Algorithm for the fermionic lines in GRACE-SUSY*
(revised version**)

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Abstract
Algorithm of constructing Feynman amplitudes in the framework of minimal supersymmetric extension of the standard model is presented, which can be easily implemented in GRACE, the program of automatic generation of Feynman amplitudes. The equivalence of our method with the prescription given by Denner et al. is proved.

1. Introduction
The fundamental theory of elementary particle physics is described by lagrangians. Given a lagrangian, the important task is to calculate the reaction rates, such as production rates, scattering cross sections or production cross sections of elementary particles, which should be confronted with experimental data to confirm that the theory based on the lagrangian is correct. As the $S$ matrix is a nonlinear functional of the lagrangian, we usually cannot calculate the reaction rates exactly. The conventional approach is, therefore, to use the perturbative expansion of the $S$ matrix in the power of the coupling constants and to calculate reaction rates to a certain order of coupling constants.

Given the order of perturbation, one draws all the Feynman diagrams consistent with the theory and write down the corresponding Feynman amplitudes. The reaction rates are obtained by squaring the sum of the amplitudes and integrating over the phase space. It is this part of the calculation that is straightforward but very tedious. For example, for the process $e^+e^- \rightarrow W^+W^-$, in the lowest order in perturbation there are only three Feynman diagrams and the computation of the cross section is rather easy. Already at the next order, however, there are as many as 140 Feynman diagrams (in a covariant gauge) and the calculation of the cross section by hand is a very

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Recently, several attempts have appeared\cite{1}\cite{2}\cite{3}\cite{4} to automatize the computation in the standard model of electroweak and QCD interactions. The most advanced one is the program called GRACE, which was developed by the Minami-Tateya Collaboration at KEK\cite{5}. The program is used, for example, to calculate the cross section of $e^+e^- \rightarrow f_1f_2\bar{f}_3\bar{f}_4$\cite{6}\cite{7}, which is the background for the $W$-pair production process $e^+e^- \rightarrow W^+W^-$ and consists of over 100 diagrams.

In this paper I would like to give an instruction how to enlarge and modify the GRACE in order to implement the supersymmetric theory (SUSY) in it. The enlarged version of GRACE is called GRACE-SUSY. The motivation of SUSY is extensively discussed in the literature (see, for example, ref.\cite{8}) and the automatic calculation of the SUSY processes is highly desired from several points of view. First, the theory of SUSY itself is very important, although at present there is no evidence of the existence of SUSY particles in Nature. Secondly, for a given process with a given order of perturbation, there are much more diagrams involved in SUSY than in the Standard Model (SM) and the introduction of automatic computation of the amplitudes is necessary when one wants to calculate various cross sections containing SUSY particles. Thirdly, expansion of the automatic computation of the amplitudes from SM to SUSY is nontrivial due to the new aspects characteristic to SUSY and it is worthwhile to study how to embed SUSY in GRACE for its own sake.

Since the new aspects of SUSY appear essentially in its fermionic sector, the discussion is concentrated on the algorithm of how to treat fermionic lines in SUSY. In section 2, the algorithm for SM is briefly discussed, then in section 3 its extension to the SUSY case is derived. Section 4 gives some comments for the case of identical particles. Some examples are shown in section 5, which show that the algorithm and the rule of GRACE-SUSY are correct. The equivalence of our method and the prescription given by Denner et al. is proved in section 6. Appendix A contains necessary mathematical formulae of charge conjugation and Majorana particles. The rule and the algorithm for GRACE-SUSY are summarized in Appendix B and Appendix C.

2 Standard Model case

The main part of this section is developed by Minami-Tateya Collaboration, KEK\cite{5}. We consider the process with $n_i$ incident spinor particles and $n_f$ outgoing spinor particles.* The matrix element is written as

$$< \text{final}|\mathcal{L}(x_1)....\mathcal{L}(x_p)|\text{initial} > = < 0|d_{n_i+n_f}....d_{n_i+1}\mathcal{L}(x_1)....\mathcal{L}(x_p)d_{n_i}^\dagger....d_{n_i+1}^\dagger|0>, \quad (1)$$

where $d_i^\dagger$ can be either $a_i^\dagger$, the creation operator of fermion, or $b_i^\dagger$ the creation operator of antifermion. Correspondingly, $d_i$ can be either $a_i$, the annihilation operator of fermion or $b_i$, the annihilation operator of antifermion. At this stage it is very important to note that the order of creation operators and annihilation operators in (1) is chosen arbitrarily but once the order is fixed we have to stick to it, because any interchange of spinor particles induces a minus sign. Tanaka and Kaneko \cite{1}\cite{2} have chosen the convention such that we assign the particle numbers 1 to $n_i$

* We use the terminology "spinor particles" as the representative of fermions, antifermions and Majorana particles.
to each of the \( n_i \) incident spinor particles, namely Dirac fermions and Dirac antifermions in this order. Likewise, we assign the particle numbers \( n_i + 1 \) to \( n_i + n_f \) to each of the \( n_f \) outgoing spinor particles, namely Dirac antifermions and Dirac fermions in this order. Note that

\[
|n_i - n_f| = \text{even},
\]

and therefore \( n_i + n_f \) is also an even integer.

### 2.1 Pairing of spinors

Since the flow of spin one-half particles is not disconnected by the interaction nor has any branch, any of the external spinor particles (independent of whether it belongs to the initial state or final state) must be pairwise connected. For a given process with \( n_i + n_f \) spinor particles, there exist \( \frac{n_i + n_f}{2} \) spinor lines in each Feynman diagram. When one of such pairs consists of two particles with particle number \( m \) and \( n \) with \( m > n \), we write it symbolically as

\[
(m, n), \quad m > n.
\]

Using the notation of (3), pairing of \( (n_i + n_f) \) spinor particles is expressed as

\[
P(a_1, a_2, \ldots, a_{n_i+n_f}) \equiv (a_1, a_2)(a_3, a_4)\ldots(a_{n_i+n_f-1}, a_{n_i+n_f}).
\]

There are \( (n_i + n_f - 1)!! \) ways of combining \( n_i + n_f \) particles into \( \frac{n_i + n_f}{2} \) pairs. Of course, not all of the \( (n_i + n_f - 1)!! \) ways of pairing spinor particles are physical or realized at the given order of the perturbation, since fermion number must be conserved along the fermion lines. For example, if both \( m \) and \( n \) belong to the initial particles, one of them is fermion and the other must be antifermion. For each physically allowed \( P \), there exists a distinct set of Feynman diagrams.

Pairing of spinor particles means that the creation and annihilation operators appearing in (1) are rearranged such that the creation and/or annihilation operators in a pair come next to each other. Symbolically, the original order of the operators becomes

\[
(n_i + n_f, n_i + n_f - 1, \ldots, 2, 1) \rightarrow \sum_P \text{Sign}(P)P,
\]

where the sum is taken over \( (n_i + n_f - 1)!! \) possible ways of pairings \( P \)'s and \( \text{Sign}(P) \) is due to the permutation of creation and annihilation operators:

\[
\text{Sign}(P) = \begin{cases} +1, & \text{even}, \\ -1, & \text{odd} \end{cases}
\]

where \( \text{even} \) and \( \text{odd} \) mean that the series \( (a_1, a_2, \ldots, a_{n_i+n_f}) \) is obtained from the initial series \( (n_i + n_f, \ldots, 2, 1) \) (see (1)) by even permutation and odd permutation, respectively. The \( \text{Sign}(P) \) is a sign of the pairing \( P \) relative to the trivial pairing, \( (n_i + n_f, n_i + n_f - 1)\ldots(4,3)(2,1) \). Since only the relative phase is observable, the absolute sign of the trivial pairing is irrelevant, and this rule is applicable even when there is no physically allowed Feynman diagram for the trivial pairing.*

* As one sees from (10) a pair \( (m, n) \) contains interaction lagrangians \( \mathcal{L}(x_1)\ldots\mathcal{L}(x_p) \). Therefore in permuting particles, some of the particles go through these lagrangians. The interaction lagrangian being bilinear in spinor fields, there arises no extra sign from shifting spinors through interaction lagrangians.
Example:  \( e^+ e^- \rightarrow e^+ e^- \) in QED

\[
\begin{array}{cccc}
\text{particle} & e^- & e^+ & e^- \\
\text{number} & 1 & 2 & 3 & 4
\end{array}
\]  

(7)

The possible \( P \) and its sign is

\[
P = \begin{cases} 
(4, 3)(2, 1) & \text{Sign}(P) = + \\
(3, 2)(4, 1) & \text{Sign}(P) = + \\
(4, 2)(3, 1) & \text{Sign}(P) = - 
\end{cases}
\]

and the number of independent pairing is indeed \((2 + 2 - 1)!! = 3\). Among the three \( P \)'s, only the first two are allowed and correspond, at tree level, to the Feynman diagrams shown in fig.1.

