POISSON LIE GROUP SYMMETRIES
FOR THE ISOTROPIC ROTATOR

G. Marmo ¹, A. Simoni ¹, A. Stern ²

1) Dipartimento di Scienze Fisiche dell’ Università di Napoli,
Mostra d’Oltremare pad. 19, 80125 Napoli, Italy.

2) Department of Physics, University of Alabama,
Tuscaloosa, Al 35487, USA.

ABSTRACT

We find a new Hamiltonian formulation of the classical isotropic rotator where left and right SU(2) transformations are not canonical symmetries but rather Poisson Lie group symmetries. The system corresponds to the classical analog of a quantum mechanical rotator which possesses quantum group symmetries. We also examine systems of two classical interacting rotators having Poisson Lie group symmetries.
1. Introduction

Recently there has been interest in examining symplectic structures which possess Poisson Lie group symmetries. [1-7] The interest is due in part to applications to classically integrable systems and in part to the claim that Poisson Lie group symmetries are the classical analog of quantum group symmetries.[6, 7, 8] To establish the connection between Poisson Lie groups and quantum groups the quantization procedure known as deformation quantization is utilized.[9] The study of classical systems with Poisson Lie group symmetries may thus provide physical insight into the corresponding quantum group invariant system.

Poisson Lie group transformations are implemented on phase space via group multiplication, and in general, they are not canonical transformations as they need not preserve the symplectic structure. However, they are defined so that invariance of the Poisson brackets follows once the parameters of the group of transformations are allowed to have certain nonzero Poisson brackets with themselves. Group multiplication is then said to correspond to a Poisson map.

Symplectic structures possessing Poisson Lie group symmetries have been constructed on spaces known as classical doubles.[1-4,7] Particle dynamics was not considered in these treatments, nor were applications to physical particle systems. In this article we write down particle Hamiltonians on the classical double corresponding to \( SL(2, C) \), and prove that the resulting dynamics is identical to that of the isotropic rotator. The Hamiltonians and the symplectic structures are parametrized by a single parameter \( \lambda \), and for a particular limiting value of \( \lambda \) one recovers the standard formulation. Away from the limiting value, we cannot express the Hamiltonian as the square of the angular momentum, and further the angular momentum does not satisfy the \( SU(2) \) algebra. The rotator does possess the usual left and right \( SU(2) \) symmetries. However for \( \lambda \) away from the limiting
value, we find that these symmetries are not canonical. Rather they are Poisson Lie group symmetries. We thus claim that it is possible to quantize the isotropic rotator (using the method of deformation quantization) so that the resulting system possesses left and right quantum group symmetries. We plan to study the quantization in a later paper.

Within the context of our example, we verify a claim\cite{5,11} that the charges generating the Poisson Lie group symmetries are group-valued; For us they are valued in the dual of $SU(2)$. Because the charges are group-valued, some novel features arise when we consider a system of two or more identical isotropic rotators. These features center around the question of what is the analog of the “total charge” for the system. The answer cannot be the sum of the individual charges since the sum is not a group operation. A more natural analog to the “total charge” for systems with Poisson Lie group symmetries is obtained by taking the group product, but this definition is not unique since the product does not in general commute. Further, the transformations generated by such charges are not the same as those generated by the total charge in the corresponding standard canonical formalism.

The outline of this article is as follows: In Sec. 2 we review the standard Hamiltonian formulation of the classical rotator. The alternative Hamiltonian formulations on the classical double $SL(2,C)$ are given in Sec. 3. The charges generating left $SU(2)$ transformations are constructed in this section. In Sec. 4 we use another coordinatization of $SL(2,C)$ to construct the charges generating right $SU(2)$ transformations. Systems of two interacting rotators are examined in Sec. 5. Here we obtain four different interactions for the rotators, each of which posses Poisson Lie group symmetries, and each of which can be thought of as deformations of the usual spin-spin interaction.
2. Standard Hamiltonian Formalism

In the standard Hamiltonian formalism for the classical rotator one has a set of angular momenta \( \ell_i, i = 1, 2, 3 \), which satisfy the \( SU(2) \) Poisson bracket algebra

\[
\{ \ell_i, \ell_j \} = c_{ij}^k \ell_k ,
\]

(1)

\( c_{ij}^k = \epsilon_{ijk} \) being the structure constants. To obtain the entire phase space we must include the analog of position variables. These variables indicate the orientation of the rotator. We denote such variables by \( g \) which here take values in the group \( G = SU(2) \). The phase space which results is known as the cotangent bundle \( T^*G \) of \( G \). In specifying the Poisson brackets for \( g \), one has that the brackets of the components of \( g \) (here represented by matrices) with themselves are zero, while the Poisson brackets of \( \ell_i \) with \( g \) are given by

\[
\{ \ell_i, g \} = i e_i g ,
\]

(2)

\( e_i, i = 1, 2, 3 \), defining a basis for the Lie algebra \( G \) associated with \( G \),

\[
[e_i, e_j] = i c_{ij}^k e_k .
\]

(3)

Eq. (2) was chosen so that \( \ell_i \) generate left translations on \( G \). The latter are the canonical transformations corresponding to spatial rotations. Canonical transformations associated with right translations on \( G \) are generated by charges \( t_i \) with \( t_i e_i = g^{-1} \ell_i e_i g \).

To determine dynamics for the system we now specify the Hamiltonian. The standard Hamiltonian for the isotropic rotator is

\[
H_0 = \frac{1}{2} \ell_i \ell_i ,
\]

(4)

where we have set the moment of inertia equal to one. The resulting system is rotationally invariant since \( \{ \ell_i, H_0 \} = 0 \). (It is also invariant under right \( SU(2) \) translations since
\{\ell_i, t_j\} = 0\) Using eq. (4) the Hamilton equations of motion for the system state are

\[ \dot{\ell}_i = 0, \quad \dot{g}g^\dagger = i\ell_i e_i . \quad (5) \]

Thus the angular momenta \(\ell_i\) are constants of the motion, while \(g\) undergoes a uniform precession.

