Interference Effects between Three Coupled Bose-Einstein Condensates

Sun Zhang and Fan Wang

Department of Physics, Nanjing University, Nanjing, 210093, China

Abstract

We study the interference effects between three weakly linked trapped Bose-Einstein condensates (BEC) as a generalization of the two-component condensates. Three coupled Gross-Pitaevskii equations (GPE) are used to describe the dynamics of the system. The nonsinusoidal oscillation is found as a generalization of the Josephson effect in superconductivity. The self-trapped effects are also predicted in three coupled BEC. Moreover, in general case, the phase diagrams of the system are closed only for some special parameters, which can be used to determine the interaction parameters between atoms in BEC.
Recently, Bose-Einstein Condensate (BEC) has been realized in a dilute and ultracold gas of trapped alkali-metal atoms. It opens a new field for studying the physics of BEC, which is hoped to improve our understanding on the fundamental concept of quantum mechanics and quantum many body problems. From the ensuing theoretical and experimental works, it is clear that the condensed Bose gas displays the crucial properties of a confined coherent matter.

Since the current understanding of BEC is mainly based on the mean-field approximation, where a macroscopic wave function is introduced as the order parameter, the study of this feature by investigating the interference phenomena should be of great importance. Such a description using a macroscopic wave function with a definite phase implies U(1) gauge symmetry broken.

There are many methods suggested theoretically for detecting the interference effects between two components of condensates. The one is the investigation using the continuous measurement theory. It shows that an interference pattern between two condensates can be built up dynamically in a single run of an experiment, even though no phases have even been assumed. Another approach is that, one can set up a potential barrier between two trapped condensates by using a far off-resonant laser. If one lowers the potential barrier produced by the laser, the atoms would tunnel through the barrier due to weak coupling between these two condensates. The interference is developed, somewhat like Josephson effect in superconductivity. Similarly, if one switches off the laser potential traps, the atoms would expanded ballistically due to the sudden disappearance of the barrier. The interference fringes can be observed. It is also interesting experimentally that the interference effects can also be tested by means of different hyperfine atomic states in a single trap.

Among these methods mentioned above, some have been realized in experiments. In Ref. [20], the authors reported that fringes were observed in the density of two overlapping condensates, released after switching off the traps. In Ref. [21], the authors measured the relative phase of two condensates in different hyperfine atomic states occupying a single trap.
using Ramsey’s method of separated oscillating fields. From experimental data, the phase locking indeed occurs for small separation between condensates, implying the broken gauge symmetry.

The interference effects of two component condensates have been studied in a variety of cases, but there are very few on three. In this paper, we would discuss the interference effects between three coupled BEC.

Any generalization on this topic would increases the difficulty significantly. The reward is that the more complex nonlinearity and the more parameters in the basic equations would bring richer dynamics and more compound structure into the model. Experimentally, a far off-resonant laser barrier can be used to split a trapped condensate into two, three or even more well-trapped condensates. Lowering the laser intensity allows atoms tunneling through the barrier. Then interference will be built up between these condensates. The optical confinement method also has the possibility to create a three component BEC in different hyperfine atomic states.

Due to the U(1) gauge symmetry broken, it is crucial to introduce a global phase to the order parameter, the macroscopic wave function Ψ(r, t) to describe BEC. Then, the wave function Ψ(r, t) at T = 0 can be factorized as

$$\Psi(r, t) = \psi(t)\Phi(r),$$

which satisfies a nonlinear Schrödinger equation, the Gross-Pitaevskii equation (GPE),

$$i\hbar \frac{\partial \Psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(r, t) + [V_{\text{trap}}(r) + g_0 |\Psi(r, t)|^2] \Psi(r, t),$$

where $V_{\text{trap}}(r)$ is the external trap potential and $g_0 = 4\pi\hbar^2 a/m$, $m$ the atomic mass and $a$ the scattering length of the atom-atom interaction. When we study the interference of three weakly coupled BEC in traps 1,2 and 3, the dynamics of the system is described by three GPE, coupled with tunneling matrix elements. It can be written as a three-state model

