VIRTUAL RESIDUE AND AN INTEGRAL FORMALISM

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Abstract. We generalize Grothendieck’s residues $Res^W_s$ to virtual cases, namely cases when the zero loci of the section $s$ has dimension larger than the expected dimension (zero). We also provide an exponential type integral formalism for the virtual residue, which can be viewed as an analogue of the Mathai-Quillen formalism for localized Euler classes.

1. Introduction

During the ’90s, physicists studied “genus zero B-twisted superconformal N=2 model” for the Landau-Ginzburg space $(\mathbb{C}^n, w)$ where $w$ is a holomorphic function on $\mathbb{C}^n$. Their “physical states” lie in a “chiral primary ring”

$$R = \mathbb{C}[x_1, \ldots, x_n]/(\partial_1 w, \ldots, \partial_n w),$$

also known as the Milnor ring. The “correlation” between given states $\{F_i\}$’s were calculated by Vafa in [Va], using path integral argument,

$$< O_1(x) \cdots O_k(x) >= \sum_p \frac{F_1(p) \cdots F_k(p)}{H(p)},$$

where $O_i(x) \in R$ is the $F_i$, $H = det(\partial_i \partial_j w)$ is the Hessian of $w$, and $p$ runs through (finitely many) closed points of the critical locus of $w$, which is assumed to be zero dimensional and nonreduced.

The form (1.1) is then known to be identical to the Grothendieck Residue

$$\sum_{p: \det(\partial_1 w, \ldots, \partial_n w) = 0} \text{Res}_{p \in \mathbb{C}^n} \frac{F_1 \cdots F_k}{(\partial_1 w, \ldots, \partial_n w)}.$$  

Such an interpretation generalizes (1.1) to the case critical locus of $w$ is nonreduced (still zero dimensional). For example, in the case $n = 5$ and $w = x_1^5 + \cdots + x_5^5$ the critical locus ($dw = 0$) = $\{x_1^4 = x_2^4 = \cdots = x_5^4 = 0\}$ is a nonreduced zero dimensional scheme. In this case the Grothendieck Residue (1.2) is understood as the correlator for a “genus zero B-twisted LG model $(\mathbb{C}^5, w = \sum x_i^5)$”.

A complete (coordinate-free) definition of the Grothendieck residue is provided in the book of Griffiths and Harris [GH Chapter 5], in which the zero locus of $dw$ needs to be assumed zero dimensional. However one may encounter the following case. Say $K_{\mathbb{P}^4}$ is the total space of the canonical line bundle $\mathcal{O}_{\mathbb{P}^4}(-5)$ of $\mathbb{P}^4$ and let $W : K_{\mathbb{P}^4} \to \mathbb{C}$ be defined by the pairing $\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \mathcal{O}_{\mathbb{P}^4}(5) \to \mathcal{O}_{\mathbb{P}^4}$

$^1$Partially supported by Hong Kong GRF grant 600711 and 16301515.

$^1$A Landau Ginzburg space is a pair $(X, W)$ of a complex manifold $X$ and a holomorphic function $W : X \to \mathbb{C}$ with compact critical locus.
with the quintic section $\sum_{i=1}^{5} x_i^5$ of $\mathbb{O}_{\mathbb{P}^4}(5)$. Usually the expression $W = p \sum_{i=1}^{5} x_i^5$ is used where $p$ stands for the local coordinate on $K_{\mathbb{P}^4}$ in the noncompact direction.

Let $\mathbb{P}^4 \subset K_{\mathbb{P}^4}$ be the zero section and let $Q$ be the quintic hypersurface \( \{ x_1^5 + \cdots + x_5^5 = 0 \} \subset \mathbb{P}^4 \). Then the form $W = p \sum_{i=1}^{5} x_i^5$ implies the critical loci ($dW = 0$) is identical to $Q$ as subschemes of $K_{\mathbb{P}^4}$, which is not zero dimensional and a "Grothendieck Residue"

$$\text{Res}_{Q \subset K_{\mathbb{P}^4}} \frac{dW}{dW},$$

expected to be responsible for the "genus zero B-twisted correlator of the Landau-Ginzburg space $(K_{\mathbb{P}^4}, W)$", is not defined to the best of authors' knowledge. On the contrary, in A side, a sequence of works [CL, CL1, CLL, CLL1, CLL2, Chi, CR, FJR, FJR1, FJR2, JKV] study mathematically all genus constructions and properties of LG spaces including $(K_{\mathbb{P}^4}, W)$ and $([\mathbb{C}^5/\mathbb{Z}_5], w)$.

In this paper we generalize the Grothendieck Residue as follows. Let $M$ be a smooth complex quasi-projective variety (usually noncompact) and let $V$ be a holomorphic bundle over $M$ with rank $V = \text{dim} M = n$. Let $s$ be a holomorphic section of $V$ with compact zero loci $Z$. Given any "weight"

$$\psi \in \Gamma (M, K_M \otimes \text{det } V)$$

the Koszul complex of $(V, s)$ associates a closed form $\eta_{\psi} \in \Omega^{n,n-1}(M - Z)$ via Griffiths-Harris’s construction ([GH, Chapter 5], also (2.2)).

We then define the residue as

$$(1.3) \quad \text{Res}_Z \psi := \int_N \eta_{\psi} \in \mathbb{C}$$

where $N$ is a real $2n - 1$ dimensional smooth compact submanifold of $M$ that "surrounds $Z"$, in the sense that $N = \partial T$ for some compact domain $T \subset M$ which contains $Z$ and is homotopically equivalent to $Z$. In Appendix we show that such $N$ always exists and (1.3) is independent of the choice of $N$, via argument extending that of Durfee [Dur].

The residue thus defined is named “Virtual Residue” when $\text{dim } Z > 0$ (note that $0 = \text{dim } M - \text{rank } V$ is the expected dimension of the Kuranishi model $(M, V, s)$). Therefore it generalizes the classical zero dimensional case of Grothendieck Residues [GH, Chapter 5]. It is the algebro-geometric model of the “genus zero B-twisted correlator” for an arbitrary LG space $(X, W)$. One simply takes $M = X, V = \Omega_X, s = dW$, and the weight $\psi$ is given as physics “observable”.

The main part of this paper is to provide an exponential type integral form of the defined virtual residue.

**Theorem 1.1.** Pick a Hermitian metric $h$ on $V$ and let $\nabla$ be its associated Hermitian connection with $\nabla^{0,1} = \overline{\partial}$. Let $\xi = -(*)_h$ be a smooth section of $V^*$ and

$$S = -|s|^2 + \overline{\partial} \xi \in \oplus_{p=0,1} \Omega^{(0,p)}(\wedge^p V^*).$$

\[2\] while certain Hodge theoretical properties of $(K_{\mathbb{P}^4}, W)$ were discussed in [HI].
Assuming polynomial growth conditions for $s$ and $\nabla s$ (Assumption [4.4]), one has

$$\text{Res}_Z \frac{\psi}{s} = (-1)^n \left(2\pi i\right)^n \int_M (\psi e^S).$$

Here $\cdot$ is the operation contracting $\det V$ with $\det V^*$ so that $\psi \cdot e^S \in \Omega^*_{M^*}.$

The formula may be viewed as an analogue of the Mathai-Quillen’s integral formalism [MQ] of the localized euler class $e_{s,\text{loc}}(V)$ for the Kuranishi model $(M, V, s)$. The virtual residue in the case $V = T_M$ (and thus $s$ is a holomorphic vector field) should be related to the holomorphic equivariant cohomology (for example [Liu] and [BR]). We also expect some purely algebraic construction can be used to construct virtual residues in arbitrary characteristic. Whether if there exists a more general algebro-geometric model governing higher genus B-twisted theory for any LG space $(X, W)$, (could it be related to a virtual residue over some moduli containing $M_{g,n}$, with the weight provided by conformal blocks?) is an interesting question.

**Convention.** In this paper, for a holomorphic bundle $V$ over a complex manifold $M$, we use $\Gamma(M, V)$ to denote the space of global holomorphic sections of the bundle $V$. Also all tensor products of bundles are over $\mathbb{C}$.

**Acknowledgment:** The authors thank Ugo Bruzzo, Jun Li, Eric Sharpe, Si Li, Qile Chen, Zheng Hua, Huijun Fan, Yongbin Ruan, Edward Witten for helpful discussions. Special thanks to Si Li for informing us the operators $T_\rho, R_\rho$ in section three.

### 2. Construction of virtual residue

#### 2.1. Classical Grothendieck residue

We review the classical setup of Grothendieck residues. Let $B$ be the ball $\{z \in \mathbb{C}^n : |z| < \epsilon\}$ and $f_1, \cdots, f_n \in \mathcal{O}(\overline{B})$ functions holomorphic in a neighborhood of the closure. Suppose a “transverse condition” is satisfied.

**Assumption 2.1.** The only common zero of $f_1, \cdots, f_n$ is the origin.

Denote $\omega$ to be a meromorphic form as following

$$\omega = \frac{g(z)dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n} \quad (g \in \mathcal{O}(\overline{B})).$$

Pick a positive $\delta << \epsilon$ and let $\Gamma$ be the real $n$-cycle defined by

$$\Gamma = \{z : |f_i(z)| = \delta\},$$

with the orientation given by $d(\text{arg} f_1) \wedge \cdots \wedge d(\text{arg} f_n) \geq 0$. Then the residue is defined as

$$\text{Res}_{\partial \delta B \omega} = \left(\frac{1}{2\pi i}\right)^n \int_{\Gamma} \omega. \tag{2.1}$$

This number is independent of the coordinate. In [CH] Lemma in Page 651 the number (2.1) is also identified as $\int_{S^{2n-1}} \eta_\omega$ where $S^{2n-1}$ is a real $2n - 1$-dimensional sphere centered at origin and contained in $B$, and $\eta_\omega$ is some closed $(n, n-1)$ form over $B$ constructed by a Koszul complex that we shall use in next subsection.
One can view the Grothendieck residue \((2.1)\) to be associated to the complex manifold \(B\), a section \((f_1, \cdots, f_n)\) of the trivial bundle \(B \times \mathbb{C}^n\) over \(B\) whose (reduced) zero loci is \(0 \in B\), and a “weight”
\[ g(z)dz_1 \wedge \cdots \wedge dz_n \in \Gamma(B, K_B \otimes \det V). \]

More generally one can consider the following.

**Griffiths-Harris Set-up.**

1. \(V\) is a holomorphic bundle over a complex manifold \(M\), \(\dim M = \text{rank} V = n\);
2. \(s \in \Gamma(M, V)\); the zero loci of \(s\) is a compact set \(Z \subset M\); \(\dim Z = 0\);
3. \(\psi \in \Gamma(M, K_M \otimes \det V)\), called “weight”.

