Detection of entanglement for multipartite quantum states

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Abstract We study genuine tripartite entanglement and multipartite entanglement of arbitrary $n$-partite quantum states by using the representations with generalized Pauli operators of a density matrices. While the usual Bloch representation of a density matrix uses three types of generators in the special unitary Lie algebra $\text{su}(d)$, the representation with generalized Pauli operators has one uniformed type of generators and it simplifies computation. In this paper, we take the advantage of this simplicity to derive useful and operational criteria to detect genuine tripartite entanglement. We also obtain a sufficient criterion to detect entanglement for multipartite quantum states in arbitrary dimensions. The new method can detect more entangled states than previous methods as backed by detailed examples.

Keywords Genuine entanglement · Correlation tensor · Generalized Pauli operators
1 Introduction

Quantum entanglement is a key resource in quantum information with wide applications in entanglement swapping [1], quantum cryptography [2] and quantum secure communication [3]. The genuine multipartite entanglement (GME) stands out with significant properties [4,5]. Thus measuring and detection of genuine multipartite entanglement for given states has been an important task in quantum computation.

A lot of methods have been presented in detecting entanglement and genuine entanglement [6–9]. For tripartite quantum states, sufficient conditions to test entanglement of three-qubit states in the vicinity of the GHZ, the W states and the PPT entangled states were found in [10]. A sufficient criterion for the entanglement of tripartite systems based on local sum uncertainty relations was proposed in [11]. The sufficient conditions for judging genuine tripartite entanglement were presented by using partial transposition and realignment of density matrices in [12]. Yang et al [13] derived a criterion for detecting genuine tripartite entanglement based on quantum Fisher information. By using the Bloch representation of density matrices and the norms of correlation tensors, the genuine tripartite entangled criteria were presented in [14,15]. The authors in [16] studied the separability criteria in tripartite and four-partite quantum system by the matrix method. The separability criteria for four-partite quantum system based on the upper bound of Bloch vectors were discussed in [17]. For higher dimensional quantum system, Chen et al [18] presented a generalized partial separability criterion of multipartite quantum systems in arbitrary dimensions. The separable criteria and $k$-separable criteria for general $n$-partite quantum states were given in [19,21].

Many of these methods have used the Bloch representation of the density matrix, which has become more complex as dimension of the quantum system increases. This is partly due to the fact that the Bloch representation is relied on the Gell-Mann basis of the special unitary Lie algebra $\mathfrak{su}(d)$ which has three kinds of basis elements: upper, diagonal and lower matrices. In view of this, perhaps using another well-known basis of the Lie algebra $\mathfrak{su}(d)$: the Weyl basis to study quantum entanglement will likely simplify some of the criteria, as the latter consists of uniformed basis elements. In Ref. [22], the authors showed that the principal basis matrix plays an essential role in the representation theory of the Yangian $Y(\mathfrak{sl}(3))$ which has a close relation with the study of entangled states in quantum information (see also for recent applications [23]).

In this paper, we study entanglement of multipartite quantum systems by using the representation with generalized Pauli operators, and we obtain several better criteria in detecting the GME than previously available tests. The paper is organized as follows. In section 2, after reviewing the representation with generalized Pauli operators of the quantum state, we construct matrices by using the correlation tensors and derive the criteria to detect entanglement and genuine tripartite entanglement. By detailed example, our results can detect more genuine entangled states. In section 3, we obtain the entanglement theorem for arbitrary $n$-partite quantum systems. Conclusions are given in section 4.

2 Genuine entanglement for tripartite quantum state

We first consider the GME for tripartite states. Let $E_{ij}$ be the $d \times d$ unit matrix with the only nonzero entry 1 at the position $(i, j)$, and let $\omega$ be a fixed $d$-th primitive root of unity. By means of the division with remainder, for certain $d_s$ and $u_s \in \{0, \cdots , d_s^2-1\}$, there exists unique integers $i$ and $j$ such that $u_s = d_s i + j$ ($0 \leq i, j \leq d_s-1$), then the generalized Pauli operators of the $s$th
$d_s$-dimensional Hilbert space $H^d_s$ are given by

$$A^{(s)}_{u_s} = A^{(s)}_{d_s,i+j} = \sum_{m=0}^{d_s-1} \omega^{im} E_{m,m+j},$$

(1)

where $\omega^{d_s} = 1$. The basis obeys the algebraic relation:

$$A^{(s)}_{d_s,i+j} A^{(s)}_{d_s,k+l} = \omega^{ik} A^{(s)}_{d_s,(i+k)+(j+l)},$$

then $(A^{(s)}_{d_s,i+j})^\dagger = \omega^{ij} A^{(s)}_{d_s,(d_s-i)+(d_s-j)}$, so $tr(A^{(s)}_{d_s,i+j} (A^{(s)}_{d_s,k+l})^\dagger) = \delta_{ik} \delta_{jl} d_s$. The trace norm (Ky Fan norm) of a rectangular matrix $A$ is defined as $\|A\|_{tr} = \sum_i \sigma_i = tr \sqrt{AA^\dagger}$, where $\sigma_i$ are the singular values of $A$ and $\|A\|_{tr} \leq \sqrt{\min\{m,n\}} \|A\|$ for any matrix $A$. Clearly $\|A\|_{tr} = \|A^\dagger\|_{tr}$.