3. Alternative Hamiltonian Formalism

We now give alternative Hamiltonian formulations of the isotropic rotator in which we modify i) the phase space, ii) the nature of the symmetries and iii) the Hamiltonian. Yet we preserve the dynamical system defined by the equations of motion (5). The alternative Hamiltonian formulations are all parametrized by a single parameter \(\lambda\), and for a particular limiting value of \(\lambda\) we recover the standard formulation described in the previous section.

i) We replace the phase space \(T^*G\) by a space known as the classical double \(D\) which we define below. \(D\) is a group which contains \(G\) along with another subgroup \(G^*\), which is the dual of \(G\). \(G^*\) has the same dimension of \(G\). Further, let \(e^i, i = 1, 2, 3\), define a basis for the Lie algebra \(G^*\) associated with \(G^*\), and \(f^{ij}_k\) be the structure constants for the algebra, ie.

\[ [e^i, e^j] = if^{ij}_k e^k . \quad (6) \]

The Lie algebra \(D\) associated with \(D\) is spanned by \(e_i\) and \(e^i\). If for the Lie bracket between \(e_j\) and \(e^i\) one takes

\[ [e^i, e_j] = ie^j_k e^k - if^{ik}_j e_k , \quad (7) \]

then there exists an invariant scalar product \(<,>\) on \(D\) such that \(e_i\) and \(e^i\) are dual to each other, ie.

\[ <e_j, e^i> = <e^i, e_j> = \delta^i_j, \quad <e_i, e_j> = <e^i, e^j> = 0 . \quad (8) \]
In this sense the group \( G^* \) is dual to \( G \), and vice versa. The algebra \( \mathcal{D} \) is determined by the relations \((3), (6)\) and \((7)\). The invariance of the scalar product \(<, >\) is with respect to the adjoint action of \( D \). If \( \alpha \in \mathcal{D} \), then under the adjoint action an infinitesimal change in \( \alpha \) is given by \( \delta, \alpha = [\epsilon, \alpha] \), \( \epsilon \) being an infinitesimal element of \( \mathcal{D} \). The invariance condition thus reads:

\[
< \delta, \alpha, \beta > + < \alpha, \delta, \beta > = 0 , \quad \alpha, \beta \in \mathcal{D} .
\] (9)

For the case of interest where \( G = SU(2) \), the dual group \( G^* \) is the group of \( 2 \times 2 \) lower triangular matrices with determinant equal to one. It is denoted by \( SB(2, C) \). The structure constants for \( G^* \) can be chosen to be \( f_{ij}^k = \epsilon_{ij\ell} \epsilon_{\ell k3} \). In the defining representation for the algebra \( \mathcal{G} \) and \( \mathcal{G}^* \), the basis \( e_i \) and \( e^i \) satisfying commutation relations \((3), (6)\) and \((7)\) can be expressed in terms of Pauli matrices \( \sigma_i \) according to:

\[
e_i = \frac{1}{2}\sigma_i , \quad e^i = \frac{1}{2}(i\sigma_i + \epsilon_{ij3}\sigma_j) .
\] (10)

For the above representation the scalar product \(<, >\) defined in \((8)\) corresponds to \( 2 \) times the imaginary part of the trace. \( e_i \) and \( e^i \) together span the \( SL(2, C) \) Lie algebra. Thus after exponentiation we have that the classical double \( D \) is \( SL(2, C) \).

We shall coordinatize the phase space \( D = \{ \gamma \} \) with variables in \( G \) and in \( G^* \). Thus let \( g \in G \) and \( g^* \in G^* \). An element \( \gamma \) in \( D \) is then labeled by \( (g^*, g) \) and can be defined by using the Iwasawa decomposition \( \gamma = g^*g \). The coordinates \( (g^*, g) \) of course do not globally cover \( D \) as, for instance, \((1, 1)\) and \((-1, -1)\) are both mapped to the identity in \( D \). Nevertheless, they serve as a useful parametrization of a finite region of \( D \).

For the Poisson brackets of \( g \) and \( g^* \) we propose the following quadratic\[10\] relations

\[
\{g_1, g_2\} = [r_{12}^*, g_1 g_2] , \quad \{g_1^*, g_2^*\} = -[r_{12} , g_1^* g_2^*] , \quad \{g_1^*, g_2\} = -g_1^* r_{12} g_2 ,
\] (11)-(13)
where we use tensor product notation. The indices 1 and 2 refer to two separate vector spaces on which the matrices act. $r_{12}$ and $r_{12}^*$ act nontrivially on both vector spaces 1 and 2, while $g_1 = g \otimes 1$, $g_2 = 1 \otimes g$, $g_1^* = g^* \otimes 1$, $g_2^* = 1 \otimes g^*$. Antisymmetry for Poisson bracket relations (11) and (12) is insured upon assuming the following conditions for matrices $r_{12}$ and $r_{12}^*$:

$$r_{12}^* = -r_{21} \quad \text{and} \quad r_{12}^* - r_{12} = \text{adjoint invariant},$$

where adjoint invariant means: $\gamma_1 \gamma_2(\text{adjoint invariant}) = (\text{adjoint invariant}) \gamma_1 \gamma_2$, for any $\gamma \in D$. Assuming these relations, antisymmetry for the remaining Poisson bracket (13) implies that

$$\{g_1, g_2^*\} = -g_2^* r_{12}^* g_1 .$$

Jacobi identities involving $g$ and $g^*$ are satisfied provided the $r$ matrices fulfill two quadratic equations,

$$[r_{13}, r_{12}] + [r_{23}, r_{12}] + [r_{23}, r_{13}] = 0 ,$$

$$[r_{23}, r_{31}] + [r_{31}, r_{12}] + [r_{12}, r_{23}] = \text{adjoint invariant} .$$

(Along with the relations obtained by interchanging the three vector spaces.) Eq. (16) is the classical Yang-Baxter equation, while (17) is a modified classical Yang-Baxter equation. We have used conditions (14) to derive (16) and (17). Two solutions to (16) and (17) are:

$$r_{12} = \lambda e^i \otimes e_i$$

or

$$r_{12} = \mu e_i \otimes e^i ,$$

where $\lambda$ and $\mu$ are constants. From $r_{12}$ and eqs. (14) we can obtain $r_{12}^*$.

