On a deformation of the nonlinear Schrödinger equation

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Received 20 November 2015
Accepted for publication 16 December 2015
Published 8 February 2016

Abstract

We study a deformation of the nonlinear Schrödinger (NLS) equation recently derived in the context of deformation of hierarchies of integrable systems. Although this new equation has not been shown to be completely integrable, its solitary wave solutions exhibit typical soliton behaviour, including near elastic collisions. We will first focus on standing wave solutions which can be smooth or peaked, then with the help of numerical simulations we will study solitary waves, their interactions and finally rogue waves in the modulational instability regime. Interestingly, the structure of the solution during the collision of solitary waves or during the rogue wave events is sharper and has larger amplitudes than in the classical NLS equation.

Keywords: solitary waves, nonlinear Schrödinger equation, nonlinear partial differential equation

(Some figures may appear in colour only in the online journal)

1. Introduction

In this work, we will study a deformation of the nonlinear Schrödinger equation, called hereafter the CH-NLS equation. The standard dimensionless form of this equation is given for a complex field \( u \) by

\[
im_t + u_{xx} + 2\sigma m(|u|^2 - \alpha^2 |u_x|^2) = 0,
\]

where \( m = u - \alpha^2 u_{xx} \) (1)

and \( \sigma = \pm 1 \) selects between the focusing and defocusing equation. Although the two equations have different properties, we will only consider the focusing case, thus we will set \( \sigma = 1 \) for the rest of this paper.

This equation was recently derived by the author in [1] when developing a theory of a deformation of hierarchies of integrable systems. When applied to the Ablowitz–Kaup–
Newell–Segur (AKNS) hierarchy of [2] this deformation led to the derivation of several known integrable equations. The most important one is the Camassa–Holm (CH) equation [3], found as a deformation of the Korteweg–de Vries (KdV) equation. The CH equation has been extensively studied in the context of shallow water waves, see [3, 4]. This has led to the development of the \( \alpha \)-model, a regularized fluid model which extends the CH equation to higher dimensions (see [5] and references therein). Other deformations of integrable equations derived in [1] include the modified CH equation [6] and other coupled equations [7] viewed as deformations of the mKdV and coupled KdV equations respectively. These equations all correspond to the deformation of the third flow of the AKNS hierarchy. However, the method of [1] can deform the previous flow of the AKNS hierarchy as well, which corresponds to the NLS equation. The CH-NLS equation (1) is the result of this systematic approach which gathers several integrable equations together in a single framework. Although most of the equations derived in this context were independently shown to be completely integrable, this deformation does not guarantee the complete integrability of any of them. For this reason, there is still no proof for the complete integrability of the CH-NLS equation (1). The question of the complete integrability of the CH-NLS equation therefore remains an open problem.

We want to mention that, although it is a very important subject, the possible link between this equation and a physical mechanism in optical fibres or with other fields of physics will not be addressed in this work. The main difficulty for finding this link is that the Hamiltonian is unknown for this equation. Indeed, the most powerful method for deriving approximated equations such that the NLS or the CH equation relies on some specific approximation in the Hamiltonian or Lagrangian of the complete physical theory. Even if a proper physical derivation of the CH-NLS equation is a difficult open problem, the properties of this equation highlighted in this work show that the CH-NLS equation is not fundamentally very different from the NLS equation, which is a model for many physical systems.

The main focus of this paper we will thus be on the solutions of this equation. We found a one parameter family of standing waves solutions with the remarkable property of having a transition between smooth and peaked shapes by varying this parameter. Although very rare in dispersive equations, such peaked solutions have already been found and studied in a generalized NLS equation by [8, 9]. We then numerically studied solitary waves by running initial value problems with hyperbolic secant initial conditions. We observed the emergence of dispersive and solitary waves. The latter can even have a breathing behaviour, typical of cubic nonlinear equations such as the KdV or NLS for example. Numerical simulations of the collisions between these solitary waves show that they keep their identity after a collision, despite the creation of high frequency dispersive waves. The existence and stability of solitary waves will not be addressed here. We refer to [10] for a recent review.

