PUISSEUX COEFFICIENTS AND PARAMETRIC DEFORMATIONS OF PLANE CURVE SINGULARITIES

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Abstract. We study deformations of plane curve cuspidal singularities from analytical point of view, obtaining some new concrete results. We show some rather unexpected properties of Puiseux coefficients treated as functions on a suitably defined parameter space. The methods used in paper are very elementary.

1. Introduction

Although deformations of singular points of plane algebraic curves have been studied for a long time, there is still more unanswered questions than the answered ones. Major part of researchers concentrate on study of deformations of the defining equation of singularity: one takes a family of polynomials in two variables, say \( F_s(x, y) \), and looks at the changes of the zero locus of \( F_s \) as the parameter \( s \) varies in an appropriate deformation space. In this setting an algebraic approach (like in [GLS]) seems to be the most natural. Yet there are some geometric properties that are not well controlled from an algebraic point of view. For example the geometric genus of the zero locus of \( F_s \) (i.e. half the first Betti number of the normalization) is not directly seen by looking at algebraic properties of \( F_s \).

We focus ourselves on deformations of cuspidal singularities that preserve the (local) geometric genus. Such case on the one hand is the most basic: these deformations are all parametric (see [Bo2]) and can be explicitly written by a concrete formula. Namely, each cuspidal singularity can be given a local parametrization \( t \rightarrow (x(t), y(t)) \in \mathbb{C}^2 \), where \( t \) is a parameter from some small disk in \( \mathbb{C} \) and \( x, y \) are (germs of) analytic functions. Now a family of singularities is nothing else as a family of maps \( (x_s(t), y_s(t)) \) defined on some small disk in \( \mathbb{C} \), where \( s \) is a parameter.

On the other hand preserving the geometric genus seems is a strong condition (very close to being \( \delta \)-constant), if imposed on a deformation defined by an equation. For example, if \( F(x, y) = 0 \) defines an isolated singularity,
then the geometric genus of a general fiber of the deformation $F(x, y) - s$ is half the Milnor number of the singularity, so it is rather far from zero.

In [Bo2] we studied parametric deformation from a topological point of view, examining mostly the variations of the Tristram–Levine signatures of the corresponding links of singular points. Another approach can be obtained by studying the dual graph of the resolution [Ke], or so–called Enriques diagrams [ACPS]. This gives some obstructions for the existence of some deformation. These approaches have some serious drawbacks: either the results are general and weak, or the obstructions are close to optimal, but then the combinatorics is complicated, and few general questions can be answered. Even the dimension of a base space of a (semi)universal parametric deformation is unknown. Conjecturally it should be so called external codimension (see [BZ2]) of the singularity, but the proof of this fact seems to be beyond the reach of presently known methods.

In this paper we use a completely different approach. As we have already mentioned, a parametric deformation can be explicitly written down by a formula (like (2.1)). Then, the topological type of a cuspidal singularity can be expressed uniquely by the following triple: the multiplicity, which can be seen as the minimum of the orders of functions (denoted by $x$ and $y$ in (2.1)) parametrising given singularity and two sets of indices: one, of those that the corresponding Puiseux coefficients must vanish and the other one of those, for whose the corresponding Puiseux coefficients must be non–zero. We want to study how the Puiseux coefficients change, as we vary the deformation parameter. It is easy to see that these coefficients depend continuously on parameters as long as the orders of parametrising functions are preserved. Yet, if the multiplicity of the singular point jumps, the Puiseux coefficients easily get out of hand.

Our approach consists of looking not at the Puiseux coefficients $c_k$, but rather at some auxiliary functions $P_k$, which are polynomials in parametrising functions $x, y$ and their derivatives (see [Bo3]). The sequence of orders of $P_k$ carries full data needed to determine the topological type of the singularity. But $P_k$ behave well in the limit $s \to 0$, so we can reread the informations about the Puiseux expansion in the limit by looking merely at the orders of $P_k$. In some cases we can show that this approach is sufficient.

Things would be much simpler if we could assume that the Puiseux polynomials are in some sense generic. Of course, this is a risky assumptions since their coefficients are rational. Quite deceptively, we prove that in the sense of Kouschnirenko–Bernstein (see [Be, Ku]) Puiseux polynomials do not have to be generic and show a concrete example. Namely we study the subset in a suitable parameter space describing given singular point and try to compute its homology class in the appropriate projective space. We can do it (in some cases) by hand, or we can assume the Kouschnirenko–Bernstein genericity condition and compute the class using mixed volumes. In many cases we get different results, which contradicts the genericity. One of the explanations of this phenomenon is the possibility of inverting the Puiseux
expansion: we can expand \( y \) in powers of \( x \) or \( x \) in powers of \( y \). We can express the Puiseux coefficients of one expansion in terms of Puiseux coefficients of the other one. The expression is polynomial and has very strong and unexpected symmetries.

The structure of the article is the following. First we define the Puiseux polynomials and show their homogeneity properties. The material in Sections 1.1 and 1.2 is rather standard and has been put here firstly for completeness and secondly for fixing the notation used throughout the article. Then we pass to families in Section 2. In particular, we solve completely the question of deformations with constant multiplicity and then, in Section 2.3, we show on a very simple example, that the Puiseux coefficients may in fact go to infinity if the multiplicity jumps. Section 3 contains the most interesting results about the Puiseux coefficients, obtained by studying ODE’s depending on a parameter. We show also that in one non–trivial case the results obtained are optimal. In Section 4 we study the subset in parameter space describing given singular point. We show the violation of the genericity conditions of Puiseux polynomials. In the last section we study more deeply the relation between two possible Puiseux expansions. We show that the polynomials relating one to another satisfy something very close to the WDVV equation, which seems to be rather unexpected feature and certainly deserves a further study.

1.1. Puiseux polynomials. Let \( p \) and \( q \) be two integers such that \( p > 1 \) and \( q > 1 \). Let us consider the linear space of pairs of polynomials

\[
\begin{align*}
x(t) &= a_0 t^p + a_1 t^{p+1} + \ldots \\
y(t) &= b_0 t^q + b_1 t^{q+1} + \ldots.
\end{align*}
\]

(1.1)

The singular point we consider is assumed to be at \((0, 0)\) and the singular parameter is \( t = 0 \). Unless specified otherwise we shall assume that \( a_0 b_0 \neq 0 \).

Remark 1.1. Because of the finite determinacy of a singular point it is unimportant whether we deal with polynomials of sufficiently high degree, formal power series or convergent power series.

Any such two polynomials parametrize a cuspidal singularity of a plane curve. We are mostly interested in the Puiseux expansion of \( y \) in fractional powers of \( x \), which is of the form

\[
y = c_0 x^{q/p} + c_1 x^{(q+1)/p} + \ldots,
\]

(1.2)

where \( c \)'s depend on \( a \)'s and \( b \)'s. For example

\[
\begin{align*}
c_0 &= a_0^{-q/p} b_0 \\
c_1 &= a_0^{-(q-1)/p} \left( b_1 - \frac{q b_0 a_1}{p a_0} \right).
\end{align*}
\]

(1.3)

The following fact is well-known and its proof shall be omitted.
Proposition 1.2. The function \( \gamma_k = c_k \cdot a_0^{(q+k)/p} \) is a polynomial in \( a \)'s and \( b \)'s. It is linear in \( b \)'s, degree \( k + 1 \) homogeneous in \( a \)'s and weighted homogeneous of degree \( k \) if the weight of variables \( a_j \) and \( b_j \), for \( j \geq 0 \), is defined to be \( j \).

