THE q–CHARACTERS AT ROOTS OF UNITY

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ABSTRACT. We consider various specializations of the untwisted quantum affine algebras at roots of unity. We define and study the q–characters of their finite-dimensional representations.

1. Introduction

The theory of q–characters of finite-dimensional representations of quantum affine algebras was developed in [FR, FM]. In those works, q was assumed to be a generic non-zero complex number (i.e., not a root of unity). In the present paper we extend the results of [FR, FM] to the case when q is a root of unity.

There are different versions of the quantum affine algebra $U_q \hat{g}$ when q is specialized to a root of unity $\epsilon$: the non-restricted specialization $\tilde{U}_\epsilon \hat{g}$ studied by Beck and Kac [BK], the restricted specialization $U_{\epsilon}^{\text{res}} \hat{g}$ studied by Chari and Pressley [CP4] (following the general definition due to Lusztig [L]), and the “small” affine quantum group $U_{\epsilon}^{\text{fin}} \hat{g}$, which is the image of the natural homomorphism $\tilde{U}_\epsilon \hat{g} \to U_{\epsilon}^{\text{res}} \hat{g}$.

It was shown in [CP4] that all irreducible finite-dimensional representations of $U_{\epsilon}^{\text{res}} \hat{g}$ (of type 1) are highest weight representations with respect to a triangular decomposition in terms of the Drinfeld generators (more precisely, these results were obtained in [CP4] in the case when $\epsilon$ is a root of unity of odd order, but we extend them here to all roots of unity). Using these results, we define the $\epsilon$–character of a $U_{\epsilon}^{\text{res}} \hat{g}$–module $V$ via the generalized eigenvalues of a commutative subalgebra of $U_{\epsilon}^{\text{res}} \hat{g}$ on $V$. We establish various properties of the $\epsilon$–character homomorphism; in particular, we show that the q–characters specialize to $\epsilon$–characters as $q \to \epsilon$.

In addition, using results of [BK] we show that all irreducible finite-dimensional representations of $U_{\epsilon}^{\text{fin}} \hat{g}$ (of type 1) are highest weight representations, describe their highest weights and define the corresponding $\epsilon$–characters. We show that each irreducible $U_{\epsilon}^{\text{fin}} \hat{g}$–module admits a unique structure of $U_{\epsilon}^{\text{res}} \hat{g}$–module. This allows us to obtain the $\epsilon$–characters of $U_{\epsilon}^{\text{fin}} \hat{g}$–modules from the $\epsilon$–characters of $U_{\epsilon}^{\text{res}} \hat{g}$–modules.

Finally, we study the quantum Frobenius homomorphism, following Lusztig’s definition [L]. Let $l$ be the order of $\epsilon^2$ and $\epsilon^* = \epsilon^l \in \{\pm 1\}$. The Frobenius homomorphism maps $U_{\epsilon}^{\text{res}} \hat{g}$ (more precisely, its modified version) either to $U_{\epsilon}^{\text{res}} \hat{g}$ or to $U_{\epsilon}^{\text{res}} \hat{g}^L$, depending on whether $l$ is divisible by the lacing number $r^\vee$ or not. Here $\hat{g}^L$ denotes the Langlands dual Lie algebra to $\hat{g}$ (it is twisted if $\hat{g}$ is not simply-laced) and we use the notation $\text{Rep} U$ for the Grothendieck ring of finite-dimensional representations of an algebra $U$. In this paper we deal only with the untwisted quantum affine algebras, and so we consider the Frobenius homomorphism in detail only in the case when $(l, r^\vee) = 1.$
The Frobenius pull-backs of irreducible finite-dimensional $U^{\text{res}}_\epsilon \hat{g}$-modules are reducible $U^{\text{res}}_\epsilon \hat{g}$-modules. We describe their $\epsilon$-characters in terms of the ordinary characters of irreducible finite-dimensional representations of the simple Lie algebra $g$.

A decomposition theorem (previously proved by Chari and Pressley [CP4] in the case when $\epsilon$ has odd order) allows one to decompose any irreducible finite-dimensional representation of $U^{\text{res}}_\epsilon \hat{g}$ as a tensor product of a Frobenius pull-back and a module which remains irreducible when restricted to $U^{\text{fin}}_\epsilon \hat{g} \subset U^{\text{res}}_\epsilon \hat{g}$. Thus, we obtain an isomorphism of vector spaces

$$\text{Rep} U^{\text{res}}_\epsilon \hat{g} \simeq \text{Rep} U^{\text{fin}}_\epsilon \hat{g} \otimes \text{Rep} U^{\text{res}}_\epsilon \hat{g}.$$ 

Similar results can be obtained for the twisted quantum affine algebras. We will describe the $q$-characters and $\epsilon$-characters of twisted quantum affine algebras in a separate publication.

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2. Definitions

2.1. **Root data.** Let $g$ be a simple Lie algebra of rank $\ell$. Let $\langle \cdot, \cdot \rangle$ be the invariant inner product on $g$, such that the square of the length of the maximal coroot $\alpha^\vee_{\text{max}}$ equals 2. The induced inner product on the dual space to the Cartan subalgebra $h$ of $g$ is also denoted by $\langle \cdot, \cdot \rangle$. Denote by $I$ the set $\{1, \ldots , \ell\}$. Let $\{\alpha_i\}_{i \in I}$, $\{\alpha_i^\vee\}_{i \in I}$, $\{\omega_i\}_{i \in I}$ be the sets of simple roots, simple coroots and fundamental weights of $g$, respectively. We denote by $\alpha^\vee_{\text{max}}$ the maximal (positive) root of $g$.

Let $r^\vee$ be the maximal number of edges connecting two vertices of the Dynkin diagram of $g$. Thus, $r^\vee = 1$ for simply-laced $g$, $r^\vee = 2$ for $B_\ell, C_\ell, F_4$, and $r^\vee = 3$ for $G_2$.

Set $r_i = \frac{(\alpha_i, \alpha_i)}{2}$. All $r_i$’s are equal to 1 for simply-laced $g$, and $r_i$’s are equal to 1 or $r^\vee$ depending on whether $\alpha_i$ is short or long, for non-simply laced $g$.

Let $C = (C_{ij})_{i,j \in I}$ be the Cartan matrix of $g$, $C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$.

Let $\hat{I} = \{0, 1, \ldots , \ell\}$ and let $(C_{ij})_{i,j \in \hat{I}}$ be the Cartan matrix of $\hat{g}$. We set $r_0 = r^\vee$.

We also fix a choice of a function $o: I \rightarrow \{\pm 1\}$ such that $o(i) \neq o(j)$ whenever $(\alpha_i, \alpha_j) \neq 0$.

2.2. **Quantum affine algebras.** Let $q$ be an indeterminate, $\mathbb{C}(q)$ the field of rational functions of $q$ and $\mathbb{C}[q,q^{-1}]$ the ring of Laurent polynomials with complex coefficients. For $n \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq 0}$ set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \left[ \begin{array}{c} n \\ m \end{array} \right]_q = \prod_{i=m+1}^{n} [i]_q \prod_{i=1}^{m} [i]_q^{-1}, \quad [m]_q! = \prod_{i=1}^{m} [i]_q.$$

Let $q_i = q^{r_i}$, $i \in \hat{I}$.

In this paper we deal exclusively with finite-dimensional representations of quantum affine algebras, all of which have level zero. The (multiplicative) central element of a quantum affine algebra acts as the identity on such representations. To simplify our
Here we use the following notation for divided powers formulas, we will impose in the definition of the quantum affine algebra the additional relation that this central element is equal to 1.

The quantum affine algebra $U_q\hat{g}$ (of level zero) in the Drinfeld-Jimbo realization \cite{Dr} is an associative algebra over $\mathbb{C}(q)$ with generators $x_i^\pm$ ($i \in \hat{I}$), $k_i^{\pm 1}$ ($i \in I$), and relations:

\begin{align*}
k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \\
k_i x_j^\pm k_i^{-1} &= q_i^{\pm C_{ij}} x_j^\pm, \\
[x_i^+, x_j^-] &= \delta_{ij} k_i - k_i^{-1} \frac{q_i - q_i^{-1}}{q_i - q_i^{-1}}, \\
\sum_{r=0}^{1-C_{ij}} (-1)^r \left[1 - \frac{C_{ij}}{r}ight] q_i (x_i^+)^r x_j^\mp (x_i^\pm)^{1-C_{ij}-r} &= 0 \quad (i \neq j).
\end{align*}

Here we use the notation $k_0 = \prod_{i \in I} k_i^{-a_i}$, where $a_{\max} = \sum_{i \in I} a_i a_i$.

The algebra $U_q\hat{g}$ has a structure of a Hopf algebra with the comultiplication $\Delta$ and the antipode $S$ given on the generators by the formulas:

\begin{align*}
\Delta(k_i) &= k_i \otimes k_i, \\
\Delta(x_i^+) &= x_i^+ \otimes 1 + k_i \otimes x_i^+, \\
\Delta(x_i^-) &= x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-, \\
S(x_i^+) &= -x_i^+ k_i, \quad S(x_i^-) = -k_i^{-1} x_i^-, \quad S(k_i^{\pm 1}) = k_i^{\mp 1}.
\end{align*}

The Hopf algebra $U_q\hat{g}$ is defined as the $\mathbb{C}(q)$–subalgebra of $U_q\hat{g}$ with generators $x_i^\pm$, $k_i^{\pm 1}$, where $i \in \hat{I}$.

Let $\sigma$ be an automorphism of the affine Dynkin diagram, i.e., $\sigma : \hat{I} \rightarrow \hat{I}$, such that $(\alpha_i, \alpha_j) = (\sigma \alpha_i, \sigma \alpha_j)$. Then we also denote by $\sigma$ an $U_q\hat{g}$ automorphism defined by the formulas

\begin{equation}
\sigma(x_i^\pm) = x_{\sigma(i)}^\pm, \quad \sigma(k_i) = k_{\sigma(i)} \quad (i \in \hat{I}).
\end{equation}

Let $T_i$, $i \in \hat{I}$, be the $\mathbb{C}(q)$–algebra automorphisms of $U_q\hat{g}$, defined by the formulas

\begin{align*}
T_i(k_j) &= k_i^{-\alpha_{ij}} k_j, \\
T_i((x_i^+)^{(n)}) &= (-1)^n q_i^{-n(n-1)} (x_i^-)^{(n)} k_i^n, \quad T_i((x_i^-)^{(n)}) = (-1)^n q_i^{n(n-1)} k_i^{-n} (x_i^+)^{(n)}, \\
T_i((x_i^+)^{(n)}) &= \sum_{r=0}^{-n\alpha_{ij}} (-1)^r q_i^{-r} (x_i^+)^{(-n\alpha_{ij} - r)} (x_i^+)^{(n)} (x_i^+)^{(r)} \quad (i \neq j), \\
T_i((x_i^-)^{(n)}) &= \sum_{r=0}^{-n\alpha_{ij}} (-1)^r q_i^r (x_i^-)^{(r)} (x_i^-)^{(n)} (x_i^-)^{(-n\alpha_{ij} - r)} \quad (i \neq j).
\end{align*}

Here we use the following notation for divided powers

\begin{equation}
(x_i^+)^r = \frac{(x_i^+)^r}{[r]_{q_i}},
\end{equation}
If \( i \in I \) then \( T_i \) induces an \( \mathbb{C}(q) \)-automorphism of subalgebra \( U_q \mathfrak{g} \) of \( \hat{U}_q \mathfrak{g} \), also denoted by \( T_i \). The automorphisms \( T_i \) are the operators \( T'_{i,-1}^m \) introduced in \([L], \S 41.1.2\).

Next, we describe the Drinfeld “new” realization of \( \hat{U}_q \mathfrak{g} \).

