THE ENTIRE CYCLIC COHOMOLOGY OF
NONCOMMUTATIVE 3-SPHERES

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ABSTRACT. In this paper, we compute the entire cyclic cohomology of noncommutative 3-spheres. First of all, we verify the Mayer-Vietoris exact sequence of entire cyclic cohomology in the framework of Fréchet $^*$-algebras. Applying it to their noncommutative Heegaard decomposition, we deduce that their entire cyclic cohomology is isomorphic to the d’Rham homology of the ordinary 3-sphere with the complex coefficients.

1. Introduction

Since Connes [5] constructed a generalization of periodic cyclic cohomology which is called entire cyclic cohomology, its explicit computation is executed only for few examples (cf. [3, 5, 12]). As a matter of fact, their entire cyclic cohomologies are nothing but their periodic ones. Recently, the first named author [16] computed that of smooth noncommutative 2-tori, which have the same property cited above.

In this paper, we firstly formulate the Mayer-Vietoris exact sequence for entire cyclic cohomology, then we apply it to compute for smooth noncommutative 3-spheres. The key idea is based on Meyer’s excision [14, 15] concerning the short exact sequences of Fréchet $^*$-algebras to obtain a noncommutative Mayer-Vietoris exact sequence for entire cyclic cohomology. To use his excision, we need to construct a bounded linear section for a short exact sequence of Fréchet $^*$-algebras. To ensure it, we reformulate the notion of metric approximation property in the framework of Fréchet $^*$-algebras to solve the lifting problem (see [4]). We then use Baum, Hajac, Matthes and Szymański’s method [1] for a Heegaard decomposition of smooth noncommutative 3-spheres since they pointed out an insufficient part of Matsumoto’s construction [13] in the case of $C^*$-algebras.

Under this circumstance, we conclude that the entire cyclic cohomology of noncommutative 3-spheres is the same as their periodic one.

Throughout this paper, $\theta$ is an irrational number in the open unit interval $(0, 1)$ and we use the notation $\mathbb{Z}_{\geq 0}$ for the set of all nonnegative integers.
2. Preliminaries

We prepare some notations and basic properties used throughout the paper. Let \( \mathfrak{A} \) be a Fréchet \( * \)-algebra or \( F^* \)-algebra and denote by \( C^\infty([0,1], \mathfrak{A}) \) the set of all \( \mathfrak{A} \)-valued smooth functions on the closed unit interval \([0,1]\) with respect to Fréchet topology. Given an element \( f \in C^\infty([0,1], \mathfrak{A}) \) and an integer \( n \geq 1 \), we write by \( f^{(n)}(t) \) its \( n \)-th derivative of \( f \) at \( t \) \( (0 < t < 1) \) and denote by \( f^{(n)}_+(0) \), \( f^{(n)}_-(1) \) the \( n \)-th derivatives at 0 or 1 as follows:

\[
\begin{align*}
  f^{(n)}_+(0) &= \lim_{t \to 0^+} f^{(n)}(t) \\
  f^{(n)}_-(1) &= \lim_{t \to 1^-} f^{(n)}(t).
\end{align*}
\]

For \( n = 0 \), we write \( f^{(0)}_+(0) = f(0) \), \( f^{(0)}_-(1) = f(1) \).

**Definition 2.1.** For a \( F^* \)-algebra \( \mathfrak{A} \), we define the suspension \( S^\infty \mathfrak{A} \) of \( \mathfrak{A} \) by

\[
S^\infty \mathfrak{A} = \{ f \in C^\infty([0,1], \mathfrak{A}) \mid f^{(n)}_+(0) = f^{(n)}_-(1) = 0 \ (n \geq 0) \}.
\]

and we also define the cone \( C^\infty \mathfrak{A} \) of \( \mathfrak{A} \) by

\[
C^\infty \mathfrak{A} = \{ f \in C^\infty([0,1], \mathfrak{A}) \mid f^{(n)}_-(1) = 0 \ (n \geq 0) \}.
\]

Then we have the following short exact sequence:

\[
0 \longrightarrow \mathfrak{I} \overset{i}{\longrightarrow} C^\infty \mathfrak{A} \overset{q}{\longrightarrow} \mathfrak{A} \longrightarrow 0,
\]

where \( q \) is defined by \( q(f) = f(0) \),

\[
\mathfrak{I} = \{ f \in C^\infty(\mathfrak{A}) \mid f(0) = 0 \}
\]

and \( i \) is the canonical inclusion. The map \( s : \mathfrak{A} \to C^\infty \mathfrak{A} \) defined by

\[
s(a)(t) = (1 - t)a \quad (a \in \mathfrak{A}, t \in [0,1])
\]

is a bounded linear section of \( q \) with respect to Fréchet topology. We need to know the entire cyclic cohomologies of \( C^\infty \mathfrak{A} \) and \( \mathfrak{I} \). We say that given two \( F^* \)-algebras \( \mathfrak{A} \) and \( \mathfrak{B} \), the map

\[
\Phi : \mathfrak{A} \to C^\infty([0,1], \mathfrak{B})
\]

is called a smooth homotopy if it is a bounded homomorphism with respect to Fréchet topology and two bounded homomorphisms \( f, g : \mathfrak{A} \to \mathfrak{B} \) are smoothly homotopic if there exists a smooth homotopy \( \Phi \) from \( \mathfrak{A} \) to \( \mathfrak{B} \) with \( \Phi_0 = f, \Phi_1 = g \).

A Fréchet algebra \( \mathfrak{A} \) is smoothly homotopic to another one \( \mathfrak{B} \) if there are two homomorphisms \( f : \mathfrak{A} \to \mathfrak{B} \) and \( g : \mathfrak{B} \to \mathfrak{A} \) such that \( g \circ f \) (resp. \( f \circ g \)) is smoothly homotopic to the identity on \( \mathfrak{A} \) (resp. \( \mathfrak{B} \)). According to Meyer [14], we know the homotopy invariance of entire cyclic cohomology in the framework of \( F^* \)-algebras:

**Proposition 2.2** ([14]). If two bounded homomorphisms are smoothly homotopic, then they induce the same map on the entire cyclic cohomology.
We also mention the following lemma:

**Lemma 2.3.** Let $S^\infty A, C^\infty A$ and $I$ be cited above, we then have that

$$HE^*(C^\infty A) = 0, \quad HE^*(I) \simeq HE^*(S^\infty A).$$

**Proof.** By Proposition 2.2, it suffices to show that $C^\infty A$ is smoothly homotopic to 0 to obtain the former isomorphism. The map

$$F : C^\infty A \to C^\infty ([0,1], C^\infty A)$$

defined by

$$F_s(f)(t) = f(s + (1 - s)t) \quad (f \in C^\infty A, \quad s, t \in [0,1])$$

gives a smooth homotopy on $C^\infty A$. Since $F_0$ is the identity on $C^\infty A$ and for any $f \in C^\infty A$,

$$F_1(f)(t) = f(1) = 0.$$

We know that $C^\infty A$ is smoothly homotopic to 0. For the latter one, we introduce the map $t \mapsto f(e^{1-1/t}) \quad (f \in C^\infty A, \quad t \in [0,1])$, which belongs to $S^\infty A$. Indeed, we note that for any $n \geq 1$, $\frac{d^n}{dt^n} f(e^{1-1/t})$ is a linear combination of some functions such as

$$f^{(k)}(e^{1-1/t}) \frac{e^{(1-1/t)}}{t^n} \quad (k, l, m \geq 1).$$

In fact, for $n = 1$, we have that

$$\frac{d}{dt} f(e^{1-1/t}) = f^{(1)}(e^{1-1/t}) \frac{e^{1-1/t}}{t^2}.$$

Suppose that the function $\frac{d^n}{dt^n} f(e^{1-1/t})$ is a linear combination of fuctions

$$f^{(k)}(e^{1-1/t}) \frac{e^{(1-1/t)}}{t^m} \quad (k, l, m \geq 1),$$

then we deduce that

$$\frac{d}{dt} \left( f^{(k)}(e^{1-1/t}) \frac{e^{(1-1/t)}}{t^m} \right) = f^{(k+1)}(e^{1-1/t}) \frac{e^{(1-1/t)}}{t^{m+2}} + l f^{(k)}(e^{1-1/t}) \frac{e^{(1-1/t)}}{t^{m+2}} - m f^{(k)}(e^{1-1/t}) \frac{e^{(1-1/t)}}{t^{m+1}},$$

so is $\frac{d^{n+1}}{dt^{n+1}} f(e^{1-1/t})$. Because of the following equalities:

$$\lim_{t \to 0+} f^{(k)}(e^{1-1/t}) \frac{e^{(1-1/t)}}{t^m} = f^{(n)}_+(0) \cdot 0 = 0$$

$$\lim_{t \to 1-0} f^{(k)}(e^{1-1/t}) \frac{e^{(1-1/t)}}{t^m} = f^{(n)}_-(1) = 0,$$

for any $f \in I$, $k, l, m \geq 1$, the function $f(e^{1-1/t})$ belongs to $S^\infty A$. Let

$$r : I \to S^\infty A.$$
be the map defined by
\[ r(f)(t) = f(e^{1-t}) \quad (f \in \mathcal{F}, \ t \in [0,1]) \]
and \( i \) the natural inclusion from \( S^\infty \mathcal{A} \) into \( \mathcal{F} \). For the proof that \( r \circ i \) is smoothly homotopic to the identity on \( S^\infty \mathcal{A} \), we use the bounded homomorphism
\[ G : S^\infty \mathcal{A} \rightarrow C^\infty([0,1], S^\infty \mathcal{A}) \]
defined by
\[ G_s(f)(t) = f(se^{1-t} + (1-s)t) \quad (f \in S^\infty \mathcal{A}, s, t \in [0,1]) \]
which gives a smooth homotopy connecting \( r \circ i \) and the identity on \( S^\infty \mathcal{A} \). We firstly show that \( G_s(f) \in S^\infty \mathcal{A} \) for any fixed \( f \in S^\infty \mathcal{A}, s \in [0,1] \). Since
\[ \frac{d}{dt} G_s(f)(t) = f'((se^{1-t} + (1-s)t)\left(\frac{s}{t^2}e^{1-t} + 1 - s\right)) \]
we know that
\[ \lim_{t \to 0^+} \frac{d}{dt} G_s(f)(t) = f'^{(1)}(0) \cdot (1-s) = 0 \]
\[ \lim_{t \to 1^-} \frac{d}{dt} G_s(f)(t) = f'^{(1)}(1) = 0. \]
For general \( n \geq 2 \), we also see that
\[ \lim_{t \to 0^+} \frac{d^n}{dt^n} G_s(f)(t) = \lim_{t \to 1^-} \frac{d^n}{dt^n} G_s(f)(t) = 0. \]
The case for \( n = 1 \) has already been shown. It suffices to show that for \( n \geq 2 \), the function \( \frac{d^n}{dt^n} G_s(f)(t) \) is a linear combination of functions like
\[ f^{(k)}(se^{1-t} + (1-s)t)e^{l(1-t)/m} \quad (k, l, m \geq 1). \]
We now calculate that
\[ \frac{d}{dt} f^{(k)}(se^{1-t} + (1-s)t)e^{l(1-t)/m} \]
\[ = f^{(k+1)}((se^{1-t} + (1-s)t)e^{l(1-t)/m} \left(\frac{s}{t^2} + 1 - s\right)) \]
\[ + f^{(k)}((se^{1-t} + (1-s)t)\left(\frac{le^{l(1-t)/m}}{m+1} - \frac{me^{l(1-t)/m}}{m+2}\right)), \]
which completes the induction process. Moreover we see that \( \frac{d^n}{dt^n} G_s \) is uniformly bounded on \([0,1]\) for each \( n \geq 1 \). We note that the function
\[ t \mapsto \frac{e^{l(1-t)/m}}{l^m} \]
is bounded on \([0,1]\) and that \( f^{(k)} \) is also bounded since \( f \in C^\infty([0,1], \mathcal{A}) \). Hence \( G \) is a smooth homotopy connecting \( r \circ i \) and the identity on \( S^\infty \mathcal{A} \) since \( G_1 = r \circ i \).
and \( G_0 \) is the identity on \( S^\infty \mathfrak{A} \). Similarly, \( i \circ r \) and the identity on \( \mathfrak{I} \) are smoothly homotopic via the smooth homotopy defined by the same way as \( G \), which implies that

\[
HE^*(\mathfrak{I}) \cong HE^*(S^\infty \mathfrak{A})
\]
as desired. \( \square \)

