THE ARGUMENT PRINCIPLE AND HOLOMORPHIC EXTENDIBILITY

Josip Globevnik

ABSTRACT Let $D$ be a bounded domain in the complex plane whose boundary consists of finitely many pairwise disjoint simple closed curves. Give $bD$ the standard orientation and let $A(D)$ be the algebra of all continuous functions on $\overline{D}$ which are holomorphic on $D$. In the paper we prove that a continuous function $f$ on $bD$ extends to a function in $A(D)$ if and only if for each $g \in A(D)$ such that $f + g \neq 0$ on $bD$ the change of argument of $f + g$ along $bD$ is nonnegative.

1. Introduction and the main result

H. Alexander and J. Wermer [AW] obtained a characterization of boundaries of analytic varieties in terms of a generalized argument principle. Their results brought new results also into the classical function theory of one variable. For instance, E. L. Stout [S1] observed that their results imply that a smooth function on the unit circle $b\Delta$ extends holomorphically through the open unit disc $\Delta$ if and only if for each polynomial $Q$ of two variables such that $Q(z,f(z)) \neq 0 (z \in b\Delta)$ the change of argument of $z \mapsto Q(z,f(z))$ around $b\Delta$ is nonnegative, and proved that this holds for continuous functions. J. Wermer [W] showed that for smooth functions on $b\Delta$ there is a better result: a smooth function $f$ on $b\Delta$ extends holomorphically through $\Delta$ if and only if for each polynomial $P$ such that $f + P \neq 0$ on $b\Delta$ the change of argument of $f + P$ around $b\Delta$ is nonnegative. For continuous functions this was proved by the author in [Gl] so that we have the following characterization of the disc algebra in terms of the argument principle:

**Theorem 1.0** [Gl] A continuous function $f$ on $b\Delta$ extends holomorphically through $\Delta$ if and only if for each polynomial $P$ such that $f + P \neq 0$ on $b\Delta$ the change of argument of $f + P$ around $b\Delta$ is nonnegative.

In the present paper we prove that the analogous theorem holds for multiply connected domains.

Let $D \subset \mathbb{C}$ be a bounded domain whose boundary consists of finitely many pairwise disjoint simple closed curves. We give $bD$ the standard orientation. Denote by $A(D)$ the algebra of all continuous functions on $\overline{D}$ which are holomorphic on $D$. Our main result is

**Theorem 1.1** A continuous function $f$ on $bD$ extends to a function in $A(D)$ if and only if for each $g \in A(D)$ such that $f + g \neq 0$ on $bD$ the change of argument of $f + g$ along $bD$ is nonnegative.

If the condition in Theorem 1.1 holds for all $g$ belonging to a dense subset of $A(D)$ then it holds for all $g \in A(D)$. Thus, since rational functions with poles outside $\overline{D}$ are dense in $A(D)$ [S2, 23], it is enough to assume that the condition in Theorem 1.1 holds for rational functions with poles outside $\overline{D}$. The only if part of the theorem is an obvious consequence of the argument principle. In fact, if $f$ admits an extension $\tilde{f} \in A(D)$ then...
the change of argument of $f + g$ along $bD$ equals $2\pi$ times the number of zeros of $\tilde{f} + g$ in $D$.

2. Preliminaries

Every bounded domain $D \subset \Phi$ whose boundary consists of finitely many pairwise disjoint simple closed curves is biholomorphically equivalent to a domain $D'$ bounded by finitely many pairwise disjoint circles [Go]. Moreover, every biholomorphic map $\Phi: D \to D'$ extends to a homeomorphism $\tilde{\Phi}: \overline{D} \to \overline{D'}$ (see the proof in [CL, pp. 46-49] which works also for multiply connected domains, bounded by finitely many pairwise disjoint simple closed curves). Thus, with no loss of generality assume that $D$ is bounded by finitely many pairwise disjoint circles.