**Theorem 2.1** ([Dr2, KT, LSS, B]). The algebra \( U_q \mathfrak{g} \) is isomorphic over \( \mathbb{C}(q) \) to the algebra with generators \( x_{i,n}^\pm (i \in I, n \in \mathbb{Z}), k_i^{\pm 1} (i \in I), h_{i,n} (i \in I, n \in \mathbb{Z} \setminus 0) \), with the following relations:

\[
\begin{align*}
  k_i k_j &= k_j k_i, \quad k_i h_{j,n} = h_{j,n} k_i, \\
  k_i x_{j,n}^\pm k_i^{-1} &= q_i^{\pm C_{ij}} x_{j,n}^\pm, \\
  [h_{i,n}, x_{j,m}^\pm] &= \frac{1}{n}[n C_{ij}] q_i x_{j,n+m}^\pm, \\
  x_{i,n+1}^\pm x_{j,m}^\pm - q_i^{\pm C_{ij}} x_{j,m}^\pm x_{i,n+1}^\pm &= q_i^{\pm C_{ij}} x_{i,n}^\pm x_{j,m+1}^\pm - x_{j,m+1}^\pm x_{i,n}^\pm, \\
  [h_{i,n}, h_{j,m}] &= 0, \\
  [x_{i,n}^+, x_{j,m}^-] &= \delta_{ij} \frac{\phi_{i,n+m}^+ - \phi_{i,n+m}^-}{q_i - q_i^{-1}}.
\end{align*}
\]

for all sequences of integers \( n_1, \ldots, n_s \), and \( i \neq j \), where \( \Sigma_s \) is the symmetric group on \( s \) letters, and \( \phi_{i,n}^\pm \)'s are determined by the formula

\[
(2.2) \quad \Phi_i^\pm(u) := \sum_{n=0}^\infty \phi_{i,n}^\pm u^{\pm n} = k_i^{\pm 1} \exp \left( \pm (q_i - q_i^{-1}) \sum_{m=1}^\infty h_{i,m} u^{\pm m} \right).
\]

Let \( U_q \mathfrak{g}^\pm \) be the subalgebras of \( U_q \mathfrak{g} \) generated by \( x_{i,n}^\pm (i \in I, n \in \mathbb{Z}) \). Let \( U_q \mathfrak{g}^0 \) be the subalgebra of \( U_q \mathfrak{g} \) generated by \( k_i^{\pm 1}, h_{i,n}^\pm (i \in I, n \in \mathbb{Z} \setminus 0) \).

In fact, according to \([L]\) (see also \([CP4, BCP]\)), the isomorphism described in Theorem 2.1 has the form

\[
x_{i,n}^\pm = o(i)^r T_{\omega_i}^r(x_{i,n}^\pm),
\]

where \( T_{\omega_i} \) is an automorphism of \( U_q \mathfrak{g} \) given by a certain composition of braid group automorphisms \( T_i \) and a diagram automorphism \( \sigma \) of type \( \circ \). We remark that in \([CP4, BCP]\) a different normalization for the braid group action is used. In this paper we follow the conventions of \([BCP], L]\).

For any \( a \in \mathbb{C}^\times \), there is a Hopf algebra automorphism \( \tau_a \) of \( U_q \mathfrak{g} \) defined on the generators by the following formulas:

\[
(2.3) \quad \tau_a(x_{i,n}^\pm) = a^n x_{i,n}^\pm, \quad \tau_a(\phi_{i,n}^\pm) = a^n \phi_{i,n}^\pm,
\]

for all \( i \in I, n \in \mathbb{Z} \). Given a \( U_q \mathfrak{g} \)-module \( V \) and \( a \in \mathbb{C}^\times \), we denote by \( V(a) \) the pull-back of \( V \) under \( \tau_a \).
2.3. Restricted integral form. Let $U_q^{\text{res}}\widehat{\mathfrak{g}}$ be the $\mathbb{C}[q,q^{-1}]$-subalgebra of $U_q\widehat{\mathfrak{g}}$ generated by $k_i^{\pm 1}$ and $(x_i^\pm)^{(n)}$, $i \in \hat{I}$, $n \in \mathbb{Z}_{>0}$. Then $U_q^{\text{res}}\widehat{\mathfrak{g}}$ is a $\mathbb{C}[q,q^{-1}]$ Hopf subalgebra of $U_q\widehat{\mathfrak{g}}$ preserved by automorphisms $T_i$, $i \in \hat{I}$.

Similarly, let $U_q^{\text{res}}\mathfrak{g}$ be the $\mathbb{C}[q,q^{-1}]$-subalgebra of $U_q\mathfrak{g}$ generated by $k_i^{\pm 1}$ and $(x_i^\pm)^{(n)}$, $i \in I$, $n \in \mathbb{Z}_{>0}$. Then $U_q^{\text{res}}\mathfrak{g}$ is a $\mathbb{C}[q,q^{-1}]$ Hopf subalgebra of $U_q\mathfrak{g}$ preserved by automorphisms $T_i$, $i \in I$.

For $i \in I$, $r \in \mathbb{Z}_{>0}$, define

$$\left[\frac{k_i}{r}\right] = \prod_{s=1}^{r} \frac{k_i q_i^{1-s} - k_i^{-1} q_i^{s-1}}{q_i^s - q_i^{-s}}.$$  

Note that $\left[\frac{k_i}{r}\right]$ is denoted by $\left[\frac{k_i;0}{r}\right]$ in $[\text{CP}4]$.

For $i \in \hat{I}$, $n \in \mathbb{Z}$, define the elements $P_{i,n} \in U_q\widehat{\mathfrak{g}}$ by

$$P_i^\pm(u) = \sum_{n=0}^{\infty} P_{i,\pm n} u^{\pm n} = \exp \left( \mp \sum_{m=1}^{\infty} \frac{h_i^{\pm m}}{|m| q_i} u^{\pm m} \right).$$

Note that

$$\Phi_i^\pm(u) = k_i^{\pm 1} \frac{P_i^\pm(uq_i^{-1})}{\Phi_i^\pm(uq_i)}.$$

Denote by $U_q^{\text{res}}\widehat{\mathfrak{g}}^\pm$ the $\mathbb{C}[q,q^{-1}]$-subalgebra of $U_q\widehat{\mathfrak{g}}$ generated by $(x_i^\pm)^{(n)}$, $i \in I$, $r \in \mathbb{Z}$, $n \in \mathbb{Z}_{>0}$. Denote $U_q^{\text{res}}\mathfrak{g}$ the $\mathbb{C}[q,q^{-1}]$-subalgebra of $U_q\mathfrak{g}$ generated by $k_i$, $\left[\frac{k_i}{r}\right]$, $i \in \hat{I}$ and $P_{i,n}$, $i \in I$, $n \in \mathbb{Z}$, $r \in \mathbb{Z}_{>0}$.

**Theorem 2.2 ([CP4], Proposition 6.1).** We have $U_q^{\text{res}}\widehat{\mathfrak{g}}^\pm \subset U_q^{\text{res}}\widehat{\mathfrak{g}}$, $U_q^{\text{res}}\mathfrak{g}^0 \subset U_q^{\text{res}}\mathfrak{g}$. The algebra $U_q^{\text{res}}\widehat{\mathfrak{g}}$ is generated by the subalgebras $U_q^{\text{res}}\mathfrak{g}^+$, $U_q^{\text{res}}\mathfrak{g}^-$ and $k_i^{\pm 1}$, $i \in I$.

Moreover, the multiplication gives an isomorphism of vector spaces

$$U_q^{\text{res}}\widehat{\mathfrak{g}}^+ \cong U_q^{\text{res}}\mathfrak{g}^+ \otimes U_q^{\text{res}}\mathfrak{g}^0 \otimes U_q^{\text{res}}\mathfrak{g}^-.$$  

(2.4)

Let $\epsilon \in \mathbb{C}$ be a primitive $s$-th root of unity, i.e., $\epsilon^s = 1$ and $\epsilon^k \neq 1$ for $k = 1, 2, \ldots, s-1$.

For $i \in \hat{I}$, let $\epsilon_i = \epsilon^{r_i}$ and let

$$l = \begin{cases} s, & s \text{ is odd} \\ s/2, & s \text{ is even} \end{cases} \quad \text{and} \quad l_i = \begin{cases} l, & l \text{ is not divisible by } r_i \\ s/r_i, & l \text{ is divisible by } r_i. \end{cases}$$

In other words, $l$ is the order of $\epsilon^2$, and $l_i$ is the order of $\epsilon_i^2$. Thus, $l$ is the minimal natural number with the property $[l]_\epsilon = 0$ and $l_i$ is the minimal natural number with the property $[l_i]_{\epsilon_i} = 0$.

Define the structure of a module over the ring $\mathbb{C}[q,q^{-1}]$ on $\mathbb{C}$ by the formula

$$p(q,q^{-1}) \mapsto p(\epsilon, \epsilon^{-1}), \quad p(q,q^{-1}) \in \mathbb{C}[q,q^{-1}].$$

Denote this module by $\mathbb{C}_\epsilon$. Set

$$U_{\epsilon}^{\text{res}}\widehat{\mathfrak{g}} = U_q^{\text{res}}\widehat{\mathfrak{g}} \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}_\epsilon,$$

$$U_{\epsilon}^{\text{res}}\mathfrak{g} = U_q^{\text{res}}\mathfrak{g} \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}_\epsilon.$$
Also set
\[
U^\text{res}_\epsilon \tilde{\mathfrak{g}}^\pm = U^\text{res}_\epsilon \tilde{\mathfrak{g}}^0 \otimes \mathbb{C}[q, q^{-1}] \mathbb{C}_\epsilon \subset U^\text{res}_\epsilon \tilde{\mathfrak{g}},
\]
\[
U^\text{res}_\epsilon \mathfrak{g} = U^\text{res}_\epsilon \tilde{\mathfrak{g}} \otimes \mathbb{C}[q, q^{-1}] \mathbb{C}_\epsilon \subset U^\text{res}_\epsilon \tilde{\mathfrak{g}}.
\]

The algebra \(U^\text{res}_\epsilon \tilde{\mathfrak{g}}\) is called the restricted specialization of \(U_q \tilde{\mathfrak{g}}\) at \(q = \epsilon\).

### 2.4. Non-restricted integral form and small affine quantum group.

Let \(\tilde{U}_q \tilde{\mathfrak{g}}\) be the \(\mathbb{C}[q, q^{-1}]\)-subalgebra of \(U_q \tilde{\mathfrak{g}}\) generated by \(k_i^\pm, i \in \hat{I}\). This is a \(\mathbb{C}[q, q^{-1}]\)-Hopf subalgebra of \(U_q \tilde{\mathfrak{g}}\) preserved by automorphisms \(T_i, i \in \hat{I}\). Using Theorem 2.1 we obtain that \(\tilde{U}_q \tilde{\mathfrak{g}}\) may also be described as the \(\mathbb{C}[q, q^{-1}]\)-subalgebra of \(U_q \tilde{\mathfrak{g}}\) generated by \(k_i^\pm, x_{i,n}^\pm (i \in \hat{I}, n \in \mathbb{Z})\), \(h_{i,n} (i \in \hat{I}, n \in \mathbb{Z}
\}(0)\). Note that \(\tilde{U}_q \tilde{\mathfrak{g}}\) is a \(\mathbb{C}[q, q^{-1}]\)-subalgebra of \(U^\text{res}_\epsilon \tilde{\mathfrak{g}}\).

Define the non-restricted specialization of \(\tilde{U}_q \tilde{\mathfrak{g}}\) at \(q = \epsilon\) by
\[
\tilde{U}_\epsilon \tilde{\mathfrak{g}} := \tilde{U}_q \tilde{\mathfrak{g}} \otimes \mathbb{C}[q, q^{-1}] \mathbb{C}_\epsilon.
\]

We have a natural homomorphism \(\tilde{U}_\epsilon \tilde{\mathfrak{g}} \to U^\text{res}_\epsilon \tilde{\mathfrak{g}}\). Denote by \(U^\text{fin}_\epsilon \tilde{\mathfrak{g}}\) the image of this homomorphism. We call \(U^\text{fin}_\epsilon \tilde{\mathfrak{g}}\) the small affine quantum group of \(\tilde{\mathfrak{g}}\) at \(q = \epsilon\). This is the subalgebra of \(U^\text{res}_\epsilon \tilde{\mathfrak{g}}\) generated by \(k_i^\pm\) and \(x_{i,n}^\pm, i \in \hat{I}\).

We remark that Chari and Pressley use in [CP4] the notation \(U^\text{fin}_\epsilon \tilde{\mathfrak{g}}\) for a larger subalgebra of \(U^\text{res}_\epsilon \tilde{\mathfrak{g}}\), which includes \(P_{i,n}, i \in \hat{I}, n \in \mathbb{Z}\) in addition to the above generators.

