REFINED REGULARITY AND STABILIZATION PROPERTIES
IN A DEGENERATE HAPTOTAXIS SYSTEM

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ABSTRACT. We consider the degenerate haptotaxis system
\[
\begin{align*}
    u_t &= (d(x)u)_{xx} - (d(x)uw_x)_x, \\
    w_t &= -ug(w),
\end{align*}
\]
edowed with no-flux boundary conditions in a bounded open interval \(\Omega \subset \mathbb{R}\). It was proposed as a basic model for haptotactic migration in heterogeneous environments. If the diffusion is degenerate in the sense that \(d\) is non-negative, has a non-empty zero set and satisfies \(\int_{\Omega} \frac{1}{d} < \infty\), then it has been shown in [12] under appropriate assumptions on the initial data that the system has a global generalized solution satisfying in particular \(u(\cdot,t) \rightharpoonup \mu_\infty\) weakly in \(L^1(\Omega)\) as \(t \to \infty\) for some positive constant \(\mu_\infty\).

We now prove that under the additional restriction \(\int_{\Omega} \frac{1}{d^2} < \infty\) we have the strong convergence \(u(\cdot,t) \to \mu_\infty\) in \(L^p(\Omega)\) as \(t \to \infty\) for any \(p \in (1,2)\). In addition, with the same restriction on \(d\) we obtain improved regularity properties of \(u\), for instance \(d/u \in L^\infty((0,\infty);L^p(\Omega))\) for any \(p \in (1,\infty)\).

1. Introduction. Tumor invasion into the healthy tissue relies on a plethora of processes. However, many types of cancer cells are only able to move if they adhere to the tissue fibers in the extracellular matrix. Hence, they migrate from places with low densities of the tissue fibers (and corresponding adhesive molecules on the fibers) to places with higher densities. This process is called haptotaxis (see e.g. [4]) and has been present in a growing number of macroscopic models for tumor invasion into the tissue (see e.g. [5] for one of the first models). Consequently, the mathematical analysis of haptotaxis systems has got growing interest during the past decade. Mathematically, these systems usually consist of a cross-diffusive parabolic PDE for the tumor cell density (modeling diffusion and haptotaxis) coupled with an ODE for the density of the tissue fibers (since the latter is a non-diffusive attractant for the tumor cells). In most of these systems the random movement of cancer cells is described by non-degenerate diffusion of Fickian type (see e.g. [1, Section 4.3] for a recent survey), only few containing degenerate diffusion (see e.g. [13]). However, in organs with very heterogeneous tissue (e.g. in the brain) recent

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modeling approaches suggest that non-Fickian diffusion operators could possibly be more adequate, among them the so-called myopic diffusion (see e. g. [2]).

In [10] a degenerate haptotaxis system involving myopic diffusion has been proposed as a basic model for describing glioma spread in heterogeneous tissue. We will study a particular one-dimensional version thereof, which has been analyzed in [12] concerning the global existence and asymptotic behavior of a generalized solution. Namely, in a bounded open interval \( \Omega \subset \mathbb{R} \), we consider the initial-boundary value problem

\[
\begin{aligned}
  u_t &= (d(x)u)_{xx} - (d(x)uw_x)_x, & x \in \Omega, & t > 0, \\
  w_t &= -ug(w), & x \in \Omega, & t > 0, \\
  (d(x)u)_x - d(x)uw_x &= 0, & x \in \partial\Omega, & t > 0, \\
  u(x, 0) &= u_0(x), & w(x, 0) &= w_0(x), & x \in \Omega,
\end{aligned}
\]

(1)

with given nonnegative functions \( d, u_0 \) and \( w_0 \) on \( \overline{\Omega} \) and \( g \) generalizing the prototypical choice \( g(s) = s, s \geq 0 \). More precisely, we assume that \( d \in C^0(\overline{\Omega}) \cap C^1((\{d > 0\}) \) is nonnegative and such that

\[
\int_{\Omega} \frac{1}{d^2} < \infty,
\]

(2)

where \( \{d > 0\} := \{x \in \overline{\Omega} \mid d(x) > 0\} \). Furthermore, let \( g \in C^2([0, \infty)) \) such that \( g(0) = 0 \) and that with some positive constants \( \gamma \) and \( \tau \) we have

\[
\gamma \leq g'(s) \leq \tau \quad \text{and} \quad \gamma s \leq g(s) \leq \tau s \quad \text{for all } s \geq 0.
\]

Finally, we choose initial data satisfying

\[
u_0, w_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0, u_0 \neq 0, \quad w_0 \geq 0, \sqrt{w_0} \in W^{1,2}(\Omega) \quad \text{and} \quad \int_{\Omega} \frac{d^2}{d} w_0 < \infty. \]

(4)

In [12] the above assumptions were prescribed, but with \( \int_{\Omega} \frac{1}{d} < \infty \) instead of (2), and it was shown that there exists a global generalized solution \( (u,w) \) to (1) in the sense of [12, Definition 2.1] such that \( u \in C^0_{\text{loc}}([0,\infty);L^1(\Omega)) \cap L^\infty((0,\infty);L^1(\Omega)), \) \( w \in C^0(\overline{\Omega} \times [0,\infty)) \cap L^\infty(\Omega \times (0,\infty)) \cap L^1_{\text{loc}}((0,\infty);W^{1,1}(\Omega)), \) (\( \Omega \) obeys conservation of mass and the solution has the asymptotic behavior \( w(\cdot,t) \to 0 \) in \( L^\infty(\Omega) \) as well as

\[
u(\cdot,t) \to \frac{\mu_\infty}{d} \quad \text{weakly in } L^1(\Omega) \quad \text{as } t \to \infty,
\]

(5)

where \( \mu_\infty := \frac{\int_{\Omega} u_0}{\int_{\Omega} d}. \)

It is the purpose of the present paper to establish a strong convergence of \( u(\cdot,t) \) to \( \frac{\mu_\infty}{d} \) in some space \( L^p(\Omega) \). To this end, it turns out that instead of requiring \( \frac{1}{d} \) belonging to \( L^1(\Omega) \) we need that it belongs also to \( L^2(\Omega) \). The latter means an additional restriction of the behavior of \( d \) near its zeros as compared to the setting from [12].

**Main results.** By requiring the generalization (6) of (2) we have the following main results:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R} \) be a bounded interval, and suppose that \( d \in C^0(\overline{\Omega}) \cap C^1((\{d > 0\}) \) is nonnegative and such that

\[
\int_{\Omega} \frac{1}{d^\lambda} < \infty \quad \text{for some } \lambda \geq 2.
\]

(6)
Moreover, let $g \in C^2([0, \infty))$ be such that $g(0) = 0$ and that (3) is valid with some $\gamma > 0$ and $\gamma > 0$, and assume that the initial data $u_0$ and $w_0$ satisfy (4). Then the global generalized solution $(u, w)$ of (1) from [12, Theorem 1.1] has the additional properties that

$$du \in L^\infty((0, \infty); L^p(\Omega)) \quad \text{for all } p \in (1, \infty)$$

and

$$u \in L^\infty((0, \infty); L^p(\Omega)) \quad \text{for all } p \in (1, \lambda)$$

as well as

$$u \in C^0(\{d > 0\} \times (0, \infty)),$$

and furthermore with $\mu_\infty := \frac{\int_\Omega u_0}{\int_\Omega d}$ we have

$$u(\cdot, t) \to \frac{\mu_\infty}{d} \text{ in } L^p(\Omega) \quad \text{for all } p \in (1, \lambda) \quad \text{as } t \to \infty. \quad (10)$$

While the proof of the regularity properties in [12] is mainly based on bounds obtained from an energy-like inequality for regularized approximations of (1), our approach to prove (7) stems from the observation that the flux term in the first equation of (1) has the form

$$(du-x - duw)_x = e^w (due - w)_x.$$

This idea was established in non-degenerate haptotaxis systems in [6, 7] and leads for a supposedly given smooth solution of (1) to the identity

$$\frac{d}{dt} \int_\Omega \frac{1}{d} (due - w)_x^p e^w = -p(p-1) \int_\Omega e^w (due - w)^{p-2} (due - w)_x^2$$

$$+ (p-1) \int_\Omega \frac{1}{d^2} g(w) e^{2w} (due - w)^{p+1}.$$

In order to rigorously prove an appropriate ODI for a regularized approximation of $\int_\Omega \frac{1}{\gamma} (due - w)_x^p e^w$, in the regularized version of the above identity we estimate the last term on the right-hand side, where $u$ appears at a high power, by using on the one hand an interpolation inequality of Gagliardo-Nirenberg type which may be viewed as a derivate of an observation originally made in [3] and on the other hand estimates provided by [12] for the approximate problems (11), see (12)–(23) below. Here we will make essential use of our overall assumption that $\frac{1}{d}$ does not only belong to $L^1(\Omega)$ but even to $L^2(\Omega)$. In addition, we will also have to adequately cope with terms stemming from the artificial diffusion introduced in the second equation in (11), in this context no longer acting in a dissipative manner, and this will be achieved by substantially relying on the boundedness properties from [12]. These ingredients will then lead to uniform $L^p$ estimates for the regularizations $d \varepsilon u_\varepsilon$ of $du$. This we will do in Section 2.1, after having stated the approximate problems (11) along with some of their important properties from [12] in the beginning of Section 2.

If in addition to the assumptions from Theorem 1.1 we require $\frac{w_0}{d} \in L^\infty(\Omega)$, then [12, Theorem 1.3] implies for a.e. $t > 0$ the existence of positive constants $C_1(t)$ and $C_2(t)$ such that

$$\frac{C_1(t)}{d(x)} \leq u(x, t) \leq \frac{C_2(t)}{d(x)} \quad \text{for a.e. } x \in \Omega.$$

Hence, we cannot expect to achieve bounds for $u_\varepsilon$ itself in $L^p(\Omega)$ for large $p$. However, for any compact set $K \subset \{d > 0\}$ we obtain uniform bounds for $u_\varepsilon$ in $L^\infty(K)$. 