Definition 1.3. The polynomial

\[
\gamma_k = c_k \cdot a_0^{k+(q+k)/p}
\]

will be called the *Puiseux polynomial* corresponding to \( (p, q) \).

Example 1.4. It is easy to compute that

\[
\gamma_1 = a_0 b_1 - \frac{q}{p} a_1 b_0
\]

\[
\gamma_2 = a_0^2 b_2 - \frac{q+1}{p} a_0 a_1 b_1 + \frac{q^2+2q+p}{2p^2} a_1^2 b_0.
\]

1.2. Description of singular points. If \( p \) and \( q \) happen to be coprime, then each pair of polynomials (1.1) parametrizes a quasi-homogeneous singularity equivalent to \( x^q - y^p = 0 \). The type of the singular point does not depend on \( a \)'s and \( b \)'s as long \( a_0 b_0 \neq 0 \). The situation is quite different in the case, where \( \gcd(p, q) > 1 \). Then, vanishing or not of Puiseux coefficients \( c_k \) may influence the topological type of the singularity.

The following lemma is standard, although its formulation may seem slightly artificial at first.

Lemma 1.5. Let \( T \) be a topological type of a cuspidal singular point and \( p, q \) such integers, that there exists a specific choice of coefficients \( a_k, b_k \) for \( k = 0, 1, \ldots \), that (1.1) parametrizes a singular point of type \( T \).

Then there exist two finite subsets of integers \( I \) and \( J \) such that (1.1) parametrizes a singularity of type \( T \) if and only if \( c_i = 0 \) for any \( i \in I \) and \( c_j \neq 0 \) for any \( j \in J \).

Example 1.6. Let \( T \) be the \( A_4 \) singularity. We can take \( p, q \) to be (2, 2), (2, 4) or (2, 5), but we cannot take it (2, 3) (this would parametrize an \( A_2 \) singularity). This explains the assumptions of the above lemma.

The sets \( I \) and \( J \) may depend on a specific choice of \( (p, q) \). In the above example, if \( (p, q) = (2, 2) \) then \( I = \{1\} \) and \( J = \{3\} \); if \( (p, q) = (2, 4) \) then \( I = \emptyset \) and \( J = \{1\} \); while if \( (p, q) = (2, 5) \) both sets \( I \) and \( J \) are empty.

Definition 1.7. The set \( I \) is called the *defining set* of the singularity (once \( p, q \) are fixed). The quantity \( \nu = \#I \) will be called the *codimension* of the singularity.

Remark 1.8. \( I \) consists of all indices of all essential Puiseux terms which must vanish. The set \( J \) is uniquely determined by \( I \). However it is sometimes convenient to have it written explicitly.
Example 1.9. For \((p, q) = (2, 2)\) and \(A_{2k}\) singularity, \(I = \{1, 3, 5, \ldots, 2k-3\}\) and \(J = \{2k-1\}\). For \((p, q) = (4, 6)\) and the singularity equivalent to \(x = t^4, y = t^6 + t^9\), \(I = \{1\}\) and \(J = \{3\}\).

Remark 1.10. In [BZ2] the codimension defined above corresponds to the subtle codimension \(\nu'\). It is related to the external codimension mentioned in Introduction by the formula \(\text{ext}_\nu = \nu' + p + q - 3 - \left\lfloor \frac{p}{q} \right\rfloor - \left\lfloor \frac{q}{p} \right\rfloor\).

2. Families of singularities

2.1. Basic definitions. When we say that we deal with a family of singular points, this means that we allow coefficients \(a_k\) and \(b_l\) in (1.1) to vary. Therefore we will consider now a family of singularities

\[
\begin{align*}
  x_s(t) &= a_0(s)t^p + a_1(s)t^{p+1} + \ldots \\
  y_s(t) &= b_0(s)t^q + b_1(s)t^{q+1} + \ldots,
\end{align*}
\]

(2.1)

where the following conditions are satisfied:

(D1) for each \(k \geq 0\), \(a_k\) and \(b_k\) are functions of a variable \(s \in D\), where \(D\) is a unit disk in a complex plane;

(D2) these functions are assumed to be analytic and the series (2.1) are convergent for any \(s\);

(D3) \(a_0b_0\) does not vanish away from \(0 \in D\);

(D4) isolated zero of polynomials \(\gamma_k\) can occur only at \(s = 0\). In other words, the function \(s \rightarrow \gamma_k(s)\) is either identically zero, or non-vanishing, or vanishing only at 0.

We can draw an immediate consequence of properties D3 and D4.

Lemma 2.1. The family is topologically equisingular over the punctured disk \(D^* = D \setminus \{0\}\).

Proof. Let us pick two points \(s_1, s_2 \in D^*\). As \(a_0(s_i) \neq 0\) (property D3) \((i = 1, 2)\), vanishing of \(c_k\) at \(s_i\) is equivalent to vanishing of \(\gamma_k\) at \(s_i\). But now, \(\gamma_k(s_1) = 0\) if and only if \(\gamma_k(s_2) = 0\) (by D4). Thus the singularities at \(s_1\) and \(s_2\) are topologically equivalent.

It is worth noticing that while the conditions D1 and D2 are merely technical, the conditions D3 and D4 can be guaranteed for any one-dimensional deformation if we restrict ourselves to a sufficiently small neighborhood of \(s = 0\) (and rescale the parameter \(s\) if we want \(D\) to be the unit disk).

Remark 2.2. Equation (2.1) can be regarded as a map of the disk \(D\) into the parameter space. Conditions D1 and D2 imply that the map is analytic, whereas Lemma 2.1 means that the punctured disk \(D^*\) is mapped into an equisingularity stratum. Conversely, an analytic disk in the parameter space can be regarded as the deformation (we may need to ensure D3 and D4...
by shrinking this disk). This approach will be important in the proof of Proposition 3.10.

**Definition 2.3.** The singularity at \( s \in D^* \) will be called a **general member** and the singularity at 0 a **special member** of the family. Passing with \( s \to 0 \) will be called a **degeneration** or a **specialization** of a singularity.

Note that the Puiseux coefficients \( c_k \) are well defined on \( D^* \) only up to a multiplicative constant (the choice of a branch of a root \( a_0(s)^{1/p} \)).

It is worth pointing out that if \( a_0(0) = 0 \) and \( s \to 0 \), \( c_k(s) \) can — but does not always have to — tend to infinity. The meaning of \( \gamma_k(0) \) is unclear. Definitely it is not a Puiseux polynomial related to the parametrization of the singularity at \( s = 0 \).

2.2. **Simplest case of deformations, when** \( a_0(0) \neq 0 \). This simple case, containing all parametric deformations with constant multiplicity is completely solved.

**Proposition 2.4.** Let us fix \( p, q \). Let \((I, J)\) and \((I', J')\) be defining pairs of two singularities. A family of singularities with general member defined by \((I, J)\), a special member defined by \((I', J')\) and \( a_0(0) \neq 0 \) exists if and only if \( I \subseteq I' \) and \( J' \cap I = \emptyset \).

