K-THEORY OF REAL GRASSMANN MANIFOLDS

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Abstract. Let $G_{n,k}$ denote the real Grassmann manifold of $k$-dimensional vector subspaces of $\mathbb{R}^n$. We compute the complex $K$-ring of $G_{n,k}$, up to a small indeterminacy, for all values of $n,k$ where $2 \leq k \leq n-2$. When $n \equiv 0 \pmod{4}, k \equiv 1 \pmod{2}$, we use the Hodgkin spectral sequence to determine the $K$-ring completely.

1. Introduction

Let $G_{n,k}$ denote the real Grassmann manifold consisting of all $k$-dimensional vector subspaces in the real vector space $\mathbb{R}^n$. We put the standard inner product on $\mathbb{R}^n$. We have the identification of $G_{n,k}$ with the homogeneous space $\text{SO}(n)/\text{S}(\text{O}(k) \times \text{O}(n-k))$ where $\text{O}(k) \times \text{O}(n-k)$ is the subgroup of the orthogonal group $\text{O}(n)$ that stabilises the subspace $\mathbb{R}^k$ spanned by the first $k$ standard basis vectors, and $S(\text{O}(k) \times \text{O}(n-k)) = \text{SO}(n) \cap (\text{O}(k) \times \text{O}(n-k))$.

In this note our aim is to compute the complex $K$-ring of $G_{n,k}$. Recall that the oriented Grassmann manifold $\tilde{G}_{n,k} \cong \text{SO}(n)/\text{SO}(k) \times \text{SO}(n-k)$ is the double cover of $G_{n,k}$ and is simply-connected, except in the case of $\tilde{G}_{2,1} \cong \mathbb{S}^1$. The description of the $K$-ring of $\tilde{G}_{n,k}$ goes back to work of Atiyah and Hirzebruch [AH] when $n$ is odd or $k$ is even. Note that in these cases, the subgroup $\text{SO}(k) \times \text{SO}(n-k)$ is connected and has rank equal to that of the whole group $\text{SO}(n)$. When $n$ is even and $k$ odd the $K$-ring was computed by Sankaran and Zvengrowski [SZ1].

The fact that $S(\text{O}(k) \times \text{O}(n-k))$ is not connected makes the determination of the ring $K(G_{n,k})$ difficult and, to the best of our knowledge, has not been carried out for $2 \leq k \leq n-2$. Note that since $G_{n,k} \cong G_{n,n-k}$, it suffices to consider the case when $k \leq n/2$. When $k = 1$, $G_{n,1}$ is the same as the real projective space $\mathbb{R}P^{n-1}$, whose $K$-ring had been determined by Adams [A].

Our aim is to express $K^*(G_{n,k}) = K^0(G_{n,k}) \oplus K^1(G_{n,k})$ in terms of generators and relations. However, we have thus far only met with partial success. We obtain complete results only

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under the assumption that $n \equiv 0 \pmod{4}$ and $k$ odd. In the remaining cases, our description is complete up to a small indeterminacy. See Theorem 1.2 below and Proposition 5.5.

We now state the two main results of this paper. The proofs will be given in §4 and §5.

**Theorem 1.1.** Let $n = 2m, k = 2s + 1, n - k = 2t + 1$ and suppose that $m = s + t + 1$ is even. Let $S$ denote the polynomial algebra $\mathbb{Z}[\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_t, \theta]$ in $s + t + 1$ variables. Then

$$K^0(G_{n,k}) = S/\mathcal{I} = \mathbb{Z}[\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_t, \theta]/\mathcal{I}$$

where the ideal $\mathcal{I}$ is generated by the following elements:

(i) $\theta^2 - 1, 2^{m-1}(\theta - 1)$,

(ii) $\sum_{0 \leq p \leq j} \lambda_p \mu_{j-p} - \binom{n}{j} \theta^j, 1 \leq j \leq m - 1, \text{ where } \lambda_{k-p} = \lambda_p, \mu_{n-k-q} = \mu_q$.

The $K^0(G_{n,k})$-module $K^1(G_{n,k})$ is the ideal generated by $\theta + 1$ in the ring $S/\tilde{\mathcal{I}}$, where $\tilde{\mathcal{I}}$ is generated by elements listed in (ii) above together with $\theta^2 - 1$.

The element $[\theta]$ in the above theorem corresponds to the complexification of the Hopf line bundle $\xi = \xi_{n,k}$ over $G_{n,k}$, which is associated to double cover $\tilde{G}_{n,k} \to G_{n,k}$. Note that since $\theta^2 - 1 \in \tilde{\mathcal{I}}$ we have $(\theta - 1) \cdot y = 0$ for all $y \in K^1(G_{n,k})$. It follows that the $S/\tilde{\mathcal{I}}$-module $K^1(G_{n,k})$ is indeed a module over $S/\mathcal{I} = K^0(G_{n,k})$-module.

Let $\gamma_{n,k}$ be the canonical (real) $k$-plane bundle over $G_{n,k}$. Denote by $\mathcal{K}_{n,k}$ the $\lambda$-subring of $K(G_{n,k})$ generated by the class $[\gamma_{n,k} \otimes \mathbb{C}]$. An algebraic description $\mathcal{K}_{n,k}$ will be given in §5.

**Theorem 1.2.** Let $2 \leq k \leq n/2$. With the above notation, the inclusion $\mathcal{K}_{n,k} \hookrightarrow K(G_{n,k})$ has finite cokernel.

The main tool needed in the proof of Theorem 1.1 is the Hodgkin spectral sequence. This will be recalled in §2. We need to compute the complex representation ring $RH_{n,k}$ of a certain subgroup $H_{n,k}$ of the spin group $\text{Spin}(n)$ and determine its structure as a module over $R\text{Spin}(n)$. The relevant subgroup $H_{n,k}$ is such that $G_{n,k} \cong \text{Spin}(n)/H_{n,k}$. This is carried out in §4 when $n \equiv 0 \pmod{4}$ and $k$ is odd. This seems rather complicated for arbitrary values of $n, k$. As an application we obtain bounds for the order of the element $[\xi \otimes \mathbb{C}] - 1 \in K(G_{n,k})$ for any $n, k$, $2 \leq k \leq n/2$.

Our proof of Theorem 1.2 uses standard arguments involving the Chern character.

The Hodgkin spectral sequence had been used to determine the $K$-theory of many compact homogeneous manifolds. Hodgkin [Ho, §12] applied it to determine the $K$-ring of most of the compact simple Lie groups which are not necessarily simply connected. Roux [R] used it to compute the $K$-ring of real Stiefel manifolds, independently of Gitler and Lam [GL],
who had determined the same using a different approach. Antoniano, et al. [AGUZ] and Barufatti and Hacon [BH] used the Hodgkin spectral sequence for computing the $K$-ring of real projective Stiefel manifolds, and Minami [Mi] for simply connected compact symmetric spaces. See also [SZ1], [SZ2].

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### 2. The Hodgkin spectral sequence

We briefly recall the Hodgkin spectral sequence here. Let $H$ be a proper closed subgroup of a compact Lie group $G$. We denote the complex representation ring of $G$ by $RG$. Let $\rho : RG \to RH$ denote the restriction homomorphism and regard $RH$ as an $RG$-module via $\rho$.

Hodgkin [Ho] established the existence of a spectral sequence, whose $E_2$-diagram is given by $\text{Tor}^*_{RG}(RH, \mathbb{Z})$, which converges to $K^*(G/H)$ when $\pi_1(G)$ is torsion-free. Here $\text{Tor}^p_A(B, M)$ denotes $\text{Tor}^p_{A}(B, M)$. In particular, $\text{Tor}^*_{RG}(RH, \mathbb{Z})$ is graded by non-positive integers.

We define the degree of an element $x \in \text{Tor}^*_{RG}(RH, \mathbb{Z})$ to be $p$.

When the rings $RG, RH,$ and $\mathbb{Z}$ are given the trivial $\mathbb{Z}_2$ grading, we obtain a $\mathbb{Z}_2$-grading on $E^p,q_2$, where $E^p,q_2 = \text{Tor}^p_{RG}(RH, \mathbb{Z})$ if $q$ is even and is zero if $q$ is odd. In particular, $0 = E^p,q_2 = E^\infty_p,q$ if $q$ is odd. The differential $d_r : E^p,q_r \to E^{p+r,q-r+1}_\infty$ vanishes when $r$ is even.

Using the multiplication in $RH$, one obtains a $\mathbb{Z}_2$-graded ring structure on $\text{Tor}^*_{RG}(RH, \mathbb{Z})$. The differential in the spectral sequence is an anti-derivation, leading to a $\mathbb{Z}$-graded ring structure on $E^*_\infty$ which is compatible with the $\mathbb{Z}_2$-graded ring $K^*(G/H)$.

If $\text{Tor}^*_{RG}(RH, \mathbb{Z})$ is generated by elements of degree at least $-2$, then the spectral sequence collapses at the $E_2$-stage and we have $\text{Tor}^*_{RG}(RH, \mathbb{Z}) \cong K^*(G/H)$. See [R].

Pittie [P] has shown that $RH$ is stably free over $RG$ if $H$ is connected, $\pi_1(G)$ is torsion-free, and the rank of $H$ equals the rank of $G$, i.e., if $H$ has a maximal torus $T \subset H$ which is maximal in $G$. Moreover, if $|W(G, T)|/|W(H, T)| > 1 + \dim T$, then $RH$ is a free $RG$-module. (Here $W(G, T)$ denotes the Weyl group of $G$ with respect to $T$.) Consequently the Hodgkin spectral sequence collapses and we have $K^*(G/H) = \text{Tor}^0_{RG}(RH, \mathbb{Z}) = RH \otimes_{RG} \mathbb{Z}$.

In case $G$ is prime to the exceptional Lie groups of type $E_6, E_7, E_8$, this was proved by Atiyah and Hirzebruch [AH], who conjectured its validity for any $G$ with $\pi_1(G)$ torsion-free.
2.1. Change of rings spectral sequence. Suppose that $G$ is simply connected so that $RG$ is a polynomial ring $\mathbb{Z}[x_1, \ldots, x_m]$. When $RH$ is not a free $RG$-module (via the restriction homomorphism), but is free over a subring $\Lambda = \mathbb{Z}[x_1, \ldots, x_r]$, then it is possible to use the change of rings spectral sequence due to Cartan and Eilenberg [CE] to compute $\text{Tor}^*_RG(RH, \mathbb{Z})$. See [R], [AGUZ, §6] and also [BH, §6] for a more detailed discussion on the use of the change of rings spectral sequence in the computation of $K(G/H)$. We now recall the Cartan-Eilenberg change of rings theorem.

Let $K$ be any ring. A $K$-algebra $\Lambda$ together with a $K$-homomorphism $\varepsilon : \Lambda \to K$ is called a supplemented $K$-algebra with augmentation $\varepsilon$. Let $(\Lambda, \varepsilon), (\Gamma, \eta)$ be supplemented $K$-algebras, and let $\varphi : \Lambda \to \Gamma$ be a $K$-algebra homomorphism such that $\eta \circ \varphi = \varepsilon$. Denote $\ker(\varepsilon)$ by $I(\Lambda)$. A $K$-algebra homomorphism $\varphi : \Lambda \to \Gamma$ is normal if the left ideal, denoted $\Gamma \cdot I(\Lambda)$, of $\Gamma$ generated by $\varphi(I(\Lambda))$, is also a right ideal of $\Gamma$ (always the case when $K$ is commutative). Then $\Omega := \Gamma/(\Gamma \cdot I(\Lambda))$ is a supplemented $K$-algebra.

