Synchronization in coupled integer and fractional-order gauss map

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Abstract. Nonlinear maps are often used to model dynamical systems in engineering and sciences. In some cases, the entire memory dictates the time evolution of the system. Fractional-order maps have been proposed to model such situations. In a spatially extended system, different parts of the system may respond to memory differently. We propose a mixed order system where one variable has a fractional-order evolution while the other variable has integer-order evolution. It models an inhomogeneous system where certain variables retain memory while certain others do not. It is also relevant to control problems where fractional or integer-order control may be useful for achieving the desired state. In two coupled gauss maps, we consider unidirectional as well as bidirectional coupling. Bifurcation diagram of the system has been studied for various values of fractional-order parameter $\alpha$. In particular, we study synchronization in these systems.

1. Introduction
Fractional calculus has a long history and Leibniz himself had proposed certain ideas regarding fractional calculus. Almost all great mathematicians have worked in it sometime or other. However, Newtonian integer-order calculus was enormously successful in several walks of science and was pursued zealously. The lesser intuitive fractional-order calculus took a backseat. It does not find its way in standard calculus textbooks. Only recently, it has emerged as an active domain of research. Fractional calculus is an excellent modelling tool for phenomena in which dynamics depends on memory. It has found numerous applications in recent past. The systems like polymers and viscous materials are modelled successfully using it [1]. Fractional calculus and systems involving fractional-order derivatives have found widespread usage in engineering and sciences [2, 3, 4] and have been used to model and simulate numerous real-world systems [5]. Coupled fractional maps for large systems on a Euclidean 1-d lattice and complex networks has been studied in [6]. These are homogeneous systems and we study large lattices. In this paper, we study inhomogeneous systems where different orders are coupled in a unidirectional or bidirectional manner.

Difference equations have played an enormous role in the development of dynamical systems theory. Several phenomena observed in differential equations have been qualitatively observed in difference equations. In certain cases, they have also helped understanding of these phenomena. With this motivation, we study discrete-time analogues of fractional differential equations. Such
fractional difference equations have been defined and their dynamical behaviour is studied numerically [7]. One of the most extensively studied phenomena in dynamical systems is that of synchronization. We have previously studied synchronization in a large number of coupled fractional maps [6]. In this work, we study an inhomogeneous system. We study a coupled system of a fractional difference equation and an integer-order difference equation. The motivation of the current work is as follows. In some inhomogeneous systems, the memory of the past could affect the different parts of the system in different ways. Some parts may forget the past quickly while some may not. These studies have implications in control problems. We study the cases in which systems with memory are used to drive systems which do not remember past and vice versa. The results can be generic. In particular, we propose a dynamical system consisting of a fractional-order gauss map and an integer-order gauss map which interacts with each other as they evolve in time. We investigate different possible ways in interactions. In particular, we study synchronization which is a simple pattern to observe. We explore the bifurcation diagram of this coupled system and study phase diagram for the synchronized state.

The paper has been outlined as follows. First, we introduce the Gauss map, which is a specific nonlinear map or difference equation which has been a test case in our investigations. The range of this map is bounded while the domain is entire real line. Thus numerical problems are avoided. We define a fractional-order Gauss map which has been introduced in [7]. Various interaction schemes of integer and fractional order maps are introduced. We study unidirectional coupling in either direction as well as bidirectional coupling. Synchronization in chaotic systems has been studied extensively in integer order systems.

We note some previous studies on synchronization between fractional and integer order continuous differential equations. Synchronization between fractional-order hyperchaotic systems and integer-order hyperchaotic systems via a sliding-mode controller is investigated in [8]. Synchronization can be of various types such as complete synchronization, antisynchronization, lag synchronization, anticipated synchronization, generalized synchronization etc. Function projective synchronization means that the master and slave systems are synchronized to a scaling function but not a constant. In [9], function projective synchronization between fractional-order chaotic systems and integer-order chaotic systems were investigated using the stability theory of fractional-order systems. A mechanism to achieve synchronization between fractional-order chaotic systems with mismatched parameters was proposed in [10]. Sun and others investigated a novel kind of compound synchronization involving integer-order and fractional-order chaotic systems [11]. It has also been studied experimentally. Chen et al. studied synchronization between a new double-wing fractional-order chaotic system and different Lorenz systems with different structures [12] using electrical circuits. We extend these studies to synchronization of maps of different orders. We study exact synchronization in these systems without any control.