Lemma 1 Let $H^d_s$ denote the $s^{th}$ $d_s$-dimensional Hilbert space. For a quantum state $\rho_1 \in H^d_1$, $\rho_1$ can be expressed as $\rho_1 = \frac{1}{d_1^2} \sum_{u_1=0}^{d_1^2-1} t_{u_1} A^{(1)}_{u_1}$, where $A^{(1)}_{u_1} = I_{d_1}$, $t_{u_1} = tr (\rho_1 (A^{(1)}_{u_1})^\dagger)$ are complex coefficients. Let $T^{(1)}$ be the column vector with entries $t_{u_1}$ for $u_1 \neq 0$, we have

$$\|T^{(1)}\|_2 ^2 \leq d_1 - 1.$$  

(2)

Proof. Since $tr (\rho_1^2) \leq 1$, we have

$$tr (\rho_1 t_{u_1}^2) = tr (\rho_1 (A^{(1)}_{u_1})^\dagger) \leq 1,$$  

(3)

namely, $\|T^{(1)}\|_2 ^2 \leq d_1 - 1$. 

For a state $\rho_{12} \in H^d_1 \otimes H^d_2$, $\rho_{12}$ has the generalized Pauli operators representation:

$$\rho_{12} = \frac{1}{d_1 d_2} \sum_{u_1=0}^{d_1^2-1} \sum_{u_2=0}^{d_2^2-1} t_{u_1,u_2} A^{(1)}_{u_1} \otimes A^{(2)}_{u_2}$$

(4)

where $A^{(s)}_{u_0} = I_{d_s}$, $s = 1, 2$, the coefficients $t_{u_1,u_2} = tr (\rho_{12} (A^{(1)}_{u_1})^\dagger \otimes (A^{(2)}_{u_2})^\dagger)$ are complex numbers. Let $T^{(1)}, T^{(2)}, T^{(12)}$ be the vectors with entries $t_{u_1,0}, t_{0,u_2}, t_{u_1,u_2}$ for $u_1, u_2 \neq 0$.

Lemma 2 Let $\rho_{12} \in H^d_1 \otimes H^d_2$ be a mixed state, we have $\|T^{(12)}\|_2 ^2 \leq d_1 d_2 (1 - \frac{1}{d_1^2} - \frac{1}{d_2^2}) + 1$.

Proof. For a pure state $\rho_{12}$, we have $tr (\rho_{12}^2) = 1$, namely

$$tr (\rho_{12}^2) = tr (\rho_{12} \rho_{12}^\dagger) = \frac{1}{d_1 d_2} (1 + \|T^{(1)}\|^2 + \|T^{(2)}\|^2 + \|T^{(12)}\|^2) = 1.$$  

(5)

By using $tr (\rho_{12}^2) = tr (\rho_1^2)$, we have $\frac{1}{d_1^2} (1 + \|T^{(1)}\|^2) = \frac{1}{d_2^2} (1 + \|T^{(2)}\|^2)$, where $\rho_1$ and $\rho_2$ are the reduced density operators on $H^d_1$ and $H^d_2$, respectively. Then

$$\frac{1}{d_1^2} (1 + \|T^{(1)}\|^2) + \frac{1}{d_2^2} (1 + \|T^{(2)}\|^2) = \frac{1}{d_1 d_2} (2 + \|T^{(1)}\|^2 + \|T^{(2)}\|^2).$$
Using the above two equations we obtain that
\[ \|T^{(12)}\|^2 = d_1 d_2 - 1 - \|T^{(1)}\|^2 - \|T^{(2)}\|^2 \]
\[ = d_1 d_2 + 1 - \frac{d_2}{d_1}(1 + \|T^{(1)}\|^2) + \frac{d_1}{d_2}(1 + \|T^{(2)}\|^2). \]
By \(\|T^{(1)}\|^2 \geq 0, \|T^{(2)}\|^2 \geq 0\), we have
\[ \|T^{(12)}\|^2 \leq d_1 d_2 + 1 - \frac{d_2}{d_1} \cdot 1 + \frac{d_1}{d_2} \cdot 1 = d_1 d_2(1 - \frac{1}{d_1^2} - \frac{1}{d_2^2}) + 1, \]
If \(\rho\) is a mixed state, then \(\rho = \sum_i p_i \rho_i\) is a convex sum of pure states, \(\sum_i p_i = 1\). Then
\[ \|T^{(12)}(\rho)\| \leq \sum_i p_i \|T^{(12)}(\rho_i)\| \leq \sqrt{d_1 d_2(1 - \frac{1}{d_1^2} - \frac{1}{d_2^2}) + 1}. \]

A general tripartite state \(\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}\) can be written as
\[ \rho = \frac{1}{d_1 d_2 d_3} \sum_{u_1=0}^{d_1-1} \sum_{u_2=0}^{d_2-1} \sum_{u_3=0}^{d_3-1} t_{u_1,u_2,u_3} A_{u_1}^{(1)} \otimes A_{u_2}^{(2)} \otimes A_{u_3}^{(3)}, \]
where \(A_{u_i}^{(f)}\) stands for that the tensor operator with \(A_{u_i}\) acting on the space \(H_i^{d_i}\), \(A_0^{(f)} = I_{d_f}\), \(t_{u_1,u_2,u_3} = tr(\rho(A_{u_1}^{(1)})^\dagger \otimes (A_{u_2}^{(2)})^\dagger \otimes (A_{u_3}^{(3)})^\dagger)\) are the complex coefficients.

