DERANDOMIZING THE LOVÁSZ LOCAL LEMMA VIA LOG-SPACE STATISTICAL TESTS

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Abstract. The Lovász Local Lemma (LLL) is a keystone principle in probability theory, guaranteeing the existence of configurations which avoid a collection $\mathcal{B}$ of “bad” events which are mostly independent and have low probability. In its simplest form, it asserts that whenever a bad-event has probability $p$ and affects at most $d$ other bad-events, and $ep(d+1) < 1$, then a configuration avoiding all $\mathcal{B}$ exists.

A seminal algorithm of Moser & Tardos (2010) gives randomized constructions based on the LLL. However, deterministic algorithms have lagged behind. Notably, prior deterministic LLL algorithms have required stringent conditions on $\mathcal{B}$; for example, they have required that events in $\mathcal{B}$ have low decision-tree complexity or depend on a small number of variables. For this reason, they can only be applied to small fraction of the numerous LLL applications in practice.

We describe an alternate deterministic parallel (NC) algorithm for the LLL, based on a general derandomization method of Sivakumar (2002) using log-space statistical tests. The only requirement here is that bad-events should be computable via a finite automaton with $\text{poly}(d)$ states. This covers most LLL applications to graph theory and combinatorics. No auxiliary information about the bad-events, including any conditional probability calculations, are required. Additionally, the proof is a straightforward combination of general derandomization results and high-level analysis of the Moser-Tardos algorithm.

We illustrate with applications to defective vertex coloring, domatic partition, and independent transversals.

1. Introduction

The Lovász Local Lemma (LLL) is a keystone principle in probability theory which asserts that if one has a probability space $\Omega$ and a set $\mathcal{B}$ of “bad-events” in $\Omega$, then under appropriate “local” conditions there is a positive probability that no event in $\mathcal{B}$ occurs. The LLL has numerous applications to combinatorics, graph theory, routing, etc. The simplest “symmetric” form of the LLL states that if each bad-event $B \in \mathcal{B}$ has probability $P_\Omega(B) \leq p$ and affects at most $d$ bad-events (including itself), then if $epd \leq 1$ then $P(\bigcap_{B \in \mathcal{B}} \overline{B}) > 0$.

Although the LLL applies to general probability spaces, in most applications a simpler variable-based form suffices; in this setting, the probability space $\Omega$ has $n$ independent variables $X_1, \ldots, X_n$ over some alphabet $\Sigma$ (the variables $X_i$ do not necessarily have the same distribution), and each bad-event $B \in \mathcal{B}$ is a boolean function $f_B$ on a subset $Y_B$ of the variables. We say that bad-events $B, B'$ affect each other (and write $B \sim B'$) if $Y_B \cap Y_{B'} \neq \emptyset$. We say that a configuration $x = x_1, \ldots, x_n$ avoids $\mathcal{B}$ if $f_B(x) = 0$ for all $B \in \mathcal{B}$. We let $m$ denote the size of $\mathcal{B}$.

The LLL, in its classical probabilistic form, only shows an exponentially small probability that a configuration avoids $\mathcal{B}$; thus, it does not give efficient algorithms. In a seminal paper [10], Moser & Tardos introduced the following simple randomized algorithm, which we refer to as the MT algorithm, giving efficient randomized constructions for nearly all LLL applications.

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Algorithm 1: The Moser-Tardos algorithm

1: Draw \( x \) from the distribution \( \Omega \)
2: \textbf{while} some bad-event is true on \( x \) \textbf{do}
3: Arbitrarily select some true bad-event \( B \)
4: For each \( i \in Y_B \), draw \( x_i \) from its distribution in \( \Omega \). (We refer to this as resampling \( B \).)

Under nearly the same conditions as the probabilistic LLL, the MT algorithm terminates in polynomial expected time. Moser & Tardos also gave a parallel (RNC) variant of this algorithm, requiring a slack compared to the LLL criterion. This algorithm is remarkably simple and general, requiring almost no special structure in the form of the bad-events.

1.1. Derandomized LLL algorithms. For deterministic algorithms, the situation is not as clean. In their original paper [10], Moser & Tardos gave a deterministic sequential algorithm for the LLL, under the assumption that \( d = O(1) \). This was strengthened by Chandrasekaran, Goyal & Haeupler [3] to give a deterministic parallel (NC) algorithm for unbounded \( d \) under a stronger LLL criterion \( epd^{1+\epsilon} \leq 1 \). We refer to this strengthened criterion \( epd^{1+\epsilon} \leq 1 \) as satisfying the LLL criterion with \( \epsilon \)-slack. (These algorithms can be extended to more general asymmetric LLL criteria, but this leads to even further technical conditions on the precise form of \( B \). To keep the discussion simple for this paper, we focus exclusively on symmetric LLL criteria.)

Concretely, the algorithm of [3] has a complexity of \( O(\log^3(mn)) \) time and \( (mn)^{O(1/\epsilon)} \) processors. In [6], Haeupler & Harris described an alternate LLL derandomization algorithm which has a slightly faster run-time of \( O(\log^2(\frac{mn}{\epsilon})) \).

These two algorithm both have an additional type of constraint, which is that the bad-events \( B \) must come from a very limited class of functions. The most straightforward application is \( k \)-SAT, in which a bad-event (a clause being false) is a monomial function. More generally, [3] allows bad-events with decision-tree complexity of order \( O(\log d) \).

