Abstract: We apply the supervariable approach to derive the proper quantum Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetries for the 1D diffeomorphism invariant model of a free scalar relativistic particle by exploiting the infinitesimal classical reparameterization (i.e. 1D diffeomorphism) symmetry of this theory. We derive the conserved and off-shell nilpotent (anti-)BRST charges and prove their absolute anticommutativity property by using the virtues of Curci-Ferrari (CF)-type restriction of our present theory. We establish the sanctity of the existence of CF-type restriction (i) by considering the (anti-)BRST symmetry transformations of the coupled (but equivalent) Lagrangians, and (ii) by proving the symmetry invariance of the Lagrangians within the framework of supervariable approach. We capture the nilpotency and absolute anticommutativity of the conserved (anti-)BRST charges within the framework of (anti-)chiral supervariable approach (ACSA) to BRST formalism. One of the novel observations of our present endeavor is the derivation of CF-type restriction by using the modified Bonora-Tonin (BT) supervariable approach (while deriving the (anti-)BRST symmetries for the target spacetime and/or momenta variables) and by symmetry considerations of the Lagrangians of the theory. The rest of the (anti-)BRST symmetries, for the other variables, are derived by using the newly proposed ACSA. We also demonstrate the existence of CF-type restriction in the proof of absolute anticommutativity of the (anti-)BRST charges.

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1 Introduction

For the covariant canonical quantization of the gauge theories [characterized by the first-class constraints in the terminology of Dirac’s prescription for the classification scheme of constraints (see, e.g. [1, 2])], one of the most intuitive, instructive and mathematically rich approaches is the Becchi-Rouet-Stora-Tyutin (BRST) formalism [3-6] which is also useful in the quantization of the diffeomorphism invariant theories. Two of the key characteristic features of the BRST formalism are the nilpotency and absolute anticommutativity properties associated with the (anti-) BRST symmetries which exist at the quantum level corresponding to an infinitesimal classical local gauge (and/or diffeomorphism) symmetry transformation. The geometrical superfield/supervariable approach [7-14] to BRST formalism provides the geometrical origin and interpretation for the above cited two key properties that are associated with the quantum gauge [i.e. (anti-) BRST] symmetries.

The usual superfield approach (USFA) to BRST formalism [7-11] exploits the idea of horizontality condition (HC) where a $(p + 1)$-form curvature tensor (i.e. field strength tensor), corresponding to a given $p$-form ($p = 1, 2, 3, ...$) gauge field, plays a pivotal role [9-11]. In fact, the application of the USFA leads to the precise derivation of (i) the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for the $p$-form gauge and associated (anti-)ghost fields for a given $p$-form gauge theory, and (ii) the (anti-)BRST invariant Curci-Ferrari (CF)-condition [9]. It does not shed any light, however, on the derivation of the (anti-)BRST symmetries for the matter fields in an interacting $p$-form gauge theory. The USFA has been systematically generalized so as to derive the (anti-)BRST symmetries for the gauge, (anti-)ghost and matter fields together by invoking the gauge invariant restrictions (GIRs) in addition to the HC. This extended version (see, e.g. [12-14]) of the USFA has been christened as the augmented version of superfield/supervariable approach (AVSA) to BRST formalism. It may be mentioned here that both the HC and GIRs are found to be consistent with each-other and, primarily, they complement each-other beautifully.

All the above developments have been achieved in the context of $p$-form gauge theories only. It has been a long-standing problem to apply the AVSA/USFA to the diffeomorphism invariant theories which are very important in the context of modern developments in the gravitational and (super)string theories. Against this backdrop, it is pertinent to point out that we have applied the (anti-)chiral superfield/supervariable approach (ACSA) (see, e.g. [15-18]) to derive the (anti-)BRST symmetries for all the relevant variables of a set of two reparameterization invariant theories [18]. These theories are the models [i.e. $(0 + 1)$-dimensional (1D) models] of a free scalar as well as a spinning relativistic particle. However, these theories are also found to be invariant under the gauge symmetry transformations. The latter symmetry transformations are found to be equivalent to the reparameterization symmetries in some specific limits (see, e.g. [19, 18] for details) where the on-shell condition and some choices are made in an ad-hoc fashion. Thus, the models considered in [18] are not purely reparameterization (i.e. 1D diffeomorphism) invariant theories. The classical gauge symmetries of these theories have been exploited in the context of their BRST quantization (see, e.g. [19] for details).
In a quite recent set of works in Refs. [20, 21], it has been shown that the Bonora-Tonin (BT) superfield formalism [9-11] can be applied to the D-dimensional diffeomorphism invariant theories provided we take into account the generalized version of the infinitesimal diffeomorphism transformations [cf. Eq. (13) below] in the superfies/supervariables that are defined on the (D, 2)-dimensional supermanifold. The latter is parametrized by the superspace coordinates \( Z^M \equiv (x^\mu, \theta, \bar{\theta}) \) where \( x^\mu (\mu = 0, 1, 2, \ldots D-1) \) are the bosonic variables and \((\theta, \bar{\theta})\) are a pair of Grassmannian variables that satisfy: \( \theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0 \).

We perform the proper super expansions of the supervariables/superfields along the \( \theta \) and \( \bar{\theta} \)-directions of the (D, 2)-dimensional supermanifold. The restrictions that are imposed on the superfies/supervariables have been called as the HC because the (super)exterior derivatives play very important roles in these restrictions. The importance of the (super)exterior derivatives becomes very clear when one derives the (anti-)BRST symmetries (corresponding to the infinitesimal diffeomorphism transformation) for the vector and metric tensors of the theory [20, 21]. In this analysis, only scalar supervariable is an exception where there happens to be no use of the (super)exterior derivatives in any kind of restrictions. Despite this, the simple and straightforward restriction that is imposed on the supervariable/superfield is still called as the HC (see, e.g. [20, 21]). In our present endeavor, we deal only with the scalar (super)variables that are defined on the (1, 2)-dimensional supermanifold.

We have applied, in our present endeavor, the modified BT-supervariable approach (MBTSA) to BRST formalism proposed in Refs. [20, 21] to a 1D diffeomorphism (i.e. reparameterization) invariant theory of a free scalar relativistic particle and obtained the proper (i.e. off-shell nilpotent and absolutely anticommuting) (anti-)BRST symmetries for the target space variables and CF-type restriction for the first time. The restriction(s) that have been imposed to derive the (anti-)BRST symmetries and CF-type restriction have been called as the HC. The existence of the CF-type restriction is the hallmark of a quantum theory when the latter is discussed within the framework of BRST formalism [22, 23]. We have applied the ACSA to obtain all the rest of the (anti-)BRST transformations for the other variables [i.e. auxiliary and (anti-)ghost variables] of our BRST invariant theory. This has led us to derive the appropriate coupled (but equivalent) Lagrangians for our theory which individually respect the (anti-)BRST transformations provided we restrict our discussions on a submanifold of the space of quantum variables (in the total quantum Hilbert space) where the CF-type restriction of our theory is satisfied.

The following key factors have been responsible for our curiosity in pursuing our present investigation. First, the diffeomorphism invariance is one of the key features of the gravitational theories in general and superstring theory in particular. Thus, it is important to apply the superfield/supervariable approach to discuss the BRST quantization of such theories. Second, our present model of the free scalar relativistic particle is a reparameterization invariant theory whose generalization is nothing but the bosonic string theory [24]. Thus, it is crucial to carry out its BRST analysis and derive the associated CF-type

\[ \text{We christen the geometrical BT-superfield/supervariable approach [9-11] as the modified BT-superfield/supervariable approach (MBTSA) to BRST formalism [20, 21] when we take into account the ordinary infinitesimal diffeomorphism/reparameterization transformation [cf. Eq. (5) below] and its generalization to the (1, 2)-dimensional superspace infinitesimal diffeomorphism/reparameterization transformation [cf. Eq. (13) below] that is defined on a (1, 2)-dimensional supermanifold.} \]
restriction. Third, this model is interesting in its own right because it is endowed with the gauge as well as reparameterization symmetries which are found to be equivalent under specific restrictions (cf. Sec. 2 below). Fourth, the gauge symmetry has been exploited for the BRST quantization in the standard literature (see, e.g. [19, 18]). We exploit the infinitesimal classical reparameterization symmetry for the BRST analysis in our present endeavor as, to the best of our knowledge, this has not been accomplished elsewhere. Finally, our present discussion is our modest first-step towards our central goal to apply the superfield approach to any arbitrary D-dimensional diffeomorphism invariant theory.

The contents of our present endeavor are organized as follows. In Sec. 2, we discuss the bare essentials of the classical gauge and reparameterization symmetries of the one (0 + 1)-dimensional (1D) model of a scalar relativistic particle and establish their equivalence. Our Sec. 3 deals with the derivation of CF-type restriction in the context of quantum nilpotent (anti-)BRST symmetries (corresponding to the classical reparameterization symmetry) within the framework of supervariable formalism where the full super expansions of the supervariables, on the suitably chosen (1, 2)-dimensional supermanifold, are taken into account. We also obtain here the (anti-)BRST symmetry transformations for the target spacetime and momenta variables. Our Sec. 4 contains the theoretical material where we derive the rest of the (anti-)BRST symmetries by using the ACSA [15-18]. Sec. 5 of our paper describes the invariance of the coupled Lagrangians in the ordinary spacetime. Sec. 6 of our present investigation is devoted to capture the invariance of the Lagrangians, off-shell nilpotency and absolute anticommutativity of the conserved (anti-)BRST charges within the framework of ACSA. Finally, in Sec. 7, we summarize our key results and point out a few future directions for further investigation(s).

Our Appendix A is devoted to the proof of the (anti-)BRST invariance of the CF-type restriction [cf. Eq. (24) below] of our theory. In our Appendix B, we provide an alternative proof for the existence of the CF-type restriction by proving the absolute anticommutativity of the conserved and nilpotent (anti-)BRST charges.

2 Preliminaries: Lagrangian Formulation and Some Continuous Symmetry Transformations

We begin with the following three equivalent Lagrangians for the free scalar relativistic particle as (see, e.g. [19], [18] for details)

\[ L_0 = m \sqrt{\dot{x}^2}, \quad L_f = p_\mu \dot{x}^\mu - \frac{e}{2} (p^2 - m^2), \quad L_s = \frac{1}{2} e \dot{x}^2 + \frac{e}{2} m^2, \]  

(1)

where \( L_0 \) is the Lagrangian with a square-root, \( L_f \) is the first-order Lagrangian and \( L_s \) is the second-order Lagrangian. The trajectory of the 1D free scalar relativistic particle is parametrized by the evolution parameter \( \tau \) and \( \dot{x}_\mu = \frac{dx_\mu}{d\tau} \) (with \( \mu = 0, 1, 2, ..., D - 1 \) are the generalized “velocities” of the free particle with momenta \( p_\mu \) and rest mass \( m \). In the above, Lagrangians \( L_f \) and \( L_s \) contain a Lagrangian multiplier variable which is called as the einbein variable and it behaves like a “gauge” variable in our theory (see, e.g. [19]). It is to be noted that the trajectory of our 1D toy model is embedded in a D-dimensional flat
Minkowskian spacetime manifold. The latter acts as the target spacetime manifold [with \(x^\mu(\mu = 0, 1, 2, 3...D - 1), \partial_\mu = \partial/\partial x^\mu\), etc.]

We shall focus, in our present investigation, on the first-order Lagrangian \(L_f\) (because \(L_0\) has a square-root and \(L_s\) has a variable in the denominator). This first-order Lagrangian is endowed with the following first-class constraints in the terminology of Dirac’s prescription for the classification scheme [1, 2], namely;

\[
\Pi(e) \approx 0, \quad -\frac{1}{2} (p^2 - m^2) \approx 0, \quad (2)
\]

where \(\Pi(e)\) is the canonical conjugate momentum w.r.t. \(e\) and \(p^2 - m^2 = 0\) is the mass-shell condition. It is evident that \(\Pi(e) \approx 0\) is the primary constraint and \(p^2 - m^2 \approx 0\) is the secondary constraint on the theory. These constraints are at the heart of the presence of a gauge symmetry transformation in the theory because the latter is generated by the following generator \((G)\)

\[
G = \xi \Pi(e) + \frac{1}{2} \xi (p^2 - m^2), \quad (3)
\]

where \(\xi(\tau)\) is the infinitesimal gauge transformation parameter. It is obvious that both the first-class constraints are present in the generator \(G\) of the gauge symmetry transformations:

\[
\delta_g x_\mu = \xi p_\mu, \delta_g p_\mu = 0, \delta_g e = \xi
\]

which are derived from the general formula \(\delta_g \phi = -i [\phi, G]\) for the generic variable \(\phi = x_\mu, p_\mu, e\). In the above derivation, we have to use the non-vanishing standard commutators \([x_\mu, p_\nu] = i \hbar \delta_\mu^\nu\) and \([e, \Pi(e)] = i \hbar\) and take the natural units \(\hbar = c = 1\). The above gauge symmetry transformations \((\delta_g)\) lead to the variation of the first-order Lagrangian \(L_f\) as

\[
\delta_g L_f = \frac{d}{d\tau} \left[ \frac{1}{2} \xi (p^2 + m^2) \right], \quad (4)
\]

thereby rendering the action integral \(S = \int_{-\infty}^{+\infty} d\tau L_f\) invariant for the physically well-defined parameter \(\xi(\tau)\) and the target space momenta variables \(p_\mu(\tau)\) which vanish-off as \(\tau \to \pm \infty\) (i.e. the limiting case when \(\tau \to \pm \infty\)).