![Feynman diagrams](image-url)

Figure 1. There are two Feynman diagrams for the process \( e^+ e^- \rightarrow e^+ e^- \) at tree level which correspond to the first two \( P \)'s in (8).

Recalling that the matrix element contains several interaction lagrangians (1) is now written as

\[
< \text{final}| \mathcal{L}(x_1) \ldots \mathcal{L}(x_p)|\text{initial} > = \sum_P \text{Sign}(P) \prod_{m > n} (m, n),
\]

(9)

where a more exact meaning of the pair \((m, n)\) is given as follows:

\[
(m, n) = \begin{cases} 
\langle 0| d_m \mathcal{L} \ldots \mathcal{L} d_n^\dagger |0 \rangle, & \text{when } m \in F, \ n \in I, \\
\langle 0| \mathcal{L} \ldots \mathcal{L} d_m^\dagger d_n^\dagger |0 \rangle, & \text{when } m, n \in I, \\
\langle 0| d_m d_n \mathcal{L} \ldots \mathcal{L} |0 \rangle, & \text{when } m, n \in F,
\end{cases}
\]

(10)

where

\[
\mathcal{L} = \bar{\Psi}(x) \Gamma \Psi(x).
\]

(11)

Here \( I \) and \( F \) stand for the initial state and final state, respectively. In (10), the first case corresponds to the scattering, the second case to the pair-annihilation and the last case to the pair-creation. Note that due to the restriction \( m > n \), there exists no combination of the form

\[
\langle 0| d_m \mathcal{L} \ldots \mathcal{L} d_n^\dagger |0 \rangle.
\]

(12)
2.2 Grace lines

In this subsection, we introduce the concept of Grace line, Grace line amplitude, M-direction and F-direction which are useful in constructing Feynman amplitudes from the pairs \((m, n)\). As (10) shows, a pair \((m, n)\) with \(m > n\) corresponds to a spinor line in a certain Feynman diagram. Let us stretch such a spinor line in a straight line and display it horizontally. Such a horizontal line is called Grace spinor line, in short Grace line, which is denoted by \(G_{p,q}\) when the particle numbers of the external spinor particles at the left end and at the right end are \(p\) and \(q\), respectively. For a given pair \((m, n)\), there exist two Grace lines, \(G_{m,n}\) and \(G_{n,m}\), depending on which particle is put on the right and which on the left.

In order to restore the one-to-one correspondence between \((m, n)\) and the Grace line, we assign \(G_{m,n}\) to \((m, n)\) and \(G_{n,m}\) to \(−(m, n)\). In fact, for annihilation and pair-creation, one can interchange two creation or annihilation operators in (10), leading to \(−(m, n)\). Therefore, it is natural to extend \((m, n)\) to the case when \(n > m\) by the following equation:

\[
(m, n) = −(n, m), \quad \text{when } n > m.
\]

For annihilation and pair-creation, (13) is consistent with the definition of \((m, n)\), (10), while for scattering, \((n, m)\) with \(m > n\) is defined only through (13). When \(d_m\) and \(d_n^\dagger\) are interchanged naively in (10), \((n, m)\) vanishes, contradicting with (13).

Our task is to find a rule of writing down, for a given Grace line \(G_{m,n}\), the corresponding expression. We call such an expression Grace spinor line amplitude, in short Grace line amplitude, and denote it by \([G_{m,n}]\). The Grace line amplitude is constructed by replacing each block of the Grace line, the external spinor wave functions, vertices and propagators, by its representation and putting it exactly at the place where it appears in the Grace line. The Grace line amplitude is the faithful representation of the Grace line. The Grace line amplitude, \([G_{m,n}]\), starts with the external spinor wave function corresponding to the particle \(n\) at the right endpoint and ends with the external spinor wave function corresponding to the particle \(m\) at the left endpoint.

Using the generalized definition of \((m, n)\), the correspondence among the pair \((m, n)\), the Grace line \(G_{m,n}\) and Grace line amplitude \([G_{m,n}]\) can be expressed as

\[
(m, n) ⇔ G_{m,n} ⇔ [G_{m,n}].
\]

Therefore, the rule of Grace line amplitude must respect the property (14) and satisfy an important requirement,

\[
[G_{m,n}] = −[G_{n,m}].
\]

A Grace line has \(k\) vertices and \(k−1\) spinor propagators when there are \(k\) interaction lagrangians in \((m, n)\) (see (10)). On each segment of the spinor line separated by the vertices, we assign two directions, the momentum direction (M-direction) and the fermionic direction (F-direction). Tanaka[2] named the direction from right to left (←) as the A-direction (amplitude direction), which is the direction in which the spinors are ordered in the Feynman amplitude when a fermion comes in and goes out after some interactions in between.
Table 1: Assignment of F-direction (shown on the Grace line by arrow) and M-direction (shown in the lower part) to the external spinor particles. Here $a, a^\dagger$ refer to fermions and $b, b^\dagger$ refer to antifermions. "•" stands for the interaction vertex and the fermion propagators between two such vertices are not explicitly shown.

**momentum direction**

The momentum direction (M-direction) is assigned in a natural way. For the external incoming particles the M-direction is the direction towards the vertex, while for the external outgoing particles the M-direction is the direction from the vertex. For the internal fermions, the M-direction is always taken in the A-direction (irrespective of whether the internal propagator is that of fermion or antifermion when seen in the A-direction).

**fermionic direction**

For the external Dirac particle, we can assign the F-direction in a natural way: for fermions the F-direction is in the M-direction and for antifermions in the opposite direction to its M-direction. Since the fermion number is conserved in the Standard Model, external particles at the both ends of the Grace line must have the same F-direction. This fact uniquely fixes the F-direction of the internal fermion propagators: The F-direction of the internal propagators must be in the same direction as that of the external particles. Therefore, in the Standard Model we can define the F-direction for each Grace line itself.

For illustration purpose, we show $(m, n)$ and the corresponding general form of Grace lines, $G_{m,n}$ in Table 1, where "•" stands for the interaction vertex. The fermion propagators between two such vertices are not explicitly shown.

### 2.3. Rule of $[G_{m,n}]$

In this subsection, I will find the Grace line rule, the rule of writing a Grace line amplitude $[G_{m,n}]$ from the corresponding Grace line $G_{m,n}$. It should be stressed that the Grace line rule is not identical to the Feynman rule.

The direct calculation of (11) gives

\[
< 0 |\mathcal{L}(x)...\mathcal{L}(z)b_m^\dagger a_n^\dagger |0 > = - \bar{v}_m \Gamma(x)...\Gamma(z)u_n , \quad (16a)
\]
\[
< 0 |a_m b_n \mathcal{L}(x)...\mathcal{L}(z) |0 > = - \bar{u}_m \Gamma(x)...\Gamma(z)v_n , \quad (16b)
\]
\[
< 0 |a_m \mathcal{L}(x)...\mathcal{L}(z)a_n^\dagger |0 > = + \bar{u}_m \Gamma(x)...\Gamma(z)u_n , \quad (16c)
\]
\[
< 0 |b_m \mathcal{L}(x)...\mathcal{L}(z)b_n^\dagger |0 > = - \bar{v}_m \Gamma(x)...\Gamma(z)v_m \\
= - v^T_m \Gamma(z)^T ... S^T \Gamma(x)^T (\bar{v}_n)^T , \quad (16d)
\]
where $\Gamma$ and $S$ stand for the vertex and the spinor propagator, respectively. The Grace lines given in Table 1 correspond exactly to each of the expressions (16). Note that all these expressions (16) correspond to a single notation $G_{m,n}$ and the Grace line rule must automatically reproduce all of four equations given in (16), depending on the nature of $m$ and $n$ (creation operator or annihilation operator; or fermion or antifermion).

Comparing (16) and Table 1, one immediately obtains the first rule of the Grace line amplitude concerning the external spinors at the ends of the Grace lines, which is summarized in Table 2. In order to explain the minus signs in (16a) and (16b) relative to (16c), one has to supplement the Table 2 with the following rule:

**Rule 1:** Multiply the Grace spinor amplitude by ($-$) for each external spinor whose M-direction is opposite to the A-direction.

This minus sign could, in principle, be put in the definition of endpoint spinors given in Table 2.

