Optimal Online Algorithms for One-Way Trading and Online Knapsack Problems: A Unified Competitive Analysis

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Abstract—We study two canonical online optimization problems under capacity/budget constraints, the fractional one-way trading problem (OTP) and the integral online knapsack problem (OKP) with an infinitesimal assumption. Under the competitive analysis framework, it is well-known that both problems have the same optimal competitive ratio. However, these two problems are investigated by distinct approaches under separate contexts in the literature. There is a gap in understanding the connection between these two problems and the nature of their online algorithm design. This paper provides a unified framework for the online algorithm design, analysis and optimality proof for both problems. We find that the infinitesimal assumption of the OKP is the key that connects the OTP in the analysis of online algorithms and the construction of worse-case instances. With this unified understanding, our framework shows its potential for analyzing other extensions of OKP and OTP in a more systematic manner.

I. INTRODUCTION

Online optimization under capacity/budget constraints is a classical and challenging problem. Two well-known examples are the one-way trading problem (OTP) and the online knapsack problem (OKP).

In the OTP, an investor plans to trade a total amount of 1 dollar into yen. The exchange rates $p_i$ arrive online and are bounded, i.e., $p_i \in [L, U]$. The investor must immediately decide how much to trade at each exchange rate. If $x_i$ dollars are traded at the $i$th exchange rate $p_i$, $p_ix_i$ is the amount of yen the investor gains. The goal is to maximize the amount of yen traded after processing the $N$th exchange rate $\sum_{i=1}^{N} p_ix_i$, while respecting the budget limit $\sum_{i=1}^{N} x_i \leq 1$. It is well-known that $(\ln(U/L)+1)$-competitive algorithms can be designed, e.g., the threat-based algorithm in [1] and the CR-Pursuit algorithm in [2].

The 0-1 knapsack problem is a classic problem in computer science, where a decision maker selects a best subset of items that maximizes the total value of the knapsack contents without exceeding the normalized capacity limit 1. Whereas in the OKP, the items arrive online. The value $v_i$ and weight $w_i$ of the $i$th item are only revealed upon its arrival. An online decision is made on whether to accept the item $(z_i = 1)$ or not $(z_i = 0)$. There exist no online algorithms with bounded competitive ratios for the OKP in the general setting [3]. However, $(\ln(U/L)+1)$-competitive algorithms can be designed [4] under the infinitesimal assumption that the weight of each item is much smaller than the capacity (i.e., $\max_i w_i \ll 1$), and the bounded value-to-weight ratio assumption (i.e., $v_i/w_i \in [L, U]$). The infinitesimal assumption is a technical simplification but has been shown to hold in practical applications like cloud computing systems [5] and widely accepted in the literature.

In this paper, we refer to the OKP with the infinitesimal assumption as OKP.

Both problems and their many variants have appeared in numerous applications, such as portfolio selection, cloud resource allocation, and keyword auctions, and thus have attracted considerable attention. A typical variant for both problems is to assume that the arrivals follow certain distributions[6][7] or come in random order[8] in order to circumvent the analysis of the worst-case scenario. For the OTP, unbounded prices [9] and interrelated prices [10] have been considered recently; for the OKP, knapsacks with unknown capacity[11] and removable items[12] are interesting generalizations.

Motivated by the gaps in understanding the nature of challenges in the online algorithm design, we aim to unify the online algorithms for the OTP and the OKP into a threshold-based algorithm, in which the competitive performance of the algorithm mainly depends on the threshold function. We provide a sufficient condition on the threshold function that can ensure a bounded competitive ratio, and design the best possible threshold function based on this sufficient condition. Finally, we derive the lower bound of the competitive ratios of the OTP and the OKP. Although all results match those in the literature, the existing works approach the results by distinct methods and lack a systematic way of designing and analyzing related problems. Instead of the results, this paper mainly focus on the analysis and proofs. Our contributions are two-fold.

- We unify the online algorithms for the OTP and the OKP into a threshold-based algorithm and show that the unified algorithm can achieve the optimal competitive ratios under a unified competitive analysis.
- We provide new proofs for the lower bound of competitive ratios for the OTP and the OKP. The connection between these two problems is founded in the construction of the worse-case instances.

