Exact penalty functions for optimal control problems II: Exact penalization of terminal and pointwise state constraints

M. V. Dolgopolik

Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, Saint Petersburg, Russia

Correspondence
M. V. Dolgopolik, Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, Bolshoi pr. V.O. 61, Saint Petersburg, 199178, Russia.
Email: maxim.dolgopolik@gmail.com

Funding information
The President of Russian Federation grant for the support of young Russian scientists, MK-3621.2019.1

Summary
The second part of our study is devoted to an analysis of the exactness of penalty functions for optimal control problems with terminal and pointwise state constraints. We demonstrate that with the use of the exact penalty function method one can reduce fixed-endpoint problems for linear time-varying systems and linear evolution equations with convex constraints on the control inputs to completely equivalent free-endpoint optimal control problems, if the terminal state belongs to the relative interior of the reachable set. In the nonlinear case, we prove that a local reduction of fixed-endpoint and variable-endpoint problems to equivalent free-endpoint ones is possible under the assumption that the linearized system is completely controllable, and point out some general properties of nonlinear systems under which a global reduction to equivalent free-endpoint problems can be achieved. In the case of problems with pointwise state inequality constraints, we prove that such problems for linear time-varying systems and linear evolution equations with convex state constraints can be reduced to equivalent problems without state constraints, provided one uses the $L^\infty$ penalty term, and Slater's condition holds true, while for nonlinear systems a local reduction is possible, if a natural constraint qualification is satisfied. Finally, we show that the exact $L^p$-penalization of state constraints with finite $p$ is possible for convex problems, if Lagrange multipliers corresponding to the state constraints belong to $L^{p'}$, where $p'$ is the conjugate exponent of $p$, and for general nonlinear problems, if the cost functional does not depend on the control inputs explicitly.

KEYWORDS
exact penalty function, fixed-endpoint problem, optimal control, state constraint, terminal constraint

1 | INTRODUCTION

The exact penalty method is an important tool for solving constrained optimization problems. Many publications have been devoted to its analysis from various perspectives (eg, References 1-12). The main idea of this method consist in the
The main goal of this two-part study is to develop a general theory of exact penalty functions for optimal control problems containing sufficient conditions for the complete or local exactness of penalty functions that can be readily verified in various particular cases. In the first part of our study\textsuperscript{45} we obtained simple sufficient conditions for the exactness

\begin{equation}
\min f(x) \text{ subject to } g_i(x) \leq 0, \quad i \in \{1, \ldots, n\},
\end{equation}

to the unconstrained optimization problem of minimizing the nonsmooth penalty function \( \Phi_\lambda(x) = f(x) + \lambda \max_{i \geq 1} \{ g_i(x), 0 \} \). Under some natural assumptions such as the coercivity of \( \Phi_\lambda \) and the validity of a suitable constraint qualification one can prove that for any sufficiently large (but finite) value of the penalty parameter \( \lambda \) the penalized problem is equivalent to the original problem in the sense that these problems have the same optimal value and the same globally optimal solutions. In this case the penalty function \( \Phi_\lambda \) is called exact. Under some additional assumptions not only globally optimal solutions, but also locally optimal solutions and stationary (critical) points of these problems coincide. In this case the penalty function \( \Phi_\lambda \) is called completely exact. Finally, if a given locally optimal solution of problem (1) is a point of local minimum of \( \Phi_\lambda \) for any sufficiently large \( \lambda \), then \( \Phi_\lambda \) is said to be locally exact at this solution.

Thus, the exactness property of a penalty function allows one to reduce (locally or globally) a constrained optimization problem to the equivalent unconstrained problem of minimizing a penalty function and, as a result, apply numerical methods of unconstrained optimization to constrained problems. However, note that exact penalty functions not depending on the derivatives of the objective function and constraints are inherently nonsmooth (eg, remark 3 in Reference 10), and one has to either utilize general methods of nonsmooth optimization to minimize exact penalty functions or develop specific methods for minimizing such functions that take into account their structure. See the works of Mäkelä et al\textsuperscript{13-15} for a survey and comparative analysis of modern nonsmooth optimization methods and software.

Numerical methods for solving optimal control problems based on exact penalty functions were developed by Maratos\textsuperscript{16} and in a series of papers by Mayne et al\textsuperscript{17-21} (see also monograph\textsuperscript{22}). An exact penalty method for optimal control problems with delay was proposed by Wong and Teo,\textsuperscript{23} and such method for some nonsmooth optimal control problems was studied in the works of Outrata et al.\textsuperscript{24-26} A continuous numerical method for optimal control problems based on the direct minimization of an exact penalty function was considered in recent paper.\textsuperscript{27} Finally, closely related methods based on Huyer and Neumaier's exact penalty function\textsuperscript{11,28,29} were developed for optimal control problems with state inequality constraints\textsuperscript{30,31} and optimal feedback control problems.\textsuperscript{32}

Despite the abundance of publications on exact penalty methods for optimal control problems, relatively little attention has been paid to an actual analysis of the exactness of penalty functions for such problems. To the best of author's knowledge, the possibility of the exact penalization of pointwise state constraints for optimal control problems was first mentioned by Luenberger;\textsuperscript{33} however, no particular conditions ensuring exact penalization were given in this paper. Lasserre\textsuperscript{34} proved that a stationary point of an optimal control problem with endpoint equality and state inequality constraints is also a stationary point of a nonsmooth penalty function for this problem (this result is closely related to the local exactness). The local exactness of a penalty function for problems with state inequality constraints was proved by Xing et al\textsuperscript{35,36} under the assumption that certain second-order sufficient optimality conditions are satisfied. First results on the global exactness of penalty functions for optimal control problems were probably obtained by Demyanov et al for a problem of finding optimal parameters in a system described by ordinary differential equations,\textsuperscript{37} free-endpoint optimal control problems,\textsuperscript{38-40} and certain optimal control problems for implicit control systems.\textsuperscript{41} However, the main results of these papers are based on the assumptions that the penalty function attains a global minimum in the space of piecewise continuous functions for any sufficiently large value of the penalty parameter, and the cost functional is Lipschitz continuous on a possibly unbounded and rather complicated set. It is unclear how to verify these assumptions in particular cases, which makes it very difficult to apply the main results of papers\textsuperscript{37-41} to real problems. To the best of author's knowledge, the only verifiable sufficient conditions for the global exactness of penalty functions for optimal control problems were obtained by Gugat\textsuperscript{42} for an optimal control of the wave equation, by Gugat and Zuazua\textsuperscript{43} for optimal control problems for general linear evolution equations, and by Jayswal and Preeti\textsuperscript{44} for an optimal control problem for a system described by a partial differential equation state inequality constraints. In papers\textsuperscript{42,43} only the exact penalization of terminal constraint was analyzed, while in article\textsuperscript{44} the exact penalization of state inequality constraints and system dynamics was proved under some restrictive convexity assumptions.

The main goal of this two-part study is to develop a general theory of exact penalty functions for optimal control problems containing sufficient conditions for the complete or local exactness of penalty functions that can be readily verified in various particular cases. In the first part of our study\textsuperscript{45} we obtained simple sufficient conditions for the exactness
of penalty functions for free-endpoint optimal control problems. This result allows one to apply numerical method for solving variational problems to free-endpoint optimal control problems.

In the second part of our study we analyze the exactness of penalty functions for problems with terminal and pointwise state constraints. In the first half of this paper, we study when a penalization of the terminal constraint is exact, that is, when the fixed-endpoint problem

\[
\min_{(x,u)} I(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt \\
\text{subject to } \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T], \quad x(0) = x_0, \quad x(T) = x_T, \quad u \in U
\]

is equivalent to the penalized free-endpoint one

\[
\min_{(x,u)} \Phi_\lambda(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt + \lambda[x(T) - x_T] \\
\text{subject to } \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T], \quad x(0) = x_0, \quad u \in U.
\]

We prove that fixed-endpoint problems for linear time-varying systems and linear evolution equations in Hilbert spaces with convex constraints on the control inputs (ie, the set \(U\) is convex) are equivalent to the corresponding penalized free-endpoint problems, if the terminal state \(x_T\) belongs to the relative interior of the reachable set. This result significantly generalizes the one of Gugat and Zuazua\textsuperscript{43} (see Remark 8 for a detailed discussion). In the case of nonlinear problems, we show that the penalty function \(\Phi_\lambda\) is locally exact at a given locally optimal solution, if the corresponding linearized system is completely controllable, and point our some general assumption on the system \(\dot{x} = f(x, u, t)\) that ensure the complete exactness of \(\Phi_\lambda\). We also present an extension of these results to the case of variable-endpoint problems.

In the second half of this paper, we study the exact penalization of pointwise state constraints, that is, we study when the optimal control problem with pointwise state inequality constraints

\[
\min_{(x,u)} I(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt \quad \text{subject to } \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T], \\
x(0) = x_0, \quad x(T) = x_T, \quad u \in U, \quad g_j(x(t), t) \leq 0 \quad \forall t \in [0, T], \quad j \in \{1, \ldots, l\}
\]

is equivalent to the penalized problem without state constraints

\[
\min_{(x,u)} \Phi_\lambda(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt + \lambda \|\max\{g_1(x(\cdot), \cdot), \ldots, g_l(\cdot), \cdot\}\|_p \\
\text{subject to } \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T], \quad x(0) = x_0, \quad x(T) = x_T, \quad u \in U
\]

for some \(1 \leq p \leq +\infty\) (here \(| \cdot | \|_p\) is the standard norm in \(L^p(0, T)\)). In the case of problems for linear time-varying systems and linear evolution equation with convex state constraints, we prove that the penalization of state constraints is exact, if \(p = +\infty\), and Slater’s condition holds true, that is, there exists a feasible point \((x, u)\) such that \(g_j(x(t), t) < 0\) for all \(t \in [0, T]\) and \(j \in \{1, \ldots, l\}\). In the nonlinear case, we prove the local exactness of \(\Phi_\lambda\) with \(p = +\infty\) under the assumption that a suitable constraint qualification is satisfied. Finally, we demonstrate that under some additional assumptions the exact \(L^p\) penalization of state constraints with finite \(p\) is possible for convex problems, if Lagrange multipliers corresponding to state constraints belong to \(L^{p'}(0, T)\), and for nonlinear problems, if the cost functional \(I\) does not depend on the control inputs \(u\) explicitly.

The paper is organized as follows. Some basic definitions and results from the general theory of exact penalty functions for optimization problems in metric spaces are collected in Section 2, so that the second part of the paper can be read independently of the first one. Section 3 is devoted to the analysis of exact penalty functions for fixed-endpoint and variable-endpoint problems, while exact penalty functions for optimal control problems with state constraints are considered in Section 4. Finally, a proof of a general theorem on completely exact penalty function from Section 2 is given in Appendix A, while Appendix B contains some useful results on Nemytskii operators that are utilized throughout the paper.
2 | EXACT PENALTY FUNCTIONS IN METRIC SPACES

In this section we recall some basic definitions and results from the theory of exact penalty functions that will be utilized throughout the article (see papers10,11 for more details). Let \((X, d)\) be a metric space, \(M, A \subseteq X\) be nonempty sets such that \(M \cap A \neq \emptyset\), and \(I : X \to \mathbb{R} \cup \{+\infty\}\) be a given function. Consider the following optimization problem:

\[
\min_{x \in X} I(x) \quad \text{subject to } x \in M \cap A. \quad (P)
\]

Here the sets \(M\) and \(A\) represent two different types of constraints, for example, pointwise and terminal constraints or linear and nonlinear constraints, etc. In what follows, we suppose that there exists a globally optimal solution \(x^*\) of the problem \((P)\) such that \(I(x^*) < +\infty\), that is, the optimal value of this problem is finite and is attained.

Our aim is to “get rid” of the constraint \(x \in M\) without losing any essential information about (locally or globally) optimal solutions of the problem \((P)\). To this end, we apply the exact penalty function technique. Let \(\varphi : X \to [0, +\infty)\) be a function such that \(\varphi(x) = 0\) iff \(x \in M\). For example, if \(M\) is closed, one can put \(\Phi(x) = \mathrm{dist}(x, M) = \inf_{y \in M} d(x, y)\). For any \(\lambda \geq 0\) define \(\Phi_\lambda(x) = I(x) + \lambda \varphi(x)\). The function \(\Phi_\lambda\) is called a penalty function for the problem \((P)\) (corresponding to the constraint \(x \in M\)), \(\lambda\) is called a penalty parameter, and \(\varphi\) is called a penalty term for the constraint \(x \in M\).

Observe that the function \(\Phi_\lambda\) is non-decreasing in \(\lambda\), and \(\Phi_\lambda(x) \geq I(x)\) for all \(x \in X\) and \(\lambda \geq 0\). Furthermore, for any \(\lambda > 0\) one has \(\Phi_\lambda(x) = I(x)\) iff \(x \in M\). Therefore, it is natural to consider the penalized problem

\[
\min_{x \in X} \Phi_\lambda(x) \quad \text{subject to } x \in A. \quad (2)
\]

Note that this problem has only one constraint \((x \in A)\), while the constraint \(x \in M\) is incorporated into the new objective function \(\Phi_\lambda\). We would like to know when this problem is, in some sense, equivalent to the problem \((P)\), that is, when the penalty function \(\Phi_\lambda\) is exact.

**Definition 1.** The penalty function \(\Phi_\lambda\) is called (globally) exact, if there exists \(x^* \geq 0\) such that for any \(\lambda \geq x^*\) the set of globally optimal solutions of the penalized problem \((2)\) coincides with the set of globally optimal solutions of the problem \((P)\).

From the fact that \(\Phi_\lambda(x) = I(x)\) for any feasible point \(x\) of the problem \((P)\) it follows that if \(\Phi_\lambda\) is globally exact, then the optimal values of the problem \((P)\) and problem \((2)\) coincide. Thus, the penalty function \(\Phi_\lambda\) is globally exact iff the problem \((P)\) and problem \((2)\) are equivalent in the sense that they have the same globally optimal solutions and the same optimal value. However, optimization methods often can find only locally optimal solutions (or even only stationary/critical points) of an optimization problem. Therefore, the concept of the global exactness of the penalty function \(\Phi_\lambda\) is not entirely satisfactory for practical applications. One needs to ensure that not only globally optimal solutions, but also local minimizers and stationary points of the problem \((P)\) and problem \((2)\) coincide. To provide conditions under which such complete exactness takes place we need to recall the definitions of the rate of steepest descent\(^{46-48}\) and inf-stationary point\(^{46,47}\) of a function defined on a metric space.

Let \(K \subseteq X\) and \(f : X \to \mathbb{R} \cup \{+\infty\}\) be given, and \(x \in K\) be such that \(f(x) < +\infty\). The quantity

\[
f_K^1(x) = \lim_{y \to x, y \in K} \inf_{y \neq x} \frac{f(y) - f(x)}{d(y, x)},
\]

is called the rate of steepest descent of \(f\) with respect to the set \(K\) at the point \(x\). If \(x\) is an isolated point of \(K\), then by definition \(f_K^1(x) = +\infty\). It should be noted that the rate of steepest descent of \(f\) at \(x\) is closely connected to the so-called strong slope \(\nabla f(x)\) of \(f\) at \(x\). See papers\(^{11,49,50}\) for some calculus rules for strong slope/rate of steepest descent, and the ways one can estimate them in various particular cases.

Let \(x^* \in K\) be such that \(f(x^*) < +\infty\). The point \(x^*\) is called an inf-stationary point of \(f\) on the set \(K\) if \(f_K^1(x^*) \geq 0\). Observe that the inequality \(f_K^1(x^*) \geq 0\) is a necessary optimality condition for the problem

\[
\min_{x \in X} f(x) \quad \text{subject to } x \in K.
\]

In the case when \(X\) is a normed space, \(K\) is convex, and \(f\) is Fréchet differentiable at \(x^*\) the inequality \(f_K^1(x^*) \geq 0\) is reduced to the standard optimality condition: \(f'(x^*)(x - x^*) \geq 0\) for all \(x \in K\), where \(f'(x^*)\) is the Fréchet derivative of \(f\) at \(x^*\).

\[\text{901}\]

\[\text{WILEY}\]

\[\text{DOLGOPOLIK}\]
Now we can formulate sufficient conditions for the complete exactness of the penalty function $\Phi_{\lambda}$. For any $\lambda \geq 0$ and $c \in \mathbb{R}$ denote $S_{\lambda}(c) = \{ x \in A | \Phi_{\lambda}(x) < c \}$. Let also $\Omega = M \cap A$ be the feasible region of $(P)$, and for any $\delta > 0$ define $\Omega_\delta = \{ x \in A | \varphi(x) < \delta \}$.

**Theorem 1.** Let $X$ be a complete metric space, $A$ be closed, $I$ and $\varphi$ be lower semicontinuous on $A$, and $\varphi$ be continuous at every point of the set $\Omega$. Suppose also that there exist $c > I^* = \inf_{x \in \Omega} I(x)$, $\lambda_0 > 0$, and $\delta > 0$ such that

1. there exists an open set $V$ such that $S_{\lambda_0}(c) \cap \Omega_\delta \subset V$ and the functional $I$ is Lipschitz continuous on $V$;
2. there exists $a > 0$ such that $\varphi'_{\lambda}(x) \leq -a$ for all $x \in S_{\lambda_0}(c) \cap (\Omega_\delta \setminus \Omega)$;
3. $\Phi_{\lambda_0}$ is bounded below on $A$.

Then there exists $\lambda^* \geq 0$ such that for any $\lambda \geq \lambda^*$ the following statements hold true:

1. the optimal values of the problem $(P)$ and problem $(2)$ coincide;
2. globally optimal solutions of the problem $(P)$ and problem $(2)$ coincide;
1. $x^* \in S_{\lambda}(c)$ is a locally optimal solution of the penalized problem $(2)$ iff $x^* \in \Omega$, and it is a locally optimal solution of the problem $(P)$;
2. $x^* \in S_{\lambda}(c)$ is an inf-stationary point of $\Phi_{\lambda}$ on $A$ iff $x^* \in \Omega$, and it is an inf-stationary point of $I$ on $\Omega$.

If the penalty function $\Phi_{\lambda}$ satisfies the four statements of this theorem, then it is said to be completely exact on the set $S_{\lambda}(c)$. The proof of Theorem 1 is given in the first part of our study.\(^{45}\)

**Remark 1.** Let us note that Theorem 1 is valid even in the case when the set $\Omega_\delta \setminus \Omega$ is empty. Moreover, if $\Omega_\delta \setminus \Omega = \emptyset$ for some $\delta > 0$, then the penalty function $\Phi_{\lambda}$ is completely exact on $S_{\lambda}(c)$ for any $c > I^*$, provided there exists $\lambda_0 \geq 0$ such that $\Phi_{\lambda_0}$ is bounded below on $A$. Indeed, in this case for any $\lambda \geq \lambda_0$ and $x \notin \Omega_\delta$ one has

$$
\Phi_{\lambda}(x) = \Phi_{\lambda_0}(x) + (\lambda - \lambda_0) \varphi(x) \geq \eta + (\lambda - \lambda_0) \delta \geq c \quad \forall \lambda \geq \lambda^* = \lambda_0 + (c - \eta)/\delta,
$$

where $\eta = \inf_{x \in A} \Phi_{\lambda_0}(x)$, which implies that $S_{\lambda}(c) \subseteq \Omega$ for any $\lambda \geq \lambda^*$. Hence taking into account the fact that $\Phi_{\lambda}(x) = I(x)$ for any $x \in \Omega$ one obtains that the first two statements of Theorem 1 hold true, and if $x^* \in S_{\lambda}(c)$ is a local minimizer/inf-stationary point of $\Phi_{\lambda}$ on $A$, then $x^* \in \Omega$ and it is a local minimizer/inf-stationary point of $I$ on $\Omega$, provided $\lambda \geq \lambda^*$. On the other hand, if $\lambda \geq \lambda^*$ and $x^* \in S_{\lambda}(c)$ is a locally optimal solution of the problem $(P)$, then for any $x$ in a neighborhood of $x^*$, either $x \in \Omega$ and $\Phi_{\lambda}(x) = I(x) \geq I(x^*) = \Phi_{\lambda}(x^*)$ or $x \notin \Omega$ and $\Phi_{\lambda}(x) \geq c > \Phi_{\lambda}(x^*)$, that is, $x^*$ is a locally optimal solution of the penalized problem $(2)$. The analogous statement for inf-stationary points is proved in a similar way.

Under the assumptions of Theorem 1 nothing can be said about locally optimal solutions of the penalized problem and inf-stationary points of $\Phi_{\lambda}$ on $A$ that do not belong to the sublevel set $S_{\lambda}(c)$. If a numerical method for minimizing the penalty function $\Phi_{\lambda}$ finds a point $x^* \notin S_{\lambda}(c)$, then this point might even be infeasible for the original problem (in this case, usually, either constraints are degenerate in some sense at $x^*$ or $I$ is not Lipschitz continuous near this point). Under more restrictive assumptions one can exclude such possibility, that is, prove that the penalty function $\Phi_{\lambda}$ is completely exact on $A$, that is, on $S_{\lambda}(c)$ with $c = +\infty$. Namely, the following theorem holds true.\(^{*}\) Its proof is given in Appendix A.

**Theorem 2.** Let $X$ be a complete metric space, $A$ be closed, $I$ be Lipschitz continuous on $A$, and $\varphi$ be lower semicontinuous on $A$ and continuous at every point of the set $\Omega$. Suppose also that there exists $a > 0$ such that $\varphi_{\lambda}(x) \leq -a$ for all $x \in A \setminus \Omega$, and the function $\Phi_{\lambda_0}$ is bounded below on $A$ for some $\lambda_0 \geq 0$. Then the penalty function $\Phi_{\lambda}$ is completely exact on $A$.

In some important cases it might be very difficult (if at all possible) to verify the assumptions of Theorems 1 and 2 and prove the complete exactness of the penalty function $\Phi_{\lambda}$. In these cases one can try to check whether $\Phi_{\lambda}$ is at least locally exact.

**Definition 2.** Let $x^*$ be a locally optimal solution of the problem $(P)$. The penalty function $\Phi_{\lambda}$ is said to be locally exact at $x^*$, if there exists $\lambda^*(x^*) \geq 0$ such that $x^*$ is a point of local minimum of the penalized problem $(2)$ for any $\lambda \geq \lambda^*(x^*)$.

\(^*\)This result as well as its applications in the following sections were inspired by a question raised by one of the reviewers of the first part of our study.

The author wishes to express his gratitude to the reviewer for raising this question.
Thus, if the penalty function $\Phi_\lambda$ is locally exact at a locally optimal solution $x^*$, then one can “get rid” of the constraint $x \in M$ in a neighbourhood of $x^*$ with the use of the penalty function $\Phi_\lambda$, since by definition $x^*$ is a local minimizer of $\Phi_\lambda$ on $A$ for any sufficiently large $\lambda$. The following theorem, which is a particular case of Theorem 2.4 and Proposition 2.7 in Reference 5, contains simple sufficient conditions for the local exactness. Let $B(x, r) = \{y \in X : d(x, y) \leq r\}$ for any $x \in X$ and $r > 0$.

**Theorem 3.** Let $x^*$ be a locally optimal solution of the problem (P). Suppose also that $I$ is Lipschitz continuous near $x^*$ with Lipschitz constant $L > 0$, and there exist $r > 0$ and $a > 0$ such that

$$\varphi(x) \geq a \text{dist}(x, \Omega) \quad \forall x \in B(x^*, r) \cap A.$$  \hspace{1cm} (3)

Then the penalty function $\Phi_\lambda$ is locally exact at $x^*$ with $\lambda^*(x^*) \leq L/a$.

Let us also point out a useful result (Corollary 2.2 in Reference 5) that allows one to easily verify inequality (3) for a large class of optimization and optimal control problems.

**Theorem 4.** Let $X$ and $Y$ be Banach spaces, $C \subseteq X$ and $K \subseteq Y$ be closed convex sets, and $F : X \to Y$ be a given mapping. Suppose that $F$ is strictly differentiable at a point $x^* \in C$ such that $F(x^*) \in K$, $DF(x^*)$ is its Fréchet derivative at $x^*$, and

$$0 \in \text{core} \left[ DF(x^*)(C - x^*) - (K - F(x^*)) \right].$$  \hspace{1cm} (4)

where “core” is the algebraic interior. Then there exist $r > 0$ and $a > 0$ such that

$$\text{dist}(F(x), K) \geq a \text{dist}(x, F^{-1}(K) \cap C) \quad \forall x \in B(x^*, r) \cap C.$$  

**Remark 2.** Let $C$, $K$, and $F$ be as in the previous theorem. Suppose that $A = C$ and $M = \{x \in X : F(x) \in K\}$. Then $\Omega = F^{-1}(K) \cap C$, and one can define $\varphi(\cdot) = \text{dist}(F(\cdot), K)$. In this case under the assumptions of Theorem 4 constraint qualification (4) guarantees that $\varphi(x) \geq a \text{dist}(x, F^{-1}(K) \cap C) = a \text{dist}(x, \Omega)$ for all $x \in B(x^*, r) \cap A$, that is, inequality (3) holds true.

In the linear case, the following nonlocal version of Robinson-Ursescu’s theorem due to Robinson (see Theorems 1 and 2 in Robinson’s original paper[22]) is very helpful for verifying inequality (3) and the exactness of penalty functions.

**Theorem 5.** (Robinson). Let $X$ and $Y$ be Banach spaces, $T : X \to Y$ be a bounded linear operator, and $C \subseteq X$ be a closed convex set. Suppose that $x^* \in C$ is such that the point $y^* = Tx^*$ belongs to the interior $\text{inter}(T(C))$ of the set $T(C)$. Then there exist $r > 0$ and $\kappa > 0$ such that

$$\text{dist}(x, T^{-1}(y) \cap C) \leq \kappa (1 + \|x - x^*\|) \|Tx - y\| \quad \forall x \in C \quad \forall y \in B(y^*, r).$$

In the following sections we employ Theorems 1 to 5 to verify complete or local exactness of penalty function for optimal control problems with terminal and state constraints.

**Remark 3.** In our exposition of the theory of exact penalty functions we mainly followed papers.\textsuperscript{10,11} A completely different approach to an analysis of the global exactness of exact penalty functions based on the Palais-Smale condition was developed by Zaslavski.\textsuperscript{9} It seems possible to apply the main results of monograph\textsuperscript{9} to obtain sufficient conditions for the global exactness of penalty functions for some optimal control problems that significantly differ from the ones obtained in this article. A derivation of such conditions lies beyond the scope of this article, and we leave it as an interesting open problem for future research.

### 3 Exact Penalization of Terminal Constraints

In this section, we analyze exact penalty functions for fixed-endpoint optimal control problems, including such problems for linear evolution equations in Hilbert spaces. Our aim is to convert a fixed-endpoint problem into a free-endpoint one by penalizing the terminal constraint and obtain conditions under which the penalized free-endpoint problem is
equivalent (locally or globally) to the original one. The main results of this section allow one to apply methods for solving free-endpoint optimal control problems to fixed-endpoint problems.

3.1 Notation

Let us introduce notation first. Denote by \( L^q_{\text{d}}(0, T) \) the Cartesian product of \( m \) copies of \( L^q(0, T) \), and let \( W^d_{1,p}(0, T) \) be the Cartesian product of \( d \) copies of \( W^1_{1,p}(0, T) \). Here \( 1 \leq q, p \leq +\infty \). As usual (e.g., monograph\(^5\)), we identify the Sobolev space \( W^{d,q}(0, T) \) with the space consisting of all those absolutely continuous functions \( x : [0, T] \to \mathbb{R} \) for which \( x \in L^q(0, T) \). The space \( L^q_{\text{d}}(0, T) \) is equipped with the norm \( \|u\|_q = (\int_0^T |u(t)|^q \, dt)^{1/q} \), when \( 1 \leq q < +\infty \) (here \( |\cdot| \) is the Euclidean norm), while the space \( L^\infty_{\text{d}}(0, T) \) is equipped with the norm \( \|u\|_\infty = \text{ess sup}_{t\in[0,T]}|u(t)| \). The Sobolev space \( W^d_{1,p}(0, T) \) is endowed with the norm \( \|x\|_{1,p} = \|x\|_p + \|\dot{x}\|_p \). Let us note that by the Sobolev imbedding theorem (e.g., theorem 5.4 in monograph\(^5\)) for any \( p \in [1, +\infty) \) there exists \( C_p > 0 \) such that

\[
\|x\|_\infty \leq C_p \|x\|_{1,p} \quad \forall x \in W^d_{1,p}(0, T),
\]

which, in particular, implies that any bounded set in \( W^d_{1,p}(0, T) \) is also bounded in \( L^\infty_{\text{d}}(0, T) \). In what follows we suppose that the Cartesian product \( X \times Y \) of normed spaces \( X \) and \( Y \) is endowed with the norm \( \|(x, y)\| = \|x\|_X + \|y\|_Y \). For any \( r \in [1, +\infty) \) denote by \( r' \in [1, +\infty) \) the conjugate exponent of \( r \), that is, \( 1/r + 1/r' = 1 \).

Let \( g : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}^k \) be a given function. We say that \( g \) satisfies the growth condition of order \((l,s)\) with \( 0 \leq l < +\infty \) and \( 1 \leq s \leq +\infty \), if for any \( R > 0 \) there exist \( C_R > 0 \) and an a.e. nonnegative function \( \omega_R \in L^l(0, T) \) such that

\[
|g(x, u, t)| \leq C_R |u|^l + \omega_R(t)
\]

for a.e. \( t \in [0, T] \) and for all \( (x, u) \in \mathbb{R}^d \times \mathbb{R}^m \) with \( |x| \leq R \).

Finally, if the function \( g = g(x, u, t) \) is differentiable, then the gradient of the function \( x \mapsto g(x, u, t) \) is denoted by \( \nabla_x g(x, u, t) \), and a similar notation is used for the gradient of the function \( u \mapsto g(x, u, t) \).

3.2 Linear time-varying systems

We start our analysis with the linear case, since in this case the complete exactness of the penalty function can be obtained without any assumptions on the controllability of the system. Consider the following fixed-endpoint optimal control problem:

\[
\begin{align*}
\min & \quad I(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt \\
\text{subject to} & \quad \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in [0, T], \quad u \in U, \quad x(0) = x_0, \quad x(T) = x_T.
\end{align*}
\]

Here \( x(t) \in \mathbb{R}^d \) is the system state at time \( t \), \( u(\cdot) \) is a control input, \( \theta : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}, A : [0, T] \to \mathbb{R}^{d \times d}, \) and \( B : [0, T] \to \mathbb{R}^{d \times m} \) are given functions, \( T > 0 \) and \( x_0, x_T \in \mathbb{R}^d \) are fixed. We suppose that \( x \in W^d_{1,p}(0, T) \), while the control inputs \( u \) belong to a closed convex subset \( U \) of the space \( L^q_{\text{d}}(0, T) \) (here \( 1 \leq p, q \leq +\infty \)).

Let us introduce a penalty function for problem (6). We will penalize only the terminal constraint \( x(T) = x_T \). Define \( X = W^d_{1,p}(0, T) \times L^q_{\text{d}}(0, T), M = \{(x, u) \in X|x(T) = x_T\}, \) and

\[
A = \{(x, u) \in X|x(0) = x_0, u \in U, \dot{x}(t) = A(t)x(t) + B(t)u(t) \text{ for a.e. } t \in [0, T]\}.
\]

Then problem (6) can be rewritten as the problem of minimizing \( I(x, u) \) subject to \( (x, u) \in M \cap A \). Define \( \varphi(x, u) = |x(T) - x_T| \). Then \( M = \{(x, u) \in X|\varphi(x, u) = 0\} \), and one can consider the penalized problem of minimizing the penalty function \( \Phi(x, u) = I(x, u) + \lambda \varphi(x, u) \) subject to \( (x, u) \in A \). Note that this is a free-endpoint problem of the form:

\[
\begin{align*}
\min & \quad \Phi(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt + \lambda \left|x(T) - x_T\right| \\
\text{subject to} & \quad \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in [0, T], \quad u \in U, \quad x(0) = x_0.
\end{align*}
\]
Our aim is to show that under some natural assumptions the penalty function $\Phi_\lambda$ is completely exact, that is, that free-endpoint problem (8) is equivalent to fixed-endpoint problem (6) for any sufficiently large $\lambda \geq 0$.

Let $I^*$ be the optimal value of problem (6). Recall that $S_\lambda(c) = \{(x, u) \in A | \Phi_\lambda(x, u) < c \}$ for any $c \in \mathbb{R}$ and $\Omega_\delta = \{(x, u) \in A | \varphi(x, u) < \delta \}$ for any $\delta > 0$. In our case the set $\Omega_\delta$ consists of all those $(x, u) \in W^d_{1,p}(0, T) \times L^m_q(0, T)$ for which $u \in U,$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad \text{for a.e. } t \in [0, T], x(0) = x_0,$$

and $|x(T) - x_T| < \delta$. Finally, denote by $R(x_0, T)$ the set that is reachable in time $T$, that is, the set of all those $\xi \in \mathbb{R}^d$ for which there exists $u \in U$ such that $x(T) = \xi$, where $x(\cdot)$ is a solution of Equation (9). Observe that the reachable set $R(x_0, T)$ is convex due to the convexity of the set $U$ and the linearity of the system. Finally, recall that the relative interior of a convex set $C \subset \mathbb{R}^d$, denoted $\text{relint}C$, is the interior of $C$ relative to the affine hull of $C$.

The following theorem on the complete exactness of the penalty function $\Phi_\lambda$ for problem (6) can be proved with the use of state-transition matrix for Equation (9). Here, we present a different and more instructive (although slightly longer) proof of this result, since it contains several important ideas related to penalty functions for optimal control problems, which will be utilized in the following sections.

**Theorem 6**. Let $q \geq p$, and the following assumptions be valid:

1. $A(\cdot) \in L^\infty_{\text{loc}}(0, T)$ and $B(\cdot) \in L^\infty_{\text{loc}}(0, T)$;
2. the function $\theta = \theta(x, u, \tau)$ is continuous, differentiable in $x$ and $u$, and the functions $\nabla_x \theta$ and $\nabla_u \theta$ are continuous;
3. either $q = +\infty$ or the functions $\theta$ and $\nabla_x \theta$ satisfy the growth condition of order $(q, 1)$, while the function $\nabla_u \theta$ satisfies the growth condition of order $(q-1, q')$;
4. there exists a globally optimal solution of problem (6), and $x_T$ belongs to the relative interior of the reachable set $R(x_0, T)$ (in the case $U = L^p_{\text{loc}}(0, T)$ this assumption holds true automatically);
5. there exist $\lambda_0 > 0$, $c > I^*$ and $\delta > 0$ such that the set $S_{\lambda_0}(c) \cap \Omega_\delta$ is bounded in $W^d_{1,p}(0, T) \times L^m_q(0, T)$, and the function $\Phi_{\lambda_0}(x, u)$ is bounded below on $A$.