It is well known that the existence of solitary waves in many nonlinear partial differential equations relies on the balance between nonlinearities and linear dispersion, as usual for the study of the CH equation. One of the few exceptions is the CH equation [3] which admits solitons even in the absence of linear dispersion. The fundamental mechanism of the CH equation is nonlinear dispersion, coming from the Helmholtz operator. By construction, the CH-NLS equation (1) has a similar nonlinear dispersion based on the Helmholtz operator but no transformation can remove its linear dispersion. The nonlinear dispersion based on the Helmholtz operator is in fact a common feature of CH type equations, a growing family of integrable equations which admit soliton solutions without linear dispersion (see [3, 6, 7, 11–13] among others). The CH-NLS equation is thus a very particular equation where the linear dispersion plays a more important role than for other equations of this type. The linear dispersion relation is given by \( \omega(\kappa) = \frac{\kappa^2}{1 + \alpha^2 \kappa^2} \). This is a bounded function so waves of high
frequencies with respect to \( \alpha \) will not propagate. This effect can be observed in the high frequency waves created during collisions or can explain the fission of initial pulses with enough high frequency components. This type of modification of the linear dispersion relation for the NLS equation has already been mathematically investigated for example by [14] in the context of short pulses in optical fibres. We will also see that the CH-NLS equation shows more extreme behaviours than the standard NLS equation and especially during collisions of solitary waves or rogue waves in the modulational instability regime.

In section 2 we will look at the conservation laws of the CH-NLS equation as well as a short discussion on its non-integrability. Peaked and smooth standing wave solutions will be presented in section 4. Numerical initial value problems meant to isolate and characterize solitary waves are in section 5 and their collision in section 6. The collisions exhibit more extreme behaviours than the collisions of NLS solitons, namely a higher amplitude and sharper structures during the collisions. The same features are found in the modulational instability, studied in the last section 7 where the CH-NLS Peregrine type solutions are higher and sharper than the NLS Peregrine solutions. A contains the split-step numerical scheme used in this work and a convergence analysis for one of the collision of solitary waves.

2. Conservation laws and symmetries

The CH-NLS equation (1) is a Hamiltonian system with Hamiltonian given by

\[
P = \frac{1}{2} \int (m u_x - m u_x) \, dx,
\]

and with the non-canonical Hamiltonian structure of the form

\[
K = \begin{bmatrix}
-2m \partial_x^{-1} m & \partial_x + 2m \partial_x^{-1} m \\
\partial_x + 2m \partial_x^{-1} m & -2m \partial_x^{-1} m
\end{bmatrix}.
\]

This Hamiltonian structure is exactly the same as the second NLS Hamiltonian structure. For the NLS equation, the Hamiltonian \( P \) corresponds to the momentum of the solution, as it is associated to the space translation invariance of the equation, but the lack of a canonical Hamiltonian structure for the CH-NLS equation makes this link unclear. As we will discuss at the end of this section, this issue is also related to the fact that the Hamiltonian corresponding to the time invariance of this equation has not yet been found. A similar feature already exists in this type of equation, as noted in [15], where the same conserved quantity arises from the space and the time invariance of the equation independently.

An important symmetry of NLS equations is invariance under phase shifts, which is also shared by the CH-NLS equation. For the NLS equation this symmetry is associated with conservation of total mass

\[
M = \int |m|^2 \, dx,
\]

which is also a conserved quantity of the CH-NLS equation. The associated flow is given by

\[
(m_t, \bar{m}_t)^T = J_0 \left( \frac{\delta M}{\delta m} \right) = (im, -i\bar{m})^T,
\]

where \( J_0 := J (\alpha = 0) \) from equation (6). However, as we will see below, this Hamiltonian structure does not produce the CH-NLS equation and the link between the conservation of mass and the phase shift symmetry seems broken, despite the existence of both properties.
The CH-NLS equation is also invariant under the following ‘scaling transformation’
\[ t \mapsto \lambda^{-2} t, \quad x \mapsto \lambda^{-1} x, \quad u(t, x) \mapsto \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad \alpha \mapsto \lambda^{-1} \alpha, \]
for arbitrary \( \lambda \). This is not a proper transformation as it only relates solutions with different values of \( \alpha \). Nevertheless this scaling can be useful to obtain new solutions and at least makes clear the importance of \( \alpha \) to fix a particular length scale in this equation.

