ERGODICITY VIA CONTINUITY

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Abstract

We show that the ergodicity of an aperiodic automorphism of a Lebesgue space is equivalent to the continuity of a certain map on a metric Boolean algebra. A related characterization is also presented for periodic and totally ergodic transformations.

MSC2010: 37A25; 28D05; 54C05

1 Introduction

The main goal of this short note is to show that the ergodicity of an aperiodic transformation $T$ of a Lebesgue space $(\Omega, \mathcal{F}, \mu)$ is equivalent to the continuity of a certain transformation associated with $T$. There are many different but equivalent definitions of ergodicity for measure preserving transformations in the literature (see [2, §2.3] for example). Yet another criterion presented here seems to be new and quite interesting.

Let $(\mathcal{F}, d)$ be the metric space of $\mu$-equivalent classes of $\mathcal{F}$-measurable sets (a set $A \in \mathcal{F}$ belongs to the class $[B]$ induced by a set $B \in \mathcal{F}$ iff $\mu(A \Delta B) = 0$). The metric $d$ is the Frechet–Nikodym metric defined as

$$d([A], [B]) = \mu(A \Delta B).$$

Let $\mathcal{N} \in \mathcal{F}$ denote the class of sets of $\mu$-measure zero. Given an automorphism $T$, for each $m \in \mathbb{N}$ define the map $\phi_T^{(m)} : \mathcal{F} \to [0, 1]$ as

$$\phi_T^{(m)}([A]) = \mu\left(\bigcup_{n=0}^{m} T^n A\right).$$

We also put

$$\phi_T([A]) = \lim_{m \to \infty} \phi_T^{(m)}([A]) = \mu\left(\bigcup_{n \geq 0} T^n A\right).$$

The sequence $\{\phi_T^{(m)}([A])\}_{m \in \mathbb{N}}$ is called the wandering rate of $A$ [1, §3.8].

It is well known that $T$ is ergodic iff $\phi_T([A]) = 1$ for every $A \notin \mathcal{N}$. It turns out equivalent to $\phi_T$ being continuous everywhere except the point $\mathcal{N}$, which is the main statement (Theorem 2) in this note. We also present related characterizations for periodic (Theorem 1) and for totally ergodic transformations (Theorem 3).
2 Continuity of the maps

2.1. Periodic transformation. The first local aim is to investigate the continuity of the transformations $\phi_T^{(m)}$, for $m \in \mathbb{N}$. It is required for studying the continuity of $\phi_T$ for periodic transformation $T$. The continuity of $\phi_T^{(m)}$ is quite easy to prove by using the methods of university measure theory courses. We give a proof of this assertion for completeness of exposition.

Lemma 1 The transformation $\phi_T^{(m)} : \mathcal{F} \to [0, 1]$ is everywhere continuous for each $m \in \mathbb{N}$.

Proof It is evident that
\[
\left( \bigcup_{n=0}^{m} T^n A \right) \triangle \left( \bigcup_{n=0}^{m} T^n B \right) \subseteq \bigcup_{n=0}^{m} (T^n A \triangle T^n B),
\]
and then
\[
|\phi_T^{(m)}([A]) - \phi_T^{(m)}([B])| \leq \mu\left( \left( \bigcup_{n=0}^{m} T^n A \right) \triangle \left( \bigcup_{n=0}^{m} T^n B \right) \right) \leq \mu\left( \bigcup_{n=0}^{m} (T^n A \triangle T^n B) \right) \leq \sum_{k=0}^{m} \mu(T^n(A \triangle B)) = (m + 1)\mu(A \triangle B).
\]
This completes the proof. It is worth noting that even the Lipschitz property of $\phi_T^{(m)}$ follows from the proof.

Recall the definitions of periodic and aperiodic transformations. A point $x \in \Omega$ is called periodic for $T$ if there exists a number $n \in \mathbb{N}$ with $T^n x = x$, and the smallest of these numbers is called the period of $x$. Denote the set of periodic points of period $n \in \mathbb{N}$ by $P_n$ and the set of aperiodic points by $P_0$. It is clear that
\[
\Omega = \bigcup_{n \geq 0} P_n.
\]
If $\mu(P_0) = 0$ then the automorphism $T$ is called almost everywhere periodic (or shortly periodic). If $\mu(P_0) = 1$ then $T$ is an aperiodic transformation.

