On Stanley’s Partition Function

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Abstract. Stanley defined a partition function $t(n)$ as the number of partitions $\lambda$ of $n$ such that the number of odd parts of $\lambda$ is congruent to the number of odd parts of the conjugate partition $\lambda'$ modulo 4. We show that $t(n)$ equals the number of partitions of $n$ with an even number of hooks of even length. We derive a closed-form formula for the generating function for the numbers $p(n) - t(n)$. As a consequence, we see that $t(n)$ has the same parity as the ordinary partition function $p(n)$ for any $n$. A simple combinatorial explanation of this fact is also provided.

Keywords: partition function, Jacobi’s triple product identity, hook length.

AMS Mathematical Subject Classifications: 05A17.

1 Introduction

This note is concerned with the partition function $t(n)$ introduced by Stanley [7, 8]. We shall give a combinatorial interpretation of $t(n)$ in terms of hook lengths and shall prove that $t(n)$ and the partition function $p(n)$ have the same parity. Moreover, we compute the generating function for $p(n) - t(n)$ and related generating functions.

We shall adopt the common notation on partitions in Andrews [1] or Andrews and Eriksson [3]. A partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r)$ of a nonnegative integer $n$ is a nonincreasing sequence of nonnegative integers such that the sum of the components $\lambda_i$ equals $n$. A part is meant to be a positive component, and the number of parts of $\lambda$ is called the length, denoted $l(\lambda)$. The conjugate partition of $\lambda$ is defined by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$, where $\lambda'_i$ $(1 \leq i \leq t, t = l(\lambda))$ is the number of parts in $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ which are greater than or equal to $i$. The number of odd parts in $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is denoted by $O(\lambda)$.

For $|q| < 1$, the $q$-shifted factorial is defined by

$$(a; q)_n := (1 - a)(1 - aq)\cdots(1 - aq^{n-1}), \quad n \geq 1,$$

and

$$(a; q)_\infty := (1 - a)(1 - aq)(1 - aq^2)\cdots.$$
Stanley \[7, 8\] introduced the partition function \(t(n)\) as the number of partitions \(\lambda\) of \(n\) such that \(O(\lambda) \equiv O(\lambda') \pmod{4}\), and obtained the following formula

\[
t(n) = \frac{1}{2} (p(n) + f(n)),
\]

where \(p(n)\) is the number of partitions of \(n\) and

\[
\sum_{n=0}^{\infty} f(n)q^n = \prod_{i \geq 1} \frac{(1 + q^{2i-1})}{(1 - q^{4i})(1 + q^{4i-2})^2}.
\]

Andrews \[2\] obtained the following closed-form formula for the generating function of \(t(n)\)

\[
\sum_{n=0}^{\infty} t(n)q^n = \frac{(q^2; q^2)_{\infty}^2 (q^{16}; q^{16})_{\infty}^5}{(q; q)_{\infty}(q^4; q^4)_{\infty}^5 (q^{32}; q^{32})_{\infty}^2}.
\]

He also derived the congruence relation

\[
t(5n + 4) \equiv 0 \pmod{5}.
\]

In this note, we shall consider the complementary partition function of \(t(n)\), namely, the partition function \(u(n) = p(n) - t(n)\), which is the number of partitions \(\lambda\) of \(n\) such that \(O(\lambda) \not\equiv O(\lambda') \pmod{4}\). We obtain a closed-form formula for the generating function of \(u(n)\) which implies that Stanley’s partition function \(t(n)\) and ordinary partition function \(p(n)\) have the same parity for any \(n\). We also present a simple combinatorial explanation of this fact. Then we derive formulas for the generating functions for the numbers \(u(4n), u(4n+1), u(4n+2)\) and \(u(4n+3)\) which are analogous to the formulas for the partition function \(t(n)\) due to Andrews \[2\]. In the last section, we find combinatorial interpretations for \(t(n)\) and \(u(n)\) in terms of hooks of even length.

