"Plane" electromagnetic wave in spatially flat Friedman universe

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Abstract

The electromagnetic theory is, to a large extend, metric independent. Before the metric is introduced, it is called premetric electrodynamics. Metric enters the constitutive relation. We consider this relation for the Friedman model of an expanding Universe and find that the magnitudes of the field quantities depend on the scale factor. This factor, however, does not enter the permeability and permittivity of the vacuum. A spatially uniform electromagnetic field is obtained for spatially flat metric. Then plane electromagnetic wave is found with uniform field playing the role of amplitudes. It turns out that the magnitudes of the frequency and the wave covector depend on the scale factor determining the redshift, but the phase velocity of the wave is constant.

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1 Introduction

Solutions to Maxwell equations in the Friedman universe have been presented in many papers [1]–[9]. They were using antisymmetric tensor $F^{\mu\nu}$ of the electromagnetic field, uniting the fields $E$ and $H$. Two Maxwell equations were written only for $F$ field with the use of space-time metric. The solutions (in analogy to quantum mechanics) were searched with a definite angular momentum, therefore the spherical harmonics have been used. There were no attempts to look for uniform fields.

We are using another approach to electromagnetism with the aid of differential forms. A way of presenting electrodynamics based on a broad use of differential forms has been proposed in the last decades, see Refs. [10]–[17]. Within this approach, a question was discussed whether electrodynamics can be formulated in such a way that the metric of space-time doesn’t enter the fundamental laws. This framework
is called premetric electrodynamics, see the papers [17–23]. Related developments can be also found in the books [24,25]. A crowning achievement of this approach is the book by Hehl and Obukhov [26] in which classical electrodynamics is introduced deductively, i.e. in the form of axioms: conservation of electric charge, magnetic flux and energy-momentum. This approach uses two electromagnetic field exterior forms, namely field strength $F$ and excitation $G$ and two separate Maxwell equations for them. At the end, the metric is introduced by the constitutive relations between $F$ and $G$.

The splitting of $F$ and $G$ onto electric and magnetic parts depends on the observer or, equivalently, on the coordinate system. Once the space-time coordinates $(t, x^1, x^2, x^3)$ are chosen, the four-dimensional two-form $F$ is expressed by electric field strength $E$ and magnetic induction $B$:

$$F = E \wedge dt + B,$$

and the four-dimensional two-form $G$ – by magnetic field strength $H$ and electric induction $D$:

$$G = -H \wedge dt + D,$$

where $d$ is the exterior derivative. After introducing the three-form of the electric current density

$$J = -j \wedge dt + \rho$$

the differential Maxwell equations are written as two equations

$$d \wedge F = 0,$$

$$d \wedge G = J.$$

Up to now all considerations are generally covariant and metric-free. They are valid in flat Minkowskian as well as in curved pseudo-Riemannian space-time. Therefore, these Maxwell’s equations represent the optimal formulation of classical electrodynamics.

The electromagnetic theory at some moment incorporates the metric of a flat or curved space-time via the constitutive relation between the excitation and the field strength. In the present paper we consider it in the form

$$G = \lambda_0 \star F,$$

where the Hodge star $\star$ is defined by the space-time metric and $\lambda_0 = \sqrt{\varepsilon_0/\mu_0}$ is the vacuum admittance, the same as in the standard Maxwell-Lorentz electrodynamics in the vacuum. It is our purpose to consider electromagnetic field in a space-time of general relativity applied to cosmology. We assume that the electromagnetic field is weak, which means that its energy momentum tensor does not influence the gravitational field.

The simplest cosmological model is the Friedman solution [27] of the Einstein equations without cosmological constant with the following square of the space-time

\[^1E, H\] are one-forms, $B, D$ are two-forms.
distance expressed in so called *comoving frame* in which galaxies have permanent positions \((x^1, x^2, x^3)\) \[^2\]

\[
ds^2 = c^2 dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

where \(a(t)\) is the scale factor, \(r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2\) and \(\theta, \phi\) are angles of spherical coordinates. The constant \(c\) is the dimensional coefficient relating units of time and space coordinates. Whether \(c\) is velocity of light in this universe, will be checked in Section 5. Since our aim is to find spatially uniform fields, we choose the spatially flat universe with \(K = 0\)

\[
ds^2 = c^2 dt^2 - a(t)^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],
\]

that is

\[
ds^2 = c^2 dt^2 - a(t)^2 \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] = c^2 dt^2 - a(t)^2 dr^2.
\]

The scalar product determined by (9) is used in Section 2 to define magnitudes of the electromagnetic field quantities, which we consider to be measurable quantities. This notion was not yet expressed in the literature. The named magnitudes turn out to depend on the scale factor.

We exploit constitutive relation in Section 3 to determine the permittivity and permeability of the vacuum. They are not functions of time; their product is constant and related to \(c\) through \(c = (\varepsilon_0 \mu_0)^{-1/2}\).

Generally, a wave is a product of the amplitude and a periodic function \(\psi\) of the phase. In fundamental physics courses, the whole spatially-temporal dependence is present in \(\psi\), hence the amplitude is constant. When the metric is time-dependent, the amplitude may depend on time, hence we look for an electromagnetic wave in two steps. In Section 4, we consider an electromagnetic field \(\tilde{F}, \tilde{G}\) which could be a counterpart of the static uniform field possible in Minkowski space-time. We can not expect the field to be static, but we choose it to be uniform, that is, independent of the spatial coordinates. We find such solutions to homogeneous Maxwell equations i.e. in the case devoid of charges and currents.

