RECASTING A BRINKMAN-BASED ACOUSTIC MODEL AS THE DAMPED BURGERS EQUATION

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Abstract. In order to gain a better understanding of the behavior of finite-amplitude acoustic waves under a Brinkman-based poroacoustic model, we make use of approximations and transformations to recast our model equation into the damped Burgers equation. We examine two special case solutions of the damped Burgers equation: the approximate solution to the damped Burgers equation and the boundary value problem given an initial sinusoidal signal. We study the effects of varying the Darcy coefficient, Reynolds number, and coefficient of nonlinearity on these solutions.

1. Introduction. It is generally understood that Darcy’s law governs poroacoustic propagation [7]:

\[ \nabla P = -\left( \frac{\mu \chi}{K} \right) \mathbf{V}, \]

where \( P \) is the intrinsic pressure, \( \mu \) is the dynamic viscosity, \( \chi \) is the porosity, \( K \) is the permeability, and \( \mathbf{V} \) is the intrinsic velocity. This expression models the behavior of the acoustic potential when considering the fluid-pore interactions alone. However, Payne et al. [8] argue that if the porosity is near unity, i.e. the fluid-pore interaction is not the dominating factor, or if a boundary or interface is present, then Brinkman’s equation should be used [7]:

\[ \nabla P = \hat{\mu} \chi \nabla^2 \mathbf{V} - \left( \frac{\mu \chi}{K} \right) \mathbf{V}. \]

Here, \( \hat{\mu} \) is the Brinkman or effective viscosity. This equation not only accounts for the fluid-pore interaction found in Darcy’s law, it also models the fluid-fluid interactions. In this work we will investigate the effects this model has on an acoustic signal.

Assuming (2) as the filtration law, Jordan [4] put forth the following weakly-nonlinear partial differential equation (PDE) as a model of poroacoustic propagation:

\[ \Box^2 \phi + \chi (Re)^{-1} \phi_{txx} - \delta \phi_t = \epsilon \partial_t \left[ (\beta - 1) \phi_t^2 + \phi_z^2 \right]. \]

Here, \( \Box^2 \equiv \partial_{xx} - \partial_{tt} \) is the 1D d’Alembertian operator, \( \phi = \phi (x,t) \) is the velocity potential, \( Re = \frac{c_e L \sigma_e}{\mu} \) is a Reynolds number, \( \delta \propto \chi \) is the dimensionless Darcy coefficient, \( \epsilon \) is the mach number, and \( \beta \) is the coefficient of nonlinearity.
Rossmanith and Puri have studied (3) in the context of a semi-infinite medium with harmonic driving at the boundary [9], as well as the evolution of an initial sinusoidal signal in a finite medium [10]. The complicated nature of this equation, however, requires further approximations. In section 2 we will assume unidirectional propagation (i.e., the right-running approximation), which will help cast (3) into the damped Burgers equation (DBE), a slight generalization of Burgers equation. This will allow us to exploit the exact solution of Burgers equation to analyze the problems considered below.

2. Mathematical formulation. Crighton [2] has shown that the unidirectional approximation (right-running) applied to Blackstock’s third order acoustic model gives rise to the classical Burgers equation. More recently, Jordan [3] asserted that applying the right running unidirectional approximation to (3) allows it to be recast as the DBE. To verify this, we rewrite the d’Alembertian in (3) to get:

\[
(\partial_x + \partial_t)(\partial_x - \partial_t)\phi + \chi(Re)^{-1}\phi_{txx} - \delta \phi_t = \epsilon \partial_t[(\beta - 1)\phi_t^2 + \phi_x^2].
\] (4)

Assuming right-running waves and employing the approximation \( \phi_x \propto -\phi_t \) in the small terms of (4), yields:

\[
(\partial_x + \partial_t)(-2\partial_t)\phi + \chi(Re)^{-1}\phi_{txx} - \delta \phi_t = 2\epsilon \beta \phi_t \phi_{tt}.
\] (5)

We then integrate equation (5) with respect to \( t \) to get:

\[
-2(\partial_x + \partial_t)\phi + \chi(Re)^{-1}\phi_{xx} - \delta \phi = \epsilon \beta (\phi_t)^2.
\] (6)

Differentiating equation (6) with respect to \( x \) and rearranging, we arrive at:

\[
u_t + (1 + \epsilon \beta u) u_x - \frac{1}{2} \chi(Re)^{-1} u_{xx} + \frac{1}{2} \delta u = 0.
\] (7)

Here, we have used the fact that \( \phi_x = u \). We set \( \hat{x} = x - t \) and \( \hat{t} = t \), which reduces (5) to

\[
u_t + \epsilon \beta uu_{\hat{x}} - \frac{1}{2} \chi(Re)^{-1} u_{\hat{xx}} + \frac{1}{2} \delta u = 0,
\] (8)

the DBE.