In general, not every real-valued harmonic function $u$ on $D$ is the real part of a holomorphic function on $D$. If there is a harmonic function $v$ on $D$ such that $u + iv$ is holomorphic on $D$ then we call $v$ a conjugate of $u$. If $D$ is simply connected then every harmonic function on $D$ has a conjugate on $D$. Let $f$ be a complex valued harmonic function on $D$. Write $f = p + iq$ with $p, q$ real. We will say that $f$ has a conjugate on $D$ if $p$ has a conjugate $r$ on $D$ and $q$ has a conjugate $s$ on $D$, and we will call the function $r - is$ a conjugate of $f$ on $D$. This happens if and only if $f = F + G$ where $F$ and $G$ are holomorphic functions on $D$. In fact, if $P = p + ir$ and $Q = q + is$ then $P$ and $Q$ are holomorphic functions on $D$ and $F = (P + iQ)/2$ and $G = (P - iQ)/2$.

Given a continuous function $\Phi$ on $bD$ there is a continuous extension of $\Phi$ to $\overline{D}$ which is harmonic on $D$ and which we will denote by $\mathcal{H}(\Phi)$; moreover, if $\Phi \in C^\infty(bD)$ then $\mathcal{H}(\Phi) \in C^\infty(\overline{D})$ [B, p.53]. $\mathcal{H}$ is a linear map from $C(bD)$ to the space of continuous functions on $\overline{D}$ which are harmonic on $D$. If $D$ is simply connected then $\mathcal{H}(\Phi)$ has a conjugate harmonic function on $D$ which is also in $C^\infty(\overline{D})$ provided that $\Phi \in C^\infty(bD)$ [B, p.91]. If $\Omega_1$ and $\Omega_2$ are bounded, simply connected domains with boundaries of class $C^\infty$ then a biholomorphic map $\Phi$ from $\Omega_1$ to $\Omega_2$ extends to a smooth map from $\overline{\Omega_1}$ to $\overline{\Omega_2}$ [B, p.28]. Applying this locally along $bD$ and using the preceding discussion we see that if $f$ is a harmonic function on $D$ that has a conjugate on $D$ then the conjugate extends smoothly to $\overline{D}$ provided that $f$ extends smoothly to $\overline{D}$. We summarize this in

**Lemma 2.1** If $f \in C^\infty(bD)$ is such that $\mathcal{H}(f)$ has a conjugate on $D$ then both $\mathcal{H}(f)$ and its conjugate extend smoothly to $\overline{D}$.

Every harmonic function $f$ on $D$ is real-analytic on $D$ so, if $f$ is holomorphic on an open subset of $D$ then it is holomorphic on $D$. We will need the following fact which can be found as an exercise in [R].

**Lemma 2.2** Let $f$ be a harmonic function on $D$ such that $z \mapsto zf(z)$ is harmonic on a nonempty open set $U \subset D$. Then $f$ is holomorphic on $D$.

**Proof.** By the preceding discussion it is enough to prove that $f$ is holomorphic on a disc $\Omega \subset D$. Since $f$ is harmonic on $D$ there is a disc $\Omega \subset U$ such that $f = P + \overline{Q}$ on $\Omega$ where $P$ and $Q$ are holomorphic functions on $\Omega$. By our assumption, the function
$z \mapsto zf(z) = zP(z) + zQ(z)$ is harmonic on $\Omega$ so
\[
\frac{\partial^2}{\partial z \partial \overline{z}} [zP(z) + zQ(z)] = 0 \quad (z \in \Omega)
\]
which implies that $\overline{Q'(z)} = 0$ $(z \in \Omega)$. Thus, $Q$ is constant on $\Omega$ and consequently $f$ is holomorphic on $\Omega$. This completes the proof.

3. A new proof in the case of a disc

The proof in [Gl] does not generalize to multiply connected domains. In this section we give a new, different proof of the theorem in the case when $D$ is a disc which we later generalize to multiply connected domains.