This is done in Section 2.2 with the help of a transformation to an inhomogeneous heat equation and the use of well-known estimates for the heat semigroup. These bounds in conjunction with standard parabolic regularity results then yield uniform interior Hölder estimates for \( u \).

As all these estimates are uniform with respect to \( \varepsilon \), by taking the limit \( \varepsilon \to 0 \) along an appropriate subsequence we finally conclude in Section 3 the properties of \( u \) claimed in Theorem 1.1. A short appendix contains the proof of the announced interpolation inequality of Gagliardo-Nirenberg type.

### 2. Refined uniform regularity properties for approximating problems

As in [12] we consider the approximating problems:

\[
\begin{cases}
  u_{\varepsilon t} = (d_{\varepsilon} u_{\varepsilon})_{xx} - (d_{\varepsilon} u_{\varepsilon})_{wx}, & x \in \Omega, \ t > 0, \\
  w_{\varepsilon t} = \varepsilon \left( d_{\varepsilon} \frac{u_{\varepsilon}}{\sqrt{g(w_{\varepsilon})}} \right)_x - u_{\varepsilon} g(w_{\varepsilon}), & x \in \Omega, \ t > 0, \\
  u_{\varepsilon x} = w_{\varepsilon x} = 0, & x \in \partial \Omega, \ t > 0, \\
  u_{\varepsilon}(x, 0) = u_{0}(x), \quad w_{\varepsilon}(x, 0) = w_{0}(x), & x \in \Omega,
\end{cases}
\]

(11)

for \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \), where \( (\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1) \) with \( \varepsilon_j \searrow 0 \) as \( j \to \infty \) as well as \( d_{\varepsilon} \) and \( w_{0} \) are defined in [12, Lemma 2.2 and Lemma 2.6]. In view of the latter references, these functions have the following properties, which we will frequently use in the sequel: For all \( \varepsilon, \varepsilon' \in (\varepsilon_j)_{j \in \mathbb{N}} \) we have

\[
d_{\varepsilon} \in C^\infty(\bar{\Omega}), \quad d_{\varepsilon} \to d \quad \text{in } L^\infty(\Omega) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0, \\
d_{\varepsilon x} \to d_{\varepsilon} \quad \text{in } L^p(\Omega; \{d > 0\}) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0 \text{ for all } p \in [1, \infty), \\
d_{\varepsilon} > 0 \quad \text{and} \quad d \leq d_{\varepsilon} \leq d_{\varepsilon'} \quad \text{in } \bar{\Omega} \quad \text{for } \varepsilon \leq \varepsilon', \\
d_{\varepsilon x} = 0 \quad \text{on } \partial \Omega, \\
d_{\varepsilon} \leq \|d\|_{L^\infty(\Omega)} + 1 \quad \text{in } \Omega, \\
\varepsilon^2 \int_\Omega \frac{d_{\varepsilon}^2}{\varepsilon^2} \leq 1 \quad \text{and} \quad \varepsilon^4 \cdot \frac{1}{\inf_{x \in \bar{\Omega}} d_{\varepsilon}(x)} \leq 1, \\
w_{0}(x) := w_{0}(x) + \varepsilon^4 \cdot \frac{1}{\inf_{x \in \bar{\Omega}} d_{\varepsilon}(x)} \quad \text{for all } x \in \bar{\Omega}, \quad \varepsilon = \varepsilon_j
\]

where \( w_{0} \in L^\infty(\Omega) \) is nonnegative and in particular satisfies \( \sqrt{w_{0}} \in W^{1,2}(\Omega) \) as well as \( \text{supp } w_{0j} \subset \{d > 0\} \) and, as \( j \to \infty \), \( w_{0j} \searrow w_{0} \) in \( \Omega \).

Furthermore, it was shown in [12, Section 4] that for any \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) there is a global classical solution \( (u_{\varepsilon}, w_{\varepsilon}) \) to (11). According to [12, Lemmas 2.7., 2.8 and 3.5], for any \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) this solution fulfills \( u_{\varepsilon} \geq 0, \ w_{\varepsilon} > 0 \) in \( \Omega \times [0, \infty) \) as well as

\[
\int_\Omega u_{\varepsilon}(\cdot, t) = \int_\Omega u_{0} \quad \text{for all } t > 0, \\
\int_\Omega w_{\varepsilon}(x, t) \leq M := \|w_{0}\|_{L^\infty(\Omega)} + 1 \quad \text{for all } x \in \Omega \text{ and } t > 0
\]

and there exists a constant \( C > 0 \) such that for any \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) we have

\[
\int_\Omega \ln(d_{\varepsilon} u_{\varepsilon}(\cdot, t)) \leq C \cdot (1 + \sqrt{\varepsilon}t) \quad \text{for all } t > 0, \\
\int_\Omega \frac{w_{\varepsilon}^2}{w_{\varepsilon}}(\cdot, t) \leq C \cdot (1 + \sqrt{\varepsilon}t) \quad \text{for all } t > 0, \\
\int_0^t \int_\Omega \frac{(d_{\varepsilon} u_{\varepsilon})^2}{d_{\varepsilon} u_{\varepsilon}} \leq C \cdot (1 + \sqrt{\varepsilon}t) \quad \text{for all } t > 0.
\]
2.1. An estimate for \(d_\varepsilon u_\varepsilon \) in \(L^p(\Omega)\). A crucial step for our asymptotic analysis consists in deriving appropriate \(\varepsilon\)-independent regularity properties of the solution component \(u_\varepsilon\) in Lebesgue spaces involving higher integrability powers. In order to prove the desired \(L^p\) estimate for \(d_\varepsilon u_\varepsilon\) for arbitrary large finite \(p\), we will rely on an interpolation using Lemma 4.1 as well as the estimates from [12] and our assumption that \(\frac{1}{2}\) belongs to \(L^2(\Omega)\) and not only to \(L^1(\Omega)\) as described in the introduction.

**Lemma 2.1.** Assume that \(\int_\Omega \frac{1}{2} < \infty\). Then for all \(p \in (1, \infty)\) there exists \(C(p) > 0\) such that for all \(\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}\) we have

\[
\|d_\varepsilon u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \leq C(p)e^{C(p)\sqrt{\varepsilon t}} \quad \text{for all } t > 0.
\]

**Proof.** Let us first follow an idea well-established in related non-degenerate frameworks ([6, 7]) to rewrite the flux in the first equation in (11) according to

\[
(d_\varepsilon u_\varepsilon)_x - d_\varepsilon u_\varepsilon w_{\varepsilon x} = e^{w_\varepsilon}(d_\varepsilon u_\varepsilon e^{-w_\varepsilon})_x, \quad x \in \Omega, \ t > 0.
\]

This, namely, suggests to test the PDE in question by \((d_\varepsilon u_\varepsilon e^{-w_\varepsilon})^{p-1}\) to obtain

\[
\frac{d}{dt} \int_\Omega \frac{1}{d_\varepsilon} \left[ (d_\varepsilon u_\varepsilon e^{-w_\varepsilon})^p \right] e^{w_\varepsilon} = p \int_\Omega \left( (d_\varepsilon u_\varepsilon e^{-w_\varepsilon})^{p-1} \cdot \left( e^{w_\varepsilon}(d_\varepsilon u_\varepsilon e^{-w_\varepsilon})_x \right) \right)_x \\
- (p-1) \int_\Omega d_\varepsilon^{p-1} u_\varepsilon^p e^{-(p-1)w_\varepsilon} \cdot \left\{ \varepsilon \left( d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(\varepsilon)}} \right)_x - u_\varepsilon g(\varepsilon) \right\} \\
= -(p-1) \int_\Omega e^{w_\varepsilon} \left( d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right)^{p-2} \cdot \left( d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right)_x \\
+ (p-1) \varepsilon \int_\Omega \left( d_\varepsilon^{p-1} u_\varepsilon^p e^{-(p-1)w_\varepsilon} \right)_x \cdot d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(\varepsilon)}} \\
+ (p-1) \int_\Omega d_\varepsilon^{p-1} u_\varepsilon^p g(\varepsilon) e^{-(p-1)w_\varepsilon} \quad \text{for all } t > 0,
\]

where the second summand on the right can be expanded so as to yield

\[
(p-1) \varepsilon \int_\Omega \left( d_\varepsilon^{p-1} u_\varepsilon^p e^{-(p-1)w_\varepsilon} \right)_x \cdot d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(\varepsilon)}} \\
= (p-1) \varepsilon \int_\Omega \left\{ \frac{1}{d_\varepsilon} e^{w_\varepsilon} \cdot \left( d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right)^p \right\}_x \cdot d_\varepsilon \frac{w_{\varepsilon x}}{\sqrt{g(\varepsilon)}} \\
= p(p-1) \varepsilon \int_\Omega e^{w_\varepsilon} \cdot \left( d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right)^{p-1} \cdot \left( d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right)_x \cdot \frac{w_{\varepsilon x}}{\sqrt{g(\varepsilon)}} \\
+ (p-1) \varepsilon \int_\Omega e^{w_\varepsilon} \cdot \left( d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right)^p \cdot \frac{w_{\varepsilon x}^2}{\sqrt{g(\varepsilon)}} \\
- (p-1) \varepsilon \int_\Omega \frac{d_{\varepsilon x}}{d_\varepsilon} e^{w_\varepsilon} \cdot \left( d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right)^p \cdot \frac{w_{\varepsilon x}}{\sqrt{g(\varepsilon)}} \quad \text{for all } t > 0.
\]