In this work, we will study the effects that the parameters in (8) have on different acoustic signals. To this end, we will, in the next section, study the approximate traveling wave solution to (8), and discuss the role of the Darcy, Brinkman, and nonlinear terms on the behavior of the solution. In section 4, we will analyze numerically the damped form of Cole’s problem [1]. We will also derive an approximate solution to (8), from an energy analysis. Lastly, in section 5, we relate our study to previous results in other fields.

3. Traveling wave solution. In order to derive an approximate traveling wave solution for (3), we transform (8) into its more widely used form. We begin by letting \( \bar{x} = \sqrt{(2Re/\chi)} \ \hat{x} \). This gives:

\[
u_t + \epsilon \beta \sqrt{\frac{2Re}{\chi}} u_{\bar{x}} - \frac{1}{2} \delta u = 0.
\] (9)

Now, defining \( \alpha = \epsilon \beta \sqrt{(2Re/\chi)} \) and \( \hat{u} = \alpha \ u \), and after some manipulation we get,

\[
\hat{u}_t + \hat{u} \hat{u}_{\bar{x}} - \hat{u}_{\bar{xx}} + \lambda \hat{u} = 0,
\] (10)
where $\lambda = \delta/2$. We make use of the approximate solution found by Malfiet [6]:

$$\hat{u} = 2e^{-\lambda \hat{t}}[(1 - y)(1 + a_3y^3(1 + y) + a_5y^5(1 + y))...],$$

(11)

where:

$$y = \tanh[\bar{x} - \frac{2}{\lambda}(1 - e^{-\lambda \hat{t}})],$$

$$a_3 = \frac{1}{3}(e^{-\lambda \hat{t}} - 1),$$

$$a_5 = -\frac{1}{60}(\lambda e^{-\lambda \hat{t}} + 8e^{-2\lambda \hat{t}} - 40e^{-\lambda \hat{t}} + 32).$$

Figure 1. $\hat{u}$ vs $\hat{x}$ with $\hat{t} = 10$, $Re = 2000$, $\chi = 0.9$, $\epsilon = 0.1$, and $\beta = 1.07$. Solid: $\lambda = 0.2$. Dashed: $\lambda = 0.1$. Dotted: $\lambda = 0.05$.

Figure 1 is a plot of $\hat{u}$ vs $\hat{x}$ for various values of lambda. Plotting $\hat{u}$ instead of $u$ effectively normalizes the amplitude across all studies. This allows us to more clearly see the effects that the various parameters have on the structure of the solution. It is clear that the location of the drop in wave amplitude is $\lambda$ dependent. Increasing $\lambda$ shifts the location to shorter distances. However, the form of the solution is unchanged. Next, we study the effects of $Re$ on the wave.

We plot $\hat{u}$ vs $\hat{x}$ for various values of $Re$ in Figure 2. It is clear that the location of the drop in wave amplitude is $Re$ dependent. Increasing $Re$, which effectively reduces the strength of the Brinkman term, delays the location of the drop to larger distances. Unlike $\lambda$ however, a prominent steepening in the wave form can be seen with decreasing $Re$.

In order to study the effects of $\beta$ on the solution, we plot $u$ vs $\hat{x}$ in Figure 3 instead of $\hat{u}$ vs $\hat{x}$, because the $\beta$ dependence is found only in $\alpha$. We note that the amplitude is decreased as a function of increasing $\beta$. The apparent steepening found in this figure is a result of the different starting amplitudes, rather than the structure of the solution.
We now wish to study the behavior of (3) using an initial sinusoidal signal.