II. A UNIFIED ALGORITHM / OUR RESULTS

A. Notations

Since the two problems have distinct sets of terms originally, we unify the notations for the brevity of problem formulation and clarify the different meanings here. Let $x_i$ denote the amount of dollars traded at the $i$th exchange rate for the OTP, while for the OKP, it represents the capacity used after processing the $i$th item $w_i z_i$. Let $b_i$ denote the exchange rate $p_i$ in the OTP, and the value-to-weight ratio of the $i$th item in the OKP, i.e., $\frac{v_i}{w_i}$. The LP that characterizes
the offline problem of the OTP under the unified notation is:

$$\begin{align*}
\text{maximize} & \quad \sum_{i=1}^{N} b_i x_i \\
\text{subject to} & \quad \sum_{i=1}^{N} x_i \leq 1 \\
& \quad x_i \geq 0, \forall i \in [N]
\end{align*}$$

(1)

The dual problem of (1) is

$$\begin{align*}
\text{minimize} & \quad \lambda \\
\text{subject to} & \quad \lambda \geq b_i, \forall i \in [N]
\end{align*}$$

(2)

Change the last constraint of (1) to $0 \leq x_i \leq w_i$, then the resulting LP serves as an upper bound of the OKP, and its dual is

$$\begin{align*}
\text{minimize} & \quad \lambda + \sum_{i=1}^{N} w_i \beta_i \\
\text{subject to} & \quad \lambda + \beta_i \geq b_i, \forall i \in [N] \\
& \quad \lambda \geq 0, \beta_i \geq 0, \forall i \in [N]
\end{align*}$$

(3)

### B. A Unified Algorithm

Both the OTP and the OKP target to allocate one budget-constrained resource sequentially. Since the current decision affects the future decision through the budget constraint, we need an estimation on the value of the remaining resource to facilitate decision-making. Our idea is to use a threshold function to estimate the marginal cost of a resource at utilization.

**Definition 1.** A threshold function $\phi(y) : [0, 1] \rightarrow [0, \infty)$ estimates the marginal cost of a resource at utilization $y$.

Given $\phi(y)$, we can estimate the pseudo-cost of allocating $x_i$ amount of resource by $\int_{y^{(i)-1}+x} y^{(i)-1} \phi(\delta)d\delta$. Our unified algorithm then decides $x_i$ that can maximizes the pseudo-revenue $b_i x_i - \int_{y^{(i)-1}+x} y^{(i)-1} \phi(\delta)d\delta$. The overall algorithm is summarized in Algorithm 1. For the OTP, $S$ is the set of positive real numbers, whereas $S$ is the set $\{0, w_i\}$ for the OKP. Note that step 3 is only necessary for the OKP, and then step 1 reduces to $x_i = \begin{cases} w_i, & b_i \geq \phi(y^{(i)-1}) \\ 0, & \text{otherwise} \end{cases}$, which corresponds to the update equation in [4]. Algorithm 1 can be easily applied in the posted-price setting by its nature.

### C. Main Results

A standard measure for the performance of an online algorithm is the competitive ratio. Under the unified notation, define an arrival instance $A$ as $\{b_i\}_{i \in [N]}$ for the OTP, and as $\{b_i, w_i\}_{i \in [N]}$ for the OKP. Given the arrival instance $A$, denote the objective value achieved by the online algorithm and the offline optimal by $\text{ALG}(A)$ and $\text{OPT}(A)$, if

$$\alpha = \max_A \frac{\text{OPT}(A)}{\text{ALG}(A)}$$

then we say the online algorithm is $\alpha$-competitive. The competitive ratio of Algorithm 1 only depends on the choice of the function $\phi$. We find the sufficient conditions of $\phi$ for Algorithm 1 to be $\alpha$-competitive as in the following theorem.

**Theorem 2** (Sufficiency). Algorithm 1 is $\alpha$-competitive for both the OTP and the OKP if $\phi$ is given by

$$\phi(y) = \begin{cases} L & y \in [0, \omega] \\ \phi(y) & y \in [\omega, 1] \end{cases},$$

where $\omega$ is a budget/capacity utilization level that satisfies $\frac{1}{\alpha} \leq \omega \leq 1$, and $\phi(y)$ is an increasing function that satisfies

$$\begin{align*}
\phi(y) & \geq \frac{1}{\alpha} \phi'(y), y \in [\omega, 1] \\
\phi(\omega) & \geq L, \phi(1) \geq U.
\end{align*}$$

(5)

As is shown in the theorem, $\phi$ is composed of two segments, one constant and the other exponential. Note that the functions used in [4][5] satisfy the conditions. However, they come up with the function from nowhere. In contrast, by the following theorem, we can characterize the performance limit over the space of eligible functions and rigorously show the function that admits the smallest (best) competitive ratio.

**Theorem 3.** Given $L$ and $U$, the best competitive ratio that can be achieved by Algorithm 1 is $(\ln \theta + 1)$, and the corresponding $\phi^*$ is unique, where $\theta = U/L$.

We show that no other online algorithm can do better than Algorithm 1 by the following theorem.

**Theorem 4.** Given $L$ and $U$, $(\ln \theta + 1)$ is the lowest possible competitive ratio for both the OTP and the OKP.

