The balanced 2-median and 2-maxian problems on a tree

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Accepted: 1 February 2023 / Published online: 20 February 2023
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Abstract
This paper deals with the facility location problems on a balancing allocation approach. Two location models are proposed, which are convex combinations of the traditional \( p \)-median and \( p \)-maxian problems together with minimizing the maximum demand level allocated to any facility. An edge deletion method with time complexity \( O(n^2) \) is represented for the balanced 2-median problem on a tree. For the balanced 2-maxian problem, it is shown the optimal solution is two end vertices of the longest path of the tree, which can be obtained in linear time.

Keywords Facility location · 2-maxian · 2-median · Balanced allocation

1 Introduction

In the \( p \)-median and \( p \)-maxian problems a connected network \( G = (V, E, w, l) \) with vertex set \( V \), edge set \( E \), weight function \( w : V \rightarrow R^+ \) and length function \( l : E \rightarrow R^+ \) are given. The weight \( w_i \) is the number of clients on vertex \( v_i \) and is also called the demand for node \( v_i \). Let \( d(v_i, v_j) \) be the distance between vertices \( v_i \) and \( v_j \). The \( p \)-median problem is to find a set of \( p \) vertices, \( X_p = \{x_1, ..., x_p\} \) called facilities, so that the sum of the weighted distances from the vertices to the closest facility in \( X_p \) is minimized:

\[
f_1(X_p) := \sum_{i=1}^{n} w_i \min_{x_j \in X_p} d(x_j, v_i).
\]
The obnoxious case of the $p$-median problem is called the $p$-maxian problem. In the $p$-maxian problem the goal is maximizing the sum of the weighted distances from the vertices to the farthest facility in $X_p$:

$$f_2(X_p) := \sum_{i=1}^{n} w_i \max_{x_j \in X_p} d(x_j, v_i).$$

Kariv and Hakimi (1979) showed that the $p$-median problem is $NP$-hard on general networks, while it can be solved in polynomial time on tree networks. Initial work on the $p$-median problem is by Hakimi (1964, 1965). Hakimi (1965) showed that at least one optimal solution of the $p$-median problem is located on vertices. Kariv and Hakimi (1979) presented an $O(p^2 n^2)$-time algorithm for this problem on a tree. The time complexity is improved to $O(p n^2)$ by Tamir (1996). In the case $p = 2$ on a tree, Gavish and Sridhar (1995) presented an $O(n \log n)$ algorithm and Burkard et al. (2000) proposed an $O(n^2)$ algorithm when the weight of vertices can be positive or negative. The positive values represent desirable facilities, while the negative values represent the undesirable facilities.

The $NP$-hardness of the $p$-maxian problem on general networks is shown by Hansen and Moon (Hansen and Moon 1988). Burkard et al. (2007) showed that the optimal solution to the 2-maxian problem on a tree lies on the two end vertices of the longest path of the tree. They also showed that the $p$-maxian problem on the tree is reduced to the 2-maxian. Based on these findings, they gave a linear time algorithm for the $p$-maxian problem on a tree. The $p$-maxian problem with an upper bound on the distance of facilities has been investigated by Nguyen et al. (2020). They presented a linear time algorithm for solving this problem.

The equity location models have been introduced in the last three decades. In these models, the locations of facilities are adjusted so that the demand attracted to each facility will be as close as possible to one another. Berman et al. (2009) formulated this problem by minimizing the maximum weights assigned to various facilities. Marin (Marin 2011) considered the balanced discrete location problem, in which the objective function is allocating clients to the facilities such that the difference between the maximum and the minimum allocated clients to the different facilities is minimized. Daskin and Tucker (2018) investigated the trade-off between minimizing the average distance and minimizing the range in assigned demand to the facilities. By adding some new constraints to the classical $p$-median problem, they presented a mathematical model for this problem. Moreover, they proposed a genetic algorithm for solving this problem. For a comprehensive study on the equity measurement in location theory, we refer the interested reader to Marsh and Schilling (1994); Eiselt and Laporte (1995).

Note that the $p$-median and $p$-maxian problems are concerned with the optimization of transportation time, while equity location models consider the minimization of the total service time. In this paper, a combination of transportation and service times is applied as the objective function. The $p$-median and $p$-maxian objective functions have been considered for evaluating the transportation time while a congestion avoidance function has been proposed to account for service time. We call these problems as
balanced $p$-median and $p$-maxian problems, respectively. To balance service times of facilities, minimizing the maximum number of clients served by facilities is applied.

In the next section, the balanced $p$-median and $p$-maxian problems are formulated. In Sect. 3, an $O(n^2)$-time algorithm for the balanced 2-median problem on a tree is represented. The balanced 2-maxian problem on a tree is investigated in Sect. 4, and a linear time algorithm is proposed for this problem.