4. Sinusoidal initial condition: Cole’s problem. Starting with (8), we define \( \tilde{x} = (\hat{x}/\epsilon \beta) \). This gives:

\[
\dot{u} + uu_{\tilde{x}} - \frac{1}{2(\epsilon \beta)^2} \chi (Re)^{-1} u_{\tilde{x}\tilde{x}} + \frac{\delta}{2} u = 0. \quad (12)
\]
Then, defining $\tilde{\nu} = \chi/[2Re(\epsilon\beta)^2]$ and $\lambda = \delta/2$, and dropping the Darcy term, we have the classical Burgers equation,

$$u_t + uu_x - \tilde{\nu}u_{xx} = 0. \quad (13)$$

We now define the initial and boundary conditions:

$$u(0, \hat{t}) = u(1, \hat{t}) = 0, \quad \text{for} \quad \hat{t} > 0; \quad (14a)$$

$$u(\tilde{x}, 0) = \sin(\pi \tilde{x}), \quad \text{for} \quad 0 < \tilde{x} < 1. \quad (14b)$$

The exact solution to this problem, as derived in [1], is,

$$u = 4\pi\tilde{\nu} \left[ \sum_{n=1}^{\infty} ne^{-\tilde{\nu}n^2\pi^2\hat{t}}I_n \left( \frac{1}{2\tilde{\nu}} \right) \sin(n\pi\tilde{x}) \right], \quad (15)$$

where $I_n(\cdot)$ denotes the modified Bessel function of the first kind of order $n$.

We now study the role that the Darcy term plays in this problem by considering the DBE:

$$u_t + uu_x - \tilde{\nu}u_{xx} + \lambda u = 0. \quad (16)$$

We solve numerically the DBE (with the Darcy term) and compare the result to the exact solution to the undamped case (15). The results are plotted in Figure 4.

![Figure 4](image-url)

**Figure 4.** $u$ vs $\tilde{x}$ with $\hat{t} = 0.4$, $\beta = 1.07$, $\chi = 0.9$, $\epsilon = 0.1$, and $Re = 200$. Solid: $\lambda = 0$. Dashed: $\lambda = 0.05$. Dotted: $\lambda = 0.4$.

Here, we have $u$ vs. $\tilde{x}$ plotted for various values of $\lambda$ with $Re = 200$, and $\beta = 1.07$, which corresponds to CO$_2$ at several thousand Kelvin [12]. There is an excellent match to the exact solution for the $\lambda = 0$ case, giving confidence in the numerical scheme. A drop in the signal’s amplitude is noted as a function of increasing $\lambda$.

Next, to investigate how $\tilde{\nu}$ affects the evolution of the signal, we plot Figure 5. We see that increasing $Re$ ($\tilde{\nu}$ is decreasing) gives rise to a steepening effect in time, leading to a blow-up for very small $\tilde{\nu}$. Also, the signal amplitude decays more slowly as a function of increasing $Re$.

In Figure 6, $\tilde{\nu}$ and $\hat{t}$ are kept constant, and the change in signal amplitude as a function of $\lambda$ is studied. It is clear that $\lambda$ is a damping term, reducing the amplitude with increasing $\lambda$. Furthermore, we see the prominent steepening effect $\tilde{\nu}$ has on
Figure 5. $u$ vs $\tilde{x}$ for various times with $\lambda = .001$, $\epsilon = 0.1$, $\beta = 1.07$. 
Figure 6. $u$ vs $\tilde{x}$ for various $\lambda$ with $\epsilon = 0.1$, $\beta = 1.07$, $\hat{t} = 0.4$. 
the signal. Note that $\lambda$ reduces this steepening, slightly shifting the maximum of the signal towards center.

4.0.1. Energy analysis. In order to develop a better understanding of (12), we will analyze its energy equation. We start by multiplying all terms in (12) by $u$ giving:

$$uu_t + uu_x - \nu uu_{xx} + \lambda u^2 = 0.$$  \hspace{1cm} (17)

Next, integrating (17) over the spatial domain results in:

$$\frac{d}{dt} \left( \frac{1}{2} \int_{\tilde{x}_1}^{\tilde{x}_2} (u^2) d\tilde{x} \right) + \frac{1}{3} [u^3(\tilde{x}_2, \hat{t}) - u^3(\tilde{x}_1, \hat{t})] + \lambda \int_{\tilde{x}_1}^{\tilde{x}_2} u^2 d\tilde{x} =$$

$$\tilde{\nu}(u \partial_{\tilde{x}} u) \bigg|_{\tilde{x}_2}^{\tilde{x}_1} - \tilde{\nu} \int_{\tilde{x}_1}^{\tilde{x}_2} (\partial_{\tilde{x}} u)^2.$$ \hspace{1cm} (18)

