Non–Relativistic Approximation of Dirac Equation for Slow Fermions Coupled to the Chameleon and Torsion Fields in the Gravitational Field of the Earth

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We analyse a non–relativistic approximation of the Dirac equation for slow fermions, coupled to the chameleon field and torsion in the spacetime with the Schwarzschild metric, taken in the weak gravitational field of the Earth approximation. We follow the analysis of the Dirac equation in the curved spacetime with torsion, proposed by Kostelecky (Phys. Rev. D 69, 105009 (2004)), and apply the Foldy–Wouthuysen transformations. We derive the effective low–energy gravitational potentials for slow fermions, coupled to the gravitational field of the Earth, the chameleon field and to torsion with minimal and non–minimal couplings.

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I. INTRODUCTION

Recently the non–relativistic approximation of the Dirac equation for slow fermions, moving in the gravitational field of the Earth, described by the Schwarzschild metric in the weak gravitational field approximation, has been analysed in Ref.[1]. As has been shown in [1], slow fermions couple to the external gravitational field through the effective low–energy gravitational potential

\[
\Phi_{\text{eff}}(\vec{r}, \vec{\nabla}, \vec{\sigma}) = m U(\vec{r}) - \frac{1 + 2\gamma}{2m} \left( U(\vec{r}) \Delta + \vec{\nabla} U(\vec{r}) \cdot \vec{\nabla} + \frac{1}{4} \Delta U(\vec{r}) \right) - i \frac{1 + 2\gamma}{4m} \vec{\sigma} \cdot (\vec{\nabla} U(\vec{r}) \times \vec{\nabla}),
\]

where \( \vec{r} \) is the radius vector of a slow fermion, \( \vec{\nabla} = \partial / \partial \vec{r} \) is the gradient and \( U(\vec{r}) \) is the gravitational potential taken in the form

\[
U(\vec{r}) = \vec{g} \cdot \vec{r} + \frac{\beta}{M_{\text{Pl}}} \phi(\vec{r}).
\]

The first term is the Newtonian gravitational potential of the Earth with the gravitational acceleration \( \vec{g} \), whereas the second one stands for the contribution of the chameleon field \( \phi(\vec{r}) \) and describes a deviation from the Newtonian gravity \( \vec{g} \). It is determined by the reduced Planck mass \( M_{\text{Pl}} = 1/\sqrt{8\pi G_N} = 2.435 \times 10^{27} \text{ eV} \), where \( G_N \) is the gravitational constant \( [4] \), and the chameleon–matter coupling constant \( \beta < 5.8 \times 10^8 \) [5]. The last term in Eq.(1) has been interpreted as the potential of the torsion–fermion interaction with the torsion field \( \vec{T} = (\beta/M_{\text{Pl}}) \vec{\nabla} \phi(\vec{r}) \), caused by the chameleon field. For the confirmation of the relation of the last term in Eq.(1) to the torsion field there have been used the results, obtained by Kostelecky et al. [6, 7]. An extension of the results, obtained in [1], as a version of the Einstein–Cartan gravitational theory with torsion, defined by the gradient of the chameleon field, has been proposed in [8].

In this paper we give a derivation of the effective low–energy gravitational potential of slow fermions, coupled to the chameleon, torsion and weak gravitational field in the spacetime defined by the Schwarzschild metric. In the Einstein frame it takes the form

\[
ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = (1 + 2U_E) dt^2 - (1 - 2U_E) d\vec{r}^2,
\]

where \( U_E \) is a gravitational potential of the Earth. The torsion field \( T^\alpha_{\mu\nu} = -T^\alpha_{\nu\mu} \) is related to the affine connection as follows [8] (see also [3])

\[
\Gamma^\alpha_{\mu\nu} = \{^\alpha_{\mu\nu}\} = -\frac{1}{2} (T^\alpha_{\mu\nu} - T^\mu_{\nu\alpha} - T^\nu_{\mu\alpha} = \{^\alpha_{\mu\nu}\} + g^{\alpha\sigma} K_{\sigma\mu\nu},
\]

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where $\{\alpha_{\mu\nu}\}$ are the Christoffel symbols.

$$\{\alpha_{\mu\nu}\} = \frac{1}{2} g^{\alpha\lambda} \left( \frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right)$$ \hspace{1cm} (5)

and $\mathcal{K}_{\alpha\mu\nu} = -(1/2)(T^\alpha_{\mu\nu} - T^\alpha_{\nu\mu} - T^\alpha_{\mu\nu})$ is the contorsion tensor. The metric tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ have the following components: $g_{00} = 1 + 2U_E$, $g_{0j} = 0$ and $g_{ij} = -(1 - 2U_E) \delta_{ij}$ and $g^{00} = 1 - 2U_E$, $g^{ij} = 0$ and $g^{0j} = -(1 + 2U_E) \delta_{ij}$, respectively, taken in the linear approximation with respect to the $U_E$-expansion.

An analogous analysis of the Dirac equation for slow fermions, moving in spacetime with the Schwarzschild metric, has been carried out by Lämmerzahl. Unlike Lämmerzahl, we take into account a dependence of torsion on a weak gravitational field approximation and torsion but without the contribution of the chameleon field, has non-minimally to the torsion field. Throughout the paper we follow the analysis of the Dirac equation in the curved spacetime with torsion.

In the gravitational theory with the chameleon field slow fermions couple to the chameleon field through the metric $\tilde{g}_{\mu\nu}$ in the Jordan frame satisfying the anticommutation relation

$$\tilde{\gamma}^\mu(x)\tilde{\gamma}^\nu(x) + \tilde{\gamma}^\nu(x)\tilde{\gamma}^\mu(x) = 2\tilde{g}^{\mu\nu}(x)$$ \hspace{1cm} (7)

and $D_\mu$ is a covariant derivative without gauge fields. For an exact definition of the Dirac matrices $\tilde{\gamma}^\mu(x)$ and the covariant derivative $D_\mu$ we follow and use a set of vierbein fields $e^\alpha_\mu(x)$ at each spacetime point $x$ defined by

$$dx^\alpha = e^\alpha_\mu(x)dx^\mu.$$ \hspace{1cm} (8)

The vierbein fields relate in an arbitrary (world) coordinate system a spacetime point $x$, which is characterized by the index $\alpha = 0, 1, 2, 3$, to a locally Minkowskian coordinate system erected at a spacetime point $x$, which is characterized by the index $\alpha = 0, 1, 2, 3$. The vierbein fields $e^\alpha_\mu(x)$ are related to the metric tensor $\tilde{g}_{\mu\nu}(x)$ by

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = \eta_{\alpha\beta} \left( e^\alpha_\mu(x) dx^\mu \right) \left( e^\beta_\nu(x) dx^\nu \right) = \eta_{\alpha\beta} e^\alpha_\mu(x) e^\beta_\nu(x) dx^\mu dx^\nu = \tilde{g}_{\mu\nu}(x) dx^\mu dx^\nu,$$ \hspace{1cm} (9)

where $\eta_{\alpha\beta}$ is the metric tensor in the Minkowski spacetime with the signature $\eta_{\alpha\beta} = (1, -1, -1, -1)$. This gives

$$\tilde{g}_{\mu\nu}(x) = \eta_{\alpha\beta} e^\alpha_\mu(x) e^\beta_\nu(x).$$ \hspace{1cm} (10)
Thus, the vierbein fields can be viewed as the square root of the metric tensor $\tilde{g}_{\mu\nu}(x)$ in the sense of a matrix equation. Inverting the relation Eq. (1) we obtain

$$\eta_{\alpha\beta} = \tilde{g}_{\mu\nu}(x)\tilde{e}^\mu_\alpha(x)\tilde{e}^\nu_\beta(x).$$  \hspace{1cm} (11)

There are also the following relations

$$\begin{align*}
\tilde{e}^\mu_\alpha(x)\tilde{e}^\beta_\mu(x) &= \delta^\beta_\alpha, \\
\tilde{e}^\mu_\alpha(x)\tilde{e}^\nu_\alpha(x) &= \delta^\nu_\alpha, \\
\tilde{e}^\mu_\alpha(x)\tilde{e}_\beta(x) &= \eta_{\alpha\beta}, \\
\tilde{e}_{\alpha\mu}(x) &= \eta_{\alpha\beta}\tilde{e}^\beta_\mu(x), \\
\tilde{e}^\alpha_\mu(x)\tilde{e}_{\alpha\nu}(x) &= \tilde{g}_{\mu\nu}(x),
\end{align*}$$

