NONEXISTENCE RESULTS FOR ELLIPTIC DIFFERENTIAL INEQUALITIES WITH A POTENTIAL ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper we are concerned with a class of elliptic differential inequalities with a potential both on $\mathbb{R}^m$ and on Riemannian manifolds. In particular, we investigate the effect of the geometry of the underlying manifold and of the behavior of the potential at infinity on nonexistence of nonnegative solutions.

1. Introduction

One of the most important and well-studied class of elliptic differential inequalities in Global Analysis, due to its ubiquitous presence in many applications, is

\[ \Delta u + V(x)u^\sigma \leq 0, \]

both on $\mathbb{R}^m$ and on general Riemannian manifolds $(M, g)$, where $\Delta$ denotes the Laplace-Beltrami operator associated to the metric and $\sigma > 1$. In particular, in many instances it is also required that the solution $u$ of the problem is positive.

The aim of this paper is to investigate in depth the influence of the geometry of the underlying complete, noncompact Riemannian manifold $(M, g)$ of dimension $m$ and of the potential $V$ on the existence of positive solutions to the class of elliptic differential inequalities

\[ \frac{1}{a(x)} \text{div} \left( a(x)|\nabla u|^{p-2}\nabla u \right) + V(x)u^\sigma \leq 0, \]

thus highlighting the interplay between analysis and geometry in this class of problems, which includes (1.1). Here and in the rest of the paper we assume that $a : M \to \mathbb{R}$ satisfies

\[ a > 0, \quad a \in \text{Lip}_{\text{loc}}(M), \]

$V > 0$ a.e. on $M$ and $V \in L^1_{\text{loc}}(M)$, and the constants $p$ and $\sigma$ satisfy $p > 1$, $\sigma > p - 1$. In our results, the geometry of $M$ appears through conditions on the growth of suitably weighted volumes of geodesic balls, involving both the potential $V$ and the function $a$. We explicitly note that some of the results we find are new also in the specific case of the model equation (1.1).

This class of problems has a very long history, particularly in the Euclidean setting, starting with the seminal works of Gidas [4] and Gidas-Spruck [5]. In those papers the authors show, among other results, that any nonnegative solution of equation (1.1) is in fact identically null if and only if $\sigma \leq \frac{m}{m-2}$, in case $V \equiv 1$ and $m \geq 3$.

We refer to the interesting papers of Mitidieri-Pohozaev [12], [13], [14], [15], [16] for a comprehensive description of results related to these (and also more general) problems on $\mathbb{R}^m$. Note that analogous results have also been obtained for degenerate elliptic equations and inequalities (see, e.g., [3], [17]), and for the parabolic companion problems (see, e.g., [16], [18], [19], [20]). Using nonlinear capacity arguments, which exploit suitably chosen test functions, Mitidieri-Pohozaev prove nonexistence of weak

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or distributional solutions of wide classes of differential inequalities in $\mathbb{R}^m$, which include also many of the examples that we consider here. In particular, they show that equation (1.1) on $\mathbb{R}^m$ does not admit any nontrivial nonnegative solution, provided that

$$\liminf_{R \to +\infty} R^{-\frac{2\sigma}{\sigma - 1}} \int_{B_R \setminus B_{\frac{R}{2}}} V^{-\frac{1}{\sigma - 1}} \, dx < \infty.$$  

For the case of equation (1.2) on $\mathbb{R}^m$, the authors show that no nonnegative, nontrivial solution exists in case $V \equiv 1$, $m > p$ and $p - 1 < \sigma \leq \frac{m(p - 1)}{m - p}$. This can again be read as a condition relating the volume growth of Euclidean balls, which depends on the dimension $m$ of the space, and the exponent of the nonlinearity in the equation.

The results in the case of a complete Riemannian manifold have a more recent history, in particular we cite the inspiring papers of Grygor’yan-Kondratiev [8] and Grygor’y an-Sun [9], whose approach originates from the work of Kurta [12]. Using a capacity argument which only exploits the gradient of the distance function from a fixed reference point, in particular the authors showed in [8] that equation (1.2), in case $p = 2$, admits a unique nonnegative weak solution provided that there exist positive constants $C, C_0$ such that for every $R > 0$ sufficiently large and every small enough $\varepsilon > 0$,

$$\int_{B_R} aV^{-\beta + \varepsilon} \, d\mu_0 \leq CR^\alpha C_0^\varepsilon (\log R)^k,$$

where $d\mu_0$ is the canonical Riemannian measure on $M$, $B_R$ is the geodesic ball centered at a point $x_0 \in M$ and

$$\alpha = \frac{2\sigma}{\sigma - 1}, \quad \beta = \frac{1}{\sigma - 1}, \quad 0 < k < \beta.$$

Let $r(x)$ denote the geodesic distance of a point $x \in M$ from a fixed origin $o \in M$. Note that condition (1.5) is satisfied if, for instance, for every $R > 0$ large enough one has

$$V(x) \leq C(1 + r(x))^{C_0}$$

and

$$\int_{B_R} V^{-\beta} \, d\mu \leq CR^\alpha (\log R)^k$$

for some positive constants $C$, $C_0$. The sharpness of the exponent $\alpha$ in this type of results is evident from the Euclidean case (1.1) with $V \equiv 1$ and $m \geq 3$, where $\alpha = m$ and the corresponding critical growth is $\sigma = \frac{m}{m - 2}$. The sharpness of the exponent $\beta$ is definitely a more delicate question and has recently been settled on a general Riemannian manifold, in case $a \equiv 1$ and $V \equiv 1$, in [9]. In that paper, using carefully chosen families of test functions, the authors showed that equation (1.1) with $V \equiv 1$ does not admit any nonnegative weak solution provided (1.5) holds with $k = \beta$.

In this work we intend to further focus our attention on these classes of differential inequalities, with the objective of adding some new results to the already very interesting overall picture. Our results concerning equation (1.1), in their simplest form, are contained in the two following theorems.

**Theorem 1.1.** Let $(M, g)$ be a complete Riemannian manifold, $\sigma > 1$, $V \in L^1_{\text{loc}}(M)$ with $V > 0$ a.e. on $M$. Define

$$\alpha = \frac{2\sigma}{\sigma - 1}, \quad \beta = \frac{1}{\sigma - 1}$$

and assume that there exist $C, C_0 > 0$ such that for every $R > 0$ sufficiently large one has

$$\int_{B_R \setminus B_{R/2}} V^{-\beta} \, d\mu \leq CR^\alpha (\log R)^\beta$$

and

$$\frac{1}{C}(1 + r(x))^{-C_0} \leq V(x) \leq C(1 + r(x))^{C_0} \quad \text{a.e. on } M.$$

Then the only nonnegative weak solution of (1.1) is $u \equiv 0$. 
Theorem 1.2. With the same notation of Theorem 1.1, assume that there exist $C_0 \geq 0$, $k \geq 0$, $\theta > 0$, $\tau > \max\{\frac{\sigma}{\sigma - p + 1}, (k + 1)\}$ such that for every sufficiently large $R > 0$

\[(1.9) \quad \int_{B_R \setminus B_{R/2}} V^{-\beta} \, d\mu \leq CR^\alpha (\log R)^k\]

and

\[(1.10) \quad V(x) \leq C (1 + r(x))^{C_0} e^{-\theta (\log r(x))^\tau} \quad \text{a.e. on } M.\]

Then the only nonnegative weak solution of (1.1) is $u \equiv 0$.

A few remarks are now in order. For equation (1.1), on $\mathbb{R}^m$, condition (1.8) is not required in the results of Mitidieri-Pohozaev. On the other hand, the potential $V(x) = (\log(2 + |x|^2))^{-1}$ and the choice of the exponent $\sigma = \frac{m}{m-2}$ with $m \geq 3$ satisfy the conditions of our Theorem 1.1 but they do not satisfy condition (1.9).

Moreover, Theorem 1.1 and Theorem 1.2 extend the result contained in [9] for problem (1.1) on a complete Riemannian manifold $(M,g)$, where only the case of a constant potential is considered, to the case of a nonconstant $V$.

On the other hand, while (as we have already noted) the exponent $\alpha = \frac{2p}{m-2}$ in the power of $R$ in conditions (1.7) and (1.9) is indeed sharp, Theorem 1.2 also shows that the sharpness of the exponent of the term $\log R$ for this type of results is a notion which is also related to the behavior of the potential $V$ at infinity. In particular, if $V$ decays at infinity faster than any power of $r(x)$, as in condition (1.10), then the critical threshold for the power of the logarithmic term in the volume growth condition (1.9) for the nonexistence of nonnegative, nontrivial solutions of (1.1) correspondingly increases to $\frac{2p}{m-2} - 1 > \beta = \frac{1}{\sigma - p - 1}$.

We explicitly note that the type of phenomenon described in Theorem 1.2 has not been pointed out before in literature, to the best of our knowledge.

The aforementioned theorems are a consequence of more general results concerning nonnegative weak solutions of inequality (1.2), showing that similar phenomena also occur for this larger class of problems, which includes the case of inequalities involving the $p$-Laplace operator. We start with a definition describing the weighted volume growth conditions on geodesic balls of $(M,g)$, that will be used in obtaining the nonexistence results for nonnegative solutions of (1.2). We recall that with $d\mu_0$ we denote the canonical Riemannian measure on $M$, while we define

\[(1.11) \quad d\mu = a \, d\mu_0\]

the weighted measure on $M$ with density $a$.

Definition 1.3. Let $p > 1$, $\sigma > p - 1$, $V > 0$ a.e. on $M$ and $V \in L^1_{\text{loc}}(M)$. Define

\[(1.12) \quad \alpha = \frac{p\sigma}{\sigma - p + 1}, \quad \beta = \frac{p - 1}{\sigma - p + 1}.\]

We introduce the following three weighted volume growth conditions:

i) We say that condition (HP1) holds if there exist $C_0 > 0$, $k \in [0,\beta)$ such that, for every $R > 0$ sufficiently large and every $\varepsilon > 0$ sufficiently small,

\[(1.13) \quad \int_{B_R \setminus B_{R/2}} V^{-\beta + \varepsilon} \, d\mu \leq CR^\alpha + C_0 \varepsilon (\log R)^k.\]

ii) We say that condition (HP2) holds if there exists $C_0 > 0$ such that, for every $R > 0$ sufficiently large and every $\varepsilon > 0$ sufficiently small,

\[(1.14) \quad \int_{B_R \setminus B_{R/2}} V^{-\beta + \varepsilon} \, d\mu \leq CR^\alpha + C_0 \varepsilon (\log R) \quad \text{and} \quad \int_{B_R \setminus B_{R/2}} V^{-\beta - \varepsilon} \, d\mu \leq CR^\alpha + C_0 \varepsilon (\log R)^3.\]
iii) We say that condition (HP3) holds if there exist $C_0 \geq 0$, $k \geq 0$, $\theta > 0$, $\tau > \max\{\frac{2-p+1}{\sigma}(k+1), 1\}$ such that, for every sufficiently large $R > 0$ and every $\varepsilon > 0$ sufficiently small,

$$\int_{B_R \setminus B_{R/2}} V^{-\beta+\varepsilon} \, d\mu \leq CR^{\alpha+C_0\varepsilon}(\log R)^k e^{-\varepsilon\theta(\log R)^\tau}.$$  

(1.15)

We explicitly note that, when $p = 2$, the definitions of $\alpha$, $\beta$ given in (1.6) and (1.12) agree.