**Proof.** First we show the 'only if' part. Let us consider a family with \( a_0(0) \neq 0 \). We can then change variables \((t, s)\) to \((\tau, s)\), where \( \tau = x_s(t)^{1/p} \). This is an analytic change near \((t, s) = (0, 0)\). In these new variables we have

\[
x_s(\tau) = \tau^p \quad y_s(\tau) = c_0(s)\tau^q + c_1(s)\tau^{q+1} + \ldots,
\]

where \( c_k(s) \) are analytic functions in \( s \). The notation \( c_k \) is on the purpose, these functions are in fact Puiseux coefficients. Now if \( c_k(s) = 0 \) for \( s \in D^* \) then \( c_k(0) = 0 \). On the other hand if \( c_k(0) \neq 0 \) then \( c_k \) cannot vanish on \( D^* \) (compare with property D4). In particular, if \( i \in I \) then \( c_i(0) = 0 \) and if \( j \in J' \) then \( c_j(s) \neq 0 \). It follows immediately that \( J' \cap I = \emptyset \). A little more care is needed to show that \( I \subseteq I' \). We omit the details.

To show the 'if' part, we consider a family given by

\[
x_s(t) = t^p \quad y_s(t) = b_0(s)t^q + b_1(s)t^{q+1} + \ldots,
\]

where

- \( b_k(s) = 0 \) if \( k \notin I \);
- \( b_k(s) = s/k! \) if \( k \in J \cap I' \);
- \( b_k(s) = 1/k! \) otherwise.

A general member has singularity defined by \((I, J)\), the special one by \((I', J')\). □

The situation is much more complex in the case \( a_0(0) = 0 \).
Definition 2.5. The number $r$ such that $a_0(0) = \cdots = a_{r-1}(0) = 0 \neq a_r(0)$ is called a jump of given family.

We begin with an elaborated example.

2.3. Case $p = 2$ and $r = 1$. This case should shed some light on the phenomena occurring in the degenerations. Let us consider the following family.

\begin{align}
  x_s(t) &= st^2 + t^3 \\
  y_s(t) &= b_2(s)t^2 + b_3(s)t^3 + \ldots,
\end{align}

Please remark that we have changed indexing of variables so we write here $b_k(s)t^k$ instead of $b_k(s)t^{k+q}$ as in (1.1). The convention adopted here seems to be more natural if we work with concrete numbers (here $q = 2$).

Assume that $b_i$’s are chosen in such a way that for $s \neq 0$ the resulting singularity is an $A_{10}$ singularity. This amounts to the fact that we have

\begin{align}
  y_s &= c_2(s)x_s + c_4(s)x_s^2 + c_6(s)x_s^3 + c_8(s)x_s^4 + c_{10}(s)x_s^5 + c_{11}(s)x_s^{11/2} + \ldots.
\end{align}

Substituting $x$ from (2.2) into (2.3) yields

\begin{align}
  y_s &= t^2sc_2(s) + t^3c_2(s) + t^4s^2c_4(s) + t^5 + 2sc_4(s) + \\
  &\quad + t^6(c_4(s) + c_6(s)s^3) + t^7 + 3s^2c_6(s) + \\
  &\quad + t^8(3sc_6(s) + s^4c_8(s)) + t^9(c_6(s) + 4s^3c_8(s)) + \\
  &\quad + t^{10}(6sc_8(s) + s^5c_{10}(s)) + \ldots
\end{align}

If all $c_2, c_4, c_6, c_8$ and $c_{10}$ remained bounded from above while $s \to 0$ then, on passing to the limit $s = 0$, all above terms with $s$ in a positive power would vanish. Then, the resulting singularity would be topologically equivalent to $\{x^b - y^3 = 0\}$, with $b \geq 11$.

In general, some $c_i$’s can diverge to infinity and the limit singularity can be less complicated.

Lemma 2.6. $sc_6(s)$ is bounded as $s \to 0$.

Proof. Assume contrary. Let us take a subsequence $s_n \to 0$ such that $|s_n c_4(s_n)| \to \infty$. To shorten the notation, we will call $c_j(s_n)$ as $c_j^n$.

Let us pick $n$ sufficiently large and consider the terms $b_8(s)$ and $b_9(s)$ (i.e. at $t^8$ and $t^9$) written in the following way

\begin{align}
  \begin{pmatrix}
    3s_n & s_n^4 \\
    1 & 4s_n^3
  \end{pmatrix}
  \begin{pmatrix}
    c_8^n \\
    c_9^n
  \end{pmatrix}
  =
  \begin{pmatrix}
    b_8^n \\
    b_9^n
  \end{pmatrix}
\end{align}

We claim that the above $2 \times 2$ matrix must have non–trivial kernel. In fact, the leading terms in $sc_6^n$ and $s^4c_8^n$ are unbounded and must mutually cancel so that $b_8$ and $b_9$ stay bounded as $s \to 0$.

The desired contradiction comes from the fact that

\[ \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 11 \neq 0. \]
From Lemma 2.6 it follows that in the limit expansion \( b \rightarrow 0 \).

**Corollary 2.7.** If a singularity \( A_{10} \) specializes to a singularity \( \{ x^b - y^3 = 0 \} \), then \( b \geq 8 \).

**Remark 2.8.** The bound \( b \geq 8 \) in Corollary 2.7 is optimal. Indeed, let us consider the family \( x_s(t) = st^2 + t^3, \) \( y_s(t) = 4s^2t^6 + 12st^7 + t^8 \). For \( s \neq 0 \) the singularity at the origin is an \( A_{10} \) singularity, while for \( s = 0 \) we have \( x(t) = t^3, y(t) = t^8 \).

When we deal with a singularity \( A_{2k} \) with arbitrary \( k \), we can argue in a similar way. Namely, for \( n = 2k \) let \( l = \lfloor \frac{n+3}{4} \rfloor \). Consider the determinant

\[
d_n := \begin{vmatrix}
\frac{k-(l-1)}{l-1} & \frac{k-(l-1)}{l} & \ldots & \frac{k-(l-1)}{2l-3} \\
\frac{k-2}{5} & \frac{k-2}{6} & \ldots & \frac{k-2}{6} \\
\frac{k-1}{3-l} & \frac{k-1}{4-l} & \ldots & \frac{k-1}{3}
\end{vmatrix}
\]

The above discussion shows that if \( d_4, d_6, \ldots, d_n \neq 0 \), then for all \( i \leq 2k - l \) such that \( 3 \mid i, b_i(s) \rightarrow 0 \) when \( s \rightarrow 0 \). Then, the singularity \( A_{2k} \) degenerates to \( (3; b) \) with \( b \geq 2k - l + 1 \). This is in agreement with Petrov’s result [Pet], see also [Bo2].

We cannot find, unfortunately, any compact formula for \( d_n \). We remark only that for larger \( p \)'s and \( r = 1 \) similar arguments can be presented, but still the determinants are difficult to compute.

3. Differential equations related to Puiseux expansion

3.1. Simple case. Let us assume that our family has the following Puiseux expansion for \( s \neq 0 \).

\[
y_s = c_0(s)x_s^{q_0/p} + c_1(s)x_s^{q_1/p} + c_2(s)x_s^{q_2/p} + \ldots.
\]

Here \( q_0 < q_1 < q_2 < \ldots \) and the Puiseux coefficients between \( x^{q_0/p} \) and \( x^{q_1+1/p} \) are supposed to vanish. Moreover \( a_0(s) \neq 0 \) for \( s \in D^* \) and the jump is exactly \( r \).

Note that Puiseux terms \( x_s^{q_i/p} \) are not necessarily essential, which is contrary to the convention we have been adapting in previous sections. The reason for this will soon be clarified in what follows.

