INVARIANT MEANS AND THE STRUCTURE OF INNER AMENABLE GROUPS

ROBIN D. TUCKER-DROB

ABSTRACT. We study actions of countable discrete groups which are amenable in the sense that there exists a mean on X which is invariant under the action of G. Assuming that G is nonamenable, we obtain structural results for the stabilizer subgroups of amenable actions which allow us to relate the first $\ell^2$-Betti number of G with that of the stabilizer subgroups. An analogous relationship is also shown to hold for cost. This relationship becomes even more pronounced for transitive amenable actions, leading to a simple criterion for vanishing of the first $\ell^2$-Betti number and triviality of cost. Moreover, for any marked finitely generated nonamenable group G we establish a uniform isoperimetric threshold for Schreier graphs $G/H$ of G, beyond which the group H is necessarily weakly normal in G.

Even more can be said in the particular case of an atomless mean for the conjugation action – that is, when G is inner amenable. We show that inner amenable groups have cost 1 and moreover they have fixed price. We establish $\mathcal{Z}_{\text{fin}}$-cocycle superrigidity for the Bernoulli shift of any nonamenable inner amenable group. In addition, we provide a concrete structure theorem for inner amenable linear groups over an arbitrary field. We also completely characterize linear groups which are stable in the sense of Jones and Schmidt. Our analysis of stability leads to many new examples of stable groups; notably, all nontrivial countable subgroups of the group $H(\mathbb{R})$, recently studied by Monod, are stable. This includes nonamenable groups constructed by Monod and by Lodha and Moore, as well as Thompson’s group $F$.

INTRODUCTION

0.A. Amenable actions. An action of a discrete group G on a set X is said to be amenable if there exists a finitely additive probability measure $m : \mathcal{P}(X) \to [0, 1]$, henceforth called a mean, defined on the powerset of X, which is invariant under the action of G. This definition goes back to von Neumann’s 1929 memoir on paradoxicality [56], where the notion of amenability of a group simpliciter was also introduced: by definition, G is amenable if the left translation action of G on itself is amenable in the above sense.

Every action of an amenable group is amenable, and for a long time this simple observation could account for most of the known examples of amenability in actions. A more systematic study of actions whose amenability could not be traced back to that of some acting group began with van Douwen’s constructions of amenable actions of the free group [55], and has continued in recent years with [39, 44, 13, 19, 10, 11, 42, 11, 14, 27, 29, 26]. A stunning recent application of amenable actions is in the article [27] of Juschenko and Monod in which the authors turn the classical implication on its head, deducing amenability

2010 Mathematics Subject Classification. Primary 37A20, 43A07; Secondary 20H20.

There are early examples of actions of nonamenable groups with asymptotic fixed points (e.g., in [13]), although the amenability of such actions was not adduced until much later (e.g., in [13, 51, 5, 25]).
of a group from that of an action, thereby providing the first examples of infinite finitely generated simple amenable groups.

If a nonamenable group $G$ acts amenably on $X$ then it is well known that this action is far from being free: any $G$-invariant mean $m$ on $X$ must concentrate on the set of points $x \in X$ whose associated stabilizer subgroup $G_x$ is nonamenable. Our first result strengthens this considerably by showing that, on a $m$-conull set, the subgroups $G_x$ are in fact so large in $G$ as to be “visible from above.” The precise statement uses the following variation of Popa’s notions of $q$-normality and $wq$-normality [18]. A subgroup $H$ of $G$ is said to be $q^\alpha$-$\text{normal}$ in $G$ if the set $\{g \in G : gHg^{-1} \cap H \text{ is nonamenable}\}$ generates $G$. The subgroup $H$ is $wq^\alpha$-$\text{normal}$ in $G$ if there exists an ordinal $\lambda$ and an increasing sequence $(H_\alpha)_{\alpha \leq \lambda}$ of subgroups of $G$, with $H_0 = H$ and $H_\lambda = G$, such that $\bigcup_{\beta < \alpha} H_\beta$ is $q^\beta$-$\text{normal}$ in $H_\alpha$ for all $\alpha \leq \lambda$. The notions of $q$-$\text{normal}$ and $wq$-$\text{normal}$ subgroups are defined in the same way, except with “nonamenable” replaced by “infinite.” It is immediate that a $wq$-normal subgroup is necessarily infinite, and a $wq^\alpha$-normal subgroup is necessarily nonamenable.

**Theorem 1.** Let $G$ be a finitely generated nonamenable group. Assume that $G$ acts amenably on $X$ and fix a $G$-invariant mean $m$ on $X$. Then $G_x$ is $wq^\alpha$-$\text{normal}$ in $G$ for $m$-almost every $x \in X$.

**Example 0.1.** The assumption of finite generation is necessary in the statement of Theorem 1. Let $G$ be a free group with free generating set $S = \{s_i\}_{0 \leq i < \infty}$, and let $G_n \leq G$ be the subgroup generated by $\{s_i\}_{0 \leq i < n}$. The action $G \acts X = \bigsqcup_{n \geq 0} G/G_n$ is amenable, although the stabilizer of any $x \in X$ is malnormal in $G$.

**Remark 0.2.** Even when $G$ is not finitely generated Theorem 1 can be applied to finitely generated subgroups, as in Corollary 2.6 below, to obtain a statement which holds for all countable groups. Corollary 2.6 also shows that Example 0.1 is in fact the prototypical obstruction to $wq^\alpha$-normality of stabilizer subgroups when $G$ is not finitely generated.

**Remark 0.3.** Theorem 1 can be strengthened. In [11] we introduce a natural hierarchy of incremental strengthenings of $wq^\alpha$-normality (see Definition 1.3). The conclusion of Theorem 1 then remains true when $wq^\alpha$-normality is replaced by any of these strengthenings. In addition, a relativized version of Theorem 1 holds; see Theorem 2.

Theorem 1 provides a means of studying measured group theoretical properties of $G$ via its amenable actions, since many such properties are known to be reflected in the structure of $wq$-normal subgroups. For example, Popa has shown that if $G$ contains a $wq$-normal subgroup whose Bernoulli shift is $\mathcal{U}_{\text{fin}}$-cocycle superrigid, then the same holds for the Bernoulli shift of $G$ [19]. In [47], Peterson and Thom show that the first $\ell^2$-Betti number, $\beta_1^{(2)}(G)$, of $G$ is bounded above by that of its $wq$-normal subgroups. An analogous statement also holds for the pseudocost, $\mathcal{P}\mathcal{C}(G)$, of $G$, see Proposition A.3. We therefore obtain the following corollary, which strengthens a theorem of Promislow [51] concerning actions of free groups.

**Corollary 2.** Let $G$ be a countable nonamenable group. Assume that $G$ acts amenably on $X$ and fix a $G$-invariant mean $m$ on $X$.

1. Suppose that $G$ is finitely generated. Then $\beta_1^{(2)}(G_x) \geq \beta_1^{(2)}(G)$ and $\mathcal{P}\mathcal{C}(G_x) \geq \mathcal{P}\mathcal{C}(G)$ for $m$-almost every $x \in X$. 
(2) In general, if \( \beta_1^{(2)}(G) > r \), then \( \beta_1^{(2)}(G_x) > r \) for \( m \)-almost every \( x \in X \). Likewise, if \( \mathcal{P}C(G) > r \) then \( \mathcal{P}C(G_x) > r \) for \( m \)-almost every \( x \in X \).

Proof of Corollary 2. Part (1) follows from Theorem 1 using Theorem 5.6 of [17] in the case of first \( \ell^2 \)-Betti number, and using Proposition A.3 in the case of pseudocost. For part (2), if \( \mathcal{P}C(G) > r \) then by Proposition A.1 there exists a finitely generated nonamenable subgroup \( H_0 \leq G \) such that \( \mathcal{P}C(H) > r \) for all \( H_0 \leq H \leq G \). By Corollary 2.6 there exists a \( G \)-map \( \varphi : X \to Y \) to a \( G \)-set \( Y \) such that \( G_x \) is \( wq^* \)-normal in \( G_\varphi(x) \) and \( H_0 \leq G_\varphi(x) \) for \( m \)-almost every \( x \in X \). Therefore, by Proposition A.3 \( \mathcal{P}C(G_x) \geq \mathcal{P}C(G_\varphi(x)) > r \) for \( m \)-almost every \( x \in X \). An analogous argument goes through for the first \( \ell^2 \)-Betti number using Corollary 5.13 of [17] in place of Proposition A.1.

0.B. An isoperimetric threshold. Let \( X \) be a \( G \)-set and for a finite subset \( S \) of \( G \) denote by \( \phi_S(X) \) the isoperimetric constant of the Schreier graph with respect to \( S \), associated with the action

\[
(0.1) \quad \phi_S(X) = \inf \left\{ \sum_{s \in S} \left| sP \setminus P \right| / |P| : P \subseteq X \text{ is finite and nonempty} \right\}.
\]

When \( S \) generates \( G \), the value \( \phi_S(G) \) is then the isoperimetric constant of the Cayley graph of \( G \) with respect to \( S \). We always have \( \phi_S(X) \leq \phi_S(G) \), and if \( S \) generates \( G \) then \( X \) is an amenable \( G \)-set if and only if \( \phi_S(X) = 0 \).

Remark 0.4. There are several variations of the definition (0.1). For example, it can often be convenient to work with the conductance constant \( h_S(X) = 1/|S| \phi_S(X) \). See [28] for a discussion in the case \( X = G \). For our purpose, any fixed multiplicative renormalization of \( \phi_S(X) \) would be suitable since our main interest will be in the ratio \( \phi_S(X) / \phi_S(G) \).

Assume now that \( S \) generates \( G \). Theorem 1 then has a surprising consequence when combined with Kazhdan’s trick. Namely, there exists a constant \( \epsilon = \epsilon_{G,S} > 0 \) such that any subgroup \( H \) of \( G \) satisfying \( \phi_S(G/H) < \epsilon \) must be \( wq^* \)-normal in \( G \). Indeed, otherwise, for each \( n \geq 0 \) there is a subgroup \( H_n \leq G \) such that \( \phi_S(G/H_n) < 2^{-n} \) but with \( H_n \) not \( wq^* \)-normal in \( G \), so the amenable action \( G \simeq \bigsqcup_n G/H_n \) contradicts Theorem 1. While this argument does not give any indication about the actual value of \( \epsilon_{G,S} \), we obtain a sharp estimate in (2.B) (See [1] for the definition of \( n \)-degree \( \mathcal{N}^X\)-\( wq \)-normality.)

Theorem 3. Let \( G \) be a nonamenable group with finite generating set \( S \) and let \( H \) be a subgroup of \( G \). If \( \phi_S(G/H) < \frac{1}{2^n} \phi_S(G) \), then \( H \) is \( wq^* \)-normal in \( G \). More generally, for each \( G \)-set \( X \), and for each nonnegative integer \( n \) we have the following implication

\[
\phi_S(G/H) < \frac{1}{2^n} \phi_S(X) \Rightarrow H \text{ is } n\text{-degree } \mathcal{N}^X\text{-}\text{wq-normal in } G.
\]

Example 0.5. Theorem 3 is sharp. Consider the free group \( \mathbb{F}_2 \) of rank 2 with free generating set \( S = \{a, b\} \). We have \( \phi_S(\mathbb{F}_2) = 1 \) [3, Example 47]. The subgroup \( H = \langle a, bab^{-1} \rangle \) is not \( wq^* \)-normal in \( \mathbb{F}_2 \) (although it is \( q \)-normal). An inspection of the Schreier graph of \( \mathbb{F}_2/H \) verifies that \( \phi_S(\mathbb{F}_2/H) = \frac{1}{2} \).
0.C. Transitive actions and a vanishing criterion. In the case of an amenable transitive action we obtain the following strengthening of Corollary 2.

Theorem 4. Let $G$ be a countable group. Assume that $G$ acts amenably and transitively on an infinite set $X$ and fix some $x \in X$. If $\beta_1^{(2)}(G_x) < \infty$ then $\beta_1^{(2)}(G) = 0$. Likewise, if $\mathcal{P}C(G_x) < \infty$ then $\mathcal{P}C(G) = 1$.

It follows that if $G$ is a group with $\beta_1^{(2)}(G) > 0$ or with $\mathcal{P}C(G) > 1$, then for any finitely generated infinite index subgroup $H \leq G$, the action $G \rtimes G/H$ is not amenable.

Theorem 4 is closely related to a vanishing criterion due to Peterson and Thom [47]. They define a subgroup $H$ of $G$ to be $s$-normal in $G$ if $gHg^{-1} \cap H$ is infinite for every $g \in G$; the notion of $ws$-normality is then obtained by iterating $s$-normality transfinitely. Theorem 5.12 of [47] states that if $G$ contains a $ws$-normal infinite index subgroup $H$ with $\beta_1^{(2)}(H) < \infty$ then $\beta_1^{(2)}(G) = 0$. Ioana (unpublished) has shown that the analogous statement also holds for cost (Ioana’s argument works for pseudocost as well). Theorem 4 would therefore follow from these results if the subgroup $G_x$ were always $ws$-normal in $G$. This turns out not to be the case however, as the following example shows.

Example 0.6. Let $K$ be a group which is isomorphic with one of its proper malnormal subgroups $K_0$ (e.g., any nonabelian free group has this property, see [4, Example 1]). Fix an isomorphism $\varphi : K \rightarrow K_0$ and let $G = \langle t, K \mid tk^{-1} = \varphi(k) \rangle$ be the associated HNN-extension. By Proposition 2 of [39], the action of $G$ on $G/K$ is amenable. However, $K$ is not $ws$-normal in $G$. To see this note that, from the semidirect product decomposition $G = (\bigcup_{n \geq 0} t^{-n}Kt^n) \rtimes \langle t \rangle$, it follows that every intermediate subgroup $K \leq L \leq G$ contains an element of the form $g = t^{-n}kt^m$, where $n, m > 0$ and $k \in K - K_0$, and clearly $gKg^{-1} \cap K = 1$.

0.D. Inner amenability. In their 1943 study of $\text{II}_1$ factors [43], Murray and von Neumann distinguished the hyperfinite $\text{II}_1$ factor from the free group factor $\mathbb{L}_F^2$ by means of property Gamma, that is, the existence of nontrivial asymptotically central sequences. In demonstrating that $\mathbb{L}_F^2$ lacks this property, Murray and von Neumann hinted at a connection with amenability [43, footnote 71], remarking that their argument, which makes ancillary use of approximately invariant measures, closely mirrors Hausdorff’s famous paradoxical division of the sphere. This connection was not made explicit however until 1975 when Effros [13] introduced the following group theoretic notion:

Definition 0.7. A group $G$ is inner amenable if the action of $G$ on itself by conjugation admits an atomless invariant mean.

Effros showed that if a group factor $L_F^G$ has property Gamma, then $G$ is necessarily inner amenable. An ICC counterexample to the converse statement was found only very recently by Vaes [54].

The proof of Theorem 1 naturally involves exploiting the tension between the nonamenability of $G$ and the amenability of the action. In the case of the conjugation action, this tension leads to remarkably strong consequences for the group theoretic and measured group theoretic structure of $G$. Many of these consequences will in fact be shown to hold in the more general setting of inner amenable pairs: If $H$ is a subgroup of $G$ then we say that the pair...
(G, H) is inner amenable if the conjugation action of H on G admits an atomless invariant mean.

0.E. The cost of inner amenable groups. We let C(G) denote the cost of G, that is, C(G) is the infimum of the costs of free probability measure preserving actions of G. We let C*(G) denote the supremum of the costs of free probability measure preserving actions of G. Then G has fixed price if C(G) = C*(G).

Theorem 5. Let G be a countable group.

(1) Suppose that G contains a wq-normal subgroup H such that (G, H) is inner amenable. Then C(G) = 1.

(2) Suppose that G is inner amenable. Then C(G) = 1 and G has fixed price.

Remark 0.8. In part (1) of Theorem 5 it would be desirable to additionally obtain that G has fixed price. The proof of part (1) shows that this holds if and only if direct products of infinite groups have fixed price, which is a well known open problem.

As a consequence of Theorem 5 we recover the result of Chifan, Sinclair, and Udrea [10, Corollary D], that inner amenable groups have vanishing first ℓ²-Betti number. Moreover, we obtain a strengthening which holds for inner amenable pairs.

Corollary 6. Let G be a countable group and suppose that G contains a wq-normal subgroup H such that the pair (G, H) is inner amenable. Then β₁^(2)(G) = 0. In particular, if G is inner amenable then β₁^(2)(G) = 0.

Proof. This follows from Theorem 5 and the inequality β₁^(2)(G) ≤ C(G) − 1 due to Gaboriau [17]. Alternatively, a direct proof may be obtained by observing that each step of the proof of Theorem 5 in §4.C has an analogue for the first ℓ²-Betti number.

In [2], Abért and Nikolov show that for a finitely generated, residually finite group G, the rank gradient of any Farber chain in G is equal to one less than the cost of the associated boundary action of G. We therefore obtain the following corollary.

Corollary 7. Let G be a finitely generated, residually finite group which is inner amenable. Then the rank gradient of any Farber chain in G vanishes. In particular, the absolute rank gradient of G vanishes.

The two main ingredients in the proof of Theorem 5(2) concern the subgroup structure of nonamenable inner amenable groups.

Theorem 8. Let G be a nonamenable inner amenable group. Then every nonamenable subgroup of G is wq-normal in G.

The next result roughly states that very large portions of G commute. To make the statement somewhat less cumbersome we define N to be the collection of all nonamenable subgroups of G.

This is a bit different from the notion, defined by Jolissaint in [24], of H being inner amenable relative to G, which amounts to amenability of the conjugation action of H on G − H. While Jolissaint’s notion does not appear anywhere else in this article, to avoid conflicting terminology we will make sure to refer to inner amenability of the pair (G, H) when referring to the notion defined in the main text.
Theorem 9. Let $G$ be a nonamenable inner amenable group. Then at least one of the following holds:

1. For every finite $\mathcal{F} \subseteq \mathcal{N}$ there exists an infinite amenable subgroup $K$ of $G$ such that $L \cap C_G(K)$ is nonamenable for all $L \in \mathcal{F}$.

2. For every finite $\mathcal{F} \subseteq \mathcal{N}$ there exists an increasing sequence $M_0 \leq M_1 \leq \cdots$ of finite subgroups of $G$, with $\lim_{n \to \infty} |M_n| = \infty$, such that $L \cap C_G(M_n)$ is nonamenable for all $L \in \mathcal{F}$, $n \in \mathbb{N}$.

3. For every finite $\mathcal{F} \subseteq \mathcal{N}$ and every $n \in \mathbb{N}$ there exist pairwise commuting nonamenable subgroups $K_0, K_1, \ldots, K_{n-1} \leq G$ such that $L \cap C_G(K_i)$ is nonamenable for all $L \in \mathcal{F}$, $i < n$. Moreover, there exists a sequence $(M_n)_{n \in \mathbb{N}}$ of finite subgroups of $G$ with $\lim_{n \to \infty} |M_n| = \infty$ such that $L \cap C_G(M_n)$ is nonamenable for all $L \in \mathcal{F}$, $n \in \mathbb{N}$.

Corollary 10. Let $G$ be a nonamenable inner amenable group. Then $G$ either contains an infinite amenable subgroup or $G$ contains finite subgroups of arbitrarily large order. In addition, $G$ contains an infinite subgroup $K$ such that $C_G(K)$ is infinite.

Proof of Corollary 10. The first statement follows easily from Theorem 9. The second statement is clear if either (1) or (3) of Theorem 9 holds. If (2) holds then $G$ contains an infinite locally finite subgroup, hence by [21] $G$ contains an infinite abelian subgroup $A$, so $A \leq C_G(A)$. □

0.F. Cocycle superrigidity. If $H$ is a subgroup of $G$, then a cocycle $w$ of a probability measure preserving action of $G$ is said to untwist on $H$ if $w$ is cohomologous to a cocycle $w'$ whose restriction to $H$ is a homomorphism. Following [49, 50], let $\mathcal{U}_{\text{fin}}$ denote the class of all Polish groups which embed as a closed subgroup of the unitary group of a finite von Neumann algebra. A free, probability measure preserving action of $G$ is said to be $\mathcal{U}_{\text{fin}}$-cocycle superrigid if every cocycle for the action which takes values in some group in $\mathcal{U}_{\text{fin}}$ untwists on the entire group $G$.

Popa’s Second Cocycle Superrigidity Theorem (Theorem 1.1 of [50]) provides general conditions for a cocycle which takes values in some group $L \in \mathcal{U}_{\text{fin}}$, to untwist on the centralizer $C_G(H)$ of a nonamenable subgroup $H$ of $G$. The following theorem, which is joint with Adrian Ioana, strengthens Popa’s theorem by showing that, under suitable conditions, the untwisting in fact occurs on the centralizer $C_{\mathcal{M}(G)}(H)$ of $H$ in the semigroup $\mathcal{M}(G)$ of all means on $G$.

Theorem 11 (with A. Ioana). Let $G \acts_{\sigma_0} (X, \mu)$ be a probability measure preserving action of a countable group $G$. Let $H \leq G$ be a nonamenable subgroup and assume that

- $\sigma_0|_H$ has stable spectral gap;
- $\sigma_0|_{C_{\mathcal{M}(G)}(H)}$ is weakly mixing (Definition 5.1);
- $\sigma_0$ is $s$-malleable.

Let $L$ be a group in $\mathcal{U}_{\text{fin}}$ and let $d_L$ denote the (compatible, bi-invariant) metric on $L$ coming from an embedding of $L$ as a closed subgroup of the unitary group of a finite von Neumann algebra. Let $w : G \times X \to L$ be a measurable cocycle with values in $L$. Then there exists a cocycle $w' : G \times X \to L$ cohomologous to $w$ such that:

1. The restriction of $w'$ to $H$ is a homomorphism;
(2) For any separable subalgebra $\mathcal{A} \subseteq \ell^\infty(G)$ there exists a closed subsemigroup $\mathcal{M}_0$ of $C_\#(G)(H)$, with $\mathcal{M}_0|\mathcal{A} = C_\#(G)(H)|\mathcal{A}$, along with a conull set $X_0 \subseteq X$ and a map $\rho : G \to L$ such that

$$\sup_{m \in \mathcal{M}_0} \sup_{x \in X_0} \int_G d_L(w'(g,x), \rho(g)) \, dm = \sup_{x \in X_0} \int_X d_L(w'(g,x), \rho(g)) \, d\mu \, dm = 0.$$  

Theorem 11 applies to the Bernoulli shift of $G$ whenever $H \leq G$ is nonamenable and the pair $(G, H)$ is inner amenable. Applying Lemma 3.5 of [15], we therefore obtain:

**Corollary 12.** Let $G$ be a countable group containing a wq-normal nonamenable subgroup $H$ such that the pair $(G, H)$ is inner amenable. Then the Bernoulli shift of $G$ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid. In particular, the Bernoulli shift of any nonamenable inner amenable group is $\mathcal{U}_{\text{fin}}$-cocycle superrigid.

Corollary 12 strengthens a result of Peterson and Sinclair [16], stating that the Bernoulli shift of $G$ is $\mathcal{U}_{\text{fin}}$-cocycle superrigid provided $G$ is nonamenable and $\text{LG}$ has property Gamma.

The case $H = G$ of Corollary 12 would follow from Popa’s theorem combined with Lemma 3.5 of [15] and Theorems 8 and 9 above, provided that alternative (2) could be dropped from the statement of Theorem 9. However, the following example exhibits an inner amenable group with the property that the centralizer of every nonamenable subgroup is finite; in particular, such a group does not satisfy either of the alternatives (1) or (3) of Theorem 9.

**Example 0.9.** Let $\mathbb{F}_2 \actson X$ be a transitive amenable action of the free group $\mathbb{F}_2$ on an infinite set $X$ with the following property: for all $u \in \mathbb{F}_2 - 1$ the set $\{ P \in \mathcal{P}_t(X) : u \cdot P = P \}$ is finite, where $\mathcal{P}_t(X)$ denotes the collection of all finite subsets of $X$. Such an action is constructed in Theorem B.1. The group $G = \mathcal{P}_t(X) \rtimes \mathbb{Z}_2 \cong \bigoplus_{x \in X} \mathbb{Z}_2 \rtimes \mathbb{F}_2$ is then inner amenable so Corollary 12 applies to $G$. The group $G$ is also finitely generated and ICC. In addition, the centralizer of any nonamenable subgroup of $G$ is finite, or equivalently, the centralizer of every infinite subgroup $H$ of $G$ is amenable. Indeed, if $(P, u) \in H$ then we have $C_G(H) \leq C_G((P, u)) \leq \mathcal{P}_t(X) \rtimes C_{\mathbb{F}_2}(u)$, which is amenable unless $u = 1$. We may therefore assume that $H \leq \mathcal{P}_t(X)$, in which case, since $H$ is infinite we have

$$C_G(H) = \mathcal{P}_t(X) \rtimes \{ u \in \mathbb{F}_2 : u \cdot P = P \text{ for all } P \in H \} = \mathcal{P}_t(X) \rtimes 1,$$

which is amenable.

**0.0. The structure of inner amenable linear groups.** In [6] we characterize inner amenability for linear groups in terms of a certain amenable characteristic subgroup of $G$. The **AC-center** of a countable group $G$ is the subgroup

$$\mathcal{A} \mathcal{C}(G) = \{ N \leq G : N \text{ is normal in } G \text{ and } G/C_G(N) \text{ is amenable} \}.$$  

The **inner radical** of $G$ is the subgroup

$$\mathcal{I}(G) = \{ N \leq G : N \text{ is normal in } G \text{ and the action } N \rtimes G \actson N \text{ is amenable} \}.$$  

Here, $N \rtimes G \actson N$ is the action where $N$ acts by left translation and $G$ acts by conjugation. The relevant properties of these subgroups are summarized in the following theorem.

**Theorem 13.** Let $G$ be a countable group.

i. $\mathcal{A} \mathcal{C}(G)$ and $\mathcal{I}(G)$ are amenable characteristic subgroups of $G$;
ii. \( \mathcal{A} \mathcal{C}(G) \leq \mathcal{I}(G) \);

iii. The actions \( \mathcal{A} \mathcal{C}(G) \rtimes G \) and \( \mathcal{I}(G) \rtimes G \) are amenable;

iv. \( G/C_G(\mathcal{A} \mathcal{C}(G)) \) is residually amenable;

v. If \( \mathcal{I}(G) \) is infinite then \( G \) is inner amenable;

vi. Let \( N \) be a normal subgroup of \( G \) with \( N \leq \mathcal{I}(G) \). Then \( \mathcal{I}(G/N) = \mathcal{I}(G)/N \);

vii. Every conjugation invariant mean on \( G/\mathcal{I}(G) \) is the projection of a conjugation invariant mean on \( G \).

