Equiangular lines and regular graphs

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Abstract

In 1973, Lemmens and Seidel posed the problem of determining $N^R_\alpha(r)$, the maximum number of equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$. Recently, this question has been almost completely settled in the case where $r$ is large relative to $1/\alpha$, with the approach relying on Ramsey’s theorem. In this paper, we use orthogonal projections of matrices with respect to the Frobenius inner product in order to overcome this limitation, thereby obtaining upper bounds on $N^R_\alpha(r)$ which significantly improve on the only previously known universal bound of Glazyrin and Yu, as well as taking an important step towards determining $N^R_\alpha(r)$ exactly for all $r, \alpha$. In particular, our results imply that $N_\alpha(r) = \Theta(r)$ for $\alpha \geq \Omega(1/r^{1/5})$.

Our arguments rely on a new geometric inequality for equiangular lines in $\mathbb{R}^r$ which is tight when the number of lines meets the absolute bound $(r+1)^2$. Moreover, using the connection to graphs, we obtain lower bounds on the second eigenvalue of the adjacency matrix of a regular graph which are tight for strongly regular graphs corresponding to $(r+1)$ equiangular lines in $\mathbb{R}^r$. Our results only require that the spectral gap is less than half the number of vertices and can therefore be seen as an extension of the Alon-Boppana theorem to dense graphs.

Generalizing to $\mathbb{C}$, we also obtain the first universal bound on $N^C_\alpha(r)$, the maximum number of complex equiangular lines in $\mathbb{C}^r$ with common Hermitian angle $\arccos(\alpha)$. In particular, we prove an inequality for complex equiangular lines in $\mathbb{C}^r$ which is tight if the number of lines meets the absolute bound $r^2$ and may be of independent interest in quantum theory. Additionally, we use our projection method to obtain an improvement to Welch’s bound.

1 Introduction

Given $n$ lines $l_1, \ldots, l_n$ passing through the origin in $\mathbb{R}^r$, we say that they are equiangular if for all $i \neq j$, the acute angle between $l_i$ and $l_j$ is the same. If we choose a unit vector $v_i$ along each line $l_i$, an equivalent definition is that there exists $\alpha \in [0, 1)$ such that $|\langle v_i, v_j \rangle| = \alpha$ for all $i \neq j$, in which case $\arccos(\alpha)$ is the common angle between any pair of lines. Moreover, this latter definition extends naturally to complex lines in $\mathbb{C}^r$. Large sets of equiangular lines arise naturally in a wide variety of different areas including elliptic geometry [3, 32], the theory of polytopes [19], frame theory [14, 39, 41], and perhaps more surprisingly, quantum theory [15, 16, 31, 32, 60, 64, 71] with connections to algebraic number theory and Hilbert’s twelfth problem [3, 4, 7]. Moreover, Godsil and Royle [34] consider the question of determining the maximum number of equiangular lines in $\mathbb{R}^r$ to be one of the founding problems of algebraic graph theory.

In this paper, we use orthogonal projections of matrices with respect to the Frobenius inner product in order to obtain bounds on various parameters of a collection of equiangular lines, such as eigenvalues/eigenvectors of a corresponding Gram matrix or the maximum degree of a corresponding graph. Moreover, our bounds are tight for maximum-sized collections of lines and
they allow us to obtain strong upper bounds on the number of real or complex equiangular lines in a given dimension with a given angle. Using the connection between equiangular lines and graphs, we also obtain tight bounds on eigenvalues of the adjacency matrix of a regular graph, which can be seen as an extension of the Alon-Boppana theorem to dense graphs. In addition, we use orthogonal projection to obtain a strengthening of the Welch bound from coding theory.

This paper is organized as follows. In Section 1.1 we introduce and provide some background. Section 1.2 and 1.3 respectively clarify and extend the main results on real equiangular lines, complex equiangular lines, and eigenvalues of regular graphs. In Section 1.4 we examine relevant notation and definitions that will be used throughout the paper. Finally, in Section 2 we apply our new inequalities to derive an improved version of our geometric inequality and apply it to obtain bounds on the second eigenvalue of a regular graph. In Section 3 and 4 we apply our new inequalities to obtain proofs of our main results on real and complex equiangular lines respectively. In Section 5, we derive an improved version of our geometric inequality and apply it to obtain bounds on the second eigenvalue of a regular graph. Finally, in Section 6 we give some concluding remarks and directions for future research.

1.1 Real equiangular lines

Define $N^R(r)$ to be the maximum number of equiangular lines in $\mathbb{R}^r$. For anyone who has cut a pizza pie into 6 equal slices using 3 cuts, it is not hard to see that $N^R(2) = 3$. In $\mathbb{R}^3$ and $\mathbb{R}^4$, the question was first studied by Haantjes [37] in 1948, who showed that $N^R(3) = N^R(4) = 6$, with an optimal configuration coming from the 6 diagonals of a regular icosahedron. The question of determining $N^R(r)$ for an arbitrary $r$ was first formally posed in 1966 by van Lint and Seidel [54]. In 1973, Gerzon (see [52]) proved the absolute bound $N^R(r) \leq R(r, 2)$, and it is known that this bound is met for $r = 2, 3, 7, 23$, see e.g. [52]. Surprisingly, we do not know if there are any other $r$ for which this is the case. Moreover, the order of magnitude of $N^R(r)$ was not even known until 2000, when de Caen [12] gave a construction showing that $N^R(r) \geq \frac{2}{3}(r+1)^2$ for all $r$ of the form $3 \cdot 2^{2^t-1} - 1$ for $t \in \mathbb{N}$.

In order to get a better understanding of $N^R(r)$, Lemmens and Seidel [52] defined $N^R_\alpha(r)$ to be the maximum number of equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$, and proved the relative bound, $N^R_\alpha(r) \leq \frac{1-\alpha^2}{1-\alpha^r} r$ for all $\alpha < 1/\sqrt{7}$. They were particularly interested in the case where $1/\alpha$ is an odd integer, due to a result of Neumann (see [52]), who showed that if $1/\alpha$ is not of this form, then $N^R_\alpha(r) \leq 2r$. They showed that $N^R_{1/3}(r) = 2r - 2$ for $r \geq 15$ and conjectured that $N^R_{1/2}(r) = \lfloor 3(r-1)/2 \rfloor$ for $r$ sufficiently large, which was later confirmed by Neumaier [57], see also [32].

Apart from results for small dimensions (see e.g. [51] for more information), there wasn’t much progress made on these questions until recently, when Bukh [10] showed that $N^R_\alpha(r) \leq 2O(1/\alpha^2) r$, in particular implying that the $N^R_\alpha(r)$ is linear in $r$ when $\alpha$ is fixed, as well as conjecturing the asymptotic value of $N^R_\alpha(r)$ as $r \to \infty$ whenever $1/\alpha$ is an odd integer. Together with Dräxler, Keevash, and Sudakov [5], we developed an approach which allowed us to show that $N^R_\alpha(r) \leq 2r - 2$ for all $\alpha \gg 1/\sqrt{\log r}$, with equality if and only if $\alpha = 1/3$. Some of the ideas we proposed were further clarified and extended by Jiang and Polyanskii [46], who also proposed a generalization of Bukh’s conjecture to any $\alpha$. Using the methods developed in [5], together with a new bound on the maximum multiplicity of the second eigenvalue of a graph, Jiang, Tidor, Yao, Zhang, and Zhao [47] were able to verify this conjecture in a strong form, showing that $N^R_\alpha(r) = \left\lfloor \frac{k}{k-1} (r-1) \right\rfloor$ for all $\alpha \geq Ck/\log \log r$, where $C$ is a constant and $k$ is the least integer such that there exists a
graph on $k$ vertices with spectral radius $\frac{1}{2}(1/\alpha - 1)$.

We note that all of these recent results crucially rely on using Ramsey’s theorem in order to bound the maximum degree of a corresponding graph, and the bounds one gets are exponential in $1/\alpha$ so that such methods cannot work when $\alpha$ is much smaller than $1/\log r$. Indeed, the best bounds in this regime are obtained using linear/semidefinite programming and Gegenbauer polynomials. For $\frac{1}{\sqrt{r+2}} \leq \alpha \leq \sqrt{\frac{3}{r+16}}$ and $\alpha \leq 1/3$, Yu [70] showed that $N^R_\alpha(r) \leq \frac{1}{2} \left( \frac{1}{\alpha^2} - 1 \right) \left( \frac{1}{\alpha^2} - 2 \right)$, and since a family of equiangular lines meeting the absolute bound $\binom{r+1}{2}$ must satisfy $\alpha^2 = \frac{1}{r+2}$ (see [52]), Yu’s bound can be seen as a refinement of the absolute bound when $\alpha$ is near $1/\sqrt{r}$. For the remaining regime $\sqrt{\frac{3}{r+16}} < \alpha < 1$, the only known bound is due to Glazyrin and Yu [33], who proved that $N^R_\alpha(r) \leq \left( \frac{2}{\alpha^2} + \frac{1}{2} \right) r + 2$ for all $\alpha \leq \frac{1}{3}$. When $\alpha \leq O(1/\sqrt{r})$, this bound gives the correct order of magnitude $O(r^2)$. However, when $\alpha \gg 1/\sqrt{\log r}$ it is evident from the previous paragraph that Glazyrin and Yu’s bound is off by a factor of $1/\alpha^2$.

In this paper, we use orthogonal projections of symmetric matrices with respect to the Frobenius inner product in order to overcome the limitations of the Ramsey-theoretic approach and obtain a bound on the maximum degree of a corresponding graph which is $O(1/\alpha^3)$, thereby allowing us to extend the ideas of [8] to the regime $\alpha \leq O(1/\sqrt{\log r})$. Our projection method allows us to prove a new geometric inequality for real equiangular lines which is tight whenever the absolute bound is met, and as a consequence, we obtain new upper bounds on $N^R_\alpha(r)$, significantly improving on the bound of Glazyrin and Yu, as well as effectively bridging the gap between $O(r^2)$ (when $\alpha \leq O(1/\sqrt{r})$) and $O(r)$ (when $\alpha \gg 1/\sqrt{\log r}$). The first bound we give below is obtained by studying the eigenvalues of the Gram matrix of a corresponding set of unit vectors.

**Theorem 1.** For all $r \in \mathbb{N}$, $\alpha \in (0, 1)$ we have

$$N^R_\alpha(r) \leq \frac{\sqrt{r}}{2\alpha^2} + \frac{(1+\alpha)r}{2\alpha}.$$  

**Remark.** Note that the terms $\frac{\sqrt{r}}{2\alpha^2}$ and $\frac{(1+\alpha)r}{2\alpha}$ in the above bound are both on the order of $\Theta(1/\alpha^3)$ when $\alpha = \Theta(1/r^{1/4})$ and moreover if $\alpha$ is far away from $1/r^{1/4}$, then only one term dominates. In particular, if $\alpha = 1/\sqrt{r} + \frac{1}{2}$ then we have $N^R_\alpha(r) \leq (1 + o(1)) \frac{\sqrt{r}}{2}$, asymptotically matching the absolute bound $\binom{r+1}{2}$.

The fact that the behavior of the upper bound in **Theorem 1** changes when $\alpha$ is near $1/r^{1/4}$ is intriguing, as it suggests that the behavior of $N^R_\alpha(r)$ may also change when $\alpha$ is near $1/r^{1/4}$. While the above bound improves on that of Glazyrin and Yu, it is only $O(r/\alpha)$ when $\alpha$ is large relative to $1/r$, which is still off by a factor of $1/\alpha$. Using our bound on the maximum degree of a corresponding graph, together with an Alon-Boppana type theorem, we obtain the following bound which is indeed $O(r)$ when $\alpha > \Omega(1/r^{1/5})$, as claimed.

**Theorem 2.** For all $r \in \mathbb{N}$, $\alpha \in (0, 1)$ we have

$$N^R_\alpha(r) \leq \max \left( \frac{2}{\alpha^3} + \frac{2}{\alpha^3(1-\alpha)^2}, \left( 2 + \frac{8\alpha^2}{(1-\alpha)^2} \right)(r+1) \right).$$

In particular, if $\alpha \to 0$ and $r \to \infty$, then $N^R_\alpha(r) \leq (1 + o(1)) \max \left( \frac{2}{\alpha^3}, 2r \right)$.

Although we do not know if $1/\alpha^5$ is the correct order of magnitude of $N^R_\alpha(r)$ when $\alpha = \Theta(1/r^{1/4})$, it is interesting to note that, despite having seemingly different proofs, **Theorem 1** and **Theorem 2** both give the bound of $O(1/\alpha^5)$ in this regime. Also, note that the two terms
in the bound of Theorem 2 are on the same order when $\alpha$ is on the order of $1/r^{1/5}$, and when $\alpha$ is significantly larger, we may apply a stronger Alon-Boppana theorem in order to obtain the following improved bound, in the same way Jiang and Polanski\textsuperscript{[40]} refined the approach of \textsuperscript{[3]}. 

**Theorem 3.** Let $r \in \mathbb{N}$, $\alpha \in (0,1)$. If $\alpha \to 0$, $r \to \infty$ and $\alpha \gg 1/r^{1/(2q+1)}$ for some integer $q \geq 2$, then we have

$$N^r_{\alpha}(r) \leq (1 + o(1)) \left(1 + \frac{1}{4\cos^2\left(\frac{\pi}{r+2}\right)}\right) r.$$ 

In particular, if $\log(r)/\log(1/\alpha) \to \infty$ then $N^r_{\alpha}(r) \leq (1 + o(1)) \frac{\sqrt{r}}{2r}$.

Regarding lower bounds, note that by taking a standard basis in $\mathbb{R}^r$ and rotating each vector the same amount towards the all ones vector, we can obtain $r$ unit vectors with pairwise inner product $\alpha$ for any $\alpha \in [0,1)$. Therefore we always have $N^r_{\alpha}(r) \geq r$, and combining this with Theorem 2 we conclude that $N^r_{\alpha}(r) = \Theta(r)$ for all $\alpha \geq \Omega(1/r^{1/5})$. Moreover, we note that all known constructions of $\Omega(r^2)$ lines in $\mathbb{R}^r$ have a common angle of $\arccos(\alpha)$ where $\alpha = \Theta(1/\sqrt{r})$. In particular, de Caen’s construction \textsuperscript{[12]} works for all dimensions $r_t = 3 \cdot 2^{2t-1} - 1$ where $t \in \mathbb{N}$. Therefore, there exist constants $C, D > 0$ and a decreasing sequence $(\alpha_t)_{t \in \mathbb{N}}$ satisfying $\alpha_t \geq \frac{C}{r_t}$ such that $N^r_{\alpha_t}(r_t) \geq \frac{D}{\alpha_t}(r_t + 1)^2 \geq \frac{D}{\alpha_t^2}$. Since $N^r_{\alpha}(r)$ is non-decreasing as a function of $r$, we conclude that

$$N^r_{\alpha_t}(r) \geq \max\left(\frac{D}{\alpha_t^2}, r\right)$$

for all $t, r \in \mathbb{N}$ such that $\alpha_t \geq C/\sqrt{r}$.

### 1.2 Complex equiangular lines

For a pair of 1-dimensional subspaces $U, V \subseteq \mathbb{C}^r$, i.e. complex lines through the origin, the quantity $\arccos|\langle u, v \rangle|$ is the same for any choice of unit vectors $u \in U, v \in V$, and so we may define this quantity to be the *Hermitian angle* between $U$ and $V$, see e.g. \textsuperscript{[62]}. Given $n$ complex lines $l_1, \ldots, l_n$ passing through the origin in $\mathbb{C}^r$, we say they are *equiangular* if there exists $\theta \in [0, \pi/2]$ such that the Hermitian angle between $l_i$ and $l_j$ is $\theta$ for all $i \neq j$. In this case, we call $\theta$ the *common Hermitian angle*. As in the real case, we define $N^C_{\alpha}(r)$ to be the maximum number of complex equiangular lines in $\mathbb{C}^r$ with common Hermitian angle $\arccos(\alpha)$ and $N^C_{\alpha}(r) = \max_{\alpha \in [0,1]} N^C_{\alpha}(r)$.

The earliest results on complex equiangular lines go back to the 1975 work of Delsarte, Goethals, and Seidel \textsuperscript{[22]}. Analogous to the real case, they proved the absolute bound $N^C_{\alpha}(r) \leq r^2$ and gave matching lower bound constructions in $\mathbb{C}^2$ and $\mathbb{C}^3$. Zauner \textsuperscript{[71]} was the first to make the connection between complex equiangular lines and quantum theory, as well as showing that $N^C_{\alpha}(r) = r^2$ for $r \leq 5$. This lead him to conjecture that $N^C_{\alpha}(r) = r^2$ for all $r \in \mathbb{N}$, and moreover that such constructions can be obtained as the orbit of some vector under the action of a Weyl-Heisenberg group. Unlike the real case, this conjecture has turned out to be true for 102 different values of the dimension $r$, including all $r \leq 40$ and as large as $r = 1299$, see \textsuperscript{[64]} and the survey paper of Fuchs, Hoang, and Stacey \textsuperscript{[32]} for more information.

Interest in collections of $r^2$ complex equiangular lines in $\mathbb{C}^r$ arose independently in quantum mechanics due to the work of Renes, Blume-Kohout, Scott, and Caves \textsuperscript{[60]}, where such objects have come to be known as symmetric, informationally complete, positive operator-valued measures, or SIC-POVMs/SICs for short. SICs turn out to be quite remarkable objects, due to the fact that they have applications in quantum state tomography \textsuperscript{[15]} and quantum cryptography \textsuperscript{[16]}, as well as being candidates for a “standard quantum measurement” in the foundations of quantum
mechanics, most notably in QBism \[31\]. Moreover, they also have applications in high-precision radar and speech recognition and it has even been suggested that a SIC in \(\mathbb{C}^{2048}\) is worth patenting (see the last paragraph of \[32\]). Finally, it is very intriguing to note that Zauner’s conjecture is related to algebraic number theory with a connection to Hilbert’s twelfth problem \[2,4,6\].

Despite all of the research on \(N_{\alpha}(r)\), very little is known about \(N_{\alpha}^C(r)\). Delsarte, Goethels, and Seidel \[22\] prove the analogous relative bound \(N_{\alpha}^C(r) \leq \frac{1}{2\alpha^2} - r\) for all \(\alpha < 1/\sqrt{r}\), and to the best of our knowledge, there aren’t any general bounds known in the remaining regime \(\alpha \geq 1/\sqrt{r}\). Our projection method generalizes to the complex setting, allowing us to obtain the first universal bound approach to obtain the first generalization of the Alon-Boppana theorem which holds for dense graphs, see e.g. \[44\]. Moreover, Alon conjectured and Friedman \[29\] proved that for any \(C\) lines in \(\mathbb{N}\), \(\alpha\) not only deep and interesting on its own, but also has extensive applications in mathematics and computer science \[44,50\], as well as applications to other sciences, see e.g. \[21,66\].