3. Toeplitz F*-Algebras

In this section, we construct smooth Toeplitz algebras based on 1-torus and to analyze them. They could be viewed as a quantization of 2-disc (cf. [1] [11]). Let \( \{ z^n \}_{n \in \mathbb{Z}} \) be the orthonormal basis of the Hilbert space \( L^2(T) \) of all square integrable functions on the 1-torus \( T \), where \( z^n(t) = t^n \) (\( t \in T, n \in \mathbb{Z} \)), and \( H^2 = H^2(T) \) the Hardy space on \( T \) which is a closed subspace of \( L^2(T) \) spanned by \( \{ z^n \}_{n \geq 0} \). For \( f \in C^\infty(T) \) of all infinitely differentiable functions on \( T \), in which we mean that the derivation is defined by

\[
\frac{d}{dt} f(t) = \lim_{r \to 0} \frac{f(e^{2\pi i} t) - f(t)}{r},
\]
we define the operator \( T_f \) for \( f \in C^\infty(T) \) by

\[
T_f \xi = Pf \xi \quad (\xi \in H^2),
\]
where \( P \) is the projection onto \( H^2 \). We consider the *-algebra \( \mathcal{P} \) generated by \( T_{z^j} (j \in \mathbb{Z}) \), namely,

\[
\mathcal{P} = \bigcup_{N \in \mathbb{Z}_{\geq 0}} \left\{ \sum_{i_j \in \mathbb{Z}, |i_j| \leq N} c_{i_1, \ldots, i_n} T_{z^{i_1}} \cdots T_{z^{i_n}} \bigg| c_{i_1, \ldots, i_n} \in \mathbb{C}, n \in \mathbb{Z}_{\geq 0} \right\}.
\]

Since \( T_f T_g - T_{fg} \) is a compact operator for any \( f, g \in C^\infty(T) \) and \( T_f \) is compact if and only if \( f = 0 \) (cf. [8]), it is easily seen by induction that for any \( T \in \mathcal{P} \), there is a unique \( f \in C^\infty(T) \) and a unique compact operator \( S \) with \( T = T_f + S \). Actually, if

\[
T = \sum_{i_j \in \mathbb{Z}, |i_j| \leq N} c_{i_1, \ldots, i_n} T_{z^{i_1}} \cdots T_{z^{i_n}} \in \mathcal{P},
\]
then \( T = T_f + S \), where

\[
f = \sum_{i_j \in \mathbb{Z}, |i_j| \leq N} c_{i_1, \ldots, i_n} z^{i_1} \cdots z^{i_n}
\]
and the compact operator \( S \) is a linear combination of the operators of the form

\[
T_{z^{l_1}} \cdots T_{z^{l_k}} (T_{z^n} T_{z^m} - T_{z^n + m}) T_{z^{l'_1}} \cdots T_{z^{l'_{k'}}} \quad (l_1, \ldots, l_k, l'_1, \ldots, l'_{k'}, n, m \in \mathbb{Z}).
\]

We show that there exists a function \( K_S(t, s) \in C^\infty(T^2) \) which is a polynomial of \( t, s \) and satisfies

\[
(S \xi)(t) = \int_T K_S(t, s) \xi(s) ds. \quad (\xi \in H^2).
\]
This function $K_S$ is called the kernel function of $S$. Given $n, m \in \mathbb{Z}$, it is easily verified that

$$
(T_{z^n}T_{z^m} - T_{z^{n+m}})\xi(t) = \left( \sum_{k \geq \max\{-m,-n\}} - \sum_{k \geq -m-n} \right) \langle \xi | z^k \rangle z^k(t)
$$

$$
= \int_T \left( \sum_{k \geq \max\{-m,-n\}} - \sum_{k \geq -m-n} \right) t^{k-s-k} \xi(s) ds,
$$

where

$$
\langle f | g \rangle = \int_T f(s) g(s) ds \quad (f, g \in L^2(T))
$$

is the usual inner product on $L^2(T)$. Then the kernel function $K_{T_{z^n}T_{z^m} - T_{z^{n+m}}}$ of $T_{z^n}T_{z^m} - T_{z^{n+m}}$ is a finite sum of the functions $t^{k-s-k}$ since there exists a finite subset $I_{n,m} \subset \mathbb{Z}$ such that

$$
K_{T_{z^n}T_{z^m} - T_{z^{n+m}}}(t, s) = \left( \sum_{k \geq \max\{-m,-n\}} - \sum_{k \geq -m-n} \right) t^{k-s-k} = \pm \sum_{k \in I_{n,m}} t^{k-s-k}
$$

(when $I_{n,m}$ is empty, we regard the function $K_{T_{z^n}T_{z^m} - T_{z^{n+m}}}(t, s) = 0$). Moreover, given $l \in \mathbb{Z}$, we compute that

$$
K_{(T_{z^n}T_{z^m} - T_{z^{n+m}})T_{z^l}}(t, s) = \int_T K_{T_{z^n}T_{z^m} - T_{z^{n+m}}}(t, r) K_{T_{z^l}}(r, s) dr
$$

$$
= \pm \int_T \sum_{k \in I_{n,m}} t^{k-r-k} \sum_{k' \geq -l} r^{k'-s-k'} dr
$$

$$
= \pm \sum_{k' \geq -l} \sum_{k \in I_{n,m}} t^{k-s-k'} \int_T r^{k'-s-k} dr
$$

$$
= \pm \sum_{k \in I_{n,m}, k \geq -l} t^{k-s-k},
$$

which implies that the kernel function $K_{(T_{z^n}T_{z^m} - T_{z^{n+m}})T_{z^l}}$ is a polynomial of $t, s$. By the similar computation, it follows that for $l, m, n \in \mathbb{Z}$, the kernel function $K_{T_{z^n}T_{z^m} - T_{z^{n+m}}}$ is also a polynomial. Then, by the inductive argument, we have that the kernel functions

$$
K_{T_{z^n} \cdots T_{z^m}}(T_{z^n}T_{z^m} - T_{z^{n+m}})T_{z^l} \cdots T_{z^l}
$$

are also polynomials, which in particular belong to $C^\infty(T^2)$.

Let $K^\infty$ be the set of all compact operators $S$ such that there exists a function $K_S \in C^\infty(T^2)$ with the property that

$$
(S\xi)(t) = \int_T K_S(t, s) \xi(s) ds \quad (\xi \in H^2, t \in T).
$$

By the above argument, it follows that for each operator $T \in \mathcal{P}$, there exist a function $f \in C^\infty(T)$ and an operator $S \in K^\infty$ with $T = T_f + S$. Since $T_g$ is
compact if and only if $g = 0$, the function $f$ and the operator $S$ are uniquely determined. We define the seminorms $\{\| \cdot \|_{k,l,m} \}$ on $\mathcal{P}$ by

$$\|T + S\|_{k,l,m} = \|f^{(k)}\|_{\infty} + \|K^{(l,m)}_S\|_{\infty} \quad (k, l, m \in \mathbb{Z}_{\geq 0}),$$

where $f^{(k)}$ is the $k$-th derivative of $f$,

$$K^{(l,m)} = \frac{\partial^{k+l+m}}{\partial t^l \partial s^m} K(t, s) \quad (K \in C^\infty(T^2)),$$

and $\| \cdot \|_{\infty}$ mean the supremum norms on the corresponding function spaces.

**Definition 3.1.** The smooth Toeplitz algebra $T^\infty$ is defined by the completion of $\mathcal{P}$ with respect to the topology induced by the seminorms $\{\| \cdot \|_{k,l,m}\}$.

Similarly as in the case of $\mathcal{P}$, we have that for any $T \in T^\infty$, there exist a function $f \in C^\infty(T)$ and an operator $S \in \mathbb{K}^\infty$ with $T = Tf + S$. In fact, if $\{T_n\}_{n \geq 1} \subset \mathcal{P}$ converges to $T$ with respect to the seminorms $\{\| \cdot \|_{k,l,m}\}$ with $T_n = T_{fn} + S_n$, we compute that

$$\|f_n^{(k)} - f_{n'}^{(k)}\|_{\infty} = \|T_{fn} - T_{fn'}\|_{k,0,0} \leq \|T_n - T_{n'}\|_{k,0,0} \to 0 \quad (n, n' \to \infty),$$

for any $k \in \mathbb{Z}_{\geq 0}$, which ensures that there exists the function $f \in C^\infty(T)$ such that $f_n \to f$ with respect to the seminorms. Alternatively, since $\{S_n\}$ is also Cauchy, we have that for any $k, l, m \in \mathbb{Z}$,

$$\|S_n - S_{n'}\|_{k,l,m} = \|K^{(l,m)}_{S_n} - K^{(l,m)}_{S_{n'}}\|_{\infty} \to 0 \quad (n, n' \to \infty).$$

Hence, we find a function $K \in C^\infty(T^2)$ with $K_{S_n} \to K$ as $n \to \infty$ with respect to Fréchet topology on $C^\infty(T^2)$. Then the operator $S$ defined by

$$S\xi(t) = \int_T K(t, s)\xi(s)ds \quad (\xi \in H^2, t \in T)$$

belongs to $\mathbb{K}^\infty$ and $S_n - S \to 0$ as $n \to \infty$ with respect to the seminorms, which implies the conclusion. It is clear by the above argument that $\mathbb{K}^\infty$ is a $^*$-ideal of $T^\infty$ and Fréchet closed.

We define a homomorphism $q : T^\infty \to C^\infty(T)$ by $q(Tf + S) = f$, which is continuous with respect to the seminorms cited before. The following lemma is already clear:

**Lemma 3.2.** We obtain the following short exact sequence as $F^*$-algebras:

$$0 \longrightarrow \mathbb{K}^\infty \overset{i}{\longrightarrow} T^\infty \overset{q}{\longrightarrow} C^\infty(T) \longrightarrow 0,$$

where $i$ is the canonical inclusion.

We next deduce the following lemma, which is a smooth version of $C^*$-algebra case:
Lemma 3.3. We have the following isomorphism:
\[ \mathbb{K}^\infty \simeq \varinjlim (M_n(\mathbb{C}), \varphi_n), \]
where the homomorphisms \( \varphi_n : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C}) \) are given by
\[ \varphi_n(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad (A \in M_n(\mathbb{C}), \, n \geq 1). \]

Proof. Let \( P_n \) \((n \geq 1)\) be the orthogonal projections on \( H^2 \) defined by
\[ P_n \xi(t) = \sum_{k=0}^{n-1} \langle \xi | z^k \rangle z^k(t) \]
\[ = \sum_{k=0}^{n-1} \left( \int_T \xi(s)s^{-k}ds \right) t^k \]
\[ = \int_T \sum_{k=0}^{n-1} t^k s^{-k} \xi(s)ds \quad (\xi \in H^2), \]
which implies that
\[ K_{P_n}(t, s) = \sum_{k=0}^{n-1} t^k s^{-k}. \]
Then \( P_n \mathbb{K}^\infty P_n \) is isomorphic to \( M_n(\mathbb{C}) \). Indeed, the kernel function \( K_{P_n}SP_n \) for \( S \in \mathbb{K}^\infty \) is calculated as follows: since
\[ K_{SP_n}(t, s) = \int_T K_S(t, r)K_{P_n}(r, s)dr \]
\[ = \int_T \sum_{k=0}^{n} r^k s^{-k} K_S(t, r)dr, \]
we have that
\[ K_{P_nSP_n}(t, s) = \int_T K_{P_n}(t, u)K_{SP_n}(u, s)du \]
\[ = \int_T \sum_{k=0}^{n-1} t^k \sum_{k'=0}^{n-1} r^{k'} s^{-k'} K_S(u, r)du \]
\[ = \int_T \int_T \sum_{k,k'=0}^{n-1} t^k u^{-k} r^{k'} s^{-k'} K_S(u, r)drdu \]
\[ = \sum_{k,k'=0}^{n-1} t^k s^{-k'} \int_T \int_T r^{k'} u^{-k} K_S(u, r)drdu \]
\[ = \sum_{k,k'=0}^{n-1} c_{k,k'} t^k s^{-k'}, \]
where
\[ c_{k,k'} = \int_T \int_T r^{k'} u^{-k} K_S(u, r)drdu \]
are the Fourier coefficients of $K_S \in C^\infty(T^2)$.

On the other hand, we define the matrix units $E_{ij}$ in what follows: when $i = j$, we define

$$E_{ii} = T_{z_i^{-1}}T_{z_i}^* - T_iT_i^*.$$  

For $i \neq j$, we define

$$E_{ij} = \begin{cases} T_{z_i^{-1}}E_{ii} & (i < j) \\ E_{jj}T_{z_j^{-1}} & (i > j). \end{cases}$$

It is not hard to see that $\{E_{ij}\}$ forms a family of matrix units. By taking $m = -n$ in the computation of the kernel function of $T_{z^m}T_{z^m} - T_{z^m}$, we have

$$K_{-z^m} = \sum_{k=0}^{n-1} t^k s^{-k}.$$  

Hence we have

$$K_{E_{ii}}(t, s) = t^{i-1}s^{-(i-1)}.$$  

More generally, we obtain that

$$K_{E_{ij}}(t, s) = t^{i-1}s^{-(i-1)}.$$  

Then $P_nK^\infty P_n$ is generated by the matrix units $\{E_{ij}\}_{i,j=1}^n$ so that it is isomorphic to $M_n(\mathbb{C})$ with the seminorms given by

$$\|\lambda_{kl}\|_{p,q} = \sup_{t,s \in T} \left| \sum_{k,l=0}^{n-1} \lambda_{kl}t^k s^l s^{-k} \right| (\lambda_{kl} \in M_n(\mathbb{C})).$$

For any $S \in K^\infty$, $\|S - P_nSP_n\|_{t,m} \to 0$ as $n \to \infty$ for any $l, m \geq 0$ since $\{c_k, k'\}$ belongs to the Schwartz space on $\mathbb{Z}^2$. Therefore,

$$\|S - P_nSP_n\|_{l,m} \to 0 \quad (n \to \infty)$$

for any $l, m \in \mathbb{Z}_{\geq 0}$. Hence, the conclusion follows.

