Asymptotic flocking for the Cucker-Smale model with
time variable time delays

ELISA CONTINELLI *

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Abstract

In this paper, we investigate a Cucker-Smale flocking model with varying time delay. We establish exponential asymptotic flocking without requiring smallness assumptions on the time delay size and the monotonicity of the influence function.

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1 Introduction

Multiagent systems have been largely studied in these last years, by virtue of their wide application to many scientific fields, such as biology [5, 13], economics [20], robotics [4, 22], control theory [7, 3, 31, 36, 1, 30], social sciences [2, 34]. In this regard, we mention the Hegselmann-Krause opinion formation model [21] and the Cucker-Smale flocking model [13]. Typically, for the solutions of such systems, the convergence to consensus, in the case of the Hegselmann-Krause model, or the exhibition of asymptotic flocking, in the case of the Cucker-Smale model, is investigated. Also, we mention [15, 8, 6] for the analysis of the kinetic version of the Cucker-Smale model. The Cucker-Smale model is presented for nonsymmetric interaction coefficients in [27].

In the theory of multiagent systems, the introduction of time delays is reasonable. For instance, in opinion formation or flocking models, a particle does not necessarily receive information from the other agents instantaneously. In the applications, it is more realistic to contemplate models in which each particle has to wait for a while before collecting all the information coming from the other particles of the system (see [23, 28]).

In this paper, we focus on a flocking model with time variable time delays. Multiagent systems with delay have already been studied in some works, among them [24, 14, 19]. Some flocking results for the delayed Cucker-Smale model have been established in [20, 9, 10, 33, 12]. In all these works, the authors require a smallness assumption on the time delay size. Also, convergence to consensus for the Hegselmann-Krause model with small time delays has been investigated in [11, 18, 29]. Very recently, a consensus result for the constant time delay case is proved in [19], without requiring upper bounds on the time delay. We also refer to [25] for...
Asymptotic flocking for the Cucker-Smale model with time variable time delays

a consensus result without any restrictions on the constant time delay but in the particular case of constant interaction coefficients. For the Cucker-Smale model with constant time delay, asymptotic flocking is achieved by [35] without assuming the smallness of the time delay size. A flocking result for the Cucker-Smale model with leadership and time delay without upper bounds is obtained by [32].

In this paper, we extend the argument of [35] to a Cucker-Smale flocking model with time-varying time delay. We succeed in improving previous flocking results by removing upper bounds on the time delay size. To be precise, we deal with a time delay function \( \tau(\cdot) : [0, +\infty) \rightarrow [0, +\infty) \) that satisfies

\[
0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t \geq 0,
\]

for some positive constant \( \bar{\tau} \), and we prove the asymptotic flocking for the solutions of the delayed Cucker-Smale model without requiring any smallness assumptions on the time delay size \( \bar{\tau} \). Moreover, we are able to weaken the monotonicity hypothesis on the influence function that is assumed in [35]. Indeed, in [35] the influence function is a continuous function of the distance between the agents’ positions that is required to be nonincreasing. Here, we rather consider an influence function that it is assumed to be just positive, bounded and continuous.

Consider a finite set of \( N \in \mathbb{N} \) particles, with \( N \geq 2 \). Let \( (x_i(t)) \in \mathbb{R}^d \) and \( (v_i(t)) \in \mathbb{R}^d \) denote the position and the velocity of the \( i \)-th particle at time \( t \), respectively. We shall denote with \( |\cdot| \) and \( \langle \cdot, \cdot \rangle \) the usual norm and scalar product in \( \mathbb{R}^d \), respectively. The interactions between the elements of the system are described by the following Cucker-Smale type model with a variable time delay

\[
\begin{align*}
\frac{d}{dt} x_i(t) &= v_i(t), & t > 0, \forall i = 1, \ldots, N, \\
\frac{d}{dt} v_i(t) &= \sum_{j: j \neq i} a_{ij}(t)(v_j(t - \tau(t)) - v_i(t)), & t > 0, \forall i = 1, \ldots, N,
\end{align*}
\]

(1.1)

with weights \( a_{ij}(t) \) of the form

\[
a_{ij}(t) := \frac{1}{N-1} \psi(|x_i(t) - x_j(t - \tau(t))|), \quad \forall t > 0, \forall i, j = 1, \ldots, N,
\]

(1.2)

where \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) is a positive function. The time delay function \( \tau : [0, \infty) \rightarrow [0, \infty) \) is assumed to be continuous and it satisfies

\[
0 \leq \tau(t) \leq \bar{\tau}, \quad \forall t \geq 0,
\]

(1.3)

for some positive constant \( \bar{\tau} \).

The initial conditions

\[
x_i(s) = x^0_i(s), \quad v_i(s) = v^0_i(s), \quad \forall s \in [-\bar{\tau}, 0], \forall i = 1, \ldots, N,
\]

(1.4)

are assumed to be continuous functions.

The influence function \( \psi \) is assumed to be continuous. Moreover, we assume that it is bounded and we denote with

\[
K := \|\psi\|_{\infty}.
\]

For existence results related to system (1.1) we refer to [16, 17]. Here, we are interested in the analysis of the asymptotic flocking of the solutions to (1.1).

For each \( t \geq -\bar{\tau} \), we define the diameters in space and in velocity

\[
d_X(t) := \max_{i,j = 1, \ldots, N} |x_i(t) - x_j(t)|,
\]

2
Asymptotic flocking for the Cucker-Smale model with time variable time delays

\[ d_V(t) := \max_{i,j=1,\ldots,N} |v_i(t) - v_j(t)|. \]

**Definition 1.1.** (Unconditional flocking) We say that a solution \( \{(x_i, v_i)\}_{i=1,\ldots,N} \) to system (1.1) exhibits **asymptotic flocking** if it satisfies the two following conditions:

1. there exists a positive constant \( d^* \) such that
   \[ \sup_{t \geq -\bar{\tau}} d_X(t) \leq d^*; \]

2. \( \lim_{t \to \infty} d_V(t) = 0. \)

Our main result is the following.

**Theorem 1.1.** (Unconditional flocking) Assume that \( \psi : \mathbb{R} \to \mathbb{R} \) is a positive, bounded, continuous function that satisfies

\[ \int_{0}^{+\infty} \min_{r \in [0,x]} \psi(r) dx = +\infty. \] (1.5)

Moreover, let \( x_i^0, v_i^0 : [-\bar{\tau}, 0] \to \mathbb{R}^d \) be continuous functions, for any \( i = 1, \ldots, N \). Then, for every solution \( \{(x_i, v_i)\}_{i=1,\ldots,N} \) to (1.1) with the initial conditions (1.4), there exists a positive constant \( d^* \) such that

\[ \sup_{t \geq -\bar{\tau}} d_X(t) \leq d^*, \] (1.6)

and there exists another positive constant \( C, \) independent of \( N, \) for which the following exponential decay estimate holds

\[ d_V(t) \leq \left( \max_{i,j=1,\ldots,N} \max_{r,s \in [-\bar{\tau},0]} |v_i(r) - v_j(s)| \right) e^{-C(t-2\bar{\tau})}, \quad \forall t \geq -\bar{\tau}. \] (1.7)

**Remark 1.2.** Let us note that, if the influence function \( \psi \) is nonincreasing, then the assumption (1.5) reduces to

\[ \int_{0}^{+\infty} \psi(x) dx = +\infty. \] (1.8)

The condition (1.8) is the one assumed in [35] in order to achieve the unconditional flocking for the solutions of the Cucker-Smale model.

The rest of this paper is organized as follows. In Section 2 we present some preliminary definitions and results that will be needed for the proof of Theorem 1.1. In Section 3 we prove our asymptotic flocking result by introducing a suitable Lyapunov functional.

## 2 Preliminaries

Let \( \{(x_i, v_i)\}_{i=1,\ldots,N} \) be solution to (1.1) under the initial conditions (1.4). In this section, we assume that the hypotheses of Theorem 1.1 are satisfied.

We now present some auxiliary lemmas that generalize and extend the analogous results in [35].
Lemma 2.1. For each \( v \in \mathbb{R}^d \) and \( T \geq 0 \), we have that
\[
\min_{j=1,\ldots,N} \min_{s \in [T-\bar{\tau},T]} \langle v_j(s), v \rangle \leq \langle v_i(t), v \rangle \leq \max_{j=1,\ldots,N} \max_{s \in [T-\bar{\tau},T]} \langle v_j(s), v \rangle,
\]
(2.1)
for all \( t \geq T - \bar{\tau} \) and \( i = 1,\ldots,N \).