Next, we turn to the vertices and propagators. We adopt the convention that when the F-direction of Grace line is the same as the A-direction, we use the usual spinor propagator $S(k)$ and the usual vertex defined by the model lagrangian.* More explicitly stated, for the model lagrangian

$$\mathcal{L} = \bar{\Psi}_A \Gamma_{A,B} \Psi_B + \bar{\Psi}_B \Gamma_{B,A} \Psi_A,$$

(17)

the propagator and the vertex function of the Dirac spinor in the Grace line amplitude are given by

\[
\begin{align*}
\bullet \leftarrow \bullet \leftarrow \bullet &= S(k) \\
A \leftarrow \bullet \leftarrow B &= \Gamma_{A,B} \\
B \leftarrow \bullet \leftarrow A &= \Gamma_{B,A}
\end{align*}
\]

(18)

Here $k$ denotes the propagator momentum, which by definition of our assignment of M-direction is along the A-direction, and in turn in the present case, in the F-direction of the Grace line. Then, one can easily write down the Grace line amplitude $[G_{m,n}]$ corresponding to the first three Grace lines depicted in Table 1, with the result which is indeed identical to the right hand side of (16a),(16b) and (16c). When the F-direction is opposite to the A-direction as is the case in (16d) we use the following rule.

**Rule 2:** When the F-direction of the Grace line is opposite to the A-direction, the propagator and the vertex function of the Grace spinor amplitudes are given by $-S^T(-k)$ and $-\Gamma^T$, respectively.

* Here we tacitly assume that the model lagrangian which defines $\Gamma_{AB}$ etc is written in terms of particles (e.g. electron) but not in terms of antiparticles (e.g. positron). This seems to be a trivial statement, but actually it is nontrivial as we soon notice when we apply the Grace line rule to SUSY in which whether a certain particle is a fermion or antifermion is a matter of convention.
where $k$ is the momentum of the propagator along the A-direction (Note that the momentum of the internal spinors is always defined along the A-direction). Schematically,

\[
\begin{align*}
\bullet & \rightarrow \rightarrow \bullet \quad = \quad -S^T(-k) \\
\leftarrow k & \\
B & \rightarrow \bullet \rightarrow A \quad = \quad -\Gamma^T_{A,B} \\
A & \rightarrow \bullet \rightarrow B \quad = \quad -\Gamma^T_{B,A}
\end{align*}
\]

(19)

Table 2 and Rules 1, 2 are all we need for writing down the Grace line amplitude for a given Grace line in the Standard Model.

To be complete, we will check that the above rules indeed respect the antisymmetry property (15). When the direction of the Grace lines depicted in Table 1 is reversed together with their momenta, we obtain the Grace lines, $G_{n,m}$. Applying the rule of Grace line amplitudes to $G_{n,m}$ the Grace line amplitudes of the reversed Grace lines are obtained with the following result:

\[
\begin{align*}
\rightarrow \bullet \bullet \bullet \bullet \rightarrow \quad & = \quad -u_n^T(-\Gamma^T)(-S^T)...(-\Gamma^T)(\bar{v}_m)^T \\
p_n \rightarrow \quad & \quad \leftarrow p_m \quad = \quad \bar{v}_m\Gamma S...\Gamma u_n, \\
(20a) & \\
\rightarrow \bullet \bullet \bullet \bullet \rightarrow \quad & = \quad -v_n^T(-\Gamma^T)(-S^T)...(-\Gamma^T)(\bar{u}_m)^T \\
p_n \leftarrow \quad & \quad \rightarrow p_m \quad = \quad \bar{u}_m\Gamma S...\Gamma v_n, \\
(20b) & \\
\rightarrow \bullet \bullet \bullet \bullet \rightarrow \quad & = \quad (-)(-)^n u_n^T(-\Gamma^T)(-S^T)...(-\Gamma^T)(\bar{u}_m)^T \\
p_n \rightarrow \quad & \quad \leftarrow p_m \quad = \quad -\bar{u}_m\Gamma S...\Gamma u_n, \\
(20c) & \\
\leftarrow \bullet \bullet \bullet \bullet \leftarrow \quad & = \quad (-)(-)^n \bar{v}_n\Gamma S...\Gamma v_m \\
p_n \leftarrow \quad & \quad \rightarrow p_m \quad = \quad \bar{v}_n\Gamma S...\Gamma v_m, \\
(20d) & \\
\end{align*}
\]

Comparing (20) with (16), one sees that the requirement $[G_{m,n}]=-[G_{n,m}]$ is indeed satisfied.

The Feynman amplitude corresponding to the Feynman diagram characterized by the partition $P$ is given by the product of $\frac{n_i+n_f}{2}$ Grace line amplitudes $[G_{m,n}]$ and the appropriate bosonic parts(purely bosonic vertices and bosonic propagators) which bridge vertices on the Grace lines.

3 SUSY case

We restrict our discussion to the minimal SUSY extension of the standard model (MSSM). It contains several new particles which do not exist in SM. They are

- charged Higgs boson, $(H^{\pm})$,
- three neutral Higgs bosons (two $CP$ even states and one $CP$ odd state),
- two charginos, $\tilde{\chi}^{\pm}_1, \tilde{\chi}^{\pm}_2$,
- four Majorana neutralinos, $\tilde{\chi}^0_1, \tilde{\chi}^0_2, \tilde{\chi}^0_3, \tilde{\chi}^0_4$,
- 7 right handed scalars and 8 left handed scalars.

For the number of the scalars cited above the colour factor is taken into account.

What is new in SUSY, in addition to the new particles, is the existence of the fermion number violating interactions which come about from two kinds of interactions. Neutralino is a Majorana
particle which behaves itself as fermion as well as as antifermion. Therefore, neutralino propagator
does not necessarily conserve fermion number. Secondly, chargino behaves as fermion at some
interaction vertices but as antifermion at some other vertices. Therefore, chargino can flip the
fermion number at certain vertices. Note that, consequently, the fermion number of charginos is
not determined by their interaction. We define in our convention that the positive charginos are
Dirac fermions.

In the MSSM case, therefore, the F-direction is not necessarily conserved along the GRACE
line and the clash of the F-directions happens either at vertices involving Majorana particle(s) or
chargino(s), or at Majorana spinor propagators. Before going to the next step, we have to recall
several important properties of Majorana particles which are useful for the ensuing discussion.
More details are discussed in Appendix A.

3.1 Majorana particle

Majorana particle behaves as particle as well as antiparticle: \( \Psi = \Psi^c = C\Psi^T \). As our conven-
tion, we fix the relative phase of the spinor wave functions such that

\[
C\bar{v}(k, \lambda)^T = u(k, \lambda), \quad C\bar{u}(k, \lambda)^T = v(k, \lambda),
\]

where \( C \) stands for the charge conjugation matrix, which satisfies, independent of the representation
of gamma matrices, the following conditions,

\[
C^\dagger = C^{-1}, \quad C = -C^T, \quad C\gamma_\mu C^{-1} = -\gamma_\mu.
\]

For Majorana particles, there are three kinds of propagators,

\[
\begin{align*}
\langle \Psi \bar{\Psi} \rangle &= \bullet \leftarrow \bullet = \frac{i}{\not{k} - m}, \\
\langle \bar{\Psi} \bar{\Psi} \rangle &= \bullet \rightarrow \bullet = C^{-1} \frac{i}{\not{k} - m}, \\
\langle \bar{\Psi} \Psi \rangle &= \bullet \leftarrow \rightarrow \bullet = \frac{i}{\not{k} - m} C^T.
\end{align*}
\]

In (23) the momentum \( k \) runs from right to left. The interaction lagrangian at Majorana-Dirac
vertices is given as

\[
\mathcal{L}_{MD} = \bar{\Psi}(M_\alpha)\Gamma_{M_\alpha,D_\beta}\Psi(D_\beta) + \bar{\Psi}(D_\alpha)\Gamma_{D_\alpha,M_\beta}\Psi(M_\beta)
\]

with \( \Gamma_{D_\alpha,M_\beta} = \gamma_0\Gamma_{M_\beta,D_\alpha}\gamma_0 \). The Majorana-Majorana interaction is given by

\[
\mathcal{L}_{MM} = \sum_{\alpha,\beta} \bar{\Psi}_L(M_\alpha)\tilde{\Gamma}_{\alpha,\beta}\Psi_L(M_\beta).
\]

Using the Majorana condition, which states

\[
C(\bar{\Psi}_\alpha L)^T = (\Psi_\alpha L)^c = \Psi_{\alpha R}, \\
(C^{-1}\bar{\Psi}_\beta L)^T = (\bar{\Psi}_\beta L)^c = \bar{\Psi}_{\beta R},
\]

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the lagrangian (25) can be recast in the following form, which can be directly applicable to the Feynman rule,

$$\mathcal{L}_{MM} = \sum_{\alpha \geq \beta} \bar{\Psi}(M_\alpha) \Gamma_{\alpha, \beta} \Psi(M_\beta).$$

(27)

### 3.2 Grace line rule for MSSM

We will modify the Grace line rule of section 2 so that Majorana particles as well as the F-direction clashing vertices are included in the rule.