2 Problem definition

As the balancing function, we consider minimizing the maximum sum of service time of clients by each facility. It is supposed the service time of clients are equal for all facilities. In the balancing model, we should find a partition $G_1 = (V_1, E_1), \ldots, G_p = (V_p, E_p)$ containing $p$ connected sub-graphs of $G$ and allocate each vertex $u \in V_j$, $j = 1, \ldots, p$ to a facility server $x_j$, so that the maximum sum of the service time with each facility is minimized. Hence, minimizing the following objective function is considered:

$$f_3 = \max \{ \sum_{v_i \in V_j} w_i, \ j = 1, \ldots, p \}. \quad (1)$$

The problem of minimizing objective function $f_3$ is considered by Berman et al. (2009). They showed this problem is $NP$-hard on general networks.

In the following lemma, we present an equivalent objective function of $f_3$ in the case $p = 2$. The obtained result would be used in the next sections.

**Lemma 1** In the case $p = 2$, the model $f_3$ can be represented as the following,

$$f_4 = | \sum_{v_i \in V_1} w_i - \sum_{v_i \in V_2} w_i |. \quad (2)$$

**Proof** Let $W = \sum_{i=1}^n w_i$, then in the case $p = 2$, the objective function $f_3 = \max \{ \sum_{v_i \in V_j} w_i, \ j = 1, \ldots, p \}$ can be represented as

$$f_3 = \max \{ \sum_{v_i \in V_1} w_i, \ \sum_{v_i \in V_2} w_i \}$$

$$= \sum_{v_i \in V_1} w_i + \sum_{v_i \in V_2} w_i + | \sum_{v_i \in V_1} w_i - \sum_{v_i \in V_2} w_i |$$

$$= W + | \sum_{v_i \in V_1} w_i - \sum_{v_i \in V_2} w_i | \cdot \frac{2}{2}.$$ 

The second equality is because, for real numbers $A$ and $B$, that $A \geq B$, we have $\max\{A, B\} = A$, and

$$\frac{A + B + |A - B|}{2} = \frac{A + B + A - B}{2} = A.$$
Moreover, since $W$ is a fixed value, it can be eliminated from the objective function. \hfill \Box

Lemma 1 yields, in the case $p = 2$, any partition of $V$ between $V_1$ and $V_2$, where $\sum_{v_i \in V_1} w_i = \sum_{v_i \in V_2} w_i$, provides an optimal solution to $\min f_3$. Note that in the balancing model just the service time is considered and locations of facilities are not essential. However, in the real applications the transportation time is also a significant factor, and it depends on locations of facilities. Therefore, we consider a combination of balancing on service and transportation times, as well.

We define the balanced $p$-median problem as finding a partition of $G$ which contains $p$ connected sub-graphs $G_1 = (V_1, E_1), \ldots, G_p = (V_p, E_p)$ and its corresponding set $X_p = \{x_1, \ldots, x_p\}$ containing $p$ vertices of $G$ so that the following objective function is minimized,

$$f_{pmed}(X_p) := \lambda \frac{1}{M_1} \sum_{j=1}^p \sum_{v_i \in V_j} w_i d(x_j, v_i) + \frac{(1-\lambda)}{M_2} f_3,$$

where $0 \leq \lambda \leq 1$ and $M_1$ and $M_2$ are normalization coefficients for the median and balanced parts of the objective function, respectively. Since the values of these two parts of objective function may not be comparable in range, the normalize coefficients are used. One way is setting $M_1$ and $M_2$ equal to the maximum values that the median and balanced parts of the objective function can be achieved, respectively. Thus, in this paper, the coefficient $M_1$ is chosen as the value of the objective function of the 1-maxian problem and $M_2$ is the total service time, i.e.

$$M_2 = \sum_{i=1}^n w_i.$$

Similarly, for the balanced $p$-maxian problem, the set $X_p$ and its corresponding partition with maximizing the following objective function is sought,

$$f_{pmax}(X_p) := \frac{\lambda}{M_3} \sum_{j=1}^p \sum_{v_i \in V_j} w_i d(x_j, v_i) - \frac{(1-\lambda)}{M_2} f_3,$$

where $M_3$ is the $p$-maxian objective value function. Note that, in these models, $1 - \lambda$ is interpreted as the servers’ balanced coefficient.

**Theorem 1** The balanced $p$-median and $p$-maxian problems on general networks are NP-hard.

**Proof** In the balanced $p$-median objective function let

$$f_3 = \max \{ \sum_{v_i \in V_j} w_i, \quad j = 1, \ldots, p \} = \sum_{v_i \in V_k} w_i,$$