In the next section, we introduce the primal-dual analysis framework, with which we prove Theorem 2. Subsequently, we prove Theorem 3 by the Gronwall’s inequality. In Section IV, we show Theorem 4 by adversarial arguments.

### III. COMPETITIVE ANALYSIS

**A. Primal-Dual Competitive Analysis**

Given the arrival instance $A$, we denote the primal and dual objective values after processing $b_n$ by $P_n(A)$ and $D_n(A)$, respectively. For simplicity, we drop the parenthesis and write $P_n$ and $D_n$ hereinafter. We briefly introduce the framework by giving the following lemma.

**Lemma 1.** An online algorithm is $\alpha$-competitive if it can determine the primal variables $x$ and construct dual variables $\lambda$ based on the primal variables such that
• (Feasible Solutions) $x$ and $\lambda$ are feasible solutions of the primal and the dual.
• (Initial Inequality) there exists an index $k \in [N] \cup \{0\}$ such that $P_k \geq \frac{1}{\alpha} D_k$.
• (Incremental Inequalities) for $i \in \{k + 1, \ldots, N\}$,
\[ P_i - P_{i-1} \geq \frac{1}{\alpha}(D_i - D_{i-1}). \]

**Proof.** The primal feasibility is trivial since any online algorithm must first produce a feasible solution to the problem. It suffices to prove $P_N \geq \frac{1}{\alpha} D_N$ since
\[ ALG = P_N \geq \frac{1}{\alpha} D_N \geq \frac{1}{\alpha} D^* \geq \frac{1}{\alpha} OPT, \]
where (a) is due to the dual feasibility, and (b) is due to the weak duality. Suppose there exists an $k$ such that $P_i - P_{i-1} \geq \frac{1}{\alpha}(D_i - D_{i-1})$ holds for all $i \in \{k + 1, \ldots, N\}$, then we have $P_N - P_k \geq \frac{1}{\alpha}(D_N - D_k)$. Combining it with the initial inequality, it leads to $P_N \geq \frac{1}{\alpha} D_N$. We thus complete the proof. \[ \square \]

Note that the primal-dual competitive analysis framework that we use are more general than those used in the existing works, in that the initial inequality starts from $k \in [N] \cup \{0\}$ rather than the original 0.

Next we show the proofs of Theorems 2 and 3 for the OTP, and highlight the differences between them and the OKP case.

**B. Analysis of OTP**

**Proof of Theorem 2 (Feasible Solutions)** First we show that the primal and dual solutions given by Algorithm 1 are feasible,
\[ x_i = \begin{cases} \phi^{-1}(b_i) - y^{(i-1)} & b_i \geq \phi(y^{(i-1)}) \\ 0 & b_i < \phi(y^{(i-1)}) \end{cases}, \]
where $\phi^{-1}(b) = \begin{cases} \omega & b = L \\ \phi^{-1}(b) & b > L \end{cases}$ ensures $\forall i, x_i \geq 0$, and $\phi(1) \geq U$ ensures $\phi^{-1}(U) \leq 1$. Since $y^{(N)} = \phi^{-1}([\max_{i \in [N]} b_i])$, we have $y_N \leq \phi^{-1}(U) \leq 1$. Thus the primal solutions are feasible. Construct the dual variables as
\[ \lambda_i = \phi(y^{(i)}). \]
Since $\phi(y)$ is non-decreasing, $\lambda_N = \phi(y^{(N)}) \geq \phi(y^{(i)}), \forall i \in [N]$. Thus $\lambda_N$ is a feasible solution to the dual.

**Initial Inequality** For the OTP, $P_0 = 0, D_0 = \phi(y^{(0)}) = L > 0$. When $k \geq 1$, the primal objective at the end of the $k$th time slot is $P_k = \sum_{i=1}^{k} b_i x_i$, while the dual objective is $D_k = \lambda_k = \phi(y^{(k)})$.

Since $y^{(0)} = 0$ and $\forall i, \phi(0) = L \leq b_i$, by (6), we have
\[ x_1 = \phi^{-1}(b_1) = \begin{cases} \omega & b_1 = L \\ \phi^{-1}(b_1) & b_1 > L. \end{cases} \]
Because $\phi(y)$ is an increasing function, we have
\[ x_1 \geq \omega \geq \frac{1}{\alpha}. \]

Since $b_1 = \phi(x_1)$, it follows that
\[ P_1 = b_1 x_1 \geq \frac{b_1}{\alpha} = \frac{1}{\alpha} \phi(x_1) = \frac{1}{\alpha} D_1. \]

(Incremental Inequalities) Next we show the incremental inequalities for $i \geq 1$. Note that when $x_i > 0, \forall i > 1$, when the behavior of the algorithm is controlled by the second segment of $\phi$, which satisfies $\phi(y) \geq \frac{1}{\alpha} \phi'(y)$ for $y \in [\omega, 1]$, and two boundary conditions $\phi(\omega) = L$ and $\phi(1) \geq U$.

