Numerical study of coorbital thermal torques on cold or hot satellites

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ABSTRACT

We evaluate the thermal torques exerted on low-mass planets embedded in gaseous protoplanetary discs with thermal diffusion, by means of high-resolution three-dimensional hydrodynamics simulations. We confirm that thermal torques essentially depend on the offset between the planet and its corotation, and find a good agreement with analytic estimates when this offset is small compared to the size of the thermal disturbance. For larger offsets that may be attained in discs with a large pressure gradient or a small thermal diffusivity, thermal torques tend toward an asymptotic value broadly compatible with results from a dynamical friction calculation in an unsheared medium. We perform a convergence study and find that the thermal disturbance must be resolved over typically 10 zones for a decent agreement with analytic predictions. We find that the luminosity at which the net thermal torque changes sign matches that predicted by linear theory within a few percents. Our study confirms that thermal torques usually supersede Lindblad and corotation torques by almost an order of magnitude for low mass planets. As we increase the planetary mass, we find that the ratio of thermal torques to Lindblad and corotation torques is progressively reduced, and that the thermal disturbance is increasingly distorted by the horseshoe flow. Overall, we find that thermal torques are dominant for masses up to an order of magnitude larger than implemented in recent models of planetary population synthesis. We finally briefly discuss the case of stellar or intermediate-mass objects embedded in discs around AGNs.

Keywords: hydrodynamics – protoplanetary discs – planet-disc interactions – galaxies: nuclei – accretion, accretion discs.

1 INTRODUCTION

Protoplanets gravitationally interact with the protoplanetary discs out of which they form. This interaction leads to a progressive change of their orbital elements, in particular of their semi-major axis. This process, known as planetary migration, has been extensively studied since the realisation by Goldreich & Tremaine (1980) that the semi-major axis could vary by large factors over the lifetime of the protoplanetary disc. Most analytic studies focused on planets embedded in isothermal or adiabatic discs (the disc’s torque is essentially the same in these two limit cases, to within a factor \( \gamma \), the adiabatic index). It was found that for realistic disc profiles, the disc’s torque is generally a negative quantity that leads to a decrease of the planet’s orbital radius with time Ward (1986); Tanaka et al. (2002). Protoplanetary discs are neither isothermal nor adiabatic, however. They are subjected to heat transport, essentially effected by radiative transfer. Thermal diffusion depends on a number of parameters, such as the disc’s temperature and density, and the opacity of its dust component. Planet forming regions of protoplanetary discs are thought to have a thermal diffusivity of same order, or larger than their kinematic viscosity Paardekooper et al. (2011). While early attempts to include thermal diffusion in theories of planet migration were limited to its impact on the non-linear corotation torque exerted on intermediate-mass planets (Masset & Casoli 2010; Paardekooper et al. 2011; Jiménez & Masset 2017), recent studies of the interaction between a low-mass planet and a disc with thermal diffusion have shown that the torque exerted on the planet differs significantly from the torque that would be exerted if the disc was adiabatic (Lega et al. 2014; Masset 2017). If, furthermore, the planet releases energy into the surrounding gas, another component of the disc’s force onto the planet appears, that scales with its luminosity. If the planet is on a circular orbit, this force leads to a torque, called heating torque, that is in general positive. Low-mass planets undergoing planetesimal or pebble accretion (or both) at rates of order \( 10^{-5} M_\oplus \text{yr}^{-1} \) can have luminosities large enough to reverse the net torque and undergo outward migration (Benítez-Llambay et al. 2015; Masset 2017). If the planet is free to move on a non-circular, non-coplanar orbit, this force can lead to a growth of the planet’s eccentricity (Masset & Velasco Romero 2017; Chrenko et al. 2017; Eklund & Masset 2017; Fromenteau & Masset 2019) and inclination (Masset & Velasco Romero 2017; Eklund & Masset 2017; Fromenteau & Masset 2019).

In the case of a non-luminous planet embedded in a disc with thermal diffusion, the torque difference with respect to the adiabatic case is called the cold thermal torque. Compared to the adiabatic case, the gas in the vicinity of the planet is colder and denser, as the

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energy arising from compressional heating diffuses away from the planet. The cold thermal torque arises from the density difference between that of the flow with thermal diffusion and that of the adiabatic flow. This density difference displays a striking similarity with the perturbation of density arising from the release of heat by the planet, except for the sign: the latter corresponds to a negative perturbation of density, centred on the planet, and exhibiting two lobes in the downstream parts of the Keplerian flow, whereas the former has same shape, but has a positive sign.

When the planet is offset from corotation as is the case when there is a radial pressure gradient, the lobes are not symmetric with respect to the planet. The lobe located on the same side of corotation as the planet is more pronounced, so that there is a net torque on the planet with same sign as that arising from this lobe. In the usual case in which the disc is slightly sub-Keplerian and the planet is outside of its corotation, the heating torque is positive (Benítez-Llambay et al. 2015), while the cold thermal torque is negative (Lega et al. 2014), in which the disc is slightly sub-Keplerian and the planet is outside of its corotation, the heating torque is positive (Benítez-Llambay et al. 2015), while the cold thermal torque is negative (Lega et al. 2014).

Masset (2017) provides analytic expressions for both torques, using linear perturbation theory. These expressions require a hierarchy of length scales: the distance $\lambda_P$ of the planet to corotation must be much smaller than the size $\lambda_T$ of the thermal disturbance, which itself must be much smaller than the pressure lengthscale $H$ of the disc. The purpose of the present work is to corroborate numerically the analytic expressions, to assess the resolution required to correctly capture thermal torques, and to identify possible deviations of the thermal torque values with that predicted by linear theory. In a somehow similar spirit, Velasco Romero & Masset (2019); Velasco-Romero & Masset (2020) have realised numerical simulations to corroborate the heating force exerted on a perturber moving in a three-dimensional medium at rest (Masset & Velasco Romero 2017). This process is more easily described than that occurring in a protoplanetary disc, due to the lack of shear and to the existence of axial symmetry. Here, we specifically focus on the more challenging problem of the release of heat in the sheared flows of differentially rotating protoplanetary discs.

In a recent work, Hankla et al. (2020) investigated the heating torque by means of numerical simulations. As they focused exclusively on the heating component, they considered a massless, point-like heat source in a shearing box. Here we adopt a different approach: we consider the full problem of a low (but non vanishing) mass planet embedded in a disc with thermal diffusion. This enables us to study not only the heating torque but also the cold thermal torque. While we do not consider a full disc, for reasons of computational cost, we consider a wedge of a spherical mesh centred on the planet, rather than a shearing box.

Our paper is organised as follows. In Section 2, we present our setup. In section 3 we summarise the results on thermal torques obtained by linear theory. In Section 4, we perform a study of the heating torque and compare our results to analytic expectations. In Section 5, we perform a study of the total torque, which uncovers the cold thermal torque. In section 6 we examine the case of larger mass planets, then we discuss in section 7 the importance of thermal torques in two different astrophysical contexts: that of low mass protoplanets embedded in protoplanetary discs, and that of stellar mass objects embedded in AGN discs, and we summarise our results in section 8.

