ON WEIGHTED COMPACTNESS OF COMMUTATOR OF SEMI-GROUP MAXIMAL FUNCTION ASSOCIATED TO SCHröDINGER OPERATORS

SHIFEN WANG AND QINGYING XUE*

ABSTRACT. Let $T^*$ be the semi-group maximal function associated to the Schrödinger operator $-\Delta + V(x)$ with $V$ satisfying an appropriate reverse Hölder inequality. In this paper, we show that the commutator of $T^*$ is a compact operator on $L^p(w)$ for $1 < p < \infty$ if $b \in \text{CMO}_0(p)(\mathbb{R}^n)$ and $w \in A_p^0(\mathbb{R}^n)$. Here, $\text{CMO}_0(p)(\mathbb{R}^n)$ denotes the closure of $C^\infty_c(\mathbb{R}^n)$ in the BMO space (which is larger than the classical BMO space) topology. The space where $b$ belongs and the weights class $w$ belongs are more larger than the usual CMO space and the Muckenhoupt $A_p$ weights class, respectively.

1. INTRODUCTION

Consider the Schrödinger operator
\[ L = -\Delta + V(x) \]
in $\mathbb{R}^n$, $n \geq 3$. Where $\Delta$ is the Laplacian operator on $\mathbb{R}^n$ and the function $V$ is a nonnegative potential belonging to certain reverse Hölder class $RH_q$ with an exponent $q > n/2$, that is, there exists a constant $C > 0$ such that
\[
\left( \frac{1}{|B(x,r)|} \int_{B(x,r)} V(y)dy \right)^{\frac{1}{q}} \leq C \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} V(y)dy \right),
\]
for every ball $B(x,r) \subset \mathbb{R}^n$. It is worth pointing out that if $V \in RH_q$ for some $q > 1$, then there exists $\epsilon > 0$, such that $V \in RH_{q+\epsilon}$ (see [12]). On the other hand, the Hölder inequality gives that $RH_{q_1} \subset RH_{q_2}$ if $q_1 \geq q_2 > 1$. Therefore, the assumption $q > n/2$ is equivalent to the case $q \geq n/2$. Throughout this paper, we always assume that $V \neq 0$ and $V \in RH_q$ with $q \geq n/2$.

The Schrödinger operator $L$ with nonnegative potentials is very useful in the study of certain subelliptic operators. For instance, by taking the partial Fourier transform in the $t$ variable, the operator $-\Delta_x - V(x)\partial_{x}^2$ is reduced to $-\Delta_x + V(x)\partial_t^2$ (See [19]). Some basic results on $L$, including certain estimates of the fundamental solutions of $L$ and the boundedness on $L^p$ of Riesz transforms $\Delta L^{-1/2}$ were obtained by Fefferman [11], Shen [20] and Zhong [31].

Attention has also been paid to the study of function spaces associated to $L$. It was Dziubański and Zienkiewicz [10] who characterized the Hardy space $H^1_L$ related to the Schrödinger operator. Later on, for $0 < p \leq 1$, the $H^p_L$ space with potentials from reverse Hölder classes were studied in [8] and [9]. Subsequently, Yang et al. [30] characterize the localized Hardy spaces by establishing the boundedness of Riesz transforms, maximal operators and endpoint estimates of fractional integrals associated with $L$.

For the classical Schrödinger operators $L$, there are many interesting results of its associated Riesz transforms, which essentially and heavily depend on the properties of $e^{-tL}$. The properties of semi-group $e^{-tL}$ such as the positivity, Gaussian estimates and off-diagonal estimates play a fundamental role in the study of Riesz transform. The maximal function defined by the semigroup

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$e^{-tL}$ (t > 0) or the Riesz transforms $\Delta L^{-1/2}$ were further generalized by Lin, et. al. [16] to the setting of Heisenberg groups.

In order to introduce more results, we need to give some definitions. The semi-group maximal function associated to the Schrödinger operator $L$ is defined by

$$T^*(f)(x) = \sup_{t > 0} |e^{-tL}f(x)| = \sup_{t > 0} \left| \int_{\mathbb{R}^n} k_t(x, y)f(y)dy \right|,$$

where $k_t$ is the kernel of the operator $e^{-tL}$. As in [20], we will use the auxiliary function $\rho$ defined for $\mathbb{R}^n$ as

$$\rho(x) = \sup_{r > 0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x, r)} V(y)dy \leq 1 \right\}.$$

**Remark 1.1.** Under the above assumptions on $V$, it is easy to get that $0 < \rho(x) < \infty$. In particular, $\rho(x) = 1$ with $V = 1$. For more details concerning the function $\rho(x)$ and its applications in studying the Schrödinger operator $L$, we refer the reader to [11, 20, 21].

For $1 < p < \infty$, the $A^0_p$ weights class is defined as follows.

**Definition 1.2.** ($A^0_p$ weights class, [2]). Let $w$ be a nonnegative, locally integrable function on $\mathbb{R}^n$. For $1 < p < \infty$, we say that a weight $w$ belongs to the class $A^0_p$ if there exists a positive constant $C$ such that for all balls $B = B(x, r)$, it holds that

$$\left( \frac{1}{|B|} \int_B w(y)dy \right) \left( \frac{1}{|B|} \int_B w(y)^{-1/(p-1)}dy \right)^{p-1} \leq C \left( 1 + \frac{r}{\rho(x)} \right)^{\theta_p}.$$

$w$ is said to satisfy the $A^0_p$ condition if there exists a constant $C$ such that for all balls $B$

$$M^0_p(w)(x) \leq Cw(x), \text{ a.e. } x \in \mathbb{R}^n,$$

where

$$M^0_p(f)(x) = \sup_{x \in B} \left( 1 + \frac{r}{\rho(x)} \right)^{\theta} \int_B |f(y)|dy.$$

**Remark 1.3.** Clearly, the classes $A^0_p$ are increasing with $\theta$, and $A^0_p \subset A^0_p$ for $1 \leq p < \infty$. Moreover, from the Remark 1.6 below, it is easy to see that $A^0_p \subset A^0_p$.