In the following we will construct some matrices out of the expansion coefficients of the density matrix \(\rho\) in \(\{0\}\). For \(f,g,h \in \{1,2,3\}\), set
\[ N_{ij}^{fg} = \text{diag}(N_1^{fg}, N_2^{fg}, \ldots, N_d^{fg}), \]
where \(N_i^{fg} = [t_{i,u_g,u_h}]\) is a \((d_g^2 - 1) \times (d_h^2 - 1)\) matrix, \(i = 1,2,\ldots,d_f - 1\). For example, when \(\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}\), \(N_2^{13} = \text{diag}(N_1^{213},N_2^{213},N_3^{213})\), where
\[ N_i^{213} = \begin{bmatrix} t_{1,1,1} & t_{1,1,2} & \cdots & t_{1,1,8} \\ t_{2,1,1} & t_{2,1,2} & \cdots & t_{2,1,8} \\ \vdots & \vdots & \ddots & \vdots \\ t_{3,1,1} & t_{3,1,2} & \cdots & t_{3,1,8} \end{bmatrix}, i = 1,2,3. \]

**Theorem 1** For a biseparable tripartite pure state \(\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}\) and \(f \neq g \neq h \in \{1,2,3\}\), we have
(i) if \(\rho\) is separable under bipartite partition \(f|gh\), then
\[ \|N_{f|gh}\|_tr \leq \sqrt{(d_f^2 - 1) \cdot \min\{d_g^2 - 1, d_h^2 - 1\}(d_f - 1)[d_g d_h(1 - \frac{1}{d_f^2} - \frac{1}{d_h^2}) + 1]} \]
(ii) if \(\rho\) is separable under bipartite partition \(g|fh\), then
\[ \|N_{g|fh}\|_tr \leq \sqrt{(d_f^2 - 1)(d_g - 1)[d_f d_h(1 - \frac{1}{d_f^2} - \frac{1}{d_h^2}) + 1]} \]
(iii) if \(\rho\) is separable under bipartite partition \(h|fg\), then
\[ \|N_{h|fg}\|_tr \leq \sqrt{(d_f^2 - 1)(d_h - 1)[d_f d_g(1 - \frac{1}{d_f^2} - \frac{1}{d_g^2}) + 1]} \]
Proof (i) If the tripartite pure state $\rho$ is separable under the bipartition $f|gh$, then it can be expressed as

$$\rho_{f|gh} = \rho_f \otimes \rho_{gh},$$

where

$$\rho_f = \frac{1}{d_f} \sum_{u_f=0}^{d_f^2-1} t_{u_f} A^{(f)}_{u_f},$$

and

$$\rho_{gh} = \frac{1}{d_g d_h} \sum_{u_g=0}^{d_g^2-1} \sum_{u_h=0}^{d_h^2-1} t_{u_g, u_h} A^{(g)}_{u_g} \otimes A^{(h)}_{u_h}.$$

Let $T^{(f)}$, $T^{(g)}$, $T^{(h)}$ and $T^{(gh)}$ be the vectors with entries $t_{u_f}$, $t_{u_g,0}$, $t_{u_h}$ and $t_{u_g, u_h}$ for $u_f, u_g, u_h \neq 0$. Then

$$N^{f|gh} = \begin{bmatrix} t_1 & t_2 & \cdots & t_{d_f^2-1} \\ t_2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ t_{d_f^2-1} & & & t_{d_f^2-1} \end{bmatrix} \otimes \begin{bmatrix} t_{11} & \cdots & t_{1(d_f^2-1)} \\ t_{21} & \cdots & t_{2(d_f^2-1)} \\ \vdots & \vdots & \vdots \\ t_{(d_f^2-1)1} & \cdots & t_{(d_f^2-1)(d_f^2-1)} \end{bmatrix}.$$

It follows from Lemma 1 and Lemma 2 that

$$\|N^{f|gh}\|_{tr} = \|t_{11} \cdots t_{1(d_f^2-1)} \|_{tr} \leq \sqrt{(d_f^2 - 1) \|T^{(f)}\| \cdot \min\{d_g^2 - 1, d_h^2 - 1\} \|T^{(gh)}\|} \leq \sqrt{(d_f^2 - 1) \cdot \min\{d_g^2 - 1, d_h^2 - 1\} (d_f - 1)[d_g d_h (1 - \frac{1}{d_f} - \frac{1}{d_h}) + 1]}.$$