The first-order Lagrangian \(L_f\) also respects an infinitesimal reparameterization symmetry \((\delta_r)\) as given below (see, e.g. [19, 18] for details)

\[
\delta_r x_\mu = \epsilon \dot{x}_\mu, \quad \delta_r p_\mu = \epsilon \dot{p}_\mu, \quad \delta_r e = \frac{d}{d\tau} (\epsilon e), \quad (5)
\]

where \(\epsilon(\tau)\) is the infinitesimal transformation parameter in \([\epsilon]: \tau \to \tau - \epsilon(\tau)\). In fact, under (5), the Lagrangian \(L_f\) transforms as: \(\delta_r L_f = \frac{d}{d\tau} (\epsilon L_f)\) thereby rendering the action integral \(S = \int_{-\infty}^{+\infty} d\tau L_f\) invariant. A close look at the gauge and reparameterization symmetry transformations demonstrates that both these continuous symmetries are equivalent on-shell (i.e. \(\dot{x}_\mu = \epsilon p_\mu, \dot{p}_\mu = 0\)) provided we identify the gauge transformation parameter \(\xi\)

\[\text{Actual reparameterization symmetry transformation is: } \tau \to \tau' = f(\tau) \text{ where } f(\tau) \text{ is a physically well-defined function of } \tau. \text{ However, this function is taken as: } f(\tau) = \tau - \epsilon(\tau) \text{ for its infinitesimal version where } \epsilon(\tau) \text{ is the infinitesimal transformation parameter.} \]
with the infinitesimal reparameterization transformation parameter $\epsilon$ as: $\xi = e\epsilon$ (where $e$ is the einbein variable that is present in our theory as a “gauge” variable and/or as a Lagrange multiplier).

In literature [19], the classical gauge symmetry ($\delta_g$) has been elevated to the quantum gauge [i.e. (anti-)BRST] symmetries for our present theory, namely:

$$s_{ab} x_\mu = \bar{c} p_\mu, \quad s_{ab} c = 0, \quad s_{ab} b = 0, \quad s_{ab} e = \bar{c},$$

$$s_b x_\mu = c p_\mu, \quad s_b c = 0, \quad s_b b = 0, \quad s_b e = \bar{c},$$

which are respected by a single Lagrangian [19]

$$L_b = p_\mu \dot{x}^\mu - \frac{1}{2} e (p^2 - m^2) + b \dot{e} + \frac{1}{2} b^2 - i \dot{\bar{c}} \bar{c},$$

where $b(\tau)$ is the Nakanishi-Lautrup type bosonic auxiliary variable, ($\bar{c})c$ are the fermionic ($c^2 = \bar{c}^2 = 0, c \bar{c} + \bar{c}c = 0$) (anti-)ghost variables and the gauge-fixing and Faddeev-Popov ghost terms have been derived from the following three explicit variations w.r.t. the (anti-)BRST symmetries $s_{(a)b}$, namely:

$$s_b s_{ab} \left[ \frac{i}{2} e^2 - \bar{c} c \right], \quad s_b \left[ - i \bar{c} \left( \dot{c} + \frac{b}{2} \right) \right], \quad s_{ab} \left[ i c \left( \dot{\bar{c}} + \frac{b}{2} \right) \right],$$

modulo some total derivatives w.r.t. the evolution parameter $\tau$. It is elementary to check that we have the following explicit (anti-)BRST symmetry transformations for the Lagrangian $L_f$, namely:

$$s_{ab} L_b = \frac{d}{d\tau} \left[ \frac{1}{2} \bar{c} (p^2 + m^2) + b \dot{c} \right], \quad s_b L_b = \frac{d}{d\tau} \left[ \frac{1}{2} e c (p^2 + m^2) + b \dot{c} \right],$$

which demonstrate that the (anti-)BRST symmetries (6) are the symmetries of the action integral $S = \int_{-\infty}^{+\infty} d\tau L_b$ because of the Gauss divergence theorem.

We end this section with the following remarks. First, we observe that there is a single Lagrangian that respects both the BRST as well as the anti-BRST symmetries corresponding to the classical gauge transformations: $\delta_g x_\mu = \xi p_\mu, \delta_g p_\mu = 0, \delta_g e = \xi$. Second, the (anti-)BRST symmetries $s_{(a)b}$ are off-shell nilpotent ($s_{(a)b}^2 = 0$) and absolutely anticommuting in nature (i.e. $s_b s_{ab} + s_{ab} s_b = 0$). As a consequence of the above observation, it can be checked that the following is true:

$$s_b s_{ab} \left[ \frac{i}{2} e^2 - \bar{c} c \right] \equiv - s_{ab} s_b \left[ \frac{i}{2} e^2 - \bar{c} c \right].$$

The above result establishes the fact that the gauge-fixing and Faddeev-Popov ghost terms are (anti-)BRST invariant due to the off-shell nilpotency ($s_{(a)b}^2 = 0$) of the fermionic (anti-)BRST symmetry transformations. Third, the above quantum (anti-)BRST symmetries are

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1The derivation of the gauge-fixing and Faddeev-Popov ghost terms [cf. Eq. (8)] is exactly same as the ones that are used for the Abelian 1-form ($A^{(1)} = dx^\mu A_\mu$) Maxwell’s $U(1)$ gauge theory where the gauge field $A_\mu$ has been replaced by the “gauge” variable $e(\tau)$ in our reparameterization (1D diffeomorphism) invariant theory for the BRST analysis.
continuous. Thus, the Noether theorem leads to the derivation of the following conserved charges as the generators for the (anti-)BRST symmetry transformations (6), namely;

\[ Q_{ab} = \frac{\bar{c}}{2} (p^2 - m^2) + b \dot{c} \equiv b \dot{c} - \dot{b} \bar{c}, \]

\[ Q_b = \frac{c}{2} (p^2 - m^2) + b \dot{c} \equiv b \dot{c} - \dot{b} c. \]

(11)

Fourth, the off-shell nilpotency and absolute anticommutativity of these charges can be proven by using the basic principle behind the continuous symmetry transformations and their generators (as the conserved Noether charges). In other words, we have the following relationships

\[ s_b Q_b = -i \{Q_b, Q_b\} = 0, \]

\[ s_{ab} Q_b = -i \{Q_b, Q_{ab}\} = 0, \]

\[ s_{ab} Q_{ab} = -i \{Q_{ab}, Q_{ab}\} = 0, \]

\[ s_b Q_{ab} = -i \{Q_{ab}, Q_b\} = 0, \]

(12)

where the l.h.s. can be computed easily by applying directly the (anti-)BRST symmetry transformations (6) on the conserved (anti-)BRST charges (11). Fifth, there is a ghost-scale symmetry (and corresponding conserved charge) in our theory and the (anti-)BRST and ghost charges obey the standard BRST algebra (see, e.g. [19] for details) establishing that the (anti-)BRST charges have the ghost numbers \((-1)\) respectively. Sixth, it is the existence of the CF-type restriction(s) that characterizes [22, 23] a quantum version of a classical gauge theory discussed and described within the framework of BRST formalism. In our present trivial Abelian gauge theory, we have the trivial CF-type restriction as: \(b + \bar{b} = 0\) where, in general, we have: \(s_{ab} c = i \bar{b}\) and \(s_{b} \bar{c} = i b\). Finally, we have seen that the BRST quantization of our model is straightforward when we take into account only the infinitesimal version of the classical gauge transformations for our whole discussion.

3 Nilpotent (Anti-)BRST Symmetries for the Target Space Variables and CF-Type Restriction: MBTSA

In the previous section, we have discussed the nilpotent (anti-)BRST symmetries, conserved (anti-)BRST charges and BRST quantization of our model by exploiting the beauty of the infinitesimal classical gauge symmetry transformations: \(\delta_g x_\mu = \xi p_\mu, \delta_g p_\mu = 0, \delta_g e = \xi\). The purpose of our present section is to exploit the infinitesimal reparameterization symmetries: \(\delta_r x_\mu = \epsilon \dot{x}_\mu, \delta_r p_\mu = \epsilon \dot{p}_\mu, \delta_r e = \frac{d}{d\tau} (\epsilon e)\) for the discussion of the corresponding (anti-)BRST symmetries and (anti-)BRST charges in the context of the BRST quantization of our 1D model of a reparameterization invariant free scalar relativistic particle. It is self-evident that the (anti-)BRST symmetry transformations for the target space variables \(x_\mu(\tau)\) and \(p_\mu(\tau)\) and einbein variable are: \(s_{ab} x_\mu = C \dot{x}_\mu, s_{ab} p_\mu = C \dot{p}_\mu, s_{ab} e = \frac{d}{d\tau} (C e)\), \(s_b x_\mu = C \dot{x}_\mu, s_b p_\mu = C \dot{p}_\mu, s_b e = \frac{d}{d\tau} (C e)\) where \((C)C\) are the fermionic \((C^2 = C^2 = 0, C C + C C = 0)\) (anti-)ghost variables corresponding to the infinitesimal parameter \(\epsilon(\tau)\) present in \(\tau \rightarrow \tau - \epsilon(\tau)\). In this section, we derive the off-shell nilpotent (anti-)BRST symmetries for
the target space variables \(x_\mu(\tau)\) and \(p_\mu(\tau)\) by using the modified BT-supervariable approach (MBTSA) to BRST formalism [20, 21] where the super diffeomorphism transformations [cf. Eq. (13) below] and the full super expansions of the supervariables along all the possible Grassmannian directions of the \((1, 2)\)-dimensional supermanifold are taken into account.

To derive the (anti-)BRST symmetries for the target space phase variables [i.e. \(x_\mu(\tau)\) and \(p_\mu(\tau)\)], first of all, we generalize the reparameterization (i.e. diffeomorphism) symmetry transformation parameter \(\tau\) [i.e. \(\tau \rightarrow \tau' = f(\tau) = \tau - \epsilon(\tau)\)] from the ordinary 1D spacetime manifold onto our suitably chosen \((1, 2)\)-dimensional supermanifold as

\[
f(\tau) \rightarrow \tilde{f}(\tau, \theta, \bar{\theta}) = \tau - \theta \bar{C}(\tau) - \bar{\theta} C(\tau) + \theta \bar{\theta} h(\tau),
\]

where the supermanifold is parameterized by \((\tau, \theta, \bar{\theta})\) and we have replaced the infinitesimal parameter \(\epsilon(\tau)\) by the fermionic (anti-)ghost variables \((\bar{C})C\) and they have been incorporated into (13) as the coefficients of \((\theta)\bar{\theta}\) due to the fact that the Grassmannian translational generators \((\partial_\theta)\partial_{\bar{\theta}}\) [along the \((\theta, \bar{\theta})\)-directions] have been shown [9, 10] to be intimately connected with the nilpotent (anti-)BRST symmetry transformations \(s_{(a)b}\). In other words, we have already incorporated the (anti-)BRST symmetry transformations \(s_{ab}\tau = -\bar{C}\) and \(s_b\tau = -C\) into the expansion (13). We have to compute the exact expression for the secondary variable \(h(\tau)\) from other consistency considerations.

According to the basic tenets of the modified BT-supervariable approach to BRST formalism, all the ordinary variables of the theory have to be generalized onto the suitably chosen \((1, 2)\)-dimensional supermanifold as the supervariables where the generalization in (13) has to be incorporated as one of the arguments of the supervariables. After that, we have to take into account the full super expansions along all the possible Grassmannian directions of the \((1, 2)\)-dimensional supermanifold. Thus, we have the following explicit generalizations for the target space variables

\[
x_\mu(\tau) \rightarrow \tilde{X}_\mu(\tilde{f}(\tau, \theta, \bar{\theta}), \theta, \bar{\theta}) = X_\mu(\tilde{f}(\tau, \theta, \bar{\theta})) + \theta \tilde{R}_\mu(\tilde{f}(\tau, \theta, \bar{\theta})) + \bar{\theta} R_\mu(\tilde{f}(\tau, \theta, \bar{\theta})) + \theta \bar{\theta} S_\mu(\tilde{f}(\tau, \theta, \bar{\theta})),
\]

\[
p_\mu(\tau) \rightarrow \tilde{P}_\mu(\tilde{f}(\tau, \theta, \bar{\theta})) = P_\mu(\tilde{f}(\tau, \theta, \bar{\theta})) + \theta \tilde{T}_\mu(\tilde{f}(\tau, \theta, \bar{\theta})) + \bar{\theta} T_\mu(\tilde{f}(\tau, \theta, \bar{\theta})) + \theta \bar{\theta} U_\mu(\tilde{f}(\tau, \theta, \bar{\theta})),
\]

where all the secondary supervariables on the r.h.s. (i.e. \(R_\mu, \tilde{R}_\mu, S_\mu, T_\mu, \tilde{T}_\mu, U_\mu\)) are function of the super diffeomorphism transformation (13). Thus, we have to take the appropriate Taylor expansion of all the above supervariables as:

\[
X_\mu(\tau - \theta \bar{C} - \bar{\theta} C + \theta \bar{\theta} h) = x_\mu(\tau) - \theta \bar{C} \dot{x}_\mu - \bar{\theta} C \dot{x}_\mu + \theta \bar{\theta} (h \dot{x}_\mu - \bar{C} C \bar{x}_\mu),
\]

\[
\theta R_\mu(\tau - \theta \bar{C} - \bar{\theta} C + \theta \bar{\theta} h) = \theta \tilde{R}_\mu(\tau) - \theta \bar{\theta} C \tilde{R}_\mu(\tau),
\]

\[
\bar{\theta} R_\mu(\tau - \theta \bar{C} - \bar{\theta} C + \theta \bar{\theta} h) = \bar{\theta} R_\mu(\tau) + \theta \bar{\theta} C \tilde{R}_\mu(\tau),
\]

\[
\theta \bar{\theta} S_\mu(\tau - \theta \bar{C} - \bar{\theta} C + \theta \bar{\theta} h) = \theta \bar{\theta} S_\mu(\tau).
\]