Section 5 is devoted to find electromagnetic field which can be treated as a "plane" wave. The fields are assumed in the form

\[
F = \psi(\Phi) \tilde{F}, \quad G = \psi(\Phi) \tilde{G},
\]

The factor \(\psi\) is a scalar function of the scalar variable \(\Phi\), describing wave-like behaviour, where \(\Phi\) represents phase of the wave. The solution is found without invoking the wave equation, only Maxwell equations with the constitutive relation are used. The \(\psi\) contains fast changes in its dependence on \(t\) and \(r\). The second factors \(\tilde{F}\) and \(\tilde{G}\) contain much slower dependence. They are taken as previously found uniform fields.

\[^2\]Here \(dt, dr\) and so on denote the ordinary differentials.
2 Magnitudes of the electromagnetic field quantities

The expression (9) defines the scalar product of vectors with the following matrix of the metric tensor $g_{\mu\nu}$:

$$
g = \begin{pmatrix}
c^2 & 0 & 0 & 0 \\
0 & -a^2 & 0 & 0 \\
0 & 0 & -a^2 & 0 \\
0 & 0 & 0 & -a^2
\end{pmatrix}.
$$

(11)

The measure (9) of distances in space-time serves to determine separations in proper time $\tau$ for time-like distances:

$$
d\tau^2 = dt^2 - \frac{a^2}{c^2}dr^2,
$$

(12)

and distances in proper length for space-like distances:

$$
dl^2 = a^2dr^2 - c^2dt^2.
$$

(13)

The increments of coordinates are not physically measured quantities for space or time separations. The metric tensor serves to calculate them as distances in space-time. The metric $g$ determines the metric magnitudes or the lengths for pure space vectors $\Delta r$

$$
||\Delta r||_g = a\sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}.
$$

(14)

As follows from (12), for pure time vectors the metric magnitude of time separation is equal to the time separation itself without any additional factor:

$$
||d\tau||_g = |dt|.
$$

(15)

The expression (9) defines also the scalar product of forms with the reciprocal metric tensor $g^{\mu\nu}$ which has the matrix inverse to (11):

$$
g^{-1} = \begin{pmatrix}
\frac{1}{c^2} & 0 & 0 & 0 \\
0 & -\frac{1}{a^2} & 0 & 0 \\
0 & 0 & -\frac{1}{a^2} & 0 \\
0 & 0 & 0 & -\frac{1}{a^2}
\end{pmatrix}.
$$

(16)

The electromagnetic field quantities $E$, $D$, $H$, $B$ are exterior forms, i.e. mappings of line or surface elements into scalars (for the operational definition of $D$ and $H$ see [17]). These mappings determine components of the forms when space coordinates are given. However, for defining magnitudes of physical quantities represented by the forms a scalar product is needed. This is analogous to the line or surface elements: the components of the named elements can be ascribed for a given

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3The expression (9) is divided by $c^2$ in order to obtain the physical dimension of time.
coordinate system, but lengths and areas can be defined only when a scalar product is introduced.

When scalar product, $g^{-1}$, for one-forms is given, a natural scalar product, $\hat{g}^{-1}$, for two-forms $K$, $L$ is introduced through the following formula written for coordinates:

$$\hat{g}^{-1}(K, L) = \frac{1}{2} K_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} L_{\alpha\beta}. \quad (17)$$

In particular, the scalar square of the electromagnetic field strength is

$$\hat{g}^{-1}(F, F) = \frac{1}{2} \left[ F_{0j} g^{00} g^{jk} F_{0k} + F_{k0} g^{kl} g^{00} F_{l0} + F_{kj} g^{kl} g^{im} F_{lm} \right]. \quad (18)$$

Similarly,

$$\hat{g}^{-1}(G, G) = \frac{1}{a^2} \left[ -\frac{1}{c^2} (E_1^2 + E_2^2 + E_3^2) + \frac{1}{a^2} (B_{12}^2 + B_{23}^2 + B_{31}^2) \right]. \quad (19)$$

If only the electric part of the field strength is present, its magnitude, as determined by the metric (16), is

$$||E||_g = \frac{1}{a} \sqrt{E_1^2 + E_2^2 + E_3^2}. \quad (20)$$

If the field strength is purely magnetic, its magnitude, determined by $g^{-1}$, is

$$||B||_g = \frac{1}{a^2} \sqrt{B_{12}^2 + B_{23}^2 + B_{31}^2}. \quad (21)$$

Similar expressions should be introduced for the electromagnetic excitations for the magnitudes if pure electric or magnetic fields exist, respectively:

$$||D||_g = \frac{1}{a^2} \sqrt{D_{12}^2 + D_{23}^2 + D_{31}^2}, \quad (22)$$

$$||H||_g = \frac{1}{a} \sqrt{H_1^2 + H_2^2 + H_3^2}. \quad (23)$$

\(^4\text{The coefficient } c \text{ is omitted in order to obtain proper physical dimension}\)
3 Constitutive relation

In what follows we use the following notation: \( e_0 = \frac{\delta}{\delta t} \), \( e_j = \frac{\delta}{\delta x^j} \) are basic vectors, \( f^0 = dt, f^i = dx^i \) are basic one-forms, \( \partial \) is the exterior derivative and \( \lfloor \) is the contraction.

The constitutive relation between two-forms \( F \) and \( G \) is proportional to the Hodge map \( \star \) given \([26, 28]\) by the formula\(^5\)

\[
G = \sqrt{\frac{\varepsilon_0}{\mu_0}} \star F = -\lambda_0 \sqrt{-\det g} f^{0123} [\hat{g}^{-1}(F)]
\]  

(24)

where the bivector in square bracket has the components

\[
[\hat{g}^{-1}(F)]^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta},
\]

(25)

and \( f^{0123} = f^0 \wedge f^1 \wedge f^2 \wedge f^3 \) is the volume-measure four-form built of the basic one-forms.