We assign the particle number 1 to $n_i$ to each of the $n_i$ incoming spinor particles, namely Dirac fermions, Dirac antifermions and Majorana particles in this order. Likewise, we assign the particle number $n_i + 1$ to $n_i + n_f$ to each of the $n_f$ outgoing spinors, namely Dirac antifermions, Dirac fermions and Majorana particles in this order.

The M-direction of the Majorana particle is assigned, like the conventional fermions, to its momentum direction for external Majorana particles and to the A-direction for internal Majorana particles. For Majorana particles which are self charge-conjugate, it is irrelevant whether one assigns the F-direction along the A-direction or opposite to the A-direction. For our convention, we define the F-direction of Majorana particles by their M-direction, which is along the A-direction for internal Majorana particles. This corresponds to regarding all the Majorana particles as fermion. Therefore, only the first type of the propagator shown in (23) appears in our Grace line rule and the clashing of the F-direction occurs only at vertices.

For the lagrangian (24), we assign the vertex function as follows (The irrelevant constant $i$ is not explicitly exhibited hereafter),

$$M_\alpha \leftarrow \bullet \leftarrow D_\beta \quad \Gamma_{M_\alpha, D_\beta},$$

$$M_\alpha \rightarrow \bullet \leftarrow D_\beta \quad (C^{-1})^T \Gamma_{M_\alpha, D_\beta},$$

$$D_\alpha \leftarrow \bullet \leftarrow M_\beta \quad \Gamma_{D_\alpha, M_\beta},$$

$$D_\alpha \leftarrow \bullet \rightarrow M_\beta \quad \Gamma_{D_\alpha, M_\beta} C.$$  

(28)

For the vertex which has an opposite F-directions, we apply the rule 2 and obtain the following rule.*

$$D_\beta \rightarrow \bullet \rightarrow M_\alpha \quad -\Gamma_{M_\alpha, D_\beta}^T,$$

$$D_\beta \rightarrow \bullet \leftarrow M_\alpha \quad -\Gamma_{M_\alpha, D_\beta}^T C^{-1},$$

$$M_\beta \rightarrow \bullet \rightarrow D_\alpha \quad -\Gamma_{D_\alpha, M_\beta}^T,$$

$$M_\beta \leftarrow \bullet \rightarrow D_\alpha \quad -C^T \Gamma_{D_\alpha, M_\beta}^T.$$  

(29)

Next, we consider the Majorana - Majorana vertices. For the lagrangian, (27), we assign the vertex functions as follows:

$$M_\alpha \leftarrow \bullet \leftarrow M_\beta \quad \Gamma_{M_\alpha, M_\beta},$$

(30)

* This can be seen by considering the processes whose external particles are conventional Dirac fermions and in which Majorana particles appear only in the internal lines. In such processes, the argument for the SM given in the section 2 is applicable.
\[ M_\alpha \rightarrow \bullet \leftarrow M_\beta \hspace{1cm} (C^{-1})^T \Gamma_{M_\alpha,M_\beta}, \quad (31) \]

\[ M_\alpha \leftarrow \bullet \rightarrow M_\beta \hspace{1cm} \Gamma_{M_\alpha,M_\beta} C, \quad (32) \]

\[ M_\beta \rightarrow \bullet \rightarrow M_\alpha \hspace{1cm} - \Gamma^T_{M_\alpha,M_\beta}, \quad (33) \]

\[ M_\beta \rightarrow \bullet \leftarrow M_\alpha \hspace{1cm} - \Gamma^T_{M_\alpha,M_\beta} C^{-1}, \quad (34) \]

\[ M_\beta \leftarrow \bullet \rightarrow M_\alpha \hspace{1cm} - CT^T \Gamma^T_{M_\alpha,M_\beta}. \quad (35) \]

Note that these Grace line rules are valid under the condition, \( \alpha \geq \beta \). When the Majorana spinor species at the vertex has an opposite order, rewriting the lagrangian (27) as

\[ \mathcal{L} = \sum_{\alpha \geq \beta} \bar{\Psi}(M_\beta) \Gamma^c_{\beta,\alpha} \Psi(M_\alpha). \quad (36) \]

where

\[ \Gamma^c_{\beta,\alpha} = CT^T \Gamma C^{-1}. \quad (37) \]

we obtain the following rule;

\[ M_\beta \leftarrow \bullet \leftarrow M_\alpha \hspace{1cm} CT^T \Gamma_{M_\alpha,M_\beta} C^{-1}, \quad (38) \]

\[ M_\beta \rightarrow \bullet \leftarrow M_\alpha \hspace{1cm} - \Gamma^T_{M_\alpha,M_\beta} C^{-1}, \quad (39) \]

\[ M_\beta \leftarrow \bullet \rightarrow M_\alpha \hspace{1cm} CT^T \Gamma_{M_\alpha,M_\beta}, \quad (40) \]

\[ M_\alpha \rightarrow \bullet \rightarrow M_\beta \hspace{1cm} - C^{-1} \Gamma_{M_\alpha,M_\beta} C, \quad (41) \]

\[ M_\alpha \rightarrow \bullet \leftarrow M_\beta \hspace{1cm} - C^{-1} \Gamma_{M_\alpha,M_\beta}, \quad (42) \]

\[ M_\alpha \leftarrow \bullet \rightarrow M_\beta \hspace{1cm} \Gamma_{M_\alpha,M_\beta} C. \quad (43) \]

Note that (31)=(42), (32)=(43), (34)=(39) and (35)=(40). These equalities provide us one of the consistency check of the Grace line rule, since the rules (31)–(35) are obtained from (30) while (39)–(43) are obtained from (38).

From (28), (29) and above equations, one finds the following general rule of how and where to put the charge conjugation matrix \( C \):

**Rule 3**

\[ \leftarrow \bullet - - - = - C(\rightarrow \bullet - - -), \]

\[ \rightarrow \bullet - - - = - C^{-1}(\leftarrow \bullet - - -), \]

\[ - - - \leftarrow = (\rightarrow - - -) C^{-1}, \]

\[ - - - \rightarrow = (\leftarrow - - -) C, \]

where \(- - -\) is a fermion line whose F-direction can be arbitrary.

Finally, a comment is in order concerning the fermion number violating chargino vertex. MSSM contains two charginos \( \tilde{\chi}_1^\pm \) and \( \tilde{\chi}_2^\pm \). Remember that we have defined the *positive* chargino as Dirac fermion. Their interactions are given by

\[ \mathcal{L}_{CD} = \bar{\Psi}(\tilde{\chi}_1^\pm) \Gamma_{C,e} \Psi(e) + \bar{\Psi}(\tilde{\chi}_1^+) \Gamma_{C,\nu} \Psi(\nu) + h.c., \]

\[ \mathcal{L}_{CM} = \bar{\Psi}(\tilde{\chi}_1^+) \Gamma_{C,M_\alpha} \Psi(M_\alpha) + h.c. \quad (44) \]
Denoting

\[ \Gamma_{e,C} = \gamma_0 \Gamma_{e,C}^\dagger \gamma_0, \quad \Gamma_{\nu,C} = \gamma_0 \Gamma_{\nu,C}^\dagger \gamma_0, \quad \Gamma_{M,\alpha} = \gamma_0 \Gamma_{M,\alpha}^\dagger \gamma_0, \]

the Grace line is represented by

\[ \tilde{\chi}_i^+ \leftarrow \bullet \leftarrow \nu \quad \Gamma_{C,\nu} \]
\[ \nu \leftarrow \bullet \leftarrow \tilde{\chi}_i^+ \quad \Gamma_{\nu,C} \]
\[ \tilde{\chi}_i^+ \rightarrow \bullet \leftarrow e^- \quad - C \Gamma_{C,e} \]
\[ e^- \leftarrow \bullet \rightarrow \tilde{\chi}_i^+ \quad \Gamma_{e,C} \]
\[ \tilde{\chi}_i^+ \leftarrow \bullet \leftarrow M_\alpha \quad \Gamma_{C,\alpha} \]
\[ M_\alpha \leftarrow \bullet \leftarrow \tilde{\chi}_i^+ \quad \Gamma_{M,\alpha} \]

(46)

The first two Grace line rules are the same as those for the conventional Dirac fermion vertex, while the subsequent two lines are new, violating the fermion number conservation.

Note that the charge conjugation matrix \( C \) appear explicitly in the Grace line rule (46). The way \( C \) appears in (46) obeys the general Grace line rule, rule 3, although the vertices corresponding to \( \Gamma_{C,e} \) and \( \Gamma_{e,C} \) do not exist in Grace line amplitudes.