Here by Young’s inequality, (20) and (3),

\[
p(p-1) \varepsilon \int_\Omega e^{w_\varepsilon} \cdot \left( d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right)^{p-1} \cdot \left( d_\varepsilon u_\varepsilon e^{-w_\varepsilon} \right)_x \cdot \frac{w_{\varepsilon x}}{\sqrt{g(\varepsilon)}}
\]
\[
\leq \frac{p(p-1)}{2} \int_{\Omega} \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 \\
+ \frac{p(p-1)\varepsilon^2}{2} \int_{\Omega} e^{2w_x} \cdot \left( d_x u_x e^{-w_x} \right)^p \cdot \frac{w_{xx}^2}{\sqrt{g(w_x)}} \\
\leq \frac{p(p-1)}{2} \int_{\Omega} \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 \\
+ \frac{p(p-1)\varepsilon^2 e^{2M}}{2\gamma} \cdot \left\{ \varepsilon \cdot \frac{1}{\inf_{x \in \Omega} d_x(x)} \right\} \cdot \| d_x u_x e^{-w_x} \|_{L^\infty(\Omega)}^p \cdot \int_{\Omega} d_x \frac{w_{xx}^2}{w_x}
\]
for all \( t > 0 \), so that since
\[
\varepsilon \cdot \frac{1}{\inf_{x \in \Omega} d_x(x)} \leq 1
\]
by (17), and since (22) warrants that with some \( c_1 > 0 \) we have
\[
\int_{\Omega} d_x \frac{w_{xx}^2}{w_x} \leq c_1 \cdot (1 + \sqrt{\varepsilon}t) \quad \text{for all} \quad t > 0,
\]
we infer that for any choice of \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1) \) and all \( t > 0 \),
\[
(p-1) \varepsilon \int_{\Omega} e^{w_x} \cdot \left( d_x u_x e^{-w_x} \right)^p \cdot \frac{w_{xx}^2}{\sqrt{g(w_x)}} \\
\leq (p-1) \int_{\Omega} \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 \\
+ c_2 \cdot (1 + \sqrt{\varepsilon}t) \cdot \| d_x u_x e^{-w_x} \|_{L^\infty(\Omega)}^p \tag{27}
\]
where \( c_2 := \frac{p(p-1)e^{2M}c_1}{2\gamma} \).

Similarly, the second last summand in (26) can be controlled according to
\[
(p-1) \varepsilon \int_{\Omega} e^{w_x} \cdot \left( d_x u_x e^{-w_x} \right)^p \cdot \frac{w_{xx}^2}{\sqrt{g(w_x)}} \\
\leq (p-1) \int_{\Omega} \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 \\
+ c_3 \cdot (1 + \sqrt{\varepsilon}t) \cdot \| d_x u_x e^{-w_x} \|_{L^\infty(\Omega)}^p \tag{28}
\]
with \( c_3 := (p-1) \int_{\Omega} \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 \\
+ c_4 \cdot (1 + \sqrt{\varepsilon}t) \cdot \| d_x u_x e^{-w_x} \|_{L^\infty(\Omega)}^p \tag{29}
\]
for all \( t > 0 \).
with \( c_4 := \frac{(p-1)\varepsilon t}{\sqrt{\varepsilon}} \), because (17) asserts that
\[
\varepsilon \cdot \left\{ \int_{\Omega} \frac{d^2_x}{d_t^3} \right\} \leq 1.
\]

In summary, (26)-(29) show that writing \( c_5 := c_2 + c_3 + c_4 \) we obtain
\[
(p-1)\varepsilon \int_{\Omega} \left( \frac{d^{p-1} u_x e^{-(p-1)w_x}}{d_t} \right) \cdot d_x \frac{u_x}{\sqrt{g(w_x)}} \leq \frac{p(p-1)}{2} \int_{\Omega} \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 \leq c_5 \cdot (1 + \sqrt{\varepsilon}t) \cdot \left\| d_x u_x e^{-w_x} \right\|_{L^\infty(\Omega)}^p \quad (30)
\]
for all \( t > 0 \), and in order to further estimate the last summand herein, employing the Gagliardo-Nirenberg inequality we pick \( c_6 > 0 \) such that
\[
\| \varphi \|_{L^\infty(\Omega)} \leq c_6 \| \varphi_x \|_{L^2(\Omega)} \| \varphi \|_{L^{\frac{2p}{p+1}}(\Omega)} + c_6 \| \varphi \|_{L^{\frac{2p}{p+1}}(\Omega)}^2
\]
and invoke Young's inequality in fixing \( c_7 > 0 \) such that
\[
ab \leq \frac{p-1}{p} a^{\frac{p+1}{p}} + c_7 b^{p+1} \quad \text{for all } a \geq 0 \text{ and } b \geq 0.
\]

Using that by (16) and (19) we know that
\[
\left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\|_{L^\frac{2}{p}(\Omega)} = \int \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \leq c_8 := (\|d\|_{L^\infty(\Omega)} + 1) \cdot \int_{\Omega} u_0 \quad \text{for all } t > 0,
\]
we thereby see that for all \( t > 0 \),
\[
c_5 \cdot (1 + \sqrt{\varepsilon}t) \cdot \left\| d_x u_x e^{-w_x} \right\|_{L^\infty(\Omega)}^p = c_5 \cdot (1 + \sqrt{\varepsilon}t) \cdot \left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\|_{L^\infty(\Omega)}^2 \leq c_5 c_6 \cdot (1 + \sqrt{\varepsilon}t) \cdot \left\{ \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\} \cdot \left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\|_{L^\frac{2p}{p+1}(\Omega)}^2 \leq \frac{p-1}{p} \left\{ \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\} \cdot \left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\|_{L^\frac{2p}{p+1}(\Omega)}^2 \leq p(p-1) \int_{\Omega} \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 \quad (32)
\]
with $c_3 := c_5^{p+1} c_6^{p+1} c_7^p + c_5 c_6^p$.

On the right-hand side of (25), we next proceed to use (3) and (20) as well as (14) to estimate

\[(p - 1) \int_\Omega d_\varepsilon e^{2u_x} g(w_x) e^{-(p-1)w_x} \leq c_{10} \left\| d_\varepsilon u_x e^{-w_x} \right\|_{L^\infty(\Omega)}^{p+1} \quad \text{for all } t > 0,
\]

noting that $c_{10} := (p - 1) \gamma M c^2 M \int_\Omega \frac{1}{x}$ is finite thanks to our overall assumption on square integrability of $\frac{1}{x}$. Here we recall that (21) provides $c_{11} > 0$ such that

$$\int_\Omega u_x \ln(d_\varepsilon u_x) \leq c_{11} \cdot (1 + \sqrt{e} t) \quad \text{for all } t > 0,$$

which by (16), (19) and (20) entails that there exists $c_{12} > 0$ such that

\[
\left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \cdot \ln \left( d_x u_x e^{-w_x} \right) \right\|_{L^{\frac{2}{p}}(\Omega)} \leq \frac{p}{2} \int_\Omega d_x u_x e^{-w_x} \cdot \ln(d_x u_x e^{-w_x}) - \frac{p}{2} \int_\Omega d_x u_x w_x e^{-w_x} \\
\leq \frac{p}{2} \left\| d_x u_x \right\|_{L^\infty(\Omega)} + 1 \int_\Omega u_x \ln(d_x u_x) + \frac{p}{2} \left\| d \right\|_{L^\infty(\Omega)} + 1) M \int_\Omega u_x \\
\leq c_{12} \cdot (1 + \sqrt{e} t) \quad \text{for all } t > 0.
\]

Therefore, applying Lemma 4.1 to $q := \frac{2}{p}$ and recalling (31) we see that with some $c_{13} \geq p + 1$ we have

