Kirkman Equiangular Tight Frames and Codes

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Abstract—An equiangular tight frame (ETF) is a set of unit vectors in a Euclidean space whose coherence is as small as possible, equaling the Welch bound. Also known as Welch-bound-equality sequences, such frames arise in various applications, such as waveform design and compressed sensing. At the moment, there are only two known flexible methods for constructing ETFs: harmonic ETFs are formed by carefully extracting rows from a discrete Fourier transform; Steiner ETFs arise from a tensor-like combination of a combinatorial design and a regular simplex. These two classes seem very different: the vectors in harmonic ETFs have constant amplitude, whereas Steiner ETFs are extremely sparse. We show that they are actually intimately connected: a large class of Steiner ETFs can be unitarily transformed into constant-amplitude frames, dubbed Kirkman ETFs. Moreover, we show that an important class of Kirkman ETFs. This connection informs the discussion of both types of ETFs are extremely sparse. We show that they are actually equality sequences, such frames arise in various applications, such as waveform design and compressed sensing. We conclude by showing that ETFs of almost arbitrary redundancy and size.

The fifth known construction method involves Steiner systems, namely a set $B$ of blocks (subsets) of a finite set $V$ which has the properties that (i) every block has the same number of points, (ii) every point is contained in the same number of blocks and (iii) any two points determine a unique block. For many years, it has been known that such systems can be used to build strongly regular graphs [12]. Moreover, strongly regular graphs with certain parameters are known to be equivalent to real ETFs [15], [28]. In [10], these ideas are distilled into a direct method for constructing real or complex ETFs via a tensor-like combination of the incidence matrix of Steiner system with a unimodular regular simplex. Like harmonic ETFs, these Steiner ETFs are extremely flexible, providing ETFs whose size and redundancy are arbitrary, up to an order of magnitude. However, whereas harmonic ETFs have constant amplitude—the entries of their frame vectors have constant modulus—Steiner ETFs are extremely sparse. This sparsity can be a detriment in applications: in radio communication and radar, constant-amplitude waveforms allow more energy to be transmitted by power-limited hardware. Moreover, though Steiner ETFs have optimal coherence, are thus good for coherence-based compressed sensing, they have terrible spark: a small number of Steiner ETF elements can be linearly dependent [10]. As such, Steiner ETFs do not satisfy compressed sensing’s Restricted Isometry Property (RIP) in a way that rivals that of random matrices.

In this paper, we provide a new method for unitarily transforming certain Steiner ETFs into constant-amplitude ETFs. This method only works when the underlying Steiner system is resolvable, meaning that its blocks $B$ can partitioned into several collections of blocks $\{B_r\}_{r \in \mathbb{R}}$, where for any $r$, the blocks in $B_r$ form a partition of $V$. Such systems were first made famous in 1850 by Kirkman’s schoolgirl problem, and as such, we dub these frames Kirkman ETFs.
In the next section, we provide the basic mathematical background on Steiner ETFs. In Section 3, we provide the Kirkman construction itself, and then use the existing literature on resolvable Steiner systems to construct several new families of constant-amplitude ETFs. It turns out that one of these “new” families—those arising from finite affine geometries—corresponds to one of the most important classes of harmonic ETFs, namely those constructed via McFarland difference sets; as discussed in the fourth section, this identification allows us, for the first time, to seriously investigate the RIP properties of these McFarland ETFs. In Section 5, we identify a real-valued constant-amplitude ETF with a self-complementary binary code, and in this context show that the Welch bound is as well-lie to the right of a 1 in the first column. In particular, for a $(2, K, V)$-Steiner system that consists of all 2-element subsets of a set of 4 elements:

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \quad (4)$$

Here, $B = 6$ and $R = 3$.

As shown in [10], every $(2, K, V)$-Steiner system generates an ETF of $N = V(R + 1)$ vectors in a space of dimension $M = B$. The main idea is to take a tensor-like combination of $B$ with a unimodular regular simplex of $R + 1$ vectors in $R$-dimensional space. For example, for the $(2, 2, 4)$-Steiner system in [2] in which $R = 3$, we can construct such a simplex by removing a row from a $4 \times 4$ matrix with orthogonal columns and unimodular entries, such as a DFT or Hadamard matrix:

$$F = \begin{bmatrix} + & + & + & - \\ + & + & - & - \\ + & - & + & - \end{bmatrix}. \quad (5)$$

Here and throughout, “+” and “−” denote 1 and −1, respectively. To construct an ETF from $B$ and $F$, we replace each of the $R$ nonzero entries in any given column of $B$ with a corresponding row from $F$, and replace each zero entry of $B$ with a $(R + 1)$-long row of zeros. We then normalize the resulting columns. In particular, Figure [1] gives the $6 \times 16$ ETF obtained by “tensoring” (4) with (5) in this fashion.

Since any finite-dimensional space always contains a unimodular regular simplex, the only restrictions on the existence of such ETFs arise from restrictions on Steiner systems themselves. For example, the $B$ and $R$ parameters of a $(2, K, V)$-Steiner system are uniquely determined by $K$ and $V$ according to the necessary relationships that

$$BK = VR, \quad R(K - 1) = V - 1. \quad (6)$$

The first identity follows from counting the total number of 1’s in the incidence matrix $B$ both row-wise and column-wise; the second follows from counting the number of 1’s in $B$ that lie to the right of a 1 in the first column. In particular, for a $(2, K, V)$-Steiner system to exist, both $R = (V - 1)/(K - 1)$ and $B = V(V - 1)/[K(K - 1)]$ must be integers.
\[ \Phi = \frac{1}{\sqrt{3}} \begin{bmatrix} + & - & + & - & + & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & + & - & + & - & + & - \\ + & + & - & 0 & 0 & 0 & 0 & + & - & + & 0 & 0 \\ 0 & 0 & 0 & + & + & - & 0 & 0 & 0 & + & + & - \\ + & - & - & + & 0 & 0 & 0 & 0 & 0 & + & - & + \\ 0 & 0 & 0 & + & - & + & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

Fig. 1. A Steiner ETF of 16 vectors in 6-dimensional space obtained by “tensoring” the incidence matrix \( \Phi \) of a \((2,2,4)\)-Steiner system with a regular simplex of 4 vectors in 3-dimensional space \( \Phi \) according to [10]. Here, the Welch bound \( \Phi \) is 1/3, and any two distinct columns have a dot product of this magnitude. Indeed, grouping the columns as \( V = 4 \) sets of \( R + 1 = 4 \) vectors (as pictured), any two distinct columns from the same set (like the first and second columns of \( \Phi \)) have a dot product of \(-1/3\), since it corresponds to a dot product of distinct columns in our regular simplex \( \Phi \). Meanwhile, any two columns from distinct sets (like the first and fifth columns of \( \Phi \)) have only one point of common support, since any two distinct points in our Steiner system determine a unique block; as such, the dot product of such columns has value \( \pm 1/3 \). Such ETFs are sparse—many of the entries of \( \Phi \) are zero—and also have low spark; the first four of these vectors are linearly dependent. In this paper, we show how to unitarily transform this matrix into a constant-amplitude ETF: a trick which “corrects” its sparsity but not its spark. Our approach will only work due to the fact that our Steiner system here is resolvable: the first and second blocks (rows) in \( \Phi \) yield a partition of the points (columns), as do its third and fourth blocks, and its fifth and sixth blocks.

Our parameters must also satisfy \( B \geq V \); known as Fisher’s inequality, this follows from the fact that the \( V \times M \) matrix \( B^TB \) is necessarily of full rank, since its off-diagonal entries are 1 while its diagonal entries are \( R = (V - 1)/(K - 1) > 1 \). These facts are important here since the parameters \( K \) and \( V \) indicate the dimensions of the resulting Steiner ETF. Indeed, in light of (6), the redundancy of a Steiner ETF is

\[
N = \frac{V(R+1)}{B} = K\frac{V(R+1)}{BK} = K\frac{V(R+1)}{VR} = K(1 + \frac{1}{R}).
\]

Since (6) and Fisher’s inequality give \( K/R = V/B \leq 1 \), the redundancy of a Steiner ETF is essentially \( K \). Moreover, for any fixed \( K \), both \( M \) and \( N \) grow quadratically with \( V \):

\[
M = B = \frac{V(V-1)}{K(K-1)} \quad \text{and} \quad N = V(R+1) = V\left(\frac{K}{K-1}+1\right).
\]

In particular, in order to build ETFs of various sizes and redundancies, we need explicit constructions of \((2, K, V)\)-Steiner systems which permit flexible, independent control of both \( K \) and \( V \). There are three known families of such systems [5], all arising from finite geometry.

