Symmetry-mixed bound-state order: extended degeneracy of $(d + ig)$-superconductivity in Sr$_2$RuO$_4$

Roland Willa$^1$

$^1$Institute for Theory of Condensed Matter, Karlsruhe Institute of Technology, Karlsruhe, Germany

We report a fluctuation-driven state of matter that develops near an accidental degeneracy point of two symmetry-distinct primary phases. Due to symmetry mixing, this bound-state order exhibits unique signatures, incompatible with either parent phase. Within a field-theoretical formalism, we derive the generic phase diagram for a system with bound-state order, study its response to strain, and evaluate analytic expressions for a specific model. Our results support the $(d + ig)$-superconducting state as a candidate for Sr$_2$RuO$_4$: Most noticeably, the derived strain-dependence is in excellent agreement with recent experiments [Hicks et al. Science (2014) and Grinenko et al. arXiv (2020)].

The evolution above a non-vanishing strain from a joint onset of superconductivity and time-reversal symmetry-breaking to two split phase transitions provides a testable prediction for this scenario.

The phase space of correlated matter is rich as electronic, magnetic, and structural degrees compete for dominance [1–11]. Surprisingly often these orders are nearly degenerate such that a tuning parameter—like pressure, chemical doping, or magnetic field—allows to change the energy balance in favor of a different ground-state. This possibility is very pronounced in multiband systems such as the iron-based materials [12–15], heavy-fermion systems [16–18], and different oxide families [19–29]. It is not immediately clear how to systematically treat the collision of electronic phases as their mutual interaction can be repulsive, attractive, or largely indifferent. Also, symmetry arguments for the classification of low-temperature phases are often obstructed by the possibility of nearly-degenerate phases. This may be particularly relevant for the decades-old quest of understanding superconductivity in Sr$_2$RuO$_4$, which has transitioned from an odd-parity p-wave superconductor [30] to the recently proposed time-reversal symmetry-breaking even-parity $(d + id)$- [26] or $(d + ig)$-superconductor [31, 32] with nearly degenerate $d_{xz}/d_{yz}$ and $d_{x^2−y^2}/g_{xy}(z^2−y^2)$ states, respectively. As the last candidate gains further experimental support [32], it remains to be understood why two distinct order parameters coincidentally appear at a degeneracy point.

A standard and seemingly reasonable approach to tackle correlated phases of matter consists in refining the analysis to symmetry sectors defined by the known parent order parameters. The latter transform according to their irreducible representation and dictate which symmetries are broken at a phase transition. This Letter introduces a fluctuation-driven phase of matter where two symmetry-distinct phases $\eta_1$ and $\eta_2$ form a two-order bound-state, see Fig. 1, while both primary phases remain absent. For real fields $\eta_1$ and $\eta_2$ the bound-state order parameter is $\mu = \langle \eta_1 \eta_2 \rangle$. In the case of two complex phases, it depends on the system’s fluctuation-free ground state. If the latter breaks time-reversal symmetry $\langle [\eta_1 + i\eta_2]\rangle$, the bound-state $\mu = \langle -i(\eta_1^* \eta_2 - \eta_1 \eta_2^*)/2 \rangle$ is also odd under time reversal, whereas it assumes the form $\mu = \langle (\eta_1^* \eta_2 + \eta_1 \eta_2^*)/2 \rangle$ in the even case. With this new order parameter transforming according to its own irreducible representation, the associated bound-state breaks a set of symmetries that is distinct from each parent phase. As described below, a bound-state order naturally appears near the accidental degeneracy of two complex phases. This observation provides crucial support towards the $(d + ig)$-scenario [31] for Sr$_2$RuO$_4$: While the appearance of a stand-alone bound-state phase may be difficult to detect, it’s existence forces the two primary phases to appear jointly, thereby extending the accidental degeneracy point to a (near-)degeneracy line. Furthermore, the bound-state order qualitatively accounts for the strain dependence reported in Refs. [33–35].

