EXISTENCE CRITERIA OF GROUND STATE SOLUTIONS FOR SCHRÖDINGER-POISSON SYSTEMS WITH A VANISHING POTENTIAL

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Abstract. In this paper, we consider the following Schrödinger-Poisson system
\[
\begin{aligned}
-\Delta u + u + K(x)\phi(x)u &= a(x)|u|^{p-2}u, \quad x \in \mathbb{R}^3, \\
-\Delta \phi &= K(x)u^2, \quad x \in \mathbb{R}^3,
\end{aligned}
\]
where \( p \in [4, 6) \), \( a(x) \geq \lim_{|x| \to \infty} a(x) = a_{\infty} > 0 \) and \( \lim_{|x| \to \infty} K(x) = 0 \).
Lack of any symmetry property of \( a \) and \( K \), we present some new sufficient conditions to guarantee the existence of a positive ground state solution of above system. Our results extend and complement the ones of [G. Cerami, G. Vaira, J. Differential Equations 248 (2010)] in which \( p \in (4, 6) \), \( a \) and \( K \) need to satisfy additional integrability conditions.

1. Introduction. This paper deals with the existence of ground state solutions for the following Schrödinger-Poisson system:
\[
\begin{aligned}
-\Delta u + u + K(x)\phi(x)u &= a(x)|u|^{p-2}u, \quad x \in \mathbb{R}^3, \\
-\Delta \phi &= K(x)u^2, \quad x \in \mathbb{R}^3,
\end{aligned}
\]  
(System (SP))
where \( p \in [4, 6) \), \( a : \mathbb{R}^3 \to \mathbb{R} \) and \( K : \mathbb{R}^3 \to \mathbb{R} \) satisfy the following assumptions:
(A) \( a(x) \geq \lim_{|x| \to \infty} a(x) = a_{\infty} > 0 \), \( \forall x \in \mathbb{R}^3 \) and \( a(x) - a_{\infty} > 0 \) on a positive measure set;
(K) \( K \in L^\infty(\mathbb{R}^3) \), \( 0 \leq K(x) \leq K_{\infty} \), \( K(x) \neq 0 \) and \( \lim_{|x| \to \infty} K(x) = 0 \).
System (SP), also known as the nonlinear Schrödinger-Maxwell system, was first introduced by Benci and Fortunato [5] as a model describing solitary waves for the nonlinear stationary Schrödinger equations interacting with the electrostatic field.

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It has a strong physical meaning because it appears in quantum mechanics models (see e.g. [7, 8, 18]) and in semiconductor theory [6, 21, 22].

In recent years there has been a large amount of work dealing with the existence of positive solutions, ground state solutions, multiple solutions and semiclassical states for systems like (SP) via variational methods. It is well known that the Poisson equation is solved by using Lax-Milgram theorem, so, for all states for systems like (SP) via variational methods. It is well known that the

\begin{equation}
-\triangle u + u + K(x)\phi_u(x)u = a(x)|u|^{p-2}u,
\end{equation}

see Section 2 for more details. Moreover, (1.1) is variational and its solutions are the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

\begin{equation}
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\,dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,dx - \frac{1}{p} \int_{\mathbb{R}^3} a(x)|u|^p\,dx.
\end{equation}

To overcome the lack of compactness of Sobolev embeddings $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ for $2 < q < 6$, many early studies were devoted to the autonomous case or to the case in which the coefficients appearing in (SP) are supposed to be radial or coercive (see e.g. [2–4, 24]); some recent contributions to (SP) have also been given looking at cases in which the coefficients satisfy periodic conditions or asymptotically periodic conditions with suitable decaying limitations (see e.g. [1, 11, 13–15, 29, 30, 32]). Note that the greatest part of previous literature focuses on the study of system (SP) with $\inf_{x \in \mathbb{R}^3} K(x) > 0$. As far as we know, there seems to be only one paper [10] considering (SP) with a vanishing potential $K$ satisfying (K), in which $p \in (4, 6)$, moreover $a$ and $K$ satisfy (A), (K) and the following integrability conditions:

(A') $a - a_\infty \in L^{6/(6-p)}(\mathbb{R}^3)$;

(K') $K \in L^2(\mathbb{R}^3)$.

In order to state clearly the results of [10], we denote by $w$ and $w_\alpha$ positive ground state solutions of the following Schrödinger equations

\begin{equation}
-\Delta u + u = a_\infty |u|^{p-2}u,
\end{equation}

and

\begin{equation}
-\Delta u + u = a(x)|u|^{p-2}u,
\end{equation}

whose energies are

\begin{equation}
m_\infty = \left(\frac{1}{2} - \frac{1}{p}\right) a_\infty \int_{\mathbb{R}^3} |w|^p\,dx \quad \text{and} \quad m_\alpha = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^3} a(x)|w_\alpha|^p\,dx,
\end{equation}

respectively. Lack of symmetry assumptions, Cerami and Vaira in [10] located the levels of $I$ in which the Palais-Smale condition can fail, giving a representation theorem for the Palais-Smale sequences, and showing that the only obstacles to the compactness are the solutions of the problem at infinity (NS$_\infty$), and based on a minimization argument constrained on the Nehari manifold, proved that (SP) has a positive ground state solution $u \in H^1(\mathbb{R}^3)$ such that $I(u) = \inf_{w \in N} I > 0$ provided

\begin{equation}
\|K\|_2^2 < \frac{(p - 2)S^4S^2}{2p} \frac{(m_\infty^{(p-4)/p} - m_\alpha^{(p-4)/p})}{(2m_\alpha^{(2p-4)/p})}
\end{equation}

or

\begin{equation}
\int_{\mathbb{R}^3} K(x)\phi_u(x)w^2\,dx < \frac{4}{p} \int_{\mathbb{R}^3} [a(x) - a_\infty]|w|^p\,dx.
\end{equation}
where
\[ \mathcal{N} = \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \langle I'(u), u \rangle = 0 \right\}, \tag{1.6} \]