Nearby each point \(p \in Z\) one can pick a local holomorphic frame \(e_1, \cdots, e_n\) for \(V\) and a local holomorphic coordinate \(z_1, \cdots, z_n\) on \(M\) to represent
\[ \psi = h(z)(dz_1 \wedge \cdots \wedge dz_n) \otimes (e_1 \wedge \cdots \wedge e_n) \quad \text{and} \quad s = s_1(z)e_1 + \cdots + s_n(z)e_n. \]

In [GH, Chapter 5, Page 731] the Grothendieck residue is defined to be
\[ \text{Res} \frac{\psi}{s} = \sum_p \text{Res}_p \left( \frac{\psi}{s} \right) = \sum_p \text{Res}_p \left( \frac{h(z)(dz_1 \wedge \cdots \wedge dz_n)}{s_1(z) \cdots s_n(z)} \right). \]

Using such formulation [GH, Chapter 5] develops properties of residues over an arbitrary complex manifold \(M\). For example the residue theorem in complex analysis is generalized.

**Theorem 2.2.** [GH] Page 731 *If \(M\) is compact, then \(\sum_{p \in Z} \text{Res}_p \left( \frac{\psi}{s} \right) = 0\).*

2.2. Virtual residue construction. Let us keep the symbol \(Z\) to denote the zero loci of the fixed section of a bundle \(V\) as before. If one gives a further thought about Assumption 2.1, it is not natural to assume that \(Z\) is zero dimensional, even when \(\text{rank} V = \dim M\). It is easy to find an example where \(Z\) has components of various dimensions.

**Example 2.3.** Consider \(V = O(2) \oplus O(2)\) over \(\mathbb{P}^2\) and
\[ s_t = (x_0 \cdot x_1, (x_0 + t(x_1 - x_2)) \cdot x_2) \in \Gamma(\mathbb{P}^2, V), \]
where \([x_0, x_1, x_2]\) is the homogeneous coordinate of \(\mathbb{P}^2\). Let
\[ L_0 = \{x_0 = 0\} \subset \mathbb{P}^2, \quad L_1 = \{x_1 = 0\} \subset \mathbb{P}^2 \quad \text{and} \quad L_2 = \{x_2 = 0\} \subset \mathbb{P}^2 \]
be three lines in \(\mathbb{P}^2\). Denote the zero loci of \(s_t\) by \(Z_t \subset \mathbb{P}^2\). Then

1. \(Z_0 = L_0 \cup \{[1, 0, 0]\}\);
2. \(Z_t = \{[0, 1, 0], [1, 0, 0], [t, 0, 1], [0, 1, 1]\}, \quad \text{for} \quad t \neq 0.\)

As \(Z\) can have positive dimensional components, we would still like to ask about residues associated to such locus. We begin with the following setup.

**Setup 2.4.**

1. \(M\) is a smooth quasi-projective variety over \(\mathbb{C}\), \(V\) is a holomorphic bundle over \(M\), \(\dim M = \text{rank} V = n\);
2. \(s \in \Gamma(M, V)\); the zero loci of \(s\) is a compact set \(Z \subset M\);
3. \(\psi \in \Gamma(M, K_M \otimes \det V)\), called the “weight”.

We denote \(U = M \setminus Z\), and let \(V_U\) be the restriction of \(V\) over \(U\). Since \(s\) is nowhere zero over \(U\), the following Koszul sequence is exact over \(U\)
\[ 0 \rightarrow K_U \xrightarrow{s} K_U \otimes V_U \xrightarrow{s^*} \cdots \xrightarrow{s^{n-1}} K_U \otimes \wedge^{n-1} V_U \xrightarrow{s^n} K_U \otimes \wedge^n V_U \rightarrow 0. \]
The exact Koszul sequence induces a homomorphism
\begin{equation}
H^0(U, K_U \otimes \wedge^n V_U) \longrightarrow H^{n-1}(U, K_U).
\end{equation}
One also has a canonical Dolbeault isomorphism
\begin{equation}
H^{n-1}(U, K_U) \cong H^n_{\bar{\partial}}(U).
\end{equation}
Applying (2.2) and (2.3) to $\psi$, and using that every $(n,n-1)$ form is $\bar{\partial}$-closed, one obtains a (unique) De-Rham cohomology class
\begin{equation}
\eta \in H^{2n-1}(U, \mathbb{C}).
\end{equation}

The compactness of $Z$ implies $Z$ has finitely many connected components $\{Z_i\}$. As each $Z_i$ is compact analytic subset of $M$, Serre’s GaGa theorem implies each $Z_i$ is compact algebraic subset of the complex variety $M$.

For every connected component $Z_i$ of $Z$ in Appendix 1 constructs a compact neighborhood $T_i$ of $Z_i$ in $M$, where $T_i \hookrightarrow M$ is a homotopy equivalence. The collection $\{T_i\}$ can be made disjoint by Corollary 5.7. Denote $\partial T_i$ be the boundary of $T_i$ whose orientation is that induced from $M$.

We then define the contribution of $Z_i$ to the residue associated from the datum $(U,M,V_U,s,\psi)$ to be
\begin{equation}
\text{Res}_{Z_i} \frac{\psi}{s} := \frac{1}{(2\pi i)^n} \int_{N_i} \eta.
\end{equation}

If $T'_i$ is another good neighborhood of $Z_i$, by Lemma 5.6 one can find a smaller good neighborhood $T$ of $Z_i$ such that $T \subset T_i \cap T'_i$. Then
\[
0 = \int_{T_i \setminus T} \bar{\partial} \eta = \int_{T_i \setminus T} d\eta = \int_{N_i} \eta - \int_{\partial T} \eta
\]
and the similar identity for $T'_i$ imply $\int_{N_i} \eta = \int_{N_i} \eta$. Hence the definition is independent of the choice of good neighborhoods.

In general we define
\begin{equation}
\text{Res} \frac{\psi}{s} = \sum_i \text{Res}_{Z_i} \frac{\psi}{s} = \frac{1}{(2\pi i)^n} \int_N \eta
\end{equation}
where $N = \cup N_i \subset M$. It vanishes whenever $M$ is compact by Stoke theorem.

**Remark 2.5.** For an arbitrary holomorphic $\psi \in \Gamma(U, K_M \otimes \det V)$, the same definition of residue is still valid.

**Example 2.6** (Unorbid Landau Ginzburg B model of genus zero). Suppose $M$ is a smooth quasi-projective variety over $\mathbb{C}$ such that $\theta : \mathcal{O}_M \rightarrow K_M^{\otimes 2}$ is an isomorphism of line bundles. Suppose $W : M \rightarrow \mathbb{C}$ is a regular function (called “superpotential”) whose critical locus $(dW = 0) \subset M$ is compact. We say $(M,W)$ is a Landau Ginzburg space.

Let $V = \Omega_M$ and $s = dW \in \Gamma(M,V)$. For each $f \in \Gamma(M, \mathcal{O}_M)$ the $\psi := f \theta \in \Gamma(M, K_M \otimes \det V)$ associates a complex number $\text{Res} \frac{\psi}{s} = \text{Res} \frac{f}{s}$. This number is understood physically the correlator for the $B$-twisted LG model $(M,W)$ where $f$ is a given observable, as mentioned in Introduction.
Remark 2.7. The Grothendieck residue $\int_{T_s} \omega$ can be viewed as a period of the domain $M \setminus Z$ as it is an integral of a holomorphic form $\omega$ over an integral homology class $\Gamma \subset M \setminus Z$. The authors are not aware of whether $\text{Res}_{\bar{s}}^\omega$ (when $\dim(s) = 0$) can be represented as some period of $M \setminus Z$.

3. Cohomology with compact support and Virtual residue

In this section we represent the virtual residue as an integration of some compactly-supported twisted Dolbeault cohomology class. As before $V$ is a holomorphic bundle over a smooth complex manifold $M$ with $rk V = dim M$ and $s$ is a holomorphic section of $V$ with compact zero loci $Z = (s = 0)$. Let $V^*$ be the dual bundle of $V$, and denote $A^{i,j}(\Lambda^k V \otimes \Lambda^l V^*)$ to be the sheaf of smooth $(i,j)$ forms on $M$ with value in $\Lambda^k V \otimes \Lambda^l V^*$.

Denote $\Omega^{i,j}(\Lambda^k V \otimes \Lambda^l V^*) := \Gamma(M, A^{i,j}(\Lambda^k V \otimes \Lambda^l V^*))$ and assign its element $\alpha$ to have degree $\sharp \alpha = i + j - k - l$. Then

$$B := \oplus_{i,j} \Omega^{i,j}(\Lambda^k V \otimes \Lambda^l V^*)$$

is a graded commutative algebra with the (wedge) product uniquely extending wedge products in $\Omega^*, \Lambda^*, \Lambda^* V$ and mutual tensor products. Denote

$$E_M = \oplus_{0 \leq i,j \leq n} E_M^{i,j} \subset B \quad \text{with} \quad E_M^{i,j} := \Omega^{(i,j)}(\Lambda^i V) = \Gamma(M, A^{n-j}(\Lambda^j V)),$$

and

$$E_{c,M} = \oplus_{0 \leq i,j \leq n} E_{c,M}^{i,j} \quad \text{with} \quad E_{c,M}^{i,j} := \{ \alpha \in E_M^{i,j} \mid \alpha \text{ has compact support} \}.$$

For $\alpha \in E_M$ we denote $\alpha_{i,j}$ to be its component in $E_M^{i,j}$. Clearly, $E_M$ is a bi-graded $C^\infty(M)$-module. Under the operations

$$\bar{\partial} : E_M^{i,j} \to E_M^{i,j+1} \quad \text{and} \quad s \wedge : E_M^{i,j} \to E_M^{i+1,j}$$

the space $E_M^{i,j}$ becomes a double complex and $E_{c,M}^{i,j}$ is a subcomplex. We shall study the cohomology of $E_{c,M}$ with respect to the following coboundary operator

$$\bar{\partial}_s := \bar{\partial} + s \wedge.$$

One checks $\bar{\partial}_s^2 = 0$ using Leibniz rule of $\bar{\partial}$ and $\bar{\partial} s = 0$.

Let us introduce more operators. Fix a Hermitian metric $h$ on $V$. Let

$$\bar{s} := \frac{(s, s)_h}{(s, s)_h} \in \Gamma(U, A^{0,0}(V^*)).$$

It associates a contraction map (c.f. [LLS]) in Appendix 2)

$$\iota_s : \Gamma(U, A^{n-i}(\Lambda^i V)) \to \Gamma(U, A^{n-i}(\Lambda^{i-1} V)).$$

To distinguish it in later calculation, we denote $T_s := \iota_s : E^{n-\ast}_U \to E^{n-1,\ast}_U$.

The injection $j : U \to M$ induces the restriction $j^* : E^{n-\ast}_M \to E^{n-\ast}_U$. Let $\rho$ be a smooth cut-off function on $M$ such that $\rho|_{U_1} = 1$ and $\rho|_{M \setminus U_2} = 0$ for some relatively compact open neighborhoods $U_1 \subset U_2 \subset U_2$ of $Z$ in $M$.

