Entanglement in the Dicke model

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Abstract

We show how an ion trap, configured for the coherent manipulation of external and internal quantum states, can be used to simulate the irreversible dynamics of a collective angular momentum model known as the Dicke model. In the special case of two ions, we show that entanglement is created in the coherently driven steady state with linear driving. For the case of more than two ions we calculate the entanglement between two ions in the steady state of the Dicke model by tracing over all the other ions. The entanglement in the steady state is a maximum for the parameter values corresponding roughly to a bifurcation of a fixed point in the corresponding semiclassical dynamics. We conjecture that this is a general mechanism for entanglement creation in driven dissipative quantum systems.

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INTRODUCTION

Generically, many body quantum systems are known to be difficult to simulate efficiently on a classical computer. This is because the quantum system may explore regions of state space with non zero entanglement, giving these systems access to a vastly larger state space than is possible classically. In an open quantum system we may, in some circumstances, be able to resort to stochastic methods, such as Monte-Carlo simulations. However, this will not be possible for open systems in which the steady state itself is entangled, as in the example we describe here. Terhal et al.\cite{Terhal} have considered the possibility of using a quantum computer to simulate open quantum systems in thermal equilibrium. Plenio et al.\cite{Plenio} have considered how decay can lead to entanglement rather than destroying it. Cabrillo et al.\cite{Cabrillo} discuss creating entanglement in two or more atoms, by driving the atoms with a weak laser pulse and detecting the spontaneous emission. In a recent paper by Arnesen et al.\cite{Arnesen}, the authors look at a situation where the spins in a Heisenberg chain with an external magnetic field show entanglement in the thermal state with non-zero temperature. In this paper we consider the case of an ion trap quantum computer realisation to simulate the irreversible dynamics of $N$ two level systems.

In the following we will investigate the use of an ion trap quantum computer as a simulation device for the so-called Dicke model\cite{Dicke}. This model includes resonance fluorescence of a set of two-level atoms driven by a resonant coherent laser field as well as a collective decay mechanism. We will first describe how a collective decay mechanism may be realised for $N$ trapped ions interacting with a collective vibrational mode when the vibrational mode is subject to controlled heating. Simulating irreversible dynamics for a trapped ion has been previously suggested by a number of authors\cite{Arnesen}. We consider the dynamics of the system relaxing from an initial state in which all ions are in the excited state. For two ions we explicitly calculate the Wooters entanglement measure (concurrence)\cite{Wooters} as a function of time, which shows that the system passes through entangled states on the way to the collective ground state. Next we add coherent driving to try to force the system into a non trivial steady state and show that, again for the case of two ions, this steady state can be partially entangled. Extending this result to many ions is not possible at present due to the lack of a general measure of entanglement of mixed states in higher dimensions. However, we can calculate the entanglement between two ions or atoms by tracing over all the other ions.
or atoms. This will show us at least whether entanglement is present. Interestingly, the maximum entanglement occurs for parameter values for which the corresponding semiclassical system undergoes a bifurcation and loss of stability of the fixed point. We conjecture that the loss of stability of a semiclassical fixed point will generically be associated with entanglement in the steady state of the full quantum system.

**THE MODEL**

In the seventies the Dicke model and cooperative effects were subjects of research in various groups (see e.g. [8, 9, 10] and references therein). The model consists of a group of two-level atoms which is placed in a volume with dimensions small compared to the wavelength associated with the atom’s two-level dipole and evolves on time scales shorter than any $\hat{J}^2$ breaking relaxation mechanism (see [8]) like an angular momentum system which has collective atomic raising and lowering operators $\hat{J}_+$ and $\hat{J}_-$ with a fixed spin quantum number $j = N/2$, where $N$ is the number of atoms.

In the rotating frame with Markov, electric dipole and rotating wave approximations and ignoring a small atomic frequency shift the master equation for the density matrix of this group of atoms under the cooperative influence of an electromagnetic field is [9, 11, 12]

$$
\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{2} \Omega \left[ \hat{J}_+ + \hat{J}_-, \hat{\rho} \right] + \frac{\gamma_A}{2} \left( 2\hat{J}_-\hat{\rho}\hat{J}_+ - \hat{J}_+\hat{\rho}\hat{J}_- - \hat{\rho}\hat{J}_+\hat{J}_- \right)
$$

(1)

where $\Omega$ is the Rabi frequency and $\gamma_A$ is the Einstein A-coefficient of each atom. This model can be solved exactly [11] and it exhibits a critical-point nonequilibrium phase transition for $\Omega/j = \gamma_A$ in the limit $\Omega, j \rightarrow \infty$ [46].