\[
c_{10} \left\| d_x u_x e^{-w_x} \right\|_{L^\infty(\Omega)}^{p+1} = c_{10} \left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{p+2} \\
\leq \frac{p - 1}{2p} \cdot \frac{1}{c_{12} \cdot (1 + \sqrt{e} t)} \cdot \left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^2 \\
\cdot \left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \cdot \ln \left( d_x u_x e^{-w_x} \right) \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{p}{2}} \\
+ c_{13}^p c_{13} \cdot (1 + \sqrt{e} t) + c_{13} \left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
\leq \frac{p - 1}{2p} \left\| \left( d_x u_x e^{-w_x} \right)^{\frac{p}{2}} \right\|_{L^{\frac{2}{p}}(\Omega)}^2 + c_{13} c_{13}^p c_{13} \cdot (1 + \sqrt{e} t) + c_{14} \\
= \frac{p(p - 1)}{8} \int_\Omega \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)^{2} + c_{13} c_{13}^p c_{13} \cdot (1 + \sqrt{e} t) + c_{14}
\]
for all $t > 0$. Together with (33), (32), (30) and (25), since $(1 + \sqrt{t})^{p+1} \leq e^{(p+1)\sqrt{t}}$ for all $t > 0$ this entails that
\[
\frac{d}{dt} \int_\Omega \frac{1}{d_x} \left( d_x u_x e^{-w_x} \right)^p e^{w_x} + \frac{p(p-1)}{8} \int_\Omega \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 \leq (c_9 + c_{13}) e^{c_{13}(1+\sqrt{t})} + c_{14} \quad \text{for all } t > 0.
\]
Since finally a Sobolev inequality associated with the embedding $W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ yields $c_{15} > 0$ such that
\[
\|\varphi\|_{L^\infty(\Omega)}^2 \leq c_{15} \|\varphi_x\|_{L^2(\Omega)}^2 + c_{15} \|\varphi\|_{L^\infty(\Omega)}^2 \quad \text{for all } \varphi \in W^{1,2}(\Omega),
\]
again by means of (31) and (14), as $\int_\Omega \frac{1}{\eta}$ is finite we can find $c_{16} > 0$ such that
\[
\int_\Omega \frac{1}{d_x} \left( d_x u_x e^{-w_x} \right)^p e^{w_x} \leq e^M \cdot \left\{ \int_\Omega \frac{1}{d_x} \left( d_x u_x e^{-w_x} \right)^p \right\}^2 \leq c_{16} \int_\Omega \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 + c_{16}
\]
for all $t > 0$ and hence
\[
\int_\Omega \left( d_x u_x e^{-w_x} \right)^{p-2} \left( d_x u_x e^{-w_x} \right)_x^2 \geq \frac{1}{c_{16}} \int_\Omega \frac{1}{d_x} \left( d_x u_x e^{-w_x} \right)^p e^{w_x} - 1
\]
for all $t > 0$. Therefore, (34) shows that for
\[
y_x(t) := \int_\Omega \frac{1}{d_x} \left( d_x u_x e^{-w_x} \right)^p e^{w_x} \quad \text{for all } t \geq 0,
\]
and $c_{17} := \frac{p(p-1)}{8c_{16}}$ as well as $c_{18} := (c_9 + c_{13}) e^{c_{13}} + c_{14}$ we have
\[
y_x(t) + c_{17} y_x(t) \leq c_{18} e^{c_{13}\sqrt{x}} \quad \text{for all } t > 0,
\]
which by an ODE comparison argument implies that
\[
y_x(t) \leq y_x(0) e^{-c_{17} t} + c_{18} \int_0^t e^{-c_{17}(t-s)} e^{c_{13}\sqrt{x}} ds = y_x(0) e^{-c_{17} t} + \frac{c_{18}}{c_{17} + c_{13}\sqrt{x}} e^{-c_{17} t} \left\{ e^{(c_{17} + c_{13}\sqrt{x})t} - 1 \right\} \leq y_x(0) e^{-c_{17} t} + \frac{c_{18}}{c_{17}} e^{c_{13}\sqrt{x}} \quad \text{for all } t > 0
\]
and thereby yields the claim, because
\[
y_x(0) = \int_\Omega d_x^{-1} u_x^p e^{-(p-1)w_x} \leq (\|d\|_{L^\infty(\Omega)} + 1)^{p-1} \|u_0\|_{L^\infty(\Omega)}^p |\Omega|
\]
by (16).

2.2. A local $L^\infty$ bound for $u_x$ in $\{d > 0\}$. We first plan to derive a bound for $u_x$ in $L^\infty(K)$ with arbitrary compact $K \subset \{d > 0\}$, which in view of the equation satisfied by the quantity $d_x u_x e^{-w_x}$ apparently does not follow from Lemma 2.1 in a trivial manner upon performing a straightforward Moser-type iteration. Fortunately, in the present one-dimensional situation an alternative approach can be based on a variable transformation which allows for a reduction to a linear inhomogeneous heat equation:
Lemma 2.2. Let $J \subset \overline{\Omega}$ be an interval and $x_0 \in J$, and given $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ let

$$
\phi_\varepsilon(x) := \int_{x_0}^x \frac{d\xi}{\sqrt{d_\varepsilon(\xi)}}, \quad x \in J,
$$

(35)
as well as $\tilde{J}_\varepsilon := \phi_\varepsilon(J)$. Then $\phi_\varepsilon \in C^\infty(J)$ is strictly increasing, and if for arbitrary $\zeta \in C^2(\tilde{J}_\varepsilon)$ we let the functions $Z_\varepsilon, W_\varepsilon$ and $H_\varepsilon$ be defined on $\tilde{J}_\varepsilon \times [0, \infty)$ by setting

$$
Z_\varepsilon(y, t) := d_\varepsilon(x)u_\varepsilon(x, t)
$$

(36)
and

$$
W_\varepsilon(y, t) := w_\varepsilon(x, t)
$$

(37)
as well as

$$
H_\varepsilon(y, t) := \zeta(y)Z_\varepsilon(y, t)
$$

(38)
with

$$
x := \phi_\varepsilon^{-1}(y)
$$

(39)
for $y \in \tilde{J}_\varepsilon$ and $t \geq 0$, then

$$
H_{\varepsilon t} = H_{\varepsilon yy} - b_{\varepsilon y}^{(1)} - b_{\varepsilon y}^{(2)} + b_{\varepsilon}^{(3)}, \quad y \in \tilde{J}_\varepsilon, \ t > 0,
$$

(40)
where

$$
b_{\varepsilon}^{(1)}(y, t) := \zeta(y)Z_\varepsilon(y, t)W_{\varepsilon y}(y, t)
$$

(41)
and

$$
b_{\varepsilon}^{(2)}(y, t) := \left(2\zeta_y(y) + D_\varepsilon(y)\zeta(y)\right)Z_{\varepsilon y}(y, t)
$$

(42)
as well as

$$
b_{\varepsilon}^{(3)}(y, t) := \left(-\zeta_{yy}(y) + \zeta_y(y)W_{\varepsilon y}(y, t) + D_\varepsilon(y)\zeta(y)W_{\varepsilon y}(y, t)\right)Z_\varepsilon(y, t)
$$

(43)
with

$$
D_\varepsilon(y) := \frac{d_\varepsilon(x)}{2\sqrt{d_\varepsilon(x)}}, \quad x = \phi_\varepsilon^{-1}(y),
$$

(44)
for $(y, t) \in \tilde{J}_\varepsilon \times (0, \infty)$. Moreover, if $J \cap \partial \Omega \neq \emptyset$, and if for some $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$ we have $\zeta_y = 0$ on $\phi_\varepsilon(J \cap \partial \Omega)$, then

$$
H_{\varepsilon y} = 0 \quad \text{on} \quad \phi_\varepsilon(J \cap \partial \Omega).
$$

(45)
Proof. The claimed regularity and monotonicity properties of $\phi_\varepsilon$ are evident from the inclusion $d_\varepsilon \in C^\infty(\overline{\Omega})$ and the positivity of $d_\varepsilon$ on $\overline{\Omega}$, as asserted by (12) and (14). To verify (40), we only need to combine (35)-(39) in computing

$$
Z_{\varepsilon y} = \sqrt{d_\varepsilon} \cdot (d_\varepsilon u_\varepsilon)_x
$$

(46)
as well as

$$
Z_{\varepsilon yy} = d_\varepsilon \cdot (d_\varepsilon u_\varepsilon)_{xx} + \frac{1}{2}d_\varepsilon \cdot (d_\varepsilon u_\varepsilon)_x = d_\varepsilon \cdot (d_\varepsilon u_\varepsilon)_{xx} + D_\varepsilon Z_{\varepsilon y}
$$

and, similarly,

$$
W_{\varepsilon y} = \sqrt{d_\varepsilon w_\varepsilon} \quad \text{as well as}
$$

$$
W_{\varepsilon yy} = d_\varepsilon w_{\varepsilon xx} + \frac{1}{2}d_\varepsilon w_\varepsilon = d_\varepsilon w_{\varepsilon xx} + D_\varepsilon W_{\varepsilon y},
$$
so that by (11),
\[ Z_{xt} = d_x u_{xt} \]
\[ = d_x \cdot (d_x u_x)_x - d_x \cdot (d_x u_x)_x w_{xx} - d_x \cdot (d_x u_x)_x w_{xxx} \]
\[ = \{Z_{xy} - D_x Z_{xy}\} - Z_{xy} W_{xy} - Z_x \cdot \{W_{xxy} - D_x W_{xy}\} \]
\[ = Z_{xy} - (Z_x W_{xy})_y - D_x Z_{xy} + D_x Z_x W_{xy} \]
in \( \mathcal{J}_\varepsilon \times (0, \infty) \). By furthermore using the identities
\[ H_{xy} = \zeta Z_{xy} + \zeta_y Z_x \]
and
\[ H_{xy} = \zeta Z_{xy} + 2\zeta_y Z_{xy} + \zeta_{yy} Z_x, \]
from this we readily derive (40). Finally, (45) is a direct consequence of (47) and the fact that due to (46), the boundary condition for \( u_x \) in (11) together with the property \( d_x \mid \partial \Omega = 0 \) achieved in (15) warrants that \( Z_{xy} = 0 \) on \( \phi_{\varepsilon}(J \cap \partial \Omega) \). \( \square \)

In consequence, deriving bounds of the desired type essentially reduces to suitably estimating the inhomogeneities in (40) on the basis of [12, Lemma 3.5] and Lemma 2.1. Indeed, this will form the core of the otherwise mainly technical reasoning in the following.