Moreover, if \( G \) is linear then

ix. \( \mathcal{A} \mathcal{C}(G) = \mathcal{I}(G) \);

x. \( G/C_G(\mathcal{I}(G)) \) is amenable;

xi. \( \mathcal{I}(G) = C_G(C_G(\mathcal{I}(G))) \);

xii. \( G/\mathcal{I}(G) \) is not inner amenable;

xiii. Every conjugation invariant mean on \( G \) concentrates on \( \mathcal{I}(G) \);

xiv. Let \( N \) be a normal subgroup of \( G \) with \( N \leq \mathcal{I}(G) \). Then ix. through xiii. all hold with \( G/N \) in place of \( G \).

Remark 0.10. Theorem 13.xi. implies that if \( G \) is linear then so are the groups \( G/\mathcal{I}(G) \) and \( G/C_G(\mathcal{I}(G)) \) (see Theorem 6.2 of [57]). It then follows from item x. and the Tits alternative that if \( G \) is additionally finitely generated, then \( \mathcal{I}(G) \) is virtually solvable.

Using Theorem 13 we are able to show that within the class of linear groups, inner amenability occurs only for the most obvious reasons: every linear inner amenable group is an amenable extension either of a group with infinite center or of a near product group in which one of the factors is infinite and amenable. More precisely, we obtain the following structure theorem for inner amenable linear groups.

**Theorem 14.** Let \( G \) be a countable linear group. Then the following are equivalent:

1. \( G \) is inner amenable.
2. \( \mathcal{I}(G) \) is infinite.
3. There exists a short exact sequence \( 1 \to N \to G \to K \to 1 \), where \( K \) is amenable and either
   - \( Z(N) \) is infinite, or
   - \( N = LM \), where \( L \) and \( M \) are commuting normal subgroups of \( G \) such that \( M \) is infinite and amenable, and \( L \cap M \) is finite.

0.H. **Stability.** A discrete probability measure preserving equivalence relation \( \mathcal{R} \) is said to be **stable** if it is isomorphic to its direct product \( \mathcal{R} \times \mathcal{R}_0 \) with the equivalence relation \( \mathcal{R}_0 \), of eventual equality on \( 2^\mathbb{N} \) equipped with the uniform product measure. A countable group \( G \) is said to be **stable** if it possesses a free ergodic probability measure preserving action which generates a stable equivalence relation. Stability was introduced by Jones and Schmidt in [25], where it was also shown that stable groups are necessarily inner amenable. The first examples of ICC inner amenable groups which are not stable were recently constructed by Kida [33]; these groups are obtained as HNN extensions of property (T) groups with infinite center. Further results of Kida from [34] show that if the center \( Z(G) \) of a group \( G \) is infinite, then the question of whether \( G \) is stable is intimately related to the question of
whether the pair \((G, Z(G))\) lacks relative property \((T)\). Using Theorem 13 in 7 we are able to completely characterize stability for linear groups in terms of relative property \((T)\).

**Theorem 15.** Let \(G\) be a countable linear group. Then the following are equivalent:

1. \(G\) is stable.
2. The pair \((G, \mathcal{F}(G))\) does not have relative property \((T)\).

**Remark 0.11.** The hypothesis that \(G\) is linear in Theorems 13ix.-xiv., 14 and 15 can be weakened: we only need to assume that \(G\) satisfies the minimal condition on centralizers, that is, every decreasing sequence \(C_G(A_0) \geq C_G(A_1) \geq \cdots\) of centralizers of subsets of \(G\) eventually stabilizes. Every linear group satisfies the minimal condition on centralizers, since centralizers of arbitrary subsets of \(GL_n(F)\) are closed in the Zariski topology.

In addition to Theorem 13 an essential component in the proof of Theorem 15 is the following extension theorem for stability (see 7.A for the definition of stability sequence).

**Theorem 16.** Let \(1 \to N \to G \to K \to 1\) be a short exact sequence of groups in which \(K\) is amenable. Assume that there exists a probability measure preserving action \(G \acts (X, \mu)\) such that the translation groupoid \(N \times (X, \mu)\) admits a stability sequence. Then \(G\) is stable.

Theorem 16 has a variety of applications outside the context of linear groups. Under each of the following hypotheses (H1)-(H6), the stability of \(G\) will be established in 7 by applying Theorem 16 to an appropriate input action of \(G\). The application of Theorem 16 to groups satisfying (H4) and the ensuing Corollary 18 were kindly suggested by Yoshikata Kida (remarking on an earlier draft of this paper), who had obtained stability of \(G\) from (H4) by different means.

**Theorem 17.** Let \(1 \to N \to G \to K \to 1\) be a short exact sequence of groups in which \(K\) is amenable. Then \(G\) is stable provided at least one of the following hypotheses holds:

1. \(N = LM\), where \(L\) and \(M\) are commuting subgroups of \(N\) which are normal in \(G\), with \(M\) amenable and \([N : L] = \infty\).
2. There exists a central subgroup \(C\) of \(N\) such that the pair \((N, C)\) does not have relative property \((T)\).
3. There exists a sequence \(L_0 \leq L_1 \leq \cdots\), of subgroups of \(N\) with \(N = \bigcup_{m \in \mathbb{N}} L_m\), and for each \(m \in \mathbb{N}\) there exists a central subgroup \(D_m\) of \(L_m\) such that the pair \((L_m, D_m)\) does not have relative property \((T)\).
4. There exists a commensurated abelian subgroup \(A\) of \(G\) such that \(N\) is the kernel of the modular homomorphism from \(G\) into the abstract commensurator of \(A\), and the pair \((N, A)\) does not have relative property \((T)\).
5. \(N\) has the Haagerup property and is asymptotically commutative, i.e., there exists an injective sequence \((c_n)_{n \in \mathbb{N}}\) in \(N\) such that each \(h \in N\) commutes with \(c_n\) for cofinitely many \(n \in \mathbb{N}\).
6. \(N\) is doubly asymptotically commutative, i.e., there exist sequences \((c_n)_{n \in \mathbb{N}}\) and \((d_n)_{n \in \mathbb{N}}\) in \(N\) such that \(c_n d_n \neq d_n c_n\) for all \(n \in \mathbb{N}\), and each \(h \in N\) commutes with both \(c_n\) and \(d_n\) for cofinitely many \(n \in \mathbb{N}\).

**Corollary 18** (Y. Kida). Let \(G\) be a generalized Baumslag-Solitar group (i.e., the Bass-Serre fundamental group of a finite graph of infinite cyclic groups), or an HNN-extension of \(\mathbb{Z}^n\) relative to an isomorphism between two finite index subgroups. Then \(G\) is stable.
Proof of Corollary 18. Suppose first that $G$ is a generalized Baumslag-Solitar group. Then any vertex group $A \leq G$ is commensurated by $G$, and if $N$ denotes the kernel of the modular homomorphism from $G$ into the abstract commensurator $\text{comm}(A)$ of $A$, then $G/N$ is abelian, since $\text{comm}(A)$ is isomorphic to $\mathbb{Q}^*$. By Corollary 1.7 of [11], $G$ has the Haagerup property, so the pair $(N, A)$ does not have property (T). The hypothesis (H4) is therefore satisfied, so $G$ is stable by Theorem 17.

The case where $G$ is an HNN-extension of $\mathbb{Z}^n$ relative to an isomorphism between two finite index subgroups is similar. The image of $G$ under the modular homomorphism into the abstract commensurator of $\mathbb{Z}^n$ is cyclic and, letting $N$ denote the corresponding kernel, the pair $(N, \mathbb{Z}^n)$ does not have property (T) since by Corollary 1.7 of [11], $G$ has the Haagerup property. Hypothesis (H4) once again holds, so $G$ is stable by Theorem 17. □

Example 0.12. (i) Let $K$ be an infinite amenable group acting on a countable set $X$, and let $H$ be any countable group. Then the restricted wreath product $H \wr X K$ is stable. This is clear if $H$ is amenable, and it follows from Theorem 17 via (H1) if $X$ is finite. In the remaining case, the group $\bigoplus_X H$ is doubly asymptotically commutative, so Theorem 17 applies to $G$ via (H6), using the short exact sequence $1 \to \bigoplus_X H \to H \wr_X K \to K \to 1$.

(ii) Let $H$ be a group which is doubly asymptotically commutative. Let $\varphi : H \to H$ be an injective homomorphism and let $G = \langle t, H \mid tht^{-1} = \varphi(h) \rangle$ be the associated ascending HNN-extension. Theorem 17 then shows that $G$ is stable, since we have a short exact sequence $1 \to N \to G \to \mathbb{Z} \to 1$ in which the group $N = \bigcup_{i \in \mathbb{N}} t^{-i} H t^i$ is doubly asymptotically commutative, and hence the hypothesis (H6) holds. Similarly, if we instead assume that $H$ is an increasing union $H = \bigcup_m H_m$, where for each $m \in \mathbb{N}$ the pair $(H_m, \mathbb{Z}(H_m))$ does not have property (T), then the any ascending HNN extension of $H$ will be stable via (H3).

Notably, Theorem 17 also applies to the group $H(\mathbb{R})$, recently studied by Monod [38], consisting of all homeomorphisms of the projective line $\mathbb{P}^1$ which fix $\infty$ and are piecewise in $\text{PSL}_2(\mathbb{R})$ with respect to a finite subdivision of $\mathbb{P}^1$. It is shown by Monod in [38] that $H(\mathbb{R})$ does not contain any nonabelian free subgroups, and Theorem 1 of [38] exhibits a family of countable nonamenable subgroups of $H(\mathbb{R})$. An explicit finitely presented nonamenable subgroup of $H(\mathbb{R})$ is constructed by Lodha and Moore in [37]. We now have the following.

Theorem 19. Every nontrivial countable subgroup of $H(\mathbb{R})$ is stable.

Since Thompson’s group $F$ is a subgroup of $H(\mathbb{R})$, Theorem 19 implies:

Corollary 20. Thompson’s group $F$ is stable. In particular, $F$ and $F \times A$ are measure equivalent, where $A$ is any amenable group.

Corollary 20 yields a new proof of the fact, due to Lück [36] and also proved by Bader, Furman, and Sauer in [3], that all $\ell^2$-Betti numbers of $F$ vanish. Indeed, Gaboriau has shown that vanishing of $\ell^2$-Betti numbers is an invariant of measure equivalence [17], and by [9] all $\ell^2$-Betti numbers of $F \times \mathbb{Z}$ vanish.
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1. Weak forms of normality

In this section we gather some facts about $wq^*$-normality (defined in \([0,\infty)\)), and we discuss several related normality conditions. We work in the following general setting. Fix an ambient group $G$ along with a nonempty collection $\mathcal{L}$ of subgroups of $G$ which is upward closed in $G$. Let $H \leq M$ be subgroups of $G$. We say that $H$ is $\mathcal{L}$-$q$-normal in $M$, denoted $H \leq_{wq}^\mathcal{L} M$, if the set $\{g \in M : gHg^{-1} \cap H \in \mathcal{L}\}$ generates $M$. We say that $H$ is $\mathcal{L}$-$wq$-normal in $M$, denoted $H \leq_{wq}^\mathcal{L} M$, if there exists an ordinal $\lambda$ and an increasing sequence $(H_\alpha)_{\alpha \leq \lambda}$ of subgroups of $M$, with $H_0 = H$ and $H_\lambda = M$, such that $\bigcup_{\beta < \alpha} H_\beta \leq_{wq}^\mathcal{L} H_\alpha$ for all $\alpha \leq \lambda$. The notions of $wq$-normality and $wq^*$-normality then correspond to taking $\mathcal{L}$ to be, respectively, the collection $\mathcal{F}$ of infinite subgroups of $G$, and the collection $\mathcal{N}$, of nonamenable subgroups of $G$. Given a $G$-set $X$, we will be interested in the collection

$$\mathcal{N}^X = \{H \leq G : H \not\rhd X \text{ is nonamenable}\}.$$ 

For $S \subseteq G$ finite and $r > 0$, the collection $\mathcal{L}^{S,r} = \{H \leq G : \phi_S(G/H) < r\}$ (where $\phi_S(G/H)$ is defined by \([0,1]\)) will also be of interest, albeit less directly than $\mathcal{N}^X$. Both of these collections are upward closed in $G$ (for $\mathcal{L}^{S,r}$ this follows from Lemma \([2.4]\) below) and both are invariant under conjugation by $G$.

The following characterization of $\mathcal{L}$-$wq$-normality, along with its proof, is a straightforward extension of \([47]\) Lemma 5.2.

**Lemma 1.1.** Let $H \leq M$ be subgroups of $G$. Then $H \leq_{wq}^\mathcal{L} M$ if and only if for any intermediate proper subgroup $H < K \leq M$ there exists $g \in M \setminus K$ such that $gKg^{-1} \cap K \in \mathcal{L}$.

Let $H \leq M$ be subgroups of $G$ and let $\hat{H}$ denote the union of all subgroups $L \leq M$ with $H \leq_{wq}^\mathcal{L} L$. We call $\hat{H}$ the $\mathcal{L}$-$wq$-closure of $H$ in $M$.

**Proposition 1.2.** $\hat{H}$ is a subgroup of $M$. Moreover, $\hat{H}$ is the unique subgroup of $M$ satisfying

(i) $H \leq_{wq}^\mathcal{L} \hat{H}$ and

(ii) $g\hat{H}g^{-1} \cap \hat{H} \not\in \mathcal{L}$ for every $g \in M \setminus \hat{H}$.

**Proof.** By Zorn’s Lemma the set $\{K \leq M : H \leq_{wq}^\mathcal{L} K\}$ contains a maximal element $L$. By definition, $L \subseteq \hat{H}$. A consequence of Lemma \([1.1]\) is that if $L_0, L_1 \leq M$ are two subgroups of $M$ with $H \leq_{wq}^\mathcal{L} L_0$ and $H \leq_{wq}^\mathcal{L} L_1$, then $H \leq_{wq}^\mathcal{L} (L_0, L_1)$. It follows that $L = \hat{H}$. Properties (i) and (ii) are immediate. If $K$ is a subgroup of $M$ satisfying properties (i) and (ii) in place of $\hat{H}$, then $H \leq K \leq \hat{H}$ by property (i) for $K$ and the definition of $\hat{H}$, and since $H \leq_{wq}^\mathcal{L} \hat{H}$, Lemma \([1.1]\) and property (ii) for $K$ imply that $K = \hat{H}$.

**Definition 1.3.** Let $\mathcal{L}_0 = \mathcal{L}$ and for each $n \geq 0$ define $\mathcal{L}_{n+1} = \{H \leq G : H \leq_{wq}^\mathcal{L} G\}$, which is upward closed by Lemma \([1.1]\) and induction. A subgroup $H$ of $G$ is said to be $n$-degree $\mathcal{L}$-$wq$-normal in $G$ if $H \in \mathcal{L}_n$.

Note that if the collection $\mathcal{L}$ is invariant under conjugation by $G$, then so are each of the collections $\mathcal{L}_n$, $n \in \mathbb{N}$. By applying this definition to the collection $\mathcal{N}^X$, for a $G$-set $X$, we obtain the sequence $\mathcal{N}_n^X$, $n \in \mathbb{N}$. If $H \in \mathcal{N}_n^G$, then we say that $H$ is $n$-degree $wq^*$-normal in $G$. Thus, $\hat{H}$ is 0-degree $wq^*$-normal in $G$ if and only if $H$ is nonamenable, and $H$ is 1-degree $wq^*$-normal in $G$ if and only if $H$ is $wq^*$-normal in $G$ in the previously defined sense.
For the next proposition, we equip the space of subgroups of $G$ with the subspace topology inherited from the product topology on $2^G$. Note that for any $G$-set $X$, the collection $\mathcal{N}^X$ is an open set, since for a subgroup $H \leq G$, nonamenability of the action $H \curvearrowright X$ is witnessed by a finite subset of $H$. The same holds for the collection $\mathcal{L}^{S,r}$ as well as for each $\mathcal{N}^X_n$, $n \in \mathbb{N}$, when $G$ is finitely generated. Taking $\mathcal{L} = \mathcal{N}^X$ in the following proposition then shows that, when $G$ is finitely generated, the transfinite sequence in the definition of $\mathcal{L}^{-}\text{wq}$-normality can be replaced by a finite sequence.

**Proposition 1.4.** Let $\mathcal{L}$ be an upward closed collection of subgroups of the countable group $G$. Suppose in addition that $\mathcal{L}$ is open in the space of subgroups of $G$.

(i) Let $H$ and $M$ be subgroups of $G$. Assume that $M$ is finitely generated and $H \leq wq M$. Then there exist subgroups $H_0, \ldots, H_n$ such that

\begin{equation}
H = H_0 \leq wq H_1 \leq wq \cdots \leq wq H_n = M.
\end{equation}

Moreover, for any such sequence $(H_i)_{i=0}^n$ there exists a sequence $(H'_i)_{i=0}^n$ with $H'_0 \leq wq H'_1 \leq wq \cdots \leq wq H'_n = M$, where $H'_i$ is finitely generated and $H'_i \leq H_i$ for all $i$.

(ii) If $G$ is finitely generated then $\mathcal{L}_n$ is open for all $n \geq 0$.

**Proof.** For $M, K \leq G$ define $f_M(K) = \langle \{g \in M : gKg^{-1} \cap K \in \mathcal{L}\} \rangle$. Then $f_M$ is monotone and, since $\mathcal{L}$ is open and upward closed, the function $f_M$ is lower semicontinuous, that is, $f_M(\liminf K_i) \leq \liminf f_M(K_i)$ for any sequence $(K_i)_{i \in \mathbb{N}}$ of subgroups of $G$, where $\liminf K_i$ denotes the subgroup of elements of $G$ which are in cofinitely many $K_i$. It follows that for any finite sequence $M_0, M_1, \ldots, M_n \leq G$, the function $f_{M_n} \circ \cdots \circ f_{M_1} \circ f_{M_0}$ is lower semicontinuous.

(i): Let $H_\omega = \bigcup_{n \in \mathbb{N}} f^n_M(H)$. Then semicontinuity of $f_M$ implies $f_M(H_\omega) = H_\omega$. This shows that $H_\omega$ is the $\mathcal{L}$-$\text{wq}$-closure of $H$ in $M$, hence $H_\omega = H$. Since $M$ is finitely generated and the sequence $f^n_M(H)$ is nondecreasing there exists an $n$ with $f^n_M(H) = M$. This shows the first part of (i). Fix now any sequence $(H_i)_{i=0}^n$ as in (1.1). Then $f_{H_n} \circ \cdots \circ f_{H_0}(H_0) = M$, so there exists a finitely generated $H'_0 \leq H_0$ with $f_{H_n} \circ \cdots \circ f_{H_0}(H'_0) = M$. Assume now that $k < n - 1$ and $f_{H_n} \circ \cdots \circ f_{H_{k+1}}(H'_k) = M$. Let $Q_0 \subseteq Q_1 \subseteq \cdots$ be a sequence of finite sets which exhaust $\{g \in H_{k+1} : gH'_k g^{-1} \cap H_k \in \mathcal{L}\}$. Then $\bigcup_k (H'_k,Q_i) = f_{H_{k+1}}(H'_k)$, so there exists some $i$ such that $f_{H_n} \circ \cdots \circ f_{H_{k+2}}(H'_k,Q_i) = M$. Take $H'_{k+1} = (H'_k,Q_i)$. The resulting groups $H'_0, H'_1, \ldots, H'_{n-1}, H'_n = M$ satisfy the conclusion of (ii) by construction.

(ii): It suffices to show $\mathcal{L}_l$ is open. This follows from (i) and semicontinuity of $f^n_G$. \qed

**Remark 1.5.** In [3], Bader, Furman, and Sauer define higher order notions of $s$-normality and establish a connection with higher $\ell^2$-Betti numbers. It seems reasonable to expect a similar connection to hold between higher degree $wq$-normality (or some variant) and higher $\ell^2$-Betti numbers, although this is largely speculative.

2. **Amenable actions**

Let $X$ be a $G$-set. Let $S \subseteq G$ be finite and let $\epsilon > 0$. A nonempty finite subset $P$ of $X$ is said to be $(S, \epsilon)$-**invariant** if $\sum_{s \in S} |sP \setminus P| < \epsilon |P|$. Equivalently, $P$ is $(S, \epsilon)$-invariant if $\sum_{s \in S} |sP \cap P| > (|S| - \epsilon) |P|$. 
Remark 2.1. Assume that $S$ generates $G$ and that every $G$-orbit has cardinality greater than $1/\epsilon$. Then any $(S,\epsilon)$-invariant set $P$ has cardinality $|P| > 1/\epsilon$. Otherwise we would have $\sum_{s \in S} |sP \setminus P| < 1$, so $P$ would be a $G$-invariant set of cardinality at most $1/\epsilon$, a contradiction.

Remark 2.2. We will make use of the observation [18, Remark 2.12] that if $P \subseteq X$ is $(S,\epsilon)$-invariant, then there exists a single $G$-orbit $X_0 \subseteq X$ such that $P \cap X_0$ is $(S,\epsilon)$-invariant.

2.A. An estimate with Følner sets. For each $n \geq 1$ we let $X^{\otimes n}$ denote the set of all $n$-tuples of distinct points in $X$ which lie in the same $G$-orbit

$$X^{\otimes n} = \{(x_0, \ldots, x_{n-1}) \in X^n : i \neq j \Rightarrow Gx_i = Gx_j \text{ and } x_i \neq x_j\}.$$ 

Then we have a natural action $G \rhd X^{\otimes n}$ under which the inclusion map $X^{\otimes n} \hookrightarrow X^n$ is a $G$-map to the diagonal product action. For a subset $P \subseteq X$ let $P^{\otimes n} = P^n \cap X^{\otimes n}$.

Lemma 2.3. Let $S$ be a finite subset of $G$. Let $n \geq 1$ and let $\epsilon > 0$. Let $P \subseteq X$ be an $(S,\epsilon)$-invariant set which is contained in a single $G$-orbit, and assume $|P| \geq n$. Then $P^{\otimes n}$ is $(S,\epsilon n)$-invariant in $X^{\otimes n}$.

Proof. For each $s \in S$ let $\epsilon_s = \frac{|sP \cap P|}{|P|}$. Then $\sum_{s \in S} \epsilon_s < \epsilon$, so it suffices to show that for all $k \leq n$ we have

$$|sP^{\otimes k} \cap P^{\otimes k}| \geq |P^{\otimes k}|(1 - k\epsilon_s). \quad (2.1)$$

If $k = 1$ then we have equality, so assume inductively that (2.1) holds, where $k < n$, and we will show that it holds with $k + 1$ in place of $k$. Note that $\epsilon_s > 0$ implies $\epsilon_s \geq 1/|P|$. It follows that $(1 - \frac{|sP \cap P|}{|P| - k})(1 - k\epsilon_s) \geq (1 - (k + 1)\epsilon_s)$, and hence

$$|sP^{\otimes (k+1)} \cap P^{\otimes (k+1)}| = |(sP \cap P)^{\otimes (k+1)}|$$

$$= |(sP \cap P - k)(sP \cap P)^{\otimes k}|$$

$$\geq |(P - \epsilon_s)(sP \cap P)^{\otimes k}|(1 - k\epsilon_s)$$

$$= (1 - \frac{|sP \cap P|}{|P| - k})(|P| - k)|P^{\otimes k}|(1 - k\epsilon_s)$$

$$\geq |P^{\otimes (k+1)}|(1 - (k + 1)\epsilon_s). \quad \square$$

2.B. Proof of Theorems 1 and 3. Assume now that $G$ is a finitely generated by $S$. Let $\phi_S(X)$ denote the isoperimetric constant of $X$ with respect to $S$, defined in (0.1).

Lemma 2.4. Let $X$ and $Y$ be $G$-sets and assume that there exists a $G$-map $\varphi : X \to Y$ from $X$ to $Y$. Then, given any $P \subseteq X$ which is $(S,\epsilon)$-invariant in $X$, we may find some $Q \subseteq \varphi(P)$ which is $(S,\epsilon)$-invariant in $Y$. In particular $\phi_S(Y) \leq \phi_S(X)$.

Proof. See the first proof in §1.2 of [20]. \quad \square

Lemma 2.5. Let $X$ be a $G$-set and let $H$ be a subgroup of $G$. Assume that the action $H \rhd X$ is amenable. Then $\phi_S(G/H) \geq \phi_S(X)$.

Proof. Let $Y_0$ denote the $G$-set $G/H \times X$ equipped with the diagonal product action of $G$. Then $\phi_S(Y_0) \geq \phi_S(X)$ by Lemma 2.4, so it suffices to show that $\phi_S(G/H) \geq \phi_S(Y_0)$ (so in fact $\phi_S(G/H) = \phi_S(Y_0)$ by Lemma 2.4). Fix a section $\sigma : G/H \to G$ for the map
Given by \( \phi \) set obtained by inducing from the \( H \)

Lemma \( n \) is \((G \to \hat{G}) \) by Lemma \( \sum \).

Proof of Theorem 1.

For each nonamenable subgroup \( \hat{G} \) has a unique \( \hat{G} \)-invariant mean on \( X \), so it suffices to show that \( \phi_S(G/H) \geq \phi_S(Y_1) \). Let \( \emptyset \neq P \subseteq G/H \) be finite. Then, for each finite \( \emptyset \neq Q \subseteq X \), we have \( P \times Q \subseteq Y_1 \), so \( \phi_S(Y_1) \) is bounded above by

\[
\sum_{s \in S} \frac{|s \cdot (P \times Q) \setminus (P \times Q)|}{|P \times Q|} = \frac{1}{|P|} \sum_{s \in S} \sum_{kH \in sP \cap P} \frac{|\rho(s, s^{-1}kH \cdot Q \setminus Q)|}{|Q|} + \sum_{s \in S} \frac{|sP \setminus P|}{|P|}.
\]

Since \( H \cap X \) is amenable, taking the infimum over all such \( Q \subseteq X \) shows that \( \phi_S(Y_1) \leq \sum_{s \in S} \frac{|sP \setminus P|}{|P|} \), so taking the infimum over \( P \) shows that \( \phi_S(Y_1) \leq \phi_S(G/H) \).

