NOTE ON THE MOTIVIC DT/PT CORRESPONDENCE AND THE MOTIVIC FLOP FORMULA

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ABSTRACT. We prove the motivic version of the DT/PT-correspondence in [1] and the motivic flop formula of the curve counting invariants in the derived category of smooth Calabi-Yau threefold DM stacks. The main method we use is Bridgeland’s Hall algebra identities and the motivic integration map of Bridgeland and Joyce from the motivic Hall algebra of the abelian category of coherent sheaves on a Calabi-Yau threefold DM stack $Y$ to the motivic quantum torus, which is a Poisson algebra homomorphism.

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1. **Introduction**

1.1. **Background on the DT/PT-correspondence.**

. **(1.1.1)** Let $Y$ be a smooth proper Calabi-Yau threefold or a smooth threefold Deligne-Mumford (DM) stack. The Donaldson-Thomas (DT) invariants of $Y$ count stable coherent sheaves on $Y$, which was constructed by R. Thomas in $[43]$ using the perfect obstruction theory $E^\bullet$ in the sense of Li-Tian $[32]$, and Behrend-Fantechi $[5]$ on the moduli space $X$ of stable sheaves over $Y$. If $X$ is proper, then the virtual dimension of $X$ is zero, and the integral

$$\text{DT}_Y = \int_{[X]_{\text{virt}}} 1$$

is the Donaldson-Thomas invariant of $X$. In general the Donaldson-Thomas invariants are defined for certain projective threefolds, and the virtual dimension is not zero and one should integrate some cohomology classes over the virtual fundamental cycle. Here we only restrict to the case of Calabi-Yau threefolds.

. **(1.1.2)** The case of ideal sheaves of $O_Y$ is interesting in the curve counting. If $I \subset O_Y$ is an ideal sheaf of a curve $C \subset Y$, then let $O_C$ be the structure sheaf of the curve. Let $X := I_n(Y, \beta)$ be the moduli space of ideal sheaves $I \subset O_Y$ with the topological data $\beta = [C] \in H_2(Y, \mathbb{Z})$ and $\chi(O_C) = n$. The Donaldson-Thomas invariants $\text{DT}_Y$ on $Y$ have been proved to have deep connections to Gromov-Witten theory and provided more deep understanding of the curve counting invariants, see $[36]$, $[37]$, $[41]$, etc.

. **(1.1.3)** Donaldson-Thomas invariants count ideal sheaves of dimension at most 1 of the Calabi-Yau 3-fold DM stack $Y$. An ideal sheaf $I$ of $Y$ may corresponds to a curve together with random points on the 3-fold DM stack $Y$. So the Donaldson-Thomas invariants do not purely count curves. In $[41]$, Pandharipande and Thomas introduced the moduli space $P_n(Y, \beta)$ of stable pairs $(F, s)$ with fixed determinant, where $F$ is a pure sheaf of dimension at most 1 and $s$ is a section. Let

$$0 \to I_C \to O_Y \xrightarrow{s} F$$

be the corresponding exact sequence. The topological data $\beta, n$ fix the invariants $[F] \in H_2(Y, \mathbb{Z}); \quad \chi(F) = n.$

In this case the stable pairs really count curves with embedded points on the curve. Pandharipande and Thomas constructed a perfect obstruction theory on the moduli space $P_n(Y, \beta)$ and defined the stable pair invariants which we call the Pandharipande-Thomas (PT) invariants. The virtual count of Pandharipande-Thomas invariant is defined as:

$$P_{n, \beta} := \int_{[P_n(Y, \beta)]_{\text{virt}}} 1.$$  

The case of stable pairs of orbifolds or threefold DM stacks is similarly defined and there is a dimensional zero virtual fundamental class on the moduli scheme (stack).
The DT/PT-correspondence conjecture in [41, Conjecture 3.3] is described as follows. The Donaldson-Thomas partition function is defined as

\[ Z^{\text{DT}, \beta}(q) := Z^{\text{DT}, \beta}(Y, q) = \sum_n I_{n, \beta} q^n \]

and \( Z^{\text{DT}, \beta}(q) \) is a Laurent series in \( q \). The reduced partition function \[36\] is given by

\[ Z^{\prime \text{DT}, \beta}(q) = \frac{Z^{\text{DT}, \beta}(q)}{Z^{\text{DT}, \beta}(Y, q)} \]

where \( Z^{\text{DT}, 0}(q) \) is the degree zero series and is proved in [6], [33] and [34] to be the MacMahon function

\[ Z^{\text{DT}, 0}(q) = M(-q)^{\chi(X)}. \]

Let \( Z^{\text{PT}, \beta}(q) \) be the Pandharipande-Thomas stable pair partition function defined as

\[ Z^{\text{PT}, \beta}(q) := Z^{\text{PT}, \beta}(Y, q) = \sum_n P_{n, \beta} q^n. \]

The moduli space \( P_n(Y, \beta) \) is empty for sufficiently negative \( n \), so \( Z^{\text{PT}, \beta}(q) \) is a Laurent series in \( q \).

The DT/PT-correspondence is the following conjecture.

**Conjecture 1.1.** (41 Conjecture 3.3)

\[ Z^{\text{DT}, \beta}(q) = Z^{\text{PT}, \beta}(q). \]

From the definition of \( Z^{\prime \text{DT}, \beta}(q) \), this conjecture is

\[ Z^{\text{PT}, \beta}(q) \cdot Z^{\text{DT}, 0}(q) = Z^{\text{DT}, \beta}(q) \]

which can be written down as

\[ \sum_m P_{n-m, \beta} \cdot I_{m, 0} = I_{n, \beta}. \]

From [41], the equation (1.1.6) can be interpreted as a wall-crossing formula for counting invariants in the bounded derived category \( D^b(X) \) of coherent sheaves.

The DT/PT-correspondence in Conjecture 1.1 was proved by Bridgeland in [9] and Toda in [44]. Toda used the method of Joyce’s wall crossing formula in the space of Bridgeland stability conditions on the derived category of \( Y \). In this paper we follow the method of Bridgeland by proving some Hall algebra identities and then applying the integration map to get the DT/PT-correspondence. The orbifold version of the DT/PT-correspondence was proved by A. Bayer in [3], but the paper is not available yet.

Similar method of Bridgeland on the Hall algebra identities works for threefold flops. J. Calabrese in [16] generalized Bridgeland’s Hall algebra identities to the threefold flop case to give the flop formula for the Donaldson-Thomas type invariants. Note that in [21] we generalize the result of Calabrese to the case of flops of threefold DM stacks.

The main goal of this paper is to generalize the above DT/PT-correspondence formula and the flop formula of Donaldson-Thomas invariants to the motivic version by using the motivic integration map from the motivic Hall algebra \( H(A') \) of the abelian category \( A' \) of coherent sheaves on \( Y \) to the motivic quantum torus of \( Y \) proved in [20, Theorem 4.16].
1.2. The motivic DT/PT-correspondence.

. (1.2.1) For the DT moduli space \( I_n(Y, \beta) \) and PT moduli space \( P_n(Y, \beta) \), they both admit symmetric obstruction theories in the sense of Behrend in [4]. We let \( X := I_n(Y, \beta) \) or \( P_n(Y, \beta) \). In [4] Behrend proves that the invariant \( I_n, \beta \) or \( P_n, \beta \) is given by
\[
\int_{[X]_{\text{virt}}} 1 = \chi(X, \nu_X),
\]
where \( \chi(X, \nu_X) \) is the weighted Euler characteristic weighted by the Behrend function \( \nu_X \) of \( X \). The Behrend function is an integer value constructible function on \( X \) defined by MacPherson’s local Euler obstructions of prime cycles on \( X \), see a survey introduction on the Behrend function in [25]. If \( Y \) is a threefold DM stack, then the moduli scheme \( I_n(Y, \beta) \) and \( P_n(Y, \beta) \) are all DM stacks. There are also symmetric obstruction theories on such DM stacks. If they are proper, the formula
\[
\int_{[X]_{\text{virt}}} 1 = \chi(X, \nu_X)
\]
is conjectured by Behrend in [4] and is proved in [22].

. (1.2.2) An interesting question proposed by K. Behrend is whether there exists a global defined perverse sheaf \( \mathcal{F} \) on the moduli scheme \( X \) such that
\[
\chi(X, \mathcal{F}) = \chi(X, \nu_X).
\]
Such an idea is true if the moduli scheme \( X \) is the critical locus of a global regular function (or holomorphic) function \( f : M \to \kappa \) on a higher smooth scheme \( M \). Then the value of the Behrend function \( \nu_X \) is given by
\[
\nu_X(P) = (-1)^{\dim(M)}(1 - \chi(\mathcal{F}_P)),
\]
where \( \mathcal{F}_P \) is the Milnor fiber of the function \( f \) at \( P \in X \). The sheaf \( \mathcal{F} \) is the perverse sheaf \( \varphi_f[-1] \) of vanishing cycles of \( f \) and it is known that
\[
\chi(X, \varphi_f[-\dim(M)]|_P) = \nu_X(P).
\]
Thus it is interesting to lift the Donaldson-Thomas invariants to the motivic level of cycles.

. (1.2.3) Let \( \mathcal{M}_k = K(\text{Var}_k)[L^{-1}] \) be the motivic ring, which is reviewed in [20] §2.1, where \( K(\text{Var}_k) \) is the Grothendieck ring of varieties. Similarly, let \( \hat{\mu} = \varprojlim \mu_n \) and let \( \mathcal{M}_k^\hat{\mu} = K^\hat{\mu}(\text{Var}_k)[L^{-1}] \) be the equivariant motivic ring, where \( K^\hat{\mu}(\text{Var}_k) \) is the equivariant Grothendieck ring of varieties.

. (1.2.4) The motivic Donaldson-Thomas theory on any Calabi-Yau three category was developed by Kontsevich-Soibelman in [30]. We follow the proposal of Joyce-Song in [26] to study the motivic Donaldson-Thomas invariants. We use the numerical K-group class \( \alpha \in K(Y) \) to index the topological data of the moduli space. Let
\[
\text{Ch} : K(Y) \to A_*(Y)
\]
be the Chern character morphism. For \( \alpha = (1, 0, -\beta, -n) \) let \( \text{DT}(\alpha) := I_n(Y, \beta) \) be the Donaldson-Thomas moduli space of ideal sheaves with topological data \( \alpha \).
Similarly let \( \PT(\alpha) := P_\alpha(Y, \beta) \) be the Pandharipande-Thomas moduli space of stable pairs with topological data \( \alpha \).

. (1.2.5) In [27], Joyce proves that both DT(\alpha) and PT(\alpha) are \( d \)-critical schemes. Let us review the definition here. Let \( X \) be a scheme, from [27] Theorem 2.1, there exists a unique coherent sheaf \( S_X \) such that it satisfies the properties in Theorem 2.1 of [27], and the is a natural decomposition \( S_X = \kappa_X \oplus S^0_X \) where \( \kappa_X \) is the constant sheaf.

An algebraic \( d \)-critical scheme over the field \( \kappa \) is a pair \((X, s)\), where \( X \) is a \( \kappa \)-scheme, locally of finite type, and \( s \in H^0(S^0_X) \). These data satisfy the following conditions: for any \( x \in X \), there exists a Zariski open neighbourhood \( R \) of \( x \) in \( X \), a smooth \( \kappa \)-scheme \( U \), a regular function \( f : U \to \mathbb{A}^1_{\kappa} \) and a closed embedding \( i : R \to U \), such that \( i(R) = \text{Crit}(f) \) as a \( \kappa \)-subscheme of \( U \), and \( i_{R, U}(s|_R) = i^{-1}(f) + i^2_{R, U} \). Here

\[
i_{R, U} : S_X|_R \to \frac{i^{-1}O_U}{i^2_{R, U}}
\]

is the morphism in [27] Theorem 2.1] fitting into the exact sequence

\[
0 \to S_X|_R \xrightarrow{i_{R, U}} \frac{i^{-1}O_U}{i^2_{R, U}} \xrightarrow{d} \frac{i^{-1}(T^1U)}{i_{R, U} \cdot i^{-1}(T^1U)}
\]

of sheaves of \( \kappa \)-vector spaces over \( R \), where \( d \) maps

\[
f + i^2_{R, U} \mapsto df + i_{R, U} \cdot i^{-1}(T^1U),
\]

and \( i_{R, U} \) is a morphism of sheaves of commutative \( \kappa \)-algebras. We call the quadruple \((R, U, f, i)\) a \( d \)-critical chart on \((X, s)\).