We shall state the theorem in the special case of commutative augmented $K$-algebras. So if $\Gamma, \Lambda$ are supplemented, any augmentation preserving $K$-homomorphism $\Gamma \to \Lambda$ is normal. In our applications, $K = \mathbb{Z}, \Gamma = RG, \Lambda$ will be a subring of $\Gamma$, and $A = RH$, where the $\Gamma$-module structure is given via the restriction homomorphism $\rho : RG \to RH$. Also, the $\Omega$-module $C$ in the statement of the theorem below will be $\mathbb{Z}$ (via the augmentation).

**Theorem 2.1.** ([CE, Theorem 6.1, Chapter XVI]) We keep the above notations. Suppose that $K$ is commutative. Suppose that $\varphi : \Lambda \to \Gamma$ is normal and that $\Gamma$ is projective as a $\Lambda$-module (via $\varphi$). Then, for any $\Gamma$-module $A$ and $\Omega$-module $C$, there exists a spectral sequence $\text{Tor}^\Omega_s(\text{Tor}^\Lambda_r(A, K), C)$ that converges to $\text{Tor}^\Gamma_s(A, C)$.

The $\Omega$-module structure on $\text{Tor}^\Lambda_r(A, K)$ arises from the functorial isomorphism $\text{Tor}^\Gamma_q(A, \Omega) = \text{Tor}^\Gamma_q(A, \Gamma \otimes_\Lambda K) \cong \text{Tor}^\Lambda_q(A, K)$. (See [CE] for details.)

3. The representation ring of $H_{n,k}$

We follow the notations of Husemoller’s book [H] closely in our description of the representation rings of the groups $SO(n)$ and $Spin(n)$.

Let $2 \leq k \leq \lceil n/2 \rceil$. Recall that $H_{n,k}$ is the inverse image of $S(O(k) \times O(n-k))$ under the double cover $\pi : Spin(n) \to SO(n)$. The identity component of $H_{n,k}$ is the group $H^0_{n,k} := \text{Spin}(k) \cdot \text{Spin}(n-k) \subset \text{Spin}(n)$ with quotient $H_{n,k}/H^0_{n,k} \cong \mathbb{Z}_2$. Although the representation ring of $H^0_{n,k}$ has been worked out in [SZ1], we shall give most of the details here in order to make the exposition self-contained. Note that $H^0_{n,k}$ is the quotient of $\text{Spin}(k) \times \text{Spin}(n-k)$
by the cyclic subgroup of order 2 generated by $(-1, -1)$. The canonical surjection $\text{Spin}(k) \times \text{Spin}(n-k) \to H_{n,k}^0$ induces a ring monomorphism $RH_{n,k}^0 \to R(\text{Spin}(k) \times \text{Spin}(n-k))$ which we regard as an inclusion. The image is generated as an abelian group by representations of $\text{Spin}(k) \times \text{Spin}(n-k)$ on which $(-1, -1)$ acts as identity. Likewise, the projection $H_{n,k}^0 \to \text{SO}(k) \times \text{SO}(n-k)$ induces a monomorphism $R(\text{SO}(k) \times \text{SO}(n-k)) \to RH_{n,k}^0$, which we regard as an inclusion, whose image is generated by representations of $\text{Spin}(k) \times \text{Spin}(n-k)$ on which the kernel of the projection acts as the identity. This allows us to describe $RH_{n,k}^0$ in a straightforward manner. The ring $R(\text{SO}(k) \times \text{SO}(n-k))$ is a polynomial ring when $n$ is even and $k$ is odd. The ring homomorphism $R\text{SO}(2r+1) \to R\text{SO}(2r)$ induced by the inclusion $\text{SO}(2r) \hookrightarrow \text{SO}(2r+1)$ is a monomorphism. Moreover, $\text{SO}(2r+1)$ is a polynomial ring in $r$ indeterminates. The ring $R\text{SO}(2r)$ is not isomorphic to a polynomial algebra; it is known that $R\text{SO}(2r)$ is generated over $R\text{SO}(2r+1)$ by an element $\lambda_r^+$ which satisfies a monic quadratic equation. As such $R\text{SO}(2r)$ is a free $R\text{SO}(2r+1)$-module of rank 2. So, for all parities of $k,n$, $R(\text{SO}(k) \times \text{SO}(n-k))$ is a free module of finite rank over a polynomial ring generated by $[k/2] + [(n-k)/2]$ indeterminates. We will show in this section that the same statement holds for $RH_{n,k}$ as well.

Before proceeding further in describing $RH_{n,k}^0$, $RH_{n,k}$, we need to introduce notations for certain natural representations of the spin and special orthogonal groups.

Set $k = 2s + \varepsilon, n-k = 2t + \eta, \varepsilon, \eta \in \{0, 1\}$ where $s,t$ are integers. Now $n = 2s + 2t + 1$ if $n$ is odd. When $n$ is even, both $k$ and $n-k$ are of same parity and $n = 2s + 2t$ or $n = 2s + 2t + 2$ according as $k$ is even or odd. Let $\lambda_1$ denote the standard $k$-dimensional complex representation of $\text{SO}(k)$. We denote by $\lambda_j \in R\text{SO}(k)$ the $j$th exterior power $\wedge^j_\mathbb{C}(\lambda_1)$, $j \leq k$. (It is understood that $\lambda_0 = 1$, the trivial representation). We have the equality

$$\lambda_j = \lambda_{k-j} \text{ in } R\text{SO}(k).$$

When $k$ is even, the Hodge star operator $\ast$ yields a splitting $\lambda_s = \lambda_s^+ + \lambda_s^-$, where $\lambda_s^+, \lambda_s^- \in R\text{SO}(2s)$ are the classes of $+1,-1$-eigenspaces when $k \equiv 0 \pmod{4}$ and are the $i,-i$-eigenspaces when $k \equiv 2 \pmod{4}$ respectively. In the case of $\text{Spin}(k)$ we have the spin representation $\Delta_s$. When $k$ is even, it splits as a sum of two half-spin representations $\Delta_s^+, \Delta_s^-$; they are distinguished by the way an element $z_0$ in the centre of $\text{Spin}(k)$ acts. (This will be made precise later.) We have the following theorem proved in [H, §10, Chapter 13]. In the case of $R\text{SO}(2s)$, our description is slightly different from the one given in Husemoller’s book op. cit., but it is readily seen that the two descriptions are equivalent.

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1We shall often use the same notation for a representation and its class in the representation ring.
Theorem 3.1. With the above notations, we have

(i) \( R\text{Spin}(2s) = \mathbb{Z}[\lambda_1, \ldots, \lambda_{s-2}, \Delta^+_s, \Delta^-_s] \),

(ii) \( R\text{Spin}(2s + 1) = \mathbb{Z}[\lambda_1, \ldots, \lambda_s, \Delta^+_s] \),

(iii) \( R\text{SO}(2s + 1) = \mathbb{Z}[\lambda_1, \ldots, \lambda_s] \), and,

(iv) \( R\text{SO}(2s) = \mathbb{Z}[\lambda_1, \lambda_2, \ldots, \lambda_s][\lambda^+_s]/\sim \) where the ideal of relations is generated by \((\lambda^+_s)^2 - a\lambda^+_s - b\) for suitable polynomials \(a, b\) in \(\lambda_j, 1 \leq j \leq s\) (with \(\mathbb{Z}\)-coefficients).

As the notation suggests, the rings \(R\text{Spin}(2s), R\text{Spin}(2s + 1), R\text{SO}(2s + 1)\) are polynomial rings in the indicated variables. Also, the elements \(\lambda_j, 1 \leq j \leq s\), in \(R\text{SO}(2s)\) are algebraically independent.

Remark 3.2. The quadratic relation that \(\lambda^+_s\) satisfies over \(\mathbb{Z}[\lambda_1, \ldots, \lambda_s]\) can be explicitly written down as follows: Set \(\lambda^-_s := \lambda_s - \lambda^+_s\). From [H, Theorem 10.3, Chapter 13], we have the relation \(\lambda^+_s \cdot \lambda^-_s = (\lambda_{s-1} + \lambda_{s-3} + \cdots)^2 - \lambda_s(\lambda_{s-2} + \lambda_{s-4} + \cdots) - (\lambda_{s-2} + \lambda_{s-4} + \cdots)^2 \in \mathbb{Z}[\lambda_1, \ldots, \lambda_s]\). Denoting the negative of the right hand side of the last equality by \(b\) and setting \(a := \lambda_s\) we have

\[
(\lambda^+_s)^2 = \lambda_s(\lambda_s - \lambda^-_s) = a\lambda^+_s + b.
\]

The inclusion \(\text{Spin}(2s) \hookrightarrow \text{Spin}(2s+1)\) induces an injective ring homomorphism \(\rho : R\text{Spin}(2s + 1) \to R\text{Spin}(2s)\) where \(\rho(\Delta_s) = \Delta^+_s + \Delta^-_s\), \(\rho(\lambda_i) = \lambda_i + \lambda_{i-1}\), \(1 \leq i \leq s\). The homomorphism \(R\text{Spin}(2s) \to R\text{Spin}(2s - 1)\) induced by the inclusion \(\text{Spin}(2s - 1) \hookrightarrow \text{Spin}(2s)\) is given by \(\lambda_j \mapsto \lambda_j + \lambda_{j-1}, 1 \leq j < s, \Delta^+_s \mapsto \Delta_{s-1}^+\). These restriction homomorphisms also yield the restrictions \(R\text{SO}(k) \to R\text{SO}(k - 1)\) for any parity of \(k\).

Recall that given any two compact Lie groups \(H_1, H_2\), we have \(R(H_1 \times H_2) = RH_1 \otimes RH_2\). We have the natural quotient homomorphisms \(\pi_0 : \text{Spin}(k) \times \text{Spin}(n-k) \to H^0_{n,k}\) and \(\pi : H^0_{n,k} \to \text{SO}(k) \times \text{SO}(n-k)\) where \(\ker(\pi_0) \cong \mathbb{Z}_2\) is generated by \((-1, -1) \in \text{Spin}(k) \times \text{Spin}(n-k)\) and \(\ker \pi \cong \mathbb{Z}_2\), by \(\pi_0(1, -1) \in H^0_{n,k}\). We shall regard the ring homomorphisms \(\pi^*_0 : RH^0_{n,k} \to R(\text{Spin}(k) \times \text{Spin}(n-k)), \pi^* : R(\text{SO}(k) \times \text{SO}(n-k)) \to RH^0_{n,k}\), which are injective, as inclusions. It is easy to see that \(RH^0_{n,k}\) is generated as an \(R(\text{SO}(k) \times \text{SO}(n-k)\))-algebra by elements \(xy \in R(\text{Spin}(k) \times \text{Spin}(n-k))\) where \(x, y\) vary over the \(R(\text{SO}(k) \times \text{SO}(n-k))\)-algebra generators of \(R(\text{Spin}(k) \times \text{Spin}(n-k))\). The following description, in Proposition 3.3, of \(RH^0_{n,k}\) is an immediate consequence of Theorem 3.1.