In the above, we have taken into account the usual key properties of the Grassmannian variables \((\theta, \bar{\theta})\): \(\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0\). In exactly similar fashion, we have to expand the secondary supervariables in the expansion for \(\tilde{P}_\mu(\tilde{f}(\tau, \theta, \bar{\theta}), \theta, \bar{\theta})\). In other words, we
have the following:

\[ P_\mu(\tau - \theta C - \bar{\theta} C + \theta \bar{\theta} h) = p_\mu(\tau) - \theta C \dot{p}_\mu - \bar{\theta} C \dot{p}_\mu + \theta \bar{\theta} (h \dot{p}_\mu - \bar{C} C \bar{p}_\mu), \]
\[ \theta \dot{T}_\mu(\tau - \theta C - \bar{\theta} C + \theta \bar{\theta} h) = \theta \dot{T}_\mu(\tau) - \theta \bar{C} \dot{T}_\mu(\tau), \]
\[ \bar{\theta} \dot{T}_\mu(\tau - \theta C - \bar{\theta} C + \theta \bar{\theta} h) = \bar{\theta} \dot{T}_\mu(\tau) + \theta \bar{\theta} \bar{C} \dot{T}_\mu(\tau), \]
\[ \theta \bar{\theta} U_\mu(\tau - \theta C - \bar{\theta} C + \theta \bar{\theta} h) = \theta \bar{\theta} U_\mu(\tau). \] (16)

Ultimately, the secondary supervariables on the r.h.s. of (14) have to be replaced by the ordinary secondary variables because they are Lorentz scalars w.r.t. the 1D spacetime manifold (i.e. 1D trajectory of the motion of the scalar relativistic particle which is embedded in a D-dimensional target flat Minkowskian spacetime manifold). As a consequence, the final expressions for the super expansions (14) are:

\[ \bar{X}_\mu(\bar{f}(\tau, \theta, \bar{\theta}), \theta, \bar{\theta}) = x_\mu(\tau) + \theta \bar{R}_\mu(\tau) + \bar{\theta} R_\mu(\tau) + \theta \bar{\theta} S_\mu(\tau), \]
\[ \equiv x_\mu(\tau) + \theta (s_b \bar{x}_\mu(\tau)) + \bar{\theta} (s_b x_\mu(\tau)) + \theta \bar{\theta} (s_b s_b x_\mu(\tau)), \]
\[ \bar{P}_\mu(\bar{f}(\tau, \theta, \bar{\theta}), \theta, \bar{\theta}) = p_\mu(\tau) + \theta \bar{T}_\mu(\tau) + \bar{\theta} T_\mu(\tau) + \theta \bar{\theta} U_\mu(\tau), \]
\[ \equiv p_\mu(\tau) + \theta (s_b p_\mu(\tau)) + \bar{\theta} (s_b p_\mu(\tau)) + \theta \bar{\theta} (s_b s_b p_\mu(\tau)). \] (17)

It is clear that we have to compute explicitly the exact values of the secondary variables \([R_\mu(\tau), \bar{R}_\mu(\tau), S_\mu(\tau), T_\mu(\tau), \bar{T}_\mu(\tau), U_\mu(\tau)]\) for the derivation of the nilpotent (anti-)BRST symmetry transformations \(s_{(a)\, b}\) for \(x_\mu(\tau)\) and \(p_\mu(\tau)\). As a side remark, we would like to lay emphasis on the fact that, for the existence of the proper (anti-)BRST symmetries, we should have \(s_b s_b x_\mu(\tau) = -s_{ab} s_b x_\mu(\tau)\) and \(s_b s_b p_\mu(\tau) = -s_{ab} s_b p_\mu(\tau)\) which lead to the absolute anticommutativity (i.e. \(s_b s_a + s_a s_b = 0\)) of the (anti-)BRST symmetry transformations \([s_{(a)\, b}]\).

At this stage, we exploit the theoretical potential and power of the “horizontality condition” (HC) for the reparameterization invariant theory and demand the following on physical ground\(^\text{b}\)

\[ \bar{X}_\mu(\bar{f}(\tau, \theta, \bar{\theta}), \theta, \bar{\theta}) \equiv X_\mu(\tau, \theta, \bar{\theta}) = x_\mu(\tau), \]
\[ \bar{P}_\mu(\bar{f}(\tau, \theta, \bar{\theta}), \theta, \bar{\theta}) \equiv P_\mu(\tau, \theta, \bar{\theta}) = p_\mu(\tau), \] (18)
due to the fact that \(x_\mu(\tau)\) and \(p_\mu(\tau)\) are scalars w.r.t. the 1D diffeomorphism transformation. For the above equality to be true, we have to collect all the expansions in (15) and (16) in a systematic and precise manner as illustrated below, namely:

\[ \bar{X}_\mu(\bar{f}(\tau, \theta, \bar{\theta}), \theta, \bar{\theta}) = x_\mu(\tau) + \theta \bar{R}_\mu - \bar{C} \dot{x}_\mu + \bar{\theta} (R_\mu - C \dot{x}_\mu) \]
\[ + \theta \bar{\theta} [S_\mu + \bar{C} \bar{R}_\mu - C R_\mu + h \dot{x}_\mu - \bar{C} C \bar{x}_\mu], \]
\[ \bar{P}_\mu(\bar{f}(\tau, \theta, \bar{\theta}), \theta, \bar{\theta}) = p_\mu(\tau) + \theta \bar{T}_\mu - \bar{C} \dot{p}_\mu + \bar{\theta} (T_\mu - C \dot{p}_\mu) \]
\[ + \theta \bar{\theta} [U_\mu + \bar{C} \bar{T}_\mu - C T_\mu + h \dot{p}_\mu - \bar{C} C \bar{p}_\mu]. \] (19)

\(^\text{b}\)A (fermionic and/or bosonic) Lorentz scalar should remain the same Lorentz scalar under any kind of spacetime and/or internal, (non-)supersymmetric, etc., transformations.
Now, we utilize the theoretical potential and power of the HC. Mathematically, this requires that: \( \ddot{X}_\mu (\ddot{f}(\tau, \theta, \dot{\theta}), \theta, \dot{\theta}) = x_\mu(\tau), \ddot{P}_\mu(\ddot{f}(\tau, \theta, \dot{\theta}), \theta, \dot{\theta}) = p_\mu(\tau) \). This leads to the determination of the secondary variables as

\[
\begin{align*}
\ddot{R}_\mu &= C \ddot{x}_\mu, & R_\mu &= C \dot{x}_\mu, & S_\mu &= C \dot{R}_\mu - C \ddot{R}_\mu + \hat{C} C \ddot{x}_\mu - h \dot{x}_\mu, \\
\ddot{T}_\mu &= \ddot{C} \ddot{p}_\mu, & T_\mu &= C \dot{p}_\mu, & U_\mu &= C \dot{T}_\mu - \ddot{C} \ddot{T}_\mu + \hat{C} C \ddot{p}_\mu - h \dot{p}_\mu.
\end{align*}
\]

Plugging in the values of \( R_\mu, \ddot{R}_\mu, T_\mu \) and \( \ddot{T}_\mu \) in the above, we obtain the following expressions for \( S_\mu(\tau) \) and \( U_\mu(\tau) \), namely:

\[
\begin{align*}
S_\mu(\tau) &= -[(\hat{C} C + \hat{C} \dot{C} + h) \dot{x}_\mu + \hat{C} C \ddot{x}_\mu], \\
U_\mu(\tau) &= -[(\hat{C} C + \hat{C} \dot{C} + h) \dot{p}_\mu + \hat{C} C \ddot{p}_\mu].
\end{align*}
\]

As argued earlier, the above expressions are also equal to \( s_b s_{ab} x_\mu \equiv -s_{ab} s_b x_\mu \) and \( s_b s_{ab} p_\mu = -s_{ab} s_b p_\mu \), respectively, where we have already derived \( s_b x_\mu = C \ddot{x}_\mu, s_b x_\mu = \hat{C} \dot{x}_\mu, s_b p_\mu = C \ddot{p}_\mu \) and \( s_b p_\mu = \hat{C} \ddot{p}_\mu \) because of the comparison with (17). In other words, as is evident from (20), the expressions for \( R_\mu, \ddot{R}_\mu, T_\mu \) and \( \ddot{T}_\mu \) imply that we have already obtained the nilpotent (anti-)BRST symmetries \( s_{(a)b} \) for the target space variables \( x_\mu(\tau) \) and \( p_\mu(\tau) \).

The nilpotency \([s_{(a)b}^2 = 0]\) properties of \( s_{(a)b} \) lead to the derivation of the (anti-)BRST symmetry transformations on the (anti-)ghost variables as:

\[
s_b C = C \dot{C}, \quad s_{ab} \bar{C} = \bar{C} \dot{C}.
\]

We assume that \( s_b C = \bar{B} \) and \( s_b \bar{C} = \bar{B} \) where \( B \) and \( \bar{B} \) are the Nakanishi-Lautrup type auxiliary variables of the theory. These transformations (i.e. \( s_b C = \bar{B}, s_{ab} C = \bar{B} \)) are the standard assumptions in the realm of BRST formalism. As a consequence of these off-shell nilpotent [i.e. \( s_{(a)b}^2 = 0 \)] transformations \( (s_b B = 0, s_b \bar{B} = 0) \), we note the following:

\[
\begin{align*}
s_b s_{ab} x_\mu &= (B - \dot{C} \bar{C}) \dot{x}_\mu - \hat{C} C \ddot{x}_\mu \equiv S_\mu(\tau), \\
-s_{ab} s_b x_\mu &= (-\bar{B} - \ddot{C} \bar{C}) \dot{x}_\mu - \hat{C} C \ddot{x}_\mu \equiv S_\mu(\tau), \\
s_b s_{ab} p_\mu &= (B - \dot{C} \bar{C}) \dot{p}_\mu - \hat{C} C \ddot{p}_\mu \equiv U_\mu(\tau), \\
-s_{ab} s_b p_\mu &= (-\bar{B} - \ddot{C} \bar{C}) \dot{p}_\mu - \hat{C} C \ddot{p}_\mu \equiv U_\mu(\tau).
\end{align*}
\]

The comparison of the above with the expressions (21) (derived from the MBTSA) leads to the derivation of the secondary variable \( h(\tau) \) as

\[
h(\tau) = \bar{B} - \dot{C} \bar{C} \equiv -B - \dot{C} C \implies B + \bar{B} + (\hat{C} C - \bar{C} \dot{C}) = 0.
\]

Thus, we have derived the celebrated Curci-Ferrari (CF)-type restriction [i.e. \( B + \bar{B} + (\hat{C} C - \bar{C} \dot{C}) = 0 \)] from the application of MBTSA to BRST formalism where it is the determination of the secondary variable \( h(\tau) \) [cf. Eq. (13)], in terms of the basic and auxiliary variables, from the requirement (i.e. \( s_b s_{ab} x_\mu = -s_{ab} s_b x_\mu \) or \( s_b s_{ab} p_\mu = -s_{ab} s_b p_\mu \)) of the absolute anticommutativity (i.e. \( s_b s_{ab} + s_{ab} s_b = 0 \)) property of \( s_{(a)b} \) that has played a crucial role.
We end this section with the following remarks. First, we note that our choice of \( s_b \bar{C} = B \) and \( s_{ab} C = \bar{B} \) implies that we have the following generalizations for the (anti-)ghost variables \((\bar{C})C\) from the 1D ordinary spacetime manifold to the \((1, 1)\)-dimensional (anti-)chiral super submanifolds of the \((1, 2)\)-dimensional supermanifold, as
\[
\begin{align*}
C(\tau) \rightarrow F^{(c)}(\tau, \theta) &= C(\tau) + \theta [\bar{B}(\tau)] \equiv C(\tau) + \theta (s_{ab}C), \\
\bar{C}(\tau) \rightarrow \bar{F}^{(ac)}(\tau, \bar{\theta}) &= \bar{C}(\tau) + \bar{\theta} [B(\tau)] \equiv \bar{C}(\tau) + \bar{\theta} (s_b\bar{C}),
\end{align*}
\]
where the superscripts \((c)\) and \((ac)\) denote the chiral and anti-chiral super expansions. This observation, in a subtle manner, explains that the (anti-)chiral supervariable approach (ACSA) to BRST formalism \([15-18]\) would be useful to us in our further discussions. Second, it can be checked that the absolute anticommutativity \((s_b s_{ab} + s_{ab} s_b = 0)\) properties, for the phase space target variables [i.e. \( x_\mu(\tau), p_\mu(\tau) \)] w.r.t. the off-shell nilpotent (anti-)BRST symmetry transformations, namely;
\[
\begin{align*}
\{ s_b, s_{ab} \} x_\mu &= [B + \bar{B} + (\dot{\bar{C}} C - \bar{C} \dot{C})] \dot{x}_\mu = 0, \\
\{ s_b, s_{ab} \} p_\mu &= [B + \bar{B} + (\dot{\bar{C}} C - \bar{C} \dot{C})] \dot{p}_\mu = 0,
\end{align*}
\]
are valid if and only if we apply the power and potential of the CF-type restriction \((24)\) from outside. Finally, we note that the requirement of the absolute anticommutativity of the (anti-)BRST symmetry transformations \(s_{ab}\) on the (anti-)ghost variables, namely;
\[
\begin{align*}
\{ s_b, s_{ab} \} C &= 0 \implies s_b B = \dot{B} C - B \dot{C}, \\
\{ s_b, s_{ab} \} \bar{C} &= 0 \implies s_{ab} B = \dot{\bar{B}} \bar{C} - B \dot{\bar{C}},
\end{align*}
\]
leads to the derivation of \(s_b \bar{B} = \dot{\bar{B}} C - \bar{B} \dot{C}\) and \(s_{ab} B = \dot{B} \bar{C} - B \dot{\bar{C}}\) which are found to be off-shell nilpotent \((|s_{ab}|^2 = 0)\) and absolutely anticommuting in nature (i.e. \(\{ s_b, s_{ab} \} B = 0, \{ s_b, s_{ab} \} \bar{B} = 0 \) without any use of the CF-type restriction.

