two solitons (two kinks), namely $\phi_\pm = \frac{a}{2}[\pm 1 + \tanh(\mu a(r \mp L))]$, which are localized at $r = -L$ and $r = +L$.

In flat two-dimensional space-time one constructs such a scalar model as the bosonic sector of a $N = 1$ SUSY model with the potential given by only the first contribution at the right hand side of the equation (3). In such a circumstance one can clearly see that the potential has two minima at $\phi = \pm a$ and one additional minimum appears at $\phi = 0$ as $b = -a$. This transition in the potential is reflected in the behavior of the soliton in (8), which is continuously deformed from one kink into a two-kink profile and finally in one of the solutions, $\phi_\pm$, for $b = -a$ (in this critical case $L$ is replaced by an arbitrary constant of integration). A first-order phase transition takes place. It is characterized by the growing of the disordered phase with the formation of two interfaces separating the disordered phase from the ordered ones, up to the complete wetting in the critical limit. The phase transition can also be seen from the behavior of the warp factor, discussed in [18]. One can see that the warp factor separates the space along the extra dimension in two similar regions, whose asymptotic behavior is an AdS$_5$ space and, as $L$ increases, a double-brane structure is triggered at a specific value of $L$. Such a double-brane structure is evident from the peaks of the Ricci scalar $R = -(8 \, d^2A/dr^2 + 20 \, (dA/dr)^2)$ just at the points the core of the branes are localized (see Figure 1 below). Moreover, in the limit $b = -a$ the branes are infinitely separated from each other, there being the formation of an extreme brane. Such an extreme case is discussed in references [13] and [21] where the brane is called a critical Bloch brane. In the next sections we focus only on the localization of massless fermions on single brane that splits into two brane, that we also call double-brane or double-wall.

3 Localization of fermions: warped space-time

In this section we analyze the localization of massless fermions on the brane described in the previous section by taking into account also the curvature of the space. The problem is approached by searching normalized solutions of the Dirac equation for fermions coupled with the scalar field by a general Yukawa coupling $\bar{\Psi} F(\phi) \Psi$, where $F(\phi)$ is a functional of the field $\phi(r)$ taken at the classical solution. The functional form of $F(\phi)$, as well as the dependence of the metric on $r$, is crucial for the possible localization of massless fermions on the brane, as we discuss below.
The fermion action defined as

$$S_{\text{fermion}} = \int d^5 x \sqrt{-g} \left( \bar{\Psi} i \Gamma^a D_a \Psi - \bar{\Psi} F(\phi) \Psi \right),$$

leads to the Dirac equation

$$[i \Gamma^a D_a - F(\phi)] \Psi = 0.$$  \hfill (10)

The gamma matrices satisfy the algebra $$\{ \Gamma^a, \Gamma^b \} = 2g^{ab},$$ and can be defined in the irreducible representation as $$\Gamma^\mu = e^{-A(y)} \gamma^\mu, \ \Gamma^5 = -i \gamma^5$$ in terms of the $4 \times 4$ gamma matrices, $$\gamma^a,$$ in flat space-time. By means of the definition of the covariant derivative $$D_a = (\partial_a + \omega_a)$$ and the metric in (2), one finds that the only non-vanishing components of the connection $$\omega_a$$ are

$$\omega_\mu = \frac{1}{2} e^A(\partial_\mu A) \gamma_\mu \gamma_5$$

and that the Dirac equation turns out to be written as

$$\{i \gamma^\mu \partial_\mu + e^{A(r)} \gamma^5 [\partial_r + 2\partial_\mu A(r)] - e^{A(r)} F(\phi) \} \Psi(x, r) = 0.$$  \hfill (11)