**Theorem 4.** For all \(r \in \mathbb{N}\), \(\alpha \in (0,1)\) we have

\[
N_{\alpha}^C(r) \leq \frac{\sqrt{7}}{\alpha^2} + \frac{r}{\alpha}.
\]

### 1.3 Eigenvalues of regular graphs

Let \(G\) be a \(k\)-regular graph whose adjacency matrix \(A\) has second and last eigenvalue \(\lambda_2\) and \(\lambda_n\), with \(m(\lambda_2)\) denoting the multiplicity of \(\lambda_2\). There is by now a lot of literature on how the size of \(\lambda_2\) (and \(\lambda_n\)) relate to other properties of \(G\), see \[11,20,21\] for more information. Inspired by the fact that \(G\) is connected if and only if the spectral gap \(k - \lambda_2 > 0\), Fiedler \[27\] studied the spectral gap as an algebraic measure of the connectivity of a graph and considered how it relates to the usual combinatorial notions of vertex and edge connectivity. It is also known that the spectral gap and \(\max(\lambda_2, -\lambda_n)\) are related to the mixing time of random walks on a graph and that this generalises to other Markov chains, see e.g. \[21,25\]. Moreover, Alon and Milman \[1,2\] showed that the spectral gap as well as \(\max(\lambda_2, -\lambda_n)\) are closely related to combinatorial notions of graph expansion, which can also be interpreted as discrete isoperimetric inequalities. In particular, Chung, Graham, and Wilson \[17\] showed that \(\max(\lambda_2, -\lambda_n)\) determines how pseudo-random a graph is. We remark that the theory of graphs having good expansion properties is not only deep and interesting on its own, but also has extensive applications in mathematics and computer science \[44,50\], as well as applications to other sciences, see e.g. \[21,66\].

Given the above, it interesting to determine how large the spectral gap \(k - \lambda_2\) can be, or equivalently, how small \(\lambda_2\) can be as a function of \(k\). If \(G\) has diameter \(D \geq 4\), the well-known Alon-Boppana theorem \[58\] gives the lower bound \(\lambda_2 \geq 2\sqrt{k - T} - (2\sqrt{k - T} - 1)/|D/2|\). As \(D \rightarrow \infty\), this bound approaches \(2\sqrt{k - 1}\), which is known to be tight due to the existence of so-called Ramanujan graphs, see e.g. \[44\]. Moreover, Alon conjectured and Friedman \[29\] proved that for any \(\varepsilon > 0\), “most” \(k\)-regular graphs \(G\) on sufficiently many vertices will satisfies \(\lambda_2(G) \leq 2\sqrt{k - 1} + \varepsilon\). Note that even though the Alon-Boppana theorem has been improved and generalized to irregular graphs (see e.g. \[28,43,45\]), it seems that all previously known results require the assumption that the graph is sufficiently sparse, i.e. having diameter \(D \geq 4\).

In this paper, we make use of the connection between graphs and equiangular lines in order to obtain the first generalization of the Alon-Boppana theorem which holds for dense \(k\)-regular graphs with no assumption on the diameter. We only require the following mild assumption on
the spectral gap of $G$,

$$k - \lambda_2 < \frac{n}{2}. \tag{1}$$

Note that any $k$-regular graph $G$ with $k < n/2$ must have $\lambda_2(G) \geq 0$ (see Section 5.1 for more information), and so it must satisfy (1). Additionally, the complement of any $k$-regular graph is $(n - 1 - k)$-regular, so if $k \geq n/2$ then (1) holds for the complement graph. Finally, the bounds we obtain are tight for any strongly regular graph corresponding to a family of equiangular lines meeting the absolute bound in $\mathbb{R}^r$.

**Theorem 5.** Let $k, n \in \mathbb{N}$ and let $G$ be a $k$-regular graph. Let $A$ be its adjacency matrix and let $\lambda_2 = \lambda_2(A), \lambda_n = \lambda_n(A)$ be its second and last eigenvalue, respectively. If the spectral gap satisfies $k - \lambda_2 < n/2$, then we have

$$2 \left( k - \frac{(k - \lambda_2)^2}{n} \right) \leq \frac{\lambda_2(\lambda_2 + 1)(2\lambda_2 + 1)}{1 - \frac{2(k - \lambda_2)}{n}} - \lambda_2(3\lambda_2 + 1),$$

and

$$-\lambda_n \leq \frac{\lambda_2(\lambda_2 + 1)}{1 - \frac{2(k - \lambda_2)}{n}} - \lambda_2,$$

with equality in both whenever $n = \binom{n - m(\lambda_2) + 1}{2} - 1$, i.e. when $G$ is a strongly regular graph corresponding to a family of $\binom{r + 1}{2}$ equiangular lines in $\mathbb{R}^r$ for $r = n - m(\lambda_2)$.

**Remark.** $k - \lambda_2 < n/2$ implies that $2 \left( k - \frac{(k - \lambda_2)^2}{n} \right) > k + \lambda_2$, so that the first bound of Theorem 5 gives an upper bound on the degree $k$.

If we further assume that the spectral gap is significantly smaller than $n/2$, we immediately obtain the following corollary.

**Corollary 6.** Let $k, n \in \mathbb{N}$ and let $G$ be a $k$-regular graph. Let $A = A(G)$ be its adjacency matrix and let $\lambda_2 = \lambda_2(A), \lambda_n = \lambda_n(A)$ be its second and last eigenvalue, respectively. If the spectral gap satisfies $k - \lambda_2 \leq (1 - \varepsilon)\frac{n}{2}$ for some constant $\varepsilon > 0$, then we have

$$\lambda_2 \geq \Omega \left( \max \left( k^{1/3}, \sqrt{-\lambda_n} \right) \right).$$

In particular, if $G$ is bipartite then $\lambda_2 \geq \Omega(\sqrt{k})$. Moreover, if the spectral gap $k - \lambda_2 = o(n)$ then

$$\lambda_2 \geq (1 - o(1)) \max \left( k^{1/3}, \sqrt{-\lambda_n} \right),$$

so that if $G$ is bipartite, we have $\lambda_2 \geq (1 - o(1))\sqrt{k}$.

Corollary 6 raises the question of whether there actually exist $k$-regular graphs satisfying $\lambda_2 \leq O(k^{1/3})$ or $\lambda_2 \leq O(\sqrt{-\lambda_n})$. Interestingly, Taylor [62] constructed a family of strongly regular graphs based on the projective unitary group $\text{PU}(2, q^2)$ which satisfy both of these conditions. In particular, for any odd prime power $q$, Taylor’s graph has $n = q^3$ vertices, degree $k = \frac{1}{2}(q^2 + 1)(q - 1)$, second eigenvalue $\lambda_2 = \frac{1}{2}(q - 1)$, and smallest eigenvalue $\lambda_n = -\frac{1}{2}(q^2 + 1)$. However, this construction has the downside that the spectral gap $k - \lambda_2 = \frac{1}{2}q^2(q - 1) = (1 - \frac{1}{2mq})\frac{n}{2}$ is very close to $n/2$, so that we cannot apply Corollary 6 to it and the bounds of Theorem 5 are far from tight for such graphs.
1.4 Notation and definitions

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}$ denote the field of real numbers, and $\mathbb{C}$ denote the field of complex numbers. We let $S^1(\alpha) = \{z \in \mathbb{C} : |z| = \alpha\}$ denote the circle of radius $\alpha$ centered at 0 in the complex plane. For all $n \in \mathbb{N}$, we define $[n] = \{1, \ldots, n\}$. Furthermore, for any quantities $f, g$ we write $f \leq O(g)$, respectively $f \geq \Omega(g)$ if $f, g$ are implicitly assumed to be monotone functions of an increasing parameter $t \in \mathbb{N}$ and there exist $C > 0, t_0 \in \mathbb{N}$ such that $f(t) \leq Cg(t)$, respectively $f(t) \geq Cg(t)$ for all $t \geq t_0$. Moreover, we write $f = \Theta(g)$ if $f \leq O(g)$ and $f \geq \Omega(g)$. Also, we write $f = o(g)$ or $f \ll g$ whenever $f(t)/g(t) \to 0$ as $t \to \infty$.

Linear algebra

For a field $F$ and a fixed $n \in \mathbb{N}$, we let $I$ denote the $n \times n$ identity matrix, let $J$ denote the $n \times n$ all ones matrix, let $1 \in \mathbb{F}^n$ denote the all ones vector, and for $i \in [n]$, let $e_i \in \mathbb{F}^n$ denote the $i$th standard basis vector. We define $\mathbb{F}^{n \times m}$ to be the space of all $n \times m$ matrices over $\mathbb{F}$, and we identify a vector $v \in \mathbb{F}^n$ with an $n \times 1$ column matrix in $\mathbb{F}^{n \times 1}$ for the purpose of computations.

Now fix $M \in \mathbb{F}^{n \times m}$. Given a function $f : \mathbb{F} \to \mathbb{F}$, we define $f(M) \in \mathbb{F}^{n \times m}$ to be the matrix obtained from $M$ by entry-wise application of $f$, so that $f(M)_{ij} = f(M_{ij})$. Moreover, we let $M^\top$ and $M^*$ denote the $m \times n$ transpose matrix and adjoint matrix, respectively. We say that $M$ is symmetric (Hermitian) if it satisfies $M = M^\top = M^*$. We let $\mathcal{S}_n = \{M \in \mathbb{R}^{n \times n} : M^\top = M\}$ denote the space of all $n \times n$ real symmetric matrices. Given a pair of vectors $u, v \in \mathbb{C}^n$, we let $\langle u, v \rangle = u^*v$ denote the standard inner product of $u$ and $v$.

Now let $M \in \mathbb{C}^{n \times n}$ be symmetric (Hermitian). By the spectral theorem for symmetric (Hermitian) matrices, there exist real eigenvalues $\lambda_i = \lambda_i(M)$ for $i \in [n]$ such that $\lambda_1 \geq \ldots \geq \lambda_n$ and an orthonormal basis of corresponding eigenvectors $u_1, \ldots, u_n \in \mathbb{R}^n$ ($\mathbb{C}^n$) such that $M = \sum_{i=1}^n \lambda_i u_i u_i^\top$, and so we define $m_M(\lambda) = |\{j : \lambda_j = \lambda\}|$ to be the multiplicity of $\lambda$. Furthermore, we say that $M$ is positive semidefinite if $\lambda_n \geq 0$. Finally, we define the Moore-Penrose generalized inverse of $M$ to be $M^\dagger = \sum_{\lambda_i > 0} \frac{1}{\lambda_i} u_i u_i^\top$.

Finite-dimensional Hilbert spaces

Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $V$ be a finite-dimensional vector space over $\mathbb{F}$. We say that $V$ is a Hilbert space\(^1\) if there exists a map $\langle \cdot, \cdot \rangle_V : V \times V \to \mathbb{F}$ called the inner product, such that

1. $\langle u, \cdot \rangle_V : V \to \mathbb{F}$ is linear for all $u \in V$,
2. $\langle u, v \rangle_V = \langle v, u \rangle_V$ for all $u, v \in V$,
3. $\langle u, u \rangle_V = 0$ implies $u = 0$ for all $u \in V$.

Given a Hilbert space $V$ with inner product $\langle \cdot, \cdot \rangle_V$, we let $||v||_V = \sqrt{\langle v, v \rangle_V}$ denote the corresponding norm. Given a finite set of vectors $\mathcal{C} = \{v_1, \ldots, v_n\} \subseteq V$, we let $M_{\mathcal{C}} \in \mathbb{F}^{n \times n}$ denote their Gram matrix (with respect to the inner product $\langle \cdot, \cdot \rangle_V$), defined by $(M_{\mathcal{C}})_{ij} = \langle v_i, v_j \rangle_V$. Moreover, for any subset $S$ of the unit ball in $\mathbb{F}$, we say that a collection of unit vectors $\mathcal{C}$ is a spherical $S$-code if $\langle u, v \rangle_V \in S$ for all $u \neq v \in \mathcal{C}$.

For any $u \in V$, we let $u^\# : V \to \mathbb{F}$ denote the adjoint map defined by $u^\#v = \langle u, v \rangle_V$ for all $v \in V$. More generally, if $U$ and $V$ are finite-dimensional Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_U$ and $\langle \cdot, \cdot \rangle_V$, and $L : U \to V$ is a linear map, then via the finite-dimensional Riesz representation

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\(^1\)The definition we give here is that of an inner product space. However, any finite-dimensional inner product space is complete and therefore, automatically a Hilbert space.
order to represent the unordered pair. We let $G$ and $L$ and let $d$ vertex set $V$.

For convenience, whenever $u$ and $v$ are vertices of a graph $G$, we write $uv$ instead of $\{u, v\}$ in order to represent the unordered pair. We let $G^C$ denote the complement of $G$, i.e. the graph with vertex set $V(G)$ such that $uv \in E(G^C)$ if and only if $uv \notin E(G)$. If $H$ is such that $V(H) \subseteq V(G)$ and $E(H) \subseteq V(G)$, then $H$ is called a subgraph of $G$ and we equivalently say that $H \subseteq G$.

For disjoint subsets $H, L \subseteq V(G)$ we let $E(H, L) = E_G(H, L)$ denote the set of edges between $H$ and $L$. For any vertex $v \in V(G)$, we let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ denote its neighborhood and let $d(v) = d_G(v) = |N_G(v)|$ denote its degree. We also define $\overline{d}(G) = \frac{1}{|V(G)|} \sum_{v \in V(G)} d(v)$ to be the average degree of $G$ and $\Delta(G) = \max_{v \in V(G)} d(v)$ to be the maximum degree of $G$. We say that a graph $G$ is $k$-regular if $d(v) = k$ for all $v \in V(G)$. We define the diameter of a graph $G$ to be the maximum over all pairs of vertices $u, v \in V(G)$ of the length of the shortest path from $u$ to $v$. We let $K_n$ denote the complete graph on $n$ vertices and for $r \geq 2$, let $K_{n_1, \ldots, n_r}$ denote the complete multipartite graph with $r$ parts of size $n_1, \ldots, n_r$.

For any $G$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$, we define the $n \times n$ adjacency matrix $A = A(G)$ via

$$A_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G) \\ 0 & \text{else.} \end{cases}$$

Since $A$ is a symmetric matrix over $\mathbb{R}$, it has a full set of real eigenvalues $\lambda_1(G) \geq \ldots \geq \lambda_n(G)$, and we equivalently say that $\lambda_1 = \lambda_1(G)$ is the $i$th eigenvalue of $G$, with multiplicity $m(\lambda_i) = m_A(\lambda_i)$. Furthermore, if $G$ is $k$-regular, then $\lambda_1 = k$ and we say that $k - \lambda_2$ is the spectral gap of $G$.

## 2 Orthogonal projections of matrices

In this section we use orthogonal projections of matrices with respect to the Frobenius inner product in order to obtain real and complex variants of some geometric inequalities for finite sets of vectors, or equivalently positive semidefinite matrices, which underlie most of the results obtained in this paper. Our motivation comes from studying real equiangular lines in $\mathbb{R}^r$, in which case we have already observed in the introduction that choosing a unit vector along each line yields a collection of unit vectors with all pairwise inner products lying in the set $\{\alpha, -\alpha\}$, i.e. a spherical $\{\alpha, -\alpha\}$-code $\mathcal{C} = \{v_1, \ldots, v_n\}$ in $\mathbb{R}^r$.

In our previous work together with Dräxl, Keevash, and Sudakov, the arguments rested upon finding a sufficiently large subset $S \subseteq \mathcal{C}$ such that $\langle u, v \rangle = \alpha$ for all $u \neq v \in S$, i.e. a large regular simplex, and then projecting the remaining vectors onto the orthogonal complement of the span of this simplex. In this paper, we overcome this barrier by working in the Hilbert space of
\( r \times r \) real symmetric matrices \( \mathcal{S}_r \) equipped with the Frobenius inner product and observing that the collection of projection matrices \( \mathcal{C}' = \{v_1v_1^\top, \ldots, v_nv_n^\top\} \) forms a large regular simplex in this space. Note that the fact that \( \mathcal{C}' \) is a simplex has been known for quite some time, as it implies that \( \mathcal{C}' \) is a linearly independent set which, together with the fact that the dimension of \( \mathcal{S}_r \) is \( \binom{r+1}{2} \), allows one to conclude Gerzon’s absolute bound. Our main contribution is the observation that one can obtain interesting geometric inequalities by orthogonally projecting a matrix onto the span of \( v_1v_1^\top, \ldots, v_nv_n^\top \) and using the fact that the Frobenius norm can only decrease. A benefit of this approach is that if \( n = \binom{r+1}{2} = \dim(\mathcal{S}_r) \), then \( v_1v_1^\top, \ldots, v_nv_n^\top \) span \( \mathcal{S}_r \), in which case our inequalities are tight.

As a first application, we observe that projecting the identity matrix \( I \) onto \( v_1v_1^\top, \ldots, v_nv_n^\top \) gives a new proof of the relative bound. In fact, the relative bound is actually a special case of the Welch bound, an important inequality in coding theory \cite{69} with applications in signal analysis and quantum information theory, see \cite{26, 53, 55}. In a more general form, it states that for any collection \( \mathcal{C} = \{v_1, \ldots, v_n\} \) of unit vectors in \( \mathbb{C}^r \), we have

\[
\sum_{i,j \in [n]} |\langle v_i, v_j \rangle|^2 \geq n^2/r,
\]

with equality if and only if \( \mathcal{C} \) forms a tight frame, i.e. \( \sum_{i=1}^n v_iv_i^\top \) is a multiple of the identity matrix.

Moreover, it is known that Welch’s bound follows from the inequality \( 0 \leq \|I - \sum_{i=1}^n c_i v_i v_i^\top\|^2 \), and thus choosing coefficients \( c_1, \ldots, c_n \in \mathbb{C}^r \) such that \( \|I - \sum_{i=1}^n c_i v_i v_i^\top\|^2 \) is minimized would clearly strengthen it. Since choosing such coefficients is equivalent to orthogonally projecting \( I \) onto the span of \( v_1v_1^\top, \ldots, v_nv_n^\top \), we are able to obtain an improvement on the Welch bound.

To state our result, we recall that for any matrix \( M \in \mathbb{C}^{n \times n} \) and function \( f : \mathbb{C} \to \mathbb{C} \), the matrix \( f(M) \) is defined by \( f(M)_{i,j} = f(M_{i,j}) \) for all \( i,j \in [n] \) and moreover, for any real symmetric matrix \( M \in \mathbb{R}^{n \times n} \) with eigenvalues \( \lambda_1, \ldots, \lambda_n \) and corresponding unit eigenvectors \( u_1, \ldots, u_n \), the Moore-Penrose generalized inverse of \( M \) is \( M^\dagger = \sum_{i: \lambda_i > 0} \frac{1}{\lambda_i} u_i u_i^\top \).