The change in the primal objective is given as follows:
\[ P_i - P_{i-1} = b_i x_i^{(a)} = \phi(y^{(i)}) x_i, \]
where (a) is due to (6) and $y^{(i)} = y^{(i-1)} + x_i$.

The change in the dual objective is given as follows:
\[ D_i - D_{i-1} = \lambda_i - \lambda_{i-1} = \phi(y^{(i)}) - \phi(y^{(i-1)}) = \phi(y^{(i)}) - \phi(y^{(i-1)}). \]

By the Cauchy mean value theorem, for every segment $[y^{(i-1)}, y^{(i)}]$, there exists a $\delta_i \in [y^{(i-1)}, y^{(i)}]$ such that
\[ \frac{\phi(y^{(i)}) - \phi(y^{(i-1)})}{y^{(i)} - y^{(i-1)}} = \phi'(\delta_i). \]

Since $\forall y \in [\omega, 1], \phi(y) \geq \frac{1}{\alpha} \phi'(y)$, and $\phi(y)$ is increasing, we have
\[ \alpha \phi(y^{(i)}) \geq \alpha \phi(\delta_i) \geq \frac{\phi(y^{(i)}) - \phi(y^{(i-1)})}{y^{(i)} - y^{(i-1)}}. \]

Because $y^{(i)} - y^{(i-1)} > 0$, we have
\[ \phi(y^{(i)}) - y^{(i)} > 0 \]
\[ \phi(y^{(i)}) - y^{(i)} \geq \frac{1}{\alpha} (\phi(y^{(i)}) - \phi(y^{(i-1)})), \]
where the LHS is $P_i - P_{i-1}$, and the RHS is $\frac{1}{\alpha}(D_i - D_{i-1})$. Thus $P_i - P_{i-1} \geq \frac{1}{\alpha} (D_i - D_{i-1})$ holds for all $i > 1$.

Therefore, Theorem 2 holds for the OTP. \[ \square \]

Theorem 3 characterizes the performance limit of Algorithm 1.

**Proof of Theorem 3 (Best Competitive Ratio)** By the differential form of the Gronwall’s Inequality[13], if there exists a $\varphi$ that satisfies
\[ \varphi(y) \geq \frac{1}{\alpha} \varphi'(y), y \in [\omega, 1], \]
where $\omega \in [\frac{L}{\alpha}, 1]$, it is bounded as follows:
\[ \varphi(y) \leq \phi(\omega) \exp \left\{ \int_{\omega}^{y} \alpha dt \right\}, y \in [\omega, 1]. \]

Substituting the first boundary condition $\phi(\omega) = L$, we have
\[ \varphi(y) \leq L \exp \left\{ (\alpha(y - \omega)) \right\}, y \in [\omega, 1]. \]

If the other boundary condition $\phi(1) \geq U$ holds, it implies
\[ L \exp \left\{ (\alpha(1 - \omega)) \right\} \geq U, \]
otherwise $\varphi(1) \leq L\exp\{ (\alpha(1 - \omega)) \} < U$, which incurs infeasibility. From the inequality above, we have

$$\omega \leq 1 - \frac{1}{\alpha} \ln \theta.$$ 

A necessary condition for $\omega \geq \frac{1}{\alpha}$ to hold is

$$1 - \frac{1}{\alpha} \ln \theta \geq \frac{1}{\alpha},$$

and thus the competitive ratio $\alpha \geq \ln \theta + 1$.

(\Phi^* and Its Uniqueness) When $\alpha$ takes the smallest possible $\alpha^* = \ln \theta + 1$, the corresponding $\phi^*$’s satisfy

$$\phi^*(y) = \begin{cases} L & y \in [0, \omega], \\
\varphi(y) & y \in [\omega, 1], \end{cases}$$

where $\omega \in [\frac{1}{\ln \theta + 1}, 1]$ and $\varphi$’s are given by

$$\begin{cases} \varphi^*(y) \geq \frac{1}{\ln \theta + 1}\varphi(y), & y \in [\omega, 1] \\
\varphi^*(\omega) = L, \varphi^*(1) \geq U. \end{cases}$$