Let \((X, s)\) be an algebraic \( d \)-critical scheme, and \( X^{\text{red}} \subset X \) the associated reduced \( \kappa \)-scheme. Then from [27] §2.5] there exists a line bundle \( K_{X,s} \) on \( X^{\text{red}} \) which we call the canonical line bundle of \((X, s)\), that is natural up to canonical isomorphism, such that if \((R, U, f, i)\) is a critical chart on \((X, s)\), there is a natural isomorphism

\[
\iota_{R, U,f,i} : (K_{X,s})|_{R^{\text{red}}} \to i^*(K^{\otimes 2}_{U})|_{R^{\text{red}}}
\]

where \( K_U \) is the canonical line bundle of \( U \).

**Definition 1.2.** Let \((X, s)\) be an algebraic \( d \)-critical scheme, and \( K_{X,s} \) the canonical line bundle of \((X, s)\). An orientation on \((X, s)\) is a choice of square root line bundle \( K_{X,s}^{1/2} \) for \( K_{X,s} \) on \( X^{\text{red}} \). I.e., an orientation of \((X, s)\) is a line bundle \( L \) over \( X^{\text{red}} \) and an isomorphism \( L^{\otimes 2} = L \otimes L \cong K_{X,s} \). A \( d \)-critical scheme with an orientation will be called an oriented \( d \)-critical scheme.

Bussi, Brav and Joyce [12] prove the following interesting result: Let \((X, \omega)\) be a \((-1)\)-shifted symplectic derived scheme over \( \kappa \) in the sense of [42], and let \( X := t_0(X) \) be the associated classical \( \kappa \)-scheme of \( X \). Then \( X \) naturally extends to an algebraic \( d \)-critical scheme \((X, s)\). The canonical line bundle \( K_{X,s} \cong \det(L_X)|_{X^{\text{red}}} \) is the determinant line bundle of the cotangent complex \( L_X \) of \( X \).
\((1.2.6)\) Let \((X, s)\) be an oriented \(d\)-critical scheme and \((R, U, f, i)\) a \(d\)-critical chart. Let \(\text{Crit}(f)\) be the critical locus of \(f\), then \(R \cong \text{Crit}(f)\). Then in \([14]\), the authors associated with this local chart a perverse sheaf of vanishing cycle
\[
S^{\phi}_{U, f} \in \mathcal{M}^\delta_X
\]
such that
\[
S^{\phi}_{U, f}|_{R_c} = L^{-\dim(U)/2} \circ ([U_c, i] - S_{U, f|_{R_c}})|_{R_c},
\]
where \(f : R \to \mathbb{A}^1\) is the function \(f\) restricted to \(R\), and \(R = \bigcup_{c \in f(R)} R_c\) and \(R_c = R \cap U_c\) with \(U_c = f^{-1}(c) \subset U\). We call \(S^{\phi}_{U, f}\) the motivic vanishing cycle of \(f\).

\(\quad\) \((1.2.7)\)

\(\quad\) \((1.2.8)\) As in \([14]\), a principal \(\mathbb{Z}_2\)-bundle \(P \to X\) is a proper, surjective, étale morphism of \(\kappa\)-schemes \(\pi : P \to X\) together with a free involution \(\sigma : P \to P\) such that the orbits of \(\mathbb{Z}_2\) are the fibers of \(\pi\).

Let \(\mathbb{Z}_2(X)\) be the abelian group of isomorphism classes \([P]\) of principal \(\mathbb{Z}_2\)-bundles \(P \to X\), with multiplication \([P] \cdot [Q] = [P \otimes \mathbb{Z}_2 Q]\) and the identity the trivial bundle \([X \times \mathbb{Z}_2]\). We know that \(P \otimes \mathbb{Z}_2 P \cong X \times \mathbb{Z}_2\), so every element in \(\mathbb{Z}_2(X)\) has order 1 or 2.

In \([14]\), the authors define the motive of a principal \(\mathbb{Z}_2\)-bundle \(P \to X\) by:
\[
Y(P) = L^{-\frac{1}{2}} \circ ([X, i] - [P, \hat{\rho}]) \in \mathcal{M}^\delta_X,
\]
where \(\hat{\rho}\) is the \(\mu_2\)-action on \(P\).

In \([14]\), for any scheme \(Y\), the authors define an ideal \(l_Y^\delta\) in \(\mathcal{M}^\delta_Y\) which is generated by
\[
\phi_+(Y(P \otimes \mathbb{Z}_2 Q) - Y(P) \otimes Y(Q))
\]
for all morphisms \(\phi : X \to Y\) and principal \(\mathbb{Z}_2\)-bundles \(P, Q\) over \(X\). Then define
\[
\overline{\mathcal{M}}_Y^\delta = \mathcal{M}^\delta_Y / l_Y^\delta.
\]
Then \(\overline{\mathcal{M}}_Y^\delta \circ\) is a commutative ring with \(\circ\) and there is a natural projection \(\prod_Y^\delta : \mathcal{M}^\delta_Y \to \overline{\mathcal{M}}_Y^\delta\).

\(\quad\) \((1.2.9)\) Let \((X, s)\) be an oriented \(d\)-critical scheme. Let \(Q_{R, U, f, i} \to R\) be the principal \(\mathbb{Z}_2\)-bundle parameterizing local isomorphisms
\[
\alpha : K_{X,s}^{1/2}|_{\text{red}} \to i^*(K_U)|_{\text{red}}
\]
with \(\alpha \otimes \alpha = i_{R,U,f,i} \cdot \tau\), where
\[
i_{R,U,f,i} : K_{X,s}|_{\text{red}} \to i^*(K_U^\otimes 2)|_{\text{red}}
\]
is the isomorphism in the definition of the canonical line bundle \(K_{X,s}\).

From \([14]\) Theorem 5.10, if \((X, s)\) is a finite type algebraic \(d\)-critical scheme with a choice of orientation \(K_{X,s}^{1/2}\), then there exists a unique motive
\[
S^{\phi}_{X,s} \in \overline{\mathcal{M}}_X^\delta
\]
such that if \((R, U, f, i)\) is a critical chart on \((X, s)\), then
\[
S^{\phi}_{X,s}|_R = i^*(S^{\phi}_{U, f}) \circ Y(Q_{R, U, f, i}) \in \overline{\mathcal{M}}_R^\delta
\]
where
\[ Y(Q,R,f,i) = \mathcal{L}^{-1/2} \odot (\mathcal{I} - [Q,R]) \in \mathcal{M}_{\hat{R}}^\phi \]
is the motive of the principal \( \mathbb{Z}_2 \)-bundle defined in §2.5 of [14].

(1.2.10) Thus for the DT and PT moduli schemes \( DT(\alpha) \) and \( PT(\alpha) \), assume that they admit orientations, i.e., there exists square roots \( K_{DT(\alpha)}^{\frac{1}{2}} \) and \( K_{PT(\alpha)}^{\frac{1}{2}} \) of the canonical line bundles \( K_{DT(\alpha)} \) and \( K_{PT(\alpha)} \), we have constructed the global motives \( S_{\phi DT}(\alpha) \) and \( S_{\phi PT}(\alpha) \) inside \( \hat{\mathcal{M}}^\phi \), which categorify the corresponding global DT and PT invariants. The motivic DT/PT-correspondence can be stated as follows: The motivic DT partition function is defined as
\[ S_{\phi DT}(q) = \sum_\alpha S_{\phi DT}(\alpha) q^\alpha. \]
The reduced partition function is given by
\[ S_{\phi DT}'(q) = \frac{S_{\phi DT}(q)}{S_{\phi DT,0}(q)}, \]
where \( S_{\phi DT,0}(q) \) is the degree zero motivic series and is calculated in [7] to be the motivic MacMahon function.

Let \( S_{\phi PT}(q) \) be the motivic Pandharipande-Thomas stable pair partition function defined as
\[ S_{\phi PT}(q) = \sum_\gamma S_{\phi PT}(\gamma) q^\gamma. \]

The motivic DT/PT-correspondence is the following:

**Theorem 1.3.**
\[ S_{\phi DT}'(q) = S_{\phi PT}(q). \]

The motivic DT/PT-correspondence for the conifold has been proved by Andrew Morrison, Sergei Mozgovoy, Kentaro Nagao and Balazs Szendroi by explicit calculations in [39]. The ADE singularity case was calculated by Sergei Mozgovoy in [40].

1.3. The motivic Flop formula. Recall that an orbifold flop (in the sense of [21]) between two smooth threefold DM stacks is given by the following diagram:

\[ \begin{array}{c}
\text{Z} \\
\uparrow f \\
\downarrow f' \\
\text{Y} \\
\uparrow \phi \\
\downarrow \psi \\
\text{S} \\
\psi' \\
\end{array} \]

where
1. \( Y \) and \( Y' \) are smooth Calabi-Yau threefold DM stacks;
2. \( S \) is a singular variety with only zero-dimensional singularities;
3. Both \( \psi \) and \( \psi' \) contract cyclic quotients of weighted projective lines \( \mathbb{P}(a_1, a_2), \mathbb{P}(b_1, b_2) \) respectively;
(4) $Z$ is the common weighted blow-up along the exceptional locus.

From Abramovich-Chen in [1] and [21], the derived categories of $Y$ and $Y'$ are equivalent for such orbifold flops using the idea of perverse point sheaves of Bridgeland. The equivalence

\begin{equation}
\Phi : D^b(Y) \to D^b(Y')
\end{equation}

is given by the Fourier-Mukai transformation $\Phi = FM$, where

\[FM(-) = f'_*(f^*(-)).\]

Moreover, the equivalence $\Phi$ also sends the abelian category of perverse sheaves to the abelian category of perverse sheaves.

\begin{equation}
\Phi(q \text{ Per}(Y)) = p \text{ Per}(Y'),
\end{equation}

where $q = -(p + 1)$.

Let $p_{\alpha} := p_{\alpha}(Y) := p \text{ Per}(Y)$. We work on the Hall algebra $H(p_{\alpha})$ of $p_{\alpha}$.

Let $K(Y)$ be the numerical K-group of $Y$, and $F_0 K(Y) \subset F_1 K(Y) \subset \cdots \subset K(Y)$ be the filtration in terms of the support of dimensions.

Let $\alpha$ be a $K$-group class in $F_1 K(Y)$, and let $\text{Hilb}^a(Y)$ be the Hilbert scheme of substacks of $Y$ with class $\alpha$. The DT-invariant is defined by

\[\text{DT}_{\alpha}(Y) = \chi(\text{Hilb}^a(Y), v_H),\]

where $v_H$ is the Behrend function in [4] of $\text{Hilb}^a(Y)$. Define the motivic DT-partition function by

\begin{equation}
S^\phi_{\text{DT}}(Y) = \sum_{\alpha \in F_1 K(Y)} S^\phi_{\text{DT}(\alpha)}(Y) q^a.
\end{equation}

The motivic PT-partition function is given by

\begin{equation}
S^\phi_{\text{PT}}(X) = \sum_{\beta \in F_1 K(Y)} S^\phi_{\text{PT}(\beta)}(X) q^\beta.
\end{equation}

We define the following motivic DT-type partition functions:

\begin{align*}
S^\phi_{\text{DT}0}(Y) &= \sum_{\alpha \in F_0 K(Y)} S^\phi_{\text{DT}(\alpha)}(Y) q^a; \\
S^\phi_{\text{DT}a}(Y) &= \sum_{\alpha \in F_1 K(Y) / F_0} S^\phi_{\text{DT}(\alpha)}(Y) q^a; \\
S^\phi_{\text{DT}\text{exc}}(Y) &= \sum_{\alpha \in F_1 K(Y) / F_0; \psi_\alpha = 0} S^\phi_{\text{DT}(\alpha)}(Y) q^{-a}; \\
S^\phi_{\text{DT}\text{exc}}(Y') &= \sum_{\alpha \in F_1 K(Y') / F_0; \psi_\alpha = 0} S^\phi_{\text{DT}(\alpha)}(Y') q^{-a};
\end{align*}

Then we have:

**Theorem 1.4.** Let $\varphi : Y \dashrightarrow Y'$ be an orbifold flop between two smooth Calabi-Yau threefold DM stacks. Then

\[\Phi_* \left( \frac{S^\phi_{\text{DT}}(Y)}{S^\phi_{\text{DT}0}(Y)} \cdot \frac{S^\phi_{\text{DT}\text{exc}}(Y)}{S^\phi_{\text{DT}0}(Y)} \right) = S^\phi_{\text{DT}}(Y') \cdot \frac{S^\phi_{\text{DT}\text{exc}}(Y')}{S^\phi_{\text{DT}0}(Y')},\]

where $\Phi_*$ is understood as sending the data $\alpha \in K(Y)$ to $\varphi(\alpha) \in K(Y')$. 
1.4. The higher rank stable pair and DT/PT-correspondence. Recently in [46], Toda studied the higher rank stable pairs and the higher rank Donaldson-Thomas invariants. The DT/PT-correspondence in higher rank case is proved by Toda in [46]. For completeness, we provide here a motivic version of Toda’s formula by using the similar method of him and Bridgeland in [2] taking into account the motivic integration map in higher rank case.