Another important symmetry of the NLS equation is the Galilean invariance, used to derive the travelling wave solutions from the standing waves. This symmetry is absent for the CH-NLS equation, as it is for the CH equation. In order to see this fact, one can easily compute the time derivative of the centre of mass \( C := \int x |m|^2 \mathrm{d}x \) and obtain
\[ \frac{d}{dt} C = i \int (\bar{u}_x u_x - u \bar{u}_x) \mathrm{d}x. \]
The right-hand side is not a conserved quantity, thus the evolution of the centre of mass does not follow a simple linear equation in the time variable, as for the NLS equation, where the right-hand side would be \( 2P \).

3. On the question of integrability of the CH-NLS equation

The CH-NLS equation (1) is only known to admit a weak zero curvature relation (ZCR), as defined in [1]. This equivalent formulation of equation (1) cannot guarantee its complete integrability. It is actually not yet known if the CH-NLS equation is completely integrable. We briefly expose here the argument against its complete integrability, although it cannot prove its non-integrability. The possible second Hamiltonian structure which could give a bi-Hamiltonian formulation for the CH-NLS equation would be
\[ J = \begin{bmatrix} 0 & -i(1 - \alpha^2 \partial_x^2) \\ i(1 - \alpha^2 \partial_x^2) & 0 \end{bmatrix}. \]
Together with \( P \) and \( M \), they will produce the travelling wave equation \( m_t = m_x \). Then, the well-known theorem of Magri would apply to produce the hierarchy of integrable equations and their associated conserved quantities. However, this argument fails because the two Hamiltonian structures \( J \) and \( K \) are not compatible. This failure arises in the Jacobi identity of \( J + K \) and does not seem to be easy to fix by modification of the Hamiltonian structures. We refer to [16] for more details on the compatibility of these Hamiltonian structures. We can therefore not predict the existence of other conserved quantities, such as the Hamiltonian itself. The explicit calculation of the Hamiltonian is another difficult task that is made even more difficult by the fact that the canonical Hamiltonian structure (equation (6)) is not compatible with equation (3), thus another structure and its associated Hamiltonian must be found. As we will see, the numerical simulations exhibit integrable behaviours of the CH-NLS equation such as solitary waves but would not be enough for a mathematical proof of complete integrability.

4. Standing wave solutions

We derive here a one parameter family of localized standing wave solutions of the CH-NLS equation (1). Although this solution cannot be promoted to a travelling wave solution because of the lack of a Galilean transformation, the standing wave solutions still exhibit interesting
properties such as peaked profiles. This type of standing waves has already been discovered and investigated by [8, 9] in the context of short pulses in optical fibres.

We recall the usual form for the standing wave solutions, given by

\[ u = a(x) e^{i\phi(t)}, \quad \text{so} \quad m = (a(x) - \alpha^2 a_{xx}(x)) e^{i\phi(t)} \quad \text{and} \quad u_t = i\phi_t u, \]

where \( \phi(t), a(x) \) are real functions to be determined. Inserting this ansatz into equation (1) and using separation of variables yields

\[ \phi_t = \frac{1}{a - \alpha^2 a_{xx}} \left( \frac{1}{2} a_{xx} + (a - \alpha^2 a_{xx})(a^2 - \alpha^2 a_x^2) \right). \]

This allows us to set \( \phi = ct \) and integrate the resulting equation after its multiplication by \( a_x \) to obtain

\[ a_x^4 + (1 + 2\alpha^2(c - a^2))a_x^2 - a^2(2c - a^2) = 0. \]