While the LLL has a vast number of applications, there is one particular genre which appears frequently in graph theory and combinatorics. In this genre, we have a graph of degree \( \Delta \), and we want to get a randomized assignment of variables which reduces the degree. As a simple example, we want to color the edges red or blue so that each vertex gets at most \( \frac{\Delta}{2}(1 + \epsilon) \) neighbors of either color, where \( \epsilon \) is some suitably small function of \( \Delta \). In such settings, a bad-event for a vertex \( v \) is typically defined by sums of independent random variables, and depends on the behavior of other vertices within some constant distance from \( v \). Thus, \( d = \Delta^{O(1)} \). We will see a number of examples of this in Section 5.

Satisfying the LLL criterion with \( \epsilon \)-slack is not hard in these constructions: because of the exponential concentration of the Chernoff bound, there is typically only a small loss in going from \( p = 1/(ed) \) to \( p = 1/(ed^c) \) for any constant \( c \). However, the bad-event here typically has decision-tree complexity of order \( \text{poly}(d) \), which is exponentially far from satisfying the requirement of the algorithm of [3]. The, the constraint of [3] is really extremely limiting; it means that most LLL applications encountered in the wild lack deterministic algorithms.

In [7], we discussed another LLL algorithm which allows a more general class of bad-events: namely, those which depend on only \( \text{polylog}(n) \) variables. The latter algorithm comes closer to covering LLL applications in practice: for example, constructions on graphs of maximum degree \( \Delta = \text{polylog}(n) \) will typically satisfy this condition. As an example, [7] gave an algorithm for defective vertex coloring in \( \tilde{O}(\Delta^2) \) time. Using additional derandomization techniques (not the LLL), [7] showed how to extend this to a general \( \tilde{O}(\log^2 n) \) time algorithm.

In additional to these restrictive conditions on the form of the bad-events, there are a few additional technical complications with the previous LLL algorithms. Typically, these algorithms assume that the underlying variables are independent Bernoulli-1/2. This is usually sufficient to
simulate other probability distributions of interest by discretization, but this step can be technically cumbersome and require problem-specific error estimates. Another complication of \cite{7} is that it requires the computation of certain conditional probabilities for the events $B \in B$; namely, one must be able to compute the probability that a bad-event $B$ becomes true, if some variables $x$ are fixed. While these issues are usually manageable in practice, they are annoying. Note that such conditions are not needed for the randomized algorithm: it only requires the ability to check if a bad-event is true or not, and to draw the variables $X$ from $\Omega$.

1.2. Our contribution and overview. In this paper, we discuss a new LLL derandomization algorithm based on a general technique of Sivakumar \cite{13} for derandomizing log-space statistical tests. We show that, if the bad-events can be computed via statistical tests on $\text{poly}(d)$ states, then one can get a deterministic LLL algorithm using $O(\log^2 mn)$ time and $\text{poly}(m, n)$ processors. More specifically, we state the main result as follows:

**Theorem 1.1.** Suppose that $\epsilon d^{1+\epsilon} \leq 1$ for some $\epsilon > 0$, and that every bad-event can be determined by a statistical test on $\eta \leq \text{poly}(d)$ states.

Then there is a deterministic parallel algorithm using $\tilde{O}(\log^2 (|\Sigma|mn) / \epsilon)$ time and $(|\Sigma|mn)^O(1/\epsilon)$ processors to find a configuration $x \in \Sigma^n$ avoiding all the bad-events in $B$.

We will give the formal definition of statistical tests in Section 3. This class of functions is quite general, and seems to cover most applications to graph theory. Although there still remain some functions which are compatible with the criterion of \cite{7} but not the criterion of this paper, such exceptions seem rare. A further advantage of our new algorithm is that it removes the restriction on the probability distribution as well as not requiring any auxiliary information about the bad-events (such as conditional probability calculations); these all come for free from the description in terms of statistical tests. We believe that for typical applications of the LLL, this method will give efficient NC algorithms in a nearly black-box way.

As an example of how this is useful, our algorithm gives straightforward algorithm for defective vertex coloring and domatic partitions. The only step needed here is to convert the standard randomized applications of the iterated LLL into deterministic applications of it. These algorithms have no restrictions on the degree of the graph.

In addition, we note that the proof strategy is a quite straightforward combination of the general Sivakumar derandomization method and some results of \cite{3} on the random behavior of the Moser-Tardos algorithm.

In order to keep this paper more readable, we have omitted a number of minor technical issues concerning the computational complexity of representing and storing the automata and probability spaces.

1.3. Outline. In Section 2 we review the analysis of the randomized Moser-Tardos algorithm, and its derandomization. Much of this section is taken from the works \cite{3, 6}.

In Section 3 we review Sivakumar’s derandomization method. We define log-space statistical tests, and some basic results on how to build probability distributions fooling them.

In Section 4 we combine these results, to show how to use statistical tests to fool the Moser-Tardos algorithm.

In Section 5 we provide some straightforward applications to graph theory problems.

2. Analysis of the randomized Moser-Tardos algorithm

In this section, we provide an overview of the probabilistic analysis underlying the Moser-Tardos algorithm. Much of this material is taken nearly verbatim from our previous paper \cite{7}, which needed to derandomize the Moser-Tardos algorithm in a similar way.

There are two key analytic techniques introduced by \cite{10} for this algorithm. The first is the idea of a resampling table. In the MT algorithm as we have presented it, the new values for each variable
are drawn in an online fashion. Instead, one can imagine a fixed table \( R \). This table records, for each variable \( i \), an infinite list of values \( R(i,1), R(i,2), \ldots \), for that variable. The entries \( R(i,j) \) are all drawn independently, and \( R(i,j) \) has the same distribution as the variable \( X_i \) drawn from \( \Omega \). When the MT algorithm begins, it sets \( x_i = R(i,1) \) for each variable \( i \); if a variable \( x_i \) needs to be resampled, it sets \( x_i = R(i,2) \), and so forth. Once we have fixed a resampling table \( R \), the MT algorithm can be executed deterministically.