By the above lemma, we deduce the following corollaries:

**Corollary 3.4.** $K^\infty$ is a simple $F^*$-algebra, which is equal to the commutator $F^*$-ideal $[T^\infty, T^\infty]$ of $T^\infty$.  

In what follows, we study briefly the $F^*$-crossed products $T^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ of $T^\infty$ by the gauge action $\alpha_\theta$ of $\mathbb{Z}$. Let $\alpha_\theta$ be the action of $\mathbb{Z}$ on $T^\infty$ defined by

$$\alpha_\theta(f) = f \circ T_f \quad (f \in C^\infty(T), n \in \mathbb{Z}),$$

where $f_\theta(z) = f(\theta z)$, which gives a $F^*$-dynamical system $(T^\infty, \mathbb{Z}, \alpha_\theta)$. We also consider the unitary operator $U_\theta$ on $H^2$ defined by

$$U_\theta \xi(t) = \xi(e^{2\pi it}) \quad (\xi \in H^2, t \in T).$$

It is easily seen that $U_\theta \xi(t) = U_\theta^{-1} \xi(t) = \xi(e^{-2\pi it})$. Then we form the $F^*$-crossed products $T^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ of the $F^*$-dynamical system $(T^\infty, \mathbb{Z}, \alpha_\theta)$, which could be viewd
as the deformation quantization \((D^2 \times S^1)_\theta\) of the solid torus \(D^2 \times S^1\). In fact, let \(T^\infty[Z]\) be the \(*\)-algebra of all finite sums
\[
 f = \sum_{n \in \mathbb{Z}, |n| \leq N} A_n U_\theta^n \quad (A_n \in T^\infty, N \in \mathbb{Z}_{\geq 0}),
\]
where its multiplication is determined by \(U_\theta A U_\theta^{-1} = \alpha_\theta(A)\) and its \(*\)-operation is given by \((AU_\theta)^* = \alpha_\theta^{-1}(A^*)U_\theta^{-1}\). For \(f = \sum A_n U_\theta^n \in T^\infty[Z]\), we induce the seminorms defined by
\[
 \|f\|_{p,q,r,s} = \sup_{n \in \mathbb{Z}} (1 + |n|^2)^p \|A_n\|_{q,r,s} \quad (p, q, r, s \in \mathbb{Z}_{\geq 0}).
\]
We define the \(F^*\)-crossed product \((D^2 \times S^1)_\theta = T^\infty \rtimes_{\alpha_\theta} \mathbb{Z}\) by the completion of \(T^\infty[Z]\) with respect to the seminorms cited above. For \(S \in \mathbb{K}^\infty\), we calculate that
\[
 \alpha_\theta(S) \xi(t) = U_\theta^S U_\theta^* \xi(t)
 = U_\theta \int_T K_S(t, s) \xi(e^{-2\pi i \theta} s) ds
 = \int_T K_S(e^{2\pi i \theta} t, e^{2\pi i \theta} s) \xi(s) ds
\]
to obtain that
\[
 K_{\alpha_\theta(S)}(t, s) = K_S(e^{2\pi i \theta} t, e^{2\pi i \theta} s).
\]
Therefore, we have \(\alpha_\theta(\mathbb{K}^\infty) = \mathbb{K}^\infty\) so that we construct a \(F^*\)-dynamical system \((\mathbb{K}^\infty, \mathbb{Z}, \alpha_\theta)\). Since
\[
 K_{\alpha_\theta(P_n)}(t, s) = P_n(e^{2\pi i \theta} t, e^{2\pi i \theta} s)
 = \sum_{k=0}^{n-1} (e^{2\pi i \theta} t)^k (e^{2\pi i \theta} s)^{-k}
 = \sum_{k=0}^{n-1} t^k s^{-k} = K_{P_n}(t, s),
\]
we have \(\alpha_\theta(P_n \mathbb{K}^\infty P_n) = P_n \mathbb{K}^\infty P_n\). Therefore, we also construct \(F^*\)-dynamical systems \((P_n \mathbb{K}^\infty P_n, \mathbb{Z}, \alpha_\theta^{(n)})\), where \(\alpha_\theta^{(n)}\) are the restrictions of \(\alpha_\theta\) on \(P_n \mathbb{K}^\infty P_n\). Let \(i_n\) be the isomorphism from \(P_n \mathbb{K}^\infty P_n\) onto \(M_n(\mathbb{C})\) defined before and \(i = \lim i_n\) the isomorphism from \(\lim P_n \mathbb{K}^\infty P_n\) onto \(\mathbb{K}^\infty\) induced by the isomorphisms \(i_n\). We write by \(\overline{\alpha_\theta}^{(n)}\) the action \(i_n \circ \alpha_\theta \circ i_n^{-1}\).

**Proposition 3.5.** We have the following isomorphism:
\[
 \mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \simeq \lim_{\overrightarrow{n}} (M_n(\mathbb{C}) \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z}, \overline{\varphi}_n),
\]
where \(\overline{\varphi}_n\) are the inclusions induced naturally by \(\varphi_n\).

**Proof.** Since \(i \circ \overline{\alpha_\theta}^{(n)} = \alpha_\theta \circ i\) for any \(n \geq 1\), we have
\[
 \mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \simeq \lim_{\overrightarrow{n}} (P_n \mathbb{K}^\infty P_n \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z}, \varphi_n),
\]
where \( \varphi_n : P_n \mathcal{K}^\infty P_n \times_{\alpha_0^{(n)}} \mathbb{Z} \to P_{n+1} \mathcal{K}^\infty P_{n+1} \times_{\alpha_0^{(n+1)}} \mathbb{Z} \) are the canonical inclusions. Moreover, since \( i_n \circ \alpha_0^{(n)} = \overline{\alpha_0^{(n)}} \circ i_n \), we find isomorphisms

\[
\psi_n : P_n \mathcal{K}^\infty P_n \times_{\alpha_0^{(n)}} \mathbb{Z} \to M_n(\mathbb{C}) \times_{\varphi_n} \mathbb{Z}.
\]

Then since \( \psi_n \circ \varphi_n = \varphi_n \), we conclude

\[
\lim (P_n \mathcal{K}^\infty P_n \times_{\alpha_0^{(n)}} \mathbb{Z}, \varphi_n) \simeq \lim (M_n(\mathbb{C}) \times_{\varphi_n} \mathbb{Z}, \varphi_n)
\]

as desired. \( \square \)

Then we construct a *-homomorphism

\[
\rho_n : M_n(\mathbb{C}) \times_{\alpha_0^{(n)}} \mathbb{Z} \to M_n(\mathbb{C}) \hat{\otimes}_\gamma S(\mathbb{Z}),
\]

where \( S(\mathbb{Z}) \) is the set of all rapidly decreasing sequences \( \{c_n\} \subseteq \mathbb{C} \) and \( \hat{\otimes}_\gamma \) means the tensor product of \( F^* \)-algebras completed by the topology induced by the seminorms defined by

\[
\left\| \sum_{j=1}^N x_j \otimes y_j \right\| = \inf \sum_{j,k,l} \| x_j \| \| y_j \|, \]

where the infimum is taken over the all representations of \( \sum_{j=1}^N x_j \otimes y_j \). Equivalently, \( M_n(\mathbb{C}) \hat{\otimes}_\gamma S(\mathbb{Z}) \) is regarded as \( S(\mathbb{Z}, M_n(\mathbb{C})) \) with the ordinary convolution as its product. For \( x \in M_n(\mathbb{C}) \times_{\alpha_0^{(n)}} \mathbb{Z} \), we define

\[
\rho_n(x) = xU_\theta^n.
\]

It is easily seen that it is an isomorphism. Moreover, since

\[
\| \rho_n(x) \|_{p,q,r,s} = \sup_{m \in \mathbb{Z}} (1 + m^2)^p \| x U_\theta^m \|_{q,r,s} = \| x \|_{p,q,r,s} \quad (p,q,r,s \in \mathbb{Z}_{\geq 0}),
\]

for any \( x = \sum x_m U_\theta^m \in \mathcal{T}^\infty \mathbb{Z} \), it is Fréchet isometry. Therefore, we have

\[
M_n(\mathbb{C}) \times_{\alpha_0^{(n)}} \mathbb{Z} \simeq M_n(\mathbb{C}) \hat{\otimes}_\gamma S(\mathbb{Z})
\]

by \( \rho_n \). Now it is immediately known that the following fact follows:

**Corollary 3.6.** The isomorphism

\[
\mathcal{K}^\infty \times_{\alpha_0} \mathbb{Z} \simeq \mathcal{K}^\infty \hat{\otimes}_\gamma C^\infty(T)
\]

holds.

**Proof.** By Proposition 3.5 and Lemma 3.3, we have that

\[
\mathcal{K}^\infty \times_{\alpha_0} \mathbb{Z} \simeq \lim (M_n(\mathbb{C}) \times_{\varphi_n} \mathbb{Z}, \varphi_n)
\]

\[
\simeq \lim (M_n(\mathbb{C}) \hat{\otimes}_\gamma S(\mathbb{Z}), \varphi_n \otimes id_{S(\mathbb{Z})})
\]

\[
\simeq \lim (M_n(\mathbb{C}), \varphi_n) \hat{\otimes}_\gamma S(\mathbb{Z})
\]

\[
\simeq \mathcal{K}^\infty \hat{\otimes}_\gamma S(\mathbb{Z}).
\]
Since $S(\mathbb{Z})$ is isomorphic to $C^\infty(T)$ sending by the Fourier transform, the conclusion follows.

We end this section by stating the following fact:

**Corollary 3.7.** We have the following short exact sequence:

$$0 \longrightarrow \mathcal{K} \otimes \gamma S(\mathbb{Z}) \xrightarrow{\tilde{i}} (D^2 \times S^1)_\theta \xrightarrow{\tilde{q}} C^\infty(T) \times_{\pi_0} \mathbb{Z} \longrightarrow 0,$$

where $\pi_\theta : C^\infty(T) \times \mathbb{Z} \to C^\infty(T)$ is the Fréchet continuous action defined by

$$\pi_\theta^n(f)(z) = f(e^{2\pi in\theta}z) \quad (f \in C^\infty(T), z \in T),$$

with a bounded linear section $\tilde{s}$ of $\tilde{q}$.

**Proof.** Since $i \circ \alpha_\theta^n = \alpha_\theta^n \circ i$ and $q \circ \alpha_\theta^n = \pi_\theta^n \circ q$ for all $n \in \mathbb{Z}$, it is clear that the desired short exact sequence holds and $\tilde{s}(fU^n_\theta) = T_fU^n_\theta$ ($f \in C^\infty(T)$, $n \in \mathbb{Z}$).

\[\square\]

### 4. Metric Approximation Property

We introduce an analogue of the notion of metric approximation property for Banach spaces [4]. Let $\mathfrak{A}$, $\mathfrak{B}$ be two Banach spaces and $\mathfrak{I} \subset \mathfrak{B}$ an $M$-ideal. In [4], the authors prove that if $\mathfrak{A}$ is separable and has the metric approximation property, then each contractive map $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{I}$ has a lift $\tilde{\varphi} : \mathfrak{A} \to \mathfrak{B}$ which is contractive and satisfies $q \circ \tilde{\varphi} = \varphi$, where $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{I}$ is the quotient map. Our purpose in this section is to define this property for $F^*$-algebras to prove lifting problem cited above. The topology on $\mathfrak{A}$ induced by its seminorms $\{\|\cdot\|_k\}_{k \geq 0}$ is same as that induced by the metric $d_{\mathfrak{A}}$ defined by

$$d_{\mathfrak{A}}(a, b) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|a - b\|_k}{1 + \|a - b\|_k} \quad (a, b \in \mathfrak{A}).$$

We say that a linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ is bounded if and only if there exists a constant $C > 0$ with

$$d_{\mathfrak{A}}(\varphi(a), 0) \leq C d_{\mathfrak{A}}(a, 0) \quad (a \in \mathfrak{A}).$$

**Definition 4.1.** Let $\mathfrak{A}$ be a $F^*$-algebra and $\{\|\cdot\|_k\}_{k \geq 0}$ its seminorms. We say that it has the metric approximation property if there exists a family of bounded linear maps $\{\theta_n\}_{n \geq 1}$ on $\mathfrak{A}$ with the following properties:

1. each $\theta_n$ has a finite rank,
2. for any $a \in \mathfrak{A}$, $d_{\mathfrak{A}}(\theta_n(a), a) \to 0$ as $n \to \infty$.

We give some examples of $F^*$-algebras with the metric approximation property. Here we note that $d_{\mathfrak{A}}(\theta_n(a), a) \to 0$ is satisfied if and only if $\|\theta_n(a) - a\|_k \to 0$ for any $k \geq 0$. 