We set the integration constant to zero as we will only look for solutions vanishing at \( \pm \infty \). The travelling wave ODE can then be derived after calculating the roots of this polynomial for \( a_x \) and reads

\[ a_x^2 = \frac{1}{2\alpha^2} \left[ -(1 + 2\alpha^2(c - a^2)) \pm \sqrt{(1 + 2\alpha^2c)^2 - 4\alpha^2a^2} \right]. \quad (7) \]

The height of the stationary wave is given as a function of \( c \) by \( a_{\text{max}} = \sqrt{2c} \), found by setting \( a_x = 0 \) in equation (7). This calculation neglected the fact that there is a square root in the equation. Taking it into account gives another limit for the height of the solution and its maximum gradient

\[ a_{\text{max}}^* = \frac{1 + 2\alpha^2c}{2\alpha}, \quad a_{x,\text{max}}^* = \sqrt{c^2 - \frac{1}{4\alpha^2}}. \quad (8) \]

We thus have two different standing wave heights which correspond to two types of solutions. If \( a_{\text{max}} < a_{\text{max}}^* \) the solution will be smooth, otherwise a peak will appear at the maximum. Indeed, the condition \( \frac{\partial a}{\partial t} = 0 \) is equivalent to having a jump in \( a_x \), thus a peaked profile. The critical value for \( c \) for which the peak appears is \( c_{\text{crit}} = \frac{1}{2\alpha} \), and if \( \alpha \to 0 \) then \( c_{\text{crit}} \to \infty \), which is compatible with the non-peaked NLS standing waves.

We display in figure 1(a) the phase portrait of the standing wave solution, given by the right–hand side of equation (7) and in figure 1(b) the standing wave solution for the same values of \( c \). We fixed \( \alpha = 0.5 \) and varied \( c \) such that the two different solutions appear. These solutions have also been tested in the numerics and remain stable, even with the peaked profiles which require higher spatial resolutions.

5. Solitary wave solutions

The main class of travelling wave solutions that will interest us here are the moving solitary waves, or travelling waves. The exact form of the travelling wave solutions of the CH-NLS equation turned out to be difficult to find for several reasons. First of all, the CH-NLS equation is not Galilean invariant thus a standing wave cannot be boosted to produce a family of travelling waves as usually done for the NLS equation. The second difficulty arises when the travelling wave ansatz is plugged into the equation as no integration can be performed. Another issue arises when trying to compute perturbations of solitons if \( \alpha \) is formally taken as a small parameter. Indeed, the equation contains three different scales: \( \alpha^0, \alpha^2 \) and \( \alpha^4 \). Each
method tried so far has failed, mainly due to their computational complexity, but we hope that more progress could be done in this direction. For now, we will concentrate on numerical solutions, which already highlight interesting properties of this equation.

Recall that for the NLS equation solitons have the form

\[ u(x, t = 0) = e^{i\nu A} \text{sech}(Ax), \]  

which can then be transformed under a Galilean boost to give the full time dependent solution. As noted before, this Galilean boost does not exist for the CH-NLS equation which makes this method useless here. An important point regarding this solution is that \( A \) and \( \nu \) are two independent free parameters which define a two parameters family of solutions. In the CH-NLS equation, the Helmholtz operator will always couple the phase and the amplitude of the solution such that the shape of the solitary wave will depend on the speed. The definition of \( A \) and \( \nu \) must therefore be adapted if one wants to characterize the CH-NLS travelling wave solutions. This fact makes the calculation of the travelling waves difficult with the usual travelling wave ansatz and we will therefore only focus here, as a first step, on numerical solutions. We want to observe solitary waves of the CH-NLS, hence we did some numerical experiments with the hyperbolic secant as initial data.