Proof of Theorem 2.1. The base case \( n = 0 \) is immediate from Lemma 2.5. Assume now that \( \phi_S(G/H) < \frac{1}{2n} \phi_S(X) \), where \( n > 0 \), and we will show that \( H \) is \( n \)-degree \( \mathcal{N}_X \)-wq-normal in \( G \). By Lemma 1.1, given \( H \leq L \leq G \), it suffices to find some \( g \in G \setminus L \) such that \( gLg^{-1} \cap L \) is \( (n - 1) \)-degree \( \mathcal{N}_X \)-wq-normal in \( G \). The quotient map \( G/H \to G/L \) is a \( G \)-map, so by Lemma 2.4 we have \( \phi_S(G/L) \leq \phi_S(G/H) < \frac{1}{2n} \phi_S(X) \). Then \( \phi_S((G/L)^{g2}) < \frac{1}{2n} \phi_S(X) \) by Lemma 2.3, so by Remark 2.2 there is some point \( x = (g_0L, g_1L) \in (G/L)^{g2} \) with \( \phi_S(G/L_x) < \frac{1}{2n} \phi_S(X) \). By the induction hypothesis, the group \( G_x = g_0Lg_0^{-1} \cap g_1Lg_1^{-1} \) is \( (n - 1) \)-degree \( \mathcal{N}_X \)-wq-normal in \( G \). Then \( g = g_1^{-1}g_0 \in G \setminus L \) and \( gLg^{-1} \cap L \) is \( (n - 1) \)-degree \( \mathcal{N}_X \)-wq-normal in \( G \).

Proof of Theorem 2.6. This follows immediately from Theorem 2.1 and the observation that for any \( \epsilon > 0 \), the set \( \{ x \in X : \phi_S(G/G_x) < \epsilon \phi_S(G) \} \) is \( m \)-conull.

2.C. An extension to infinitely generated groups. Example 0.11 shows that a direct translation of Theorem 1 does not hold in the general infinitely generated setting. However, a refined version of Theorem 1 still holds in general. In what follows, for each \( G \)-set \( X \) let \( X_0 = \{ x \in X : G_x \text{ is nonamenable} \} \).

Corollary 2.6. Let \( G \) be a nonamenable group. For each \( G \)-set \( X \) there is a \( G \)-map \( \varphi_X : X \to \hat{X} \) to a \( G \)-set \( \hat{X} \) with the following properties:

(i) \( G_x \) is \( wq \)-normal in \( G_{\varphi_X(x)} \) for all \( x \in X_0 \);

(ii) If \( m \) is any \( G \)-invariant mean on \( X \), then for any finitely generated subgroup \( H \) of \( G \) we have \( H \leq G_{\varphi_X(x)} \) for \( m \)-almost every \( x \in X \).

Moreover, this assignment can be made functorial: if \( \psi : X \to Z \) is a \( G \)-map then there exists a unique \( G \)-map \( \hat{\psi} : \hat{X} \to \hat{Z} \) with \( \varphi_Z \circ \hat{\psi} = \hat{\psi} \circ \varphi_X \).

Proof. For each nonamenable subgroup \( H \leq G \) let \( \hat{H} \) denote the \( wq \)-closure of \( H \) in \( G \) (see 3). For \( x \in X \) let \( O(x) \) denote the \( G \)-orbit of \( x \). Let \( G \) act on the set \( X_0 = \{ (\hat{G}_x, O(x)) : x \in X_0 \} \) by conjugating the first coordinate, and let \( \hat{X} = \hat{X}_0 \sqcup X \setminus X_0 \). Define \( \varphi_X : X \to \hat{X} \) by \( \varphi_X(x) = (\hat{G}_x, O(x)) \) for \( x \in X_0 \), and \( \varphi_X(x) = x \) for \( x \in X \setminus X_0 \). Then \( \varphi_X \) is a \( G \)-map, and for each \( x \in X_0 \) we have \( G_{\varphi_X(x)} = \hat{G}_x \) since \( \hat{G}_x \) is self-normalizing. This verifies (i). For (ii), let \( m \) be a \( G \)-invariant mean on \( X \) and let \( H \leq G \) be finitely generated. After making
$H$ larger we may assume that $H$ is nonamenable. By Theorem 4 $H_x$ is $wq^*$-normal in $H$ for $m$-almost every $x \in X$. For each such $x$, since $H_x \leq G_x$, we have $H \leq \hat{G}_x = G_{\varphi_x}(x)$.

If $\psi : X \to Z$ is a $G$-map, then we must show that $\varphi_Z(\psi(x))$ only depends on $\varphi_X(x)$. This is clear for $x \in X \setminus X_0$. Suppose now that $(\hat{G}_{x_0}, O(x_0)) = (\hat{G}_{x_1}, O(x_1))$ where $x_0, x_1 \in X_0$. Find $g \in G$ with $gx_0 = x_1$. Then $\hat{G}_{x_0} = \hat{G}_{x_1} = g\hat{G}_{x_0}g^{-1}$, so $g \in \hat{G}_{x_0} \leq \hat{G}_{\psi(x_0)}$. It follows that $\hat{G}_{\psi(x_1)} = \hat{G}_{\psi(x_0)}g^{-1} = \hat{G}_{\psi(x_0)}$, hence $\varphi_Z(\psi(x_0)) = \varphi_Z(\psi(x_1))$. \hfill $\Box$

3. Transitive amenable actions

3.A. Weak normality for groupoids. To prove Theorem 4 we need an extension of the results of [17] on weakly normal inclusions of discrete probability measure preserving (p.m.p.) groupoids. We adopt the notation and conventions for discrete p.m.p. groupoids from [47, §6], and we will need a few additional definitions.

Let $(G, \mu)$ be a discrete p.m.p. groupoid. We do not distinguish two subgroupoids $H$ and $K$ of $G$ if they agree off of a $\mu$-null set. Recall that a local section of $G$ is a measurable map $\phi : \text{dom}(\phi) \to G$, with $\text{dom}(\phi) \subseteq G^0$ and $s(\phi x) = x$ for all $x \in \text{dom}(\phi)$, such that the assignment $\phi^0 : x \mapsto r(\phi x)$ is injective. We do not distinguish two local sections whose domains and values agree off of a $\mu$-null set. Let $[[G]]$ denote the collection of all local sections of $G$. The inverse of $\phi \in [[G]]$ is the local section $\phi^{-1} : \text{ran}(\phi^0) \to G$ given by $\phi^{-1}(y) = \phi((\phi^0)^{-1}y)^{-1}$. The composition of two local sections $\phi, \psi \in [[G]]$ is the local section $\phi \circ \psi : ((\psi^0)^{-1}(\text{ran}(\psi^0) \cap \text{dom}(\phi^0)) \to G, x \mapsto \phi(\psi^0(x))\psi(x)$.

We equip $[[G]]$ with the separable complete metric $d(\phi, \psi) = \mu(\text{dom}(\phi) \triangle \text{dom}(\psi)) + \mu(\{x \in \text{dom}(\phi) \cap \text{dom}(\psi) : \phi(x) \neq \psi(y)\})$. A consequence of separability of the metric $d$ is that if $\Phi$ is any subset of $[[G]]$ then up to a $\mu$-null set there is a unique smallest subgroupoid $K$ of $G$ with $\Phi \subseteq [[K]]$; we call $K$ the subgroupoid generated by $\Phi$ and denote it by $\langle \Phi \rangle$.

For measurable subsets $R \subseteq G$ and $A \subseteq G^0$ we let $R_A = \{\gamma \in R : s(\gamma), r(\gamma) \in A\}$. For $\phi \in [[G]]$ and $\gamma \in G_{\text{ran}(\phi^0)}$ let $\gamma^\phi = \phi^{-1}(r(\gamma))\gamma\phi^{-1}(s(\gamma))^{-1} \in G_{\text{dom}(\phi^0)}$. The $q$-normalizer of $R$ in $G$ is the set

$$Q_G(R) = \{\phi \in [[G]] : (R_A)^\phi \cap R_{(\phi^0)^{-1}A} \text{ has infinite measure for all non-null } A \subseteq \text{ran}(\phi^0)\}.$$

A subgroupoid $\mathcal{H}$ of $G$ is said to be $q$-normal in $G$ if $Q_G(\mathcal{H})$ generates $G$. As usual, we obtain the corresponding notion of $wq$-normality by iterating $q$-normality transfinity. Then the analogue of Lemma 11 holds: $\mathcal{H}$ is $wq$-normal in $G$ if and only if for every intermediate proper subgroupoid $\mathcal{H} \subseteq K \subseteq G$ there exists a local section $\phi \in [[G]] \setminus [[K]]$ with $\phi \in Q_G(K)$. While Theorem 6.9 of [17] is stated for $wq$-normal subgroupoids, we note that the proof holds more generally for $wq$-normal subgroupoids.

**Theorem 3.1** (cf. [17, Theorem 6.9]). Let $\mathcal{H}$ be a subgroupoid of the discrete p.m.p. groupoid $(G, \mu)$. If $\mathcal{H}$ is $wq$-normal in $G$ then the restriction map $H^1(G, U(G, \mu)) \to H^1(\mathcal{H}, U(\mathcal{H}, \mu))$ is injective.

**Proof.** The proof of Theorem 6.9 of [17] shows that if $c$ is a $G$-cocycle with values in $U(G, \mu)$ which vanishes on $[[\mathcal{H}]]$, then $c$ vanishes on $Q_G(\mathcal{H})$, and therefore on $\langle Q_G(\mathcal{H}) \rangle$ since the set where $c$ vanishes is closed under compositions and inverses, and $c$ respects countable decompositions. The theorem follows. (We note the following minor correction to the proof of Theorem 6.9 of [17]: using the notation from that proof, the fact that $\mathcal{H}_A$ is $s$-normal in
$G_A$ is irrelevant to the proof; what is being used is that $(\chi_A\psi)^{-1}\mathcal{H}_A(\chi_A\psi) \cap \mathcal{H}_G$ has infinite measure, which holds since $\mathcal{H}$ is $s$-normal in $G$ and hence $\psi \in \mathcal{Q}_G(\mathcal{H})$. The rest of the proof remains unchanged after replacing $A$ by $(\psi^0)^{-1}A$ in the appropriate places.) □

3.B. Recurrence and normality. Let $G \curvearrowright (Y, \nu)$ be a free probability measure preserving action of $G$. We let $\mathcal{R}^G$ denote the orbit equivalence relation generated by this action. Then $(\mathcal{R}^G, \nu)$ is a discrete p.m.p. groupoid so that the notation and terminology of §3.A applies. In this case, we will identify each local section $\phi \in [[\mathcal{R}^G]]$ with the corresponding partial isomorphism $\phi^0$ of $(Y, \nu)$, and we identify elements of $G$ with their image in $[[\mathcal{R}^G]]$.

For each subset $P \subseteq G$ let $R^P \subseteq \mathcal{R}^G$ denote the graph $R^P = \{(y, ky) : y \in Y, k \in P\}$. Define the sets

$$Q(P) = \{g \in G : (\forall n)(\exists\text{distinct } k_0, \ldots, k_{n-1} \in G) \ (k_i^{-1}k_j \in P \cap P^g \text{ for all } i < j < n)\}$$

$$L(P) = \{g \in G : (\forall n)(\exists\text{distinct } k_0, \ldots, k_{n-1} \in G) \ (k_i^{-1}k_j \in P \cap g^{-1}P \text{ for all } i < j < n)\}.$$ 

For a subgroup $H \leq G$ we then have $R^H = \mathcal{R}^H$ and $Q(H) = \{g \in G : H \cap H^g \text{ is infinite}\}$.

Lemma 3.2. Let $G \curvearrowright (Y, \nu)$ be a free probability measure preserving action of $G$. Let $A \subseteq Y$ be measurable and let $P = \{1\}$ be a subset of $G$.

(i) If $g \in Q(P)$ then $g_{|[A \cap \nu^{-1}A]} \in Q_{\mathcal{R}^G}(R^P_A)$. In particular, if $H$ is an infinite subgroup of $G$ then $R^H_A$ is $q$-normal in the equivalence relation generated by $R^Q_{\mathcal{R}^G}(H)$.

(ii) Let $d^R_{A^P}, d^L_{A^P} : Y \times Y \rightarrow \mathbb{N} \cup \{\infty\}$ denote the extended graph metrics on $R^P_A$ and $R^{L(P)}_A$ respectively. Then $d^R_{A}(x, y) \leq 2d^L_{A}(x, y)$ for almost every $(x, y) \in \mathcal{R}^G$. In particular, if $L(P) = G$ then $R^P_A$ generates $\mathcal{R}^G_A$.

Proof of Lemma 3.2 (i) Fix $g \in Q(P)$. It suffices to show that for almost every $y \in A \cap g^{-1}A$, the set $\{z \in A : (z, y), (gz, gy) \in R^P_A\}$ is infinite. Suppose toward a contradiction that there exists an $m > 0$ such that the set

$$C = \{y \in A \cap g^{-1}A : |\{(z \in A : (z, y), (gz, gy) \in R^P_A)\}| < m\}$$

has positive measure, say $\nu(C) = \epsilon > 0$. By the Poincaré recurrence theorem there exists some $n \in \mathbb{N}$, depending only on $\epsilon$ and $m$, such that if $(C_i)_{i < n}$ is any sequence of measurable sets in $Y$, each with $\nu(C_i) \geq \epsilon$, then there exists $i_0 < i_1 < \cdots < i_m < n$ with $\nu(\bigcap_{j \leq m} C_{i_j}) > 0$. Using this $n$, let $(k_i)_{i < n}$ be a sequence as in the definition of $g \in Q(P)$. By our choice of $n$ there exists $i_0 < i_1 < \cdots < i_m < n$ with $\nu(\bigcap_{j \leq m} k_j C) > 0$. For each $0 \leq j < m$ let $h_j = k_{i_j}^{-1}k_{i_{j+1}}$ so that $h_j^{-1} \in P \cap P^g$ and $\nu(C \cap \bigcup_{j < m} h_j C) > 0$, and the elements $h_0, \ldots, h_{m-1}$ are pairwise distinct. Fix $y \in C \cap \bigcap_{j \leq m} h_j C$ and fix any $i < m$ and put $h = h_i$. Then $y, h^{-1}y \in C \subseteq A \cap g^{-1}A$, so $y, h^{-1}y, gy, gh^{-1}y \in A$. Moreover, $h^{-1} \in P$ and $gh^{-1}g^{-1} \in P$, so it follows that $(h^{-1}y, y) \in R^P_A$ and $(gh^{-1}y, gy) = ((gh^{-1}g^{-1})y, gy) \in R^P_A$. This shows that $\{h_j^{-1}y\}_{j=0}^{m-1} \subseteq \{z \in A : (z, y), (gz, gy) \in R^P_A\}$, which contradicts that $y \in C$.

(ii) It suffices to show that $d^R_{A}(gy, y) \leq 2$ for all $g \in L(P)$ and almost every $y \in A \cap g^{-1}A$. Suppose toward a contradiction that there exists some $g \in L(P)$ such that the set

$$D = \{y \in A \cap g^{-1}A : d^R_{A}(gy, y) > 2\}$$

has positive measure. Let $n \in \mathbb{N}$ be so large that $\frac{1}{n} < \nu(D)$. Let $(k_i)_{i < n}$ be a sequence as in the definition of $g \in L(P)$. Then $\nu(k_i D) = \nu(D) > \frac{1}{n}$ for all $0 \leq i < n$, so there exists
\( i < j < n \) with \( \nu(k_i D \cap k_j D) > 0 \). Let \( k = k_i^{-1} k_j \), so that \( k \in P \cap g^{-1} P \), and the set \( D_0 := D \cap kD \) is non-null. Fix \( y \in D_0 \). Then we have \( y, gy, k^{-1} y \in A \) and \( k, gk \in P \), so \((k^{-1}y, y) = (k^{-1}y, (gk)k^{-1}y) \in R^P_A \) and \((k^{-1}y, y) = (k^{-1}y, k(k^{-1}y)) \in R^P_A \). This shows that \( d^P_A(gy, y) \leq 2 \), which contradicts that \( y \in D \). \( \square \)

Example 0.6 shows that there are nonamenable groups having a transitive amenable action \( G \rhd X \) such that \( G_x \) is not \( ws \)-normal in \( G \). The next Lemma shows that \( G_x \) is still very close to being \( s \)-normal in \( G \). Recall that a subset \( B \) of \( G \) is said to be \textbf{thick} in \( G \) if for every finite subset \( F \subseteq G \) the intersection \( \bigcap_{g \in F} gB \) is nonempty (equivalently: infinite). Observe that if \( B \subseteq G \) is thick then \( L(B) = G \) since given \( g \in G \) we can define \( k_0 = 1 \) and inductively let \( k_{n+1} \) be any element of \( \big( \bigcap_{i \leq n} k_i (B \cap g^{-1}B) \big) \setminus \{k_0, \ldots, k_n\} \), so that \( (k_n)_{n \geq 0} \) witnesses that \( g \in L(B) \).

\textbf{Lemma 3.3.} Let \( G \rhd X \) be a transitive amenable action of a nonamenable group \( G \). Fix any element \( x \in X \) and let \( H = G_x \). Then \( Q(H) \) is thick in \( G \). In particular, \( L(Q(H)) = G \).

\textbf{Proof.} Since \( H \) is a subgroup of \( G \) we have \( Q(H) = \{g \in G : H \cap H^g \text{ is infinite}\} \). Note that \( H \) is nonamenable since \( G \) is nonamenable and the action \( G \rhd G/H \) is amenable. Let \( m \) be a \( G \)-invariant mean on \( G/H \). Then \( m \) is also \( H \)-invariant, so we obtain

\begin{equation}
(3.1) \quad m(\{gH \in G/H : H \cap gH^{-1}g \text{ is nonamenable}\}) = 1.
\end{equation}

Let \( \pi : G \to G/H \) be the projection map \( \pi(g) = gH \). Then (3.1) implies that \( m(\pi(Q(H))) = 1 \). If \( g \in G \) then by \( G \)-invariance of \( m \) we have \( m(\pi(gQ(H))) = m(\pi(\{gH \in G/H : H \cap gH^{-1}g \text{ is nonamenable}\})) = 1 \). Therefore, for any finite subset \( F \) of \( G \) we have \( m(\pi(\bigcap_{g \in F} gQ(H))) = m(\pi(\bigcap_{g \in F} gQ(H))) = 1 \), where the equality \( \pi(\bigcap_{g \in F} gQ(H)) = \bigcap_{g \in F} \pi(gQ(H)) \) follows from \( Q(H) \) being a union of left cosets of \( H \). In particular, \( \bigcap_{g \in F} gQ(H) \neq \emptyset \). \( \square \)

\textbf{3.C. Proof of Theorem 4.} Using Theorem 3.1 and Lemmas 3.2 and 3.3 we can now argue as in Theorem 5.12 of [17].

\textbf{Proof of Theorem 4.} We can of course assume that \( G \) is nonamenable. Let \( H = G_x \) and assume that \( \beta^{(2)}_1(H) < \infty \). Since \( H \) is infinite index in \( G \) there exists a free ergodic p.m.p. action \( G \rhd (Y, \nu) \) of \( G \) whose restriction to \( H \) has a continuum of ergodic components. Such an action may be obtained, e.g., by coinducing from any free p.m.p. action of \( H \) with a continuum of ergodic components. Fix \( n \geq 1 \) and let \( A_0, \ldots, A_{n-1} \) be a partition of \( Y \) into \( H \)-invariant sets of equal measure. Let \( \mathcal{R}_n = \bigcup_{i<n} \mathcal{R}^G_{A_i} \). Observe that \( \mathcal{R}^H \subseteq \mathcal{R}_n \subseteq \mathcal{R}^G \). For each \( i < n \) let \( \mathcal{S}_i \) be the equivalence relation generated by \( R^Q_{A_i} \). Then by Lemma 3.2(i), \( \mathcal{R}^H \) is \( q \)-normal in \( \bigcup_{i<n} \mathcal{S}_i \). By Lemma 3.3 we have \( L(Q(H)) = G \), so Lemma 3.2(ii) implies that \( \mathcal{S}_i = \mathcal{R}^G_{A_i} \) for each \( i < n \), and hence \( \bigcup_{i<n} \mathcal{S}_i = \mathcal{R}_n \). This shows that \( \mathcal{R}^H \) is \( q \)-normal in \( \mathcal{R}_n \) and hence

\[ \beta^{(2)}_1(\mathcal{R}_n) \leq \beta^{(2)}_1(\mathcal{R}^H) \]

by Theorem 3.1. Since \( G \rhd (Y, \nu) \) is ergodic, each of the sets \( A_i \) is a complete section for \( \mathcal{R}^G \), so \( [\mathcal{R}^G : \mathcal{R}_n] = n \). By Corollary 3.16 and Proposition 5.11 of [17] we therefore have

\[ \beta^{(2)}_1(G) = \frac{1}{n} \beta^{(2)}_1(\mathcal{R}^G) = \frac{1}{n} \beta^{(2)}_1(\mathcal{R}_n) \leq \frac{1}{n} \beta^{(2)}_1(\mathcal{R}^H) = \frac{1}{n} \beta^{(2)}_1(H). \]

Since \( n \geq 1 \) was arbitrary and \( \beta^{(2)}_1(H) < \infty \), we conclude that \( \beta^{(2)}_1(G) = 0 \).
Assume now that \( \mathscr{P}(H) < \infty \) and let \( H \curvearrowright (Y, \nu_0) \) be a free p.m.p. action of \( H \) with \( \mathscr{P}(R^H_{\nu_0}) < \infty \). After taking the product of this action with an identity action of \( H \) on an atomless probability space, we may assume that \( H \curvearrowright (Y, \nu) \) has continuous ergodic decomposition. Let \( G \approx (Y, \nu) \) be the induced action, which is free and ergodic. Since \( H \curvearrowright (Y, \nu) \) factors onto \( H \curvearrowright (Y_0, \nu_0) \), we have \( \mathscr{P}(\mathcal{R}^H) \leq \mathscr{P}(\mathcal{R}^H_{\nu_0}) < \infty \). The proof now proceeds as above, using Proposition A.2 in place of Theorem 3.1 and Proposition 25.7 of [32] in place of Proposition 5.11 of [17] (the proof of Proposition 25.7 of [32] works just as well for pseudocost as for cost).

4. The cost of inner amenable groups

4.A. Proof of Theorem 8 We will often use the following well-known classical fact, which is a weakening of Theorem 1.

**Lemma 4.1.** Let \( G \approx X \) be an amenable action of a nonamenable group \( G \) and let \( m \) be a \( G \)-invariant mean on \( X \). Then \( G_x \) is nonamenable for \( m \)-almost every \( x \in X \).

The following simple consequence will be very useful.

**Lemma 4.2.** Assume that the pair \( (G, H) \) is inner amenable and let \( m \) be an \( H \)-conjugation invariant mean on \( G \). Let \( \mathcal{F} \) be a finite collection of nonamenable subgroups of \( H \). Then

\[
m(\{g \in G : L \cap C_G(g) \text{ is nonamenable for all } L \in \mathcal{F}\}) = 1.
\]

**Proof.** It suffices to prove the lemma in the case where \( \mathcal{F} = \{L\} \) is a singleton. This follows from Lemma 4.1 by taking \( X = G \) along with the conjugation action \( L \curvearrowright X \).

Theorem 8 is an immediate consequence of the following more detailed analysis.

**Theorem 4.3.** Assume that the pair \( (G, H) \) is inner amenable and that \( H \) is nonamenable. For each nonamenable subgroup \( L \leq H \) let \( K_L = \langle \{g \in G : L \cap C_G(g) \text{ is nonamenable}\} \rangle \).

(i) Let \( L \) be a nonamenable subgroup of \( H \). Then \( L \leq q^* \langle HK_L \rangle \leq q^* HK_L \). In particular, \( L \) is \( wq^* \)-normal in \( HK_L \), and \( H \) is \( q^* \)-normal in \( HK_L \).

(ii) Every \( H \)-conjugation invariant mean \( m \) on \( G \) concentrates on \( HK_L \). In particular, \( m \) concentrates on the \( wq^* \)-closure of \( H \) in \( G \).

**Proof.** Fix an atomless \( H \)-conjugation invariant mean \( m \) on \( G \). Let \( L \leq H \) be nonamenable. Let \( S_L = \{g \in G : L \cap C_G(g) \text{ is nonamenable}\} \). Then \( m(S_L) = 1 \) by Lemma 4.2, and \( S_L \subseteq \{g \in G : gLg^{-1} \cap L \text{ is nonamenable}\} \) implies that \( L \leq q^* LK_L \). Since \( S_L \subseteq K_L \) we have \( m(LK_L) = 1 \), so \( m(hLK_Lh^{-1} \cap LK_L) = 1 \) for all \( h \in H \), and since \( m \) is atomless this shows that \( LK_L \leq q^* \langle H, K_L \rangle \). Then \( \langle H, K_L \rangle \leq q^* HK_L \) follows from \( H \leq q^* HK_L \).

4.B. Proof of Theorem 9 We begin with the version of Theorem 9 for inner amenable pairs. Let \( \mathcal{N}(H) \) denote the collection of all nonamenable subgroups of \( H \).

**Theorem 4.4.** Assume that the pair \( (G, H) \) is inner amenable and that \( H \) is nonamenable. Then at least one of the following holds:

1. For every finite \( \mathcal{F} \subseteq \mathcal{N}(H) \) there exists an infinite amenable subgroup \( K \) of \( G \) such that \( L \cap C_G(K) \text{ is nonamenable for all } L \in \mathcal{F} \).
(2) For every finite \( \mathcal{F} \subseteq \mathcal{N}(H) \) there exists an increasing sequence \( M_0 \leq M_1 \leq \cdots \) of finite subgroups of \( G \), with \( \lim_{n \to \infty} |M_n| = \infty \), such that \( L \cap C_G(M_n) \) is nonamenable for all \( L \in \mathcal{F}, \ n \in \mathbb{N} \).

(3) For every finite \( \mathcal{F} \subseteq \mathcal{N}(H) \) there exists a nonamenable subgroup \( K \) of \( G \) such that \( L \cap C_G(K) \) is nonamenable for all \( L \in \mathcal{F} \).

**Proof.** Assume that neither (1) nor (2) holds, as witnessed by the collections \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) respectively. We will show that (3) holds. Toward this end, fix \( \mathcal{F} \subseteq \mathcal{N}(H) \) finite. We may assume that \( \mathcal{F}_1 \cup \mathcal{F}_2 \subseteq \mathcal{F} \). Fix an atomless \( H \)-conjugation invariant mean \( m \) on \( G \). Let \( M_0 = \{ e \} \) and for each \( L \in \mathcal{F} \) let \( \varphi_0(L) = L \). Assume for induction that \( \varphi_i(L) \geq \cdots \geq \varphi_n(L), L \in \mathcal{F} \), and \( M_0 \leq \cdots \leq M_n \) have been defined with \( \varphi_n(L) \) nonamenable and commuting with \( M_n \) for all \( L \in \mathcal{F} \). If \( M_n \) is infinite then we stop; otherwise, since \( m \) is atomless we have \( m(M_n) = 0 \), so by Lemma 4.2 there exists \( g \in G \) such that \( \varphi_{n+1}(L) := \varphi_n(L) \cap C_G(g) \) is nonamenable for all \( L \in \mathcal{F} \). The induction continues with \( M_{n+1} := (M_n,g) \).