Then there exists $\lambda^* \geq 0$ such that for any $\lambda \geq \lambda^*$ the penalty function $\Phi_\lambda$ for problem (6) is completely exact on $S_\lambda(x_0)$.

**Proof**. Our aim is to employ Theorem 1. To this end, note that from the essential boundedness of $A(\cdot)$ and $B(\cdot)$, and the fact that $p \leq q$ it follows that the function $(x, u) \mapsto \dot{x}(\cdot) - A(\cdot)x(\cdot) - B(\cdot)u(\cdot)$ continuously maps $X$ to $L^p_{\text{loc}}(0, T)$. Hence taking into account inequality (5) and the fact that $U$ is closed by our assumptions one obtains that the set $A$ is closed (see equality (7)). By applying inequality (5) one gets that

$$|\varphi(x, u) - \varphi(y, v)| = ||x(T) - x_T| - |y(T) - x_T|| \leq |x(T) - y(T)| \leq C_p||x - y||_{1,p} \quad \forall (x, u), (y, v) \in X,$$

that is, the function $\varphi$ is continuous. By theorem 7.3 of Reference 55 the growth condition on the function $\theta$ guarantees that the functional $I(x, u)$ is correctly defined and finite for any $(x, u) \in X$, while by proposition 4 of Reference 45 the growth conditions on $\nabla_x \theta$ and $\nabla_u \theta$ ensure that the functional $I(x, u)$ is Lipschitz continuous on any bounded subset of $X$. Hence, in particular, it is Lipschitz continuous on any bounded open set containing the set $S_{\lambda_0}(c) \cap \Omega_\delta$ (such bounded open sets exist, since $S_{\lambda_0}(c) \cap \Omega_\delta$ is bounded by our assumption). Thus, by Theorem 1 it remains to check that there exists $a > 0$ such that $\varphi_A^1(x, u) \leq -a$ for any $(x, u) \in S_{\lambda_0}(c) \cap \Omega_\delta$.

Let $(x, u) \in S_{\lambda_0}(c) \cap \Omega_\delta$ be such that $\varphi(x, u) > 0$, that is, $x(T) \neq x_T$. Choose any $(\hat{x}, \hat{u}) \in \Omega = M \cap A$ (recall that $M$ is not empty, since by our assumption problem (6) has a globally optimal solution). By definition $\dot{x}(T) = x_T$. Put $\Delta x = (\hat{x} - x)/\sigma$ and $\Delta u = (\hat{u} - u)/\sigma$, where $\sigma = ||\dot{x} - \dot{\hat{x}}||_{1,p} + ||\ddot{u} - \ddot{\hat{u}}||_{1,q} > 0$. Then $||\Delta x, \Delta u||_{1,p} = ||\Delta x||_1 + ||\Delta u||_{1,q} = 1$. From the linearity of the system and the convexity of the set $U$ it follows that for any $a \in [0, \sigma]$ one has $(x + a\Delta x, u + a\Delta u) \in A$. Furthermore, note that $(x + a\Delta x)(T) = x(T) + a\sigma^{-1}(x_T - x(T))$. Hence

$$\varphi_A^1(x, u) \leq \lim_{a \to 0^+} \frac{\varphi(x + a\Delta x, u + a\Delta u) - \varphi(x, u)}{a||\Delta x, \Delta u||_1} = \lim_{a \to 0^+} \frac{(1 - a\sigma^{-1})|x(T) - x_T| - |x(T) - x_T|}{a} = -\frac{1}{\sigma}|x(T) - x_T|.$$
Therefore, it remains to check that there exists $C > 0$ such that for any $(x, u) \in S_{\delta}(c) \cap (\Omega \setminus \Omega)$ one can find $(\hat{x}, \hat{u}) \in \Omega$ satisfying the inequality

$$||x - \hat{x}||_{1,p} + ||u - \hat{u}||_q \leq C|x(T) - x_T|.$$  \hspace{1cm} (10)

Then $\varphi_A(x, u) \leq -1/C$ for any $(x, u) \in S_{\delta}(c) \cap (\Omega \setminus \Omega)$, and the proof is complete.

Firstly, let us check that inequality (10) follows from a seemingly weaker inequality, which is easier to prove. Let $(x_1, u_1) \in A$ and $(x_2, u_2) \in A$. Then for any $t \in [0, T)$ one has $x_1(t) - x_2(t) = \int_0^t (A(r)(x_1(r) - x_2(r)) + B(r)(u_1(r) - u_2(r))) dr$. By applying Hölder’s inequality one gets that for any $t \in [0, T]

$$|x_1(t) - x_2(t)| \leq ||B(\cdot)||_{1/q'} T^{1/q'} ||u_1 - u_2||_q + ||A(\cdot)||_p \int_0^t |x_1(r) - x_2(r)| dr.$$ 

Hence by the Grönwall-Bellman inequality one obtains that $|x_1 - x_2|_\infty \leq L_0 ||u_1 - u_2||_q$ for all $(x_1, u_1) \in A$ and $(x_2, u_2) \in A$, where $L_0 = ||B(\cdot)||_{1/q'} T^{1/q'} (1 + T||A(\cdot)||_p e^{T||A(\cdot)||_p})$. Consequently, by applying the equality

$$\dot{x}_1(t) - \dot{x}_2(t) = A(t)(x_1(t) - x_2(t)) + B(t)(u_1(t) - u_2(t)),$$

Hölder’s inequality, and the fact that $q \geq p$ one obtains

$$||\dot{x}_1 - \dot{x}_2||_p \leq T^{1/p} ||A(\cdot)||_p ||x_1 - x_2||_\infty + ||B(\cdot)||_q T^{p/q} ||u_1 - u_2||_q \leq \left(T^{1/p} ||A(\cdot)||_p L_0 + T^{p/q} ||B(\cdot)||_q \right) ||u_1 - u_2||_q,$$

that is, $||x_1 - x_2||_{1,p} \leq L ||u_1 - u_2||_q$ for some $L > 0$ depending only on $A(\cdot), B(\cdot), T, p$, and $q$. Therefore, it is sufficient to check that there exists $C > 0$ such that for any $(x, u) \in S_{\delta}(c) \cap (\Omega \setminus \Omega)$ one can find $(\tilde{x}, \tilde{u}) \in \Omega$ satisfying the inequality

$$||u - \tilde{u}||_q \leq C|x(T) - x_T|,$$  \hspace{1cm} (11)

(cf. inequality (10)). Let us prove inequality (11) with the use of Robinson’s theorem (Theorem 5).

Introduce the linear operator $T : L^m_q(0, T) \to \mathbb{R}^d$, $Tv = h(T)$, where $h \in W^d_{1,p}(0, T)$ is a solution of

$$\dot{h}(t) = A(t)h(t) + B(t)v(t), h(0) = 0.$$  

(12)

For any $v \in L^m_q(0, T)$ a unique absolutely continuous solution $h$ of this equation defined on $[0, T]$ exists by theorem 1.1.3 of Reference 56. By applying Hölder’s inequality and the fact that $q \geq p$ one gets that $||\dot{h}||_p \leq T^{1/p} ||A(\cdot)||_p ||h||_\infty + ||B(\cdot)||_q T^{p/q} ||v||_q$, which implies that $h \in W^d_{1,p}(0, T)$, and the linear operator $T$ is continuously defined. Let us check that it is bounded. Indeed, fix any $v \in L^m_q(0, T)$ and the corresponding solution $h$ of Equation (12). For all $t \in [0, T]$ one has

$$||h(t)|| = \left|\int_0^t (A(\tau)h(\tau) + B(\tau)v(\tau)) d\tau \right| \leq ||B(\cdot)||_q T^{1/q'} ||v||_q + ||A(\cdot)||_p \int_0^t |h(\tau)| d\tau.$$ 

which with the use of the Grönwall-Bellman inequality implies that $|h(T)| \leq L_0 ||v||_q$, that is, the operator $T$ is bounded.

Fix any feasible point $(x, u) \in \Omega$ of problem (6), that is, $\dot{x}_i(t) = A(t)x_i(t) + B(t)u_i(t)$ for a.e. $t \in [0, T], u_i \in U, x(0) = x_0$, and $x(T) = x_T$. Observe that for any $(x, u) \in A$ one has $x(0) = x_{i}(0) = 0$ and $\dot{x}_i(t) = A(t)(x(t) - x_{i}(t)) + B(t)(u(t) - u_{i}(t))$ for a.e. $t \in [0, T]$, which implies that $x(T) = (x(T) - x_{T}) + x_T = T(u - u_{*}) + x_T$ (see equalities (7) and (12)). Consequently, one has

$$R(x_0, T) = x_T + T(U - u_{*}).$$  

(13)

Define $X_0 = \operatorname{cl} \operatorname{span}(U - u_{*})$ and $Y_0 = \operatorname{span} T(U - u_{*})$. Note that $Y_0$ is closed as a subspace of the finite dimensional space $\mathbb{R}^d$. Moreover, $T(X_0) = Y_0$. Indeed, it is clear that the operator $T$ maps $\operatorname{span}(U - u_{*})$ onto span $T(U - u_{*})$. If $u \in X_0$, then there exists a sequence $\{u_n\} \subset \operatorname{span}(U - u_{*})$ converging to $u$. From the boundedness of the operator $T$ it follows that $T(u_n) \to T(u)$ as $n \to \infty$, which implies that $T(u) \in Y_0$ due to the closedness of $Y_0$ and the fact that $\{T(u_n)\} \subset Y_0$ by definition. Thus, $T(X_0) = Y_0$. 


Finally, introduce the operator $T_0 : X_0 \to Y_0$, $T_0(u) = T(u)$ for all $u \in X_0$. Clearly, $T_0$ is a bounded linear operator between Banach spaces. Recall that by our assumption $x_T \in \text{relint } R(x_0, T)$. By the definition of relative interior it means that $0 \in \text{int } T_0(U - u_*)$ (see equality (13)). Therefore by Robinson’s theorem (Theorem 5 with $C = U - u_*, x^* = 0$, and $y = 0$) there exists $\kappa > 0$

$$\text{dist } (u - u_*, T_0^{-1}(0) \cap (U - u_*)) \leq \kappa (1 + \|u - u_*\|) |T_0(u - u_*)| \quad \forall u \in U. \quad (14)$$

With the use of this inequality we can easily prove inequality (11). Indeed, fix any $(x, u) \in S(x_0) \cap (\Omega_q \setminus \Omega_\delta)$. Note that $T_0(u - u_*) = x(T) - x_T \neq 0$, since $(x, u) \notin \Omega$. By inequality (14) there exists $v \in U - u_*$ such that $T_0(v) = 0$ and

$$\|u - u_* - v\| \leq 2\kappa (1 + \|u - u_*\|) |x(T) - x_T|. \quad (15)$$

Define $\tilde{u} = u_* + v$, and let $\hat{x}$ be the corresponding solution of original system (9). Then $\hat{x}(T) = x_T$, since $(x, u_*) \in \Omega$ by definition and $T(v) = 0$, which yields $(\hat{x}, \tilde{u}) \in \Omega$. Furthermore, by inequality (15) one has

$$\|u - \tilde{u}\| \leq 2\kappa (1 + \|u - u_*\|) |x(T) - \hat{x}(T)|. \quad (16)$$

By our assumption the set $S(x_0) \cap (\Omega_q \setminus \Omega_\delta)$ is bounded, which implies that there exists $C > 0$ such that $2\kappa (1 + \|u - u_*\|) \leq C$ for all $(x, u) \in S(x_0) \cap (\Omega_q \setminus \Omega_\delta)$. Thus, for all such $(x, u)$ there exists $(\hat{x}, \tilde{u}) \in \Omega$ satisfying the inequality $\|u - \tilde{u}\| \leq C(x(T) - \hat{x}(T))$, that is, inequality (11) holds true, and the proof is complete.

**Remark 4.** Let $1 < p \leq q < +\infty$, and the function $\theta(x, u, t)$ be convex in $u$ for all $x \in \mathbb{R}^d$ and $t \in [0, T]$. Then under assumptions 1, 2, 3, and 5 of Theorem 6 a globally optimal solution of problem (6) exists iff $x_T \in \text{relint } R(x_0, T)$. Indeed, if $x_T \in \text{Relint } R(x_0, T)$, then the sublevel set $\{x \in \mathbb{R}^d | \theta(x, u, t) < c \} \subset S(x_0) \cap \Omega_q$ is nonempty and bounded due to the fact that $c > I_*$. Therefore there exists a bounded sequence $\{x_n, u_n\} \subset \Omega_q$ such that $I(x_n, u_n) \to I_*$ as $n \to \infty$. From the fact that the spaces $W^{1,p}_{C_0}(0, T)$ and $L^q_{\text{loc}}(0, T)$ are reflexive, provided $1 < p, q < +\infty$, it follows that there exists a subsequence $(x_{n_k}, u_{n_k})$ weakly converging to some $(x^*, u^*) \in X$. Since the imbedding of $W^{1,p}_{C_0}(0, T)$ into $(C(\overline{D}, \mathbb{R}^m))^d$ is compact (eg, theorem 6.2 in monograph [54]), without loss of generality one can suppose that $x_{n_k}$ converges to $x^*$ uniformly on $[0, T]$. Utilizing this result, as well as the facts that the system is linear and the set $U$ of admissible control inputs is convex and closed, one can readily verify that $(x^*, u^*) \in \Omega$. Furthermore, the convexity of the function $u \mapsto \theta(x, u, t)$ ensures that $I(x^*, u^*) \leq \lim_{k \to \infty} I(x_{n_k}, u_{n_k}) = I_*$ (see section 7.3.2 in monograph [55] and Reference 57), which implies that $(x^*, u^*)$ is a globally optimal solution of problem (6).

**Remark 5.** Let us note that Assumption 1 of Theorem 6 is satisfied, in particular, if the set $U$ is bounded or there exist $C > 0$ and $\omega \in L^1_b(0, T)$ such that $\theta(x, u, t) \geq C\|u\|^q + o(t)$ for all $x \in \mathbb{R}^d$, $u \in \mathbb{R}^m$, and a.e. $t \in (0, T)$. Indeed, with the use of this inequality one can check that for any $c > I_*$ there exists $K > 0$ such that for all $(x, u) \in S(x_0)$ one has $\|u\| \leq K$ (if $U$ is bounded, then this inequality is satisfied by definition). Then by applying the Grönwall-Bellman inequality one can easily check that the set $S(x_0)$ is bounded for all $c > I_*$, provided $q \geq p$. Moreover, with the use of the boundedness of the set $S(x_0)$ and the growth condition of order $(q, 1)$ on the function $\theta$ one can easily check that the penalty function $\Phi_{x_0}$ is bounded below on $A$ for all $\lambda_0 \geq 0$.

The following example demonstrates that in the general case Theorem 6 is no longer true, if the assumption that $x_T$ belongs to the relative interior of the reachable set $R(x_0, T)$ is dropped.

**Example 1.** Let $d = m = 2$, $p = q = 2$, and $T = 1$. Define $U = \{u \in L^2(0, 1)|u(t) \in Q \text{ for a.e. } t \in (0, 1)\}$, where $Q = \{(u = (u^1, u^2)^T) \in \mathbb{R}^d | u^1 + u^2 \leq 1, (u^1 - u^2)^2 \leq u^1 + u^2\}$. Note that the set $U$ of admissible control inputs is closed and convex, since, as is easy to see, $Q$ is a closed convex set. Consider the following optimal control problem:

$$\min I(x, u) = \int_0^1 (u^2(t) - u_1^2(t)) \, dt \quad \text{s.t.} \quad \begin{cases} x^1 = 0 \\ x^2 = u^1 + u^2 \end{cases} \quad t \in [0, 1], \quad u \in U, \quad x(0) = x(1) = 0. \quad (16)$$

Let us show at first that in this case $R(x_0, T) = \{x \in \mathbb{R}^2 | x^1 = 0, x^2 \in [0, 1]\}$ (note that $x_0 = 0$ and $T = 1$), which implies that $x_T = 0 \notin \text{relint } R(x_0, T) = \{x \in \mathbb{R}^2 | x^1 = 0, x^2 \in (0, 1)\}$. Indeed, by the definitions of the sets $U$ and $Q$ for any $u \in U$
one has
\[ x^2(1) = \int_0^1 (u^1(t) + u^2(t)) \, dt \leq \int_0^1 dt = 1, \quad x^2(1) = \int_0^1 (u^1(t) + u^2(t)) \, dt \geq \int_0^1 (u^1(t) - u^2(t))^2 \, dt \geq 0, \]
that is, \( x^2(1) \in [0, 1] \). Furthermore, for any \( s \in [0, 1] \) one has \( x^2(1) = s \) for \( u^1(t) \equiv (s + \sqrt{s})/2 \) and \( u^2(t) \equiv (s - \sqrt{s})/2 \) (note that \( u_0 \in U \)). Thus, \( R(x_0, T) = \{0\} \times [0, 1] \), and \( x_T \not\in \text{relint} \ R(x_0, T) \). Note that all other assumptions of Theorem 6 are satisfied.

Let us check that the penalty function \( \Phi_\lambda(x, u) = I(x, u) + \lambda |x(1)| \) for problem (16) is not globally exact. Firstly, note that the only feasible point of problem (16) is \((x^*, u^*)\) with \( x^*(t) \equiv 0 \) and \( u^*(t) = 0 \) for a.e. \( t \in [0, 1] \). Indeed, fix any feasible point \((x, u) \in \Omega \). From the terminal condition \( x(1) = 0 \) and the definition of \( Q \) it follows that

\[ 0 = x^2(1) = \int_0^1 (u^1(t) + u^2(t)) \, dt \geq \int_0^1 (u^1(t) - u^2(t))^2 \, dt \geq 0, \]
which implies that \( u^1(t) = u^2(t) \) for a.e. \( t \in [0, 1] \). Furthermore, by the definition of \( Q \) one has \( u^1(t) + u^2(t) \geq 0 \) for a.e. \( t \in (0, T) \), which yields \( u^1(t) = -u^2(t) \) for a.e. \( t \in [0, 1] \). Therefore \( u(t) = 0 \) for a.e. \( t \in [0, 1] \), \( x(t) \equiv 0 \), and \( \Omega = \{(x^*, u^*)\} \).

Arguing by reductio ad absurdum, suppose that the penalty function \( \Phi_\lambda(x, u) = I(x, u) + \lambda |x(1)| \) for problem (16) is globally exact. Then there exists \( \lambda > 0 \) such that \((x^*, u^*)\) is a globally optimal solution of the problem

\[ \min \Phi_\lambda(x, u) = \int_0^1 (u^2(t) - u^1(t)) \, dt + \lambda |x(1)| \quad \text{s.t.} \quad \begin{cases} \dot{x}^1 = 0 \\ \dot{x}^2 = u^1 + u^2 \end{cases} \quad t \in [0, 1], \quad u \in U. \quad x(0) = 0. \]

Fix any \( s \in (0, 1) \), and define \( u = u_s \in U \) (recall that \( u^1_s(t) \equiv (s + \sqrt{s})/2 \) and \( u^2_s(t) \equiv (s - \sqrt{s})/2 \)). For the corresponding solution \( x_s(t) \) one has \( x^2_s(1) = s \), and

\[ \Phi_\lambda(x_s, u_s) = -\sqrt{s} + \lambda s \geq 0 = \Phi_\lambda(x^*, u^*), \]
which is impossible for any sufficiently small \( s \in (0, 1) \). Thus, the penalty function \( \Phi_\lambda \) is not globally exact.

Remark 6. It should be noted that the assumption \( x_T \in \text{relint} \ R(x_0, T) \) is not necessary for the exactness of the penalty function \( \Phi_\lambda \) for problem (6). For instance, the interested reader can check that if in Example 1 the system has the form \( \dot{x}^1 = u^1 \) and \( \dot{x}^2 = u^2 \), then the penalty function \( \Phi_\lambda(x, u) = \int_0^1 (u^2(t) - u^1(t)) \, dt + \lambda |x(1)| = x^2(1) - x^1(1) + \lambda |x(1)| \) is completely exact, despite the fact that in this case \( R(x_0, T) = Q \) and \( x_T = 0 \not\in \text{relint} \ R(x_0, T) \). We pose an interesting open problem to find necessary and sufficient conditions for the complete exactness of the penalty function \( \Phi_\lambda \) for problem (6) (at least in the time-invariant case). In particular, it seems that in the case when \( U = \{u \in L^q_2(0, T) | u(t) \in Q \quad \text{for a.e.} \quad t \in [0, T]\} \) and \( Q \) is a convex polytope, the assumption \( x_T \in \text{relint} \ R(x_0, T) \) in Theorem 6 can be dropped.

Let us finally note that in the case when the set \( U \) of admissible control inputs is bounded, one can prove the complete exactness of the penalty function \( \Phi_\lambda \) for problem (6) on \( A \). In other words, one can prove that free-endpoint problem (8) is completely equivalent to fixed-endpoint problem (6) in the sense that these problems have the same optimal value, the same globally/locally optimal solutions, and the same inf-stationary points.

Theorem 7. Let \( q \geq p \), assumptions 1 to 4 of Theorem 6 be valid, and suppose that the set \( U \) is bounded. Then the penalty function \( \Phi_\lambda \) for problem (6) is completely exact on \( A \).

Proof. By our assumption there exists \( K > 0 \) such that \( \|u\|_q \leq K \) for any \( u \in U \). Choose any \((x, u) \in A \). Then by definition

\[ x(t) = x_0 + \int_0^t (A(\tau)x(\tau) + B(\tau)u(\tau)) \, d\tau \quad \text{for all} \quad t \in [0, T], \]

which by Hölder's inequality implies that

\[ \|x(t)\| \leq \|x_0\| + \|B(\cdot)\|_\infty T^{1/q'}\|u\|_q + \|A(\cdot)\|_\infty \int_0^t |x(\tau)| \, d\tau. \]

Consequently, by applying the Grönwall-Bellman inequality and the fact that \( \|u\|_q \leq K \) one obtains that \( \|x\|_\infty \leq C \) for some \( C > 0 \) depending only on \( K, A(\cdot), B(\cdot), T, \) and \( q \). Hence by Hölder's inequality and the definition of the set \( A \).
that is, the set $A$ is bounded in $X$ and in $L^2_{\infty}(0, T) \times L^4_{\infty}(0, T)$. Therefore, both $I$ and $\Phi_A$, for any $\lambda \geq 0$, are bounded below on $A$ due to the fact that the function $\theta$ satisfies the growth conditions of order $(q, 1)$ (see Assumption 1 of Theorem 6). Now, arguing in the same way as in the proof of Theorem 6, but replacing $S_{\phi_i}(c) \cap \Omega_\delta$ with $A$ and utilizing Theorem 2 instead of Theorem 1, one obtains the desired result.

### 3.3 Linear evolution equations

Let us demonstrate that Theorems 6 and 7 can be easily extended to the case of optimal control problems for linear evolution equations in Hilbert spaces. In this section we use standard definitions and results on control problems for infinite dimensional systems that can be found, for example, in monograph 58.

Let $H$ and $U$ be complex Hilbert spaces, $T$ be a strongly continuous semigroup on $H$ with generator $A : D(A) \to H$, and let $B$ be an admissible control operator for $T$ (see definition 4.2.1 of Reference 58). Consider the following fixed-endpoint optimal control problem:

$$
\min_{(x,u)} I(x,u) = \int_0^T \theta(x(t), u(t), t) \, dt
$$

subject to $\dot{x}(t) = Ax(t) + Bu(t), \ t \in [0, T], \ u \in U, \ x(0) = x_0, \ x(T) = x_T. \tag{17}
$$

Here $\theta : H \times U \times [0, T] \to \mathbb{R}$ is a given function, $T > 0$ and $x_0, x_T \in H$ are fixed, and $U$ is a closed convex subset of the space $L^2((0, T); U)$ consisting of all those measurable functions $u : (0, T) \to U$ for which $\|u\|_{L^2((0, T); U)} = \int_0^T \|u(t)\|_U^2 \, dt < +\infty$.

Let us introduce a penalty function for problem (17). As in the previous section, we only penalize the terminal constraint $x(T) = x_T$. For any $t \geq 0$ let $F_t u = \int_0^t T_{t-s} Bu(s) \, ds$ be the input map corresponding to $(A, B)$. By proposition 4.2.2 of Reference 58, $F_t$ is a bounded linear operator from $L^2((0, T); U)$ to $H$. Furthermore, by applying Proposition 4.2.5 of Reference 58 one obtains that for any $u \in L^2((0, T); U)$ the initial value problem

$$
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{18}
$$

has a unique solution $x \in C([0, T]; H)$ given by

$$
x(t) = T_t x_0 + F_t u \quad \forall t \in [0, T]. \tag{19}
$$

Define $X = C([0, T]; H) \times L^2((0, T); U), M = \{(x, u) \in X | x(T) = x_T\}$, and

$$
A = \{(x, u) \in X | x(0) = x_0, u \in U, \text{ and equality (19) holds true}\}.
$$

Then problem (17) can be rewritten as the problem of minimizing $I(x, u)$ subject to $(x, u) \in M \cap A$. Introduce the penalty term $\varphi(x, u) = \|x(T) - x_T\|_H$. Then $M = \{(x, u) \in X | \varphi(x, u) = 0\}$, and one can consider the penalized problem of minimizing the penalty function $\Phi_A(x, u) = I(x, u) + \lambda \varphi(x, u)$ subject to $(x, u) \in A$, which is a free-endpoint problem of the form:

$$
\min_{(x,u)} \Phi_A(x,u) = I(x,u) + \lambda \varphi(x,u) = \int_0^T \theta(x(t), u(t), t) \, dt + \lambda \|x(T) - x_T\|_H
$$

subject to $\dot{x}(t) = Ax(t) + Bu(t), t \in [0, T], u \in U, x(0) = x_0. \tag{20}$

Denote by $R(x_0, T)$ the set that is reachable in time $T$, that is, the set of all those $\xi \in H$ for which there exists $u \in U$ such that $x(T) = \xi$, where $x(\cdot)$ is defined in equality (19). Observe that by definition $R(x_0, T) = F_T(U) + T_T x_0$, which implies that the reachable set $R(x_0, T)$ is convex due to the convexity of the set $U$. 

(see equality (7)) one obtains that

$$
\|\dot{x}\|_p = \|A(\cdot)x(\cdot) + B(\cdot)u(\cdot)\|_p \leq T^{1/p} \|A(\cdot)\|_\infty C + \|B(\cdot)\|_\infty T^{\frac{p}{2}} K \quad \forall (x, u) \in A,
$$

$$
\int
$$
Our aim is to show that under a natural assumption on the reachable set \( R(x_0, T) \) the penalty function \( \Phi_\lambda \) is completely exact, that is, for any sufficiently large \( \lambda \geq 0 \) free-endpoint problem (20) is equivalent to fixed-endpoint problem (17).

In this finite dimensional case we assumed that \( x_T \in \text{relint} \ R(x_0, T) \). In the infinite dimensional case we will use the same assumption, since to the best of author's knowledge it is the weakest assumption allowing one to utilize Robinson's theorem. However, recall that the relative interior of a convex subset of a finite dimensional space is always nonempty, but this statement is no longer true in infinite dimensional spaces (see References 59 and 60). Thus, in the finite dimensional case the condition \( x_T \in \text{relint} \ R(x_0, T) \) simply restricts the location of \( x_T \) in the reachable set, while in the infinite dimensional case it also imposes the assumption (relint \( R(x_0, T) \neq \emptyset \)) on the reachable set itself. For the sake of completeness recall that the relative interior of a convex subset \( C \) of a Banach space \( Y \), denoted relint\( C \), is the interior of \( C \) relative to the closed affine hull of \( C \).

**Theorem 8.** Let the following assumptions be valid:

1. \( \theta \) is continuous, and for any \( R > 0 \) there exist \( C_R > 0 \) and an a.e. nonnegative function \( \omega_R \in L^1(0, T) \) such that
   \[
   |\theta(x, u, t)| \leq C_R \|u\|_{\mathcal{H}}^2 + \omega_R(t) \quad \text{for all } x \in \mathcal{H}, u \in \mathcal{U}, \text{ and } t \in (0, T) \text{ such that } \|x\|_{\mathcal{H}} \leq R;
   \]
2. either the set \( U \) is bounded in \( L^2((0, T), \mathcal{H}) \) or there exist \( C_1 > 0 \) and \( \omega \in L^1(0, T) \) such that \( \theta(x, u, t) \geq C_1 \|u\|_{\mathcal{H}}^2 + \omega(t) \) for all \( x \in \mathcal{H}, u \in \mathcal{U}, \text{ and } t \in [0, T] \);
3. \( \theta \) is differentiable in \( x \) and \( u \), the functions \( \nabla_x \theta \) and \( \nabla_u \theta \) are continuous, and for any \( R > 0 \) there exist \( C_R > 0 \), and a.e. nonnegative functions \( \omega_R \in L^1(0, T) \) and \( \eta_R \in L^2(0, T) \) such that
   \[
   \|\nabla_x \theta(x, u, t)\|_{\mathcal{H}} \leq C_R \|u\|_{\mathcal{H}}^2 + \omega_R(t), \quad \|\nabla_u \theta(x, u, t)\|_{\mathcal{H}} \leq C_R \|u\|_{\mathcal{H}}^2 + \eta_R(t),
   \]
   for all \( x \in \mathcal{H}, u \in \mathcal{U}, \text{ and } t \in (0, T) \text{ such that } \|x\|_{\mathcal{H}} \leq R;
4. there exists a globally optimal solution of problem (17), relint \( R(x_0, T) \neq \emptyset \) and \( x_T \in \text{relint} \ R(x_0, T) \).

Then for all \( c \in \mathbb{R} \) there exists \( \lambda^*(c) \geq 0 \) such that for any \( \lambda \geq \lambda^*(c) \) the penalty function \( \Phi_\lambda \) for problem (17) is completely exact on the set \( S_\lambda(c) \).

**Proof.** Our aim is to apply Theorem 1. It is easily seen that assumption 1 ensures that the functional \( I(x, u) \) is correctly defined and finite for any \( (x, u) \in X \). In turn, from Assumption 2 it follows that for any \( c \in \mathbb{R} \) and \( \lambda \geq 0 \) there exists \( K > 0 \) such that \( \|u\|_{L^2((0, T); \mathcal{H})} \leq K \) for any \( (x, u) \in S_\lambda(c) \), and the penalty function \( \Phi_\lambda \) is bounded below on \( A \) for all \( \lambda \geq 0 \) (if \( U \) is bounded, then this fact follows from Assumption 1). Hence taking into account equality (19), and the facts that
   \[
   \|F_t\| \leq \|F_T\| \text{ for any } t \leq T \quad \text{and} \quad \|F_T\| \leq M_o e^{ct} \text{ for all } t \geq 0 \text{ and for some } \omega \in \mathbb{R} \text{ and } M_o \geq 1
   \]
by Proposition 2.1.2 of Reference 58 one obtains that \( \|x\|_{C([0, T]; \mathcal{H})} \leq M_{\text{max}} e^{ct} \|x_0\| + \|F_T\| K \), that is, the set \( S_\lambda(c) \) is bounded in \( X \) for any \( \lambda \geq 0 \) and \( c \in \mathbb{R} \).

Observe that the penalty term \( \varphi \) is continuous on \( X \), since by the reverse triangle inequality one has
   \[
   |\varphi(x, u) - \varphi(y, v)| = \|\varphi(x, u) - \varphi(y, v)\|_{\mathcal{H}} \leq \|\varphi(x, u) - \varphi(y, v)\|_{\mathcal{H}} \leq \|x - y\|_{C([0, T]; \mathcal{H})}
   \]
for all \( (x, u), (y, v) \in X \). Furthermore from equality (19), the closedness of the set \( U \), and the fact that \( F_t \) continuously maps \( L^2((0, T); \mathcal{H}) \) to \( \mathcal{H} \) by Proposition 4.2.2 of Reference 58 it follows that the set \( A \) is closed.

Let us check that Assumption 3 ensures that the functional \( I \) is Lipschitz continuous on any bounded subset of \( X \) (in particular, on any bounded open set containing the set \( S_\lambda(c) \)). Indeed, fix any \( (x, u) \in X, (h, v) \in X \), and \( \alpha \in (0, 1) \). By the mean value theorem for a.e. \( t \in (0, T) \) there exists \( a(t) \in (0, \alpha) \) such that
   \[
   \frac{1}{\alpha} (\theta(x(t) + a(t)h(t), u(t) + a(v(t), t) - \theta(x(t), u(t), t)) = \langle \nabla_x \theta(x(t) + a(t)h(t), u(t) + a(t)v(t), h(t)), h(t) \rangle + \langle \nabla_u \theta(x(t) + a(t)h(t), u(t) + a(t)v(t), t), v(t) \rangle.
   \]

The right-hand side of this equality converges to \( \langle \nabla_x \theta(x(t), u(t), t), h(t) \rangle + \langle \nabla_u \theta(x(t), u(t), t), v(t) \rangle \) as \( \alpha \to 0 \) for a.e. \( t \in (0, T) \) due to the continuity of the gradients \( \nabla_x \theta \) and \( \nabla_u \theta \). Furthermore, by inequality (21) there exist \( C_R > 0 \), and a.e.
nonnegative functions $\omega_R \in L^1(0, T)$ and $\eta_R \in L^2(0, T)$ such that

$$
\|\langle \nabla_x \theta(x(t) + ah(t), u(t) + av(t), t), h(t) \rangle \| \leq (4C_R(\|u(t)\|_{\mathcal{X}}^2 + \|v(t)\|_{\mathcal{X}}^2) + \omega_R(t)) \|h\|_{C[0,T; \mathcal{X}]}
$$

$$
\|\langle \nabla_x \theta(x(t) + ah(t), u(t) + av(t), t), v(t) \rangle \| \leq (C_R(\|u(t)\|_{\mathcal{X}} + \|v(t)\|_{\mathcal{X}}) + \eta_R(t)) \|v(t)\|_{\mathcal{Y}},
$$

for a.e. $t \in (0, T)$ and all $\alpha \in [0, 1]$. Note that the right-hand sides of these inequalities belong to $L^1(0, T)$ and do not depend on $\alpha$. Therefore, integrating equality (22) from 0 to $T$ and passing to the limit with the use of Lebesgue's dominated convergence theorem one obtains that the functional $I$ is Gâteaux differentiable at every point $(x, u) \in X$, and its Gâteaux derivative has the form

$$
I'(x, u)[h, v] = \int_0^T (\langle \nabla_x \theta(x(t), u(t), t), h(t) \rangle + \langle \nabla_u \theta(x(t), u(t), t), v(t) \rangle) \, dt.
$$

Hence and from inequality (21) it follows that for any $R > 0$ and $(x, u) \in X$ such that $\|x\|_{C([0,T]; \mathcal{X})} \leq R$ there exist $C_R > 0$, and a.e. nonnegative functions $\omega_R \in L^1(0, T)$ and $\eta_R \in L^2(0, T)$ such that

$$
\|I'(x, u)\| \leq C_R \|u\|^2_{L^2([0,T]; \mathcal{Y})} + \|\omega_R\|_{L^1(0,T)} + C_R \|u\|_{L^1([0,T]; \mathcal{Y})} + \|\eta_R\|_{L^2(T)}.
$$

Therefore, the Gâteaux derivative of $I$ is bounded on bounded subsets of the space $X$, which, as is well-known and easy to check, implies that the functional $I$ is Lipschitz continuous on bounded subsets of $X$.