We define the degree of an operator to be its change on the total degree of elements in $E_{c,M}(E_U)$. Then $\bar{\partial}$ and $T_s$ are of degree 1 and $-1$ respectively, and $[\bar{\partial}, T_s] = \bar{\partial} T_s + T_s \bar{\partial}$ is of degree 0. Consider two operators introduced in [LLS] page 11

$$T_\rho : E_M \to E_{c,M} \quad \text{with} \quad T_\rho(\alpha) := \rho \alpha + (\bar{\partial} \rho) T_s \frac{1}{1 + [\bar{\partial}, T_s]}(j^* \alpha)$$

(3.1)
and

\[ R_\rho : E_M \to E_M \quad R_\rho (\alpha) := (1 - \rho) T_s \frac{1}{1 + [\bar{\partial}, T_s]} (j^* \alpha). \]

Here as an operator

\[ \frac{1}{1 + [\bar{\partial}, T_s]} := \sum_{k=0}^{\infty} (-1)^k [\bar{\partial}, T_s]^k \]

is well-defined since \([\bar{\partial}, T_s]^k(\alpha) = 0\) whenever \(k > n\). Clearly \(T_\rho\) is of degree zero and \(R_\rho\) is of degree \(-1\). Also \(R_\rho(E_{c,M}) \subset E_{c,M}\) by definition.

**Lemma 3.1.** \([\bar{\partial}_s, R_\rho] = 1 - T_\rho\) as operators on \(E_M\).

**Proof.** It is direct to check that

\[ [s \wedge, T_s] = 1 \quad \text{on } E_U. \]

Moreover,

\[ [P, [\bar{\partial}, T_s]] = 0 \]

for \(P\) being \(s \wedge, \bar{\partial}\) or \(T_s\). Therefore, we have

\[
[\bar{\partial}_s, R_\rho] = \left[ \bar{\partial}_s, 1 - \rho \right] T_s \frac{1}{1 + [\bar{\partial}, T_s]} j^* + (1 - \rho) [\bar{\partial}_s, T_s] \frac{1}{1 + [\bar{\partial}, T_s]} j^* \\
= -[\bar{\partial} \rho] T_s \frac{1}{1 + [\bar{\partial}, T_s]} j^* + (1 - \rho) j^* \\
= -[\bar{\partial} \rho] T_s \frac{1}{1 + [\bar{\partial}, T_s]} j^* + (1 - \rho) = 1 - T_\rho. 
\]

\[ \square \]

**Proposition 3.2.** The embedding \((E_{c,M}, \bar{\partial}_s) \rightarrow (E_M, \bar{\partial}_s)\) is a quasi-isomorphism.

**Proof.** By Lemma 3.1 \(H^*(E_M/E_{c,M}, \bar{\partial}_s) \equiv 0\), and thus the proposition follows. \[ \square \]

We define the trace map via integrating its component in \(\Omega^{(n,n)}_M\), namely

\[ \text{tr} : E_{c,M} \to \mathbb{C}, \quad \text{tr}(\alpha) := \int_M \alpha_{0,n}. \]

By definition we have \(\text{tr}(\bar{\partial}_s \alpha) = 0\) and \(\text{tr}(s \wedge \alpha) = 0\), which imply that the trace map is well defined on the cohomology

\[ \text{tr} : H^*(E_{c,M}, \bar{\partial}_s) \to \mathbb{C}. \]

Therefore Proposition 3.2 induces a trace map

\[ \text{tr} : H^*(E_M, \bar{\partial}_s) \xrightarrow{\cong} H^*(E_{c,M}, \bar{\partial}_s) \to \mathbb{C} \]

where the first isomorphism is the inverse of that induced from Proposition 3.2.

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3 As a notation convention, we always denote \([,]\) for the graded commutator, that is for operators \(A, B\) of degree \(|A|\) and \(|B|\), the bracket is given by

\[ [A, B] = AB - (-1)^{|A||B|} BA. \]
Proposition 3.3. Let $V$ be a holomorphic bundle over a smooth quasi projective complex manifold $M$ with $\text{rk} V = \dim M = n$, and $s$ is a holomorphic section of $V$ with $Z = (s = 0)$ compact. Let $\psi \in \Gamma(M, K_M \otimes \wedge^n V)$, then $[\psi] \in H^*(E_M, \partial s)$. Suppose $\psi = \alpha + \partial s \sigma$ for some $\alpha \in H^*(E_{c,M}, \partial s)$ and $\sigma \in E_M$. Then

$$\text{Res} \frac{\psi}{s} = \frac{(-1)^n}{(2\pi i)^n} \text{tr}(\psi) = \frac{(-1)^n}{(2\pi i)^n} \int_M \alpha_{0,n}.$$  

Proof. Consider the Dolbeault resolution of the Koszul exact sequence. Recall the contraction map $\iota_\beta : \Gamma(U, A^{n,q}(\wedge^p V)) \to \Gamma(U, A^{n,q}(\wedge^{p-1} V))$ where one easily checks $(s \wedge \iota_\beta s)\alpha = \alpha$, for each $\alpha \in \Gamma(U, A^{n,q}(\wedge^p V))$. Let

$$\beta_0 = \iota_\beta s, \quad \beta_k = -\iota_\beta \partial \beta_{k-1} = (-\iota_\beta \partial) \iota_\beta \psi.$$  

By (3.3) we have

$$\begin{align*}
s \wedge \beta_0 &= \psi \\
s \wedge \beta_1 &= s \wedge (-\iota_\beta \partial \beta_0) = -\partial \beta_0 + \iota_\beta s \wedge \partial \beta_0 \\
&= -\partial \beta_0 - \iota_\beta \partial \psi = -\partial \beta_0 \\
&\vdots \\
s \wedge \beta_{n-1} &= -\partial \beta_{n-2}.
\end{align*}$$

This implies $\partial s(\sum \beta_k) = \psi$. One obtains $\partial \beta_{n-1} = 0$ because $\partial \beta_{n-1} \in \Omega^{n,n}_U$ and

$$s \wedge \partial \beta_{n-1} = \partial(s \wedge \beta_{n-1}) = -\partial(\partial \beta_{n-2}) = 0.$$  

By zigzag the $\eta$ constructed in (2.4) is identical to the class $[(-1)^{n-1} \beta_{n-1}] \in H^{(n,n-1)}(U)$. Thus by definition (2.5)

$$\text{Res} \frac{\psi}{s} = \frac{(-1)^{n-1}}{(2\pi i)^n} \int_N \beta_{n-1},$$

where $N$ is finite disjoint union of $N_i$ with $N_i = \partial T_i$ for good neighborhoods $T_i$'s of $Z_i$'s respectively (c.f. (2.6)). We may assume $\{T_i\}$'s are disjoint, and denote $T = \cup T_i$. Then $N = \partial T$. Let $T'$ be another good neighborhood of $Z$ in $M$, such that $T' \subset T$ as in Lemma 5.3 in Appendix 1. Consider a smooth cut-off function $l$, which is zero on $T'$ and identical to one outside $T$. Set $\sigma' = l \sum_{k=0}^{n-1} \beta_k$ and $\alpha' = \psi - \partial s \sigma'$. Then $\partial s \alpha' = \partial s \psi - \partial_\beta \partial_\alpha \sigma' = 0$ and

$$\begin{align*}
\alpha' &= \psi - \partial_\alpha \sigma' \\
&= \psi - \partial(l \sum_{k=0}^{n-1} \beta_k) - s \wedge (l \sum_{k=0}^{n-1} \beta_k) \\
&= \psi - \partial(l \sum_{k=0}^{n-1} \beta_k) - \partial(l \sum_{k=0}^{n-1} \beta_k) - s \wedge (l \sum_{k=0}^{n-1} \beta_k) \\
&= \psi - \partial(l \sum_{k=0}^{n-1} \beta_k) - l \psi \\
&= (1 - l) \psi - \partial(l \sum_{k=0}^{n-1} \beta_k).
\end{align*}$$
We then get a decomposition $\psi = \alpha' + \bar{\partial}_s \sigma'$. Because $\alpha'$ has compact support, by the definition of the trace map (3.4)

$$
\frac{1}{(2\pi i)^n} \int_M \alpha_{0,n} = \frac{1}{(2\pi i)^n} \int_M \alpha'_{0,n}
$$

$$
= -\frac{1}{(2\pi i)^n} \int_M \bar{\partial}(l_{\beta_{n-1}})
$$

$$
= -\frac{1}{(2\pi i)^n} \int_M (\bar{\partial}l)_{\beta_{n-1}}
$$

$$
= -\frac{1}{(2\pi i)^n} \int_{T\setminus T'} \bar{\partial}(l_{\beta_{n-1}})
$$

$$
= -\frac{1}{(2\pi i)^n} \int_{T\setminus T'} (\bar{\partial}l)_{\beta_{n-1}}
$$

using $\bar{\partial}(l_{\beta_{n-1}}) = d(l_{\beta_{n-1}})$ and $l|_{\partial T'} = 0$. Thus

$$
\frac{1}{(2\pi i)^n} \int_M \alpha_{0,n} = (-1)^n \text{Res} \frac{\psi}{s}.
$$

4. Exponential type Integral form of Virtual Residue

We aim to give a natural integral representation for the virtual residue $\text{Res} \frac{\psi}{s}$. To do so we need to put metric on bundle and manifold and make suitable boundedness conditions. First let us be in the situation where $V$ is a holomorphic bundle over a noncompact smooth complex manifold $M$ with $\text{rk} V = \dim M$ and $s$ is a holomorphic section of $V$ with compact zero loci $Z = \{s = 0\}$.

We pick a reference point $\nu \in M$ and fix it once for all. We pick a hermitian metric $h$ on $V$ and assume $M$ admits a complete Hermitian metric $g$ such that there exists $C > 0, \lambda > 1$ making

(4.1) $\text{vol}(B(r)) \leq C r^\lambda \quad \forall \ r > 0$,

where $B(r) := \{z \in M | d(z, \nu) \leq r\}$.

Denote $\mathcal{A}^{i,j}(\wedge^k V \otimes \wedge^l V^*)$ to be the sheaf of smooth $(i,j)$ forms on $M$ valued in $\wedge^k V \otimes \wedge^l V^*$. The Hermitian metrics of $M$ and $V$ induce a metric on the bundle which corresponds to the sheaf $\oplus_{i,j,k,l} \mathcal{A}^{i,j}(\wedge^k V \otimes \wedge^l V^*)$ (c.f. Appendix 2). Denote this metric by $(\cdot, \cdot)_z$ for $z \in M$ and set $|\alpha|_z = \sqrt{(\alpha, \alpha)_z}$.

**Definition 4.1.** We say $\alpha \in \Gamma(\mathcal{A}^{i,j}(\wedge^k V \otimes \wedge^l V^*))$ is rapidly decreasing if for all $m \geq 0$,

$$
\sup_{z \in M} (1 + d^2(z, \nu))^m |\alpha|_z < \infty,
$$

where $d(z, \nu)$ denotes the distance between $z$ and $\nu$.

