A CRITERION FOR PERFECTOID FIELDS

EHSAN SHAHOSEINI AND KIRAN S. KEDLAYA∗

Abstract. The tilting correspondence is a fundamental property of perfectoid fields. In this note, we show that the tilting construction can also be used to detect perfectoid fields among nonarchimedean fields. In particular, for $K$ a complete subfield of $\mathbb{C}_p$ (a completed algebraic closure of $\mathbb{Q}_p$), $K$ is perfectoid if and only if its tilt is not algebraic over $\mathbb{F}_p$. We also include some conjectures on APF (arithmetically profinite) extensions, perfectoid fields, and their relations.

Keywords: Perfectoid fields, Fargues-Fontaine curve, APF extensions.

1. Introduction

The theory of perfectoid fields, developed by Scholze [8] and Kedlaya–Liu [6], has its roots in the works of Fontaine–Wintenberger on APF (arithmetically profinite) extensions [3, 4]. Let us recall the definition of a perfectoid field.

Definition 1.1. [8 Definition 3.1], [6 Definition 3.5.1] By a nonarchimedean field, we will mean a field complete with respect to a nonarchimedean absolute value, which for convenience we do allow to be the trivial absolute value unless specified. For $K$ a nonarchimedean field, let $K^\circ$ denote the valuation subring of $K$.

Fix a prime number $p$. A perfectoid field is a nontrivially valued nonarchimedean field $K$ of residue characteristic $p > 0$ whose associated rank-1 valuation is nondiscrete, such that the Frobenius is surjective on $K^\circ/p$.

One important feature of such fields is the following.

Lemma 1.2. [8 Lemma 3.2] The value group of a perfectoid field is $p$-divisible.

Our main result is a criterion for detecting perfectoid fields based on a key construction in the theory dating back to Fontaine’s work.

Definition 1.3. For $K$ a nonarchimedean field of residue characteristic $p$, the tilt of $K$ is initially defined as the multiplicative group

$$K^\flat := \lim_{x \to x^p} K.$$ 

A standard calculation shows that defining addition via the formula

$$(x_n) + (y_n) := \left( \lim_{m \to \infty} (x_m + y_m)^{p^m} \right)_n$$

promotes $K^\flat$ to a field of characteristic $p$, which moreover is complete with respect to the absolute value

$$|(x_n)| := |x_0|$$

and thereby a nonarchimedean field (possibly with trivial absolute value).

2020 Mathematics Subject Classification. Primary 14G45, Secondary 11S20, 11S15, 14H60.

∗Corresponding author.
For context, we point out the tilting correspondence which plays a central role.

**Theorem 1.4.** [8 Theorem 3.7], [5 Theorem 1.5.6], [6 Theorem 3.5.3, Lemma 3.5.4, Theorem 3.5.6] Let $K$ be a perfectoid field.

(i) Every finite extension of $K$ is again perfectoid.

(ii) The tilting functor induces an equivalence of categories between the category of finite extensions of $K$ and the category of finite extensions of $K^\flat$. Also, this equivalence preserves degrees.

Our main result shows that tilting can also be used to detect perfectoid subfields of a nonarchimedean field which is not itself known to be perfectoid. Somewhat surprisingly, the proof of this is not “elementary”: it requires a mild use of the geometry of the Fargues–Fontaine curve associated to a perfectoid field [2].

**Theorem 1.5 (Theorem 2.5).** Let $K$ be a nonarchimedean field of residue characteristic $p$.

(a) If $K^\flat$ is trivially valued, then $K$ does not contain a perfectoid field.

(b) If $K^\flat$ is not trivially valued, then there is a maximal perfectoid subfield $K_1$ of $K$, and the inclusion $K_1 \to K$ induces an isomorphism $K_1^\flat \cong K^\flat$.

This result specializes to a cleaner statement when $K$ is contained in $\mathbb{C}_p$, a completed algebraic closure of $\mathbb{Q}_p$.

**Theorem 1.6 (Theorem 2.7).** Let $K$ be a complete subfield of $\mathbb{C}_p$ such that $K^b$ is not algebraic over $\mathbb{F}_p$. Then $K$ is a perfectoid field.

Our work is motivated by the question of whether Lemma 1.2 admits the following partial converse: is any complete subfield of $\mathbb{C}_p$ with $p$-divisible value group a perfectoid field? We discuss this and some related questions in the last section.

2. Results

For $F'/F$ an extension of nonarchimedean fields, let $\text{Aut}(F'/F)$ denote the group of continuous automorphisms of $F'$ over $F$.

**Lemma 2.1.** Let $F$ be a nonarchimedean field (of any characteristic). Let $F'$ be the completion of an algebraic closure $F_1$ of $F$. Then there is a natural isomorphism $\text{Aut}(F'/F) \cong \text{Gal}(F_1/F)$, which becomes a homeomorphism if we equip $\text{Aut}(F'/F)$ with the compact-open topology and $\text{Gal}(F_1/F)$ with the usual profinite topology.