Fix any $\alpha \geq 0$ and $c > \inf_{(x, u) \in I} I(x, u)$. By Theorem 1 it remains to check that there exists $a > 0$ such that $\varphi_A^1(x, u) \leq -a$ for any $(x, u) \in S_A(c)$ such that $\varphi(x, u) > 0$. Choose any such $(x, u)$ and $(\hat{x}, \hat{u}) \in \Omega$. Note that $x(T) \neq x_T$ due to the inequality $\varphi(x, u) > 0$. Define $\Delta x = (\hat{x} - x)/\sigma$ and $\Delta u = (\hat{u} - u)/\sigma$, where $\sigma = \|\hat{x} - x\|_{C([0,T]; \mathcal{X})} + \|\hat{u} - u\|_{L^1([0,T]; \mathcal{Y})} > 0$. Then $\|(\Delta x, \Delta u)\|_X = \|\Delta x\|_{C([0,T]; \mathcal{X})} + \|\Delta u\|_{L^1([0,T]; \mathcal{Y})} = 1$. Due to the linearity of the system and the convexity of the set $U$, for any $\alpha \in [0, \sigma]$ one has $(x + \alpha \Delta x, u + \alpha \Delta u) \in A$, $(x + \alpha \Delta x)(T) = x(T) + \alpha \sigma^{-1}(x_T - x(T))$, since $x(T) = x_T$ by definition. Hence

$$
\varphi_A^1(x, u) \leq \lim_{\alpha \to 0} \frac{\varphi(x + \alpha \Delta x, u + \alpha \Delta u) - \varphi(x, u)}{\alpha \|(\Delta x, \Delta u)\|_X} = \lim_{\alpha \to 0} \frac{(1 - a\sigma^{-1})\|x(T) - x_T\|_{\mathcal{X}} - \|x(T) - x_T\|_{\mathcal{X}}}{\alpha} = -\frac{1}{\sigma} \|x(T) - x_T\|_{\mathcal{X}}.
$$

Therefore, it remains to check that there exists $C > 0$ such that for any $(x, u) \in S_A(c) \setminus \Omega$ one can find $(\hat{x}, \hat{u}) \in \Omega$ satisfying the inequality

$$
\|x - \hat{x}\|_{C([0,T]; \mathcal{X})} + \|u - \hat{u}\|_{L^2([0,T]; \mathcal{Y})} \leq C \|x(T) - x_T\|_{\mathcal{X}}. \quad (23)
$$

Then $\varphi_A^1(x, u) \leq -1/C$ for any such $(x, u)$, and the proof is complete.

From equality (19) and the inequality $\|F_T\| \leq \|F_T\|_t$, $t \in [0, T]$ (see formula (4.2.5) of Reference 58), it follows that for any $(x, u) \in A$ and $(\hat{x}, \hat{u}) \in A$ one has $\|x - \hat{x}\|_{C([0,T]; \mathcal{X})} \leq \|F_T\| \|u - \hat{u}\|_{L^2([0,T]; \mathcal{Y})}$. Consequently, it is sufficient to check that there exists $C > 0$ such that for any $(x, u) \in S_A(c) \setminus \Omega$ one can find $(\hat{x}, \hat{u}) \in \Omega$ satisfying the inequality

$$
\|u - \hat{u}\|_{L^2([0,T]; \mathcal{Y})} \leq C \|x(T) - x_T\|_{\mathcal{X}}. \quad (24)
$$

To this end, fix any $(x, u) \in \Omega$, and denote by $\mathcal{T} : \text{cl span}(U - u_*) \to \text{cl span} F_T(U - u_*)$ the mapping such that $\mathcal{T}(u) = F_T(u)$ for any $u \in \text{cl span}(U - u_*)$. Note that $\mathcal{T}$ is correctly defined, since the operator $F_T$ maps $\text{cl span}(U - u_*)$ to $\text{cl span} F_T(U - u_*)$. Indeed, by definition $F_T(\text{span}(U - u_*)) \subseteq \text{span} F_T(U - u_*)$. If $u_0 \in \text{cl span} U$, then there exists a sequence $\{u_n\} \subset \text{span}(U - u_*)$ converging to $u_0$. Due to the continuity of $F_T$ the sequence $\{F_T u_n\}$ converges to $F_T u_0$, which yields $F_T u_0 \in \text{cl span} F_T(U - u_*)$.

Observe that $\mathcal{T}$ is a bounded linear operator between Banach spaces, since the operator $F_T$ is bounded. Furthermore, by (19) for any $u \in U$ one has $F_T(u - u_*) = x(T) - x_T$, which implies that $\mathcal{T}(U - u_*) = F_T(U - u_*) = R(x_0, T) - x_T$. Therefore, by Assumption 4 one has $0 \in \text{int} \mathcal{T}(U - u_*)$, since the closed affine hull of $R(x_0, T)$ coincides with $\text{cl span} F_T(U - u_*) + x_T$ due to the fact that $0 \in F_T(U - u_*)$. Hence by Robinson’s theorem (Theorem 5 with $C = U - u_*, x^* = 0$, and $y = 0$) there exists $\kappa > 0$ such that

$$
\|u - u_*\|_{L^2([0,T]; \mathcal{Y})} \leq \kappa \left(1 + \|u - u_*\|_{L^2([0,T]; \mathcal{Y})}\right) \|\mathcal{T}(u - u_*)\|_{\mathcal{X}} \quad \forall u \in U. \quad (25)
$$
Fix any \((x, u) \in S_j(c) \setminus \Omega\) (ie, \(x(T) \neq x_T\)). Then taking into account the fact that \(T(u - u_*) = x(T) - x_T\) and utilizing inequality (25) one obtains that there exists \(v \in U - u_*\) such that \(T(v) = 0\) and

\[
\|u - u_* - v\|_{L^2([0, T]; \mathcal{H})} \leq 2 \kappa \left(1 + \|u - u_*\|_{L^2([0, T]; \mathcal{H})}\right) \|x(T) - x_T\|_{\mathcal{H}}.
\]  

(26)

Define \(\hat{u} = u_* + v \in U\), and let \(\hat{x}\) be the corresponding solution of Equation (18). Then \(\hat{x}(T) - x_T = T(\hat{u} - u_*) = T(v) = 0\), that is, \((\hat{x}, \hat{u}) \in \Omega\). Note that \(C := \sup_{(x,u) \in S_j(c)} 2 \kappa (1 + \|u - u_*\|_{L^2([0, T]; \mathcal{H})}) < +\infty\) due to the boundedness of the set \(S_j(c)\). Consequently, by inequality (26) one for any \((x, u) \in S_j(c) \setminus \Omega\) there exists \((\hat{x}, \hat{u}) \in \Omega\) such that \(\|u - \hat{u}\|_{L^2([0, T]; \mathcal{H})} \leq C \|x(T) - x_T\|_{\mathcal{H}}\), that is, inequality (24) holds true, and the proof is complete.

\[\text{Remark 7.}\] Let us note that for the validity of the assumption \(x_T \in \text{relint} \mathcal{R}(x_0, T)\) in the case \(\text{Im}(F_T) = \mathcal{H}\) it is sufficient to suppose that \(x_T\) belongs to the interior of \(\mathcal{R}(x_0, T)\), while in the case \(U = L^2((0, T); \mathcal{U})\) this assumption is satisfied iff the image of the input map \(F_T\) is closed.

\[\text{Remark 8.}\] Recall that system (18) is called exactly controllable using \(L^2\)-controls in time \(T\), if for any initial state \(x_0 \in \mathcal{H}\) and for any final state \(x_T \in \mathcal{H}\) there exists \(u \in L^2((0, T); \mathcal{U})\) such that for the corresponding solution \(x\) of system (18) one has \(x(T) = x_T\). It is easily seen that this system is exactly controllable using \(L^2\)-controls in time \(T\) iff the input map \(F_T\) is surjective, that is, \(\text{Im}(F_T) = \mathcal{H}\). Thus, in particular, in Theorem 8 it is sufficient to suppose that system (18) is exactly controllable and \(x_T \in \text{int} \mathcal{R}(x_0, T)\). If, in addition, \(\text{int} U \neq \emptyset\), then it is sufficient to suppose that system (18) is exactly controllable and there exists a feasible point \((x_*, u_*)\) of problem (17) such that \(u_* \in \text{int} U\).

The exactness of the penalty function \(\Phi_{\lambda}\) for problem (17) with \(\theta(x, u, t) = \|u\|_{L^2}^2 / 2\) and no constraints on the control inputs (ie, \(U = L^2((0, T); \mathcal{U})\)) was proved by Gugat and Zuazua\(^{43}\) under the assumption that system (18) is exactly controllable, and the control \(u\) from the definition of exact controllability satisfies the inequality

\[
\|u\|_{L^2((0, T); \mathcal{H})} \leq C (\|x_0\| + \|x_T\|),
\]

(27)

for some \(C > 0\) independent of \(x_0\) and \(x_T\). Note that our Theorem 8 significantly generalizes and strengthens theorem 1 of Reference 43, since we consider a more general objective function and convex constraints on the control inputs, impose a less restrictive assumption on the input map \(F_T\) (instead of exact controllability it is sufficient to suppose that \(\text{Im}(F_T)\) is closed), and demonstrate that inequality (27) is, in fact, redundant.

Let us also extend Theorem 7 to the case of optimal control problems for linear evolution equations.

\[\text{Theorem 9.}\] Let all assumptions of Theorem 8 be valid, and suppose that either the set \(U\) of admissible control inputs is bounded in \(L^2((0, T); \mathcal{U})\) or the function \((x, u) \mapsto \theta(x, u, t)\) is convex for all \(t \in [0, T]\). Then the penalty function \(\Phi_{\lambda}\) for problem (17) is completely exact on \(A\).

\[\text{Proof.}\] Suppose at first that the set \(U\) is bounded. Recall that the input map \(F_t\) continuously maps \(L^2((0, T); \mathcal{U})\) to \(\mathcal{H}\) by Proposition 4.2.2\(^{58}\) and \(\|F_t\| \leq \|F_T\|\) for any \(t \leq T\) (see formula (4.2.5)\(^{58}\)). Note also that by proposition 2.1.2 of Reference 58 there exist \(\omega \in \mathbb{R}\) and \(M_\omega \geq 1\) such that \(\|T_t\| \leq M_\omega e^{\omega t}\) for all \(t \geq 0\).

Fix any \((x, u) \in A\). By our assumption there exists \(K > 0\) such that \(\|x\|_{C([0, T]; \mathcal{H})} \leq K\) for any \(u \in U\). Hence \(\|x\|_{C([0, T]; \mathcal{H})} \leq M_{\max}e^{\omega t}M_\omega\|x_0\| + \|F_T\|K\) due to equality (19), and the bounds on \(\|T_t\|\) and \(\|F_t\|\). Thus, the set \(A\) is bounded in \(X\). Now, arguing in the same way as in the proof of Theorem 8, but replacing \(S_j(c)\) with \(A\) and utilizing Theorem 2 instead of Theorem 1, one obtains the required result.

Suppose now that the function \((x, u) \mapsto \theta(x, u, t)\) is convex. Then the functional \(I(x, u)\) and the penalty function \(\Phi_{\lambda}\) are convex. Hence with the use of the fact that the set \(A\) is convex one obtains that any point of local minimum of \(\Phi_{\lambda}\) on \(A\) is also a point of global minimum of \(\Phi_{\lambda}\) on \(A\). Furthermore, any inf-stationary point of \(\Phi_{\lambda}\) on \(A\) is also a point of global minimum of \(\Phi_{\lambda}\) on \(A\). Indeed, \(x^*\), \(u^*\) be an inf-stationary point of \(\Phi_{\lambda}\) on \(A\). Arguing by reductio ad absurdum, suppose that \((x^*, u^*)\) is not a point of global minimum. Then there exists \((x_0, u_0) \in A\) such that \(\Phi_{\lambda}(x_0, u_0) < \Phi_{\lambda}(x^*, u^*)\). By applying the convexity of \(\Phi_{\lambda}\) one gets that

\[
\Phi_{\lambda}(x^* + \alpha(x_0 - x^*), u^* + \alpha(u_0 - u^*)) \leq \Phi_{\lambda}(x^*, u^*) + \alpha(\Phi_{\lambda}(x_0, u_0) - \Phi_{\lambda}(x^*, u^*)) \quad \forall \alpha \in [0, 1].
\]
Consequently, one has
\[
(\Phi_\lambda)'(x^*, u^*) \leq \lim_{\alpha \to +0} \inf_{x} \frac{\Phi_\lambda(x^* + \alpha(x_0 - x^*), u^* + \alpha(u_0 - u^*)) - \Phi_\lambda(x^*, u^*)}{\alpha \| (x^*, u^*) - (x_0, u_0) \|_X} \leq \frac{\Phi_\lambda(x_0, u_0) - \Phi_\lambda(x^*, u^*)}{\| (x^*, u^*) - (x_0, u_0) \|_X} < 0
\]
which is impossible, since by the definition of inf-stationary point \((\Phi_\lambda)'(x^*, u^*) \geq 0\).

Similarly, any point of local minimum/inf-stationary point of \(I\) on \(\Omega\) is a globally optimal solution of problem (17) due to the convexity of \(I\) and \(\Omega\). Therefore, in the convex case the penalty function \(\Phi_\lambda\) is completely exact on \(A\) if and only if it is globally exact. In turn, the global exactness of this function follows from Theorem 8.

\[\text{3.4 Nonlinear systems: Complete exactness}\]

Now we turn to the analysis of nonlinear finite dimensional fixed-endpoint optimal control problems of the form:

\[
\min I(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt
\]
subject to \( \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T], \quad x(0) = x_0, \quad x(T) = x_T, \quad u \in U. \tag{28}\]

Here \(\theta : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}\) and \(f : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}^d\) are given functions, \(x_0, x_T \in \mathbb{R}^d\), and \(T > 0\) are fixed, \(x \in W_{1,p}^d(0, T), U \subseteq L_q^m(0, T)\) is a nonempty closed set, and \(1 \leq p, q \leq +\infty\).

As in the case of linear problems, we penalize only the terminal constraint \(x(T) = x_T\). To this end, define \(X = W_{1,p}^d(0, T) \times L_q^m(0, T), M = \{(x, u) \in X \mid x(T) = x_T\}\), and

\[
A = \{(x, u) \in X \mid x(0) = x_0, \quad u \in U, \quad x(t) = f(x(t), u(t), t) \text{ for a.e. } t \in [0, T]\}. \tag{29}\]

Then problem (28) can be rewritten as the problem of minimizing \(I(x, u)\) subject to \((x, u) \in M \cap A\). As in the previous sections, define \(\varphi(x, u) = |x(T) - x_T|\). Then \(M = \{(x, u) \in X \mid \varphi(x, u) = 0\}\), and one can consider the penalized problem

\[
\min \Phi_\lambda(x, u) = I(x, u) + \lambda \varphi(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt + \lambda|\varphi(x(T) - x_T| \]
subject to \( \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T], \quad x(0) = x_0, \quad u \in U. \tag{30}\]

which is a nonlinear free-endpoint optimal control problem.

The nonlinearity of the systems makes an analysis of the exactness of the penalty function \(\Phi_\lambda(x, u)\) a very challenging problem. Unlike the linear case, it does not seem possible to obtain any easily verifiable conditions for the complete exactness of this function. Therefore, the main goal of this section is to understand what properties the system \(\dot{x} = f(x, u, t)\) must have for the penalty function \(\Phi_\lambda(x, u)\) to be completely exact.

In the linear case, the main assumption ensuring the complete exactness of the penalty function was \(x_T \in \text{relint } R(x_0, T)\). Therefore, it is natural to expect that in the nonlinear case one must also impose some assumptions on the reachable set of the system \(\dot{x} = f(x, u, t)\). Moreover, in the linear case we utilized Robinson's theorem, but there are no nonlocal analogues of this theorem in the nonlinear case. Consequently, we must impose an assumption that allows one to avoid the use of this theorem.

Thus, to prove the complete exactness of the penalty function \(\Phi_\lambda\) in the nonlinear case we need to impose two assumptions on the controlled system \(\dot{x} = f(x, u, t)\). The first one does not allow the reachable set of this system to be, roughly speaking, too “wild” near the point \(x_T\), while the second one ensures that this system is, in a sense, sensitive enough with respect to the control inputs. It should be mentioned that the exactness of the penalty function \(\Phi_\lambda\) for problem (28) can be proved under a much weaker assumption that imposes some restrictions on the reachable set and sensitivity with respect to the control inputs simultaneously. However, for the sake of simplicity we split this rather complicated assumption in two assumptions that are much easier to understand and analyze.

Denote by \(R(x_0, T) = \{ \xi \in \mathbb{R}^d \mid \exists (x, u) \in A : \xi = x(T)\}\) the set that is reachable in time \(T\). We obviously suppose that \(x_T \in R(x_0, T)\). Also, we exclude the trivial case when \(x_T\) is an isolated point of \(R(x_0, T)\), since in this case the penalty...
function $\Phi_x$ is completely exact on $S_x(c)$ for any $c \in \mathbb{R}$ iff $\Phi_x$ is bounded below on $A$ due to the fact that in this case $\Omega_x \setminus \Omega = \emptyset$ for any sufficiently small $\delta > 0$ (see Remark 1).

**Definition 3.** One says that the set $R(x_0, T)$ has the **negative tangent angle property** near $x_T$, if there exist a neighbourhood $\mathcal{O}(x_T)$ of $x_T$ and $\beta > 0$ such that for any $\xi \in \mathcal{O}(x_T) \cap R(x_0, T)$, $\xi \neq x_T$, there exists a sequence $\{\xi_n\} \subset R(x_0, T)$ converging to $\xi$ and such that

$$\left\langle \frac{\xi - x_T}{|\xi - x_T|}, \frac{\xi_n - \xi}{|\xi_n - \xi|} \right\rangle \leq -\beta \quad \forall n \in \mathbb{N}. \quad (31)$$

One can easily see that if $x_T$ belongs to the interior of $R(x_0, T)$ or if there exists a neighborhood $\mathcal{O}(x_T)$ of $x_T$ such that the intersection $\mathcal{O}(x_T) \cap R(x_0, T)$ is convex, then the set $R(x_0, T)$ has the negative tangent angle property near $x_T$ (take as $\{\xi_n\}$ any sequence of points from the segment $\text{co}[x_T, \xi]$ converging to $\xi$ and put $\beta = 1$). However, this property holds true in a much more general case. In particular, $x_T$ can be the vertex of a cusp.

The negative tangent angle property excludes the sets that, roughly speaking, are “very porous” near $x_T$ (i.e., sets having an infinite number of “holes” in any neighborhood of $x_T$) or are very wiggly near this point (like the graph of $y = x \sin(1/x)$ near $(0, 0)$). Furthermore, bearing in mind the equality

$$\{\xi \in \mathbb{R}^d | \exists (x, u) \in \Omega_x \setminus \Omega : \xi = x(T)\} = \{\xi \in R(x_0, T) | 0 < |\xi - x_T| < \delta\},$$

the definition of the rate of steepest descent, and the fact that $\varphi(x, u) = |x(T) - x_T|$ one can check that for the validity of the inequality $\varphi'_x(x, u) \leq -a$ for all $(x, u) \in \Omega_x \setminus \Omega$ and some $a, \delta > 0$ it is necessary that there exists $\beta > 0$ such that for any $\xi \in R(x_0, T)$ lying in a neighborhood of $x_T$ inequality $(31)$ holds true.

Indeed, suppose that $\varphi'_x(x, u) \leq -a$ for all $(x, u) \in \Omega_x \setminus \Omega$ and some $a, \delta > 0$. Let $\xi \in R(x_0, T)$ satisfy the inequalities $0 < |\xi - x_T| < \delta$, and $(x, u) \in \Omega_x \setminus \Omega$ be such that $x(T) = \xi$. By the definition of $\varphi'_x(x, u)$ there exists a sequence $\{(x_n, u_n)\} \subset A$ converging to $(x, u)$ and such that

$$\frac{-2a}{3} \geq \frac{\varphi(x_n, u_n) - \varphi(x, u)}{\|x_n - x, u_n - u\|_x} = \frac{|x_n(T) - x_T| - |x(T) - x_T|}{\|x_n - x, u_n - u\|_x}$$

for all $n \in \mathbb{N}$. Hence with the use of inequality (5) one obtains that $0 < |x_n(T) - x_T| < \delta$ for any sufficiently large $n$, and there exists $n_0 \in \mathbb{N}$ such that inequality $(31)$ is satisfied with $\xi_n = x_{n + n_0}(T), n \in \mathbb{N}$, and $\beta = a/3C_p$. Thus, the negative tangent angle property is closely related to the validity of assumption 2 of Theorem 1.

**Definition 4.** Let $K \subset A$ be a given set. One says that the property $(S)$ is satisfied on the set $K$, if there exists $C > 0$ such that for any $(x, u) \in K$ one can find a neighborhood $\mathcal{O}(x(T)) \subset \mathbb{R}^d$ of $x(T)$ such that for all $\hat{x}_T \in \mathcal{O}(x(T)) \cap R(x_0, T)$ there exists a control input $\hat{u} \in U$ that steers the system from $x(0) = x_0$ to $\hat{x}_T$ in time $T$, and

$$\|u - \hat{u}\|_q + |x - \hat{x}|_{1,p} \leq C|x(T) - \hat{x}(T)|,$$

where $\hat{x}$ is a trajectory corresponding to $\hat{u}$, that is, $(\hat{x}, \hat{u}) \in A$.

Let $K \subset A$ be a given set. Recall that the set $A$ consists of all those pairs $(x, u) \in X$ for which $u \in U$, and $x$ is a solution of $\dot{x} = f(x, u, t)$ with $x(0) = x_0$ (see equality (29)). Roughly speaking, the property $(S)$ is satisfied on $K$ iff for any $(x, u) \in K$ and any reachable end-point $\hat{x}_T \in R(x, T)$ lying sufficiently close to $x(T)$ one can reach $\hat{x}_T$ by slightly changing the control input $u$ in such a way that the corresponding trajectory stays in a sufficiently small neighbourhood of $x(T)$ (more precisely, the magnitude of change of $u$ and $x$ must be proportional to $|x(T) - \hat{x}_T|$). Note that the property $(S)$ implicitly appeared in the proofs of Theorems 6 and 8 (cf. inequalities (10) and (23)) and was proved with the use of Robinson’s theorem.

**Remark 9.** Let the function $(x, u) \mapsto f(x, u, t)$ be locally Lipschitz continuous uniformly for all $t \in (0, T)$. Suppose also that the set $A$ is bounded in $L^p_{\infty}(0, T) \times L^p_{\infty}(0, T)$, that is, the control inputs and corresponding trajectories of the system are uniformly bounded. By definition $|x_1(t) - x_2(t)| = \int_0^T |f(x_1(r), u_1(r), r) - f(x_2(r), u_2(r), r)| \, dr$ for any $(x_1, u_1), (x_2, u_2) \in A$. 

$$\int_0^T \int_0^T \int_0^T |f(x_1(r), u_1(r), r) - f(x_2(r), u_2(r), r)| \, dr \, dt \leq C|x_1(t) - x_2(t)|.$$
Therefore, due to the boundedness of $A$ and the Lipschitz continuity of $f$ there exists $L > 0$ such that

$$|x_1(t) - x_2(t)| \leq L \int_0^t |x_1(\tau) - x_2(\tau)| \, d\tau + L \int_0^t |u_1(\tau) - u_2(\tau)| \, d\tau \leq L \int_0^t |x_1(\tau) - x_2(\tau)| \, d\tau + LT^{1/p} \|u_1 - u_2\|_q,$$

for all $t \in [0, T]$. Hence with the use of the Grönwall-Bellman inequality one can easily check that there exists $L_1 > 0$ such that $\|x_1 - x_2\|_\infty \leq L_1 \|u_1 - u_2\|_q$ for any $(x_1, u_1), (x_2, u_2) \in A$. Then by applying the inequality

$$|\hat{x}_1(t) - \hat{x}_2(t)| \leq |f(x_1(t), u_1(t), t) - f(x_2(t), u_2(t), t)| \leq L|x_1(t) - x_2(t)| + L|u_1(t) - u_2(t)|,$$

and Hölder’s inequality (here we suppose that $q \geq p$) one obtains that there exists $L_2 > 0$ such that $\|x_1 - x_2\|_p \leq L_2 \|u_1 - u_2\|_q$ for all $(x_1, u_1), (x_2, u_2) \in A$. In other words, the map $u \mapsto x_u$, where $x_u$ is a solution of $\dot{x} = f(x, u, t)$ with $x(0) = x_0$, is Lipschitz continuous on $U$. Therefore, under the assumptions of this remark inequality (32) in the definition of the property $(S)$ can be replaced with the inequality $\|u - \hat{u}\|_q \leq C|x(T) - \hat{x}(T)|$.

A detailed analysis of the property $(S)$ lies beyond the scope of this paper. Here we only note that the property $(S)$ is, in essence, a reformulation of the assumption that the mapping $u \mapsto x_u(T)$ is metrically regular on the set $K$ (here $x_u$ is a solution of $\dot{x} = f(x, u, t)$ with $x(0) = x_0$). Thus, it seems possible to apply general results on metric regularity\textsuperscript{49,51,61,62} to verify whether the property $(S)$ is satisfied in particular cases. Our aim is to show that this property along with the negative tangent angle property ensures that the penalty function $\Phi_\lambda$ for fixed-endpoint problem (28) is completely exact. Denote by $I^*$ the optimal value of this problem.

**Theorem 10.** Let the following assumptions be valid:

1. $\theta$ is continuous and differentiable in $x$ and $u$, and the functions $\nabla_x \theta$, $\nabla_u \theta$, and $f$ are continuous;
2. either $q = +\infty$ or $\theta$ and $\nabla_x \theta$ satisfy the growth condition of order $(q, 1)$, $\nabla_u \theta$ satisfies the growth condition of order $(q - 1, q')$;
3. there exists a globally optimal solution of problem (28);
4. the set $R(x_0, T)$ has the negative tangent angle property near $x_T$;
5. there exist $\hat{\lambda}_0 > 0$, $c > T^*$, and $\delta > 0$ such that the set $S_{\hat{\lambda}_0}(c) \cap \Omega_\delta$ is bounded in $W^d_{1,p}(0, T) \times L^m_q(0, T)$, the property $(S)$ is satisfied on $S_{\hat{\lambda}_0}(c) \cap (\Omega_\delta \backslash \Omega)$, and the function $\Phi_\lambda(x, u)$ is bounded below on $A$.

Then there exists $\lambda^* \geq 0$ such that for any $\lambda \geq \lambda^*$ the penalty function $\Phi_\lambda$ for problem (28) is completely exact on $S_{\lambda}(c)$.

**Proof.** As was noted in the proof of Theorem 6, the growth conditions on the function $\theta$ and its derivatives ensure that the functional $I$ is Lipschitz continuous on any bounded open set containing the set $S_{\lambda}(c) \cap \Omega_\delta$. The continuity of the penalty term $\varphi(x, u) = |x(T) - x_T|$ can also be verified in the same way as in the proof of Theorem 6.

Let us check that the set $A$ is closed. Indeed, choose any sequence $\{(x_n, u_n)\} \subset A$ converging to some $(x^*, u^*) \in X$. Recall that the set $U$ is closed and by definition $\{u_n\} \subset U$. Therefore $u^* \in U$. By inequality (5) the sequence $x_n$ converges to $x^*$ uniformly on $[0, T]$, which, in particular, implies that $x(0) = x_0$. Note also that by definition $\{x_n\}$ converges to $\hat{x}_n$ in $L^d_p(0, T)$, while $\{u_n\}$ converges to $u^*$ in $L^m_q(0, T)$. As is well known (eg, theorem 2.20 in Reference 55), one can extract subsequences $\{x_{n_k}\}$ and $\{u_{n_k}\}$ that converge almost everywhere. From the fact that $(x_{n_k}, u_{n_k}) \in A$ it follows that $\hat{x}_{n_k}(t) = f(\hat{x}_{n_k}(t), u_{n_k}(t), t)$ for a.e. $t \in (0, T)$. Consequently, passing to the limit as $k \to \infty$ with the use of the continuity of $f$ one obtains that $\hat{x}_n(t) = f(\hat{x}(t), u_n(t), t)$ for a.e. $t \in (0, T)$, that is, $(\hat{x}, u^*) \in A$, and the set $A$ is closed. Thus, by Theorem 1 it remains to check that there exists $0 < \eta \leq \delta$ such that $\varphi_A^*(x, u) \leq -\alpha$ for any $(x, u) \in S_{\lambda}(c) \cap (\Omega_\delta \backslash \Omega)$.

Let $0 < \eta \leq \delta$ be arbitrary, and fix $(x, u) \in S_{\lambda}(c) \cap (\Omega_\delta \backslash \Omega)$. By definition one has $0 < \varphi(x, u) = |x(T) - x_T| < \eta$. Decreasing $\eta$, if necessary, and utilizing the negative tangent angle property one obtains that there exist $\beta > 0$ (independent of $(x, u)$) and a sequence $\{\xi_n\} \subset R(x_0, T)$ converging to $x(T)$ such that

$$\frac{\langle x(T) - x_T, \xi_n - x(T) \rangle}{|x(T) - x_T| \|\xi_n - x(T)\|} \leq -\beta \quad \forall n \in \mathbb{N}. \quad (33)$$
By applying the property $(S)$ one obtains that there exists $C > 0$ (independent of $(x, u)$) such that for any sufficiently large $n \in \mathbb{N}$ one can find $(x_n, u_n) \in A$ satisfying the inequality

$$\sigma_n := \|u - u_n\|_q + \|x - x_n\|_p, \leq C[x(T) - x_n(T)],$$  \hspace{1cm} (34)

and such that $x_n(T) = \xi_n$. By the definition of rate of steepest descent one has

$$\varphi_A(x, u) \leq \liminf_{n \to \infty} \frac{\varphi(x_n, u_n) - \varphi(x, u)}{\sigma_n}.$$ 

Taking into account the equality

$$\varphi(x_n, u_n) - \varphi(x, u) = \left(\frac{x(T) - x_T}{|x(T) - x_T|}, \xi_n - x(T)\right) + o(|\xi_n - x(T)|),$$

where $o(|\xi_n - x(T)|)/|\xi_n - x(T)| \to 0$ as $n \to \infty$ and inequality (33) one obtains that

$$\varphi_A(x, u) \leq \liminf_{n \to \infty} \left(-\frac{\xi_n - x(T)}{\sigma_n} + \frac{o(|\xi_n - x(T)|)}{\sigma_n}\right).$$

By applying the inequality $|x(T) - x_n(T)| \leq T^{1/q'}\|\dot{x} - \dot{x}_n\|_p \leq T^{1/q'} \sigma_n$ one gets that $o(|\xi_n - x(T)|)/\sigma_n \to 0$ as $n \to \infty$. Hence with the use of inequality (34) one obtains that $\varphi_A(x, u) \leq -\beta/\sigma$, and the proof is complete.

Remark 10. It is worth noting that in Example 1, (i) the functions $\theta$ and $f$ satisfy all assumptions of Theorem 10, (ii) there exists a globally optimal solution, (iii) the set $R(x_0, T)$ has the negative tangent angle property near $x_T$, and (iv) the set $A$ is bounded and the penalty function $\Phi_A$ is bounded below on $A$ for any $\lambda \geq 0$. However, this penalty function is not globally exact. Therefore, by Theorem 10 one can conclude that in this example the property $(S)$ is not satisfied on $S_{\lambda_0}(c) \cap (\Omega_0 \setminus \Omega)$ for any $\lambda_0 \geq 0$, $c > I^*$, and $\delta > 0$, when $x_T = 0$. Arguing in a similar way to the proof of Theorem 8 and utilizing Robinson’s theorem one can check that the property $(S)$ is satisfied on $S_{\lambda_0}(c) \cap (\Omega_0 \setminus \Omega)$ for some $\lambda_0 \geq 0$, $c > I^*$, and $\delta > 0$, provided $x_T \in \{0\} \times (0, 1)$. Thus, although the property $(S)$ might seem independent of the end-point $x_T$, the validity of this property on the set $S_{\lambda_0}(c) \cap (\Omega_0 \setminus \Omega)$ depends on the point $x_T$ and, in particular, its location in the reachable set $R(x_0, T)$.

### 3.5 Nonlinear systems: Local exactness

Although Theorem 10 gives a general understanding of sufficient conditions for the complete exactness of the penalty function $\Phi_A$ for problem (28), its assumptions cannot be readily verified for any particular problem. Therefore, it is desirable to have at least verifiable sufficient conditions for the local exactness of this penalty function. Our aim is to show a connection between the local exactness of the penalty function $\Phi_A$ for problem (28) and the complete controllability of the corresponding linearized system. This result serves as an illuminating example of how one can apply Theorems 3 and 4 to verify the local exactness of a penalty function.

Recall that the linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$  \hspace{1cm} (35)

with $x \in \mathbb{R}^d$ and $u \in \mathbb{R}^m$ is called *completely controllable* using $L^q$-controls in time $T$, if for any initial state $x_0 \in \mathbb{R}^d$ and any final state $x_T \in \mathbb{R}^d$ one can find $u \in L^q_T(0, T)$ such that there exists an absolutely continuous solution $x$ of Equation (35) with $x(0) = x_0$ defined on $[0, T]$ and satisfying the equality $x(T) = x_T$.

