The duplicube graph - a hybrid of structure and randomness

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Abstract

Connect two copies of a given graph $G$ by a perfect matching. What are the properties of the graphs obtained by recursively repeating this procedure? We show that this construction shares some of the structural properties of the hypercube, such as a simple routing scheme and small edge expansion. However, when the matchings are uniformly random, the resultant graph also has similarities with a random regular graph, including: a smaller diameter and better vertex expansion than the hypercube; a semicircle law for its eigenvalues; and no non-trivial automorphisms. We propose a simple deterministic matching which we believe could provide a derandomization.

1 Introduction and construction

The Boolean hypercube $Q_n$ is the graph whose vertex set is $V(Q_n) = \{0, 1\}^n$ and whose edge set is $E(Q_n) = \{\{x, y\} \mid x$ and $y$ differ by exactly one coordinate\}. One appeasing property of the hypercube graph is its recursive construction: starting with $Q_1$ as a single edge, $Q_n$ is given by the Cartesian graph product $Q_n = Q_1 \Box Q_{n-1}$; essentially, the Cartesian product with an edge amounts to matching together the corresponding vertices of two disjoint copies of $Q_{n-1}$. See Figure 1.1 for the first steps of this process.

![Figure 1.1: The recursive construction of the hypercube. The color of an edge indicates in which step it was created. Newly created edges are in bold.](image)

Generalizing this procedure, we introduce the duplicube graph, which is obtained by iteratively applying perfect matchings between the vertices of two copies of the original graph.

**Definition 1.** Let $G = (V, E)$ be a finite graph, and let $\sigma$ be a permutation of the vertices $V$. The **duplimatching** operation with base $G$ and matching $\sigma$, denoted $\sigma \diamond G$, produces a graph $\sigma \diamond G = (V', E')$
defined as follows. For \( i = 0, 1 \), let \( V^i = \{ (x, i) \mid x \in V \} \) be the vertex sets of two disjoint copies of \( G \), and \( E^i = \{ (x, i), (y, i) \mid (x, y) \in E \} \) be their edge sets. Then \( \sigma \circ G \) has vertex set
\[
V' = V^0 \cup V^1
\]
and edge set
\[
E' = E^0 \cup E^1 \cup \{ (x, 0), (\sigma(x), 1) \mid x \in V \}.
\]
Alternatively, if \( A \in \mathbb{R}^{m \times m} \) is the adjacency matrix of \( G \), and \( P \) is the \( m \times m \) permutation matrix representing \( \sigma \), then the adjacency matrix of \( \bar{G} \) is given by
\[
\begin{pmatrix}
A & P \\
PT & A
\end{pmatrix}.
\]

**Definition 2.** Let \( n \geq 0 \) and let \( \bar{\sigma} = (\sigma_k)_{k=1}^{\infty} \) be a sequence of permutations, where each \( \sigma_k \) is a permutation on \( \{0, 1\}^k \). The duplicube graph of generation \( n \) with permutations \( \bar{\sigma} \), denoted \( G_n = G_n(\bar{\sigma}) \), is defined as follows. For \( n = 0 \), \( G_0 \) is an isolated vertex, labeled by \( \emptyset \). For \( n = 1 \), \( G_1 \) is a single edge, i.e. \( V(G_1) = \{0, 1\} \) and \( E(G_1) = \{\{0, 1\}\} \), while for \( n > 1 \), \( G_n \) is given by
\[
G_n = \sigma_{n-1} \circ G_{n-1}.
\]
Alternatively, \( G_n \) can also be described as follows. The vertex set is \( V(G_n) = \{0, 1\}^n \), and for every \( k \in [n] \), the vertex \( x = (x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) \) is connected to \( y = (\sigma_{k-1}(x_1, \ldots, x_{k-1}), 1, x_{k+1}, \ldots, x_n) \). See Figure 1.2 for the first steps of this process.

![Figure 1.2: An example of the recursive construction of the duplicube, using random matchings. The color of an edge indicates in which step it was created. Newly created edges are in bold.](image)

A duplicube graph \( G_n \) is an \( n \)-regular graph with \( N = 2^n \) vertices. When \( \sigma_k \) is the identity permutation for every \( k \), then \( G_n \) is just the Boolean hypercube graph \( Q_n \). The hypercube has diameter \( n \), has poor vertex- and edge-expansion (relative to the fact that its degree grows with the graph size; see Section 2.2), and a random-walk mixing time of order \( \Theta(n \log n) \) [11]. Many other geometric and structural properties of the hypercube are known (e.g. distances between vertices [8] and isoperimetric inequalities for various sets [18, 9, 3]).

Another well-researched class of \( n \)-regular graphs are the uniformly random regular graphs. With probability \( 1 - o(1) \), a random \( n \)-regular graph on \( 2^n \) vertices has diameter \( \Theta(n / \log n) \) [4], has high edge-expansion [1] and a random-walk mixing time of order \( \Theta(n / \log n) \) [13]. Further, its eigenvalues follow a semicircle distribution [6].

For fixed \( n \) and \( N \to \infty \), the uniform distribution over \( n \)-regular graphs on \( N \) vertices can be approximated by adding \( n \) successive random perfect matchings on \( N \) isolated vertices, where the \( i \)-th matching is uniform over all matchings on previously-unmatched pairs of vertices [7, Theorem 8]. In contrast, consider the random dupligraph \( G_n = G_n(\bar{\sigma}) \), where \( \sigma_k \) is a uniformly random permutation on \( \{0, 1\}^k \) for all \( k \). It consists of a union of \( n \) independent matchings as well, but these matchings are not uniformly random. For example, the last matching is a uniformly random matching only between the two copies of \( G_{n-1} \), while
earlier matchings consist of copies of smaller matchings and therefore have even stronger dependencies between the edges. In this sense, the random duplicube $G_n$ is a hybrid between the structure of the Boolean hypercube and the randomness of a random $n$-regular graph. It is therefore natural to ask how its various geometric and structural properties compare to those of the hypercube and random $n$-regular graphs.

2 Our results

In this work, we study the diameter, expansion, eigenvalues, and symmetries of $G_n$.

2.1 The diameter

For a graph $G = (V, E)$, let $d_G : V^2 \to \mathbb{R}$ be the graph distance between two vertices. The diameter of a graph is the maximum distance in the graph, i.e. $D(G) := \max \{d_G(x, y) \mid x, y \in V\}$. An immediate result shows that the diameter of the hypercube $Q_n$ has the worst possible diameter out of all duplicube graphs.

**Proposition 3.** For every choice of permutations $\sigma_k$, we have $D(G_n) \leq D(Q_n) = n$.

*Proof.* By induction. For $n = 1$, it is clear. In the general case, let $x, y \in V(G_n)$, and denote $x = (\hat{x}, x_n)$, $y = (\hat{y}, y_n)$. If $x_n = y_n$, then $\hat{x}, \hat{y} \in V(G_{n-1})$, and $d_{G_n}(x, y) = d_{G_{n-1}}(\hat{x}, \hat{y}) \leq n - 1$. Otherwise, $x$ is connected to some $(x', 1 - x_n)$, and

$$d_{G_n}(x, y) \leq 1 + d_{G_{n-1}}(x', \tilde{y}) \leq n.$$

The following lower bound is also immediate.

**Proposition 4.** For every choice of permutations $\sigma_k$, we have $D(G_n) \geq (n - 1) / \log_2 n$.

*Proof.* If $G_n$ has diameter $d$, then the ball $B(v, d)$ of radius $d$ around any vertex $v$ must contain the entire graph. Since the graph is $n$-regular, the number of vertices in this ball is smaller than $2n^d$, and we get

$$2^n = |B(v, d)| \leq 2n^d,$$

yielding

$$d \geq \frac{n - 1}{\log_2 n}.$$

When the permutations $\sigma_k$ are chosen uniformly at random, we show that the diameter can be asymptotically better than that of Proposition 3, by at least a $\log \log n$ factor. The following theorem is proved in Section 4.2.

**Theorem 5.** For every integer $n_0 \geq 3$, there exists a constant $C > 0$ such that for every $n \geq n_0$,

$$D(G_n) \leq C \frac{n}{\log \log n}$$

with probability at least

$$1 - \sum_{k=n_0}^{n} 2^{-k}.$$

As an immediate corollary, we have:
Corollary 6. For every function \( f : \mathbb{N} \to \mathbb{R} \) such that \( \lim_{n \to \infty} f(n) = \infty \), we have

\[
\lim_{n \to \infty} \mathbb{P} \left( D(G_n) \leq f(n) \frac{n}{\log \log n} \right) = 1.
\]

Remark 7. The proof of Proposition 3 also gives a simple routing scheme between any two vertices \( x, y \): when at \( x \), let \( k \in [n] \) be the largest index such that \( x_k \neq y_k \), and go along the edge created by \( \sigma_{k-1} \). Thus, we always have a local routing scheme which gives a good approximation to the diameter (and indeed, the average distance between two vertices). Contrast this with general random \( n \)-regular graphs, where there is no known local easy way to find an approximation to the minimal path between two vertices.

2.2 Vertex expansion

Let \( G = (V, E) \) be any graph. For a set \( S \subseteq V \), let \( \partial S \) be its set of neighbors, i.e. \( \partial S = \{ x \notin S \mid \exists y \in S \text{ such that } \{x, y\} \in E \} \).

Definition 8 (vertex expander). Let \( 0 < \eta < 1 \) and \( \alpha > 0 \), and let \( G = (V, E) \) be a graph. A set \( S \subseteq V \) is said to have \( \alpha \)-expansion if \( |\partial S| \geq \alpha |S| \). The graph \( G \) is an \( (\eta, \alpha) \)-vertex-expander if \( S \) has \( \alpha \)-expansion for all \( S \subseteq V \) of size \( |S| \leq \eta |V| \).

The hypercube \( Q_n \) is not a very good vertex expander for any constants \( \alpha > 0 \). To see this, fix some constant \( \eta > 0 \). There is some \( \rho > 0 \) so that the ball \( S = \{ x \in \{0, 1\}^n \mid \sum x_i \leq n/2 - \rho \sqrt{n} - 1 \} \) has size \( (n + o(1)) 2^n \). However, its boundary is \( \partial S = \{ x \in \{0, 1\}^n \mid \sum x_i = n/2 - \rho \sqrt{n} \} \) and has size at most \( \left( \frac{n}{\sqrt{n}} \right)^2 = 2^n (1 + o(1)) / \sqrt{n} \). Thus, \( Q_n \) cannot have an expansion factor \( \alpha \) asymptotically larger than \( \frac{1}{\sqrt{n}} \) for any constant \( \eta \). The duplicate graph, on the other hand, achieves constant expansion with high probability. The following theorem is proved in Section 4.3.

Theorem 9. Let \( \eta \in (0, 1) \). There exists a constant \( \alpha > 0 \) such that the following holds. Let \( F_n \) be the event that \( G_n \) is not an \( (\eta, \alpha) \)-vertex expander, i.e. there exists a set \( S \subseteq V_n \) with \( |S| \leq \eta |V_n| \) such that \( |\partial S| < \alpha |S| \). Then

\[
\lim_{n \to \infty} \mathbb{P} \left( \bigcup_{n > n_0} F_n \right) = 0. \quad (2.2)
\]

Remark 10. It is also possible to talk about edge expanders, and compare the size of a set \( S \) to the number of edges connecting it to \( \partial S \). Both \( Q_n \) and \( G_n \) are not very good edge expanders (for any choice of permutations \( \sigma_k \)); see Section 3 for more details.

Remark 11. In random \( d \)-regular graphs, balls of any constant radius \( r \) around an individual vertex are trees with high probability (even when \( d \) is logarithmic in the number of vertices). Such sets are very poorly connected – the vertex expansion is of order \( 1/d^r \) (consider cutting the \( d \)-ary tree in half at the central vertex). However, in the duplicate, a ball of radius \( r \) contains a copy of \( G_r \), which, by the theorem above, has good vertex expansion with arbitrarily high probability for large \( r \).

2.3 Eigenvalues

Let \( A \in \mathbb{R}^{n \times m} \) be a symmetric matrix, whose eigenvalues are \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \). Let \( \mu^A := \frac{1}{m} \sum_{i=1}^{m} \delta_{\lambda_i} \) be the uniform measure over the eigenvalues of \( A \), where \( \delta_s \) is the Dirac-delta distribution centered at \( s \).

Let \( \text{Adj}(Q_n) \) be the adjacency matrix of the hypercube \( Q_n \). The \( n^2 \) eigenvalues and eigenvectors of \( \text{Adj}(Q_n) \) are well understood; the following is well known [5, Section 1.4.6].

Fact 12. For every integer \( d \in [0, n] \), the adjacency matrix \( \text{Adj}(Q_n) \) has eigenvalue \( n - 2d \) with multiplicity \( \binom{n}{d} \).