We shall use the notation \(\mu_j \in R\text{SO}(n-k)\) for the element represented by the \(j\)th exterior power of the standard representation of \(\text{SO}(n-k)\). Also \(\Delta'_t\) and \(\Delta'^+_t\) will denote the spin and half-spin representations of \(\text{Spin}(n-k)\) respectively. Thus \(R(\text{SO}(k) \times \text{SO}(n-k))\) contains the polynomial subring \(\mathbb{Z}[\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t]\).
Proposition 3.3. We keep the above notations. Let \( R := R(\SO(k) \times \SO(n-k)) \). Then

\[
RH_{n,k}^0 = \begin{cases} 
R[\Delta_j \Delta'_j], & \text{if } k = 2s + 1, n - k = 2t + 1, \\
R[\Delta_j (\Delta'_j)^{\pm}], & \text{if } k = 2s + 1, n - k = 2t, \\
R[\Delta_j^{\pm} \Delta'_j], & \text{if } k = 2s, n - k = 2t + 1, \\
R[\Delta_j^{\pm} (\Delta'_j)^{\pm}], & \text{if } k = 2s, n - k = 2t.
\end{cases}
\]

Moreover, the squares of the indicated generators belong to \( R \).

Notations: We shall denote by \( \Delta_{s,t} \) the element \( \Delta_s \Delta'_s \in R(\Spin(k) \times \Spin(n-k)) \). Also \( \Delta_{s,t}^{\varepsilon,\eta} \) will denote \( \Delta_s \cdot (\Delta'_s)^{\varepsilon,\eta} \), \( \varepsilon, \eta \in \{+, -\} \). Also, we shall use upper case letters \( \Lambda_j, 1 \leq j \leq m \), etc., to denote the generators of \( R\Spin(n) \) and similarly \( \lambda_1, \ldots, \lambda_s \) (resp. \( \mu_1, \ldots, \mu_t \)) to denote generators of \( R\Spin(k) \) (resp. \( R\Spin(n-k) \)) as in Theorem 3.1.

Next we turn our attention to the representation ring of \( H_{n,k} \). Recall that \( 2 \leq k \leq n/2 \) and so \( n \geq 4 \). First we analyse when the exact sequence

\[ 1 \to H_{n,k}^0 \to H_{n,k} \to Z \to 1 \]  

(1)

splits. Evidently, the sequence splits if and only if there exists an element \( z_0 \in H_{n,k} \subset \Spin(n) \) of order 2 such that \( z_0 \not\in H_{n,k}^0 \). Taking \( z_0 := e_1 e_2 e_3 e_n \in C_n \), we see that \( z_0^2 = 1 \) and \( z_0 \in H_{n,k} \setminus H_{n,k}^0 \), so the short exact sequence (1) splits. Here \( C_n \) denotes the Clifford algebra of the quadratic space \( (\mathbb{R}^n, -|| \cdot ||^2) \) and \( e_1, \ldots, e_n \) denote the standard basis vectors of \( \mathbb{R}^n \). So \( H_{n,k} \cong H_{n,k}^0 \times \mathbb{Z}_2 \).

Suppose that \( H_{n,k} = H_{n,k}^0 \times Z \) and let \( z_0 \) be the generator of \( Z \cong \mathbb{Z}_2 \). Then \( \pi(H_{n,k}^0) \times \pi(Z) = \pi(H_{n,k}) = S(\SO(k) \times \SO(n-k)) \) is isomorphic to the product \( \SO(k) \times \SO(n-k) \times \{ \pm I_n \} \). In particular, \( n \) is even and \( k \) is odd and \( z_0 \in Z \) maps to \( -I_n \). So, the order 2 element \( z_0 \) is in the centre of \( \Spin(n) \). It follows that \( n \equiv 0 \pmod{4}, k \equiv 1 \pmod{2} \).

When \( n \equiv 0 \pmod{4}, k \equiv 1 \pmod{2} \), we may take \( z_0 = e_1 e_2 \cdots e_n \in H_{n,k} \). Then \( z_0 \) is in the centre of \( H_{n,k} \) and \( z_0 \not\in H_{n,k}^0 \) and so \( H_{n,k} \) is the direct product \( H_{n,k}^0 \times Z \).

Thus \( H_{n,k} \cong H_{n,k}^0 \times \mathbb{Z}_2 \) if and only if \( n \equiv 0 \pmod{4}, k \equiv 1 \pmod{2} \).

Using Proposition 3.3, we obtain the following.

Proposition 3.4. We keep the above notations. Let \( k = 2s + 1, n - k = 2t + 1, s + t \) odd. Let \( f_{s,t} \in R := R(\SO(k) \times \SO(n-k)) \) be the element such that \( \Delta_{s,t}^2 = f_{s,t} \) and let \( \theta \) be the class of the unique non-trivial one-dimensional representation of \( H_{s,t} \). Then

\[
RH_{n,k} = RH_{n,k}^0 \otimes RZ = R[\Delta_{s,t}, \theta]/\langle \theta^2 - 1, \Delta_{s,t}^2 - f_{s,t} \rangle.
\]

(2)

In particular, \( RH_{n,k} \) is a free \( R \)-module with basis \( \{1, \theta, \Delta_{s,t}, \theta \Delta_{s,t}\} \).
Writing $\lambda_0 = 1 = \mu_0$, $f_{s,t} \in R$ can be expressed as a polynomial in $\lambda_p, \mu_q \in R, 0 \leq p \leq s, 0 \leq q \leq t$ as follows (see [H, Theorem 10.3, Chapter 14].)

$$f_{s,t} = \Delta^2_{s,t} = \Delta^2_s \cdot (\Delta_t^2) = (\sum_{0 \leq p \leq s} \lambda_p)(\sum_{0 \leq q \leq t} \mu_q) = \sum_{0 \leq r \leq s+t} (\sum_{p+q=r} \lambda_p \mu_q).$$  \hspace{1cm} (3)

4. THE RESTRICTION HOMOMORPHISM $R\text{Spin}(n) \to RH_{n,k}$

Throughout this section we assume that $k = 2s + 1, n - k = 2t + 1$ so that $n = 2m$, where $m := s + t + 1$. Also we shall assume that $s + t$ is odd so that $n \equiv 0 \pmod{4}$. Hence $H_{n,k} = H_{n,k}^0 \times Z$ where $Z \cong \mathbb{Z}_2$ is generated by $z_0 = e_1 \cdots e_n \in \text{Spin}(n)$.

The double covering $\phi : \text{Spin}(n) \to \text{SO}(n)$ is defined as $\phi(u)(x) = u x u^*, \, x \in \mathbb{R}^n$, where $*$ is (the restriction to $\text{Spin}(n)$ of) the anti-involution of the Clifford algebra $C_n$, uniquely defined by the requirement: $v^* = v, \, v \in \mathbb{R}^n$. We refer the reader to [H] concerning the spin group and its representation ring.

**Maximal tori.** Set $\omega(\theta_1, \ldots, \theta_m) := \prod_{1 \leq j \leq m}(\cos 2\pi \theta_j + \sin 2\pi \theta_j e_{2j-1} e_{2j}) \in \text{Spin}(n)$ where $\theta_j \in \mathbb{R}$. Then $\tilde{T} := \{\omega(\theta_1, \ldots, \theta_m) \in \text{Spin}(n) \mid \theta_j \in \mathbb{R}, 1 \leq j \leq m\} \cong (\mathbb{S}^1)^m$ is a maximal torus of $\text{Spin}(n)$. Its image in $\text{SO}(n)$ is the standard maximal torus $T := \text{SO}(2) \times \cdots \times \text{SO}(2)$ whose elements restrict to rotations on $\mathbb{R} e_{2j-1} + \mathbb{R} e_{2j}, 1 \leq j \leq m$. In fact $\phi(\omega(\theta_1, \ldots, \theta_m)) = D(2\theta_1, \ldots, 2\theta_m) \in T$ where $D(t_1, \ldots, t_m)$ restricts to the positive rotation by angle $2\pi t_j$ on the oriented vector subspace $\mathbb{R} e_{2j-1} + \mathbb{R} e_{2j}, 1 \leq j \leq m$, the orientation being given by the ordering $e_{2j-1}, e_{2j}$ of the basis elements.

Let $\mathbb{T}$ be the ‘standard torus’ $(\mathbb{S}^1)^m = (\mathbb{R}/\mathbb{Z})^m$. One has a homomorphism $\omega : \mathbb{T} \to \tilde{T}$ defined by $(\theta_1, \ldots, \theta_m) \mapsto \omega(\theta_1, \ldots, \theta_m)$. Note that $\omega(\theta_1 + \varepsilon_1/2, \ldots, \theta_m + \varepsilon_m/2) = (-1)^{\varepsilon} \omega(\theta_1, \ldots, \theta_m)$ where $\varepsilon_j \in \{0, 1\}$ for all $j$, and $\varepsilon = \sum_{1 \leq j \leq m} \varepsilon_j$. In particular $\ker(\omega) \cong (\mathbb{Z}_2)^{m-1}$. The kernel of $\phi \circ \omega : \mathbb{T} \to T$ is readily seen to be $\mathbb{Z}_2^m \cong \{-1, 1\}^m \subset \mathbb{T}$.

Since $n$ is even and $k$ is odd, the rank of $H_{n,k}^0$ equals $m - 1 = \text{rank}(\text{Spin}(n)) - 1$. In this case, $\tilde{T}_0 := H_{n,k}^0 \cap \tilde{T} = \{\omega(\theta_1, \ldots, \theta_m) \in \tilde{T} \mid \theta_{s+1} = 0\}$ is a maximal torus of $H_{n,k}^0$. Also, we observe that the element $z_0 = e_1 \cdots e_n$, the generator of $Z$, belongs to $\tilde{T}$. Let $T_0 = \pi(\tilde{T}_0) = T \cap (\text{SO}(k) \times \text{SO}(n-k))$ which is a maximal torus of $\text{SO}(k) \times \text{SO}(n-k)$.

The representation rings of $\tilde{T}, \tilde{T}_0, T, T_0$ are viewed as subrings of $R\mathbb{T}$ as follows: Let $u_j : \mathbb{T} \to \mathbb{S}^1$ be the $j$th projection, regarded as a character. We also denote the corresponding 1-dimensional representation of $T$ by the same symbol $u_j$. Then $R\mathbb{T} = \mathbb{Z}[u_1^\pm 1, \ldots, u_m^\pm 1]$, $R\tilde{T} = \mathbb{Z}[u_1^\pm 2, \ldots, u_m^\pm 2, u_1 \cdots u_m]$, and, $RT = \mathbb{Z}[u_1^\pm 2, \ldots, u_m^\pm 2]$, both regarded as subrings of
In order to determine the restriction homomorphism $H \to R$ the homomorphism $\times T$ where $\mu$ by an abuse of notation. These group homomorphisms induce homomorphisms of rings appropriately restricting its domain and co-domain, will all be denoted by the same symbol $R$. The inclusion $\rho$ of $RH$ is contained in $\rho : RH \to R$. The inclusion $\rho$ restriction homomorphism $RH$ the homomorphisms in the first row are induced by respective inclusions of groups.

