Hyperbolic thermostat and Hamilton’s Harnack inequality for the Ricci flow

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Abstract

In this paper, we will recover the Hamilton’s Harnack inequality for the Ricci flow from the viewpoint of “Hyperbolic thermostat”.

1 Introduction

The Ricci flow was introduced by Hamilton as a tool to solve the Poincaré conjecture. We mean by the Ricci flow a pair \((M, g)\) of a smooth manifold \(M\) and an evolving Riemannian metric \(g = g(x, t)\) obeying the evolution equation

\[
\frac{\partial g(x, t)}{\partial t} = -2\text{Ric}_g(x, t).
\]  

(1.1)

The Ricci flow equation is invariant under the action the group \(\text{Diff}(M)\) of diffeomorphism of \(M\), which means that the Ricci flow is a gauge theory with gauge group \(\text{Diff}(M)\). This implies, via the (contracted) second Bianchi identity, that the Ricci flow equation is not parabolic, but only weakly parabolic. This causes a difficulty on the existence of the solution to the Ricci flow for a given smooth initial metric. In [9], Hamilton proved the short time existence on closed manifold, which later was greatly simplified by DeTurck [8].

During 1980-1990’s, it was inevitable to put some assumption on the curvature of the initial or the solution metric for the study of the Ricci flow. Nevertheless, Hamilton [9] was able to establish a program (Hamilton program) toward proving Thurston’s geometrization conjecture by studying the
time-global solution metric for the Ricci flow with arbitrary initial metric. Recall that Thurston's geometrization conjecture claims that any closed 3-manifold is decomposed into pieces each having one of the eight maximal model geometries. Here, the decomposition means first the connected sum decomposition into prime components and second the torus decomposition of each prime component into pieces having one of the eight maximal model geometries.

According to Hamilton’s program, the occurrence of the decomposition into the prime components is the effect under the formation of singularities in finite time for the Ricci flow. The difficulty of the proof of the “no local collapsing property” under the formation of singularities in finite time for the Ricci flow had been the major difficulty against the progress of Hamilton’s program. In this direction, Hamilton’s Harnack inequality [8], which compares the curvature of the Ricci flow at two points in the space-time, was the most prominent result in the study of the Ricci flow obtained in the period 1980-1990’s. In fact, it was believed to play an essential role in the study of the structure of the finite-time singularities of the Ricci flow. Since the Ricci flow equation is only weakly parabolic due to its diffeomorphism invariance, the concept of a self-similar solution, i.e., a special solution to the Ricci flow equation which evolves under a 1-parameter family of diffeomorphisms (coupled with scalings) makes sense and such a solution is named a Ricci soliton. Therefore, the Ricci soliton should play an essential role in the proof of Harnack type inequality for the Ricci flow. This is the basic idea of Hamilton’s proof of the Harnack inequality [8]. In fact, Hamilton proved the Harnack inequality by applying the maximum principle to the Harnack expression which was constructed from the equation of the gradient expanding Ricci soliton. Hamilton’s Harnack inequality is “mysterious” in the sense that it is proved under the assumption of the nonnegative curvature operator, while the expanding Ricci soliton generalizes the Einstein metrics with negative Ricci curvature.

It was Perelman [11] who introduced the \( W \)-entropy and used its monotonicity under the Ricci flow to prove the no local collapsing property of the finite-time singularities of the Ricci flow. Moreover, Perelman was able to establish the propagation of the no local collapsing property to the space-time by introducing the reduced volume and proving their monotonicity under the Ricci flow. This way, Perelman was able to prove Thurston’s geometrization
conjecture. It was remarkable that Perelman combined the $W$-entropy / the reduced volume and Hamilton’s Harnack inequality in the analysis of the finite-time singularities (the determination of the structure of the ancient solution with no collapsing condition).

In [11] Perelman introduced the concept of the “Riemannian geometric thermostat” and developed a statistical theory following the standard formalism of the statistical mechanics. It seems that it was the way how Perelman discovered these functionals having the monotonicity under the Ricci flow. The theory of the thermostat is a heuristic framework which produces basic quantities such as the reduced volume and the $W$-entropy of the Ricci flow. This also gives hints (the concept of the $L$-length) to rigorous proofs of their basic properties. By developing the $L$-geometry, i.e., the comparison geometry based on the $L$-length, Perelman was able to give rigorous proofs to results obtained by heuristic arguments based on the Riemannian geometric thermostat.

Perelman proposed the correct position where Hamilton’s Harnack inequality lives, i.e., the $L$-geometry which emerges from the Riemannian geometric thermostat. Therefore, it is a conceptually interesting problem to search for a reason for Hamilton’s Harnack inequality in the framework of the theory of the Riemannian geometric thermostat.

The purpose of this paper is to propose a geometric interpretation to Hamilton’s Harnack inequality. There are several known results in this direction: Chow - Chu [5] and Cabezas-Rivas - Topping [3]. In particular, the last paper introduced the Canonical Expanding Solitons which connects Hamilton’s result and Brendle’s result [2] and induces some new Harnack inequalities for the Ricci flow.

In this paper, we introduce a variation of Riemannian geometric thermostat namely the hyperbolic thermostat. It turns out that the full curvature tensor of the hyperbolic thermostat gives rise to the exact Hamilton’s Harnack expression. Hence, we recover Hamilton’s Harnack inequality by formally applying the preservation principle under the Ricci flow. It is well known that the positivity of some curvature is preserved under the Ricci flow. In [3], Cabezas-Rivas and Topping recovered Hamilton’s Harnack inequality in this direction. In the present paper, we will recover Hamilton’s Harnack in-
equality by direct calculation and applying the maximum principle.

Hyperbolic thermostat and the Canonical Expanding Solitons induced Harnack inequality. On the other hand, the Canonical Shrinking Solitons introduced by [4] and [15] recover some results under the Ricci flow which were discovered by Perelman [13] (e.g. the monotonicity of $W$-entropy). In fact, the monotonicity of $W$-entropy under the Ricci flow is induced from the viewpoint of Perelman’s thermostat by formally applying a comparison theorem for the total scalar curvature (See section 5). Hence, we hope that all of results from the Canonical Solitons are interpreted by the results from thermostat.

This paper is organized as follows. In Section 2 we recall the basic set up for the Riemannian geometry and basic properties of the Ricci flow. In Section 3 we introduce the hyperbolic thermostat $(\tilde{M}, \tilde{g})$ and see that the Harnack expression for the Ricci flow $g(t)$ appears as the full curvature tensor of the hyperbolic thermostat metric $\tilde{g}(t)$. The differential equations of the components of curvature tensor is derived in section 3.2. In section 4 we state and prove the main theorem which is equivalent to Hamilton’s Harnack inequality, by applying the maximum principle along a submanifold $\bar{M}$ in $\tilde{M}$ equipped with a degenerate metric $\bar{g}$ as a section of $\text{Sym}^2(T^*\bar{M})$. Here, $\tilde{M}$ is a potentially infinite dimensional manifold and $\bar{M}$ is a $(\dim M + 1)$-dimensional manifold. In section 5 we prove the comparison theorem for the total scalar curvature on geodesic sphere which holds if the manifold is complete and Ricci flat. The monotonicity of $W$-entropy is recovered from a viewpoint of Riemannian’s geometric thermostat by formally applying this comparison theorem.

2 Preparation

In this section, we define fundamental quantities for a Riemannian manifold. Moreover, we deduce some basic properties for the Ricci flow. In this paper, we adopt the convention of curvatures on [14] as stated below.
2.1 Riemannian geometry

Let \((M, g)\) be a Riemannian manifold, and \(X, Y, Z, W\) vector fields independent of time \(t\) on \(M\). We define the full curvature tensor \(Rm\), Ricci curvature \(Ric\), and scalar curvature \(R\):

\[
Rm(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z ,
\]

\[
Rm(X, Y, Z, W) := \langle Rm(X, Y)Z, W \rangle ,
\]

\[
Ric(X, Y) := tr Rm(X, \cdot, Y, \cdot) ,
\]

\[
R := tr Ric ,
\]

where \(\langle \cdot, \cdot \rangle\) is the inner product with respect to \(g\), \(\nabla\) is the Levi-Civita connection with respect to \(g\), and \([X, Y] := \nabla_X Y - \nabla_Y X\) is the Lie bracket. These components are expressed as

\[
R_{ijkl} \frac{\partial}{\partial x^l} = Rm\left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} ,
\]

\[
R_{ijkl} = g_{pl}R_{ijkl}^p ,
\]

\[
R_{ij} = g^{kl}R_{ikjl} ,
\]

\[
R = g^{ij}R_{ij} ,
\]

where \((x^1, \cdots, x^n)\) are local coordinates on \(M\) and \(\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^n\) is a local frame of \(TM\) consisting of coordinate vector fields.

The full curvature tensor has some symmetry properties:

\[
R_{ijkl} = -R_{ijkl} = -R_{ijlk} = R_{klij} ,
\]

\[
R_{ijkl} + R_{jikl} + R_{kijl} = 0 ,
\]

\[
\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0 .
\]

The second and third equations are well-known as the first and the second Bianchi identities, respectively. By tracing the second Bianchi identity, we have

\[
\nabla^i R_{ijk} = \nabla_j R_{ki} - \nabla_k R_{ji} ,
\]

where \(\nabla^i := g^{ij} \nabla_j\). We trace again to get the twice contracted second Bianchi identity:

\[
\nabla^i R_{ij} = \frac{1}{2} \nabla_j R .
\]
We note that the first and second Bianchi identities are equivalent to the diffeomorphism invariance of the curvature tensor [11]. For arbitrary tensor $A$, we have the equation which deduce the commutation equation for Levi-Civita connection:

$$\nabla_X \nabla_Y A - \nabla_Y \nabla_X A = -R(X,Y)A .$$

This equation is called the Ricci identity. The following two examples for the tensors $R_{ij}$ and $\nabla_p R_{ij}$ are important to know how geometric quantities evolve under the Ricci flow.

$$\begin{align*}
\nabla_m \nabla_n R_{ij} - \nabla_n \nabla_m R_{ij} &= R_{mni}^p R_{pj} + R_{mnj}^p R_{ip} \ , \\
\nabla_m \nabla_n \nabla_p R_{ij} - \nabla_n \nabla_m \nabla_p R_{ij} &= R_{mnp}^q R_{qij} + R_{mn}^j R_{iq} + R_{mnj}^p \nabla_p R_{iq} .
\end{align*}$$

(2.4)

We use these identities in the proof of the equations (2.9). By the symmetric properties of the full curvature tensor, we naturally define the curvature operator $\mathcal{R}: \wedge^2 T^*M \rightarrow \wedge^2 T^*M$ as follows:

$$Rm(X, Y, Z, W) = \langle \mathcal{R}(X \wedge Y), Z \wedge W \rangle .$$

Our sign convention is that $R_{ijij} > 0$ on the round sphere.

### 2.2 The evolution equations for the curvature under the Ricci flow

In this section, we will see how geometric quantities evolve when the metric evolves under the Ricci flow. Details can be found in, for instance, [14, Chapter 2].

Let $(M, g(t))$ be a Ricci flow, i.e.

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t)$$

as described in (1.1). The curvature tensor evolves as follows:

$$\frac{\partial}{\partial t} Rm(X, Y, W, Z) = (\Delta Rm)(X, Y, W, Z) + Q(X, Y, W, Z) + F(X, Y, W, Z)$$

(2.5)
where
\[ Q(X, Y, W, Z) := 2(B(X, Y, W, Z) - B(X, Y, Z, W)) + B(X, W, Y, Z) - B(X, Z, Y, W), \]
\[ F(X, Y, W, Z) := -Ric(R(X, Y)W, Z) + Ric(R(X, Y)Z, W) - Ric(R(W, Z)X, Y) + Ric(R(W, Z)Y, X), \]
\[ B(X, Y, W, Z) := \langle Rm(X, \cdot, Y, \cdot), Rm(W, \cdot, Z, \cdot) \rangle, \]
and \( \Delta \) is Laplace-Beltrami operator \( \Delta := g^{ij} \nabla_i \nabla_j \). This equation is also expressed as
\[
\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2R_{imjn} R_{k \ell}^{\ m \ n} - 2R_{imjn} R_{\ell k}^{\ m \ n}
+ 2R_{imkn} R_{ij}^{\ m \ n} - 2R_{imkn} R_{ij}^{\ m \ k}
- R_{ij}^{\ m \ l} R_{ijkm} + R_{ij}^{\ m \ k} R_{ijlm} - R_{ij}^{\ m \ l} R_{ijlk} + R_{ij}^{\ m \ l} R_{ijkl} .
\] (2.6)

By taking the trace, we have the evolution equation for the Ricci tensor:
\[
\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} - 2R_{ik}^{\ k} R_{kj} + 2R_{ijkl} R^{kl} .
\] (2.7)

Moreover, by taking the trace again, we have the evolution equation for the scalar curvature:
\[
\frac{\partial R}{\partial t} = \Delta R + 2R_{ij}^{\ ij} R_{ij} .
\]

We set
\[
P_{ijk} := \nabla_i R_{jk} - \nabla_j R_{ik} ,
\]
\[
M_{ij} := \Delta R_{ij} + 2R_{ikjl} R^{kl} - R_{ik}^{\ k} R_{kj} - \frac{1}{2} \nabla_i \nabla_j R + \frac{1}{2t} R_{ij} .
\] (2.8)