Example 4.2. For an integer $n \geq 2$, let $F_n$ be the free group with $n$-generators. Given $g \in F_n$, we denote by $|g|$ its word length, and for $f \in C[F_n]$ and an integer $k \in \mathbb{Z}_{\geq 0}$, we define seminorms by

$$
\|f\|_k = \sup_{g \in F_n} (1 + |g|)^k |f(g)|.
$$

The Schwartz space $S(F_n)$ is defined by the completion of $C[F_n]$ with respect to the above seminorms. For $f \in S(F_n)$, we define a bounded operator $\lambda(f)$ on the Hilbert space $l^2(F_n)$ by the convolution with $f$, that is,

$$(\lambda(f)\xi)(g) = (f * \xi)(g) = \sum_{h \in F_n} f(h)\xi(h^{-1}g) \quad (g \in F_n, \xi \in l^2(F_n)).$$

on which the seminorms are defined by

$$
\|\lambda(f)\|_k = \|f\|_k \quad (k \in \mathbb{Z}_{\geq 0}).
$$

This definition is well-defined. Indeed, if $\lambda(f) = 0$, then $\lambda(f)\delta_e = 0$, where $e \in F_n$ is the unit and the element $\delta_e \in l^2(F_n)$ is defined by

$$
\delta_e(g) = \begin{cases} 
1 & (g = e) \\
0 & (g \neq e).
\end{cases}
$$

Hence, for any $g \in F_n$, we have that

$$
0 = (\lambda(f)\delta_e)(g) = (f * \delta_e)(g) = \sum_{h \in F_n} f(h)\delta_e(h^{-1}g) = f(g).
$$

Therefore, $\lambda(f) = 0$ leads $f = 0$, which implies that the seminorms are well-defined.

We define the $F^*$-algebra $C^*_r(F_n)^\infty$ by the completion of the $*$-algebra generated by the bounded operators $\lambda(f) (f \in S(F_n))$. Here we claim that

$$
C^*_r(F_n)^\infty = \{\lambda(f) | f \in S(F_n)\}.
$$

In fact, it is clear that $\lambda(f^*) = \lambda(f^*)$ for any $f \in S(F_n)$, where

$$
f^*(g) = \overline{f(g^{-1})} \in S(F_n).
$$

For any $T \in C^*_r(F_n)^\infty$, there exits a family $\{\lambda(f_n)\}_{n \geq 1}$ ($f_n \in S(F_n)$) which converges to $T$ with respect to the seminorms cited above. Then, for any $k \in \mathbb{Z}_{\geq 0}$, we have that

$$
\|f_n - f_m\|_k = \|\lambda(f_n) - \lambda(f_m)\|_k \to 0 \quad (n, m \to \infty),
$$

which implies that there exists a function $f \in S(F_n)$ which is the limit of $\{f_n\}$. Thus, for any $k \in \mathbb{Z}_{\geq 0}$, we have that

$$
\|T - \lambda(f)\|_k \leq \|T - \lambda(f_n)\|_k + \|\lambda(f_n) - \lambda(f)\|_k \to 0 \quad (n \to \infty).
$$
We construct a family of finite dimensional bounded linear maps \( \{ \theta_k \} \) on \( C^*_r(\mathbb{F}_n)^\infty \) in what follows. Given an integer \( k \geq 1 \), let \( E_k = \{ g \in \mathbb{F}_n \mid |g| \leq k \} \) and \( \chi_k \) the function on \( \mathcal{S}(\mathbb{F}_n) \) defined by

\[
\chi_k(g) = \begin{cases} 
1 & (g \in E_k) \\
0 & (g \notin E_k).
\end{cases}
\]

Since the number of elements of \( E_k \) is finite for each \( k \geq 1 \), the linear maps \( \psi_n : \mathbb{F}_n \to \mathbb{F}_n \) defined by

\[
\psi_k(\lambda(f)) = \lambda(\chi_k f)
\]

have finite ranks. We define the finite linear bounded maps \( \theta_k : \mathbb{F}_n \to \mathbb{F}_n (k \geq 1) \) by

\[
\theta_k(\lambda(f)) = \lambda(e^{-|\cdot|/k} \chi_k f).
\]

Then for any \( l \geq 0 \) and \( f \in C^*_r(\mathbb{F}_n)^\infty \), we compute that

\[
\| \lambda(f) - \theta_k(\lambda(f)) \|_l \\
\leq \| \lambda(f) - \lambda(e^{-|\cdot|/k} f) \|_l + \| \lambda(e^{-|\cdot|/k} f) - \lambda(e^{-|\cdot|/k} \chi_k f) \|_l \\
= \| f - e^{-|\cdot|/k} f \|_l + \| e^{-|\cdot|/k} f - e^{-|\cdot|/k} \chi_k f \|_l \\
= \sup_{g \in \mathbb{F}_n} \left( (1 + |g|) f(g)(1 - e^{-|g|/k}) \right) + \sup_{g \in \mathbb{F}_n} \left( (1 + |g|) e^{-|g|/k}(1 - \chi_k(g)) \right) \\
\leq \| f \|_l \sup_{g \in \mathbb{F}_n} \left( 1 - e^{-|g|/k} \right) + \sup_{|g| \geq k+1} \left( (1 + |g|) e^{-|g|/k} \right) \\
\to 0 \quad (k \to \infty).
\]

Therefore, \( C^*_r(\mathbb{F}_n)^\infty \) has the metric approximation property.

**Example 4.3.** According to [16], the smooth noncommutative 2-torus \( T^2_\theta \) is isomorphic to the Fréchet inductive limit

\[
\lim C^\infty(T) \hat{\otimes}_\gamma (M_{p_n}(\mathbb{C}) \oplus M_{q_n}(\mathbb{C})).
\]

We show that it also has the metric approximation property. As a preparation, we verify that the Fréchet algebra \( C^\infty(T) \hat{\otimes}_\gamma M_q(\mathbb{C}) \) has the metric approximation property. It suffices to show that \( C^\infty(T) \) has this property since if it had this property with a family \( \{ \theta^{(q)}_n \} \) of bounded linear maps there, the family \( \{ \theta^{(q)}_n \otimes I_q \} \) would be the desired one for \( C^\infty(T) \otimes M_q(\mathbb{C}) \), where \( I_q \) is the identity map on \( M_q(\mathbb{C}) \). For \( f \in C^\infty(T) \), we define the maps \( \theta^{(q)}_n : C^\infty(T) \to C^\infty(T) \) by

\[
\theta^{(q)}_n(f) = \sum_{|l| \leq n} \hat{f}(l) z^l \quad (n \geq 1),
\]

where \( \hat{f}(l) \) are the Fourier coefficients and \( z \in C^\infty(T) \) is the canonical generator defined by \( z(t) = t \ (t \in T) \). Then it is clear that they are of finite rank. For
$f \in C^\infty(T)$ and $k \in \mathbb{Z}_{\geq 0}$,

$$
\|f - \theta_n^{(j)}(f)\|_k = \|f^{(k)} - (\theta_n(f))^{(k)}\|_\infty
$$

$$
= \sup_{m \in \mathbb{Z}} \left| \hat{f}^{(k)}(m) - \sum_{|l| \leq n} \hat{f}(l)(2\pi il)^k \delta_l(m) \right|
$$

$$
= \sup_{m \in \mathbb{Z}} \left| \sum_{|l| \geq n+1} \hat{f}(l)(2\pi il)^k \delta_l(m) \right|
$$

$$
= \sup_{m \in \mathbb{Z}, |m| \geq n+1} \left| \hat{f}(m)(2\pi m)^k \right|
$$

$$
\to 0 \ (n \to \infty),
$$

where $\delta_l(m) = 0 (m \neq l), = 1 (m = l)$, since $\{\hat{f}(l)\}_{l \in \mathbb{Z}}$ is a rapidly decreasing sequence by the hypothesis $f \in C^\infty(T)$. Hence $C^\infty(T)$ has the metric approximation property.

We turn to show briefly that $T^2_\theta$ also has this property. For any $x \in T^2_\theta$, we define the sequence $\{x_n\}$ by

$$
x_n = e_1^{(n)} x e_1^{(n)} + e_2^{(n)} x e_2^{(n)} \ (n \geq 1),
$$

where $e_j^{(n)} (j = 1, 2)$ are the projections such that

$$
e_1^{(n)} x e_1^{(n)} \in C^\infty(T) \otimes M_{p_n}(\mathbb{C}), \quad e_2^{(n)} x e_2^{(n)} \in C^\infty(T) \otimes M_{q_n}(\mathbb{C})
$$

for any $x \in T^2_\theta$ (see [10]). We define the linear maps $\Phi_n$ on $T^2_\theta$ by

$$
\Phi_n(x) = \theta_n^{(p_n)}(e_1^{(n)} x e_1^{(n)}) + \theta_n^{(q_n)}(e_2^{(n)} x e_2^{(n)})
$$

It is easily seen that $\Phi_n(x) \to x$ with respect to the seminorms on $T^2_\theta$ (see [10]), hence to the metric $d$ as well. Therefore, $T^2_\theta$ has the metric approximation property.

By the similar argument for $C^\infty(T)$, the operation of taking suspension preserves the metric approximation property.

**Corollary 4.4.** If a $F^*$-algebra $\mathfrak{A}$ has the metric approximation property, so does its suspension $S^\infty\mathfrak{A}$.

**Proof.** It suffices to show that the $F^*$-algebra

$$
C^\infty_0(0, 1) = \{ f \in C^\infty(0, 1) | f^+(n)(0) = f^-(n)(1) = 0 \ (n \in \mathbb{Z}_{\geq 0}) \}
$$

has the metric approximation property. For any integer $j \geq 1$, we put

$$
f_j(t) = e^{-\frac{\pi it}{j-1}} \in C^\infty_0(0, 1).
$$

Let $\{\xi_j\}_{j=1}^\infty$ be the orthogonal family of $C^\infty_0(0, 1)$ obtained by Schmidt orthogonalization of $\{f_j\}$. Then we define the linear maps $\theta_n : C^\infty_0(0, 1) \to C^\infty(0, 1)$
by
\[ \theta_n(f)(t) = \sum_{j=1}^{n} \langle f|\xi_j \rangle \xi_j(t) \quad (\xi \in C_0^\infty(0,1), \ t \in (0,1), \ n \geq 1). \]

It is easily seen that the images of \( \theta_n \) are included in \( C_0^\infty(0,1) \). By the similar argument for \( C^\infty(T) \), we obtain the conclusion. \( \square \)

For a \( F^* \)-algebra \( \mathfrak{A} \), by \( \mathfrak{A}^* \) we denote the set of all bounded linear functionals on \( \mathfrak{A} \), where we say a linear functional \( \varphi \) on \( \mathfrak{A} \) is bounded if and only if
\[ \| \varphi \| = \sup_{a \in \mathfrak{A} \setminus \{0\}} \frac{|\varphi(a)|}{d_\mathfrak{A}(a,0)} < \infty. \]

Before we proceed to show the lifting problem, we need the following lemma:

**Lemma 4.5.** Let \( \mathfrak{A}, \mathfrak{B} \) be two \( F^* \)-algebras. Suppose that \( \mathfrak{I} \) is an \( F^* \)-ideal of \( \mathfrak{B} \) and that \( L, N \) are finite dimensional subspaces of \( \mathfrak{A} \) with \( L \subset N \). We consider the following diagram of bounded linear maps:
\[
\begin{array}{ccc}
L & \xrightarrow{\iota} & N \\
\Psi \downarrow & & \downarrow q \\
\mathfrak{B} & & \mathfrak{B}/\mathfrak{I},
\end{array}
\]
where \( q \) and \( \iota \) are the quotient map and the natural inclusion respectively, and suppose that
\[ d_\mathfrak{B}(q \circ \Psi(a) - \varphi(a), 0) \leq \varepsilon d_\mathfrak{A}(a,0) \quad (a \in L). \]
for a positive constant \( \varepsilon > 0 \). Then there is a bounded linear map \( \varphi' : N \to \mathfrak{B}/\mathfrak{I} \) with the property that
\[
\begin{align*}
\{ \varphi = q \circ \varphi' \\
\| \varphi'(a), 0 \| \leq d_\mathfrak{A}(a,0) \quad (a \in N) \\
\| \varphi'(a) - \Psi(a), 0 \| \leq 6 \varepsilon \quad (a \in L) \}
\end{align*}
\]