Proof. First of all, we can note that the inequalities in (2.1) are satisfied for every \( t \in [T - \bar{\tau}, T] \). Now, let \( T \geq 0 \). Given a vector \( v \in \mathbb{R}^d \), we set
\[
M = \max_{j=1,\ldots,N} \max_{s \in [T-\bar{\tau},T]} \langle v_j(s), v \rangle.
\]

For all \( \epsilon > 0 \), let us define
\[
K^\epsilon := \left\{ t > T : \max_{i=1,\ldots,N} \langle v_i(s), v \rangle < M + \epsilon, \forall s \in [T, t) \right\}.
\]

By continuity, we have that \( K^\epsilon \neq \emptyset \). Thus, denoting with
\[
S^\epsilon := \sup K^\epsilon,
\]
it holds that \( S^\epsilon > T \).

We claim that \( S^\epsilon = +\infty \). Indeed, suppose by contradiction that \( S^\epsilon < +\infty \). Note that, by definition of \( S^\epsilon \), it turns out that
\[
\max_{i=1,\ldots,N} \langle v_i(t), v \rangle < M + \epsilon, \ \forall t \in (T, S^\epsilon),
\]
(2.2)
and
\[
\lim_{t \to S^\epsilon} \max_{i=1,\ldots,N} \langle v_i(t), v \rangle = M + \epsilon.
\]
(2.3)

For all \( i = 1,\ldots,N \) and \( t \in (T, S^\epsilon) \), we compute
\[
\frac{d}{dt} \langle v_i(t), v \rangle = \frac{1}{N-1} \sum_{j:j \neq i} \psi(|x_i(t) - x_j(t - \tau(t))|)(v_j(t - \tau(t)) - v_i(t), v).
\]
Notice that, being \( t \in (T, S^\epsilon) \), then \( t - \tau(t) \in (T - \bar{\tau}, S^\epsilon) \) and
\[
\langle v_j(t - \tau(t)), v \rangle < M + \epsilon, \ \forall j = 1,\ldots,N.
\]
(2.4)

Moreover, (2.2) implies that
\[
\langle v_i(t), v \rangle < M + \epsilon,
\]
so that
\[
M + \epsilon - \langle v_i(t), v \rangle \geq 0.
\]

Combining this last fact with (2.4) and by recalling that \( \psi \) is bounded, we can write
\[
\frac{d}{dt} \langle v_i(t), v \rangle \leq \frac{1}{N-1} \sum_{j:j \neq i} \psi(|x_i(t) - x_j(t - \tau(t))|)(M + \epsilon - \langle v_i(t), v \rangle)
\]
\[
\leq K(M + \epsilon - \langle v_i(t), v \rangle), \ \forall t \in (T, S^\epsilon).
Asymptotic flocking for the Cucker-Smale model with time variable time delays

Then, from the Gronwall’s inequality we get

$$\langle v_i(t), v_i(T) \rangle \leq e^{-K(t-T)} \langle v_i(T), v \rangle + K(M + \epsilon) \int_t^T e^{-K(t-s)} ds$$

$$= e^{-K(t-T)} \langle v_i(T), v \rangle + (M + \epsilon)(1 - e^{-K(t-T)})$$

$$\leq e^{-K(t-T)} M + M + \epsilon - Me^{-K(t-T)} - \epsilon e^{-K(t-T)}$$

$$= M + \epsilon - \epsilon e^{-K(t-T)}$$

$$\leq M + \epsilon - \epsilon e^{-K(S^\epsilon - T)},$$

for all $t \in (T, S').$ We have so proved that, $\forall i = 1, \ldots, N,$

$$\langle v_i(t), v \rangle \leq M + \epsilon - \epsilon e^{-K(S^\epsilon - T)}, \quad \forall t \in (T, S').$$

Thus, we get

$$\max_{i=1, \ldots, N} \langle v_i(t), v \rangle \leq M + \epsilon - \epsilon e^{-K(S^\epsilon - T)}, \quad \forall t \in (T, S'). \tag{2.5}$$

Letting $t \rightarrow S^\epsilon$ in (2.5), from (2.3) we have that

$$M + \epsilon \leq M + \epsilon - \epsilon e^{-K(S^\epsilon - T)} < M + \epsilon,$$

which is a contradiction. Thus, $S' = +\infty,$ which means that

$$\max_{i=1, \ldots, N} \langle v_i(t), v \rangle < M + \epsilon, \quad \forall t > T.$$

From the arbitrariness of $\epsilon$ we can conclude that

$$\max_{i=1, \ldots, N} \langle v_i(t), v \rangle \leq M, \quad \forall t > T,$$

from which

$$\langle v_i(t), v \rangle \leq M, \quad \forall t > T, \forall i = 1, \ldots, N,$$

which proves the second inequality in (2.1).

Now, to prove the other inequality, let $v \in \mathbb{R}^d$ and define

$$m = \min_{j=1, \ldots, N} \min_{s \in [T^- T]} \langle v_j(s), v \rangle.$$

Then, for all $i = 1, \ldots, N$ and $t > T,$ by applying the second inequality in (2.1) to the vector $-v \in \mathbb{R}^d$ we get

$$-\langle v_i(s), v \rangle = \langle v_i(t), -v \rangle \leq \max_{j=1, \ldots, N} \max_{s \in [T^- T]} \langle v_j(s), -v \rangle$$

$$= - \min_{j=1, \ldots, N} \min_{s \in [T^- T]} \langle v_j(s), v \rangle = -m,$$

from which

$$\langle v_i(s), v \rangle \geq m.$$

Thus, also the first inequality in (2.1) is fulfilled. $\square$

We now introduce some notation.
Definition 2.1. We define

\[
D_0 = \max_{i,j=1,\ldots,N} \max_{s,t \in [-\bar{\tau},0]} |v_i(s) - v_j(t)|,
\]

and in general, \(\forall n \in \mathbb{N},\)

\[
D_n := \max_{i,j=1,\ldots,N} \max_{s,t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]} |v_i(s) - v_j(t)|.
\]

Notice that inequality (1.7) can be written as

\[
d_V(t) \leq D_0 e^{-C(t-2\bar{\tau})}, \quad \forall t \geq -\bar{\tau}.
\]

Let us denote with \(N_0 := \mathbb{N} \cup \{0\}.\)

Lemma 2.2. For each \(n \in N_0\) we have that

\[
D_{n+1} \leq D_n.
\]  

(2.6)

Proof. Let \(n \in N_0.\) If \(D_{n+1} = 0,\) then of course

\[
D_n = \max_{i,j=1,\ldots,N} \max_{s,t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]} |v_i(s) - v_j(t)| \geq 0 = D_{n+1}.
\]

So, suppose that \(D_{n+1} > 0.\) Let \(i, j = 1, \ldots, N, t_1, t_2 \in [n\bar{\tau}, n\bar{\tau} + \bar{\tau}]\) be such that

\[
D_{n+1} = \max_{i,j=1,\ldots,N} |v_i(t_1) - v_j(t_2)|.
\]

We set

\[
v = \frac{v_i(t_1) - v_j(t_2)}{|v_i(t_1) - v_j(t_2)|}.
\]

Then \(v\) is a unit vector and, by using (2.1) with \(T = n\bar{\tau}\) and the Cauchy-Schwarz inequality, we have that

\[
D_{n+1} = \langle v_i(t_1) - v_j(t_2), v \rangle = \langle v_i(t_1), v \rangle - \langle v_j(t_2), v \rangle
\]

\[
\leq \max_{l,k=1,\ldots,N} \max_{s,t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]} \langle v_l(s), v \rangle - \min_{l,k=1,\ldots,N} \min_{s \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]} \langle v_l(s), v \rangle
\]

\[
\leq \max_{l,k=1,\ldots,N} \max_{s,t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]} \langle v_l(s) - v_k(t), v \rangle
\]

\[
= \max_{l,k=1,\ldots,N} \max_{s,t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]} |v_l(s) - v_k(t)| = D_n.
\]

With an analogous argument, one can find a bound on \(|v_i(t)|\), uniform with respect to \(t\) and \(i = 1, \ldots, N.\) Indeed, we have the following lemma.

Lemma 2.3. For every \(i = 1, \ldots, N,\) we have that

\[
|v_i(t)| \leq R_0^V, \quad \forall t \geq -\bar{\tau},
\]  

(2.7)

where

\[
R_0^V := \max_{i=1,\ldots,N} \max_{s \in [-\bar{\tau},0]} |v_i(s)|.
\]
Now, assume $t \geq -\bar{\tau}$. Note that, if $|v_i(t)| = 0$, then trivially $R^0_V \geq 0 = |v_i(t)|$. So, we can assume $|v_i(t)| > 0$. We define

$$v = \frac{v_i(t)}{|v_i(t)|}. $$

Then, $v$ is a unit vector for which we can write

$$|v_i(t)| = \langle v_i(t), v \rangle.$$ 

Since $t \geq -\bar{\tau}$, we can apply (2.7) for $T = 0$ and we get

$$|v_i(t)| \leq \max_{j=1,\ldots,N} \max_{s \in [-\bar{\tau},0]} \langle v_j(s), v \rangle \leq \max_{j=1,\ldots,N} \max_{s \in [-\bar{\tau},0]} |v_j(s)| |v| = \max_{j=1,\ldots,N} \max_{s \in [-\bar{\tau},0]} |v_j(s)| = R^0_V.$$ 

Thus, (2.7) is fulfilled.