(12)

which are useful for the derivation of the Dirac equation and calculation of the Dirac Hamilton operator of fermions with mass $m$ (see the Appendix). In terms of the vierbein fields the Dirac matrices $\gamma^\mu(x)$ are defined by

$$\gamma^\mu(x) = \tilde{e}^\mu_\alpha(x)\gamma^\alpha,$$ \hspace{1cm} (13)

where $\gamma^\alpha$ are the Dirac matrices in the Minkowski spacetime. A covariant derivative $D_\mu$ we define as

$$D_\mu = \partial_\mu - \tilde{\Gamma}_\mu(x),$$ \hspace{1cm} (14)

where $\tilde{\Gamma}_\mu(x)$ is the spin affine connection. In terms of the spin connection $\tilde{\omega}_{\mu\alpha\beta}(x)$ it is given by

$$\tilde{\Gamma}_\mu(x) = \frac{i}{4}\tilde{\omega}_{\mu\alpha\beta}\sigma^{\alpha\beta},$$ \hspace{1cm} (15)

where $\sigma^{\alpha\beta} = (i/2)(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)$ are the Dirac matrices in the Minkowski spacetime. The derivation of the Dirac equation in the curved spacetime with the metric tensor $\tilde{g}_{\mu\nu}(x)$ and torsion we have carried out in the Appendix. The result is

$$\left( i\tilde{e}^\mu_\lambda(x)\gamma^\lambda D_\mu - \frac{1}{2} i\tilde{\Gamma}^\alpha_\beta\gamma_\beta \tilde{e}^\alpha_\mu(x)\tilde{e}^\mu_\lambda(x)\gamma^\lambda - \frac{1}{2} i\tilde{\omega}_{\mu\alpha\beta}(x)\tilde{e}^\alpha_\lambda(x) \left( \eta^{\lambda\beta}\gamma^\lambda + \frac{1}{4} i [\sigma^{\alpha\beta}, \gamma^\lambda] \right) - m \right) \psi(x) = 0,$$ \hspace{1cm} (16)

where $[\sigma^{\alpha\beta}, \gamma^\lambda] = \sigma^{\alpha\beta}\gamma^\lambda - \gamma^\lambda\sigma^{\alpha\beta}$. The Dirac equation Eq. (16) agrees well with Eq. (18) of Ref. [14]. The spin connection $\tilde{\omega}_{\mu\alpha\beta}(x)$ is related to the vierbein fields and the affine connection as follows

$$\tilde{\omega}_{\mu\alpha\beta}(x) = -\eta_{\alpha\beta}\left( \partial_\mu\tilde{e}^\alpha_\nu(x) - \tilde{\Gamma}^\alpha_\mu\nu(x)\tilde{e}^\beta_\nu(x) \right)\tilde{e}^\nu_\beta(x).$$ \hspace{1cm} (17)

The vierbein fields $\tilde{e}^\alpha_\mu(x)$ in the Jordan frame are related to the vierbein fields $e^\alpha_\mu(x)$ in the Einstein frame by

$$e^\alpha_\mu(x) = f\tilde{e}^\alpha_\mu(x), \quad e^\beta_\alpha(x) = f^{-1}\tilde{e}^\beta_\alpha(x).$$ \hspace{1cm} (18)

For the Einstein–frame metric Eq. (1) the vierbein fields are equal to

$$\begin{align*}
e^\alpha_\mu(x) &= (1 + U_E)\delta^\alpha_\mu, \quad e^0_\mu(x) = (1 - U_E)\delta^0_\mu, \\
e^1_\mu(x) &= (1 - U_E)\delta^1_\mu, \quad e^2_\mu(x) = (1 + U_E)\delta^2_\mu,\end{align*}$$ \hspace{1cm} (19)

where we have kept only the linear terms in the $U_E$–expansion. Expanding the conformal factor $f = e^{\beta\phi/M_{Pl}}$ in powers of $\beta\phi/M_{Pl}$ we define the vierbein fields $\tilde{e}^\alpha_\mu(x)$ in the Jordan frame as

$$\begin{align*}
\tilde{e}^0_\alpha(x) &= (1 + U_+ )\delta^0_\alpha, \quad \tilde{e}^0_\alpha(x) = (1 - U_+ )\delta^0_\alpha, \\
\tilde{e}^1_\alpha(x) &= (1 - U_- )\delta^1_\alpha, \quad \tilde{e}^2_\alpha(x) = (1 + U_- )\delta^2_\alpha.
\end{align*}$$ \hspace{1cm} (20)

The potentials $U_\pm$ are determined by

$$U_\pm = U_E \pm \frac{\beta}{M_{Pl}} \phi,$$ \hspace{1cm} (21)
where we have kept only the linear contributions of the chameleon field and the gravitational field of the Earth. In such an approximation the spin affine connection $\tilde{\Gamma}_\mu(x)$ is defined by the affine connection as follows

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2} \delta^\alpha_\gamma \left( \frac{\partial \tilde{g}^\gamma_{\alpha\lambda}}{\partial x^\mu} + \frac{\partial \tilde{g}^\gamma_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \tilde{g}^\gamma_{\nu\lambda}}{\partial x^\mu} \right) - \frac{1}{2} \left( T^\alpha_{\mu\nu} - T^\mu_{\alpha\nu} - T^\nu_{\alpha\mu} \right),$$

(22)

where $\tilde{g}_{00} = 1 + 2U_+, \tilde{g}_{0j} = \tilde{g}_{j0} = 0, \tilde{g}_{ij} = (1 - 2U_-) \eta_{ij}$ and $\eta_{ij} = -\delta_{ij}$ determine the Schwarzschild metric in a weak gravitational field approximation modified by the contribution of the chameleon field. In the linear approximation for the definition of the contribution of the torsion field we have set the conformal factor $f = 1$. Thus, Eq. (19) with the vierbein fields, given by Eq. (20), and the spin affine connection, determined in terms of the affine connection Eq. (22), is the Dirac equation for fermions, coupled to the torsion field $T^\alpha_{\mu\nu}$ in the spacetime with the Schwarzschild metric: $\tilde{g}_{00} = 1 + 2U_+, \tilde{g}_{0j} = \tilde{g}_{j0} = 0$ and $\tilde{g}_{ij} = -(1 - 2U_-) \delta_{ij}$, modified by the contribution of the chameleon field. The Dirac equation Eq. (16) realizes also a minimal coupling for the torsion–fermion (matter) interactions.

III. DIRAC HAMILTON OPERATOR FOR SLOW FERMIONS IN THE GRAVITATIONAL FIELD OF THE EARTH WITH CHAMELEON AND TORSION FIELDS

In the Appendix we have derived the general expression for the Dirac Hamilton operator for fermions in the curved space time with the chameleon field and torsion. For the derivation of an effective gravitational potential for slow fermions we have approximated such a Hamilton operator keeping only the linear order contributions of interacting fields. The Dirac equation for slow fermions in the standard form is

$$i \frac{\partial \psi}{\partial t} = H \psi,$$

(23)

where $H = H_0 + \delta H$ and $H_0 = \gamma^0 \hat{m} - i \gamma^0 \vec{\gamma} \cdot \vec{\nabla}$ is the Hamilton operator of free fermions, whereas $\delta H$ defines the interactions of fermions with gravitational, chameleon and torsion fields

$$\delta H = (\hat{c}^0_0(x) - 1) \gamma^0 \hat{m} - i (\hat{c}^0_0(x) - 1) \gamma^0 \gamma^j \frac{\partial}{\partial x^j} - i \hat{c}^0_0(x)(\hat{c}^j_j(x) - \delta^j_0) \gamma^0 \gamma^j + \frac{1}{2} i \eta^{\mu\nu} \left( \partial_{\mu} \hat{c}^0_0(x) - \{ \hat{c}^0_0(x) - \{ \hat{c}^0_0(x) \}^\mu_{\nu} \right) \gamma^0 \gamma^j$$