The tensors \( P_{ijk} , M_{ij} , \) and \( R_{ijkl} \) are essentially parts of the curvature tensor \( \tilde{R}_{abcd} \) on the hyperbolic thermostat \((\tilde{M}, \tilde{g}_{ab})\) which we will define in section 3. These are first introduced by Hamilton as Hamilton’s Harnack expression. Here, we compute the deformation of these tensors when the metric \( g(t) \) evolves under the Ricci flow. The following lemma will be used to compute the evolution equation for \( \tilde{R}_{abcd} \).
Lemma 2.1 (Hamilton [9]). Let \((M, g(t))\) be a Ricci flow, and \(P_{ijk}, M_{ij}\) tensors defined as above. Then, we have

\[
\left(\frac{\partial}{\partial t} - \Delta\right) P_{ijk} = 2R_{imjn}P_{k}^{mn} + 2R_{imkn}P_{j}^{mn} + 2R_{jmkn}P_{i}^{mn} - 2R_{n}^{m}\nabla_{m}R_{ijk}^{n}
\]
\[+ R_{i}^{m}P_{mjk} + R_{j}^{m}P_{imk} + R_{k}^{m}P_{ijm},
\]

\[
\left(\frac{\partial}{\partial t} - \Delta\right) M_{ij} = 2R_{imjn}M_{mn} + 2R_{mn}^{im}[\nabla_{m}P_{nij} + \nabla_{n}P_{mji}]
\]
\[+ 2P_{imon}P_{j}^{mn} - 4P_{imon}P_{jm}^{mn} + 2R_{m}^{m}R_{imj} - \frac{1}{2t^{2}}R_{ij}
\]
\[+ R_{i}^{m}M_{mj} + R_{j}^{m}M_{im}.
\]

(2.9)

The first equation is shown by using the formula (2.4) and

\[P_{ijk} + P_{jki} + P_{kij} = 0.\]

To get the second equation, we use the formula

\[M_{ij} = \nabla^{u}P_{uj} + R_{ijkl}R_{kl}^{u} + \frac{1}{2t}R_{ij}.\]

Remark 2.2. In [9], Hamilton shows this lemma by using the vector field \(D_{t}\) on the orthonormal frame bundle of the Ricci flow where \(D_{t}\) is defined by

\[D_{t} := \frac{\partial}{\partial t} + R_{a}^{b}\nabla_{a}.\]

Here, \(\nabla_{a}\) is obtained by modifying \(\frac{\partial}{\partial t}\) by vatical vector field so that \(D_{t}\) is tangent to the orthonormal frame bundle of the Ricci flow. The advantage of using \(D_{t}\) instead of \(\frac{\partial}{\partial t}\) is that the frame \(\{e_{a}\}\) which is a local orthonormal frame at \(t = 0\) behaves like a local orthonormal frame for all time \(t\) under the Ricci flow. However, we don’t use \(D_{t}\) in this paper because we will deal with the parameter \(t\) not only as a time but also as a part of local coordinate system \(t := x^{0}\) on \((\tilde{M}, \tilde{g}_{ab})\).

3 Riemannian geometric thermostat

In this section, we canonically construct hyperbolic thermostat from arbitrary Ricci flow following Perelman’s Riemannian geometric thermostat. Moreover, we derive the evolution equation of the curvature tensor.
3.1 Spherical thermostat

Here, we describe Riemannian geometric thermostat which was introduced by Perelman in [13]. We only check the full curvature tensor of the thermostat in this section. We will see the other properties in section 5.

Theorem 3.1 (Perelman [13]). Let \((M, g_{ij}(\tau))\) be a complete backward Ricci flow (i.e., \(\frac{\partial g_{ij}}{\partial \tau} = 2R_{ij}(\tau)\)) for \(\tau \in (0, T] \subset \mathbb{R}^+\) with bounded curvature, and \((S^N, g_{\alpha\beta})\) a sphere with constant sectional curvature \(\frac{1}{2N}\) for \(N \in \mathbb{N}\). We define \(\hat{M} = M \times S^N \times (0, T]\), and a metric \(\hat{g}\) on \(\hat{M}\) as follows:

\[
\hat{g}_{ij} = g_{ij}, \quad \hat{g}_{\alpha\beta} = \tau g_{\alpha\beta}, \quad \hat{g}_{00} = R + \frac{N}{2\tau}, \quad \hat{g}_{i0} = \hat{g}_{0i} = 0,
\]

where \(i, j\) are coordinate indices on the \(M\) factor, \(\alpha, \beta\) are coordinate indices on the \(S^N\) factor, 0 represent the index of the scale coordinate \(\tau\), \(R\) is the scalar curvature with respect to the metric \(g_{ij}\). Then, \((\hat{M}, \hat{g})\) is Ricci flat up to errors of order \(\frac{1}{N}\).

Ricci flatness can be shown by the fundamental computation in the same way as the proof of Theorem 3.2. We also compute the components of full curvature tensor:

\[
\hat{R}_{ijkl} = R_{ijkl}, \quad \hat{R}_{ij0k} = \nabla_i R_{jk} - \nabla_j R_{ik}, \quad \hat{R}_{0i0j} = \Delta R_{ij} + 2R_{ikj}R_{kl} - R^k_i R_{kj} - \frac{1}{2} \nabla_i \nabla_j R - \frac{1}{2\tau} R_{ij},
\]

up to errors of order \(\frac{1}{N}\). We will see that by setting \(\tau = -t\), the full curvature tensor is equal to Hamilton’s Harnack expression [9], up to errors of order \(\frac{1}{N}\). Hence, we would like to prove Hamilton’s Harnack inequality from the view point of the thermostat by using the basic property of the Ricci flow that the weak positivity of curvature operator is preserved (maximum principle). However, since \(\tau > 0\) and Hamilton’s Harnack inequality holds with \(t > 0\), the Harnack expression which appears in the thermostat has opposite sign at terms where \(\tau\) appears. This motivates the hyperbolic thermostat we will introduce in the next section.
3.2 Hyperbolic thermostat

**Theorem 3.2.** Let \((M, g_{ij}(t))\) be a complete (forward) Ricci flow for \(t \in (0, T] \subset \mathbb{R}^+\) with bounded curvature, and \((\mathbb{H}^N, g_{\alpha\beta})\) be a N-dimensional hyperbolic space with constant sectional curvature \(-\frac{1}{2N}\) for \(N \in \mathbb{N}\). We define \(\tilde{M} = M \times \mathbb{H}^N \times (0, T]\), and a metric \(\tilde{g}\) on \(\tilde{M}\) as follows:

\[
\tilde{g}_{ij} = g_{ij}, \quad \tilde{g}_{\alpha\beta} = tg_{\alpha\beta}, \quad \tilde{g}_{00} = R - \frac{N}{2t}, \quad \tilde{g}_{i\alpha} = \tilde{g}_{0i} = \tilde{g}_{0\alpha} = 0,
\]

where \(i, j\) are coordinate indices on the \(M\) factor, \(\alpha, \beta\) are coordinate indices on the \(\mathbb{H}^N\) factor, 0 represent the index of the time coordinate \(t\), \(R\) is the scalar curvature with respect to the metric \(g_{ij}\). Then, \((\tilde{M}, \tilde{g})\) is Ricci flat up to errors of order \(\frac{1}{N}\). Moreover, Hamilton’s Harnack expression appears as the full curvature operator with respect to the metric \(\tilde{g}_{ab}\).

Note that \(\tilde{M}\) has potentially infinitesimal dimension because later we take the limit \(N \to \infty\), and \(\tilde{g}_{00}\) is negative for sufficiently large \(N > 0\). Hence, \(\tilde{g}\) is a Lorentzian metric on \(\tilde{M}\) in this situation. When we see the tensor \(\tilde{g}\) as a metric on \(T^*\tilde{M}\) rather than \(T\tilde{M}\), it degenerates in a limit \(N \to \infty\). Then, this metric converges to a weakly positive definite tensor in this limit.

**proof of Theorem 3.1.** We prove Theorem 3.1 using basic computations in Riemannian geometry. We use the Koszul formula for the Christoffel symbols,

\[
\tilde{\Gamma}^a_{bc} = \frac{1}{2} \tilde{g}^{ad} \left( \frac{\partial \tilde{g}_{bd}}{\partial x^c} + \frac{\partial \tilde{g}_{cd}}{\partial x^b} - \frac{\partial \tilde{g}_{bc}}{\partial x^d} \right)
\]

(3.1)

to compute all kinds of \(\tilde{\Gamma}^a_{bc}\), where the indices \(a, b, c\) represent either \(i\) (the
Proposition 3.3. The orthonormal frames \( \{ \frac{\partial}{\partial x^i} \}_{j=1}^n \subset \{ \frac{\partial}{\partial x^\alpha} \}_{\alpha=1}^{n+N+1} \) at \( t = 0 \) are orthonormal for all time up to errors of order \( \frac{1}{N} \).

The covariant derivative \( \tilde{\nabla} \frac{\partial}{\partial x^i} \) plays a role like the vector field \( D_t \) as mentioned Remark 2.2.

By the definition of \( \tilde{g}_{ab} \), the other components of Christoffel symbol are clearly
vanish:
\[
\begin{align*}
\tilde{\Gamma}^{\alpha}_{0\beta} &= \frac{1}{2} \tilde{g}^{\alpha\beta} \left( \frac{\partial \tilde{g}_{0\beta}}{\partial x^0} + \frac{\partial \tilde{g}_{0\alpha}}{\partial x^0} - \frac{\partial \tilde{g}_{00}}{\partial x^\beta} \right) = 0, \\
\tilde{\Gamma}^{0}_{\alpha\beta} &= \frac{1}{2} \tilde{g}^{0\beta} \left( \frac{\partial \tilde{g}_{0\alpha}}{\partial x^0} + \frac{\partial \tilde{g}_{0\alpha}}{\partial x^0} - \frac{\partial \tilde{g}_{00}}{\partial x^\beta} \right) = 0, \\
\tilde{\Gamma}^{i}_{\alpha\beta} &= \frac{1}{2} \tilde{g}^{ij} \left( \frac{\partial \tilde{g}_{\beta j}}{\partial x^\alpha} + \frac{\partial \tilde{g}_{\alpha j}}{\partial x^\beta} - \frac{\partial \tilde{g}_{ij}}{\partial x^\beta} \right) = 0, \\
\tilde{\Gamma}^{\alpha}_{i\beta} &= \frac{1}{2} \tilde{g}^{\alpha\gamma} \left( \frac{\partial \tilde{g}_{i\gamma}}{\partial x^\beta} + \frac{\partial \tilde{g}_{i\gamma}}{\partial x^\beta} - \frac{\partial \tilde{g}_{i\gamma}}{\partial x^\beta} \right) = 0.
\end{align*}
\]

We compute the Ricci tensor with respect to \( \tilde{g} \) by taking the trace of the curvature tensor. First, by using the standard formula for the curvature tensor:
\[
\tilde{R}_{abcd} = \tilde{g}_{df} \left( \frac{\partial \tilde{\Gamma}^f_{ac}}{\partial x^b} - \frac{\partial \tilde{\Gamma}^f_{bc}}{\partial x^a} + \tilde{\Gamma}^e_{ac} \tilde{\Gamma}^f_{be} - \tilde{\Gamma}^e_{bc} \tilde{\Gamma}^f_{ae} \right),
\]
we have
\[
\tilde{R}_{ijkl} = \tilde{g}_{lm} \left( \frac{\partial \tilde{\Gamma}^m_{ik}}{\partial x^j} - \frac{\partial \tilde{\Gamma}^m_{jk}}{\partial x^i} + \tilde{\Gamma}^e_{ik} \tilde{\Gamma}^m_{je} - \tilde{\Gamma}^e_{jk} \tilde{\Gamma}^m_{ie} \right)
= R_{ijkl} + \tilde{g}_{lm} \left( \tilde{\Gamma}^{0}_{0k} \tilde{\Gamma}^{m}_{0j} - \tilde{\Gamma}^{0}_{0j} \tilde{\Gamma}^{m}_{0k} \right)
= R_{ijkl} + \tilde{g}_{lm} \left\{ \left( R - \frac{N}{2t} \right)^{-1} R_{ik} (-R_{jk}) - \left( R - \frac{N}{2t} \right)^{-1} R_{jk} (-R_{ik}) \right\}
= R_{ijkl} - \left( R - \frac{N}{2t} \right)^{-1} (R_{ik} R_{jl} + R_{jk} R_{il}),
\]
(3.2)
\[
\bar{R}_{ij0k} = \bar{g}_{km} \left( \frac{\partial \bar{\Gamma}_m^i}{\partial x^j} - \frac{\partial \bar{\Gamma}_m^j}{\partial x^i} + \bar{\Gamma}_i^l \bar{\Gamma}_m^{jl} - \bar{\Gamma}_j^l \bar{\Gamma}_m^{li} \right) \\
= \bar{g}_{km} \left\{ \frac{\partial}{\partial x^j} \left( -R_m^i \right) - \frac{\partial}{\partial x^i} \left( -R_m^j \right) + \Gamma_m^i \left( -R^n_j \right) - \Gamma_m^n \left( -R^i_j \right) \right\} \\
+ \bar{g}_{km} \left\{ \frac{1}{2} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^j} \left( -R_m^j \right) - \frac{1}{2} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} \left( -R_m^n \right) \right\} \\
= \nabla_i R_{jk} - \nabla_j R_{ik} + \frac{1}{2} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} R_{ik} - \frac{\partial R}{\partial x^j} R_{jk} ,
\]
(3.3)
\[
\bar{R}_{m0j} = \bar{g}_{0i} \left( \frac{\partial \bar{\Gamma}_i^0}{\partial x^0} - \frac{\partial \bar{\Gamma}_0^i}{\partial x^0} + \bar{\Gamma}_0^i \bar{\Gamma}_0^{0j} - \bar{\Gamma}_0^j \bar{\Gamma}_0^{0i} \right) \\
= \bar{g}_{0i} \left( \frac{\partial}{\partial t} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^j} \right) - \frac{\partial}{\partial x^i} \left( \frac{1}{2} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^j} \right) \\
+ \bar{\Gamma}_0^i \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^j} - \frac{1}{2} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} + \frac{1}{2} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} \\
- \frac{1}{2} \frac{\partial^2 R}{\partial x^i \partial x^j} + \frac{1}{2} \Gamma_0^m \frac{\partial R}{\partial x^m} + \frac{1}{4} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} \\
+ R_m^m R_{im} - \frac{1}{4} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} \\
= \frac{\partial R_{ij}}{\partial t} - \frac{1}{2} \left( \frac{\partial^2 R}{\partial x^i \partial x^j} - \Gamma_m^i \frac{\partial R}{\partial x^m} + R_m^m R_{im} \right) + \frac{1}{2} \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} \\
- \frac{1}{4} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} - \frac{\partial R}{\partial x^i} R_{ij} + \frac{1}{4} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} \\
= \Delta R_{ij} + 2R_{ik} R^{kl} - \frac{1}{2} \nabla_i \nabla_j R - R_m^m R_{im} \\
- \frac{1}{4} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} - \frac{\partial R}{\partial x^i} R_{ij} + \frac{1}{4} \left( R - \frac{N}{2t} \right) \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} ,
\]
(3.4)
where \( \nabla \) is the Levi-Civita connection associated to the metric \( g_{ij} \). Here, we have used the evolution equation for the Ricci tensor (2.7).
We see that the other components vanish up to errors of order $\frac{1}{\mathcal{N}}$:

