Privately Answering Classification Queries in the Agnostic PAC Model

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Abstract

We revisit the problem of differentially private release of classification queries. In this problem, the goal is to design an algorithm that can accurately answer a sequence of classification queries based on a private training set while ensuring differential privacy. We formally study this problem in the agnostic PAC model and derive a new upper bound on the private sample complexity. Our results improve over those obtained in a recent work [BTT18] for the agnostic PAC setting. In particular, we give an improved construction that yields a tighter upper bound on the sample complexity. Moreover, unlike [BTT18], our accuracy guarantee does not involve any blow-up in the approximation error associated with the given hypothesis class.

Given any hypothesis class with VC-dimension \( d \), we show that our construction can privately answer up to \( m \) classification queries with average excess error \( \alpha \) using a private sample of size \( \approx \frac{d}{\alpha^2} \max \left( 1, \sqrt{\frac{m\alpha^3}{2}} \right) \). Using recent results on private learning with auxiliary public data, we extend our construction to show that one can privately answer any number of classification queries with average excess error \( \alpha \) using a private sample of size \( \approx \frac{d}{\alpha^2} \max \left( 1, \sqrt{\frac{d\alpha}{2}} \right) \). Our results imply that when \( \alpha \) is sufficiently small (high-accuracy regime), the private sample size is essentially the same as the non-private sample complexity of agnostic PAC learning.

1 Introduction

In this paper, we revisit the problem of answering a sequence of classification queries in the agnostic PAC model under the constraint of \((\epsilon, \delta)\)-differential privacy. An algorithm for this problem is given a private training dataset \( S = \{(x_1, y_1), \ldots, (x_n, y_n)\} \) of \( n \) i.i.d. binary-labeled examples drawn from some unknown distribution \( D \) over \( X \times Y \), where \( X \) denotes an arbitrary data domain (space of feature-vectors) and \( Y \) denotes a set of binary labels (e.g., \( \{0, 1\} \)). The algorithm is also given as input some hypothesis class \( H \subseteq \{0, 1\}^Y \) of binary functions mapping \( X \) to \( Y \). The algorithm accepts a sequence of classification queries given by a sequence of i.i.d. feature-vectors \( Q = (\tilde{x}_1, \tilde{x}_2, \ldots) \), drawn from the marginal distribution of \( D \) over \( X \), denoted as \( D_X \). Here, the feature-vectors defining the set of queries \( Q \) do not involve any privacy constraint. The queries are also assumed to arrive one at a time, and the algorithm is required to answer the current query \( \tilde{x}_j \) by predicting a label \( \hat{y}_j \) for it before seeing the next query. The goal is to answer up to a given number \( m \) of queries (which is a parameter of the problem) such that, (i) the entire process of answering the \( m \) queries is \((\epsilon, \delta)\)-differentially private, and (ii) the average excess error in the predicted labels does not exceed some desired level \( \alpha \in (0, 1) \); specifically, \( \frac{1}{m} \sum_{j=1}^{m} 1(\hat{y}_j \neq \tilde{y}_j) \leq \alpha + \min_{h \in \mathcal{H}} \text{err}(h; D) \), where \( \tilde{y} \) is the corresponding (hidden) true label, and \( \min_{h \in \mathcal{H}} \text{err}(h; D) \) is the approximation error associated with \( \mathcal{H} \), i.e., the least possible true (population) error that can be attained by a hypothesis in \( \mathcal{H} \) (see Section 2 for formal definitions).

One could argue that a more direct approach for differentially private classification would be to design a differentially private learner that, given a private training set as input, outputs a classifier that is safe to publish and then can be used to answer any number of classification queries. However, there
are several pessimistic results that either limit or eliminate the possibility of differentially private learning even for elementary problems such as one-dimensional thresholds [BNSV15, ALMM18]. Therefore, it is natural to study the problem of classification-query release under differential privacy as an alternative approach.

A recent formal investigation of this problem was carried out in [BTT18]. This recent work gives an algorithm based on a combination of two useful techniques from the literature on differential privacy, namely, the sub-sample-and-aggregate technique [NRS07, ST13] and the sparse-vector technique [DR14]. The algorithm in [BTT18], hereafter denoted as \( A_{SubSamp} \), assumes oracle access to a generic, non-private (agnostic) PAC learner \( B \) for \( H \). In this work, we give non-trivial improvements over the results of [BTT18] in the agnostic PAC setting. More details on the comparison to [BTT18] are given in the “Related work” section below. Our improvements are in terms of the attainable accuracy guarantees and the associated private sample complexity bounds in the agnostic setting. These improvements are achieved via importing new ideas and techniques from literature (particularly, the elegant agnostic-to-realizable reduction technique of [BNS15]) to provide an improved construction for the one that appeared in [BTT18].

**Main results**

In this work, we formally study algorithms for classification queries release under differential privacy in the agnostic PAC model. We focus on the sample complexity of such algorithms as a function of the privacy and accuracy parameters as well as the number of queries to be answered.

- We give an algorithm for this problem that is well-suited for the agnostic setting. Our algorithm is a two-stage construction that is based on a careful combination of the relabeling technique of [BNS15] and the private classification algorithm \( A_{SubSamp} \) of [BTT18] (see “Techniques” section below).

- We show that our construction provides significant improvements over the results of [BTT18] for the agnostic setting:
  - The error guarantees in [BTT18] involves a constant blow-up (a multiplicative factor > 2) in the approximation error \( \min_{\mathcal{H}} \text{err}(h; D) \) associated with the given hypothesis class \( \mathcal{H} \). Using our construction, we give a standard excess error guarantee that does not involve such a blow-up.
  - We show that our construction can answer up to \( m \) queries with average excess error \( \alpha \) using a private sample whose size \( \approx \text{VC}(\mathcal{H})/\alpha^2 \cdot \max(1, \sqrt{m} \alpha^{3/2}) \) (assuming \( \epsilon \) is a constant, e.g. 0.1), where \( \text{VC}(\mathcal{H}) \) is the VC-dimension of \( \mathcal{H} \). Note that this implies that we can answer up to \( \approx 1/\alpha^5 \) queries with private sample size that is essentially the same as the standard non-private sample complexity of agnostic PAC learning. i.e., that many queries can be answered with essentially no additional cost due to privacy.
  - Using recent results of [ABM19] on the sample complexity of semi-private learners (introduced in [BNS13]), we show that our construction immediately leads to an universal private classification algorithm that can answer any number of classification queries using a private sample of size \( \approx \text{VC}(\mathcal{H})/\alpha \cdot \max(1, \sqrt{\text{VC}(\mathcal{H})} \alpha) \), which is independent of the number of queries. This implies that in the high accuracy regime \( \alpha < 1/\sqrt{\text{VC}(\mathcal{H})} \), we can privately answer any number of classification queries with private sample size that is essentially the same as the standard non-private sample complexity of agnostic PAC learning.

**Techniques:** Our algorithm is a two-stage construction. In the first stage, the input training set is pre-processed once and for all via a relabeling procedure due to [BNS15] in which the labels are replaced with the labels generated by an appropriately chosen hypothesis in the given hypothesis class \( \mathcal{H} \). This step allows us to reduce the agnostic setting to a realizable one. In the second stage, we first sample a new training set from the empirical distribution of the relabeled set in the first stage, then feed it to \( A_{SubSamp} \) of [BTT18] together with other appropriately chosen input parameters.
Related work

Our results are most closely related to [BTT18]. In [BTT18], Bassily et al. provide formal accuracy guarantees for their algorithm in both the realizable and agnostic settings of the PAC model. However, the accuracy guarantees they provide for the agnostic setting is far from optimal. In particular, their guarantees involves a constant blow-up in the approximation error $\min_{h \in \mathcal{H}} \text{err}(h; \mathcal{D})$, which would limit or eliminate the utility of their construction in scenarios where the approximation error is not negligible. In fact, in most typical scenarios in practice, the approximation error associated with the hypothesis (model) class is a non-negligible constant, (e.g., the test error attained by some state-of-the-art neural networks on benchmark datasets can be as large as 5%, or 10%). Our improved construction avoids this blow-up in the approximation error.