**Definition 4.2.** We say $\alpha \in \Gamma(\mathcal{A}^{i,j}(\wedge^k V \otimes \wedge^l V^*))$ is tempered if there exists an $m \geq 0$, such that

$$
\sup_{z \in M} (1 + d^2(z, \nu))^{-m} |\alpha|_z < \infty,
$$

**Remark 4.3.** By triangle inequality one may show both definitions are independent of the choice of the base point $\nu$, but we will not need this.

We make the following assumption.

**Assumption 4.4.** (1) The section $s$ is tempered;
(2) Fix a Hermitian connection $\nabla$ on $V$ with $\nabla^{0,1} = \overline{\partial}$. The induced $\nabla$s is tempered;

(3) There is a constant $C_0 > 0$ and a compact subset $Y$ of $M$ with $T \subset Y$, where $T$ is a good neighborhood of $Z$ in $M$ (c.f. Definition 5.5 and Corollary 5.7), such that

$$|s|^2(z) \geq C_0(1 + d^2(z, \nu)), \quad \forall z \in M \setminus Y.$$  

In short the assumption says that $s$ has polynomial growth and $\nabla$s has at most polynomial growth near $\partial M$.

Remark 4.5. If there is a holomorphic bundle $V$ over a smooth complex projective variety $M$, and a section $s$ of $\mathbb{P}(V \oplus \mathcal{O}_M)$ such that $M = (\tilde{s} \neq \infty) \subset M$ and $V = V|M$, then one can construct $g, h$ satisfying (11) and Assumption 4.4. This provides a lot of examples. We omit the proof as it is not needed in this paper.

Lemma 4.6. If $s$ satisfies the above Assumption 4.4, then $e^{-|s|^2}$ is rapidly decreasing, and $\beta \land \alpha$ is rapidly decreasing if $\beta \in \Omega^{(i,j)}(\wedge^k V)$ is tempered and $\alpha \in \Omega^{(i,m)}(\wedge^e V)$ is rapidly decreasing, or $\beta \in \Omega^{(i,j)}(\wedge^k V^*)$ is tempered and $\alpha \in \Omega^{(i,m)}(\wedge^e V^*)$ is rapidly decreasing.

Proof. For arbitrary $m \geq 0$, we have

$$\sup_{z \in M}(1 + d^2(z, \nu))^m e^{-|s|^2} \leq \max\{\sup_{z \in Y}(1 + d^2(z, \nu))^m e^{-|s|^2}, \sup_{z \in M \setminus Y}(1 + d^2(z, \nu))^m e^{-C_0(1 + d^2(z, \nu))}\} < \infty,$$

by Assumption 4.4. Thus $e^{-|s|^2}$ is rapidly decreasing. For $\beta \land \alpha$ we apply Lemma 4.3 which gives a positive number $D$ such that for all $z \in M$

$$|\beta \land \alpha|(z) \leq D \cdot |\beta|(z)|\alpha|(z).$$

Then for arbitrary $m \geq 0$,

$$\sup_{z \in M}(1 + d^2(z, \nu))^m |\beta \land \alpha|(z) < D \cdot \sup_{z \in M}(1 + d^2(z, \nu))^m |\beta|(z)|\alpha|(z) < \infty.$$  

Thus $\beta \land \alpha$ is rapidly decreasing. \hfill \Box

The contraction operator (defined in Appendix 2 (6.3)) and the dbar operator

$$\iota_s : \mathcal{A}^{0,q}(\wedge^p V^*) \rightarrow \mathcal{A}^{0,q}(\wedge^{p-1} V^*), \quad \overline{\partial} : \mathcal{A}^{0,q}(\wedge^p V^*) \rightarrow \mathcal{A}^{0,q+1}(\wedge^p V^*)$$

define $\overline{\partial} + \iota_s$ that acts on

$$F_M^{p,q} = \Omega^{(0,q)}(\wedge^p V^*) := \Gamma(M, \mathcal{A}^{0,q}(\wedge^p V^*)).$$

Clearly $\oplus_{p,q}\Omega^{(0,q)}(\wedge^p V^*)$ is a graded subalgebra of $\mathcal{B}$, and the action of $\overline{\partial} + \iota_s$ on $\oplus_{p,q}\Omega^{(0,q)}(\wedge^p V^*)$ satisfies Leibniz rule:

$$(\overline{\partial} + \iota_s)(\alpha \beta) = ((\overline{\partial} + \iota_s)\alpha)\beta + (-1)^{\tilde{q}\alpha}(\overline{\partial} + \iota_s)\beta.$$  

Let $\psi$ be a holomorphic section of $K_M \otimes \det V$. By using the contraction operator defined in Appendix 2 (6.2), we have the following map:

$$\psi : F_M^{p,q} \rightarrow E_M^{n-p,q} \quad u \mapsto \psi \cdot u.$$  

Lemma 4.7. If $\psi$ is a tempered holomorphic section of $K_M \otimes \det V$, and $u \in F_M^{p,q}$ is rapidly decreasing, then $\psi \cdot u$ is also rapidly decreasing.
Proof. By Lemma 4.8 there exists a constant $k$ such that for every $z \in M$
\[
(\psi, u, \psi, u)(z) \leq k \cdot (u, u)(z)(\psi, \psi)(z).
\]
Because $\psi$ is tempered there exists $m' \geq 0$ such that $|\psi| \leq C'(1 + d^2(z, \nu))^m'$, where $C'$ is a constant. Then for arbitrary $m$
\[
\sup_{z \in M}(1 + d^2(z, \nu))^m |\psi, u|(z) \leq k \cdot \sup_{z \in M}(1 + d^2(z, \nu))^m |u|(z)|\psi|(z)
\]
\[
\leq k \cdot C' \sup_{z \in M}(1 + d^2(z, \nu))^{m+m'} |u|(z)
\]
\[
< \infty.
\]
Thus $\psi, u$ is rapidly decreasing. \qed

For $\beta \in B$, its exponential is defined as $e^\beta := 1 + \beta + \frac{\beta^2}{2} + \cdots$, which is a finite sum by degree reason. Let $\xi = -(s, s)_h \in \Omega^{(0,0)}(V^*)$. We define
\[
S = (\overline{\partial} + i_s)\xi = |s|^2 + \overline{\partial}\xi \in \oplus_{p=0,1} \Omega^{(0,p)}(\wedge^p V^*).
\]
Then $e^S$ is an element in $\oplus_p \Omega^{(0,p)}(\wedge^p V^*)$. Therefore $e^S$ is rapidly decreasing. Let $s$ satisfies the Assumption 4.4, then $\xi, \overline{\partial}\xi$ are tempered. Hence $e^S \in \oplus_{p} F_{M}^{p,p}$ and $\overline{\partial}e^S \in \oplus_{p} F_{M}^{p,p+1}$ are both rapidly decreasing.

Proof. Let $z \in M$ be an arbitrary point. Then by formula 1.20 in page 63 of [Wu], there exists a local holomorphic frame $\{e_i\}$ of $V$ around $z$, such that for any $i, j$
\[
(e_i, e_j)(z) = \delta^j_i \quad \text{and} \quad \nabla e_i(z) = 0.
\]
Let $e^i$ be the dual local frame of $V^*$. Locally represent $s = \sum_i s_i e_i$, so $\xi(z) = -\sum_i \nabla s_i e_i$. Then
\[
(\xi, \xi)(z) = \sum_i |s_i|^2 = (s, s)(z).
\]
Varying $z \in M$ we see $\xi$ is tempered because $s$ is tempered. Then
\[
(\nabla s)(z) = \sum_i (ds_i e_i + s_i \nabla e_i)(z) = \sum_i (ds_i)(z) e_i
\]
implies
\[
(\nabla s, \nabla s)(z) = \left( \sum_i (ds_i)(z) e_i, \sum_j (ds_j)(z) e_j \right) = \sum_i (ds_i, ds_i)(z).
\]
Hence
\[
\overline{\partial}\xi(z) = -\overline{\partial}[\sum_i (e_i, s) e^i](z) = -\sum_i \overline{\partial}(e_i, s)(z) e^i
\]
\[
= -\sum_i (e_i, \nabla s)(z) e^i = -\sum_i (e_i, \sum_j ds_j e_j)(z) e^i = -\sum_i ds_i(z) e^i.
\]
Hence we have
\[
(\overline{\partial}\xi, \overline{\partial}\xi)(z) = \left( \sum_i ds_i(z) e^i, \sum_j ds_j(z) e^j \right) = \sum_i (ds_i, ds_i)(z) = \sum_i (ds_i, ds_i)(z) = (\nabla s, \nabla s)(z).
\]
Varying $z \in M$ we see $\overline{\partial}\xi$ is tempered because $\nabla s$ is tempered by assumption.
By Lemma 6.3 arbitrary power of $\overline{\partial} \xi$ is also tempered. By Lemma 4.6 $e^{-|s|^2}$ is rapidly decreasing. Therefore using Lemma 4.9
\[ e^S = e^{-|s|^2} (1 + \overline{\partial} \xi + \frac{(\overline{\partial} \xi)^2}{2!} + \cdots + \frac{(\overline{\partial} \xi)^n}{n!}) \in \oplus_p \Omega^{(0,p)}(\wedge^p V^*) \]
is rapidly decreasing.

Using the formula (6.1) in Appendix 2, we have
\[ \overline{\partial}|s|^2(z) = -\overline{\partial} <s, \xi> (z) = <s, \overline{\partial} \xi>. \]

By Lemma 6.4 one has $(\overline{\partial}|s|^2, \overline{\partial}|s|^2) \leq (s, s)(\overline{\partial} \xi, \overline{\partial} \xi)$. Then $\overline{\partial}|s|^2$ is tempered as $s$ and $\overline{\partial} \xi$ are tempered. Apply Lemma 6.3 and Lemma 4.6 again
\[ \overline{\partial}e^S = -\overline{\partial}|s|^2e^{-|s|^2} (1 + \overline{\partial} \xi + \frac{(\overline{\partial} \xi)^2}{2!} + \cdots + \frac{(\overline{\partial} \xi)^n}{n!}) \]
is rapidly decreasing, for $\overline{\partial}|s|^2$ and $\overline{\partial} \xi$ are tempered and $e^{-|s|^2}$ is rapidly decreasing. \hfill \Box

**Lemma 4.9.** Let $s$ and $Y$ be in the Assumption 4.4 and $T_s$ is defined in section 3. Suppose $\psi$ is a tempered holomorphic section of $K_M \otimes \det V$, and $p$ is a smooth function with compact support. One can find positive constants $\mu$ and $C_1$ so that
\[ |(\overline{\partial} \rho) T_s(\overline{\partial} T_s)^k(\psi, e^S)|(z) \leq C_1 |\overline{\partial} \rho| e^{-|s|^2} (1 + d^2(z, \nu))^{\mu} |s|^2, \forall z \in M \setminus Y. \]