**Theorem 7.** Let \( f : \mathbb{C} \to \mathbb{C} \) be defined by \( f(x) = |x|^2 \). Let \( \mathcal{C} = \{v_1, \ldots, v_n\} \) be a collection of unit vectors in \( \mathbb{C}^r \) and let \( M = M_{\mathcal{C}} \) be the corresponding Gram matrix. Then

\[
\mathbb{I}^\top f(M) \mathbb{I} \leq r,
\]

with equality if and only if \( I \) is in the span of \( v_1v_1^\top, \ldots, v_nv_n^\top \). Moreover

\[
\mathbb{I}^\top f(M) \mathbb{I} \geq \frac{n^2}{r} \left( 2 - \frac{\mathbb{I}^\top f(M) \mathbb{I}}{r} \right),
\]

and if \( f(M) \) is invertible, then we have equality if and only if \( f(M) \mathbb{I} = \mathbb{I} \mathbb{I} \).

Observe that in the preceding theorem, \( \mathbb{I}^\top f(M) \mathbb{I} = \sum_{i,j \in [n]} |\langle v_i, v_j \rangle|^2 \), so that the latter inequality is indeed a strengthening of the Welch bound. Moreover, it follows that if \( \mathcal{C} \) is a tight frame, then both inequalities are tight. Before giving the proof, we recall that for a linear map \( L : \mathbb{C}^n \to \mathbb{C}^{r \times s} \) where \( \mathbb{C}^{r \times s} \) is equipped with the Frobenius inner product, the adjoint \( L^\# : \mathbb{C}^{r \times s} \to \mathbb{C}^n \) satisfies \( \langle L^\# M, v \rangle = \langle M, Lv \rangle_F \) for all \( v \in \mathbb{C}^n, M \in \mathbb{C}^{r \times s} \), and for a single matrix \( M \in \mathbb{C}^{r \times s} \), the adjoint \( M^\# : \mathbb{C}^{r \times s} \to \mathbb{C}^s \) satisfies \( M^\# L = \langle M, L \rangle_F \) for all \( L \in \mathbb{C}^{s \times r} \). Finally, in all of the proofs in this section, we will make use of the fact that for any \( a, b, c, d \in \mathbb{C}^r \),

\[
\langle ab^*, cd^* \rangle_F = \text{tr}((ab^*)^* cd^*) = \text{tr}(a^* cd^* b) = \langle a, c \rangle \langle d, b \rangle.
\]
Proof of Theorem 7. Let \( \mathcal{C} = \{v_1, \ldots, v_n\} \), and define \( W_i = v_i v_i^\top \) for all \( i \in [n] \). Observe that for all \( i, j \in [n] \), the Frobenius inner product
\[
(W_i, W_j)_F = \langle v_i, v_j \rangle = \langle v_i, v_j \rangle = |\langle v_i, v_j \rangle|^2.
\]
Therefore, if we consider the linear map \( \mathcal{W} : \mathbb{C}^n \to \mathbb{C}^{r \times r} \) defined by \( \mathcal{W} v_i = W_i \) for all \( i \in [n] \), then we have \( \mathcal{W}^\# \mathcal{W} = f(M) \). Now we compute that \( ||I||_F^2 = tr(I) = r \) and \( \langle I, W_i \rangle_F = tr(v_i v_i^\top) = ||v_i||^2 = 1 \) for all \( i \in [n] \), so that \( \mathcal{W}^\# I = I \). Moreover, it is well known (see e.g. [8]) that \( \mathcal{P} = \mathcal{W}(\mathcal{W}^\# \mathcal{W})^\dagger \mathcal{W}^\# : \mathbb{C}^{r \times r} \to \mathbb{C}^{r \times r} \) is the orthogonal projection (with respect to the Frobenius inner product) onto the range of \( \mathcal{W} \). Indeed, one can verify that \( \mathcal{P}^2 = \mathcal{P} = \mathcal{P}^\# \) and \( \mathcal{P} \mathcal{W} = \mathcal{W} \).

Since \( \mathcal{P} \) is a contraction map, we conclude
\[
r = ||I||_F^2 \geq ||\mathcal{P} I||_F^2 = \langle I, \mathcal{P} I \rangle_F = I^\# \mathcal{W}(\mathcal{W}^\# \mathcal{W})^\dagger \mathcal{W}^\# I = \mathbb{I}^T f(M)^\dagger \mathbb{I}.
\]
Moreover, because \( \mathcal{P} \) is a projection, we have equality above if and only if \( \mathcal{P} I = I \), which occurs if and only if \( I \) lies in the span of \( W_1, \ldots, W_n \).

Now observe that \( \mathcal{P} I \) is the unique minimizer of the expression \( ||I - X||_F^2 \) over all \( X \) in the range of \( \mathcal{W} \). Therefore, we obtain
\[
r - \mathbb{I}^T f(M) \mathbb{I} = ||I||_F^2 - ||\mathcal{P} I||_F^2 = ||I - \mathcal{P} I||_F^2 \leq \left|\left| I - \frac{r}{n} \mathcal{W} \mathbb{I} \right|\right|_F^2 = I^\# I - \frac{2r}{n} I^\# \mathcal{W} \mathbb{I} + \frac{r^2}{n^2} \mathbb{I}^T \mathcal{W}^\# \mathcal{W} \mathbb{I}
\]
\[
= r - \frac{2r}{n} \mathbb{I} + \frac{r^2}{n^2} \mathbb{I}^T f(M) \mathbb{I} = \frac{r^2}{n^2} \mathbb{I}^T f(M) \mathbb{I} - r,
\]
which is equivalent to the desired inequality.

Furthermore, let us assume that \( f(M) \) is invertible, so that \( \mathcal{P} I = \mathcal{W}(\mathcal{W}^\# \mathcal{W})^\dagger \mathcal{W}^\# I = \mathcal{W} f(M)^{-1} \mathbb{I} \). By the uniqueness of \( \mathcal{P} I \), the previous inequality is tight if and only if \( \mathcal{P} I = \frac{r}{n} \mathcal{W} \mathbb{I} \). Thus if \( \mathcal{P} I = \frac{r}{n} \mathcal{W} \mathbb{I} \), then we conclude
\[
\frac{r}{n} f(M) \mathbb{I} = \frac{r}{n} \mathcal{W} \mathbb{I} \mathcal{W} \mathbb{I} = \mathcal{W} \mathcal{P} I = \mathcal{W} \mathcal{W} f(M)^{-1} \mathbb{I} = \mathcal{W} \mathcal{W} \mathbb{I} \mathbb{I} = \mathcal{W} \mathbb{I} \mathbb{I} = \mathbb{I}.
\]
On other hand, if \( f(M) \mathbb{I} = \mathbb{I} \) then \( f(M)^{-1} \mathbb{I} = \frac{r}{n} \mathbb{I} \) and applying \( \mathcal{W} \) yields \( \mathcal{P} I = \mathcal{W} f(M)^{-1} \mathbb{I} = \frac{r}{n} \mathcal{W} \mathbb{I} \), as desired.

Remark. In the previous proof, one could have equivalently defined the matrices \( W_1, \ldots, W_n, I \) to be vectors in \( \mathbb{C}^{r^2} \) and defined \( \mathcal{W} \) to be the \( r^2 \times n \) matrix with the \( i \)th column being \( W_i \). In this setting, the Frobenius inner product would become the standard inner product in \( \mathbb{C}^{r^2} \) and the adjoint maps \( \mathcal{W}^\#, I^\# \) would become the adjoint matrices \( \mathcal{W}^*, I^* \).

Returning to our main application of equiangular lines, let \( \mathcal{C} = \{v_1, \ldots, v_n\} \) be a spherical \( \{\alpha, -\alpha\} \)-code in \( \mathbb{R}^r \), let \( V \in \mathbb{R}^{r \times n} \) be the matrix with \( i \)th column \( v_i \), and let \( x, y \in \mathbb{R}^n \), so that \( Vx \) and \( Vy \) are arbitrary vectors in the span of \( \mathcal{C} \). We project symmetric matrices of the form \( Vx(Vy)^\top + Vy(Vx)^\top \) onto the span of \( v_1 v_1^\top, \ldots, v_n v_n^\top \), in order to obtain the following geometric inequality.
Theorem 8. Let $\alpha \in (0, 1)$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. For any spherical \{\alpha, -\alpha\}-code $\mathcal{C}$ in $\mathbb{R}^r$ with corresponding Gram matrix $M = M_\mathcal{C}$ and for all $x, y \in \mathbb{R}^n$, we have

$$\frac{1 - \alpha^2}{2} \langle x, Mx \rangle \langle y, My \rangle + \langle x, My \rangle^2 + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} \langle Mx, My \rangle^2 \geq \langle f(Mx), f(My) \rangle,$$

with equality whenever $|\mathcal{C}| = \binom{n+1}{2}$.

By choosing $x, y$ appropriately (usually a standard basis vector or an eigenvector) in Theorem 8 we obtain new geometric information about our original vectors, such as bounds on eigenvalues and eigenvectors of their Gram matrix or bounds on the degrees of a corresponding graph. Although one may be tempted to project asymmetric matrices like $Vx(Vy)^\top$, a linear combination of real symmetric matrices must necessarily be symmetric. Thus, if we would like our projection-based inequality to have a chance of being an equality, we must limit ourselves to symmetric matrices.

One benefit of our approach is that it generalizes to the setting of complex equiangular lines, in which case we have a collection of unit vectors $\mathcal{C} = \{v_1, \ldots, v_n\}$ in $\mathbb{C}^r$ satisfying $|\langle v_i, v_j \rangle| = \alpha$ for all $i \neq j$, or equivalently a spherical $S^1(\alpha)$-code where $S^1(\alpha)$ is the circle of radius $\alpha$ centered at 0 in the complex plane. As in the real case, it is well known that the collection $\mathcal{C}' = \{v_1 v_1^*, \ldots, v_n v_n^*\}$ forms a regular simplex with respect to the Frobenius inner product. However, unlike in the real case, a linear combination of complex Hermitian matrices need not be Hermitian when complex coefficients are used. Indeed, even for a single Hermitian matrix $H$ we have $(iH)^* = -iH \neq iH$. Therefore, $\mathcal{C}'$ is capable of spanning the entire space $\mathbb{C}^r \times \mathbb{C}^r$ of $r \times r$ complex matrices and so, unlike in the real case, we project (not necessarily Hermitian) matrices of the form $Vx(Vy)^*$ to obtain the following.

Theorem 9. Let $\alpha \in (0, 1)$ and let $f : \mathbb{C} \to \mathbb{R}$ be defined by $f(x) = |x|^2$. For any spherical $S^1(\alpha)$-code $\mathcal{C}$ in $\mathbb{C}^r$ with corresponding Gram matrix $M = M_\mathcal{C}$ and for all $x, y \in \mathbb{C}^n$, we have

$$(1 - \alpha^2) \langle x, Mx \rangle \langle y, My \rangle + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} |\langle Mx, My \rangle|^2 \geq \langle f(Mx), f(My) \rangle,$$

with equality whenever $|\mathcal{C}| = r^2$.

Remark. As noted in Section 1.2, the case of equality in Theorem 9 is of particular interest because sets of $r^2$ equiangular lines in $\mathbb{C}^r$ are important objects in quantum theory known as SICs, and Zauner’s conjecture implies that they exist for every $r$. In particular, it is well known that a set of $r^2$ equiangular lines in $\mathbb{C}^r$ with common angle $\arccos(\alpha)$ must satisfy $\alpha = 1/\sqrt{r+1}$ (see e.g. [22]), in which case Theorem 9 yields the equality

$$r \langle x, Mx \rangle \langle y, My \rangle + \frac{1}{r} |\langle Mx, My \rangle|^2 = (r + 1) \langle f(Mx), f(My) \rangle.$$

Notice that the inequality in Theorem 9 is slightly simpler than that of Theorem 8 with the crucial difference in the real case coming from the limitation of working with symmetric matrices. In view of this, together with the fact that various calculations over $\mathbb{C}$ restrict naturally to $\mathbb{R}$, we first present a proof of Theorem 9.

Proof of Theorem 9. Let $n = |\mathcal{C}|$, $\mathcal{C} = \{v_1, \ldots, v_n\}$, and as in the proof of Theorem 7 define $W_i = v_i v_i^*$ for $i \in [n]$ and let $\mathcal{W} : \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$ be the corresponding linear map defined by $\mathcal{W}e_i = W_i$ for $i \in [n]$. Therefore $\mathcal{W}^\# \mathcal{W} = f(M) = (1 - \alpha^2)I + \alpha^2 J$, which is invertible with $((1 - \alpha^2)I + \alpha^2 J)^{-1} = \frac{1}{1 - \alpha^2} \left( I - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} J \right)$, so that in particular $\text{rk}(\mathcal{W}^\# \mathcal{W}) = n$. Now let
$V \in \mathbb{C}^{r \times n}$ be the matrix with $i$th column $v_i$ so that $V^*V = M$. Given $x, y \in \mathbb{C}^n$, we define $X = Vx(Vy)^*$ and observe that

$$||X||_F^2 = \langle Vx(Vy)^*, Vx(Vy)^* \rangle_F = \langle Vx, Vx \rangle \langle Vy, Vy \rangle = \langle x, Mx \rangle \langle y, My \rangle,$$

as well as

$$\langle W_i, X \rangle_F = \langle v_i, Vx \rangle \langle Vy, v_i \rangle = \langle Ve_i, Vx \rangle \langle Vy, Ve_i \rangle = \langle e_i, Mx \rangle \langle My, e_i \rangle = \langle (Mx)_i(My)_i \rangle$$

for all $i \in [n]$. Thus if we define $u \in \mathbb{C}^n$ by $u_i = (Mx)_i(My)_i$, then we have $\mathcal{W}^#X = u$. Similarly to the proof of Theorem 7, we have that $\mathcal{P} = \mathcal{W}(\mathcal{W}^# \mathcal{W})^{-1} \mathcal{W}^# : \mathbb{C}^{r \times r} \rightarrow \mathbb{C}^{r \times r}$ is the orthogonal projection (with respect to the Frobenius inner product) onto the range of $\mathcal{W}$, and we thus conclude

$$\langle x, Mx \rangle \langle y, My \rangle = ||X||_F^2 \geq ||\mathcal{P}X||_F^2 = \langle X, \mathcal{P}X \rangle_F = X^\# \mathcal{W}(\mathcal{W}^# \mathcal{W})^{-1} \mathcal{W}^#X = u^*((1 - \alpha^2)I + \alpha^2 J)^{-1}u$$

$$= \frac{1}{1 - \alpha^2} \left( u^* I u - \frac{\alpha^2}{\alpha^2n + 1 - \alpha^2} u^* Ju \right)$$

$$= \frac{1}{1 - \alpha^2} \left( ||u||^2 - \frac{\alpha^2}{\alpha^2n + 1 - \alpha^2} \langle 1, u \rangle^2 \right).$$

It therefore remains to determine $||u||^2$ and $\langle 1, u \rangle$, and so we compute

$$||u||^2 = \sum_{i=1}^{n} (|f(Mx)_i|^2((My)_i|^2 = \langle f(Mx), f(My) \rangle,$$

and

$$\langle 1, u \rangle = \sum_{i=1}^{n} (My)_i(Mx)_i = \sum_{i=1}^{n} y^* Me_i e_i^* Mx = y^* M \left( \sum_{i=1}^{n} e_i e_i^* \right) Mx = y^* M Mx = \langle My, Mx \rangle,$$

as desired.

Finally, note that $n = \text{rk}(\mathcal{W}^# \mathcal{W}) = \text{rk}(\mathcal{W})$ and $\mathbb{C}^{r \times r}$ has dimension $r^2$. Thus if $n = r^2$, then $\text{rk}(\mathcal{W}) = r^2$, so that the range of $\mathcal{W}$ is $\mathbb{C}^{r \times r}$ and thus $\mathcal{P}$ is the identity map, giving equality above.

**Proof of Theorem 8** We proceed as in the proof of Theorem 9 except with $\mathbb{C}$ replaced by $\mathbb{R}$ and $\mathbb{C}^{r \times r}$ replaced by the space of real symmetric matrices $\mathcal{S}$.

Let $n = |\mathcal{E}|$ and $\mathcal{E} = \{v_1, \ldots, v_n\}$. For each $i \in [n]$, we define $W_i = v_i v_i^\top$ and note that it is symmetric. As before, the corresponding linear map $\mathcal{W} : \mathbb{R}^n \rightarrow \mathcal{S}$ satisfies $\mathcal{W}^# \mathcal{W} = (1 - \alpha^2)I + \alpha^2 J$ and $\mathcal{P} = \mathcal{W}(\mathcal{W}^# \mathcal{W})^{-1} \mathcal{W}^# : \mathcal{S} \rightarrow \mathcal{S}$ is the orthogonal projection onto the span of $W_1, \ldots, W_n$. On the other hand, $Vx(Vy)^\top$ is not necessarily symmetric and so we define $X = \frac{1}{2} (Vx(Vy)^\top + Vy(Vx)^\top)$ so that $X$ is symmetric. Following the previous proof, for $i \in [n]$ we compute $\langle W_i, Vx(Vy)^\top \rangle_F = \langle W_i, Vy(Vx)^\top \rangle_F = \langle (Mx)_i(My)_i \rangle$. Thus if we define $u \in \mathbb{R}^n$ by $u_i = (Mx)_i(My)_i$, then we have $\langle W_i, X \rangle = u_i$ for $i \in [n]$, and therefore $\mathcal{W}^#X = u$. As in the
previous proof, we conclude
\[
||X||^2_F \geq ||\mathcal{P}X||^2_F = u^*((1 - \alpha^2)I + \alpha^2 J)^{-1} u
\]
\[
= \frac{1}{1 - \alpha^2} \left( f(Mx) - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} (Mx, My)^2 \right).
\]
Now it suffices to compute
\[
4||X||^2_F = \langle Vx(Vy)^T, Vx(Vy)^T \rangle_F + \langle Vy(Vx)^T, Vy(Vx)^T \rangle_F
\]
\[
+ \langle Vx(Vx)^T, Vx(Vy)^T \rangle_F + \langle Vy(Vx)^T, Vy(Vx)^T \rangle_F
\]
\[
= 2 \langle Vx, Vx \rangle \langle Vy, Vy \rangle + 2 \langle Vx, Vy \rangle \langle Vx, Vy \rangle
\]
\[
= 2 \left( \langle x, Mx \rangle \langle y, My \rangle + \langle x, My \rangle^2 \right).
\]
Finally, note that \( \mathcal{S} \) has dimension \( \binom{r + 1}{2} \) and as in the previous proof, \( \text{rk}(\mathcal{W}) = n \). Thus if \( n = \binom{r + 1}{2} \), then the range of \( \mathcal{W} \) is \( \mathcal{S} \), so that \( \mathcal{P} \) is the identity map and we have equality above.

