Guaranteed Lower Eigenvalue Bound of Steklov Operator with Conforming Finite Element Methods

Qin Li · Meiling Yue · Xuefeng Liu

Received: date / Accepted: date

Abstract For the eigenvalue problem of the Steklov differential operator, an algorithm based on the conforming finite element method (FEM) is proposed to provide guaranteed lower bounds for the eigenvalues. The proposed algorithm utilizes the a priori error estimation for FEM solution to nonhomogeneous Neumann problems, which is solved by constructing the hypercircle for the corresponding FEM spaces and boundary conditions. Numerical examples are also shown to confirm the efficiency of our proposed method.

Keywords Steklov eigenvalue problems · Nonhomogeneous Neumann problems · Finite element methods · Hypercircle · Guaranteed lower eigenvalue bounds

Mathematics Subject Classification (2010) 65N30 · 65N25 · 65L15 · 65B99

The research has been supported by the National Natural Science Foundations of China (No. 11426639), Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientist (B) 26800090 and Grant-in-Aid for Scientific Research (C) 18K03411. This work was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

Qin Li
School of Mathematics and Statistics, Beijing Technology and Business University, Beijing, 100048, P. R. China
E-mail: liqin@lsec.cc.ac.cn

Meiling Yue
School of Mathematics and Statistics, Beijing Technology and Business University, Beijing, 100048, P. R. China
E-mail: yuemeiling@lsec.cc.ac.cn

Xuefeng Liu
Graduate School of Science and Technology, Niigata University, 8050 Ikarashi 2-no-cho, Nishi-ku, Niigata City, Niigata, 950-2181, Japan
E-mail: xflin@math.sc.niigata-u.ac.jp
1 Introduction

To evaluate lower bounds of the eigenvalues for differential operators is a fundamental problem in numerical analysis. For the eigenvalue approximation based on the finite element method (FEM), there are two new approaches for this aim in the past decade.

(1) The qualitative error estimation, e.g., convergence order, of approximate eigenvalues has a long history in the research of error estimation theories of FEM. A new approach in this scope is about the asymptotical lower eigenvalue bounds from nonconforming finite element methods. That is, the approximate eigenvalues tend to the exact eigenvalues from below if the mesh is fine enough; see, e.g., [13,32] and the references therein.

(2) Another new approach, in the scope of quantitative error estimation, is to provide explicit bounds for the eigenvalues. Early results of [11,12,27,29] require the \textit{a priori} information of eigenvalue, for example, rough bound for certain eigenvalues. A fully computable explicit eigenvalue bound without any additional conditions is proposed by Liu in a sequence of papers ([25,23,24,30,33]), which utilizes the finite element method and the error estimation for the corresponding projection operators. The idea of Liu’s approach traces back to the work of Birkhoff [6], Kikuchi [14] and Kobayashi [15,16].

Our paper considers the explicit eigenvalue bound and can be regarded as an extension of the work of Liu’s approach. In [25,24], the conforming FEM approximation to Laplacian eigenvalue problem is considered, where the projection error for the homogenous boundary value problem is estimated by the hypercircle method. In [23], this approach is applied to non-conforming FEMs, which has an advantage that the projection error estimation can be easily obtained by considering the local interpolation error estimation. Here, in this paper, we follow the frame of [24] to consider the Steklov eigenvalue problem with conforming FEM, while the associated boundary condition is nonhomogenous.

The Steklov eigenvalue problem is one of the important eigenvalue problems about differential operators; see [3,4,17] for a systematic introduction of background and applications. Below is a short review of the numerical approaches to the eigenvalues of Steklov eigenvalue problems. [8,10,18,22] discuss the qualitative error estimation by conforming FEM for Steklov eigenvalue problems, based on which [5,21,28] study more efficient algorithms such as two-grid and multilevel methods to solve Steklov eigenvalue problems. [1] and [26] discuss the a posteriori error estimates with conforming FEM and nonconforming Crouzeix-Raviart FEM, respectively. Especially, in [19,31], the asymptotical lower bounds for Steklov eigenvalue problems are discussed along with nonconforming finite elements such as Crouzeix-Raviart finite element. Meanwhile, in [33], Liu’s method is utilized to obtain the explicit lower bounds of eigenvalues for Steklov eigenvalue problems by using the Crouzeix-Raviart finite element.
In this paper, we adopt the hypercircle method with conforming finite element to get guaranteed lower bounds of eigenvalues for Steklov eigenvalue problems. In Theorem 5, we obtain lower bound for the $k$-th eigenvalue $\lambda_k$ as follows.

\[ \lambda_k \geq \frac{\lambda_{k,h}}{1 + M_h^2 \lambda_{k,h}}. \]

Here, $\lambda_{k,h}$ is the approximate eigenvalue from conforming FEM and $M_h$ is a computable quantity that tends to zero as mesh is refined. The efficiency of our proposed method is compared with the one in [33] through numerical results.

**Objective eigenvalue problem** Without loss of generality, we are concerned with the following model problem

\[-\Delta u + u = 0 \quad \text{in} \; \Omega; \quad \frac{\partial u}{\partial n} = \lambda u \quad \text{on} \; \Gamma = \partial \Omega \]  

(1.1)

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, $\frac{\partial}{\partial n}$ is the outward normal derivative on boundary $\partial \Omega$.

Throughout this paper, we use the standard notation (see, e.g., [27]) for the Sobolev spaces $H^m(\Omega)$ ($m \geq 0$). The Sobolev space $H^0(\Omega)$ coincides with $L^2(\Omega)$. Denote by $\|v\|_{L^2}$ or $\|v\|_0$ the $L^2$ norm of $v \in L^2(\Omega)$; $|v|_{m, \Omega}$ and $\|v\|_{m, \Omega}$ the seminorm and norm in $H^m(\Omega)$, respectively. Symbol $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$. The space $H(\text{div}, \Omega)$ is defined by

\[ H(\text{div}, \Omega) := \{ q \in (L^2(\Omega))^2 \mid \text{div} \; q \in L^2(\Omega) \}. \]

A weak formulation of the above problem is as follows: Find $\lambda \in \mathbb{R}$ and $u \in V = H^1(\Omega)$ such that $\|u\|_b = 1$ and

\[ a(u, v) = \lambda b(u, v) \quad \forall v \in V, \tag{1.2} \]

where

\[ a(u, v) = \int_{\Omega} (\nabla u \nabla v + uv) dx, \quad b(u, v) = \int_{\partial \Omega} uv ds, \quad \|u\|_b = \sqrt{b(u, u)}. \]

Evidently the bilinear form $a(\cdot, \cdot)$ is symmetric, continuous and coercive over the product space $H^1(\Omega)$.