From (11)-(13) we can compute Poisson brackets for elements $\gamma$ in $D$. Using $\gamma = g^* g$, we find

$$\{\gamma_1, \gamma_2\} = -\gamma_1 \gamma_2 r_{12}^* - r_{12} \gamma_1 \gamma_2 .$$
This symplectic structure has been studied in [4]. In [4] it was shown that using solution (18) the corresponding symplectic two form is a deformation of the symplectic two form for $T^*G$. In what follows, we shall be interested in only solution (18) for this reason.

It is easy to show that symplectic structure defined by (11)-(13) is a deformation of the Poisson brackets for a rigid body. The deformation parameter is $\lambda$. It is clear from (11) and (18) that the brackets of components of $g$ with themselves are zero in the limit $\lambda \to 0$. To recover (1) and (2) from (12) and (13) in this limit, one must also introduce the parameter $\lambda$ in $g^* = g^*(\lambda)$. We define $g^*(\lambda)$ as follows:

$$ g^*(\lambda) = e^{i\lambda e_i} . $$

Upon expanding $g^*(\lambda)$ in $\lambda$ and keeping the first order terms in eq. (13), we obtain

$$ i\lambda e^i \otimes \{ e_i, g \} = -\lambda e^i \otimes e_i g , $$

which is equivalent to eq. (2). By keeping second order terms in eq. (12), we obtain

$$ -\lambda^2 e^i \otimes e^j \{ e_i, e_j \} = -i\lambda^2 \ell_k \left[ e^i \otimes e_i , e^k \otimes 1 + 1 \otimes e^k \right] $$

$$ = -\lambda^2 \ell_k c_{ij}^k e^i \otimes e^j , $$

which is equivalent to eq. (1).

ii) One feature of the symplectic structure defined by (11)-(13) is the existence of Poisson Lie group transformations. In general Poisson-Lie group transformations are not canonical transformations as they need not preserve the symplectic structure. However, the symplectic structure can be made to be invariant under such transformations if we let the parameters of the transformations have nonzero Poisson brackets with themselves.

Among transformations of this type for our system are the right transformations of $G$ on $D = \{ \gamma \}$,

$$ \gamma \to \gamma h , \ h \in G $$

(24)
and the left action of $G^*$ on $D = \{ \gamma \}$,

$$\gamma \rightarrow h^* \gamma ,\ h^* \in G^*. \tag{25}$$

In terms of the coordinates $(g^*, g)$ this implies

$$g \rightarrow gh ,\ g^* \rightarrow g^*, \tag{26}$$

for the former and

$$g \rightarrow g ,\ g^* \rightarrow h^* g^*, \tag{27}$$

for the latter. By themselves transformations (26) and (27) do not preserve the Poisson brackets (11)-(13). But (11)-(13) can be made to be invariant under (26) if we insist that $h$ has the following Poisson bracket with itself

$$\{h_1, h_2\} = [ r_{12}^*, h_1 h_2 ] , \tag{28}$$

and zero Poisson brackets with $g$ and $g^*$. Then $SU(2)$ right multiplication is a Poisson map and (26) corresponds to a Poisson Lie group transformation. For (27) to be a Poisson Lie group transformation, $h^*$ must have the following Poisson bracket with itself

$$\{h_1^*, h_2^*\} = -[ r_{12}^* , h_1^* h_2^* ] , \tag{29}$$

and zero Poisson brackets with $g$ and $g^*$. Since the right-hand-sides of (28) and (29) vanish in the limit $\lambda \rightarrow 0$, the transformations (26) and (27) become canonical in the limit.

We note that Poisson brackets (11)-(13) are invariant under the simultaneous action of both $G$ and $G^*$ via (26) and (27). For this we assume that

$$\{h_1^*, h_2\} = 0 . \tag{30}$$

By comparing with eq. (13) we conclude that the algebra of the observables $g$ and $g^*$ is different from the algebra of the symmetries parametrized by $h$ and $h^*$. 

8
If we consider the infinitesimal version of the right action of G, then from eq. (28) we find that the Poisson bracket algebra of the corresponding infinitesimal parameters is isomorphic to $G^*$. Similarly, from eq. (29) we find that the Poisson bracket algebra of the infinitesimal parameters for left $G^*$ transformations is isomorphic to $G$.

Two additional Poisson Lie group transformations exist for this system and they too become canonical in the limit $\lambda \to 0$. They correspond to the left action of $G$ on $D = \{\gamma\}$,

$$\gamma \to f\gamma, \; f \in G$$

(31)

and the right action of $G^*$ on $D$,

$$\gamma \to \gamma f^*, \; f^* \in G^*.$$

(32)

Because we decompose $\gamma$ with that an element $g^*$ of $G^*$ on the left and an element $g$ of $G$ on the right, these transformations have a complicated action on the coordinates $(g^*, g)$. Below we shall examine only the former set of transformations, ie. the left action of $G$ on $D$, because as we shall show they are deformations of the ordinary rotations of the rotator.