To be precise, for any prime power \( q \) and \( j \geq 1 \), there exist affine geometry-based Steiner systems with \( K = q \) and \( V = q^{j+1} \). For \( j \geq 2 \), there also exist projective geometry-based systems with \( K = q + 1 \) and \( V = (q^{j+1} - 1)/(q - 1) \). In either case, varying \( q \) and \( j \) controls the redundancy and size of the ETF, respectively. The third family is Denniston designs [5], in which \( K = 2^i, V = 2^{i+j} + 2^i - 2^j \) for some \( 2 \leq i < j \) which arise from maximal arcs in projective spaces [6]. Importantly, both the affine and Denniston designs are resolvable [11]. As we now discuss, this means we can transform them into constant-amplitude ETFs.

III. KIRKMAN ETFS

In this section, we introduce a method for unitarily transforming certain Steiner ETFs, like the one depicted in Figure 1 into constant-amplitude ETFs. This method requires the underlying Steiner system to be resolvable, meaning its blocks \( B \) can be partitioned into disjoint subcollections \( \{B_i\}_{i \in \mathbb{R}} \) so that the blocks in any given \( B_i \) form a partition for \( V \). For example, the \((2, 2, 4)\)-Steiner system given in (4) is resolvable: its first and second blocks (rows) form a partition for our underlying set of \( V = 4 \) points, as do its third and fourth, and its fifth and sixth. The main idea of this new method is to multiply the synthesis matrix of a resolvable Steiner ETF, like Figure 1 by a block-Hadamard/DFT matrix to obtain a constant-amplitude ETF; see Figure 2.

Not every Steiner system is resolvable. Indeed, if any subset of the blocks \( B \) forms a partition of \( V \), then \( K \) must divide \( V \); each block contains \( K \) points and there are \( V \) points total. This requirement alone prohibits the famous \((2,3,7)\)-Steiner system known as the Fano plane from being resolvable. When coupled with the previous restriction that \( K - 1 \) divides \( V - 1 \), this new condition subsumes the previous requirement that \( K(K-1) \) divides \( V(V-1) \). Moreover, since we necessarily have \( V \equiv 1 \mod K - 1 \) and \( V \equiv 0 \mod K \) where \( K \) is relatively prime to \( K - 1 \), the Chinese Remainder Theorem gives that these two conditions are equivalent to having \( V \equiv K \mod K(K-1) \). For resolvable designs, it also turns out [11] that Fisher’s inequality can be strengthened to Bose’s condition that \( B \geq V + R - 1 \).

Nevertheless, many Steiner systems are resolvable, such as those arising from affine geometries over finite fields and Denniston designs [11]. It seems to be an open question whether or not projective geometries with \( K = q + 1 \) and \( V = (q^{j+1} - 1)/(q - 1) \) are resolvable when \( j \) is odd. At least in some cases, the answer is yes: when \( q = 2 \) and \( j = 2 \), this is Kirkman’s schoolgirl problem. Since this famous problem is so closely associated with resolvable Steiner systems, we refer to the constant-amplitude ETFs that arise from such systems as Kirkman ETFs.

We now formally verify that every resolvable Steiner system generates a (constant-amplitude) Kirkman ETF that is unitarily equivalent to a (sparse) Steiner ETF. Here, as usual, the quickest way to verify that certain vectors form an ETF is to show they satisfy the Welch bound (2) with equality. In this Steiner-system-induced setting where \( M = B \) and \( N = V(R+1) \), the lower bound itself is simply \( 1/R \); noted in [10], this can be most easily seen by making repeated use of the identities (6) to show \( M(N-1)/(N-M) = R^2 \). This is a special case of a known necessary integrality condition [26]: if all the entries in an ETF are suitably-normalized roots of unity, then \( M(N-1)/(N-M) \) is necessarily an integer.

Before stating the result, it is helpful to introduce some notation. Note that in any resolvable \((2, K, V)\)-Steiner system, the number of blocks in any single partition is \( V/K \). Since the total number of blocks is \( B = VR/K \), the number of distinct partitions of \( V \) is \( R \). As such, we enumerate these
partitions using some R-element indexing set \( S \), and write our blocks as the disjoint union \( B = \bigcup_{r \in R} B_r \). Here, for any \( r \in R \), the \( S := V/K \) blocks that lie in \( B_r \) form a partition for \( V \), and we index them with some \( S \)-element indexing set \( S \). To be precise, for any \( r \in R \), let \( B_r = \{ b_{rs} \}_{s \in S} \) where \( V = \sqcup_{s \in S} b_{rs} \). We now state and prove our first main result:

**Theorem 1:** Let \((V,B)\) be a resolvable \((2,K,V)\)-Steiner system: let \( \{ B_r \}_{r \in R} \) be a partition of \( B \) where for any \( r \), \( B_r = \{ b_{rs} \}_{s \in S} \) is a partition of \( V \). Let \( \{ f_u \}_{u=0}^R \) be an element regular simplex in \( C^R \) and let \( \{ h_s \}_{s \in S} \) be a unimodular orthogonal basis for \( C^S \). Then letting \( M = R \times S \) and \( N = V \times \{ 0, \ldots, R \} \), the \( (R+1) \times 2 \) vectors \( \{ \varphi_{u,v} \}_{(u,v) \in N} \) form a Steiner ETF for the \( B \)-dimensional space \( C^M \):

\[
\varphi_{u,v}(r,s) := R^{-\frac{1}{2}} \sum_{r,s \in S} f_u(r) \cdot h_s(v) \cdot h_s(\pi(v)) \cdot s^*.
\]

Moreover, applying a unitary operator to this Steiner ETF yields the Kirkman ETF \( \{ \psi_{u,v} \}_{(u,v) \in N} \) defined by

\[
\psi_{u,v}(r,s) := B^{-\frac{1}{2}} f_u(r) h_{s(r,v)}(s),
\]

where for any \( r \in R \) and \( v \in V \), \( s(r,v) \) denotes the unique \( s \in S \) such that \( v \in b_{rs} \).

**Proof:** We first prove that \( \{ \varphi_{u,v} \}_{(u,v) \in N} \) is an ETF. To do this, it suffices to show that each \( \varphi_{u,v} \) is unit norm and that the inner product of any distinct two of these vectors has modulus equal to the Welch bound \( 1/R \). In general, we have

\[
\langle \varphi_{u,v}, \varphi_{u',v'} \rangle = \sum_{r,s \in S} \varphi_{u,v}(r,s) \varphi_{u',v'}(r,s)^*.
\]

In the special case where \( v = v' \), note that for any \( r \in R \), the fact that \( \{ b_{rs} \}_{s \in S} \) is a partition of \( V \) implies there is exactly one \( s \in S \) such that \( v \in b_{rs} \). In light of (7), this fact reduces (9) in this case to

\[
\langle \varphi_{u,v}, \varphi_{u',v'} \rangle = \frac{1}{R} \sum_{r \in R} f_u(r) f_{u'}(r)^* = \frac{1}{R} \langle f_u, f_{u'} \rangle.
\]