We conclude this section by discussing the connection between our projection method and the classical linear programming approach of Delsarte, Goethels, and Seidel \[22, 23\]. Observe that if we let \( \mathcal{C} = \{v_1, \ldots, v_n\} \) be a spherical \( \{\alpha, -\alpha\}\)-code in \( \mathbb{R}^r \) and project each \( v_i v_i^\top \) onto the orthogonal complement (with respect to the Frobenius inner product) of the identity \( I \), then the resulting collection of matrices \( \mathcal{C}' = \{v_1 v_1^\top - \frac{1}{r}I, \ldots, v_i v_i^\top - \frac{1}{r}I\} \) has the Gram matrix \( M_{\mathcal{C}'} = g(M) \) where \( g(x) = x^2 - \frac{1}{r} \) is a scaled version of the second Gegenbauer polynomial. In addition to being generalizations of the Chebyshev and Legendre polynomials, the significance of Gegenbauer polynomials goes back to a well-known result of Schoenberg \[63\], who proved that a function \( f : [-1, 1] \rightarrow \mathbb{R} \) has the property that \( f(M_{\mathcal{C}}) \) is positive semidefinite for any finite set of unit vectors \( \mathcal{C} \) in \( \mathbb{R}^r \), if and only if \( f \) is a nonnegative linear combination of Gegenbauer polynomials. Indeed, inequalities of the form \( 1^\top f(M_{\mathcal{C}}) 1 \geq 0 \) underlie the approach of Delsarte, Goethels, and Seidel and in view of Schoenberg’s theorem, our approach can also be reduced to using such inequalities. However, the \( f \) that we would need is determined implicitly by the computation of orthogonal projections of certain matrices. Moreover, we believe that it is more natural and transparent to work directly with matrices with respect to the Frobenius inner product and therefore, we do not attempt to give a Delsarte-style proof in this paper.

We also note that a symmetric matrix which is orthogonal to the identity \( I \) may be viewed as a traceless symmetric 2-tensor and by considering higher-order traceless symmetric tensors with respect to the Frobenius inner product, one may obtain \( G_k^r(M) \) as a Gram matrix where \( G_k^r \) is the \( k \)th Gegenbauer polynomial for any \( k \in \mathbb{N} \). Indeed, traceless symmetric tensors are in one to one correspondence with harmonic homogeneous polynomials and for \( \mathbb{R}^3 \), the traceless symmetric tensor approach to spherical harmonics is more well known in the physics literature, see e.g. \[18, 12, 59, 72\]. However, we could not find a standard reference text for this approach in higher dimensions and indeed, Guth \[30\] has written lecture notes on such an approach for Legendre polynomials for his course on electromagnetism, motivated by the fact that he also did not know of an appropriate reference.
3 Equiangular lines in \( \mathbb{R}^r \)

In this section we will prove our main theorems regarding real equiangular lines. Now let \( r \in \mathbb{N} \) and let \( \mathcal{L} = \{\ell_1, \ldots, \ell_n\} \) be a set of \( n \) equiangular lines in \( \mathbb{R}^r \). As noted in the previous section, if we choose a unit vector \( v_i \in \ell_i \) along the \( i \)th line, the resulting collection \( \mathcal{C} = \{v_1, \ldots, v_n\} \) forms a spherical \( \{\alpha, -\alpha\} \)-code in \( \mathbb{R}^r \) for some \( \alpha \in [0, 1) \), in which case we say that \( \mathcal{C} \) represents \( \mathcal{L} \). For convenience, in this section we shall always let \( n = |\mathcal{C}| \) and label the vectors in \( \mathcal{C} \) as \( v_1, \ldots, v_n \).

Our approach will be to apply Theorem 8 to the Gram matrix \( M = M_\mathcal{C} \) of a spherical \( \{\alpha, -\alpha\} \)-code \( \mathcal{C} \) which represents a given set of equiangular lines in order to obtain bounds on eigenvalues and eigenvectors of \( M \), as well as bounds on the maximum degree of a corresponding graph. This will allow us to obtain upper bounds on \( N_\alpha(r) \), the maximum number of equiangular lines in \( \mathbb{R}^r \) with common angle \( \arccos(\alpha) \). Our methods build on and generalize some of the ideas appearing in \[3\].

3.1 Spectral bounds

In this subsection, we will obtain bounds on eigenvalues and eigenvectors of the Gram matrix \( M = M_\mathcal{C} \) of a spherical \( \{\alpha, -\alpha\} \)-code \( \mathcal{C} \) in order to prove Theorem 1. To this end, we first apply Theorem 8 with \( x \) and \( y \) being orthogonal eigenvectors corresponding to some positive eigenvalues \( \lambda, \mu \) of \( M \), to obtain the following lemma.

**Lemma 10.** Let \( \alpha \in (0, 1) \) and let \( \mathcal{C} \) be a spherical \( \{\alpha, -\alpha\} \)-code in \( \mathbb{R}^r \) with corresponding Gram matrix \( M = M_\mathcal{C} \). If \( x, y \in \mathbb{R}^n \) are orthogonal unit eigenvectors of \( M \) with nonzero eigenvalues \( \lambda, \mu \), then

\[
\sum_{i=1}^{n} x_i^2 y_i^2 \leq \frac{1 - \alpha^2}{2\lambda\mu},
\]

with equality whenever \( n = \binom{r+1}{2} \).

**Proof.** We compute that \( \langle x, Mx \rangle = \lambda, \langle y, My \rangle = \mu, \langle x, My \rangle = \mu \langle x, y \rangle = 0, \) and \( \langle Mx, My \rangle = \lambda\mu \langle x, y \rangle = 0 \). Furthermore, we let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^2 \) and observe that \( f(Mx) = \lambda^2 f(x), f(My) = \mu^2 f(y), \) so that we have \( \langle f(Mx), f(My) \rangle = \lambda^2\mu^2 \langle f(x), f(y) \rangle = \lambda^2\mu^2 \sum_{i=1}^{n} x_i^2 y_i^2 \).

Therefore, applying Theorem 8 and dividing by \( \lambda^2\mu^2 \) gives the desired inequality, with the case of equality following via Theorem 8 as well. \( \square \)

Note that if we have a lower bound on \( \min_{i \in [n]} x_i^2 \) in Lemma 10, this can be used to obtain an upper bound on the eigenvalue \( \mu \). To this end, we state the following useful lemma which follows from Theorem 8 by letting \( y = e_i \) be a standard basis vector.

**Lemma 11.** Let \( \alpha \in (0, 1) \) and let \( \mathcal{C} \) be a spherical \( \{\alpha, -\alpha\} \)-code in \( \mathbb{R}^r \) with corresponding Gram matrix \( M = M_\mathcal{C} \). For all \( i \in [n] \) and \( x \in \mathbb{R}^n \), we have

\[
\frac{1 - \alpha^2}{2\alpha^2} \langle x, Mx \rangle^2 + \frac{1}{\alpha^2n + 1 - \alpha^2} (M^2x)^2 \geq \langle x, M^2x \rangle,
\]

with equality whenever \( n = \binom{r+1}{2} \).

**Proof.** Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^2 \) and observe that \( f(Me_i) = \alpha^2 \mathbb{1} + (1 - \alpha^2)e_i \).

Moreover, observe that \( \langle f(z), \mathbb{1} \rangle = \sum_{j=1}^{n} z_j^2 = \|z\|^2 \) for all \( z \in \mathbb{R}^n \) and thus

\[
\langle f(Mx), f(Me_i) \rangle = \alpha^2 \langle f(Mx), \mathbb{1} \rangle + (1 - \alpha^2) \langle f(Mx), e_i \rangle = \alpha^2 \|Mx\|^2 + (1 - \alpha^2)(Mx)_i^2.
\]
Moreover, we have that \( \langle e_i, Me_i \rangle = 1 \), \( \langle x, Me_i \rangle = (Mx)_i \), and \( \langle Mx, Me_i \rangle = (M^2x)_i \). Thus applying Theorem 8 with \( y = e_i \), we conclude

\[
\frac{1 - \alpha^2}{2} (\langle x, Mx \rangle + (Mx)_i^2) + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} (M^2x)_i^2 \geq \alpha^2 ||Mx||^2 + (1 - \alpha^2)(Mx)_i^2.
\]

The desired inequality now follows by subtracting \( (1 - \alpha^2)(Mx)_i^2 \) from both sides, dividing by \( \alpha^2 \), and noting that \( ||Mx||^2 = \langle x, M^2x \rangle \). If \( n = \binom{r+1}{2} \) then we have equality in Theorem 8 and therefore, we have equality above.

Now we apply Lemma 11 with \( x \) being the unit eigenvector corresponding to \( \lambda_1(M) \) in order to obtain a lower bound on \( x_i^2 \) for any \( i \in [n] \), provided that \( \lambda_1(M) > \frac{1 - \alpha^2}{2\alpha^2} \). We then combine this with Lemma 10 to obtain an upper bound on the second largest eigenvalue \( \lambda_2(M) \).

**Lemma 12.** Let \( \alpha \in (0, 1) \) and let \( C \) be a spherical \( \{\alpha, -\alpha\} \)-code in \( \mathbb{R}^r \) with corresponding Gram matrix \( M = M_C \). Let \( x \) be a unit eigenvector corresponding to the largest eigenvalue \( \lambda_1 = \lambda_1(M) \). If \( \lambda_1 > \frac{1 - \alpha^2}{2\alpha^2} \), then \( \frac{\lambda_1^2}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{2\alpha^2} > 0 \) and for all \( i \in [n] \) we have

\[
x_i^2 \geq \frac{1 - \frac{1 - \alpha^2}{\alpha^2 n + 1 - \alpha^2}}{ \frac{\lambda_1^2}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{2\alpha^2} },
\]

and moreover the second largest eigenvalue satisfies

\[
\lambda_2(M) \leq \frac{1 - \alpha^2}{2} \frac{\lambda_1^2}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{2\alpha^2 \lambda_1},
\]

with equality in both whenever \( n = \binom{r+1}{2} \).

**Proof.** Let \( i \in [n] \). Since \( x \) is a unit eigenvector corresponding to \( \lambda_1 \), we apply Lemma 11 to obtain

\[
\frac{1 - \alpha^2}{2\alpha^2} (\lambda_1 - \lambda_1 x_i^2) + \frac{1}{\alpha^2 n + 1 - \alpha^2} \lambda_1^4 x_i^2 \geq \lambda_1^2.
\]

Dividing by \( \lambda_1^2 \) and rearranging terms, we have

\[
\left( \frac{\lambda_1^2}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{2\alpha^2} \right) x_i^2 \geq 1 - \frac{1 - \alpha^2}{2\alpha^2 \lambda_1},
\]

where the right hand side is positive by assumption. Therefore \( \frac{\lambda_1^2}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{2\alpha^2} \) is also positive and we may divide by it to conclude

\[
x_i^2 \geq \frac{1 - \frac{1 - \alpha^2}{\alpha^2 n + 1 - \alpha^2}}{ \frac{\lambda_1^2}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{2\alpha^2} },
\]

Now that we have a universal lower bound on \( x_i^2 \) over all \( i \in [n] \), we may apply Lemma 10 with \( y \) being a unit eigenvector corresponding to \( \lambda_2 = \lambda_2(M) \) in order to obtain

\[
\frac{\lambda_1^2}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{2\alpha^2} \leq \sum_{k=1}^{n} x_i^2 y_i^2 \leq \frac{1 - \alpha^2}{2\lambda_1 \lambda_2},
\]

15
which implies the desired upper bound on $\lambda_2$. Finally, $n = \binom{r+1}{2}$ implies we have equality in Lemma 10 and Lemma 11 in which case we have equality everywhere above.

In order to make use of Lemma 12 we first give an upper bound on $\lambda_1(M)$.

**Lemma 13.** Let $\alpha \in (0, 1)$ and let $C$ be a spherical $\{\alpha, -\alpha\}$-code in $\mathbb{R}^r$ with corresponding Gram matrix $M = M_C$. Then

$$\lambda_1(M) \leq \alpha n + 1 - \alpha.$$  

**Proof.** Let $x$ be a unit eigenvector corresponding to $\lambda_1(M)$. Using Cauchy-Schwarz, we obtain

$$\lambda_1(M) = \sum_{i,j \in [n]} M_{i,j} x_i x_j \leq \sum_{i,j \in [n]} |M_{i,j}| |x_i||x_j| = (1 - \alpha) \sum_{i=1}^n |x_i|^2 + \alpha \left( \sum_{i=1}^n |x_i| \right)^2$$

$$\leq (1 - \alpha) \sum_{i=1}^n |x_i|^2 + \alpha n \sum_{i=1}^n |x_i|^2$$

$$= 1 - \alpha + \alpha n.$$

We now note that, provided $\lambda_1(M) > \frac{(1+\epsilon)}{2\alpha^2}$ for some fixed $\epsilon > 0$, Lemma 12 together with Lemma 13 implies that $\lambda_2(M) \leq O(1/\alpha)$. Using this idea, together with Lemma 13 and the fact that $n = \text{tr}(M) = \sum_{i=1}^r \lambda_i(M)$, we therefore obtain the following upper bound on $n$.

**Lemma 14.** Let $\alpha \in (0, 1)$ and let $C$ be a spherical $\{\alpha, -\alpha\}$-code in $\mathbb{R}^r$ with corresponding Gram matrix $M = M_C$. If $\lambda_1 = \lambda_1(M) > \frac{1-\alpha^2}{2\alpha^2}$ then

$$n < \frac{1 + \alpha}{2\alpha - \frac{1-\alpha^2}{\alpha \lambda_1}} (r - 1) + 1.$$  

**Proof.** Applying Lemma 13 we have

$$\lambda_1(\alpha^2 n + 1 - \alpha^2) \leq \frac{\alpha n + 1 - \alpha}{\alpha^2 n + 1 - \alpha^2} < \frac{1}{\alpha},$$

and since $\lambda_1 > \frac{1-\alpha^2}{2\alpha^2}$, we may apply Lemma 12 to obtain

$$\lambda_2(M) < \frac{1 - \alpha^2}{2 \left( 1 - \frac{1-\alpha^2}{\alpha^2 \lambda_1} \right)} \quad \text{and} \quad \lambda_1 < \frac{1 - \alpha}{\alpha^2 n + 1 - \alpha^2} \frac{1}{2 \left( 1 - \frac{1-\alpha^2}{\alpha^2 \lambda_1} \right)}.$$

Since $\text{rk}(M)$ is at most $r$, we have that $\text{tr}(M) = \sum_{i=1}^r \lambda_i(M)$, and so we may apply Lemma 13 once again in order to conclude the desired bound

$$(1 - \alpha)(n - 1) = n - (\alpha n + 1 - \alpha) \leq n - \lambda_1 = \text{tr}(M) - \lambda_1 = \sum_{i=2}^r \lambda_i(M) < \frac{1 - \alpha^2}{2\alpha - \frac{1-\alpha^2}{\alpha \lambda_1}} (r - 1).$$

Note that the bound in Lemma 14 implies $n \leq O\left(\frac{\sqrt{r}}{\alpha} \right)$, provided that $\lambda_1(M)$ is slightly larger than $\frac{1}{2\alpha}$. On the other hand, when $\lambda_1(M)$ is not too much larger than $1/\alpha^2$, we may use the following simple lemma in order to conclude that $n \leq O\left(\frac{\sqrt{r}}{\alpha^2} \right)$.

16
Lemma 15. Let \( \alpha \in (0, 1) \), and let \( \mathcal{C} \) be a spherical \( \{\alpha, -\alpha\}\)-code in \( \mathbb{R}^r \) with corresponding Gram matrix \( M = M_C \). Then we have
\[
n < \frac{\lambda_1(M)}{\alpha} \sqrt{r}.
\]

Proof. We have that
\[
\alpha^2 n^2 < n(\alpha^2 n + 1 - \alpha^2) = \text{tr}(M^2) = \sum_{i=1}^{r} \lambda_i(M)^2 \leq \lambda_1(M)^2 r,
\]
so dividing by \( \alpha^2 \) and taking a square root gives the desired bound. \( \square \)

Using Lemma 14 and Lemma 15, we are now able to give a proof of Theorem 1.

Proof of Theorem 1. Let \( \mathcal{C} \) be a spherical \( \{\alpha, -\alpha\}\)-code representing a set of \( n = N^R_\alpha(r) \) equiangular lines in \( \mathbb{R}^r \), and let \( M = M_C \) be the corresponding Gram matrix. Let \( \lambda_1 = \lambda_1(M) \) and \( t = \alpha^2(1+\alpha)\sqrt{r} \). Note that \( \frac{(1+t)\sqrt{r}}{2\alpha} = \frac{\sqrt{r}}{2\alpha} + \frac{(1+\alpha)\sqrt{r}}{2\alpha} \), so it will suffice to show that \( n < \frac{(1+t)\sqrt{r}}{2\alpha} \).

To this end, first suppose that \( \lambda_1 \leq \frac{(1+t)\sqrt{r}}{2\alpha} \). In this case, we apply Lemma 15 to obtain the desired bound
\[
n < \frac{\lambda_1 \sqrt{r}}{\alpha} \leq \frac{(1+t)\sqrt{r}}{2\alpha^3}.
\]

Otherwise, we must have \( \lambda_1 > \frac{(1+t)\sqrt{r}}{2\alpha} \) so that \( 2\alpha - \frac{\alpha^2}{\alpha_1} > 2\alpha \left( 1 - \frac{1}{1+t} \right) = 2\alpha \frac{1}{1+t} \). In this case, we may apply Lemma 14 to obtain the desired
\[
n < \frac{(1+\alpha)(1+t)(r-1)}{2\alpha t} + 1 < \frac{(1+\alpha)(1+t)}{2\alpha t} \cdot r = \frac{(1+t)\sqrt{r}}{2\alpha^3}.
\]

\( \square \)

3.2 Degree bounds

In this section, we will obtain bounds on the maximum degree of a corresponding graph in order to prove Theorem 2 and Theorem 3. To this end, we first make the connection between equiangular lines and graphs more explicit. More specifically, given a spherical \( \{\alpha, -\alpha\}\)-code \( \mathcal{C} \), we associate a corresponding graph \( G = G_\mathcal{C} \) having vertex set \( V(G) = \mathcal{C} \) and edge set \( E(G) = \{uv : u, v \in \mathcal{C} \text{ and } \langle u, v \rangle = -\alpha\} \), i.e. the graph with the vectors of \( \mathcal{C} \) as vertices and an edge between any pair of vertices whose inner product is \(-\alpha\). Note that we will refer to the elements of \( \mathcal{C} \) as vertices or vectors depending on the context. Now if we let \( A = A(G) \) be the adjacency matrix of \( G \) and let \( M = M_\mathcal{C} \) be the Gram matrix of \( \mathcal{C} \), then
\[
M = \alpha J - 2\alpha A + (1 - \alpha)I. \tag{2}
\]

We will also need to choose the spherical code which represents our given family of equiangular lines more carefully. We say that \( \mathcal{C} \subseteq \mathbb{R}^r \) is a restricted spherical \( \{\alpha, -\alpha\}\)-code if the corresponding graph \( G_\mathcal{C} \) has an isolated vertex. Moreover, we will always assume for convenience that this vertex is \( v_1 \), so that \( \langle v_i, v_1 \rangle = \alpha \) for all \( 2 \leq i \leq n \). Now we show that for any family of equiangular lines, we can always find a restricted spherical \( \{\alpha, -\alpha\}\)-code that represents it.