The following proposition on the continuity of $\phi_T$ for periodic automorphisms $T$ is a corollary of Lemma 1.

Proposition 1 Let $T$ be a periodic automorphism of a probability space $(\Omega, \mathcal{F}, \mu)$. Then the transformation $\phi_T$ is everywhere continuous.

Proof Since $\sum_{n=1}^{\infty} \mu(P_n) = 1$, for arbitrary $\varepsilon > 0$ there exists a number $n_0 = n_0(\varepsilon) \geq 1$ such that
\[
\sum_{n=n_0+1}^{\infty} \mu(P_n) < \varepsilon/2.
\]
For every $A \in \mathcal{F}$, put
$$A_{n_0} = A \cap \left( \bigcup_{n=1}^{n_0} P_n \right),$$
and then
$$\phi_T([A]) = \phi_T([A_{n_0}]) + \phi_T([A \setminus A_{n_0}]) = \phi_T^{(n_0-1)}([A_{n_0}]) + \phi_T([A \setminus A_{n_0}]).$$
As soon as
$$d([A],[B]) = \mu(A_{n_0} \triangle B_{n_0}) + \mu((A \setminus A_{n_0}) \triangle (B \setminus B_{n_0}) < \varepsilon/2n_0!,$$
the last calculation in the proof of Lemma 1 yields
$$|\phi_T([A]) - \phi_T([B])| \leq \phi_T^{(n_0-1)}([A_{n_0}]) - \phi_T^{(n_0-1)}([B_{n_0}]) +$$
$$\mu\left( \bigcup_{n \geq 0} T^n(A \setminus A_{n_0}) \triangle \bigcup_{n \geq 0} T^n(B \setminus B_{n_0}) \right) \leq$$
$$< \varepsilon/2 + \mu\left( \bigcup_{n=n_0+1}^{\infty} P_n \right) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$
This completes the proof.

The converse to Proposition 1 is discussed in the next subsection.

2.2. The points of continuity. The following statements describe in detail all points of continuity of $\phi_T$.

**Lemma 2** Let $T$ be an automorphism of a Lebesgue space $(\Omega, \mathcal{F}, \mu)$. If $\phi_T([A]) = 1$ then $[A]$ is a continuity point of $\phi_T$.

**Proposition 2** Suppose that $T$ is an automorphism of a Lebesgue space $(\Omega, \mathcal{F}, \mu)$ and $\mu(P_0) > 0$. Then $\mathcal{N}$ is a discontinuity point of $\phi_T$. Moreover, if $T$ is aperiodic then $\phi_T([A]) < 1$ iff $[A]$ is a discontinuity point of $\phi_T$.

Before proving these assertions, we remark that Propositions 1 and 2 together imply the following characterization of periodic automorphisms of a Lebesgue space.

**Theorem 1** An automorphism $T$ of a Lebesgue space $(\Omega, \mathcal{F}, \mu)$ is periodic iff $\phi_T$ is everywhere continuous.

**Proof** (of Lemma 2) Consider the partition
$$\Omega = \bigcup_{\alpha \in I} \Omega_\alpha$$
of $\Omega$ into ergodic components $\Omega_\alpha$ where $I$ is the set of indices (see [5] for example). Express the condition $\phi_T([A]) = 1$ in terms of this ergodic decomposition:
$$A = \bigcup_{\alpha \in I} A_\alpha, \quad A_\alpha = A \cap \Omega_\alpha.$$
and
\[ \mu\left( \bigcup_{\alpha \in I} \Omega_{\alpha} \right) = 1 = \phi_T([A]) = \mu\left( \bigcup_{n \geq 0} T^n A \right) = \mu\left( \bigcup_{n \geq 0} \bigcup_{\alpha \in I} T^n A_{\alpha} \right) = \mu\left( \bigcup_{\alpha \in I} \bigcup_{n \geq 0} T^n A_{\alpha} \right). \]

It follows that
\[ 0 = \mu\left( \left( \bigcup_{\alpha \in I} \Omega_{\alpha} \right) \setminus \left( \bigcup_{n \geq 0} T^n A_{\alpha} \right) \right) = \mu\left( \bigcup_{\alpha \in I} \left( \Omega_{\alpha} \setminus \left( \bigcup_{n \geq 0} T^n A_{\alpha} \right) \right) \right). \]