**Collective driving**

How do we get a similar master equation to Eq. (1) in an ion trap? The coherent evolution is easy: We just shine the same laser at the carrier frequency on all the ions at the same time, thus forcing each ion to undergo Rabi oscillations at the same frequency. If we start initially with all the ions in their electronic ground state $|g\rangle$, the ions will not leave the $j = N/2$ space. From there we can then define collective angular momentum operators in
the following way
\[ \hat{J}_- = \sum_{i=1}^{N} \hat{\sigma}_-^{(i)} \]  
\[ \hat{J}_+ = \sum_{i=1}^{N} \hat{\sigma}_+^{(i)}, \]
where the raising and lowering operators for each ion are defined by \( \hat{\sigma}_- = |g\rangle \langle e| \) and \( \hat{\sigma}_+ = |e\rangle \langle g| \). With this the Hamiltonian for simultaneous resonant driving of all the ions can be written as
\[ \hat{H} = \hbar \frac{\Omega^2}{2} (\hat{J}_+ + \hat{J}_-), \]
where \( \Omega \) is the Rabi frequency for the electronic transition.

**Cooperative damping**

For the collective decay mechanism we need to couple the ions equally to the same heat reservoir. In this paper we will argue that the reservoir may be taken to be the centre-of-mass vibrational mode. It is subject to heating and we assume that it is in a thermal state. To couple the ions to the vibrational mode we need another laser which, again, illuminates all the ions at the same time, but which is detuned from the carrier frequency to the red by the trap frequency so that the electronic state of each atom gets coupled simultaneously to the centre-of-mass vibrational mode. This is described by a Hamiltonian for the \( i \)th ion of the form
\[ \hat{H}_{\text{red}}^{(i)} = \hbar \Omega_2 \left( \hat{\sigma}_+^{(i)} \hat{a} + \hat{\sigma}_-^{(i)} \hat{a}^\dagger \right), \]
where we have introduced the bosonic annihilation operator \( \hat{a} \) for the vibrational mode and the coupling constant is \( \Omega_2 = \eta \Omega_0 \). The parameter \( \eta^2 = E_r/\hbar M \omega_0 \) is the Lamb-Dicke parameter with \( E_r \) the recoil kinetic energy of the atom, \( \omega_0 \) is the trap vibrational frequency, and \( M \) is the effective mass for the centre-of-mass mode. The Lamb-Dicke limit assumes \( \eta \ll 1 \), which is easily achieved in practice. The frequency \( \Omega_0 \) is the effective Rabi frequency for the electronic transition involved. This sideband transition is used to efficiently remove thermal energy from the vibrational degree of freedom. If the rate of this cooling process can overcome heating due to external fluctuations in the trap potential, the ion may eventually
be prepared in the vibrational ground state. However, in general the vibrational state will reach a thermal mixture, 

\[ \hat{\rho}_v = \frac{1}{Z} \exp[-\hbar \omega_0 \hat{a} \hat{a}^\dagger/k_B T] \]

where \( Z = \text{Tr}(\exp[-\hbar \omega_0 \hat{a} \hat{a}^\dagger/k_B T]) \) at some effective temperature \( T \). If the heating and cooling rates are such that the system relaxes at a rate, \( \alpha \), large compared with any other time scale for ion motion, the ion can effectively be regarded as interacting with a thermal reservoir at temperature \( T \). We can also arrange that the associated rate of energy dissipation is small, \( \gamma_A \ll \omega_0 \) which simply requires that the coupling to the vibrational degree of freedom is weak. Finally we assume that the temperature of the vibrational degree of freedom is such that \( \gamma_A \ll k_B T/\hbar \). Under these assumptions we may eliminate the description of the vibrational motion from the dynamics and obtain a master equation for the electronic state \( \hat{\rho}^{(i)} \) of the ion

\[
\frac{d\hat{\rho}^{(i)}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}^{(i)}] + \gamma_A \bar{n} \mathcal{D}[\hat{\sigma}_+^{(i)}] \hat{\rho}^{(i)} + \gamma_A (\bar{n} + 1) \mathcal{D}[\hat{\sigma}_-^{(i)}] \hat{\rho}^{(i)} ,
\]

where the superoperator is defined by

\[
\mathcal{D}[\hat{A}] \hat{\rho} = \hat{A} \rho \hat{A}^\dagger - \frac{1}{2} (\hat{A}^\dagger \hat{A} \hat{\rho} + \hat{\rho} \hat{A}^\dagger \hat{A}) ,
\]

and where \( \hat{H} \) is the Hamiltonian for any other reversible electronic dynamics and \( \bar{n} \) is the mean thermal occupation number of the vibrational degree of freedom. In what follows we assume that the cooling is very efficient and set \( \bar{n} = 0 \). At any time we may turn off the cooling lasers, thus reducing \( \gamma_A \) suddenly to zero. The master equation, Eq.(6), looks like the master equation for a single two level atom interacting with the many body radiation field via spontaneous emission and thermal absorption and we will discuss this case in the next section. In the case considered here, however, we ignore spontaneous emission which is justified if the electronic states coupled by the lasers correspond to a long lived dipole forbidden transition. Thus the irreversible dynamics of the electronic state is due entirely to the interaction with the phonons associated with the vibrational degree of freedom.