To ease the separation of variables one can resort to the chiral decomposition ($x$ stands for the four space-time coordinates)

$$\Psi(x, r) = \sum_n \psi_{Ln}(x) \alpha_{Ln}(r) + \sum_n \psi_{Rn}(x) \alpha_{Rn}(r),$$

where $$\psi_{Ln(R)n}(x)$$ are the chiral modes which satisfy $$\gamma^5 \psi_{Ln}(x) = -\psi_{Ln}(x), \ \gamma^5 \psi_{Rn}(x) = \psi_{Rn}(x)$$

and also the four-dimensional massive Dirac equations

$$i \gamma^\mu \partial_\mu \psi_{Ln(R)n} = m_n \psi_{R(L)n}.$$  \hfill (9)

Then, we find the following differential equations for the $r$-dependent scalar parts of the spinor

$$\alpha'_{Rn} + [2A' - F(\phi)] \alpha_{Rn} = -m_n e^{-A} \alpha_{Ln},$$

and

$$\alpha'_{Ln} + [2A' + F(\phi)] \alpha_{Ln} = m_n e^{-A} \alpha_{Rn},$$

where the prime stands for the first-derivative with respect to $r$. The equations above can be put in a more familiar form, namely

$$R_n' - F(\phi) R_n = -m_n e^{-A} L_n \text{ and } L_n' + F(\phi) L_n = m_n e^{-A} R_n;$$

by using the redefinitions $$\alpha_{Rn}(r) = e^{-2A(r)} R_n(r)$$ and $$\alpha_{Ln}(r) = e^{-2A(r)} L_n(r).$$

Equations (15) are equivalent to the equations for the components of a spinor describing a massless fermion in 1+1 dimensions subject to a mixing of scalar and vector potentials. The time-independent equation for a fermion under such potentials can be written as

$$H \psi(r) = E \psi(r),$$

with the Dirac Hamiltonian given (in natural units) by

$$H = \sigma_2 p + V_\phi(r) \sigma_1 + V_s(r),$$

where $p = -id/dr$ is the momentum operator, $\sigma_1$ and $\sigma_2$ are the two off-diagonal Pauli matrices,

$$V_\phi(r) = -F(\phi) \text{ (note that } F(\phi) \text{ is a function of } r \text{ is the scalar potential and } V_s(r) = m_n e^{-A(r)} \text{ is the vector potential. Analogously, } L_n(r) \text{ and } R_n(r) \text{ play the role of the upper and lower components, for the fermion zero-mode in 1+1 dimensions. As a matter of fact, the scalar potential can be seen as a position-dependent fermion mass. As one knows, many examples of such systems were already solved in the literature [22], but only a few potentials may support bound states.}$$
In the brane world scenario, by its turn, massive \((m_n \neq 0)\) as well as massless \((m_n = 0)\) fermions might be localized inside the brane, depending on the functional coupling \(F(\phi)\). The issue of localization of massive modes on the brane is not going to be discussed in this section, even though, we would like to remark that the analysis of the possible massive spectrum of bound states is not as simple as the search for localized massless modes [21], [23]. Moreover, normalizable \(\alpha_{R(L)n}(r)\)—functions is guaranteed if they satisfy the orthonormalization relations

\[
\int_{-\infty}^{\infty} e^{3A(r)} \alpha_{Ln}(r) \alpha_{Lm}(r) dr = \int_{-\infty}^{\infty} e^{3A(r)} \alpha_{Rn}(r) \alpha_{Rm}(r) dr = \delta_{nm},
\]

\[
\int_{-\infty}^{\infty} e^{3A(r)} \alpha_{Ln}(r) \alpha_{Rm}(r) dr = 0.
\]

From these relations one can note that the warp factor is crucial to determine the localization of whatever mode.