(ii) If the tripartite pure state $\rho$ is separable under the bipartition $g|fh$, then it can be expressed as

$$\rho_{g|fh} = \rho_g \otimes \rho_{fh},$$

where

$$\rho_g = \frac{1}{d_g} \sum_{u_g=0}^{d_g^2-1} t_{u_g} A^{(g)}_{u_g},$$

and

$$\rho_{fh} = \frac{1}{d_f d_h} \sum_{u_f=0}^{d_f^2-1} \sum_{u_h=0}^{d_h^2-1} t_{u_f, u_h} A^{(f)}_{u_f} \otimes A^{(h)}_{u_h}.$$

Then, we have $N_i^{f|gh} = T^{(g)} \cdot (T_i^{(fh)})^t$, and

$$\|N^{f|gh}\|_{tr} = \|T^{(g)}\| \cdot \sum_{i=1}^{d_f^2-1} \|T_i^{(fh)}\| \leq \sqrt{d_f^2 - 1} \|T^{(g)}\| \|T^{(fh)}\| \leq \sqrt{(d_f^2 - 1)(d_g - 1)[d_f d_h (1 - \frac{1}{d_f} - \frac{1}{d_h}) + 1]}.$$
where $T_i^{(fh)}$ is the vector with entries $t_{i,u,h}, i = 1, \cdots, d_f^2 - 1$, $t$ stands for transpose.

(iii) Using similar method, if $\rho$ is separable under the bipartition $h|fg$, we have $N_i^{fh} = T_i^{(fh)} (T^{(h)})^t, i = 1, \cdots, d_f^2 - 1$ and

$$
\|N_i^{fh}\|_{tr} \leq \sqrt{(d_f^2 - 1)(d_h - 1)[d_f d_g(1 - \frac{1}{d_f^2} - \frac{1}{d_g^2}) + 1]}.
$$

Now we consider genuine tripartite entanglement. A mixed state is said to be genuine multipartite entangled if it cannot be written as a convex combination of biseparable states. Let $T(\rho) = \frac{1}{3}(\|N^{123}\|_{tr} + \|N^{213}\|_{tr} + \|N^{312}\|_{tr})$, we define

$$
Q_1 = \text{Max}\left\{ \sqrt{(d_1^2 - 1) \cdot \min\{d_2^2 - 1, d_3^2 - 1\}(d_1 - 1)[d_2 d_3(1 - \frac{1}{d_1^2} - \frac{1}{d_2} + 1)}],
\sqrt{(d_1^2 - 1)(d_2 - 1)[d_1 d_3(1 - \frac{1}{d_1^2} - \frac{1}{d_3}) + 1]},
\sqrt{(d_1^2 - 1)(d_3 - 1)[d_1 d_2(1 - \frac{1}{d_1^2} - \frac{1}{d_2}) + 1]} \right\}.
$$

$$
Q_2 = \text{Max}\left\{ \sqrt{(d_2^2 - 1) \cdot \min\{d_1^2 - 1, d_3^2 - 1\}(d_2 - 1)[d_1 d_3(1 - \frac{1}{d_2^2} - \frac{1}{d_3}) + 1)},
\sqrt{(d_2^2 - 1)(d_1 - 1)[d_2 d_3(1 - \frac{1}{d_2^2} - \frac{1}{d_3}) + 1]},
\sqrt{(d_2^2 - 1)(d_3 - 1)[d_2 d_1(1 - \frac{1}{d_2^2} - \frac{1}{d_1}) + 1]} \right\}.
$$

$$
Q_3 = \text{Max}\left\{ \sqrt{(d_3^2 - 1) \cdot \min\{d_1^2 - 1, d_2^2 - 1\}(d_3 - 1)[d_1 d_2(1 - \frac{1}{d_1^2} - \frac{1}{d_2}) + 1]},
\sqrt{(d_3^2 - 1)(d_1 - 1)[d_3 d_2(1 - \frac{1}{d_3^2} - \frac{1}{d_2}) + 1]},
\sqrt{(d_3^2 - 1)(d_2 - 1)[d_3 d_1(1 - \frac{1}{d_3^2} - \frac{1}{d_1}) + 1]} \right\}.
$$

We have the following theorem.

**Theorem 2** A mixed state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}$ is genuine tripartite entangled if $T(\rho) > \frac{1}{3}(Q_1 + Q_2 + Q_3)$. 