The construction in [BTT18] can answer up to $m$ queries with average excess error $\alpha + O(\gamma)$ (where $\gamma = \min_{h \in \mathcal{H}} \text{err}(h; \mathcal{D})$ is the approximation error) using a private sample of size $\approx \frac{\text{VC}(\mathcal{H})}{\alpha^2} \cdot \max(1, \sqrt{\frac{m}{\alpha}})$ (follows from [BTT18, Theorem 3.5]). Given our results discussed in the “Main results” section above, it follows that our sample complexity bound is tighter than that of [BTT18] by roughly a factor of $\max(1, \min(\sqrt{\frac{m}{\alpha}}, 1/\alpha))$. In particular, our bound is tighter by roughly a factor of $\frac{1}{\alpha}$ for $\frac{1}{2} \leq m < \frac{1}{\alpha}$, and it is tighter by roughly a factor of $\frac{1}{m}$ for $m \geq \frac{1}{\alpha}$. Equivalently, for the same private sample size, our construction can answer roughly a factor of $\sqrt{\frac{1}{\alpha^2}}$ more queries than that of [BTT18].

Bassily et al. [BTT18] also extend their construction to provide a semi-private learner that can finally produce a classifier. This is done by answering a sufficiently large number of queries then applying the knowledge transfer technique using the new training set formed by the set of answered queries. The output classifier can then be used to answer any subsequent queries, and hence, their extended construction provides a universal private classification algorithm. Their private sample complexity bound for this task is $\approx \text{VC}(\mathcal{H})^{3/2}/\alpha^{3/2}$ (see [BTT18, Theorem 4.3]). Given our results in the “Main results” section above, our universal private classification algorithm yields a private sample complexity bound that is tighter by roughly a factor of $\min\left(\sqrt{\frac{\text{VC}(\mathcal{H})}{\alpha}}, \frac{1}{\alpha}\right)$.

Other related works: The problem of differentially private classification has been considered directly or indirectly in several previous works, e.g., [HCB16] [PAE17] [PSM+18] [DF18]. Reference [DF18] considers the problem of differentially private classification in the single-query setting, and gives upper bounds on the private sample complexity for that problem in the PAC model. Our results imply that the bound shown in [DF18] for the agnostic setting is sub-optimal. In the single-query setting (i.e., $m = 1$), our bound is essentially optimal as it nearly matches the standard non-private sample complexity of the agnostic PAC model.

2 Preliminaries

Notation

For classification tasks we denote the space of feature vectors by $\mathcal{X}$, the set of labels by $\mathcal{Y}$, and the data universe by $\mathcal{U} = \mathcal{X} \times \mathcal{Y}$. A function $h : \mathcal{X} \to \mathcal{Y}$ is called a hypothesis and it labels data points in the feature space $\mathcal{X}$ by either 0 or 1 i.e. $\mathcal{Y} = \{0, 1\}$. A set of hypotheses $\mathcal{H} \subseteq \{0, 1\}^\mathcal{X}$ is called a hypothesis class. The VC dimension of $\mathcal{H}$ is denoted by $\text{VC}(\mathcal{H})$. We use $\mathcal{D}$ to denote a distribution defined over the space of feature vectors and labels $\mathcal{U} = \mathcal{X} \times \mathcal{Y}$, and $\mathcal{D}_\mathcal{X}$ to denote the marginal distribution over $\mathcal{X}$. A sample dataset of $n$ i.i.d. draws from $\mathcal{D}$ is denoted by $\mathcal{S} = \{(x_1, y_1), \cdots, (x_n, y_n)\}$, where $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$.

Expected error: The expected error of a hypothesis $h : \mathcal{X} \to \mathcal{Y}$ with respect to a distribution $\mathcal{D}$ over $\mathcal{U}$ is defined by $\text{err}(h; \mathcal{D}) \triangleq \mathbb{E}_{(x, y) \sim \mathcal{D}}[1(h(x) \neq y)]$. The excess expected error is defined as $\text{err}(h; \mathcal{D}) = \min_{h \in \mathcal{H}} \text{err}(h; \mathcal{D})$.

Empirical error: The empirical error of a hypothesis $h : \mathcal{X} \to \mathcal{Y}$ with respect to a labeled set $\mathcal{S}$ is denoted by $\tilde{\text{err}}(h; \mathcal{S}) \triangleq \frac{1}{n} \sum_{i=1}^{n} 1(h(x_i) \neq y_i)$.
The problem of minimizing the empirical error on a dataset is known as Empirical Risk Minimization (ERM). We use \( h_S^{\text{ERM}} \) to denote the hypothesis that minimizes the empirical error with respect to a dataset \( S \),
\[
\arg\min_{h \in H} \text{err}(h; S).
\]

**Expected disagreement:** The expected disagreement between a pair of hypotheses \( h_1 \) and \( h_2 \) with respect to a distribution \( \mathcal{D}_X \) is defined as \( \text{dis}(h_1, h_2; \mathcal{D}_X) \triangleq \mathbb{E}_{x \sim \mathcal{D}_X} [1(h_1(x)) \neq h_2(x)] \).

**Empirical disagreement:** The empirical disagreement between a pair of hypotheses \( h_1 \) and \( h_2 \) w.r.t. an unlabeled dataset \( S_u = \{x_1, \ldots, x_n\} \) is defined as
\[
\text{dis}(h_1, h_2; S_u) \triangleq \frac{1}{n} \sum_{i=1}^{n} 1(h_1(x_i)) \neq h_2(x_i)).
\]

**Realizable setting:** In the realizable setting of the PAC model, there exists a hypothesis \( h^* \in \mathcal{H} \) such that \( \text{err}(h^*; \mathcal{D}) = 0 \) i.e., the true labeling function is assumed to be in \( \mathcal{H} \). In this setting, the distribution \( \mathcal{D} \) can be completely described by \( \mathcal{D}_X \) and the hypothesis \( h^* \in \mathcal{H} \). Such a distribution \( \mathcal{D} \) is called **realizable** by \( \mathcal{H} \). Hence, for realizable distributions, the expected error of a hypothesis \( h \) will be denoted as \( \text{err}(h; (\mathcal{D}_X, h^*)) \).

**Definitions**

Next, we define the notion of differential privacy. For any two datasets \( S, S' \in \mathcal{U}^n \), we denote the symmetric difference between them by \( S \triangle S' \).

**Definition 2.1 ((\( \varepsilon, \delta \))-Differential Privacy [DKM+06, DMNS06]).** A (randomized) algorithm \( M : \mathcal{U}^n \to \mathcal{R} \) is \( (\varepsilon, \delta) \)-differentially private if for all pairs of datasets \( S, S' \in \mathcal{U}^n \) s.t. \( |S \triangle S'| = 1 \), and every measurable \( O \subseteq \mathcal{R} \), we have with probability at least \( 1 - \delta \) over the coin flips of \( M \), that:
\[
\Pr(M(S) \in O) \leq e^\varepsilon \cdot \Pr(M(S') \in O) + \delta.
\]

When \( \delta = 0 \), it is known as pure differential privacy, and parameterized only by \( \varepsilon \) in this case.

We study private classification algorithms that take as input a private labeled dataset \( S \sim \mathcal{D}^n \), and a sequence of classification queries \( Q = (\tilde{x}_1, \ldots, \tilde{x}_m) \sim \mathcal{D}_X^m \), defined by \( m \) unlabeled feature-vectors from \( \mathcal{X} \) (where \( m \) is an input parameter), and output a corresponding sequence of predictions, i.e., labels, \( (\tilde{y}_1, \ldots, \tilde{y}_m) \). Here, we assume that the classification queries come one at a time and the algorithm is required to generate a label for the current query before seeing and responding to the next query. The goal is: i) after answering \( m \) queries the algorithm should satisfy \( (\varepsilon, \delta) \)-differential privacy, and ii) the labels generated should be \( (\alpha, \beta) \)-accurate with respect to a hypothesis class \( \mathcal{H} \subseteq \{0, 1\}^\mathcal{X} \), a notion of accuracy which we formally define shortly. We give a generic description of the above classification paradigm in Algorithm 1 below (denoted as \( \mathcal{A}_{\text{PrivClass}} \)).