When coupled with the fact that \( \{ f_u \}_{u=0}^R \) is a unimodular regular simplex, this implies that the \( \varphi_{u,v} \)'s have unit norm and satisfy \( \langle \varphi_{u,v}, \varphi_{u',v'} \rangle = 1/R \) whenever \( u \neq u' \), as needed. To show that we also have \( \langle \varphi_{u,v}, \varphi_{u',v'} \rangle = 1/R \) whenever \( u \neq u' \), recall that since \( \{ V, B \} \) is a \( (2,K,V) \)-Steiner system, there is exactly one block \( b = b_{v0,s0} \) that contains both \( v \) and \( v' \). Again recalling (7), this means that there is only one nonzero summand of (9), yielding

\[
\langle \varphi_{u,v}, \varphi_{u',v'} \rangle = \frac{1}{R} \langle f_u(r_0), f_{u'}(r_0) \rangle = \frac{1}{R}. \]

Thus, \( \{ \varphi_{u,v} \}_{(u,v) \in N} \) is an ETF, as claimed. For the second conclusion, note that by (8), \( \{ \psi_{u,v} \}_{(u,v) \in N} \) is:

\[
\frac{1}{R} \sum_{r \in R} \sum_{s \in S} f_u(r) h_{s(r,v)}(s) f_{u'}(r) h_{s(r,v')}^*(s)^* = \frac{1}{R} \sum_{r \in R} f_u(r) f_{u'}(r)^* (h_{s(r,v)}, h_{s(r,v')}). \]

Fixing \( r \) for the moment, note that since \( \{ h_s \}_{s \in S} \) is a unimodular orthogonal basis for the space \( C^S \) of dimension \( S = V/K = 2/R \), we have that \( \langle h_{s(r,v)}, h_{s(r,v')} \rangle = 0 \) when \( s(r,v) = s(r,v') \) and is otherwise zero. Since \( s(r,v) \) denotes the unique \( s \in S \) such that \( v \in b_{rs} \), this implies

\[
\langle h_{s(r,v)}, h_{s(r,v')} \rangle = \frac{1}{R} \sum_{s \in S} 1_{br,s}(v) 1_{br,s}(v'), \]

where \( 1_{br,s} : V \rightarrow \{ 0, 1 \} \subseteq C \) is the characteristic function of the block \( b_{rs} \). For every \( r \in R \), substituting (11) into (10) and then recalling (7) gives that

\[
\langle \psi_{u,v}, \psi_{u',v'} \rangle = \frac{1}{R} \sum_{r \in R} \sum_{s \in S} f_u(r) f_{u'}(r)^* 1_{br,s}(v) 1_{br,s}(v')
\]

\[
= \sum_{r \in R} \sum_{s \in S} \varphi_{u,v}(r,s) \varphi_{u',v'}(r,s)^*
\]

\[
= \langle \varphi_{u,v}, \varphi_{u',v'} \rangle. \]
Since (12) holds for all \((u,v),(u',v') \in \mathcal{N}\), we know that \(\{\psi_{u,v}\}_{(u,v) \in \mathcal{N}}\) is also an ETF, and moreover, is obtained by applying a unitary transformation to \(\{\varphi_{u,v}\}_{(u,v) \in \mathcal{N}}\). In truth, this transformation is a block-unitary transform, but we do not need this specificity for the work that follows.

For the remainder of this section, we consider the ramifications of Theorem 1 on the existence of constant-amplitude ETFs. In particular, we first describe Kirkman ETFs that arise from known flexible families of resolvable \((2,K,V)\)-Steiner systems, meaning they permit independent control of \(K\) and \(V\). We then describe some inflexible families, meaning that \(K\) uniquely determines \(V\), or vice versa. Finally, we conclude this section with a discussion of the known asymptotic existence results for resolvable Steiner systems.

For each of these families, we state whether or not constant-amplitude ETFs with those parameters have been found before. To be clear, the existence of Steiner ETFs of these sizes was already noted in [10]. However, before Theorem 1, the only known method for constructing constant-amplitude ETFs was to use difference sets [30], [8]. We also do our best to answer a deeper question: whether or not a given Kirkman ETF is actually a harmonic ETF in disguise. Note that this would necessarily imply that there exist difference sets \(D\) of \(M = B\) elements in Abelian groups \(G\) of order \(N = V(R + 1)\). This, in turn, requires that

\[
\Lambda := \frac{M(M-1)}{N-1} = \frac{V(V-K)}{K(K-1)}
\]

is an integer, since \(\Lambda\) is the number of times any nonzero element of \(G\) may be written as a difference of two elements in \(D\). However, every Kirkman ETF automatically satisfies this integrality condition: if a resolvable \((2,K,V)\)-Steiner system exists, then \(V \equiv K \mod K(K-1)\); writing \(V = W[K(K-1)] + 1\), gives \(\Lambda = W[W(K-1)+1]\). Moreover, this implies that the degree \(M - \Lambda\) of such a difference set is necessarily the perfect square \(M - \Lambda = [W(K-1)+1]^2\), meaning that the necessary conditions of the Bruck-Ryster-Chowla Theorem are automatically satisfied whenever \(N\) is even [11]. As such, trying to show that a given Kirkman ETF is not harmonic can quickly lead to hard, open problems concerning the existence of difference sets.

A. Flexible Kirkman ETFs

1) Affine geometries over finite fields: For any \(j \geq 1\) and prime power \(q\), there exists a resolvable \((2,K,V)\)-Steiner system with \(K = q + 1\) and \(V = q^{j+1}\), where \(F_q\) is the finite field of order \(q\). Meanwhile, the blocks \(B\) are affine lines in this space, namely sets of the form \(\{au + v : a \in F_q\}\) for some direction vector \(u \in F_q^{j+1}\setminus\{0\}\) and initial point \(v \in F_q^{j+1}\). These systems play an important role in the theory of the next section, and we describe them more fully there. For now, the most important things to note are that (i) these systems are easy to construct explicitly, meaning the construction of Theorem 1 can be truly implemented; (ii) the resulting Kirkman ETFs consist of \(N\) vectors in an \(M\)-dimensional space where

\[
M = q^j \left(\frac{q^{j+1} - 1}{q - 1}\right), \quad N = q^{j+1} \left(\frac{q^{j+1} - 1}{q - 1} + 1\right);
\]

and that (iii) the redundancy and size of this ETF can be controlled by manipulating \(q\) and \(j\), respectively. The existence of constant-amplitude ETFs with these parameters is not new: harmonic ETFs with dimensions (13) can be constructed with McFarland difference sets [8]. In the next section, we show this is not a coincidence: we prove that every McFarland harmonic ETF is a Kirkman ETF, and as such, is unitarily equivalent to a low-sparse, sparse Steiner ETF.

2) Denniston designs: For any positive integers \(i\) and \(j\), \(i \leq j\), there exists a resolvable \((2,K,V)\)-Steiner system with \(K = 2^i\) and \(V = 2^{i+1} + 2^i - 2\). The construction is nontrivial: one constructs a maximal arc in the projective plane of order \(2^i\) using an irreducible quadratic form [6], and then constructs a resolvable design in terms of this arc [11]. The resulting Kirkman ETF has \(M = (2^i+1)(2^i+1 - 2^{i+1})\) vectors in a space of dimension \(N = 2^i(2^i + 2)(2^i + 1 - 2^{i+1})\).

Note that when \(i = j\), these designs have the same parameters as an affine geometry where \(q = 2^i\). Meanwhile, for \(i < j\), the constant-amplitude ETFs generated by these designs seem to be new. For example, when \(i = 2\) and \(j = 3\), we find that there exists a constant-amplitude ETF of \(N = 280\) vectors in a space of dimension \(M = 63\); such an ETF is not found in the existing literature [30], [8]. Such ETFs might be harmonic: we did not find any examples of \(i\) and \(j\) for which it is known that there cannot exist an \(M\)-element difference set in an Abelian group of order \(N\); since \(N\) is even, the Bruck-Ryster-Chowla Theorem is toothless. We leave a more thorough investigation of this problem for future research.