From the argument of compact operators (see, e.g., [8]), we know the eigenvalue problem (1.2) has an eigenvalue sequence $\{\lambda_j\}$:

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty, \]

and the associated eigenfunctions $u_1, u_2, \ldots, u_j, \ldots$, where $b(u_i, u_j) = \delta_{ij}$ and $a(u_i, u_j) = \lambda_i \delta_{ij}$. 

**Finite element approximation** Let $T_h$ be a shape regular triangulation of the domain $\Omega$. For each element $K \in T_h$, denote by $h_K$ the longest edge length of $K$ and define the mesh size $h$ by

$$h := \max_{K \in T_h} h_K.$$  

The finite element space $V^h$ consists of piecewise linear and continuous functions:

$$V^h := \{v_h \in V : v_h|_K \in P_1(K) \quad \forall K \in T_h\}$$

where $P_1(K)$ is the space of polynomials of degree $\leq 1$ on $K$.

The conforming finite element approximation of (1.2) is defined as follows: Find $\lambda_h (> 0) \in \mathbb{R}$ and $u_h \in V^h$ such that

$$a(u_h, v_h) = \lambda_h b(u_h, v_h) \quad \forall v_h \in V^h.$$  

Let $n := \dim(V^h)$ and $n_0 := n - \dim(V^h \cap H_0^1(\Omega))$. The eigenvalue problem (1.3) has eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{n_0,h} < \infty \quad (n_0 < n)$$

and the corresponding eigenfunctions $u_{1,h}, u_{2,h}, \cdots, u_{n_0,h}$, where $b(u_{i,h}, u_{j,h}) = \delta_{ij}, 1 \leq i, j \leq n_0$.

The rest of the paper is organized as follows. In section 2, we discuss the corresponding nonhomogeneous Neumann problems for the model problem (1.1) by the conforming linear finite element and obtain the explicit a priori error estimates. With the basis of the results in the previous section and maximum-minimum principle, explicit lower bounds of the eigenvalues for (1.1) are given in section 3. In section 4, numerical results are shown to verify the theorem results. We make a conclusion in the last section with an appendix.

**2 Finite element approximations of the corresponding nonhomogeneous Neumann problems**

The following boundary value problem and its FEM approach will play an important role in bounding the eigenvalues of the Steklov operator.

$$\begin{cases}
-\Delta u + u = 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = f, & \text{on } \Gamma = \partial \Omega.
\end{cases}$$

(2.1)

A weak formulation of the above problem is as follows: Given $f \in \mathbb{L}^2(\Gamma)$, find $u \in V = H^1(\Omega)$ such that

$$a(u, v) = b(f, v) \quad \forall v \in V.$$  

(2.2)
The conforming finite element approximation of \([2,2]\) is defined as follows: Find \(u_h \in V^h\) such that
\[
a(u_h, v_h) = b(f, v_h) \quad \forall v_h \in V^h. \tag{2.3}
\]
Define the projection \(P_h : V \to V^h\) as follows.
\[
a(u - P_h u, v_h) = 0 \quad \forall v_h \in V^h.
\]
Thus, \(u_h = P_h u\).

In this section, the following classical finite element spaces will be used in constructing the \textit{a priori} estimate. Define by \(E_h\) the set of edges of the triangulation and \(E_h, \Gamma\) the one on the boundary of the domain. Define \(T^b_h\) the set of elements of \(T_h\) having at least one edge on \(\partial \Omega\).

(i) Piecewise function spaces \(X^h\) and \(X^h_{\Gamma}\):
\[
X^h := \{v \in L^2(\Omega) : v|_K \in P_1(K) \quad \forall K \in T_h\}
\]
\[
X^h_{\Gamma} := \{v \in L^2(\Gamma) : v|_e \in P_1(e) \quad \forall e \in E_h, \Gamma\}
\]
where \(P_1(e)\) is the space of polynomials of degree \(\leq 1\) on the edge \(e\).

(ii) Raviart-Thomas FEM space \(W^h\) with order one \([9]\):
\[
W^h := \left\{p_h \in H(\text{div}, \Omega) \mid p_h = \begin{pmatrix} a_K \\ b_K \\ c_K \end{pmatrix} \left( \begin{array}{c} x \\ y \end{array} \right), \right. \\
a_K, b_K, c_K \in P_1(K) \ \text{for} \ K \in T_h\}
\]
The space \(W^h_{f_h}\) is a shift of \(W^h\) corresponding to \(f_h \in X^h_{\Gamma}\):
\[
W^h_{f_h} := \{p_h \in W^h \mid p_h \cdot \mathbf{n} = f_h \in X^h_{\Gamma} \ \text{on} \ \Gamma\}.
\]
Notice that the following relation holds for current space settings.
\[
V^h \subset \text{div}(W^h) = X^h.
\]

2.1 The Hypercircle

In this subsection, we first present the hypercircle which can be used to facilitate the error estimate. The argument about \textit{a priori} estimation here can be regarded as a special case discussed in [20], while the space setting here will lead to a more concise \textit{a priori} estimation for the FEM solution.

**Theorem 1** Given \(f_h \in X^h_{\Gamma}\), let \(\tilde{u} \in V\) and \(\tilde{u}_h \in V^h\) be solutions to the following variational problems, respectively,
\[
a(\tilde{u}, v) = b(f_h, v) \quad \forall v \in V \tag{2.4}
\]
\[
a(\tilde{u}_h, v_h) = b(f_h, v_h) \quad \forall v_h \in V^h. \tag{2.5}
\]
Then, for \( p_h \in W_h \) satisfying \( \text{div } p_h = \tilde{u}_h \), we have the hypercircle
\[
\| \nabla \tilde{u}_h - p_h \|_{L^2}^2 = \| \tilde{u} - \tilde{u}_h \|_{H^1(\Omega)}^2 + \| \nabla \tilde{u} - p_h \|_{L^2}^2 + \| \tilde{u} - \tilde{u}_h \|_{L^2}^2
\]  
(2.6)
and thus the following computable error estimate holds:
\[
\| \tilde{u} - \tilde{u}_h \|_{H^1(\Omega)} \leq \kappa_h \| f_h \|_b
\]  
(2.7)
where \( \kappa_h \) is defined by
\[
\kappa_h := \max_{f_h \in X^h \setminus \{0\}} \min_{p_h \in W_h, \text{div } p_h = \tilde{u}_h} \frac{\| \nabla \tilde{u}_h - p_h \|_0}{\| f_h \|_b}.
\]  
(2.8)

