QUOTIENTS OF CONTINUOUS CONVEX FUNCTIONS ON NONREFLEXIVE BANACH SPACES

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Abstract. On each nonreflexive Banach space $X$ there exists a positive continuous convex function $f$ such that $1/f$ is not a d.c. function (i.e., a difference of two continuous convex functions). This result together with known ones implies that $X$ is reflexive if and only if each everywhere defined quotient of two continuous convex functions is a d.c. function. Our construction gives also a stronger version of Klee’s result concerning renormings of nonreflexive spaces and non-norm-attaining functionals.

A function on a Banach space $X$ is called a d.c. function if it can be represented as a difference of two continuous convex functions (all functions considered in this note are real-valued). Thus the system of all d.c. functions on $X$ is the smallest vector space containing all continuous convex functions. Moreover, it is well-known, and not difficult to show, that it is even an algebra and a lattice (see, e.g., [2, III.2]). While an everywhere defined quotient $g/f$ of two d.c. functions on a finite-dimensional Banach space is always d.c. (cf. [1] Corollary]), the situation is completely different for infinite-dimensional spaces: by [7, Corollary 5.7], on each infinite-dimensional Banach space there exists a positive d.c. function such that $1/f$ is not d.c.

The following natural question arises:

is the quotient $g/f$ of two continuous convex functions on $X$ d.c. if $f \neq 0$?

Quite surprisingly, the answer is affirmative for all reflexive spaces $X$; indeed, it is proved in [7, Remark 3.5(i)] that $1/f$ ($f \neq 0$ continuous and convex) is d.c. on $X$ whenever $X$ is reflexive. The main aim of this note is to show that the above question has a negative answer for each nonreflexive Banach space $X$.

The following criterion for non-d.c. functions (cf. [7, Lemma 4.1]) suggests how to construct a counterexample.

**Lemma 1.** Let $X$ be a Banach space and $h : X \to \mathbb{R}$ be a function. If there exist sets $M \subset X$ of arbitrarily small diameter such that $h$ is unbounded on $M$, then $h$ is not a d.c. function.

If there exists a continuous convex function $f$ on $X$ such that

\begin{equation}
    f > 0, \text{ and there exist sets } M \subset X \text{ of arbitrarily small diameters with } \inf f(M) = 0,
\end{equation}

then $1/f$ is not a d.c. function by Lemma 1. (Of course, such an $f$ cannot exist if $X$ is reflexive since, in this case, $f$ attains its minimum on any closed ball.)

To construct $f$, it might seem natural to proceed by finding an $x^* \in X^*$ such that

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We denote by \( B \) \( \| \cdot \|_e \) the closed unit ball in a Banach space \( X \). By \( \| \cdot \|_e \) we denote the corresponding dual norm on \( X^* \) (the topological dual of \( X \)).

Let us start by fixing some notations. We consider only Banach spaces over the reals \( \mathbb{R} \). We denote by \( B_X \) or \( B_{(X,\|\cdot\|)} \) the closed unit ball in a Banach space \( X \) endowed with a norm \( \| \cdot \| \). By \( \| \cdot \|_e \) we denote the corresponding dual norm on \( X^* \) (the topological dual of \( X \)).

In what follows, we consider \( X \oplus \mathbb{R} \) equipped with the maximum norm, and we identify \( x \in X \) with \( (x,0) \in X \oplus \mathbb{R} \) (and so \( X \) with \( X \times \{0\} \)).

**Lemma 2.** Let \( X \) be a nonreflexive Banach space. Then there exists a nonempty bounded convex set \( C \subset X \oplus \mathbb{R} \) such that

(a) \( \varphi(x) := \text{dist}(x,C) > 0 \) for every \( x \in X \), and

(b) for each \( \varepsilon > 0 \) there is a set \( M_\varepsilon \subset X \) with \( \text{diam} M_\varepsilon < \varepsilon \) and \( \inf \varphi(M_\varepsilon) = 0 \).