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Recall first that the CH-NLS equation contains a length scale \( \alpha \), which can be converted into a scale in wavelength given by \( 1/\alpha \). Provided that the spectral width of the initial data is small compared to \( 1/\alpha \), namely that \( |k_0 \pm \Delta \kappa| \ll 1/\alpha \), where \( \Delta \kappa \) is the spectral width of the solution and \( k_0 \) its maximum wavelength, the solution will behave as an NLS soliton. In contrast, the higher order terms of the CH-NLS equation will start to play an important role and other type of solutions and behaviour will be observed, such as breather type solutions. Note that in the case of our initial condition in equation (9) we have the simple relations \( \Delta \kappa = 1/A \) and \( k_0 = c \). The numerical experiments displayed on figure 2(a) are snapshots at \( t = 150 \) with initial profile given by equation (9) with parameters \( A = 1 \) and \( \nu = 1, 2, 3, 4, 5 \). We also used \( \alpha = 0.3 \) such that the numerical scheme could converge with realistic resolutions. In each case, one or more solitary waves will emerge from the initial profile as well as dispersive radiations on both sides. In order to illustrate the fission of the initial pulse, we plotted in figure 2(b) a simulation from a larger initial pulse with \( A = 1.5 \) which undergoes an
initial fission into two solitary waves and dispersive radiations. The trigger of this mechanism can be explained by the linear dispersion relation. The high frequency components of the initial pulse propagate slower that the low frequencies, thus splitting the high and low frequencies of the solution. The nonlinear dispersion is then responsible for the formation of the slower solitary wave. In order to confirm this analysis, we ran the same simulation without the Helmholtz operator in the linear dispersion and another without the nonlinear dispersion and neither of them exhibit this fission mechanism. Another interesting feature which happened in most of our simulations is that the solitary waves at large times are not pure travelling waves, but are of breather type, namely they exhibit periodic motions in time. They contain substructures easily identifiable in figure 2(a) which periodically oscillate as the solitary wave moves. The four solitary waves displayed in this figure have approximately the same phase gradient (roughly corresponding to a $\nu$ in equation (9)), which makes the breathing phenomenon as well as the speed of the solitary wave dependent on the shape of the envelope of the solitary wave and not just on the spectral peak position, as for the NLS. Indeed, the sharper the solitary wave, the faster. Although the solutions of equation (1) are very rich, we will not further pursue this investigation here as we are missing the analytical solutions of the equation and because precise numerically based results about the properties of the solutions are out of the scope of this short study.

6. Collision of solitary waves

Another interesting phenomenon of equations admitting solitary waves that we will explore in this section is the interaction between solitary waves. For integrable equations it is guaranteed that the solitons will keep their identity after the collision but this is not certain for the CH-NLS equation. Nevertheless, we numerically observed collisions where the solitary waves conserve their identity, despite the creation of high frequency waves.

We have run three different collisions, with solitary wave extracted from initial value problems as in the previous section. We used three different types of solitary waves for this experiment. The first solitary waves in figure 3(a) have a narrow spectrum near the zero wavelength. The second solitary waves in figure 3(b) are sharper than the first, thus have a

![Figure 2. Numerical solutions of the CH-NLS equation are displayed in both panels. The left panel displays a selection of stable solitary waves. They are not travelling waves as they periodically oscillate in time. The right panel displays the space–time plot of a larger initial condition which undergoes fission and produces two solitary waves as well as dispersive radiations.](image)
larger spectrum. The last solitary waves in figure 3(c) have a similar profile to the first, but with a spectrum centred at higher frequencies. The first collision is therefore similar to a typical NLS soliton collision whereas the others differ. The spectrum of last two type of solitary waves contain higher wavelengths, closer to the scale $1/\alpha$, thus the effect of nonlinear dispersion is more apparent than for the first collision. The main difference is that the amplitude and sharpness of the structures created during the collision are bigger than for a standard NLS collision. This corresponds to the excitation of higher wavelengths than for the NLS equation. This shows that the nonlinear dispersion of the CH-NLS is an important factor for such collision behaviours. The collisions of soliton can be used to describe rogue waves events, and the collision of the CH-NLS equation would correspond to higher amplitude rogue waves. The collision is also not completely elastic and we observed a small amount of high frequency dispersive waves at the location of the collision. Despite this fact, the solitary waves survive after a collision, an effect which is not common for non-integrable equations. Our numerical scheme is actually not able to perfectly resolve the collision (see appendix A for the convergence study). Thus, no more can be said about these higher frequency waves apart from the fact that they are purely dispersive. Finally we want to note that the phase shift in position of the solitary wave after the collision is also present, as for the NLS equation.