**Algorithm 1 \( \mathcal{A}_{\text{PrivClass}} \): Private Classification-Query Release Algorithm**

**Input:** Private dataset: \( S \in (\mathcal{X} \times \mathcal{Y})^n \), upper bound on the number of queries: \( m \), sequence of classification queries: \( Q = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m) \), hypothesis class: \( \mathcal{H} \), privacy parameters \( \varepsilon, \delta > 0 \), accuracy: \( \alpha \), and failure probability: \( \beta \)

\[
\begin{align*}
1: & \text{ for } j = 1, \ldots, m \text{ do} \\
2: & \quad \tilde{y}_j \leftarrow \text{PrivLabel}(S, \mathcal{H}, \tilde{x}_j) \{ \text{Generic procedure that, given } S, \mathcal{H} \text{ and current query } \tilde{x}_j, \text{ generates a label } \tilde{y}_j \} \\
3: & \text{ Output } \tilde{y}_j
\end{align*}
\]

The algorithm \( \mathcal{A}_{\text{PrivClass}} \) invokes a procedure \( \text{PrivLabel} \), which is a generic classification procedure that given the input private training set \( S \) and the knowledge of hypothesis class \( \mathcal{H} \), it generates a label for an input query (feature-vector) \( \tilde{x} \in \mathcal{X} \).

**Definition 2.2 ((\( \varepsilon, \delta, \alpha, \beta, n, m \))-Private Classification-Query Release Algorithm).** Let \( \varepsilon, \delta, \alpha, \beta \in (0, 1) \). Let \( \mathcal{H} \) be a hypothesis class \( \mathcal{H} \subseteq \{0, 1\}^\mathcal{X} \). A randomized algorithm \( \mathcal{A} \) (whose generic format is described in Algorithm 1) is said to be an \( (\varepsilon, \delta, \alpha, \beta, n, m) \)-PCQR (private classification-query release) algorithm for \( \mathcal{H} \), if the following conditions hold:
1. For any sequence $Q \in X^m$, $A$ is $(\epsilon, \delta)$-differentially private with respect to its input dataset.

2. For every distribution $D$ over $X \times Y$, given a dataset $S \sim D^n$ and a sequence $V \triangleq ((\tilde{x}_1, \tilde{y}_1), \ldots, (\tilde{x}_m, \tilde{y}_m)) \sim D^m$ (where $\tilde{x}_i$’s are the queried feature-vectors in $Q$ and $\tilde{y}_i$’s are their true hidden labels), $A$ is $(\alpha, \beta)$-accurate with respect to $H$, where our notion of $(\alpha, \beta)$-accuracy is defined as follows: With probability at least $1 - \beta$ over the choice of $S$, $V$, and the internal randomness in PrivLabel (Step 2 in Algorithm 1), we have

$$\frac{1}{m} \sum_{j=1}^{m} 1(\hat{y}_j \neq \tilde{y}_j) \leq \alpha + \gamma,$$

where $\gamma \triangleq \min_{h \in H} \text{err}(h; D)$.

In the realizable setting, we have an analogous definition where $\gamma = 0$. In this case, we say that the algorithm is a PCQR algorithm for $H$ in the realizable setting.

**Definition 2.3** ($\alpha$-cover for a hypothesis class). A family of hypotheses $\tilde{H}$ is said to form an $\alpha$-cover for a hypothesis class $H \subseteq \{0, 1\}^X$ with respect to distribution $D_X$ if for every $h \in H$ there exists a $\tilde{h} \in \tilde{H}$ such that $\mathbb{E}_{x \sim D_X} [1(h(x) \neq \tilde{h}(x))] \leq \alpha$.

2.1 Previous work on private classification-query release [BTT18]

In [BTT18], they give a construction for a PCQR algorithm (referred to as $A_{\text{SubSamp}}$), which combines the sub-sample-aggregate framework [NRS07, ST13] with the sparse-vector technique [DR14]. Bassily et al. [BTT18] provide formal privacy and accuracy guarantees with sample complexity bounds for $A_{\text{SubSamp}}$ in both the realizable and agnostic settings of the PAC model. Here, we briefly describe the algorithm $A_{\text{SubSamp}}$ (Algorithm 2 below), and restate the privacy and accuracy guarantees.

The input to $A_{\text{SubSamp}}$ is a private labeled dataset $S = \{(x_1, y_1), \ldots, (x_n, y_n)\}$, a sequence of classification queries $Q = (\tilde{x}_1, \ldots, \tilde{x}_m)$, and a generic non-private PAC learner $B$ for a hypothesis class $H$. The algorithm outputs a sequence of private labels $(\hat{y}^{\text{priv}}_1, \ldots, \hat{y}^{\text{priv}}_m)$. The key idea in $A_{\text{SubSamp}}$ is as follows: first, it arbitrarily splits $S$ into $k$ equal-sized sub-samples $S_1, \ldots, S_k$ for appropriately chosen $k$. Each of those sub-samples is used to train $B$. Hence, we obtain an ensemble of $k$ classifiers $h_{S_1}, \ldots, h_{S_k}$. Next for each input query $\tilde{x}_i \in Q$, the votes $(h_{S_1}(\tilde{x}_i), \ldots, h_{S_k}(\tilde{x}_i))$ are computed. It then applies the distance-to-instability test [ST13] on the difference between the largest count of votes and the second largest count. If the majority vote is sufficiently stable, $A_{\text{SubSamp}}$ returns the majority vote as the predicted label for $\tilde{x}_i$; otherwise, it returns a random label. The sparse-vector framework is employed to efficiently manage the privacy budget over the $m$ queries. In particular, by employing the sparse-vector technique, the privacy budget of $A_{\text{SubSamp}}$ is only consumed by those queries where the majority vote is not stable. Algorithm $A_{\text{SubSamp}}$ takes an input cut-off parameter $T$, which represents a bound on the total number of “unstable queries” the algorithm can answer before it halts in order to ensure $(\epsilon, \delta)$-differential privacy.
Algorithm 2 \( \mathcal{A}_{\text{SubSamp}} \) [BTT18]: Private Classification via subsample-aggregate and sparse-vector

**Input:** Private dataset: \( S \), upper bound on the number of queries: \( m \), sequence of classification queries: \( Q = \{ \tilde{x}_1, \ldots, \tilde{x}_m \} \), oracle access to non-private learner: \( B \), unstable query cutoff: \( T \), privacy parameters: \( \epsilon, \delta > 0 \), failure probability: \( \beta \)

1. \( c \leftarrow 0 \), \( \lambda \leftarrow \frac{\sqrt{3T} \log(2/\delta)}{\sqrt{n}} \) and \( k \leftarrow 34 \sqrt{2^\lambda \cdot \log(4mT / \min(\delta, \beta/2))} \)
2. \( w \leftarrow 2 \lambda \cdot \log(2m/\delta) \), \( \tilde{w} \leftarrow w + \text{Lap}(\lambda) \) \( \{ \text{Lap}(b) \text{ denotes the Laplace distribution with scale } b \} \)
3. Split \( S \) into \( k \) non-overlapping chunks \( S_1, \ldots, S_k \).
4. for \( j \in [k] \) do
   5. \( h_{S_j} \leftarrow B(S_j) \)
   6. for \( i \in [m] \) and \( c \leq T \) do
      7. \( F_i \leftarrow \{ h_{S_i}(x_i), \ldots, h_{S_k}(x_i) \} \) \( \{ \text{For every } y \in \{0,1\}, \text{let } ct(y) = \# \text{ times } y \text{ appears in } F_i \} \)
      8. \( \hat{q}_{xi} \leftarrow \arg\max_{y \in \{0,1\}} [ct(y)] \), \( \text{dist}_{\hat{q}_{xi}} \leftarrow \text{largest } ct(y) - \text{second largest } ct(y) \)
      9. \( y^\text{priv}_i \leftarrow \mathcal{A}_{\text{stab}}(S, \hat{q}_{xi}, \text{dist}_{\hat{q}_{xi}}, \tilde{w}, \frac{1}{\sqrt{m}}) \) \( \{ \text{Stability test for } \hat{q}_{xi}, \text{ given by Algorithm 3 below.} \} \)
     10. if \( y^\text{priv}_i = \perp \), then \( c \leftarrow c + 1 \), \( \tilde{w} \leftarrow w + \text{Lap}(\lambda) \)
11. Output \( y^\text{priv}_i \)

Algorithm 3 \( \mathcal{A}_{\text{stab}} \) [ST13]: Private estimator for \( f \) via distance to instability

**Input:** Dataset: \( S \), function: \( f : U^n \rightarrow R \), distance to instability: \( \text{dist}_f : U^n \rightarrow R \), threshold: \( \Gamma \), privacy parameter: \( \epsilon > 0 \)

1. \( \text{dist} \leftarrow \text{dist}_f(S) + \text{Lap}(1/\epsilon) \)
2. if \( \text{dist} > \Gamma \), then output \( f(S) \), else output \( \perp \)

Next, we restate the results of [BTT18] for the realizable and agnostic settings.