4 \ ((Anti-)BRST Symmetry Transformations for other Variables of the Theory: ACSA

As has been pointed out earlier, we have already utilized the (anti-)chiral supervariable approach (ACSA) to determine the (anti-)BRST symmetry transformations: \( s_{ab} C = \bar{B} \) and \( s_b \bar{C} = B \) [cf. Eq. \((25)\)] which are primarily assumed in the BRST approach. In this section, we apply the ACSA to BRST formalism to derive the rest of the off-shell nilpotent (anti-)BRST symmetry transformations (besides our derivations in the previous section which are:
\( s_b x_\mu = C \dot{x}_\mu, \ s_{ab} x_\mu = \bar{C} \dot{x}_\mu, \ s_b p_\mu = \bar{C} \dot{p}_\mu, \ s_{ab} p_\mu = C \dot{p}_\mu, \ s_b \bar{C} = B, \ s_{ab} C = \bar{B} \)). Towards this objective in mind, we generalize the 1D ordinary variables \([e(\tau), C(\tau), \bar{B}(\tau), B(\tau)]\) onto a \((1, 1)\)-dimensional anti-chiral super-submanifold of the general \((1, 2)\)-dimensional supermanifold as
\[
\begin{align*}
e(\tau) \rightarrow E(\tau, \bar{\theta}) &= e(\tau) + \bar{\theta} f_1(\tau), \\
\bar{B}(\tau) \rightarrow \bar{B}(\tau, \bar{\theta}) &= \bar{B}(\tau) + \bar{\theta} f_3(\tau), \\
B(\tau) \rightarrow B(\tau, \bar{\theta}) &= B(\tau) + \bar{\theta} f_2(\tau), \\
C(\tau) \rightarrow F(\tau, \bar{\theta}) &= C(\tau) + \bar{\theta} b_1(\tau),
\end{align*}
\]
where the secondary variables \((f_1, f_2, f_3)\) are fermionic and \(b_1\) is bosonic in nature because of the fermionic \((\theta^2 = 0)\) nature of \(\theta\). It is evident that, in the limit \(\theta = 0\), we get back our ordinary variables \([e(\tau), C(\tau), B(\tau), \bar{B}(\tau)]\) from the above super expansions. Furthermore, it should be noted that our \((1, 1)\)-dimensional anti-chiral super-submanifold is parameterized by \((\tau, \bar{\theta})\) where the evolution parameter \(\tau\) is bosonic and \(\bar{\theta}\) is fermionic.  

To determine the secondary variables, in terms of the basic and auxiliary variables of the theory, we have to exploit one of the basic tenets of the ACSA which states that the quantum gauge (i.e. BRST) invariant quantities should be independent of the Grassmannian variable \(\bar{\theta}\). In this context, we note:

\[
  s_b (\bar{C} \dot{x}_\mu) = 0, \quad s_b (e \dot{C} + \dot{e} C) = 0, \quad s_b (\dot{B} C - B \dot{C}) = 0, \quad s_b B = 0. \tag{29}
\]

The above interesting BRST-invariant quantities, generalized onto a \((1, 1)\)-dimensional anti-chiral super-submanifold, should be independent of \(\bar{\theta}\). In other words, we have the validity of the following equalities, namely:

\[
  F(\tau, \bar{\theta}) \dot{X}_\mu^{(ha)}(\tau, \bar{\theta}) = C(\tau) \dot{x}_\mu(\tau), \quad \dot{B}(\tau, \bar{\theta}) = B(\tau), \tag{30}
\]

\[
  E(\tau, \bar{\theta}) \dot{F}(\tau, \bar{\theta}) + \dot{E}(\tau, \bar{\theta}) F(\tau, \bar{\theta}) = e(\tau) \dot{C}(\tau) + \dot{e}(\tau) C(\tau), \tag{31}
\]

where \(X_\mu^{(ha)}(\tau, \bar{\theta})\) is the anti-chiral limit of the full super expansion that has been obtained in the previous section, namely:

\[
  X_\mu^{(h)}(\tau, \theta, \bar{\theta}) = x_\mu(\tau) + \theta (\bar{C} \dot{x}_\mu) + \bar{\theta} (C \dot{x}_\mu) + \theta \bar{\theta} \left[ - \{ (\bar{B} + \dot{\bar{C}} C) \dot{x}_\mu + \bar{C} C \dot{x}_\mu \} \right] \\
  \equiv x_\mu(\tau) + \theta (\bar{C} \dot{x}_\mu) + \bar{\theta} (C \dot{x}_\mu) + \theta \bar{\theta} \left[ (B - \bar{C} \dot{C}) \dot{x}_\mu - \bar{C} C \dot{x}_\mu \right]. \tag{32}
\]

In the above, the superscript \((h)\) denotes that the supervariable \(X_\mu^{(h)}(\tau, \theta, \bar{\theta})\) has been obtained after the application of the HC. In other words, we have the following anti-chiral limiting case, namely,

\[
  X_\mu^{(ha)}(\tau, \bar{\theta}) = x_\mu(\tau) + \bar{\theta} [C(\tau) \dot{x}_\mu(\tau)], \tag{33}
\]

where the superscript \((ha)\) denotes the anti-chiral limit of the super expansion \((31)\) that has been obtained after the application of the HC in the previous section. The substitutions, from \((31)\) and \((28)\) into the first entry of Eq. \((30)\), leads to \(b_1(\tau) = C(\tau) \dot{C}(\tau)\). The BRST invariance of the Nakanishi-Lautrup auxiliary variable (i.e. \(s_b B = 0\)) implies that \(f_2(\tau) = 0\). Thus, we have the following super expansions:

\[
  F^{(b)}(\tau, \bar{\theta}) = C(\tau) + \bar{\theta} (C \dot{C}) \equiv C(\tau) + \bar{\theta} (s_b C(\tau)), \tag{34}
\]

\[
  \dot{B}^{(b)}(\tau, \bar{\theta}) = B(\tau) + \dot{\theta}(0) \equiv B(\tau) + \dot{\theta} (s_b B(\tau)). \tag{35}
\]

A close look at the above equation demonstrates that we have already obtained the BRST symmetry transformations: \(s_b C = C \dot{C}\) and \(s_b B = 0\) as the coefficients of \(\dot{\theta}\) in the expansions \((33)\) where the superscript \((b)\) denotes the supervariables that have been obtained after the applications of the BRST-invariant (i.e. quantum gauge invariant) restrictions \((29)\).
It should be noted that \( s_b C = C \dot{C} \) can also be derived from the restriction corresponding to the invariance \( s_b (C \dot{p}_\mu) = 0 \) on the \((1, 1)\)-dimensional anti-chiral super-submanifold. However, for the sake of brevity, we have not discussed it here. In the rest of the restrictions in (30), we use the final expressions from (33) to obtain the exact expressions for the secondary variables as:

\[
\begin{align*}
f_1(\tau) &= e(\tau) \dot{\tau} + \dot{e}(\tau) C(\tau), \\
f_3(\tau) &= \dot{B}(\tau) C(\tau) - B(\tau) \dot{C}(\tau).
\end{align*}
\]  

As a consequence, we have the following super expansions for some of the supervariables [cf. Eq. (28)], namely:

\[
\begin{align*}
E^{(b)}(\tau, \theta) &= e(\tau) + \theta (\dot{e} C + e \dot{C}) \equiv e(\tau) + \theta (s_b e(\tau)), \\
\tilde{B}^{(b)}(\tau, \theta) &= \tilde{B}(\tau) + \theta (\tilde{B} C - \tilde{B} \dot{C}) \equiv \tilde{B}(\tau) + \theta (s_b \tilde{B}(\tau)),
\end{align*}
\]

where the superscript \((b)\) stands for the expansions that have been obtained after the applications of the BRST (i.e. quantum gauge) invariance listed in (29). It is straightforward to note that we have already derived the BRST transformations: \( s_b B = 0, s_b C = C \dot{C}, s_b e = e \dot{C} + \dot{e} C \) and \( s_b \tilde{B} = \tilde{B} C - \tilde{B} \dot{C} \) as the coefficients of \( \theta \)-Grassmannian variable in the super expansions of equations (33) and (35). In other words, we observe that all the BRST symmetry transformations \( (s_b) \) for all the variables of our theory have been obtained in equations (25), (33) and (35) besides the target space variables that have been obtained earlier by exploiting the theoretical strength of MBTSA (cf. Sec. 3).

To derive the anti-BRST symmetry transformations \( (s_{ab}) \) for the variables \( (B, e, \tilde{C}, \tilde{B}) \), we note that the following quantities (that are present in the round brackets) are anti-BRST invariant, namely:

\[
s_{ab} B = 0, \quad s_{ab} (\tilde{B} \dot{C} - B \dot{C}) = 0, \quad s_{ab} (e \dot{C} + \dot{e} C) = 0, \quad s_{ab} (\tilde{C} \dot{x}_\mu) = 0.
\]  

According to the basic tenets of ACSA, the above quantities must be independent of the Grassmannian variable \( \theta \) when they are generalized onto a \((1, 1)\)-dimensional chiral super-submanifold of the \((1, 2)\)-dimensional supermanifold on which our theory is generalized. Towards this aim in mind, we generalize the 1D variables \( (e, B, \tilde{B}, \tilde{C}) \) onto the chosen \((1, 1)\)-dimensional chiral super-submanifold as the following super expansions, namely:

\[
\begin{align*}
e(\tau) &\longrightarrow E(\tau, \theta) = e(\tau) + \theta \tilde{f}_1(\tau), \\
B(\tau) &\longrightarrow \tilde{B}(\tau, \theta) = B(\tau) + \theta \tilde{f}_2(\tau), \\
\tilde{C}(\tau) &\longrightarrow \tilde{F}(\tau, \theta) = \tilde{C}(\tau) + \theta \tilde{b}_1(\tau), \\
\tilde{B}(\tau) &\longrightarrow \tilde{B}(\tau, \theta) = \tilde{B}(\tau) + \theta \tilde{f}_3(\tau),
\end{align*}
\]

where \( (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) \) are fermionic and \( \tilde{b}_1(\tau) \) is the bosonic secondary variables because of the fermionic \( (\theta^2 = 0) \) nature of the Grassmannian variable \( \theta \) which characterizes the \((1, 1)\)-dimensional chiral super-submanifold besides the evolution bosonic parameter \( \tau \) of our 1D diffeomorphism invariant system.

As a first-step, let us compute the secondary variable \( \tilde{b}_1(\tau) \) in terms of the basic variables of the theory. The anti-BRST invariance we use is: \( s_{ab} (\tilde{C} \dot{x}_\mu) = 0 \). In other words, we have
the validity of the following restriction on the chiral supervariables according to the basic tenets of ACSA, namely;

\[ F(\tau, \theta) X^{(hc)}_\mu(\tau, \theta) = \dot{C}(\tau) \dot{x}_\mu(\tau), \quad (38) \]

where \( X^{(hc)}_\mu \) is the chiral limit of the full super expansion (31) that has been obtained for \( X^{(h)}_\mu(\tau, \theta, \bar{\theta}). \) To be precise, the latter has been derived in the previous section. Mathematically, the above chiral limit implies the following:

\[ X^{(hc)}_\mu(\tau, \theta) = x_\mu(\tau) + \theta (\bar{C}(\tau) \dot{x}_\mu(\tau)). \quad (39) \]

Plugging in the expansions from (37) and (39), we obtain the expression for \( \bar{b}_1(\tau) = \bar{C}(\tau) \dot{C}(\tau). \) Thus, we have already obtained

\[ \bar{F}^{(ab)}(\tau, \theta) = \bar{C}(\tau) + \theta (\dot{C} \dot{\bar{C}}) \equiv \bar{C}(\tau) + \theta (s_{ab} \bar{C}(\tau)), \quad (40) \]

where the coefficient of \( \theta \) is nothing but the anti-BRST symmetry transformation for the \( \bar{C}(\tau) \) variable as\(^5\) \( s_{ab} \bar{C} = \bar{C} \dot{\bar{C}} \) and the superscript \((ab)\) denotes the supervariable that has been obtained after the application of the specific anti-BRST invariant restriction in (36). Against the backdrop of the above derivation, we can derive the other anti-BRST symmetry transformations by using the anti-BRST (i.e. quantum gauge) invariant quantities (36) and using the super expansions (37) and (40). In other words, we have the following restrictions

\[ \bar{\tilde{B}}(\tau, \theta) = \bar{B}(\tau), \quad \hat{\tilde{B}}(\tau, \theta) \bar{F}^{(ab)}(\tau, \theta) - \tilde{B}(\tau, \theta) \hat{F}^{(ab)}(\tau, \theta) = \bar{B}(\tau) \bar{C}(\tau) - B(\tau) \dot{\bar{C}}(\tau), \]

\[ E(\tau, \theta) \hat{F}^{(ab)}(\tau, \theta) + \tilde{E}(\tau, \theta) \bar{F}^{(ab)}(\tau, \theta) = e(\tau) \dot{\bar{C}}(\tau) + \dot{\bar{e}}(\tau) \bar{C}(\tau), \quad (41) \]

which lead to the precise determination of the secondary variables as follows:

\[ \bar{f}_0(\tau) = 0, \quad \bar{f}_1(\tau) = e(\tau) \dot{\bar{C}} + \dot{\bar{e}}(\tau) \bar{C}(\tau), \quad \bar{f}_2(\tau) = \bar{B}(\tau) \bar{C}(\tau) - B(\tau) \dot{\bar{C}}(\tau). \quad (42) \]