Particularly, for the massless mode one finds that

\[
\alpha_{R0}(r) = N_{R0} \exp[-2A(r) + \int r F(r') dr'], \quad \alpha_{L0}(r) = N_{L0} \exp[-2A(r) - \int r F(r') dr'],
\]

where we have used \(F(r) = F(\phi(r))\) and \(N_{R(L)0}\) are constants of normalization that are found according to the normalization relations

\[
|N_{R0}|^2 \int_{-\infty}^{\infty} e^{-A(r)+2 \int r F(r') dr'} dr = |N_{L0}|^2 \int_{-\infty}^{\infty} e^{-A(r)-2 \int r F(r') dr'} dr = 1.
\]

One can note that the normalization conditions are determined by the asymptotic behavior of the integrands in the expressions above. In most of the examples dealing with thick branes the factor \(\exp[-A(r)]\) goes asymptotically to infinity given the asymptotic behavior of the warp factor, that is \(\exp[2A(r \to \pm \infty)] \to 0\), hence one has to choose \(F(\phi(r))\) in such a way that the decreasing of \(\exp[\pm \int r F(r') dr']\) is faster than the increasing of \(\exp[-A(r)]\), but this choice does not guarantee the normalization of both chiralities, because of the different signs, \(\pm\), in the exponent, that is, if \(\alpha_{R0}(r)\) is normalizable, \(\alpha_{L0}(r)\) is not and vice-versa. Moreover, the factor \(\exp[-2A(r)]\) is symmetric around the core of the brane whereas the kink solution is odd under the reversion of the \(r\) coordinate around the core of the brane, which is usually chosen at \(r = 0\), hence one has to choose \(F(\phi)\) as an odd function on \(r\), in order to have the normalizable chiral mode even in \(r\) and with a peak on the brane.

We have analyzed the behavior of the massless chiral modes for two cases, namely: \(F(\phi) = \eta \phi(r)\) and \(F(\phi) = -\eta W_{\phi\phi}\), where \(\eta > 0\) is a coupling constant and \(W_{\phi\phi}\) is the second-derivative of the superpotential \(\mathcal{S}(\phi)\) with respect to \(\phi\) taken at the two-kink configuration in (7). The first case is the simplest Yukawa coupling of fermions to a scalar real field, while the second one is inspired on the coupling of fermions and bosons within a \(N = 1\).
SUSY model, which is also considered in the next section. This last functional coupling is the one which provides the correct localization on the brane. In fact, the simplest Yukawa coupling also provides a localized massless left-handed mode, $\alpha_{L_0}(r)$, but it does not follows the brane splitting, that is, while the double-wall is formed, the peak of the wavefunction is midway between the two walls, there being a small probability density to find such a mode on the core of the walls themselves. On the other hand, in the case $F(\phi) = -\eta W_{\phi\phi}$, one has $\alpha_{L_0}(r)$ with peaks on the branes, signalizing a great probability for the massless left-handed mode to be found just on the branes and a small probability to be found on the bulk and between the walls. Those behaviors can be seen from Figure 2, where the profiles of $\alpha_{L_0}(r)$ are shown for different values of $L$. Figure 2 should be confronted with Figure 1.

4 Localization of fermions: flat space-time

In this section we adopt the usual analysis to find fermion bound states under the action of the scalar field whose classical configuration is given in (7). Particularly, we focus on the scenario proposed in [1], that is, a brane (or domain wall) immersed in a five-dimensional flat space-time. The action for the fermion field is given in (9) with $g_{ab} = \eta_{ab}$ the Mikowskian metric, $D_a \equiv \partial_a$ and the irreducible form of the gamma matrices, $\Gamma^\mu = \gamma^\mu$, is going to be used. The chiral decomposition (12) can also be used to separate the four space-time variables, $x^\mu$, from the variable $r$. Now, the $r$-dependent functions appearing in the chiral decomposition obey the following equations

$$\alpha'_{Rn} - F(\phi)\alpha_{Rn} = -m_n\alpha_{Ln}, \quad (19)$$

and

$$\alpha'_{Ln} + F(\phi)\alpha_{Ln} = m_n\alpha_{Rn}. \quad (20)$$

Particularly, for the massless mode one finds

$$\alpha_{R_0}(r) = N_{R_0} \exp\left[+ \int_r^\infty F(r') dr'\right]$$

$$\alpha_{L_0}(r) = N_{L_0} \exp\left[- \int_r^\infty F(r') dr'\right]. \quad (21)$$

Now, the normalization of the massless modes depends on the asymptotic behavior of $\int_r^\infty F(r') dr'$ only, hence one usually has a unpaired (isolated) chiral (left-handed or right-handed) zero-mode, which is the main feature for having fermion number fractionization, as shown in [7].