Note that if \(l = 1\), we have: \(\tilde{U}_1 \tilde{\mathfrak{g}} = U^\text{fin}_1 \tilde{\mathfrak{g}} = U^\text{res}_1 \tilde{\mathfrak{g}}\). Hence from now on, when dealing with \(\tilde{U}_\epsilon \tilde{\mathfrak{g}}\) or \(U^\text{fin}_\epsilon \tilde{\mathfrak{g}}\), we will always assume that \(l > 1\).

**Lemma 2.3.** For \(l > 1\), the following relations hold in \(U^\text{fin}_\epsilon \tilde{\mathfrak{g}}\): \((x_i^\pm)^l_i = 0 (i \in \hat{I})\), \((x_{i,n}^\pm)^l_i = 0 (i \in \hat{I}, n \in \mathbb{Z})\), \(h_{i,m_l}/[m_l]_{\tilde{\mathfrak{g}}} = 0 (i \in \hat{I}, n \in \mathbb{Z}\{0\})\).

**Proof.** This follows from the fact that the elements \((x_i^\pm)^l_i\), \((x_{i,n}^\pm)^l_i\), and \(h_{i,m_l}/[m_l]_{\tilde{\mathfrak{g}}}\), belong to \(U^\text{res}_\epsilon \tilde{\mathfrak{g}}\). \(\square\)

In particular, \(U^\text{fin}_\epsilon \tilde{\mathfrak{g}}\) is generated by \(k_i^\pm\), \(x_{i,n}^\pm (i \in \hat{I}, n \in \mathbb{Z})\), and \(h_{i,n} (i \in \hat{I}, n \in \mathbb{Z}\{0\})\).

### 2.5. Finite-dimensional representations of \(U^\text{res}_\epsilon \tilde{\mathfrak{g}}\).

A finite-dimensional representation of \(U_q \tilde{\mathfrak{g}}\) is said to be of type 1 if \(k_i, i \in \hat{I}\), act by semi-simple operators with eigenvalues in \(q_{\mathbb{Z}}\). A finite-dimensional representation \(V\) of \(U^\text{res}_\epsilon \tilde{\mathfrak{g}}\) is said to be of type 1 if \(V\) has a basis of eigenvectors of the elements \(k_i\). Hence the Grothendieck rings of the type
1 finite-dimensional representations of $U_q\hat{\mathfrak{g}}$ and $U^{\text{res}}_\epsilon\hat{\mathfrak{g}}$ carry ring structures. We denote these rings by $\text{Rep}U_q\hat{\mathfrak{g}}$ and $\text{Rep}U^{\text{res}}_\epsilon\hat{\mathfrak{g}}$, respectively.

A vector $v$ in a $U_q\hat{\mathfrak{g}}$-module (resp., $U^{\text{res}}_\epsilon\hat{\mathfrak{g}}$-module) $V$ is called a highest weight vector if $U_q\hat{\mathfrak{g}}^+v = 0$ and $U_q\hat{\mathfrak{g}}\cdot v = \mathbb{C}(q)\cdot v$ (resp., $U^{\text{res}}_\epsilon\hat{\mathfrak{g}}^+v = 0$ and $U^{\text{res}}_\epsilon\hat{\mathfrak{g}}\cdot v = \mathbb{C}\cdot v$). If in addition $V = U_q\hat{\mathfrak{g}}\cdot v$ (resp., $V = U^{\text{res}}_\epsilon\hat{\mathfrak{g}}\cdot v$), then $V$ is called a highest weight representation.

**Theorem 2.4.** Every irreducible finite-dimensional representation of $U^{\text{res}}_\epsilon\hat{\mathfrak{g}}$ is a highest weight representation. An irreducible highest weight representation $V$ with highest weight vector $v$ is finite-dimensional if and only if there exists an $I$-tuple of polynomials with constant term 1,

$$P = (P_i(u))_{i \in I}, \quad P_i(u) = \prod_{j=1}^{j_i} (1 - a_{ij}u), \quad a_{ij} \in \mathbb{C},$$

such that

$$k_i v = \epsilon_i^{\deg P_i} v, \quad \begin{bmatrix} k_i \\ l_i \end{bmatrix} v = \begin{bmatrix} \deg P_i \\ l_i \end{bmatrix}_\epsilon v,$$

$$P_i^+(u) v = P_i(u) v, \quad P_i^-(u) v = P_i(u) v,$$

where $P_i(u) = \prod_{j=1}^{j_i} (1 - (a_{ij}u)^{-1})$.

**Proof.** In [CP4], Proposition 8.1 and Theorem 8.2, the theorem is proved when $s$ is odd and $(s, r') = 1$, using the triangular decomposition (2.4) for $U^{\text{res}}_\epsilon\hat{\mathfrak{g}}$. The same proof applies in general as the decomposition (2.4) does not depend on the choice of $\epsilon$. \qed

The $I$-tuple of polynomials $P$ is called the highest weight of $V$. We write $V = V(P)$ for the irreducible representation $V$ with highest weight $P$. We refer to the polynomials $P_i(u)$ as Drinfeld polynomials.

There is an analogous description of irreducible finite-dimensional $U_q\hat{\mathfrak{g}}$-modules given as highest weight representations given in [CP2, CP3]. Namely, irreducible $U_q\hat{\mathfrak{g}}$-modules are also classified by highest weights $P$, which are $I$-tuples of polynomials in $\mathbb{C}(q)[u]$ with constant coefficient 1. We denote the irreducible $U_q\hat{\mathfrak{g}}$-module with highest weight $P$ by $V(P)_q$.

Let $\Lambda = \text{span}_\mathbb{Z} \{\omega_1, \ldots, \omega_l\}$ be the weight lattice of $\mathfrak{g}$. Recall that $\Lambda$ carries a partial order. Namely, for $\lambda, \lambda' \in \Lambda$, we say that $\lambda \geq \lambda'$ if $\lambda = \lambda' + \sum_{i \in I} n_i \alpha_i$, where $n_i \in \mathbb{Z}_{\geq 0}$. This induces a partial order on the set of $I$-tuples of polynomials: we say that $P \geq Q$ if $\sum_{i \in I} \omega_i \deg P_i \geq \sum_{i \in I} \omega_i \deg Q_i$.

The next proposition shows that all irreducible finite-dimensional representations of $U^{\text{res}}_\epsilon\hat{\mathfrak{g}}$ can be obtained as subquotients of specializations of $U_q\hat{\mathfrak{g}}$-modules, and moreover, the corresponding decomposition matrix is triangular with respect to the above partial order.

**Proposition 2.5.** Let $V(P)_q$ be an irreducible highest weight $U_q\hat{\mathfrak{g}}$-module of dimension $d$ with highest vector $v$ and highest weight $P$, such that $P_i(u) \in \mathbb{C}[q, q^{-1}, u]$ ($i \in I$). Then $V(P)_q^{\text{res}} := U^{\text{res}}_q\hat{\mathfrak{g}}\cdot v$ is a free $\mathbb{C}[q, q^{-1}]$-module of rank $d$, and

$$V(P)_q^{\text{res}} := V(P)_q^{\text{res}} \otimes_{\mathbb{C}[q, q^{-1}]} \mathbb{C}_\epsilon.$$
is a $U^\text{res}_\epsilon\widehat{\mathfrak{g}}$–module of dimension $d$. Moreover, the map
\[ e : \text{Rep} U_q\widehat{\mathfrak{g}} \to \text{Rep} U^\text{res}_\epsilon\widehat{\mathfrak{g}}, \]
\[ [V(P)_q] \to [V(P)_\epsilon^\text{res}] \]
is a surjective ring homomorphism, and
\[
(V(P)_\epsilon^\text{res}) = [V(P)] + \sum_{Q < P, Q \neq \epsilon} m_{Q,P}[V(Q)], \quad m_{Q,P} \in \mathbb{Z}_{\geq 0},
\]
where $P_\epsilon$ is obtained from $P$ by substituting $q = \epsilon$.

Proof. The module $V(P)_q^\text{res}$ is a free $\mathbb{C}[q, q^{-1}]$–module of rank $d$ by Lemma 4.6 i) in [CP5].

Since $V^\text{res}_q\widehat{\mathfrak{g}}$ is a Hopf subalgebra of $U_q\widehat{\mathfrak{g}}$, the map $e$ is a homomorphism of rings. Since the degrees of the polynomials $P_i(u)$ determine the highest weight of $V(P)_q$, considered as a $U_q\widehat{\mathfrak{g}}$–module, we obtain that any $V^\text{res}_\epsilon\widehat{\mathfrak{g}}$–subquotient occurring in $[V(P)_\epsilon^\text{res}]$ has highest weight less than or equal to $P$ with respect to our partial order. This proves the surjectivity of $e$ and formula (2.5). □

Given an irreducible $U^\text{res}_\epsilon\widehat{\mathfrak{g}}$–module $V$, we call the $U^\text{res}_\epsilon\widehat{\mathfrak{g}}$–module $V^\text{res}_\epsilon$ the specialization of the module $V$.

2.6. Finite-dimensional representations of $U^\text{fin}_\epsilon\widehat{\mathfrak{g}}$. A finite-dimensional representation $V$ of $U^\text{fin}_\epsilon\widehat{\mathfrak{g}}$ is said to be of type 1 if $k_i, i \in I$, act on $V$ semi-simply with eigenvalues in $\mathbb{C}$.

A vector $v$ in a $U^\text{fin}_\epsilon\widehat{\mathfrak{g}}$–module $V$ is called a highest weight vector if $x_i^+v = 0$ for all $i \in I, n \in \mathbb{Z}$, and
\[
(2.6) \quad \Phi_i^\pm(u) = \Psi_i^\pm(u)v, \quad \Psi_i^\pm(u) \in \mathbb{C}[[u^{\pm 1}]] \quad (i \in I).
\]

If in addition $V = U^\text{fin}_\epsilon\widehat{\mathfrak{g}} \cdot v$, then $V$ is called a highest weight representation with highest weight $(\Psi_i^\pm(u))_{i \in I}$. Introduce the notation $\Psi^\pm_i(u) = \Psi_i^+(u)/\Psi_i^+(0), \Psi^-_i(u) = \Psi^-_i(u)/\Psi^-_i(\infty)$ (these are the eigenvalues of $k_i^{\pm 1}\Phi_i^\pm(u)$ on $v$). By Lemma 2.3, the elements $h_i, ml_i$ act on $V$ by 0. Therefore $\Psi_i^\pm(u)$ necessarily satisfies the property
\[
\prod_{j=0}^{l_i-1} \Psi_i^{\pm}(ue^{2j}) = 1 \quad (i \in I), \tag{2.7}
\]
cf. Section 6.5 of [BK]. Note that the commuting elements $h_{i,n}, n \neq ml_i$, are algebraically independent in $U^\text{res}_\epsilon\widehat{\mathfrak{g}}$, and hence in $U^\text{fin}_\epsilon\widehat{\mathfrak{g}}$. Therefore for any choice of $(\Psi_i^\pm(u))_{i \in I}$ satisfying the condition (2.7), there exists a unique irreducible highest weight representation with highest weight $(\Psi_i^\pm(u))_{i \in I}$.

We will say that a polynomial $P(u) \in \mathbb{C}[u]$ is $l$–acyclic if it is not divisible by $(1 - au^l)$ (equivalently, the set of roots of $P(u)$ does not contain a subset of the form \( \{a, ae^2, \ldots, ae^{2l-2}\}, a \in \mathbb{C}^\times \), where $e^2$ is a primitive root of unity of order $l$).

The following statement essentially follows from the results of Beck and Kac [BK].
Theorem 2.6. Every irreducible finite-dimensional type 1 representation of $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$ is a highest weight representation. An irreducible highest weight representation $V$ with highest weight $(\Psi_{i}^{\pm}(u))_{i \in I}$ is finite-dimensional if and only if there exists an $I$–tuple of polynomials with constant term 1, $P = (P_{i}(u))_{i \in I}$, where $P_{i}(u)$ is $l_{i}$–acyclic, such that

$$\Psi_{i}^{\pm}(u) = \gamma_{i} \epsilon_{i}^{\deg P_{i}(ue_{i}^{-1})} P_{i}(ue_{i})^{-1}, \quad i \in I,$$

where $\gamma_{i} = 1$, if $\epsilon_{i}$ has odd order, $\gamma_{i} = \pm 1$, if $\epsilon_{i}$ has even order, and by the rational function appearing in the right hand side we understand its expansion in $u^{\pm 1}$.