B. Inflexible Kirkman ETFs

1) Round-robin tournaments: For any positive integer \(V\), consider the \((2,2,V)\)-Steiner system that consists of every two-element subset of \(V = \{1, \ldots, V\}\), such as the \((2,2,4)\)-Steiner system whose incidence matrix is given in [4]. When \(V\) is even, this system is resolvable via the famous round-robin schedule, which is sometimes used in tournament competitions, as it ensures that each competitor faces all others exactly once while letting the entire tournament be as quick as possible. The resulting family of constant-amplitude Kirkman ETFs is inflexible, since the redundancy \(N/M\) of any such frame is essentially two: \(M = V(V-1)/2\). When \(V\) is even, ETFs with these parameters are well-known [8], arising from McFarland difference sets in Abelian groups isomorphic to \(Z_2^{j+1}\). However, when \(V\) is even but not a power of 2, some of the resulting Kirkman ETFs do not arise from difference sets. In particular, there does not exist a difference set of \(M = 45\) elements in an Abelian group of order \(N = 100\) [10], and so the \((2,2,10)\)-Round Robin Kirkman ETF is not harmonic. In Section V we exploit these ideas to build new examples of real-valued constant-amplitude ETFs provided there exists a Hadamard matrix of size \(V\); this leads to new examples of (nonlinear) binary codes that achieve the Grey-Rankin bound.

2) Kirkman’s Schoolgirl Problem: For any positive integer \(V \equiv 3 \mod 6\), there exists a resolvable \((2,3,V)\)-Steiner (triple) system [21]. The resulting Kirkman ETFs have an
approximate redundancy of 3, consisting of \(V(V + 1)/2\) vectors in a space of dimension \(V(V - 1)/6\). At least some of these frames are new constant-amplitude ETFs; when \(V = 15\), for example (Kirkman’s original problem), the resulting ETF consists of 120 vectors in a space of dimension 35, and there does not exist an Abelian difference set with those parameters [16].

3) Three-dimensional projective geometries: For any prime power \(q\), a resolvable \((2, q + 1, q^3 + q^2 + q + 1)\)-Steiner system exists [17]. The resulting family of Kirkman ETFs is inflexible: though both \(M = (q^2 + 1)/(q^2 + q + 1)\) and \(N = (q^2 + q + 2)/(q^2 + q + 1)\) can grow arbitrarily large, the single parameter \(q\) determines both the size and redundancy of such frames. Note that when \(q = 2\) this design is a \((2,3,15)\)-Steiner triple system which, as noted above, generates an ETF which is not harmonic. As such, at least some of these frames are new constant-amplitude ETFs.

To be clear, projective geometries generate a flexible family of Steiner ETFs: for any \(j \geq 2\) the projective geometry with parameters \(K = q^j + 1\) and \(V = (q^{j+1} - 1)/(q - 1)\) generates a Steiner ETF [10] with dimensions

\[
M = \frac{(q^j - 1)(q^{j+1} - 1)}{(q + 1)(q - 1)^2}, \quad N = \frac{q^{j+1} - 1}{q - 1} \left( \frac{q^j - 1}{q - 1} - 1 \right),
\]

and varying \(q\) and \(j\) independently generates ETFs of various sizes and redundancies. However, we could only find references to projective geometries being resolvable—and thus able to generate Kirkman ETFs—in special cases, like here where \(j = 3\). Note that in order to be resolvable we need \(K\) to divide \(V\) which requires \(j\) to be odd; it seems to be an open question whether such systems are resolvable for odd \(j \geq 5\).

4) Unitals: For any prime power \(q\), there exists a resolvable \((2, q + 1, q^3 + 1)\)-Steiner system [18]. This family, like the last, is inflexible, since \(q\) determines both \(M = q^2(q^2 - q + 1)\) and \(N = (q^2 + 1)(q^3 + 1)\). At least some of these are new constant-amplitude ETFs: taking \(q = 3\) yields a constant-amplitude ETF of 280 vectors in a space of dimension 63 which, as noted earlier, is not in the literature. We did not find any examples of any such ETFs which are provably not harmonic; note that whenever \(q\) is odd, \(N\) is even and so the conditions of the Bruck-Ryser-Chowla Theorem are automatically satisfied.

C. Existence results for Kirkman ETFs

A remarkable fact about resolvable Steiner systems is that they are known to exist asymptotically. That is, for any \(K \geq 2\) it is known that there exists a positive integer \(V_0(K)\) such that for any \(V \geq V_0(K)\) which satisfies \(V \equiv K \mod (K - 1)\), there exists a resolvable \((2, K, V)\)-Steiner system [22]. Thus, for any such \(V\), we know there exists a constant-amplitude ETF of \(N\) \(V(2V + K + 2)/(K(K - 1))\) vectors in a space of dimension \(M = V(V - 1)/(K(K - 1))\). In particular, for any \(K \geq 2\), there exist constant-amplitude ETFs whose redundancy \(N/M\) is arbitrarily close to \(K\). In contrast, all known examples of harmonic ETFs have redundancies which are essentially a power of a prime. Unfortunately, the methods used to demonstrate the existence of these systems are not constructive in any practical sense. Nevertheless, these existence results encourage the search for explicit constructions of constant-amplitude ETFs with arbitrary redundancies.

IV. Connecting Kirkman ETFs and harmonic ETFs

In this section, we show that an important class of harmonic ETFs, namely those generated by McFarland difference sets, can also be constructed as Kirkman ETFs via Theorem 4. As a corollary, we find that harmonic ETFs from this particular family have Steiner representations, making them less desirable for certain compressed-sensing-related applications. This result also demonstrates how truly rare it is to discover a new flexible family of ETFs.

To be precise, as noted in the introduction, there are only three known approaches for constructing nontrivial ETFs: via conference matrices [25], difference sets [24, 30, 8] and Steiner systems [10]. In the previous section, we refined the Steiner-based approach so as to produce constant-amplitude Kirkman ETFs. Moreover, nearly all of the particular instances of these constructions are inflexible. Indeed, ETFs generated by conference matrices have redundancy \(\frac{N}{M} = 2\); surveying [10], [8] as well as a comprehensive list of Abelian difference sets [16], we see that harmonic ETFs generated by Paley, Hadamard, twin prime power and Davis-Jedwab-Chen difference sets all have an approximate redundancy of 2, while those generated by other cyclotomic or Spence difference sets have approximate redundancies of either 3, 4 or 8; as noted earlier, families of Steiner systems with a fixed \(K\) yield ETFs whose approximate redundancy is \(K\), and those produced from unital designs are also inflexible.

There are only five known flexible families of ETFs: harmonic ETFs arising from (i) Singer difference sets [30] and (ii) McFarland difference sets [8], and Steiner ETFs arising from (iii) affine geometries, (iv) Denniston designs and (v) projective geometries [10]. Also, as shown in the previous section, classes (ii) and (iv) arise from resolvable Steiner systems, meaning we can apply a unitary operator to them to produce constant-amplitude ETFs. In this section, we show that modulo such unitaries, (ii) is a special case of (iii). That is, we show that in truth there are only four known flexible families of ETFs: three of these four, namely (i), (ii/iii) and (iv), have constant-amplitude representations; another three of these four, namely (ii/iii), (iv) and (v), have very sparse representations.

To show that every McFarland harmonic ETF is a special example of an affine Kirkman ETF, let \(j \geq 1\), let \(q\) be a prime power, and consider \(G \times V\) where \(G\) is any Abelian group of order \((q^{j+1} - 1)/(q - 1)\) and \(V\) is the additive group of finite field \(F_{q^{j+1}}\). We form a harmonic ETF over this group via the approach of [8] by letting \(D\) be a McFarland difference set in \(G \times V\). McFarland’s approach [18] is clever, and is eerily similar to Goethals and Seidel’s method for constructing strongly regular graphs from Steiner systems [12]; though made independently, we show these two approach are related, since both can lead to the same ETFs.