1.5. The outline for the proof. We follow the same method of Hall algebra identities of Bridgeland in [9] and [16] to prove Theorem 1.3 and Theorem 1.4. In [20, Theorem 4.16] we prove that the integration map from the motivic Hall algebra \( H(\mathcal{A}) \) of the abelian category of coherent sheaves on \( Y \) to the motivic quantum torus is a Poisson algebra homomorphism. Thus applying the integration map for the global motives of the DT and PT moduli schemes constructed in [14], we get the motivic version of the DT/PT-correspondence and the motivic flop formula in Theorem 1.3 and Theorem 1.4. The higher rank case invariants of Toda is proved in a similar way.

1.6. Brief outline. In §2 we prove the motivic DT/PT-correspondence, and finishing the proof of Theorem 1.3. We prove Theorem 1.4 in §3. Finally in §4 we talk about the higher rank case of the DT/PT-correspondence.

Convention. We work over an algebraically closed field \( \kappa \) throughout the paper. We use \( L \) to represent the Lefschetz motive \( [\mathbb{A}_1^1] \). We use \( Y \) to represent a quasi-projective threefold DM stack, and \( X \) the DT or PT moduli space of stable objects in the derived category.

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2. The proof of Theorem 1.3

2.1. Motivic Hall algebras.

(2.1.1) In this section we review the definition and construction of motivic Hall algebra of Joyce and Bridgeland in [28], [10]. We define the integration map from motivic Hall algebra to the motivic quantum torus.

(2.1.2) We define the Grothendieck ring of stacks of finite type.

Definition 2.1. The Grothendieck ring of stacks \( K(\text{St} / \kappa) \) is defined to be the \( \kappa \)-vector space spanned by isomorphism classes of Artin stacks of finite type over \( \kappa \) with affine stabilizers, modulo the relations:

1. For every pair of stacks \( X_1 \) and \( X_2 \) a relation:
   \[ [X_1 \sqcup X_2] = [X_1] + [X_2]; \]

2. For any geometric bijection \( f : X_1 \to X_2 \), \( [X_1] = [X_2]; \)

3. For any Zariski fibrations \( p_i : X_i \to Y \) with the same fibers, \( [X_1] = [X_2]. \)
Let \([A_1] = \mathbb{L}\), the Lefschetz motive. If \(S\) is a stack of finite type over \(\kappa\), we define the relative Grothendieck ring of stacks \(K(\text{St} / S)\) as follows:

**Definition 2.2.** The relative Grothendieck ring of stacks \(K(\text{St} / S)\) is defined to be the \(\kappa\)-vector space spanned by isomorphism classes of morphisms

\([X \xrightarrow{f} S]\),

with \(X\) an Artin stack over \(S\) of finite type with affine stabilizers, modulo the following relations:

1. for every pair of stacks \(X_1\) and \(X_2\) a relation:

\([X_1 \sqcup X_2 \xrightarrow{f_1 \cup f_2} S] = [X_1 \xrightarrow{f_1} S] + [X_2 \xrightarrow{f_2} S];\)

2. for any diagram:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{g} & X_2 \\
\downarrow{f_1} & & \downarrow{f_2} \\
\downarrow{S} & & \downarrow{S}
\end{array}
\]

where \(g\) is a geometric bijection, then \([X_1 \xrightarrow{f_1} S] = [X_2 \xrightarrow{f_2} S];\)

3. for any pair of Zariski fibrations

\(X_1 \xrightarrow{h_1} \mathcal{Y}; \quad X_2 \xrightarrow{h_2} \mathcal{Y}\)

with the same fibers, and \(g : \mathcal{Y} \to S\), a relation

\([X_1 \xrightarrow{\text{gh}_1} S] = [X_2 \xrightarrow{\text{gh}_2} S].\)

(2.1.3) The motivic Hall algebra in \([10]\) is defined as follows. Let \(\mathcal{M}\) be the moduli stack of coherent sheaves on \(Y\). It is an algebraic stack, locally of finite type over \(\kappa\). The motivic Hall algebra is the vector space

\(H(\mathcal{A}) = K(\text{St} / \mathcal{M})\)

equipped with a non-commutative product given by the role:

\([X_1 \xrightarrow{f_1} \mathcal{M}] \ast [X_2 \xrightarrow{f_2} \mathcal{M}] = [Z \xrightarrow{\text{gh}} \mathcal{M}],\)

where \(h\) is defined by the following Cartesian square:

\[
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{h} & \mathcal{M} \\
\downarrow{(a_1, a_2)} & & \downarrow{b} \\
X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & \mathcal{M} \times \mathcal{M},
\end{array}
\]

with \(\mathcal{M}^{(2)}\) the stack of short exact sequences in \(\mathcal{A}\), and the maps \(a_1, a_2, b\) send a short exact sequence

\[0 \to A_1 \to B \to A_2 \to 0\]

to sheaves \(A_1, A_2,\) and \(B\) respectively. Then \(H(\mathcal{A})\) is an algebra over \(K(\text{St} / \kappa)\).

2.2. The integration map.
In this section we define the integration map from the motivic Hall algebra to the motivic quantum torus.

Recall that in §3 of [10], there exists maps of commutative rings:
\[ K(\text{Sch} / \kappa) \to K(\text{Sch} / \kappa)[L^{-1}] \to K(\text{St} / \kappa), \]
where \( K(\text{Sch} / \kappa) \) is the Grothendieck ring of schemes of finite type over \( \kappa \). Since \( H(A) \) is an algebra over \( K(\text{St} / \kappa) \), define a \( K(\text{Sch} / \kappa)[L^{-1}] \)-module
\[ H_{\text{reg}}(A) \subset H(A) \]
to be the span of classes of maps \([X \xrightarrow{f} M] \) with \( X \) a scheme. An element of \( H(A) \) is regular if it lies in \( H_{\text{reg}}(A) \). Then from Theorem 5.1 of [10], the Hall algebra product preserves the regular elements in \( H_{\text{reg}}(A) \).

For our purpose, we define a \( K(\text{Sch} / \kappa)[L^{-1}] \)-module
\[ H_{d-\text{Crit}}(A) \subset H(A) \]
to be the span of classes of maps \([X \xrightarrow{f} M] \) with \( (X,s) \) an algebraic \( d \)-critical scheme in the sense of Joyce [27]. Since \( X \) is a scheme, the module
\[ H_{d-\text{Crit}}(A) \subset H_{\text{reg}}(A). \]
The following is a generalization of Theorem 5.1 of [10]:

**Theorem 2.3.** ([20, Theorem 4.12]) The sub-module of \( d \)-critical elements of \( H(A) \) is closed under the convolution product:
\[ H_{d-\text{Crit}}(A) \ast H_{d-\text{Crit}}(A) \subset H_{d-\text{Crit}}(A) \]
and is a \( K(\text{Sch} / \kappa)[L^{-1}] \)-algebra. Moreover, the quotient
\[ H_{\text{ssc},d-\text{Crit}}(A) = H_{d-\text{Crit}}(A) / (L-1)H_{d-\text{Crit}}(A) \]
is a commutative \( K(\text{Sch} / \kappa) \)-algebra.

The algebra \( H_{\text{ssc},d-\text{Crit}}(A) \) is called semi-classical Hall algebra for the elements of \( d \)-critical schemes. In [10], Bridgeland also defines a Poisson bracket on \( H(A) \) by:
\[ \{f, g\} = \frac{f \ast g - g \ast f}{L-1}. \]
This bracket preserves the subalgebra \( H_{d-\text{Crit}}(A) \).

We define the motivic quantum torus. Let \( K(Y) = K(A) \) be the Grothendieck group of the category \( A \). Let \( E, F \in k(A) \) and let
\[ \chi(E,F) = \sum_i (-1)^i \dim_\kappa \text{Ext}^i(E,F). \]
So \( \chi(\cdot, \cdot) \) is a bilinear form on \( K(A) \), which is called the Euler form. The numerical Grothendieck group is the quotient
\[ N(Y) = K(A) / K(A)^{\perp}, \]
where $K(Y)^\perp$ means the Euler form zero subgroup. Let $\Gamma \subset N(Y)$ denote the monoid of effective classes, which is to say the classes of the form $[E]$ with $E$ a sheaf.

**Remark 2.4.** The stack $\mathcal{M}$ split into disjoint union of open and closed substacks

$$\mathcal{M} = \bigsqcup_{a \in \Gamma} \mathcal{M}_a$$

where $\mathcal{M}_a \subset \mathcal{M}$ is the stack of objects of class $a \in \Gamma$. And $\mathcal{M}_a \subset \mathcal{M}$ implies that $K(\text{St} / \mathcal{M}_a) \subset K(\text{St} / \mathcal{M})$.

Also the Hall algebra

$$H(\mathcal{A}) = \bigoplus_{a \in \Gamma} H(\mathcal{A})_a$$

and $H(\mathcal{A})$ is a graded algebra with respect to the convolution product.

. (2.2.6) Let $\mathcal{M}_k^\beta$ be the equivariant ring of motives and consider

$$\mathcal{M}_k^\beta = \mathcal{M}_k^\beta[\mathbb{L}^{-1/2}, (\mathbb{L}^i - 1)^{-1}, i \in \mathbb{N}_{>0}].$$

Let $\overline{\mathcal{M}}_{k, \text{loc}}$ be the ring $\mathcal{M}_k^\beta / I_k^\beta$ with the product $\odot$ reviewed in (1.2.6).

**Definition 2.5.** Define

$$\overline{\mathcal{M}}_{k, \text{loc}}[\tilde{\Gamma}] = \bigoplus_{a \in \Gamma} \overline{\mathcal{M}}_{k, \text{loc}} \cdot x^a$$

to be the ring generated by symbols $x^a$ for $a \in \Gamma$, with product defined by:

$$x^a \star x^\beta = \frac{\mathbb{L}}{\mathbb{L} - 1} x^{(a, \beta)} \cdot x^{a + \beta}.$$

Even the Euler form is skew-symmetric, this ring is not commutative due to the factor of the Lefschetz motive.

For classes $a, \beta \in \Gamma$, we define

$$\text{Ext}^i_{E,F}(a, \beta) = \text{Ext}^i(E, F)$$

for $[E] = a, [F] = \beta$, where $E, F \in \mathcal{A}$.

Since different representatives $E', F'$ of $a, \beta$ may have different Extension groups, let $e^i := \dim(\text{Ext}^i(E, F))$ we calculate:

$$\sum_{i=0}^3 (-1)^{i+1} \frac{\text{dim Ext}^i_{E,F}(a, \beta)}{\mathbb{L} - 1}$$

$$= \frac{\mathbb{L}^{\chi(a, \beta)} - 1}{\mathbb{L} - 1} + \frac{\mathbb{L}^{\alpha(1 - \mathbb{L} - \mathbb{L}^2 - \mathbb{L}^3 - \mathbb{L}^4)}}{\mathbb{L}^{\chi(a, \beta)} - 1} + \cdot \text{Term}_{E,F}$$

for any $E, F \in \mathcal{A}$ such that $[E] = a, [F] = \beta$. Let

$$\overline{\mathcal{M}}_{k, \text{loc}}[\tilde{\Gamma}] = \overline{\mathcal{M}}_{k, \text{loc}}[\tilde{\Gamma}]_{\text{Term}_{E,F}}$$

be the localization ring on all such terms $\text{Term}_{E,F}$. 
Then we use $\text{Ext}^i(\alpha, \beta)$ to represent the extension group for any representatives. The Poisson bracket is given by:

$$\{x^\alpha, x^\beta\} = \frac{\sum_{i=0}^{3} (-1)^{i+1} \dim \text{Ext}^i(\alpha, \beta)}{\mathbb{L} - 1} \cdot x^\alpha + \beta$$

over $\overline{M}^0_{k, \text{loc}}[\Gamma]$.