We view the resampling table \( R \) as a function \( R : [n] \times \mathbb{Z}_+ \to \Sigma \). We define a slice to be a set \( W \subseteq [n] \times \mathbb{Z}_+ \) with the property that each \( i \in [n] \) has at most one \( j \in \mathbb{Z}_+ \) with \((i,j) \in W \). For such a slice \( W \), sorted as \( W = \{ (i_1,j_1), \ldots, (i_k,j_k) \} \) with \( i_1 < i_2 < \cdots < i_k \), we define the projection \( \pi_{W} \) by setting \( \pi_{W}(R) = (R(i_1,j_1), \ldots, R(i_k,j_k)) \).

The other key idea introduced by Moser & Tardos is the witness tree, which represents a possible execution path for the MT algorithm leading to a given resampling. A witness tree here is a type of rooted tree, whose nodes are labeled by \( \pi \). For a node \( \tau \) in a witness tree \( \tau \), we let \( L(\tau) \subseteq \mathcal{B} \) denote the label of \( \tau \).

The witness tree generation process is explained in great detail in [10], which we recommend as an introduction. As a brief summary, suppose that we run the MT algorithm for \( t \) steps (not necessarily to completion), and that at each time \( s \leq t \) we resample bad-event \( B_s \). Suppose we want to explain why \( B_t \) was resampled at time \( t \). To do so, we form a witness tree \( \hat{\tau}_t \) by first placing a root node labeled \( B_t \), and then going in time for times \( s = t-1, t-2, \ldots, 1 \); at each time \( s < t \), we look in \( \hat{\tau}_t \) to find if there is some node \( v' \) with \( L(v') \sim B_s \). If so, we place a node \( v \) labeled by \( B_s \) in the tree as a child of \( v' \); if there are multiple choices for \( v' \), we always choose the one of greatest depth (if there are multiple choices at greatest depth, we break the tie arbitrarily).

If the MT algorithm runs for \( T \) time-steps, then this produces \( T \) distinct values \( \hat{\tau}_1, \ldots, \hat{\tau}_T \). We say that a witness tree \( \tau \) appears if \( \hat{\tau}_t = \tau \) for any value \( t \).

**Definition 2.1** (Weight and size of witness tree). For a witness tree \( \tau \), we define the size of \( \tau \) as the number of nodes in \( \tau \) and we define the weight of \( \tau \) as \( w(\tau) = \prod_{v \in \tau} P_\Omega(L(v)) \).

The most important result of [10], which explains why the MT algorithm works, is the Witness Tree Lemma:

**Lemma 2.2** ([10]). The probability that a witness tree \( \tau \) appears during the execution of the MT algorithm is at most \( w(\tau) \).

To prove this Lemma, [10] shows that \( \tau \) imposes certain conditions on the resampling table \( R \).

**Lemma 2.3** ([10]). For any witness tree \( \tau \), there is a set of pairwise disjoint slices \( W_v \), indexed by nodes \( v \in \tau \), such that a necessary condition for \( \tau \) to appear is

\[
\forall v \in \tau \quad f_{L(v)}(\pi_{W_v}(R)) = 1
\]

**Proof.** For each node \( v \in \tau \) and each \( i \in [n] \), let \( u_{i,v} \) denote the number of nodes \( v' \) which are at greater depth than \( v \) and which have \( i \in Y_{L(v')} \). Define \( W_v = \{ (i, u_{i,v}+1) \mid i \in Y_{L(v)} \} \).

**Definition 2.4.** We say that a witness tree \( \tau \) is compatible with \( R \) if the condition holds for \( \tau \) and \( R \). We define \( G(\tau, R) \) to be the event that \( \tau \) is compatible with \( R \).

Lemma 2.2 follows from Lemma 2.3 as the entries of \( R \) are independently drawn from the original distribution \( \Omega \), each event \( f_{L(v)}(\pi_{W_v}(R)) = 1 \) has probability \( P_\Omega(L(v)) \); furthermore, since the sets \( W_v \) are disjoint, these events are all independent. Therefore, for any \( \tau, R \) we have

\[
P(G(\tau, R)) = w(\tau)
\]

The analysis of [6] is based on an extension of the witness tree to a more general object referred to as a collectible witness DAG (CWD). These objects represent in a sense all the ways the MT algorithm could require a long execution time. This requires a great deal of notation to define
properly, but the important point for us is that each CWD \( \tau \) satisfies Lemma 2.3 in the same way a witness tree does. We will not discuss the (technical) differences between witness trees and CWD’s; please see [6] for further details.

For any set \( \mathcal{T} \) of CWD’s and a resampling table \( R \), we define \( \mathcal{T}^R \subseteq \mathcal{T} \) to be the set of CWD’s \( \tau \in \mathcal{T} \) which satisfy \( G(\tau, R) \). We summarize some key results of [6] which are relevant to us.