**Theorem 11.** Let $U = L^m_T(0, T)$, $q \geq p$, and $(x^*, u^*)$ be a locally optimal solution of problem (28). Let also the following assumptions be valid:

1. $\theta$ and $f$ are continuous, differentiable in $x$ in $u$, and the functions $\nabla x \theta, \nabla u \theta, \nabla x f, \text{ and } \nabla u f$ are continuous;
2. either \( q = +\infty \) or \( \theta \) and \( \nabla_x \theta \) satisfy the growth condition of order \( (q, 1) \), \( \nabla u \theta \) satisfies the growth condition of order \( (q - 1, q') \), \( f \) and \( \nabla_x f \) satisfy the growth condition of order \( (q/p, p) \), and \( \nabla_u f \) satisfies the growth condition of order \( (q/s, s) \) with \( s = qp/(q - p) \) in the case \( q > p \), and \( \nabla_x f \) does not depend on \( u \) in the case \( q = p \);

3. the linearized system \( \dot{h}(t) = A(t)h(t) + B(t)v(t) \) with \( A(t) = \nabla_x f(x^*(t), u^*(t), t) \) and \( B(t) = \nabla_u f(x^*(t), u^*(t), t) \) is completely controllable using \( L^q \)-controls in time \( T \).

Then the penalty function \( \Phi_\lambda \) for problem (28) is locally exact at \( (x^*, u^*) \).

Proof. As noted in the proof of Theorem 6, the growth conditions on the function \( \theta \) and its derivatives ensure that the functional \( I \) is Lipschitz continuous on any bounded subset of \( X \) (in particular, in any bounded neighborhood of \( (x^*, u^*) \)).

For any \( (x, u) \in X \) define

\[
F(x, u) = \begin{pmatrix} \dot{x}(\cdot) - f(x(\cdot), u(\cdot), \cdot) \\ x(T) \end{pmatrix}, \quad K = \begin{pmatrix} 0 \\ x_T \end{pmatrix}.
\]

Our aim is to apply Theorem 4 with \( C = \{(x, u) \in X|x(0) = x_0\} \) to the operator \( F \). Then one gets that there exists \( a > 0 \) such that

\[
\text{dist}(F(x, u), K) \geq a \text{ dist}(x, u), F^{-1}(K) \cap C)
\]

for any \( (x, u) \in C \) in a neighbourhood of \( (x^*, u^*) \). Hence taking into account the facts that \( \text{dist}(F(x, u), K) = |x(T) - x_T| = \phi(x, u) \) for any \( (x, u) \in A \), and \( F^{-1}(K) \cap C \) coincides with the feasible set \( \Omega \) of problem (28) one obtains that \( \phi(x) \geq a \text{ dist}(x, u), \Omega \) for any \( (x, u) \in A \) in a neighbourhood of \( (x^*, u^*) \). Then by applying Theorem 3 one obtains the desired result.

By Theorem 20 (see Appendix B) the growth conditions on the function \( f \) and its derivatives guarantee that the nonlinear operator \( F \) maps \( X \) to \( L^q_T(0, T) \times \mathbb{R}^d \), is strictly differentiable at \( (x^*, u^*) \), and its Fréchet derivative at this point has the form

\[
DF(x^*, u^*)[h, v] = \begin{pmatrix} h(T) - A(T)h(T) - B(v(T)) \end{pmatrix},
\]

where \( A(t) = \nabla_x f(x^*(t), u^*(t), t) \) and \( B(t) = \nabla_u f(x^*(t), u^*(t), t) \). Observe also that \( C - (x^*, u^*) = \{(h, v) \in X|h(0) = 0\} \) and \( K - F(x^*, u^*) = (0, 0)^T \), since \( x^*(0) = x_0 \) and \( x^*(T) = x_T \) by definition. Consequently, the regularity condition (4) from Theorem 4 takes the form: for any \( \omega \in L^q_T(0, T) \) and \( h_T \in \mathbb{R}^d \) there exists \( (h, v) \in X \) such that

\[
\dot{h}(t) = A(t)h(t) + B(t)v(t) + \omega(t) \quad \text{for a.e.} \quad t \in (0, T), h(0) = 0, h(T) = h_T.
\]

Let us check that this condition holds true. Then by applying Theorem 4 we arrive at the required result.

Fix any \( \omega \in L^q_T(0, T) \) and \( h_T \in \mathbb{R}^d \). Let \( h_1 \) be an absolutely continuous solution of the equation \( \dot{h}_1(t) = A(t)h_1(t) + \omega(t) \) with \( h_1(0) = 0 \) defined on \( [0, T] \) (the existence of such solution follows from theorem 1.1.3 of Reference 56). From the fact that \( \nabla_x f \) satisfies the growth condition of order \( (q/p, p) \) in the case \( q < +\infty \) it follows that \( A(\cdot) \in L^q_T(0, T) \) (in the case \( q = +\infty \) one obviously has \( A(\cdot) \in L^q_T(0, T) \)). Hence \( h_1 \in W^q_T(0, T) \), since \( h_1 \) is absolutely continuous and the right-hand side of the equality \( \dot{h}_1(t) = A(t)h_1(t) + \omega(t) \) belongs to \( L^q_T(0, T) \).

Let \( v \in L^q_T(0, T) \) be such that an absolutely continuous solution \( h_2 \) of the system \( \dot{h}_2(t) = A(t)h_2(t) + B(t)v(t) \) with \( h_2(0) = 0 \) satisfies the equality \( h_2(T) = -h_1(T) + h_T \). Note that such \( v \) exists due to the complete controllability assumption. By applying the fact that \( \nabla_x f \) satisfies the growth condition of order \( (q/s, s) \) one obtains that \( B(\cdot) \in L^q_T(0, T) \) in the case \( p < q < +\infty \), which with the use of Hölder inequality implies that \( B(\cdot)v(T) \in L^q_T(0, T) \) (in the case \( q = +\infty \) one obviously has \( B(\cdot)v(\cdot) \in L^q_T(0, T) \)), while in the case \( p = q < +\infty \) one has \( B(\cdot)v(\cdot) \in L^q_T(0, T) \), since \( \nabla_x f \) does not depend on \( u \), and \( B(\cdot)v(\cdot) \in L^q_T(0, T) \). Therefore \( h_2 \in W^q_T(0, T) \) by virtue of the fact that the right-hand side of \( \dot{h}_2(t) = A(t)h_2(t) + B(t)v(t) \) belongs to \( L^q_T(0, T) \). It remains to note that the pair \( (h_1 + h_2, v) \) belongs to \( X \) and satisfies equalities (36).

Remark 11. From the proof of Theorem 11 it follows that under the assumption of this theorem the penalty function

\[ \Psi_\lambda(x, u) = I(x, u) + \lambda \left[ |x(T) - x_T| + \left( \int_0^T |x(t) - f(x(t), u(t), t)|^p \, dt \right)^{1/p} \right], \]

is locally exact at \((x^*, u^*)\), that is, \((x^*, u^*)\) is a point of local minimum of this penalty function on the set \(\{(x, u) \in X| x(0) = x_0\}\) for any sufficiently large \(\lambda\).

Remark 12. Let us note that Theorem 11 can be extended to the case of problems with convex constraints on control inputs, but in this case the complete controllability assumption must be replaced by a much more restrictive assumption. Namely, let \(U \subset L_q^m(0, T)\) be a closed convex set, \((x^*, u^*)\) be a locally optimal solution of problem (28), and \(\hat{h}(t) = A(t)\hat{h}(t) + B(t)v(t)\) be the corresponding linearized system. Define \(C = \{(x, u) \in X| x(0) = x_0, u \in U\}\) and \(K = (0, x_T)^T\). One can easily see that in this case the regularity condition (4) takes the form: for any \(\omega \in L_p^d(0, T)\) and \(h_T \in \mathbb{R}^d\) there exists \((v, \bar{v}) \in X\) such that \(v \in \text{cone}(U - u^*)\) and

\[ \hat{h}(t) = A(t)\hat{h}(t) + B(t)v(t) + \omega(t) \quad \text{for a.e. } t \in (0, T), \quad h(0) = 0, \quad h(T) = h_T, \]

where \(\text{cone}(U - u^*) = \cup_{x \geq 0} a(U - u^*)\) is the cone generated by the set \(U - u^*\). If \(u^* \in \text{int}U\), then \(\text{cone}(U - u^*) = L_q^m(0, T)\), and this regularity condition is equivalent to the complete controllability of the linearized system. However, if \(u^* \notin \text{int}U\), then one must suppose that for any initial state \(x_0 \in \mathbb{R}^d\) and for any final state \(x_T \in \mathbb{R}^d\) one can find \(u \in \text{cone}(U - u^*)\) such that there exists an absolutely continuous solution \(x\) of Equation (35) with \(x(0) = x_0\) defined on \([0, T]\) and satisfying the equality \(x(T) = x_T\), that is, the linearized system must be completely controllable using control inputs from \(\text{cone}(U - u^*)\). If this assumption is satisfied, then arguing in the same way as in the proof of Theorem 11 one can verify that the penalty function \(\Phi_\lambda\) for problem (28) is locally exact at \((x^*, u^*)\).

It should be noted that the complete controllability of the linearized system is not necessary for the local exactness of the penalty function \(\Phi_\lambda(x, u)\), as the following simple example shows.

Example 2. Let \(d = m = 1\) and \(p = q = 2\). Consider the following fixed-endpoint optimal control problem:

\[ \min I(u) = -\int_0^T u(t)^2 \, dt \quad \text{s.t.} \quad \dot{x}(t) = x(t) + u(t)^2, \quad t \in [0, T], \quad x(0) = x(T) = 0, \quad u \in L^2(0, T). \quad (37) \]

Solving the differential equation one obtains that \(x(t) = \int_0^t e^{-\tau} u(\tau)^2 \, d\tau\) for all \(t \in [0, T]\), which implies that the only feasible point of this problem is \((x^*, u^*)\) with \(x^*(t) \equiv 0\) and \(u^*(t) = 0\) for a.e. \(t \in [0, T]\). Thus, \((x^*, u^*)\) is a globally optimal solution of this problem. The linearized system at this point has the form \(\dot{h} = h\). Clearly, it is not completely controllable, which renders Theorem 11 inapplicable. Let us show that, nevertheless, the penalty function \(\Phi_\lambda\) for problem (37) is globally exact.

Indeed, in this case the penalized problem has the form

\[ \min \Phi_\lambda(x, u) = -\int_0^T u(t)^2 \, dt + \lambda |x(T)| \quad \text{s.t.} \quad \dot{x}(t) = x(t) + u(t)^2, \quad t \in [0, T], \quad x(0) = 0, \quad u \in U. \]

With the use of the fact that \(x(t) = \int_0^t e^{-\tau} u(\tau)^2 \, d\tau\) one gets that

\[ \Phi_\lambda(x, u) = -\int_0^T u(t)^2 \, dt + \lambda \int_0^T e^{T-t} u(t)^2 \, dt \geq -\int_0^T u(t)^2 \, dt + \lambda \int_0^T u(t)^2 \, dt, \]

for any \(u \in U\). Therefore, for all \(\lambda \geq 1\) one has \(\Phi_\lambda(x, u) \geq 0 = \Phi_\lambda(x^*, u^*)\) for any feasible point of the penalized problem, that is, the penalty function \(\Phi_\lambda\) for problem (37) is globally exact.

Remark 13. As Theorem 11 demonstrates, the local exactness of the penalty function \(\Phi_\lambda\) for problem (28) is implied by the complete controllability of the corresponding linearized system. It should be noted that a similar result can be proved in the case of complete exactness, but one must assume some sort of uniform complete controllability of linearized systems.
is completely controllable using $L^q$-controls, then there exists $C > 0$ such that for any $x_T$ one can find $u \in L^m_q(0, T)$ with $\|u\|_q \leq C|x_T|$ such that for the corresponding solution $x(\cdot)$ of Equation (38) with $x(0) = 0$ one has $x(T) = x_T$. In other words, one can steer the state of system (38) from the origin to any point $x_T$ in time $T$ with the use of a control input whose $L^q$-norm is proportional to $|x_T|$. Denote the greatest lower bound of all such $C$ by $C_T(A(\cdot), B(\cdot))$. In the case when system (38) is not completely controllable we put $C_T(A(\cdot), B(\cdot)) = +\infty$. Then one can say that the nonlinear system $\dot{x} = f(x, u, t)$ is uniformly completely controllable in linear approximation on a set $K \subseteq A$, if there exists $C > 0$ such that $C_T(∇_u f(x(\cdot), u(\cdot), \cdot), ∇_x f(x(\cdot), u(\cdot), \cdot)) \leq C$ for any $(x, u) \in K$. With the use of general results on non-local metric regularity one can check that under some natural assumptions on the function $f$ uniform complete controllability in linear approximation on a set $K \subseteq A$ guarantees that the property $(S)$ is satisfied on this set, provided $U = L^m_q(0, T)$. Hence, by applying Theorem 10 one can prove that uniform complete controllability in linear approximation of the nonlinear system $\dot{x} = f(x, u, t)$ implies that the penalty function $Φ_\lambda$ for problem (28) is completely exact. A detailed proof of this result lies beyond the scope of this paper, and we leave it to the interested reader.

### 3.6 Variable-endpoint problems

Let us briefly outline how the main results on the exact penalization of terminal constraints from previous sections (in particular, Theorems 11 and 6) can be extended to the case of variable-endpoint problems of the form

$$\min I(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt + \zeta(x(T)) \quad \text{subject to} \quad \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T],$$

$$x(0) = x_0, \quad g_i(x(T)) \leq 0 \quad \forall i \in I, \quad g_k(x(T)) = 0 \quad \forall k \in J, \quad u \in U. \quad (39)$$

Here $θ : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}$, $ζ : \mathbb{R}^d \to \mathbb{R}$, $f : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}^d$, and $g_i : \mathbb{R}^d \to \mathbb{R}$ are given functions, $i \in I \cup J, I = \{1, \ldots, I_1\}, J = \{I_1 + 1, \ldots, I_2\}, x_0 \in \mathbb{R}^d$, and $T > 0$ are fixed, $x \in W^d_{1,p}(0, T)$, $U \subseteq L^m_q(0, T)$ is a nonempty closed set, and $1 \leq p, q \leq +\infty$.

We penalize only the endpoint constraints. To this end, define $X = W^d_{1,p}(0, T) \times L^m_q(0, T)$, $M = \{(x, u) \in X | g_i(x(T)) \leq 0, \ i \in I, \ g_k(x(T)) = 0, \ k \in J\}$, and

$$A = \{(x, u) \in X | x(0) = x_0, \ x(t) = f(x(t), u(t), t) \ \text{for a.e.} \ \ t \in [0, T]\}. \quad \text{Then problem (39) can be rewritten as the problem of minimizing} \ I(x, u) \ \text{subject to (x, u) \in M \cap A. Define}$

$$φ(x, u) = \sum_{i \in I} \max\{g_i(x(T)), 0\} + \sum_{k \in J} |g_k(x(T))|. \quad \text{Then M = \{(x, u) \in X | φ(x, u) = 0\}, and one can consider the penalized problem}$

$$\min Φ_\lambda(x, u) = I(x, u) + λφ(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt + \zeta(x(T)) + λ \left( \sum_{i \in I} \max\{g_i(x(T)), 0\} + \sum_{k \in J} |g_k(x(T))| \right)$$

$$\text{subject to } \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T], \quad x(0) = x_0, \quad u \in U,$$

which is a nonlinear free-endpoint optimal control problem.

For any $x \in \mathbb{R}^d$ denote $I(x) = \{i \in I | g_i(x) = 0\}$. Let the functions $g_i$, $i \in I \cup J$, be differentiable. Recall that one says that the Mangasarian-Fromovitz constraint qualifications (MFCQ) holds at a point $x_T \in \mathbb{R}^d$, if the gradients $∇g_k(x_T), k \in J$, are linearly independent, and there exists $h \in \mathbb{R}^d$ such that $⟨∇g_k(x_T), h⟩ = 0$ for any $k \in J$, and $⟨∇g_i(x_T), h⟩ < 0$ for any
i \in I(x_T)$. Let us show that the complete controllability of the linearized system along with MFCQ guarantee the local exactness of the penalty function $\Phi_i$ for problem (39).

**Theorem 12.** Let $U = L_q^d(0, T)$, $q \geq p$, and $(x^*, u^*)$ be a locally optimal solution of problem (39). Suppose also that Assumptions 1, 2, and 3 of Theorem 11 are satisfied, $\zeta$ is locally Lipschitz continuous, the functions $g_i$, $i \in I \cup J$ are continuously differentiable in a neighbourhood of $x^*(T)$, and MFCQ holds true at the point $x^*(T)$. Then the penalty function $\Phi_i$ for problem (39) is locally exact at $(x^*, u^*)$.

**Proof.** For any $(x, u) \in X$ define

$$F(x, u) = \left( \dot{x} - f(x, u), \cdot \right), \quad K = \left( \begin{array}{c} 0 \\ \mathbb{R}_+^l \times \{0_{l_1-l_i} \} \end{array} \right),$$

where $g(\cdot) = (g_1(\cdot), \ldots, g_l(\cdot))^T$, $\mathbb{R}_+ = (-\infty, 0]$ and $0_{l_1-l_i}$ is the zero vector of dimension $l_2 - l_1$. Let us apply Theorem 4 with $C = \{(x, u) \in X | x(0) = x_0\}$ to the operator $F$. Then arguing in the same way as in the proof of Theorem 11 one arrives at the required result.

With the use of Theorem 20 (see Appendix B) one obtains that the nonlinear operator $F$ maps $X$ to $L_p^d(0, T) \times \mathbb{R}_+^l$, is strictly differentiable at $(x^*, u^*)$, and its Fréchet derivative at this point has the form

$$DF(x^*, u^*)[h, v] = \begin{pmatrix} \dot{h} - A(\cdot)h - B(\cdot)v(h) \\ Vg(x^*(T))h(T) \end{pmatrix}, \quad A(t) = \nabla_x f(x^*(t), u^*(t), t), \quad B(t) = \nabla_u f(x^*(t), u^*(t), t).$$

Hence bearing in mind the fact that $C - (x^*, u^*) = \{(h, v) \in X | h(0) = 0\}$, since $x^*(0) = x_0$, one gets that the regularity condition (4) from Theorem 4 takes the form $0 \in \text{core}(K(x^*, u^*))$ with

$$K(x^*, u^*) = \left\{ \begin{pmatrix} \dot{h} - A(\cdot)h - B(\cdot)v(h) \\ Vg(x^*(T))h(T) \end{pmatrix} \Bigg| (h, v) \in X, h(0) = 0 \right\},$$

(40)

where $K_0 = (\mathbb{R}_+^l - \{g_1(x^*(T)), \ldots, g_l(x^*(T))\} \times \{0_{l_1-l_i}\}$.

Indeed, denote $g_i(\cdot) = (g_{i1}(\cdot), \ldots, g_{il}(\cdot))^T$. By MFCQ the matrix $Vg_i(x^*(T))$ has full row rank. Therefore, by the open mapping theorem (see formula (0.2) of Reference 61) there exists $\eta > 0$ such that for any $y_2 \in \mathbb{R}^l_{-l_i}$ one can find $h_2 \in \mathbb{R}^d$ with $|h_2| \leq \eta |y_2|$ satisfying the equality $Vg_i(x^*(T))h_2 = y_2$.

Fix any $\alpha \in \mathbb{R}^d(0, T)$, $r_2 > 0$, and $y_2 \in B(0_{l_2-l_1}, r_2)$. Then there exists $h_2 \in \mathbb{R}^d$ with $|h_2| \leq \eta r_2$ such that $Vg_i(x^*(T))h_2 = y_2$. By MFCQ there exists $h_1 \in \mathbb{R}^d$ such that $Vg_i(x^*(T))h_1 = 0$ and $Vg_i(x^*(T))h_1 < 0$ for any $i \in I(x^*(T))$. Taking into account the fact that $g_i(x^*(T)) < 0$ for any $i \not\in I(x^*(T))$, one obtains that there exists $\alpha > 0$ such that $\langle Vg_i(x^*(T)), ah_1 + g_i(x^*(T)) \rangle < 0$ for all $i \in I$. Furthermore, decreasing $r_2$, if necessary, one may suppose that

$$\max_{|h| \leq \eta r_2} \langle Vg_i(x^*(T)), ah_1 + h \rangle + g_i(x^*(T)) < 0 \quad \forall i \in I.$$

(41)

Observe also that $Vg_i(x^*(T))(ah_1 + h_2) = y_2$.

Denote $h_T = ah_1 + h_2$. Arguing in the same way as in the proof of Theorem 11 and utilizing the complete controllability assumption one can verify that there exists $(h, v) \in X$ such that

$$\dot{h}(t) = A(t)h(t) + B(t)v(t) + \omega(t) \quad \text{for a.e. } t \in (0, T), \quad h(0) = 0, \quad h(T) = h_T.$$

Consequently, one has $L_p^d(0, T) \times (r_1, +\infty)^j_1 \times B(0_{l_2-l_1}, r_2) \subset K(x^*, u^*)$, where

$$r_1 = \max_{i \in I} \left( \max_{|h| \leq \eta r_2} \left\langle \nabla g_i(x^*(T)), ah_1 + h \right\rangle + g_i(x^*(T)) \right) < 0,$$

(see equality (40) and inequality (41)). Thus, $0 \in \text{core}(K(x^*, u^*))$, and the proof is complete.

Let us now show how one can extend Theorem 6 to the case of variable-endpoint problems. Theorem 8 can be extended to the case of variable-endpoint problems for linear evolution equations in a similar way. Let, as in the proof of the previous
Theorem 13. Let \( q \geq p \), the functions \( g_i, i \in I \) and the set \( U \) be convex, the functions \( g_k, k \in J \) be affine, and the following assumptions be valid:

1. \( f(x, u, t) = A(t)x + B(t)u \) for some \( A(\cdot) \in L^{\text{bd}}_{\infty}(0, T) \) and \( B(\cdot) \in L^{\text{bd,m}}_{\infty}(0, T) \);
2. the function \( \zeta \) is locally Lipschitz continuous, the function \( \theta = \theta(x, u, t) \) is continuous, differentiable in \( x \) and \( u \), and the functions \( \nabla_x \theta \) and \( \nabla_u \theta \) are continuous;
3. either \( q = +\infty \) or the functions \( \theta \) and \( \nabla_x \theta \) satisfy the growth condition of order \((q, 1)\), while the function \( \nabla_u \theta \) satisfies the growth condition of order \((q - 1, q')\);
4. there exists a globally optimal solution of problem (39), and the following Slater condition holds true: \( \exists 0 \in \text{relint} g_i(R_j(x_0, T)) \) and there exists a feasible point \((\hat{x}, \hat{u}) \in \Omega \) such that \( g_i(\hat{x}(T)) < 0 \) for all \( i \in I \);
5. there exist \( \lambda_0 > 0, c > I^* \) and \( \delta > 0 \) such that the set \( S_{\lambda_0}(c) \cap \Omega_\delta \) is bounded in \( W^d_{1, p}(0, T) \times L^d_q(0, T) \), and the function \( \Phi_{\lambda_0}(x, u) \) is bounded below on \( A \).

Then there exists \( \lambda^* \geq 0 \) such that for any \( \lambda \geq \lambda^* \) the penalty function \( \Phi_{\lambda} \) for problem (39) is completely exact on \( S_{\lambda}(c) \).

Proof. Arguing in almost the same way as in the proof of Theorem 6 and utilizing Theorem 1 one obtains that it is sufficient to check that there exists \( a > 0 \) such that \( \varphi^1_A(x, u) \leq -a \) for all \((x, u) \in S_{\lambda_0}(c) \cap (\Omega_\delta \setminus \Omega)\). Fix any \((x, u)\), and define \( I_+(x, u) = \{i \in I | g_i(\hat{x}(T)) > 0\} \). Let us consider two cases.

Case I. Suppose that \( I_+(x, u) \neq \emptyset \). Define \( (\Delta x, \Delta u) = (\hat{x} - x, \hat{u} - u) \), where \((\hat{x}, \hat{u})\) is from Slater’s condition. Observe that \((x + \Delta x, u + \Delta u) \in A \) for any \( a \in [0, 1] \) due to the convexity of the set \( U \) and the linearity of the system. Furthermore, by virtue of the convexity of the functions \( g_i, i \in I \), for any \( a \in [0, 1] \) one has
\[
g_i(x(T) + a \Delta x(T)) \leq a g_i(\hat{x}(T)) + (1-a) g_i(x(T)), \quad \eta = \max_{i \in I} g_i(\hat{x}(T)) < 0. \tag{42}
\]
Consequently, for any \( i \not\in I_+(x, u) \) one has \( g_i(x(T) + a \Delta x(T)) < 0 \) for all \( a \in [0, 1] \) by inequality (42), while for any \( i \in I_+(x, u) \) one has \( g_i(x(T) + a \Delta x(T)) \geq 0 \) for any sufficiently small \( a \) due to the fact that a convex function defined on a finite dimensional space is continuous in the interior of its effective domain (e.g., theorem 3.5.3 of Reference 63). Moreover, \( g_i(x(T) + a \Delta x(T)) = (1-a) g_i(x(T)) \), since the functions \( g_i, k \in J \), are affine and \( g_i(\hat{x}(T)) = 0 \) (recall that \((\hat{x}, \hat{u}) \in \Omega\)). Hence with the use of inequality (42) one obtains that
\[
\varphi_A^1(x, u) \leq \lim_{a \to 0^+} \frac{\varphi(x + a \Delta x, u + a \Delta u)}{a \|\Delta x, \Delta u\|_X} = \frac{1}{\|\Delta u, \Delta x\|_X} \lim_{a \to 0^+} \frac{1}{a} \left( \sum_{i \in I_+(x(T))} (g_i(x(T) + a \Delta x(T)) - g_i(x(T))) \right) \]
\[
+ \sum_{k \in J^*} ((1-a)|g_k(x(T))| - |g_k(x(T))|) \leq \frac{1}{\|\Delta u, \Delta x\|_X} \left( \sum_{i \in I_+(x(T))} (\eta - g_i(x(T))) - \sum_{k \in J^*} |g_k(x(T))| \right) \leq \frac{\eta}{\|\Delta u, \Delta x\|_X}.
\]
From the fact that the set \( S_{\lambda_0}(c) \cap \Omega_\delta \) is bounded it follows that there exists \( C > 0 \) (independent of \((x, u)\)) such that \( \|\Delta u, \Delta x\|_X = ||\hat{x} - x, \hat{u} - u||_X \leq C \). Thus, \( \varphi_A^1(x, u) \leq \eta/C < 0 \), and the proof of the first case is complete.

Case II. Let now \( I_+(x, u) = \emptyset \). Note that \( g_i(x(T)) \neq 0 \), since \((x, u) \not\in \Omega\). Choose any \((\tilde{x}, \tilde{u}) \in \Omega \) and define \( (\Delta x, \Delta u) = (\tilde{x} - x, \tilde{u} - u) \). Then, as in the first case, for any \( a \in [0, 1] \) one has \((x + a \Delta x, u + a \Delta u) \in A\), and
\[
g_i(x(T) + a \Delta x(T)) \leq a g_i(\hat{x}(T)) + (1-a) g_i(x(T)) \leq 0 \quad \forall i \in I
\]
due to the convexity of the functions \( g_i \) and the fact that \( I_+(x, u) = \emptyset \). In addition, \( g_j(x(T) + a \Delta x(T)) = (1-a) g_j(x(T)) \) for all \( a \in [0, 1] \), since the functions \( g_k, k \in J \), are affine and \( (\tilde{x}, \tilde{u}) \in \Omega \). Therefore
\[
\varphi_A^1(x, u) \leq \lim_{a \to 0^+} \frac{\varphi(x + a \Delta x, u + a \Delta u)}{a \|\Delta x, \Delta u\|_X} = \frac{1}{\|\Delta u, \Delta x\|_X} \lim_{a \to 0^+} \sum_{k \in J} ((1-a)|g_k(x(T))| - |g_k(x(T))|) \]
\[
= - \frac{1}{\|\Delta u, \Delta x\|_X} \sum_{k \in J} |g_k(x(T))| \leq - \frac{|g_r(x(T))|}{\|\Delta u, \Delta x\|_X} < 0.
\]
Thus, it remains to show that there exists $C > 0$ such that for any $(x, u) \in S_{\alpha}(c) \cap (\Omega_0 \setminus \Omega)$ such that $I_+ (x, u) = \emptyset$ one can find $(\bar{x}, \bar{u}) \in \Omega$ satisfying the inequality

$$||(\Delta x, Du)||_X = ||x - \bar{x}||_{1,p} + ||u - \bar{u}||_q \leq C|g_r(x(T))|. \quad (43)$$

Then $\varphi^A_\lambda (x, u) \leq -1/C$, and the proof is complete.

As was shown in the proof of Theorem 6, there exists $L > 0$ (depending only on $A(\cdot), B(\cdot), T, p, \text{and} q$) such that $||x_1 - x_2||_{1,p} \leq L||u_1 - u_2||_q$ for any $(x_1, u_1), (x_2, u_2) \in A$. Therefore, instead of inequality (43) it suffices to prove the validity of the inequality

$$||u - \bar{u}||_q \leq C|g_r(x(T))|. \quad (44)$$

Moreover, from inequality (5) and the inequality $||x_1 - x_2||_{1,p} \leq L||u_1 - u_2||_q$ it follows that the map $u \mapsto x_u(T)$ is Lipschitz continuous, where $x_u$ is a solution of the system $\dot{x}_u(\cdot) = A(\cdot)x_u(\cdot) + B(\cdot)u(\cdot)$ such that $x_u(0) = x_0$. Hence with the use of the fact that the functions $g_i$ are convex and continuous one obtains that the set $U_t = \{ u \in U | g_i(x_u(T)) \leq 0, i \in I \}$ is closed and convex.

Let us prove inequality (44) with the use of Robinson’s theorem. Note that the function $g_i(\cdot) - g_i(0)$ is linear, since the functions $g_k, k \in I$, are affine. Define the linear operator $\mathcal{T} : L^m_q(0, T) \to \mathbb{R}^{k-I}$, $\mathcal{T} v = g_i(\hat{h}(T)) - g_i(0)$, where $\hat{h}$ is a solution of the differential equation

$$\hat{h}(t) = A(t)\hat{h}(t) + B(t)v(t), \quad \hat{h}(0) = 0, \quad t \in [0, T].$$

As was shown in the proof of Theorem 6, the mapping $v \mapsto \mathcal{T} v$ is continuous, which implies that the linear operator $\mathcal{T}$ is bounded.

Fix any $(x, u) \in \Omega$. By definition for all $(x, u) \in A$ one has $\dot{x}(t) - x_u(T) = A(t)(x(t) - x_u(t)) + B(t)(u(t) - u_u(t))$ for a.e. $t \in [0, T], x(0) - x_u(0) = 0, g_r(x_u(T)) = 0$, and

$$\mathcal{T}(u - u_u) = g_i(x(T) - x_u(T)) - g_i(0) = g_i(x(T)) - g_i(0) - (g_i(x_u(T)) - g_i(0)) = g_i(x_u(T)).$$

Therefore $g_i(R_t(x_0, T)) = \mathcal{T}(U_t - u_u)$. Define $X_0 = \text{cl span}(U_t - u_u)$ and $Y_0 = \text{span}(\mathcal{T}(U_t - u_u))$. Note that $Y_0$ is a closed subspace of $\mathbb{R}^{k-I}$ and $\mathcal{T}(X_0) = Y_0$. Finally, introduce the operator $T_0 : X_0 \to Y_0, T_0(v) = \mathcal{T}(v)$ for all $v \in X_0$. Clearly, $T_0$ is a bounded linear operator between Banach spaces. Moreover, by Slater’s condition $0 \in \text{relint} g_i(R_t(x_0, T)) = \text{int} T(U_t - u_u)$. Consequently, by Robinson’s theorem (Theorem 5 with $C = U_t - u_u, x^* = 0, y = y^* = 0$) there exists $\kappa > 0$ such that

$$\text{dist} (u - u_u, T^{-1}_0(0) \cap (U_t - u_u)) \leq \kappa \left(1 + ||u - u_u||_q\right) |T_0(u - u_u)|, \quad \forall u \in U_t.$$

Recall that $T_0(u - u_u) = g_i(x_u(T))$, since $(x, u) \in \Omega$. Thus, for any $(x, u) \in S_{\alpha}(c) \cap (\Omega_0 \setminus \Omega)$ such that $I_+(x, u) = \emptyset$ one can find $v \in U_t - u_u$ such that $T_0(v) = 0$ and

$$||u - u_u - v||_q \leq 2\kappa \left(1 + ||u - u_u||_q\right)|g_r(x(T))|.$$

Define $\bar{u} = u_u + v$, and let $\bar{x}$ be the corresponding solution of the original system, that is, $(\bar{x}, \bar{u}) \in A$. Then $g_i(\bar{x}_u(T)) \leq 0$ for all $i \in I$, since $\bar{u} \in U_t$, and $g_i(\bar{x}_u(T)) = \mathcal{T}(\bar{u} - u_u) = \mathcal{T}(v) = 0$, that is, $g_i(\bar{x}_u(T)) = 0$ and $(\bar{x}, \bar{u}) \in \Omega$. Moreover, one has $\|u - \bar{u}\|_q \leq 2\kappa(1 + ||u - u_u||_q)|g_r(x(T))|$. By our assumption the set $S_{\alpha}(c) \cap (\Omega_0 \setminus \Omega)$ is bounded. Therefore there exists $C > 0$ such that $2\kappa(1 + ||u - u_u||_q) \leq C$ for any $(x, u) \in S_{\alpha}(c) \cap (\Omega_0 \setminus \Omega)$ such that $I_+(x, u) = \emptyset$. Thus, for all such $(x, u)$ there exists $(\bar{x}, \bar{u}) \in \Omega$ satisfying inequality (44), and the proof is complete.

**Remark 14.** Note that in the case when there are no equality constraints Slater’s condition takes an especially simple form. Namely, it is sufficient suppose that there exists a feasible point $(\bar{x}, \bar{u}) \in \Omega$ such that $g_i(\bar{x}_u(T)) < 0$ for all $i \in I$.

**Remark 15.** Let us briefly discuss how one can extend Theorem 10 to the case of nonlinear variable-endpoint problems. In the case when there are no equality constraints and the inequality constraints are differentiable, one has to replace the negative tangent angle property with the assumption that there exists $\beta > 0$ such that for any $\xi \in R(x_0, T)$ satisfying the
inequalities $0 < \sum_{i \in I} \max \{g_i(\xi), 0\} < \delta$ one can find a sequence $\{\xi_n\} \subset R(x_0, T)$ converging to $\xi$ such that

$$
\left\langle V g_i(\xi), \frac{\xi_n - \xi}{|\xi_n - \xi|} \right\rangle \leq -\beta \quad \forall n \in \mathbb{N} \quad \forall i \in I : g_i(\xi) \geq 0.
$$

Then arguing in essentially the same way as in the proof of Theorem 10 one can show that the penalty function $\Phi_\lambda$ for problem (39) is completely exact on $S_\lambda(c)$ for any sufficiently large $\lambda$. In the general case, a similar but more cumbersome assumption must be imposed on both equality and inequality constraints.