\[
\tilde{R}_{a\beta\gamma\delta} = \tilde{g}_{\delta e} \left( \frac{\partial \tilde{\Gamma}_{\alpha e}^{\gamma}}{\partial x^\beta} - \frac{\partial \tilde{\Gamma}_{\gamma e}^{\beta}}{\partial x^\alpha} + \tilde{\Gamma}_{\alpha e}^{\gamma} \tilde{\Gamma}_{\beta e}^{\gamma} - \tilde{\Gamma}_{\beta e}^{\gamma} \tilde{\Gamma}_{\gamma e}^{\beta} \right) \\
= t R_{a\beta\gamma\delta} + \frac{1}{4} \left( R - \frac{N}{2t} \right)^{-1} \left( g_{\beta\gamma} g_{\alpha\delta} - g_{\alpha\gamma} g_{\beta\delta} \right) \\
= - \frac{1}{4} \frac{2t}{N} \left( R - \frac{N}{2t} \right)^{-1} \left( g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \right) ,
\]

\[
\tilde{R}_{a\beta \gamma i} = \tilde{g}_{ij} \left( \frac{\partial \tilde{\Gamma}_{\alpha j}^{\gamma}}{\partial x^\beta} - \frac{\partial \tilde{\Gamma}_{\beta j}^{\gamma}}{\partial x^\alpha} + \tilde{\Gamma}_{\alpha j}^{\gamma} \tilde{\Gamma}_{\beta j}^{\gamma} - \tilde{\Gamma}_{\beta j}^{\gamma} \tilde{\Gamma}_{\alpha j}^{\gamma} \right) = 0 ,
\]

\[
\tilde{R}_{a\beta \gamma 0} = \tilde{g}_{00} \left( \frac{\partial \tilde{\Gamma}_{\alpha 0}^{\gamma}}{\partial x^\beta} - \frac{\partial \tilde{\Gamma}_{\beta 0}^{\gamma}}{\partial x^\alpha} + \tilde{\Gamma}_{\alpha 0}^{\gamma} \tilde{\Gamma}_{\beta 0}^{\gamma} - \tilde{\Gamma}_{\beta 0}^{\gamma} \tilde{\Gamma}_{\alpha 0}^{\gamma} \right) \\
= \tilde{g}_{00} \left\{ - \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \\
- \frac{1}{2} \Gamma_{\alpha\gamma}^{\delta} \left( R - \frac{N}{2t} \right)^{-1} g_{\beta\delta} + \frac{1}{2} \Gamma_{\beta\gamma}^{\delta} \left( R - \frac{N}{2t} \right)^{-1} g_{\alpha\delta} \right\} \\
= - \frac{1}{2} \left( \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} + \frac{1}{2} g^{\delta e} \left( \frac{\partial g_{\gamma e}}{\partial x^\alpha} + \frac{\partial g_{\alpha e}}{\partial x^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial x^e} \right) \right) g_{\beta\delta} \\
= 0 ,
\]

\[
\tilde{R}_{a\beta ij} = \tilde{g}_{jk} \left( \frac{\partial \tilde{\Gamma}_{\alpha j}^{\beta k}}{\partial x^i} - \frac{\partial \tilde{\Gamma}_{\beta k}^{\alpha j}}{\partial x^i} + \tilde{\Gamma}_{\alpha j}^{\beta k} \tilde{\Gamma}_{\beta k}^{\alpha j} - \tilde{\Gamma}_{\beta k}^{\alpha j} \tilde{\Gamma}_{\alpha j}^{\beta k} \right) = 0 ,
\]

\[
\tilde{R}_{a\beta 0i} = \tilde{g}_{ij} \left( \frac{\partial \tilde{\Gamma}_{\alpha j}^{\beta 0}}{\partial x^i} - \frac{\partial \tilde{\Gamma}_{\beta 0}^{\alpha j}}{\partial x^i} + \tilde{\Gamma}_{\alpha j}^{\beta 0} \tilde{\Gamma}_{\beta 0}^{\alpha j} - \tilde{\Gamma}_{\beta 0}^{\alpha j} \tilde{\Gamma}_{\alpha j}^{\beta 0} \right) = 0 ,
\]

\[
\tilde{R}_{ai\beta j} = \tilde{g}_{jk} \left( \frac{\partial \tilde{\Gamma}_{a j}^{\beta k}}{\partial x^i} - \frac{\partial \tilde{\Gamma}_{\beta k}^{a j}}{\partial x^i} + \tilde{\Gamma}_{a j}^{\beta k} \tilde{\Gamma}_{\beta k}^{a j} - \tilde{\Gamma}_{\beta k}^{a j} \tilde{\Gamma}_{a j}^{\beta k} \right) \\
= \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} g_{\alpha\beta} R_{ij} ,
\]

\[
\tilde{R}_{a\beta 0j} = \tilde{g}_{ij} \left( \frac{\partial \tilde{\Gamma}_{a j}^{\beta 0}}{\partial x^0} - \frac{\partial \tilde{\Gamma}_{\beta 0}^{a j}}{\partial x^0} + \tilde{\Gamma}_{a j}^{\beta 0} \tilde{\Gamma}_{\beta 0}^{a j} - \tilde{\Gamma}_{\beta 0}^{a j} \tilde{\Gamma}_{a j}^{\beta 0} \right) \\
= \frac{1}{4} \left( R - \frac{N}{2t} \right)^{-1} g_{\alpha\beta} \frac{\partial R}{\partial x^i} ,
\]
\[ \tilde{R}_{\alpha;0} = g_{00} \left( \frac{\partial \tilde{\Gamma}^0_{\alpha \beta}}{\partial x^0} - \frac{\partial \tilde{\Gamma}^0_{\beta \alpha}}{\partial x^0} + \tilde{\Gamma}^e_{\alpha \beta} \tilde{\Gamma}^0_{0 e} - \tilde{\Gamma}^e_{0 \beta} \tilde{\Gamma}^0_{0 e} \right) \]

\[ = g_{00} \left\{ \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \left( \frac{\partial R}{\partial t} + \frac{N}{2t^2} \right) g_{\alpha \beta} \right. \]

\[ - \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} g_{\alpha \beta} + \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \left( \frac{\partial R}{\partial t} + \frac{N}{2t^2} \right) \]

\[ - \frac{1}{2 \delta_\beta} \left\{ - \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} g_{\alpha \gamma} \right\} \]

\[ = \frac{1}{4} \left( R - \frac{N}{2t} \right)^{-1} \left( \frac{\partial R}{\partial t} + \frac{R}{t} \right) g_{\alpha \beta} , \]

\[ \tilde{R}_{\alpha i j k} = \tilde{g}_{k l} \left( \frac{\partial \tilde{\Gamma}^l_{\alpha j}}{\partial x^i} - \frac{\partial \tilde{\Gamma}^l_{\alpha i}}{\partial x^j} + \tilde{\Gamma}^e_{\alpha j} \tilde{\Gamma}^l_{i e} - \tilde{\Gamma}^e_{i j} \tilde{\Gamma}^l_{\alpha e} \right) = 0 , \]

\[ \tilde{R}_{\alpha i 0 j} = \tilde{g}_{00} \left( \frac{\partial \tilde{\Gamma}^0_{\alpha j}}{\partial x^i} - \frac{\partial \tilde{\Gamma}^0_{\alpha i}}{\partial x^j} + \tilde{\Gamma}^e_{\alpha j} \tilde{\Gamma}^0_{i e} - \tilde{\Gamma}^e_{i j} \tilde{\Gamma}^0_{\alpha e} \right) = 0 , \]

\[ \tilde{R}_{\alpha 0 j k} = \tilde{g}_{j k} \left( \frac{\partial \tilde{\Gamma}^k_{\alpha i}}{\partial x^0} - \frac{\partial \tilde{\Gamma}^k_{\alpha 0 i}}{\partial x^i} + \tilde{\Gamma}^e_{\alpha i} \tilde{\Gamma}^k_{0 e} - \tilde{\Gamma}^e_{0 i} \tilde{\Gamma}^k_{\alpha e} \right) = 0 , \]

\[ \tilde{R}_{\alpha 0 0 j} = \tilde{g}_{00} \left( \frac{\partial \tilde{\Gamma}^0_{\alpha i}}{\partial x^0} - \frac{\partial \tilde{\Gamma}^0_{\alpha 0 i}}{\partial x^0} + \tilde{\Gamma}^e_{\alpha i} \tilde{\Gamma}^0_{0 e} - \tilde{\Gamma}^e_{0 i} \tilde{\Gamma}^0_{\alpha e} \right) = 0 . \]

Since the Ricci tensors are defined by \( \tilde{R}_{ab} := \tilde{g}^{cd} \tilde{R}_{abcd} \), we have

\[ \tilde{R}_{00} = \tilde{g}^{ij} \tilde{R}_{0i0j} + \tilde{g}^{\alpha \beta} \tilde{R}_{0\alpha0\beta} \]

\[ = g^{ij} \left\{ \Delta R_{ij} + 2 R_{ikjl} R^{kl} - \frac{1}{2} \nabla_i \nabla_j R - R^m_{\ j} R^l_{\ m l} \right. \]

\[ - \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \left( \frac{\partial R}{\partial t} + \frac{N}{2t^2} \right) R_{ij} + \frac{1}{4} \left( R - \frac{N}{2t} \right)^{-1} \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} \}

\[ + \frac{1}{t} g^{\alpha \beta} \frac{1}{4t} g_{\alpha \beta} + \frac{1}{4} \left( R - \frac{N}{2t} \right)^{-1} \frac{\partial R}{\partial t} + \frac{N}{2t^2} \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} \}

\[ = \frac{1}{2} \Delta R + |Ric|_g^2 + \frac{1}{4} \left( R - \frac{N}{2t} \right)^{-1} |\nabla R|_g + \frac{N}{4t^2} - \frac{1}{2} \frac{\partial R}{\partial t} - \frac{N}{4t^2} \]

\[ = \frac{1}{4} \left( R - \frac{N}{2t} \right)^{-1} |\nabla R|_g , \]
\[
\tilde{R}_{0i} = g^{jk} \tilde{R}_{0jk} + g^{\alpha\beta} \tilde{R}_{0\alpha\beta} \\
= g^{jk} \tilde{R}_{ijk} + g^{\alpha\beta} \tilde{R}_{0\alpha\beta} \\
= g^{jk} \left\{ \nabla_i R_{kj} - \nabla_k R_{ij} + \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \left( \frac{\partial R}{\partial x^k} R_{ij} - \frac{\partial R}{\partial x^i} R_{kj} \right) \right\} \\
+ \frac{1}{t} g^{\alpha\beta} \frac{1}{4} \left( R - \frac{N}{2t} \right)^{-1} \frac{\partial R}{\partial x^i} g_{\alpha\beta} \\
= \nabla_i R - g^{jk} \nabla_k R_{ji} + \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \left( \frac{\partial R}{\partial x^j} R^i_j - \frac{\partial R}{\partial x^i} R \right) \\
+ \frac{N}{4t} \left( R - \frac{N}{2t} \right)^{-1} \frac{\partial R}{\partial x^i} \right] \text{[from the contracted Bianchi identity (2.3)]} \\
= \nabla_i R - \frac{1}{2} \nabla_i R + \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \frac{\partial R}{\partial x^j} R^j_i - \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \frac{\partial R}{\partial x^i} \left( R - \frac{N}{2t} \right) \\
= \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \frac{\partial R}{\partial x^i} R^j_i, \\
\tilde{R}_{0\alpha} = g^{00} \tilde{R}_{0aoj} + g^{\beta\gamma} \tilde{R}_{0\beta\alpha\gamma} = 0, \\
\tilde{R}_{i\alpha} = g^{00} \tilde{R}_{i0a0} + g^{jk} \tilde{R}_{ijk} + g^{\beta\gamma} \tilde{R}_{i\beta\alpha\gamma} = 0, \\
\tilde{R}_{\alpha\beta} = g^{00} \tilde{R}_{a0j0} + g^{ij} \tilde{R}_{a\beta j} + g^{\gamma\delta} \tilde{R}_{\alpha\gamma\delta} \\
= \frac{1}{4t} \left( R - \frac{N}{2t} \right)^{-1} g_{\alpha\beta} + \frac{1}{4} \left( \frac{\partial R}{\partial t} + \frac{N}{2t^2} \right) \left( R - \frac{N}{2t} \right)^{\frac{1}{2}} g_{\alpha\beta} \\
+ \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} R g_{\alpha\beta} - \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} \left( \frac{N}{2t} g_{\alpha\beta} - g_{\alpha\beta} \right) \\
- \frac{1}{4t} \left( R - \frac{N}{2t} \right)^{-1} \left( N g_{\alpha\beta} - g_{\alpha\beta} \right) \\
= \left( R - \frac{N}{2t} \right)^{-2} g_{\alpha\beta} \left\{ \frac{R}{4t} - \frac{N}{8t^2} + \frac{1}{4} \left( \frac{\partial R}{\partial t} + \frac{N}{2t^2} \right) + \frac{R^2}{2t} - \frac{R}{4t} - \frac{R^2}{2} + \frac{RN}{2t} \right. \\
- \frac{N^2}{8t^2} + \frac{R^2}{2N} - \frac{RN}{2Nt} + \frac{N}{8t^2} - \frac{RN}{4t} + \frac{N^2}{8t^2} + \frac{R}{4t} - \frac{N}{8t^2} \left\} \\
= \frac{1}{4} \left( R - \frac{N}{2t} \right)^{-2} \left( \frac{\partial R}{\partial t} + \frac{2R^2}{N} \right) g_{\alpha\beta},
\]

\[16\]
\begin{align*}
\tilde{R}_{ij} &= \tilde{g}^{00} \tilde{R}_{i0j0} + \tilde{g}^{kl} \tilde{R}_{ikjl} + \tilde{g}^{\alpha\beta} \tilde{R}_{i\alpha j\beta} \\
&= \left( R - \frac{N}{2t} \right)^{-1} \left( \Delta R_{ij} + 2R_{ijkkl} - \frac{1}{2} \nabla_i \nabla_j R - R^m_{j} R_{im} \right) \\
&\quad - \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-2} \left( \frac{\partial R}{\partial t} R_{ij} - \frac{1}{2} \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} + \frac{N}{2t^2} R_{ij} \right) \\
&\quad - \left( R - \frac{N}{2t} \right)^{-1} \left( R_{ij} R - R^m_{i} R_{mj} \right) + R_{ij} + \frac{N}{2t} \left( R - \frac{N}{2t} \right)^{-1} R_{ij} \\
&= \left( R - \frac{N}{2t} \right)^{-1} \left( \Delta R_{ij} + 2R_{ijkkl} R^{kl} - \frac{1}{2} \nabla_i \nabla_j R \right) \\
&\quad + \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-2} \left( \frac{1}{2} \frac{\partial R}{\partial x^i} \frac{\partial R}{\partial x^j} - \left( \frac{\partial R}{\partial t} + \frac{N}{2t} \right) R_{ij} \right) .
\end{align*}

If we take a limit as \( N \to \infty \), then we see all of the components of Ricci tensor are zero. Moreover, one can see that the norm (could be negative since the metric \( \tilde{g} \) is Lorentzian metric) of Ricci tensor is also zero up to errors of order \( \frac{1}{N} \).