**Lemma 2.4.** For every $i, j = 1, \ldots, N$, we get

$$|x_i(t - \tau(t)) - x_j(t)| \leq 2\bar{\tau}R^0_V + 4M^0_X + d_X(t - \bar{\tau}), \; \forall t \geq 0, \tag{2.8}$$

where

$$M^0_X := \max_{i=1,\ldots,N} \max_{s \in [-\bar{\tau},0]} |x_i(s)|.$$ 

**Proof.** Given $i, j = 1, \ldots, N$ and $t \geq 0$, we have

$$|x_i(t - \tau(t)) - x_j(t)| \leq |x_i(t - \tau(t)) - x_i(t - \bar{\tau})| + |x_i(t - \bar{\tau}) - x_j(t - \bar{\tau})| + |x_j(t - \bar{\tau}) - x_j(t)| \leq |x_i(t - \tau(t)) - x_i(t - \bar{\tau})| + d_X(t - \bar{\tau}) + |x_j(t - \bar{\tau}) - x_j(t)|. \tag{2.9}$$

Now, assume $t > \bar{\tau}$. Then both $t - \bar{\tau}, t - \tau(t) > 0$ and from inequalities (1.3) and (2.7) we get

$$|x_i(t - \tau(t)) - x_i(t - \bar{\tau})| = \left| \int_{t-\bar{\tau}}^{t-\tau(t)} v_i(s) \, ds \right| \leq \int_{t-\bar{\tau}}^{t-\tau(t)} |v_i(s)| \, ds \leq R^0_V (t - \tau(t) - t + \bar{\tau}) \leq \bar{\tau}R^0_V,$$

and

$$|x_j(t - \bar{\tau}) - x_j(t)| = \left| - \int_{t-\bar{\tau}}^{t} v_j(s) \, ds \right| \leq \int_{t-\bar{\tau}}^{t} |v_j(s)| \, ds \leq \bar{\tau}R^0_V.$$ 

Thus, (2.9) becomes

$$|x_i(t - \tau(t)) - x_j(t)| \leq 2\bar{\tau}R^0_V + d_X(t - \bar{\tau}).$$

On the contrary, assume that $t \leq \bar{\tau}$. Then $t - \bar{\tau} \leq 0$ and from (1.3) and (2.7) we get

$$|x_j(t - \bar{\tau}) - x_j(t)| = \left| x_j(t - \bar{\tau}) - x_j(0) - \int_{0}^{t} v_j(s) \, ds \right| \leq |x_j(t - \bar{\tau}) - x_j(0)| + \int_{0}^{t} |v_j(s)| \, ds \leq 2M^0_X + tR^0_V \leq 2M^0_X + \bar{\tau}R^0_V.$$ 

7
Note that our assumption, \( t \leq \bar{\tau} \), does not imply that \( t - \tau(t) \leq 0 \). So we can distinguish two cases.

If \( t - \tau(t) > 0 \), then
\[
|x_i(t - \tau(t)) - x_i(t - \bar{\tau})| = |x_i(0) + \int_0^{t - \tau(t)} v_i(s) \, ds - x_i(t - \bar{\tau})| \\
\leq |x_i(0) - x_i(t - \bar{\tau})| + \int_0^{t - \tau(t)} |v_i(s)| \, ds \\
\leq 2M_X^0 + (t - \tau(t))R_\nu^0 \leq 2M_X^0 + R_\nu^0,
\]
and (2.8) becomes
\[
|x_i(t - \tau(t)) - x_j(t)| \leq 4M_X^0 + 2\bar{\tau}R_\nu^0 + d_X(t - \bar{\tau}).
\]

On the other hand, if \( t - \tau(t) \leq 0 \), we have
\[
|x_i(t - \tau(t)) - x_i(t - \bar{\tau})| \leq 2M_X^0,
\]
and we can write
\[
|x_i(t - \tau(t)) - x_j(t)| \leq 4M_X^0 + \bar{\tau}R_\nu^0 + d_X(t - \bar{\tau}).
\]

We have so proved that, in all cases,
\[
|x_i(t - \tau(t)) - x_j(t)| \leq 4M_X^0 + 2\bar{\tau}R_\nu^0 + d_X(t - \bar{\tau}),
\]
which proves (2.8). \( \square \)

In the following, given \( t \geq -\bar{\tau}, i, j = 1, \ldots, N \) and a vector \( v \in \mathbb{R}^d \), we shall denote with
\[
d^{(ij)}_v(t) := \langle v_i(t) - v_j(t), v \rangle.
\]

The next lemma is analogous to Lemma 3.4 in [35]. We give the proof for the reader’s convenience.

**Lemma 2.5.** For all \( i, j = 1, \ldots, N \), unit vector \( v \in \mathbb{R}^d \) and \( n \in \mathbb{N}_0 \), we have that
\[
d^{(ij)}_v(t) \leq e^{-K(t-t_0)}d^{(ij)}_v(t_0) + (1 - e^{-K(t-t_0)})D_n, \tag{2.10}
\]
for all \( t \geq t_0 \geq n\bar{\tau} \). Moreover, for each \( n \in \mathbb{N}_0 \) it holds
\[
D_{n+1} \leq e^{-K\bar{\tau}}d_{\nu}(n\bar{\tau}) + (1 - e^{-K\bar{\tau}})D_n. \tag{2.11}
\]

**Proof.** Given \( n \in \mathbb{N}_0 \), for each \( v \in \mathbb{R}^d \) unit vector, let denote with
\[
M = \max_{l = 1, \ldots, N} \max_{s \in [n\bar{\tau}, n\bar{\tau}]} \langle v_l(s), v \rangle,
\]
\[
m = \min_{l = 1, \ldots, N} \min_{s \in [n\bar{\tau}, n\bar{\tau}]} \langle v_l(s), v \rangle.
\]

Then \( M - m \leq D_n \). We claim that, for all \( i, j = 1, \ldots, N, t \geq t_0 \geq n\bar{\tau} \),
\[
\langle v_i(t), v \rangle \leq e^{-K(t-t_0)}(v_i(t_0), v) + (1 - e^{-K(t-t_0)})M,
\]
\[
\langle v_j(t), v \rangle \geq e^{-K(t-t_0)}(v_j(t_0), v) + (1 - e^{-K(t-t_0)})m. \tag{2.12}
\]
So, fix \(i, j = 1, \ldots, N\) and \(t \geq t_0 \geq n\bar{\tau}\). Then, being \(t \geq 0\), we have

\[
\frac{d}{dt} \langle v_i(t), v \rangle = \sum_{l \neq i} a_{il}(t) (\langle v_l(t - \tau(t)), v \rangle - \langle v_i(t), v \rangle)
\]

(2.13)

We recall that \(a_{il}(t) = \frac{1}{N} \psi(|x_i(t) - x_l(t - \tau(t))|)\). Thus, being \(\psi\) a bounded function, we can write \(a_{il}(t) \leq \frac{1}{N} K\). Furthermore, \(t \geq n\bar{\tau}\), which implies that \(t - \tau(t) \geq n\bar{\tau} - \bar{\tau}\). Then, by virtue of (2.1), we have that

\[
m \leq \langle v_k(t - \tau(t)), v \rangle \leq M, \quad m \leq \langle v_k(t), v \rangle \leq M, \quad \forall k = 1, \ldots, N.
\]

So, combining all these facts, (2.13) becomes

\[
\frac{d}{dt} \langle v_i(t), v \rangle = \sum_{l \neq i} a_{il}(t) (\langle v_l(t - \tau(t)), v \rangle - M + M - \langle v_i(t), v \rangle) \leq \sum_{l \neq i} a_{il}(t) (M - \langle v_i(t), v \rangle) \leq \frac{K}{N - 1} \sum_{l \neq i} (M - \langle v_i(t), v \rangle) = K (M - \langle v_i(t), v \rangle).
\]

Then, from the Gronwall’s inequality with \(t \geq t_0\) we get

\[
\langle v_i(t), v \rangle \leq e^{-\int_{t_0}^{t} K ds} \langle v_i(t_0), v \rangle + \int_{t_0}^{t} K M e^{-\int_{s_0}^{s} K ds} \langle v_i(t_0), v \rangle + M e^{-K(t-t_0)} (e^{K(t-t_0)} - 1) = e^{-K(t-t_0)} \langle v_i(t_0), v \rangle + (1 - e^{-K(t-t_0)}) M.
\]

Hence, it holds

\[
\langle v_i(t), v \rangle \leq e^{-K(t-t_0)} \langle v_i(t_0), v \rangle + (1 - e^{-K(t-t_0)}) M, \quad (2.14)
\]

for every \(i = 1, \ldots, N, t \geq t_0 \geq n\bar{\tau}\) and unit vector \(v \in \mathbb{R}^d\), which proves the first inequality in (2.12).