**Notation 1.** From now on, the order of a function, denoted \( \text{ord} \), will always be its order at \( t = 0 \) with respect to \( t \). The order may depend on \( s \), if the function does.

Let us divide (3.1) by \( x_s^{q_0/p} \), differentiate both parts with respect to \( t \) and multiply it again by \( x_s^{1+q_0/p} \) (cf. [BZ1], proof of Lemma 3.2). We obtain

\[
\dot{y}_s x_s - \frac{q_0}{p} y_s \dot{x}_s = \frac{q_1 - q_0}{p} c_1(s) \dot{x}_s x_s^{q_1/p} + \frac{q_2 - q_0}{p} c_2(s) \dot{x}_s x_s^{q_2/p} + \ldots
\]
Let us denote
\begin{equation}
(3.3) \quad P_1(s, t) = \dot{y}_s x_s - \frac{q_0}{p} y_s \dot{x}_s.
\end{equation}

As \( x \sim t^p \) and so \( x^{q_1/p} \sim t^{q_1} \) we get that
\[
\text{ord} \ P_1(s, t) = q_1 + p - 1,
\]
for \( s \neq 0 \).

But \( P_1(s, t) \to P_1(0, t) \) for \( s \to 0 \) uniformly in \( t \) (in some neighborhood of \( 0 \)). Therefore
\begin{equation}
(3.4) \quad \text{ord} \ P_1(0, t) \geq q_1 + p - 1.
\end{equation}

Now let us put \( s = 0 \) and regard the equation (3.3) as an ordinary differential equation with unknown function \( y_0 \). Solving it we get at \( s = 0 \):
\begin{equation}
(3.5) \quad y_0 = x_0^{q_0/p} \left( \int_0^t P_1(0, u)x_0^{-q_0/p-1} \, du + C \right),
\end{equation}
with \( C \) an integration constant.

**Lemma 3.1.** If \( q_0(p + r)/p \) is not an integer then \( C = 0 \).

**Proof.** In this case the r.h.s. of (3.5) is analytic around \( t = 0 \) iff \( C = 0 \). \( \square \)

**Lemma 3.2.** If \( q_0/p \) is integer, we can assume that \( C = 0 \).

**Proof.** If \( q_0/p = n \) we can apply the change of coordinates \( y_s \to y_s - Cx_s^q \) (for \( s \in D \)). \( \square \)

**Corollary 3.3.** If either \( q_0(p + r)/p \notin \mathbb{Z} \) or \( q_0/p \notin \mathbb{Z} \) then
\[
\text{ord} \ y_0 \geq q_1 - r.
\]

If \( q_0(p + r)/p \in \mathbb{Z} \), then (3.5) admits a non-unique analytic solutions, because the solution to the corresponding homogeneous equation \( \dot{y}x - \frac{q_0}{p} y \dot{x} = 0 \) is equal to \( Cx_0^{q_0/p} \) is analytic near zero.

But then we can choose such \( c_0 \) that \( y_0 - c_0 x_0^{q_0/p} \) has order at least \( q_1 - r \). This case correspond to the Puiseux expansion \( y_0 = c_0 x_0^{q_0/p} + c_1 x_0^{(q_1-r)/(p+r)} + \ldots. \)

The above procedure gives some restrictions for possible Puiseux terms in the limit. We illustrate them in the following example.

**Example 3.4.** Assume that the multiplicity sequence of the singularity at \( s \neq 0 \) is \((9; 17)\), so that the Puiseux expansion is
\[
y_s = c_0(s) x_s^{9/9} + c_1(s) x_s^{17/9} + \ldots.
\]

Assume that at \( s = 0 \) the order of \( x \) is 10. Then, by Corollary 3.3, the order of \( y_0 \) is at least 16.
Remark 3.5. One could be tempted to do the following trick in this case. If $y_s = c_0(s)x_s^{q_0/p}x_0^{q_1/p} + \cdots$, we apply changes $y_s \to y_s - c_0(s)x_s$ so that the resulting new $y$ has the order $17$ as $t = 0$. What is wrong is that nothing can prevent $c_0$ from escaping to infinity as $s \to 0$ (see Section 2.3). There are examples (see Proposition 3.10) that the order of $y_0$ is precisely $16$.

Observe also that in this case vanishing of the inessential Puiseux term $c_0(s)$ for $s = 0$ would lead to an increment of the minimal order of $y_0$ (to $17$), in other words lead to the vanishing of an essential Puiseux term in the expansion of $y_0$ in the powers of $x_0$.

3.2. Slight generalization. The above method admits further improvements. Here we follow closely [Bo3]. Let us consider the equation (3.2). Let us divide both sides by $x^{q_1/p}s$, differentiate them with respect to $t$ and multiply again by $x^{q_1/p+1}s$. We obtain

$$(3.6) \quad P_2(s, t) = \frac{q_2 - q_0}{p} \frac{q_2 - q_1}{p} c_2(s)x_s^{q_2/p}x_0^{q_1/p} + \cdots,$$

where

$$(3.7) \quad P_2(s, t) = x\dot{x}P'_1 - \left(\frac{q_1}{p}x^2 + \ddot{x}x\right) P_1.$$

Here and in the following $P'_1$ means $\frac{\partial}{\partial t}P_1(s, t)$.

We can repeat this procedure of dividing, differentiating and multiplying several times. The reader may easily verify the following formula valid for $n \geq 2$

$$(3.8) \quad P_n(s, t) = \prod_{k=0}^{n-1} \frac{q_n - q_k}{p} c_n(s)x_s^{q_{n-1}}x_0^{q_n/p} + \cdots,$$

where $P_n$ are defined inductively by the formula

$$(3.9) \quad P_{n+1}(s, t) = x\dot{x}P'_n - \left(\frac{q_n}{p}x^2 + (2n - 1)\ddot{x}x\right) P_n.$$

An analogue of equation (3.4) is

$$(3.10) \quad \text{ord } P_n(s, t) \geq q_n + (2n - 1)(p - 1).$$

The inequality for the orders is valid for $s \neq 0$ by virtue of (3.8) (as $\text{ord } x^{2n-1} = (2n - 1)(p - 1)$ and $\text{ord } x^{q_n/p} = q_n$). It holds also for $s = 0$ because $P_n(s, t) \to P_n(0, t)$ uniformly in $t$ (in some neighborhood of $t = 0$) if $s \to 0$.

As before we can treat the equation (3.9) as the ordinary differential equation with the known function $P_{n+1}(0, t)$ and unknown $P_n(0, t)$. We get the following solution

$$(3.11) \quad P_n(0, t) = x_0^{q_n/p}x_0^{2n-1}\left(\int_0^t P_{n+1}(0, u)x(u)^{-2n}x(u)^{-q_n/p-1}du + C\right).$$
The requirement that $P_n(0,t)$ is analytic near $t = 0$ implies the following result.

**Lemma 3.6.** If $q_n(p + r)/p$ is not an integer then $C = 0$. Moreover, if $q_n/p$ is integer, we can still perform a change of coordinates so that $C = 0$.

