Dynamical behavior connection of the gluon distribution and the proton structure function at small-$x$

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We make a critical study of the relationship between the singlet structure function $F_2^S$ and the gluon distribution function $G(x,Q^2)$ proposed in Refs. [4-8], which is frequently used to extract the gluon distribution from the proton structure function. We show that a simple relation is not generally valid in the simplest state. We completed this relation by using a laplace-transform method and hard-Pomeron behavior at LO and NLO at small-$x$. Our study show that this relation is dependence to the splitting functions and initial conditions at $Q^2 = Q_0^2$ and running coupling constant at NLO. The resulting analytic expression allow us to predict the proton structure function with respect to the gluon distributions and to compare the results with H1 data and a QCD analysis fit. Comparisons with other results are made and predictions for the proposed best approach are also provided.

1. Introduction

The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [1 – 3] are fundamental tools to study the $\ln Q^2$ and $x$ evolutions of structure functions, where $x$ and $Q^2$ are Bjorken scaling and the square of the four-vector momentum exchange in deep inelastic scattering (DIS) process respectively. The measurements of the $F_2(x,Q^2)$ structure functions by DIS processes in the small-$x$ region have opened up a new era in parton density measurements inside hadrons. The structure function reflects the momentum distributions of partons in a nucleon. It is also important to know the gluon distribution inside a hadron at small-$x$ because gluons are expected to be dominant in this region. On the other hand, the gluon distribution functions cannot be measured directly through experiments. It is, therefore, important to measure the gluon distribution $G(x,Q^2)$ directly using the proton structure function $F_2(x,Q^2)$. This expectation has led to an approximate phenomenological scheme, as in the past two decades some authors [4-8] reported an ansatz between the gluon distribution function and singlet structure function. The commonly used relation is

$$G(x,Q^2) = K(x)F_2^S(x,Q^2),$$

where $K(x)$ is a parameter to be chosen from the experimental data and those assumed $K(x) = k, ax^b$ or $ce^{dx}$ where $k, b, a, c$ and $d$ are constants. Authors used a Taylor expansion for the gluon and singlet functions at low-$x$ in solving DGLAP evolution equations with applying Eq.1 to the distribution functions. As, Eq.1 is a relationship between singlet structure function and gluon distribution function was proposed in order to facilitate the extraction of the gluon density from the data.

In this paper we deduce the general relations between the proton structure function and the gluon distribution function with analytical methods at leading order (LO) and next-to-leading order (NLO). However, a relation between singlet structure function and gluon distribution function can be determined by simultaneous solutions of coupled DGLAP evolution equations of singlet structure functions and gluon distribution functions. We demonstrate here that the validity of this relation crucially depends on the splitting functions at LO and running coupling constant at NLO. We derive the master equation to extract the relation between the gluon distribution and the proton structure function, by using a Laplace-transform technique at LO and also a hard Pomeron behavior for the gluon distribution up to next-to-leading order (NLO). Our purpose here is to improve the situation with an approximation equation at small-$x$ at LO and NLO. Section 2 outlines the theory and formalism while section 3 is devoted to results and discussions.

2. Compact Formula

The DGLAP evolution equations for the singlet quark structure function and the gluon density have the forms

$$\frac{d}{d\ln Q^2} \begin{bmatrix} q(x,Q^2) \\ g(x,Q^2) \end{bmatrix} = \begin{bmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{bmatrix} \otimes \begin{bmatrix} q(x,Q^2) \\ g(x,Q^2) \end{bmatrix}$$

which emphasized that quark and gluon densities are coupled. The convolution, defined as $P_{ij} \otimes f_i = \int_x^1 \frac{dy}{y} P_{ij}(\frac{x}{y})f_i(y,Q^2)$, express the possibility that a parton $i$ with momentum fraction $x$ may originate from the
branching of a parent parton $j$ of the higher momentum fraction $y$ ($P_{ij}$ is the splitting function).