**Lemma 2.4** (Realizable Setting: follows from Theorems 3.2 & 3.4 in [BTT18]). Let \( \epsilon, \delta, \alpha, \beta \in (0, 1) \). Let \( \mathcal{H} \) be a hypothesis class with \( \text{VC}(\mathcal{H}) = d \). Suppose that \( B \in \mathcal{A}_{\text{SubSamp}} \) is a PAC learner for \( \mathcal{H} \). Let \( D \) be any distribution over \( U \) that is realizable by \( \mathcal{H} \). There is a setting for the cut-off parameter \( T = \max \left( 1, \tilde{O}(m \alpha) \right) \) such that \( \mathcal{A}_{\text{SubSamp}} \) is an \((\epsilon, \delta, \alpha, \beta, n, m)\)-PCQR algorithm for \( \mathcal{H} \) in the realizable setting where the private sample size is \( n = \tilde{O}(\frac{d}{\epsilon \alpha} \cdot \max(1, \sqrt{m \alpha})) \).

In the agnostic setting, the accuracy guarantee of [BTT18] is not compatible with Definition 2.2. the accuracy guarantee therein has a sub-optimal dependency on the approximation error, \( \gamma \) (where \( \gamma \equiv \min_{h \in \mathcal{H}} \text{err}(h; D) \)). In particular, their result entails a blow-up in \( \gamma \) by a constant factor (\( \approx 3 \)). This significantly limit the applicability of this result in scenarios where \( \gamma \geq\alpha \). In fact, in practical scenarios, it is typical to have non-negligible approximation error, which is a constant that does not depend on the sample size. For example, a class of neural networks may have \( \gamma = 0.1 \) (i.e., test accuracy cannot exceed 90%) but its excess error can be, say, \( \alpha = 10^{-8} \) (for a large enough sample).

**Lemma 2.5** (Agnostic Setting: follows from Theorems 3.2 & 3.5 in [BTT18]). Let \( \epsilon, \delta, \alpha, \beta \in (0, 1) \). Let \( \mathcal{H} \) be a hypothesis class with \( \text{VC}(\mathcal{H}) = d \). Suppose \( B \in \mathcal{A}_{\text{SubSamp}} \) is an agnostic PAC learner for \( \mathcal{H} \). Let \( D \) be any distribution over \( U \), and let \( \gamma = \min_{h \in \mathcal{H}} \text{err}(h; D) \). Let \( S \sim D^n \) denote the input private sample to \( \mathcal{A}_{\text{SubSamp}} \). Let \( V \equiv ((\tilde{x}_1, \tilde{y}_1), \ldots, (\tilde{x}_m, \tilde{y}_m)) \sim D^m \), where \( \tilde{x}_i \)'s are the queried feature-vectors in \( Q \) and \( \tilde{y}_i \)'s are their true (hidden) labels. Let \( y^\text{priv}_1, \ldots, y^\text{priv}_m \) denote the output labels of \( \mathcal{A}_{\text{SubSamp}} \). There is a setting for the cut-off parameter \( T = \max \left( 1, \tilde{O}(m(\alpha + \gamma)) \right) \) such that: 1) \( \mathcal{A}_{\text{SubSamp}} \) is \((\epsilon, \delta)\)-differentially private with respect to the input training set; 2) when the private sample is of size \( n = \tilde{O}(\frac{d}{\epsilon \alpha} \cdot \max(1, \sqrt{m \alpha})) \), then with probability at least \( 1 - \beta \) over \( S, V \) and the randomness in \( \mathcal{A}_{\text{SubSamp}} \), we have:

\[
\frac{1}{m} \sum_{j=1}^{m} 1(y_j^\text{priv} \neq \hat{y}_j) \leq \alpha + 3 \gamma.
\]
3 Private Release of Classification Queries in the Agnostic PAC Setting

In this section, we give the main results of this paper. We give an improved construction for the private classification-query release algorithm in [BTT18] in the general agnostic setting. Our construction can privately answer up to \( m \) queries with excess classification error \( \epsilon \), and input sample size \( \tilde{O} \left( \frac{\text{VC}(\mathcal{H}) \cdot \max \left(1, \sqrt{\frac{\text{VC}(\mathcal{H})}{\epsilon^2}} \right)}{\epsilon} \right) \). (where \( \tilde{O} \) hides log factors of \( m, 1/\alpha, 1/\delta, 1/\beta \)). Assuming \( \epsilon = \Theta(1) \), it follows that we can answer up-to \( \sim 1/\alpha^3 \) queries with a private sample whose size is essentially the same as the standard non-private sample complexity of agnostic PAC learning. Comparing to the result of [BTT18] for the agnostic setting, where the private sample size is \( \approx 1/\alpha^3 \), it follows that we can answer up-to \( \approx 1/\alpha^3 \) queries with a private sample whose size is essentially the same as the standard non-private sample complexity of agnostic PAC learning.

Comparing to the result of [BTT18] for the agnostic setting, where the private sample size is \( \approx 1/\alpha^3 \), it follows that we can answer up-to \( \approx 1/\alpha^3 \) queries with a private sample whose size is essentially the same as the standard non-private sample complexity of agnostic PAC learning.

Overview

Our construction is made up of two phases. The first phase is a pre-processing phase in which the input private sample, \( S \) is relabeled using a “good” hypothesis \( h \in \mathcal{H} \). This phase is a reenactment of the elegant technique due to Beimel et al. [BNS15], which was called LabelBoost Procedure therein. In this phase \( h \) can be considered as if it is the true labeling hypothesis and so we can reduce the agnostic setting to the realizable setting. By construction \( h \) is chosen such that it is close to the ERM hypothesis \( h^{\text{ERM}} \) (where \( S' \) is a subset of \( S \)). As the chosen input sample size is sufficiently large, \( h^{\text{ERM}} \) is a good hypothesis, i.e., it attains low excess error.

Now as we reduced the problem to the realizable setting, in the next phase we invoke the techniques in [BTT18]. In the second phase, the relabeled training set \( S'' \) is used to provide input training examples to \( \mathcal{A}_{\text{SubSamp}} \) of [BTT18] (described in Section 2.4). Note that \( S'' \) is no longer i.i.d., and hence we sample with replacement from \( S'' \) to form a new dataset. Algorithm \( \mathcal{A}_{\text{SubSamp}} \) then uses this new training set to privately generate labels for a sequence of classification queries. We need to carefully calibrate the privacy parameters of \( \mathcal{A}_{\text{SubSamp}} \) according to the input sample size, the target accuracy guarantee, and also the fact that input to \( \mathcal{A}_{\text{SubSamp}} \) is a re-sampled version of \( S'' \) and may contain multiple copies of the elements in \( S'' \). Note also that the distribution of the input dataset is no longer the true distribution \( D \) but the empirical distribution of \( S'' \). We give a careful analysis of the overall construction where we show that this re-sampling step does not impact our desired accuracy guarantees.

3.1 From the agnostic to the realizable setting: A generic reduction

In this section, we describe the pre-processing step mentioned earlier to reduce the agnostic setting to the realizable setting. This is done via adopting the relabeling technique devised by Beimel et al. in [BNS15]. We denote this procedure here as \( \mathcal{A}_{\text{Relabel}} \) (given by Algorithm 4 below). We briefly describe the algorithm \( \mathcal{A}_{\text{Relabel}} \) below, and state the privacy and accuracy guarantees for it.