The formation of a bound-state order in low-
dimensional systems is reminiscent of the vestigial orders discussed in the iron pnictide materials, which have successfully captured the nematic phases in proximity of spin-density wave instabilities [36–43]. This concept has also been considered for symmetry-related $p_x \pm ip_y$ superconductors [44]. In that context, the two order parameters belong to the same irreducible representation which imposes additional constraints on the phenomenology. Here, in contrast, the primary orders generically develop at different transition temperatures and allow to study a possible formation of a bilinear bound-state away from the phase degeneracy.

Within a Ginzburg-Landau description, the free energy density of two nearly-degenerate complex order parameters $\eta_1$ and $\eta_2$ can be cast in terms of an expansion including all symmetry-allowed contributions. Keeping the qualitative discussion on a general level, specific physical implications will be discussed for the superconducting orders $\eta_1 \sim d_{x^2-y^2}$ and $\eta_2 \sim g_{xy}(x^2-y^2)$ proposed for Sr$_2$RuO$_4$ [31]. The problem’s free energy density takes the form

$$
\mathcal{F} = \frac{r_0}{2} (\eta_1^* \eta_1 + \eta_2^* \eta_2) - \frac{x}{2} (\eta_1^* \eta_1 - \eta_2^* \eta_2) + \frac{u}{8} (\eta_1^* \eta_1 - \eta_2^* \eta_2)^2 + \frac{u}{8} (\eta^*_1 \eta_1 - \eta^*_2 \eta_2)^2 + \mathcal{F}_C
$$

(1)

with $u_{\pm} = u \pm (g + \lambda)$, the phenomenological interaction parameters $u$, $g$, and $\lambda$, and a quadratic form $\mathcal{F}_C$ of gauge-invariant gradient terms. A discussion of the possible ground-states is provided in Ref. [31]. Focusing on the case of interest, $u_{\pm}, g > 0$ provides a ground-state that breaks time-reversal symmetry, where $\eta_1$ and $\eta_2$ have a relative phase shift of $\pm \pi/2$. The two phases are degenerate for $x = 0$, hence $x$ provides a natural parameter away from that point. As, by assumption, the two order parameters belong to different irreducible representations, terms $\propto (\eta_1^* \eta_2 \pm \eta_2^* \eta_1)$ are symmetry-forbidden.

The assumptions $u = u_{\pm}$ and $\mathcal{F}_C = \frac{1}{2} \sum_j (\nabla \eta_j^*) (\nabla \eta_j)$ significantly simplify the results, without affecting the qualitative findings. When appropriate, non-trivial effects of relaxing these constraints shall be mentioned.

The spontaneous condensation of the bound-state order parameter necessitates an attractive interaction channel between the parent phases (here provided by $g > 0$). For the candidate $(d + ig)$-superconductor, the parent phase $d_{x^2-y^2} [g_{xy}(x^2-y^2)]$ belongs to the $B_{1g}$ $(A_{2g})$ representation of the $D_{4h}$ point group, see Table I. In this case the bound-state $\mu = -i(\eta_1^* \eta_2 - \eta_2^* \eta_1)/2$ belongs to the $B_{2g}$ representation. It preserves the $U(1)$ gauge and is odd under time reversal.

An established route to evaluate the condensation condition of a bound-state consists in replacing $\eta_j$ by an $N$-dimensional vector field $\eta_j$ and treating the above action in a large-$N$ limit. With the identity,

$$
1 = \int D\sigma_m D\phi_m \exp \left\{ -\int q \sigma_m(q) |\phi_m(q) - m(q)| \right\}
$$