**Remark 1.4.** The following are sufficient conditions that imply the above weighted volume growth conditions for geodesic balls in $M$.

i) Suppose that there exist $C_0 > 0$, $k > 0$ such that

$$V(x) \leq C(1 + r(x))^{C_0}$$

(1.16)

and

$$\int_{B_R \setminus B_{R/2}} V^{-\beta} \, d\mu \leq CR^\alpha(\log R)^k$$

(1.17)

for every $R > 0$ sufficiently large; then condition (1.13) holds.

ii) Suppose that there exists $C_0 > 0$ such that

$$\frac{1}{C}(1 + r(x))^{-C_0} \leq V(x) \leq C(1 + r(x))^{C_0}$$

(1.18)

and

$$\int_{B_R \setminus B_{R/2}} V^{-\beta} \, d\mu \leq CR^\alpha(\log R)^\beta$$

(1.19)

for every $R > 0$ sufficiently large; then condition (1.14) holds.

iii) Suppose there exist $C_0 \geq 0$, $k \geq 0$, $\theta > 0$, $\tau > \max\{\frac{2-p+1}{\sigma}(k+1), 1\}$ such that

$$V(x) \leq C(r(x))^{C_0} e^{-\theta(\log r(x))^{\tau}}$$

(1.20)

and

$$\int_{B_R \setminus B_{R/2}} V^{-\beta} \, d\mu \leq CR^\alpha(\log R)^k$$

(1.21)

for every $R > 0$ sufficiently large; then condition (1.15) holds.

We can now state our main theorem.

**Theorem 1.5.** Let $p > 1$, $\sigma > p - 1$, $V \in L^1_{\text{loc}}(M)$ with $V > 0$ a.e. on $M$ and $a \in \text{Lip}_{\text{loc}}(M)$ with $a > 0$ on $M$. If $u \in W^{1,p}_{\text{loc}}(M) \cap L^\sigma_{\text{loc}}(M, V d\mu_0)$ is a nonnegative weak solution of (1.2), then $u \equiv 0$ on $M$ provided that one of the conditions (HP1), (HP2) or (HP3) holds (see Definition 1.3).

In the particular case $p = 2$, from Theorem 1.5 we can also derive nonexistence criteria for nonnegative weak solutions of the semilinear inequality

$$\frac{1}{a(x)} \text{div}(a(x) \nabla u) + b(x)u + V(x)u^\sigma \leq 0 \quad \text{on} \ M.$$  

(1.22)

We refer the reader to Section 4 for a precise description of the results concerning inequality (1.22).

The rest of the paper is organized as follows. In Section 2 we state and prove some preliminary technical results, that we put to use in Section 3, where we give the proof of Theorem 1.5. In Section 4 we describe in more detail nonexistence results for nontrivial nonnegative weak solutions of (1.22). Finally in Section 5 we collect some counterexamples to Theorem 1.5 for the case $p = 2$, showing that the weighted volume growth conditions that we assume on geodesic balls are in many cases sharp.

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2. Preliminary results

For any relatively compact domain $\Omega \subset M$ and $p > 1$, $W^{1,p}(\Omega)$ is the completion of the space of Lipschitz functions $w : \Omega \to \mathbb{R}$ with respect to the norm

$$
\|w\|_{W^{1,p}(\Omega)} = \left( \int_{\Omega} |\nabla w|^p \, d\mu_0 + \int_{\Omega} |w|^p \, d\mu_0 \right)^{\frac{1}{p}}.
$$

For any function $u : M \to \mathbb{R}$ we say that $u \in W^{1,p}(\Omega)$ if for every relatively compact domain $\Omega \subset M$ one has $u|\Omega \in W^{1,p}(\Omega)$.

**Definition 2.1.** Let $p > 1$, $\sigma > p - 1$, $V > 0$ a.e. on $M$ and $V \in L^1(M)$. We say that $u$ is a weak solution of equation (1.2) if $u \in W^{1,p}_{\text{loc}}(M) \cap L^p_\sigma(M, V d\mu_0)$ and for every $\varphi \in W^{1,p}(M) \cap L^\infty(M)$, with $\varphi \geq 0$ a.e. on $M$ and compact support, one has

$$
(2.1) \quad - \int_M a(x)|\nabla u|^{p-2} \left( \nabla u, \nabla \left( \frac{\varphi}{a(x)} \right) \right) \, d\mu_0 + \int_M V(x) u^\sigma \varphi \, d\mu_0 \leq 0 \quad \text{on } M.
$$

**Remark 2.2.** We note that, by (1.2), $u \in W^{1,p}_{\text{loc}}(M) \cap L^p_\sigma(M, V d\mu)$ is a weak solution of (2.2) if and only if it is a weak solution of

$$
\text{div} \left( a(x)|\nabla u|^{p-2}\nabla u \right) + a(x) V(x) u^\sigma \leq 0 \quad \text{on } M,
$$

i.e. if and only if for every $\psi \in W^{1,p}(M) \cap L^\infty(M)$, with $\psi \geq 0$ a.e. on $M$ and compact support, one has

$$
(2.2) \quad - \int_M |\nabla u|^{p-2} \left( \nabla u, \nabla \psi \right) \, d\mu_0 + \int_M V(x) u^\sigma \psi \, d\mu_0 \leq 0 \quad \text{on } M,
$$

where $d\mu$ is the measure on $M$ with density $a$, as defined in (1.1).

Indeed, given any nonnegative $\varphi \in W^{1,p}(M) \cap L^\infty(M)$ with compact support, one can choose $\varphi = a\psi$ as a test function in (2.1) in order to obtain (2.2). Similarly, given any nonnegative $\varphi \in W^{1,p}(M) \cap L^\infty(M)$ with compact support, one can insert $\psi = \frac{1}{\sigma} \, \varphi$ in (2.2) and find (2.1).

The following two lemmas will be crucial ingredients in the proof the Theorem 1.5.

**Lemma 2.3.** Let $s \geq \frac{p-1}{\sigma-p+1}$ be fixed. Then there exists a constant $C > 0$ such that for every $t \in (0, \min \{1, p-1\})$, every nonnegative weak solution $u$ of equation (1.2) and every function $\varphi \in \text{Lip}(M)$ with compact support and $0 \leq \varphi \leq 1$ one has

$$
(2.3) \quad \frac{1}{p} \int_M \varphi^s u^{-t-1}|\nabla u|^p \chi_\Omega \, d\mu_0 + \frac{1}{p} \int_M V u^s \varphi^s \, d\mu_0 \leq \frac{C t^{(s-1)p}}{(s-p+1)} \int_M V^{-\frac{s-1}{p-1}} |\nabla \varphi|^{p(s-1)} \, d\mu_0,
$$

where $\Omega = \{x \in M : u(x) > 0\}$, $\chi_\Omega$ is the characteristic function of $\Omega$ and $d\mu$ is the measure on $M$ with density $a$, as defined in (1.1).

**Proof.** Let $\eta > 0$ and $u_\eta = u + \eta$, then $u_\eta \in W^{1,p}_{\text{loc}}(M) \cap L^p_\sigma(M, V d\mu)$. Define $\psi = \varphi^s u_\eta^{-t}$, then $\psi$ is an admissible test function for equation (2.2), with

$$
\nabla \psi = s \varphi^{s-1} u_\eta^{-t} \nabla \varphi - t \varphi^s u_\eta^{-t-1} \nabla u \quad \text{a.e. on } M.
$$

Indeed, $\text{supp} \, \psi = \text{supp} \, \varphi$, $0 \leq \psi \leq \eta^{-t}$ so that $\psi \in L^\infty(M)$ and $\psi \in W^{1,p}(M)$ with

$$
\int_M |\nabla \psi|^p \, d\mu \leq 2^{p-1} \left[ \int_M \varphi^{(s-1)p} u_\eta^{-tp} |\nabla \varphi|^p \, d\mu_0 + \int_M \varphi^s u_\eta^{-(t+1)p} |\nabla u|^p \, d\mu_0 \right]
$$

$$
\leq 2^{p-1} \left[ \int_{\text{supp} \, \varphi} |\nabla \varphi|^p \, d\mu_0 + t \eta^{-p} \int_M V u^s \varphi^s \, d\mu_0 \right] + \int_{\text{supp} \, \varphi} |\nabla u|^p \, d\mu_0 < +\infty.
$$

Equation (2.2) then gives

$$
(2.4) \quad \int_M \varphi^s u_\eta^{-t-1} |\nabla u|^p \, d\mu + \int_M V u^s u_\eta^{-t} \varphi^s \, d\mu \leq \frac{s}{t} \int_M \varphi^{s-1} u_\eta^{-t} |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \, d\mu_0.
$$
Now we estimate the right-hand side of (2.4) using Young’s inequality, obtaining
\[
s \int_M \varphi^{q-1} u_{\eta}^{-t} \nabla u |\nabla \varphi|^{-2} \langle \nabla u, \nabla \varphi \rangle \, d\mu \leq s \int_M \varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p-1} |\nabla \varphi| \, d\mu
\]
\[
\leq \int_M \left( \frac{t}{p} \frac{\varphi^{q-1} u_{\eta}^{-t} \left(1+\frac{1}{p-1}\right)}{p-1} |\nabla u|^{p-1} \left( s \frac{\varphi^{q-1} u_{\eta}^{-t} \left(1+\frac{1}{p-1}\right)}{p-1} |\nabla \varphi| \right) \right) \, d\mu
\]
\[
\leq \frac{p-1}{p} \int_M t \varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} \, d\mu + \frac{1}{p} \int_M s^{p} t^{-(p-1)} \varphi^{q-1} u_{\eta}^{-t} |\nabla \varphi|^{p} \, d\mu.
\]
From (2.4) we have
\[
(2.5) \quad \frac{t}{p} \int_M \varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} \, d\mu + \int_M V u^{s} u_{\eta}^{-t} \varphi^{s} \, d\mu \leq \frac{1}{p} \int_M s^{p} t^{-(p-1)} \varphi^{q-1} u_{\eta}^{-t} |\nabla \varphi|^{p} \, d\mu.
\]
We exploit again Young’s inequality on the right-hand side of (2.5), with
\[
q = \frac{\sigma - t}{p - t - 1}, \quad q' = \frac{q}{q - 1} = \frac{\sigma - t}{\sigma - p + 1}, \quad \delta = \frac{p - 1}{p}
\]
obtaining
\[
\frac{1}{p} \int_M s^{p} t^{-(p-1)} \varphi^{q-1} u_{\eta}^{-t} |\nabla \varphi|^{p} \, d\mu = \int_M \left( \delta t \varphi^{q-1} u_{\eta}^{-t} V^{\frac{q'}{q}} \right) \left( \delta t \varphi^{q-1} u_{\eta}^{-t} V^{\frac{q'}{q}} \right) \left( \frac{\varphi^{q-1} u_{\eta}^{-t} V^{\frac{q'}{q}}}{\varphi^{q-1} u_{\eta}^{-t} V^{\frac{q'}{q}}} \right) |\nabla \varphi|^{p} \, d\mu
\]
\[
\leq \frac{1}{p} \delta \int_M u^{\sigma-\varphi(t-1)} V^{\frac{q'}{q}} \varphi^{q-1} u_{\eta}^{-t} |\nabla \varphi|^{p} \, d\mu \leq \frac{1}{p} \delta \int_M u^{\sigma-\varphi(t-1)} V^{\frac{q'}{q}} |\nabla \varphi|^{p} \, d\mu
\]
\[
\leq \frac{1}{p} \delta \int_M u^{\sigma-\varphi(t-1)} V^{\frac{q'}{q}} \, d\mu + \int_M C\varphi^{q-1} u_{\eta}^{-t} V^{\frac{q'}{q}} \, d\mu.
\]
Substituting in (2.5) we have
\[
(2.6) \quad I = \frac{t}{p} \int_M \varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} \, d\mu + \int_M V u^{s} u_{\eta}^{-t} \varphi^{s} \, d\mu - \frac{p - 1}{p} \int_M V u^{\sigma-\varphi(t-1)} \varphi^{s} \, d\mu
\]
\[
\leq C\varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} \, d\mu + \int_M V u^{\sigma-\varphi(t-1)} \varphi^{s} \, d\mu.
\]
Since \( \nabla u = 0 \) a.e. on the set \( M \setminus \Omega \), see [5] Lemma 7.7, we have
\[
I = \int_M \left[ \frac{t}{p} \varphi^{q-1} u_{\eta}^{-t} V^{\frac{q'}{q}} + V u^{s} u_{\eta}^{-t} \varphi^{s} \right] \varphi(t-1) \, d\mu - \int_M \frac{p - 1}{p} V \varphi(t-1) \, d\mu.
\]
Now note that \( \frac{t}{p} \varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} + V u^{s} u_{\eta}^{-t} \varphi^{s} = \frac{p - 1}{p} \varphi(t-1) \, d\mu \) converges a.e. in \( M \) to the function
\[
\frac{t}{p} \varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} + \frac{1}{p} V u^{s} u_{\eta}^{-t} \varphi^{s} \chi_{\Omega}
\]
as \( \eta \to 0^{+} \). By an application of Fatou’s lemma and using (2.6) we obtain
\[
\int_M \frac{t}{p} \varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} \chi_{\Omega} \, d\mu + \int_M \frac{1}{p} V u^{s} u_{\eta}^{-t} \varphi^{s} \, d\mu
\]
\[
= \int_M \left[ \frac{t}{p} \varphi^{q-1} u_{\eta}^{-t} V^{\frac{q'}{q}} + \frac{1}{p} V u^{s} u_{\eta}^{-t} \varphi^{s} \right] \chi_{\Omega} \, d\mu
\]
\[
\leq \liminf_{\eta \to 0^{+}} \frac{t}{p} \varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} + V u^{s} u_{\eta}^{-t} \varphi^{s} \, d\mu - \frac{p - 1}{p} \varphi(t-1) \, d\mu
\]
\[
\leq C\varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} \, d\mu + \liminf_{\eta \to 0^{+}} \int_M \frac{p - 1}{p} V \varphi(t-1) \, d\mu
\]
\[
= C\varphi^{q-1} u_{\eta}^{-t} |\nabla u|^{p} \, d\mu,
\]
that is inequality (2.3).
Lemma 2.4. Let $s \geq \frac{2\sigma}{p-1}$ be fixed. Then there exists a constant $C > 0$ such that for every nonnegative weak solution $u$ of equation (1.12), every function $\varphi \in \text{Lip}(M)$ with compact support and $0 \leq \varphi \leq 1$ and every $t \in (0, \min\{1, p-1, \frac{\sigma}{2(p-1)}\})$ one has