Lemma 2.5 ([6]). Suppose that \( \epsilon 2^{1+\epsilon} \leq 1 \) for \( \epsilon > 0 \). Let \( K = \frac{c \log(mn/\epsilon)}{\epsilon \log d} \) for some constant \( c > 0 \).
There is a set \( \mathcal{T} \) of CWD’s with the following properties:

1. \( |\mathcal{T}| \leq (mn/\epsilon)^{O(1/\epsilon)} \)
2. Each \( \tau \in \mathcal{T} \) has size at most \( 2K \).
3. If every \( \tau \in \mathcal{T}^R \) has size less than \( K \), then an assignment avoiding \( B \) can be found using \((K \log n + K \log(mn|\mathcal{T}^R|) + \log^2 |\mathcal{T}^R|) \) time and \( \text{poly}(K, m, |\mathcal{T}^R|) \) processors.
4. \( \sum_{\tau \in \mathcal{T}} w(\tau) < 1/2 \).
5. \( \sum_{\tau \in \mathcal{T}} w(\tau) < O(m) \).
6. The set \( \mathcal{T} \) can be enumerated with using \( O\left( \frac{\log^2(mn/\epsilon)}{\epsilon} \right) \) time and \( (mn/\epsilon)^{O(1/\epsilon)} \) processors.

Note that although the resampling table \( R \) nominally has infinite dimension, Lemma 2.5 only makes reference to the first \( K \) rows of it. Thus, we can assume throughout that the resampling table uses has \( O(nK) \) entries in total.

3. Fooling log-space statistical tests

In this section, we summarize a general method of Sivakumar [13] for derandomizing certain probabilistic constructions. This method is based on a construction on Nisan [11, 12] to fool finite automata. Let us consider a typical application of the zeroth-moment probabilistic method.

We have the probability space \( \Omega \) over \( \Sigma^n \). Suppose further we know that when \( X \sim \Omega \) we have \( P(E_1(X)) + \cdots + P(E_m(X)) < 1/2 \). In this case, there is a simple randomized algorithm to find \( x \in \Sigma^n \) avoiding all the events; we want to turn this into a deterministic algorithm.

Now suppose that each \( E_i \) can be computed by an automaton \( F_i \) on some state-space \( A \), where \( |A| = \eta \). Namely, at each time step \( t = 1, \ldots, n \), the automaton \( F_i \) maintains a state variable \( a_{i,t} \in A \), which is initially set to some designated start state \( a_{i,0} = 0 \). When it receives as input an entry \( x_t \in \Sigma \) and updates its state \( a_{i,t+1} \) to

\[
a_{i,t+1} = F_i(a_{i,t}, x_t)
\]

At the end of the process, \( a_{i,n} \) is an indicator variable for the event \( E_i(x) \). Note that the function \( F \) receives as input the current time \( t \), which is not counted as part of its state space. See [13] for further details on the computational complexity aspects of this construction.

The automata \( F_i \) are often referred to as “log-space statistical tests”. The reason for this terminology is that the input variables \( x_1, \ldots, x_n \) are thought of as an incoming data stream, and so each automaton computes some test statistic (in this case, binary-valued) of the data. Having a polynomial state space is of course equivalent to using logarithmically means storage bits.

We emphasize here that the functions \( F_1, \ldots, F_m \) must all process the input variable \( x_1, \ldots, x_n \) in the same order. This order may be arbitrary, but it must be shared by all \( m \) automata. This can be a significant limitation of the method; certain types of events will only have finite-space automatia if the inputs are processed in a certain order. While the method of Sivakumar can be used for a single such automaton, it may not be possible to use it for a family of them.

We abuse notation, so that \( F \) refers to multi-step transitions as well. Namely, given a \( r = (r_1, \ldots, r_k) \in \Sigma^k \), we define \( F^k_i(s, t, r) \) to be the final state obtained by starting at state \( s \) and time \( t \) and proceeding through \( k \) iterations with inputs \( x_t, \ldots, x_{t+k} = r_1, \ldots, r_k \) respectively.
As a simple example, let us consider a bad-event $E_i$ defined by a sum of independent random variables, e.g. $E_i$ is an indicator that $\sum_{j \in U_i} x_j \geq c_i$ for some threshold $c_i$. We can implement this as a finite-state automaton by maintaining a counter $a_{i,t}$ to be the running sum $\sum_{j \leq t, j \in U_i} x_j$. When we receive input $x_t$, we update as

$$F_i(a, t, x_t) = a + x_t[t \in U_i]$$

Because the events $E_1, \ldots, E_m$ are determined by finite-state automata $F_1, \ldots, F_m$, we know that

$$\sum_{i=1}^m P_{X \sim \Omega}(F^n_i(0, 0, X)) \leq 1/2$$

However, the distribution $\Omega$ may have exponential-size support, so we cannot efficiently search for a satisfying $x \in \Sigma^n$. The main result of Nisan is that, if the state-space $A$ has size at most $\eta = \text{poly}(n)$, then it is possible to get a much smaller distribution $D$ over $\Sigma^n$, of size roughly $\text{poly}(n)$, with the property that $F^n_i(0, 0, X)$ has a similar distribution for $X \sim D$ as for $X \sim \Omega$.

**Definition 3.1.** Distribution $D$ fools the automaton $F$ to error $\epsilon$ if, for any state $s \in A$, we have

$$|P_{X \sim D}(F^n(0, 0, r) = s) - P_{X \sim \Omega}(F^n(0, 0, r) = s)| \leq \epsilon$$

If we get a distribution $D$ which fools the automata to $\epsilon = 1/(4m)$, we then get

$$P_{X \sim D}(\sum_{i=0}^m E_i(X)) = \sum_{i=0}^m P_{X \sim D}(F^n_i(0, 0, X) = 1) \leq \sum_{i=0}^m P_{X \sim \Omega}(F^n_i(0, 0, X) = 1) + \frac{1}{2m} \leq 3/4$$

In particular, there exists some $x$ in the support of $D$ which avoids all the events $E_i$; we can exhaustively search this support with $\text{poly}(n)$ work to find it. Furthermore, the distribution $D$ can be constructed efficiently and in parallel, using only $\text{poly}(n)$ processors and $O(\log^2 n)$ time.