(24)

to linear order of interacting fields, where $\hat{c}^{\alpha\beta}_{\mu\nu}$ is the Levi–Civita tensor such as $\epsilon^{0123} = +1$ and $\epsilon^{0j\ell m} = \epsilon^{j\ell m}$ and $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$ is the Dirac matrix $\gamma^5$. We would like to note that below because of the proportionality of the vierbein fields to Kronecker tensors Eq. (20) we do not distinguish the indices in the Minkowski spacetime $\hat{\alpha}$ and the indices in the curved spacetime $\alpha$. This is also confirmed by the use of the weak gravitational, torsion and chameleon field approximation, where the gravitational, torsion and chameleon fields appear in the form of interactions with Dirac fermions in the Minkowski spacetime. Using the vierbein fields Eq. (20) and the relation

$$- \frac{1}{2} i \eta^{\mu\nu} \left( \partial_{\mu} \hat{c}^0_0(x) - \{ \hat{c}^0_0(x) \}^\mu_{\nu} \right) \gamma^0 \gamma^j = i \frac{3}{2} \frac{\partial U_-}{\partial t} - \frac{1}{2} i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} U_+ + i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} U_-$$

(25)

we transcribe the Hamilton operator $\delta H$ into the form

$$\delta H = i \frac{3}{2} \frac{\partial U_-}{\partial t} + \gamma^0 U_+ m - i \left( U_+ + U_- \right) \gamma^0 \vec{\gamma} \cdot \vec{\nabla} - \frac{1}{2} i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} U_+ + i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} U_- - \frac{1}{4} \gamma^5 \mathcal{K} - \frac{1}{4} \Sigma \cdot \vec{B},$$

(26)

where we have denoted $\Sigma = \gamma^0 \gamma^5$, which is the $4 \times 4$–diagonal matrix with elements defined by the $2 \times 2$ Pauli matrices $\mathcal{K}$, and

$$\mathcal{K} = \frac{1}{2} \epsilon^{\ell m} T_{\ell m},$$

$$\vec{B} \gamma^j = \frac{1}{2} \epsilon^{\ell m} (T_{\ell m0} + T_{m0\ell} + T_{0m\ell}),$$

(27)

which are the time and space components of the axial 4–vector $B^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} T_{\beta\mu\nu} = (\mathcal{K}, \vec{B})$, respectively.

For the subsequent derivation of the low–energy Hamilton operator of slow fermions we have to make the standard transformations of the wave function of slow fermions and the Hamilton operator (13, 17) (see also (1))

$$\psi(x) = (1 + \frac{3}{2} U_-) \psi'(x)$$

$$H' = H_0 + \delta H = \frac{3}{2} [H_0, U_-] - i \frac{3}{2} \frac{\partial U_-}{\partial t},$$

(28)
where we have kept only the linear contributions of gravitational, chameleon and torsion fields. For the Hamilton operator $H'$ we obtain the following expression

$$H' = H_0 + \delta H',$$  \hfill (29)

where $\delta H'$ is given by

$$\delta H' = \gamma^0 m U_+ - i (U_+ + U_-) \gamma^0 \vec{\gamma} \cdot \vec{\nabla} - \frac{i}{2} \gamma^0 \vec{\gamma} \cdot \vec{\nabla} (U_+ + U_-) - \gamma^5 \frac{1}{4} K - \frac{1}{4} \vec{\Sigma} \cdot \vec{B}.$$  \hfill (30)

The Dirac equation in its standard form reads

$$i \frac{\partial \psi}{\partial t} = H' \psi'.$$  \hfill (31)

The Hamilton operator Eq. (30) agrees well with those calculated by Lämmerzahl (see Eq. (10) of Ref. 12) and Obukhov, Silenko, and Teryaev (see Eq. (2.21) of Ref. 13). However, these authors did not take into account the contributions of the chameleon field.

IV. EFFECTIVE HAMILTON OPERATOR FOR SLOW FERMIIONS IN THE GRAVITATIONAL FIELD OF THE EARTH WITH CHAMELEON AND TORSION FIELDS

For the derivation of the low-energy approximation of the Dirac equation Eq. (31) we use the Foldy–Wouthuysen (FW) transformation 19. The aim of the FW transformation is to delete all odd operators, which are proportional to $\vec{\gamma}$, $\gamma^0 \vec{\gamma}$, $\vec{\Sigma}$, and $\gamma^0 \vec{\Sigma}$. As a result, the final low-energy Hamilton operator should be expressed in terms of the even operators only, which are proportional to $\gamma^0$, $\vec{\Sigma} = \gamma^0 \vec{\gamma} \vec{\gamma}$ and $\gamma^0 \vec{\Sigma}$, respectively. For the elimination of odd operators we perform the unitary transformation 19, 20

$$H_1 = e^{+i S_1} H' e^{-i S_1} - i e^{i S_1} \frac{\partial}{\partial t} e^{-i S_1} = H' - \frac{\partial S_1}{\partial t} + i \left[ S_1, H' \right] - \frac{i}{2} \left[ S_1, \frac{\partial S_1}{\partial t} \right] + \ldots$$  \hfill (32)

The time derivative appears because of a time dependence of the chameleon and torsion fields. Then, following 19 we take the operator $S_1$ in the form

$$S_1 = -\frac{i}{2 m} \gamma^0 \left[ (1 + U_+) i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} - \frac{i}{2} \gamma^0 \vec{\gamma} \cdot \vec{\nabla} (U_+ + U_-) - \frac{1}{4} \gamma^5 K \right] =$$

$$= -\frac{1}{2 m} (1 + U_-) \vec{\gamma} \cdot \vec{\nabla} - \frac{1}{4 m} \vec{\gamma} \cdot \vec{\nabla} (U_+ + U_-) + \frac{i}{8 m} \gamma^0 \gamma^5 K.$$  \hfill (33)

The time derivative of $S_1$ and the commutators in Eq. (32) are equal to

$$\frac{\partial S_1}{\partial t} = -\frac{i}{2 m} \frac{\partial U_+}{\partial t} \vec{\gamma} \cdot \vec{\nabla} + \frac{i}{2 m} \gamma^0 \gamma^5 \frac{\partial K}{\partial t},$$

$$i \left[ S_1, H' \right] = i \left( 1 + U_+ + U_- \right) \gamma^0 \vec{\gamma} \cdot \vec{\nabla} \left( \vec{\nabla} U_+ + \vec{\nabla} U_- \right) + \frac{i}{2 m} \vec{\gamma} \cdot \vec{\nabla} (3U_+ + 4U_-) \cdot \vec{\nabla}$$

$$- \frac{\gamma^0}{2 m} i \left( \vec{\nabla} U_+ + \vec{\nabla} U_- \right) \cdot \vec{\nabla} + i \gamma^0 \vec{\gamma} \cdot \vec{\nabla} U_+ - \frac{\gamma^0}{2 m} \left( 1 + U_+ + 2U_- \right) \Delta - \frac{\gamma^0}{2 m} \left( U_+ + 2U_- \right) \Delta$$

$$+ \frac{1}{4} \gamma^5 K - \frac{i}{8 m} \gamma^0 \vec{\Sigma} \cdot \vec{\nabla} - \frac{i}{8 m} \gamma^0 \vec{\Sigma} \cdot \vec{\nabla} K,$$

$$+ \frac{i}{4 m} \gamma^0 \vec{\Sigma} \cdot \left( \vec{B} \times \vec{\nabla} \right) - \frac{i}{4 m} \gamma^0 \vec{\Sigma} \cdot \vec{\nabla} K,$$

$$\frac{i}{2} \left[ S_1, \frac{\partial S_1}{\partial t} \right] = \frac{\gamma^0}{2 m} \left( 1 + U_+ + 2U_- \right) \Delta + \frac{\gamma^0}{2 m} \vec{\nabla} U_+ \cdot \vec{\nabla} + \frac{\gamma^0}{8 m} \Delta \left( 3U_+ + 2U_- \right)$$

$$+ \frac{\gamma^0}{4 m} i \left( \vec{\nabla} U_+ + \vec{\nabla} U_- \right) \cdot \vec{\nabla} - \frac{i}{4 m} \gamma^0 \vec{\Sigma} \cdot \vec{\nabla} - \frac{i}{8 m} \gamma^0 \vec{\Sigma} \cdot \vec{\nabla} K.$$  \hfill (34)