2 PROBLEM SETUP
We present here the different parts of our setup: the disc, the planet, and we present the code used to solve the equations of hydrodynamics.

2.1 Protoplanetary disc
We consider a three-dimensional, non self-gravitating inviscid gaseous disc whose motion is governed by the following equations: the equation of continuity which reads
\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \] the equation of conservation of momentum which reads
\[ \partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p \mathbf{I}) = -\nabla p - \rho \nabla \Phi, \] and the equation of evolution of the density of internal energy $\epsilon$, which reads:
\[ \partial_t \epsilon + \nabla \cdot (\epsilon \mathbf{v}) = -p \nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{F}_H + \mathbf{S}. \]
In these equations $\rho$, $\mathbf{v}$ and $\Phi$ denote the density, the velocity and the gravitational potential, respectively. The source term for the energy (arising from the heat release of the planet) is denoted with $\mathbf{S}$ and $\mathbf{l}$ represents the unit tensor. Here $\mathbf{F}_H$ is the heat flux, given by:
\[ \mathbf{F}_H = -\chi \rho \nabla \left( \frac{\epsilon}{\rho} \right), \]
where $\chi$ is the thermal diffusivity. In the planet forming regions of protoplanetary discs, the heat flux usually arises from radiative diffusion in the optically thick regions between the disc’s photospheres. In that case the thermal diffusivity depends on the temperature, density and opacity of the gas (e.g. Jiménez & Masset 2017, Eq. 34). Here we adopt a constant thermal diffusivity, which allows a clean comparison with analytic results. This approximation is valid for weakly perturbed parts of the flow, i.e. essentially outside of the planetary Bondi radius. It is therefore justified in the present work, as most of the planets considered here have a Bondi radius much smaller than the thermal disturbance.

The gas pressure $p$ obeys the equation of state of ideal gases:
\[ p = (\gamma - 1) \epsilon, \]
where $\gamma$ is the adiabatic index. Unless otherwise stated we use in the following $\gamma = 7/5$, appropriate for the diatomic molecules that constitute most of the gaseous disc. We neglect the radiation pressure, which is negligible in the cold, dense environment of protoplanetary discs.

We use spherical coordinates $(r, \theta, \phi)$, where $r$ is the radial distance from the star, $\theta$ is the polar angle ($\theta = \pi/2$ at the midplane of the disc) and $\phi$ is the azimuthal angle. We assume different key quantities of the disc to be power laws of the radial distance $r$. The aspect ratio $h \equiv H/r$, where $H$ is the vertical scale height of the disc and $r$ the distance to the central star, obeys the power law:
\[ h = h_p \left( \frac{r}{r_p} \right)^f, \]
where $f$ is the flaring index, $r_p$ is the orbital radius of the planet and $h_p$ is the aspect ratio at the planet’s location. In all the simulations presented here, we have $h_p = 0.05$. Eq. (6) implies that the temperature is also a power law of the distance to the star:
\[ T(r) = T_0 \left( \frac{r}{r_p} \right)^{-\beta}, \]
where \( T_0 \) is the temperature at the planet’s location and the exponent \( \beta \) is related to the flaring index through:

\[
\beta = 1 - 2f.
\]

Similarly, the surface density is also chosen to be a power law of \( r \):

\[
\Sigma(r) = \Sigma_0 \left( \frac{r}{r_p} \right)^{-\alpha},
\]

where \( \Sigma_0 \) is the surface density at \( r = r_p \).

### 2.2 Planet and stellar potentials

The gravitational potential \( \Phi \) due to the central star and the planet is given by

\[
\Phi = \Phi_S + \Phi_p,
\]

where

\[
\Phi_S = -\frac{GM_s}{r},
\]

\( M_s \) being the mass of the central star and \( G \) the gravitational constant, and

\[
\Phi_p = -\frac{GM_p}{\sqrt{r^2 + e^2}} + \frac{GM_p \cos \phi \sin \theta}{r_p^2}
\]

are respectively the stellar and planetary potential. In Eq. (12), \( M_p \) is the planet mass, \( r' = |r - r_p| \) is the distance to the planet, \( \phi \) is the azimuth with respect to the planet’s direction, \( \theta \) is the colatitude and \( e \) is a softening length used to avoid computational problems arising from a divergence of the potential in the vicinity of the planet. The second term on the right hand side of Eq. (12) is the indirect term arising from the reflex motion of the star. For the lowest mass planets considered in this work, we have performed simulations with two different values of the softening length: \( e_1 = 0.1H \) and \( e_2 = 5 \times 10^{-3}H \). The former is comparable to the size of the radial disturbance, while the latter is comparable to the mesh resolution, and is typically an order of magnitude smaller than the size of the thermal disturbance. In all cases the back torque exerted on the planet by the perturbed disc was evaluated using the softening length \( e_2 \). We have found that thermal torques thus evaluated do not depend on the softening length, and that there are only minute differences in the perturbation of the density field arising from heat release. All runs with larger mass planets as well as those focused on the study of the cold torque were performed using a softening length \( e_2 \).

### 2.3 Code and mesh domain

To numerically solve equations (1-5), we use the hydrodynamic public code FARGO3D\(^1\) (Benítez-Llambay & Masset 2016) with orbital advection enabled (Masset 2000). We have implemented the energy diffusion equation as an additional source step corresponding to the following differential equation:

\[
\partial_t e = \nabla \cdot \mathbf{F}_H
\]

where \( \mathbf{F}_H \) is given by equation (4).

In our simulations we adopt parameters for the pressure lengthscale and for the thermal diffusivity that are typical of those found at a few astronomical units in protoplanetary discs (e.g. Bitsch et al. 2015; Lega et al. 2015; Benítez-Llambay et al. 2015). As a consequence the size of the thermal disturbance \( \lambda_c \) is significantly smaller than the pressure lengthscale \( H \) (Masset 2017). This is in contrast with the work of Hankla et al. (2020), who considered a thermal disturbance comparable in size to the pressure lengthscale, which allowed them to satisfy largely the scale separation between the distance \( x_p \) of the planet to corotation, and \( \lambda_c \), i.e. they have \( x_p \ll \lambda_c \ll H \), whereas in our case we have \( x_p \ll \lambda_c \ll H \). Namely, we take the following values in all our runs:

\[
\chi = 10^{-3} r_p^2 \Omega_p, \quad \Omega_p \quad \text{is the planet’s frequency. At the planet’s location we have:}
\]

\[
\lambda_c = 2.2 \cdot 10^{-3} r_p \approx \frac{H}{23},
\]

Resolving this lengthscale over a few zones while simulating a full disc would have a prohibitive computational cost, especially for parameter space explorations such as those presented here. For this reason we restrict our computational domain in the three coordinates. In the \( r \)-direction we cover the interval \( [r_{\text{min}}, r_{\text{max}}] = [0.9 r_p, 1.1 r_p] \), in the \( \theta \)-direction we use \([\theta_{\text{min}}, \theta_{\text{max}}] = [-\frac{\pi}{2}, -\frac{\pi}{2}]\) (hence we simulate only one hemisphere of the disc and use reflecting boundary conditions at the midplane). Finally our azimuthal extent is \([\phi_{\text{min}}, \phi_{\text{max}}] = [-\frac{\pi}{3}, \frac{\pi}{3}]\). The cell numbers in each direction are \((N_r, N_{\theta}, N_{\phi}) = (880, 64, 1536)\). Recast in pressure lengthscale our box size is \((4H, H, 7H)\), while its resolution is \((2.3 \cdot 10^{-4} r_p, 7.8 \cdot 10^{-4} r_p, 2.3 \cdot 10^{-4} r_p)\). To avoid reflections on the radial boundaries of our computational box, we use damping boundary conditions as in de Val-Borro et al. (2006), the width of the inner damping ring being \( 3 \cdot 10^{-2} r_p \) and that of the outer ring being \( 3.5 \cdot 10^{-2} r_p \), the damping timescale at the edge of each damping ring being \( 1/20 \) of the local orbital period.