Proof. If $\rho$ is a biseparable, one has $\rho = \sum_i \alpha_i \rho_i^1 \otimes \rho_i^2 + \sum_j r_j \rho_j^2 \otimes \rho_j^3 + \sum_k s_k \rho_k^3 \otimes \rho_k^4$ with $0 \leq \alpha_i, r_j, s_k \leq 1$ and $\sum_i \alpha_i + \sum_j r_j + \sum_k s_k = 1$. By Theorem 1, we have that

$$T(\rho) = \frac{1}{3}(\|N^{1|23}(\rho)\|_{tr} + \|N^{2|13}(\rho)\|_{tr} + \|N^{3|12}(\rho)\|_{tr})$$

$$= \frac{1}{3}(\|N^{1|23}(\rho)\|_{tr} + \|N^{2|13}(\rho)\|_{tr} + \|N^{3|12}(\rho)\|_{tr})$$

$$+ \sum_j r_j \rho_j^2 \otimes \rho_j^3 + \sum_k s_k \rho_k^3 \otimes \rho_k^4)\|_{tr} + \|N^{2|13}(\rho)\|_{tr} + \|N^{3|12}(\rho)\|_{tr})$$

$$\leq \frac{1}{3}\left(\sum_i \alpha_i \|N^{1|23}(\rho)\|_{tr} + \sum_j r_j \|N^{1|23}(\rho)\|_{tr} + \sum_k s_k \|N^{1|23}(\rho)\|_{tr})\|_{tr} + \sum_j \|N^{2|13}(\rho)\|_{tr} + \sum_k \|N^{2|13}(\rho)\|_{tr})\|_{tr} + \sum_j \|N^{3|12}(\rho)\|_{tr} + \sum_k \|N^{3|12}(\rho)\|_{tr})\|_{tr}$$

$$= \frac{1}{3}(\sum_i \alpha_i \rho_i^1 + \sum_j \rho_j^2 + \sum k \rho_k^3)$$

$$= \frac{1}{3}(Q_1 + Q_2 + Q_3)$$

(15)

Consequently, if $T(\rho) > \frac{1}{3}(Q_1 + Q_2 + Q_3)$, $\rho$ is genuine tripartite entangled. □

Next we consider the permutational invariant state $\rho$, i.e. $\rho = \rho^p = ppm^p$ for any permutation $p$ of the qudits. A biseparable permutational invariant state can be written as $\rho = \sum_i \alpha_i \rho_i^1 \otimes \rho_i^2 + \sum_j r_j \rho_j^2 \otimes \rho_j^3 + \sum k \rho_k^3 \otimes \rho_k^4$, where $0 \leq \alpha_i, r_j, s_k \leq 1$. Set $d_1 = d_2 = d_3 = d$, we have the following corollary.

Corollary 1. If a permutational invariant mixed state is biseparable, then we have

$$T(\rho) = \frac{1}{3}(\|N^{1|23}\|_{tr} + \|N^{2|13}\|_{tr} + \|N^{3|12}\|_{tr}) \leq J_1.$$  

Therefore if $T(\rho) > J_1$, $\rho$ is genuine tripartite entangled. Here

$$J_1 = \frac{(d-1)^2 \sqrt{(d-1)(d^2-1)} + 2(d^2-1)\sqrt{d-1}}{3}.$$  

Example 1. Consider the mixed three-qubit $W$ state,

$$\rho = \frac{1-x}{8} I_8 + x |W\rangle \langle W|,$$  

$0 \leq x \leq 1,$

(16)

where $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ and $I_8$ is the $8 \times 8$ identity matrix. Let $f_1(x) = T(\rho) - J_1 = 5x - (2 + \sqrt{3})$, using Corollary 1 we have $\rho$ is genuine entangled if $f_1(x) > 0$, i.e. $0.7464 < x \leq 1$. Theorem 2 in [15] implies that $\rho$ is genuine entangled if $g_1(x) = 3.26x - \frac{4\sqrt{3}}{10} > 0$, i.e. $0.791 < x \leq 1$. Our corollary can detect more genuine entangled, see the comparison in Fig. 1.


Fig. 1 $f_1(x)$ from our result (solid straight line), $g_1(x)$ from Theorem 2 in [15] (dash-dot straight line).

3 Entanglement for multipartite quantum state

Now we consider entanglement of $n$-partite quantum systems. Let \( \{ A_s^{(u)} \} \) \((u_s = 0, \ldots, d_s^2 - 1)\) be the generalized Pauli operators of the $s$th $d_s$-dimensional Hilbert space $H_s^{d_s}$. Any quantum state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}$ has the generalized Pauli operators representation:

$$
\rho = \frac{1}{d_1 d_2 \cdots d_n} \sum_{u_1} d_1^{2} - 1 \sum_{u_2} d_2^{2} - 1 \cdots \sum_{u_n} d_n^{2} - 1 A_{u_1}^{(1)} \otimes A_{u_2}^{(2)} \otimes \cdots \otimes A_{u_n}^{(n)}
$$