Thus, the unitary transformation Eq. (32) yields the following Hamilton operator of slow fermions

$$H_1 = \gamma^0 m - \frac{\gamma^0}{2 m} \Delta + \gamma^0 m U_+ + \frac{i}{2} \gamma^0 \vec{\gamma} \cdot \vec{\nabla} U_+ - \frac{\gamma^0}{2 m} \vec{\nabla} U_+ \cdot \vec{\nabla} + \frac{\gamma^0}{2 m} (U_+ + 2U_-) \Delta - \frac{\gamma^0}{8 m} \Delta (U_+ + 2U_-)$$

$$- \frac{\gamma^0}{4 m} i \vec{\Sigma} \cdot \left( \vec{\nabla} U_+ + \vec{\nabla} U_- \right) - \frac{1}{4} \vec{\Sigma} \cdot \vec{B} + \frac{i}{4 m} \gamma^0 \vec{\Sigma} \cdot \vec{\nabla} + \frac{i}{8 m} \gamma^0 \vec{\Sigma} \cdot \vec{\nabla} K$$

$$+ \frac{1}{4 m} \vec{\gamma} \cdot \left( \vec{B} \times \vec{\nabla} \right) - \frac{i}{8 m} \vec{\gamma} \cdot \vec{\nabla} K,$$  \hfill (35)
where we have neglected the contributions of order $1/m^2$. In order to remove odd operators in the Hamilton operator $H_1$ we perform the second unitary transformation \[\gamma \] 

$$H_2 = e^{+iS_2} H_1 e^{-iS_2} - i e^{iS_2} \frac{\partial}{\partial t} e^{-iS_2} = H_1 - \frac{\partial S_2}{\partial t} + i \left[ S_2, H_1 - \frac{1}{2} \frac{\partial S_2}{\partial t} \right] + \frac{i^2}{2} \left[ S_2, \left[ S_2, H_1 - \frac{1}{3} \frac{\partial S_2}{\partial t} \right] \right] + \ldots, \quad (36)$$

where the operator $S_2$ is equal to

$$S_2 = \frac{i}{2m} \gamma^0 \left( \frac{i}{2} \gamma^0 \gamma^7 \cdot \nabla U_+ + \frac{1}{4m} \gamma^0 (\vec{B} \times \nabla) - \frac{1}{8m} \gamma^7 \cdot \text{rot} \vec{B} + \frac{i}{8m} \gamma^0 \gamma^5 \text{div} \vec{B} + \frac{1}{2m} \frac{\partial U_-}{\partial t} \gamma^7 \cdot \nabla - \frac{i}{8m} \gamma^0 \gamma^5 \frac{\partial K}{\partial t} \right) = \frac{1}{4m} \gamma^7 \cdot \nabla U_+ - \frac{i}{8m} \gamma^0 \gamma^7 (\vec{B} \times \nabla) + \frac{i}{16m} \gamma^0 \gamma^7 \cdot \text{rot} \vec{B} + \frac{1}{4m} \gamma^5 \text{div} \vec{B} - \frac{1}{4m} \frac{\partial U_-}{\partial t} i \gamma^0 \gamma^7 \cdot \nabla = \frac{1}{16m} \gamma^5 \frac{\partial K}{\partial t}. \quad (37)$$

Keeping only the contributions to order $1/m$ for the time derivative of $S_2$ and the commutators we obtain the following expressions

$$\frac{\partial S_2}{\partial t} = \frac{1}{4m} \gamma^7 \cdot \nabla \frac{\partial U_+}{\partial t},$$

$$i \left[ S_2, H_1 - \frac{1}{2} \frac{\partial S_2}{\partial t} \right] = -\frac{i}{2} \gamma^0 \gamma^7 \cdot \nabla U_+ - \frac{1}{4m} \gamma^7 \cdot (\vec{B} \times \nabla) + \frac{1}{8m} \gamma^0 \gamma^7 \cdot \text{rot} \vec{B} - \frac{i}{8m} \gamma^0 \gamma^5 \text{div} \vec{B},$$

$$\frac{i^2}{2} \left[ S_2, \left[ S_2, H_1 - \frac{1}{3} \frac{\partial S_2}{\partial t} \right] \right] = 0. \quad (38)$$

Thus, after two unitary transformation the effective Hamilton operator of slow fermions takes the form

$$H_2 = \gamma^0 m - \frac{\gamma^0}{2m} \Delta + \gamma^0 m U_+ - \frac{\gamma^0}{2m} \nabla (U_+ + 2U_-) \cdot \nabla - \gamma^0 \left( U_+ + 2U_- \right) \Delta - \frac{\gamma^0}{8m} \Delta (U_+ + 2U_-) - \frac{\gamma^0}{4m} i \Sigma^2 \cdot \text{rot} (\vec{B} \times \nabla) + \frac{i}{4m} \gamma^0 \gamma^7 \cdot \nabla \frac{\partial U_+}{\partial t}. \quad (39)$$

The last term in Eq. (39), which is the last remained odd operator after two FW transformations, we delete by the third FW transformation

$$H_3 = e^{+iS_3} H_2 e^{-iS_3} - i e^{iS_3} \frac{\partial}{\partial t} e^{-iS_3} = H_2 - \frac{\partial S_3}{\partial t} + i \left[ S_3, H_3 - \frac{1}{2} \frac{\partial S_3}{\partial t} \right] + \frac{i^2}{2} \left[ S_3, \left[ S_3, H_3 - \frac{1}{3} \frac{\partial S_3}{\partial t} \right] \right] + \ldots \quad (40)$$

with the operator $S_3$, given by

$$S_3 = -\frac{i}{2m} \gamma^0 \left( -\frac{1}{4m} \gamma^7 \cdot \nabla \frac{\partial U_+}{\partial t} \right) = \frac{i}{8m^2} \gamma^0 \gamma^7 \cdot \nabla \frac{\partial U_+}{\partial t}. \quad (41)$$

Neglecting the contributions of the terms of order $1/m^2$ we obtain the low–energy reduction of the Dirac Hamilton operator for slow fermions

$$H_3 = \gamma^0 m - \frac{\gamma^0}{2m} \Delta + \gamma^0 m U_+ - \frac{\gamma^0}{2m} \nabla (U_+ + 2U_-) \cdot \nabla - \gamma^0 \left( U_+ + 2U_- \right) \Delta - \frac{\gamma^0}{8m} \Delta (U_+ + 2U_-) - \frac{\gamma^0}{4m} i \Sigma^2 \cdot \text{rot} (\vec{B} \times \nabla) + \frac{i}{4m} \gamma^0 \gamma^7 \cdot \nabla \frac{\partial U_+}{\partial t}. \quad (42)$$

Following the standard procedure \[\gamma \], removing the mass term $\gamma^0 m$ and skipping intermediate calculations we arrive at the Schrödinger–Pauli equation for the large component $\Psi(\vec{r}, t)$ of the Dirac wave function of slow fermions

$$i \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left( -\frac{1}{2m} \Delta + m U_E + \Phi_{\text{ngp}_{-\text{ch}}} + \Phi_{\text{mct}} \right) \Psi(\vec{r}, t), \quad (43)$$
where \( \Phi_{\text{ngr-ch}} \) and \( \Phi_{\text{mt}} \) are the effective low–energy gravitational potentials

\[
\Phi_{\text{ngr-ch}} = m(U_+ - U_E) - \frac{1}{2m} \left( \frac{U_+ + 2U_-}{2} \cdot \nabla - \frac{1}{2m} (U_+ + 2U_-) \Delta - \frac{1}{8m} \Delta(U_+ + 2U_-) \right)
- \frac{i}{4m} \bar{\sigma} \cdot \left( \nabla(U_+ + 2U_-) \times \nabla \right)
\]

and

\[
\Phi_{\text{mt}} = - \frac{1}{4} \bar{\sigma} \cdot \bar{B} + \frac{i}{4m} \kappa \bar{\sigma} \cdot \nabla + \frac{1}{8m} \bar{\sigma} \cdot \nabla \kappa,
\]

(44)

(45)

describing interactions of slow fermions with the chameleon and torsion field, respectively, in the spacetime with the Schwarzschild metric taken in the weak gravitational field approximation. The contributions of the potentials \( \Phi_{\text{ngr-ch}} \) and \( \Phi_{\text{mt}} \) provide a deviation from the Newtonian potential of the Earth \( mU_E \). The abbreviation \( \text{(ngr-ch)} \) means the Newtonian gravitational potential with chameleon coupling, whereas the abbreviation \( \text{(mt)} \) stands for the minimal torsion coupling.

