Research Article

Fundamental Results of Conformable Sturm-Liouville Eigenvalue Problems

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Received 7 May 2017; Accepted 6 August 2017; Published 14 September 2017

Academic Editor: Abdelalim Elsadany

We suggest a regular fractional generalization of the well-known Sturm-Liouville eigenvalue problems. The suggested model consists of a fractional generalization of the Sturm-Liouville operator using conformable derivative and with natural boundary conditions on bounded domains. We establish fundamental results of the suggested model. We prove that the eigenvalues are real and simple and the eigenfunctions corresponding to distinct eigenvalues are orthogonal and we establish a fractional Rayleigh Quotient result that can be used to estimate the first eigenvalue. Despite the fact that the properties of the fractional Sturm-Liouville problem with conformable derivative are very similar to the ones with the classical derivative, we find that the fractional problem does not display an infinite number of eigenfunctions for arbitrary boundary conditions. This interesting result will lead to studying the problem of completeness of eigenfunctions for fractional systems.

1. Introduction and Preliminaries

Fractional calculus is old as the Newtonian calculus [1–3]. The name fractional was given to express the integration and differentiation up to arbitrary order. Traditionally, there are two approaches to define the fractional derivative. The first approach, Riemann-Liouville approach, is to iterate the integral with respect to certain weight function and replace the iterated integral by single integral through Leibniz-Cauchy formula and then fractionalize by using the Gamma function. In this approach, the arbitrary order Riemann-Liouville results from the integrating measure \( dt \) and the Hadamard fractional integral results from the integrating measure \( dt/t \). The second approach, Grünwald-Letinkov approach, is to iterate the limit definition of the derivative to get a quantity with certain binomial coefficient and then fractionalize by using the Gamma function instead of the factorial in the binomial coefficient. In case of the Riemann-Liouville and Caputo fractional derivatives, a singular kernel of the form \( (t−s)^\alpha \) is generated for \( 0 < \alpha < 1 \) to reflect the nonlocality and the memory in the fractional operator. Through history, hundreds of researchers did their best to develop the theory of fractional calculus and generalize it, either by obtaining more general fractional derivatives with different kernels or by defining the fractional operator on different time scales such as the discrete fractional difference operators (see [4–7] and the references therein) and \( q \)-fractional operators (see [8] and the references therein).

In 2014 [9], Khalil et al. introduced the so-called conformable fractional derivative by modifying the limit definition of the derivative by inserting the multiple \( t^{1−\alpha} \), \( 0 < \alpha < 1 \) inside the definition. The word fractional there was used to express the derivative of arbitrary order although no memory effect exists inside the corresponding integral inverse operator. This conformable (fractional) derivative seems to be kind of local derivative without memory. An interesting application of the conformable fractional derivative in Physics was discussed in [10], where it has been used to formulate an Action Principle for particles under frictional forces. Despite the many nice properties the conformable derivative has, it has the drawbacks that when \( \alpha \) tends to zero we do not obtain the original function and the conformable integrals inverse operators are free of memory...
and do not have a semigroup property. It is most likely to call them conformable derivatives or local derivatives of arbitrary order. In connection with this, at the end of reference [11], the author asked whether it is possible to fractionalize the conformable (fractional) derivative by using conformable (fractional) integrals of order $0 < \alpha \leq 1$ or by iterating the conformable derivative. The first part, Riemann-Liouville approach, was answered in [12, 13], where the author iterated the (conformable) integral with weight $t^\rho - 1$, $\rho \neq 0$ to define generalized fractional integrals and derivatives that unify Riemann-Liouville fractional integrals and derivatives ($\rho = 1$) and derivatives together with Hadamard fractional integrals and derivatives. Actually, the limiting case of that generalization is when $\rho \rightarrow 0^+$ leads to Hadamard type. However, the Grünwald-Letnikov approach for conformable derivatives is still open. The conformable time-scale fractional calculus of order $0 < \alpha < 1$ is introduced in [14] and has been used to develop the fractional differentiation and fractional integration. After then, many authors got interested in this type of derivatives for their many nice behaviors [10, 15–18]. Motivated by the need of some new fractional derivatives with nice properties and that can be applied to more real world modeling, some authors introduced very recently new kinds of fractional derivatives whose kernel is nonsingular. For the fractional derivatives with exponential kernels we refer to [19]. For fractional derivatives of nonsingular Mittag-Leffler type of derivatives for their many nice behaviors [10, 15–18].