In particular, the hypercube's largest eigenvalue is \( n \), while its second largest eigenvalue is \( n - 2 \). Thus, its normalized spectral gap, defined as \( \frac{1}{n} (\lambda_1 - \lambda_2) \), is \( \frac{2}{n} \). The same gap is achieved for the graphs \( G_n \), regardless of the choice of \( \bar{\sigma} \). The following proposition is proved in Section 4.4.
Proposition 13. Let $A_n$ be the adjacency matrix of $G_n$. Then $\lambda_1 = n$ and $\lambda_2 = n - 2$.

A consequence of Fact 12 is that $\mu^{\text{Adj}(Q_n)}$ is the probability measure of a $\{\pm 1\}$ Binomial random variable with $n$ trials and success probability $1/2$. By the central limit theorem, we then have that

$$
\mu^{\text{Adj}(Q_n)}/\sqrt{n} \to \Gamma
$$

weakly, where $\Gamma$ is the standard Gaussian distribution on $\mathbb{R}$. Unlike the spectral gap, this property is not preserved for the random duplicate graph. In fact, the spectrum of $A = \text{Adj}(G_n)$ behaves like that of a random $n$-regular graph.

Theorem 14. Let $\mu_n = \mu^A/\sqrt{n}$. Then the random measure $\mu_n$ converges weakly to the semicircle law $\mu_{\text{circ}}$ in probability, i.e. the absolutely continuous measure whose probability density function is

$$
f_{\text{circ}}(x) = \begin{cases} 
\frac{2}{\sqrt{\pi}} \sqrt{1-x^2} & x \in [-2,2], \\
0 & x \notin [-2,2].
\end{cases}
$$

The above theorem follows from the following lemma, which states that the number of short cycles in the neighborhood of any vertex in $G_n$ is small. Essentially, this means that $G_n$ is almost locally treelike. For a vertex $v$ and positive integer $k$, let $\theta(v,k)$ denote the number of cycles of length no more than $k$ containing $v$, and $B(v,k)$ denote the ball of radius $k$ around $v$.

Lemma 15. Let $v \in V_n$ and let $k > 0$ be an integer. There exists a constant $C > 0$ which depends only on $k$ such that the following holds. Let $m_0 > 0$ be an integer, and let

$$
F_v = \bigcup_{u \in B(v,k)} \{ \theta(v,k) \leq C m_0^{k+1} \}.
$$

Then

$$
P[F_v] \geq 1 - C 2^{-m_0} m_0^{2k+2} n^{2k+1}.
$$

Theorem 14 and Lemma 15 are proven in Section 4.4.

Remark 16. A classical theorem by McKay [17] states that a regular graph on $N$ vertices has a limiting semicircle law if, for every $k$, the number of $k$-cycles in the graph is $o(N)$. This result cannot be directly used in the case of the duplicate: for example, each vertex is guaranteed to be in a 4-cycle, so there are at least $N/4$ 4-cycles in every duplicate (in fact, we conjecture that in $G_n$, for every $k$, each vertex is in a constant number of $k$-cycles in expectation). Lemma 15 is the main technical component in our proof of Theorem 14.

2.4 Asymmetry of $G_n$

Let $G = (V, E)$ be any graph. A function $\varphi : V \to V$ is called an automorphism of $G$ if $\{x,y\} \in E \iff \{\varphi(x),\varphi(y)\} \in E$. The set of all automorphisms of a graph is denoted by $\text{Aut}(G)$, and always contains the trivial automorphism -- the identity function $\text{Id}$.

It is well known that for the hypercube, $|\text{Aut}(Q_n)| = n!2^n$, and every automorphism $\varphi(x)$ is of the form $\varphi(x_1, \ldots, x_n) = (x_{\pi(1)} + b_1, \ldots, x_{\pi(n)} + b_n)$ for some permutation $\pi \in S_n$ and $b \in \{0,1\}^n$. On the other hand, a random regular graph of degree $n$ on $2^n$ vertices is almost surely asymmetric, i.e. almost surely has no non-trivial automorphisms [16, Corollary 3.5]. This is also true for the duplicate.

Theorem 17.

$$
\lim_{n \to \infty} P[\text{Aut}(G_n) = \{\text{Id}\}] = 1.
$$

The proof of Theorem 17 is found in Section 4.5.
2.5 Different base graphs

The duplicube graph is the result of repeatedly applying the duplimatching operation on a single vertex. It is also possible to start with any base graph \( H \), and define \( G_n^H = \sigma_{n-1} \circ \ldots \circ \sigma_0 \circ H \), where \( \sigma_k \) is a permutation on \( 2^k |V(H)| \) vertices. When each \( \sigma_k \) is a uniformly random permutation on \( 2^k |V(H)| \) elements, we denote the resulting random graph by \( G_n^H \). In this case, none of the main results concerning the diameter, expansion, and eigenvalues are severely affected. This is because as \( n \to \infty \), the vast majority of the edges meeting each vertex are those created by the duplimatching operation.

**Corollary 18** (Corollary to Theorem 5). Let \( H \) be a finite connected graph. For every integer \( n_0 \geq 3 \), there exists a constant \( C > 0 \) (which may depend on \( H \)) such that for all \( n \geq n_0 \),

\[
D \left( G_n^H \right) \leq C \frac{n}{\log \log n}
\]

with probability at least

\[
1 - \sum_{k=n_0}^{n} 2^{-k}.
\]

**Proof sketch.** Consider the duplimatching operation \( G_k^H = \sigma_k \circ G_k^H \). By contracting each copy of \( H \) in \( G_{k+1}^H \) to a single vertex and using Hall’s marriage theorem, there exists a set \( S \) of \( 2^k \) edges induced by \( \sigma_k \) which comprise a perfect matching between the copies of \( G \) in the two duplicates of \( G_k^H \). Such a set \( S \) naturally induces a permutation on \( \{0, 1\}^k \). For a given \( \sigma_k \), let \( \pi_k \) be chosen uniformly at random among all such induced permutations. Then \( \pi_k \) is a uniform random permutation on \( \{0, 1\}^k \), and the graph \( G_n(\pi) \) is connected (this can be shown to happen with probability tending to 1 as \( n \to \infty \)), then use that as the base graph.

**Remark 19.** If \( H \) is not connected, simply apply the duplimatching operation several times first until \( G_n^H \) is connected (this can be shown to happen with probability tending to 1 as \( n \to \infty \)), then use that as the base graph.

**Corollary 20** (Corollary to Theorem 9). Let \( H \) be a finite connected graph. Let \( \eta \in (0, 1) \). There exists a constant \( \alpha > 0 \) such that the following holds. Let \( F_n \) be the event that \( G_n^H \) is not an \((\eta, \alpha)\)-vertex expander. Then

\[
\lim_{n_0 \to \infty} P \left[ \bigcup_{n > n_0} F_n \right] = 0.
\]

The proof of the above corollary is essentially identical to that of Theorem 9. The latter only uses the edges created by the last three duplimatching operations, and so equation (2.2) still holds for \( G_n^H \) as well for \( n \geq 3 \).

**Corollary 21** (Corollary to Theorem 14). Let \( \mu_n = \mu_{\circ l(G_n^H)}/\sqrt{n} \). Then \( \mu_n \) converges weakly to the semicircle law \( \mu_{\circ l} \) in probability, i.e. the absolutely continuous measure whose probability density function is

\[
f_{\circ l}(x) = \begin{cases} \frac{2}{\sqrt{4-x^2}} & x \in [-2, 2] \\ 0 & x \notin [-2, 2] \end{cases}.
\]

**Proof sketch.** We will assume for simplicity that \( |V(H)| = 2^d \) for some integer \( d \). We can couple \( G_n^H \) with \( G_{n+d} \) by observing that \( G_{n+d} = G_n^H \circ G_d \), and using the same permutations \( \sigma_k \) for \( G_n^H \) and \( G_d \). Since all the edges due to the permutations are the same for \( G_n^H \) and \( G_n^H \circ G_d \), their adjacency matrices differ by no more than \( c := |V(H)| \) entries at each row, and all the eigenvalues of the matrix \( \Delta = \text{Adj} \left( G_n^H \right) - \text{Adj} \left( G_n^H \circ G_d \right) \)
are bounded by $c$. Denoting $A := \text{Adj} \left(G_n^{G_d}\right)$, for every integer $k > 0$ we have

$$\left| \sum_{i=1}^{2^{n+d}} \lambda_i \left(G_n^{G_d}\right)^k - \sum_{i=1}^{2^{n+d}} \lambda_i \left(G_n^H\right)^k \right| = \left| \text{Tr} \left(A^k\right) - \text{Tr} \left((A + \Delta)^k\right) \right| = \left| \text{Tr} \left(P (A, \Delta)\right) \right|,$$

where $P$ is a polynomial of degree $k$ for which in every monomial, $A$ has total degree at most $k - 1$. By Von Neumann’s trace inequality [15, eq H.10], if $A_1, \ldots, A_m$ are $N \times N$ symmetric matrices, then

$$\sum_{i=1}^{N} \lambda_i (A_1 \cdots A_m) \leq \sum_{i=1}^{N} \lambda_i (A_1) \cdots \lambda_i (A_m),$$

and so the trace of every monomial in $P$ is bounded above by $c^k \sum_{i=1}^{2^{n+d}} \left| \lambda_i (A)^{k-1} \right|$. Thus the difference in the normalized moments of $G_n^{G_d}$ and $G_n^H$ is bounded by

$$\frac{n + d}{2n + d} \left| \sum_{i=1}^{2^{n+d}} \lambda_i \left(G_n^{G_d}\right)^k - \sum_{i=1}^{2^{n+d}} \lambda_i \left(G_n^H\right)^k \right| \leq C (k) \frac{n + d}{2n + d} \sum_{i=1}^{2^{n+d}} \left| \lambda_i (A)^{k-1} \right|$$

(Cauchy-Schwarz) \leq C (k) \left( \frac{1}{2n + d} \sum_{i=1}^{n-1} \lambda_i (A) \right)^2 \left( \frac{1}{2n + d} \sum_{i=1}^{n-1} \lambda_i (A)^2 \right)^{2-k}.$$ 

By the proof of Theorem 14, $\frac{1}{2n + d} \sum_{i=1}^{n-1} \lambda_i (A)^2 k - 4$ converges to a constant in probability as $n \to \infty$, which means that the sum on the right-hand side above converges to 0 in probability. This implies that the $k$-th moments of the empirical distribution of the eigenvalues of $G_n^H$ converge to those of the semicircle law.

Finally, the proof of Theorem 17 applies equally well to $G_n^H$.

**Corollary 22** (Corollary to Theorem 17). Let $H$ be a finite connected graph. Then

$$\lim_{n \to \infty} P \left[ \text{Aut} \left(G_n^H\right) = \{\text{Id}\} \right] = 1.$$

### 3 Remarks and further directions

1. Is there a simple-to-define, non-random sequence of permutations $\bar{\sigma} = (\sigma_k)_{k=1}^{\infty}$ for which the duplicube $G_n (\bar{\sigma})$ has improved geometric properties over the hypercube (say, a shorter diameter, or better expansion)? If the diameter is better, is there a simple routing scheme for it? We conjecture that this is possible, and suggest looking at arithmetic rules for permutation. More concretely, we propose the following construction, inspired by the Margulis expanders [14]. For $x \in \{0, 1\}^k$, let $y$ be the integer represented by the first $\lfloor k/2 \rfloor$ bits of $x$, let $z$ be the integer represented by the last $\lfloor k/2 \rfloor$ bits of $z$, and write $x = (y, z)$. When $k$ is even, let $\sigma_k (y, z) = (y, z + y \mod 2^{\lfloor k/2 \rfloor})$ and when $k$ is odd, let $\sigma_k (y, z) = (y + z \mod 2^{\lfloor k/2 \rfloor}, z)$. Is it true that $D (G_n (\bar{\sigma})) = o (n)$?

2. As an intermediate step towards derandomization, it is also natural to ask about distributions on permutations with smaller entropy than the uniform measure on $S_n$. In particular, are there small subgroups of $S_n$ (of sublinear index), so that if $\sigma_k$ is sampled uniformly form these subgroups, the resulting duplicube has similar properties to $G_n$?

3. Given a sequence of permutations $(\sigma_k)_{k=1}^{\infty}$, is there a good local routing scheme that gives a better approximation than Proposition 3 to the shortest path between two vertices?
4. The duplimatching model can be readily extended to $d$-dimensional hypergraphs: at every step, create $d$ copies of the graph, and connect the vertices of the $d$ copies by a perfect matching of $d$-hyperedges. What can be said about the resultant hypergraph?

5. The graph $G_n$ is, in general, not a good edge-expander. One reason for this are cuts across the matchings $\sigma_k$ for large $k$. For example, the two copies of $G_{n-1}$ in $G_n$ each have $2^{n-1}$ vertices, and are connected by $2^{n-1}$ edges, giving an isoperimetric ratio of 1. This is not so large for a graph whose degree is $n$. Is it true that these types of cuts are the only obstacle for having a large boundary? More generally, what can we say about the geometric properties of a set with small edge boundary? Is it true that $G_n$ has some form of small set expansion?