After the replacement \( U_+ \rightarrow U \) and \( U_- \rightarrow \gamma U \) the potential \( \Phi_{\text{ngr-ch}} \) in Eq.(44) coincides with the effective low–energy gravitational potential calculated in [3] (see also Eq. (20) of Ref. [1]). The difference, appearing in the effective coupling constants, can be explained as follows. The chameleon field in [3] has been added to the weak gravitational field in the form Eq.(2), whereas in this paper we have used a relation \( g_{\mu\nu} = f^2 \tilde{g}_{\mu\nu} \), connecting the metric tensor \( \tilde{g}_{\mu\nu} \) in the Jordan frame with the metric tensor \( g_{\mu\nu} \) in the Einstein frame through the conformal factor \( f = e^{\beta \phi/M}\). This leads to the contributions of the chameleon field to the metric tensor \( \tilde{g}_{\mu\nu} \) as it is shown in Eq. (21).

The first two terms in the effective low–energy gravitational potential \( \Phi_{\text{mt}} \), given by Eq.(15), were obtained by Lämmerzahl [12], where the derivatives of the torsion field were neglected. In turn, Obukhov, Silenko, and Teryaev [13] kept the derivatives of the torsion field and the all terms of the effective low–energy gravitational potential \( \Phi_{\text{mt}} \) can be found in the effective Hamilton operator \( H_{FW}^{(3)} \) of Ref. [13] by expanding this operator in powers of \( 1/m \).

V. NON–MINIMAL COUPLING OF TORSION WITH SLOW FERMIONS

The torsion tensor field \( T_{\sigma \mu \nu} \), being a third–order tensor antisymmetric with respect to \( \mu \) and \( \nu \) indices \( T_{\sigma \mu \nu} = -T_{\sigma \nu \mu} \), is defined by 24 independent components. They can be represented in the following irreducible form [6]

\[
T_{\sigma \mu \nu} = \frac{1}{3} (g_{\mu \nu} E_{\sigma} - g_{\sigma \nu} E_{\mu} + \frac{i}{3} \epsilon_{\sigma \mu \nu \beta} B^{\beta} + M_{\sigma \mu \nu},
\]

(46)

where the 4–vector and axial 4–vector fields \( E_{\sigma} \) and \( B^{\alpha} \) fields, respectively, possessing 4 independent components each, are defined by

\[
E_{\nu} = g^{\sigma \nu} T_{\sigma \mu \nu} , \quad B^{\alpha} = \frac{1}{2} \epsilon^{\alpha \mu \nu \sigma} T_{\sigma \mu \nu}.
\]

(47)

For the definition of the vector field \( B^{\alpha} \) in terms of the torsion tensor field \( T_{\sigma \mu \nu} \) we have used the relation \( \epsilon^{\alpha \mu \nu \beta} \epsilon_{\sigma \mu \nu \beta} = -6 \delta^{\alpha \beta} \) [18]. The residual 16 independent components can be attributed to the tensor field \( M_{\sigma \mu \nu} \), which obeys the constraints \( g^{\mu \nu} M_{\sigma \mu \nu} = \epsilon^{\alpha \mu \nu \beta} M_{\sigma \mu \nu} = 0 \) [6].

As we have shown above the effective low–energy potential of slow fermions, coupled minimally to the torsion, contains only the axial 4–vector torsion components \( B^{\alpha} \). The 4–vector \( T_{\sigma \alpha \mu} = E_{\mu} \) and tensor \( M_{\sigma \mu \nu} \) torsion components have no minimal couplings with Dirac fermions. The most general phenomenological Lagrangian of Dirac fermions coupled to torsion has been proposed by Kostelecky, Russell, and Tasson [6]. We would like to mention that the non–minimal torsion–matter couplings have been recently discussed by Puetzfeld and Obukhov [21].

In this section we derive the effective low–energy potential for slow fermions, coupled to the \( E_{\nu} \) field and tensor \( M_{\sigma \mu \nu} \) torsion components. Following Kostelecky, Russell, and Tasson [6] we consider the torsion–fermion Lagrangian

\[
\delta \mathcal{L}_{\text{nt}} = \frac{1}{4} \kappa_1 \sqrt{-\tilde{g}} \bar{\psi} \tilde{T}^{\alpha \mu \nu}(x) (\tilde{e}^{\nu}_{\alpha}(x) \bar{\psi}(x) \gamma^\lambda \psi(x)) + \frac{\kappa_2}{8m} \sqrt{-\tilde{g}} i \bar{\psi} \tilde{T}^{\alpha \mu \nu}(x) (\tilde{e}^{\mu}_{\alpha}(x) \bar{\psi}^{\nu}(x) \left( \bar{\psi}(x) \sigma^\alpha \tilde{D}_\nu \psi(x) - (\bar{\psi}(x) \tilde{D}_\nu) \sigma^\alpha \psi(x) \right)) + \frac{\kappa_3}{8m} \sqrt{-\tilde{g}} i \bar{\psi} \tilde{M}^{\lambda \mu \nu}(x) (\tilde{e}^{\mu}_{\lambda}(x) \bar{\psi}^{\nu}(x) \left( \bar{\psi}(x) \sigma^\lambda \tilde{D}_\nu \psi(x) - (\bar{\psi}(x) \tilde{D}_\nu) \sigma^\lambda \psi(x) \right)),
\]

(48)
where $\kappa_j$ ($j = 1, 2, 3$) are phenomenological dimensionless coupling constants and the abbreviation (nmt) means the non–minimal torsion coupling. The phenomenological Lagrangian Eq. (48) of non–minimal torsion couplings is a generalization of the phenomenological Lagrangian, proposed by Kostelecky, Russell, and Tasson [21], on a curved spacetime with the chameleon field. We would like to note that from the general phenomenological Lagrangian, proposed in [21], we have taken only the interactions, which are invariant under parity (P), charge–parity (CP) and time–reversal (T) transformations, and neglected the contribution of the non–derivative axial 4–vector torsion–fermion interaction, since it is already taken into account as the minimal torsion–fermion coupling.

To linear order approximation of interacting fields the Hamilton operator, corresponding to the phenomenological Lagrangian Eq. (48), takes the form

$$\delta H_{\text{nmt}} = -\frac{1}{4} \kappa_1 \mathcal{E}_\mu \gamma^0 \gamma^\mu - \frac{\kappa_2}{4m} i \mathcal{E}_\mu \gamma^0 \sigma^{\mu\nu} \partial_{\nu} - \frac{\kappa_2}{8m} i \partial_{\nu} \mathcal{E}_\mu \gamma^0 \sigma^{\mu\nu} - \frac{\kappa_3}{4m} i \mathcal{M}_\mu \gamma^0 \sigma^{\mu\nu} \partial_{\nu} - \frac{\kappa_3}{8m} i \partial_{\nu} \mathcal{M}_\mu \gamma^0 \sigma^{\mu\nu}. $$

Using the relations for the Dirac matrices

$$\gamma^0 \sigma^{\mu\nu} = i (\gamma^0 \nu \gamma^\mu - \gamma^0 \mu \gamma^\mu) - \epsilon^{\mu\nu\rho\sigma} \gamma^\rho \gamma^\sigma, $$

$$\gamma^0 \gamma^k \gamma^j = \gamma^0 \eta^{jk} + i \epsilon^{jkl} \gamma^0 \Sigma_k, $$

where $\Sigma_k = \gamma^0 \gamma^5$, we arrive at the Hamilton operator of the non–minimal torsion–fermion (matter) couplings

$$\delta H_{\text{nmt}} = -\kappa_1 \frac{1}{4} \mathcal{E}_0 + \kappa_1 \frac{1}{4} \gamma^0 \mathcal{E} \cdot \mathcal{E} + \frac{\kappa_2}{4m} \gamma^0 \mathcal{E} \cdot \partial \mathcal{E} + \frac{\kappa_2}{8m} \gamma^0 \mathcal{E} \cdot \partial \mathcal{E} + \frac{\kappa_3}{4m} \mathcal{E}_0 \gamma^0 \mathcal{E} + \frac{\kappa_3}{8m} \mathcal{E}_0 \gamma^0 \mathcal{E} + \frac{\kappa_3}{4m} \partial_{\mu} \mathcal{M}_\nu \gamma^0 \mathcal{E} + \frac{\kappa_3}{8m} \partial_{\mu} \mathcal{M}_\nu \gamma^0 \mathcal{E} + \frac{\kappa_3}{4m} \mathcal{M}_\nu \gamma^0 \mathcal{E} \cdot \partial_{\mu} \mathcal{M}_\nu \gamma^0 \mathcal{E}, $$