(2)

interaction terms are brought to a quadratic form in the primary fields, where $\phi_m$ is a generalized Hubbard-Stratonovich field associated to a real field $m$. The functional integration over $\sigma_m$ imposes $\delta(\phi_m - m)$. The notation $\int_q$ abbreviates the momentum-integral and accounts for the system’s anisotropy. With the three Hubbard-Stratonovich fields $\phi_j = \eta_j^2$ and $\mu = -i(\eta_1^* \eta_2 - \eta_2^* \eta_1)/2$, the partition function transforms to

$$
Z \propto \int D\sigma_1 D\sigma_2 D\sigma_\mu D\phi_1 D\phi_2 D\mu e^{-N S_{\text{eff}}}, \quad \text{with}
$$

(3)

$$
S_{\text{eff}} = \sum_{i,j=\{1,2\}} \langle \eta_i^* \rangle (G_0^{-1})_{ij} \langle \eta_j \rangle + \int_q \text{tr}[\log(G_q^{-1})]
$$

$$
+ \frac{u}{4} (\phi_1^2 + \phi_2^2) - \frac{g}{2} \mu^2 + i\sigma_1 \phi_1 + i\sigma_2 \phi_2 + 2i\sigma_\mu \mu,
$$

(4)

and

$$
G_q^{-1} = \frac{1}{2} \begin{pmatrix} r_0 - x + q^2 - 2i\sigma_1 & i(2\sigma_\mu) \\ -i(2\sigma_\mu) & r_0 + x + q^2 - 2i\sigma_2 \end{pmatrix}. \quad \text{(5)}
$$

Here, the quadratic fluctuations of the fields $\eta_j$ around the mean value $\langle \eta_j \rangle$ have been integrated out. Furthermore, $\phi_j$ and $\mu$ are assumed uniform in space. The above action captures all possibilities of forming primary and/or bound-state phases [45]. The large-$N$ limit now allows to search for saddle-point solutions to the remaining fields. Minimizing the action for $\phi_j$ and $\mu$ provides $2i\sigma_j = -u\phi_j$ and $2i\sigma_\mu = gy$. The saddle-point conditions

| $\Gamma \downarrow$ | $E$ | $2C_2^+$ | $C_2^-$ | $2C_2'$ | $I$ | $2IC_4^+$ | $IC_4^-$ | $2IC_4'$ | $2IC_4''$ |
|-------|-----|--------|--------|--------|----|----------|--------|----------|----------|
| $A_{1g}$ | 1   | 1      | 1      | 1      | 1  | 1        | 1      | 1        | 1        |
| $A_{2g}$ | 1   | 1      | -1     | -1     | 1  | 1        | -1     | -1       | -1       |
| $B_{1g}$ | 1   | -1     | 1      | -1     | 1  | 1        | -1     | -1       | -1       |
| $B_{2g}$ | 1   | -1     | 1      | -1     | 1  | 1        | -1     | -1       | -1       |

TABLE I. Excerpt of the character table (top) and the product table (bottom) for the $D_{4h}$ point group. The product table specifies to which irreducible representation $\Gamma(\eta_1,\eta_2)$ the bound-state of two primary phases $\eta_j$ belongs. The colored cells correspond to the relevant orders for $(d + ig)$ superconductivity in Sr$_2$RuO$_4$ [31].
FIG. 2. Susceptibilities $\chi_1(T)$ (blue), $\chi_2(T)$ (red), $\chi_0(T)$ (yellow) upon approaching the ordered phase. Far from the phase degeneracy [a], the order $\langle \eta_1 \rangle$ sets in first, followed by the instability in the two remaining sectors. Near the degeneracy [b], the symmetry-mixed bound-state $\mu$ appears (yellow) while the parent orders remain zero. A renormalization of the susceptibilities induces a simultaneous divergence of $\chi_1$ and $\chi_2$.

then read

$$r_1 = r_{0,1} + u \langle \eta_1^* \rangle \langle \eta_1 \rangle + 2u \int_q \frac{r_1 + q^2}{(r_1 + q^2)(r_2 + q^2) - (g\mu)^2},$$

(6)

$$r_2 = r_{0,2} + u \langle \eta_2^* \rangle \langle \eta_2 \rangle + 2u \int_q \frac{r_1 + q^2}{(r_1 + q^2)(r_2 + q^2) - (g\mu)^2},$$