Proof. Theorem 6.3 of [BK] describes the so-called diagonal finite-dimensional representations of the non-restricted specialization $\widehat{U}_{e} \mathfrak{g}$ (although $l$ is assumed in [BK] to be odd, the proof of Theorem 6.3 does not depend on this restriction). It states in particular that these representations are highest weight representations. Since we have a surjective homomorphism $\widehat{U}_{e} \mathfrak{g} \to U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$, any irreducible $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$–module gives rise to an irreducible $\widehat{U}_{e} \mathfrak{g}$–module. By Lemma 2.8, an irreducible $\widehat{U}_{e} \mathfrak{g}$–module obtained by pull-back from an irreducible $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$–module is diagonal. Therefore we obtain that every irreducible finite-dimensional $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$–module is a highest weight module.

Furthermore, according to Theorem 6.3 of [BK], the irreducible highest weight representation of $\widehat{U}_{e} \mathfrak{g}$ with highest weight $(\Psi_{i}^{\pm}(u))_{i \in I}$ is finite-dimensional if and only if $\Psi_{i}^{\pm}(u)$ is a rational function

$$f_{i}(u) = \frac{P_{i}^{(1)}(u)}{P_{i}^{(2)}(u)}, \quad i \in I,$$

where $P_{i}^{(1)}(u), P_{i}^{(2)}(u)$ are two polynomials in $u$ of equal degrees with non-zero constant coefficients; $f_{i}(u)$ is regular at 0 and $\infty$, and $f_{i}(0) = f_{i}(\infty)^{-1}$.

In addition, the highest weight of an irreducible $\widehat{U}_{e} \mathfrak{g}$–module obtained by pull-back from an irreducible $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$–module of type 1 satisfies formula (2.7) and the condition $\Psi_{i}^{\pm}(0) \in \epsilon_{i} l_{i}$.

Furthermore, according to Theorem 6.3 of [BK], the irreducible highest weight representation of $\widehat{U}_{e} \mathfrak{g}$ with highest weight $(\Psi_{i}^{\pm}(u))_{i \in I}$ is finite-dimensional if and only if $\Psi_{i}^{\pm}(u)$ is a rational function

$$f_{i}(u) = \frac{P_{i}^{(1)}(u)}{P_{i}^{(2)}(u)}, \quad i \in I,$$

where $P_{i}^{(1)}(u), P_{i}^{(2)}(u)$ are two polynomials in $u$ of equal degrees with non-zero constant coefficients; $f_{i}(u)$ is regular at 0 and $\infty$, and $f_{i}(0) = f_{i}(\infty)^{-1}$.

In addition, the highest weight of an irreducible $\widehat{U}_{e} \mathfrak{g}$–module obtained by pull-back from an irreducible $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$–module of type 1 satisfies formula (2.7) and the condition $\Psi_{i}^{\pm}(0) \in \epsilon_{i} l_{i}$. But then for each $i \in I$ there exists a unique acyclic polynomial $P_{i}(u)$, such that formula (2.8) holds. This completes the proof.

Proposition 2.7. Let $\mathbf{P} = (P_{i}(u))_{i \in I}$ be an $I$–tuple of polynomials with constant term 1, such that $P_{i}(u)$ is $l_{i}$–acyclic for each $i \in I$. Then the irreducible $U_{e}^{\text{res}}\widehat{\mathfrak{g}}$–module $V(\mathbf{P})$ remains irreducible when restricted to $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$.

Proof. This statement is proved in Theorem 9.2 of [CP] in the case when the order of $\epsilon$ is odd (although the algebra $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$ in [CP] includes the elements $P_{i, n}$, the same proof works for the algebra without the elements $P_{i, n}$, as in our definition of $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$). For general root of unity $\epsilon$, one repeats the proof of [CP] replacing $l$ by $l_{i}$, where appropriate.

Remark 2.8. Let $I_{ev} = \{ i \in I | \epsilon_{i} \text{ has even order} \}$. The algebra $U_{e}^{\text{fin}}\widehat{\mathfrak{g}}$ has $2^{\#I_{ev}}$ non-isomorphic one-dimensional type 1 representations, on which the $\epsilon_{i}$’s and $f_{i}$’s act by zero, $k_{i} \in I \setminus I_{ev}$ act identically, and $k_{i} \in I_{ev}$ act by multiplication by $\pm 1$ (indeed, if $\epsilon_{i}$ is a root of unity of an even order $2l_{i}$, then $\epsilon_{i}^{l_{i}} = -1$). By Theorem 2.3, any
irreducible $U_{\ell, \epsilon}^{\text{fin} \hat{g}}$-module of type 1 is isomorphic to the tensor product of an irreducible module with the highest weight of the form \((2.8)\), where $\gamma_i = 1$ for all $i \in I$, and a one-dimensional module. Note that in contrast to $U_{\ell, \epsilon}^{\text{fin} \hat{g}}$, the algebra $U_{\ell, \epsilon}^{\text{res} \hat{g}}$ has only one one-dimensional type 1 representation for any $\epsilon$ (namely, the trivial representation).

Proposition 2.7 implies that any irreducible $U_{\ell, \epsilon}^{\text{fin} \hat{g}}$-module may be extended to a $U_{\ell, \epsilon}^{\text{res} \hat{g}}$-module, although this $U_{\ell, \epsilon}^{\text{res} \hat{g}}$-module may not be of type 1. Furthermore, an irreducible $U_{\ell, \epsilon}^{\text{fin} \hat{g}}$-module can be extended to a $U_{\ell, \epsilon}^{\text{res} \hat{g}}$-module in a unique way. This follows from the Decomposition Theorem 5.4 (which is proved in this paper under the assumption that $(l, r^\vee) = 1$).

### 3. The $\epsilon$-characters

#### 3.1. Definition

Let $V$ be a finite-dimensional representation of $U_{\ell, \epsilon}^{\text{res} \hat{g}}$. We have a commutative subalgebra generated by the elements $k_i$, $P^{\pm}_{i,n}$ ($i \in I$, $n \in \mathbb{Z}$) acting on $V$. Let $v_1, \ldots, v_d$ be a basis of common generalized eigenvectors of these elements. In what follows we look at the corresponding generalized eigenvalues.

**Lemma 3.1.** The common eigenvalues of $P^+_{i,n}(u)$ are rational functions of the form

$$\Gamma^+_{i,n}(u) = \frac{\prod_{j=1}^{j_{in}} (1 - a_{inj}u)}{\prod_{m=1}^{m_{in}} (1 - b_{inm}u)},$$

where $j_{in}, m_{in}$ are non-negative integers and $a_{inj}, b_{inm}$ are non-zero complex numbers. Moreover, the generalized eigenvalues of $P^-_{i,n}(u)$ are

$$\Gamma^-_{i,n}(u) = \frac{\prod_{j=1}^{j_{in}} (1 - (a_{inj}u)^{-1})}{\prod_{m=1}^{m_{in}} (1 - (b_{inm}u)^{-1})},$$

and we have

$$k_i v_n = q^{j_{in} - m_{in}} v_n, \quad \left[\begin{array}{c} k_i \\ l_i \end{array}\right]\ v_n = \left[\begin{array}{c} j_{in} - m_{in} \\ l_i \end{array}\right]_{\epsilon_i} v_n.$$

**Proof.** In the case of $U_q \hat{g}$ the analogous statement follows from Proposition 1 in [FR]. Lemma 3.1 then follows from Proposition 2.3.

The $q$-character of a $U_q \hat{g}$-module $V$ defined in [FR] (see also [FM]) encodes the generalized eigenvalues of $P^+_{i,n}(u)$ on $V$. It is denoted by $\chi_q(V)$ and it takes values in the polynomial ring $\mathbb{Z}[\prod_{i,a} Y_{i,a}^{\pm 1} a \in \mathbb{C}^\times]$.

Similarly, we define the $\epsilon$-character of a finite-dimensional $U_{\ell, \epsilon}^{\text{res} \hat{g}}$-module $V$, denoted by $\chi_{\epsilon}(V)$, to be the element of the ring $\mathbb{Z}[\prod_{i,a} Y_{i,a}^{\pm 1} a \in \mathbb{C}^\times]$ equal to

$$\chi_{\epsilon}(V) = \sum_{n=1}^{d} \prod_{i \in I} \left( \prod_{j=1}^{j_{in}} Y_{i,a_{inj}} \prod_{m=1}^{m_{in}} Y_{i,b_{inm}}^{-1} \right).$$

**Theorem 3.2.** The map

$$\chi_{\epsilon} : \text{Rep} U_{\ell, \epsilon}^{\text{res} \hat{g}} \rightarrow \mathbb{Z}[\prod_{i,a} Y_{i,a}^{\pm 1} a \in \mathbb{C}^\times], \quad V \mapsto \chi_{\epsilon}(V).$$
is an injective homomorphism of rings. Moreover, for any irreducible finite-dimensional
$U_q\hat{g}$-module $V$, \( \chi_\epsilon(V^\text{res}) \) is obtained from \( \chi_q(V) \) by setting $q$ equal to \( \epsilon \).

**Proof.** The first part follows from Theorem 3 in [FR] and Proposition 2.5. The second
part follows from Proposition 2.3. □

In particular, we obtain that the ring $\text{Rep}_{U_q^\text{res}}\hat{g}$ is commutative.

A monomial $m \in \mathbb{Z}[Y_{i,a}^\pm \in I]$ is called dominant if it does not contain factors $Y_{i,a}$ in
negative powers. The monomial in $\chi_\epsilon(V)$ corresponding to the highest weight vector
is always dominant. We call it the highest weight monomial. The $i$-th fundamental
representation $V_{\omega_i}(a)$ is by definition the irreducible $U_q^\text{res}\hat{g}$-module whose highest weight
monomial is $Y_{i,a}$.

Theorems 2.4 and 3.2 imply that the map

\[
\text{Rep}_{U_q^\text{res}}\hat{g} \rightarrow \mathbb{Z}[t_{i,a} \in I],
\]

\[
V_{\omega_i}(a) \rightarrow t_{i,a}
\]

is an isomorphism of rings (cf. the analogous statement in the case of $\text{Rep}_{U_q}\hat{g}$ in [FR],
Corollary 2).

### 3.2. Properties of the $\epsilon$-characters

Using the definition, Proposition 2.3 and Theorem 3.2, we obtain that the $\epsilon$-characters satisfy combinatorial properties similar to $q$-characters. In this section we list these properties.

We have the ordinary character homomorphism $\chi : \text{Rep}_{U_q^\text{res}}\hat{g} \rightarrow \mathbb{Z}[Y_{i,a}^\pm \in I]$: if $V = \bigoplus \mu V_\mu$ is the weight decomposition of $V$, then $\chi(V) = \sum_{\mu} \dim V_\mu \cdot y^\mu$, where for $\mu = \sum_{i \in I} \mu_i \omega_i$ we set $y^\mu = \prod_{i \in I} y_i^{\mu_i}$.

Define the ring homomorphism $\beta : \mathbb{Z}[Y_{i,a}^\pm \in I] \rightarrow \mathbb{Z}[y_{i,a}^\pm \in I]$ by the rule $\beta(Y_{i,a}^\pm) = y_{i,a}^\pm$. We will say that a monomial $m \in \mathbb{Z}[Y_{i,a}^\pm \in I]$ has weight $\mu$ if $\beta(m) = y^\mu$.