By the Gronwall’s inequality, we have

$$\varphi^*(y) \leq L\exp\{ (\ln \theta + 1)(y - \omega) \} \tag{8}$$

where $(a)$ is due to $\omega \geq \frac{1}{\ln \theta + 1}$. Then we have

$$\varphi^*(1) \leq L\exp(\ln \theta) = L\theta = U.$$

Combining with the second boundary condition $\varphi^*(1) \geq U$, we have $\varphi^*(1) = U$. Substituting it into (8), we have

$$L\exp\{ (\ln \theta + 1)(1 - \omega) \} \geq U,$$

$$1 - \omega \geq \frac{\ln \theta}{\ln \theta + 1},$$

$$\omega \leq \frac{1}{\ln \theta + 1}.$$

Because $\omega \geq \frac{1}{\ln \theta + 1}$, we have $\omega = \frac{1}{\ln \theta + 1}$. Therefore, the solution space of (7) is equivalent to the solution space of the following differential equation counterparts with equality boundary conditions:

$$\begin{cases} u(y) \geq \frac{1}{\ln \theta + 1}u'(y), & y \in [\frac{1}{\ln \theta + 1}, 1] \\
u\left(\frac{1}{\ln \theta + 1}\right) = L, u(1) = U. \end{cases} \tag{9}$$

The differential equation counterpart is as follows:

$$\begin{cases} v(y) = \frac{1}{\ln \theta + 1}v'(y), & y \in [\frac{1}{\ln \theta + 1}, 1] \\
v\left(\frac{1}{\ln \theta + 1}\right) = L, v(1) = U. \end{cases} \tag{10}$$

Thus the differential equation holds.

Based on the online decision rule (4), the online algorithm will accept the first $k$ items, where $\sum_{i=1}^{k} w_i = \omega$. Also note that $\lambda_i = L, \forall i \in [N]$. Thus the dual feasibility holds.

Take integral of $u'$ over $[\frac{1}{\ln \theta + 1}, 1]$, we have

$$\int_{\frac{1}{\ln \theta + 1}}^{1} u' = u\left|^{1}_{\frac{1}{\ln \theta + 1}} = U - L, \right.$$ 

Meanwhile, it can be expressed as

$$\int_{\frac{1}{\ln \theta + 1}}^{1} u' = \int_{\frac{1}{\ln \theta + 1}}^{1} u' + \int_{\frac{1}{\ln \theta + 1}}^{1} u' \tag{11}$$

$$\leq \int_{\frac{1}{\ln \theta + 1}}^{1} v' = v\left|^{1}_{\frac{1}{\ln \theta + 1}} = U - L, \right.$$ 

which shows $U - L < U - L$. Thus $u(y) = v(y)$ for $y \in [\frac{1}{\ln \theta + 1}, 1]$. In conclusion, the optimal $\phi^*$ achieving competitive ratio $(\ln \theta + 1)$ is unique and

$$\phi^*(y) = \begin{cases} L & y \in [0, \frac{1}{\ln \theta + 1}] \\
e^{(\ln \theta + 1)y} & y \in \left(\frac{1}{\ln \theta + 1}, 1\right]. \end{cases} \tag{12}$$

C. Analysis of OKP

We highlight the differences in the analysis of the OKP. The primal feasibility holds trivially and the dual variables are constructed as follows:

$$\lambda = \lambda_N, \quad \beta_i = \begin{cases} b_i - \lambda_i & x_i = w_i \\
0 & x_i = 0 \end{cases},$$

where $\lambda_i = \phi(y^{(i-1)})$. When $x_i = w_i$, based on the decision-making rule (4), we must have $b_i \geq \phi(y^{(i-1)})$. Therefore, $\beta_i \geq 0, \forall i \in [N]$. The constraint of the dual problem is

$$\lambda + \beta_i - b_i = \begin{cases} \lambda - \lambda_i \geq 0 & x_i = w_i \\
\lambda - b_i \geq 0 & x_i = 0. \end{cases}$$

Thus the dual feasibility holds.

The unique solution to (12) is $v(y) = L e^{(\ln \theta + 1)y}$. By the Gronwall’s inequality, any feasible solution of (9) is bounded by $v(y)$ from above. Next, we are going to show that the solution of (9) is unique and is exactly $v(y)$.

Suppose $u$ is a feasible solution to (9) and $u(y) < v(y)$ for $y \in I$, where $I \subset \left[\frac{1}{\ln \theta + 1}, 1\right]$. We know that for any $y \in \left[\frac{1}{\ln \theta + 1}, 1\right]$,

$$v' = (\ln \theta + 1)v,$$

so for $y \in I$, we have

$$v' = (\ln \theta + 1)v > (\ln \theta + 1)u \geq u'.$$
where $(a)$ is due to the fact $\phi'(y^{i-1}) = \phi(y^{i-1}) - \phi(y^{i-1})$, and $w_i = y^i - y^{i-1}$ (using the infinitesimal weight assumption). Combining the ODE (5) with the fact that $b_i \geq \phi(y^{i-1})$, the incremental inequality holds for $i \in [N]$. Thus Theorem 2 holds for the OKP. Note that the proof of Theorem 3 holds generally for the two problems.