(7)

$$2\mu = i(\langle \eta_1 \rangle \langle \eta_2^* \rangle - \langle \eta_1^* \rangle \langle \eta_2 \rangle) + \int_q \frac{4g\mu}{(r_1 + q^2)(r_2 + q^2) - (g\mu)^2},$$

(8)

where $r_{0,1} = r_0 - x$, $r_{0,2} = r_0 + x$ are the bare, and $r_j = r_{0,j} + 2u\phi_j$ the fluctuation-renormalized masses. The possibility of having non-zero primary phases imposes the additional constraints [45]

$$r_1 \langle \eta_1 \rangle = -i g\mu \langle \eta_2 \rangle,$$

(9)

$$r_2 \langle \eta_2 \rangle = i g\mu \langle \eta_1 \rangle.$$  

(10)

The susceptibility of the order parameters is evaluated by coupling conjugate fields $h_j$ and $h_\mu$ to the primary and bound-state orders. A physical implementation of $h_\mu$ is discussed below. Following a similar derivation as before one finds in the absence of primary phases

$$\chi_j \equiv \left. \frac{\partial \langle \eta_j \rangle}{\partial h_j} \right|_{h_j \to 0} = \frac{1}{r_j} \frac{r_1 r_2}{r_1 r_2 - g^2 \mu^2},$$

(11)

$$\chi_\mu \equiv \left. \frac{\partial \mu}{\partial h_\mu} \right|_{h_\mu \to 0} = \frac{K_\mu(r_1, r_2)}{1 - g K_\mu(r_1, r_2)},$$

(12)

dependent on $h_\mu$ in a similar manner. The longitudinal susceptibilities, where

$$K_\mu(r_1, r_2) \equiv \int_q \frac{2}{(r_1 + q^2)(r_2 + q^2)},$$

(13)

and $r_1$, $r_2$ satisfy Eqs. (6) and (7) for $\mu = 0$. Once one primary phase develops [here $\langle \eta_1 \rangle$] the remaining susceptibilities are modified to

$$\chi_2 = \frac{1}{r_2} \frac{1 - g\mu(0, r_2)}{1 - g\mu(0, r_2) - g(\langle \eta_1 \rangle \langle \eta_2 \rangle)/r_2},$$

(14)

$$\chi_\mu = \frac{K_\mu(0, r_2) + g\mu(\langle \eta_1 \rangle \langle \eta_2 \rangle)}{1 - g K_\mu(0, r_2) - g(\langle \eta_1 \rangle \langle \eta_2 \rangle)/r_2},$$

(15)

Contemplating Eqs. (6)-(10)—which determine the phase diagram for the three orders $\langle \eta_1 \rangle$, $\langle \eta_2 \rangle$, and $\mu$ and the two renormalized masses $r_1$ and $r_2$—allows to establish the following generic observations:

The bound-state order appears purely as a fluctuation phenomenon, as indicated by the momentum integrals coupling the different equations. As for the vestigial phases in iron-based systems, sufficiently strong fluctuations are required to trigger such a phase [36–43]. In fact, for large masses $r_j$ the integrals in Eqs. (6)-(8) tend to be small. As a corollary, the bound-state phase is expected in the vicinity of a phase degeneracy where the renormalized masses $r_1$ and $r_2$ become small.

With each bilinear product of two of the three orders acting as a conjugate field to the third one, see Eqs. (8)-(10), the presence of solely two orders is excluded. The phase diagram therefore exhibits at most two ordering transitions. Starting from the high-temperature phase, the system can undergo a sequence of transitions into ordered phases following one of three scenarios:

**Scenario 1: Ordering of a primary phase** Far from a phase degeneracy one primary order appears with the subordinate one remaining zero. Let $\langle \eta_1 \rangle$ be the order to condensate first when $r_1$ vanishes. As phase interactions are still inactive, this transition line coincides with the conventional $T_{c1}$. If the above assumption $u_+ = u_-$ is relaxed, the transition is affected by a shift $\propto (u_+ - u_-).$
With fluctuations $T_{c1}^0$ is shifted down with respect to the bare $T_{c1}$ where $T_{c0,1} = 0$. As the transition coincides with a divergent susceptibility $\chi_1$, Eq. (11), it is of second order, see Fig. 2a). At the phase boundary, the set of symmetries associated with irreducible representation $\Gamma(\eta_1)$ is spontaneously broken.