Write $\Delta = \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}$. Throughout this section, $D = \Delta$, Denote by $Z$ the identity function: $Z(z) = z$ $(z \in bD)$. Assume that $f \in C(b\Delta)$ does not extend to a function from the disc algebra $A(\Delta)$. Then there is an $a \in \Delta$ such that
\[
\mathcal{H}[(Z-a)f](a) \neq 0. \quad (3.1)
\]
To see this, suppose for a moment that $\mathcal{H}[(Z-a)f](a) = 0$ $(a \in \Delta)$. This implies that $\mathcal{H}(Zf)(a) = a\mathcal{H}(f)(a)$ $(a \in \Delta)$. In particular, the function $a \mapsto a\mathcal{H}(f)(a)$ is harmonic on $\Delta$. Since $\mathcal{H}(f)$ is harmonic on $D$ Lemma 2.2 implies that $\mathcal{H}(f)$ is holomorphic on $\Delta$ so $f$ extends holomorphically through $\Delta$, a contradiction. This proves that there is an $a \in \Delta$ such that (3.1) holds.

With no loss of generality, replacing $f$ with $e^{i\omega}f$, $\omega \in \mathbb{R}$, if necessary, we may assume that
\[
\Re\{\mathcal{H}[(Z-a)f](a)\} = \beta \neq 0. \quad (3.2)
\]
For easier understanding we complete the proof first under the additional assumption that $f$ is smooth. Suppose for a moment that $f$ is smooth. The function $z \mapsto \Re\{\mathcal{H}[(Z-a)f](z)\} - \beta$ is continuous on $\overline{\Delta}$, harmonic on $\Delta$ and has smooth boundary values $\Re[(z-a)f(z)] - \beta$ $(z \in b\Delta)$. Hence by Lemma 2.1 there is a function $g \in A(\Delta)$ such that
\[
\Re[g(z)] = \Re\{\mathcal{H}[(Z-a)f](z)\} - \beta \quad (z \in \overline{\Delta}).
\]
By (3.2) we have $\Re g(a) = \Re\{\mathcal{H}[(Z-a)f](a)\} - \beta = 0$ so by adding an imaginary constant to $g$ if necessary we may assume that $g(0) = 0$ so $g(z) = (z-a)h(z)$ $(z \in \overline{\Delta})$ where $h \in A(\Delta)$. Consider the function $z \mapsto G(z) = (z-a)f(z) - g(z)$ $(z \in b\Delta)$. We have $\Re[G(z)] = \beta$ $(z \in b\Delta)$ which, since $\beta \neq 0$, implies that $G \neq 0$ on $b\Delta$ and that the change of argument of $G$ around $b\Delta$ equals zero. Since $a \in \Delta$ the change of argument of $z \mapsto (z-a)$ around $b\Delta$ equals $2\pi$. Since $G(z) = (z-a)[f(z) - h(z)]$ $(z \in b\Delta)$ it follows that $f - h \neq 0$ on $b\Delta$ and that the change of argument of $f - h$ around $b\Delta$ is negative. Since $h \in A(\Delta)$ this completes the proof in the special case when $f$ is smooth.

In general, we have to approximate $f$ by smooth functions as follows: Let $f_1$ be a smooth function on $b\Delta$ such that
\[
|(z-a)[f(z) - f_1(z)]| < |\beta|/4 \quad (z \in b\Delta). \quad (3.3)
\]
Write \( \beta_1 = \Re\{\mathcal{H}[(Z - a)f_1](a)\} \). By (3.3) and by the maximum principle for the real harmonic function \( \Re\{\mathcal{H}[(Z - a)(f_1 - f)]\} \) we have

\[
|\beta_1 - \beta| < |\beta|/4.
\] (3.4)

The function \( z \mapsto \Re\{\mathcal{H}[(Z - a)f_1](z)\} - \beta_1 \) is continuous on \( \overline{D} \), harmonic on \( \Delta \) and has smooth boundary values \( \Re\{z - a)f_1(z)\} - \beta_1 \ (z \in b\Delta) \). Hence by Lemma 2.1 there is a function \( g_1 \in A(\Delta) \) such that

\[
\Re[g_1(z)] = \Re\{\mathcal{H}[(Z - a)f_1](z)\} - \beta_1 \ (z \in \overline{\Delta}).
\] (3.5)

Clearly \( \Re[g_1(a)] = \Re\{\mathcal{H}[(Z - a)f_1](a)\} - \beta_1 = 0 \) so by adding an imaginary constant to \( g_1 \) if necessary we may assume that \( g_1(a) = 0 \) so that \( g_1(z) = (z - a)h(z) \ (z \in \overline{\Delta}) \) where \( h \in A(\Delta) \).