2. Integer-order Gauss Map and its Fractional-order Version

Gauss map is a nonlinear difference equation. It is given as,

\[ f(x) = \exp(-\nu x^2) + \beta \]  

where \( \nu \) and \( \beta \) are the parameters. \( \nu \) determines the width of the peak and \( \beta \) determines its position. We fix the value of \( \nu = 7.5 \) while \( \beta \) is our control parameter.

Like fractional differential equations, discrete fractional difference equations can be defined in many ways [7, 13, 14]. We choose the definition given by [7]. According to the definition given by Deshpande and Daftardar-Gejji [7], the evolution of fractional-order gauss map is given as,

\[ x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t - j + \alpha)}{\Gamma(t - j + 1)} [f(x(j-1)) - x(j-1)] \]  

where,

\[ f(x) = \exp(-7.5(x(t-1))^2) + \beta \]  

(3)

### 2.1. Integer-order gauss map driving evolution of fractional-order gauss map

The evolution of integer-order and fractional order gauss maps are given as,

\[ x_1(t) = f(x_1(t-1)) = \exp(-7.5(x_1(t-1))^2) + \beta \]  

(4)

\[ x_2(t) = x_2(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)} [f(x_2(j-1)) - x_2(j-1)] \]  

(5)

A system in which integer-order gauss map drives the evolution of fractional-order gauss map is defined as follows,

\[ x_2(t) = x_2(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)} G(x_2(j-1), x_1(j-1)) \]  

(6)

where \( G(a, b) = (1 - \epsilon)f(a) + \epsilon f(b) - a \) and variable \( x_1 \) evolves according to eq. 4.

In the above definition, the evolution of fractional-order gauss map is driven by integer-order gauss map according to eq. 6. But integer-order gauss map evolves independently in time. We explore the system by plotting bifurcation diagram and phase plots which show basin of attraction for the synchronized state. The synchronized state is characterized by the situation when the absolute difference between variable values of both integer and fractional-order gauss map sub-systems is less than \( \Delta \). Here we have taken the value of \( \Delta \) equal to 0.01. We vary coupling strength \( \epsilon \) and parameter \( \beta \). We also vary fractional-order parameter, \( \alpha \) and study for random initial conditions. Figure 3 shows the phase-space diagram for this system for random initial conditions.

![Figure 1. Bifurcation diagram for single fractional-order gauss map for fractional-order parameter, \( \alpha = 0.4 \).](image1)

![Figure 2. Bifurcation diagram for single integer-order gauss map.](image2)

We plot the bifurcation diagram for the system for various values of coupling strength, \( \epsilon \). We use the fixed initial conditions for the evolution of the system. Figure 4 shows the bifurcation diagram for fractional-order gauss map of the system for \( \epsilon = 0.2 \) whereas for \( \epsilon = 0 \), the fractional-order gauss map of the system reduces to a single fractional-order map shown in Figure 1. The
Figure 3. Phase-space diagram showing parameter values for which synchronization is observed for integer-order gauss map sub-system driving fractional-order gauss map sub-system for $\alpha = 0.4$.

Figure 4. Bifurcation diagram for integer-order gauss map sub-system driving fractional-order gauss map sub-system for the evolution of fractional-order gauss map for $\epsilon = 0.2$ and fractional-order parameter $\alpha = 0.4$.

bifurcation diagram for integer-order gauss map of the system which is independent of $\epsilon$ is the same as that for a single integer-order gauss map as shown in Figure 2. Thus integer-order component which drives the fractional-order component isn’t affected by an increase in the coupling strength while fractional-order component which is driven by integer-order component is affected significantly.