From the equations (3.2), (3.3), (3.4), when we take the limit \( N \to \infty \), we have

\begin{align*}
\tilde{R}_{ijkl} &\longrightarrow R_{ijkl} , \\
\tilde{R}_{ij0k} &\longrightarrow \nabla_i R_{jk} - \nabla_j R_{ik} , \\
\tilde{R}_{i0j0} &\longrightarrow \Delta R_{ij} + 2R_{ijkkl} R^{kl} - R^m_{j} R_{im} - \frac{1}{2} \nabla_i \nabla_j R + \frac{1}{2t} R_{ij} .
\end{align*}

Hence, Hamilton’s Harnack expression appears as the components of the full curvature tensor.

**Remark 3.4.** Note that these equations hold when we consider them as a function on \( \tilde{M} \). We can’t see the right hand side of these equations as the tensor of \( \tilde{M} \). For example, we have

\[ \tilde{R}_{i\alpha j\beta} = R_{i\alpha j\beta} \]

by the symmetric property of the curvature tensor of course. But, the right hand side of the following equation

\[ \tilde{R}_{\alpha i\beta j} = \frac{1}{2} \left( R - \frac{N}{2t} \right)^{-1} g_{\alpha\beta} R_{ij} , \]

17
can’t be commute with $i$ and $\alpha$ because it is not well-defined. Hence, we should be careful when we compute tensors on $\tilde{M}$.

We consider the differential equations for the coefficients of the curvature $\tilde{R}_{ijkl}$, $\tilde{R}_{ij0k}$, $\tilde{R}_{0ij0}$. At first, we compute the Laplacian of $\tilde{R}_{abcd}$. We define the Laplacian with respect to $\tilde{g}$,

$$\tilde{\Delta} := \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b .$$

(3.9)

In fact, the term including $\tilde{g}^{00}$ goes to zero up to errors of order $\frac{1}{N}$ since $g^{00}$ is of magnitude $O(\frac{1}{N})$. Using the formula for the tensor $\tilde{A} \in \otimes^4 T^* \tilde{M}$,

$$\tilde{\nabla}_k \tilde{\nabla}_l \tilde{A}_{abcd} = \frac{\partial^2 \tilde{A}_{abcd}}{\partial x^k \partial x^l} - \frac{\partial \Gamma^m_{la}}{\partial x^k} \tilde{A}_{mbcd} - \frac{\partial \Gamma^m_{lb}}{\partial x^k} \tilde{A}_{amcd} - \frac{\partial \Gamma^m_{lc}}{\partial x^k} \tilde{A}_{abcd} - \frac{\partial \Gamma^m_{ld}}{\partial x^k} \tilde{A}_{amcd}$$

(3.10)

and taking trace with respect to the metric $\tilde{g}$, we have

$$\tilde{\Delta} \tilde{R}_{ijkl} = \tilde{g}^{ab}[\tilde{\nabla}_a \tilde{\nabla}_b \tilde{R}_{ijkl}]$$

(3.11)

$$= g^{mn} [\tilde{\nabla}_m \tilde{\nabla}_n \tilde{R}_{ijkl}] + \tilde{g}^{\alpha \beta} [\tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{R}_{ijkl}]$$

$$= g^{mn} [\tilde{\nabla}_m \tilde{\nabla}_n \tilde{R}_{ijkl}] - \tilde{g}^{\alpha \beta} [\tilde{\Gamma}_{\alpha \beta} \tilde{\nabla}_0 \tilde{R}_{ijkl}]$$

$$= \Delta R_{ijkl} - \tilde{\nabla}_0 \tilde{R}_{ijkl} .$$
\[ \hat{\Delta} R_{ij0} = g^{mn} \left[ \nabla_m \nabla_n \tilde{R}_{ij0} \right] + \hat{\varphi}^{\alpha \beta} \left[ \nabla_\alpha \nabla_\beta \tilde{R}_{ij0} \right] \]

\[ = g^{mn} \left[ \nabla_m \nabla_n \tilde{R}_{ij0} - \frac{\partial \Gamma^p_{lm}}{\partial x^m} \tilde{R}_{lpj0} - \frac{\partial \Gamma^p_{lm}}{\partial x^m} \tilde{R}_{mpj0} - \frac{\partial \Gamma^p_{ln}}{\partial x^m} \tilde{R}_{ipj0} - \frac{\partial \Gamma^p_{ln}}{\partial x^m} \tilde{R}_{ipj0} \right. \]

\[ + \tilde{\Gamma}_{mn} \tilde{R}_{ij0} + \tilde{\Gamma}_{mn} \tilde{R}_{ipj0} + \tilde{\Gamma}_{mn} \tilde{R}_{mpj0} + \tilde{\Gamma}_{mn} \tilde{R}_{ipj0} + \tilde{\Gamma}_{mn} \tilde{R}_{ipj0} \]

\[ + \tilde{\Gamma}_{mn} \tilde{R}_{ij0} + \tilde{\Gamma}_{mn} \tilde{R}_{ipj0} + \tilde{\Gamma}_{mn} \tilde{R}_{ipj0} + \tilde{\Gamma}_{mn} \tilde{R}_{ipj0} \]

\[ - \tilde{\varphi}^{\alpha \beta} \left[ \hat{\varphi}^{\alpha \beta} \nabla_\alpha \nabla_\beta \tilde{R}_{ij0} - 2 \hat{\varphi}^{\alpha \beta} \nabla_\alpha \nabla_\beta \tilde{R}_{ij0} \right] \]

\[ = g^{kl} \left[ \nabla_k \nabla_l M_{ij} + \nabla_k P^l_{pij} + \nabla_k R^p_{P_{pij}} \right] \]

\[ - R^l_k \nabla_k P_{pij} - R^l_k \nabla_k P_{pij} - R^l_k \nabla_k P_{pij} \]

\[ + \tilde{\Gamma}_{kl} \tilde{R}_{lpj0} + \tilde{\Gamma}_{kl} \tilde{R}_{mpj0} \]

\[ - \nabla_0 \tilde{R}_{0ij0} + \frac{1}{2} \nabla^p R \left[ P_{pij} + P_{pji} \right] \]

\[ + 2 R^q_{\ n} \nabla_q P_{pij} + \nabla_q P_{pij} + 2 R^p_k R^k_{qj} \]
\[
\begin{align*}
\tilde{\Delta} R_{ijkl} &= \Delta R_{ijkl} - \tilde{\nabla}_0 \tilde{R}_{ijkl} , \\
\tilde{\Delta} R_{ij0k} &= \Delta P_{ijk} - \tilde{\nabla}_0 \tilde{R}_{ij0k} + \frac{1}{2}(\nabla^m R) R_{ijmk} + 2R^m l R_{ijkl} + \frac{1}{2t} P_{ijk} , \\
\tilde{\Delta} R_{i0j0} &= \Delta M_{ij} - \tilde{\nabla}_0 \tilde{R}_{i0j0} + \frac{1}{2} \nabla^m R( P_{mij} + P_{mji} ) \\
&+ 2R^{mn}(\nabla_n P_{mij} + \nabla_n P_{mji}) + 2R^m l R_{ikn} R_{injm} + \frac{1}{t} M_{ij} - \frac{1}{2t^2} R_{ij} ,
\end{align*}
\]
up to errors of order \( \frac{1}{N} \), where
\[
\tilde{\Delta} \tilde{R}_{abcd} := (\tilde{\Delta} \tilde{R}^m)( \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d} ) .
\] (3.13)

On the other hand, the curvature tensor \( \tilde{R}_{abcd} \) evolves in the direction \( \mathbb{R}^+ \) as follows:
\[
\begin{align*}
\tilde{\nabla}_0 \tilde{R}_{ijkl} &= \frac{\partial}{\partial t} R_{ijkl} + R^m_i R_{mijkl} + R^m_j R_{imjkl} + R^m_k R_{ijlkm} + R^m_l R_{ijkm} , \\
\tilde{\nabla}_0 \tilde{R}_{ij0k} &= \frac{\partial}{\partial t} P_{ijk} + R^m_i P_{mjk} + R^m_j P_{imk} \\
&+ R^m_k P_{ijm} + \frac{1}{2}(\nabla^m R) R_{ijmk} + \frac{1}{2t} P_{ijk} , \\
\tilde{\nabla}_0 \tilde{R}_{i0j0} &= \frac{\partial}{\partial t} M_{ij} + R^m_i M_{mj} + R^m_j M_{im} + \frac{1}{2} \nabla^m R( P_{mij} + P_{mji} ) + \frac{1}{t} M_{ij} ,
\end{align*}
\]
up to errors of order \( \frac{1}{N} \), where
\[
\tilde{\nabla}_0 \tilde{R}_{abcd} := (\nabla_0 \tilde{R}^m)( \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d} ) .
\] (3.15)
Indeed, choosing an orthonormal frames on $M \times \mathbb{H}^N$ at a point, we compute

\[
\hat{\nabla}_0 \hat{R}_{ijkl} = \frac{\partial}{\partial t} \hat{R}_{ijkl} - \hat{\Gamma}^{m}_{0i} \hat{R}_{mkl} - \hat{\Gamma}^{m}_{0j} \hat{R}_{imk} - \hat{\Gamma}^{m}_{0k} \hat{R}_{ijm} + \frac{1}{2} (\nabla^m R) \hat{R}_{ijkl} ,
\]

\[
\hat{\nabla}_0 \hat{R}_{ij0k} = \frac{\partial}{\partial t} \hat{R}_{ij0k} - \hat{\Gamma}^{m}_{0i} \hat{R}_{mj0k} - \hat{\Gamma}^{m}_{0j} \hat{R}_{im0k} - \hat{\Gamma}^{m}_{0k} \hat{R}_{i0jm} + \frac{1}{2} (\nabla^m R) \hat{R}_{ij0k} ,
\]

\[
\hat{\nabla}_0 \hat{R}_{i0j0} = \frac{\partial}{\partial t} \hat{R}_{i0j0} - \hat{\Gamma}^{m}_{0i} \hat{R}_{m0j0} - \hat{\Gamma}^{m}_{0j} \hat{R}_{i0m0} - \hat{\Gamma}^{m}_{00} \hat{R}_{ij0a} + \frac{1}{2} (\nabla^m R) \hat{R}_{i0j0} .
\]

From the equations (3.12), (3.14), (2.6), and Lemma 2.1, we have

\[
\tilde{\Delta} \tilde{R}_{ijkl} = -2 R^{m}_{ijn} R^{m n}_{k l} + 2 R^{m}_{ijmn} R^{m n}_{l k} - 2 R^{m}_{imkn} R^{m n}_{j l} + 2 R^{m}_{imln} R^{m n}_{j k} ,
\]

\[
\tilde{\Delta} \tilde{R}_{ij0k} = -2 R^{m}_{ijn} P^{mn}_{k} + 2 R^{m}_{ijkn} P^{mn}_{j} - 2 R^{m}_{imkn} P^{mn}_{j} ,
\]

\[
\tilde{\Delta} \tilde{R}_{i0j0} = -2 R^{m}_{ijn} M^{m n} + 2 P^{mn}_{j} M^{m n} + 4 P^{mn}_{j} P^{nm} + \frac{1}{2} (\nabla^m R) \hat{R}_{ij0k} .
\]

up to errors of order $\frac{1}{N}$. On the other hand, if a Riemannian manifold is Ricci flat, the evolution equation (2.6) can be described as

\[
\Delta R_{ijkl} = -2 R^{m}_{ijn} R^{m n}_{k l} + 2 R^{m}_{ijmn} R^{m n}_{l k} - 2 R^{m}_{imkn} R^{m n}_{j l} + 2 R^{m}_{imln} R^{m n}_{j k} .
\]

When we apply this equation to the components of the full curvature tensor $\tilde{R}_{abcd}$, we get the equation which is equivalent to (3.16).

We consider the curvature operator as a section of $\text{Sym}^2(\wedge^2 T^* \tilde{M})$. The curvature operator is a symmetric operator on the space of 2-forms $\tilde{U}^{ab}$ on $\tilde{M}$ defined by

\[
\tilde{R}m(\tilde{U}, \tilde{U}) = \tilde{R}_{abcd} \tilde{U}^{ab} \tilde{U}^{cd}
\]

where the Harnack expression

\[
Z = \tilde{R}_{ijkl} \tilde{U}^{ij} \tilde{U}^{kl} + 2 \tilde{R}_{ij0k} \tilde{U}^{ij} \tilde{U}^{0k} + \tilde{R}_{0i0j} \tilde{U}^{0i} \tilde{U}^{0j}
\]

21
appears as a part of $\tilde{Rm}(\tilde{U}, \tilde{U})$. They are essentially same in the sense that all terms other than those involved in the Harnack expression is of magnitude $O(\frac{1}{N})$ as $N \to \infty$. However, if $(\tilde{U}^{ab})$’s with at least one index from the $\mathbb{H}^N$ factor are chosen independent of $N$, then $\tilde{Rm}(\tilde{U}, \tilde{U})$ diverges as $N \to \infty$ since the dimension of $\tilde{M}^{n+N+1}$ goes to $\infty$. Therefore, we are forced to consider the restriction to $\tilde{M} = M \times \mathbb{R}^+$ which we will discuss in the next section.