Proof Rewriting \( \nabla \tilde{u}_h - p_h \) by \((\nabla \tilde{u}_h - \nabla \tilde{u}) + (\nabla \tilde{u} - p_h)\), we have
\[
\| \nabla \tilde{u}_h - p_h \|_{L^2}^2 = \| \nabla \tilde{u}_h - \nabla \tilde{u} \|_{L^2}^2 + \| \nabla \tilde{u} - p_h \|_{L^2}^2 + 2(\nabla \tilde{u}_h - \nabla \tilde{u}, \nabla \tilde{u} - p_h).
\]
Furthermore, the Green theorem and the Neumann boundary conditions setting lead to
\[
(\nabla \tilde{u}_h - \nabla \tilde{u}, \nabla \tilde{u} - p_h) = (\tilde{u}_h - \tilde{u}, -\tilde{u} + \text{div } p_h)
= (\tilde{u}_h - \tilde{u}, -\tilde{u} + \tilde{u}_h) = \| \tilde{u} - \tilde{u}_h \|_{L^2}^2.
\]
Notice that to make the above equality hold, the \( H^1 \) regularity of \( \tilde{u} \) and \( \tilde{u}_h \) is enough. Then, we get the desired hypercircle (2.6). Based on the result (2.6), it is easy to get
\[
\| \tilde{u} - \tilde{u}_h \|_{H^1(\Omega)} \leq \| \nabla \tilde{u}_h - p_h \|_{L^2}.
\]  
(2.9)
By further varying \( f_h \) in \( X_h^h \), we draw the conclusion about \( \kappa_h \).

Remark 1 In Theorem 3.3 of [20], a more general case such that \( \text{div } p_h - \tilde{u}_h \neq 0 \) is discussed. Since the Raviart-Thomas space \( W_h \) in this paper has a higher order, one can find \( p_h \in W_h \) such that \( \text{div } p_h = \tilde{u}_h \) for \( \tilde{u}_h \in V_h \).

2.2 Explicit A Priori Error Estimates

We first quote an explicit bound for the constant in trace theorem.

Lemma 1 ([20]) Let \( e \) be an edge of triangle element \( K \). Define function space
\[
V_e(K) := \{ v \in H^1(K) \mid \int_e v ds = 0 \}.
\]
Given \( u \in V_e(K) \), we have the following inequality related to the trace theorem:
\[
\| u \|_{L^2(e)} \leq C(K) |u|_{H^1(K)}, \quad C(K) := 0.574 \left( \frac{|e|}{|K|} \right)^{1/2} h_K \leq 0.8118 \frac{h_K}{\sqrt{H_K}}.
\]  
(2.10)
Here, \( H_K \) denotes the height of triangle \( K \) respect to edge \( e \).
Let us introduce a projection operator $\pi_{h,\Gamma} : L^2(\Gamma) \mapsto X^h_\Gamma$: Given $f \in L^2(\Gamma)$, $\pi_{h,\Gamma} f \in X^h_\Gamma$ satisfies
\[
   b(f - \pi_{h,\Gamma} f, v_h) = 0 \quad \forall v_h \in X^h_\Gamma.
\]

**Theorem 2** Let $u$ and $\tilde{u}$ be solutions to (2.2) and (2.4), respectively, with $f_h$ taken as $f_h := \pi_{h,\Gamma} f$. Then, the following error estimate holds:
\[
   \|u - \tilde{u}\|_{H^1(\Omega)} \leq C_h \|(I - \pi_{h,\Gamma}) f\|_b,
\]
where $I$ is the identity operator and $C_h$ takes the maximum of $C(K)$ over the boundary elements:
\[
   C_h := \max_{K \in T_h} C(K). \tag{2.11}
\]

**Proof** Setting $v = u - \tilde{u}$ in (2.2) and (2.4), we have
\[
   a(u - \tilde{u}, u - \tilde{u}) = b(f - f_h, u - \tilde{u}) = b((I - \pi_{h,\Gamma}) f, (I - \pi_{h,\Gamma})(u - \tilde{u})).
\]
From the Schwartz inequality and Lemma 1 we get
\[
   \|u - \tilde{u}\|_{H^1(\Omega)}^2 \leq \|(I - \pi_{h,\Gamma}) f\|_b \|(I - \pi_{h,\Gamma})(u - \tilde{u})\|_b \leq C_h \|(I - \pi_{h,\Gamma}) f\|_b \|u - \tilde{u}\|_{H^1(\Omega)}
\]
which implies the conclusion.

Now, we are ready to formulate and prove the explicit a priori error estimate.

**Theorem 3** Let $u$ and $u_h$ be solutions to (2.2) and (2.3), respectively. Then, the following error estimates hold:
\[
   \|u - u_h\|_{H^1(\Omega)} \leq M_h \|f\|_b \tag{2.12}
\]
\[
   \|u - u_h\|_b \leq M_h \|u - u_h\|_{H^1(\Omega)} \leq M^2_h \|f\|_b \tag{2.13}
\]
with $M_h := \sqrt{C^2_h + \kappa^2_h}$.

**Proof** The estimation in (2.12) can be obtained by applying (2.7) and Theorem 2
\[
   \|u - u_h\|_{H^1(\Omega)} \leq \|u - \tilde{u}_h\|_{H^1(\Omega)} \leq \|u - \tilde{u}\|_{H^1(\Omega)} + \|\tilde{u} - \tilde{u}_h\|_{H^1(\Omega)} \leq C_h \|(I - \pi_{h,\Gamma}) f\|_b + \kappa_h \|f_h\|_b \leq \sqrt{C^2_h + \kappa^2_h} \|f\|_b = M_h \|f\|_b.
\]
The error estimate (2.13) can be obtained by the classic Aubin-Nitsche duality technique.
2.3 Computation of \( \kappa_h \)

The quantity \( \kappa_h \) defined in (2.8) is not easily to evaluate directly. In the practical computation, we turn to give a upper bound for \( \kappa_h \). To give an estimation of the quantity \( \kappa_h \), the following two sub-problems for a given \( f_h \in X_h^\Gamma \) will be needed.