**Proof.** Only the second part of the proof requires some comments. If $q_n/p = k \in \mathbb{Z}$, we apply the change $y_s \to y_s - \tilde{C}x_s^k$. Such a change induces, by virtue of formulae (3.3) and (3.9) the change $P_l \to P_l - C_l x_s^{k + 2i - 1}$, where $C_l$ depends linearly on $\tilde{C}$. It is now clear that picking a suitable $\tilde{C}$ we can ensure that $C = 0$. \qed

**Proposition 3.7.** Assume that for all $i = 1, \ldots, n$ either $q_i/p \in \mathbb{Z}$ or $q_i r/p \notin \mathbb{Z}$. Then we have

$$\text{ord } y_0 \geq \max_{i=2,\ldots,n} q_i - (2i - 1)r.$$  

**Proof.** If $D = 0$ in (3.11) then

$$\text{ord } P_n(t,0) \geq \text{ord } P_{n+1}(t,0) - 2r.$$  

The statement follows now by an easy induction on $n$. \qed

**Remark 3.8.** The conditions $q_i r/p \notin \mathbb{Z}$ are automatically satisfied if $r$ and $p$ are coprime.

**Example 3.9.** Let $N > 0$ and consider a deformation with $p = 16$ and the Puiseux expansion of a generic fiber

$$y_s = c_0 x_s^{16/16} + c_1 x_s^{20/16} + \cdots + c_N x_s^{1+N/4} + c_{N+1} x_s^{(4N+17)/16} + \ldots.$$  

Suppose that the order of $x_0$ is 17, so $r = 1$. Then by Proposition 3.7

$$\text{ord } y_0 \geq 17 + 2N.$$  

It is interesting to compare the Milnor numbers. The general member $s \neq 0$ has $\mu_s = 282 + 12N$, while the degenerate one has $256 + 16N$. Hence, for large $N$ it is easy to see that Proposition 3.7 provides much better estimate than semicontinuity of Milnor numbers.

The method described above yields a nice criterion for determining possible degeneracies of singularities. Some improvements should be, nevertheless, done. In a deformation of $A_{2k}$ singularity into a singularity with the characteristic pair $(3,b)$ (see Section 2.3), the order of $y_0$ must be much larger than predicted by Proposition 3.7. This additional jump of orders of $y_0$ and, probably, also $P_k$ is still to be understood.

**3.3. Exactness in a simple case.** In one case we can show explicitly that there are no additional jumps of orders. Assume that a general member has the Puiseux expansion

$$(3.12)\quad y_s = c_0 x_s^{q_0/p} + c_1 x_s^{q_1/p} + \ldots.$$  

If the order of $x$ jumps by $r$, Corollary [3.3] implies that the order of $y_0$ is at least $q_1 - r$. We shall show that without other assumptions this is an optimal bound.

**Proposition 3.10.** Assume that $q_1 - q_0 > r$. Then there exists a deformation as in (2.1) such that the order of $y_0$ is precisely $q_1 - r$.

**Proof.** Let $d = q_1 - q_0$ and $q = q_0$. Consider a vector space $V = V_x \oplus V_y$ of pairs of polynomials
\[
\begin{align*}
  x(t) &= a_0 t^p + a_1 t^{p+1} + a_2 t^{p+2} + \cdots + a_{d+1} t^{p+d} \\
  y(t) &= b_0 t^q + b_1 t^{q+1} + b_2 t^{q+2} + \cdots + b_{d+1} t^{q+d}.
\end{align*}
\]
To simplify notation let us denote
\[
h_{m,n} = m - \frac{q}{p}.
\]
Remember that $P_1 = \dot{y} - \frac{2}{p} \dot{x} y \dot{\dot{x}}$. The requirement that ord $P_1 = q_1 + p - 1$ translates into a system of equations
\[
F_1(a, b) = \cdots = F_{d-1}(a, b) = 0 \neq F_d(a, b),
\]
where
\[
F_i(a, b) = \sum_{i+j=l} h_{ij} a_i b_j.
\]
For $k \leq d$ define
\[
\Sigma_k = \{(a, b) \in V : F_1(a, b) = \cdots = F_k(a, b) = 0\}.
\]

**Lemma 3.11.** For all $k < d$, $\Sigma_k$ is smooth away from the set $\{a_0 = 0\}$.

**Proof of Lemma [3.11]** The matrix of partial derivatives of function $F = (F_1, \ldots, F_k)$ with respect to $b$ variables is
\[
\begin{pmatrix}
  h_{10} a_1 & h_{01} a_0 & 0 & \cdots & \\
  h_{20} a_2 & h_{11} a_1 & h_{02} a_0 & 0 & \cdots & \\
  h_{30} a_3 & h_{21} a_2 & h_{12} a_1 & h_{03} a_0 & 0 & \cdots & \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
  h_{k,0} a_k & h_{k-1,1} a_{k-1} & \cdots & h_{0,k} a_0 & \\
\end{pmatrix}
\]
This is a $k \times (k + 1)$ matrix. If $a_0 \neq 0$ it is obvious that its rows are linearly independent. The lemma follows from the implicit function theorem. \qed

The next thing we need is the structure of the set $N_{d-1} = \Sigma_{d-1} \cap \{a_0 = 0\}$.

**Lemma 3.12.** The set $N_{d-1}$ is an union of sets $N_{d-1,0}, \ldots, N_{d-1,d-1}$, where
\[
N_{d-1,k} = \{a_0 = a_1 = \cdots = a_k = b_0 = \cdots = b_{d-2-k} = 0\}.
\]
In particular set $N_{d-1}$ is a codimension one subset in $\Sigma_{d-1}$. 

Proof. We shall prove slightly more. Namely let

\[ N_{l,k} = \{a_0 = \cdots = a_k = 0 = b_0 = \cdots = b_{l-1-k}\}. \]

Then the set \( N_l = \Sigma_I \cap \{a_0 = 0\} \) is a union of \( N_{l,k} \)'s for \( k = 0, \ldots, l \).

For \( l = 1 \) the statement is trivial. Assume it has been proved for \( l - 1 \). Consider the equation \( F_l = 0 \) on \( N_{l-1,k} \). From the definition of this space we infer (see (3.15)) that the only monomial in \( F_l \) that does not vanish identically is

\[ h_{k+1,d-k-1}a_{k+1}b_{d-k-1}. \]

In fact in all other \( a_i b_j \)'s either \( i \leq k \) or \( j \leq l - k - 2 \). It follows that

\[(3.17) \quad N_{l-1,k} \cap \{F_l = 0\} = N_{l,k} \cap N_{l,k+1},\]

and the induction step is proved. \( \square \)

**Corollary 3.13.** The set \( N_{d-1} = \Sigma_{d-1} \cap \{a_0 = 0\} \) is not contained in \( \Sigma_d \).

**Proof.** From (3.17) we infer that \( F_d \) does not vanish identically on any subset \( N_{d-1,k} \). \( \square \)

**Finishing the proof of Proposition 3.10.** The rest of the proof is easy. Consider the set \( N_{d-1,r-1} \) in \( \Sigma_{d-1} \). There exists a small smooth disk \( D \) in \( \Sigma_{d-1} \) omitting \( \Sigma_d \) and intersecting \( N_{d-1,r-1} \) in one point disjoint from \( \Sigma_d \). Hence \( a_{r-p} \neq 0 \) at this point (otherwise \( F_d = 0 \)).