## 2 The generating function formula

We shall derive a formula for the partition function \(u(n) = p(n) - t(n)\). The proof is similar to Andrews’ proof of (1.3) for \(t(n)\). As a consequence, one sees that \(t(n)\) and \(p(n)\) have the same parity for any \(n\). This fact also has a simple combinatorial interpretation. We shall also compute the generating functions for the numbers \(u(4n), u(4n+1), u(4n+2)\) and \(u(4n+3)\).

**Theorem 2.1** We have

\[
\sum_{n=0}^{\infty} u(n)q^n = \frac{2q^2(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2 (q^{32}; q^{32})_{\infty}^2}{(q; q)_{\infty}(q^4; q^4)_{\infty}^5 (q^{16}; q^{16})_{\infty}} \tag{2.5}
\]
Proof. By (2.18) and (1.2), we find
\[ \sum_{n=0}^{\infty} u(n)q^n = \frac{1}{2} \left( \frac{1}{(q; q)_{\infty}} - \frac{(-q; q^2)^{\infty}}{(q^4; q^4)^{\infty}(-q^2; q^4)^{\infty}} \right) \]

\[ = \frac{1}{2} \left( \frac{(-q; q^2)^{\infty}}{(q^4; q^4)^{\infty}(-q^2; q^4)^{\infty}} - \frac{(-q; q^2)^{\infty}}{(q^4; q^4)^{\infty}(-q^2; q^4)^{\infty}} \right) \]

\[ = \frac{(-q; q^2)^{\infty}}{2(q^4; q^4)^{\infty}(-q^2; q^4)^{\infty}} \left( (q^4; q^4)^{\infty}(-q^2; q^4)^{\infty} - (q^4; q^4)^{\infty}(q^2; q^4)^{\infty} \right). \]

Using Jacobi’s triple product identity [11 p.10]
\[ \sum_{n=-\infty}^{\infty} z^n q^{n^2} = \frac{(-zq; q^2)^{\infty}(-q/z; q^2)^{\infty}(q^2; q^2)^{\infty}}{(q; q)_{\infty}}, \]

we see that
\[ (q^4; q^4)^{\infty}(-q^2; q^4)^{\infty} = \sum_{n=-\infty}^{\infty} q^{2n^2} \]

and
\[ (q^4; q^4)^{\infty}(q^2; q^4)^{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}. \]

Clearly,
\[ \sum_{n=-\infty}^{\infty} q^{2n^2} - \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} = 2 \sum_{n=-\infty}^{\infty} q^{2(n^2+1)} \]

It follows that
\[ \sum_{n=0}^{\infty} u(q)q^n = \frac{(-q; q^2)^{\infty}}{(q^4; q^4)^{\infty}(-q^2; q^4)^{\infty}} \sum_{n=-\infty}^{\infty} q^{2(n^2+1)} \]

\[ = \frac{q^2(-q; q^2)^{\infty}}{(q^4; q^4)^{\infty}(-q^2; q^4)^{\infty}} \sum_{n=-\infty}^{\infty} q^{8n^2+8n}. \]

Jacobi’s triple product identity yields
\[ \sum_{n=-\infty}^{\infty} q^{8n^2+8n} = (-q^{16}; q^{16})_{\infty}(-1; q^{16})_{\infty}(q^{16}; q^{16})_{\infty}. \]

Observe that
\[ (-1; q^{16})_{\infty} = 2(-q^{16}; q^{16})_{\infty}. \]

