An optical analog of quantum optomechanics

B M Rodríguez-Lara and H M Moya-Cessa

Instituto Nacional de Astrofísica, Óptica y Electrónica Calle Luis Enrique Erro No. 1, Sta. Ma. Tonantzintla, Pue. CP 72840, México

E-mail: bmlara@inaoep.mx

Received 19 November 2014
Accepted for publication 8 December 2014
Published 8 June 2015

Abstract
We use an optical analog of a quantum optomechanical system to show that non-trivial effective couplings, such as column isolation and diagonal coupling can be engineered in a two-dimensional array of nearest-neighbour-coupled waveguides by an adequate selection of refractive indices and nearest-neighbour couplings.

Keywords: optical lattices, field-mirror interaction, quantum-classical analogies

1. Introduction

Recently, Man’ko, Man’ko and Mendes [1] have shown possible applications of classical optics systems in the engineering of quantum gates. Optical analogs of quantum processes [2] may become a valuable resource not only in quantum computing simulation [3], but also in the design of integrated optics devices [4–6]. In this spirit, a plethora of optical analogs to quantum mechanical systems and the algebraic methods to solve them provide a valuable toolbox for optical designers. Here, we want to show a quantum analogy that gives a way to engineer non-trivial effective couplings in a two-dimensional (2D) array of nearest-neighbour-coupled waveguides. This may open new avenues for the use of quantum mechanics in classical optics. For this, we will use a 2D array of photonic waveguides described by the differential set

\[-i\hbar \frac{\partial}{\partial z} \mathcal{E}_{j,k} = \left( \delta k + \omega_m j \right) \mathcal{E}_{j,k} + kg \left( \sqrt{j+1} \mathcal{E}_{j+1,k} + \sqrt{j} \mathcal{E}_{j-1,k} \right) + d \left( \sqrt{k+1} \mathcal{E}_{j,k+1} + \sqrt{k} \mathcal{E}_{j,k-1} \right),\]

where the field amplitude at the \((j, k)\)th waveguide is \(\mathcal{E}_{j,k}\) and \(z\) is the propagation parameter. This photonic lattice is a 2D semi-infinite array of waveguides where the effective refractive index, \(\delta k + \omega_m j\), depends on both the horizontal and vertical positions in the array, \(j\) and \(k\), in that order, where the constant vertical and horizontal base couplings \(d\) and \(g\) are real. A primer on arrays of coupled waveguides can be found in [7]. The array may be made finite as will be shown later. This photonic lattice is a 2D generalization of the Glauber–Fock oscillator lattice [8] with some modifications. Here, the constant parameter set from the waveguides, \(\{\delta, \omega_m, d, g\}\), are taken as fabrication parameters, but they will be linked to a quantum optomechanical model in the following.

In order to show that this 2D array of nearest-neighbour-coupled waveguides can show non-trivial effective couplings, we need to rewrite it as a Schrödinger equation, \(i\hbar \frac{\partial}{\partial \psi_j} = \mathcal{H} \psi_j\), where the Hamiltonian is given by

\[\mathcal{H} = \delta \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} + g \hat{a}^\dagger \hat{b} \left( \hat{b}^\dagger + \hat{b} \right) + d \left( \hat{a}^\dagger + \hat{a} \right).\]

It is straightforward to go from the Schrödinger equation to the differential set in (1) if we perform the variable change \(z = -t\) and the wavefunction decomposition \(\psi_j = \sum_l E_{m,n} \left| m \right> \left< l \right| \psi_l\), where the state \(\left| m \right> \psi_l\) is the \(m\)th Fock state of the \(q\) oscillator, and the field at the \((j, k)\)th waveguide is \(E_{j,k}\) with \(j, k = 0, 1, 2, \ldots\) and \(E_{j,k|j-1|k-1} = 0\). The Hamiltonian in (2) describes an optomechanical model composed of a driven cavity coupled to a mechanical oscillator [9–12] where the detuning between the cavity field mode, described by the creation (annihilation) operators \(\hat{a}^\dagger (\hat{a})\), and the classical pump field frequencies is given by \(\delta = \omega_p - \omega_p\), the frequency of the mechanical oscillator, described by the creation (annihilation) operators \(\hat{b}^\dagger (\hat{b})\), is \(\omega_m\), and the parameters \(d\) and \(g\) are the strength of the pump field and the
linear coupling between the field and the mechanical oscillator, respectively. To our knowledge, it is not possible to provide a closed form analytic result for these equivalent models, but some approximations can be made in the quantum model. The parameters from these approximations will allow the array of nearest-neighbour-coupled waveguides to behave as an experimentally unfeasible lattice with certain non-trivial effective couplings, e.g. column isolation, diagonal couplings, and second- and higher-order diagonal couplings.