**Remark 2.6.** In practice, later on we will always fix to the coherent sheaves supported at most dimension one. Then in this case if we have $E, F$ such that $E = F$, then $\dim \text{Ext}^i(\alpha, \beta) = \dim \text{Ext}^3 - i(\alpha, \beta)$ by Serre duality. So we don't need any modified terms, since the Euler form is zero and also $\sum_{i=0}^{3} (-1)^{i+1} \cdot \mathbb{L}^\dim \text{Ext}^i(\alpha, \beta) = 0$.

If we have two classes $\alpha \neq \beta$ and coherent sheaves $[E] = \alpha, [F] = \beta$ such that they all support on dimension one, then we have $E \neq F$, and

$$\text{Ext}^2(E, F) = \text{Ext}^3(E, F) = 0$$

and the extra factor is just $\mathbb{L}^\dim \text{Ext}^1(E, F)$.

We define the integration map. Let

$$I : H_{\text{ssc}, d-\text{Crit}}(\mathcal{A}) \to \overline{M}^0_{k, \text{loc}}[\Gamma]$$

be the map defined by: for any element $[Z \to \mathcal{M}] \in H_{\text{ssc}, d-\text{Crit}}(\mathcal{A})$, let

$$t : Z \to Z$$

be the map from the algebraic $d$-critical scheme $Z$ to the corresponding $d$-critical Artin stack $\mathcal{Z}$, which is a smooth morphism, see for instance [13, Theorem 5.14]. Then

$$I([Z \to \mathcal{M}]) = \left( \int t^* S_Z^\phi \right) \cdot x^\alpha \in \overline{M}^0_{k, \text{loc}}[\Gamma]$$

where $\int : \overline{M}^0_{Z, \text{loc}} \to \overline{M}^0_{k, \text{loc}}$ is the pushforward of motives.

**Remark 2.7.** From [13, Theorem 5.14], for any $d$-critical scheme $Z$, it is enough to choose a $d$-critical Artin stack $Z$ such that $Z \to Z$ is a smooth morphism.

**Remark 2.8.** Let $v_Z$ be the Behrend function on $Z$ which is the pullback $i^* v_M$ from $i : Z \to \mathcal{M}$. Then taking cohomology of the perverse sheaf $t^* S_Z^\phi$ we get the weighted Euler characteristic $\chi(Z, t^* v_Z)$. This is the map $I$ in [10, Theorem 5.2].

**Theorem 2.9.** ([20, Theorem 4.16]) The map $I$ in (2.2.7) is a Poisson algebra homomorphism.

**Remark 2.10.** Theorem 2.9 generalizes the result of Bridgeland in [10, Theorem 5.2] to the motivic level.

**Remark 2.11.** The proof of Theorem 2.9 relies on the motivic Behrend function identities in [20]. The Euler characteristic level of these identities was originally proved for coherent sheaves by Joyce-Song [26]. These identities were recently proved by V. Bussi [15] using algebraic method and also works in characteristic $p$. In [19] we study these formulas using non-archimedean spaces.
2.3. Some Hall algebra identities from Bridgeland.

. (2.3.1) (Stable pairs) For the Calabi-Yau threefold stack $Y$, generalizing the definition of Pandharipande-Thomas [41], a stable pair $[\mathcal{O}_Y \xrightarrow{s} F]$ is an object in $D^b(Y)$, such that

1. $\dim \text{Supp}(F) \leq 1$ and $F$ is pure;
2. $\text{Coker}(s)$ is zero dimensional.

The stable pairs lies in the heart of a $t$-structure constructed in [9]. As in [9], let

$$\mathcal{P} := \text{Coh}_0(Y) \subset \mathcal{A} := \text{Coh}(Y)$$

be the sub-category consisting of sheaves supported on dimension zero. Let

$$Q = \{ E \in \mathcal{A} | \text{Hom}(P, E) = 0 \text{ for } P \in \mathcal{P} \}.$$ 

Then $(\mathcal{P}, Q)$ is a torsion pair:

1. if $P \in \mathcal{P}$ and $Q \in Q$, then $\text{Hom}_\mathcal{A}(P, Q) = 0$;
2. Every $E \in \mathcal{A}$ fits into a short exact sequence

$$0 \to P \to E \to Q \to 0$$

with $P \in \mathcal{P}$ and $Q \in Q$.

A new $t$-structure on $D^b(Y) = D(\mathcal{A})$ is defined by tilting the standard $t$-structure, see §2.2 of [9]. The heart $\mathcal{A}^\#$ of this new $t$-structure is given by:

$$\mathcal{A}^\# = \{ E \in D(\mathcal{A}) | H_0(E) \in Q, H_1(E) \in \mathcal{P}, H_i(E) = 0 \text{ for } i \notin \{0, 1\} \}.$$ 

We have $Q = \mathcal{A} \cap \mathcal{A}^\#$ and $\mathcal{O}_Y \in \mathcal{A}^\#$. Bridgeland [9] proves the following result:

**Proposition 2.12.** A stable pair $[\mathcal{O}_Y \xrightarrow{s} F]$ is an epimorphism $\mathcal{O}_Y \to F$ in $\mathcal{A}^\#$ with $\dim \text{Supp}(F) \leq 1$ and $F \in Q$.

Fixing $[\mathcal{O}_Y] = \beta \in K(Y)$, let $\text{PT}^\beta(Y)$ be the moduli stack of stable pairs, parameterizing the objects $[\mathcal{O}_Y \xrightarrow{s} F]$ satisfying the conditions in the definition. From [3], it is represented by a scheme $\text{PT}^\beta(Y)$.

. (2.3.2) Recall that the infinite-type Hall algebra $H_\infty(\mathcal{A})$ is defined as $L(\text{St}_\infty / \mathcal{M})$ by only assuming the stacks $\mathcal{X}$ locally of finite type.

As in [9], the substack $\mathcal{P} \subset \mathcal{M}, Q \subset \mathcal{M}$ give rise to the elements

$$1_P, 1_Q \in H_\infty(\mathcal{A}).$$ 

Also we have:

$$\mathcal{H} = [\text{Hilb}_Y \xrightarrow{\eta} \mathcal{M}] \in H_\infty(\mathcal{A}),$$

which parametrizes the DT moduli spaces.

. (2.3.3) From [9], we fix notations for some particular elements in $H_\infty(\mathcal{A})$. Let $\mathcal{N} \subset \mathcal{M}$ be an open substack, then we write

$$1_{\mathcal{N}} = [\mathcal{N} \to \mathcal{M}] \in H_\infty(\mathcal{A})$$

where $i : \mathcal{N} \to \mathcal{M}$ is the inclusion map.
(2.3.4) Recall the stack \( \mathcal{M}(O) \) of framed sheaves in [9, §2.3], which parameterizes coherent sheaves equipped with a section. An object of \( \mathcal{M}(O) \) lying over a scheme or DM stack \( S \) is a pair \((E, \gamma)\) such that \( \gamma : \mathcal{O}_{S \times Y} \to E \) is a section of an \( S \)-flat coherent sheaf \( E \). There is a morphism

\[
q : \mathcal{M}(O) \to \mathcal{M}
\]
defined by forgetting about the section.

We have

\[
\mathcal{H}^\# = [\text{Hilb}_Y \overset{q}{\to} \mathcal{M}(O) \to \mathcal{M}] \in H_\infty(\mathcal{A})
\]
parametrizes the PT moduli spaces.

(2.3.5) (Laurent subsets) As in [9] and [16], we need to introduce Laurent elements in the numerical Grothendieck group \( K(Y) \). The reason to do this is that the infinite-type Hall algebra \( H_\infty(\mathcal{A}_{\leq 1}) \) is too big to support an integration map and we have to work on spaces of locally finite type.

Definition 2.13. A subset \( S \subset \Gamma \) is called Laurent if for each \( \beta \in N_1(Y) \) the set of integers \( n \) for which \((\beta, n) \in S\) is bounded below.

Let \( \Lambda \) denote the set of Laurent subsets of \( \Gamma \). Then the system of subsets satisfies the following properties:

1. if \( S, T \in \Lambda \) then so is \( S + T = \{\alpha + \beta : \alpha \in S, \beta \in T\} \).
2. if \( S, T \in \Lambda \) and \( \alpha \in \Gamma \) there are only finitely many decompositions \( \alpha = \beta + \gamma \) with \( \beta \in S \) and \( \gamma \in T \).

As in [9, §5.2], for any \( \Gamma \)-graded associative algebra \( R \), the \( \Lambda \)-completion \( R_\Lambda \) is defined to be the vector space of formal series:

\[
\sum_{(\gamma, \delta, n)} x_{(\gamma, \delta, n)}
\]
with \( x_{(\gamma, \delta, n)} \in R_{x_{(\gamma, \delta, n)}} \) and \( x_{(\gamma, \delta, n)} = 0 \) outside a Laurent subset. The product is defined by:

\[
x \cdot y = \sum_{a \in \mathbb{R} \Delta} \sum_{a_1 + a_2 = a} x_{a_1} \cdot y_{a_2}.
\]

Then the integration map \( I : H_{\text{ssc}}(\mathcal{A}_{\leq 1}) \to \overline{\mathcal{M}}^\delta_k[\Gamma] \) induces a morphism on the completions:

\[
(2.3.6) \quad I_\Lambda : H_{\text{ssc}}(\mathcal{A}_{\leq 1})_\Lambda \to \overline{\mathcal{M}}^\delta_k[\Gamma]_\Lambda.
\]

Proposition 2.14. ([9 Proposition 6.5]) We have the following Hall algebra identity:

\[
\mathcal{H}_{\leq 1} \star \mathbb{I}_p = \mathcal{H}_0 \star \mathbb{I}_p \star \mathcal{H}^\#.
\]

Proof. The proof is the same as in [9 Proposition 6.5] by using the Hall algebra identities developed in section 4 of [9]. Note that in the preprint [3], Bayer proves this for smooth three dimensional Calabi-Yau orbifolds.

2.4. Finishing the proof.
(2.4.1) According to [9, §6.1], let $H$ be an ample divisor on $Y$. Let $\gamma \in \Gamma$, and define the slope

$$\mu(\gamma) = \frac{\text{Ch}_3(\gamma)}{\text{Ch}_2(\gamma) \cdot H} \in (-\infty, \infty].$$

A nonzero object $E \in \mathcal{A}_{\leq 1}$ is (Gieseker or Simpson) semistable if $\mu(A) \leq \mu(E)$ for any nonzero subject $A \subset E$.

Let $SS(\gamma) \subset \mathcal{A}_{\leq 1}$ be the open stack whose $\kappa$-valued points are semistable sheaves of class $\gamma$. Note that if $\text{Ch}_2(\gamma) = 0$, then $\gamma$ has slope $+\infty$. For a fixed $I \subset (-\infty, +\infty]$, define $SS(I) \subset \mathcal{A}_{\leq 1}$ to be the full subcategory consisting of zero objects together with those sheaves whose Harder-Narasimhan factors all have slope in $I$. Here any nonzero sheaf $E \in \mathcal{A}_{\leq 1}$ has a unique Harder-Narasimhan filtration:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_i = E_i / E_{i-1}$ are semistable with

$$\mu(F_1) > \mu(F_2) > \cdots > \mu(F_n).$$

We have

$$\mathcal{P} = SS(\infty), \quad \mathcal{Q} \cap \mathcal{A}_{\leq 1} = SS(-\infty, +\infty).$$

For any $\mu \in (-\infty, +\infty]$, we have the famous Joyce’s no pole theorem:

**Theorem 2.15.** (Joyce)

$$\mathbf{1}_{SS(\mu)} = \exp(\epsilon_\mu) \in H(\mathcal{A}_{\leq 1})_\Lambda,$$

where $\eta_\mu = [(\mathbb{A}^1_\kappa)^* \cdot \epsilon_\mu] \in H_{\text{reg}}(\mathcal{A}_{\leq 1})_\Lambda$ is a regular element.

