Generalized thermodynamics and kinetic equations: Boltzmann, Landau, Kramers and Smoluchowski

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Abstract

We propose a formal extension of thermodynamics and kinetic theories to a larger class of entropy functionals. Kinetic equations associated to Boltzmann, Fermi, Bose and Tsallis entropies are recovered as a special case. This formalism first provides a unifying description of classical and quantum kinetic theories. On the other hand, a generalized thermodynamical framework is justified to describe complex systems exhibiting anomalous diffusion. Finally, a notion of generalized thermodynamics emerges in the context of the violent relaxation of collisionless stellar systems and two-dimensional vortices due to the existence of Casimir invariants and incomplete relaxation. A thermodynamical analogy can also be developed to analyze the nonlinear dynamical stability of stationary solutions of the Vlasov and 2D Euler-Poisson systems. On general grounds, we suggest that generalized entropies arise due to the existence of “hidden constraints” that modify the form of entropy that we would naively expect. Generalized kinetic equations are therefore “effective” equations that are introduced heuristically to describe complex systems.

1 Introduction

Standard kinetic equations satisfy two fundamental properties linked to the first and second principles of thermodynamics: the conservation of energy (and mass) and the increase of entropy (H-theorem). These properties are shared in particular by the Boltzmann and by the Landau equations which are at the basis of the kinetic theory of dilute gases and neutral plasmas [1]. When the system is in contact with a thermostat, instead of being isolated, the conservation of energy and the increase of entropy (microcanonical description) are replaced by the decrease of free energy $F = E - TS$ at fixed temperature (canonical description). This is the proper description of Brownian motion that is analyzed in terms of stochastic processes and Fokker-Planck equations (Kramers, Smoluchowski,...) [2]. In the standard framework, the functional increasing monotonically with time is the Boltzmann entropy $S_B[f] = - \int f \ln f d^3r d^3v$ or the Boltzmann free energy $J_B[f] = S_B[f] - \beta E[f]$.

In a recent paper [3], we have proposed to develop a generalized thermodynamical formalism for a larger class of functionals that we called generalized entropies. They can be written $S[f] = - \int C(f) d^3r d^3v$ where $C(f)$ is a convex function, i.e. $C''(f) > 0$. Boltzmann, Fermi, Bose, and Tsallis entropies are particular functionals of the above form. On general grounds,
what we mean by generalized thermodynamics is the extension of the usual variational principle of classical thermodynamics (maximization of Boltzmann entropy $S_B[f]$ at fixed mass $M$ and energy $E$) to a larger class of functionals. This variational principle arises in many problems of physics (or biology, economy,...) for various reasons that do not have necessarily a direct relation to thermodynamics. In many cases, it is relevant to develop a thermodynamical analogy and to use the same vocabulary as in standard thermodynamics. This allows one to transpose directly the standard methods developed in ordinary thermodynamics to a new context.

In [3] we have proposed a generalized class of Fokker-Planck equations associated to this generalized thermodynamical framework. This formalism is interesting to develop because it leads to a unified description of known kinetic theories (classical and quantum) and it also generates new types of kinetic equations. In that respect, it can be of interest in applied mathematics and theoretical physics. This formalism can also have important physical applications. For example, these generalized Fokker-Planck equations can account for a process of anomalous diffusion in complex media. They arise naturally from ordinary Fokker-Planck equations by assuming that the diffusion coefficient is a function of the density. Generalized thermodynamics can also be relevant for the violent relaxation of stellar systems and two-dimensional (2D) vortices. In that context, generalized entropies (also called $H$-functions) emerge due to the existence of fine-grained constraints (Casimir invariants) that modify the form of entropy that we would naively expect. Generalized Fokker-Planck equations can provide a simple small-scale parametrization of turbulence (mixing) in stellar dynamics and 2D hydrodynamics. They can also serve as powerful numerical algorithms to compute arbitrary nonlinearly dynamically stable solutions of the 2D Euler-Poisson or Vlasov-Poisson systems. Indeed, the condition of nonlinear dynamical stability can be put in a form analogous to a condition of generalized thermodynamical stability [4, 5, 6, 7]. This is a striking illustration of the thermodynamical analogy mentioned above. Since the notion of generalized thermodynamics can have different interpretations, it is relevant to work at a general level and develop a formalism without explicit reference to a precise context. Then, a justification of this generalization and a physical interpretation of the results must be given in each case.

In an early work [8], we observed that classical and quantum Fokker-Planck equations can be obtained from a phenomenological Maximum Entropy Production Principle (MEPP) [9] by maximizing the rate of entropy (resp. free energy) production at fixed mass and energy (resp. temperature). This variational approach, closely related to the linear thermodynamics of Onsager, is the most natural extension of the equilibrium thermodynamical principle in which we maximize entropy (resp. free energy) at fixed mass and energy (resp. temperature). In [3], we generalized this principle to a larger class of entropy functionals and obtained generalized Fokker-Planck equations. We also “guessed” a form of generalized Landau equation consistent to the generalized Kramers equation obtained with the MEPP. Recently, we found that a similar generalization of kinetic theory was attempted by Kaniadakis [11] from a different point of view. His approach consists in generalizing the assumptions that are made at the start to derive the Boltzmann and the Fokker-Planck equations. This amounts to modifying the form of the transition probabilities that arise in the dynamical process. Such generalized transition probabilities are relevant for quantum particles (fermions and bosons) and possibly, also, in the physics of complex media. The MEPP approach, on the other hand, is purely thermodynamical and exploits at best the first and second principles of thermodynamics (possibly extended to generalized functionals) in a viewpoint reminiscent of Jaynes’ ideas.

In this paper, we show that the two approaches lead to equivalent kinetic equations. In Sec. 2 we use Kaniadakis approach to derive the generalized Landau equation from the generalized Boltzmann equation in a weak deflexion approximation. In Sec. 3 we show that the generalized Landau equation can also be obtained by coarse-graining the Vlasov equation in the context
of the violent relaxation of stellar systems. In Sec. 4 we establish the main properties of
the generalized Landau equation (conservation laws, generalized H-theorem,...). In Sec. 5
we derive the generalized Kramers equation from the generalized Landau equation in a test
particle approach and a thermal bath approximation. The generalized Smoluchowski equation
is in turn derived from the generalized Kramers equation in a hydrodynamical limit. In Secs. 6
and 7 we determine explicit expressions of the diffusion coefficient for Boltzmann, Fermi and
Tsallis distributions. In Sec. 8 we use the generalized Kramers equation to derive generalized
truncated distribution functions accounting for an escape of particles above a limit energy.

2 Generalized kinetic equations

2.1 The generalized Boltzmann equation

The ordinary Boltzmann equation can be written in the form

\[ \frac{df}{dt} = \int d^3v_1 \, dΩ \, w(v, v_1; v', v'_1) \left\{ f(v') f(v'_1) - f(v) f(v_1) \right\}, \]

where \( dΩ \) is the element of solid angle and \( w(v, v_1; v', v'_1) \) is the density probability of a collision
transforming the velocities \( v, v_1 \) in \( v', v'_1 \) or the converse (the abbreviations \( f(v'), f(v'_1), f(v),
\( f(v_1) \) stand for \( f(r, v', t), f(r, v'_1, t), f(r, v, t), f(r, v', t) \)). The material derivative is \( d/dt = \partial/\partial t + v \partial/\partial r + F \partial/\partial v \) where \( F = -\nabla \Phi \) is a mean-field force acting on the particles. The Boltzmann equation conserves the mass

\[ M = \int \rho d^3r, \]

the energy

\[ E = \int f \frac{v^2}{2} d^3rd^3v + \frac{1}{2} \int \rho \Phi d^3r, \]

the angular momentum

\[ L = \int f r \times v d^3rd^3v, \]

and the impulse

\[ P = \int f v d^3rd^3v, \]

where \( \rho(r, t) \) is the spatial density. In addition, the Boltzmann entropy

\[ S = - \int f \ln f d^3rd^3v, \]

satisfies a H-theorem, i.e. \( \dot{S} \geq 0 \) with \( \dot{S} = 0 \) if, and only if, the distribution \( f(r, v) \) is the
Boltzmann distribution

\[ f_{eq}(r, v) = Ae^{-\beta \epsilon'}, \]

where \( \epsilon' = \frac{v^2}{2} + \Phi + \Omega \cdot (r \times v) + U \cdot v \) is the energy of a particle by unit of mass.
Recently, Kaniadakis [10] has proposed the generalization

\[
\frac{df}{dt} = \int d^3v_1 \, d\Omega \, w(v, v_1, v_1') \left\{ a(f') b(f) a(f_1) b(f_1') - a(f) b(f') a(f_1) b(f_1') \right\},
\]

where the functions \(a(f)\) and \(b(f)\) are somewhat arbitrary (we have noted \(f = f(v)\), \(f_1 = f(v_1)\), \(f' = f(v')\) and \(f_1' = f(v_1')\)). This generalization encompasses the case of quantum particles (fermions and bosons) with exclusion or inclusion principles. This generalization could also be relevant in the case of complex systems for which the transition probabilities are not simply the product of distribution functions. This can happen when we are not in the strict conditions of validity of the ordinary Boltzmann equation. It is also possible that Eq. (8) is just an effective kinetic equation accounting for “hidden constraints” in complex media.

In the following, we shall consider the situation in which the potential of interaction between particles is Coulombian (or Newtonian). In that case, each encounter provokes a weak deflexion of the particles trajectory and it is of order to consider the weak deflexion limit of the Boltzmann equation. Classically, this leads to the so-called Landau equation which forms the basis of the kinetic theory of neutral plasmas [1] and stellar systems [11]. Our aim is this section is to derive a generalized Landau equation from the generalized Boltzmann equation proposed by Kaniadakis. This generalization is essentially formal. In the following, we follow the classical derivation of the Landau equation reported in the monograph of Balescu [1]. Therefore, we shall omit the calculations that are identical to the classical case and refer to [1] for more details.

### 2.2 The weak deflexion approximation

First, it is convenient to write the velocities of the particles before and after the collision as

\[
v' = v + \Delta,
\]

\[
v_1' = v_1 - \Delta,
\]

where \(\Delta\) is the velocity deviation. We can now express the probability of a collision in terms of new variables as [11]:

\[
w(v, v_1; v', v_1') \rightarrow w(v + \frac{\Delta}{2}, v_1 - \frac{\Delta}{2}; \Delta).
\]

The generalized Boltzmann equation can thus be rewritten

\[
\frac{df}{dt} = \int d^3v_1 \, d\Omega \, w(v + \frac{\Delta}{2}, v_1 - \frac{\Delta}{2}; \Delta) \left\{ a[f(v + \Delta)] b[f(v)] a[f(v_1 - \Delta)] b[f(v_1)] - a[f(v)] b[f(v + \Delta)] a[f(v_1)] b[f(v_1 - \Delta)] \right\}.
\]

In the weak deflexion limit \(|\Delta| \ll |v|, |v_1|\), we can expand the r.h.s. of Eq. (12) in Taylor series. Using

\[
w(v + \frac{\Delta}{2}, v_1 - \frac{\Delta}{2}; \Delta) \approx w(v, v_1; \Delta) + \frac{1}{2} \partial^\mu \left\{ \frac{\partial w(v, v_1; \Delta)}{\partial v_1^\mu} - \frac{\partial w(v, v_1; \Delta)}{\partial v^\mu} \right\} + \ldots,
\]

\[
f(v + \Delta) = f(v) + \Delta^\mu \frac{\partial f(v)}{\partial v^\mu} + \frac{1}{2} \partial^\mu \Delta^\nu \frac{\partial^2 f(v)}{\partial v_1^\mu \partial v_1^\nu} + \ldots,
\]
we get
\[
\frac{df}{dt} = \frac{1}{2} \int d^3v_1 \, d\Omega \Delta^\mu \Delta^\nu \left\{ w(v, v_1; \Delta) \left[ ab(b_1a' - a_1b') \frac{\partial^2 f_1}{\partial v^\mu \partial v^\nu} + ab(b_1a'' - a_1b'') \frac{\partial f_1}{\partial v^\mu} \frac{\partial f_1}{\partial v_1^\nu} \right] \\
+ a_1b_1(ba' - b'a) \frac{\partial f}{\partial v^\mu} \right\}
\]

where \(a = a[f(v)], a_1 = a[f(v_1)], a' = a'[f(v)], a_1' = a'[f(v_1)]\) etc. Integrating the last term by parts, the foregoing expression simplifies in
\[
\frac{df}{dt} = \frac{1}{2} \int d^3v_1 \, d\Omega \Delta^\mu \Delta^\nu \left\{ w(v, v_1; \Delta) \left[ a_1b_1(ba' - b'a) \frac{\partial f}{\partial v^\mu} \right] \right\}
\]

Equation (16) can be written more compactly as
\[
\frac{df}{dt} = \frac{\partial}{\partial v^\mu} \int d^3v_1 \, K^{\mu \nu} \left\{ a_1b_1(ba' - b'a) \frac{\partial f}{\partial v^\nu} - ab(b_1a' - b_1a) \right\}
\]

where we have defined
\[
K^{\mu \nu} = \frac{1}{2} \int d\Omega \, w(v, v_1; \Delta) \Delta^\mu \Delta^\nu.
\]

Equation (17) can be rewritten
\[
\frac{df}{dt} = \frac{\partial}{\partial v^\mu} \int d^3v_1 \, K^{\mu \nu} \left\{ g(f_1)h(f) \frac{\partial f}{\partial v^\nu} - g(f)h(f_1) \frac{\partial f_1}{\partial v^\nu} \right\}
\]

This will be called the generalized Landau equation. The tensor \(K^{\mu \nu}\) can be calculated explicitly in the linear trajectory approximation \([1]\). The result can be expressed as
\[
K^{\mu \nu} = \frac{A}{u} \left( \delta^{\mu \nu} - \frac{u^\mu u^\nu}{u^2} \right).
\]

where \(u = v_1 - v\) is the relative velocity and \(A\) is a constant (in the plasma case \(A = (e^4/8\pi m^2c_0^2) \ln(L_{\text{Debye}}/L_{\text{min}})\) and in the gravitational case \(A = 2\pi G^2m \ln(L_{\text{max}}/L_{\text{min}})\)).