The size of the box considered is too small to properly capture the Lindblad and corotation torques. However, when our focus is on the heating torque, we subtract a “cold run” from a “hot run”, i.e. a run without energy release by the planet from a run with a luminous planet. Our set of parameters is such that the flow is weakly perturbed over the whole mesh, so that the perturbation arising from the planet’s gravity is nearly independent of that due to its luminosity. By subtracting both runs we are therefore essentially left with the thermal disturbance, which fits comfortably in our computational domain.

Similarly, when our focus is on the cold thermal torque, we subtract an adiabatic run from a “cold run”, i.e. a run with a non-luminous planet and thermal diffusion. Again, we are left essentially with a thermal disturbance of size \( \lambda_c \), which is properly captured on our domain. Arguably thermal diffusion may also alter the Lindblad torque, and the corotation torque under some circumstances. We will discuss these issues later.

In all our simulations, the planet is held on a fixed circular orbit and is located at the intersection between cell interfaces in azimuth, radius and colatitude, so that it lies at the centre of an eight-cell cube. One eighth of the energy released is added to these cells at each timestep, as in Benítez-Llambay et al. (2015).

### 3 SUMMARY OF RESULTS FROM LINEAR THEORY

The expressions for the cold and heating thermal torques are respectively (Masset 2017):

\[
\Gamma_{\text{cold thermal}} = -1.61 \frac{r_p}{\lambda_c} \Gamma_0
\]

\[
\Gamma_{\text{thermal}} = -1.61 \frac{r_p}{\lambda_c} \Gamma_0
\]
and
\[
\Gamma_{\text{thermal}} = 1.61 \frac{\gamma - 1}{\gamma} \frac{x_p}{\lambda_c} \frac{L}{L_c} \Gamma_0.
\]
(17)

where
\[
L_c = \frac{4\pi G M_p x \rho_0}{\gamma},
\]
(18)
\[
\rho_0 = \frac{\Sigma_0}{\sqrt{\pi H}}
\]
being the unperturbed midplane density at \( r = r_p \), and
\[
\Gamma_0 = \Sigma_0 \rho_0^2 \left( \frac{M_p}{M_\star} \right)^2 \frac{\Omega_p}{\Omega_\star} h^{-3},
\]
(19)

Eqs. (16) and (17) have been obtained in the limit \( |x_p| \ll \lambda_c \). The distance \( x_p \) of the planet to its corotation is given by:
\[
x_p \approx \eta h_p^2 r_p,
\]
(20)
where \( \eta \) is a function of the density and temperature gradients:
\[
\eta = \frac{\alpha + \beta + 3}{6}.
\]
(21)
The size \( \lambda_c \) of the thermal disturbance is given by:
\[
\lambda_c = \frac{\lambda}{\sqrt{\left(3/2\right) \Omega_p \gamma}}.
\]
(22)
The sum of equations (16) and (17) gives the net or total thermal torque:
\[
\Gamma_{\text{total, thermal}} = 1.61 \frac{\gamma - 1}{\gamma} \frac{x_p}{\lambda_c} \frac{L}{L_c} \left( \frac{L}{L_c} - 1 \right) \Gamma_0.
\]
(23)

As said above, these expressions are valid in the limit \( x_p/\lambda_c \to 0 \).

When, on the contrary the distance to corotation is large compared to the size of the disturbance, the force acting on the planet should tend to that worked out by Masset & Velasco Romero (2017), which has the expression:
\[
F_{\text{heat}} = \frac{\gamma (\gamma - 1) G M_p L}{2 \gamma \lambda_c^3}.
\]
(24)
where \( \lambda_c \) is the adiabatic sound speed and where the \( \infty \) sign conveys the fact that the distance to corotation is large in units of \( \lambda_c \). Eq (24) gives the heating force acting on a perturber with low Mach number in a medium with negligible shear. Note that this force can only be attained if the Keplerian flow can still be largely subsonic for \( x_p \gg \lambda_c \), i.e. if there exists \( x_p \gg \lambda_c \) such that we have \( \Omega_p x_p \ll c_s \).
The force of Eq. (24) is therefore attained if the corotation offset satisfies the double hierarchy:
\[
\lambda_c \ll x_p \ll H.
\]
(25)

Using Eqs. (18), (19) and the relationship between the adiabatic sound speed \( c_s \) and the pressure lengthscale which reads:
\[
H = \frac{c_s}{\gamma \Omega},
\]
(26)
we can write the corresponding, asymptotic torque \( r_p F_{\text{heat}} \) as:
\[
\Gamma^\infty = \sqrt{2\pi} \frac{\gamma - 1}{\gamma} \frac{L}{L_c} \Gamma_0.
\]
(27)
Therefore, for a large planet offset, the heating torque normalized to \( \Gamma_0 (L/L_c) \) should tend toward the constant value \( \sqrt{2\pi} \frac{\gamma - 1}{\gamma} \approx 0.72 \)
(for \( \gamma = 7/5 \) as here) or 1.00 (for \( \gamma = 5/3 \)).

4 HEATING TORQUE

In this section we study numerically the density perturbation arising from the release of heat by the planet and its diffusion in the disc, for the case of a planet centred on corotation then for the case of a planet offset from corotation. The runs used to obtain the maps of perturbation of surface density are performed using the softening length \( \epsilon_1 \). Next we turn to the impact of the perturbation of density on the torque. As explained in section 2.3, for each set of parameters we perform two runs: one with a luminous planet, one with the planet’s luminosity switched off, and subtract the results.

4.1 Case of a planet centred on corotation

We first consider the case of a planet embedded in a strictly Keplerian disc that has \( \alpha = -2 \) and \( \beta = 1 \). In that case, the planet’s orbit and corotation coincide: \( x_p = 0 \), and no torque is expected from the thermal disturbance. In that case we verify that the perturbation of surface density corresponds to that predicted by linear theory for a planet centred on corotation (Masset 2017). We represent on the left plot of Fig. 1 the integral \( \sigma^{(0)} \) along the \( \theta \)-direction of the difference of density fields between a hot and cold run. This field matches satisfactorily the predictions of linear theory, both in shape and amplitude: the right plot of Fig. 1 shows that both fields coincide to within \( \pm 5 \% \) over the domain represented, except in the rightmost part of the map, where the discrepancy reaches \( \sim 10 \% \), and at the innermost cells. At the distance at which the departure between the two fields exceeds \( 5 \% \), the field value is about one order of magnitude smaller than its value at the distance \( \lambda_c \) from the planet. We finally comment that the isocontours from linear theory, which has been formulated in the shearing sheet formalism, are symmetric by construction. In contrast, a small asymmetry of the contours from the numerical simulation is apparent.