Bridgeland [9] proves the following result:

**Corollary 2.16.** (Bridgeland) For any $\mu \in (-\infty, +\infty]$ the element $\mathbf{1}_{SS(\mu)}$ in $H(\mathcal{A}_{\leq 1})_\Lambda$ is invertible and the automorphism

$$\text{Ad}_{\mathbf{1}_{SS(\mu)}} : H(\mathcal{A}_{\leq 1})_\Lambda \to H(\mathcal{A}_{\leq 1})_\Lambda$$

preserves the subring of regular elements. The induced Poisson automorphism of $H_{d-crit,\text{ssc}}(\mathcal{A}_{\leq 1})_\Lambda$ is:

$$\text{Ad}_{\mathbf{1}_{SS(\mu)}} = \exp(\eta_\mu, -).$$

(2.4.2) From [9, §6.4], we have

$$\mathbf{1}_\mathcal{P} = \mathbf{1}_{SS(\infty)} = \exp(\epsilon_\infty)$$

with $\epsilon_\infty \in (\mathbb{L} - 1)^{-1}H_{\text{reg}}(\mathcal{A}_{\leq 1})_\Lambda$. Hence we get the following identity in $H_{\text{reg}}(\mathcal{A}_{\leq 1})_\Lambda$:

$$\mathcal{H}_{\leq 1} = \mathcal{H}_0 \cdot \exp\{\eta_\infty,-\}(\mathcal{H}^\#_{\leq 1}).$$

Now applying the motivic integration map in (2.3.6) and note from Remark 2.6 the Poisson brackets vanish and we have:

$$I_\Lambda(\mathcal{H}_{\leq 1}) = I_\Lambda(\mathcal{H}_0) \cdot I_\Lambda(\mathcal{H}^\#_{\leq 1}).$$

Note that

$$I_\Lambda(\mathcal{H}_{\leq 1}) = \mathcal{S}_{DT}(q)$$

$$I_\Lambda(\mathcal{H}_0) = \mathcal{S}_{DT,0}(q)$$
\[ I_{\Lambda}(H_{\geq 1}) = S_{PT}(\eta) \]

The theorem is proved. □

3. The proof of Theorem [14]

3.1. Some notations of perverse sheaves.

(3.1.1) Fix a smooth Calabi-Yau threefold stack \( Y \), denote by \( \mathcal{A} := \text{Coh}(Y) \) the abelian category of coherent sheaves over \( Y \). Let \( D^b(Y) := D(\mathcal{A}) = D^b(\text{Coh}(Y)) \) be the bounded derived category of coherent sheaves over \( Y \). The abelian category \( \text{Coh}(Y) \) is the heart of the standard \( t \)-structure of \( D^b(\text{Coh}(Y)) \).

Let \( \varphi : Y \rightarrow Y' \) be an orbifold flop of Calabi-Yau threefold stacks, i.e. there exists a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & Y' \\
\downarrow{\psi} & & \downarrow{\psi'} \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
p \in \mathcal{S} & & \\
\end{array}
\]

This orbifold flop satisfies the following properties:

1. \( \varphi \) and \( \varphi' \) are proper, birational and an isomorphism in codimension one;
2. \( S \) is projective and only has zero dimensional singular locus;
3. the dualising sheaf of \( Y \) is trivial, i.e. \( \omega_S = \mathcal{O}_S \);
4. \( R\psi_*\mathcal{O}_Y = \mathcal{O}_S ; R\psi'_*\mathcal{O}_{Y'} = \mathcal{O}_S ; \)
5. \( \dim_{\mathbb{Q}} N^1(Y/S)_Q = 1 \), so is \( \dim_{\mathbb{Q}} N^1(Y'/S)_Q \).

where \( N^1(Y/S)_Q = N^1(Y/S)_Z \otimes \mathbb{Q} \) and \( N^1(Y/S) \) is the group of divisors on \( Y \) modulo numerical equivalence over \( S \). Similar results hold for \( N^1(Y'/S) \).

(3.1.2) (Perverse \( t \)-structure on \( \mathcal{X} \)) Let \( \pi : Y \rightarrow \overline{Y} \) be the map to its coarse moduli space, so that we have the following diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & \overline{Y} \\
\downarrow{\pi} & & \downarrow{\overline{\psi}} \\
\rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow \\
S. & & \\
\end{array}
\]

As in [1] there are two sub-categories of \( D^b(Y) \):

\[
\begin{align*}
B &= \{ L\pi^*C \in D^b(Y)| C \in D^b(\overline{Y}) \}; \\
C_2 &= \{ C \in D^b(Y)| R\pi_*C = 0 \}.
\end{align*}
\]

The pair \((B, C_2)\) gives a semiorthogonal decomposition on \( D^b(Y) \). On the category \( C_2 \), there is a standard \( t \)-structure which is induced from the standard \( t \)-structure on \( D^b(Y) \).

Recall from [8], for the map \( \overline{\psi} : \overline{Y} \rightarrow S \), there is a perverse \( t \)-structure \( t(-1) \) and the heart of this \( t \)-structure is denoted by \( \text{Per}^{-1}(\overline{Y}/S) \).
Definition 3.1. The derived functor $R\pi_\ast$ has right adjoint $\pi_!$ and the left adjoint $L\pi^\ast$. Denote by $t(p,0)$ the t-structure obtained by gluing: the perverse t-structure $t(p)$ on $D^b(\overline{Y})$, and the standard t-structure on $C_2$. We denote by the heart of this t-structure by $\text{Per}^p(Y/S) := \text{Per}^p_!(Y/S)$. Usually we take $p = -1,0$ and we always denote by $\text{Per}(Y/S) := \text{Per}^{-1}(Y/S)$.

Recall that in [1], the perverse sheaf is classified as follows: An object $E$ in $D^b(Y)$ is a “perverse sheaf” i.e. $E \in \text{Per}(Y/S)$ if:

1. $R\pi_\ast E$ is a perverse sheaf for $\overline{\psi} : \overline{Y} \to S$ and $\pi : Y \to \overline{Y}$ is the map to its coarse moduli space;
2. $\text{Hom}(E,C) = 0$ for all $C$ in $C_2^{>0}$ and $\text{Hom}(D,E) = 0$ for all $D$ in $C_2^{<0}$.

Then Lemma 3.3.1 of [1] classifies all perverse coherent sheaves:

Lemma 3.2. An object $E \in D^b(Y)$ is a perverse sheaf if and only if the following conditions are satisfied:

1. $H_i(E) = 0$ unless $i = 0$ or $1$;
2. $R^1\psi_\ast H_0(E) = 0$ and $R^0\psi_\ast H_1(E) = 0$;
3. $\text{Hom}(\pi_\ast H_0(E), C) = 0$ for any sheaf $C$ on $Y$ satisfying $\overline{\psi}_\ast C = R^1\psi_\ast C = 0$;
4. $\text{Hom}(D, H_1(E)) = 0$ for any sheaf $D$ in $C_2$.

Recall that in [1], the perverse sheaves can be obtained by tilting a torsion pair. We say that an object $E \in D(\mathcal{A})$ connects to $C_2$, denoted by $E|C_2$ if $E$ satisfies the conditions: $\text{Hom}(E,C) = 0$ for all $C$ in $C_2^{>0}$ and $\text{Hom}(D,E) = 0$ for all $D$ in $C_2^{<0}$.

Let

$$\mathcal{C} = \{ E \in \text{Coh}(\overline{Y}) | R\overline{\psi}_\ast E = 0 \}$$

and let

$$0^T = \{ T \in \mathcal{A} | R^1\overline{\psi}_\ast (R\pi_\ast T) = 0 ; T|C_2 \};$$

$$-1^T = \{ T \in \mathcal{A} | R^1\overline{\psi}_\ast (R\pi_\ast T) = 0, \text{Hom}(T,C) = 0, T|C_2 \};$$

$$0^F = \{ F \in \mathcal{A} | R^0\overline{\psi}_\ast (R\pi_\ast T) = 0, \text{Hom}(C,F) = 0, F|C_2 \};$$

$$-1^F = \{ F \in \mathcal{A} | R^0\overline{\psi}_\ast (R\pi_\ast T) = 0, F|C_2 \}$$

Then $(p^T, p^F)$ is a torsion pair on $\mathcal{A}$ for $p = -1,0$ and a tilt of $\mathcal{A}$ with respect to the torsion pair is the category of perverse coherent sheaves $p\mathcal{A} := \text{Per}^p(Y/S)$. Then every element $E \in p\mathcal{A}$ fits into the exact sequence:

$$(3.1.3) \quad F[1] \hookrightarrow E \twoheadrightarrow T$$

with $F \in p^F$ and $T \in p^T$.

From Bridgeland [3] and Abramovich-Chen [1], the category of perverse sheaves forms a heart of t-structure on $D^b(Y)$. Usually there are actually two perversities $p = -1,0$.

\[ \textbf{(3.1.4) (Derived equivalence)} \] Let $\varphi : Y \to Y'$ be an orbifold flop, in this section we prove, following the method of [3], [1], that there is an equivalence between derived categories:

$$(3.1.5) \quad \Phi : D^b(Y) \to D^b(Y')$$

by the Fourier-Mukai transformation and

$$\Phi(\text{Per}^{-1}(Y/S)) = \text{Per}^0(Y'/S).$$
3.2. The main Hall algebra identity.

. (3.2.1) From [21] §5.5, we define:

\[ \begin{align*}
Q_{\text{exc}} &= \{ Q \in Q_{\leq 1} | \dim \text{Supp} R\psi_*Q = 0 \}; \\
p_{\mathcal{A}_{\leq 1}} &= \{ E \in p_{\mathcal{A}_{\leq 1}} | \dim \text{Supp} R\psi_*E = 0 \}; \\
p_{\mathcal{T}_{\text{exc}}} &= p_{\mathcal{T}} \cap p_{\mathcal{A}_{\text{exc}}}, \\
p_{\mathcal{T}, \bullet} &= p_{\mathcal{T}_{\text{exc}}} \cap Q_{\text{exc}},
\end{align*} \]

where \( \psi : Y \to S \) is the contraction map. Hence inside \( \text{Hilb}^\#_{\leq 1}(Y) \), there is an open subscheme \( \text{Hilb}^\#_{\text{exc}}(Y) \), parameterizing quotients \( O_Y \to F \) in \( \mathcal{A}_{\leq 1} \) such that \( F \in p_{\mathcal{T}, \bullet} \). Its Hall algebra element is denoted by \( \mathcal{H}^\#_{\text{exc}} \in H_{\infty}(\mathcal{A}_{\leq 1}) \).

. (3.2.2) (Perverse Hilbert scheme)

Let \( \mathcal{A}_{\leq 1} \subset \mathcal{A} \) be the full sub-category consisting of sheaves with support of \( \dim \leq 1 \). Similarly, \( p_{\mathcal{A}_{\leq 1}} \subset p_{\mathcal{A}} \) is the full sub-category consisting of perverse sheaves with support of \( \dim \leq 1 \). Let \( H_{\infty}(\mathcal{A}_{\leq 1}) (H_{\infty}(p_{\mathcal{A}_{\leq 1}})) \) be the corresponding sub-Hall algebra.

The first element in our formula is

\[ \mathcal{H}_{\leq 1} \in H_{\infty}(\mathcal{A}_{\leq 1}), \]

the Hilbert scheme of \( X \), which parameterizes quotients

\[ O_Y \to F \]

in \( \mathcal{A}_{\leq 1} \). Let \( \mathcal{M}_{\leq 1} \subset \mathcal{M} \) be the moduli stack of coherent sheaves with support \( \dim \leq 1 \). Then \( \mathcal{H}_{\leq 1} \) is given by the morphism \( \text{Hilb}_{\leq 1}(Y) \to \mathcal{M}_{\leq 1} \).

Remark 3.3. If \( O_Y \to E \) is a quotient in \( \mathcal{A}_{\leq 1} \), then \( E \in p_{\mathcal{T}} \). This is because \( E \in p_{\mathcal{T}} \), and the quotient of torsion is torsion. So the morphism

\[ \text{Hilb}_{\leq 1}(Y) \to \mathcal{M}_{\leq 1} \]

factors through the element \( p_{\mathcal{T}, \leq 1} \), which is represented by \( [p_{\mathcal{T}} \to \mathcal{M}_{\leq 1}] \). Hence \( \mathcal{H}_{\leq 1} \in H_{\infty}(p_{\mathcal{A}_{\leq 1}}) \), since \( p_{\mathcal{T}, \leq 1} \in p_{\mathcal{M}_{\leq 1}} \).