**Definition 1.4.** ($BMO_p(\mathbb{R}^n)$ space, [1]). For $\theta > 0$, we defined the class $BMO_p(\mathbb{R}^n)$ of locally integrable functions $f$ such that

$$\frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f_B|dy \leq C \left( 1 + \frac{r}{\rho(x)} \right)$$

for all $x \in \mathbb{R}^n$ and $r > 0$, where $f_B = \frac{1}{|B|} \int_B f$. A norm for $f \in BMO_p(\mathbb{R}^n)$, denoted by $\|f\|_{BMO_p(\mathbb{R}^n)}$, is given by the infimum of the constants satisfying (1.4), after identifying functions that differ upon a constant. Clearly $BMO(\mathbb{R}^n) \subset BMO_p(\mathbb{R}^n) \subset BMO_{p'}(\mathbb{R}^n)$ for $0 < \theta < \theta'$.

The commutator of $T^*$ with $b \in BMO_p(\mathbb{R}^n)$ is defined by

$$T^*_b(f)(x) = \sup_{t > 0} \left| \int_{\mathbb{R}^n} k_t(x, y)(b(x) - b(y))f(y)dy \right|.$$

In 2011, Bongioanni, Harboure and Salinas [1] considered the $L^p(\mathbb{R}^n)(1 < p < \infty)$ boundedness of the commutators of Riesz transforms related to $L$ with $BMO_p(\mathbb{R}^n)$ functions. In another paper, they [2] established the weighted boundedness for the semi-group maximal function, Riesz transforms, fractional integrals and Littlewood-Paley functions related to $L$ with weights belong to $A^0_p$ (see definition 1.2) class which includes the Muckenhoupt weight class $A_p$.

**Theorem A** ([2]). For $1 < p < \infty$, the operators $T^*$ is bounded on $L^p(w)$ when $w \in A^0_p$. 

Recently, Tang [21] considered the weighted norm inequalities for $T^*_b$.

**Theorem B** ([21]). Let $1 < p < \infty$, $w \in A^p_\theta$ and $b \in \text{BMO}_\theta(\rho)(\mathbb{R}^n)$, then there exists a constant $C$ such that

$$||T^*_b(f)||_{L^p(w)} \leq C||b||_{\text{BMO}_\theta(\rho)}||f||_{L^p(w)}.$$

This paper is devoted to studying the weighted compactness for commutators of semi-group maximal function related to Schrödinger operators. Before stating our main results, we recall some background for the compactness of the commutators of some classical operators. Given a locally integrable function $b$, the commutator $[b, T]$ is defined by

$$[b, T](f)(x) = bTf(x) - T(bf)(x).$$

In [1978], Uchiyama [25] first studied the compactness of commutators and showed that the commutator $[b, T_\Omega]$ is compact on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \text{CMO}(\mathbb{R}^n)$, where $T_\Omega$ is a singular integral operator with rough kernel $\Omega \in \text{Lip}_1(S^{n-1})$ and $\text{CMO}(\mathbb{R}^n)$ is the closure of $C_C^\infty(\mathbb{R}^n)$ in the BMO($\mathbb{R}^n$) topology.

Since then, the study on the compactness of commutators of different operators has attracted much more attention. Krantz and Li applied the compactness characterization of the commutator $[b, T_\Omega]$ to study Hankel type operators on Bergman space in [14] and [15]. Wang [26] showed the compactness of the commutator of fractional integral operator form $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. In 2009, Chen and Ding [5] proved the compactness of singular integrals with variable kernels is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \text{CMO}(\mathbb{R}^n)$ and they also established the compactness of Littlewood-Paley square functions in [6]. Later on, Chen, Ding and Wang [7] obtained the compactness of commutators for Marcinkiewicz Integral in Morrey Spaces. Recently, Liu, Wang and Xue [18] showed the compactness of the commutator of oscillatory singular integrals with Hölder class kernels of non-convolutional type. We refer the reader to [3,4,17,23,24,27,29] for the compactness of commutators of multilinear operators.

The above compactness results are all concerned with the space $\text{CMO}(\mathbb{R}^n)$. However, Theorem B shows that the $L^p$ boundedness holds for more larger space BMO$_\theta(\rho)(\mathbb{R}^n)$, rather than $\text{BMO}(\mathbb{R}^n)$ and the weights class $A^p_\theta$ is more larger than $A_p$ weights class. Let $\text{CMO}_\theta(\rho)(\mathbb{R}^n)$ be the closure of $C_C^\infty(\mathbb{R}^n)$ in the BMO$_\theta(\rho)(\mathbb{R}^n)$ topology. Then, it is quite natural to ask the following question:

**Question 1.5.** Is the operator $T^*_b$ compact from $L^p(w)$ to $L^p(w)$ when $w \in A^p_\theta$ and $b$ belongs to the space $\text{CMO}_\theta(\rho)(\mathbb{R}^n)$?

The main purpose of this paper is to give a firm answer to the above question. Our result is as follows:

**Theorem 1.1.** Let $1 < p < \infty$, $w \in A^p_\theta$ and $b \in \text{CMO}_\theta(\rho)(\mathbb{R}^n)$. If $w$ satisfies the following condition

$$(1.6) \quad \lim_{A \to +\infty} A^{-np+n} \int_{|x| > 1} \frac{w(Ax)}{|x|^{np}} dx = 0,$$

then the operator $T^*_b$ defined by (1.1) is a compact operator from $L^p(w)$ to $L^p(w)$.