Consider the function \( z \mapsto G(z) = (z - a)f(z) - g_1(z) \ (z \in b\Delta) \). By (3.5) we have \( \Re G(z) = \beta + \Re[(z - a)(f(z) - f_1(z))] + (\beta_1 - \beta) \ (z \in b\Delta) \) which, by (3.3) and (3.4) implies that

\[
|\Re G(z) - \beta| < |\beta|/4 + |\beta|/4 = |\beta|/2 \ (z \in b\Delta)
\]

so

\[
\beta - |\beta|/2 < \Re G(z) < \beta + |\beta|/2 \ (z \in \Delta),
\]

which, since \( \beta \neq 0 \), implies that \( G \neq 0 \) on \( b\Delta \) and that the change of argument of \( G \) around \( b\Delta \) is zero. We now repeat the reasoning from the proof in the smooth case to conclude that \( f - h \neq 0 \) on \( b\Delta \) and that the change of argument of \( f - h \) around \( b\Delta \) is negative. Since \( h \in A(\Delta) \) this completes the proof.

4. Harmonic functions and their conjugates

The main problem in generalizing the proof in Section 3 to multiply connected domains \( D \) is that in general, a harmonic function on \( D \) has no conjugate on \( D \).

We have assumed that \( D \subset \subset \mathfrak{C} \) is a domain bounded by pairwise disjoint circles. Denote these circles by \( \Gamma_1, \Gamma_2, \cdots, \Gamma_n \) where \( \Gamma_n \) is the boundary of the unbounded component of \( \mathfrak{C} \setminus \overline{D} \). For each \( k, 1 \leq k \leq n \), the harmonic measure function \( \omega_k \) is the continuous function on \( \overline{D} \), harmonic on \( D \) which satisfies \( \omega_k \equiv 1 \) on \( \Gamma_k \) and \( \omega_k \equiv 0 \) on \( \Gamma_j, \ 1 \leq j \leq n, j \neq k \). By the preceding discussion each \( \omega_k, 1 \leq k \leq n \), is smooth on \( \overline{D} \). We have \( \sum_{k=1}^{n} \omega_k \equiv 1 \) on \( \overline{D} \).

For each \( k, 1 \leq k \leq n - 1 \), let \( \gamma_k \) be a circle with the same center as \( \Gamma_k \) and with \( a \) a slightly larger radius, and let \( \gamma_n \) be a circle with the same center as \( \Gamma_n \) and with a slightly smaller radius so that the circles \( \gamma_k, 1 \leq k \leq n \) bound a domain \( D' \), slightly smaller than \( D \), whose closure is contained in \( D \). We give each \( \gamma_k \) the orientation induced by the standard orientation of \( bD' \).

Let \( u \) be a real-valued harmonic function on \( D \). A conjugate \( v \) of \( u \) has to satisfy the Cauchy-Riemann equations

\[
\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.
\]
This system is always solvable for $v$ locally. It is solvable for $v$ on $D$ if and only if
\[
\int_{\gamma_k} \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) = 0 \quad (1 \leq k \leq n - 1). \tag{4.1}
\]

Since
\[
2 \int_{\gamma_k} \frac{\partial u}{\partial z} dz = i \int_{\gamma_k} \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) \quad (1 \leq k \leq n - 1)
\]
(4.1) holds if and only if
\[
\int_{\gamma_k} \frac{\partial u}{\partial z} dz = 0 \quad (1 \leq k \leq n - 1). \tag{4.2}
\]