### 3.3 The differential equations for the coefficients of the curvature tensors on the space-time

As is observed at the end of the previous section, the quantity $\tilde{Rm}(\tilde{U}, \tilde{U})$ diverges as $N \to \infty$ unless we do not introduce $N$-dependence in the part of $(\tilde{U}^{ab})$’s which include at least one index from the $\mathbb{H}^N$-part. Therefore, we should introduce an $N$-dependence on such $\tilde{U}^{ab}$’s so that the contribution from this part becomes negligible as $N \to \infty$. Therefore, we choose to work on the restriction to the slice $\tilde{M} := M \times \mathbb{R}^+$ defined by the condition that the $\mathbb{H}^N$-component being constant. We define the metric $\tilde{g}$ on $\tilde{M}$ as follows:

$$
\tilde{g}_{00} = R - \frac{N}{2t}, \quad \tilde{g}_{ij} = g_{ij}, \quad \tilde{g}_{0i} = 0,
$$

where $i, j$ are coordinate indices on the $M$ factor, and 0 represent the the index of the time coordinate $t$. Then, $(\tilde{M}, \tilde{g})$ is not Ricci flat up to errors of order $\frac{1}{N}$. However, The full curvature tensor of the metric $\tilde{g}$ gives Hamilton’s Harnack expression. In the same way as section 3.2, we see that how the curvature tensor $\tilde{R}^{abcd}$ evolves in the direction $\mathbb{R}^+$. Note that we compute at a point $x^i \in M$ and we often write “$A = B$” that means “$A$ is equal to $B$ up to errors of order $\frac{1}{N}$”. Let $P_{ijk}$, $M_{ij}$ be as defined (2.8). Then, we get the
following:

\[ \bar{\nabla}_0 \bar{R}_{ijkl} = \frac{\partial}{\partial t} R_{ijkl} + R^m_i R_{mjkl} + R^m_j R_{imkl} + R^m_k R_{ijml} + R^m_l R_{ijkl} , \]

\[ \bar{\nabla}_0 \bar{R}_{ij0k} = \frac{\partial}{\partial \ell} P_{ijk} + R^m_i P_{mjk} + R^m_j P_{imk} + R^m_k P_{ijm} + \frac{1}{2}(\nabla^m R) R_{ijmk} + \frac{1}{2t} P_{ijk} , \]

\[ \bar{\nabla}_0 \bar{R}_{0ij0} = \frac{\partial}{\partial \ell} M_{ij} + R^m_i M_{mj} + R^m_j M_{im} + \frac{1}{2} \nabla^m R (P_{mij} + P_{mji}) + \frac{1}{t} M_{ij} , \]

up to errors of order \( \frac{1}{N} \), where

\[ \bar{\nabla}_0 \bar{R}_{abcd} := (\bar{\nabla}_0 \bar{R}^m_i) \left( \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d} \right) . \]  

Moreover, computing the Laplacian of \( \bar{R}_{abcd} \) by using the formula (3.10), we have

\[ \bar{\Delta} \bar{R}_{ijkl} = \Delta R_{ijkl} , \]

\[ \bar{\Delta} \bar{R}_{ij0k} = \Delta P_{ijk} + \frac{1}{2}(\nabla^m R) R_{ijmk} + 2 R^m_i \nabla_i R_{ijmk} , \]

\[ \bar{\Delta} \bar{R}_{0ij0} = \Delta M_{ij} + \frac{1}{2} \nabla^m R [P_{mij} + P_{mji}] + 2 R^m_i \nabla_i [P_{mij} + \nabla_i P_{mji}] + 2 R^m_i R_{kn} R_{mjm} , \]

up to errors of order \( \frac{1}{N} \), where

\[ \bar{\Delta} \bar{R}_{abcd} := (\bar{\Delta} \bar{R}^m_i) \left( \frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^c}, \frac{\partial}{\partial x^d} \right) . \]  

up to errors of order \( \frac{1}{N} \), where
Indeed,
\[
\Delta R_{ijkl} = g^{ab}[\nabla_a \nabla_b R_{ijkl}]
\]
\[
= g^{mn}[\nabla_m \nabla_n R_{ijkl}]
\]
\[
= g^{mn}\left[\nabla_m \nabla_n R_{ijkl} - \frac{\partial}{\partial x^m} \Gamma^p_{ab} \frac{\partial}{\partial x^k} R_{ijkl} - \frac{\partial}{\partial x^m} \nabla^p R_{ijkl} - \frac{\partial}{\partial x^m} \nabla^p R_{ijkl}
\right.
\]
\[
+ \Gamma^p_{mn} \Gamma^q_{p0} R_{ijkl} + \Gamma^p_{mn} \Gamma^q_{n0} R_{ijkl} + \Gamma^p_{mj} \Gamma^q_{n0} R_{ipk} + \Gamma^p_{n0} \Gamma^q_{nk} R_{ipq} + \Gamma^p_{n0} \Gamma^q_{k0} R_{iqp}
\left.
\right]
\]
\[
= g^{mn}[\nabla_m \nabla_n P_{ijkl} + (\nabla_m R_{ij}) R_{ijkl} + R_{n} \nabla_m R_{ijkl} - R_{m} \nabla_n R_{ijkl}]
\]
\[
= \Delta P_{ijkl} + \frac{1}{2} (\nabla^m R) R_{ijkl} + 2 R^m \nabla_l R_{ijkl}
\],
\[
\Delta R_{i0j0} = g^{kl}[\nabla_k \nabla_l R_{i0j0}]
\]
\[
= g^{kl}\left[\nabla_k \nabla_l R_{i0j0} - \frac{\partial}{\partial x^k} \Gamma^p_{lj0} R_{ipj0} - \frac{\partial}{\partial x^l} \Gamma^p_{lj0} R_{ipj0}
\right.
\]
\[
- \Gamma^p_{lj0} \frac{\partial}{\partial x^k} R_{ipj0} - \Gamma^p_{lj0} \frac{\partial}{\partial x^k} R_{ipj0} - \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0} + \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0}
\]
\[
+ \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0} + \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0} + \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0} + \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0}
\]
\[
+ \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0} + \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0} + \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0} + \Gamma^p_{l0j} \frac{\partial}{\partial x^k} R_{ipj0}
\]
\[
\left.
\right]
\]
\[
= g^{kl}[\nabla_k \nabla_l M_{ij} + \nabla_k R^p_{ij} P_{pj} + \nabla_k R^p_{ij} P_{pj} - R^p_{kij} \nabla_l P_{pj} - R^p_{kij} \nabla_l P_{pj} - R^p_{kij} \nabla_l P_{pj} + R^p_{kij} \nabla_l P_{pj}]
\]
\[
= \Delta M_{ij} + \frac{1}{2} \nabla^p R[P_{pj} + P_{pj}]
\]
\[
+ 2 R^p_{ij} [\nabla_q P_{pj} + \nabla_q P_{pj}] + 2 R^p_{kij} R^{kq}_{pqj}  
\]
up to errors of order $\frac{1}{\bar{N}}$. Here, we have used the contracted second Bianchi identity (2.3) again. We see that
\[
\Delta R_{ijkl} = \Delta R_{ijkl}
\]
holds, which follows from $\Gamma^0_{ij} = 0$ up to errors of order $\frac{1}{\bar{N}}$. The difference between the equation (3.11) and (3.22) is whether the derivative of the curvature tensor in the direction of $\mathbb{R}^+$ appears or not. We can see the reason
in the computation of (3.11) (the \(\nabla_0\)-covariant derivative stems from by the twice covariant derivative in the \(\mathbb{H}^N\) direction via the formula (3.10)).

From the equations (3.18) and (3.20), we have

\[
(\nabla_0 - \Delta)\bar{R}_{ijkl} = \left(\frac{\partial}{\partial t} - \Delta\right) R_{ijkl}
+ R^m_i R_{mjkl} + R^m_j R_{imkl} + R^m_k R_{ijml} + R^m_l R_{ijkm} ,
\]

\[
(\nabla_0 - \Delta)\bar{R}_{i0j0} = \left(\frac{\partial}{\partial t} - \Delta\right) P_{ij} + R^m_i P_{mj} + R^m_j P_{im} + R^m_i P_{jm}
+ \frac{1}{2t} P_{ijk} - 2R^{lm} \nabla_l R_{ijmn} ,
\]

\[
(\nabla_0 - \Delta)\bar{R}_{i0j0} = \left(\frac{\partial}{\partial t} - \Delta\right) M_{ij} + R^m_i M_{mj} + R^m_j M_{im}
+ \frac{1}{t} M_{ij} - 2R^{mn} (\nabla_n P_{mij} - \nabla_n P_{mji}) - 2R^m_k R^{kn} R_{lmjn} .
\]

From (2.6) and (2.9), we have the following proposition:

**Proposition 3.5.**

\[
(\nabla_0 - \Delta)\bar{R}_{ijkl} = 2R_{imjn} R^m_{kn} - 2R_{imjn} R^m_{kn}
+ 2R_{imkn} R^m_{jn} - 2R_{imkn} R^m_{jn} ,
\]

\[
(\nabla_0 - \Delta)\bar{R}_{i0j0} = 2R_{imjn} P^m_{kn} + 2R_{imkn} P^m_{jn} + 2R_{imkn} P^m_{jn} + \frac{1}{2t} P_{ijk} ,
\]

\[
(\nabla_0 - \Delta)\bar{R}_{i0j0} = 2R_{imjn} M^m_{kn} + 2P^m_{mn} P^m_{jn} - 4P^m_{mn} P^m_{jn} - \frac{1}{2t^2} R_{ij} + \frac{1}{t} M_{ij} ,
\]

up to errors of order \(\frac{1}{N}\).

### 4 Proof of Hamilton’s Harnack inequality

Recall that for any 2-form \(\bar{U}^{ab}\) on \(M\), we can write

\[
Rm(\bar{U}, \bar{U}) = R_{abcd} \bar{U}^{ab} \bar{U}^{cd} .
\]

Here, we prove that the curvature tensor \(Rm(\bar{U}, \bar{U})\) is weakly positive when the curvature operator \(R_{ijkl} U^{ij} U^{kl}\) on \(M\) is positive. First, we can pick a 2-form \(\bar{U}^{ab}\) such that
\[
\begin{aligned}
\left\{
\begin{array}{l}
\nabla_i \bar{U}^{0j} = 0 \\
(\bar{\nabla}_0 - \bar{\Delta}) \bar{U}^{ij} = 0
\end{array}
\right.
\tag{4.2}
\end{aligned}
\]

at a point on \( \bar{M} \). Then, we have

\[
(\bar{\nabla}_0 - \bar{\Delta}) \bar{R} m(\bar{U}, \bar{U}) = ( (\bar{\nabla}_0 - \bar{\Delta}) \bar{R}_{ijkl}) \bar{U}^{ij} \bar{U}^{kl} + 2((\bar{\nabla}_0 - \bar{\Delta}) \bar{R}_{jok}) \bar{U}^{ij} \bar{U}^{0k} + ((\bar{\nabla}_0 - \bar{\Delta}) \bar{R}_{0i0j}) \bar{U}^{0i} \bar{U}^{0j} - 4(\bar{\nabla}^p \bar{R}_{ijkl})(\bar{\nabla}^p \bar{U}^{ij}) \bar{U}^{kl} - 2\bar{R}_{ijkl}(\bar{\nabla}^p \bar{U}^{ij})(\bar{\nabla}^p \bar{U}^{kl}) - 4(\bar{\nabla}^p \bar{R}_{ij0k})(\bar{\nabla}^p \bar{U}^{ij}) \bar{U}^{0k} + 2\bar{R}_{ij0k} \bar{U}^{ij}(\bar{\nabla}_0 - \bar{\Delta}) \bar{U}^{0k} + 2\bar{R}_{0i0j} (\bar{\nabla}_0 - \bar{\Delta}) \bar{U}^{0i} \bar{U}^{0j}
\tag{4.3}
\end{aligned}
\]

where \((\bar{\nabla}_0 - \bar{\Delta}) \bar{R} m(\bar{U}, \bar{U}) = \nabla_0 (\bar{R} m(\bar{U}, \bar{U})) - \Delta (\bar{R} m(\bar{U}, \bar{U}))\).