\[
(2.7) \int_M \varphi^s u^\sigma V \, dm \leq Ct^{-\frac{p-1}{p}} \left( \int_{M \setminus K} V^{-\frac{(t+1)(p-1)}{(t+1)(p-1)\sigma+1}} |\nabla \varphi|^\frac{\sigma}{\sigma+1} \, dm \right)^{\frac{p}{p-1}} \left( \int_{M \setminus K} \varphi^s u^\sigma V \, dm \right)^{\frac{p-1}{p}}
\]

with $K = \{x \in M : \varphi(x) = 1\}$ and $dm$ is the measure on $M$ with density $a$, as defined in (1.11).

**Proof of Lemma 2.4.** Under our assumptions $\psi = \varphi^s$ is a feasible test function in equation (2.2). Thus we obtain

\[
(2.8) \int_M \varphi^s u^\sigma V \, dm \leq \int_M s \varphi^{s-1} |\nabla u|^{p-1} \langle \nabla u, \nabla \varphi \rangle \, dm \leq \int_M s \varphi^{s-1} |\nabla u|^{p-1} |\nabla \varphi| \, dm.
\]

Now let $\Omega = \{x \in M : u(x) > 0\}$ and let $\chi_\Omega$ be the characteristic function of $\Omega$. Since $\nabla u = 0$ a.e. on the set $M \setminus \Omega$, through an application of Hölder’s inequality we obtain

\[
(2.9) \int_M s \varphi^{s-1} |\nabla u|^{p-1} |\nabla \varphi| \, dm = \int_M s \varphi^{s-1} |\nabla u|^{p-1} \chi_\Omega |\nabla \varphi| \, dm
\]

\[
= s \int_M \left( \varphi^\frac{s-1}{p-1} |\nabla u|^{p-1} \chi_\Omega \right) \left( \varphi^\frac{s-1}{p-1} |\nabla \varphi| \right) \, dm
\]

\[
\leq s \left( \int_M \varphi^{s-1} |\nabla u|^{p-1} \chi_\Omega \, dm \right)^{\frac{p-1}{p}} \left( \int_M \varphi^s u^\sigma |\nabla \varphi|^\sigma \, dm \right)^{\frac{1}{p}}.
\]

Moreover from equation (2.3) we deduce

\[
(2.10) \int_M \varphi^s |\nabla u|^p |\nabla \varphi|^\sigma \chi_\Omega \, dm \leq Ct^{-1} \int_M V^{-\frac{(p-1)\sigma}{p+1}} |\nabla \varphi|^\frac{\sigma(p+1)}{p+1} \, dm,
\]

with $C > 0$ depending on $s$. Thus from (2.8), (2.9) and (2.10) we obtain

\[
(2.11) \int_M \varphi^s u^\sigma V \, dm \leq C \left( t^{-1} \int_M V^{-\frac{(p-1)\sigma}{p+1}} |\nabla \varphi|^\frac{\sigma(p+1)}{p+1} \, dm \right)^{\frac{p}{p+1}} \left( \int_M \varphi^s u^\sigma |\nabla \varphi|^\sigma \, dm \right)^{\frac{1}{p}}.
\]

Now we use again Hölder’s inequality with exponents

\[
q = \frac{\sigma}{(t+1)(p-1)}, \quad q' = \frac{q}{q-1} = \frac{\sigma}{\sigma - (t+1)(p-1)}
\]

to obtain

\[
\int_M \varphi^{s-p}(p-1)(t+1)|\nabla \varphi|^p \, dm
\]

\[
= \int_{M \setminus K} \left( \varphi^s u^{(p-1)(t+1)} V^\frac{p-1}{p} \right) \left( \varphi^{s-p} V^{-\frac{1}{p}} |\nabla \varphi|^p \right) \, dm
\]

\[
\leq \left( \int_{M \setminus K} \varphi^s u^\sigma V \, dm \right)^{\frac{p+1}{p}} \left( \int_{M \setminus K} \varphi^{s-p} u^{(p-1)(t+1)} V^{-\frac{(p-1)\sigma}{p+1}} |\nabla \varphi|^\sigma \, dm \right)^{\frac{p-1}{p}}.
\]

Substituting into (2.11) we get

\[
\int_M \varphi^s u^\sigma V \, dm \leq Ct^{-\frac{p-1}{p}} \left( \int_M V^{-\frac{(p-1)\sigma}{p+1}} |\nabla \varphi|^\frac{\sigma(p+1)}{p+1} \, dm \right)^{\frac{p}{p-1}} \left( \int_{M \setminus K} \varphi^s u^\sigma V \, dm \right)^{\frac{p-1}{p}}.
\]
Now inequality (2.7) immediately follows from the previous relation, by our assumptions on \( s, t \) and since \( 0 \leq \varphi \leq 1 \).

From Lemma 2.4 we immediately deduce

**Corollary 2.5.** Under the same assumptions of Lemma 2.4 there exists a constant \( C > 0 \), independent of \( u, \varphi \) and \( t \), such that

\[
\int_M \varphi^n u^\alpha V \, d\mu \leq C t^{-\frac{p-1}{\sigma} - \frac{(p-1)^2}{\sigma(p+1)}} \left( \int_M V^{-\frac{p-1}{\sigma}} |\nabla \varphi|^{\frac{p(p-1)}{\sigma(p+1)}} \, d\mu \right)^{\frac{p-1}{p}} \left( \int_M V^{-\frac{t(1+p)(p-1)}{\sigma}} |\nabla \varphi|^{\frac{p(1+p)}{\sigma}} \, d\mu \right)^{\frac{t}{p}}.
\]

**Proof.** Inequality (2.12) easily follows from (2.7), since \( s \geq \frac{p\alpha}{\sigma - (1+p)(p-1)} \) and \( 0 \leq \varphi \leq 1 \) on \( M \).

3. PROOF OF THEOREM 1.5

We divide the proof of Theorem 1.5 in three cases, depending on which of the conditions (HP1), (HP2) or (HP3) is assumed to hold (see Definition 1.3).

**Proof of Theorem 1.5.** (a) Assume that condition (HP1) holds (see (1.13)). Let \( r(x) \) be the distance of \( x \in M \) from a fixed origin \( o \), for any fixed \( R > 0 \) sufficiently large let \( t = \frac{1}{\log R} \) and denote by \( B_R \) the metric ball centered at \( o \) with radius \( R \). Fix any \( C_1 \geq \frac{C_0 + p + 2}{p\sigma} \) with \( C_0 \) as in condition (1.13), define for \( x \in M \)

\[
\varphi(x) = \begin{cases} 
1 & \text{for } r(x) < R, \\
\left( \frac{r(x)}{R} \right)^{-C_1 t} & \text{for } r(x) \geq R
\end{cases}
\]

and for \( n \in \mathbb{N} \)

\[
\eta_n(x) = \begin{cases} 
1 & \text{for } r(x) < nR, \\
2 - \frac{r(x)}{nR} & \text{for } nR \leq r(x) \leq 2nR, \\
0 & \text{for } r(x) \geq 2nR.
\end{cases}
\]

Let

\[
\varphi_n(x) = \eta_n(x) \varphi(x) \quad \text{for } x \in M,
\]

then \( \varphi_n \in \text{Lip}(M) \) with \( 0 \leq \varphi_n \leq 1 \), we have

\[
|\nabla \varphi_n| \leq C_1 |\nabla \varphi| + C_1 |\nabla \eta_n| \quad \text{a.e. in } M
\]

and for every \( a \geq 1 \)

\[
|\nabla \varphi_n|^a \leq 2^{a-1} |(\nabla \varphi|^a + \varphi^a |\nabla \eta_n|^a) \quad \text{a.e. in } M.
\]

Now we use \( \varphi_n \) in formula (2.3) of Lemma 2.3 with any fixed \( s \geq \frac{p\sigma}{\sigma - p + 1} \) and deduce that, for some positive constant \( C \) and for every \( n \in \mathbb{N} \) and every small enough \( t > 0 \), we have

\[
\int_M V^{s-t} \varphi_n^a \, d\mu \leq C t^{-\frac{(p-1)a}{\sigma - p + 1}} \int_M V^{-\frac{s}{\sigma - p + 1}} |\nabla \varphi|^\frac{p(a-t)}{\sigma - p + 1} \, d\mu
\]

\[
= C t^{-\frac{(p-1)a}{\sigma - p + 1}} \int_M V^{-\frac{s}{\sigma - p + 1}} |\nabla \varphi|^\frac{p(a-t)}{\sigma - p + 1} \, d\mu
\]

\[
\leq C t^{-\frac{(p-1)a}{\sigma - p + 1}} \left[ \int_M V^{-\frac{s}{\sigma - p + 1}} |\nabla \varphi|^\frac{p(a-t)}{\sigma - p + 1} \, d\mu + \int_{B_{2n} \setminus B_{n}} V^{-\frac{s}{\sigma - p + 1}} \varphi^a |\nabla \eta_n|^\frac{a-t}{\sigma - p + 1} \, d\mu \right]
\]

\[
\leq C t^{-\frac{(p-1)a}{\sigma - p + 1}} [I_1 + I_2],
\]
where

\[ I_1 := \int_{M \setminus B_R} V^{-\beta + \frac{p(\sigma - t)}{\sigma - p + 1}} |\nabla \varphi|^\frac{p(\sigma - t)}{\sigma - p + 1} \, d\mu, \]

\[ I_2 := \int_{B_{2aR} \setminus B_{aR}} \varphi^\frac{p(\sigma - t)}{\sigma - p + 1} |\nabla \eta| \frac{p(\sigma - t)}{\sigma - p + 1} V^{-\beta + \frac{p(\sigma - t)}{\sigma - p + 1}} \, d\mu. \]