where we have used that $\mathcal{T}_\alpha^a = (-\mathcal{E})_\mu^\alpha$. Skipping standard intermediate Foldy–Wouthuysen calculations we arrive at the effective low–energy gravitational potential for slow fermions, non–minimally coupled to torsion:

$$\Phi_{\text{nmt}} = -\kappa_1 \frac{1}{4} \mathcal{E}_0 - i \frac{\kappa_1}{2m} \mathcal{E} \cdot \nabla - i \frac{\kappa_1}{4m} \mathcal{E} \cdot \nabla + \frac{\kappa_1}{4m} \mathcal{E} \cdot \nabla + \frac{\kappa_2}{4m} \mathcal{E} \cdot \nabla + \frac{\kappa_2}{8m} \mathcal{E} \cdot \nabla + \frac{\kappa_3}{4m} \mathcal{E} \cdot \nabla + \frac{\kappa_3}{8m} \mathcal{E} \cdot \nabla + \frac{\kappa_3}{4m} \mathcal{E} \cdot \nabla + \frac{\kappa_3}{8m} \mathcal{E} \cdot \nabla + \frac{\kappa_3}{4m} \mathcal{E} \cdot \nabla + \frac{\kappa_3}{8m} \mathcal{E} \cdot \nabla, $$

where we have neglected the contributions of order $O(1/m^2)$ and denoted $\sigma_{\ell} = (-\mathcal{E})_\mu^\ell$.

The important peculiarity of the effective low–energy potential Eq. (52) is the appearance of non–spin–torsion–fermion interactions with the coupling constants proportional to $\kappa_1$. This agrees well with the observation, obtained by Puetzfeld and Obukhov [21]. In turn, the torsion–fermion interactions, proportional to the coupling constants $\kappa_2$ and $\kappa_3$, are only spin–torsion–fermion ones.

The interactions, proportional to $\kappa_2$, do not contradict the definition of the vector torsion components in terms of the scalar field $\mathcal{E} \sim \nabla \varphi$, where $\varphi$ is a scalar field. Since in this case $\text{rot} \mathcal{E} = 0$, the spin–chameleon–matter interaction in Eq. (48) becomes equivalent to the phenomenological non–minimal torsion–matter derivative coupling of the vector torsion components in Eq. (48). This agrees well the analysis of the 4–vector torsion components, carried out in [18]. The tensor torsion components, described by $\mathcal{M}_\mu \nu$, possess only spin–torsion–matter couplings.

Thus, the Schrödinger–Pauli equation of slow fermions, coupled to the gravitational field of the Earth, the chameleon field and torsion, is given by

$$i \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left( -\frac{1}{2m} \Delta + m U_E + \Phi_{\text{ngr–ch}} + \Phi_{\text{met}} + \Phi_{\text{nmt}} \right) \Psi(\vec{r}, t).$$

This equation can be, in principle, used for the analysis of the fine–structure and transition frequencies of quantum gravitational states of ultracold neutrons [22, 23].
We have analysed the low–energy approximation of the Dirac equation for slow fermions, coupled to the chameleon field and torsion in the spacetime with the Schwarzschild metric taken in the weak Newtonian gravitational field of the Earth approximation. The aim of this analysis is addressed to the derivation of an effective low–energy gravitational potential for slow fermions coupled to gravitational and chameleon fields and torsion with minimal and non–minimal couplings. The obtained effective low–energy gravitational potential \( \Phi_{\text{eff}} = \Phi_{\text{grav-ch}} + \Phi_{\text{matt}} + \Phi_{\text{nmt}} \) for slow fermions (neutrons), coupled to the gravitational field of the Earth, the chameleon field and torsion with minimal and non–minimal torsion–fermion couplings, can be, in principle, investigated experimentally in the terrestrial laboratories in the qBounce experiments \([22, 23]\) (see also \([7]\)), the quantum ball experiments \([24–26]\) and neutron interferometry \([27, 28]\).

We have reproduced the main structure of the effective low–energy gravitational potential for slow fermions in the weak gravitational field of the Earth and the chameleon field, which was calculated in \([1]\). The distinction of our potential Eq.(44) from the potential Eq.(20) of Ref.\([1]\) is in the coupling constants of the interactions of order \(1/m\). Describing the chameleon–fermion interactions in terms of the Jordan–frame metric we become that the time–time potential Eq.(44) from the potential Eq.(20) of Ref.\([1]\) is in the coupling constants of the interactions of order \(1/m\) and \(10^{-10}\) GeV at 68 % of C.L.. For example, the upper bound of the product \(\xi_j M^j\) for the axial–vector torsion components in the minimal torsion–matter coupling approach, are: \(|B_T| < 1.0 \times 10^{-27} \text{GeV}, |B_X| < 7.0 \times 10^{-32} \text{GeV}, |B_Y| < 8.4 \times 10^{-32} \text{GeV} \) and \(|B_Z| < 3.4 \times 10^{-30} \text{GeV} \) (see Eq.(6) of Ref.\([6]\)), where we have used the relation \(B = 3J\). They agree well with the estimates, given by Lämmerzahl \([12]\) and by Obukhov, Silenko, and Teryaev \([13]\). In turn, Kostelecky et al. \([6]\) have given also the estimates of the vector and tensor torsion components. The largest value \(10^{-26}\) appears for the time–space–space torsion tensor \(|\xi_j | M^j | < 10^{-26}\). It is important to note that estimates, carried out in \([1]\), have been done for constant torsion components. This has led to the disappearance of the products \(\xi^j | M^j \) and \(\xi^j | E^j \) where coupling constants \(\xi^j \) and \(\xi^j \) are equal to \(\xi^j = \kappa_j/4\) and \(\xi^j = \kappa_j/4m\), respectively, from the superpositions of torsion–matter couplings.

The obtained effective low–energy torsion–fermion potential Eq.(45) agrees well with the effective low–energy torsion–fermion potentials, derived by Lämmerzahl \([12]\) at the neglect of the derivatives of the torsion field (see Eq.(22) of Ref.\([12]\)), and by Obukhov, Silenko, and Teryaev \([13]\) by expanding the effective potential operator \(\mathcal{H}_{\text{FW}}^{(3)}\) in powers of \(1/m\) to order \(1/m\).

In addition to the effective low–energy potential Eq.(45), caused by the minimal torsion–fermion couplings, we have derived the effective low–energy gravitational potential \(\Phi_{\text{grav}}\) of the non–minimal torsion–fermion couplings. An interesting peculiarity of this effective low–energy potential \(\Phi_{\text{grav}}\) is an appearance of some non–spin–torsion–fermion couplings, caused by non–minimal non–derivative couplings of vector torsion components. This agrees well with the results, obtained by Puetzfeld and Obukhov \([21]\). In turn, in the low–energy approximation the phenomenological derivative non–minimal torsion–fermion interactions of vector \(E^j\) and tensor \(M^{j\mu}\) torsion components possess only spin–torsion–fermion couplings. Therewith the derivative non–minimal torsion–fermion couplings of vector torsion components agree well with the hypothesis, proposed in \([1]\) and developed in \([8]\), that the vector torsion components can be induced by the chameleon field. In this case the spin–chameleon–fermion potential in Eq.(42) can be treated as a low–energy approximation of the corresponding phenomenological derivative torsion–fermion interaction of vector torsion components in Eq.(45). As regards the torsion tensor components \(M^{j\mu}\), we have to note that, according Eq.(52), at low–energies only time–space–space \(M^{0jk}\) and space–space–space \(M^{i\mu i}\) torsion components can be, in principle, observable.

The numerical estimates of the upper bounds of the axial–vector torsion components have been carried out by Lämmerzahl \([12]\), Kostelecky, Russell, and Tasson \([8]\), and Obukhov, Silenko, and Teryaev \([13]\). In turn, Kostelecky et al. \([6]\) have given also the estimates of the vector and tensor torsion components. The results, obtained by Kostelecky and Obukhov \([8]\), for the axial–vector torsion components in the minimal torsion–matter coupling approach, are: \(|B_T| < 1.0 \times 10^{-27} \text{GeV}, |B_X| < 7.0 \times 10^{-32} \text{GeV}, |B_Y| < 8.4 \times 10^{-32} \text{GeV} \) and \(|B_Z| < 3.4 \times 10^{-30} \text{GeV} \) (see Eq.(6) of Ref.\([6]\)), where we have used the relation \(B = 3J\). They agree well with the estimates, given by Lämmerzahl \([12]\) and by Obukhov, Silenko, and Teryaev \([13]\).