If $u$ is a real valued harmonic function on $D$ then there are real constants $c_1, c_2, \ldots, c_{n-1}$ such that the harmonic function $u + \sum_{j=1}^{n-1} c_j \omega_j$ has a conjugate on $D$. For this to happen we must have
\[
\int_{\gamma_k} \frac{\partial}{\partial z} \left[ u + \sum_{j=1}^{n-1} c_j \omega_j \right] dz = 0 \quad (1 \leq k \leq n - 1),
\]
that is,
\[
\sum_{j=1}^{n-1} c_j \int_{\gamma_k} \frac{\partial \omega_j}{\partial z} dz = -\int_{\gamma_k} \frac{\partial u}{\partial z} dz \quad (1 \leq k \leq n - 1). \tag{4.3}
\]

The system (4.3) has a unique solution since the matrix
\[
\left[ \int_{\gamma_k} \frac{\partial \omega_j}{\partial z} dz \right]_{1 \leq j, k \leq n-1} = \left[ \int_{\Gamma_k} \frac{\partial \omega_j}{\partial z} dz \right]_{1 \leq j, k \leq n-1}
\]
is known to be nonsingular [B, p.82]. (In the last equality we used the fact that all functions $\frac{\partial \omega_j}{\partial z}$ are smooth on $\overline{D}$ and holomorphic on $D$.) The Poisson formula implies that given $\varepsilon > 0$ and a compact set $K \subset D$ there is a $\delta > 0$ such that $|\frac{\partial u}{\partial z}| < \varepsilon$ on $K$ whenever $u$ is a real harmonic function on $D$ such that $|u| < \delta$ on $D$. Thus, given $\varepsilon > 0$ there is a $\delta > 0$ such that
\[
\left| \int_{\gamma_k} \frac{\partial u_1}{\partial z} dz - \int_{\gamma_k} \frac{\partial u}{\partial z} dz \right| < \varepsilon \quad (1 \leq k \leq n - 1)
\]
provided that $u_1$ is a real harmonic function on $D$ satisfying $|u_1(z) - u(z)| < \delta$ ($z \in D$).

The preceding discussion gives

**Lemma 4.1** Given a harmonic function $f$ on $D$ there is a unique $(n-1)$-tuple $c_1(f), \ldots, c_{n-1}(f)$ of complex numbers such that $f + \sum_{j=1}^{n-1} c_j(f) \omega_j$ has a conjugate on $D$. These numbers depend continuously on $f$ in the sup norm on $D$.

**5. Proof of Theorem 1.1, Part 1**

Let $D$ be a domain bounded by pairwise disjoint circles $\Gamma_1, \ldots, \Gamma_n$ where $\Gamma_n$ is the boundary of the unbounded component of $\mathcal{C} \setminus \overline{D}$. Section 3 contains the proof of Theorem 1.1 in the case $n = 1$ so assume that $n \geq 2$. 