4.2 Case of a planet offset from corotation

We then turn to the case of a planet offset from corotation. Namely, we check that the derivative of the perturbation of surface density with respect to \( x_p \), in \( x_p = 0 \), is in agreement with that predicted by linear theory. For this purpose we study two cases (corresponding to four runs in total) with two small values of \( x_p \), with same absolute value and different signs. In this section only we used a resolution in radius of 768 cells instead of 880. We take in one case \( (\alpha, \beta) = (-1.6875008, 1.0) \), which yields a corotation offset \( x_p^0 \) exactly equal to the radial resolution, and in the other case \( (\alpha, \beta) = (-2.3124992, 1.0) \), which yields a corotation offset \( -x_p^0 \). Prior to subtracting the runs with different offsets, we shift the maps one cell in the radial direction, outward for that with an offset of \( x_p^0 \) and inward for the other one, so that their corotations are superimposed.

We show in the left plot of Fig. 2 the map \( |\sigma^{(0)}(x_p^0^\pm) - \sigma^{(0)}(-x_p^0)|/(2|x_p^0|) \) and compare it to analytical expectations. The agreement between numerics and analytics is satisfactory, although the mismatch between the actual and expected isocontours is larger than in the map of Fig. 1, as can be seen in the right plot of Fig. 2. We see that the value from numerical simulations coincides to within \( 5 \% \) with that expected from linear theory up to a distance \( \sim 1.5 \lambda_c \) from the planet (except for the two innermost cells). At the edge of the region represented, the relative discrepancy becomes large, while the field is about two orders of magnitude smaller than at the distance \( \lambda_c \) from the planet.
Normalised perturbation

Figure 1. Left: perturbation of surface density \( \sigma'(0) \) normalized to \( \gamma(y-1)L_c/\chi c_s^2 \). Solid lines: isocontours of the fields obtained after 14 orbital periods of the planet. Dashed lines: isocontours of the perturbation predicted by linear theory. Here \( x = r - r_p \) and \( y = r_p \phi \). This map is obtained by subtracting two runs (a hot one and a cold one) with a planet centred on corotation, as described in section 4.1. Right: cut of the absolute value of \( \sigma'(0) \) along the axis \( y = 0 \). The black circles represent the values measured in the simulation, while the solid black line represents the value expected from linear theory. The squares (in purple in the electronic version of this manuscript) represent the ratio of the value measured in the numerical simulation to the value expected from linear theory. The light horizontal band delineates the region where these values coincide to within \( \pm 5\% \).

4.3 Torque dependency on the gradients of surface density and temperature

Linear theory predicts that the heating torque, for a given disc and planet, only depends on the distance of the planet from corotation. This distance, in turn, depends on a non unique manner on the slopes of surface density and temperature \( \sigma \) and \( \beta \). Here, by varying these two parameters, we consider a wide range of values for \( x_p \), from zero or largely smaller than \( \lambda_c \), to larger than \( \lambda_c \). Note that despite the marked scale separation between \( \lambda_c \) and \( H \) (see Eq. 15), there is hardly enough room between these two values to accommodate a value of \( x_p \) that simultaneously fulfills the two requirements of scale separation given by Eq. (25), so we expect the torque estimate at large corotation offset given by Eq. (27) to be only approximate.

On the other hand, in the limit \( |x_p| \ll \lambda_c \), Eq. (17) shows that the heating torque normalized to \( \Gamma_0(L_c/L_c) \) should be a linear function of \( x_p/\lambda_c \) with slope \( 1.61 \sum_{-1}^{1} = 0.46 \) (for \( \gamma = 7/5 \)) or 0.64 (for \( \gamma = 5/3 \)).

We have performed a variety of runs to check these expectations. In a first exploration, we have run 18 cases with values of \( \sigma \) and \( \beta \) chosen arbitrarily in the interval \([-2, 2] \) interval and a planet with a luminosity close to the critical luminosity \( L_c \) (namely \( L = 0.965L_c \)) and checked that the heating torque is indeed a one-to-one function of \( \eta \).

Fig. 3 shows the comparison between the results of our runs and Eqs. (17) and (27). All points fall on a curve to a high level of approximation, which confirms that the heating torque does not have a dependency on the gradients of surface density and temperature other than that borne by the corotation offset.

We also have explored the regime of large corotation offsets with runs additional to those mentioned above. Achieving a large corotation offset with a power law disc requires large gradients.
of surface density or temperature, which may lead to problems of numerical stability. Instead, we have explored this regime using a strictly Keplerian disc (with $\alpha = -2$ and $\beta = 1$) and have artificially moved the planet to larger orbital radii, while keeping its orbital frequency. In this specific case only, the planet is no longer at the intersection of cell interfaces, and the energy release is performed according to the prescription described in Eklund & Masset (2017). These extra runs are represented with filled circles in Fig. 3. We have checked the validity of our method by choosing for the leftmost of these points an offset equal to the largest one obtained in the runs with varying values of $\alpha$ and $\beta$, and found that the torque thus obtained does indeed coincide with that obtained from sub-Keplerian discs.

We perform a second order polynomial fit of the data from sub-Keplerian discs of the form:

$$\frac{\Gamma}{(L/L_c)\Gamma_0} = a \left( \frac{x_p}{L_c} \right)^2 + b \frac{x_p}{L_c} + c,$$

and find a slope at the origin $b \approx 0.44$, about 5% below the value of 0.46 expected from analytics. We also find $a \approx -0.083$ and $c \approx -4.8 \times 10^{-3}$, this last value being close to zero, as expected for a vanishing corotation offset.

There is a slight dispersion of the torque measurements with respect to the polynomial fit. This may be due to a dependence of the heating on the disc gradients other than that borne by the corotation offset. These effects are minute, however: we find that the residual has an r.m.s. of $4.5 \times 10^{-3} \Gamma_0(L/L_c)$, with extreme values of $(-5) \times 10^{-3} \Gamma_0(L/L_c)$.

### 4.4 Convergence study

We analyse the effect of the mesh resolution on the heating torque by varying the number of zones in all directions from 0.1 to 0.9 times that of the fiducial resolution, by steps of 0.1, while maintaining the size of the box. In addition, we consider an extra resolution of 1.2 times the fiducial one. As our original number of zones in radius is a multiple of 20, the radial number of zones is an even number in all our cases. This is not necessarily true for the number of zones in azimuth. When it is odd, the planet is at the centre of a zone in azimuth, and the heat is then released in a four-zone layer instead of within an eight-zone cube, following the general prescription of Eklund & Masset (2017).