In case the negative charginos are defined as Dirac fermions (as is preferred by some authors), rewriting the lagrangian (44) as follows,

\[ L_{CD} = \bar{\Psi}(\tilde{\chi}^- + i) \Gamma_{e,C} \Psi(\tilde{\chi}^- + i) + \bar{\Psi}(\bar{\nu}) \Gamma_{\nu,C} \Psi(\bar{\nu}^-) + hc, \]
\[ L_{CM} = \bar{\Psi}(M_\alpha) \Gamma_{\alpha,M,C} \Psi(\tilde{\chi}^- + i) + hc, \]

where

\[ \Gamma_{A,B}^\dagger \equiv C \Gamma_{T,B,A}^C \]

(47)

one obtains the corresponding Grace line rule:

\[ \nu \rightarrow \bullet \leftarrow \tilde{\chi}_i^- \quad - C^{-1} \Gamma_{\nu,C} \]
\[ \tilde{\chi}_i^- \leftarrow \bullet \rightarrow \nu \quad \Gamma_{e,C} \]
\[ e^- \rightarrow \bullet \rightarrow \tilde{\chi}_i^+ \quad \Gamma_{e,C} \]
\[ \tilde{\chi}_i^- \rightarrow \bullet \rightarrow e^- \quad \Gamma_{e,C} \]
\[ M_\alpha \leftarrow \bullet \leftarrow \tilde{\chi}_i^- \quad \Gamma_{M,\alpha} \]
\[ \tilde{\chi}_i^- \leftarrow \bullet \leftarrow M_\alpha \quad \Gamma_{M,\alpha} \]

(48)

The second term of \( L_{CD} \) now violates the fermion number conservation, which can be explicitly understood from (49).

The rule (49) is equivalent to the rule (46) and therefore, (49) is derived from (46) without referring to the lagrangian (47). For example, the Grace line rule of the second line of (49) is derived from (46) as follows,

\[ \tilde{\chi}_i^- \leftarrow \bullet \rightarrow \nu = -C(\tilde{\chi}_i^+ \rightarrow \bullet \rightarrow \nu ) = -C(-\Gamma_{\nu,C}) = \Gamma_{C,\nu} \]

(50)

where rule 2, rule 3 and (37) are used.
Note that the expression of the Grace line amplitudes is independent of the convention of whether one assigns the fermion number +1 for positive charginos or negative charginos. The differences at the vertices, (46) vs. (49), are compensated by the chargino propagators and external spinors.

4. Identical Particles

For the vertex consisting of two identical Majorana particles ($\alpha = \beta$ in (27) and (30)–(43)), twelve Grace line rules in (30)–(43) are not independent from each other, but the first six rules are identical to the second six ones. Therefore, the expression of the first six vertices is found among the second six ones, namely, (30)=(38), (31)=(39), etc. Consequently, the Majorana-Majorana vertex function $\Gamma_{\alpha,\alpha}$ must satisfy the following condition,

$$\Gamma_{\alpha,\alpha} = C \Gamma_{\alpha,\alpha}^T C^{-1}, \quad \alpha \text{ not summed}. \quad (51)$$

This is equivalent to saying that there are no vector current nor tensor current in the Majorana bilinear form, a well known property of Majorana particles, as

$$\Gamma = (1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5, \sigma_{\mu\nu}) \quad \rightarrow \quad \Gamma^\epsilon \equiv C \Gamma C^{-1} = (1, \gamma_5, -\gamma_{\mu}, \gamma_\mu \gamma_5, -\sigma_{\mu\nu}). \quad (52)$$

See also the item 2 of Appendix C. In fact, the MSSM lagrangian $L_{MM}$ has a following form for the diagonal Majorana-Majorana interaction part (see e.g. [8]),

$$L_{MM} = c_Z \bar{\Psi}_M \gamma_\mu \gamma_5 Z^\mu \Psi_M + c_H \bar{\Psi}_M (1 + b_H \gamma_5) H \Psi_M + c_G \bar{\Psi}_M (1 + b_G \gamma_5) G \Psi_M, \quad (53)$$

where $H$ represents any of the three neutral Higgs particles and $G$ is a neutral Goldstone boson. In addition, we have to multiply the vertex functions by a statistical factor 2.

5 Example

In this section let us consider two examples in which F-direction is not conserved on a Grace line. The first example is the Majorana-Dirac-fermion annihilation process with two vertices and one Majorana propagator,

$$< 0 | L_{MM} L_{MD} c^\dagger_m a_n | 0 > = -\bar{v}_m \begin{pmatrix} \Gamma_{m,p}^c \\ \Gamma_{m,p} \end{pmatrix} S_p \Gamma_{p,n} u_n, \quad (54)$$

where $p$ stands for the Majorana species of the propagator $S$ and two vertex functions correspond to the case $m \geq p$ and $m \leq p$, respectively. There are four possible ways of assigning the F-direction
on the Majorana propagator.

\[
\tilde{\chi}_m \rightarrow \bullet \leftarrow \bullet \leftarrow D_n \quad \rightarrow \quad (-) u_m^T \left( (C^{-1})^T \Gamma_{m,p} \right) S_p(k) \Gamma_{p,n} u_n,
\]

\[
p_m \rightarrow \leftarrow k \leftarrow p_n
\]

\[
\tilde{\chi}_m \rightarrow \bullet \rightarrow \bullet \leftarrow D_n \quad \rightarrow \quad (-) \bar{v}_m \left( \Gamma_{m,p} \Gamma_{p,m} C^{-1} \right) S_p(k) \Gamma_{p,n} u_n,
\]

\[
p_m \rightarrow \leftarrow k \leftarrow p_n
\]

\[
\tilde{\chi}_m \leftarrow \bullet \rightarrow \bullet \leftarrow D_n \quad \rightarrow \quad (-) \bar{v}_m \left( \Gamma_{m,p} \Gamma_{p,m} C^{-1} \right) S_p(k) \Gamma_{p,n} u_n.
\]

\[
p_m \rightarrow \leftarrow k \leftarrow p_n
\]

Upon using (21), (37) and

\[-CS(-k)^T(C^{-1})^T = CS(-k)^TC^{-1} = C \frac{-k^T + m}{k^2 - m^2 + i\epsilon} C^{-1} = S(k),\]  

(56)

it is easy to prove that the four expressions of (55) are indeed identical to (54). Our convention of assigning the M-direction and F-direction to Majorana particles chooses the first assignment of

(55).

The next example, \(\tilde{\chi}_i \rightarrow e^- \tilde{\nu}_L\), contains the fermion number violating chargino interaction. The lagrangian is given by (44). The Grace line for this process is depicted as follows,

\[
e^- \leftarrow \bullet \rightarrow \chi_i^+ \quad p_m \leftarrow \leftarrow p_n
\]

(57)

From (46) and Table 2, one finds

\[\mathcal{M} = \bar{u}(p_m)(\Gamma_{e,C} C) \bar{\nu}^T(p_n) = \bar{u}(p_m)\Gamma_{e,C} u(p_n).\]  

(58)

This can be explicitly evaluated as follows,

\[< \bar{\nu}_L e^- | \mathcal{L} | \tilde{\chi}_i^- >= < \bar{\nu}_L e^- | \bar{\Psi}(e) \Gamma_{e,C} \Psi(\tilde{\chi}_i^-) | \tilde{\chi}_i^- >
\]

\[= < \bar{\nu}_L | a_e \bar{\Psi}(e) \Gamma_{e,C} \Psi^+(\tilde{\chi}_i^+) b^+_\chi | 0 >
\]

\[= \bar{u}(e) \Gamma_{e,C} u(\tilde{\chi}_i)
\]

(59)

At first sight it seems odd that spinor \(u(\tilde{\chi}_i)\) appears in the amplitude in spite of the antifermion in the initial state. We should recall, however, that whether chargino behaves like fermion or antifermion is not determined by the convention we adopt but by the nature of the interaction vertex.
6. Comparison with Denner’s method

Denner et al. [9] have introduced the lagrangian written in terms of the charge conjugated state in order to eliminate the charge conjugation matrix $C$ from the Feynman amplitudes. Although for our purpose we don’t find any particular advantage in adopting their method, it is instructive to compare our method with theirs. The concept of orientation (fermion flow) which they have introduced in order to specify whether the conventional vertices are used or the charge conjugated ones are used corresponds to our two ways of displaying Grace line. Although as our convention we have decided to stick to $G_{m,n}$ with $m > n$, we can equally use the Grace line $G_{n,m}$ with $m > n$. The main point of ref. [9] is that all the charge conjugation matrices appearing in the Grace line amplitudes can be combined with the vertex function $\Gamma$ to form the charge conjugated one $\Gamma^c$ and can be eliminated completely.