For a function $f: (0, \infty) \rightarrow \mathbb{R}$ the (conformable) fractional derivative of order $0 < \alpha \leq 1$ of $f$ at $t > 0$ was defined by

$$D^\alpha_a f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon (t - a)^{-\alpha}) - f(t)}{\varepsilon}, \quad (1)$$

and the fractional derivative at $a$ is defined as $(D^\alpha_a f)(a) = \lim_{\varepsilon \to 0^+} (D^\alpha_a f)(t)$. The corresponding conformable (fractional) integral of order $0 < \alpha < 1$ and starting from $a$ is defined by

$$(I^\alpha_a f)(x) = \int_a^x f(t) \, d_\alpha (t) = \int_a^x f(t) (t - a)^{\alpha - 1} \, dt. \quad (2)$$

It is to be noted that the author used this modified conformable integral in order to extend it to left-right concept and confirm it by the $Q$-operator and obtain a left-right integration by parts version. Otherwise the integral can be given by $(I^\alpha_a f)(x) = \int_a^x f(t) t^{\alpha - 1} \, dt$. It was shown in [9, 11] that $(I^\alpha_a D^\alpha_a f)(x) = f(x) - f(a)$ and $(D^\alpha_a I^\alpha_a f)(x) = f(x)$. For the higher order case and other details such as the product rule, chain rule, and integration by parts, we refer the reader to [9, 11].

### 2. Main Results

In this paper we consider the fractional extension of the Sturm-Liouville eigenvalue problem

$$D^\alpha_a (p(x) D^\alpha_a y) + q(x) y = -\lambda w(x) y, \quad \frac{1}{2} < \alpha \leq 1, \quad a < x < b,$$

where $p, D^\alpha_a p, q$ and the weight functions $w$ are continuous on $(a, b)$, $p(x) > 0$, and $w(x) > 0$, on $[a, b]$, and the fractional derivative $D^\alpha_a$ is the conformable fractional derivative. We discuss (3) with boundary conditions

$$c_1 y(a) + c_2 y'(a) = 0, \quad c_2^2 + c_1^2 > 0,$$

$$r_1 y(b) + r_2 y'(b) = 0, \quad r_1^2 + r_2^2 > 0. \quad (4)$$

We say that $y$ is $2\alpha$-continuously differentiable on $[a, b]$, if $D^\alpha_a D^\alpha_a y$ is continuous on $[a, b]$, and $y \in C^{2\alpha}[a, b]$, if $y \in C^2[a, b]$ and is $2\alpha$-continuously differentiable on $[a, b]$. Let

$$L(y, \alpha) = D^\alpha_a (p(x) D^\alpha_a y) + q(x) y;$$

then the fractional Sturm-Liouville eigenvalue problem (3) can be written as

$$L(y, \alpha) = -\lambda w(x) y. \quad (6)$$

The following is a generalized result of the well-known Lagrange identity.

**Theorem 1** (fractional Lagrange identity). Letting $y_1, y_2$ be $2\alpha$-continuously differentiable on $[a, b]$, then the following holds true:

$$\int_a^b (y_2 L(y_1, \alpha) - y_1 L(y_2, \alpha)) \, d\alpha (x)$$

$$= [p(x)(y_2 D^\alpha_a y_1 - y_1 D^\alpha_a y_2)]_a^b. \quad (7)$$

**Proof.** We have

$$y_2 L(y_1, \alpha) - y_1 L(y_2, \alpha)$$

$$= y_2 D^\alpha_a (p(x) D^\alpha_a y_1) + q(x) y_1 y_2$$

$$- y_1 D^\alpha_a (p(x) D^\alpha_a y_2) - q(x) y_1 y_2$$

$$= y_2 D^\alpha_a (p(x) D^\alpha_a y_1) - y_1 D^\alpha_a (p(x) D^\alpha_a y_2). \quad (8)$$
Proof. Since loss of generality, then \( \alpha \) \( \in \mathbb{R} \), which proves the result.