**Remark 1.6.** We give some comments about Theorem 1.1:

1. The weights class in Theorem 1.1 is more larger than the classical Muckenhoupt weights class $A_p$. In fact, if $w \in A_p$, the classical Muckenhoupt weights class, then the condition (1.6) holds. Let $0 < \gamma < \theta$ and $w(x) = (1 + |x|)^{-\gamma}$, it is easy to see that $w$ satisfies (1.6) and $w(x) \notin A_p$ ($1 \leq p < \infty$), but $w \in A^p_\gamma$ ($1 < p < \infty$) provided that $V = 1$ (see [21]).

2. Obviously, the space $\text{CMO}_\theta(\rho)(\mathbb{R}^n)$ where $b$ belongs is more larger than $\text{CMO}(\mathbb{R}^n)$ space.
The paper is organized as follows. In section 2 we give some definitions and preliminary lemmas, which are the main ingredients of our proofs. In section 3 we will give the proof of Theorem 1.1 via smooth truncated techniques. The domain of integration will be divided into several cases. In actuality some cases are combinable, but various subcases also arise, which increases the difficulty we need to deal with.

Throughout the paper, the letter C or c, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. A ∼ B means that there exists constants C1 > 0 and C2 > 0 such that C2B ≤ A ≤ C1B.

2. Preliminaries

We first recall some notation. Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, $|E|$ will denote the Lebesgue measure of $E$. If $B = B(x, r)$ is a ball in $\mathbb{R}^n$ and $\lambda$ is a real number, then $\lambda B$ shall stand for the ball with the same center as $B$ and radius $\lambda$ times that of $B$. A weight $w$ is a non-negative measurable function on $\mathbb{R}^n$. The measure associated with $w$ is the set function given by $w(E) = \int_E w(x) dx$. For $0 < p < \infty$ we denote by $L^p(w)$ the space of all Lebesgue measurable function $f(x)$ such that

$$\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}.$$

The auxiliary function $\rho$ enjoys the following property.

**Lemma 2.1.** ([20]). There exists $k_0 \geq 1$ and $C > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$C^{-1} \rho(x) \left( 1 + \frac{|x-y|}{\rho(x)} \right)^{-k_0} \leq \rho(y) \leq C \rho(x) \left( 1 + \frac{|x-y|}{\rho(x)} \right)^{k_0 \eta^{-p}}.$$

In particular, $\rho(x) \sim \rho(y)$ if $|x-y| < C \rho(x)$.

$A_p^\theta$ weights class has some properties analogy to $A_p$ weights class for $1 \leq p < \infty$.

**Lemma 2.2.** ([2] [22]). Let $1 < p < \infty$ and $w \in A_p^\infty = \bigcup_{\theta \geq 0} A_p^{\theta}$. Then

(i) If $1 \leq p_1 < p_2 < \infty$, then $A_p^{\theta_1} \subset A_p^{\theta_2}$.

(ii) $w \in A_p^{\theta_1}$ if and only if $w^{1/p_1} \in A_{p_1}^{\theta_1}$, where $1/p + 1/p' = 1$.

(iii) If $w \in A_p^{\infty}$, $1 < p < \infty$, then there exists $\varepsilon > 0$ such that $w \in A_p^{\infty - \varepsilon}$.

It should be pointed out that (iii) of Lemma 2.2 was proved by Bongioanni, Harboure and Salinas in [2].

For convenience, we write $\Psi_0(B) = (1 + r/\rho(x_0))^\theta$, if $B = B(x, r_0)$. Then $M_p^\theta$ can be rewritten as

$$M_p^\theta(f)(x) = \sup_{x \in B} \frac{1}{\Psi_0(B)|B|} \int_B |f(y)| dy,$$

and the following result holds:

**Lemma 2.3.** ([22]). Let $1 < p < \infty$ and suppose that $w \in A_p^{\theta}$. If $p < p_1 < \infty$, then

$$\int_{\mathbb{R}^n} |M_p^\theta f(x)|^{p_1} w(x) dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^{p_1} w(x) dx.$$

By the Lemma 2.3, $M_p^\theta$ may not be bounded on $L^p(w)$ for all $w \in A_p^{\theta}$ and $1 < p < \infty$. So we need the variant maximal operator $M_{V, \eta}$ defined by

$$M_{V, \eta} f(x) = \sup_{x \in B} \frac{1}{\Psi_0(B)|B|} \int_B |f(y)| dy, \quad 0 < \eta < \infty.$$

We have the following Lemma.
Lemma 2.4. ([22]). Let $1 < p < \infty$, $p' = p/(p - 1)$ and suppose that $w \in A_p^{\rho, \theta}$. Then there exists a constant $C > 0$ such that
\[ \|Mv_p f\|_{L^p(w)} \leq C\|f\|_{L^p(w)}. \]

We also need the following properties of the kernel $k_i$.

Lemma 2.5. ([8], [13]). For every $N$, there is a constant $C_N$ such that
\[ 0 < k_i(x, y) \leq C_N r^{-\frac{\rho}{\rho - 1}} e^{-\frac{x - y}{\rho}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}. \]

Lemma 2.6. ([9]). There exists $0 < \delta < 1$ and a constant $c > 0$ such that for every $N > 0$ there is a constant $C_N > 0$ so that, for all $|h| \leq \sqrt{t}$
\[ |k_i(x + h) - k_i(x, y)| \leq C_N \frac{|h|}{\sqrt{t}} \delta r^{-\frac{\rho}{\rho - 1}} e^{-\frac{\rho |h|}{\rho - 1}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}. \]

We end this section by introducing the general weighted version of Frechet-Kolmogorov theorems, which was proved by Xue, Yabuta and Yan in [29].