**Proof.** By definition $e^S$ can be written as $e^S = \sum w_i$, where $w_i = e^{-|s|^2} \frac{(\overline{\partial} \xi)^i}{i!} \in F_M^i$. First we claim for arbitrary $k \in \mathbb{N}$ one has
\[ (\overline{\partial} T_s)^k(\psi, w_i) = \psi, (s \wedge (\overline{\partial} s)^{k-1} \wedge \overline{\partial} w_i). \]

We prove it by induction. For $k = 1$, by Lemma 6.4 and Lemma 6.2
\[ \overline{\partial} T_s(\psi, w_i) = (s \wedge (\overline{\partial} s) \wedge w_i) = (s \wedge (\overline{\partial} s) \wedge w_i). \]

Assuming (4.3) holds for $k = l - 1$, then
\[ \overline{\partial} T_s(\psi, w_i) = (s \wedge (\overline{\partial} s)^{l-1} \wedge w_i) = (s \wedge (\overline{\partial} s)^{l-1} \wedge w_i). \]

This proves the claim. Therefore
\[ T_s(\overline{\partial} T_s)^k(\psi, w_i) = \psi, (s \wedge (\overline{\partial} s)^{k-1} \wedge w_i)) = (s \wedge (\overline{\partial} s)^{k-1} \wedge w_i). \]

Over $M \setminus Z$ we have $\bar{s} = -\frac{\xi}{|s|^2}$ and thus an identity
\[ s \wedge (\overline{\partial} s)^{k} \wedge w_i = (\frac{s}{|s|^2}) \wedge (\overline{\partial} s) \wedge w_i = \overline{\partial} s \wedge (\overline{\partial} s)^{k} \wedge w_i, \quad \overline{\partial} s \wedge (\overline{\partial} s)^{k} \wedge w_i. \]

Assumption 4.4 (3) implies $|s|^2(z) \geq C_0$ for $z \in M \setminus Y$. Since $\xi$ and $\overline{\partial} \xi$ are tempered by Lemma 4.8, $\xi(\overline{\partial} \xi)^l$ is also tempered by Lemma 6.3. By Lemma 6.3
and Lemma 6.5 there exists a positive number $C'$ independent of $z \in M \setminus Y$, such that
\[
|\overline{\partial}^n \partial (\overline{\partial} T_s \partial T_s)^k \psi \omega| (z) \leq \sum_i |\overline{\partial}^n \partial (\psi \omega (s \wedge (\overline{\partial} s)^k \wedge w_i)) (z)
\leq C' \sum_i |\psi| |\overline{\partial}^n \partial s \wedge (\overline{\partial} s)^k \wedge w_i| (z)
\leq C_1 |\overline{\partial}^n \partial e^{-(l^2 (1 + d^2 (z, \nu)))^1} (z).
\]
for some positive $\nu$ and $C_1$ independent of $z \in M \setminus Y$. \hfill \Box

**Lemma 4.10.** $e^S$ is $(\overline{\partial} + \partial^0)$-closed, and $1 - e^S$ is $(\overline{\partial} + \partial^0)$-exact.

**Proof.** The first assertion is from $(\overline{\partial} + \partial^0)^2 = 0$ and the Leibniz rule of $\overline{\partial} + \partial^0$. Use the similar identity $(\overline{\partial} + \partial^0) e^S = 0$ ($t$ is a variable) and $e^t - 1 = \int_0^1 e^{tx} dt$, we have
\[
e^S - 1 = (\overline{\partial} + \partial^0) \int_0^1 \xi e^{S} dt,
\]
and the exactness follows. \hfill \Box

Since $e^S$ lies in $\otimes_p \Omega^{0,p}(\Lambda^q V^*)$, the objects $\psi_\omega e^S$ (and hence $\psi_\omega (1 - e^S)$) lie in $\otimes_p \Omega^{q,p}(\Lambda^q V^*)$, a subspace of $E_M$ defined in previous section.

**Lemma 4.11.** If $\psi$ is a holomorphic section of $K_M \otimes \det V$, then $\psi_\omega (1 - e^S)$ is $\overline{\partial}^0$-exact, and $\psi_\omega e^S$ is $\overline{\partial}^0$-closed.

**Proof.** Denote $e^{S} = e^{-t^2 s^2} \sum_{k=0}^n t^k \beta_k$, where $\beta_k = (\overline{\partial} s)^k \in F_M^{k,k}$. By previous Lemma one may represent $1 - e^S = (\overline{\partial} + \partial^0)\omega$ with
\[
\omega = \sum_{p,q} \omega_{p,q}, \quad \omega_{p,q} \in F_M^{q,p}
\]
where the sum runs over integers $p, q \in [0, n]$. By Lemma 6.1 and Lemma 6.2 we have
\[
\psi_\omega (\overline{\partial} w_{p,q}) = \partial (\psi_\omega w_{p,q}) \quad \text{and} \quad \psi_\omega (\partial w_{p,q}) = s \wedge (\psi_\omega w_{p,q})
\]
Together we obtain
\[
\psi_\omega (1 - e^S) = \sum \psi_\omega ([\overline{\partial} + \partial^0] \omega_{p,q}] = (\overline{\partial} + s \wedge) \sum \psi_\omega \omega_{p,q}
\]
is exact with respect to the operator $\partial^0 := \overline{\partial} + s \wedge$.

Using $\psi = \psi_\omega e^S + \psi_\omega (1 - e^S)$ and that $\psi$ is $\overline{\partial}^0$-closed, we have
\[
\partial^0 (\psi_\omega e^S) = \partial^0 (\psi - \psi_\omega (1 - e^S)) = 0
\]
\hfill \Box

**Lemma 4.12.** For each rapidly decreasing $\alpha \in E_M^{*n}$ one has
\[
|\int_M \alpha_{\omega,n}| \leq \int_M |\alpha| d\text{vol}_M < \infty.
\]

**Proof.** By definition of rapidly decreasing, there exists a constant $D$ and an $l > \lambda + 2$ ($\lambda$ is as in (4.1)) such that
\[
|\alpha| < D (1 + d^2 (z, \nu))^{-l}, \forall z \in M.
\]
Hence
\[ |\int_M \alpha_{n,1}| \leq \int_M |\alpha_{n,1}| d\text{vol}_M \leq \int_M |\alpha| d\text{vol}_M \leq D \int_M (1 + d^2(z, \nu))^{-l} d\text{vol}_M.\]

Recall \( B(\rho) := \{ z \in M | d(z, \nu) \leq \rho \} \). By (4.1) and completeness of \( g \) on \( M \)
\[
\int_M (1 + d^2(z, \nu))^{-l} d\text{vol}_M = \lim_{k \to \infty} \int_{B(k)} (1 + d^2(z, \nu))^{-l} d\text{vol}_M \\
= \sum_k \int_{B(k) \setminus B(k-1)} (1 + d^2(z, \nu))^{-l} d\text{vol}_M \\
\leq \int_{B(1)} (1 + d^2(z, \nu))^{-l} d\text{vol}_M + C \sum_{k=2}^{\infty} (1 + (k - 1)^2)^{-l} k^l \\
< \infty.
\]

where the last series converges because \( l > \lambda + 2 \). This proves the claim. 

Using Proposition 3.3 we obtain an exponential type integral presentation of virtual residues.

**Theorem 4.13.** Suppose \( s \) satisfies Assumption 4.4. Then for each tempered holomorphic section \( \psi \) of \( K_M \otimes \det V \), the contraction \( \psi \langle e^S \rangle \) is rapidly decreasing and, if \( M \) is quasi-projective, one has
\[
\text{Res} \frac{\psi}{s} = \frac{(-1)^n}{(2\pi i)^n} \int_M \psi \langle e^S \rangle.
\]

**Proof.** By Lemma 4.7 and Lemma 4.8 \( \psi \langle e^S \rangle \) is rapidly decreasing. By the completeness of the metric, there exists an exhaustive sequence of compact subsets \( \{ K_i \} \) of \( M \), \( M = \bigcup K_i \), and smooth functions \( \rho_i \) such that
\[
\rho_i = 1 \quad \text{in a neighborhood of} \quad K_i, \quad \text{Supp} \rho_i \subset K_{i+1}^0 \quad 0 \leq \rho_i \leq 1 \quad \text{and} \quad |d\rho_i| \leq 2^{-i},
\]
see Lemma 2.4 in page 366 of [Dem]. Choosing \( c \) big enough such that \( Z \subset Y \subset K_c \), where \( Y \) is compact as in Assumption 4.4, then \( M = \bigcup_{j \geq c} K_j \). By the definition of \( T_\rho \) in (3.1) and Lemma 3.1, we have
\[
[\overline{\partial}_s, R_\rho](\psi \langle e^S \rangle) = \psi \langle e^S \rangle - T_\rho(\psi \langle e^S \rangle)
\]
and pointwise convergence
\[
\lim_{j \to \infty} T_\rho_j(\psi \langle e^S \rangle) = \psi \langle e^S \rangle.
\]

Thus we may write
\[
\psi = T_\rho(\psi \langle e^S \rangle) + \overline{\partial}_s(R_\rho(\psi \langle e^S \rangle)) + \psi_j(1 - e^S),
\]
where \( \psi_j(1 - e^S) \) is also \( \overline{\partial}_s \)-exact by Lemma 4.11 and \( T_\rho(\psi \langle e^S \rangle) \) is compactly supported by definition (5.1). Apply Proposition 5.3
\[
\int_M T_\rho(\psi \langle e^S \rangle) = (-2\pi i)^n \text{Res} \frac{\psi}{s}.
\]

Use constants \( \mu, C_1 \) in Lemma 4.9, we define a smooth positive function on \( M \)
\[
G(z) = |\psi \langle e^S \rangle| + (n + 1)C_1 e^{-|s|^2} (1 + d^2(z, \nu))^\mu(z).
\]
By Lemma \textbf{[4.7]} and Lemma \textbf{[4.8]} \( \psi e^S \) is rapidly decreasing. Because \( e^{-|s|^2} \) is rapidly decreasing and \( (1 + d^2(z, \nu))^\mu \) is tempered, we know \( G(z) \) is also rapidly decreasing. By Lemma \textbf{[4.12]}

\[
\int_M G(z) d\text{vol}_M < \infty.
\]

Therefore \( G(z) \in L^1(M) \), where \( L^1(M) \) is the function space with the norm \( \|\beta\| := \int_M |\beta| d\text{vol}_M \) (c.f. the definition in [Dem, page 288]).