We again have analyzed the behavior of the massless modes by setting $F(\phi) = \eta \phi(r)$ and $F(\phi) = -\eta W_{\phi\phi}$, with $\eta > 0$. In the first case we have

$$\alpha_{R_0}(r) = 0 \text{ and } \alpha_{L_0}(r) = N_{L_0} (\cosh 2L + \cosh 2r)^{-\eta/2}. \quad (22)$$

As in the previous section, the function $\alpha_{L_0}(r)$ is symmetric in $r$ and has a peak at $r = 0$, hence the massless left-handed mode is localized in the region between the branes with a very small probability density at the core of the branes.
For $F(\phi) = -\eta W_{\phi\phi}$ one finds

$$\alpha_{R_0}(r) = 0 \text{ and } \alpha_{L_0}(r) = N_{L_0} \left( \text{sech}^2(r + L) + \text{sech}^2(r - L) \right)^{\eta}. \quad (23)$$

From Figure 3 we can note that $\alpha_{L_0}(r)$ is symmetric in $r$, has no nodes and exhibits peaks at the cores of the branes. One can also note that the probability density to find the left-handed massless mode in the midway between the branes decreases as $L$ increases.

The adequate behavior of the massless mode is sufficient enough to consider the functional coupling $F(\phi) = -\eta W_{\phi\phi}$ as very convenient and has motivated us to analyze the consequences of such a coupling on the localization of massive modes on the split brane in flat space-time. As one knows, equations (19) and (20) can be decoupled to two second-order differential equations, namely

$$-\alpha''_{Rn} + U_R(r)\alpha_{Rn} = m_n^2\alpha_{Rn},$$

$$-\alpha''_{Ln} + U_L(r)\alpha_{Ln} = m_n^2\alpha_{Ln}, \quad (24)$$

where $U_R(r) = (\eta W_{\phi\phi})^2 - \eta W_{\phi\phi}'$ and $U_L(r) = (\eta W_{\phi\phi})^2 + \eta W_{\phi\phi}'$ in the case $F(\phi) = -\eta W_{\phi\phi}$. It is also known that the equations above are time-independent Schrödinger equations, whose corresponding Hamiltonians are superpartners of each other, that is, one has a quantum mechanics supersymmetry. This is formally true whatever is the functional coupling $F(\phi)$, but in the case considered here such supersymmetry seems to be a reflection of a supersymmetry at the fundamental level. In other words, one can note that $r$-dependent part of the excitations of the scalar field (branons), picked up to quadratic terms on the fundamental Lagrangian density in flat space-time (now with $V(\phi) = W_{\phi\phi}^2/2$), obeys a time-dependent Schrödinger equation similar to the one obeyed by $\alpha_{Ln}(r)$ with $\eta = 1$, that is, with an effective potential given by $U_{eff}(r) = (W_{\phi\phi})^2 + W_{\phi\phi}'$, resulting identical mass spectrum for branons (bosonic excitations) and fermions.

In Figure 4 it is shown the form of the potentials $U_R(r)$ and $U_L(r)$ for a specific value of $L$ and $\eta = 1$. For values of $L$ close to zero, $U_L(r)$ is a single well potential, which starts to be deformed into a double-well potential as $L$ approaches to a critical value $L_c$, that is determined by the condition $U''_L(r = 0) = 0$; for $L \gtrapprox L_c$, dimples are observed around $r = \pm L$, and a remarkable double-well is observed for the value of $L = l$ corresponding to $U_L(r = 0) = 0$. It worth mentioning that, although the bottom of the double-well is slightly raised as $L$ increases, the width of the double-well potential also increases, signaling the possible entrapment of a massive state, besides the, always present, massless one. This possible appearance of a massive bound state can also be seen from the behavior of $U_R(r)$, which is a single well potential whose bottom is above zero for $L < l$ and equals to zero for $L = l$, that is, the deepness and width of $U_R(r)$ increase as $L$ increases.