By developing a kinetic theory of 2D point vortices \([12, 13]\), we have derived a kinetic equation of the form
\[
\frac{\partial P}{\partial t} + \langle V \rangle \nabla P = \frac{N\gamma^2}{8} \frac{\partial}{\partial r^\mu} \int d^3r_1 K^{\mu \nu} \delta(\mathbf{r} \cdot \mathbf{v}) \left( P_1 \frac{\partial P}{\partial r^\nu} - P \frac{\partial P_1}{\partial r_1^\nu} \right).
\]
where

\[ K^{\mu\nu} = \frac{\xi^2 \delta^{\mu\nu} - \xi^\mu \xi^\nu}{\xi^2}, \]

and \( \xi = r_1 - r, \ v = \langle V \rangle(r_1, t) - \langle V \rangle(r, t) \). This equation conserves all the constraints of the point vortex model and increases the Boltzmann entropy \((H\text{-theorem})\). It is reminiscent of the Landau equation but it differs from the Landau equation due to the \( \delta \)-function which takes into account the conservation of energy \( E = \frac{1}{2} \int \omega \psi d^2r \) where \( \psi (r, t) \) is the stream-function. In addition, its physical interpretation and derivation is completely different from that of the Landau equation. We can heuristically propose a formal extension of this equation to the more general form

\[ \frac{\partial P}{\partial t} + \langle V \rangle \nabla P = \frac{N\gamma^2}{8} \frac{\partial}{\partial r^\mu} \int d^2r_1 K^{\mu\nu} \delta (\xi \cdot v) \left( g(P_e)h(P) \frac{\partial P}{\partial r^\nu} - g(P)h(P_e) \frac{\partial P_1}{\partial r_1^\nu} \right). \]

### 2.3 Generalized entropy

We shall say that a kinetic equation possesses a generalized microcanonical thermodynamical structure if it conserves mass and energy and increases continuously a functional of the form

\[ S = - \int C(f) d^3r d^3v, \]

where \( C(f) \) is a convex function. By analogy with ordinary thermodynamics, the functional \( \delta S = \beta \delta E - \alpha \delta M + \beta \Omega \delta L + \beta U \delta P = 0 \),

where we have also accounted for the conservation of angular momentum and impulse, we find that the equilibrium distribution function is given by

\[ C'(f_{eq}) = -\beta \left( \frac{v^2}{2} + \Phi \right) + \beta \Omega (r \times v) + \beta U v - \alpha. \]

As we shall see, the generalized Landau equation \( \text{(20)} \) possesses a microcanonical thermodynamical structure. The generalized entropy can be expressed in terms of the functions \( a(f) \) and \( b(f) \) as

\[ S = - \int C(f) d^3r d^3v, \quad \text{with} \quad C'(f) = \ln \left[ \frac{a(f)}{b(f)} \right]. \]

From Eqs. \( \text{(28)} \) and \( \text{(19)} \), we obtain the relation

\[ h(f) = g(f) C''(f), \]

which leads to the identities

\[ a(f) = \sqrt{g(f)} e^{\frac{1}{2} C'(f)} , \quad b(f) = \sqrt{g(f)} e^{-\frac{1}{2} C'(f)} . \]

If we know \( a \) and \( b \), we can obtain \( g \) and \( h \) from Eq. \( \text{(19)} \) and \( C \) from Eq. \( \text{(28)} \). Alternatively, if we know \( h \) and \( g \), we obtain \( C''(f) \) from Eq. \( \text{(24)} \) and deduce \( a \) and \( b \) from Eq. \( \text{(30)} \) (up to a multiplicative factor). If we only specify \( C(f) \), we cannot obtain \( a \) and \( b \) individually but only the ratio \( a/b \). In the next section, we shall consider simplified forms of the generalized Landau equation where everything is determined by the specification of the generalized entropy \( S[f] \), or equivalently by the function \( C(f) \).
2.4 Simplified forms of the generalized Landau equation

We shall first impose that

\[ g(f) = f, \quad h(f) = fC''(f), \]

where \( C \) is a convex function. In that case, Eq. (30) takes the form

\[ \frac{df}{dt} = \frac{\partial}{\partial v^\mu} \int d^3v_1 \ K^{\mu\nu} f f_1 \left\{ C''(f) \frac{\partial f}{\partial v^\nu} - C''(f_1) \frac{\partial f_1}{\partial v^\nu} \right\}. \]

This generalized Landau equation was first written in [3]. From Eqs. (30) and (31) we find that \( a(f) \) and \( b(f) \) are related to \( C(f) \) by

\[ a(f) = \sqrt{fe^{\frac{1}{2}C'(f)}}, \quad b(f) = \sqrt{fe^{-\frac{1}{2}C'(f)}}. \]

It is interesting to consider particular cases of the generalized Landau equation (32). For the Boltzmann entropy \( C(f) = f \ln f \), we recover the ordinary Landau equation

\[ \frac{df}{dt} = \frac{\partial}{\partial v^\mu} \int d^3v_1 \ K^{\mu\nu} \left\{ f_1 \frac{\partial f}{\partial v^\nu} - f \frac{\partial f_1}{\partial v^\nu} \right\}, \]

which is the weak deflexion limit of the ordinary Boltzmann equation (1) corresponding to

\[ a(f) = f, \quad b(f) = 1, \]

in Eq. (8). For the Tsallis entropy \( C(f) = \frac{1}{q-1}(f^q - f) \), we obtain the \( q \)-Landau equation

\[ \frac{df}{dt} = \frac{\partial}{\partial v^\mu} \int d^3v_1 \ K^{\mu\nu} \left\{ f_1 \frac{\partial f^q}{\partial v^\nu} - f \frac{\partial f^q_1}{\partial v^\nu} \right\}, \]

which is associated to the \( q \)-Boltzmann equation (8) with

\[ a(f) = \sqrt{fe^{\frac{1}{2q-1}(q^q - 1)}}, \quad b(f) = \sqrt{fe^{-\frac{1}{2q-1}(q^q - 1)}}. \]

Instead of Eq. (31), we can impose the relations

\[ h(f) = 1, \quad g(f) = 1/C''(f). \]

In that case, Eq. (20) reduces to

\[ \frac{df}{dt} = \frac{\partial}{\partial v^\mu} \int d^3v_1 \ K^{\mu\nu} \left\{ \frac{1}{C''(f_1)} \frac{\partial f}{\partial v^\nu} - \frac{1}{C''(f)} \frac{\partial f_1}{\partial v^\nu} \right\}. \]

This alternative form was also written in [3]. From Eqs. (30) and (38), we find that \( a(f) \) and \( b(f) \) are related to \( C(f) \) by

\[ a(f) = \frac{1}{\sqrt{C''(f)}} e^{\frac{1}{2}C'(f)}, \quad b(f) = \frac{1}{\sqrt{C''(f)}} e^{-\frac{1}{2}C'(f)}. \]

For the entropy \( C(f) = f \ln f + (\eta_0 - \mu f) \ln(\eta_0 - \mu f) \), we get

\[ \frac{df}{dt} = \frac{\partial}{\partial v^\mu} \int d^3v_1 \ K^{\mu\nu} \left\{ f_1(\eta_0 - \mu f_1) \frac{\partial f}{\partial v^\nu} - f(\eta_0 - \mu f) \frac{\partial f_1}{\partial v^\nu} \right\}. \]
For $\mu = 1$, the foregoing equation describes the case of fermions accounting for the Pauli exclusion principle. For $\mu = 1$, it describes the case of bosons. For intermediate values of $\mu$ it describes a intermediate quantum statistics interpolating between the Bose and Fermi ones. The generalized Landau equation (41) is the weak deflexion limit of the generalized Boltzmann equation (8) with

$$a(f) = f, \quad b(f) = \eta_0 - \mu f.$$  

(42)

Although this extension of kinetic theory is interesting on a formal point of view, we do not claim that it is relevant to plasma physics and stellar dynamics (except in the quantum case). Indeed, in neutral plasmas where the interaction is short-ranged due to Debye shielding the diffusion is normal and the ordinary Landau equation is rigorously valid. On the other hand, in collisional stellar systems, the diffusion is only slightly anomalous due to logarithmic divergences [14]. Therefore, the ordinary Landau equation remains marginally valid when logarithmic divergences are properly regularized [15]. There can be corrections to the ordinary Landau equation due to memory effects and spatial delocalization [16] (similar effects arise in the kinetic theory of point vortices [12]). However, this does not apparently justify a rigorous notion of generalized thermodynamics (even if deviations to the Maxwellian distribution may be expected for intermediate times). For large times, ordinary thermodynamics (based on the Boltzmann entropy) is rigorously justified for collisional stellar systems (and point vortices) in a suitable thermodynamic limit although these systems are non-extensive and non-additive (see [17] for a more complete discussion). Therefore, there does not seem to be any justification of a generalized thermodynamics for Hamiltonian systems of point particles in the infinite time limit (collisional relaxation). At the present time, it is not clear to which systems the generalized kinetic theory developed previously could apply (with the exception of quantum particles). This is an open problem left for future investigations. Our guess is that generalized kinetic equations can serve as effective equations in the case of complicated systems. In the following section, we show that a notion of generalized thermodynamics also emerges in the context of the violent relaxation of collisionless stellar systems (and other Hamiltonian systems with long-range interactions) for a completely different reason. Generalized kinetic equations can find physical applications in that context.

3 Violent relaxation of collisionless stellar systems

3.1 The Vlasov-Poisson system

For most stellar systems, including the important class of elliptical galaxies, the encounters between stars are completely negligible [11, 17] and the galaxy dynamics is described by the self-consistent Vlasov-Poisson system

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} f + \mathbf{F} \cdot \frac{\partial}{\partial \mathbf{v}} f = 0,$$

(43)

$$\Delta \Phi = 4\pi G \int f d^3\mathbf{v}.$$  

(44)

Here, $f(\mathbf{r}, \mathbf{v}, t)$ denotes the distribution function (defined such that $f d^3\mathbf{r} d^3\mathbf{v}$ gives the total mass of stars with position $\mathbf{r}$ and velocity $\mathbf{v}$ at time $t$), $\mathbf{F}(\mathbf{r}, t) = -\nabla \Phi$ is the gravitational force (by unit of mass) experienced by a star and $\Phi(\mathbf{r}, t)$ is the gravitational potential related
to the star density $\rho(r, t) = \int f d^3v$ by the Newton-Poisson equation (44). The Vlasov equation (43) simply states that, in the absence of encounters, the distribution function $f$ is conserved by the flow in phase space. This can be written $df/dt = 0$ where $d/dt = \partial/\partial t + U_6 \nabla_6$ is the material derivative and $U_6 = (v, F)$ is a generalized velocity field in the 6-dimensional phase space $(r, v)$ (by definition, $\nabla_6 = (\partial/\partial r, \partial/\partial v)$ is the generalized nabla operator). Since the flow is incompressible, i.e. $\nabla_6 U_6 = 0$, the hypervolume of a “fluid” particle is conserved. Since, in addition, a fluid particle conserves the distribution function, this implies that the total mass (or hypervolume) of all phase elements with phase density between $f$ and $f + \delta f$ is conserved. This is equivalent to the conservation of the Casimir integrals $I_h = \int h(f) d^3r d^3v$ for any continuous function $h(f)$. This is also equivalent to the conservation of the moments $M_n = \int f^n d^3r d^3v$, which include in particular the total mass $M = \int f d^3r d^3v$. It is also straightforward to check that the Vlasov-Poisson system conserves the total energy $E$, the angular momentum $L$ and the impulse $P$.