Now, to prove the second inequality in (2.12), let \(j = 1, \ldots, N, t \geq t_0 \geq n\bar{\tau}\) and a unit vector \(v \in \mathbb{R}^d\). Then, we can apply (2.14) to the unit vector \(-v \in \mathbb{R}^d\) and we get

\[
\langle v_j(t), -v \rangle \leq e^{-K(t-t_0)} \langle v_j(t_0), -v \rangle + (1 - e^{-K(t-t_0)}) \left( \max_{l=1,\ldots,N} \max_{s \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]} \langle v_l(s), -v \rangle \right),
\]

from which

\[
\langle v_j(t), v \rangle \geq e^{-K(t-t_0)} \langle v_j(t_0), v \rangle + (1 - e^{-K(t-t_0)}) \left( -\max_{l=1,\ldots,N} \max_{s \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]} \langle v_l(s), -v \rangle \right) = e^{-K(t-t_0)} \langle v_j(t_0), v \rangle + (1 - e^{-K(t-t_0)}) m.
\]
Therefore (2.12) holds true.
Now, from (2.12), for each \( i, j = 1, \ldots, N \), \( v \in \mathbb{R}^d \) unit vector and \( t \geq t_0 \geq n\bar{\tau} \), we have

\[
\begin{align*}
\bar{d}^{(ij)v}(t) &= \langle v_i(t) - v_j(t), v \rangle = \langle v_i(t)(t), v \rangle - \langle v_j(t), v \rangle \\
&\leq e^{-K(t-t_0)} \langle v_i(t_0), v \rangle + (1 - e^{-K(t-t_0)})M - e^{-K(t-t_0)} \langle v_j(t_0), v \rangle - (1 - e^{-K(t-t_0)})m \\
&= e^{-K(t-t_0)} \langle v_i(t_0) - v_j(t_0), v \rangle + (1 - e^{-K(t-t_0)})(M - m) \\
&= e^{-K(t-t_0)} \bar{d}^{(ij)v}(t_0) + (1 - e^{-K(t-t_0)})(M - m).
\end{align*}
\]

Then, by recalling that \( M - m \leq D_n \), we finally get

\[
\bar{d}^{(ij)v}(t) \leq e^{-K(t-t_0)} \bar{d}^{(ij)v}(t_0) + (1 - e^{-K(t-t_0)})D_n,
\]

which proves (2.10).
Finally, we prove (2.11). Let \( i, j = 1, \ldots, N \) and \( t_1, t_2 \in [n\bar{\tau}, n\bar{\tau} + \bar{\tau}] \) be such that

\[
D_{n+1} = |v_i(t_1) - v_j(t_2)|.
\]

Note that, if \( D_{n+1} = 0 \), then trivially

\[
e^{-K\bar{\tau}}d_v(n\bar{\tau}) + (1 - e^{-K\bar{\tau}})D_n \geq 0 = D_{n+1}.
\]

So we can assume \( D_{n+1} > 0 \) and we define the unit vector

\[
v = \frac{v_i(t_1) - v_j(t_2)}{|v_i(t_1) - v_j(t_2)|}.
\]

By applying (2.12) with \( t_0 = n\bar{\tau} \leq t_1, t_2 \), we get

\[
\langle v_i(t_1), v \rangle \leq e^{-K(t_1-n\bar{\tau})} \langle v_i(n\bar{\tau}), v \rangle + (1 - e^{-K(t_1-n\bar{\tau})})M \\
= e^{-K(t_1-n\bar{\tau})} \langle v_i(n\bar{\tau}), v \rangle + (1 - e^{-K(t_1-n\bar{\tau})})M \\
= e^{-K\bar{\tau}} \langle v_i(n\bar{\tau}), v \rangle + (1 - e^{-K\bar{\tau}})M,
\]

where we used the fact that \( t_1 \leq n\bar{\tau} + \bar{\tau} \) and \( \langle v_i(n\bar{\tau}), v \rangle - M \leq 0 \), and

\[
\langle v_j(t_2), v \rangle \geq e^{-K(t_2-n\bar{\tau})} \langle v_j(n\bar{\tau}), v \rangle + (1 - e^{-K(t_2-n\bar{\tau})})m \\
= e^{-K(t_2-n\bar{\tau})} \langle v_j(n\bar{\tau}), v \rangle - (1 - e^{-K(t_2-n\bar{\tau})})m \\
\geq e^{-K\bar{\tau}} \langle v_j(n\bar{\tau}), v \rangle - (1 - e^{-K\bar{\tau}})m,
\]

where we used the fact that \( t_2 \leq n\bar{\tau} + \bar{\tau} \) and \( \langle v_j(n\bar{\tau}), v \rangle - m \geq 0 \).
As a consequence, it holds
\[ D_{n+1} = \langle v_i(t_1) - v_j(t_2), v \rangle = \langle v_i(t_1), v \rangle - \langle v_j(t_2), v \rangle \leq e^{-K_\tau} \langle v_i(n\bar{\tau}), v \rangle + \langle v_j(n\bar{\tau}), v \rangle - (1 - e^{-K_\tau})m \]
\[ = e^{-K_\tau} \langle v_i(n\bar{\tau}) - v_j(n\bar{\tau}), v \rangle + (1 - e^{-K_\tau})(M - m) \leq e^{-K_\tau} |v_i(n\bar{\tau}) - v_j(n\bar{\tau})| |v| + (1 - e^{-K_\tau})(M - m) \leq e^{-K_\tau} d_V(n\bar{\tau}) + (1 - e^{-K_\tau})D_n, \]
which concludes our proof.

Now, we give the following definition.

**Definition 2.2.** We define
\[ \phi(t) := \min \left\{ e^{-K_\tau}, \frac{e^{-2K_\tau}}{\bar{\tau}} \right\}, \]
where
\[ \psi_t = \min \left\{ \psi(r) : r \in \left[ 0, 2\bar{\tau}R_0^2 + 4M_0 + \max_{s \in [-\bar{\tau}, \bar{\tau}]} d_X(s) \right] \}, \]
for all \( t \geq -\bar{\tau} \).

By definition, being \( \psi \) a positive function, we have that \( \psi_t > 0 \), for all \( t \geq -\bar{\tau} \). Thus, the function \( \phi \) is positive too.

**Remark 2.6.** Let us note that from estimate (2.8), for all \( t \geq 0 \) and \( i, j = 1, \ldots, N \), it holds that
\[ \psi(|x_i(t') - x_j(t' - \tau(t'))|) \geq \psi_{t-\bar{\tau}}, \]
from which
\[ \psi(|x_i(t') - x_j(t' - \tau(t'))|) \geq e^{K_\tau} \phi(t - \bar{\tau}). \] \hfill (2.15)

**Lemma 2.7.** For each integer \( n \geq 2 \), we have that
\[ D_{n+1} \leq \left( 1 - e^{-K_\tau} \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} \phi(s) ds \right) D_{n-2}. \] \hfill (2.16)

**Proof.** We first show that, for each \( n \geq 2 \),
\[ d_V(n\bar{\tau}) \leq \left( 1 - \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} \phi(s) ds \right) D_{n-2}. \] \hfill (2.17)

To this aim, let \( n \geq 2 \). Note that, if \( d_V(n\bar{\tau}) = 0 \), by definition of \( \phi \) we have that
\[ \left( 1 - \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} \phi(s) ds \right) D_{n-2} \geq \left( 1 - \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} e^{-2K_\tau} \frac{1}{\bar{\tau}} ds \right) D_{n-2} \]
\[ = \left( 1 - \frac{e^{-2K_\tau}}{\bar{\tau}} (n\bar{\tau} - \bar{\tau} - n\bar{\tau} + 2\bar{\tau}) \right) D_{n-2} \]
\[ = \left( 1 - e^{-2K_\tau} \right) D_{n-2} \geq 0 = d_V(n\bar{\tau}). \]
Asymptotic flocking for the Cucker-Smale model with time variable time delays

So we can assume \( d_V(n\bar{\tau}) > 0 \). Moreover, let \( i, j = 1, \ldots, N \) be such that

\[
d_V(n\bar{\tau}) = |v_i(n\bar{\tau}) - v_j(n\bar{\tau})|.
\]

We set

\[
v = \frac{v_i(n\bar{\tau}) - v_j(n\bar{\tau})}{|v_i(n\bar{\tau}) - v_j(n\bar{\tau})|}.
\]

Then \( v \) is a unit vector for which we can write

\[
d_V(n\bar{\tau}) = \langle v_i(n\bar{\tau}) - v_j(n\bar{\tau}), v \rangle = d_V^{(ij)}(n\bar{\tau}).
\]

At this point, we distinguish two cases.