Lemma 2.7. ([29]). Let $w$ be a weight on $\mathbb{R}^n$. Assume that $w^{-1/(pn - 1)}$ is also a weight on $\mathbb{R}^n$ for some $p_0 > 1$. Let $0 < p < \infty$ and $\mathcal{F}$ be a subset in $L^p(w)$, then $\mathcal{F}$ is sequentially compact in $L^p(w)$ if the following three conditions are satisfied:

(i) $\mathcal{F}$ is bounded, i.e., $\sup_{f \in \mathcal{F}} \|f\|_{L^p(w)} < \infty$;

(ii) $\mathcal{F}$ uniformly vanishes at infinity, i.e.,
\[ \lim_{N \to \infty} \sup_{f \in \mathcal{F}} \int_{|x| > N} |f(x)|^p w(x) dx = 0; \]

(iii) $\mathcal{F}$ is uniformly equicontinuous, i.e.,
\[ \lim_{|h| \to 0} \sup_{f \in \mathcal{F}} \int_{\mathbb{R}^n} |f(\cdot + h) - f(\cdot)|^p w(x) dx = 0. \]

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. We shall prove Theorem 1.1 via smooth truncated techniques. First, we introduce the following smooth truncated function. Let $\Phi \in C^\infty([0, \infty))$ satisfy
\[ 0 \leq \Phi \leq 1 \quad \text{and} \quad \Phi(x) = \begin{cases} 1, & x \in [0, 1], \\ 0, & x \in [2, \infty). \end{cases} \]

For any $\gamma > 0$, let
\[ k_{\gamma, \gamma}(x, y) = k_i(x, y) \left(1 - \Phi(\gamma^{-1}|x - y|)\right). \]

Define
\[ T^{\gamma}_e f(x) = \sup_{\gamma > 0} \left| \int_{\mathbb{R}^n} k_{\gamma, \gamma}(x, y) f(y) dy \right|. \]

and
\[ T^{\gamma}_w f(x) = \sup_{\gamma > 0} \left| \int_{\mathbb{R}^n} k_{\gamma, \gamma}(x, y)(b(x) - b(y)) f(y) dy \right|. \]
For any $b \in C_c^\infty(\mathbb{R}^n)$ and $\gamma, \eta > 0$, by (3.2), (3.4) and lemma 2.5 with $N = \eta$, one has

$$\left| T_b^* f(x) - T_{b, \gamma}^* f(x) \right| \leq C \gamma \sup_{\tau > 0} \int_{|x-y| < \tau} t^{-n/2} e^{-\frac{|x-y|^2}{2\tau^2}} \left( 1 + \frac{\sqrt{\tau}}{\rho(x)} \right)^{-\eta} |f(y)| dy$$

$$\leq C \gamma \left\{ \sup_{\sqrt{\tau} < \gamma} \int_{|x-y| < \sqrt{\tau}} t^{-n/2} e^{-\frac{|x-y|^2}{2\tau^2}} \left( 1 + \frac{\sqrt{\tau}}{\rho(x)} \right)^{-\eta} |f(y)| dy \right.$$

$$+ \sup_{\sqrt{\tau} < \gamma \leq |x-y| < 2\gamma} t^{-n/2} e^{-\frac{|x-y|^2}{2\tau^2}} \left( 1 + \frac{\sqrt{\tau}}{\rho(x)} \right)^{-\eta} |f(y)| dy \right.$$ (3.5)

$$\left. + \sup_{\sqrt{\tau} \geq |x-y| < 2\gamma} t^{-n/2} e^{-\frac{|x-y|^2}{2\tau^2}} \left( 1 + \frac{\sqrt{\tau}}{\rho(x)} \right)^{-\eta} |f(y)| dy \right\} =: C \gamma \{ J_1 + J_2 + J_3 \}.$$

One may obtain

$$J_1 \leq \sup_{\sqrt{\tau} < \gamma} t^{-n/2} \left( 1 + \frac{\sqrt{\tau}}{\rho(x)} \right)^{-\eta} \int_{|x-y| < \sqrt{\tau}} |f(y)| dy \leq CM_{V, \eta} f(x).$$

and

$$J_3 \leq 2^{2\eta} \sup_{\sqrt{\tau} \geq |x-y| < 2\gamma} \left( 1 + \frac{2\gamma}{\rho(x)} \right)^{-\eta} \int_{|x-y| < 2\gamma} |f(y)| dy \leq CM_{V, \eta} f(x).$$

It remains to estimate $J_2$. Using the estimate $e^{-x} \leq \frac{C}{|x|^M}$ with $M > n + \eta$ and splitting to annuli, it follows that

$$J_1 \leq \sum_{k=1}^{\infty} \left( \frac{M-n}{2} \right)^{2k} \left( 1 + \frac{\sqrt{\tau}}{\rho(x)} \right)^{-\eta} \int_{|x-y| \sim 2^k \sqrt{\tau}} |f(y)| dy$$

$$\leq \sum_{k=1}^{\infty} \left( \frac{2^{-k(M-n-\eta)}}{2^{k} \sqrt{\tau}} \right)^{\eta} \int_{|x-y| < 2^k \sqrt{\tau}} |f(y)| dy \leq CM_{V, \eta} f(x).$$

Combining (3.8) with (3.5), (3.6) and (3.7) may lead to

$$\left| T_b^* f(x) - T_{b, \gamma}^* f(x) \right| \leq C \gamma M_{V, \eta} f(x).$$