The infinitesimal version of (31) is given by variations $\delta \gamma = \iota^b e_b \gamma$. It in turn leads to the following variations $\delta$ on $g$ and $g^*$

$$\delta g = i\epsilon^b (ad g^*)_b^a e_a g, \quad \delta g^* = i\epsilon^b \left( e_b g^* - (ad g^*)_b^a g^* e_a \right),$$

(33)

where $ad g^*$ denotes an element of the adjoint representation of $G^*$,

$$(ad g^*)_b^a = < g^* e_a g^{* -1}, e_b >,$$

and $\epsilon^b$ are the infinitesimal parameters of the transformation. Just as the infinitesimal parameters of the right action of $G$, satisfied a Poisson bracket algebra which was isomorphic to $G^*$, the same must be true of the infinitesimal parameters $\epsilon^a$ if the symplectic structure defined in (11)-(13) is to be invariant under (31). More specifically,

$$\{\epsilon^a, \epsilon^b\} = -\lambda f^{ab} \epsilon^c, \quad \{\epsilon^a, g\} = \{\epsilon^a, g^*\} = 0.$$

(34)
To show invariance we note that the Leibniz rule does not apply for $\delta$ acting on a Poisson bracket. For example, in computing $\delta\{g_1, g_2\} = \{g_1 + \delta g_1, g_2 + \delta g_2\} - \{g_1, g_2\}$ we have

$$
\delta\{g_1, g_2\} = \{\delta g_1, g_2\} + \{g_1, \delta g_2\} - \{\epsilon^b, \epsilon^d\}(ad g^*)^a_b (ad g^*)^c_d e_a \otimes e_c g_1 g_2.
$$

(35)

The last term in (35) cannot be ignored since like the other terms it is first order in $\epsilon^a$. Using (34) we can then show that the right-hand-side of (35) is equal to $[ r^*_1, \delta g_1 g_2 + g_1 \delta g_2 ]$, and hence (11) is invariant.

To show that transformations (33) are deformations of the ordinary rotations of a rigid body we expand $g^* = g^*(\lambda)$ in powers of $\lambda$ around $\lambda = 0$. To lowest order, (33) reduce to

$$
\delta g \rightarrow i \epsilon^a e_a g = \epsilon^a \{\ell_a, g\}, \quad \delta \ell_a \rightarrow -c^c_{ab} \epsilon^b \ell_c = \epsilon^b \{\ell_b, \ell_a\},
$$

(36)

where we used (1) and (2) to compute the Poisson brackets. From eq. (34) the parameters $\epsilon^a$ have zero Poisson brackets with everything when $\lambda \rightarrow 0$ and hence (33) are canonical transformations in the limit. Eqs. (36) show that the limiting transformations are generated by the angular momenta $\ell_a$, and therefore correspond to rotations of the rigid body.

It has been noted[5, 11] that the charges which generate Lie Poisson transformations take values in a group. This will now be made evident for our example. For arbitrary $\lambda$, we can express the variations (33) according to

$$
\delta g_1 = \frac{i}{\lambda} \epsilon^a <\{g_1, g_2^*\}g_2^{-1} g_1 \otimes e_a >_2, \quad \delta g_1^* = \frac{i}{\lambda} \epsilon^a <\{g_1^*, g_2^*\}g_2^{-1} g_1 \otimes e_a >_2,
$$

(37)

(38)

where $<,>_2$ indicates that the scalar product is taken on vector space 2. Eq’s (37) and (38) reduce to (36) in the limit $\lambda \rightarrow 0$. From the chain rule it follows that any function $\mathcal{F} = \mathcal{F}(g^*, g)$ of the coordinates $g^*$ and $g$ undergoes the variation

$$
\delta \mathcal{F} = \frac{i}{\lambda} \epsilon^a <\{\mathcal{F}_1, g_2^*\}g_2^{-1} g_1 \otimes e_a >_2,
$$

(39)
under the left action of $G$ on $D$, where $\mathcal{F}_1 = \mathcal{F} \otimes 1$. The variation of any function $\mathcal{F}$ on $D$ due to the left action of $G$ can therefore be obtained by computing the Poisson bracket of $\mathcal{F}$ with $g^*$. In this sense $g^*$ can be thought of as the charges generating the left action of $G$ on $D$. These charges are valued in $G^*$. The charges generating the right action of $G$ on $D$ also take values in $G^*$. (We construct them in Sec. 4.) Conversely, the charges associated with the action of $G^*$ on $D$ take values in $G$.

iii) Just as the Hamiltonian $H_0$ on $T^*G$ [Cf. eq. (4)] describing a rigid body is invariant under both the left and right actions of $SU(2)$, we can insist that the corresponding “deformed” Hamiltonian $H(\lambda)$ on $D$ be invariant under both the left and right actions of $SU(2)$. Let us also insist that $H(\lambda = 0) = H_0$. One possible Hamiltonian consistent with these requirements is

$$H(\lambda) = \frac{1}{2\lambda^2} (Tr \gamma \gamma^\dagger - 2) = \frac{1}{2\lambda^2} (Tr g^* g^{*\dagger} - 2),$$

(40)

where we now take $\gamma$ and $g^*$ to be matrices in the defining representation of $SL(2, C)$ and $SB(2, C)$ respectively, with the generators $e_i$ and $e^i$ given by (10). For the Hermitean conjugates of the generators we have

$$e_i^\dagger = e_i, \quad e^i = \frac{1}{2}(-i\sigma_i + \epsilon_{ij3}\sigma_j).$$

(41)

The 2 in parenthesis in (40) was subtracted so that (40) reduces to the standard rigid body Hamiltonian (4) when $\lambda \to 0$, ie. $H(\lambda = 0) = H_0$. Although $H(\lambda)$ is invariant under either the left or right actions of $G$, it is not invariant under either the left or right actions of the dual group $G^*$. Thus dynamics breaks the $SB(2, C)$ Lie Poisson symmetry of the symplectic structure.