We claim that this process stops at some stage, i.e., there is some \( n > 0 \) such that \( M_n \) is infinite. Otherwise, if the process never stops, we would obtain an infinite sequence \( M_0 \leq M_1 \leq \cdots \) of finite groups such that \( \varphi_n(L) \leq L \cap C_G(M_n) \) for all \( L \in \mathcal{F}_2 \), contradicting our choice of \( \mathcal{F}_2 \). Let \( n \) be the stage at which the process stops and take \( K := M_n \). Then for each \( L \in \mathcal{F} \) we have \( \varphi_n(L) \leq L \cap C_G(K) \), so \( L \cap C_G(K) \) is nonamenable. Then \( K \) must be nonamenable since \( \mathcal{F}_1 \subseteq \mathcal{F} \).

We can now prove Theorem 9.

**Proof of Theorem 9.** Assume that neither (1) nor (2) of Theorem 9 holds and fix \( \mathcal{F} \subseteq \mathcal{N} \) finite and \( n \in \mathbb{N} \) toward the goal of verifying (3). We already know that the pair \( (G,G) \) satisfies alternative (3) of Theorem 4.4. We may therefore find a nonamenable subgroup \( K_0 \leq G \) such that the group \( \psi_0(L) := L \cap C_G(K_0) \) is nonamenable for all \( L \in \mathcal{F} \). Let \( k \geq 0 \) and assume for induction that we have defined the nonamenable subgroups \( \psi_k(L), L \in \mathcal{F} \), and nonamenable \( K_0, K_1, \ldots, K_k \) such that

- \( K_i \) and \( K_j \) commute for all \( 0 \leq i < j \leq k \),
- \( \psi_k(L) \leq L \) and \( \psi_k(L) \) and \( K_i \) commute for all \( L \in \mathcal{F} \) and \( 0 \leq i \leq k \).

Now apply alternative (3) of Theorem 4.4 to \( (G,G) \), using the finite collection \( \{ \psi_k(L) : L \in \mathcal{F} \} \cup \{ K_i^k : 0 \leq i \leq k \} \), to obtain a nonamenable subgroup \( K_{k+1}^k \) of \( G \) such that \( \psi_{k+1}(L) := \psi_k(L) \cap C_G(K_{k+1}^k) \) is nonamenable for all \( L \in \mathcal{F} \), and \( K_i^{k+1} := K_i^k \cap C_G(K_{k+1}^k) \) is nonamenable for all \( i \leq k \). This continues the induction. For each \( i < n \) let \( K_i = K_i^{n-1} \). Then \( K_0, K_1, \ldots, K_{n-1} \) are the desired subgroups for the first part of (3). For the second statement, we may assume that \( \mathcal{F} \) contains a subset \( \mathcal{F}_1 \) witnessing that alternative (1) fails. For each \( 0 \leq i < n \) inductively let \( g_i \) be any element of \( K_i \setminus \langle g_0, \ldots, g_{i-1} \rangle \); this set is nonempty since \( K_i \) is nonamenable and \( \langle g_0, \ldots, g_{i-1} \rangle \) is abelian. The group \( M_n := \langle g_0, g_1, \ldots, g_n \rangle \) is an abelian group which commutes with \( \psi(L) \) for all \( L \in \mathcal{F} \), hence \( L \cap C_G(M_n) \) is nonamenable for all \( L \in \mathcal{F} \). Then \( M_n \) is finite since \( \mathcal{F}_1 \subseteq \mathcal{F} \). By construction, \( M_n \) has order \( |M_n| \geq 2^n \). \( \square \)

4.4. Proof of Theorem 5.

**Lemma 4.5.** Let \( M \) be a finite normal subgroup of \( G \). Then \( |M|(|\mathcal{C}^*(G) − 1|) \leq |\mathcal{C}^*(G/M) − 1| \).
Proof. Let $G \curvearrowright (X, \mu)$ be a free p.m.p. action of $G$. Let $Y \subseteq X$ be a measurable transversal for $R^M_X$, so $\mu(Y) = \frac{1}{|M|}$. Let $\mu_Y$ be the normalized restriction of $\mu$ to $Y$. Then $G/M$ acts on $(Y, \mu_Y)$ by the rule $gM \cdot y_0 = y_1$ if and only if $gMY_0 = MY_1$. This is an action since $M$ is normal in $G$, and it is free and measure preserving. Fix $\epsilon > 0$ and let $R$ be a graphing of $\mathcal{R}^G_Y$ with $\mathcal{C}_\mu(R) < \mathcal{C}^*(G/M) + \epsilon$. Let $T$ be a treeing of $R^M_X$. Then $\mathcal{C}_\mu(T) = 1 - \frac{1}{|M|}$ and $T \cup R$ is a graphing of $\mathcal{R}^G_X$, hence

\[
\mathcal{C}_\mu(R^G_X) \leq \mathcal{C}_\mu(T \cup R) = \mathcal{C}_\mu(T) + \mu(Y)\mathcal{C}_\mu(R) < 1 - \frac{1}{|M|} + \frac{1}{|M|}((\mathcal{C}^*(G/M) + \epsilon) = 1 + \frac{\mathcal{C}^*(G/M) - 1}{|M|} + \epsilon/|M|.
\]

Since $\epsilon > 0$ was arbitrary this shows that $\mathcal{C}_\mu(R^G_X) \leq 1 + \frac{\mathcal{C}^*(G/M) - 1}{|M|}$. Since this holds for all free p.m.p. actions $G \curvearrowright (X, \mu)$ the proof is complete. \qed

Lemma 4.6. Let $G$ be a nonamenable group.

1. There exists a finitely generated nonamenable subgroup $H \leq G$ with $\mathcal{C}^*(H) \leq 2$.
2. Let $M$ be a finite normal subgroup of $G$. Then there exists a finitely generated nonamenable subgroup $H \leq G$ containing $M$ with $\mathcal{C}^*(H) \leq 1 + 1/|M|$.

Proof. (1): Let $F$ be a finite subset of $G$ which is minimal (under inclusion) with respect to the property that $\langle F \rangle$ is nonamenable. Take $H = \langle F \rangle$. Let $G \curvearrowright (X, \mu)$ be a free p.m.p. action of $H$. By minimality of $F$, for any $g \in F$ the group $K = \langle F \setminus \{g\}\rangle$ is amenable, hence $\mathcal{C}_\mu(R^H_X) \leq \mathcal{C}_\mu(R^K_X) + \mathcal{C}_\mu(R^G_X) \leq 2$.

(2): By applying part (1) to $G/M$ we may find a finitely generated nonamenable subgroup $H \leq G$ containing $M$ such that $\mathcal{C}^*(H/M) \leq 2$. Then by Lemma 4.5 we have

\[
\mathcal{C}^*(H) \leq 1 + \frac{\mathcal{C}^*(H/M) - 1}{|M|} \leq 1 + \frac{1}{|M|}.
\]

Proof of Theorem 4.1. (1): If $H$ is amenable then $\mathcal{C}^*(H) = 1$, so $\mathcal{P}^\mathcal{C}^*(H) = 1$, hence $\mathcal{P}^\mathcal{C}^*(G) = 1$ by Proposition A.3 and thus $\mathcal{C}^*(G) = 1$, so we are done. Assume now that $H$ is nonamenable. Fix an $H$-conjugation invariant mean $m$ on $G$. If $L$ is any subgroup of $G$ such that $L \cap H$ is nonamenable, then Theorem 4.3 implies that $(L \cap H) \leq_{wq} HKH \leq_{wq} G$, so $L \leq_{wq} G$, and hence $\mathcal{P}^\mathcal{C}^*(G) \leq \mathcal{P}^\mathcal{C}^*(L)$ by Proposition A.3. This shows that

\[
\mathcal{P}^\mathcal{C}^*(G) = \inf\{\mathcal{P}^\mathcal{C}^*(L) : L \leq G \text{ and } L \cap H \text{ is nonamenable}\}.
\]

Apply Theorem 4.4 and take $\mathcal{F} = \{H\}$. If alternative (1) holds then the subgroup $L = (H \cap C_G(K))K$ has fixed price 1, and $L \cap H$ is nonamenable, so $\mathcal{P}^\mathcal{C}^*(G) = 1$ by (4.1), and hence $\mathcal{C}^*(G) = 1$. If alternative (2) holds then by Lemma 4.6 we may find a sequence $(L_n)_{n \in \mathbb{N}}$ of nonamenable subgroups $L_n \leq (H \cap C_G(M_n))M_n$ with $\mathcal{P}^\mathcal{C}^*(L_n) \leq 1 + 1/|M_n| \to 1$ as $n \to \infty$. Since $L_n \cap H$ is nonamenable for all $n \in \mathbb{N}$ we conclude once again that $\mathcal{P}^\mathcal{C}^*(G) = 1$ and hence $\mathcal{C}^*(G) = 1$. Finally, suppose that alternative (3) holds and let $L = (H \cap C_G(K))K$. Then $\mathcal{C}(L) = 1$ since $H \cap C_G(K)$ commutes with $K$ and both groups are infinite [16]. Since $L \leq_{wq} G$ it follows from Proposition A.3 that $\mathcal{P}^\mathcal{C}(G) = 1$ and hence $\mathcal{C}(G) = 1$.

(2): We may assume that $G$ is nonamenable, and by the proof of part (1) we may assume that alternative (3) of Theorem 4 holds. Then using the sequence $(M_n)_{n \in \mathbb{N}}$ we obtain that $\mathcal{C}^*(G) = 1$ as in the proof for alternative (2) above. \qed
5. Cocycle superrigidity

5.A. The space of means. Let $\mathcal{M} = \mathcal{M}(G)$ denote the space of means on $G$. Then $\mathcal{M}$ is a weak$^\ast$-closed subset of the unit ball of $\ell^\infty(G)^\ast$, hence by the Banach-Alaoglu Theorem $\mathcal{M}$ is compact in the weak$^\ast$-topology. Let $\mathcal{P} = \mathcal{M} \cap \ell^1(G)$ denote the collection of all probability vectors on $G$. Then $\mathcal{P}$ is a weak$^\ast$-dense subset of $\mathcal{M}$.

For $g \in G$ and $m \in \mathcal{M}$ we define the means $gm$ and $mg$ respectively by $(gm)(A) = m(g^{-1}A)$ and $(mg)(A) = m(Ag^{-1})$ for $A \subseteq G$. The assignments $(g,m) \mapsto gm$ and $(m,g) \mapsto mg$ define left and right actions respectively of $G$ on $\mathcal{M}$ which commute. The convolution of two means $m$ and $n$ on $G$ is defined to be the mean $m \ast n = \int_{g \in G} gm \, dm(g)$. This gives $\mathcal{M}$ the structure of a semigroup in which multiplication is weak$^\ast$-continuous in the left variable. We identify $G$ with the collection of point masses $\{\delta_g\}_{g \in G} \subseteq \mathcal{P}$. Then we have $gm = \delta_g \ast m$ and $mg = m \ast \delta_g$.

For a subgroup $H \leq G$, let $C_{\mathcal{M}(G)}(H)$ denote the subset of $\mathcal{M}$ consisting of all means on $G$ which are invariant under conjugation by $H$. Observe that $C_{\mathcal{M}(G)}(H)$ is a closed, convex, subsemigroup of $\mathcal{M}$ with $C_G(H) \subseteq C_{\mathcal{M}(G)}(H)$.

Lemma 5.1. Let $H$ be a subgroup of $G$, let $m \in C_{\mathcal{M}(G)}(H)$, and let $\mathcal{A} \subseteq \ell^\infty(G)$ be a separable subalgebra. Then there exists a sequence $(p_n)_{n \in \mathbb{N}}$ in $\mathcal{P}$ such that $\lim_n \|hp_nh^{-1} - p_n\|_1 = 0$ for all $h \in H$, and $\lim_n p_n(\phi) = m(\phi)$ for all $\phi \in \mathcal{A}$.

Proof. Fix finite sets $S \subseteq H$ and $\mathcal{A}_0 \subseteq \mathcal{A}$, along with $\epsilon > 0$. It suffices to show that there exists some $p \in \mathcal{P}$ with $\sup_{s \in S} \|sp - p\|_1 < \epsilon$ and $\sup_{\phi \in \mathcal{A}_0} |p(\phi) - m(\phi)| < \epsilon$. Since $\mathcal{P}$ is weak$^\ast$-dense in $\mathcal{M}$, the convex set $K_0 = \{p \in \mathcal{P} : \sup_{\phi \in \mathcal{A}_0} |p(\phi) - m(\phi)| < \epsilon\}$ contains $m$ in its weak$^\ast$-closure. Since $m \in C_{\mathcal{M}(G)}(H)$, the convex subset $\{(sp - p)_{s \in S} : p \in K_0\}$ of $\ell^1(G)^\mathcal{A}$ contains $0 \in \ell^1(G)^\mathcal{A}$ in its weak closure, hence in its norm closure by Mazur’s Theorem. This implies that there exists $p \in K_0$ such that $\sup_{s \in S} \|sp - p\|_1 < \epsilon$. □

5.B. Weak mixing for subsemigroups of $\mathcal{M}$. Let $\mathcal{H}$ be a Hilbert space and let $\varphi : G \to \mathcal{B}(\mathcal{H})$, $g \mapsto \varphi_g$, be a map from $G$ into the bounded linear operators on $\mathcal{H}$ whose image is contained in the unit ball of $\mathcal{B}(\mathcal{H})$. We extend $\varphi$ to a map $\mathcal{M} \to \mathcal{B}(\mathcal{H})$, by taking $\varphi_m = \int_G \varphi_g \, dm$, i.e., $\varphi_m$ is the unique bounded linear operator satisfying $\langle \varphi_m \xi, \eta \rangle = \int_{g \in G} \langle \varphi_g \xi, \eta \rangle \, dm(g)$ for all $\xi, \eta \in \mathcal{H}$. In particular, each unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ of $G$ extends to a map $\mathcal{M} \to \mathcal{B}(\mathcal{H})$ by taking $\pi_m = \int_G \pi_g \, dm$ for each $m \in \mathcal{M}$.

Proposition 5.2. Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation of $G$. Then the extended map $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ has the following properties:

i. $\pi$ is an affine semigroup homomorphism.

ii. For each $m \in \mathcal{M}$ we have $\pi_m^* = \pi_m$, where $\hat{m}(A) = m(A^{-1})$ for $A \subseteq G$.

iii. For each $m \in \mathcal{M}$ the operator $\pi_m$ is a contraction, i.e., $\|\pi_m\|_\infty \leq 1$.

iv. $\pi$ is continuous when $\mathcal{M}$ is given the weak$^\ast$-topology and when $\mathcal{B}(\mathcal{H})$ is given the weak operator topology.

v. $\{\pi_m\}_{m \in \mathcal{M}} \subseteq W^*(\pi(G))$.

Proof. Properties i. through iv. follow from the definitions, and v. follows from iv. □
If \( \pi \) and \( \kappa \) are unitary representations of \( G \) on \( \mathcal{H} \) and \( \mathcal{K} \) respectively, then for \( m \in \mathcal{M} \), the operators \((\pi \otimes \kappa)_m = \int_G \pi_g \otimes \kappa_g \, dm\) and \( \pi_m \otimes \kappa_m \) are generally distinct. We will only make use of the operator \((\pi \otimes \kappa)_m\).

**Definition 5.3.** Let \( \pi \) be a unitary representation of \( G \) and let \( \mathcal{M}_0 \) be a subsemigroup of \( \mathcal{M} \). We say that \( \pi|_{\mathcal{M}_0} \) is **weakly mixing** if \((\pi \otimes \pi)|_{\mathcal{M}_0} \) has no nonzero invariant vectors.

The next proposition extends several well-known characterizations of weak mixing from the group setting to the setting of subsemigroups of \( \mathcal{M} \).

**Proposition 5.4.** Let \( \pi : G \to \mathcal{U}(\mathcal{H}) \) be a unitary representation of \( G \) and let \( \mathcal{M}_0 \) be a subsemigroup of \( \mathcal{M} \). Then the following are equivalent:

i. \( \pi|_{\mathcal{M}_0} \) is weakly mixing.

ii. \((\pi \otimes \kappa)|_{\mathcal{M}_0} \) has no nonzero invariant vectors for every unitary representation \( \kappa \) of \( G \).

iii. For every finite \( F \subseteq \mathcal{H} \) and \( \epsilon > 0 \) there exists \( m \in \mathcal{M}_0 \) such that

\[
\int_{g \in G} |\langle \pi_g \xi, \eta \rangle|^2 \, dm(g) < \epsilon
\]

for all \( \xi, \eta \in F \).

iv. There is no nonzero finite dimensional subspace \( L \) of \( \mathcal{H} \) such that \( P_L = \int_G \pi_g P_L \pi_g^* \, dm \) for all \( m \in \mathcal{M}_0 \). Here, \( P_L \) denotes the orthogonal projection onto \( L \).

**Proof.** Using the properties in Proposition 5.2, the proof is a routine extension of the proof for the case \( \mathcal{M}_0 = G \) (see, e.g., [15]). \( \square \)

**Example 5.5.** Let \( \lambda : G \to \mathcal{U}(\ell^2(G)) \) be the left regular representation of \( G \). Then \( \lambda \) is a mixing representation of \( G \), so if \( m \) is any atomless mean on \( G \) then \( \lambda_m = 0 \) in \( \mathcal{B}(\ell^2(G)) \).

It follows that if \( \mathcal{M}_0 \) is a subsemigroup of \( \mathcal{M} \) whose weak*-closure contains a mean which is atomless, then \( \lambda|_{\mathcal{M}_0} \) is weakly mixing.

The next proposition will be used to show that weak mixing for the Koopman representation associated to a p.m.p. action of \( G \) behaves as expected.

**Proposition 5.6.** Let \( G \curvearrowright (X, \mu) \) be a p.m.p. action of \( G \) and let \( \kappa \) denote the associated Koopman representation on \( L^2(X, \mu) \). Let \( \mathcal{M}_0 \) be a subsemigroup of \( \mathcal{M} \). Then the collection \( \{A \subseteq X : \kappa_m(1_A) = 1_A \text{ for all } m \in \mathcal{M}_0\} \) is a \( \| \cdot \|_2 \)-norm closed sigma subalgebra of the measure algebra of \( (X, \mu) \). Furthermore, a function \( \xi \in L^2(X, \mu) \) is \( \kappa|_{\mathcal{M}_0} \)-invariant if and only if \( 1_A \) is \( \kappa|_{\mathcal{M}_0} \)-invariant for every \( \xi \)-measurable set \( A \subseteq X \).

**Proof.** A function \( \xi \in L^2(X, \mu) \) is \( \kappa_m \)-invariant if and only if \( \int_G \|\kappa_g(\xi) - \xi\|_2^2 \, dm = 0 \). Therefore, if \( f_0, f_1 \in L^\infty(X, \mu) \) are both \( \kappa_m \)-invariant, then

\[
\int_G \|\kappa_g(f_0f_1) - f_0f_1\|_2^2 \, dm \leq \|f_1\|_\infty^2 \int_G \|\kappa_g(f_0) - f_0\|_2^2 \, dm + \|f_0\|_\infty^2 \int_G \|\kappa_g(f_1) - f_1\|_2^2 \, dm = 0,
\]

hence \( f_0f_1 \) is also \( \kappa_m \)-invariant.

Assume that \( \xi \in L^2(X, \mu) \) is \( \kappa|_{\mathcal{M}_0} \)-invariant. It suffices to show that sets of the form \( A_r = \{x \in X : \xi(x) \geq r\} \), \( r \in \mathbb{R} \), are \( \kappa|_{\mathcal{M}_0} \)-invariant. Suppose toward a contradiction that \( A_r \) is not \( \kappa_m \)-invariant for some \( r \in \mathbb{R} \) and \( m \in \mathcal{M}_0 \). Then we have \( \int_G \mu(A_r \setminus gA_r) \, dm > 0 \), so there is some \( \epsilon > 0 \) such that \( m(D_\epsilon) > 0 \) where \( D_\epsilon = \{g \in G : \mu(A_r \setminus gA_r) > \epsilon\} \).
Find $\delta > 0$ such that $\mu(A_{r-\delta} \setminus A_r) = \mu(\{x \in X : r > \xi(x) \geq r - \delta\}) < \epsilon/2$. Then for $g \in D$, we have $\mu(A_r \setminus gA_{r-\delta}) > \epsilon/2$ and hence $\|\xi - \kappa_g(\xi)\|_2^2 \geq \delta \epsilon/2$. Therefore, $0 = \int_{g \in D_r} \|\xi - \kappa_g(\xi)\|_2^2 dm \geq m(D_r) \delta \epsilon/2 > 0$, a contradiction. For the reverse implication, approximate $\xi$ in $\|\cdot\|_2$-norm by $\xi$-measurable simple functions.

**Definition 5.7.** Let $G \rtimes (X, \mu)$ be a p.m.p. action of $G$ and let $\kappa$ denote the associated Koopman representation on $L^2(X, \mu)$. Let $\mathcal{M}_0$ be a subsemigroup of $\mathcal{M}$. We say that $\sigma|_{\mathcal{M}_0}$ is ergodic if every $\kappa|_{\mathcal{M}_0}$-invariant function in $L^2(X, \mu)$ is essentially constant. We say that $\sigma|_{\mathcal{M}_0}$ is weakly mixing if $(\sigma \otimes \sigma)|_{\mathcal{M}_0}$ is ergodic.

**Proposition 5.8.** Let $G \rtimes (X, \mu)$ be a p.m.p. action of $G$ and let $\kappa$ denote the associated Koopman representation on $L^2(X, \mu)$. Let $\mathcal{M}_0$ be a subsemigroup of $\mathcal{M}$. Then the following are equivalent:

i. $(\sigma \otimes \sigma)|_{\mathcal{M}_0}$ is weakly mixing;

ii. $(\sigma \otimes \rho)|_{\mathcal{M}_0}$ is ergodic for every ergodic p.m.p. action $\rho$ of $G$;

iii. The restriction of $\kappa|_{\mathcal{M}_0}$ to $L^2(X, \mu) \otimes \mathbb{C}1_X$ is weakly mixing.

**Proof.** This follows from Propositions 5.4 and 5.6.

5.C. **Proof of Theorem 11**

**Proof.** We may assume that $L$ is a closed subgroup of the unitary group $\mathcal{U}(N)$ of some finite von Neumann algebra $N$, and that $d_L$ comes from the $\|\cdot\|_2$-norm on $N$. Let $A = L^\infty(X, \mu)$ and view $w$ as a cocycle $w : G \to \mathcal{U}(A \otimes N)$ for the action $\sigma = \sigma_0 \otimes \id_N$, i.e., satisfying $w_{g \sigma} = w_g \sigma_0(wh)$. We will use Popa’s setup from Theorem 4.1 of [50], with $A$ here taking the place of $P$. Namely, let $\tilde{A} = A \rtimes \mathbb{R}$, let $\tilde{\sigma} = \sigma_0 \otimes \sigma_0 \otimes \id_N$, let $M = (A \otimes N) \rtimes_{\tilde{\sigma}} G$ and let $\tilde{M} = (\tilde{A} \otimes N) \rtimes_{\tilde{\sigma}} G$. We view $M$ as a subalgebra of $\tilde{M}$ so that the canonical unitaries $\{u_g\}_{g \in G} \subset M$ implement $\sigma$ and $\tilde{\sigma}$ on $M$ and $\tilde{M}$ respectively. We let $\tau$ denote the trace on $\tilde{M}$. Let $\{\beta\} \cup \{\alpha_t\}_{t \in \mathbb{R}} \subset \Aut(\tilde{A})$ denote the $s$-malleable deformation, and we extend $\beta$ and $\alpha_t$, $t \in \mathbb{R}$, to automorphisms of $\tilde{M}$ by taking $\beta(x) = \alpha_t(x) = x$ if $x \in N \rtimes LG$. Let $\tilde{\kappa}_g = w_g \kappa_\sigma$, $g \in G$, so that $g \mapsto \tilde{\kappa}_g$ is a homomorphism. Let $\tilde{\pi}$ denote the representation of $G$ on $L^2(\tilde{M}) = L^2(M) \otimes L^2(A)$ determined by $\tilde{\pi}(g)((xu_h) \otimes y) = \Ad(\tilde{\kappa}_g)(xu_h) \otimes (\sigma_0)_g(y)$, for $x \in A \otimes N$, $y \in A$, $g, h \in G$.

**Claim 5.9.** $\lim_{t \to 0} \left( \sup_{m \in C_{\kappa}(G)(H)} \int_{g \in G} \|\alpha_t(\tilde{\kappa}_g) - \tilde{\kappa}_g\|_2^2 dm(g) \right) = 0$.

**Proof of Claim 5.9.** Fix $\epsilon > 0$. It suffices to show that there exists $t_\epsilon > 0$, along with $S \subset H$ finite and $\delta > 0$, such that if $p \in \mathcal{P}$ satisfies $\sup_{s \in S} \|sp^S - p\|_1 < \delta$ then for all $0 < t < t_\epsilon$ we have $\int_G \|\alpha_t(\tilde{\kappa}_g) - \tilde{\kappa}_g\|_2^2 dp(g) < \epsilon$, since the claim will then follow using Lemma 5.1. Since $H \rtimes \mathcal{M}_0 (X, \mu)$ has stable spectral gap, there exists a finite set $S \subset H$ and $\delta_0 > 0$ such that if $\eta \in L^2(\tilde{M})$ is a unit vector satisfying $\sup_{s \in S} \|\tilde{\pi}(s)\eta - \eta\|_2 < \delta_0$, then $\|\eta - e(\eta)\|_2 < \epsilon/2$, where $e : L^2(\tilde{M}) \to L^2(M)$ denotes the orthogonal projection. Since $S$ is finite, there exists $t_\delta > 0$ such that for all $0 < t_0 \leq t_1$ we have $\sup_{s \in S} \|\alpha_{t_0}(\tilde{\kappa}_s) - \tilde{\kappa}_s\|_2 < \delta_0/4$. Let $t_\epsilon = 2t_1$ and fix $t_0$ with $0 < t_0 < t_1$. Let $\delta = \delta_0^2/4$ and fix $p \in \mathcal{P}$ with $\sup_{s \in S} \|sp^S - p\|_1 < \delta$.