**Lemma 2.3.** Suppose that \( \int_{\Omega} \frac{1}{x} \, dx < \infty \), and let \( K \subset \{d > 0\} \) be compact. Then there exists \( C(K) > 0 \) such that whenever \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \),
\[ \|u_x(\cdot, t)\|_{L^\infty(K)} \leq C(K) e^{C(K)\sqrt{t}} \quad \text{for all } t > 0. \]
(48)

**Proof.** We evidently only need to consider the situation when \( K \subset \{d > 0\} \) is a compact interval, and hence contained in a single connected component \( J \subset \overline{\Omega} \) of \( \{d > 0\} \). Here we first concentrate on the case when \( J \subset \Omega \), in which since then \( J \) is open, by compactness of \( K \) we can find points \( a_0, a, b_0 \) and \( b \) in \( \Omega \) fulfilling
\[ K \subset (a_0, b_0) \subset (a, b) \subset [a, b] \subset J. \]
(49)
Now fixing any \( x_0 \in J \), for \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) we let \( \phi_\varepsilon \) and \( \mathcal{J}_\varepsilon \) be taken from Lemma 2.2, and observe that since \( d_\varepsilon \rightarrow d \) in \( L^\infty(\Omega) \) by (12) and hence \( \phi_\varepsilon(x) \rightarrow \int_{x_0}^x \frac{d_\varepsilon}{\sqrt{d_\varepsilon(\xi)}} \)
uniformly with respect to \( x \in [a, b] \subset \{d > 0\} \) as \( \varepsilon = \varepsilon_j \searrow 0 \), due to (49) it is possible to choose real numbers \( \tilde{a}_0, \tilde{a}, \tilde{b}_0 \) and \( \tilde{b} \) as well as \( \varepsilon^{(1)} \in (0, 1) \) such that for any \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) fulfilling \( \varepsilon < \varepsilon^{(1)} \) we have
\[ \phi_\varepsilon(K) \subset [\tilde{a}_0, \tilde{b}_0] \subset (\tilde{a}, \tilde{b}) \subset [\tilde{a}, \tilde{b}] \subset \phi_\varepsilon([a, b]). \]
(50)
Now writing \( G_0 := (\tilde{a}_0, \tilde{b}_0) \) and \( G := (\tilde{a}, \tilde{b}) \), we fix a cut-off function \( \zeta \in C^0_c(G) \) satisfying \( 0 \leq \zeta \leq 1 \) in \( G \) as well as \( \zeta \equiv 1 \) in \( G_0 \), and thereupon take \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) such that \( \varepsilon < \varepsilon^{(1)} \) as well as \( Z_\varepsilon, W_\varepsilon, H_\varepsilon, D_\varepsilon \) and \( b^{(i)}_\varepsilon, i \in \{1, 2, 3\} \), as introduced in Lemma 2.2. Then letting \( A \) denote the realization of the operator \(-\cdot_{yy}\) under homogeneous Neumann boundary conditions in the fixed domain \( G \), we may use (40) to represent \( H_\varepsilon \) according to
\[ H_\varepsilon(\cdot, t) = e^{-\min(1,t)A} H_\varepsilon(\cdot, (t - 1)_+) - \int_{(t - 1)_+}^t e^{-(t-s)A} b^{(1)}_\varepsilon(\cdot, s) \, ds \]
\[ - \int_{(t - 1)_+}^t e^{-(t-s)A} b^{(2)}_\varepsilon(\cdot, s) \, ds + \int_{(t - 1)_+}^t e^{-(t-s)A} b^{(3)}_\varepsilon(\cdot, s) \, ds \]
(51)
for \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon^{(1)}) \) and \( t > 0 \), where by the maximum principle, (38), (36) and (16),

\[
\| e^{-\min(1,t)A} \mathcal{H}_e(\cdot, (t-1)_+) \|_{L^\infty(G)} = \| e^{-tA} \mathcal{H}_e(\cdot, 0) \|_{L^\infty(G)} \\
\leq \| \mathcal{H}_e(\cdot, 0) \|_{L^\infty(G)} \\
\leq \| d_e u_0 \|_{L^\infty(\Omega)} \\
\leq c_1 := (\| d \|_{L^\infty(\Omega)} + 1) \| u_0 \|_{L^\infty(\Omega)} \quad \text{for all } t \leq 1,
\]

and where by a known regularization property of \((e^{-tA})_{t \geq 0}\) (\cite{11}), there exists \( c_2 > 0 \) such that

\[
\| e^{-\min(1,t)A} \mathcal{H}_e(\cdot, (t-1)_+) \|_{L^\infty(G)} = \| e^{-A} \mathcal{H}_e(\cdot, t-1) \|_{L^\infty(G)} \\
\leq c_2 \| \mathcal{H}_e(\cdot, t-1) \|_{L^1(G)} \\
\leq c_2 \| \mathcal{Z}_e(\cdot, t-1) \|_{L^1(G)} \quad \text{for all } t > 1.
\]

To further estimate the latter, we substitute \( y = \phi_e(x) \) and recall (19) in deriving

\[
\| \mathcal{Z}_e(\cdot, t-1) \|_{L^1(G)} = \int_G \mathcal{Z}_e(y, t-1)dy \\
= \int_{\phi_e^{-1}(G)} d_e(x) u_e(x, t-1) \frac{dx}{\sqrt{d_e(x)}} \\
\leq (\| d \|_{L^\infty(\Omega)} + 1)^{\frac{1}{2}} \int_\Omega u_e(x, t-1)dx \\
= c_3 := (\| d \|_{L^\infty(\Omega)} + 1)^{\frac{1}{2}} \int_\Omega u_0 \quad \text{for all } t > 1,
\]

whence (52) and (53) imply that

\[
\| e^{-\min(1,t)A} \mathcal{H}_e(\cdot, (t-1)_+) \|_{L^\infty(G)} \leq \max\{c_1, c_2c_3\} \quad \text{for all } t > 0. \tag{54}
\]

We next fix an arbitrary \( p \in (\frac{3}{2}, 2) \) and employ further known smoothing estimates for the Neumann heat semigroup \((\mathcal{H}_e)_{e \geq 0}\) to obtain \( c_4 > 0 \) and \( c_5 > 0 \) with the property that for all \( \tau \in (0, 1) \) we have

\[
\| e^{-\tau A} \varphi \|_{L^\infty(G)} \leq c_4 \tau^{-\frac{1}{2}} \frac{1}{p+1} \| \varphi \|_{L^p(G)} \quad \forall \varphi \in C^1(\bar{G}) \text{ such that } \varphi|_{\partial G} = 0 \tag{55}
\]

and

\[
\| e^{-\tau A} \varphi \|_{L^\infty(G)} \leq c_5 \tau^{-\frac{1}{2}} \| \varphi \|_{L^p(G)} \quad \text{for all } \varphi \in C^0(\bar{G}). \tag{56}
\]

Therefore,

\[
\left\| \int_{(t-1)_+}^t e^{-(t-s)A} \mathcal{B}_e(\cdot, s)ds \right\|_{L^\infty(G)} \\
\leq c_4 \int_{(t-1)_+}^t (t-s)^{-\frac{1}{2}} \frac{1}{p+1} \| b_e^{(1)}(\cdot, s) \|_{L^p(\Omega)}ds \\
\leq c_4 c_6 \| b_e^{(1)} \|_{L^\infty((0,t):L^p(G))} \quad \text{for all } t > 0 \tag{57}
\]
with \( c_6 := \int_0^1 \sigma^{-\frac{1}{p+1}} \frac{\sigma}{\sigma} d\sigma = \frac{2p}{p+1} \), and by means of the Hölder inequality we see that

\[
\left\| \int_{(t-1)^+}^t e^{-(t-s)A}b^{(2)}(\cdot, s)ds \right\|_{L^\infty(G)} \\
\leq c_5 \int_{(t-1)^+}^t (t-s)^{-\frac{1}{p}} \left\| b^{(2)}(\cdot, s) \right\|_{L^p(G)} ds \\
\leq c_5 \cdot \left\{ \int_{(t-1)^+}^t (t-s)^{-\frac{1}{p}} ds \right\}^{\frac{p-1}{p}} \cdot \left\{ \int_{(t-1)^+}^t \left\| b^{(2)}(\cdot, s) \right\|_{L^p(G)}^p ds \right\} \frac{1}{p} \\
\leq c_5 c_7 \left\| b^{(2)} \right\|_{L^p(G \times ((t-1)^+, t))} \quad \text{for all } t > 0
\]

with

\[
c_7 := \left\{ \int_0^1 \sigma^{-\frac{1}{p+1}} \frac{\sigma}{\sigma} d\sigma \right\}^{\frac{p-1}{p}} = \left( \frac{2(p-1)}{2p-3} \right)^{\frac{p-1}{p}},
\]

and

\[
\left\| \int_{(t-1)^+}^t e^{-(t-s)A}b^{(3)}(\cdot, s)ds \right\|_{L^\infty(G)} \\
\leq c_5 \int_{(t-1)^+}^t (t-s)^{-\frac{1}{p}} \left\| b^{(3)}(\cdot, s) \right\|_{L^p(G)} ds \\
\leq c_5 c_8 \left\| b^{(3)} \right\|_{L^\infty((0, t); L^p(G))} \quad \text{for all } t > 0,
\]

where \( c_8 := \int_0^1 \sigma^{-\frac{1}{p+1}} \frac{\sigma}{\sigma} d\sigma = \frac{2p}{2p-1} \). To prepare an appropriate further estimation of the right-hand sides in (57), (58) and (59), using that \( \phi^{-1}_\varepsilon(G) \subset [a, b] \) we infer from the uniform positivity of \( d \) in \([a, b]\) and the fact that \( d_\varepsilon \geq d \) for all \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) that there exists \( \delta > 0 \) such that for each \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \),

\[
d_\varepsilon(x) \geq \delta \quad \text{for all } x \in [a, b].
\]

For arbitrary \( q \in (1, \infty) \), we thus obtain that

\[
\int_G Z^q_\varepsilon(y, t) dy = \int_{\phi^{-1}_\varepsilon(G)} \left( d_\varepsilon(x) u_\varepsilon(x, t) \right)^q \frac{dx}{\sqrt{d_\varepsilon(x)}} \\
\leq \frac{1}{\sqrt{d}} \int_a^b (d_\varepsilon u_\varepsilon)^q \quad \text{for all } t > 0,
\]

so that from Lemma 2.1 it follows that for any such \( q \) we can find \( c_9(q) > 0 \) fulfilling

\[
\| Z_\varepsilon(\cdot, t) \|_{L^q(G)} \leq c_9(q) e^{c_9(q) \sqrt{\tau t}} \quad \text{for all } t > 0.
\]