The estimates of the vector \(\xi^j E^j\) and to the tensor \(\xi^j M^{j\mu}\) torsion components from derivative torsion–fermion couplings, multiplied by the corresponding phenomenological coupling constants \(\xi^j\), vary over the range \((10^{-31} \text{–} 10^{-26})\). For example, the upper bound of the product \(|\xi^j M^{j\mu}| \) for different components of the torsion tensor components \(M^{j\mu}\) from \(10^{-31}\) to \(10^{-26}\) (see Table I of Ref.\([4]\)). The largest value \(10^{-26}\) appears for the time–space–space torsion tensor component \(|\xi^j M^{j\mu}| < 10^{-26}\). It is important to note that estimates, carried out in \([1]\), have been done for constant torsion components. This has led to the disappearance of the products \(\xi^j E^j\) and \(\xi^j E^j\) where coupling constants \(\xi^j\) and \(\xi^j\) are equal to \(\kappa_j/4\) and \(\kappa_j/4m\), respectively, from the superpositions of torsion–matter couplings.

VI. CONCLUSIVE DISCUSSION
Standard Model Extension (SME) \cite{34}. This gives a chance that the contributions of these torsion–fermion interactions can be estimated from the experimental data in the terrestrial laboratories, where a space–time dependence of torsion is taken into account. The experimental upper bound $|\zeta| < 5.4 \times 10^{-18}$ GeV, reported by Lehnert, Snow, and Yan \cite{21}, is by twelve orders of magnitude larger compared to the estimate $|\zeta| < 10^{-27}$ GeV, which can be obtained from Table I of Ref. \cite{6}. Completing our discussion we would like to note that we have considered the low–energy approximation of a part of the phenomenological relativistic invariant torsion–fermion interactions only. The analysis of the low–energy approximation of the rest of the phenomenological torsion–fermion interactions, proposed by Kostelecky, Russell, and Tasson \cite{12}, we are planning to perform in our forthcoming publication.

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VIII. APPENDIX A: COVARIANT DERIVATION OF THE DIRAC EQUATION IN THE CURVED SPACETIME WITH THE CHAMELEON FIELD AND TORSION

For the investigation of the dynamics of fermions, coupled to the gravitational field with torsion and the chameleon field through the Jordan metric tensor $\tilde{g}_{\mu\nu}$, we follow Kostelecky \cite{14} and define by the action

$$S_\psi = \int d^4x \sqrt{-\tilde{g}} \left( i \frac{1}{2} \bar{\psi}(x) \gamma^\mu \tilde{D}_\mu \psi(x) - m \bar{\psi}(x) \psi(x) \right).$$  \hspace{1cm} (A-1)

The definition $\bar{\psi}(x) \gamma^\mu(x) \tilde{D}_\mu \psi(x)$ should be understood as follows \cite{14}

$$\bar{\psi}(x) \gamma^\mu(x) \tilde{D}_\mu \psi(x) = \tilde{e}^\mu_\alpha(x) \left( \bar{\psi}(x) \gamma^\alpha \tilde{D}_\mu \psi(x) - (\bar{\psi} \tilde{D}_\mu) \gamma^\alpha \psi(x) \right),$$  \hspace{1cm} (A-2)

where $\left( \bar{\psi} \tilde{D}_\mu \right)$ means

$$\left( \bar{\psi} \tilde{D}_\mu \right) = \partial_\mu \bar{\psi} - \bar{\psi}(x) \gamma^\alpha \tilde{\Gamma}_\mu^\alpha \gamma^\beta.$$  \hspace{1cm} (A-3)

For the derivation of the covariant Dirac equation in the curved spacetime we rewrite the action Eq. (A-1) in terms of the vierbein fields and the spin connection. We get

$$S_\psi = \int d^4x \sqrt{-\tilde{g}} \left( i \frac{1}{2} \tilde{e}_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \left( \partial_\mu \psi(x) - \frac{1}{4} i \bar{\omega}_{\mu\alpha\beta}(x) \sigma^{\alpha\beta} \psi(x) \right) \right.$$

$$- i \frac{1}{2} \tilde{e}_\lambda^\mu(x) \left( \partial_\mu \bar{\psi}(x) + \frac{1}{4} i \bar{\omega}_{\mu\alpha\beta}(x) \bar{\psi}(x) \sigma^{\alpha\beta} \right) \gamma^\lambda \psi(x) - m \bar{\psi}(x) \psi(x) \left. \right).$$  \hspace{1cm} (A-4)

Thus, the Lagrangian of the fermion in the curved spacetime is equal to

$$L_\psi = \sqrt{-\tilde{g}} \left( i \frac{1}{2} \tilde{e}_\lambda^\mu(x) \bar{\psi}(x) \gamma^\lambda \left( \partial_\mu \psi(x) - \frac{1}{4} i \bar{\omega}_{\mu\alpha\beta}(x) \sigma^{\alpha\beta} \psi(x) \right) \right.$$

$$- i \frac{1}{2} \tilde{e}_\lambda^\mu(x) \left( \partial_\mu \bar{\psi}(x) + \frac{1}{4} i \bar{\omega}_{\mu\alpha\beta}(x) \bar{\psi}(x) \sigma^{\alpha\beta} \right) \gamma^\lambda \psi(x) - m \bar{\psi}(x) \psi(x) \right).$$  \hspace{1cm} (A-5)

The equation of motion of the fermion in the curved spacetime or the Dirac equation is

$$\partial_\mu \frac{\delta L_\psi}{\delta \partial_\mu \psi} = \frac{\delta L_\psi}{\delta \psi},$$  \hspace{1cm} (A-6)

where

$$\frac{\delta L_\psi}{\delta \partial_\mu \psi} = -\sqrt{-\tilde{g}} i \frac{1}{2} \tilde{e}_\lambda^\mu(x) \gamma^\lambda \psi(x),$$

$$\frac{\delta L_\psi}{\delta \bar{\psi}} = \sqrt{-\tilde{g}} i \frac{1}{2} \tilde{e}_\lambda^\mu(x) \gamma^\lambda \left( \partial_\mu \psi(x) - \frac{1}{4} i \bar{\omega}_{\mu\alpha\beta}(x) \sigma^{\alpha\beta} \psi(x) \right)$$

$$+ \sqrt{-\tilde{g}} \frac{1}{8} \tilde{e}_\lambda^\mu(x) \bar{\omega}_{\mu\alpha\beta}(x) \sigma^{\alpha\beta} \gamma^\lambda \psi(x) - \sqrt{-\tilde{g}} m \psi(x).$$  \hspace{1cm} (A-7)
Substituting Eq. (A-8) into Eq. (A-9) we obtain the Dirac equation in the following form

\[
(i \tilde{\sigma}^a_{\lambda}(x)\gamma^\lambda D^a + \frac{1}{2} \frac{i}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} \tilde{e}^a_{\lambda}(x)) \gamma^\lambda + \frac{1}{8} \tilde{e}^a_{\lambda}(x) \tilde{\omega}_{\mu \alpha \beta}(x)[\sigma^{\alpha \beta}, \gamma^\lambda] - m) \psi(x) = 0. 
\]  

(A-8)

The second term in the brackets can be transformed as follows

\[
\left( \frac{1}{2} \frac{i}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} \tilde{e}^a_{\lambda}(x)) \gamma^\lambda \right) = \frac{1}{2} \frac{i}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} \tilde{e}^a_{\lambda}(x)) \gamma^\lambda = \frac{1}{2} \frac{i}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} \tilde{e}^a_{\lambda}(x)) \gamma^\lambda = \frac{1}{2} \left( - \tilde{\Gamma}^\alpha_{\mu \alpha}(x) \tilde{e}^a_{\lambda}(x) \gamma^\lambda - \tilde{\omega}_{\mu \alpha \beta}(x) \tilde{e}^a_{\lambda}(x) \gamma^\lambda \right) 
\]