Below $T_{c1}$, the primary phase $\langle \eta_1 \rangle$ follows a typical behavior $\langle \eta_1 \rangle \propto (T_{c1} - T)^{1/2}$, while $r_1$ is pinned to zero, see Eq. (9). The remaining orders appear simultaneously as the susceptibilities in Eqs. (14) and (15) diverge. This second-order transition further symmetries—according to the irreducible representation $\Gamma(\eta_2)$—are broken. The above ordering mechanism always precedes the onset driven by a disappearance of $r_2$ in Eq. (14).

Scenario 2: Appearance of bound-state order By definition of the phase degeneracy, the two primary phases appear simultaneously at $x = 0$ in the absence of phase interactions. In the vicinity of that point both $r_1$ and $r_2$ are small and the bound-state $\mu$ can appear as a stand-alone phase. Its onset is determined as the bound-state susceptibility diverges, see Eq. (12), i.e.

$$1 - gK_u(r_1, r_2) = 0.$$

To allow for a non-zero solution to $\mu$ without the appearance of the primary phases, the balance between the right- and left-hand side of Eq. (8) must be guaranteed by a momentum integral. This highlights particularly well the fluctuation-driven origin of the bound-state phase. Upon entering this phase, the system breaks the symmetries associated with the irreducible representation $\Gamma(\eta_1 \eta_2)$. As a corollary and curious consequence, all point-group symmetries that are broken by both primary orders remain preserved in the bound-state phase. For the superconducting $d_{x^2−y^2}$ and $g_{xy}(x^2−y^2)$ states of Sr$_2$RuO$_4$, this observation applies to two-fold rotations about the diagonal axes [110] and [-110] (see $C''_2$ in Table 1). As the bound-state order breaks time-reversal symmetry it does not directly couple to the lattice and hence, is not a nematic state. At the same time it is also insensitive to a magnetic field. However, the order $\mu$ can be induced by applying an external magnetic field $H_c$ (along the crystallographic $c$ axis) together with $B_{1g}$ strain through a coupling $\epsilon_{B_{1g}}H_c\mu$. This implies that $h_\mu = \epsilon_{B_{1g}}H_c \langle \eta_1 \rangle$ is a conjugate field to $\mu$.

For a finite $\mu$, the primary phases now appear simultaneously when the criterion $r_1r_2 = (\mu u)^2$ is first met, i.e., as both susceptibilities $\chi_j$ in Eq. (11) jointly diverge. The evolution of the three susceptibilities $\chi_j$ and $\chi_\mu$ is shown in Fig. 2b), where the bound-state phase appears first and renormalizes the primary susceptibilities, Eq. (11), to force a joint transition. At this second-order transition, both orders $\langle \eta_j \rangle$ develop and break the remaining symmetries associated with $\Gamma(\eta_j)$. For the test case, this implies a full superconducting $(d+ig)$ state. The bound-state is limited in phase space by a maximal attractive interaction strength $g^* = u/2$ and a maximal distance $x^*(g)$ away from the degeneracy point, see Fig. 3. For larger interactions $g > g^*$, the line $x^*(g)$ separates between a joint transition (when $|x| < x^*$, scenario 3 below), or split transitions (when $|x| > x^*$, scenario 1 of the two primary phases. The Supplemental Material specifies the analytic model from which this phase space is computed.