The first step in forming a McFarland difference set is to parametrize the distinct hyperplanes in \(F_{q^{j+1}}\), regarded as the
that the characters of the direct product known that the characters of any finite Abelian group form a group into the unit circle in the complex plane. It is well known that this trace is a nontrivial linear functional, and so its null space

\[ S = \{ v \in \mathbb{F}_{q^2} : \text{tr}_{q^2/q}(v) = 0 \} \]

is one example of a hyperplane in \( \mathbb{F}_{q^2} \), and thus has cardinality \( S = q^2 \). To find the remaining hyperplanes, let \( \gamma \) be a primitive element of \( \mathbb{F}_{q^2} \), meaning \( \gamma \) generates its cyclic multiplicative group. Since the mappings \( v \mapsto \text{tr}_{q^2/q}(\gamma^{-r}v) \) are distinct for every \( r = 0, \ldots, q^2+1 \), every nontrivial linear functional on \( \mathbb{F}_{q^2} \) can be represented this way. Moreover, since the nonzero elements of \( \mathbb{F}_q \) form a \( (q-1) \)-element subgroup of this multiplicative group of \( \mathbb{F}_{q^2} \), these functionals are distinct, even modulo scalar multiplication, provided we restrict the exponent \( r \) of \( \gamma \) to the set \( \{0, \ldots, R-1\} \). As such, the \( R \) distinct hyperplanes in \( \mathbb{F}_{q^2} \) can be written as the null spaces of the mappings \( v \mapsto \text{tr}_{q^2/q}(\gamma^{-r}v) \), that is, as \( \{\gamma^rS\}_{r \in \mathbb{R}} \) where \( S \) is the canonical hyperplane \( (14) \).

These hyperplanes in hand, we are ready to construct a McFarland difference set \( D \) in \( G \times V \), where \( G \) is any Abelian group of order \( R+1 \) and \( V \) is the additive group of \( \mathbb{F}_{q^2} \). To be precise, letting \( \{g_r\}_{r=0}^R \) be any enumeration of \( G \) and letting \( R := \{0, \ldots, R-1\} \), McFarland [18] showed that

\[ D = \{(g, v) : \exists r \in R \text{ such that } g = g_r, v \in \gamma^rS\} \]

is a difference set in \( G \times V \). Moreover, as discussed in [8], these difference sets, like all Abelian difference sets, yield harmonic ETFs. Our goal is to show that these McFarland harmonic ETFs can also be constructed by applying Theorem 1 in the special case where the underlying resolvable Steiner system is an affine geometry. To accomplish this, we next find explicit expressions for the harmonic ETF that arises from \( (15) \).

To be precise, [8] gives that the restrictions of the characters of \( G \times V \) to \( D \), suitably normalized, form an ETF for \( \mathbb{C}^D \). Here, a character of a finite Abelian group is a homomorphism from that group into the unit circle in the complex plane. It is well known that the characters of any finite Abelian group form a unimodular orthogonal basis over that group, and moreover, that the characters of the direct product \( G \times V \) are simply the tensor products of the characters of \( G \) with those of the additive group of \( \mathbb{F}_{q^2} \).

We denote the characters of \( (R+1) \)-element group \( G \) as \( \{\chi_u\}_{u=0}^R \). To form the characters of \( V \), usually called the additive characters of \( \mathbb{F}_{q^2} \), recall that \( \mathbb{F}_{q^2} \) is an extension of \( \mathbb{F}_q \), which in turn is an extension of its base field \( \mathbb{F}_p = (1) \cong \mathbb{Z}_p \), where the prime \( p \) is the characteristic of \( \mathbb{F}_q \). As such, we have another trace function \( \text{tr}_{q/p} : \mathbb{F}_{q^2} \to \mathbb{F}_p \). Moreover, it is well known that the characters \( \{e_v\}_{v \in V} \) of \( V \) can be formulated in terms of this trace: \( e_v(v') := \exp \left( \frac{2\pi i}{p} \text{tr}_{q/p}(v') \right) \) for all \( v' \in V \). Overall, for any \( u = 0, \ldots, R+1 \) and \( v \in V \), we see that the \( (u, v) \)-th character of \( G \times V \) is the function \( \chi_{(u,v)}(g) := \chi_u(g)\chi_v(g) \).

To form an ETF with the approach of [8], we restrict the domain of these characters to the difference set \( (15) \), and then normalize the resulting functions. Note that since \( D \) is parametrized in terms of \( R \) and \( S \) with \( (g, v) = (g_r, \gamma^r s) \), we regard these restricted characters as functions over \( R \times S \); for any \( u = 0, \ldots, R+1 \) and \( v \in V \), consider \( \psi_{u,v} \in \mathbb{C}^{R \times S} \).

\[ \psi_{u,v}(r, s) := D^{-\frac{1}{2}} \chi_u(g_r) \exp \left( \frac{2\pi i}{p} \text{tr}_{q/p}(\gamma^r s) \right) \]
Note that for any $s \in S$, multiplying (14) by $s \delta^{-1}$ and then applying the linear functional $\text{tr}_{q/p}$ yields:

$$\text{tr}_{q/p}(v' r s) = \text{tr}_{q/p}(s(r, v) \delta^{-1} + t(r, v) s) = \text{tr}_{q/p}(s(r, v) \delta^{-1}) + t(r, v) \text{tr}_{q/p}(s).$$  \hspace{1cm} (18)

At this point, we introduce a third trace $\text{tr}_{q/p} : F_q \to F_q$, which complements the other two. Recalling $F_p \subseteq F_q \subseteq F_{q^2}$, it is well known that these three traces satisfy the nice property that $\text{tr}_{q/p} = \text{tr}_{q/p} \circ \text{tr}_{q/p}$. In particular, recalling (14) and the linearity of the trace, any $s \in S$ satisfies

$$\text{tr}_{q/p}(s) = \text{tr}_{q/p}(\text{tr}_{q/p}(s)) = \text{tr}_{q/p}(0) = 0.$$  \hspace{1cm} (19)

As such, (19) reduces to $\text{tr}_{q/p}(v' r s) = \text{tr}_{q/p}(s(r, v) \delta^{-1})$, implying that the formula (16) for our McFarland ETF can be rewritten as

$$\psi_{a,v}(r, s) = D^{-1} \chi_a(g_r) \exp\left(\frac{2\pi i}{p} \text{tr}_{q/p}(s(r, v) \delta^{-1})\right).$$

Comparing this to (8), showing that this McFarland harmonic ETF is a Kirkman ETF boils down to showing two claims: (i) that $\{h_v\}_{v \in S}$, $h_v(s) := \exp\left(\frac{2\pi i}{p} \text{tr}_{q/p}(s' \delta^{-1})\right)$ is a unimodular orthogonal basis for $C^S$ and (ii) that the means of identifying $s(r, v)$ from a given $r$ and $v$ according to (17) corresponds to a resolvable Steiner system.

The truth of the first claim arises from the orthogonality of the additive characters of $F_{q^2 + 1}$. Indeed, for any $s' \neq s''$, the fact that $(s' - s'') \delta^{-1} \neq 0$ gives that

$$0 = \sum_{v \in V} \text{tr}_{q/p}(s' \delta^{-1})$$

Decomposing any $v \in V$ as $v = s + t \delta$ where $s \in S$, $t \in F_q$ and then using (19) in the case where “$s$” is $s' - s'' \in S$ gives

$$\text{tr}_{q/p}(s' \delta^{-1}) = \text{tr}_{q/p}(s'' \delta^{-1}).$$

Substituting this into (20) then gives our first claim:

$$0 = \sum_{s \in S} \sum_{t \in F_q} \text{tr}_{q/p}(s' \delta^{-1})$$

For the second claim, we let (17) define a block design. To be precise, let $B = \{b_{r,v} : (r,v) \in \mathcal{R} \times S\}$ be a set of subsets of $\mathcal{V} = F_{q^2 + 1}$. Where for any $r = 0, \ldots, n-1$ and $s \in S$ we say that $v \in b_{r,s}$ if and only if there exists a $t \in F_q$ such that $v' r s \delta = s + t \delta$ solved for $v$ reveals the $(r, s)$ block to be

$$b_{r,v} = \{s' \delta^{-1} \in \mathcal{V} : s' \delta^{-1} \in \mathcal{V}\}.$$  \hspace{1cm} (21)

Recalling that for any element of $F_{q^2 + 1}$ there exists exactly one $s \in S$ and $t \in F_q$ so that it can be written as $s + t \delta$, we see that for any fixed $r \in \mathcal{R}$, there exists exactly one $s = (r, v)$ such that $v \in b_{r,s}$. As such, every $v \in \mathcal{V}$ is contained in exactly $R$ blocks and moreover, for any fixed $r \in \mathcal{R}$, $B_r = \{b_{r,s} : s \in S\}$ forms a partition of $\mathcal{V}$. Also, every block $b_{r,s}$ contains the same number of points, namely the $K = q$ points that arise from the various choices of $t$.