## 4 | EXACT PENALIZATION OF POINTWISE STATE CONSTRAINTS

Let us now turn to the analysis of the exactness of penalty functions for optimal control problems with pointwise state constraints. In this case the situation is even more complicated than in the case of problems with terminal constraints. It seems that verifiable sufficient conditions for the complete exactness of a penalty function for problems with state constraints can be obtained either under very stringent assumptions on the controllability of the system or in the case of linear systems and convex state constraints. Furthermore, a penalty term for state constraints can be designed with the use of the $L^p$-norm with any $1 \leq p \leq +\infty$. The smooth norms with $1 < p < +\infty$ and the $L^1$-norm are more appealing for practical applications, while, often, one can guarantee exact penalization of state constraints only in the case $p = +\infty$.

### 4.1 | A Counterexample

We start our analysis of state constrained problems with a simple counterexample that illuminates the difficulties of designing exact penalty functions for state constraints. It also demonstrates that in the case when the functional $I(x, u)$ explicitly depends on control it is apparently impossible to define an exact penalty function for problems with state equality constraints within the framework adopted in our study.

**Example 3.** Let $d = 2$, $m = 1$, and $p = q = 2$. Define $U = \{ u \in L^2(0, T) \}^d | u(t) \in [-1, 1] \text{ for a.e. } t \in (0, T) \}$, and consider the following fixed-endpoint optimal control problem with state equality constraint:

$$
\min_I(u) = -\int_0^T u(t)^2 dt \quad \text{s.t.} \quad \begin{cases}
\dot{x}_1 = 1 \\
\dot{x}_2 = u
\end{cases} \quad t \in [0, T], \quad x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x(T) = \begin{pmatrix} T \\ 0 \end{pmatrix}, \quad u \in U, \quad g(x(t)) \equiv 0,
$$

where $g(x_1, x_2) = x_2^2$. The only feasible point of this problem is $(x^*, u^*)$ with $x^*(t) \equiv (t, 0)^T$ and $u^*(t) = 0$ for a.e. $t \in [0, T]$. Thus, $(x^*, u^*)$ is a globally optimal solution of this problem.

We would like to penalize the state equality constraint $g(x(t)) = x_2^2(t) = 0$. One can define the penalty term in one of the following ways:

$$
\varphi(x) = \left( \int_0^T |g(x(t))|^r dt \right)^{1/r}, \quad 1 \leq r < +\infty, \quad \varphi(x) = \max_{t \in [0, T]} |g(x(t))|, \quad \varphi(x) = \int_0^T |g(x(t))|^\alpha dt, \quad 0 < \alpha < 1.
$$

Clearly, all these functions are continuous with respect to the uniform metric, which by inequality (5) implies that they are continuous on $W^1_0(0, T)$. Therefore, instead of choosing a particular function $\varphi$, we simply suppose that $\varphi : W^1_0(0, T) \to [0, +\infty)$ is an arbitrary function, continuous with respect to the uniform metric, and such that $\varphi(x) = 0$ if and only if $g(x(t)) \equiv 0$. One can consider the penalized problem

$$
\min_{\lambda, u} \Phi_\lambda(x, u) = -\int_0^T u(t)^2 dt + \lambda \varphi(x) \quad \text{s.t.} \quad \begin{cases}
\dot{x}_1 = 1 \\
\dot{x}_2 = u
\end{cases} \quad t \in [0, T], \quad x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x(T) = \begin{pmatrix} T \\ 0 \end{pmatrix}, \quad u \in U. \quad (45)
$$

Observe that the goal function $I$ is Lipschitz continuous on any bounded subset of $L^2(0, T)$ by proposition 4 of Reference 45, and the set

$$
A = \{ (x, u) \in X | x(0) = (0, 0)^T, x(T) = (T, 0)^T, u \in U, \dot{x}_1 = 1, \dot{x}_2 = u \text{ for a.e. } t \in [0, T] \},
$$
is obviously closed in \( X = W_{1,2}^2(0, T) \times L^2(0, T) \). Consequently, penalized problem (45) fits the framework of Section 2. However, the penalty function \( \Phi_\lambda \) is not exact regardless of the choice of the penalty term \( \phi \).

Indeed, arguing by reductio ad absurdum, suppose that \( \Phi_\lambda \) is globally exact. Then there exists \( \lambda \geq 0 \) such that \( \Phi_\lambda(x, u) \geq \Phi_\lambda(x^\ast, u^\ast) \) for all \( (x, u) \in A \). For any \( n \in \mathbb{N} \) define

\[
  u_n(t) = \begin{cases} 
    1, & \text{if } t \in \left[ \frac{T(2k-1)}{2n}, \frac{T(2k-1)}{2n} \right), \ k \in \{1, 2, \ldots, n\}, \\
    -1, & \text{if } t \in \left[ \frac{T(2k-1)}{2n}, \frac{T(2k)}{2n} \right), \ k \in \{1, 2, \ldots, n\},
  \end{cases}
\]

that is, \( u_n \) takes alternating values \( \pm 1 \) on the segments of length \( T/2n \). For the corresponding trajectory \( x_n \) one has \( x(0) = (0, 0)^T, \ x(T) = (T, 0)^T \) (i.e., \( (x_n, u_n) \in A \)), and \( \|x_n\|_\infty = T/2n \). Therefore, \( \phi(x_n) \to 0 \) as \( n \to \infty \) due to the continuity of the function \( \phi \) with respect to the uniform metric. On the other hand, \( I(u_n) = -T \) for all \( n \in \mathbb{N} \), which implies that \( \Phi_\lambda(x_n, u_n) \to -T \) as \( n \to \infty \). Consequently, \( \Phi_\lambda(x_n, u_n) < 0 = \Phi_\lambda(x^\ast, u^\ast) \) for any sufficiently large \( n \in \mathbb{N} \), which contradicts our assumption. Thus, the penalty function \( \Phi_\lambda \) is not globally exact for any penalty term \( \phi \) that is continuous with respect to the uniform metric.

The previous example might lead one to think that linear penalty functions for state constrained optimal control problems cannot be exact. Our aim is to show that in some cases exact penalization of state constraints (especially, state inequality constraints) is nevertheless possible, but one must utilize the highly nonsmooth \( L^\infty \)-norm to achieve exactness. Furthermore, we demonstrate that exact \( L^p \)-penalization with finite \( p \) is possible in the case when either the problem is convex and Lagrange multipliers corresponding to state constraints belong to \( L^p(0, T) \) or the functional \( I(x, u) \) does not depend on the control inputs explicitly.

\section*{4.2 Linear Evolution Equations}

We start with the convex case, that is, with the case when the controlled system is linear and state inequality constraints are convex. The convexity of constraints, along with widely known Slater’s conditions from convex optimization, allows one to prove the complete exactness of \( L^\infty \)-penalty function under relatively mild assumptions. The main results on exact penalty functions in this case can be obtained for both linear time varying systems and linear evolution equations in Hilbert spaces. For the sake of shortness, we consider only evolution equations.

Let, as in Section 3.3, \( \mathcal{H} \) and \( \mathcal{U} \) be complex Hilbert spaces, \( T \) be a strongly continuous semigroup on \( \mathcal{H} \) with generator \( A: D(A) \to \mathcal{H} \), and let \( B \) be an admissible control operator for \( T \). For any \( t \geq 0 \) denote by \( F_t u = \int_0^t \mathcal{T}_{t-s}Bu(\sigma) \, d\sigma \) the input map corresponding to \( (A, B) \). Then, as was pointed out in Section 3.3, for any \( u \in L^2((0, T); \mathcal{U}) \) the initial value problem \( \dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0 \) with \( x_0 \in \mathcal{H} \) has a unique solution \( x \in C([0, T]; \mathcal{H}) \) given by

\[
  x(t) = \mathcal{T}_t x_0 + F_t u \quad \forall t \in [0, T].
\]  

Consider the following fixed-endpoint optimal control problem with state constraints:

\[
  \min_{(x,u)} I(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt \quad \text{subject to} \quad \dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0, T], \\
  x(0) = x_0, \quad x(T) = x_T, \quad u \in U, \quad g_j(x(t), t) \leq 0 \quad \forall t \in [0, T], \quad j \in J.
\]  

Here \( \theta: \mathcal{H} \times \mathcal{U} \times [0, T] \to \mathbb{R} \) and \( g_j: \mathcal{H} \times [0, T] \to \mathbb{R}, \ j \in J = \{1, \ldots, l\} \), are given functions, \( T > 0 \) and \( x_0, x_T \in \mathcal{H} \) are fixed, and \( U \subseteq L^2((0, T); \mathcal{U}) \) is a closed convex set.

Let us introduce a penalty function for problem (47). Our aim is to penalize the state inequality constraints \( g_j(x(t), t) \leq 0 \). To this end, define \( \tilde{X} = C([0, T]; \mathcal{H}) \times L^2((0, T); \mathcal{U}), \ M = \{(x, u) \in X | g_j(x(t), t) \leq 0 \ \forall t \in [0, T], \ j \in J\}, \) and

\[
  A = \{(x, u) \in \tilde{X} | x(0) = x_0, x(T) = x_T, u \in U, \ \text{and equality (46) holds true}\}.
\]

Then problem (47) can be rewritten as the problem of minimizing \( I(x, u) \) subject to \( (x, u) \in M \cap A \). Introduce the penalty term \( \phi(x, u) = \sup_{t \in [0, T]} \{g_1(x(t), t), \ldots, g_l(x(t), t), 0\} \). Then \( M = \{(x, u) \in \tilde{X} | \phi(x, u) = 0\} \), and one can consider the
penalized problem of minimizing $\Phi_A$ over the set $A$, which is a fixed-endpoint problem without state constraints of the form:

$$\min_{(x,u)} \int_0^T \theta(x(t), u(t), t) \, dt + \lambda \sup_{t \in [0,T]} \{ g_1(x(t), t), \ldots, g_d(x(t), t), 0 \}$$

subject to $x(t) = Ax(t) + Bu(t), \quad t \in [0, T], \quad u \in U, \quad x(0) = x_0, \quad x(T) = x_T.$

Note that after discretization in $t$ this problem becomes a standard minmax problem with convex constraints, which can be solved via a wide variety of existing numerical methods of minimax optimization or nonsmooth convex optimization in the case when the function $(x, u) \mapsto \theta(x, u, t)$ is convex. Our aim is to show that this fixed-endpoint problem is equivalent to problem (47), provided Slater's condition holds true, that is, provided there exists a control input $\hat{u} \in U$ such that for the corresponding solution $\hat{x}$ (see equality (46)) one has $\hat{x}(T) = x_{T}$ and $g_j(\hat{x}(t), t) < 0$ for all $t \in [0, T]$ and $j \in J$.

**Theorem 14.** Let the following assumptions be valid:

1. $\theta$ is continuous, and for any $R > 0$ there exist $C_R > 0$ and an a.e. nonnegative function $\omega_R \in L^1(0, T)$ such that $|\theta(x, u, t)| \leq C_R \|u\|^2_{\mathcal{U}} + \omega_R(t)$ for all $x \in \mathcal{X}$, $u \in \mathcal{U}$, and $t \in (0, T)$ such that $\|x\|_{\mathcal{X}} \leq R$;
2. either the set $U$ is bounded in $L^2((0, T); \mathcal{U})$ or there exist $C_1 > 0$ and $\omega \in L^1(0, T)$ such that $\theta(x, u, t) \geq C_1 \|u\|^2_{\mathcal{U}} + \omega(t)$ for all $x \in \mathcal{X}$, $u \in \mathcal{U}$, and for a.e. $t \in (0, T)$;
3. $\theta$ is differentiable in $x$ and $u$, the functions $\nabla_x \theta$ and $\nabla_u \theta$ are continuous, and for any $R > 0$ there exist $C_R > 0$, and a.e. nonnegative functions $\omega_R \in L^1(0, T)$ and $\eta_R \in L^2(0, T)$ such that

$$\|\nabla_x \theta(x, u, t)\|_{\mathcal{X}} \leq C_R \|u\|^2_{\mathcal{U}} + \omega_R(t), \quad \|\nabla_u \theta(x, u, t)\|_{\mathcal{U}} \leq C_R \|u\|_{\mathcal{U}} + \eta_R(t),$$

for all $x \in \mathcal{X}$, $u \in \mathcal{U}$, and $t \in (0, T)$ such that $\|x\|_{\mathcal{X}} \leq R$;
4. there exists a globally optimal solution of problem (47);
5. the functions $g_j(x, t), j \in J$, are convex in $x$, continuous jointly in $x$ and $t$, and Slater’s condition holds true.

Then for all $c \in \mathbb{R}$ there exists $\lambda^*(c) \geq 0$ such that for any $\lambda \geq \lambda^*(c)$ the penalty function $\Phi_A$ for problem (47) is completely exact on the set $S_A(c)$.

**Proof.** Almost literally repeating the first part of the proof of Theorem 8 one obtains that the assumptions on the function $\theta$ and its derivatives ensure that the functional $I(x, u)$ is Lipschitz continuous on any bounded subset of $X$, the set $S_A(c)$ is bounded in $X$ for all $c \in \mathbb{R}$ and $\lambda \geq 0$, and the penalty function $\Phi_A$ is bounded below on $A$. In addition, the set $A$ is closed by virtue of the closedness of the set $U$ and the fact that the input map $F_i$ is a bounded linear operator from $L^2((0, T); \mathcal{U})$ to $\mathcal{X}$ (see (46)). Finally, the mappings $x \mapsto g_j(x(\cdot), \cdot), j \in J$, and the penalty term $\varphi$ are continuous by Proposition 2 and Corollary 2 (see Appendix B).

Fix any $\lambda \geq 0$ and $c \in \mathbb{R}$. By applying Theorem 1 one gets that it remains to verify that there exists $a > 0$ such that

$$\varphi_A(x, u) \leq -a$$

for any $(x, u) \in S_A(c) \cap (\Omega_A \setminus \Omega)$ (i.e., $(x, u) \in S_A(c)$ and $0 < \varphi(x, u) < \delta$).

Fix any $\delta > 0$ and $(x, u) \in S_A(c) \cap (\Omega_A \setminus \Omega)$, and let a pair $(\hat{x}, \hat{u})$ be from Slater’s condition. Denote $\sigma = \|\hat{x} - x\|_{\mathcal{X}} = \|\hat{x} - x\|_{C([0,T];\mathcal{X})} + \|\hat{u} - u\|_{L^2((0,T);\mathcal{U})}$. Note that there exists $R > 0$ (independent of $(x, u)$) such that $\sigma \leq R$, since the set $S_A(c)$ is bounded. Furthermore, $\sigma > 0$, since $(\hat{x}, \hat{u}) \in \Omega$ by definition.

Define $\Delta x = (\hat{x} - x)/\sigma$ and $\Delta u = (\hat{u} - u)/\sigma$. Observe that $\|\Delta x, \Delta u\|_X = 1$, and $(x + a \Delta x, u + a \Delta u) \in A$ for any $a \in [0, \sigma]$ due to the convexity of the set $U$, and the linearity of the set $x_A = \mathcal{A}x + Bu$. Define $J(x) = \{ j \in J| \exists t \in [0, T] : g_j(x(t), t) = \varphi(x, u) \}$. Note that the supremum in the definition of $\varphi(x, u)$ is attained due to the continuity of $g_j$. Thus, the set $J(x)$ is nonempty. Observe also that there exists $\epsilon > 0$ such that $g_j(x(t), t) < \varphi(x, u) - \epsilon$ for any $t \in [0, T]$ and $j \not\in J(x)$, since for any fixed $j$ and $t \in [0, T]$ one has $g_j(x(t), t) < \varphi(x, u)$, and the function $g_j(x(\cdot), \cdot)$ is continuous. Moreover, by applying the continuity of the functions $x \mapsto g_j(x(\cdot), \cdot)$ and $\varphi$ one obtains that there exists $a_1 > 0$ such that for any $a \in [0, a_1]$ one has $g_j(x(t) + a \Delta x(t), t) < \varphi(x, u) - \epsilon/2$ for all $t \in [0, T]$ and $j \not\in J(x)$, while $\varphi(x + a \Delta x, u + a \Delta u) > \varphi(x, u) - \epsilon/2$. Therefore, for any $a \in [0, a_1]$ one has

$$\varphi(x + a \Delta x, u + a \Delta u) = \max_{j \in J(x)} \max_{t \in [0,T]} \{ g_j(x(t) + a \Delta x(t), t), 0 \}.$$  

(48)
For any \( j \in J(x) \) define \( T_j(x) = \{ t \in [0, T] | g_j(x(t), t) \geq \varphi(x, u)/2 \} \). From the definition of \( J(x) \) and the fact that \( \varphi(x, u) > 0 \) it follows that the sets \( T_j(x) \) are nonempty, and for any \( j \in J(x) \) there exists \( t_j \in T_j(x) \) such that \( g_j(x(t_j), t_j) = \varphi(x, u) \). By applying the continuity of the mappings \( x \mapsto g_j(x, \cdot) \) once again one gets that there exists \( \alpha_2 > 0 \) such that for any \( j \in J(x) \) and \( \alpha \in [0, \alpha_2] \) one has \( g_j(x(t) + \alpha \Delta x(t), t) > 0 \) for any \( t \in T_j(x) \), \( g_j(x(t_j) + \alpha \Delta x(t_j), t_j) > \varphi(x, u)/3 \), while \( g_j(x(t) + \alpha \Delta x(t), t) < 2\varphi(x, u)/3 \) for any \( t \notin T_j(x) \). Hence and from equality (48) it follows that

\[
\varphi(x + \alpha \Delta x, u + \alpha \Delta u) = \max_{j \in J(x)} \sup_{t \in T_j(x)} g_j(x(t) + \alpha \Delta x(t), t) \quad \forall \alpha \in [0, \min\{\alpha_1, \alpha_2\}]
\]  

(49)

With the use of the convexity of the functions \( g_j(x, t) \) in \( x \) one obtains that

\[
g_j(x(t) + \alpha \Delta x(t), t) \leq \frac{\alpha}{\sigma} g_j(x(t), t) + \left(1 - \frac{\alpha}{\sigma}\right) g_j(x(t), t) \quad \forall \alpha \in [0, \sigma],
\]

for any \( j \in J(x) \) and \( t \in T_j(x) \), which implies that

\[
g_j(x(t) + \alpha \Delta x(t), t) \leq \frac{\alpha \eta}{\sigma} + \varphi(x, u) \quad \forall \alpha \in [0, \sigma],
\]

(50)

where \( \eta = \max_{j \in [0, T]} \{g_j(x(t), t)|j \in J\} < 0 \) due to Slater’s condition. Taking the supremum over all \( t \in T_j(x) \), and then over all \( j \in J(x) \), and utilizing equality (49) one obtains that

\[
\varphi(x + \alpha \Delta x, u + \alpha \Delta u) - \varphi(x, u) \leq \frac{\alpha \eta}{\sigma} \quad \forall \alpha \in [0, \min\{\alpha_1, \alpha_2, \sigma\}].
\]

Dividing this inequality by \( \alpha \) and passing to the limit superior as \( \alpha \to +0 \) one finally gets that

\[
\varphi_A^+(x, u) \leq \limsup_{\alpha \to +0} \frac{\varphi(x + \alpha \Delta x, u + \alpha \Delta u)}{\alpha} \leq \frac{\eta}{\sigma} \leq \frac{\eta}{R} < 0
\]

where both \( \eta \) and \( R \) are independent of \((x, u) \in S_j(c) \cap (\Omega_\delta \setminus \Omega)\). Thus, \( \varphi_A^+(x, u) \leq -\eta/R \) for any \((x, u) \in S_j(c) \cap (\Omega_\delta \setminus \Omega) \), and the proof is complete.

**Corollary 1.** Let all assumptions of Theorem 14 be valid. Suppose also that either the set \( U \) is bounded in \( L^2((0, T); \mathcal{U}) \) or the function \((x, u) \mapsto \vartheta(x, u, t)\) is convex for all \( t \in [0, T] \). Then the penalty function \( \Phi_A \) for problem (47) is completely exact on \( A \).

**Proof.** If the set \( U \) is bounded, then by the first part of the proof of Theorem 9 the set \( A \) is bounded in \( X \). Therefore, arguing in the same way as in the proof of Theorem 14, but replacing the set \( S_j(c) \) with \( A \) and utilizing Theorem 2 instead of Theorem 1 we arrive at the required result.

If the function \((x, u) \mapsto \vartheta(x, u, t)\) is convex, then, as was shown in the proof of Theorem 9, the penalty function \( \Phi_A \) for problem (47) is completely exact on \( A \) if and only if it is globally exact. It remains to note that its global exactness follows from Theorem 14.

**Remark 16.** Theorem 14 and Corollary 1 can be easily extended to the case of problems with inequality constraints of the form \( g_j(x, u) \leq 0 \), where \( g_j : C([0, T], \mathbb{R}) \times L^2((0, T); \mathcal{U}) \to \mathbb{R} \) are continuous convex functions. In particular, one can consider the integral constraint \( \|u\|_{L^2((0, T), \mathcal{U})} \leq C \) for some \( C > 0 \). In this case one can define \( \varphi(x, u) = \max\{g_1(x, u), \ldots, g_\delta(x, u), 0\} \), while Slater’s condition takes the form: there exists a feasible point \((\hat{x}, \hat{u})\) such that \( g_j(\hat{x}, \hat{u}) < 0 \) for all \( j \).

**Remark 17.** It should be noted that Theorem 14 and Corollary 1 can be applied to problems with distributed \( L^\infty \) state constraints. For instance, suppose that \( \mathcal{H} = W^{1,2}(0, 1) \), and let the constraints have the form \( b_1 \leq x(t, r) \leq b_2 \) for all \( t \in [0, T] \) and a.e. \( r \in (0, 1) \). Then one can define \( g_1(x(t)) = \esssup_{r \in (0, 1)} x(t, r) - b_2 \) and \( g_2(x(t)) = \esssup_{r \in (0, 1)} -(x(t, r) + b_1) \) and consider the state constraints \( g_1(x(\cdot)) \leq 0 \) and \( g_2(x(\cdot)) \leq 0 \). One can easily check that both functions \( g_1 \) and \( g_2 \) are convex and continuous. Slater’s condition in this case takes the form: there exists \((\hat{x}, \hat{u}) \in \Omega \) such that \( b_1 + \epsilon \leq \hat{x}(t, r) \leq b_2 - \epsilon \) for some \( \epsilon > 0 \), for all \( t \in [0, T] \), and a.e. \( r \in (0, 1) \).

Observe that Slater’s condition imposes some restriction on the initial and final states. Namely, Slater’s conditions implies that \( g_j(x_0, 0) < 0 \) and \( g_j(x_T, T) < 0 \) for all \( j \in J \) (in the general case only the inequalities \( g_j(x_0, 0) \leq 0 \) and \( g_j(x_T, T) \leq 0 \)
hold true). Let us give an example demonstrating that in the case when the strict inequalities are not satisfied, the penalty function \( \Phi_j \) for problem (47) need not be exact. For the sake of simplicity, we consider a free-endpoint finite dimensional problem. As one can readily verify, Theorem 14 remains valid in the case of free-endpoint problems.

**Example 4.** Let \( d = 2, m = 1, p = q = 2 \). Define \( U = \{ u \in L^2(0, T) | u(t) \geq 0 \text{ for a.e.} t \in (0, T), ||u||_2 \leq 1 \} \), and consider the following free-endpoint optimal control problem with the state inequality constraint:

\[
\min_{u} I(u) = - \int_0^T u(t)^2 dt \quad \text{s.t.} \quad \begin{cases} x_1^2 = 1, \\ x_2 = u \\ t \in [0, T], \quad x(0) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \quad u \in U, \quad g(x(t)) \leq 0, \end{cases}
\]

where \( g(x^1, x^2) = x^2. \) The only feasible point of this problem is \((x^*, u^*)\) with \( x^*(t) \equiv (t, 0)^T \) and \( u^*(t) = 0 \) for a.e. \( t \in [0, T]. \) Thus, \((x^*, u^*)\) is a globally optimal solution of this problem. Note also that the function \( \theta(x, u, t) = -(u)^2 \) satisfies the assumptions of Theorem 14. Furthermore, in this case the set \( A \) is obviously bounded in \( X, \) and the penalty function \( \Phi_j(x, u) = I(u) + \lambda \varphi(x) \) with \( \varphi(x) = \max_{t \in [0, T]} \{g(x(t)), 0\} \) is bounded below on \( A. \)

Observe that \( g(x(0)) = 0, \) which implies that Slater's condition does not hold true. Let us check that the penalty function \( \Phi_j \) is not exact. Arguing by reductio ad absurdum, suppose that \( \Phi_j \) is globally exact. Then there exists \( \lambda^* \geq 0 \) such that for any \( \lambda \geq \lambda^* \) and \( (x, u) \in A \) one has \( \Phi_j(x, u) \geq \Phi_j(x^*, u^*). \) For any \( n \in \mathbb{N} \) define \( u_n(t) = n, \) if \( t \in [0, 1/n^2], \) and \( u_n(t) = 0, \) if \( t > 1/n^2. \) Then \( ||u_n||_2 = 1, \) and \((x_n, u_n) \in A, \) where \( x_n(t) = (t, \min\{nt, 1/n\})^T \) is the corresponding trajectory of the system. Observe that \( I(u_n) = -1 \) and \( \varphi(x_n) = 1/n \) for any \( n \in \mathbb{N}. \) Consequently, \( \Phi_j(x_n, u_n) < 0 = \Phi_j(x^*, u^*) \) for any sufficiently large \( n \in \mathbb{N}, \) which contradicts our assumption. Thus, the penalty function \( \Phi_j \) is not globally exact. Moreover, one can easily see that the penalty function \( \Phi_j(x) = I(u) + \lambda \varphi(x) \) is not globally exact for any penalty term \( \varphi \) that is continuous with respect to the uniform metric.

In the case when not only the state constraints but also the cost functional \( I \) is convex, one can utilize the convexity of the problem to prove that the exact \( L^p \)-penalization of state constraints with any \( 1 \leq p < +\infty \) is possible, provided Lagrange multipliers corresponding to the state constraints are sufficiently regular. Indeed, let \((x^*, u^*)\) be a globally optimal solution of problem (47), and let \( E(x^*, u^*) \subset \mathbb{R} \times (C[0, T])^j \) be a set of all those vectors \((y_0, y_1, \ldots, y_j)\) for which one can find \((x, u) \in A \) such that \( I(x, u) - I(x^*, u^*) < y_0 \) and \( g(x(t), t) \leq y_j(t) \) for all \( t \in [0, T] \) and \( j \in J. \) The set \( E(x^*, u^*) \) has nonempty interior due to the fact that \((0, +\infty) \times (C_+[0, T])^j \subset E(x^*, u^*) \) (put \((x, u) = (x^*, u^*), \) where \( C_+[0, T] \) is the cone of nonnegative functions. Observe also that \( 0 \not\in E(x^*, u^*), \) since otherwise one can find a feasible point \((x, u) \) of problem (47) such that \( I(x, u) < I(x^*, u^*), \) which contradicts the definition of \((x^*, u^*). \) Furthermore, with the use of the convexity of \( I \) and \( g_j \) one can easily check that the set \( E(x^*, u^*) \) is convex. Therefore, by applying the separation theorem (eg. theorem V.2.8 of Reference 64) one obtains that there exist \( \mu_0 \in \mathbb{R} \) and continuous linear functionals \( \psi_j \) on \( C[0, T], \) \( j \in J, \) not all zero, such that \( \mu_0 \psi_0 + \sum_{j=1}^l \psi_j(y_j) \geq 0 \) for all \((y_0, y_1, \ldots, y_l) \in E(x^*, u^*). \) Taking into account the fact that \((0, +\infty) \times (C_+[0, T])^j \subset E(x^*, u^*) \) one obtains that \( \mu_0 \geq 0 \) and \( \psi_j(y_j) \geq 0 \) for any \( y \in C_+[0, T] \) and \( j \in J. \) Consequently, utilizing the Riesz-Markov-Kakutani representation theorem (see theorem IV.6.3 of Reference 64) and bearing in mind the definition of \( E(x^*, u^*) \) one gets that there exist regular Borel measures \( \mu_j \) on \( [0, T], \) \( j \in J, \) such that

\[
\mu_0 I(x, u) + \sum_{j=1}^l \int_{[0, T]} g_j(x(t), t) d\mu_j(t) \geq \mu_0 I(x^*, u^*) \quad \forall (x, u) \in A. \tag{51}
\]

If Slater's condition holds true, then obviously \( \mu_0 > 0, \) and we suppose that \( \mu_0 = 1. \) Any collection \((\mu_1, \ldots, \mu_l)\) of regular Borel measures on \([0, T]\) satisfying inequality (51) with \( \mu_0 = 1 \) is called Lagrange multipliers corresponding to the state constraints of problem (47). Let us note that one has to suppose that Lagrange multipliers are Borel measures, since if one replaces \( C[0, T] \) in the definition of \( E(x^*, u^*) \) with \( L^p([0, T]), 1 \leq p < +\infty, \) then the set \( E(x^*, u^*), \) in the general case, has empty interior, which makes the separation theorem inapplicable.

**Theorem 15.** Let assumptions 1, 4, and 5 of Theorem 14 be valid, and let the function \((x, u) \mapsto \theta(x, u, t)\) be convex for any \( t \in [0, T]. \) Suppose, in addition, that for some \( 1 \leq p < +\infty \) there exist Lagrange multipliers \((\mu_1, \ldots, \mu_l) \) such that the Borel measures \( \mu_j \) are absolutely continuous with respect to the Lebesgue measure, and their Radon-Nikodym derivatives belong to \( L^p([0, T]). \) Then there exists \( \lambda^* \geq 0 \) such that for any \( \lambda \geq \lambda^* \) the penalty function

\[
\Phi_j(x, u) = I(x, u) + \lambda \sum_{j=1}^l \left( \int_0^T \max\{g_j(x(t), t), 0\}^p dt \right)^{1/p},
\]
for problem (47) is completely exact on A.

Proof. Let \((x^*, u^*)\) be a globally optimal solution of problem (47), and \(h_j\) be the Radon-Nikodym derivative of \(\mu_j\) with respect to the Lebesgue measure, \(j \in J\). Denote \(\lambda_0 = \max_{j \in J} \| h_j \|_{p'}\). Then by applying inequality (51) and Hölder’s inequality one obtains that

\[
\Phi_\lambda(x^*, u^*) = I(x^*, u^*) \leq I(x, u) + \sum_{j=1}^l \int_0^T g_j(x(t), t) h_j(t) \, dt \leq I(x, u) + \sum_{j=1}^l \int_0^T \max\{g_j(x(t), t), 0\} |h_j(t)| \, dt
\]

\[
\leq I(x, u) + \lambda_0 \sum_{j=1}^l \left( \int_0^T \max\{g_j(x(t), t), 0\}^p \, dt \right)^{1/p} \leq \Phi_\lambda(x, u)
\]

for any \((x, u) \in A\) and \(\lambda \geq \lambda_0\). Hence, as is easy to see, the penalty function \(\Phi_\lambda\) is globally exact. Now, bearing in mind the convexity of \(\Phi_\lambda\) and arguing in the same way as in the proof of Theorem 9 one arrives at the required result. 

4.3 | Nonlinear systems: Local exactness

Let us now turn to general nonlinear optimal control problems with state constraints of the form:

\[
\min I(x, u) = \int_0^T \theta(x(t), u(t), t) \, dt \quad \text{subject to} \quad \dot{x}(t) = f(x(t), u(t), t), \quad t \in [0, T],
\]

\[
x(0) = x_0, x(T) = x_T, u \in U, g_j(x(t), t) \leq 0 \quad \forall t \in [0, T], j \in J. \tag{52}
\]

Here \(\theta : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}, f : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}^d, g_j : \mathbb{R}^d \times [0, T] \to \mathbb{R}, j \in J = \{1, \ldots, l\},\) are given functions, \(x_0, x_T \in \mathbb{R}^d,\) and \(T > 0\) are fixed, \(x \in W_{1,p}^d(0, T),\) and \(U \subseteq L_q^m(0, T)\) is a closed set of admissible control inputs.

Let \(X = W_{1,p}^d(0, T) \times L_q^m(0, T)\). Define \(M = \{(x, u) \in X|g_j(x(t), t) \leq 0 \text{ for all } t \in [0, T], j \in J\}\) and

\[
A = \{(x, u) \in X|u \in U, x(0) = x_0, x(T) = x_T, \dot{x}(t) = f(x(t), u(t), t) \text{ for a.e. } t \in (0, T)\}.
\]

Then problem (52) can be rewritten as the problem of minimizing \(I(x, u)\) over the set \(M \cap A\). Define \(\varphi(x, u) = \sup_{t \in [0, T]} \{g_1(x(t), t), \ldots, g_l(x(t), t), 0\}\). Then \(M = \{(x, u) \in X|\varphi(x, u) = 0\}\), and one can consider the penalized problem of minimizing the penalty function \(\Phi_\lambda\) over the set \(A\). Our first goal is to obtain simple sufficient conditions for the local exactness of the penalty function \(\Phi_\lambda\).

**Theorem 16.** Let \(U = L_q^m(0, T), q \geq p,\) and \((x^*, u^*)\) be a locally optimal solution of problem (52). Let also the following assumptions be valid:

1. \(\theta\) and \(f\) are continuous, differentiable in \(x\) in \(u\), and the functions \(\nabla_x \theta, \nabla_u \theta, \nabla_x f, \) and \(\nabla_u f\) are continuous;
2. either \(q = +\infty\) or \(\theta\) and \(\nabla_x \theta\) satisfy the growth condition of order \((q, 1), \) \(\nabla_u \theta\) satisfies the growth condition of order \((q - 1, q'),\) \(f\) and \(\nabla_x f\) satisfy the growth condition of order \((q/p, p),\) and \(\nabla_u f\) satisfies the growth condition of order \((q/p, s)\) with \(s = qp/(q - p)\) in the case \(q > p,\) and \(\nabla_u f\) does not depend on \(u\) in the case \(q = p;\)
3. \(g_j, j \in J,\) are continuous, differentiable in \(x\), and the functions \(\nabla_x g_j, j \in J,\) are continuous.