If the external laser field on each ion is identical (in amplitude and phase) the interaction Hamiltonian is

\[
\hat{H}_I = \hbar \Omega_2 (\hat{a} \hat{J}_+ + \hat{a}^\dagger \hat{J}_-) ,
\]

where \( \hat{J}_+ \) and \( \hat{J}_- \) are defined in Eqs. (3) and (2). For the case of a linear ion trap, with separately addressable ions, identical laser fields could easily be obtained by splitting the cooling laser into multiple beams. In this way we can simulate an angular momentum system.
with quantum number \( j = N/2 \). This imposes a permutation symmetry on the system which reduces the effective Hilbert space dimension from \( 2^N \) to \( 2^N + 1 \). Thus an exponentially large portion of the available Hilbert space, i.e. all the states with \( j < N/2 \), is not used in this simulation. However, it is easy to generate the relevant unitary transformations to simulate the \( j = N/2 \) angular momentum quantum system.

It is not trivial to keep the vibrational mode in a thermal state of fixed temperature. One way of doing this was recently suggested by Kielpinski \textit{et al}. \[13\]: They put one ion, which is a different species of all the other ions, in the centre of a string of ions, so that they have an odd number of ions in the trap. Through this centre ion, which they can cool at will without disturbing the coherence of the other ions, the sympathetically cool all the ions and are thus able to keep the string of ions at a well defined temperature. The authors conclude that such a scheme of sympathetic cooling is “well within the reach of current experimental technique” \[13\]. We assume for further calculations that the centre-of-mass mode is kept in such a thermal state by the outlined technique. In the following derivation of the master equation we do not explicitly put the laser cooling into the equation. We just assume that the vibrational state instantly, i.e. on a time scale fast compared to all the other processes involved, relaxes back into the thermal state.

With all these assumptions we get the master equation describing the collective motion of the density matrix of all the ions

\[
\frac{\partial \hat{\rho}}{\partial t} = -i \frac{\Omega}{2} \left[ \hat{J}_+ + \hat{J}_-, \hat{\rho} \right] + \gamma_A \frac{\bar{n}}{2} \left( 2 \hat{J}_+ \hat{\rho} \hat{J}_- - \hat{J}_- \hat{J}_+ \hat{\rho} - \hat{\rho} \hat{J}_- \hat{J}_+ \right) + \gamma_A \frac{\bar{n} + 1}{2} \left( 2 \hat{J}_- \hat{\rho} \hat{J}_+ - \hat{J}_+ \hat{J}_- \hat{\rho} - \hat{\rho} \hat{J}_+ \hat{J}_- \right) ,
\]

(9)

where \( \bar{n} \) is the mean phonon number of the vibrational centre-of-mass mode and \( \gamma_A = 2\Omega^2 \eta^2 \).

Note that for \( \bar{n} = 0 \) this equation is identical to Eq. (1) for the Dicke model.

\textbf{STEADY STATE AND ENTANGLEMENT FOR} \( j = 1 \)

With two ions we have \( j = 1 \) and from the master equation, Eq. (9), we can write down the equation of motion for the components of the three by three density matrix of the state of the system, taking into account that \( \text{Tr}(\hat{\rho}) = 1 \) and that \( \hat{\rho} \) is Hermitian. Getting the steady state is then a matter of simple algebra.
Once we have determined the steady state of the $j = 1$ system, we can rewrite this state in the underlying two-qubit basis. What we are interested in is the change of entanglement in the system as the parameters $\gamma$ and $\bar{n}$ change. The entanglement of two qubits is well defined \[7, 14, 15\], and we choose the concurrence \[7\] as a measure for it.

A numeric evaluation of the concurrence leads to the plot in Fig. 1. What we see is that we can get a certain amount of entanglement in the steady state of a coherently driven system that is coupled to a thermal reservoir. This is remarkable as the steady state is independent of the initial state, which can be unentangled. The coherent evolution alone does not lead to any entanglement for an initially unentangled state either, as it only consists of (simultaneously acting) single qubit rotations and no coupling between the qubits is present. Thus the entanglement is due to the cooperative decoherence in the system acting together with the coherent evolution.

**ZERO TEMPERATURE CASE.**

The analysis here is restricted to the case $j = 1$. For $j > 1$ and $\bar{n} \neq 0$, numerical methods will need to be employed to derive the steady state, however, in this case another problem will arise due to the fact that there is currently no measure of entanglement for $N$ coupled qubits. Nevertheless, other phase transitions analogous to the one in the Dicke-model (see e.g. \[8, 9\] and references therein) will appear.