Scenario 3: First order transition to fully ordered state As a third possibility, the system may undergo a first-order transition by discontinuously jumping to a finite $\mu$ and triggering a continuous appearance of $\langle \eta_1 \rangle$ and $\langle \eta_2 \rangle$. Here, the onset of all three phases is not accompanied by a divergent susceptibility. This scenario becomes relevant for large values $g/u$ in agreement with the simultaneous appearance of both primary phases obtained in a mean-field analysis, see also discussion below. In Sr$_2$RuO$_4$, the coherence lengths are relatively large and a quantitative distinction between scenario 2 (with a slim bound-state phase) may be experimentally challenging. Qualitatively however, both scenarios provide finite tuning range over which the breaking of the $U(1)$ gauge and time-reversal symmetry (almost) coincide, hence transforming the degeneracy point to an extended (near)-degeneracy line.

A meaningful discussion of the system’s response to external strain requires specific assumptions on the phases $\langle \eta \rangle$. The following one is tailored to the test case Sr$_2$RuO$_4$. Noting that $A_{1g}$ strain merely rescales all couplings and $B_{2g}$ strain modifies the system’s ground state away from a pure $(d+ig)$ case [by coupling to $(d^*g+dg^*)$], the treatment is further narrowed to the interesting strain sector $B_{1g}$. While a bare linear coupling is prohibited by symmetry, strain couples to the bound-state order in combination with an external field $H_c$ along

FIG. 4. Qualitative dependence of the transition temperatures for a $(d+ig)$-superconductor to an external $B_{1g}$-strain with (left) and without (right) magnetic field $H_c$. In the first case, the bound-state order is induced by $h_\mu = \epsilon_{B_{1g}} H_c$ and imposes a joint ordering of the $d$ and $g$ component. Without $H_c$, the system first develops a bound-state followed by a full $(d+ig)$-superconducting state. In this sequence the strain-dependence is indirect, i.e., through $r_2(e)$ in Eq. (8). Above critical strain the superconducting (blue) and the time-reversal-symmetry breaking (red) transitions split. The superconducting transition temperature now directly depends on $r_1(e)$ via Eq. (6).
c: The combination $\epsilon_{B_{1g}} H_{\perp \mu}$ is symmetry-allowed and breaks time-reversal symmetry at any temperature. The number of phase transitions is then reduced to one, see Fig. 4, where both superconducting orders appear. In the absence of magnetic fields, $B_{1g}$ strain renormalizes the masses to $r_1(\epsilon_{B_{1g}}) \approx r_1(0) + r_2(0) \epsilon_{B_{1g}}^2$. While it is possible to treat the generic case, it is instructive and reasonable to assume that strain dominantly couples to one primary order [we set $r_2(0) < 0$ and $r_2(0) = 0$]. Near the degeneracy the system features two distinct regimes of strain response, see Fig. 4: At first, strain weakly affects the superconducting onset temperature [both with/without upstream bound-state order] through Eq. (8). At larger strain, the system is pushed towards a split transition where the superconductivity precedes time-reversal symmetry breaking at a transition directly dictated by the strain-dependence of $r_1(\epsilon_{B_{1g}})$, see Eq. (6). This behavior is in excellent agreement with recent experiments [33–35]. As a falsifiable prediction, $\mu$SR experiments under $B_{1g}$ strain should resolve the splitting of an (almost) joint superconducting and time-reversal-symmetry breaking transition for strain below $\sim 0.05\%$ (based on the kink-like feature of $T_c$ observed in Ref. [33]) into two separate transitions for larger strain.

In conclusion, this Letter presents a fluctuation-driven phase of matter that is the symmetry-mixed bound-state of two complex primary orders $\eta_j$. This order parameter naturally emerges as a stand-alone phase near the degeneracy point of the parent orders. As this bound-state phase develops, it breaks a set of symmetries associated with the irreducible representation $\Gamma(\eta_1,\eta_2)$ rather than those dictated by the parent phases. Consequently the emergent phase preserves all point-group symmetries that are broken by both parent phases. The results presented here demonstrate that the symmetry sectors of parent phases give a too narrow view on the possible electronic states of matter in the vicinity of degeneracy points.