This method means that whenever the events $E_1, \ldots, E_m$ are computed by automata with polynomial-sized state-space (equivalently, by an algorithm with logarithmic space), we can almost completely match the randomized zeroth-moment method. For example, when the $E_1, \ldots, E_m$ are defined by sums of independent random variables, then we can deterministically achieve discrepancy bounds on them comparable to the Chernoff bounds. By contrast, other derandomization methods (for example, drawing the bits $X$ from a $k$-wise independent probability space for some constant value of $k$) would achieve exponentially weaker discrepancy bounds.

A further advantage of this method is that does not require any conditional probability calculations with respect to $E_i$; the only thing needed is to be able to efficiently compute $F_i$ for given input bits.

The original algorithm of [11, 12] did not give precise bounds on the complexity. Later work of [9, 8] analyzed the algorithm further and optimized its complexity. In order to get optimized parameters for our MT algorithm, we quote a version of [8].

**Theorem 3.2 (8).** Suppose that the variables $X_1, \ldots, X_n$ are iid Bernoulli-$1/2$ (i.e. $\Sigma = \{0, 1\}$ and $\Omega$ is the uniform distribution). Let $\phi = \max(\eta, n, m, 1/\epsilon)$. There is a deterministic parallel algorithm to find a distribution $D$ of support size $|D| = \text{poly}(\phi)$ which fools the automata $F_1, \ldots, F_m$ to error $\epsilon$. The algorithm has a complexity of $\text{poly}(\phi)$ processors and $\tilde{O}(\log \phi \log n)$ time.

It is convenient to to allow other possibilities for the alphabet $\Sigma$ and for the probability distribution $\Omega$.

**Theorem 3.3.** Suppose that the variables $X_1, \ldots, X_n$ are independently, but not necessarily identical, drawn from an alphabet $\Sigma$. Let $\phi = \max(\eta, n, m, 1/\epsilon, |\Sigma|)$. There is a deterministic parallel algorithm to find a distribution $D$ of support size $|D| = \text{poly}(\phi)$ which fools the automata $F_1, \ldots, F_m$ to error $\epsilon$. The algorithm has a complexity of $\text{poly}(\phi)$ processors and $\tilde{O}(\log \phi \log n)$ time.
Proof. Let $\sigma = |\Sigma|$. We may assume without loss of generality that $\sigma$ is a power of two; if not, then add dummy values to $\Sigma$ which have probability zero. Let $\Omega'$ be the probability space obtained by quantizing the probability distribution on each variable $X_i$ to multiple of $2^{-b}$. A simple induction on $t$ shows that for $t = 1, \ldots, n$ and any state $s \in A$, we have

$$|P_{X \sim \Omega}(F_i^t(0,0,X) = s) - P_{X \sim \Omega'}(F_i^t(0,0,X))| \leq 2^{-bt}$$

By putting $b = \Omega(\log(2n/\epsilon))$, we therefore ensure that $\Omega'$ fools $F_1, \ldots, F_m$ to error $\epsilon/2$.

We can encode the probability distribution $\Omega'$ by replacing each variable $X_i$ with $k = \log_2 \sigma + b$ independent variables $X_{i1}, \ldots, X_{ik}$, wherein each $X_{i1}, \ldots, X_{ik}$ is an independent Bernoulli-$1/2$. Also, we can simulate automaton $F$ on the original variables $X_1, \ldots, X_n$ by an automaton $F'$ on the expanded variables $X_{ij}$, by adding $2^k$ additional states (to track the running values $X_{i1}, \ldots, X_{ik}$). Thus, the state-space of automaton $F'$ has size $\eta' = 2^k \eta \leq \sigma \eta \text{poly}(n/\epsilon)$, and the number of variables has increased to $n' = nb \log \sigma$.

We apply Theorem 4.3 to the automata $F'$, which generates a distribution $D$ over $\{0,1\}^{n'}$ fooling $F_1', \ldots, F_m'$ to error $\epsilon/2$. This distribution $D$ can be interpreted as a distribution over $\Sigma^{n'}$. Overall, it fools $F_1, \ldots, F_m$ to error $\epsilon$.

The distribution $|D|$ has support $\text{poly}(\eta', n', m, 1/\epsilon) = \text{poly}(\phi)$, and the processor count also satisfies this bound. The runtime is $O(\log(\phi) \log(n')) = \tilde{O}(\log \phi \log n)$. \qed

4. Fooling the Moser-Tardos Algorithm

We now show how to combine the automata-fooling algorithm with our analysis of the Moser-Tardos algorithm. We assume that every bad-event $B \in \mathcal{B}$ has an associated automaton $F_B$ on a state-space of size $\eta$, which takes as input the variables $x_1, \ldots, x_n$ in that order and outputs whether $B$ is true on $x_1, \ldots, x_n$.

Proposition 4.1. For a CWD $\tau$ of size $k$, there is an automaton $F_\tau$ with state-space $\eta^k$ to determine $G(\tau, R)$.

Proof. We will process the entries of $R$ as $R(i,j)$ where $i = 1, \ldots, n$ and $j = 1, \ldots, K$ (in that order). For each node $v \in \tau$, we maintain a copy of automaton $F_{L(v)}$ and a corresponding state variable $a_v$. So the state-space of $F_\tau$ can be written as $(a_v \mid v \in \tau)$, which has size $\eta^k$.