Ultimately, we have the following super expansions in their full blaze of glory

\[ X^{(hc)}_\mu(\tau, \theta) = x_\mu(\tau) + \theta (\bar{C} \dot{x}_\mu) \equiv x_\mu(\tau) + \theta (s_{ab} x_\mu(\tau)), \]

\[ P^{(hc)}_\mu(\tau, \theta) = p_\mu(\tau) + \theta (\bar{C} \dot{p}_\mu) \equiv p_\mu(\tau) + \theta (s_{ab} p_\mu(\tau)), \]

\[ E^{(ab)}(\tau, \theta) = e(\tau) + \theta (e \dot{\bar{C}} + \dot{\bar{e}} \bar{C}) \equiv e(\tau) + \theta (s_{ab} e(\tau)), \]

\[ F^{(ab)}(\tau, \theta) = C(\tau) + \theta (B) \equiv C(\tau) + \theta (s_{ab} C(\tau)), \]

\[ F^{(ab)}(\tau, \theta) = \bar{C}(\tau) + \theta (\dot{\bar{C}} \dot{\bar{C}}) \equiv \bar{C}(\tau) + \theta (s_{ab} \bar{C}(\tau)), \]

\[ \tilde{B}^{(ab)}(\tau, \theta) = \bar{B}(\tau) + \theta (\dot{\bar{B}} \dot{\bar{C}} - \dot{\bar{B}} \dot{\bar{C}}) = \bar{B}(\tau) + \theta (s_{ab} \bar{B}(\tau)), \]

\[ \tilde{B}^{(ab)}(\tau, \theta) = \tilde{B}(\tau) + \theta (0) \equiv \tilde{B}(\tau) + \theta (s_{ab} \tilde{B}(\tau)). \quad (43) \]

\(^5\)Exactly similar kinds of exercise can be performed with the variable \( p_\mu(\tau) \) and, from the restriction \( s_{ab} (\bar{C} \dot{p}_\mu) = 0, \) we can obtain the anti-BRST symmetry transformation of the anti-ghost variable as: \( s_{ab} \bar{C} = \bar{C} \dot{\bar{C}}. \) For the sake of brevity, however, we have not discussed it explicitly here.
5 Lagrangian Formulation: Reparameterization Symmetry and Corresponding (Anti-)BRST Symmetry Transformations

In this section, we elevate the classical reparameterization symmetry $\tau \to \tau' = \tau - \epsilon(\tau)$ to its quantum counterparts within the framework of BRST formalism. In this context, the nilpotent (anti-)BRST symmetries (that have been derived in the previous section) help in finding out the gauge-fixing and Faddeev-Popov (FP) ghost terms in the following manner:

$$-s_{ab} s_b \left[ \frac{e^2}{2} - \frac{C C}{2} \right] = \bar{B} \left[ e \dot{e} + 2 \dot{C} \dot{C} + \ddot{C} C \right] - \frac{B^2}{2} - e^2 \dddot{C} - e \dddot{C} \dot{C} - \dddot{C} \dddot{C} C,$$

$$s_b s_{ab} \left[ \frac{e^2}{2} - \frac{C C}{2} \right] = -B \left[ e \dot{e} + 2 \dot{C} C + \dddot{C} C \right] - \frac{B^2}{2} - e^2 \dddot{C} - e \dddot{C} C - \dddot{C} \dddot{C} C. \quad (44)$$

As a consequence of (44), we have the following (anti-)BRST invariant coupled (but equivalent) Lagrangians for our theory, namely;

$$L_B = p_\mu \dot{x}^\mu - \frac{e}{2} \left( p^2 - m^2 \right) + \bar{B} \left( e \dot{e} + 2 \dot{C} \dot{C} + \dddot{C} C \right)$$

$$- \frac{B^2}{2} - e^2 \dddot{C} - e \dddot{C} \dot{C} - \dddot{C} \dddot{C} C,$$

$$L_B = p_\mu \dot{x}^\mu - \frac{e}{2} \left( p^2 - m^2 \right) - B \left( e \dot{e} + 2 \dot{C} C + \dddot{C} C \right)$$

$$- \frac{B^2}{2} - e^2 \dddot{C} - e \dddot{C} C - \dddot{C} \dddot{C} C. \quad (45)$$

We point out that the pure FP-ghost part (i.e. $- \dddot{C} \dddot{C} C$) of the Lagrangians (45) remains the same. Furthermore, because of the off-shell nilpotency $[s_{(a)b}^2 = 0]$ of the (anti-)BRST symmetries $s_{(a)b}$, it is straightforward to note that $L_B$ would be anti-BRST invariant and $L_B$ would be BRST invariant [cf. Eq. (44)]. To corroborate the latter statement, we note the sanctity of the following equation:

$$s_{ab} L_{\bar{B}} = \frac{d}{d\tau} \left[ \dddot{C} L_f + e^2 \dddot{B} \dddot{C} + e \dot{e} \dddot{B} C \dddot{C} C - \dddot{B} \dddot{C} - B^2 \dddot{C} \right],$$

$$s_b L_B = \frac{d}{d\tau} \left[ C L_f - e^2 B \dddot{C} C - e \dddot{B} C - B \dddot{C} C + B^2 C \right], \quad (46)$$

which render the action integrals $S_1 = \int_{-\infty}^{+\infty} d\tau L_{\bar{B}}$ and $S_2 = \int_{-\infty}^{+\infty} d\tau L_B$ of our theory (described by the coupled Lagrangian densities $L_{\bar{B}}$ and $L_B$) (anti-)BRST invariant, respectively, for the physically well-defined variables which vanish-off as $\tau \to \pm \infty$. In the

---

It should be noted that we have taken the same combination of variables in the square bracket (44) which has been taken in Sec. 2, in the context of BRST quantization, corresponding to the gauge symmetry modulo a factor of i. The latter has been taken for the sake of brevity.

**We are sure that $L_B$ and $L_B$ would be (anti-)BRST invariant because the first-order Lagrangian $L_f$ [i.e. the first two terms of (45)] transforms to a total derivative under the infinitesimal reparameterization [i.e. diffeomorphism transformations (5)] (cf. Sec. 2). As a consequence, under the nilpotent $[s_{(a)b}^2 = 0]$ (anti-)BRST symmetry transformations, $L_f$ would transform as: $s_{ab} L_f = \frac{d}{d\tau}(\dddot{C} L_f)$, $s_b L_f = \frac{d}{d\tau}(C L_f)$.**
above, the first-order Lagrangian $L_f$ is same as defined in Sec. 2 and the full nilpotent (anti-)BRST transformations (for our 1D theory of a scalar relativistic particle) are as follows:

\[
\begin{align*}
    s_{ab} x_\mu &= \bar{C} \dot{x}_\mu, 
    s_{ab} p_\mu = \bar{C} \dot{p}_\mu, 
    s_{ab} C &= \bar{B}, 
    s_{ab} \bar{C} = \bar{C} \dot{C}, \\
    s_{ab} e &= \frac{d}{d\tau}(\bar{C} e), 
    s_{ab} \bar{B} = 0, 
    s_{ab} B = \dot{B} \bar{C} - B \dot{C}, \\
    s_b x_\mu &= C \dot{x}_\mu, 
    s_b p_\mu = C \dot{p}_\mu, 
    s_b C &= C \dot{C}, 
    s_b \bar{C} = B, \\
    s_b e &= \frac{d}{d\tau}(C e), 
    s_b \bar{B} = \dot{B} C - B \bar{C}.
\end{align*}
\]

The above transformations are off-shell nilpotent [$s_{ab}^2 = 0$] and absolutely anticommuting in nature. The absolute anticommutativity ($s_b s_{ab} + s_{ab} s_b = \{s_b, s_{ab}\} = 0$) property is true for all variables of our theory, namely:

\[
\begin{align*}
    \{s_b, s_{ab}\} x_\mu &= [B + \bar{B} + (\dot{C} C - \bar{C} \dot{C})] \dot{x}_\mu = 0, \\
    \{s_b, s_{ab}\} p_\mu &= [B + \bar{B} + (\dot{C} C - \bar{C} \dot{C})] \dot{p}_\mu = 0, \\
    \{s_b, s_{ab}\} e &= \frac{d}{d\tau} \left[ \{B + \bar{B} + (\dot{C} C - \bar{C} \dot{C})\} e \right] = 0, \\
    \{s_b, s_{ab}\} C &= 0, \quad \{s_b, s_{ab}\} \bar{C} = 0, \\
    \{s_b, s_{ab}\} B &= 0, \quad \{s_b, s_{ab}\} \bar{B} = 0.
\end{align*}
\]

provided we impose the (anti-)BRST invariant CF-type restriction: $B + \bar{B} + \dot{C} C - \bar{C} \dot{C} = 0$ from outside on our theory which is, obviously, a physical requirement because it remains invariant under the (anti-)BRST symmetry transformations (47). In other words, it can be checked that $s_{ab} [B + B + (\dot{C} C - \bar{C} \dot{C})] = 0$. Furthermore, it can be checked that the CF-type restriction also remains invariant under a set of discrete symmetry transformations: $B \rightarrow -\bar{B}$, $\bar{B} \rightarrow -B$, $C \rightarrow \pm i \bar{C}$, $\bar{C} \rightarrow \pm i C$.

As claimed earlier, the equivalence of the coupled Lagrangians $L_B$ and $L_{\bar{B}}$ w.r.t. the off-shell nilpotent (anti-)BRST symmetries can be corroborated by the following explicit observations when we apply $s_b$ on $L_B$ and $s_{ab}$ on $L_B$, namely:

\[
\begin{align*}
    s_{ab} L_B &= \frac{d}{d\tau} \left[ \bar{C} \dot{L}_f + e^2 (\dot{\bar{C}} C \dot{C} - B \dot{B}) - B^2 \bar{C} + e \dot{\bar{C}} (\dot{\bar{C}} C C - B \bar{C}) + (2 B - \bar{B}) \dot{\bar{C}} C \bar{C} \right] \\
    &+ \left[ B + \bar{B} + \dot{\bar{C}} C - \bar{C} \dot{C} \right] \left[ 2 B \dot{\bar{C}} + 2 \dot{\bar{C}} C \dot{C} + \bar{C} \bar{C} C + e C \dot{C} \right] \\
    &+ \frac{d}{d\tau} \left[ B + \bar{B} + \dot{\bar{C}} C - \bar{C} \dot{C} \right] \times (B \bar{C} + e^2 \dot{\bar{C}}), \\
    s_b L_B &= \frac{d}{d\tau} \left[ C \dot{L}_f + e^2 (\dot{C} C \bar{C} + \bar{B} \dot{C} - \bar{B}^2 C + e (\dot{C} C C + \bar{B} \bar{C}) - (2 \bar{B} - B) \bar{C} C \bar{C} \right] \\
    &+ \left[ B + \bar{B} + \dot{C} C - C \dot{C} \right] \left[ 2 B \dot{\bar{C}} - 2 \dot{C} C \dot{C} + C \dot{C} C - e \dot{C} \right] \\
    &+ \frac{d}{d\tau} \left[ B + \bar{B} + \dot{C} C - C \dot{C} \right] \times (B \bar{C} - e^2 \dot{\bar{C}}).
\end{align*}
\]

In other words, we note that both the Lagrangians respect both the nilpotent (anti-)BRST symmetries [cf. Eq. (47)] provided we take into account the validity of the CF-type restriction: $B + B + (\dot{C} C - \bar{C} \dot{C}) = 0$. Thus, it is crystal clear that the absolute anticommutativity
property as well as the *equivalence* of the Lagrangians $L_B$ and $L_{\bar{B}}$ are true if and only if the CF-type restriction is taken into account. It is also evident that, under the validity of the *latter*, we have the following explicit expressions for symmetry transformations

$$ s_{ab} L_B = \frac{d}{d\tau} \left[ C L_f + c^2 (\dot{C} \dot{C} \dot{C} - B \dot{\dot{C}}) - B^2 C \right. $$

$$ + e \dot{C} (\dot{C} \dot{C} \dot{C} - B \dot{\dot{C}}) + (2 B - \bar{B}) \dot{\dot{C}} \dot{C} \dot{C} \right], $$

$$ s_b L_{\bar{B}} = \frac{d}{d\tau} \left[ C L_f + c^2 (\ddot{C} \ddot{C} \ddot{C} + B \dddot{C}) - B^2 C, \right. $$

$$ + e \dot{C} (\ddot{C} \ddot{C} \ddot{C} + B \dddot{C}) - (2 \dddot{B} - B) \dddot{C} \ddot{C} \dot{C} \right], $$

which render the action integrals $S_1 = \int d\tau L_B$ and $S_2 = \int d\tau L_{\bar{B}}$ of our theory (anti-)BRST invariant for the physically well-defined variables that vanish-off as $\tau \to \pm \infty$ when our theory is restricted to respect the (anti-)BRST invariant CF-type restriction: $B + \bar{B} + \dot{C} C - C \dot{C} = 0$.