**Proof.** Since $F_1$ is algebraically closed, it must be the full integral closure of $F$ in $F'$; hence every element of $\text{Aut}(F'/F)$ carries $F_1$ into itself. We thus obtain a restriction homomorphism $\text{Aut}(F'/F) \to \text{Gal}(F_1/F)$. This homomorphism is injective because $F_1$ is dense in $F'$; it is surjective because every element of $\text{Gal}(F_1/F)$ extends continuously to $F'$ (because the absolute value on $F$ extends uniquely to $F'$, so this extension is preserved by the Galois action). \qed

**Lemma 2.2.** Let $F$ be a nonarchimedean field (of any characteristic). Let $F'$ be the completion of an algebraic extension $F_1$ of $F$. Then every complete perfect subfield of $F'$ containing $F$ is the completion of the perfect closure of some subextension of $F_1$. 


Proof. We first prove the claim assuming that $F_1$ is algebraically closed, in which case so is $F'$ (by the continuity of roots of a polynomial over $F_1$ as one varies the coefficients). Let $E$ be a complete perfect subfield of $F'$ containing $F$. Since $F'$ is algebraically closed, it also contains an algebraic closure of $E$, which contains $F_1$ and hence is dense in $F'$. By the Ax–Sen–Tate theorem $[11]$, $F$ is the fixed subfield of $F'$ under $\operatorname{Aut}(F'/F)$ and $E$ is the fixed subfield of $F'$ under $\operatorname{Aut}(F'/E)$.

Since the inclusion $\operatorname{Aut}(F'/E) \to \operatorname{Aut}(F'/F)$ is a continuous map between two compact topological spaces, its image is necessarily closed. Via Lemma 2.4, $\operatorname{Aut}(F'/E)$ corresponds to a closed subgroup of $\operatorname{Aut}(F'/F) \cong \operatorname{Gal}(F_1/F)$; let $E_0$ be the fixed field of the latter. By Ax–Sen–Tate again, the fixed field of $F'$ under $\operatorname{Aut}(F'/E)$ is equal to the completion of the perfect closure of $E_0$, completing the proof under the assumption that $F_1$ is algebraically closed.

To treat the general case, let $F_2$ be an algebraic closure of $F_1$ and let $F''$ be the completion of $F'$. Then the previous argument shows that every complete perfect subfield $E$ of $F'$ containing $F$ is the completion of the perfect closure of some subextension $E_0$ of $F_2$. By Lemma 2.4

$$\operatorname{Gal}(F_2/F_1) \cong \operatorname{Aut}(F''/F) \subseteq \operatorname{Aut}(F'/E) \cong \operatorname{Gal}(F_2/E_0);$$

from the Galois correspondence, we deduce that $E_0 \subseteq F_1$. □

Lemma 2.3. For $K_0 \to K$ an inclusion of perfectoid fields and $L$ a nonarchimedean field containing $K_0$, any morphism $K_0 \to L^o$ lifts to an inclusion $K \to L$.

Proof. The original morphism restricts to a morphism $K_0^o \to L^o$. Applying the Witt vector functor then yields a morphism $W(K_0^o) \to W(L^o)$. Let $I$ be the kernel of $W(K_0^o) \to K_0^o$; by the tilting equivalence $[3]$ Theorem 1.5.1, $W(K_0^o)/IW(K_0^o)$ is isomorphic to $K^o$ whereas $W(L^o)/IW(L^o)$ maps to $L^o$. We thus end up with a map $K^o \to L^o$, which must be an inclusion. □

Lemma 2.4. Let $K$ be a nonarchimedean field of mixed characteristic $(0,p)$. If $K^p$ is not trivially valued, then $K$ contains a perfectoid subfield.

Proof. Let $K_0$ be the completion of $\mathbb{Q}_p(\mu_{p^\infty})$, which is a perfectoid field; let $F_0$ be its tilt. The group of continuous automorphisms of $K_0$ over $\mathbb{Q}_p$ is naturally isomorphic to $\mathbb{Z}_p^\times$. We may assume that $K$ does not contain $K_0$, as otherwise there is nothing to check.

By hypothesis, $K^p$ contains an element $t^p = (t_n)_n$ such that $0 < |t_0| < 1$. Let $K_1 \subseteq K$ be the completion of $\mathbb{Q}_p(t_0, t_1, \ldots)$ and let $K_2$ be the completion of the compositum $K_0K_1$. Put $G = \operatorname{Gal}(K_2/K_0(t_0))$, $H = \operatorname{Gal}(K_2/K_0(t_0))$: using Kummer theory, we obtain an embedding of the exact sequence

$$1 \to H \to G \to G/H \to 1$$

into

$$1 \to \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times \to 1.$$ 

Since we are assuming that $K_0 \not\subseteq K$, each of these inclusions has finite index.