IV. LOWER BOUNDS

In this section, we show that the lower bound of the OTP and that of the OKP coincide. Denote Algorithm 1 with $\phi^*$ by ALG$^*$. We follow the same approach, first present the proofs for the OTP, and call attention to the differences for the OKP case.

A. Lower Bounds of OTP

Below we find the family of the worst-case sequences under which ALG$^*$ incurs a ratio of $\ln \theta + 1$.

Lemma 2. Given $L$ and $U$, the family of the worst-case sequences of ALG$^*$ in the OTP are denoted by $\{\hat{b}_k\}_{k \in \mathbb{N}}$, where $\hat{b}_k = \hat{b}_i + \epsilon_i$, $b_i \in [L, U]$ and the rates satisfy

$$\hat{b}_1 = L, \hat{b}_i = \hat{b}_{i-1} + \epsilon_{i-1}, i > 1, \lim_{k \to \infty} \hat{b}_k = U,$$

where $\epsilon_i$s are infinitesimal positive values. The amount traded by ALG$^*$ at the exchange rate $\hat{b}_i$ is denoted by $\hat{x}_i$ and satisfy

$$\hat{x}_1 = \frac{1}{\ln \theta + 1}, \hat{x}_i = \frac{\ln \hat{b}_i / \ln \theta + 1}{\hat{x}_{i-1}}, i > 1, \lim_{k \to \infty} \sum_{i=1}^{k} \hat{x}_i = 1.$$

Proof. The proof of Lemma 2 is in the Appendix.

Proof of Theorem 2 Let ALG be any online algorithm different from ALG$^*$. We show that ALG cannot achieve a competitive ratio smaller than $\ln \theta + 1$ by using an adversarial argument.

Let $\delta = \{L, L + \epsilon_1, \ldots, U\}$. First present $\hat{b}_1 = L$ to ALG. If ALG exchanges $x'_1 < \hat{x}_1 = 1/\ln \theta + 1$, then we end the sequence. If so, ALG cannot achieve $\ln \theta + 1$, because

$$\frac{\text{OPT}(\hat{b}_1)}{\text{ALG}(\hat{b}_1)} = \frac{1}{x'_1} > \ln \theta + 1.$$

Thus we can assume that ALG spends an amount $x'_1 \geq \hat{x}_1$, in this case, we continue and present $\hat{b}_2$ to ALG. In general, if after processing the $k$th exchange rate, the total amount of dollar spent is less than $\sum_{i=1}^{k} \hat{x}_i$, we immediately end the sequence. Otherwise, we continue to present $\hat{b}_{k+1}$, etc.

Let $f(k) = \sum_{i=1}^{k} (x'_i - \hat{x}_i)$. Let $\mathcal{K} = \{k \in \mathbb{N} | f(k) < 0\}$, denote the minimum in $\mathcal{K}$ as $j$, we have

$$x'_1 \geq \hat{x}_1,$$

$$x'_1 + x'_2 \geq \hat{x}_1 + \hat{x}_2, \ldots$$

$$\sum_{i=1}^{j-1} x'_i \geq \sum_{i=1}^{j-1} \hat{x}_i,$$

$$\sum_{i=1}^{j} x'_i \geq \sum_{i=1}^{j} \hat{x}_i,$$

$$\sum_{i=1}^{j} x'_i < \sum_{i=1}^{j} \hat{x}_i.$$

Thus ALG could gain more by spending exactly the same as ALG$^*$ at the first $(j - 1)$ exchange rates and by spending

$$x'_j = x'_j + \sum_{i=1}^{j-1} (x'_i - \hat{x}_i)$$

at the $j$th exchange rate. Since $x'_j < \hat{x}_j$, ALG cannot guarantee the competitive ratio of $\ln \theta + 1$. If $f(k) \geq 0$ for all $k \in \mathbb{N}^+$, we have

$$\liminf_{k \to \infty} f(k) \geq 0, \limsup_{k \to \infty} f(k) \geq 0.$$

Since ALG cannot exceed the capacity limit, we have

$$\lim_{k \to \infty} \sum_{i=1}^{k} x'_i \leq 1,$$

and also have $\lim_{k \to \infty} \sum_{i=1}^{k} \hat{x}_i = 1$, therefore we have

$$\lim_{k \to \infty} f(k) \leq 0.$$

For an infinite sequence $f(k)$, the limit exists iff

$$\limsup_{k \to \infty} f(k) = \liminf_{k \to \infty} f(k) = \lim f(k),$$

so we have $\lim_{k \to \infty} f(k) = 0$. By the Abel transformation, we have

$$\sum_{i=1}^{k} \hat{b}_i (x'_i - \hat{x}_i) = \sum_{i=1}^{k-1} f(i) (\hat{b}_i - \hat{b}_{i+1}) + f(k) \hat{b}_k$$

$$\leq f(k) \hat{b}_k,$$

where (1) is due to the monotonicity of $\{\hat{b}_i\}$.

