NON-EXISTENCE OF SOME APPROXIMATELY SELF-SIMILAR SINGULARITIES FOR THE LANDAU, VLASOV-POISSON-LANDAU, AND BOLTZMANN EQUATIONS

JACOB BEDROSSIAN, MARIA PIA GUALDANI, AND STANLEY SNELSON

Abstract. We consider the homogeneous and inhomogeneous Landau equation for very soft and Coulomb potentials and show that approximate Type I self-similar blow-up solutions do not exist under mild decay assumptions on the profile. We extend our analysis to the Vlasov-Poisson-Landau system and to the Boltzmann equation without angular cut-off.

1. Introduction

We consider the inhomogeneous Landau equation:

\begin{equation}
\partial_t f + v \cdot \nabla_x f = Q_L(f, f) := \text{tr}(\bar{a}^f D_v^2 f) + \bar{c}^f f,
\end{equation}

where, for \( f : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \),

\[
\bar{a}^f(t, x, v) := a_\gamma \int_{\mathbb{R}^3} \Pi(v_*) |v_*|^\gamma f(t, x, v - v_*) \, dv_*,
\]

\[
\bar{c}^f(t, x, v) := \begin{cases} 
  c_\gamma \int_{\mathbb{R}^3} |v_*|^\gamma f(t, x, v - v_*) \, dv_*, & \gamma > -3, \\
  f, & \gamma = -3,
\end{cases}
\]

and \( \Pi(z) := \left( I - \frac{z \otimes z}{|z|^2} \right) \). The constants \( a_\gamma \) and \( c_\gamma \) are positive and only depend on \( \gamma \). The constant \( \gamma \) belongs to the range of very soft potentials, i.e. \( \gamma \in [-3, -2] \). For \( \gamma \leq -2 \) the Landau collision operator \( Q_L \) shares several similarities with the semilinear operator \( \Delta f + f^2 \) and a question naturally arises: do smooth solutions to (1.1) stay bounded for all times or do they become unbounded after a finite time? We say that a solution \( f \) blows up at a time \( T < +\infty \) if it is well defined for all \( 0 < t < T \), and if

\[
\lim_{t \to T^-} \|f(t, x, v)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} = +\infty.
\]

We would call \( T \) the blow-up time for \( f \). This question of regularity versus singularity formation for (1.1) is, at the present day, still unanswered.

The existence of smooth solutions to the inhomogeneous Landau equation (1.1) for very soft potentials is known for a short time, \[44, 46, 47\], and for long times under simplifying assumptions on the initial data. For example, when the initial data is sufficiently close to a

---

**Acknowledgments**

JB was partially supported by National Science Foundation RNMS #1107444 (Ki-Net).

MG is partially supported by the DMS-NSF 2019335 and would like to thanks NCTS Mathematical division Taipei for their kind hospitality.

SS was partially supported by a Ralph E. Powe Award from ORAU.

This project began at the AIM Workshop “Nonlocal differential equations in collective behavior.” The authors would like to thank AIM for their kind hospitality.
Maxwellian equilibrium state, solutions exist globally and converge to equilibrium [11]. Solutions are also known to exist when initial data are near vacuum in the cases of moderately soft potentials [59] and hard potentials [18]. Recently, several new studies concerning regularity and continuation criteria have appeared; these results are based on conditional assumptions on the hydrodynamic quantities, see [37, 11, 45, 21, 57]. The situation for Vlasov-Poisson-Landau is less well-studied; see [42, 72, 19] for results near global Maxwellians and [30] for results near some local Maxwellian data.

The available literature on the homogeneous version of (1.1) for very soft potentials is larger. In [5], [74], and later, in [28], the authors show global existence of weak solutions. Recently, it was proven that for short time weak solutions become instantaneously regular and smooth, see [71] and [40]. The question of whether they stay smooth for all time or become unbounded after a finite time is, however, also in this case, still open. Recent research has also produced several conditional results. This includes uniqueness results in [34] for solutions that belong to the space \( L^1(0, T, L^\infty(\mathbb{R}^3)) \) and in [22] for solutions in \( L^\infty(0, T, L^p(\mathbb{R}^3)) \) with \( p > \frac{3}{2} \); and regularity results for solutions in \( L^\infty(0, T, L^p(\mathbb{R}^d)) \) with \( p > \frac{d}{2} \) [71, 40]. We also mention the long time asymptotic results for weak solutions from [13] and [12].

In the very recent manuscript [29] the authors studied behavior of solutions in the space \( L^\infty(0, T, \dot{H}(\mathbb{R}^3)) \). They show that for general initial data there exists a time \( T^* \) after which the weak solution belongs to \( L^\infty((T^*, +\infty), H^1(\mathbb{R}^3)) \). This result is in accordance with the one in [36]; in [36] the authors showed that the set of singular times for weak solutions has Hausdorff measure at most \( \frac{1}{2} \). On the other hand, global existence of bounded smooth solutions has been shown for an isotropic modification of the Landau equation \( \partial_t f = tr(\tilde{a}f \Delta f) + f^2 \) in [39].

For the non-cutoff Boltzmann equation, existence theory is in a roughly similar stage as the Landau equation. Global existence is known for initial data that is close to equilibrium [28]. In the space homogeneous case, solutions are known to exist globally when \( \gamma + 2s \geq 0 \) [43]. (The parameter \( s \) will be defined in Section 5 below.) See also [58, 65] for global existence of measure-valued solutions in the homogeneous setting, which regularize in some cases. Short-time existence for the inhomogeneous equation was established in various regimes in, e.g. [2, 8, 45]. There is also a program of conditional regularity (see [70, 52, 50, 53]) that gives \( C^\infty \) smoothness in the case \( \gamma + 2s \in [0, 2] \), provided the mass, energy, and entropy densities remain under control.