Let us prove that two methods are equivalent by showing that the Grace line amplitude can be brought in the form that is obtained from the prescription given by Denner et al. This is done by showing (1) charge conjugation matrices, transposed propagators and transposed vertices can fully be eliminated in favour of $\Gamma^c = CT^T C^{-1}$ and $S(k) = CS^T (-k) C^{-1}$ (see (56)). (2) The resulting expression contains $\Gamma^c$ at the proper vertices. (3) The overall sign of the Feynman amplitude * is given by $(-)^{P+V}$ when expressed without charge conjugation matrix $C$. Here $P$ is due to the permutation of spinors and $V$ is the number of $v$ and $\bar{v}$ in the Feynman amplitude.

First we note that as our convention the M-direction of the internal spinors in a Grace line is, irrespective of Majorana or Dirac particle, always taken in the A-direction and that the F-direction of the internal Majorana particles on the Grace line is also taken in the A-direction. Consequently, there are only two sources of the charge conjugation matrix; from the external spinor wave functions and from the clashing vertices (vertices where two F-directions are not same). From the propagator we don’t get any $C$. With the use of (21), we rewrite Table 2 in the form which has no transposed wave functions, which is shown in Table 3. Note that the minus signs coming from the rule 1 are now included in Table 3.

From Table 3 we notice that the charge conjugation matrix appears in connection with external spinors whose F-direction is opposite to the A-direction.

At vertices, the charge conjugation matrix appears when two F-directions clash. From rule 3 one finds,

$$ p \leftarrow \bullet \rightarrow q = \hat{\Gamma} C \quad with \quad \hat{\Gamma} = \begin{cases} \Gamma_{D,M} & \text{when } (D,M) \\ \Gamma^c_{M,D} & \text{when } (M,D) \\ \Gamma_{\alpha,\beta} & \text{when } (M_\alpha, M_\beta) \\ \Gamma^c_{\beta,\alpha} & \text{when } (M_\beta, M_\alpha) \\ \Gamma_{e,C_i} & \text{when } (e, C_i) \end{cases} $$

* Here we refer to the Feynman amplitude and not the Grace line amplitude, since the sign of the Grace line amplitude $[G_{p,q}]$ is not independent from the partition (4) and therefore from $\text{sign}(P)$. 

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{F-direction} & \textbf{M-direction} & $\leftarrow \bullet$ & $\rightarrow \bullet$ & $\bullet \leftarrow$ & $\bullet \rightarrow$ \\
\hline
$\leftarrow$ & $\bar{u}(p)$ & $-\bar{u}(p) C^\dagger$ & $u(p)$ & $C^\dagger u(p)$ \\
$\rightarrow$ & $-\bar{v}(p)$ & $\bar{v}(p) C^\dagger$ & $-v(p)$ & $-C^\dagger v(p)$ \\
\hline
\end{tabular}
\caption{Spinor assignment at left endpoint and at right endpoint.}
\end{table}
\[ p \rightarrow \bullet \leftrightarrow q = -C^{-1}\hat{\Gamma} \quad \text{with} \quad \hat{\Gamma} = \begin{cases} \Gamma_{M,D} & \text{when } (M,D) \\ \Gamma_{C_{D,M}} & \text{when } (D,M) \\ \Gamma_{\alpha,\beta} & \text{when } (M_{\alpha}, M_{\beta}) \\ \Gamma_{\beta,\alpha} & \text{when } (M_{\beta}, M_{\alpha}) \\ \Gamma_{C_{i,e}} & \text{when } (C_{i}, e) \end{cases} \quad (61) \]

Here the charge conjugated vertex function is defined by
\[ \Gamma_{p,q}^c = C^{T\Gamma_{q,p}}C^{-1}, \quad (37) = (62) \]
and the vertex functions, \( \Gamma_{MD}, \Gamma_{\alpha\beta} \) etc are defined by our lagrangian (24), (27) and (44), namely,
\[ \Gamma_{A,B} = A \leftarrow \bullet \leftarrow B. \quad (63) \]

The rules (60) and (61) are summarized as follows: \( C \) appears to the right of \( \hat{\Gamma} \) when two F-directions depart from the vertex, and \( -C^{-1} \) appears to the left of \( \hat{\Gamma} \) when two F-direction clash at the vertex. \( \Gamma^c \) is used when the ”reversed” vertex appears. Note that the word ”reversed” is understood in the sense that the order of the particle species (in the case of Dirac particles also their F-direction) is reversed compared with the order appearing in the model lagrangian. Therefore, it is important to fix the lagrangian we use for the definition of \( \Gamma_{AB} \). In particular, for the chargino-Dirac fermion interactions, which violate fermion number conservation, we have to decide which of the two equivalent forms, namely, (44) or (47), we adopt for our lagrangian. If we had adopted the lagrangian (47) instead of (44), then \( \Gamma^c \) in (47) would have been considered as \( \Gamma \) that is used in (60) and (61). See the footnote between rule 1 and rule 2.

In order to see where the charge conjugation matrices appear and how they are eliminated in the Grace line in which the F-direction is not conserved at some vertices, it is sufficient to consider the following three spinor chains, each of which is a section of a certain Grace line,
\[
\cdots \leftarrow \bullet \rightarrow \bullet \cdots \rightarrow \bullet \leftarrow \cdots \quad (64a) \\
\rightarrow \bullet \rightarrow \bullet \cdots \rightarrow \bullet \leftarrow \cdots \quad (64b) \\
\cdots \leftarrow \bullet \rightarrow \bullet \cdots \rightarrow \bullet \rightarrow \quad (64c)
\]
Here \( \cdots \) means that there exist propagators to the left or to the right of the chain, namely the external spinors are not involved in this sequence of spinors and \( \bullet \bullet \bullet \bullet \bullet \) stands for the vertices and propagators with F-direction opposite to the A-direction. The spinor chain of the type (64a) appears somewhere inside the Grace line. When the F-directions of the spinors at both ends of the Grace line are in the A-direction, this kind of the spinor chain is the only possible source of the charge conjugation matrix. If the F-direction of the left endpoint is opposite to the A-direction, the spinor chain of the type (64b) produces charge conjugation matrices, while if the F-direction of the right endpoint is opposite to the A-direction, the spinor chain of the type (64c) produces charge conjugation matrices. The shortest chain of the type (64a) is the chain with one propagator and that of the type (64b) and (64c) is the chain without propagators.

The Grace line amplitude corresponding to (64) is given by
\[
\cdots \hat{\Gamma}C(-S^T)(-\Gamma^T)(-S^T)(-C^{-1})\hat{\Gamma} \cdots = \cdots \hat{\Gamma}S\Gamma^c \cdots S\hat{\Gamma} \quad (65a) \\
\left\{ \begin{array}{c} -\bar{u} \\
\bar{v} \end{array} \right\} C(-\Gamma^T)(-S^T)(-S^T)(-C^{-1})\hat{\Gamma} \cdots = \left\{ \begin{array}{c} \bar{u} \\
-\bar{v} \end{array} \right\} \Gamma^c S \cdots S\hat{\Gamma} \cdots \quad (65b) \\
\cdots \hat{\Gamma}C(-S^T)(-\Gamma^T)(-S^T)(-C^{-1}) \left\{ \begin{array}{c} u \\
-\bar{v} \end{array} \right\} = \cdots \hat{\Gamma}S \cdots S\Gamma^c \left\{ \begin{array}{c} u \\
-\bar{v} \end{array} \right\} \quad (65c)
\]
proves that charge conjugation matrices and transposed vertices as well as the transposed propagators have completely disappeared from the Grace line amplitude in favour of the appearance of \( \Gamma^c \). In addition, (65) also proves that the sign of the Grace line amplitude is given by \( (-)^{V_g} \), where \( V_g \) is the total number of \( v \) and \( \bar{v} \) appearing in a Grace line. When combined with the sign coming from the permutation of spinors, one finds that the sign of the amplitude is given* by \( m_{\text{sign}}(P) (-)^V \) with \( V = \sum V_g \), in accordance with Denner et al. Further, (60), (61) and (65) tell us that we should use the \( \Gamma^c \) at the clashing vertex which has a reserved order of spinors compared to the vertex which defines \( \Gamma \) and at the non-clashing vertex when the F-direction is opposite to the A-direction. This completes the proof of the equivalence of the two methods.