**Proposition 2.** If \( y \in C^1[0, 1] \) and \( y'(x_0) = 0 \), for some \( x_0 \in [a, b] \), then \( (D^\alpha_y)(x_0) = 0 \).

**Proof.** Since \( y \in C^1[0, 1] \), then \( (D^\alpha_y)(x) = (x - a)^{1-\alpha}y'(x) \), and the result follows for \( a < x_0 \leq b \). If \( x_0 = a \), we have \( (D^\alpha_y)(a) = \lim_{x \to a^-}(x - a)^{1-\alpha}y'(x) = 0 \).

**Proposition 3.** Let \( y_1 \) and \( y_2 \) in \( C^1[a, b] \), which satisfy the boundary conditions (4); then it holds that

\[
\int_a^b \left[ p(x) (y_2 D^\alpha_a y_1 - y_1 D^\alpha_a y_2) \right] db.
\]

**Proof.** Since \( y_1 \in C^1[a, b] \), then \( D^\alpha_a y_1 = (x - a)^{1-\alpha}y'_1(x) \). Similarly, \( D^\alpha_a y_2 = (x - a)^{1-\alpha}y'_2(x) \). We have

\[
\left[ p(x) (y_2 D^\alpha_a y_1 - y_1 D^\alpha_a y_2) \right] db.
\]

Thus,

\[
y_2 (b) (D^\alpha_a y_1) (b) - y_1 (b) (D^\alpha_a y_2) (b) = - \frac{r_2}{r_1} y'_2 (b)
\]

\[
(D^\alpha_a y_1) (b) + \frac{r_2}{r_1} y'_1 (b) (D^\alpha_a y_2) (b)
\]

\[
= - \frac{r_2}{r_1} \left( y'_2 (b) (b - a)^{1-\alpha} y_1 (b) - y'_1 (b) (b - a)^{1-\alpha} y_2 (b) \right) = 0.
\]

Analogously,

\[
y_2 (a) (D^\alpha_a y_1) (a) - y_1 (a) (D^\alpha_a y_2) (a) = 0,
\]

which proves the result.

**Definition 4.** We say that \( f \) and \( g \) are \( \alpha \)-orthogonal with respect to the weight function \( \mu(x) \geq 0 \), if

\[
\int_a^b \mu(x) f(x) g(x) dx = 0.
\]

**Theorem 5.** The eigenfunctions of the fractional eigenvalue problem (3)-(4) corresponding to distinct eigenvalues are \( \alpha \)-orthogonal with respect to the weight function \( w(x) \).

**Proof.** Let \( \lambda_1 \) and \( \lambda_2 \) be two distinct eigenvalues and \( y_1 \) and \( y_2 \) are the corresponding eigenfunctions. We have

\[
L (y_1, \alpha) = - \lambda_1 w(x) y_1,
\]

\[
L (y_2, \alpha) = - \lambda_2 w(x) y_2.
\]

Multiplying (16) by \( y_2 \) and (17) by \( y_1 \) and subtracting the two equations yield

\[
y_2 L (y_1, \alpha) - y_1 L (y_2, \alpha) = - (\lambda_1 - \lambda_2) w(x) y_1 y_2.
\]

Performing the fractional integral \( D^\alpha_a \) and using the fractional Lagrange identity we have

\[
- (\lambda_1 - \lambda_2) \int_a^b \mu(x) y_1 y_2 d\alpha(x)
\]

\[
= \int_a^b (y_2 L (y_1, \alpha) - y_1 L (y_2, \alpha)) dx
\]

\[
= \int_a^b \left[ p(x) (y_2 D^\alpha_a y_1 - y_1 D^\alpha_a y_2) \right] db = 0,
\]

by virtue of Proposition 3. Since \( \lambda_1 \neq \lambda_2 \), we have \( \int_a^b w(x) y_1 y_2 d\alpha(x) = 0 \), and the result is obtained.