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where \( k \in \{1, \ldots, p\} \). Therefore, the balanced objective function (3) would be rewritten as follows:

\[
f_{p\text{med}}(X_p) = \frac{\lambda}{M_1} \sum_{j=1}^{p} \sum_{v_i \in V_j} w_i d(x_j, v_i) + \frac{(1 - \lambda)}{M_2} \sum_{v_i \in V_k} w_i
\]

\[
= \frac{(1 - \lambda)}{M_2} \left( \sum_{j=1, j \neq k}^{p} \sum_{v_i \in V_j} \bar{\lambda} w_i d(x_j, v_i) + \sum_{v_i \in V_k} \bar{\lambda} w_i (d(x_k, v_i) + \frac{1}{\lambda}) \right)
\]

\[
= \sum_{j=1, j \neq k}^{p} \sum_{v_i \in V_j} \bar{w}_i d(x_j, v_i) + \sum_{v_i \in V_k} \bar{w}_i \bar{d}(x_k, v_i)
\]

where \( \bar{\lambda} = \frac{\lambda / M_1}{(1 - \lambda) / M_2} \) and \( \bar{w}_i = \frac{1 - \lambda}{M_2} \lambda w_i \), for \( i = 1, \ldots, n \), and

\[
\bar{d}(x_j, v_i) = \begin{cases} 
  d(x_j, v_i) & \text{if } j \neq k \text{ and } v_i \in V_j \\
  d(x_k, v_i) + \frac{1}{\lambda} & \text{if } v_i \in V_k.
\end{cases}
\]

This problem formulation proposes a new \( p \)-median problem, which is also \( NP \)-hard on general graphs. The proof for the balanced \( p \)-maxian problem is similar. Therefore, the balanced \( p \)-median and \( p \)-maxian problems are \( NP \)-hard on general graphs. \( \square \)

In the rest of this paper, we focus on the case that the underlying network is a tree. Let \( T = (V, E) \) be the given tree.

### 3 Balanced 2-median problem on a tree

In this section, the balanced 2-median problem on a tree network is considered. Using Lemma 1, for the balanced 2-median problem, a partition of \( T \) containing two subtrees \( T_1 = (V_1, E_1) \) and \( T_2 = (V_2, E_2) \) should be found so that the following objective function is minimized.

\[
f_{p\text{med}}(X_2) := \frac{\lambda}{M_1} \left( \sum_{v_i \in T_1} w_i d(v_i, x_1) + \sum_{v_i \in T_2} w_i d(v_i, x_2) \right)
\]

\[
+ \frac{1 - \lambda}{M_2} \left| \sum_{v_i \in T_1} w_i - \sum_{v_i \in T_2} w_i \right|.
\]  

In the classical 2-median problem each client is allocated to the closer facility where \( x_i \in V_i \) for \( i = 1, 2 \). However, in the balanced case, each client may be allocated either
to the closer or to the farther facility. The following example shows that the customers may not be assigned to the closer median, in the optimal solution to Problem (5).

**Example 1** Consider the tree depicted in Fig. 1, where the weights and service times of all vertices are equal to one. Then $M_1 = 18$ and $M_2 = 6$. The solution of the balanced 2-median problem for the case $\lambda = \frac{1}{2}$ is $\{v_2, v_4\}$. If $v_3$ is allocated to $v_4$ then the objective function value is $\frac{1}{2} \left( \frac{4}{18} + \frac{2}{6} \right) = 0.2778$ while allocating $v_3$ to $v_2$ results in the objective function value equal to $\frac{1}{2} \left( \frac{5}{18} + 0 \right) = 0.1389$. Therefore, the optimal solution is obtained by assigning $v_3$ to $v_2$ instead of the closer node $v_4$.

Since, locations of facilities do not affect the balancing part of objective function the following property holds.

**Lemma 2** There exists an optimal solution $X^* = \{x^*_1, x^*_2\}$ to the balanced 2-median problem on a tree network, so that $x^*_j \in V_j$, for $j = 1, 2$.

**Proof** Obviously, we should consider the median part of the objective function. Let $X = \{x_1, x_2\}$ be a set that contains two vertices of $T$ so that $x_1 \not\in V_1$. Let $u$ be the closest node in $V_1$ to $x_1$, then since all nodes in the sub-tree induced by $V_1$ are connected to $x_1$ via $u$, by moving from $x_1$ toward $u$, the objective function will be reduced or unchanged. Thus $f_{pmed}(X') \geq f_{pmed}(X)$, where $X' = \{u, x_2\}$. □

A well-known method to solve the classical 2-median problem on a tree network is edge deletion method (see (Oudjit and Stallmann 2021) and references therein). In this method, by deleting any edge the medians of the obtained sub-trees are found and the best one is chosen as the solution to the 2-median problem. This method can also be extended to solve the balanced 2-median problem.

**Theorem 2** The optimal solution to the balanced 2-median problem on a tree can be found in $O(n^2)$ time.

**Proof** Note that dividing the given tree into two sub-trees is necessary to solve the problem. Moreover, although the clients may not be allocated to the closer facility, based on Lemma 2, there exist an optimal solution $X^* = \{x^*_1, x^*_2\}$ so that $x^*_j \in V_j$, $j = 1, 2$, where all vertices in $V_1$ are assigned to $x^*_1$ and all vertices in $V_2$ are assigned to $x^*_2$. Therefore, after dividing the tree into two sub-trees, the balanced part of the
objective function is fixed and it suffices to solve a 1-median problem in each sub-tree. The edge deletion method would be applied as follows.