Let $\sigma : \tilde{T} \hookrightarrow \text{Spin}(n)$ be the inclusion. We have the following commutative diagram where the homomorphisms in the first row are induced by respective inclusions of groups.

$$
\begin{align*}
\text{RSpin}(n) \otimes RZ & \hookrightarrow \tilde{R} \otimes RZ & \tilde{R} \otimes RZ & \hookrightarrow R_{0} \otimes RZ & R_{0} \otimes RZ \\
\uparrow \mu^{*} & \mu^{*} \uparrow & \mu^{*} & \mu^{*} & \mu^{*} \\
\text{RSpin}(n) & \cong \tilde{T} & \tilde{T} & \tilde{R}_{0} \otimes RZ & R_{0} \otimes RZ
\end{align*}
$$

(4)

The inclusion $\sigma^{*} : \text{RSpin}(n) \hookrightarrow \tilde{R}$ is via the identification of $\text{RSpin}(n)$ with the invariant subgroup of $\tilde{R}$ under the action of the Weyl group $W(\text{Spin}(n), \tilde{T})$. Similarly we have the inclusion $RH_{0} \hookrightarrow R_{0}$ which in turn induces $RH_{0} = R_{0} \otimes RZ$. Moreover, $\mu^{*}(\text{RSpin}(n))$ is contained in $RH_{0} \subset R_{0}$ since $H_{0} \subset \text{Spin}(n)$. This allows one to describe the restriction homomorphism $\rho : \text{RSpin}(n) \to RH_{0}$ easily, once $\mu^{*} : \tilde{R} \to R_{0}$ is determined. This we shall carry out below, with $\theta$ as in Proposition 3.4.

Routine computation, using $n = 2m$, $m$ even, yields that

$$
u_{1} \cdots \nu_{m}(z_{0}) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{8} \\ \theta(z_{0}) & \text{if } n \equiv 4 \pmod{8}. \end{cases}
$$

(5)
When \( t \in \tilde{T}_0 \), we have \( u_{s+1}^2(t) = 1 \) and so \( u_1 \ldots u_m \) restrict to \( u_1 \cdots u_s \cdot v_1 \cdots v_t \) on \( \tilde{T}_0 \). Therefore

\[
\mu^*(u_j^{\pm2}) = \begin{cases} 
\theta u_j^{\pm2}, & 1 \leq j \leq s, \\
\theta, & j = s + 1, \\
\theta v_{s+1}^{\pm2}, & s + 1 < j \leq m,
\end{cases}
\]

and,

\[
\mu^*(u_1 \cdots u_m) = \begin{cases} 
\prod_{1 \leq j \leq s} u_j \cdot \prod_{1 \leq j \leq t} v_j, & n \equiv 0 \pmod{8}, \\
\theta \prod_{1 \leq j \leq s} u_j \cdot \prod_{1 \leq j \leq t} v_j, & n \equiv 4 \pmod{8}.
\end{cases}
\]

Let \( e_j(x_1, \ldots, x_s) \) denote the \( j \)th elementary symmetric polynomial in \( x_1, \ldots, x_p \). Recall that \( \sigma^*(\Lambda_j) = e_j(u_1^2, u_1^{-2}, \ldots, u_m^2, u_m^{-2}) \). So, for \( 1 \leq j \leq m \), we have

\[
\rho(\Lambda_j) = \mu^*(e_j(u_1^2, u_1^{-2}, \ldots, u_m^2, u_m^{-2})) = \theta^i e_j(u_1^2, u_1^{-2}, \ldots, u_m^2, u_m^{-2}, 1, v_1^2, v_1^{-2}, \ldots, v_t^2, v_t^{-2}) = \theta^j \cdot \sum_{p+q=j} e_p(u_1^2, u_1^{-2}, \ldots, u_m^2, u_m^{-2}, 1) \cdot e_q(v_1^2, v_1^{-2}, \ldots, v_t^2, v_t^{-2}, 1)
\]

for a suitable element \( f_j = f_j(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_s) \in R \) since \( \lambda_p = \lambda_{k-p}, \mu_q = \mu_{n-k-q} \).

Using Equations (6) and (7) we obtain that if \( \varepsilon_j \in \{1, -1\} \), then

\[
\mu^*(u_1^{\varepsilon_1} \cdots u_m^{\varepsilon_m}) = \theta^e u_1^{\varepsilon_1} \cdots u_m^{\varepsilon_m},
\]

where \( \eta_j = \varepsilon_{s+1+j} \), and the value of \( \varepsilon \in \{0, 1\} \) is obtained as follows: \( \varepsilon \equiv \sum_{1 \leq j \leq m} \varepsilon_j \pmod{2} \) if \( n \equiv 0 \pmod{8} \) and \( \varepsilon \equiv 1 + \sum_{1 \leq j \leq m} \varepsilon_j \pmod{2} \) if \( n \equiv 4 \pmod{8} \).

The following proposition now follows immediately from equations (8), (9), and the definitions of \( \Delta^\pm_m, \Delta_{s,t} \).

**Proposition 4.1.** Let \( n = 2m \equiv 0 \pmod{4}, k = 2s + 1, n - k = 2t + 1 \). With the above notations, the restriction homomorphism \( \rho : R\text{Spin}(n) \to RH_{n,k} \) is defined by \( \rho(\Lambda_j) = \Lambda'_j = \theta^j \sum_{p+q=j} \lambda_p \mu_q = \theta^i f_j, 1 \leq j \leq m - 1, \rho(\Delta^+_m) = \theta^e \Delta_{s,t}, \rho(\Delta^-_m) = \theta^{1+e} \Delta_{s,t} \) where \( \varepsilon = 0, 1 \) according as \( n \equiv 0 \pmod{8} \) or \( n \equiv 4 \pmod{8} \) respectively.

The ring \( R' := \mathbb{Z}[\theta^p \lambda_p, \theta^q \mu_q], 1 \leq p \leq s, 1 \leq q \leq t] \subset RH_{n,k} \) is mapped to the polynomial ring \( \mathbb{Z}[\lambda_p, \mu_q], 1 \leq p \leq s, 1 \leq q \leq t] = R = R(SO(k) \times SO(n - k)) \) by an automorphism of the ring \( R[\theta] \) since \( \theta \) is invertible. It follows that \( R' \) is a polynomial ring in \( s + t = m - 1 \) indeterminates. Evidently, \( R'[\theta] = R[\theta] \).
Lemma 4.2. Let \( n = 2m \equiv 0 \pmod{4} \), \( k = 2s + 1 \), \( n-k = 2t+1 \). Let \( R'[\theta] = R[\theta] \subset RH_{n,k} \). Then \( R'[\theta] \) is a free \( \Lambda' \)-module of rank \( 2(\frac{m-1}{s}) \) where \( \Lambda' := \mathbb{Z}[\Lambda'_1, \ldots, \Lambda'_{m-1}] \subset RH_{n,k} \). In particular, \( \Lambda'_1, \ldots, \Lambda'_{m-1} \) are algebraically independent. Also \( RH_{n,k} = R[\theta, \Delta_{s,t}] \) is a free module of rank \( 4\left(\frac{m-1}{s}\right) \) over \( \mathbb{Z}[\Lambda_1, \ldots, \Lambda_{m-1}] \) via \( \rho \).

Proof. Since \( R = R(\text{SO}(k) \times \text{SO}(n-k)) \) is a polynomial algebra in \( s + t = m - 1 \) indeterminates, the algebraic independence of \( \Lambda'_1, \ldots, \Lambda'_{m-1} \) would follow once we show that \( R[\theta] \cong R \oplus R \) is a finitely generated free \( \Lambda' \)-module.

First note that \( \Lambda'[\theta] \) is free as a \( \Lambda' \)-module with basis \{1, \theta\}.

Next we will show that \( R[\theta] \subset RH_{n,k} \) is free as a \( \Lambda'[\theta] \)-module of rank \( \left(\frac{m-1}{s}\right) \). Let \( \Lambda_0 = \mathbb{Z}[f_1, \ldots, f_{m-1}] \). Then \( \Lambda'[\theta] = \Lambda_0[\theta] = \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{Z}[\theta] \). Since \( R[\theta] = R \otimes_{\mathbb{Z}} \mathbb{Z}[\theta] \), it suffices to show that \( R \) is free as a module over \( \Lambda_0 \subset R \), of rank \( \left(\frac{m-1}{s}\right) \).

Denote by \( \rho_0 : R\text{Spin}(n) \to RH_{n,k} \to RH^0_{n,k} \) the restriction homomorphism induced by the inclusion \( H_{n,k} \hookrightarrow H_{n,k} \hookrightarrow \text{Spin}(n) \). Then \( \Lambda_0 = \rho_0(\Lambda) \) and \( \rho_0(\Lambda) \subset R \subset R[\Delta_{s,t}] \). Then \( R \) is free as a \( \Lambda_0 \)-module (see [SZ1, Lemma 2.6]). We give a proof for the sake of completeness.

Let \( z_j = e_j(u_1^2 + u_1^{-2}, \ldots, u_m^2 + u_m^{-2}) \), \( x_p = e_p(u_1^2 + u_1^{-2}, \ldots, u_s^2 + u_s^{-2}) \), and \( y_q = e_q(v_2^2 + v_2^{-2}, \ldots, v_t^2 + v_t^{-2}) \). Then \( \mathbb{Z}[z_1, \ldots, z_m] = \mathbb{Z}[\Lambda_1, \ldots, \Lambda_m] \). Indeed, since \( \Lambda_1, \ldots, \Lambda_m \) are expressible as symmetric polynomials in \( u_j^2 + u_j^{-2}, 1 \leq j \leq m \), they are expressible as polynomials in \( z_1, \ldots, z_m \). Conversely, since \( z_1, \ldots, z_m \in \mathbb{Z}[u_1^2, u_1^{-2}, \ldots, u_m^2, u_m^{-2}] \) are invariant under the permutations of the variables \( u_1^2, \ldots, u_m^2 \) as well as the involutions \( u_j^2 \mapsto u_j^{-2} \) for every \( j \), we see that the \( z_j \) belong to the subring of \( \mathbb{Z}[u_1^2, u_1^{-2}, \ldots, u_m^2, u_m^{-2}] \) fixed by the group \( \mathbb{Z}_2 \times S_n \).

This fixed subring equals \( \mathbb{Z}[\Lambda_1, \ldots, \Lambda_m] \); see [H, §10, Ch. 13]. So each \( z_j \) is expressible as a polynomial in the \( \Lambda_i \).

The same argument shows that \( \mathbb{Z}[\lambda_1, \ldots, \lambda_s] = \mathbb{Z}[x_1, \ldots, x_s] \) and \( \mathbb{Z}[\mu_1, \ldots, \mu_t] = \mathbb{Z}[y_1, \ldots, y_t] \).

Consequently, \( R = \mathbb{Z}[\lambda_p, \mu_q; 1 \leq p \leq s, 1 \leq q \leq t] \subset RH^0_{n,k} \).

