ZEILBERGER TO THE RESCUE

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Abstract. We provide both human and computer (even better collaboration between the two) proofs to four recent American Mathematical Monthly problems, namely problems #11897, #11899, #11916, and #11928. We also show that problem 11928 may lead to interesting combinatorial identities.

Dedicated to Herbert S. Wilf, 1931-2012.

We will demonstrate that Zeilberger’s creative telescoping proof methods coupled with human touch proves most of monthly problems involving the binomial coefficients. Problem 11928 leads to the following curious identity involving the Catalan number.

\[
\sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} C(i+j) = \sum_{k=0}^{n+m} \binom{n+m}{k} C(k),
\]

where \( C(n) \) is the \( n \)th Catalan number.

Problem #11897. Proposed by P. Dalyay, Szeged, Hungary. Prove for \( n \geq 0 \), that

\[
\sum_{k,l=0}^{k+l=n} \frac{1}{k+1} \binom{2k}{k} \binom{2l+2}{l+1} = 2 \binom{2n+2}{n}.
\]

First solution using generating functions: First we recall a theorem from product of power series, namely

Theorem [S. H. Wilf, Generatingfunctionology, p. 36] If \( f \) and \( g \) are ordinary power series generating functions for sequences \( \{a_n\} \) and \( \{b_n\} \), then \( fg \) is the ordinary power series generating function for the sequence

\[
\left\{ \sum_{s+t=n; s,t \geq 0} a_s b_t \right\}_{n=0}^{\infty}.
\]

Our solution will make use of the following well known formulas ( [4], pp 52-54)
\[
\frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{k=0}^{\infty} \frac{1}{k + 1} \binom{2k}{k} x^k
\]

(1)

\[
\frac{1}{\sqrt{1 - 4x}} = \sum_{k=0}^{\infty} \binom{2k}{k} x^k
\]

(2)

\[
\frac{1}{\sqrt{1 - 4x}} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right)^k = \sum_{n=0}^{\infty} \binom{2n + k}{n} x^n
\]

(3)

From (2), with change of summation variable, we get

\[
\sum_{j=0}^{\infty} \binom{2j + 2}{j + 1} x^j = \sum_{j=1}^{\infty} \binom{2j}{j} x^{j-1} = \frac{1}{x} \left( \frac{1}{\sqrt{1 - 4x}} - 1 \right) = \frac{1 - \sqrt{1 - 4x}}{x\sqrt{1 - 4x}}.
\]

(4)

Therefore, combining (1) and (4) with the theorem, the right-side of the sum in question has ordinary power series generating function

\[
2 \frac{1}{\sqrt{1 - 4x}} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right)^2.
\]

Finally, the identity follows from (3).

**Second (pocket size) proof using Gosper’s decision procedure [2]**. This time rewrite the sum in the form

\[
\sum_{k=0}^{n} \frac{1}{k + 1} \binom{2k}{k} \binom{2(n - k + 1)}{n - k + 1} = 2 \binom{2n + 2}{n},
\]

and let \( F(n, k) \) be the summand. By Gosper’s algorithm, the hypergeometric term

\[
G(k) = \frac{(-2n + 2k - 3) (k + 1) \binom{2k}{k} \binom{2n - 2k + 2}{n-k+1} k!}{(k + 1)! (n + 2)!}
\]

is an anti-difference of \( F(n, k) \), that is,

\[
F(n, k) = G(k + 1) - G(k).
\]

Now adding both sides over \( k \) for \( 0 \leq n \leq n \), we end up with the identity above, namely

\[
\sum_{k=0}^{n} \frac{1}{k + 1} \binom{2k}{k} \binom{2(n - k + 1)}{n - k + 1} = G(n + 1) - G(0) = 2 \binom{2n + 2}{n}.
\]
Problem #11899. Proposed by J. Sorel, Romania. Show that for any positive integer \( n \),
\[
\sum_{k=0}^{n} \binom{2n}{k} \binom{2n+1}{k} + \sum_{k=n+1}^{2n+1} \binom{2n}{k-1} \binom{2n+1}{k} = \left( \frac{4n+1}{2n} \right) + \left( \frac{2n}{n} \right)^2.
\]

We start by observing that the second sum on the right-hand side is equal to the first sum. To see this, re-write the second sum as
\[
\sum_{k=n+1}^{2n+1} \binom{2n}{k-1} \binom{2n+1}{k} = \sum_{m=0}^{n} \binom{2n}{m} \binom{2n+1}{m},
\]
and make the change of variable \( m = 2n+1-k \) to obtain
\[
\sum_{k=n+1}^{2n+1} \binom{2n}{k-1} \binom{2n+1}{k} = \sum_{m=0}^{n} \binom{2n}{m} \binom{2n+1}{m}.
\]