**Theorem 6.** The eigenvalues of the fractional eigenvalue problem (3)-(4) are real.

**Proof.** Let \( y \) be a solution to the fractional Sturm-Liouville eigenvalue problem (3)-(4). Taking the complex conjugate of (3)-(4) and using the fact that \( p(x), q(x) \) and \( w(x) \) are real valued functions, we have

\[
L (\overline{y}, \alpha) = D^\alpha_a \left( p(x) D^\alpha_a \overline{y} + q(x) \overline{y} \right)
\]

\[
= - \lambda w(x) \overline{y},
\]

\[
c_1 \overline{y} (a) + c_2 \overline{y}' (a) = 0,
\]

\[
r_1 \overline{y} (b) + r_2 \overline{y}' (b) = 0.
\]

Applying analogous steps to the proofs of Theorem 5 and Proposition 3 with \( y_1 = y \) and \( y_2 = \overline{y} \), we have

\[
- (\lambda - \overline{\lambda}) \int_a^b w(x) \left| y(x) \right|^2 d\alpha(x)
\]

\[
= \int_a^b (\overline{y} L (y, \alpha) - y L (\overline{y}, \alpha)) dx
\]

\[
= \int_a^b \left[ p(x) (y D^\alpha_a \overline{y} - \overline{y} D^\alpha_a y) \right] db = 0,
\]

and thus \( \lambda = \overline{\lambda} \) which completes the proof.
**Definition 7.** Let $f$ and $g$ be $\alpha$-differentiable; the fractional Wronskian function is defined by

$$W_{\alpha}(f, g) = fD^\alpha g - gD^\alpha f.$$  \hfill (22)

**Theorem 8.** Let $y_1$ and $y_2$ be $2\alpha$-continuously differentiable on $[a, b]$, and they are linearly independent solutions of (3); then

$$W_{\alpha}(y_1, y_2) = W_{\alpha}(y_1, y_2)(a) p(a)/p(x).$$  \hfill (23)

**Proof.** Applying the product rule one can easily verify that

$$D^\alpha W_{\alpha}(y_1, y_2) = y_1D^\alpha D^\alpha y_2 - y_2D^\alpha D^\alpha y_1.$$  \hfill (24)

Analogously, applying the product rule to (3) yields

$$D^\alpha D^\alpha y = -\frac{1}{p}(D^\alpha pD^\alpha y + (q + \lambda w)y).$$  \hfill (25)

Substituting the last equation in (24) yields

$$D^\alpha W_{\alpha}(y_1, y_2) = -\frac{y_1}{p}(D^\alpha pD^\alpha y_2 + (q + \lambda w)y_2)
+ \frac{y_2}{p}(D^\alpha pD^\alpha y_1 + (q + \lambda w)y_1)
+ \frac{D^\alpha p}{p}(y_2D^\alpha y_1 - y_1D^\alpha y_2)
= -\frac{D^\alpha p}{p}W_{\alpha}(y_1, y_2).$$  \hfill (26)

One can easily verify that the solution of the above fractional differential equation is

$$W_{\alpha}(y_1, y_2) = \frac{c}{p}$$  \hfill (27)

where $c$ is constant. Now, $W_\alpha(y_1, y_2)(a) = c/p(a)$, and thus $c = W_\alpha(y_1, y_2)(a)p(a)$, and hence the result. \hfill (28)

**Theorem 9.** The eigenvalues of the fractional eigenvalue problem (3)-(4) are simple.

**Proof.** Let $y_1$ and $y_2$ be two eigenfunctions for the same eigenvalue $\lambda$. From (18) we have

$$0 = y_2L(y_1, \alpha) - y_1L(y_2, \alpha)
= y_2D^\alpha(p(x)D^\alpha y_1) - y_1D^\alpha(p(x)D^\alpha y_2)
= y_2(D^\alpha pD^\alpha y_1 + pD^\alpha D^\alpha y_1)
- y_1(D^\alpha pD^\alpha y_2 + pD^\alpha D^\alpha y_2)
= p[y_2D^\alpha D^\alpha y_1 - y_1D^\alpha D^\alpha y_2]
+ D^\alpha p[y_2D^\alpha y_1 - y_1D^\alpha y_2]
= D^\alpha (p[y_2D^\alpha y_1 - y_1D^\alpha y_2]).$$  \hfill (29)