**Case I.** Assume that there exists \( t_0 \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}] \) such that \( d_V^{(ij)}(t_0) < 0 \). Note that

\[
(1 - \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau}} \phi(s) ds) \geq 1 - e^{-2K\bar{\tau}}.
\]

Then, by using (2.10) with \( n\bar{\tau} \geq t_0 \geq n\bar{\tau} - 2\bar{\tau} \), we have

\[
d_V^{(ij)}(n\bar{\tau}) \leq e^{-K(n\bar{\tau} - t_0)} d_V^{(ij)}(t_0) + (1 - e^{-K(n\bar{\tau} - t_0)}) D_{n-2} < (1 - e^{-K(n\bar{\tau} - t_0)}) D_{n-2} \leq (1 - e^{-2K\bar{\tau}}) D_{n-2} \leq \left(1 - \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau}} \phi(s) ds\right) D_{n-2}.
\]

**Case II.** Assume that \( d_V^{(ij)}(t) \geq 0 \), for every \( t \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}] \). We set

\[
M = \max_{l=1,\ldots,N, \bar{s} \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau} - \bar{\tau}]} \max_{s \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau} - \bar{\tau}]} \langle v_l(s), v \rangle,
\]

\[
m = \min_{l=1,\ldots,N, \bar{s} \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau} - \bar{\tau}]} \min_{s \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau} - \bar{\tau}]} \langle v_l(s), v \rangle.
\]

Then, \( M - m \leq D_{n-1} \). Notice that, from (2.15), for each \( l, k = 1, \ldots, N \) and \( t \geq 0 \),

\[
a_{lk}(t) \geq e^{-K\bar{\tau}} \phi(t - \bar{\tau}) \frac{\bar{\tau} - \bar{\tau}}{N - 1}.
\]

(2.18)

Thus, for every \( t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}] \), it comes that

\[
\frac{d}{dt} d_V^{(ij)}(t) = \sum_{l \neq i} a_{il}(t) \langle v_l(t - \tau(t)) - v_i(t), v \rangle + \sum_{l \neq i} a_{jl}(t) \langle v_j(t) - v_l(t - \tau(t)), v \rangle
\]

\[
= \sum_{l \neq i} a_{il}(t) ((v_l(t - \tau(t)), v) - M + M - \langle v_i(t), v \rangle + \sum_{l \neq i} a_{jl}(t) ((v_j(t), v) - m + m - \langle v_l(t - \tau(t), v) \rangle
\]

\[
:= S_1 + S_2.
\]

We recall that \( \psi \) is bounded and that, from (2.1),

\[
m \leq \langle v_k(s), v \rangle \leq M, \quad \forall s \geq n\bar{\tau} - 2\bar{\tau}, \forall k = 1, \ldots, N.
\]

12
Combining these facts with (2.18), for every $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$, it holds that $t, t - \tau(t) \geq n\bar{\tau} - 2\bar{\tau}$ and we can write
\[
S_1 = \sum_{l:l \neq i} a_{il}(t)((v_l(t - \tau(t)), v_i) - M) + \sum_{l:l \neq i} a_{il}(t)(M - (v_l(t), v_i))
\]
\[
\leq \frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} \sum_{l:l \neq i} ((v_l(t - \bar{\tau}(t)), v_i) - M) + \frac{K}{N - 1} \sum_{l:l \neq i} (M - (v_l(t), v_i))
\]
\[
= \frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} \sum_{l:l \neq i} ((v_l(t - \bar{\tau}(t)), v_i) - M) + K(M - (v_l(t), v_i)),
\]
and
\[
S_2 = \sum_{l:l \neq j} a_{jl}(t)((v_j(t), v_i) - m) + \sum_{l:l \neq j} a_{jl}(t)(m - (v_l(t - \tau(t)), v_i))
\]
\[
\leq \frac{K}{N - 1} \sum_{l:l \neq j} ((v_j(t), v_i) - m) + \frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} \sum_{l:l \neq j} (m - (v_l(t - \tau(t)), v_i))
\]
\[
= K((v_j(t), v_i) - m) + \frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} \sum_{l:l \neq j} (m - (v_l(t - \tau(t)), v_i)).
\]

Hence, we get
\[
S_1 + S_2 \leq K(M - (v_l(t), v_i) + (v_j(t), v_i) - m)
\]
\[
+ e^{K\bar{\tau}\phi(t - \bar{\tau})} \sum_{l:l \neq i,j} ((v_l(t - \bar{\tau}(t)), v_i) - M + m - (v_l(t - \tau(t)), v_i))
\]
\[
+ \frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} (v_j(t - \bar{\tau}(t)), v_i) - M + m - (v_l(t - \tau(t)), v_i))
\]
\[
= K(M - m - d_V^{(ij)}(t)) + \frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} (N - 2)(m - M)
\]
\[
+ \frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} (m - M - d_V^{(ij)}(t - \tau(t))).
\]

Note that, being $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$, it holds that $t - \tau(t) \in [n\bar{\tau} - 2\bar{\tau}, n\bar{\tau}]$. Therefore, from our assumption, we have $d_V^{(ij)}(t - \tau(t)) \geq 0$, from which follows that
\[
\frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} (m - M - d_V^{(ij)}(t - \tau(t)) \leq \frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} (m - M).
\]

Thus, taking into account of the fact that $M - m \leq D_{n-1}$, we get
\[
\frac{d}{dt} d_V^{(ij)}(t) \leq K(M - m - d_V^{(ij)}(t)) + e^{K\bar{\tau}\phi(t - \bar{\tau})}(N - 2)(m - M) + \frac{e^{K\bar{\tau}\phi(t - \bar{\tau})}}{N - 1} (m - M)
\]
\[
= K(M - m - d_V^{(ij)}(t)) + e^{K\bar{\tau}\phi(t - \tau)}(m - M)
\]
\[
= (K - e^{K\bar{\tau}\phi(t - \tau)})(M - m) - Kd_V^{(ij)}(t)
\]
\[
\leq (K - e^{K\bar{\tau}\phi(t - \tau)})D_{n-1} - Kd_V^{(ij)}(t),
\]

13
for every $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$. Then, from the Gronwall’s inequality, for every $t \in [n\bar{\tau} - \bar{\tau}, n\bar{\tau}]$, we have

$$d_V^{(ij)}(t) \leq e^{-\mu_{n\bar{\tau} - \bar{\tau}}} K d_V^{(ij)}(n\bar{\tau} - \bar{\tau}) + D_{n-1} \int_{n\bar{\tau} - \bar{\tau}}^{t} (K - e^{K\tau}\phi(s - \bar{\tau}))e^{-\int_{n\bar{\tau} - \bar{\tau}}^{s} K dv - \int_{n\bar{\tau} - \bar{\tau}}^{s} K dv} ds$$

$$= e^{-K(t-n\bar{\tau} + \bar{\tau})} d_V^{(ij)}(n\bar{\tau} - \bar{\tau}) + D_{n-1} \int_{n\bar{\tau} - \bar{\tau}}^{t} (K - e^{K\tau}\phi(s - \bar{\tau}) e^{-K(t-s)} ds$$

$$= e^{-K(t-n\bar{\tau} + \bar{\tau})} d_V^{(ij)}(n\bar{\tau} - \bar{\tau}) + D_{n-1} \left(1 - e^{-K(t-n\bar{\tau} + \bar{\tau})}e^{-K\bar{\tau}} ds + D_{n-1} \int_{n\bar{\tau} - \bar{\tau}}^{t} e^{-K(t-s)} \phi(s - \bar{\tau}) ds \right)$$

In particular, for $t = n\bar{\tau}$ it holds

$$d_V^{(ij)}(n\bar{\tau} - \bar{\tau}) \leq D_{n-1} \left(1 - e^{-K(t-n\bar{\tau} + \bar{\tau})}e^{-K\bar{\tau}} ds + D_{n-1} \int_{n\bar{\tau} - \bar{\tau}}^{t} e^{-K(t-s)} \phi(s - \bar{\tau}) ds \right)$$

Notice that $d_V^{(ij)}(n\bar{\tau} - \bar{\tau}) \leq D_{n-1}$ and that

$$e^{K\tau} \int_{n\bar{\tau} - \bar{\tau}}^{n\tau} \phi(s - \bar{\tau}) ds \geq \int_{n\bar{\tau} - \bar{\tau}}^{n\tau} \phi(s - \bar{\tau}) ds.$$

So we can write

$$d_V^{(ij)}(n\bar{\tau}) \leq e^{-K\tau} D_{n-1} + D_{n-1} \left(1 - e^{-K\tau} - \int_{n\bar{\tau} - \bar{\tau}}^{n\tau} \phi(s - \bar{\tau}) ds \right)$$

$$= D_{n-1} \left(1 - \int_{n\bar{\tau} - \bar{\tau}}^{n\tau} \phi(s - \bar{\tau}) ds \right).$$

Then, with a change of variable, we get

$$d_V^{(ij)}(n\bar{\tau}) \leq D_{n-1} \left(1 - \int_{n\bar{\tau} - \bar{\tau}}^{n\tau} \phi(s) ds \right),$$

and, being $D_{n-1} \leq D_{n-2}$, we can conclude that

$$d_V^{(ij)}(n\bar{\tau}) \leq \left(1 - \int_{n\bar{\tau} - 2\bar{\tau}}^{n\tau - \bar{\tau}} \phi(s) ds \right) D_{n-2}.$$

Therefore, (2.17) holds true.