(a) Find \( \tilde{u}_h \in V_h \) such that
\[
a(\tilde{u}_h, v_h) = b(f_h, v_h) \quad \forall v_h \in V_h.
\]

(b) Let \( \tilde{u}_h \) be the solution of (a). Find \( p_h \in W_h^0 \) and \( \rho_h \in X_h \), \( c \in \mathbb{R} \) such that
\[
\begin{align*}
\left\{ \begin{array}{l}
(p_h, \tilde{p}_h) + (\rho_h, \text{div} \tilde{p}_h) + (\rho_h, d) = 0 & \forall \tilde{p}_h \in W_h^0, \forall d \in \mathbb{R} \\
(\text{div} p_h, \tilde{q}_h) + (c, \tilde{q}_h) = (\tilde{u}_h, \tilde{q}_h) & \forall \tilde{q}_h \in X_h
\end{array} \right.
\]
where \( W_h^0 := \{ \tilde{p}_h \in W_h | \tilde{p}_h \cdot n = 0 \in X_h^\Gamma \text{ on } \Gamma \} \).

For each given \( f_h \), there exist unique solution \( \tilde{u}_h \) and \( p_h \) to the sub-problems (a) and (b). By using the mapping from \( f_h \) to \( \tilde{u}_h \) and \( p_h \), let us introduce the quantity \( \bar{\kappa}_h \), which works as an upper bound of \( \kappa_h \):
\[
\bar{\kappa}_h := \max_{f_h \in X_h^\Gamma \setminus \{0\}} \frac{\|\nabla \tilde{u}_h - p_h\|_0}{\|f_h\|_b}.
\]

According to the definition of \( \bar{\kappa}_h \), it is required to find \( f_h \) that maximizes the value of \( \|\nabla \tilde{u}_h - p_h\|_0/\|f_h\|_b \), which can be by solving an eigenvalue problem of matrices. For detailed solution of this eigenvalue problem, we refer to ([24]), where an analogous problem is described.

**Remark 2** The setting of problem (b) implies \( c = 0 \). In fact, setting \( v_h = 1 \) in the problem (a), we have
\[
\int_\Omega \tilde{u}_h dx = \int_{\partial \Omega} f_h ds = \int_{\partial \Omega} p_h \cdot n ds = \int_\Omega \text{div } p_h dx.
\]
By taking \( \tilde{q}_h = 1 \) in the problem (b), we can easily check that \( c = 0 \).

3 Lower bounds of eigenvalues

The theorem on lower eigenvalue bound of Steklov eigenvalue problem has been discussed in [33]. Here, we deduce the same eigenvalue bounds as in [33] but with a different proof. The main idea in [33] is to utilize the min-max principle of eigenvalue problem, while here we turn to the max-min principle. Moreover, the setting of eigenvalue problem in (3.1), rather than (1.2), helps to make the proof more concise in the sense that complicated space decomposition in [33] is not needed.
As a preparation to the proof in Theorem 4, we consider the eigenvalue problem in a different formulation. Let us consider the operator $D^{-1} : L^2(\Gamma) \to H^1(\Omega)$ such that for $f \in L^2(\Gamma)$, $D^{-1}f = u$ satisfies the variational equation

$$a(D^{-1}f, v) = b(f, v) \quad \forall v \in H^1(\Omega).$$

Let $\gamma$ be the trace operator $\gamma : H^1(\Omega) \to L^2(\Gamma)$. From the theory of compact self-adjoint operators, we know that $D^{-1} \cdot \gamma : H^1(\Omega) \to H^1(\Omega)$ has the zero eigenvalues along with the associated eigenspace as $H^1_0(\Omega)$, and the positive eigenvalue sequence $\{\mu_j\}$ listed in decreasing order

$$\mu_k > 0, \quad \mu_1 \geq \mu_2 \geq \cdots, \quad \lim_{k \to \infty} \mu_k = 0.$$

The weak formulation of the eigenvalue problem for $D^{-1} \cdot \gamma$ is given by: Find $u \in H^1(\Omega)$ and $\mu \geq 0$ such that,

$$b(u, v) = \mu a(u, v) \quad \forall v \in H^1(\Omega). \quad (3.1)$$

The eigenfunctions of $D^{-1} \cdot \gamma$ or (3.1) form an orthonormal basis of $H^1(\Omega)$.

As for the relation between the eigen problem of $D^{-1} \cdot \gamma$ and the one defined in (1.2), we have that the non-zero eigenvalues $\mu_j$’s are given by the reverse of $\lambda_j$, i.e., $\mu_j = 1/\lambda_j$.

**Finite element approximation** The approximation of the eigenvalue problem (3.1) over $V^h$ is given by: Find $\mu_h(\geq 0) \in \mathcal{R}$ and $u_h \in V^h$ such that

$$b(u_h, v_h) = \mu_h a(u_h, v_h) \quad \forall v_h \in V^h. \quad (3.2)$$

Notice that discretized system (3.2) just has $n_0$ non-zero eigenvalues. The approximate eigenvalues are given by

$$0 = \mu_{n_0,h} = \mu_{n_0-1,h} = \cdots = \mu_{n_0+1,h} < \mu_{n_0,h} \leq \cdots \leq \mu_{2,h} \leq \mu_{1,h}.$$

For the non-zero $\mu_{j,h}$’s, we have $\mu_{j,h} = 1/\lambda_{j,h}$.

Denote the Rayleigh quotient over $V$ by $R(\cdot)$ as follows: for any $v \in V$, $\|v\|_b \neq 0$,

$$R(v) := \frac{b(v, v)}{a(v, v)}.$$

The following maximum-minimum principle will play an important role in the lower eigenvalue bound estimation.

$$\mu_k = \max_{V_k \subset V, \dim V_k = k} \min_{u \in V_k} R(u),$$

$$\mu_{k,h} = \max_{S^h \subset V^h, \dim(S^h) = k} \min_{u \in S^h} R(u).$$

**Theorem 4** An explicit upper bound for $\mu_k$ is given by

$$\mu_k \leq \mu_{k,h} + M^2_k, \quad k = 1, 2, 3, \cdots n.$$
Proof If $\mu_k \leq M_k^2$, then the conclusion $\mu_k \leq M_k^2 + \mu_{k,h}$ holds obviously. In the rest of the proof, we only consider the case that $0 < M_k^2 < \mu_k$.

Let $E_k$ be the space spanned by the eigenfunctions \( \{ w_i \}_{i=1}^k \). For $w \in E_k(\subset V)$, $w \neq 0$, it is easy to see $\mu_k \leq \frac{\| w \|_b^2}{\| w \|_2^2}$ and hence,

\[
\| w \|_b \geq \sqrt{\mu_k} \| w \|_1 > M_k \| w \|_1.
\]

Let $w_h = P_h w$. From (2.13), we have

\[
\| w - w_h \|_b \leq M_k \| w - w_h \|_1.
\] (3.3)

By applying the triangle inequality and the above two inequalities, we have

\[
\| w_h \|_b \geq \| w \|_b - \| w - w_h \|_b \geq \| w \|_b - M_k \| w - w_h \|_1 \geq \| w \|_b - M_k \| w \|_1 > 0.
\] (3.4)

The fact that $P_h w \neq 0$ for $w \neq 0$ implies $\dim(P_h E_k) = \dim(E_k) = k$.