Thus

$$p[y_2D^\alpha y_1 - y_1D^\alpha y_2] = c,$$  \hfill (30)

and since $y_1$ and $y_2$ satisfy the same boundary conditions, we have $c = 0$ and

$$y_2D^\alpha y_1 - y_1D^\alpha y_2 = 0.$$  \hfill (31)

Since $W_\alpha(y_1, y_2) = 0$, and $y_1$ and $y_2$ are both solutions to the fractional eigenvalue problem (3)-(4), then they are linearly dependent. \hfill (32)

**Theorem 10** (fractional Rayleigh Quotient). The eigenvalues $\lambda$ of problem (3) satisfy

$$\lambda = \frac{\int_a^b p(D^\alpha y)\^2 dx}{\int_a^b w y^2 dx} \quad \text{for} \quad y \neq 0.$$  \hfill (33)

**Proof.** Multiplying (3) by $y$ and integrating yields

$$\int_a^b yD^\alpha(D^\alpha y) dx = \int_a^b q(x)y^2 dx - \int_a^b pD^\alpha y \| y \|_2^2 dx.$$  \hfill (34)

Integrating the first integral by parts we have

$$\int_a^b yD^\alpha y \|_2^b - \int_a^b p(D^\alpha y)^\|_2^2 dx + \int_a^b q(x)y^2 dx.$$  \hfill (35)

which proves the result. \hfill (36)

**Corollary 11.** Letting $y \in C^1[a, b]$ and $q(x) \leq 0$, then the eigenvalues of (3) associated with homogeneous boundary conditions of Dirichlet or Neumann type are nonnegative.

**Proof.** Since the boundary conditions are of Dirichlet or Neumann type then it holds that

$$yD^\alpha y \|_2^b = 0.$$  \hfill (37)

Then the result is directly obtained from the fractional Rayleigh Quotient as $q(x) \leq 0$. \hfill (38)

Now if $y$ is a stationary function for

$$F^\alpha_a(y) = \int_a^b F(y, D^\alpha_a y, x) dx (\alpha),$$  \hfill (39)

then it holds that, see [10],

$$\frac{\partial F}{\partial y}(y, D^\alpha_a y, x) - D^\alpha_a \left( \frac{\partial F}{\partial \alpha}(y, D^\alpha_a y, x) \right) = 0,$$  \hfill (40)
the fractional Euler equation. We remark here that the above equation is a necessary condition for a stationary point and not sufficient. In the following we show that the fractional Sturm-Liouville eigenvalue problem (3)-(4) is equivalent to the following:

(i) Finding the stationary function \( y(x) \) of

\[
F[y] = \int_a^b \left( p(D_a^y y)^2 - qy^2 \right) (x-a)^{\alpha-1} \, dx,
\]

subject to \( G[y] = 1 \), where

\[
G[y] = \int_a^b wy^2 (x-a)^{\alpha-1} \, dx.
\]

To find the stationary of \( F[y] \) subject to \( G[y] = 1 \), we first find the stationary value \( y \) of \( K[y] = F[y] - \lambda G[y] \) and then eliminate \( \lambda \) using \( G \) and integrating yields

\[
-2qy - 2\lambda wy - D_a^\alpha (2pD_a^\alpha y) = 0,
\]

or

\[
D_a^\alpha (pD_a^\alpha y) + qy = \lambda wy,
\]

which is the fractional Sturm-Liouville problem. Moreover, multiplying (3) by \( y \) and integrating yields

\[
\int_a^b yD_a^\alpha (pD_a^\alpha y) (x-a)^{\alpha-1} \, dx + \int_a^b qy^2 (x-a)^{\alpha-1} \, dx
= -\lambda \int_a^b wy^2 (x-a)^{\alpha-1} \, dx.
\]

Performing integration by parts of the first integral yields

\[
pyD_a^\alpha y|_a^b - \int_a^b p(D_a^\alpha y)^2 (x-a)^{\alpha-1} \, dx
+ \int_a^b qy^2 (x-a)^{\alpha-1} \, dx
= -\lambda \int_a^b wy^2 (x-a)^{\alpha-1} \, dx.
\]