Now, we are able to prove (2.16). Indeed, for each $n \geq 2$, from (2.11) and (2.16), it immediately follows that

$$D_{n+1} \leq e^{-K\tau} D_V(n\bar{\tau}) + (1 - e^{-K\tau}) D_n$$

$$\leq e^{-K\tau} \left(1 - \int_{n\tau - 2\bar{\tau}}^{n\tau - \bar{\tau}} \phi(s) ds \right) D_{n-2} + (1 - e^{-K\tau}) D_{n-2}$$

$$= \left(1 - e^{-K\tau} \int_{n\tau - 2\bar{\tau}}^{n\tau - \bar{\tau}} \phi(s) ds \right) D_{n-2}.$$
3 Proof of Theorem 1.1

Proof. Let \(\{(x_i, v_i)\}_{i=1,...,N}\) be solution to (1.1) under the initial conditions (1.3). Following Rodriguez Cartabia [35], we introduce the function \(D : [-\bar{\tau}, \infty) \to [0, \infty)\), defined as

\[
D(t) := \begin{cases} 
D_0, & t \in [-\bar{\tau}, 2\bar{\tau}], \\
D(n\bar{\tau}) \left(1 - e^{-K\bar{\tau}} \int_{n\bar{\tau}}^{t} \phi(s)ds \right)^{1/2}, & t \in (n\bar{\tau}, n\bar{\tau} + \bar{\tau}], \ n \geq 2.
\end{cases}
\]

By construction, \(D\) is continuous and nonincreasing. Moreover, we claim that

\[ D_n \leq D(t), \tag{3.1} \]

for all \(n \in \mathbb{N}_0\) and \(t \in [-\bar{\tau}, n\bar{\tau}]\). To prove this, we first show that, for each \(n \geq 3\),

\[
1 - e^{-K\bar{\tau}} \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} \phi(s)ds \leq \frac{D(n\bar{\tau} + \bar{\tau})}{D(n\bar{\tau} - 2\bar{\tau})}. \tag{3.2}
\]

So, let \(n \geq 3\). We split

\[
1 - e^{-K\bar{\tau}} \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} \phi(s)ds = \left(1 - e^{-K\bar{\tau}} \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} \phi(s)ds \right)^{1/2} \left(1 - e^{-K\bar{\tau}} \int_{n\bar{\tau} - \bar{\tau}}^{n\bar{\tau}} \phi(s)ds \right)^{1/2} \left(1 - e^{-K\bar{\tau}} \int_{n\bar{\tau}}^{n\bar{\tau} + \bar{\tau}} \phi(s)ds \right)^{1/2}.
\]

Now, it is easy to see that \(\phi\) is a nonincreasing function. Thus, for each \(m \geq n\),

\[
\int_{m\bar{\tau} - 2\bar{\tau}}^{m\bar{\tau} - \bar{\tau}} \phi(s)ds \leq \int_{m\bar{\tau} - \bar{\tau}}^{m\bar{\tau}} \phi(s)ds.
\]

So we can write

\[
1 - e^{-K\bar{\tau}} \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} \phi(s)ds \leq \left(1 - e^{-K\bar{\tau}} \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} \phi(s)ds \right)^{1/2} \left(1 - e^{-K\bar{\tau}} \int_{n\bar{\tau} - \bar{\tau}}^{n\bar{\tau}} \phi(s)ds \right)^{1/2} \left(1 - e^{-K\bar{\tau}} \int_{n\bar{\tau}}^{n\bar{\tau} + \bar{\tau}} \phi(s)ds \right)^{1/2}.
\]

from which (3.2) holds true.

At this point, we are able to prove (3.1). By induction, if \(n \leq 2\), from Lemma 2.2 we can immediately say that

\[ D_n \leq D_0 = D(t), \]

for all \(t \in [-\bar{\tau}, 2\bar{\tau}]\). So we can assume that (3.1) holds for each \(2 < m \leq n\) and prove it for \(n + 1\). From the induction hypothesis and by using again Lemma 2.2 we have

\[ D_{n+1} \leq D_n \leq D(t), \]

for all \(t \in [-\bar{\tau}, n\bar{\tau}]\). On the other hand, for all \(t \in (n\bar{\tau}, n\bar{\tau} + \bar{\tau}]\), being \(n > 2\), from (2.10) we get

\[
D_{n+1} \leq \left(1 - e^{-K\bar{\tau}} \int_{n\bar{\tau} - 2\bar{\tau}}^{n\bar{\tau} - \bar{\tau}} \phi(s)ds \right) D_{n-2}.
\]

15
Asymptotic flocking for the Cucker-Smale model with time variable time delays

From the induction hypothesis or from the base case, $D_{n-2} \leq D(t)$, for each $t \in [-\bar{\tau}, n\bar{\tau} - 2\bar{\tau}]$. So, in particular, $D_{n-2} \leq D(n\bar{\tau} - 2\bar{\tau})$. Therefore, combining this with (3.2) and with the fact that $D$ is nonincreasing, we have that

$$
D_{n+1} \leq \frac{D(n\bar{\tau} + \bar{\tau})}{D(n\bar{\tau} - 2\bar{\tau})} D_{n-2} \leq \frac{D(n\bar{\tau} + \bar{\tau})}{D(n\bar{\tau} - 2\bar{\tau})} D(n\bar{\tau} - 2\bar{\tau}) = D(n\bar{\tau} + \bar{\tau}) \leq D(t),
$$

for all $t \in (n\bar{\tau}, n\bar{\tau} + \bar{\tau})$, which proves (3.1).

Now, notice that, for almost all time

$$
\left| \frac{d}{dt} d_X(t) \right| \leq d_Y(t).
$$

To see this, let $i, j = 1, \ldots, N$ be such that $d_X(t) = |x_i(t) - x_j(t)|$. Obviously, if $|x_i(t) - x_j(t)| = 0$, then

$$
\left| \frac{d}{dt} d_X(t) \right| = 0 \leq d_Y(t).
$$

So we can assume $|x_i(t) - x_j(t)| > 0$. Notice that

$$
\frac{d}{dt} (d_X(t))^2 = \frac{d}{dt} |x_i(t) - x_j(t)|^2 = 2 |x_i(t) - x_j(t)| \frac{d}{dt} |x_i(t) - x_j(t)|
$$

$$
= 2 |x_i(t) - x_j(t)| \frac{d}{dt} d_X(t),
$$

with $|x_i(t) - x_j(t)| > 0$, since otherwise $d_X(\cdot)$ wouldn’t be differentiable at $t$. Also,

$$
\frac{d}{dt} (d_X(t))^2 = 2 \langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle,
$$

so that

$$
|x_i(t) - x_j(t)| \frac{d}{dt} d_X(t) = \langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle.
$$