According to the basic concepts behind the Noether theorem, the above continuous symmetries [i.e. (anti-)BRST symmetries] lead to the derivation of conserved and nilpotent (anti-)BRST charges. The *equivalent* expressions, for the conserved BRST charge, are

$$ Q_b^{(1)} = B \ddot{C} \dot{C} \dot{C} - B^2 C - B e^2 \dot{C} - B e \dot{C} + \frac{1}{2} e C (p^2 - m^2), $$

$$ Q_b^{(2)} = e^2 (\dot{B} \dot{C} - B \dot{\dot{C}} + \dot{\dot{C}} \dot{C} \dot{C}) + B \ddot{C} \dot{C} \dot{C} - B^2 C - B e \dot{C}, $$

$$ Q_b^{(3)} = e^2 (\dot{B} \dot{C} - B \dot{\dot{C}} + \dot{\dot{C}} \dot{C} \dot{C}), $$

$$ Q_b^{(4)} = e^2 (\dot{B} \dot{C} - B \dot{\dot{C}} + \dot{\dot{C}} \dot{C} \dot{C}) + e^2 \dddot{C} C \dddot{C} + 2 e \dddot{C} C \dddot{C}, $$

$$ \equiv s_b[e^2 (\dot{\dot{C}} C - \dddot{C})], $$

$$ Q_b^{(5)} = e^2 (\dot{B} \dot{C} - B \dot{\dot{C}} + 2 \dot{\dot{C}} \dddot{C}) + 2 e \dddot{C} C \dddot{C}, $$

$$ \equiv s_b(e^2 C \dddot{C}). $$

Similarly, the *equivalent* forms of the conserved anti-BRST charge are:

$$ Q_b^{(1)} = \dddot{B} \dddot{C} \dddot{C} - B^2 C + B e \dddot{C}, $$

$$ Q_b^{(2)} = e^2 (\ddot{B} \ddot{C} - B \dddot{C} \dddot{C} + \dddot{C} \dot{C} \dot{C}) + \dddot{B} \ddot{C} \dot{C} \dot{C} - B^2 C + B e \dddot{C}, $$

$$ Q_b^{(3)} = e^2 (\dddot{B} \ddot{C} - B \dddot{C} \dddot{C} + \dddot{C} \dot{C} \dot{C}), $$

$$ Q_b^{(4)} = e^2 (\dddot{B} \ddot{C} - B \dddot{C} \dddot{C} - e^2 \dddot{C} \ddot{C} C - 2 e \dddot{C} \dddot{C}), $$

$$ \equiv s_b[e^2 (\dddot{C} \dot{C} - \dddot{C} \dddot{C})], $$

$$ Q_b^{(5)} = e^2 (\dddot{B} \ddot{C} - B \dddot{C} + 2 \dddot{C} \dddot{C}) + 2 e \dddot{C} \dddot{C} \dddot{C}, $$

$$ \equiv s_b(e^2 \dddot{C} \dddot{C}). $$

The conservation law (i.e. $Q^{(r)}_{(a)b} = 0, r = 1, 2, 3, 4, 5$) can be proven in a straightforward manner by using the equations of motion derived from the Lagrangians $L_B$ and $L_{\bar{B}}$ [cf.
Eqs. (53), (54) below. We would like to point out that the expressions \(Q^{(1)}_b\) and \(Q^{(1)}_{ab}\) have been derived by using the direct mathematical form of the Noether theorem. However, the other equivalent forms for the charges have been obtained by using the equations of motion (EOM) for the Lagrangians \(L_B\) and \(L_B\). In fact, the precise forms of EOM from \(L_B\) are:

\[
\begin{align*}
\dot{p}_\mu &= 0, \quad \dot{x}_\mu = e p_\mu, \quad B + 2 \dot{C} C + e \dot{e} + \ddot{C} \dot{C} = 0, \\
e \dot{B} - e \dot{C} \dot{C} + e \dot{C} C - \frac{1}{2}(p^2 - m^2) &= 0, \\
\dot{B} - 2 \dot{B} C - 3 e \dot{e} \ddot{C} - e^2 \dot{C} - e \ddot{e} C - e^2 C + 2 \ddot{C} C \dot{C} + C C \dot{C} &= 0.
\end{align*}
\]

In exactly similar fashion, the exact forms of EOM from \(L_B\) are:

\[
\begin{align*}
\dot{p}_\mu &= 0, \quad \dot{x}_\mu = e p_\mu, \quad \ddot{B} - 2 \ddot{C} C - e \ddot{e} - \ddot{\dot{C}} C = 0, \\
e \ddot{B} - e \ddot{C} \dot{C} + e \ddot{C} C + \frac{1}{2}(p^2 - m^2) &= 0, \\
\ddot{B} - 2 \ddot{B} C - 3 e \ddot{e} \dddot{C} + e^2 \dddot{C} + e \dddot{e} C + e^2 \dddot{C} + C C \dddot{C} + 2 \dddot{C} C \dddot{C} &= 0.
\end{align*}
\]

The above EOMs (53) and (54) can be used, in a straightforward fashion, to prove that all the (anti-)BRST charges, listed in (52) and (51), are conserved (i.e. \(\dot{Q}^{(r)}_a = 0, r = 1, 2, ..., 5\)), primarily, due to the basic concepts behind Noether’s theorem.

We have expressed the conserved and off-shell nilpotent (anti-)BRST charges in various forms [cf. Eqs. (52), (51)] because all the forms have their own significance. For instance, a close look at the \(Q^{(4)}_{ab}\) establishes the nilpotency of the charges as it can be seen that:

\[
\begin{align*}
s_b Q^{(4)}_b &= -i \{ Q^{(4)}_b , Q^{(4)}_b \} = 0 \quad \Rightarrow \quad (Q^{(4)}_b)^2 = 0 \quad \iff \quad s_b^2 = 0, \\
s_{ab} Q^{(4)}_{ab} &= -i \{ Q^{(4)}_{ab} , Q^{(4)}_{ab} \} = 0 \quad \Rightarrow \quad (Q^{(4)}_{ab})^2 = 0 \quad \iff \quad s_{ab}^2 = 0.
\end{align*}
\]

Thus, it is crystal clear (from the above equation) that the nilpotency of the (anti-)BRST symmetries are very intimately connected with the off-shell nilpotency of the (anti-)BRST charges. The expressions for the equivalent conserved (anti-)BRST charges \(Q^{(2,3)}_{(a)b}\) are the intermediate steps for obtaining the BRST exact form of \(Q^{(4)}_b\) and anti-BRST exact form of \(Q^{(4)}_{ab}\). Furthermore, we would like to mention that the expressions for the conserved charges \(Q^{(5)}_{(a)b}\) have been obtained from \(Q^{(4)}_{(a)b}\) by using the beauty and strength of the CF-type restriction: \(B + B + \dot{C} C - \ddot{C} \dot{C} = 0\). The expressions for the conserved (anti-)BRST charges \(Q^{(5)}_{(a)b}\) are very interesting for us because they encode in themselves the absolute anticommutativity property

\[
\begin{align*}
s_{ab} Q^{(5)}_b &= -i \{ Q^{(5)}_b , Q^{(5)}_b \} = 0 \quad \iff \quad s_{ab}^2 = 0, \\
s_b Q^{(5)}_{ab} &= -i \{ Q^{(5)}_{ab} , Q^{(5)}_b \} = 0 \quad \iff \quad s_b^2 = 0.
\end{align*}
\]

where we have used the basic principle behind the connection between the continuous symmetry transformations \(s_{(a)b}\) and their generators as the conserved Noether (anti-)BRST
charges. We would like to lay emphasis on the fact that it is the power and potential of the
CF-type restriction that has enabled us to express the conserved BRST charge \( Q^{(5)}_b \) as an anti-BRST exact quantity and the conserved anti-BRST charge \( Q^{(5)}_{ab} \) as the BRST exact object. In our Appendix B, we discuss more about the absolute anticommutativity of the nilpotent (anti-)BRST conserved charges and the existence of (anti-)BRST invariant CF-type restriction on our theory.

In some sense, the above exercise is a reflection of our observations in Eq. (48) where we have shown that the absolute anticommutativity property \( s_b s_{ab} + s_{ab} s_b = 0 \) of the (anti-)BRST symmetries \( s_{(a)b} \) are true only on a submanifold, in the space of quantum variables, which is defined by the CF-type equation: \( B + \bar{B} + \dot{\bar{C}} \bar{C} - \bar{C} \dot{\bar{C}} = 0 \). Since, the nilpotency and absolute anticommutativity properties are very sacrosanct in the BRST formalism, the requirement of the latter property for the conserved charges, in our present discussion, leads to the derivation of the CF-type restriction (24) which was also derived from the modified BT-supervariable approach (MBTSA) to BRST formalism (cf. Sec. 3). In other words, we take directly the help of the CF-type restriction to recast the conserved (anti-)BRST charges in a specific form \([e.g. \left(Q^{(5)}_{(a)b}\right)]\) such that the BRST charge is expressed as an anti-BRST exact quantity (and the anti-BRST charge as the BRST exact form). At this juncture, it is crystal clear that the absolute anticommutativity of (i) the nilpotent (anti-)BRST symmetries \([cf. \text{Eq. (48)}]\), and (ii) the conserved and nilpotent (anti-)BRST charges \([cf. \text{Eq. (56)}]\) owe their origin to the CF-type restriction:

\[
B + \bar{B} + \dot{\bar{C}} \bar{C} - \bar{C} \dot{\bar{C}} = 0
\]
on our 1D diffeomorphism invariant scalar relativistic particle.

### 6 Invariance of the Lagrangians, Nilpotency and Anticommutativity of the Conserved Charges: ACSA

We now capture the (anti-)BRST invariance of the coupled Lagrangians within the framework of ACSA to BRST formalism and thereby prove the existence of the CF-type restriction (24) on our theory from the point of view of the symmetry considerations\(^\dagger\). In this context, first of all, we generalize the BRST invariant Lagrangian \( L_B \) to its counterpart super Lagrangian \( \tilde{L}^{(ac)}_B \) on the (1, 1)-dimensional anti-chiral super sub-manifold of the general (1, 2)-dimensional supermanifold (on which our theory is generalized) as follows

\[
L_B \quad \longrightarrow \quad \tilde{L}^{(ac)}_B = P^{(h) \mu}_\tau (\tau, \bar{\theta}) X^{(h) \mu} (\tau, \bar{\theta}) - \frac{1}{2} E^{(b) \mu} (\tau, \bar{\theta}) P^{(h) \mu}_\tau (\tau, \bar{\theta}) \left( \frac{P^{(h) \mu}_\tau (\tau, \bar{\theta}) P^{(h) \mu}_\tau (\tau, \bar{\theta})}{2} - m^2 \right)
- \frac{1}{2} \tilde{B}^{(b) \mu} (\tau, \bar{\theta}) \tilde{B}^{(b) \mu} (\tau, \bar{\theta}) - \frac{1}{2} \tilde{B}^{(b) \mu} (\tau, \bar{\theta}) \tilde{B}^{(b) \mu} (\tau, \bar{\theta}) \left( E^{(b) \mu} (\tau, \bar{\theta}) E^{(b) \mu} (\tau, \bar{\theta}) \right)
+ 2 F^{(b) \mu} (\tau, \bar{\theta}) F^{(b) \mu} (\tau, \bar{\theta}) + F^{(b) \mu} (\tau, \bar{\theta}) F^{(b) \mu} (\tau, \bar{\theta})
\]

\(^\dagger\)To be precise, we actually capture the (anti-)BRST invariance of the coupled (but equivalent) Lagrangians \( L_B \) and \( \tilde{L}^{(ac)}_B \) [cf. Eq. (46)]. Furthermore, we also describe our observations in equation (49) in the language of the ACSA which establishes the existence of the CF-type restriction: \( B + \bar{B} + \dot{\bar{C}} \bar{C} - \bar{C} \dot{\bar{C}} = 0 \) on our theory in terms of the invariance of the action integrals.


\[ L_B \longrightarrow \tilde{L}^{(c)}_B = P^{(hc)}(\tau, \theta) X^{(hc)}(\tau, \theta) \]

\[- E^{(b)}(\tau, \theta) E^{(b)}(\tau, \theta) \dot{F}^{(b)}(\tau, \theta) \dot{\bar{F}}^{(b)}(\tau, \theta) \]

\[- E^{(b)}(\tau, \theta) E^{(b)}(\tau, \theta) \dot{\bar{F}}^{(b)}(\tau, \theta) F^{(b)}(\tau, \theta) \]

\[- \bar{F}^{(b)}(\tau, \theta) \dot{\bar{F}}^{(b)}(\tau, \theta) F^{(b)}(\tau, \theta) \dot{F}^{(b)}(\tau, \theta), \]

(57)

where it can be noted that we have \( \tilde{B}^{(b)}(\tau, \bar{\theta}) = B(\tau) \) because of the fact that \( s_b \bar{B} = 0 \). Thus, even though, we have written \( \tilde{B}^{(b)}(\tau, \bar{\theta}) \) in the above equation (57), it is actually an ordinary Nakanishi-Lautrup type auxiliary variable \( B(\tau) \) of our theory [cf. Eq. (45)]. Now we are in the position to capture the BRST invariance of the Lagrangian \( L_B \) [cf. Eq. (46)] in the language of ACSA as:

\[ \frac{\partial}{\partial \bar{\theta}} \tilde{L}^{(ac)}_B = \frac{d}{d\tau} \left[ C L_f - e^2 B \dot{C} - e \dot{e} B C - B \dot{C} C - B^2 C \right] \equiv s_b L_B. \]  

(58)

Geometrically, it implies that the anti-chiral super Lagrangian \( \tilde{L}^{(ac)}_B \) is a combination of the suitable (super)variables such that its translation along the \( \theta \)-direction of the (1, 1)-dimensional anti-chiral super sub-manifold produces a total “time” derivative in the ordinary space thereby rendering the action integral, in the ordinary space, invariant under the BRST symmetry transformations \( s_b \) due to the Gauss divergence theorem. It should be noted that the BRST transformations \( s_b \) is identified with the translational generator \( \partial_{\bar{\theta}} \) [9-11] on the anti-chiral super sub-manifold [of the general (1, 2)-dimensional supermanifold on which our 1D system of a scalar non-supersymmetric free relativistic particle is generalized].