In view of (2.10), we get
\[ \sum_{n=0}^{\infty} u(q)q^n = \frac{2q^2(-q^{16}; q^{16})_{\infty}(-q^{16}; q^{16})_{\infty}(-q; q^2)^{\infty}(q^{16}; q^{16})_{\infty}}{(q^4; q^4)^{\infty}(-q^2; q^4)^{\infty}} \]

\[ = \frac{2q^2(q^{32}; q^{32})_{\infty}(-q; q^2)^{\infty}(-q^{16}; q^{16})_{\infty}}{(q^4; q^4)^{\infty}(-q^2; q^4)^{\infty}}. \]
Now,
\[ (-q; q^2)_\infty = \frac{(q^2; q^2)_\infty}{(q; q)_\infty (q^4; q^4)_\infty}, \quad (2.13) \]
\[ (q^4; q^8)_\infty = \frac{(q^4; q^4)_\infty}{(q^8; q^8)_\infty} \quad (2.14) \]
and
\[ (-q^{16}; q^{16})_\infty = \frac{(q^{32}; q^{32})_\infty}{(q^{16}; q^{16})_\infty}. \quad (2.15) \]
Consequently,
\[ \sum_{n=0}^\infty u(q)q^n = \frac{2q^2(q^{32}; q^{32})_\infty (q^8; q^8)_\infty (q^2; q^2)_\infty (q^{32}; q^{32})_\infty}{(q^4; q^4)_\infty (q^4; q^4)_\infty (q; q)_\infty (q^4)_{\infty} (q^{16}; q^{16})_\infty} = \frac{2q^2(q^2; q^2)_\infty (q^8; q^8)_\infty (q^{32}; q^{32})_\infty}{(q; q)_\infty (q^4; q^4)_\infty (q^{16}; q^{16})_\infty}. \]
This completes the proof. \[ \square \]

**Corollary 2.2** For \( n \geq 0 \),
\[ t(n) \equiv p(n) \pmod{2}. \]

We remark that there is a simple combinatorial explanation of the above parity property. First, we observe that for any partition \( \lambda \) of \( n \),
\[ O(\lambda) \equiv O(\lambda') \pmod{2} \quad (2.16) \]
because we have both \( O(\lambda) \equiv n \pmod{2} \) and \( O(\lambda') \equiv n \pmod{2} \). By the definition of \( u(n) \) and the relation (2.16), we see that \( u(n) \) equals the number of partitions of \( n \) such that
\[ O(\lambda) - O(\lambda') \equiv 2 \pmod{4}. \quad (2.17) \]
Suppose \( \lambda \) is a partition counted by \( u(n) \). From (2.17) it is evident that its conjugation \( \lambda' \) is also counted by \( u(n) \). Once more, from (2.17) we deduce that \( O(\lambda) \) and \( O(\lambda') \) are not equal, so that \( \lambda \) is different from \( \lambda' \). Thus we reach the conclusion that \( u(n) \) must be even, and so \( t(n) \) has the same parity as \( p(n) \) since \( p(n) = t(n) + u(n) \).

From (1.1) it follows that
\[ u(n) = p(n) - t(n) = \frac{p(n) - f(n)}{2}. \quad (2.18) \]
So we have the following congruence relation.

**Corollary 2.3** For \( n \geq 0 \),
\[ f(n) \equiv p(n) \pmod{4}. \]
Theorem 2.1 enables us to derive the generating functions for \( u(4n + i) \), where \( i = 0, 1, 2, 3 \). Andrews [2] has obtained formulas for the generating functions of \( t(4n + i) \) for \( i = 0, 1, 2, 3 \).

Theorem 2.4 We have

\[
\sum_{n=0}^{\infty} u(4n) q^n = 2q^2(q^{16}; q^{16})_\infty (-q; q^{16})_\infty (-q^{15}; q^{16})_\infty V(q)
\]

\[
\sum_{n=0}^{\infty} u(4n + 1) q^n = 2q(q^{16}; q^{16})_\infty (-q^3; q^{16})_\infty (-q^{13}; q^{16})_\infty V(q)
\]

\[
\sum_{n=0}^{\infty} u(4n + 2) q^n = 2(q^{16}; q^{16})_\infty (-q^7; q^{16})_\infty (-q^9; q^{16})_\infty V(q)
\]

\[
\sum_{n=0}^{\infty} u(4n + 3) q^n = 2(q^{16}; q^{16})_\infty (-q^5; q^{16})_\infty (-q^{11}; q^{16})_\infty V(q)
\]

where

\[ V(q) = \frac{(q^2; q^2)_\infty (q^8; q^8)_\infty}{(q; q)_\infty^5 (q^4; q^4)_\infty} \]