As the first step in performing the map \( (24) \) we write the time-space components of \( (25) \)

\[
[\hat{g}^{-1}(F)]^{0j} = -g^{00} g^{jk} F_{0k} = - \frac{1}{c^2 a^2} F_{0j},
\]

(26)

and the space-space components

\[
[\hat{g}^{-1}(F)]^{jk} = g^{jl} g^{km} F_{lm} = \frac{1}{a^4} F_{jk}.
\]

(27)

Now we express the bivector in square bracket of \( (24) \) by the basic bivectors \( e_{\mu\nu} = e_\mu \wedge e_\nu \):

\[
\hat{g}^{-1}(F) = \frac{1}{a^2} \left[ -\frac{1}{c^2} \left(F_{01} e_{01} + F_{02} e_{02} + F_{03} e_{03}\right) + \frac{1}{a^2} \left(F_{12} e_{12} + F_{23} e_{23} + F_{31} e_{31}\right) \right].
\]

The second step in map \( (24) \) is the contraction with the basic four-form – the coefficients do not change, only the basic bivectors change into complementary basic two-forms:

\[
f^{0123} [\hat{g}^{-1}(F)] = \frac{1}{a^2} \left[ \frac{1}{c^2} \left(F_{01} f^{23} + F_{02} f^{31} + F_{03} f^{12}\right) - \frac{1}{a^2} \left(F_{12} f^{03} + F_{23} f^{01} + F_{31} f^{02}\right) \right].
\]

The last step is multiplication by the numerical factor \(-\sqrt{\frac{\varepsilon_0}{\mu_0}} \sqrt{-\det g} = -\sqrt{\frac{\varepsilon_0}{\mu_0}} c a^3\):

\[
G = -\sqrt{\frac{\varepsilon_0}{\mu_0}} c a^3 \frac{1}{a^2} \left[ \frac{1}{c^2} \left(F_{01} f^{23} + F_{02} f^{31} + F_{03} f^{12}\right) - \frac{1}{a^2} \left(F_{12} f^{03} + F_{23} f^{01} + F_{31} f^{02}\right) \right].
\]

\(^5\)Also an alternative Hodge map exists \([28]\) given by \( \star F = -(-\det g)^{-1} \hat{g} (e_{0123} | F) \) where \( e_{0123} = e_0 \wedge e_1 \wedge e_2 \wedge e_3 \) is the volume quadrivector built of the basic vectors. This formula, however, leads to the same result for \( G = \sqrt{\varepsilon_0/\mu_0} \star F \).
\[ = -\sqrt{\frac{\varepsilon_0}{\mu_0}} \left[ \frac{\alpha}{c} \left( F_{01} f^{23} + F_{02} f^{31} + F_{03} f^{12} \right) - \frac{c}{\alpha} \left( F_{12} f^{03} + F_{23} f^{01} + F_{31} f^{02} \right) \right]. \]

We use \( c = 1/\sqrt{\varepsilon_0 \mu_0} \):

\[ G = -\varepsilon_0 a \left( F_{01} f^{23} + F_{02} f^{31} + F_{03} f^{12} \right) + \frac{1}{\mu_0 a} \left( F_{12} f^{03} + F_{23} f^{01} + F_{31} f^{02} \right). \]  

(28)

The coefficients in front of appropriate basic two forms are interpreted as components of the two-form \( G = D - H \wedge dt \). For instance, the coefficient in front of the basic two-form \( f^{23} \) is \( D_{23} \), i.e.

\[ D_{23} = -\varepsilon_0 a F_{01} = \varepsilon_0 a F_{10} = \varepsilon_0 a E_1, \]

therefore we obtain:

\[ D_{23} = \varepsilon_0 a E_1, \quad D_{31} = \varepsilon_0 a E_2, \quad D_{12} = \varepsilon_0 a E_3, \]  

(29)

\[ H_1 = \frac{1}{\mu_0 a} B_{23}, \quad H_2 = \frac{1}{\mu_0 a} B_{31}, \quad H_3 = \frac{1}{\mu_0 a} B_{12}. \]  

(30)

One could be prone to treat the coefficients at the right-hand sides as the permittivity and (the inverse of) permeability of the vacuum, respectively. The components \( D_{ij}, B_{ij}, E_i, H_i \), however, are not directly measurable, one should rather compare magnitudes of the fields. In this purpose we insert relations (29) into (22):

\[ \|D\|_g = \frac{1}{a^2} \sqrt{\varepsilon_0^2 a^2 (E_1^2 + E_2^2 + E_3^2)} = \frac{\varepsilon_0}{a} \sqrt{E_1^2 + E_2^2 + E_3^2} \]  

(31)

and use (20)

\[ \|D\|_g = \varepsilon_0 \|E\|_g. \]  

(32)

We see that the permittivity of the vacuum does not depend on time and is equal to the electric constant. We similarly arrive at the relation

\[ \|B\|_g = \mu_0 \|H\|_g \]  

(33)

which allows us to claim that the permeability of the vacuum also does not depend on time and is equal to the magnetic constant. The product \( \varepsilon \mu = \varepsilon_0 \mu_0 \) does not depend on time.

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6This can be compared with the relations given (in different unit system where permeability and permittivity have the same physical dimension) in 4, 5

\[ \varepsilon_{ij} = \mu_{ij} = (\det g)^{1/2} (g^{ij}/g_{00}), \]

which for the metric (16) yield \( \varepsilon_{ij} = \mu_{ij} = a \delta_{ij} \), which is also proportional to \( a \).
4 Spatially uniform electromagnetic field

We are going to find field strength $\tilde{F}$ as a simple two-form, i.e. an exterior product of two one-forms with the following combinations of basic one-forms $f^\mu$:

$$k = \xi(t) f^0 + k_j f^j, \quad h = \zeta(t) f^0 + h_i f^i,$$

where $k_j, h_i = \text{const}$ and summation is present over repeated indices $i, j = 1, 2, 3$. Thus

$$\tilde{F} = k \wedge h = \xi(t) f^0 \wedge h_i f^i + k_j f^j \wedge \zeta(t) f^0 + k_j h_i f^{ji},$$

$$\tilde{F} = (\xi h_i - \zeta k_i) f^{0i} + (k_1 h_2 - k_2 h_1) f^{12} + (k_2 h_3 - k_3 h_2) f^{23} + (k_3 h_1 - k_1 h_3) f^{31}. \quad (35)$$

The first term describes the electric field, the three other – magnetic one. From the exterior derivative

$$d = f^0 \frac{\partial}{\partial t} + f^i \frac{\partial}{\partial x^i}$$

only first term gives nonzero contribution on $F$ because no dependence on spatial coordinates is present in (35). In this manner we obtain

$$d \wedge F = f^0 \frac{d}{dt} (\xi h_i - \zeta k_i) \wedge f^{0i} = (\xi' h_i - \zeta' k_i) f^0 \wedge f^{0i} = 0. \quad (36)$$

First Maxwell equation (11) is satisfied.