We now discuss the anti-BRST invariance of our theory. To capture the anti-BRST symmetry invariance [cf. Eq. (46)] of the Lagrangian \( L_B \), first of all, we generalize the latter to the (1, 1)-dimensional chiral super sub-manifold [of the general (1, 2)-dimensional supermanifold on which our system of a 1D ordinary free non-supersymmetric scalar relativistic particle is considered] as

\[ L_B \longrightarrow \tilde{L}^{(c)}_B = P^{(hc)}(\tau, \theta) X^{(hc)}(\tau, \theta) \]

\[- \frac{1}{2} E^{(ab)}(\tau, \theta) [P^{(hc)}(\tau, \theta) P^{(hc)}(\tau, \theta) - m^2] \]

\[- \frac{1}{2} \tilde{B}^{(ab)}(\tau, \theta) \tilde{B}^{(ab)}(\tau, \theta) + \tilde{B}^{(ab)}(\tau, \theta) [E^{(ab)}(\tau, \theta) \dot{E}^{(ab)}(\tau, \theta)] \]

\[ + 2 \tilde{F}^{(ab)}(\tau, \theta) \tilde{F}^{(ab)}(\tau, \theta) + \tilde{F}^{(ab)}(\tau, \theta) F^{(ab)}(\tau, \theta) \]

\[- E^{(ab)}(\tau, \theta) E^{(ab)}(\tau, \theta) \dot{\tilde{F}}^{(ab)}(\tau, \theta) \dot{F}^{(ab)}(\tau, \theta) \]

\[- E^{(ab)}(\tau, \theta) \dot{E}^{(ab)}(\tau, \theta) F^{(ab)}(\tau, \theta) \dot{F}^{(ab)}(\tau, \theta) \]

\[- \tilde{F}^{(ab)}(\tau, \theta) \dot{\tilde{F}}^{(ab)}(\tau, \theta) F^{(ab)}(\tau, \theta) \dot{F}^{(ab)}(\tau, \theta), \]

(59)

where it can be noted that \( \tilde{B}^{(ab)}(\tau, \theta) = \tilde{B}(\tau) \) because of the fact that \( s_{ab} \tilde{B} = 0 \). Thus, even though, we have written \( \tilde{B}^{(ab)}(\tau, \theta) \) in our chiral super Lagrangian \( \tilde{L}^{(c)}_B \), it is actually an ordinary variable \( \tilde{B}(\tau) \). The anti-BRST invariance of the above chiral super Lagrangian can be expressed, in terms of the translational generator along the Grassmannian \( \theta \)-direction
(with the input $\partial_{\theta} \rightarrow s_{ab}$), as:

$$\frac{\partial}{\partial \theta} \tilde{L}_{B}^{(c)} = \frac{d}{d\tau} \left[ \bar{C} L_I + e^2 \bar{B} \dot{\bar{C}} + e \dot{\bar{B}} \dot{C} - (B + \bar{B} + \dot{\bar{C}} - \dot{C}) \right] \equiv s_{ab} L_B,$$

(60)

where $\partial_{\theta}$ is the translational generator [9-11] along the Grassmannian (i.e. $\theta$) direction of the $(1, 1)$-dimensional chiral super sub-manifold of the general $(1, 2)$-dimensional supermanifold. Once again, we note that, geometrically, the chiral super Lagrangian $\tilde{L}_{B}^{(c)}$ is a specific combination of the appropriate chiral (super)variables such that its translation along the $\theta$-direction of the chiral super sub-manifold generates a total “time” derivative (in the ordinary space) thereby rendering the action integral (in the ordinary space) invariant under the anti-BRST symmetry transformations ($s_{ab}$) due to the Gauss divergence theorem. In the language of ACSA to BRST formalism, we note that the super action (in the $\bar{\theta}$-direction of the chiral super sub-manifold) generates a total “time” derivative (with the input:

\[ B + \bar{B} + \dot{\bar{C}} C - \dot{C} \bar{C} = 0 \]

within the framework of the ACSA to BRST formalism by considering the anti-BRST invariance of the a chiral super Lagrangian $\tilde{L}_{B}^{(c)}$ and BRST invariance of the anti-chiral super Lagrangian $\tilde{L}_{B}^{(ac)}$. This is due to the fact that, as claimed in our earlier discussions (cf. Sec. 5), both the Lagrangians $L_B$ and $\tilde{L}_B$ respect both the symmetries provided the theory is considered on a sub-manifold of the space of quantum variables where the CF-type restriction is satisfied. Towards this goal in mind, we note the following generalization:

\[ L_B \rightarrow \tilde{L}_{B}^{(ac)} = \sum_{(h)} P_{(h)}^{(a)}(\tau, \bar{\theta}) X^{(h)}(\tau, \bar{\theta}) \]

- $\frac{1}{2} E^{(b)}(\tau, \bar{\theta}) \left[ \sum_{(h)} P_{(h)}^{(a)}(\tau, \bar{\theta}) P^{(h)}(\tau, \bar{\theta}) - m^2 \right]$

- $\frac{1}{2} \bar{B}^{(b)}(\tau, \bar{\theta}) \bar{B}^{(b)}(\tau, \bar{\theta}) E^{(b)}(\tau, \bar{\theta}) \dot{E}^{(b)}(\tau, \bar{\theta})$

+ $2 \bar{F}^{(b)}(\tau, \bar{\theta}) \dot{F}^{(b)}(\tau, \bar{\theta}) + \dot{F}^{(b)}(\tau, \bar{\theta}) F^{(b)}(\tau, \bar{\theta})$

- $E^{(b)}(\tau, \bar{\theta}) E^{(b)}(\tau, \bar{\theta}) \dot{F}^{(b)}(\tau, \bar{\theta}) \dot{F}^{(b)}(\tau, \bar{\theta})$

- $E^{(b)}(\tau, \bar{\theta}) \dot{E}^{(b)}(\tau, \bar{\theta}) \dot{F}^{(b)}(\tau, \bar{\theta}) \dot{F}^{(b)}(\tau, \bar{\theta})$

- $\bar{F}^{(b)}(\tau, \bar{\theta}) \dot{F}^{(b)}(\tau, \bar{\theta}) F^{(b)}(\tau, \bar{\theta}) F^{(b)}(\tau, \bar{\theta})$

(61)

In the above, it should be noted that we have generalized the perfectly anti-BRST invariant Lagrangian $L_B$ to its counterpart anti-chiral super Lagrangian $\tilde{L}_{B}^{(ac)}$ on the $(1, 1)$-dimensional anti-chiral super sub-manifold [of the general $(1, 2)$-dimensional supermanifold]. We are in the position now to apply a derivative $\partial_{\theta}$ w.r.t. $\theta$ on the above super Lagrangian which yields the following (with the input: $s_{b} \leftrightarrow \partial_{\theta}$):

\[ \frac{\partial}{\partial \theta} \tilde{L}_{B}^{(ac)} = \frac{d}{d\tau} \left[ C L_I + e^2 (\bar{C} C \dot{\bar{C}} + \bar{B} \dot{C}) + e \dot{\bar{B}} \dot{C} C - \dot{B} C \bar{C} \right] \equiv s_{b} L_B. \]

(62)
The above equation leads to the derivation of the CF-type restriction in the sense that the anti-chiral super Lagrangian \( \tilde{L}^{(ac)}_B \), when operated by \( \partial_{\theta} \), produces a total “time” derivative plus terms that vanish-off on the submanifolds of the space of variables which is defined by the CF-type restriction: \( B + \bar{B} + \bar{C} \bar{C} - C \dot{C} \). With the identification: \( s_b \leftrightarrow \partial_{\bar{b}} \), it is clear that we have obtained the same relationship as given in equation (49) in the ordinary space for the BRST symmetry transformation of \( L_B \) (i.e. \( s_b L_B \)).

In exactly similar fashion, we can generalize the perfectly BRST invariant Lagrangian \( L_B \) to its counterpart chiral super Lagrangian \( \tilde{L}^{(c)}_B \) as follows:

\[
L_B \longrightarrow \tilde{L}^{(c)}_B = P^{(hc)}(\tau, \theta) \dot{X}^{(hc)}(\tau, \theta) \\
- \frac{1}{2} E^{(ab)}(\tau, \theta) \left[ P^{(hc)}(\tau, \theta) P^{(hc)}(\tau, \theta) - m^2 \right] \\
- \frac{1}{2} \tilde{B}^{(ab)}(\tau, \theta) \tilde{B}^{(ab)}(\tau, \theta) - \tilde{\tilde{B}}^{(ab)}(\tau, \theta) \left[ E^{(ab)}(\tau, \theta) \tilde{E}^{(ab)}(\tau, \theta) \right] \\
+ 2 \tilde{F}^{(ab)}(\tau, \theta) F^{(ab)}(\tau, \theta) + \tilde{F}^{(ab)}(\tau, \theta) \tilde{F}^{(ab)}(\tau, \theta) \\
- E^{(ab)}(\tau, \theta) \tilde{E}^{(ab)}(\tau, \theta) \tilde{F}^{(ab)}(\tau, \theta) \\
- E^{(ab)}(\tau, \theta) \tilde{E}^{(ab)}(\tau, \theta) F^{(ab)}(\tau, \theta) \\
- F^{(ab)}(\tau, \theta) \tilde{F}^{(ab)}(\tau, \theta) F^{(ab)}(\tau, \theta) \tilde{F}^{(ab)}(\tau, \theta),
\]

where all the symbols and notations have been clarified earlier. At this juncture, we apply a Grassmannian derivative \( \partial_{\theta} \) on the above super Lagrangian which yields the following:

\[
\frac{\partial}{\partial \theta} \tilde{L}^{(c)}_B = \frac{d}{dt} \left[ \tilde{C} L_f + e^2 (\dot{\tilde{C}} \tilde{C} \bar{C} - B \dot{\tilde{C}}) + e \dot{e} (\dot{\tilde{C}} \bar{C} C - B \bar{C}) \\
+ (2 B - \bar{B}) \bar{C} \tilde{C} C - B^2 \bar{C} \right] + (B + \bar{B} + \tilde{C} \bar{C} - C \dot{C}) \\
\times \left( 2 B \dot{\tilde{C}} + 2 \tilde{C} \dot{\tilde{C}} \tilde{C} + \bar{C} \bar{C} C + e \dot{e} \tilde{C} \right) \\
+ \frac{d}{dt} \left[ B + \bar{B} + \dot{\tilde{C}} C - \tilde{C} \dot{C} \right] (B \bar{C} + e^2 \dot{\tilde{C}}) \equiv s_{ab} L_B.
\]

Thus, we note that we have derived the observation that has been made in equation (49). In other words, the ACSA to BRST formalism leads to the derivation of the CF-type restriction when we consider the anti-BRST invariance of the perfectly BRST invariant Lagrangian \( L_B \) as well as the BRST invariance of the perfectly anti-BRST invariant Lagrangian \( L_B \) of our theory.

At this stage, we would like to capture the off-shell nilpotency as well as the absolute anticommutativity of the conserved (anti-)BRST charges [cf. Eqs. (55), (56)] within the framework of the ACSA to BRST formalism. Towards this goal in mind, we note that, out of the equivalent expressions for the conserved (anti-) BRST charges quoted in (52) and (51), one set of the conserved charges \( Q^{(4)}_{(a)b} \) have been expressed in the (anti-)BRST exact forms. Keeping in mind the identifications: \( s_b \leftrightarrow \partial_{\bar{b}}, s_{ab} \leftrightarrow \partial_{\theta} \), we note the following:

\[
Q^{(4)}_{b} = \frac{\partial}{\partial \theta} \left[ E^{(b)}(\tau, \theta) E^{(b)}(\tau, \theta) \{ \tilde{F}^{(b)}(\tau, \theta) F^{(b)}(\tau, \theta) - F^{(b)}(\tau, \theta) \tilde{F}^{(b)}(\tau, \theta) \} \right],
\]
the following relationships:

As a consequence, it is straightforward to point out the fact that we have the validity of the exactness of the conserved (anti-)BRST charges, the BRST charge $\partial$.

In other words, we note that the nilpotency (i.e. $\partial^2 = 0$) of the translational generators $(\partial_\theta, \partial_b)$ along the $(\theta, \bar{\theta})$-directions of the $(1, 1)$-dimensional chiral and anti-chiral super sub-manifolds [of the general $(1, 2)$-dimensional supermanifold] are responsible for capturing the off-shell nilpotency of the conserved (anti-)BRST charges $Q^{(4)}_{ab}$. To be more precise, we further point out that the off-shell nilpotency of the conserved BRST charge $Q^{(4)}_b$ is connected with the nilpotency (i.e. $\partial^2_b = 0$) of the translational generator $\partial_b$ along the $\theta$-direction of the $(1, 1)$-dimensional anti-chiral super submanifold. However, the off-shell nilpotency of the conserved anti-BRST charge $Q^{(4)}_{ab}$ is intimately connected with the nilpotency (i.e. $\partial^2_{\bar{\theta}} = 0$) of the translational generator $\partial_{\bar{\theta}}$ along the $\bar{\theta}$-direction of the $(1, 1)$-dimensional chiral super submanifold of the general $(1, 2)$-dimensional supermanifold.

We concentrate, finally, on the proof of the absolute anticommutativity [cf. Eq. (56)] of the conserved and nilpotent (anti-)BRST charges within the framework of the ACSA to BRST formalism. In this context, we note that, from the list of the equivalent forms of the conserved (anti-)BRST charges, the BRST charge $Q^{(5)}_b$ has been expressed as the anti-BRST exact quantity. On the other hand, the conserved anti-BRST charge $Q^{(5)}_{ab}$ has been written in the BRST exact form. With the identifications: $s_b \leftrightarrow \partial_{\bar{\theta}}, s_{ab} \leftrightarrow \partial_b$, we observe the sanctity of the following:

$$Q^{(5)}_b = \frac{\partial}{\partial \theta} \left[ E^{(ab)}(\tau, \theta) E^{(ab)}(\tau, \bar{\theta}) F^{(ab)}(\tau, \theta) \tilde{F}^{(ab)}(\tau, \bar{\theta}) \right],$$

$$\equiv \int d\theta \left[ E^{(ab)}(\tau, \theta) E^{(ab)}(\tau, \bar{\theta}) F^{(ab)}(\tau, \theta) \tilde{F}^{(ab)}(\tau, \bar{\theta}) \right].$$

$$Q^{(5)}_{ab} = \frac{\partial}{\partial \theta} \left[ E^{(b)}(\tau, \theta) E^{(b)}(\tau, \bar{\theta}) \dot{F}^{(b)}(\tau, \theta) \tilde{\dot{F}}^{(b)}(\tau, \bar{\theta}) \right],$$

$$\equiv \int d\theta \left[ E^{(b)}(\tau, \theta) E^{(b)}(\tau, \bar{\theta}) \dot{F}^{(b)}(\tau, \theta) \tilde{\dot{F}}^{(b)}(\tau, \bar{\theta}) \right].$$

As a consequence, it is straightforward that the followings results are true, namely:

$$\partial_\theta Q^{(5)}_b = 0 \iff \partial^2_\theta = 0 \iff s^2_b = 0,$$

$$\partial_\theta Q^{(5)}_{ab} = 0 \iff \partial^2_{\bar{\theta}} = 0 \iff s^2_{ab} = 0.$$

Thus, it is crystal clear that, in the ordinary space, the above equation is equivalent to equation (56) where we have proven the absolute anticommutativity of the conserved and
off-shell nilpotent (anti-)BRST charges. In the terminology of the ACSA to BRST formalism, we note that the absolute anticommutativity of the BRST charge with the anti-BRST charge is deeply connected with the nilpotency (i.e. $\partial^2 = 0$) of the translational generator $\partial_\theta$ along the Grassmannian direction $\theta$ of the $(1, 1)$-dimensional chiral super sub-manifold of the general $(1, 2)$-dimensional supermanifold on which our 1D theory is generalized. This should be contrasted with our earlier observation of the off-shell nilpotency of the BRST charge (within the framework of ACSA to BRST formalism) where it is the nilpotency (i.e. $\partial^2 \bar{\theta} = 0$) of the translational generator $\partial_{\bar{\theta}}$ along the Grassmannian direction $\bar{\theta}$ of the $(1, 1)$-dimensional anti-chiral super sub-manifold that plays a decisive role. Similar kinds of statements could be made for the proof of the absolute anticommutativity of the anti-BRST charge with the BRST charge. However, for the sake of brevity, we do not wish to make any statement, in this regard, at this juncture of our discussion.