\( S \) and \( \bar{S} \) are the best constants for the embedding of \( H^1(\mathbb{R}^3) \) and \( D^{1,2}(\mathbb{R}^3) \) in \( L^6(\mathbb{R}^3) \), respectively. It is worth mentioning that integrability conditions \((A')\) and \((K')\) play crucial roles in showing the convergence of the minimizing sequences of \( \inf_{\mathcal{N}} I \), moreover, the method used in [10] is only available for the case \( 4 < p < 6 \), and does not work for \( p = 4 \).

Motivated by [10], we shall remove integrability conditions \((A')\) and \((K')\), when \( 4 \leq p < 6 \), \( a \) and \( K \) satisfy \((A)\) and \((K)\), we present some sufficient conditions to guarantee the existence a ground state solution of \((SP)\). Different from the minimization argument of [10], based on the non-Nehari manifold approach developed by Tang [27, 28], we find a minimizing Cerami sequence for \( I \) on \( \mathcal{N} \), and overcome the difficulties due to the lack of compactness by introducing some refined estimates instead of using the global compactness lemma of [10].

**Theorem 1.1.** Let \( 4 < p < 6 \), \((A)\) and \((K)\) hold. Furthermore assume either
\[ K^2_{\infty} < \frac{3 \sqrt{\pi} (p - 2) S_0^4}{16 \sqrt{2} p m_a} \left( \frac{m_{\infty}}{m_a} \right)^{(p - 4)/p} - 1, \tag{1.7} \]

where \( S_0 = \inf_{H^1(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|_{L^{2/\alpha}}^2}{\|u\|_{L^p}^p} \), \( m_{\infty} \) and \( m_a \) are given by (1.3); or
\[ \int_{\mathbb{R}^3} K(x) \phi_w(x) w^2 \, dx \leq \theta_0 a_{\infty} \int_{\mathbb{R}^3} |w|^p \, dx, \tag{1.8} \]

where \( w \) is a positive solution of \((NS_{\infty})\) and \( \theta_0 \) satisfies
\[ (1 + \theta_0) \left[ 1 + \frac{p - 4}{2(p - 2)} \theta_0 \right]^{(p - 2)/2} = \frac{1}{a_{\infty} \|w\|_p} \int_{\mathbb{R}^3} a(x) |w|^p \, dx. \tag{1.9} \]

Then \((SP)\) has a positive ground state solution \( u_0 \in H^1(\mathbb{R}^3) \) such that \( I(u_0) = \inf_{\mathcal{N}} I > 0 \).

**Theorem 1.2.** Let \( p = 4 \), \((A)\) and \((K)\) hold. Furthermore assume either
\[ \int_{\mathbb{R}^3} K(x) \phi_{w_a}(x) w_a^2 \, dx < \int_{\mathbb{R}^3} a(x) w_a^4 \, dx \quad \text{and} \quad K^2 \leq \frac{3 \sqrt{\pi} S_0^4}{8 \sqrt{2} \left( \frac{1}{m_a} - \frac{1}{m_{\infty}} \right)}; \tag{1.10} \]

or
\[ \int_{\mathbb{R}^3} K(x) \phi_w(x) w^2 \, dx < \int_{\mathbb{R}^3} [a(x) - a_{\infty}] w^4 \, dx, \tag{1.11} \]

where \( w_a \) and \( w \) are positive ground state solutions of \((NS_a)\) and \((NS_{\infty})\), respectively, \( m_{\infty} \) and \( m_a \) are given by (1.3). Then \((SP)\) has a positive ground state solution \( u_0 \in H^1(\mathbb{R}^3) \) such that \( I(u_0) = \inf_{\mathcal{N}} I > 0 \).