2. The 2D waveguide lattice

Let us start working on the optomechanical model with an standard approach that uses a coherent displacement [11, 12],

$$D(\hat{a}) = e^{(\hat{a}^\dagger - \hat{a})^2},$$

(3)

over the full Hamiltonian (2) and yields,

$$\hat{H}_D = \hat{D}(g\hat{a}^\dagger \hat{a} / \omega_m)\hat{H}\hat{D}(g\hat{a}^\dagger \hat{a} / \omega_m),$$

$$= \delta\hat{a}^\dagger \hat{a} - \frac{g^2}{\omega_m}(\hat{a}^\dagger \hat{a})^2 + \omega_m \hat{b}^\dagger \hat{b}$$

+ $$d \left[ \hat{a}^\dagger e^{\hat{a}^\dagger} + \hat{a} e^{-\hat{a}} \right].$$

(4)

At this point, we can use the time-dependent rotation

$$\hat{U}_0(t) = e^{i\omega_m(t)}(\delta\hat{a}^\dagger + \omega_m \hat{b}^\dagger \hat{b})$$

and the new formulation of the Schrödinger equation has an effective Hamiltonian given by

$$\hat{H}_I = \hat{H}_0(-t) \left[ \hat{H}_D - \delta\hat{a}^\dagger \hat{a} - \omega_m \hat{b}^\dagger \hat{b} \right] \hat{U}_0(t),$$

(6)

$$= -\frac{g^2}{\omega_m}(\hat{a}^\dagger \hat{a})^2$$

+ $$\omega_m \hat{b}^\dagger \hat{b}$$

+ $$d \left[ \hat{a}^\dagger e^{\hat{a}^\dagger} \hat{b} e^{-\hat{a}^\dagger} + \hat{a} e^{-\hat{a}^\dagger} \hat{b}^\dagger e^{\hat{a}^\dagger} \right].$$

(7)

$$= -\frac{g^2}{\omega_m}(\hat{a}^\dagger \hat{a})^2$$

+ $$\omega_m \hat{b}^\dagger \hat{b}$$

+ $$d \left[ \hat{a}^\dagger e^{\hat{a}^\dagger} \hat{b} e^{-\hat{a}^\dagger} + \hat{a} e^{-\hat{a}^\dagger} \hat{b}^\dagger e^{\hat{a}^\dagger} \right].$$

(8)

with $\beta = g / \omega_m$. Please keep in mind that the Schrödinger equation $i\hbar \partial_\phi = \hat{H}_I(\phi)$ with $\hat{H}_I$ and $|\phi\rangle = \hat{U}_0(-t)\hat{D}(g\hat{a}^\dagger \hat{a} / \omega_m)|\phi\rangle$, plus all other definitions made before, is still identical to the differential set (1) describing the starting generalization of the 2D Glauber–Fock oscillator.