Then Lemma 2.4 with $p' \leq \eta < \infty$ gives that

$$\| T_b^* f - T_{b, \gamma}^* f \|_{L^p(w)} \leq C \gamma \| f \|_{L^p(w)},$$

which implies that

$$\lim_{\gamma \to 0} \| T_b^* f - T_{b, \gamma}^* f \|_{L^p(w)} = 0.$$

On the other hand, if $b \in \text{CMO}_0(\mathbb{R})$, then for any $\epsilon > 0$, there exists $b_\epsilon \in C_c^\infty(\mathbb{R}^n)$ such that $\| b - b_\epsilon \|_{\text{BMO}(\mathbb{R})} < \epsilon$, so that

$$\| b - b_\epsilon \|_{\text{BMO}(\mathbb{R})} < \epsilon, \quad \text{so that}$$

$$\| T_b^* f - T_{b, \gamma}^* f \|_{L^p(w)} \leq \| T_{b-\epsilon}^* f \|_{L^p(w)} \leq C \| b - b_\epsilon \|_{\text{BMO}(\mathbb{R})} \| f \|_{L^p(w)} \leq C \epsilon.$$

Thus, to prove $T_b^* f$ is compact on $L^p(w)$ for any $b \in \text{CMO}_0(\mathbb{R})$, it suffices to prove that $T_b^* f$ is compact on $L^p(w)$ for any $b \in C_c^\infty(\mathbb{R}^n)$. By (3.9) and [28], it suffices to show that $T_{b, \gamma}^*$ is compact for any $b \in C_c^\infty(\mathbb{R}^n)$ when $\gamma > 0$ is small enough. To this end, for arbitrary bounded set $F$ in $L^p(w)$, let

$$\mathcal{F} = \{ f \in F : T_{b, \gamma}^* f = f \}.$$

Then, we need to show that for $b \in C_c^\infty(\mathbb{R}^n)$, $\mathcal{F}$ satisfies the conditions (i)-(iii) of Lemma 2.7.
From the definition of $k_{r,\gamma}$, we know that $0 < k_{r,\gamma}(x, y) \leq k_{1}(x, y)$, then $T_{q}^{*}f(x) \leq T^{*}(|f|)(x)$ and $T_{h,\gamma}^{*}f(x) \leq T^{*}(|f|)(x)$. Hence, the boundedness of $T_{q}^{*}$ and $T_{h,\gamma}^{*}$ also holds. Thus, we have
\[
\sup_{f \in F} \|T_{h,\gamma}^{*}f\|_{L^{p}(w)} \leq C \sup_{f \in F} \|f\|_{L^{p}(w)} \leq C,
\]
which yields the fact that the set $\mathcal{F}$ is bounded.

Assume $b \in C^{\infty}_{c}(\mathbb{R}^{n})$ and $\text{supp}(b) \subset B(0, R)$, where $B(0, R)$ is the ball of radius $R$ center at origin in $\mathbb{R}^{n}$. For any $|x| > A > 2R$, $w \in A_{p}^{\theta}$, $1 < p < \infty$ and $f \in F$. By Lemma 2.5 and the estimate $e^{-\frac{|x|^{2}}{8R}} \leq C \frac{R}{|x|^{\frac{n}{2}-1}}$, we have
\[
|T_{b,\gamma}^{*}f(x)| \leq \sup_{t > 0} \int_{|y| < R} k_{t}(x, y)|b(y)f(y)|\frac{dy}{|y|^{n}}
\leq C \sup_{t > 0} t^{-n/2} \left(1 + \frac{1}{\sqrt{t}}\right)^{-N} \int_{|y| < R} e^{-\frac{|y|^{2}}{8R}}|f(y)|\frac{dy}{|y|^{n}}
\leq C|x|^{-n} \int_{|y| < R} |f(y)|\frac{dy}{|y|^{n}}
\leq C|x|^{-n} \|f\|_{L^{p}(w)} \left(\int_{|y| < R} w^{-p/p}(y)\frac{dy}{|y|^{n}}\right)^{1/p'}.
\]
Therefore
\[
\int_{|x| > A} |T_{b,\gamma}^{*}f(x)|^{p}w(x)\frac{dx}{|x|^{n}} \leq C \int_{|x| > A} \frac{w(x)}{|x|^{np}}\frac{dx}{|x|^{n}}
= C \sum_{j=0}^{\infty} \int_{2^{j+1}A < |x| < 2^{j+1}A} \frac{w(x)}{|x|^{np}}\frac{dx}{|x|^{n}}
= CA^{-np+n} \int_{|x| > 1} \frac{w(Ax)}{|x|^{np}}\frac{dx}{|x|^{n}}.
\]
This together with the condition (1.6) yields that
\[
\lim_{A \to \infty} \int_{|x| > A} |T_{b,\gamma}^{*}f(x)|^{p}w(x)\frac{dx}{|x|^{n}} = 0,
\]
whenever $f \in F$.

It remains to show that the set $\mathcal{F}$ is uniformly equicontinuous. It suffices to verify that
\[
\lim_{|h| \to 0} \|T_{b,\gamma}^{*}f(h + \cdot) - T_{b,\gamma}^{*}f(\cdot)\|_{L^{p}(w)} = 0,
\]
holds uniformly for $f \in F$.