Given a private labeled dataset \( S \sim D^n \) as input, \( \mathcal{A}_{\text{Relabel}} \) randomly chooses a subset \( S' \) of \( S \) of size \( n' \), where \( n' \approx cn \). Let \( S_u \) denote the unlabeled version of \( S' \), i.e., \( S_u = \{x_1, \ldots, x_{n'}\} \). Given a hypothesis class \( \mathcal{H} \) with VC(\( \mathcal{H} \)) = \( d \), \( \mathcal{A}_{\text{Relabel}} \) generates the set of all dichotomies on \( S_u \) that are realized by \( \mathcal{H} \). This is denoted as \( \prod_{\mathcal{H}}(S_u) = \{(h(x_1), \ldots, h(x_{n'})) : h \in \mathcal{H}\} \). It then chooses a finite subset \( \tilde{H} \subseteq \mathcal{H} \) such that each dichotomy in \( \prod_{\mathcal{H}}(S_u) \) is represented by one of the hypotheses in \( \tilde{H} \). We note that \( \tilde{H} \) forms an \( \alpha \)-cover for \( \mathcal{H} \). Also note that by Sauer’s lemma [Sau72], the size of \( \tilde{H} \) is \( O \left( (n'/d)^d \right) \). Finally, \( \mathcal{A}_{\text{Relabel}} \) chooses a hypothesis \( \hat{h} \) using the exponential mechanism with privacy parameter \( \epsilon' = 1 \) and a score function \( q(S', h) = -\epsilon' \cdot \text{err}(h; S') \). Then, \( \hat{h} \) is used to relabel \( S_u \), and finally output a labeled set \( S'' \).
Algorithm 4 $A_{\text{Relabel}}$: Relabel Procedure

**Input:** Training dataset $S \in (\mathcal{X} \times \mathcal{Y})^n$, Hypothesis class $\mathcal{H}$, parameter $\epsilon \leq 1$.

1. $\widetilde{H} \leftarrow \emptyset$, $n' \leftarrow \frac{\epsilon}{\varepsilon} \exp(\epsilon + \delta) n$.
2. Sample $n'$ random elements without replacement from $S$ and add to $S'$. Let $S_u = \{x_1, \ldots, x_n'\}$ be the unlabeled version of $S'$.
3. For every $(y_1, \ldots, y_n') \in \prod_{h \in \mathcal{H}}(S_u) = \{(h(x_1), \ldots, h(x_n')) : h \in \mathcal{H}\}$, add to $\widetilde{H}$ any arbitrary hypothesis $h \in \mathcal{H}$ s.t. $h(x_i) = y_i, \forall i \in [n']$.
4. Use the exponential mechanism with inputs $S'$, $\widetilde{H}$, privacy parameter $\epsilon = 1$, and a score function $q(S', h) \triangleq -\Delta(h; S')$ to select $h$ from $\widetilde{H}$.
5. Relabel $S_u$ using $\hat{h}$, and denote this relabeled dataset as $S''$.
6. Output $S''$.

**Lemma 3.1.** Let $\epsilon \leq 1$. Let $A$ be an $(\epsilon, \delta)$-differentially private. Let $B$ be an algorithm that on an input dataset $S$ runs $A_{\text{Relabel}}$ on $S$ and input parameter $\epsilon$, then invokes $A$ on the output dataset. Then, $B$ is $(\epsilon, \delta)$-differentially private.

**Proof.** The proof follows from a straightforward combination of [BNS15, Lemma 4.1] and privacy amplification by sampling [KLN08, LQS12]. For completeness, we give an outline here. Fix the randomness in dataset $S'$ due to sampling in Step 2 of $A_{\text{Relabel}}$. In this case, using [BNS15, Lemma 4.1], any algorithm $B$ that on input a dataset $S'$ applies $A$ on the output of $A_{\text{Relabel}}$ is $(\epsilon + 3, 4\epsilon\delta)$-differentially private. Hence, $B$ is $(\epsilon + 3, 4\epsilon\delta)$-differentially private. Now we take into account the randomness due to sampling in Step 2. By privacy amplification due to sampling [KLN08, LQS12], it follows that $B$ is $(\epsilon, \delta)$-differentially private.

The following lemma establishes the accuracy of the hypothesis $\hat{h}$ selected by the exponential mechanism in Step 4 of $A_{\text{Relabel}}$. In particular, the lemma asserts that the expected error of $\hat{h}$ is close to that of the ERM hypothesis $h^{\text{ERM}}_{S'}$.

**Lemma 3.2.** Let $\mathcal{H}$ be a hypothesis class with $\text{VC}(\mathcal{H}) = d$. Let $\alpha, \beta \in (0, 1)$. Let $\epsilon \leq 1$ be the input parameter to $A_{\text{Relabel}}$. Let $\mathcal{D}$ be an arbitrary distribution over $\mathcal{X} \times \mathcal{Y}$, and $S \sim \mathcal{D}^n$ be an input dataset to $A_{\text{Relabel}}$, where $n \geq 256 \frac{d(\alpha + \log(1/\alpha))}{2(\alpha^2 + \log (1/\alpha) + \log(3/\beta))}$. With probability at least $1 - \beta$, hypothesis $\hat{h}$ (generated in Step 4 of $A_{\text{Relabel}}$) satisfies the following:

$$\text{err} (\hat{h}; \mathcal{D}) - \text{err} (h^{\text{ERM}}_{S'}, \mathcal{D}) \leq \alpha,$$

where $h^{\text{ERM}}_{S'}$ is the ERM hypothesis w.r.t. the sample $S'$ generated in Step 2 of $A_{\text{Relabel}}$.

**Proof.** Note that the score function for the exponential mechanism is $-\Delta(h; S')$ whose global sensitivity is $1/n'$. Now, by using the standard accuracy guarantees of exponential mechanism [MT07] (and the fact that its instantiated here with privacy parameter $= 1$), w.p. $\geq 1 - \beta/3$ we have

$$\text{err} (\hat{h}; S') - \text{err} (h^{\text{ERM}}_{S'}, S') \leq \frac{2}{n'} \left( d \log \left( \frac{d+\log(3/\beta)}{2\alpha^2} \right) + \log(3/\beta) \right).$$

Given the value of $n$ in the lemma statement, we have $n' \geq 256 \frac{d(\alpha + \log(1/\alpha))}{2(\alpha^2 + \log (1/\alpha) + \log(3/\beta))}$. Using this setting of $n'$ together with Sauer’s Lemma [Sau72] to bound the size of $\tilde{H}$, it follows that:

$$\text{err} (\hat{h}; S') - \text{err} (h^{\text{ERM}}_{S'}, S') \leq \frac{2}{n'} \left( d \log \left( \frac{en'}{d} \right) + \log(3/\beta) \right) \leq \frac{80\alpha^2 \left( d \log(1/\alpha) + \log(3/\beta) \right)}{256(d + \log(3/\beta))} \leq \frac{\alpha}{3}.$$ (1)

Given the bound on $n'$ and the fact that $S' \sim \mathcal{D}^n$, by a standard uniform convergence argument from learning theory [SSBD14], we have the following generalization error bounds. With probability
$\geq 1 - 2\beta/3$, we have:

$$|\text{err}(\widehat{h}; D) - \text{err}(\widehat{h}; S')| \leq \alpha/3,$$

(2)

$$|\text{err}(h^{\text{ERM}}_S; D) - \text{err}(h^{\text{ERM}}_S; S')| \leq \alpha/3$$

(3)

Putting (1)-(3) together, we conclude that w.p. $\geq 1 - \beta$, we have $\text{err}(\widehat{h}; D) - \text{err}(h^{\text{ERM}}_S; D) \leq \alpha$. This completes the proof. □

3.2 A Private Classification-Query Release Algorithm

In this section, we describe the overall PCQR algorithm (Algorithm 5 below) that combines the two techniques given by $A_{\text{Relabel}}$ and $A_{\text{SubSamp}}$. As previously, Algorithm 5 (denoted by $A_{\text{AgPrivCl}}$) takes as input a private dataset $S \sim D_n$, the total number of queries $m$, and a sequence of classification queries $Q = (x_1, \ldots, x_m) \sim D_X^n$. Together with these, $A_{\text{AgPrivCl}}$ also has oracle access to a non-private PAC learner $B_{\text{PAC}}$ for a hypothesis class $\mathcal{H}$ (in the realizable setting). Note that, dataset $S''$ (output by $A_{\text{Relabel}}$) is relabeled using hypothesis $\widehat{h} \in \mathcal{H}$. In order to ensure that our input to the next stage is i.i.d., we sample $n'$ (size of $S''$) points with replacement from the empirical distribution of $S''$ to form a new dataset $\tilde{S}$. Next, we invoke $A_{\text{SubSamp}}$ in the realizable setting with dataset $\tilde{S}, m, Q$, and $B_{\text{PAC}}$ as inputs. We set the cut-off parameter of $A_{\text{SubSamp}}$ (the maximum number of allowable “unstable” queries) as $T = \tilde{O}(m\alpha)$, where $\alpha$ is the accuracy parameter of $B_{\text{PAC}}$. The privacy parameters to $A_{\text{SubSamp}}$ are set to $(\tilde{\epsilon}, \tilde{\delta})$, where $\tilde{\epsilon}, \tilde{\delta}$ will be specified later. This is needed to ensure $(\epsilon, \delta)$-differential privacy for the entire construction. Finally we output the sequence of private labels $\{y^{\text{priv}}_1, \ldots, y^{\text{priv}}_m\}$ generated by $A_{\text{SubSamp}}$.