7. Modulational instability

We will end this study with some considerations on the modulational instability (MI) [17, 18], a fundamental mechanism of the focussing NLS equation and which is also present in the focussing CH-NLS equation. We will follow the standard derivation of the MI gain for the NLS equation. We first perturb the plane wave solution of equation (1) with $|\epsilon| \ll \beta$ as

$$ u(x) = \beta e^{i(2\beta x)} \Rightarrow u_t(x, t) = (\beta + \epsilon(x, t)) e^{2\beta x}, $$

and then obtain the linearized equation for $\epsilon$

$$ i\epsilon_t = -i\epsilon^2 \epsilon_{xx} + \epsilon_{xx} + 2\beta^2 (\epsilon + \bar{\epsilon}) = 0. $$

The growth rate of the MI denoted $g$ can be found using $\epsilon$ of the form

$$ \epsilon = \epsilon_+ e^{-i\omega t + i\alpha x} + \epsilon_- e^{i\omega t - i\alpha x} $$

and is given by
The regularization comes only from the convolution terms of the linear dispersion, hence the MI behaviour will be similar to the NLS equation, but with a smaller growth rate. This effect is typical of nonlocal interactions in NLS equations, see for instance \[19\]. We plotted in figure 4(a) the gain \(g\) for \(\alpha = 0, 0.2, 0.5\) with \(\beta = 1\). The effect of the nonlocality only reduces the gain, not the size of the instability region as in \[19\].

After that the instabilities have developed, nonlinearities take over and the solution evolves toward a train of pulses. It happens that rogue wave events sometimes emerge, with the Peregrine solution being one of the most frequent. We refer to \[20\] for a recent numerical study of the appearance of rogue waves in the NLS equation. We have performed similar experiments with the CH-NLS and extracted candidates for Peregrine solutions for \(\alpha = 0, 0.12, 0.18\). The initial condition for all the simulations were a plane wave of amplitude 1 and of two periods in a periodic domain of length 50. The instabilities developed naturally from the noise induced by the numerical errors. We observed a delay in the instability growth, as predicted by equation \(11\) and the formation of rogue events but the main result is the amplitude of the rogue wave events. We displayed the best candidates of Peregrine solutions in figure 4(b) for \(\alpha = 0, 0.12, 0.18\). Similarly to the simulations of collisions, the nonlinear effects are stronger than for the NLS and numerically difficult to capture, hence the low value of \(\alpha\) that we could use. The shapes of the solutions are similar to the standard Peregrine solution, but the amplitudes are higher with larger values of \(\alpha\). The standard factor of 3 for the ratio of the amplitude of the Peregrine solution with the background amplitude is obviously recovered with \(\alpha = 0\) and reaches values as large as 12 with only \(\alpha = 0.18\). At this stage, no more can be said about the relation between \(\alpha\) and the Peregrine amplitude as we would need a more reliable numerical scheme and a statistical analysis, as done by \[20\] for example.

\[
g(\kappa) = \text{Real} \left( \frac{\kappa \sqrt{2\beta^2 - \kappa^2}}{1 + \alpha^2 \kappa^2} \right) \tag{11}
\]
8. Summary and open problems

We presented here our latest results on the CH-NLS equation (1), a deformation of the NLS equation, previously derived in [1]. We first exposed the Hamiltonian structure, the conserved quantities such as the mass and momentum of this equation and discussed its lack of complete integrability. Further investigations of the integrability of the CH-NLS are needed, but would have to face technical difficulties, owing to the Helmholtz operator.