2.2. Fractional-order gauss map driving evolution of Integer-order gauss map
In this case, fractional-order gauss map drives the evolution of integer-order gauss map. So evolution of integer-order map depends upon the evolution of fractional-order gauss map but the evolution of fractional-order map is independent of the evolution of the integer-order map. Fractional-order gauss map evolves according to Equation 7 and integer-order gauss map evolves according to Equation 8.

\[
x_2(t) = x_2(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t-j+\alpha)}{\Gamma(t-j+1)} [f(x_2(j-1)) - x_2(j-1)]
\]

\[
x_1(t) = (1-\epsilon)f(x_1(t-1)) + \epsilon f(x_2(t-1))
\]

We plot the phase-space diagram as before shown in Figure 5. The bifurcation diagram for fractional-order component is the same as that for a single fractional-order gauss map as shown in Figure 1. The fractional-order component remains unchanged even as we change the coupling strength. Integer-order component changes with the corresponding change in coupling strength, $\epsilon$. The bifurcation diagram for $\epsilon = 0.2$ is as shown in Figure 6. Again for $\epsilon = 0$, integer-order component of the system reduces to a single integer-order gauss map as shown in Figure 2.

2.3. Bidirectional coupling of integer and fractional order gauss maps
We now couple fractional-order gauss map and integer-order gauss map in such a way that both affect the evolution of each other. The equation for evolution for integer and fractional-order, gauss map is given in Equation 9 and 10.
Figure 5. Phase-space diagram showing parameter values for which synchronization is observed for fractional-order gauss map sub-system driving integer-order gauss map sub-system for $\alpha = 0.4$.

\begin{align*}
x_1(t) &= x_1(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \frac{\Gamma(t - j + \alpha)}{\Gamma(t - j + 1)} G(x_1(j - 1), x_2(j - 1)) \\
x_2(t) &= (1 - \epsilon)f(x_2(t - 1)) + \epsilon f(x_1(t))
\end{align*}

(9)

and

(10)

where $G(a, b) = (1 - \epsilon)f(a) + \epsilon f(b) - a$.

The phase-space diagram for the system is given in Figure 7. We find that as the fractional-order parameter $\alpha$ is increased the range of synchronization also increases. We give the phase-plots for $\alpha = 0.6$ and 0.8 in Figure 8 and 9.

Figure 6. Bifurcation diagram for fractional-order gauss map sub-system driving integer-order gauss map sub-system for the evolution of integer-order gauss map for $\epsilon = 0.2$.

Figure 7. Phase-space diagram showing parameter values for which synchronization is observed for mutually driven gauss map system for $\alpha = 0.4$.

The bifurcation diagrams for the system is given in Figure 10 and 11. From the bifurcation diagram shown in Figure 10, we conclude that the fractional-order component changes with the change in coupling strength. Also from Figure 11 we conclude that the integer-order component changes with change in coupling strength.
Figure 8. Phase-space diagram showing parameter values for which synchronization is observed for mutually driven gauss map system for $\alpha = 0.6$.

Figure 9. Phase-space diagram showing parameter values for which synchronization is observed for mutually driven gauss map system for $\alpha = 0.8$.

Figure 10. Bifurcation diagram for mutually driven gauss map system for evolution of fractional-order map for $\epsilon = 0.2$ and fractional-order parameter $\alpha = 0.4$.

Figure 11. Bifurcation diagram for mutually driven gauss map system for evolution of integer-order map for $\epsilon = 0.2$.

3. Conclusion
This study gives a glimpse of the interaction between fractional and integer-order nonlinear maps in time. A similar study with multiple alternate fractional and integer-order coupled maps could be done to explore spatiotemporal interaction. The studies can have important consequences in the control of dynamical systems. The fractional-order parameter $\alpha$ is another control parameter which could be varied and the system behaviour could be studied. We observe that for larger values of $\alpha$, synchronization is obtained over a larger parameter regime in phase space. We can also see that integer order driving fractional-order has less parameter area where synchronization is obtained. This system studies coupled system of fractional-order and integer-order system and the studies can be extended to spatiotemporal systems where there is a quenched disorder in the fractional parameter $\alpha$. Such a mixed order system has not been studied before. The computation-intensive nature of the fractional-order system, as it retains
and uses the information of all previous states to compute each further state. This feature significantly hampers numerical studies of asymptotic states which can be inferred after a long time. Studies in this low dimensional system can give a useful indication of possibilities in a high dimensional system involving many maps.

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