Suppose finally that the linearized system

\[
\dot{h}(t) = A(t)h(t) + B(t)u(t), \quad A(t) = \nabla_x f(x^*(t), u^*(t), t), \quad B(t) = \nabla_u f(x^*(t), u^*(t), t), \tag{53}
\]

is completely controllable using \(L^q\)-controls in time \(T, A(\cdot) \in L_{\text{ad}}^{\infty}(0, T),\) and there exists \(v \in L_q^0(0, T)\) such that the corresponding solution \(h\) of Equation (53) with \(h(0) = 0\) satisfies the condition \(h(T) = 0,\) and for any \(j \in J\) one has

\[
\langle \nabla_x g_j(x^*(t), t), h(t) \rangle < 0 \quad \forall t \in [0, T] : g_j(x^*(t), t) = 0. \tag{54}
\]
Then the penalty function \( \Phi_\lambda \) for problem (52) is locally exact at \((x^*, u^*)\).

Proof. By [Reference 45, propositions 3 and 4] the growth conditions on the function \( \theta \) and its derivatives ensure that the functional \( I \) is Lipschitz continuous in any bounded neighbourhood of \((x^*, u^*)\). Introduce a nonlinear operator \( F : X \to L^2_0(0, T) \times \mathbb{R}^d \times (C[0, T])^l \) and a closed convex set \( K \subset L^2_0(0, T) \times \mathbb{R}^d \times (C[0, T])^l \) as follows:

\[
F(x, u) = \left( \frac{\dot{x}(\cdot) - f(x(\cdot), u(\cdot), \cdot)}{x(T)}, \frac{g(x(\cdot), \cdot)}{x_T} \right), \quad K = \left( \frac{0}{x_T} \right)_{(C_-, [0, T])^l}.
\]

Here \((C[0, T])^l\) is the Cartesian product of \( l \) copies of the space \( C[0, T] \) of real-valued continuous functions defined on \([0, T]\) endowed with the uniform norm \( g(\cdot) = (g_1(\cdot), \ldots, g_l(\cdot))^T \), and \( C_-, [0, T] \subset C[0, T] \) is the cone of nonpositive functions. Our aim is to apply Theorem 4 with \( C = \{(x, u) \in X | x(0) = x_0\} \) to the operator \( F \). Then one obtains that there exists \( a > 0 \) such that \( \text{dist}(F(x, u), K) \geq \text{adist}((x, u), F^{-1}(K) \cap C) \) for any \((x, u) \in C \) in a neighborhood of \((x^*, u^*)\). Consequently, taking into account the facts that the set \( F^{-1}(K) \cap C \) coincides with the feasible region of problem (52), and

\[
\text{dist}(F(x, u), K) = \sum_{j=1}^l \max_{t \in [0, T]} \| g(x(t), t, 0) \| \leq l \varphi(x, u) \quad \forall (x, u) \in A
\]

one obtains that \( \varphi(x, u) \geq (a/l) \text{dist}((x, u), \Omega) \) for any \((x, u) \in A \) in a neighborhood of \((x^*, u^*)\). Hence by applying Theorem 3 we arrive at the required result.

By Theorems 20 and 19 (see Appendix B) the growth conditions on the function \( f \) and its derivative guarantee that the mapping \( F \) is strictly differentiable at \((x^*, u^*)\), and its Fréchet derivative at this point has the form

\[
DF(x^*, u^*)[h, v] = \left( \frac{\dot{h}(\cdot) - A(\cdot)h(\cdot) - B(\cdot)v(\cdot)}{h(T)} \right)_{\nabla g(x^*(\cdot), \cdot)h(\cdot)},
\]

where \( A(\cdot) \) and \( B(\cdot) \) are defined in equalities (53). Observe also that \( C - (x^*, u^*) = \{(h, v) \in X | h(0) = 0\} \), since \( x^*(0) = x_0 \).

Consequently, the regularity condition (4) from Theorem 4 takes the form \( 0 \in \text{core}K(x^*, u^*) \) with

\[
K(x^*, u^*) = \left\{ \left( \frac{\dot{h}(\cdot) - A(\cdot)h(\cdot) - B(\cdot)v(\cdot)}{h(T)} \right)_{\nabla g(x^*(\cdot), \cdot)h(\cdot)} \right\} \quad \forall (h, v) \in X, h(0) = 0 \).
\]

Let us check that this condition is satisfied. Indeed, define \( X_0 = \{(h, v) \in X | h(0) = 0\} \), and introduce the linear operator \( E : X_0 \to L^2_0(0, T), E(h, v) = \dot{h}(\cdot) - A(\cdot)h(\cdot) - B(\cdot)v(\cdot) \). This operator is surjective and bounded, since the linear differential equation \( E(h, 0) = w \) has a unique solution for any \( w \in L^2_0(0, T) \) by theorem 1.1.3 of Reference 56, and by Hölder’s inequality one has

\[
\|E(h, v)\|_p \leq \|h\| + \|A(\cdot)\|_\infty \|h\| + \|B(\cdot)\|_s \|v\|_q \leq C\|\langle h, v\rangle\|_X,
\]

where \( C = \max \{1 + \|A(\cdot)\|_\infty, \|B(\cdot)\|_s\} \), and \( s = +\infty \) in the case \( p = s \) (note that \( \|B(\cdot)\|_s \) is finite due to the growth condition on \( V_{af} \); see the proof of Theorem 20).

Consequently, by the open mapping theorem there exists \( \eta_1 > 0 \) such that

\[
\text{dist}(E(h, v), E^{-1}(w)) \leq \eta_1 \|w - E(h, v)\|_p \quad \forall (h, v) \in X_0, w \in L^2_0(0, T).
\]

(see formula (0.2) in Reference 61). Taking \((h, v) = (0, 0)\) in the previous inequality one gets that for any \( w \in L^2_0(0, T) \) there exists \( v_1 \in L^p_0(0, T) \) such that the solution \( h_1 \) of the perturbed linearized equation

\[
\dot{h}_1(t) = A(t)h_1(t) + B(t)v_1(t) + w(t), \quad h_1(0) = 0, \quad t \in [0, T],
\]

satisfies the inequality \( \|(h_1, v_1)\|_X \leq (\eta_1 + 1)\|w\|_p \).

Introduce the operator \( T : L^p_0(0, T) \to \mathbb{R}^d, Tv = h(T) \), where \( h \) is a solution of Equation (53) with the initial condition \( h(0) = 0 \). Arguing in a similar way to the proof of Theorem 6 (recall that \( A(\cdot) \in L^{\text{loc}}_\infty(0, T) \) one can check that the operator
$\mathcal{T}$ is bounded, while the complete controllability assumption implies that it is surjective. Hence by the open mapping theorem there exists $\eta_2 > 0$ such that

$$\text{dist}(v, \mathcal{T}^{-1}(h_\mathcal{T})) \leq \eta_2 |h_\mathcal{T} - \mathcal{T}(v)| \quad \forall v \in L^m_{q}(0, T), h_\mathcal{T} \in \mathbb{R}^d.$$ 

Taking $v = 0$ one obtains that for any $h_\mathcal{T} \in \mathbb{R}^d$ there exists $v_2 \in L^m_{q}(0, T)$ such that $\|v_2\|_q \leq (\eta_2 + 1)|h_\mathcal{T}|$, where $h_\mathcal{T}$ is a solution of Equation (53) with $v = v_2$ satisfying the conditions $h_2(0) = 0$ and $h_2(T) = h_\mathcal{T}$. Furthermore, by applying the Grönewall-Bellman and Hölder’s inequalities, and the fact that

$$|h_2(t)| \leq \|B(\cdot)\|_s \|v_2\|_q + \|A(\cdot)\|_{\infty} \int_0^t |h_2(r)| \, dr \quad \forall t \in [0, T),$$

one can verify that $\|h_2\|_{1,p} \leq L \|v\|_q$ for some $L > 0$ (see Remark 9). Therefore there exists $\eta_3 > 0$ such that for any $h_\mathcal{T} \in \mathbb{R}^d$ one can find $v_2 \in L^m_{q}(0, T)$ satisfying the inequality $\|(h_2, v_2)\|_X \leq \eta_3 |h_\mathcal{T}|$, where $h_2$ is a solution of Equation (53) with $v = v_2$ such that $h_2(0) = 0$ and $h_2(T) = h_\mathcal{T}$.

Choose $r_1, r_2 > 0$, $w \in L^d_{q}(0, T)$ with $\|w\|_p \leq r_1$, and $h_\mathcal{T} \in \mathbb{R}^d$ with $|h_\mathcal{T}| \leq r_2$. As we proved earlier, there exists $(h_1, v_1) \in X$ satisfying Equation (56) and such that $\|(h_1, v_1)\|_X \leq (\eta_1 + 1) \|w\|_p \leq (\eta_1 + 1)r_1$. By inequality (5) one has $\|h_1\|_{\infty} \leq C_p \|h_1\|_{1,p} \leq C_p (\eta_1 + 1)r_1$, for some $C_p > 0$ independent of $h_1$. Furthermore, there exists $(h_2, v_2) \in X_0$ satisfying Equation (53), and such that $h(0) = h_2(0), h(T) = h_2(T)$, and $\|(h_2, v_2)\|_X \leq \eta_3 |h_\mathcal{T} - h_2(T)|$. Hence, in particular, one gets that

$$\|h_2\|_{\infty} \leq C_p \eta_3 |h_\mathcal{T} - h_2(T)| \leq C_p \eta_3 |h_\mathcal{T}| + C_p \eta_3 \|h_1\|_{\infty} \leq C_p \eta_3 r_2 + C_p^2 \eta_3 (\eta_1 + 1)r_1.$$ 

Finally, by our assumption there exists $(h_1, v_1) \in X_0$ satisfying Equation (53) and inequality (54) and such that $h_3(T) = 0$. For any $j \in J$ denote $T_j = \{t \in [0, T]: g(x^*(t), t) = 0\}$. Clearly, the sets $T_j$ are compact, which implies that for any $j \in J$ there exists $\gamma_j > 0$ such that $\langle \nabla_x g_j(x^*(t), t), h_3(t) \rangle \leq -\gamma_j$ for all $t \in T_j$ due to inequality (54) and the continuity of the functions $\nabla_x g_j, x^*$, and $h_3$. With the use of the compactness of the sets $T_j$ one obtains that for any $j \in J$ there exists a set $\mathcal{O}_j \subset [0, T]$ such that $\mathcal{O}_j$ is open in $[0, T]$, $T_j \subset \mathcal{O}_j$, and $\langle \nabla_x g_j(x^*(t), t), h_3(t) \rangle \leq -\gamma_j/2$ for all $t \in \mathcal{O}_j$. On the other hand, for any $j \in J$ there exists $\gamma_j > 0$ such that $g_j(x^*(t), t) \leq -\gamma_j$ for any $t \in [0, T] \setminus \mathcal{O}_j$, since by definition $g_j(x^*(t), t) < 0$ for all $t \in T_j$ and the set $[0, T] \setminus \mathcal{O}_j$ is compact.

Note that for any $a > 0$ the pair $(ah_3, av_3)$ belongs to $X_0$ and satisfies Equation (53) and the equality $ah_3(T) = 0$. Choosing a sufficiently small $a > 0$ one can suppose that $\langle \nabla_x g_j(x^*(t), t), ah_3(t) \rangle < \gamma_j$ for all $t \in [0, T] \setminus \mathcal{O}_j$ and $j \in J$, while for any $t \in \mathcal{O}_j$ one has $\langle \nabla_x g_j(x^*(t), t), ah_3(t) \rangle \leq -a \gamma_j/2$. Thus, replacing $(h_1, v_3)$ with $(ah_3, av_3)$, where $a > 0$ is small enough, one can suppose that

$$\langle \nabla_x g_j(x^*(t), t), h_3(t) \rangle + g_j(x^*(t), t) < 0 \quad \forall t \in [0, T] \quad \forall j \in J.$$ 

With the use of the continuity of $g_j, \nabla_x g_j, x^*$, and $h_3$ one obtains that there exists $r_3 > 0$ such that

$$\langle \nabla_x g_j(x^*(t), t), h_3(t) \rangle + g_j(x^*(t), t) \leq -r_3 \quad \forall t \in [0, T] \quad \forall j \in J.$$ 

Choosing $r_1 > 0$ and $r_2 > 0$ sufficiently small one gets that for any $j \in J$

$$\langle \nabla_x g_j(x^*(t), t), h_1(t) + h_2(t) + h_3(t) \rangle + g_j(x^*(t), t) \leq -\frac{r_3}{2} \quad \forall t \in [0, T],$$

since $\|h_1\|_{\infty}$ and $\|h_2\|_{\infty}$ can be made arbitrarily small by a proper choice of $r_1$ and $r_2$.

Define $h = h_1 + h_2 + h_3$ and $v = v_1 + v_2 + v_3$. Then $(h, v) \in X$, $h(0) = 0$, $h(T) = h_\mathcal{T}$, $(h, v)$ satisfies Equation (56), and inequality (57) holds true. Therefore, $(w, h_\mathcal{T}, y)^T \in K(x^*, u^*)$ for any $y = (y_1, \ldots, y_l)^T \in (\mathbb{C}[0, T])^l$ such that $\|y\|_{\infty} \leq r_3/2$ for all $j \in J$ (see equality (55)). In other words, $B(0, r_1) \times B(0, r_2) \times B(0, r_3/2) \subset K(x^*, u^*)$, that is, $0 \in \text{int} K(x^*, u^*)$, and the proof is complete.

Remark 18. (i) From inequality (54) it follows that $g_j(x_0, 0) < 0$ and $g_j(x_T, T) < 0$ for all $j \in J$, since $h(0) = h(T) = 0$ in inequality (54). Furthermore, the assumption that there exists a control input $v$ such that the corresponding solution $h$ of
the linearized system satisfies inequality (54) is, roughly speaking, equivalent to the assumption that there exists \((h, v) \in X\) such that for any sufficiently small \(\alpha \geq 0\) the point \((x_\alpha, u_\alpha) = (x + ah + r_1(a), u + av + r_2(a))\) is feasible for problem (52) for some \((r_1(a), r_2(a)) \in X\) such that \(\| (r_1(a), r_2(a)) \|_X / \alpha \to 0\) as \( \alpha \to +0 \), and \(g_j(x_\alpha(t), t) < 0\) for all \(t \in [0, T]\), \(j \in J\) and for any sufficiently small \(a\). Thus, assumption inequality (54) is, in essence, a local version of Slater's condition in the nonconvex case.

(ii) It should be noted that in the case when there is no terminal constraint the complete controllability assumption and the assumptions that the equality \(h(T) = 0\) holds true for \(h\) satisfying inequality (54) can be dropped from Theorem 16.

(iii) One might want to use the cone \(L'(0, T)_- = \{ x \in L'(0, T) | x(t) \leq 0 \text{ for a.e. } t \in (0, T) \}\) instead of \(C_-\{0, T\}\) in the proof of Theorem 16 in order verify the local exactness of the penalty function for problem (52) with the penalty term \(\varphi(x, u) = \sum_{i=1}^l \max_{0 \leq t \leq T} [g_i(x(t), t, 0)]^r dt, 1 \leq r < +\infty\). However, note that the cone \(L'(0, T)_-\) has empty algebraic interior, and for that reason an attempt to apply Theorem 4 leads to incompatible assumptions on the state constraints and the linearized system. Indeed, in this case the regularity condition (4) from Theorem 4 takes the form

\[
0 \in \text{core}\left\{ \left( \begin{array}{c} \dot{h}(\cdot) - A(\cdot)h(\cdot) - B(\cdot)v(\cdot) \\ \nabla g(x(\cdot), \cdot)h(\cdot) \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \text{ for } (h, v) \in X, h(0) = 0 \right\} .
\]

Hence, in particular, \(0 \in \text{core} K_0(x^\alpha)\), where \(K_0(x^\alpha)\) is the union of the cones \(\{ \nabla g(x^\alpha(\cdot), \cdot)h(\cdot) + g(x^\alpha(\cdot), \cdot) \} - (L'(0, T)_-)\) with \(h\) being a solution of Equation (53) for some \(v \in L_q^m(0, T)\) such that \(h(0) = h(T) = 0\). However, for the function \(y(t) = -t^{1/2}\) one obviously has \(y \in L'(0, T)\) and \(ay \not\in K_0(x^\alpha)\) for any \(a > 0\) (for the sake of simplicity we assume that \(l = 1\)), since the function \(\nabla g(x^\alpha(\cdot), \cdot)h(\cdot) + g(x^\alpha(\cdot), \cdot)\) is continuous and \(h(0) = 0\). Thus, \(0 \not\in \text{core} K_0(x^\alpha)\), and Theorem 4 cannot be applied.

4.4 Nonlinear systems: Complete exactness

Now we turn to the derivation of sufficient conditions for the complete exactness of the penalty function \(\Phi_3\) for problem (52). As in the case of terminal constraints, the derivation of easily verifiable conditions for exact penalization of pointwise state constraints does not seem possible in the nonlinear case. Therefore, our main goal, once again, is not to obtain easily verifiable conditions, but to understand what kind of general properties the nonlinear system and state constraints must possess to ensure exact penalization. To this end, we directly apply Theorem 1 in order to obtain general sufficient conditions for complete exactness. Then we consider a particular case in which one can obtain more readily verifiable sufficient conditions for the complete exactness of the penalty function.

Recall that the contingent cone to a subset \(C\) of a normed space \(Y\) at a point \(x \in C\), denoted by \(K_C(x)\), consists of all those vectors \(v \in Y\) for which there exist sequences \(\{v_n\} \subset Y\) and \(\{a_n\} \subset (0, +\infty)\) such that \(v_n \to v\) and \(a_n \to 0\) as \(n \to \infty\), and \(x + a_nv_n \in C\) for all \(n \in \mathbb{N}\). It should be noted that in the case \(U = L_q^m(0, T)\) by the Lyusternik-Graves theorem (eg, paper\(^{51}\)) for any \((x, u) \in A\) one has

\[
K_A(x, u) = \{ (h, v) \in X | h(t) = \nabla v_f(x(t), u(t), t)h(t) + \nabla u_f(x(t), u(t), t)v(t) \text{ for a.e. } t \in (0, T), h(0) = h(T) = 0 \},
\]

provided the linearized system is completely controllable, and the assumptions of Theorem 20 hold true. In the case when there is no terminal constraint, the complete controllability assumption and the condition \(h(T) = 0\) are redundant.

For any \((x, u) \in X\) denote \(\phi(x, t) = \max_{j \in J} [g_j(x(t), t)\}, 0\). Then \(\varphi(x, u) = \max_{t \in [0, T]} \phi(x(t), t)\). Define \(T(x) = \{ t \in [0, T] | \phi(x(t), t) = \varphi(x, u) \}\). Define \(\text{J}(x, t) = \{ j \in J | g_j(x(t), t) = \varphi(x, u) \}\). Clearly, \(\text{J}(x, t) \neq \emptyset\) iff \(t \in T(x)\).

Let \(I^*\) be the optimal value of problem (52). Note that the set \(\Omega_0 = \{ (x, u) \in A | \varphi(x, u) < \delta \}\) consists of all those trajectories \(x(\cdot)\) of the system \( \dot{x} = f(x, u, t), u \in U, x(0) = x_0, x(T) = x_T\), which satisfy the perturbed state constraints \(g_j(x(t), t) < \delta\) for all \(t \in [0, T]\) and \(j \in J\).

**Theorem 17.** Let the following assumptions be valid:

1. \(\theta\) is continuous and differentiable in \(x\) and \(u\), the functions \(g_j, j \in J\), are continuous, differentiable in \(x\), and the functions \(\nabla x \theta, \nabla u \theta, \nabla x g_j\), and \(f\) are continuous;
2. either \(q = +\infty\) or \(\theta\) and \(\nabla x \theta\) satisfy the growth condition of order \((q, 1), while \(\nabla u \theta\) satisfies the growth condition of order \((q - 1, q')\);
3. there exists a globally optimal solution of problem (52);
4. there exist $\lambda_0 > 0$, $c > T^*$, and $\delta > 0$ such that the set $S_{\lambda_0}(c) \cap \Omega_\delta$ is bounded in $X$, and the function $\Phi_{\lambda_0}$ is bounded below on $A$;
5. there exists $a > 0$ such that for any $(x,u) \in S_{\lambda_0}(c) \cap (\Omega_\delta \setminus \Omega)$ one can find $(h,v) \in K_A(x,u)$ such that for any $t \in T(x)$ one has

$$\langle \nabla_x g_j(x(t),t), h(t) \rangle \leq -a \|(h,v)\|_X \quad \forall j \in J(x,t).$$

Then there exists $\lambda^* \geq 0$ such that for any $\lambda \geq \lambda^*$ the penalty function $\Phi_\lambda$ for problem (52) is completely exact on $S_{\lambda}(c)$.

**Proof.** By propositions 3 and 4 of Reference 45 the functional $I$ is Lipschitz continuous on any bounded open set containing the set $S_{\lambda}(c) \cap \Omega_\delta$ due to the growth conditions on the function $\theta$ and its derivatives. Arguing in the same way as in the proof of Theorem 10 one can easily verify that the continuity of the function $f$ along with the closedness of the set $U$ ensures that the set $A$ is closed. The continuity of the penalty term $\varphi$ on $X$ follows from Corollary 2 (see Appendix B).

Thus, by Theorem 1 it is sufficient to check that there exists $a > 0$ such that $\varphi_\lambda^*(x,u) \leq -a$ for all $(x,u) \in S_{\lambda}(c) \cap (\Omega_\delta \setminus \Omega)$. Our aim is to show that assumption 5 ensures the validity of this inequality.

Fix any $(x,u) \in S_{\lambda}(c) \cap (\Omega_\delta \setminus \Omega)$, and let $(h,v) \in K_A(x,u)$ be from assumption 5. Then by the definition of contingent cone there exist sequences $(h_n,v_n) \in X$ and $\{\alpha_n\} \subset (0, +\infty)$ such that $(h_n,v_n) \rightarrow (h,v)$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, and for all $n \in \mathbb{N}$ one has $(x + \alpha_nh_n,u + \alpha_nv_n) \in A$.

Denote $\phi_\lambda(x) = \max_{(t)} g_j(x(t),t)$. Observe that $\varphi(x,u) = \max_{(t)} \phi_\lambda(x)\text{, since } \varphi(x,u) > 0$ (recall that $(x,u) \not\in \Omega$). It is well-known (eg, sections 4.4 and 4.5 of Reference 63) that the following equality holds true:

$$\lim_{n \rightarrow \infty} \frac{\phi_\lambda(x + \alpha_nh_n) - \phi_\lambda(x)}{\alpha_n} = \max_{t \in T(x)} \langle \nabla_x g_j(x(t),t), h(t) \rangle, \quad T_j(x) = \{t \in [0,T] | g_j(x(t),t) = \phi_\lambda(x)\}.$$ 

Hence by the Danskin-Demyanov theorem (eg, theorem 4.4.3 of Reference 63) one has

$$\lim_{n \rightarrow \infty} \frac{\varphi(x + \alpha_nh_n,u + \alpha_nv_n) - \varphi(x,u)}{\alpha_n} = \max_{j \in J(x)} \max_{t \in T_j(x)} \langle \nabla_x g_j(x(t),t), h(t) \rangle, \quad J(x) = \{j \in J | \phi_\lambda(x) = \varphi(x,u)\}.$$ 

Observe that $T(x) = \bigcup_{j \in J(x)} T_j(x)$, and $j \in J(x)$ for some $t \in T(x)$ iff $j \in J(x)$ and $t \in T_j(x)$. Consequently, by applying assumption (1) one obtains that

$$\varphi_\lambda^*(x,u) \leq \lim_{n \rightarrow \infty} \frac{\varphi(x + \alpha_nh_n,u + \alpha_nv_n) - \varphi(x,u)}{\alpha_n\|(h_n,v_n)\|_X} = \frac{1}{\|(h,v)\|_X} \max_{j \in J(x)} \max_{t \in T_j(x)} \langle \nabla_x g_j(x(t),t), h(t) \rangle \leq -a,$$

and the proof is complete. \( \blacksquare \)

Clearly, the main assumption ensuring that the penalty function $\Phi_\lambda$ for problem (52) is completely exact is assumption 5. This assumption can be easily explained in the case $l = 1$, that is, when there is only one state constraint. Roughly speaking, in this case assumption 5 means that if a trajectory $x(\cdot)$ of the system $\dot{x} = f(x,u,t)$ with $x(0) = x_0$ and $x(T) = x_T$ slightly violates the constraints (ie, $g_1(x(t),t) < \delta$ for all $t$), then by changing the control input $u$ in such a way that the endpoint conditions $x(0) = x_0$ and $x(T) = x_T$ remain to hold true one must be able to slightly shift the trajectory $x(t)$ in a direction close to $-\nabla_x g_1(x(t),t)$ at those points $t$ for which the constraint violation measure $\phi(x(t),t) = \max\{0,g_1(x(t),t)\}$ is the largest. However, note that this shift must be uniform for all $(x,u) \in S_{\lambda}(c) \cap (\Omega_\delta \setminus \Omega)$ in the sense that inequality (58) must hold true for all those $(x,u)$. The validity of this inequality in a neighborhood of a given point can be verified with the use of the same technique as in the proof of Theorem 16. Namely, one can check that in the case $U = L^2_0(0,T)$ inequality (58) holds true in a neighbourhood of a given point $\hat{t}$, provided there exists a solution $(h,v)$ of the corresponding linearized system such that $h(0) = h(T) = 0$ and $\langle \nabla_x g_j(\hat{x}(t),t), h(t) \rangle < 0$ for all $t \in T(x)$ and $j \in J(x)$. Consequently, the main difficulty in verifying assumption 5 stems from the fact that the validity of inequality (58) must be checked not locally, but on the set $S_{\lambda}(c) \cap (\Omega_\delta \setminus \Omega)$. Let us briefly discuss a particular case in which one can easily verify that assumption 5 holds true.
Example 5. Suppose that the system is linear, that is, \( f(x, u, t) = A(t)x + B(t)u \), the set \( U \) of admissible control inputs is convex, the functions \( g_j(x, t) \) are convex in \( x \), and Slater’s condition holds true, that is, there exists \((\hat{x}, \hat{u}) \in A \) such that \( g_j(\hat{x}(t), t) < 0 \) for all \( t \in [0, T] \) and \( j \in J \). Choose any \((x, u) \in S_{\Omega_j}(c) \cap (\Omega_q \setminus \Omega) \). For any \( n \in \mathbb{N} \) define \( a_n = 1/n \), \((h, v) = (\hat{x} - x, \hat{u} - u)\), and \((x_n, u_n) = a_n(\hat{x}, \hat{u}) + (1 - a_n)(x, u) + a_n(h, v)\). Then \((x_n, u_n) \in A \) for all \( n \in \mathbb{N} \) and \((h, v) \in K_A(u) \) due to the convexity of the set \( U \) and the linearity of the system. Fix any \( j \in J \) and \( t \in [0, T] \) such that \( g_j(x(t), t) \geq 0 \). Due to the convexity of \( g_j(x, t) \) in \( x \) one has

\[
\langle \nabla_x g_j(x(t), t), a_n h(t) \rangle \leq g_j(x(t) + a_n h(t), t) - g_j(x(t), t) \\
\leq a_n g_j(\hat{x}(t), t) + (1 - a_n)g_j(x(t), t) - g_j(x(t), t) \leq a_n \eta,
\]

where \( \eta = \max_{t \in [0, T]}(\nabla_x g_j(\hat{x}(t), t)) \). Note that \( \eta < 0 \) by Slater’s condition. Consequently, for any \( t \in T(x) \) and \( j \in J(x, t) \) one has

\[
\langle \nabla_x g_j(x(t), t), h(t) \rangle \leq \frac{\eta}{\| (h, v) \|_X} \| (h, v) \|_X.
\]

By assumption 1 of Theorem 17 the set \( S_{\Omega_j}(c) \cap (\Omega_q \setminus \Omega) \) is bounded. Therefore, there exists \( C > 0 \) such that \( \| (h, v) \|_X = \| (\hat{x} - x, \hat{u} - u) \|_X \leq C \) for all \((x, u) \in S_{\Omega_j}(c) \cap (\Omega_q \setminus \Omega) \). Hence assumption 5 of Theorem 17 is satisfied with \( a = |\eta|/C \).

Remark 19. Apparently, assumption 5 of Theorem 17 holds true in a much more general case than the case of optimal control problems for linear systems with convex state constraints. In particular, it seems that in the case when \( j = 1 \) (i.e., there is only one state constraint), \( g_1(x_0, 0) < 0, g_1(x_T, T) < 0 \), and \( \inf \| \nabla_x g_1(x, t) \| > 0 \), where the infimum is taken over all those \( t \in [0, T] \) and \( x \in \mathbb{R}^d \) for which \( 0 < g_1(x, t) < \delta \), assumption 5 of Theorem 17 is satisfied under very mild assumptions on the system. On the other hand, if either initial or terminal states lie on the boundary of the feasible region (i.e., either \( g_1(x_0, 0) = 0 \) or \( g_1(x_T, T) = 0 \)), then assumptions 5 cannot be satisfied. A detailed analysis of these conditions lies outside the scope of this article, and we leave it as a challenging open problem for future research.

Remark 20. One can easily extend the proof of Theorem 17 to the case when the penalty term \( \phi \) is defined as \( \phi(x, u) = \phi(x(\cdot), \cdot) \| r \) for some \( r \in (1, +\infty) \), where \( \phi(x, t) = \max_{j \in J} \max_{t \in [0, T]} g_j(x, t, 0) \) (i.e., the state constraints are penalized via the \( L^r \)-norm). In this case assumption 5 takes the following form: there exists \( a > 0 \) such that for any \((x, u) \in S_{\Omega_j}(c) \cap (\Omega_q \setminus \Omega) \) one can find \((h, v) \in K_A(u) \) satisfying the inequality

\[
\frac{1}{\phi(x, u)^{-1}} \int_0^T \phi(x(t), t)^{-1} \max_{j \in J} \| \nabla_x g_j(x(t), t), h(t) \| \ dt \leq -a \| (h, v) \|_X.
\]

where \( J_0(x, t) = \{ j \in J \cup \{ 0 \} \} \| g_j(x, t) = \phi(x, t) \| \) and \( g_0(x, t) \equiv 0. \) However, the author failed to find any optimal control problems for which this assumptions can be verified.

4.5 Nonlinear systems: A different view

As Examples 3 and 4 demonstrate, penalty functions for problems with state constraints may fail to be exact due to the fact that the penalty term \( \phi \), unlike the cost functional \( I(x, u) \), does not depend on the control inputs \( u \) explicitly. In the case when \( I \) does not explicitly depend on \( u \), one can utilize a somewhat different approach and obtain stronger results on the exactness of penalty functions for state constrained problems. Furthermore, this approach serves as a proper motivation to consider a general theory of exact penalty functions in the metric space setting (as it is done in Section 2), but not in the normed space setting.

Consider the following variable-endpoint optimal control problem with state inequality constraints:

\[
\min I(x) = \int_0^T \theta(x(t), t) \ dt + \zeta(x(T)) \text{ subject to } x(t) = f(x(t), u(t), t), \quad t \in [0, T], \\
x(0) = x_0, \quad x(T) \in S_T, \quad u \in U, \quad g_j(x(t), t) \leq 0 \quad \forall t \in [0, T], \forall j \in J.
\]

Here \( \theta : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}, \ \zeta : \mathbb{R}^d \rightarrow \mathbb{R}, \ f : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \rightarrow \mathbb{R}^d, \) and \( g_j : \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}, \ j \in J = \{ 1, \ldots, l \}, \) are given functions, \( x_0 \in \mathbb{R}^d \) and \( T > 0 \) are fixed, while \( S_T \subseteq \mathbb{R}^d \) and \( U \subseteq L^m_q(0, T) \) are closed sets. It should be noted that
with the use of the standard time scaling transformation time-optimal control problems can be recast as problems of the form (60).

We will treat problem (60) as a variational problem, not as an optimal control one. To this end, fix some \( p \in (1, +\infty] \), and define

\[
X = \left\{ x \in (C[0, T])^d \, | \, \exists u \in U : x(t) = x_0 + \int_0^t f(x(\tau), u(\tau), \tau) \, d\tau \quad \forall t \in [0, T] \right\},
\]

that is, \( X \) is the set of trajectories of the controlled system under consideration. We equip \( X \) with the metric \( d_X(x, y) = \|x - y\|_p + |x(T) - y(T)| \). Define \( A = \{ x \in X | x(T) \in S_T \} \) and \( M = \{ x \in X | g_j(x(t), t) \leq 0 \, \forall t \in [0, T], j \in J \} \). Then problem (52) can be rewritten as the problem of minimizing \( J(x) \), \( x \in X \), over the set \( M \cap A \). Observe that the set \( A \) is closed in \( X \) due to the facts that the set \( S_T \) is closed, and if a sequence \( \{x_n\} \) converges to some \( x \) in the metric space \( X \), then \( \{x_n(T)\} \) converges to \( x(T) \). Let us also point out simple sufficient conditions for the metric space \( X \) to be complete.

**Proposition 1.** Let the function \( f \) be continuous and \( U = \{ u \in L^p_{\infty}(0, T) | u(t) \in Q \, \text{ for a.e. } t \in (0, T) \} \) for some compact convex set \( Q \subset \mathbb{R}^m \). Suppose also that the set \( f(x, Q, t) \) is convex for all \( x \in \mathbb{R}^d \) and \( t \in [0, T] \), and for any \( u \in U \) a solution of \( \dot{x} = f(x, u, t) \) with \( x(0) = x_0 \) is defined on \( [0, T] \). Then \( X \) is a complete metric space and a compact subset of \((C[0, T])^d\).

**Proof.** Under our assumptions the space \( X \) consists of all solutions of the differential inclusion \( \dot{x} \in F(x, t), x(0) = x_0 \), with \( F(x, t) = f(x, Q, t) \) by Filippov’s theorem (eg. theorem 8.2.10 of Reference 65). Furthermore, by theorem 2.7.6 of Reference 56 the set \( X \) is compact in \( (C[0, T])^d \).

Let \( \{x_n\} \subset X \) be a Cauchy sequence in \( X \). Since \( X \) is compact in \( (C[0, T])^d \), there exists a subsequence \( \{x_{n_k}\} \) uniformly converging to some \( x^* \in X \), which obviously implies that \( \{x_{n_k}\} \) converges to \( x^* \) in \( X \). Hence with the use of the fact that \( x_n \) is a Cauchy sequence in \( X \) one can easily check that \( x_{n_k} \) converges to \( x^* \) in \( X \). Thus, \( X \) is a complete metric space.