To obtain Hamilton’s equations of motion for the system we need the Poisson brackets of $g^{*\dagger}$ with the dynamical variables $g$ and $g^*$. These brackets are not already determined by (11)-(13). We can make this more explicit by applying the two dimensional representation
for $SB(2, C)$, parametrizing $g^*$ by a real and a complex number according to

$$g^* = \left( \begin{array}{cc} y & 1/y \\ z & 1/y \end{array} \right), \quad y \in \mathbb{R}, \quad z \in \mathbb{C}.$$

Upon substituting this form into (12) using (10) we can obtain the Poisson bracket of for instance $y$ with $z$, \{ $y, z$ \} = i\lambda y z, but the Poisson bracket of $z$ with its complex conjugate is undetermined. Therefore the Poisson brackets of the matrix components $g^{*\dagger}$ with $g^*$ are not fully determined.

To specify the Poisson brackets of $g^{*\dagger}$ with the dynamical variables, we note that the group property and the algebra are preserved under

$$g^* \rightarrow g^{*\dagger-1}, \quad e^a \rightarrow e^{a\dagger}.$$

We shall require that the Poisson brackets of $g$ and $g^*$ are preserved under this mapping as well. Applying the mapping to (12) and (13) we obtain

$$\{g_1^{*\dagger-1}, g_2^*\} = -\lambda \left[ e^{i\dagger} \otimes e_i, g_1^{*\dagger-1} g_2^* \right], \quad (42)$$

$$\{g_1^{*\dagger-1}, g_2\} = -\lambda g_1^{*\dagger-1} (e^{i\dagger} \otimes e_i) g_2. \quad (43)$$

From these relations and \{ $g^{*\dagger-1}, g^{*\dagger}, \cdot$ \} = 0 we easily obtain that

$$\{g_1^{*\dagger}, g_2^*\} = \lambda \left( g_1^{*\dagger} (e^{i\dagger} \otimes e_i) g_2^* - g_2^* (e^{i\dagger} \otimes e_i) g_1^{*\dagger} \right) \quad (44)$$

$$\{g_1^{*\dagger}, g_2\} = \lambda (e^{i\dagger} \otimes e_i) g_1^{*\dagger} g_2. \quad (45)$$

For the equation of motion for $g^*$ we must find that it is a constant of the motion since $g^*$ is the charge associated with the $SU(2)$ left symmetry. Hamilton’s equation of motion gives

$$\dot{g}^* = \{ H(\lambda), g^* \} = \frac{1}{2\lambda} \left( e_i g^* Tr[g^* g^{*\dagger} (e^{i\dagger} - e^i)] - g^* e_i Tr[g^{*\dagger} g^* (e^{i\dagger} - e^i)] \right)$$

$$= -\frac{i}{4\lambda} \left( \sigma_i g^* Tr[g^* g^{*\dagger} \sigma_i] - g^* \sigma_i Tr[g^{*\dagger} g^* \sigma_i] \right), \quad (46)$$
since we may replace $e^{i\xi} - e^{i\xi}$ by $-i\sigma_i$ in the defining representation of $SL(2, C)$. But it is not hard to show that the right-hand-side of (46) is zero. Since both $g^*g^\dagger$ and $g^\dagger g^*$ are Hermitean we can express them as a linear combination of the Pauli matrices and the unit matrix. Upon substituting this form into (46) we get the desired result, ie.

$$\dot{g}^* = 0 .$$

(47)

For Hamilton’s equation of motion for $g$ we get

$$\dot{g} = \{H(\lambda), g\} = \frac{1}{2\lambda} e_i g \, \text{Tr}[g^*g^\dagger(e^{i\xi} - e^{i\xi})] = -\frac{i}{4\lambda} \sigma_i g \, \text{Tr}[g^*g^\dagger g^*\sigma_i] ,$$

or

$$\dot{gg}^\dagger = -\frac{i}{2\lambda} (g^*g^\dagger)_{tt}$$

(49)

where $A_{tt}$ denotes the traceless part of the matrix $A$, ie. $A_{tt} = A - \frac{1}{2}Tr[A] \times 1_{2\times2}$.

The right-hand-side of (49) is a traceless Hermitean matrix. From (47) it is also a constant matrix. Hence $g$ undergoes a uniform precession, and we obtain the same dynamics as that of the isotropic rotator. This is despite the fact that the Hamiltonian and symplectic structure of the deformed system defined in (11)-(13) differ from that of the standard Hamiltonian formulation of the isotropic rotator.

The result that the system described here is isotropic is at first sigh t surprising because if we expand the Hamiltonian (40) to second order in $\lambda$, we get an anisotropic looking term

$$H(\lambda) = \frac{1}{2} \ell_i \ell_i \left(1 + \frac{\lambda^2}{12}(\ell_3)^2\right) + O(\lambda^4) .$$

(50)

But $\ell_i$ in $g^* = g^*(\lambda)$ cannot be interpreted as the angular momenta of the rotator for the deformed system. Rather,

$$L_i = -\frac{1}{2\lambda} Tr \sigma_i g^* g^\dagger$$

(51)

plays that role since we can rewrite the equations of motion (47) and (49) according to

$$\dot{L}_i = 0 , \quad \dot{gg}^\dagger = iL_i e_i .$$

(52)
which is identical to (5). Unlike in the standard formalism we cannot express the Hamiltonian $H(\lambda)$ as the square of the angular momentum. This is seen by comparing (40) with $L_i L_i$. Further, $L_i = L_i(\lambda)$ does not satisfy an $SU(2)$ algebra, that is
\[ \{ L_i(\lambda), L_j(\lambda) \} \neq c^k_{ij} L_k(\lambda) \] (except in the limit $\lambda \to 0$, where $L_i$ coincides with $\ell_i$). To prove this it is sufficient to consider the case of small (but not zero) $\lambda$. By expanding $g^* g^\dagger$ to second order in $\lambda$, we find the following expression for $L_i(\lambda)$
\[ L_i(\lambda) = \ell_i - \frac{\lambda}{2} f^{ij} k^j_3 \ell_k + O(\lambda^2) = \ell_i + \frac{\lambda}{2} (\ell^2 \delta_{3i} - \ell_3 \ell_i) + O(\lambda^2) . \] (54)

Now use eq. (1) to prove the result (53). (For this we can show that there are no corrections of order $\lambda$ to the $SU(2)$ algebra (1) coming from the expansion of $g^*(\lambda)$ in (12).)

