The behavior of the sextic coupling for the three-dimensional $\varphi^4$ theory

Gíno N. J. Añaños ¹ and Marcelo P. S. Pinheiro ²

Instituto de Física Teórica-IFT
Universidade Estadual Paulista
Rua Pamplona 145, São Paulo, SP 01405-900 Brazil

Abstract

In this work we use the lattice regularization method to study the behavior of the six point renormalized coupling constant defined at zero momentum for the three-dimensional $\varphi^4$ theory in the intermediate and strong coupling domain. The result is in good agreement with the corresponding study in the Ising limit.

¹gananos@ift.unesp.br
²pinheiro@ift.unesp.br
1 Introduction

We study in this paper the behavior of the six point renormalized coupling constant defined at zero momentum for the \((\varphi^4)_{d=3}\) theory for a large variation of the bare coupling constant in the symmetric phase. For consistent we also show the behavior of the four point renormalized coupling constant. In such regime the perturbation theory doesn’t work and the nonperturbative approach is mandatory. As an analytic alternative approach in studying this theory in the strong coupling regime it is used the strong coupling expansion series [1, 2, 3]. The basic idea of this method consists in factoring out the kinematical parts of the Lagrange from the path integral and the result of evaluating the remaining non-Gaussian in close form is a formal expansion of the vacuum functional as a series in inverse power of coupling constant. It is possible from this expansion to extract a set of simple diagrammatic, which can be used to compute the n-point Green’s functions of the theory. For the \((\varphi^4)_{d=3}\) theory it can be shown from the analytic study in the strong coupling limit the renormalized coupling constant approaches to the asymptotic limit. The same thing can be said for the six point renormalized coupling constant. From the point of view of numerical study a relative little work has been done in order to simulate the general theory in the domain of intermediate to strong coupling constant. For this reason we were motivated to prepare this paper. This work is a continuation of previous paper [4] where the six point renormalized coupling constant behavior of the \((\varphi^4)_{d=2}\) theory was analyzed. The other important reason is that \((\varphi^4)_{d=3}\) theory admits a non-trivial continuum limit [3, 5, 6] contrary to \((\varphi^4)_{d=4}\). Much more effort has been done in the application of renormalization group and high temperature expansion to the
theory \((\varphi^4)_{d=3}\) in the Ising limit[7]. The techniques that have been used to determine the proper \(n(n \leq 8)\)-point Green functions (1PI) include high-temperature lattice expansion, Monte Carlo methods and \(\epsilon\) expansion. For the case of \(d = 3\) a complete list of references is given in [8]. Bender and Boettcher [9] did the analytic study for renormalized sextic coupling. This study takes into account the strong coupling calculation on hypercubic lattice in \(d\)-dimensions, which were invented to obtain the continuum limit. In two dimensions Sokolov and Orlov used renormalization group expansion and Padé-Borel-Leroy resumation technique to get \(g_6\) [10]. In the literature there are a few works available of the sextic coupling constant using the approach of lattice Monte Carlo. The first work was performed by Wheater [11] and later in the Ising limit by Tsypin [8]. So it seems appropriate the nonperturbative study using the lattice Monte Carlo technique [12] to get not only the asymptotic value of the higher order renormalized coupling constant but also the behavior on the intermediate and strong regime. Here as usual we consider \(\bar{h} = c = 1\).

2 Higher order coupling constants

We consider in the continuum the \(\phi^4\) theory in \(d\)-dimensions Euclidean space in the presence of a source \(J\), which the bare action is given by:

\[
S[\phi, J] = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{4!} \phi^4 - J\phi \right].
\]

We introduce the vacuum persistent functional \(Z[J]\):

\[
Z[J] = \int D[\phi] \exp(-S[\phi, J])
\]

2
and from this we define the generating functional $W[J]$ for the connected Green functions, by writing

$$ W[J] = \ln Z[J]. \quad (3) $$

The vacuum expectation value of the field $\phi_c$ is given by

$$ \phi_c = \langle \phi \rangle = \frac{\delta}{\delta J(x)} W[J] \bigg|_{J=0} \quad (4) $$

and the connected $n$-point Greens functions $G_n(x_1 \ldots x_n)$ is obtained from the generating functional (3),

$$ G_n(x_1 \ldots x_n) = \frac{\delta}{\delta J(x_1)} \ldots \frac{\delta}{\delta J(x_n)} \ln Z[J] \bigg|_{J=0}. \quad (5) $$