From the equations (3.24), we have

\[
((\bar{\nabla}_0 - \bar{\Delta}) \bar{R}_{ijkl}) \bar{U}^{ij} \bar{U}^{kl} = (2R_{imjn} R_k^{\cdot mn} - 2R_{imjn} R_l^{\cdot mn} + 2R_{imkn} R_j^{\cdot mn} - 2R_{imkn} R_j^{\cdot mn}) \bar{U}^{ij} \bar{U}^{kl}
\]

\[
= R_{ijmn} R_k^{\cdot mn} \bar{U}^{ij} \bar{U}^{kl} + 4R_{imkn} R_j^{\cdot mn} \bar{U}^{ij} \bar{U}^{kl}
\]

\[
((\bar{\nabla}_0 - \bar{\Delta}) \bar{R}_{ij0k}) \bar{U}^{ij} \bar{U}^{0k} = (2R_{imjn} P_k^{\cdot mn} + 2R_{imkn} P_j^{\cdot mn} + 2R_{imkn} P_j^{\cdot mn} + \frac{1}{2t} P_{ijk}) \bar{U}^{ij} \bar{U}^{0k}
\]

\[
= (2R_{imjn} P_k^{\cdot mn} + 4R_{imkn} P_j^{\cdot mn} + \frac{1}{2t} P_{ijk}) \bar{U}^{ij} \bar{U}^{0k}
\]

\[
((\bar{\nabla}_0 - \bar{\Delta}) \bar{R}_{0i0j}) \bar{U}^{0i} \bar{U}^{0j} = (2R_{imjn} M^{mn} + 2P_{imn} P_j^{\cdot mn} - 4P_{imn} P_j^{\cdot mn} - \frac{1}{2t^2} R_{ij} + \frac{1}{t} M_{ij}) \bar{U}^{0i} \bar{U}^{0j}
\]

\[
= (2R_{imjn} M^{mn} + P_{imi} P_{jm}^{\cdot mn} - 2P_{imi} P_{jm}^{\cdot mn} - \frac{1}{2t^2} R_{ij} + \frac{1}{t} M_{ij}) \bar{U}^{0i} \bar{U}^{0j}.
\]

Indeed, we can see that the following equations hold:

\[
2R_{imjn} R_k^{\cdot mn} - 2R_{imjn} R_l^{\cdot mn} = R_{imjn} R_k^{\cdot mn} + R_{jmin} R_l^{\cdot mn} - R_{imjn} R_k^{\cdot mn} - R_{jmin} R_k^{\cdot mn}
\]

\[
= (R_{imjn} - R_{jmin})(R_k^{\cdot mn} - R_l^{\cdot mn})
\]

\[
= R_{ijmn} R_k^{\cdot mn},
\tag{4.5}
\]

26
\[ 2P_{imn}P_j^{mn} - 2P_{imn}P_j^{nm} = P_{imn}P_j^{mn} + P_{min}P_m^{jn} - P_{imn}P_j^{mn} - P_{min}P_j^{nm} \]
\[ = (P_{nim} + P_{imn})(P_j^{mn} + P_j^{nm}) \]
\[ = P_{mni}P_{mn}^{ij}. \] (4.6)

Here we have used the formula \( P_{ijk} + P_{jki} + P_{kij} = 0 \). Then, we have the following lemma:

**Lemma 4.1.** If a 2-form \( \bar{U}^{ab} \) satisfies

\[
\begin{align*}
\bar{\nabla}_k \bar{U}^{ij} &= \frac{1}{4t} (\delta_k^i \bar{U}^{0j} - \delta_k^j \bar{U}^{0i}) , \\
\bar{\nabla}_i \bar{U}^{0j} &= 0 , \\
(\bar{\nabla}_0 - \bar{\Delta}) \bar{U}^{ij} &= 0 , \\
(\bar{\nabla}_0 - \bar{\Delta}) \bar{U}^{0i} &= \frac{1}{2t} \bar{U}^{0i} ,
\end{align*}
\] (4.7)

at a point on \( \bar{M} \), then we have

\[
(\bar{\nabla}_0 - \bar{\Delta}) \bar{R}m(\bar{U}, \bar{U}) = 2R_{ikjl}M^{klt} \bar{U}^{0i} \bar{U}^{0j} - 2P_{ikl}P_{jlt} \bar{U}^{0i} \bar{U}^{0j} + 8R_{iklm}P_{jlm} \bar{U}^{0i} \bar{U}^{0k} + 4R_{imkn}P_{jkn} \bar{U}^{0i} \bar{U}^{0l} \]
\[ + [P_{ijm} \bar{U}^{0m} + R_{ijmn} \bar{U}^{mn}] [P_{knm} \bar{U}^{0n} + R_{klpq} \bar{U}^{pq}] \] (4.8)

up to errors of order \( \frac{1}{N} \) where \( \bar{\delta}^i_j := \bar{g}^{ia} \bar{g}_{ja} \).

**Proof.** We can pick a 2-form \( U^{ab} \) such that

\[
\bar{\nabla}_k U^{ij} = \frac{1}{4t} (\delta_k^i U^{0j} - \delta_k^j U^{0i}) \] (4.9)

holds at a point. By using the formula \( \nabla^p R_{pijk} = P_{jki} \) from the second Bianchi identity (2.2), We have

\[
-4(\bar{\nabla}^p \bar{R}_{ijkl})(\bar{\nabla}^p \bar{U}^{ij}) \bar{U}^{kl} = -4(\bar{\nabla}^p \bar{R}_{ijkl})\left\{ \frac{1}{4t} (\delta_p^i \bar{U}^{0j} - \delta_p^j \bar{U}^{0i}) \right\} \bar{U}^{kl} \]
\[ = -\frac{1}{t} \bar{\nabla}^p R_{pijkl} \bar{U}^{0j} \bar{U}^{kl} + \frac{1}{t} \bar{\nabla}^p R_{pikl} \bar{U}^{0i} \bar{U}^{kl} \]
\[ = -\frac{1}{t} P_{kij} \bar{U}^{0j} \bar{U}^{kl} + \frac{1}{t} P_{kili} \bar{U}^{0i} \bar{U}^{kl} \]
\[ = -2t P_{ijkl} \bar{U}^{ij} \bar{U}^{0k} \] (4.10)
We can compute all other terms in the equation (4.3) in a similar way:

\[-2\bar{R}_{ijkl}(\nabla_p \bar{U}^{ij})(\nabla^p \bar{U}^{kl})
\]
\[
= -2R_{ijkl}\left\{\frac{1}{4t}(\delta_p^i \bar{U}^{0j} - \delta_p^j \bar{U}^{0i})\right\}\left\{\frac{1}{4t}((g^{kp}\bar{U}^{0l} - g^{lp}\bar{U}^{0k})\right\}
\]
\[
= -\frac{1}{8t^2}R_{ijkl}\left\{g^{ik}\bar{U}^{0j}\bar{U}^{0l} - g^{il}\bar{U}^{0j}\bar{U}^{0k} - g^{jk}\bar{U}^{0i}\bar{U}^{0l} + g^{jl}\bar{U}^{0i}\bar{U}^{0k}\right\}
\]
\[
= -\frac{1}{8t^2}\{R_{ji}\bar{U}^{0i}\bar{U}^{0j} + R_{jk}\bar{U}^{0j}\bar{U}^{0k} + R_{il}\bar{U}^{0i}\bar{U}^{0l} + R_{lk}\bar{U}^{0i}\bar{U}^{0k}\}
\]
\[
= -\frac{1}{2t^2}R_{ij}\bar{U}^{0i}\bar{U}^{0j}.
\]

(4.11)

\[-4(\nabla^p \bar{R}_{ij0k})(\nabla_p \bar{U}^{ij})\bar{U}^{0k}
\]
\[
= -4\nabla^p \bar{R}_{ij0k}(\nabla_p \bar{U}^{ij})\bar{U}^{0k} + 4\bar{R}^m_{p0k}\bar{R}_{ijmk}(\nabla_p \bar{U}^{ij})\bar{U}^{0k}
\]
\[
= -4\nabla^p P_{ijk}\left\{\frac{1}{4t}(\delta_p^i \bar{U}^{0j} - \delta_p^j \bar{U}^{0i})\right\}\bar{U}^{0k} - 4R^m_{p0k}\bar{R}_{ijmk}\left\{\frac{1}{4t}(\delta_p^i \bar{U}^{0j} - \delta_p^j \bar{U}^{0i})\right\}\bar{U}^{0k}
\]
\[
= -\frac{1}{t}\nabla^i P_{ijk}\bar{U}^{0j}\bar{U}^{0k} + \frac{1}{t}\nabla^j P_{ijk}\bar{U}^{0k}\bar{U}^{0i}
\]
\[
+ \frac{1}{t}R^m_{pjmk}\bar{U}^{0j}\bar{U}^{0k} - \frac{1}{t}R^m_{pimk}\bar{U}^{0i}\bar{U}^{0k}
\]
\[
= -\frac{2}{t}\nabla^p P_{pji}\bar{U}^{0i}\bar{U}^{0j} - \frac{2}{t}R^m_{pimj}\bar{U}^{0i}\bar{U}^{0j}.
\]

(4.12)

From the equation (4.10), (4.11), (4.12), and (4.3), we have

\[
(\nabla_0 - \Delta)\bar{R}m(\bar{U}, \bar{U}) = (\nabla_0 - \Delta)\bar{R}_{ijkl}(\nabla_0 - \Delta)\bar{R}_{ij0k}\bar{U}^{0i}\bar{U}^{0j} + 2((\nabla_0 - \Delta)\bar{R}_{ij0k}\bar{U}^{0i}\bar{U}^{0j}
\]
\[
+ (\nabla_0 - \Delta)\bar{R}_{0i0j})\bar{U}^{0i}\bar{U}^{0j}
\]
\[
- \frac{2}{t}P_{ijk}\bar{U}^{0i}\bar{U}^{0j} - \frac{1}{2t^2}R_{ij}\bar{U}^{0i}\bar{U}^{0j}
\]
\[
- \frac{2}{t}\nabla^p P_{pji}\bar{U}^{0i}\bar{U}^{0j} - \frac{2}{t}R^m_{pimj}\bar{U}^{0i}\bar{U}^{0j}
\]
\[
+ 2\bar{R}_{ij0k}(\nabla_0 - \Delta)\bar{U}^{0k} + 2\bar{R}_{0i0j}(\nabla_0 - \Delta)\bar{U}^{0i}\bar{U}^{0j}.
\]

Moreover, if we assume that

\[
(\nabla_0 - \Delta)\bar{U}^{0i} = \frac{1}{2t}\bar{U}^{0i}
\]

(4.14)
at a point, then we have

\[ 2 \bar{R}_{ij0k} \bar{U}^{ij}(\bar{\nabla}_0 - \bar{\Delta}) \bar{U}^{0k} = \frac{1}{t} P_{ijk} \bar{U}^{ij} \bar{U}^{0k} \]

\[ 2 \bar{R}_{0i0j} (\bar{\nabla}_0 - \bar{\Delta}) \bar{U}^{0i} \bar{U}^{0j} = \frac{1}{t} M_{ij} \bar{U}^{0i} \bar{U}^{0j}. \]

(4.15)

Hence, we have

\[
(\bar{\nabla}_0 - \bar{\Delta}) \bar{R} m(\bar{U}, \bar{U}) = ((\bar{\nabla}_0 - \bar{\Delta}) \bar{R}_{ijkl}) \bar{U}^{ij} \bar{U}^{kl} + 2((\bar{\nabla}_0 - \bar{\Delta}) \bar{R}_{ij0k}) \bar{U}^{ij} \bar{U}^{0k} \\
- \frac{2}{t} P_{ijk} \bar{U}^{ij} \bar{U}^{0k} - \frac{1}{2t^2} R_{ij} \bar{U}^{0i} \bar{U}^{0j} \\
- \frac{2}{t} \bar{\nabla}^p P_{p ij} \bar{U}^{0i} \bar{U}^{0j} - \frac{2}{t} R_{pimj} \bar{U}^{0i} \bar{U}^{0j} \\
+ \frac{1}{t} P_{ijk} \bar{U}^{ij} \bar{U}^{0k} + \frac{1}{t} M_{ij} \bar{U}^{0i} \bar{U}^{0j} \\
- (\bar{\nabla}_0 - \bar{\Delta}) \bar{R}_{0i0j}) \bar{U}^{0i} \bar{U}^{0j} - \frac{1}{t} P_{ijk} \bar{U}^{ij} \bar{U}^{0k} \\
- \frac{1}{t} \bar{\nabla}^p P_{p ij} + \frac{1}{t} R_{pimj} \bar{U}^{0i} \bar{U}^{0j}. \]

(4.16)

Here, we have used the formula

\[ M_{ij} = \bar{\nabla}^p P_{p ij} + R_{ikjl} R^{kl} + \frac{1}{2t} R_{ij}. \]

(4.17)

From (4.4), the component of \( \bar{U}^{ij} \bar{U}^{0k} \) in the right hand side of the equation (4.16) is expressed as

\[
2 \left( 2 R_{imjn} P_{mn}^k + 4 R_{imkn} P_{jn}^m + \frac{1}{2t} P_{ijk} \right) \bar{U}^{ij} \bar{U}^{0k} - \frac{1}{t} P_{ijk} \bar{U}^{ij} \bar{U}^{0k} \\
= (4 R_{imjn} P_{mn}^k + 8 R_{imkn} P_{jn}^m) \bar{U}^{ij} \bar{U}^{0k}. \]

(4.18)

Similarly, the component of \( \bar{U}^{0i} \bar{U}^{0j} \) becomes:

\[
\left( 2 R_{imjn} M_{mn}^n + P_{mnj} P_{j}^m - 2 P_{imn} P_{j}^m - \frac{1}{2t^2} R_{ij} + \frac{1}{t} M_{ij} \right) \bar{U}^{0i} \bar{U}^{0j} \\
- \left( \frac{1}{t} \bar{\nabla}^p P_{p ij} + \frac{1}{t} R_{pimj} \right) \bar{U}^{0i} \bar{U}^{0j} \\
= (2 R_{imjn} M_{mn}^n + P_{mnj} P_{j}^m - 2 P_{imn} P_{j}^m) \bar{U}^{0i} \bar{U}^{0j}. \]

(4.19)
Hence, we have
\[
(\nabla_0 - \bar{\Delta})\bar{R}m(\bar{U}, \bar{U}) = (R_{ijmn}R_{kl}^{mn} + 4R_{imkn}R_{j}^{m n})\bar{U}^{ij}\bar{U}^{kl} + (4R_{imjn}P_{j}^{mn}k + 8R_{imkn}P_{j}^{m n})\bar{U}^{ij}\bar{U}^{0k} + (2R_{imjn}M_{mn}^{j} + P_{mni}P_{mn}^{j} - 2P_{imn}P_{j}^{nm})\bar{U}^{0l}\bar{U}^{0j}.
\]
(4.20)

Since we can write
\[
R_{ijmn}R_{kl}^{mn}\bar{U}^{ij}\bar{U}^{kl} + 4R_{imjn}P_{j}^{mn}k\bar{U}^{ij}\bar{U}^{0k} + P_{mni}P_{mn}^{j}\bar{U}^{0i}\bar{U}^{0j} = [P_{ijm}\bar{U}^{0m} + R_{ijmn}\bar{U}^{mn}][P_{kln}\bar{U}^{0n} + R_{klpq}\bar{U}^{pq}],
\]
(4.21)

We have thus proved Lemma 4.1.