$$i\hbar \frac{\partial \psi_1}{\partial t} = (E_1 + U_1 N_1)\psi_1 - K \psi_2,$$
\[ i\hbar \frac{\partial \psi_2}{\partial t} = (E_2^0 + U_2 N_2) \psi_2 - K \psi_1 - K \psi_3, \tag{4} \]
\[ i\hbar \frac{\partial \psi_3}{\partial t} = (E_3^0 + U_3 N_3) \psi_3 - K \psi_2, \tag{5} \]

with
\[ \psi_{1,2,3} = \sqrt{N_{1,2,3}} \exp(i\theta_{1,2,3}(t)), \tag{6} \]

where \( N_{1,2,3} \) and \( \theta_{1,2,3} \) are the atom numbers and (time-dependent) phases for BEC in trap 1, 2 and 3, respectively. \( K \) is the coupling matrix element between trap 1 and 2 or trap 2 and 3 (for simplicity, we only consider the case, that these two coupling matrix elements are the same and there is no coupling between 1 and 3). \( E_{1,2,3}^0 \) are the zero-point energies in each well, \( U_{1,2,3} N_{1,2,3} \) are proportional to the atomic self-interaction energies, induced by the overlap of the spatial part of the macroscopic wave function, and the total number of atoms \( N_T = N_1 + N_2 + N_3 \) is a constant.

From Eqs. (3), (4), (5) and (6), we get four independent equations in terms of the phase difference and the atom population imbalance
\[ \hbar \frac{\partial \phi_1}{\partial t} = (E_1^0 - E_2^0) + (U_1 N_1 - U_2 N_2) + K \sqrt{\frac{N_1 - N_2}{N_1 N_2}} \cos \phi_1 + K \sqrt{\frac{N_3}{N_2}} \cos(\phi_1 - \phi_2), \tag{7} \]
\[ \hbar \frac{\partial \phi_2}{\partial t} = (E_1^0 - E_3^0) + (U_1 N_1 - U_3 N_3) - K \sqrt{\frac{N_2}{N_1}} \cos \phi_1 + K \sqrt{\frac{N_3}{N_2}} \cos(\phi_1 - \phi_2), \tag{8} \]
\[ \frac{\hbar}{2} \frac{\partial (N_1 - N_2)}{\partial t} = -2K \sqrt{N_1 N_2} \sin \phi_1 - K \sqrt{N_2 N_3} \sin(\phi_1 - \phi_2), \tag{9} \]
\[ \frac{\hbar}{2} \frac{\partial (N_1 - N_3)}{\partial t} = -K \sqrt{N_1 N_2} \sin \phi_1 + K \sqrt{N_2 N_3} \sin(\phi_1 - \phi_2), \tag{10} \]

where \( \phi_1 = \theta_2 - \theta_1 \) and \( \phi_2 = \theta_3 - \theta_1 \).

In order to get an understanding on the novel phenomena of interference effects between three trapped BEC, let us consider an ideal case: the totally symmetric case, in which, \( U_1 = U_2 = U_3 \equiv U, \ E_1^0 = E_2^0 = E_3^0 \), and the atom population imbalances of traps 1, 2 and of traps 2, 3 are the same, that is \( N_1 - N_2 = N_3 - N_2 \), or \( N_1 = N_3 \). Then, we can simplify Eqs. (7), (8), (9) and (10) as
\[ \frac{\hbar}{\partial t} \frac{\partial \phi_1}{\partial t} = U(N_1 - N_2) + K \frac{N_1 - N_2}{\sqrt{N_1 N_2}} \cos \phi_1 + K \frac{1}{\sqrt{N_1 N_2}} \cos \phi_1, \]  
\tag{11} \]

\[ \frac{\hbar}{2} \frac{\partial (N_1 - N_2)}{\partial t} = -3K \sqrt{N_1 N_2} \sin \phi_1. \]  
\tag{12} \]

After rescaling the time variable and the population imbalance, \( 2Kt/\hbar \to t, \ z \equiv \frac{N_1 - N_2}{NT} \), Eqs. (11) and (12) can be written as

\[ \frac{\partial z}{\partial t} = -\sqrt{1 - z - 2z^2} \sin \phi, \]  
\tag{13} \]

\[ \frac{\partial \phi}{\partial t} = \Lambda z + \frac{1}{2} \frac{4z + 1}{\sqrt{1 - z - 2z^2}} \cos \phi, \]  
\tag{14} \]

where \( \Lambda \equiv \frac{UN_T}{2K} \), \( \phi \equiv \phi_1 \).