By (3.1), (3.2) and assumption (HP1) with \( \varepsilon = \frac{\beta}{\sigma - p + 1} \), see equation (1.13), for every \( n \in \mathbb{N} \) and every small enough \( t > 0 \) we have

\[ I_2 \leq \left( \sup_{B_{2aR} \setminus B_{aR}} \varphi \right)^\frac{p(\sigma - t)}{\sigma - p + 1} \left( \frac{nR}{R} \right)^\frac{\beta}{\sigma - p + 1} \int_{B_{2aR} \setminus B_{aR}} V^{-\beta + \frac{p(\sigma - t)}{\sigma - p + 1}} \, d\mu \]

\[ \leq C \left( \frac{nR}{R} \right)^\frac{\beta}{\sigma - p + 1} \left( \int_{B_{2aR} \setminus B_{aR}} V^{-\beta + \frac{p(\sigma - t)}{\sigma - p + 1}} \, d\mu \right)^\frac{\beta}{\sigma - p + 1} \]

\[ \leq C n^\alpha + \frac{C_0}{\alpha} \left( C_1 t^1 \right) R^\alpha - \frac{p(\sigma - t)}{\sigma - p + 1} \left( \log(2nR) \right)^k. \]

By our choice of \( C_1 \), for every small enough \( t > 0 \)

\[ \alpha + \frac{C_0 t}{\sigma - p + 1} - \frac{p(\sigma - t)}{\sigma - p + 1} (C_1 t + 1) = \frac{t(C_0 - p\sigma C_1 + pC_1 t + p)}{\sigma - p + 1} \leq - \frac{t}{\sigma - p + 1} < 0. \]

Moreover, since \( t = \frac{1}{\log R^k} \), we have

\[ R^\alpha - \frac{C_0 t}{\sigma - p + 1} = R^\alpha e^{-\frac{C_0 t}{\sigma - p + 1}} = R^\alpha e^{-\frac{C_0 t}{\sigma - p + 1}} \log R = e^{-\frac{C_0 t}{\sigma - p + 1}}. \]

In view of (3.5) and (3.6) for \( R > 1 \) large enough, and thus \( t = \frac{1}{\log R^k} \) small enough, we obtain

\[ I_2 \leq C n^\alpha - \frac{p(\sigma - t)}{\sigma - p + 1} \left( \log(2nR) \right)^k. \]

In order to estimate \( I_1 \) we recall that if \( f : [0, \infty) \to [0, \infty) \) is a nonnegative decreasing function and (1.13) holds, then for any small enough \( \varepsilon > 0 \) and any sufficiently large \( R > 1 \) we have

\[ \int_{M \setminus B_R} f(r(x))(V(x))^{-\beta + \varepsilon} \, d\mu \leq C \int_0^{+\infty} f(r) r^{\alpha + C_0 \varepsilon - 1} (\log r)^k \, dr \]

for some positive constant \( C \), see [9, formula (2.19)]. Moreover, there holds

\[ |\nabla \varphi| \leq C_1 t R^{C_1 t - r - C_1 t - 1}. \]

Thus, using (3.19) - (3.9),

\[ I_1 \leq \int_{M \setminus B_R} V^{-\beta + \frac{p(\sigma - t)}{\sigma - p + 1}} (R^C_1 t C_1 t r - C_1 t - 1) \left( \frac{\varphi}{\varphi - e^{-\frac{p(\sigma - t)}{\sigma - p + 1}}} \right) \, d\mu \]

\[ \leq C \int_0^{+\infty} R\left( R^{C_1 t} \right)^{C_1 t} (1 + C_1) \frac{\varphi}{\varphi - e^{-\frac{p(\sigma - t)}{\sigma - p + 1}}} \, dr \]

\[ \leq C \int_0^{+\infty} R\left( R^{C_1 t} \right)^{C_1 t} (1 + C_1) \frac{\varphi}{\varphi - e^{-\frac{p(\sigma - t)}{\sigma - p + 1}}} \, dr. \]

Now note that

\[ R^{C_1 t} = e^{\frac{p(\sigma - t)}{\sigma - p + 1}} < e^{\frac{p(\sigma - t)}{\sigma - p + 1}} \]

and that by our choice of \( C_1 \) we have

\[ a := - \frac{p(\sigma - t)}{\sigma - p + 1} (C_1 t + 1) + \alpha + C_0 \frac{t}{\sigma - p + 1} = \frac{t}{\sigma - p + 1} (pC_1 t - p\sigma C_1 + p + C_0) \leq - \frac{t}{\sigma - p + 1} < 0. \]
Then, by the above inequalities and performing the change of variables $\xi := |a| \log r$, we get
\begin{equation}
(3.10) \quad I_1 \leq C t^{\frac{p(\sigma - t)}{\sigma - p + 1}} \int_1^\infty r^{-\frac{p(\sigma - t)}{\sigma - p + 1}} (C_{1 t+1} + a) \frac{1}{r} \frac{dr}{r} d\mu
\end{equation}
\begin{align*}
&\leq C |a|^{-(k+1)} t^{\frac{p(\sigma - t)}{\sigma - p + 1}} \int_0^\infty e^{-\xi k} d\xi \\
&\leq C \left( \frac{t}{\sigma - p + 1} \right)^{-k-1} \left( \frac{p(\sigma - t)}{\sigma - p + 1} \right) \int_0^\infty e^{-\xi k} d\xi \\
&\leq C t^{\frac{p(\sigma - t)}{\sigma - p + 1}} \frac{k}{k - 1}.
\end{align*}
By (3.4), (3.7) and (3.10)
\begin{equation}
(3.11) \quad \int_{B_R} V u^{\sigma-t} d\mu \leq \int_M V u^{\sigma-t} \varphi^*_n d\mu
\end{equation}
\begin{align*}
&\leq C t^{-\frac{(n-1)\alpha}{\alpha} \log (2nR)^k t^\frac{p(\sigma - t)}{\sigma - p + 1} - k - 1.}
\end{align*}
Since $R > 1$ is large and fixed, and thus $t = \frac{1}{\log R} < 1$ is also fixed, taking the lim inf as $n \to \infty$ in (3.11) we obtain
\begin{equation}
(3.12) \quad \int_{B_R} V u^{\sigma-t} d\mu \leq C t^{\frac{p(\sigma - t)}{\sigma - p + 1}} \frac{k}{k - 1}.
\end{equation}
Observe that, for each small enough $t > 0,
\begin{align*}
\frac{p(\sigma - t)}{\sigma - p + 1} - k - 1 - \frac{(p - 1)\sigma}{\sigma - p + 1} &= \frac{p - 1}{\sigma - p + 1} - k - \frac{pt}{\sigma - p + 1} = \beta - k - \frac{pt}{\sigma - p + 1} \geq \delta_\ast > 0.
\end{align*}
Then, for any fixed sufficiently small $t > 0$, we have
\begin{align*}
\int_M V u^{\sigma-t} \chi_{B_R \setminus \varepsilon} \ d\mu &= \int_{B_R} V u^{\sigma-t} \ d\mu \leq C t^{\delta_\ast},
\end{align*}
By Fatou’s Lemma, taking the lim inf as $t \to 0^+$ in the previous inequality we obtain
\begin{align*}
\int_M V u^{\sigma} \ d\mu \leq 0,
\end{align*}
which implies $u \equiv 0$ in $M$.

(b) Assume that condition (HP2) holds (see (1.14)). Let the functions $\varphi$, $\eta_n$ and $\varphi_n$ be defined on $M$ as in formulas (3.1), (3.2) and (3.3) with $R > 1$ large enough, $t = \frac{1}{\log R}$, $C_1 \geq \max \left\{ C_0, C_0^{\frac{1}{p-1}}, C_0 + C_0 \alpha \right\}$ and $C_0$ as in condition (1.14). We now apply formula (2.12), using the family of functions $\varphi \in \operatorname{Lip}_0(M)$ and any fixed $s \geq \frac{2p\sigma}{\sigma - p + 1}$, and thus we have
\begin{align*}
\left( \int_M \varphi_n^s \ u^s \ V \ d\mu \right)^{\frac{1}{p(\sigma - t)}} &\leq C t^{-\frac{p(\sigma - t)}{\sigma - p + 1}} \left( \int_M V^{-\frac{s(\sigma - t)}{p(\sigma - t)}} \ |\nabla \varphi_n|^\frac{p(\sigma - t)}{p(\sigma - t)} \ d\mu \right)^{\frac{1}{p(\sigma - t)}}
\end{align*}
We now need to estimate
\begin{equation}
(3.13) \quad \int_M V^{-\frac{\beta - 1}{\alpha} \frac{(p(\sigma - t))}{p(\sigma - t)}} |\nabla \varphi|^\frac{p(\sigma - t)}{p(\sigma - t)} \ d\mu \quad \text{and} \quad \int_M V^{-\frac{(\beta - 1)(p - 1)}{p(\sigma - t)}} |\nabla \varphi|^\frac{p(\sigma - t)}{p(\sigma - t)} \ d\mu.
\end{equation}
Arguing as in the previous proof of the theorem under the validity of condition (HP1), with the only difference that the condition $k < \beta$ there is replaced here by $k = \beta$, using (1.14) we can deduce that
\begin{equation}
(3.14) \quad \int_M V^{-\frac{\beta - 1}{\alpha} \frac{(p(\sigma - t))}{p(\sigma - t)}} |\nabla \varphi_n|^\frac{p(\sigma - t)}{p(\sigma - t)} \ d\mu \leq C \left[ n^{-\frac{\beta - 1}{\alpha} \frac{1}{p(\sigma - t)}} (\log (2nR))^\beta q + t^{\frac{p(\sigma - t)}{\sigma - p + 1} - \beta - 1} \right].
\end{equation}
In order to estimate the second integral in (3.13) we start by defining \( \Lambda = \frac{(p-1)\sigma}{(\sigma-p+1)(\sigma-(t+1)(p-1))} \), and we note that

\[
(3.15) \quad \frac{(p-1)\sigma}{(\sigma-p+1)^2} t < \Lambda < \frac{2(p-1)\sigma}{(\sigma-p+1)^2} t < \varepsilon^*
\]

for every small enough \( t > 0 \), and that

\[
\frac{(t+1)(p-1)}{\sigma - (t+1)(p-1)} = \beta + \Lambda \quad \text{and} \quad \frac{p\sigma}{\sigma - (t+1)(p-1)} = \alpha + \Lambda p,
\]

with \( \alpha, \beta \) as in Definition 1.3. By our definition of the functions \( \varphi_n \), for every \( n \in \mathbb{N} \) and every small enough \( t > 0 \) we have

\[
(3.16) \quad \int_M V^{-\beta-\Lambda} |\nabla \varphi_n|^{\alpha+\Lambda p} d\mu \leq C \left( \int_M V^{-\beta-\Lambda} \varphi_n^{\alpha+\Lambda p} |\nabla \varphi|^{\alpha+\Lambda p} d\mu + \int_M V^{-\beta-\Lambda} \varphi^{\alpha+\Lambda p} |\nabla \eta_n|^{\alpha+\Lambda p} d\mu \right)
\]

\[
\leq C \left( \int_{M \setminus B_n} V^{-\beta-\Lambda} |\nabla \varphi|^{\alpha+\Lambda p} d\mu + \int_{B_{2n} \setminus B_n} V^{-\beta-\Lambda} \varphi^{\alpha+\Lambda p} |\nabla \eta_n|^{\alpha+\Lambda p} d\mu \right) := C(I_1 + I_2).
\]