The method of approximate determination a relation between the gluon and structure function is based on the simplification of the convolutions $P_{ij} \otimes f_j$ by the Laplace transforms [9-12] and other methods based on the behavior of the gluon distribution such as the hard Pomeron and the expatiating methods [13-15]. Here we present a general solution of the DGLAP evolution equations at low-$x$, as the gluons are expected to be dominant. Therefore we can neglect the quark singlet part to the evolution equations and also the non-singlet contribution $F_{NS}^{S}$ can be ignored safely at small-$x$ in the DGLAP equations.

Complete solution of the decoupling DGLAP evolution equations for a relation between gluon and singlet functions can be discussed at section 2.2.

The LO DGLAP equations for the singlet and gluon functions can be written as

$$\frac{4 \pi}{\alpha_s^{LO}(Q^2)} \frac{\partial F_2^S(x, Q^2)}{\partial \ln Q^2} \simeq 2n_f x \int_x^1 G(z, Q^2)(1 - \frac{x}{z}) \left[ 2 \frac{x^2}{z^2} \frac{dz}{z^2} \right],$$

and

$$\frac{4 \pi}{\alpha_s^{LO}(Q^2)} \frac{\partial G(x, Q^2)}{\partial \ln Q^2} \simeq \frac{33 - 2n_f}{3} G(x, Q^2) \left[ 12 G(x, Q^2) \ln \frac{1 - x}{x} + 12 x \int_x^1 \frac{G(z, Q^2) - G(x, Q^2)}{z - x} \frac{dz}{z^2} \right] + 12 x \int_x^1 G(z, Q^2) \frac{dz}{z^2}.$$

Here $\alpha_s^{LO}(Q^2)$ is given by the LO form

$$\alpha_s^{LO} = \frac{4 \pi}{(11 - \frac{2}{3} n_f \ln(Q^2/\Lambda^2))},$$

where $n_f$ being the number of active quark flavors ($n_f = 4$) and $\Lambda$ is the QCD cut-off parameter.

2.1. Laplace Transform method

Authors in Ref.[9-12] uses a somewhat unusual application transform, in which those first introduce the variable

$$v \equiv \ln(1/x),$$

into the coupled DGLAP evolution equations, then obtained the coupled equations in the Laplace-space variable $s$, as we can be written these equations at small-$x$ for our limit

$$\frac{\partial F_2^S}{\partial \ln Q^2}(s, Q^2) \simeq \frac{\alpha_s^{LO}(Q^2)}{4\pi} \Theta_F(s) G(s, Q^2),$$

and

$$\frac{\partial G}{\partial \ln Q^2}(s, Q^2) \simeq \frac{\alpha_s^{LO}(Q^2)}{4\pi} \Phi_G(s) G(s, Q^2),$$

where $F(s) = \mathcal{L}[F(v); s] = \int_0^\infty F(v) e^{-sv} dv$ and $G(s) = \mathcal{L}[G(v); s] = \int_0^\infty G(v) e^{-sv} dv$ ($\hat{F}(v) \equiv F(e^{-v}), \hat{G}(v) \equiv G(e^{-v})$). The coefficient functions $\Phi_G(s)$ and $\Theta_F(s)$ are given by [9-12]

$$\Theta_F(s) = 2n_f \left( \frac{1}{1 + s} - \frac{2}{3 + s} + \frac{2}{3 + s} \right),$$

and

$$\Phi_G(s) = \frac{33 - 2n_f}{3} + 12 \left( \frac{1}{s + 1} - \frac{1}{2 + s} + \frac{1}{2 + s} \right) \left[ 1 + \frac{1}{3 + s} - \psi(1 + s - \gamma_E) \right],$$

where $\psi(x)$ is the digamma function and $\gamma_E = 0.5772156...$ is Euler’s constant.