Thus, the performance gap between ALG and ALG$^*$ for this infinite exchange rate sequence is

$$\lim_{k \to \infty} \sum_{i=1}^{k} \hat{b}_i (x'_i - \hat{x}_i) \leq \lim_{k \to \infty} f(k) \hat{b}_k = 0.$$

Therefore, any online algorithm for the OTP cannot achieve a better competitive ratio than ALG$^*$. The lowest possible competitive ratio is $\ln \theta + 1$.

B. Lower Bounds of OKP

We show that with a slight modification to $\{\hat{b}_k\}_{k \in \mathbb{N}}$, they are also the worst-case sequences for the OKP.

Consider a family of the value-to-weight ratio sequences $\{b_k\}$ indexed by $b \in [L, U]$. $b_k$ is composed of a continuum of subsequences, with the ratios in the $i$th subsequence all being $\hat{b}_i, i \in \mathbb{N}^+$, where $\hat{b}_i \leq b$ and is given in Lemma 2. The length of each subsequence is large enough so that it can fulfill the capacity of the knapsack even if presented alone. Note that given $b_k$, the resource allocation strategy is analogous to the OTP case. The offline optimal solution is to only select from the subsequence with $\hat{b}_i = b$ until reaching the capacity limit, whereas ALG$^*$ will select a value-to-weight ratio as long as it is no less than $\phi^*(y)$, where $y$ is the current capacity utilization level. Therefore $\{b_k\}_{b \in [L, U]}$ are the worst-case sequences for the OKP.
In regard to the proof of Theorem 4, one can replace the worst sequence \( \delta \) with \( I_{L,U} \), present a substitute instead of an arrival at a time to ALG, and act adversely in the same way in response to the decisions made by ALG, and the results still hold.

**APPENDIX**

**Proof of Lemma 2** Denote any strictly-increasing sequence with length \( k \) by \( \delta_k = \{b_1, \ldots, b_k\} \). We can simply focus on the strictly-increasing sequences, because ALG\( \phi \) only trades something when the current exchange rate is the highest ever observed. Any normal sequences can be transformed into a strictly-increasing sequence by keeping the exchange rate higher than all of its predecessors and omitting the rest, and the optimal in hindsight is not affected by this transformation. By (6), we have

\[
\begin{align*}
x_1 &= \phi^{-1}(b_1) = \frac{\ln(b_1 e/L)}{\ln \theta + 1}, \\
x_i &= \phi^{-1}(b_i) - \phi^{-1}(b_{i-1}) = \frac{\ln(b_i/b_{i-1})}{\ln \theta + 1}, \quad i \geq 2.
\end{align*}
\]

Denote the total amount of yen ALG\( \phi \) trades for \( \delta_k \) by ALG(\( \delta_k \)) and the offline optimal one by OPT(\( \delta_k \)). We have OPT(\( \delta_k \)) = \( p_k \), and

\[
\text{ALG}(\delta_k) = \sum_{i=1}^{k} b_i x_i = \frac{b_1 \ln(b_1 e/L) + \sum_{i=2}^{k} b_i \ln(b_i/b_{i-1})}{\ln \theta + 1}.
\]

Let \( r_k(b_1, \ldots, b_k) = \frac{b_1 \ln(b_1 e/L) + \sum_{i=2}^{k} b_i \ln(b_i/b_{i-1})}{\ln \theta + 1} \). So the competitive ratio for ALG\( \phi \), can be expressed as

\[
\text{OPT}(\delta_k) = \frac{\ln \theta + 1}{\text{ALG}(\delta_k)} = \min_{\{b_1, \ldots, b_k\}} r_k(b_1, \ldots, b_k).
\]

Because ALG\( \phi \) can achieve \( \ln \theta + 1 \) with \( \phi^* \) by Theorem 3 we know that \( r_k(b_1, \ldots, b_k) \) is minimized by \( \{b_1, \ldots, b_k\} \) that minimize \( r_k(b_1, \ldots, b_k) \) for each \( k \). When \( k = 1 \), \( r_1(b_1) = \ln(b_1 e/L) \geq 1 \), so \( \delta_1 = \{L\} \), \( \hat{x_1} = \frac{1}{\ln \theta + 1} \), and

\[
\text{OPT}(\delta_1) = \text{ALG}(\delta_1) = \ln \theta + 1.
\]