**Corollary 1.3.** Let \( 4 \leq p < 6 \). Suppose that \( a \) and \( K \) satisfy the following assumptions:

\((A1)\) \( a \in L^\infty(\mathbb{R}^3) \), \( \lim_{|x| \to \infty} a(x) = a_{\infty} > 0 \), and there exist \( \alpha_2 \in (0, 4) \) and \( \beta_2 > 0 \) such that
\[ a(x) \geq a_{\infty} + \beta_2 e^{-\alpha_2 |x|}, \quad \forall \ x \in \mathbb{R}^3; \]

\((K1)\) \( K \in L^\infty(\mathbb{R}^3) \), and there exist \( \alpha_1 > 2 \) and \( \beta_1 > 0 \) such that
\[ 0 \leq K(x) \leq \beta_1 e^{-\alpha_1 |x|}, \quad \forall \ x \in \mathbb{R}^3. \]

Then \((SP)\) has a positive ground state solution \( u_0 \in H^1(\mathbb{R}^3) \) such that \( I(u_0) = \inf_{\mathcal{N}} I > 0 \).
Throughout the paper we make use of the following notations:
• \(L^s(\mathbb{R}^3)(1 \leq s < \infty)\) denotes the Lebesgue space with the norm \(\|u\|_s = (\int_{\mathbb{R}^3}|u|^s dx)^{1/s}\);
• \(H^1(\mathbb{R}^3)\) denotes the usual Sobolev space equipped with the inner product and norm \((u,v) = \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + uv)dx, \|u\| = (u,u)^{1/2}, \forall u,v \in H^1(\mathbb{R}^3)\);
• For any \(x \in \mathbb{R}^3\) and \(r > 0\), \(B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}\);
• \(C_1, C_2, \cdots\) denote positive constants possibly different in different places.

### 2. Preliminaries

It is well known that (SP) can be reduced to a single equation with a non-local term. Namely, for any \(Ku^2 \in L^1_{loc}(\mathbb{R}^3)\) such that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dx dy < \infty,
\]
the distribution solution
\[
\phi_u(x) = \int_{\mathbb{R}^3} \frac{K(x)u^2(y)}{|x-y|} dy = \frac{1}{|x|} * Ku^2
\]
\[(2.1)\]
of the Poisson equation
\[-\Delta \phi = Ku^2, \quad x \in \mathbb{R}^3
\]
belongs to \(D^{1,2}(\mathbb{R}^3)\) and is the unique weak solution in \(D^{1,2}(\mathbb{R}^3)\) (see e.g. [25] for more details), and
\[
\int_{\mathbb{R}^3} \nabla \phi_u \cdot \nabla v dx = \int_{\mathbb{R}^3} K(x)u^2v dx, \quad \forall v \in H^1(\mathbb{R}^3), \tag{2.2}
\]
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(x)K(y)}{|x-y|} u^2(x)u^2(y) dx dy = \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 dx. \tag{2.3}
\]
Moreover, \(\phi_u(x) > 0\) for \(u \neq 0\) under (K). By using Hardy-Littlewood-Sobolev inequality (see [19] or [20, page 98]), we have the following inequality:
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)v(y)|}{|x-y|} dx dy \leq \frac{8\sqrt{2}}{3\sqrt{\pi}}\|u\|_{6/5}\|v\|_{6/5}, \quad u,v \in L^{6/5}(\mathbb{R}^3). \tag{2.4}
\]

Formally, the solutions of (SP) are then the critical points of the reduced functional \(I\). Under assumptions (A) and (K), \(I\) is a well-defined class \(C^1\) functional, and that
\[
\langle I'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx + \int_{\mathbb{R}^3} [K(x)\phi_u(x)u - a(x)|u|^{p-2}u] v dx. \tag{2.5}
\]

Hence if \(u \in H^1(\mathbb{R}^3)\) is a critical point of \(I\), then the pair \((u,\phi_u)\) is a solution of (SP). For the sake of simplicity, in many cases we just say \(u \in H^1(\mathbb{R}^3)\), instead of \((u,\phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\), is a weak solution of (SP).

**Lemma 2.1.** [12, Lemma 2.1] Assume that \(4 \leq p < 6\), (A) and (K) hold. Then
\[
I(u) \geq I(tu) + \frac{1-t^4}{4}\langle I'(u), u \rangle + \frac{(1-t^2)^2}{4}\|u\|^2, \quad \forall u \in H^1(\mathbb{R}^3), \quad t \geq 0. \tag{2.6}
\]

**Corollary 2.2.** Assume that \(4 \leq p < 6\), (A) and (K) hold. Then
\[
I(u) = \max_{t \geq 0} I(tu), \quad \forall u \in \mathcal{N}. \tag{2.7}
\]
Lemma 2.3. Assume that (A) and (K) hold.

(i) If \(4 < p < 6\), then for any \(u \in H^1(\mathbb{R}^3) \setminus \{0\}\), there exists \(t_u > 0\) such that \(t_u u \in \mathcal{N}\);

(ii) If \(p = 4\), then for any \(u \in \Lambda\), there exists \(t_u > 0\) such that \(t_u u \in \mathcal{N}\).

Proof. By a standard argument, we can deduce that (i) holds. Here, we just prove (ii) holds. First, we show that \(\Lambda \neq \emptyset\). From (2.4) and Sobolev imbedding theorem, there exists \(C_1 > 0\) such that \(\int_{\mathbb{R}^3} K(x) \phi_u u^2 dx \leq C_1 \|u\|^4\) for all \(u \in H^1(\mathbb{R}^3)\). For any fixed \(u \in H^1(\mathbb{R}^3) \setminus \{0\}\), set \(u_t(x) = u(tx)\) for \(t > 0\). Since \(a\) and \(K\) are bounded, we have

\[
\int_{\mathbb{R}^3} [K(x)\phi(u_t)u_t^2 - a(x)|u_t|^4] \, dx \leq C_1 t^{-1} \|u\|^4 - a_\infty t \|u\|^4, \tag{2.8}
\]

which implies

\[
\int_{\mathbb{R}^3} [K(x)\phi(u_t)u_t^2 - a(x)|u_t|^4] \, dx \to -\infty, \quad \text{as} \quad t \to +\infty.
\]