The term $d \left[ \hat{a}^\dagger e^{\hat{a}^\dagger} \hat{b} e^{-\hat{a}^\dagger} + \hat{a} e^{-\hat{a}^\dagger} \hat{b}^\dagger e^{\hat{a}^\dagger} \right]$ in Hamiltonian $\hat{H}_I$ is equivalent to that found in a trapped-ion setup [13]. Thus, we can use a standard approach of ion-trap quantum electrodynamics and work in the regime $d \ll \omega_m$; that is, the couplings between waveguides are weak compared to the refractive indices. While current experimental realizations implement couplings and refractive indices of the same order of magnitude [8, 14–16], it may be hard but feasible to implement in the laboratory the regime we require. Once in this regime, we can choose the refractive indices so that the pump-cavity field detuning is set to $\delta = \omega_{lm}$ with $l = 0, \pm 1, \pm 2, \ldots$, and obtain an approximate Hamiltonian [13].

$$\hat{H}_U(l \geq 0) \approx -\frac{g^2}{\omega_m} (\hat{a}^\dagger \hat{a})^2$$

+ $$d e^{-\frac{l^2}{2}} \left\{ \hat{a}^\dagger \left( \frac{\hat{b} \hat{b}^\dagger}{\hat{b} \hat{b}^\dagger + 1} \right)^l \right\} L^{0\beta}_{l\bar{b}b}(\beta^2)(-\beta \hat{b})^l$$

+ $$\hat{a} \left( -\beta \hat{b} \right)^l \left( \frac{\hat{b} \hat{b}^\dagger}{\hat{b} \hat{b}^\dagger + 1} \right) L^{0\beta}_{l\bar{b}b}(\beta^2),$$

(9)

for positive values of $l$. Here the function $L^\alpha_{n\beta}(x)$ stands for generalized Laguerre polynomials [17]. For the sake of simplicity, we will restrict ourselves to this example. Then, under the condition $d \ll \omega_m$, if we choose the refractive indices in the lattice to fulfill $\delta = 0$, the horizontal and vertical modes of the 2D nearest-neighbour-coupled lattice should remain effectively uncoupled, as strange as it may sound, due to the quantum analog,

$$\hat{H}_0 = \hat{H}_U(l = 0),$$

(10)

$$= \frac{g^2}{\omega_m}(\hat{a}^\dagger \hat{a})^2 - d\beta e^{-\frac{l^2}{2}}(\hat{a} + \hat{a}^\dagger),$$

(11)

that approximately describes the dynamics in the photonic lattice. If we choose refractive indices that lead to $\delta = \omega_{lm}$, then we have an approximate effective Hamiltonian,

$$\hat{H}_I = \hat{H}_U(l = 1),$$

(12)

$$= \frac{g^2}{\omega_m}(\hat{a}^\dagger \hat{a})^2 - d\beta e^{-\frac{l^2}{2}}(\hat{a}^3 + \hat{a}^\dagger),$$

(13)

which tells us that $E_{j,k}$ will effectively couple to $E_{j\pm 1,k \pm 2}$; i.e., the nearest-neighbour lattice has been engineered to behave as a diagonal coupled lattice. If we choose a parameter $\delta = 2\omega_{lm}$, then the effective Hamiltonian is

$$\hat{H}_2 = \hat{H}_U(l = 2),$$

(14)

$$= \frac{g^2}{\omega_m}(\hat{a}^\dagger \hat{a})^2 - d\beta e^{-\frac{l^2}{2}}(\hat{a}^4 + \hat{a}^\dagger),$$

(15)

and the nearest-neighbour coupling lattice will behave as if the second-nearest diagonal neighbours were coupled; i.e., $E_{j,k}$ will effectively couple to $E_{j \pm 1,k \pm 2}$. Figure 1 shows a diagram exemplifying these cases. Note that by choosing $g \ll \omega_m$ and staying close to the first waveguides, we could neglect the importance of the quadratic term $(\hat{a}^\dagger \hat{a})^2$, and that

\footnote{Note that the subindex of the Laguerre polynomials depends on the operator $\hat{b} \hat{b}^\dagger$. It makes sense when it is applied to a number state (any initial state may be developed in number states). The same applies for the factorial number operator.}
experimental considerations should be taken into account for the definition of the parameter ranges, as stronger couplings for waveguides far from the zeroth waveguide will induce second- and higher-neighbour couplings.