(17)

where $A_s^{(u)} = I_{d_s} \quad (s = 1, \ldots, n)$, $t_{u_1, u_2, \ldots, u_n} = \text{tr}(\rho (A_{u_1}^{(1)})^\dagger \otimes (A_{u_2}^{(2)})^\dagger \otimes \cdots \otimes (A_{u_n}^{(n)})^\dagger)$ are complex coefficients. Let $T^{(l_1 \ldots l_k)}$ be the vectors with entries $t_{u_{l_1}, u_{l_2}, \ldots, u_{l_k}}$, $u_{l_1}, \ldots, u_{l_k} \neq 0$ and $1 \leq l_1 < \cdots < l_k \leq n$. We have

$$
\| T^{(1)} \|^2 = \sum_{u_1 = 1}^{d_1^{2} - 1} \cdot \sum_{u_1 = 1}^{d_1^{2} - 1} \cdots \sum_{u_1 = 1}^{d_1^{2} - 1} t_{u_1, u_2, \ldots, u_n}^* t_{u_1, u_2, \ldots, u_n},
$$

$$
\| T^{(l_1 \ldots l_k)} \|^2 = \sum_{s=1}^{k} \sum_{u_{l_s} = 1}^{d_{l_s}^{2} - 1} \cdots \sum_{u_{l_s} = 1}^{d_{l_s}^{2} - 1} t_{u_{l_1}, \ldots, u_{l_k}}^* t_{u_{l_1}, \ldots, u_{l_k}} \cdot 0,
$$

$$
\| T^{(12 \ldots n)} \|^2 = \sum_{s=1}^{n} \sum_{u_{l_s} = 1}^{d_{l_s}^{2} - 1} t_{u_{l_1}, \ldots, u_{l_n}}^* t_{u_{l_1}, \ldots, u_{l_n}},
$$

where $*$ represents the conjugate. Set

$$
A_1 = \| T^{(1)} \|^2 + \cdots + \| T^{(n)} \|^2,
$$

$$
A_2 = \| T^{(12)} \|^2 + \cdots + \| T^{(n-1), n} \|^2,
$$

where $A_1$ and $A_2$ are the entanglement measures for the $n$-partite quantum state $\rho$. The entanglement measures $A_1$ and $A_2$ provide a quantitative way to characterize the entanglement and separability of multipartite quantum states.
\begin{equation}
A_n = \|T^{(1\cdots n)}\|^2.
\end{equation}

**Lemma 3** Let \( \rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n} \) \( (n \geq 2) \) be a \( n \)-partite pure quantum state. Then

\begin{equation}
\|T^{(1\cdots n)}\|^2 \leq \frac{d_1 \cdots d_n (n - 1 - \sum_{s=1}^{n} \frac{1}{d_s}) + 1}{n - 1}. \tag{18}
\end{equation}

**Proof** It’s enough to show the lemma for a pure state \( \rho \), where we have \( \text{tr}(\rho^2) = 1 \) and \( \text{tr}(\rho_{i_1}^2) = \text{tr}(\rho_{i_2}^2) \) for any distinct indices \( i_1, \ldots, i_n \in \{1, 2, \ldots, n\} \). Here \( \rho_{i_1} \) and \( \rho_{i_2} \) are the reduced states for the subsystem \( H_{i_1}^{d_{i_1}} \) and \( H_{i_2}^{d_{i_2}} \otimes \cdots \otimes H_{i_n}^{d_{i_n}} \). Therefore, we have

\begin{equation}
\text{tr}(\rho^2) = \frac{1}{d_1 d_2 \cdots d_n} (1 + A_1 + \cdots + A_n) = 1, \tag{19}
\end{equation}

and

\begin{equation}
\frac{1}{d_{i_1}} (1 + \|T^{(i_1)}\|^2) = \frac{1}{d_{i_2} \cdots d_{i_n}} (1 + \|T^{(i_2)}\|^2 + \cdots + \|T^{(l_{i_n})}\|^2 + \cdots + \|T^{(l_{i_2} \cdots l_{i_n})}\|^2). \tag{20}
\end{equation}

Since

\begin{equation}
\sum_{i_1=1}^{n} \frac{1}{d_{i_1}} \text{tr}(\rho_{i_1}^2) = \sum_{i_1=1}^{n} \frac{1}{d_{i_1}} \text{tr}(\rho_{i_2}^2), \tag{19}
\end{equation}

we get that

\begin{equation}
\sum_{i_2=1}^{n} \frac{1}{d_{i_1} d_{i_2}} (1 + \|T^{(i_2)}\|^2) = \frac{1}{d_1 \cdots d_n} [n + (n - 1) A_1 + (n - 2) A_2 + \cdots + A_{n-1}] + 1, \tag{21}
\end{equation}

Therefore,

\begin{equation}
A_1 = \frac{d_1 \cdots d_n}{n - 1} \sum_{s=1}^{n} \frac{1}{d_s^2} (1 + \|T^{(s)}\|^2) - \frac{n - 2}{n - 1} A_2 - \frac{n - 3}{n - 1} A_3 - \cdots - \frac{1}{n - 1} A_{n-1}. \tag{21}
\end{equation}

Substituting (21) into (19), we get

\begin{equation}
A_n = d_1 \cdots d_n - 1 - \frac{1}{n - 1} \left( d_1 \cdots d_n \sum_{s=1}^{n} \frac{1}{d_s^2} (1 + \|T^{(s)}\|^2) - n \right) - \frac{1}{n - 1} A_2
\end{equation}

\begin{equation}
= \frac{2}{n - 1} A_3 - \cdots - \frac{n - 2}{n - 1} A_{n-1} - \frac{d_1 \cdots d_n (n - 1 - \sum_{s=1}^{n} \frac{1}{d_s}) + 1}{n - 1} \tag{22}
\end{equation}