Let $F_2 \subseteq K_2$ be the completion of the perfect closure of $F_0(t^p)$; then $F_2$ is a perfect extension of $F_0$, so we may untilt it to obtain a perfectoid extension $K'_2$ of $K_0$ contained in $K_2$. Since $K'_2$ also contains $t_n$ for each $n$, it must equal $K_2$; consequently, $K_2$ is perfectoid and $K'_2 = F_2$. In particular, the tilting correspondence (Theorem 1.4) gives an action of $G$ on $F_2$. Let $F_1$ be the fixed field of $H$ acting
on $F_2$, which contains (but need not equal; see Remark 2.8) the completed perfect closure of $\mathbb{F}_p((t^p))$.

Let $X_0, X_1, X_2$ be the Fargues–Fontaine curves associated to $F_0$, $F_1$, $F_2$, respectively. (For this discussion we may use FF curves in either the “adic” or “schematic” sense.) The untilt $K_0$ of $F_0$ corresponds to a closed point $x \in X_0$; let $\mathcal{I}_0 \subset \mathcal{O}_{X_0}$ be the ideal sheaf on $X_0$ corresponding to the closed immersion $x \hookrightarrow X_0$. Let $\mathcal{I}_2$ be the pullback of $\mathcal{I}_0$ to $X_2$.

We now apply v-descent for vector bundles on Fargues–Fontaine curves [7, Theorem 4.7.4] to deduce that the inclusion $\mathcal{I}_2 \rightarrow \mathcal{O}_{X_2}$, being equivariant for an action of $H$ on both sides, descends to an inclusion $\mathcal{I}_1 \rightarrow \mathcal{O}_{X_1}$ of line bundles on $X_1$. Via this inclusion, $\mathcal{I}_1$ is an ideal sheaf on $X_1$ defining a closed immersion $x_1 \hookrightarrow X_1$ of a degree-1 divisor, which then corresponds to an untilt $L_1$ of $F_1$. By Lemma 2.5, the map $F_1 \rightarrow K^\flat$ induces an inclusion $L_1 \hookrightarrow K$, proving the claim.

\begin{center}
\textbf{Theorem 2.5.} Let $K$ be a nonarchimedean field of mixed characteristic $(0,p)$.
\begin{enumerate}[(a)]
\item If $K^\flat$ is trivially valued, then $K$ does not contain a perfectoid field.
\item If $K^\flat$ is not trivially valued, then there is a unique maximal perfectoid subfield $K_1$ of $K$, and the inclusion $K_1 \hookrightarrow K$ induces an isomorphism $K_1^\flat \cong K^\flat$.
\end{enumerate}
\end{center}

\begin{proof}
Part (a) is evident from its contrapositive: if $K$ contains a perfectoid subfield $K_0$, then $K^\flat$ contains $K_0^\flat$ and the latter is not trivially valued.

To establish (b), we first establish the existence of a perfectoid subfield $K_1$ with $K_1^\flat \cong K^\flat$. By hypothesis, we may apply Lemma 2.4 to obtain a perfectoid subfield $K_0$ of $K$. The tilting equivalence [5, Theorem 1.5.1] then lifts $K^\flat$ to a perfectoid field $K_1$ containing $K_0$; by Lemma 2.3, the isomorphism $K_1^\flat \cong K^\flat$ induces an inclusion $K_1 \hookrightarrow K$.

Now notice that given any perfectoid subfield $K_0$ of $K$, we may repeat the previous logic to obtain a perfectoid subfield $K_1'$ of $K$ with $(K_1')^\flat = K^\flat = K_1^\flat$. For $t^p = (t^n)_n \in K^\flat$, we have $t_0 \in K_1 \cap K_1'$; since each of $K_1$ and $K_1'$ is topologically generated over $\mathbb{Q}_p$ by such elements, we conclude that $K_1 \subseteq K_1'$ and vice versa.
\end{proof}

\begin{center}
\textbf{Remark 2.6.} Let $K$ be the completion of $\mathbb{C}_p(t)$ for the Gauss norm. Then $K^\flat = \mathbb{C}_p^\flat$ is not trivially valued but $K$ is not perfectoid.
\end{center}

\begin{center}
\textbf{Theorem 2.7.} Let $K$ be a complete subfield of $\mathbb{C}_p$ such that $K^\flat$ is not algebraic over $\mathbb{F}_p$. Then $K$ is a perfectoid field.
\end{center}

\begin{proof}
By Lemma 2.4, $K$ contains some perfectoid field $K_0$. By Lemma 2.2, $K$ is the completion of an algebraic extension of $K_0$. By the tilting correspondence (Theorem 1.4) and the fact that a completed direct union of perfectoid fields is perfectoid, we deduce that $K$ is perfectoid.
\end{proof}

\begin{center}
\textbf{Remark 2.8.} We caution the reader not to read too much into Theorem 2.5 while it is true that every finite-index subfield of a perfectoid field is perfectoid [6, Proposition 3.5.9], this does not imply that every finite-index perfectoid subfield of $K^\flat$ untilts to a perfectoid subfield of $K$. We briefly recall an example from [3] to illustrate this.