Let $T_1$ and $T_2$ be the two sub-trees of $T$ which are obtained by deleting edge $e \in E$. Also, let $m_1$ and $m_2$ be the 1-median solutions for $T_1$ and $T_2$, respectively. The value of objective function $f_{pmed}(.)$ is calculated as follows:

$$f_{pmed}(m_1, m_2) := \frac{\lambda}{M_1} \left( \sum_{v_i \in T_1} w_i d(v_i, m_1) + \sum_{v_i \in T_2} w_i d(v_i, m_2) \right)$$

$$+ \frac{1 - \lambda}{M_2} | \sum_{v_i \in T_1} w_i - \sum_{v_i \in T_2} w_i |.$$

(6)

Then, the pairs of medians corresponding to the minimum value of Problem (6) are chosen as the solution to the balanced 2-median problem. Therefore, the time complexity of this method is $O(n^2)$.

4 Balanced 2-maxian problem on a tree

In this section, we consider the balanced 2-maxian problem on the tree $T$. The goal of the balanced 2-maxian model, is finding a partition of $T$, containing two sub-trees $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ so that the following objective function is maximized.

$$f_{pmax}(X_2) = \frac{\lambda}{M_3} \left( \sum_{v_i \in T_1} w_i d(v_i, x_1) + \sum_{v_i \in T_2} w_i d(v_i, x_2) \right)$$

$$- \frac{1 - \lambda}{M_2} | \sum_{v_i \in T_1} w_i - \sum_{v_i \in T_2} w_i |.$$

(7)

In the case $\lambda = 1$, the balanced 2-maxian problem reduces to the ordinary 2-maxian problem. Burkard et al. (2007) showed that the two end vertices of a longest path of the tree is the solution for the 2-maxian problem. The longest path is a path that the sum of lengths of its edges is maximum.

In the traditional 2-maxian problem $x_i \notin T_i$ for $i = 1, 2$. We consider this property as an assumption in the balanced 2-maxian problem i.e. suppose $x_1 \in T_2$ and $x_2 \in T_1$. Note that the parameter $\lambda$ is referred to the priority or importance of two terms in the balanced 2-maxian problem, hence, we provide a range in which the 2-maxian term is more important and identifies the first priority. Based on this selection, then a threshold is provided for the $\lambda$ where properties of the classical 2-maxian problem would be extended to the proposed balanced 2-maxian problem.
Let us rewrite the balanced 2-maxian problem as follows:

\[
f_{p_{\text{max}}}^{2}(X_2) = \tilde{\lambda} \left( \sum_{v_i \in T_1} w_i d(v_i, x_1) + \sum_{v_i \in T_2} w_i d(v_i, x_2) \right) - | \sum_{v_i \in T_1} w_i - \sum_{v_i \in T_2} w_i |
\]

where \( \tilde{\lambda} = \frac{\lambda}{M_2} \). Since \( \frac{1-\lambda}{M_2} \) is a fixed value, it is removed from (9), hereafter. Now consider an assignment \( A_1 \) where \( x_1 \in T_2 \) and \( x_2 \in T_1 \) and a reassignment \( A_2 \) so that \( x_1 \in T_1 \) and \( x_2 \in T_2 \) and assignment of all other nodes remain unchanged compared with assignment \( A_1 \). Next, some thresholds for \( \tilde{\lambda} \) are provided so that the objective function 2-maxian decreases, when assignment \( A_2 \) is used instead of \( A_1 \). Let \( w(x_1) \) and \( w(x_2) \) be the weights of vertices \( x_1 \) and \( x_2 \), respectively and for each sub-tree \( T' \) of \( T \), \( W(T') = \sum_{v_i \in T'} w_i \).

Let \( f_{p_{\text{max}}}^{1} \) and \( f_{p_{\text{max}}}^{2} \) be 2-maxian objective functions corresponding to assignments \( A_1 \) and \( A_2 \), respectively.

\[
f_{p_{\text{max}}}^{1}(X_2) = \tilde{\lambda} \left( \sum_{v_i \in T_1} w_i d(v_i, x_1) + \sum_{v_i \in T_2} w_i d(v_i, x_2) \right) - | \sum_{v_i \in T_1} w_i - \sum_{v_i \in T_2} w_i |
\]

\[
= \tilde{\lambda}(w(x_1) + w(x_2))d(x_1, x_2) + K
- |(w(x_2) + W(T_1 \setminus \{x_1\})) - (w(x_1) + W(T_2 \setminus \{x_2\}))|
\]

\[
f_{p_{\text{max}}}^{2}(X_2) = \tilde{\lambda} \left( \sum_{v_i \in T_1} w_i d(v_i, x_1) + \sum_{v_i \in T_2} w_i d(v_i, x_2) \right) - | \sum_{v_i \in T_1} w_i - \sum_{v_i \in T_2} w_i |
\]

\[
= K - |(w(x_1) + W(T_1 \setminus \{x_1\})) - (w(x_2) + W(T_2 \setminus \{x_2\}))|
\]

where

\[
K = \tilde{\lambda} \left( \sum_{v_i \in T_1 \setminus \{x_1\}} w_i d(v_i, x_1) + \sum_{v_i \in T_2 \setminus \{x_2\}} w_i d(v_i, x_2) \right).
\]