As an illustration, we compare the two methods in two examples. We start with the decay of a scalar \( (\Phi) \) into a Dirac \( (D) \) and a Majorana \( (M) \) particle, an example discussed by Denner et al.[9]. Suppose that the underlying lagrangian is given by

\[
\mathcal{L} = \bar{\Psi}(D)\Gamma_{D,M}\Psi(M)\Phi + \bar{\Psi}(M)\Gamma_{M,D}\Psi(D)\Phi^*. \tag{66}
\]

Assign number 1 to the Dirac particle and number 2 to the Majorana particle. There is only one pair \((2,1)\) and the Grace line and the corresponding Grace line amplitude \([G_{2,1}]\) is given by

\[
M \leftarrow \bullet \rightarrow D \quad \rightarrow \quad \begin{bmatrix} \Gamma_{2,1} \end{bmatrix} = (-)^{\bar{u}(p_m)}(-C^T\Gamma_{D,M}^T)(\bar{u}(p_d))^T
\]

\[
\begin{array}{c}
p_m \leftarrow \rightarrow p_d \\
\end{array}
\]

\[
= -\bar{u}(p_m)\Gamma_{D,M}^c\bar{v}(p_d). \tag{67}
\]

This corresponds to the case of Fig.4b of ref.[9].** The difference of the overall sign is irrelevant. The amplitude corresponding to Fig.4c of ref.[9], whose orientation is from Majorana to Dirac, corresponds to \([G_{1,2}]\). The explicit construction based on our Grace line rule gives

\[
D \leftarrow \bullet \rightarrow M \quad \rightarrow \quad \begin{bmatrix} \Gamma_{1,2} \end{bmatrix} = (-)^{\bar{u}(p_d)}(-C_{D,M}\Gamma_{1,2})^T
\]

\[
\begin{array}{c}
p_d \leftarrow \rightarrow p_m \\
\end{array}
\]

\[
= -\bar{u}(p_d)\Gamma_{D,M}v(p_m), \tag{68}
\]

which indeed coincides with the result given by Denner et al., and which also proves that \([G_{1,2}] = -[G_{2,1}]\).

The next example discussed by Denner et al.[9] is the scattering \( D_aD_b \rightarrow \Phi_c\Phi_d \), where it is assumed that \( D_a \) and \( D_b \) are not identical particles. There is only one Feynman diagram which has a Majorana propagator, but there are two possible orientations as shown in Figure 2.

---

* At first sight, the value of \( V = \sum V_g \) seems not unique since the external Majorana can be taken as particle as well as antiparticle. Note, however, that the assignment of the spinor wave functions is unique, provided one consistently uses Table 2. As shown in Table 2, when the external particle has the M-direction which is opposite to the A-direction, one has to use \( v \) or \( \bar{v} \), depending on whether the external particle is at the right endpoint or at the left endpoint, but independent of the F-direction. Since the assignment of \( u \) and \( v \) to each external spinor is unique, the value \( V \), the total number of \( v \) and \( \bar{v} \) appearing in the Feynman amplitude is also unique.

** It seems there is a misprint in (3.1) of ref.[9]. For the decay of \( \Phi \), one should use the third term of the second line of (2.1). Then, Denner’s \( k^*\Gamma_i \) corresponds to our \( \Gamma_{D,M} \) and \( k^*\Gamma_i^c \) to \( \Gamma_{D,M}^c \).
Corresponding to (3.3a) and (3.3b) of ref.[9], we have

\[ [G_{a,b}] = (-)u_a^T(-\Gamma_{M,D}^T C^{-1})S(p_b - p_d)\Gamma_{M,D}u_b \]
\[ p_a \rightarrow \leftarrow p_b \]
\[ = -\bar{v}_a\Gamma_{M,D}S(p_b - p_d)\Gamma_{M,D}u_b, \quad (69) \]

\[ [G_{b,a}] = (-)u_b^T(-\Gamma_{M,D}^T C^{-1})S(p_a - p_c)\Gamma_{M,D}u_a \]
\[ p_b \rightarrow \leftarrow p_a \]
\[ = -\bar{v}_b\Gamma_{M,D}S(p_a - p_c)\Gamma_{M,D}u_a, \quad (70) \]

respectively. Note that \( p_a + p_b = p_c + p_d \).

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Appendix A. Charge conjugation and Majorana particles

Dirac spinor field operators are expanded in terms of the creation and annihilation operators:

\[ \Psi(x) = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3 2p_0} \left[ a(p\lambda)u(p\lambda)e^{-i\mathbf{p}\cdot\mathbf{x}} + b^\dagger(p\lambda)v(p\lambda)e^{i\mathbf{p}\cdot\mathbf{x}} \right], \]

\[ \bar{\Psi}(x) = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3 2p_0} \left[ b(p\lambda)\bar{u}(p\lambda)e^{-i\mathbf{p}\cdot\mathbf{x}} + a^\dagger(p\lambda)\bar{v}(p\lambda)e^{i\mathbf{p}\cdot\mathbf{x}} \right], \]  

The unitary operator \( \mathcal{C} \), called as charge conjugation operator, is defined by the following properties:

\[ \mathcal{C}a^\dagger(p, \lambda)|0> = \epsilon^* b^\dagger(p, \lambda)|0>, \]

\[ \mathcal{C}^2a^\dagger(p, \lambda)|0> = a^\dagger(p, \lambda)|0>, \]

\[ \mathcal{C}|0> = |0>. \]  

From these three conditions, it follows:

\[ \mathcal{C}a^\dagger(p, \lambda)\mathcal{C}^{-1} = \epsilon^* b^\dagger(p, \lambda), \]

\[ \mathcal{C}b^\dagger(p, \lambda)\mathcal{C}^{-1} = \epsilon a^\dagger(p, \lambda), \]

\[ \mathcal{C}a(p, \lambda)\mathcal{C}^{-1} = eb(p, \lambda), \]

\[ \mathcal{C}b(p, \lambda)\mathcal{C}^{-1} = \epsilon^* a(p, \lambda), \]

\[ \mathcal{C}\Psi(x)\mathcal{C}^{-1} = \epsilon \Psi^c(x), \]

\[ \mathcal{C}\bar{\Psi}(x)\mathcal{C}^{-1} = \epsilon^* \bar{\Psi}^c(x), \]

where

\[ \Psi(x)^c = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3 2p_0} \left[ b(p\lambda)\bar{u}(p\lambda)e^{-i\mathbf{p}\cdot\mathbf{x}} + a^\dagger(p\lambda)\bar{v}(p\lambda)e^{i\mathbf{p}\cdot\mathbf{x}} \right], \]

\[ \bar{\Psi}(x)^c = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3 2p_0} \left[ a(p\lambda)\bar{v}(p\lambda)e^{-i\mathbf{p}\cdot\mathbf{x}} + b^\dagger(p\lambda)\bar{u}(p\lambda)e^{i\mathbf{p}\cdot\mathbf{x}} \right]. \]  

Using the four-by-four unitary matrix \( \mathcal{C} \) (charge conjugation matrix), which satisfies the conditions,

\[ \mathcal{C}\bar{u}(p, \lambda)^T = v(p, \lambda), \quad \mathcal{C}\bar{v}(p, \lambda)^T = u(p, \lambda), \]

\( \Psi(x) \) and its charge conjugation partner \( \Psi^c(x) \) are related as follows:

\[ \Psi^c = \mathcal{C}\bar{\Psi}, \quad \bar{\Psi}^c = \Psi^T(C^T)^{-1}, \]

and

\[ \mathcal{C}\Psi(x)^T \mathcal{C} = \epsilon \Psi^c(x), \]

\[ \mathcal{C}\bar{\Psi}(x)^T \mathcal{C} = \epsilon^* \bar{\Psi}^c(x). \]

The charge conjugation matrix \( \mathcal{C} \) is subject to several conditions. First, from the normalization condition, \( \bar{u}u = -\bar{v}v = 1 \) and (A.5), one finds that \( \mathcal{C} \) is antisymmetric,

\[ \mathcal{C}^T = -\mathcal{C}. \]
Noting that $u(p, \lambda)$ and $v(p, \lambda)$ are the solutions of Dirac equations, one finds that $C$ must fulfill the following conditions,

$$C\gamma^{T}_{\mu}C^{-1} = -\gamma_{\mu}. \quad (A.9)$$

This condition is obtained also from the requirement that

$$(i\partial - eA - m)\Psi(x) = 0, \quad (A.10)$$

leads to the following equation of motion for $\Psi^{c}$,

$$(i\partial + eA - m)\Psi^{c}(x) = 0. \quad (A.11)$$

From the consistency between two equations in (A.5) or in (A.6), and the unitarity of $C$, $C^{\dagger} = C^{-1}$, one finds again (A.8).