Thus,

$$
|x_i(t) - x_j(t)| \left| \frac{d}{dt} d_X(t) \right| = |\langle v_i(t) - v_j(t), x_i(t) - x_j(t) \rangle| \leq |v_i(t) - v_j(t)||x_i(t) - x_j(t)|,
$$

from which, dividing by $|x_i(t) - x_j(t)|$, we get

$$
\left| \frac{d}{dt} d_X(t) \right| \leq |v_i(t) - v_j(t)| \leq d_Y(t).
$$

Therefore, for almost all time

$$
\frac{d}{dt} \max_{s \in [-\bar{\tau}, t]} d_X(s) \leq \left| \frac{d}{dt} d_X(t) \right| \leq d_Y(t).
$$  (3.3)
Next, let $\mathcal{L} : [-\bar{\tau}, \infty) \to [0, \infty)$ be the function given by
\[
\mathcal{L}(t) := D(t) + \frac{e^{-K\bar{\tau}}}{3} \int_0^{2\bar{\tau}} R_0^0 + 4M_0^0 + \max_{s \in [-\bar{\tau}, t]} d_X(s) \min \left\{ e^{-K\tau} \min_{\sigma \in [0, r]} \psi(\sigma) \frac{e^{-2K\tau}}{\bar{\tau}} \right\} dr,
\]
for all $t \geq -\bar{\tau}$. By definition, $\mathcal{L}$ is continuous. In addition, for each $n \geq 2$ and for a.e. $t \in (n\bar{\tau}, n\bar{\tau} + \bar{\tau})$, we have that
\[
\frac{d}{dt} \mathcal{L}(t) = \frac{d}{dt} D(t) + \frac{e^{-K\bar{\tau}}}{3} \min \left\{ e^{-K\tau} \psi_t, \frac{e^{-2K\tau}}{\bar{\tau}} \right\} \frac{d}{dt} \max_{s \in [-\bar{\tau}, t]} d_X(s)
\]
and from (3.3) we get
\[
\frac{d}{dt} \mathcal{L}(t) \leq \frac{d}{dt} D(t) + \frac{e^{-K\bar{\tau}}}{3} \phi(t) d\nu(t).
\]
Now, for a.e. $t \in (n\bar{\tau}, n\bar{\tau} + \bar{\tau})$, with $n \geq 2$, we compute
\[
\frac{d}{dt} D(t) = -\frac{1}{3} D(n\bar{\tau}) \left( 1 - e^{-K\bar{\tau}} \int_{n\bar{\tau}}^t \phi(s) ds \right)^{\frac{3}{2}} e^{-K\bar{\tau}} \phi(t).
\]
Thus, for each $n \geq 2$ and for a.e. $t \in (n\bar{\tau}, n\bar{\tau} + \bar{\tau})$,
\[
\frac{d}{dt} \mathcal{L}(t) \leq \frac{e^{-K\bar{\tau}}}{3} \phi(t) \left( d\nu(t) - \frac{D(n\bar{\tau})}{\left( 1 - e^{-K\bar{\tau}} \int_{n\bar{\tau}}^t \phi(s) ds \right)^{\frac{3}{2}}} \right)
\]
\[
\leq \frac{e^{-K\bar{\tau}}}{3} \phi(t) (d\nu(t) - D(n\bar{\tau})).
\]
Lastly, we can note that $d\nu(t) \leq D(n\bar{\tau})$, since $d\nu(t) \leq D_{n+1}$ and $D_{n+1} \leq D(n\bar{\tau})$ from inequality (3.1). Then, we get
\[
\frac{d}{dt} \mathcal{L}(t) \leq 0,
\]
for a.e. $t \in (n\bar{\tau}, n\bar{\tau} + \bar{\tau})$ and for each $n \geq 2$. Integrating (3.4) over $(2\bar{\tau}, t)$ for $t > 2\bar{\tau}$ it comes that
\[
\mathcal{L}(t) \leq \mathcal{L}(2\bar{\tau}).
\]
Therefore, from (3.3), it holds
\[
\frac{e^{-K\bar{\tau}}}{3} \int_0^{t+2\bar{\tau}} R_0^0 + 4M_0^0 + \max_{s \in [-\bar{\tau}, t]} d_X(s) \min \left\{ e^{-K\bar{\tau}} \min_{\sigma \in [0, r]} \psi(\sigma) \frac{e^{-2K\bar{\tau}}}{\bar{\tau}} \right\} dr \leq \mathcal{L}(2\bar{\tau}),
\]
for all $t \geq 2\bar{\tau}$. Letting $t \to \infty$ in (3.6), we finally get
\[
\frac{e^{-K\bar{\tau}}}{3} \int_0^{\infty} R_0^0 + 4M_0^0 + \max_{s \in [-\bar{\tau}, \infty)} d_X(s) \min \left\{ e^{-K\bar{\tau}} \min_{\sigma \in [0, r]} \psi(\sigma) \frac{e^{-2K\bar{\tau}}}{\bar{\tau}} \right\} dr \leq \mathcal{L}(2\bar{\tau}).
\]
Asymptotic flocking for the Cucker-Smale model with time variable time delays

Finally, since the function $\psi$ satisfies property (1.5), from (3.7), we can conclude that there exists a positive constant $d^*$ such that

$$\bar{\tau} R^0 + 2 M^0 X + \sup_{s \in [-\bar{\tau}, \infty)} d_X(s) \leq d^*.$$  (3.8)

Indeed, assume by contradiction that

$$\bar{\tau} R^0 + 2 M^0 X + \sup_{s \in [-\bar{\tau}, \infty)} d_X(s) = +\infty.$$  (3.9)

Then, equation (3.7) reads as

$$\int_0^{+\infty} \min \left\{ e^{-K\bar{\tau}} \min_{\sigma \in [0, r]} \psi(\sigma), \frac{e^{-2K\bar{\tau}}}{\bar{\tau}} \right\} dr \leq L(2\bar{\tau})$$  (3.10)

Now, two different situations can occur.

Case I) Assume that, for all $r \in [0, +\infty)$,

$$\frac{e^{-2K\bar{\tau}}}{\bar{\tau}} \leq e^{-K\bar{\tau}} \min_{\sigma \in [0, r]} \psi(\sigma).$$

Thus,

$$\int_0^{+\infty} \min \left\{ e^{-K\bar{\tau}} \min_{\sigma \in [0, r]} \psi(\sigma), \frac{e^{-2K\bar{\tau}}}{\bar{\tau}} \right\} dr = \int_0^{+\infty} \frac{e^{-2K\bar{\tau}}}{\bar{\tau}} dr = +\infty,$$

which is in contradiction with (3.10).

Case II) Assume that there exists $r_1 \in [0, +\infty)$ such that

$$e^{-K\bar{\tau}} \min_{\sigma \in [0, r_1]} \psi(\sigma) < \frac{e^{-2K\bar{\tau}}}{\bar{\tau}}.$$

Note that, for all $r \geq r_1$, it holds that

$$\min_{\sigma \in [0, r]} \psi(\sigma) \leq \min_{\sigma \in [0, r_1]} \psi(\sigma),$$

from which

$$e^{-K\bar{\tau}} \min_{\sigma \in [0, r]} \psi(\sigma) < \frac{e^{-2K\bar{\tau}}}{\bar{\tau}}, \quad \forall r \geq r_1.$$

Thus, using (1.5) we can write

$$\int_0^{+\infty} \min \left\{ e^{-K\bar{\tau}} \min_{\sigma \in [0, r]} \psi(\sigma), \frac{e^{-2K\bar{\tau}}}{\bar{\tau}} \right\} dr \geq \int_{r_1}^{+\infty} \min \left\{ e^{-K\bar{\tau}} \min_{\sigma \in [0, r]} \psi(\sigma), \frac{e^{-2K\bar{\tau}}}{\bar{\tau}} \right\} dr$$

$$= e^{-K\bar{\tau}} \int_{r_1}^{+\infty} \min \left\{ e^{-K\bar{\tau}} \psi(\sigma), \frac{e^{-2K\bar{\tau}}}{\bar{\tau}} \right\} dr = +\infty.$$

Hence, also in this case we get a contradiction.

As a consequence, in all the two possible situations we get a contradiction and we deduce the existence of a positive constant $d^*$ for which inequality (3.8) is fulfilled.

Finally, we define

$$\phi^* := \min \left\{ e^{-K\bar{\tau}} \psi, \frac{e^{-2K\bar{\tau}}}{\bar{\tau}} \right\}.$$
Asymptotic flocking for the Cucker-Smale model with time variable time delays

where

\[ \psi_\ast = \min_{r \in [0,d^*]} \psi(r). \]

Note that \( \phi^\ast > 0 \), being \( \psi \) a positive function. Also, from (3.8), it comes that

\[ \psi_\ast \leq \min \left\{ \psi(r) : r \in \left[ 0, \bar{T} \right] \right\} = \psi_t, \]

for all \( t \geq -\bar{T} \). Thus, we get

\[ \phi^\ast \leq \phi(t), \quad \forall t \geq -\bar{T}. \]

This implies that, for each \( n \geq 2 \)

\[ \frac{1}{3} \left( 1 - e^{-K\bar{T}} \right) \frac{\phi^\ast + \bar{T}}{\phi^\ast + \bar{T}} \leq \left( 1 - e^{-K\bar{T}} \right) \frac{\phi^\ast + \bar{T}}{\phi^\ast + \bar{T}}, \quad (3.11) \]

with \( \left( 1 - e^{-K\bar{T}} \phi^\ast + \bar{T} \right)^{\frac{1}{3}} < 1 \).