Consequently, for each \( J \subseteq I \), we have
\[ \mu\left( \bigcup_{\alpha \in J} \Omega_{\alpha} \setminus \left( \bigcup_{n \geq 0} T^n A_{\alpha} \right) \right) = 0. \]

It is equivalent to
\[ \mu\left( \bigcup_{\alpha \in J} \Omega_{\alpha} \right) = \mu\left( \bigcup_{\alpha \in J} \bigcup_{n \geq 0} T^n A_{\alpha} \right). \tag{1} \]

On the set \( I \) of indices consider the family of measures \( \{ \nu_C \}_{C \in \mathcal{F}} \) defined as
\[ \nu_C(J) = \mu\left( \bigcup_{\alpha \in J} C_{\alpha} \right), \quad J \subseteq I. \]

We claim that (1) guarantees the equivalence of the probability measure \( \nu_{\Omega} \) and the measure \( \nu_A \). It is clear that \( \nu_A \ll \nu_{\Omega} \). Suppose that the opposite is false. Then there exists a set \( J \subset I \) with
\[ \nu_A(J) = 0, \quad \text{but} \quad \nu_{\Omega}(J) > 0. \]

It follows that
\[ 0 = \mu\left( \bigcup_{n \geq 0} T^n \left( \bigcup_{\alpha \in J} A_{\alpha} \right) \right) = \mu\left( \bigcup_{n \geq 0} \bigcup_{\alpha \in J} T^n A_{\alpha} \right) = \mu\left( \bigcup_{\alpha \in J} \Omega_{\alpha} \right) = \nu_{\Omega}(J) > 0, \]

which is a contradiction. Consequently, for each \( \varepsilon > 0 \) there exists \( \delta = \delta(A, \varepsilon) > 0 \) such that \( \nu_A(J) < \delta \) implies \( \nu_{\Omega}(J) < \varepsilon \).

Now we are ready to prove the continuity of \( \phi_T \) at the point \([A]\). For arbitrary \( \varepsilon > 0 \), take \( \delta > 0 \) as in the previous discussion. For \( B \in \mathcal{F} \) with \( \mu(B) > 0 \), because the set \( \bigcup_{n \in \mathbb{Z}} T^n B \) is invariant under \( T \), there exists a set \( J = J(B) \subset I \) of indices such that
\[ \bigcup_{\alpha \in J} \Omega_{\alpha} = \bigcup_{n \in \mathbb{Z}} T^n B \quad (\text{mod } \mu). \]

This yields
\[ \mu\left( \bigcup_{\alpha \in J} \Omega_{\alpha} \right) = \mu\left( \bigcup_{n \in \mathbb{Z}} T^n B \right) = \mu\left( \bigcup_{n \geq 0} T^n B \right) \]

and
\[ \mu\left( \bigcup_{\alpha \in J \setminus I} B_{\alpha} \right) = 0. \]
Assume that \( d([A], [B]) < \delta \). We have
\[
d([A], [B]) = \mu(A \Delta B) = \mu\left( \bigcup_{\alpha \in J} (A_\alpha \Delta B_\alpha) \right) =
\mu\left( \bigcup_{\alpha \in I \setminus J} A_\alpha \right) + \mu\left( \bigcup_{\alpha \in J} (A_\alpha \Delta B_\alpha) \right) = \nu_A(I \setminus J) + \mu\left( \bigcup_{\alpha \in J} (A_\alpha \Delta B_\alpha) \right) < \delta,
\]
which implies \( \nu_A(I \setminus J) < \delta \) and, hence, \( \nu_I(I \setminus J) < \varepsilon \). It follows that
\[
\phi_T([A]) - \phi_T([B]) = 1 - \mu\left( \bigcup_{n \geq 0} T^n B \right) =
\mu\left( \bigcap_{\alpha \in J} \Omega_\alpha \right) - \mu\left( \bigcup_{\alpha \in J} \Omega_\alpha \right) =
\nu_I(I \setminus J) < \varepsilon,
\]
completing the proof.