**Lemma 3.3.** The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Rep}_{U_q^\text{res}}\hat{g} & \xrightarrow{\chi} & \mathbb{Z}[Y_{i,a}^\pm \in I] \\
\downarrow \text{res} & & \downarrow Y_{i,a} \rightarrow y_i \\
\text{Rep}_{U_q^\text{res}}\hat{g} & \xrightarrow{\chi} & \mathbb{Z}[y_{i,a}^\pm \in I]
\end{array}
\]

Given a subset $J$ of $I$, denote by $U_{r_{r_{J}}}\hat{g}$ the subalgebra of $U_q^\text{res}\hat{g}$ generated by $k_i, P_{i,r}, (x_{i,r}^\pm)^{(n)}$, $i \in J$, $r \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$. Let $\beta_J : \mathbb{Z}[Y_{i,a}^\pm \in I] \rightarrow \mathbb{Z}[Y_{i,a}^\pm \in J]$ be a
ring homomorphism defined by the rule $\beta_J(Y_{i,a}^\pm) = Y_{i,a}^\pm$ if $i \in J$ and $\beta_J(Y_{i,a}^\pm) = 1$ if $i \notin J$. Then we have:

**Lemma 3.4.** The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Rep}_{U_q^\text{res}}\hat{g} & \xrightarrow{\chi} & \mathbb{Z}[Y_{i,a}^\pm \in I] \\
\downarrow \text{res} & & \downarrow Y_{i,a} \rightarrow 1, i \notin J \\
\text{Rep}_{U_q^\text{res}}\hat{g}_J & \xrightarrow{\chi_J} & \mathbb{Z}[Y_{i,a}^\pm \in J]
\end{array}
\]

Define $A_{i,a} \in \mathbb{Z}[Y_{j,b}^{\pm 1}]_{j \in I}$ by the formula

(3.1)

$$A_{i,a} = Y_{i,a \epsilon^{-1}} \left( \prod_{j: j \neq i, C_{ji} \neq 0} A_{i,j,a} \right) Y_{i,a \epsilon}, \quad A_{i,j,a} = \begin{cases} Y_{j,a}^{-1} & \text{if } C_{ji} = -1, \\ Y_{j,a}^{-1}Y_{j,a \epsilon}^{-1} & \text{if } C_{ji} = -2, \\ Y_{j,a \epsilon}^{-1} & \text{if } C_{ji} = -3. \end{cases}$$

Thus $\beta(A_{i,a})$ corresponds to the simple root $\alpha_i$.

Using Theorem 4.1 in [FM] and Proposition 2.5, we obtain:

**Proposition 3.5.** Let $V$ be an irreducible $U^{\text{res}}_{\hat{\theta}}$ module. The $\epsilon$-character of $V$ has the form

$$\chi_\epsilon(V) = m_+ (1 + \sum_p a_p M_p),$$

where $m_+$ is the highest weight monomial and for each $p$, $M_p$ is a product of $A^{-1}_{i,c}, i \in I, c \in \mathbb{C}^\times, a_p$ is a nonnegative integer.

Define the rings $\mathcal{K}_i, i \in I$ by the formula

$$\mathcal{K}_i = \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j \in I, j \neq i} \otimes \mathbb{Z}[Y_{i,b} + Y_{i,b}A_{i,c,i}]_{b \in \mathbb{C}^\times}. $$

Set

$$\mathcal{K} = \bigcap_{i \in I} \mathcal{K}_i \subset \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j \in I}.$$

Corollary 5.7 in [FM] and Proposition 2.5 imply:

**Proposition 3.6.** The image of the $\epsilon$-character homomorphism equals $\mathcal{K}$. Therefore

$$\chi_\epsilon : \text{Rep} U^{\text{res}}_{\hat{\theta}} \to \mathcal{K}$$

is a ring isomorphism.

### 3.3. The $\epsilon$-characters of the small quantum group $U^{\text{fin}}_{\hat{\theta}}$.

The $\epsilon$-characters of finite-dimensional $U^{\text{fin}}_{\hat{\theta}}$-modules are defined similarly to the $\epsilon$-characters of $U^{\text{res}}_{\hat{\theta}}$-modules. Namely, we consider the generalized eigenvalues $\Psi_\pm(u)$ of the series $\Phi_\pm(u)$ on a given finite-dimensional $U^{\text{fin}}_{\hat{\theta}}$-module $V$. By Proposition 2.5 and Remark 2.8, any irreducible $U^{\text{fin}}_{\hat{\theta}}$-modules may be represented as the tensor product of the restriction of a type one $U^{\text{res}}_{\hat{\theta}}$-module and a one-dimensional sign representation. Therefore we find from Lemma 3.1 that the generalized eigenvalues $\Psi_\pm(u)$ have the form (2.8), where

$$P_i(u) = \prod_{j=1}^{h_i} (1 - a_{ij} u)$$

is an $l_i$-acyclic polynomial for each $i \in I$, and for each $i \in I$ the sign $\gamma_i$ in (2.8) is the same for all generalized eigenvectors. We attach to a set of common generalized eigenvalues of this form, the monomial $\prod_{i \in I} P_i(u) Y_{i,a_{ij}}$.

Let $\text{Rep} U^{\text{fin}}_{\hat{\theta}}$ be the quotient of the Grothendieck ring of $U^{\text{fin}}_{\hat{\theta}}$ by the ideal generated by the elements of the form $[V] - 1$, where $[V]$ is any non-trivial one-dimensional $U^{\text{fin}}_{\hat{\theta}}$-module. Introduce the notation

$$Y_{i,a} := \prod_{j=0}^{l_i - 1} Y_{i,a \epsilon^{2j}}.$$
Then we obtain an injective ring homomorphism
\[ \chi_\epsilon^\text{fin} : \text{Rep} U^{\text{fin}}_\epsilon \rightarrow \mathbb{C}[Y_{i,a}]_{i \in I}/(Y_{i,b} - 1)_{i \in I}. \]

Let \( \pi \) be the ring homomorphism
\[ \pi : \mathbb{C}[Y_{i,a}]_{i \in I} / (Y_{i,b} - 1)_{i \in I}, Y_{i,a} \rightarrow Y_{i,a}. \]
Then we have the following

**Lemma 3.7.** Let \( V \) be a finite dimensional \( U^{\text{res}}_\epsilon \hat{\mathfrak{g}} \) module with \( \epsilon \)-character \( \chi_\epsilon(V) \). Then the following diagram is commutative. \( \chi_\epsilon^{\text{fin}}(V|_{U^{\text{fin}}_\epsilon \hat{\mathfrak{g}}}) = \pi(\chi_\epsilon(V)). \)

Since all irreducible \( U^{\text{fin}}_\epsilon \hat{\mathfrak{g}} \)-modules may be obtained by restriction from irreducible \( U^{\text{res}}_\epsilon \hat{\mathfrak{g}} \)-modules, we see that the study of \( \epsilon \)-characters of \( U^{\text{fin}}_\epsilon \hat{\mathfrak{g}} \)-modules is reduced to the study of \( \epsilon \)-characters of \( U^{\text{res}}_\epsilon \hat{\mathfrak{g}} \)-modules.

### 4. The Frobenius homomorphism

In this section we introduce the Frobenius homomorphism following Lusztig [1], §35. Because in the case of roots of unity of even order this homomorphism is not discussed in detail in the literature (except for [1]), we first give a brief overview.

Recall that we denote by \( \epsilon \) a fixed primitive root of unity of order \( s \). If \( s \) is an odd number, relatively prime to \( r^\nu \), then we expect that there exists a Frobenius homomorphism \( U^{\text{res}}_\epsilon \hat{\mathfrak{g}} \rightarrow U \hat{\mathfrak{g}} \), although we have not been able to locate such a homomorphism in the literature (such a homomorphism has been constructed by Lusztig in [1] in the case of \( U_q \mathfrak{g} \) with the above restrictions on \( s \)). If \( s \) is even, but not divisible by 4 and by \( r^\nu \) (so that \( l \) is odd and not divisible by \( r^\nu \)), then presumably there exists a homomorphism \( U^{\text{res}}_\epsilon \hat{\mathfrak{g}} \rightarrow U^{\text{res}}_{l^{-1}} \hat{\mathfrak{g}} \). If \( s \) is even and the above additional conditions are not satisfied, the construction becomes more complicated.

The important aspect of Lusztig’s definition of the Frobenius homomorphism is that it uses the modified quantized enveloping algebra \( \hat{U}^{\text{res}}_\epsilon \hat{\mathfrak{g}} \) (it is denoted by \( \hat{U} \) in [1]) instead of \( U^{\text{res}}_\epsilon \hat{\mathfrak{g}} \). This appears to be necessary in order to give a uniform definition of the Frobenius homomorphism for roots of unity of odd and even orders (subject to some mild restrictions mentioned below). In this section we apply Lusztig’s definition of the Frobenius homomorphism using the modified quantized enveloping algebra \( \hat{U}^{\text{res}}_\epsilon \hat{\mathfrak{g}} \). However, for the purposes of our paper the difference between \( \hat{U}^{\text{res}}_\epsilon \hat{\mathfrak{g}} \) and \( U^{\text{res}}_\epsilon \hat{\mathfrak{g}} \) is inessential, because the category of finite-dimensional \( U^{\text{res}}_\epsilon \hat{\mathfrak{g}} \)-modules of type 1 is equivalent to the category of unital \( \hat{U}^{\text{res}}_\epsilon \hat{\mathfrak{g}} \)-modules.

#### 4.1. The modified quantized enveloping algebra

Recall that we denote by \( \Lambda \) the weight lattice of \( \mathfrak{g} \). For \( \lambda \in \Lambda \),
\[ U_\lambda = U_q \hat{\mathfrak{g}}/ \sum_{i \in I} U_q \hat{\mathfrak{g}}(k_i - q^{(\alpha_i, \lambda)}). \]
Here we set \((\alpha_0, \lambda) = -(\alpha_{\text{max}}, \lambda)\). Let \( 1_\lambda \in U_\lambda \) be the image of \( 1 \in U_q \hat{\mathfrak{g}} \) in \( U_\lambda \). The space \( U_\lambda \) is a left \( U_q \hat{\mathfrak{g}} \)-module. For \( g \in U_q \hat{\mathfrak{g}} \), we denote by \( g1_\lambda \) the image of \( g \) in \( U_\lambda \).
This is a $C(q)$–algebra with multiplication given by

$$1_\lambda 1_\mu = \delta_{\lambda, \mu} 1_\lambda,$$
$$x_i^\pm 1_\lambda = 1_{\lambda \pm x_i^\pm}.$$

Now we compare the algebra $\hat{U}_q\mathfrak{g}$ with the modified quantized enveloping algebra $\hat{U}$ defined in [L]. The latter is assigned to a root datum, which consists of two lattices $X, Y$, a set $I$, a pairing $\langle, \rangle : Y \times X \to \mathbb{Z}$, a bilinear form $\cdot : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$, and embeddings $I \hookrightarrow X, I \hookrightarrow Y$ satisfying the conditions listed in [L], §§1.1.1, 2.2.1.

We take as $I$ the set $\hat{I}$ of vertices of the Dynkin diagram of $\hat{g}$, as $X$ the weight lattice $\Lambda$ of $\mathfrak{g}$ (spanned by $\omega_i, i \in \hat{I}$), and as $Y$ the coroot lattice $Q^\vee$ of $\mathfrak{g}$ (spanned by $\alpha_i^\vee, i \in I$). The pairing between $X$ and $Y$ is defined by the formula $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij}$ for all $i, j \in I$. The bilinear form on $\mathbb{Z}[\hat{I}]$ is defined by the formula $i \cdot j = (\alpha_i, \alpha_j)$ for $i, j \neq 0$, $0 \cdot j = -(\alpha_{\text{max}}, \alpha_j)$ for $j \neq 0$ and $0 \cdot 0 = (\alpha_{\text{max}}, \alpha_{\text{max}})$. The embedding $\hat{I} \hookrightarrow X$ is defined by the formula $i \to \alpha_i, i \neq 0; 0 \to -\alpha_{\text{max}}$. The embedding $\hat{I} \hookrightarrow Y$ is defined by the formula $i \to \alpha_i^\vee, i \neq 0; 0 \to -\alpha_{\text{max}}^\vee$. Attached to these data are associative algebras over $\mathbb{C}(q)$, $U$ and $\hat{U}$, defined in [L] (see §3.1.1, §23.1.1, and Corollary 33.1.5). Straightforward and explicit comparison gives:

Lemma 4.1. The algebra $U$ is isomorphic to the algebra $U_q\hat{g}$ extended by the elements $k_i^{1/n_i}, i \in \hat{I}$. The algebra $\hat{U}$ is isomorphic to the algebra $\hat{U}_q\hat{g}$.

A representation of $\hat{U}_q\hat{g}$ is called unital if $\sum_{\lambda \in \Lambda} 1_\lambda$ acts on it as the identity (note that the infinite sum $\sum_{\lambda \in \Lambda} 1_\lambda$ is a well-defined operator on any $\hat{U}_q\hat{g}$–module). We will consider only finite-dimensional unital $\hat{U}_q\hat{g}$–modules. These are the finite-dimensional $\hat{U}_q\hat{g}$–modules, which do not contain subspaces of positive dimension on which $\hat{U}_q\hat{g}$ acts by 0.