**Proof.** Since \( X \) is nonreflexive, by [3, Theorem 1] (see, e.g., [4, Theorem 10.3] or [5, Theorem 1.13] for simpler proofs) there exist unit vectors \( \{e_i\}_{i=1}^\infty \) in \( X \) and unit functionals \( \{e_i^*\}_{i=1}^\infty \) in \( X^* \) such that

\[ e_i^*(e_j) = 0 \quad \text{if} \quad i > j, \quad \text{and} \quad e_i^*(e_j) > 1/2 \quad \text{if} \quad i \leq j. \]

Set \( e_\infty := (0,1) \in X \oplus \mathbb{R} \), and let \( f_i \in (X \oplus \mathbb{R})^* \) be the extension of \( e_i^* \) for which \( f_i(e_\infty) = 1 \). Clearly \( \|f_i\|_* = 2 \). For \( 0 < k < n \) in \( \mathbb{N} \), we define

\[ x_{k,n} := 2e_k + 2k e_n + \frac{1}{n} e_\infty. \]

Clearly

\[ f_i(x_{k,n}) \geq 1 \quad \text{for} \quad 1 \leq i \leq k, \]

\[ f_i(x_{k,n}) \geq \frac{1}{k} \quad \text{for} \quad k < i \leq n, \quad \text{and} \]

\[ f_i(x_{k,n}) = \frac{1}{n} \quad \text{for} \quad i > n. \]
We define $C := \text{conv}\{x_{k,n} : 0 < k < n, \ k, n \in \mathbb{N}\}$ and $X_0 := \overline{\text{span}}\{e_j : j \in \mathbb{N}\}$.

To prove (a), we need to show $C \cap X = \emptyset$. Since clearly $C \cap X \subset X_0$, it is sufficient to show that $C \cap X_0 = \emptyset$. So, suppose to the contrary that an $x_0 \in C \cap X_0$ is given. As $\|f_i\|_* = 2$ and $\lim_{i \to \infty} f_i(e_j) = 0$ for each $j \in \mathbb{N}$, it is easy to check that $\lim_{i \to \infty} f_i(x) = 0$ for every $x \in X_0$. So, we may find natural numbers $i_1 < i_2 < i_3$ such that

$$f_{i_1}(x_0) < \frac{1}{3}, \quad i_1 f_{i_2}(x_0) < \frac{1}{3}, \quad i_2 f_{i_3}(x_0) < \frac{1}{3}.$$  

Since $x_0 \in C$ and $f_{i_1}$, $f_{i_2}$, $f_{i_3}$ are continuous, we can find $c \in C$ so close to $x_0$ that

$$f_{i_1}(c) < \frac{1}{3}, \quad i_1 f_{i_2}(c) < \frac{1}{3}, \quad i_2 f_{i_3}(c) < \frac{1}{3}.$$  

Since $c \in C$, we can assign to each $(k, n)$ with $1 \leq k < n$ a number $\alpha_{k,n} \geq 0$ so that $\sum \alpha_{k,n} = 1$, the set $\{(k, n) : \alpha_{k,n} \neq 0\}$ is finite, and $c = \sum \alpha_{k,n} x_{k,n}$.

Using subsequently (4), (5), and (6), we obtain

$$f_{i_1}(c) = \sum \alpha_{k,n} f_{i_1}(x_{k,n}) \geq \sum_{k > i_1 \atop n > k} \alpha_{k,n},$$

$$f_{i_2}(c) = \sum \alpha_{k,n} f_{i_2}(x_{k,n}) \geq \sum_{k < i_1 \atop n \geq i_2} \frac{1}{k} \alpha_{k,n} \geq \frac{1}{i_1} \sum_{k < i_1 \atop n \geq i_2} \alpha_{k,n},$$

$$f_{i_3}(c) = \sum \alpha_{k,n} f_{i_3}(x_{k,n}) \geq \sum_{k < i_1 \atop n < i_2} \frac{1}{n} \alpha_{k,n} \geq \frac{1}{i_2} \sum_{k < i_1 \atop n < i_2} \alpha_{k,n}.$$  

Using (9), (10), (11) and (8), we easily obtain $\sum \alpha_{k,n} < 1$, which is a contradiction.