One can also observe that $U_L(r) = 2(2 - 3 \text{sech}^2 r)$ and $U_R(r) = 2(2 - \text{sech}^2 r)$ for $L = 0$ and $\eta = 1$; such that the first potential admits two bound states and the later admits only one bound state. The fundamental state of $U_L(r)$ for $L = 0$ and $\eta = 1$ is $\alpha_{L_0}(r) \approx \text{sech}^2 r$, whereas the first excited state is $\alpha_{L_1}(r) \approx \text{sech} \ t\tan hr$ and the fundamental state of $U_R(r)$ for $L = 0$ and $\eta = 1$ is $\alpha_{R_0}(r) \approx \text{sech} r$. Moreover, from the expression (23) with $\eta = 1$, one can construct an antisymmetric function as $\alpha_{L_1}(r) \sim \text{sech}^2(r + L) - \text{sech}^2(r - L)$ as an approximate expression for the first excited state of $U_L(r)$ when $L >> l$. This approximation
for the first excited state is commonly used to approach the discrete spectrum of double well potentials [24], [25].

We have used the results described above, together with the Numerov method to analyze the behavior of \( \alpha_L(r) \), \( \alpha_R(r) \) and the eigenvalue \( m_1^2 \) at intermediary values of \( L \). Those behaviors are shown in Figures 5 and 6. From them one can see that \( \alpha_R(r) \) is mainly distributed on the region between the branes, hence there is a small probability for the massive right-handed mode being observed inside the wells where the Universe(s) would be realized, whereas the probability density associated with \( \alpha_L(r) \) is pronounced just on the cores of the branes. In summary, at least one massive left-handed mode is localized on the branes. The eigenvalue \( m_1^2 \) decreases smoothly as \( L \) increases, that is an expected result when one is dealing with double-well potentials in non-relativistic quantum mechanics. That way, one can construct a mixed left-handed massive state with both, the fundamental and the first excited state, which are quasi-degenerate for very large values of \( L \), namely

\[
\Psi_{L,\text{mix}}(t,r) = N \left( \alpha_{L_0}(r) + \alpha_{L_1}(r)e^{-im_1t} \right) \chi_L,
\]

where \( \chi_L \) is a constant spinor which satisfies \( \gamma^5 \chi_L = -\chi_L \). We have tried to be cautious when proposing this mixed state, since we are assuming that there is a rest reference frame for the particle in such a mixed state. With this reasoning, the Dirac equation \( i\gamma^\mu \partial_\mu \Psi_{L,\text{mix}} = m_n \Psi_R \) is satisfied, since \( m_1 \gamma^0 \chi_L e^{-im_1t} = m_1 \chi_R e^{-im_1t} (\gamma^5 \chi_R = \chi_R) \) and there is no right-handed massless state, neither inside nor outside the branes. The probability density associated with this mixed state is given by

\[
\rho(r,t) = |N|^2 \left[ \alpha_{L_0}(r)^2 + \alpha_{L_1}(r)^2 + 2\alpha_{L_0}(r)\alpha_{L_1}(r) \cos(m_1t) \right],
\]

which is an oscillating probability density with period of oscillation \( T = \hbar/m_1c^2 \), such that a tiny mass implies a long tunneling time. In this scenario the fermion tunnels from one brane to the other, being likely found on both branes, but not simultaneously. As \( L \) increases, other massive localized states can be realized inside the branes. In fact, we have found numerically that there is room to one more massive state in the present case. The appearance of a tower of localized massive states is very dependent on the deepness and width of the double-well effective potential \( U_L(r) \), that is ultimately dependent on the superpotential \( W(\phi) \) whose classical solution is a kink that can be continuously deformed into two-kink solution. In the next section we comment on the construction of such models.