For obtain an general explicit form between the gluon distribution and the proton structure function at small-$x$, rewrite Eqs.7 and 8 in $s$ space as

$$\frac{\partial G}{\partial \ln Q^2}(s, Q^2) = \frac{\Phi_G(s)}{\Theta_F(s)} \frac{\partial F_2^S}{\partial \ln Q^2}(s, Q^2).$$

or

$$\frac{\partial G}{\partial \ln Q^2}(s, Q^2) = h(s) \frac{\partial F_2^S}{\partial \ln Q^2}(s, Q^2).$$

In the above equation we used the following property for Laplace transformation

$$\mathcal{L}^{-1}[h(s) \frac{\partial F_2^S(s, Q^2)}{\partial \ln Q^2}; v] = \int_0^v \frac{\partial \hat{F}_2^S(w, Q^2)}{\partial \ln Q^2} \tilde{H}(v - w) dw,$$

where $\tilde{H}(v) \equiv \mathcal{L}^{-1}[h(s), v]$. The calculation of $\tilde{H}(v)$, using Eqs.9 and 10, for LO is straightforward and given by

$$\tilde{H}(v) \equiv \frac{9}{4} + \frac{13}{8} \delta(v) + \frac{25}{24} \delta'(v) - e^{(\sqrt{7}/2)} [\frac{\sqrt{7}}{4} \sin(\frac{\sqrt{7}}{2} v) + \frac{13}{3} \cos(\frac{\sqrt{7}}{2} v)].$$

Here we neglecting the some terms at small-$x$, as $12 \int_0^\infty \frac{\partial G}{\partial \ln Q^2}(w, Q^2) \ln(1 - e^{-e^{-v}}) dw \to 0$. Therefore we obtain an explicit solution for the derivatives of the gluon distribution in terms of the integral

$$\frac{\partial \hat{G}(v, Q^2)}{\partial \ln Q^2} = \int_0^v \frac{\partial \hat{F}_2^S(w, Q^2)}{\partial \ln Q^2} \tilde{H}(v - w) dw.$$
Transforming back into $x$-space, finally we have an approximative approach to the relation between the gluon distribution and singlet structure function at low-$x$ by the following form

$$G(x, Q^2) = G(x, Q_0^2) + \frac{13}{8} F\mathcal{F}(x)$$

$$+ \frac{9}{4} \int_x^1 \frac{dz}{z} \frac{dz}{x} \left( \frac{\sqrt{\pi}}{2} \ln \frac{1}{x} \right)$$

$$- \int_x^1 \frac{dz}{z} \frac{dz}{x} \left( \frac{\sqrt{\pi}}{2} \ln \frac{1}{x} \right)$$

$$+ \frac{13}{3} \cos(\sqrt{\frac{\pi}{2}} \ln \frac{z}{x}) \frac{dz}{x}.$$

(16)

where $F\mathcal{F}(x) = F_{2s}(x, Q^2) - F_{2s}(x, Q_0^2)$ and $F\mathcal{F}(z) = F_{2s}(z, Q^2) - F_{2s}(z, Q_0^2)$. Therefore the gluon distribution can be expressed into the singlet structure function by Eq.(16) with respect to the initial conditions.

This result is general with respect to the approximated limit for the coupled DGLAP evolution equations at small-$x$, and its the simplest answer to the relation between the gluon distribution and singlet structure function by using a Laplace-transform method.

### 2.2. Hard-Pomeron behavior

With respect to the Regge-like behavior of the gluon distribution at small-$x$, we would like to get a simplest formula to extract the gluon distribution with respect to the proton singlet structure function [13-15]. Authors in Refs.[16-17] shown a simple relation between the gluon and $F_2$ at small-$x$ based on the coupled integro-differential equations as can be converted into more simple linear relations between the gluon distribution and structure function and its derivatives with respect to $\ln Q^2$. The authors results in Refs.[16-17] are different from Eq.(1) as it was proposed in the literature [4-8] to isolate the gluon distribution only with the singlet structure function.