Now using Equation (6) we obtain

\[
\rho_0(z_j) = \sum_{p+q=j} x_p y_q + 2 \sum_{p+q=j-1} x_p y_q, 1 \leq j \leq m - 1, \quad (10)
\]

and \( \rho_0(z_m) = 2x_m y_t \), where it is understood that \( z_0 = x_0 = y_0 = 1 \). Set \( z'_1 := z_1 - 2 \), and, inductively, \( z'_r := z_r - 2z'_{r-1}, 2 \leq r < m \), so that \( \rho_0(z'_r) = \sum_{p+q=r} x_p y_q, 1 \leq r \leq m - 1 \). Then \( \mathbb{Z}[z'_1, \ldots, z'_{m-1}] = \mathbb{Z}[z_1, \ldots, z_{m-1}] = \Lambda_0 \). Moreover, we have

\[
\rho_0(z'_j) = \sum_{p+q=j} x_p y_q, 1 \leq j \leq m - 1. \quad (11)
\]
The proof that $R$ is a free $\Lambda_0$-module of rank $(m-1)$ is now completed using some well-known facts concerning the cohomology of classifying spaces $BU(s)$ of the unitary group $U(s)$, as we shall now explain. We regard $R = \mathbb{Z}[x_1, \ldots, x_s, y_1, \ldots, y_t]$ as a graded ring where $|x_p| = 2p, |y_q| = 2q$. Then $\Lambda_0 = \mathbb{Z}[z_1', \ldots, z_{m-1}']$ is a graded subring where $|z_r'| = 2r$. We may identify $R$ with $H^*(B(U(s) \times U(t)); \mathbb{Z})$ and $\Lambda_0$ with $H^*(BU(s+t); \mathbb{Z})$ so that the inclusion $\Lambda_0 \hookrightarrow R$ corresponds to the homomorphism induced by the the projection of the fibre bundle $B(U(s) \times U(t)) \to BU(s+t)$ with fibre the complex Grassmann manifold $\mathbb{C}G_{s+t,s} = U(s+t)/U(s) \times U(t)$. The Grassmann manifold bundle is totally non-cohomologous to zero (with $\mathbb{Z}$-coefficients) and so by the Leray-Hirsch theorem $H^*(B(U(s) \times U(t)); \mathbb{Z})$ is a free $H^*(BU(s+t); \mathbb{Z})$-module of rank equal to rank$(H^*(\mathbb{C}G_{s+t,s}); \mathbb{Z})) = \binom{s+t}{s}$.

Next we note that irrespective of whether $n \equiv 0$ or 4 (mod 8), we have $\rho((\Delta_m^+)^2 - (\Delta_m^-)^2) = 0$ and $\rho(\Delta_m^+ \Delta_m^-) = \theta \Delta_{s,t}^2 = \theta f_{s,t}$. We have the following consequence of Lemma 4.2.

**Lemma 4.4.** The elements $\Lambda'_1, \ldots, \Lambda'_{m-2}, \rho(\Delta_m^+), \Lambda'_m \in RH_{n,k}$ are algebraically independent. As a module over $\Lambda := \mathbb{Z}[\Lambda'_1, \ldots, \Lambda_{m-2}, \Delta_{m}^+] \subset R\operatorname{Spin}(n)$, $RH_{n,k}$ is free of rank $2\binom{m-1}{s}$ with basis $\mathcal{B}_0 \cup \mathcal{B}_\theta$.

**Proof.** Since $\rho(\Delta_m^+)^2 = \Delta_{s,t}^2 = f_{s,t}$, it suffices to show that $\Lambda'_1, \ldots, \Lambda'_{m-2}, f_{s,t}$ are algebraically independent in $RH_{n,k}$. Note that $\Delta_{m-1} \cdot \Delta_m = \Lambda_1 + \Lambda_{m-3} + \cdots + \Lambda_1$ in $R\operatorname{Spin}(n)$; see [H, Theorem 10.3, Chapter 14]. So

$$f_{s,t} = \theta \rho(\Delta_m^+ \Delta_m^-) = \Lambda'_m + \Lambda'_{m-3} + \cdots + 1.$$

Since $\Lambda_1, \ldots, \Lambda_{m-1}$ are algebraically independent, it follows that $\Lambda'_1, \ldots, \Lambda'_{m-2}, f_{s,t}$ are also algebraically independent. Moreover, we have $\Lambda'_1[\rho(\Delta_m^+)] = \rho(\Lambda) \cong \Lambda$.

Let $\mathcal{B}$ be a basis for $R[\theta] = R[\theta]$ over $\Lambda' = \mathbb{Z}[\Lambda'_1, \ldots, \Lambda'_{m-1}]$. Note that we may take $\mathcal{B}$ to be $\mathcal{B}_0 \cup \mathcal{B}_\theta$ by Remark 4.3. Then $\mathcal{B}$ is a basis for $R[\theta, \rho(\Delta_m^+)] = RH_{n,k}$ over $\Lambda'[\rho(\Delta_m^+)] \cong \Lambda$. In view of Lemma 4.2, we conclude that $RH_{n,k}$ is a free module over $\Lambda$ of rank $2\binom{m-1}{s}$. □
Let $\delta_m = \Delta^+_m - \Delta^-_m$. Then $R\text{Spin}(n) = \Lambda[\delta_m]$ with $\Lambda$ as in Lemma 4.4. Note that $\rho((\Delta^+_m)^2 - (\Delta^-_m)^2) = 0$ and $\rho(\Delta^+_m \cdot \Delta^-_m) = \theta \Delta^*_m = \theta f_{s,t}$. So the following equations hold in $R\text{H}_{n,k}$:

$$\rho((\Delta^+_m)^2) = \rho(\delta^2_m - 2\Delta^+_m \delta^-_m) = 0, \quad \text{and} \quad \rho(\Delta^+_m)\rho(\delta_m) + (\theta - 1) \cdot f_{s,t} = 0. \quad (12)$$

4.1. Computation of $\text{Tor}^*_{R\text{Spin}(n)}(R\text{H}_{n,k}, \mathbb{Z})$. We shall apply the change of rings spectral sequence (§2.1) to compute $\text{Tor}^*_{R\text{Spin}(n)}(R\text{H}_{n,k}, \mathbb{Z})$. In the notation of Theorem 2.1, we let $\Gamma = R\text{Spin}(n), A = R\text{H}_{n,k}, K = C = \mathbb{Z}$ and $\Lambda = \mathbb{Z}[\Lambda_1, \ldots, \Lambda_{m-2}, \Delta^+_m] \subset \Gamma = R\text{Spin}(n)$. Then $A$ is a free $\Lambda$-module via the restriction homomorphism, in view of Lemma 4.4. Hence setting

$$B := \text{Tor}^A_*(R\text{H}_{n,k}, \mathbb{Z}),$$

we have, with $\varepsilon \in \{0, 1\}$ as in Proposition 4.1,

$$B_q = \text{Tor}^A_q(R\text{H}_{n,k}, \mathbb{Z}) = \begin{cases} R\text{H}_{n,k}/\langle A'_j - \binom{n}{j} \rangle, & 1 \leq j \leq m - 2; \theta^q \Delta^*_{s,t} - 2^{m-1}, \text{ if } q = 0, \\ 0, & \text{if } q \neq 0. \end{cases} \quad (13)$$

Thus

$$B = B_0 = R\text{H}_{n,k}/\langle A'_j - \binom{n}{j} \rangle, 1 \leq j \leq m - 2; \theta^q \Delta^*_{s,t} - 2^{m-1}. \quad (14)$$

Recall the basis $B = B_0 \cup B_0\theta$ of $R\text{H}_{n,k}$ over $\Lambda$ given in Lemma 4.4. (See Remark 4.3 for the definition $B_0$.) Under the natural projection $\eta : R\text{H}_{n,k} \to B$, the subring $\rho(\Lambda)$ maps to $\mathbb{Z}$ and $B$ to a $\mathbb{Z}$-basis $\overline{B} = \overline{B}_0 \cup \overline{B}_0\theta$ where $\overline{B}_0 = \eta(B_0)$. It is readily seen that $|\overline{B}| = |B|$. We summarise this observation as a lemma.

**Lemma 4.5.** The set $\overline{B}$ is a $\mathbb{Z}$-basis for $B$. Thus $B$ is free abelian of rank $2^{(m-1)_s}$. By Theorem 2.1, the change of rings spectral sequence collapse and we have $\text{Tor}^*_{\Omega}(A, \mathbb{Z}) \cong \text{Tor}^*_{\Omega}(B, \mathbb{Z})$, where $\Omega = R\text{Spin}(n)/\langle A_j - \binom{n}{j}, \Delta^+_m - 2^{m-1} \rangle = \mathbb{Z}[\delta_m]$ and $\delta_m = \Delta^+_m - \Delta^-_m$.

Since $\Omega$ is a polynomial ring, one can use the Koszul resolution to compute $\text{Tor}^*_{\Omega}(B, \mathbb{Z})$. The $\Omega$-module structure on $B$ is obtained via the algebra homomorphism $\bar{\rho} : \Omega \to B$ defined by $\rho : R\text{Spin}(n) \to R\text{H}_{n,k}$. In view of Proposition 4.1, we have $\bar{\rho}(\delta_m) = \epsilon' (\theta - 1) \Delta^*_{s,t}$, where the value of $\epsilon' \in \{1, -1\}$ depends on the value of $n$ modulo $8$. The Koszul resolution of $\mathbb{Z}$ is

$$0 \to \Omega \cdot \delta \xrightarrow{d} \Omega \xrightarrow{\bar{\epsilon}} \mathbb{Z} \to 0.$$ 

Here $\bar{\epsilon}$ is the augmentation defined by $\bar{\epsilon}(\delta_m) = 0$ and $d(\delta) = \delta_m$. Tensoring with the $\Omega$-module $B$ we obtain the following chain complex whose homology is $\text{Tor}^*_{\Omega}(B, \mathbb{Z})$:

$$0 \to B\delta \xrightarrow{\delta} B \to 0.$$
where \( \bar{d}(\delta) = \bar{d}(1 \cdot \delta) = \bar{\rho}(\delta_m) = \epsilon'(\theta - 1)\Delta_{s,t} \in B \). In particular \( \text{Tor}_q^\Omega(B, \mathbb{Z}) = 0 \) if \( q \geq 2 \), \( \text{Tor}_1^\Omega(B, \mathbb{Z}) = \ker(\bar{d}), \text{Tor}_0^\Omega(B, \mathbb{Z}) = B/\langle (\theta - 1)\Delta_{s,t} \rangle \).

We set

\[
\tilde{B} := \text{Tor}_0^\Omega(B, \mathbb{Z}) = B/\langle (\theta - 1)\Delta_{s,t} \rangle.
\]

Recall from Equation (8) that \( \Lambda_j = \theta^j f_j \) where \( f_j = \sum_{0 \leq p < j} \lambda_p \mu_{j-p} \in RH_{n,k}, 1 \leq j \leq m - 1 \), and

\[
\lambda_p = \lambda_{k-p} \text{ and } \mu_q = \mu_{n-k-q} \text{ when } p > s, q > t.
\]

Denote by \( \eta : RH_{n,k} \to B \) the canonical quotient map and by \( \bar{\eta} : RH_{n,k} \to \tilde{B} \) the composition \( RH_{n,k} \xrightarrow{\eta} B \to \tilde{B} \) where \( B \to \tilde{B} \) is the canonical quotient map. If \( x \in RH \), we shall denote \( \eta(x) \in B \) by the same symbol \( x \) and we shall denote \( \bar{\eta}(x) \in \tilde{B} \) by \( [x] \).