In what follows, we fix $\gamma \in (0, \frac{1}{2})$ and $|h| < \frac{\gamma}{4}$. Then
\[
|T_{b,\gamma}^{*}f(x + h) - T_{b,\gamma}^{*}f(x)| \leq \sup_{t > 0} \int_{\mathbb{R}^{n}} |k_{t,\gamma}(x + h, y) - k_{t,\gamma}(x, y)| |b(x + h) - b(y)||f(y)|\frac{dy}{|y|^{n}}
+ \sup_{t > 0} \int_{\mathbb{R}^{n}} k_{t,\gamma}(x, y) |b(x + h) - b(x)||f(y)|\frac{dy}{|y|^{n}}
=: I(x) + II(x).
\]
For $II(x)$, it holds that
\[
II(x) = |b(x + h) - b(x)| \sup_{t > 0} \int_{\mathbb{R}^{n}} |k_{t,\gamma}(x, y)||f(y)|\frac{dy}{|y|^{n}}
\leq C|h| T_{q}^{*}(|f|)(x).
\]
Then, by the $L^{p}(w)$-bounds of $T_{q}^{*}$, we have
\[
\|II\|_{L^{p}(w)} \leq C|h||f||_{L^{p}(w)}.
\]
For $I(x)$, we decompose it into two parts

$$I(x) \leq \sup_{\sqrt{T} \geq |h|} \int_{\mathbb{R}^n} |k_{i,\gamma}(x+h,y) - k_{i,\gamma}(x,y)||b(x+h) - b(y)||f(y)|dy$$

$$+ \sup_{\sqrt{t} < |h|} \int_{\mathbb{R}^n} |k_{i,\gamma}(x+h,y) - k_{i,\gamma}(x,y)||b(x+h) - b(y)||f(y)|dy$$

$$=: I_1(x) + I_2(x).$$

**Contribution of $I_1$.** For $I_1(x)$, if $|h| \leq \sqrt{T}$, then by lemma 2.5 and lemma 2.6, we have

$$|k_{i,\gamma}(x+h,y) - k_{i,\gamma}(x,y)| \leq |k_i(x+h) - k_i(x,y)| + |k_i(x+h) - k_i(x,y)|\phi'(\gamma^{-1}|x+h-y|)$$

$$+ k_i(x,y)|\phi'(\gamma^{-1}|x+h-y|) - \phi'(\gamma^{-1}|x-y|)|$$

$$\leq C \left( \frac{|h|}{\sqrt{T}} \right)^{\delta} \frac{t^{-\frac{N}{2}} e^{-\frac{\epsilon_t}{T}}} \left( 1 + \frac{\sqrt{T}}{\rho(x)} + \frac{\sqrt{T}}{\rho(y)} \right)^{-N}$$

$$+ C \frac{|h|}{\gamma} t^{-\frac{N}{2}} e^{-\frac{\epsilon_t}{T}} \left( 1 + \frac{\sqrt{T}}{\rho(x)} + \frac{\sqrt{T}}{\rho(y)} \right)^{-N}.$$  

Therefore, we have

$$I_1(x) \leq C \sup_{\sqrt{T} \geq |h|} \left( \left( \frac{|h|}{\sqrt{T}} \right)^{\delta} + \frac{|h|}{\gamma} \right) t^{-\frac{N}{2}} e^{-\frac{\epsilon_t}{T}} \left( 1 + \frac{\sqrt{T}}{\rho(x)} \right)^{-N}$$

$$\times |b(x+h) - b(y)||f(y)|dy$$

$$\leq C \sup_{\sqrt{T} \geq |h|} \left( \int_{|x-y| < \sqrt{T}} + \int_{|x-y| \geq \sqrt{T}} \right) \left( \left( \frac{|h|}{\sqrt{T}} \right)^{\delta} + \frac{|h|}{\gamma} \right) t^{-\frac{N}{2}} e^{-\frac{\epsilon_t}{T}}$$

$$\times \left( 1 + \frac{\sqrt{T}}{\rho(x)} \right)^{-N} |b(x+h) - b(y)||f(y)|dy$$

$$+ C \sup_{|h| \leq \sqrt{T} < 1} \left( \int_{|x-y| < \sqrt{T}} + \int_{|x-y| \geq \sqrt{T}} \right) \left( \left( \frac{|h|}{\sqrt{T}} \right)^{\delta} + \frac{|h|}{\gamma} \right) t^{-\frac{N}{2}} e^{-\frac{\epsilon_t}{T}}$$

$$\times \left( 1 + \frac{\sqrt{T}}{\rho(x)} \right)^{-N} |b(x+h) - b(y)||f(y)|dy$$

$$=: I_{11}(x) + I_{12}(x) + I_{13}(x) + I_{14}(x).$$

Now, we are in the position to estimate the above four terms.

For $I_{11}(x)$, if $\sqrt{T} \geq 1$, then $t^{-\delta/2} \leq 1$. Taking $N = \Theta \eta$ for any $\Theta, \eta > 0$, then we have

$$I_{11}(x) \leq C \gamma^{-1} (|h|^\delta + |h|) \sup_{\sqrt{T} \geq 1} \int_{|x-y| < \sqrt{T}} t^{-\frac{\delta}{2}} \left( 1 + \frac{\sqrt{T}}{\rho(x)} \right)^{-\Theta \eta} |f(y)|dy$$

$$\leq C \gamma^{-1} (|h|^\delta + |h|) \sup_{\sqrt{T} \geq 1} \int_{|x-y| < \sqrt{T}} \left( 1 + \frac{\sqrt{T}}{\rho(x)} \right)^{-\Theta \eta} |f(y)|dy$$

$$\leq C \gamma^{-1} (|h|^\delta + |h|) \mathcal{M}_{V,\eta} f(x).$$

In order to estimate $I_{12}(x)$, we need the following inequality: for any $M > 0$, there exists a constant $C > 0$, such that