7 Conclusions

In our present endeavor, we have applied the BT-superfield/supervariable approach [9-11] in its modified form where the infinitesimal diffeomorphism transformation has been consistently taken into account [20, 21]. First of all, we have generalized the 1D infinitesimal diffeomorphism (i.e. reparameterization) transformation: $\tau \to \tau' = \tau - \epsilon(\tau)$ to its counterpart superspace infinitesimal reparameterization [cf. Eq. (13)] on the $(1, 2)$-dimensional supermanifold where the (anti-)ghost variables ($\bar{C}$) appear as the coefficients of the Grassmannian variables. This superspace reparameterization transformation has been incorporated into the supervariables [defined on the $(1, 2)$-dimensional supermanifold] and, then only, the super expansions along all possible Grassmannian directions of the above supermanifold have been taken into account in our present endeavor. After that, we have applied the HC [cf. Eq. (18)] to obtain the quantum (anti-)BRST transformations corresponding to the classical infinitesimal reparameterization transformation: $\tau \to \tau' = \tau - \epsilon(\tau)$ of our 1D theory. We have christened this approach as the modified BT-supervariable/superfield approach (MBTSA) to BRST formalism [20, 21].

One of the highlights of our present investigation is the derivation of the CF-type restriction: $B + \bar{B} + \bar{C} C - \bar{C} \bar{C} = 0$ by exploiting the power and potential of the MBTSA which has also led to the derivation of the off-shell nilpotent (anti-)BRST symmetries for the target space variables. The (anti-)BRST symmetry transformations for the other variables of our theory have been derived by using the newly proposed ACSA to BRST formalism [15-18] where the (anti-)BRST invariant restrictions on the supervariables have played a decisive role. We have also provided the proof of the existence of the CF-type restrictions on our theory by considering (i) the symmetry invariance of the coupled (but equivalent) Lagrangians in the ordinary space, (ii) the (anti-)BRST invariance of the super Lagrangians by exploiting the potential and power of the ACSA to BRST formalism in the superspace, and (iii) the requirement of the proof of the absolute anticommutativity of the conserved (anti-)BRST charges. We have established that the absolute anticommutativity of the (anti-)BRST symmetries (as well as corresponding conserved charges) and equivalence of the coupled (but equivalent) Lagrangians owe their origin to the (anti-)BRST invariant
CF-type restriction (cf. Appendix A below).

In our present endeavor, we have applied the MBTSA to derive the proper nilpotent (anti-)BRST symmetry transformations for the phase space variables $x_\mu(\tau)$ and $p_\mu(\tau)$ of the target space. Rest of the (anti-)BRST symmetries for the other variables of our theory have been derived by using the ACSA. One of the novel observations of our present endeavor is the proof of the off-shell nilpotency and absolute anticommutativity of the conserved (anti-)BRST charges within the framework of ACSA. In this context, one interesting result is the observation that the absolute anticommutativity of the BRST charge with the anti-BRST charge is deeply connected with the nilpotency ($\partial^2_\theta = 0$) of the translational generator $\partial_\theta$ along the $\theta$-direction of the chiral super sub-manifold of the general (1, 2)-dimensional supermanifold. However, the absolute anticommutativity of the anti-BRST charge with the BRST charge is intimately connected with the nilpotency ($\partial^2_{\bar{\theta}} = 0$) of the Grassmannian translational generator $\partial_{\bar{\theta}}$ along the $\bar{\theta}$-direction of the anti-chiral super submanifold of the general (1, 2)-dimensional supermanifold. Thus, in some sense, the ACSA distinguishes between the chiral and anti-chiral super sub-manifolds as far as the proof of the absolute anticommutativity property.

As a closing remark on our present investigation, we would like to lay emphasis on the fact that the BRST quantization of a 1D free scalar relativistic particle is now a standard text-book material [19] where the infinitesimal gauge symmetry transformations: $\delta_g x_\mu = \xi p_\mu$, $\delta_g p_\mu = 0$, $\delta_g c = \xi$ have been exploited (cf. Sec. 2 for details). The central theme of our present endeavor is, however, to perform the consistent BRST quantization of our system by exploiting the full classical infinitesimal reparameterization symmetry transformations (5). This has been accomplished in the hope that our present understanding would help us to go to the higher dimensional diffeomorphism invariant theories [e.g. (super)strings and gravitational theories] where our theoretical method would be useful. It is gratifying to mention, in this context, that the reparameterization invariant model of a 1D scalar relativistic particle has already been systematically generalized to its counterpart bosonic string in [24]. Another interesting point which we would like to mention is the observation that the nature and form of the CF-type restriction is universal in the sense that we have obtained the same restriction in the BRST quantization of the 1D spinning (i.e. supersymmetric) relativistic particle [25] and a non-relativistic (as well as non-supersymmetric) free particle [26].

We have discussed the (anti-)BRST symmetries and BRST quantization of the D-dimensional diffeomorphism invariant theories with scalars, contravariant as well as covariant vectors and metric tensor as well as its inverse, etc. (see, e.g. [21]). This has enabled us to discuss the (anti-)BRST symmetries for the affine connection, curvature tensor, curvature scalar, etc., for the gravitational theories. Our future plan is to discuss the BRST quantization of the diffeomorphism invariant theories [e.g. (super)string theories, gravitational theories, etc.] which are very popular at the frontier level of research in the domain of theoretical high energy physics. It is gratifying that we have taken a modest step in this direction in our earlier and recent works [27, 28] for a 2D diffeomorphism invariant model of bosonic string and derived the CF-type restrictions as: $B^a + B^a + i (C^b \partial_a C^a + C^b \partial_b C^a) = 0$ (with $a, b = 0, 1$) which are the 2D version of the universal CF-type restrictions for the D-dimensional diffeomorphism invariant theory [20, 21].
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Appendix A: On the (Anti-)BRST Invariance of the CF-Type Restriction: ACSA

One of the key consequences of the geometrical BT-superfield/supervariable approach [9-11] is the observation that it leads to the derivation of the (anti-)BRST invariant CF-type restriction in the context of gauge theories. This is also true when we apply the modified version of BT-supervariable approach [20, 21] to our 1D reparameterization/diffeomorphism invariant theory. It is straightforward to check that the CF-type restriction: $B + \tilde{B} + \dot{C} \dot{C} = 0$ changes under the (anti-)BRST symmetry transformations (47) as:

$$s_b [B + \tilde{B} + \dot{C} \dot{C} - \bar{C} \dot{C}] = \left[ \frac{d}{d \tau} (B + \tilde{B} + \dot{C} \dot{C} - \bar{C} \dot{C}) \right] \bar{C} - \left( B + \tilde{B} + \dot{C} \dot{C} - \bar{C} \dot{C} \right) \dot{C},$$

$$s_{ab} [B + \tilde{B} + \dot{C} \dot{C} - \bar{C} \dot{C}] = \left[ \frac{d}{d \tau} (B + \tilde{B} + \dot{C} \dot{C} - \bar{C} \dot{C}) \right] \bar{C} - \left( B + \tilde{B} + \dot{C} \ddot{C} - \bar{C} \dot{C} \dot{C} \right) \dot{C}. \quad (A.1)$$

Thus, it can be noted that the (anti-)BRST invariance of the above CF-type restriction is valid only on the submanifold of the space of quantum variables which is defined by the CF-type restriction (i.e. $B + \tilde{B} + \dot{C} \dot{C} - \bar{C} \dot{C} = 0$) itself. This establishes the fact that, at the quantum level, the CF-type restriction is a physical constraint. In fact, our whole quantum 1D non-supersymmetric system of the relativistic scalar particle is defined on the submanifold of space of variables where the CF-type restriction is satisfied which, ultimately, leads to the existence of the coupled (but equivalent) Lagrangians. Furthermore, it is also responsible for the absolute anticommutativity ($s_b s_{ab} + s_{ab} s_b = 0$) of the off-shell nilpotent ($s_{(a)b} = 0$) (anti-)BRST symmetry transformations $s_{(a)b}$.

The above observation can be captured within the framework of ACSA to BRST formalism, too. For instance, it can be checked that:

$$\frac{\partial}{\partial \bar{\theta}} \left[ \tilde{B}^{(b)} (\tau, \bar{\theta}) + \tilde{B}^{(b)} (\tau, \bar{\theta}) \right] = \left[ \frac{d}{d \tau} (B + \tilde{B} + \dot{C} \ddot{C} - \bar{C} \dot{C} \dot{C}) \right] \bar{C} - \left( B + \tilde{B} + \dot{C} \ddot{C} - \bar{C} \dot{C} \dot{C} \right) \dot{C} \equiv s_b [B + \tilde{B} + \dot{C} \ddot{C} - \bar{C} \dot{C} \dot{C}],$$

$$\frac{\partial}{\partial \bar{\theta}} \left[ \tilde{B}^{(ab)} (\tau, \bar{\theta}) + \tilde{B}^{(ab)} (\tau, \bar{\theta}) + \tilde{F}^{(ab)} (\tau, \bar{\theta}) \right] \equiv s_{ab} [B + \tilde{B} + \dot{C} \ddot{C} - \bar{C} \dot{C} \dot{C}]. \quad (A.2)$$
In the above, we have used the following
\[ \tilde{B}^{(b)}(\tau, \tilde{\theta}) = B(\tau), \quad \tilde{B}^{(ab)}(\tau, \theta) = \tilde{B}(\tau), \quad (A.3) \]
due to the observation that \( s_b B = 0, s_{ab} \tilde{B} = 0 \). It is very interesting to note that the CF-type restriction: \( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} = 0 \) is a physical constraint on the quantum theory because it is an (anti-)BRST invariant quantity on a submanifold of the space of variables which is defined by the CF-type equation.

**Appendix B: On an Alternative Proof of the Existence of CF-Type Restriction: Anticommutativity of the (Anti-)BRST Charges**

We have provided the proof of the existence of the CF-type restriction in our Sec. 6 by exploiting the virtues of symmetry consideration within the framework of ACSA to BRST formalism. We concentrate, in this Appendix, on an alternative exploiting the virtues of symmetry consideration within the framework of ACSA to BRST formalism. We have provided the proof of the existence of the CF-type restriction in our Sec. 6 by imposing the CF-type restriction on the expressions for the conserved (anti-)BRST charges \( Q^{(5)}_{(ab)} \) in Eqs. (51) and (52) to recast them in the exact forms w.r.t. BRST and anti-BRST symmetries which have enabled us to prove the absolute anticommutativity. To achieve the above goal in an alternative manner, we directly apply the BRST symmetry transformations \( (s_b) \) on the expression for the conserved anti-BRST charge \( Q^{(4)}_{ab} \) as follows:

\[ s_b Q^{(4)}_{ab} = e^2 \left[ \frac{d}{d\tau} \left\{ \left( \frac{d}{d\tau} \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \right) \tilde{C} C + B \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \right. \right. \]
\[ \left. \left. - \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \dot{\tilde{C}} C \right\} - 2 \dot{B} \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \right] \]
\[ + 2 e \dot{e} \tilde{C} \left[ \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \dot{\tilde{C}} - \left\{ \frac{d}{d\tau} \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \right\} \tilde{C} \right]. \quad (B.1) \]

It is evident, from the above, that every term on the r.h.s. is zero provided we impose the (anti-)BRST invariant CF-type restriction: \( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} = 0 \) from outside. In other words, the absolute anticommutativity of the (anti-)BRST charges [hidden in the expression \( s_b Q^{(4)}_{ab} \equiv -i \{ Q^{(4)}_{ab}, Q^{(4)}_{ab} \} = 0 \) on the l.h.s. of (B.1)] is true only in the space of variables where the CF-type restriction is satisfied.

To corroborate the above statement, we now apply the anti-BRST symmetry transformation \( (s_{ab}) \) on the BRST charge \( Q^{(4)}_b \) to obtain the following

\[ s_{ab} Q^{(4)}_b = e^2 \left[ \frac{d}{d\tau} \left\{ \left( \frac{d}{d\tau} \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \right) \tilde{C} C - B \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \right. \right. \]
\[ \left. \left. - \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \dot{\tilde{C}} C \right\} + 2 \dot{B} \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \right] \]
\[ - 2 e \dot{e} \tilde{C} \left[ \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \dot{\tilde{C}} - \left\{ \frac{d}{d\tau} \left( B + \bar{B} + \dot{\tilde{C}} C - \bar{C} \dot{\tilde{C}} \right) \right\} \tilde{C} \right]. \quad (B.2) \]

which, once again, demonstrates clearly that the absolute anticommutativity of the (anti-)BRST charges is true only when the entire theory is considered on a submanifold in the space of quantum variables where the (anti-)BRST invariant CF-type restriction is satisfied.
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