The Majorana particle is characterized by the condition,

$$\Psi = \Psi^{c}. \quad (A.12)$$

Consequently, the Majorana field $\Psi$ is expanded as

$$\Psi(x) = \sum_{\lambda} \int \frac{d^{3}p}{(2\pi)^{3}2p_{0}} [c(p\lambda)u(p\lambda)e^{-ipx} + c^{\dagger}(p\lambda)v(p\lambda)e^{ipx}],$$

$$\bar{\Psi}(x) = \sum_{\lambda} \int \frac{d^{3}p}{(2\pi)^{3}2p_{0}} [c(p\lambda)\bar{v}(p\lambda)e^{-ipx} + c^{\dagger}(p\lambda)\bar{u}(p\lambda)e^{ipx}], \quad (A.13)$$

There is no vector current nor tensor current for Majorana particles:

$$\bar{\Psi}_{M}\gamma_{\mu}\Psi_{M} = \bar{\Psi}_{M}\sigma_{\mu\nu}\Psi_{M} = 0, \quad (A.14)$$

since

$$C\gamma^{T}_{\mu}C^{-1} = -\gamma_{\mu}, \quad C\sigma^{T}_{\mu\nu}C^{-1} = -\sigma_{\mu\nu}. \quad (A.15)$$
Appendix B. Summary of Grace line rule

1. Fix the underlying theory and determine the model lagrangian which defines the vertex functions,
\[ \Gamma_{\xi,\eta} = \xi \leftarrow \bullet \leftarrow \eta \]
with \((\xi, \eta) = (D_i, D_j), (M_\alpha, D_i), (D_i, M_\alpha), (C_i, D), (D, C_i)\) and \((M_\alpha, M_\beta)\) with \(\alpha > \beta\). Only these vertex functions appear in the Grace line amplitudes. In particular, the lagrangian for Majorana-Majorana vertex must be brought in the form of (27).

2. Assign particle numbers 1 to \(n_i\) to \(n_i\) incoming particles (in the order of Dirac, anti-Dirac and Majorana particles) and \(n_i + 1\) to \(n_i + n_f\) to \(n_f\) outgoing particles (in the order of anti-Dirac, Dirac and Majorana particles).

3. Draw Feynman diagrams for a given process and the order of perturbation based on the underlying model lagrangian.

   For each Feynman diagram:

4. Read out \(P = (a_1, a_2)(a_3, a_4) \ldots (a_{n_i+n_f-1}, a_{n_i+n_f})\) from the diagram and determine \(\text{Sign}(P)\).

   Note that the pair \((m, n)\) must be arranged such that \(m > n\). If there are several Feynman diagrams which give the same \(P\), count and treat them separately.

5. Draw \(\frac{n_i + n_f}{2}\) Grace lines \(G_{m,n}\) corresponding to the pair \((m, n)\) with \(m > n\).

   For each Grace line \(G_{m,n}\):

6. Assign the F-direction and M-direction to each segment of the Grace line based on the rule given in section 3. The M-direction is the same as the physical momentum direction for external particles, and for internal particles it is in the same direction as the A-direction. The F-direction of the Dirac particle is defined by the direction of fermion number flow. The F-direction of Majorana particles is in the same direction of their M-direction.

7. For external spinors, assign spinor wave functions based on the rule given in Table 2.

8. Put \((-)\) for each external spinor when its M-direction is opposite to the A-direction.

9. Use the propagator \(S(k)\) or \(-S(-k)^T\), depending on whether the F-direction is in or opposite to the A-direction.

10. Use the vertex function given by (18) and (19) for Dirac-Dirac vertices, (28) and (29) for Dirac-Majorana vertices, (30)-(43) for Majorana-Majorana vertices and (46) for the vertices including charginos.

11. Finally, multiply \(\frac{n_i + n_f}{2}\) Grace line amplitudes \([G_{m,n}]\) and the appropriate bosonic parts which connect vertices on the Grace lines in order to obtain an amplitude for each Feynman diagram, and sum up the resulting amplitudes over all the possible partitions \(P\) (corresponding to all Feynman diagrams):

\[ M = \sum_P \text{Sign}(P)M_P, \quad M_P = \Pi_{(m,n)}[G_{m,n}](\text{bosonic part})_{m,n}. \]
Appendix C. Grace line rule without charge conjugation matrix

1. Fix the underlying theory and determine the model lagrangian which defines the vertex functions,

\[ \Gamma_{\xi,\eta} = \xi \leftarrow \bullet \leftarrow \eta \]

with \((\xi, \eta) = (D_i, D_j), (M_\alpha, D_i), (D_i, M_\alpha), (C_i, D), (D, C_i)\) and \((M_\alpha, M_\beta)\) with \(\alpha > \beta\). For Majorana-Majorana vertex, bring the lagrangian in the form of (27) and neglect the F-direction of Majorana lines.

2. Prepare \(\Gamma^c \equiv C \Gamma^T C^{-1}\) for each vertex function \(\Gamma\). Note that \(\Gamma^c = (1, \gamma_5, -\gamma_\mu, \gamma_\mu \gamma_5, -\sigma_{\mu \nu})\) for \(\Gamma = (1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5, \sigma_{\mu \nu})\).

3. Assign particle numbers 1 to \(n_i\) incoming particles (in the order of Dirac, anti-Dirac and Majorana particles) and \(n_i + 1\) to \(n_i + n_f\) outgoing particles (in the order of anti-Dirac, Dirac and Majorana particles).

4. Draw Feynman diagrams for a given process and the order of perturbation based on the underlying model lagrangian.

For each Feynman diagram:

5. Read out \(P = (a_1, a_2)(a_3, a_4)......(a_{n_i+n_f-1}, a_{n_i+n_f})\) from the diagram and determine \(\text{Sign}(P)\). Note that the pair \((m, n)\) must be arranged such that \(m > n\). If there are several Feynman diagrams which give the same \(P\), count and treat them separately.

6. Draw \(\frac{n_i+n_f}{2}\) Grace lines \(G_{m,n}\) corresponding to the pair \((m, n)\) with \(m > n\).

For each Grace line \(G_{m,n}^\cdot\):

7. Assign the M-direction to each segment of the Grace line based on the rule given in section 3. The M-direction is the same as the physical momentum direction for external particles, and it is in the same direction as the A-direction for internal particles.

8. Assign the F-direction to each segment of the Grace line. First, assign the F-direction to Dirac particles, which is taken in the direction of fermion number flow. Then assign the F-direction to Majorana particles starting from the right endpoint, so that the F-directions do not clash as much as possible (that is, until the Dirac particle propagator which has a different F-direction from that of right-end spinor appears.) Repeat the procedure for the next Majorana propagator. If the right endpoint spinor is Majorana particle, start with the left endpoint Dirac spinor. If both of the endpoint spinors are Majorana particles, start with a Dirac spinor in the propagator. If there is no Dirac spinor in a Grace line, assign the F-direction of the Grace line along the A-direction.

9. For external spinors, assign spinor wave functions based on the rule given in Table 4.

| F-direction | ←− • | →− • | • ← | • → |
|------------|-------|-------|-----|-----|
| ←−         | \(\bar{u}(p)\) | \(\bar{u}(p)\) | \(u(p)\) | \(u(p)\) |
| →−         | \(\bar{v}(p)\) | \(\bar{v}(p)\) | \(v(p)\) | \(v(p)\) |

Table 4: Spinor assignment at left endpoint and at right endpoint.

10. For propagators, use \(S(k)\) independent of the F-directions of the spinors. For the vertex function, use \(\Gamma_{\xi,\eta}^c\) for \(\xi \leftarrow \bullet \leftarrow \eta\) and \(\Gamma_{\eta,\xi}^c = C \Gamma_{\xi,\eta}^T C^{-1}\) for \(\eta \rightarrow \bullet \rightarrow \xi\). For clashing vertices,
use \( \hat{\Gamma} \), where the rules \((60)\) and \((61)\) are to be applied. By construction there are no clashing vertices for \((M_\alpha, M_\beta)\).

11. Multiply the Grace line amplitude so constructed by factor \((-)^{V_g}\) where \(V_g\) is the number of \(v\) and \(\bar{v}\) appearing in the Grace line. This procedure can be neglected since \(\sum V_g\) is common to all Feynman diagrams in a given process. Here, the sum is taken over the Grace lines in the Feynman diagram we are concerned with.

12. Finally, multiply \(\frac{n_i + n_f}{2}\) Grace line amplitudes \([G_{m,n}]\) and the appropriate bosonic parts which connect vertices on the Grace lines in order to obtain an amplitude for each Feynman diagram, and sum up the resulting amplitudes over all the possible partitions \(P\) (corresponding to all Feynman diagrams):

\[
\mathcal{M} = \sum_P \text{Sign}(P) \mathcal{M}_P, \quad \mathcal{M}_P = \Pi_{(m,n)}[G_{m,n}](\text{bosonic part})_{m,n}.
\]