$$e^{-\frac{\epsilon_t}{|x-y|^M}} \leq C \frac{t^{\frac{M}{2}}}{|x-y|^M}.$$
Using (3.17) with $M > n + \theta \eta$, splitting into annuli, we obtain
\begin{equation}
I_{12}(x) \leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{|x-y| \geq \sqrt{\gamma}} t^{\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| \geq \sqrt{\gamma}} \frac{|f(y)|}{|x-y|^M} dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{\sqrt{\gamma} \leq 1} \sum_{k=1}^{\infty} 2^{-k(M-n-\theta \eta)} \int_{|x-y| < 2^k \sqrt{\gamma}} |f(y)| dy
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{\sqrt{\gamma} \leq 1} \sum_{k=1}^{\infty} 2^{-k(M-n-\theta \eta)} \int_{|x-y| < 2^k \sqrt{\gamma}} |f(y)| dy
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{|x-y| \leq \sqrt{\gamma}} t^{\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| < \sqrt{\gamma}} \frac{|f(y)|}{|x-y|^M} dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{|x-y| \leq \sqrt{\gamma}} t^{\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| < \sqrt{\gamma}} \frac{|f(y)|}{|x-y|^M} dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{|x-y| < \sqrt{\gamma}} t^{\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| < \sqrt{\gamma}} \frac{|f(y)|}{|x-y|^M} dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{|x-y| < \sqrt{\gamma}} t^{\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| < \sqrt{\gamma}} \frac{|f(y)|}{|x-y|^M} dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{|x-y| < \sqrt{\gamma}} t^{\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| < \sqrt{\gamma}} \frac{|f(y)|}{|x-y|^M} dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{|x-y| < \sqrt{\gamma}} M_{V, \eta} f(x).
\end{equation}

If $\sqrt{\gamma} < 1$, then $t^{-\delta/2} < t^{-1/2}$. For any $\theta, \eta > 0$, taking $N = \theta \eta$. For $I_{13}(x)$, if $|h| \leq \sqrt{\gamma}$, $|x-y| < \sqrt{\gamma}$ and $b \in C^\infty_c(\mathbb{R}^n)$, then
\begin{equation}
|b(x+h) - b(y)| \leq C|x+h-y| \leq C(|x-y| + |h|) \leq C\sqrt{\gamma}.
\end{equation}

Then, it follows that
\begin{equation}
I_{13}(x) \leq C|h|^\delta \sup_{|h| \leq \sqrt{\gamma}} t^{-\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| < \sqrt{\gamma}} |b(x+h) - b(y)||f(y)| dy}
+ C\gamma^{-1}|h| \sup_{|h| \leq \sqrt{\gamma}} t^{-\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| < \sqrt{\gamma}} |f(y)| dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{|h| \leq \sqrt{\gamma}} t^{-\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| < \sqrt{\gamma}} |f(y)| dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) M_{V, \eta} f(x).
\end{equation}

For $I_{14}(x)$, if $|x-y| < 2^k \sqrt{\gamma}$, $k = 1, 2, \ldots$, and $|h| \leq \sqrt{\gamma}$, $b \in C^\infty_c(\mathbb{R}^n)$, then
\begin{equation}
|b(x+h) - b(y)| \leq C|x+h-y| \leq C(|x-y| + |h|) \leq C2^k \sqrt{\gamma},
\end{equation}

which combining with (3.17) for $M > n + 1 + \theta \eta$ yields that
\begin{equation}
I_{14}(x) \leq C|h|^\delta \sup_{|h| \leq \sqrt{\gamma}} t^{-\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| \geq \sqrt{\gamma}} |f(y)| \frac{|b(x+h) - b(y)|}{|x-y|^M} dy}
+ C\gamma^{-1}|h| \sup_{|h| \leq \sqrt{\gamma}} t^{-\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| \geq \sqrt{\gamma}} |f(y)| \frac{|b(x+h) - b(y)|}{|x-y|^M} dy}
\leq C|h|^\delta \sup_{|h| \leq \sqrt{\gamma}} t^{-\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| \geq \sqrt{\gamma}} |f(y)| \frac{|b(x+h) - b(y)|}{|x-y|^M} dy}
+ C\gamma^{-1}|h| \sup_{|h| \leq \sqrt{\gamma}} t^{-\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| \geq \sqrt{\gamma}} |f(y)| \frac{|b(x+h) - b(y)|}{|x-y|^M} dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) \sup_{|h| \leq \sqrt{\gamma}} t^{-\frac{M-n}{2} \left(1 + \frac{\sqrt{\gamma}}{\rho(x)}\right)^{\theta \eta} \int_{|x-y| \geq \sqrt{\gamma}} |f(y)| \frac{|b(x+h) - b(y)|}{|x-y|^M} dy}
\leq C\gamma^{-1}(|h|^\delta + |h|) M_{V, \eta} f(x).
\end{equation}

Sum up (3.15), (3.16), (3.18), (3.19) and (3.20) in all, we get
\begin{equation}
I_1(x) \leq C\gamma^{-1}(|h|^\delta + |h|) M_{V, \eta} f(x).
\end{equation}
Contribution of $I_2$. Next we will estimate $I_2(x)$. When $|x - y| < \frac{\delta}{2}$ and $|h| < \frac{\delta}{4}$, then $|x + h - y| < \frac{3\delta}{4}$. Hence $\varphi(\gamma^{-1}|x + h - y|) = 1 = \varphi(\gamma^{-1}|x - y|)$. This implies $k_{\gamma}(x + h, y) = 0 = k_{\gamma}(x, y)$. For $I_2(x)$, we decompose it as follows:

$$I_2(x) \leq \sup_{\sqrt{\gamma} < |h| < \rho(x)} \int_{|x - y| \geq \frac{\delta}{2}} |k_{\gamma}(x + h, y) - k_{\gamma}(x, y)||b(x + h) - b(y)||f(y)|dy$$

$$+ \sup_{\sqrt{\gamma} < |h| \leq |h|} \int_{|x - y| \geq \frac{\delta}{2}} |k_{\gamma}(x + h, y) - k_{\gamma}(x, y)||b(x + h) - b(y)||f(y)|dy$$