Let $Q = \{u_g\}_{g \in G}$. Then we may identify $L^2(G)$ with $L^2(Q) \subset L^2(M)$ via $g \mapsto \tilde{\kappa}_g$. For each $q \in \mathcal{P}$, let $\eta_q = \sum_{g \in G} q(g)^{1/2} \tilde{\kappa}_g \in L^2(Q)$. We have $\tau(\alpha_{t_0}(\tilde{\kappa}_g)\tilde{\kappa}_h) = 0$ for all $g \neq h$, which
implies that \( \|\alpha_{t_0}(\tilde{u}_g)\xi - \tilde{u}_g\xi\|_2 = \|\alpha_{t_0}(\tilde{u}_g) - \tilde{u}_g\|_2 = \|\xi\alpha_{t_0}(\tilde{u}_g) - \xi\tilde{u}_g\|_2 \) for every unit vector \( \xi \in L^2(Q) \), and therefore \( \sup_{s \in S} \|\alpha_{t_0}(\tilde{u}_s)\eta - \eta\alpha_{t_0}(\tilde{u}_s)\|_2 < \delta_0/2 + \sup_{s \in S} \|\eta_{ps^{-1}} - \eta_p\|_2 < \delta_0 \).

By replacing \( t_0 \) by \( -t_0 \) and applying \( \alpha_{t_0} \) we obtain \( \sup_{s \in S} \|\tilde{\pi}(s)\alpha_{t_0}(\eta) - \alpha_{t_0}(\eta)\|_2 < \delta_0 \).

Our choice of \( \delta_0 \) then implies that \( \|\alpha_{t_0}(\eta_p) - \epsilon(\alpha_{t_0}(\eta_p))\|_2 < \epsilon^{1/2}/2 \), and hence by Popa's Transversality Lemma [50, Lemma 2.1],

\[
\int_G \|\alpha_{t_0}(\tilde{u}_g) - \tilde{u}_g\|^2 \, dm \leq 4 \int_G \|\alpha_{t_0}(\tilde{u}_g) - \epsilon(\alpha_{t_0}(\tilde{u}_g))\|^2 \, dm = 4 \|\alpha_{t_0}(\eta_p) - \epsilon(\alpha_{t_0}(\eta_p))\|^2 < \epsilon.
\]

\( \square \) [Claim 5.9]

Fix \( \epsilon > 0 \). By Claim 5.9 there exists \( t_\epsilon > 0 \) such that \( \int_G \|\alpha_t(\tilde{u}_g) - \tilde{u}_g\|_2^2 \, dm < \epsilon \) for all \( 0 \leq t \leq t_\epsilon \) and all \( m \in C.(G)(H) \). Fix such a \( t \) of the form \( t = 2^{-n} \) for some \( n \in \mathbb{N} \). Since \( C.(G)(H) \) is convex and weak*-compact, the set \( K = \{ \int_G \tilde{u}_g\alpha_t(\tilde{u}_g)^* \, dm : m \in C.(G)(H) \} \) is a convex and weakly closed subset of the unit ball of \( A \otimes N \), so there is a unique element \( x \in K \) of minimal \( \|\cdot\|_2 \)-norm. For each \( m \in C.(G)(H) \) we have \( \int_G \tilde{u}_g x \alpha_t(\tilde{u}_g)^* \, dm \in K \) since \( C.(G)(H) \) is a semigroup, and \( \|\int_G \tilde{u}_g x \alpha_t(\tilde{u}_g)^* \, dm\|_2 \leq \|x\|_2 \), hence \( x = \int_G \tilde{u}_g x \alpha_t(\tilde{u}_g)^* \, dm \), and therefore \( \int_G \|\tilde{u}_g x - x \alpha_t(\tilde{u}_g)\|_2^2 \, dm = 0 \), or equivalently \( \int_G \|w_g \tilde{g}(x) - x \alpha_t(\tilde{u}_g)\|_2^2 \, dm = 0 \).

**Claim 5.10.** Let \( v = \int_G \tilde{u}_g \alpha_t(\tilde{u}_g)^* \, dm \) for all \( m \in C.(G)(H) \).

**Proof.** Fix \( m \in C.(G)(H) \). For \( \xi \in \ker(x) \) we claim that \( \langle \int_G \alpha_t(\tilde{u}_g)^* \, dm, \xi \rangle \in \ker(x) \), or equivalently, \( \langle \int_G \alpha_t(\tilde{u}_g)^* \, dm, \xi \rangle = 0 \) for all \( y \in L^2(\tilde{M}) \). It is enough to show this for \( y \in \tilde{M} \), in which case we have

\[
|\langle \int_G \alpha_t(\tilde{u}_g)^* \, dm, \xi \rangle| = |\int_G \langle x \alpha_t(\tilde{u}_g)^* \xi, y \rangle \, dm| = |\int_G \langle (\tilde{u}_g x \alpha_t(\tilde{u}_g)^* - x) \xi, \tilde{u}_g y \rangle \, dm| \leq \|\xi\|_2 \int_G \|\tilde{u}_g x \alpha_t(\tilde{u}_g)^* - x\|_2 \, dm = 0.
\]

Since \( \ker(x) = \ker(v) \), it follows that for every \( \xi \in \ker(x) \) and \( y \in L^2(\tilde{M}) \) we have \( \int_G \langle \tilde{u}_g x \alpha_t(\tilde{u}_g)^* \xi, y \rangle \, dm = 0 \) (by first considering \( y \in \tilde{M} \)), and hence \( v \) and \( \int_G \tilde{u}_g \alpha_t(\tilde{u}_g)^* \, dm \) agree on \( \ker(x) \).

Assume now that \( \xi \in \text{ran}(\|x\|) \), say \( \xi = |x| \xi_0 \) for some \( \xi_0 \in L^2(\tilde{M}) \). Note that \( x^* x = \int_G \alpha_t(\tilde{u}_g)^* x^* x \alpha_t(\tilde{u}_g) \, dm \), hence \( |x| = \int_G \alpha_t(\tilde{u}_g)^* |x| \alpha_t(\tilde{u}_g) \, dm \). Therefore, for any \( y \in \tilde{M} \) we have

\[
\int_G \langle \tilde{u}_g x \alpha_t(\tilde{u}_g)^* \xi, y \rangle \, dm = \int_G \langle \tilde{u}_g x \alpha_t(\tilde{u}_g)^* |x| \xi_0, y \rangle \, dm = \int_G \langle \tilde{u}_g v |x| \alpha_t(\tilde{u}_g)^* \xi_0, y \rangle \, dm = \int_G \langle \tilde{u}_g x \xi_0, y \rangle \, dm = \int_G \langle v \xi, y \rangle \, dm.
\]

We conclude that \( v \) and \( \int_G \tilde{u}_g \alpha_t(\tilde{u}_g)^* \, dm \) agree on \( \text{ran}(\|x\|) = \ker(x)^\perp \), hence on \( L^2(\tilde{M}) \). \( \square \)

The rest of the argument proceeds as in Theorem 4.1 of [50]. By our choice of \( t \), we have

\[
\|x - 1\|_2^2 \leq \sup_{m \in C.(G)(H)} \int_G \|\tilde{u}_g m(\tilde{u}_g)^* - 1\|_2^2 \, dm < \epsilon,
\]
so \(\|v - 1\|_2^2 < 16\epsilon\). Since \(\sigma_{0,\mathcal{A}(G)}(H)\) is weakly mixing, we can apply Popa’s doubling procedure \(n\) times starting at \(t = 2^{-n}\) to obtain a partial isometry \(v_1 \in \tilde{A} \otimes N\) with \(\|v_1\|_2^2 = \|v\|_2^2 > 1 - 4\epsilon^{1/2}\), and satisfying \(\int_G \|w_g \sigma_g(v_1) - v_1 \alpha_1(w_g)\|_2^2 \, dm = 0\) for all \(m \in \mathcal{A}(G)(H)\). We can therefore extend \(v_1\) to a unitary \(u_1 \in \tilde{A} \otimes N\) which satisfies \(\sup_{m \in \mathcal{A}(G)(H)} \int_G \|w_g \sigma_g(u_1) - u_1 \alpha_1(w_g)\|_2^2 \, dm < 8\epsilon^{1/2}\). Since \(\epsilon > 0\) was arbitrary, the proof of Lemma 2.12.2 of [9] shows that there exists a unitary \(u \in \tilde{A} \otimes N\) such that \(\int_G \|w_g \sigma_g(u) - u \alpha_1(w_g)\|_2^2 \, dm = 0\) for all \(m \in \mathcal{A}(G)(H)\). We now apply Lemma C.6 to conclude that there exist unitaries \(u' \in A \otimes 1 \otimes N\), \(v' \in 1 \otimes A \otimes N\) with \(u = u'^* v'\), and such that \(u'\), viewed as a map \(X \to \mathcal{U}(N)\), takes values in the closed subgroup \(L\). For \(g \in G\) we define \(w_g' = u'^* w_g \sigma_g(u')\), so that the cocycle \(w'\) is cohomologous to \(w\).

Let \(\mathcal{A}\) be a separable subalgebra of \(\ell^\infty(G)\). We apply the second part of Lemma C.3 to obtain a closed subsemigroup \(\mathcal{M}_0 \subseteq \mathcal{A}(G)(H)\) with \(\mathcal{M}_0|\mathcal{A} = \mathcal{A}(G)(H)|\mathcal{A}\) along with a map \(\rho : G \to \mathcal{U}(N)\) such that

\[
\sup_{m \in \mathcal{A}_0} \int_G \|w_g' - \rho_g\|_2^2 \, dm = 0. \tag{5.1}
\]

We may assume that the algebra \(\mathcal{A}\) contains the function \(|\varphi|^2\) for every matrix coefficient \(\varphi\) associated to the Koopman representation of \(G\) on \(L^2(X, \mu)\), and hence that \(\sigma_{0,\mathcal{A}_0}\) is weakly mixing. Lemma C.5 now shows that the cocycle \(w'\) satisfies property (2) in the statement of Theorem 11.

For the rest of the argument, we continue to work with a subsemigroup \(\mathcal{M}_0\) of \(\mathcal{A}(G)(H)\) with \(\sigma_{0,\mathcal{M}_0}\) weakly mixing, and which satisfies (5.1). Fix any \(h \in H\). Then equation (5.1) implies that \(\sup_{m \in \mathcal{M}_0} \int_G \|\sigma_g(w_{h^{-1}gh}) - \rho_{h^{-1}gh}\|_2^2 \, dm = 0\) since each \(m \in \mathcal{M}_0\) is invariant under conjugation by \(H\). Using the identity \(w_g' \sigma_h(w_{h^{-1}gh}) = w_g' \sigma_g(w_h')\), it follows that for \(m \in \mathcal{M}_0\) we have

\[
\int_G \|w_g' \rho_{h^{-1}gh} - \rho_g \sigma_g(w_h')\|_2^2 \, dm \\
\leq \int_G \|w_g' \rho_{h^{-1}gh} - w_g' \sigma_h(w_{h^{-1}gh})\|_2^2 \, dm + \int_G \|w_g' \sigma_g(w_h') - \rho_g \sigma_g(w_h')\|_2^2 \, dm = 0.
\]

Therefore, since \(\sigma_{0,\mathcal{M}_0}\) is weakly mixing, if we view \(w_h'\) as a map from \(X\) to \(L\), then Lemma C.2 implies that \(w_h'\) is almost surely constant. This concludes the proof. \(\square\)

6. The AC-center, the inner radical, and linear groups

6.A. Proof of Theorem 13, parts i. through viii.

Proof of Theorem 13, parts i. through viii. We begin with the statements involving \(\mathcal{A}(G)\). It is clear that \(\mathcal{A}(G)\) and \(\mathcal{G}(G)\) are characteristic subgroups of \(G\). If \(N_0\) and \(N_1\) are normal subgroups of \(G\) with both \(G/C_G(N_0)\) and \(G/C_G(N_1)\) amenable, then \(G/C_G(N_0N_1) = G/(C_G(N_0) \cap C_G(N_1))\) is amenable. Therefore, \(\mathcal{A}(G)\) may be written as an increasing union \(\mathcal{A}(G) = \bigcup_{i \in \mathbb{N}} N_i\), with each \(N_i\) normal in \(G\) and \(G/C_G(N_i)\) amenable. It follows that \(G/C_G(\mathcal{A}(G))\) is residually amenable since \(\mathcal{C}_G(\mathcal{A}(G)) = \bigcap_{i \in \mathbb{N}} C_G(N_i)\). Each of the groups \(N_i\) is amenable since \(N_i/Z(N_i)\) is isomorphic to a subgroup of the amenable group \(G/C_G(N_i)\). This shows that \(\mathcal{A}(G)\) is amenable. Moreover, for each \(i \in \mathbb{N}\), the
action \( N_i \times G \cong N_i \) is amenable since it descends to an action of the amenable group \( N_i \times (G/C_G(N_i)) \). If \( m_i \) is a \( N_i \times G \)-invariant mean on \( N_i \), then any accumulation point of \( \{ m_i \}_{i \in \mathbb{N}} \) in the space of means on \( G \) will be a mean witnessing that the action \( \mathcal{A}^c(G) \times G \cong \mathcal{A}^c(G) \) is amenable. It follows that \( \mathcal{A}^c(G) \leq \mathcal{I}(G) \).

To prove the remaining statements involving \( \mathcal{I}(G) \) we will use the following lemma.

**Lemma 6.1.** Let \( H \) and \( K \) be normal subgroups of \( G \).

1. Assume that \( H \leq K \) and that the actions \( H \times G \cong H \) and \( K/H \times G/H \cong K/H \) are both amenable, with invariant means \( m_H \) and \( m_{K/H} \) respectively. Then the action \( K \times G \cong K \) is amenable with invariant mean \( m_K = \int_{KH \subseteq G/H} km_H \, dm_{K/H} \).
2. Assume that the actions \( H \times G \cong H \) and \( \text{K} \times G \cong K \) are both amenable, with invariant means \( m \) and \( n \) respectively. Then the action \( HK \times G \cong HK \) is amenable with invariant mean \( m * n \).

**Proof of Lemma 6.1** Since \( m_H \) is invariant under left translation by \( H \), for each \( g \in G \) the mean \( gm_H \) only depends on the coset \( gH \in G/H \). The mean \( m_K \) is therefore well-defined and it is straightforward to verify that it is \( K \times G \)-invariant. This shows (1), and (2) can either be deduced from (1) or verified directly.

It follows from Lemma 6.1 (2) that \( \mathcal{I}(G) \) may be written as an increasing union \( \mathcal{I}(G) = \bigcup_{i \in \mathbb{N}} M_i \), where each each \( M_i \) is normal in \( G \) and the action \( M_i \times G \cong M_i \) is amenable. If \( m_i \) is an invariant mean for the action \( M_i \times G \cong M_i \), then any accumulation point \( m \) of \( \{ m_i \}_{i \in \mathbb{N}} \) will be an invariant mean for the action \( \mathcal{I}(G) \times G \cong \mathcal{I}(G) \). In particular, \( m \) witnesses that \( \mathcal{I}(G) \) is amenable, and if \( \mathcal{I}(G) \) is infinite then \( m \) also witnesses that \( G \) is inner amenable. The proof of i. through v. is now complete.

vi. Let \( \pi : G \to G/N \) denote the projection map. Then the image under \( \pi \) of an \( \mathcal{I}(G) \times G \)-invariant mean on \( \mathcal{I}(G) \) is an \( \mathcal{I}(G)/N \times G/N \)-invariant mean on \( \mathcal{I}(G)/N \). This shows that \( \mathcal{I}(G)/N \leq \mathcal{I}(G/N) \). The reverse containment then follows by applying part (1) of Lemma 6.1 to the groups \( H = \mathcal{I}(G) \) and \( K = \pi^{-1}(\mathcal{I}(G)/N) \).

vii. Part vi. implies that \( \mathcal{I}(G/\mathcal{I}(G)) = 1 \), and this in turn implies that \( G/\mathcal{I}(G) \) is ICC since every finite conjugacy class in \( G/\mathcal{I}(G) \) is contained in \( \mathcal{A}^c(G/\mathcal{I}(G)) \leq \mathcal{I}(G/\mathcal{I}(G)) = 1 \).

viii. This in fact holds more generally with \( \mathcal{I}(G) \) replaced by any normal subgroup \( N \) of \( G \) for which \( N \times G \cong N \) is amenable. To see this, fix an invariant mean \( n \) for the action \( N \times G \cong N \). As in Lemma 6.1 we obtain a well-defined map

\[
m \mapsto m * n = \int_{gN \in G/N} gm \, dm(gN)
\]

taking means on \( G/N \) to means on \( G \). This map is a section for the projection map on means, and since \( n \) is invariant under conjugation by \( G \), this map takes conjugation invariant means on \( G/N \) to conjugation invariant means on \( G \).

**6.B. Proof of Theorem 13 parts ix. through xiv.** The second half of Theorem 13 will be deduced from the following spectacular theorem of S.G. Dani from [12], which appears to have been overlooked since its publication in 1985. In what follows, if \( G \cong X \) is an action of a group \( G \) then for \( A \subseteq X \) let \( \text{stab}_G(A) \) denote the pointwise stabilizer of \( A \) in \( G \), and for \( D \subseteq G \) let \( \text{fix}_X(D) \) denote the set of points in \( X \) which are fixed by every element of \( D \).
Theorem 6.2 (Theorem 1.1 of [12]). Let $G \curvearrowright X$ be an amenable action of a group $G$ on a set $X$ and let $m$ be a $G$-invariant mean on $X$. Suppose that the action satisfies the following two conditions:

1. For every subset $A \subseteq X$ there exists a finite $A_0 \subseteq A$ such that $\text{stab}_G(A) = \text{stab}_G(A_0)$.
2. For every subset $D \subseteq G$ there exists a finite $D_0 \subseteq D$ such that $\text{fix}_X(D) = \text{fix}_X(D_0)$.

Then there exists a normal subgroup $N$ of $G$ such that $G/N$ is amenable and $m(\text{fix}_X(N)) = 1$.

Proof of Theorem 13 parts ix. through xiv. Assume that $G$ is linear. We first show that any conjugation invariant mean $m$ on $G$ must concentrate on a normal subgroup $M$ of $G$ such that $G/C_G(M)$ is amenable. Consider the conjugation action $G \curvearrowright G$. For $A, D \subseteq G$ we have $\text{stab}_G(A) = C_G(A)$, and $\text{fix}_X(D) = C_G(D)$. Conditions (1) and (2) of Theorem 6.2 are therefore satisfied since $G$ satisfies the minimal condition on centralizers (see Remark 0.11).

We conclude that there exists a normal subgroup $N$ of $G$ such that $G/N$ is amenable, and $m(C_G(N)) = 1$. Take $M = C_G(N)$. Then $N \leq C_G(M)$, so $G/C_G(M)$ is amenable, as was to be shown. This also shows that xiii. holds.

ix. and x. By part iii., we may find an invariant mean $m$ for the action $\mathcal{I}(G) \rtimes G \curvearrowright \mathcal{I}(G)$. Since $m$ is conjugation invariant there exists a normal subgroup $M$ of $G$ with $G/C_G(M)$ amenable and $m(M) = 1$. Then $M \leq \mathcal{A}(G) \leq \mathcal{I}(G)$ and, since $m$ is invariant under left translation by $\mathcal{I}(G)$, we have equality $M = \mathcal{I}(G)$.

x. Parts x. and ii. show that $\mathcal{I}(G) \leq C_G(C_G(\mathcal{I}(G))) \leq \mathcal{A}(G) \leq \mathcal{I}(G)$.

xii. This follows from vii. and xii.

xiv. From vii., we have $C_{G/N}(\mathcal{I}(G)/N) = C_{G/N}(\mathcal{I}(G)/N) \geq C_G(\mathcal{I}(G))N/N$, hence $(G/N)/C_{G/N}(\mathcal{I}(G)/N)$ is amenable, and $\mathcal{I}(G)/N \leq \mathcal{A}(G/N)$. Part ii. gives the reverse inclusion. It follows as in x. above that $\mathcal{I}(G)/N$ coincides with its double centralizer. The group $(G/N)/\mathcal{I}(G)/N = (G/N)/(\mathcal{I}(G)/N) \cong G/\mathcal{I}(G)$ is not inner amenable by xii. Now let $m_0$ be a conjugation invariant mean on $G/N$, and let $m_1$ denote the projection of $m_0$ to $G/\mathcal{I}(G)$. Then $m_1$ is a conjugation invariant mean on $G/\mathcal{I}(G)$, so by viii., $m_1$ is the projection of $m_0$ to $G/\mathcal{I}(G)$. By xiii., $m$ concentrates on $\mathcal{I}(G)$, hence $m_1$ is the point mass at the identity, and therefore $m_0$ concentrates on $\mathcal{I}(G)/N = \mathcal{I}(G/N)$. \qed

6.C. Proof of Theorem 14.

Proof of Theorem 14. The implication (1) $\Rightarrow$ (2) follows from Theorem 13 xiii., and (2) $\Rightarrow$ (1) is Theorem 13 v. Assume now that (2) holds and let $N = C_G(\mathcal{I}(G)).\mathcal{I}(G)$. Then $G/N$ is amenable by Theorem 13 x., and $Z(N) = C_G(\mathcal{I}(G)) \cap \mathcal{I}(G)$ by Theorem 13 xi., so (3) follows. If (3) holds then in the first alternative $Z(N)$ infinite and $Z(N) \leq \mathcal{A}(G) \leq \mathcal{I}(G)$, and in the second alternative $M$ is infinite and $M \leq \mathcal{A}(G) \leq \mathcal{I}(G)$, so (2) holds either way. \qed

7. Stability

7.A. Kida’s stability criterion. In this section we employ the notation from §3.A. Let $(\mathcal{G}, \mu)$ be a discrete p.m.p. groupoid and let $[\mathcal{G}]$ denote the full group of $\mathcal{G}$, i.e., the collection of all local sections with domain equal to all of $\mathcal{G}$.

Definition 7.1. A sequence $(T_n)_{n \in \mathbb{N}}$ in $[\mathcal{G}]$ is said to be asymptotically central if
Remark 7.2. Suppose that $G = G \rtimes (X, \mu)$ is the translation groupoid associated to a p.m.p. action $G \actson (X, \mu)$ of a countable group $G$. We view each $T \in [G]$ as a map from $X$ to $G$, so that $T^0(x) = T(x) \cdot x$ for $x \in X$. We will make use of the observation [34, §3.1] that in this situation, a sequence $(T_n)_{n \in \mathbb{N}}$ in $[G]$ is asymptotically central if and only if it satisfies (i) along with

$$(ii') \mu \{x \in X : T_n(g \cdot x) = g T_n(x) g^{-1}\} \to 1$$

for all $g \in G$.

Likewise, a sequence $(A_n)_{n \in \mathbb{N}}$ of measurable subsets of $X$ is asymptotically invariant for $G$ if and only if $\mu(g \cdot A_n \triangle A_n) \to 0$ for all $g \in G$.

The following theorem, due to Kida [34], provides an important criterion for demonstrating stability of a group.

Theorem 7.3 (Theorem 1.4 of [34]). Let $G$ be a countable group and suppose that there exists a p.m.p. action $G \actson (X, \mu)$ of $G$ whose associated translation groupoid $G \rtimes (X, \mu)$ admits a stability sequence. Then $G$ is stable.

7.B. Proof of Theorem 16

Proof of Theorem 16. Let $G \actson (X, \mu)$ be a p.m.p. action of $G$ such that $N \rtimes (X, \mu)$ admits a stability sequence. Let $K \actson (Z, \eta)$ be a free p.m.p. action of $K$ and let $G \actson (X, \mu) \otimes (Z, \eta)$ be the diagonal product action, where $G$ acts on the second coordinate via the quotient map to $K$. In what follows we will often identify an element of $G$ with its image in $K$.

By Theorem 7.3 it suffices to show that the translation groupoid $G \rtimes (X, \mu) \otimes (Z, \eta)$ admits a stability sequence. We will construct such a sequence $(T_n)_{n \in \mathbb{N}}$ which is moreover contained in $[N \rtimes (X, \mu) \otimes (Z, \eta)]$. Let $F_0 \subseteq F_1 \subseteq \cdots$ be an exhaustion of $G$ by finite subsets. By Theorem 3.1 of [3], since $K$ is amenable, for each $n \geq 0$ we may find a measurable function $\varphi_n : Z \to K$ such that $\eta(C_n) > 1 - 2^{-n}$, where $C_n = \{z \in Z : (\forall g \in F_n) \varphi_n(g \cdot z) = g \varphi_n(z)\}$. Let $1_K \in Q_0 \subseteq Q_1 \subseteq \cdots$ be an exhaustion of $K$ by finite subsets such that for each $n \geq 0$ we have $\eta(D_n) > 1 - 2^{-n}$, where $D_n = \{z \in Z : \varphi_n(z) \in Q_n\}$. Let $Z_n = \bigcap_{m \geq n} C_m \cap D_m$, so that $Z_0 \subseteq Z_1 \subseteq \cdots$, and $\eta(\bigcup_n Z_n) = 1$. After ignoring a null set we may assume that $\bigcup_n Z_n = Z$.

Fix a section $\sigma : K \to G$ for the map $G \to K$ with $\sigma(1_K) = 1_G$, and let $\rho : G \times K \to N$ be the associated Schreier cocycle $\rho(g, k) = \sigma(gk)^{-1} g \sigma(k) \in N$.

By assumption, the groupoid $N \rtimes (X, \mu)$ admits a stability sequence $(S_i)_{i \in \mathbb{N}}$. After replacing $(S_i)_{i \in \mathbb{N}}$ by a subsequence if necessary we may assume that there exists an asymptotically invariant sequence $(B_i)_{i \in \mathbb{N}}$ for $N \rtimes (X, \mu)$ such that $\lim_i \mu(S_i^0 B_i \triangle B_i) > 0$. Fix a sequence $B_0 \subseteq B_1 \subseteq \cdots$ of finite algebras of measurable subsets of $X$ whose union generates the measure algebra of $X$. By moving to a subsequence $(S_{i_n})_{n \in \mathbb{N}}$ and $(B_{i_n})_{n \in \mathbb{N}}$, which we will call $(S_n)_{n \in \mathbb{N}}$ and $(B_n)_{n \in \mathbb{N}}$ respectively, we may ensure that

$$(C1) \quad \mu(S^0_n (\sigma(k)^{-1} \cdot A) \triangle \sigma(k)^{-1} \cdot A) < 1/n$$

for all $k \in Q_n$ and $A \in B_n$. 

\(\textbf{(C2)}\) \(\mu(W_n) > 1 - 2^{-n}\), where 
\[W_n = \{x : S_n(\rho(g,k) \cdot (\sigma(k)^{-1} \cdot x)) = \rho(g,k)S_n(\sigma(k)^{-1} \cdot x)\rho(g,k)^{-1} \text{ for all } g \in F_n, \; k \in Q_n\}.
\]

\(\textbf{(C3)}\) \(\mu(\rho(g,k)^{-1}B_n \triangle B_n) < 1/n\) for all \(g \in F_n, \; k \in Q_n\).