Therefore, the identity to be shown is equivalent to
\[
\sum_{k=0}^{n} 2^{\binom{2n}{k}} \binom{2n+1}{k} = \left( \frac{4n+1}{2n} \right) + \left( \frac{2n}{n} \right)^2.
\]

If \( w(n) \) is the sum on the left-hand side, then application of Zeilberger’s creative telescoping method \[3\] (go to Maple and type \( \text{ZeilbergerRecurrence}(F(n, k), n, k, w, 0..n) \), where \( F(n, k) \) is the summand on the left-hand side of (3) ) yields that the sum satisfies the non-homogeneous linear recurrence
\[
(2n^2 + 5n + 3)w(n+1) - (32n^2 + 64n + 30)w(n) = -\frac{(n+1)(16n^2 + 38n + 18)}{n^2} \left( \frac{2n}{n+1} \right)^2.
\]

Now it is a routine exercise to show that the right-hand side also satisfies this recurrence. Verify that both sides equal to 2 for \( n = 0 \) to complete the proof.

Problem #11916. Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Universita di Roma " Tor Vergata," Rome, Italy. Show that if \( n, r, \) and \( s \) are positive integers, then
\[
\binom{n+r}{n} \sum_{k=0}^{s-1} \binom{r+k}{r-1} \binom{n+k}{n} = \binom{n+s}{n} \sum_{k=0}^{r-1} \binom{s+k}{s-1} \binom{n+k}{n}.
\]

First re-write the identity as
\[
\sum_{k=0}^{s-1} \binom{n+r}{n} \binom{r+k}{r-1} \binom{n+k}{n} = \sum_{k=0}^{r-1} \binom{n+s}{n} \binom{s+k}{s-1} \binom{n+k}{n} \tag{1}.
\]
We use the Wilf-Zeilberger Method to prove the identity. Denote the sum on the left-side of (1) by \( f(n) \) and on the right-side by \( g(n) \). Then, \( f(n) \) and \( g(n) \) are solutions of the first-order non-homogeneous difference equation

\[
w(n) - (n + 1)w(n + 1) = -\binom{n + s}{n} \binom{s + r}{s} \binom{n + r}{r} \frac{sr}{n + 1}.
\]

To see this, call the summand on the left-side of (1) \( F_1(n, k) \) and on the right-side \( F_2(n, k) \). Also define two companion functions

\[
G_1(n, k) := F_1(n, k) \frac{(k + 1)k}{n + 1},
\]

and

\[
G_2(n, k) := F_2(n, k) \frac{(k + 1)k}{n + 1}.
\]

Then, first check that \( nF_1(n, k) - (n + 1)F_1(n + 1, k) = G_1(n, k + 1) - G_1(n, k) \) and \( nF_2(n, k) - (n + 1)F_2(n + 1, k) = G_2(n, k + 1) - G_2(n, k) \), and sum the first of these equations from \( k = 0 \) to \( k = s - 1 \) and the second equation from \( k = 0 \) to \( k = r - 1 \). Now show that \( G_1(n, 0) = G_2(n, 0) = 0 \) and \( G_1(n, s) = G_2(n, r) \), which equals the right-side of the non-homogeneous difference equation. This establishes that \( f(n) \) and \( g(n) \) satisfy the difference equation.

Finally, since both \( f(n) \) and \( g(n) \) satisfy the first-order difference equation with the initial condition \( f(1) = g(1) = r(r + 1)\binom{s + r}{s-1} \), we must have \( f(n) = g(n) \) for all \( n \geq 1 \) and any positive integers \( r \) and \( s \).

**Problem #11928.** Proposed by Hideyuki Ohtsuka, Saitama, Japan. For positive integers \( n \) and \( m \) and for a sequence \( \langle a_i \rangle \), prove

\[
\sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} a_{i+j} = \sum_{k=0}^{n+m} \binom{n+m}{k} a_k,
\]

and

\[
\sum_{i<j} \binom{n}{i} \binom{n}{j} \frac{i+j}{n} = \sum_{i<j} \binom{n}{i} \binom{n}{j}^2.
\]

For the first identity, using Vandermonde’s convolution, we can rewrite the single sum on the right side as
\[
\sum_{k=0}^{n+m} \binom{n+m}{k} a_k = \sum_{k=0}^{n+m} \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} a_k
\] (8)

\[
= \sum_{i=0}^{n+m} \sum_{k=i}^{m+n} \binom{n}{i} \binom{m}{k-i} a_k
\] (9)

\[
= \sum_{i=0}^{n+m} \sum_{j=i}^{m+n} \binom{n}{i} \binom{m}{j} a_{i+j}
\] (10)

\[
= \sum_{i=0}^{n+m} \sum_{j=0}^{n+m-i} \binom{n}{i} \binom{m}{j} a_{i+j}
\] (11)

\[
= \sum_{i=0}^{n+m} \sum_{j=0}^{n+m-i} \binom{n}{i} \binom{m}{j} a_{i+j}
\] (12)

\[
= \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} a_{i+j}
\] (13)