Following the energy analysis of the undamped Burgers equation found in [1], the physical significance of the terms in (18) are as follows:

$$\frac{d}{dt} \left( \frac{1}{2} \int_{\tilde{x}_1}^{\tilde{x}_2} (u^2) d\tilde{x} \right) = \text{total rate of change of kinetic energy in the system},$$

$$\frac{1}{3} [u^3(\tilde{x}_2, \hat{t}) - u^3(\tilde{x}_1, \hat{t})] = \text{net flux of kinetic energy out across the boundaries},$$

$$\lambda \int_{\tilde{x}_1}^{\tilde{x}_2} u^2 d\tilde{x} = \text{dissipation by pore interaction},$$

$$\left( u \partial_{\tilde{x}} u \right) \bigg|_{\tilde{x}_2}^{\tilde{x}_1} = \text{rate of work done on the system at the boundaries},$$

$$\tilde{\nu} \int_{\tilde{x}_1}^{\tilde{x}_2} (\partial_{\tilde{x}} u)^2 = \text{total dissipation of energy by viscosity}.$$  

It is worth noting that the pore interaction term is a new contribution made by the Darcy term.

For our particular study, we recast (18) and make use of our particular boundary conditions to give:

$$\left( \frac{d}{dt} + 2\lambda \right) K = -\tilde{\nu} \int_{0}^{1} (\partial_{\tilde{x}} u)^2,$$ \hspace{1cm} (19)

where $K = (1/2) \int_{0}^{1} (u^2) d\tilde{x}$ is the kinetic energy. Noting that $\left( \frac{d}{dt} + 2\lambda \right)$ is the relaxation operator, which admits an exponential kernel, the following ansatz is suggested:

$$u = e^{-mt} \check{u},$$ \hspace{1cm} (20)

where $\check{u}$ denotes the solution of (13) given (14).

Applying (20) to (19) we get,

$$\frac{d}{dt} K + (2\lambda - 2m) K = -\tilde{\nu} \int_{0}^{1} (\partial_{\tilde{x}} \check{u})^2,$$ \hspace{1cm} (21)

which is exactly the energy equation for the classical Burgers equation when $m = \lambda$. Thus, (20) satisfies the energy equation (19).

Unfortunately, our ansatz does not satisfy the DBE. However, it does provide a practical approximation. Applying (20) to the DBE (16), with $m = \lambda$, results in:

$$\check{u}_t + \check{u} \check{u}_x - \tilde{\nu} \check{u}_{xx} = -(e^{\lambda \hat{t}} - 1) \check{u} \check{u}_x.$$ \hspace{1cm} (22)
The left-hand side of (22) is zero from (13). Thus, this solution approximately satisfies the DBE when $\hat{t} \ll 1/\lambda$. Figure 7 is a comparison of the numerical solution to the DBE and the approximate solution (20). We see that the approximate solution fits very well for small times. However, Figure 8 shows that when $\hat{t}$ is of the order $1/\lambda$, this approximate solution does not hold. This is an example which demonstrates that although a function can satisfy the energy equation, it does not necessarily satisfy the full PDE.
5. Relation to other fields. Korsunskii [5] found an approximate solution to (16) in relation to magnetoacoustic waves in an electrically conducting fluid. For \( \lambda \ll 1 \), he found that the approximate solution, bounded in the limit \( \tilde{x} \to \infty \), can be written in the form:

\[
\hat{u} = e^{\lambda \hat{t}} \left[ D - A \tanh \left( \frac{A e^{-\lambda \hat{t}}}{2\tilde{\nu}} \left( \tilde{x} - D \frac{1 - e^{-\lambda \hat{t}}}{\lambda} \right) \right) \right], \tag{23}
\]

where \( A \) and \( D \) are constants. This approximate solution is valid for \( \hat{t} \ll 1/\lambda \).