When we process $R(i,j)$, we determine if there is any $v \in \tau$ such that $(i,j) \in W_v$ — there can be at most one such $v$. If there is none, then $a$ does not update. If there is one such $v$, we update $a_v$ by modifying coordinate $a_v$ to $a'_v = F_{L(v)}(a_v, i, R(i,j))$.

At the end of the process, we determine if $\tau$ is compatible with $R$ by checking that $a_v = \text{TRUE}$ for every $v \in \tau$. \qed

Let $\mathcal{T}$ be the set of CWD’s coming from Lemma 2.5. Our goal is to find a resampling table $R$ which is satisfies property (T3).

Lemma 4.2. Suppose that $\epsilon > 1/n$ and $\eta \leq \text{poly}(d)$. There is a distribution $D$ over resampling tables, with support size $|D| \leq |\Sigma|^{O(1)}(mn)^{O(1/\epsilon)}$ that fools all the automata $F_\tau$ for $\tau \in \mathcal{T}$ to error $\delta = \frac{1}{\log n}$. Furthermore this distribution $D$ can be generated using $|\Sigma|^{O(1)}(mn)^{O(1/\epsilon)}$ processors and $\tilde{O}(\log^{2}(mn|\Sigma|)/\epsilon)$ time.

Proof. By Lemma 2.5, each $\tau \in \mathcal{T}$ has at most $2K \leq O\left(\frac{\log(mn)}{\epsilon \log n}\right)$ nodes. (Due to our assumption on $\epsilon$, we can omit the $1/\epsilon$ term from the logarithms). By Proposition 4.1, each automaton $F_\tau$ has a state-space of size $\psi = \eta^{O\left(\frac{\log(mn)}{\epsilon \log n}\right)}$. Since $\eta \leq \text{poly}(d)$, we have $\psi \leq (mn)^{O(1/\epsilon)}$. There are $|\mathcal{T}| \leq (mn)^{O(1/\epsilon)}$ such automata altogether. Also, the resampling table $R$ involves at $O(nK) \leq \text{poly}(m,n)$ entries.
Let us define $\alpha = |\Sigma|(mn)^{1/\epsilon}$. So $\psi$ and $|T|$ are both bounded by $\text{poly}(\alpha)$. Therefore, Theorem 3.3 produces a distribution $D$ to fool all such $F_\tau$ to error $\delta$ with size $|D| = \text{poly}(\alpha)$. The total complexity of generating $D$ is $\text{poly}(\alpha)$ processors and $\tilde{O}(\log \alpha \times \log (nK)) \leq \tilde{O}(\frac{\log^2(|\Sigma|mn)}{\epsilon})$ time.

This gives us the following crisp LLL derandomization result. Notably, the only condition on the bad-events $\mathcal{B}$ is that they are computable in $O(\log d)$ space.

**Theorem 4.3.** Suppose that $\epsilon \text{poly} d^{1+\epsilon} \leq 1$ for some $\epsilon > 0$ and $\eta \leq \text{poly}(d)$. Then there is a deterministic parallel algorithm using $\tilde{O}(\frac{\log^2(|\Sigma|mn)}{\epsilon})$ time and and $(|\Sigma|mn)^O(1/\epsilon)$ processors to find a configuration $x \in \Sigma^n$ avoiding all the bad-events in $\mathcal{B}$.

**Proof.** If $\epsilon < 1/n$, then we simply search the entire space, which takes $|\Sigma|^n \leq (|\Sigma|mn)^O(1/\epsilon)$ time. Otherwise, apply Lemma 4.2 to generate a distribution $D$ fooling all the automata $F_\tau$ for $\tau \in \mathcal{T}$, where $\mathcal{T}$ is defined in Lemma 2.5. This step takes $\tilde{O}(\frac{\log^2(|\Sigma|mn)}{\epsilon})$ time and $(|\Sigma|mn)^O(1/\epsilon)$ processors.

Let us define the following quantity $S(R)$, which is a function of the resampling table $R$:

$$S(R) = \frac{1}{C_m} \sum_{\tau \in \mathcal{T}} [G(\tau, R)] + \sum_{\tau \in \mathcal{T}} [G(\tau, R)]$$

for some constant $C > 0$.

When $R$ is drawn from $\Omega$, then $P(G(\tau, R)) = w(\tau)$. Therefore, Properties (T4) and (T5) show that $\mathbb{E}_{R \sim \Omega}[S(R)] \leq 3/4$. Since $G(\tau, R)$ is computed by automaton $F_\tau$ which is fooled by $D$ to error $\delta = \frac{1}{10|\mathcal{T}|}$, we have

$$P_{\tau \sim D}[G(\tau, R)] \leq w(\tau) + \frac{1}{10|\mathcal{T}|}$$

Therefore, we see that

$$\mathbb{E}_{R \sim D}[S(R)] \leq \frac{1}{C_m} \sum_{\tau \in \mathcal{T}} P_{\tau \sim D}(G(\tau, R)) + \sum_{\tau \in \mathcal{T}} P_{\tau \sim D}(G(\tau, R))$$

$$\leq \frac{1}{C_m} \sum_{\tau \in \mathcal{T}} (w(\tau) + \frac{1}{10|\mathcal{T}|}) + \sum_{\tau \in \mathcal{T}} (w(\tau) + \frac{1}{10|\mathcal{T}|})$$

$$\leq 3/4 + 1/10 < 1$$

In particular, there exists some $R$ in the support of $D$ with $S(R) < 1$. We can search $D$ in parallel to find it in $(|\Sigma|mn)^O(1/\epsilon)$ processors and $\tilde{O}(\frac{\log(|\Sigma|mn)}{\epsilon})$ time.