We consider two values for the ratio $x_p/L_c$: 0.15 and 0.95, that are realised with the pairs of values $(\alpha, \beta) = (-1.607, 1.0)$ and $(0.5, 1.0)$, respectively. The lowest of these two ratios satisfies reasonably well the requirement $x_p \ll L_c$. However, the value of the torque in this case is small, and may slightly differ from that given by linear theory owing to the slight dispersion mentioned in the previous section.

Conversely, the largest of these two ratios falls in the regime where the torque squarely departs from the analytic estimate obtained assuming $x_p \ll L_c$. In this case, the residual dispersion with respect to the polynomial fit is a smaller fraction of the torque, but there is no analytic formula that predicts what the torque value should be.

Figure (4) shows the value of the heating torque normalized to the theoretical value (Eq. 17) as a function of the radial resolution $\Delta x$ normalized to the size of the disturbance $L_c$. The vertical, red dashed line represents our fiducial resolution, which resolves $L_c$ over 9.6 cells in radius and azimuth and 2.8 cells in colatitude. We can see that the maximum value of the heating torque obtained from the simulations for the case when $x_p/L_c = 0.15$ is $\approx 70\%$ of the value predicted by linear theory, while it is $\approx 80\%$ of the theoretical value for $x_p/L_c = 0.95$. We also see that the torque value does not vary monotonically with the resolution. In particular, the maximum value for the heating torque is obtained for the fiducial resolution. When we increase the resolution in all directions by the factor 1.2, we obtain a value slightly smaller than that obtained for the fiducial resolution, for the two values of $x_p/L_c$ considered.

The fact that the torque value is $\approx 20\%$ smaller than the theoretical prediction for $x_p/L_c = 0.95$ is largely attributable to the fact that $x_p$ is not small compared to $L_c$. Using the second order fit performed in section 4.3, and noting $X = x_p/L_c \approx 0.95$, we have $(ax^2 + bx) x = 0.82$ (note that we discard the small value of the constant term $c$ here): the expected value is approximately $20\%$ smaller than the linearly extrapolated value $bX$. This fit has been obtained for the finite, fiducial resolution, but it is reasonable to expect that the drop between the linearly extrapolated value and the actual value is relatively independent of resolution and of the order of $20\%$.

The reason for the discrepancy between numerics and the theoretical formula in the other case ($x_p/L_c = 0.15$) is less clear. Here, the expected value of the torque is only $0.46 \times 0.15 \Gamma_0 \approx 0.07 \Gamma_P$; this small value compounds any minute variation that the torque may have with respect to the analytic expectation. A mismatch of $30\%$ on the torque value therefore suggests that effects neglected in the formulation of Masset (2017), such as the dependence on the temperature gradient or curvature effects might be as large as $0.02 \Gamma_0$, comparable in magnitude to, but larger than the maximal dispersion with respect to the polynomial fit that we measured in section 4.3.

Anticipating on the behaviour at larger mass that will be explored in section 6, we also note that the mass considered in this study is only a factor of $\sim 6$ smaller than the critical mass beyond which thermal effects are cut off Velasco-Romero & Masset (2020).
We now turn to the case of the total torque. We consider a disc in the expression of the heating torque.

Although it is unlikely that such a small mass would result in a $30\%$ cut off, it may still account for a few percents difference with respect to the value expected from linear theory.

We also note that for this case, the corotation offset $x_p$ is just marginally larger than the mesh resolution and the softening length even at the largest resolution (the softening length $\varepsilon_2$ scales with the resolution in this convergence study). A study with resolution even higher than that considered in the present work is warranted to disentangle finite resolution effects from additional, minor terms in the expression of the heating torque.

5 FULL TORQUE

We now turn to the case of the total torque. We consider a disc with same parameters as those described in section 2, with $\alpha = 0$ and $\beta = 0$, and a planet of mass $10^{-7}M_\odot$ as previously. We have therefore in such disc:

$$\frac{x_p}{\lambda_c} \approx 0.57$$

(29)

We perform several runs in which we vary the planet’s luminosity from 0 to $2L_c$. We also perform an additional run with a non-luminous planet in an adiabatic disc (with exact same parameters except that it has $\chi = 0$). We plot in Fig. 5 the torque obtained in the different runs as a function of time, over the first 5 orbital periods.

The torque are presented with two different normalization: in terms of $\Gamma_0$ on the left axis, which naturally arises from the formulation of Eqs. (16)-(17), and in terms of $\Gamma_0h$ on the right axis$^2$, which follows the scaling of Lindblad and corotation torques.

In addition, as is customary in the literature, we multiply the torque by the adiabatic index $\gamma$. This figure helps to grasp the importance of thermal torques. When those are not present, the normalized torque (right axis) is usually comprised between $-2$ and $+2$ (e.g. Lega et al. 2015), whereas here it spans a much broader range from $-7$ to $+46$.

We show on that figure the torque expected in the adiabatic disc, according to the torque formulae of Jiménez & Masset (2017). Not surprisingly, the torque that we measure is smaller: it is not possible to correctly capture the Lindblad and corotation torques with our small computational domain. However, thermal disturbances largely fit within our domain, so that our estimate of thermal torques, obtained by subtraction of two runs, is not affected by this effect.

We see that the different curves display an offset nearly constant in time between each other past one orbital period: this is the time it takes to establish thermal torques. We also see that the torque obtained for $L = L_c$ is nearly identical to the torque in an adiabatic disc, as can be expected from Eqs. (16) and (17): for $L = L_c$, thermal torques are expected to cancel out.

Fig. 6 shows the time averaged value of the torques of Fig. 5 as a function of the planetary luminosity. We see that the cold thermal torque measured in our runs is in correct agreement with the value predicted by Eq. (16). Although we do not have undertaken a systematic study of this torque as a function of $x_p/\lambda_c$ as we did in section 4 for the heating torque, it is natural to expect that this torque follows the same deviation from the proportionality law in $x_p/\lambda_c$, since the diffusion and advection equation governing the evolution of the thermal lobes is identical for the cold and heating torques (Masset 2017), and since a non-luminous planet with a

\[\text{Following Masset (2017) we use for } \Gamma_0 \text{ the scaling } \Sigma L^2 r_p^3 q^2 h^{-3} \text{ of the one-sided Lindblad torque instead of the more commonly used reference } \Gamma_0 h = \Sigma L^2 r_p^3 q^2 h^{-2} \text{ which scales with the differential Lindblad torque.}

\[\]
large corotation offset is subjected to the force given by Eq. (24) with $L = -L_c$ (Velasco Romero & Masset 2019; Velasco-Romero & Masset 2020). For the not so small value of $X \equiv x_p / \lambda_c$ used in our calculations (see Eq. 29), we expect a drop of the cold thermal torque of order $(a X^2 + b X) / b X = 1 + a x_p / b \lambda_c \approx 0.89$. We report the value of the expected cold thermal torque thus corrected in Fig. 6 as a thick vertical line. Our measurement is marginally smaller, by $\sim 10\%$, than this expected value.

We also report in Fig. 6 the value of the luminosity required for the thermal torques to cancel out (i.e. for the net torque to be equal to the adiabatic torque). We find $L = 0.94 L_c$. The fact that this value is not exactly $L_c$ as expected from Eqs. (16) and (17) can be attributable to the resolution used in our runs, and also possibly to the fact that the introduction of a finite thermal diffusivity in the disc yields corrections to the Lindblad torque in addition to the appearance of the cold thermal torque (Masset 2017). These corrections should be minute, however, as the introduction of a finite thermal diffusivity induces a torque change in good agreement with that expected from the cold thermal torque.