Inequality \( f_{p_{\text{max}}}^{1}(X_2) \geq f_{p_{\text{max}}}^{2}(X_2) \) holds true provided that the following inequality is satisfied:

\[
\lambda(w(x_1) + w(x_2))d(x_1, x_2) - |w(x_2) + W(T_1 \setminus \{x_1\}) - (w(x_1) + W(T_2 \setminus \{x_2\}))| \geq - |w(x_1) + W(T_1) - (w(x_2) + W(T_2))|.
\]
Let
\[ k_1 = w(x_2) + W(T_1 \setminus \{x_1\}) - (w(x_1) + W(T_2 \setminus \{x_2\})) , \]
\[ k_2 = w(x_1) + W(T_1 \setminus \{x_1\}) - (w(x_2) + W(T_2 \setminus \{x_2\})) . \]

Four cases may arise as follows:

1. \( k_1 \) and \( k_2 \geq 0 \) \( \Rightarrow \bar{\lambda}(w(x_1) + w(x_2))d(x_1, x_2) \geq 2(w(x_2) - w(x_1)) \).
2. \( k_1 \) and \( k_2 \leq 0 \) \( \Rightarrow \bar{\lambda}(w(x_1) + w(x_2))d(x_1, x_2) \geq 2(w(x_2) - w(x_1)) \).
3. \( k_1 \geq 0 \) and \( k_2 \leq 0 \) \( \Rightarrow \bar{\lambda}(w(x_1) + w(x_2))d(x_1, x_2) \geq 2(W(T_1 \setminus \{x_1\}) - W(T_2 \setminus \{x_2\})) \).
4. \( k_1 \leq 0 \) and \( k_2 \geq 0 \) \( \Rightarrow \bar{\lambda}(w(x_1) + w(x_2))d(x_1, x_2) \geq 2(W(T_2 \setminus \{x_2\}) - W(T_1 \setminus \{x_1\})) \).

If \( \bar{\lambda} \) is chosen so that the above cases are met, then actually the assignment \( A_1 \) is better compared to the assignment \( A_2 \).

Note that for solving the balanced 2-maxian problem, the tree should be divided into two sub-trees. Furthermore, all vertices in \( V_1 \) are allocated to \( x_1 \) while all vertices in \( V_2 \) are assigned to \( x_2 \). Therefore, this problem would be solved using the edge deletion method as given in Burkard et al. (2000) which is presented for the 2-median problem on a tree with positive and negative weights. This method is developed in detail in Sect. 4.1.

### 4.1 The edge deletion method

Let \( T_1 \) and \( T_2 \) be two sub-trees of tree \( T \), which are obtained by deleting an edge. For \( i = 1, \ldots, n \), we define the following weights:

\[ w^1_i = \begin{cases} w_i & \text{if } v_i \in T_2 \\ 0 & \text{otherwise} \end{cases} \]
\[ w^2_i = \begin{cases} w_i & \text{if } v_i \in T_1 \\ 0 & \text{otherwise} \end{cases} \]

Then the set \( X = \{x_1, x_2\} \) is considered which maximizes the following objective function.

\[ f_{p\max}(X) = \frac{\lambda}{M_3} \left( \sum_{v_i \in T} w^1_i d(v_i, x_1) + \sum_{v_i \in T} w^2_i d(v_i, x_2) \right) - \frac{1 - \lambda}{M_2} \left| \sum_{v_i \in T} w^1_i - \sum_{v_i \in T} w^2_i \right| . \]

Similar to the edge deletion method of Burkard et al. (2000), for each edge we should find \( x_1 \in T_2 \) and \( x_2 \in T_1 \), along with their objective function values. Then the best pair is chosen as the optimal solution. Thus, the problem would be solved in \( O(n^2) \)
running time. However, in the next section, a linear time algorithm for solving the balanced 2-maxian problem on a tree network is presented.

4.2 A linear time method

In this section, the time complexity of the balanced 2-maxian problem is improved to the linear time. We start by showing that the optimal solution is on the set of leaf nodes.

Lemma 3  Let $T$ be a tree. Then an optimal solution for the balanced 2-maxian problem is achieved on two leaves of $T$.

Proof  Let $X = \{x_1, x_2\}$ be the solution to balanced 2-maxian problem and $V_1$ and $V_2$ be the sets of vertices, which are assigned to $x_1$ and $x_2$, respectively. Let $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ be two sub-trees of $T$ which are obtained by deleting an edge so that $x_1 \in T_2$ and $x_2 \in T_1$. If either $x_1$ or $x_2$ is not a leaf node, then we consider its adjacent vertex. Let $x_1$ be an inner vertex and $u \in T_2$ be an adjacent vertex to $x_1$, out of the path connecting $x_1$ to $x_2$. Then for all vertices $v_i \in T_1$, $d(v_i, u) = d(v_i, x_1) + d(x_1, u) \geq d(u, x_1)$.

