Application of continuous symmetry groups for numerical solutions of two-phase filtration problems

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Abstract. The method of numerical solution calculations for difference schemes on the basis of continuous symmetry groups has been presented for the two-phase flow equation. This method gives an opportunity to calculate numerical solutions using transformations of a symmetry group and known particular numerical solutions what allows to achieve an acceleration of serial numerical calculations by several orders. The families of invariant difference schemes have been obtained for one of generalized Buckley-Leverett equations - the Rapoport-Leas equation. A particular nonlinear equation of multiphase flow in fractured porous media with certain coefficients from the mentioned families is analyzed numerically by the presented method.

1. Introduction

Many problems in modeling are not solved rigorously and have to be replaced by some numerical algorithms such as solving of correspondent difference equations. But for using these algorithms many times for different problems, one must be sure that this particular numerical method is still reliable. That is why there is a need to research them using different methods, which come from different modern fields of mathematics. Also, many problems in mathematical modeling have a great degree of uncertainty and different variants of optimal solutions. That is why there is a need to conduct multivariate calculations. However, this number of calculations demands fast and effective numerical calculation methods.

There are two possible ways of the calculation time decreasing: improvements of hardware and using more advanced methods for theoretical analysis of algorithms. Improvements of numerical algorithms have to be gained by using new methods for their analysis. It is the only way to increase both the speed and the effectiveness of numerical calculations in many fields. And one of these new methods of analysis is the theory of continuous symmetries for difference equations.

This article is intended to demonstrate the method of fast obtaining of numerical solutions using continuous symmetries for difference schemes when conducting a large number of numerical calculations for the Rapoport-Leas equation with some certain types of defining coefficients for relative permeability and capillary pressure.

2. Continuous symmetries of differential and difference equations

There are different types of equations, which can be analyzed using continuous symmetries: differential equations [13], difference equations [6], differential-difference equation [24] and others. There are some classical books about group analysis of differential equation such as [8], [13], [19], [21]. One of the first articles about applications of continuous groups to discrete equations...
was [14] where continuous symmetries were applied to discrete dynamical systems. It gave an impulse for further studying of discrete dynamical systems: [15], [18] and others. Also, specialists started to study difference equations: [3], [5], [11], [12] and others.

2.1. The method of difference scheme construction with the preservation of continuous symmetries

The method of difference schemes construction preserving continuous symmetries of an original differential equation can be found in [6]. There are two approaches: the first one uses definitions of finite-difference derivatives to prolong the action of a continuous group and the second one uses values of independent and dependent variables in points on a mesh as the basis for all calculations. The method in this article uses the second approach for further results of possible invariant difference schemes.

Invariant difference schemes for a differential equation are constructed by the following steps:

- the identification of continuous symmetries for a differential equation;
- the expression of a differential equation in terms of differential invariants;
- the construction of difference invariants for a symmetry group;
- the approximation of differential invariants via difference invariants;
- the notation of equations via approximations of differential invariants;
- the seeking of functions for the definition of a mesh.

2.2. The method of numerical solution calculations using continuous symmetries

The method of numerical solution calculations is based on the fact that a transformation from a continuous symmetry group transforms solutions into solutions. This approach is applied for differential equations [8] and can be applied for difference schemes as well [6]. The step by step description of the developed method is presented below [17]:

- the obtaining of an invariant difference scheme, i.e. a scheme with a continuous symmetry group, for a differential equation;
- the obtaining of a set of initial and boundary conditions, which can be transformed into each other by the continuous symmetry group;
- the calculating of one particular solution using an invariant difference scheme according to initial and boundary conditions, which allows to obtain a set of solutions for the gained set of initial and boundary conditions;
- the seeking of parameters for the continuous group of symmetry, which allows to transform the particular solution conditions into the set of conditions;
- the transformations of the particular numerical solution into the set of numerical solutions with conditions from the set of conditions.

Errors for numerical solutions can be controlled and decreased when solutions are generated by symmetry groups [17]. Boundary and initial conditions must be transformed one into another by using a symmetry group if we want to obtain a family of solutions. Similar problems are discussed in, for example, [7].

3. Results for the Rapoport-Leas equation

3.1. Results of group classification for the Rapoport-Leas equation

The partial differential equation of two-phase flow in one-dimensional porous media within the generalized Buckley-Leverett model [23] (also known as the Rapoport-Leas equation) is
considered in this article. This equation can be presented as

$$\frac{\partial S}{\partial t} + \frac{K}{\mu_0 \phi} \frac{\partial}{\partial x} \left( K_o(S) f(S) \frac{dP_c}{dS} \frac{\partial S}{\partial x} \right) + \frac{V}{\phi} \frac{df}{dS} \frac{\partial S}{\partial x} = 0,$$

(1)

where $S(t, x)$ - water saturation, $\mu_w$ and $\mu_o$ - constant viscosities of water and oil respectively, $\phi$ - porosity, $P_c(S)$ - capillary pressure, $V = V_o + V_w = \text{const}$ - total flow, $f(S)$ - fractional flow rate of water, $K_w(S)$ - relative permeability for water, $K_o(S)$ - relative permeability for oil.