Thus, in order to see that $B$ is a resolvable $(2, K, V)$-Steiner system over $\mathcal{V}$, all that remains to be shown is that any two distinct $v, v' \in V$ determine a unique block. This gets to the heart of an affine geometry over a finite field: the blocks are affine lines (21), which are determined by a nonzero direction vector $\gamma^{-1}$, which is only unique up to nonzero scalar multiples, along with an initial point $s \delta^{-1}$ which lies in some hyperplane. Any two distinct points $v, v'$ determine a direction $v' - v \neq 0$; so that $\{\gamma^r : r \in F_q\}$ represent every nonzero element of $F_{q^2 + 1}$ modulo scalar multiplication, we know there exists a unique $t_0 \in \mathcal{R}$ and $t_0 \in F_q$ such that $v' - v = t_0 \gamma^{-r_0}$. Moreover, for this particular $r$, we know there exists unique $s, s' \in S$ and $t, t' \in F_q$ such that $v' = s \gamma^{-r_0} \delta + t \gamma^{-r_0}$ and $v' = s' \gamma^{-r_0} \delta + t' \gamma^{-r_0}$, respectively. Combining these facts gives

$$v' = (v' - v) + v$$

$$= t_0 \gamma^{-r_0} + (s \gamma^{-r_0} \delta + t \gamma^{-r_0})$$

$$\delta^{-1} + (t + t_0) \gamma^{-r_0},$$

at which point the uniqueness gives $s' = s$ and $t' = t + t_0$. Since $s' = s$, these two points $v$ and $v'$ are contained in the same block $b_{r,s} = b_{r,s'}$. Also, this common block is unique: if $v, v' \in b_{r,s}$ are distinct, then $v' - v$ uniquely determines $r$; knowing $v$ and $s$, is always uniquely determined. We summarize these results in the following theorem, which is the second main result of this paper.

Theorem 2: Let $j \geq 1$ and let $q$ be a power of a prime $p$. Let $\mathcal{R} = \{0, \ldots, R-1\}$ where $R = (q^{j+1} - 1)/(q-1)$, and let $S$ be the hyperplane (14). Let $\{\xi_u\}_{u \in \mathcal{R}}$ be the characters of an Abelian group $\mathcal{G} = \{g_r\}_{r \in \mathcal{R}}$ and let $\gamma$ be a primitive element of $F_{q^2 + 1}$, whose additive group is denoted $\mathcal{V}$.

Then the harmonic ETF generated by the McFarland difference set (15), namely the vectors $\{\psi_u\}_{u \in \mathcal{R}}$ defined in (16), is an example of a Kirkman ETF constructed by Theorem 1. To be precise, taking any $\delta \in F_{q^2 + 1}$ such that $\text{tr}_{q/p}(\delta) = 1$, the blocks $B = \{b_{r,s} : r = 0, \ldots, R-1\}$ defined in (21) are a resolvable $(2, q, q^{j+1})$-Steiner system (affine geometry) which generates this same ETF, provided we let $f_u(r) = \chi_u(g_r)$ and $h_u(s) := \exp\left(\frac{2\pi i}{p} \text{tr}_{q/p}(s' \delta^{-1})\right)$.

We emphasize that this result tells us nothing new about the existence of ETFs. Rather, the significance of Theorem 2 is that it provides (sparse) Steiner ETF representations for one of the only two known flexible classes of harmonic ETFs; as we now describe, this has ramifications on the use of such ETFs for compressed sensing.

A. Kirkman ETFs and the Restricted Isometry Property

Given $L \leq M \leq N$ and $\delta < 1$, the vectors $\{\varphi_n\}_{n \in \mathcal{N}}$ in $\mathbb{C}^M$ have the $(L, \delta)$-Restricted Isometry Property (RIP) if for any $L$-element subset $L$ of $\mathcal{N}$, the eigenvalues of the Gram matrix of $\{\varphi_n\}_{n \in L}$ lie in $[1 - \delta, 1 + \delta]$. That is, for all such $L$, we want $|\mathbf{P}_L \mathbf{X} - \mathbf{I}_L|_2 \leq \delta$ where $\mathbf{P}_L := \sum_{n \in L} \mathbf{y}(n) \varphi_n$ is the corresponding restricted synthesis operator. In essence, an $(L, \delta)$-RIP matrix $\mathbf{P}$ has the property that any $L$-element subset of its columns are nearly orthonormal.

Though other paradigms exist, this property is undeniably central to compressed sensing. For a given $M$ and $N$, the goal is to design $\{\varphi_n\}_{n \in \mathcal{N}}$ so that it is $(L, \delta)$-RIP for $L$ being as large as possible, subject to the constraint that $\delta$ is sufficiently small compared to 1. To date, the most successful examples of such matrices are given by random matrix theory;
such random constructions typically yield matrices $\Phi$ that, with high probability, are $(L, \delta)$-RIP for $L$ on the order of $M/\text{polylog}(N)$. This is in stark contrast to nearly all deterministic constructions of such matrices which, with the exception of [3], are only provably $(L, \delta)$-RIP for $L$ on the order of $M^2$. In the compressed sensing literature, this is known as the square-root bottleneck. These facts are common knowledge, and are more thoroughly explained in [2].

For most deterministic constructions, it is unknown whether this bottleneck is due to a lack of good proof techniques or more seriously, is due to a fault in the construction itself. To be precise, the Gershgorin Circle Theorem gives

$$
\|\Phi_L^* \Phi_L - I\|_2 \leq \max_{n \in \mathcal{L}} \sum_{n' \in \mathcal{L}, n' \neq n} |\langle \varphi_n, \varphi_{n'} \rangle| \leq (L-1)\mu, \tag{22}
$$

where $\mu$ is the coherence of $\{\varphi_n\}_{n \in \mathcal{N}}$. To use this fact to prove that $\{\varphi_n\}_{n \in \mathcal{N}}$ is $(L, \delta)$-RIP, we thus want to choose $L$ such that $(L-1)\mu \leq \delta < 1$. In the case where $\{\varphi_n\}_{n \in \mathcal{N}}$ is a sequence of unit vectors with redundancy $\rho := N/M$, the Welch bound [2] then yields the bottleneck:

$$
L - 1 \leq \frac{\mu}{\rho} < \frac{(M(N-1))^{1/2}}{\rho} = \left(\frac{\rho M - 1}{\rho-1}\right)^{1/2} \leq \left(\frac{\rho}{\rho-1}\right)^{1/2} M^2. \tag{23}
$$

As such, in order to push beyond this bottleneck, we need to first find vectors for which the bounds in (22) are too coarse, and then find a better way for estimating the eigenvalues of the resulting submatrices. These are hard problems since the Gershgorin Circle Theorem, though easily proven, yields bounds which are surprisingly sharp.

Indeed, the bounds in (22) are good in the case where $\{\varphi_n\}_{n \in \mathcal{N}}$ is a Steiner ETF. To see this, recall from Section III that the Welch bound of any such ETF is $1/R$ and so (23) becomes $L - 1 < R$. That is, for any $L \leq R$, there exists $\delta < 1$ such that $\{\varphi_n\}_{n \in \mathcal{N}}$ is $(L, \delta)$-RIP. Remarkably, the converse of this fact is also true. To elaborate, note that if $\{\varphi_n\}_{n \in \mathcal{N}}$ is $(L, \delta)$-RIP for some fixed $L \leq M$ and $\delta < 1$, then at the very least, any $L$-element subset of $\{\varphi_n\}_{n \in \mathcal{N}}$ is linearly independent. This means its spark—the number of vectors in its smallest linearly dependent subcollection—is at least $L + 1$. However, as noted in [10], the spark of any Steiner ETF is at most $R + 1$. Indeed, in the special case where the underlying Steiner system is resolvable, note that for any fixed $v \in \mathcal{V}$, the subcollection $\{\varphi_{i,v} \}_{i \in \mathcal{O}}$ of the Steiner ETF [7] defined in Theorem 1 is only supported over the indices $(r,v)$ of those blocks $b_{r,v}$ which contain $v$. Since there are $R + 1$ such vectors but only $R$ such blocks, these vectors are necessarily linearly dependent. As such, if $\{\varphi_n\}_{n \in \mathcal{N}}$ is $(L, \delta)$-RIP for some $\delta < 1$ then $L + 1 \leq \text{spark}(\{\varphi_n\}_{n \in \mathcal{N}}) \leq R + 1$. In summary, a Steiner ETF is $(L, \delta)$-RIP for some $\delta < 1$ if and only if $L \leq R$.