Therefore,

$$\sum_{v_i \in T_1} w_i d(v_i, u) \geq \sum_{v_i \in T_1} w_i d(v_i, x_1).$$

So by the relocation of facilities toward leaves, the maxian term of the objective function is not decreased, while the balancing term remains unchanged.  \[\square\]

Lemma 4  If $T_1$ and $T_2$ are sub-trees induced by the deletion of an edge, $x_1 \in T_2$ and $x_2 \in T_1$ are leaves and $d(x_1, x_2)$ is not the longest distance between two leaves, then there exist either a vertex $u \in T_1$ so that $d(u, x_1) \geq d(x_1, x_2)$ or a vertex $u' \in T_2$ so that $d(u', x_2) \geq d(x_1, x_2)$.

Proof  Let $P$ be the path between $x_1$ and $x_2$, and $P'$ be a longest path of $T$. Let $a$ and $b$ be the end vertices of $P'$. First, consider the case that $P \cap P' \neq \emptyset$. Then let $v_a \in P$ and $v_b \in P$ be the closest vertices of $P$ to the leaves $a$ and $b$, respectively. We consider two cases, where the leaves $a$ and $b$ are in the same or different sub-trees.

If $a, b \in T_1$ (see Fig. 2a), then either

$$d(a, v_a) \geq d(v_a, x_2) \quad \text{or} \quad d(b, v_b) \geq d(v_b, x_2).$$

Otherwise, the path $P'$ is not a longest path. Therefore, either

$$d(a, x_1) = d(a, v_a) + d(v_a, x_1) \geq d(v_a, x_2) + d(v_a, x_1) = d(x_1, x_2),$$

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The vertices \( a \) and \( b \) are on the same sub-trees.

or

\[
d(b, x_1) = d(b, v_b) + d(v_b, x_1) \geq d(v_b, x_2) + d(v_b, x_1) = d(x_1, x_2).
\]

Now consider the other case where \( a \in T_1 \) and \( b \in T_2 \) (see Fig. 2b). Then either

\[
d(a, v_a) \geq d(v_a, x_2) \quad \text{or} \quad d(b, v_b) \geq d(v_b, x_1).
\]

Otherwise, the path \( P' \) is not the longest one. So

\[
d(a, x_1) \geq d(x_1, x_2), \quad \text{or} \quad d(b, x_2) \geq d(x_1, x_2).
\]

The other cases, where \( a, b \in T_2 \) and \( a \in T_2, \ b \in T_1 \) would similarly be proved.

Second consider the case \( P \cap P' = \emptyset \). In this case, either \( P' \in T_1 \) or \( P' \in T_2 \). W.l.o.g suppose \( P' \in T_1 \). Let \( v \in P' \) be the closest vertex in \( P' \) to \( P \). Then either

\[
d(a, v) \geq d(v, x_2) \quad \text{or} \quad d(b, v) \geq d(v, x_2).
\]

Otherwise, the path \( P' \) is not a longest path. Therefore, either

\[
d(a, x_1) = d(a, v) + d(v, x_1) \geq d(v, x_2) + d(v, x_1) \geq d(x_1, x_2),
\]

or

\[
d(b, x_1) = d(b, v) + d(v, x_1) \geq d(v, x_2) + d(v, x_1) \geq d(x_1, x_2).
\]

\( \square \)

**Theorem 3** There is an optimal solution to the balanced 2-maxian problem, on the end vertices of a longest path of the tree \( T \).

**Proof** By Lemma 3 there is an optimal solution on the leaves of the tree \( T \). Let \( x_1 \) and \( x_2 \) be two leaves of \( T \), that the path connecting them is not a longest path of \( T \). Let \( T_1 \) and \( T_2 \) be two sub-trees of \( T \) obtained by deleting edge \((v_r, v_s)\), which contain \( x_2 \) and \( x_1 \), respectively. Let also the objective function of \( x_1 \) and \( x_2 \) corresponding to this
partition be less than or equal to other partitions. The vertices in \(T_1\) are allocated to \(x_1 \in T_2\), while the vertices in \(T_2\) are allocated to \(x_2 \in T_1\). Since the path connecting \(x_1\) and \(x_2\) is not a longest path, then Lemma 4 yields either there is a vertex \(u \in T_2\) that \(d(u, x_2) \geq d(x_1, x_2)\) or there is a vertex \(u' \in T_1\) that \(d(u', x_1) \geq d(x_1, x_2)\). W.I.o.g., let there exist \(u \in T_2\) where \(d(u, x_2) \geq d(x_1, x_2)\).

\[
d(u, v_r) + d(v_r, x_2) = d(u, x_2) \geq d(x_1, x_2) = d(x_2, v_r) + d(v_r, x_1),
\]

and consequently

\[
d(u, v_r) \geq d(x_1, v_r).
\]

Hence, for each \(v_i \in T_1\),

\[
d(v_i, x_1) = d(v_i, v_r) + d(v_r, x_1) \leq d(v_i, v_r) + d(v_r, u) = d(v_i, u).
\]

So

\[
\sum_{v_i \in T_1} w_i d(v_i, u) \geq \sum_{v_i \in T_1} w_i d(v_i, x_1).
\]

Since, the partitions remain unchanged, then the objective function will not be decreased by choosing \(u\) instead of \(x_1\).