3.2 The metaequilibrium state

The Vlasov-Poisson system develops very complex filaments as a result of a mixing process in phase space. If we introduce a coarse-graining procedure, the coarse-grained distribution function $f(r, v, t)$ will reach a metaequilibrium state $\tilde{f}(r, v)$ on a very short timescale, of the order of the dynamical time. This process is known as “phase mixing” and “violent relaxation” (11). Lynden-Bell (15) has tried to describe this metaequilibrium state in terms of statistical mechanics. If $\rho(r, v, \eta)$ denotes the density probability of finding the value $f = \eta$ of distribution function in $(r, v)$, then the mixing entropy is given by

$$S[\rho] = -\int \rho \ln \rho \, d^3r d^3v d\eta.$$  

(46)

It can be obtained by a standard combinatorial analysis (15) or by using the concept of Young measures and large deviations (19). Assuming ergodicity (which may not be realized in practice, see below) the statistical equilibrium state is obtained by maximizing $S[\rho]$ while conserving mass $M$, energy $E$ and all the Casimirs (or moments $M_n$). The optimal equilibrium state can be written (15, 4)

$$\rho(r, v, \eta) = \frac{1}{Z} \chi(\eta) e^{-(\beta\epsilon + \alpha)\eta},$$  

(47)

where $\epsilon = \frac{v^2}{2} + \Phi$ is the energy of a star by unit of mass, $\chi(\eta) \equiv \exp(-\sum_{n>1} \alpha_n \eta^n)$ accounts for the conservation of the fragile moments $M_{n>1} = \int \rho \eta^n d^3r d^3v$ and $\alpha, \beta$ are Lagrange multipliers for $M$ and $E$ (robust integrals). The partition function $Z$ is determined by the local normalization $\int \rho d\eta = 1$ and the equilibrium coarse-grained field is given by $\tilde{f} = \int \rho \eta d\eta$. This can be written (4)

$$\tilde{f} = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon} = F(\beta\epsilon + \alpha) = \tilde{f}(\epsilon).$$  

(48)

We emphasize that Eq. (18) is obtained after two successive coarse-grainings. Intrinsically, a galaxy is a collection of $N$ point-like stars and the exact distribution function $f_{\text{exact}} =$
\[ \sum_{i=1}^{N} m_i \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{v} - \mathbf{v}_i) \] is a sum of \( \delta \)-functions. As is customary in statistical mechanics, we consider a smooth distribution function \( f(\mathbf{r}, \mathbf{v}, t) \) which is the statistical average of \( f_{\text{exact}} \), i.e. \( f = \langle f_{\text{exact}} \rangle \). In the collisionless regime valid for times \( t \ll t_{\text{coll}} \sim \frac{N}{m N} t_{\text{dyn}} \) (\( t_{\text{coll}} \) is the timescale of collisional relaxation and \( t_{\text{dyn}} \) the dynamical time), this function \( f(\mathbf{r}, \mathbf{v}, t) \) satisfies the Vlasov equation \( \text{(43)} \). However, the Vlasov-Poisson system develops a mixing process and it is relevant to introduce another coarse-graining (at a much larger scale) and define \( f(\mathbf{r}, \mathbf{v}, t) \) as the local average of \( f(\mathbf{r}, \mathbf{v}, t) \) on a phase space cell of volume \( \epsilon_r^3 \epsilon_v^3 \). The quantity appearing in Eq. \( \text{(48)} \) is the most probable form of \( f \) at statistical equilibrium (in the sense of Lynden-Bell), assuming ergodicity (i.e. efficient mixing).

From Eq. \( \text{(47)} \), it is easy to show (see the equivalent proof in \[20\]) that

\[ f'(\epsilon) = -\beta f_2, \quad f_2 \equiv \int \rho(\eta - \bar{f})^2 d\eta > 0, \]

where \( f_2 \) is the centered local variance of the distribution \( \rho(\mathbf{r}, \mathbf{v}, \eta) \). Therefore, \( \bar{T} = \bar{T}(\epsilon) \) is a decreasing function of the stellar energy (assuming \( \beta > 0 \)). Since \( \bar{T}(\epsilon) \) is monotonic, the coarse-grained distribution \( \text{(48)} \) extremizes a “generalized entropy” \( \text{(50)} \)

\[ S[f] = -\int C(f) d^3 \mathbf{r} d^3 \mathbf{v}, \]

at fixed mass \( M \) and energy \( E \), where \( C(f) \) is a convex function, i.e. \( C'' > 0 \). Indeed, introducing Lagrange multipliers and writing the variational principle in the form

\[ \delta S - \beta \delta E - \alpha \delta M = 0, \]

we find that

\[ C'(\bar{T}) = -\beta \epsilon - \alpha. \]

Since \( C' \) is a monotonically increasing function of \( f \), we can inverse this relation to obtain

\[ \bar{T} = F(\beta \epsilon + \alpha) = \bar{T}(\epsilon), \]

where \( F(x) = (C')^{-1}(-x) \). Equation \( \text{(53)} \) can be compared to Eq. \( \text{(48)} \). From the identity

\[ \bar{T}'(\epsilon) = -\beta / C''(\bar{T}), \]

resulting from Eq. \( \text{(52)} \), \( \bar{T}(\epsilon) \) is a monotonically decreasing function of energy if \( \beta > 0 \). Therefore, for any Gibbs state of the form \( \text{(47)} \), there exists a generalized entropy of the form \( \text{(50)} \) that \( \bar{T} \) extremizes (at fixed \( E, M \)). It can be shown furthermore that \( \bar{T} \) maximizes this functional. We note that \( C(f) \), hence the generalized entropy \( \text{(50)} \), is a non-universal function which depends on the initial conditions. In general, it is not the ordinary Boltzmann entropy \( S_B[\bar{T}] = -\int \bar{T} \ln \bar{T} d^3 \mathbf{r} d^3 \mathbf{v} \) due to fine-grained constraints (Casimirs) that modify the form of entropy that we would naively expect. We emphasize that maximizing the multi-levels Boltzmann entropy \( S[\rho] \) at fixed mass \( M \), energy \( E \) and with an infinite number of fine-grained constraints \( M_n \) (Casimirs) gives the same result for the coarse-grained field \( \bar{T} \) as maximizing a certain generalized entropy \( S[\bar{T}] \) (non-universal) while conserving only mass \( M \) and energy \( E \) (robust constraints). The existence of “hidden constraints” (here the Casimir invariants that are not accessible at the coarse-grained scale) is the physical reason for the occurrence of “generalized entropy” functionals in a problem. We can either work with the Boltzmann entropy and take into account all the constraints imposed by the dynamics, or keep only the constraints
that are the most directly accessible to the observations and change the form of entropy. This clearly leads to an indetermination which appears in the parameter $q$ of Tsallis or more generally in our function $C(f)$. We expect this idea of “hidden constraints” to be very general and of fundamental importance.

The above statistical approach rests on the assumption that the evolution is ergodic. In reality, this is not the case. It has been understood since the beginning [18] that violent relaxation is incomplete so that the Boltzmann entropy (46) is not maximized in the whole available phase space (this is independant on the fact that it has no maximum!). However, the metaequilibrium state reached by the system as a result of incomplete violent relaxation is always a nonlinearly dynamically stable solution of the Vlasov-Poisson system (on the coarse-grained scale). If $\mathcal{F} = \mathcal{F}(\epsilon)$, which is a particular case of the Jeans theorem [11], it maximizes a functional of the form (50) at fixed mass and energy. In this dynamical context, $S[f]$ is called a $H$-function [5]. This functional depends on the initial conditions (for the same reasons as before) and also on the strength of mixing (if the mixing is complete, $C(f)$ can be predicted by the statistical theory of Lynden-Bell). Boltzmann and Tsallis functionals are particular $H$-functions corresponding to isothermal and polytropic distribution functions. They are not good models of incomplete relaxation for elliptical galaxies [7]. A better model is a composite model that is isothermal in the core and polytropic in the halo. Since the variational principle determining the nonlinear dynamical stability of a collisionless stellar system (maximization of a $H$-function at fixed mass and energy) is similar to the usual thermodynamical variational principle (maximization of the Boltzmann entropy at fixed mass and energy) we can use a thermodynamical analogy to analyze the dynamical stability of collisionless stellar systems [7, 3]. In this analogy, the $H$-function can be called a “generalized entropy”. We believe that this thermodynamical analogy is the correct interpretation of the notion of “generalized thermodynamics” introduced by Tsallis in the context of stellar systems (and 2D turbulence). We emphasize that the maximization of a $H$-function (e.g., Tsallis entropy) at fixed mass and energy is a condition of nonlinear dynamical stability, not a condition of thermodynamical stability.

### 3.3 Heuristic approach of violent relaxation

Violent relaxation is a very complicated concept because of the presence of fine-grained constraints (Casimirs). The Casimirs differ from robust constraints such as mass and energy because they are altered by the coarse-graining procedure since $\overline{\mathcal{F}}_n \neq \mathcal{F}^n$ for $n \neq 1$. Therefore, they can be determined only from the initial conditions (which are not mixed) at $t = 0$, say. Unfortunately, in practice, we do not know the initial conditions (e.g., the initial state that gave rise to an elliptical galaxy) so that we do not know the Casimirs. Only mass and energy (robust constraints) can be determined at all times $t \geq 0$ since $\overline{M} = M$ and $\overline{E} \simeq E$ [7]. We thus have to deal with a limited amount of information on the system. Therefore, we cannot make predictions because we do not know all the constraints on the system. If we want to make some predictions, we have two possibilities: (i) the first possibility is to “guess” the initial conditions (consistent with the information that we have on the system) and determine the corresponding equilibrium state. We can then study how the equilibrium state depends on the initial conditions (for given $E$ and $M$). (ii) the second possibility is to “guess” the generalized entropy that is maximized by the system at equilibrium. We can then study how the equilibrium state depends on the form of $C(f)$ and whether generalized entropies presenting the same properties can be regrouped in “classes of equivalence” [3].

Each approach has its own advantages and drawbacks. The first approach is more complicated because, for realistic initial conditions, we have to account for an infinity of constraints (Casimirs) in addition to mass and energy. This clearly leads to practical difficulties. In ad-
dition, it is not clear whether all these constraints are physically relevant because they can be altered by non-ideal effects as discussed in [7, 8]. Finally, this description assumes that the mixing is complete which is not the case in practice. We shall therefore prefer the second approach where we have only two robust constraints $M$ and $E$. The other constraints are taken into account implicitly in the form of the generalized entropy that we consider. This is a much more convenient approach. In addition, this approach is directly connected to the problem of dynamical stability that we discussed previously. It has therefore a lot of attractive advantages.

3.4 The quasilinear theory

Basically, a collisionless stellar system is described in a self-consistent mean field approximation by the Vlasov-Poisson system (43)(44). In principle, these coupled equations determine completely the evolution of the distribution function $f(r, v, t)$. However, as discussed in Sec. 3.2 we are not interested in practice by the finely striated structure of the flow in phase space but only by its macroscopic, i.e. smoothed-out, structure. Indeed, the observations and the numerical simulations are always realized with a finite resolution. Moreover, the “coarse-grained” distribution function $\bar{f}(r, v, t)$ is likely to converge towards an equilibrium state contrary to the exact distribution $f$ which develops smaller and smaller scales.