**Lemma 4.6.** We keep the above notations. The following relations hold in \( \tilde{B} \):

(a) \( 2^{m-1}(\theta) - 1 = 0 \), \( [\Delta_{s,t}] = 2^{m-1} \),

(b) \( \sum_{0 \leq p \leq j} [\lambda_p] [\mu_{j-p}] = [f_j] = (\binom{n}{j}) [\theta^j], 1 \leq k \leq m - 1 \), (where \( [\lambda_p] = [\lambda_{k-p}], [\mu_q] = [\mu_{n-k-q}] \)),

(c) \( [\Delta_{s,t}^2] = (\sum_{0 \leq p \leq s} [\lambda_p]) (\sum_{0 \leq q \leq t} [\mu_q]) = [f_{s,t}] = 2^{2m-2} \).

**Proof.** (a). We have, by Proposition 4.1, \( \rho(\Delta^+_m) = \theta^\delta \Delta_{s,t} \), in \( RH_{n,k} \) where \( \epsilon \in \{0, 1\} \) depending on the value of \( n \) modulo 8. Since \( ([\theta] - 1)[\Delta_{s,t}] = 0 \) in \( \tilde{B} \), irrespective of the value of \( \epsilon \) we have \( \bar{\eta} \circ \rho(\Delta^+_m) = [\Delta_{s,t}] \) in \( \tilde{B} \). On the other hand, since \( \Delta^+_m = 2^{m-1} \) in \( \Omega \), we obtain that \( 2^{m-1} = \eta(\rho(\Delta^+_m)) = \theta^\delta \Delta_{s,t} \) in \( B \). It follows that \( [\Delta_{s,t}] = 2^{m-1} \) and so \( 2^{m-1}(\theta) - 1 = 0 \).

(b). It is clear that, when \( 1 \leq j \leq m - 2 \), the relation \( f_j = \bar{\rho}(\Lambda_j) \theta^j = (\binom{n}{j}) \theta^j \) holds in \( B \) and hence in \( \tilde{B} \) using \( \theta^2 = 1 \). Since \( \Delta^+_m \Delta^-_m = \sum_{1 \leq j \leq m} \Lambda_{2j-1} \) in \( R\text{Spin}(n) \), and since \( \bar{\eta} \circ \rho(\Delta^+_m) = [\Delta_{s,t}] = [\theta][\Delta_{s,t}] = 2^{m-1} \) in \( \tilde{B} \), applying \( \bar{\eta} \circ \rho \) we obtain the following equations in \( \tilde{B} \):

\[
2^{2(m-1)} = \bar{\eta} \circ \rho(\Delta^+_m \Delta^-_m) = \bar{\eta} \circ \rho(\sum_{1 \leq j \leq m} \Lambda_{2j-1}) = [f_{m-1}] - \binom{2m}{m-1} + \sum_{1 \leq j < m/2} \binom{2m}{2j-1} = [f_{m-1}] - \binom{2m}{m-1} + 2^{2m}/4
\]

since \( \sum_{1 \leq j < m/2} \binom{2m}{2j-1} = (1/2) \sum_{1 \leq j \leq m} \binom{2m}{2j-1} = 2^{2m}/4 \). Hence \( [f_{m-1}] = \binom{2m}{m-1} \).

(c). Since \( \Delta^2_{s,t} = f_{s,t} \) holds in \( B \), and since \( [\Delta_{s,t}] = 2^{m-1} \) holds in \( \tilde{B} \), we see that \( [f_{s,t}] = 2^{2m-2} \) in \( \tilde{B} \).

**Remark 4.7.** It turns out that the relation (c) is a consequence of relations (a), (b). Indeed, recalling that \( [\lambda_p] = [\lambda_{k-p}], [\mu_q] = [\mu_{n-k-q}] \) in \( \tilde{B} \) and also that \( k = 2s + 1, n - k = 2t + 1 \), we
have

\[ f_{s,t} = [\Delta_{s,t}^2] = (\sum_{0 \leq p \leq s} [\lambda_p])(\sum_{0 \leq q \leq t} [\mu_q]) \]

\[ = \frac{1}{4}(\sum_{0 \leq p \leq s} [\lambda_p])(\sum_{0 \leq q \leq n-k} [\mu_q]) \]

\[ = \frac{1}{4}\sum_{0 \leq r \leq n}(\sum_{0 \leq j \leq r} [\lambda_j][\mu_{r-j}]) \]

\[ = \frac{1}{4}\sum_{0 \leq r \leq n}(\text{n choose r})[\theta]^r, \text{ using (b)}, \]

\[ = \left(\frac{1}{4}\right)(1 + [\theta])^n. \]

Since \([\theta]^2 = 1\), we have \((1 + [\theta])^2 = 2(1 + [\theta]). So (1 + [\theta])^r = 2^{r-1}(1 + [\theta]) for all \(r \geq 1\). Therefore, since \(n = 2m \geq 4\), we have

\[ (1/4)(1 + [\theta])^n = (1/4)(1 + [\theta])^3 \cdot (1 + [\theta])^{n-3} \]

\[ = (1 + [\theta]) \cdot (1 + [\theta])^{2m-3} \]

\[ = (1 + [\theta])^{2m-2} \]

\[ = 2^{2m-3}(1 + [\theta]) \]

\[ = 2^{2m-2}, \]

using \(2^{2m-3}[\theta] = 2^{2m-3}\). Therefore \(f_{s,t} = 2^{2m-2}\).

**Lemma 4.8.** With the above notations, the rank of the abelian group \(\mathcal{B}\) equals \(\binom{m-1}{s}\). Moreover, the torsion subgroup of \(\mathcal{B}\) is generated as a \(\mathcal{B}\)-module by \((\theta - 1)\). In particular, any torsion element has order \(2^r\) for some \(r \leq m - 1\).

**Proof.** In view of Lemma 4.5, the set \(\mathcal{B}_0 \cup \mathcal{B}_0(\theta - 1)\) is a basis for \(\mathcal{B}\). Under the quotient map \(\mathcal{B} \to \mathcal{B}\), the abelian group \(\mathcal{B}_0\) generated by \(\mathcal{B}_0\) projects isomorphically on a summand of \(\mathcal{B}_0\). Since \(2^{m-1}([\theta] - 1) = 0\), the subgroup \(C\) of \(\mathcal{B}\) is generated by \(([\theta] - 1)\mathcal{B}_0\) consists only of elements whose (additive) order divides \(2^{m-1}\). This completes the proof. \(\square\)

We now turn to \(\text{Tor}_1^\Omega(B, \mathcal{Z}) = \ker(\bar{d} : B\delta \to B)\). Since \(\bar{d}(\delta) = \pm(\theta - 1)\Delta_{s,t}\), \(\ker(\bar{d})\) is the \(\mathcal{B}\)-submodule \(J \cdot \delta\) where \(J \subset \mathcal{B}\) is the annihilator ideal of \((\theta - 1)\Delta_{s,t} \in \mathcal{B}\). It is clear that \((\theta + 1) \in J\) since \(\theta^2 - 1 = 0\). We claim that \(J\) equals the ideal generated by \(\theta + 1\). In order to see this, let \(x \in J\) and let \(\mathcal{B}_0 = \{b_j\}\). Write \(x = \sum y_j b_j + \theta \sum z_j b_j\) where \(y_j, z_j \in \mathcal{Z}\). Since \(x \in J\), multiplying by \((\theta - 1)\Delta_{s,t}\), and using the relations \(\Delta_{s,t} = 2^{m-1}\theta^\varepsilon\) (where the value of \(\varepsilon \in \{0, 1\}\) depends on the parity of \(m\) and \(\theta(\theta - 1) = 1 - \theta\) in \(\mathcal{B}\), we obtain that

\[ 2^{m-1}(\theta - 1)\theta^\varepsilon \sum y_j b_j - 2^{m-1}\theta^\varepsilon(\theta - 1) \sum z_j b_j = 0. \]

Since \(\mathcal{B}\) is a free abelian group, and since \(\theta^\varepsilon\) is invertible in \(\mathcal{B}\), the above equation can be rewritten as \(-(\sum (y_j - z_j)b_j) + \theta(\sum (y_j - z_j)b_j) = 0\). This implies that \(y_j = z_j\) for all \(j\). Therefore \(x = (\theta + 1)(\sum y_j b_j) \in J\).

We are now ready to prove Theorem 1.1.
Proof of Theorem 1.1: The Hodgkin spectral sequence \( \text{Tor}^*_R(Spin(n), RH_{n,k}, \mathbb{Z}) \) converges to \( K^*(G_{n,k}) \). Since \( \text{Tor}^*_R(Spin(n), RH_{n,k}, \mathbb{Z}) \cong \text{Tor}^*_Q(B, \mathbb{Z}) \), and since \( \text{Tor}^*_Q(B, \mathbb{Z}) \) is generated by degree \(-1\) elements, by the discussion in §2 we obtain that \( K^0(G_{n,k}) = \text{Tor}^0_Q(B, \mathbb{Z}) = B \) and \( K^{-1}(G_{n,k}) = \text{Tor}^1_Q(B, \mathbb{Z}) = \text{Ann}(\theta - 1) \subset B \). The theorem now follows from Equation (14), Lemma 4.6, and the above discussion that describes \( \text{Ann}(\theta - 1) \). \( \square \)

Let \( \xi = \xi_{n,k} \) be the Hopf line bundle over \( G_{n,k} \). It is associated to the double cover \( \tilde{G}_{n,k} \rightarrow G_{n,k} \). If \( \eta \) is a real vector bundle, we denote by \( \eta^\mathbb{C} \) the complexification of \( \eta \). Note that \( \eta^\mathbb{C} \), regarded as a real vector bundle via restriction of scalars, is isomorphic to \( \eta \oplus \eta \). See [MS, p. 176].

Proposition 4.9. Let \( n = 2m, k = 2s + 1 \). If \( n \equiv 0 \pmod{4}, k \equiv 1 \pmod{2} \) and \( k(n - k) < 2^m \), then \( 2^{m-1}\xi \cong 2^{m-1}\epsilon_R \) where \( n = 2m \) but \( [2^{m-2}\xi] \neq 2^{m-2} \) in \( KO(G_{n,k}) \). If \( n \equiv 0 \pmod{8} \) and \( k(n - k) < 2^{m-1} \), then \( 2^{m-1}\xi \cong 2^{m-1}\epsilon_R \) provided \( k(n - k) < 2^{m-1} \).

Proof. Since \( 2^{m-1}[\xi^\mathbb{C}] = 2^{m-1}\theta = 2^{m-1} \in K(G_{n,k}) \), it follows that \( 2^m[\xi] = 2^m \in KO(G_{n,k}) \). If \( \dim G_{n,k} = k(n - k) < 2^m = \text{rank}(2^m\xi) \), then equality of the classes of the vector bundles \( [2^m\xi] \) and \( [2^m\epsilon_R] = 2^m \) in \( KO(G_{n,k}) \) implies the isomorphism of the vector bundles: \( 2^{m-1}\xi \cong 2^{m-1}\epsilon_R \). See [H, Theorem 1.5, Chapter 8].