If we treat \( z(t) \) and \( \phi(t) \) as a pair of canonically conjugate variables, it is not difficult to construct a Hamiltonian

\[ H = \Lambda z^2 - \sqrt{1 - z - 2z^2} \cos \phi, \]  
\tag{15} \]

satisfying the canonical equations,

\[ \dot{z} = -\frac{\partial H}{\partial \phi}, \]  
\tag{16} \]

\[ \dot{\phi} = \frac{\partial H}{\partial z}. \]  
\tag{17} \]

In Eq. (15), \( z < 0.5 \), consistent with the totally symmetric conditions. The fluctuations of the phases and the atom numbers must be small and has not been included here.

Now, let us discuss Eqs. (13) and (14) in more details.

(1) zero-phase mode. With initial conditions \( z(0) = 0.3 \) and \( \phi(0) = 0 \), we can solve Eqs. (13) and (14) numerically in zero-phase mode. Some results are given in Fig. 1. \( \Lambda \) takes on the values 10, 20, 38, 38.25 and 39 for Fig. 1(a), 1(b), 1(c), 1(d) and 1(e), respectively. The anharmonic nonsinusoidal oscillations are the generalized sinusoidal Josephson oscillations. \( z \) is asymmetric about its zero point because of the existence of the third trap, \( \langle z(t) \rangle \neq 0 \). As \( \Lambda \) increases, the oscillations around \( z = 0 \) become anharmonic [Figs. 1(a), 1(b) and 1(c)]. When \( \Lambda \) exceeds a critical value \( \Lambda_c = 38.25 \), the populations oscillates around some nonzero
values, and \( z(t) > 0 \) [Fig. 1(e)]. This population imbalance self-trapped effect is somewhat like the pendulum bob swing over the \( \phi = \pi \) vertical orientation for sufficiently large initial values. This state can be achieved from different approaches. In zero-phase mode, there is a pair of eigenfunctions of GPE, \( z = 0 \) and \( \phi = \pi \), and the ground state energy is \( E = 1 \). If the self-trapped effect takes place, \( z \neq 0 \) when \( \phi = \pi \), then

\[
H(z(0), \phi(0)) = \frac{\Lambda}{2} z(0)^2 - \sqrt{1 - z(0) - 2z(0)^2 \cos \phi(0)} > 1,
\]

so the critical values depend on the following condition

\[
\Lambda_c = 2 \left(1 + \sqrt{1 - z(0) - 2z(0)^2 \cos \phi(0)} \right) / z(0)^2.
\]

Here, \( z(0) = 0.3 \) and \( \phi(0) = 0 \), so \( \Lambda_c = 38.25 \).

(2) \( \pi \)-phase mode. If we select the initial conditions as \( \phi(0) = \pi \) and \( z(0) = 0.3 \), we can discuss the interference dynamics of \( \pi \)-phase oscillations from Eqs. (13) and (14). The numerical results are shown in Fig. 2. It is clear that the nonsinusoidal oscillations in \( \pi \)-phase mode are similar to that in zero-phase mode [Fig. 2(a)], but the self-trapped effects are not.