Now we use condition (1.14) with \( \varepsilon = \Lambda \), and we obtain

\[
I_2 = \int_{B_{2n} \setminus B_n} V^{-\beta-\Lambda} \varphi^{\alpha+\Lambda p} |\nabla \varphi|^{\alpha+\Lambda p} d\mu \leq \left( \sup_{B_{2n} \setminus B_n} \varphi \right)^{\alpha+\Lambda p} \left( \frac{1}{nR} \right)^{\alpha+\Lambda p} \left( \int_{B_{2n} \setminus B_n} V^{-\beta-\Lambda} d\mu \right)
\]

\[
\leq C n^{-\alpha + \Lambda p} C_{1t} \left( \frac{1}{nR} \right)^{\alpha + \Lambda p} (2nR)^{\alpha + \Lambda p} \left( \log (2nR) \right)^\beta
\]

\[
\leq C n^{-\alpha + \Lambda p} C_{1t-p\Lambda + C_0 \Lambda} R^{-p\Lambda + C_0 \Lambda} \left( \log (2nR) \right)^\beta.
\]

By our definition of \( C_1, \Lambda \) and by relation (3.15), we easily find

\[
(3.17) \quad -C_1 (\alpha + \Lambda p) t - \Lambda p + C_0 < -\frac{p\alpha \sigma C_1}{\sigma - (t+1)(p-1)} + \frac{\sigma (p-1)C_0}{\sigma - (t+1)(p-1)(\sigma - p + 1)}
\]

\[
\leq \frac{\sigma (p-1)C_0}{\sigma - (t+1)(p-1)(\sigma - p + 1)} < 0,
\]

for any small enough \( t > 0 \). Moreover by (3.15), since \( t = \frac{1}{\log R} \), we have

\[
R^{-p\Lambda + C_0 \Lambda} \leq R^{C_0 \Lambda} \leq R^{\frac{2(p-1)C_0 t}{(\sigma - p + 1)}} = e^\frac{2(p-1)C_0}{(\sigma - p + 1)\log R}.
\]

Thus, for any sufficiently large \( R > 0 \),

\[
(3.18) \quad I_2 \leq C n^{-\frac{\sigma C_0}{\sigma - (\sigma - p + 1)\log (2nR)}} \beta.
\]

In order to estimate \( I_1 \) we note that if \( f : [0, \infty) \to [0, \infty) \) is a nonnegative decreasing function and (1.14) holds, then for any small enough \( \varepsilon > 0 \) and any sufficiently large \( R > 1 \) we have

\[
(3.19) \quad \int_{M \setminus B_n} f(r(x))(V(x))^{-\beta - \varepsilon} d\mu \leq C \int_{B_n} f(r) r^{2(p-1)C_1 t} (\log r)^\beta dr
\]

for some positive constant \( C \), see (3.19) and [3] formula (2.19). Thus, noting that \( |\nabla \varphi| \leq C_1 t R^{C_1 t} \), a.e. on \( M \) and using (3.17), for every small enough \( t > 0 \) we have

\[
I_1 \leq \int_{M \setminus B_n} V^{-\beta-\Lambda} (C_1 t R^{C_1 t} - C_1 t^{-1})^{\alpha+\Lambda p} d\mu
\]

\[
\leq C \int_{M \setminus B_n} R^{(\alpha + \Lambda p)C_1 t} (1 + C_1)^{\alpha + \Lambda p} [t^{(\alpha + \Lambda p)C_1 t} \varphi^{\alpha + \Lambda p} (\log r)^\beta] dr
\]
Now, since \( t = \frac{1}{\log R} \), by relation (3.15) we have
\[
R^{(\alpha + \Lambda p)\xi} t = e^{(\alpha + \Lambda p)\xi t} \leq e^{(\alpha + \alpha s)p\xi t},
\]
moreover, as we noted already in (3.17), for \( t > 0 \) small enough
\[
b = -(\alpha + \Lambda p)(C_1 t + 1) + \alpha + C_0 \Lambda < -\frac{C_0 t \sigma}{(\sigma - p + 1)^2} < 0.
\]
With the change of variables \( \xi = |b| \log r \), using the previous relations we find
\[
(3.20) \quad I_1 \leq C t^{\alpha + \Lambda p} \int_1^\infty r^k (\log r)^{\beta} \frac{dr}{r} = C t^{\alpha + \Lambda p} |b|^{-\beta - 1} \left( \int_0^\infty e^{-\xi} \xi^\beta d\xi \right)
\]
From equations (3.16), (3.18) and (3.20) it follows that
\[
\text{for every sufficiently small } n \text{ uniformly in } t,
\]
By taking the lim inf as \( n \to +\infty \) we get
\[
\left( \int_{B_R} u^\sigma V d\mu \right)^{1 - \frac{(p + 1)(\sigma - 1)}{p \sigma}} \leq C t^{-\frac{p - 1}{p} - \frac{(p - 1)(\sigma - 1)}{p \sigma}} \left[ n \left( \frac{p - 1}{\sigma - 1} - \frac{\alpha + \Lambda p}{\alpha + \Lambda p} \right) \left( \frac{p - 1}{\sigma - 1} - \frac{\beta + 1}{\beta - 1} \right) \log (2nR) \right]^{\frac{\beta - 1}{\beta - 1}}
\]
for every sufficiently small \( t > 0 \), with \( t = \frac{1}{\log R} \). But
\[
-\frac{p - 1}{p} - \frac{(p - 1)(\sigma - 1)}{p \sigma} + \frac{(p - 1)(\sigma - 1)}{\sigma - 1} = \frac{(\beta + 1)(p - 1)}{\beta - 1} + 1 - \frac{\beta + 1}{\alpha + \Lambda p} = -\frac{(p - 1)^2}{p(\sigma - p + 1)},
\]
hence for every small enough \( t > 0 \) we have
\[
\left( \int_{B_{1/t}} u^\sigma V d\mu \right)^{1 - \frac{(p + 1)(\sigma - 1)}{p \sigma}} \leq C t^{-\frac{(p - 1)(\sigma - 1)}{p(\sigma - p + 1)}} \leq C,
\]
that is
\[
\int_{B_{1/t}} u^\sigma V d\mu \leq C
\]
uniformly in \( t \), for \( t > 0 \) sufficiently small. By taking the limit for \( t \to 0^+ \) we deduce
\[
(3.22) \quad \int_M u^\sigma V d\mu < +\infty,
\]
and thus \( u \in L^\sigma(M, V\mu) \). Now we exploit inequality (2.7) with the cutoff function \( \varphi_n \), and using again (3.13) and (3.21) we obtain

\[
\int_M \varphi_n^s u^\sigma V\,d\mu \leq Ct^{-\frac{p-1}{p\rho}} \left[ n^{-\frac{1}{p\rho}} \left( \log(2nR) \right)^\beta + \int M \varphi_n^s u^\sigma V\,d\mu \right] \left( \int M \varphi_n^s u^\sigma V\,d\mu \right)^{\frac{p-1}{p}}.
\]

Using previous inequalities and taking the lim inf as \( n \to +\infty \) we get

\[
\int_{M} u^\sigma V\,d\mu \leq C \left( \int M \varphi_n^s u^\sigma V\,d\mu \right)^{\frac{p-1}{p}} \leq C \left( \int M \varphi_n^s u^\sigma V\,d\mu \right)^{\frac{p-1}{p}}.
\]

uniformly for \( t > 0 \) sufficiently small, with \( t = \frac{1}{\log R} \). Since \( u \in L^\sigma(M, V\mu) \),

\[
\int_{M} u^\sigma V\,d\mu \to 0 \quad \text{as} \quad R \to +\infty.
\]

Moreover \( \frac{(t+1)(p-1)}{p\rho} \to \frac{p-1}{p\rho} > 0 \) as \( R \to +\infty \). It follows that

\[
\int_{M} u^\sigma V\,d\mu = \lim_{R \to +\infty} \int_{M} u^\sigma V\,d\mu = 0,
\]

which implies \( u \equiv 0 \) on \( M \).

(c) Assume that condition (HP3) holds (see (1.15)). Consider the functions \( \varphi \), \( \eta_n \), and \( \varphi_n \) defined in (3.1), (3.2) and (3.3), with \( R > 0 \) large enough, \( t = \frac{1}{\log R} \), \( C_1 \geq \frac{C_0 + p + 2}{p\rho} \) and \( C_0 \) as in condition (1.15). Arguing as in the previous proof of the theorem under the assumption of the validity of (HP1), by formula (2.3) with any fixed \( s \geq \frac{p-1}{p\rho+1} \), we see that

\[
\int_{M} V^{\beta} \varphi_n^s \,d\mu \leq Ct^{-\frac{p-1}{p\rho}} \left[ \int M V^{\beta} |\nabla \varphi_n|^{\frac{p(\sigma-1)}{p\rho}} \,d\mu \right] \leq C t^{\frac{p-1}{p\rho}} \left[ \int_{M} V^{\beta} |\nabla \varphi_n|^{\frac{p(\sigma-1)}{p\rho}} \,d\mu + \int B_{2nR}\setminus B_{nR} V^{\beta} \varphi_n^{\frac{p(\sigma-1)}{p\rho}} |\nabla \eta_n|^{\frac{p(\sigma-1)}{p\rho}} \,d\mu \right]
\]

\[
:= Ct^{\frac{p-1}{p\rho}} (I_1 + I_2),
\]

for some positive constant \( C \) and for every \( n \in \mathbb{N} \) and every small enough \( t > 0 \). Now, recalling the definitions of \( \varphi \) and \( \eta_n \), by condition (1.19) with \( \varepsilon = \frac{1}{p\rho+1} \), for every small enough \( t > 0 \) we have

\[
I_2 \leq \sup_{B_{2nR}\setminus B_{nR}} \varphi \left( \frac{1}{nR} \right)^{\frac{p(\sigma-1)}{p\rho}} \int_{B_{2nR}\setminus B_{nR}} V^{\beta} \,d\mu 
\]

\[
\leq C n^{-\frac{p(\sigma-1)}{p\rho+1}} C_1 t \left( \frac{1}{nR} \right)^{\frac{p(\sigma-1)}{p\rho}} (2nR)^{\alpha} \frac{C_0 + p + 2}{p\rho} (\log(2nR))^k e^{-\frac{p-1}{p\rho+1}(\log(2nR))^\tau}
\]

\[
= C n^{-\frac{p(\sigma-1)}{p\rho+1}} C_1 t (2nR)^{\alpha} \frac{C_0 + p + 2}{p\rho} (\log(2nR))^k e^{-\frac{p-1}{p\rho+1}(\log(2nR))^\tau}.
\]
Note that, since \( t = \frac{1}{\log R} \), we have

\[
R^{C_0 t + \epsilon} = e^{C_0 t + \epsilon \log R} = e^{\epsilon \frac{C_0 + p \sigma}{\log R}}
\]

and that by our choice of \( C_1 \), if \( t > 0 \) is sufficiently small, we have \( (C_0 - p \sigma C_1 + p C_1 t + p) < -1 \). Thus we conclude that

(3.24)

\[
I_2 \leq C n^{-\frac{\epsilon}{\log R}} (\log (2nR))^k.
\]