Lemma 16. Let \( \mathcal{L} \) be a family of equiangular lines in \( \mathbb{R}^r \) with common angle \( \arccos(\alpha) \). Then there exists a restricted spherical \( \{\alpha, -\alpha\}\)-code \( \mathcal{C} \) that represents \( \mathcal{L} \).

Proof. As previously discussed, choosing a unit vector along each line of \( \mathcal{L} \) yields a spherical \( \{\alpha, -\alpha\}\)-code \( \mathcal{C} = \{v_1, \ldots, v_n\} \) representing \( \mathcal{L} \). We may negate (known in the literature as
switching) each \(v_i\) for which \(\langle v_i, v_1 \rangle = -\alpha\), to obtain \(C'\). Since this operation also negates the corresponding inner products, \(C'\) is a restricted spherical \(\{\alpha, -\alpha\}\)-code. Moreover, negating a vector doesn’t change the line it spans, so that \(C'\) still represents \(L\), as desired.

**Remark.** The type of switching argument used in the previous lemma was also used by Glazyrin and Yu [32] to obtain derived sets of vectors for their universal bound on \(N^R_2(r)\).

Our goal will now be to show that any sufficiently large collection of equiangular lines can be represented by a spherical \(\{\alpha, -\alpha\}\)-code whose corresponding graph has a maximum degree of at most \(O(1/\alpha^4)\). We first apply Lemma 11 with \(x = e_1\) and \(i \geq 2\) in order to obtain bounds on the degree of the \(i\)th vertex of the graph corresponding to a restricted spherical code.

**Lemma 17.** Let \(\alpha \in (0, 1)\) and let \(C\) be a restricted spherical \(\{\alpha, -\alpha\}\)-code in \(\mathbb{R}^r\) with corresponding graph \(G = G_C\). For all \(i \geq 2\), the degree \(d(v_i)\) of the \(i\)th vertex satisfies

\[
(n - 2d(v_i) + \frac{2}{\alpha} - 2)^2 \geq \left(n + \frac{1}{\alpha^2} - 1\right) \left(n - \frac{1}{\alpha^2} - 1\right) \left(\frac{1}{\alpha^2} - 3\right),
\]

with equality whenever \(n = \binom{r+1}{2}\).

**Proof.** Observe that \(Me_1 = \alpha \mathbb{1} + (1 - \alpha)e_1\), so that \((Me_1)_i = \alpha\) and

\[(M^2e_1)_i = (Me_1)^TMe_i = (\alpha \mathbb{1} + (1 - \alpha)e_1)^TMe_i = \alpha \langle \mathbb{1}, Me_i \rangle + \alpha(1 - \alpha).
\]

Moreover, we have that \(\langle e_1, M^2e_1 \rangle = ||Me_1||^2 = \alpha^2n + 1 - \alpha^2\) and \(\langle e_1, Me_1 \rangle = 1\). Applying Lemma 11 with \(x = e_1\), we therefore obtain

\[
\alpha^2n + 1 - \alpha^2 \leq \frac{1 - \alpha^2}{2\alpha^2(1 - \alpha^2)} + \frac{(\alpha \langle \mathbb{1}, Me_i \rangle + \alpha(1 - \alpha))^2}{\alpha^2n + 1 - \alpha^2}.
\]

Multiplying by \(n + \frac{1}{\alpha^2} - 1\) and rearranging, we equivalently have

\[
\langle \mathbb{1}, Me_j \rangle + (1 - \alpha)^2 \geq (\alpha^2n + 1 - \alpha^2) \left(n - \frac{1}{\alpha^2} - 1\right) \left(\frac{1}{\alpha^2} - 3\right).
\]

Now observe that \(\langle \mathbb{1}, Me_i \rangle = 1 - \alpha + \alpha(2d(v_i))\), so that dividing by \(\alpha^2\) gives the desired bound. If \(n = \binom{r+1}{2}\) then equality in the above also follows from Lemma 11.

In order to make use of the preceding lemma, we observe that it implies that if \(n\) is larger than \(\frac{1}{\alpha^4}\), then the vertices of the corresponding graph can be partitioned into those having high degree \(H\) and low degree \(L\), as follows.

**Lemma 18.** Let \(\alpha \in (0, 1)\), let \(C\) be a restricted spherical \(\{\alpha, -\alpha\}\)-code in \(\mathbb{R}^r\) with corresponding graph \(G = G_C\), and suppose that \(n > \frac{1}{4} \left(\frac{1}{\alpha^2} - 1\right) \left(\frac{1}{\alpha^2} - 3\right)\). If we let \(H = \{v \in C : d(v) > \frac{1}{2} + \frac{1}{\alpha} - 1\}\) and \(L = C \setminus H\), then for all \(v \in H\) we have

\[
d(v) > n - \frac{1}{4} \left(\frac{1}{\alpha^2} - 1\right) \left(\frac{1}{\alpha^2} - 3\right) + \frac{1}{\alpha} - 1.
\]

and for all \(v \in L\) we have

\[
d(v) < \frac{1}{4\alpha^2} - \left(\frac{1}{\alpha} - \frac{1}{2}\right)^2.
\]
Proof. First suppose \( d(v_i) \leq \frac{n}{2} + \frac{1}{\alpha} - 1 \). Since \( n > \frac{1}{2} \left( \frac{1}{\alpha^2} - 1 \right) \left( \frac{1}{\alpha^2} - 3 \right) \) and \( \alpha < 1 \), we have

\[
n - 2d(v_i) + \frac{2}{\alpha} - 2 \geq \sqrt{n - 1} \left( \frac{1}{\alpha^2} - 1 \right) \left( \frac{1}{\alpha^2} - 3 \right) \geq n - \frac{1}{2} \left( \frac{1}{\alpha^2} - 1 \right) \left( \frac{1}{\alpha^2} - 3 \right),
\]

so that we conclude

\[
d(v_i) < \frac{1}{\alpha} - 1 + \frac{1}{4} \left( \frac{1}{\alpha^2} - 1 \right) \left( \frac{1}{\alpha^2} - 3 \right) = \frac{1}{4\alpha^4} - \left( \frac{1}{\alpha} - \frac{1}{2} \right)^2.
\]

On the other hand, if \( d(v_i) \leq \frac{n}{2} + \frac{1}{\alpha} - 1 \) then \( n + 2d(v_i) - \frac{2}{\alpha} + 2 > n - \frac{1}{2} \left( \frac{1}{\alpha^2} - 1 \right) \left( \frac{1}{\alpha^2} - 3 \right) \) so that

\[
d(v_i) > n - \frac{1}{2} \left( \frac{1}{\alpha^2} - 1 \right) \left( \frac{1}{\alpha^2} - 3 \right) + \frac{1}{\alpha} - 1.
\]

Proof. Let \( M = M_\varnothing \) be the Gram matrix corresponding to \( \varnothing \) and let \( L = \varnothing \setminus H \) to be the set of low degree vertices so that via Lemma 18, we have \( d(v) < \frac{1}{n} \) for all \( v \in L \) and \( d(v) > n - \frac{1}{\alpha} + \frac{4}{\alpha} - 1 \geq \frac{3}{4}n + \frac{1}{\alpha} - 1 \) for all \( v \in H \). As in Lemma 17, we have \( e_1^T M 1 = an + 1 - \alpha - 2d(v_i) \) for all \( i \in [n] \), so that

\[
1^T M 1 = n(an + 1 - \alpha) - 2an \sum_{v \in L} d(v) - 2n \sum_{v \in H} d(v) \leq n(an + 1 - \alpha) - |H| \left( \frac{3}{2} \alpha n + 2(1 - \alpha) \right),
\]

and since \( 1^T M 1 \geq 0 \), we conclude \( |H| \leq \frac{n(an + 1 - \alpha)}{\frac{3}{2} \alpha n + 2(1 - \alpha)} \leq \frac{3}{4}n \). We now count \( |E(H, L)| \), the number of edges between \( H \) and \( L \), in two ways. On the one hand, we have that \( |E(H, L)| \leq \sum_{v \in L} d(v) \leq (n - |H|) \left( \frac{1}{4\alpha^4} \right) \), and on the other hand

\[
|E(H, L)| \geq \sum_{v \in H} d(v) - 2|E(H)| \geq |H| \left( n - \frac{1}{4\alpha^4} \right) - |H|^2.
\]

Putting it together and subtracting \( \frac{|H|^3}{4\alpha^4} \), we obtain \( |H| (n - |H|) \leq \frac{n}{4\alpha^4} \), and dividing by \( n \) we therefore conclude

\[
\frac{|H|}{3} \leq |H| \left( 1 - \frac{|H|}{n} \right) \leq \frac{1}{4\alpha^4}.
\]

Now that we have this improved bound on \( |H| \) we apply the above inequality once again to obtain

\[
|H| \left( 1 - \frac{3}{4\alpha^4} \right) \leq |H| \left( 1 - \frac{|H|}{n} \right) \leq \frac{1}{4\alpha^4}.
\]
Lemma 16 implies that any family of equiangular lines can be represented by a restricted spherical code. Since Lemma 19 implies that the set of high degree vertices $H$ of the corresponding graph is small, we now apply another switching in order obtain a spherical $\{\alpha, -\alpha\}$-code whose corresponding graph has a maximum degree of at most $O(1/\alpha^4)$, as desired.

Lemma 20. Let $\alpha \in (0, 1)$ and let $\mathcal{L}$ be a family of $n$ equiangular lines in $\mathbb{R}^r$ with common angle $\arccos(\alpha)$. If $n \geq 1/\alpha^4$ then there exists a spherical $\{\alpha, -\alpha\}$-code $\mathcal{C}$ representing $\mathcal{L}$ whose corresponding graph $G_\mathcal{C}$ has maximum degree

$$\Delta(G_\mathcal{C}) \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}}.$$

Proof. By Lemma 16 there exists a restricted spherical $\{\alpha, -\alpha\}$-code $\mathcal{C}$ that represents $\mathcal{L}$. Let $H = \{v_i \in [n] : d(v_i) > n/2 + 1/\alpha - 1\}$ be the high degree vertices of the corresponding graph $G_\mathcal{C}$ and let $L = \mathcal{C} \setminus H$. Then via Lemma 18 we have $d_G(v) < \frac{1}{4\alpha^4}$ for all $v \in L$ and $d_G(v) > n - \frac{1}{4\alpha^4}$ for all $v \in H$. Moreover, Lemma 20 implies $|H| \leq \frac{1}{4\alpha^4 - \frac{3}{n}}$.

Now define $\mathcal{C}' = \{-v : v \in H\} \cup L$, i.e. the spherical $\{\alpha, -\alpha\}$-code obtained from $\mathcal{C}$ by negating each vector of $H$. As in the proof of Lemma 16, $\mathcal{C}'$ still represents $\mathcal{L}$. Observe that the graph $G' = G_\mathcal{C}'$, corresponding to $\mathcal{C}'$, is obtained from $G$ by flipping all edges between $H$ and $L$. Therefore, for each vertex $v \in L$, its degree can only increase by $|H|$ and so

$$d_{G'}(v) \leq d_G(v) + |H| \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}},$$

Moreover, for each vertex $v \in H$ the degree of $-v$ in $G'$ is $d_{G'}(-v) = |L \setminus N_G(v)| + |H \cap N_G(v)|$. Since $d_G(v) > n - \frac{1}{4\alpha^4}$, we have that $|L \setminus N_G(v)| \leq \frac{1}{4\alpha^4}$, and moreover $|H \cap N_G(v)| \leq |H| \leq \frac{1}{4\alpha^4 - \frac{3}{n}}$, so that we conclude

$$d_{G'}(-v) \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}},$$

as desired. $\square$

Now that we have accomplished our goal of bounding on the maximum degree of the corresponding graph, we will use spectral methods as in $\cite{3}$ in order to obtain bounds analogous to the Alon-Boppana theorem, which we then use to obtain upper bounds on $n$. We therefore obtain a proof of Theorem 2 and furthermore, by using a stronger Alon-Boppana type theorem as in $\cite{46}$, we also conclude Theorem 3.

One of our main tools is the following inequality for a symmetric matrix, which follows immediately via Cauchy-Schwarz applied to its nonzero eigenvalues. See $\cite{3}$ for more information about the history of this inequality and its applications.

Lemma 21. Let $M \in \mathcal{S}_n$ be a real symmetric $n \times n$ matrix. Then

$$\text{tr}(M)^2 \leq \text{tr}(M^2) \text{rk}(M).$$

Proof. Let $r = \text{rk}(M)$ and note that $\lambda_i(M) = 0$ for all $i > r$, so that $\text{tr}(M) = \sum_{i=1}^r \lambda_i(M)$ and $\text{tr}(M^2) = \sum_{i=1}^r \lambda_i(M)^2$. The result now follows via Cauchy-Schwarz. $\square$

Applying Lemma 21 to the matrix $M_\mathcal{C} - \alpha J$, we immediately obtain the following bound on $n$ in terms of the average degree of the corresponding graph.
Lemma 22. Let $\alpha \in (0,1)$ and let $C$ be a spherical $\{\alpha, -\alpha\}$-code in $\mathbb{R}^r$ with corresponding graph $G_C$ having average degree $d = \overline{d}(G_C)$. Then

$$n \leq \left(1 + \left(\frac{2\alpha}{1-\alpha}\right)^2 \overline{d}\right) (r+1).$$

Proof. Let $A = A(G_C)$ be the adjacency matrix of the corresponding graph and define $B = M - \alpha J$. By the subadditivity of rank, we have $\text{rk}(B) \leq \text{rk}(M) + \text{rk}(-\alpha J) \leq r + 1$. Moreover, using (2) we also have $B = (1-\alpha)I - 2\alpha A$, so that $\text{tr}(B) = (1-\alpha)n$ and

$$\text{tr}(B^2) = \sum_{i,j=1}^n B_{i,j}^2 = (1-\alpha)^2 n + 4\alpha^2 \sum_{i=1}^n d(v_i) = ((1-\alpha)^2 + 4\alpha^2 \overline{d}) n.$$ 

Clearly $B$ is symmetric and so we may apply Lemma 21 to conclude

$$(1-\alpha)^2 n^2 \leq \text{tr}(B^2) \text{rk}(B) \leq ((1-\alpha)^2 + 4\alpha^2 \overline{d}) n(r+1).$$

Dividing by $(1-\alpha)^2 n$ now yields the desired bound.

Note that Lemma 20 already implies that if $n \geq 1/\alpha^4$ then $\overline{d} \leq \Delta \leq O(1/\alpha^4)$, so that via Lemma 22 we have $n \leq O(r/\alpha^2)$. However, this is not enough to prove Theorem 2 and Theorem 3 and so we now make use of Alon-Boppana type arguments in order to obtain sharper upper bounds on $\Delta$. To this end, we will need to define a new graph parameter which can be seen as an extension of the second eigenvalue of a regular graph.

For any graph $G$ on $n$ vertices with adjacency matrix $A = A(G)$, we define

$$\mu(G) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T Ax}{x^T x},$$

i.e. the maximum Rayleigh quotient over all vectors orthogonal to the all ones vector $1$. Note that by the Courant min-max principle, the second largest eigenvalue $\lambda_2(G) \leq \mu(G)$. Moreover, when $G$ is regular, $1$ is an eigenvector of $A$ corresponding to $\lambda_1(G)$ (see Section 5.1) and therefore $\lambda_2(G) = \mu(G)$. Using (2), we immediately obtain the following characterization of $\mu$ for the graph corresponding to a spherical $\{\alpha, -\alpha\}$-code.

Lemma 23. Let $\alpha \in (0,1)$ and let $C$ be a spherical $\{\alpha, -\alpha\}$-code in $\mathbb{R}^r$ with corresponding graph $G = G_C$. Then

$$\mu(G) \leq \frac{1}{2} \left(\frac{1}{\alpha} - 1\right),$$

with equality whenever $n \geq r + 2$.

Proof. Let $M = M_C$ be the corresponding Gram matrix and $A = A(G)$ be the adjacency matrix of the corresponding graph. Let $x \in \mathbb{R}^n \setminus \{0\}$ such that $x \perp 1$. Then $Jx = 0$ and so via (2) we have

$$0 \leq x^T M x = (1-\alpha)x^T I x - 2\alpha x^T A x,$$

so that dividing by $x^T x$ we obtain $\frac{x^T Ax}{x^T x} \leq \frac{1-\alpha}{2\alpha}$. Since $x$ was arbitrary, we conclude that $\mu(G) \leq \frac{1-\alpha}{2\alpha}$ as desired.

---

The maximum in the definition of $\mu$ is indeed attained since it is equivalent to maximizing $x^T Ax$ over the unit sphere $\{x \in \mathbb{R}^n : ||x|| = 1\}$, which is compact.
Moreover, if $n \geq r + 2$ then since $\text{rk}(M) \leq r$, the nullspace $\{ x : Mx = 0 \}$ of $M$ has dimension at least 2 and therefore must contain a nonzero vector $x$ which is orthogonal to 1. Using (2), we conclude

$$0 = x^T M x = (1 - \alpha) x^T x - 2 \alpha x^T A x,$$

which implies $\mu(G) \geq \frac{x^T A x}{x^T x} = \frac{1 - \alpha}{2\alpha}$, so that we have the equality $\mu(G) = \frac{1 - \alpha}{2\alpha}$. \hfill $\square$

Since we have just obtained an upper bound on $\mu$, we now show how to bound it from below by a function of its average or maximum degree, as in the Alon-Boppana theorem. In particular, we employ Friedman and Tillich’s approach of starting with a subgraph $H$ and “projecting out the constant” \cite{31}, as demonstrated in the following lemma.

**Lemma 24.** Let $G$ be a graph on $n$ vertices with maximum degree $\Delta = \Delta(G)$ and average degree $\bar{d} = \bar{d}(G)$. For any subgraph $H \subseteq G$, we have

$$\mu(G) \geq \lambda_1(H) - \frac{2\Delta - \bar{d}}{n} |H|.$$

**Proof.** Let $A = A(G)$ be the adjacency matrix corresponding to $G$ and let $x$ be a unit eigenvector corresponding to $\lambda_1(H)$, so that $x \in \mathbb{R}^{|H|}$ and by the Perron-Frobenius theorem all entries of $x$ are nonnegative. Observe that we may extend $x$ to $\mathbb{R}^n$ by adding 0s such that $x^T A x = \lambda_1(H)$.