As in the standard formulation of the isotropic rotator the equations of motion are invariant under $SU(2) \times SU(2)$ transformations. The latter were canonical transformations in the standard formulation, whereas they correspond to Poisson Lie group symmetries for the formulation presented here. To see that the equations of motion (47) and (49) are invariant under transformations (24) and (31), let us rewrite them as equations for the $SL(2,C)$ group element $\gamma = g^* g$. We get:
\[ \dot{\gamma} \gamma^{-1} = -i \left( \frac{\lambda}{2} \gamma \gamma^\dagger \right)_{tt} . \] (55)

Eq. (55) is unchanged under $\gamma \to f \gamma h, \ h, f \in SU(2)$.

4. Alternative Parametrization of Phase Space

In the preceding Hamiltonian formalisms, the isotropic rotator is invariant under both left and right $SU(2)$ transformations. The left $SU(2)$ Poisson Lie group transformations
are generated by $g^*$, the general form for infinitesimal transformations given by (39). What are the generators of the right $SU(2)$ Poisson Lie group transformations?

To answer this question it is easiest to introduce a new parametrization of the phase space $D$. It corresponds to decomposing any element $\gamma \in D$ with an element $\tilde{g}$ of $G$ on the left and an element $\tilde{g}^*$ of $G^*$ on the right. Thus locally we have

$$\gamma = g^* g = \tilde{g} \tilde{g}^* .$$

The symplectic structure on $D$ given by (20) is recovered if for the coordinates $\tilde{g}$ and $\tilde{g}^*$ we take the following Poisson brackets

$$\{ \tilde{g}_1 , \tilde{g}_2 \} = - [ r_{12}^* , \tilde{g}_1 \tilde{g}_2 ] ,$$

$$\{ \tilde{g}_1^* , \tilde{g}_2^* \} = [ r_{12} , \tilde{g}_1^* \tilde{g}_2^* ] ,$$

$$\{ \tilde{g}_1^* , \tilde{g}_2 \} = - \tilde{g}_2 r_{12} \tilde{g}_1^* .$$

In terms of the coordinates $\tilde{g}$ and $\tilde{g}^*$, the left action of $SU(2)$ on $D$ now has a simple form

$$\tilde{g} \to f \tilde{g} , \quad \tilde{g}^* \to \tilde{g}^* ,$$

the symplectic structure (57)-(59) being invariant provided

$$\{ f_1 , f_2 \} = - [ r_{12}^* , f_1 f_2 ] .$$

The infinitesimal form of (61) agrees with (34) upon setting $f = 1 + i \epsilon^a e_a$.

On the other hand, right $SU(2)$ transformations have a more complicated action on $\tilde{g}$ and $\tilde{g}^*$. Infinitesimal variations are of the form

$$\delta \tilde{g} = i \eta^b (ad \, \tilde{g}^{*\! -1})_b^a \tilde{g} e_a , \quad \delta \tilde{g}^* = i \eta^b \left( \tilde{g}^{*\! -1} e_b - (ad \, \tilde{g}^{*\! -1})_b^a e_a \tilde{g}^* \right) ,$$

where $\eta^b$ are the infinitesimal parameters and, in order for the symplectic structure be invariant, they satisfy

$$\{ \eta^a , \eta^b \} = \lambda f_c^{ab} \eta^c , \quad \{ \eta^a , \tilde{g} \} = \{ \eta^a , \tilde{g}^* \} = 0 .$$
Eq. (63) is the infinitesimal version of (28) with \( h = 1 + i\eta \epsilon_a \). These variations are generated by \( \tilde{g}^* \), as any function \( F = F(\tilde{g}^*, \tilde{g}) \) of the coordinates \( \tilde{g}^* \) and \( \tilde{g} \) undergoes the variation

\[
\delta F_1 = \frac{i}{\lambda} \eta^a < \tilde{g}_2^{-1} \{ F_1, \tilde{g}_2^* \}, 1 \otimes \epsilon_a >_2 ,
\]

(64)

under the right action of \( SU(2) \) on \( D \). Thus just as with the left action of \( SU(2) \), the charges \( \tilde{g}^* \) generating the right action take values in the dual group \( SB(2, C) \).

Finally, we note that the invariant Hamiltonain (40) for the isotropic rotator can be written solely in terms of the generators \( \tilde{g}^* \) of right \( SU(2) \) transformations,

\[
H(\lambda) = \frac{1}{2\lambda^2} (Tr\tilde{g}^*\tilde{g}^{*\dagger} - 2) .
\]

(65)

5. System of Two Rotators

In the previous sections we examined the charges \( g^* \) and \( \tilde{g}^* \) which generate the left and right \( SU(2) \) Poisson Lie group symmetries of the isotropic rotator. They were both valued in the dual of \( SU(2) \). In general, Poisson Lie group symmetries are generated by group-valued charges. When we put together two identical systems possessing Poisson Lie group symmetries, what are the associated charges for the combined system?