(A-9)

where we have used the relation

\[
\tilde{\Gamma}^\alpha_{\mu \alpha}(x) = \tilde{\omega}_{\mu \alpha \beta}(x) + \tilde{K}^\alpha_{\mu \alpha}(x) = \frac{1}{2 \sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} + \tilde{T}^\alpha_{\mu \alpha}(x)) 
\]

(A-10)

and the properties of the contorsion tensor \( \tilde{K}_{\alpha \mu \nu} = - \frac{1}{2} (\tilde{T}_{\alpha \mu \nu} - \tilde{T}_{\alpha \nu \mu} - \tilde{T}_{\gamma \mu \nu}) \)

\[ i) \tilde{K}_{\alpha \mu \nu} = - \tilde{T}_{\alpha \mu \nu}, \]

\[ ii) \tilde{g}^{\mu \nu} \tilde{K}_{\alpha \mu \nu} = - \tilde{T}^{\alpha}_{\nu \mu}, \]

\[ iii) \tilde{g}^{\alpha \nu} \tilde{K}_{\alpha \mu \nu} = 0, \]

\[ iv) \tilde{g}^{\alpha \nu} \tilde{K}_{\alpha \mu \nu} = \tilde{T}^{\alpha}_{\mu \nu}. \]

(A-11)

Thus, the Dirac equation in the curved spacetime takes the form

\[
\left( i \tilde{\sigma}^a_{\lambda}(x)\gamma^\lambda D^a + \frac{1}{2} i \tilde{T}^\alpha_{\mu \alpha}(x) \tilde{e}^a_{\lambda}(x) \gamma^\lambda - \frac{1}{2} i \tilde{\omega}_{\mu \alpha \beta}(x) \tilde{e}^a_{\lambda}(x) \gamma^\lambda + \frac{1}{4} i \sigma^{\alpha \beta}, \gamma^\lambda\right) \psi(x) = 0 
\]

(A-12)

and agrees well with Eq. (18) of Ref. [14]. From Eq. (A-12) we derive the Dirac Hamilton operator

\[
H = H_0 + \delta H 
\]

(A-13)

where \( H_0 = \gamma^0 m - i \gamma^0 \hat{\nabla} \cdot \vec{\nabla} \) and \( \delta H \) is given by

\[
\delta H = (\tilde{e}^0_\alpha(x) - 1) \gamma^0 m - i (\tilde{e}^0_\alpha(x) - 1) \gamma^0 \gamma^j \delta^j_{\beta} \frac{\partial}{\partial x^\beta} - i \tilde{\omega}_{\mu \alpha \beta}(x) \gamma^\beta \gamma^\lambda \frac{\partial}{\partial x^\lambda} + \frac{1}{2} i \tilde{\omega}_{\mu \alpha \beta}(x) \tilde{e}^a_{\lambda}(x) \gamma^\beta \gamma^\lambda + \frac{1}{2} \left( \tilde{e}^a_{\lambda}(x) \gamma^\beta \gamma^\lambda + i \sigma^{\alpha \beta}, \gamma^\lambda\right)
\]

(A-14)

which is valid for Dirac fermions in curved spacetimes with diagonal metric tensors, related to the vierbein fields with vanishing non-diagonal time–space (space–time) components. For the calculation of the Hamilton operator Eq. (A-14) it is convenient to use the following relations

\[
\gamma^\lambda \sigma^{\alpha \beta} = i (\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha) - \epsilon^{\lambda \alpha \beta \rho} \gamma^\rho, \\
\sigma^{\alpha \beta} \gamma^\lambda = i (\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha) - \epsilon^{\lambda \alpha \beta \rho} \gamma^\rho, \\
\sigma^{\alpha \beta}, \gamma^\lambda = -2 \epsilon^{\lambda \alpha \beta \rho} \gamma^\rho.
\]

(A-15)

Substituting Eq. (A-15) into Eq. (A-14) we arrive at the following Hamilton operator

\[
\delta H = (\tilde{e}^0_\alpha(x) - 1) \gamma^0 m - i (\tilde{e}^0_\alpha(x) - 1) \gamma^0 \gamma^j \delta^j_{\beta} \frac{\partial}{\partial x^\beta} - i \tilde{\omega}_{\mu \alpha \beta}(x) \gamma^\beta \gamma^\lambda \frac{\partial}{\partial x^\lambda} + \frac{1}{4} i \sigma^{\alpha \beta}, \gamma^\lambda\right) \tilde{e}^a_{\lambda}(x) \gamma^\beta \gamma^\lambda + \frac{1}{4} i \sigma^{\alpha \beta}, \gamma^\lambda\right)
\]

(A-16)

Then, the spin connection \( \tilde{\omega}_{\mu \alpha \beta}(x) \) is defined in terms of the vierbein fields and the affine connection as follows

\[
\tilde{\omega}_{\mu \alpha \beta}(x) = -\eta_{\alpha \beta} \left( \partial_{\mu} \tilde{e}^a_{\lambda}(x) - \Gamma^a_{\mu \nu}(x) \tilde{e}^a_{\lambda}(x) \right) \tilde{e}^a_{\lambda}(x).
\]

(A-17)
As a result, the Hamilton operator Eq. (A-16) reads
\[
\delta H = (c_0^T(x) - 1)\gamma^0\tilde{m} - i(c_0^T(x) - 1)i\gamma^0\gamma^\alpha\beta\gamma^\beta (\frac{\partial}{\partial x^\mu}) - i\tilde{c}_0^T(x)(c_0^T(x) - 1)\beta(\gamma^0\gamma^\alpha\beta) (\frac{\partial}{\partial x^\mu})
\]
\[
+ \frac{1}{2}i\tilde{T}_{\alpha\mu}(x)c_0^T(x)c_0^T(x)\gamma^0\gamma^\lambda - \frac{1}{2}i\tilde{T}_{\alpha\mu}(x)c_0^T(x)\tilde{c}_0^T(x)\gamma^0\gamma^\lambda \times \frac{\gamma^0(\tilde{\lambda}\tilde{\alpha}\tilde{\gamma} - \frac{1}{2}i\tilde{\epsilon}^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\gamma}})}{(A-18)}.
\]
Keeping only the linear contributions of interacting fields we rewrite the Hamilton operator \(\delta H\) as follows
\[
\delta H = (c_0^T(x) - 1)\gamma^0\tilde{m} - i(c_0^T(x) - 1)i\gamma^0\gamma^\alpha\beta\gamma^\beta (\frac{\partial}{\partial x^\mu}) - i\tilde{c}_0^T(x)(c_0^T(x) - 1)\beta(\gamma^0\gamma^\alpha\beta) (\frac{\partial}{\partial x^\mu})
\]
\[
+ \frac{1}{2}i\tilde{T}_{\alpha\mu}(x)c_0^T(x)c_0^T(x)\gamma^0\gamma^\lambda + \frac{1}{2}i\eta^{\mu\nu}\tilde{C}_{\lambda\mu\nu}(x)\gamma^0\gamma^\lambda + \frac{1}{4}i\epsilon^{\tilde{\alpha}\tilde{\beta}\mu\nu}\tilde{C}_{\tilde{\lambda}\mu\nu}(x)\gamma^0\gamma^\lambda.
\]
(A-19)

Using the properties of the contorsion tensor Eq. (A-11) we get
\[
\delta H = (c_0^T(x) - 1)\gamma^0\tilde{m} - i(c_0^T(x) - 1)i\gamma^0\gamma^\alpha\beta\gamma^\beta (\frac{\partial}{\partial x^\mu}) - i\tilde{c}_0^T(x)(c_0^T(x) - 1)\beta(\gamma^0\gamma^\alpha\beta) (\frac{\partial}{\partial x^\mu})
\]
\[
- \frac{1}{8}i\epsilon^{\tilde{\alpha}\tilde{\beta}\mu\nu}\tilde{T}_{\tilde{\lambda}\mu\nu}(x)\gamma^0\gamma^\lambda.
\]
(A-20)
The term \(\frac{1}{8}i\tilde{T}_{\alpha\mu\nu}(x)\gamma^0\gamma^\lambda\) is cancelled in agreement with the analysis of the Dirac equation, carried out by Kostelecky [14]. This implies that the axial 4-vector torsion components can possess only the minimal torsion–fermion couplings for the Dirac fermions.

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