There are two kinds of self-trapped effects in \( \pi \)-phase mode \( \langle z(t) \rangle < z(0) \) and \( \langle z(t) \rangle > z(0) \) as given in Figs. 2(d) and 2(f). According to the degenerate GPE eigenstates that break the \( z \) symmetry, we can get the critical value of \( \Lambda_c \) for these two different self-trapped effects

\[
\Lambda_c = \frac{4z(0) + 1}{2z(0)\sqrt{1 - z(0) - 2z(0)^2}},
\]

here, \( z(0) = 0.3 \), so \( \Lambda_c = 5.08 \). If \( \Lambda \) exceeds the value of \( \Lambda_c \), the system goes from the first kind of self-trapped state [Fig. 2(d)] into the second kind of self-trapped state [Fig. 2(f)].

(3) \( z-\phi \) phase diagrams. \( z \) and \( \phi \) are two canonically conjugate variables in Eqs. (13) and (14), their dynamical behavior in \( z-\phi \) phase diagrams are shown in Fig. 3. In Figure 3(a), \( \Lambda = 10 \) and \( \phi(0) = 0 \), it is the zero-phase mode. For \( z(0) = 0.20 \), and 0.40, the trajectories are closed, asymmetric curves, corresponding to the generalized nonsinusoidal oscillations. For \( z(0) = 0.487 \), it is the critical case (here, for the zero-phase mode, we should solve the critical value of \( z \) from Eq. (15) with \( \Lambda = 10 \)). And for \( z(0) = 0.498 \), the
running self-trapped case, \( z(t) \) is locked but \( \phi(t) \) is unlocked, \(-\infty < \phi(t) < \infty\). In Fig. 3(b), \( \Lambda = 7 \) and \( \phi(0) = \pi \), the \( \pi \)-phase mode, the system is self-trapped for all values of \( z(0) \). For \( z(0) = 0.08 \), it is the running-mode self-trapped case, \( \phi \) is unbounded, \(-\infty < \phi(t) < \infty\), while for larger \( z(0) \), above a critical value of \( z_c = 0.111 \) according to Eq. (20) (here, for the \( \pi \)-phase mode, we should solve the critical value of \( z \) from Eq. (20) with \( \Lambda = 7 \)), \( \phi(t) \) is also localized around \( \pi \). So in Fig. 3(b), there are two kinds of self-trapped effects for \( z(0) = 0.08 \) and for \( z(0) = 0.111, 0.30 \) and \( 0.37 \) in \( \pi \)-phase mode, respectively.

The self-trapped effects in coupled BEC are the results of the nonlinearity of interatomic interaction in GPE and the long-range quantum coherence of a macroscopic number of atoms.

(4) \( z-\phi \) phase diagrams, the general case. From Eqs. (7), (8), (9) and (10) generally, after rescaling the atom population imbalance variables \( z_1 \equiv \frac{N_1 - N_3}{N_T} \) and \( z_2 \equiv \frac{N_1 - N_3}{N_T} \), and time \( 2Kt/\hbar \to t \), we can get four coupled dynamical equations

\[
\frac{\partial \phi_1}{\partial t} = \frac{1}{2} \sqrt{\frac{3z_1}{z_1 + z_2 + 1}} \cos \phi_1 + \frac{1}{2} \sqrt{\frac{z_1 + 1 - 2z_2}{z_2 + 1 - 2z_1}} \cos(\phi_1 - \phi_2)
\]

\[
+ \frac{1}{3} \Delta(z_1 + z_2 + 1) - \frac{1}{3} \Delta(z_2 + 1 - 2z_1), \tag{21}
\]

\[
\frac{\partial \phi_2}{\partial t} = -\frac{1}{2} \sqrt{\frac{z_2 + 1 - 2z_1}{z_1 + z_2 + 1}} \cos \phi_1 + \frac{1}{2} \sqrt{\frac{z_2 + 1 - 2z_1}{z_1 + 1 - 2z_2}} \cos(\phi_1 - \phi_2)
\]

\[
+ \frac{1}{3} \Delta(z_1 + z_2 + 1) - \frac{1}{3} \Delta(z_1 + 1 - 2z_2), \tag{22}
\]

\[
\frac{\partial z_1}{\partial t} = -\frac{2}{3} \sqrt{\frac{(z_1 + z_2 + 1)(z_2 + 1 - 2z_1)}{z_1 + 1 - 2z_2}} \sin \phi_1
\]