In order to find the excitation field, we insert the components of $\tilde{F}$ from (35) into (28):

$$\tilde{G} = -\varepsilon_0 a \left[ (\xi h_1 - \zeta k_1) f^{23} + (\xi h_2 - \zeta k_2) f^{31} + (\xi h_3 - \zeta k_3) f^{12} \right]$$

$$+ \frac{1}{\mu_0 a} \left[ (k_1 h_2 - k_2 h_1) f^{03} + (k_2 h_3 - k_3 h_2) f^{01} + (k_3 h_1 - k_1 h_3) f^{02} \right]. \quad (37)$$

We calculate its exterior derivative

$$d \wedge \tilde{G} = -\varepsilon_0 \left\{ \left[ (a \xi)' h_1 - (a \zeta)' k_1 \right] f^{023} + \left[ (a \xi)' h_2 - (a \zeta)' k_2 \right] f^{031} + \left[ (a \xi)' h_3 - (a \zeta)' k_3 \right] f^{012} \right\}$$

$$+ \frac{d}{dt} \left( \frac{1}{\mu_0 a} \right) f^0 \wedge \left[ (k_1 h_2 - k_2 h_1) f^{03} + (k_2 h_3 - k_3 h_2) f^{01} + (k_3 h_1 - k_1 h_3) f^{02} \right]. \quad (38)$$

The second term is zero because of exterior product of $f^0$ with $f^{0j}$. In order to ensure vanishing of the first term we assume $a(t) \zeta(t) = A = \text{const}$, $a(t) \xi(t) = C = \text{const}$, that is

$$\zeta(t) = \frac{A}{a(t)}, \quad \xi(t) = \frac{C}{a(t)}.$$

In such a case

$$d \wedge \tilde{G} = 0. \quad (40)$$

The second (homogeneous) Maxwell equation is satisfied.

In this manner we have found

$$k = \frac{C}{a(t)} f^0 + k_j f^j, \quad h = \frac{A}{a(t)} f^0 + h_i f^i \quad (41)$$
and
\[
\tilde{F} = k \land h = \left[ \frac{C}{a(t)} f^0 + k_j f^j \right] \land \left[ \frac{A}{a(t)} f^0 + h_j f^j \right]
\] (42)

with the following expressions for the components of the uniform electromagnetic field:
\[
\tilde{F} = \frac{1}{a(t)} (C h_i - A k_i) f^{0i} + (k_1 h_2 - k_2 h_1) f^{12} + (k_2 h_3 - k_3 h_2) f^{23} + (k_3 h_1 - k_1 h_3) f^{31}. \quad (43)
\]

\[
\tilde{G} = -\varepsilon_0 [((C h_1 - A k_1) f^{23} + (C h_2 - A k_2) f^{31} + (C h_3 - A k_3) f^{12})
\]
\[
+ \frac{1}{\mu_0 a(t)} [(k_1 h_2 - k_2 h_1) f^{03} + (k_2 h_3 - k_3 h_2) f^{01} + (k_3 h_1 - k_1 h_3) f^{02}]
\] (44)

with two arbitrary constants \(A, C\). Since \(k_j\) and \(h_i\) are constant, the fields \(\tilde{F}, \tilde{G}\) do not depend on spatial coordinates, hence the electromagnetic field is uniform in space. Magnitudes of the electric and magnetic parts are
\[
||\tilde{E}||_g = \frac{1}{a^2} \sqrt{(C h_1 - A k_1)^2 + (C h_2 - A k_2)^2 + (C h_3 - A k_3)^2}, \quad (45)
\]
\[
||\tilde{B}||_g = \frac{1}{a^2} \sqrt{(k_1 h_2 - k_2 h_1)^2 + (k_2 h_3 - k_3 h_2)^2 + (k_3 h_1 - k_1 h_3)^2} \quad (46)
\]

We see that both fields decrease for expanding universe with the same rate.

In the case of \(k_j = 0\) only electric field is present:
\[
\tilde{F} = \frac{C}{a} h_i f^{0i}
\]

and similarly for \(h_j = 0\). If one wants the electric field to vanish, one may assume \(C h_i = A k_i\), but for \(C \neq 0, A \neq 0\) the spatial parts of the one-forms \(k\) and \(h\) are parallel, so the three last terms in (43) also are zero and \(\tilde{F} = 0\) which is inappropriate. Thus we choose \(A = C = 0\), that is, \(\xi = \zeta = 0\) and then
\[
\tilde{F} = k_j f^j \land h_i f^i
\]

expresses the pure magnetic field. Notice that components of this field are constant, but the magnitude – according to (46) – is not.

## 5 Plane electromagnetic wave

An important notion of any wave is its phase. Let us ponder what is phase of the plane wave in the Minkowski space-time. It is the expression
\[
\Phi(x) = k_\mu x^\mu = k^\nu \eta_{\nu\mu} x^\mu, \quad (47)
\]
where $\eta_{\nu\mu} = \text{diag}\{c^2, -1, -1, -1\}$ is the Minkowski metric tensor and $k_\mu = \text{const.}$

Which collection of constant numbers: $k_\mu$ or $k^\nu$ is more important in this expression? In the language of differential geometry it is the first one because it establishes components of a one-form. The level surfaces of the phase, that is, loci of points satisfying $\Phi(x) = \text{const}$, are planes and this is the reason why the wave is called plane.