**Proof.** This lemma is an analogy of Lemma 2.5 in [4]. Let \( D' \) and \( K \) be the closed unit ball of \( L_{\diamond,\gamma} \mathfrak{B} \) and \( N_{\diamond,\gamma} \mathfrak{B} \) respectively, that is,
\[
\begin{align*}
D' &= \{ \varphi : \mathfrak{B} \to L \mid \| \varphi \|_{L_{\diamond,\gamma} \mathfrak{B}^*} \leq 1 \} \subset L_{\diamond,\gamma} \mathfrak{B}^* \\
K &= \{ \varphi : \mathfrak{B} \to N \mid \| \varphi \|_{N_{\diamond,\gamma} \mathfrak{B}^*} \leq 1 \} \subset N_{\diamond,\gamma} \mathfrak{B}^*,
\end{align*}
\]
where
\[ \| \varphi \|_{L_{\diamond,\gamma} \mathfrak{B}^*} = \sup_{a \in \mathfrak{B} \setminus \{0\}} \frac{d_\mathfrak{B}(\varphi(a),0)}{d_\mathfrak{B}(a,0)} \]
and \( \| \cdot \|_{N_{\diamond,\gamma} \mathfrak{B}^*} \) is defined by the similar way for \( \| \cdot \|_{L_{\diamond,\gamma} \mathfrak{B}^*} \), and \( \text{Aff}_T(D') \) the set of all affine functions \( \psi \) on \( D' \) such that \( \psi(\alpha \varphi) = \alpha \psi(\varphi) \) for all \( \alpha \in T, \varphi \in D' \). It is clear that \( \mathfrak{I} \) is a \( M \)-ideal of \( \mathfrak{B} \) and the equality
\[ \mathfrak{B}^* = \mathfrak{I}^\perp \oplus \mathfrak{I}^* \]
holds as a linear space, where \( J \) is the annihilator of \( I \). Let \( e : B^* \to J \) be the natural projection and \( W \) the image of \( N \hat{\otimes}_\gamma B^* \) via \( 1 \otimes e \), which is equal to \( N \hat{\otimes}_\gamma J \). Then \( D' \) is mapped weak\(^*\) homeomorphically to \( D' \cap W \) through the natural embedding \( \iota \otimes 1 : \hat{\otimes}_\gamma B^* \to N \hat{\otimes}_\gamma B^* \). We may identify the closed unit ball of \( N \hat{\otimes}_\gamma B^* \) with \( F = K \cap W \). We also identify the closed unit ball of \( L \hat{\otimes}_\gamma B^* \) with \( D' \cap W' \), where \( W' = (1 \otimes e)(\hat{\otimes}_\gamma B^*) = L \hat{\otimes}_\gamma J \). It is verified by the same argument in the proof of Lemma 2.5 in [4] that

\[
(i \otimes 1)(1 \otimes e) = (1 \otimes e)(i \otimes 1)
\]

and

\[
(i \otimes 1)(D' \cap W') = D \cap (i \otimes 1)(W') = D \cap W = D \cap F.
\]

Thus we have the following diagram of restrictions

\[
\begin{array}{c}
\text{Aff}_T(D \cap F) \quad \text{Aff}_T(F) \\
\uparrow \quad \uparrow \\
\text{Aff}_T(D) \quad \text{Aff}_T(K).
\end{array}
\]

Since \( 1 \otimes e : \hat{\otimes}_\gamma B^* \to \hat{\otimes}_\gamma J \) maps \( D' \) onto \( D' \cap W \), \( D \) is mapped onto \( D \cap F \) by [1] and [2] and \( D \) satisfies the condition of Lemma 2.1 in [4]. Therefore, with the diagram \( \& \), we obtain the conclusion by the same argument of Lemma 2.5 in [4]. □

**Proposition 4.6.** Let \( \mathfrak{A}, \mathfrak{B} \) be two \( F^*\)-algebras and \( \mathfrak{J} \subset \mathfrak{B} \) an \( F^*\)-ideal. If \( \mathfrak{A} \) is separable and has the metric approximation property, then for any bounded linear map \( \varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{J} \), there exists a bounded linear map \( \Phi : \mathfrak{A} \to \mathfrak{B} \) with the property that \( q \circ \Phi = \varphi \), where \( q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J} \) is the quotient map.

**Proof.** This proof is inspired by that of Theorem 2.6 in [4]. We fix a sequence \( \{a_n\}_{n \in \mathbb{N}} \subset \mathfrak{A} \) dense in \( \mathfrak{A} \). We construct recursively the pairs \( \{(L_n, \theta_n)\}_{n \in \mathbb{N}} \) which consist of increasing finite dimensional subspaces \( L_n \subset \mathfrak{A} \) with \( a_n \in L_n \) for any \( n \in \mathbb{N} \) and bounded linear maps \( \theta_n : \mathfrak{A} \to L_n \) with the property that for any \( a \in L_{n-1} \), the inequalities

\[
d_{\mathfrak{A}}(a, \theta_n(a)) \leq \frac{1}{2^n}
\]

are satisfied. We put \( L_0 = \{0\} \) and \( \theta_0 = 0 \). We suppose that for some \( n \in \mathbb{N} \) the pairs \( (L_0, \theta_0), \cdots, (L_n, \theta_n) \) with the above properties are given. By the approximation property of \( \mathfrak{A} \), there exists a bounded linear map \( \theta_{n+1} : \mathfrak{A} \to \mathfrak{A} \) such that for each \( a \in L_n \), the inequality

\[
d_{\mathfrak{A}}(a, \theta_{n+1}(a)) \leq \frac{1}{2^{n+1}}
\]

holds. We define the subspace \( L_{n+1} \) of \( \mathfrak{A} \) by

\[
L_{n+1} = L_n + \theta_{n+1}(\mathfrak{A}) + \mathbb{C}a_{n+1}.
\]
Then we have the desired pairs \( \{(L_n, \theta_n)\}_{n \in \mathbb{Z}_{\geq 0}} \). We note that \( \bigcup_{n \in \mathbb{Z}_{\geq 0}} L_n \) is dense in \( \mathfrak{A} \) with respect to Fréchet topology.

Next we inductively define a family of bounded linear maps
\[
\Psi_n : L_n \to \mathfrak{B} \quad (n \in \mathbb{Z}_{\geq 0})
\]
such that
\[
q \circ \Psi_n(a) = \varphi(a) \quad (a \in L_n).
\]
Putting \( \Psi_0 = 0 \), suppose that for some \( n \in \mathbb{Z}_{\geq 0} \), bounded linear maps \( \Psi_0, \cdots, \Psi_n \) satisfying (4) are constructed. Then we have that for any \( a \in L_{n-1} \),
\[
d_{\mathfrak{B}/\mathfrak{F}}(q \circ \Psi_n \circ \theta_n(a), \varphi(a)) = d_{\mathfrak{B}/\mathfrak{F}}(q \circ \theta_n(a), \varphi(a))
\leq C d_\mathfrak{F}(\theta_n(a), a)
\leq \frac{C}{2^n}.
\]
By Lemma 4.5 we find a bounded map \( \Psi_{n+1} : L_{n+1} \to \mathfrak{B} \) such that \( \varphi = q \circ \Psi_{n+1} \) on \( L_{n+1} \) and that for any \( a \in L_{n-1} \) with
\[
d_{\mathfrak{B}}(\Psi_{n+1}(a), \Psi_n \circ \theta_n(a)) \leq \frac{6C}{2^n}.
\]
Therefore, we compute that
\[
d_{\mathfrak{B}}(\Psi_{n+1}(a), \Psi_n(a)) \leq \frac{6C}{2^n} + d_{\mathfrak{B}}(\Psi_n(a), \Psi_n \circ \theta_n(a))
\leq \frac{6C}{2^n} + d_\mathfrak{B}(a, \theta_n(a))
\leq \frac{6C + 1}{2^n} \quad (a \in L_{n-1}).
\]
Hence for a fixed integer \( n_0 \in \mathbb{Z}_{\geq 0} \), we have for all \( n \geq n_0 \),
\[
d_{\mathfrak{B}}(\Psi_{n+1}(a), \Psi_n(a)) \leq \frac{6C + 1}{2^n}.
\]
Thus, for a fixed integer \( n_0 \in \mathbb{Z}_{\geq 0} \), the family of bounded linear maps \( \{\Psi_n\} \) converges to some \( \Psi^{(n_0)} : L_{n_0-1} \to \mathfrak{B} \). Therefore, we have the bounded linear map
\[
\Psi : \bigcup_{n \in \mathbb{Z}_{\geq 0}} L_n \to \mathfrak{B}
\]
such that \( \Psi|_{L_n} = \Psi^{(n)} (n \in \mathbb{Z}_{\geq 0}) \), and we can extend it to that on the closure of \( \bigcup_{n \in \mathbb{Z}_{\geq 0}} L_n \) which is equal to \( \mathfrak{A} \). This completes the proof.

5. Mayer-Vietoris Exact Sequence

This section is devoted to proving Mayer-Vietoris exact sequence for the entire cyclic cohomology. We firstly give a short proof of Bott periodicity for the entire cyclic cohomology by using the following Meyer’s excision for the entire cyclic theory [14]:

\[
\Psi_n : L_n \to \mathfrak{B} \quad (n \in \mathbb{Z}_{\geq 0})
\]
Proposition 5.1. Let
$$0 \rightarrow K \overset{i}{\rightarrow} P \overset{q}{\rightarrow} Q \rightarrow 0$$
be a short exact sequence of $F^*$-algebras with a bounded linear section $s$ of $q$. Then the following 6-terms exact sequence:
$$
\begin{array}{cccccccc}
HE^{ev}(Q) & \xrightarrow{q^*} & HE^{ev}(P) & \xrightarrow{i^*} & HE^{ev}(K) \\
\uparrow & & & & \downarrow \\
HE^{od}(K) & \leftarrow & HE^{od}(P) & \leftarrow & HE^{od}(Q)
\end{array}
$$
holds.

This yields the following fact, which has been already shown by Brodzki and Plymen [3] using bivariant entire homology and cohomology theory:

Lemma 5.2 (Bott periodicity for entire cyclic cohomology). For a $F^*$-algebra $\mathfrak{A}$,

$$HE^{ev}(S^\infty \mathfrak{A}) \simeq HE^{od}(\mathfrak{A}), \quad HE^{od}(S^\infty \mathfrak{A}) \simeq HE^{ev}(\mathfrak{A}).$$

Proof. By the exact sequence cited above, we have the following exact diagram:

$$
\begin{array}{cccccccc}
HE^{ev}(\mathfrak{A}) & \rightarrow & HE^{ev}(C^\infty \mathfrak{A}) & \rightarrow & HE^{ev}(\mathfrak{A}) \\
\uparrow & & & & \downarrow \\
HE^{od}(\mathfrak{A}) & \leftarrow & HE^{od}(C^\infty \mathfrak{A}) & \leftarrow & HE^{od}(\mathfrak{A})
\end{array}
$$

By Lemma 2.3 we deduce the conclusion. \qed

In what follows, we show an entire cyclic cohomology version of Mayer-Vietoris exact sequence. Before stating it, we review briefly the fibered product of $F^*$-algebras, which is an noncommutative analogue of the connected sum of two manifolds. Let $\mathfrak{A}_1, \mathfrak{A}_2$ and $\mathfrak{B}$ be $F^*$-algebras and $f_j : \mathfrak{A}_j \rightarrow \mathfrak{B}$ ($j = 1, 2$) epimorphisms.

Definition 5.3. \{(a_1, a_2) \in \mathfrak{A}_1 \oplus \mathfrak{A}_2 \mid f_1(a_1) = f_2(a_2)\} is called the fibered product of $(\mathfrak{A}_1, \mathfrak{A}_2)$ along $(f_1, f_2)$ over $\mathfrak{B}$, which we denote by $\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2$. Let $g_j$ be the projections of $\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2$ onto $\mathfrak{A}_j$ ($j = 1, 2$).

Theorem 5.4 (Mayer-Vietoris Exact Sequence for entire cyclic cohomology). In the situation of Definition 5.3, suppose that $\mathfrak{B}$ has the metric approximation property and separable. Then we have that the following exact diagram:

$$
\begin{array}{cccccccc}
HE^{ev}(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2) & \rightarrow & HE^{od}(\mathfrak{B}) & \xrightarrow{-f_1^* + f_2^*} & HE^{od}(\mathfrak{A}_1) \oplus HE^{od}(\mathfrak{A}_2) \\
\downarrow g_1^* + g_2^* & & & & \downarrow g_1^* + g_2^* \\
HE^{ev}(\mathfrak{A}_1) \oplus HE^{ev}(\mathfrak{A}_2) & \leftarrow & HE^{ev}(\mathfrak{B}) & \leftarrow & HE^{od}(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2)
\end{array}
$$

holds.
Proof. We write
\[ C = \{ (h_1, h_2) \in C^\infty(A_1 \oplus C^\infty(A_2) \mid f_1 \circ (h_1)^{(n)}(0) = (-1)^n f_2 \circ (h_2)^{(n)}(0) (n \in \mathbb{Z}_{\geq 0}) \} \]
and define a map \( q : C \to A_1 \#_{\mathcal{A}_2} \) by
\[ q(h_1, h_2) = (h_1(0), h_2(0)). \]
It is easily verified that the following sequence:
\[ 0 \longrightarrow \mathcal{I} \overset{i}{\longrightarrow} C \overset{q}{\longrightarrow} A_1 \#_{\mathcal{A}_2} \longrightarrow 0 \]
is exact, where
\[ \mathcal{I} = \{ (h_1, h_2) \in C \mid h_j(0) = 0 (j = 1, 2) \} \]
and \( i \) is the canonical inclusion. Then there exists a bounded linear section \( s \) of \( q \) defined by
\[ s(a_1, a_2) = ((1 - t)a_1, (1 - t)a_2). \quad ((a_1, a_2) \in A_1 \#_{\mathcal{A}_2}, t \in [0, 1]) \]
Then by Proposition 5.1, we have the following exact diagram:
\[
\begin{array}{ccc}
HE^{ev}(\mathcal{I}) & \longrightarrow & HE^{ev}(C) \\
\uparrow & & \downarrow \\
HE^{od}(\mathcal{I}) & \leftarrow & HE^{od}(C)
\end{array}
\]
Moreover, repeating the argument cited above, \( \mathcal{I} \) is smoothly homotopic to \( S^\infty A_1 \oplus S^\infty A_2 \). More precisely, we define the map \( r : \mathcal{I} \to S^\infty A_1 \oplus S^\infty A_2 \) by
\[ r(h_1, h_2)(t) = (h_1(e^{1-1/t}), h_2(e^{1-1/t})), \quad ((h_1, h_2) \in C) \]
and let \( i : S^\infty A_1 \oplus S^\infty A_2 \to \mathcal{I} \) be the natural inclusion. It follows by the same argument discussed above that since the functions \( t \mapsto h_j(e^{1-1/t}) \) are in \( S^\infty A_j \) (\( j = 1, 2 \)) and using the maps
\[ G_1 : \mathcal{I} \to C^\infty([0, 1], \mathcal{I}) \]
\[ G_2 : S^\infty A_1 \oplus S^\infty A_2 \to C^\infty([0, 1], S^\infty A_1 \oplus S^\infty A_2) \]
defined by
\[ (G_j)_s(h_1, h_2)(t) = (h_1(se^{1-1/t} + (1 - s)t), h_2(se^{1-1/t} + (1 - s)t)) \quad (j = 1, 2), \]
\( \mathcal{I} \) is smoothly homotopic to \( S^\infty A_1 \oplus S^\infty A_2 \). Hence we conclude that
\[ HE^*(\mathcal{I}) \simeq HE^*(S^\infty A_1) \oplus HE^*(S^\infty A_2). \]
Now we define the map $\Psi : C \to S^\infty \mathfrak{B}$ by

$$
\Psi(h_1, h_2)(t) = \begin{cases} 
  f_1 \circ h_1(1 - 2t) & (t \in [0, 1/2]) \\
  f_2 \circ h_2(2t - 1) & (t \in [1/2, 1])
\end{cases}.
$$