If we decompose the distribution function and the gravitational potential in a mean and fluctuating part ($f = \bar{f} + \tilde{f}$, $\Phi = \bar{\Phi} + \tilde{\Phi}$) and take the local average of the Vlasov equation (43), we readily obtain an equation of the form

$$\frac{\partial \bar{f}}{\partial t} + v \frac{\partial \bar{f}}{\partial r} + \bar{F} \frac{\partial \bar{f}}{\partial v} = -\frac{\partial \bar{J}}{\partial v},$$

for the “coarse-grained” distribution function with a diffusion current $\bar{J} = \bar{f} \tilde{F}$ related to the correlations of the “fine-grained” fluctuations. Any systematic calculation of the diffusion current starting from the Vlasov equation (43) must necessarily introduce an evolution equation for $\tilde{f}$. This equation is simply obtained by subtracting Eq. (55) from Eq. (43). This yields

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial r} + \tilde{F} \frac{\partial \tilde{f}}{\partial v} = -\tilde{F} \frac{\partial \bar{f}}{\partial v} - \bar{F} \frac{\partial \tilde{f}}{\partial v} + \bar{F} \frac{\partial \bar{f}}{\partial v}.$$

To go further, we need to implement some approximations. In the sequel, we shall develop a quasilinear theory (see also [21, 22, 17]). This will provide a precise theoretical framework to analyze the process of “collisionless relaxation” in stellar systems. The essence of the quasilinear theory is to assume that the fluctuations are weak and neglect the nonlinear terms in Eq. (56) altogether. In that case, Eqs. (55) and (56) reduce to the coupled system

$$\frac{\partial \bar{f}}{\partial t} + L \bar{f} = -\frac{\partial \bar{J}}{\partial v},$$

$$\frac{\partial \tilde{f}}{\partial t} + L \tilde{f} = -\bar{F} \frac{\partial \bar{f}}{\partial v},$$

where $L = v \frac{\partial}{\partial r} + \bar{F} \frac{\partial}{\partial v}$ is the advection operator in phase space. Physically, these equations describe the coupling between a subdynamics (here the small scale fluctuations $\tilde{f}$) and a macrodynamics (described by the coarse-grained distribution function $\bar{f}$). Due to the strong simplifications implied by the neglect of nonlinear terms in Eq. (56), the quasilinear theory only
describes the late quiescent stages of the violent relaxation process, when the fluctuations have weaken (gentle relaxation). Although this is essentially an asymptotic theory, it is of importance to develop this theory in detail since it provides an explicit expression for the effective “collision operator” which appears on a coarse-grained scale.

Introducing the Greenian

\begin{equation}
G(t_2,t_1) \equiv \exp\left\{-\int_{t_1}^{t_2} dt L(t)\right\},
\end{equation}

we can immediately write down a formal solution of Eq. (58), namely

\begin{equation}
\tilde{f}(r,v,t) = G(t,0) \tilde{f}(r,v,0) - \int_{0}^{t} ds \tilde{G}(t,t-s) \tilde{F}(r,t-s) \frac{\partial \tilde{f}}{\partial v}(r,v,t-s).
\end{equation}

Although very compact, this formal expression is in fact extremely complicated. Indeed, all the difficulty is encapsulated in the Greenian \(G(t,t-s)\) which supposes that we can solve the smoothed-out Lagrangian flow

\begin{equation}
\frac{dr}{dt} = v, \quad \frac{dv}{dt} = F,
\end{equation}

between \(t\) and \(t-s\). In practice, this is impossible and we will have to make some approximations.

The objective now is to substitute the formal result (60) back into Eq. (57) and make a closure approximation in order to obtain a self-consistant equation for \(\tilde{f}(r,v,t)\). If the fluctuating force \(\tilde{F}\) were external to the system, we would simply obtain a diffusion equation

\begin{equation}
\frac{\partial \tilde{f}}{\partial t} + L\tilde{f} = \frac{\partial}{\partial v^\mu} \left(D^{\mu\nu} \frac{\partial \tilde{f}}{\partial v^\nu}\right),
\end{equation}

with a diffusion coefficient given by a Kubo formula

\begin{equation}
D^{\mu\nu} = \int_{0}^{t} ds \tilde{F}^\mu(r,t) \tilde{F}^\nu(r,t-s).
\end{equation}

However, in the case of the Vlasov-Poisson system, the gravitational force is produced by the distribution of matter itself and this coupling will give rise to a friction term in addition to the pure diffusion. Indeed, we have

\begin{equation}
\tilde{F}(r,t) = \int F(r' \rightarrow r) \tilde{f}(r',v',t) d^3r' d^3v',
\end{equation}

where

\begin{equation}
F(r' \rightarrow r) = G \frac{r' - r}{|r' - r|^3},
\end{equation}

represents the force (by unit of mass) created by a star in \(r'\) on a star in \(r\) (Newton’s law). Therefore, considering Eqs. (60) and (64), we see that the fluctuations of the distribution function \(\tilde{f}(r,v,t)\) are given by an iterative process: \(\tilde{f}(t)\) depends on \(\tilde{F}(t-s)\) which itself depends on \(\tilde{f}(t-s)\) etc... We shall solve this problem perturbatively in an expansion in powers of the gravitational constant \(G\). This is the equivalent of the “weak coupling approximation” in plasma physics.
According to Eq. (64), we have

$$f(r, v, t)\tilde{F}^\mu(r, t) = \int d^3r' d^3v' F^\mu(r' \to r)\tilde{f}(r', v', t)\tilde{f}(r, v, t).$$

(66)

On the other hand, according to Eqs. (60) and (64), we have

$$\tilde{f}(r, v, t) = G(t, 0)\tilde{f}(r, v, 0) - \int_0^t ds \int d^3r'' d^3v'' G(t, t - s)$$

$$\times F''(r'' \to r)\tilde{f}(r'', v'', t - s)\frac{\partial\tilde{f}}{\partial v''}(r, v, t - s),$$

(67)

and a similar expression for $\tilde{f}(r', v', t)$. Substituting the foregoing expansion in Eq. (66), we find that the term of order $G^2$ is

$$f(r, v, t)\tilde{F}^\mu(r, t) = -\int_0^t ds \int d^3r'' d^3v'' \int d^3r' d^3v' F^\mu(r' \to r)$$

$$\times \left\{ G'(t, 0)\tilde{G}(t, t - s)F''(r'' \to r)\tilde{f}(r'', v'', t - s)\frac{\partial\tilde{f}}{\partial v''}(r, v, t - s) + G(t, 0)\tilde{G}'(t, t - s)\tilde{f}(r', v', 0)\tilde{f}(r'', v'', t - s)\frac{\partial\tilde{f}}{\partial v''}(r', v', t - s) \right\}.\tag{68}$$

Using

$$G'(t, 0) = G'(t, t - s)G(t - s, 0),$$

(69)

and

$$f(r', v', t - s) = G'(t - s, 0)\tilde{f}(r', v', 0) + O(G),$$

(70)

we obtain after some rearrangements

$$\frac{\partial\tilde{f}}{\partial t} + L\tilde{f} = \frac{\partial}{\partial v^\mu} \int_0^t ds \int d^3r' d^3v' d^3r'' d^3v'' F^\mu(r' \to r)\tilde{G}'(t, t - s)G(t, t - s)$$

$$\times \left\{ F''(r'' \to r)\tilde{f}(r'', v'', t - s)\tilde{f}(r', v', t - s)\frac{\partial\tilde{f}}{\partial v''}(r, v, t - s) + F''(r'' \to r')\tilde{f}(r', v', t - s)\tilde{f}(r'', v'', t - s)\frac{\partial\tilde{f}}{\partial v''}(r', v', t - s) \right\}.\tag{71}$$

In this expression, the Greenian $G$ refers to the fluid particle $r(t), v(t)$ and the Greenian $G'$ to the fluid particle $r'(t), v'(t)$. To close the system, it remains for one to evaluate the correlation function $\tilde{f}(r, v, t)\tilde{f}(r', v', t)$. We shall assume that the mixing in phase space is sufficiently efficient so that the scale of the kinematic correlations is small with respect to the coarse-graining mesh size. In that case,

$$\tilde{f}(r, v, t)\tilde{f}(r', v', t) = \epsilon_r^3\epsilon_v^3\delta(r - r')\delta(v - v')f_2(r, v, t),$$

(72)

where $\epsilon_r$ and $\epsilon_v$ are the resolution scales in position and velocity respectively and

$$f_2 \equiv \bar{f}^2 = (\tilde{f} - \bar{f})^2 = \bar{f}^2 - \bar{f}^2.$$

(73)

is the local variance of the fine-grained fluctuations.
3.5 The closure approximation

We are now led to a closure problem. Indeed, in order to obtain a self-consistent kinetic equation for $\overline{f}$, we need to determine the variance $f_2$. If the initial condition in phase space consists of patches of uniform distribution function $f = \eta_0$ surrounded by vacuum $f = 0$ (two-levels approximation), then $\overline{f^2} = \overline{\eta_0 f}$ and, therefore, $f_2 = \overline{f(\eta_0 - f)}$. This leads to a generalized Landau equation of the form \cite{11} as for fermions \cite{21, 22, 17}. For more complicated initial conditions (multi-levels case), the strategy would be to write down a kinetic equation for $\rho(r, v, \eta, t)$, the density probability of finding the value $f = \eta$ in $(r, v)$ at time $t$. This extension can be performed along the lines sketched in \cite{22} and the resulting equation for $\rho(r, v, \eta, t)$ can be closed. However, this approach leads to a system of $N$ coupled equations (one for each level $\eta$) which is not convenient to solve when $N \gg 1$. We could alternatively obtain a hierarchy of equations for the moments $\overline{f}$, $\overline{f^2}$,..., $\overline{f^n}$,... but we would then encounter a closure problem. The equations obtained by this method are complicated because they take into account the conservation of all the Casimirs.

In this paper, we propose a closure approximation that leads to a simpler kinetic equation. While not being exact, this equation preserves the robust features of the process of violent relaxation and is amenable to an easier numerical implementation. Its main interest is to go beyond the two-levels approximation while leaving the problem tractable. The idea is to observe that Eqs. \cite{19} and \cite{54} lead to the important relation

\begin{equation}
(74) \quad f_2 = \frac{1}{C''(\overline{f})}.
\end{equation}

This relation is valid at equilibrium but we propose to use it as a closure approximation in Eq. \cite{72}. This is expected to be a reasonable approximation if we are close to equilibrium, which is in fact dictated by the quasi-linear approximation. Of course, this procedure assumes that we know the function $C(f)$ in advance. This is the case if we have already determined the equilibrium state and we want to describe the dynamics close to equilibrium. This is also the case in the heuristic approach of violent relaxation discussed in Sec. \ref{sec:3.3} where we have to “guess” the form of $C(f)$ that is relevant to our system (or try different functionals).

If we substitute Eqs. \cite{73}-\cite{74} in Eq. \cite{71} and carry out the integrations on $r''$ and $v''$, we obtain

\begin{equation}
(75) \quad \frac{\partial \overline{F}}{\partial t} + L\overline{F} = \epsilon_r^2 \epsilon_v^2 \frac{\partial}{\partial \nu''} \int_0^t ds \int d^3 \mathbf{r}' d^3 \mathbf{v}' F''(\mathbf{r}' \rightarrow \mathbf{r})_t F''(\mathbf{r}' \rightarrow \mathbf{r})_{t-s} \times \left\{ \frac{1}{C''(\overline{f})} \frac{\partial \overline{F}}{\partial \nu''} - \frac{1}{C''(\overline{f})} \frac{\partial \overline{F}}{\partial \nu''} \right\}_{t-s}.
\end{equation}

We have written $\overline{F}_{t-s} \equiv \overline{F}(\mathbf{r}'(t-s), \mathbf{v}'(t-s), t-s)$, $\overline{F}_{t-s} \equiv \overline{F}(\mathbf{r}(t-s), \mathbf{v}(t-s), t-s)$, $F''(\mathbf{r}' \rightarrow \mathbf{r})_t \equiv F''(\mathbf{r}'(t \rightarrow \mathbf{r}(t))$ and $F''(\mathbf{r}' \rightarrow \mathbf{r})_{t-s} \equiv F''(\mathbf{r}'(t-s) \rightarrow \mathbf{r}(t-s))$ where $\mathbf{r}(t-s)$ and $\mathbf{v}(t-s)$ are the position and velocity at time $t-s$ of the stellar fluid particle located in $\mathbf{r} = \mathbf{r}(t)$, $\mathbf{v} = \mathbf{v}(t)$ at time $t$. They are determined by the characteristics \cite{61} of the smoothed-out Lagrangian flow.