The equation (1) can be written in the form

$$\frac{\partial S}{\partial t} - \frac{\partial}{\partial x} \left( A(S) \frac{\partial S}{\partial x} \right) + \frac{\partial B(S)}{\partial S} \frac{\partial S}{\partial x} = 0,$$

(2)

which is similar to the correspondent heat transfer equations. Different heat transfer equations are well studied using continuous groups of transformations [10], [20].

The equation (2) has two one-parameter groups of symmetries for arbitral functions of relative permeability and capillary pressure: the groups of translations for time and space because time and spatial variables do not appear explicitly in the equation. If the mentioned functions are defined as [16]

$$K_w(S) = S^N, \quad K_o(S) = (1 - S^N), \quad P_c(S) = -\frac{P_1}{N} \ln \left| \frac{S^N}{S^{N-1}} \right| + P_2,$$

(3)

where $P_1$ and $P_2$ - scaling coefficients defining relations for capillary pressure, $N$ - a dimensionless parameter fulfilling conditions $N \neq 1$ and $N \in \mathbb{R}$, the equation (1) is transformed into the nonlinear equation below:

$$\frac{\partial S}{\partial t} - \frac{\partial}{\partial x} \left( A(S) \frac{\partial S}{\partial x} \right) - \beta S^{N-1} \frac{\partial S}{\partial x} = 0, \quad \alpha = \frac{KP_1}{\mu \phi}, \quad \beta = -\frac{VN}{\phi}.$$

(4)

The equation (4) has the three-parameters group

$$\bar{t} = e^{-a_1} t + a_2, \quad \bar{x} = x + a_3, \quad \bar{S} = e^{a_1} S, \quad a_1, a_2, a_3 \in \mathbb{R}$$

(5)

with the infinitesimal operators [10]

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = -t \frac{\partial}{\partial t} + \frac{S}{N - 1} \frac{\partial}{\partial S}.$$

(6)

3.2. Families of invariant difference schemes

Differential equations of type (2) are well studied by the group analysis. This type of equations has group classifications, which can be found in [10], [20] as it is mentioned above. These classifications are used for the current section for constructions of families of invariant difference schemes for differential equations of the type (2), which represents the physical problems (1).

The purpose of this section is to present differential equations (by their coefficients) and their invariant difference schemes (by their difference invariants) with coefficients, which can be found in real physical cases for two-phase flow in porous media. It means that the following results in Table 1 are not intended to present all cases from the known classifications. There are similar results for other types of heat transfer equations, which can be found in [2], [4].

In Table 1, $p_i \in \mathbb{Z}$ and $c_j \in \mathbb{R}$ are arbitral constants, operators $X_1$ and $X_2$ are from (6). The equation (4) and all further calculation results belong to the case 1 of Table 1 with coefficients
Table 1. Invariant difference schemes

| No. | $A$ | $B$ | Operators | Difference invariants |
|-----|-----|-----|-----------|-----------------------|
| 1   | $A = c_1 S^{c_2}$, $c_2 \neq -1$ | $B = c_1 c_3 S^{c_2+1}$ | $X_1, X_2, X_3 = -t \frac{\partial}{\partial t} + \frac{S}{c_2} \frac{\partial}{\partial S}$ | $J_1 = (t^{n+1} - t^{n+2}) \times (S_{k+p_4}^{n+3} - S_{k+p_4}^{n+2})$, $J_2 = S_{k+p_4}^{n+3} / S_{k+p_4}^{n+2}$, $J_3 = x_{k+p_4} - x_{k+p_4}$ |
| 2   | $A = \frac{c_3}{S}$ | $B = c_1 c_3 \ln S$ | $X_1, X_2, X_3 = -t \frac{\partial}{\partial t} - S \frac{\partial}{\partial S}$ | $J_1 = \left( \frac{S_{k+p_4}^{n+3}}{S_{k+p_4}^{n-1}} \right)^3$, $J_2 = S_{k+p_4}^{n+3} / S_{k+p_4}^{n+2}$, $J_3 = x_{k+p_4} - x_{k+p_4}$ |
| 3   | $A = c_1 e^{c_2 S}$ | $B = c_1 c_3 e^{c_2 S}$ | $X_1, X_2, X_3 = -t \frac{\partial}{\partial t} + \frac{1}{c_2} \frac{\partial}{\partial S}$ | $J_1 = e^{c_2 S} S_{k+p_4}^{n+3}$, $J_2 = S_{k+p_4}^{n+3} / S_{k+p_4}^{n+2}$, $J_3 = x_{k+p_4} - x_{k+p_4}$ |

(3) where $c_1 = \alpha$, $c_2 = N - 1$, $c_3 = \beta / \alpha$. All cases from Table 1 can have certain coefficients with certain physical meanings in the field of two-phase flow in porous media and can be used for constructions of wide sets of invariant difference schemes.