In more general space-time the phase $\Phi$ needs not be a linear function of coordinates, but still should be a scalar quantity. Its outer derivative

$$k = d\Phi = \frac{\partial \Phi}{\partial t} f^0 + \frac{\partial \Phi}{\partial x^1} f^1 + \frac{\partial \Phi}{\partial x^2} f^2 + \frac{\partial \Phi}{\partial x^3} f^3,$$

with components $k_\mu = \partial_\mu \Phi$, is the one-form still called wave covector, but the components need not be constants. We expect, however, that there exist coordinates of the space-time in which at least spatial components $k_i, i \in \{1, 2, 3\}$ are constant. (Of course, $k^i = g^{i\mu} k_\mu$ are not constant.) For the time-harmonic wave, the component $\frac{\partial \Phi}{\partial t} = k_0 = \omega$ should be interpreted as the circular frequency of the plane wave.

We seek a solution of homogeneous Maxwell equations in the form

$$F(t, r) = \psi(\Phi) \tilde{F}(t, r),$$

$$G(t, r) = \psi(\Phi) \tilde{G}(t, r),$$

where two-forms $\tilde{F}$ and $\tilde{G}$ satisfy homogeneous Maxwell equations:

$$d \wedge \tilde{F} = 0, \quad d \wedge \tilde{G} = 0.$$

We take them from previous section as uniform fields. The factor $\psi$ is a scalar function of the scalar variable $\Phi$, describing wave-like behaviour. It contains fast changes in its dependence on $t$ and $r$. The second factors contain much slower dependence. In the flat space-time, $\tilde{F}$ and $\tilde{G}$ would be simply constant two-forms. Usually $\psi$ is taken as a combination of sine and cosine functions of $\Phi$, which is tantamount to assume that the wave is time-harmonic. We present our reasoning without this assumption. For time-harmonic wave, $\tilde{F}$, $\tilde{G}$ play the role of amplitudes. The presence of the same function $\psi$ in front of $\tilde{F}$ and $\tilde{G}$ expresses the synchronicity of changes of the field strength $F$ and excitation $G$.

The exterior derivatives of (49) and (50) are

$$d \wedge F = \psi'(\Phi) d\Phi \wedge \tilde{F},$$

$$d \wedge G = \psi'(\Phi) d\Phi \wedge \tilde{G}.$$
If the space is devoid of charges and currents, the two Maxwell equations (4), (5) are homogeneous and yield
\[ d\Phi \wedge \tilde{F} = 0, \]  
\[ d\Phi \wedge \tilde{G} = 0. \]  
Equation (52) implies that the two-form \( \tilde{F} \) can be factorized in the exterior product containing \( d\Phi \) as one of its factors:
\[ \tilde{F} = d\Phi \wedge h, \]  
where \( dh = 0 \); the best way to fulfil this condition is to assume \( h_\mu = \text{const.} \) Two-form (54) satisfies \( \iota d\Phi \tilde{F} = 0 \), hence the first equation (51) is fulfilled.

It would be easy to solve (53) by substitution \( \tilde{G} = d\Phi \wedge m \) with a one-form \( m \) satisfying \( dm = 0 \), but also the constitutive relation must be satisfied. We apply now the map (24) (for brevity we introduce the notation \( \lambda = \lambda_0 \sqrt{-\det \hat{g}} \)) to (54):
\[ \tilde{G} = -\lambda f^{0123} [ \hat{g}^{-1}(\tilde{F}) ] = -\lambda f^{0123} [ \hat{g}^{-1}(d\Phi \wedge h) ] = -\lambda f^{0123} [ g^{-1}(d\Phi) \wedge g^{-1}(h) ]. \]

We write this down as
\[ \tilde{G} = -\lambda f^{0123} (u \wedge v). \]  
where we introduced two vectors
\[ u = g^{-1}(d\Phi), \quad v = g^{-1}(h). \]  

We have to check whether two-form (55) satisfies condition (53). Due to the identity
\[ k \wedge [ f^{0123} (u \wedge v) ] = -f^{0123} [ \{ k \}(u \wedge v) ] \]
valid for any one-form \( k \), we substitute \( d\Phi = k \) and write the condition (53) as
\[ d\Phi \wedge \tilde{G} = k \wedge \tilde{G} = \lambda f^{0123} [ k ](u \wedge v) = 0. \]  
Since the contraction with four-form is invertible, the expression in square bracket must be zero
\[ k](u \wedge v) = k(\mu) u^\mu - k(\nu) v^\nu = 0 \]  
where \( k(u) = k_\mu u^\mu \) is the value of one-form \( k \) on vector \( u \), and similarly for \( k(v) \). Eq. (58) indicates that the vectors \( u \) and \( v \) are parallel for nonzero scalars \( k(u) \) and \( k(v) \), and this along with (55) would imply that \( \tilde{G} = 0 \) which is undesirable. Thus the equalities
\[ d\Phi(u) = k_\mu u^\mu = 0, \quad d\Phi(v) = k_\nu v^\nu = 0 \]  
are necessary to satisfy condition (57) and, therefore, (53). Obviously, they are also sufficient. We know from (56) that \( u = g^{-1}(k) \), i.e. \( u^\mu = g^{\mu\alpha} k_\alpha \), similarly \( v^\nu = g^{\nu\beta} h_\beta \), so the conditions (52) are \( k_\mu g^{\mu\alpha} k_\alpha = 0, \) \( k_\nu g^{\nu\beta} h_\beta = 0 \), or
\[ g^{-1}(d\Phi, d\Phi) = 0, \quad g^{-1}(d\Phi, h) = 0. \]
Thus we conclude that equation (53) is satisfied if and only if the one-forms $d\Phi, h$ from factorization (54) satisfy conditions (60). The first one says that the wave covector $k$ is orthogonal to itself. Let us look at this closely.