. (3.2.3) (Framed coherent sheaves) Let \( B \subset \mathcal{A} \) be a sub-category. We denote by \( \mathcal{M} \) the element of \( H_{\infty}(\mathcal{A}) \) represented by the inclusion of stacks \( B \subset \mathcal{M} \), which is an open immersion. (Similar for \( \mathcal{A}_{\leq 1} \) and \( p_{\mathcal{A}_{\leq 1}} \).)

Following §2.3 of [29], we define \( \mathcal{M}^O_{\leq 1} \), the stack of framed coherent sheaves, which parametrizes coherent sheaves with a fixed section \( O_Y \to E \). Then \( \text{Hilb}_{\leq 1}(Y) \) is an open subscheme of \( \mathcal{M}^O_{\leq 1} \) by considering a surjective section \( [O_Y \to E] \in \text{Hilb}_{\leq 1}(Y) \). We also have a forgetful morphism:

\[ \mathcal{M}^O_{\leq 1} \to \mathcal{M}_{\leq 1} \]
by taking $[\mathcal{O}_Y \to E]$ to $E \in \mathcal{M}_{\leq 1}$. Given any open substack $B \subset \mathcal{M}_{\leq 1}$, we have a Cartesian diagram:

(3.2.4) 

\[ \begin{array}{ccc}
B & \to & \mathcal{M}_{\leq 1}^O \\
\downarrow & & \downarrow \\
B & \to & \mathcal{M}_{\leq 1}
\end{array} \]

and $1_B^O \in H_{\infty}(\mathcal{A}_{\leq 1})$

Similarly if $pB \subset \mathcal{P}_{\leq 1}$ is an open stack, then we have similar diagram as in (3.2.4) and an element $1_B^P \in H_{\infty}(\mathcal{P}_{\leq 1})$.

Finally let $p\text{Hilb}_{\leq 1}(Y/S)$ be the “perverse Hilbert scheme” parametrizing quotients of $\mathcal{O}_Y$ in $p\mathcal{A}_{\leq 1}$. Then we have an element $p\mathcal{H}_{\leq 1} \in H_{\infty}(p\mathcal{A}_{\leq 1})$.

(3.2.5) Recall that there is a duality functor $D : DB(Y) \to DB(Y)$ by

\[ E \mapsto R\text{Hom}_Y(E, \mathcal{O}_Y)[2]. \]

Let $D' := D[1]$ be the functor of $D$ shifted by one. Then we have the following Hall algebra identity in [21]:

Proposition 3.4.

(3.2.6) $p\mathcal{H}_{\leq 1} \ast 1_{p\mathcal{F}[1]} = 1_{p\mathcal{F}[1]} \ast D'(\mathcal{H}^{\#}_{\text{exc}}) \ast \mathcal{H}_{\leq 1}$.

3.3. Laurent elements and a complete Hall algebra.

(3.3.1) In this section we enlarge the definition of the Hall algebra, as in §4.2 of [9] and [16]. For the stack $\mathcal{M}$, define infinite-type Grothendieck group $L(\text{St}_{\infty}/S)$ by the symbols $[X \to S]$, but with $X$ only assumed to be locally of finite type over $S$. Then we need to drop the relation (1) in Definition 2.2. The infinite-type Hall algebra is then

\[ H_{\infty}(\mathcal{A}) = L(\text{St}_{\infty}/\mathcal{M}) \]
\[ H_{\infty}(p\mathcal{A}) = L(\text{St}_{\infty}/p\mathcal{M}). \]

Remark 3.5. By working on infinite-type Hall algebra, we may not have integration map $I$ in (2.2.7). We will have such an integration map $I$ in the Laurent Hall algebra $H_{\Lambda} \subset H_{\infty}$, and $H(\mathcal{A}) \subset H_{\Lambda}$.

(3.3.2) Recall that for the contraction $\psi : Y \to S$, we have

\[ N_1(Y/S) \hookrightarrow N_1(Y) \twoheadrightarrow N_1(S). \]

So we have

\[ N_1(Y) = N_1(Y/S) \oplus N_1(S). \]

We can index elements in $N_{\leq 1}(Y) = N_1(Y) \oplus N_0(Y) = N_1(S) \oplus N_1(Y/S) \oplus N_0(Y)$ by $(\gamma, \delta, n)$. Recall that we have a Chern character map:

\[ [E] \in F_1 K(Y) \hookrightarrow (\text{Ch}_2(E), \text{Ch}_3(E)) \in N_1(Y) \oplus N_0(Y). \]
Let \( p\Delta \subset \mathcal{F}_1 K(\mathbb{P}\mathcal{A}) \cong N_1(Y) \oplus N_0(Y) \) be the image of the Chern character map of \( p\mathcal{A}_\leq 1 \). Then the Hall algebra \( H(\mathbb{P}\mathcal{A}_\leq 1) \) is graded by \( p\Delta \). Let \( \mathscr{C} \subset N_1(Y/S) \) be the effective curve classes in \( Y \) contracted by \( \psi \).

**Definition 3.6.** Let \( L \subset p\Delta \) be a subset. We call \( L \) to be Laurent is the following conditions hold:

1. for any \( \gamma \), there exists an \( n(\gamma, L) \) such that for all \( \delta, n \), with \( (\gamma, \delta, n) \in L \), we have \( n \geq n(\gamma, L) \);
2. for all \( \gamma, n \), there exists a \( \delta(\gamma, n, L) \in \mathscr{C} \), such that for all \( \delta \) with \( (\gamma, \delta, n) \in L \), one has \( \delta \leq \delta(\gamma, n, L) \).

Let \( \Lambda \) be the set of all Laurent subsets of \( p\Delta \). The set \( \Lambda \) satisfies the following properties as in Lemma 3.10 of [16]:

1. If \( L_1, L_2 \in \Lambda \), then \( L_1 + L_2 \in \Lambda \);
2. If \( \alpha \in p\Delta \) and \( L_1, L_2 \in \Lambda \), then there exist only finitely many decompositions \( \alpha = a_1 + a_2 \) with \( a_i \in L_i \).

**The \( \Lambda \)-completion** \( H(\mathbb{P}\mathcal{A}_\leq 1)_\Lambda \).

Recall the algebra:

\[
\kappa_\nu[\mathbb{P}\Delta] = \bigoplus_{\alpha \in p\Delta} x^\alpha.
\]

The integration map is given by:

\[
I : H_{\text{sec}}(\mathbb{P}\mathcal{A}_\leq 1) \rightarrow \kappa_\nu[\mathbb{P}\Delta].
\]

For any \( p\Delta \)-graded associative algebra \( R \), the \( \Lambda \)-completion \( R_\Lambda \) is defined to be the vector space of formal series:

\[
\sum_{(\gamma, \delta, n)} x_{(\gamma, \delta, n)}
\]

with \( x_{(\gamma, \delta, n)} \in R_{x_{(\gamma, \delta, n)}} \), and \( x_{(\gamma, \delta, n)} = 0 \) outside a Laurent subset. The product is defined by:

\[
x \cdot y = \sum_{\alpha \in p\Delta} \sum_{a_1 + a_2 = \alpha} x_{a_1} \cdot y_{a_2}.
\]

Then the integration map \( I : H_{\text{sec}}(\mathbb{P}\mathcal{A}_\leq 1) \rightarrow \kappa_\nu[\mathbb{P}\Delta] \) induces a morphism on the completions:

\[
I_\Lambda : H_{\text{sec}}(\mathbb{P}\mathcal{A}_\leq 1)_\Lambda \rightarrow \kappa_\nu[\mathbb{P}\Delta]_\Lambda.
\]

**Elements in** \( H(\mathbb{P}\mathcal{A}_\leq 1)_\Lambda \).

Let \( \mathcal{G} \) be an algebraic stack of locally of finite type over \( \kappa \), such that \( [\mathcal{G} \rightarrow \mathbb{P}\mathcal{M}_\leq 1] \) is a map to \( \mathbb{P}\mathcal{M}_\leq 1 \). For \( \alpha \in p\Delta \), the preimage of \( \mathbb{P}\mathcal{M}_\alpha \) is denoted by \( \mathcal{G}_\alpha \).

The element

\[
[\mathcal{G} \rightarrow \mathbb{P}\mathcal{M}_\leq 1] \in H_\infty(\mathbb{P}\mathcal{A}_\leq 1)
\]

is Laurent if \( \mathcal{G}_\alpha \) is a stack of finite type for all \( \alpha \in p\Delta \), and \( \mathcal{G}_\alpha \) is empty for \( \alpha \) outside a Laurent subset.

Then following results are due to Calabrese in [16].

**Proposition 3.7.** The elements

\[
1_{\mathcal{P}\mathcal{F}[1]}, \quad 1^Q_{\mathcal{P}\mathcal{F}[1]}, \quad \mathcal{P}\mathcal{H}_\leq 1, \quad \mathcal{H}_\leq 1
\]

are all Laurent.
The stability condition \( \mu \) and note that the set of \( \mu \) is a weak stability condition in sense of Definition 3.5 of [26]. Then we want to apply the motivic integration map to the identity (3.2.6). We cancel the term \( \mathcal{H}_{\leq 1} \) is bounded. That the element \( \mathcal{H}_{\leq 1} \) is is from the fact that once fixing numerical data \((\gamma, \delta, n)\), Riemann-Roch tells us that the subset \( a \) is bounded. That the element \( \mathcal{H}_{\leq 1} \) is is from the fact that once we fix \( \gamma, n \), varying \( \delta \) then the corresponding perverse Hilbert scheme is of finite type. The case of \( \mathcal{H}_{\leq 1} \) is from the Hall algebra identity:
\[
\mathcal{H}_{\leq 1} \mathcal{E} \mathcal{F} [1] = \mathcal{E} \mathcal{F} [1] \mathcal{H}_{\leq 1}
\]
in [21] Theorem 5.3].

3.4. Finishing the proof.

(3.4.1) Let \( \mathcal{E} \mathcal{F} \subseteq \mathcal{E} \mathcal{F} \otimes N_{1}(Y) \otimes N_{0}(Y) \) be the image of the Chern character map of \( \mathcal{E} \mathcal{F} \leq 1 \). Then we have the motivic integration map
\[
I_{\Lambda} : H_{d-Crit,ssc}(\mathcal{E} \mathcal{F} \leq 1)_{\Lambda} \rightarrow \mathcal{M}_{X,loc}[\mathcal{E} \mathcal{F}]
\]
given by
\[
[X \rightarrow \mathcal{M}] \mapsto \mathcal{S}_{X}^{\mathcal{E} \mathcal{F}}.
\]

(3.4.2) The proof of the formula in the Theorem is similar to the proof of [21] Theorem 1.3. We want to apply the motivic integration map to the identity (3.2.6). Note that \( \mathcal{H}_{\leq 1}, \mathcal{H}_{\leq 1}^{\#}, \mathcal{H}_{\leq 1} \in H_{d-Crit,ssc}(\mathcal{E} \mathcal{F} \leq 1)_{\Lambda} \), but \( \mathcal{E} \mathcal{F} [1] \) does not.

We cancel the term \( \mathcal{E} \mathcal{F} [1] \) from (3.2.6) after applying the motivic integration map. As in [16, 21 §5.8], define a stability condition \( \mu \) by:
\[
(0, \delta, n) \mapsto \begin{cases}
1, & \delta \geq 0; \\
2, & \delta < 0.
\end{cases}
\]
The stability condition \( \mu \) is a weak stability condition in sense of Definition 3.5 of [26]. Then the set of \( \mu \)-semistable objects of slope \( \mu = 2 \) is \( \mathcal{E} \mathcal{F} [1] \), and the set of \( \mu \)-semistable objects of slope \( \mu = 1 \) is \( \mathcal{E} \mathcal{F}^{exc} \). Let \( \epsilon := \log(1_{\mathcal{E} \mathcal{F} [1]}) \), then \( \eta = (L - 1) \cdot \epsilon \in H_{reg}(\mathcal{E} \mathcal{F} \leq 1) \) and inside \( H(\mathcal{E} \mathcal{F} \leq 1)_{\Lambda}, 1_{\mathcal{E} \mathcal{F} [1]} = \exp(\epsilon) \). The automorphism
\[
\text{Ad}_{\mathcal{E} \mathcal{F} [1]} : H(\mathcal{E} \mathcal{F} \leq 1)_{\Lambda} \rightarrow H(\mathcal{E} \mathcal{F} \leq 1)_{\Lambda}
\]
preserves the regular elements and the induced Poisson automorphism of \( H_{d-Crit,ssc}(\mathcal{E} \mathcal{F} \leq 1) \) is \( \text{Ad}_{\mathcal{E} \mathcal{F} [1]} = \exp(\eta, -) \).