Then this disk represents a specialization of a singularity (cf. Remark 2.2)

\[ y = c_0 x^{q_0/p} + c_1 x^{q_1/p} + \ldots \]

to the singularity with order of \( x \) equal to \( p + r \) and order of \( y \) equal to \( q_1 - r \) (because if \( (a,b) \in N_{d-1,r-1} \) then \( b_0 = b_1 = \cdots = b_{q_1-q_0-r-1} = 0 \) so the order of \( y \) jumps to \( q_1 - r \)). \( \square \)

The above proof admits a minor but important generalization. Namely, let us consider a topological type \( T \) of the singularity that can be parametrized by \( x = t^{p+r} + \ldots \) and \( y = b_{q_1-q_0-r} t^{q_1-r} + \ldots \). If we enlarge the number of parameters in (3.13) (keeping the notation unchanged) then we can relate to \( T \) some subset (like \( B_{f,d} \) in Section 4) of \( N_{d-1,r-1} \). Then, choosing a small disk \( D \) as above we can find a deformation with a general member as in (3.12) specializing to \( T \). Hence we get a corollary.

**Corollary 3.14.** A singularity that can be parametrized as \( x = t^{p+r} + \ldots \), \( y = t^{q'} + \ldots \) can be obtained as a specialization of the family with a general member (3.12) with \( q_1 - q_0 > r \) if and only if \( q' \geq q_1 - r \).

4. **Class of the set of parametric singularities**

Let us be given a topological type of singularity given as quadruple \( p,q,I,J \), where \( (I,J) \) is a defining pair of the singularity. Let \( d = \max\{j \in J\} \) and consider the space of polynomials \( V \) as in (3.13). Then the polynomials \( \gamma_k \) can be regarded as functions on the space \( V \). These functions are
homogeneous polynomials in variables \(a\) and \(b\), so they descend to the projectivisation of \(V\) defined by

\[\mathbb{P}V = V/\sim,\]

where \((a_0, \ldots, a_d, b_0, \ldots, b_d) \sim (\lambda a_0, \ldots, \lambda a_d, \mu b_0, \ldots, \mu b_d)\) for any \(\lambda, \mu \in \mathbb{C} \setminus \{0\} \).

The pair \((I, J)\) naturally defines two submanifolds of parametric singularities

\[A_I = \{\gamma_i = 0 \text{ for } i \in I\},\]
\[B_{IJ} = \{\gamma_i = 0 \text{ for } i \in I, \gamma_j \neq 0 \text{ for } j \in J\}\]

\(B_{IJ}\) is the set we want to study. Especially, understanding of \(\{a_0 = 0\} \cap B_{IJ}\) is equivalent to understanding parametric deformations. Unfortunately, \(B_{IJ}\) is not compact, because of the conditions \(\gamma_j \neq 0\) on \(B_{IJ}\). On the other hand there is a serious problem with \(A_I\).

**Lemma 4.1.** For any \(I\) and \(J\), the hyperplane \(\{a_0 = 0\} \cap \{a_1 b_0 = 0\}\) belongs to \(A_I\).

**Proof.** We will show that \(\gamma_m = 0\) whenever \(a_0 = 0\) and \(a_1 b_0 = 0\). By Proposition 1.2 all monomials occurring in \(\gamma_m\) are of the form

\[b_k a_0^{n_0} \cdots a_m^{n_m},\]

where

\[n_0 + n_1 + n_2 + \cdots + n_m = m,\]
\[k + n_1 + 2n_2 + \cdots + mn_m = m.\]

If \(n_0 = 0\) then \(k = n_2 = \cdots = n_m = 0\). Hence the only monomial in \(\gamma_m\) without \(a_0\) is \(b_0 a_1^m\). \(\square\)

Therefore \(A_I\) does not even have the same expected codimension.

**Lemma 4.2.** \(A_I\) is smooth of proper dimension away from the set \(\{a_0 = 0\}\).

**Proof.** Let \(I = \{i_1, \ldots, i_\nu\}\). Consider the matrix of partial derivatives

\[D\gamma = \begin{pmatrix}
\frac{\partial \gamma_{i_1}}{\partial a_0} & \cdots & \frac{\partial \gamma_{i_1}}{\partial a_k} & \frac{\partial \gamma_{i_1}}{\partial b_0} & \cdots & \frac{\partial \gamma_{i_1}}{\partial b_k} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \gamma_{i_\nu}}{\partial a_0} & \cdots & \frac{\partial \gamma_{i_\nu}}{\partial a_k} & \frac{\partial \gamma_{i_\nu}}{\partial b_0} & \cdots & \frac{\partial \gamma_{i_\nu}}{\partial b_k}
\end{pmatrix},\]

for \((a, b) \in \mathbb{P}V \setminus \{a_0 = 0\}\). As the derivative of \(\gamma_{i_k}\) with respect to \(b_{i_k}\) is proportional to \(a_0^{i_k}\), the \((k + 1 + i_1)\)-st,\((k + 1 + i_2)\)-nd up to \((k + 1 + i_\nu)\)-th columns of \(D\gamma\) (corresponding to derivatives over \(b_{i_1}, \ldots, b_{i_\nu}\)) form an upper triangular matrix with \(a_0\) to some power on the diagonal. Hence \(\text{rank } D\gamma = d\) as long as \(a_0 \neq 0\). By implicit function theorem \(A_I\) is smooth of codimension \(d\) away from \(\{a_0 = 0\}\). \(\square\)
Definition 4.3. The variety
\[ R_I = \text{closure of } A_I \setminus \{a_0 = 0\}. \]
will be called the defining variety of the singularity given by \( p, q, I, J \).

Example 4.4. If \( I = \{1, 2, \ldots, \nu\} \), then \( R_I \) coincides with the projectivization of \( \Sigma_\nu \subset \nu \). In fact, neither of them has a component lying entirely in \( \{a_0 = 0\} \), they are both smooth away from \( \{a_0 = 0\} \) and they both (if \( a_0 = 0 \)) describe the locus of curves with expansion \( y = c_0 x^{q/p} + c_1 x^{(q+\nu+1)/p} \).

The knowledge of the intersection of \( R_I \) with the line \( \{a_0 = 0\} \) is precisely what we need in order to study degeneracies with a non-zero jump. For example in Section 3.3 we explicitly described the locus of \( \Sigma_{d-1} \) at \( \{a_0 = 0\} \) (Lemma 3.12), what allowed to completely solve the problem of deformations with \( I = \{1, 2, \ldots, \nu\} \).

Unfortunately, in general it is very difficult to compute even the class of the set \( R_I \) in \( H^{2\nu}(\nu) \). Thus, the following partial result should be of interest.

Let \( H_x, H_y \in H^2(\nu) \) be the classes respectively of \( \{a_0 = 0\} \) and \( \{b_0 = 0\} \). The cohomology of \( \nu \) is then a polynomial algebra spanned by \( H_x \) and \( H_y \) with relations \( H_x^{d+1} = H_y^{d+1} = 0 \). We can write in \( H^{2\nu} \)
\[ [R_I] = r_0 H_x^\nu + r_1 H_x^\nu H_y + \cdots + r_\nu H_y^\nu \]
for some unknown parameters \( r_0, \ldots, r_\nu \).

Proposition 4.5. We have \( r_\nu = 1 \).

Proof. Let us intersect \( R_I \) with a generic plane \( L \) in the class \( H^2_x H^{d-\nu}_y \) This intersection is a finite set of points. By picking a sufficiently generic plane we may assume that no such point lies on \( a_0 b_0 = 0 \). The proof will be finished when we have shown that there is precisely one such point.