Now we define $y = x - \frac{(x, \mathbbm{1})}{n} \mathbbm{1}$, i.e. the projection of $x$ onto the orthogonal complement of $\mathbbm{1}$. Observe that $y^T y = \|x\|^2 - \frac{(x, \mathbbm{1})^2}{n} \leq 1$ and moreover, if we let $V(G) = \{v_1, \ldots, v_n\}$ then

$$y^T A y = x^T A x - 2 \frac{(x, \mathbbm{1})}{n} \mathbbm{1}^T A \mathbbm{1} = \lambda_1(H) - 2 \frac{(x, \mathbbm{1})}{n} \sum_{i=1}^n x_i d(v_i) + \frac{(x, \mathbbm{1})^2}{n^2} \sum_{i=1}^n d(v_i)$$

$$\geq \lambda_1(H) - 2 \frac{(x, \mathbbm{1})}{n} \Delta \sum_{i=1}^n x_i + \frac{(x, \mathbbm{1})^2}{n^2} \bar{d}$$

$$= \lambda_1(H) - \frac{2\Delta - \bar{d}}{n} (x, \mathbbm{1})^2.$$

Additionally, note that since $x$ has at most $|H|$ nonzero entries, we have via Cauchy-Schwarz that $(x, \mathbbm{1}) \leq \|x\| \sqrt{|H|} = \sqrt{|H|}$. Therefore, we conclude the desired bound

$$\mu(G) \geq \frac{y^T A y}{y^T y} \geq y^T A y \geq \lambda_1(H) - \frac{2\Delta - \bar{d}}{n} |H|.$$

The previous lemma suggests finding a small subgraph $H$ with $\lambda_1(H)$ large and indeed, in the proof of the Alon-Boppana theorem one takes $H$ to be a ball centered at a vertex. For the proof of Theorem 2 we will take $H = K_{1,t}$, i.e. the graph consisting of a vertex connected to $t$ other vertices, where $t \leq \Delta$ will be chosen appropriately depending on how large $\Delta$ is relative to $n$. We first show that $\lambda_1(K_{1,t}) = \sqrt{t}$.

**Lemma 25.** For all $t \in \mathbb{N}$, we have $\lambda_1(K_{1,t}) = \sqrt{t}$.

**Proof.** Let $H = K_{1,t}$ and assume without loss of generality that $V(H) = \{v_1, v_2, \ldots, v_{t+1}\}$ and $v_{t+1} v_i \in E(H)$ for all $1 \leq i \leq t$. Let $A = A(H)$ be the corresponding adjacency matrix and observe that for all $x \in \mathbb{R}^{t+1}$, $x^T A x = 2x_{t+1} \sum_{i=1}^t x_i$. Note that $\lambda_1(H)$ is the maximum of $x^T A x$ over all $x \in \mathbb{R}^{t+1}$ satisfying $\|x\| = 1$. It is a straightforward calculation to verify that the maximum
occurs when \( x_i = 1/\sqrt{2t} \) for \( i \in [t] \) and \( x_{t+1} = 1/\sqrt{2} \), in which case we have \( x^\top A x = \sqrt{t} \), as desired.

Now we combine Lemma 23 and Lemma 25 to obtain the desired lower bound on \( \mu \).

**Lemma 26.** Let \( G \) be a graph on \( n \) vertices with maximum degree \( \Delta = \Delta(G) \). For all \( t \in \mathbb{N} \) with \( t \leq \Delta \), we have
\[
\mu(G) \geq \sqrt{t} - \frac{2\Delta(t+1)}{n}.
\]

**Proof.** Observe that we can always find a copy of \( K_{1,t} \) as a subgraph of \( G \). Indeed, just consider a vertex \( v \) having degree \( d(v) = \Delta \), together with \( t \) of its neighbors. Letting \( H \) be this subgraph and applying Lemma 24 and Lemma 25, we obtain the desired
\[
\mu(G) \geq \lambda_1(H) - \frac{2\Delta}{n} |H| \geq \sqrt{t} - \frac{2\Delta(t+1)}{n}.
\]

Finally, we apply the above lemmas in order to prove Theorem 2.

**Proof of Theorem 2.** By Lemma 20 there exists a spherical \( \{\alpha, -\alpha\} \)-code \( \mathcal{C} \) representing a set of \( n = N_{\alpha}^R(r) \) equiangular lines in \( \mathbb{R}^r \), such that the corresponding graph \( G = G_{\mathcal{C}} \) has maximum degree \( \Delta = \Delta(G) \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - 3/\pi} \). Our goal will be to show that
\[
n \leq \max \left( \frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)}, \left( 2 + \frac{8\alpha^2}{(1-\alpha)^2} \right) (r+1) \right),
\]
and so we henceforth assume that \( n > \frac{2}{\alpha^5} + \frac{2}{\alpha^3(1-\alpha)} \), so that we have
\[
\Delta \leq \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{n}} < \frac{1}{4\alpha^4} + \frac{1}{4\alpha^4 - \frac{3}{2\alpha^5}} = \frac{1}{2\alpha^4} + \frac{3}{8\alpha^5} (1 - \frac{3}{8\alpha}) < \frac{1}{2\alpha^4} + \frac{3}{20\alpha^3}.
\]
We will show that in this case, we must have \( 4\Delta^{3/2} < n \).

To this end, let us assume for sake of contradiction that \( 4\Delta^{3/2} \geq n \). We first establish the following inequality,
\[
n \leq 4\Delta \left( \frac{1}{\alpha} - 1 - \frac{2\alpha}{1-\alpha} \right).
\]

Indeed, observe that if \( \frac{4\Delta}{n} \geq \frac{\alpha}{1-\alpha} \) then we are done, so suppose instead that \( \frac{4\Delta}{n} < \frac{\alpha}{1-\alpha} \). Now define \( t = \left\lceil \frac{n^2}{16\Delta^2} \right\rceil \) and note that \( t \leq \Delta \), so that we may apply Lemma 26 to obtain
\[
\mu(G) \geq \sqrt{t} - \frac{2\Delta(t+1)}{n} \geq \frac{n}{4\Delta} - \frac{2\Delta \left( \frac{n^2}{16\Delta^2} + 2 \right) n}{8\Delta} = \frac{n}{8\Delta} - \frac{4\Delta}{n} > \frac{n}{8\Delta} - \frac{\alpha}{1-\alpha}.
\]

On the other hand, Lemma 23 gives \( \mu(G) \leq \sqrt{t} \left( \frac{1}{\alpha} - 1 - \frac{2\alpha}{1-\alpha} \right) \), so that we conclude \( \frac{1}{2} \left( \frac{n}{8\Delta} - \frac{\alpha}{1-\alpha} \right) \geq \frac{n}{8\Delta} \), which is equivalent to (3). Using our bound on \( \Delta \) together with (3), we obtain
\[
n \leq 4 \left( \frac{1}{2\alpha^4} + \frac{3}{20\alpha^3} \right) \left( \frac{1}{\alpha} - 1 - \frac{2\alpha}{1-\alpha} \right) \leq 2 \frac{\alpha}{\alpha^5} + \left( \frac{1}{2\alpha^4} + \frac{3}{20\alpha^3} \right) \frac{2\alpha}{1-\alpha} = \frac{2\alpha}{\alpha^5} + \frac{1 + \frac{3}{10\alpha}}{(1-\alpha)\alpha^3}
\]
\[
< \frac{2}{\alpha^5} + \frac{2}{(1-\alpha)\alpha^3},
\]

23
a contradiction.

Now that we have \(4\Delta^{3/2} < n\), we apply Lemma 23 and Lemma 26 with \(t = \Delta\) in order to obtain
\[
\frac{1}{2} \left( \frac{1}{\alpha} - 1 \right) \geq \sqrt{\Delta} - \frac{2\Delta(\Delta + 1)}{n} \geq \sqrt{\Delta} - \frac{2\Delta(\Delta + 1)}{4\Delta^{3/2}} = \frac{1}{2} \left( \sqrt{\Delta} - \frac{1}{\sqrt{\Delta}} \right) \geq \frac{1}{2} (\sqrt{\Delta} - 1),
\]
where in the last inequality we use that \(\Delta \geq 1\) (otherwise \(G\) is the empty graph, so that \(\mathcal{C}\) is a spherical \(\{\alpha\}\)-code in \(\mathbb{R}^\ell\) and therefore we trivially have \(|\mathcal{C}| \leq r\). It follows that \(\Delta \leq 1/\alpha^2\) and so applying Lemma 23 and Lemma 26 with \(t = \Delta\) once again, we obtain
\[
\frac{1}{2} \left( \frac{1}{\alpha} - 1 \right) \geq \sqrt{\Delta} - \frac{2\Delta^{1/2}(\Delta + 1)}{n} \geq \sqrt{\Delta} - \frac{2\Delta^{1/2}(\Delta + 1)}{4\Delta^{3/2}} = \sqrt{\Delta} - \frac{\alpha}{\sqrt{\Delta}} + \frac{\alpha}{\sqrt{\Delta}} = \sqrt{\Delta} - \alpha,
\]
and therefore \(\Delta \leq \left(\frac{1}{\alpha^2} + \alpha\right)^2\). Finally, using the fact that the average degree \(\bar{d}(G) \leq \Delta\), we apply Lemma 22 to obtain the desired bound
\[
n \leq \left(1 + \left(\frac{2\alpha}{1 - \alpha}\right)^2 \bar{d}\right) (r + 1) \leq \left(1 + \left(1 + \frac{2\alpha^2}{1 - \alpha}\right)^2\right) (r + 1)
\[
\leq \left(2 + \frac{8\alpha^2}{(1 - \alpha)^2}\right) (r + 1).
\]

In the previous proof, we used Lemma 26 with \(H\) being a ball of radius 1, but in order to prove Theorem 3, we will take \(H\) to be a ball of some radius \(q \geq 2\). More specifically, for all \(q \in \mathbb{N}\) and for any graph \(G\) with vertex \(v \in V(G)\), we define \(G(v, q)\) to be the subset of vertices for which there exists a path from \(v\) of length at most \(q\), i.e. the ball of radius \(q\) centered at \(v\) in \(G\). We first recall the well-known fact that \(|G(v, q)| \leq \sum_{i=0}^{\lfloor \frac{q}{2}\rfloor} \Delta^i = \frac{\Delta^{\lfloor \frac{q}{2}\rfloor + 1} - 1}{\Delta - 1}\). Indeed, there can be at most \(\Delta^i\) vertices at a distance of exactly \(i\) from \(v\). We will also make use of the following lemma of Jiang [45] (see also Jiang and Polyanskii [46]) showing that for any \(q \in \mathbb{N}\), every graph has a ball of radius \(q\) whose first eigenvalue is large.

Lemma 27 (Jiang [45]). Let \(q \in \mathbb{N}\) and let \(G\) be a graph on \(n\) vertices with average degree \(\bar{d} = \bar{d}(G) \geq 1\). Then there exists a vertex \(v_0 \in V(G)\) such that
\[
\lambda_1(G(v_0, q)) \geq 2\sqrt{\bar{d} - 1} \cos\left(\frac{\pi}{q + 2}\right).
\]

Proof of Theorem 3 By Lemma 20 there exists a spherical \(\{\alpha, -\alpha\}\)-code \(\mathcal{C}\) representing a set of \(n = N_{\alpha}^R(r)\) equiangular lines in \(\mathbb{R}^r\), such that the corresponding graph \(G = G_{\mathcal{C}}\) has maximum degree \(\Delta = \Delta(G) \leq \frac{1}{4\alpha} + \frac{1}{4\alpha - 3/\alpha^2}\). Since we must have \(n \geq r\) (see the end of Section 1.1) and \(q \geq 2\), it follows that \(n >> 1/\alpha^{2q+1} \geq 1/\alpha^5\). In particular, we may apply the same argument as in the proof of Theorem 2 to conclude that \(\Delta \leq \left(\frac{1}{\alpha^5} + \alpha\right)^2\).

Our goal is to now use Lemma 23 in order to obtain a stronger bound on \(\bar{d}\). If \(\bar{d} \leq 1\) then we are done, and otherwise we may apply Lemma 27 to obtain a vertex \(v_0\) such that \(H = G(v_0, q)\), i.e. the ball of radius \(q\) centered at \(v_0\), satisfies \(\lambda_1(H) \geq 2\sqrt{\bar{d} - 1} \cos\left(\frac{\pi}{q + 2}\right)\). As previously noted, we have \(|H| \leq O(\Delta^q)\) and thus \(\Delta |H| \leq O(1/\alpha^{2q+2})\). Therefore, we may apply Lemma 23 and
Lemma 24 in order to conclude
\begin{equation*}
\frac{1 - \alpha}{2\alpha} \geq \mu(G) \geq \lambda_1(H) - \frac{2\Delta|H|}{n} \geq 2\sqrt{d - 1} \cos\left(\frac{\pi}{q + 2}\right) - o(1),
\end{equation*}
so that \(d \leq \frac{1 + o(1)}{16\alpha^2 \cos^2\left(\frac{\pi}{q + 2}\right)}\). To complete the proof, we now apply Lemma 22 to obtain
\begin{equation*}
n \leq \left(1 + \frac{4\alpha^2}{(1 - \alpha)^2} d\right) (r + 1) \leq (1 + o(1)) \left(1 + \frac{1}{4 \cos^2\left(\frac{\pi}{q + 2}\right)}\right) r.
\end{equation*}

\section{Equiangular lines in \(\mathbb{C}^r\)}

In this section, the goal will be to obtain a universal upper bound on \(N^\mathbb{C}_r(\alpha)\), the maximum number of complex equiangular lines in \(\mathbb{C}^r\) with common angle \(\arccos(\alpha)\), thereby proving Theorem 4. This can be seen as a complex version of Theorem 1 and indeed the proof proceeds analogously, except with Theorem 8 replaced by Theorem 9. Indeed, each lemma in this section has a corresponding lemma in Section 3.1 and as such, we skip details in the following proofs whenever they follow analogously to the real case.

Let \(r \in \mathbb{N}\) and let \(\mathcal{L} = \{\ell_1, \ldots, \ell_n\}\) be a set of \(n\) equiangular lines in \(\mathbb{C}^r\). As noted in Section 2, if we choose a complex unit vector \(v_i \in \ell_i\) along the \(i\)th line, the resulting collection \(\mathcal{C} = \{v_1, \ldots, v_n\}\) forms a spherical \(S^1(\alpha)\)-code, where \(S^1(\alpha)\) is the circle of radius \(\alpha\) centered at 0 in \(\mathbb{C}\). As in Section 3, we say that \(\mathcal{C}\) represents \(\mathcal{L}\) and for convenience, we shall always let \(n = |\mathcal{C}|\) and label the vectors in \(\mathcal{C}\) as \(v_1, \ldots, v_n\).

As in Section 3.1, we proceed by obtaining bounds on eigenvalues and eigenvectors of the Gram matrix \(M_C\) corresponding to a spherical \(S^1(\alpha)\)-code \(\mathcal{C}\). We first apply Theorem 9 with \(x\) and \(y\) being orthogonal eigenvectors corresponding to some positive eigenvalues \(\lambda, \mu\) of \(M\), to obtain the following lemma analogous to Lemma 10.

**Lemma 28.** Let \(\alpha \in (0, 1)\) and let \(\mathcal{C}\) be a spherical \(S^1(\alpha)\)-code in \(\mathbb{C}^r\) with corresponding Gram matrix \(M = M_\mathcal{C}\). If \(x, y \in \mathbb{C}^r\) are orthogonal unit eigenvectors of \(M\) with nonzero eigenvalues \(\lambda, \mu\), then
\begin{equation*}
\sum_{i=1}^n |x_i|^2 |y_i|^2 \leq \frac{1 - \alpha^2}{\lambda \mu},
\end{equation*}
with equality whenever \(n = r^2\).

**Proof.** Similarly to the proof of Lemma 10, the result follows by applying Theorem 9 and dividing by \(\lambda^2 \mu^2\), with the case of equality also following via Theorem 9.

Applying Theorem 9 with \(y = e_i\) being a standard basis vector, we obtain the following lemma, which is analogous to Lemma 11.

**Lemma 29.** Let \(\alpha \in (0, 1)\) and let \(\mathcal{C}\) be a spherical \(S^1(\alpha)\)-code in \(\mathbb{C}^r\) with corresponding Gram matrix \(M = M_\mathcal{C}\). For all \(i \in [n]\) and \(x \in \mathbb{C}^n\), we have
\begin{equation*}
\frac{1 - \alpha^2}{\alpha^2} (\langle x, Mx \rangle - |(Mx)_i|^2) + \frac{1}{\alpha^2 n + 1 - \alpha^2} |(M^2x)_i|^2 \geq \langle x, M^2x \rangle,
\end{equation*}
with equality whenever \(n = r^2\).
Proof. Similarly to the proof of Lemma 11 we apply Theorem 9 with $y = e_i$ to conclude

$$(1 - \alpha^2) \langle x, Mx \rangle + \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} |(M^2 x)_i|^2 \geq \alpha^2 |Mx|^2 + (1 - \alpha^2) |(Mx)_i|^2,$$

which is equivalent to the desired bound. Furthermore, if $n = r^2$ then we have equality in Theorem 9 and therefore, we have equality above.

We now apply Lemma 28 and Lemma 11 in order to obtain the following, which is analogous to Lemma 12.

Lemma 30. Let $\alpha \in (0, 1)$ and let $C$ be a spherical $S^1(\alpha)$-code in $\mathbb{C}^r$ with corresponding Gram matrix $M = M_{\mathcal{C}}$. Let $x$ be a unit eigenvector corresponding to the largest eigenvalue $\lambda_1 = \lambda_1(M)$. If $\lambda_1 > \frac{1 - \alpha^2}{\alpha^2 n + 1 - \alpha^2}$, then

$$x_i^2 \geq \frac{1 - \frac{1 - \alpha^2}{\alpha^2 \lambda_1} - \frac{1 - \alpha^2}{\alpha^2 \lambda_1}}{\frac{1}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{\alpha^2}},$$

and moreover the second largest eigenvalue satisfies

$$\lambda_2(M) \leq (1 - \alpha^2) \frac{\lambda_1}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{\alpha^2 \lambda_1},$$

with equality in both whenever $n = r^2$.