The impulse function giving the field amplitude at the \((p,q)\)th waveguide for an input at the \((r, s)\)th waveguide is given by

\[
I_{(p,q),(r,s)} = \sum_{j=0}^{\infty} e^{i (\delta p + \alpha r)} \langle q | D \left( -\frac{g p}{o_m} \right) | j \rangle \langle j | D \left( \frac{g r}{o_m} \right) | s \rangle, \quad p \neq r,
\]

\[
\times \langle q | D \left( -\frac{g p}{o_m} \right) | l \rangle \langle l | D \left( \frac{g r}{o_m} \right) | s \rangle, \quad p = r,
\]

where we can recover the terms given by

\[
\langle j | \hat{D} (\beta) | k \rangle = e^{-\frac{\beta^2}{2}} \sqrt{\frac{k!}{j!}} \beta^{j-k} L^j_k \left( \beta^2 \right),
\]

\[
e^{\frac{\beta^2}{2}} \sqrt{\frac{j!}{k!}} (-\beta)^{k-j} L^j_k \left( \beta^2 \right) \langle j \rangle \langle k \rangle,
\]

via reference [18] where \(L^j_k (x)\) stands for the generalized Laguerre polynomials [17]. This impulse function is just the impulse function of a series of isolated Glauber–Fock oscillator times that comprise a phase factor for each column of the array. We already know from the harmonic oscillator analog of the Glauber–Fock oscillator lattice that the oscillation frequency between different single-waveguide inputs is the same [19].

Thus, if we introduce a horizontal light line, the propagating field in each excited waveguide should oscillate back to the exact state at almost the same time. Figure 2 shows the exact numerical propagation in a finite nearest-neighbour-coupled lattice composed by 10 000 waveguides in a 100 \(\times\) 100 array described by the differential set (1). The lattice size is adequate to keep the propagating field away from the end boundary, and the parameters are chosen to fulfill \(\delta = 0, \alpha = 0.01 \omega_0\). The initial field distribution, \(\delta_{j,k} = \delta_{j,10} e^{-i(\alpha^2/2) \gamma (j) \alpha^2/2}\), corresponds to a quantum analog where the mechanical oscillator is in the tenth Fock state and the field oscillator is in a coherent state, \((10, \alpha)\), with the coherent parameter \(\alpha = \sqrt{10}\). This initial field distribution looks like a horizontal light line (see figure 2(b)) impinging the \(j = 10\) row of waveguides with a Gaussian intensity distribution centered at the \((j = 10, k = 10)\) waveguide.

Figure 2(a) shows the fidelity between the original input field and the propagating field, \(F (z) = |E(0) \cdot E(z)|^2\), where the field amplitudes vector is defined as \(E = \{\delta_{0,0}, \delta_{0,1}, \ldots, \delta_{1,0}, \delta_{1,1}, \ldots\}\). The fidelity is equal to one when the propagated state is identical to the initial state. Figures 2(b) to 2(g) show the field intensity in the two-dimensional lattice at positions \(z = 0\), figure 2(b), \(z = \pi/2\), figure 2(c), \(z = \pi\), figure 2(d), \(z = 3\pi/2\), figure 2(e), \(z = 2\pi\), figure 2(f), and \(z = 4\pi\), figure 2(g). We can see from the fidelity, figure 2(a), that the initial state is recovered after a propagation distance of \(z = 2\pi\). Also, the light intensity at each waveguide shows us that each set of waveguide columns behave as an oscillator and the initial field, figure 2(b), performs a periodic oscillation and returns to its original state at the propagation distance \(z = 2\pi\), figure 2(g), as expected from the approximate impulse function in (17).

Of course, we are only engineering effective column isolation in the array of nearest-neighbour-coupled waveguides. Thus, light will not be perfectly isolated to just a single waveguide column in the 2D waveguide array. Figure 3 shows the propagation of an initial field distribution \(\epsilon_{j,k} = \delta_{j,10} e^{-i(\alpha^2/2) \gamma (j) \alpha^2/2}\) that corresponds to the mechanical oscillator in a coherent state and to the field in a Fock state, \((10, \alpha)\), with coherent parameter \(\alpha = \sqrt{10}\), and looks like an initial vertical light line impinging at the \(k = 10\) column of waveguides with a Gaussian distribution centered at the \((j = 10, k = 10)\) waveguide. If true column oscillation were achieved by the optical simulator, the fidelity should remain constant, but we see that, in reality, it oscillates with an almost complete reconstruction of the initial light.