As the question of whether or not solutions to the Landau and Boltzmann equations exhibit finite-time singularities remains an open challenge, it is natural to narrow down the search to certain kinds of singularities. Our goal is to investigate, and eliminate the existence of, one particular breakdown mechanism, which is usually called approximately self-similar blowup. Self-similar singularities are very common in nonlinear partial differential equations and can come in many different forms; see [31, 7] for many examples and detailed discussions. Here we consider a singularity to be (approximately) self-similar if the solution is of the following form

\begin{equation}
(1.3) \quad f(t, x) = \frac{1}{\mu(t)^s} g \left( \frac{x}{\lambda(t)} \right) + \mathcal{E}(t, x),
\end{equation}

where \( \mathcal{E} \) is some error (possibly zero) which is less singular than the self-similar term and where \( \mu(t), \lambda(t) \) are rates such that \( \lim_{t \to T} \mu(t) = 0 \) and \( \lim_{t \to T} \lambda(t) = 0 \). The function \( g \) is called the “(inner) profile”. In the literature, self-similar singularities are roughly divided into two kinds (terminology dating from at least [8]): Type I self-similar, in which the blow-up rate is determined by dimensional analysis (i.e. the scaling symmetry of the equations); and Type II self-similar, in which the rate is determined also by other additional effects, for example by an
eigenvalue problem associated to the inner blow-up profile. In this context, two well-studied equations are the semilinear heat equation and the Keller-Segel system. The semilinear heat equation, despite its simplicity, displays both types of singularities; see reviews in [61, 67, 24] and the references therein. The Keller-Segel equations displays type II self-similar finite time and infinite time singularities [27, 85, 68], and with nonlinear diffusion, can display type I self-similar singularities [5]. Another semilinear parabolic PDE studied in this context are the incompressible Navier-Stokes equations; finite-energy type I self-similar solutions were ruled out in [66, 73]; see also [14, 69]. Self-similar singularities are also intensely studied in the setting of dispersive equations, such as for example, the nonlinear Schrödinger equations [62] and wave equations.

One significant difference between the Landau and Boltzmann equations and the semilinear equations discussed above (and many quasilinear problems too) is a two-parameter scaling symmetry. That is, if \(f\) is a solution to either (1.1), or (1.6), then for any \(\alpha \in \mathbb{R}\) and \(\lambda > 0\), so is
\[
 f_{\lambda, \alpha}(t, x, v) := \lambda^{\alpha+3+\gamma} f(\lambda^\alpha t, \lambda^{1+\alpha} x, \lambda v).
\]

This provides for a much wider and subtle class of potential type I singularities (and likely type II as well) than equations with a one-parameter scaling symmetry. These kinds of two-parameter symmetry groups are common in fluid mechanics and kinetic theory. Some examples include the Burgers equation, which undergoes type I self-similar shock formation [20, 31], and the isentropic, compressible Euler equations, for which there are many self-similar finite-time singularities, including implosions [60, 64] and shocks [9, 10, 23, 24]. Another example are the incompressible Euler equations, for which the existence or smooth finite-time singular solutions remains open. Type I self-similar singularities have been ruled out under a variety of decay and/or integrability conditions on the profile [14, 15, 17, 16], nevertheless, there is strong numerical evidence that type I self-similar singularity formation is possible, at least along the boundary [60]. Moreover, in Hölder regularity (as opposed to smooth), there does exist type I self-similar finite-time blow-up solutions [32, 33]. Smooth type I self-similar blow-up solutions have also been constructed for some toy models of the Euler equations, such as the Choi-Kiselev-Yao (CKY) model [49] and the de Gregorio model [20]. Finally, the four-dimensional gravitational Vlasov-Poisson equations have a family of type I self-similar finite-time singularities [55, 56]. One significant difference that Landau and Boltzmann equations (with singular cut-off) from all of the examples just discussed is the presence of hypoelliptic (or parabolic, if homogeneous) smoothing. For example, this is likely to rule out the kind of regularity-dependent blow-up dynamics observed in Burgers and Euler [26, 32, 33].

In light of the rich number and types of blow-up profiles found in similar equations, it makes sense to narrow down the search for potential singularities by eliminating them one at the time. This work can be considered a first study in this direction, endeavoring to rule out as many kinds of Type I singularities as possible. Our first main result is summarized in the following statement, which will be presented and discussed in detail in the next section:

**Main Theorem Summary** Let \(\gamma \in [-3, -2]\) and let \(f\) be a smooth solutions to (1.1) with mass and kinetic energy locally bounded, namely
\[
f \geq 0, \quad f \in C^\infty((0, T) \times \mathbb{R}^3_x \times \mathbb{R}_v^3), \quad \forall R > 0, \quad \sup_{0 < t < T} \int_{|x| \leq R} \int_{|v| \leq R} (1 + |v|^2) f \, dv < \infty,
\]
for any \(T > 0\). Then, if \(f\) has the form
\[
f(t, x, v) = \phi(t, x, v) + \frac{1}{(T-t)^{1+\theta(3+\gamma)}} g \left( \frac{x}{(T-t)^{1+\theta}}, \frac{v}{(T-t)^{\theta}} \right),
\]
for any \(T > 0\).
with $-1 < \theta < 1/2$, $g$ smooth, $\phi$ not too singular as $t \nearrow T$, and $g$ bounded and satisfying mild decay conditions, then we must have $g \equiv 0$.

**Corollary.** Let $\gamma \in [-3, -2]$ and let $f$ be a smooth solutions to the homogeneous Landau equation

$$\partial_t f = \text{tr}(\bar{a}_f D^2 f) + \bar{c}_f f.$$  
Then, if $f$ has the form

$$f(t, v) = \phi(t, v) + \frac{1}{(T - t)^{1+\theta(3+\gamma)}} g \left( \frac{v}{(T - t)^{\theta}} \right),$$

with $1/|\gamma| < \theta < 1/2$, $g$ smooth, $\phi$ not too singular as $t \nearrow T$, and $g$ bounded and satisfying mild decay conditions, then we must have $g \equiv 0$.

In the second part of our manuscript we extend our blow-up analysis to the Vlasov-Poisson-Landau system ($\gamma = -3$):

$$\partial_t f + v \cdot \nabla_x f - \nabla_x E \cdot \nabla_v f = Q_L(f, f),$$
(1.5)

$$- \Delta_x E = \pm 4\pi \int_{\mathbb{R}^3} f(t, x, v) \, dv,$$

and to the non-cutoff Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = Q_B(f, f) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)[f(v')f(v') - f(v_*)f(v)] \, d\sigma \, dv_*, \quad (1.6)$$

(See Section 5 for the definitions of $B(v - v_*, \sigma)$, $v'$, and $v_*$.) For both models we similarly rule out existence of solutions of the form (1.4), see Theorem 5.1 and Theorem 4.1.