Let $V$ be a finite-dimensional unital representation of $\hat{U}_q\hat{g}$. Then using the projectors $1_\lambda$, we obtain a decomposition $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$. We define a $\hat{U}_q\hat{g}$–module structure on $V$ by the rule $g \cdot v = g 1_\lambda \cdot v$ for $g \in \hat{U}_q\hat{g}$, $v \in V_\lambda$.

Conversely, if $V$ is a type 1 finite-dimensional $\hat{U}_q\hat{g}$–module, then we have the weight decomposition $V = \bigoplus_{\lambda} V_\lambda$. Define the action of $\hat{U}_q\hat{g}$ in $V$ by letting $1_\lambda$ act as the identity on $V_\lambda$ and by zero on $V_\mu$, $\mu \neq \lambda$.

Hence we obtain the following result (cf. [L], §23.1.4):

Lemma 4.2. The category of finite-dimensional $U_q\hat{g}$–modules of type 1 is equivalent to the category of finite-dimensional unital $\hat{U}_q\hat{g}$–modules.

Note that strictly speaking $\hat{U}_q\hat{g}$ is not a Hopf algebra. However, we define tensor product of $\hat{U}_q\hat{g}$–modules using Lemma [1], see also [L], §23.1.5.

The automorphisms $T_i$ on $U_q\hat{g}$ induce automorphisms of $\hat{U}_q\hat{g}$, which we also denote by $T_i$. 


4.2. Specialization to the root of unity. Next we define $\hat{U}_q^\text{res} \hat{g}$ as the $\mathbb{C}[q,q^{-1}]$-subalgebra of $\hat{U}_q \hat{g}$, generated by the elements $(x_i^\pm)^{(n)} 1_\lambda, i \in I, n \geq 0, \lambda \in \Lambda$. Note that we have the following relations in $\hat{U}_q^\text{res} \hat{g}$.

$$(x_i^\pm)^{(n)} 1_\lambda = 1_{\lambda \pm n a_i} (x_i^\pm)^{(n)}.$$  

Then we set $\hat{U}_q^\text{res} \hat{g} = \hat{U}_q \hat{g} \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}_\epsilon$.

**Lemma 4.3.** The algebras $\hat{U}_q^\text{res} \hat{g}$ and $\hat{U}_\epsilon^\text{res} \hat{g}$ are generated by $(x_i^\pm)^{(n)} 1_\lambda, i \in I, r \in \mathbb{Z}, n \geq 0, \lambda \in \Lambda$.

If $J \subset I$, then the maps

$$\hat{U}_q^\text{res} \hat{g}_J \rightarrow \hat{U}_q^\text{res} \hat{g}, \quad \hat{U}_\epsilon^\text{res} \hat{g}_J \rightarrow \hat{U}_\epsilon^\text{res} \hat{g},$$

$$(x_i^\pm)^{(n)} 1_\lambda \mapsto (x_i^\pm)^{(n)} 1_\lambda, \quad i \in J, r \in \mathbb{Z}, n \in \mathbb{Z}_{>0}$$

extend to injective algebra homomorphisms. Here we extended a weight $\lambda$ of $\hat{g}_J$ to a weight of $\hat{g}$ by the rule $\langle \alpha_i^\vee, \lambda \rangle = 0$ if $i \in I \setminus J$.

**Proof.** This lemma follows from Theorem 2.2 and the commutation relations in $U_{\epsilon^s} \hat{g}$.

4.3. Definition of the Frobenius homomorphism. Recall that $\epsilon^s = 1, l = s$ if $s$ is odd and $l = s/2$ if $s$ is even.

From now on, we impose the following restrictions on $s$. First we assume that $s > 2$ (if $s = 1, 2$ then the Frobenius homomorphism is just the identity map). Next, in this section we exclude from consideration the case $\hat{g} = \hat{s}_2 \mathfrak{sl}_{2n+1}$ because of condition 35.1.2 b) in [L]. We will deal with it in Section 4.4. Finally, due to condition 35.1.2 a) in [L] we assume that $l \neq 2, 4$ in the case $r^\vee = 2$, and $l \neq 2, 3, 6$ in the case $r^\vee = 3$.

In what follows a weight $\lambda \in \Lambda$ is called $l$-admissible if $\langle \alpha_i^\vee, \lambda \rangle$ is divisible by $l_i$ for all $l$. In other words, $\lambda$ is $l$-admissible if $\lambda = \sum_{i \in I} a_i \omega_i$, where each $a_i$ is divisible by $l_i$. Denote the lattice of all $l$-admissible weights by $\Lambda_l$, and for each $\lambda = \sum_{i \in I} a_i \omega_i \in \Lambda_l$, denote by $\lambda/l$ the weight $\sum_{i \in I} (a_i/l_i) \omega_i$.

In [L], §35.1.6 an algebra $\hat{U}^*$ over $\mathbb{C}[q,q^{-1}]$ is introduced. It is defined in the same way as $\hat{U}$ with respect to the Cartan datum dual to the Cartan datum used in the definition of $\hat{U}$ (see [L], §2.2.5). Let $R$ be a $\mathbb{C}[q,q^{-1}]$-module, on which the operator of multiplication by $q$ has order $s$. Set $R \hat{U} = \hat{U} \otimes_{\mathbb{C}[q,q^{-1}]} R, R \hat{U}^* = \hat{U}^* \otimes_{\mathbb{C}[q,q^{-1}]} R$. Then by Theorem 35.1.9 in [L], there exists an algebra homomorphism $R \hat{U} \rightarrow R \hat{U}^*$.

In the case at hand this result translates as follows. If $l$ is relatively prime to $r^\vee$, define $^* \hat{U}_\epsilon \hat{g}$ to be $\hat{U}_\epsilon^\text{res} \hat{g}$, where $\epsilon^s = \epsilon^{l^\vee}$. If $l$ is divisible by $r^\vee$, define $^* \hat{U}_\epsilon \hat{g}$ to be $\hat{U}_\epsilon^\text{res} \hat{g}^L$, where $\epsilon^s = \epsilon^{l^\vee/r^\vee}$. Here $\hat{g}^L$ is the twisted affine Kac-Moody algebra, whose Cartan matrix is the transpose of the Cartan matrix of $\hat{g}$. The algebra $\hat{U}_\epsilon^\text{res} \hat{g}^L$ is defined in the same way as $\hat{U}_\epsilon^\text{res} \hat{g}$ above (using the Drinfeld-Jimbo realization, with the generators labeled in a way compatible with the labeling of the generators of $\hat{U}_\epsilon^\text{res} \hat{g}$).

Then there exists an algebra homomorphism

$${\text{Fr}} : \hat{U}_\epsilon^\text{res} \hat{g} \rightarrow ^* \hat{U}_\epsilon \hat{g},$$

$$(x_i^\pm)^{(m)} 1_\lambda \mapsto (x_i^\pm)^{(m/l_i)} 1_{\lambda/l_i}, \quad m/l_i \in \mathbb{Z}, \lambda \in \Lambda_l,$$

$$(x_i^\pm)^{(m)} 1_\lambda \mapsto 0, \quad \text{otherwise}.$$
Here and below we put a bar over the elements of the target algebra to avoid confusion.

The homomorphism $\text{Fr}$ is called the quantum Frobenius homomorphism.

**Remark 4.4.** In [L], the algebra $U_q\hat{\mathfrak{g}}$ is defined over $\mathbb{Q}(q)$ and the algebra $U^{\text{res}}_{\epsilon}\hat{\mathfrak{g}}$ is defined over $\mathbb{Z}[q,q^{-1}]$. However, all of the results of [L] referred to above, remain valid if we replace $\mathbb{Q}(q)$ by $\mathbb{C}(q)$ and $\mathbb{Z}[q,q^{-1}]$ by $\mathbb{C}[q,q^{-1}]$.

The defining relations of the algebra $\mathcal{U}$ used in [L] are strictly speaking different from those used here. Namely, the relations between the generators $x_i^+, i \in \hat{I}$ (and $x_i^-, i \in \hat{I}$) are described in [L] in terms of a certain bilinear form (see [L], §§1.2.3, 1.2.4) instead of the quantum Serre relations that we use here. However, it is known that the quantum Serre relations are included into these relations (see [L], §33.2) unless $\hat{\mathfrak{g}} = sl_{2n+1}$ which we have excluded from consideration in this subsection.

The homomorphism $\text{Fr}$ is the composition of the homomorphism $\hat{U}^{\text{res}}_{\epsilon}\hat{\mathfrak{g}} \to \hat{\mathcal{U}}$ with the Frobenius homomorphism from [L], Theorem 35.1.9 (when $R = \mathbb{C}$).

In order to avoid twisted quantum affine algebras, we assume from now on that $l$ is relatively prime to $r^\vee$. We will describe the $q$-characters of twisted quantum affine algebras and the Frobenius homomorphism involving them in a separate paper.

Since $l$ is relatively prime to $r^\vee$, we have: $\epsilon_i = \epsilon$ and $l_i = l$ for all $i \in \hat{I}$. Therefore $\Lambda_\ell = l \cdot \Lambda$ and $\lambda/\ell = \lambda/l$ for $\lambda \in \Lambda_\ell$.

Note that $\epsilon^* = \epsilon^l$ is equal to 1 if $l$ is odd and $s = l$, or if $l$ is even, and to $-1$ if $l$ is odd and $s = 2l$. Note also that $-\epsilon^* = (-\epsilon)^l$. By Proposition 33.2.3 in [L], the algebras $\hat{U}^{\text{res}}_{\epsilon\ell}\hat{\mathfrak{g}}$ and $\hat{U}^{\text{res}}_{-\ell}\hat{\mathfrak{g}}$ are isomorphic if $\hat{\mathfrak{g}} \neq sl_{2n+1}$.

In the same way as in the proof of Lemma 4.2 in [L] we obtain the following result (see the beginning of Section 34.3 for the definition of $U^{\text{res}}_{\epsilon}\hat{\mathfrak{g}}$–modules of type 1).

**Lemma 4.5.** The category of finite-dimensional $U^{\text{res}}_{\epsilon}\hat{\mathfrak{g}}$–modules of type 1 is equivalent to the category of finite-dimensional unital $U^{\text{res}}_{\epsilon}\hat{\mathfrak{g}}$–modules.

Therefore, the quantum Frobenius homomorphism induces a map of Grothendieck rings

$$\text{Fr}^*: \text{Rep} U^{\text{res}}_{\epsilon}\hat{\mathfrak{g}} \to \text{Rep} U^{\text{res}}_{\epsilon}\hat{\mathfrak{g}}.$$ 

By [L], §35.1.10, this map is a ring homomorphism.

**Lemma 4.6.** We have $\text{Fr}((x^{\pm}_{i,r})^{(m)}\mathbb{I}_\lambda) = 0$ if $m$ is not divisible by $l$ or if $\lambda \notin \Lambda_\ell$. For $\lambda \in \Lambda_\ell$,

$$\text{Fr}((x^{\pm}_{i,r})^{(nl)}\mathbb{I}_\lambda) = \begin{cases} (\hat{x}^{\pm}_{i,r})^{(nl)}\mathbb{I}_{\lambda/l}, & l \text{ is odd} \\ \sigma(i)^{nl}(\hat{x}^{\pm}_{i,r})^{(nl)}\mathbb{I}_{\lambda/l}, & l \text{ is even}. \end{cases}$$

**Proof.** We use the fact that the automorphisms $T_i, i \in \hat{I}$, commute with the Frobenius homomorphism (see [L], §41.1.9), and so do the automorphisms $\sigma$ of the Dynkin diagram.
of \( \hat{g} \) (see Section 2.2). Hence the automorphisms \( T_{\varpi_i}, i \in I \), also commute with the Frobenius homomorphism. From this we find:

\[
\text{Fr}((x_{i,r}^\pm)^{(m)}1_\lambda) = \text{Fr}(\vartheta(i)^{mr}T_{\varpi_i}^{sr}(x_{i,r}^\pm)^{(m)}1_{\omega_i^r(\lambda)}) = o(i)^{mr}T_{\varpi_i}^{sr}\text{Fr}((x_{i,r}^\pm)^{(m)}1_{\omega_i^r(\lambda)}).
\]

So, it is zero if \( m \) is not divisible by \( l \) or if \( \lambda \not\in \Lambda_i \). If \( m = nl \) we obtain

\[
\text{Fr}((x_{i,r}^\pm)^{(m)}1_\lambda) = o(i)^{mr}T_{\varpi_i}^{sr}(x_{i,r}^\pm)^{(n)}1_{\omega_i^r(\lambda)/l} = o(i)^{mr+nr}(x_{i,r}^\pm)^{(n)}1_{\lambda/l}.
\]

If \( l \) is odd then \( m \) and \( n \) have the same parity, and all the signs cancel. If \( l \) is even, then \( m \) is also even, and we acquire the sign \( o(i)^{mr} \).

\[\square\]

4.4. **The case** \( \hat{g} = \widehat{sl}_{2n+1} \). Let now \( \hat{g} = \widehat{sl}_{2n+1} \). We define the Frobenius homomorphism \( \text{Fr} : \hat{U}_e \cdot \widehat{sl}_{2n+1} \to \hat{U}_e \cdot \widehat{sl}_{2n+1} \) using the generators \( (x_{i,r}^\pm)^{(m)} \) as in Lemma 4.6

\[
\text{Fr}((x_{i,r}^\pm)^{(m)}1_\lambda) = \begin{cases} (x_{i,r}^\pm)^{(n)}1_{\lambda/l} & \text{if } l \text{ is odd, } \lambda \in \Lambda, \\ o(i)^{mr}(x_{i,r}^\pm)^{(n)}1_{\lambda/l} & \text{if } l \text{ is even, } \lambda \in \Lambda \end{cases}
\]

and \( \text{Fr}((x_{i,r}^\pm)^{(m)}1_\lambda) = 0 \) if \( m \) is not divisible by \( l \) or \( \lambda \not\in \Lambda_i \).