(3.22)

$$:= I_{21}(x) + I_{22}(x).$$

For $I_{21}(x)$, since $|x - y| > \frac{\delta}{2} > 2|h|$ and $\sqrt{\gamma} < |h| < \rho(x)$, then $|x + h - y| \sim |x - y|$, $|h|/\rho(x) < 1$ and $(|h|/\sqrt{\gamma})^M > 1$ for any $M > 0$. Choosing $M > n + 1 + \theta \eta$ and using Lemma 2.5 and (3.17), we get

$$I_{21}(x) = \sup_{\sqrt{\gamma} < |h| < \rho(x)} \int_{|x - y| \geq \frac{\delta}{2}} |k_{\gamma}(x + h, y) - k_{\gamma}(x, y)||b(x + h) - b(y)||f(y)|dy$$

$$\leq C \sup_{\sqrt{\gamma} < |h| < \rho(x)} \int_{|x - y| \geq 2|h|} t^{-\frac{\eta}{2}} \left( \frac{|h|}{\sqrt{\gamma}} \right) M^{-\eta} e^{-\frac{\lambda_1|y|^2}{t}} |x - y||f(y)|dy$$

(3.23)

$$\leq C |h|^{M-n} \sup_{\sqrt{\gamma} < |h| < \rho(x)} \sum_{k=2}^{\infty} \int_{|x - y| \geq 2^k|h|} |f(y)|dy$$

$$\leq C |h|^{M-n} \sup_{\sqrt{\gamma} < |h| < \rho(x)} \sum_{k=2}^{\infty} \frac{2^{-k(M-n-1-\theta \eta)}}{(2^k|h|)^n2^{k\theta \eta}} \int_{|x - y| < 2^k|h|} |f(y)|dy$$

$$\leq C |h| |M|^{\nu \eta} f(x).$$

Finally, it remains to consider $I_{22}(x)$. Since $b \in C_{\infty}^\omega([\mathbb{R}^n])$, $|x - y| \geq 2|h|$, $\sqrt{\gamma} < |h|$, then $|h|/\sqrt{\gamma} > 1$ and $|b(x + h) - b(x)| \leq C|x - y|$. In addition, if $|x - y| < 2^l \rho(x)$, $l = 1, 2, \cdots$, then by lemma 2.1, we have

$$\rho(y) \leq C \frac{h_{\rho(x)}}{\rho(x)} \rho(x).$$

Then, it follows that

$$\left(1 + \frac{\sqrt{\gamma}}{\rho(x)} + \frac{\sqrt{\gamma}}{\rho(y)}\right)^{-N} \leq 2 \left(1 + \frac{\sqrt{\gamma}}{\rho(y)}\right)^{-N} \leq C_N \left(1 + \frac{\sqrt{\gamma}}{\rho(y)}\right)^{-N}. $$

Choosing $M, N$ such that $N > M > n + 1 + (k_0 + 1)\theta \eta$, and applying lemma 2.5 and (3.17) again, we obtain

$$I_{22}(x) \leq C |h| \sup_{\sqrt{\gamma} < |h| < \rho(x)} \int_{|x - y| \geq 2^l \rho(x)} t^{-\frac{n+1}{2}} e^{-\frac{\lambda_1|y|^2}{t}} \left(1 + \frac{\sqrt{\gamma}}{\rho(y)}\right)^{-N} |x - y||f(y)|dy$$

(3.24)

$$\leq C |h| \sup_{\sqrt{\gamma} < |h| < \rho(x)} \sum_{l=2}^{\infty} \left(1 + \frac{2^{\frac{n+1}{2}}}{\rho(x)} \sqrt{\gamma} \right)^{-N} \int_{|x - y| > 2^l \rho(x)} |f(y)|dy$$

$$\leq C |h| \sum_{l=2}^{\infty} \left(1 + \frac{2^{\frac{n+1}{2}}}{\rho(x)} \sqrt{\gamma} \right)^{-N} \frac{2^{-l(M-n-1-\theta \eta)}}{(2^l \rho(x))^{n2^{l\theta \eta}}}.$$
\[ \int \int_{|x-y|<2^k p(x)} |f(y)|dy \leq C|h| \sup_{\nu \geq |h|} \sum_{k=2}^{\infty} \frac{2^{-k}M_{\nu+1} - \theta \eta}{(2^\nu p(x))^{\nu/2^k}} \int_{|x-y|<2^k p(x)} |f(y)|dy \leq C|h|M_{\nu \gamma} f(x). \]

Inequality (3.24) together with (3.22) and (3.23) gives that

(3.25) \[ I_2(x) \leq C|h|M_{\nu \gamma} f(x). \]

Therefore, by (3.13) and (3.21) we have

\[ I(x) \leq C(|h| + |h|^{\delta})M_{\nu \gamma} f(x). \]

By Lemma 2.4 for any \( p' \leq \eta < \infty \), it holds that

(3.26) \[ \|I\|_{L^p(w)} \leq C(|h| + |h|^{\delta})\|M_{\nu \gamma} f\|_{L^p(w)} \leq C(|h| + |h|^{\delta})\|f\|_{L^p(w)}. \]

From (3.11), (3.12) and (3.26), we get

\[ \|\mathcal{T}_{h,\gamma}^* f(h + \cdot) - \mathcal{T}_{h,\gamma}^* f(\cdot)\|_{L^p(w)} \leq C(|h| + |h|^{\delta})\|f\|_{L^p(w)}, \]

which yields (3.10) and finishes the proof of Theorem 1.1. \( \square \)

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