This phenomenology provides support for a $(d + ig)$-superconducting state in Sr$_2$RuO$_4$: While the exact degeneracy between the $d$- and $g$-component is rather improbable, fluctuations allow for a time-reversal symmetry breaking transition into a bound-state phase to precede (scenario 2) or coincide (scenario 3) with the superconducting transition over a finite tuning range away from the degeneracy point. Furthermore, the results under $B_{1g}$ strain exhibit two distinct regimes of parabolic strain dependence in good agreement with recent experiments [33–35]. As a testable consequence, the bound-state order should be observed in Kerr experiments [46] when induced by simultaneously applying $B_{1g}$-strain and a magnetic field $H_c$ along $c$.

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To determine for which doping range the bound-state order parameters have the onset of the composite order coincides with the tuned integrals take the explicit form favorable, where we find that for three-dimensional anisotropic systems qualitatively consistent, be treated analytically. A valid path to describe phases as the primary phases appear at the first time a solution to the bare parameters have been rescaled to $T - T_{c,j} = r_{0,j} + u \int q^{-2}$ such that in the absence of phase interactions the primary phases appear at $T_{c,j}$, see Eqs. (17) and (18) with $\mu = 0$. An explicit evaluation of the integrals yields

$$K_j(r_1, r_2, \mu) = \frac{\Omega_0 \pi}{2 \cdot 2^{1/4}} \left\{ \frac{\int_{\mathbf{q}} \frac{2(r_2 + q^2)}{(r_1 + q^2)(r_2 + q^2) - (g\mu)^2} - \frac{2}{q^2}}{r_1 r_2 - g^2 \mu^2} \sqrt{(r_1 - r_2)^2 + 4g^2 \mu^2} \right\}$$

$$K_\mu(r_1, r_2, \mu) = \frac{\Omega_0 \pi 2^{1/4}}{2 \cdot 2^{1/4}} \left\{ \frac{\int_{\mathbf{q}} \frac{2(r_2 + q^2)}{(r_1 + q^2)(r_2 + q^2) - (g\mu)^2} - \frac{2}{q^2}}{r_1 r_2 - g^2 \mu^2} \sqrt{(r_1 - r_2)^2 + 4g^2 \mu^2} \right\}$$

The function $K_\mu(r_1, r_2)$ in the main text is related to its three-argument cousin via $K_\mu(r_1, r_2) \equiv K_\mu(r_1, r_2, 0)$. In absence of the composite order, i.e. $\mu = 0$, the functions assume the simple form

$$K_j(r_1, r_2, 0) = -\frac{\Omega_0 2\pi (r_j / 4)^{1/4}}{r_1 - r_2}$$

$$K_\mu(r_1, r_2, 0) = \frac{\Omega_0 2\pi (r_1 / 4)^{1/4} - (r_2 / 4)^{1/4}}{r_1 - r_2}$$

To determine for which doping range the bound-state order appears before the onset temperatures $T_{c1}$ and $T_{c2}$ we evaluate the singularity in $\chi_\mu$, i.e. for $1 = gK_\mu(r_1, r_2, 0)$. We find that for $|x| < x^*$ the bound-state order can be favorable, where $x^* > 0$ is the critical doping for which the onset of the composite order coincides with $T_{c1}$ for $x = -x^*$ it coincides with $T_{c2}$. Solving the coupled set of equations (17)-(19), with $x = x^*$ and $T = 1 + x^*$, yields

$$r_2 = (4\pi^4 g^4 \Omega_0^4)^{1/3}$$

$$x^* = \frac{1}{2} (4\pi^4 g^4 \Omega_0^4)^{1/3} g^{1/3} (g + u).$$

The bare parameters $r_{0,j}$ have been rescaled to $T - T_{c,j} = r_{0,j} + u \int q^{-2}$ such that in the absence of phase interactions the primary phases appear at $T_{c,j}$, see Eqs. (17) and (18) with $\mu = 0$. An explicit evaluation of the integrals yields