Now from (3.2.6) we have that:
\[
\mathcal{H}_{\leq 1} = \mathcal{D}^{\prime}(\mathcal{H}_{exc}^{\#}) \cdot \exp{\{\eta, -\}}(\mathcal{H}_{\leq 1}).
\]
Applying the motivic integration map and note that \( \text{Ext}^{i}(E, F) = 0 \) since \( \dim(\text{supp } E) \leq 1, \dim(\text{supp } F) \leq 1 \), the Poisson brackets vanish and we have:
\[
I_{\Lambda}(\mathcal{H}_{\leq 1}) = I_{\Lambda}(\mathcal{D}^{\prime}(\mathcal{H}_{exc}^{\#}) \cdot I_{\Lambda}(\mathcal{H}_{\leq 1}).
\]
So let
\[
\mathcal{S}^{\phi}_{\mathcal{D}^{\prime}}(Y) = I_{\Lambda}(\mathcal{H}_{\leq 1}),
\]
and note that \( I_{\Lambda}(\mathcal{H}_{\leq 1}) = \mathcal{S}^{\phi}_{\mathcal{D}^{\prime}}(Y) \), we have
\[
\mathcal{S}^{\phi}_{\mathcal{D}^{\prime}}(Y) = I_{\Lambda}(\mathcal{D}^{\prime}(\mathcal{H}_{exc}^{\#}) \cdot \mathcal{S}^{\phi}_{\mathcal{D}^{\prime}}(Y).
\]
Now by the motivic DT/PT correspondence,
\[ I_\Lambda(D'(\mathcal{H}_{\text{exc}}^\#)) = S_{\text{PT}}^\phi(Y). \]
The flop\[ \varphi : Y \to Y' \]
gives the equivalence:
\[ \Phi_* (S_{\text{DT}}^\phi(Y)) = S_{\text{DT}}^\phi(Y'). \]
So we are done.

\[ \square \]

4. Higher rank DT and stable pair invariants of Toda

4.1. Higher rank invariants.

. \textbf{(4.1.1)} In this section we prove the motivic DT/PT correspondence for higher rank DT and stable pair invariants introduced by Toda in [46].

. \textbf{(4.1.2)} Let \( Y \) be a quasi-projective smooth Calabi-Yau DM stack. Fix a sufficiently large ample divisor \( \omega \) on \( Y \) such that for any divisor class \( D \in H^2(Y) \), \( \omega^2 \cdot D \) is an integer. Similarly as in [46], let
\[ \Gamma := \text{Im}(\text{Ch} : K_0(Y) \to \bigoplus_{* \in \mathbb{Z}} H^{2*}(Y)) \]
be the image of the Chern character map. We assume that \( v = (r, D, -\beta, -n) \in \Gamma \) and \( (r, D \cdot \omega^2) \) coprime.

. \textbf{(4.1.3)} For any coherent sheaf \( E \) on \( X \), the slope \( \mu_\omega \) is defined by:
\[ \mu_\omega(E) := \frac{c_1(E) \cdot \omega^2}{\text{rank}(E)} \in \mathbb{Q} \cup \{\infty\}. \]
The sheaf \( E \) is called \( \mu_\omega \)-(semi)stable if for any nontrivial subsheaf \( F \subset E \),
\[ \mu_\omega(F) < (\leq) \mu_\omega(E/F). \]
Since \( (D \cdot \omega^2, r) = 1 \), the moduli space \( \mathcal{M}_{\text{DT}}(a) \) for \( a \in K_0(Y) \) with \( \text{Ch}(a) = (r, D, -\beta, -n) \in \Gamma \) of \( \mu_\omega \)-semistable sheaves is a projective scheme, see [17]. The moduli scheme admits a symmetric obstruction theory of Behrend in [4] and
\[ \text{DT}(a) = \int_{\mathcal{M}_{\text{DT}}(a)} \chi(\mathcal{M}_{\text{DT}}(a), v_{\mathcal{M}_{\text{DT}}}). \]

. \textbf{(4.1.4)} Let \( a_{(r,D)} \subset K(Y) \) be all the K-group classes such that
\[ \text{Ch}(a_{(r,D)}) = (r, D, -\beta, -n). \]
The moduli scheme \( \mathcal{M}_{\text{DT}}(a) \) is naturally a \( d \)-critical scheme in the sense of Joyce in [27]. Assume that there is an orientation data \( K_{\mathcal{M}_{\text{DT}}(a)}^\phi \) in the sense of [12], the global motive
\[ S_{\text{DT}}^\phi(a_{(r,D)}) = \sum_{a \in a_{(r,D)}} S_{\mathcal{M}_{\text{DT}}(a)}^\phi \cdot x^a. \]
The higher rank DT invariants and stable pair invariants was introduced in Lo’s work \cite{35} in the study of Bayer’s polynomial stability conditions \cite{2}.

**Definition 4.1.** \cite{35, 46} An object \( I^• \in D^b(\text{Coh}(Y)) \) is called PT (semi)stable if the following conditions are satisfied:

1. \( H^i(I^•) = 0 \) for \( i \neq 0, 1 \);
2. \( H^0(I^•) \) is \( \mu_\omega \)-(semi)stable and \( H^1(I^•) \) is zero dimensional;
3. \( \text{Hom}(Q[-1], I^•) = 0 \) for any zero dimensional sheaf \( Q \).

As explained in \cite{46}, if \( E \) is a locally free \( \mu_\omega \)-(semi)stable sheaf on \( Y \) and \( F \) is a pure dimensional one sheaf and \( s : E \to F \) is a morphism such that it is surjective in dimension one. Then

\[
I^• = (E \to F) \in D^b(\text{Coh}(Y))
\]

is a PT stable object. When \( E = O_Y \), this is the PT stable pair in \cite{41}.

**4.2. Tiling of the abelian category \( \text{Coh}(Y) \) by Toda.**

**4.2.1** We first recall Toda’s tiling in \cite{46}. For an interval \( I \subset \mathbb{R} \cup \{ \infty \} \), set

\[
\text{Coh}_I(Y) = \left\{ E \in \text{Coh}(Y) | E \text{ is } \mu_\omega \text{ semistable and } \mu_\omega(E) \in I \right\} \cup \{0\}.
\]

Fix \((r, D, -\beta, -n)\), set

\[
\mu := \frac{D \cdot \omega^2}{r} \in \mathbb{Q}.
\]

From the existence of Harder-Narasimhan filtrations with respect to the \( \mu_\omega \)-stability, we have the torsion pair:

\[
\text{Coh}(Y) = \left\langle \text{Coh}_{\geq \mu}(Y), \text{Coh}_{\leq \mu}(Y) \right\rangle.
\]

For any \( E \in \mathcal{A}_\mu \), we have \( \text{rank}(E) \cdot D \omega^2 - c_1(E) \cdot \omega^2 \cdot r = 0 \). Let

\[
B := \{ E \in \mathcal{A}_\mu | \text{rank}(E) \cdot D \omega^2 - c_1(E) \cdot \omega^2 \cdot r = 0 \}.
\]

Then \( B_\mu \) is an abelian subcategory of \( \mathcal{A}_\mu \) and

\[
B_\mu = \left\langle \text{Coh}_\mu(X), \text{Coh}_{\leq 1}(X)[-1] \right\rangle,
\]
The Hall algebra product on $\mathcal{C}_{\omega < 1}(Y) = \{ F \in \text{Coh}(Y) \mid \dim \text{supp}(F) \leq 1 \}$.

Let $\mathfrak{p}_\omega$ be the slope function on $\mathcal{C}_{\omega < 1}(Y)$:

$$ \mathfrak{p}_\omega(F) = \frac{\text{Ch}_3(F)}{\text{Ch}_2(F) \cdot \omega}. $$

Then we can define $\mathfrak{p}_\omega$-stability on $\mathcal{C}_{\omega < 1}(Y)$. Set

$$ \mathcal{C}_I := \left\{ F \in \mathcal{C}_{\omega < 1}(Y) \mid F \text{ is } \mathfrak{p}_\omega \text{-semistable and } \mathfrak{p}_\omega(F) \in I \right\} \cup \{0\}.$$

Then from [46, Definition 3.5],

$$ B_\mu = \left\langle \text{Coh}_\mu(Y), \mathcal{C}_\infty, \mathcal{C}_{[0, \infty)}, \mathcal{C}_{< 0} \right\rangle,$$

where the bracket means the following: let us denote the elements inside the bracket by order $F_1, F_2, F_3, F_4$. Then

1. $\text{Hom}(F_i, F_j) = 0$, where $F_i \in \mathcal{F}_i, F_j \in \mathcal{F}_j, i < j$;
2. Any $E \in \mathcal{A}$, there exists a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ such that $E_i/E_{i-1} \in \mathcal{F}_i$.

Remark 4.2. The category $\mathcal{C}_\infty$ consists of $Q[-1]$ for zero dimensional sheaves $Q$.

4.3. Completion of Hall algebras.

. (4.3.1) In [46], Toda constructed certain completions of Hall algebras $H(\mathcal{A}_\mu)$. By [31] [3], there exists a function

$$ l : H^0(Y) \oplus H^2(Y) \oplus H^4(Y) \to \mathbb{Q}_{>0} $$

such that for any torsion free $\mu_\omega$-semistable sheaf $E$ we have

$$ \text{Ch}_3(E) \leq l(\text{Ch}_0(E), \text{Ch}_1(E), \text{Ch}_2(E)). $$

For a fixed $(r, D)$, Toda in [46] defines

$$ \Gamma_{r, D} := \{(r, D, -\beta, -n) \in \Gamma | \omega \beta \geq -\left(\frac{D \cdot \omega^2}{2r \omega^3}\right), n \geq -l(r, D, -\beta)\} $$

be the Bogomolov inequality:

$$ (\text{Ch}_1(E) \cdot \omega^2)^2 \geq 1 \text{Ch}_0(E) \omega^3 \cdot \text{Ch}_2(E) \omega. $$

. (4.3.2) Let

$$ \Gamma_{\#} := \{(0, 0, -\beta, -n) \in \Gamma | \beta \geq 0, n \geq 0\}. $$

Then the completions of Hall algebras are defined by:

$$ \hat{H}_{r, D}(\mathcal{A}_\mu) := \prod_{v \in \Gamma_{r, D}} H_v(\mathcal{A}_\mu); \quad \hat{H}_\#(\mathcal{A}_\mu) := \prod_{v \in \Gamma_{\#}} H_v(\mathcal{A}_\mu). $$

The Hall algebra product on $H(\mathcal{A}_\mu)$ induces the Hall algebra product on $\hat{H}_\#(\mathcal{A}_\mu)$ by [46] Lemma 3.9] below:

Lemma 4.3. (1) For any $E \in \mathcal{D}_\mu := \langle \text{Coh}_\mu(Y), \mathcal{C}_\infty, \mathcal{C}_{[0, \infty)} \rangle$ with $(\text{Ch}_3(E), \text{Ch}_2(E)) = (r, D)$, we have $\text{Ch}(E) \in \Gamma_{r, D}$;

(2) For any $E \in \mathcal{D}_\mu$ with $\text{Ch}_0(F) = 0$, we have $\text{Ch}(F) \in \Gamma_{\#}$;

(3) For any $v \in \Gamma_{r, D}$ and $v' \in \Gamma_{\#}$, $v + v' \in \Gamma_{r, D}$;
For $v \in \Gamma_\mu$, there is only a finite number of ways to write it as $v_1 + v_2 + \ldots + v_l$
for $v_i \in \Gamma_\mu \setminus \{0\};$

(5) For $v \in \Gamma_\mu$, there is only a finite number of ways to write it as $v_1 + v_2 + \ldots + v_l + v_{l+1}$ for $v_i \in \Gamma_\mu \setminus \{0\}$ and $v_{l+1} \in \Gamma_\mu$.