Let \( f \) be a continuous function on \( bD \) which does not extend holomorphically through \( D \). Define
\[
A(a, f) = \mathcal{H}[(Z - a)f](a) = \mathcal{H}(Zf)(a) - a\mathcal{H}(f)(a) \quad (a \in \overline{D}).
\]
Since \( \mathcal{H}(f) \) is not holomorphic Lemma 2.2 implies that
\[
\{a \in D: A(a, f) = 0\} \text{ is a closed, nowhere dense subset of } D. \tag{5.1}
\]
There are constants \( c_k(f), 1 \leq k \leq n - 1 \), such that
\[
\mathcal{H}(f)(z) + \sum_{k=1}^{n-1} c_k(f) \omega_k(z) = P_f(z) + Q_f(z) \quad (z \in D) \tag{5.2}
\]
where \( P_f \) and \( Q_f \) are holomorphic functions on \( D \). Similarly, there are constants \( d_k(f), 1 \leq k \leq n - 1 \), such that
\[
\mathcal{H}(Zf)(z) + \sum_{k=1}^{n-1} d_k(f) \omega_k(z) = R_f(z) + S_f(z) \quad (z \in D) \tag{5.3}
\]
where \( R_f \) and \( S_f \) are holomorphic functions on \( D \). We know that the constants \( c_k(f) \) and \( d_k(f), 1 \leq k \leq n - 1 \), are determined uniquely and, by the maximum principle for harmonic functions depend continuously on \( f \in C(bD) \). We have
\[
c_k(e^{i\omega}f) = e^{i\omega}c_k(f), \quad d_k(e^{i\omega}f) = e^{i\omega}d_k(f) \quad (1 \leq k \leq n - 1, \, \omega \in \mathbb{R}) \tag{5.4}
\]
and
\[
A(a, e^{i\omega}f) = e^{i\omega}A(a, f) \quad (\omega \in \mathbb{R}, \, a \in D). \tag{5.5}
\]
Define
\[
\Phi_{a,f}(z) = \sum_{j=1}^{n-1} [d_j(f) - ac_j(f)][\omega_j(z) - \omega_j(a)] - A(a, f) \quad (z \in D).
\]
For each \( a \in D \) the function \( \Phi_{a,f} \) is smooth on \( \overline{D} \) and harmonic on \( D \). By the preceding discussion for each \( a \in D \) the harmonic function
\[
z \mapsto \mathcal{H}[(Z - a)f](z) + \Phi_{a,f}(z) \quad (z \in D)
\]
vanishes at \( a \) and has a conjugate on \( D \), that is, it is of the form \( F_{a,f} + \overline{G_{a,f}} \) where \( F_{a,f} \) and \( G_{a,f} \) are holomorphic on \( D \).

6. The function \( \Phi_{a,f} \)

Note that for each \( a \in D \) the function \( \Phi_{a,f} \) is smooth on \( \overline{D} \), harmonic on \( D \) and constant on each component \( \Gamma_j, 1 \leq j \leq n, \) of \( bD \).

**Lemma 6.1** There is an \( a \in D \) such that \( \Phi_{a,f}(z) \neq 0 \) \( (z \in bD) \).
Proof. Recall that for each \( a \in D \) the function \( \Phi_{a,f}|\Gamma_k \) is constant for each \( k, \, 1 \leq k \leq n \); we have to prove that for some \( a \in D \) these constants are all different from 0. We shall prove that

\[
\text{for each } k, \, 1 \leq k \leq n, \text{ the set } \{ a \in D : \Phi_{a,f}|\Gamma_k = 0 \} \text{ is a closed subset of } D \text{ with empty interior.} \quad (6.1)
\]

Assume that we have done this. Then \( \bigcup_{k=1}^n \{ a \in D : \Phi_{a,f}|\Gamma_k = 0 \} \) is a closed subset of \( D \) with empty interior which implies that there is an open dense subset of \( D \) of those \( a \) for which \( \Phi_{a,f}(z) \neq 0 \) \( (z \in bD) \) which will complete the proof. It remains to prove (6.1).

Let \( 1 \leq k \leq n - 1 \). On \( \Gamma_k \) the function \( \Phi_{a,f} \) is equal to the constant \( -A(a,f) + \sum_{j=1, j \neq k}^{n-1} [d_j(f) - ac_j(f)].[-\omega_j(a)] + [d_k(f) - ac_k(f)].[1 - \omega_k(a)] \). Since \( a \mapsto A(a,f) \) is continuous on \( D \) it follows that \( \{ a \in D : \Phi_{a,f}|\Gamma_k = 0 \} \) is a closed subset of \( D \). Suppose that it contains a disc \( U \). Then

\[
A(a,f) = \sum_{j=1, j \neq k}^{n-1} [d_j(f) - ac_j(f)].[-\omega_j(a)] + [d_k(f) - ac_k(f)].[1 - \omega_k(a)] \quad (6.2)
\]

for all \( a \in U \). Since both sides of (6.2) are real-analytic in \( a \) on \( D \) it follows that (6.2) holds for all \( a \in D \). Since both sides of (6.2) are continuous in \( a \) on \( \overline{D} \) it follows that (6.2) holds for all \( a \in bD \). However, \( A(a,f) = 0 \) \( (a \in bD) \) so

\[
\sum_{j=1, j \neq k}^{n-1} [d_j(f) - ac_j(f)].[\omega_j(a)] = [d_k(f) - ac_k(f)].[1 - \omega_k(a)] \quad (a \in bD). \quad (6.3)
\]