6 TOWARD LARGER PLANETARY MASSES

We have seen in the previous sections how numerical estimates of thermal torques are in reasonable agreement with analytical expectations. For this purpose, we have used a very small planetary mass (about one third of the mass of Mars, for a solar-like central star), which allowed a clean comparison with linear theory. In order to incorporate thermal torques in models of planetary formation and migration, one needs reliable torque expressions in the mass range for which it is crucial to correctly predict migration, i.e. in the Earth mass range and above. There is a critical mass that features prominently in analysis of thermal forces, which reads:

$$M_c = \frac{\chi^2}{G}.$$  
(30)

When the perturber’s mass is larger than this critical mass, not all the energy released by the perturber contributes to heat its surroundings, resulting in thermal forces smaller than their nominal values (Velasco-Romero & Masset 2020). Differently said, if the thermal diffusivity is much smaller than $GM_c / \epsilon_x$, the flow in the planet’s vicinity tends to behave adiabatically and thermal forces become unimportant. Velasco-Romero & Masset (2020) have investigated in detail the cut-off of thermal forces (both the cold force and the heating force) in the context of (negative) dynamical friction. No such study has been undertaken in the context of the present work, that of planets on circular orbits in a sheared flow. Given the considerable numerical endeavour that such study represents, as one needs to resolve adequately the Bondi sphere of low mass objects, we do not present here a systematic study of thermal torques for objects in the Earth mass range. Instead, we discuss some of the potential complications that arise in this mass range, and illustrate them with selected runs.

6.1 Interaction with the horseshoe flow: scaling laws

The perturbation of density in the lobes that leads to the torque expressions of section 3 arises from an advection-diffusion equation, where advection occurs with the unperturbed Keplerian flow. When the perturbation of velocity in the vicinity of the planet is comparable to that of the unperturbed flow over the length scale of the thermal disturbance $\sim \Omega_p \lambda_c$, one can expect a significant distortion of the thermal lobes, and consequently a change in the torque. The speed at which horseshoe U-turns of low mass planets are executed is, in order of magnitude (Baruteau & Masset 2008):

$$v_U \sim \frac{GM_p}{\Omega_p H^2} \sim R_p \Omega_p.$$  
(31)

A significant change in the shape of the thermal lobes and the torque they exert on the planet is therefore to be expected when:

$$v_U \gtrsim \Omega_p \lambda_c$$  
(32)

which translates into, using Eqs. (30) and (31):

$$M_p \gtrsim \frac{\chi^2}{G} \sqrt{\frac{\chi}{\Omega}} \sim M_c \frac{H}{\lambda_c}.$$  
(33)

We note that this critical mass is not the planet mass for which the width of the horseshoe region, which is $r_p [M_p / (M_p + h)]$ for a low-mass planet (Lega et al. 2015; Masset & Benítez-Llambay 2016), becomes comparable to the extent of the thermal disturbance. This occurs when:

$$M_p \sim \frac{\chi^2 M \epsilon_x}{\Omega_p R_p} \sim M_c.$$  
(34)

In other words, when a planet reaches the critical mass of Eq. (30), the radial size of the horseshoe region becomes comparable to that of the thermal disturbance. Yet Eq. (33) shows that planetary masses at least a factor of $H / \lambda_c$ larger are required to yield a significant distortion of their thermal lobes. The width of the horseshoe region refers to the width reached at large azimuthal distance from the planet. At small distances corresponding to the size of thermal lobes, the horseshoe region is considerably more narrow. The horseshoe region of a planet that marginally fulfills $M_p \gtrsim M_c$ is therefore considerably more narrow, near the planet, than the thermal disturbance, and the lobes are not significantly distorted.

Noting that the critical mass of Eq. (30) can also be written as $M_{th}(\lambda_c / H)^2$, where $M_{th}$, the thermal mass, is given by:

$$M_{th} = M_* \epsilon_x^3 \frac{\chi^2}{\Omega^2}.$$  
(35)

we can enumerate the following set of cases, in which for brevity...
we denote with $\theta \equiv \lambda_c/H$ the (generally small) dimensionless ratio of thermal to pressure length scales:

(i) If $M_p < \theta^2 M_\text{th}$, thermal torques have a value compatible with that given by linear theory (Eqs. 16-17).

(ii) If $\theta^2 M_\text{th} < M_p < \theta M_\text{th}$, thermal torques are below the value predicted by linear theory, as they are in the cut-off regime. The extent of the horseshoe region is comparable to or larger than that of the thermal disturbance, but the perturbation of velocity of the horseshoe flow should not significantly alter the shape of the lobes, so that the magnitude of the thermal torque should be that given by linear theory, reduced by a cut-off factor of magnitude similar to that investigated by Velasco-Romero & Masset (2020).

(iii) If $\theta M_\text{th} < M_p < \theta^2 M_\text{th}$, the thermal torques are significantly cut-off. Besides, the horseshoe flow has a magnitude comparable to or larger than that of the unperturbed disc over the thermal disturbance, with a strong impact both on the shape of the lobes and the magnitude of the thermal torques. Depending on the value of $\theta$ and the value of the planet's luminosity, thermal torques may be irrelevant compared to the Lindblad and corotation torque in this mass regime.

(iv) Finally, the case of a planetary mass in excess of the thermal mass is beyond the scope of this work. It has been investigated thoroughly in the literature. In this regime thermal torques should be irrelevant.

6.2 Some illustrative runs

We present here four runs which fall in the different regimes detailed above. The disc parameters are the same in those of section 2, so that we have: $\theta = 0.043$, $M_\text{th} = 1.25 \cdot 10^{-3} M_\odot$, hence we have:

$$\theta M_\text{th} = 5.4 \cdot 10^{-6} M_\odot$$
$$\theta^2 M_\text{th} = 2.4 \cdot 10^{-7} M_\odot.$$

We consider planet masses of $10^{-7} M_\odot$ [which falls marginally in the regime (i) of section 6.1], $10^{-6} M_\odot$ [which is on the lower side of regime (ii)], $3 \cdot 10^{-6} M_\odot$ [which falls rather on the upper side of regime (ii)] and $10^{-5} M_\odot$, which falls at the frontier between regime (ii) and (iii). The disc has $\alpha = 3/2$ and $\beta = 1.24$, so that $x_p/\lambda_c = 1.38$. For this parameter, we expect, from Fig. 3, that the normalized heating torque for a low mass planet is about 0.45. We recover this result in Fig. 7. The normalized torque for the other planets has a smaller value. In addition, for these masses, large fluctuations appear a few orbits after the insertion of the planet, and almost immediately for the largest mass, due to the appearance of vortices on the edge of the horseshoe region. As the heating torque plateaus in less than one orbital period, we use the temporal window (materialised by a grey band in Fig. 7) from 1.5 to 5 orbits to obtain time averaged values, except for the largest mass where the early onset of large fluctuations precludes any time averaging. We find a normalized value of 0.27 for the planet of mass $10^{-6} M_\odot$, i.e. about 60% of the low-mass value, and a normalized value of 0.11 for the planet of mass $3 \cdot 10^{-6} M_\odot$ (i.e. an Earth mass planet if the central star has a solar mass), which represents 25% of the low-mass value. Fig. 8 shows the density perturbation arising from heat release for these three planets, with same values for the isocontours.