Thus, from the maximum-minimum principle,

\[
\mu_{k,h} \geq \min_{w_h \in P_h E_k} \frac{\| w_h \|_2^2}{\| w_h \|_1^2} = \min_{w \in E_k} \frac{\| P_h w \|_2^2}{\| P_h w \|_1^2} \geq \min_{w \in E_k} \frac{\| w \|_b - M_k \| w - w_h \|_1}{\| w \|_b/\mu_k - \| w - w_h \|_1^2}.
\]

Notice that (3.3) implies $\| w - w_h \|_1 \leq 1/M_k$ for $\| w \|_b = 1$. Define $g(t)$ by

\[
g(t) := (1 - M_k t)^2/(1/\mu_k - t^2), \quad 0 \leq t \leq 1/M_k.
\]

Since $g(t)$ has the minimal value at $t = M_k/\mu_k(< 1/M_k)$, we have

\[
\mu_{k,h} \geq \min_{w \in E_k, \| w \|_b = 1} g(\| w - w_h \|_1) \geq \min_{w \in E_k, \| w \|_b = 1} g(M_k/\mu_k) = \mu_k - M_k^2.
\]

which implies the final conclusion.

Notice that for $k = 1, 2, 3, \ldots, n_0$, $\mu_k \geq \mu_{k,h} > 0$. Hence, $\mu_k = 1/\lambda_k$ and $\mu_{k,h} = 1/\lambda_{k,h}$. As a direct result, Theorem 4 leads to the core goal of this paper as follows.

**Theorem 5** From the conforming eigenvalue approximation $\lambda_{k,h}$, we have the following lower eigenvalue bounds for the leading eigenvalues of (1.2).

\[
\lambda_k \geq \frac{\lambda_{k,h}}{1 + M_k^2 \lambda_{k,h}}, \quad k = 1, 2, 3, \ldots, n_0.
\] (3.5)

4 Numerical Examples

In this section, we apply the proposed eigenvalue estimation (3.5) to the problem (1.1) on both the unit square domain $\Omega = (0, 1) \times (0, 1)$ and the L-shaped domain $\Omega = (0, 2) \times (0, 2) \setminus [1, 2] \times [1, 2]$. Also, the existing method of [33] based on the non-conforming FEM is utilized to compare the efficiency with each other.
4.1 Preparation

The explicit values of the exact eigenvalues for both domains are not available. For the unit square domain, the following high-precision estimation with trustable significant digits are used as a nice approximation to true eigenvalues \cite{31}.

\begin{equation*}
(\text{unit square}) \quad \lambda_1 \approx 0.240079, \quad \lambda_2 \approx 1.49230.
\end{equation*}

In case of the L-shaped domain, the cubic conforming FEM with the mesh size $h = \sqrt{2}/256$ provides a high-precision approximation to eigenvalues:

\begin{equation*}
(\text{L-shaped domain}) \quad \lambda_1 \approx 0.3414160, \quad \lambda_2 \approx 0.6168667, \quad \lambda_3 \approx 0.9842784.
\end{equation*}

For both domains, the uniform meshes are adopted. The eigenvalue estimation \cite{33} provides a guaranteed lower eigenvalue bound:

\begin{equation*}
\hat{\lambda}_{k,h} := \frac{\lambda_{k,h}}{1 + M_h^2 \lambda_{k,h}}, \quad M_h = \sqrt{C_h^2 + \kappa_h^2},
\end{equation*}

where $\lambda_{k,h}$ denotes the $k$-th conforming finite element solution and the quantity $C_h$ in estimating $M_h$ is given by

\begin{equation*}
C_h := 0.8118 \max_{K \in T_h} \frac{h_K}{\sqrt{H_K}} \leq \hat{C}_h := 0.966 \sqrt{h_K}.
\end{equation*}

The eigenvalue estimation from Theorem 3.8 of \cite{33} has the formula as follows.

\begin{equation}
\hat{\lambda}_{k,h} := \frac{\hat{\lambda}_{k,h}}{1 + \hat{C}_h^2 \lambda_{k,h}},
\end{equation}

where $\hat{\lambda}_{k,h}$ denotes the $k$-th approximate eigenvalue from the Crouzeix-Raviart FEM. Particularly, for the uniform mesh used here, $\hat{C}_h$ is estimated by

\begin{equation*}
\hat{C}_h := 0.6711 \max_{K \in T_h} \frac{h_K}{\sqrt{H_K}} + \frac{0.1893}{\sqrt{\hat{\lambda}_{1,h}}} \max_{K \in T_h} h_K \leq \hat{C}_h := 0.7981 \sqrt{h_K} + \frac{0.1893}{\sqrt{\hat{\lambda}_{1,h}}} h_K.
\end{equation*}
4.2 Computation results for two domains

Sample uniform triangular meshes for two domains are displayed in Figure 1, where the mesh size for the unit square is $h = \sqrt{2}/8$ and the one for the L-shaped domain is $h = \sqrt{2}/4$.

For the unit square domain, the eigenvalue estimations (3.5) for the leading 3 eigenvalues are displayed in Table 1, while the results based on the non-conforming FEM ([33]) are displayed in Table 2. The results for the L-shaped domain are displayed in Table 3 and 4. Figure 2 and Figure 3 describe the relation between the absolute errors and the degrees of freedom (DOF) over the unit square and L-shaped domains, respectively. Here, the DOF of (3.5) is counted as the the dimension of the linear conforming FEM space $V_h$, while the one for [33] is the dimension of the Crouzeix-Raviart FEM space.

Let us also introduce the total errors by

$$
\text{Error-(3.5)} := |\lambda_1 - \lambda_{1,h}| + |\lambda_2 - \lambda_{2,h}| + |\lambda_3 - \lambda_{3,h}|
$$

$$
\text{Error-(4.1)} := |\lambda_1 - \hat{\lambda}_{1,h}| + |\lambda_2 - \hat{\lambda}_{2,h}| + |\lambda_3 - \hat{\lambda}_{3,h}|
$$

The relation between the total errors and the degrees of freedom (DOF) is displayed in Figure 4.