Proof. Similarly to the proof of Lemma 12 we let $i \in [n]$ and apply Lemma 29 to obtain

$$\left( \frac{\lambda_1^2}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{\alpha^2} \right) |x_i|^2 \geq 1 - \frac{1 - \alpha^2}{\alpha^2 \lambda_1}.$$

Since the right-hand side of the above is positive, the left-hand side must also be positive and we may divide by it to conclude

$$|x_i|^2 \geq \frac{1 - \frac{1 - \alpha^2}{\alpha^2 \lambda_1}}{\frac{1}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{\alpha^2}}.$$

Since the above holds for all $i \in [n]$, we now apply Lemma 28 with $y$ being a unit eigenvector corresponding to $\lambda_2 = \lambda_2(M)$ in order to obtain

$$\frac{1 - \frac{1 - \alpha^2}{\alpha^2 \lambda_1} - \frac{1 - \alpha^2}{\alpha^2}}{\frac{1}{\alpha^2 n + 1 - \alpha^2} - \frac{1 - \alpha^2}{\alpha^2}} \leq \sum_{k=1}^{n} |x_i|^2 |y_i|^2 \leq \frac{1 - \alpha^2}{\lambda_1 \lambda_2},$$

which yields the desired upper bound on $\lambda_2$. Finally, $n = r^2$ implies we have equality in Lemma 28 and Lemma 29 in which case we have equality everywhere above.

As in Lemma 13 we give an upper bound on $\lambda_1(M)$.

Lemma 31. Let $\alpha \in (0, 1)$ and let $C$ be a spherical $S^1(\alpha)$-code in $\mathbb{C}^r$ with corresponding Gram matrix $M = M_{\mathcal{C}}$. Then

$$\lambda_1(M) \leq \alpha n + 1 - \alpha.$$

Proof. The proof is identical to that of Lemma 13.

26
When \( \lambda_1(M) \) is bigger than \( \frac{1}{\alpha^2} \), we give the following upper bound on \( n \), in analogy to Lemma 14.

**Lemma 32.** Let \( \alpha \in (0, 1) \) and let \( \mathcal{C} \) be a spherical \( S^1(\alpha) \)-code in \( \mathbb{C}^r \) with corresponding Gram matrix \( M = M_{\mathcal{C}} \). If \( \lambda_1 = \lambda_1(M) > \frac{1-\alpha^2}{\alpha^2} \) then

\[
 n < \frac{1 + \alpha}{\alpha - \frac{1-\alpha^2}{\alpha^2} \alpha}(r - 1) + 1.
\]

**Proof.** As in the proof of Lemma 14, we apply Lemma 31 to obtain

\[
 \lambda_1 \alpha^2 \frac{n + 1}{\alpha} \leq \left( 1 - \frac{1-\alpha^2}{\alpha^2} \right) \alpha \left( 1 - \frac{1-\alpha^2}{\alpha^2} \right) \alpha \frac{\lambda_1}{\alpha - \frac{1-\alpha^2}{\alpha^2} \alpha} = \frac{1 - \alpha^2}{\alpha - \frac{1-\alpha^2}{\alpha^2} \alpha}.
\]

We now apply Lemma 31 once again in order to conclude the desired bound

\[
 (1 - \alpha)(n - 1) = n - (\alpha n + 1 - \alpha) \leq \text{tr}(M) - \lambda_1 = \sum_{i=2}^{r} \lambda_i(M) < \frac{1 - \alpha^2}{\alpha - \frac{1-\alpha^2}{\alpha^2} \alpha}(r - 1).
\]

When \( \lambda_1(M) \) is not too big, we give the following bound, which is analogous to Lemma 15.

**Lemma 33.** Let \( \alpha \in (0, 1) \), and let \( \mathcal{C} \) be a spherical \( S^1(\alpha) \)-code in \( \mathbb{C}^r \) with corresponding Gram matrix \( M = M_{\mathcal{C}} \). Then we have

\[
 n < \frac{\lambda_1(M)}{\alpha} \sqrt{r}.
\]

**Proof.** The proof is identical to that of Lemma 15. 

Using Lemma 32 and Lemma 33, we now give a proof of Theorem 4.

**Proof of Theorem 4.** Analogous to the proof of Theorem 1, we let \( \mathcal{C} \) be a spherical \( S^1(\alpha) \)-code representing a set of \( n = N^\alpha_{\mathcal{C}}(r) \) equiangular lines in \( \mathbb{C}^r \), and let \( M = M_{\mathcal{C}} \) be the corresponding Gram matrix. Also let \( \lambda_1 = \lambda_1(M) \) and \( t = \alpha^2(1 + \alpha) \sqrt{r} \). If \( \lambda_1 \leq \frac{1 + t}{\alpha^2} \), then applying Lemma 33 yields

\[
 n < \frac{\lambda_1 \sqrt{r}}{\alpha} \leq \frac{(t + 1) \sqrt{r}}{\alpha^3}.
\]

Otherwise, \( \lambda_1 > \frac{1 + t}{\alpha^2} \) so that \( \alpha - \frac{1-\alpha^2}{\alpha^2} > \alpha \left( 1 - \frac{1}{1+t^2} \right) = \alpha \frac{1}{1+t^2} \), and thus applying Lemma 32 yields

\[
 n < \frac{(1 + \alpha)(1 + t)}{\alpha t}(r - 1) + 1 < \frac{(1 + \alpha)(1 + t)}{\alpha t} r = \frac{(1 + t) \sqrt{r}}{\alpha^3}.
\]

5 Eigenvalues of regular graphs

In this section, we obtain our main results on the eigenvalues of the adjacency matrix of regular graphs, thereby obtaining a generalization of the Alon-Boppana theorem to dense regular graphs.
5.1 What is known?

In this subsection, we briefly discuss some of the well-known results about eigenvalues of regular graphs. Let $G$ be a $k$-regular graph with corresponding adjacency matrix $A$ and eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. It is known that the Perron-Frobenius theorem implies that $\lambda_1 = k$ with $1$ as a corresponding eigenvector and that the multiplicity of $\lambda_1$, $m(\lambda_1)$, is equal to the number of connected components of $G$, see e.g. [34]. Moreover, we have that $-\lambda_n \leq k$ with equality if and only if $G$ is bipartite. Note that for regular graphs, questions about $\lambda_2$ and $m(\lambda_2)$ are essentially equivalent to that of $\lambda_n$ and $m(\lambda_n)$. Indeed, the complement $G^c$ of $G$ is $n-k-1$ regular and satisfies $A(G) + A(G^c) = J - I$, from which it follows that $\lambda_n(G) = -\lambda_2(G^c) - 1$ and $m_{A(G^c)}(\lambda_2) = m_{A(G)}(-\lambda_n)$. Also note that if $G$ is disconnected, its adjacency matrix has a block diagonal form with one block for each connected component, so that its eigenvalues are the union of the eigenvalues of each of its connected components, taken with multiplicity. Thus it suffices to study the eigenvalues of connected graphs.

It is also well known that $\lambda_2(K_n) = -1$ for any complete graph $K_n$, that $\lambda_2(K_{n_1, \ldots, n_r}) = 0$ for any complete multipartite graph $K_{n_1, \ldots, n_r}$, and that $\lambda_2(H) > 0$ for any other nontrivial connected graph $H$, see e.g. [21] p. 163. Beyond this, there is a line of research characterizing graphs $H$ with $\lambda_2(H) \leq c$ or $\lambda_2(H) = c$ for small constant $c$ such as $1/3, 1, \sqrt{2} - 1, (\sqrt{5} - 1)/2, 2$, see [20]. There are also important results on characterizing graphs $H$ with the smallest eigenvalue $\lambda_n(H) \geq -c$ for small $c$, such as the classical result for $c = 2$ [13, 34] and the more recent partial generalization to $c = 3$ [49].

5.2 Lower bounds for $\lambda_2$

In this subsection, we will prove Theorem 5 and Corollary 6 which can be seen as generalizations of the Alon-Boppana theorem to dense graphs. Our approach will be to convert the given regular graph into a corresponding system of real equiangular lines and then apply the methods of Section 2. Indeed, the asymptotic bounds of Corollary 6 can be obtained directly from Theorem 8 via the approach used in Lemma 12. However, if we are a bit more careful, we can use the connection to equiangular lines to prove inequalities which are tight whenever our graph is a strongly regular graph corresponding to a collection of $(r+1)$ equiangular lines in $\mathbb{R}^r$. To this end, we first derive an improved version of Theorem 8 in the special case where $y = 1$ is an eigenvector of $M$.

**Theorem 34.** Let $\alpha \in (0, 1)$ and let $G$ be a spherical $\{\alpha, -\alpha\}$-code in $\mathbb{R}^r$ with corresponding Gram matrix $M = M_G$. If $1$ is an eigenvector of $M$ with corresponding eigenvalue $\lambda \neq 0$, then for all $x \in \mathbb{R}^r$ we have

$$
\frac{(1-\alpha^2)n}{2\lambda} \langle x, Mx \rangle + \left( \frac{\lambda^2}{n} - 1 \right) \langle x, 1 \rangle^2 \geq \langle x, M^2x \rangle,
$$

with equality whenever $n = \binom{r+1}{2} - 1$.

**Proof.** Let us suppose we are in the setting of the proof of Theorem 8 so that $G = \{v_1, \ldots, v_n\}$ and we have $W_i = v_iv_i^T$ for $i \in [n]$, with the corresponding linear map $\Psi : \mathbb{R}^n \to \mathcal{S}_r$ defined by $\Psi e_i = W_i$ for $i \in [n]$ and $\Psi^\# : \mathcal{S}_r \to \mathcal{S}_r$ being the orthogonal projection onto the span of $W_1, \ldots, W_n$.

Now let $y \in \mathbb{R}^n$ and define $X_y = \frac{1}{2} (V_1 y (V_1)^T + V_1 (V y)^T)$. Following the proof of Theorem 8 we compute

$$
\|X_y\|^2_\mathcal{F} = \frac{1}{2} \left( \langle y, M y \rangle \langle 1, 1 \rangle + \langle y, M 1 \rangle^2 \right) = \frac{1}{2} \left( \lambda n \langle y, M y \rangle + \lambda^2 \langle y, 1 \rangle^2 \right),
$$

(4)
and \( \|\mathcal{P}_X\|_F^2 = \frac{1}{1-\alpha^2} \left( \lambda^2 \langle y, M^2 y \rangle - \frac{\alpha^2 \lambda^4}{\alpha^2 n + 1 - \alpha^2} \langle y, \mathbb{1} \rangle^2 \right) \), where \( f \) is defined by \( f(t) = t^2 \). Now we observe that \( f(M\mathbb{1}) = f(\lambda\mathbb{1}) = \lambda^2 \mathbb{1} \), so that \( \langle f(My), f(M\mathbb{1}) \rangle = \lambda^2 \langle f(My), \mathbb{1} \rangle = \lambda^2 \|My\|^2 = \lambda^2 \langle y, M^2 y \rangle \). Therefore, we conclude that

\[
\|\mathcal{P}_X\|_F^2 = \frac{1}{1-\alpha^2} \left( \lambda^2 \langle y, M^2 y \rangle - \frac{\alpha^2 \lambda^4}{\alpha^2 n + 1 - \alpha^2} \langle y, \mathbb{1} \rangle^2 \right). 
\]

(5)

We first let \( y = \mathbb{1} \), so that via \( \|X_1\|_F^2 = \lambda^2 n^2 \) and

\[
\|\mathcal{P}_X\|_F^2 = \frac{1}{1-\alpha^2} \left( \lambda^4 n - \frac{\alpha^2 \lambda^4 n^2}{\alpha^2 n + 1 - \alpha^2} \right) = \frac{\lambda^4 n}{1-\alpha^2} \left( 1 - \frac{\alpha^2 n}{\alpha^2 n + 1 - \alpha^2} \right) = \frac{\lambda^4 n}{\alpha^2 n + 1 - \alpha^2}.
\]

We now claim that \( X_1 - \mathcal{P}X_1 \neq 0 \). Indeed, otherwise we would have

\[
0 = \|X_1 - \mathcal{P}X_1\|_F^2 = \|X_1\|_F^2 - \|\mathcal{P}X_1\|_F^2 = \lambda^2 n^2 - \frac{\lambda^4 n}{\alpha^2 n + 1 - \alpha^2} = \lambda^2 n \left( n - \frac{\lambda^2}{\alpha^2 n + 1 - \alpha^2} \right),
\]

which implies \( \lambda^2 = n(\alpha^2 n + 1 - \alpha^2) \). However, we also have \( \sum_{i=1}^n \lambda_i(M^2) = \text{tr}(M^2) = n(\alpha^2 n + 1 - \alpha^2) \), so that all eigenvalues of \( M \) except \( \lambda \) must be 0 and thus \( M = \frac{\lambda}{n} \mathbb{1} \mathbb{1}^T \), which yields a contradiction since \( \frac{\lambda}{n} = M_{1,1} = 1 \) while \( \frac{\lambda}{n} = |M_{1,2}| = \alpha < 1 \).

Therefore, we may define

\[
Z = \frac{X_1 - \mathcal{P}X_1}{\sqrt{\|X_1\|_F^2 - \|\mathcal{P}X_1\|_F^2}} = \frac{X_1 - \mathcal{P}X_1}{\lambda \sqrt{n^2 - \frac{\lambda^4 n}{\alpha^2 n + 1 - \alpha^2}}}
\]

and observe that \( Z \) is orthogonal to \( W_1, \ldots, W_n \) and satisfies \( \|Z\|_F = 1 \). It follows that \( \mathcal{P}' = \mathcal{P} + ZZ^\# \) is the orthogonal projection onto the span of \( W_1, \ldots, W_n, Z \) and moreover,

\[
\|X_1\|_F^2 \geq \|\mathcal{P}'X_1\|_F^2 = \|\mathcal{P}X_1\|_F^2 + \langle Z, X_1 \rangle_F^2.
\]

Now we apply \( \|X_1\|_F^2 \) with \( y = x \) to obtain

\[
\|X_1\|_F^2 = \frac{1}{2} \left( \lambda n \langle x, Mx \rangle + \lambda^2 \langle x, \mathbb{1} \rangle^2 \right),
\]

as well as

\[
\|\mathcal{P}X_1\|_F^2 = \frac{1}{1-\alpha^2} \left( \lambda^2 \langle x, M^2 x \rangle - \frac{\alpha^2 \lambda^4}{\alpha^2 n + 1 - \alpha^2} \langle x, \mathbb{1} \rangle^2 \right),
\]

so that it remains to compute \( \langle Z, X_1 \rangle_F \). To this end, we have

\[
\langle X_1, Vx(V\mathbb{1})^T \rangle_F = \langle V\mathbb{1}, Vx \rangle \langle V\mathbb{1}, V\mathbb{1} \rangle = \langle M\mathbb{1}, x \rangle \langle \mathbb{1}, M\mathbb{1} \rangle = \lambda^2 n \langle x, \mathbb{1} \rangle
\]

and similarly \( \langle X_1, V\mathbb{1}(Vx)^T \rangle_F = \lambda^2 n \langle x, \mathbb{1} \rangle \), so that \( \langle X_1, X_1 \rangle_F = \lambda^2 n \langle x, \mathbb{1} \rangle \). Moreover, for all \( i \in [n] \) and \( y \in \mathbb{R}^n \) we have

\[
\langle W_i, V\mathbb{1}(Vy)^T \rangle_F = \langle v_i, V\mathbb{1} \rangle \langle v_i, Vy \rangle = \langle e_i, M\mathbb{1} \rangle \langle e_i, My \rangle = \lambda(My)_i,
\]

and similarly \( \langle W_i, V\mathbb{1}(Vy)^T \rangle_F = \lambda(My)_i \), so that \( \mathcal{W}X_y = \lambda My \). Now recalling \( \mathcal{W}' = (1 - \mathcal{W}) \mathcal{W} \), we conclude that...
\( \alpha^2 I + \alpha^2 J \), it follows that
\[
\langle \mathcal{F}X_1, X_x \rangle = X_1^\dagger \mathcal{W}^\dagger (\mathcal{W}^\dagger \mathcal{W})^{-1} \mathcal{W}^\dagger X_x = \lambda^2 (M 1) \mathcal{T} ((1 - \alpha^2 I + \alpha^2 J)^{-1} M x
\]
\[
= \frac{\lambda^3}{1 - \alpha^2} \mathcal{T} \left( I - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} J \right) M x
\]
\[
= \frac{\lambda^3}{1 - \alpha^2} \left( 1 - \frac{\alpha^2 n}{\alpha^2 n + 1 - \alpha^2} \right) \| x \|^2 M x
\]
\[
= \frac{\alpha^2 n + 1 - \alpha^2}{\alpha^2 n + 1 - \alpha^2} \langle x, 1 \rangle.
\]

Therefore, we conclude
\[
\langle Z, X_x \rangle = \langle X_1, X_x \rangle - \langle \mathcal{F}X_1, X_x \rangle
\]
\[
= \frac{\lambda^2 n}{\lambda^2 n} \langle x, 1 \rangle - \frac{\lambda^4}{\alpha^2 n + 1 - \alpha^2} \langle x, 1 \rangle
\]
\[
= \frac{\lambda^2 n}{\lambda^2 n} \langle x, 1 \rangle - \frac{\lambda^4}{\alpha^2 n + 1 - \alpha^2} \langle x, 1 \rangle
\]
\[
= \frac{\lambda^2 n}{\lambda^2 n} \langle x, 1 \rangle - \frac{\lambda^4}{\alpha^2 n + 1 - \alpha^2} \langle x, 1 \rangle.
\]

Putting everything together, we finally obtain
\[
\frac{1}{2} \left( \lambda n \langle x, M x \rangle + \lambda^2 \langle x, 1 \rangle^2 \right) \geq \frac{1}{2} \left( \lambda^2 \langle x, M^2 x \rangle - \frac{\alpha^2 \lambda^4}{\alpha^2 n + 1 - \alpha^2} \langle x, 1 \rangle^2 \right)
\]
\[
+ \frac{\lambda^2}{n} \left( n^2 - \frac{\lambda^2 n}{\alpha^2 n + 1 - \alpha^2} \right) \langle x, 1 \rangle^2.
\]

Now multiplying by \( \frac{1 - \alpha^2}{\lambda} \) and collecting terms corresponding to \( \langle x, 1 \rangle^2 \), we conclude
\[
\frac{(1 - \alpha^2)n}{2 \lambda} \langle x, M x \rangle
\]
\[
\geq \langle x, M^2 x \rangle + \left( \frac{1 - \alpha^2}{2} - \frac{\alpha^2}{\alpha^2 n + 1 - \alpha^2} \frac{\lambda^2}{\alpha^2 n + 1 - \alpha^2} + (1 - \alpha^2) \left( 1 - \frac{\lambda^2}{n(\alpha^2 n + 1 - \alpha^2)} \right) \right) \langle x, 1 \rangle^2
\]
\[
= \langle x, M^2 x \rangle - \left( \frac{\lambda^2}{n} - \frac{1 - \alpha^2}{2} \right) \langle x, 1 \rangle^2,
\]

as desired. Moreover, as in the proof of Theorem 8, we note that \( \mathcal{F} \) has dimension \( \binom{n + 1}{2} \) so that if \( n = \binom{r + 1}{2} - 1 \), then since \( \text{rk}(\mathcal{W}) = n \) and \( Z \) is orthogonal to \( W_1, \ldots, W_n \), we have that \( W_1, \ldots, W_n, Z \) span \( \mathcal{F} \) and thus \( \mathcal{F}' \) is the identity map, giving equality above. \( \square \)

Remark. The inequality of Theorem 34 is equivalent to the matrix \( \frac{1 - \alpha^2}{2 \lambda} M + \left( \frac{\lambda^2}{n} - \frac{1 - \alpha^2}{2} \right) J - M^2 \) being positive semidefinite.