Thus, taking \(v = T u_\Gamma\) for \(T\) large, we have \(v \in \Lambda\). Hence, \(\Lambda \neq \emptyset\), moreover, by (2.5), one has \(\mathcal{N} \subset \Lambda\). Next, let \(u \in \Lambda\) be fixed and define a function \(g(t) := \langle I'(tu), tu \rangle\) on \([0, \infty)\). It is easy to see that \(g(0) = 0\), \(g(t) > 0\) for \(t > 0\) small and \(g(t) < 0\) for \(t\) large due to \(u \in \Lambda\). Therefore, there exists \(t_u > 0\) so that \(g(t_u) = 0\) and \(t_u u \in \mathcal{N}\).

From Corollary 2.2 and Lemma 2.3, we can obtain the following lemma.

Lemma 2.4. Assume that (A) and (K) hold.

(i) If \(4 < p < 6\), then \(c = \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t \geq 0} I(tu) > 0\).

(ii) If \(p = 4\), then \(c = \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in \Lambda} \max_{t \geq 0} I(tu) > 0\).

Similar to the proof of [12, Lemma 2.5], we have the following lemma.

Lemma 2.5. Assume that \(4 \leq p < 6\), (A) and (K) hold. Then there exist a constant \(c_* \in (0, c)\) and a sequence \(\{u_n\} \subset H^1(\mathbb{R}^3)\) such that

\[
I(u_n) \to c_*, \quad \|I'(u_n)\|(1 + \|u_n\|) \to 0. \tag{2.9}
\]

To find a ground state solution of (SP), we define the energy functionals on \(H^1(\mathbb{R}^3)\) of (NS)_a and (NS)_\infty as follows:

\[
I_a(u) = \frac{1}{2} \|u\|^2 - \frac{1}{p} \int_{\mathbb{R}^3} a(x)|u|^p \, dx \tag{2.10}
\]

and

\[
I_\infty(u) = \frac{1}{2} \|u\|^2 - \frac{a_\infty}{p} \int_{\mathbb{R}^3} |u|^p \, dx, \tag{2.11}
\]

and denote the corresponding Nehari manifolds by

\[
\mathcal{N}_a = \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \|u\|^2 = \int_{\mathbb{R}^3} a(x)|u|^p \, dx \right\} \tag{2.12}
\]

and

\[
\mathcal{N}_\infty = \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : \|u\|^2 = a_\infty \|u\|^p \right\}. \tag{2.13}
\]
We state, in the two following propositions, some known results about the existence of positive solutions of (NS$_\infty$) and (NS$_{a}$) that are useful in the sequel.

**Proposition 1.** ([16, 17]) (NS$_\infty$) has a positive, ground state, solution $w \in \mathcal{N}_\infty$, radially symmetric about the origin, unique up to translations, decaying exponentially, together its derivatives, as $|x| \to +\infty$. Furthermore, for any $\varepsilon > 0$, there exist constants $A_\varepsilon > 0$ and $B_0 > 0$ such that

$$A_\varepsilon e^{-(1+\varepsilon)|x|} \leq w(x) \leq B_0 e^{-|x|}, \quad \forall \ x \in \mathbb{R}^3$$

and

$$I_\infty(w) = m_\infty = \inf \{ I_\infty(u) : u \in \mathcal{N}_\infty \}.$$ 

**Proposition 2.** ([9, 10]) Let (A) holds. Then (NS$_{a}$) has a positive, ground state, solution $w_a \in \mathcal{N}_a$. Furthermore,

$$I_a(w_a) = m_a = \inf \{ I_a(u) : u \in \mathcal{N}_a \} > m_\infty.$$ 

(2.14)

### 3. Proof of the results

In this section, we give the proofs of Theorems 1.1 and 1.2 and Corollary 1.3.

**Proof of Theorem 1.1.** From Lemma 2.5, there exists $\{ u_n \} \subset H^1(\mathbb{R}^3)$ such that

$$I(u_n) \to c^* \in (0, c], \quad ||I'(u_n)||((1 + ||u_n||) \to 0.$$ 

(3.1)

By (3.1), one has

$$c^* + o(1) = I(u_n) - \frac{1}{4} \langle I'(u_n), u_n \rangle \geq \frac{1}{4} ||u_n||^2,$$

which implies $\{ u_n \}$ is bounded in $H^1(\mathbb{R}^3)$. Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$. Next, we prove $\bar{u} \neq 0$.