The small-$x$ region of DIS offers a unique possibility to explore the Regge limit of pQCD. This theory is successfully described by the exchange of a particle with appropriate quantum numbers and the exchange particle is called a Regge pole. Phenomenologically, the Regge pole approach to DIS implies that the structure functions are sums of powers in $x$, modulus logarithmic terms, each with a $Q^2$-dependent residue factor. This model gives the following parametrization of the DIS structure function $F_2(x, Q^2)$ at small $x$, $F_2(x, Q^2) = A(Q^2)x^{-\delta}$, that the singlet part of the structure function is controlled by Pomeron exchange at small $x$. The rapid rise in $Q^2$ of the structure functions was considered as a sign of departure from the standard Regge behavior. In principle, the HERA data should determine the small-$x$ behavior of the gluon and sea-quark distribution. Roughly speaking, the data on the singlet part of the structure function $F_2$ constrain the sea quarks and the data on the slope $dF_2/d\ln Q^2$ determine the gluon density. In the DGLAP formalism, the gluon splitting functions are singular as $x \rightarrow 0$. Thus, the gluon distribution will become large as $x \rightarrow 0$, and its contribution to the evolution of the parton distribution becomes dominant. In particular, the gluon will drive the quark singlet distribution, and, hence, the structure function $F_2$ becomes large as well, the rise increasing in steepness as $Q^2$ increases [18-21]. Therefore, the small $x$ limit corresponds to a study of a partonic system inside of a nucleon which is predominantly formed by gluons. This strong rise can eventually violate unitarity and so it has to be tamed by screening effects. However when the density becomes large enough, the gluons start interacting with each others and then their further evolution is non-linear. This happens reduce the growth of gluon distribution and called parton saturation [22-31]. Therefore, the linear evolution equation in this case is modified by non-linear term description gluon recombination. An important point in the gluon saturation approach is the $x$-dependent saturation scale $Q_s^2(x)$. This scaling argument leads to the conclusion that $\gamma^p$ cross section, which is a priori function of two independent variable ($x$ and $Q^2$), is a function of only variable $\tau = \frac{Q^2}{Q_s^2(x)}$ where the saturation scale is given by $Q_s^2(x) = \frac{Q_0^2}{\Lambda_{QCD}^2}$ and its known as geometrical scaling. Here $Q_0$ and $x_0$ are free parameters and exponent $\lambda$ is a dynamical quantity of the order $\lambda \sim 0.3$, although one can take into account phenomenologically where exponent $\lambda$ has an effective $Q^2$ dependence $\lambda(\Lambda_{QCD}, Q^2)$. For $Q^2 < Q_s^2(x)$ such a scaling is natural, whereas for large $Q^2 > Q_s^2(x)$ it is a consequence of hard-pomeron behavior from hard diffraction [22-34]. At small $x$, $Q_s^2(x) > \Lambda_{QCD}^2$ and the approach based on pQCD is fully justified and results are based on the phenomenon of geometric scaling. All results to DIS data from HERA for $x < 0.01$ show that geometrical scaling was found in the data from different experiments. In the limit of high energy, pQCD consistently predicts that the high gluon density should form a Color Glass Condensate (CGC), where the interaction probability in DIS becomes large and this is characterized by a hard saturation scale $Q_s(x)$ which grows rapidly with $1/x$ [22-31]. In this region, the non-linear saturation dynamics is incorporated into the CGC model. As, it is valid only for $Q^2$ less than or of the order of the saturation momentum, which is at most several $GeV^2$, while the fit result to SGK [25] model extends up to $Q^2$ of the order of several hundred $GeV^2$. Indeed, the extended scaling at $Q^2 > Q_s^2$ arises from the general non-linear evolution equations in the kinematical range. The validity of these evolution equations in the present of saturation has been estimated as $Q_s^2(x) < Q^2 < Q_0^2(x)/\Lambda_{QCD}^2$. This means that the geometric scal-
ing for all momenta \( Q^2 \) have to satisfy this inequality as, \( \ln(Q^2/Q_0^2) = \ln(q^2/A^2_{QCD}) \). At soft momenta \( Q^2 < Q_0^2 \) this scaling is an expected consequence of saturation and at high momenta \( 1 < \ln(Q^2/Q_0^2) \approx \ln(q^2/A^2_{QCD}) \) it rather corresponds to a regime where parton densities are small, and linear evolution equations apply.