**Remark 4.2.** (1) We note that the equation (4.8) is the same as Hamilton [9, Theorem 4.1] if we set
\[
W^i := \bar{U}^{0i}.
\]
Indeed, Hamilton proved the following equation for Harnack expression \(Z\):
\[
(D_t - \Delta)Z = 2R_{ikjl}M_{kl}^{i}W^j - 2P_{klj}P_{j}^{lk}W^iW^j + 8R_{iklm}P_{j}^{lm}U^{ij}W^k + 4R_{imkn}R_{j}^{m n}U^{ij}\bar{U}^{kl} + [P_{ijm}W^m + R_{ijmn}U^{mn}][P_{kln}W^n + R_{klpq}U^{pq}]
\]
(4.23)
at a point where
\[
\begin{align*}
\nabla_k U^{ij} &= \frac{1}{2}(R^i_k W^j - R^j_k W^i) + \frac{1}{4t}(\delta^i_k W^j - \delta^j_k W^i), \\
\nabla_i W^j &= 0, \\
(D_t - \Delta)U^{ij} &= 0, \\
(D_t - \Delta)W^i &= \frac{1}{t}W^i.
\end{align*}
\]
(4.24)

He gave the geometric interpretation to the way of the extension of 2-form \(U^{ij}\) in the direction \(M\) from the following three equations:
\[
\begin{align*}
U_{ij} &= \frac{1}{2}(W_i X_j - W_j X_i), \\
\nabla_i W^j &= R_{ij} + \frac{1}{2t}g_{ij} \quad (gradient \; expanding \; soliton), \\
\nabla_i X^j &= 0.
\end{align*}
\]
(4.25)
Indeed, we compute
\[ \nabla_k U^{ij} = \frac{1}{2} (R_k^i X_j - R_k^j X_i) + \frac{1}{4t} (\delta^i_k X^j - \delta^j_k X^i) . \]  
\text{(4.26)}

Hamilton derived the formulae (4.24) by using a gradient expanding soliton as a model. Since we use the hyperbolic thermostat as a model in our situation, our extension of 2-form (4.7) is the \( R_{ij} = 0 \) version of (4.24).

On the other hand, Our extension of \( \bar{U}^{0i} \) in the direction of \( \mathbb{R}^+ \) forces all components of the right hand side of (4.3) in to be the products in two of \( R_{ijkl}, P_{ijk}, M_{ij} \) like \( R_{imjn} U^{0m} U^{0n} \). If we extend 2-form \( \bar{U}^{0k} \) like Hamilton in our situation, we see that the right hand side of (4.3) becomes the sum of squares like in (4.8) and the additional terms of the form
\[ -\frac{1}{t} [P_{ijk} \bar{U}^{ij} \bar{U}^{0k} + M_{ij} \bar{U}^{0k} \bar{U}^{0j}] \]
which may be negative. For the interpretation of the extra terms, we differentiate the second equation of (4.25) and make \( \nabla_i \nabla_j W^k - \nabla_j \nabla_i W^k \) to get
\[ \nabla_i R_{jk} - \nabla_j R_{ik} = -R_{ijk} W^l . \]

We differentiate again to get
\[ \nabla_i \nabla_j R_{kl} - \nabla_i \nabla_k R_{jl} = -\nabla_i R_{jklm} W^m - R_i^m R_{jklm} - \frac{1}{2t} R_{jkl} . \]

If we take the trace of the equation, we have
\[ P_{bij} X^i X^j W^k + M_{ij} X^i X^j = 0 \]
for all vector \( X^i \). Hence, by setting \( U_{ij} = \frac{1}{2} (X_i \wedge W_j - X_j \wedge W_i) \), \( \bar{U}^{0i} = W^i \), and \( \bar{U}^{ij} = U^{ij} \), we have
\[ P_{ijk} \bar{U}^{ij} \bar{U}^{0k} + M_{ij} \bar{U}^{0k} \bar{U}^{0j} = 0 \]
which holds under the second equation of (4.25), i.e., the equation of the gradient expanding soliton.

(2) We recall a remark by Hamilton (see Hamilton [9, Lemma 4.5]). We see that if \( \bar{R}m(\bar{U}, \bar{U}) \) is weakly positive at a point, then the first and the
second terms of the right hand side of this equation is the sum of squares of linear forms. Indeed, if the curvature tensor $\bar{R}_{ijkl}\bar{U}^{ij}\bar{U}^{kl} + 2\bar{R}_{ij0k}\bar{U}^{ij}\bar{U}^{0k} + \bar{R}_{0i0j}\bar{U}^{0i}\bar{U}^{0j}$ is weakly positive, the tensor is expressed as $[\bar{X}_i\bar{U}^{ij} + \bar{X}_k\bar{U}^{0k}]^2$ by setting $\bar{R}_{abcd}\bar{U}^{ab}\bar{U}^{cd} = (\bar{X}_{ab}\bar{U}^{ab})(\bar{X}_{cd}\bar{U}^{cd})$. Then, we can write

$$
2R_{ijkl}M^{kl}\bar{U}^{0i}\bar{U}^{0j} - 2P_{ikl}P_{j}\bar{U}^{ij}\bar{U}^{0j} + 8R_{ikm}P_{j}^{lm}\bar{U}^{ij}\bar{U}^{0k} + 4R_{imkn}R_{j}^{m}\bar{U}^{ij}\bar{U}^{kl} = \sum_{M,N}(\bar{X}^{ik}_{M}\bar{X}_{0k}^{N}\bar{U}^{0i} - \bar{X}^{N}_{jk}\bar{X}_{0k}^{M}\bar{U}^{0j} - 2\bar{X}^{M}_{ik}\bar{X}^{N}_{jk}\bar{U}^{ij})^2.
$$

Indeed, we have

$$
\sum_{M,N}(\bar{X}^{ik}_{M}\bar{X}_{0k}^{N}\bar{U}^{0i} - \bar{X}^{N}_{jk}\bar{X}_{0k}^{M}\bar{U}^{0j} - 2\bar{X}^{M}_{ik}\bar{X}^{N}_{jk}\bar{U}^{ij})^2 = (\bar{X}_{im}\bar{X}_{jn})(\bar{X}_{0m}\bar{X}_{0n})\bar{U}^{0i}\bar{U}^{0j} + (\bar{X}_{0m}\bar{X}_{0n})(\bar{X}_{im}\bar{X}_{jn})\bar{U}^{0i}\bar{U}^{0j} + 4(\bar{X}_{im}\bar{X}_{jn})(\bar{X}_{km}\bar{X}_{ln})\bar{U}^{ij}\bar{U}^{kl} - 2(\bar{X}_{im}\bar{X}_{jn})(\bar{X}_{0m}\bar{X}_{0n})\bar{U}^{0i}\bar{U}^{0j} + 4(\bar{X}_{0m}\bar{X}_{jn})(\bar{X}_{im}\bar{X}_{kn})\bar{U}^{0k}\bar{U}^{ij} - 4(\bar{X}_{im}\bar{X}_{kn})(\bar{X}_{jm}\bar{X}_{0n})\bar{U}^{ij}\bar{U}^{0k} + 4R_{imjn}M^{mn}\bar{U}^{0i}\bar{U}^{0j} + 4R_{imkn}R_{j}^{m}\bar{U}^{ij}\bar{U}^{kl} + 8R_{imkn}P_{j}^{mn}\bar{U}^{0i}\bar{U}^{0j} - 2P_{imjn}M^{mn}\bar{U}^{0i}\bar{U}^{0j} + 8R_{imkn}P_{j}^{mn}\bar{U}^{ij}\bar{U}^{0k}$$

(4.27)

On the other hand, the third term is clearly a weakly positive quadratic form.

The main theorem is as follows:

**Theorem 4.3.** Assume that the manifold $(M, g_{ij})$ has a weakly positive curvature operator in the setting of theorem 3.2, then $\bar{R}m(\bar{U}, \bar{U})$ is weakly positive for all two-forms $\bar{U}$.

**Proof.** At first we prepare some functions for considering perturbation of $\bar{R}m$ to make $\bar{R}m$ very positive near $t = 0$. In [16], W. X. Shi constructed a smooth function $f \geq 1$ on $M$ which satisfies the following properties:

$$
f(X) \to \infty \quad (X \to \infty)$$

all the covariant derivatives of $f$ are bounded .

If the manifold $M$ is compact, we take $f = 1$. Next, we construct two functions $\phi$ on the space-time $\bar{M}$, $\psi$ on $M$ by using the function $f$. we define

$$
\phi = e^{A t} f(X) \quad , \quad \psi = e^{B t} .
$$

(4.28)
Then, by choosing constants $C$, $\epsilon$, $\delta$, $A$, and $B$, there are some relations between $\phi$ and $\psi$ as described [9, Lemma 5.2]:

\[
\begin{align*}
(\nabla_0 - \Delta) \phi &> C \phi, \\
\nabla_0 \psi &> C \psi, \\
\phi &\geq C \psi.
\end{align*}
\]

In fact, to prove the main theorem, we need the other properties of these functions if the manifold is noncompact. In the case, for any constant $C$ and $\eta > 0$ we can pick two functions $\phi$ and $\psi$ such that

\[
\begin{align*}
\psi &\leq \eta \text{ and } \psi \geq \delta \text{ everywhere}, \\
\phi &\leq \eta \text{ on a compact set avoiding } \tau = 0, \\
\phi &\geq \epsilon
\end{align*}
\]

for some $\delta > 0$ and $\epsilon > 0$. We take the limit $\eta \to \infty$ after we use the maximal principle.

By using these functions, we consider the perturbation $Rm$ of $\tilde{Rm}$. We define $Rm$ as follows:

\[
\hat{Rm}(\bar{U}, \bar{U}) = R_{abcd} \bar{U}^a \bar{U}^b + \frac{1}{t} \phi g_{ij} \bar{U}^{0i} \bar{U}^{0j} + \frac{1}{2} \psi (g_{ik} g_{jl} - g_{il} g_{jk}) \bar{U}^{ij} \bar{U}^{kl}. \tag{4.31}
\]

Then, we have

\[
(\nabla_0 - \Delta) \hat{Rm} = (\nabla_0 - \Delta) \tilde{Rm} + \frac{1}{t} \left[ (\nabla_0 - \Delta) \phi - \frac{1}{t} \phi \right] |W|^2 + \frac{1}{t} \phi g_{ij} \bar{U}^{0i} (\nabla_0 - \Delta) \bar{U}^{0j} + (\nabla_0 \psi) \bar{U}^2 - \psi |\nabla_k \bar{U}^{ij}|^2 \tag{4.32}
\]

where $|W|^2 = g_{ij} \bar{U}^{0i} \bar{U}^{0j}$ and $|\nabla_k \bar{U}^{ij}|^2 = g_{im} g_{jn} \nabla_k \bar{U}^{ij} \nabla_n \bar{U}^{lm}$.

Under the assumption (4.7) at a point, we have

\[
(\nabla_0 - \Delta) \hat{Rm} \geq (\nabla_0 - \Delta) \tilde{Rm} + \frac{1}{t} \left[ (\nabla_0 - \Delta) \phi - C \psi \right] |W|^2 + (\nabla_0 \psi) |U|^2 \tag{4.33}
\]

from the assumption which the curvature tensor $R_{ijkl}$ is bounded. Note that the coefficient of $|W|^2$ of above equation is weakly positive by using (4.29) and choosing a sufficiently large constant $C > \frac{1}{t}$.
Recall that the definition of $\hat{R}m$. We put
\[
\hat{R}_{ijkl} = R_{ijkl} + \frac{1}{2} \psi (g_{ik}g_{jl} - g_{il}g_{jk}) ,
\]
\[
\hat{M}_{ij} = M_{ij} + \frac{1}{t} \phi g_{ij} .
\]
(4.34)

Then, we can write
\[
(\bar{\nabla}_0 - \bar{\Delta}) \hat{R}m(\bar{U}, \bar{U}) \geq 2 \hat{R}_{ijkl} \hat{M}^{kl} \bar{U}^{0i} \bar{U}^{0j} - 2 P_{ikl} P^l_{j} \bar{U}^{0i} \bar{U}^{0j}
\]
\[+ 8 \hat{R}_{iklm} P^l_{m} \bar{U}^{ij} \bar{U}^{0k} + 4 \hat{R}_{lmn} \hat{R}_{ij}^{m n} \bar{U}^{kl} \bar{U}^{jk}
\]
\[+ [P_{ijm} \bar{U}^{0m} + \hat{R}_{ijmn} \bar{U}^{mn}] [P_{klm} \bar{U}^{0m} \bar{U}^{kn} + \hat{R}_{klpq} \bar{U}^{pq}]
\]
\[+ \frac{1}{t} \left[(\nabla_0 - \Delta) \phi - \frac{C}{t} \psi - C \phi \right] |W|^2 + [\nabla_0 \psi - C \phi] |U|^2 .
\]
(4.35)

Indeed, we have we substitute the following equation by the definition (4.31) and the properties (4.29),
\[
(\nabla_0 - \Delta) \hat{R}m(\bar{U}, \bar{U}) \geq 2 \hat{R}_{ijkl} \hat{M}^{kl} \bar{U}^{0i} \bar{U}^{0j} - 2 P_{ikl} P^l_{j} \bar{U}^{0i} \bar{U}^{0j}
\]
\[+ 8 \hat{R}_{iklm} P^l_{m} \bar{U}^{ij} \bar{U}^{0k} + 4 \hat{R}_{lmn} \hat{R}_{ij}^{m n} \bar{U}^{kl} \bar{U}^{jk}
\]
\[+ [P_{ijm} \bar{U}^{0m} + \hat{R}_{ijmn} \bar{U}^{mn}] [P_{klm} \bar{U}^{0m} \bar{U}^{kn} + \hat{R}_{klpq} \bar{U}^{pq}]
\]
\[+ \frac{1}{t} \left[(\phi + \psi + \phi \psi) |W|^2 + C \psi |U||W|
\]
\[+ C (\psi^2 + \psi) |U|^2 .
\]
(4.36)

for the equation (4.33). We see that the last two terms of the right hand side of the equation (4.35) is weakly positive by using (4.29).