Proof. By Theorem 2.1 we find

\[
\sum_{n=0}^{\infty} u(n) q^n = \frac{2q^2(q^2; q^2)_\infty^2}{(q; q)_\infty} V(q^4)
\]

\[
= \frac{2q^2(q^2; q^2)_\infty}{(q^2; q^2)_\infty} V(q^4)
\]

Since

\[
\frac{1}{(q; q^2)_\infty} = (-q; q)_\infty
\]

and

\[
(q^2; q^2)_\infty = (q; q)_\infty (-q; q)_\infty
\]

we have

\[
\sum_{n=0}^{\infty} u(n) q^n = 2q^2(q; q)_\infty (-q; q)_\infty (-q; q)_\infty V(q^4)
\]

\[
= q^2(q; q)_\infty (-1; q)_\infty (-q; q)_\infty V(q^4).
\]

Using Jacobi’s triple product identity, we get

\[
(q; q)_\infty (-1; q)_\infty (-q; q)_\infty = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}.
\]

(2.21)
Thus we have
\[ \sum_{n=0}^{\infty} u(n)q^n = q^2 \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} V(q^4) = 2q^2 \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} V(q^4). \] (2.22)

It is easy to check that
\[ \sum_{n=0}^{\infty} \frac{n(n+1)}{2} = \sum_{n=-\infty}^{\infty} q^{2n^2-n}. \] (2.23)

In view of (2.22), we get
\[ \sum_{n=0}^{\infty} u(n)q^n = 2q^2 \sum_{n=-\infty}^{\infty} q^{2n^2-n} V(q^4) \]
\[ = 2q^2 \sum_{i=0}^{3} \sum_{k=-\infty}^{\infty} q^{2(4k+i)^2-(4k+i)} V(q^4). \] (2.24)

For \( i = 0 \), extracting the terms of the form \( q^{4j+2} \) in (2.24) for \( j = 0, 1, 2, \ldots \), we obtain
\[ \sum_{n=0}^{\infty} u(4n + 2)q^{4n+2} = 2q^2 \sum_{j=-\infty}^{\infty} q^{32j^2-4j} V(q^4). \]

Again, Jacobi’s triple product identity gives
\[ \sum_{j=-\infty}^{\infty} q^{32j^2-4j} = (q^{64}; q^{64})_\infty (-q^{28}; q^{64})_\infty (-q^{36}; q^{64})_\infty. \] (2.25)

Hence we get
\[ \sum_{n=0}^{\infty} u(4n + 2)q^{4n+2} = 2q^2(q^{64}; q^{64})_\infty (-q^{28}; q^{64})_\infty (-q^{36}; q^{64})_\infty V(q^4), \]

which simplifies to
\[ \sum_{n=0}^{\infty} u(4n + 2)q^n = 2(q^{16}; q^{16})_\infty (-q^7; q^{16})_\infty (-q^9; q^{16})_\infty V(q). \]

The remaining cases can be verified using similar arguments. This completes the proof. \( \blacksquare \)
3 Combinatorial interpretations for \( t(n) \) and \( u(n) \)

In [7, Proposition 3.1], Stanley found three partition statistics that have the same parity as \( (\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))/2 \), and gave several combinatorial interpretations for \( t(n) \). We shall present combinatorial interpretations of partition functions \( t(n) \) and \( u(n) \) in terms of the number of hooks of even length. For the definition of hook lengths, see Stanley [6, p. 373]. A hook of even length is called an even hook. The following theorem shows that the number of even hooks has the same parity as \( (\mathcal{O}(\lambda) - \mathcal{O}(\lambda'))/2 \).