Next, we set

\[ C = \frac{1}{3\bar{T}} \ln \left( \frac{1}{1 - e^{-K\bar{T}} \phi^\ast + \bar{T}} \right) > 0. \]

Notice that \( C \) is a constant independent of \( N \). Moreover, we have

\[ \left( 1 - e^{-K\bar{T}} \phi^\ast + \bar{T} \right)^{\frac{1}{3}} \leq e^{-C\bar{T}}, \quad \forall n \geq 2. \]

(3.12)

Now we claim that, for each \( n \geq 2 \), it holds

\[ \mathcal{D}(n\bar{T}) \leq D_0 e^{-C(n-2)\bar{T}}. \]

(3.13)

Indeed, by induction, if \( n = 2 \) then trivially \( \mathcal{D}(2\bar{T}) = D_0 \) and the claim holds. So suppose (3.13) holds true for \( n > 2 \) and prove it for \( n + 1 \). From the induction hypothesis and by recalling of (3.12), we can write

\[ \mathcal{D}(n\bar{T} + \bar{T}) = \mathcal{D}(n\bar{T}) \left( 1 - e^{-K\bar{T}} \right) \frac{\phi^\ast + \bar{T}}{\phi^\ast + \bar{T}} \leq D_0 e^{-C(n-2)\bar{T}} e^{-C\bar{T}} = D_0 e^{-C(n+1-2)\bar{T}}. \]

Hence, from (3.1) and (3.13) it follows that, for each \( t > 2\bar{T} \), if \( t \in (n\bar{T}, n\bar{T} + \bar{T}) \), for some \( n \geq 2 \),

\[ d_V(t) \leq D_{n+1} \leq \mathcal{D}(n\bar{T} + \bar{T}) \leq D_0 e^{-C(n+1-2)\bar{T}} \leq D_0 e^{-C(t-2\bar{T})}. \]

Thus, combining this with the fact that, for all \([\bar{T}, 2\bar{T}]\),

\[ d_V(t) \leq D_0 \leq D_0 e^{-C(t-2\bar{T})}, \]

we can conclude that estimate (1.7) holds too.
Asymptotic flocking for the Cucker-Smale model with time variable time delays

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References

[1] Aydogdu, A., Caponigro, M., McQuade, S., Piccoli, B., Pouradier Duteil, N., Rossi, F., and Trélat, E. (2017). *Interaction network, state space, and control in social dynamics*. Active particles. Vol. 1. Advances in theory, models, and applications, 99–140. Birkhäuser/Springer, Cham.

[2] Bellomo, N., Herrero, M.A., and Tosin, A. (2013). On the dynamics of social conflict: Looking for the Black Swan, *Kinet. Relat. Models*, 6, 459–479.

[3] Borzì, A., and Wongkaew, S. (2015). Modeling and control through leadership of a refined flocking system, *Math. Models Methods Appl. Sci.*, 25, 255–282.

[4] Bullo, F., Cortés, J., and Martínez, S. (2009). *Distributed control of robotic networks: a mathematical approach to motion coordination algorithms*. Princeton series in applied mathematics. Princeton University Press, Princeton.

[5] Camazine, S., Deneubourg, J. L., Franks, N.R., Sneyd, J., Theraulaz, G., and Bonabeau, E. (2001). *Self-Organization in Biological Systems*, Princeton University Press, Princeton, NJ.

[6] Cañizo, J. A., Carrillo, J.A., and Rosado, J. (2011). A well-posedness theory in measures for some kinetic models of collective motion, *Math. Mod. Meth. Appl. Sci.*, 21(3), 515–539.

[7] Caponigro, M., Fornasier, M., Piccoli, B., and Trélat, E. (2013). Sparse stabilization and optimal control of the Cucker-Smale model, *Math. Control Relat. Fields*, 3(4), 447–466.

[8] Carrillo, J.A., Fornasier, M., Rosado, J., and Toscani, G. (2010). Asymptotic flocking dynamics for the kinetic Cucker-Smale model, *SIAM J. Math. Anal.*, 42, 218–236.

[9] Choi, Y.-P., and Haskovec, J. (2017). Cucker-Smale model with normalized communication weights and time delay, *Kinet. Relat. Models*, 10, 1011–1033.

[10] Choi, Y.-P., and Li, Z. (2018). Emergent behavior of Cucker-Smale flocking particles with heterogeneous time delays, *Appl. Math. Lett.*, 86, 49–56.

[11] Choi, Y.-P., Paolucci, A., and Pignotti, C. (2021). Consensus of the Hegselmann-Krause opinion formation model with time delay, *Math. Methods Appl. Sci.*, 44, 4560–4579.
Asymptotic flocking for the Cucker-Smale model with time variable time delays

[12] Choi, Y.-P., and Pignotti, C. (2019). Emergent behavior of Cucker-Smale model with normalized weights and distributed time delays, Netw. Heterog. Media, 14, 789-804.

[13] Cucker, F., and Smale, S. (2007). Emergent behaviour in flocks, IEEE Transactions on Automatic Control, 52, 852-862.

[14] Dong, J.-G., Ha, S.-Y., Doheon, K., and Jeongho, K. (2019). Time-delay effect on the flocking in an ensemble of thermomechanical Cucker-Smale particles, J. Differential Equations, 266, 2373-2407.

[15] Ha, S.-Y., and Tadmor, E. (2009). From particle to kinetic and hydrodynamic descriptions of flocking, Kinet. Relat. Models, 1(3), 415-435.

[16] Halanay, A. (1966). Differential Equations, Accademic Press Inc., 23, Elvesier.

[17] Hale, J. K., and Lunel, S.M. V. (1993). Introduction to functional differential equations, Applied Mathematical Sciences, 99, Springer.

[18] Haskovec, J. (2021). A simple proof of asymptotic consensus in the Hegselmann-Krause and Cucker-Smale models with normalization and delay, SIAM J. Appl. Dyn. Syst., 20, 130-148.

[19] Haskovec, J. (2021). Direct proof of unconditional asymptotic consensus in the Hegselmann-Krause model with transmission-type delay, Bull. Lond. Math. Soc., 53, 1312-1323.

[20] Haskovec, J., and Markou, I. (2020). Asymptotic flocking in the Cucker-Smale model with reaction-type delays in the non-oscillatory regime, Kinet. Relat. Models, 13.

[21] Hegselmann,R., and Krause, U. (2002). Opinion dynamics and bounded confidence models, analysis, and simulation, J. Artif. Soc. Soc. Simul., 5, 1-24.

[22] Jadbabaie, A., Lin, J., and Morse, A. S. (2003). Coordination of groups of mobile autonomous agents using nearest neighbor rules, IEEE Trans. Automat. Control, 48, 988-1001.

[23] Kuang Y. (1993). Delay Differential Equations with Applications in Population Dynamics, Math. Sci. Engrg., 191, Academic Press Inc.

[24] Liu, Y., and Wu, J. (2014). Flocking and asymptotic velocity of the Cucker-Smale model with processing delay, J. Math. Anal. Appl., 415, 53-61.

[25] Lu, J., Ho, D. W. C., and Kurths, J. (2009). Consensus over directed static networks with arbitrary finite communications delays, Phys. Rev. E, 80, 066121, 7 pp.

[26] Marsan, G. A., Bellomo, N., and Egidi, M. (2008). Towards a mathematical theory of complex socio-economical systems by functional subsystems representation, Kinet. Relat. Models, 1, 249-278.

[27] Motsch, S., and Tadmor, E. (2011). A new model for self-organized dynamics and its flocking behavior, J. Stat. Phys., 144, 923-947.

[28] Niculescu, S.-I. (2001). Delay Effects on Stability: A Robust Control Approach, Lect. Notes Control Inf. Sci., 269, Springer-Verlag London.
Asymptotic flocking for the Cucker-Smale model with time variable time delays

[29] Paolucci, A. (2021). Convergence to consensus for a Hegselmann-Krause-type model with distributed time delay, *Minimax Theory Appl.*, 6, 379–394.

[30] Piccoli, B., Pouradier Duteil, N., and Trélat, E. (2019). Sparse control of Hegselmann-Krause models: Black hole and declustering, *SIAM J. Control Optim.*, 57, 2628–2659.

[31] Piccoli, B., Rossi, F., and Trélat, E. (2015). Control to flocking of the kinetic Cucker-Smale model. *SIAM J. Math. Anal.*, 47:4685–4719.

[32] Pignotti, C., and Reche Vallejo, I. (2018). Flocking estimates for the Cucker-Smale model with a time lag and hierarchical leadership, *J. Math. Anal. Appl.*, 464, 1313–1332.

[33] Pignotti, C., and Trélat, E. (2018). Convergence to consensus of the general finite-dimensional Cucker-Smale model with time-varying delays, *Commun. Math. Sci.*, 16, 2053–2076.

[34] Pilyugin, S. Y., and Campi, M. C. (2019). Opinion formation in voting processes under bounded confidence, *Netw. Heterog. Media*, 14, 617–632.

[35] Rodríguez Cartabia, M. (2022). Cucker-Smale model with time delay, *Discrete Contin. Dynam. Systems*, 42, 2409–2432.

[36] Wongkaew, S., Caponigro, M., and Borzì, A. (2015). On the control through leadership of the Hegselmann-Krause opinion formation model. *Math. Models Methods Appl. Sci.*, 25, 565–585.