6. We show that with high probability $G_n$ is a good vertex expander. However, to our knowledge there is no efficient way to verify that a given graph is a vertex expander: assuming the Small-Set-Expansion Hypothesis, it is hard to even approximate the vertex expansion of a graph in polynomial time [12]. Is it possible to exploit the structure of the duplicube to verify this property in time $\text{poly}(2^n)$?

7. By using the same coupon-collector argument as for the hypercube, the mixing time of the lazy simple random walk of any duplicube is $O(n \log n)$. On the other hand, if an $n$-th generation edge is never refreshed, then the random walk stays constrained to one half of the graph, and so the mixing time must also be $\omega(n)$. What is the mixing time for the lazy simple random walk on $G_n$? Is it $o(n \log n)$ with high probability?

8. Is it possible to remove edges from $G_n$ and obtain a (near) constant-degree graph, while maintaining good vertex expansion? Is it possible to approach the vertex expansion of a constant-degree random regular graph in this way?

9. Replace every vertex of $Q_n$ by an $n$-cycle, obtaining a graph $\text{CCC}_n$; this is known as the cube-connected-cycle [19]. As $n \to \infty$, it is well known that $\text{CCC}_n$ converges in the Benjamini-Schramm sense [2] to the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{Z}$. We conjecture that the Benjamini-Schramm limit of the duplicube-connected cycle, obtained by replacing every vertex of $G_n$ by an $n$-cycle, is the 3-regular tree: as $n \to \infty$, a vertex chosen at random from this graph corresponds to a high-generation edge with high probability, and these should not be part of many small cycles.

10. Although Theorem 17 shows that random permutations lead to an asymmetric graph, in general different choices of $\tilde{\sigma}$ can lead to different automorphism groups. Can we relate properties of the automorphism group of $G_n(\tilde{\sigma})$ with properties of $\tilde{\sigma}$? In particular, can we find large families of $\tilde{\sigma}$ so that $G_n(\tilde{\sigma})$ is vertex-transitive? As a non-trivial example, consider the permutations $\sigma_k = \text{Id}$ for $k \neq 2$. There are two essentially different possibilities for $\sigma_2$: the first is $\sigma_2 = \text{Id}$, leading to the hypercube $Q_n$. The second is the matching between a pair of 4-cycles which sends an edge to a non-edge. This leads to a vertex-transitive graph that is not isomorphic to $Q_n$. Can we find a (perhaps random) vertex-transitive $G_n(\tilde{\sigma})$ with improved geometric properties over the hypercube?

11. The argument in Theorem 9 only uses the edges of the last three generations of the duplicube. On the other hand, such an argument could not hold while using only the edges of the last two generations, since the graph induced by the edges of the last two generations is a union of cycles. In fact, we believe that when $\sigma_i = \text{Id}$ for $i < n - 2$, the resultant graph does not have constant vertex-expansion with high probability. In light of this, it is natural to ask: for an integer $k > 0$, what are the properties of the duplicube graph $G_n(\tilde{\sigma})$, where $\sigma_i = \text{Id}$ for $i < n - k$, and $\sigma_i$ is uniformly random for $i \geq n - k$? What happens when $k$ grows slowly to infinity with $n$? This is a natural interpolation between the hypercube $Q_n$ and the completely random duplicube $G_n$.

12. The hypercube $Q_n$ induces a partial order on its vertices in a natural way: $x \leq y$ if $x_i \leq y_i$ for every $i$. This natural partial order has applications (see e.g. [20, Chapter 6]). The duplicube induces a similar partial order inductively: given the order on $G_{n-1}$, extend it to $G_n$ by having $(x,0) < (\sigma_{n-1}(x),1)$
for all \( x \in V_{n-1} \), and by keeping the original order within \( G_{n-1} \) in both copies. It can be verified that this is indeed a partial order. What are the properties of this partial order as a function of \( \delta \)? Are there any combinatorial applications to the partial order produced by the duplicube?

13. The hypercube \( Q_n \) is bipartite, and hence always 2-colorable. On the other hand, the chromatic number \( \chi \) of random \( d \)-regular graphs (of constant degree \( d \)) is known to take only one of two possible values with high probability, and satisfies \( 2\chi \log \chi \approx d \) [10]. What is the chromatic number of \( G_n \)? It can be shown that it is at least 3 with high probability, and so \( G_n \) is in general not bipartite (to see this, consider the case where \( G_n \) is bipartite, and look at the probability that \( \sigma_n \) induces an odd cycle in \( G_{n+1} \)).

4 Proofs

4.1 Notation and definitions

All logarithms are in base \( e \) unless otherwise noted. For two sequences \( f (n), g (n) \), we write \( f = o (g) \) if \( \lim |f (n)| / |g (n)| \to 0 \). For natural numbers \( n, k \in \mathbb{N} \), we set \( N = 2^n \) and \( K = 2^k \). The set of numbers \( 1, \ldots, n \) is denoted by [\( n \)].

We denote the vertex set of \( G_n \) by \( V_n := \{0, 1\}^n \). For two sets \( S, \bar{S} \subseteq V_n \), write \( S \sim \bar{S} \) if \( x \in S \) and \( y \in \bar{S} \) with \( \{x, y\} \in E (G_n) \), and \( S \not\sim \bar{S} \) otherwise. We say that the edges between two disjoint sets of vertices \( A, B \subseteq V_n \) constitute a matching if every vertex in \( A \cup B \) is adjacent to at most one such edge.

Let \( x, y \in V_n \). The generation number of \( x \) and \( y \), denoted by \( \gamma (x, y) \), is defined as

\[
\gamma (x, y) := n - \max \{1 \leq s \leq n \mid x_i = y_i \forall i \geq s\},
\]

i.e. \( n \) minus the longest common suffix of \( x \) and \( y \). If \( \{x, y\} \in E (G_n) \) is an edge, then that edge is due to the permutation \( \sigma_{\gamma (x, y) - 1} \). Supposing that \( \gamma (x, y) = k \), we then say that \( x \) and \( y \) are \( k \)-neighbors. Every vertex \( x \) has exactly one \( k \)-neighbor for every \( k \in [n] \); we denote it by \( N_k (x) \).

For an integer \( r > 0 \) and vertex \( v \in V_n \), denote by

\[
B (v, r) := \{z \in V_n \mid \exists \text{a path of at most } r \text{ edges from } v \text{ to } z\}
\]

the ball of radius \( r \) around \( v \), and by

\[
B_{\leq k} (v, r) := \{z \in V_n \mid \exists \text{ a path } P \text{ of at most } r \text{ edges from } v \text{ to } z \text{ s.t. } \gamma (x, y) < k \forall \{x, y\} \in E (P) \}
\]

the \( r \)-neighborhood of \( v \) obtained by paths which only use edges of generations smaller than \( k \).

For \( 1 \leq s < n \), the graph \( G_n \) contains multiple disjoint copies of graphs \( G_s \). Indeed, let \( z \in \{0, 1\}^{n-s} \), and define

\[
V_n^z := \{(y, z) \in V_n \mid y \in \{0, 1\}^s\}.
\]  

(4.1)

Then the induced graph on \( V_n^z \) is isomorphic to \( G_s \). The sets \( V_n^z \) are disjoint for different \( z \), and partition the vertices of \( G_n \). For a vertex \( x \in V_n \), let \( I_s (x) \) be the set \( V_n^z \) which contains \( x \); it is the set of all vertices in \( G_n \) which share a suffix with \( x \) of size at least \( n - s \), i.e.,

\[
I_s (x) := \{y \in V_n \mid \gamma (x, y) \leq s\}.
\]

Note that \( |I_s (x)| = 2^s \). See Figure 4.1 for a visual aid. Finally, for a set \( S \subseteq V_n \), we denote by \( \partial_k S \) the boundary due to the first \( k \) generations of edges, i.e.

\[
\partial_k S = \{x \not\in S \mid \exists y \in S, \{x, y\} \in E (G_n), \gamma (x, y) \leq k\}.
\]

We often write \( \partial S \) instead of \( \partial_0 S \) for brevity.
Figure 4.1: Each large rectangle represents the same graph $G_n$, with the same vertex $x$ highlighted. The partition into copies of $G_s$ is shown for $s \in \{n-1, n-2, n-3\}$ while highlighting $I_s(x)$.

4.2 The diameter

Proof of Theorem 5. The proof is by induction. For the base cases, by Proposition 3, for all $n \leq n_0$ we have

$$D(G_n) \leq n \leq \log \log n_0 \frac{n}{n \log \log n},$$

and so (2.1) holds with probability 1 with $C = \log \log n_0$.

For the induction step, let $n > n_0$. By increasing $C$, we may assume that $n_0$ is larger than any given global constant; this will ensure that inequalities which hold only when $n$ is large enough indeed hold. Denote by $E_k$ the event that $D(G_k) \leq C \frac{k}{\log \log k}$, and assume that $E_1, \ldots, E_{n-1}$ hold. Let $x, y \in V_n$. If $x_n = y_n$, i.e., the two vertices are in the same half of the graph $G_n$, then by the induction hypothesis, $D(G_{n-1}) \leq C \frac{n-1}{\log \log (n-1)}$, and we certainly have $d_{G_n}(x, y) \leq C \frac{n}{\log \log n}$.

For the case $x_n = 1 - y_n$, i.e., the two vertices are in opposite sides of the graph $G_n$, we’ll show that for a not-too-large radius, the spheres around $x$ and $y$ contain enough vertices, so that with high probability there is an edge between them induced by $\sigma_{n-1}$.

Given a vertex $v \in V_n$ and any integers $t \geq s \geq 0$, let $M(v, s, t) = \{(N_k(v), k) \mid s \leq k \leq t\}$ be the set of neighbors of $v$ whose edge to $v$ was added at times $s \leq k \leq t$, along with their generation number. Note that for $(z, k) \in M(v, s, t)$, the set $I_k(z)$ is contained in $I_{t+1}(v)$, and that since each $k$-neighbor is added at a different generation, the sets $\{I_k(z)\}_{(z,k) \in M(v,s,t)}$ are all mutually disjoint (see Figure 4.2). We can therefore iteratively apply the function $M(v, s, t)$ to obtain a large set of disjoint vertices.

Figure 4.2: The entire rectangle represents the graph $I_{t+1}(v)$. Each neighbor $N_k(v)$ is contained in $I_k(N_k(v))$, and these $I_k(N_k(v))$ are all disjoint.
More formally, let $s, \ell > 0$ be integers, and consider a subset $S_\ell(x)$ of the sphere of radius $\ell$ around $x$, defined as follows:

$$S_0(x) = \{(x, n-1)\}$$

$$S_\ell(x) = \bigcup_{(z,t) \in S_{\ell-1}(x)} M(z,s,t-1).$$

By the remark above, the sets $\{I_s(z)\}_{(z,t) \in S_\ell(x)}$ are all disjoint, and so the set $U(x) := \cup_{(z,t) \in S_{\ell}(x)} I_s(z)$ has cardinality $2^s |S_\ell(x)|$. Define $S_\ell(y)$ and $U(y)$ similarly. Write the values of $s$ and $\ell$ as $s = \frac{n}{2} - \frac{1}{2} \alpha(n)$ and $\ell = \frac{n}{\beta(n)}$, for some functions $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ to be chosen later. Assuming that there is a vertex $u \in U(x)$ which is connected to $v \in U(y)$, the distance between $x$ and $y$ can be bounded as follows:

$$d_{G_n}(x,y) \leq 2\ell + 2D(G_s) + 1,$$

(4.2)

where $\ell$ bounds the distance to go from $x$ to a vertex $z$ in $S_\ell(x)$, $D(G_s)$ bounds the distance from $z$ to $u$, and $1$ is the distance from $u$ to $v$ (see Figure 4.3).

Figure 4.3: If $U(x)$ is connected to $U(y)$, we have a path from $x$ to $y$. The red dotted lines represent an optimal path within $G_s$.

Since we assume that $D(G_s) \leq C \frac{n}{\log \log n}$, we have

$$2D(G_s) \leq 2C \frac{n}{\log \log \left(\frac{n}{2} - \frac{1}{2} \alpha(n)\right)} \leq C \frac{n - \alpha(n)}{\log \log \left(\frac{n}{4}\right)} = C \frac{n \left(1 - \frac{\alpha(n)}{n}\right)}{\log \log n + \log \left(1 - \frac{\log 4}{\log n}\right)}$$

$$\leq C \frac{n \left(1 - \frac{\alpha(n)}{n}\right)}{\log \log n - \frac{2\log 4}{\log n}} = C \frac{n \left(1 - \frac{\alpha(n)}{n}\right)}{\log \log n \left(1 + \frac{6}{\log n \log \log n - \frac{\alpha(n)}{n}}\right)}.$$

Choosing $\alpha(n) = \frac{17n}{\log n}$ then gives

$$2D(G_s) \leq C \frac{n}{\log \log n \left(1 - \frac{1}{\log n}\right)}.$$
Choosing also $\beta(n) = \frac{\log 2}{\log 18} \log n \log \log n$, so that $\ell = \frac{18}{\log 2} \frac{n}{\log n \log \log n}$, by (4.2) we have that

$$d_G(n, y) \leq \frac{36}{\log 2} \frac{n}{\log n \log \log n} + C \frac{n}{\log \log n} \left(1 - \frac{1}{\log n}\right) + 1$$

(for $C$ large enough) $\leq C \frac{n}{\log \log n}$.