The overall physical picture is dependence to the different regions in the \((x, Q^2)\)-plane. For \( Q^2 < Q_0^2(x) \) the linear evolution is strongly perturbed by nonlinear effects where the parton system becomes dense and the saturation corrections start to play an important role. In this region the dipole cross section is bounded by an energy independent corrections start to play an important role. In this region the dipole cross section is bounded by an energy independent value, as the dipole cross section was proposed [22-31] to have the form \( \sigma_{\text{dipole}}(x, r) = \sigma_0 \left(1 - \exp(-r^2 Q_0^2(x)/4)\right) \) which impose the unitarity condition \( \sigma_{\text{dipole}} \leq \sigma_0 \) for large dipole sizes \( r \). At small-\( r \) region, the dipole cross section is related to the gluon density where it is valid in the double logarithmic approximation. This geometrical scaling holds until the line boundary where \( Q^2 = Q_0^2(x) \). As the gluon density is \( xy(x, Q^2 = Q_0^2(x)) = r^{\alpha_s} x^{-\lambda} \), and the parameter \( r_0 \) specifies the normalization along the critical line. Thus, the saturation scale is an intrinsic characteristic of a dense gluon system. For \( Q^2 > Q_0^2(x) \) the nonlinear screening effects can be neglected and evolution of parton densities is governed by the linear DGLAP equations [35-37]. Therefore, the validity of our method only holds in the kinematic region \( Q^2 \gg Q_0^2/\Lambda_{QCD}^2 \). Hence, as \( x \) gets smaller, the gluon distribution grows rapidly and \( \lambda \to \delta \) where \( \delta \) is the hard pomeron exponent. So, the dipole cross section extracted from DIS data with assuming a hard pomeron dependence, as \( \sigma \sim x^{-\delta} \). Therefore we study the DGLAP evolution upon the geometrical scaling in the region \( Q^2 > Q_0^2(x) \) with solving the linear DGLAP evolution equation starting from the gluon distribution satisfying the hard-pomeron behavior. The gluon distribution at small-\( x \) increase with decreasing \( x \) as

\[
G(x, Q^2) = f(Q^2) x^{-\delta}.
\]  

The form \( x^{-\delta} \) of the gluon parameterization at small \( x \) is suggested by Regge behavior, but because the conventional Regge exchange is that of a soft Pomeron, with \( \delta \sim 0 \), we may also allow a hard Pomeron with \( \delta = 0.5 \) [18-21]. Based on the hard Pomeron behavior for the gluon distribution, let us put Eq.(17) in Eqs.(3) and (4). Let us introduce the variable \( y = \frac{Q^2}{x} \). After doing the integration over \( y \), Eqs.(3) and (4) can be rewritten as

\[
\frac{4\pi}{\alpha_s^\text{LO}(Q^2)} \frac{\partial F_2^g(x, Q^2)}{\partial \ln Q^2} \simeq 2 n_f G(x, Q^2) \times \int_x^1 z^\delta (1 + 2y + 2y^2) dy,  
\]  

and

\[
\frac{4\pi}{\alpha_s^\text{LO}(Q^2)} \frac{\partial G(x, Q^2)}{\partial \ln Q^2} = \frac{33 - 2n_f}{3} G(x, Q^2) + 12 G(x, Q^2) \ln \frac{1 - x}{x} + 12 G(x, Q^2) \int_x^1 (y^{1+\delta} - 1) \frac{dy}{y(1-y)} + 12 G(x, Q^2) \int_x^1 y^\delta (y - 2 + y - y^2) dy.
\]

Consequently

\[
\frac{\partial G(x, Q^2)}{\partial \ln Q^2} = g_1 \frac{\partial F_2^g(x, Q^2)}{p_1 \partial \ln Q^2},
\]

where

\[
p_1 = 2 n_f \int_x^1 z^\delta (1 - 2y + 2y^2) dy,
\]

and

\[
g_1 = \frac{33 - 2n_f}{3} + 12 \ln \frac{1 - x}{x} + 12 \int_x^1 (y^{1+\delta} - 1) \frac{dy}{y(1-y)} + 12 \int_x^1 y^\delta (y - 2 + y - y^2) dy.
\]

Eq.(20) is independent of the running coupling constant \( (\alpha_s(Q^2)) \) at LO. After successive integrations of both sides of Eq.(20), and some rearranging, we find an simplest equation which determine \( G(x, Q^2) \) in terms of \( F_2^g(x, Q^2) \). Consequently

\[
G(x, Q^2) = \frac{g_1}{p_1} F_2^g(x, Q^2) + \frac{\alpha_s(Q^2)}{2\pi} [F_2^g(x, Q_0^2) - F_2^g(x, Q^2)].
\]