Arguing by contradiction, suppose that $\bar{u} = 0$, i.e. $u_n \to 0$ in $H^1(\mathbb{R}^3)$, and so $u_n \to 0$ in $L^s_{loc}(\mathbb{R}^3)$, $2 \leq s < 6$ and $u_n \to 0$ a.e. in $\mathbb{R}^3$. Note that

$$I_\infty(u) = I(u) - \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{u}(x)u^2 dx + \frac{1}{p} \int_{\mathbb{R}^3} [a(x) - a_\infty]|u|^p dx$$

and

$$\langle I'_\infty(u), v \rangle = \langle I'(u), v \rangle - \int_{\mathbb{R}^3} K(x)\phi_{u}(x)uv dx + \int_{\mathbb{R}^3} [a(x) - a_\infty]|u|^{p-2}uv dx, \quad \forall \ v \in H^1(\mathbb{R}^3).$$

(3.2)

(3.3)

By (A), (K), (3.1), (3.2) and (3.3), one has

$$I_\infty(u_n) \to c^*, \quad ||I'_\infty(u_n)||((1 + ||u_n||) \to 0.$$ 

(3.4)

By a standard argument, we can prove that there exists $\{ k_n \} \subset \mathbb{Z}^3$, going if necessary to a subsequence, such that

$$\int_{B_2(k_n)} |u_n|^2 dx > \frac{\delta}{2} > 0.$$ 

Let us define $v_n(x) = u_n(x + k_n)$ so that

$$\int_{B_2(0)} |v_n|^2 dx > \frac{\delta}{2}$$

(3.5)

Then

$$I_\infty(v_n) \to c, \quad ||I'_\infty(v_n)||((1 + ||v_n||) \to 0.$$ 

(3.6)
From (3.5), there exists \( \bar{v} \in H^1(\mathbb{R}^3) \setminus \{0\} \) such that, up to a subsequence, \( v_n \to \bar{v} \) in \( H^1(\mathbb{R}^3) \), \( v_n \to \bar{v} \) in \( L^p_{\text{loc}}(\mathbb{R}^3) \), \( 2 \leq s < 6 \) and \( v_n \to \bar{v} \) a.e. in \( \mathbb{R}^3 \). A standard argument shows that \( I_{\infty}^* (\bar{v}) = 0 \) and \( I_{\infty} (\bar{v}) \leq c \). Since \( \bar{v} \in \mathcal{N}_\infty \), one has \( c \geq I_{\infty} (\bar{v}) \geq m_\infty \).

First assume that (1.7) holds. Let \( w_a \) be given by Proposition 1. Applying Lemma 2.3, there exists \( t_0 > 0 \) such that \( t_0 w_a \in \mathcal{N} \), and so \( \langle I'(t_0 w_a), t_0 w_a \rangle = 0 \) and \( I(t_0 w_a) \geq c \). By (2.5) and (2.12), we have

\[
0 = t_0^2 \|w_a\|^2 + t_0^4 \int_{\mathbb{R}^3} K(x) \phi_{w_a}(x) w_a^2 \, dx - t_0^p \int_{\mathbb{R}^3} a(x) |w_a|^p \, dx
\]

which implies

\[
1 \leq t_0 \leq \left[ 1 + \frac{\int_{\mathbb{R}^3} K(x) \phi_{w_a}(x) w_a^2 \, dx}{\|w_a\|^2} \right]^{1/(p-4)}. \tag{3.7}
\]

By (2.3) and (2.4), one has

\[
\int_{\mathbb{R}^3} K(x) \phi_{w_a}(x) w_a^2 \, dx \leq C_0 K_\infty^2 \|w_a\|^4_{12/5} \leq C_0 K_\infty^2 S_0^{-4} \|w_a\|^4,
\tag{3.9}
\]

where \( C_0 = 8 \sqrt[3]{2}/3 \sqrt[4]{3} \). Substituting (3.9) into (3.8) and using (2.14), one obtain

\[
t_0 \leq \left[ 1 + C_0 K_\infty^2 S_0^{-4} \|w_a\|^2 \right]^{1/(p-4)} = \left[ 1 + 2pC_0 K_\infty^2 m_a \frac{t_0^p m_a}{(p - 2) S_0^4} \right]^{1/(p-4)}. \tag{3.10}
\]

In view of (1.2), (2.12) and (3.7), we have

\[
c \leq I(t_0 w_a) = \frac{t_0^2}{2} \|w_a\|^2 + \frac{t_0^4}{4} \int_{\mathbb{R}^3} K(x) \phi_{w_a}(x) w_a^2 \, dx - \frac{t_0^p}{p} \int_{\mathbb{R}^3} a(x) |w_a|^p \, dx
\]

\[
= \frac{t_0^2}{4} \|w_a\|^2 + \left( \frac{1}{4} - \frac{1}{p} \right) t_0^p \int_{\mathbb{R}^3} a(x) |w_a|^p \, dx
\]

\[
\leq t_0^p \left( \frac{1}{2} - \frac{1}{p} \right) \|w_a\|^2 = t_0^p m_a. \tag{3.11}
\]

Then it follows from (1.7), (3.10) and (3.11) that

\[
m_\infty \leq c \leq t_0^p m_a \leq \left[ 1 + \frac{2pC_0 K_\infty^2 m_a}{(p - 2) S_0^4} \right]^{p/(p-4)} m_a < m_\infty. \tag{3.12}
\]

This contradiction shows that \( u \neq 0 \).