---

**Figure 1.** A schematic showing the (a) nearest-neighbour-couplings optical lattice described by (1) and the non-trivial effective couplings engineered by choosing parameters \(d \ll o_m\) and \(\delta = lo_m\) with (b) \(l = 0\), (c) \(l = 1\) and (d) \(l = 2\).
distribution at $z = 2\pi$, figure 3(a). As we are only engineering effective column isolation, a negligible portion of the light leaks to the adjacent waveguide columns, as can be seen in figure 3(c), which shows the light intensity in the waveguide array after a propagation distance of $z = 2\pi$.

3. Conclusions

In summary, we have shown that analogies between optical and quantum systems may help the optical designer even in those cases where an exact solution cannot be procured. We chose as an example a 2D array of nearest-neighbour-coupled waveguides that is the optical analog of a quantum opto-mechanical system and use a trapped-ion cavity quantum electrodynamics method to approximate the dynamics in a given regime. Such an approach allows us to engineer certain non-trivial effective couplings between waveguides that are otherwise experimentally unfeasible; e.g. uncoupling between columns or effective diagonal coupling. For example, we calculate an approximate impulse function for the nearest-neighbour photonic lattice with a parameter set that produces effective column isolation; i.e., each and every column of the array behaves as an independent Glauber–Fock oscillator. Finally, we provide a numerically exact propagation in the original 2D nearest-neighbour lattice that supports our prediction.

References

[1] Man’ko M A, Man’ko V I and Vilela Mendes R 2001 Phys. Lett. A 288 132–8
[2] Longhi S 2011 Appl. Phys. B 104 453–68
[3] Spreeuw R J C 2001 Phys. Rev. A 63 062302
[4] El-Ganainy R, Eisfeld A, Levy M and Christodoulides D N 2013 Appl. Phys. Lett. 103 161105
[5] Rodríguez-Lara B M, Moya-Cessa H M and Christodoulides D N 2014 Phys. Rev. A 89 013802
[6] Perez-Leija A, Keil R, Kay A, Moya-Cessa H, Nolte S, Kwek L-C, Rodríguez-Lara B M, Szameit A and Christodoulides D N 2013 Phys. Rev. A 87 012309
[7] Rodriguez-Lara B M, Soto-Eguibar F and Christodoulides D N 2015 Quantum optics as a tool for photonic lattice design Phys. Scr. 90 074004
[8] Perez-Leija A, Keil R, Szameit A, Abouraddy A F, Moya-Cessa H and Christodoulides D N 2012 Phys. Rev. A 85 013848
[9] Mancini S, Man’ko V I and Tombesi P 1997 Phys. Rev. A 55 3042–50
[10] Pace A F, Collett M J and Walls D F 1993 Phys. Rev. A 47 3173–89
[11] Xu G F and Law C K 2013 Phys. Rev. A 87 053849
[12] Xu X W, Wang H, Zhang J and Liu Y X 2013 Phys. Rev. A 88 063819
[13] Moya-Cessa H, Soto-Eguíbar F, Vargas-Martínez J M, Juárez-Amaro R and Zúñiga-Segundo A 2012 Phys. Rep. 513 229–61
[14] Pertsch T, Dannberg P, Elflein W, Bräuer A and Lederer F 1999 Phys. Rev. Lett. 83 4752–5
[15] Morandotti R, Peschel U, Aitchison J S, Eisenberg H S and Silberberg Y 1999 Phys. Rev. Lett. 83 4756–9
[16] Dreisow F, Szameit A, Heinrich M, Pertsch T, Nolte S, Tunnermann A and Longhi S 2009 Phys. Rev. Lett. 102 076802
[17] Edélyi A 1953 Higher Transcendental Functions vol 1 (New York: McGraw-Hill)
[18] Wünsche A 1991 Quantum Opt. 3 359–83
[19] Rodríguez-Lara B M, Aleahmad P, Moya-Cessa H M and Christodoulides D N 2014 Opt. Lett. 39 2083–5