At last, we using the maximum principle for the differential equation (4.35) in order to prove main theorem. Since the manifold $(M, g_{ij})$ has a weakly positive curvature operator and a bounded curvature so that $P_{ijk}, M_{ij}$ are bounded, we can write
\[
\hat{R}m(\bar{U}, \bar{U}) = Rm(\bar{U}, \bar{U}) + \frac{1}{t} \phi |W|^2 + \psi |U|^2
\]
\[\geq \left( \frac{1}{t} \phi - C \right) |W|^2 - C |W||U| + \psi |U|^2
\]
(4.37)
If $t > 0$ is sufficiently small or $C, \hat{R}m$ is strictly positive. We would like to prove this positivity is preserved for all time $t$.

Assume that $\bar{R}(U,U)$ is equals to zero at a point $(x_0, t_0)$ of the space-time first where $U = U^{ab} \in \wedge^2 T^*_x \bar{M}$. we can extend $U$ to a 2-form $\bar{U}^{ab}$ on $\bar{M}$, satisfying

\[
\bar{\nabla}_k \bar{U}^{ij} = \frac{1}{4t}(\delta_k^i \bar{U}^{0j} - \delta_k^j \bar{U}^{0i})
\]

\[
\bar{\nabla}_i \bar{U}^{0j} = 0
\]

\[
(\bar{\nabla}_0 - \bar{\Delta}) \bar{U}^{ij} = 0
\]

\[
(\bar{\nabla}_0 - \bar{\Delta}) \bar{U}^{0k} = \frac{1}{2t} \bar{U}^{0k}
\]

at the point $(x_0, t_0) \in \bar{M}$. Then, we see $\bar{\Delta} \hat{R}m$ is nonpositive and the right hand side of the equation (4.35) is strictly positive at the point. Hence, we have $\nabla_0 \hat{R}m > 0$ at the point from the equation (4.35). This implies $\hat{R}m$ must be negative at a short time before. This is a contradiction. \hfill \Box

5 The monotonicity of $\mathcal{W}$–entropy

In this section we state the heuristic argument to recover the monotonicity of $\mathcal{W}$-entropy from a view point of his thermostat. $\mathcal{W}$–entropy is defined by

\[
\mathcal{W}(g, f, \tau) = \int [\tau(R + |\nabla f|^2) + f - n] (4\pi \tau)^{-\frac{n}{2}} e^{-f} dV_g,
\]

where $dV_g$ is Riemannian volume form with respect to $g$, and $f$ is a smooth function on $M^n$.

In the setting of Perelman’s thermostat as defined section 1, we define a diffeomorphism on $\bar{M}$ as follows:

\[
\phi: (x^i, y^\alpha, \tau) \rightarrow (x^i, y^\alpha, \tau(1 - \frac{2f}{N}))
\]

where $f$ is a function on $\bar{M}$ independent on $S^N$. Then we have

\[
\frac{\partial \phi^0}{\partial \tau} = 1 - \frac{2f}{N} - \frac{2\tau}{N} \frac{\partial f}{\partial \tau}, \quad \frac{\partial \phi^0}{\partial x^i} = -\frac{2\tau}{N} \nabla_i f, \quad \frac{\partial \phi^0}{\partial x^0} = \delta^i_j,
\]

\[
\frac{\partial \phi^a}{\partial y^\alpha} = \delta^a_\alpha, \quad \frac{\partial \phi^a}{\partial x^i} = \frac{\partial \phi^a}{\partial \tau} = \frac{\partial \phi^a}{\partial \tau} = \frac{\partial \phi^0}{\partial y^\alpha} = \frac{\partial \phi^a}{\partial y^\alpha} = 0.
\]
Hence, we have
\[(\phi^* \tilde{g})_{ij} = \tilde{g}_{ij} + \left(\frac{N}{2\tau(1 - \frac{2f}{N})} + R\left(\frac{4\tau^2}{N^2} \nabla_i f \nabla_j f\right)\right)\]
\[
= \tilde{g}_{ij} + \left(\frac{N}{2\tau(1 - \frac{2f}{N})} + R\left(\frac{4\tau^2}{N^2} \nabla_i f \nabla_j f\right)\right)
\]
up to errors of order \(\frac{1}{N}\) where \((\phi^* \tilde{g})_{ab} = \tilde{g}_{cd} \frac{\partial \phi^c}{\partial x^a} \frac{\partial \phi^d}{\partial x^b}\).

Moreover, we set \(\psi_\tau: (\tilde{M}, \phi^* \tilde{g}) \longrightarrow (\tilde{M}, \psi_\tau(\phi^* \tilde{g}))\) are the 1-parameter family of diffeomorphisms generated by a vector field \(X(\tau) = \nabla f\). Let \(m\) is a measure on \(\tilde{M}\) satisfies
\[dm = (4\pi \tau)^{-\frac{n}{2}} e^{-f} dV_{\phi^* \tilde{g}}. \tag{5.1}\]
If we choose a function \(f\) such that \(dm\) is independent on \(\tau\), then the function \(f\) satisfies
\[
\frac{\partial f}{\partial \tau} = \frac{1}{2} \text{tr} \frac{\partial \phi^* \tilde{g}}{\partial \tau} - \frac{n}{2\tau}. \tag{5.2}\]
Meanwhile, \(g^m := \psi_\tau(\phi^* \tilde{g})\) satisfies
\[
\frac{\partial g^m}{\partial \tau} = 2Ric(g^m) + 2Hess_{g^m}(f). \tag{5.3}\]
and a function \(f^m := f \circ \psi_\tau\) on \(\tilde{M}\) satisfies
\[
\frac{\partial f^m}{\partial \tau} = \Delta f + R - \frac{n}{2\tau} - |\nabla f|^2 \tag{5.4}\]
Hence, under the evolution equations (5.3), (5.4), we have

\[
g_{\alpha\beta} = (\phi^* \tilde{g})_{\alpha\beta} = (\phi^* \tilde{g})_{00} + (\phi^* \tilde{g})_{ij} \frac{\partial \phi^i}{\partial \tau} \frac{\partial \phi^j}{\partial \tau} + 2(\phi^* \tilde{g})_{i0} \frac{\partial \phi^i}{\partial \tau} \frac{\partial \phi^0}{\partial \tau}
\]

\[
= (\phi^* \tilde{g})_{00} - |\nabla f|^2
\]

\[
= \frac{1}{\tau} \left( \frac{N}{2} - [\tau(2\Delta f - |\nabla f|^2 + R) + f - n] \right)
\]

\[
g^m_{\alpha\beta} = (1 - \frac{2f}{N}) \tilde{g}_{\alpha\beta}, \quad g^m_{i0} = g^m_{a0} = g^m_{ia} = 0.
\]

One can see that the integrand of \( W \)-functional appears a part of \( g^m_{00} \). To clarify the geometric interpretation of that functional in \((\tilde{M}, g^m_{00})\), we consider the hypersurface with respect to \( \tau = \text{const} \). We compute the curvature tensor of \( g^m_{00} \),

\[
R_{ijkl} = R_{ijkl}
\]

\[
R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - |\nabla f|^2 \frac{N}{2} \tau (g_{\alpha\gamma} g_{\beta\delta} - g_{\beta\gamma} g_{\alpha\delta})
\]

\[
R_{\alpha\beta} = \frac{\tau}{N} g_{\alpha\beta} \nabla_i \nabla_j f
\]

up to errors of order \( \frac{1}{N} \). Hence,

\[
R^m = R + \frac{N}{2\tau} + \frac{f}{\tau} - |\nabla f|^2 + 2f = \frac{N}{2\tau} + \frac{1}{\tau} \left[ \tau(2\Delta f - |\nabla f|^2 + R) + f \right] \quad (5.5)
\]

In [13], Perelman describe that the \( W \)-functional is essentially total scalar curvature of this hypersurface as above.

Furthermore, we compute the volume form with respect to \( g^m_{00} \). Let \( U_\alpha, X_i \) be a local coordinate on \( S^N, M \) respectively, then we have

\[
\sqrt{\det(g^m_{\alpha\beta})} \sqrt{\det(g^m_{ij})} \prod_{\alpha=1}^N U^*_\alpha \prod_{i=1}^n X^*_i
\]

\[
= (1 - \frac{2f}{N}) \frac{N}{\tau} \tau^\frac{N}{2} \sqrt{\det(g_{\alpha\beta})} \sqrt{\det(g_{ij})} \prod_{\alpha=1}^N U^*_\alpha \prod_{i=1}^n X^*_i
\]

\[
= \tau^{\frac{N}{2}} e^{-f} \sqrt{\det(g_{\alpha\beta})} \sqrt{\det(g_{ij})} \prod_{\alpha=1}^N U^*_\alpha \prod_{i=1}^n X^*_i
\]

37
up to errors of order $\frac{1}{N}$. The third equation is deduced by the binomial theorem with respect to $(1 - \frac{2f}{N})^N$ and Taylor expansion with respect to $e^{-2f}$ for large $N$. One can see that the volume form with respect to $g_{00}^m$ is equal to $\tau_N^N e^{-f}$ times the standard volume form on $M \times S^N$.

To prove the monotonicity of $W$-functional, we consider the total scalar curvature of geodesic sphere on $(\tilde{M}, g_{00}^m)$. The key of the proof is the following lemma:

**Lemma 5.1.** Let $(X, G)$ be a complete Ricci flat Riemannian manifold and $r$ a distance function on $X$. Then,

$$\frac{\partial}{\partial r} \log \int_{S_X(r)} R_{S_X(r)} dS_X \leq \frac{\partial}{\partial r} \log \int_{S_{R^n}(r)} R_{S_{R^n}(r)} dS_{R^n}.$$

**Proof.** Let $h_{ij}$ be the second fundamental form. From Gauss equation (see, [7], (1.91),

$$(Rm_{S_X(r)})_{ijkl} = (Rm_X)_{ijkl} - h_{il}h_{jk} + h_{ik}h_{jl}.$$ 

Hence,

$$(Ric_{S_X(r)})_{jl} = (Ric_X)_{jl} - (Rm_X)_{njln} - h_{il}h_{lj} + Hh_{jl}.$$ 

where $H$ is the mean curvature. Moreover, we have

$$R_{S_X(r)} = R_X - 2(Ric_X)_{nn} - |h|^2 + H^2.$$ 

Since $X$ is Ricci flat, we have

$$R_{S_X(r)} = -|h|^2 + H^2.$$ 

Hence, we can express the scalar curvature of $S_X(r)$ by the second fundamental form only. In the same way, the scalar curvature of $S_{R^n}(r)$ can be expressed because $R^n$ is flat. Using Bishop-Gromov theorem, the ratio of the total scalar curvature of $S_X(r)$ and $S_{R^n}(r)$ is non-increasing with respect to $r$. \hfill \Box

From (5.5),

$$\int_{S_M(r)} R^m dS_M$$

$$= \frac{N + 2n}{2\tau} \int_\tau^{\frac{N}{2}} e^{-f} \sqrt{\det(g_{\alpha\beta})} \sqrt{\det(g_{ij})} \prod_{\alpha=1}^{N} U_{\alpha}^* \prod_{i=1}^{n} X_i^* + (4\pi)^{\frac{n}{2}} \tau^{\frac{N}{2} + \frac{3}{2} - 1} W.$$ 

$$= (C_1(N, n) + (4\pi)^{\frac{n}{2}} W) \tau^{\frac{N}{2} + \frac{3}{2} - 1}.$$
where $C_i(N,n)$ are constants depending on $N$ and $n$. Meanwhile, we have

$$\int_{S^{n-1}(r)} R_{S^n(r)} dS^n = \frac{(N+n)(N+n-1)}{r^2} \int dS = C_2(N,n) r^{N+n-2}.$$ 

Hence, we see that the $W$-functional is increasing for $\tau$ by applying this lemma to $(\tilde{M}, g^m)$.

References

[1] C. Böhm and B. Wilking, Manifolds with positive curvature operators are space forms. Ann. of Math. (2) 167 (2008), no. 3, 1079-1097.

[2] S. Brendle, A generalization of Hamilton’s differential Harnack inequality for the Ricci flow. J. Differential Geom. 82 (2009), 207-227.

[3] Esther Cabezas-Rivas and Peter Topping, THE CANONICAL EXPANDING SOLITON AND HARNACK INEQUALITIES FOR RICCI FLOW, Preprint (2009).

[4] Esther Cabezas-Rivas and Peter Topping, THE CANONICAL SHRINKING SOLITON ASSOCIATED TO A RICCI FLOW, Preprint (2008), http://www.warwick.ac.uk/~maseq.

[5] B. Chow and S-C. Chu, A geometric interpretation of Hamilton’s Harnack inequality for the Ricci flow. Math. Res. Lett. 2 (1995) 701-718.

[6] B. Chow and D. knopf, New Li-Yau-Hamilton inequalities for the Ricci Flow via the space-time approach. J. Differential Geom. 60 (2002), 1-54.

[7] Bennett Chow, Peng Lu, and Lei Ni, Hamilton’s Ricci Flow. AMS and Science Press, 2008.

[8] D. DeTurck, Deforming metrics in the direction of their Ricci tensors. In Collected papers on Ricci flow. Edited by H. D. Cao, B. Chow, S. C. Chu and S. T. Yau. Series in Geometry and Topology, 37. International Press, 2003.

[9] R. Hamilton, The Harnack estimate for the Ricci flow. J. Differential Geom. 37 (1993) 225-243.
[10] R. Hamilton, The formation of singularities in the Ricci flow. Surveys in differential geometry, Vol. II (Cambridge, MA, 1993) 7-136, Internat. Press, Cambridge, MA, 1995.

[11] J. Kazdan, Another proof of Bianchi’s identity in Riemannian geometry, Proc. Am. Math. Soc., 81(1981), 341-342.

[12] B. Kleiner and J. Lott, Notes on Perelman’s papers, math.DG/0605667

[13] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, math.DG/0211159v1

[14] P. M. Topping, LECTURES ON THE RICCI FLOW, 2006, http://www.maths.warwick.ac.uk/topping/RFnotes.html.

[15] P. M. Topping, L-optimal transportation for Ricci flow. J. Reine Angew. Math. 636 (2009) 93-122.

[16] W.-X. Shi, Ricci deformation of the metric on complete noncompact Riemannian manifolds. J. Differential Geometry 30 (1989), 303-394.