Let us briefly comment on the admissible values of $\theta$. Define the self-similar variables

$$y = \frac{x}{(T - t)^{1+\theta}}, \quad w = \frac{v}{(T - t)^{\theta}}.$$  

In these variables, our ansatz becomes

$$f(t, x, v) = \phi \left( t, (T - t)^{1+\theta} y, (T - t)^{\theta} w \right) + \frac{1}{(T - t)^{1+\theta(3+\gamma)}} g(y, w).$$

We consider all $\theta$ that satisfy at the same time $1 + \theta > 0$ and $1 + \theta(3 + \gamma) > 0$ for all $\gamma \in [-3, -2]$, i.e. we want a solution that forms a singularity a point in space. This implies $\theta > -1$. In the case of the homogeneous Landau equations we additionally have the requirement $1/|\gamma| < \theta$ because otherwise, our ansatz violates conservation of mass and is therefore not an admissible solution.

To motivate the upper bound on $\theta$ that appears in the our results, we recall from [40, 71], that if $f$ is a solution to the homogeneous Landau equations which belongs to $L^\infty(0, T, L^q_2(\mathbb{R}^3))$ for some $q > 3/(5 + \gamma)$ then $f$ is uniformly bounded. Hence, it is natural to require blow up in all of $L^\infty_2(\mathbb{R}^3)$ with $3/(5 + \gamma) < q \leq +\infty$ at $x = 0$. This motivates the requirement $\theta < \frac{1}{2}$, which also appears in the proof in order to control error terms coming from the interaction of $\phi$ and $g$ near the singularity.

For the Boltzmann equation, we will take the same ansatz, with $\theta > -1$ for the same reasons mentioned above. The upper restriction on $\theta$ that arises from our proof in the Boltzmann case is $1/(2s)$ rather than $1/2$. Note that $2s$ is the order of the diffusion generated by the collision operator, whereas the Landau collision operator gives rise to diffusion of order 2. For
homogeneous Boltzmann, conservation of mass also holds, which rules out values of $\theta$ smaller than $1/|\gamma|$ in our ansatz.

**Remark 1.1.** Note that $\theta < 0$ and $\theta > 0$ each correspond to very qualitatively different blow-up scenarios. In $\theta > 0$ the distribution function is forming a singularity at $v = 0$; that is, many particles are slowing to a halt near the singularity. For $\theta < 0$, many particles are being accelerated to unbounded velocities near the point of singularity. Due to the conservation of energy, the latter kind of singularity cannot occur in the homogeneous equations. However there is, a priori, no reason why such a singularity cannot occur in the inhomogeneous equations, such as in (1.5) or (1.1). In fact, precisely this kind of approximately Type I self-similar singularity with accelerating particles occurs in the 4-dimensional gravitational Vlasov equation [55, 56].

**2. Main results on the Landau equation**

To properly formulate our results, we need to choose a proper class of solutions. In that class, we will show that breakdown mechanisms of the form (1.4) will not occur. The conditions we impose on $\phi$ and $g$ in (1.4) are mild, but somewhat tedious to state. For convenience, by shifting time, we will take $t = 0$ to be the potential blowup time, and assume that $f$ is defined for $(t, x, v) \in (-T, 0) \times \mathbb{R}^3 \times \mathbb{R}^3$ for some $T > 0$. Hence, we write

$$f(t, x, v) = \phi(t, x, v) + \frac{1}{(-t)^{1+\theta(3+\gamma)}} g \left( \frac{x}{(-t)^{1+\theta}}, \frac{v}{(-t)^{\theta}} \right),$$

or

$$f(t, x, v) = \phi \left( t, (-t)^{1+\theta} y, (-t)^{\theta} v \right) + \frac{1}{(-t)^{1+\theta(3+\gamma)}} g(y, w),$$

in the self-similar variables

$$y := \frac{x}{(-t)^{1+\theta}}, \quad w := \frac{v}{(-t)^{\theta}}.$$

The first condition is that the mass and kinetic energy of $f$ are locally bounded,

$$f \geq 0, \quad \forall R > 0, \quad \sup_{-T < t < 0} \int_{|x| < R} \int \left( 1 + |v|^2 \right) f \, dv \, dx < \infty,$$

in particular, we do not require the solution to decay as $x \to \infty$. Our analysis is, therefore, valid also for periodic domains and homogeneous solutions (that is, $x$-independent solutions).

The second condition is that the singularity occurs only at the blow-up space-time point $(t, x) = (0, 0)$:

$$f \in C^\infty((-T, 0] \times \mathbb{R}^3 \times \mathbb{R}^3 \setminus \{0\} \times \{0\} \times \mathbb{R}^3).$$

The third condition concerns the inner profile $g$. For all $1 \leq p \leq \infty$, $0 \leq j \leq 2$, $0 \leq \ell \leq 1$, we require

$$g \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3), \quad D^\ell_x D^\ell_y g \in L^p_{y, loc} L^1_w.$$

**Remark 2.1.** In fact, one can use the slightly weaker condition $g \in L^\infty_{y, loc} L^1_w \cap L^\infty_{y, loc} L^p_w \cap C^\infty$ for some $p > \frac{3}{\theta + \gamma}$, however for simplicity of exposition we will use the stronger assumption (2.5).

The fourth condition will assure that, near the singularity, the contribution of $\phi$ is small compared to $g$ in the natural self-similar frame. In this regard, the function $\phi$ is allowed to form a singularity at a rate that is ‘sub-critical’ with respect to the scaling. First note that

$$D^\ell_v D^\ell_x f(v, x, t) = D^\ell_v D^\ell_x \phi + (-t)^{-1-\theta(3+\gamma)} (-t)^{(1+\theta)-j \theta} D^\ell_v D^\ell_y g(y, w).$$
We would like to compare
\[ \sup_{|y| \leq R} \| D^i_y D^j_x \phi \|_{L^p} \]
with (using the definition of the self-similar coordinates (2.2))
\[ (-t)^{-1-\theta(3+\gamma) - \ell(1+\theta) - j\theta} \sup_{|y| \leq R} \| D^i_y D^j_x g \|_{L^p} = (-t)^{-1-\theta(3+\gamma) + 3\theta/p - \ell(1+\theta) - j\theta} \sup_{|y| \leq R} \| D^i_y D^j_x g \|_{L^p}. \]

Let’s consider first \( \ell = j = 0; \) we compare the terms
\[ \sup_{|y| \leq R} \| \phi \|_{L^p} \text{ vs } (-t)^{-1-\theta(3+\gamma) + 3\theta/p} \sup_{|y| \leq R} \| g \|_{L^p}. \]

If \( \theta \) and \( p \) are such that \(-1 - \theta(3 + \gamma) + 3\theta/p < 0\), we enforce \( \phi \) to satisfy
\[ (2.6) \quad \lim_{t \to 0} (-t)^{1+\theta(3+\gamma) - 3\theta/p} \sup_{|y| \leq R} \| \phi \|_{L^p} = 0. \]

In this case, \( \sup_{|y| \leq R} \| f \|_{L^p} \) blows up with a rate \((-t)^{-1-\theta(3+\gamma) + 3\theta/p} \) dictated by \( g \) and the \( \phi \) contribution is at least slightly less singular. This case happens for all \( p \geq 1 \) for \( \theta < \frac{1}{|\gamma|} \) and for \( p > \frac{3\theta}{1+\theta(3+\gamma)} \) if \( \theta \geq \frac{1}{|\gamma|} \). Note that this condition with \( p = \infty \), implies \( g \geq 0 \), by taking \( t \to 0 \).