Note that if \(d(v_i, v_j) > 0\) for \(i, j = 1, \ldots, n\) then by Theorem 3 the optimal solution is the end vertices of a longest path. Since, the longest path of a tree would be obtained in a linear time (see e.g. (Handler 1973)), then the optimal solution to the balanced 2-maxian problem would also be obtained in linear time. However, to calculate the optimal objective function value, the corresponding objective function should be computed by deleting any edge on the longest path, which would be performed in \(O(n^2)\) running time. On the other side, using a recursive computation, the best edge for deleting on a path can be found in a linear time. Thus, if we aggregate the vertices weight of the tree on the longest path, the running time would be improved. These ideas lead us to the following algorithm.

**Algorithm B2-maxian.**

**Input:** A tree \(T\), with edge length \(l\) and vertex weight \(w\).

**Output:** The solution to the balanced 2-maxain problem on the tree \(T\).

1. Set \(M_2 = \sum_{i=1}^n w_i\) and \(M_3\) the value of objective function of 2-maxian problem.
2. Find \(P\), the longest path of the tree \(T\), and rename its vertices by \(\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_m\).
3. For \(i = 1, \ldots, m\) do
   a. Let \(T_i\) be the sub-tree of \(T\) containing \(\hat{v}_i\), which is obtained by deleting the (at most) two edges of \(P\) incident with \(\hat{v}_i\) (see Fig. 3).
   b. Set \(\hat{w}_i = \sum_{j \in T_i} w_j\).
4. Find the vertex \(\hat{v}_r \in P\) in which \(\sum_{i=1}^{r-1} \hat{w}_i < \frac{M_2}{2}\) and \(\sum_{i=1}^r \hat{w}_i \geq \frac{M_2}{2}\).

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5. For \( i = 1, \ldots, m \), let \( e_i = (\hat{v}_i, \hat{v}_{i+1}) \).

6. Calculate the objective functions of the balanced 2-maxian problem obtained by deleting edges \( e_1 \) and \( e_r \), as follows:

\[
\begin{align*}
fe_1 &= \frac{\lambda}{M_3} \left( \sum_{i=2}^{m} \hat{w}_i d(\hat{v}_i, \hat{v}_1) + \hat{w}_1 d(\hat{v}_1, \hat{v}_m) \right) - \frac{1 - \lambda}{M_2} \left| \sum_{i=2}^{m} \hat{w}_i - \hat{w}_1 \right|, \\
fer &= \frac{\lambda}{M_3} \left( \sum_{i=1}^{r} \hat{w}_i d(\hat{v}_i, \hat{v}_m) + \sum_{i=r+1}^{m} \hat{w}_i d(\hat{v}_i, \hat{v}_1) \right) - \frac{1 - \lambda}{M_2} \left| \sum_{i=1}^{r} \hat{w}_i - \sum_{i=r+1}^{m} \hat{w}_i \right|.
\end{align*}
\]

7. For \( i = 2, \ldots, r - 1 \), let

\[
fe_i = fe_{i-1} + \frac{\lambda \hat{w}_i}{M_3} \left( d(\hat{v}_i, \hat{v}_m) - d(\hat{v}_i, \hat{v}_1) \right) + \frac{2(1 - \lambda) \hat{w}_i}{M_2}.
\]

8. For \( i = r + 1, \ldots, m \), let

\[
fe_i = fe_{i-1} + \frac{\lambda \hat{w}_i}{M_3} \left( d(\hat{v}_i, \hat{v}_m) - d(\hat{v}_i, \hat{v}_1) \right) - \frac{2(1 - \lambda) \hat{w}_i}{M_2}.
\]

9. Let \( e^* = \arg\min_{i=1,\ldots,m} \{fe_i\} \).

10. The solution to the balanced 2-maxian problem is \( X^* = \{\hat{v}_1, \hat{v}_m\} \), and the value of objective function is obtained by deleting edge \( e^* \).