Equation \cite{75} is a non Markovian integrodifferential equation: the value of $\overline{F}$ in $\mathbf{r}, \mathbf{v}$ at time $t$ depends on the value of the whole field $\overline{F}(\mathbf{r}', \mathbf{v}', t-s)$ at earlier times. If the decorrelation time $\tau$ is short, we can make a Markov approximation and replace the bracket at time $t-s$ by its value taken at time $t$. Noting furthermore that the integral is dominated by the contribution of field stars close to the star under consideration (i.e. when $\mathbf{r}' \rightarrow \mathbf{r}$), we shall make a local
approximation and replace $C''(\mathcal{F})$ and $\frac{\partial \mathcal{F}}{\partial v^\nu}$ by their values taken at $r$. In that case, the foregoing equation simplifies in

$$\frac{\partial \mathcal{F}}{\partial t} + L\mathcal{F} = \epsilon_r^2 \epsilon_v^2 \frac{\partial}{\partial \nu^\mu} \int_0^t ds \int d^3r' d^3v' F^\mu(r' \rightarrow r) F^\nu(r' \rightarrow r)t-s \cdot \left\{ \frac{1}{C''(\mathcal{F})} \frac{\partial \mathcal{F}}{\partial v^\nu} - \frac{1}{C''(\mathcal{F})} \frac{\partial \mathcal{F}}{\partial v'^\nu} \right\},$$

(76)

where, now, $\mathcal{F} = \mathcal{F}(r,v,t)$. The explicit reference to the past evolution of the system is only retained in the memory function

$$\int_0^t ds \int d^3r' F^\mu(r' \rightarrow r)_t F^\nu(r' \rightarrow r)t-s.$$ 

This function can be calculated explicitly if we assume that, between $t-s$ and $t$, the stars follow linear trajectories, so that $v(t-s) = v$ and $r(t-s) = r - vs$. This leads to the generalized Landau equation

$$\frac{\partial \mathcal{F}}{\partial t} + L\mathcal{F} = \frac{\partial}{\partial v^\mu} \left\{ D \left[ \frac{\partial \mathcal{F}}{\partial v^\nu} + \beta f^\nu v_t \right] \right\},$$

(77)

where $K^\mu\nu$ is the tensor

$$K^\mu\nu = 2\pi G^2 \epsilon_r^2 \epsilon_v^2 \ln \Lambda \frac{1}{u} \left( \delta^\mu\nu - \frac{u^\mu u^\nu}{u^2} \right),$$

(78)

and $u = v' - v$, $\ln \Lambda = \ln(R/\epsilon_r)$. This equation applies to inhomogeneous systems but, as a result of the local approximation, the effect of inhomogeneity is only retained in the advective term.

By using a different approach based on a Maximum Entropy Production Principle (MEPP), Chavanis, Sommeria & Robert have proposed a relaxation equation for $\rho(r,v,\eta,t)$ of the form

$$\frac{\partial \rho}{\partial t} + L\rho = \frac{\partial}{\partial v^\mu} \left\{ D \left[ \frac{\partial \rho}{\partial v^\nu} + \beta(\eta - \mathcal{F})\rho v \right] \right\}. $$

(79)

From this equation, we can deduce a hierarchy of equations for the moments $\mathcal{F}^n = \int \rho^n d^3r d^3v d\eta$. The equation for the first moment $\mathcal{F}$ is

$$\frac{\partial \mathcal{F}}{\partial t} + L\mathcal{F} = \frac{\partial}{\partial v^\mu} \left\{ D \left[ \frac{\partial \mathcal{F}}{\partial v^\nu} + \beta f^\nu v_t \right] \right\}. $$

(80)

If we close the hierarchy of equations with Eq. (74), we obtain a self-consistent equation of the form

$$\frac{\partial \mathcal{F}}{\partial t} + L\mathcal{F} = \frac{\partial}{\partial v^\mu} \left\{ D \left[ \frac{\partial \mathcal{F}}{\partial v^\nu} + \frac{\beta}{C''(\mathcal{F})} v_t \right] \right\}. $$

(81)

Note that this equation can also be derived from Eq. (77) by replacing $\mathcal{F}$ by its equilibrium value. Then, the diffusion coefficient $D$ can be explicitly evaluated (see in the two-levels approximation). Furthermore, Chavanis et al. proposed to let $\beta$ depend on time, i.e.
\(\beta = \beta(t)\), so as to conserve energy. This heuristic “microcanonical description” is more adapted to the context of the violent relaxation than a “canonical description” with fixed \(\beta\).

By developing a quasi-linear theory of 2D turbulence for the Euler-Poisson system [24], we have derived a kinetic equation of the form

\[
\frac{\partial \omega}{\partial t} + \mathbf{u} \nabla \omega = \frac{\epsilon^2 \tau}{8\pi^2} \frac{\partial}{\partial r} \int d^2 \mathbf{r}' K^{\mu \nu} \left\{ \frac{1}{C''(\omega)} \frac{\partial \omega}{\partial r} - \frac{1}{C''(\omega)} \frac{\partial \omega'}{\partial r'} \right\},
\]

(82)

where \(K^{\mu \nu}\) is the tensor

\[
K^{\mu \nu} = \frac{1}{\xi} \left( \delta^{\mu \nu} - \frac{\xi^\mu \xi^\nu}{\xi^2} \right),
\]

(83)

and \(\xi = \mathbf{r}' - \mathbf{r}\). This equation is valid in the two-levels approximation of the statistical theory \(\omega = \{\sigma_0, 0\}\). Once again, the multi-levels case would lead to \(N\) coupled differential equations for \(\rho(r, \sigma, t)\), the density probability of finding the vorticity level \(\omega = \sigma\) in \(r\) at time \(t\). Alternatively, if we implement a closure approximation of the form \(\omega^2 = 1/C''(\omega)\), similar to Eq. (74), we obtain a simpler equation

\[
\frac{\partial \omega}{\partial t} + \mathbf{u} \nabla \omega = \nabla \left\{ D \left[ \nabla \omega + \beta \omega \nabla \psi \right] \right\}.
\]

(84)

which takes into account several features of the process of “inviscid relaxation” in 2D turbulence.

Unfortunately, as discussed in [24], Eqs. (82) and (84) do not conserve energy. Non-markovian effects may be necessary to restore the conservation of energy.

By using a Maximum Entropy Production Principe (MEPP), Robert & Sommeria [9] have proposed a relaxation equation for the local distribution of vorticity \(\rho(r, \sigma, t)\) of the form

\[
\frac{\partial \rho}{\partial t} + \mathbf{u} \nabla \rho = \nabla \left\{ D \left[ \nabla \rho + \beta \rho \nabla (\sigma - \rho) \nabla \psi \right] \right\}.
\]

(85)

From this equation, we can deduce a hierarchy of equations for the moments \(\bar{\omega}^n = \int \rho \sigma^n d\sigma\). For the first moment \(\bar{\omega}\), we have

\[
\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \nabla \bar{\omega} = \nabla \left\{ D \left[ \nabla \bar{\omega} + \beta \bar{\omega} \nabla \psi \right] \right\}.
\]

(86)

Kazantsev et al. [25] proposed to close the hierarchy of equations with a Gaussian approximation. The resulting equation converges to a state of minimum enstrophy. More generally, we propose to close the hierarchy of equations by the relation \(\omega^2 = 1/C''(\bar{\omega})\) so that

\[
\frac{\partial \bar{\omega}}{\partial t} + \mathbf{u} \nabla \bar{\omega} = \nabla \left\{ D \left[ \nabla \bar{\omega} + \beta \frac{\bar{\omega}}{C''(\bar{\omega})} \nabla \psi \right] \right\}.
\]

(87)

The function \(C(\bar{\omega})\) is a free function which has to be adapted to the context (see Sec. 3.3). This indetermination reflects the fundamental observation that there is no universal form of entropy \(S[\omega]\) in 2D hydrodynamics [3]. The enstrophy [26], the Fermi-Dirac entropy [27], the Boltzmann entropy [28] (leading to a sinh-Poisson equation) or the functional proposed by Ellis et al. [6] (leading to a Gamma law for the vorticity fluctuations) are particular functionals \(S[\omega]\) which prove to be more relevant than others in a specific context (geophysical flows [25], Jupiter’s Great Red Spot [29], 2D turbulence [30] and jovian atmosphere [6] respectively). None of them has a universal domain of validity but generalized entropies can possibly be regrouped
in “classes of equivalence” \[3\]. Tsallis entropies, sometimes arising in 2D hydrodynamics \[31\], are special because they are due to incomplete relaxation. They are particular H-functions associated to a dynamical equilibrium, not to a thermodynamical equilibrium (they are not true entropies). The self-confinement of 2D vortices can be explained by a lack of mixing \[32\]. It can be taken into account in the statistical theory by using relaxation equations with a space-dependant diffusion coefficient related to the local fluctuations of the vorticity \[33\]. The same arguments can be invoked to account for the confinement of galaxies in astrophysics \[8\].

In conclusion, the generalized kinetic equations introduced previously can provide a simple parametrization of turbulence (mixing) in the context of the violent relaxation of stellar systems and 2D vortices. They can take into account incomplete relaxation thanks to a varying diffusion coefficient \[33, 8\]. In addition, due to the thermodynamical analogy discussed in Sec. 3.2, they can also serve as powerful numerical algorithms to compute nonlinearly dynamically stable solutions of the Vlasov-Poisson or 2D Euler-Poisson systems. This is a great practical interest of these equations independently of the statistical theory.

4 Properties of the generalized Landau equation

4.1 Conservation laws

In this section, we derive the conservation laws satisfied by the generalized Landau equation \[20\]. Let us first introduce the current of diffusion

\[
J_f^\mu = -\int d^3v_1 K_{\mu\nu} \left\{ g(f_1)h(f) \frac{\partial f}{\partial v_\nu} - g(f)h(f_1) \frac{\partial f_1}{\partial v_1^\nu} \right\}.
\]

The time variation of energy can be written

\[
\dot{E} = \int \frac{\partial f}{\partial t} \left( \frac{v^2}{2} + \Phi \right) d^3r d^3v = \int \mathbf{J}_f \cdot \mathbf{v} d^3r d^3v,
\]

where we have used an integration by parts. Introducing the current \[88\] in Eq. \[89\], we obtain

\[
\dot{E} = -\int d^3r d^3\mathbf{v} d^3v_1 K_{\mu\nu} v_\mu \left\{ g(f_1)h(f) \frac{\partial f}{\partial v_\nu} - g(f)h(f_1) \frac{\partial f_1}{\partial v_1^\nu} \right\}.
\]

Interchanging the dummy variables \(\mathbf{v}\) and \(\mathbf{v}_1\), we get

\[
\dot{E} = \int d^3r d^3\mathbf{v} d^3v_1 K_{\mu\nu} v_1^\mu \left\{ g(f_1)h(f) \frac{\partial f}{\partial v_\nu} - g(f)h(f_1) \frac{\partial f_1}{\partial v_1^\nu} \right\},
\]

where we have exploited the symmetrical form of the diffusion current. Taking the half sum of the last two expressions, we find

\[
\dot{E} = \frac{1}{2} \int d^3r d^3\mathbf{v} d^3v_1 K_{\mu\nu} (v_\mu - v_1^\mu) \left\{ g(f_1)h(f) \frac{\partial f}{\partial v_\nu} - g(f)h(f_1) \frac{\partial f_1}{\partial v_1^\nu} \right\}.
\]

Noting that

\[
K_{\mu\nu} u^\nu = 0,
\]

according to Eq. \[21\], we finally establish that \(\dot{E} = 0\). Therefore, the generalized Landau equation conserves energy. We can show by a similar procedure that it also conserves angular momentum and impulse.
4.2 Generalized H-theorem

The rate of production of generalized entropy is given by

\[ \dot{S} = - \int C'(f) \frac{\partial f}{\partial t} d^3 \mathbf{r} d^3 \mathbf{v} = - \int C''(f) J^\mu J^\nu \frac{\partial f}{\partial v^\mu} d^3 \mathbf{r} d^3 \mathbf{v} = - \int \frac{h(f)}{g(f)} J^\mu \frac{\partial f}{\partial v^\mu} d^3 \mathbf{r} d^3 \mathbf{v}, \]

where we have used Eq. (29) to get the last equality. Inserting the current (88) in Eq. (94), we obtain

\[ \dot{S} = \int d^3 \mathbf{r} d^3 \mathbf{v} d^3 \mathbf{v}_1 h \frac{\partial f}{\partial v^\nu} K^{\mu \nu} \left\{ g_1 h \frac{\partial f}{\partial v^\nu} - gh_1 \frac{\partial f_1}{\partial v^\nu} \right\}. \]

Interchanging the dummy variables \( \mathbf{v} \) and \( \mathbf{v}_1 \), we get

\[ \dot{S} = - \int d^3 \mathbf{r} d^3 \mathbf{v} d^3 \mathbf{v}_1 h \frac{\partial f_1}{\partial v^\nu} K^{\mu \nu} \left\{ g_1 h \frac{\partial f}{\partial v^\nu} - gh_1 \frac{\partial f_1}{\partial v^\nu} \right\}. \]

Taking the half sum of the foregoing equations we find that

\[ \dot{S} = \frac{1}{2} \int d^3 \mathbf{r} d^3 \mathbf{v} d^3 \mathbf{v}_1 \frac{1}{gg_1} \left\{ g_1 h \frac{\partial f}{\partial v^\nu} - gh_1 \frac{\partial f_1}{\partial v^\nu} \right\} K^{\mu \nu} \left\{ g_1 h \frac{\partial f}{\partial v^\nu} - gh_1 \frac{\partial f_1}{\partial v^\nu} \right\}. \]

Noting that \( X^\mu K^{\mu \nu} X^\nu = X^2 - (X \cdot \mathbf{u})^2 / u^2 \geq 0 \), we conclude that \( \dot{S} \geq 0 \). Therefore, the generalized Landau equation satisfies a generalized \( H \)-theorem.