Recall in definition \textbf{[3.1]} \( T_s(\partial, \partial_T)^k \) because \( T_s^2 = 0 \). Therefore

\[
T_{\rho_j}(\psi e^S) = \rho_j(\psi e^S) + (\partial \rho_j) \sum_{k=0}^{n} (-1)^k T_s(\partial_T)^k(\psi e^S).
\]

Take absolute value and use Lemma \textbf{[4.9]} one sees at arbitrary \( z \in M \setminus Y \)

\[
|T_{\rho_j}(\psi e^S)| \leq |\psi e^S| + (n + 1)C_1|\partial \rho_j| e^{-|s|^2}(1 + d^2(z, \nu))^\mu(z) \leq G(z).
\]

When \( z \in Y \) one has \( \partial \rho_j(z) = 0 \) because \( Y \subset K_c \). Thus the same inequality holds for arbitrary \( z \in M \). Then by Lemma \textbf{[4.12]}, \textbf{[4.3]} and Lebesgue dominated convergence theorem [Royden page 376], we have

\[
\int_M (\psi e^S) = \lim_{j \to \infty} \int_M T_{\rho_j}(\psi e^S) = (-2\pi i)^n \text{Res}_z \frac{\psi}{s}.
\]

\( \square \)

**Proposition 4.14.** Suppose \( \psi \) is a tempered holomorphic section of \( K_M \otimes \text{det}V \), and \( s \) satisfies the Assumption \textbf{[4.4]} Then as \( e^{tS} \) is rapidly decreasing for \( t > 0 \) by Lemma \textbf{[4.7]} and Lemma \textbf{[4.8]} we have that \( \int_M (\psi e^{tS}) \) is independent of \( t \) for \( t > 0 \).

**Proof.** Applying Leibniz rule to \( \partial + t_s \) and Lemma \textbf{[4.10]} one has

\[
\frac{d(\psi e^{tS})}{dt} = \psi_j(S e^{tS}) = \psi_j((\partial + t_s)(\xi e^{tS}) = \psi_j((\partial + t_s)(\xi e^{tS})).
\]

Lemma \textbf{[6.1]} and Lemma \textbf{[6.2]} imply that

\[
\psi_j((\partial + t_s)(\xi e^{tS})) = \partial_j(\psi_j(\xi e^{tS})).
\]

Note that here \( \psi_j(\xi e^{tS}) \) is in \( \oplus_s \Omega^{(n,n)}(\land^{n-1} q V^*) \).

By lemma \textbf{[4.8]} \( \partial, \partial_T \) are tempered, and \( e^{tS} \) is rapidly decreasing. Then Lemma \textbf{[4.6]} implies \( e^{tS} \) is rapidly decreasing and that

\[
(\partial + t_s)(\xi e^{tS}) = ((\partial + t_s)(\xi)) e^{tS} = (\partial(\xi) e^{tS} - |s|^2 e^{tS}
\]

is also rapidly decreasing. Then Lemma \textbf{[4.7]} shows \( \psi_j(\xi e^{tS}) \) and \( \psi_j((\partial + t_s)(\xi e^{tS})) \) are rapidly decreasing. Use Lemma \textbf{[4.12]}

\[
\int_M |\psi_j(\xi e^{tS})| d\text{vol}_M < \infty \quad \text{and} \quad \int_M |\psi_j((\partial + t_s)(\xi e^{tS}))| d\text{vol}_M < \infty.
\]

Therefore for \( \varpi \) to be the component of \( \psi_j(\xi e^{tS}) \) in \( \Omega^{(n,n-1)} \) one has

\[
\int_M |\varpi| d\text{vol}_M \leq \int_M |\psi_j(\xi e^{tS})| d\text{vol}_M < \infty,
\]

\[
\int_M |d\varpi| d\text{vol}_M = \int_M |\partial \varpi| d\text{vol}_M \leq \int_M |\partial_s(\psi_j(\xi e^{tS}))| d\text{vol}_M = \int_M |\psi_j((\partial + t_s)(\xi e^{tS}))| d\text{vol}_M < \infty.
\]
Apply [Gal, p141 Thm] one concludes \( \int_M d\omega = 0 \), and thus \( \int_M d(\psi \mathcal{J}(\xi e^{tS})) = 0 \). Using that \( s \wedge (\psi \mathcal{J}(\xi e^{tS})) \) has no component in \( \Omega^{(n,n)} \), we have

\[
\int_M \frac{d(\psi e^{tS})}{dt} = \int_M \mathcal{J}_M(\psi \mathcal{J}(\xi e^{tS})) = \int_M \mathcal{J} \omega = \int_M d\omega = 0.
\]

We finally claim \( \frac{d}{dt} \int_M (\psi e^{tS}) \) = \( \int_M \frac{d(\psi e^{tS})}{dt} \). One first writes \( e^{tS} = e^{-t|s|^2} \sum_{k=0}^{n} t^k \beta_k \) with \( \beta_k = \frac{e^{\epsilon k}}{k!} e^{tS} \in F_M^k \). Because \( e^{-t|s|^2} \beta_k \) is rapidly decreasing and \( \mathcal{J}_M(\xi) \), \( |s|^2 \) are tempered, \( e^{-t|s|^2} (\mathcal{J}_M(\xi)) \beta_k \) and \( e^{-t|s|^2} |s|^2 \beta_k \) are rapidly decreasing. As \( \psi \mathcal{J} \) preserves the rapidly-decreasing property, there exists a constant \( C_2 \) such that, for \( t > t_0 > 0 \)

\[
\left| \frac{d(\psi e^{tS})}{dt} \right| = e^{-t|s|^2} - \sum_{k=0}^{n} t^k |s|^2 (\psi \mathcal{J}(\beta_k)) + \sum_{k=1}^{n} t^{k-1} \psi \mathcal{J}(\mathcal{J}(\xi)(\beta_{k-1})) |s|^2
\]

\[
\leq C_2 e^{-\frac{t}{2}|s|^2} \leq C_2 e^{-\frac{t}{2}|s|^2},
\]

where \( e^{-\frac{t}{2}|s|^2} \) is in \( L^1(M) \) (integrable). This implies

\[
\frac{d}{dt} \int_M \psi e^{tS} = \int_M \frac{d(\psi e^{tS})}{dt} = 0.
\]

\[\square\]

**Corollary 4.15.** Under the same condition and for \( t > 0 \) let \( s = t \cdot s \) associates \( \xi, S \) as in (4.2), then \( \int_M \psi (e^{St}) = t^{-n} \int_M \psi e^S \).

**Proof.** One checks \( (\psi e^{St})_{n,n} = t^{-n}(\psi e^{2St})_{n,n} \), and then applies the previous proposition. \[\square\]

The identity corresponds to \( \operatorname{Res} \frac{\psi}{s} = \frac{1}{t^n} \operatorname{Res} \frac{\psi}{s} \) for \( n = \dim M \) (residue taken near compact connected components of \( s = 0 \)).

5. **Appendix 1: The existence of a good algebraic neighborhood**

5.1. **Durfee’s construction.** We first recall several results of Durfee [Dur] on constructing neighborhoods of real algebraic sets.

A subset of some Euclidean space \( \mathbb{R}^n \) is called real algebraic if it is the common zero set of some finite collection of real polynomials.

**Definition 5.1** (Dur, Definition 1.1). Let \( N \) be an algebraic set in \( \mathbb{R}^n \) and let \( Y \) be a compact algebraic subset of \( N \) \( \setminus Y \) nonsingular. An (algebraic) rug function for \( Y \) in \( N \) is a proper (real) polynomial function \( \beta : N \to \mathbb{R} \) such that \( \beta(x) \geq 0 \) for \( x \in N \) and \( \beta^{-1}(0) = Y \).

By [Dur] Corollary 1.3] every set \( Y \) in Definition 5.1 has a rug function, and by [Dur] Lemma 1.4] every rug function has finitely many critical values.

**Definition 5.2** (Dur, Definition 1.5). Let \( N, Y \) be as in Definition 5.1. A subset \( T \subseteq N \) with \( Y \subseteq T \subseteq N \) is called an algebraic neighborhood of \( Y \) in \( N \) if \( T = \beta^{-1}(0, \delta) \), for some rug function \( \beta \) of \( Y \) in \( N \), and some positive number \( \delta \) smaller than all nonzero critical values of \( \beta \).

Hence for every \( N, Y \) in Definition 5.1] one can obtain an algebraic neighborhood of \( Y \) in \( N \) for every rug function of \( Y \subseteq N \).
Proposition 5.3 (Dur Proposition 1.6). Let $T$ be an algebraic neighborhood of $Y$ in $N$. Then the inclusion $Y \subset T$ is a homotopy equivalence.

We refine [Dur] Lemma 2.2 as follows.

Lemma 5.4. There is a regular embedding of real manifolds (c.f. [Ch Def. 3.1])

$h : \mathbb{C}^n \to \mathbb{R}^{2(n+1)^2}$ where $h(\mathbb{C}^n)$ is a smooth real algebraic subset of $\mathbb{R}^{2(n+1)^2}$, and for every complex algebraic subset $B \subset \mathbb{C}^n$ its image $h(B)$ is also a real algebraic subset of $\mathbb{R}^{2(n+1)^2}$.

Proof. The first assertion follows from the construction in [Dur] Lemma 2.2 and we explain the detail as follows. Let $z_0, \ldots, z_n$ be complex homogeneous coordinates for $\mathbb{CP}^n$ and let $\{w_{ij}\}_{0 \leq i, j \leq n}$ be complex coordinates for $\mathbb{C}^{(n+1)^2} \cong \mathbb{R}^{2(n+1)^2}$.

Define $h : \mathbb{CP}^n \to \mathbb{C}^{(n+1)^2}$ by sending $z = [z_0, \ldots, z_n]$ to the point $w = (w_{ij})_{ij} \in \mathbb{C}^{(n+1)^2}$ given by

$$w_{ij} = \frac{z_i z_j}{z_0 z_1 + \cdots + z_n z_n}.$$  

Clearly $h$ is a well-defined smooth map between real manifolds.

Let

$$R_k := \{w = (w_{ij})_{ij} \in \mathbb{C}^{(n+1)^2} | w_{kk} \neq 0\},$$

and define $\pi_k : R_k \to \mathbb{CP}^n$ by $\pi_k(w) = [w_0, \ldots, w_n]$. Then $\pi_j$ is a smooth map.

Let $U_k = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n | z_k \neq 0\}$ and let $h_k : U_k \to R_k$ be the restriction of $h$ on $U_k$. Then $\pi_k \circ h_k = \text{Id}$ by definition. This implies $h_k$ and $dh_k|_p$ are injective for each $p \in U_k$ and every $k$. As $\{U_k\}_{k=0}^n$ covers of $\mathbb{CP}^n$, we know $dh_k|_p$ is injective at every point $p \in \mathbb{CP}^n$.

We claim $h$ is injective. If $h(z) = h(z')$ for $z = [z_0, \ldots, z_n]$ and $z' = [z'_0, \ldots, z'_n]$, then comparing coordinate $w_{kk}$ one deduces $z_k, z'_k$ are both zero or both nonzero, for each $k$. Therefore $z, z' \in R_k$ for some $k$. Applying $\pi_k$ to $h_k(z) = h_k(z')$ one has $z = z'$.

The above properties of $h$ and the compactness of $\mathbb{CP}^n$ implies $h$ is a regular embedding by [Ch] Theorem 3.5]. Therefore $h$ induces a homeomorphism from $\mathbb{CP}^n$ to $h(\mathbb{CP}^n)$.

