Temperature dependence of critical velocities and long time decay of supercurrents

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Landau gave the first understanding of persistent currents in $^4$He by relating this phenomenon with properties of the quasiparticle spectrum of the system \[1\]. Bogoliubov explained the linear dispersion of the quasiparticle spectrum at small momenta with the assumption of BEC fraction in a superfluid [2], and thus supported Landau’s theory. This microscopic picture of superfluid is, however, not invoked in understanding some later important observations of superfluids, such as, *A) the temperature dependence of critical velocities \[3, 4\], and *B) the long time decay of supercurrents \[5\]. It is natural to wish that a theory of superfluidity can account for these observations besides explaining superfluidity, for the purpose of a unified picture to describe superfluidity and its related properties. In this paper, we show that a rather simple microscopic theory of superfluidity can accomplish such a wish. We also make comparisons between this theory and Iordanskii-Langer-Fisher (ILF) theory \[6\], which is formulated to understand the temperature dependence of critical velocities and in which multiple assumptions of vortex rings are essential.

This microscopic theory relates superfluidity with the properties of the many-body spectrum of a superfluid. The main conclusions of the theory are the following: i) The many-body dispersion spectrum of a superfluid \[E = E(P)\], where \(E\) is the lowest many-body eigen-energy at given momentum \(P\), is not a monotonic function of \(P\); there exist energy barriers in the many-body dispersion spectrum which separate and prevent some current-carrying states from decaying \[7, 8, 9\] (see Fig. 1). ii) the existence of the energy barriers is due to Bose exchange symmetry \[9\]. iii) The height of barriers will decrease with increasing momentum (velocity) of the corresponding supercurrents; and beyond a certain velocity, the barriers disappear and the system dissipates its momentum like a normal system \[7, 9\]. We shall show that with this knowledge of the many-body spectrum, the observations (*A, *B) can be naturally explained. We first use the spectrum of a one-dimensional superfluid to illustrate these observations, and later discuss higher dimensional superfluids of which the gross structures of the spectra are also the same as what specified above.

We consider \(N\) Bose particles moving in a ring with a radial size of \(R\). The Hamiltonian has the form of

\[
H = -\sum_{i=1}^{N} \frac{\hbar^2}{2MR^2} \frac{\partial^2}{\partial \theta_i^2} + \sum_{i<j}^N V(\theta_i - \theta_j),
\]

where \(\theta_i\) is the angular coordinate of the \(i\)-th particle, \(M\) is the mass of a particle and \(V\) is the repulsive interparticle interaction. \(V\) can be either short-range or zero-range. The many-body spectrum can be classified by their angular momenta \(L\) (in following we replace \(L\) by a corresponding momentum \(P = L/R\), considering the equivalent system moving in a straight line with periodic boundary condition). Due to Galileo invariance, the full spectrum can be obtained if the part at momentum regime \(0 \leq P \leq Nh/R\) is known \[7, 9\]. Specifically, if one denotes the \(n\)-th level at momentum \(P\) by \(e_n(P)\),
then
\[ e_n(P + kNh/R) = e_n(P) + ((P + kNh/R)^2 - P^2)/2NM, \]
where \( k \) is an integer.

The many-body dispersion spectrum, as a function of \( P \) at regime \( 0 \leq P \leq Nh/R \), relative to the ground state energy, can be written in a form of
\[ E(P) \equiv E_{ex}(P) + hP/2MR, \]
where \( hP/2MR \) is a (trivial) part of kinetic energy, and where \( E_{ex}(P) \) is symmetric (in \( E - P \) plane) to the line \( P = Nh/2R \) and reaches its maximum at \( P = Nh/2R \). Numerical results [9, 10] suggest that \( E_{ex}(P) \) is parabolic, i.e., \( E_{ex}(P) = v_c(Nh/R - P)/N(R/n) \), we shall use this form of \( E_{ex}(P) \) for discussions.

With the knowledge of \( E_{ex}(P) \), one can obtain the full many-body dispersion spectrum and realize the following feature of this spectrum. i) for a positive integer \( k < v_cMR/h \), there is an energy barrier in the dispersion spectrum within the momentum regime \((k - 1)Nh/R \leq P \leq kNh/R \). ii) the height of this barrier is \( \Delta = \Delta_0(1 - 1/(v_cMR/h))^{2}, \) where \( \Delta_0 = v_cNh/4R \) (see Fig. 1).

Each valley of the dispersion spectrum can 'trap' metastable states of the system given that the temperature is lower enough. For a certain valley, we consider the quasi thermal equilibrium distribution of many-body levels in the region plotted in Fig. 2, then the possibility for the system staying at the levels lower than the energy barrier \( \Delta \) is that
\[ \gamma = \sum_{i} e^{-(e_i - e_o)/k_BT} \quad (e_i - e_o < \Delta) \quad (4) \]
Where \( i, j \) refers to levels in the region, \( k_B \) is the Boltzmann constant, and \( e_o \) is the lowest eigen-energy at the valley, i.e., the local minimum of the dispersion spectrum.