All that remains is to bound the probability of the event $\{U(x) \sim U(y) \forall x, y \in V_n\}$ from below. We do this using a union bound. The number of vertices in $S_\ell(x)$ can readily be seen to be

$$\sum_{k_1 = s}^{n-1} \sum_{k_2 = s}^{k_1-1} \cdots \sum_{k_{\ell-1} = s}^{1} \sum_{k_{\ell} = s}^{n-s-1} k_{\ell-1} \cdots k_{k_{\ell-1}} = \sum_{k_1 = 1}^{n-s-1} \sum_{k_2 = 1}^{k_1-1} \cdots \sum_{k_{\ell} = 1}^{1} 1.$$

This is the number of decreasing positive integer sequences of length $\ell$, whose maximum entry is bounded by $n - s - 1$. Since every choice of $\ell$ integers can be ordered in a unique fashion, we have

$$|S_\ell(x)| = \binom{n - s - 1}{\ell} = \binom{n}{\ell} - \binom{n}{\ell - 1} - \binom{n}{\ell - 2} \cdots - \binom{n}{1} \geq \binom{n}{\ell} \frac{n/\beta(n)}{n/\beta(n)} \geq \binom{n}{\ell} \frac{n/\beta(n)}{2} \geq \exp\left(\log \beta(n) - \log 2\right) \frac{n}{\beta(n)} \geq \frac{n \log \beta(n)}{2\beta(n)}.$$

(Assume $n_0$ large so that $\log \beta(n) \geq 2 \log 2 \geq \exp\left(\frac{n \log \beta(n)}{2\beta(n)}\right)$.

The collection $U(x) = \cup_{(z, t) \in S_\ell(x)} I_s(z)$ has size at least

$$|U(x)| = 2^s |S_\ell(x)| \geq 2^{\frac{n}{2} - \frac{1}{2} \alpha(n) + \frac{\log 2}{2} \frac{n \log \beta(n)}{\beta(n)}}.$$

Denoting $U = |U(x)| = |U(y)|$, the probability that the sets $U(x)$ and $U(y)$ are disconnected at the $n$-th step is therefore bounded from above by

$$\mathbb{P}[U(x) \neq U(y)] = \frac{2^{n-1} - U}{2^n} \leq \frac{2^{n-1} - U}{2^n} \frac{2^n}{2^n - U} \leq \frac{U - 1}{2^{n-1} - U} \leq \frac{1}{2^{n-1} - U}.$$  

(A.M.-G.M. inequality) $\leq \frac{2^{n-1} - \frac{3U - 1}{2}}{2^{n-1} - U} \leq \left(1 - \frac{U - 1}{2^{n-1} - U}\right)^U \leq \left(1 - \frac{1}{2^n}\right)^U \leq \exp\left(-U^2/2^n + U/2^n\right) \leq \exp\left(-2^{-\alpha(n) + \log 2 - \log \beta(n) + o(1)}\right)$.
Plugging in our choice of \( \alpha(n) \) and \( \beta(n) \), we get
\[
\Pr[U(x) \neq U(y)] \leq \exp \left( -2 \left( -\frac{17n}{\log n} + \log 2 \cdot \log \frac{\log 2}{\log \log n} + o(1) \right) \right)
\leq \exp \left( -2 \left( -\frac{17n}{\log n} + \frac{18n}{\log n} (1+o(1)) \right) \right)
= \exp \left( -2 \frac{18n}{\log n} (1+o(1)) \right)
\leq 2^{-3n}.
\]

As there are no more than \( 2^{2n} \) choices for the pairs \( x, y \), this gives
\[
\Pr[\exists x, y \text{ s.t. } U(x) \neq U(y)] \leq 2^{-3n} \cdot 2^{2n} = 2^{-n}.
\]

We have thus shown that
\[
\Pr[E_n \mid E_1, \ldots, E_{n-1}] \geq 1 - 2^n.
\]

The theorem then follows, recalling that \( \Pr[E_k] = 1 \) for \( k \leq n_0 \). \( \square \)

### 4.3 Vertex expansion

The proof of Theorem 9 relies on the observation that a set \( S \subseteq V_n \) sampled uniformly at random will be an \( \alpha \)-vertex expander with high probability (for some small constant \( \alpha > 0 \)), since a constant fraction of the edges of \( \sigma_{n-1} \) will go from \( S \) to its complement. This alone is not enough, since there are always sets of the form \( S = S_0 \cup S_1 \), where \( S_0 \subseteq V_n^0 \) (recall (4.1) for the definition of \( V_n^0 \)) and \( S_1 = \{N_n(x) \mid x \in S_0\} \). To overcome this, we look at edges coming from the last three permutations, \( \sigma_{n-1}, \sigma_{n-2}, \sigma_{n-3} \), and bound the number of sets \( S \subseteq V_n \) so that the boundary that comes from \( \sigma_{n-3} \)-edges isn’t large enough. Afterwards we apply a union bound over these sets to bound the probability that they have a small \( \sigma_{n-1} \)- and \( \sigma_{n-2} \)-boundary.

More precisely, sets which have a small contribution to their boundary at the \( k \)-th generation are defined as follows.

**Definition 23** (Badly-matched sets). Let \( x \in \{0,1\}^k \). Let \( A \subseteq V_n^0 \), \( B \subseteq V_n^1 \). We say that \( A, B \) are \((k, \alpha)\)-badly-matched if
\[
\frac{2 \cdot |x \in A \mid N_k(x) \in B|}{|A| + |B|} \geq (1 - \alpha).
\]

**Remark 24.** If \( A, B \) are badly-matched, then \(||A| - |B|| \leq \alpha (|A| + |B|)\). This is because if, say, \(|A| > |B| + \alpha (|A| + |B|)\) then even when all edges from \( B \) go into \( A \) there will still be \( \alpha (|A| + |B|) \) edges between \( A \) and \( V_k^1 \setminus B \). This implies that
\[
2 |x \in A \mid N_k(x) \in B| \leq 2 |B| < (1 - \alpha) (|A| + |B|) .
\]

If \( A, B \) are not badly-matched, then \( |\partial_k (A \cup B)| > \alpha |A \cup B| \), since \(|\partial_k (A \cup B)| = |A| + |B| - 2 |x \in A \mid N_k(x) \in B|\), so the set \( A \cup B \) has \( \alpha \)-expansion. If \( A, B \) are \((k, \alpha)\)-badly-matched, then they are also \((k, \alpha')\)-badly-matched for every \( \alpha' \geq \alpha \).

As alluded to above, we start by bounding the possible number of badly-matched sets in generation \( n - 2 \), for any permutation \( \sigma_{n-3} \); this is the content of Proposition 25. We then bound the probability that said badly-matched sets are also badly-matched in generations \( n - 1 \) and \( n \); this is the content of Proposition 26. The last claim we need for the proof is that sets of size \( O(n) \) have non-trivial vertex expansion regardless of the permutation. The proofs of all assertions are found at the end of the section.

Recall that for an integer \( k > 0 \), we set \( K := 2^k \). In addition, denote by \( H(x) = -x \log x - (1-x) \log(1-x) \) the binary entropy function.
**Proposition 25.** There exists a function $\delta : \mathbb{R} \to \mathbb{R}$ with $\lim_{x \to 0} \delta(x) = 0$, that depends on $\eta$, such that the following holds. Let $\alpha > 0$, and let $k, j > 0$ be integers so that $j \leq \eta K$. Then for any permutation $\sigma_{k-1}$, the number of $(k, \alpha)$-badly-matched sets $A \subseteq V_k^0$ and $B \subseteq V_k^1$ such that $(1-\alpha)\frac{j}{2} \leq |A|, |B| \leq (1+\alpha)\frac{j}{2}$ is smaller than
\[
5\alpha^3 K^3 2^{\frac{j}{2}} (1+\delta(\alpha)) H(\frac{j}{2}) + j \delta(\alpha).
\]

**Proposition 26.** There exists a function $\delta : \mathbb{R} \to \mathbb{R}$ with $\lim_{x \to 0} \delta(x) = 0$ that depends on $\eta$, such that the following holds. Let $\alpha > 0$, and let $k, j > 0$ be integers so that $j \leq \eta K$. Then $A \subseteq V_k^0$ and $B \subseteq V_k^1$ be such that $(1-\alpha)\frac{j}{2} \leq |A|, |B| \leq (1+\alpha)\frac{j}{2}$. If the permutations $\sigma$ are uniformly random, then
\[
P[A, B \text{ are } (k, \alpha)\text{-badly-matched}] \leq 3\alpha K^2 2^{-(1-\delta(\alpha)) H(\frac{j}{2}) + j \delta(\alpha)}.
\]

**Claim 27.** Let $c > 3$. Then there is some $n_0 \in \mathbb{N}$ so that for every $n > n_0$ and every $S \subseteq V_n$ so that $|S| \leq cn$,
\[
|\partial S| \geq \frac{1}{c^2} |S|.
\]

**Proof of Theorem 9.** Let $\alpha > 0$. We wish to bound
\[
P[F_n] \leq \sum_{j=0}^{\eta N} P[\exists S \text{ s.t. } |S| = j \text{ and } |\partial S| < \alpha |S|] \leq \frac{\eta N}{n}.
\]

By Claim 27, it is enough to start this sum with $j = cn$, as long as $c = \frac{1}{\alpha}$. The constant $c$ will be determined at the end of the proof. Thus the right-hand side of (4.4) is equal to
\[
\sum_{j=cn}^{\eta N} P[\exists S \text{ s.t. } |S| = j \text{ and } |\partial S| < \alpha |S|].
\]

Let $S \subseteq V_n$ with $|S| = j$. For $x \in \{0, 1\}$, denote $S_x = S \cap V_n^x$. If the sets $S_0, S_1$ are not $(n, \alpha)$-badly-matched, then by Remark 24, the edges from $\sigma_{n-1}$ are enough to guarantee a large boundary, i.e. the set $S$ has $\alpha$-expansion. This happens in particular when $|S_1| - |S_0| \geq \alpha |S|$. We may thus restrict ourselves to $S$ that satisfy $|S_i| \geq \frac{1}{2} (1 - \alpha) |S|$. Thus
\[
(1-\alpha)\frac{j}{2} \leq |S_i| \leq (1+\alpha)\frac{j}{2}
\]
for every $i \in \{0, 1\}$. Similarly, if $S_{i0}, S_{i1}$ are not $(n-1, 3\alpha)$-badly-matched for any $i \in \{0, 1\}$, then $|\partial_{n-1} (S_{i0} \cup S_{i1})| \geq 3\alpha |S_i| \geq \frac{3}{2} \alpha (1+\alpha) |S|$, which is larger than $\alpha |S|$ for $\alpha$ small enough. This happens in particular when $||S_{i1}|-|S_{i0}|| \geq 3\alpha |S_i|$. We may thus further restrict ourselves to $S$ that satisfy $|S_{ij}| \geq \frac{1}{2} (1 - 3\alpha) |S_i|$, which means that
\[
(1 - 3\alpha)(1-\alpha)\frac{j}{4} \leq |S_{ij}| \leq (1 + 3\alpha)(1+\alpha)\frac{j}{4}
\]
for every $i, j \in \{0, 1\}$.