Denote $\mathcal{S} \subset \mathbb{C}^{(n+1)^2}$ to be the real algebraic set of points $(w_{ij})_{ij}$ satisfying

$$w_{ij} w_{kl} = w_{il} w_{kj} \quad 0 \leq i, j, k, l \leq n$$

$$\sum_{i=0}^n w_{ii} = 1$$

$$w_{ij} = w_{ji} \quad 0 \leq i, j \leq n.$$  

One easily verifies $h(\mathbb{C}^n) \subset \mathcal{S}$. For each $w = (w_{ij})_{ij} \in \mathcal{S}$, the identity $\sum_{i=0}^n w_{ii} = 1$ implies $w_{kk} \neq 0$ for some $k$. Then the $ij$-coordinate of $h([w_0, w_k, \ldots, w_n])$ is

$$w_{ij} = \frac{w_{ik} w_{jk}}{\sum_{l=0}^n w_{ik} w_{lk}} = \frac{w_{ij} w_{kk}}{\sum_{l=0}^n w_{ij} w_{kl}} = \frac{w_{ij} w_{kk}}{w_{kk} (\sum_{l=0}^n w_{il})} = w_{ij},$$

Thus $h([w_0, w_k, \ldots, w_n]) = w$. This implies $h(\mathbb{C}^n) = \mathcal{S}$.

Suppose $B \subset \mathbb{CP}^n$ is a complex projective subvariety. We claim $h(B)$ is real algebraic. First if $B = (z_k = 0) \subset \mathbb{CP}^n$ for some $k$ then one easily checks $h(B) = (w_0 = \cdots = w_{kn} = 0) \cap h(\mathbb{C}^n)$ is real algebraic. Secondly it is enough to prove the assertion by assuming $B \cap U_k \neq \emptyset$ for all $k = 0, \ldots, n$. The reason is if $B \subset (z_k = 0)$ for some $k$ then one can find a complex subvariety $B' \subset \mathbb{CP}^n$, 

$B' \cap (z_k = 0) = B$, and $B' \cup U_j \neq \emptyset$ for all $j$. Then $h(B) = h(B') \cap h((z_k = 0))$ is an intersection of two real algebraic subsets and thus is also real algebraic.

We now assume $B \cup U_k \neq \emptyset$ for all $k$. For convenience let us view $h$ as an inclusion. Observe for each $k = 0, \ldots, n$, the functions $\{z_{ik} := w_{ik}/w_{kk}\}$(for $i = 0, \ldots, n, i \neq k$) give affine coordinate of $U_k$. One can find finitely many complex polynomials $f_\ell, g_\ell$, and $r_\ell \in \mathbb{N}$ such that $f_\ell (w_0, \ldots, w_n)/w_{kk}^{r_\ell} = g_\ell (z_{0k}, \ldots, z_{nk})$ and $\{g_\ell\}_\ell$ generates the ideal of $B \cap U_k$ in $U_k$. Let $C_k$ be the common zero set of $\{f_\ell (w_0, \ldots, w_n)/w_{kk}^{r_\ell}\}_\ell$ on $\mathbb{R}^{2(n+1)^2}$. Then $C_k$ is a real algebraic subset of $\mathbb{R}^{2(n+1)^2}$ such that $C_k \cap U_k = B \cap U_k$ because $w_{kk}|U_k$ is nowhere vanishing. Since $B$ is complex subvariety of $\mathbb{CP}^n$, $B$ equals the closure of $B \cap U_k$ and thus $B \subset C_k$. Hence $B$ is contained in the real algebraic set $C := C_0 \cap \cdots C_n$. Then $C \cap U_k \subset C_k \cap U_k \subset B$ (for each $k$) implies $C \cap \mathbb{CP}^n = B$. As an intersection of two real algebraic sets $B$ is also a real algebraic set.

If $B$ is a complex algebraic subset of $\mathbb{CP}^n$, one writes $B = B_1 \cup \cdots \cup B_m$ where each $B_i$ is complex projective subvariety of $\mathbb{CP}^n$. Then $h(B) = \cup_j h(B_j)$ is real algebraic. This implies the second statement. \hfill $\square$

5.2. Good neighborhoods of subvarieties. We now turn to the complex projective cases.

**Definition 5.5.** Let $M$ be a smooth quasi-projective variety over $\mathbb{C}$, and $X$ is an compact connected complex algebraic subset of $M$. A “good neighborhood” of $X$ in $M$ is a compact neighborhood $T$ of $X$ in $M$ whose boundary $\partial T = T - T$ is a compact submanifold of $M$, and the inclusion $X \subset T$ is a homotopy equivalence.

If $\{X_1\}_1$’s are disjoint compact connected complex algebraic subsets of $M$, a “good neighborhood” of $X = \cup_j X_j$ in $M$ is a collection of good neighborhoods $T_j$ of $X_j$ in $M$ (for each $j$) such that $\{T_j\}_j$’s are disjoint.

**Lemma 5.6.** Let $M$ be a smooth quasi-projective subvariety of $\mathbb{CP}^n$, and $X \subset M$ is a compact connected algebraic subset of $M$. Then (1) For each open neighborhood $U \subset M$, $X$ has a good neighborhood contained in $U$; (2) If $T_1$ and $T_2$ are two good neighborhoods of $X \subset M$ then there exists another good neighborhood $T \subset M$ such that $T \subset T_1 \cap T_2$.

**Proof.** Let $\overline{M} \subset \mathbb{CP}^n$ be a smooth projective compactification of $M$, then $X = \overline{X} \subset \overline{M}$, for $X$ is compact. By Lemma [5.4] there is an embedding of smooth manifolds $h : \mathbb{CP}^n \rightarrow \mathbb{R}^{2(m+1)^2}$, such that $h(X) \subset h(\overline{M}) \subset \mathbb{R}^{2(m+1)^2}$ are real algebraic subsets. Then $h(X)$ is compact and $h(\overline{M})$ is smooth. Thus we can use [Dun], Corollary 1.3, Lemma 1.4 as stated in previous subsection to find a rug function $\beta$ for $h(X) \subset h(\overline{M})$ such that $\beta^{-1}(0) = h(X)$ and $\beta$ have finitely many critical values.

For the first assertion. Denote $D = h(\overline{M}) - h(U)$; then $D$ is a compact subset of $h(\overline{M})$ and $h(X) \cap D = \emptyset$. Thus $D$ is a compact subset of $\mathbb{R}$ and $0 \notin \beta(D)$. Find disjoint open subsets $V_1$ and $V_2$ of $\mathbb{R}$ with $0 \in V_1$ and $\beta(D) \subset V_2$. Choose $\delta$ smaller than all the nonzero critical values of $\beta$ such that $[0, \delta] \subset V_1$, and set $T_\delta = \beta^{-1}(0, \delta)]$.

By construction $T_\delta \cap D = \emptyset$ implies $T_\delta \subset h(U)$. Also $T_\delta$ is compact because $\beta$ is proper. By implicit function theorem $\partial T_\delta = \beta^{-1}(\delta)$ is a smooth compact submanifold of $\overline{M}$. And by Proposition [5.8] the inclusion $h(X) \subset T_\delta$ is a homotopy
equivalence. Since \( h : M \to h(M) \) is a homeomorphism of smooth manifolds, the above properties imply that \( h^{-1}(T_\delta) \) is a good neighborhood of \( X \subset M \) and is contained in \( U \).

We prove the second assertion. Denote \( Z_1 = h(\overline{M}) \setminus h(T_1) \). Then \( Z_1 \) is a compact subset of \( h(\overline{M}) \) and \( h(X) \cap Z_1 = \emptyset \). Thus \( \beta(Z_1) \) is a compact subset of \( \mathbb{R} \) and \( 0 \notin \beta(Z_1) \). Find disjoint open subsets \( V_1 \) and \( V_2 \) of \( \mathbb{R} \) with \( 0 \in V_1 \) and \( \beta(Z_1) \subset V_2 \). Pick interval \([0, \delta_1] \subset V_1 \) and set \( T_{\delta_1} = \beta^{-1}([0, \delta_1]) \). Then \( T_{\delta_1} \cap Z_1 = \emptyset \), so \( T_{\delta_1} \subset h(T_1) \). Similarly there exists a \( \delta_2 \) such that \( T_{\delta_2} = \beta^{-1}([0, \delta_2]) \) satisfies \( T_{\delta_2} \subset h(T_2) \). Choosing \( \delta = \min\{\delta_1, \delta_2\} \), then \( T_{\delta} \) is contained in \( h(T_1) \) and \( h(T_2) \). Let \( T = h^{-1}(T_\delta) \), then \( T \subset T_1 \cap T_2 \) is the requested good neighborhood. \( \Box \)

An argument analogous to that in the above proof gives the following.

**Corollary 5.7.** Every subset \( X \) of \( M \) which is a finite union of compact connected complex subvarieties admits a good neighborhood in \( M \).

### 6. Appendix 2: Operators and metrics on exterior algebra \( \mathcal{B} \)

Let \( V \) be a rank \( n \) holomorphic bundle over a complex manifold \( M \). Recall in section 3 \( \mathcal{B} := \oplus_{i,j,k,l} \Omega^{(i,j)}(\wedge^i V \otimes \wedge^j V^*) \) is a graded commutative algebra extending the wedge products of \( \Omega^* \), \( \wedge^* V \) and \( \wedge^* V^* \). The degree of \( \alpha \in \Omega^{(i,j)}(\wedge^k V \otimes \wedge^l V^*) \) is \( \omega \alpha := i + j + k - l \). We briefly \( A^0(\wedge^k V \otimes \wedge^l V^*) = \Omega^{(0,0)}(\wedge^k V \otimes \wedge^l V^*) \).

Set \( \kappa : \mathcal{B} \to \Omega^* \) which sends \( \omega(e \otimes e') \) for \( \omega \in \Omega^{(i,j)}, e \in \wedge^k V, e' \in \wedge^l V^* \) to \( \omega < e, e' > \), where \( <, > \) is the dual pairing between \( \wedge^k V, \wedge^l V^* \) and \( e, e' > = 0 \) when \( k \neq \ell \). We further extend the pairing \( < \alpha, \beta > := \kappa(\alpha \beta) \) for \( \alpha, \beta \in \mathcal{B} \). It is direct to verify

\[
\overline{\alpha} < \alpha, \beta > = < \overline{\alpha}, \overline{\beta} > + (-1)^{|\alpha|} < \alpha, \overline{\beta} > .
\]

We now define three different types of contraction maps. Given \( u \in \Omega^{(i,j)}(\wedge^k V) \) and \( k \geq \ell \), we define

\[
uu_{\ell,j} : \Omega^{(p,q)}(\wedge^l V^*) \to \Omega^{(p+i+q+j)}(\wedge^{k-\ell} V)
\]

where for \( \theta \in \Omega^{(p,q)}(\wedge^l V^*) \), the \( u, \theta \) is determined by

\[
< u, \theta, \nu^* > = (-1)^{(i+j)+l(p+q)} u^\alpha \theta^\beta < u, \theta \wedge \nu^* >, \quad \forall \nu^* \in A^0(\wedge^{k-\ell} V^*).
\]

Given \( \alpha \in A^0(V) \), we define

\[
\iota_\alpha : \Omega^{(i,j)}(\wedge^k V^*) \to \Omega^{(i,j)}(\wedge^{k-1} V^*)
\]

where for \( w \in \Omega^{(i,j)}(\wedge^k V^*) \), the \( \iota_\alpha(w) \) is determined by

\[
< \nu, \iota_\alpha(w) > = < \alpha \wedge \nu, w >, \quad \forall \nu \in A^0(\wedge^{k-1} V).
\]

For above \( \alpha, \theta \) and \( w \) one has \( \iota_\alpha(w \wedge \theta) = \iota_\alpha(w) \wedge \theta + (-1)^{|w|} \theta w \wedge \iota_\alpha(\theta) \).