Next, assume (1.8) holds. Let \( w \) be given by Proposition 1,

\[
\alpha = \|w\|^2 = a_\infty \int_{\mathbb{R}^3} |w|^2 \, dx, \quad \beta = \int_{\mathbb{R}^3} a(x) |w|^p \, dx, \quad \gamma = \int_{\mathbb{R}^3} K(x) \phi_w(x) w^2 \, dx.
\]

Then (1.8) implies that \( \gamma \leq \theta_0 \alpha \). Hence, it follows from (1.10) that

\[
\left( 1 + \frac{2}{\alpha} \right) \left[ 1 + \frac{p - 4}{2(p - 2)} \gamma \right] \frac{(p-2)/2}{\alpha} \leq \frac{\beta}{\alpha},
\]

which yields

\[
\alpha + \gamma < \beta \tag{3.13}
\]

and

\[
\left( \frac{\alpha + \gamma}{\beta} \right)^{2/(p-2)} \left[ \frac{\alpha}{4} + \left( \frac{1}{4} - \frac{1}{p} \right) (\alpha + \gamma) \right] \leq \left( \frac{1}{2} - \frac{1}{p} \right) \alpha. \tag{3.14}
\]
Note that \( w \) is a positive solution of \( \text{(NS}_\infty \text{)} \) satisfying \( m_\infty = \left(\frac{1}{2} - \frac{1}{p}\right) \alpha \). Applying Lemma 2.3, there exists \( t = t_w > 0 \) such that \( tw \in \mathcal{N} \), and so \( \langle I'(tw), tw \rangle = 0 \) and \( I(tw) \geq c \). From (2.5), we have

\[
0 = \tilde{t}^2 \|w\|^2 + \tilde{t}^4 \int_{\mathbb{R}^3} K(x) \phi_{w_a}(x) w^2 dx - \tilde{t}^p \int_{\mathbb{R}^3} a(x) |w|^p dx = \alpha \tilde{t}^2 + \gamma \tilde{t}^4 - \beta \tilde{t}^p. \tag{3.15}
\]

Combining (3.13) with (3.15), we have

\[
0 < \tilde{t} \leq \left(\frac{\alpha + \gamma}{\beta}\right)^{1/(p-2)} < 1. \tag{3.16}
\]

Then it follows from (1.2), (1.8), (3.15) and (3.16) that

\[
m_\infty \leq c \leq I(\tilde{t}w) = \frac{\tilde{t}^2}{2} \|w\|^2 + \frac{\tilde{t}^4}{4} \int_{\mathbb{R}^3} K(x) \phi_{w_a}(x) w^2 dx - \frac{\tilde{t}^p}{p} \int_{\mathbb{R}^3} a(x) |w|^p dx = \frac{\tilde{t}^2}{4} \|w\|^2 + \left(\frac{1}{4} - \frac{1}{p}\right) \beta \tilde{t}^p = \frac{1}{4} \alpha \tilde{t}^2 + \left(\frac{1}{4} - \frac{1}{p}\right) (\alpha \tilde{t}^2 + \gamma \tilde{t}^4) \leq \left(\frac{\alpha + \gamma}{\beta}\right)^{2/(p-2)} \left[\frac{\alpha}{4} + \left(\frac{1}{4} - \frac{1}{p}\right) (\alpha + \gamma)\right] \leq \left(\frac{1}{2} - \frac{1}{p}\right) \alpha = m_\infty. \tag{3.17}
\]

This contradiction shows that \( \bar{u} \neq 0 \).

It is easy to check that \( I'(\bar{u}) = 0 \) and \( I(\bar{u}) = c = \inf_{\mathcal{N}} I \). Moreover, by a standard argument [23, 31], we can deduce that \( \bar{u} > 0 \). Hence, \( \bar{u} \in H^1(\mathbb{R}^3) \) is a positive ground state solution of \( \text{(SP)} \) with \( I(\bar{u}) = \inf_{\mathcal{N}} I > 0 \).

**Proof of Theorem 1.2.** As in the proof of Theorem 1.1, there exists a bounded sequence \( \{u_n\} \subset H^1(\mathbb{R}^3) \) such that (3.1) holds. Passing to a subsequence, we have \( u_n \rightharpoonup \bar{u} \) in \( H^1(\mathbb{R}^3) \). Next, we prove \( \bar{u} \neq 0 \). Arguing by contradiction, suppose that \( \bar{u} = 0 \). In the same way as that of Theorem 1.1, we have \( c \geq m_\infty \).

First assume that (1.10) holds. Let \( w_a \) be given by Proposition 2. Since \( w_a \in \Lambda \), applying Lemma 2.3, there exists \( t_0 = t_{w_a} > 0 \) such that \( t_0 w_a \in \mathcal{N} \), and so \( \langle I'(t_0 w_a), t_0 w_a \rangle = 0 \) and \( I(t_0 w_a) \geq c \). By (2.5) and (2.12), we have

\[
0 = t_0^2 \|w_a\|^2 + t_0^4 \int_{\mathbb{R}^3} K(x) \phi_{w_a}(x) w_a^2 dx - t_0^4 \int_{\mathbb{R}^3} a(x) |w_a|^4 dx = t_0^2 (1 - t_0^2) \|w_a\|^2 + t_0^4 \int_{\mathbb{R}^3} K(x) \phi_{w_a}(x) w_a^2 dx. \tag{3.18}
\]