4.3 Equilibrium distribution

Taking the derivative of Eq. (27) with respect to \( \mathbf{v} \), we get

\[ C''(f_{eq}) \frac{\partial f_{eq}}{\partial \mathbf{v}} = -\beta (\mathbf{v} - \Omega \times \mathbf{r} - \mathbf{U}). \]

Inserting this relation in the current (88) and using Eq. (29), we obtain

\[ J^\mu_f = -\beta \int d^3 \mathbf{v}_1 K^{\mu \nu} g g_1 (v^\nu_{1} - v^\nu), \]

which vanishes identically in virtue of Eq. (93). Therefore, an extremum of the generalized entropy at fixed mass, energy, angular momentum and impulse is a stationary solution of the generalized Landau equation. Alternatively, a stationary solution satisfies \( \dot{S} = 0 \) hence \( J_f = 0 \). We can show that this condition implies that \( f_{eq} \) is of the form (27). The proof is similar to the one given for the ordinary Landau equation.

Finally, considering the linear stability of a stationary solution of the generalized Landau equation, we can derive the general relation \[ 2\lambda^2 J = \delta^2 \dot{S} \geq 0, \] connecting the growth rate \( \lambda \) of the perturbation (such that \( \delta f \sim e^{\lambda t} \)) to the second order variations of the free energy \( J = S - \beta E \) and the second order variations of the rate of entropy production \( \delta^2 \dot{S} \geq 0 \). This aesthetic relation implies that a stationary solution of the generalized Landau equation is linearly stable if, and only if, it is an entropy \textit{maximum} at fixed mass and energy \[ 3 \].
5 The thermal bath approximation

The generalized Landau equation (20) can be put in a form reminiscent of a Fokker-Planck equation

\[ \frac{df}{dt} = \frac{\partial}{\partial v^\mu} \left[ D^{\mu\nu} h(f) \frac{\partial f}{\partial v^\nu} + g(f) \eta^\mu \right], \]

by introducing a diffusion tensor

\[ D^{\mu\nu} = \int K^{\mu\nu} g(f_1) d^3v_1, \]

and a friction term

\[ \eta^\mu = -\int K^{\mu\nu} h(f_1) \frac{\partial f_1}{\partial v_1^\nu} d^3v_1. \]

The ordinary Fokker-Planck equation [2] can be written

\[ \frac{df}{dt} = \frac{\partial^2}{\partial v^\mu \partial v^\nu} (f \zeta^{\mu\nu}) + \frac{\partial}{\partial v^\mu} (f \zeta^\mu), \]

where

\[ \zeta^{\mu\nu} = \frac{1}{2} \frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{\Delta t}, \quad \zeta^\mu = -\frac{\langle \Delta v^\mu \rangle}{\Delta t}, \]

are the first (friction) and second (diffusion) moments of the velocity deviation. An equivalent form of the Fokker-Planck equation is

\[ \frac{df}{dt} = \frac{\partial}{\partial v^\mu} \left[ D^{\mu\nu} \frac{\partial f}{\partial v^\nu} + \left( \frac{\partial \zeta^{\mu\nu}}{\partial v^\nu} + \zeta^\mu \right) f \right]. \]

For the ordinary Landau equation, we have \( g(f) = f \) et \( h(f) = 1 \). Therefore, Eq. (101) becomes

\[ \frac{df}{dt} = \frac{\partial}{\partial v^\mu} \left[ D^{\mu\nu} \frac{\partial f}{\partial v^\nu} + f \eta^\mu \right]. \]

By comparing with the Fokker-Planck equation (106), we obtain the classical relations

\[ \zeta^{\mu\nu} = D^{\mu\nu}, \quad \frac{\partial \zeta^{\mu\nu}}{\partial v^\nu} + \zeta^\mu = \eta^\mu. \]

According to Eq. (108), there is a difference between \( \eta \) and \( \langle \Delta v \rangle \) due to the velocity dependance of the diffusion tensor \( D^{\mu\nu} \). Noting that

\[ \frac{\partial D^{\mu\nu}}{\partial v^\nu} = \int \frac{\partial K^{\mu\nu}}{\partial v^\nu} f_1 d^3v_1 = -\int \frac{\partial K^{\mu\nu}}{\partial v^\nu_1} f_1 d^3v_1 = \int K^{\mu\nu} \frac{\partial f_1}{\partial v^\nu_1} d^3v_1 = -\eta^\mu, \]

we find that Eq. (108) yields

\[ \eta = -\frac{1}{2} \frac{\langle \Delta v \rangle}{\Delta t}. \]

Therefore, in the classical case, the vector \( -\eta \) represents half the friction force.
The generalized Landau equation (101) can be rewritten

\[
\frac{df}{dt} = \frac{\partial^2}{\partial v^\mu \partial v^\nu} (D^{\mu \nu} H(f)) + \frac{\partial}{\partial v^\nu} \left( g(f) \eta^\mu - H(f) \frac{\partial D^{\mu \nu}}{\partial v^\nu} \right),
\]

where \( H \) is a primitive of \( h \). By comparing with the ordinary Fokker-Planck equation (104), we get

\[
\zeta^{\mu \nu} = D^{\mu \nu} \frac{H(f)}{f}, \quad \zeta^\mu = \frac{g(f)}{f} \eta^\mu - \frac{H(f)}{f} \frac{\partial D^{\mu \nu}}{\partial v^\nu}.
\]

If we impose the relations \( g(f) = f \) and \( h(f) = fC''(f) \) of Sec. 2.4, Eq. (112) can be rewritten

\[
\frac{1}{2} \frac{\langle \Delta v^\mu \Delta v^\nu \rangle}{\Delta t} = D^{\mu \nu} f \left[ \frac{C(f)}{f} \right]'.
\]

This relation was noted in [3]. In the particular case of the Tsallis entropy, Eq. (113) reduces to the form considered by Borland [34]. In the Borland approach, generalized Fokker-Planck equations arise because the transition probabilities \( \zeta^{\mu \nu} \) and \( \zeta^\mu \) depend explicitly on the distribution function. A different approach is followed by Kaniadakis [10]. Starting from a kinetic interaction principle (KIP), he obtains a generalized Fokker-Planck equation of the form

\[
\frac{df}{dt} = \frac{\partial}{\partial v^\nu} \left[ h(f) \zeta^{\mu \nu} \frac{\partial f}{\partial v^\nu} + g(f) \left( \frac{\partial \zeta^{\mu \nu}}{\partial v^\nu} + \zeta^\mu \right) f \right].
\]

This corresponds to his Eq. (17) in our notations. Comparing with our equation (101), we find that the classical relations (108) are preserved in this generalized framework. This is much more aesthetic than Eq. (112). This is also more physical because we would expect that the transition moments depend only on the form of interaction, not explicitly on the distribution function. Besides, the other approach leads to ambiguity because, comparing Eqs. (101) and (106), we get

\[
\zeta^{\mu \nu} = D^{\mu \nu} h(f), \quad \frac{\partial \zeta^{\mu \nu}}{\partial v^\nu} + \zeta^\mu = \frac{g(f)}{f} \eta^\mu,
\]

which differ from Eq. (112) except in the classical case.

Physically, the Landau equation describes the statistical evolution of a system of self-interacting particles in a mean-field approximation. This description treats all the particles on the same footing and conserves the energy of the whole system. This corresponds to a microcanonical description. We can alternately view the Landau equation as describing the evolution of a test particle (described by \( f(v,t) \)) interacting with field particles (described by \( f(v_1,t) \)). In the so-called thermal bath approximation (canonical description), we consider that the field particles are in statistical equilibrium and replace \( f(v_1,t) \) by their equilibrium distribution \( f_{eq}(v_1) \) given by Eq. (27). This approximation transforms an integro-differential equation (Landau) into a differential equation (Kramers). Combining Eqs. (103), (38) and (29), we get

\[
\eta^\mu = \beta \int K^{\mu \nu} g(f_1) w^\nu d^3v_1,
\]

with \( w = v - \Omega \times r - U \). Now, using Eqs. (38) and (102), we have equivalently

\[
\eta^\mu = \beta \int K^{\mu \nu} g(f_1) w^\nu d^3v_1 = \beta D^{\mu \nu} w^\nu.
\]
This relation can be interpreted as a generalized Einstein relation. In the test particle approach, the kinetic equation (101) thus becomes

\[
\frac{df}{dt} = \partial_{\nu} \left\{ D^{\mu \nu} \left[ h(f) \frac{\partial f}{\partial v^\mu} + \beta g(f) w^\nu \right] \right\}.
\]

This equation will be called the generalized Kramers equation. Assuming for simplicity that the diffusion is isotropic, taking \( \Omega = U = 0 \) and imposing the relations of Eq. (31), we recover the generalized Kramers equation

\[
\frac{df}{dt} = \partial_{\nu} \left\{ D \left[ f^{\prime \prime}(f) \frac{\partial f}{\partial v} + \beta f v \right] \right\},
\]

proposed in [3]. In the present approach, the generalized Kramers equation is derived from the generalized Landau equation by a systematic procedure. This makes possible to determine explicitly the expression of the diffusion coefficient (see Secs. 6-7) which was left unspecified by the MEPP approach [3].

We recall also that in the long time limit (or strong friction limit \( \xi = D \beta \to +\infty \)), the generalized Kramers equation leads to the generalized Smoluchowski equation

\[
\frac{\partial \rho}{\partial t} = \nabla \left[ \frac{1}{\xi} (\nabla p + \rho \nabla \Phi) \right],
\]

where \( p(\rho) \) has the interpretation of a pressure [3]. The generalized Smoluchowski equation was first introduced in [8]. The gravitational (i.e. attracting) Smoluchowski-Poisson and generalized Smoluchowski-Poisson systems have been studied in [35, 36] (see also [37] for a connexion with mathematical results). We shall say that a kinetic equation has a generalized canonical thermodynamical structure if it increases the generalized free energy \( J = S - \beta E \) at fixed inverse temperature \( \beta \) and mass \( M \) (the function \( J \) differs from the usual free energy \( F = E - TS \) by a factor \( -\beta \)). We have shown that the generalized Kramers equation and the generalized Smoluchowski equation possess such a thermodynamical structure [3]. We can formally obtain a microcanonical formulation of the generalized Kramers and Smoluchowski equations by letting the inverse temperature \( \beta \) depend on time, i.e. \( \beta = \beta(t) \), so as to conserve energy. With this modification, the generalized Kramers and Smoluchowski equations have a microcanonical thermodynamical structure [3].

6 Diffusion coefficient and dynamical friction

6.1 Generalized Rosenbluth potentials

In order to evaluate the expressions of diffusion coefficient and friction, it is convenient to introduce auxiliary functions which are called the Rosenbluth potentials [11]. Noting that

\[
\frac{\partial^2 u}{\partial v^\mu \partial v^\nu} = \frac{u^2 \delta^{\mu \nu} - u^\mu u^\nu}{u^3},
\]

we can rewrite the diffusion tensor (102) in the form

\[
D^{\mu \nu} = A \frac{\partial^2 \chi}{\partial v^\mu \partial v^\nu}(v),
\]

where

\[
\chi(v) = \int g(f_1) |v - v_1| d^3v_1,
\]

\[
\frac{\partial^2 u}{\partial v^\mu \partial v^\nu} = \frac{u^2 \delta^{\mu \nu} - u^\mu u^\nu}{u^3},
\]

we can rewrite the diffusion tensor (102) in the form

\[
D^{\mu \nu} = A \frac{\partial^2 \chi}{\partial v^\mu \partial v^\nu}(v),
\]

where

\[
\chi(v) = \int g(f_1) |v - v_1| d^3v_1,
\]

\[
\frac{\partial^2 u}{\partial v^\mu \partial v^\nu} = \frac{u^2 \delta^{\mu \nu} - u^\mu u^\nu}{u^3},
\]

we can rewrite the diffusion tensor (102) in the form

\[
D^{\mu \nu} = A \frac{\partial^2 \chi}{\partial v^\mu \partial v^\nu}(v),
\]

where

\[
\chi(v) = \int g(f_1) |v - v_1| d^3v_1.
is the generalized Rosenbluth potential associated to the diffusion. Writing the friction term in the form

\[ \eta^\mu = - \int K^{\mu \nu} \frac{\partial H(f_1)}{\partial v_1^\nu} d^3v_1, \]

and integrating by parts, we obtain

\[ \eta^\mu = \int \frac{\partial K^{\mu \nu}}{\partial v_1^\nu} H(f_1) d^3v_1. \]

Noting that

\[ \frac{\partial K^{\mu \nu}}{\partial v_1^\nu} = 2 \frac{u^\nu}{u^3} = -2A \frac{\partial}{\partial v^\mu} \left( \frac{1}{u} \right), \]

we can rewrite the friction term as

\[ \eta^\mu = -2A \frac{\partial}{\partial v^\mu} (v), \]

where

\[ \lambda(v) = \int \frac{H(f_1)}{|v - v_1|} d^3v_1, \]

is the generalized Rosenbluth potential associated to the friction (we need to impose that \(H(f_1)\) tends to zero for \(|v_1| \to +\infty\) to make the integral well defined; this fixes the constant of integration in \(H\)). We note also that

\[ \frac{\partial D^{\mu \nu}}{\partial v^\nu} = \int \frac{\partial K^{\mu \nu}}{\partial v_1^\nu} g(f_1) d^3v_1 = -\int \frac{\partial K^{\mu \nu}}{\partial v_1^\nu} g(f_1) d^3v_1 = 2A \frac{\partial}{\partial v^\mu} (v), \]

where

\[ \sigma(v) = \int \frac{g(f_1)}{|v - v_1|} d^3v_1, \]

is the potential associated to the velocity dependance of the diffusion coefficient. In the usual thermodynamical framework where \(g(f) = H(f) = f\), the potentials \(\lambda\) and \(\sigma\) coincide.