Given \( \gamma \in A^0(V^*) \), we also define

\[
\iota_\gamma : \Omega^{(i,j)}(\wedge^k V) \to \Omega^{(i,j)}(\wedge^{k-1} V)
\]

where for \( v \in \Omega^{(i,j)}(\wedge^k V) \), the \( \iota_\gamma(v) \) is determined by

\[
< \iota_\gamma(v), w > = (-1)^{|i+j|} < v, \gamma \wedge w >, \quad \forall w \in A^0(\wedge^{k-1} V^*).
\]

**Lemma 6.1.** Given \( u \in \Omega^{(i,j)}(\wedge^k V) \), and \( \theta, \alpha, \gamma \) as above, one has \( \alpha \wedge (u, \theta) = u, \iota_\alpha(\theta) \) and \( \iota_\gamma(u, \theta) = \iota_\gamma(u \wedge \theta) \)
Proof. For arbitrary \( w \in A^0(\wedge^{n-\ell+1}V^*) \) one calculates (using \( \theta \wedge w = 0 \))
\[
< \alpha \wedge (u_\ell \theta), w > = (-1)^{l+j+q+p} < u_\ell \theta, t_\alpha(w) > \\
= (-1)^{l+j+q+p+l(1-\ell)+q+q} < u_\ell \theta, \alpha \wedge t_\alpha(w) > \\
= (-1)^{l+j+q+p+l(1-\ell)+q+q} < u_\ell \theta, \alpha \wedge t_\alpha(w) > \\
= (-1)^{l+j+q+p+l(1-\ell)+q+q} < u_\ell \theta, \alpha \wedge w > \\
< u_\ell \theta(\alpha(w)), w > .
\]

The proof of the second identity is similar and we omit the proof.
\[\square\]

Lemma 6.2. For \( u \in \Omega^{(i,j)}(\wedge^k V), \theta \in \Omega^{(p,q)}(\wedge^l V^*), \ k \geq l \) and smooth form \( \alpha \in \Omega^{(a,b)}(M) \), we have
\[
\alpha \wedge (u_\ell \theta) = u_\ell (\alpha \theta) \quad \text{and} \quad \overline{\partial}(u_\ell \theta) = (-1)^{pl} \overline{\partial}(u_\ell) \theta + u_\ell (\overline{\partial}(\theta)).
\]

Proof. Let \( \{e_i\} \) be a holomorphic local frame of \( V \) and \( \{e^i\} \) is the dual frame of \( V^* \). Denote \( \{e_I\} \) and \( \{e^I\} \) to be the induced frames of \( \wedge^k V \) and \( \wedge^{k-1} V^* \). Then
\[
\overline{\partial}(u_\ell \theta) = \overline{\partial}( \sum_{< u_\ell \theta \wedge e^I > > e_I} ) = (-1)^{(l+q+q_1)} \sum_{< u_\ell \theta \wedge e^I > > e_I} \overline{\partial}(u_\ell \theta, \wedge e^I) > e_I \\
+ (-1)^{(l+q+q_1)} \sum_{< u_\ell \theta \wedge e^I > > e_I} \overline{\partial}(u_\ell \theta) > e_I \\
= (-1)^{l+q+q_1} < u_\ell \theta, \wedge e^I > e_I \\
= (-1)^{l+q+q_1} u_\ell \theta(\alpha(w)) + u_\ell (\overline{\partial}(\theta)).
\]

The proof of the first identity is similar and we omit the proof.
\[\square\]

Now we study some simple metric inequalities on \( B \). Let \( h \) be a fixed hermitian metric over \( V \). For arbitrary holomorphic local frame \( \{e_i\} \) of \( V \) with \( \{t^I\} \) its dual frame of \( V^* \), one represents \( h = \sum h_{j_0} t^I_j \). The induced metric \( h^* \) on \( V^* \) can be written as \( h^* = \sum h_{j_0} e_i \otimes \bar{e}_j \), where \( \sum h_{j_0} h_{j_0} = \delta^I_j \).

As in [Wa], page 79 Ex 13], the induced metric \( h_\wedge V \) on \( \wedge^k V \) is
\[
(h_\wedge V)_j^k = (\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k)_\wedge V := \det[h(\alpha_i, \beta_j)].
\]

Similarly \( h^* \) induced metrics \( h^*_\wedge V \) on \( \wedge^l V^* \) and \( h^*_\wedge V \otimes h^*_\wedge V \) on \( \wedge^l V \otimes \wedge^l V^* \). The induced metric on \( B = \oplus_i \wedge_i, \wedge^{(i)}(\wedge^k V \otimes \wedge^l V^*) \) would be denoted by \( (,\cdot) \) and \( |\alpha|^2 := (\alpha, \alpha) \) for \( \alpha \in B \).

Lemma 6.3. For each \( \alpha \in \Omega^{(i,j)}(\wedge^l V) \) and \( \beta \in \Omega^{(k,m)}(\wedge^r V) \) one has
\[
(\alpha \beta, \alpha \beta) \leq c \cdot c' (\alpha, \alpha) \cdot (\beta, \beta),
\]
where \( c, c' \) are ranks of the bundles correspond to \( \Omega^{(i,j)}(\wedge^l V) \) and \( \Omega^{(k,m)}(\wedge^r V) \).

Proof. Let \( \{e_i\} \) be a local orthogonal frame of \( V \). Denote the induced frame of \( \wedge^l V \) and \( \wedge^r V \) as \( \{e_I\} \) and \( \{e^I\} \). Let \( a_1, \cdots, a_n, b_1, \cdots, b_m \) be (unitary) frame for \( \Omega^{(1,0)}(\Omega^{(i,j)}) \). Their induced (unitary) frame of \( \Omega^{(i,j)} \) and \( \Omega^{(k,m)} \) are denoted \( \{f_K\} \) and \( \{f^L\} \) respectively. Then the subset of nonzero elements in \( \{f_K \wedge f_L\}_{K,L} \) is part of the induced unitary frame of \( \Omega^{(i+k,j+m)} \).

Write
\[
\alpha = \sum_{K,I} a_{KI} f_K \otimes e_I \quad \text{and} \quad \beta = \sum_{L,J} b_{LJ} f_L \otimes e_J
\]
where \( \alpha_{KI}, \beta_{LJ} \) are functions. Then we have
\[
\alpha \beta = (-1)^{(l+k+m)} \sum \alpha_{KI} \beta_{LJ} f_K \wedge f_L e_I \otimes e_J.
\]

Using that any two elements in the collection \( \{ f_K \wedge f_L e_I \otimes e_J \}_{KLIJ} \) has metric pairing to be 0, 1 or \(-1\), we have
\[
|\langle \alpha \beta, \alpha \beta \rangle| = |(\sum_{KLIJ} \alpha_{KI} \beta_{LJ} f_K \wedge f_L e_I \otimes e_J)|
\leq \sum_{K} \sum_{LJ} |\alpha_{KI}| |\alpha_{KI}| |\beta_{LJ}| = (\sum_{K} |\alpha_{KI}|^2)(\sum_{LJ} |\beta_{LJ}|^2)
\]
(Cauchy Inequality) \leq (c \sum_{KI} |\alpha_{KI}|^2)(c' \sum_{LJ} |\beta_{LJ}|^2) = c \cdot c' |\langle \alpha, \alpha \rangle| |\langle \beta, \beta \rangle|

\]

Lemma 6.4. For arbitrary \( \alpha \in \Omega^{(0,0)}(\wedge^k V) \) and \( \beta \in \Omega^{(p,q)}(\wedge^k V^*) \) one has
\[
\langle \alpha, \beta \rangle >, \langle \alpha, \beta \rangle \leq (\alpha, \alpha)(\beta, \beta).
\]

Proof. Let \( \{e_i\} \) be a unitary frame of \( V \) inducing unitary frame \( \{e^I\} \) for \( \wedge^k V (\wedge^k V^*) \). Then \( \langle e_I, e^J \rangle = \langle e^I, e^J \rangle = \delta^I_J \). Write \( \alpha = \sum_I x_I e_I \) and \( \beta = \sum_I y_I e^I \), where \( x_I \)'s are (local) functions and \( y_I \)'s are local sections of \( \Omega^{(p,q)} \). Then
\[
|\langle \alpha, \beta \rangle|^2 = \sum_I x_I y^*_I \leq (\sum_I |x_I|^2)(\sum_I |y^*_I|^2) = (\alpha, \alpha)(\beta, \beta).
\]

Lemma 6.5. For \( u \in \Omega^{(n,0)}(\wedge^k V) \) and \( v^* \in \Omega^{(0,q)}(\wedge^l V^*) \) with \( k \geq l \), one has
\[
\langle u, v^* \rangle, \langle u, v^* \rangle \leq b \cdot c^2 (u, u)(v^*, v^*),
\]
where \( b, c \) are ranks of the bundles correspond to \( \Omega^{(0,q)} \otimes \wedge^l V^*, \wedge^{k-l} V^* \).

Proof. Let \( \{e_i\} \) be a local unitary frame of \( V \), and \( \{e^I\} \) be its dual frame of \( V^* \). The induced frame of \( \wedge^{k-l} V \) and \( \wedge^{k-l} V^* \) are denoted as \( \{e_I\} \) and \( \{e^I\} \). Since \( \Omega^{(n,0)} \) is of rank one we write \( u = x \otimes y \), where \( x, y \) are sections of \( \Omega^{(n,0)}, \wedge^k V \) respectively. Then \( \langle u, u \rangle = \langle x, x \rangle \langle y, y \rangle \) and
\[
u^*_I = \sum_I \langle u, v^*_I, e^I \rangle e^I = (-1)^{n+l+k(n+k)+\frac{(l-1)}{2}} \sum_I \langle u, v^* \wedge e^I \rangle e^I.
\]
Using \( \langle u, v^* \wedge e^I \rangle = x \langle y, v^* \wedge e^I \rangle \), Lemma 6.3 and Lemma 6.4 imply
\[
|\langle u, v^* \rangle|^2 = \sum_I \langle x \rangle^2 \langle y \rangle^2 |v^* \wedge e^I|^2 \leq |x|^2 (b \cdot c) \sum_I |y|^2 |v^*|^2 |v^*|^2 \leq b c^2 |u|^2 |v^*|^2.
\]

\]
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