When \( n \equiv 0 \pmod{8} \), the representations \( \Delta^+_m, \Delta^-_m \in RSpin(n) \) are real, that is, they arise as complexification of real representations \( \Delta^+_m, \Delta^-_m \) of \( Spin(n) \). See [H, §12, Chapter 13]. Evidently \( \theta \) is real. Indeed \( \theta = \chi \otimes_R \mathbb{C} \) of \( H_{n,k} \) where \( \chi : H_{n,k} \rightarrow O(1) \) is defined by the projection \( H_{n,k} \rightarrow H_{n,k}/H_{n,k}^0 \cong O(1) \).

The line bundle associated to \( \chi \) is isomorphic to \( \xi \) whereas the bundle associated to \( \Delta^-_{m,R} \) equals the trivial real vector bundle of rank \( 2^{m-1} \). This can be shown to imply that \( 2^{m-1}[\xi] = 2^{m-1} \in KO(G_{n,k}) \). As before, this leads to the isomorphism \( 2^{m-1}\xi \cong 2^{m-1}\epsilon_R \) when \( k(n - k) < 2^{m-1} \). \( \square \)

As for the torsion part of \( K^0(G_{n,k}) \), it has no \( p \)-torsion for any odd prime \( p \). For any \( n, k \), the element \( [\Lambda^k(\gamma_{n,k}^\mathbb{C})]-1 = [\xi^\mathbb{C}]-1 \in K(G_{n,k}) \) generates a finite cyclic subgroup of order \( 2^r \) for some \( r \). There are the obvious inclusions \( i : G_{n,k} \rightarrow G_{n+1,k+1}, j : G_{n,k} \rightarrow G_{n+1,k} \) which have the property that \( i^*(\gamma_{n+1,k+1}) \cong \gamma_{n,k} \oplus \epsilon_R \) and \( j^*(\gamma_{n+1,k}) = \gamma_{n,k} \).

Theorem 4.10. Suppose that \( n = 4l + j, k = 2s + \varepsilon, 1 \leq j \leq 3, \varepsilon \in \{0, 1\} \). Let \( 2^r \) be the order of \( [\xi^\mathbb{C}] \in K(G_{n,k}) \). Then \( 2l - 1 \leq r \leq 2l + 1 \).

Proof. Suppose \( \varepsilon = 1 \). Then we have inclusions \( G_{4l,k} \hookrightarrow G_{4l+j,k} \hookrightarrow G_{4l+4,k} \) where \( j_1^*(\xi_{4l+4,k}) = \xi_{n,k}, j_0^*(\xi_{n,k}) = \xi_{4l,k} \). The bounds for \( r \) now follow from Theorem 1.1.
When $\varepsilon = 0$, we use the inclusions $G_{4l,2s-1} \overset{i_0}{\hookrightarrow} G_{n,k} \overset{i_1}{\hookrightarrow} G_{4l+4,2s+1}$. When $s = 1$, $G_{4l,2s-1} = \mathbb{R}P^{4l-1}$ and the order of the bundle $[\xi^C]-1$ is known to be $2^{2l-1}$ from the work of Adams [A, Theorem 7.3]. Now we proceed exactly as in the case $\varepsilon = 1$. 

5. $K$-theory of $G_{n,k}$ for arbitrary values of $n, k$

In this section we shall prove Theorem 1.2. The proof will make use of the Chern character $\text{ch} : K^*(G_{n,k}) \otimes \mathbb{Q} \to H^*(G_{n,k}; \mathbb{Q})$. We begin by recalling, in Theorem 5.1 and the following paragraph, the rational cohomology algebra of the Grassmann manifolds. We refer the reader to [MS, §15] for the definition and properties of Pontrjagin classes. We shall write $k = 2s + \varepsilon, n - k = 2t + \eta$ where $\varepsilon, \eta \in \{0, 1\}$ so that $n = 2s + 2t + \varepsilon + \eta$.

We denote by $\beta_{n,k}$ the canonical $(n-k)$-plane bundle over $G_{n,k}$ whose fibre over $L \in G_{n,k}$ is the vector space $L^+ \subset \mathbb{R}^n$. We have $\gamma_{n,k} \oplus \beta_{n,k} \cong n\varepsilon \mathbb{R}$, and, (denoting the complexification $\gamma_{n,k} \otimes \mathbb{C}$ by $\gamma^C_{n,k}$ etc.,) we obtain

$$\gamma^C_{n,k} \oplus \beta^C_{n,k} = n\varepsilon \mathbb{C}. \quad (15)$$

Let $p_j = p_j(\gamma_{n,k}) \in H^{4j}(G_{n,k}; \mathbb{Z}[1/2]), 1 \leq j \leq s$, be the $j$th (rational) Pontrjagin class of $\gamma_{n,k}$, and let $q_j = p_j(\beta_{n,k}), 1 \leq j \leq t$. Since $\gamma_{n,k} \oplus \beta_{n,k} \cong n\varepsilon \mathbb{R}$, we have, for $1 \leq r \leq s + t$,

$$\sum_{0 \leq j \leq s} p_j q_{r-j} = 0, \quad (16)$$

where it is understood that $p_0 = q_0 = 1, p_i = 0, q_j = 0$ if $i > s, j > t$. In fact, the cohomology algebra $H^*(G_{n,k}; \mathbb{Z}[1/2])$ has the following description. It can be derived from the known description of $H^*(\tilde{G}_{n,k}; \mathbb{Z}[1/2])$ as the fixed subring under the action of the deck transformation group of the double covering $\tilde{G}_{n,k} \to G_{n,k}$. We refer the reader to [MS, Theorem 15.9] for the description of $H^*(\tilde{G}_{n,k}; \mathbb{Z}[1/2])$.

**Theorem 5.1.** With the above notations, we have

$$H^*(G_{n,k}; \mathbb{Z}[1/2]) = \mathbb{Z}[1/2][p_1, \ldots, p_s; q_1, \ldots, q_t, v_{n-1}] / J \quad (17)$$

where degree of $v_{n-1} = n - 1$, and the ideal $J$ is generated by the following elements:

(i) $\sum_{0 \leq j \leq r} p_j q_{r-j}, 1 \leq r \leq s + t$,

(ii) $v_{n-1}$ if $n$ is odd or $k$ is even; $v_{n-1}^2$ if $n$ is even and $k$ odd.

As a consequence we note that $H^*(G_{n,k}; \mathbb{Z})$ has no $p$-torsion except when $p = 2$.

Denote by $P_{n,k} \subset H^*(G_{n,k}; \mathbb{Q})$ the even-graded subalgebra, namely, $H^{\text{ev}}(G_{n,k}; \mathbb{Q}) = \bigoplus_{r \geq 0} H^{2r}(G_{n,k}; \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_s; q_1, \ldots, q_t]/\sim$. Then $P_{n,k}$ depends only on $s, t$ and not on the values of $\varepsilon, \eta \in \{0, 1\}$, and, $\dim_{\mathbb{Q}} P_{n,k} = \binom{s+t}{s}$. Moreover, $P_{n,k} = H^*(G_{n,k}; \mathbb{Q})$, except when $n = 2s + 2t + 2$.
is even and \( k = 2s + 1 \) is odd. When \( n = 2s + 2t + 2, k = 2s + 1 \), we have \( H^{\text{odd}}(G_{n,k}; \mathbb{Q}) = v_{n-1}P_{n,k} \cong P_{n,k} \) as a \( P_{n,k} \)-module. We have a natural \( \mathbb{Z}_2 \)-gradation on \( H^*(G_{n,k}; \mathbb{Q}) \) defined by the parity of the degree.

Recall the Chern character map \( \text{ch} : K^*(G_{n,k}) \otimes \mathbb{Q} \to H^*(G_{n,k}; \mathbb{Q}) \), which is an isomorphism of \( \mathbb{Z}_2 \)-graded rings. So \( K^0(G_{n,k}) \) has rank equal to \( \dim_{\mathbb{Q}} P_{n,k} = \binom{s+t}{s} \). In case \( n \) is odd or \( k \) is even, we have \( H^{\text{odd}}(G_{n,k}; \mathbb{Q}) = 0 \) and so \( K^1(G_{n,k}) \) is a finite abelian group. When \( n \) is even and \( k \) is odd, \( K^1(G_{n,k}) \) has rank equal to that of \( K^0(G_{n,k}) \).

We now turn to the proof of Theorem 1.2. We shall denote by \( \phi \) the inclusion map \( K_{n,k} \hookrightarrow K(G_{n,k}) \).

**Proof of Theorem 1.2** The inclusion \( \phi : K_{n,k} \hookrightarrow K(G_{n,k}) \) induces an inclusion \( \phi \otimes 1 : K_{n,k} \otimes \mathbb{Q} \to K(G_{n,k}) \otimes \mathbb{Q} \).

We need to show that the composition \( \text{ch} \circ (\phi \otimes 1) : K_{n,k} \otimes \mathbb{Q} \to P_{n,k} \) is surjective. Note that, in view of Equation (16), \( P_{n,k} \) is generated by \( p_j, 1 \leq j \leq s \). So we need only show that the \( p_j \in P_{n,k} \) are in the image of \( \text{ch} \circ (\phi \otimes 1) \).

We have a formal expression of \( p_j = p_j(\gamma_{n,k}) \) in terms of the Chern ‘roots’ \( x_j, -x_j, 1 \leq j \leq s \), of \( \gamma_{n,k}^C \) given as \( p_j = (-1)^j e_j(x_j^2, \ldots, x_j^2), 1 \leq j \leq s \), where \( e_j \) denotes the \( j \)-th elementary symmetric polynomial in the indicated arguments. (See [MS, §15].) From the definition of Chern character we have \( \text{ch}(\gamma_{n,k}^C) = k + 2 \sum_{m \geq 1} \sum_{1 \leq j \leq s} x_j^{2m} / (2m)! = k + 2 \sum_{m \geq 1} u_m / (2m)! \)

where \( u_m := \sum_{1 \leq m \leq s} x_j^{2m} \) for \( m \geq 1 \). The symmetric polynomials can be expressed as polynomials in the power sums over \( \mathbb{Q} \) and so we have

\[
(-1)^j p_j = u_j / j + F_j(u_1, \ldots, u_{j-1}), \quad 1 \leq j \leq s, \tag{18}
\]

where \( u_0 = k \) and \( F_j(u_1, \ldots, u_{j-1}) \in H^{2j}(G_{n,k}; \mathbb{Q}) \) is a suitable polynomial in \( u_1, \ldots, u_{j-1} \). So it suffices to show that the \( u_j \) are in the image of \( \text{ch} \circ (\phi \otimes 1) \). To see this, it is convenient to use the Adams operations \( \psi^r \). Note that \( K_{n,k} \) contains \( A_j(\gamma_{n,k}^C) \) and so it also contains \( \psi^r(\gamma_{n,k}^C) \) for all integers \( r \geq 1 \) since the \( \psi^r \) can be expressed (with \( \mathbb{Z} \)-coefficients) in terms of the exterior power operations. Although \( \psi^r(\gamma_{n,k}^C) \) is only a virtual bundle, its Chern characters are easy to compute since \( rx_j, -rx_j \) are its Chern roots. Thus, writing \( d = \lfloor k(n-k)/2 \rfloor \), we have, for \( r \in \mathbb{Z} \),

\[
v_r := \text{ch}([\psi^r(\gamma_{n,k}^C)] - k) = 2 \sum_{m \geq 1} (\sum_{1 \leq j \leq s} r^{2m} x_j^{2m} / (2m)! ) = 2 \sum_{1 \leq m \leq d} r^{2m} u_m / (2m)!. \tag{19}
\]