**Lemma 4.7.** These formulas give rise to a well-defined homomorphism of algebras. Moreover, the induced map \( \text{Fr}^* : \text{Rep} \hat{U}_e \cdot \widehat{g} \to \text{Rep} \hat{U}_e \cdot \widehat{g} \) is a ring homomorphism.

**Proof.** We embed the Dynkin diagram of \( \widehat{sl}_{2n+1} \) into the Dynkin diagram of \( \widehat{sl}_{2n+2} \) in such a way that the numbers \( o(i) \) coincide. By Lemma 4.3, we have the corresponding embeddings \( \hat{U}_e \cdot \widehat{sl}_{2n+1} \to \hat{U}_e \cdot \widehat{sl}_{2n+2} \) and \( \hat{U}_e \cdot \widehat{sl}_{2n+1} \to \hat{U}_e \cdot \widehat{sl}_{2n+2} \). It follows from Lemma 4.6 that the image of \( \hat{U}_e \cdot \widehat{sl}_{2n+1} \) under the Frobenius homomorphism \( \hat{U}_e \cdot \widehat{sl}_{2n+2} \to \hat{U}_e \cdot \widehat{sl}_{2n+2} \) is contained in \( \hat{U}_e \cdot \widehat{sl}_{2n+1} \). Hence we obtain an algebra homomorphism \( \hat{U}_e \cdot \widehat{sl}_{2n+1} \to \hat{U}_e \cdot \widehat{sl}_{2n+1} \), given by the above formula. The lemma follows. \[\square\]

By a direct computation similar to the one used in the proof of Lemma 4.6, we obtain the following formulas for the Frobenius homomorphism in the case of \( \hat{U}_e \cdot \widehat{sl}_{2n+1} \) in terms of the Drinfeld-Jimbo generators.

**Lemma 4.8.** Let \( \hat{g} = \widehat{sl}_{2n-1} \), then

\[
\text{Fr}((x_{i,r}^\pm)^{(m)}1_\lambda) = (\bar{x}_{i}^\pm)^{(m/l)}1_{\lambda/l}, \quad m/l \in \mathbb{Z}, \lambda \in \Lambda, i \in \hat{I},
\]

\[
\text{Fr}((x_{i,r}^\pm)^{(m)}1_\lambda) = 0, \quad \text{otherwise}.
\]

5. Frobenius homomorphism and \( \epsilon \)-characters

5.1. **Properties of the Frobenius homomorphism.** Recall the automorphism \( \tau_\alpha \) defined by formula (2.3). It gives rise to an automorphism of \( \hat{U}_e \cdot \widehat{g} \) in an obvious way. Using Lemma 4.6, we obtain:

**Lemma 5.1.** We have: \( \text{Fr} \circ \tau_\alpha = \tau_\alpha \circ \text{Fr.} \) In particular, \( \text{Fr} \circ \tau_{\epsilon} = \text{Fr} \).
Denote by $\tau_0^*$ the endomorphism of $\text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}}$ induced by $\tau_0$. The next lemma follows from Lemmas 4.3 and 4.4.

**Lemma 5.2.** Let $J \subset I$ correspond to a Dynkin subgraph of $\mathfrak{g}$ and let $\mathfrak{g}_J$ be the corresponding Lie algebra. If the functions $o(i)$ for diagrams $I$ and $J$ coincide or $l$ is odd, then the following diagram is commutative.

\[
\begin{array}{ccc}
\text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}} & \xrightarrow{\text{Fr}^*} & \text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}} \\
\downarrow \text{res} & & \downarrow \text{res} \\
\text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}}_J & \xrightarrow{\text{Fr}^*_J} & \text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}}_J
\end{array}
\]

If the functions $o(i)$ for diagrams $I$ and $J$ do not coincide and $l$ is even then $\text{Fr}^* \circ \tau_{-1}^* = \tau_0 \circ \text{Fr}^*$ and the following diagram is commutative.

\[
\begin{array}{ccc}
\text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}} & \xrightarrow{\text{Fr}^*} & \text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}} \\
\downarrow \text{res} & & \downarrow \text{res} \\
\text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}}_J & \xrightarrow{\text{Fr}^* \circ \tau_{-1}^*} & \text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}}_J
\end{array}
\]

The algebras $\hat{U}_{\epsilon}^{\text{res}} \hat{\mathfrak{g}}$, $\hat{U}_{\epsilon}^{\text{res}} \mathfrak{g}$ and the corresponding Frobenius map are defined similarly to the affine case. Then the following property follows immediately from the definitions in terms of the Drinfeld-Jimbo generators.

**Lemma 5.3.** The following diagram is commutative:

\[
\begin{array}{ccc}
\text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}} & \xrightarrow{\text{Fr}^*} & \text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}} \\
\downarrow \text{res} & & \downarrow \text{res} \\
\text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}} & \xrightarrow{\text{Fr}^*} & \text{Rep} \, U_{\epsilon}^{\text{res}} \hat{\mathfrak{g}}
\end{array}
\]

5.2. **The decomposition theorem.** For a polynomial $P(u)$ with constant term 1, define polynomials with constant term 1, $P^0(u)$ and $P^1(u)$, to be such that $P(u) = P^0(u)P^1(u)$, $P^0(u)$ is not divisible by $1 - au^l$, for any $a \in \mathbb{C}^\times$ and $P^1(u) = R(u')$ for some polynomial $R$. If $\mathbf{P} = (P_i(u))_{i \in I}$, we write: $\mathbf{P}^0 = (P^0_i(u))_{i \in I}$ and $\mathbf{P}^1 = (P^1_i(u))_{i \in I}$.

Let $V$ be a $\hat{U}_{\epsilon}^{\text{res}} \mathfrak{g}$ module. We call the $\hat{U}_{\epsilon}^{\text{res}} \mathfrak{g}$-module $\text{Fr}^*(V)$ obtained by pull-back of $V$ via the quantum Frobenius homomorphism the Frobenius pull-back of $V$.

**Theorem 5.4.** Let $V(\mathbf{P})$ be an irreducible $\hat{U}_{\epsilon}^{\text{res}} \hat{\mathfrak{g}}$ module with Drinfeld polynomials $\mathbf{P}$. Then $V(\mathbf{P}) \simeq V(\mathbf{P}^0) \otimes V(\mathbf{P}^1)$. Moreover $V(\mathbf{P}^1)$ is the Frobenius pull-back of an irreducible $\hat{U}_{\epsilon}^{\text{res}} \hat{\mathfrak{g}}$-module and $V(\mathbf{P}^0)$ is irreducible over $U_{\epsilon}^{\text{fin}} \mathfrak{g}$.

**Proof.** This theorem is proved in [CP4], Theorems 9.1–9.3, in the case when $\epsilon$ is a root of unity of odd order (i.e., $s = l$ and $l$ is odd). The proof given in [CP4] goes through for other $l$ (under our assumption that $l$ is relatively prime to $r^s$) with the following changes: Lemma 9.5 of [CP4] should be replaced by Lemma 4.6 and the diagram (49) of [CP4] in the case of even $l$ should be replaced by the identity $\text{Fr} \circ ev_b = ev_{(b)} \circ \text{Fr}$ (which is proved in the same way as the commutativity of the diagram (49)). Finally,
the proof uses the description of the Drinfeld polynomials of the Frobenius pull-backs of irreducible $U_q^{\text{res}}$–modules. These Drinfeld polynomials are determined in Theorem 5.7 below.

Theorem 5.4 and Proposition 5.6 imply:

**Corollary 5.5.** There is an isomorphism of vector spaces

$$\text{Rep}U_q^{\text{res}} \overset{\cong}{\to} \text{Rep}U_q^{\text{res}} \otimes \text{Rep}U_q^{\text{fin}} \hat{g}.$$  

The decomposition in Corollary 5.3 is not a decomposition of rings. However, the Frobenius map $\text{Fr}^*: \text{Rep}U_q^{\text{res}} \hat{g} \to \text{Rep}U_q^{\text{res}} \hat{g}$ is a ring homomorphism. Therefore we have a natural ring structure on the quotient of $\text{Rep}U_q^{\text{res}} \hat{g}$ by its ideal generated by the augmentation ideal of $\text{Rep}U_q^{\text{res}} \hat{g}$. By Corollary 5.3, this quotient is isomorphic to $\text{Rep}U_q^{\text{fin}} \hat{g}$ as a vector space. We call the induced multiplication on $\text{Rep}U_q^{\text{fin}} \hat{g}$ the *factorized tensor product*. It would be interesting to extend it to the level of the category of finite-dimensional representations of $U_q^{\text{res}} \hat{g}$.

The statement of Theorem 5.4 remains true in the case when $l$ is relatively prime to $r^V$. However, in the course of proving it we need to use information about the Frobenius homomorphism which in that case takes values in the (modified) twisted quantum affine algebra. This case will be discussed in a separate paper.

### 5.3. The $\epsilon$–characters of the Frobenius pull-backs

Recall the notation

$$Y_{i,a} := \prod_{j=0}^{l-1} Y_{i,a^j}.$$  

Note that because $l$ is relatively prime to $r^V$, and hence by $r_i$, we have: $Y_{i,a} = \prod_{j=0}^{l-1} Y_{i,a^j}$. The monomial $Y_{i,a}$ corresponds to the polynomial $(1 - a^l u^l)$.

Note that we have a homomorphism $\chi_{\epsilon^*}: \text{Rep}U_q^{\text{res}} \hat{g} \to \mathbb{Z}[\hat{Y}_{i,a}]_{i \in I}^\ell$.

**Lemma 5.6.** Let $V$ be an irreducible representation of $U_q^{\text{res}} \hat{s}\hat{l}_2$ with Drinfeld polynomial $P(u)$. Then the Drinfeld polynomial of $\text{Fr}^*(V)$ is $P(u) = \tilde{P}(u^l)$, if $l$ is odd, and $\tilde{P}(a^l u)\bar{u}$, if $l$ is even.

Moreover, $\chi_{\epsilon}(\text{Fr}^*(V))$ is obtained from $\chi_{\epsilon^*}(V)$ by replacing $\hat{Y}_{1,a^l}^{\pm 1}$ with $\hat{Y}_{1,a}^{\pm 1}$, if $l$ is odd, and with $\hat{Y}_{1,a^l}^{\pm 1}$, where $\theta(1) = (1 - a(1))/2$, if $l$ is even.

**Proof.** Note that if $a^l = b^l$ then $(a/b)^l = 1$ and $Y_{i,a} = Y_{i,b}$. Therefore the rule described in the theorem is unambiguously defined.