Now let $R$ have $S(R) < 1$. This resampling table $R$ satisfies property (T3) and must have $|\mathcal{T}^R| \leq O(m)$. Therefore, by Property (T3), we get an assignment avoiding $\mathcal{B}$ in $\tilde{O}(\frac{\log^2(|\Sigma|mn)}{\epsilon})$ time and poly($m$) processors.

5. Applications of the LLL

We next describe a few simple applications of our LLL derandomization result.

5.1. **Domatic partition.** We begin by considering the problem of domatic partition: given a graph $G$, partition the vertices $V$ into $r$ dominating sets $V_1, \ldots, V_r$. Equivalently, we want to $r$-color the vertices, such that every vertex sees every color in its neighborhood. This is classic application of the iterated LLL [4]. For simplicity we focus on the case in which $G$ is $k$-regular. The proof of Theorem 5.1 is adapted from [7] with a few minor modifications.
Theorem 5.1. Let $\eta > 0$ be any fixed constant. There is some constant $K = K_\eta$ with the following property. If $G$ is $k$-regular with $k > K$, then $G$ has a domatic partition of size $c \geq (1 - \eta)\frac{k}{\log k}$, which can be found using $O(\log^2 n)$ time and poly$(n)$ processors.

Proof. We follow the iterated LLL construction of [4]. Here, the color of each vertex is an ordered pair $\chi(v) = (\chi_1(v), \chi_2(v))$, where $\chi_1$ is chosen from $c_1 = k/\log^3 k$ colors, and $\chi_2$ is chosen from $c_2 = (1 - \eta)\log^2 k$ colors. In the first phase of the LLL, we will select $\chi_1$ and the second phase will select $\chi_2$. Each vertex chooses its colors uniformly at random among $[c_1], [c_2]$ respectively.

Let us first analyze Phase I. For each vertex $v$ and each color $j \in [c_1]$, define $N_j(v)$ to the set of neighbors $w$ with $\chi_1(w) = j$ and let $X_{v,j} = |N_j(v)|$. The expected value of $X_{v,j}$ is $\mu = \log^3 k$. For each vertex $v$ and each color $j \in [c_1]$, we have a bad-event $B_{v,j}$ that $X_{v,j} \leq t_0$ or $X_{v,j} \geq t_1$, where $t_0 = \mu - \phi \log^2 k$ and $t_1 = \mu + \phi \log^2 k$ and $\phi$ is a large constant.

For $\phi$ sufficiently large, the Chernoff bound shows that $B_{v,j}$ has probability at most $p \leq k^{-5}$. Furthermore, each bad-event $B_{v,j}$ affects $B_{v',j'}$ only if dist$(v, v') \leq 2$, so in the sense of the LLL we have $d \leq k^4$. It is easy to construct a log-space statistical test $B_{v,j}$: namely, vertex $v$ simply counts the number of neighbors which have taken color $j$. The statespace of this test is $k$, which is poly($d$).

We next turn to Phase II. For each vertex $v$, each $j \in [c_1]$, and each $j' \in [c_2]$, we have a bad-event $B_{v,j,j'}$ that there is no $w \in N_j(v)$ with $\chi_2(w) = j'$; if all such bad-events are avoided then the resulting coloring $(\chi_1(v), \chi_2(v))$ gives a domatic partition. The only dependencies now are between bad-events $B_{v,j,j'}$ and $B_{w,j,j''}$ where $v, w$ share a neighbor $u$ with $\chi_1(u) = j$, so $d \leq t_2 k + 2$ and $p \leq (1 - 1/c_2)^{10}$. Again, constructing log-space statistical tests is straightforward: for each $v, j, j'$, we simply have a flag to detect whether any neighbor of $v$ has taken color $j, j'$. The statespace of this test is just 2.

Set $\epsilon = \eta/2, \phi = 10$. It is straightforward to verify that the criterion $epd^{1+\epsilon} < 1$ is satisfied when $k$ is sufficiently large. Our deterministic LLL algorithms avoids the bad-events in each phase using $O(\log^2 n)$ time and poly$(n)$ processors.

We note that in Theorem 5.1, the runtime is polylog$(n)$ for every value of $k$; by contrast, in [7], the corresponding algorithm had a runtime which was linear in $k$.

5.2. Defective vertex coloring. A $k$-defective vertex $c$-coloring of a graph $G = (V, E)$, is an assignment of colors to the vertices such that every vertex $v$ has at most $k$ neighbors with the same color as $v$. This generalizes proper vertex coloring, in that a proper vertex coloring is a 0-defective coloring. There is a standard construction for using the iterated LLL to get a defective coloring; see [3, 7] for a detailed description. This iterated LLL construction boils down to a one-round degree-splitting procedure.

Theorem 5.2. There is an absolute constant $K$ with the following property. Given a graph $G$ of maximum degree $\Delta$ and an integer parameter $j \leq \log_2(\frac{\Delta}{\log_2 \Lambda})$, there is a deterministic parallel algorithm with $O(\log^2 n)$ time and poly$(n)$ processors to $2^j$-color the vertices, so that each vertex $v$ has at most $(\Delta/2^j)(1 + K \sqrt{(2/\Delta) \log \Delta})$ neighbors with the same color as $v$.