For the case of two identical systems possessing canonical symmetries the associated charges for the combined system are sums of the charges for the individual systems. But clearly we cannot apply the same procedure to the case of systems possessing Poisson Lie group symmetries since the sum is not a group operation. A more natural analog to the “total charge” for systems with Poisson Lie group symmetries is obtained by taking the group product. However, such a definition is not unique since the product does not in general commute. Thus if we put together two isotropic rotators then there are two distinct analogs to the “total charge” associated with say left \( SU(2) \), and they generate two
distinct sets of Poisson Lie group transformations. Further, both of these transformations have the novel feature that the individual rigid bodies are rotated by different amounts, and the rotation of one of them depends on the coordinates of the other. Thus neither set of transformations corresponds to the rotations which would be generated by the total angular momentum of the corresponding standard canonical formalism (although they all coincide in the limit $\lambda \to 0$). We show these results in what follows.

Say that $\gamma^{(1)}$ and $\gamma^{(2)}$ denote $SL(2, C)$ elements corresponding to two distinct rigid rotators 1 and 2, while and $g^{*^{(1)}}$ and $g^{*^{(2)}}$ are the generators of the left $SU(2)$ Poisson Lie groups for the two respective systems. Thus, $\gamma^{(A)}$ and $g^{*^{(A)}}$ satisfy the Poisson brackets:

\[
\{ \gamma^{(A)}_1, \gamma^{(A)}_2 \} = -\gamma^{(A)}_1 \gamma^{(A)}_2 r^*_{12} - r_{12} \gamma^{(A)}_1 \gamma^{(A)}_2 ,
\]

(66)

\[
\{ g^{*^{(A)}}, g^{*^{(B)}}, 2 \} = -[ r_{12}, g^{*^{(A)}} g^{*^{(B)}}, 2 ] ,
\]

(67)

\[
\{ \gamma^{(A)}_1, g^{*^{(A)}}, 2 \} = -r^*_{12} \gamma^{(A)}_1 g^{*^{(A)}}, 2 , \text{ for } A = 1, 2 ,
\]

(68)

and

\[
\{ \gamma^{(A)}_1, \gamma^{(B)}_2 \} = \{ g^{*^{(A)}}, g^{*^{(B)}}, 2 \} = \{ \gamma^{(A)}_1, g^{*^{(B)}}, 2 \} = 0 , \text{ for } A \neq B ,
\]

(69)

Generalizing formula (39), the left $SU(2)$ Poisson Lie group generated by $g^{*^{(A)}}$ induces the variation

\[
\delta^{(A)} \mathcal{F}_1 = \frac{i}{\lambda} e^a \langle \mathcal{F}_1, g^{*^{(A)}}, 2 \rangle g^{*^{(A)}}, 2^{-1} , e_a \rangle_2 .
\]

(70)

on an arbitrary function $\mathcal{F}$ of $\gamma^{(1)}$ and $\gamma^{(2)}$. Upon identifying $\mathcal{F}$ with the dynamical variables $\gamma^{(A)}$, we then have that

\[
\delta^{(A)} \gamma^{(A)} = i e^a e_a \gamma^{(A)} , \text{ for } A = 1, 2 ,
\]

(71)

and

\[
\delta^{(A)} \gamma^{(B)} = 0 , \text{ for } A \neq B ,
\]

(72)

Now define the product charge $g^{*^{(12)}} = g^{*^{(1)}} g^{*^{(2)}}$. Upon applying formula (69) with
\[(A) = (12)\), it induces the following variations of \(\gamma^{(A)}\):

\[
\delta^{(12)} \gamma^{(1)} = i e^a e_a \gamma^{(1)} , \tag{73}
\]
\[
\delta^{(12)} \gamma^{(2)} = i e^a (\text{ad } g^{*(1)})_a b e_b \gamma^{(2)} , \tag{74}
\]

Thus rigid rotors 1 and 2 are transformed differently under the action of \(g^{*(12)}\). Furthermore, since \(g^{*(1)}\) corresponds to coordinates for rotator 1, we find that the transformation of rotator 2 depends on the coordinates of rotator 1.

Similarly, let us define the product charge \(g^{*(21)} = g^{*(2)} g^{*(1)}\). Transformations of rotator 1 induced by \(g^{*(21)}\) depend on the coordinates of rotator 2. These transformations are given by (73) and (74) with the indices 1 and 2 interchanged.

If the dynamics for the system is such that we have two noninteracting isotropic rotators, i.e., the Hamiltonian for the combined system is just the sum of two free Hamiltonians \(H^{(1)}\) and \(H^{(2)}\),

\[
H^{(A)} = \frac{1}{2\lambda^2} (\text{Tr} g^{*(A)} g^{*(A)\dagger} - 2) , \tag{75}
\]

then charges \(g^{*(12)}\) and \(g^{*(21)}\) generate two distinct Poisson Lie group symmetries. When \(\lambda \to 0\), these charges have an identical limit and it just corresponds to the total angular momentum for the combined system. Thus only when \(\lambda \to 0\), do \(g^{*(12)}\) and \(g^{*(21)}\) generate the usual rotation symmetry for the combined system.

Interactions can be introduced which break one of the above Poisson Lie group symmetries, leaving the other intact. For example, the interaction Hamiltonian term

\[
H^{(12)} = \frac{1}{2\lambda^2} (\text{Tr} g^{*(12)} g^{*(12)\dagger} - 2) , \tag{76}
\]

is not invariant under \(SU(2)\) transformations generated by \(g^{*(21)}\). On the other hand, (76) is invariant under \(SU(2)\) transformations generated by \(g^{*(12)}\). To see this we only need to use the result that the group product preserves the Poisson brackets, i.e., it is a Poisson map. Similarly, the interaction term

\[
H^{(21)} = \frac{1}{2\lambda^2} (\text{Tr} g^{*(21)} g^{*(21)\dagger} - 2) , \tag{77}
\]
is invariant under $SU(2)$ transformations generated by $g^{*(21)}$, but it is not invariant under $SU(2)$ transformations generated by $g^{*(12)}$. Of course, neither (76) or (77) are invariant under independent left $SU(2)$ transformations of 1 and 2 generated by $g^{*(1)}$ and $g^{*(2)}$, respectively.