At any small but finite \( T \), \( \gamma \) is smaller than unity. It is with some possibilities that some states, which are near the edge of the valley regime and close to the next lower valley (thick bars in Fig. 3), can be reached, and subsequently these states can decay with ease to the states in the next lower valley, i.e., they are kind of 'doorway' states. This process is then repeated and eventually the system transfers from this valley to the next valley. When \( \gamma \) is the close to one, this transferring process is slow, which, nevertheless, leads to the decay of the supercurrent at the large time scale [3, 4].

When \( \gamma \) exceed a certain value \( \gamma_c \) close to unity, one can ignore the long time slow decay of supercurrents if only relatively short time scale is concerned, the value of \( \gamma \) depends on the many-body spectra, \( T \) and \( \Delta_k \). If one roughly estimates that \( \gamma = \gamma_c \) at \( T = c\Delta_k \) at the \( k-th \) valley, where \( c \) is a constant, one then finds the highest possible velocity of supercurrents at \( T \) is given by \( v_c(T) = v_c(1 - \sqrt{T/T_\lambda}) \) where \( T_\lambda \) is the transition temperature \( T_\lambda \) [11]. One shall note that the local spectra at different valleys are not the same, for example, the shapes of the valleys become more asymmetric with valleys further away from the ground state regime. For this reason, the rough temperature-critical-velocity relation could be rendered to take a form of
\[ v_c(T) = v_c(1 - \sqrt{\alpha(T)T/\alpha(\lambda)T_\lambda}) \quad (5) \]
where \( \alpha(T) \) is a slowly varying function of \( T \).

We thus illustrated the theoretical pictures of two observations (\( *A, *B \)) using the many-body spectrum. One could note that the pictures are direct and unavoidable. A spectrum of a system is a fundamental property of the system, and one can map physical processes to the transferring processes among the many-body levels of the system.

Previously, ILF theory is constructed to explain the temperature dependence of critical velocities. It is interesting to compare our theory with ILF theory. One can see the following agreements between them: a) that there are energy barriers to prevent the decay of currents at short time scale; b) the energy barriers can be 'overcome' by thermal excitations, which leads to long time decay of supercurrents; c) the energy barrier is a decreasing function of the velocity of the supercurrent.

The differences between two theories are the following: a) ILF involves multiple assumptions of vortex rings, such as their sizes, their dynamics, the creation and annihilation of vortex rings. Our theory suggests that these assumptions are not necessary [12]; b) ILF theory determines the heights of the energy barriers, using energetics of vortices, and suggests the heights decrease linearly as the function of the velocity of the supercurrent, i.e., \( \Delta(v) = \Delta_0 - p_\alpha v \) [13]. These 'conclusions' lead to inadequacies of the theory in its quantitative description of some systems [14]. They also lead to inadequacies in
quantitative description at the temperature regime far below the transition point \([4]\). Within our theory, the height of energy barriers are naturally determined by the many-body spectra for which all low-lying eigenstates including the many-body dispersion states are relevant.

We shall discuss higher dimensional cases. We consider for example a system of \(N\) particles in a tube with periodic boundary condition. The section area \(\sigma\) of a tube could be much smaller than \(R^2\), where \(R\) is the length of tube divided by \(2\pi\), but is order(s) of magnitude larger than \(a^2\), where \(a\) is the average interparticle distance. Again, with the knowledge of dispersion spectra in the momentum regime \(0 \leq P \leq Nh/R\) \([7, 9, 15]\), one can derive the full dispersion spectra, particularly all local minima and the energy barrier height associated with each minimum, due to the Galileo invariance.

Beside the primary type of minimum at \(P = Nh/R\), with the corresponding many-body state in principle allows a fraction of BEC, in \([15]\) we also find some local minima at \(P = h/a, 2h/a, \ldots\), the existence of these local minima has sensitive dependences on boundary conditions and on the value of \(N\) (see Fig. 3). It also requires strong interaction. Once these conditions are not satisfied, that type of minima at \(P = h/a, 2h/a, \ldots\) may not exist or be very shallow, and only the primary type of minimum plays an essential role in superfluidity.

In conclusion, we illustrate that a microscopic theory of superfluidity, based on the properties of a superfluid’s many-body spectrum, explains naturally the temperature dependence of critical velocities and the long time decay of supercurrents. We emphasize that these pictures of a superfluid are direct and unavoidable, due to that physical processes can be viewed as changes of occupation probability of each eigen-level and the transfers among them.