When \( k = 2 \),

\[
r_2(b_1, b_2) = \frac{b_1 \ln(b_1 e/L) + b_2 \ln(b_2/b_1)}{b_2}.
\]

The first order derivatives are

\[
\frac{\partial \text{OPT}(\delta_k)}{\partial b_i} = \frac{\partial \text{OPT}(\delta_k)}{\partial b_j} = \frac{b_2 - b_1 \ln(b_1 e/L)}{b_2^2}.
\]

We notice that \( \partial r_2/\partial b_1 \) and \( \partial r_2/\partial b_2 \) cannot be zero simultaneously. This means that \( r_2 \) has no critical points, and the minimum value of \( r_2 \) on \([L, U] \times [L, U] \) must be on one of the four boundary points. It turns out that \( r_2 \) reaches minimum when \( (b_1, b_2) = (L, L) \). We need to find a close neighbor to \((L, L)\) with \( b_2 > b_1 \) and whose value does not increase too much. Notice that \( \partial r_2/\partial b_1|_{(L,L)} > 0, \partial r_2/\partial b_2|_{(L,L)} = 0 \), thus increasing \( b_2 \) to \( b_2 + \epsilon \) with infinitesimal positive \( \epsilon \) should incur the least inaccuracy. So \( \hat{\delta}_2 = \{L, L + \epsilon\} \) and \( \frac{\text{OPT}(\delta_k)}{\text{ALG}(\delta_k)} \rightarrow (\ln \theta + 1)^{-1} \) as \( \epsilon \rightarrow 0^+ \). For general \( k \geq 3 \),

\[
r_k(b_1, \ldots, b_k) = b_1 \ln(b_1 e/L) + \sum_{i=2}^{k} b_i \ln(b_i/b_{i-1})
\]

The first order derivatives are:

\[
\frac{\partial r_k}{\partial b_1} = \frac{\ln(b_1 e/L) + 1 - b_2/b_1}{b_2},
\]

\[
\frac{\partial r_k}{\partial b_i} = \frac{b_k - b_{k-1}r_{k-1}(b_1, \ldots, b_{k-1})}{b_k^2},
\]

\[
\frac{\partial r_k}{\partial b_i} = \frac{b_i - b_{i-1}/b_i}{b_k}, i = 2, \ldots, k - 1.
\]

Notice something in common that, \( \frac{\partial r_k}{\partial b_k} \) and \( \frac{\partial r_k}{\partial b_{k-1}} \) cannot be zero at the same time, and \( r_k \) reaches minimum when \( b_i = L, i \in [k] \). The increasing sequence closest to the minimum point is \([L, L + \epsilon_1, \ldots, L + \sum_{i=1}^{k-1} \epsilon_i] \), where \( \epsilon_i \)s are infinitesimal positive values and we have \( \frac{\text{OPT}(\delta_k)}{\text{ALG}(\delta_k)} \rightarrow (\ln \theta + 1)^{-1} \sum_{i=1}^{k-1} \epsilon_i \rightarrow 0^+ \). Actually, each \( \hat{\delta}_k \) is the prefix of \( \hat{\delta}_m, m \geq k \). From these observations, we claim that as long as the exchange rate sequence increases slowly enough from \( L \), it is the worst-case sequence for ALG\( \phi^* \).

To verify the claim, let \( \hat{b}_k \) be \( L + \sum_{i=1}^{k-1} \epsilon_i \), we have

\[
\text{OPT}(\hat{\delta}_k) = \sum_{i=1}^{k} \hat{b}_i x_i = \frac{L}{\ln \theta + 1} + \sum_{i=2}^{k} \hat{b}_i \left( \frac{\ln(\hat{b}_i) - \ln(\hat{b}_{i-1})}{\ln \theta + 1} \right) = \frac{L}{\ln \theta + 1} + \int_{L}^{\hat{b}_k} \frac{\gamma}{\ln \theta + 1} d\gamma = \frac{\hat{b}_k}{\ln \theta + 1},
\]

and thus \( \frac{\text{OPT}(\hat{\delta}_k)}{\text{ALG}(\hat{\delta}_k)} = \max_{\{b_1, \ldots, b_k\}} \frac{\text{OPT}(\hat{\delta}_k)}{\text{ALG}(\hat{\delta}_k)} = \ln \theta + 1 \). Since the exchange rate is upper bounded by \( U \), by the monotone convergence theorem, we have

\[
\lim_{k \to \infty} \hat{b}_k = U,
\]

and thus \( \lim_{k \to \infty} \sum_{i=1}^{k} \hat{x}_i = 1 \).

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