We observe that this equation demonstrates the close relation between \( G(x, Q^2) \) and \( F_2^g(x, Q^2) \) at small-\( x \) into the initial conditions at \( Q_0^2 \) at LO by using a hard-Pomeron behavior for the gluon distribution. The NLO corrections are add to LO, as the splitting functions \( P_{ij} \) s are the LO and NLO Altarelli- Parisi splitting kernels by the following form

\[
P_{ij}(x, \alpha_s(Q^2)) = P_{ij}^{\text{LO}}(x) + \frac{\alpha_s(Q^2)}{2\pi} P_{ij}^{\text{NLO}}(x).
\]

The next-to-leading order is the standard approximation for most important processes. The corresponding one- and two-loop splitting functions have been known for a long time. Also, the NNLO corrections can be need to be included, in order to obtain a quantitatively reliable predictions for hard processes at present and future high-energy colliders [38].
The running coupling constant $\frac{\alpha_s}{\pi}$ has the form in the LO and NLO respectively

$$\frac{\alpha_s^{\text{LO}}}{2\pi} = \frac{2}{\beta_0 \ln \frac{Q^2}{\Lambda^2}}, \quad (25)$$

and

$$\frac{\alpha_s^{\text{NLO}}}{2\pi} = \frac{2}{\beta_0 \ln \frac{Q^2}{\Lambda^2}}[1 - \beta_1 \ln \ln \frac{Q^2}{\Lambda^2}], \quad (26)$$

where $\beta_0 = \frac{1}{3}(33 - 2N_f)$ and $\beta_1 = 102 - \frac{32}{3}N_f$ are the one-loop and two-loop corrections to the QCD $\beta$-function. Therefore the DGLAP evolution equations have these behavior at NLO with respect to the hard-Pomeron behavior at small-$x$, as we have

$$\frac{dG(x, Q^2)}{d\ln Q^2} = \frac{\alpha_s}{4\pi}[g_1 + (\alpha_s/4\pi)g_2]G(x, Q^2), \quad (27)$$

and

$$\frac{dF_2^S(x, Q^2)}{d\ln Q^2} = \frac{\alpha_s}{4\pi}[p_1 + (\alpha_s/4\pi)p_2]G(x, Q^2), \quad (28)$$

where

$$g_2 = 2(12C_F n_f T_R - 46C_A n_f T_R)\frac{1}{9\delta}$$

$$+ 2[n_f T_R(-\frac{61}{9}C_F + \frac{172}{72}C_A + \frac{1643}{54} \zeta(2)) - \frac{22}{3} \zeta(2)]\frac{1}{1 + \delta}, \quad (29)$$

and

$$p_2 = \frac{\alpha_s}{4\pi}\frac{80C_A N_f T_R}{9\delta}(1 - x^\delta), \quad (30)$$

where $p_2$ is the NLO kernel after doing the integration based on the hard Pomeron behavior at Eq.28 according to the NLO splitting function in Appendix. Therefore the close relation between the gluon distribution and singlet structure functions at NLO, when the coupling is fixed, is given by

$$G(x, Q^2) = k F_2^S(x, Q^2) + [G(x, Q^2_0) - k F_2^S(x, Q^2_0)], \quad (31)$$

where

$$k = \frac{g_1 + \alpha_s/4\pi g_2}{p_1 + \alpha_s/4\pi p_2}. \quad (32)$$

We now pass to the more realistic case with running coupling. In this case the relation between the distribution functions takes the form

$$G(x, Q^2) = G(x, Q^2_0) + \int_{Q^2_0}^{Q^2} k \frac{dF_2^S(x, Q^2)}{d\ln Q^2} d\ln Q^2. \quad (33)$$