Finally, if there is are sets $S_{ij0}, S_{ij1}$ for some $i, j \in \{0, 1\}$ that are not $(n - 2, 5\alpha)$-badly-matched, then for $\alpha$ small enough we have $|\partial_{n-2} (S_{ij0} \cup S_{ij1})| \geq \alpha |S|$, and we can assume that
\[
(1 - 5\alpha)(1 - 3\alpha)(1-\alpha)\frac{j}{8} \leq |S_{ij}| \leq (1 + 5\alpha)(1 + 3\alpha)(1+\alpha)\frac{j}{8}.
\]
In particular, this happens when
\[
(1 - 10\alpha)\frac{j}{8} \leq |S_{ij}| \leq (1 + 10\alpha)\frac{j}{8}.
\]
Thus, to bound \( P \left[ \exists S \text{ s.t. } |S| = j \text{ and } |\partial S| < \alpha |S| \right] \), we only need to consider sets \( S \) whose all four pairs \( S_{ij0}, S_{ij1} \) are \((n-2,10\alpha)\) badly-matched; any other set has \( \alpha \)-expansion by Remark 24. By Proposition 25, with \( k = n-2 \), for each \( i, j \in \{0,1\} \) there are at most \( 5000\alpha^3 N^3 2^{N(1+\delta(10\alpha))}H \left( \frac{1}{\alpha} \right) + 4j(10\alpha) \) sets \( S_{ij} \) such that \( S_{ij0}, S_{ij1} \) are \((n-2,10\alpha)\)-badly-matched, so there are at most

\[
5000\alpha^3 N^3 2^{N(1+\delta(10\alpha))}H \left( \frac{1}{\alpha} \right) + 4j(10\alpha)
\]

possible sets to consider. Thus

\[
P \left[ \exists S \text{ s.t. } |S| = j \text{ and } |\partial S| < \alpha |S| \right] \leq 5000\alpha^4 N^{12} 2^{N(1+\delta(10\alpha))}H \left( \frac{1}{\alpha} \right) + 4j(10\alpha).
\]

(4.7)

\[
\max_S P \left[ S \text{ does not have } \alpha\text{-expansion} \right].
\]

(4.8)

where \( S \) is restricted as above. To bound the probability, observe that for any fixed \( S \subseteq V_n \),

\[
P \left[ S \text{ does not have } \alpha\text{-expansion} \right] \leq P \left[ S_0, S_1 \text{ are } (n,\alpha)\text{-badly-matched} \right]
\]

\[
\text{and } S_{00}, S_{01} \text{ are } (n-1, 3\alpha)\text{-badly-matched}.
\]

As the event \( \{S_0, S_1 \text{ are badly-matched} \} \) depends only on the permutation \( \sigma_{n-1} \) and \( \{S_{00}, S_{11} \text{ are badly-matched} \} \) depends only on the permutation \( \sigma_{n-2} \), these two events are independent. By the relations (4.5) and (4.6), we can apply Proposition 26, yielding

\[
P \left[ S \text{ does not have } \alpha\text{-expansion} \right] = P \left[ S_0, S_1 \text{ are } (n,\alpha)\text{-badly-matched} \right]
\]

\[
\cdot P \left[ S_{00}, S_{01} \text{ are not } (n-1, 3\alpha)\text{-badly-matched} \right].
\]

\[
\leq 3\alpha N^2 2^{-\frac{N(1-\delta(\alpha)) H \left( \frac{1}{\alpha} \right) + \delta(\alpha) j}{4}} \cdot 3\alpha N^2 2^{-\frac{N(1-\delta(3\alpha)) H \left( \frac{1}{\alpha} \right) + \delta(3\alpha) j}{4}}.
\]

For simplicity, in the next inequalities we unify all the expressions of the form \( \delta(\alpha) \) appearing in the exponents to \( \delta(\alpha) \) (that goes to 0 as \( \alpha \to 0 \)). Using (4.7), we get that

\[
P \left[ \exists S \text{ s.t. } |S| = j \text{ and } |\partial S| < \alpha |S| \right] \leq 9 \cdot 5000^4 \alpha^{14} N^{16} 2^{\delta(\alpha) j} \left( 2^\frac{j}{4} \right)^{N H \left( \frac{1}{\alpha} \right)}.
\]

Note that we abused notation. Plugging this back in (4.4) we get

\[
P \left[ F_n \right] \leq 9 \cdot 5000^4 \alpha^{14} N^{16} \sum_{j=cn}^{\eta N} 2^{\delta(\alpha) j} \left( 2^\frac{j}{4} \right)^{N H \left( \frac{1}{\alpha} \right)}.
\]

(4.9)

We take \( \alpha \) so that \( \delta(\alpha) < \frac{1}{4} \) and get that \( 2^\frac{j}{4} \geq 2^j \). In addition, we use the well known inequality \( H(x) \geq 4x(1-x) \) to bound the right-hand side of (4.9) from above by

\[
9 \cdot 5000^4 \alpha^{14} N^{16} \sum_{j=cn}^{\eta N} 2^{\delta(\alpha) j} 2^{-\frac{j}{4}(1-\eta)} = 9 \cdot 5000^4 \alpha^{14} N^{16} \sum_{j=cn}^{\eta N} 2^{-\frac{j}{4}(1-\eta-2\delta(\alpha))}.
\]

Finally, by taking \( \alpha \) so that \( 1-\eta-2\delta(\alpha) \geq \frac{1-\eta}{2} \) and taking \( c \) so that \( 2^{c n/2} \left( \frac{1-\eta}{2} \right) \geq N^{16} = 2^{16n} \) we get that the sum on the right-hand side is at most

\[
9 \cdot 5000^4 \alpha^{14} N^{16} 2^{-\frac{c n/2 (1-\eta)}{4}} \sum_{j=\frac{5}{2} n}^{\infty} 2^{-\frac{j}{4}(1-\eta)} \leq 9 \cdot 5000^4 \alpha^{14} \sum_{j=\frac{5}{2} n}^{\infty} 2^{-\frac{j}{4}(1-\eta)}
\]

\[
\frac{9}{1-2^{-\frac{(1-\eta) n}{4}}} \cdot 5000^4 \alpha^{14} 2^{-\frac{c n/2 (1-\eta)}{4}}.
\]

Thus,

\[
P \left[ \bigcup_{n > n_0} F_n \right] \leq \sum_{n > n_0} P \left[ F_n \right] \leq \frac{9}{1-2^{-\frac{(1-\eta) n}{4}}} \cdot 5000^4 \alpha^{14} \sum_{n=n_0+1}^{\infty} 2^{-\frac{c n/2 (1-\eta)}{4}}.
\]

This series converges, hence this probability goes to 0 as \( n_0 \to \infty \).
Proof of Proposition 25. We can count the subsets \( A \subseteq V_k^0, B \subseteq V_k^1 \) by first choosing a set of edges of the \( k \)-th matching that are connected to \( A \cup B \). For each chosen edge \( \{x, y\} \) where \( x \in V_k^0, y \in V_k^1 \) we decide whether \( x \in A, y \in B \), or \( x \in A, y \notin B \) or \( x \notin A, y \in B \). For sets that are \((k, \alpha)\)-badly matched, our count yields the following.

1. The number of edges that are adjacent to \( A \cup B \) is at least \((1 - \alpha)\frac{\ell}{2}\) (the lower bound is achieved when \(|A| = |B| = (1 - \alpha)\frac{\ell}{2}\) and \(N_k(A) = B\)). It is at most \((1 + \alpha)\frac{\ell}{2} + 2\alpha(1 + \alpha)\frac{\ell}{2}\) (since there could be at most \((1 + \alpha)\frac{\ell}{2}\) that cross from \( A \) to \( B \), and no more than \(2\alpha(1 + \alpha)\frac{\ell}{2}\) additional edges that are adjacent to only one of \( A, B \), since \(A, B\) are supposed to be \((k, \alpha)\)-badly matched). Since \((1 + \alpha)\frac{\ell}{2} + 2\alpha(1 + \alpha)\frac{\ell}{2} \leq (1 + 4\alpha)\frac{\ell}{2}\) for \(\alpha \leq \frac{1}{2}\), using the relation \((\frac{\ell}{2})^2 \leq 2^nH(k/n)\), the number of possible choices for edges adjacent to \( A \cup B \) is at most

\[
\sum_{\ell=(1-\alpha)\frac{\ell}{2}}^{(1+4\alpha)\frac{\ell}{2}} \left(\frac{K}{\ell}\right)^{1/2} \leq \sum_{\ell=(1-\alpha)\frac{\ell}{2}}^{(1+4\alpha)\frac{\ell}{2}} 2^{\frac{K}{\ell}H(\frac{\ell}{2\ell})} \leq \sum_{\ell=(1-\alpha)\frac{\ell}{2}}^{(1+4\alpha)\frac{\ell}{2}} 2^{\frac{K}{\ell}H(\frac{\ell}{2\ell}) + \frac{K}{\ell}(H(\frac{\ell}{2\ell}) - H(\frac{j}{\ell}))}. \tag{4.10}
\]

By Lagrange’s mean-value theorem, \(\frac{K}{\ell} \left(H\left(\frac{\ell}{K}\right) - H\left(\frac{j}{\ell}\right)\right) = \left(\ell - \frac{j}{\ell}\right) H'(\xi) = -\left(\ell - \frac{j}{\ell}\right) \log \frac{\xi}{\ell}\) for some \(\xi\) between \(\frac{\ell}{K}\) and \(\frac{j}{\ell}\). Thus we can write \(\xi = c'\frac{j}{\ell}\) for some \((1 - \alpha) \leq c' \leq (1 + 4\alpha)\), and we have

\[
\left|\left(\ell - \frac{j}{\ell}\right) H'(\xi)\right| \leq 4\alpha j |H'(\xi)| = 4\alpha j |\log \xi - \log (1 - \xi)|. \tag{4.11}
\]

We now bound the logarithms. Since \(\xi = c'\frac{j}{\ell} \leq c'\eta\), we have that \(-\log (1 - \xi) \leq -\log (1 - c'\eta)\); the quantity on the right-hand side is just a constant (provided that \(\alpha\) is small enough so that \((1 + 4\alpha)\eta \leq 1\)). For \(|\log \xi|\), we have

\[
4\alpha j |\log \xi| = 4\alpha j \left|\log \left(c'\frac{j}{K}\right)\right| \leq 4\alpha j |\log c'| + 4\alpha j \log \frac{j}{K} = 4\alpha j |\log c'| + 8\left(\frac{K}{2}\frac{j}{\alpha K} \log \frac{j}{K}\right) \leq 4\alpha j |\log c'| + 8\left(\frac{K}{2}\alpha H\left(\frac{j}{K}\right)\right).
\]

Thus, there exists a constant \(c > 0\) (that depends on \(\eta\)) such that

\[
\left|\left(\ell - \frac{j}{\ell}\right) H'(\xi)\right| \leq c\alpha j + c\alpha \frac{K}{2} H\left(\frac{j}{K}\right).
\]

for some \(c > 0\) that depends on \(\eta\). Thus the left-hand side in (4.10) is at most

\[
5\alpha K \left(\frac{K}{2}\right)^{2} H\left(\frac{j}{K}\right) + c\alpha j. \tag{4.12}
\]

2. Then we choose out of the edges adjacent to \( A \cup B \) the edges that touch \( A \) only, and the edges that touch \( B \) only. As \( A, B \) are \((k, \alpha)\)-badly matched, at least a \((1 - \alpha)\)-fraction of the edges must touch both \( A \) and \( B \), so no more than an \(\alpha\)-fraction of the edges are available to touch only one of the sets. Assuming that \(\alpha < 1/4\), the number of possibilities (for a given edge set chosen in the previous step) is at most

\[
\sum_{\ell_A=0}^{\alpha(1+4\alpha)\frac{\ell}{2}} \left(\frac{1}{\ell_A}\right) \sum_{\ell_B=0}^{(1+4\alpha)\frac{\ell}{2}} \left(\frac{1}{\ell_B}\right) \leq \left(\frac{1}{\alpha(1+4\alpha)\frac{\ell}{2}}\right)^2 \leq \alpha^2 K^2 2^{2jH(\alpha)}. \tag{4.13}
\]
Multiplying (4.12) and (4.13), and setting $\delta(\alpha) = 2H(\alpha) + c\alpha$, the number of badly-matched pairs is bounded by
\[
5\alpha^3 K^3 2^{\frac{2}{3} \delta(\alpha) H(\frac{1}{K}) + \delta(\alpha) j}.
\]

**Proof of Proposition 26.** To bound the probability in (4.3), we go over all possible subsets $A' \subseteq A$ and sum the probability that the set of outgoing edges from $A'$ is some set $B' \subseteq B$. Since $A$ and $B$ both have sizes in the interval $[(1 - \alpha)\frac{2}{j}, (1 + \alpha)\frac{2}{j}]$, the size of $A', B'$ should be at least $(1 - \alpha)(1 - \alpha)\frac{2}{j} \geq (1 - 2\alpha)\frac{2}{j}$. The probability is bounded by
\[
\sum_{\ell = (1 - 2\alpha)\frac{2}{j}}^{(1 + \alpha)\frac{2}{j}} \sum_{A' \subseteq A, B' \subseteq B} \mathbb{P}[N_k(A') = B'] = \sum_{\ell = (1 - 2\alpha)\frac{2}{j}}^{(1 + \alpha)\frac{2}{j}} \sum_{A' \subseteq A, B' \subseteq B} \frac{1}{K^2 2^{\frac{2}{3} H(\frac{\ell}{K^2})}} \leq \sum_{\ell = (1 - 2\alpha)\frac{2}{j}}^{(1 + \alpha)\frac{2}{j}} \sum_{A' \subseteq A, B' \subseteq B} \frac{K}{2} 2^{\frac{2}{3} H(\frac{\ell}{K^2})} \leq 2^{2(1 + \alpha)\frac{2}{j} H(\frac{\ell}{K^2})} \frac{K}{2} \sum_{\ell = (1 - 2\alpha)\frac{2}{j}}^{(1 + \alpha)\frac{2}{j}} 2^{\frac{2}{3} H(\frac{\ell}{K^2})}.
\]

By Lagrange’s mean-value theorem, we write
\[
\frac{K}{2} H\left(\frac{\ell}{K^2}\right) = \frac{K}{2} H\left(\frac{j}{K}\right) + \frac{K}{2} \left(H\left(\frac{\ell}{K^2}\right) - H\left(\frac{j}{K}\right)\right) = \frac{K}{2} H\left(\frac{j}{K}\right) + \left(\ell - \frac{j}{2}\right) H'(\xi)
\]
for some $\xi$ between $\frac{\ell}{K^2}$ and $\frac{j}{K^2}$. As $|\ell - \frac{j}{2}| \leq \alpha j$, we bound $(\ell - \frac{j}{2}) H'(\xi)$ by $c\alpha H'(\xi)$. Write $\xi = \frac{c'}{K}$ for some $1 - 2\alpha \leq c' \leq 1 + 2\alpha$. Thus (similar to (4.11) in the proof of the previous proposition)
\[
\left|\left(\ell - \frac{j}{2}\right) H'(\xi)\right| \leq c\alpha j + c\alpha \left(\frac{K}{2}\right) H\left(\frac{j}{K}\right)
\]
for some constant $c > 0$ which only depends on $\eta$. Thus (4.14) is at most
\[
2^{2(1 + \alpha)\frac{2}{j} H(\frac{\ell}{K^2})} \frac{K}{2} \sum_{\ell = (1 - 2\alpha)\frac{2}{j}}^{(1 + \alpha)\frac{2}{j}} 2^{\frac{2}{3} (1 - \delta(\alpha)) H(\frac{1}{K}) + \delta(\alpha) j} \leq 3\alpha K^2 2^\frac{2}{3} (1 - \delta(\alpha)) H(\frac{1}{K}) + \delta(\alpha) j,
\]
where $\delta(\alpha) = \max\left\{(1 + \alpha) H\left(\frac{1 - 2\alpha}{1 + \alpha}\right), c\alpha\right\}$.