### 6.2 Isotropic distribution of velocities

When the velocity distribution of the field particles is isotropic, i.e. \(f_1 = f(v_1)\), we can obtain more explicit expressions for the Rosenbluth potentials. For reasons of symmetry, the Rosenbluth potentials depend only on \(v = |v|\). To determine \(\chi(v)\) and \(\lambda(v)\), we use the identity

\[ \frac{1}{|v - v_1|} = \sum_{l=0}^{+\infty} \frac{v_<^l v_>^l}{v_{l+1}^l} P_l(\cos \gamma), \]

where \(v_<\) and \(v_>\) denote the smallest and largest value of \(v\) and \(v_1\), \(P_l(x)\) is a Legendre polynomial and \(\gamma\) is the angle between \(v\) and \(v_1\). Then, we have

\[ \lambda(v) = 2\pi \sum_{l=0}^{+\infty} \int_0^{+\infty} \frac{v_<^l v_>^l}{v_{l+1}^l} H[f(v_1)]dv_1 \int_0^{\pi} P_l(\cos \gamma) \sin \gamma d\gamma. \]
With the change of variable $x = \cos \gamma$, we get

$$\lambda(v) = 2\pi \sum_{l=0}^{+\infty} \int_{0}^{+\infty} \frac{v_<^2 v^l}{v_>^{l+1}} H[f(v_1)]dv_1 \int_{-1}^{+1} P_l(x)dx.$$  

Using the identity $\int_{-1}^{+1} P_l(x)dx = 2\delta_{l0}$, the foregoing expression can be simplified in

$$\lambda(v) = 4\pi \int_{0}^{+\infty} \frac{v^2}{v_>} H[f(v_1)]dv_1 = 4\pi \left[ \frac{1}{v} \int_{0}^{v} v^2 H[f(v_1)]dv_1 + \int_{v}^{+\infty} v_1 H[f(v_1)]dv_1 \right].$$

Similarly, we have

$$\sigma(v) = 4\pi \left[ \frac{1}{v} \int_{0}^{v} v^2 g[f(v_1)]dv_1 + \int_{v}^{+\infty} v_1 g[f(v_1)]dv_1 \right].$$

To determine $\chi(v)$, we write $|v - v_1| = (v^2 + v_1^2 - 2vv_1 \cos \gamma)/|v - v_1|$ and we use the identity $\chi(v_1) = 4\pi \left[ \frac{1}{v} \int_{0}^{v} v^2 g[f(v_1)]dv_1 + \int_{v}^{+\infty} v_1 g[f(v_1)]dv_1 \right].$. We then obtain

$$\chi(v) = 2\pi \sum_{l=0}^{+\infty} \int_{0}^{+\infty} \frac{v_<^2 v^l}{v_>^{l+1}} g[f(v_1)]dv_1 \int_{0}^{\pi} (v^2 + v_1^2 - 2vv_1 \cos \gamma) P_l(\cos \gamma) \sin \gamma d\gamma.$$  

With the change of variables $x = \cos \gamma$, we get

$$\chi(v) = 2\pi \sum_{l=0}^{+\infty} \int_{0}^{+\infty} \frac{v_<^2 v^l}{v_>^{l+1}} g[f(v_1)]dv_1 \int_{-1}^{+1} (v^2 + v_1^2 - 2vv_1 x) P_l(x)dx.$$  

Using the identity $\int_{-1}^{+1} xP_l(x)dx = \frac{2}{3}\delta_{l1}$, we obtain after some manipulations

$$\chi(v) = \frac{4\pi v}{3} \left[ \int_{0}^{v} \left( 3v_1^2 + v^4 \right) g[f(v_1)]dv_1 + \int_{v}^{+\infty} \left( \frac{3v^3}{v} + vv_1 \right) g[f(v_1)]dv_1 \right].$$

### 6.3 The diffusion coefficient

We are now in a position to determine an explicit expression for the diffusion tensor $D^{\mu\nu}$ that is valid for an arbitrary isotropic distribution function of the field particles. Starting from the identity

$$\frac{\partial^2 \chi}{\partial v^{\mu} \partial v^{\nu}} = \frac{v^\mu v^\nu}{v^2} \left( \frac{d^2 \chi}{dv^2} - \frac{1}{v} \frac{d\chi}{dv} \right) + \frac{1}{v} \frac{d\chi}{dv} \delta^{\mu\nu},$$

the diffusion coefficient (122) can be put in the form

$$D^{\mu\nu} = \left( D_{\parallel} - \frac{1}{2} D_{\perp} \right) \frac{v^\mu v^\nu}{v^2} + \frac{1}{2} D_{\perp} \delta^{\mu\nu},$$

where

$$D_{\perp} = 2A \frac{1}{v} \frac{d\chi}{dv},$$

and

$$D_{\parallel} = A \frac{d^2 \chi}{dv^2},$$
are the diffusion coefficients in the directions perpendicular and parallel to the velocity of the test particle. To see that, we consider a system of coordinates where the z-axis is taken in the direction of $v$ so that $v_x = v_y = 0$ and $v_z = v$. In this system of coordinates, all the non-diagonal elements of $D^\mu\nu$ vanish while $D_{xx} = D_{yy} = \frac{1}{2}D_\perp$ and $D_{zz} = D_\parallel$. According to formula (138), we have explicitly

$$D_\perp = \frac{8\pi}{3} A \frac{1}{v} \left[ \int_0^v \left(3v^2 - \frac{v_1^4}{v^2}\right) g[f(v_1)] dv_1 + 2v \int_{v_1}^{+\infty} v_1 g[f(v_1)] dv_1 \right],$$

$$D_\parallel = \frac{8\pi}{3} A \frac{1}{v} \left[ \int_0^v v_1^4 g[f(v_1)] dv_1 + v \int_{v_1}^{+\infty} v_1 g[f(v_1)] dv_1 \right].$$

### 6.4 The dynamical friction

The friction term (127) can be simplified similarly. For an isotropic velocity distribution,

$$\frac{\partial \lambda}{\partial v^\mu} = \frac{d\lambda}{dv^\mu},$$

so that

$$\boldsymbol{\eta} = -2A \frac{1}{v} \frac{d\lambda}{dv} \mathbf{v}.$$ 

Using Eq. (134), we get

$$\boldsymbol{\eta} = 8\pi A \frac{\mathbf{v}}{v^3} \int_0^v v_1^2 H[f(v_1)] dv_1.$$

Similarly, we have

$$\frac{\partial D^{\mu\nu}}{\partial v^\nu} = -8\pi A \frac{\mathbf{v}}{v^3} \int_0^v v_1^2 g[f(v_1)] dv_1.$$

We note that when the field particles have an isotropic velocity distribution, the dynamical friction experienced by the test particle

$$\langle \mathbf{F} \rangle_{\text{friction}} = -8\pi A \frac{\mathbf{v}}{v^3} \int_0^v v_1^2 \left[ \frac{g(f)}{f} H(f_1) + \frac{H(f)}{f} g(f_1) \right] dv_1,$$

is parallel and opposite to its velocity $\mathbf{v}$. Moreover, it is due only to field particles with velocity $v_1 < v$. In fact, we can obtain this result without calculation. Indeed, according to Eqs. (127) et (128), we have

$$\boldsymbol{\eta} = 2A \int \frac{\mathbf{v} - \mathbf{v}_1}{|\mathbf{v} - \mathbf{v}_1|^3} H[f_1] d^3\mathbf{v}_1,$$

and a similar expression for $\partial D^{\mu\nu}/\partial v^\nu$. Now, Eq. (150) is analogous to the gravitational force created in $\mathbf{v}$ by a distribution of mass with density $H[f(v_1)]$ where $\mathbf{v}$ plays the role of the position $\mathbf{r}$. According to Newton’s law, the gravitational force created in $\mathbf{r}$ by an isotropic distribution of mass depends only on the mass interior to $r$ and is given by an expression equivalent to Eq. (147).
7 Explicit results for typical distribution functions

7.1 Isothermal distributions

We shall now obtain explicit expressions of diffusion coefficient and friction force for typical distribution functions. First, we consider the case of ordinary thermodynamics based on the Boltzmann entropy. In the thermal bath approximation, the field stars are at statistical equilibrium with the isothermal distribution

\[ f_{eq}(v_1) = \rho \left( \frac{\beta}{2\pi} \right)^{3/2} e^{-\beta v_1^2/2}. \]

For the Boltzmann entropy \( C(f) = f \ln f \), we have \( g(f) = H(f) = f \). The diffusion coefficient and the friction coefficient can then be calculated explicitly by substituting Eq. (151) in Eqs. (143), (144) and (147), and performing the integrals. Introducing the notation \( X = \sqrt{\beta v^2/2} \) and the function

\[ G(X) = \frac{2}{\sqrt{\pi}} \frac{1}{X^2} \int_0^X t^2 e^{-t^2} dt = \frac{1}{2X^2} \left[ \text{erf}(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2} \right], \]

where

\[ \text{erf}(X) = \frac{2}{\sqrt{\pi}} \int_0^X e^{-t^2} dt, \]

is the error function, we find after some elementary calculations that

\[ D_{||} = 2A\rho G(X) \frac{1}{v}, \]

\[ D_\perp = 2A\rho [\text{erf}(X) - G(X)] \frac{1}{v}, \]

\[ \eta = 2A\rho \beta G(X) \frac{v}{v}. \]

These results are well-known \[11\] and are recalled here for sake of completeness. We note that \( D_{||} \sim v^{-3} \) for \( |v| \rightarrow +\infty \). On the other hand, combining Eqs. (110), (156) and (154) we note that the dynamical friction can be written in the form

\[ \langle \mathbf{F} \rangle_{\text{friction}} = -2\eta = -2\beta D_{||} v, \]

so that the Einstein relation reads \( \xi = 2\beta D_{||} \) (the factor 2 is due to the velocity dependance of the diffusion coefficient). Quite generally, in the thermal bath approximation, we have

\[ \eta = \beta D_{||} v, \]

which results from Eqs. (117) and (140).
7.2 Fermi-Dirac distributions

We now consider the case of quantum particles (fermions) described by the Fermi-Dirac entropy. In the thermal bath approximation, the field particles are at statistical equilibrium with the Fermi-Dirac distribution

\[ f_{eq}(v_1) = \frac{\eta_0}{1 + \lambda e^{\beta v_1^2/2}}. \]

The parameter \( \lambda \) is related to the density \( \rho \) by

\[ \rho = \frac{4\pi \sqrt{2} \eta_0}{\beta^{3/2}} I_{1/2}(\lambda), \]

where

\[ I_n(t) = \int_0^{+\infty} \frac{x^n}{1 + te^{x}} dx, \]

is the Fermi integral of order \( n \). For the Fermi-Dirac entropy, the functions \( g \) and \( H \) are given by \( g(f) = f(1 - f/\eta_0) \) and \( H(f) = f \). The diffusion coefficient and the friction coefficient can be calculated explicitly by substituting Eq. (159) in Eqs. (143), (144) and (147), and performing the integrals. Introducing the notation \( X = \sqrt{\beta v^2/2} \) and the incomplete Fermi integral

\[ I_n(t, X) = \frac{1}{I_n(t)} \int_0^{X^2} \frac{x^n}{1 + te^{x}} dx, \]

we find after elementary calculations that

\[ D_{||} = A\rho \frac{1}{X^2} I_{1/2}(\lambda, X) \frac{1}{v}, \]

\[ D_{\perp} = A\rho \left[ \frac{I_{-1/2}(\lambda)}{I_{1/2}(\lambda)} I_{-1/2}(\lambda, X) - \frac{1}{X^2} I_{1/2}(\lambda, X) \right] \frac{1}{v}, \]

\[ \eta = A\rho \frac{\beta}{X^2} I_{1/2}(\lambda, X) \frac{v}{v} = \beta D_{||} v. \]