Generalizing to derivatives, we assume that for every \( R > 0 \), \( 0 \leq i \leq 1, 0 \leq j \leq 2, 0 \leq \ell \leq 1 \), the function \( \phi \) satisfies
\[ (2.7) \quad \lim_{t \to 0} (-t)^{1+\theta(3+\gamma) - \frac{3\theta}{p} + i + (1+\theta)\ell + j} \sup_{|y| \leq R} \left\| \partial^i_y \partial^j_x \partial^\ell_x \phi \left( t, \left( -t \right)^{1+\theta} y, \cdot \right) \right\|_{L^p} = 0. \]

Our main result for the Landau equation is summarized in the following theorem:

**Theorem 2.2.** Let \( \gamma \in [-3, -2], -1 < \theta < 1/2 \). Let \( f \) be a smooth solution of the Landau equation (1.1) that satisfies (2.3) and (2.4). Assume that \( \phi \) satisfies (2.7). For \( g \), assume it satisfies (2.5) and that there exist \( h \) and \( q \) such that
\[ g(y, w) = q(w) + h(y, w), \]
with
\[ (1 + |y| + |w|) h \in L^1_{y,w}(\mathbb{R}^6) \quad \text{and} \quad q \in L^1_w(\mathbb{R}^3). \]

Finally, if \( \theta = \pm 1/3 \) we additionally assume that
\[ (1 + |y| |w|^2 + |w|^3) h \in L^1_{y,w}(\mathbb{R}^6) \quad \text{and} \quad (1 + |w|^2) q \in L^1_w(\mathbb{R}^3). \]

Then, for any solution to the Landau equation (1.1) of the form
\[ f(t, x, v) = \phi(t, x, v) + \frac{1}{(-t)^{1+\theta(3+\gamma)}} \frac{x}{(-t)^{1+\theta}}, \quad \text{for } \theta \neq 0, \]
we must have \( g \equiv 0 \) and hence no approximate self-similar singularity of this type can occur.

**Remark 2.3.** Note that the inhomogeneous problem could have a self-similar profile with \( q \neq 0 \). In other equations, there are type I self-similar singularities with inner profiles that do not decay at infinity (although the solution does), such as in the semilinear heat equation [74], and even profiles grow at infinity, such as in shock formation in Burgers [26, 31] and in the CKY model [49]. Numerical evidence suggests that such singularities exist also in the incompressible Euler equations [60]. At the current moment, we do not know how to classify potential singularities with inner profiles that grow at infinity.
Remark 2.4. If one knows a priori that \( g \in L^1_{w,v} \), then it suffices to use \((1 + |w|)g \in L^1_{g,v} (\mathbb{R}^6)\) and \((1 + |w|^3)h \) if \( \theta = \pm 1/3 \).

Next, we specialize our analysis to the homogeneous Landau equation. Our next theorem shows essentially that if any solution to the homogeneous Landau equation has a singularity, such singularity is either (i) not of Type I self-similar, or (ii) is of Type I self-similar with a profile \( g \not\in L^1(\mathbb{R}^3) \).

**Theorem 2.5.** Let \( \gamma \in [-3,-2) \) and \( 1/|\gamma| \leq \theta < 1/2 \). Assume that \( f = f(v,t) \) has finite mass and second moment and satisfies

\[
f \in C^\infty((-T,0) \times \mathbb{R}^3), f \in C^\infty((-T,0] \times \mathbb{R}^3).
\]

Assume that \( \phi \) satisfies (2.7), and that \( g = g(v) \) is such that

\[
g \in C^\infty(\mathbb{R}^3), \quad g \in L^1_{w,v}.
\]

If \( \theta = 1/3 \), then additionally assume that \( g \) satisfies \((1 + |w|^2)g \in L^1_{w,v} \).

Then, for any solution to the homogeneous Landau equation of the form

\[
f(t,v) = \phi(t,v) + \frac{1}{(-t)^{1+\theta(3+\gamma)}} g \left( \frac{v}{(-t)^\theta} \right),
\]
we must have \( g \equiv 0 \), and hence no approximate self-similar singularity of this type can occur.

The outline of the paper is as follows: in Section 3 we prove Theorem 2.2 and Theorem 2.5, in Section 4 we investigate the Vlasov-Landau-Poisson system and in Section 5 the Boltzmann equation.

2.1. **Notation.** We will employ the notation \( \langle \cdot \rangle = \sqrt{1 + |\cdot|^2} \) throughout. When we say \( t \to 0 \), we always mean that \( t \) increases to 0 through negative values. We will write \( A \lesssim B \) when \( A \leq CB \) for some universal constant \( C \). When integrals appear with no domain, it is assumed that the domain of integration is \( \mathbb{R}^3 \). Similarly, norms such as \( \| \cdot \|_{L^p} \) are over \( \mathbb{R}^3 \), unless stated otherwise.

3. **Proof of Theorem 2.2**

3.1. **Preliminary lemmas.** First, we recall important global estimates on the coefficients in (1.1). The proof is standard, but we include a sketch for the readers’ convenience.