The effective action $\Gamma[\phi_c]$ is defined by a functional Legendre transform of $W[J]$:  

$$ \Gamma[\phi_c] = \int d^d x \phi_c(x) J(x) - W[J]. \quad (6) $$

It is well known that the effective action is the generating functional of one-particle-irreducible (1PI) vertices, in particular the functional $\Gamma[\phi]$ has a Taylor expansion in powers of $\phi$ at $\phi = 0$;

$$ \Gamma[\phi_c] = \sum_n \frac{1}{n} \int d^d x_1 \ldots d^d x_n \Gamma^{(n)}(x_1, \ldots, x_n) \phi_c(x_1) \ldots \phi_c(x_n). \quad (7) $$

Here $\Gamma^{(n)}(x_1, \ldots, x_n)$ is the proper $n$-point Green functions (1PI). Now we consider a source $J$ which is constant and uniform in Euclidean space-time. This implies that $\phi$ is a constant independent of space-time. We define the Fourier transform of $\Gamma^{(n)}(x_1, \ldots, x_n)$ by

$$ \Gamma^{(n)}(x_1, \ldots, x_n) = \int \frac{dk_1}{(2\pi)^d} \ldots \int \frac{dk_n}{(2\pi)^d} \left[ (2\pi)^d \delta(\sum_{i=1}^n k_i) \tilde{\Gamma}^{(n)}(k_1, \ldots, k_n) \right]. \quad (8) $$
Thus, for constant \( \phi(x) = \phi \) we have

\[
\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \bar{\Gamma}^{(n)}(0,0,\ldots,0)\phi^n (2\pi)^d \delta(0) \equiv (2\pi)^d \delta(0) U(\phi) \tag{9}
\]

which is the defining equation for the effective potential \( U(\phi) \). Thus, the Taylor coefficients of the effective potential are the 1PI vertices \( \bar{\Gamma}^{(n)}(0,0,\ldots,0) \) evaluated at zero external momentum. The renormalized fourier transform \( \bar{\Gamma}^{(n)}_r(0) \) proper \( n \)-point Green functions is obtained as follows: the wave-function renormalization is obtained from the Fourier transform connected Green function of two-points from:

\[
Z^{-1} = \left. \frac{d\tilde{G}_2(p^2)}{dp^2} \right|_{p^2=0} \tag{10}
\]

and the renormalized mass \( m_r^2 \) is defined by

\[
m_r^2 = Z \tilde{G}_2^{-1}(p^2) \bigg|_{p^2=0} . \tag{11}
\]

In general the renormalized \( \bar{\Gamma}^{(n)}_r(0) \) proper \( n \)-point Green’s functions are given by

\[
\bar{\Gamma}^{(n)}_r(0) = Z^{n/2} \bar{\Gamma}^{(n)}(0) , \tag{12}
\]

and it follows that the renormalized effective potential can be written as,

\[
U_r = \sum_{n=1}^{\infty} \frac{\bar{\Gamma}^{(n)}_r(0)}{n!} \phi^n_r
\]

where \( \phi_r = Z^{-1/2} \phi \). From here we see that \( U_r \) is the generating function of one particle irreducible renormalized Green’s function at zero external momentum on all legs. Since we are doing the study of the theory in the symmetric phase, we have \( \bar{\Gamma}^{(2n+1)}_r(0) = 0 \). The particular interest to us is the
renormalized coupling constant $\tilde{\Gamma}^{(4)}(0)$ and the renormalized sextic coupling constant $(\tilde{\Gamma}^{(6)}(0))$, which can be expressed in terms of Fourier transform of connected Green functions as follows:

\[ \tilde{\Gamma}^{(4)}(0) = -Z^2(\tilde{G}_2^{-1}(p^2))^{4}\tilde{G}_4(p^2)\bigg|_{p=0} \]  

(14)

and

\[ \tilde{\Gamma}^{(6)}(0) = -Z^3(\tilde{G}_2^{-1}(p^2))^{6}\left(\tilde{G}_6(p^2) - 10\tilde{G}_4^2(p^2)\tilde{G}_2^{-1}(p^2)\right)\bigg|_{p=0}. \]  

(15)

The quantities that will be extracted from lattice Monte Carlo simulation are the dimensionless renormalized zero momentum scattering amplitudes $g_{2n}$ defined by,

\[ g_{2n} = \frac{\tilde{\Gamma}^{(2n)}(0)}{m_{2n}^{2n-nd+d}(2n)!}. \]  

(16)