The second equality is by reversing the order of summation; the third equality is by change of variable of summation \((j = k - i)\); and (5) and (6) follow from \(\binom{n}{k} = 0\) for \(k > n\). This completes the proof of the first identity.
If we take $a_i = \binom{i}{n}$, then the first identity leads to

$$\sum_{j=0}^{n} \sum_{i=0}^{m} \binom{n}{i} \binom{m}{j} \binom{i+j}{n} = \sum_{k=0}^{n+m} \binom{n+m}{k} \binom{k}{n}. $$

Using the identity $\binom{n}{k} \binom{k}{m} = \binom{n}{m} \binom{n-k}{m-k}$, the right side evaluates to

$$\sum_{k=0}^{n+m} \binom{n+m}{k} \binom{k}{n} = \sum_{k=0}^{n+m} \binom{n+m}{n} \binom{n+m-n}{k-n} = \sum_{k=0}^{n+m} \binom{n+m}{n} \binom{m}{k-n} = \binom{n+m}{n} \sum_{j=0}^{m} \binom{m}{j} = \binom{n+m}{n} 2^m. $$

This gives the following nice identity: For positive integers $m$ and $n$,

$$\sum_{j=0}^{n} \sum_{i=0}^{m} \binom{n}{i} \binom{m}{j} \binom{i+j}{n} = \binom{n+m}{n} 2^m. \quad (14)$$

To prove the second identity, taking $m = n$ in (7), we get

$$\sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{i} \binom{n}{j} \binom{i+j}{n} = \sum_{k=0}^{2n} \binom{2n}{k} \binom{k}{n}. \quad (15)$$

The left side of (8) can be written as

$$\sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{i} \binom{n}{j} \binom{i+j}{n} = \sum_{0 \leq i < j \leq n} \binom{n}{i} \binom{n}{j} \binom{i+j}{n} + \sum_{0 \leq j < i \leq n} \binom{n}{i} \binom{n}{j} \binom{i+j}{n} + \sum_{0 \leq i = j \leq n} \binom{n}{i} \binom{n}{j} \binom{i+j}{n}. $$

Using the symmetry in $i$ and $j$, we can simplify this sum to

$$\sum_{j=0}^{n} \sum_{i=0}^{n} \binom{n}{i} \binom{n}{j} \binom{i+j}{n} = 2 \sum_{0 \leq i < j \leq n} \binom{n}{i} \binom{n}{j} \binom{i+j}{n} + \sum_{i=0}^{n} \binom{n}{i} \binom{2i}{n}. $$
On the other hand using symmetry in $i$ and $j$ and the identity $\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$, we can write the right side of (8) as

$$\sum_{k=0}^{2n} \binom{2n}{k} \binom{k}{n} = \binom{2n}{n} 2^n = \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n}{i}^2 \binom{n}{j}^2 = 2 \sum_{0 \leq i < j \leq n} \binom{n}{i}^2 \binom{n}{j}^2 + \sum_{0 \leq i = j \leq n} \binom{n}{i}^2 \binom{n}{j} = 2 \sum_{0 \leq j < i \leq n} \binom{n}{i} \binom{n}{j} \binom{n}{j}^2 + \sum_{i=0}^{n} \binom{n}{i}^3.$$ 

To complete the proof of the second identity, we must show that

$$\sum_{i=0}^{n} \binom{n}{i} \binom{2i}{n} = \sum_{i=0}^{n} \binom{n}{i}^3.$$

We accomplish that we appeal to Zeilberger’s creative telescoping method [3]. Denote the left and right side by $a_n$ and $b_n$ respectively. Then both sequences start with 1, 2, 10, 56, 346, 2252 and satisfy the second order recurrence (computed using the Zeilberger algorithm)

$$(n + 2)^2 w(n + 2) - (7n^2 + 21n + 16)w(n + 1) - 8(n + 1^2)w(n) = 0.$$ 

Therefore, $a(n) = b(n)$ for all positive integers $n$. This completes the proof of the second identity.

Remark: This shows that with careful choice of $\{a_i\}$, one can obtain (perhaps a nontrivial) binomial identities. For example, if we take $a_i$ is the $i^{th}$ Catalan number, then we get

$$\sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \frac{1}{i + j + 1} \binom{2(i + j)}{i + j} = \sum_{k=0}^{n + m} \binom{n + m}{k} \frac{1}{k + 1} \binom{2k}{k}.$$ 

Remark: Proofs of Problem 11899 and Problem 11916 are also provided in [1] as a special case of a general theorem. Here we provided direct proof to these problems.