![Graph showing \( \hat{u} \) vs \( \tilde{x} \) with \( \chi = 0.9, Re = 100, \hat{t} = 7, \epsilon = 0.1, \beta = 1.7, \lambda = 0.05, A = 0.8, \) and \( D = 0.8 \). Solid: Series DBE solution. Dashed: Approximate analytical solution.](image)

**Figure 9.** \( \hat{u} \) vs \( \tilde{x} \) with \( \chi = 0.9, Re = 100, \hat{t} = 7, \epsilon = 0.1, \beta = 1.7, \lambda = 0.05, A = 0.8, \) and \( D = 0.8 \). Solid: Series DBE solution. Dashed: Approximate analytical solution.

We compare the solution in (23) with the series solution (13) in Figure 9. We have used a low value of \( \lambda = 0.05 \) and a time of \( \hat{t} = 7 \). We find a close match to our numerical work.

Soluyan and Khokhlov [11] studied a similar equation in the context of relaxing media. Beginning with (7) and dropping the Brinkman term, we get,

\[
u_t + (1 + \epsilon \beta u) u_x + \frac{1}{2} \delta u = 0. \tag{24}
\]

Now, replacing \( x \) with \( t \), and vice-versa, as well as defining \( \delta/2 = \lambda \), we have the inviscid damped Burgers equation found in [11]:

\[
u_t + (1 + \epsilon \beta u) u_t + \lambda u = 0. \tag{25}
\]

Soluyan and Khokhlov [11] found that the exact solution to (25) has the form,

\[
\omega t = \arcsin(ue^{\lambda x}) + \frac{\epsilon \beta \omega}{\lambda} (1 - e^{-\lambda x})ue^{\lambda x} + \omega x, \tag{26}
\]

where \( \omega = \pi \) in the context of this study.
6. **Discussion.** In this paper, we have used analytical and numerical techniques to study the behavior of the right-running approximation to the Brinkman-based poroacoustic model given in (3); that is, we investigated, in the poroacoustic context, the DBE and studied the roles of the Reynolds number, Darcy coefficient, and the coefficient of nonlinearity $\beta$ under two special settings: the traveling wave solution, and the sinusoidal initial condition. Based on these analyses, we report the following:

6.1. **Traveling wave.**

1. As shown in Figure 1, the transition region is $\lambda$ dependent. An increase in $\lambda$ leads to a shift in the location of the drop of the signal amplitude to shorter distances.
2. We note that an increase in $Re$ shifts the drop of the amplitude of the signal to larger distances, as shown in Figure 2. Also noted in Figure 2, increasing $Re$ weakens the steepness caused by the nonlinearity.
3. The nonlinear term allows for a steepening effect. However, this steepening is a result of the different starting amplitudes for the various values of $\beta$ used in the study, as seen in Figure 3. Once the amplitudes are normalized, the $\beta$ effect is masked.
4. Figure 9 shows that the closed form solution found in (23) closely matches the series solution to the DBE in (11), for $\lambda \ll 1$.

6.2. **Cole’s problem.**

1. Figure 6 shows that as $\lambda$ is increased, a dampening in the amplitude of the initial sinusoidal signal is noted. Furthermore, $\lambda$ mitigates the steepening effect caused by the nonlinear term, shifting the maximum amplitude back towards the midpoint.
2. Figure 5 shows that increasing $Re$ (decreasing $\tilde{\nu}$) slows the decay in the signal.
3. Figure 5 shows how the nonlinear term, through $\tilde{\nu}$, affects the evolution of an initial sinusoidal pulse. As $\beta$ is increased, $\tilde{\nu}$ is decreased, which leads to a prominent steepening in the signal. This effect is also noted in Figure 6.
4. Figures 7 and 8 show that the approximate solution found in (20) holds well for the DBE for $\hat{t} \ll 1/\lambda$.

6.3. **Relation to other fields.** We compared our study of the DBE to related research in other fields. For the special case of $\lambda \ll 1$, the traveling wave series solution in (11) is found to be in agreement with the corresponding closed form approximation found in the field of magnetoacoustics [5]. We have also related our study of the DBE to a similar study in the field of relaxing media. Soluyan and Khokhlov [11] found an analytical solution to the inviscid DBE in the semi-infinite domain under harmonic driving, whereas we investigated the DBE in the context of the evolution of an initial sinusoidal signal on a finite domain.

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