Similarly, We get the singlet structure function evolution at NLO, as

$$F_2^S(x, Q^2) = F_2^S(x, Q^2_0) + \int_{Q^2_0}^{Q^2} k' \frac{dG(x, Q^2)}{d\ln Q^2} d\ln Q^2, \quad (34)$$

where

$$k' = \frac{p_1 + \alpha_s/4\pi p_2}{g_1 + \alpha_s/4\pi g_2}. \quad (35)$$

The expansion of the results from NLO to NNLO approximation can be done easily. Here we used our approximation approach to obtained a simplest relation between the gluon distribution and singlet structure function. The complete calculation of the DGLAP evolution equations, when the singlet quark distribution is essentially driven by the generic instability of the gluon distribution, can be down numerically for shown that what is the best relation between the distribution functions at LO up to NNLO. In a resent paper [38] the distribution functions have been obtained by solving decoupling DGLAP evolution equations at LO up to NNLO with respect to the hard pomeron behavior for the parton distributions at low-$x$. So in the next section we try to do this comparison for the distribution functions using available results at NNLO. Eqs.16, 23 and 31-34 are our results for connection between the gluon distribution and singlet structure function at small-$x$ by using the Laplace-transform and the hard-Pomeron behavior (LO up to NLO) respectively. Therefore we show that Eq.1 is not generally true and its validity crucially depends on the splitting functions and the initial conditions.

3.Results and Discussions

In order to show our results we computed the gluon distribution function on the l.h.s of formulas (16, 23 and 31) at small-$x$ with respect to the initial conditions according to the Block distribution [9-12,39-40]. This distribution represent the spectrum of possible behavior of the proton structure function and gluon distribution in the region $x > 0.00001$ and $0.11 < Q^2 \leq 1200 GeV^2$.

We begin by illustration the use of the analytical expression in Eq.16 to derive $G(x, Q^2)$ from $F_2^{p^p}(x, Q^2)$ in the case of Refs.[9-12,39-40]. We take the published initial distributions as our basic input at $Q^2_0 = 1 GeV^2$, and use this distribution to calculate the proton structure function needed in $FF(x)$. Then, we solve Eq.16 for the gluon distribution by this $F_2^{p^p}(x, Q^2)$ and compared the results with the published gluon distributions. In Fig.1 we show the LO $x$-space results for the gluon distribution for two representative values of $Q^2$. The curves are the published Block [9-12] gluon distribution,
functions and initial conditions in a general model by
elor relations can be estimated with respect to the splitting
gluon distribution and singlet structure function is not
comparable with other results.

In Fig. 2 we present results for the gluon distribution
at LO and NLO using the hard-Pomeron behavior for
the gluon distribution function. This seems to indicate
that the gluon distribution is dominated at small-x by
hard-Pomeron exchange. This powerful approach to
the small-x data for \( G(x, Q^2) \) extends the Regge phe-
nomenology that is so successful for hadronic processes.
The hard intercept is \( \delta = 0.437 \) and we choose \( \Lambda \) such
that \( \alpha_s(M_Z^2) = 0.116 \), this gives \( \Lambda^{N\text{LO}}_{n_f=4} = 400 \text{MeV} \)
[19-21]. We compared our results by published Block
[9-12] gluon distribution, Donnachie and Landshoff (DL)
[18-21], GRV-HO [41-42] and GJR parameterization
[43]. As can be seen, the values of the gluon distribu-
tion function increase as \( x \) decreases. This is because
the hard-Pomeron exchange defined by DL model is
expected to hold in the small-x limit. Comparing our
results in Figs. 1 and 2 with other results indicates that
our global solution (Eq.16) and hard-Pomeron solution
(Eqs. 23, 31) at the simplest case are compatible with
other phenomenological models and this is the reason
why the approximate relation (1) is not valid at small-x.