We apply the metric tensor (16) to the one-form (48):

$$\frac{\omega^2}{c^2} - \frac{1}{a^2} (k_1^2 + k_2^2 + k_3^2) = 0,$$

hence

$$\omega^2 = \frac{c^2}{a^2} (k_1^2 + k_2^2 + k_3^2) \quad (61)$$

and

$$\omega = \pm \frac{c}{a} \sqrt{k_1^2 + k_2^2 + k_3^2} = \pm \frac{c}{a} |k|. \quad (62)$$

We see that $k_i$ can be constants, but $\omega$ can not. By comparing (62) with (48) we obtain

$$\frac{\partial \Phi}{\partial x^i} = k_i, \quad \frac{\partial \Phi}{\partial t} = \pm \frac{c}{a(t)} |k|. \quad (63)$$

After introducing the indefinite integral

$$b(t) = \int \frac{c}{a(t)} dt \quad (64)$$

we are allowed to write down

$$\Phi(x) = \pm b(t)|k| + k_i x^i \quad (65)$$

as a phase of the plane electromagnetic wave. By taking $h$ as purely spatial one-form orthogonal to $k$ we obtain

$$\tilde{F} = \left[ \pm \frac{c}{a(t)} |k| f^0 + k_1 f^1 + k_2 f^2 + k_3 f^3 \right] \wedge h. \quad (66)$$

which is particular uniform electromagnetic field (12), with $A = 0$, found in previous section.

The substitution of the phase (64) into (54) and (49) yields the explicit field strength of the plane electromagnetic wave:

$$F(t, r) = \psi \left( \pm b(t)|k| + k_i x^i \right) \left[ \pm \frac{c}{a(t)} |k| f^0 + k_i f^i \right] \wedge h_j f^j. \quad (67)$$

The electric part of it, according to (11) is determined by the first term in square bracket:

$$E \wedge dt = E \wedge f^0 = \pm \frac{c}{a(t)} \psi |k| f^0 \wedge h,$$

hence the electric field one-form is

$$E(t, r) = c \psi(t, r) \frac{a(t)}{a(t)} |k|h. \quad (68)$$
The magnetic part of (66) is the magnetic induction two-form

\[ B(t, r) = \psi(t, r) \left[ (k_1 h_2 - k_2 h_1) f^{12} + (k_2 h_3 - k_3 h_2) f^{23} + (k_3 h_1 - k_1 h_3) f^{31} \right] \]  

(68)

To write them down in terms of traditional vectors we simply equate the one-form components \( E_i \) to vector components \( E^i \):

\[ E^i = E_i \]

and the two-form components \( B_{ij} \) to pseudovector components \( B^i \):

\[ B^1 = B_{23}, \quad B^2 = B_{32}, \quad B^3 = B_{12}. \]

Then the expressions (67, 68) assume the form

\[ \vec{E}(t, r) = \mp \frac{c \psi(t, r)}{a(t)} |\vec{k}| \vec{h}, \]  

(69)

\[ \vec{B}(t, r) = \psi(t, r) \vec{k} \times \vec{h}. \]  

(70)

Formula (69) describes a linearly polarized wave with coordinate dependence \( \psi(t, r) a(t)^{-1} \) and amplitude parallel to \( \vec{h} \), establishing the polarization direction.

The scale factor \( a \) is present only in the electric part, but when calculating magnitudes according to (20, 21), the scale factor appears with the same power in both fields:

\[ ||E||_g = \frac{|c \psi|}{a^2} |\vec{k}| |\vec{h}|, \quad ||B||_g = \frac{|\psi|}{a^2} |\vec{k}| |\vec{h}|. \]

For the sake of completeness, we shall find the two-form \( G \). We repeat here formula (28):

\[ G = -\varepsilon_0 a \left( F_{01} f^{23} + F_{02} f^{31} + F_{03} f^{12} \right) + \frac{1}{\mu_0 a} \left( F_{12} f^{03} + F_{23} f^{01} + F_{31} f^{02} \right). \]  

(71)

We find from (66)

\[ F_{0j} = \pm \psi \frac{c |\vec{k}|}{a} h_j, \quad F_{ij} = \psi (k_i h_j - k_j h_i) \]

and insert into (71)

\[ G(t, r) = \psi \left( \pm b(t) |\vec{k}| + k_i x^i \right) \left\{ \mp \sqrt{\varepsilon_0/\mu_0} |\vec{k}| (h_1 f^{23} + h_2 f^{31} + h_3 f^{12}) \right. \]

\[ - \frac{1}{\mu_0 a(t)} \left[ (k_1 h_2 - k_2 h_1) f^3 + (k_2 h_3 - k_3 h_2) f^1 + (k_3 h_1 - k_1 h_3) f^2 \right] \wedge f^0 \}. \]

(72)

The second term in curly bracket determines the magnetic field one-form

\[ H(t, r) = -\psi(t, r) \frac{c |\vec{k}|}{\mu_0 a(t)} (k_1 h_2 - k_2 h_1) f^3 + (k_2 h_3 - k_3 h_2) f^1 + (k_3 h_1 - k_1 h_3) f^2, \]

(73)

whereas the first term is connected with the electric induction two-form

\[ D(t, r) = \mp \psi(t, r) \sqrt{\varepsilon_0/\mu_0} |\vec{k}| (h_1 f^{23} + h_2 f^{31} + h_3 f^{12}) \]

(74)
6 Magnitudes of some observable quantities

We now consider the measurable quantities of the phase, i.e. the magnitudes determined by the metric (16):

\[ ||\omega||_g = |\omega| = \frac{c}{a(t)} \sqrt{k_1^2 + k_2^2 + k_3^2}, \]  

(75)

where \(|\omega|\) is the absolute value of the real number \(\omega\), and

\[ ||k||_g = \frac{1}{a(t)} \sqrt{k_1^2 + k_2^2 + k_3^2}. \]

(76)

We see that the measured circular frequency and the measured magnitude of the wave covector decrease with time when the Universe expands. This corresponds to the observation that the light from distant galaxies is shifted to the red end of spectrum which is called the redshift. By comparing the magnitudes (75) and (76) we obtain

\[ ||\omega||_g = c ||k||_g, \]  

(77)

which means that the phase velocity \(||\omega||_g/||k||_g\) of the plane wave is constant in time and equals \(c\).