(4.3.3) Similarly we have the Hall algebra of regular elements as in [9, 46]:

$$\tilde{H}^{reg}_{r,D}(\mathcal{A}_\mu) \subset \tilde{H}_{r,D}(\mathcal{A}_\mu),$$

and

$$\tilde{H}^{reg}_{\#}(\mathcal{A}_\mu) \subset \tilde{H}_{\#}(\mathcal{A}_\mu).$$

The semiclassical quotients are:

$$\tilde{H}^{reg}_{r,D}(\mathcal{A}_\mu) = \tilde{H}^{reg}_{r,D}(\mathcal{A}_\mu) / (L - 1) \cdot \tilde{H}^{reg}_{r,D}(\mathcal{A}_\mu),$$

and

$$\tilde{H}^{reg}_{\#}(\mathcal{A}_\mu) = \tilde{H}^{reg}_{\#}(\mathcal{A}_\mu) / (L - 1) \cdot \tilde{H}^{reg}_{\#}(\mathcal{A}_\mu)$$

and there are induced Poisson brackets on them.

(4.3.4) Since $\mathcal{A}_\mu$ is an abelian category, let $M_\mu$ be the moduli stack of objects in $\mathcal{A}_\mu$. From [13], $M_\mu$ is a derived Artin stack. Let

$$\tilde{H}^{d-\text{Crit}}_{r,D}(\mathcal{A}_\mu); \quad \tilde{H}^{d-\text{Crit}}_{\#}(\mathcal{A}_\mu)$$

be the Hall subalgebras of the corresponding $d$-critical elements in $M_\mu$. Then we have:

$$\tilde{H}^{d-\text{Crit,ssc}}_{r,D}(\mathcal{A}_\mu) = \tilde{H}^{d-\text{Crit}}_{r,D}(\mathcal{A}_\mu) / (L - 1) \cdot \tilde{H}^{d-\text{Crit}}_{r,D}(\mathcal{A}_\mu),$$

and

$$\tilde{H}^{d-\text{Crit,ssc}}_{\#}(\mathcal{A}_\mu) = \tilde{H}^{d-\text{Crit}}_{\#}(\mathcal{A}_\mu) / (L - 1) \cdot \tilde{H}^{d-\text{Crit}}_{\#}(\mathcal{A}_\mu).$$

4.4. Applying the integration map.

(4.4.1) Recall from [46 §3.4],

$$\tilde{H}_{\#}(\mathcal{A}_\mu) = \prod_{v \in \Gamma_\mu} H_0(\text{Coh}_{\leq 1}(Y)[-1]).$$

For any $\gamma \in \tilde{H}_{\#}(\mathcal{A}_\mu)$ with zero $H_0(\mathcal{A}_\mu)$-component, from [?, Theorem 8.7], [9, Theorem 6.3],

$$(L - 1) \cdot \log(1 + \gamma) \in \tilde{H}^{reg}_{\#}(\mathcal{A}_\mu).$$

For any interval $I \subset \mathbb{R}_{\geq 0}$, the stack of objects decomposes into

$$\text{Obj}(C_I) = \bigsqcup_{\pi \in \Gamma_\mu, \pi \not= 0} \text{Obj}_\pi(C_I)$$

and each component $\text{Obj}_\pi(C_I)$ is a finite type stack. Let

$$\delta(C_I) := [\text{Obj}(C_I) \to M_\mu] \in \tilde{H}_{\#}(\mathcal{A}_\mu).$$

Then

$$\varepsilon(C_I) := \log(\delta(C_I)) \in \tilde{H}_{\#}(\mathcal{A}_\mu)$$

and

$$(L - 1) \cdot \varepsilon(C_I) \in \tilde{H}^{reg}_{\#}(\mathcal{A}_\mu)$$

by Joyce’s no pole theorem.
Consider the motivic integration map:

\[ I^# : \hat{H}^d_{\text{Crit,ssc}}(\omega_\mu) \to \hat{\mathcal{M}}^d_{k,\text{loc}}[\Gamma^#], \]

where

\[ \hat{\mathcal{M}}^d_{k,\text{loc}}[\Gamma^#] = \bigoplus_{a \in \Gamma^#} \hat{\mathcal{M}}^d_{k,\text{loc}} \cdot x^a \]

and

\[ I([Z \to \mathcal{M}_\mu]) = \int_Z t^* S_Z^\phi \in \hat{\mathcal{M}}^d_k \]

where \( t_Z : Z \to \mathcal{Z} \) is the map from \( Z \) to the corresponding derived Artin stack \( \mathcal{Z} \).

**Remark 4.4.** The moduli scheme \( Z \) is the coarse moduli space of the objects in \( \mathcal{M}_\mu \) under some stability conditions and \( Z \) is the corresponding d-critical Artin stack in the sense of [13].

**Proposition 4.5.** For any interval \( I \subset \mathbb{R}_{\geq 0} \cup \{\infty\} \),

\[ I^#(\mathcal{C}_I^\mu) = \sum_{\nu \in \Gamma^#; \mathcal{P}_{\omega}(\nu) = \mathcal{P}} [N_\nu] \cdot x^\nu \]

where \( N_\nu \) is the global motive of \( \mathcal{P}_{\omega} \)-semistable sheaves \( F \in \text{Coh}_{\leq 1}(X) \) with \( \text{Ch}(F) = \nu \), and

\[ \lim_{L^{1/2} \to (-1)} N_\nu = -N_\nu \]

which is the invariant in [16, Remark 3.14].

**Proof.** By the existence of Harder-Narasimhan filtration with respect to the \( \mathcal{P}_{\omega} \)-stability, we have

\[ \delta(C_I) = \prod_{\mathcal{P} \in I} \mathcal{P}(\mathcal{C}_I). \]

Then by taking the logarithm of both sides and multiplying \( (L - 1) \) we get:

\[ \mathcal{C}(C_I) = \sum_{\mathcal{P} \in I} \mathcal{C}(\mathcal{C}_I) + \{ \text{nested Poisson brackets in } \mathcal{C}(\mathcal{C}_I) \}. \]

For any \( \nu_i \in \Gamma^# \), \( \text{Ext}^s(\nu_i, \nu_j) = 0 \) for all \( s \) by dimensional reasons, then the Poisson brackets vanish and

\[ I^#(\mathcal{C}_I^\mu) = \sum_{\mathcal{P} \in I} I^#(\mathcal{C}_I^\mu). \]

\[ \square \]
Theorem 4.6. We have

\[ S^\phi_{\text{DT}}(r, D) = \exp \left( \sum_{n > 0} \frac{L^{nr-1}L^n - 1}{L - 1} [N_n] x^n \right) \cdot S^\phi_{\text{PT}}(r, D) \].

Proof. Recall that from [46, §3.7], let

\[ B_\mu = \langle \text{Coh}_\mu(Y), C_\infty, C_{(0, \infty)}, C_{<0} \rangle. \]

Then we have

\[ \langle \text{Coh}_\mu(Y), C_\infty \rangle = \langle C_\infty, \text{Coh}_\mu(Y) \rangle, \]

where \( \text{Coh}_\mu(Y) \) is the category of PT-semistable objects \( I^* \) with \( \mu_\omega(I^*) = \mu \).

Let

\[ \delta_{\text{DT}}(r, D) := [M_{\text{DT}}(r, D) \to \text{Obj}(\alpha_\mu)] \in \tilde{H}_{r, D}(\alpha_\mu) \]

and

\[ \delta_{\text{PT}}(r, D) := [M_{\text{PT}}(r, D) \to \text{Obj}(\alpha_\mu)] \in \tilde{H}_{r, D}(\alpha_\mu). \]

Inside \( \tilde{H}_{r, D}(\alpha_\mu) \), the following identity was proved in [46]:

\[ \delta_{\text{DT}}(r, D) \star \delta(C_\infty) = \delta(C_\infty) \star \delta_{\text{PT}}(r, D). \]

Then we have

\[ \delta_{\text{DT}}(r, D) = \exp(\epsilon(C_\infty)) \star \delta_{\text{PT}}(r, D) \star \exp(\epsilon(C_\infty))^{-1}. \]

By Baker-Cambell-Hausdorff formula:

\[ \delta_{\text{DT}}(r, D) = \exp(\text{Ad}(\epsilon(C_\infty))) \cdot \delta_{\text{PT}}(r, D). \]

Hence multiplying \( (L - 1) \) on both sides we get:

\[ S^\phi_{\text{DT}}(r, D) = \exp(\text{Ad}(\epsilon(C_\infty))) \cdot S^\phi_{\text{DT}}(r, D). \]

Applying the motivic integration map \( I \):

\[ S^\phi_{\text{DT}}(r, D) = \exp \left( \sum_{n > 0} \frac{L^{nr-1}L^n - 1}{L - 1} [N_n] x^n \right) \cdot S^\phi_{\text{PT}}(r, D). \]

\[ \square \]

(4.4.6) We know that the invariant \( N_n \) counts semistable sheaves \( \mathcal{F} \in \text{Coh}_{<0}(X) \) with \( \chi(\mathcal{F}) = n \).

Proposition 4.7. We have:

\[ \exp \left( \sum_{n > 0} \frac{L^{nr-1}L^n - 1}{L - 1} [N_n] x^n \right) = S^{\phi}_Y(-q) = \exp \left( \frac{-t \cdot r \cdot [Y]_{\text{vir}}}{(1 + L^{1/2}t)(1 + L^{-1/2}t)} \right), \]

where \( [Y]_{\text{vir}} = L \frac{\dim(Y)}{2} [Y] \).
Proof. This is the rank $r$ version of the formula in \cite{12}. In the paper \cite{12}, Behrend, Bryan and Szendroi defined the global motive of the Hilbert scheme $\text{Hilb}^n(Y)$ of points on $Y$ by

$$\text{[Hilb}^n(Y)]_{\text{virt}} = \sum_\alpha \text{[Hilb}^n_\alpha(Y)]_{\text{virt}}$$

and

$$\text{[Hilb}^n_\alpha(Y)]_{\text{virt}} = \pi_{G_\alpha} \left( \prod_i [Y^{\alpha_i} \setminus \Delta] \cdot \prod_i \text{[Hilb}^i(\mathbb{A}^3_{\kappa})_{\text{virt}}^{\alpha_i}] \right)$$

where the motivic class $\prod_i [Y^{\alpha_i} \setminus \Delta]$ and $\prod_i \text{[Hilb}^i(\mathbb{A}^3_{\kappa})_{\text{virt}}^{\alpha_i}]$ carry $G_\alpha$-actions, and

$$\pi_{G_\alpha}: \tilde{\mathcal{M}}_{k\alpha}^G \to \mathcal{M}_k$$

is the projection. Here the motive

$$\text{[Hilb}^n_\alpha(Y)]_{\text{virt}} = \text{[Y]} : \text{[Hilb}^n(\mathbb{A}^3_{\kappa})_{\text{virt}}^{\alpha}]$$

and $\text{[Hilb}^n(\mathbb{A}^3_{\kappa})_{\text{virt}}^{\alpha}]$ can be defined by the global motive

$$\text{[Hilb}^n(\mathbb{A}^3_{\kappa})] = L^{-n} \varphi_f$$

where

$$f: M \to \kappa$$

is the global function on a smooth variety $M$ such that $Z(df) = \text{Hilb}^n(\mathbb{A}^3_{\kappa})$, see \cite{12}. Then using the power structure on the motivic ring, \cite{12} deduces that

$$\exp \left( \sum_{n > 0} L^{n-1} \frac{L^n - 1}{L - 1} [N_n] x^n \right) = S^1_{A^3_{\kappa}}(-q)$$

$$= \exp \left( \frac{-t \cdot [Y]_{\text{virt}}}{(1 + L^{1/2} t)(1 + L^{-1/2} t)} \right)$$

$$= S^1_{A^3_{\kappa}}(-q)(Y)$$

$$= \left( \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} (1 - L^{2k+\frac{m}{2}} (-t)^{m-1}) \right)_{[Y]}.$$  

For higher rank $r$, the degree zero higher rank pair of Toda on $A^3_{\kappa}$ is a $r$ copies of stable pairs, and the degree zero stable pairs of PT is the same as the degree zero DT invariants, i.e. the Hilbert scheme of points on $Y$. So a direct analysis implies that

$$S^r_{A^3_{\kappa}} = (S^1_{A^3_{\kappa}})^r$$

and

$$S^r_{Y} = (S^r_{A^3_{\kappa}})_{[Y]_{\text{virt}}} = (S^1_{A^3_{\kappa}})^r_{[Y]_{\text{virt}}}.$$  

\qed
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