Since

\[
yD_a^\alpha y|_a^b = 0,
\]

we have

\[
\lambda = \int_a^b \frac{p(D_a^\alpha y)^2 - qy^2}{y^2} (x-a)^{\alpha-1} \, dx = \int_a^b \frac{\alpha^2 (1 - 2x^\alpha)^2 x^{\alpha-1}}{(x^\alpha - x^{2\alpha})^2 x^{\alpha-1}} \, dx
= 10\alpha^2.
\]
So, we obtain an upper estimate \( \lambda_1 = 10\alpha^2 \), which is comparable with the exact eigenvalue \( \lambda_1 = \pi^2 \alpha^2 \). However, this upper bound can be improved by choosing a trial function
\[
\psi(x) = x^a (1 - x^a) + a (x^a (1 - x^a))^3,
\]
with parameter \( a \) and then choosing \( a \) to minimize the fractional Rayleigh Quotient. Direct calculations show that
\[
\int_0^1 (D_0^a \psi)^2 x^{a-1} dx = \frac{\alpha}{105} (35 + 2a (a + 7)),
\]
\[
\int_0^1 \psi^2 x^{a-1} dx = \frac{21 + a (a + 9)}{630a}.
\]
Thus, the fractional Rayleigh Quotient will produce
\[
\text{FR}(a, a) = \frac{630 (35 + a (a + 7))}{105 (21 + a (a + 9))}.
\]
The minimum value of
\[
R(a) = \frac{630 (35 + a (a + 7))}{105 (21 + a (a + 9))}
\]
is 9.86975 and occurs at \( a = 1.13314 \cdots \). Hence, an upper estimate \( \lambda_1 = 9.86975\alpha^2 \) is obtained which is very close to the exact one.

**Example 2.** Consider the fractional eigenvalue problem (3)-(4) with \( p = 1, q = 0, w = 1, 0 < x < 1 \) and with boundary condition \( y(0) = y'(0) = 0, y'(1) = 0 \). The eigenfunctions are
\[
\phi_n = a_n \sin(\lambda_n x^a) + b_n \cos(\lambda_n x^a).
\]
We choose \( a_n = 0 \), so that \( \phi'_n = \lambda_n a_n x^{a-1} a_n \cos(\lambda_n x^a) - \lambda_n a_n x^{a-1} b_n \sin(\lambda_n x^a) \) is defined at \( x = 0 \). Thus, \( \phi_n = b_n \cos(\lambda_n x^a) \), and applying the boundary conditions we have \( \phi_0 = 0 \). That is, the problem possesses no eigenfunctions for \( 1/2 < \alpha < 1 \).

**Remark 3.** It is well-known that the regular Sturm-Liouville eigenvalue problem with integer derivative possesses an infinite number of eigenvalues. This result is not valid for the fractional one as shown in the previous example. However, the fractional Sturm-Liouville equation in (3) can be discussed with fractional boundary conditions of the type
\[
c_i y(a) + c_i (D_0^a y)(a) = 0, \quad c_i^2 + c_i^2 > 0,
\]
\[
r_1 y(b) + r_1 (D_0^a y)(b) = 0, \quad r_1^2 + r_2^2 > 0.
\]
We believe that the above fractional eigenvalue problem possesses an infinite number of eigenvalues and we left it for a future work.

**4. Conclusion**

We have considered a regular conformable fractional Sturm-Liouville eigenvalue problem. We proved that the eigenvalues are real and simple and the eigenfunctions are orthogonal. We also established the fractional Wronskian result for any two linearly independent solutions of the problem. We obtained a fractional Rayleigh Quotient and applied a fractional variational principle to show that the minimum value of the Quotient is obtained at an eigenfunction. This result is used to estimate the first eigenvalue and the presented example illustrates the efficiency of the result. We illustrated by an example that the existence of eigenfunctions is not guaranteed unlike the result for the regular Sturm-Liouville eigenvalue problem. Most of the obtained results are analogous for the ones of regular Sturm-Liouville eigenvalue problems and they open the door for establishing other results such as the countability of eigenfunctions and completeness of eigenfunctions which are essential in solving fractional differential equations by fractional eigenfunction expansion.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Acknowledgments**

The first author gratefully acknowledges the support of the United Arab Emirates University under the Grant 31S239-UPAR(1) 2016.

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