Comparison between the low-mass case and the second case (with $M_p = 10^{-6} M_\odot$) shows that the isocontours are less extended in this second case: with same luminosity, this more massive planet heats less its surroundings than the low-mass planet, which is in agreement with the smaller normalized torque. Although there is some difference in the shape of the isocontours of these two cases, we believe that the torque reduction for this second case arises primarily from the fact that $M_p > M_\odot$, in much the same way as the heating force is cut off for when $M_p \geq M_\odot$ in studies of (negative) dynamical friction Velasco-Romero & Masset (2020): when $M > M_\odot$, not all the energy deposited in the gas by the planet ends up as an excess of internal energy in a heated region near the perturber (a hot trail in that work, a two-lobe pattern here). In the third case, both trends are confirmed: the isocontours are even less extended, and their distortion with respect to the other cases is now evident. Studying the torque decay for masses in excess of $M_\odot$ is therefore a twofold problem, which requires an assessment of the decrease of the heating efficiency of the nearby gas and of the distortion of the thermal lobes by the horseshoe flow. Finally, for the largest mass considered, we do not find any lobed structure. A localised, hot region is found, associated to a vortex that drifts in the horseshoe region. There is no sizeable heating torque, even one orbital period after the insertion of the planet in the disc.

7 DISCUSSION

7.1 Relevance of thermal torques in different contexts

There has been two recent attempts to include and assess the impact of thermal torques in models of planetary population synthesis. Guilera et al. (2019) find that the heating torque has a dramatic impact on scenarios of planetary migration and formation, as low mass embryos undergo a sustained phase of outward migration before superseding the critical mass, at which point they reverse their migration toward the central object. Baumann & Bitsch (2020) perform an analysis with prescriptions for the thermal torques similar to those of Guilera et al. (2019), and find them to have a negligible impact on the resulting planetary populations. Baumann & Bitsch (2020) claim that the different outcome between their work and that of Guilera et al. (2019) stems from the use, in the latter, of a large, constant pebble accretion rate of $10^{-3} M_\odot \text{yr}^{-1}$, thereby overestimating the luminosity of the embryos. This claim, however, is not correct, as such a large accretion rate was used only to produce Fig. 6 of that paper, while elsewhere the accretion rate was determined using standard prescriptions for pebble accretion (Lam-
Figure 8. Perturbation of density arising from heat release (obtained by subtracting a hot run and a cold run), integrated in colatitude and normalized to \( y/(y - 1) L_c/\lambda_c^2 \) at \( t = 2 \) orbits, for the four planet masses considered in the text. The vertical dashed line shows the corotation. The isocontours have same value in the four plots. They start at 0.01 and are in geometric sequence with ratio \( \sqrt{2} \).

It is therefore unclear where the difference between apparently similar setups comes from. We stress, however, that both works assume that thermal torques vanish abruptly as soon as the planetary mass supersedes the critical mass of Eq. (30) (or even this critical mass divided by \( \sqrt{2} \) in the case of Baumann & Bitsch (2020), who use the isothermal sound speed to estimate \( M_c \)). This is a very stringent assumption, at odds with the findings of section 6. While this can be regarded as a conservative assumption in the case of Guilera et al. (2019), as it goes against their conclusion, such is not the case for Baumann & Bitsch (2020). Consider the run with an Earth-mass planet of section 6.2. Fig. 7 shows that a heating torque of magnitude \( \sim 2 \Gamma_\theta h \) is exerted on the planet if it has the critical luminosity \( L_c \), so that the net torque changes sign\(^3\) for a luminosity \( L \sim (1 + y^{-1}) L_c \). Specializing to the case of a solar mass central star and an orbital radius of \( r_p = 5.2 \) au, the data used in our runs translates to a critical luminosity as low as \( L_c = 7.8 \times 10^{25} \) erg s\(^{-1}\). Making the simplifying assumption that the planetary luminosity is given by

\[
L = \frac{G M_p M_p}{R_p},
\]

the luminosity \( \sim (1 + y^{-1}) L_c \) required to reverse migration corresponds to a mass doubling time \( M_p/M_p \sim 8.9 \cdot 10^5 \) yrs, which largely exceeds the mass doubling time for an Earth mass embryo subjected to planetesimal or pebble accretion. In other words, a luminous Earth mass embryo embedded in our fiducial disc, despite having a mass five times larger than the critical mass (or six times larger than the threshold considered by Baumann & Bitsch (2020)), is still subjected to a vigorous heating torque that drives an outward migration at a rate substantially faster than that given by usual torque formulae that neglect thermal torques.

The magnitude of thermal torques in the regime of large masses \( M_p > M_c \) still warrants further work, and will probably be best tackled through high resolution numerical simulations, due to the need to resolve the Bondi sphere, where the flow is highly non-linear, and to the intricacies linked to the interaction with the horseshoe flow. Nonetheless, it is clear that thermal torques play a far more important role for forming planets than envisioned in the early work mentioned above. We also note that thermal torques fluctuate for the planet masses larger than critical. Further work is also warranted to determine whether the time averaged value of thermal torques coincides with that measured at early time, and how the magnitude of these fluctuations depend on the disc’s parameter, in particular the viscosity. Chrenko & Lambrechts (2019). Note that complex migratory behaviours, arising from a fluctuating heating torque on exorted on super-Earths, have also been found by Chrenko & Lambrechts (2019) when the opacity of the disc is not constant.

### 7.2 On the relevance of thermal torques in AGN discs

We now turn to a more speculative discussion about the migration of stellar or intermediate-mass objects in the discs surrounding Active Galactic Nuclei (AGN). The considerations outlined below are speculative mostly because we apply the formulae for thermal torques, which have been derived in discs where the radiation pressure is negligible, to media dominated by radiation pressure (Sirko & Goodman 2003). Note that previous work on migration in AGN discs has also used classical torque formulae at Lindblad and corotation resonances, derived for equations of state that are not that of radiation pressure dominated flows (e.g. Secunda et al. 2019).

The importance of heating torques for the migration of massive stars or accreting compact objects in the accretion discs has been discussed by Hankla et al. (2020). Here we give a few simple relationships\(^4\) which help assess quickly the importance of thermal torques (be it the heating torque or the cold thermal torque) in the different regions of these discs. We base our discussion on the fiducial model of Sirko & Goodman (2003, namely the model presented in Fig. 2 of that work).

Thermal torques dominate over Lindblad and corotation torques when the ratio of the pressure lengthscale \( H \) to the size of the disturbance \( x_c \) is large on the one hand, and when the mass of the perturber is not large compared to the critical mass given by Eq. (30). Both quantities (the size of the disturbance and the critical mass) depend on the thermal diffusivity. Over the whole radial

\(^3\) We assume here that the Lindblad plus corotation torques amount to \(-2 \Gamma_\theta h/y\).