From the computational results for two domains and the comparison between the bound (3.5) and the one from [33], we can draw the conclusion that

(1) The conforming FEM can lead not only the guaranteed upper bounds $\lambda_{k,h}$ but also lower bounds $\underline{\lambda}_{k,h}$ of the eigenvalues $\lambda_k$. However, the non-conforming FEM in [33] merely provide the guaranteed lower eigenvalue bounds.

(2) Both the lower eigenvalue bounds proposed in this paper and the one in [33] have a sub-optimal convergence rate, compared with the theoretical estimation for the approximate eigenvalues.

(3) With the same degree of freedom (DOF), the lower bound in (3.5) gives slightly better estimation than the one from the non-conforming FEM. However, to obtain the bound (3.5), one has to pay more efforts to solve matrix problem of larger scale to obtain $\tilde{\kappa}_h$.

5 Conclusion

In this paper, we propose a method to obtain the guaranteed lower bound of Steklov eigenvalue based on the technique of the hypercircle method. Also, the proposed eigenvalue bounds here utilizing the linear conforming FEM is compared with the one from the non-conforming FEM. In the future research, we will apply the obtained eigenvalue bounds to give sharp bound for the constants in numerical analysis of FEM. For example, the constant $C(K)$ in (2.10) can be solved by solving the corresponding Steklov eigenvalue problem.
| $h$   | $\sqrt{2}/4$ | $\sqrt{2}/8$ | $\sqrt{2}/16$ | $\sqrt{2}/32$ |
|-------|--------------|--------------|--------------|--------------|
| $\bar{\kappa}_h$ | 0.2891 | 0.2642 | 0.1443 | 0.1021 |
| $C_h$ | 0.5740 | 0.4059 | 0.2870 | 0.2029 |
| $M_h$ | 0.6427 | 0.4544 | 0.3208 | 0.2272 |
| $\lambda_{1,h}$ | 0.2404841 | 0.2401798 | 0.2401042 | 0.2400854 |
| $\Delta_{1,h}$ | 0.218753 | 0.228833 | 0.2343144 | 0.2371468 |
| $\lambda_{2,h}$ | 1.527151 | 1.502305 | 1.494918 | 1.492966 |
| $\Delta_{2,h}$ | 0.936415 | 1.146662 | 1.295596 | 1.386153 |

(Note: $\lambda_{2,h} = \lambda_{3,h}$, $\Delta_{2,h} = \Delta_{3,h}$)

| $h$   | $\sqrt{2}/4$ | $\sqrt{2}/8$ | $\sqrt{2}/16$ | $\sqrt{2}/32$ |
|-------|--------------|--------------|--------------|--------------|
| $\hat{C}_h$ | 0.6110176 | 0.4038323 | 0.2714162 | 0.1848489 |
| $\hat{\lambda}_{1,h}$ | 0.2404829 | 0.2401793 | 0.2401041 | 0.2400853 |
| $\hat{\Delta}_{1,h}$ | 0.2206705 | 0.2311264 | 0.235931 | 0.2381318 |
| $\hat{\lambda}_{2,h}$ | 1.460229 | 1.483297 | 1.489892 | 1.491678 |
| $\hat{\Delta}_{2,h}$ | 0.9450309 | 1.19438 | 1.342541 | 1.419335 |

(Note: $\hat{\lambda}_{2,h} = \hat{\lambda}_{3,h}$, $\hat{\Delta}_{2,h} = \hat{\Delta}_{3,h}$)

| $h$   | $\sqrt{2}/2$ | $\sqrt{2}/4$ | $\sqrt{2}/8$ | $\sqrt{2}/16$ |
|-------|--------------|--------------|--------------|--------------|
| $\bar{\kappa}_h$ | 0.5106 | 0.3633 | 0.2591 | 0.1847 |
| $C_h$ | 0.8118 | 0.5740 | 0.4059 | 0.2870 |
| $M_h$ | 0.9590 | 0.6793 | 0.4815 | 0.3413 |
| $\lambda_{1,h}$ | 0.3443305 | 0.3421498 | 0.3416010 | 0.3414626 |
| $\lambda_{2,h}$ | 0.6513041 | 0.6299816 | 0.6217140 | 0.6186763 |
| $\lambda_{3,h}$ | 1.0278736 | 0.9968693 | 0.9876317 | 0.9851393 |
| $\Delta_{1,h}$ | 0.5283638 | 0.6827630 | 0.8035932 | 0.8837230 |

Appendix

This appendix gives the details on the computation of $\bar{\kappa}_h$ in Section 2.3. Given $f_h \in X_h^0$, denote by $\bar{u}_h \in V_h$ and $(p_h, y_h, c) \in W_h \times X_h \times \mathcal{R}$ the solutions of the two sub-problems (a) and (b) in Section 2.3.

By selecting $v_h = \bar{u}_h$ in (a) and $\bar{q}_h = \bar{u}_h$ in (b), we have

$$\langle \nabla \bar{u}_h, \nabla \bar{u}_h \rangle = b(f_h, \bar{u}_h) - (\bar{u}_h, \bar{u}_h), \quad \langle \text{div} p_h, \bar{u}_h \rangle = (\bar{u}_h - c, \bar{u}_h). \quad (5.1)$$

By applying Green’s formula to $\langle \nabla \bar{u}_h, p_h \rangle$ and the second equality in (5.1), we have

$$\langle \nabla \bar{u}_h, p_h \rangle = -(\bar{u}_h, \text{div} p_h) + b(\bar{u}_h, f_h) = -(\bar{u}_h, \bar{u}_h - c) + b(\bar{u}_h, f_h).$$
Thus,

\[
\| \nabla \tilde{u}_h - p_h \|_0^2 = (\nabla \tilde{u}_h - p_h, \nabla \tilde{u}_h - p_h) \\
= (\nabla \tilde{u}_h, \nabla \tilde{u}_h) - 2(p_h, \nabla \tilde{u}_h) + (p_h, p_h) \\
= b(f_h, \tilde{u}_h) - (\tilde{u}_h, \tilde{u}_h) + 2(\tilde{u}_h, \tilde{u}_h - c) - 2b(\tilde{u}_h, f_h) + (p_h, p_h) \\
= -b(f_h, \tilde{u}_h) - (\tilde{u}_h, \tilde{u}_h) + 2(\tilde{u}_h, \tilde{u}_h - c) + (p_h, p_h) \\
= -b(f_h, \tilde{u}_h) + (\tilde{u}_h, \tilde{u}_h) - 2(\tilde{u}_h, c) + (p_h, p_h).
\]

\[\text{Table 4} \quad \text{Quantities in the eigenvalue estimation} \quad (L\text{-shaped domain})\]

| $h$ | $\sqrt{2}/2$ | $\sqrt{2}/4$ | $\sqrt{2}/8$ | $\sqrt{2}/16$ |
|-----|--------------|--------------|--------------|--------------|
| $C_h$ | 0.8997886    | 0.5889361    | 0.3928155    | 0.2659045    |
| $\lambda_{1,h}$ | 0.3425959    | 0.3416846    | 0.3414799    | 0.3414316    |
| $\lambda_{2,h}$ | 0.2682036    | 0.3054704    | 0.3243874    | 0.3333834    |
| $\lambda_{3,h}$ | 0.9608929    | 0.9769290    | 0.9821661    | 0.9837098    |
| $\lambda_{4,h}$ | 0.5404476    | 0.7296185    | 0.8529063    | 0.9197389    |