Recall that

$$\text{Fr}^*: \text{Rep}U_q^{\text{res}} \hat{s}\hat{l}_2 \to \text{Rep}U_q^{\text{res}} \hat{s}\hat{l}_2,$$

as well as $\chi_{\epsilon}$ and $\chi_{\epsilon^*}$, are ring homomorphisms. Since the ring $U_q^{\text{res}} \hat{s}\hat{l}_2$ is generated by the fundamental representation $V_{\omega_1}(a)$, it is sufficient to prove the lemma when $V = V_{\omega_1}(a)$.

In this case the Drinfeld polynomial is $\tilde{P}(u) = 1 - au$. Let $v$ be the highest weight vector in $V$ and $\text{Fr}^*(V_{\omega_1}(a))$. By Lemma 5.1 in [CP4], we obtain:

$$(-1)^m \epsilon^{-m^2} k^m P_m v = (x_{1,0})_m^+ (x_{1,1})_m^- v.$$
(Note that $P_m$ in this paper differs from $P_m$ in \cite{CP2} by the factor $q^{-m}$.) So, $P_mv = 0$ unless $m$ is divisible by $l$.

If $m = nl$, then for odd $l$ we obtain:

$$(x^+_{1,0})^{(m)}(x^-_{1,1})^{(m)}v = (\bar{x}^+_{1,0})^{(n)}(\bar{x}^-_{1,1})^{(n)}v = (-1)^m(q^2)^{-n^2+n}\bar{P}_nv.$$

We have: $\bar{P}_nv = 0$, unless $n = 1$, and $\bar{P}_1v = -av$. In addition, $kv = \epsilon_1^2v, kv = \epsilon_1v$.

Hence we find: $P_m = 0$, unless $m = l$, and $P_1v = -av$.

For even $l$ we obtain:

$$(x^+_{1,0})^{(m)}(x^-_{1,1})^{(m)}v = o(0)^n(\bar{x}^+_{1,0})^{(n)}(\bar{x}^-_{1,1})^{(n)}v = (-1)^n\epsilon(1)^{-n^2+n\bar{k}}\bar{P}_nv.$$

We have: $\bar{P}_nv = 0$, unless $n = 1$, and $\bar{P}_1v = -av$. Hence we find that $P_m = 0$, unless $m = l$, and $P_1v = -o(1)av$.

It remains to determine the $\epsilon$-character of $\Fr^*(V_{\omega_1}(\delta'))$. Consider the case when $l$ is odd. Since the Drinfeld polynomial of this module is $1 - a'\bar{u}', we find that the highest weight monomial is $Y_{1,a}$. By Proposition 5.3 and Theorem 3.2, the other monomial in it must be obtained by specialization of a monomial in the $q$-character of the irreducible $U_{\hat{\frak{sl}}_2}$-module with the highest weight monomial $\prod_{l=0}^{n-1}Y_{1,\sigma_l}$. Moreover, by Lemma 5.3, the degree of this monomial must be equal to $-l$. There is only one monomial of such degree, i.e., the lowest weight monomial, and it is equal to $\prod_{l=0}^{n-1}Y_{1,\sigma_l}$. Therefore we obtain that the $\epsilon$-character of $\Fr^*(V_{\omega_1}(\delta'))$ is equal to $Y_{1,a} + Y_{1,a}^{-1}$. On the other hand, we have: $\chi_\epsilon^*(V_{\omega_1}(\delta')) = Y_{1,a}' + Y_{1,a}'^{-1}$, cf. Section 5.4, and the second assertion of the lemma follows for odd $l$.

If $l$ is even, we obtain by the same argument that the $\epsilon$-character of $\Fr^*(V_{\omega_1}(\delta'))$ is equal to $Y_{1,a}^\theta(1) + Y_{1,a}^{-\theta(1)}$. This proves the second assertion of the lemma for even $l$.

We can now obtain the description of the $\epsilon$-characters of Frobenius pull-backs for general $\frak{g}$. Recall that we are under the assumption that $l$ is relatively prime to $\epsilon^\nu$.

Set $\theta(i) = (1 - o(i))/2$.

**Theorem 5.7.** Let $V$ be an irreducible representation of $U_{\epsilon^*}^{\res}\frak{g}$ with Drinfeld polynomials $P_i(u), i \in I$. Then the $i$-th Drinfeld polynomial of $\Fr^*(V)$ is equal to $P_i(u^l)$, if $l$ is odd, and to $P_i(o(i)u^l)$ if $l$ is even.

Moreover, if $l$ is odd, then $\chi_\epsilon(\Fr^*(V))$ is obtained from $\chi_\epsilon^*(V)$ by replacing $\bar{Y}_{i,a}^{\pm1}$ with $Y_{i,a}^{\pm1}$. If $l$ is even, then $\chi_\epsilon(\Fr^*(V))$ is obtained from $\chi_\epsilon^*(V)$ by replacing $\bar{Y}_{i,a}^{\pm1}$ with $Y_{i,a}^{\pm1}$.

**Proof.** Restricting to the subalgebras $U_{\epsilon^*}^{\res}\frak{sl}_2 \subset U_{\epsilon^*}^{\res}\frak{g}$ and using Lemmas 5.2 and 5.3, we obtain the statement of the theorem from Lemma 5.6.

**Theorem 5.7** allows us to write down explicit formulas for the $\epsilon$-characters of Frobenius pull-backs from the $\epsilon^*$-characters of irreducible $U_{\epsilon^*}^{\res}\frak{g}$-modules, which we now set out to determine.
5.4. The $\epsilon^*$-characters of irreducible $U_{1,\epsilon}^{\text{res}}\hat{g}$-modules. First, we consider the case when $\epsilon^* = 1$ (i.e., $l$ is odd and $s = l$, or $l$ is even). There is a surjective homomorphism $U_{1,\epsilon}^{\text{res}}\hat{g} \to U\hat{g}$ (where $U\hat{g}$ is the universal enveloping algebra of $\hat{g} = g[t, t^{-1}]$), which sends the generators $(x_i^\pm)^{(m)}$ of $U_{1,\epsilon}^{\text{res}}\hat{g}$ to the corresponding generators of $U\hat{g}$. In fact, this map gives rise to an isomorphism between $U\hat{g}$ and the quotient of $U_{1,\epsilon}^{\text{res}}\hat{g}$ by the ideal generated by the central elements $(K_i - 1), i \in I$ (see [CP2], Proposition 9.3.10).

Hence the category of finite-dimensional type 1 $U_{1,\epsilon}^{\text{res}}\hat{g}$-modules is equivalent to the category of finite-dimensional $\hat{g}$-modules, on which the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} \subset \hat{g}$ acts diagonally (we call them weight modules).

The description of irreducible finite-dimensional representations of $\hat{g}$ is as follows [CP1]. Consider the “evaluation homomorphism” $\phi_a : \hat{g} = g[t, t^{-1}] \to g$ corresponding to evaluating a Laurent polynomial in $t$ at a point $a \in \mathbb{C}^\times$. For an irreducible $\mathfrak{g}$-module $V_\lambda$ with highest weight $\lambda$, let $V_\lambda(a)$ be its pull-back under $\phi_a$ to an irreducible representation of $\hat{g}$. Then $V_\lambda(a_1) \otimes \ldots \otimes V_\lambda(a_n)$ is irreducible if $a_i \neq a_j, \forall i \neq j$, and these are all irreducible finite-dimensional representations of $\hat{g}$ up to an isomorphism.

Therefore in order to obtain $\chi_1(V)$ for an arbitrary irreducible $U_{1,\epsilon}^{\text{res}}\hat{g}$-module, it suffices to know $\chi_1(V_\lambda(a))$. Those are found by explicitly computing the image of $P_i^\pm(u)$ under the evaluation homomorphism. The answer is the following. Let $\chi(V_\lambda)$ be the ordinary character of the $\mathfrak{g}$-module $V_\lambda$, considered as a polynomial in $y_i^{\pm 1}, i \in I$, as in Section 4.3.

**Lemma 5.8.** $\chi_1(V_\lambda(a))$ is obtained from $\chi(V_\lambda)$ by replacing each $y_i^{\pm 1}$ with $Y_i^{\pm 1}$.

Combining this statement with Theorem 5.7, we obtain a complete description of the $\epsilon$-characters of irreducible Frobenius pull-backs in the case when $\epsilon^* = 1$.

Next, consider the case when $\epsilon^* = -1$ (i.e., $s = 2l$ and $l$ is odd). Introduce a function $\psi : I \to \{\pm 1\}$ by the rule: $\psi(i) \neq \psi(j)$, whenever $i \neq j$, $(\alpha_i, \alpha_j) \neq 0$, and $\min\{r_i, r_j\} = 1$. We fix this function by the requirement that $\alpha(i) = \psi(i)$ if $r_i = 1$.

We say that a monomial $m \in \mathbb{Z}[Y_i^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ is supported at $a_0 \in \mathbb{C}^\times$ if $m \in \mathbb{Z}[Y_i^{\pm 1}]_{i \in I, a_0 \psi(i) \in k \in \mathbb{Z}}$. Any monomial can be written in a unique way as a product of monomials supported at some $a_1, \ldots, a_k \in \mathbb{C}^\times$, $a_i \neq a_j$.

**Lemma 5.9.** Let $V$ be an irreducible finite-dimensional $U_{-1}^{\text{res}}\hat{g}$-module (of type 1), such that the highest weight monomial in $\chi_{-1}(V)$ is the product $m_1 \ldots m_k$, where $m_i$ is a monomial supported at $a_i \in \mathbb{C}^\times$ and has weight $\lambda_i$. Assume that all $\lambda_i$ are distinct, $a_i \neq a_j$. Then $V \simeq V_1 \otimes \ldots \otimes V_k$, where $V_i$ is an irreducible finite-dimensional $U_{-1}^{\text{res}}\hat{g}$-module with highest weight monomial $m_i$.

Moreover, let $m_i$ have weight $\lambda_i$. Then $\chi_{-1}(V_i)$ is obtained from the ordinary character $\chi(V_\lambda)$ of the irreducible $\mathfrak{g}$-module $V_\lambda$, by replacing each $y_i^{\pm 1}$ with $Y_i^{\pm 1}$.

**Proof.** According to [4], Proposition 33.2.3, the algebras $U_{-1}^{\text{res}}\hat{g}$ and $U_{1,\epsilon}^{\text{res}}\hat{g}$ are isomorphic, if $\mathfrak{g} \neq \mathfrak{sl}_{2n+1}$. Therefore the category of type 1 finite-dimensional $U_{-1}^{\text{res}}\hat{g}$-modules is isomorphic to the category of weight $\mathfrak{g}$-modules. The statement of the lemma is obtained by combining this fact with Theorem 5.2 and the following result about $\chi_q$-characters, which follows from [FM], Corollary 6.4:
Set $\varphi(i) = (1 + \psi(i))/2$. Suppose that $V$ is an irreducible $U_q\widehat{\mathfrak{g}}$–module, such that the highest weight monomial $m$ in $\chi_q(V)$ belongs to $\mathbb{Z}[Y_{i,a_0q^{\varphi(i)}+2k}]_{i \in I}^{k \in \mathbb{Z}}$. Then all other monomials in $\chi_q(V)$ belong to $\mathbb{Z}[Y_{i,a_0q^{\varphi(i)}+2k}]_{i \in I}^{k \in \mathbb{Z}}$.

In the case of $\mathfrak{g} = \mathfrak{sl}_{2n+1}$, we use the inclusion $U_q_{-1}\mathfrak{sl}_{2n+1} \rightarrow U_q_{-1}\mathfrak{sl}_{2n+2}$ and Lemma 5.3 follows from Lemma 3.4.

Combining this lemma with Theorem 5.7, we obtain a complete description of the $\epsilon$–characters of irreducible Frobenius pull-backs in the case when $\epsilon^* = -1$.

These results together with Theorem 5.4 reduce the problem of calculating the $\epsilon$–characters of irreducible finite-dimensional representations of $U_{\epsilon^*}^{\text{res}}\widehat{\mathfrak{g}}$ to the problem of calculating $\epsilon$–characters of those representations which remain irreducible when restricted to the subalgebra $U_{\epsilon^*}^{\text{fin}}\widehat{\mathfrak{g}}$.

In particular, we obtain explicit formulas for the $\epsilon$–characters of all irreducible finite-dimensional $U_{\epsilon^*}^{\text{fin}}\mathfrak{sl}_2$–modules (cf. [CP4], Theorem 9.6). We conjecture that in the case when $s > 2$ the $\epsilon$–characters of irreducible modules can be constructed by applying the algorithm described in Section 5.5 of [FM].

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