We now apply Theorem 34 to obtain a useful lemma which is similar to Lemma 12.

Lemma 35. Let \( \alpha \in (0, 1) \) and let \( C \) be a spherical \( \{\alpha, -\alpha\} \)-code in \( \mathbb{R}^r \) with corresponding Gram matrix \( M = M_C \). If \( 1 \) is an eigenvector of \( M \) with corresponding eigenvalue \( \lambda \neq 0 \) then
\[
\lambda^2 \geq n(\alpha^2 n + 1 - \alpha^2) - \frac{1 - \alpha^2}{2} n \left( \frac{n}{\lambda} - 1 \right),
\]

30
and for any eigenvalue \( \mu \) of \( M \) corresponding to an eigenvector orthogonal to \( \mathbf{1} \), we have

\[
\mu \leq \frac{(1 - \alpha^2)n}{2\lambda},
\]

with equality in both whenever \( n = \binom{r+1}{2} - 1 \).

**Proof.** Note that \( \langle e_1, M^2e_1 \rangle = \alpha^2n + 1 - \alpha^2 \). Thus applying Theorem 34 with \( x = e_1 \), we obtain

\[
\frac{1 - \alpha^2}{2} \left( \frac{n}{\lambda} - 1 \right) + \frac{\lambda^2}{n} = \frac{(1 - \alpha^2)n}{2\lambda} + \frac{\lambda^2}{n} - \frac{1 - \alpha^2}{2} \geq \alpha^2n + 1 - \alpha^2,
\]

which is equivalent to the first bound.

For the second bound, if \( \mu = 0 \) then it holds trivially. Otherwise, we let \( x \) be a unit eigenvector corresponding to \( \mu \). Since \( \langle x, \mathbf{1} \rangle = 0 \), we have via Theorem 34 that

\[
(1 - \alpha^2)n\mu \geq \mu^2.
\]

Dividing by \( \mu \) gives the desired bound. Moreover, \( n = \binom{r+1}{2} - 1 \) implies we have equality in Theorem 34 and therefore also in both of the above bounds. \( \square \)

**Remark.** The bound on \( \mu \) of Lemma 35 also follows directly from Lemma 10 with \( x \) being a unit eigenvector corresponding to \( \mu \) and \( y = \frac{1}{\sqrt{n}} \mathbf{1} \).

Using Lemma 35, we are now able to obtain the desired bounds on the eigenvalues of regular graphs and prove Theorem 5. For this purpose, recall that \( m(\lambda) \) is defined to be the multiplicity of \( \lambda \) for the adjacency matrix \( A \).

**Proof of Theorem 5.** First note that if \( k = 0 \) then both inequalities hold trivially. Assume now that \( k \geq 1 \). We claim that \( \lambda_2 > 0 \). Indeed, if \( G \) is connected then, as mentioned in Section 3.1, it is well known that \( \lambda_2 > 0 \) provided that \( G \) is not a complete graph or a complete multipartite graph. To this end, one can immediately verify that the spectral gap of a complete graph on \( n \) vertices is \( n \) and that the spectral gap of a regular complete multipartite graph with \( t \geq 2 \) parts of size \( n/t \) is \( n - n/t \geq n/2 \), contradicting our assumption. Otherwise, if \( G \) isn’t connected then via the Perron-Frobenius theorem we must have \( \lambda_2 = k \geq 1 \).

Now for each \( i \in [n] \) we let \( \lambda_i = \lambda_i(A) \) be the \( i \)th largest eigenvalue of \( G \). By the spectral theorem for symmetric matrices, we have \( A = \sum_{i=1}^{n} \lambda_i u_i u_i^\top \) for an orthonormal basis of eigenvectors \( u_1, \ldots, u_n \in \mathbb{R}^n \). As previously mentioned, the Perron-Frobenius theorem implies \( \lambda_1 = k \) and \( u_1 \) can be taken to be a scalar multiple of \( \mathbf{1} \). Since \( \| \mathbf{1} \| = \sqrt{n} \) we may take \( u_1 = \frac{1}{\sqrt{n}} \mathbf{1} \). Since \( J = \mathbf{1} \mathbf{1}^\top \), we therefore have

\[
A = \frac{k}{n} J + \sum_{i=2}^{n} \lambda_i u_i u_i^\top.
\]

Let \( U = (u_1, \ldots, u_n) \) be the \( n \times n \) matrix with \( u_1, \ldots, u_n \) as columns. Since \( u_1, \ldots, u_n \) form an orthonormal basis, \( U \) is an orthogonal matrix and so we have

\[
I = U U^\top = \sum_{i=1}^{n} u_i u_i^\top = \frac{1}{n} J + \sum_{i=2}^{n} u_i u_i^\top.
\]
We now define \( \alpha = \frac{1}{2k+1} \) and note that \( 0 < \alpha < 1 \) since \( \lambda_2 > 0 \). We define the matrix
\[
M = \alpha J - 2\alpha A + (1 - \alpha)I
\]
and observe that \( M_{i,i} = 1 \) and \( |M_{i,j}| = \alpha \) for all \( i \neq j \in [n] \). Moreover, we have that
\[
M = \frac{1}{2\lambda_2 + 1} (J - 2A + 2\lambda_2 I) = \frac{1}{2\lambda_2 + 1} \left( \left( 1 - \frac{2(k - \lambda_2)}{n} \right) J + 2 \sum_{i=2}^{n} (\lambda_2 - \lambda_i) u_i u_i^T \right),
\]
which implies that \( M \) is positive semidefinite and \( \mathbb{1} \) as an eigenvector with corresponding eigenvalue \( \lambda = \frac{n - 2(k - \lambda_2)}{2\lambda_2 + 1} > 0 \). Moreover, this implies that the dimension of the null space of \( M \) is precisely \( m(\lambda_2) \), so that \( r = \text{rk}(M) = n - m(\lambda_2) \). It follows that there exist vectors \( v_1, \ldots, v_n \in \mathbb{R}^r \) such that \( M \) is their Gram matrix, i.e. \( M_{i,j} = \langle v_i, v_j \rangle \) for all \( i, j \in [n] \). Therefore, applying Lemma 35 we obtain
\[
\frac{(n - 2(k - \lambda_2))^2}{(2\lambda_2 + 1)^2} = \lambda^2 \geq \frac{n(\alpha^2 n + 1 - \alpha^2) - \frac{(1 - \alpha^2)n^2}{2\lambda}}{2} + \frac{(1 - \alpha^2)n}{2}
\]
\[
= \frac{n(n + (2\lambda_2 + 1)^2 - 1)}{(2\lambda_2 + 1)^2} - \frac{((2\lambda_2 + 1)^2 - 1)n^2}{2(2\lambda_2 + 1)(n - 2(k - \lambda_2))} + \frac{(2\lambda_2 + 1)^2 - 1)n}{2(2\lambda_2 + 1)^2}.
\]
Multiplying by \( \frac{(2\lambda_2 + 1)^2}{n} \), we therefore have
\[
n - 4k + 4\lambda_2 + \frac{4(2\lambda_2 + 1)^2}{n} = \frac{(n - 2(k - \lambda_2))^2}{n} \geq n + (2\lambda_2 + 1)^2 - 1 - \frac{(2\lambda_2 + 1)((2\lambda_2 - 1)^2 - 1)n + (2\lambda_2 + 1)^2 - 1)}{2(n - 2(k - \lambda_2))}
\]
\[= n + 4\lambda_2^2 + 4\lambda_2 - \frac{(2\lambda_2 + 1)(4\lambda_2^2 + 4\lambda_2)n}{2(n - 2(k - \lambda_2))} + 2\lambda_2^2 + 2\lambda_2
\]
\[= n - \frac{2\lambda_2(2\lambda_2 + 1)(\lambda_2^2 + 1)}{1 - \frac{2(k - \lambda_2)}{n}} + 6\lambda_2^2 + 6\lambda_2,
\]
which implies the desired bound
\[
2k - \frac{2(k - \lambda_2)^2}{n} \leq \frac{\lambda_2(2\lambda_2 + 1)(\lambda_2 + 1)}{1 - \frac{2(k - \lambda_2)}{n}} - \lambda_2(3\lambda_2 + 1).
\]
Moreover, it follows from the spectral decomposition of \( M \) that \( u_n \) is an eigenvector orthogonal to \( \mathbb{1} \) with corresponding eigenvalue \( \mu = \frac{n - 2(k - \lambda_2)}{2\lambda_2 + 1} \), so that Lemma 35 also implies
\[
\frac{2(\lambda_2 - \lambda_n)}{2\lambda_2 + 1} = \mu \leq \frac{(1 - \alpha^2)n}{2\lambda} = \frac{(2\lambda_2 + 1)^2 - 1)n}{2(2\lambda_2 + 1)(n - 2(k - \lambda_2))}.
\]
Multiplying by \( \frac{(2\lambda_2 + 1)^2}{2} \) and subtracting \( \lambda_2 \) we conclude
\[
-\lambda_n \leq \frac{\lambda_2(\lambda_2 + 1)}{1 - \frac{2(k - \lambda_2)}{n}} - \lambda_2,
\]
as desired. Finally, \( n = \binom{n - m(\lambda_2)}{2} - 1 = \binom{r + 1}{2} - 1 \) implies we have equality in both bounds of
Lemma 35 and therefore also in both of the above.

Proof of Corollary 6. We first suppose that there exists a constant \( \varepsilon > 0 \) such that \( k - \lambda_2 \leq (1 - \varepsilon) \frac{n}{2} \), or equivalently \( 1 - \frac{2(k - \lambda_2)}{n} \geq \varepsilon \). Via Theorem 5 and the remark following it, we have

\[
    k < 2 \left( k - \frac{(k - \lambda_2)^2}{n} \right) \leq \frac{\lambda_2(2\lambda_2 + 1)(\lambda_2 + 1)}{\varepsilon} - \lambda_2(3\lambda_2 + 1) \leq O(\lambda_2^3),
\]

and therefore \( \lambda_2 \geq \Omega(k^{1/3}) \). Also via Theorem 5 we have

\[
    -\lambda_n \leq \frac{\lambda_2(\lambda_2 + 1)}{\varepsilon} - \lambda_2 \leq O(\lambda_2^2),
\]

so that \( \lambda_2 \geq \Omega(\sqrt{-\lambda_n}) \). Moreover, if \( G \) is bipartite then \( \lambda_n = -k \) and so \( \lambda_2 \geq \Omega(\sqrt{k}) \).

Now let us further suppose that \( k - \lambda_2 = o(n) \), so that \( \frac{k - \lambda_2}{n} = o(1) \). In this case, Theorem 5 implies

\[
    (1 - o(1))2k + o(1)\lambda_2 = 2(k - o(1)(k - \lambda_2)) = 2 \left( k - \frac{(k - \lambda_2)^2}{n} \right) \leq \frac{\lambda_2(2\lambda_2 + 1)(\lambda_2 + 1)}{1 - o(1)} - \lambda_2(3\lambda_2 + 1)
\]

\[
= (1 + o(1))(2\lambda_2^3 + 3\lambda_2^2 + \lambda_2 - 3\lambda_2^2 - \lambda_2)
\]

\[
= (1 + o(1))2\lambda_2^3,
\]

so that \( \lambda_2 \geq (1 - o(1))k^{1/3} \). Again using Theorem 5 we have

\[
    -\lambda_n \leq \frac{\lambda_2(\lambda_2 + 1)}{1 - o(1)} - \lambda_2 \leq (1 + o(1))\lambda_2^2,
\]

so that \( \lambda_2 \geq (1 - o(1))\sqrt{-\lambda_n} \). Finally, if \( G \) is bipartite we have \( \lambda_n = -k \) so that \( \lambda_2 \geq (1 - o(1))\sqrt{k} \).

6 Concluding remarks

In this section, we make some concluding remarks regarding questions left open in this paper, as well suggesting directions for future research. Although any significant improvements to any of our upper bounds or matching lower bound constructions would be interesting, in the following we focus more on directions in which our projection method can be applied or generalized.

1. Regarding real equiangular lines, one of the most interesting questions is to determine how far the approach of Jiang, Tidor, Yao, Zhang, and Zhao [47], in which they determine \( N^\alpha_\alpha(r) \) exactly for \( \alpha \geq \Omega(1/\log \log r) \), can be extended. Indeed, their method relied on two ingredients: A bound on the maximum degree \( \Delta \) of a corresponding graph \( G \) which only depends on \( \alpha \), and a sublinear bound on the multiplicity \( m(\lambda_2) \) of the second eigenvalue \( \lambda_2 \) of any connected graph with maximum degree at most \( \Delta \). One of the limitations to extending their approach was that their bound on \( \Delta \) is at best exponential in \( 1/\alpha \), and since we are able to obtain a bound which is polynomial in \( 1/\alpha \), we overcome this limitation and
thereby take a step towards exactly determining \( N_{\alpha}^R(r) \) in the regime \( \alpha > \Omega(1/r^c) \) for some constant \( c \).

The only remaining limitation is that the bound obtained in [47] on the second eigenvalue multiplicity \( m(\lambda_2) \) for a connected graph with maximum degree \( \Delta \) is \( O(\Delta (n/\log\log n)) \). Note that for normalized adjacency matrices and therefore also for regular graphs, McKenzie, Rasmussen, and Srivastava [56] improve on the approach of [47] to improve the upper bound to \( O(\Delta (n/\log n)) \) for some constant \( c \). However, they give reason to suggest that their methods cannot improve the bound beyond \( O(\Delta (n/\log n)) \). Moreover, Haiman, Schildkraut, Zhang, and Zhao [40] give a construction of a graph with max degree \( \Delta = 4 \) and \( O(\sqrt{n/\log n}) \), and they also point out that for bounded degree \( (\Delta = O(1)) \) graphs, \( \sqrt{n} \) is a natural barrier for group representation based constructions such as theirs.

We also point out that a related question was considered by Colin de Verdière [68] regarding the maximum multiplicity of the second smallest eigenvalue over all generalized Laplacian operators on a Reimann surface \( S \). He conjectured that the maximum is precisely \( \text{chr}(S) - 1 \), where \( \text{chr}(S) \) is the chromatic number of the surface. Moreover, via Heawood’s formula [61] it is known that \( \text{chr}(S) = \left(7 + \sqrt{48g(S) + 1}\right)/2 \) where \( g(S) \) is the genus of \( S \). In view of this, Tlusty [67] formulates an analogous statement in the setting of graphs: Given a graph \( G \), the maximum multiplicity of the second eigenvalue of a weighted Laplacian matrix on \( G \) is \( \left(7 + \sqrt{48g(G) + 1}\right)/2 - 1 \), where \( g(G) \) is the genus of the graph. Moreover, he proves this claim for a class of graphs including paths, cycles, complete graphs, and their Cartesian products. Now note that via the generalized Euler’s formula, we have \( 2 - 2g(G) \geq -|E(G)| \) and thus if Tlusty’s statement were to be true for say, the unweighted Laplacian \( kI - A \) of a \( k \)-regular graph \( G \) on \( n \) vertices with adjacency matrix \( A \), then we would be able to conclude that the multiplicity of the second eigenvalue \( \lambda_2 \) of \( A \) is

\[
m_A(\lambda_2) \leq \left(7 + \sqrt{24|E(G)| + 49}\right)/2 - 1 = O\left(\sqrt{kn}\right).
\]

In particular if \( k \) is fixed, then this would give a bound of \( O(\sqrt{n}) \), matching the natural barrier discussed in the previous paragraph and giving evidence to suggest that such a bound could be correct for bounded degree graphs.

2. Another natural question is whether the graph-based methods used in Section 3.2 for real case can be extended to complex equiangular lines. We remark that starting with the Gram matrix \( M \) of a spherical \( S^1(\alpha) \)-code in \( C^r \), one may use [2] to define a matrix \( A \) which is like a complex generalization of an adjacency matrix, and moreover that the methods of Section 3.2 can be generalized to give an upper bound on \( \max_{\alpha \in [n]} \text{Re}(\alpha_i^* A \alpha) \), a quantity analogous to the maximum degree. However, one would also need an appropriate generalization of the Alon-Boppana type argument which is applicable to \( A \), or at least to its real part \( \text{Re}(A) \) (which corresponds to a weighted graph).

Since we have a way of generalizing the maximum degree bound to complex equiangular lines, it would be interesting to determine if the results of Jiang, Tidor, Yao, Zhang, and Zhao [47] can be extended in order to exactly determine \( N_{\alpha}^C(r) \) when \( r \) is large relative to \( 1/\alpha \). We note that this would require complex versions of the bound on the multiplicity of the second largest eigenvalue, as well as a complex version of the Perron-Frobenius theorem.

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4For a graph \( G \) with vertex set \( V(G) = [n] \), a symmetric matrix \( L \in \mathbb{R}^{n \times n} \) is called a weighted Laplacian of \( G \) if \( L \mathbf{1} = 0 \) and for all \( i \neq j \), \( L_{i,j} = 0 \) when \( ij \notin E(G) \) and \( L_{i,j} < 0 \) when \( ij \in E(G) \).
3. We observe that the projection methods used in this paper can be applied to other matrices associated to a graph, such as the unweighted Laplacian, in order to obtain new spectral inequalities. Indeed, one only needs a positive semidefinite matrix and any matrix can be made as such by adding a sufficient multiple of the identity.

4. It would be interesting to obtain generalizations of our results to arbitrary spherical $L$-codes in $\mathbb{R}^r$ with $|L| = s$, i.e. to $s$-distance sets for $s \geq 2$, as well as for $L = [-1, \alpha]$, i.e. the classical spherical codes question which is equivalent to packing spherical caps on a sphere in $\mathbb{R}^r$.

5. We expect that our approach should extend beyond lines to higher-order equiangular subspaces with respect to different notions of angle, which are described in [6]. In particular, we predict that our methods can be generalized to equiangular subspaces with respect to the chordal distance.

6. The Welch bound is actually the first in a family of higher-order Welch bounds. By generalizing our projection method to higher-order tensors, we expect that it is possible to obtain an improvement to all of the Welch bounds in the same way as we have done for the first bound.

7. Note that spherical $\{\alpha, -\alpha\}$-codes correspond to signed complete graphs, and so it would be interesting to generalize and apply our methods to signed graphs and more generally, to unitary signings of graphs. In particular, Koolen, Cao, and Yang [48] have recently used a Ramsey-theoretic approach analogous to [5] in order to study signed graphs, and thus our methods can be applied to extend their result.

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