**Lemma 3.1.** Let \( \gamma \in [-3,-2] \). With \( \hat{a}^h \) and \( \hat{c}^h \) defined as in (1.2), for any \( 1 \leq p < \frac{3}{\gamma+2} \), there exists \( C > 0 \) such that

\[
|\hat{a}^h(v)| \leq C \|h\|_{L^1(\mathbb{R}^3)}^{1+\frac{\gamma}{2}(\gamma+2)} \|h\|_{L^{\frac{p}{\gamma+2}}(\mathbb{R}^3)}^{-(\gamma+2)p/3}.
\]

Moreover, for any \( 1 \leq q < \frac{3}{\gamma+5} \), there exists \( C > 0 \) such that

\[
|\hat{a}^h(v)| \leq C \|h\|_{L^q(\mathbb{R}^3)}^{\frac{3}{2}(\gamma+5)} \|h\|_{L^\infty(\mathbb{R}^3)}^{1-(\gamma+5)\frac{q}{2}}.
\]

For any \( 1 \leq p < \frac{3}{\gamma+1} \), there exists \( C > 0 \) such that

\[
|\partial_v \hat{a}^h(v)| \leq C \|h\|_{L^1(\mathbb{R}^3)}^{1+\frac{\gamma}{2}(\gamma+1)} \|h\|_{L^{\frac{p}{\gamma+1}}(\mathbb{R}^3)}^{-(\gamma+1)p/3}.
\]
Finally, for any $1 \leq p < \frac{-3}{\gamma}$ and $\gamma \in (-3, -2]$

\[(3.4) \quad |e^h(v)| \leq C\|h\|_{L^1(\mathbb{R}^3)}^{1 + \frac{p}{3}}\|h\|_{L^{\frac{3}{p}}(\mathbb{R}^3)}^{-\gamma p / 3}, \quad \gamma \in (-3, -2].\]

**Proof.** For $s \in (-3, 0)$, splitting the integral into $|v_*| \leq R$ and $|v_*| > R$, applying Hölder’s inequality on each, we have for $1 \leq p < 3/(3 - s) < r \leq \infty$

\[(3.5) \quad \left| \int_{\mathbb{R}^3} h(v - v_*) |v_*|^s \, dv_* \right| \lesssim R^{s+3-3/p} \|h\|_{L^p} + R^{s+3-3/r} \|h\|_{L^r}.\]

Optimizing in $R$ gives the estimates (3.1), (3.2), and (3.4) (using also $|\Pi(v_*)| \leq 1$). For (3.3), first one integrates by parts and uses \( |\partial_v(\Pi(v_*)|v_*|^{\gamma+2})| \lesssim |v_*|^\gamma + 1 \) before applying again (3.5). \( \square \)

The next lemma ensures that the formal identity $\int_{\mathbb{R}^3} Q_L(g, g)(1 + |w|^2) \, dw = 0$ and entropy dissipation inequality $\int_{\mathbb{R}^3} \log g Q_L(g, g) \, dw \leq 0$ are valid under our assumptions on $g$.

**Lemma 3.2.** Let $\chi \in C^\infty(B(0, 2))$ be a smooth cut-off function, such that $\chi(|x|) = 1$ for $|x| \leq 1$. With $g \in L^1_w \cap L^\infty_w(\mathbb{R}^3) \cap C^\infty$, we have

- \[ \lim_{R \to \infty} \int_{\mathbb{R}^3} \chi\left( \frac{w}{R} \right) Q_L(g, g) \, dw = 0. \]

- **If, in addition, $g$ satisfies $|w|^2 g \in L^1_w$ we have**

\[ \lim_{R \to \infty} \int_{\mathbb{R}^3} \chi\left( \frac{w}{R} \right) |w|^2 Q_L(g, g) \, dw = 0. \]

- **If, in addition, $g$ satisfies**

\[(3.6) \quad g \log g \in L^1_w, \quad \nabla \sqrt{g} \in L^2_w,\]

then we have

\[ \lim_{R \to \infty} \int_{\mathbb{R}^3} \chi\left( \frac{w}{R} \right) \log g Q_L(g, g) \, dw \leq 0. \]

**Proof.** Define

\[ \chi_R(w) := \chi(w/R); \]

notice that

\[(3.7) \quad |\nabla^j \chi_R| \lesssim R^{-j}, \]

and moreover, the derivatives are only supported in the region $w \approx R$.

Since $Q_L$ can be written as

\[ Q_L(g, g) = \nabla_v \cdot \left( \int_{\mathbb{R}^3} \Pi(v_*)|v_*|^{\gamma+2}[g(v - v_*)\nabla_v g(v) - g(v)\nabla_v g(v - v_*)] \, dv_* \right), \]
introduction by parts gives
\[
\int_{\mathbb{R}^3} \chi_R Q_L(g,g) \, dw = - \int_{\mathbb{R}^3} \nabla \chi_R \\
\cdot \left( \int_{\mathbb{R}^3} \Pi(|w|) \langle g(w) - g(w) \rangle \, dw \right) \, dw \\
= 2 \int_{\mathbb{R}^3} g \nabla \chi \cdot \text{div}_w \tilde{a}^g(w) \, dw \\
+ \int_{\mathbb{R}^3} g \nabla^2 \chi : \tilde{a}^g \, dw.
\]

Since \( g \in L^p_w \) for any \( p \geq 1 \), thanks to (3.2) and (3.3) the integrals are bounded uniformly in \( R \) and hence as \( R \to +\infty \), we have by (3.7), \( \lim_{R \to \infty} \int_{\mathbb{R}^3} \chi \left( \frac{w}{R} \right) Q_L(g,g) \, dw = 0 \).

Similarly, integration by parts yields
\[
\int_{\mathbb{R}^3} \chi_R |w|^2 Q_L(g,g) \, dw = \frac{2}{R} \int_{\mathbb{R}^3} g |w|^2 \nabla \chi_R \cdot \text{div}_w \tilde{a}^g(w) \, dw \\
+ 4 \int_{\mathbb{R}^3} g \chi_R \text{div}_w \tilde{a}^g(w) \cdot w \, dw \\
+ \frac{1}{R^2} \int_{\mathbb{R}^3} g |w|^2 \nabla^2 \chi : \tilde{a}^g(w) \, dw + 2 \int_{\mathbb{R}^3} g \chi_R Tr[\tilde{a}^g] \, dw \\
+ \int_{\mathbb{R}^3} g \sum_{i,j} \tilde{a}^g_{i,j} (2 w_j \partial_i \chi_R + 2 w_i \partial_j \chi_R) \, dw.
\]

All of the terms involving derivatives of the cutoff function vanish as \( R \to \infty \) by the same arguments as used in the previous case. Since
\[
2g \text{div}_w \tilde{a}^g(w) \cdot w + g Tr[\tilde{a}^g] = 2 \frac{g(w)}{|z - w|} \nabla g(z) \cdot w + \frac{g(w)g(z)}{|w - z|},
\]
integration by parts yields
\[
\int_{\mathbb{R}^3} 2g \chi_R (\text{div}_w \tilde{a}^g(w) \cdot w + g Tr[\tilde{a}^g]) \, dw = \int_{\mathbb{R}^6} \chi_R \frac{g(w)g(z)}{|w - z|} \left[ \frac{1}{|z - w|} - \frac{|z| - |w|^2}{|z - w|^3} \right] \, dz \, dw \\
- 2 \int_{\mathbb{R}^3} \nabla \chi_R \cdot wg \tilde{a}^g \, dw \\
= -2 \int_{\mathbb{R}^3} \nabla \chi_R \cdot wg \tilde{a}^g \, dw.
\]