3 Simulation results

A discrete version of action related to Monte Carlo simulation can be written as

\[ S[\phi, J] = \frac{a^{d-2}}{2} \sum_{x,\mu} (\phi_0(x + e_\mu) - \phi_0(x))^2 + \frac{a^d}{2} \sum_x m_0^2\phi_0(x)^2 + \sum_x \frac{g_0}{4!}\phi_0(x)^4, \]  

\[ S[\phi, J] = \frac{1}{2} \sum_{x,\mu} (\phi(x + e_\mu) - \phi(x))^2 + \frac{1}{2} \sum_x m^2\phi(x)^2 + \sum_x \frac{g}{4!}\phi(x)^4. \]  

(17)
As usually we impose periodic boundary condition on fields:

$$\phi(n + L_\mu) = \phi(n) \quad \text{for all } \mu ,$$

(19)

We use the standard Metropolis algorithm combined with the Wolff single cluster method [13] which is used to avoid the trapping into meta stable states due to the underlying Ising dynamics. We use the cluster algorithm using the embedded dynamics for $\phi^4$ theory, according to the action, [14]

$$S_{\text{Ising}} = -\sum_x \sum_{\hat{\mu}} |\phi(x + \hat{\mu}) \phi(x)| s(x + \hat{\mu}) s(x) ,$$

(20)

where $s(x) = \text{sign}(\phi(x))$. Statistical errors are evaluated taking into account the autocorrelation time in the statistical sample generated by the Monte Carlo simulation. For simulation it is used $32^3$ lattice. We choose correlation length $\xi_r = 1/m_r$ as $1 << \xi_r << L$ to minimize finite size effects.

For numerical simulation purpose it is appropriated to work with the equivalence Langrange:

$$S[\phi] = 2\kappa \sum_{x,\mu} \phi(x) \phi(x + e_\mu) + \sum_x \phi(x)^2 + \sum_x \lambda (\phi(x)^2 - 1)^2 ,$$

(21)

where

$$m^2 = \frac{1 - 2\lambda}{\kappa} - 2d \quad g = \frac{6\lambda}{\kappa^2} .$$

(22)

The most part of our effort it is to find the values of $\lambda$ and $\kappa$ for different values of the bare coupling constants $g$ for a fixed value of the renormalized mass, which is found within few percent. As we vary the coupling constant to a greater value we have to put more negative the bare mass $m^2$. Considering the translational invariance of the correlations functions, one can choose to
approximate the momentum derivation in eq. (10) by variation of $\tilde{G}_2(p^2)$ across one lattice spacing and in one direction in order to calculate the renormalized mass $m_r$

$$m_r^2 = \left( \frac{L}{2\pi} \right)^2 \left[ \frac{\langle \tilde{\phi}(0)^2 \rangle - \langle |\tilde{\phi}(p)|^2 \rangle}{\langle |\tilde{\phi}(p)|^2 \rangle} \right],$$

(23)

where here $\tilde{\phi}$ is the Fourier transform of the field and $p = (\frac{2\pi}{L}, 0)$ is the smallest available non-zero momentum.

To figure out $g_4$ and $g_6$ values from simulation we used lattice version of eqs. (14) and (15),

$$g_4(4!) = \frac{\tilde{G}_4(p^2) \tilde{G}_2^{-2}(p^2)}{\xi_r^d} \bigg|_{p^2=0} = -\frac{<\tilde{\phi}(0)^2 > - 3 <\tilde{\phi}(0)^2 >=^2}{<\phi(0)^2 >^2 \xi_r^d},$$

(24)

$$g_6(6!) = \frac{10\tilde{G}_6(p^2)\tilde{G}_2^{-4}(p^2) - \tilde{G}_6(p^2)\tilde{G}_2^{-3}(p^2)}{\xi_r^{2d}} \bigg|_{p^2=0} = 10g_4^2 - \frac{-15 <\tilde{\phi}(0)^4 > +30 <\tilde{\phi}(0)^2 >^3}{<\phi(0)^2 >^3 \xi_r^{2d}}.$$  

(25)