Formally introduce the penalty term

\[
\varphi(x) = \|\phi(x(\cdot), \cdot)\|_p \quad \forall x \in X, \quad \phi(x, t) = \max\{g_1(x(t), t), \ldots, g_l(x(t), t), 0\} \quad \forall x \in \mathbb{R}^d, t \in [0, T],
\]

(note that here \( p \) is the same as in the definition of metric in \( X \)). Then \( M = \{ x \in X | \varphi(x) = 0 \} \), and one can consider the penalized problem of minimizing the penalty function \( \Phi_\lambda(x) = I(x) + \lambda \varphi(x) \) over the set \( A \), which in the case \( p < +\infty \) can be formally written as follows:

\[
\min \Phi_\lambda(x) = \int_0^T \theta(x(t), t) \, dt + \lambda \left( \int_0^T \max\{g_1(x(t), t), \ldots, g_l(x(t), t), 0\}^p \, dt \right)^{1/p} + \zeta(x(T))
\]

subject to \( x(t) = x_0 + \int_0^t f(x(\tau), u(\tau), \tau) \, d\tau, \quad t \in [0, T], \quad x(T) \in S_T, \quad u \in U, \quad x \in (C[0, T])^d \). (61)

Note, however, that due to our choice of the space \( X \) and the metric in this space the notions of locally optimal solutions/inf-stationary points of this problem (and problem (60)) are understood in a rather specific sense. In particular, \((x^*, u^*)\) is a locally optimal solution of this problem iff for any feasible point \((x, u)\) satisfying the inequality \( \|x - x^*\| + |x(T) - x^*(T)| < r \) for some \( r > 0 \) one has \( \Phi_\lambda(x) \geq \Phi_\lambda(x^*) \). It should be mentioned that any locally optimal solution/inf-stationary point of problem (60) (or (61)) in \( X \) is also a locally optimal solution/inf-stationary point of problem (60) (or (61)) in the space \( W^1_p(0, T) \times L^m_q(0, T) \), but the converse statement is not true. In a sense, one can say that our choice of the underlying space \( X \) in this section reduces the number of locally optimal solutions/inf-stationary points (and, as a result, leads to the weaker notion of the complete exactness of \( \Phi_\lambda \) than in the previous section).

Let us derive sufficient conditions for the complete exactness of the penalty function \( \Phi_\lambda \) for problem (60). To conveniently formulate these conditions, define \( g_0(x, t) \equiv 0 \). Then \( \phi(x, t) \equiv \max\{g_j(x(t), t) | j \in J \cup \{0\} \} \). For any \( x \in \mathbb{R}^d \) and \( t \in [0, T] \) let \( J(x, t) = \{ j \in J \cup \{0\} | \phi(x, t) = g_j(x, t) \} \). Finally, suppose that the functions \( g_j \) are differentiable in \( x \), and define the subdifferential \( \partial_x \phi(x, t) \) of the function \( x \mapsto \phi(x, t) \) as follows:

\[
\partial_x \phi(x, t) = \text{co} \left\{ \nabla_x g_j(x, t) | j \in J(x, t) \right\}.
\]

(62)

Let us point out that \( \partial_x \phi(x, t) \) is a convex compact set, and \( \partial_x \phi(x, t) = \{0\} \), if \( g_j(x, t) < 0 \) for all \( j \in J \).
Denote by $I^*$ the optimal value of problem (60), and recall that $\Omega_\delta = \{x \in A | \varphi(x) < \delta \}$. Observe that in the case $p = +\infty$ the set $\Omega_\delta$ consists of all those trajectories $x(\cdot)$ of the system that satisfy the perturbed constraints $g_j(x(t), t) < \delta$ for all $t \in [0, T]$ and $j \in J$. In the case $p < +\infty$ the set $\Omega_\delta$ consists of all those trajectories $x(\cdot)$ for which there exists $w \in L^p(0, T)$ with $|w(t)| < \delta$ such that $g_j(x(t), t) - w(t) \leq 0$ for all $t \in [0, T]$ and $j \in J$, which implies that at every point $t \in [0, T]$ the violation of the state constraints can be arbitrarily large, that is, $\phi(x(t), t)$ can be arbitrarily large as long as $\|\phi(x(\cdot), \cdot)\|_p < \delta$.

To avoid the usage of some complicated and restrictive assumptions on the problem data, we prove the following theorem in the simplest case when the set $X$ is compact in $(C[0, T])^d$. This assumption holds true, in particular, if the assumptions of Proposition 1 are satisfied.

**Theorem 18.** Let $p \in (1, +\infty)$ and the following assumptions be valid:

1. $\zeta$ is locally Lipschitz continuous, $\theta$ and $g_j, j \in J$ are continuous, differentiable in $x$, and the functions $\nabla x \theta$ and $\nabla x g_j, j \in J$, are continuous;
2. the set $X$ is compact in $(C[0, T])^d$, and there exists a feasible point of problem (60);
3. there exist a $\rho > 0$ and $\eta > 0$ such that for any $x \in A \setminus \Omega$ one can find a sequence of trajectories $\{x_n\} \subseteq A$ converging to $x$ in the space $X$ such that $|x_n(T) - x(T)| \leq \eta |x_n - x|_p$ for all $n \in \mathbb{N}$, the sequence $(x_n - x)/|x_n - x|_p$ converges to some $h \in L^p(0, T)$, and

$$\int_0^T \phi(x(t), t)^{p-1} \max_{v \in \Omega, \phi(x(t), t)} \langle v, h(t) \rangle \, dt \leq -a \varphi(x)^{p-1},$$

(63)

in the case $1 < p < +\infty$, while

$$\langle \nabla x g_j(x(t), t), h(t) \rangle \leq -a \quad \forall t \in [0, T], j \in J : \varphi(x) = g_j(x(t), t),$$

(64)

in the case $p = +\infty$.

Then the penalty function $\Phi_\lambda$ for problem (60) is completely exact on $A$.

**Proof.** The functional $I$ is obviously continuous with respect to the uniform metric, which with the use of the fact that $X$ is compact in $(C[0, T])^d$ implies that the penalty function $\Phi_\lambda$ is bounded below on $X$ for any $\lambda \geq 0$. Moreover, the set $X$ is bounded in $(C[0, T])^d$. Hence, by applying the mean value theorem and the fact that the function $\zeta$ is locally Lipschitz continuous one obtains that there exist $L_\zeta > 0$ such that

$$|I(x) - I(y)| \leq \left| \int_0^T \theta(x(t), t) \, dt - \int_0^T \theta(y(t), t) \, dt \right| + |\zeta(x(T)) - \zeta(y(T))|$$

$$\leq \int_0^T \sup_{x \in [0, 1]} |\nabla_x \theta(x(t) + a(y(t) - x(t)), t)| \cdot |x(t) - y(t)| \, dt + L_\zeta |x(T) - y(T)| \leq \max\{T^{1/p} K, L_\zeta\} d_X(x, y),$$

for all $x, y \in X$, where $K = \max\{|\nabla x \theta(z, t)| : |z| \leq R, t \in [0, T]\}$ and $R > 0$ is such that $\|x\|_\infty \leq R$ for all $x \in X$. Thus, the functional $I$ is Lipschitz continuous on $X$.

Let a sequence $(\{x_n, u_n\})$ of feasible points of problem (60) be such that $I(x_n)$ converges to the optimal value $I^*$ of this problem (recall that we assume that at least one feasible point exists). Since the set $X$ is compact in $(C[0, T])^d$, one can extract a subsequence $\{x_{n_k}\}$ uniformly converging to some $x^* \in X$. From the uniform convergence, the continuity of the functions $g_j$, and the closedness of the set $S_T$, it follows $x^*(T) \in S_T$ and $g_j(x^*(\cdot), \cdot) \leq 0$ for all $j \in J$. Furthermore, by the definition of $X$ there exists $u^* \in U$ such that $x^*(t) = x_0 + \int_0^t f(x^* (\tau), u^*(\tau), \tau) \, d\tau$ for all $t \in [0, T]$, which implies that $(x^*, u^*)$ is a feasible point of problem (60). Taking into account the fact that the functional $I$ is continuous with respect to the uniform metric one obtains that $I(x^*) = I^*$. Thus, $(x^*, u^*)$ is a globally optimal solution of problem (60), that is, this problem has a globally optimal solution.

Let us check that the penalty term $\varphi$ is continuous on $X$. Indeed, arguing by reductio ad absurdum, suppose that $\varphi$ is not continuous at some point $x \in X$. Then there exists $\epsilon > 0$ and a sequence $\{x_n\} \subset X$ converging to $x$ in the space $X$ such that $|\varphi(x_n) - \varphi(x)| \geq \epsilon$ for all $n \in \mathbb{N}$. By applying the compactness of $X$ one obtains that there exists a subsequence $\{x_{n_k}\}$ uniformly converging to some $\bar{x} \in X$. Clearly, $\{x_{n_k}\}$ also converges to $\bar{x}$ in the space $X$, which implies that $\bar{x} = x$. Utilizing
the uniform convergence of \{x_{n_k}\} to x and the continuity of the functions \(g_j\) one can easily prove that \(\varphi(x_{n_k}) \to \varphi(x)\) as \(k \to \infty\) (see Corollary 2), which contradicts our assumption. Therefore, the penalty term \(\varphi\) is continuous on \(X\).

Thus, by Theorem 2 it remains to check that there exists \(a > 0\) such that \(\varphi'_{A}(x) \leq -a\) for all \(x \in A \setminus \Omega\). Our aim is to show that this inequality is implied by assumption 3.

The case \(p < +\infty\). Fix any \(x \in A \setminus \Omega\), and let \(\{x_{n}\} \subset A\) and \(h\) be from assumption 3. Define \(a_n = ||x_n - x||_p\) and \(h_n = (x_n - x)/||x_n - x||_p\). Then \(x_n = x + a_nh_n\). Let us verify that

\[
\lim_{n \to \infty} \frac{\varphi(x + a_nh_n) - \varphi(x)}{a_n} = \frac{1}{\varphi(x)^{p-1}} \int_0^T \phi(x(t), t)^{p-1} \max_{v \in \partial_x \varphi(x(t), t)} \langle v, h(t) \rangle \, dt. \tag{65}
\]

Then by applying inequality (63) and the inequality \(|x_n(T) - x(T)| \leq \eta||x_n - x||_p\) one gets that

\[
\varphi'_{A}(x) \leq \liminf_{n \to \infty} \frac{\varphi(x_n) - \varphi(x)}{d_X(x_n, x)} = \liminf_{n \to \infty} \frac{a_n}{d_X(x_n, x)} \frac{\varphi(x + a_nh_n) - \varphi(x)}{a_n} \leq - \frac{a}{1 + \eta},
\]

and the proof is complete.

Instead of proving equality (65), let us check that

\[
\lim_{n \to \infty} \frac{\varphi(x + a_nh_n)^p - \varphi(x)^p}{a_n} = \int_0^T \phi(x(t), t)^{p-1} \max_{v \in \partial_x \varphi(x(t), t)} \langle v, h(t) \rangle \, dt. \tag{66}
\]

Then taking into account the facts that \(\varphi(x) > 0\) (recall that \(x \not\in \Omega\)), and the function \(\omega(s) = s^{1/p}\) is differentiable at any point \(s > 0\) one obtains that equality (65) holds true.

To prove equality (66), note at first that the multifunction \(t \mapsto \partial_x \varphi(x(t), t)\) is upper semi-continuous and thus measurable by proposition 8.2.1 of Reference 65, which by theorem 8.2.11 of Reference 65 implies that the function \(t \mapsto \max_{v \in \partial_x \varphi(x(t), t)} \langle v, h(t) \rangle\) is measurable. Arguing by reductio ad absurdum, suppose that equality (66) does not hold true. Then there exist \(\epsilon > 0\) and a subsequence \(\{n_k\}, k \in \mathbb{N}\), such that

\[
\left| \frac{\varphi(x + a_{n_k}h_{n_k})^p - \varphi(x)^p}{a_{n_k}} - \int_0^T \phi(x(t), t)^{p-1} \max_{v \in \partial_x \varphi(x(t), t)} \langle v, h(t) \rangle \, dt \right| \geq \epsilon, \tag{67}
\]

for all \(k \in \mathbb{N}\). Since \(h_n\) converges to \(h\) in \(L^p_d(0, T)\), one can find a subsequence of the sequence \(\{h_{n_k}\}\), which we denote once again by \(\{h_{n_k}\}\), that converges to \(h\) almost everywhere. Hence by the Danskin-Demyanov theorem (eg, theorem 4.4.3 of Reference 63) for a.e. \(t \in (0, T)\) one has \(\lim_{k \to \infty} \omega_k(t) = 0\), where

\[
\omega_k(t) = \frac{\phi(x(t) + a_{n_k}h_{n_k}(t), t)^p - \phi(x(t), t)^p}{a_{n_k}} - \phi(x(t), t)^{p-1} \max_{v \in \partial_x \varphi(x(t), t)} \langle v, h(t) \rangle.
\]

With the use of a nonsmooth version of the mean value theorem (eg, proposition 2 of Reference 66) one obtains that for any \(k \in \mathbb{N}\) and a.e. \(t \in (0, T)\) there exist \(\beta_k(t) \in (0, 1)\) and \(v_k(t) \in \partial_x \varphi(x(t) + \beta_k(t)(x_{n_k}(t) - x(t)), t)\) such that

\[
\omega_k(t) = \left( \phi(x(t) + \beta_k(t)(x_{n_k}(t) - x(t)), t) \right)^{p-1} \langle v_k(t), h_{n_k}(t) \rangle - \phi(x(t), t)^{p-1} \max_{v \in \partial_x \varphi(x(t), t)} \langle v, h(t) \rangle.\]

Consequently, bearing in mind the facts that the set \(X\) is compact in \((C[0, T])^d\) and the functions \(g_j\) and \(V_{xg_j}\) are continuous (see equality (62)) one obtains that there exists \(C > 0\) such that \(|\omega_k(t)| \leq C|h_{n_k}(t)| + C|h(t)|\) for all \(k \in \mathbb{N}\) and a.e. \(t \in (0, T)\). The sequence \(\{h_{n_k}\}\) converges to \(h\) in \(L^p_d(0, T)\), which by Hölder’s inequality implies that it converges to \(h\) in \(L^p(0, T)\). By the “only if” part of Vitali’s theorem characterizing convergence in \(L^p\)-spaces (eg, theorem III.6.15 of Reference 64) and the absolute continuity of the Lebesgue integral for any \(\epsilon > 0\) one can find \(\delta(\epsilon) > 0\) such that for any Lebesgue measurable set \(E \subset [0, T]\) with \(\mu(E) < \delta(\epsilon)\) (here \(\mu\) is the Lebesgue measure) one has \(\int_E h_k \, d\mu < \epsilon/2\) and \(\int_E h \, d\mu < \epsilon/2\). Consequently, \(\int_E \omega_k \, d\mu < \epsilon\), provided \(\mu(E) < \delta(\epsilon)\). Hence bearing in mind the fact \(\omega_k(t) \to 0\) for a.e. \(t \in (0, T)\) and passing to the limit with the use of “if” part of the Vitali theorem one obtains that \(\lim_{k \to \infty} \int_0^T |\omega_k(t)| \, dt = 0\), which contradicts inequality (67). Thus, equality (66) holds true, and the proof of the case \(p < +\infty\) is complete.
The case \( p = +\infty \). The proof of this case coincides with the derivation of the inequality \( \varphi^1_A(x, u) \leq -a \) within the proof of Theorem 17.

Remark 21. (i) Let us note that one can define

\[
\varphi(x) = \left( \sum_{j=1}^l \int_0^T \max\{ g_j(x(t), t), 0 \}^p \, dt \right)^{1/p}
\]

or

\[
\varphi(x) = \sum_{j=1}^l \left( \int_0^T \max\{ g_j(x(t), t), 0 \}^p \, dt \right)^{1/p}, 1 < p < +\infty,
\]

and easily obtain corresponding sufficient conditions for the complete exactness of the penalty function \( \Phi_A \), which are very similar, but not identical, in all three cases.

(ii) It should be mentioned that the term \( |x(T) - y(T)| \) was introduced into the definition of the metric \( d(x, y) = \|x - y\|_p + |x(T) - y(T)| \) in \( X \) to ensure that the functional \( I \) is Lipschitz continuous on \( A \). In the case of problems with the cost functional of the form \( I(x) = \int_0^T \theta(x(t), t) \, dt \) one can define \( d(x, y) = \|x - y\|_p \) and drop the inequality \( |x(T) - x(T)| \leq \eta \|x_n - x\|_p \) from assumption 3 of Theorem 18. Note that the closedness of the set \( A = \{ x \in X | x(T) \in S_T \} \) in the case when \( X \) is equipped with the metric \( d(x, y) = \|x - y\|_p \) can be easily proved under the assumption that \( X \) is compact in \((C[0, T], \mathbb{R}^d)^d\), since in this case the topologies on \( X \) generated by the metrics \( d(x, y) = \|x - y\|_p \) and \( d(x, y) = \|x - y\|_\infty \) coincide.

At first glance, assumption 3 of Theorem 18 might seem very similar to assumption 5 of Theorem 17 and inequality (59). In particular, arguing in the same way as in Example 5 one can check that in the case \( p = +\infty \) inequality (64) is satisfied, provided the system is linear, the state constraints are convex, and Slater’s condition holds true. However, there is one important difference. In assumption 3 of Theorem 18 one does not need to care about control inputs corresponding to the sequence \( \{x_n\} \), as well as the derivatives of \( x_n \), which makes this assumption significantly less restrictive, then assumption 5 of Theorem 17.

Remark 22. Let us point out a particular case in which assumption 3 of Theorem 18 can be reformulated in a more convenient form. Suppose that \( p < +\infty, l = 1 \) (i.e., there is only one state constraint), and there exist \( a_1, a_2 > 0 \) such that \( a_1 \leq \|\nabla g_1(x, t)\| \leq a_2 \) for all \( t \in [0, T] \) and \( x \in \mathbb{R}^d \) satisfying the inequality \( g_1(x, t) > 0 \). In this case assumption 3 of Theorem 18 is satisfied, if there exists \( \eta > 0 \) such that for any \( x \in A \backslash \Omega \) one can find a sequence of trajectories \( \{x_n\} \subset A \) converging to \( x \) such that \( |x_n(T) - x(T)| \leq \eta \|x_n - x\|_p \) for all \( n \in \mathbb{N} \), and the sequence \( \{x_n - x\}/\|x_n - x\|_p \) converges to \( h = y/\|y\|_p \) with \( y(t) = -\varphi(x(t), t)\nabla g_1(x(t), t) \) for all \( t \in [0, T] \). Indeed, by applying the inequalities \( a_1 \leq \|\nabla g_1(x, t)\| \leq a_2 \) and the fact that \( \varphi(x) = \|\varphi(x, \cdot, \cdot)\|_p \) one obtains

\[
\int_0^T \varphi(x(t), t)^{p-1} \max_{v \in \mathcal{D}, \varphi(v(x(t), t))} \langle v, h(t) \rangle \, dt = -\frac{1}{\int_0^T \varphi(x(t), t)^p \|\nabla g_1(x(t), t)\|^2 \, dt} \left( \int_0^T \varphi(x(t), t)^p \|\nabla g_1(x(t), t)\|^p \, dt \right)^{1/p} \leq -\frac{a_1^2 \varphi(x)^p}{a_2} \varphi(x) = -\frac{a_1^2}{a_2} \varphi(x)^{p-1},
\]

that is, inequality (63) holds true. This assumption, in essence, means that for any trajectory \( x \) violating the state constraint \( g_1(x, t) \leq 0 \) one has to be able to find a sequence of control inputs that shift the trajectory \( x \) along the ray \( x_n(t) = x(t) - a \varphi(x(t), t) \nabla g_1(x(t), t), a \geq 0 \) (recall that \( \varphi(x(t), t) = \max\{g_1(x(t), t), 0\} \)), that is, the trajectory is shifted only at those points where the state constraint is violated. It is easily seen that for any \( t \in [0, T] \) satisfying the inequality \( g_1(x(t), t) > 0 \) and for any sufficiently small \( a > 0 \) one has \( g_1(x_n(t), t) < g_1(x(t), t) \), that is, \( x \) is shifted toward the feasible region. Thus, one can say that assumption 3 of Theorem 18 is an assumption on the controllability of the system \( \dot{x} = f(x, u, t) \) with respect to the state constraints.

It should be noted that Theorem 18 is mainly of theoretical interest, since it does not seem possible to verify assumption 3 for any particular classes of optimal control problems appearing in practice. Nevertheless, let us give a simple and illuminating example of a problem in which this assumption is satisfied.

Example 6. Let \( d = 2 \) and \( m = 1 \). Define \( U = \{ u \in L^\infty(0, T) ||u||_\infty \leq 1 \} \), and consider the following variable-endpoint optimal control problem with the state inequality constraint:

\[
\min I(x) = \int_0^T \theta(x(t), t) \, dt + \zeta(x(T)) \quad \text{s.t.} \quad \begin{cases} \dot{x}_1 = 1 \\ \dot{x}_2 = u \\ x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x(T) \in S_T, \quad u \in U, \quad g(x(t)) \leq 0, \end{cases}
\]
where \( g(x^1, x^2) = x^2 \), the functions \( \theta \) and \( \zeta \) satisfy the assumptions of Theorem 18, and \( S_T = \{ T \} \times [-\beta, 0] \) for some \( \beta \geq 0 \). Then \((x, u)\) with \( x(t) \equiv (t, 0)^T \) and \( u(t) \equiv 0 \) is a feasible point of this problem. Furthermore, by Proposition 1 the space \( X \) of trajectories of the system under consideration is compact in \((C[0, T])^d\). Thus, it remains to check that assumption 3 of Theorem 18 holds true. We will verify this assumptions with the use of the idea discussed in Remark 22.

Note that \( g(x(0)) = 0 \), that is, Slater’s condition is not satisfied. Fix any \( x \in A \setminus \Omega \), and let \( x \) corresponds to a control input \( u \in U \). For any \( n \in \mathbb{N} \) define

\[
\begin{align*}
\dot{u}_n(t) &= \begin{cases} u(t), & \text{if } x^2(t) \leq 0, \\ \left(1 - \frac{1}{n}\right)u(t), & \text{if } x^2(t) > 0, \end{cases} \\
\dot{x}_n(t) &= \begin{cases} x^2(t) - \frac{t}{n}\max(x^2(t), 0), & \text{if } x^2(t) \leq 0, \\ x^2(t) - \frac{t}{n}\max(x^2(t), 0), & \text{if } x^2(t) > 0. \end{cases}
\end{align*}
\]

Observe that \( x_n \) is a trajectory of the system corresponding to the control input \( u_n \), and for any \( n \in \mathbb{N} \) one has \( u_n \in U \), \( x_n(0) = x(0) = (0, 0)^T \), \( x(T) \in S_T \) by the definition of \( A \), and \( x_n(T) \in S_T \), since by the definition of \( S_T \) one has \( x^2(T) \leq 0 \), which implies that \( x_n(T) = x(T) \). Hence, in particular, \( x_n \in A \) and \( |x_n(T) - x(T)| = 0 \leq ||x_n - x||_p \) for all \( n \in \mathbb{N} \). The sequence \( \{x_n\} \) obviously converges to \( x \) in \( X \). Furthermore, note that \((x_n - x)/||x_n - x||_p = h \) with \( h(\cdot) = (0, -\max(x^2(\cdot), 0)/\phi(x))^T \) for all \( n \), which obviously implies that the sequence \( \{(x_n - x)/||x_n - x||_p\} \) converges to \( h \), and

\[
\int_0^T \phi(x(t), t)^{p-1} \max_{v \in A_\phi, \phi(x(t), t) \neq 0} \langle v, h(t) \rangle \, dt = -\frac{1}{\phi(x)} \int_0^T \max[x^2(t), 0]^p \, dt = -\phi(x)^{p-1},
\]

that is, assumption 3 of Theorem 18 is satisfied with \( a = 1 \) and any \( \eta > 0 \). Thus, by Theorem 18 one can conclude that for any \( 1 < p < +\infty \) there exists \( \lambda^* \geq 0 \) such that for any \( \lambda \geq \lambda^* \) the penalized problem

\[
\min \Phi_\lambda(x) = \int_0^T \theta(x(t), t)dt + \lambda \left( \int_0^T \max[x^2(t), 0]^p \, dt \right)^{1/p} + \zeta(x(T)) \quad \text{s.t.} \quad \begin{cases} \dot{x}^1 = 1 \\ \dot{x}^2 = u \quad x(0) = \left(0 \atop 0\right), x(T) \in S_T, u \in U \end{cases}
\]

is equivalent to problem (68) in the sense that these problems have the same optimal value, the same globally optimal solutions, as well as the same locally optimal solutions and in-stationary points with respect to the pseudometric \( d((x_1, u_1), (x_2, u_2)) = ||x_1 - x_2||_p + |x_1(T) - x_2(T)| \) in \( W^2_{1,\infty}(0, T) \times L^\infty(0, T) \).

Theorem 18 can be easily extended to the case of problems with state equality constraints. Namely, suppose that there is a single state equality constraint: \( g(x(t), t) = 0 \) for all \( t \in [0, T] \). Then one can define \( \phi(x) = ||g(x(\cdot), \cdot)||_p \) for \( 1 < p < +\infty \). Arguing in a similar way to the proof of Theorem 18 one can verify that this theorem remains to hold true for problems with one state equality constraint, if one replaces inequality (63) with the following one:

\[
\int_0^T |g(x(t), t)|^{p-1} \operatorname{sign}(g(x(t), t)) \langle \nabla_2 g(x(t), t), h(t) \rangle \, dt \leq -a \phi(x)^{p-1}. \tag{69}
\]

As in Remark 22, one can verify that this inequality is satisfied for \( h = y/\|y\|_p \) with \( y = -g(x(t), t)\nabla_2 g(x(t), t) \), provided \( 0 < a_1 \leq \|\nabla_2 g(x, t)\| \leq a_2 \) for all \( x \) and \( t \). Let us utilize this result to demonstrate that exact penalization of state equality constraints is possible, if the cost functional \( I \) does not depend on the control inputs explicitly (cf, Example 3 with which we started our analysis of state constrained problems).

**Example 7.** Let \( d = 2 \) and \( m = 2 \). Define \( U = \{u = (u^1, u^2)^T \in L^2_{\infty}(0, T)||u||_\infty \leq 1 \} \), and consider the following variable-endpoint optimal control problem with state equality constraint:

\[
\min I(x) = \int_0^T \theta(x(t), t)dt + \zeta(x(T)) \quad \text{s.t.} \quad \begin{cases} \dot{x}^1 = u^1 \\ \dot{x}^2 = u^2 \quad x(0) = 0, \quad x(T) \in S_T, \quad u \in U, \quad g(x(t)) = x^1(t) + x^2(t) = 0 \quad \forall t \in [0, T]. \end{cases}
\]

Here \( S_T \) is a closed subset of the set \( \{x \in \mathbb{R}^2|x^1 + x^2 = 0\} \) such that \( 0 \in S_T \), while \( \theta \) and \( \zeta \) satisfy the assumptions of Theorem 18. Note that \((x, u)\) with \( x(t) \equiv 0 \) and \( u(t) \equiv 0 \) is a feasible point of problem (70). Furthermore, by Proposition 1 the space \( X \) of trajectories of the system under consideration is compact in \((C[0, T])^d\). Thus, as one can easily verify, there exists a globally optimal solution of problem (70).
Let us check that inequality (69) holds true for any \( x \in A \setminus \Omega \), that is, for any trajectory violating the state equality constraint. Indeed, fix any such \( x \), and let \( x \) corresponds to a control input \( u \in U \). For any \( n \in \mathbb{N} \) define

\[
u_n = u - \frac{g(u)}{n} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x_n = x - \frac{g(x)}{n} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Clearly, \( x_n \) is a trajectory of the system corresponding to \( u_n \), and for any \( n \in \mathbb{N} \) one has \( x_n(0) = 0 \) and \( x_n(T) = x(T) \in S_T \), since by our assumptions \( x \in A = \{ x \in X | x(T) \in S_T \} \) and \( g(\xi) = 0 \) for any \( \xi \in S_T \). Note also that \( u_n \in U \) for any \( n \in \mathbb{N} \) due to the facts that

\[ |u_n^i(t)| = \left| u^1(t) - \frac{u^1(t) + u^2(t)}{n} \right| \leq \frac{n-1}{n} |u^1(t)| + \frac{1}{n} |u^2(t)| \leq \frac{n-1}{n} + \frac{1}{n} = 1 \quad \text{for a.e. } t \in (0, T), \]

and the same inequality holds true for \( u_n^j \). Thus, \( x_n \in A \) and \( |x_n(T) - x(T)| = 0 \leq ||x_n - x||_p \) for all \( n \). Moreover, for any \( n \in \mathbb{N} \) one has \( (x_n - x)/||x_n - x||_p = h \) with \( h = (-g(x)/\sqrt{2} \phi(x), -g(x)/\sqrt{2} \phi(x))^T \), which obviously implies that the sequence \( \{(x_n - x)/||x_n - x||_p \} \) converges to \( h \), and

\[
\int_0^T |g(x(t))|^{p-1} \text{sign}(g(x(t)))(\nabla_x g(x(t)), h(t)) \ dt = -\frac{2}{\sqrt{2} \phi(x)} \int_0^T |g(x(t))|^p \ dt = -\sqrt{2} \phi(x)^{p-1},
\]

that is, inequality (69) is satisfied with \( a = \sqrt{2} \). Thus, one can conclude that for any \( 1 < p < +\infty \) the penalized problem

\[
\min \Phi_\lambda(x) = \int_0^T \theta(x(t), t) dt + \lambda \left( \int_0^T |x^1(t) + x^2(t)|^p dt \right)^{1/p} + \zeta(x(T)) \text{s.t. } \begin{cases} \dot{x}^1 = u^1 \\ \dot{x}^2 = u^2 \end{cases} \ x(0) = 0, \ x(T) \in S_T, \ u \in U,
\]

is equivalent to problem (70) for any sufficiently large \( \lambda \).

5 | CONCLUSIONS

In the second paper of our study we analyzed the exactness of penalty functions for optimal control problems with terminal and pointwise state constraints. We proved that penalty functions for fixed-endpoint optimal control problems for linear time-varying systems and linear evolution equations are completely exact, if the terminal state belongs to the relative interior of the reachable set. In the nonlinear case, the local exactness of the penalty function can be ensured under the assumption that the linearized system is completely controllable, while the complete exactness of the penalty function can be achieved under certain assumptions on the reachable set and the controllability of the system, which require further investigation.

We also proved that penalty functions for variable-endpoint optimal control problems for linear time-varying systems with convex terminal constraints are completely exact, if Slater’s condition holds true. In the case of nonlinear variable-endpoint problems, the local exactness of a penalty function was proved under the assumptions that the linearized system is completely controllable, and the well-known MFCQ holds true for terminal constraints.

In the case of problems with pointwise state inequality constraints, we showed that penalty functions for such problems for linear time-varying systems and linear evolution equations with convex state constraints are completely exact, if the \( L^\infty \) penalty term is used, and Slater’s condition holds true. In the nonlinear case we proved the local exactness of the \( L^\infty \) penalty function under the assumption that a suitable constraint qualification is satisfied, which resembles MFCQ. We also proved that the exact \( L^p \) penalization of pointwise state constraints with finite \( p \) is possible for convex problems, if Lagrange multipliers corresponding to state constraints belong to \( L^p(0, T) \), and for nonlinear problems, if the cost functional does not depend on the control inputs explicitly and some additional assumptions are satisfied.

A reason that the exact \( L^p \) penalization of state constraints with finite \( p \) requires more restrictive assumption is indirectly connected to the Pontryagin maximum principle. Indeed, if the penalty function with \( L^p \) penalty term is locally exact at a locally optimal solution \((x^*, u^*)\), then by definition for any sufficiently large \( \lambda \geq 0 \) the pair \((x^*, u^*)\) is a locally
optimal solution of the penalized problem without state constraints:

$$\min \Phi_p(x) = \int_0^T \theta(x(t), u(t), t) \, dt + \lambda \left( \int_0^T \max\{g_1(x(t), t), \ldots, g_l(x(t), t), 0\}^p \, dt \right)^{1/p}$$

subject to $\dot{x}(t) = f(x(t), u(t), t) \, dt$, $t \in [0, T]$, $x(0) = x_0$, $x(T) = x_T$, $u \in U$.

It is possible to derive optimality conditions for this problem in the form of the Pontryagin maximum principle for the original problem, in which Lagrange multipliers corresponding to state constraints necessarily belong to $L^p[0, T]$, if $p < +\infty$. Therefore, for the exactness of the $L^p$ penalty function for state constraints with finite $p$ it is necessary that there exists Lagrange multipliers corresponding to state constraints that belong to $L^p[0, T]$. If no such multipliers exist, then the exact $L^p$-penalization with finite $p$ is impossible.

Although we obtained a number of results on exact penalty functions for optimal control problems with terminal and pointwise state constraints, a further research in this area is needed. In particular, it is interesting to find verifiable sufficient conditions under which assumptions of Theorems 10, 17, and 18 on the complete exactness of corresponding penalty functions hold true in the nonlinear case. Moreover, the main results of our study can be easily extended to nonsmooth optimal control problems. In particular, one can suppose that the integrand $\theta$ is only locally Lipschitz continuous in $x$ and $u$, and impose the same growth conditions on the Clarke subdifferential (or some other suitable subdifferential), as we did on the derivatives of this functions. Also, it seems worthwhile to analyze connections between necessary/sufficient optimality conditions and the local exactness of penalty functions (cf, the papers of Xing et al. and sections 4.6.2 and 4.7.2 in Reference 22).

It should be noted that the general results on exact penalty functions that we utilized throughout our study are based on completely independent assumptions on the cost functional and constraints (see Theorems 1 and 2). This approach allowed us to consider counterexamples in which the cost functionals were unrealistic from a practical point of view (cf, Example 3). Therefore, it seems profitable to obtain new general results on the exactness of penalty functions with the use of assumptions that are based on the interplay between the cost functional and constraints (cf, such conditions for Huyer and Neumaier’s penalty function in the finite dimensional case in Reference 29).

Finally, for obvious reasons in this two-part study we restricted our consideration to several “classical” problems. Our goal was not to apply the theory of exact penalty functions to as many optimal control problems as possible, but to demonstrate the main tools, as well as merits and limitations, of this theory on several standard problems, in the hope that it will help the interested reader to apply the exact penalty function method to the optimal control problem at hand.