$$K_1(r_1, r_2, \mu) \equiv \int_{\mathbf{q}} \frac{2(r_2 + q^2)}{(r_1 + q^2)(r_2 + q^2) - (g\mu)^2} - \frac{2}{q^2}$$

$$K_2(r_1, r_2, \mu) \equiv \int_{\mathbf{q}} \frac{2(r_2 + q^2)}{(r_1 + q^2)(r_2 + q^2) - (g\mu)^2} - \frac{2}{q^2}$$

$$K_\mu(r_1, r_2, \mu) \equiv \int_{\mathbf{q}} \frac{2}{(r_1 + q^2)(r_2 + q^2) - (g\mu)^2}.$$
$T_{\mu}(x)$ for $x = 0$ and in the vicinity of $x^*$. We find

$$T_{\mu}(0) = T_{c0} + \frac{1}{4} (\pi^4 \Omega_0^4)^{1/3} g^{1/3} (g + 4u)$$

$$(33)$$

$$= T^* + \frac{1}{8} (4\pi^4 \Omega_0^4)^{1/3} g^{1/3} [2^{1/3} (4u + g) - 4(u + g)]$$

and

$$T_{\mu}(x) \approx T^* - \frac{2g - u}{2(g - u)} (x^* - |x|) \quad \text{for} \quad 1 - |x|/x^* \ll 1.$$  

$$(34)$$

The slope of $T_{\mu}(x)$ at $x^*$ changes sign for $g = g^* \equiv u/2$, while at this interaction strength the transition $T_{\mu}(0)|_{g=g^*}$ at $x = 0$ is still shadowed by $T_{c1}$. This tells that, upon reducing $g$, the bound-state phase first becomes favorable near $x^*(g)$ and not at the degeneracy point. The bound-state order appears in the entire doping range $|x| < x^*$ once $T_{\mu}(0) \geq T_{c1} = T^*$. Solving $T_{\mu}(0) = T^*$ from Eq. (33), this condition is first satisfied for $g = g_c \equiv 4u(2^{1/3} - 1)/(4 - 2^{1/3}) \approx 0.38u$. In the intermediate parameter range $g \in [g_c, g^*]$ the composite phase exists in two split lobes [defined by the condition $T_{\mu}(x) > T^*$], one on each side of the degeneracy line.

The bound-state order occupies the largest temperature range at the degeneracy point, $\delta T = T_{\mu}(0) - T^*$, and for the parameter $g_m = [(2^{1/3} - 1)/(4 - 2^{1/3})]u$, as obtained from maximizing $\delta T$ using Eq. (33).

For $x > x^*$ the primary order $\langle \eta_1 \rangle$ appears below $T_{c1} = T_{c0} + x$ and the second order follows when the susceptibility $\chi_{\mu}$ diverges. While the latter assumes the form in Eq. (15) [main text], for most practical purposes, it is well approximated by

$$\chi_{\mu} \approx K_{\mu}(0, r_2, 0)[1 - gK_{\mu}(0, r_2, 0)]^{-1}.$$  

$$(35)$$

i.e. by neglecting the small correction term $\propto \langle \eta_1 \rangle^2/r_2$. The condition for the second phase transition then translates to $T_{c2,\mu}(x) \approx T_{c2}(x) + 2x^* = T^* - (x - x^*)$, near $x^*$. 

The condition for the transition $T_{\mu}(x)$ becomes favorable near $x^*(g)$ and not at the degeneracy point. The bound-state order appears in the entire doping range $|x| < x^*$ once $T_{\mu}(0) \geq T_{c1} = T^*$. Solving $T_{\mu}(0) = T^*$ from Eq. (33), this condition is first satisfied for $g = g_c \equiv 4u(2^{1/3} - 1)/(4 - 2^{1/3}) \approx 0.38u$. In the