The expressions above are prohibited to use in the weak coupling constant regime due to the large statistical errors. However we notice that eqs. (24) and (25) are efficient in the intermediate and strong coupling regime where the statistical errors are reasonable [15]. We also notice that the statistical errors increase with value $(L/\xi_r)$. The values in this simulation we find that for the renormalized coupling constant and sextic coupling constant are consistent with the prediction of analytic and numerical methods in the Ising limit. Our prediction for $g_6 = 2.03 \pm 0.072$ in the Ising limit is in good agreement with value obtained by Tsypin [8], $g_6 = 2.05 \pm 0.15$. In figure fig. (1) we present the result of simulations for a large range of the bare coupling constant for
$g_6$ and $g_4$. In order to be consistent we also include in figure fig.(2) the result of simulations of $64^2$ lattice in two dimensions which was taken from previous work [4]. We observe both $g_4$ and $g_6$, for two and three dimensions, approach to asymptotic constant value. This behavior is in according with numerical and analytical predictions. From both figures we also can see that the variation of both quantities are relative small considering the large variation on $g$. This means the behavior of effective potential in the strong coupling regime does not change significatively for large variation of the coupling constant. For the computation of renormalized higher orders $g_{2n}$ ($n > 3$) it is difficult due to large statistical errors even in the regime of strong coupling. To avoid this problem it is useful to look at methods, which calculate the connected Green’s functions directly. One of this method was discussed by Drumond et al [16, 17]. They write down for an action with a source term, the Langevin equation describing the stochastic evolution of the field in the theory. Differentiating both sides of the Langevin equation successively with respect to the source field gives a set of slave equations, which describe the stochastic evolution of the estimators of Grenn’s function of higher orders [18]. The interesting study would be the $(\phi^4)_3$ theory in the broken phase for finite coupling constant [19].

4 Conclusions

In this paper we studied on the lattice the behavior of the renormalized sextic coupling for the $(\phi^4)_{d=3}$ theory at intermediate and strong coupling constant domain. We performed a large variation, from intermediate to strong coupling, of the bare coupling constant and notice that the
Figure 1: The behavior of $g_4$ and $g_6$ with the bare coupling constant for $d = 3$.

Proper six-point Green functions (1PI) $g_6$ evaluated at zero external momentum has a regular behavior. This quantity has a definite asymptotic value in the limit of $g \to \infty$. Its qualitative behavior is similar to the two dimensions theory. We remark that in the expansion of effective potential in powers of the field the $\phi^6$ term is not negligible. Finally our results for $g_4$ and $g_6$ are in good agreement with the values available in the literature.
5 Acknowledgements

This paper was supported by FAPESP under contract number 03/12271-7

References

[1] C. M. Bender, F. Cooper, G. S. Guralnik and D. Sharp Phys. Rev. D 19, 1865 (1979).

[2] C. M. Bender, F. Cooper, G. S. Guralnik and D. Sharp, Phys. Rev. D 23, 2976 (1981).
[3] G.A. Baker, L. P. Benofy, F. Cooper and D. Preston, Nucl. Phys. B 210, 273 (1982).

[4] G. N. J. Ananos, hep-lat/0512024.

[5] B. Freedman, P. Smolensky and D. Weingarten, Phys. Lett. B bf 113 481 (1982).

[6] F. Cooper and B. Freedman, Nucl. Phys. B 239 459 (1984).

[7] M. Campostrini, A. Pelissetto, P. Rossi and E. Vicari, Phys. Rev. E 65, 066127 (2002) [arXiv:cond-mat/0201180].

[8] M. M. Tsypin, Phys. Rev. Lett. 73, 2015 (1994).

[9] C. M. Bender and S. Boettcher, Phys. Rev. D 51, 1875 (1995) [arXiv:hep-th/9405043].

[10] A. I. Sokolov and E.V. Orlov, Phys. Rev. B 58, 2395 (1998) [arXiv:cond-mat/9804008].

[11] J. F. Wheater, Phys. Lett. 136B, 402 (1983).

[12] I. Montvay, I. Munster, Quantum fields on the lattice, (University of Cambridge Press, 1994).

[13] U. Wolff, Phys. Rev. Lett. 62, 331 (1989)

[14] R. C. Brower and P. Tamayo, Phys. Rev. Lett. 62, 1087 (1989).

[15] A. Ardekani and A. G. Williams, Phys. Rev. E 57, 6140 (1998) [arXiv:hep-lat/9705021].

[16] I. T. Drummond, S. Duane and R. R. Horgan, Nucl. Phys. B 220, 119 (1983).
[17] I. T. Drummond, S. Duane and R. R. Horgan, Nucl. Phys. B 280, 25 (1987).

[18] R. A. Weston, Phys. Lett. B 219, 315 (1989).

[19] T. Munehisa and Y. Munehisa, Z. Phys. C 45, 329 (1989).