If \( a \in \Gamma_j, \, 1 \leq j \leq n - 1, j \neq k \), then \( \omega_j(a) = 1 \) and \( \omega_i(a) = 0 \) for all \( i, \, 1 \leq i \leq n, \, i \neq j \), so (6.3) implies that

\[
d_j(f) - ac_j(f) = d_k(f) - ac_k(f) \quad (a \in \Gamma_j, \, 1 \leq j \leq n - 1, j \neq k). \quad (6.4)
\]

If \( a \in \Gamma_n \) then \( \omega_j(a) = 0 \) for all \( j, \, 1 \leq j \leq n - 1 \), including \( k \), so (6.3) gives

\[
d_k(f) - ac_k(f) = 0 \quad (a \in \Gamma_n). \quad (6.5)
\]

Now, (6.5) implies that \( d_k(f) = c_k(f) = 0 \) which, by (6.4) gives \( d_j(f) = c_j(f) = 0 \) \( (1 \leq j \leq n - 1, \, j \neq k) \) so, by (6.2) it follows that \( A(a,f) = 0 \) for every \( a \in D \) which contradicts (5.1). This proves that \( \{ a \in D : \Phi_{a,f}|\Gamma_k = 0 \} \) has empty interior for each \( k, \, 1 \leq k \leq n - 1 \).

Let \( k = n \). We have

\[
\Phi_{a,f}|\Gamma_n = -A(a,f) + \sum_{j=1}^{n-1} [d_j(f) - ac_j(f)].[-\omega_j(a)]
\]

As before, the continuity of \( a \mapsto A(a,f) \) on \( D \) implies that the set \( \{ a \in D : \Phi_{a,f}|\Gamma_n = 0 \} \) is closed. Suppose that it has nonempty interior. As before, we get

\[
0 = A(a,f) = \sum_{j=1}^{n-1} [d_j(f) - ac_j(f)].[-\omega_j(a)] \quad (a \in bD) \quad (6.6)
\]
It follows that \( d_j(f) - ac_j(f) = 0 \) \((a \in \Gamma_j, \ 1 \leq j \leq n-1)\) which implies that \( d_j(f) = c_j(f) = 0 \) \((1 \leq j \leq n-1)\) so again \( A(a, f) \equiv 0 \) \((a \in D)\) which contradicts (5.1). This completes the proof.

7. Proof Theorem 1.1, Part 2

By Lemma 6.1 there is an \( a \in D \) such that the constants \( \Phi_{a,f}|_{\Gamma_k}, \ 1 \leq k \leq n, \) are all different from 0. By (5.4) and (5.5) we have \( \Phi_{a,e^i\omega f} = e^{i\omega f} \Phi_{a,f} \) \((\omega \in \mathbb{R})\) so replacing \( f \) by \( e^{i\omega f} \) if necessary we may assume with no loss of generality that

\[
(\Re \Phi_{a,f})|_{\Gamma_k} = \beta_k \neq 0 \quad (1 \leq k \leq n). \tag{7.1}
\]

To make the proof easier to understand we first complete it under the assumption that \( f \) is smooth. Assume that \( f \) is smooth. In this case, by Lemma 2.1,

\[
\mathcal{H}[(Z-a)f](z) + \Phi_{a,f}(z) = F_{a,f} + \overline{G_{a,f}(z)} \quad (z \in \overline{D})
\]

where \( F_{a,f} \) and \( G_{a,f} \) belong to \( A(D) \) so

\[
\Re \{ \mathcal{H}[(Z-a)f](z) + \Phi_{a,f}(z) \} = \Re [g(z)] \quad (z \in \overline{D})
\]

where \( g = (F_{a,f} + G_{a,f})/2 \in A(D) \). Clearly \( \Re [g(a)] = 0 \) so by adding an imaginary constant to \( g \) if necessary we may assume that \( g(0) = 0 \) so \( g(z) = (z-a)h(z) \) \((z \in \overline{D})\) where \( h \in A(D) \).