In order to estimate \( I_1 \) we note that if \( f : [0, \infty) \to [0, \infty) \) is a nonnegative decreasing function and holds, then for any small enough \( \epsilon > 0 \) and any sufficiently large \( R > 0 \) we have

(3.25)

\[
\int_{M \setminus B_R} f(r(x))(V(x))^{-\beta + \epsilon} \, d\mu \leq C \int_{\frac{R}{2}}^{+\infty} f(r)e^{-\epsilon (r \theta (\log r)^\tau)} \, dr
\]

for some positive fixed constant \( C \). Indeed, by the monotonicity of the involved functions, using condition (1.15) we obtain in a similar way as [9] formula (2.19)

\[
\int_{M \setminus B_R} f(r(x))(V(x))^{-\beta + \epsilon} \, d\mu = \sum_{i=0}^{+\infty} \int_{B_{2i+1} \setminus B_{2i} \cap B_{2R}} f(r(x))(V(x))^{-\beta + \epsilon} \, d\mu
\]

\[
\leq \sum_{i=0}^{+\infty} f(2^i R) \int_{B_{2i+1} \setminus B_{2i} \cap B_{2R}} V^{-\beta + \epsilon} \, d\mu
\]

\[
\leq C \sum_{i=0}^{+\infty} f(2^i R) e^{-\epsilon \theta (\log (2^{i+1} R))^{-\tau}} (2^{i+1} R)^{\alpha + C_0 \epsilon} (\log (2^{i+1} R))^k
\]

\[
\leq C \sum_{i=0}^{+\infty} f(2^i R) e^{-\epsilon \theta (\log (2^{i+1} R))^{-\tau}} (2^{i+1} R)^{\alpha + C_0 \epsilon} (\log (2^{i+1} R))^k
\]

\[
\leq C \sum_{i=0}^{+\infty} \int_{2^{i+1} R}^{2^{i+2} R} f(r)e^{-\epsilon \theta (\log r)^\tau} r^{\alpha + C_0 \epsilon - 1} (\log r)^k \, dr
\]

\[
= C \int_{\frac{R}{2}}^{+\infty} f(r)e^{-\epsilon \theta (\log r)^\tau} r^{\alpha + C_0 \epsilon - 1} (\log r)^k \, dr.
\]

Now, since for a.e. \( x \in M \) we have

\[
|\nabla \varphi(x)| \leq C_1 t R^{C_1 t} (r(x))^{-C_1 t - 1},
\]

using (3.25) with \( \epsilon = \frac{t}{\sigma + p - 1} \), we obtain that for every small enough \( t > 0 \) with \( t = \frac{1}{\log R} \)

\[
I_1 \leq \int_{M \setminus B_R} V^{-\beta + \frac{\sigma - 1}{\log R}} (C_1 t R^{C_1 t} (r(x))^{-C_1 t - 1}) \frac{1}{\varphi(x)} \, d\mu
\]

\[
\leq C \int_{\frac{R}{2}}^{+\infty} R^\frac{\sigma - 1}{\sigma + p - 1} C_1^\frac{\sigma - 1}{\sigma + p - 1} \frac{1}{\varphi(x)} \, dr
\]

\[
\leq C \int_{\frac{R}{2}}^{+\infty} R^\frac{\sigma - 1}{\sigma + p - 1} (C_0 - p \sigma C_1 + p C_1 t + p) \frac{1}{\varphi(x)} \, dr.
\]

Note that \( C_1^\frac{\sigma - 1}{\sigma + p - 1} \leq (1 + C_1)^{\frac{\sigma - 1}{\sigma + p - 1}} \) and that

\[
R^\frac{\sigma - 1}{\sigma + p - 1} = R^\frac{\sigma - 1}{\sigma + p - 1} \leq e^{\frac{\sigma - 1}{\sigma + p - 1}} \leq e^{\frac{\sigma - 1}{\sigma + p - 1}}.
\]

Thus, with the change of variable \( r = e^\xi \), we deduce

\[
I_1 \leq C I_1^\frac{\sigma - 1}{\sigma + p - 1} \int_{1}^{+\infty} e^{-\frac{\sigma - 1}{\sigma + p - 1} (C_0 - p \sigma C_1 + p C_1 t + p)} (\log r)^k e^{-\frac{\sigma - 1}{\sigma + p - 1} (\log r)^\tau} r^{-1} \, d\xi.
\]

\[
= C I_1^\frac{\sigma - 1}{\sigma + p - 1} \int_{0}^{+\infty} e^{-\frac{\sigma - 1}{\sigma + p - 1} (C_0 - p \sigma C_1 + p C_1 t + p) \xi} \xi^{k} e^{-\frac{\sigma - 1}{\sigma + p - 1} \xi^\tau} \, d\xi.
\]
Now recall that by our choice of $C_1$, for $t > 0$ small enough, we have $(C_0 - p_\sigma C_1 + p C_1 t + p) < 0$. Hence, setting $\rho = \left(\frac{t_0}{\sigma - p + 1}\right)^{\frac{1}{t}}$, we have

\begin{equation}
I_1 \leq C t^{\frac{\rho(\sigma - t)}{\sigma - p + 1}} \int_0^{+\infty} \xi^k e^{-\frac{\rho t}{\sigma - p + 1}} d\xi = C t^{\frac{\rho(\sigma - t)}{\sigma - p + 1}} \int_0^{+\infty} \rho^k e^{-\rho^t} d\rho \leq C t^{\frac{\rho(\sigma - t)}{\sigma - p + 1}} + \frac{k + 1}{\sigma - p + 1}.
\end{equation}

From (3.23), (3.24) and (3.26) we conclude that for every $n \in \mathbb{N}$ and every small enough $t = \frac{1}{\log R} > 0$ we have

\begin{equation}
\int_{B_n} V u^{\sigma - t} d\mu \leq \int_{M} V u^{\sigma - t} \varphi_n^s \, d\mu \leq C t^{-\frac{(n+1)p}{\sigma - p + 1} + \frac{p \sigma}{\sigma - p + 1} + \frac{k + 1}{(\sigma - p + 1)}} + n^{-\frac{p}{(\sigma - p + 1)}(\log(2nR))^k}
\end{equation}

for some fixed positive constant $C$. Passing to the limit as $n \to +\infty$ in the previous relation yields

\begin{equation}
\int_{B_n} V u^{\sigma - t} d\mu \leq C t^{-\frac{(n+1)p}{\sigma - p + 1} + \frac{p \sigma}{\sigma - p + 1} + \frac{k + 1}{(\sigma - p + 1)}}.
\end{equation}

Now note that by our assumptions on $\tau, k$ we have

\begin{equation}
-\frac{(p - 1)\sigma + p(\sigma - t)}{\sigma - p + 1} = \frac{\sigma}{\sigma - p + 1} - \frac{(p \tau + 1)}{\sigma - p + 1} \geq \frac{1}{2} \frac{\sigma}{\sigma - p + 1} - \frac{k + 1}{\tau} \geq 0
\end{equation}

for every small enough $t = \frac{1}{\log R} > 0$. Thus (3.27) yields

\begin{equation}
\int_{B_{1/t}} V u^{\sigma - t} d\mu \leq C t^{\delta_*}
\end{equation}

for every small enough $t > 0$. Passing to the lim inf as $t$ tends to $0^+$ in (3.28), we conclude by an application of Fatou's Lemma that

\begin{equation}
\int_{M} V u^{\sigma} d\mu = 0,
\end{equation}

so that $u \equiv 0$ on $M$.

### 4. A PROBLEM WITH LOWER ORDER TERMS

In this subsection we consider the semilinear equation

\begin{equation}
\frac{1}{a(x)} \text{div} (a(x) \nabla u) + b(x)u + V(x)u^\sigma \leq 0 \quad \text{on } M.
\end{equation}

We start with the following lemma.

**Lemma 4.1.** Let $u \in W^{1,2}_{\text{loc}}(M) \cap L^p_{\text{loc}}(M, V d\mu_0)$ be a nonnegative weak solution of (4.1), with a satisfying $L^\infty$, $\sigma > 1$, $V > 0$ a.e. on $M$, $V \in L^1_{\text{loc}}(M)$ and $b \in L^\infty_{\text{loc}}(M)$. Assume there exists a weak solution $z > 0, z \in \text{Lip}_{\text{loc}}(M)$ of

\begin{equation}
\frac{1}{a(x)} \text{div} (a(x) \nabla z) + b(x)z \geq 0 \quad \text{on } M.
\end{equation}

Then $w := \frac{z}{2} \in W^{1,2}_{\text{loc}}(M) \cap L^p_{\text{loc}}(M, V d\mu_0)$ is a nonnegative weak solution of

\begin{equation}
\frac{1}{a(x)z^2(x)} \text{div} (a(x)z^2(x) \nabla w) + V(x)z^{\sigma - 1}(x)w^\sigma \leq 0 \quad \text{on } M.
\end{equation}

**Proof.** By our assumptions, for every $\varphi \in W^{1,2}(M) \cap L^\infty(M)$ with compact support and $\varphi \geq 0$ a.e. on $M$ we have

\begin{equation}
-\int_{M} \langle \nabla u, \nabla \varphi \rangle \, d\mu + \int_{M} bu \varphi \, d\mu + \int_{M} V u^\sigma \varphi \, d\mu \leq 0,
\end{equation}

\begin{equation}
-\int_{M} \langle \nabla z, \nabla \varphi \rangle \, d\mu + \int_{M} bz \varphi \, d\mu \geq 0.
\end{equation}

We explicitly note that, by our assumptions, all the integrals in (4.4) and (4.5) are finite. Moreover, by a density argument, we easily see that inequality (4.5) also holds for every \( \varphi \in W^{1,2}(M) \) with compact support and \( \varphi \geq 0 \) a.e. on \( M \), not necessarily bounded.

Now we fix \( \psi \in W^{1,2}(M) \cap L^{\infty}(M) \) with compact support and \( \psi \geq 0 \) a.e. on \( M \), and use \( \varphi = z\psi \in W^{1,2}(M) \cap L^{\infty}(M) \) as a test function in (4.4) and \( \varphi = u\psi \in W^{1,2}(M) \) as a test function in (4.5).

Subtracting the resulting inequalities one finds

\[
(4.6) \quad - \int_M \langle \nabla u, \nabla \psi \rangle z \, d\mu + \int_M \langle \nabla z, \nabla \psi \rangle u \, d\mu + \int_M Vu^\tau \psi z \, d\mu \leq 0.
\]

Since \( w = \frac{\psi}{z} \in W^{1,2}(M) \cap L^{\infty}(M, V \, d\mu) \) with

\[
\nabla w = \frac{1}{z} \nabla u - \frac{u}{z^2} \nabla z \quad \text{a.e. on} \ M,
\]

inequality (4.6) becomes

\[
- \int_M \langle \nabla w, \nabla \psi \rangle a_z^2 \, d\mu_0 + \int_M (V u^\tau - w^\tau) a_z^2 \, d\mu_0 \leq 0.
\]

Then, see also Remark (2.2), \( w \) is a nonnegative weak solution of (4.3).

Combining Lemma (4.1) with Theorem (1.5), one can easily obtain the following nonexistence results for nontrivial nonnegative weak solutions of equation (4.1).