We then studied a class of solutions which do not move, the so-called standing wave solutions. We explicitly found the phase space ODE but only studied the standing waves in the case where the wave vanishes at infinity, thus the non-periodic case. The main finding was the existence of a transition from smooth to peaked solutions by varying the only parameter of this class of solutions. Unfortunately the lack of Galilean invariance of the CH-NLS equation does not allow the standing wave solutions to be promoted to travelling waves. The analytical solution for the travelling wave is actually difficult to compute, hence we performed numerical experiments in order to find and study them. From initial conditions given by the NLS solitons, we observed the emergence of solitary waves and dispersive radiations. Most of the solitary waves exhibit a periodic motion in time, similar to the evolution of a breather of the NLS equation. The structures of these oscillations were not studied in detail as this would require analytical solutions or more systematic numerical studies.

We then studied the collision of solitary waves with the help of numerics and found that the general behaviour is similar to the NLS collisions but with sharper structures of higher amplitudes. This fact is interesting as it is linked to the study of rogue waves, extreme events which are known to appear in nature and are usually described by NLS type equations. Apart from collisions of solitary waves, rogue waves can be found by studying the nonlinear regime of the modulational instability, which is still present in the CH-NLS equation. We also found rogue wave events of Peregrine type, but sharper and with higher amplitudes than NLS Peregrine solutions.

We finally want to mention that in most of our numerical experiments, the solutions converged well when increasing the space and time resolutions, except at the location of the extreme events such as the collisions of solitary waves or the Peregrine solutions. We did a brief analysis of this fact, but we leave the development of more accurate numerical schemes for future works.

Acknowledgments

I want to particularly thank A Hone and J Wang for having spotted a mistake in the proposed proof of complete integrability of the CH-NLS equation and D D Holm for helpful comments, suggestions and ideas during all stages of the creation of this paper. I am also grateful to J Elgin, Y Kodama, B Xia, Z Qiao, M Picasso and T Ratiu for fruitful and thoughtful discussions during the course of this work. I am thankful to the Imperial College High Performance Computing Service for their HPX facilities where the simulations of this work were run. I gratefully acknowledge partial support from an Imperial College London Roth Award and from the European Research Council Advanced Grant 267382 FCCA.

1 http://www.imperial.ac.uk/admin-services/ict/self-service/research-support/hpc/.
Appendix A. Numerical scheme

The numerical scheme used for this work is the split-step spectral method commonly used for solving the NLS equation. The idea is to solve the equation in two steps, with the following decomposition

\[ m_t = imu_{xx} = Lm, \quad \text{then} \quad m_t = im(|u|^2 - \alpha^2 |u_x|^2) = Nm. \]  

(A.1)

The scheme is based on the second order approximation of the formal solution, also called Strang splitting:

\[ m(t + dt) = e^{iL + N} dt m(t) \Rightarrow m(t + dt) \approx e^{L dt / 2} e^{N dt} e^{L dt / 2} m(t), \]  

(A.2)

for a timestep \( dt \). The derivatives in the nonlinear operator are computed with a fourth order centred finite difference scheme and the linear step is computed with the analytic solution in the Fourier space. We display in figure A1 the convergence tests performed with the collision of solitary wave of figure 3(b). The mass is exactly conserved by this scheme so we only plotted the absolute error in the conservation of the momentum \( P \) in the left panel. We used the absolute error as the total momentum is zero. The collision is clearly difficult to resolve and led to a shift in the total momentum. This is confirmed in the right panel where one can see that the convergence of the error before the collision is of order two but is much smaller after the collision. The convergence at the collision is also good, but the issue arises just after the collision, when the dispersive high frequency waves are created. This fact rises questions about the existence and properties of these waves, as the momentum is supposed to be perfectly conserved by the CH-NLS equation. Better numerical schemes should then be considered in order to study these high frequency waves.

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