**Algorithm 5 $A_{\text{AgPrivCl}}$: Private Agnostic-PAC Classification-Query Release Algorithm**

**Input:** Private dataset: $S \in (X \times Y)^n$, upper bound on the number of queries: $m$, sequence of classification queries: $Q = (x_1, \ldots, x_m)$, a hypothesis class: $\mathcal{H}$, oracle access to non-private learner: $B_{\text{PAC}}$ for $\mathcal{H}$, privacy parameters $\epsilon, \delta > 0$, accuracy parameter $\alpha$, and failure probability: $\beta$

1. $T \leftarrow \max \left(1, \frac{1}{8} m\alpha + \frac{1}{4} \sqrt{3m\alpha \log(\frac{1}{\epsilon})} \right),$

2. $\tilde{\epsilon} \leftarrow \frac{1}{\log(2/m)} \min \left(\sqrt{m^{3/2}} \log \left(\frac{1}{\epsilon}\right), \epsilon\right), \tilde{\delta} \leftarrow \frac{\delta}{e^{\tilde{\epsilon} \log(2/\tilde{\delta})}}$

3. Run $S'' \leftarrow A_{\text{Relabel}}(S, \mathcal{H}, \epsilon, \delta)$

4. $\tilde{S} \leftarrow$ Uniformly sample $n'$ points from $S''$ with replacement.

5. Output $(y^{\text{priv}}_1, \ldots, y^{\text{priv}}_m) \leftarrow A_{\text{SubSamp}}(\tilde{S}, m, Q, B_{\text{PAC}}, T, \tilde{\epsilon}, \tilde{\delta}, \beta)$

We formally state the main result of this paper in the following theorem.

**Theorem 3.3.** Let $\mathcal{H}$ be a hypothesis class with $\text{VC}(\mathcal{H}) = d$. For any $\epsilon, \delta, \alpha, \beta \in (0, 1)$, $A_{\text{AgPrivCl}}$ (Algorithm 5) is an $(\epsilon, \delta, \alpha, \beta, n, m)$-PCQR algorithm for $\mathcal{H}$, where private sample size

$$n = O \left( \frac{d + \log \left( \frac{1}{\alpha \beta} \right) \log^3 \left( \frac{1}{\epsilon} \right) \log \left( \frac{m\alpha}{\min(3/2, 2d)} \right)}{\epsilon \alpha^2} \right),$$

max \left(1, \sqrt{m^{3/2}} \log \left(\frac{1}{\epsilon}\right) \right),$$

and number of queries $m = \Omega \left( \frac{\log(1/\alpha \beta)}{\epsilon \alpha} \right)$.

**Proof.** We will prove the theorem via the following two lemmas that establish the privacy and accuracy guarantees of $A_{\text{AgPrivCl}}$. □

**Lemma 3.4** (Privacy Guarantee of $A_{\text{AgPrivCl}}$). $A_{\text{AgPrivCl}}$ is $(\epsilon, \delta)$-differentially private (with respect to its input dataset).
Proof. Let \( R(\cdot) \) denote the uniform sampling procedure in Step 3 in \( A_{\text{AgPrivCl}} \); that is, Step 3 can be written as \( \hat{S} \leftarrow R(S'') \). Note that Steps 3\&4 in \( A_{\text{AgPrivCl}} \) can now be expressed as a composition \( R \circ \text{SubSamp} \), where \( R \circ \text{SubSamp}(\cdot) \triangleq \text{SubSamp}(R(\cdot)) \).

In order to prove that \( A_{\text{AgPrivCl}} \) is \((\epsilon, \delta)\)-differentially private, it suffices to show that \( R \circ \text{SubSamp} \) is \((\epsilon, \delta)\)-differentially private. Note that the input to \( R \circ \text{SubSamp} \) dataset \( S'' \), is output by \( A_{\text{AgPrivCl}} \). Hence, it follows from Lemma 3.1 that if \( R \circ \text{SubSamp} \) is \((\epsilon, \delta)\)-differentially private, then \( A_{\text{AgPrivCl}} \) is \((\epsilon, \delta)\)-differentially private. Next, we show that \( R \circ \text{SubSamp} \) is \((\epsilon, \delta)\)-differentially private with respect to \( S'' \).

Let \( S''_1 \) and \( S''_2 \) be neighboring datasets. W.l.o.g., assume that \( S''_1 \) and \( S''_2 \) differ in index \( j \in [n'] \). Let \( r \) be the number of times the \( j \)-th index is sampled by \( R \). By property of \( R \), and Chernoff bound, w.p. \( 1 - \delta/2, r \leq \log(2/\delta) \).

Using the result in [BTT18] Theorem 3.1, \( \text{SubSamp} \) is \((\epsilon, \delta)\)-differentially private with respect to \( \hat{S} \). Conditioned on \( r \leq \log(\frac{1}{\epsilon}) \) and by the notion of group privacy we have, \( R \circ \text{SubSamp}(S') \) is \((\epsilon r, \epsilon r^\epsilon, \delta)\)-differentially private. From the above high probability bound on the event \( r \leq \log(\frac{1}{\epsilon}) \), we conclude that \( R \circ \text{SubSamp} \) is \((\min(\frac{m}{n}, \alpha^{3/2} \log(\frac{1}{\epsilon}), \epsilon), \delta)\)-differentially private. This implies that, \( A_{\text{AgPrivCl}} \) is \((\epsilon, \delta)\)-differentially private.

Lemma 3.5 (Accuracy Guarantee of \( A_{\text{AgPrivCl}} \)). Let \( \mathcal{H} \) be a hypothesis class with VC(\( \mathcal{H} \)) = \( d \). Let \( B_{\text{PAC}} \) (invoked by \( \text{SubSamp} \)) be a \( \text{PAC} \) learner for \( \mathcal{H} \) (in the realizable setting). Let \( D \) be any distribution over \( X \times Y \). Let \( S \sim D^n \) denote the input private sample to \( A_{\text{AgPrivCl}} \), where

\[
n = 2048 \left( d + \log \left( \frac{m}{\alpha} \right) \right)^{3/2} \log \left( \frac{\min(\delta, \beta/2)}{\epsilon} \right) \max \left( 1, \frac{\sqrt{m} \alpha^{3/2} \log(\frac{1}{\epsilon})}{\epsilon} \right) (3 + \exp(\epsilon + 3)) \epsilon^{-\alpha^2},
\]

and \( m \geq 8 \frac{\log(1/\alpha^\beta)}{\alpha} \). Let \( (\tilde{y}_1, \ldots, \tilde{y}_m) \) denote the corresponding true (hidden) labels for \( Q \). Then, w.p. at least \( 1 - \beta \) (over the choice of \( S, Q \), and the randomness in \( A_{\text{AgPrivCl}} \)), we have:

\[
\frac{1}{m} \sum_{j=1}^{m} 1(y_j^{\text{priv}} \neq \tilde{y}_j) \leq \alpha + \gamma,
\]

where \( \gamma \leq \min_{h \in \mathcal{H}} \text{err}(h; D) \).

In the proof of Lemma 3.5 we will use the following claim. We defer the proof of this claim after the proof of the lemma.