\( \square \)

Let \( \rho \) be a \( n \)-partite state \( \rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n} \) represented as (17), for real number \( \alpha, \beta \) and distinct indices \( l_1, \ldots, l_n \in \{1, 2, \ldots, n\} \), set

\begin{equation}
N^{l_1 \cdots l_{k-1} |l_k \cdots l_n} = \alpha S_{0}^{l_1 \cdots l_{k-1} |l_k \cdots l_n} + \beta S_{0}^{l_1 \cdots l_{k-1} |l_k \cdots l_n}, \tag{23}
\end{equation}

for \( k = 1, 2, \ldots, \lfloor n/2 \rfloor \), the smallest integer less or equal to \( n/2 \). Let \( T^{(l_1 \cdots l_k)} \) be the \( (d_{l_1}^2 - 1) \cdots (d_{l_k}^2 - 1) \)-dimensional column vector with entries \( t_{u_1 u_2 \cdots u_k l_1 \cdots l_k} \) associated with the generalized Pauli operators representation of \( \rho \), and define \( S_{0}^{l_1 \cdots l_{k-1} |l_k} \) to be the block matrix \( S_{0}^{l_1 \cdots l_{k-1} |l_k} = \).
Fix $\alpha, \beta$ as above. If the $n$-partite state $\rho \in H_{l_1}^{d_1} \otimes H_{l_2}^{d_2} \otimes \cdots \otimes H_{l_n}^{d_n}$ is separable under the bipartition $l_1 \cdots l_{k-1}|l_k \cdots l_n$, then we have that

(i) $\|N^{l_1|l_2 \cdots l_n}\|_{tr} \leq M_1$;

(ii) $\|N^{l_1 \cdots l_{k-1}|l_k \cdots l_n}\|_{tr} \leq M_{l_1 \cdots l_{k-1}}$ ($k \geq 3$);

where

$$M_1 = \sqrt{d_1 - 1} \left( |\alpha| \sqrt{d_2 - 1 + |\beta|} \right),$$

$$M_{l_1 \cdots l_{k-1}} = \frac{d_1 \cdots d_{k-1}(k - 2 - \sum_{s=2}^{k-1} d_s^{-2}) + 1}{k - 2} \left( |\alpha| \sqrt{d_k - 1 + |\beta|} \right).$$

Proof  (i) If the $n$-partite mixed state is separable under the bipartition $l_1|l_2 \cdots l_n$, it can be expressed as

$$\rho_{l_1|l_2 \cdots l_n} = \sum_s p_s \rho_{l_1}^{(s)} \otimes \rho_{l_2 \cdots l_n}^{(s)}, \quad 0 < p_s \leq 1, \sum_s p_s = 1,$$

where

$$\rho_{l_1}^{(s)} = \frac{1}{d_{l_1}} \sum_{u_{l_1} = 0}^{d_{l_1} - 1} A_{u_{l_1}}^{(l_1)},$$

$$\rho_{l_2 \cdots l_n}^{(s)} = \frac{1}{d_{l_2} \cdots d_{l_n}} \sum_{q=2}^{l_2} \sum_{u_{l_q} = 0}^{d_{l_q} - 1} A_{u_{l_q}}^{(l_q)} \otimes \cdots \otimes A_{u_{l_n}}^{(l_n)}.$$
By Lemma 1 and Lemma 3, we have

$$\|\mathcal{A}^{l_1} | l_{2 \cdots l_n} \|_{tr} \leq \sum \alpha \| T^{(l_1)}_s | | T^{(l_2)}_s \| + |\beta| | T^{(l_3)}_s \| \| T^{(l_2 \cdots l_n)}_s \|$$

$$\leq \sqrt{d_{l_1} - 1} \left( |\alpha| \sqrt{d_{l_2} - 1} + |\beta| \right) \frac{d_{l_2} \cdots d_{l_n} (n - 2 - \sum_{s=2}^{n} \frac{1}{d_s}) + 1}{n - 2}$$

$$= M_{l_1},$$

where we have used $\| A + B \|_{tr} \leq \| A \|_{tr} + \| B \|_{tr}$ for matrices $A$ and $B$ and $\| \langle a | b \rangle \|_{tr} = \| | a \rangle \| | \langle b | \|$ for vectors $| a \rangle$ and $| b \rangle$.