Hence, by the assumptions on \( g \) and (3.7), we can pass to the limit \( R \to +\infty \) and get
\[
\lim_{R \to \infty} \int_{\mathbb{R}^3} \chi \left( \frac{w}{R} \right) |w|^2 Q_L(g,g) \, dw = 0.
\]
For the entropy inequality, we begin with
\[
\int_{\mathbb{R}^3} \chi_R \log gQ_L(g, g) \, dw = \int_{\mathbb{R}^3} (2g \ln g - g) \nabla \chi_R \cdot \text{div}_w \tilde{a}^g(w) \, dw \\
+ \int_{\mathbb{R}^3} (g \ln g - g) \nabla^2 \chi_R : \tilde{a}^g(w) \, dw \\
- \int_{\mathbb{R}^3} \chi_R \left[ \left\langle \tilde{a}^g(w) \frac{\nabla g}{\sqrt{g}}, \frac{\nabla g}{\sqrt{g}} \right\rangle - \nabla g \cdot \text{div}_w \tilde{a}^g(w) \right] \, dw
\]
\[
= \int_{\mathbb{R}^3} (2g \ln g - 2g) \nabla \chi_R \cdot \text{div}_w \tilde{a}^g(w) \, dw \\
+ \int_{\mathbb{R}^3} (g \ln g - g) \nabla^2 \chi_R : \tilde{a}^g(w) \, dw \\
- \int_{\mathbb{R}^3} \chi_R \left[ \left\langle \tilde{a}^g(w) \frac{\nabla g}{\sqrt{g}}, \frac{\nabla g}{\sqrt{g}} \right\rangle - g \tilde{c}^g(w) \right] \, dw,
\]
using the identity
\[
\int_{\mathbb{R}^3} \chi_R \nabla g \cdot \text{div}_w \tilde{a}^g(w) \, dw = - \int_{\mathbb{R}^3} g \nabla \chi_R \cdot \text{div}_w \tilde{a}^g(w) \, dw \\
+ \int_{\mathbb{R}^3} \chi_R g \tilde{c}^g(w) \, dw.
\]
Using, once more, \((3.2), (3.3), g \in L^p\) for any \(p \geq 1\), \((3.7)\) and this time also \((3.6)\), we conclude that the first two integrals vanish as \(R \to +\infty\). Moreover,
\[
\left\langle \tilde{a}^g(w) \frac{\nabla g}{\sqrt{g}}, \frac{\nabla g}{\sqrt{g}} \right\rangle - g \tilde{c}^g(w)
\]
is a \(L^1\) function, thanks to \((3.2)\) and \((3.6)\) for the first term, and \((3.4)\) for the second. Hence, dominated convergence theorem allows to pass to the limit
\[
\lim_{R \to \infty} \int_{\mathbb{R}^3} \chi_R \log gQ_L(g, g) \, dw = - \int_{\mathbb{R}^3} \left[ \left\langle \tilde{a}^g(w) \frac{\nabla g}{\sqrt{g}}, \frac{\nabla g}{\sqrt{g}} \right\rangle - g \tilde{c}^g(w) \right] \, dw.
\]
The thesis follows by noticing that the integral on the right hand side can be rewritten as
\[
\int_{\mathbb{R}^3} \left[ \left\langle \tilde{a}^g(w) \frac{\nabla g}{\sqrt{g}}, \frac{\nabla g}{\sqrt{g}} \right\rangle - g \tilde{c}^g(w) \right] \, dw
\]
\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{g(w)g(w_*)}{|w - w_*|^{\gamma - 2}} \left\langle \Pi(w - w_*) \left( \frac{\nabla g}{g} - \frac{\nabla g_*}{g_*} \right), \left( \frac{\nabla g}{g} - \frac{\nabla g_*}{g_*} \right) \right\rangle \, dw \, dw_* \geq 0.
\]

3.2. Proof of Theorem 2.22. As first step, we plug ansatz \((2.21)\) into \((1.1)\) and change to self-similar variables \(y\) and \(w\). The left-hand side of \((1.1)\) transforms as follows:
\[
\partial_t f + v \cdot \nabla x f = \partial_t \phi + v \cdot \nabla x \phi + \frac{1}{(-t)^{2 + \theta(3 + \gamma)}} g + \frac{1}{(-t)^{1 + \theta(3 + \gamma)}} \left( \frac{\partial y}{\partial t} \nabla_y g + \frac{\partial w}{\partial t} \cdot \nabla_w g + v \cdot \nabla x g \right)
\]
\[
= \partial_t \phi + v \cdot \nabla x \phi + \frac{1}{(-t)^{2 + \theta(3 + \gamma)}} \left( g + (1 + \theta) y \cdot \nabla y g + \theta w \cdot \nabla w g + w \cdot \nabla y g \right).
\]
Here, and throughout the proof, all terms involving $g$ are evaluated at $(y, w)$, and terms involving $\phi$ are evaluated at $(t, (-t)^{1+\theta}g, (-t)^{\theta}w)$, unless otherwise noted. Moreover, we have

$$a^f = a^\phi + \frac{1}{(-t)^{1+\theta(3+\gamma)}} \int_{\mathbb{R}^3} \Pi(v - v_*) |v - v_*|^2 g \left( \frac{x}{(-t)^{1+\theta}}, \frac{v_*}{(-t)^\theta} \right) \, dv_*$$

$$= a^\phi + (-t)^{-\gamma-1} \int_{\mathbb{R}^3} \Pi((-t)^\theta(w - w_*)) |(-t)^\theta(w - w_*)|^2 g(y, w_*) \, dw_*$$

$$= a^\phi + (-t)^{2\theta-1} \int_{\mathbb{R}^3} \Pi(w - w_*) |w - w_*|^2 g(y, w_*) \, dw_*$$

$$= a^\phi + (-t)^{2\theta-1} a^\theta,$$

and, for $\gamma > -3$,

$$\bar{c}^f = \bar{c}^\phi + \frac{1}{(-t)^{1+\theta(3+\gamma)}} c_\gamma \int_{\mathbb{R}^3} |v - v_*|^2 g \left( \frac{x}{(-t)^{1+\theta}}, \frac{v_*}{(-t)^\theta} \right) \, dv_*$$