For $\bar{n} = 0$ we can compare our results for the steady state to the ones calculated by Puri
and Lawande [16] (see also Lawande et al. [17]). They calculate the steady state to be

$$\hat{\rho}_S = \frac{1}{D} \sum_{m,n=0}^2 (g^*)^{-m}(g)^{-n}\hat{j}_m^n\hat{j}^m_n,$$

(10)

where

$$D = \sum_{k=0}^2 H_{2,k}|g|^{-2k}$$

(11)

is a normalization constant, $g = i/\gamma$, where $\gamma = \gamma_A/\Omega$ as defined above, and

$$H_{2,m} = \frac{(2 + m + 1)!(m!)^2}{(2 - m)!(2m + 1)!}.$$  

(12)

With this we can write the density matrix of the steady state in matrix form as

$$\hat{\rho}_S = \frac{1}{D} \begin{pmatrix}
1 & -i\sqrt{2}\gamma & -2\gamma^2 \\
(i\sqrt{2}\gamma & 1 + 2\gamma^2 & -i\sqrt{2}\gamma - i2\sqrt{2}\gamma^3 \\
-2\gamma^2 & i\sqrt{2}\gamma + i2\sqrt{2}\gamma^3 & 1 + 2\gamma^2 + 4\gamma^4
\end{pmatrix},$$

(13)

where we have calculated $D$ as

$$D = 3 + 4\gamma^2 + 4\gamma^4.$$  

(14)

This gives the same result as above if we set $\bar{n} = 0$. The concurrence for this special case is

![Plot of the concurrence as a measure of entanglement depending on the parameters |g| = 1/\gamma = \Omega/\gamma_A and 1/|g| = \gamma (dashed line) with \bar{n} = 0, i.e. for the Dicke-model.](image)

FIG. 2: Plot of the concurrence as a measure of entanglement depending on the parameters $|g| = 1/\gamma = \Omega/\gamma_A$ and $1/|g| = \gamma$ (dashed line) with $\bar{n} = 0$, i.e. for the Dicke-model.

plotted in figure 2, but this time we plot it against $|g| = 1/\gamma$ as well. Thus the Dicke-model as a special case of our model shows entanglement in the steady state.
For $N > 2$ up to date no definite measure of entanglement exists. But we can calculate the entanglement between just two of the $N$ ions at least for the Dicke model, where the temperature of the bath is zero. The steady is then given by \[11, 16, 17\]

\[
\hat{\rho}_S = \frac{1}{D} \sum_{l=0}^{2j} \sum_{l'=0}^{2j'} \left( \frac{\hat{J}_-}{g} \right)^l (\hat{J}_+)^{l'}.
\]

By writing this as a sum of states with angular momentum $j_1 = 1$ and $j_2 = j - 1$ we can trace over the part of the Hilbert space with $j_2 = j - 1$ and thus get the density matrix in the steady state for just two ions (or atoms in the original Dicke model). From there we can again calculate the concurrence. This time we plot it against the relative Rabi frequency $\Omega_r = \Omega/(j\gamma)$. We note that the maxima of the entanglement occur close to the critical point in the cooperative limit $j, \Omega \to \infty$ of the Dicke model, i.e. around $\Omega_r = 1$. The two-ion entanglement is not the real measure of entanglement in the system. Thus we cannot take the cooperative limit as the two-ion entanglement goes to zero in this limit. However, we note from the plots in Fig. 3 that the maximum value of the two-ion entanglement indeed does move closer to the point $\Omega_r = 1$ for increasing $N$.

**FIG. 3:** Plot of the two-ion concurrence as a measure of entanglement depending on the parameters $\Omega_r = \Omega/(j\gamma_A)$ for (a) $j = 1$, (b) $j = 4$, (c) $j = 16$, and (d) $j = 64$.

**CONCLUSION**

In this paper we have demonstrated how the steady state of a dissipative many body system, driven far from equilibrium, may exhibit non zero quantum entanglement. This result is significant for two reasons. Firstly the steady state is a mixed state and the study of quantum entanglement for mixed states is a very active field of enquiry \[18\]. It
immediately raises the question of whether the entanglement can be distilled and used as a resource for some quantum communication or computation task [19]. Secondly the maximum entanglement occurs at the same parameter values for which the semiclassical dynamics of the system undergoes a bifurcation of the fixed point corresponding to the quantum steady state. At the bifurcation point the time constant associated with the fixed point goes to zero as the bifurcation is approached. This is reminiscent of a phenomenon that characterises quantum phase transitions, in which a morphological change in the ground state, as a parameter is varied, is associated with a frequency gap tending to zero [20, 21]. We conjecture that the association between the bifurcation of a fixed point of the semiclassical description and the maximum of entanglement will be a general feature of dissipative many body systems driven far from equilibrium.

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