which, together with (3.9), implies

\[
t_0 = \frac{\|w_a\|^2}{\|w_a\|^2 - \int_{\mathbb{R}^3} K(x) \phi_{w_a} w_a^2 dx} \leq \frac{\|w_a\|^2}{\|w_a\|^2 - C_0 K_\infty^{-1} S_0^{-1} \|w_a\|^4}. \tag{3.19}
\]
In view of (1.2), (2.12) and (3.18), we have
\[
  c \leq I(t_0w_a) = \frac{t_0^2}{2} \|w_a\|^2 + \frac{t_0^4}{4} \int_{\mathbb{R}^3} K(x)\phi_{\omega a}(x)w_a^2\,dx - \frac{t_0^4}{4} \int_{\mathbb{R}^3} a(x)|w_a|^4\,dx \\
  = \frac{t_0^2}{2} \|w_a\|^2 = t_0^4 m_a.
\] (3.20)

Then it follows from (1.10) and (3.20) that
\[
m_\infty \leq c \leq t_0^4 m_a \leq \frac{m_a}{1 - C_0 K_2 S_0^{-4} m_a} < m_\infty.
\] (3.21)

This contradiction shows that \( \tilde{u} \neq 0 \).

Next, assume (1.11) holds. Let \( w \) be given by Proposition 1,
\[
  \alpha_0 = \|w\|^2 = a_\infty \int_{\mathbb{R}^3} |w|^4\,dx, \quad \beta_0 = \int_{\mathbb{R}^3} a(x)|w|^4\,dx, \quad \gamma_0 = \int_{\mathbb{R}^3} K(x)\phi_{\omega}(x)w^2\,dx.
\]

Then (1.11) leads to
\[
  \alpha_0 + \gamma_0 < \beta_0.
\] (3.22)

Note that \( w \) is a positive solution of (NS\( _{\infty} \)) satisfying \( m_\infty = \frac{1}{4} \alpha_0 \). Since \( w \in \Lambda \), applying Lemma 2.3, there exists \( t = t_w > 0 \) such that \( tw \in \mathcal{N} \), and so \( (P'(tw), tw) = 0 \) and \( I(tw) \geq c \). From (2.5), we have
\[
  0 = t^2 \|w\|^2 + \hat{t}^4 \int_{\mathbb{R}^3} K(x)\phi_{\omega}(x)w^2\,dx - \hat{t}^4 \int_{\mathbb{R}^3} a(x)|w|^4\,dx = \alpha_0 \hat{t}^2 + \gamma_0 \hat{t}^4 - \beta_0 \hat{t}^4,
\] (3.23)

which, together with (3.22), implies
\[
  0 < \hat{t} = \frac{\alpha_0}{\beta_0 - \gamma_0} < 1.
\] (3.24)

Then it follows from (1.2), (1.8), (3.23) and (3.24) that
\[
  m_\infty \leq c \leq I(tw) = \frac{1}{4} t_0^2 \alpha_0 < \frac{1}{4} \alpha_0 = m_\infty.
\] (3.25)

This contradiction shows that \( \tilde{u} \neq 0 \).

It is easy to check that \( I'(\tilde{u}) = 0 \) and \( I(\tilde{u}) = c = \inf_{\mathcal{N}} I \). Moreover, by a standard regularity argument [31], we can deduce that \( \tilde{u} > 0 \). Hence, \( \tilde{u} \in H^1(\mathbb{R}^3) \) is a positive ground state solution of (SP) with \( I(\tilde{u}) = \inf_{\mathcal{N}} I > 0 \). \( \square \)

**Proof of Corollary 1.3.** By Proposition 1, one has
\[
  A_1 e^{-2|x|} \leq w(x) \leq B_0 e^{-|x|}, \quad \forall x \in \mathbb{R}^3.
\] (3.26)

First, we consider \( 4 < p < 6 \). In this case, since \( 0 < \alpha_2 < 4 \), we can choose \( \zeta > \alpha_2^{-1} \ln \left[ \beta_2 A_1 \left( \int_{\mathbb{R}^3} e^{-(\alpha_2 + 2p)|x|}\,dx \right) / a_\infty \|w\|^p_p \right] \) such that
\[
  \frac{5^5(4\pi)^{5/3}}{6^6(\alpha_1 - 2)^5} C_0 \beta_2^2 B_0^4 e^{-4\zeta} \leq \frac{a_\infty C_3}{p} e^{-\alpha_2 \zeta} \int_{\mathbb{R}^3} |w|^p\,dx.
\] (3.27)

Let \( e_1 = (1, 0, 0) \) and \( w_\zeta(x) = w(x - \zeta e_1) \). Then \( w_\zeta(x) \) is a solution of (NS\( _{\infty} \)), and \( I_\infty(w_\zeta) = m_\infty = \inf \{ I_\infty(u) : u \in \mathcal{N}_\infty \} \). Thus, it follows from (K1), (2.3), (2.4) and
(3.26) that
\[
\int_{\mathbb{R}^3} K(x)\phi_{w_{\zeta}}(x)w_{\zeta}^2\,dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(x)}{|x-y|} w_{\zeta}^2(x)w_{\zeta}^2(y)\,dx\,dy \\
\leq C_0 \left( \int_{\mathbb{R}^3} |K(x)w_{\zeta}^2(x)|^{6/5}\,dx \right)^{5/3} \\
\leq C_0\beta_1^2B_0^4 \left( \int_{\mathbb{R}^3} e^{-6(\alpha_1|x|+2|x-\zeta_1|)/5}\,dx \right)^{5/3} \\
\leq C_0\beta_1^2B_0^4e^{-4\zeta} \left( \int_{\mathbb{R}^3} e^{-6(\alpha_2-2)|x|/5}\,dx \right)^{5/3} \\
= \frac{5^5(4\pi)^{5/3}}{6^5(\alpha_1-2)^5} C_0\beta_1^2B_0^4 e^{-4\zeta}.
\]