Proof. Consider the random process in which each vertex independently and uniformly selects a color (represented as a $j$-bit string). For each possible color $i = 1, \ldots, 2^j$ and each vertex $v$, there is a bad-event $B_{i,v}$ that vertex $v$ has color $i$ and it also has too many neighbors of color $i$. Thus there are $m = n2^j$ bad-events, and each bad-event involves at most $\Delta 2^j$ variables. The number of neighbors of each color is a Binomial variable with mean at most $\Delta 2^{-j}$. Note that $B_{i,v} \sim B_{i,v'}$ iff $v$ and $v'$ are at distance at most 2 in $G$. So in the sense of the LLL we have $d \leq 2^j \Delta^2$. A simple argument using the Chernoff bound shows that $p \leq 2^{-j} \Delta^{-10}$ for $K$ sufficiently large. Therefore, we have $epd^{1+\epsilon} < 1$ for $\epsilon = 1/2$. 
In order to apply Theorem 4.3, we need to show how to determine a bad-event $B_{i,v}$ via log-space statistical tests. In order to do so, we maintain a counter for the number of neighbors of $v$ with color $i$, as well as a flag for whether vertex $v$ has itself chosen color $i$. The total state-space required is $2^{i+1} \Delta$, which is indeed polynomial in $d$.

**Corollary 5.3.** Given a graph $G$ of maximum degree $\Delta$ and integer parameter $h \leq \Delta$, there is a deterministic parallel algorithm with $O(\log^2 n)$ time and poly($n$) processors to get an $h$-defective coloring of $G$ with $O(\Delta/h)$ colors.

**Proof.** Repeatedly apply Theorem 5.2 at each stage converting the given graph $G$ into $2^j$ separate graphs of degree $\Delta/2^j$. After $O(\log^* \Delta)$ rounds, this gives a constant-degree graph. At that point, simply use a proper vertex coloring. \hfill \Box

This result is also shown in [7], but the proof is different and the runtime is $\tilde{O}(\Delta \log n)$. This is only polylogarithmic for $\Delta = \text{polylog}(n)$. Consequently, [7] needs a separate derandomization procedure to handle the case of large $\Delta$.

### 5.3. Independent transversals

As another representative example, let us consider a classic combinatorial problem known as the independent transversal, first considered in [2] and since attracting a long line of research. In this setting, we are given a graph $G = (V, E)$ along with a partition of the vertices as $V = V_1 \cup \cdots \cup V_r$, wherein each $V_i$ has size $b$. The goal is to select a subset of the vertices $I$ such that $I$ is an independent set, and for every $i = 1, \ldots, r$ we have $|I \cap V_i| = 1$.

A classic application of the LLL [1] is to show that an independent transversal exists if the graph degree $\Delta$ satisfies $b > 2e\Delta$. Via the Moser-Tardos algorithm, this immediately gives randomized algorithms to find an independent transversal under the same conditions. Unfortunately, this algorithm cannot be derandomized directly, since it does not satisfy the LLL criterion with $\epsilon$-slack for any constant value of $\epsilon$. In [5], Fischer & Ghaffari described how to modify the LLL construction to allow $\epsilon$-slack for arbitrary positive $\epsilon$, while satisfying a weaker condition $b > C\Delta$ for some large constant $C$. This is done via a vertex-splitting procedure, which can also be used for our LLL derandomization. We provide a sketch.

**Lemma 5.4.** When $b > C\Delta > C^2$ for some absolute constant $C$, there is a deterministic parallel algorithm using $\tilde{O}(\log^2 n)$ time and poly($n$) processors to generate a vertex set $V' \subseteq V$ such that the induced subgraph $G[V']$ has maximum degree $\Delta' = \log^2 \Delta (1 + \frac{1}{\log \Delta})$ and each component $V_i$ has $|V_i' \cap V| \geq b' = C(1 - \frac{1}{\log \Delta})\Delta'$.

**Proof.** By discarding excess vertices, we may assume that $b = C\Delta$. Consider the random process of retaining each vertex independently with probability $\frac{\log^2 \Delta}{\Delta}$. (This can be quantized to a multiple of $\Delta^{-10}$ without appreciably changing any probabilities, so that each decision about a vertex $v$ uses poly($\Delta$) bits.) We have a bad-event for each vertex $v$ that its degree exceeds $\Delta'$ and a bad-event for each block that its size is less than $b'$. One may observe here that each bad-event affects at most $d = \text{poly}(C\Delta)$ others, and has probability $p \leq 2^{-\Omega(\log^2 \Delta)}$. For $C$ sufficiently large and $\Delta > C$, this satisfies the LLL criterion with $\epsilon$-slack for $\epsilon = 1/2$.

To apply Theorem 4.3, we need to determine if a bad-event has occurred. These are both easy to do using log-space statistical tests. For each vertex $v$, we simply count the number of retained neighbors; for each block $i$, we simply count the number of retained $v \in V_i$. Both of these use $O(\Delta)$ state-space. \hfill \Box

**Theorem 5.5.** When $b > C'\Delta$ for some absolute constant $C'$, there is a deterministic parallel algorithm using $\tilde{O}(\log^2 n)$ time and poly($n$) processors to find an independent transversal of $G$.

**Proof.** Let $C$ be the constant of Lemma 5.4 and take $C = 2C'$. Repeatedly apply Lemma 5.4 to reduce the parameters $b, \Delta$ until $\Delta < C$. This takes $O(\log^* n)$ rounds, and the ratio $b/\Delta$ decreases.
overall by at most a factor of 2. Thus, at the end of this process, we have reduced to a graph with parameters $b', \Delta'$ and $b' > C\Delta$. Next, consider the random process wherein we select a single vertex from each block. We apply the LLL, wherein a bad-event is that two adjacent vertices are selected. One can verify that $p = 1/b'$ and $d = 2\Delta'$. This satisfies $epd^{1+\epsilon} < 1$ for some constant $\epsilon > 0$, given that $b', \Delta'$ are themselves constant. So we can again apply Theorem 4.3.

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