\[
-\frac{1}{3} \sqrt{(z_2 + 1 - 2z_1)(z_1 + 1 - 2z_2)} \sin(\phi_1 - \phi_2), \tag{23}
\]

\[
\frac{\partial z_2}{\partial t} = -\frac{1}{3} \sqrt{\frac{(z_1 + z_2 + 1)(z_2 + 1 - 2z_1)}{z_2 + 1 - 2z_2}} \sin \phi_1
\]

\[
+\frac{1}{3} \sqrt{(z_2 + 1 - 2z_1)(z_1 + 1 - 2z_2)} \sin(\phi_1 - \phi_2). \tag{24}
\]

Here, for simplicity, symmetric parameters are also considered, \( U_1 = U_2 = U_3 \equiv U \), \( E_1^0 = E_2^0 = E_3^0 \), and \( \Delta \equiv \frac{U N_T}{2K} \). Besides the nonsinusoidal oscillations and self-trapped effects, we can find another novel phenomenon between three trapped BEC. With some initial conditions \((\phi_1(0), \phi_2(0), z_1(0), z_2(0) \) and \( \Delta \)), the trajectory of \( z-\phi \) phase-diagrams \((z_1-\phi_1,
$z_2 - \phi_2$ are not closed. They are unclosed for general initial conditions, but closed for some special initial values. As shown in Fig. 4, we fix the values of $\phi_1(0) = 0$, $\phi_2(0) = 0$, $z_1(0) = 0.3$ and $z_2(0) = 0.1$, and let $\Lambda$ varied. When $\Lambda = 2$ [Fig. 4(a) and 4(b)] and $\Lambda = 9$ [Fig. 4(e) and 4(f)], the $z-\phi$ phase-diagrams are unclosed. But for $\Lambda = 6.8$ [Fig. 4(c) and 4(d)], they are convergent and closed. These phase-diagrams exhibit a transition from unclosure to closure, then to unclosure again during changing the values of $\Lambda$. Only a special value of $\Lambda$ can make phase-diagrams close under some values of $\phi_1(0)$, $\phi_3(0)$, $z_1(0)$ and $z_2(0)$. This property can be used to determine the value of $\Lambda$ in experiments.

This feature of phase-diagrams originates from two aspects. One is the nonlinearity of the interatomic interaction. It is known that the phase-diagram of a nonlinearity system is always not closed. Eqs. (21), (22), (23) and (24) are four coupled nonlinear equations, so the phase-diagrams are generally unclosed. The other is the interaction between three trapped BEC. The interference of BEC in a macroscopic number of atoms determine the properties of phase-diagrams. The detail study on this complicated problem would be a special topic elsewhere.

In summary, the interference of three trapped BEC induces population nonsinusoidal oscillations which is a generalization of sinusoidal Josephson effect in superconductivity. The quantum self-trapped effects occur in three BEC for both zero-phase mode (one kind of self-trapped state) and $\pi$-phase mode (two kinds of self-trapped states) when parameters exceed some critical values. Any observation of the predicted interference effects would unambiguously prove the existence of gauge symmetry broken and the relative phase between condensates. This consideration, concerning the interference between three Bose condensates, allows the numerical simulation of some recent experiments. From calculations, we have inferred that the convergence of the phase diagrams is sensitive to the adjustment of the system parameters. The phase-diagrams are not closed in general cases, but for some special conditions, the phase-diagrams are convergent and closed, which can be used to determine the interaction parameter between atoms of BEC in experiments.
REFERENCES