Let \(X_n = \bigcap_{m \geq n} W_m\), so that \(X_0 \subseteq X_1 \subseteq \cdots\), and \(\mu(\bigcup_n X_n) = 1\). After ignoring a null set we may assume that \(\bigcup_n X_n = X\). For each \(n \in \mathbb{N}\) and \((x,z) \in X \times Z\) define 
\[T_n(x,z) = \sigma(\varphi_n(z))S_n(\sigma(\varphi_n(z))^{-1} \cdot x)\sigma(\varphi_n(z))^{-1} \in N.
\]

Then \(T_n^0\) is an automorphism of \((X,\mu) \otimes (Z,\eta)\), since for each fiber \((X_z,\mu_z) := (X,\mu) \otimes (\{z\},\delta_z)\), the restriction of \(T_n^0\) to \(X_z\) is an automorphism \(T_{n,z}^0 : (X_z,\mu_z) \to (X_z,\mu_z)\). We now verify that \((T_n)_{n \in \mathbb{N}}\) satisfies properties (i)-(iv) of Definition [7.1] with respect to the groupoid \(G := G \times (X,\mu) \otimes (Z,\eta)\).

(i): It suffices to prove (i) for rectangles \(D = A \times C\), with \(A \in \bigcup_n B_n\). For \(z \in Z \setminus C\) we have \(\mu_z(T_n^0(A \times C)_z \Delta (A \times C)_z) = 0\) for all \(n\). For \(z \in C\), if \(n\) is large enough then \(A \in B_n\) and \(z \in Z_n\), so \(\varphi_n(z) \in Q_n\), hence 
\[\mu_z(T_n^0(A \times C)_z \Delta (A \times C)_z) = \mu(\sigma(\varphi_n(z)) \cdot S_n^0(\sigma(\varphi_n(z))^{-1} \cdot A) \Delta A) = \mu(S_n^0(\sigma(\varphi_n(z))^{-1} \cdot A) \Delta A) < 1/n\]
by \(\textbf{(C1)}\). Therefore, \(\mu(\otimes \eta)(T_n^0(A \times C)_z \Delta (A \times C)_z) = \int_Z \mu_z(T_n^0(A \times C)_z \Delta (A \times C)_z) \, d\eta \to 0\) as \(n \to \infty\).

(iii): Fix \(g \in G\). For all large enough \(n \in \mathbb{N}\) we have \(g \in F_n\), so if \((x,z) \in X_n \times Z_n\) then \(\varphi_n(g \cdot z) = g \cdot \varphi_n(z)\) and \(\varphi_n(z) \in Q_n\), hence by \(\textbf{(C2)}\), \(\rho(g,\varphi_n(z))^{-1}S_n(\rho(g,\varphi_n(z)) \cdot \sigma(\varphi_n(z))^{-1} \cdot x)\rho(g,\varphi_n(z)) = S_n(\sigma(\varphi_n(z))^{-1} \cdot x)\). Therefore, for all large enough \(n\), if \((x,z) \in X_n \times Z_n\) then we have 
\[T_n(g \cdot (x,z)) = T_n(g \cdot x, g \cdot z) = \sigma(\varphi_n(g \cdot z))S_n(\sigma(\varphi_n(z))^{-1} \cdot (g \cdot x))\sigma(\varphi_n(g \cdot z))^{-1} = \sigma(g \cdot \varphi_n(z))S_n(\sigma(g \cdot \varphi_n(z))^{-1} \cdot (g \cdot x))\sigma(g \cdot \varphi_n(z))^{-1} = g\sigma(\varphi_n(z))\rho(g,\varphi_n(z))^{-1}S_n(\rho(g,\varphi_n(z)) \cdot \sigma(\varphi_n(z))^{-1} \cdot x)\rho(g,\varphi_n(z))\sigma(\varphi_n(z))^{-1} = g\sigma(\varphi_n(z))S_n(\sigma(\varphi_n(z))^{-1} \cdot x)\sigma(\varphi_n(z))^{-1}g^{-1} = gT_n(x,z)g^{-1},
\]
and since \(X \times Z = \bigcup_n (X_n \times Z_n)\) the proof of (iii) is complete.

For each \(n \in \mathbb{N}\) define the set \(A_n = \{(x,z) \in X \times Z : x \in \sigma(\varphi_n(z)) \cdot B_n\}\). We will verify (iii) and (iv) using the sequence \((A_n)_{n \in \mathbb{N}}\).

(iii): Fix \(g \in G\). For all large enough \(n \in \mathbb{N}\) we have \(g \in F_n\), so for \((x,z) \in X \times Z_n\) we have \((x,z) \in g^{-1} \cdot A_n \Leftrightarrow x \in \sigma(\varphi_n(z)) \cdot \rho(g,\varphi_n(z))^{-1} \cdot B_n\). Hence, by \(\textbf{(C3)}\),
\[\mu(\otimes \eta)(g^{-1} \cdot A_n \triangle A_n) \leq \eta(Z \setminus Z_n) + \int_Z \mu(\rho(g,\varphi_n(z))^{-1} \cdot B_n \triangle B_n) \, d\eta < \eta(Z \setminus Z_n) + 1/n \to 0\]
as \(n \to \infty\). This shows that \((A_n)_{n \in \mathbb{N}}\) is asymptotically invariant for \(G \rhd (X,\mu) \otimes (Z,\eta)\).

(iv): We have \(T_n^0A_n = \{(x,z) \in X \times Z : x \in \sigma(\varphi_n(z)) \cdot S_n^0B_n\}\). It follows that \(\mu(\otimes \eta)(T_n^0A_n \triangle A_n) = \mu(S_n^0B_n \triangle B_n) \neq 0\). This completes the proof. \(\square\)
Remark 7.4. Let $1 \to N \to G \to K \to 1$ be a short exact sequence in which $K$ is amenable. Let $G \curvearrowright (X, \mu)$ be a p.m.p. action of $G$ and let $K \curvearrowright (Z, \eta)$ be a free p.m.p. action of $K$. Let $G \curvearrowright (X, \mu) \otimes (Z, \eta)$ be the diagonal product action where $G$ acts on $(Z, \eta)$ via the quotient map to $K$. The above proof constructs a map which takes asymptotically central sequences in $[N \ltimes (X, \mu)]$ to asymptotically central sequences in $[G \ltimes (X, \mu) \otimes (Z, \eta)]$, and which moreover takes stability sequences for $N \ltimes (X, \mu)$ to stability sequences for $G \ltimes (X, \mu) \otimes (Z, \eta)$. In addition, it follows from the construction that if $(S_n)_{n \in \mathbb{N}}$ is a sequence in $[N \ltimes (X, \mu)]$ witnessing that the outer automorphism group of $[\mathcal{R}_X^N]$ is not Polish, then the image of $(S_n)_{n \in \mathbb{N}}$ under this map will be a sequence witnessing that the outer automorphism group of $[\mathcal{R}_X^G]$ is not Polish (see [31, Chapter I, §7]).

Remark 7.5. In [52], Schmidt raises the question of whether every inner amenable group $G$ possesses a free ergodic p.m.p. action $G \curvearrowright (X, \mu)$ which generates an orbit equivalence relation $\mathcal{R}_X^G$ for which the outer automorphism group of the full group $[\mathcal{R}_X^G]$ is not Polish, or equivalently, for which the full group $[\mathcal{R}_X^G]$ contains an asymptotically central sequence $(T_n)_{n \in \mathbb{N}}$ with $\lim \inf_n \mu \{x \in X : T_n x \neq x\} > 0$. See also Problem 9.3 of [31]. We can now provide a positive answer to Schmidt’s question when $G$ is linear, using Theorems [13], [14] and [15]. This is straightforward if $G$ is stable, so we may assume that $G$ is inner amenable, but not stable. Then the group $N = C_G(\mathcal{A}(G))$ has infinite center $C$ (see Remark 7.11), and by Theorem 13 the group $K = G/N$ is amenable. Since $C$ is a countable abelian group, it possesses a free p.m.p. action $C \curvearrowright (Y, \nu)$ which is compact (for example, using a countable dense subset of $\hat{C}$, inject $C$ as a subgroup of $\mathbb{T}^\mathbb{N}$ and let $C$ act by translation on $\mathbb{T}^\mathbb{N}$ equipped with Haar measure). Let $G \curvearrowright (X, \mu) = (Y, \nu)^{G/C}$ be the coinduced action. This is a free weakly mixing action of $G$, and the restriction of this action to $C$ is an infinite diagonal product of compact actions of $C$, hence is itself a compact action. It follows that there exists a sequence $(c_n)_{n \in \mathbb{N}}$ in $C - 1$ which converges to the identity automorphism in the group $\text{Aut}(X, \mu)$ equipped with the weak topology. The sequence $(c_n)_{n \in \mathbb{N}}$ is then asymptotically central in $[\mathcal{R}_X^N]$, and since $C$ acts freely, the sequence $(c_n)_{n \in \mathbb{N}}$ witnesses that the outer automorphism group of $[\mathcal{R}_X^N]$ is not Polish. Let $K \curvearrowright (Z, \eta)$ be a free ergodic action of $K$, and let $G \curvearrowright (X, \mu) \otimes (Z, \eta)$ be the diagonal product action where $G$ acts on $(Z, \eta)$ via the quotient map to $K$. This action of $G$ is free and ergodic, and as observed in Remark 7.4 above, the construction in the proof of Theorem 16 yields a sequence $(T_n)_{n \in \mathbb{N}}$ witnessing that the outer automorphism group of $[\mathcal{R}_X^G]$ is not Polish.

Remark 7.6. A variation of the proof of Theorem 16 shows that if a group $G$ contains a finite index subgroup $H$ which is stable, then $G$ is stable. Take a p.m.p. action $H \curvearrowright (X_0, \mu_0)$ such that $H \ltimes (X_0, \mu_0)$ admits a stability sequence $(S_n)_{n \in \mathbb{N}}$, and let $(B_n)_{n \in \mathbb{N}}$ be a sequence of asymptotically invariant sets for the action with $\lim \mu_0(S_n^\alpha A_n \Delta A_n) = 0$. Let $G \curvearrowright (X, \mu)$ be the induced action, i.e., $X = X_0 \times G/H$, $\mu$ is the product of $\mu_0$ with normalized counting measure, and $g \cdot (x_0, kH) = (\rho(g, kH) \cdot x_0, gkH)$ for $g \in G$, $x_0 \in X_0$, $kH \in G/H$, where $\rho(g, kH) = \sigma(gkH)^{-1}g\sigma(kH)$ is $H$ and $\sigma : G/H \to G$ is a section for the projection map $G \to G/H$ with $\sigma(1H) = 1$. Define $T_n \in [G \ltimes (X, \mu)]$ by $T_n(x_0, kH) = \sigma(kH)S_n(x_0)\sigma(kH)^{-1} \in G$, and define $A_n = B_n \times G/H$. Then, using the sequence $(A_n)_{n \in \mathbb{N}}$, an argument similar to the proof of Theorem 16 shows that $(T_n)_{n \in \mathbb{N}}$ is a stability sequence for $G \ltimes (X, \mu)$. 
7.C. Proof of Theorem \[17\] Throughout this subsection we work under the assumption that \(1 \to N \to G \to K \to 1\) is a short exact sequence in which \(K\) is amenable.

Lemma 7.7. Suppose that \(N = LM\), where \(L\) and \(M\) are commuting normal subgroups of \(N\) such that \(M\) is amenable and \([N : L] = \infty\). Let \(N/M \curvearrowright (X, \mu)\) and \(N/L \curvearrowright (Y, \nu)\) be free p.m.p. actions of \(N/M\) and \(N/L\) respectively. Let \(N\) act on \((X, \mu)\) and \((Y, \nu)\) via the quotient maps to \(N/M\) and \(N/L\) respectively, and let \(N \curvearrowright (X, \mu) \otimes (Y, \nu)\) be the diagonal product action. Then the translation groupoid \(N \ltimes (X, \mu) \otimes (Y, \nu)\) admits a stability sequence.

Proof. Let \(C = L \cap M\) and let \(M_0 = M/C\). The action \(M \curvearrowright (Y, \nu)\) descends to a free action \(M_0 \curvearrowright (Y, \nu)\). Since the group \(M_0\) is amenable, the equivalence relation \(R^{N_0}_1\) is treeable. Therefore, by Theorem 1.1 of \[35\], the p.m.p. groupoids \(C \times M_0 \ltimes (Y, \nu)\) and \(M \ltimes (Y, \nu)\) are isomorphic. Here, \(C \times M_0 \ltimes (Y, \nu)\) is the translation groupoid associated to the action \(C \times M_0 \curvearrowright (Y, \nu)\), where \(C\) acts trivially. Since \(M_0\) is amenable and acts freely on \((Y, \nu)\), the groupoid \(M_0 \ltimes (Y, \nu)\) admits a stability sequence \((S_n)_{n \in \mathbb{N}}\) \[25\]. For each \(n \in \mathbb{N}\) define \(S_n' \in [C \times M_0 \ltimes (Y, \nu)]\) by \(S_n'(y) = (1_C, S_n(y)) \in C \times M_0\). Then \((S_n')_{n \in \mathbb{N}}\) is a stability sequence for \(C \times M_0 \ltimes (Y, \nu)\). The image of this sequence under the above isomorphism is then a stability sequence \((T_n)_{n \in \mathbb{N}}\) for \(M \ltimes (Y, \nu)\). Define \(\hat{T}_n \in [M \ltimes (X, \mu) \otimes (Y, \nu)]\) by \(\hat{T}_n(x, y) = T_n(y)\). Then \((\hat{T}_n)_{n \in \mathbb{N}}\) is a stability sequence for \(N \ltimes (X, \mu) \otimes (Y, \nu)\). \(\square\)

Proof of stability from hypothesis \((H1)\). Let \(G/M \curvearrowright (X, \mu)\) and \(G/L \curvearrowright (Y, \nu)\) be free p.m.p. actions of \(G/M\) and \(G/L\) respectively. Then \(G\) acts on \((X, \mu)\) and \((Y, \nu)\) via the quotient maps to \(G/M\) and \(G/L\) respectively. Let \(G \curvearrowright (X, \mu) \otimes (Y, \nu)\) be the diagonal product of these actions. By Lemma 7.7, the groupoid \(N \ltimes (X, \mu) \otimes (Y, \nu)\) admits a stability sequence, hence \(G\) is stable by Theorem \[16\]. \(\square\)

Proof of stability from hypothesis \((H6)\). Let \(N^* = N - \{1\}\). Let \(G\) act on \(N^*\) by conjugation and consider the corresponding generalized Bernoulli action \(G \curvearrowright (X, \mu) = ([0,1]^{N^*}, \lambda^{N^*})\) given by \((g \cdot x)(h) = x(g^{-1}hg)\). Let \((c_n)_{n \in \mathbb{N}}\) and \((d_n)_{n \in \mathbb{N}}\) be sequences witnessing that \(N\) is doubly asymptotically commutative. The proof of Proposition 9.8 of \[31\] shows that the sequence \((c_n)_{n \in \mathbb{N}}\), viewed as a sequence in \([N \ltimes (X, \mu)]\), is a stability sequence for \(N \ltimes (X, \mu)\). Theorem \[16\] then implies that \(G\) is stable. \(\square\)

For the rest of the proof, we first note that \((H3)\) follows from each of the hypotheses \((H2)\), \((H4)\), and \((H5)\). This is obvious for \((H2)\). Assume now that \((H4)\) holds, so that \(A \cap C_G(g)\) has finite index in \(A\) for all \(g \in N\). Then we can find a decreasing sequence \(A = A_0 \supseteq A_1 \supseteq \cdots\) of finite index subgroups of \(A\) such that \(N = \bigcup_m C_N(A_m)\). Then for all \(m \in \mathbb{N}\) the pair \((C_N(A_m), A_m)\) does not have property \((T)\), since \(A_m\) is finite index in \(A\) and \((N, A)\) does not have property \((T)\). This shows that \((H3)\) holds. Finally, assume that \((H5)\) holds. After moving to a subsequence of \((c_n)_{n \in \mathbb{N}}\), we may assume that each of the subgroups \(A_i = \{\{c_n : n \geq i\}\}, i \in \mathbb{N}\), is abelian, so that \(N = \bigcup_i C_N(A_i)\), where the union is increasing. Then each of the groups \((C_N(A_i), A_i)\) does not have property \((T)\), since \(A_i\) is infinite and \(N\) has the Haagerup property. This verifies \((H3)\). It remains to deduce stability of \(G\) from \((H3)\).

Proof of stability from hypothesis \((H3)\). We may assume that \(N\) is not doubly asymptotically commutative, since otherwise we are done by the proof of stability from \((H6)\). Let
Claim 7.8. There exists an increasing sequence \( H_0 \leq H_1 \leq \ldots \), of finitely generated subgroups of \( N \) with \( N = \bigcup_{m \in \mathbb{N}} H_m \), along with sequences \( C_0 \geq C_1 \geq \ldots \), and \( (N_m)_{m \in \mathbb{N}} \) of subgroups of \( N \) such that, for all \( m \in \mathbb{N} \),

1. \( C_m, H_m \leq N_m \), and \( C_m = C_N(H_m) = Z(N_m) \),
2. The pair \( (N_m, C_m) \) does not have relative property (T),
3. \( qH_mg^{-1} \leq H_{m+1} \) for all \( g \in F_{m+1} \),
4. \( C_m \geq g^{-1}C_{m+1} \) for all \( g \in F_{m+1} \).

Proof of Claim 7.8. Since \( N \) is not doubly asymptotically commutative there exists a finitely generated subgroup \( H_0 \leq N \) such that \( C_N(H_0) \) is abelian. After moving to a subsequence of \( (L_m)_{m \in \mathbb{N}} \) if necessary we may assume that \( H_0 \leq L_0 \). We may then extend \( H_0 \) to a sequence \( H_0 \leq H_1 \leq H_2 \leq \ldots \), of finitely generated subgroups of \( N \) with \( N = \bigcup_{m \in \mathbb{N}} H_m \), and \( H_m \leq L_m \) for all \( m \in \mathbb{N} \). After moving to a further subsequence if necessary we may assume that property 3. is satisfied for all \( m \in \mathbb{N} \). Let \( C_m = C_N(H_m) \), so that \( C_m \) is an abelian group containing \( D_m \). Then the pair \( (H_mC_m, D_m) \) does not have property (T), since \( (H_mD_m, D_m) \) does not have property (T) and \( (H_mC_m)/(H_mD_m) \) is amenable [23]. It follows that \( (H_mC_m, C_m) \) does not have property (T). Let \( N_m = H_mC_m \). Then \( C_m \leq Z(N_m) \leq C_H(N_m) \leq C_H(H_m) = C_m \), so that both 1. and 2. are satisfied, and 4. follows from 1. and 3.

Fix an increasing sequence \( S_0 \subseteq S_1 \subseteq \ldots \) of finite sets such that \( S_m \) generates \( H_m \) for all \( m \geq 0 \). Note that if \( \pi_m \) is an irreducible unitary representation of \( N_m \), then Schur’s Lemma implies that \( \pi_m(c) \) is a scalar multiple of the identity for all \( c \in C_m \), since \( C_m = Z(N_m) \). Property 4. of the claim then shows that \( \pi_m(g^{-1}c) \) is also a scalar for all \( c \in C_n, g \in F_n, n > m \). The following is based on Lemma 4.1 of [34].

Lemma 7.9. There exist sequences \( (\pi_m)_{m \in \mathbb{N}}, (\xi_m)_{m \in \mathbb{N}}, \) and \( (c_m)_{m \in \mathbb{N}} \), where for each \( m \in \mathbb{N} \), \( \pi_m \) is an irreducible unitary representation of \( N_m \), \( \xi_m \in \mathcal{H}_{\pi_m} \) is a unit vector, and \( c_m \) is an element of \( C_m \), such that

1. \( \| \pi_m(s)\xi_m - \xi_m \| < 2^{-m} \) for all \( s \in S_m \),
2. \( |\pi_m(c_m) - 1| > 1 \),
3. \( |\pi_j(g^{-1}c_mg) - 1| < 2^{-m} \) for all \( g \in F_m \) and \( j < m \).

Proof. Let \( m \geq 0 \) and assume inductively that we have already found \( (\pi_j)_{j < m} \), \( (\xi_j)_{j < m} \), and \( (c_j)_{j < m} \). By compactness there exists a nonempty finite set \( P \subseteq C_m \) such that for each \( c \in C_m \) there exists some \( d \in P \) with \( \sup_{j < m} \sup_{g \in F_m} |\pi_j(g^{-1}cg) - \pi_j(g^{-1}dg)| < 2^{-m} \). Since \( (N_m, C_m) \) does not have property (T), we may find an irreducible unitary representation \( \pi_m \) of \( N_m \) which has no nonzero \( C_m \)-invariant vector, satisfying \( \sup_{d \in P} |\pi_m(d) - 1| < 1/3 \), along with a unit vector \( \xi_m \in \mathcal{H}_{\pi_m} \), such that \( \| \pi_m(s)\xi_m - \xi_m \| < 2^{-m} \) for all \( s \in S_m \). Since \( \pi_m(C_m) \) is nontrivial, there exists some \( d_m \in C_m \) satisfying \( |\pi_m(d_m) - 1| > 4/3 \). By our choice of \( P \) there exists \( d \in P \) such that \( \sup_{j < m} \sup_{g \in F_m} |\pi_j(g^{-1}d_mg) - \pi_j(g^{-1}dg)| < 2^{-m} \). Let \( c_m = d^{-1}d_m \). Then \( \sup_{j < m} \sup_{g \in F_m} |\pi_j(g^{-1}c_mg) - 1| < 2^{-m} \), and

\[
|\pi_m(c_m) - 1| = |\pi_m(d_m) - \pi_m(d)| \geq |\pi_m(d_m) - 1| - |\pi_m(d) - 1| > 4/3 - 1/3 = 1.
\]
For each $m \in \mathbb{N}$ let $\sigma_m : G/N_m \to G$ be a section for the projection map $G \to G/N_m$ with $\sigma_m(1_{N_m}) = 1$, and let $\rho_m : G \times G/N_m \to N_m$ be the corresponding Schreier cocycle, $\rho_m(g,h_{N_m}) = \sigma_m(gh_{N_m})^{-1}g\sigma_m(h_{N_m})$. Let $\pi_m = \text{Ind}^G_{N_m}(\pi_m)$ be the induced representation, i.e., $\pi_m$ is the representation of $G$ on $\mathcal{H}_m = \mathcal{H}_{\pi_m} \otimes \ell^2(G/N_m)$ given by $\pi_m(g)(\xi \otimes \delta_{h_{N_m}}) = \pi_m(\rho_m(g,k))(\xi) \otimes \delta_{h_{N_m}}$. Let $\tilde{\xi}_m = \xi_m \otimes \delta_1$. Hence there exists an infinite finitely generated abelian group $G$ toward a contradiction that $\mathcal{H}_m \neq 0$. Therefore for all $j \in \mathbb{N}$ and $\eta \in H_j$. Let $G \curvearrowright (\Omega, \nu)$ denote the Gaussian action associated to the representation $\bigoplus_m \tilde{\pi}_m$. As in the proof of Theorem 1.1.(i) of [34], we conclude that the sequence $(\xi_m)_{m \in \mathbb{N}}$, viewed as a sequence in the full group $[\mathcal{N} \times (\Omega, \nu)]$, is a stability sequence for $\mathcal{N} \times (\Omega, \nu)$. We can now apply Theorem [10] to conclude that $G$ is stable. \hfill \Box

The following proposition shows that, aside from the relative property (T) condition, the hypothesis (H4) has a natural expression in terms of a conjugation invariant mean on $G$ which concentrates on $A$, as long as we assume that $A$ is finitely generated.

**Proposition 7.10.** Let $G$ be a countable group. Then the following are equivalent:

1. There exists an atomless conjugation invariant mean $m$ on $G$ and an infinite finitely generated abelian group $A \leq G$ with $m(A) > 0$.

2. There exists an infinite finitely generated abelian group $G \leq G$ with $|G : \text{comm}_G(A)| < \infty$ such that $\text{comm}_G(A)/N$ is amenable, where $N$ is the kernel of the modular homomorphism from $\text{comm}_G(A)$ into the abstract commensurator of $A$.

Furthermore, the statement (1'), obtained from (1) by replacing "$m(A) > 0"$ with "$m(A) = 1", is equivalent to the statement (2'), obtained from (2) by replacing "$|G : \text{comm}_G(A)| < \infty$" with "$G = \text{comm}_G(A)".

**Proof.** (2) $\Rightarrow$ (1): Let $(A_n)_{n \in \mathbb{N}}$ enumerate the finite index subgroups of $A$, and for each $n \in \mathbb{N}$ choose a nonidentity element $a_n \in \bigcap_{i \leq n} A_i$. Let $m_0$ be an accumulation point of $(\delta_{a_n})_{n \in \mathbb{N}}$ in the space of means on $G$. Then $m_0$ concentrates on $A$, and is invariant under conjugation by $N$ since each $g \in N$ commutes with $a_n$ for finitely many $n \in \mathbb{N}$. Let $G_0 = \text{comm}_G(A)$ and let $m_{G_0/N}$ be a translation-invariant mean on $G_0/N$. Then the mean $m_1 = \frac{1}{|G_0|} \sum_{g \in G_0/N} gm_0 g^{-1} \frac{d m_{G_0/N}}{d m_{G_0/N}}$ is invariant under conjugation by $G_0$, and $m_1$ concentrates on $A$ since each of the means $gm_0 g^{-1}$, $g \in G_0$, concentrates on $A$. Finally, the mean $m = \frac{1}{|G : G_0|} \sum_{g \in G_0/N} gm_0 g^{-1} dm_{G_0/N}$ is invariant under conjugation by $G$ and satisfies $m(A) > 0$. This also shows the implication (2') $\Rightarrow$ (1').

(1) $\Rightarrow$ (2): We may assume that $A$ has minimal rank among all $m$-non-null finitely generated abelian subgroups of $G$. If $g \in G$ is such that $m(gAg^{-1} \cap A) > 0$, then $gAg^{-1} \cap A$ has the same rank as $A$, so $gAg^{-1} \cap A$ has finite index in $A$. Let $G_0 = \text{comm}_G(A)$ and suppose toward a contradiction that $[G : G_0] = \infty$. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence with $g_i G_0 \neq g_j G_0$ for all $i \neq j$. Then, for all $i \neq j$, the group $g_i^{-1} g_j G_0 \cap A$ does not have finite index in $A$, hence $0 = m((g_i^{-1} g_j G_0 \cap A) = m(g_j G_0 \cap g_j G_0^{-1} g_j G_0^{-1} g_i G_0^{-1} \cap A)$. Therefore for all $n > 0$ we have $1 \geq m((\bigcup_{i \leq n} g_i G_0^{-1} g_i^{-1} = \sum_{i \leq n} m(g_i G_0^{-1} G_0) = n \cdot m(A)$, a contradiction, since $m(A) > 0$. This shows that $[G : G_0] < \infty$. 