**Proof (of Proposition 2)** Without loss of generality, assume that the automorphism \( T \) is aperiodic. Suppose that \( \phi_T([A]) < 1 \), and then the set \( B = \bigcup_{n \in \mathbb{Z}} T^n A \) of measure \( \mu(B) < 1 \) is \( T \)-invariant.

The restriction of \( T \) to \( \Omega \setminus B \) is aperiodic and preserves the probability measure \( \mu_{\Omega \setminus B} \).

Applying the Rokhlin–Halmos lemma (see [3] for example), we find that for \( \varepsilon > 0 \) and \( n_0 \geq 1 \) there exists a set \( E \subset \Omega \setminus B \) such that the sets \( T^k E \) for \( 0 \leq k \leq n_0 - 1 \) are disjoint and satisfy the inequality
\[
\mu_{\Omega \setminus B}\left( \bigcup_{k=0}^{n_0-1} T^k E \right) > 1 - \varepsilon.
\]
It is clear that \( \mu_{\Omega \setminus B}(E) < \frac{1}{n_0} \). Put \( C = A \cup E \). Then
\[
d([A], [C]) = \mu(A \Delta C) = \mu(E) < \frac{1}{n_0} \mu(\Omega \setminus B)
\]
and
\[
\phi_T([C]) = \mu\left( \bigcup_{n \geq 0} T^n C \right) = \mu\left( \bigcup_{n \geq 0} T^n A \right) + \mu\left( \bigcup_{n \geq 0} T^n E \right) \geq
\phi_T([A]) + \mu\left( \bigcup_{k=0}^{n_0-1} T^k E \right) > \phi_T([A]) + (1 - \varepsilon) \mu(\Omega \setminus B).
\]
In this way, taking \( \varepsilon = 1/2 \) and sufficiently large \( n_0 \geq 1 \) we obtain \( d([A], [C]) \) is small enough but
\[
|\phi_T([A]) - \phi_T([C])| > \mu(\Omega \setminus B)/2.
\]
(2)
This proves that $\phi_T$ is discontinuous at the point $[A]$ with $\phi_T([A]) < 1$.

Now, if $[A]$ is a discontinuity point of $\phi_T([A])$ then Lemma 2 tells us that $\phi_T([A]) < 1$. The proof is complete.

### 2.3. Ergodic and totally ergodic transformations.

The following characterization of ergodic transformations is also a direct corollary of Proposition 2.

**Theorem 2** An aperiodic automorphism $T$ of a Lebesgue space $(\Omega, \mathcal{F}, \mu)$ is ergodic iff $\phi_T : \mathcal{F} \to [0, 1]$ is continuous everywhere except the point $N$.

**Proof** If $T$ is ergodic then $\phi_T([A]) = 1$ for $\mu(A) > 0$. Consequently, Lemma 2 shows that $\phi_T$ is continuous at such points. However, Proposition 2 states that $N$ is a discontinuity point.

Now, if $T$ is not ergodic then there exists an $\mathcal{F}$-measurable $T$-invariant set $A$ with $0 < \mu(A) < 1$. It follows that $\phi_T([A]) = \mu(A) < 1$ and Proposition 2 implies that $\phi_T$ has a discontinuity at $[A]$.

As another application of Proposition 2, we discuss here a characterization of totally ergodic transformations, which means that the powers $T^n$ for all $n \in \mathbb{N}$ are ergodic transformations.

Define a new map $\phi_T^\ast : \mathcal{F} \to [0, 1]$ as

$$\phi_T^\ast([A]) = \inf_{m \in \mathbb{N}} \phi_T^{\ast m}([A]).$$

**Theorem 3** An aperiodic automorphism $T$ of a Lebesgue space $(\Omega, \mathcal{F}, \mu)$ is totally ergodic iff $\phi_T^\ast : \mathcal{F} \to [0, 1]$ is continuous everywhere except the point $N$.

**Proof** If $T$ is totally ergodic then $\phi_T^\ast([A]) = 1$ for all $k \geq 1$ and all $A \notin N$. Hence, $\phi_T^\ast([A]) = 1$, and therefore $\phi_T^\ast$ is continuous at that point. Indeed, for arbitrary $\varepsilon > 0$ we take $0 < \delta < \mu(A)$. Then the inequality $d([A], [B]) < \delta$ implies that $\mu(B) > 0$ and therefore

$$|\phi_T^\ast([A]) - \phi_T^\ast([B])] = |1 - 1| = 0 < \varepsilon.$$

It is evident, that the class $N$ is a discontinuity point of $\phi_T^\ast$.

