Degree-correlation, omniscience, and randomized immunization protocols in finite networks

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Abstract

Many naturally occurring networks have a power-law degree distribution as well as a non-zero degree correlation. Despite this, most studies analyzing the efficiency of immunization strategies in networks have concentrated only on power-law degree distribution and ignored degree correlation. This study looks specifically at the effect degree-correlation has on the efficiency of several immunization strategies in scale-free networks. Generally, we found that positive degree correlation raises the number of immunized individuals needed to stop the spread of the infection. Importantly, we found that in networks with positive degree correlation, immunization strategies that utilize knowledge of initial popularity actually perform worse on average than random immunization strategies.

1 Introduction

Since the seminal paper of Barabasi and Albert [1] showing how networks built through growth and the preferential attachment of nodes naturally have a power-law degree distribution, work in network science has focused on networks following a power-law distribution instead of a binomial distribution. Most real world networks have been found to follow a power-law distribution [10]. In this study we test the efficiency of immunization strategies in randomly-generated networks that not only follow a power-law degree distribution, but also have varying levels of degree-correlation. Even though many naturally occurring networks have non-zero degree correlation, few of the previous studies of immunization strategies on networks have included this important aspect of realism [3] [4] [6] [7] [8] [12] [14] (however, see [9]). For example, in [2] [5], the authors study the effect of node deletion, both random and targeted, on the sizes of the connected components of power-law networks; they do not consider the effect of degree correlation.
For simplicity, we imagined a disease with perfect transmission efficiency such that any individual within a population that shares a transmission vector with an infected individual will always become infected. Therefore, in the network model, individuals are represented as nodes with the edges representing direct transmission vectors; thus all nodes with a connected path to an infected node also become infected. If the network is connected, that means that if one node is infected, then the entire network will be infected. Immunization is then modeled by deletion of nodes from the network. When enough individuals are immunized—that is, when enough nodes are deleted—the network becomes disconnected and part of the network will become isolated from the infection, effectively stopping the spread of the disease. As immunization progresses through time, the network is decomposed into more and more, smaller and smaller connected components. Each of these connected components, if harboring an infected individual, will become entirely infected, but the disease will not spread to the other components. Thus, the size of the largest connected component represents the worst case scenario for the outbreak of the disease. Setting the cutoff for an acceptable number of people to be infected is arbitrary and does not qualitatively affect our results.

\section*{2 Results}

One hundred different 1000-node networks were constructed via preferential attachment. For each such network, a rewiring algorithm was applied to copies of the network to produce degree-correlation coefficients of -0.3, -0.2, -0.1, 0.0, 0.1, 0.2, and 0.3 (see Section 3.2). Three further copies of each rewired network were then created and subjected to three different immunization protocols, whereby we “immunized” individuals by deleting the nodes that represent them until the largest connected component had less than or equal to 20 nodes:

\textbf{Method 1} (Random) Choose individuals to immunize uniformly at random, until no connected component of the network has size exceeding 20

\textbf{Method 2} (Initial Popularity) Immunize individuals (delete notes) in decreasing order of node degree in the original network, until no connected component of the network has size exceeding 20

\textbf{Method 3} (Omniscient) Immunize individuals (delete notes) in decreasing order of node degree, recalculating the degree of every node every 10 deletions, until no connected component of the network has size exceeding 20

One may think of these three methods as “no knowledge”, “knowledge of initial popularity rank”, and “complete knowledge of the network structure”, respectively. The results of the analysis are displayed in Figure 1.

As expected, positive degree correlation raises the number of immunized individuals needed to stop the spread of the infection \cite{11, 9}. Not
surprisingly, under certain conditions, when you have knowledge of the initial popularity (method 2), using that knowledge to inform your immunization strategy outperforms a random immunization strategy (method 1) as previously found \[2, 5\]. However, under sufficiently high positive degree-correlation, immunizing based on initial popularity rank (method 2) actually performs worse than random immunization (method 1). In networks with sufficiently high positive degree correlation, in order for knowledge of popularity to be effectively utilized, the popularity rank must be continually updated as nodes are removed from the network (method 3). Hu and Tang \[9\] showed previously that recalculating the popularity does give better performance than method 2, and the difference is greater in networks with positive degree correlation than those with negative degree correlation, however, due to the narrow scope of their chosen correlation coefficients (0.0579 and -0.0441), they did not discover the extent to which this performance difference is manifest. Additionally, they did not compare the performance of the initial popularity strategy to a random immunization strategy, thus missing the important observation that method 2 can actually perform worse than method 1 in networks with sufficiently high positive correlation. It is worth noting that studies analyzing the effectiveness of various immunization strategies most often use the default preferential attachment model, which has a degree-correlation of zero; according to our results, this happens to be the condition under which there is the largest difference in performance between method 1 and method 2. See Figure 1 (a).

In order to estimate the variability in our samples, we performed bootstrap resampling by taking 1,000,000 samples of size 100 (with replacement) from our original data sets for Methods 1 and 2, \[r = 0.3\]. Histograms for the averages of each resample are in Figure 1 (b). The corresponding resampling intervals are non-overlapping at the 94.5% confidence level.

3 Methods

3.1 Network generation

We used the Python library networkx to generate our networks. Specifically, we used the function barabasi.albert_graph(n, m) with 1000 nodes \((n = 1000)\) and with each added node attaching to three existing nodes \((m = 3)\).

3.2 A rewiring algorithm

We consider the algorithm proposed in \[13\] to give an existing network positive or negative degree correlation without changing the degree distribution.

1. Choose two edges \(e_1, e_2\) uniformly at random, rejecting pairs of edges incident to a single node. Suppose the nodes are \(n_1, n_2, n_3, n_4\) with \(\text{deg}(n_1) \leq \text{deg}(n_2) \leq \text{deg}(n_3) \leq \text{deg}(n_4)\).
2. (a) (positive correlation) Check to see that \( n_1 \) is not adjacent to \( n_2 \), and that \( n_3 \) is not adjacent to \( n_4 \); if so, delete \( e_1, e_2 \) and add \( n_1n_2 \) and \( n_3n_4 \) to the network; else proceed.

(b) (negative correlation) Check to see that \( n_1 \) is not adjacent to \( n_4 \), and that \( n_2 \) is not adjacent to \( n_3 \); if so, delete \( e_1, e_2 \) and add \( n_1n_4 \) and \( n_2n_3 \) to the network; else proceed.

3. Repeat a prescribed number of times.

![Figure 1](image)

**Figure 1**: (a) Data and averages for different correlation levels. (b) Resampling histogram for Method 1 (in red) and Method 2 (in blue) for \( r = 0.3 \) with \( 10^6 \) resamples

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