\hfill \Box
Let $N = \{g \in G_0 : [A : A \cap C_G(g)] < \infty\}$ be the kernel of the modular homomorphism $G_0 \to \text{comm}(A)$, and let $\varphi : G_0 \to G_0/N$ be the projection to the group $G_0/N$. Suppose toward a contradiction that $\varphi(G_0)$ is nonamenable. For a subgroup $B \leq A$ let $N(B) = \{g \in G_0 : [B : B \cap C_G(g)] < \infty\}$. Then $N(B)$ is a subgroup of $G_0$ with $N \leq N(B)$ for all $B \leq A$. Since $A$ is a finitely generated abelian group it satisfies the maximal condition on subgroups. Let $B_0 \leq A$ be a maximal subgroup from the collection $\{B \leq A : \varphi(N(B)) \text{ is nonamenable}\}$; this collection is nonempty since it contains the trivial group by hypothesis. Since $N(A) = N$, the group $B_0$ has infinite index in $A$, and hence $m(B_0) = 0$. Observe that for any $a \in A - B_0$, if we let $B_1 = \langle a, B_0 \rangle$, then the group $\varphi(N(B_1))$ is amenable by maximality of $B_0$, and we have $C_{N(B_0)}(a) \leq N(B_1)$, so the group $\varphi(C_{N(B_0)}(a))$ is amenable. Let $Y$ be the saturation of $A - B_0$ under conjugation by $N(B_0)$. Then $\varphi(C_{N(B_0)}(y))$ is amenable for all $y \in Y$. It follows that for all $y \in Y$, the translation action $C_{N(B_0)}(y) \curvearrowright G_0/N$ is amenable. In addition, the conjugation action $N(B_0) \curvearrowright Y$ is amenable since $m(Y) > 0$. It follows (using Theorem 3 for example) that the translation action $N(B_0) \curvearrowright G_0/N$ is amenable, and hence the group $\varphi(N(B_0))$ is amenable, a contradiction.

For $(1') \Rightarrow (2')$, we take $A$ to have minimal rank among all $m$-conull finitely generated abelian subgroups of $G$. Then for all $g \in G$ the group $gAg^{-1} \cap A$ is $m$-conull, so $gAg^{-1} \cap A$ has finite index in $A$, and therefore $G = \text{comm}_G(A)$. The rest proceeds as above. \hfill $\square$

7.D. Proof of Theorem 15

Proof of Theorem 15 (1) $\Rightarrow$ (2): Suppose first that $G$ is stable as witnessed by the free ergodic action $G \curvearrowright (X, \mu)$. Then $G \ltimes (X, \mu)$ admits a stability sequence $(T_n)_{n \in \mathbb{N}}$ [25]. Let $U$ be a nonprincipal ultrafilter on $\mathbb{N}$ and for $D \subseteq G$ let $m(D) = \lim_{n \to U} \mu(\{x \in X : T_n(x) \in D\})$. Then $m$ is a conjugation invariant mean on $G$, so $m(\mathscr{I}(G)) = 1$ by Theorem 13.iii. We may therefore assume without loss of generality that $(T_n)_{n \in \mathbb{N}}$ is contained in $[\mathscr{I}(G) \ltimes (X, \mu)]$. It follows that every subgroup of $G$ containing $\mathscr{I}(G)$ is stable. Let $N = C_G(\mathscr{I}(G))\mathscr{I}(G)$ and note that $\mathscr{I}(N) = \mathscr{I}(G)$ since $G/N$ is amenable. If $[N : C_G(\mathscr{I}(G))] = \infty$, then the image of $\mathscr{I}(G)$ in the amenable group $G/C_G(\mathscr{I}(G))$ is infinite, so the pair $(G/C_G(\mathscr{I}(G)), N/C_G(\mathscr{I}(G)))$ does not have property (T), and hence $(G, \mathscr{I}(G))$ does not have property (T). We may therefore assume that $[N : C_G(\mathscr{I}(G))] < \infty$. This implies that the group $Z(N) = C_G(\mathscr{I}(G)) \cap \mathscr{I}(G)$ has finite index in $\mathscr{I}(G) = \mathscr{I}(N)$. Suppose toward a contradiction that $(G, \mathscr{I}(G))$ has property (T). Since the group $G/N$ is amenable, the pair $(N, \mathscr{I}(G))$ has property (T), hence $(N, Z(N))$ has property (T) [23]. The group $N$ is stable since $\mathscr{I}(G) \leq N$, so Theorem 1.1.2 of [34] implies that $N/Z(N)$ is stable, and in particular $N/Z(N)$ is inner amenable [25]. This is a contradiction since, by Theorem 13, every conjugation invariant mean on $N/Z(N)$ concentrates on the group $\mathscr{I}(N/Z(N)) = \mathscr{I}(N)/Z(N)$, which is finite. We conclude that the pair $(G, \mathscr{I}(G))$ does not have property (T).

(2) $\Rightarrow$ (1): Assume that $(G, \mathscr{I}(G))$ does not have property (T). It follows that $\mathscr{I}(G)$ is infinite. Let $L = C_G(\mathscr{I}(G))$, let $M = \mathscr{I}(G)$, and let $N = LM$ so that $Z(N) = Ln \cap M$. The group $G/N$ is amenable by Theorem 13. If $[N : L] = \infty$ then $G$ hypothesis (H1) holds, so $G$ is stable by part Theorem 17. If $[N : L] < \infty$ then $(N, Z(N))$ does not have property (T), since $(N, \mathscr{I}(G))$ does not have property (T) and $[\mathscr{I}(G) : Z(N)] = [N : L]$. This shows that hypothesis (H2) holds, so $G$ is stable by Theorem 17. \hfill $\square$
Remark 7.11. It follows from Theorems 13, 14, and 15 that a linear group $G$ is inner amenable but not stable if and only if the group $\mathcal{I}(G)$ is infinite and is finite index over its center $C = C_G(\mathcal{I}(G)) \cap \mathcal{I}(G)$, and the the pair $(C_G(\mathcal{I}(G)), C)$ has property (T).

7.E. Groups of piecewise projective homeomorphisms. Justin Moore has observed that an adaptation of the arguments of Brin-Squier [7] and Monod [38] shows the following

Lemma 7.12. Let $G$ be a countable subgroup of $H(\mathbb{R})$. Then the second derived subgroup $G''$ is either abelian or doubly asymptotically commutative.

Proof. Assume first that $G$ is finitely generated. Then the set $U = \mathbb{P}^1 \setminus \text{fix}(G)$ has finitely many connected components, each of which is an open interval. If $V \subseteq U$ is a union of a subset of these connected components then let $\varphi_V : G \to H(\mathbb{R})$ denote the homomorphism which sends $g \in G$ to the map $\varphi_V(g)$ which coincides with $g$ on $V$ and which is the identity elsewhere. In what follows, we fix an orientation of $\mathbb{P}^1 \setminus \{\infty\}$.

Claim 7.13. For any compact subset $K \subseteq U$ there exists an element $g \in G$ such that $g(K) \cap K = \emptyset$.

Proof of Claim 7.13. By induction on the number $n$ of connected components of $U$. If $n = 1$ then it suffices to show that for any $p \in U$ we have $\sup_{g \in G} g(p) = \sup U$. Suppose otherwise and let $q = \sup_{g \in G} g(p) < \sup U$. Then $q \in U$, so we may find some $g \in G$ with $g(q) \neq q$, and after replacing $g$ by $g^{-1}$ if necessary we may assume that $g(q) > q$. If $(g_n)_{n \in \mathbb{N}}$ is any sequence in $G$ with $g_n(p) \to q$ then $q \geq g(g_n(p)) \to g(q) > q$, a contradiction. Assume now that $U$ has $n + 1$ connected components and fix one such component $V$. After making $K$ larger if necessary we may assume that $K \cap V$ is a closed interval. Apply the base of the induction to the group $\varphi_V(G)$ to obtain a group element $h \in G$ with $h(K \cap V)$ disjoint from $K \cap V$. Since $K \cap V$ is an interval, after replacing $h$ by $h^{-1}$ if necessary, this means that $\inf h(K \cap V) > \sup(K \cap V)$. Let $L = (K \setminus V) \cup h(K \setminus V)$. Then $L$ is a compact subset of $U \setminus V$, so we may apply the induction hypothesis to the group $\varphi_{U \setminus V}(G)$ to obtain a group element $f \in G$ satisfying $f(L) \cap L = \emptyset$. After replacing $f$ by $f^{-1}$ if necessary we may assume that $f(p) \geq p$, where $p = \inf h(K \cap V)$. Take $g = fh$. Then $\inf g(K \cap V) = f(p) \geq p > \sup(K \cap V)$, so $g(K \cap V)$ is disjoint from $K \cap V$. In addition, $g(K \setminus V) \cap (K \setminus V) = f(h(K \setminus V)) \cap (K \setminus V) \subseteq f(L) \cap L = \emptyset$. Since $V$ is $G$-invariant, this shows that $g(K) \cap K = \emptyset$. □

Assume that $G''$ is nonabelian and fix two non-commuting elements $c_0, d_0 \in G''$. As shown in Lemma 14 of [38], the closure of the support of any element of $G''$ is a compact subset of $U$. Fix a finite subset $Q \subseteq G''$ and let $K$ be the union of the closures of the supports of all elements of $Q \cup \{c_0, d_0\}$. Apply the claim to find an element $g \in G$ with $g(K) \cap K = \emptyset$. Let $c = gc_0g^{-1}$ and let $d = gd_0g^{-1}$ so that $c, d \in G''$ and $cd \neq dc$. Then $c$ and $d$ both commute with each element of $Q$ since the support of $c$ and of $d$ are disjoint from the support of each element of $Q$. This shows that $G''$ is doubly asymptotically commutative.

When $G$ is not finitely generated we may write $G$ as an increasing union $G = \bigcup_n G_n$ with each $G_n$ finitely generated. Then $G'' = \bigcup_n G_n''$, so if $G''$ is nonabelian then $G_n''$ is nonabelian for all large enough $n$. Now note that double asymptotic commutativity is preserved by directed unions. □
Proof of Theorem 14. By [38], \( H(\mathbb{R}) \) is torsionfree, so any nontrivial amenable subgroup of \( H(\mathbb{R}) \) is infinite, hence stable. If \( G \) is a nonamenable subgroup of \( H(\mathbb{R}) \) then \( G'' \) is nonabelian, hence doubly asymptotically commutative by Lemma 7.12. Hypothesis (H6) holds, using the short exact sequence \( 1 \to G'' \to G \to G/G'' \to 1 \), hence Theorem 17 shows that \( G \) is stable.

Acknowledgements: The author warmly thanks Justin Moore for explaining Lemma 7.12, Adrian Ioana for helpful conversations and for allowing the inclusion of the joint Theorem 11, and Yoshikata Kida for pointing out (H4) from Theorem 17 and Corollary 18 and for a number of insightful comments on an earlier draft of this paper which helped improve the statement of Theorem 17. The author was supported by NSF grant DMS 1303921.

Appendices

APPENDIX A. PSEUDOCOST

Pseudocost is a modification of cost which was defined and studied by the author in [33] in order to extend several properties of cost for finitely generated groups to the non-finitely generated setting. The pseudocost of a p.m.p. equivalence relation \( R \) on \((X, \mu)\) is defined as \( \mathcal{PC}(R) = \inf (\mathcal{R}_n) \liminf_n \mathcal{C}(\mathcal{R}_n) \), where the infimum is taken over all increasing exhaustive sequences \((\mathcal{R}_n)_n \subseteq \mathcal{R}\) of subequivalence relations of \( R \). For a group \( G \), the values \( \mathcal{PC}(G) \) and \( \mathcal{PC}^*(G) \) are defined to be the infimum and supremum respectively, of the pseudocosts of orbit equivalence relations generated by free p.m.p. actions of \( G \). The inequality \( \mathcal{PC}(R) \leq \mathcal{C}(R) \) always holds, and it is shown in [33] that equality holds whenever \( \mathcal{C}(R) \) is finite (hence \( \mathcal{PC}(G) = \mathcal{C}(G) \) and \( \mathcal{PC}^*(G) = \mathcal{C}^*(G) \) whenever \( G \) is finitely generated), and that \( \mathcal{PC}(R) = 1 \) if and only if \( \mathcal{C}(R) = 1 \). Using Gaboriau’s inequality \( \beta_1^2(R) \leq \mathcal{C}(R) - 1 \) from [17], it is easy to see that in fact \( \beta_1^2(R) \leq \mathcal{PC}(R) - 1 \).

Proposition A.1. Let \( G \) be a countable group and suppose that \( \mathcal{PC}(G) > r \). Then there exists a finitely generated subgroup \( G_0 \leq G \) such that \( \mathcal{PC}(H) > r \) for all intermediate subgroups \( G_0 \leq H \leq G \). In particular, if \((H_n)_n \subseteq \mathcal{P} \) is a sequence of subgroups of \( G \) with \( G = \liminf_n H_n \) then \( \mathcal{PC}(G') \leq \liminf_n \mathcal{PC}(H_n) \).

Proof. Suppose there is no such subgroup \( G_0 \). Then we may find an increasing exhaustive sequence \((G_n)_n \subseteq \mathcal{P} \) of finitely generated subgroups of \( G \) and for each \( n \geq 1 \) an intermediate subgroup \( G_n \leq G' \leq G \) with \( \mathcal{PC}(G') \leq r \). For each \( n \geq 1 \) let \( G' \bowtie (X_n, \mu_n) \) be a free p.m.p. action with \( \mathcal{PC}(\mathcal{R}_{X_n}^{G_n}) \leq r \). Then, setting \( H_0 = 1 \), by Lemma 6.14.(4) of [33], for each \( n \geq 1 \) we may find a finitely generated group \( H_n \) with \( (G_n, H_{n-1}) \leq H_n \leq G' \), along with an equivalence relation \( R_n \) with \( \mathcal{R}_{X_n}^{(G_n, H_{n-1})} \subseteq \mathcal{R}_n \subseteq \mathcal{R}_{X_n}^{H_n} \) and \( \mathcal{C}(\mathcal{R}_n) < r + 1/n \). Let \( G \bowtie (X, \mu) = \prod_n (X_n, \mu_n)^{G/G_n} \) be the diagonal product of the coinduced actions. For each \( n \geq 1 \) the action \( G' \bowtie (X, \mu) \) factors onto \( G_n' \bowtie (X_n, \mu_n) \) via the projection \( p_n : (X, \mu) \to (X_n, \mu_n) \), so the equivalence relation \( \tilde{R}_n = \mathcal{R}_{X_n}^{G_n'} \cap (p_n \times p_n)^{-1}(\mathcal{R}_n) \) satisfies \( \mathcal{R}_{X_n}^{(G_n, H_{n-1})} \subseteq \tilde{R}_n \subseteq \mathcal{R}_{X_n}^{H_n} \) and \( \mathcal{C}(\tilde{R}_n) \leq \mathcal{C}(\mathcal{R}_n) < r + 1/n \). It follows that \( \mathcal{PC}(G) \leq \mathcal{PC}(\mathcal{R}_{X_n}^{G_n}) \leq \liminf_n \mathcal{C}(\mathcal{R}_n) \leq r \), a contradiction. \( \square \)
The next proposition is a minor modification of arguments due to Furman, Gaboriau, and Kechris (see [16, VI.24.(3)] and [32, Lemma 24.7, Proposition 35.4]).

**Proposition A.2.** Let \( S \) be a wq-normal (see [2.4]) subequivalence relation of the p.m.p. equivalence relation \( R \) on \((X, \mu)\). Then \( \mathcal{P}^*(R) \leq \mathcal{P}(S) \).

**Proof.** It suffices to deal with the case where \( S \) is q-normal in \( R \). Then we may find a countable subset \( (\phi_n)_{n \geq 1} \) of \( Q_R(S) \) which generates \( R \). Let \( A_n \) and \( B_n \) denote the domain and range respectively of \( \phi_n \). Given \( \epsilon > 0 \), for each \( n \geq 1 \) the equivalence relation \( S_{B_n}^n \cap S_{A_n} \) on \( A_n \) is aperiodic, so it has a measurable complete section \( C_n \subseteq A_n \) with \( \mu(C_n) < \epsilon / 2^n \). Then for each \( n \geq 1 \) and \( x \in A_n \) there exists a path from \( x \) to \( \phi_n(x) \) in \( S \cup \{(y, \phi_n(y)) : y \in C_n\} \), namely there is some \( y \in C_n \) with \( (x, y) \in S_{B_n}^n \cap S_{A_n} \), so that \( (x, y), (\phi_n(x), \phi_n(y)) \in S \), and hence the path from \( x \) to \( y \) to \( \phi_n(y) \) to \( \phi_n(x) \) works. It follows that \( R \) is generated by \( S \) along with \( (\phi_n|C_n)_{n \in \mathbb{N}} \). Therefore, any increasing exhaustion \( (S_i)_{i \in \mathbb{N}} \) of \( S \) gives rise to a corresponding exhaustion \( (R_i)_{i \in \mathbb{N}} \) of \( R \) with \( C(R_i) \leq C(S_i) + \sum_{n \geq 1} \mu(C_n) < C(S_i) + \epsilon \), and hence \( \mathcal{P}^*(R) \leq \mathcal{P}(S) + \epsilon \). Letting \( \epsilon \to 0 \) completes the proof. \( \square \)

**Proposition A.3.** Let \( H \) be a wq-normal subgroup of \( G \). Then for any free p.m.p. action \( G \curvearrowright (X, \mu) \), we have \( \mathcal{P}(S_R^X) \leq \mathcal{P}(S_H^X) \). In addition, we have \( \mathcal{P}^*(G) \leq \mathcal{P}^*(H) \) and \( \mathcal{P}(G) \leq \mathcal{P}(H) \).

**Proof.** The first two inequalities follow from Proposition [A.2] and the inequality \( \mathcal{P}(G) \leq \mathcal{P}(H) \) is then obtained by coinduction. \( \square \)

**Appendix B. A strongly almost free amenable action of the free group**

In [55], van Douwen constructs an amenable faithful action of a free group which is almost free, i.e., every nonidentity element fixes at most finitely many points. The property that we need for Example [0.9] is somewhat stronger however; let us say that an action of a group \( G \) on a set \( X \) is strongly almost free if the associated action of \( G \) on the collection \( \mathcal{P}(X) \), of all finite subsets of \( X \), is almost free. Equivalently, this means that each nonidentity element of \( G \), when viewed as a permutation on \( X \), contains only finitely many finite cycles in its cycle decomposition.

**Theorem B.1.** Let \( G \) be a free group of rank \( r \in \{2, 3, \ldots, \infty\} \). There exists a transitive amenable action of \( G \) on a countable set \( X \) which is strongly almost free.

For the proof, we will assume that \( G \) is finitely generated; the proof of the infinitely generated case is similar. Before presenting the proof we establish some notation. Fix a free generating set \( S \) for \( G \). The construction of the action will use the formalism of S-digraphs; the reader is referred to [30] for background.

Given a connected folded S-digraph \( \Gamma \), let \( \Gamma^* \) denote the unique connected folded S-digraph extending \( \Gamma \) which is \( 2|S| \)-regular and satisfies \( \text{Core}(\Gamma^*, v) = \text{Core}(\Gamma, v) \) for all \( v \in V(\Gamma) \). We let \( G \) act on \( V(\Gamma^*) \) in the natural way, i.e., \( g \cdot v \) is the terminus of the unique path in \( \Gamma^* \) with origin \( v \) and label \( g \). If we fix a vertex \( v_0 \in V(\Gamma) \), then \( \Gamma^* \) is isomorphic to the Schreier graph of the action \( G \curvearrowright G / G_{v_0} \) with respect to the generating set \( S \). By a cycle in \( \Gamma \) we mean a path in \( \Gamma \) whose origin and terminus coincide. A cycle \( c \) in \( \Gamma \) is cyclically reduced if its label is a cyclically reduced word in \( S \cup S^{-1} \). If a cycle \( c \) in \( \Gamma^* \) is cyclically reduced then...
it is contained in $\text{Core}(\Gamma, v)$ for any $v \in V(\Gamma^*)$, so taking $v \in V(\Gamma)$ shows that $c$ is contained in $\Gamma$. We make two observations:

(a) If $p$ is a reduced path in $\Gamma^*$ with origin and terminus in $\Gamma$, then $p$ is contained in $\Gamma$.

Proof. Otherwise, let $p_0$ be a reduced path of minimal length starting at $u \in V(\Gamma)$ and ending at $v \in V(\Gamma)$, and which is not contained in $\Gamma$. Since $\Gamma$ is connected there is a reduced path $p_1$ from $v$ to $u$ in $\Gamma$. Then the concatenation of $p_0$ followed by $p_1$ is a cycle at $u \in V(\Gamma)$ which is cyclically reduced by minimality of $p_0$, and hence is contained in $\Gamma$, a contradiction. \hfill $\square$

(b) If $w$ is a cyclically reduced word in $S \cup S^{-1}$, and if the orbit of $x \in V(\Gamma^*)$ under $w$ is finite, then $x \in V(\Gamma)$.

Proof. If $w^k \cdot x = x$ for some $x \in V(\Gamma^*)$, $k \geq 1$, then the cycle rooted at $x$ with label $w^k$ is cyclically reduced, hence is contained in $\Gamma$. \hfill $\square$

Proof of Theorem 3.1. For $n \in \mathbb{N}$, let $C_n$ denote the collection of all nonempty cyclically reduced words on $S \cup S^{-1}$ of length at most $n$. We will construct a sequence $\Gamma_n, n \in \mathbb{N}$, of finite, folded $S$-digraphs such that for all $n$:

1. $\Gamma_n$ contains a vertex of degree strictly less than $2|S|$;
2. $V(\Gamma_n)$ contains a set which is $(S, 1/n)$-invariant for the action $G \curvearrowright V(\Gamma^*_n)$;
3. If $n \geq 1$ then $\Gamma_{n-1} \subseteq \Gamma_n$, and for each $w \in C_{n-1}$, every finite orbit of $w$ in $V(\Gamma^*_n)$ is contained in $V(\Gamma_{n-1})$.

Assume first that such a sequence has been constructed and we will complete the proof. Take $\Gamma_\infty = \bigcup_n \Gamma_n$. Then $V(\Gamma^*_\infty)$ is infinite by (1), and the action $G \curvearrowright V(\Gamma^*_\infty)$ is amenable by property (2). Property (3) ensures that if $w \in C_m$ for some $m \in \mathbb{N}$, then every finite cycle in the cycle decomposition of $w$ in $V(\Gamma^*_\infty)$ is contained in the finite set $V(\Gamma_m)$, so $w$ fixes only finitely many finite subsets of $V(\Gamma^*_\infty)$. The theorem then follows, since every $g \in G$ is conjugate to a cyclically reduced word.

To define the sequence $(\Gamma_n)_{n \in \mathbb{N}}$ we start by taking $\Gamma_0$ to consist of a single vertex with no edges. Assume inductively that $\Gamma_{n-1}$ has been defined satisfying (1), (2), and (3). Let $N$ be a finite index normal subgroup of $G$ with $C_n \cap N = \emptyset$. Then the group $G/N$ is abelian-by-finite and it is torsionfree by Theorem 2 of [22]. Let $\Delta_n$ be the Schreier graph for the action $G \curvearrowright G/N'$ with respect to the generating set $S$, with root vertex $1N' \in G/N'$. Since $G/N'$ is amenable, there exists a natural number $k > 0$ such that the $(k-1)$-ball in $\Delta_n$ contains a set which is $(S, 1/n)$-invariant. Let $B_k$ denote the induced subgraph on the $k$-ball in $\Delta_n$. Fix $u \in V(\Gamma_{n-1})$ having degree strictly less than $2|S|$; by symmetry we may assume that there exists $s \in S$ such that $u$ has no outgoing edge in $\Gamma_{n-1}$ with label $s$. Since $G/N'$ is torsionfree and $s \notin N'$, there exists a vertex $v \in V(B_k)$ which has no incoming edge in $B_k$ with label $s$. Let $\Gamma_n$ be the graph obtained from the disjoint union of $\Gamma_{n-1}$ and $B_k$ by attaching a directed path $p$ from $u$ to $v$ of length $2n$, with each edge in $p$ having label $s$ (so $|V(\Gamma_n)| = |V(\Gamma_{n-1})| + |V(B_k)| + 2n - 1$). Properties (1) and (2) are immediate. Fix now $w \in C_{n-1}$ and we will verify (3). Each orbit of $w$ in $V(\Gamma^*_n)$ which meets $V(\Gamma^*_n) \setminus V(\Gamma_n)$ is infinite by (b). Let $O$ be an orbit of $w$ which is contained in $V(\Gamma_n)$. By (a), for each $x \in O$, the path $p_x$ in $\Gamma^*_n$ having origin $x$ and label $w$ is contained in $\Gamma_n$. Therefore, since $w$ is cyclically reduced and has length less than $n$, if $O$ contains a vertex $x \in V(p) \setminus \{u, v\}$
then either $p_x$ or $p_{w^{-1}x}$ is contained in $p$, and hence $w = s^i$ for some $1 \leq |i| < n$. But then $w^k \cdot x = s^{ik} \cdot x \neq x$ for all $k \geq 1$, contradicting that $O$ is finite. It follows that $O$ is contained either in $V(\Gamma_{n-1})$ or in $V(B_k)$. But $O$ cannot be contained in $V(B_k)$ since $w \notin N'$ and $G/N'$ is torsionfree. So $O$ is contained in $V(\Gamma_{n-1})$. This completes the proof of (3).

\[ \square \]

APPENDIX C. Untwisting cocycles in mean

In this appendix we prove analogues of three results from [15] which are used in the proof of Theorem [11]. We use the notation from [5.A] and [5.B].

Definition C.1. Let $q : (X, \mu) \to (Y, \nu)$ be an extension of p.m.p. actions $G \acts (X, \mu)$ and $G \acts (Y, \nu)$ of $G$, and let $\mathcal{M}_0$ be a subsemigroup of $\mathcal{M} = \mathcal{M}(G)$. We say that $\sigma_{\mathcal{M}_0}$ is ergodic relative to $(Y, \nu)$ if every $\kappa_{\mathcal{M}_0}$-invariant function in $L^2(X, \mu)$ is contained in $L^2(Y, \nu)$. Let $X_2^X = \{(x_0, x_1) \in X^2 : q(x_0) = q(x_1)\}$, and let $\mu_\nu^2 = \int_Y \mu_y \otimes \mu_y \, d\nu(y)$, where $\{\mu_y\}_{y \in Y}$ denotes the disintegration of $\mu$ via the map $q$. The product action $G \acts (X_2^X, \mu_\nu^2)$ is then an extension of $G \acts (Y, \nu)$ via the map $X_2^X \to Y$, $(x_0, x_1) \mapsto q(x_0) = q(x_1)$. We say that $\sigma_{\mathcal{M}_0}$ is weakly mixing relative to $(Y, \nu)$ if $\sigma_{\mathcal{M}_0}^2$ is ergodic relative to $(Y, \nu)$.