The energy density of the electromagnetic field is given by the formula

\[ w = \frac{1}{2} (E \wedge D + H \wedge B). \]  

(78)

Hence, we calculate the needed products

\[ E \wedge D = \frac{\psi^2 c |k|^2 |h|^2}{\mu_0 a(t)} f^{123}, \]

\[ H \wedge B = \frac{\psi^2 c |k \times h|^2}{\mu_0 a(t)} f^{123} \]

where \(f^{123} = f^1 \wedge f^2 \wedge f^3\) is the basic three-form. We choose \(k \perp h\), then

\[ H \wedge B = \frac{\psi^2 c |k|^2 |h|^2}{\mu_0 a(t)} f^{123} = E \wedge D, \]

so the energy density three-form is

\[ w = \frac{\psi^2 c |k|^2 |h|^2}{\mu_0 a(t)} f^{123}. \]  

(79)

A magnitude of the spatial three-form is its coordinate in front of \(f^{123}\), multiplied by \(\sqrt{-g^{11} g^{22} g^{33}} = \sqrt{a^{-6}} = a^{-3}\) which is

\[ ||w||_g = \frac{\psi^2 c |k|^2 |h|^2}{\mu_0 a(t)^4}. \]  

(80)

\(^8\)The coefficient \(c\) is omitted after first equality in order to obtain appropriate physical dimension
How can one interpret factor $a^4$ in the denominator? Let us look at the electromagnetic wave as a collection of photons. Let in a cube with edge $l$ be $N$ photons in time $t_0$ [when $a(t_0) = 1$], so the concentration of photons is $n_0 = N/l^3$. In another time $t$ of the Universe the cube has its edge with length $a(t)l$, hence the volume is $a(t)^3l^3$. Therefore, the concentration of photons changes with time:

$$n(t) = \frac{n_0}{a(t)^3}. \quad (81)$$

Each photon has its energy proportional to $||\omega||$: $\varepsilon = \hbar ||\omega||$. According to (75) $||\omega(t)||_g = \frac{\omega_0}{a(t)}$, so the energy of single photon also changes with time:

$$\varepsilon(t) = \frac{\hbar \omega_0}{a(t)}. \quad (82)$$

The energy density of the photons is the product of (81) and (82):

$$||w(t)|| = \hbar \frac{n_0}{a(t)^3} \frac{\omega_0}{a(t)} = \hbar n_0 \frac{\omega_0}{a(t)^4}. \quad (83)$$

Now the factor $a^{-4}$ in (80) becomes natural.

7 Conclusion

We have considered the electromagnetic field in an expanding universe described by the spatially flat Friedman model. We take the field to be weak, which means that its energy momentum tensor does not influence the gravitational field.

The scalar product matrices ($g$ for vectors, $g^{-1}$ for one-forms) govern not only distances in space-time but also magnitudes of other physical quantities. We have presented the magnitudes of electromagnetic field quantities $E$, $D$, $B$, $H$ – the scale factor is present there. The matrix $g^{-1}$ enters the constitutive relation between $F$ and $G$ which implies that the scale factor is present in the relations between components of $D$ and $E$ on the one hand and between those of $B$ and $H$ on the other. But after comparing the magnitudes of these quantities it turns out that the permittivity and the permeability of the vacuum are constant and are the same as in Maxwell-Lorentz electrodynamics.

The time variation of the permittivity and permeability of the vacuum was concluded in the literature from relations not between magnitudes but between components of $D$ and $E$ and those of $B$ and $H$ \[4, 5, 8\]. One of the authors \[8\] even speculated on the time variation of the fine structure constant in which permittivity is present. In our opinion this is superfluous.

First aim of the paper is obtaining explicit solution of the homogeneous Maxwell’s equations in the form of uniform field allowing its dependence only on time. It turns out that for the scale factor $a$ magnitudes of both electric and magnetic parts of the field depend on time as $1/a$. [Compare eqs. (20) and (31).]

The main task is finding explicit solution of the Maxwell’s equations in the form of a plane electromagnetic wave without invoking the wave equation. It was claimed
in [4, 5] that equations of electromagnetic field in presence of gravity can be interpreted as the Maxwell equations in flat space-time but in a medium characterized by permittivities depending on metric coefficients. In the present approach, the Maxwell equations are metric independent and the metric enters only the constitutive relations. It turned out that the permittivities are the same as in the flat Minkowski space-time.

In the literature about electromagnetic waves in gravitational fields, two approaches occur. In first one (see [4]), the light rays are considered and the fields themselves are found in a kind of “Born approximation”. Our solution is presented without any approximation. In second approach (see [5, 7, 9]), a time-harmonic solutions with a definite parity and angular momentum are found. Of course, time-harmonic plane wave can be represented in the basis of spherical harmonics by the Gegenbauer expansion. This is, however quite long way: (i) find the spherical waves in not so short derivation, (ii) find the limit of the Gegenbauer series.

Our proposed way is not limited to time-harmonic waves and is shorter: consider fields in the form (49,50) with synchronous dependence $\psi$ on the phase function $\Phi$. The factors $\tilde{F}, \tilde{G}$ are slowly changing fields which can be treated as amplitudes of the wave. Our assumption is that the phase of the wave has constant spatial components of the wave covector, so we are allowed to call it a plane wave. The Maxwell equations lead to the conditions (52,53) for the amplitudes $\tilde{F}, \tilde{G}$. First equation can be solved by substitution (54), the second one, combined with the constitutive equation, implies the conditions (60). The first of them imposes a relation between time and space components of the wave covector $k$. The plane wave field strength is

$$F(\tau, r) = \psi(\Phi) k \wedge h,$$

where the phase $\Phi$ is expressed by (64) and $h$ is the polarization one-form.