Consider the function \( z \mapsto G(z) = (z-a)f(z) - g(z) \) \((z \in bD)\). We have \( \Re [G(z)] = -\Re \Phi_{a,f}(z) \) \((z \in bD)\). By (7.1) for each \( k, \ 1 \leq k \leq n, \) the expression on the right is a nonzero constant on \( \Gamma_k \) which implies that for each \( k, \ 1 \leq k \leq n, \) the change of argument of \( z \mapsto G(z) = (z-a)[f(z) - h(z)] \) along \( \Gamma_k \) equals 0. Since \( a \in D \) the change of argument of \( z \mapsto (z-a) \) along each \( \Gamma_k, \ 1 \leq k \leq n-1, \) is zero, and the change of argument of \( z \mapsto (z-a) \) along \( \Gamma_n \) is \( 2\pi \). Thus, the change of argument of \( z \mapsto f(z) - h(z) \) along \( \Gamma_n \) equals \( -2\pi \). So, the change of argument of \( z \mapsto f(z) - h(z) \) along \( bD \) is negative. Since \( h \in A(D) \) this completes the proof in the case when \( f \) is smooth.

In the case of general \( f \) we have to approximate \( f \) by smooth functions. We already know that the constants \( c_k(f) \) and \( d_k(f) \) depend continuously on \( f \in C(bD) \). Further, for our fixed \( a \in D \) the maximum principle for harmonic functions implies that \( A(a, f) = \mathcal{H}[(Z-a)f](a) \) also depends continuously on \( f \in C(bD) \). It follows that \( \Phi_{a,f_1} \) is uniformly arbitrarily close to \( \Phi_{a,f} \) on \( \overline{D} \) provided that \( f_1 \in C(bB) \) is sufficiently close to \( f \). Fix \( \varepsilon, \)

\[
0 < \varepsilon < (1/4) \min\{ |\beta_k|; \ 1 \leq k \leq n \} \tag{7.3}
\]

and let \( f_1 \) be a smooth function on \( bD \) which is so close to \( f \) that

\[
|\Phi_{a,f_1}(z) - \Phi_{a,f}(z)| < \varepsilon \quad (z \in \overline{D}) \tag{7.4}
\]

and

\[
|(z-a)[f_1(z) - f(z)]| < \varepsilon \quad (z \in bD). \tag{7.5}
\]
As before, since $f_1$ is smooth, Lemma 2.1 applies to show that
\[
\Re\{H[(Z-a)f_1](z) + \Phi_{a,f_1}(z)\} = \Re[g(z)] \quad (z \in \overline{D}) \tag{7.6}
\]
where $g \in A(D)$ satisfies $g(a) = 0$. Consider the function $z \mapsto G(z) = (z - a)f(z) - g(z)$. We have $\Re[G(z)] = \Re\{(z - a)[f(z) - f_1(z)]\} + \Re[\Phi_{a,f}(z)] + \Re[\Phi_{a,f_1}(z) - \Phi_{a,f}(z)]$ so by (7.4) and (7.5) it follows that
\[
|\Re[G(z)] - \Re[\Phi_{a,f}(z)]| < 2\varepsilon \quad (z \in bD)
\]
so by (7.1) it follows that
\[
|\Re[G(z)] - \beta_k| < 2\varepsilon \quad (z \in \Gamma_k, \ 1 \leq k \leq n),
\]
which, by (7.3), implies that
\[
\beta_k - |\beta_k|/2 < \Re[G(z)] < \beta_k + |\beta_k|/2 \quad (z \in \Gamma_k, \ 1 \leq k \leq n).
\]
Since $\beta_k \neq 0$ ($1 \leq k \leq n$), it follows that for each $k$, $1 \leq k \leq n$, the change of argument of $G$ along $\Gamma_k$ is zero. Now we conclude the proof as in the smooth case.

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Institute of Mathematics, Physics and Mechanics
University of Ljubljana
Ljubljana, Slovenia
josip.globevnik@fmf.uni-lj.si