To prove (b), consider an arbitrary $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ with $4/k_0 < \varepsilon$ and set $M_\varepsilon := \{2e_{k_0} + (2/k_0)e_n : n > k_0\}$. Then clearly $\text{diam} \ M_\varepsilon \leq 4/k_0 < \varepsilon$. The other property of $M_\varepsilon$ also holds, since, for each $n > k_0$,

$$\inf \varphi(M_\varepsilon) = \text{dist}(M_\varepsilon, C) \leq \|((2e_{k_0} + (2/k_0)e_n) - (2e_{k_0} + (2/k_0)e_n + (1/n)e_\infty)\| = 1/n.$$  

□

Remark 3.  
(i) To obtain $C$ with the weaker property $\inf_{x \in X} \varphi(x) = 0$ instead of (b) in Lemma 2, it is sufficient to put $C := \text{conv}\{2e_k + (1/k)e_\infty : k \in \mathbb{N}\}$, and the proof becomes simpler.
(ii) Setting $C := \text{conv}\{2e_k + \frac{2}{k}e_n + \frac{2}{n}e_m + \frac{1}{m}e_\infty : 0 < k < n < m, \ k, n, m \in \mathbb{N}\}$, an easy modification of the proof of Lemma 2 gives the following property (b$^2$) which is slightly stronger than (b):

(b$^2$) there exist sets $M \subset X$ of arbitrarily small diameter such that $M$ contains sets $A$ of arbitrarily small diameter with $\inf \varphi(A) = 0$.

(Analogously, using indices $0 < k_1 < \cdots < k_{p+1}$ in the definition of $C$, it is possible to obtain the corresponding iterated property (b$^p$).)

Now, we are ready to state the following main result of the present paper.

Theorem 4. The following properties of a Banach space $X$ are equivalent.
(a) \(X\) is nonreflexive.
(b) There is a continuous convex function \(f: X \to (0, \infty)\) such that \(1/f\) is not representable as a difference of two continuous convex functions.
(c) There is a decreasing sequence \(\{C_n\}_{n=1}^\infty\) of bounded closed convex subsets of \(X\) such that
\[
\bigcap_{n=1}^\infty C_n = \emptyset, \quad \text{and} \quad \bigcap_{n=1}^\infty (C_n + \varepsilon B_X) \neq \emptyset \quad \text{for every } \varepsilon > 0.
\]

Proof. If \(X\) is nonreflexive, take \(f := \varphi\) where \(\varphi\) is as in Lemma \(2\). By Lemma \(1\) \(1/f\) is not d.c. on \(X\). On the other hand, if \(X\) is reflexive and \(f\) is a positive continuous convex function, then \(1/f\) is d.c. on \(X\) by \(7\) Remark 3.5(i)]. Thus (a) and (b) are equivalent.

Let us show that (a) and (c) are equivalent. If \(X\) is nonreflexive, let \(\varphi\) be again the function from Lemma \(2\). The sets \(C_n := \{x \in X : \varphi(x) \leq 1/n\}, \ n \in \mathbb{N}\), are nonempty, closed, convex, bounded (since the set \(C\) in Lemma \(2\) is bounded) and their intersection is empty. Let \(\varepsilon > 0\). By the properties of \(\varphi\), there exists \(x \in X\) such that, for each \(n\), there is \(y \in B(x, \varepsilon)\) with \(\varphi(y) \leq 1/n\), i.e. \(y \in C_n\). In other words, \(x \in \bigcap_{n=1}^\infty (C_n + \varepsilon B_X)\). Hence (a) implies (c). On the other hand, if \(X\) is reflexive, then each decreasing sequence \(\{C_n\}\) of nonempty closed bounded convex subsets of \(X\) has a nonempty intersection since each \(C_n\) is weakly compact. \(\square\)

Let us conclude our paper with the promised strengthening of a result from \(4\).