5 Conclusions

We have studied the mechanism that leads to the localization of massless fermions on a split brane in the cases of warped and flat geometries. The brane is immersed in a five-dimensional space-time and is defined by the behavior of a scalar field coupled with gravity in the case of warped space-time. The nonpolynomial potential of the self-interacting scalar field which generates the split brane was introduced before in reference [18], but any other model which has deformable solitons as minimal energy configurations could be used as well, for example, a convenient \( \phi^6 \) polynomial potential [26].

The case of flat geometry is more manageable than the case of warped geometry, not only because we obtain the localized modes in a simple way, but also, and mainly, because it
allows us to understand why the coupling of fermions with the scalar field should be chosen in such a way that a supersymmetry in a fundamental level is realized. The most convenient coupling, which preserves the supersymmetry, also leads to the expected behavior of the fermion wavefunctions, that is, the ground-state wavefunction follows the brane splitting. As the brane splits into two branes the wavefunction is also split, with peaks at the cores of the two branes. Although the simplest Yukawa coupling of fermions to scalar fields also provides localization of massless fermions, the corresponding massless fermion wavefunction present a peak just between the two branes, such that the observation of massless fermions is suppressed inside the branes. Hence, one concludes the supersymmetry inspired coupling as the most adequate one to study the entrapment of fermions by a split brane.

We also note that eventual massive localized states are more difficult to be found in the case of warped geometry than in the case of flat geometry. In the later case we have also discussed the tunneling of a massive fermion between the branes, which lasts until the branes are infinitely separated from each other. That critical limit is described by another model, obtained from the starting one by taking the limit $b \to -a$ in (6), that is $V(\phi) = 2\mu^2(\phi^2 - a|\phi|)^2$, whose solution is one of the single kinks $\phi = \frac{\phi^0}{2}[\pm 1 + \tanh(\mu a(r \mp r_0))]$, with $r_0$ a reference point where the core of the defect is localized. In this case we found that the energy gap between the first localized states is large enough such that they can not be considered as quasi-degenerate. It can also be observed that the number of massive localized states depends on the deepness and width of the quantum mechanics effective potentials, which are defined by the field theory model one chooses to deal with. We have mentioned above that others models whose classical solution exhibit a two-kink profile (split brane) can afford a tower of localized massive fermions states. A class of such models, called deformed models, have been proposed [27], as deformation of others known models. We notice that those models can also be constructed, together with new ones, from the deformation of zero modes excitations of well know models whose classical solutions are single kinks. This proposal is being analyzed in more detail and will be reported later elsewhere.

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Figure 1: Ricci scalar for $L = 0.01$ (dashed line), $L = 1.6$ (thin solid line) and $L = 4.5$ (thick solid line) evidences the formation of a double-wall structure as $L$ increases.
Figure 2: $\alpha_{L_0}(r)$ (warped geometry) in the cases $F(\phi) = \phi(r)$ (upper) and $F(\phi) = -W_{\phi\phi}$ (lower), for $L = 0.01$ (dashed line), $L = 1.6$ (thin solid line) and $L = 4.5$ (thick solid line).
Figure 3: $\alpha_{L_0}(r)$ (flat space-time) in the case $F(\phi) = -W_{\phi\phi}$, for $L = 1.6$ (thin solid line) and $L = 4.5$ (thick solid line).

Figure 4: Effective potentials of equations (23) with $L = 1.5$. $U_L(r)$ (solid line), $U_R(r)$ (dashed line).
Figure 5: $\alpha_{L_1}(r)$ (upper) and $\alpha_{R_1}(r)$ (lower) in the case $F(\phi) = -W_{\phi \phi}$, for $L = 1.6$ (thin solid line) and $L = 4.5$ (thick solid line), in flat space-time.
Figure 6: The eigenvalue of the first excited state in (24) against $L$. 