Fig. 2 Absolute errors of eigenvalue bounds for the unit square (Left: $|\lambda_i - \lambda_{i,h}|$; Right: $|\lambda_i - \hat{\lambda}_{i,h}|$ ($i = 1, 2, 3$))

Fig. 3 Absolute errors of eigenvalue bounds for the L-shaped domain (Left: $|\lambda_i - \lambda_{i,h}|$; Right: $|\lambda_i - \hat{\lambda}_{i,h}|$ ($i = 1, 2, 3$))

Let the basis functions of the FEM spaces be
\[ V^h = \text{span}\{\phi_i\}_{i=1}^n, \quad W^h = W^h_0 \oplus W^h_b, \quad W^h_b = \text{span}\{\psi_{i,(0)}\}_{i=1}^\ell, \]
\[ W^h_0 = \text{span}\{\psi_{i,(b)}\}_{i=1}^\ell, \quad X^h = \text{span}\{q_i\}_{i=1}^m, \quad X^h_b = \text{span}\{\varphi_i\}_{i=1}^s. \]

Define matrices \( P^{(0)}_{\ell \times \ell}, P^{(b)}_{\ell \times \ell}, P^{(0b)}_{\ell \times \ell}, S_{m \times s}, D_{n \times s}, G_{s \times s}, J_{n \times n}, L_{m \times n}, N^{(0)}_{m \times \ell}, N^{(0b)}_{m \times \ell} \) with their element as
\[ P^{(0)}_{i,j} = (\psi_{i,(0)}, \psi_{j,(0)}), \quad P^{(b)}_{i,j} = (\psi_{i,(b)}, \psi_{j,(b)}), \quad P^{(0b)}_{i,j} = (\psi_{i,(0)}, \psi_{j,(b)}), \]
\[ S_{i,j} = (\nabla \phi_i, \nabla \phi_j), \quad D_{i,j} = b(\phi_i, \phi_j), \quad G_{i,j} = b(\varphi_i, \varphi_j), \quad J_{i,j} = (\phi_i, \phi_j), \]
\[ L_{i,j} = (q_i, \phi_j), \quad N^{(0)}_{i,j} = (q_i, \text{div}\psi_{j,(0)}), \quad N^{(0b)}_{i,j} = (q_i, \text{div}\psi_{j,(b)}). \]

Let us represent the functions, \( f_h \in X^h, p_h \in W^h_b, \rho_h \in X^h, \tilde{u}_h \in V^h \), by the column vectors, \( g \in \mathcal{R}^s, x \in \mathcal{R}^{\ell \times \ell}, z \in \mathcal{R}^m, y \in \mathcal{R}^n \), respectively, that is,
\[ f_h = (\varphi_1, \cdots, \varphi_s) \cdot g, \quad p_h = (\psi_{1,(0)}, \cdots, \psi_{b,(0)}, \psi_{1,(b)}, \cdots, \psi_{\ell,(b)}) \cdot x, \]
\[ \rho_h = (q_1, \cdots, q_m) \cdot z, \quad \tilde{u}_h = (\phi_1, \cdots, \phi_n) \cdot y. \]

Problems (a) and (b) become
\[ (a) \quad Sy + Jy = Dg, \quad (b) \quad \left\{ \begin{array}{lcl}
PX + (N^{(0)})^Tz &=& O_{\ell \times 1} \\
Nx &=& Lg,
\end{array} \right. \]
where \( O_{\ell \times \ell} \) denotes a null matrix with \( \ell \) rows and \( \ell \) columns,
\[ P = \left( P^{(0)} P^{(0b)} \right) \text{ and } N = \left( N^{(0)} N^{(0b)} \right). \]
Then, the solutions of (a) and (b) are as follows:
\[ y = Kg, \quad \left\{ \begin{array}{lcl}
x &=& H_1LKg \\
z &=& H_2LKg
\end{array} \right. \]
where \( K = (S + J)^{-1}D, \ A := \left( \begin{array}{c}
P \\
N \\
O_{m \times m}
\end{array} \right), \ H_1 := A^{-1}(1 : \hat{p} + 1 : \hat{p} + n) \)
denotes the first \( (\hat{p}+\hat{b}) \) rows and last \( m \) columns of \( A^{-1} \), and \( H_2 := A^{-1}(\hat{p}+1 : \)
\( \hat{p} + m, \hat{p} + 1 : \hat{p} + m \) denotes the last \( m \) rows and last \( m \) columns of \( A^{-1} \). From (5.2), we have
\[
\| \nabla \tilde{u}_h - p_h \|_0^2 = -g^T D^T y + y^T J y + x^T Q x
\]
\[
= -g^T D^T K g + g^T K^T J K g + g^T K^T L^T H_1^T Q H_1 L K g
\]
\[
= g^T B g
\]
where
\[
Q := \begin{pmatrix} P^{(0)}(0) & P^{(0)}(b) \\ (P^{(b)})^T & P^{(b)} \end{pmatrix}, \\
B := -D^T K + K^T J K + K^T L^T H_1^T Q H_1 L K.
\]
Thus, from the definition of \( \hat{\kappa}_h \) in Section 2.3, we have
\[
\hat{\kappa}_h = \max_{g \in \mathbb{R}^s} \left( \frac{g^T B g}{g^T G g} \right)^{1/2}
\]
which means that the quantity \( \hat{\kappa}_h \) is the square root of the maximum eigenvalue of the eigenvalue problem
\[
B g = \lambda G g.
\]