Moreover, if \( q \in (1, 2) \), then by (16) and the Hölder inequality we see that for all \( t > 0 \),

\[
\int_t^{t+1} \int_G |Z_{\varepsilon y}(y, s)|^q dy ds \\
= \int_t^{t+1} \int_{\phi^{-1}_\varepsilon(G)} d_\varepsilon^q(x) \left( d_\varepsilon u_\varepsilon(x, s) \right)^q \frac{dx}{\sqrt{d_\varepsilon(x)}} ds \\
\leq \int_t^{t+1} \int_a^b d_\varepsilon^q \cdot \left\{ \left( \frac{d_\varepsilon u_\varepsilon}{d_\varepsilon u_\varepsilon} \right)^q \cdot (d_\varepsilon u_\varepsilon)^2 \right\} \frac{1}{q} \cdot (d_\varepsilon u_\varepsilon)^{\frac{2q}{q-2}}.
\]
\[ \leq \left( \|d\|_{L^\infty(\Omega)} + 1 \right)^{\frac{2q}{q+1}} \cdot \left\{ \int_t^{t+1} \int_{\Omega} \frac{(d_\varepsilon u_\varepsilon)^2}{d_\varepsilon u_\varepsilon} \right\}^{\frac{q}{2}} \cdot \left\{ \int_t^{t+1} \int_{\Omega} (d_\varepsilon u_\varepsilon)^{\frac{q}{q-1}} \right\}^{\frac{q-2}{2}}, \]

where (23) yields \( c_{10} > 0 \) such that

\[ \int_t^{t+1} \int_{\Omega} \frac{(d_\varepsilon u_\varepsilon)^2}{d_\varepsilon u_\varepsilon} \leq c_{10} \cdot (1 + \sqrt{t}) \quad \text{for all } t > 0. \]

Again employing Lemma 2.1, we thus conclude that for each \( q \in (1, 2) \) there exists \( c_{11}(q) > 0 \) satisfying

\[ \|Z_{xy}\|_{L^q(G \times (t,t+1))} \leq c_{11}(q) e^{c_{11}(q) \sqrt{t}} \quad \text{for all } t > 0. \]  

(62)

Proceeding similarly, using (20) and (60) we estimate

\[ \int_G W_{xy}^2(y,t) dy = \int_{\phi_\varepsilon^{-1}(G)} \sqrt{d_\varepsilon} w_{\varepsilon x}^2 dx \]

\[ \leq \int_a^b \left\{ d_\varepsilon \frac{w_{\varepsilon x}}{w_\varepsilon} \right\} \cdot \frac{w_\varepsilon}{\sqrt{d_\varepsilon}} \]

\[ \leq \frac{M}{\sqrt{\delta}} \int_{\Omega} d_\varepsilon w_{\varepsilon x}^2 \frac{w_\varepsilon}{\sqrt{d_\varepsilon}} \quad \text{for all } t > 0, \]

so that from (22) we readily infer the existence of \( c_{12} > 0 \) satisfying

\[ \|W_{xy}(\cdot,t)\|_{L^2(G)} \leq c_{12} e^{c_{12} \sqrt{t}} \quad \text{for all } t > 0. \]  

(63)

Finally, relying on the local convergence properties of \((d_\varepsilon,x)_{\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}} \) expressed in (13), given any \( q \in (1, \infty) \) we can find \( \varepsilon^{(2)}(q) \in (0, \varepsilon^{(1)}) \) and \( c_{13}(q) > 0 \) such that

\[ \int_G |D_\varepsilon(y)|^q dy = \int_{\phi_\varepsilon^{-1}(G)} \frac{1}{\sqrt{d_\varepsilon}} \frac{d_\varepsilon}{2\sqrt{d_\varepsilon}} |d_\varepsilon|^q dx \]

\[ \leq \frac{1}{2^{q/2}} \int_a^b |d_\varepsilon|^q \]

\[ \leq c_{13}(q) \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \text{ such that } \varepsilon < \varepsilon^{(2)}(q). \]  

(64)

Now going back to (57), using (61), (63) and the fact that \(|\zeta| \leq 1\), by the Hölder inequality we find that therein for each \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) such that \( \varepsilon < \varepsilon^{(1)} \) we have

\[ \|\delta_\varepsilon^{(1)}(\cdot,t)\|_{L^p(G)} = \|\zeta Z_\varepsilon W_{xy}\|_{L^p(G)} \]

\[ \leq \|Z_\varepsilon\|_{L^{2p}(G)} \|W_{xy}\|_{L^2(G)} \]

\[ \leq c_9 \left( \frac{2p}{2-p} \right) c_9 \left( \frac{2p}{2-p} \right) e^{c_9(\varepsilon^{(2)}(q) \sqrt{t})}.c_{12} e^{c_{12} \sqrt{t}} \quad \text{for all } t > 0, \]

and that hence in view of (57) there exists \( c_{14} > 0 \) such that

\[ \left\| \int_t^{t+1} e^{-(t-s)A_\varepsilon^{(1)}(\cdot,s)} ds \right\|_{L^\infty(G)} \leq c_{14} e^{c_{14} \sqrt{t}} \quad \text{for all } t > 0. \]  

(65)

Likewise, fixing any \( r \in (p,2) \) we may combine (62) with (64) to see that whenever \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) is such that \( \varepsilon < \varepsilon^{(2)}(\frac{pr}{r-p}) \),

\[ \|\delta_\varepsilon^{(2)}\|_{L^p(G \times (t,t+1))} = \|2\zeta_\varepsilon Z_{xy} + D_\varepsilon \zeta Z_{xy}\|_{L^p(G \times (t,t+1))} = \]
\[ \leq 2\|\zeta_y\|_{L^p(G)} \|Z_{xy}\|_{L^p(G \times (t,t+1))} + \|D_\xi\|_{L^\frac{p}{p-1}(G)} \|Z_{xy}\|_{L^p(G \times (t,t+1))} \]

\[ \leq 2\|\zeta_y\|_{L^\infty(G)} \cdot c_{11}(p)e^{c_{11}(p)\sqrt{\tau t}} + c_{15} \left( \frac{p}{r-p} \right) \cdot c_{11}(r)e^{c_{11}(r)\sqrt{\tau t}} \quad \text{for all } t > 0 \]

and thus, by (58),

\[ \left\| \int_{(t-1)_+}^t e^{-(t-s)A\beta(2)}(\cdot,s)ds \right\|_{L^\infty(G)} \leq c_{15}e^{c_{15}\sqrt{\tau t}} \quad \text{for all } t > 0 \]  

(66)

with some suitably large \( c_{15} > 0 \). Finally, (61), (63) and (64) imply that for \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) fulfilling \( \varepsilon < \varepsilon(2)\left( \frac{4p}{2-p} \right) \) we have

\[ \|b_\varepsilon^{(3)}(\cdot,t)\|_{L^p(G)} \]

\[ = \| - \zeta_{yy}Z_\varepsilon + \zeta_y W_{xy}Z_\varepsilon + D_\xi \zeta W_{xy}Z_\varepsilon \|_{L^p(G)} \]

\[ \leq \|\zeta_{yy}\|_{L^\infty(G)} \|Z_\varepsilon\|_{L^p(G)} + \|\zeta_y\|_{L^\infty(G)} \|W_{xy}\|_{L^2(G)} \|Z_\varepsilon\|_{L^\frac{4p}{p-1}(G)} \]

\[ + \|D_\xi\|_{L^\frac{4p}{p-1}(G)} \|W_{xy}\|_{L^2(G)} \|Z_\varepsilon\|_{L^\frac{4p}{p-1}(G)} \]

\[ \leq \|\zeta_{yy}\|_{L^\infty(G)} \cdot c_9(p)e^{c_9(p)\sqrt{\tau t}} + \|\zeta_y\|_{L^\infty(G)} \cdot c_{12}e^{c_{12}\sqrt{\tau t}} \cdot c_9 \left( \frac{2p}{2-p} \right) e^{c_9\left( \frac{4p}{2-p} \right)\sqrt{\tau t}} \]

\[ + c_{13} \left( \frac{4p}{2-p} \right) \cdot c_{12}e^{c_{12}\sqrt{\tau t}} \cdot c_9 \left( \frac{4p}{2-p} \right) e^{c_9\left( \frac{4p}{2-p} \right)\sqrt{\tau t}} \quad \text{for all } t > 0, \]

by (59) meaning that there exists \( c_{16} > 0 \) with the property that for any such \( \varepsilon \),

\[ \left\| \int_{(t-1)_+}^t e^{-(t-s)A\beta(3)}(\cdot,s)ds \right\|_{L^\infty(G)} \leq c_{16}e^{c_{16}\sqrt{\tau t}} \quad \text{for all } t > 0. \]

(67)

In summary, (51), (54), (65), (66) and (67) entail the existence of \( c_{17} > 0 \) such that writing \( \varepsilon(3) := \min\left\{ \varepsilon(2)\left( \frac{pr}{r-p} \right), \varepsilon(2)\left( \frac{4p}{2-p} \right) \right\} \), for each \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) satisfying \( \varepsilon < \varepsilon(3) \) we have

\[ \|H_\varepsilon(\cdot,t)\|_{L^\infty(G)} \leq c_{17}e^{c_{17}\sqrt{\tau t}} \quad \text{for all } t > 0 \]

and hence

\[ \|Z_\varepsilon(\cdot,t)\|_{L^\infty(G_0)} \leq c_{17}e^{c_{17}\sqrt{\tau t}} \quad \text{for all } t > 0 \]

due to the fact that \( \zeta \equiv 1 \) in \( G_0 \). Transforming back to the original variables, again by (60) we thus see that for any such \( \varepsilon \) and each \( t > 0 \),

\[ \delta\|u_\varepsilon(\cdot,t)\|_{L^\infty(K)} \leq \|d_\varepsilon u_\varepsilon(\cdot,t)\|_{L^\infty(K)} \leq \|d_\varepsilon u_\varepsilon(\cdot,t)\|_{L^\infty(\phi_\varepsilon^{-1}(G_0))} \]

\[ = \|Z_\varepsilon(\cdot,t)\|_{L^\infty(G_0)} \leq c_{17}e^{c_{17}\sqrt{\tau t}}, \]

because \( K \subset \phi_\varepsilon^{-1}(G_0) \subset [a,b] \) by (50). This proves (48) in the case when \( J \subset \Omega \), whereas the situation when \( J \cap \partial\Omega \neq \emptyset \) can be dealt with quite similarly, by e.g. fixing, for convenience, \( x_0 \in J \cap \partial\Omega \) in (35), and choosing \( \zeta \) to be identically equal to 1 near the boundary point \( 0 = \phi_\varepsilon(x_0) \).