We now combine this fact with the main results of this section and the previous one to prove, for the first time, that some harmonic ETFs are not good RIP matrices. Indeed, Theorem 2 states that every ETF generated by a McFarland difference set—one of only two known flexible constructions of harmonic ETFs—is, in fact, a Kirkman ETF. Moreover, Theorem 1 states that any Kirkman ETF can be obtained by applying a unitary transformation to a Steiner ETF; it is well known that such transforms preserve RIP. Together, we have:

**Corollary I:** If $\{\varphi_n\}_{n \in \mathcal{N}}$ is any Steiner or Kirkman ETF for $C^M$, then it is $(L, \delta)$-RIP for some $\delta < 1$ if and only if $L \leq R = \sqrt{(\rho M - 1)/(\rho - 1)}$, where $\rho = N/M$. In particular, it is impossible to surpass the square-root bottleneck using harmonic ETFs generated from McFarland difference sets.

It remains an open problem whether or not there exists an ETF which is a good RIP matrix for values of $L$ which are larger than numbers on the order of $M^2$. However, in light of Corollary 1 there is only one known flexible class of ETFs left to investigate, namely the harmonic ETFs generated by Singer difference sets. These difference sets are cyclic, meaning these ETFs are obtained by extracting $M = (q^j - 1)/(q - 1)$ rows from a standard DFT matrix of size $N = (q^{j+1} - 1)/(q - 1)$ where $j \geq 2$ and $q$ is some prime power. Here, there are some reasons for hope: when $j = 2$, such ETFs are numerically erasure-robust frames and as such, cannot be sparse in any basis [9]. Moreover, Singer harmonic ETFs are full spark—their spark is $M + 1$—when $N$ is prime, such as when $j = 2$ and $q = 2$. But even this can fail when $N$ is but a prime power, such as when $j = 4$ and $q = 3$ [1].

Also, there are inflexible families of non-Steiner ETFs whose RIP characteristics bear further study. One example of these are Paley ETFs which are constructed by modifying a quadratic-residue-based harmonic ETF into a redundancy-two ETF in the manner of [23]. There at least, spark is not the issue: any Paley ETF is $(L, \delta)$-RIP for all $L \leq M$ for some $\delta < 1$ [2]. However, it is unknown how this $\delta$ behaves as a function of $L$, $M$ and $N$; this is related to longstanding open problems regarding the clique numbers of Paley graphs [2]. These problems are nontrivial, and it is much easier to prove that a given set of vectors is not $(L, \delta)$-RIP than to prove that it is. Put simply, Corollary 1 does not tell you where to find good RIP matrices but rather, where not to look.

### V. HADAMARD ETFS AND THE GREY-RANKIN BOUND

In this section, we apply the results of Sections III and IV to produce new examples of certain types of optimal binary codes. An $(M, N)$-binary code is a set of $N$ codewords (vectors) in $Z_2^M$, that is, a sequence $\{c_n\}_{n=1}^N$ of $M \times 1$ vectors whose entries lie in $Z_2 := \{0, 1\}$. The distance of such a code is the minimum pairwise Hamming distance between any two codewords, namely $\text{dist}(\{c_n\}_{n=1}^N) := \min_{n \neq n'} d(c_n, c_{n'})$ where $d(c, c')$ counts the number of entries of $c, c' \in Z_2^M$ that differ. The Grey-Rankin bound [15] is an upper bound on the number of codewords $N$ one can have with a given distance $D$ in a space with given dimension $M$. We show that the Grey-Rankin bound is equivalent to a special case of the Welch bound, and then exploit this equivalence, using coding theory to prove new results in frame theory, and vice versa.

To be precise, the Grey-Rankin bound only applies to self-complementary codes, that is, codes in which the complement $c_{n+1}(m) := c_n(m) + 1 \mod 2$ of any codeword $c_n$ also lies in the code. In the work that follows, it is convenient for us to regard the second half of these $(M, 2N)$-binary codes as the complements of the first half, namely $c_{n+N} = c_n + 1$
for all $n = 1, \ldots, N$. Denoting the distance of such a code as $\Delta$, the Grey-Rankin bound states:

$$2N \leq \frac{8\Delta(M - \Delta)}{M - (M - 2\Delta)}$$

(24)

provided the right-hand side is positive; since the self-complementarity of the code guarantees that $2\Delta \leq M$, this positivity is equivalent to having $2\Delta > M - M^2$.

We rederive (24) by applying the Welch bound (2) to frames whose entries are all $\pm M^{-\frac{1}{2}}$. To be clear, we can exponentiate any codeword $c_n \in \mathbb{Z}_2^M$ to form a corresponding unit norm vector $\varphi_n \in \mathbb{R}^M$, $\varphi_n(m) := M^{-\frac{1}{2}}(-1)^{c_n(m)}$. Under this identification, the Euclidean distance between any two of these real vectors can be written in terms of the Hamming distance between their corresponding codewords:

$$\|\varphi_n - \varphi_{n'}\|^2 = \frac{1}{M} \sum_{m=1}^{M} (-1)^{c_n(m)} - (-1)^{c_{n'}(m)}$$

$$= \frac{1}{M} \sum_{m=1}^{M} \left\{\begin{array}{lcl}4, & c_n(m) \neq c_{n'}(m) \\0, & c_n(m) = c_{n'}(m)\end{array}\right.$$  

$$= \frac{2}{M} \|c_n - c_{n'}\|.$$  

We also have $\|\varphi_n - \varphi_{n'}\|^2 = 2(1 - \langle \varphi_n, \varphi_{n'} \rangle)$, and solving for the inner product gives

$$\langle \varphi_n, \varphi_{n'} \rangle = \frac{1}{2M} (M - 2 \|c_n - c_{n'}\|).$$

(25)

Grey himself used this identification in his derivation of (24). However, Grey’s argument relies on prior work by Rankin [20] concerning the packing of spherical caps, whereas we instead make use of Welch’s bound (2).

The mathematical novelty here is debatable: Rankin’s work is a forerunner to Welch’s bound. In fact, a little simplification reveals Equation 26 of [20] to be equivalent to the real-variable version of the Welch bound; since it predates [29] by nearly two decades, one can argue that (2) should be called the “Rankin-Welch” bound. From this perspective, our work below serves to modernize Grey’s original argument. This itself has value: unlike Rankin’s work, the Welch bound is widely studied. Also, as seen from (3), the Welch bound can be quickly proven from basic principles. Most importantly, by using the Welch bound to streamline Grey’s approach, we allow the large body of existing ETF/WE literature to be quickly and directly applied to open problems in coding theory.