**Proposition 4.2.** Assume there exists a weak solution \( z > 0, \ z \in \text{Lip}_{\text{loc}}(M) \) of equation (4.2) and let \( a \) satisfy (1.3), \( \sigma > 1, \ V > 0 \) a.e. on \( M, \ V \in L^1(M) \) and \( b \in L^{\infty}(M) \). Then any nonnegative weak solution \( u \in W^{1,2}(M) \cap L^{\infty}(M, V \, d\mu) \) of (4.1) is identically null, provided one of the following conditions holds:

i) there exist \( C_0 > 0, \ k \in [0, \frac{1}{\sigma - 1}) \) such that, for every large enough \( R > 0 \) and every \( \varepsilon > 0 \) sufficiently small,

\[
\int_{B_R} V^{-\frac{1}{\sigma} + \varepsilon} a_z^2 \, d\mu_0 \leq CR^\frac{2}{\sigma} + C_0 \varepsilon (\log R)^k, \quad \text{or}
\]

ii) there exists \( C_0 > 0 \) such that, for every large enough \( R > 0 \) and every \( \varepsilon > 0 \) sufficiently small,

\[
\int_{B_R} V^{-\frac{1}{\sigma} + \varepsilon} a_z^2 \, d\mu_0 \leq CR^\frac{2}{\sigma} + C_0 \varepsilon (\log R)^k \quad \text{and}
\]

\[
\int_{B_R} V^{-\frac{1}{\sigma} - \varepsilon} a_z^2 \, d\mu_0 \leq CR^\frac{2}{\sigma} + C_0 \varepsilon (\log R)^k \quad \text{or}
\]

iii) there exist \( C_0 \geq 0, \ k \geq 0, \ \theta > 0, \ \tau > \max\{\frac{\sigma - 1}{\sigma}(k + 1), 1\} \) such that, for every large enough \( R > 0 \) and every \( \varepsilon > 0 \) sufficiently small,

\[
\int_{B_{2R} \setminus B_R} V^{-\frac{1}{\sigma} + \varepsilon} a_z^2 \, d\mu_0 \leq CR^\frac{2}{\sigma} + C_0 \varepsilon (\log R)^k \mathcal{e}^{-\theta(\log R)^\tau}.
\]

We now proceed to describe a general setting where one can indeed produce the desired auxiliary function \( z \), in the particular case when \( a \equiv 1 \) on \( M \).

Let us fix a point \( o \in M \) and denote by \( \text{Cut}(o) \) the cut locus of \( o \). For any \( x \in M \setminus [\text{Cut}(o) \cup \{o\}] \), one can define the polar coordinates with respect to \( o \), see e.g. [7]. Namely, for any point \( x \in M \setminus [\text{Cut}(o) \cup \{o\}] \) there correspond a polar radius \( r(x) := \text{dist}(x, o) \) and a polar angle \( \theta \in \mathbb{S}^{m-1} \) such that the shortest geodesics from \( o \) to \( x \) starts at \( o \) with the direction \( \theta \) in the tangent space \( T_o M \). Since we can identify \( T_o M \) with \( \mathbb{R}^m \), \( \theta \) can be regarded as a point of \( \mathbb{S}^{m-1} \).

The Riemannian metric in \( M \setminus [\text{Cut}(o) \cup \{o\}] \) in the polar coordinates reads

\[
ds^2 = dr^2 + A_{ij}(r, \theta) d\theta^i \, d\theta^j,
\]
where \((\theta^1, \ldots, \theta^{m-1})\) are coordinates in \(S^{m-1}\) and \((A_{ij})\) is a positive definite matrix. It is not difficult to see that the Laplace-Beltrami operator in polar coordinates has the form

\[
\Delta = \frac{\partial^2}{\partial r^2} + F(r, \theta) \frac{\partial}{\partial r} + \Delta S_r,
\]

where \(F(r, \theta) := \frac{\partial}{\partial r} \left( \log \sqrt{A(r, \theta)} \right)\), \(A(r, \theta) := \det(A_{ij}(r, \theta))\), \(\Delta S_r\) is the Laplace-Beltrami operator on the submanifold \(S_r := \partial B(o, r) \setminus \text{Cut}(o)\).

\(M\) is a manifold with a pole, if it has a point \(o \in M\) with \(\text{Cut}(o) = \emptyset\). The point \(o\) is called pole and the polar coordinates \((r, \theta)\) are defined in \(M \setminus \{o\}\).

A manifold with a pole is a spherically symmetric manifold or a model, if the Riemannian metric is given by

\[
d s^2 = dr^2 + \psi^2(r) d\theta^2,
\]

where \(d\theta^2\) is the standard metric in \(S^{m-1}\), and

\[
\psi \in \mathcal{A} := \left\{ f \in C^\infty((0, \infty)) \cap C^1([0, \infty)) : f'(0) = 1, f(0) = 0, f > 0 \text{ in } (0, \infty) \right\}.
\]

In this case, we write \(M \equiv M_\psi\); furthermore, we have \(\sqrt{A(r, \theta)} = \psi^{m-1}(r)\), so the boundary area of the geodesic sphere \(\partial S_R\) is computed by

\[
\omega_m := \int_0^R S(\xi) d\xi.
\]

Moreover we have

\[
\Delta = \frac{\partial^2}{\partial r^2} + (n - 1) \frac{\psi'}{\psi} \frac{\partial}{\partial r} + \frac{1}{\psi^2} \Delta_{S^{m-1}},
\]

or equivalently

\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'}{S} \frac{\partial}{\partial r} + \frac{1}{\psi^2} \Delta_{S^{m-1}},
\]

where \(\Delta_{S^{m-1}}\) is the Laplace-Beltrami operator in \(S^{m-1}\).

Observe that for \(\psi(r) = r\), \(M = \mathbb{R}^m\), while for \(\psi(r) = \sinh r\), \(M\) is the \(m\)-dimensional hyperbolic space \(\mathbb{H}^m\).

Let us recall some useful comparison results for sectional and Ricci curvatures, that will be used in the sequel. For any \(x \in M \setminus \left[ \text{Cut}(o) \cup \{o\} \right]\), denote by \(\text{Ric}_o(x)\) the Ricci curvature at \(x\) in the direction \(\frac{\partial}{\partial r}\). Let \(\omega\) denote any pair of tangent vectors from \(T_x M\) having the form \((\frac{\partial}{\partial r}, X)\), where \(X\) is a unit vector orthogonal to \(\frac{\partial}{\partial r}\). Denote by \(K_\omega(x)\) the sectional curvature at the point \(x \in M\) of the 2-section determined by \(\omega\). If \(M \equiv M_\psi\) is a model manifold, then for any \(x = (r, \theta) \in M \setminus \{o\}\)

\[
K_\omega(x) = -\frac{\psi''(r)}{\psi(r)},
\]

and

\[
\text{Ric}_o(x) = -(m - 1) \frac{\psi''(r)}{\psi(r)}.
\]

Observe moreover that (see \([10, 11, 7\text{ Section 15}]\)), if \(M\) is a manifold with a pole \(o\) and

\[
K_\omega(x) \leq -\frac{\psi''(r)}{\psi(r)} \quad \text{for all } x = (r, \theta) \in M \setminus \{o\},
\]

for some function \(\psi \in \mathcal{A}\), then

\[
F(r, \theta) \geq (m - 1) \frac{\psi'(r)}{\psi(r)} \quad \text{for all } r > 0, \theta \in S^{m-1}.
\]
On the other hand, if $M$ is a manifold with a pole $o$ and
\begin{equation}
\text{Ric}_o(x) \geq -(m-1)\frac{\psi''(r)}{\psi(r)} \quad \text{for all } x = (r, \theta) \in M \setminus \{o\},
\end{equation}
for some function $\psi \in \mathcal{A}$, then
\begin{equation}
\mathcal{F}(r, \theta) \leq (m-1)\frac{\psi'(r)}{\psi(r)} \quad \text{for all } r > 0, \theta \in S^{m-1}.
\end{equation}

We have the following

**Lemma 4.3.** Let $M$ be a manifold with a pole $o$ and $b \in L^2_{\text{loc}}(M)$. Let $b_0 : \mathbb{R}^+ \to \mathbb{R}$ be such that
\begin{equation}
b(r, \theta) \geq b_0(r) \quad \text{for all } x = (r, \theta) \in M \setminus \{o\}.
\end{equation}
Assume that $\psi \in \mathcal{A}$, that $\zeta : \mathbb{R}^+ \to \mathbb{R}$ is a positive weak solution in $\text{Lip}_{\text{loc}}(\mathbb{R}^+)$ of
\begin{equation}
(\psi^{m-1}\zeta')' + b_0\psi^{m-1}\zeta \geq 0 \quad \text{in } \mathbb{R}^+,
\end{equation}
and that either
\begin{enumerate}[(A)]
\item $\psi$ satisfies condition 4.10 and $\zeta$ is nondecreasing,
\item $\psi$ satisfies condition 4.12 and $\zeta$ is nonincreasing.
\end{enumerate}
Then $z(x) := \zeta(r(x)) \in \text{Lip}_{\text{loc}}(M)$ is a positive weak solution of 4.2, with $a \equiv 1$ on $M$.

**Proof.** In case condition (A) holds, the result is an easy consequence of 4.7, 4.11, the monotonicity of $\zeta$ and condition 4.14. Similarly, when condition (B) holds, the result follows immediately as before, using 4.13 in place of 4.11. □

We refer the interested reader to the stimulating paper of Bianchini, Mari, Rigoli [1] for results concerning the existence of a positive solution of 4.15 and its precise asymptotic behavior as $r$ tends to $+\infty$. These combined with Lemma 4.3 and Proposition 4.2 give somehow explicit nonexistence results for equation 4.1.

5. Counterexamples

In this section, we will produce three counterexamples to the previous nonexistence results, all in the particular case of equation 1.1. Here we follow a similar approach as one finds in [7] and [9]. In the sequel, $\alpha = \frac{2m}{\sigma-1}$ and $\beta = \frac{1}{\sigma-1}$ as in Definition 1.3 with $p = 2$, while $M$ will always denote a model manifold with a pole $o$ and metric given by (4.7). Set $B_R \equiv B_R(o)$ and $r \equiv r(x) = \text{dist}(x, o)$ for any $x \in M$.

Let $\text{spec}(-\Delta)$ be the spectrum in $L^2(M)$ of the operator $-\Delta$. Note that (see [7, Section 10])
\begin{equation}
\text{spec}(-\Delta) \subseteq [0, \infty).
\end{equation}
Denote by $\bar{\lambda}(M)$ the bottom of $\text{spec}(-\Delta)$, that is
\begin{equation}
\bar{\lambda}(M) := \inf \text{spec}(-\Delta).
\end{equation}
By [2], for each fixed $x \in M$, there holds
\begin{equation}
\bar{\lambda}(M) \leq \frac{1}{4} \left\lfloor \limsup_{R \to +\infty} \frac{\log \mu(B_R(x))}{R} \right\rfloor^2.
\end{equation}
For any $\rho > 0$, let $\lambda_\rho$ be the first Dirichlet eigenvalue of the Laplace operator in $B_\rho$, that is the smallest number $\lambda_\rho$ for which the problem
\begin{equation}
\begin{cases}
\Delta u + \lambda u = 0 & \text{in } B_\rho, \\
u = 0 & \text{on } \partial B_\rho.
\end{cases}
\end{equation}
has a non-zero solution. Indeed, $\lambda_\rho$ coincides with the bottom of the spectrum of the operator $-\Delta$ in $L^2(B_\rho)$ with domain $C^0_0(B_\rho)$. It is easily checked, see e.g. [9], that $\lambda_\rho \geq 0$; moreover, $\lambda_{\rho_1} \geq \lambda_{\rho_2}$ if $\rho_1 < \rho_2$, and $\lambda_\rho \to \lambda(M)$ as $\rho \to \infty$.

In the sequel we shall make use of the following result (see [9]).