In order to compare our results with the experimental
data, using Eq. 34 for the evolution of the proton
structure function with respect to the gluon distribution
function. We show a plot of the proton structure
function in Fig. 3 for values of \( Q^2 = 8.5 \text{ GeV}^2 \) and
20 \text{ GeV}^2, compared to the values measured by the H1
collaboration [45-46] and a QCD fit based on ZEUS data
[39-40]. For each \( Q^2 \), there is a cross-over point for both
the curves where both the predictions are numerically
equal. As we want to have a good comparison between
our results and others, we have to include the singlet
distribution functions in DGLAP evolution equations.
However, as there is yet no such simple relation between
the singlet structure function and gluon distribution at
LO up to NNLO, we rather appeal to the numerical
results of Ref. [38]. In Fig. 4 we show the ratio
\( \frac{G(x, Q^2)}{F_2(x, Q^2)} \) at \( Q^2 = 20 \text{ GeV}^2 \). In this figure we show that this ratio
is hardly negligible. In order to have more accurate
solution for the proton structure function, we need to a
best global fit for this ratio, using NNLO analysis data in
Fig. 4. We compared our results for the proton structure
function at NNLO with H1 data [45-46] and GJR
parameterization [43] and also the gluon distribution
function at \( Q^2 = 20 \text{ GeV}^2 \) in Fig. 5. It is clear from this
figure for \( F_2 \) and \( G \) that our results, at NNLO analysis
and considering of the singlet parton distribution, are
comparable with other results.

In conclusion, the simple relation (1) between the
 gluon distribution and singlet structure function is not
generally valid at small-x. We show that the gluon
distribution can be estimated with respect to the splitting
functions and initial conditions in a general model by
using a Laplace-transform method and hard-Pomeron
model at LO and NLO. Therefore our results at simplest
approach lead to different results from those at Refs. [4-8].
Further, we need the singlet structure function at the
DGLAP evolution equations for the numerical relation
between the gluon distribution and single structure
function. Moreover we proposed one general numerical
approach at NNLO for this connection and conclude
that this numerical approach is agreeing with others
results.

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Appendix

The NLO splitting function for the singlet structure
function is as follows

\[
p_{qg}^2 = 2C_F N_f T_R \{ 4 - 9x - (1 - 4x) \ln x - (1 - 2x) \ln^2 x \\
+ 4 \ln(1 - x) + [2 \ln^2 \left( \frac{1 - x}{x} \right) - 4 \ln \left( \frac{1 - x}{x} \right) - \frac{2}{3} \pi^2] \\
+ 10 \{ P_{qg}(x) \} + 2C_A N_f T_R \{ \frac{182}{9} \frac{14}{9} + \frac{40}{9x} \\
+ \left( \frac{36}{3} \ln(1 - x) - 2 + 8x \right) \ln^2 x \\
+ 2P_{qg}(-x) S_2(x) + \left[ - \ln^2 x + \frac{44}{3} \ln x - 2 \ln^2 (1 - x) \\
+ 4 \ln(1 - x) + \frac{\pi^2}{3} - \frac{218}{9} \right] P_{qg}(x) \} 
\]

(36)

where \( P_{qg}(x) = x^2 + (1 - x)^2 \) and \( S_2(x) = \int_0^x \int_0^1 \frac{dz}{z} \ln \left( \frac{1 - z}{z} \right) \). The small-x limit of the NLO splitting
function for the evolution of the singlet quark is then [44]

\[
p_{qg}^2 = \alpha_s \frac{80C_A N_f T_R}{4\pi} \frac{1}{9x}. 
\]

(37)

where the casimir operators of colour SU(3) are defined
as \( C_A = 3 \), \( C_F = \frac{4}{3} \) and \( T_R = \frac{1}{2} \).

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FIG. 1: The gluon distribution, $G(x,Q^2)$, for $Q^2 = 20$ and $200\text{GeV}^2$ by using a Laplace-transform. Our results compared to the Block model [9-12], DL model [18-21], GRV-HO parameterization [41-42] and GJR parameterization [43].

FIG. 2: The same as Fig.1 but for a hard-Pomeron behavior at LO and NLO.

FIG. 3: The proton structure function, $F_2(x,Q^2)$, for $Q^2 = 8.5$ and $20\text{GeV}^2$ with respect to a hard-pomeron behavior. Our results compared to the Block model [9-12], GJR parameterization [43] and H1 data [45-46].

FIG. 4: The ratio $G(x,Q^2)$ at LO up to NNLO when we consider the complete form of the DGLAP evolution equations.

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FIG. 5: Numerical estimates at $Q^2 = 20 \text{ GeV}^2$, for the proton structure function and gluon distribution function at NNLO analysis and at the complete form of the DGLAP evolution equations, compared with other results.

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