\[ \square \]

2.3. Local Hölder regularity of \( u_\varepsilon \) in \( \{ d > 0 \} \). Now with the above boundedness information at hand, we may invoke standard parabolic regularity to obtain the announced interior Hölder regularity property.
Lemma 2.4. Assume that \( \int_{\Omega} \frac{1}{x} < \infty \). Then for each compact \( K \subset \{ d > 0 \} \) and any \( \tau \in (0, 1) \) there exist \( \theta(K, \tau) \in (0, 1) \) and \( C(K, \tau) > 0 \) with the property that for any \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) we have

\[
\|u_\varepsilon\|_{C^0(K_\varepsilon)} \leq C(K, \tau) \quad \text{for all } t \in \left( \tau, \frac{1}{\sqrt{\varepsilon}} \right). \tag{68}
\]

Proof. We write the first equation in (11) in the form

\[
uex = \left( a_\varepsilon(x, t, u_{\varepsilon x}) \right)_x, \quad x \in \Omega, \ t > 0,
\]

where

\[a_\varepsilon(x, t, \xi) := d_\varepsilon(x)\xi + d_{\varepsilon x}(x)u_\varepsilon(x, t) - d_x(x)u_\varepsilon(x, t)w_{\varepsilon x}(x, t), \quad x \in \Omega, \ t > 0, \ \xi \in \mathbb{R}. \tag{69}\]

Once again assuming without loss of generality that \( K \) is an interval, in case of \( K \subset \Omega \) we can fix an open interval \( \Omega_0 \subset \Omega \) such that \( K \subset \Omega_0 \subset \bar{\Omega}_0 \subset \{ d > 0 \} \), so that since \( \bar{\Omega}_0 \) still is a compact subinterval of \( \{ d > 0 \} \), using (13) and (14) we obtain positive constants \( c_1, c_2 \) and \( c_3 \) such that for all \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \),

\[c_1 \leq d_\varepsilon(x) \leq c_2 \quad \text{for all } x \in \Omega_0 \tag{70}\]

and

\[
\int_{\Omega_0} d_{\varepsilon x}^2 \leq c_3. \tag{71}
\]

Moreover, employing Lemma 2.3 and (22) we see that for some \( c_4 > 0 \) and \( c_5 > 0 \) and any \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \) we have

\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega_0)} \leq c_4 \quad \text{for all } t \in \left( 0, \frac{1}{\sqrt{\varepsilon}} \right) \tag{72}
\]

as well as

\[
\int_{\Omega} \frac{d_\varepsilon}{d_\varepsilon x} \frac{u_{\varepsilon x}^2}{w_\varepsilon} \leq c_5 \quad \text{for all } t \in \left( 0, \frac{1}{\sqrt{\varepsilon}} \right), \tag{73}
\]

where the latter along with (70) and (20) entails that

\[
\int_{\Omega_0} \frac{w_{\varepsilon x}^2}{w_\varepsilon} \leq \frac{M}{c_1} \int_{\Omega_0} \frac{d_\varepsilon}{d_\varepsilon x} \frac{u_{\varepsilon x}^2}{w_\varepsilon} \leq \frac{M c_5}{c_1} \quad \text{for all } t \in \left( 0, \frac{1}{\sqrt{\varepsilon}} \right). \tag{74}
\]

Writing

\[
\psi_\varepsilon^{(0)}(x, t) := \frac{d_\varepsilon^2}{d_\varepsilon x}(x)u_\varepsilon^2(x, t) + d_\varepsilon(x)u_\varepsilon^2(x, t)w_{\varepsilon x}^2(x, t)
\]

and

\[
\psi_\varepsilon^{(1)}(x, t) := |d_{\varepsilon x}(x)|u_\varepsilon(x, t) + d_\varepsilon(x)u_\varepsilon(x, t)|w_{\varepsilon x}(x, t)|
\]

for \( x \in \Omega, t > 0 \) and \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \), from (70), (71), (72) and (74) we thus infer that for any such \( \varepsilon \) we have

\[
\int_{\Omega_0} |\psi_\varepsilon^{(0)}(\cdot, t)| \leq \frac{c_1 c_4^2}{c_1} + c_2 c_4^2 + \frac{M c_5}{c_1} \quad \text{for all } t \in \left( 0, \frac{1}{\sqrt{\varepsilon}} \right)
\]

and

\[
\int_{\Omega_0} |\psi_\varepsilon^{(1)}(\cdot, t)|^2 \leq 2 \cdot \left\{ c_3 c_4^2 + c_2 c_4^2 + \frac{M c_5}{c_1} \right\} \quad \text{for all } t \in \left( 0, \frac{1}{\sqrt{\varepsilon}} \right).
\]
Since by Young’s inequality and (70),
\[ a_\varepsilon(x, t, \xi) \cdot \xi \geq \frac{1}{2} d_\varepsilon(x) \xi^2 - \psi^{(0)}_\varepsilon(x, t) \]
\[ \geq \frac{c_1}{2} \xi^2 - \psi^{(0)}_\varepsilon(x, t) \quad \text{for all } x \in \Omega_0, t > 0 \text{ and } \xi \in \mathbb{R}, \]
and since (70) moreover warrants that
\[ |a_\varepsilon(x, t, \xi)| \leq d_\varepsilon(x) |\xi| + \psi^{(1)}_\varepsilon(x, t) \]
\[ \leq c_2 |\xi| + \psi^{(1)}_\varepsilon(x, t) \quad \text{for all } x \in \Omega_0, t > 0 \text{ and } \xi \in \mathbb{R}, \]
in view of the boundedness property (72) the inequality in (68) follows from a
standard result on interior Hölder regularity of bounded solutions to scalar parabolic
equations ([9, Theorem 1.1]).

The case \( K \cap \partial \Omega \neq \emptyset \) can be dealt with in a similar way, again relying on [9, Theorem 1.3].

3. Proof of Theorem 1.1. The uniform estimates of the solutions to the approximate problems (11) proved in Section 2 imply the following refined regularity
properties of the generalized solution to (1).

Lemma 3.1. Suppose that \( \int_\Omega \frac{1}{d} < \infty \), and let \((u, w)\) denote the global generalized solution of (1) from [12, Theorem 1.1]. Then
\[ du \in L^\infty((0, \infty); L^p(\Omega)) \quad \text{for all } p \in (1, \infty) \]  
(75)
and
\[ u \in C^0(\{d > 0\} \times (0, \infty)). \]  
(76)
Moreover, for all \( p \in (1, \infty) \) there exists \( C(p) > 0 \) such that
\[ \|d(\cdot, t)\|_{L^p(\Omega)} \leq C(p) \quad \text{for all } t > 0, \]  
(77)
and given any compact \( K \subset \{d > 0\} \) and \( \tau > 0 \) one can find \( \theta(K, \tau) \in (0, 1) \) and \( C(K, \tau) > 0 \) such that
\[ \|u(\cdot, t)\|_{C^{\theta(K, \tau)}(K)} \leq C(K, \tau) \quad \text{for all } t > \tau. \]  
(78)

Proof. In view of the pointwise approximation property of \((u_{\varepsilon_j})_{j \in \mathbb{N}}\) stated in [12, Lemma 6.1], i.e. \( u_{\varepsilon_j} \to u \) a.e. in \( \Omega \times (0, \infty) \), both (75) and (77) directly result on taking \( \varepsilon = \varepsilon_j \to 0 \) in Lemma 2.1, whereas (76) and (78) can easily be deduced from Lemma 2.4.

Finally, we are in the position to prove our main result.

Proof of Theorem 1.1. In view of \( \lambda \geq 2 \) and the boundedness of \( \Omega \), (6) implies (2),
where the latter property does not depend on \( \lambda \) any more. Hence, the regularity
properties (7) for any \( p \in (1, \infty) \) and (9) have precisely been established in Lemma
3.1 already.