Let $K$ be the completion of $\mathbb{Q}_p(\mu_{p^{\infty}})$, so that $K^\flat$ is isomorphic to the completed perfect closure of $\mathbb{F}_p((t^p))$. For every positive integer $m$ coprime to $p$, let $F_m$ be the completed perfect closure of $\mathbb{F}_p((t^{p^m}))$, which is a subfield of $K^\flat$ of index $m$. If
$F_m$ were to untilt to a perfectoid subfield $K_m$ of $K$, then by Lemma 2.2, $K_m$ would be the completion of an algebraic extension of $\mathbb{Q}_p$ contained in $K$. By Lemma 2.2 again, the integral closure of $\mathbb{Q}_p$ in $K$ is in fact equal to $\mathbb{Q}_p(\mu_{p^\infty})$, so $K_m$ would have to be the completion of some subfield of $\mathbb{Q}_p(\mu_{p^\infty})$ of finite index. However, from the Galois group of the latter, we see that no such subfield exists unless $m$ divides $p - 1$.

3. Conjectures

In this section, we state some further conjectures about perfectoid fields.

**Conjecture 3.1.** Let $K$ be a subfield of $\overline{\mathbb{Q}}_p$ with $p$-divisible value group. Then, $K$ contains an APF extension in the sense of [3].

**Remark 3.2.** Since the completion of an APF extension is a perfectoid field, Conjecture 3.1 would imply (using Lemma 2.2) that any complete subfield of $\mathbb{C}_p$ with $p$-divisible value group is perfectoid.

It is well-known that if $K$ is an APF extension with finite residue field $k$, then $\hat{K}^\flat \simeq k((1/p^\infty))$. Our next conjecture (inspired by a question raised by Jared Weinstein during the 2017 Arizona Winter School) says that the converse is also true:

**Conjecture 3.3.** Let $K$ be a subfield of $\overline{\mathbb{Q}}_p$ with finite residue field $k$ such that $\hat{K}$ is a perfectoid field. If $\hat{K}^\flat \simeq k((1/p^\infty))$, then $K$ is an APF extension.

**Remark 3.4.** In Conjecture 3.3 if we drop the condition that $k$ be finite, we must allow the possibility that $K$ is an infinite unramified extension of an APF extension.

The following conjecture is a possible workaround for the issue raised in Remark 2.8.

**Conjecture 3.5.** Let $K$ be a perfectoid field. Then every finite-index subfield of $K^\flat$ is isomorphic (as a nonarchimedean field, forgetting the inclusion into $K^\flat$) to the tilt of some finite-index subfield of $K$.

**References**

1. J. Ax, Zeros of polynomials over local fields—the Galois action, J. Algebra 15 (1970), 417–428.
2. L. Fargues, J-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge $p$-adique. With a preface by Pierre Colmez. Astérisque 416 (2018), xiii+382 pp.
3. J-M. Fontaine, J-P. Wintenberger, Le "corps des normes" de certaines extensions algébriques de corps locaux. C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 6, A367–A370.
4. J-M. Fontaine, J-P. Wintenberger, Extensions algébrique et corps des normes des extensions $\text{APF}$ des corps locaux. C. R. Acad. Sci. Paris Sér. A-B 288 (1979), no. 6, A441–A444.
5. K.S. Kedlaya, New methods for $(\phi, \Gamma)$-modules. Research in Math. Sciences 2 (2015).
6. K.S. Kedlaya, R. Liu Relative $p$-adic Hodge theory: foundations. Astérisque 371 (2015), 239 pp.
7. K.S. Kedlaya and R. Liu, Relative $p$-adic Hodge theory II: Imperfect period rings, arXiv:1602.06899v3 (2019).
8. P. Scholze, Perfectoid spaces. Publ. Math. Inst. Hautes Études Sci. 116 (2012), 245–313.
9. The Nonarchimedean Scottish Book, Problem 38

Department of Mathematics, Tarbiat Modares University, 14115-134, Tehran, Iran
Email address: ehsan.shahoseini@modares.ac.ir

Department of Mathematics, University of California San Diego, 9500 Gilman Drive 0112, La Jolla, CA 92093, USA
Email address: kedlaya@ucsd.edu