Claim 3.6. Let \( \mathcal{H} \) be a hypothesis class with VC(\( \mathcal{H} \)) = \( d \). Let \( S_u \) be an an unlabeled training set of size \( n_u \), where \( n_u \geq 10 \frac{d + \log(1/\alpha^\beta)}{\alpha} \). Then, with probability at least \( 1 - \beta' \) for any \( h_1, h_2 \in \mathcal{H} \), we have

\[
\left| \text{dis}(h_1, h_2; D_X) - \text{dis}(h_1, h_2; S_u) \right| \leq \alpha. \quad \text{(Recall that \( \text{dis}(h_1, h_2; D_X) \) and \( \text{dis}(h_1, h_2; S_u) \) are the expected and empirical disagreement rates, respectively, as defined in Section 2)}
\]

Proof. Consider the description of \( A_{\text{Relabel}} \) in Algorithm 4. Note that, hypothesis \( \hat{h} \in \mathcal{H} \) selected in Step 4 of \( A_{\text{Relabel}} \) is used to generate labels of \( S'' \) (output dataset of \( A_{\text{Relabel}} \)). Note that the size of \( S'' \) is \( n' = 2048 \left( \frac{d + \log(\frac{1}{\epsilon})}{\alpha} \right) \log(2/\delta) \). Hence, the empirical distribution induced by \( S'' \) in \( A_{\text{AgPrivCl}} \) dataset \( \tilde{S} \) (input to \( \text{SubSamp} \)) is created by \( n' \) i.i.d. draws from \( D_{S'} \). Note that \( \text{err}(\hat{h}; D_{S'}) = 0 \).

From the description of \( \text{SubSamp} \) (Algorithm 3), \( \text{SubSamp} \) splits \( \tilde{S} \) into \( k \) equal-sized sub-samples, where \( k = \tilde{O} \left( \frac{n'}{\alpha} \right) \). Here \( T \) is the input cut-off parameter of \( \text{SubSamp} \) whose setting is given in Step 1 of \( A_{\text{AgPrivCl}} \). Note that since \( m = \Omega \left( \frac{\log(1/\alpha^\beta)}{\alpha} \right) \), we have \( T = O(m\alpha) \). Each sub-sample is then fed separately as an input to \( B_{\text{PAC}} \). For each input sub-sample, \( B_{\text{PAC}} \) outputs a classifier \( h_j, j \in [k] \). Hence we end up with an ensemble of classifiers \( h_1, \ldots, h_k \). Note that the
size of the input sub-sample to $B_{PAC}$ is $\frac{n'}{k}$. Observe that

$$n' = 2048 \left(\frac{d + \log(\frac{m}{\alpha})}{\alpha^2}\right) \log^{3/2}\left(\frac{1}{\epsilon}\right) \log\left(\frac{m \alpha}{\min(\delta, \beta / 2)}\right) \cdot \max\left(1, \sqrt{\frac{m \alpha^{3/2} \log\left(\frac{1}{\epsilon}\right)}{\delta}}\right),$$

and the number of sub-samples $k$ is set in Step II of $A_{SubSamp}$ as follows

$$k = O\left(\frac{\sqrt{m \alpha \log\left(\frac{1}{\epsilon}\right) \cdot \log\left(\frac{m \alpha}{\min(\delta, \beta / 2)}\right)}}{\alpha}\right).$$

Hence, using the setting of $\epsilon$ in Step II of $A_{AgPrivCl}$, we have

$$\frac{n'}{k} = \Omega\left(\frac{\left(d + \log\left(\frac{m}{\alpha}\right)\right) \cdot \epsilon \cdot \min\left(1, \sqrt{\frac{m \alpha^{3/2} \log\left(\frac{1}{\epsilon}\right)}{\delta}}\right)}{\alpha}\right) \cdot \max\left(1, \sqrt{\frac{m \alpha^{3/2} \log\left(\frac{1}{\epsilon}\right)}{\delta}}\right)$$

$$= \Omega\left(\frac{\left(d + \log\left(\frac{m}{\alpha}\right)\right) \log\left(\frac{1}{\epsilon}\right)}{\alpha}\right) = \Omega\left(\frac{\left(d + \log\left(\frac{16k}{\epsilon}\right)\right) \log\left(\frac{1}{\epsilon}\right)}{\alpha}\right).$$

By standard results in learning theory, it is easy to see that the size of the input sub-sample to $B_{PAC}$ is sufficient for $B_{PAC}$ to PAC-learn $H$ with respect to $D_{S''}$ with accuracy $\frac{\alpha}{2k}$ and confidence $\frac{1}{2\alpha}$. Fix any $j \in [k]$. Using the above fact about $B_{PAC}$ with probability at least $1 - \frac{1}{n'}$, we have $\text{err}(h_j; D_{S''}) \leq \frac{\alpha}{2k}$. Since $D_{S''}$ is the empirical distribution of $S''$, equivalently, we have $\text{dis}(h_j, \tilde{h}; S_u) \leq \frac{\alpha}{2k}$, where $S_u$ is the unlabeled version of $S''$. Note that the size of $S''$ is $n' \geq 1560 (d + \log\left(\frac{16k}{\epsilon}\right))$. Hence, by Claim 3.6 it follows that w.p. $\geq 1 - \frac{1}{2k}$, $\text{dis}(h_j, \tilde{h}; D_X) \leq \alpha/12$. Equivalently, w.p. $\geq 1 - \frac{d}{2k}$.

$$\text{err}\left(h_j; (D_X, \tilde{h})\right) \leq \frac{\alpha}{12}$$

From the above and the fact that the queries in $Q$ are i.i.d from $D_X$, we invoke the same counting argument in the proof of [BYT18] Theorem 3.2 to show that w.p. $\geq 1 - \frac{1}{2}$, the output labels of $A_{SubSamp}$ satisfy:

$$\frac{1}{m} \sum_{i=1}^{m} \mathbf{1}(y^\text{priv}_i \neq \tilde{h}(\tilde{x}_i)) \leq \frac{\alpha}{4} \quad (4)$$

Let $h^{\text{ERM}}_{S'}$ denote the ERM hypothesis with respect to the dataset $S'$ constructed in Step II of $A_{Relabel}$. Note that Lemma 3.2 implies that w.p. $\geq 1 - \beta/4$

$$\text{err}(\tilde{h}; D) - \text{err}(h^{\text{ERM}}_{S'}, D) \leq \alpha/4.$$

Since the queries and their true labels $((\tilde{x}_1, \tilde{y}_1), \ldots, (\tilde{x}_m, \tilde{y}_m))$ are drawn i.i.d. from $D$, then by Chernoff’s bound and the fact that $m \geq 8 \log\left(\frac{1/\beta}{\alpha}\right)$, we get that w.p. $\geq 1 - \beta/2$ we have

$$\frac{1}{m} \sum_{i=1}^{m} \mathbf{1}(\tilde{h}(\tilde{x}_i) \neq \tilde{y}_i) = \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}(h^{\text{ERM}}_{S'}(\tilde{x}_i) \neq \tilde{y}_i) \leq \alpha/2. \quad (5)$$

Moreover, from the bound on $n'$ and using a basic fact from learning theory, w.p. $\geq 1 - \beta/8$, the ERM hypothesis $h^{\text{ERM}}_{S'}$ satisfies

$$\text{err}(h^{\text{ERM}}_{S'}; D) \leq \alpha/8 + \gamma,$$
where \( \gamma = \min_{h \in \mathcal{H}} \text{err}(h; \mathcal{D}) \). Again, since \( (\tilde{x}_1, \tilde{y}_1), \ldots, (\tilde{x}_m, \tilde{y}_m) \) are i.i.d. from \( \mathcal{D} \), then by Chernoff’s bound and the fact that \( m \geq \frac{8 \log(1/\beta)}{\alpha} \), w.p. \( \geq 1 - \beta/4 \), we have

\[
\frac{1}{m} \sum_{i=1}^{m} 1(h_{\text{ERM}}(\tilde{x}_i) \neq \tilde{y}_i) \leq \alpha/4 + \gamma.
\]

Now, using (4), (5), and (6) together with a simple application of the triangle inequality and the union bound, we conclude that, w.p. \( \geq 1 - \beta \), we have

\[
\frac{1}{m} \sum_{j=1}^{m} 1(y_j^{\text{priv}} \neq \tilde{y}_j) \leq \alpha + \gamma.
\]

We now prove Claim 3.6 stated earlier. Note that the proof technique of this claim is similar to that of [ABMT19, Lemma 3.3].