**End of algorithm**

Note that, the steps 2 to 4 of the algorithm, aggregate the weights of vertices of \( T \) on the longest path which can performed in a linear time. Steps 6 to 9, iteratively find the best edge for deletion on the longest path. To show correctness of the relations in Steps 7 and 8, consider that for \( i = 2, \ldots, r - 1 \), we have \( \sum_{j=1}^{i} \hat{w}_j \leq \sum_{j=i+1}^{m} \hat{w}_j \), thus

\[
fe_i = \frac{\lambda}{M_3} \left( \sum_{j=1}^{i} \hat{w}_j d(\hat{v}_j, \hat{v}_m) + \sum_{j=i+1}^{m} \hat{w}_j d(\hat{v}_j, \hat{v}_1) \right) + \frac{1 - \lambda}{M_2} \left( \sum_{j=1}^{i} \hat{w}_j - \sum_{j=i+1}^{m} \hat{w}_j \right)
\]
Fig. 4 A tree with 9 vertices

Fig. 5 The path $\hat{P}$ for the tree in Fig. 4

$$
\begin{align*}
\frac{\lambda}{M_3} &\left( \sum_{j=1}^{i-1} \hat{w}_j d(\hat{v}_j, \hat{v}_m) + \hat{w}_i d(\hat{v}_i, \hat{v}_m) + \sum_{j=i}^{m} \hat{w}_j d(\hat{v}_j, \hat{v}_1) - \hat{w}_i d(\hat{v}_i, \hat{v}_1) \right) \\
+ \frac{1 - \lambda}{M_2} &\left( \sum_{j=1}^{i-1} \hat{w}_j + \hat{w}_i - \sum_{j=i}^{m} \hat{w}_j + \hat{w}_i \right) \\
&= f_{e_{i-1}} + \frac{\lambda}{M_3} \hat{w}_j (d(\hat{v}_i, \hat{v}_m) - \hat{w}_i d(\hat{v}_i, \hat{v}_1)) + \frac{2 \hat{w}_i (1 - \lambda)}{M_2}.
\end{align*}
$$

The proof for the cases $i = r + 1, \ldots, m$ are the same. In these cases, $\sum_{j=1}^{i} \hat{w}_j \geq \sum_{j=i+1}^{m} \hat{w}_j$. Using these recursive formulations the objective function can be computed in a linear time.

Therefore, the following theorem holds.

**Theorem 4** The balanced 2-maxian problem on a tree could be solved in a linear time.

In the following example, we explain the steps of the Algorithm B2-maxin for a given tree.

**Example 2** Consider the tree depicted in Fig. 4. For this tree, we obtain $M_2 = 23$ and $M_3 = 213$. Let $\lambda = 0.5$. The longest path is $P = v_1, v_2, v_3, v_4, v_5, v_6$. Thus, the solution of balanced 2-maxian is $X^* = \{v_1, v_6\}$.

To calculate the value of objective function, we should find the best edge for deleting. By aggregating the weights of vertices on path $P$, the path $\hat{P}$ is obtained which is shown in Fig. 5.

Since $\sum_{i=1}^{3} \hat{w}_i = 11 < \frac{23}{2}$ and $\sum_{i=1}^{4} \hat{w}_i = 18 \geq \frac{23}{2}$, then, $r = 4$. Therefore, we should first calculate the value of objective functions by deleting edges $e_1 = (\hat{v}_1, \hat{v}_2)$ and $e_4 = (\hat{v}_4, \hat{v}_5)$. Using step 6 of the algorithm, we obtain $f_{e_1} = \frac{163}{426} - \frac{19}{46}$ and $f_{e_4} = \frac{209}{426} - \frac{13}{46}$.
Then using step 7, the values $f_{e_2}$ and $f_{e_3}$ are obtained as follows:

\[
\begin{align*}
  f_{e_2} &= f_{e_1} + \frac{18}{426} + \frac{4}{46} = \frac{181}{426} - \frac{15}{46}, \\
  f_{e_3} &= f_{e_2} + \frac{21}{426} + \frac{14}{46} = \frac{202}{426} - \frac{1}{46}.
\end{align*}
\]

The value $f_{e_5}$, should be calculated using step 8,

\[
\begin{align*}
  f_{e_5} &= f_{e_4} + -\frac{6}{426} - \frac{6}{46} = \frac{188}{426} - \frac{19}{46}.
\end{align*}
\]

Comparing these values results in the best edge for deleting is $e_3 = (\hat{v}_3, \hat{v}_4)$.

5 Summary and conclusion

In this paper, two balanced models of the $p$-median and $p$-maxian problems on a tree have been investigated. In the balanced $p$-median problem, the objective function is a combination of balance on clients’ allocation to the facilities and the median problem, while in the balanced $p$-maxian problem the objective function is balancing on clients’ allocation and maxian problem. Based on the edge deletion method, an $O(n^2)$ algorithm is proposed for the balanced 2-median problem on a tree. Furthermore, it is shown that the optimal solution of the balanced 2-maxian problem, is the leaf nodes of the longest path of the tree. Then a linear time algorithm is presented to obtain the balanced 2-maxian objective function.

Other balancing functions such as minimizing the mean servicing time by facilities can be considered in the future works.

Acknowledgements The authors would like to thank the referees for their helpful comments.

Funding The authors have not disclosed any funding.

Data availability Enquiries about data availability should be directed to the authors.

Declarations

Conflicts of interest The authors declared that they have no conflict of interest with any Organization/Institute.

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