**Proposition 5.** Let \(Y\) be a nonreflexive Banach space and \(0 \neq y^* \in Y^*\). Then there exists an equivalent norm \(|\cdot|\) on \(Y\) such that
\[
\begin{align*}
(a) \quad & y^* \text{ does not attain its norm on } B(1), \\
(b) \quad & \text{for each } \varepsilon > 0, \text{ there is } M_\varepsilon \subset B(1) \text{ such that } \text{diam } M_\varepsilon < \varepsilon \text{ and } \sup y^*(M_\varepsilon) = |y^*|_*.
\end{align*}
\]

Proof. Set \(X := \{y \in Y : y^*(y) = 0\}\) and choose \(e \in Y\) with \(y^*(e) = 1\). Up to renorming, we may suppose that the norm on \(Y\) satisfies
\[
||y|| = \max\{|y - y^*(y)e|, |y^*(y)|\}
\]
for all \(y \in Y\). In this way we may identify \(Y\) with \(X \oplus_\infty \mathbb{R}\) so that \(y^*((x, t)) = t\) for \((x, t) \in X \times \mathbb{R}\).

As \(Y\) is not reflexive, \(X\) is not reflexive, either. Let \(\varphi\) be the function on \(X\) given by Lemma \(2\). Choose \(\alpha > \varphi(0)\) and set
\[
A = \{x \in X : \varphi(x) < \alpha\}.
\]
By the properties of \(\varphi\) the set \(A\) is bounded. Therefore we can choose \(r > 0\) such that \(A \subset B(0, r)\). Choose \(\beta > \sup \varphi(B(0, r))\); it is possible as \(\varphi\) is 1-Lipschitz. Further define
\[
D = \{(x, t) \in X \times \mathbb{R} : x \in B(0, r), \ t = \varphi(x) - \beta\},
\]
\[
C = \overline{\text{conv}} (D \cup (-D)).
\]
Then \(C\) is clearly a bounded closed convex symmetric set. Further, \(0 \in \text{int } C\), as \(0 \in A\) and \(A \times (\alpha - \beta, \beta - \alpha) \subset C\). It follows that there exists an equivalent norm \(|\cdot|\) on \(X \times \mathbb{R}\) such that \(C\) is the closed unit ball in this norm. We will show that this norm has the required properties.

We have
\[
-|y^*|_* = \inf y^*(C) = \inf y^*(D \cup (-D)) = \inf y^*(D) = \inf \{\varphi(x) - \beta : x \in B(0, r)\} = -\beta,
\]
as clearly \( \inf \varphi(B(0,r)) = \inf \varphi(X) = 0 \). Thus \( |y^*|_* = \beta \).

Next we are going to show that \( y^* \) does not attain its norm on \( C \). Suppose it does. Then there is a point \( z = (x_0, -\beta) \in C \). Note that
\[
C \subset \{(x,t) \in X \times \mathbb{R} : x \in B(0,r) \& t \geq \varphi(x) - \beta \}.
\]
The reason is that the set on the righthand side is closed and convex and it contains both \( D \) and \( -D \). It follows that \( z \) belongs to the set on the righthand side, i.e. \( -\beta \geq \varphi(x_0) - \beta \). So \( \varphi(x_0) \leq 0 \), a contradiction.

It remains to show the assertion (b). Let \( \varepsilon > 0 \) be given. By the properties of \( \varphi \) we can choose a set \( P_\varepsilon \subset A \) such that \( \text{diam} \, P_\varepsilon < \varepsilon \) and \( \inf \varphi(P_\varepsilon) = 0 \). (Note that \( \varphi \geq \alpha \) outside of \( A \).) Now set
\[
P_\varepsilon^* := \{(x,t) \in X \times \mathbb{R} : x \in P_\varepsilon, \, t = \varphi(x) - \beta \}.
\]
Then clearly \( P_\varepsilon^* \subset C \) and
\[
\inf_{z \in P_\varepsilon^*} y^*(z) = -\beta = -|y^*|_*.
\]
As \( \varphi \) is 1-Lipschitz with respect to \( \| \cdot \| \), we get that \( \| \cdot \| - \text{diam} \, P_\varepsilon^* < \varepsilon \). Set \( M_\varepsilon := -P_\varepsilon^*/K \), where \( K > 0 \) is such that \( | \cdot | \leq K \| \cdot \| \) on \( X \times \mathbb{R} \). Then \( M_\varepsilon \) has all required properties and the proof is complete. \( \square \)

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