We have to verify that it is well-defined. Since $f_1 \circ h_1(0) = f_2 \circ h_2(0)$ by the definition of $C$, it is continuous at $t = 1/2$. For $n = 1$, we compute that

$$
\begin{align*}
\lim_{t \to 1/2^+ 0} \frac{\Psi(h_1, h_2)(t) - \Psi(h_1, h_2)(1/2)}{t - 1/2} &= \lim_{t \to 1/2^+ 0} \frac{f_2 \circ h_2(2t - 1) - f_2 \circ h_2(0)}{t - 1/2} \\
&= f_2 \left( \lim_{t \to 1/2^+ 0} \frac{h_2(2t - 1) - h_2(0)}{t - 1/2} \right) \\
&= 2f_2 \left( \lim_{\varepsilon \to 0^+} \frac{h_2(\varepsilon) - h_2(0)}{\varepsilon} \right) \\
&= 2f_2 \circ (h_2)^{(1)}(0)
\end{align*}
$$

and similarly, we compute that

$$
\begin{align*}
\lim_{t \to 1/2^- 0} \frac{\Psi(h_1, h_2)(t) - \Psi(h_1, h_2)(1/2)}{t - 1/2} &= \lim_{t \to 1/2^- 0} \frac{f_1 \circ h_1(1 - 2t) - f_1 \circ h_1(0)}{t - 1/2} \\
&= -2f_1 \left( \lim_{\varepsilon \to 0^+} \frac{h_1(\varepsilon) - h_1(0)}{\varepsilon} \right) \\
&= -2f_1 \circ (h_1)^{(1)}(0) = 2f_2 \circ (h_2)^{(1)}(0).
\end{align*}
$$

Thus $\Psi(h_1, h_2)$ is differentiable once at $t = 1/2$. Suppose that it is differentiable $n$-times at $t = 1/2$. Here we note that

$$
\Psi^{(n)}(h_1, h_2)(t) = \begin{cases} 
  (-2)^n f_1 \circ h_1^{(n)}(1 - 2t) & (t \in (0, 1/2)) \\
  2^n f_2 \circ h_2^{(n)}(2t - 1) & (t \in (1/2, 1))
\end{cases}
$$

and that

$$
\Psi^{(n)}(h_1, h_2)(1/2) = (-2)^n f_1 \circ (h_1)^{(n)}(0) = 2^n f_2 \circ (h_2)^{(n)}(0)
$$

by our hypothesis of induction. Then we compute that

$$
\begin{align*}
\lim_{t \to 1/2^+ 0} \frac{\Psi(h_1, h_2)^{(n)}(t) - \Psi(h_1, h_2)^{(n)}(1/2)}{t - 1/2} &= 2^n \lim_{t \to 1/2^+ 0} \frac{f_2 \circ h_2^{(n)}(2t - 1) - f_2 \circ (h_2)^{(n)}(0)}{t - 1/2} \\
&= 2^n f_2 \left( \lim_{t \to 1/2^+ 0} \frac{h_2^{(n)}(2t - 1) - (h_2)^{(n)}(0)}{t - 1/2} \right) \\
&= 2^n f_2 \left( \lim_{\varepsilon \to 0^+} \frac{h_2^{(n)}(\varepsilon) - (h_2)^{(n)}(0)}{\varepsilon} \right) \\
&= 2^{n+1} f_2 \circ (h_2)^{(n+1)}(0).
\end{align*}
$$
Alternatively, we compute that
\[
\lim_{t \to 1/2-0} \frac{\Psi(h_1, h_2)^{(n)}(t) - \Psi(h_1, h_2)^{(n)}(1/2)}{t - 1/2} = (-2)^n \lim_{t \to 1/2-0} \frac{f_1 \circ h_1^{(n)}(1 - 2t) - f_1 \circ (h_1)^{(n)}(0)}{t - 1/2}
\]
\[
= (-2)^n \cdot (-2) f_1 \left( \lim_{\varepsilon \to 0^+} \frac{h_1^{(n)}(\varepsilon) - (h_1)^{(n)}(0)}{\varepsilon} \right)
\]
\[
= (-2)^{n+1} f_1 \circ (h_1)^{(n+1)}(0) = 2^{n+1} f_2 \circ (h_2)^{(n+1)}(0).
\]

Therefore, \( \Psi(h_1, h_2) \) is differentiable \((n+1)\)-times for each \((h_1, h_2) \in C\), which ends the process of induction so that \( \Psi \) is well-defined. Since \( f_1 \) and \( f_2 \) are surjective, it is easily verified that \( \Psi \) is surjective. In fact, we canonically can lift them on \( S^\infty \mathcal{A}_j \), which are denoted by \( f_j \) \((j = 1, 2)\). Now given a \( h \in S^\infty \mathcal{B} \), we find \( \tilde{h}_j \in S^\infty \mathcal{A}_j \) with

\[
f_j(\tilde{h}_j(t)) = h(t) \quad (j = 1, 2).
\]

Putting \( h_1(t) = \tilde{h}_1((1 - t)/2) \), \( h_2(t) = \tilde{h}_2((1 + t)/2) \) \((0 \leq t \leq 1)\), we then check that

\[
\Psi(h_1, h_2)(t) = \begin{cases} 
  f_1(h_1(1 - 2t)) & (0 \leq t \leq 1/2) \\
  f_2(h_2(2t - 1)) & (1/2 \leq t \leq 1)
\end{cases}
\]
\[
= \begin{cases} 
  f_1 \left( \tilde{h}_1 \left( 1 - 2 \frac{1 - t}{2} \right) \right) & (0 \leq t \leq 1/2) \\
  f_2 \left( \tilde{h}_2 \left( 2 \frac{1 + t}{2} - 1 \right) \right) & (1/2 \leq t \leq 1)
\end{cases}
\]
\[
= \begin{cases} 
  f_1(\tilde{h}_1(t)) & (0 \leq t \leq 1/2) \\
  f_2(\tilde{h}_2(t)) & (1/2 \leq t \leq 1)
\end{cases}
\]
\[
= h(t),
\]

which implies that \( \Psi \) is surjective. As it is clear that its kernel is \( C^\infty \mathcal{I}_1 \oplus C^\infty \mathcal{I}_2 \), where

\[
\mathcal{J}_j = \text{Ker} f_j \quad (j = 1, 2),
\]

we obtain the following short exact sequence:

\[
0 \longrightarrow C^\infty \mathcal{I}_1 \oplus C^\infty \mathcal{I}_2 \longrightarrow C \longrightarrow \Psi \longrightarrow S^\infty \mathcal{B} \longrightarrow 0.
\]

Since \( \mathcal{B} \) has the metric approximation property, so does \( S^\infty \mathcal{B} \) by Corollary 4.4. Writing \( \mathcal{J} = C^\infty \mathcal{I}_1 \oplus C^\infty \mathcal{I}_2 \), the inverse map

\[
\overline{\Psi}^{-1} : S^\infty \mathcal{B} \to C/\mathcal{J}
\]

of the isomorphism \( \overline{\Psi} \) induced by \( \Psi \) has a bounded lift

\[
\overline{\Psi}^{-1} : S^\infty \mathcal{B} \to C
\]
satisfying $\Psi^{-1} \circ q = \Psi^{-1}$ by Proposition 4.6 since $\Psi$ preserves each seminorms, where $q$ is the quotient map from $C$ onto $C/\mathfrak{J}$. Hence it is verified that $\Psi^{-1}$ is a bounded linear section of $\Psi$ since we compute that

$$\Psi \circ \Psi^{-1} = \Psi \circ q \circ \Psi^{-1} = \Psi \circ \Psi^{-1} = id_{S^{\infty} \mathfrak{B}}.$$ 

Therefore, we apply the above exact sequence (5) to Proposition 5.1 to obtain the following exact diagram:

$$\begin{array}{ccc}
HE^{\text{ev}}(S^{\infty} \mathfrak{B}) & \longrightarrow & HE^{\text{ev}}(C) \longrightarrow \ HE^{\text{ev}}(C^{\infty} \mathfrak{J}_{1} \oplus C^{\infty} \mathfrak{J}_{2}) \\
\uparrow & & \downarrow \\
HE^{\text{od}}(C^{\infty} \mathfrak{J}_{1} \oplus C^{\infty} \mathfrak{J}_{2}) & \longleftarrow & HE^{\text{od}}(C) & \longleftarrow & HE^{\text{od}}(S^{\infty} \mathfrak{B}).
\end{array}$$

Since $HE^{*}(C^{\infty} \mathfrak{J}_{1} \oplus C^{\infty} \mathfrak{J}_{2}) = 0$, we have that

$$\begin{align*}
HE^{\text{ev}}(C) & \simeq HE^{\text{ev}}(S^{\infty} \mathfrak{B}) \simeq HE^{\text{od}}(\mathfrak{B}) \\
HE^{\text{od}}(C) & \simeq HE^{\text{od}}(S^{\infty} \mathfrak{B}) \simeq HE^{\text{ev}}(\mathfrak{B})
\end{align*}$$

by the Bott periodicity (Lemma 5.2).

Summing up, we get the desired exact diagram in what follows:

$$\begin{array}{ccc}
HE^{\text{ev}}(\mathfrak{A}_{1} \# \mathfrak{A}_{2}) & \longrightarrow & HE^{\text{od}}(\mathfrak{B}) \longrightarrow \ HE^{\text{od}}(\mathfrak{A}_{1}) \oplus HE^{\text{od}}(\mathfrak{A}_{2}) \\
\uparrow & & \downarrow \\
HE^{\text{ev}}(\mathfrak{A}_{1}) \oplus HE^{\text{ev}}(\mathfrak{A}_{2}) & \longleftarrow & HE^{\text{ev}}(\mathfrak{B}) & \longleftarrow & HE^{\text{od}}(\mathfrak{A}_{1} \# \mathfrak{A}_{2}).
\end{array}$$

We consider the restriction $\Phi : S^{\infty} \mathfrak{A}_{1} \oplus S^{\infty} \mathfrak{A}_{2} \to S^{\infty} \mathfrak{B}$ of $\Psi$. We see that it is $C^{\infty}$-homotopic to $\Pi : S^{\infty} \mathfrak{A}_{1} \oplus S^{\infty} \mathfrak{A}_{2} \to \mathfrak{B}$ defined by

$$\Pi(h_{1}, h_{2})(t) = -\chi_{0,1/2}(t)(f_{1} \circ h_{1})(t) + \chi_{1,1/2}(t)(f_{2} \circ h_{2})(t)$$

for $(h_{1}, h_{2}) \in S^{\infty} \mathfrak{A}_{1} \oplus S^{\infty} \mathfrak{A}_{2}, t \in [0,1]$. To see this, we note that for a Fréchet continuous homomorphism $f : \mathfrak{A}_{1} \# \mathfrak{A}_{2} \to \mathfrak{B}$, we have

$$\bar{f}^{*} = -f^{*} : HE^{*}(S^{\infty} \mathfrak{B}) \to HE^{*}(\mathfrak{A}_{1} \# \mathfrak{A}_{2})$$

by [14], where $\bar{f} : \mathfrak{A}_{1} \# \mathfrak{A}_{2} \to S^{\infty} \mathfrak{B}$ is the homomorphism defined by

$$\bar{f}(a)(t) = f(a)(1-t) \quad (a \in \mathfrak{A}, t \in [0,1]).$$

Indeed, we prepare the map $\Theta : S^{\infty} \mathfrak{A}_{1} \oplus S^{\infty} \mathfrak{A}_{2} \to C^{\infty}([0,1], S^{\infty} \mathfrak{B})$ defined by

$$\Theta_{s}(h_{1}, h_{2})(t) = \begin{cases} 
  f_{1} \circ h_{1}(1-2t/(1+s)) & (0 \leq t \leq 1/2) \\
  f_{2} \circ h_{2}(2t/(1+s) - (1-s)/(1+s)) & (1/2 \leq t \leq 1),
\end{cases}$$

so that it is a smooth homotopy between $\Psi$ and the homomorphism given by

$$(h_{1}, h_{2}) \mapsto \left(t \mapsto \chi_{0,1/2}(t)(f_{1} \circ h_{1})(1-t) + \chi_{1,1/2}(t)(f_{2} \circ h_{2})(t)\right).$$
Therefore, we have the homotopy equivalence of $\Psi$ and $\Pi$. Considering the following commutative diagram:

\[
\begin{array}{ccc}
HE^*(C) & \longrightarrow & HE^*(S^\infty A_1 \oplus S^\infty A_2) \\
\cong \downarrow \Psi^* & & \downarrow \\
HE^*(S^\infty A_1) & \longrightarrow & HE^*(S^\infty A_1 \oplus S^\infty A_2)
\end{array}
\]

we conclude that the right upper horizontal map and the left lower horizontal map in the diagram (6) are both $\Pi^* = -f_1^* + f_2^*$. Finally, since the following diagram

\[
\begin{array}{ccc}
HE^*(S^\infty A_1) \oplus HE^*(S^\infty A_2) & \longrightarrow & HE^*(A_1 \# A_2) \\
\cong \downarrow & & \\
HE^*(A_1) \oplus HE^*(A_2) & \longrightarrow & HE^*(A_1 \# A_2)
\end{array}
\]

is commutative, the vertical maps in the diagram (6) are both $g_1^* + g_2^*$. This completes the proof. □