On the other hand, it follows from (A1) and (3.26) that
\[
\frac{1}{a_{\infty}||w_{\zeta}||_p} \int_{\mathbb{R}^3} a(x)|w_{\zeta}|^p\,dx = \frac{1}{a_{\infty}||w||_p} \int_{\mathbb{R}^3} a(x)|w_{\zeta}|^p\,dx \\
\geq 1 + \frac{\beta_2}{a_{\infty}||w||_p} \int_{\mathbb{R}^3} e^{-\alpha_2|x|}|w_{\zeta}|^p\,dx \\
\geq 1 + \frac{\beta_2A_1}{a_{\infty}||w||_p} \int_{\mathbb{R}^3} e^{-\alpha_2|x|-2p|x-\zeta_1|}\,dx \\
= 1 + \frac{\beta_2A_1 e^{-\alpha_2\zeta}}{a_{\infty}||w||_p} \int_{\mathbb{R}^3} e^{-(\alpha_2+2p)|x|}\,dx \\
\geq 1 + C_3 e^{-\alpha_2\zeta},
\]
where \( C_3 = \beta_2A_1 \left( \int_{\mathbb{R}^3} e^{-(\alpha_2+2p)|x|}\,dx \right) / a_{\infty}||w||_p^p \). By (1.9) and (3.29), one has
\[
1 + C_3 e^{-\alpha_2\zeta} \leq \frac{1}{a_{\infty}||w_{\zeta}||_p} \int_{\mathbb{R}^3} a(x)|w_{\zeta}|^p\,dx = (1+\theta_0) \left[ 1 + \frac{p-4}{2(p-2)} \theta_0 \right]^{(p-2)/2} \leq (1+\theta_0)^{p/2},
\]
which implies that \( \theta_0 \geq \frac{C_3}{p} e^{-\alpha_2\zeta} \). Jointly with (3.27) and (3.28), we have
\[
\int_{\mathbb{R}^3} K(x)\phi_{w_{\zeta}}(x)w_{\zeta}^2\,dx \leq \theta_0 a_{\infty} \int_{\mathbb{R}^3} |w_{\zeta}|^p\,dx.
\]
Therefore, if \( 4 < p < 6 \), then Corollary 1.3 follows directly from Theorem 1.1.

Next, we consider \( p = 4 \). Let
\[
\zeta_0 = \frac{2}{4-\alpha_2} \ln \left[ \frac{5^5(4\pi)^{5/3}C_0\beta_1^2B_0^4}{6^5(\alpha_1-2)^5} \int_{\mathbb{R}^3} e^{-(\alpha_2+8)|x|}\,dx \right]
\]
and \( w_{\zeta_0}(x) = w(x-\zeta_0\zeta_1) \). Then \( w_{\zeta_0}(x) \) is a solution of (NS\(_{\infty} \)), and \( I_{\infty}(w_{\zeta_0}) = m_\infty = \text{inf}\{I_{\infty}(u) : u \in N_{\infty}\} \). As in (3.28), we have
\[
\int_{\mathbb{R}^3} K(x)\phi_{w_{\zeta_0}}(x)w_{\zeta_0}^2\,dx \leq \frac{5^5(4\pi)^{5/3}}{6^5(\alpha_1-2)^5} C_0\beta_1^2B_0^4 e^{-4\zeta_0}.
\]
On the other hand, it follows from (A1) and (3.26) that
\[
\int_{\mathbb{R}^3} |a(x) - a_{\infty}| w_{\zeta_0}^4 \, dx \geq \beta_2 \int_{\mathbb{R}^3} e^{-\alpha_2 |x|} |w_{\zeta_0}|^4 \, dx
\]
\[
\geq \beta_2 A_1 \int_{\mathbb{R}^3} e^{-\alpha_2 |x| - 8|x - \zeta_0|} \, dx
\]
\[
= \beta_2 A_1 \int_{\mathbb{R}^3} e^{-\alpha_2 |x + \zeta_0| - 8|x|} \, dx
\]
\[
\geq \beta_2 A_1 e^{-\alpha_2 \zeta_0} \int_{\mathbb{R}^3} e^{-(\alpha_2 + 8)|x|} \, dx,
\]
(3.32)
Combining (3.30), (3.31) and (3.32), we have
\[
\int_{\mathbb{R}^3} K(x) \phi_{w_{\zeta_0}}(x) w_{\zeta_0}^2 \, dx < \int_{\mathbb{R}^3} |a(x) - a_{\infty}| w_{\zeta_0}^4 \, dx.
\]
Therefore, Corollary 1.3 follows directly from Theorem 1.2. \qed

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