**Theorem 3.1** For any partition \( \lambda \) of \( n \), \( \mathcal{O}(\lambda) \equiv \mathcal{O}(\lambda') \) (mod 4) if and only if \( \lambda \) has an even number of even hooks.

**Proof.** We use induction on \( n \). It is clear that Theorem 3.1 holds for \( n = 1 \). Suppose that it is true for all partitions of \( n \). We aim to show that the conclusion also holds for all partitions of \( n + 1 \). Let \( \lambda \) be a partition of \( n + 1 \) and \( v = (i, j) \) be any an inner corner of the Young diagram of \( \lambda \), that is, the removal of the square \( v \) gives a Young diagram of a partition of \( n \). Let \( \lambda^- \) denote the partition obtained by removing the square \( v \) from the Young diagram of \( \lambda \). We use \( H_e(\lambda) \) to denote the number of squares with even hooks in the Young diagram of \( \lambda \). We claim that

\[ H_e(\lambda) \equiv H_e(\lambda^-) \pmod{2} \iff \lambda_i \equiv \lambda'_j \pmod{2}. \]  

(3.26)

Let \( T(\lambda, v) \) denote the set of all squares in the Young diagram of \( \lambda \) which are in the same row as \( v \) or in the same column as \( v \). After removing the square \( v \) from the Young diagram of \( \lambda \), the hook lengths of the squares in \( T(\lambda, v) \) have decreased by one. Meanwhile, the hook lengths of other squares remain the same. Furthermore, if \( \lambda_i \) and \( \lambda'_j \) have the same parity, then the number of squares in \( T(\lambda, v) \) is even. This implies that the parity of the number of squares in \( T(\lambda, v) \) of even hook lengths coincides with the parity of the number of squares in \( T(\lambda, v) \) with odd hook lengths. Similarly, for the case when \( \lambda_i \) and \( \lambda'_j \) have different parities, it can be shown that the number of squares in \( T(\lambda, v) \) of even hook length is of opposite parity to the number of squares in \( T(\lambda, v) \) of odd hook length. Hence we arrive at (3.26).

By the inductive hypothesis, we see that \( \mathcal{O}(\lambda^-) \equiv \mathcal{O}((\lambda^-)' \pmod{4}) \) if and only if \( H_e(\lambda^-) \) is even. For any inner corner \( v = (i, j) \) of \( \lambda \), if \( \lambda_i \equiv \lambda'_j \pmod{2} \), then \( \mathcal{O}(\lambda) \equiv \mathcal{O}(\lambda') \pmod{4} \) if and only if \( \mathcal{O}(\lambda^-) \equiv \mathcal{O}((\lambda^-)' \pmod{4}) \). By (3.26), we find that in this case, \( H_e(\lambda) \) and \( H_e(\lambda^-) \) have the same parity. Thus the assertion holds for any partition \( \lambda \) of \( n + 1 \). The case that \( \lambda_i \not\equiv \lambda'_j \pmod{2} \) can be justified in the same manner. This completes the proof. 

From Theorem 3.1, we obtain a combinatorial interpretation for Stanley’s partition function \( t(n) \), which can be recast as a combinatorial interpretation for \( u(n) \).

**Theorem 3.2** The partition function \( t(n) \) is equal to the number of partitions of \( n \) with an even number of even hooks, and the partition function \( u(n) \) is equal to the number of partitions of \( n \) with an odd number of even hooks.
Combining Theorem 2.1 and Theorem 3.2 we have the following consequence.

**Corollary 3.3** For any $n$, the number of partitions of $n$ with an odd number of even hooks is always even.

Since $f(n) = t(n) - u(n)$, we see that $f(n)$ can be interpreted as the signed counting of partitions of $n$ with respect to the number of even hooks, as formally stated below.

**Theorem 3.4** The function $f(n)$ equals the number of partitions of $n$ with an even number of even hooks minus the number of partitions of $n$ with an odd number of even hooks.

**Acknowledgments.** This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

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