**Proof of Claim 3.6.** For \( S_u \sim \mathcal{D}_X^{n_u} \), define the event

\[
\text{Bad} = \{ \exists h_1, h_2 \in \mathcal{H} : |\text{dis}(h_1, h_2; \mathcal{D}_X) - \hat{\text{dis}}(h_1, h_2; S_u)| > \alpha \}
\]

We will show that

\[
P_{S_u \sim \mathcal{D}_X^{n_u}}[\text{Bad}] \leq 2 \left( \frac{en_u}{d} \right)^{2d} \exp \left( -n_u \alpha^2/8 \right).
\]

By using a standard manipulation, we will show that this is bounded by \( \beta' \) when \( n_u \) is as given in the statement of the claim.

Let \( \mathcal{H}_\Delta \) be a hypothesis class defined as \( \mathcal{H}_\Delta \triangleq \{ h_1 \Delta h_2 : h_1, h_2 \in \mathcal{H} \} \). Here the function \( h_1 \Delta h_2 : X \rightarrow \{0, 1\} \) is defined as

\[
\forall x \in X, \ h_1 \Delta h_2(x) \triangleq 1(h_1(x) \neq h_2(x)).
\]

Let \( G_{\mathcal{H}_\Delta} \) denote the growth function of \( \mathcal{H}_\Delta \); i.e. for any number \( t \),

\[
G_{\mathcal{H}_\Delta}(t) \triangleq \max_{V : |V| = t} \left| \prod_{h \in \mathcal{H}_\Delta}(V) \right|,
\]

where \( \prod_{h \in \mathcal{H}_\Delta}(V) \) is the set of all dichotomies that can be generated by \( \mathcal{H}_\Delta \) on a set \( V \) of size \( t \). Now for any set \( V \) of size \( t \), every dichotomy in \( \prod_{h \in \mathcal{H}_\Delta}(V) \) is determined by a pair of dichotomies in \( \prod_{h \in \mathcal{H}_\Delta}(V) \), and thus we get \( |\prod_{h \in \mathcal{H}_\Delta}(V)| \leq |\prod_{h \in \mathcal{H}_\Delta}(V)|^2 \). Hence, by Sauer’s Lemma \( G_{\mathcal{H}_\Delta}(t) \leq G_H(t) \leq \left( \frac{4t}{d} \right)^{2d} \). Let \( h_0 \) be the all-zero hypothesis. Note that \( h_0 \in \mathcal{H}_\Delta \). Now, using a standard VC-based uniform convergence argument we have,

\[
P_{S_u \sim \mathcal{D}_X^{n_u}}[\exists h_1, h_2 \in \mathcal{H} : |\text{dis}(h_1, h_2; \mathcal{D}_X) - \hat{\text{dis}}(h_1, h_2; S_u)| > \alpha] \leq 2G_{\mathcal{H}_\Delta} \exp \left( -n_u \alpha^2/8 \right) \leq 2 \left( \frac{en_u}{d} \right)^{2d} \exp \left( -n_u \alpha^2/8 \right)
\]

Note that the inequality in the third line is non-trivial, and is used unanimously in VC-based uniform convergence bounds (see e.g., [SSBD14]).
4 Privately Answering Any Number of Classification Queries

In this section, we describe an universal PCQR algorithm that can answer any number of queries with private sample size that is independent of the number of queries. The main idea is that after answering a number of queries \( \approx \frac{\VC(H)}{\alpha} \), we can use the feature-vectors defining those queries as an auxiliary “public” dataset. Recall that as defined earlier in our problem statement, the set of queries themselves do not entail any privacy constraints. We can then invoke the framework of semi-private learning introduced in [BNS13], where such auxiliary public dataset can be exploited to finally generate a classifier that is safe to publish. In particular, a semi-private learner takes as input two types of datasets: a private labeled dataset, and another auxiliary public dataset. The algorithm needs to satisfy differential privacy only with respect to the private dataset.

A recent construction of a semi-private learner is given in [ABM19] Algorithm 1 (referred to as \( A_{SSPP} \)), where it is shown that it suffices to have a public unlabeled dataset of size \( \approx \frac{\VC(H)}{\alpha} \) drawn i.i.d. from \( D_X \), and a private labeled training set of \( \approx \frac{\VC(H)}{\alpha} \) drawn i.i.d. from \( D \). \( A_{SSPP} \) outputs a classifier \( h_{\text{priv}} \in H \) such that \( \text{err}(h_{\text{priv}}; D) - \min_{h \in H} \text{err}(h; D) \leq \alpha \) w.r.t \( D \). Hence, \( A_{SSPP} \) outputs a classifier that can be used to answer any number of classification queries. The main idea of the construction in [ABM19] is that the public unlabeled dataset can be used to create a finite \( \alpha \)-cover for \( H \), and hence, reducing the task of privately learning \( H \) to the task of learning a finite sub-class of \( H \) (the \( \alpha \)-cover).

Using this result, we can extend our construction in Section 3.2 to allow for privately answering any number of classification queries using a private training set whose size is independent of the number of queries. In Algorithm 6 below (denoted as \( A_{\text{UnvPrivCl}} \)), we describe our universal PCQR algorithm.

\begin{algorithm}[H]
\caption{\( A_{\text{UnvPrivCl}} \): Universal Private Classification-Query Release Algorithm}
\begin{algorithmic}[1]
\STATE Input: Private dataset: \( S \in U^n \), number of queries: \( m \), sequence of classification queries: \( Q = (\tilde{x}_1, \ldots, \tilde{x}_m) \), hypothesis class: \( H \), oracle access to a non-private PAC learner for \( H \): \( B_{\text{PAC}} \), privacy parameters \( \epsilon, \delta > 0 \), accuracy parameter \( \alpha \), and failure probability \( \beta \).
\STATE 1: \( m_\alpha \leftarrow O \left( \frac{d \log(1/\alpha) + \log(1/\delta)}{\alpha} \right) \)
\STATE 2: \( T_{\text{pub}} \leftarrow (\tilde{x}_1, \ldots, \tilde{x}_{m_\alpha}) \)
\STATE 3: Output \( (y^\text{priv}_1, \ldots, y^\text{priv}_{m_\alpha}) \leftarrow A_{\text{AgPrivCl}}(S, m_\alpha, T_{\text{pub}}, H, B_{\text{PAC}}, \epsilon, \delta, \alpha, \beta) \)
\STATE 4: \( h_{\text{priv}} \leftarrow A_{SSPP}(S, T_{\text{pub}}, H, \epsilon) \) \{As appears in [ABM19] Section 3\}
\FOR {j = m_\alpha + 1, \ldots, m}
\STATE Output \( y^\text{priv}_j \leftarrow h_{\text{priv}}(\tilde{x}_j) \)
\ENDFOR
\end{algorithmic}
\end{algorithm}

We finally formalize this observation in the following theorem.

\textbf{Theorem 4.1.} Let \( H \) be any hypothesis class with \( \VC(H) = d \). For any \( \epsilon, \delta, \alpha, \beta \in (0, 1) \) and any \( m < \infty \), \( A_{\text{UnvPrivCl}} \) is an \((\epsilon, \delta, \alpha, \beta, n, m)\)-PCQR algorithm for \( H \) with private sample size

\[ n = O \left( \frac{d + \log \left( \frac{m_\alpha}{\alpha} \right)}{\epsilon} \log^{3/2} \left( \frac{1}{\alpha} \right) \log \left( \frac{m_\alpha^\alpha}{\text{min}(1, \frac{1}{2})} \right) \max \left( 1, \frac{\sqrt{d} \log^{3/2} \left( \frac{1}{\alpha} \right)}{\epsilon} \right) \right), \]

where \( m_\alpha = O \left( \frac{d \log(1/\alpha) + \log(1/\delta)}{\alpha} \right) \) (as set in Step 1). In particular, when \( \alpha \leq \frac{1}{\sqrt{d}} \), it would suffice to have a private sample of size \( n = O \left( \frac{d}{\epsilon \alpha^2} \right) \).

\section*{Acknowledgement}

The authors would like to thank Uri Stemmer, Amos Beimel, and Kobbi Nissim for pointing us to their elegant LabelBoost procedure in [BNS15] which is the crux of the pre-processing step of our
algorithm. We are also grateful to them for the several insightful discussions we had about this line of research.

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