We obtain the equation \( 2uM = v \) where \( M = (m_{ij}) \) is the \( d \times d \) matrix defined as \( m_{ij} = j^{2i} \), and \( u = (u_1/2!, u_2/4!, \ldots, u_d/(2d)! ) \), \( v = (v_1, \ldots, v_d) \) are regarded as (row) vectors in the
\(d\)-fold direct sum \((H^{ev}(G_{n,k}, \mathbb{Q}))^d\). Since \(M\) is invertible and since the \(v_j\) are in the image of \(\text{ch} \circ (\phi \otimes 1)\), it follows that the \(u_j/(2j)!\) are also in the image of \(\text{ch} \circ (\phi \otimes 1)\) for \(1 \leq j \leq d\). So \(u_1, \ldots, u_s\) are in the image of \(\text{ch} \circ (\phi \otimes 1)\). This completes the proof. \(\square\)

We conclude by giving, in Proposition 5.5, a description of \(K_{n,k}\) as a quotient of a ring \(K_{n,k}\), explicitly described in terms of generators and relations, with finite kernel. It seems plausible that \(K_{n,k}\) is isomorphic to \(K_{n,k}\) but we have not been able to prove this.

The operator \(\Lambda_t = \sum_{r \geq 0} \Lambda^r t^r\), which is a formal power series in the indeterminate \(t\) whose coefficients are exterior power operators, has the property \(\Lambda_t(\omega_0 \oplus \omega_1) = \Lambda_t(\omega_0).\Lambda_t(\omega_1)\) for any two complex vector bundles \(\omega_0, \omega_1\). So we have \(\Lambda_t(\gamma^n_{n,k}) \cdot \Lambda_t(\beta^n_{n,k}) = (1 + t)^n\) since \(\Lambda_t(\epsilon^n_c) = (1 + t)\). Equivalently, for any \(r \geq 1\), we have

\[
\sum_{p+q=r} \Lambda^p(\gamma^n_{n,k}) \otimes \Lambda^q(\beta^n_{n,k}) = \binom{n}{r}. \tag{20}
\]

We know that \(2^r \xi^n_c\) is stably trivial for some \(r\) where \(\xi = \xi_{n,k}\) denotes the Hopf line bundle over \(G_{n,k} = SO(n)/S(O(k) \times O(n-k))\). By Theorem 4.10, one may take \(r = m+1\). We let \(\nu\) be the least positive integer for which this happens. Then \((1 - [\xi^n_c])^{\nu+1} = 2^\nu(1 - [\xi^n_c]) = 0\) in \(K(G_{n,k})\).

Note that \(\xi = \Lambda^k(\gamma^n_{n,k}) = \Lambda^{n-k}(\beta^n_{n,k})\) is associated to the character \(\chi : S(O(k) \times O(n-k)) \to O(1)\) defined as \(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \det(A)\). We let \(\theta\) be the complexification of \(\chi\) so that \(\xi^n_c\) is associated to \(\theta\). We shall denote \([\xi^n_c] \in K(G_{n,k})\) by \([\theta]\).

For any real vector space \(V\) of dimension \(k\), one has a functorial non-degenerate bilinear pairing \(\Lambda^p(V) \times \Lambda^{k-p}(V) \to \Lambda^k(V)\) defined as \((u, v) \mapsto u \wedge v\). If \(V\) is an inner product space, then we have the induced inner product \(\Lambda^q(V) \times \Lambda^q(V) \to \mathbb{R}\) defined as \((u_1 \wedge \cdots \wedge u_q, v_1 \wedge \cdots \wedge v_q) \mapsto \det((u_i, v_j))\). Thus, we obtain a natural isomorphism \(\Lambda^p(V) \cong \Lambda^{k-p}(V) \otimes \Lambda^k(V)\). This yields an isomorphism \(\Lambda^p(\gamma^n_{n,k}) \cong \Lambda^{k-p}(\gamma^n_{n,k}) \otimes \xi^n_c\) of real vector bundles. See [MS, §2]. A similar isomorphism holds for \(\beta^n_{n,k}\) as well. Complexifying we obtain the following isomorphisms for \(1 \leq p \leq k, 1 \leq q \leq n-k:\)

\[
\Lambda^p(\gamma^n_{n,k}) \cong \xi^n_c \otimes \Lambda^{k-p}(\gamma^n_{n,k}), \quad \Lambda^q(\beta^n_{n,k}) \cong \xi^n_c \otimes \Lambda^{n-k-q}(\beta^n_{n,k}). \tag{21}
\]

We are now ready to define the ring \(K_{n,k}\).

**Definition 5.2.** Let \(A = \mathbb{Z}[\theta]/(\theta^2 - 1, 2^\nu(1 - \theta))\). Then \(A \cong \mathbb{Z} \oplus \mathbb{Z}_{2^\nu}(1 - \theta)\). Write \(k = 2s + \varepsilon, n-k = 2t + \eta\) where \(\varepsilon, \eta \in \{0,1\}\) so that \(n = 2s + 2t + \varepsilon + \eta\). We define \(K_{n,k} := A[\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_{n-k}]/I\), the quotient of the polynomial algebra over \(A\) where the ideal \(I\) is generated by the following elements:

(i) \(\lambda_{k-p} - \theta \lambda_p, \quad \mu_{k-q} - \theta \mu_q\) for \(1 \leq p \leq k, 1 \leq q \leq n-k,\)

(ii) \(Q_r(\lambda, \mu) - \binom{n}{r}\) for \(1 \leq r \leq n\) where \(Q_r(\lambda, \mu) := \sum_{p+q=r, 0 \leq p \leq k, 0 \leq q \leq n-k} \lambda_p \mu_q\), for \(1 \leq r \leq n\).
Remark 5.3. (a) The \(A\)-algebra \(K_{n,k}\) is generated by \(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t\). This is immediate from the relations 5.2(ii).

(b) In fact, using the relations 5.2 (ii), (and (a)), we see that \(\mu_1 = n - \lambda_1\), and, if \(2 \leq r \leq t\), then \(\mu_r\) can be expressed in terms of the \(\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_{r-1}\) (with coefficients in \(A\)). So, by induction, the \(\mu_r\) can be expressed in terms of \(\lambda_1, \ldots, \lambda_s\). Hence \(K_{n,k}\) is generated by \(\lambda_p, 1 \leq p \leq s\).

(c) One has a ring homomorphism \(f : A \rightarrow \mathbb{Z}\) which maps \(\theta\) to \(1\) with kernel the ideal \(A(1 - \theta)\).

Set \(\bar{K}_{n,k} := K_{n,k} \otimes_A \mathbb{Z} = K_{n,k}/(1 - \theta)K_{n,k} = \mathbb{Z}[\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t]/I_0\) where \(I_0\) is the ideal generated by the elements listed in Definition 5.2 (ii), where \(\lambda_p = \lambda_{k-p}, \mu_q = \mu_{n-k-q}\) for \(p > s, q > t\).

**Lemma 5.4.** One has the following isomorphisms of rings:

\[
\bar{K}_{2s+2t+2,2s+1} \xrightarrow{\beta_0} \bar{K}_{2s+2t+1,2s+1} \xrightarrow{\beta_1} \bar{K}_{2s+2t,2s},
\]

where, \(\alpha_0(\lambda_p) = \lambda_p, \alpha_0(\mu_q) = \mu_q + \mu_{q-1}\), and, \(\alpha_1(\lambda_p) = \lambda_p + \lambda_{p-1}, \alpha_1(\mu_q) = \mu_q\), for all \(p \leq k, q \leq n - k\). (It is understood that \(\lambda_0 = 1 = \mu_0\).) An abelian group \(\bar{K}_{n,k}\) is free of rank \((s+t)\) where \((n,k) = (2s+2t+2,2s+1), (2s+2t+1,2s+1), (2s+2t,2s)\).

**Proof.** It is readily verified that \(\alpha_0, \alpha_1\) are surjective homomorphisms. We need to show that they are injective as well.

Consider \(\beta_0 : \bar{K}_{2s+2t+1,2s+1} \rightarrow \bar{K}_{2s+2t+2,2s+1}\), and, \(\beta_1 : \bar{K}_{2s+2t,2s} \rightarrow \bar{K}_{2s+2t+1,2s+1}\) defined as follows: for \(p \leq s, q \leq t\),

\[
\beta_0(\lambda_p) = \lambda_p, \quad \beta_0(\mu_q) = \sum_{0 \leq j \leq q} (-1)^{q-j} \mu_j,
\]

\[
\beta_1(\lambda_p) = \sum_{0 \leq j \leq p} (-1)^{p-j} \beta_1(\lambda_j), \quad \beta_1(\mu_q) = \mu_q.
\]

Straightforward verification, using the identity \(\sum_{0 \leq j \leq r} (-1)^{r-j} {\binom{n}{r-j}} = {\binom{n-1}{r}}\), shows that \(\beta_0\) and \(\beta_1\) are well-defined homomorphisms of rings. Again, these are surjective, since the generators \(\lambda_p\) (resp. \(\mu_q\)) are in the image of \(\beta_0\) (resp. \(\beta_1\)).

We claim that \(\alpha_0, \beta_0\) (resp. \(\alpha_1, \beta_1\)) are inverses of each other. Indeed, \(\beta_0 \circ \alpha_0(\lambda_p) = \lambda_p\) for all \(p \leq s\) and \(\alpha_0 \circ \beta_0(\lambda_p) = \lambda_p\) for all \(p\). By Remark 5.3(b) above, our claim follows. Similarly \(\alpha_1, \beta_1\) are inverses of each other.

For the last assertion, we need only consider the case \((n,k) = (2s+2t+2,2s+1)\). The ring \(\bar{K}_{2s+2t+2,2s+1}\) is isomorphic to the quotient ring \(R/I \cong H^*(G_{s+t,s}; \mathbb{Z})\) considered in Remark 4.3(ii). Hence \(\bar{K}_{2s+2t+2,2s+1}\) is a free abelian group of rank \((s+t)\).

**Proposition 5.5.** One has a surjective homomorphism of rings \(\kappa : K_{n,k} \rightarrow \mathbb{K}_{n,k}\) with finite kernel, defined as \(\kappa(\lambda_j) = [\Lambda^j(\gamma^C_{n,k})], 1 \leq j \leq k\).
Proof. In view of Equations (20) and (21), $\kappa$ is a well-defined ring homomorphism. Clearly $\kappa(\lambda_j) = [\Lambda^j(\gamma_{n,k}^C)]$ for all $j$ and so, by the definition of $\mathcal{K}_{n,k}$, $\kappa$ is surjective. Since both $K_{n,k}, \mathcal{K}_{n,k}$ have the same (finite) rank, it follows that ker($\kappa$) is finite. □

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