Assume now that $T$ is not totally ergodic. Take the smallest $k_0 \geq 1$ such that $T^{k_0}$ is not ergodic. Therefore all powers $T^{nk_0}$ for $n \geq 1$ are not ergodic either (because the invariant sets of $T^{k_0}$ are invariant under $T^{nk_0}$ for all $n \geq 1$). It is clear that there are only finitely many, at most $k_0$, such sequences of non-ergodic transformations $\{T^{nk}\}_{n \geq 1}$. Denote by $\mathcal{K}$ the finite set of possible values of $k$. Thus, $k_0 \in \mathcal{K}$ and $|\mathcal{K}| \leq k_0$. Put $\kappa = \prod_{k \in \mathcal{K}} k$. Take a nontrivial invariant set $A \notin N$ of the transformation $T^\kappa$. We claim that it is a discontinuity point of $\phi_T^\kappa$.

By Theorem 2, all transformations $\phi_{T^{nk}}$ for $n \geq 1$ (being non-ergodic) are discontinuous at $[A]$. Considering (2), we conclude that for sufficiently small $\delta > 0$ and all $n \geq 1$ there exist some sets $E_n \subset \Omega \setminus A$ such that $B_n = A \cup E_n$ satisfy

$$d([A], [B_n]) = \mu(E_n) < \frac{\delta}{2^n} \quad \text{and} \quad \phi_{T^{nk}}([B_n]) - \phi_{T^{nk}}([A]) > \frac{1}{2^n} \mu(\Omega \setminus A).$$
For the set \( B = A \cup \bigcup_{n \geq 1} E_n \) it is easy to see that \( d([A], [B]) = \mu(\bigcup_{n \geq 1} E_n) < \delta \), and for all \( n \geq 1 \),
\[
\phi_{T^n}(|B|) - \phi_{T^n}(|A|) \geq \phi_{T^n}(|B_n|) - \phi_{T^n}(|A|) > \frac{1}{2} \mu(\Omega \setminus A). \tag{3}\]

Considering the value \( \phi^*_T([B]) \), we conclude that
\[
\phi^*_T([B]) = \inf_{m \in \mathbb{N}} \phi_{T^m}(|B|) = \min_k \inf_{n \in \mathbb{N}} \phi_{T^k_n}(|B|)
\]
for some \( k' = k'(B) \in \mathcal{K} \). The second equality is true because for the ergodic transformation \( T^m \) we have \( \phi_{T^m}([B]) = 1 \), and therefore the infimum is reached on non-ergodic transformations.

For arbitrary \( \varepsilon > 0 \) there exists a number \( n' \in \mathbb{N} \) such that
\[
\phi^*_T([B]) = \inf_{n \in \mathbb{N}} \phi_{T^{k'n'}}([B]) \geq \phi_{T^{k'n'}}([B]) - \varepsilon. \tag{4}\]

Taking into account the estimates (3) and (4), the monotonicity property
\[
\phi_{T^{k'n'}} \geq \phi_T,
\]
and the equality
\[
\phi^*_T([A]) = \phi_{T^{k'n'}}([A]) = \mu(A),
\]
we obtain
\[
\phi^*_T([B]) - \phi^*_T([A]) \geq \phi_{T^{k'n'}}([B]) - \varepsilon - \phi^*_T([A]) \geq
\geq \phi_{T^{k'n'}}([B]) - \phi_{T^{k'n'}}([A]) - \varepsilon > \frac{1}{2} \mu(\Omega \setminus A) - \varepsilon.
\]

Take sufficiently small \( \varepsilon > 0 \) so that the expression \( \frac{1}{2} \mu(\Omega \setminus A) - \varepsilon \) is positive. Then the last estimate guarantees that \( \phi^*_T \) is discontinuous at \([A]\). The proof is complete.

In conclusion, we remark that it would be interesting to find a related characterization for transformations with a different type of mixing property (see [4] for example).

ACKNOWLEDGMENTS. The work was supported by the program of fundamental scientific research of SB RAS № I.1.2., project № 0314-2016-0005.

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