**Proposition 5.1.** Let $\sigma > 1$, $r_0 > 0$, $A \in C^1((r_0, \infty))$ with $A > 0$ and \( \int_{r_0}^\infty \frac{dr}{A(r)} < \infty \). Let $B \in C((r_0, \infty))$ be such that
\[
\int_{r_0}^\infty |\gamma(r)|^\sigma B(r) dr < \infty,
\]
where
\[
\gamma(r) := \int_r^\infty \frac{d\xi}{A(\xi)} \quad \text{for } r \geq r_0.
\]
Then the equation
\[
(A(r)y')' + B(r)y^\sigma = 0 \quad \text{for } r > R_0,
\]
for $R_0 > r_0$ sufficiently large, admits a positive solution $y(r)$ such that $y(r) \sim \gamma(r)$ as $r \to \infty$.

**Example 5.2.** Let $\psi \in A$, see [9], with
\[
\psi(r) := \begin{cases}
 r & \text{if } 0 \leq r < 1, \\
 [r^{\alpha-1}(\log r)^{\beta_0}]^{\frac{1}{\alpha-1}} & \text{if } r > 2;
\end{cases}
\]
here $\beta_0 > \beta$. Let $0 < \delta < \beta_0 - \beta$ and define
\[
V(x) \equiv V(r) := (\log(2 + r))^{\delta} \quad \text{for all } x \in M.
\]
For any $R > 0$ sufficiently large we have
\[
S(R) = \omega_m R^{\alpha-1}(\log R)^{\beta_0}, \quad \mu(B_R) \simeq CR^\alpha(\log R)^{\beta_0};
\]
thus, thanks to (5.1), we have $\lambda(M) = 0$. Moreover, there holds
\[
(5.3) \quad \int_{B_R} V^{-\beta}(x) \, d\mu \geq CR^\alpha(\log R)^{\beta_0 - \delta},
\]
with $\beta_0 - \delta > \beta$. Hence, in view of (5.3), neither condition (1.18) nor condition (1.19) is satisfied. Furthermore, observe that (1.18) holds true, while (1.19) fails. This is essentially due to the choice of $\psi$.

Note that for any $r_0 > 0$,
\[
(5.4) \quad \int_{r_0}^\infty \frac{d\xi}{S(\xi)} < \infty;
\]
moreover, for $r > 0$ sufficiently large,
\[
\gamma(r) := \int_r^\infty \frac{d\xi}{S(\xi)} \simeq \frac{C}{r^{\alpha-2}(\log r)^{\beta_0}}.
\]
Hence for $r_0 > 0$ large enough
\[
(5.5) \quad \int_{r_0}^\infty |\gamma(r)|^\sigma V(r) S(r) \, dr \leq C \int_{r_0}^\infty \frac{[\log(2 + r)]^\beta (\log r)^{\beta_0}}{r^{\alpha-2}(\log r)^{\beta_0} \sigma} \, dr
\leq C \int_{r_0}^\infty (\log r)^{\beta_0 (1-\sigma) + \frac{\delta}{\sigma}} \frac{dr}{r} < \infty.
\]
In view of (5.4)–(5.5), from Proposition 5.1 with $A(r) = S(r)$ and $B(r) = S(r)V(r)$, we have that there exists a solution $y = y(r)$ of
\[
y''(r) + \frac{S'(r)}{S(r)} y'(r) + V(r) |y(r)|^\sigma = 0, \quad r > R_0,
\]
for some $R_0 > r_0$. Furthermore, $y(r) > 0$ for all $r \in [R_0, \infty)$ and $y(r) \sim \gamma(r)$ as $r \to \infty$.

Now for any $\rho > 0$, let $v_\rho$ be the solution to the eigenvalue problem (5.2) with $\lambda = \lambda_\rho$, which we can assume is normalized by setting $v_\rho(0) = 1$. Thus we have that $v_\rho \equiv v_\rho(r)$, that $0 < v_\rho(r) \leq 1$ and that
\( v_\rho(r) \) is decreasing for \( r \in [0, \rho] \). For any \( \rho > R_0 \), define \( m := \inf_{(R_0, \rho)} \frac{y(r)}{v_\rho(r)} \) and for any fixed \( \xi \in (R_0, \rho) \) let
\[
\tilde{u}(x) := \begin{cases} mv_\rho(r) & \text{in } B_\xi, \\
y(r) & \text{on } \partial B_\xi. \end{cases}
\]
Since \( \lambda(M) = 0 \), as in [9] one can prove that for some \( \xi \in (R_0, \rho) \) we have \( \tilde{u} \in C^1(M) \), and thus \( \tilde{u} \in W^{1}_{\text{loc}}(M) \). Moreover
\[
\Delta (mv_\rho) + \frac{\lambda_\rho}{m^2} (mv_\rho)'' = 0 \quad \text{in } B_\rho, \\
\Delta y + Vy'' = 0 \quad \text{in } B_{R_0}.
\]
Define \( M_\rho = \max_{B_\rho} V \) and \( \delta = \min \{ 1, m^{-1} \lambda_\rho M^{-\frac{1}{\sigma - 1}} \} \), then, since \( V > 0 \) on \( M \), \( \delta > 0 \) and on \( B_\rho \) we have
\[
\Delta (\delta mv_\rho) + V (\delta mv_\rho)'' \leq \Delta (\delta mv_\rho) + M_\rho (\delta mv_\rho)'' \leq \Delta (\delta mv_\rho) + \frac{\lambda_\rho}{(\delta m)^{\frac{1}{\sigma - 1}}} (\delta mv_\rho)'' = 0.
\]
On the other hand on \( B_{R_0} \) we have
\[
\Delta (\delta y) + V (\delta y)'' \leq \delta \Delta y + \delta Vy'' = 0.
\]
Thus we see that the function \( u = \delta \tilde{u} \) is positive and satisfies
\[
\Delta u + Vu'' \leq 0 \quad \text{on } M.
\]

**Example 5.3.** Let \( \psi \in A \) with
\[
\psi(r) := \begin{cases} r & \text{if } 0 \leq r < 1, \\
[r^{\alpha - 1} (\log r)^{\beta}] \frac{1}{m^{\frac{1}{\sigma - 1}}} & \text{if } r > 2. \end{cases}
\]
Let \( \delta > 0 \) and define \( V(x) \equiv V(r) := (\log (2 + r))^{-\frac{\beta}{\sigma}} \) for all \( x \in M \).

For any \( R > 0 \) sufficiently large we have
\[
S(R) = \omega_\alpha R^{\alpha - 1} (\log R)^{\beta}, \quad \mu(B_R) \approx CR^\alpha (\log R)^{\beta},
\]
thus, thanks to (5.1), we conclude that \( \lambda_1(M) = 0 \). Moreover, there holds
\[
\int_{B_R} V^{-\beta}(x) \, d\mu \geq CR^\alpha (\log R)^{\beta + \delta}.
\]
Observe that in view of (5.7), neither condition (1.13) nor condition (1.14) is satisfied. Moreover, note that (1.18) holds, while (1.19) fails. This is essentially due to the choice of \( V \).

For any \( r_0 > 0 \) we have
\[
\int_{r_0}^{\infty} \frac{d\xi}{S(\xi)} < \infty;
\]
moreover, for \( r > 0 \) large enough,
\[
\gamma(r) := \int_{r}^{\infty} \frac{d\xi}{S(\xi)} \simeq \frac{C}{r^{\alpha - 2} (\log r)^{\beta}}.
\]
Hence for \( r_0 > 0 \) large enough
\[
\int_{r_0}^{\infty} \left[ \frac{\log(2 + r)}{r^{\sigma(n-2)}} \right]^{\frac{1}{\beta}} \frac{1}{r} \, dr < \infty.
\]

In view of (5.8)-(5.9), from Proposition 5.1 with \( A(r) = S(r) \) and \( B(r) = S(r) V(r) \), we have that there exists a solution \( y = y(r) \) of (5.6), for some \( R_0 > 0 \). Furthermore, \( y(r) > 0 \) for all \( r \in [R_0, \infty) \) and \( y(r) \sim \gamma(r) \) as \( r \to \infty \). Since \( \lambda(M) = 0 \) and \( V > 0 \) on \( M \), by the same arguments as in the previous Example 5.2 we can construct \( u \in C^1(M) \), with \( u = u(r) > 0 \) on \( M \), which satisfies
\[
\Delta u + Vu^\sigma \leq 0 \quad \text{in} \ M.
\]

**Example 5.4.** Let \( \psi \in \mathcal{A} \) with
\[
\psi(r) := \begin{cases} 
\frac{r}{e^{\sqrt{r}}} & \text{if } 0 \leq r < 1, \\
 e^{\sqrt{r}} & \text{if } r > 2.
\end{cases}
\]

For any sufficiently large \( R > 0 \) we have
\[
S(R) = \omega_m e^{(m-1)\sqrt{r}}, \quad \mu(B_R) \approx C e^{(m-1)\sqrt{R}} [m - 1] \sqrt{R - 1},
\]
for some \( C > 0 \). Note that \( \lambda(M) = 0 \) by (5.1), since
\[
\limsup_{R \to \infty} \frac{\log(\mu(B_R))}{R} = 0.
\]

Let
\[
\eta = \frac{m-1}{\beta} = (m-1)(\sigma-1), \quad \theta = \frac{\sigma+1}{\sigma-1}
\]
and define
\[
V(x) = V(r) := e^{\eta \sqrt{r}} (1 + r)^{-\frac{\theta}{2}} \quad \text{for all } x \in M.
\]

Then for \( \varepsilon > 0 \) small enough and \( R > 0 \) sufficiently large we have
\[
\int_{B_R} V^{-\beta + \varepsilon}(x) \, d\mu \geq C \int_{2}^{R} e^{\eta \sqrt{r}} (1 + r)^{\theta - \varepsilon/\beta} \, dr \geq C e^{\varepsilon \sqrt{r}}.
\]

On the other hand
\[
\int_{B_R} V^{-\beta - \varepsilon}(x) \, d\mu \leq C \int_{0}^{R} e^{-\eta \sqrt{r}} (1 + r)^{\theta + \varepsilon/\beta} \, dr \leq C R^{\theta + \varepsilon/2} = C R^{\varepsilon/2}.
\]

Observe that, in view of (5.10), neither condition (1.13) nor the first inequality in condition (1.14) is satisfied. On the other hand, by (5.11) the second inequality in (1.14) holds. This is essentially due to the choice of \( V \). Note moreover that only the second inequality in (1.19) is not satisfied, while the first inequality in (1.18) and (1.19) hold.

Note that for any \( r_0 > 0 \)
\[
\int_{r_0}^{\infty} \frac{d\xi}{S(\xi)} < \infty;
\]

moreover, for \( r > 0 \) sufficiently large,
\[
\gamma(r) := \int_{r}^{\infty} \frac{d\xi}{S(\xi)} \approx Ce^{-(m-1)\sqrt{r}} \sqrt{r}.
\]

Hence
\[
\int_{r_0}^{\infty} [\gamma(r)]^\theta V(r) S(r) \, dr \leq C \int_{r_0}^{\infty} e^{-(m-1)(\sigma-1)-\eta \sqrt{r} - \frac{\theta}{2}} \, dr < \infty.
\]

In view of (5.12)-(5.13), from Proposition 5.1 with \( A(r) = S(r) \) and \( B(r) = S(r) V(r) \), we have that there exists a solution \( y = y(r) \) of (5.6), for some \( R_0 > 0 \). Furthermore, \( y(r) > 0 \) for all \( r \in [R_0, \infty) \) and \( y(r) \sim \gamma(r) \) as \( r \to \infty \). Since \( \lambda(M) = 0 \) and \( V > 0 \) on \( M \), by the same arguments as in Example 5.2
above, we can construct $u \in C^1(M)$ with $u = u(r) > 0$ on $M$, which is a weak solution of
\[ \Delta u + Vu^\sigma \leq 0 \quad \text{in } M. \]

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