Taking into account that $D_2^2 g = D_2^2 \phi + (-t)^{-(1+(5+\gamma)\theta)} D_{w}^2 g$, the right-hand side of (1.1) becomes

$$Q_L(f, f) = Q_L(\phi, \phi) + \frac{1}{(-t)^{1+\theta(5+\gamma)\theta}} \text{tr}(\bar{a}^\phi D_2^2 g) + (-t)^{2\theta-1} \text{tr}(a^\theta D_{w}^2 g) + \frac{1}{(-t)^{1+\theta(3+\gamma)}} \bar{c}^\phi g + \frac{1}{(-t)^{1+\theta(3+\gamma)}} \bar{c}^\theta g$$

$$+ \frac{1}{(-t)^{1+\theta(3+\gamma)}} \text{tr}(\bar{a}^\theta D_{w}^2 g) + \frac{1}{(-t)^{2+\theta(3+\gamma)}} \bar{c}^\phi g + \frac{1}{(-t)^{2+\theta(3+\gamma)}} \bar{c}^\theta g.$$
The second factor vanishes for any $\gamma$ and $\theta$, thanks to (2.6) with $p = \infty$. For the first term, we still use (2.6) if $p > \frac{3\theta}{1 + \theta (3 + \gamma)}$. Hence, we need

$$\frac{3\theta}{1 + \theta (3 + \gamma)} < \frac{3}{\gamma + 3},$$

which is fulfilled if $\theta < \frac{1}{2}$.

Now let us turn to the term involving $c^\phi$. We have for any $1 \leq p < 3/(3 + \gamma)$ by (3.4),

$$\left| c^\phi \right| \lesssim \| \phi \|_{L^p_\gamma} \| \phi \|_{L^\infty_\gamma}^{\theta/(\gamma + 3)}.$$  

Hence, by assumption (2.5),

$$(-t) \left| c^\phi \right| \lesssim \left( (-t)^{1 + \theta (3 + \gamma) - \frac{3\theta}{p}} \| \phi \|_{L^p_\gamma} \right)^{\theta/(\gamma + 3)} \left( (-t)^{1 + \theta (\gamma + 3)} \| \phi \|_{L^\infty_\gamma} \right)^{1 - \theta/(\gamma + 3)}.$$  

Analogous to above, the second factor vanishes for any $\gamma$ and $\theta$, thanks to (2.6) with $p = \infty$. For the first one, we still use (2.6) if $p > \frac{3\theta}{1 + \theta (3 + \gamma)}$. Hence, we need once more

$$\frac{3\theta}{1 + \theta (3 + \gamma)} < \frac{3}{\gamma + 3},$$

which is satisfied for any $\gamma$ and $\theta$.

Finally, by assumption (2.5) and (3.4) for $g$, we get

$$(-t)^{1 + \theta (3 + \gamma)} |\bar{c}^\phi| \lesssim (\bar{c}^\phi),$$

which vanishes for any $\gamma$ and $\theta$, thanks again to (2.6) with $p = \infty$. This completes the proof of the lemma. \hfill \Box

**Lemma 3.4.** For any $R_1, R_2 > 0$, we have.

$$\lim_{t \to 0} \sup_{|y| \leq R_1} \sup_{|w| \leq R_2} |E_2| = 0. \quad (3.12)$$

**Proof.** First, as $|c^\phi| + |\bar{c}^\phi| \lesssim 1$ by assumption (2.5),

$$|E_2| \lesssim (-t)^{1 + \theta (3 + \gamma)} |\phi| + (-t)^{1 + \theta (5 + \gamma)} |D_v^2 \phi| + (-t)^{1 + \theta (3 + \gamma)} (1 + \theta) \left| (-t)^{-\theta} \partial_t \phi + w \cdot \nabla_x \phi \right| + |(-t)^{2 + \theta (3 + \gamma)} Q_L (\phi, \phi)|.$$  

The first three terms on the right hand side converge to zero in $L_\text{loc}^\infty (\mathbb{R}^6)$ directly by assumption (2.7). We now look at the collision term

$$(-t)^{2 + \theta (3 + \gamma)} Q_L (\phi, \phi) = (-t)^{2 + \theta (3 + \gamma)} [\text{tr}(\bar{a}^\phi D_v^2 \phi) + \bar{c}^\phi].$$  

Analogously as in the previous lemma, we write

$$|(-t)^{2 + \theta (3 + \gamma)} \text{tr}(\bar{a}^\phi D_v^2 \phi)| \lesssim \left( (-t)^{1 + \theta (3 + \gamma) - 3\theta/p} \| \phi \|_{L^p_\gamma} \right)^{(\gamma + 5)/4} \left( (-t)^{1 + \theta (3 + \gamma)} \| \phi \|_{L^\infty_\gamma} \right)^{1 - (\gamma + 5)/4} \left( (-t)^{1 + \theta (3 + \gamma) + 2\theta} \| D_v^2 \phi \|_{L^\infty_\gamma} \right).$$  

The second and third factor vanish for any $\gamma$ and $\theta$, thanks to (2.6) with $p = \infty$. The first term is identical to the one in (3.10) and vanishes for $p < \frac{3}{\gamma + 7}$ and $\theta < \frac{1}{2}$. To estimate the last term
we conclude that (3.13) holds pointwise for all \((y,w)\) satisfies in the sense of distributions. However, using the regularity and decay assumptions for \(g\) is the trivial one. Our next step is to show that the only admissible solution to

\[
\Box \phi = 0
\]

\(p \leq \gamma \) with \(1 \leq p < \frac{3}{\gamma + 3}\). For the first term we still use (2.6) with \(\frac{3}{\gamma + 3} < p < \frac{3}{\gamma + 3}\). This completes the proof of the lemma.

Having shown that the dominant terms in (3.8) are

\[
g + (1 + \theta) y \cdot \nabla_y g + \theta w \cdot \nabla_w g + w \cdot \nabla_y g - \Omega_{L,w}(g,g),
\]

our next step is to show that the only admissible solution to

\[
0 = g + (1 + \theta) y \cdot \nabla_y g + \theta w \cdot \nabla_w g + w \cdot \nabla_y g - \Omega_{L,w}(g,g),
\]

is the trivial one \(g \equiv 0\).