**Proof of Claim 27.** For every vertex $v \in V_n$, the second neighborhood of $v$, $A_n(v) := \{u \in V_n \mid d(v, u) = 2\}$, is of size at least $\binom{n}{2}$. This can be seen by induction. The base case for $n = 2$ is clear. Assume without loss of generality that $v \in V_n^0$ and partition $A_n(v) = (A_n(v) \cap V_n^0) \cup (A_n(v) \cap V_n^1)$. Note that in the copy of $G'_{n-1}$ whose vertex set is $V_0$, the second neighborhood of $v$ is $A_{n-1}(v) = A_n(v) \cap V_n^0$. Thus, by the induction hypothesis, $|A_{n-1}(v) \cap V_n^0| \geq \binom{n-1}{2}$. In addition, $A_n(v) \cap V_n^1$ contains the neighborhood of $N_n(v)$ inside $V_n^1$, which is of size $n-1$. Summing up sizes we get $|A_n(v)| \geq \binom{n-1}{2} + \binom{n-1}{2} = \binom{n}{2}$. Note that for the hypercube $Q_n$ we have strict equality.
Now fix $S \subseteq V_n$ of size at most $cn$ and let $v \in S$. If a $\frac{1}{c}$-fraction of the neighborhood of $v$ is not in $S$ then $|\partial S| \geq \frac{1}{c} n \geq \frac{1}{2} |S|$. Otherwise, at least $(1 - \frac{1}{c})$-fraction of $v$’s neighbors are inside $S$. Denote these vertices as $T := N(v) \cap S$. Thus $|\partial S| \geq |N(T)| - |S|$. Since the neighborhood of the neighborhood of $v$ is $A_n(v) \cup \{v\}$, the neighborhood of $T$ is of size at least $|A_n(v)| - n |N(v) \setminus S| \geq \left(\frac{n}{2}\right) - \frac{1}{c} n^2 \geq \frac{1}{7} n^2$.

Hence $|\partial S| \geq \frac{1}{7} n^2 - cn \geq \frac{1}{7c} |S|$ for a large enough $n$. □

### 4.4 Eigenvalues

**Proof of Proposition 13.** Since $G_n$ is $n$-regular, its largest eigenvalue $\lambda_1$ is $n$, and its corresponding eigenvector $f_n : \{0, 1\}^n \to \mathbb{R}$ satisfies $f_n(x) = 1$. To show that $\lambda_2 \geq n - 2$, let $g_n : \{0, 1\}^n \to \mathbb{R}$ be given by $g_n(x) = (-1)^{x_n}$, i.e. $g_n$ takes value 1 on the first copy of $G_{n-1}$ in $G_n$, and $-1$ on the second copy. Then $A_n g_n = (n-2) g_n$.

The proof that $\lambda_2 \leq n - 2$ is by induction. The claim clearly holds for $n = 1$, where $G_1$ is just an edge. Assume it holds for all $k \leq n - 1$, and let $h : \{0, 1\}^n \to \mathbb{R}$ be an eigenvector of $A_n$ that is orthogonal to both $f_n$ and $g_n$. If we write $h(x) = \begin{cases} h_0(x_1, \ldots, x_{n-1}) & x_n = 0 \\ h_1(x_1, \ldots, x_{n-1}) & x_n = 1, \end{cases}$

for some functions $h_i : \{0, 1\}^{n-1} \to \mathbb{R}$, then both $h_0$ and $h_1$ are orthogonal to $f_{n-1}$, and by the induction hypothesis, we have $h_i^T A_{n-1} h_i \leq (n-3) \|h_i\|^2_2$. Using the recursive matrix representation (1.1) of the duplicube graph, we can write

$$h^T A_n h = \begin{pmatrix} h_0^T & h_1^T \end{pmatrix} \begin{pmatrix} A_{n-1} & P \\ P^T & A_{n-1} \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \end{pmatrix},$$

where $P$ is the $2^{n-1} \times 2^{n-1}$ permutation matrix representing $\sigma_{n-1}$. Explicitly opening the products, we get

$$h^T A_n h = \begin{pmatrix} h_0^T & h_1^T \end{pmatrix} \begin{pmatrix} A_{n-1} h_0 + P h_1 \\ P^T h_0 + A_{n-1} h_1 \end{pmatrix} = h_0^T A_{n-1} h_0 + h_1^T P h_1 + P h_0 h_0 + h_1^T A_{n-1} h_1$$

$$\leq (n-3) \|h_0\|^2_2 + 2 h_0^T P h_1 + (n-3) \|h_1\|^2_2$$

$$= (n-3) \|h_2\|^2_2 + 2 h_0^T P h_1 \leq (n-3) \|h_2\|^2_2 + 2 \|h_0\|_2 \|h_1\|_2$$

$$\leq (n-3) \|h_2\|^2_2 + \|h_0\|^2_2 + \|h_1\|^2_2 = (n-2) \|h_2\|^2.$$

□

**Proof of Lemma 15.** In the following, $C$ is a constant depending on $k$ whose value may change from instance to instance. A set of edges $F \subseteq E(G_n)$ is said to be “finalized at generation $m$” if for every edge $\{x, y\} \in F$, $\gamma(x, y) \leq m$, and there exists at least one edge such that $\gamma(x, y) = m$. For a given $u \in V_n$, let $w = N_m(u)$ be its $m$-neighbor, let $E_m(u)$ be event that there exists a cycle of length no more than $k$ which contains the edge $\{u, w\}$ and is finalized at generation $m$.

We will now bound the probability of the event $E_m(u)$. Since $I_{m-1}(u) \neq I_{m-1}(w)$, i.e. $u$ and $w$ are found on different copies of $V_{m-1}$, in order for a cycle of length $\leq k$ to exist, there must also be an $m$-generation edge going from $I_{m-1}(w)$ back to $B_{<m}(u, k)$; otherwise, any path starting with the edge $\{u, w\}$ cannot reach $u$ again. In fact, this edge must be reachable from $w$ in at most $k$ steps. Let $W$ be the set of all $z \in I_{m-1}(w)$ such that there exists a simple path $P = (x_1, \ldots, x_t)$ with the following properties:
1. \( P \) is a shortest path from \( w \) to \( z \), and \( t \leq k \).
2. \( \gamma (x_i, x_{i+1}) \leq m \) for all \( i = 1, \ldots, t - 1 \).
3. \( \gamma (x_{t-1}, x_t) < m \).
4. \( P \) does not contain the edge \( \{u, w\} \).

In other words, \( W \) is the set of all vertices in \( I_{m-1}(w) \) which can be reached from \( w \) by a path of at most \( k \) edges of generation at most \( m \), and which can still send out an \( m \)-generation edge without backtracking. If there are no edges from \( W \) to \( B_{\leq m} (u,k) \), then there is no cycle of length \( \leq k \) which contains \( \{u, w\} \) (see Figure 4.4 for a graphical depiction).

![Figure 4.4](image)

Figure 4.4: There can be a cycle containing the edge \( \{u, w\} \) only if there is an \( m \)-generation edge crossing from some \( z \in W \) to \( B_{\leq m} (u,k) \). Since both \( B_{\leq m} (u,k) \) and \( W \) are small in comparison to \( I_{m-1}(u) \), the probability of this happening is small.

Given \( z \in W \), the probability that \( N_m (z) \in B_{\leq m} (u,k) \) depends only on the \( m \)-generation edges used in the path \( P \). Since \( \sigma_{m-1} \) is uniform, we can bound this probability by

\[
\mathbb{P}[N_m (z) \in B_{\leq m} (u,k) \mid z \in W] \leq \frac{|B_{\leq m} (u,k)|}{\max \{1, 2^{m-1} - k\}} \leq C \frac{m^{k+1}}{2^m}
\]

for some \( C > 0 \) which depends on \( k \) (we subtract \( k \) in the denominator, since in the worst case the path from \( w \) to \( z \) has at most \( k \) \( m \)-generation edges from \( I_{m-1}(w) \) to \( I_{m-1}(u) \setminus B_{\leq m} (u,k) \)). Since there are at most \( m^{k+1} \) vertices in \( W \), taking the union bound gives

\[
\mathbb{P}[E_m (u)] \leq C \frac{m^{2k+2}}{2^m}.
\]

Letting \( E_m = \cup_{u \in B(v, 2k)} E_m (u) \), we then have

\[
\mathbb{P}[E_m] \leq C \frac{m^{2k+2}}{2^m} n^{2k+1}.
\]

In particular, there exists a constant \( C > 0 \) such that

\[
\sum_{m > m_0} \mathbb{P}[E_m] \leq C 2^{-m_0} m_0^{2k+2} n^{2k+1}.
\]

If a vertex \( z \in B(v,k) \) is part of a cycle of length at most \( k \) which is finalized at generation \( m \), then necessarily there exists some \( u \in B(v, 2k) \) such that \( E_m (u) \) holds. Thus, if \( E_m \) holds for every \( m > m_0 \),
then $z$ can only be contained in cycles of length at most $k$ which are finalized at generation $\leq m_0$. The number of such cycles is bounded by
\[
\sum_{m=1}^{m_0} \sum_{i=1}^{k} m^i \leq \sum_{m=1}^{m_0} km^k \leq Cm_0^{k+1}
\]
for some constant $C > 0$. The probability of $F_v$ is then lower bounded by
\[
P[F_v] \geq P[\bigcap_{m > m_0} E_m]
\]
\[
= 1 - P[\bigcup_{m > m_0} E_m]
\]
\[
\geq 1 - \sum_{m > m_0} P[E_m]
\]
\[
\geq 1 - C2^{-m_0} m_0^{2k+2} 2^{k+1}
\]
as needed.

\[\square\]

Proof of Theorem 14. We use the moment method. While the main technique is classical (see e.g. [17]), we write the proof in full for completeness.

Proving that $\mu_n$ converges weakly to $\mu_{\text{circ}}$ in probability means that for every continuous function $f : \mathbb{R} \to \mathbb{R}$, we have convergence in probability of the expected value of $f$:
\[
\int_{\mathbb{R}} f d\mu_n \xrightarrow{P} \int_{\mathbb{R}} f d\mu_{\text{circ}} \quad (4.15)
\]
as $n \to \infty$. By the Weierstrass theorem, every continuous function on a closed interval can be arbitrarily well-approximated by a finite-degree polynomial. Since $\mu_{\text{circ}}$ is supported on a bounded interval, it suffices to show (4.15) for functions of the form $f_k = x^k$, i.e. showing that the $k$-th moments of $\mu_n$ converge to the $k$-th moments of $\mu_{\text{circ}}$. These moments are known, and are given by
\[
\int_{\mathbb{R}} x^k d\mu_{\text{circ}} = \begin{cases} 
C_{k/2} & \text{k is even} \\
0 & \text{k is odd}
\end{cases}
\]
where $C_m$ is the $m$-th Catalan number, and is equal to the number of ordered rooted trees with $m$ edges. We will first show that $\mathbb{E} \int_{\mathbb{R}} x^k d\mu_n \to \int_{\mathbb{R}} x^k d\mu_{\text{circ}}$, and then show that $\text{Var} \left( \int_{\mathbb{R}} x^k d\mu_n \right) \to 0$; by Chebyshev’s inequality, this implies the desired convergence in probability.