Let us pick affine coordinates on \( \nu \setminus \{a_0 b_0 = 0\} \) (which still we denote by \( a_1, \ldots, a_d, b_1, \ldots, b_d \)). Then, the plane \( L \) is given by \( d \) equations of the form
\[ \theta_0 a_1 + \cdots + \theta_d a_d = 0 \]
and \( d - \nu \) equations of the form
\[ \theta'_0 b_1 + \cdots + \theta'_d b_d = 0 \]
Observe also that \( R_I \) in the affine part can be presented by
\[ P_i(a_1, \ldots, a_{i-1}, b_1, \ldots, b_{i-1}) + \frac{q}{p} a_{i_1} = b_{i_1} \]
\[ \cdots \]
\[ P_{i\nu}(a_1, \ldots, a_{i\nu-1}, b_1, \ldots, b_{i\nu-1}) + \frac{q}{p} a_{i\nu} = b_{i\nu}, \]
where the polynomials $P_i$ are defined by
\[
\gamma_i(1, a_1, \ldots, a_d, 1, b_1, \ldots, b_d) = b_i - \frac{q}{p}a_i - P_i(a_1, \ldots, a_{i-1}, b_1, \ldots, b_{i-1}).
\]

We want to show that (4.2), (4.3) and (4.4) have a unique solution. This is self-evident: (4.2) uniquely determine $a_1, \ldots, a_d$. Then (4.3) and (4.4) become a system of linear equations on $b$’s. Using genericity of $L$ we can easily see that this system is non-degenerate. \hfill \Box

**Proposition 4.6.** In the class $[R_I]$ we have also $r_0 = 1$.

**Proof.** Let us write the Puiseux expansion in the opposite direction $x = g_0 y^{p/q} + g_1 y^{(p+1)/q} + \ldots$ (see Section 5). The functions $g_i$ are of the form $b_0^{(p+i)/q-i} \delta_i(a, b)$, where $\delta_i$ is a polynomial in $a$ and $b$. The issue is that the set $\{\delta_i(a, b) = 0\} \setminus \{a_0b_0 = 0\}$ and $\{\gamma_i(a, b) = 0\} \setminus \{a_0b_0 = 0\}$ have both the same geometrical interpretation. Thus they agree, as they are both smooth and thus reduced. Hence the classes in $H^{2\nu}$ of $R_I$ and $R'_I = \{\delta_i(a, b) = 0\} \setminus \{a_0b_0 = 0\}$ are the same. The proposition follows from Proposition 4.5. \hfill \Box

We can prove in a similar way that the coefficients $r_k$ are symmetric i.e. $r_k = r_{\nu-k}$, at least if $I$ is a defining set of some singularity.

On the other hand we could attempt to compute the coefficient $r_0$ from the Kouschnirenko–Bernstein algorithm (see [F1, Section 5.5]). Their method is roughly speaking as follows: pick a generic plane in the class $H^m H^{2d-m}$ some $m$. Such plane is given by the set of $m$ equations of the form (4.2) and $2d - m - \nu$ equations of the form (4.3).

Let $N_i$ be the Newton polytope of the polynomial $\gamma_i$, $N_a$ and $N_b$ the Newton polytopes of the polynomials in (4.2), (4.3). If $\gamma_i$ were generic polynomials with Newton polytope $N_i$ then the coefficient $r_{d-m}$ would be equal to mixed volume of the polytopes
\[
(N_{i_1}, \ldots, N_{i_\nu}, N_a, \ldots, N_a, N_b, \ldots, N_b)
\]
multiplied by $(2l)!$. This volume can be computed easily by a computer (we use here Proposition 1.2 to determine the Newton polytope $N_i$), and in simple cases also by hand. Already in the case $I = \{1, 3, 5\}$ this can be easily shown to contradict Proposition 4.6. One of the reasons of the non-genericity of polynomials $c_i$ is the existence of “inverse” Puiseux polynomials $d_i$ as stated in the proof of Proposition 4.6. We may write down this result as it is rather important.

**Corollary 4.7.** The Puiseux polynomials violate the Kouschnirenko–Bernstein genericity condition.

This makes the computation of coefficients $r_k$ in general case apparently hopeless.
5. Reverse Puiseux coefficients

Up to now we were mostly concerned in the Puiseux expansion of \( y \) in powers of \( x \) (1.2). However, we can calculate also the expansion of \( x \) in powers of \( y \)

\[
(5.1) \quad x = y^{p/q} + g_1 x^{(p+1)/q} + \ldots
\]

We shall here assume that \( a_0 = b_0 = 1 \).

The map \( G: (c_1, \ldots) \rightarrow (g_1, \ldots) \) is a polynomial map. It is homogeneous of degree 1 if \( \deg c_i = i \) and \( \deg g_j = j \). \( G \) is determined uniquely, up to a choice of the root of unity. We can choose this root by requiring that \( g_i = c_i + R_i(c_1, \ldots, c_{i-1}) \). Then the coefficients of polynomials \( R_i \) are rational numbers.

We are interested in the derivative of \( G \). In order to calculate it we introduce some new notation. Let

\[
\begin{align*}
  u &= x^{1/p} \\
  w &= y^{1/q} \\
  z &= (1 + g_1 w + \ldots)^{1/p} \\
  G &= g_1 w + g_2 w^2 + \ldots \\
  G_i &= \frac{\partial g_1}{\partial c_i} w + \frac{\partial g_2}{\partial c_i} w^2 + \ldots \\
  G_{ij} &= \frac{\partial^2 g_1}{\partial c_i \partial c_j} w + \frac{\partial^2 g_2}{\partial c_i \partial c_j} w^2 + \ldots \\
  G_{ijk} &= \frac{\partial^3 g_1}{\partial c_i \partial c_j \partial c_k} w + \frac{\partial^3 g_2}{\partial c_i \partial c_j \partial c_k} w^2 + \ldots
\end{align*}
\]

We will attempt to calculate \( G_{ij} \) and show its various symmetries. We will always treat \( g_j \)'s as functions of \( c_i \)'s and \( u \). First let us write

\[
(5.2) \quad w^q = u^q + c_1 u^{q+1} + c_2 u^{q+2} + \ldots \\
       u^p = w^p + g_1 w^{p+1} + g_2 w^{p+2} + \ldots
\]

Differentiating the second equation of (5.2) over \( c_i \) we obtain

\[
0 = \frac{\partial w^p}{\partial c_i} + g_1 \frac{\partial w^{p+1}}{\partial c_i} + g_1 \frac{\partial w^{p+1}}{\partial c_i} w^{p+1} + g_2 \frac{\partial w^{p+2}}{\partial c_i} + g_2 \frac{\partial w^{p+2}}{\partial c_i} w^{p+2} + \ldots
\]

Thus

\[
-w^p \left( \frac{\partial g_1}{\partial c_i} w + \frac{\partial g_2}{\partial c_i} w^2 + \ldots \right) = \left( pw^{p-1} + g_1 (p + 1) w^p + \ldots \right) \frac{\partial w}{\partial c_i}.
\]

But

\[
pw^{p-1} + g_1 (p + 1) w + \ldots = \frac{\partial}{\partial w} (w^p z^p) = pw^{p-1} z^p + pw^p z^{p-1} \frac{\partial z}{\partial w}.
\]
Hence

\[ (5.3) \quad -w^p G_i = p w^{p-1} z^{p-1} \left( z + w \frac{\partial z}{\partial w} \right) \frac{\partial w}{\partial c_i}. \]

But \( w = (u^q + c_1 u^{q+1} + \ldots)^{1/q} = u^q (1 + c_1 u + \ldots)^{1/q} \). Hence

\[ (5.4) \quad \frac{\partial w}{\partial c_i} = \frac{i}{q} w^i u^i (1 + c_1 u + \ldots)^{1/q-1} = \frac{1}{q} w^{1-q} u^{q+i}. \]