\(^4\) Despite the different nature of the objects considered in this part, we stick to the notation introduced in section 2, so that \( M_\star \) is the mass of the central object and \( M_p \) that of one of its satellites.
range considered by Sirko & Goodman (2003), the total pressure is dominated by the radiation pressure, hence the thermal diffusivity has order of magnitude

\[ \chi \sim \tilde{\ell} c, \]

where \( \tilde{\ell} \) is the photon’s mean free path and \( c \) the speed of light. Writing \( \tilde{\ell} \sim \ell / \tau \), where \( \tau \) is the optical depth, and using Eq. (22), we are led to:

\[ \frac{\lambda_c}{H} \sim \frac{1}{\sqrt{\hbar \tau}} \left( \frac{r}{R_s} \right)^{1/4}, \]

where \( R_s \) is Schwarzschild’s radius. We plot this ratio on the left part of Fig. 9. We see that \( \lambda_c \) is smaller than the disc’s thickness up to \( \gtrsim 10^4 \) Schwarzschild’s radii, or approximately one tenth of a parsec. Up to this radius, thermal torques may therefore supersede Lindblad and corotation torques, whereas they should be negligible beyond that distance.

Next we turn to an estimate of the critical mass. Using Eqs. (30) and (37), we can write:

\[ M_c = \frac{\gamma \epsilon s}{\kappa} = \frac{H c s}{\tau G} = \frac{c^2}{G M c} \frac{\epsilon}{\kappa} = M_b \left( \frac{1}{\hbar \tau} \left( \frac{r}{R_s} \right) \right)^{1/2} = M_s \left( \frac{h^2}{\tau} \right) \left( \frac{r}{R_s} \right)^{1/2}, \]

We plot this mass on the right side of Fig. 9. Over the domain where we have seen that \( \lambda_c < H \) (i.e. for \( r \lesssim 10^4 R_s \)), this critical mass is comprised between a few solar masses and \( 10^4 M_\odot \), the minimum being reached at \( r \sim 10^3 R_s \sim 10^{-2} \) pc. This indicates that thermal torques should be important for stellar mass objects within 0.1 pc of the central object, while they should be important for intermediate-mass black holes only on a fraction of this domain. Whether thermal torques induce inward or outward migration depends on which of the cold or heating torque dominates, i.e. whether the object’s luminosity is sub- or super-critical. Using Eqs. (18) and (37), we can express the critical luminosity as:

\[ L_c = \frac{4 \pi G M_p \epsilon s \rho_0}{\gamma} \sim \frac{4 \pi G M_p c}{\kappa}, \]

where \( \kappa \sim 1/(\ell \rho_0) \) is the opacity. The right hand side of Eq. (40) is the Eddington luminosity of the perturber. The critical luminosity is therefore of the order of Eddington’s luminosity. Objects radiating at luminosities comparable to Eddington’s could be subjected to a large heating torque and possibly to a net, positive thermal torque, whereas for less luminous objects the net thermal torque would be negative, inducing inward migration.

Some cautionary remarks are in order:

- Given the large variations of \( h \) and \( \tau \), markedly different results are to be expected if the model’s parameters are changed. The above discussion is highly specific to the canonical model of Sirko & Goodman (2003).
- We reiterate that the analytic expressions for thermal torques have been obtained for a medium in which the radiation pressure is negligible. Further studies are warranted to assess whether they take a similar expression in discs dominated by radiation pressure.
- In the formulation of thermal forces in protoplanetary discs, the momentum injected into the medium by the luminous object is neglected, as the critical luminosity is of order of magnitudes smaller than Eddington’s (Velasco-Romero & Masset 2020). Such is not the case here, and the injection of momentum should be taken into account in a formulation specific to AGN discs.

8 CONCLUSIONS

We have performed numerical simulations of low-mass planets embedded in discs with thermal diffusion, aimed at checking analytical formulae of thermal torques. We have confirmed that thermal torques depend essentially on the distance \( x_p \), between the planet and corotation. We have also found a satisfactory agreement between analytics and numerics when the distance of the planet to its corotation is much smaller than the characteristic size of the thermal disturbance, and we have confirmed that when the corotation offset becomes comparable to the disturbance’s size, thermal torques tend to plateau toward a value given by a dynamical friction calculation. We have performed a convergence study and found that the thermal disturbance must be resolved over typically 10 zones for an agreement at the 20 – 30 % level between analytics and numerics. We have found a relatively large discrepancy (30 %) for the low value of \( x_p \) that we used in the convergence study. The expected torque value is then accordingly small, which compounds any minute additional dependence of the thermal torque that has been neglected in the analytic study of Masset (2017), such as a dependence on the temperature gradient, on the surface density gradient, or curvature effects. The somehow relatively large residual found in that case may signal such additional, small contributions to the heating torque not captured in the analytic study, or simply that even higher resolutions than the ones used here are required for a better agreement.

We have also evaluated the cold thermal torque, formerly called the cold finger effect (Lega et al. 2014), and found it to be in good agreement with analytic estimates. We have checked that the luminosity at which the net thermal torque switches from negative to positive is within a few percents of the critical luminosity given by Eq. (18).

We have also studied thermal torques for larger planet masses, that supersede the critical mass \( M_c \) (Eq. 30). We find a decay of the torque, arising from a lower heating efficiency of the nearby gas. Furthermore, we also find that as the planetary mass increases, the horseshoe flow distorts the heated lobes. This can have an impact on the value of the thermal torque, but we have not quantified this effect. Overall, we find that thermal torques become insignificant for planetary masses one order of magnitude larger than the critical mass.

From our analysis we conclude that thermal torques can be largely dominant over the Lindblad and corotation torques for objects in the Earth mass range. They are much more relevant for scenarios of planet migration and formation than considered in previous work where they were artificially and abruptly cancelled when the planetary mass becomes larger than \( M_c \). A systematic and quantitative assessment of the torque decay in the large mass regime requires very high resolution calculations, that resolve the Bondi radius of embedded planets. These are beyond the scope of the present work.

It should also be kept in mind that objects with a luminosity larger than the critical luminosity of Eq. (18) are not only subjected to a positive thermal torque but they also experience a growth of eccentricity and inclination (Eklund & Masset 2017; Fromenteau & Masset 2019). The migration path of such objects cannot be reduced to a simple outward motion given by the torque’s value in this regime, and its characterisation requires significant further work.

We finally apply our findings to stellar or intermediate-mass objects embedded in discs around AGNs. This exercise should be taken with a pinch of salt, however, as these discs are usually dominated by radiation pressure rather than gas pressure. Neither usual
Figure 9. Left: ratio of the thermal to pressure length scale as a function of radial distance in the fiducial disc of Sirko & Goodman (2003). This ratio needs to be small for thermal torques to be dominant. Right: critical mass as a function of radial distance in this disc. Thermal torques are cut-off for masses above the critical mass, as discussed in section 6.

torque formulae at Lindblad or corotation resonances, nor the formulae for thermal torques have been derived for radiation dominated flows.

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DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

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