Consequently, it remains to show that (6) implies (8) and (10) for any \( p \in (1, \lambda) \).
To achieve this, we let \( p \in (1, \lambda) \) be given and then infer from (7) that with some \( c_1 > 0 \) we have
\[ \int_\Omega (du)^\frac{\lambda}{\lambda-p} \leq c_1 \quad \text{for all } t > 0. \]  
(79)
By means of the Hölder inequality, this firstly implies that
\[ \int_\Omega u^p = \int_\Omega \frac{1}{p} \cdot (du)^p \leq \left\{ \int_\Omega \frac{1}{d^\lambda} \right\}^{\frac{p}{\lambda}} \cdot \left\{ \int_\Omega (du)^\frac{\lambda}{\lambda-p} \right\}^{\frac{\lambda-p}{\lambda}} \leq \left\{ \int_\Omega \frac{1}{d^\lambda} \right\}^{\frac{p}{\lambda}} \cdot c_1^\frac{\lambda-p}{\lambda} \]
for all $t > 0$ and thereby, thanks to (6), already establishes (8). Secondly, given any $\eta > 0$ we may use (6) along with the fact that $p < \lambda$ in choosing a relatively open subset $\Omega_0$ of $\Omega$ satisfying $\{d = 0\} \subset \Omega_0$ and

$$2^{p-1}c_1^{\frac{\lambda-p}{\lambda}} \cdot \left\{ \int_{\Omega_0} \frac{1}{d^\lambda} \right\}^{\frac{p}{\lambda}} \leq \frac{\eta^p}{3}$$  \hspace{1cm} (80)$$

as well as

$$2^{p-1} \mu_\infty \int_{\Omega_0} \frac{1}{d^p} \leq \frac{\eta^p}{3}.$$  \hspace{1cm} (81)$$

Then since $K := \Omega \setminus \Omega_0$ is compact, Lemma 3.1 applies so as to show that in view of the Arzelà-Ascoli theorem the semi-orbit $(u(\cdot, t))_{t \geq 1}$ is relatively compact in $C^0(K)$ and that hence, thanks to the outcome of [12, Theorem 1.2], namely (5), we have

$$u(\cdot, t) \rightarrow \frac{\mu_\infty}{d} \text{ in } L^\infty(K) \quad \text{as } t \rightarrow \infty,$$

so that we can fix $t_0 > 1$ such that

$$\left\| u(\cdot, t) - \frac{\mu_\infty}{d} \right\|_{L^\infty(K)} \leq \eta$$

for all $t > t_0$.\hspace{1cm} (82)

Now in the inequality

$$\int_{\Omega} |u(\cdot, t) - \frac{\mu_\infty}{d}|^p = \int_{\Omega_0} |u(\cdot, t) - \frac{\mu_\infty}{d}|^p + \int_{\Omega \setminus \Omega_0} |u(\cdot, t) - \frac{\mu_\infty}{d}|^p$$

$$\leq 2^{p-1} \int_{\Omega_0} u^p + 2^{p-1} \mu_\infty \int_{\Omega_0} \frac{1}{d^p} + \int_{\Omega \setminus \Omega_0} |u(\cdot, t) - \frac{\mu_\infty}{d}|^p,$$

$t > 0$, according to (80) and again the Hölder inequality we have

$$2^{p-1} \int_{\Omega_0} u^p \leq 2^{p-1} \cdot \left\{ \int_{\Omega_0} \frac{1}{d^\lambda} \right\}^{\frac{p}{\lambda}} \cdot \left\{ \int_{\Omega_0} (du)^{p\frac{\lambda}{p}} \right\}^{\frac{1}{\lambda-p}}$$

$$\leq 2^{p-1} \cdot \left\{ \int_{\Omega_0} \frac{1}{d^\lambda} \right\}^{\frac{p}{\lambda}} \cdot c_1^{\frac{\lambda-p}{\lambda}}$$

$$\leq \frac{\eta^p}{3} \quad \text{for all } t > 0,$$

whereas (81) warrants that

$$2^{p-1} \mu_\infty \int_{\Omega_0} \frac{1}{d^p} \leq \frac{\eta^p}{3}.$$\hspace{1cm} (83)

As (82) ensures that apart from that we have

$$\int_{\Omega \setminus \Omega_0} |u(\cdot, t) - \frac{\mu_\infty}{d}|^p \leq \left\| u(\cdot, t) - \frac{\mu_\infty}{d} \right\|^p_{L^\infty(\Omega \setminus \Omega_0)} |\Omega \setminus \Omega_0| \leq \frac{\eta^p}{3} \quad \text{for all } t > t_0,$$

it follows from (83) that

$$\left\| u(\cdot, t) - \frac{\mu_\infty}{d} \right\|_{L^p(\Omega)} \leq \left( \frac{\eta^p}{3} + \frac{\eta^p}{3} + \frac{\eta^p}{3} \right)^\frac{1}{p} = \eta \quad \text{for all } t > t_0,$$

thereby verifying (10). \hspace{1cm} □
4. Appendix: A refined interpolation inequality. We prove the following interpolation inequality of Gagliardo-Nirenberg type which is based on an observation originally made in [3].

**Lemma 4.1.** Let $q > 0$. Then there exist $C(q) > 0$ and $\Lambda(q) > 0$ such that for any choice of $\eta \in (0, 1)$ we have

$$
\|\varphi\|_{L^q(\Omega)}^{q+2} \leq \eta \|\varphi\|_{L^q(\Omega)}^q \cdot \left\| \varphi \ln |\varphi| \right\|_{L^q(\Omega)} + C(q) \|\varphi\|_{L^q(\Omega)}^{q+2} + C(q)e^{\frac{\Lambda(q)}{\eta}} \tag{84}
$$

for all $\varphi \in W^{1,2}(\Omega)$.

**Proof.** Following the argument in [3], we first invoke the Gagliardo-Nirenberg inequality to find $c_1 \geq 1$ such that

$$
\|\psi\|_{L^q(\Omega)}^{q+2} \leq c_1 \|\psi\|_{L^q(\Omega)}^q \|\psi\|_{L^q(\Omega)}^q + c_1 \|\psi\|_{L^q(\Omega)}^q \tag{85}
$$

for all $\psi \in W^{1,2}(\Omega)$. (85) easily have

For fixed $\eta \in (0, 1)$, we then let

$$
N := \exp\left\{ \frac{2^{q+3}c_1}{\eta} \right\} > 1
$$

and introduce $\zeta \in W^{1,\infty}(\mathbb{R})$ by defining $\zeta(\xi) := 0$ for $\xi \in [-N, N]$, $\zeta(\xi) := |\xi|$ for $|\xi| \geq 2N$ and $\zeta(\xi) := 2(|\xi| - N)$ for $N \leq |\xi| < 2N$. Then given $\varphi \in W^{1,2}(\Omega)$, we evidently have

$$
\|\varphi - \zeta(\varphi)\|_{L^\infty(\Omega)} \leq 2N
$$

and furthermore

$$
\|\zeta(\varphi)\|_{L^q(\Omega)}^q \leq \int_{\{\varphi\geq N\}} |\varphi|^q \leq \frac{1}{\ln N} \int_{\Omega} |\varphi|^q \ln |\varphi| = \frac{1}{\ln N} \|\varphi\|_{L^q(\Omega)} \ln |\varphi| \|_{L^q(\Omega)}.
$$

Since $(a + b)^{q+2} \leq 2^{q+1}(a^{q+2} + b^{q+2})$ for all $a \geq 0$ and $b \geq 0$, (85) thus entails that

$$
\|\varphi\|_{L^q(\Omega)}^{q+2} \leq 2^{q+1} \|\zeta(\varphi)\|_{L^q(\Omega)}^{q+2} + 2^{q+1} \|\varphi - \zeta(\varphi)\|_{L^q(\Omega)}^{q+2}
$$

$$
\leq 2^{q+1} c_1 \|\zeta(\varphi)\|_{L^q(\Omega)}^q \|\varphi\|_{L^q(\Omega)}^q + 2^{q+1} c_1 \|\varphi\|_{L^q(\Omega)}^{q+2}
$$

$$
+ 2^{q+1} (2N)^{q+2}
$$

$$
\leq 2^{q+3} \frac{c_1}{\ln N} \|\varphi\|_{L^q(\Omega)}^q \|\varphi\|_{L^q(\Omega)} \ln |\varphi| \|_{L^q(\Omega)}^q + 2^{q+1} c_1 \|\varphi\|_{L^q(\Omega)}^{q+2}
$$

$$
+ 2^{q+3} N^{q+2},
$$

because $\|\varphi\|_{L^\infty(\mathbb{R})} = 2$ and $|\zeta(\xi)| \leq |\xi|$ for all $\xi \in \mathbb{R}$. In view of our definition of $N$, this proves (84) with $C(q) := \max\{2^{q+1}c_1, 2^{3q+3}\}$ and $\Lambda(q) := 2^{q+3}c_1(q + 2)$. \qed

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**REFERENCES**

[1] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, *Math. Models Methods Appl. Sci.*, 25 (2015), 1663–1763.

[2] J. Belmonte-Beitia, T. E. Woolley, J. G. Scott, P. K. Maini and E. A. Gaffney, Modelling biological invasions: Individual to population scales at interfaces, *J. Theoret. Biol.*, 334 (2013), 1–12.
[3] P. Biler, W. Hebisch and T. Nadzieja, The Debye system: Existence and large time behavior of solutions, *Nonlinear Anal.*, 23 (1994), 1189–1209.

[4] S. B. Carter, Haptotaxis and the mechanism of cell motility, *Nature*, 213 (1967), 256–260.

[5] M. A. J. Chaplain and G. Lolas, Mathematical modelling of cancer invasion of tissue: Dynamic heterogeneity, *Netw. Heterog. Media*, 1 (2006), 399–439.

[6] L. Corrias, B. Perthame and H. Zaag, Global solutions of some chemotaxis and angiogenesis systems in high space dimensions, *Milan J. Math.*, 72 (2004), 1–28.

[7] M. A. Fontelos, A. Friedman and B. Hu, Mathematical analysis of a model for the initiation of angiogenesis, *SIAM J. Math. Anal.*, 33 (2002), 1330–1355.

[8] K. Fujie, A. Ito, M. Winkler and T. Yokota, Stabilization in a chemotaxis model for tumor invasion, *Discrete Cont. Dyn. Syst.*, 36 (2016), 151–169.

[9] M. M. Porzio and V. Vespri, Holder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, *J. Differential Equations*, 103 (1993), 146–178.

[10] C. Surulescu and M. Winkler, Global weak solutions to a strongly degenerate haptotaxis model, *Commun. Math. Sci.*, 15 (2017), 1581–1616.

[11] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Differential Equations*, 248 (2010), 2889–2905.

[12] M. Winkler, Singular structure formation in a degenerate haptotaxis model involving myopic diffusion, *J. Math. Pures Appl.*, 112 (2018), 118–169.

[13] A. Zhigun, C. Surulescu and A. Uatay, Global existence for a degenerate haptotaxis model of cancer invasion, *Z. Angew. Math. Phys.*, 67 (2016), Art. 146, 29 pp.

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