Since $\mu_n$ is just the empirical measure of the eigenvalues of $A/\sqrt{n}$, we have
\[
\int_{\mathbb{R}} x^k d\mu_n = \frac{1}{2^n} \sum_{i=1}^{2^n} \left( \frac{\lambda_i}{\sqrt{n}} \right)^k = \frac{1}{2^n} \text{Tr} \left( \frac{A}{\sqrt{n}} \right)^k
\]
\[
= \frac{1}{2^n n^{k/2}} \sum_{i_1, \ldots, i_k = 1}^{2^n} A_{i_1i_2} A_{i_2i_3} \cdots A_{i_{k-1}i_k} A_{i_ki_1}.
\]
For a fixed $i_1$, the sum $\sum_{i_2, \ldots, i_k} A_{i_1i_2} \cdots A_{i_{k-1}i_k}$ is the number of walks of length $k$ in $G_n$ that start and end at the vertex $i_1$. Let $X_v(k)$ be the simple random walk on $G_n$ which starts at vertex $v$. Then, since $G_n$ is $n$-regular, the number of simple random walks of length $k$ is $n^k$, and we have
\[
\int_{\mathbb{R}} x^k d\mu_n = \frac{1}{2^n n^{k/2}} \sum_{i=1}^{2^n} \mathbb{P} \left[ X_v(k) = i \right], \quad (4.16)
\]
where the probability is over the randomness induced by the random walk. Taking expectations over the measure induced by the permutations, we thus have, for any \( v \in V_n \),

\[
E \int x^k d\mu_n = \frac{1}{2n} n^{k/2} 2^n E [P \left[ X_v (k) = v \right]] = n^{k/2} E [P \left[ X_v (k) = v \right]].
\]

In the following, \( C \) is a constant depending on \( k \) whose value may change from instance to instance. Let \( m_0 = 8 (k + 1) \log_2 n \). By Lemma 15, with probability greater than \( 1 - C n^{-k} \), the event \( F_v \) holds, i.e. every vertex in \( B (v, k) \) is contained in no more than \( C m_0^{-1} \) cycles of length at most \( k \). By conditioning on \( F_n \), we have

\[
E [P \left[ X_v (k) = v \right]] = E [P \left[ X_v (k) = v \right] \mid F_n] P [F_n] + E [P \left[ X_v (k) = v \right] \mid F_n^c] P [F_n^c].
\]

The second term on the right-hand side is bounded below by 0 and above by

\[
E [P \left[ X_v (k) = v \right] \mid F_n^c] P [F_n^c] \leq P \left[ F_n^c \right] \leq C n^{-k} = o \left( n^{-k/2} \right).
\]

Since \( P \left[ F_n \right] = 1 - o (1) \), we then have

\[
E [P \left[ X_v (k) = v \right]] = (1 + o (1)) E [P \left[ X_v (k) = v \right] \mid F_n] + o \left( n^{-k/2} \right).
\]

To bound this term, we will count the number of random walks that return to the origin.

A step \((X_v (t), X_v (t + 1))\) is said to be a forward step if \( d_{G_n} (v, X_v (t)) < d_{G_n} (v, X_v (t + 1)) \), and a backward step if \( d_{G_n} (v, X_v (t)) \geq d_{G_n} (v, X_v (t + 1)) \). By analyzing the combinatorics of forward and backward steps, it was shown by McKay [17, Lemma 2.1] that in an \( n \)-regular graph where every ball \( B (v, k) \) has no cycles at all,

\[
\# \left\{ \text{Walks of length } k \text{ which return to the origin} \right\} = (1 + o (1)) n^{k/2} C_{k/2}.
\]  

(4.17)

We now show that under \( F_n \), the number of walks in \( G_n \) is of the same magnitude. Let \( \ell \) be the number of forward steps of the walk \( X_v (t) \) which are part of a cycle of length no larger than \( k \), and suppose that \( X_v (k) = v \).

If \( \ell = 0 \), then the walk must make \( k/2 \) forward steps and \( k/2 \) backward steps, since it returns to the origin. This means that \( k \) must be even, and the walk traces out a rooted tree with \( k/2 \) edges. Since the number of cycles with at most \( k \) edges is no larger than \( C (\log n)^{k+1} \), there are at least least \( n - C (\log n)^{k+1} - 1 \) choices for every forward step. By (4.17), the total number of walks with \( \ell = 0 \) is then equal to

\[
(1 + o (1)) n^{k/2} C_{k/2}
\]

when \( k \) is even, and 0 when \( k \) is odd.

If \( \ell > 0 \), then the walk makes \( \ell \) forward steps which are part of a cycle, and no more than \( k/2 - \ell \) forward steps which are not part of a cycle. There are no more than \( k \) backward steps, and each such step has no more than \( C (\log n)^{k+1} + 1 \) options. In total, the number of such walks is then bounded above by

\[
(1 + o (1)) n^{k/2 - \ell} \left( C (\log n)^{k+1} \right)^{k+\ell} = O \left( n^{k/2 - \ell} (\log n)^{4k} \right).
\]

Altogether, since the total number of walks of length \( k \) is \( n^k \), we have

\[
n^{k/2} E [P \left[ X_v (k) = v \right]] = n^{k/2} (1 + o (1)) \frac{1}{n^k} \left( n^{k/2} C_{k/2} + \sum_{\ell=1}^{k/2} O \left( n^{k/2 - \ell} (\log n)^{4k} \right) \right) = (1 + o (1)) C_{k/2}.
\]

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Lemma 28. The proof of Theorem 17 relies on the following lemma, whose proof we postpone to the end of this section.

4.5 Asymmetry

as needed.

All that is left is to show that the variance is small. By (4.16), the second moment of $\int x^k d\mu_n$ is given by

$$
\mathbb{E} \left[ \left( \int x^k d\mu_n \right)^2 \right] = \mathbb{E} \left[ \left( \frac{1}{2n} n^{k/2} \sum_{i=1}^{2n} \mathbb{P} [X_i (k) = i] \right)^2 \right]
= \mathbb{E} \left[ \frac{1}{2n} n^{k} \sum_{i,j=1}^{2n} \mathbb{P} [X_i (k) = i] \mathbb{P} [X_j (k) = j] \right].
$$

Set $m_0 = 16 (k + 1) \log_2 n$. Recall that for a vertex $v \in V_n$, $F_v$ is the event that each vertex in $B(v, k)$ is contained in no more than $C (m_0 + 1)^{k+1}$ cycles of length no more than $k$. Denote $F_{i,j} = F_i \cap F_j$. By Lemma 15, $\mathbb{P} [F_{i,j}] \geq 1 - 2C^2m_0n^{2k+2}n^{2k+1}$. By the of total probability, we have

$$
\mathbb{E} \left[ \left( \int x^k d\mu_n \right)^2 \right] = \frac{1}{2^{2n}} \sum_{i,j=1}^{2n} \mathbb{E} [n^k \mathbb{P} [X_i (k) = i] \mathbb{P} [X_j (k) = j] | F_{i,j}] \mathbb{P} [F_{i,j}]
+ \frac{1}{2^{2n}} \sum_{i,j=1}^{2n} \mathbb{E} [n^k \mathbb{P} [X_i (k) = i] \mathbb{P} [X_j (k) = j] | F_{i,j}^c] \mathbb{P} [F_{i,j}^c].
$$

The second term on the right-hand-side is bounded below by 0 and above, due to the choice of $m_0$, by $o(1)$. Thus

$$
\mathbb{E} \left[ \left( \int x^k d\mu_n \right)^2 \right] = (1 + o(1)) \frac{1}{2^{2n}} \sum_{i,j=1}^{2n} \mathbb{E} [n^k \mathbb{P} [X_i (k) = i] \mathbb{P} [X_j (k) = j] | F_{i,j}] + o(1).
$$

Using the same path-counting argument as above, by (4.17) we have that under $F_{i,j}$,

$$
\mathbb{E} n^k \mathbb{P} [X_i (k) = i] \mathbb{P} [X_j (k) = j] = (1 + o(1)) C_{k/2}^2,
$$

and taking the sum over all $i$ and $j$ shows that

$$
\mathbb{E} \left[ \left( \int x^k d\mu_n \right)^2 \right] = (1 + o(1)) \mathbb{E} \int x^k d\mu_n,
$$

which implies that $\text{Var} \left( \int x^k d\mu_n \right) \to 0$.

4.5 Asymmetry

The proof of Theorem 17 relies on the following lemma, whose proof we postpone to the end of this section.

Lemma 28. There exists a constant $C > 0$ such that the probability that there exists a decomposition of $V_n$ into two disjoint subsets other than $V_n^0 \cup V_n^1$ such that the edges between them form a matching is smaller than $Cn2^{-n}$.

Proof of Theorem 17. Let $X_n$ be the number of automorphisms of $G_n$. We partition these permutations into three kinds:

1. Automorphisms of $G_n$ that swap between $V_n^0$ and $V_n^1$. Let $W_n$ be the number of these automorphisms.

2. Automorphisms of $G_n$ that preserve both $V_n^0$ and $V_n^1$. Let $Y_n$ be the number of these automorphisms.

Note that $Y_n \geq 1$, since it always counts the trivial automorphism.
3. Automorphisms of $G_n$ that replace a proper subset $A_0 \subseteq V_n^0$ with a proper subset $A_1 \subseteq V_n^1$ of the same size (so that $V_n^0 \setminus A_0$ and $V_n^1 \setminus A_1$ stay inside $V_n^0$ and $V_n^1$, respectively). Let $Z_n$ be the number of these automorphisms.

If $\varphi$ is a non-trivial automorphism of the third kind, then the edges between $A_0$ and $V_n^0 \setminus A_0$ form a matching, and the edges between $A_1$ and $V_n^1 \setminus A_1$ form a matching (since, e.g. if there is a vertex $v \in A_0$ connected by more than one edge to $V_n^0 \setminus A_0$, then $\varphi(v)$ will have more than one edge across the main cut). But then, letting $A := A_0 \cup A_1$ and $B = V_n \setminus A$, we get that the edges between $A$ and $B$ form a matching as well, giving a partition $V_n = A \cup B$ with a matching between them. By Lemma 28, the probability that such a matching exists and, therefore, that there is a non-trivial automorphism swapping $A_0$ and $A_1$ is bounded by $O(n2^{-n})$. Thus, denoting by $F$ the event $F := \{ \exists n \in [n/20, n - 1] \text{ s.t. } Z_n > 0 \}$, we have

$$\mathbb{P}[F] = O\left(n^2 2^{-n/20}\right).$$

(4.18)

We turn to bound $Y_n, W_n$. For brevity, we abbreviate $\sigma := \sigma_{n-1}$. In the first two types, the values of an automorphism $\varphi$ on $V_n^0$ determines the value of $\varphi$ on all $V_n$. Explicitly, in the first case, for every $v \in V_{n-1}$, if we denote $\varphi(v, 0) = (\varphi_0(v), 1)$ and $\varphi(v, 1) = (\varphi_1(v), 0)$, then we must have $\varphi_0(v) = \sigma\varphi_1\sigma(v)$. For automorphisms of the second kind, we have similarly $\varphi_0 = \sigma^{1} \varphi_1 \sigma$. In both cases it must be that $\varphi_0, \varphi_1 \in \text{Aut}(G_{n-1})$. So in particular, $W_n, Y_n \leq X_{n-1}$, and

$$W_n + Y_n \leq 2X_{n-1}.$$  

(4.19)

We first bound $\mathbb{P}[W_n \geq 1 | X_{n-1}]$. By Markov’s inequality, this is at most $\mathbb{E}[W_n | X_{n-1}]$. Write out $W_n = \sum_{\varphi \in \text{Aut}(G_{n-1})} \sum_{\varphi_0 \in \text{Aut}(G_{n-1})} 1_{\varphi_1 = \sigma \varphi_0 \sigma} \cdot 1_{\varphi_0 \in \text{Aut}(G_{n-1})} \cdot 1_{\varphi_0 = \sigma \varphi_1 \sigma}$, and in particular we have

$$\mathbb{E}[W_n | X_{n-1}] \leq X_{n-1}^2 \max_{\varphi_0, \varphi_1} \{ \mathbb{P}[\varphi_0 = \sigma \varphi_1 \sigma] \mid \varphi_0, \varphi_1 \text{ bijections of } V_{n-1} \}.$$

So we need to bound $\mathbb{P}[\varphi_0 = \sigma \varphi_1 \sigma]$. Denote $A_0 = V_{n-1}$. For $v_0 \in A_0$, let $E_{v_0}$ be the event that $\varphi_0(v_0) = \sigma \varphi_1 \sigma(v_0)$. In order for $E_{v_0}$ to hold, we must have either i) $\varphi_1 \sigma(v_0) = v_0$ or ii) $\varphi_0(v_0) = \sigma \varphi_1 \sigma(v_0)$. The probability of the first is $1/2^{n-1}$, while the probability of the second given that $\varphi_1 \sigma(v_0) \neq v_0$ is $\frac{1}{2^{n-1}}$. In particular $E_{v_0}$ holds with probability no greater than $2/ (2^{n-1} - 1)$. Conditioned on $E_{v_0}$, the permutation $\sigma$ is a uniform permutation over the set $A_1 = A_0 \setminus \{v_0, \varphi_1 \sigma(v_0)\}$, with $|A_1| \geq |A_0| - 2$. By iteratively conditioning on $E_{v_0}, E_{v_0}, \ldots$, where $v_i \in A_i$, $i = 0, \ldots, 2^{n/3} - 1$, we have that $\mathbb{P}[\varphi_0 = \sigma \varphi_1 \sigma] \leq \frac{2}{2^{n-1} - 2^{n/3} + 1} \leq 2^{-\frac{2}{2^{n-1} - 2^{n/3} + 1}} \leq 2^{-\frac{2}{2^{n/3}}}$. Hence

$$\mathbb{E}[W_n | X_{n-1}] \leq 2^{-\frac{2}{2^{n/3}}} \cdot X_{n-1}^2.$$  

(4.20)

Next we bound $\mathbb{P}[Y_n > 1 | X_{n-1}, F^c]$. Although the equation $\varphi_0 = \sigma^{-1} \varphi_1 \sigma$ seems similar to the analogous equation $\varphi_0 = \sigma \varphi_1 \sigma$ for $W_n$, we shouldn’t expect the same argument to hold, since (for example) even if $X_{n-1} = 1$, we expect $W_n = 0$, whereas $Y_n \geq 1$ always since it counts the identity. The problem lies with automorphisms with small conjugacy classes. For a given $\varphi_1$ and uniformly random $\sigma$, the element $\sigma^{-1} \varphi_1 \sigma$ is a uniform element in the conjugacy class of $\varphi_1$. The probability $\mathbb{P}_\sigma[\varphi_0 = \sigma^{-1} \varphi_1 \sigma]$ is then bounded by one over the size of the conjugacy class of $\varphi_0$ (it is 0 if $\varphi_0$ and $\varphi_1$ are not conjugate). The following claim, whose proof is found at the end of the section, shows that under $F^c$, these classes must be large.