6. The Entire Cyclic Cohomology of Noncommutative 3-spheres

In [1], Heegaard-type quantum 3-spheres with 3-parameters are constructed as $C^*$-algebras. With their construction in mind, we define noncommutative 3-spheres in the framework of $F^*$-algebras as follows; given an irrational number $\theta$ with $0 < \theta < 1$, let $T^2_\theta$ be the smooth noncommutative 2-torus with unitary generators $u_\theta, v_\theta$ subject to $u_\theta v_\theta = e^{2\pi i \theta} v_\theta u_\theta$. There exists an isomorphism $\gamma_\theta : T^2_\theta \to T^2_\theta$ satisfying

$$\gamma_\theta(u_{-\theta}) = v_\theta, \quad \gamma_\theta(v_{-\theta}) = u_\theta$$

by their universality. We consider the following two $F^*$-crossed products:

$$(D^2 \times S^1)_\theta = T^\infty \rtimes_{\alpha_\theta} \mathbb{Z}, \quad (D^2 \times S^1)_{-\theta} = T^\infty \rtimes_{\alpha_{-\theta}} \mathbb{Z}$$

defined before. We define two epimorphisms $f_j (j = 1, 2)$ such as

$$f_1 : (D^2 \times S^1)_{-\theta} \to T^2_\theta, \quad f_2 : (D^2 \times S^1)_{-\theta} \to T^2_\theta$$

by $f_1 = \bar{\gamma}_+, f_2 = \gamma_\theta \circ \bar{\gamma}_-$, where $\bar{\gamma}_\pm$ are the epimorphisms from $T^\infty \rtimes_{\alpha_{\pm \theta}} \mathbb{Z}$ onto $C^\infty(T) \rtimes_{\pi_{\pm \theta}} \mathbb{Z} = T^2_\theta$ respectively.

**Definition 6.1.** Given an irrational number $\theta$, the noncommutative 3-sphere $S^3_\theta$ is defined by the fibered product $(D^2 \times S^1)_{-\theta} \# (D^2 \times S^1)_{-\theta}$ of $(D^2 \times S^1)_{-\theta} (D^2 \times S^1)_{-\theta}$ along $(f_1, f_2)$ over $T^2_\theta$.

First of all, we compute the entire cyclic cohomology of $(D^2 \times S^1)_{-\theta}$. We note that the isomorphism $C^\infty(T) \rtimes_{\pi_\theta} \mathbb{Z} \simeq T^2_\theta$ holds and that by Lemma 4.3 in [16], we have

$$HE^*(C^\infty(T) \rtimes_{\pi_\theta} \mathbb{Z}) \simeq HE^*(T^2_\theta) = HP^*(T^2_\theta),$$
where $HP^*$ is the functor of periodic cyclic cohomology. According to Connes [5], we know the generators of $HP^*(T^2_\theta)$ as follows:

$$HP^{ev}(T^2_\theta) = \mathbb{C}[\tau_\theta] \oplus \mathbb{C}[\tau^{(2)}_\theta],$$

$$HP^{od}(T^2_\theta) = \mathbb{C}[\tau^{(1)}_\theta] \oplus \mathbb{C}[\tau^{(2)}_\theta].$$

where $\tau_\theta$ is the unique normalized trace on $T^2_\theta$ and

$$\tau'_\theta(a_0, a_1, a_2) = \tau_\theta(\delta^{(1)}_\theta(a_1)\delta^{(2)}_\theta(a_2) - \delta^{(2)}_\theta(a_1)\delta^{(1)}_\theta(a_2)),$$

$$\tau^{(j)}_\theta(a_0, a_1) = \tau_\theta(\delta^{(j)}_\theta(a_1)) \quad (j = 1, 2),$$

where $\delta^{(j)}_\theta$ are the derivations on $T^2_\theta$ such that

$$\delta^{(1)}_\theta(u_\theta) = 2\pi i u_\theta, \quad \delta^{(1)}_\theta(v_\theta) = 0, \quad \delta^{(2)}_\theta(u_\theta) = 0, \quad \delta^{(2)}_\theta(v_\theta) = 2\pi i v_\theta.$$

**Proposition 6.2.**

$$HE^{ev}((D^2 \times S^1)_\theta) = \mathbb{C}[\tau_\theta \circ \tilde{q}], \quad HE^{od}((D^2 \times S^1)_\theta) = \mathbb{C}[\tau^{(1)}_\theta \circ \tilde{q}].$$

**Proof.** We remember the following short exact sequence:

$$0 \longrightarrow \mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \overset{i}{\longrightarrow} (D^2 \times S^1)_\theta \overset{\tilde{q}}{\longrightarrow} C^\infty(T) \rtimes_{\pi_\theta} \mathbb{Z} \longrightarrow 0$$

appeared in Corollary 3.7. Hence we apply the above exact sequence to Proposition 5.1 to obtain the following exact diagram:

$$HE^{ev}(C^\infty(T) \rtimes_{\pi_\theta} \mathbb{Z}) \overset{\tilde{q}}{\longrightarrow} HE^{ev}((D^2 \times S^1)_\theta) \overset{i}{\longrightarrow} HE^{ev}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z})$$

$$HE^{od}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \overset{i}{\longleftarrow} HE^{od}((D^2 \times S^1)_\theta) \overset{\tilde{q}}{\longleftarrow} HE^{od}(C^\infty(T) \rtimes_{\pi_\theta} \mathbb{Z}).$$

Alternatively, we have by Corollary 3.6 and [12] that

$$HE^e(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \simeq HE^e(\mathbb{K}^\infty \tilde{\otimes}_\gamma C^\infty(T))$$

$$= H^0_{DR}(T; \mathbb{C}),$$

which implies that

$$HE^{ev}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \simeq \mathbb{C}, \quad HE^{od}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \simeq \mathbb{C}.$$

Therefore, we have the following exact diagram:

$$\mathbb{C}^2 \overset{\tilde{q}}{\longrightarrow} HE^{ev}((D^2 \times S^1)_\theta) \overset{i}{\longrightarrow} \mathbb{C}$$

(7)

$$\mathbb{C} \overset{i}{\longleftarrow} HE^{od}((D^2 \times S^1)_\theta) \overset{\tilde{q}}{\longleftarrow} \mathbb{C}^2.$$

We note that there exists an element $[(\tilde{\psi}_{2k+1})] \in HE^{od}((D^2 \times S^1)_\theta)$ with the property that

$$(\tilde{\psi}_{2k+1}) = (\tilde{\psi}, 0, 0, \cdots),$$
and

\[ B\tilde{\psi} = \tau_\theta \circ \tilde{q}, \quad b\tilde{\psi} = 0, \]

where \( b, B = AB_0 \) are the operations defined by Connes [5]. Indeed, we define \( \tilde{\psi} \) by

\[ \tilde{\psi}(x,y) = \tau_\theta \circ \tilde{q}(x\tilde{\delta}_\theta^{(2)}(y)) \quad (x,y \in (D^2 \times S^1)_\theta), \]

where \( \tilde{\delta}_\theta^{(2)} \) is the derivation on \((D^2 \times S^1)_\theta \) induced by

\[ \tilde{\delta}_\theta^{(2)} \left( \sum_{n \in \mathbb{Z}} A_n U^n_\theta \right) = \sum_{n \in \mathbb{Z}} 2\pi i\theta n A_n U^n_\theta \]

for any \( \sum_{n \in \mathbb{Z}} A_n U^n_\theta \in T^\infty[\mathbb{Z}] \). We note that \( \tilde{\delta}_\theta^{(2)} \) is Fréchet continuous since

\[
\left\| \tilde{\delta}_\theta^{(2)} \left( \sum_{n \in \mathbb{Z}} A_n U^n_\theta \right) \right\|_{p,q,r,s} \leq 2\pi \theta \sup_{n \in \mathbb{Z}} (1 + n^2)^p \|2\pi i\theta n A_n\|_{q,r,s} \]

for any \( p,q,r,s \in \mathbb{Z}_{\geq 0} \). In this case, let \( 1 \in T^\infty \times_{\alpha_0} \mathbb{Z} \) be the unit. It is clear that \( \tilde{\delta}_\theta^{(2)}(1) = 0 \). Then by the definition of \( b \) and \( B \), we have that

\[
B\tilde{\psi}(x) = \tilde{\psi}(1,x) + \tilde{\psi}(x,1)
\]

\[
= \tau_\theta \circ \tilde{q}(x\tilde{\delta}_\theta^{(2)}(1)) + \tau_\theta \circ \tilde{q}(1\tilde{\delta}_\theta^{(2)}(x))
\]

\[
= \tau_\theta \circ \tilde{q}(\tilde{\delta}_\theta^{(2)}(x)) \quad (x \in (D^2 \times S^1)_\theta).
\]

We note that for any \( f \in C^\infty(T) \times_{\tau_\theta} \mathbb{Z} \),

\[
\tau_\theta(f) = \int_T f(0)(t) dt.
\]

Thus, we obtain that

\[
\tau_\theta(\tilde{\delta}_\theta(x)) = \int_T \tilde{q}(\tilde{\delta}_\theta(x))(0)(t) dt
\]

\[
= \int_T q(x(0))(t) dt = \tau_\theta \circ \tilde{q}(x)
\]

for any \( x \in (D^2 \times S^1)_\theta \), which implies that \( \tilde{q}^*[\tau_\theta] = 0 \). Hence, \( \ker \tilde{q}^* \neq 0 \) so that the left vertical map of (7) is not 0, therefore, injective.

Similarly, we show that the right vertical map is also injective. Since \( \theta \) is an irrational number, the set \( \{e^{2\pi i\theta n} \in \mathbb{C} \mid n \in \mathbb{Z}\} \) is dense in \( T \). Hence, for all \( r \in [0,1] \), there exists an sequence \( \{N_j\}_j \subset \mathbb{Z} \) with \(|\{\theta N_j\} - r| \to 0 \) as \( j \to \infty \), where

\[
\{x\} = x - \max_{x \geq k, k \in \mathbb{Z}} \{x\} \quad (x \in \mathbb{R}).
\]
We consider the family \( \{U_{\theta N_j}\} \) of unitary operators on \( H^2 \). Since we see that for any \( \xi \in H^2 \),

\[
\|(U_{\theta N_j} - U_{\theta N_k})\xi\|^2_{H^2} = \|(U_{\theta}^{N_j} - U_{\theta}^{N_k})\xi\|^2_{H^2} = \int_T |\xi(e^{2\pi i \theta N_j} t) - \xi(e^{2\pi i \theta N_k} t)|^2 dt = \int_T |\xi(e^{2\pi i \theta (N_j - N_k) t}) - \xi(t)|^2 dt \to 0 \quad (j, k \to \infty)
\]

by the Lebesgue dominated convergence theorem, we obtain that \( \{U_{\theta N_j}\} \) has the strong limit \( U_r \). It is easily seen that \( U_r \xi(t) = \xi(e^{2\pi i r t}) \) \( (\xi \in H^2, t \in T) \). Moreover, we define the operator \( h_\theta \) on \( H^2 \) by

\[
h_\theta \xi(t) = 2\pi \sum_{j=0}^{\infty} \{j\theta\} c_j t^j \left( \xi(t) = \sum_{j=0}^{\infty} c_j t^j \in H^2 \right).
\]