But \( u = wz \), hence

\[ \frac{\partial w}{\partial c_i} = \frac{1}{q} w^{i+1} z^{q+i}. \]

Finally

\[ (5.5) \quad G_i = \frac{-p}{q} w^{i} z^{p+q+i-1} \left( z + w \frac{\partial z}{\partial w} \right). \]

In order to compute \( G_{ij} \) we differentiate (5.5) over \( c_j \). We get

\[ \frac{\partial G_i}{\partial c_k} = \frac{-p}{q} w^{i-1} z^{p+q+i-2} \left[ \left( (i+1) wz \frac{\partial z}{\partial c_k} + iz^2 \right) \frac{\partial w}{\partial c_k} \right] \]

\[ + \left( (p+q+i) wz + (p+q+i-1) w^2 \frac{\partial z}{\partial c_k} \right) \frac{\partial z}{\partial c_k} + w^2 z \frac{\partial^2 z}{\partial c_k \partial w} \right]. \]

To compute \( \frac{\partial z}{\partial c_k} \) observe that similarly as in (5.4) we have

\[ \frac{\partial z}{\partial g_i} = \frac{1}{p} w^{i-1} z^{1-p} \]

Hence, by the chain rule

\[ \frac{\partial z}{\partial c_k} = \sum_{l=1}^{\infty} \frac{\partial z}{\partial g_l} \frac{\partial g_l}{\partial c_k} = \frac{1}{p} z^{1-p} \sum_{l=1}^{\infty} w^{l} \frac{\partial g_l}{\partial c_k} = -\frac{1}{q} w^{k} z^{q+k} \left( w + z \frac{\partial z}{\partial w} \right). \]

Now it is rather straightforward to see that

\[ \frac{\partial^2 z}{\partial c_k \partial w} = -\frac{1}{q} w^{k-1} z^{q+k-1}. \]

\[ \times \left[ k z^2 + (q + 2 + 2k) wz \frac{\partial z}{\partial w} + (q + k) w^2 \left( \frac{\partial z}{\partial w} \right)^2 + w^2 z \frac{\partial^2 z}{\partial w^2} \right]. \]

Then we get the following result

\[ G_{ik} = \frac{p}{q} w^{i+k} z^{p+q+2i+k}. \]

(5.6)

\[ \frac{\partial z}{\partial c_k} = \frac{i}{q} w^i u^i (1 + c_1 u + \ldots)^{1/q-1} \]

\[ + \left( (p+q+i+k) z^2 + (2p+3q+2i+2k+1) wz \frac{\partial z}{\partial w} + \right. \]

\[ + (p+2q+i+k-1) w^2 \left( \frac{\partial z}{\partial w} \right)^2 + w^2 z \frac{\partial^2 z}{\partial w^2} \right]. \]
Proposition 5.1. The generating functions \( G_{ik}, \ G_{ikm}, \ldots \) depend only on the sum \( k + l, \ k + l + m, \ldots \). In other words, the derivative \( \frac{\partial^r g_r}{\partial c_1^{r_1} \ldots \partial c_n^{r_n}} \), where \( r = \sum r_i \) depends only on the sum \( \sum i r_i \).

Proof. For \( r = 2 \) (or \( G_{ik} \)) the proposition follows from (5.6). We will show only the case \( r = 3 \) or \( G_{ikm} \). Observe that \( G_{ikm} = \frac{\partial}{\partial c_m} G_{1,i,i+k-1} = G_{1,m,i+k-1} \). But then, since \( G_{ikm} \) is symmetric in \( i, k \) and \( m \) we have \( G_{ikm} = \frac{\partial}{\partial c_1} G_{m,i+k-1} = \frac{\partial}{\partial c_1} G_{1,i,i+k+m-2} = G_{1,1,i+k+m-2} \). The proposition follows.

The dependence of \( \frac{\partial^r g_r}{\partial c_1^{r_1} \ldots \partial c_m^{r_m}} \) only on the sum \( k + l + m \) is an interesting feature. First we can show that the functions \( g_i \), when suitably modified, can satisfy WDVV equation (see [Du] for necessary definition and context). Namely let us pick arbitrary integer \( N > 2 \). Let us define an \( N \times N \) matrix \( \eta \) by

\[
\eta_{ab} = \begin{cases} 1 & \text{when } a + b = N + 1 \\ 0 & \text{otherwise} \end{cases}
\]

Corollary 5.2. The function \( g_{N+3} \) satisfies the WDVV equation of the form

\[
\sum_{\sigma,\tau=1}^{N} \frac{\partial^3 g_{N+3}}{\partial c_\alpha c_\beta c_\sigma} \eta_{\alpha \sigma} \frac{\partial^3 g_{N+3}}{\partial c_\mu c_\nu c_\tau} = \sum_{\sigma,\tau=1}^{N} \frac{\partial^3 g_{N+3}}{\partial c_\alpha c_\beta c_\tau} \eta_{\sigma \tau} \frac{\partial^3 g_{N+3}}{\partial c_\mu c_\nu c_\sigma}
\]

for any \( \alpha, \beta, \mu, \nu = 1, \ldots, N \).

Proof. Let

\[
H_s = \frac{\partial^3 g_{N+3}}{\partial c_\alpha \partial c_\beta \partial c_\sigma},
\]

where \( a + b + c = s \). Proposition 5.1 ensures that \( H_s \) is well-defined. Moreover, \( H_k = 0 \) for \( k \leq 2 \) and \( k > N + 3 \). Let us also define

\[
a_1 = \alpha + \beta, \ a_2 = \mu + \nu, \ a_3 = \alpha + \nu.
\]

The statement is trivial when \( a_2 = a_3 \). So let us assume \( a_2 > a_3 \) and \( a_1 \leq a_3 \).

We need to prove that

\[
\sum_{i=1}^{N+1} H_{a_1 + i} H_{a_2 + N + 1 - i} = \sum_{i=1}^{N+1} H_{a_3 + i} H_{a_1 + a_2 - a_3 + N + 1 - i}.
\]

Substituting \( i = j + a_1 - a_3 \) on the right hand side we get

\[
\sum_{i=1}^{N+1} H_{a_1 + i} H_{a_2 + N + 1 - i} = \sum_{j=a_3-a_1+1}^{N+a_3-a_1+1} H_{a_1 + j} H_{a_2 + N + 1 - j}.
\]

Now, for \( j > N + 1 \) we have \( a_1 + j > N + 3 \) so \( H_{a_1 + j} = 0 \). On the other hand, for \( i \leq a_3 - a_1 \) we have \( a_2 + N + 1 - i \geq N + 1 + a_1 + a_2 - a_3 \geq N + a_1 + 2 \geq N + 4 \), so \( H_{a_2 + N - i} = 0 \). This ends the proof. \( \square \)
Observe that for fixed set of variables $c_1, \ldots, c_N$, all functions $g_k$ for $k \leq N + 2$ satisfy the WDVV equation.

**Remark 5.3.** It is possible that if we take $g_M$ for $M$ larger than $N + 3$, but restrict ourselves to the subset $c_1 = \cdots = c_n = 0$ for suitably chosen $n$, we still get a solution to the WDVV equation (we should only change the matrix $\eta$). We note also that the Corollary 5.2 is valid for an arbitrary choice of $p$ and $q$.

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