Returning to the argument itself, taking the maximums of both sides of (26) over all $n, n' = 1, \ldots, 2N$, $n \neq n'$ gives

$$\max_{n, n' \in \{1, \ldots, 2N\}} \langle \varphi_n, \varphi_{n'} \rangle = \frac{M - 2\Delta}{M}.$$  

(26)

Moreover, the self-complementarity of the code $\{c_n\}_{n=1}^{2N}$ gives that $\varphi_{n+N} = -\varphi_n$ for all $n = 1, \ldots, N$. As such, the left-hand side of (26) can be rewritten as

$$\mu = \max_{n, n' \in \{1, \ldots, N\}} |\langle \varphi_n, \varphi_{n'} \rangle| = \frac{M - 2\Delta}{M}.$$  

(27)

At this point, the Welch bound (2) gives

$$\left(\frac{N - M}{M(N - 1)}\right)^2 \leq \frac{M - 2\Delta}{M}.$$  

(28)
are closed under addition. Moreover, it is known that a linear GRBE code with $N \geq 2$ must either have dimensions $M = 2j^2 + 1$ and $2N = 2j^2 + 2$ for some $j \geq 1$—meaning its corresponding ETF is a real-valued constant-amplitude regular simplex—or alternatively, dimensions $M = 2(2j^2 + 1)$ and $2N = 2j^2 + 3$ for some $j \geq 1$ [19]. As such, we find that:

**Corollary 2:** If there exists a real-valued harmonic ETF of $N$ vectors in an $M$-dimensional space with $M \geq 2$, then either (i) the ETF is a regular simplex of $N = 2j^2 + 1$ vectors for some $j \geq 1$ or (ii) the dimensions of the ETF are of the form (29).

This corollary illustrates how coding theory can be used to find new results in frame theory. However, from the point of view of coding theory itself, this corollary is disappointing: GRBE codes with these parameters are already known to exist [19]. It is here that the not-necessarily-harmonic ETFs of Theorem 1 truly shine: we can find Kirkman ETFs that lie outside the confines of Corollary 2; these ETFs are necessarily non-harmonic, and the resulting codes are necessarily nonlinear.

To be precise, in order to construct real-valued constant-amplitude ETFs using Theorem 1, we want both our unimodular regular simplex $\{ f_u \}_{u=1}^R$ as well as our unimodular orthogonal basis $\{ h_s \}_{s \in S}$ to be real-valued. Since such a simplex necessarily extends to a real unimodular orthogonal basis, we in particular want Hadamard matrices of size $R+1$ and $S = B/R = V/K$. It is well-known that this requires both $R+1$ and $V/K$ to either be 2 or divisible by 4; the Hadamard conjecture posits that these necessary conditions are sufficient. Also, recall that in order for a resolvable $(2, K, V)$-Steiner system to exist, we necessarily have $V \equiv K \mod (K - 1)$. Writing $V = W(K - 1) + K$ for some $W \geq 1$, we want

$$R + 1 = W(K - 1) + K$$

(30)

to either be 2 or divisible by 4. Note that since $W \geq 1$ and $K \geq 2$, we cannot have $R = 2$, and so this condition on $R$ is equivalent to having $W(K - 1) + 2 \equiv 0 \mod 4$. Meanwhile, if $W(K - 1) + 2$ then we necessarily have $K = 2$ and $W = 1$; the resulting $(2, 2, 4)$-Steiner system (4) yields the 6×32 code of Figure 3 as discussed above, linear GRBE codes with these parameters are well-known, and can be generated as McFarland harmonic ETFs, letting $j = 1$ in (29). As such, we also assume $W(K - 1) + 1 \equiv 0 \mod 4$. At this point, subtracting $W(K - 1) + 1$ from $WK + 2$, we see that these two necessary conditions are equivalent to having $W \equiv 3 \mod 4$ and $K \equiv 2 \mod 4$. Combining the above discussion with Theorems 1 and 5 gives the following result:

**Corollary 3:** Given $K \equiv 2 \mod 4$ and $W \equiv 3 \mod 4$, let $V = K[W(K - 1) + 1]$. Then both parameters in (30) are divisible by 4, and if there exist Hadamard matrices of these sizes and there also exists a resolvable $(2, K, V)$-Steiner system, then there exists a real-valued Kirkman ETF with

$$M = (W + 1)[W(K - 1) + 1]$$

$$N = K(W + 2)[W(K - 1) + 1]$$

meaning there exists a $(M, 2N)$-self-complementary code that achieves the Grey-Rankin bound (24).

To explore the consequences of this result, we first consider the case where $K = 2$. Recall from Section III that $(2, 2, V)$-Steiner systems are resolvable as a round-robin tournament for any even $V \geq 4$. Here, $V = K[W(K - 1) + 1] = 2W + 2$ for some $W \equiv 3 \mod 4$. To make the resulting Kirkman ETF real-valued, we want Hadamard matrices of size $R + 1 = 2W + 2 = V$ and $V/K = W + 1 = V/2$. It thus suffices for there to exist a Hadamard matrix of size $V/2$, since we can take the tensor product of it with the canonical Hadamard matrix of size 2 to form one of size $V$. When $V$ is a power of 2, the ETFs produced by Corollary 3 have the same dimensions as those produced by the real-valued harmonic ETFs of (29). However, when $V$ is not a power of 2, Corollary 2 tells us that these Kirkman ETFs cannot be harmonic. For example, letting $W = 11$ yields $V = 24$, and we know there exists a Hadamard matrix of size $V/2 = 12$. As such, there exists a real-valued $276 \times 576$ Kirkman ETF that cannot be harmonic, and the resulting $276 \times 1152$ GRBE code is not linear. There are an infinite number of nonharmonic Kirkman ETFs of this type:

at the very least, Paley’s quadratic-residue based construction of Hadamard matrices gives the existence of such an ETF whenever $V \equiv 3 \mod 4$ is a prime power.

For $K \equiv 2 \mod 4$ such that $K > 2$, the true implications of Corollary 3 are harder to ascertain. We could not find any explicit infinite families of resolvable $(2, K, V)$-Steiner systems for such values of $K$ in the literature. For $K = 6$ in particular, we need $V \equiv 6 \mod 30$; it is known [11] that a resolvable $(2, 6, V)$-Steiner system (i) does not exist for $V = 36$, (ii) may or may not exist for $V = 66$ and $V = 96$, (iii) does exist for $V = 126$ (unital design), $V = 156$ (projective geometry) and $V = 186$. Unfortunately, the only one of these values of $V$ that satisfies the hypotheses of Corollary 3 is $V = 96$. If
a resolvable 
\((2, 6, 96)\)-Steiner system does exist it would, to
our knowledge, give the first example of a GRBE code whose
redundancy \(2N/M = 3840/304\) is not approximately 4. For
\(K = 10\) and larger, the minimum corresponding \(V\) which
could satisfy the assumptions of Corollary \([3]\) is \(V = 280\),
which lies beyond the range of the tables of known resolvable
designs we encountered.

At this point, we turn to asymptotic existence results. Recall
that for any \(K\), there exists \(V_0(K)\) such that for all \(V \geq V_0(K)\)
with \(V \equiv K \mod K - 1\), there exists a resolvable
\((2, K, V)\)-Steiner system \([22]\). As such, if the Hadamard
conjecture is true, then for any \(K \equiv 2 \mod 4\), there exists
\(W_0(K)\) such that for all \(W \geq W_0(K)\) with \(W \equiv 3 \mod 4\),
there exists a real-valued Kirkman ETF whose parameters are
given by Corollary \([3]\). In particular, if the Hadamard conjecture
is true, for any \(K \equiv 2 \mod 4\) there exists real-valued constant-
amplitude ETFs and GRBE codes whose redundancies are
approximately \(K\) and \(2K\), respectively.

VI. CONCLUSIONS AND FUTURE WORK

We now have a method for transforming certain Steiner
ETFs into constant-amplitude ETFs, as desired for certain
waveform design applications. We also now know that an
important class of previously discovered harmonic ETFs arise
in this fashion, making them less attractive for deterministic
compressed sensing. Finally, we have seen how the problem
of constructing a real-valued constant-amplitude ETF is equiva-
 lent to that of constructing a type of optimal binary code,
allowing us to apply results from one area to the other. Several
important questions remain open: To what degree do Singer
harmonic ETFs satisfy RIP? Are Denniston Kirkman ETFs
harmonic? More generally, can we use these results along with
ideas from resolvable designs to build new examples of
difference sets, or vice versa? Do there exist nontrivial
real-valued constant-amplitude ETFs whose redundancy is not
essentially 2? Equivalently, do there exist nontrivial GRBE
codes whose redundancy is not essentially 4?

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