Difference schemes constructed from the invariants of Table 1 should be analyzed separately for stability if these difference schemes are used for numerical calculations. Invariants from Table 1 allow to choose meshes, which have time step choice depending on saturation values in some points of meshes. Time steps can be chosen constant if we choose the mentioned values of saturation from some particular point of meshes. Spatial variable steps must be constant for all points of meshes because of the invariant $J_3$ of all three classes from the table.

3.3. Numerical results of calculations

This section is intended to apply the results of previous sections: the definition of one invariant difference scheme of the Rapoport-Leas equation, which is from the case 1 of Table 1 and can be written as

$$
S_k^{n+1} = S_k^n + \beta \frac{\Delta t}{\Delta x} \left( S_k^n \right)^{N-1} \left( S_{k+1}^{n+1} - S_k^{n+1} \right) + 
+ \alpha \frac{\Delta t}{\Delta x} \left( \left( S_k^n \right)^{N-1} \left( S_{k+1}^{n+1} - S_k^{n+1} \right) - \left( S_k^{n+1} \right)^{N-1} \left( S_k^{n+1} - S_{k-1}^{n+1} \right) \right),
$$

(7)

and calculations of numerical solutions using the presented method and this invariant difference scheme. Table 2 presents parameters for the difference scheme (7), which are chosen for the sake of demonstrations of the developed method, for the sake of the presence of both convective and diffusive parts of the equation (4) and for the sake of demonstrations of practical applications for the method. The choice of initial and boundary conditions caused by the desire to check the convergence of the calculated solutions using the known accurate solution [22] for (4). Thus, these conditions are calculated using the mentioned accurate solution.

Chosen boundary and initial conditions together with the difference scheme (7) refer to the problem where drainage (oil is displacing water) of some natural fractured reservoir (fractures and porous matrix) is taking place with the constant liquid velocity. The parameters from Table 1 and functions from (3) with $N = 0.75$ can be found in the results of experiments, for example, in [1] and [9] for fractured porous media. The form of used relative permeability
Table 2. Values of parameters for numerical calculations

| Parameter            | Units       | Values     |
|----------------------|-------------|------------|
| Porosity $\phi$      | -           | 0.25       |
| Absolute permeability $K$ | mD       | 500        |
| Viscosity $\mu$      | mPa·s       | 1.4        |
| Scaling parameter $P_1$ | atm      | 5          |
| Scaling parameter $P_2$ | atm      | 33         |
| Liquid velocity $V$  | m/s         | 0.00005    |
| Parameter $N$        | -           | 0.75       |
| Length of calculation region $L$ | m     | 80         |
| Maximum time $T$     | s           | 200000     |
| Initial water saturation for left boundary $S_L$ | -   | 0.2        |
| Initial water saturation for right boundary $S_R$ | -   | 0.6        |
| Number of spatial steps $N_x$ | pcs | 40        |
| Number of time steps $N_t$ | pcs | 200000    |

curves is close to X-curves what refers to fractured porous media. For all calculations of this section, water saturation and relative permeability values change between 0 and 1 but functions for relative permeability from (3) can be scaled by the changing of variables using residual saturation points. Also, relative permeability can be scaled simultaneously for water and oil what is allowed by the form of the equation (4) and functions (3).

Figure 1 shows the generation process of five numerical solutions with values of initial left boundary water saturation $S_L = 0.32, 0.28, 0.24, 0.16, 0.12$ on the basis of the calculated solution and parameters $a_1 \neq 0, a_2 = 0, a_3 = 0$. It means using the one-parameter group of time and saturation dilations from the continuous group (5). The comparison with the known particular solution of the equation (4) shows the sufficient accuracy and the convergence of the used difference scheme with the set parameters. The relative error is less than 1 percent for all points of the mesh.

Figure 1. Time (left) and spatial (right) saturation profiles for generated solutions.

Obtained results showed that the gain in speed of calculations of the five solutions using the symmetry group in comparison with calculations of the same number of solutions using a difference scheme is several orders. In fact, the speed of the generating method for numerical solutions depends substantially on the time of calculations of the first solution where the difference scheme and some classical method for systems of linear equations are used [17].
4. Conclusions
The described method of numerical solution generations for the Rapoport-Leas equation can be applied for the increasing of the accuracy and the speed of calculations for two-phase filtration problems. The results can be used for the development and applications of simplified express techniques, which solve various problems of the development of oil and gas fields.

Unconditional advantages of the presented method can be formulated: the increasing of the calculation speed by orders because a numerical solution should be calculated only once for a family of some initial and boundary conditions; the opportunity to control accuracy for further numerical calculations because errors of generated (transformed) solutions depend on the error of the preliminary calculated numerical solution. Shortcomings of the method are connected with the following: one should narrow a family of equations using specific forms of coefficients in equations because it allows them to have continuous symmetries; it is hard to deal with arbitral initial and boundary conditions because all generated solutions must have connections for conditions by transformations of a continuous symmetry group. But these shortcomings can be neglected in conditions of high degrees of uncertainties of real initial data.

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