Since \( 0 \leq \{j\theta\} \leq 1 \), it is easily verified that \( h_\theta \) is a bounded self-adjoint positive operator on \( H^2 \) and \( U_{\theta r} = e^{i r h_\theta} \) for \( r \in [0, 1] \) by Stone’s theorem. Taking again a family \( \{N_j\}_{j \in Z_{\geq 0}} \subset Z \) with \( |e^{2\pi i \theta N_j} - e^{2\pi i r}| \to 0 \) as \( j \to \infty \), we have that

\[
\|\alpha_{\theta N_j}(x)\xi - \alpha_{\theta N_k}(x)\xi\|_{H^2} = \|U_{\theta N_j} xU_{-\theta N_j} \xi - U_{\theta N_k} xU_{-\theta N_k} \xi\|_{H^2} \leq \|U_{\theta N_j} x(U_{-\theta N_j} - U_{-\theta N_k})\xi\|_{H^2} + \|(U_{\theta N_j} - U_{\theta N_k}) xU_{-\theta N_k} \xi\|_{H^2} \to 0 \quad (x \in \mathcal{T}^\infty, \xi \in H^2)
\]

since the operation of product is strongly continuous. Therefore, it follows that \( \alpha_r(x) = U_r xU_{-r} \) for \( x \in \mathcal{B}(H^2) \). We write

\[
\tilde{\delta}^{(1)}_\theta (x) = h_\theta x - x h_\theta = \text{ad}(h_\theta)(x) \quad (x \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} Z)
\]

so that

\[
e^{ir\tilde{\delta}^{(1)}_\theta} = e^{i \text{rad}(h_\theta)} = \alpha_{\theta r} \quad (r \in [0, 1]).
\]

We now extend the homomorphism \( \tilde{q} : \mathcal{T}^\infty \rtimes_{\alpha_\theta} Z \to C^\infty(T) \rtimes_{\pi_\theta} Z \) to that from the strong closure of \( \mathcal{T}^\infty \rtimes_{\alpha_\theta} Z \) onto that of \( C^\infty(T) \rtimes_{\pi_\theta} Z \) faithfully acting on \( L^2(T) \) because of the simplicity of \( T^2_\theta = C^\infty(T) \rtimes_{\pi_\theta} Z \), that is, that from \( \mathcal{B}(H^2) \) onto \( L^\infty(T) \rtimes_{\pi_\theta} Z \). We also extend the trace \( \tau_\theta \) on \( T^2_\theta \) to that on \( L^\infty(T) \rtimes_{\pi_\theta} Z \). We use the same letters for their extensions. Then, we have that \( \tilde{q} \circ \tilde{\delta}^{(1)}_\theta = \delta^{(1)}_\theta \circ \tilde{q} \) on \( \mathcal{B}(H^2) \). Under the above preparation, we define the linear functional \( \varphi_0 \) on \( \mathcal{T}^\infty \rtimes_{\alpha_\theta} Z \) by

\[
\varphi_0(a) = -\tau_\theta \circ \tilde{q}(a h_\theta) \quad (a \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} Z).
\]
Lemma 6.3. We have the following equalities:

\[ (b \varphi_0)(a, b) = \varphi_0(ab) - \varphi_0(ba) \]

which means that \[ \tau \circ q = 0 \in HE^{od}(T^\infty \times_{\alpha_0} \mathbb{Z}). \] Hence, we have \( \text{ker} \, \overline{q} \neq 0 \) so that the right vertical map of (7) is also injective.

Summing up, we obtain the following exact diagram:

\[
\begin{array}{ccc}
\mathbb{C}^2 & \xrightarrow{\overline{q}} & HE^{ev}(\{(D^2 \times S^1)_\theta\}) \xrightarrow{0} \mathbb{C} \\
\uparrow & & \downarrow \\
\mathbb{C} & \leftarrow_{0} & HE^{od}(\{(D^2 \times S^1)_\theta\}) \leftarrow_{\overline{q}^*} \mathbb{C}^2.
\end{array}
\]

(8)

to conclude that

\[ HE^*((D^2 \times S^1)_\theta) \simeq \mathbb{C}^2 / \mathbb{C} \simeq \mathbb{C} \]

as required. Moreover, we easily seen that \( \overline{q}^* \neq 0 \). Hence \( \overline{q}^*[\tau_\theta'] = [\tau_\theta' \circ \overline{q}] \) and \( \overline{q}^*[\tau_\theta^{(2)}] = [\tau_\theta^{(2)} \circ \overline{q}] \) are the generators of corresponding entire cyclic cohomology.

We need the following lemma to end up the main result:

**Lemma 6.3.** We have the following equalities:

1. \( \tau_\theta \circ \gamma_\theta = \tau_{-\theta} \) and \( \tau_\theta' \circ \gamma_\theta = -\tau_{-\theta}' \).
2. \( \tau_\theta^{(1)} \circ \gamma_\theta = \tau_{-\theta}^{(2)} \) and \( \tau_\theta^{(2)} \circ \gamma_\theta = \tau_{-\theta}^{(1)} \).

**Proof.** Since \( \tau_\theta \circ \gamma_\theta \) is a normalized trace on \( T^2_\theta \), it follows by uniqueness that \( \tau_\theta \circ \gamma_\theta = \tau_{-\theta} \). We firstly verify that

\[ \delta_\theta^{(1)} \circ \gamma_\theta = \delta_{-\theta}^{(2)}, \quad \delta_\theta^{(2)} \circ \gamma_\theta = \delta_{-\theta}^{(1)}. \]
In fact, it is sufficient to verify these equalities for generators. We compute that
\[
\delta_\theta^{(j)} \circ \gamma_\theta(u_{-\theta}) = \delta_\theta^{(j)}(v_\theta) = \begin{cases} 
0 & (j = 1) \\
2\pi i v_\theta & (j = 2) 
\end{cases}
\]
\[
\delta_\theta^{(j)} \circ \gamma_\theta(v_{-\theta}) = \delta_\theta^{(j)}(u_\theta) = \begin{cases} 
2\pi i u_\theta & (j = 1) \\
0 & (j = 2). 
\end{cases}
\]

We then deduce that
\[
\tau_\theta'((\gamma_\theta(b_0), \gamma_\theta(b_1), \gamma_\theta(b_2))) \\
= \tau_\theta((\gamma_\theta(b_0)((\delta_\theta^{(1)} \circ \gamma_\theta(b_1))\delta_\theta^{(2)} \circ \gamma_\theta(b_2)) - (\delta_\theta^{(2)} \circ \gamma_\theta(b_1))\delta_\theta^{(1)} \gamma_\theta(b_2)))) \\
= \tau_\theta((\gamma_\theta(b_0)(\delta_\theta^{(2)}(b_1)\delta_\theta^{(1)}(b_2) - \delta_\theta^{(1)}(b_1)\delta_\theta^{(2)}(b_2)))) \\
= -\tau_\theta((\gamma_\theta(b_0)(\delta_\theta^{(2)}(b_1)\delta_\theta^{(1)}(b_2) - \delta_\theta^{(1)}(b_1)\delta_\theta^{(2)}(b_2)))) \\
= -\tau_\theta((\gamma_{-\theta}(b_0)(\delta_{-\theta}^{(2)}(b_1)\delta_{-\theta}^{(1)}(b_2) - \delta_{-\theta}^{(1)}(b_1)\delta_{-\theta}^{(2)}(b_2)))) \\
(b_0, b_1, b_2 \in T^2_{-\theta}).
\]
Moreover, for \(b_0, b_1 \in T^2_{-\theta}\), we calculate that
\[
\tau_\theta^{(1)} \circ \gamma_\theta(b_0, b_1) = \tau_\theta^{(1)}((\gamma_\theta(b_0)\delta_\theta^{(1)}(\gamma_\theta(b_1)))) \\
= \tau_\theta^{(1)}((\gamma_\theta(b_0)\delta_\theta^{(2)}(b_1))) \\
= \tau_{-\theta}^{(2)}(b_0, b_1).
\]
Similarly we have that \(\tau_\theta^{(2)} \circ \gamma_\theta = \tau_{-\theta}^{(1)}\).

Under the above preparation, we determine the entire cyclic cohomology of noncommutative 3-spheres \(S^3_\theta\). By Theorem 5.4, we have the following exact diagram:
\[
\begin{array}{ccccccccc}
HE^v(S^3_\theta) & \rightarrow & HE^0(T^2_{\theta}) & \rightarrow & HE^0(T^2_{\theta} \oplus G^1_{-\theta} \oplus G^1_{\theta}) \\
\downarrow g_1^+ + g_2^+ & & \downarrow g_1^+ + g_2^+ & & \downarrow g_1^+ + g_2^+ \\
G^0_{-\theta} \oplus G^0_{\theta} & \rightarrow & HE^v(T^2_{\theta}) & \leftarrow & HE^0(T^2_{\theta}) & \leftarrow & HE^0(S^3_\theta),
\end{array}
\]
where \(G^{0,2}_{\pm \theta} = HE^v((D^2 \times S^1)_{\pm \theta})\), \(G^{1,2}_{\pm \theta} = HE^0((D^2 \times S^1)_{\pm \theta})\) respectively. By Proposition 6.2 and the description in its proof, the above diagram becomes the following one:
\[
\begin{array}{ccccccccc}
HE^v(S^3_\theta) & \rightarrow & \mathbb{C}^2 & \rightarrow & \mathbb{C}^2 \\
\downarrow g_1^+ + g_2^+ & & \downarrow g_1^+ + g_2^+ & & \downarrow g_1^+ + g_2^+ \\
\mathbb{C}^2 & \rightarrow & \mathbb{C}^2 & \rightarrow & HE^0(S^3_\theta),
\end{array}
\]
We describe precisely the maps \(-f_1^* + f_2^*\) to compute \(HE^*(S^3_\theta)\). For the even case, we check the map
\[
-f_1^* + f_2^* : HP^v(T^2_{\theta}) = \mathbb{C}[\tau_\theta] \oplus \mathbb{C}[\tau_\theta'] \rightarrow \mathbb{C}[\tau_\theta' \circ q_\theta] \oplus \mathbb{C}[\tau_{-\theta} \circ q_{-\theta}] = G^0_{\theta} \oplus G^0_{-\theta}.
\]
We have \( f_1^* [\tau_\theta] = [\tau_\theta \circ \tilde{q}] = 0 \) by the calculation in Proposition 6.2 and \( f_1^* [\tau_\theta'] = [\tau_\theta' \circ \tilde{q}] \). Alternatively, it follows from Lemma 6.3 that \( f_2^* [\tau_\theta] = [\tau_\theta \circ \tilde{q}] = 0 \) by the same reason for the case of \( f_1^* \) and that \( f_2^* [\tau_\theta'] = [\tau_\theta' \circ \tilde{q}] = -[\tau_\theta' \circ \tilde{q}] \) by Lemma 6.3. On the other hand, for the odd case, we consider the map

\[-f_1^* + f_2^* : H^\text{odd}(T^2_\theta) = \mathbb{C}[\tau_\theta^{(1)}] \oplus \mathbb{C}[\tau_\theta^{(2)}] \to \mathbb{C}[\tau_\theta^{(2)} \circ \tilde{q}] \oplus \mathbb{C}[\tau_{-\theta}^{(2)} \circ \tilde{q}] = G^1_\theta \oplus G^1_{-\theta}.\]

Similarly we compute that

\[ f_1^* [\tau_\theta^{(2)}] = [\tau_\theta^{(2)} \circ \tilde{q}] \]
\[ f_1^* [\tau_\theta^{(1)}] = [\tau_\theta^{(1)} \circ \tilde{q}] = 0 \]
\[ f_2^* [\tau_\theta^{(1)}] = [\tau_\theta^{(1)} \circ \gamma_\theta \circ \tilde{q}] = [\tau_{-\theta}^{(2)} \circ \tilde{q}] \]

and

\[ f_2^* [\tau_\theta^{(2)}] = [\tau_\theta^{(2)} \circ \gamma_\theta \circ \tilde{q}] = [\tau_{-\theta}^{(1)} \circ \tilde{q}] = 0 \]

by Lemma 6.3.

Therefore, we have the following exact diagram:

\[
\begin{array}{cccc}
HE^{\text{ev}}(S^3_\theta) & \longrightarrow & \mathbb{C}^2 & \xrightarrow{(\lambda, \mu) \mapsto (-\mu, \lambda)} & \mathbb{C}^2 \\
\uparrow & & \downarrow 0 \\
\mathbb{C}^2 & \leftarrow & \mathbb{C}^2 & \xleftarrow{(\lambda, \mu) \mapsto (-\mu, -\mu)} & HE^{\text{odd}}(S^3_\theta),
\end{array}
\]

by which we conclude that

\[
HE^{\text{ev}}(S^3_\theta) \simeq \text{coker}\{ \mathbb{C} \oplus \mathbb{C} \ni (\lambda, \mu) \mapsto (-\mu, -\mu) \in \mathbb{C} \oplus \mathbb{C} \} \simeq \mathbb{C},
\]

\[
HE^{\text{odd}}(S^3_\theta) \simeq \text{ker}\{ \mathbb{C} \oplus \mathbb{C} \ni (\lambda, \mu) \mapsto (-\mu, -\mu) \in \mathbb{C} \oplus \mathbb{C} \} \simeq \mathbb{C}.
\]

This completes our computation of the entire cyclic cohomology of noncommutative 3-spheres.

**Theorem 6.4.** The entire cyclic cohomology of noncommutative 3-spheres is isomorphic to the d’Rham homology of the ordinary 3-spheres with complex coefficients.

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