(ii) If $\rho$ is separable under the bipartition $l_1 \cdots l_{k-1} | l_k \cdots l_n$, it can be expressed as

$$\rho_{l_1 \cdots l_{k-1} | l_k \cdots l_n} = \sum_s p_s \rho_{l_1 \cdots l_{k-1}}^s \otimes \rho_{l_k \cdots l_n}^s, 0 < p_s \leq 1, \sum_s p_s = 1,$$

where

$$\rho_{l_1 \cdots l_{k-1}}^s = \frac{1}{d_{l_1} \cdots d_{l_{k-1}}} \sum_{p=1}^{k-1} \sum_{u_i=0}^{d_p-1} t_{u_1 \cdots u_{k-1}}^s A_{w_1}^{(l_1)} \cdots A_{w_{k-1}}^{(l_{k-1})},$$

$$\rho_{l_k \cdots l_n}^s = \frac{1}{d_{l_k} \cdots d_{l_n}} \sum_{q=k}^{n} \sum_{u_i=0}^{d_q-1} t_{u_k \cdots u_n}^s A_{w_k}^{(l_k)} \cdots A_{w_n}^{(l_n)}.$$

Then,

$$S^{l_1 \cdots l_{k-1} | l_k} = \sum_s p_s T^{(l_1 \cdots l_{k-1})}_s (T^{(l_k)}_s)^t,$$

$$S^{l_1 \cdots l_{k-1} | l_k \cdots l_n} = \sum_s p_s T^{(l_1 \cdots l_{k-1})}_s (T^{(l_k \cdots l_n)}_s)^t.$$

Similarly, we get

$$\|\mathcal{A}^{l_1 \cdots l_{k-1} | l_k \cdots l_n} \|_{tr} \leq \sum_s p_s \left( |\alpha| \| T^{(l_1 \cdots l_{k-1})}_s \| \| T^{(l_k)}_s \| + |\beta| \| T^{(l_1 \cdots l_{k-1})}_s \| \| T^{(l_k \cdots l_n)}_s \| \right)$$

$$\leq \sqrt{\frac{d_{l_1} \cdots d_{l_{k-1}} (k - 2 - \sum_{s=2}^{k-1} \frac{1}{d_s}) + 1}{k - 2} |\alpha| \sqrt{d_{l_k} - 1} + |\beta| \right) \frac{d_{l_k} \cdots d_{l_n} (n - k - \sum_{s=k}^{n} \frac{1}{d_s}) + 1}{n - k}$$

$$= M_{l_1 \cdots l_{k-1}}.$$ 

\[\square\]

**Example 2** Consider the quantum state $\rho \in H_1^3 \otimes H_2^3 \otimes H_3^3$,

$$\rho = \frac{1 - x}{18} I_{18} + x |\varphi\rangle \langle \varphi|,$$

(32)

where $|\varphi\rangle = \frac{1}{\sqrt{2}} \left( |(10) + |21\rangle |0\rangle + |(00) + |11\rangle + |22\rangle |1\rangle \right)$, $0 \leq x \leq 1$, $I_{18}$ is the $18 \times 18$ identity matrix. By Theorem 3 (i), we can determine the range of $x$ where $\rho$ is surely entangled. Table [I] shows that when $\alpha = 0$, $\beta = 1$, our criterion detects the entanglement for $0.3405 < x \leq 1$, which is better than the result $0.35 \leq x \leq 1$ given in [25].
Table 1 The entanglement regions of $\rho$ as given by Theorem 3.

| $\alpha, \beta$ | $\|N_{2|13}\|_{tr}$ | The range of entanglement |
|-----------------|----------------|--------------------------|
| $\alpha = 1, \beta = 1$ | 10.5292x | $0.4852 < x \leq 1$ |
| $\alpha = 1, \beta = 2$ | 18.4650x | $0.3909 < x \leq 1$ |
| $\alpha = 0, \beta = 1$ | 9.1321x | $0.3405 < x \leq 1$ |

Example 3 Consider the four-qubit state $\rho \in H_1^2 \otimes H_2^2 \otimes H_3^2 \otimes H_4^2$, $\rho = x|\psi\rangle\langle\psi| + \frac{1-x}{16}I_{16}$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, $0 \leq x \leq 1$, $I_{16}$ is the $16 \times 16$ identity matrix. Using Theorem 3 (i) with $\alpha = 1, \beta = 1$, we set $f_2(x) = \|N_{1|234}\|_{tr} - (1 + \frac{\sqrt{11}}{2}) = (4 + \sqrt{2})x - (1 + \frac{\sqrt{11}}{2})$, $\rho$ is not separable under the bipartition $l_1|l_2l_3l_4$ for $f_2(x) > 0$, i.e. $0.6179 < x \leq 1$, while according to Theorem 3 in [17], $\rho$ is not separable under the bipartition $l_1|l_2l_3l_4$ for $g_2(x) = 9x^2 - 4 > 0$, i.e. $0.6667 < x \leq 1$. Fig. 2 shows that our method detects more entanglement.

4 Conclusions

By adopting the representation with generalized Pauli operators of density matrices, we have come up with several general tests to judge genuine entanglement for tripartite quantum systems. Our approach starts with some finer upper bounds for the norms of correlation tensors by using the generalized Pauli operators presentation, then we have obtained the entanglement criteria for genuine tripartite quantum states based on certain matrices constructed by the correlation tensor of the density matrices. We also conducted conclusion to detect entanglement in arbitrary
Detection of entanglement for multipartite quantum states. Compared with previously available criteria, ours can detect more situations, and these are explained in details with several examples.

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**Data Availability Statements** All data generated or analysed during this study are available from the corresponding author on reasonable request.

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