Let $\mathcal{G}_{\text{inv}} \supseteq \mathcal{G}_{\text{fin}}$ denote the class of Polish groups admitting a compatible bi-invariant metric. For the rest of this section we work under the following hypotheses:

- $q : (X, \mu) \to (Y, \nu)$ is an extension of p.m.p. actions $G \acts (X, \mu)$ and $G \acts (Y, \nu)$;
- $\mathcal{M}_0$ is a subsemigroup of $\mathcal{M}$, and $\sigma_{\mathcal{M}_0}$ is relatively weakly mixing over $(Y, \nu)$;
- $L$ is a group in the class $\mathcal{G}_{\text{inv}}$ with compatible bi-invariant bounded metric $d_L$.

The following lemma is the analogue of Lemma 3.2 of [15] translated to our context.

Lemma C.2. Let $\Phi : X \to L$ be a measurable map. Suppose that there exist measurable maps $\rho_0, \rho_1 : G \times Y \to L$ such that

\[
\sup_{m \in \mathcal{M}_0} \int_X \int_Y d_L(\rho_0(g^{-1}, q(x))\Phi(x), \Phi(g^{-1}x)\rho_1(g^{-1}, q(x))) \, d\mu \, dm = 0.
\]

Then there exists a measurable map $\phi : Y \to L$ such that $\Phi(x) = \phi(q(x))$ almost everywhere.

Proof. Let $\kappa$ denote the Koopman representation of $G$ on $L^2(X_2^X, \mu_\nu^2)$ associated to the action $G \acts (X_2^X, \mu_\nu^2)$. Let $f : X_2^X \to [0, 1]$ be the map $f(x_0, x_1) = d_L(\Phi(x_0), \Phi(x_1))$. The hypothesis (3.1), along with bi-invariance of $d_L$ and the triangle inequality imply that for $m \in \mathcal{M}_0$ we have $\|f\|_2^2 \leq \langle \kappa_m(f), f \rangle$, and so $\kappa_m(f) = f$. Therefore, $f$ is $Y$-measurable since $\sigma_{\mathcal{M}_0}$ is weakly mixing relative to $(Y, \nu)$. Lemma 3.1 of [15] now implies that $\Phi$ is $Y$-measurable.

The following is the analogue of Theorem 3.4 of [15].

Lemma C.3. Let $\alpha, \beta : G \times X \to L$ be measurable cocycles. Suppose that $F : X_2^X \to L$ is a measurable map satisfying

\[
\sup_{m \in \mathcal{M}_0} \int_X \int_{X_2^X} d_L(\alpha(g^{-1}, x_0)F(x_0, x_1), F(g^{-1}x_0, g^{-1}x_1)\beta(g^{-1}, x_1)) \, d\mu^2 \, dm = 0.
\]

Then there exist measurable maps $\phi, \psi : X \to L$ such that $F(x_0, x_1) = \phi(x_0)\psi(x_1)^{-1}$ for $\mu^2$-almost every $(x_0, x_1) \in X_2^X$.
Assume in addition that $\mathcal{M}_0 \subseteq C_{\mathcal{M}(G)}(H)$ for some subgroup $H \leq G$, and let $\phi, \psi : X \to L$ be any measurable maps with $F(x_0, x_1) = \phi(x_0)\psi(x_1)^{-1}$ for almost every $(x_0, x_1) \in X^2$. Then for any separable subalgebra $\mathcal{A} \subseteq C^\infty(G)$, there exists a semigroup $\mathcal{M}_1 \subseteq C_{\mathcal{M}(G)}(H)$ with $\mathcal{M}_1|\mathcal{A} = \mathcal{M}_0|\mathcal{A}$, along with a conull set $X_0 \subseteq X$, and measurable maps $\rho_\alpha, \rho_\beta : G \times Y \to L$, such that

$$\sup_{m \in \mathcal{M}_1} \sup_{x \in X_0} \int_G d_L(\phi(g^{-1}x)^{-1}\alpha(g^{-1}, x)\phi(x), \rho_\alpha(g^{-1}, q(x))) \, dm = 0,$$

and likewise for $\beta$ and $\rho_\beta$ in place of $\alpha$ and $\rho_\alpha$. If $\mathcal{M}_0$ is weak$^*$-closed then we may take $\mathcal{M}_1$ to be weak$^*$-closed as well.

**Proof.** The proof of the first part is exactly as in the proof of Theorem 3.4 of [15] except that we use Lemma C.2 in place of Lemma 3.2 of [15]. For the second part, let $\alpha'(g, x) = \phi(gx)^{-1}\alpha(g, x)\phi(x)$ and let $\beta'(g, x) = \psi(gx)^{-1}\beta(g, x)\psi(x)$, so that for all $m \in \mathcal{M}_0$ we have $\int_G \int_{X^2} d_L(\alpha'(g^{-1}, x_0), \beta'(g^{-1}, x_1)) \, d\mu d\nu = 0$. Since $\mathcal{A}$ is separable, the image $\mathcal{M}_0|\mathcal{A}$ of $\mathcal{M}_0$ in the dual space $\mathcal{A}^*$ is separable and metrizable in the weak$^*$-topology. Fix a countable subset $\{m_i\}_{i \in I} \subseteq \mathcal{M}_0$ whose image in $\mathcal{A}^*$ is weak$^*$-dense in $\mathcal{M}_0|\mathcal{A}$. For each $i \in I$ let $(p_{i,n})_{n \in \mathbb{N}}$ be a sequence in $P$ given by Lemma 5.1 applied to $m_i$. We may assume that the function $g \mapsto \int_{X^2} d_L(\alpha'(g^{-1}, x_0), \beta'(g^{-1}, x_1)) \, d\mu$ is in $\mathcal{A}$, so that for all $i \in I$ we have

$$\lim_{n \to \infty} \int_G \int_{X^2} d_L(\alpha'(g^{-1}, x_0), \beta'(g^{-1}, x_1)) \, d\mu d\nu, d\pi_{i,n} = 0.$$

For each $i \in I$, the sequence of functions $(x_0, x_1) \mapsto \int_G d_L(\alpha'(g^{-1}, x_0), \beta'(g^{-1}, x_1)) \, d\pi_{i,n}$ therefore converges to 0 in measure, so we may find a subsequence $(p'_{i,n})_{n \in \mathbb{N}}$ of $(p_{i,n})_{n \in \mathbb{N}}$ on which the convergence is pointwise almost everywhere. Since $I$ is countable we may find a $G$-invariant conull set $Z \subseteq X^2$ such that for every $(x_0, x_1) \in Z$ and every $i \in I$ we have

$$\lim_{n \to \infty} \int_G \int_{X^2} d_L(\alpha'(g^{-1}, x_0), \beta'(g^{-1}, x_1)) \, d\pi_{i,n} = 0.$$

Let $y \mapsto s(y) \in q^{-1}(y)$ be a measurable function with $(x, s(y)) \in Z$ and $(s(y), x) \in Z$ for $\nu$-almost every $y \in Y$ and $\mu_y$-almost every $x \in q^{-1}(y)$ and define $\rho_\alpha(g, y) := \beta'(g, s(y))$ and $\rho_\beta(g, y) := \alpha'(g, s(y))$.

For each $i \in I$ let $m'_i \in \mathcal{M}$ be an accumulation point of the sequence $(p_{i,n})_{n \in \mathbb{N}}$. Then $m'_i \in C_{\mathcal{M}(G)}(H)$ and $m'_i|\mathcal{A} = m_i|\mathcal{A}$ by our choice of the sequence $(p_{i,n})_{n \in \mathbb{N}}$. Let $\mathcal{M}_0'$ be a subset of the weak$^*$-closure of $\{m'_i : i \in I\}$ with $\mathcal{M}_0'|\mathcal{A} = \mathcal{M}_0|\mathcal{A}$, and let $\mathcal{M}_1$ be the semigroup generated by $\mathcal{M}_0'$. Then $\mathcal{M}_1 \subseteq C_{\mathcal{M}(G)}(H)$, and $\mathcal{M}_1|\mathcal{A} = \mathcal{M}_0|\mathcal{A}$, since $\mathcal{M}_0$ is a semigroup. Let $\mathcal{M}_Z$ denote the collection of means $m$ on $G$ satisfying

- $\int_G d_L(\alpha'(g^{-1}, x_0), \beta'(g^{-1}, x_1)) \, dm = 0$ for all $(x_0, x_1) \in Z$,
- $\int_G \int_X d_L(\alpha'(g^{-1}, k^{-1}x), \beta'(g^{-1}, k^{-1}s(q(x)))) \, d\mu \, dm = 0$ for all $k \in G$,
- $\int_G \int_X d_L(\alpha'(g^{-1}, k^{-1}s(q(x))), \beta'(g^{-1}, k^{-1}x)) \, d\mu \, dm = 0$ for all $k \in G$.

Observe that $\{m'_i\}_{i \in I} \subseteq \mathcal{M}_Z$ by (3.4), since $(k^{-1}x, k^{-1}s(q(x))) \in Z$ for $\mu$-almost every $x \in X$ and all $k \in G$. In addition, $\mathcal{M}_Z$ is weak$^*$-closed, and $\mathcal{M}_Z$ is a
semigroup since $Z$ is $G$-invariant. This shows \ref{3.3}, and for \ref{3.2}, take $X_0 \subseteq X$ to be a $G$-invariant conull set such that $(x, s(q(x))) \in Z$ for all $x \in X_0$. If $\mathcal{M}_0$ is weak*-closed then we can replace $\mathcal{M}_1$ by its closure. \hfill \Box

The combination of the following two lemmas are the analogues of Lemma 3.6 of \cite{15}. Let $M$ be a closed subgroup of $L$. Let $\alpha : G \times X \to M$ be a measurable cocycle, and let $\Phi : X \to L$ be a measurable map. Suppose that there exists a measurable map $\rho : G \times Y \to L$ such that

$$\sup_{m \in \mathcal{M}_0} \int_G \int_X d_L(\rho(g^{-1}, q(x)), \Phi(g^{-1}x)\alpha(g^{-1}, x)\Phi(x)^{-1}) \, d\mu \, dm = 0.$$ 

Then there exists a measurable map $\phi_0 : Y \to L/M$ such that $\Phi(x)M = \phi_0(q(x))$ almost everywhere.

\textbf{Proof.} This is proved as in Lemma 3.6 of \cite{15}. \hfill \Box

\textbf{Lemma C.5.} Assume that $\mathcal{M}_0 \subseteq C_{\mathcal{H}(G)}(H)$ for some subgroup $H \leq G$. Let $M$ be a closed subgroup of $L$, let $\hat{\alpha} : G \times X \to M$ be a measurable cocycle, and let $\hat{\rho} : G \times Y \to L$ be a measurable map. Suppose that

$$\sup_{m \in \mathcal{M}_0} \int_G \int_X d_L(\hat{\rho}(g^{-1}, q(x)), \hat{\alpha}(g^{-1}, x)) \, d\mu \, dm = 0.$$

Then for any separable subalgebra $\mathcal{A} \subseteq \ell^\infty(G)$, there exists a semigroup $\mathcal{M}_1 \subseteq C_{\mathcal{H}(G)}(H)$ with $\mathcal{M}_1|\mathcal{A} = \mathcal{M}_0|\mathcal{A}$, along with a conull set $X_0 \subseteq X$ and a measurable map $\rho' : G \times Y \to M$ such that

\begin{equation}
\tag{3.5}
\sup_{m \in \mathcal{M}_1} \sup_{x \in X_0} \int_G d_L(\rho'(g^{-1}, q(x)), \hat{\alpha}(g^{-1}, x)) \, dm = 0
\end{equation}

\begin{equation}
\tag{3.6}
\text{and } \sup_{m \in \mathcal{M}_1} \int_G d_L(\rho'(g^{-1}, q(x)), \hat{\alpha}(g^{-1}, x)) \, d\mu \, dm = 0.
\end{equation}

If $\mathcal{M}_0$ is weak*-closed then we may take $\mathcal{M}_1$ to be weak*-closed as well.

\textbf{Proof.} Let $\{m_i\}_{i \in I} \subseteq \mathcal{M}_0$ and $(p_{i,n})_{n \in \mathbb{N}}$ be as in the proof of Lemma \ref{C.3} We may assume that the function $g \mapsto \int_X d_L(\hat{\rho}(g^{-1}, q(x)), \hat{\alpha}(g^{-1}, x)) \, d\mu$ is in $\mathcal{A}$, so that for all $i \in I$ we have

$$\lim_{n \to \infty} \int_G \int_X d_L(\hat{\rho}(g^{-1}, q(x)), \hat{\alpha}(g^{-1}, x)) \, d\mu \, dp_{i,n} = 0.$$ 

We may therefore find, for each $i \in I$, a subsequence $(p'_{i,n})_{n \in \mathbb{N}}$ of $(p_{i,n})$, along with a $G$-invariant conull set $X_0 \subseteq X$ such that for all $x \in X_0$ and $i \in I$ we have

$$\lim_{n \to \infty} \int_G d_L(\hat{\rho}(g^{-1}, q(x)), \hat{\alpha}(g^{-1}, x)) \, dp'_{i,n} = 0.$$ 

Let $y \mapsto s(y) \in q^{-1}(y)$ be a measurable function such that $s(q(x)) \in X_0$ for all $x \in X_0$. For each $i \in I$ let $m'_i \in \mathcal{M}$ be an accumulation point of the sequence $(p'_{i,n})_{n \in \mathbb{N}}$. We obtain $\mathcal{M}_1$ from $\{m'_i\}_{i \in I}$ as in the proof of Lemma \ref{C.3} Let $\mathcal{M}_{X_0}$ denote the collection of all means $m$ on $G$ such that

- $\int_G d_L(\hat{\alpha}(g^{-1}, ks(q(x))), \hat{\alpha}(g^{-1}, kx)) \, dm = 0$ for all $x \in X_0$ and $k \in G;
Then $M_{X_0}$ is a weak$^*$-closed subsemigroup of $\mathcal{M}$ containing $\{m_i^0\}_{i \in I}$. Therefore, $\mathcal{M}_1 \subseteq \mathcal{M}_{X_0}$, so (3.5) and (3.6) follow by taking $\rho'(g, x) = \hat{\alpha}(g, s(q(x)))$. If $\mathcal{M}_0$ is weak$^*$-closed then we can replace $\mathcal{M}_1$ by its closure.

The following consequence of Lemmas C.3 and C.4 is used in the proof of Theorem 11:

**Lemma C.6.** Assume that $(Y, \nu)$ is reduced to a point. Let $M$ be a closed subgroup of $L$, and let $\alpha : G \times X \to M$ and $\beta : G \times X \to L$ be measurable cocycles. Suppose that $F : X^2 \to L$ is a measurable map satisfying

$$
\sup_{m \in \mathcal{M}_0} \int_G \int_{X^2} d_L(\alpha^{-1}(x_0)F(x_0, x_1), F(g^{-1}x_0, g^{-1}x_1)\beta(g^{-1}, x_1)) d\mu^2 d\mathcal{M} = 0.
$$

Then there exist measurable maps $\phi : X \to M$ and $\psi : X \to L$ such that $F(x_0, x_1) = \phi(x_0)^{-1}\psi(x_1)$ for $\mu^2$-almost every $(x_0, x_1) \in X^2$.

**Proof.** Apply the first part of Lemma C.3 to obtain measurable maps $\Phi, \Psi : X \to L$ such that $F(x_0, x_1) = \Phi(x_0)^{-1}\Psi(x_1)$. Let $\kappa$ denote the Koopman representation of $G$ on $L^2(X, \mu)$, and for $\xi, \eta \in L^2(X, \mu)$ let $\varphi_{\xi, \eta} \in L^\infty(G)$ be the matrix coefficient $\varphi_{\xi, \eta}(g) = \langle \kappa_g \xi, \eta \rangle$. The collection $\mathcal{A}_\kappa = \{||\varphi_{\xi, \eta}||^2 : \xi, \eta \in L^2(X, \mu)\}$ is then a separable subset of $L^\infty(G)$. Applying the second part of Lemma C.3 (with $H = 1$), we obtain a semigroup $\mathcal{M}_1 \subseteq \mathcal{M}$ with $\mathcal{M}_1|\mathcal{A}_\kappa = \mathcal{M}_0|\mathcal{A}_\kappa$, along with a map $\rho : G \to L$ such that

$$
(3.7) \quad \sup_{m \in \mathcal{M}_0} \int_G \int_X d_L(\Phi^{-1}(g^{-1}x)\alpha(g^{-1}, x)\Phi(x)^{-1}, \rho(g^{-1})) d\mu d\mathcal{M} = 0.
$$

Since $\kappa|\mathcal{M}_0$ is weakly mixing on $L^2(X, \mu) \supseteq C_1X$ and $\mathcal{M}_1|\mathcal{A}_\kappa = \mathcal{M}_0|\mathcal{A}_\kappa$, it follows from Proposition 5.4 iii. that $\kappa|\mathcal{M}_1$ is weakly mixing on $L^2(X, \mu) \supseteq C_1X$. Therefore, (3.7) shows that the hypotheses of Lemma C.4 are satisfied, so there exists some $\ell_0 \in L$ such that $\Phi(x)M = \ell_0 M$ almost everywhere. Let $\phi(x) = \ell_0^{-1} \Phi(x) \in M$ and let $\psi(x) = \ell_0^{-1} \Psi(x)$. Then $\phi(x_0)^{-1}\psi(x_1) = \Phi(x_0)^{-1}\Psi(x_1) = F(x_0, x_1)$ for almost every $(x_0, x_1) \in X^2$. \qed

**References**

[1] Miklós Abért and Gábor Elek. Hyperfinite actions on countable sets and probability measure spaces. In *Dynamical systems and group actions*, volume 567 of *Contemp. Math.*, page 116. Amer. Math. Soc., Providence, RI, 2012.

[2] Miklós Abért and Nikolay Nikolov. Rank gradient, cost of groups and the rank versus Heegaard genus problem. *J. Eur. Math. Soc. (JEMS)*, 14(no. 5):16571677, 2012.

[3] Uri Bader, Alex Furman, and Roman Sauer. Weak notions of normality and vanishing up to rank in $L^2$-cohomology. *International Mathematics Research Notices*, 12:3177–3189, 2014.

[4] Gilbert Baumslag, Alexei Myasnikov, and Vladimir Remeslennikov. Malnormality is decidable in free groups. *International Journal of Algebra and Computation*, 9(06):687–692, 1999.

[5] Erik Bédos and Pierre de la Harpe. Moyennabilité intérieure des groupes: définitions et exemples. *Enseign. Math. (2)*, 32(1-2):139–157, 1986.

[6] Lewis Bowen and Robin D Tucker-Drob. On a co-induction question of Kechris. *Israel Journal of Mathematics*, 194(1):209–224, 2013.

[7] Matthew G Brin and Craig C Squier. Groups of piecewise linear homeomorphisms of the real line. *Inventiones mathematicae*, 79(3):485–498, 1985.
[8] Tullio G Ceccherini-Silberstein, Rostislav I Grigorchuk, and Pierre De La Harpe. Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces. Proceedings of the Steklov Institute of Mathematics, 224:68–111, 1999.

[9] Jeff Cheeger and Mikhael Gromov. $L^2$-cohomology and group cohomology. Topology, 25(2):189–215, 1986.

[10] Ionut Chifan, Thomas Sinclair, and Bogdan Udrea. Inner amenability for groups and central sequences in factors. arXiv preprint arXiv:1307.5002, 2013.

[11] Yves Cornulier and Alain Valette. On equivariant embeddings of generalized baumslag–solitar groups. Geometriae Dedicata, pages 1–17, 2012.

[12] SG Dani. A note on invariant finitely additive measures. Proceedings of the American Mathematical Society, 93(1):67–72, 1985.

[13] Edward G Effros. Property Gamma and inner amenability. Proceedings of the American Mathematical Society, 47(2):483–486, 1975.

[14] Pierre Fima. Amenable, transitive and faithful actions of groups acting on trees. arXiv preprint arXiv:1202.6467, 2012.

[15] Alex Furman. On Popa’s Cocycle Superrigidity Theorem. Int. Math. Res. Not. IMRN, (no. 19):rnm073, 2007.

[16] Damien Gaboriau. Coût des relations déquivalence et des groupes. Inventiones mathematicae, 139(1):41–98, 2000.

[17] Damien Gaboriau. Invariants $\ell^2$ de relations déquivalence et de groupes. Publications Mathématiques de L’IHÉS, 95(1):93–150, 2002.

[18] Yair Glasner and Nicolas Monod. Amenable actions, free products and a fixed point property. Bulletin of the London Mathematical Society, 39(1):138–150, 2007.

[19] Rostislav Grigorchuk and Volodymyr Nekrashevych. Amenable actions of nonamenable groups. Journal of Mathematical Sciences, 140(3):391–397, 2007.

[20] Mikhail Gromov. Entropy and isoperimetry for linear and non-linear group actions. Groups, Geometry, and Dynamics, 1:499–593, 2008.

[21] Philip Hall and CR Kulatilaka. A property of locally finite groups. Journal of the London Mathematical Society, 1(1):235–239, 1964.

[22] Graham Higman. Finite groups having isomorphic images in every finite group of which they are homomorphic images. The Quarterly Journal of Mathematics, 6(1):250–254, 1955.

[23] Paul Jolissaint. On property (t) for pairs of topological groups. Enseignement mathematique, 51(1-2):31, 2005.

[24] Paul Jolissaint. Relative inner amenability, relative property gamma and non-Kazhdan groups. arXiv preprint arXiv:1312.3497, 2013.

[25] V. Jones and K. Schmidt. Asymptotically invariant sequences and approximate finiteness. Amer. J. Math., 109(1):91–114, 1987.

[26] Kate Juschenko and Mikael de la Salle. Invariant means of the wobbling group. arXiv preprint arXiv:1301.4736, 2013.

[27] Kate Juschenko and Nicolas Monod. Cantor systems, piecewise translations and simple amenable groups. Ann. of Math., 178(2):775–787, 2013.

[28] Kate Juschenko and Tatiana Nagnibeda. Small spectral radius and percolation constants on nonamenable Cayley graphs. arXiv preprint arXiv:1206.2183, 2012.

[29] Kate Juschenko, Volodymyr Nekrashevych, and Mikael de la Salle. Extensions of amenable groups by recurrent groupoids. arXiv preprint arXiv:1305.2637, 2013.

[30] Ilya Kapovich and Alexei Myasnikov. Stallings foldings and subgroups of free groups. Journal of Algebra, 248(2):608–668, 2002.

[31] A.S. Kechris. Global aspects of ergodic group actions, volume Mathematical Surveys and Monographs, 160. Amer. Math. Soc., Providence, RI, 2010.

[32] A.S. Kechris and B.D. Miller. Topics in orbit equivalence, volume Lecture Notes in Mathematics, 1852. Springer-Verlag, Berlin, 2004.
Yoshikata Kida. Inner amenable groups having no stable action. *Geometriae Dedicata*, pages 1–8, 2012.

Yoshikata Kida. Stable actions of central extensions and relative property (T). *arXiv preprint arXiv:1309.3739*, 2013.

Yoshikata Kida. Splitting in orbit equivalence, treeable groups, and the Haagerup property. *arXiv preprint arXiv:1403.0688*, 2014.

Wolfgang Lck. *L2-invariants: theory and applications to geometry and K-theory*. Springer-Verlag, Berlin, 2002.

Yash Lodha and J Tatch Moore. A geometric solution to the von Neumann-Day problem for finitely presented groups. *Preprint arxiv*, 2013.

Nicolas Monod. Groups of piecewise projective homeomorphisms. *Proceedings of the National Academy of Sciences*, 110(12):4524–4527, 2013.

Nicolas Monod and Sorin Popa. On co-amenability for groups and von neumann algebras. *C. R. Math. Acad. Sci. Soc. R. Can.*, 25(3):82–87, 2003.

Soyoung Moon. Amenable actions of amalgamated free products. *Groups, Geometry, and Dynamics*, 4(2):309–332, 2010.

Soyoung Moon. Amenable actions of amalgamated free products of free groups over a cyclic subgroup and generic property. *Ann. Math. Blaise Pascal*, 18(2):211–229, 2011.

Soyoung Moon. Permanence properties of amenable, transitive and faithful actions. *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 18(2):287–296, 2011.

FJ Murray and John von Neumann. On rings of operators iv. *Ann. Math.*, 44:716–808, 1943.

Vladimir Pestov. On some questions of eymard and bekka concerning amenability of homogeneous spaces and induced representations. *C. R. Math. Acad. Sci. Soc. R. Can.*, 25(3):76–81, 2003.

Jesse Peterson. Lecture notes on ergodic theory. Available at: http://www.math.vanderbilt.edu/ peters10/teaching/Spring2011/math390LectureNotes.html, 2011.

Jesse Peterson and Thomas Sinclair. On cocycle superrigidity for Gaussian actions. *Ergodic Theory and Dynamical Systems*, 32(1):249, 2011.

Jesse Peterson and Andreas Thom. Group cocycles and the ring of affiliated operators. *Inventiones mathematicae*, 185(3):561–592, 2011.

Sorin Popa. Some computations of 1-cohomology groups and construction of non-orbit-equivalent actions. *J. Inst. Math. Jussieu*, 5(2):309–322, 2006.

Sorin Popa. Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups. *Inventiones mathematicae*, 170(2):243–295, 2007.

Sorin Popa. On the superrigidity of malleable actions with spectral gap. *Journal of the American Mathematical Society*, 21(4):981–1000, 2008.

David Promislow. Nonexistence of invariant measures. *Proceedings of the American Mathematical Society*, 88(1):89–92, 1983.

Klaus Schmidt. Some solved and unsolved problems concerning orbit equivalence of countable group actions. In *Proceedings of the conference on ergodic theory and related topics, II (Georgenthal, 1986)*, pages 171–184, 1986.

Robin D. Tucker-Drob. Shift-minimal groups, fixed price 1, and the unique trace property. *arXiv preprint arXiv:1211.6395*, 2012.

Stefaan Vaes. An inner amenable group whose von Neumann algebra does not have property Gamma. *Acta mathematica*, 208(2):389–394, 2012.

Eric K van Douwen. Measures invariant under actions of $F_2$. *Topology and its Applications*, 34(1):53–68, 1990.

J. von Neumann. Zur allgemeinen theorie des messes. *Fundamenta Mathematicae*, 13(1):73–116, 1929.

Bertram AF Wehrfritz. *Infinite linear groups*. Springer, 1969.