In the limit $\lambda \to 0$, both $SU(2)$ transformations get identified and they are generated by the total angular momentum. Furthermore, the Hamiltonian interactions $H^{(12)}$ and $H^{(21)}$ approach the same limit when $\lambda \to 0$. For this set $g^{*(A)} = g^{*(A)}(\lambda) = e^{i\lambda\ell_i^{(A)}e_i}$ and expand (76) and (77) to lowest order in $\lambda$ to obtain

$$H^{(12)}, H^{(21)} \to \frac{1}{2}(\ell_i^{(1)} + \ell_i^{(2)})^2.$$  \hspace{1cm} (78)

Thus in the limit $\lambda \to 0$, we recover the usual spin-spin interaction from both $H^{(12)}$ and $H^{(21)}$.

In the above discussion we did not consider the possibility of right $SU(2)$ Poisson Lie group symmetries. Eq. (26) showed that the coordinates $g^*$ for a single rotator are unchanged by right $SU(2)$ transformations. Therefore all of the Hamiltonians considered so far, including the interaction Hamiltonians (76) and (77), which are functions solely of $g^{*(A)}$ and $g^{*(A)^T}$, are invariant under right $SU(2)$ transformations. In Sec. 4 we denoted the generators of the right $SU(2)$ Poisson Lie group symmetry for a single isotropic rotator by $\tilde{g}^*$. Let $\tilde{g}^{*(1)}$ and $\tilde{g}^{*(2)}$ be the generators of the right $SU(2)$ Poisson Lie groups transformations corresponding to two distinct rigid rotators 1 and 2. Then we can define another set of product charges $\tilde{g}^{*(12)} = \tilde{g}^{*(1)}\tilde{g}^{*(2)}$ and $\tilde{g}^{*(21)} = \tilde{g}^{*(2)}\tilde{g}^{*(1)}$ associated with right transformations. As with the corresponding left transformations, the rigid rotators 1 and 2 are rotated differently under the action of $\tilde{g}^{*(12)}$ and $\tilde{g}^{*(21)}$, and the transformation of one of the rotators depends on the coordinates of the other. $\tilde{g}^{*(12)}$ and $\tilde{g}^{*(21)}$ generate symmetries for all the previously discussed systems. As before, we can introduce new interactions which break one of the symmetries and leaves the other intact. In analogy
with (76) and (77), such interaction terms are
\[
\tilde{H}^{(12)} = \frac{1}{2\lambda^2} (Tr \tilde{g}^{*\dagger(12)} \tilde{g}^{(12)} - 2), \quad \text{and} \quad \tilde{H}^{(21)} = \frac{1}{2\lambda^2} (Tr \tilde{g}^{*\dagger(21)} \tilde{g}^{(21)} - 2). \tag{79}
\]
Now these interactions are invariant under left Poisson Lie group transformations generated by \(g^{*\dagger(1)}\) and \(g^{*\dagger(2)}\). Furthermore, they approach the same limit as did \(H^{(12)}\) and \(H^{(21)}\) when \(\lambda \to 0\), given in (78).

We have therefore found four different deformations of the usual spin-spin coupling. They correspond to the Hamiltonian terms: \(H^{(12)}\), \(H^{(21)}\), \(\tilde{H}^{(12)}\) and \(\tilde{H}^{(21)}\). All of these interactions possess Poisson Lie group symmetries.

**Acknowledgements**

A. Stern was supported in part by the Department of Energy, USA under contract number DE-FG-05-84ER-40141. A. Stern wishes to thank the group in Naples for their hospitality while this work was in progress. We are grateful for discussions with S. Rajeev. Without his invaluable assistance this work would not have been possible. We are also grateful for discussions with P. Vitale and I. Yakushin.

**References**

[1] V.G. Drinfel’d, Sov. Math. Doklady 27 (1983) 68; in Proc. Int. Congr. Math. (Berkekey), vol. 1 Academic Press, New York, 1986.

[2] M.A. Semenov-Tian-Shansky, Publ. RIMS, Kyoto University 21, no. 6 (1985) 1237; Theor. Math. Phys. 93 (1992) 302 (in Russian).

[3] S. Majid, Pacific J. Math 141 (1990) 311.

[4] A. Yu. Alekseev and A. Z. Malkin, Paris preprint PAR- LPTHE 93-08.
[5] A. Yu. Alekseev and I. T. Todorov, Vienna preprint. The undeformed version of the system discussed in this paper was examined in A. P. Balachandran, S. Borchardt and A. Stern, Phys. Rev. D17 (1978) 3247.

[6] J. Avan and M. Bellon, Phys. Lett. B213 (1988) 459.

[7] O. Babelon and D. Bernard, Phys. Lett. B260 (1991) 81; Commun. Math. Phys. 149 (1992) 279; Int. J. of Mod. Phys. A8 (1993) 507.

[8] For reviews see, L. Takhtajan in Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory, M-L. Ge and B-H. Zhao (eds.) (World Scientific, 1990); S. Majid, Int. J. Mod. Phys. 5 (1990) 1; T. Tjin, Int. J. Mod. Phys. 7 (1992) 6175.

[9] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, Ann. of Phys. 111 (1978) 61; 111.

[10] E. K. Sklyanin, Funct. Analy. Appl. 16 (1982) 263; Ya. I. Granovski, I. M. Lutzenko, A. S. Zhedanov, Ann. of Phys. 217 (1992) 1; J. Grabowski, G. Marmo, A. M. Perelomov, Vienna preprint.

[11] S. G. Rajeev, private communication.