ACKNOWLEDGEMENTS
The author wishes to express his thanks to the coauthor of the first part of this study A.V. Fominyh for many useful discussions on exact penalty functions and optimal control problems that led to the development of this paper. The results presented in this article were supported by the President of Russian Federation grant for the support of young Russian scientists (grant number MK-3621.2019.1).

ORCID
M. V. Dolgopolik https://orcid.org/0000-0003-0677-0797

REFERENCES
1. Evans JP, Gould FJ, Tolle JW. Exact penalty functions in nonlinear programming. Math Program. 1973;4:72-97.
2. Han SP, Mangasarian OL. Exact penalty functions in nonlinear programming. Math Program. 1979;17:251-269.
3. Di Pillo G, Grippo L. An exact penalty function method with global convergence properties for nonlinear programming problems. Math Program. 1986;36:1-18.
4. Di Pillo G, Grippo L. On the exactness of a class of nondifferentiable penalty functions. J Optim Theory Appl. 1988;57:399-410.
5. Di Pillo G, Grippo L. Exact penalty functions in constrained optimization. SIAM J Control Optim. 1989;27:1333-1360.
6. Burke JV. An exact penalization viewpoint of constrained optimization. SIAM J Control Optim. 1991;29:968-998.
7. Di Pillo G. Exact penalty methods. In: Spedicato E, ed. Algorithms for Continuous Optimization. Dordrecht, EC: Springer; 1994:209-253.
8. Di Pillo G, Facchinei F. Exact Barrier function methods for Lipschitz programs. Appl Math Optim. 1995;32:1-31.
9. Zaslavski AJ. Optimization on Metric and Normed Spaces. New York, NY: Springer; 2010.
10. Dolgopolik MV. A unifying theory of exactness of linear penalty functions. Optimization. 2016;65:1167-1202.
11. Dolgopolik MV. A unifying theory of exactness of linear penalty functions II: parametric penalty functions. Optimization. 2017;66:1577-1622.
APPENDIX A. PROOF OF THEOREM 2

Observe that under the assumptions of Theorem 2 assumptions of Theorem 1 are satisfied for any \( c \in \mathbb{R} \) and \( \delta > 0 \). Therefore, by this theorem there exists \( \lambda^* \geq 0 \) such that for any \( \lambda \geq \lambda^* \) the optimal values and globally optimal solutions of the problem \((P)\) and problem \((2)\) coincide.

Let \( L > 0 \) be a Lipschitz constant of \( I \) on \( A \), and fix any \( x \in A \setminus \Omega \). By our assumption \( \varphi^1_{\lambda}(x) \leq -a < 0 \). By the definition of the rate of steepest descent there exists a sequence \( \{x_n\} \subset A \) converging to \( x \) and such that \( \varphi(x_n) - \varphi(x) < -ad(x_n, x)/2 \) for all \( n \in \mathbb{N} \). Therefore

\[
\Phi_\lambda(x_n) - \Phi_\lambda(x) = I(x_n) - I(x) + \lambda (\varphi(x_n) - \varphi(x)) \leq \left(L - \frac{a}{2}\right) d(x_n, x),
\]

for any \( n \in \mathbb{N} \), which implies that \( (\Phi_\lambda)^1_{\lambda}(x) < 0 \) for all \( \lambda > 2L/a \) and \( x \in A \setminus \Omega \). Thus, if \( x^* \in A \) is an inf-stationary point of local minimum of \( \Phi_\lambda \) on \( A \) and \( \lambda > 2L/a \), then \( x^* \in \Omega \). Here we used the fact that any point of local minimum of \( \Phi_\lambda \) on \( A \) is also an inf-stationary point of \( \Phi_\lambda \) on \( A \), since \( (\Phi_\lambda)^1_{\lambda}(x) \geq 0 \) is a necessary condition for local minimum.

Fix any \( \lambda > 2L/a \). Let \( x^* \in A \) be a point of local minimum of the penalized problem \((2)\). Then \( x^* \in \Omega \). Hence bearing in mind the fact that by definition \( \Phi_\lambda(x) = I(x) \) for any \( x \in \Omega \) one obtains that \( x^* \) is a locally optimal solution of the problem \((P)\).

Let now \( x^* \in \Omega \) be a locally optimal solution of \((P)\). Clearly, \( x^* \in S_{\lambda}(c) \) for any \( c > \Phi_\lambda(x^*) \). Hence by lemma 1 of Reference 45 there exists \( r_1 > 0 \) such that \( \varphi(x) \geq a \) dist\((x, \Omega)\) for all \( x \in B(x^*, r_1) \cap A \). Furthermore, by lemma 2 and Remark 11 of Reference 45 there exists \( r_2 > 0 \) such that \( I(x) - I(x^*) \geq -L \) dist\((x, \Omega)\) for any \( x \in B(x^*, r_2) \cap A \). Consequently, for any \( x \in B(x^*, r) \cap A \) with \( r = \min\{r_1, r_2\} \) one has

\[
\Phi_\lambda(x) - \Phi_\lambda(x^*) = I(x) - I(x^*) + \lambda (\varphi(x) - \varphi(x^*)) \geq (-L + \lambda a) \text{ dist}\((x, \Omega)\) \geq 0.
\]
that, $x^*$ is a locally optimal solution of the penalized problem (2). Thus, locally optimal solutions of the problem $(P)$ and problem (2) coincide for any $\lambda > 2L/a$.

Let now $x^* \in A$ be an inf-stationary point of $\Phi_\lambda$ on $A$. Then $x^* \in \Omega$. By definition $\Phi_\lambda(x) = I(x)$ for any $x \in \Omega$, which yields $I_{\Omega}^1(x^*) = (\Phi_\lambda)_{1/\lambda}^1(x^*) \geq (\Phi_\lambda)_{1/\lambda}^1(x^*) \geq 0$, that is, $x^*$ is an inf-stationary point of $I$ on $\Omega$.

Let finally $x^* \in \Omega$ be an inf-stationary point of $I$ on $\Omega$. By the definition of the rate of steepest descent there exists a sequence $\{x_n\} \subset A$ converging to $x^*$ such that

$$(\Phi_\lambda)_{1/\lambda}^1(x^*) = \lim_{n \to \infty} \frac{\Phi_\lambda(x_n) - \Phi_\lambda(x^*)}{d(x_n, x^*)}.$$ 

If there exists a subsequence $\{x_{n_k}\} \subset \Omega$, then by the fact that $\varphi(x) = 0$ for all $x \in \Omega$ one gets that

$$(\Phi_\lambda)_{1/\lambda}^1(x^*) = \lim_{k \to \infty} \frac{\Phi_\lambda(x_{n_k}) - \Phi_\lambda(x^*)}{d(x_{n_k}, x^*)} = \lim_{k \to \infty} \frac{I(x_{n_k}) - I(x^*)}{d(x_{n_k}, x^*)} \geq I_{\Omega}^1(x^*) \geq 0.$$ 

Thus, one can suppose that $\{x_n\} \subset A \\Omega$.

Choose any $L' \in (L, \lambda a)$. By applying lemmas 1 and 2 from Reference 45 one obtains that

$$\Phi_\lambda(x_n) - \Phi_\lambda(x^*) = I(x_n) - I(x^*) + \lambda (\varphi(x_n) - \varphi(x^*)) \geq -L' \text{dist}(x_n, \Omega) - (L' - L)d(x_n, x^*) + \lambda \text{dist}(x_n, \Omega) \geq -L'd(x_n, x^*) + \lambda \text{dist}(x_n, \Omega) \geq -(L' - L)d(x_n, x^*),$$

for any sufficiently large $n$. Dividing this inequality by $d(x_n, x^*)$, and passing to the limit as $n \to \infty$ one obtains that $(\Phi_\lambda)_{1/\lambda}^1(x^*) \geq -(L' - L)$, which implies that $(\Phi_\lambda)_{1/\lambda}^1(x^*) \geq 0$ due to the fact that $L' \in (L, \lambda a)$ was chosen arbitrarily. Consequently, $x^*$ is an inf-stationary point of $\Phi_\lambda$ on $A$. Thus, inf-stationary points of $\Phi_\lambda$ on $A$ coincide with inf-stationary points of $I$ on $\Omega$ for any $\lambda > 2L/a$, and the proof is complete.

**APPENDIX B. SOME PROPERTIES OF NEMYTSKII OPERATORS**

For the sake of completeness, in this appendix we give complete proofs of several well-known results on continuity and differentiability of Nemytskii operators (cf, monograph 16). Firstly, we prove some auxiliary results related to state constraints of optimal control problems.

**Proposition 2.** Let $(Y, d)$ be a metric space, and $g : Y \times [0, T] \to \mathbb{R}$ be a continuous function. Then the operator $G(\gamma)(\cdot) = g(\gamma(\cdot), \cdot)$ continuously maps $C([0, T]; \gamma) \to C([0, T])$.

**Proof.** Choose any $x \in C([0, T]; \gamma)$. Due to the continuity of $g$, for any $t \in [0, T]$ and $\epsilon > 0$ there exists $\delta(t) > 0$ such that for all $y \in Y$ and $r \in [0, T]$ with $d(y, x(t)) + |t - r| < \delta(t)$ one has $|g(y, r) - g(x(t), t)| < \epsilon/2$. The set $K = \{(x(t), t) \in Y \times \mathbb{R} | t \in [0, T]\}$ is compact as the image of the compact set $[0, T]$ under a continuous map. Therefore, there exist $N \in \mathbb{N}$ and $\{t_1, \ldots, t_N\} \subset [0, T]$ such that $K \subset \bigcup_{k=1}^{N} B(x(t_k), t_k, \delta(t_k))/2$. Define $\delta = \min_{k=1}^{N} \delta(t_k)/2$.

Now, choose any $t \in [0, T]$ and $x \in C([0, T]; \gamma)$ such that $\|x - x\|_{C([0, T]; \gamma)} < \delta$. By definition one has $d(x(t), t) < \delta$. Furthermore, there exists $k \in \{1, \ldots, N\}$ such that $(x(t), t) \in B(x(t_k), t_k, \delta(t_k))/2$, which due to the definition of $\delta$ implies that $(x(t), t) \in B((x(t_k), t_k, \delta(t_k))$. Hence by the definition of $\delta(t_k)$ one has

$$\|g((x(t), t) - g(x(t), t)) \leq |g((x(t), t) - g(x(t_k), t_k))| + |g(x(t_k), t_k) - g(x(t), t)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which yields $\|g(x(t), \cdot) - g(x(t), \cdot)\|_{\infty} < \epsilon$ due to the fact that $t \in [0, T]$ is arbitrary. Thus, the operator $G$ continuously maps $C([0, T]; \gamma) \to C([0, T])$.

**Corollary 2.** Let $(Y, d)$ be a metric space, and $g_j : Y \times [0, T] \to \mathbb{R}$, $j \in J = \{1, \ldots, l\}$, be continuous functions. Then the function $\varphi : C([0, T]; \gamma) \to \mathbb{R}$, $\varphi(x) = \max_{t \in [0, T]} \max_{j \in J} g_j(x(t), t)$ is continuous.

**Proof.** Fix any $x \in C([0, T]; \gamma)$. By Proposition 2 for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $x \in B(x, \delta)$ one has $\|g_j(x(t), \cdot) - g_j(x(t), \cdot)\|_{\infty} < \epsilon$ for all $j \in J$. Consequently, for any such $x$ one has

$$g_j(\tilde{x}(t), t) \leq g_j(x(t), t) + \epsilon \leq \varphi(x) + \epsilon \quad \forall t \in [0, T], j \in J.$$
Taking the supremum, at first, over all \( j \in J \), and then over all \( t \in [0, T] \) one obtains that \( \varphi(\bar{x}) \leq \varphi(x) + \epsilon \). Arguing in the same way but swapping \( \bar{x} \) with \( x \) one obtains that \( \varphi(x) \leq \varphi(\bar{x}) + \epsilon \). Therefore, \( |\varphi(x) - \varphi(\bar{x})| < \epsilon \), provided \( \|x - \bar{x}\|_{C[0,T]:Y} \leq \delta \), that is, \( \varphi \) is continuous.

**Theorem 19.** Let a function \( g : \mathbb{R}^d \times [0, T] \to \mathbb{R} \), \( g = g(x, t) \), be continuous, differentiable in \( x \), and the function \( \nabla_x g \) be continuous. Then for any \( p \in [1, +\infty] \) the Nemytskii operator \( G(x) = g(x, \cdot) \cdot \) maps \( W^d_{1,p}(0, T) \) to \( C[0, T] \), is continuously Fréchet differentiable on \( W^d_{1,p}(0, T) \), and its Fréchet derivative has the form \( DG(x)[h] = \nabla_x g(x, \cdot) h(\cdot) \) for all \( x, h \in W^{d,p}(0, T) \).

**Proof.** Recall that we identify \( W^d_{1,p}(0, T) \) with the space of all those absolutely continuous functions \( x : [0, T] \to \mathbb{R}^d \) for which \( \dot{x} \in L^p_{1,p}(0, T) \) (e.g., monograph \(^{53}\)). Hence bearing in mind the fact the function \( g \) is continuous one obtains that for any \( x \in W^d_{1,p}(0, T) \) one has \( g(x, \cdot) \in C[0, T] \), that is, the operator \( G \) maps \( W^d_{1,p}(0, T) \) to \( C[0, T] \). Let us check that this operator is Fréchet differentiable.

Fix any \( x, h \in W^d_{1,p}(0, T) \). By the mean value theorem for any \( t \in [0, T] \) one has

\[
\frac{1}{\alpha} \left( g(x(t) + \alpha h(t), t) - g(x(t), t) \right) = \nabla_x g(x(t), t) h(t) \left| h(t) \right|_{\infty}.
\]

By Proposition 2 the function \( x \mapsto \nabla_x g(x, \cdot, \cdot) \) continuously maps \( C[0, T] \) to \( (C[0, T])^d \), which by inequality (5) implies that it continuously maps \( W^d_{1,p}(0, T) \) to \( (C[0, T])^d \). Consequently, the right-hand side of inequality (B1) converges to zero uniformly on \([0, T]\) as \( \alpha \to +0 \). Thus, one has

\[
\lim_{\alpha \to 0} \left\| G(x + \alpha h) - G(x) - \nabla_x g(x, \cdot, \cdot) h(\cdot) \right\|_{\infty} = 0,
\]

that is, the operator \( G \) is Gâteaux differentiable, and its Gâteaux derivative has the form \( DG(x)[h] = \nabla_x g(x, \cdot, \cdot) h(\cdot) \). Note that the map \( DG(\cdot) \) is continuous, since the nonlinear operator \( x \mapsto \nabla_x g(x, \cdot, \cdot) \) continuously maps \( W^d_{1,p}(0, T) \) to \( (C[0, T])^d \) by Proposition 2 and inequality (5). Hence, as is well known, the operator \( G \) is continuously Fréchet differentiable, and its Fréchet derivative coincides with the Gâteaux one.

Let us also prove the differentiability of the Nemytskii operator \( F(x, u) = x(\cdot) - f(x(\cdot), u(\cdot, \cdot)) \) associated with the nonlinear differential equation \( \dot{x} = f(x, u, t) \).

**Theorem 20.** Let a function \( f : \mathbb{R}^d \times \mathbb{R}^m \times [0, T] \to \mathbb{R}^d \), \( f = f(x, u, t) \), be continuous, differentiable in \( x \) in \( u \), and the functions \( \nabla_x f \) and \( \nabla_u f \) be continuous. Suppose also that \( q \geq p \geq 1 \), and either \( q = +\infty \) or \( f \) and \( \nabla_u f \) satisfy the growth condition of order \( q/p, p \), while \( \nabla_x f \) satisfies the growth condition of order \( q/s, s \) with \( s = q(p - q)/q > p \) in the case \( q > p \), and \( \nabla_u f \) does not depend on \( u \) in the case \( q = p \). Then the nonlinear operator \( F(x, u) = (x(\cdot) - f(x(\cdot), u(\cdot, \cdot), x(T)) \) maps \( X = W^d_{1,p}(0, T) \times L^q_u(0, T) \) to \( L^p(0, T) \times \mathbb{R}^d \), is continuously Fréchet differentiable, and its Fréchet derivative has the form

\[
DF(x, u)[h, v] = \left( \frac{\dot{h}(\cdot) - A(\cdot) h(\cdot) - B(\cdot) v(\cdot)}{h(T)} \right), \quad A(t) = \nabla_x f(x(t), u(t), t), \quad B(t) = \nabla_u f(x(t), u(t), t),
\]

for any \((x, u) \in X\).

**Proof.** Let us prove that the Nemytskii operator \( F_0(x, u) = f(x(\cdot), u(\cdot, \cdot)) \) maps \( X \) to \( L^p_u(0, T) \), is continuously Fréchet differentiable, and its Fréchet derivative has the form

\[
DF_0(x, u)[h, v] = A(\cdot) h(\cdot) + B(\cdot) v(\cdot), \quad A(t) = \nabla_x f(x(t), u(t), t), \quad B(t) = \nabla_u f(x(t), u(t), t),
\]

for any \((x, u) \in X\) and \((h, v) \in X\). With the use of this result one can easily prove that the conclusion of the theorem holds true.

Fix any \((x, u) \in X\). By inequality (5) there exists \( R > 0 \) such that \( \|x\|_{\infty} \leq R \). Then by the growth condition on the function \( f \) there exist \( C_R > 0 \) and an a.e. nonnegative function \( \omega_R \in L^p(0, T) \) such that

\[
|f(x(t), u(t), t)|^p \leq (C_R|u(t)|^q + \omega_R(t))^p \leq 2^p \left( C_R^p|u(t)|^q + \omega_R(t)^p \right).
\]
for a.e. \( t \in (0, T) \). Observe that the right-hand side of this inequality belongs to \( L^1(0, T) \). Therefore, \( F_0(x, u) = f(x(\cdot), u(\cdot), 0) \in L^2_{\text{loc}}(0, T) \), that is, the operator \( F_0 \) maps \( X \) to \( L^2_{\text{loc}}(0, T) \). Now we turn to the proof of the Fréchet differentiability of this operator. Let us consider two cases.

**Case** \( q = +\infty \). Fix any \( (x, u) \in X \), \( (h, v) \in X \) and \( a \in (0, 1) \). By the mean value theorem for a.e. \( t \in (0, T) \) one has

\[
\frac{1}{a} \left| f(x(t) + ah(t), u(t) + av(t), t) - f(x(t), u(t), t) - a\nabla f(x(t), u(t), t)h(t) - a\nabla u f(x(t), u(t), t)v(t) \right| \\
\leq \sup_{\eta \in (0, a)} \sup_{t \in [0, T]} \left| \nabla_x f(x(t) + \eta h(t), u(t) + \eta v(t), t) - \nabla_x f(x(t), u(t), t) \right| |h(t)| \\
+ \sup_{\eta \in (0, a)} \sup_{t \in [0, T]} \left| \nabla_u f(x(t) + \eta h(t), u(t) + \eta v(t), t) - \nabla_u f(x(t), u(t), t) \right| |v(t)|.
\]

(B3)

With the use of the facts that all functions \( x, h, u, \) and \( v \) are essentially bounded on \([0, T]\), and the functions \( \nabla_x f \) and \( \nabla_u f \) are uniformly continuous one the compact set \( \mathcal{B}(\mathbf{0}_d, \|x\|_\infty + \|h\|_\infty) \times \mathcal{B}(\mathbf{0}_m, \|u\|_\infty + \|v\|_\infty) \times [0, T] \) (here \( \mathbf{0}_d \) is the zero vector from \( \mathbb{R}^d \)) one can verify that the right-hand side of inequality (B3) converges to zero as \( a \to +0 \). Observe also that \( A(\cdot) = \nabla_x f(x(\cdot), u(\cdot), \cdot) \in L^{\text{ess}}_{\text{loc}}(0, T) \) and \( B(\cdot) = \nabla_u f(x(\cdot), u(\cdot), \cdot) \in L^{\text{ess}}_{\text{loc}}(0, T) \) due to the continuity of \( \nabla_x f \) and \( \nabla_u f \) and the essential boundedness of \( x \) and \( u \). Hence, as it is easy to check, the mapping \((h, v) \mapsto A(\cdot)h(\cdot) + B(\cdot)v(\cdot)\) is a bounded linear operator from \( X \) to \( L^{\text{ess}}_{p}(0, T) \) (and, therefore, to \( L^{\text{ess}}_{p}(0, T) \)). Thus, one has

\[
\lim_{a \to 0} \left\| \frac{1}{a} \left( F_0(x + ah, u + av) - F_0(x, u) \right) - DF_0(x, u)[h, v] \right\|_p = 0,
\]

where \( DF_0(x, u)[h, v] \) is defined as in equality (B2) (here \( 1/p = 0 \), if \( p = +\infty \)). Consequently, the Nemytskii operator \( F_0 \) is Gâteaux differentiable at every point \( (x, u) \in X \), and its Gâteaux derivative is defined in equality (B2).

Let us check that the Gâteaux derivative \( DF_0(\cdot) \) is continuous on \( X \). Then, as is well-known, \( F_0 \) is continuously Fréchet differentiable on \( X \), and its Fréchet derivative coincides with \( DF_0(\cdot) \). Fix any \( (x, u) \in X \) and \( (x', u') \in X \). For any \( (h, v) \in X \) one has

\[
\left\| DF_0(x, u)[h, v] - DF_0(x', u')[h, v] \right\|_p \leq T^{1/p} \left\| DF_0(x, u)[h, v] - DF_0(x', u')[h, v] \right\|_{\infty} \\
\leq T^{1/p} \text{ess sup}_{t \in [0, T]} \left| \nabla_x f(x(t), u(t), t) - \nabla_x f(x'(t), u'(t), t) \right| |h(t)| + T^{1/p} \text{ess sup}_{t \in [0, T]} \left| \nabla_u f(x(t), u(t), t) - \nabla_u f(x'(t), u'(t), t) \right| |v(t)|.
\]

Hence with the use of inequality (5) one obtains that there exists \( C_p > 0 \) (depending only on \( p \) and \( T \)) such that

\[
\left\| DF_0(x, u) - DF_0(x', u') \right\| \leq T^{1/p} C_p \text{ess sup}_{t \in [0, T]} \left| \nabla_x f(x(t), u(t), t) - \nabla_x f(x'(t), u'(t), t) \right| \\
+ T^{1/p} \text{ess sup}_{t \in [0, T]} \left| \nabla_u f(x(t), u(t), t) - \nabla_u f(x'(t), u'(t), t) \right|.
\]

Utilizing this inequality and taking into account the fact that the functions \( \nabla_x f \) and \( \nabla_u f \) are continuous one can verify via a simple \( \varepsilon - \delta \) argument that \( \left\| DF_0(x, u) - DF_0(x', u') \right\| \to 0 \) as \( (x', u') \to (x, u) \) in \( X \) (cf, the proof of Proposition 2). Thus, the mapping \( DF_0(\cdot) \) is continuous, and the proof of the case \( q = +\infty \) is complete.

**Case** \( q < +\infty \). Fix any \( (x, u) \in X \), \( (h, v) \in X \) and \( a \in (0, 1] \). By the mean value theorem

\[
\frac{1}{a} \left| f(x(t) + ah(t), u(t) + av(t), t) - f(x(t), u(t), t) - a\nabla f(x(t), u(t), t)h(t) - a\nabla u f(x(t), u(t), t)v(t) \right|^p \\
\leq 2^p \sup_{\eta \in (0, a)} \left| \nabla_x f(x(t) + \eta h(t), u(t) + \eta v(t), t) - \nabla_x f(x(t), u(t), t) \right|^p |h(t)|^p \\
+ 2^p \sup_{\eta \in (0, a)} \left| \nabla_u f(x(t) + \eta h(t), u(t) + \eta v(t), t) - \nabla_u f(x(t), u(t), t) \right|^p |v(t)|^p,
\]

(B4)

for a.e. \( t \in (0, T) \). Our aim is to apply Lebesgue’s dominated convergence theorem.

The right-hand side of inequality (B4) converges to zero as \( a \to 0 \) for a.e. \( t \in (0, T) \) due to the continuity of \( \nabla_x f \) and \( \nabla_u f \). By applying inequality (5), and the facts that \( a \in (0, 1] \) and \( \nabla_x f \) satisfies the growth condition of order \( (q/p, p) \) one
obtains that there exist $C_R > 0$ and an a.e. nonnegative function $\omega_R \in L^p(0, T)$ such that
\[
\sup_{\eta \in (0, a)} |\nabla_{uf}(x(t) + \eta h(t), u(t) + \eta v(t), t) - \nabla_{uf}(x(t), u(t), t)|^p |h(t)|^p \leq 2^p \sup_{\eta \in (0, a)} |\nabla_{uf}(x(t) + \eta h(t), u(t) + \eta v(t), t)|^p C_p \|h\|_{L^p}^p
+ 2^p \sup_{\eta \in (0, a)} |\nabla_{uf}(x(t), u(t), t)|^p C_p \|h\|_{L^p}^p \leq 2^p \left( \left( C_R 2^{q/s}(\|u(t)\|^{q/s} + \|v(t)\|^{q/s}) + \omega_R(t) \right)^p + \left( C_R \|u(t)\|^{q/s} + \omega_R(t) \right)^p \right) |v(t)|^p,
\]
for a.e. $t \in (0, T)$. Observe that the right-hand side of this inequality belongs to $L^1(0, T)$ and does not depend on $\alpha$, that is, the first term in the right-hand side of inequality (B4) can be bounded above by a function from $L^1(0, T)$ that is independent of $\alpha$.

Let us now estimate the second term in the right-hand side of inequality (B4). Let $q > p$. Bearing in mind the fact that $\nabla_{uf}$ satisfies the growth condition of order $(q/s, s)$ one obtains that there exists $C_R > 0$ and an a.e. nonnegative function $\omega_R \in L^p(0, T)$ such that
\[
\sup_{\eta \in (0, a)} |\nabla_{uf}(x(t) + \eta h(t), u(t) + \eta v(t), t) - \nabla_{uf}(x(t), u(t), t)|^p |v(t)|^p \leq 2^p \left( \left( C_R 2^{q/s}(\|u(t)\|^{q/s} + \|v(t)\|^{q/s}) + \omega_R(t) \right)^p + \left( C_R \|u(t)\|^{q/s} + \omega_R(t) \right)^p \right) |v(t)|^p,
\]
for a.e. $t \in (0, T)$. Let us check that the right-hand side of this inequality belongs to $L^1(0, T)$. Indeed, by applying Hölder’s inequality of the form
\[
\left( \int_0^T |y_1(t)|^p |y_2(t)|^p \, dt \right)^{1/p} \leq \|y_1\|_q \|y_2\|_q, \tag{B5}
\]
(here we used the fact that $(q/p)' = s/p$) one gets that
\[
\left( \int_0^T \left| C_R 2^{q/s}(\|u(t)\|^{q/s} + \|v(t)\|^{q/s}) + \omega_R(t) \right|^p \, dt \right)^{1/p} \leq \left( \left( C_R 2^{q/s}(\|u\|_q^{q/s} + \|v\|_q^{q/s}) + \|\omega_R\|_s \right) \|v\|_q \right) < +\infty.
\]
Thus, the last term in the right-hand side of inequality (B4) can also be bounded above by a function from $L^1(0, T)$ that does not depend on $\alpha$.

Finally, recall that in the case $q = p$ the function $\nabla_{uf}$ does not depend on $u$, which implies that it satisfies the growth condition of order $(0, +\infty)$, that is, for any $R > 0$ there exists $C_R > 0$ such that $|\nabla_{uf}(x, u, t)| \leq C_R$ for a.e. $t \in (0, T)$ and for all $(x, u) \in \mathbb{R}^d \times \mathbb{R}^m$ with $|x| \leq R$. Therefore, as is easy to check, in this case there exists $C > 0$ (that does not depend on $\alpha$) such that
\[
\sup_{\eta \in (0, a)} |\nabla_{uf}(x(t) + \eta h(t), u(t) + \eta v(t), t) - \nabla_{uf}(x(t), u(t), t)|^p |v(t)|^p \leq C |v(t)|^q,
\]
for a.e. $t \in (0, T)$. The right-hand side of this inequality obviously belongs to $L^1(0, T)$.

Thus, the right-hand side of inequality (B4) can be bounded above by a function from $L^1(0, T)$ that does not depend on $\alpha$. Furthermore, from the growth conditions on $\nabla_{uf}$ and $\nabla_{uf}$ it follows that $A(\cdot) = \nabla_{uf}(x(\cdot), u(\cdot), \cdot) \in L^{s,\text{loc}}(0, T)$ and $B(\cdot) = \nabla_{uf}(x(\cdot), u(\cdot), \cdot) \in L^{s,\text{loc}}(0, T)$, which, as is easily seen, implies that the mapping $(h, v) \mapsto A(\cdot)h(\cdot) + B(\cdot)v(\cdot)$ is a bounded linear operator from $X$ to $L^p(0, T)$ (here $s = +\infty$ in the case $p = q$). Therefore, integrating inequality (B4) from 0 to $T$ and passing to the limit as $\alpha \to 0$ with the use of Lebesgue’s dominated convergence theorem one obtains that
\[
\lim_{\alpha \to 0} \left\| \frac{1}{\alpha} \left( F_0(x + ah, u + av) - F_0(x, u) \right) - DF_0(x, u)[h, v] \right\|_p = 0,
\]
where $DF_0(x, u)[h, v]$ is defined as in equality (B2). Thus, the Nemyskii operator $F_0$ is Gâteaux differentiable at every point $(x, u) \in X$, and its Gâteaux derivative is given by equality (B2). Let us check that this derivative is continuous.
Then one can conclude that $F_0$ is continuously Fréchet differentiable on $X$, and its Fréchet derivative coincides with $DF_0(\cdot)$.

Indeed, choose any $(x, u) \in X$ and $(x', u') \in X$. With the use of equality (B2) and Hölder’s inequality of the form of inequality (B5) one obtains

$$\|DF_0(x, u)[h, v] - DF_0(x', u')[h, v]\|_p \leq \|\nabla f(x(\cdot), u(\cdot), \cdot) - \nabla f(x'(\cdot), u'(\cdot), \cdot)\|_p \|[h]\|_\infty + \|\nabla f(x(\cdot), u(\cdot), \cdot) - \nabla f(x'(\cdot), u'(\cdot), \cdot)\|_q \|v\|_q,$$

for any $(h, v) \in X$ (in the case $q = p$ we put $s = \infty$). Hence taking into account that $\nabla f$ is affine in control.

Taking into account the absolute continuity of the Lebesgue integral and the fact that $\mu < \delta(\epsilon)$ one can suppose that $\int_E |u|^{q_1/p} d\mu < \epsilon$. Therefore, choosing a sufficiently small $\epsilon > 0$ one can make the integral $\int_E \nabla f(x(t), u(t)) |u(t)|^{q_1/p} d\mu(t)$ for all $k \in \mathbb{N}$ and measurable sets $E \subset (0, T)$ with $\mu(E) < \delta(\epsilon)$. Consequently, by the “if” part of Vitali’s theorem on convergence in $L_p^d(0, T)$ one can conclude that $\nabla f(x(\cdot), u(\cdot), \cdot)$ converges to $\nabla f(x(\cdot), u(\cdot), \cdot)$ in $L_p^d(0, T)$, which contradicts inequality (B6).

It should be noted that all functions $\{\nabla f(x_n(\cdot), u_n(\cdot), \cdot)\}$ belong to $L_p^d(0, T)$ due to the fact that $\nabla f$ satisfies the growth condition of order $(q/p, p)$.

By inequality (5) the sequence $\{x_n\}$ converges to $x$ uniformly on $[0, T]$, which implies that $\|x_n\|_\infty \leq R$ for all $k \in \mathbb{N}$ and some $R > 0$. The sequence $\{u_n\}$ converges to $u$ in $L_p^\infty(0, T)$. Hence, as is well-known, there exists a subsequence, which we denote again by $\{u_n\}$, that converges to $u$ almost everywhere. Consequently, by the continuity of $\nabla f$ the sequence $\{\nabla f(x_n(t), u_n(t), t)\}$ converges to $\nabla f(x(t), u(t), t)$ for a.e. $t \in (0, T)$.

The sequence $\{u_n\}$ converges to $u$ in $L_p^\infty(0, T)$. Therefore, by the “only if” part of Vitali’s theorem characterizing convergence in $L^p$ spaces (e.g. theorem III.6.15 of Reference 64) for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for any Lebesgue measurable set $E \subset (0, T)$ with $\mu(E) < \delta(\epsilon)$ one has $\int_E |u_n|^{q_1/p} d\mu < \epsilon$ for all $k \in \mathbb{N}$. Hence by applying the fact that $\nabla f$ satisfies the growth condition of order $(q/p, p)$ one obtains that there exist $C_R > 0$ and an a.e. nonnegative function $\omega_R \in L_p^\infty(0, T)$ such that for any measurable set $E \subset (0, T)$ with $\mu(E) < \delta(\epsilon)$ one has

$$\int_E |\nabla f(x_n(t), u_n(t), t)|^p d\mu(t) \leq \int_E \omega_R |u(t)|^{q_1/p} d\mu(t) + \int_E |\nabla f(x_n(t), u_n(t), t)|^p d\mu(t) \leq 2^p \left( C_R^p \epsilon + \int_E |\nabla f(x(t), u(t), t)|^p d\mu(t) \right).$$

Taking into account the absolute continuity of the Lebesgue integral and the fact that $\omega_R \in L_p^\infty(0, T)$, and decreasing $\delta(\epsilon)$, if necessary, one can suppose that $\int_E \omega_R^p d\mu < \epsilon$. Therefore, choosing a sufficiently small $\epsilon > 0$ one can make the integral $\int_E |\nabla f(x_n(t), u_n(t), t)|^p d\mu(t)$ arbitrarily small for all $k \in \mathbb{N}$ and measurable sets $E \subset (0, T)$ with $\mu(E) < \delta(\epsilon)$. Consequently, by the “if” part of Vitali’s theorem on convergence in $L_p^d(0, T)$ one can conclude that $\nabla f(x(\cdot), u(\cdot), \cdot)$, $\nabla f(x(\cdot), u(\cdot), \cdot)$ in $L_p^d(0, T)$, which contradicts inequality (B6).

Note that in the case $p = q < +\infty$ one must prove that the sequence $\{\nabla f(x_n(\cdot), u_n(\cdot), \cdot)\}$ converges in $L_p^{\infty}(0, T)$, while $\{u_n\}$ converges only in $L_q^\infty(0, T)$ with $q < +\infty$. That is why in this case one must assume that $\nabla f$ does not depend on $u$, that is, $f$ is affine in control.