1 M. H. Anderson et al., Science 269 (1995) 198.

2 K. B. Davis et al., Phys. Rev. Lett. 75 (1995) 3969.

3 C. C. Bradley et al., Phys. Rev. Lett. 75 (1995) 1687.

4 A. J. Leggett, F. Sols, Found. Phys. 21 (1991) 253.

5 For a review, see F. Dalfovo et al., Rev. Mod. Phys. 71 (1999) 463.

6 J. Javanainen, S. M. Yoo, Phys. Rev. Lett. 76 (1996) 161.

7 M. Lewenstein, L. You, Phys. Rev. Lett. 77 (1996) 3489.

8 F. Dalfovo, L. Pitaevskii and S. Stringari, Phys. Rev. A 54 (1996) 4213.

9 M. Naraschewski et al., Phys. Rev. A 54 (1996) 2185.

10 H. Wallis et al., Phys. Rev. A 55 (1997) 2109.

11 G. J. Milburn et al., Phys. Rev. A 55 (1997) 4318.

12 J. I. Cirac et al., Phys. Rev. A 54 (1996) 3714.

13 T. Wang, M. J. Collett and D. F. Walls, Phys. Rev. A 54 (1996) 3718.

14 M. W. Jack, M. J. Collett and D. F. Walls, Phys. Rev. A 54 (1996) 4625.

15 A. Smerzi et al., Phys. Rev. Lett. 79 (1997) 4950.

16 S. Raghavan et al., Phys. Rev. A 59 (1999) 620.

17 W. Hoston, L. You, Phys. Rev. A 53 (1996) 4254.

18 A. Röhrle et al., Phys. Rev. Lett. 78 (1997) 4143.

19 J. Williams et al., Phys. Rev. A 59 (1999) 31.

20 M. R. Andrews et al., Science 275 (1997) 637.
21 D. S. Hall et al., Phys. Rev. Lett. 81 (1998) 1543.

22 D. S. Hall et al., Phys. Rev. Lett. 81 (1998) 1539.

23 M. Matthews et al., Phys. Rev. Lett. 81 (1998) 243.

24 N. Ramsey, Molecular Beams, Clarendon Press, Oxford, 1956.

25 J. Stenger et al., e-print cond-mat/9901072.

26 I. Zapata et al., Phys. Rev. A 57 (1998) 28

27 L. P. Pitaevskii, Sov. Phys. JETP 13 (1961) 451; E. P. Gross, Nuovo Cimento 20 (1961) 454.

28 R. P. Feynman, Statistical Mechanics, Addison-Wesley, Redwood City, 1972.

29 A. Barone, G. Paterno, Physics and Applications of the Josephson Effect, Wiley, New York, 1982.

30 D. Kaplan, L. Glass, Understanding Nonlinear Dynamics, Springer-Verlag, Berlin, 1995.

31 S. Zhang, to be submitted.
FIGURES

FIG. 1. Rescaled population imbalance $z(t)$ versus dimensionless time variable $t$, with initial conditions $z(0) = 0.3$ and $\phi(0) = 0$ in zero-phase mode. The interaction parameter $\Lambda$ takes the values (a) 10, (b) 20, (c) 38, (d) 38.25 and (e) 39.

FIG. 2. Rescaled population imbalance $z(t)$ versus dimensionless time variable $t$, with initial conditions $z(0) = 0.3$ and $\phi(0) = \pi$ in $\pi$-phase mode. The interaction parameter $\Lambda$ takes the values (a) 2, (b) 4, (c) 5, (d) 5.05 (e) 5.08 and (f) 5.1.

FIG. 3. Phase diagrams of population imbalance $z$ versus phase difference $\phi$. (a) and (b) are corresponding to zero-phase mode $\phi(0) = 0$ with $\Lambda = 10$ and $\pi$-phase mode $\phi(0) = \pi$ with $\Lambda = 7$, respectively. The values of $z(0)$ are as given.

FIG. 4. Phase diagrams of population imbalance versus phase difference in general case, with $\phi_1(0) = 0$, $\phi_2(0) = 0$, $z_1(0) = 0.3$ and $z_2(0) = 0.1$. (a), (c) and (e) for $z_1 - \phi_1$ phase diagrams, while (b), (d) and (f) for $z_2 - \phi_2$ phase diagrams. The dimensionless time $-65 < t < 65$. $\Lambda$ takes the values 2 for (a) and (b), 6.8 for (c) and (d), 9 for (e) and (f).