References

1. Armentano, M.G., Padra, C.: A posteriori error estimates for the steklov eigenvalue problem. Appl. Numer. Math. 58(5), 593–601 (2008)
2. Babuška, I., Osborn, J.: Eigenvalue Problems, Finite Element Methods (Part 1), Handbook of Numerical Analysis, Vol. II. Elsevier Science Publishers B.V., North-Holland (1991)
3. Bergman, S., Schiffer, M.: Kernel functions and elliptic differential equations in mathematical physics. Academic Press, New York (1953)
4. Bermúdez, A., Rodríguez, R., Santamarina, D.: A finite element solution of an added mass formulation for coupled fluid-solid vibrations. Numer. Math. 87(2), 201–227 (2000). DOI 10.1007/s002110000175. URL https://doi.org/10.1007/s002110000175
5. Bi, H., Zhang, Y., Yang, Y.: Two-grid discretizations and a local finite element scheme for a non-selfadjoint stekloff eigenvalue problem. Comput. Math. Appl. (2018)
6. Birkhoff, G., De Boor, C., Swartz, B., Wendroff, B.: Rayleigh-ritz approximation by piecewise cubic polynomials. SIAM J. Numer. Anal. 87(2), 201–227 (1966)
7. Boffi, D.: Finite element approximation of eigenvalue problems. Acta Numer. 19, 1–120 (2010). DOI 10.1017/S0962492910000012. URL https://doi.org/10.1017/S0962492910000012
8. Bramble, J.H., Osborn, J.: Approximation of steklov eigenvalues of non-selfadjoint second order elliptic operators. In: The mathematical foundations of the finite element method with applications to partial differential equations, pp. 387–408. Elsevier (1972)
9. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods, Springer Series in Computational Mathematics, vol. 15. Springer (1991). DOI 10.1007/978-1-4612-3172-1. URL https://doi.org/10.1007/978-1-4612-3172-1
10. Calo, F., Colton, D., Meng, S., Monk, P.: Stekloff eigenvalues in inverse scattering. SIAM J. Appl. Math. 76(4), 1737–1763 (2016). DOI 10.1137/16M1058704. URL https://doi.org/10.1007/16M1058704
11. Carstensen, C., Gallistl, D.: Guaranteed lower eigenvalue bounds for the biharmonic equation. Numer. Math. 126(1), 33–51 (2014). DOI 10.1007/s00211-013-0559-z. URL https://doi.org/10.1007/s00211-013-0559-z
12. Carstensen, C., Gedicke, J.: Guaranteed lower bounds for eigenvalues. Math. Comput. 83(290), 2605–2629 (2014). DOI 10.1090/S0025-5718-2014-02833-0. URL https://doi.org/10.1090/S0025-5718-2014-02833-0
13. Hu, J., Huang, Y., Lin, Q.: Lower bounds for eigenvalues of elliptic operators: By non-conforming finite element methods. J. Sci. Comput. 61(1), 196–221 (2014). DOI 10.1007/s10915-014-9821-5. URL https://doi.org/10.1007/s10915-014-9821-5

14. Kikuchi, F., Liu, X.: Estimation of interpolation error constants for the p0 and p1 triangular finite elements. Comput. Method Appl. M. 196(37-40), 3750–3758 (2007)

15. Kobayashi, K.: On the interpolation constants over triangular elements (in japanese). Kyoto University Research Information Repository 1733, 58–77 (2011)

16. Kobayashi, K.: On the interpolation constants over triangular elements. Appl. Math. pp. 110–124 (2015)

17. Kuznetsov, N., Kulczycki, T., Kwaśnicki, M., Nazarov, A., Poborchi, S., Polterovich, I., Siudeja, B.: The legacy of vladimir andreevich steklov. Notices of the AMS 61(1), 190 (2014)

18. Li, M., Lin, Q., Zhang, S.: Extrapolation and superconvergence of the steklov eigenvalue problem. Adv. Comput. Math. 33(1), 25–44 (2010). DOI 10.1007/s10444-009-9118-7. URL https://doi.org/10.1007/s10444-009-9118-7

19. Li, Q., Lin, Q., Xie, H.: Nonconforming finite element approximations of the steklov eigenvalue problem and its lower bound approximations. Appl. Math. 58(2), 129–151 (2013)

20. Li, Q., Liu, X.: Explicit finite element error estimates for nonhomogeneous neumann problems. Appl. Math. 63(3), 367–379 (2018)

21. Li, Q., Yang, Y.: A two-grid discretization scheme for the steklov eigenvalue problem. J. Appl. Math. Comput. 36(1-2), 129–139 (2011)

22. Liu, J., Sun, J., Turner, T.: Spectral indicator method for a non-selfadjoint steklov eigenvalue problem. J. Sci. Comput. 79(3), 1814–1831 (2019). DOI 10.1007/s10915-019-00913-6. URL https://doi.org/10.1007/s10915-019-00913-6

23. Liu, X.: A framework of verified eigenvalue bounds for self-adjoint differential operators. Appl. Math. Comput. 267, 341–355 (2015). DOI 10.1016/j.amc.2015.03.048. URL https://doi.org/10.1016/j.amc.2015.03.048

24. Liu, X., Oishi, S.: Verified eigenvalue evaluation for the laplacian over polygonal domains of arbitrary shape. SIAM J. Numer. Anal. 51(3), 1634–1654 (2013). DOI 10.1137/120878446. URL https://doi.org/10.1137/120878446

25. Liu, X.F., Oishi, S.: Verified eigenvalue evaluation for Laplace operator on arbitrary polygonal domain max and max-min principle. RIMS Kokyuroku 1733, 31–39 (2011)

26. Sebestová, I., Voichodský, T.: Two-sided bounds for eigenvalues of differential operators with applications to friedrichs, poincaré, trace, and similar constants. SIAM J. Numer. Anal. 52(1), 308–329 (2014). DOI 10.1137/13091467X. URL https://doi.org/10.1137/13091467X

27. Xie, H.: A type of multilevel method for the steklov eigenvalue problem. IMA J. Numer. Anal. 34(2), 592–608 (2014)

28. Xie, H., Xie, M., Yin, X., Yue, M.: Computable error estimates for a nonsymmetric eigenvalue problem. East Asian J. Appl. Math. 7(3), 583–602 (2017)

29. Xie, M., Xie, H., Liu, X.: Explicit lower bounds for stokes eigenvalue problems by using nonconforming finite elements. Japan J. Indust. Appl. Math. 35(1), 335–354 (2018)

30. You, C., Xie, H., Liu, X.: Guaranteed eigenvalue bounds for the steklov eigenvalue problem. SIAM J. Numer. Anal. 57(3), 1395–1410 (2019). DOI 10.1137/18M1189592. URL https://doi.org/10.1137/18M1189592