The magnitudes of the circular frequency and of the wave covector depend only on time, decreasing by the factor $a^{-1}$. Since both decrease by the same factor, the phase velocity of the wave is constant and equal to $c$. In the literature about electromagnetic waves in expanding universe only Mashhoon [5] has shown classically that the frequency of the wave depends on the metric.\footnote{Predominantly, the explanation of the decrease of frequency is done by passing to the quantum picture, namely by considering a photon moving through space-time endowed with the Friedman metric, see [29], Sec 12.6. The photon loses its energy $E$ by the factor $a^{-1}$ hence, by the Planck relation $E = h\omega$, the same concerns its frequency.} The energy density of the electromagnetic wave changes in time with the factor $a^{-4}$, which can be confirmed also by consideration of the concentration of photons and the density of their energy.

The decreasing of frequency is connected with the observed redshift of light arriving from distant galaxies. People not working in cosmology think that galaxies run away and the redshift is a result of the Doppler effect. The Friedman metric, however, is derived under assumption that the matter (i.e. the galaxies) rests in the chosen coordinate frame. In other words, the coordinates, in which the scale factor $a$ is a function of time only, are distinguished by the fact that the galaxies do not move. The distance between them grows because the space is expanding. Our result
shows that the redshift is only a manifestation of the expansion. One may ask the question: why the light is shifted to the red if the galaxies do not move? The answer is: because the light from distant objects travels very long in time, the scale factor increases during the travel and the light frequency is diminished by this factor.

One could ponder on question whether similar reasoning can be performed for general metric of the Friedman model. In such a case the metric tensor is not so simple as in eq. (11), hence in the constitutive equation (28), in addition to the time-dependent factor $a$, also another space-dependent factors must be present. Therefore, the exterior derivative (38) would be much more complicated and it is not sure whether spatially uniform electromagnetic field exists in this situation.

References

[1] E. Schrödinger: “The proper vibrations of the expanding Universe”, *Physica*, 6(1939)899-912.

[2] L. Infeld and A. Schild: “A new approach to kinematic cosmology”, *Phys. Rev* 68 (1945) 250-272.

[3] L. Infeld and A. Schild: “A new approach to kinematic cosmology – (B)” *Phys. Rev*. 70 (1946) 410-425.

[4] J. Plebański: “Electromagnetic waves in gravitational fields”, *Phys. Rev*. 118 (1960) 1396-1408.

[5] B. Mashhoom: “Electromagnetic waves in an expanding universe”, *Phys. Rev.* D 8 (1973) 4297-4302.

[6] S. Malin: “Maxwell’s equations in an expanding universe”, *J. Math. Phys.* 18,9(1977) 1788-1790.

[7] Yaobing Deng and Philip. D. Mannheim: “Perfect Maxwell fluids in the standard Cosmology”, *Gen. Rel. Grav.* 20,10(1988)969-987.

[8] William Q. Sumner: “On the variation of vacuum permittivity in Friedman universes”, *Astroph. J.* 429 (1994) 491-498.

[9] Nader Haghihgipour: “On the asymptotic character of electromagnetic waves in a Friedmann-Robertson-Walker universe” *Gen. Rel. Grav.* 37,2(2005)327-342.

[10] H. Grauert and I. Lieb: *Differential und Integralrechnung*, vol. 3, Springer Verlag, Berlin 1968.

[11] Charles Misner, Kip Thorne and John Archibald Wheeler: *Gravitation*, Freeman and Co., San Francisco 1973, Sec. 2.5.
[12] Theodore Frankel: *Gravitational Curvature. An Introduction to Einstein’s Theory*, Freeman and Co., San Francisco 1979.

[13] William L. Burke: *Spacetime, Geometry, Cosmology*, University Science Books, Mill Valley 1980.

[14] G.A. Deschamps: “Electromagnetics and differential forms”, *Proc. IEEE* **69**(1981)676.

[15] William L. Burke: *Applied Differential Geometry*, Cambridge University Press, Cambridge 1985.

[16] Roman Ingarden and Andrzej Jamiołkowski: *Classical Electrodynamics*, Elsevier, Amsterdam 1985.

[17] Bernard Jancewicz: “A variable metric electrodynamics. The Coulomb and Biot-Savart laws in anisotropic media”, *Ann. Phys (NY)* **245**(1996)227.

[18] C. Truesdell and R.A Toupin: “The classical field theories”, in *Handbuch der Physik*, vol.III/1, S. Flügge, editor, Springer Verlag, Berlin 1960, pp. 226-793.

[19] R.A. Toupin: “Elasticity and electro-magnetics”, in: *Non-Linear Continuum Theories. CIME Conference, Bressanone, Italy 1965*, C. Truesdell and G. Grioli, coordinators, pp. 203-342.

[20] E.J. Post *Formal Structure of Electromagnetics*, North-Holland, Amsterdam 1962, and Dover, New York 1997.

[21] E.J. Post: “The constitutive map and some of ramifications”, *Ann. Phys. (NY)* **71** (1972) 497-518.

[22] A. Kovetz: *Electromagnetic Theory*, Oxford Univ. Press, Oxford 2000.

[23] José de Jesus Cruz Guzman and Zbigniew Oziewicz: “Frölicher-Nijenhuis algebra and four Maxwell’s equations for non-inertial observer” *Bulletin de la Société de Sciences et de Lettres de Lódź*, vol. **53**, Série Recherches sur les Deformations **39**(2003)107-160.

[24] I.V. Lindell: *Differential Forms in Electromagnetics*, IEEE Press, Piscataway NJ and Wiley-Interscience, 2004.

[25] P. Russer: *Electromagnetics, Microwave Circuit and Antenna Design for Communications Engineering*, Artech House, Boston 2003.

[26] F.W. Hehl and Yu.N. Obukhov: *Foundations of Classical Electrodynamics: Charge, Flux and Metric*. Birkhäuser, Boston 2003.

[27] Marek Demiański: *Relativistic Astrophysics*, Pergamon Press and PWN, Oxford and Warsaw 1985. (First Polish edition: Warsaw 1978.)
[28] Zbigniew Oziewicz: “Classical field theory and analogy between Newton’s and Maxwell’s equations” Found. Phys. 24 (1994) 1379-1402.

[29] Bernard F. Schutz: A First Course in General Relativity, Cambridge Univ. Press, Cambridge 1985.