Claim 29. Assume that $F^c$ occurs. Then the conjugacy class for every $\text{Id} \neq \varphi \in \text{Aut}(G_{n-1})$ has size at least $2^{\frac{1}{2}n^2/4}$.

As in the case of $W_n$, we have

$$\mathbb{P}[Y_n > 1 | X_{n-1}, F^c] \leq \mathbb{E}[Y_n - 1 \mid X_{n-1}, F^c] \leq \mathbb{E}[X_{n-1}^2 \mid F^c] \max_{\varphi_0, \varphi_1 \in \text{Aut}(G_{n-1}) \setminus \{\text{Id}\}} \mathbb{P}_\sigma[\varphi_0 = \sigma^{-1} \varphi_1 \sigma] \leq \mathbb{E}[X_{n-1}^2 \mid F^c] 2^{-\frac{1}{2}n^2/4},$$

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where the last inequality is due to Claim 29, since the maximum is taken over elements with a conjugacy class of size at least $2^{\frac{1}{6}n^{2/3}}$.

Finally, under $F^c$, the only possible automorphisms for $m \in [n/20, n-1]$ are of the first two kinds, and by (4.19) we have

$$X_{n-1} \leq 2^{n-n/20-1}X_{n/20} \leq 2^{19n/20} \left(\frac{2n}{20}\right)! \leq 2^{n + \frac{1}{6}n2^{2/20}}.$$

(4.21)

Thus

$$\mathbb{P}[X_n > 1] \leq \mathbb{P}[F] + \mathbb{P}[F^c \cap \{X_n > 1\}]$$

$$\leq \mathbb{P}[F] + \mathbb{P}[Z_n > 0] + \mathbb{P}[F^c \cap \{W_n > 0\}] + \mathbb{P}[F^c \cap \{Y_n > 1\}]$$

$$\leq \mathbb{P}[F] + \mathbb{P}[Z_n > 0] + \mathbb{P}[W_n > 0, X_{n-1} \leq 2^{n + \frac{1}{6}n2^{2/20}}, F^c] + \mathbb{P}[Y_n > 1, X_{n-1} \leq 2^{n + \frac{1}{6}n2^{2/20}}, F^c]$$

$$\leq \mathbb{P}[F] + \mathbb{P}[Z_n > 0] + \frac{2^{2n+\frac{1}{6}n2^{2/20}}}{2^{(n-7)2^{2/3}}} + \frac{2^{2n+\frac{1}{6}n2^{2/20}}}{2^{4n2^{4/4}}} \rightarrow 0, \quad n \to \infty$$

as needed.

**Proof of Lemma 28.** With start with some preliminaries which will be of use later on in the proof. Let $V_n = A \sqcup B$ be a uniformly random partition of $V_n$ into two halves of equal size, and let $V_n = A' \sqcup B'$ be a partition where $A'$ is a binomial random subset of $V_n$ with success probability $1/2$. The difference between these two random partitions can be quantified as follows: for any arbitrary set $\Sigma$ of equal-sized partitions of $V_n$, we have

$$\mathbb{P}\{(A, B) \in \Sigma\} = \mathbb{P}\{(A', B') \in \Sigma \mid |A'| = 2^{n-1}\} \leq \frac{\mathbb{P}\{(A', B') \in \Sigma\}}{\mathbb{P}\{|A'| = 2^{n-1}\}}.$$  

(4.22)

The denominator in the right-hand side can be approximated by the de Moivre-Laplace limit theorem, which states that

$$\mathbb{P}\{|A'| = 2^{n-1}\} = \frac{1 + o(1)}{\sqrt{\pi}} 2^{(1-n)/2}.$$  

(4.23)

Note that $V_n$ contains a vertex-disjoint union of $2^{n-2}$ copies of $C_4$, the cycle of length 4, and let $\Sigma$ be the set of all equal-sized partitions which do not separate two diagonals of any of these cycles. By the Chernoff bound, $\mathbb{P}\{|A' \sqcup B' \in \Sigma\} \leq e^{-2^{-n-6}},$ and so by (4.22) and (4.23), we have

$$\mathbb{P}\{(A, B) \in \Sigma\} \leq e^{-2^{-n-6}} \frac{\sqrt{\pi}}{1 + o(1)} 2^{(n-1)/2} \leq c \cdot 2^{n-2^{-n-6}}$$  

(4.24)

for some constant $c > 0$. Let us now choose $C$ so large that the lemma is true for all $n \leq n_0$ for some large enough $n_0$. We proceed by induction on $n$. Assume that the lemma is true for $G_{n-1}$. For a decomposition $V_n = A \sqcup B$ other than $V_n^0 \cup V_n^1$, let $E_n (A, B)$ be the event that the edges between $A$ and $B$ form a matching. Let $p_n := \mathbb{P}\{\exists A, B \text{ s.t. } E_n (A, B)\}$. We will show that $p_{n-1} \leq C(n-1) 2^{-(n-1)}$ implies $p_n \leq Cn2^{-n}$.

Fix a decomposition $V_n = A \sqcup B$ and set $A_j = V_n^j \cap A$ and $B_j = V_n^j \cap B$ for $j = 0, 1$, so that $A_0 \cup B_0 = V_n^0$ and $A_1 \cup B_1 = V_n^1$. We consider three cases.

1. If both cuts coincide with the cuts of the $(n-1)$-th generation (i.e. $A_0 \cup B_0 = V_n^{00} \cup V_n^{10}$ and $A_1 \cup B_1 = V_n^{01} \cup V_n^{11}$), then for $A \sqcup B$ to induce a matching, all vertices of $A_0$ should send the edges of the last generation to $A_1$, and all vertices of $B_0$ should send the edges of the last generation to $B_1$. The probability of this event is exactly $\left(\frac{2^{n-1}}{2^{n-2}}\right)^{-1}$.

2. If both cuts differ from the $(n-1)$-th generation cuts, then assume first that $A_0$ is empty. By connectivity of $G_{n-1}$, there exists an edge between some vertex $a \in A_1$ and a vertex $b \in B_1$. Since $a$ also sends an edge to $B_0$ (induced by $\sigma_{n-1}$), in this case the edges do not form a matching. We can therefore assume that all of $A_0, B_0, A_1, B_1$ are non-empty. Assume without loss of generality that $|B_1| \geq |A_1|$. Let $a \in A_0$ be a vertex that sends an edge to $B_0$ (again, such a vertex exists by connectivity of $G_{n-1}$).
Since $|B_1| \geq |A_1|$, the probability that there are no edges from $a$ to $B_1$ induced by $\sigma_{n-1}$ is at most $1/2$, so with probability greater than $1/2$ we do not get a matching. We get that
\[
P \left[ \exists A, B \text{ s.t. } \{ A_0 \cup B_0 \neq V_{n}^{00} \cup V_{n}^{10}, A_1 \cup B_1 \neq V_{n}^{01} \cup V_{n}^{11} \} \cap E_n (A, B) \right] \leq \frac{1}{2}p_{n-1}.
\]

3. Finally, if, say, the cut $A_0 \cup B_0$ coincides with the respective $(n-1)$-th generation cut $V_{n}^{00} \cup V_{n}^{10}$, and $A_1 \cup B_1 \neq V_{n}^{01} \cup V_{n}^{11}$, then $A_1 \cup B_1$ should divide the set $V_1$ into halves; otherwise (say, if $|A_1| > |B_1|$), $B_0$ sends at least one $n$-th generation edge to $A_1$, and so there is a vertex in $B_0$ with at least two neighbors in $A$, and we do not get a matching. Moreover, we may also claim that the $n$-th generation edges form a matching between $A_0$ and $A_1$, and between $B_0$ and $B_1$ (since there is a matching between $A_0$ and $B_0$, $A_0$ cannot have an edge with $B_1$, and $B_0$ cannot have an edge with $A_1$). Then the desired probability is exactly the probability that the cut $A_1 \cup B_1$ of $G_{n-1}$ forms a matching. Since the ends of the edges of the matching between $V_{n}^{0}$ and $V_{n}^{1}$, with first vertices in sets $V_{n}^{0}$ and $V_{n}^{10}$, form a decomposition of $G_{n-1}$ into halves which is independent of $G_{n-1}$ itself, by (4.24), the latter probability is bounded by $c \cdot 2^{n-1/2^n-1}$.

Putting all these together, we get
\[
p_n \leq \frac{1}{2}p_{n-1} + \left( \frac{2^{n-1}}{2^{n-2}} \right)^{-1} + c \cdot 2^{n-2^{n-1}} < Cn2^{-n}
\]
for $n > n_0$ large enough. \hfill \qed

**Proof of Claim 29.** Assume that $F^c$ holds, i.e. every automorphism of $G_m$ either swaps or preserves $V_{m}^{0}$, $V_{m}^{1}$ for $m \in [n/20, n-1]$. We first show that every non-identity $\varphi \in Aut(G_{n-1})$ has at least $2^{19n/20}$ points that are not fixed.

Let $m \in [n/20, n-1]$, and let $\psi \in Aut(G_{m})$. Since $F^c$ holds, $\psi$ either swaps or preserves $V_{m}^{0}$ and $V_{m}^{1}$, and so can be represented by the pair $(\psi_0, \psi_1)$ as above. If it swaps $V_{m}^{0}$ and $V_{m}^{1}$, then it has no fixed points. Hence, if $\psi$ has any fixed points, it must be preserving, and its fixed points are a union of the fixed points of $\psi_0$ and $\psi_1$. In this case $\psi_0$ and $\psi_1$ are conjugate, so they have the same number of fixed points; in particular, the number of fixed points (resp. non-fixed points) of $\psi$ is equal to twice the number of fixed points (resp. non-fixed points) of $\psi_0$.

Thus by induction, if $\varphi \in Aut(G_{n-1})$ has any fixed points, then the number of non-fixed points is equal to $2^{n-1-n/20}$ times the number of non-fixed points of any of its restrictions $\psi := \varphi|_{\mathbb{Z}^2}$, where $z \in \{0, 1\}^{n-1-n/20}$. If $\psi$ is the identity, then $\varphi$ is the identity also. Otherwise, $\psi$ has at least $2$ non-fixed points, and so $\varphi$ has at least $2^{19n/20}$ non-fixed points on $G_{n-1}$.

Next we get our bound for the size of the conjugacy class of $\varphi$. Recall that we can express $\varphi$ as a composition of disjoint cycles.

1. If $\varphi$ has a cycle of length $m \geq 2^{2n/5} + 1$, then the number of conjugacy classes is bounded below by the number of conjugacy classes where (say) $1$ is in such a cycle. The number of such permutations is at least $\binom{2^{n-1}}{m-1} \cdot (m-1)!$ (since we need to choose $m-1$ more elements, and then order them in a cycle together with $1$). This is $(2^{n-1} - 1) \cdot (2^{n-1} - 2) \cdot \ldots \cdot (2^{n-1} - m) \geq 2^{(n-2)2^{n/5}}$.

2. Otherwise, the maximal cycle length is at most $2^{2n/5}$. We have at least $2^{19n/20}$ points which are not fixed, so there are at least $r = 2^{11n/20}$ cycles. The number of conjugacy classes is then lower-bounded by the number of conjugacy classes where all the elements $1, 2, \ldots, r$ are in distinct cycles, and $r+1, r+2, \ldots, 2r$ are in the same distinct cycles. For every fixed choice of cycles for the first $r$ elements, there are $r!$ ways to choose where to put $r+1, r+2, \ldots, 2r$. This is at least $2^{11n/20} \geq 2^{4n2^{n/4}}$. \hfill \qed
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