These equations were previously derived in [23] in a less neat form. For \( t \to +\infty \), we have the equivalent

\[ I_n(t) \sim \frac{1}{t} \Gamma(n + 1). \]

On the other hand,

\[ I_n(+\infty, X) = \frac{2}{\Gamma(n + 1)} \int_0^{X} y^{2n+1} e^{-y^2} dy. \]

Therefore,

\[ I_{1/2}(+\infty, X) = 2X^2 G(X), \quad I_{-1/2}(+\infty, X) = \text{erf}(X). \]

In the non-degenerate limit \( \lambda \to +\infty \), Eqs. (163)-(165) return the results (154)-(156) valid for classical particles.
7.3 Tsallis distributions

We finally consider the case of generalized thermodynamics based on Tsallis entropy. In the thermal bath approximation, the field stars are at statistical equilibrium with the polytropic distribution

\[ f_{eq}(v_1) = A \left( \frac{\lambda - v_1^2}{2} \right)^{-\frac{1}{q-1}}, \]

if \( v \leq v_m = \sqrt{2\lambda} \) and \( f = 0 \) otherwise (we restrict ourselves to \( q \geq 1 \)). The polytropic index \( n \) is related to the \( q \)-parameter \([7]\) by

\[ n = \frac{3}{2} + \frac{1}{q-1}. \]

The parameters \( A \) and \( \lambda \) are related to the generalized temperature \( \beta \) and to the density \( \rho \) by the relations

\[ A = \left[ \frac{(q-1)\beta}{q} \right]^{-\frac{1}{q-1}}, \quad \rho = 4\pi \sqrt{2} A \lambda B \left( \frac{3}{2}, n - \frac{1}{2} \right), \]

where

\[ B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \]

is the beta-function. For the Tsallis entropy, the \( g \) and \( H \) functions are \( g(f) = f \) and \( H(f) = f^q \). The diffusion coefficient and the friction coefficient can be calculated explicitly by substituting Eq. (169) in Eqs. (143), (144) and (147) and performing the integrals. Introducing the notation \( X = \sqrt{(n+1)n^2/2\lambda} \) and the incomplete beta-function

\[ B_X(m, n) = \frac{1}{B(m, n)} \int_0^{\frac{X^2}{n+1}} x^{m-1}(1-x)^{n-1} dx, \]

if \( X \leq \sqrt{n+1} \) and \( I_X(m, n) = 1 \) otherwise, we find after elementary calculations that

\[ D_{||} = A\rho \frac{1}{X^2} B_X \left( \frac{3}{2}, n + \frac{1}{2} \right) \frac{1}{v}, \]

\[ D_{\perp} = A\rho \left[ 2B_X \left( \frac{1}{2}, n + \frac{1}{2} \right) - B_X \left( \frac{3}{2}, n + \frac{1}{2} \right) \frac{1}{X^2} \right] \frac{1}{v}, \]

\[ \eta = A\rho \frac{\beta}{X^2} B_X \left( \frac{3}{2}, n + \frac{1}{2} \right) \frac{v}{v} = \beta D_{||}v. \]

For \( n \to +\infty \), we have the equivalent

\[ B(m, n) \sim \frac{\Gamma(m)}{n^m}. \]

On the other hand,

\[ B_X(m, +\infty) = \frac{2}{\Gamma(m)} \int_0^X y^{2m-1} e^{-y^2} dy. \]

Therefore,

\[ B_X \left( \frac{1}{2}, +\infty \right) = \text{erf}(X), \quad B_X \left( \frac{3}{2}, +\infty \right) = 2X^2 G(X). \]

In the limit \( n \to +\infty \), corresponding to \( q \to 1 \), Eqs. (174)-(176) return the results \([164]-[166]\) obtained in the context of ordinary thermodynamics.
8 Truncated models accounting for a permanent escape of particles

In this section, we shall derive generalized truncated distribution functions accounting for an escape of particles above a limit energy $\epsilon_m$. For simplicity we shall work in the thermal bath approximation. We thus describe the evolution of the distribution function by the generalized Kramers equation

\begin{equation}
\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left[ K \frac{\partial f}{\partial \mathbf{v}} + \beta g(f) \mathbf{v} \right],
\end{equation}

where we have neglected anisotropy, taken $D^{\mu\nu} = D_{||} \delta^{\mu\nu}$ in order to respect the Einstein relation \((158)\) and used the asymptotic expression $D_{||} \sim v^{-3}$ of the diffusion coefficient valid for high velocities. We assume that high energy particles are removed by a tidal field (for globular clusters, this is the gravitational attraction of a nearby galaxy). We seek therefore a stationary solution of equation (180) of the form $f = f(\epsilon)$ satisfying the boundary condition $f(\epsilon_m) = 0$, where $\epsilon = \frac{\mathbf{v}^2}{2} + \Phi$ is the energy of a particle and $\epsilon_m$ is the escape energy above which $f = 0$. Using the identity $\frac{\partial}{\partial \epsilon}(\frac{\mathbf{v}^3}{2}) = 0$ (valid for large $|\mathbf{v}|$), we obtain

\begin{equation}
\frac{d}{d\epsilon} \left[ h(f) \frac{df}{d\epsilon} + \beta g(f) \right] = 0,
\end{equation}

or, equivalently,

\begin{equation}
h(f) \frac{df}{d\epsilon} + \beta g(f) = -J,
\end{equation}

where $J$ is a constant of integration representing physically a current of diffusion. If $J = 0$, the foregoing equation reduces to

\begin{equation}
C''(f) \frac{df}{d\epsilon} + \beta = 0,
\end{equation}

where we have used Eq. (29). After integration, we recover the usual equilibrium distribution

\begin{equation}
C'(f_{eq}) = -\beta \epsilon - \alpha.
\end{equation}

If $J \neq 0$, Eq. (182) accounts for an escape of particles at a constant rate $J$. The system is therefore not truly static since it looses gradually particles but we can consider that it passes by a succession of quasi-stationary states that are solution of Eq. (182). Equation (182) is a first order differential equation which can be integrated as

\begin{equation}
\int_0^t \frac{h(t)dt}{g(t) + J/\beta} = \beta(\epsilon_m - \epsilon).
\end{equation}

Let us consider particular cases of this equation. For classical particles described by the Boltzmann entropy, Eq. (185) reduces to

\begin{equation}
\int_0^t \frac{dt}{t + J/\beta} = \beta(\epsilon_m - \epsilon).
\end{equation}

The integral is readily performed and we obtain the Michie-King model

\begin{equation}
f = A(e^{-\beta \epsilon} - e^{-\beta \epsilon_m}),
\end{equation}
where we have set $A = (J/\beta)e^{\beta \epsilon_m}$. The Michie-King model describes the tidally truncated structure of globular clusters in astrophysics [11]. For quantum particles described by the Fermi-Dirac entropy, Eq. (182) reduces to

$$\frac{df}{d\epsilon} + \beta f (1 - f/\eta_0) = -J.$$  

This is a Riccatti equation that can be solved analytically. Assuming that degeneracy is negligible for energies close to the escape energy, we get

$$f = \eta_0 e^{-\beta \epsilon - e^{-\beta \epsilon_m}}.$$  

This truncated distribution was previously derived in [23]. It could describe the case of galactic halos (e.g., massive neutrinos in dark matter models) limited in extension by tidal forces. It could also be of interest for collisionless stellar systems with Lynden-Bell’s interpretation of degeneracy. Finally, for systems of particles described by the Tsallis entropy, Eq. (185) reduces to

$$\int_0^{\beta f/J} \frac{t^{q-1}dx}{1 + t} = \frac{(J/\beta)^{1-q}}{q} \beta (\epsilon_m - \epsilon).$$

For $q = 1$, we recover Eq. (186). If we introduce the function

$$\phi_q(x) = \int_0^x \frac{t^{q-1}dx}{1 + t},$$

we can rewrite Eq. (190) in the form

$$f = \frac{J}{\beta} \phi_q^{-1} \left[ \frac{(J/\beta)^{1-q}}{q} \beta (\epsilon_m - \epsilon) \right].$$

For $q = 1$, we recover the Michie-King model. A detailed study of these truncated models will be given elsewhere.

9 Conclusion

In this paper, we have shown that standard kinetic equations (Boltzmann, Landau, Kramers, Smoluchowski,...) can be generalized so that they satisfy a $H$-theorem for an arbitrary functional of the form $S = -\int C(f) d^3r d^3v$, where $C(f)$ is a convex function. Boltzmann, Fermi, Bose and Tsallis entropies are particular functionals of the above form. These generalized kinetic equations have a thermodynamical structure which corresponds either to a microcanonical (Boltzmann, Landau) or a canonical (Kramers, Smoluchowski) description. One important conclusion of our work is that Tsallis entropy does not play any special role in this generalized thermodynamical formalism except that leading to simple distributions [3]. Therefore, the question that naturally emerges is whether other arguments can give Tsallis entropy a fundamental justification or whether Tsallis entropy is just a simple functional (associated to power-laws) extending Boltzmann entropy and providing a good fit of several observed phenomena. This is clearly an important point to be settled in the future.

There are at least two distinct notions of generalized thermodynamics. Generalized thermodynamics and kinetic equations can arise in complex media when the transition probabilities
have an expression different from the one we would naively expect \[10\]. We believe that generalized kinetic equations are essentially effective equations attempting to take into account “hidden constraints” that are not directly accessible to the observer or that are difficult to formalize. If this idea is correct, it means that it will never be possible to justify these equations from first principle (except in toy models). This also explains naturally why there is some indetermination in the theory, either in Tsallis $q$-parameter or more generally in the function $C(f)$. The important point in this context is to understand why standard kinetic theory breaks up. To our point of view, it is not sufficient to say “since the system is non-extensive, Boltzmann entropy is not correct and Tsallis entropy must be used instead”. This argument is too simplistic and does not bring any physical insight in the problem. The main interest of Tsallis nonextensive thermodynamics (in addition to the development of a consistent formalism) is to show that the naive Boltzmann description fails in many systems. However, the reason of this failure has to be understood in each case and this is the main challenge.

A notion of generalized thermodynamics also emerges in the context of the violent relaxation of stellar systems and 2D vortices (or other Hamiltonian systems with long-range interactions) described by Vlasov-type equations. In that context, we can explicitly illustrate the notion of “hidden constraints” that we mentioned previously. According to rigorous statistical mechanics \[18\] \[19\], the metaequilibrium state resulting from complete violent relaxation is obtained by maximizing the Boltzmann entropy $S[\rho]$ for the fine-grained distribution $\rho(r,v,\eta)$ while conserving mass, energy and an infinity of additional constraints played by the Casimirs. It turns out that the coarse-grained distribution function $\overline{f}(r,v)$ also maximizes a functional $S[\overline{f}] = -\int C(\overline{f})d^3r d^3v$ while conserving only mass and energy (robust constraints) \[7\] \[3\]. This functional, which could be called a generalized entropy, is non-universal due to fine-grained constraints that depend on the initial conditions. The Casimirs represent “hidden constraints” because, in practice, we just know the coarse-grained field and we do not have access to the initial conditions. In case of complete violent relaxation $S[\overline{f}]$ is never Tsallis entropy since $\overline{f} > 0$ according to Lynden-Bell’s theory. However, violent relaxation is incomplete in general. The only thing we know for sure is that the metaequilibrium state reached by the system is a nonlinearly dynamically stable stationary solution of the Vlasov equation on a coarse-grained scale. This implies in many cases that it maximizes a $H$-function $S[\overline{f}] = -\int C(\overline{f})d^3r d^3v$ at fixed mass and energy \[7\]. There, $C(\overline{f})$ depends on the initial conditions and on the efficiency of mixing. Tsallis entropy is a particular $H$-function. Since the dynamical stability criterion is similar to a generalized thermodynamical criterion, we believe that it is this thermodynamical analogy that justifies the consideration of Tsallis functionals (they are not true entropies!) in 2D turbulence and stellar dynamics. However, we stress that Tsallis entropy has no fundamental justification in that context \[32\] and that, indeed, most vortices and galaxies are not polytropic \[7\] \[3\]. Finally, we emphasize that the true statistical equilibrium state resulting from encounters between stars or between point vortices (collisional relaxation) is described, in a suitable thermodynamic limit, by the ordinary Boltzmann entropy although the system is non-extensive and non additive. The peculiarities due to the absence of entropy maximum in gravitational systems correspond to important physical processes (evaporation and gravothermal catastrophe) and not to a break up of thermodynamics \[7\]. We hope that this critical discussion will help to clarify the different notions of “generalized thermodynamics” that appeared in the recent literature.

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