**Proof of Theorem 2.2.** We multiply (3.8) by a general smooth test function \(\psi(y,w)\) with compact support in \(\mathbb{R}^6\), and take the limit \(t \to 0\). Thanks to (3.9) and (3.12), we conclude that \(g\) satisfies

\[
(3.13)
\]

in the sense of distributions. However, using the regularity and decay assumptions for \(g\) (2.6), we conclude that (3.13) holds pointwise for all \((y,w) \in \mathbb{R}^6\).

The rest of the theorem is devoted to showing that the only solution to (3.13) is the trivial one, \(g \equiv 0\). The proof varies depending on the value of \(\theta\). We distinguish three cases, \(\theta \neq \pm \frac{1}{3}\), \(\theta = 1/3\) and \(\theta = -1/3\).

- Let \(\theta \neq 1/3, \theta \neq -1/3\). Let \(\chi_R(w) = \chi(w/R)\) be a cutoff function and \(\varphi \in C_0^\infty(B(0,1))\) a smooth function such that \(\int_{\mathbb{R}^3} \varphi(y)dy = 1\). For \(y_0 \in \mathbb{R}^3\) define \(\varphi_{y_0}(y) := \varphi(y + y_0)\). We multiply (3.13) by \(\chi_R(w)\varphi(y + y_0)\) for some \(y_0 \in \mathbb{R}^3\) and integrate in \(\mathbb{R}^6\). Recall the decomposition \(g = h(y,w) + q(w)\). Integration by parts yields

\[
(1 - 3\theta) \int_{\mathbb{R}^6} \chi_R g \varphi_{y_0} dw dw dy - \theta \int_{\mathbb{R}^6} g \varphi_{y_0} y \cdot \nabla_w \chi_R dw dy - (1 + \theta) \int_{\mathbb{R}^6} h \chi_R y \cdot \nabla_y \varphi_{y_0} dw dy
\]

\[
- \int_{\mathbb{R}^6} h \chi_R w \cdot \nabla_y \varphi_{y_0} dw dy - 3(1 + \theta) \int_{\mathbb{R}^6} h \chi_R \varphi_{y_0} dw dy
\]

\[
= \int_{\mathbb{R}^6} \varphi \chi_R Q_{L,w}(g,g) dw dy.
\]

We first take the limit \(R \to +\infty\). Note that \(|w \cdot \nabla_w \chi_R| \lesssim 1\) by (3.7) and \(w \cdot \nabla_w \chi_R\) converges to zero pointwise. Therefore, by the dominated convergence theorem, the second term vanishes. For the collision term, we use Lemma 3.2. The remaining terms converge by the dominated convergence theorem and the assumptions on \(h\) and \(q\). Therefore, we obtain

\[
(1 - 3\theta) \int_{\mathbb{R}^6} (q + h) \varphi_{y_0} dw dy - 3(1 + \theta) \int_{\mathbb{R}^6} h \varphi_{y_0} dw dy
\]

\[
- \int_{\mathbb{R}^6} h w \cdot \nabla_y \varphi_{y_0} dw dy - (1 + \theta) \int_{\mathbb{R}^6} h y \cdot \nabla_y \varphi_{y_0} dw dy = 0.
\]
Next, we perform the limit \( y_0 \to \infty \). Thanks to the assumption \((1 + |y| + |w|)h \in L^1(\mathbb{R}^6)\), all of the terms involving \( h \) vanish as \( y_0 \to \infty \) by the dominated convergence theorem. Hence, the above identity reduces to

\[
(1 - 3\theta) \int_{\mathbb{R}^3} q(w) \, dw = 0.
\]

Since \( q \geq 0 \), we conclude that \( q \equiv 0 \). The condition \( g \geq 0 \) and \( q = 0 \) implies \( h \geq 0 \). We now go back to (3.13) with \( q = 0 \), multiply it by \( \chi_{R_1}(y)\chi_{R_2}(w) \) with \( \chi_{R_1}(y) = \chi(y/R_1) \) and \( \chi_{R_2}(w) = \chi(w/R_2) \) and integrate in \( \mathbb{R}^6 \). Similarly as above, we first take the limit \( R_2 \to +\infty \) and get

\[
-2(1 + 3\theta) \int_{\mathbb{R}^6} \chi_{R_1} h \, dw \, dy - (1 + \theta) \int_{\mathbb{R}^6} h y \cdot \nabla \chi_{R_1} \, dw \, dy - \int_{\mathbb{R}^6} h w \cdot \nabla y \chi_{R_1} \, dw \, dy = 0.
\]

Using the assumption that \((1 + w)h \in L^1(\mathbb{R}^6)\), we can pass to the limit \( R_1 \to +\infty \) in the above equation by the dominated convergence theorem as we used above (in particular that \( y \cdot \nabla \chi_{R_1} \) is uniformly bounded and converges pointwise to zero) and obtain

\[
-2(1 + 3\theta) \int_{\mathbb{R}^6} h \, dw \, dy = 0,
\]

which implies \( h \equiv 0 \).

- Let \( \theta = 1/3 \). As before, let \( \chi_{R}(w) = \chi(w/R) \) a cutoff function and \( \varphi \in C^\infty_0(B(0, 1)) \) a smooth function such that \( \int_{\mathbb{R}^3} \varphi(y) \, dy = 1 \) and take \( \varphi_{y_0}(y) = \varphi(y + y_0) \). This time we multiply (3.13) with \( \chi_{R}(w)|w|^2\varphi_{y_0} \) for some \( y_0 \in \mathbb{R}^3 \) and integrate in \( \mathbb{R}^6 \). We obtain

\[
-2 \int_{\mathbb{R}^6} g \varphi_{y_0} |w|^2 \chi_{R} \, dw \, dy - \frac{1}{3} \int_{\mathbb{R}^6} g \varphi_{y_0} |w|^2 w \cdot \nabla_{w} \chi_{R} \, dw \, dy - 2 \int_{\mathbb{R}^6} h \chi_{R} \varphi_{y_0} |w|^2 \, dw \, dy - \frac{2}{3} \int_{\mathbb{R}^6} |w|^2 h \chi_{R} \chi_{R} \cdot \nabla_{y} \varphi_{y_0} \, dy \, dw - \int_{\mathbb{R}^6} |w|^2 \chi_{R} h w \cdot \nabla_{y} \varphi_{y_0} \, dw \, dy = \int_{\mathbb{R}^6} \varphi_{y_0} \chi_{R}(w)|w|^2Q_L(g, g) \, dw \, dy.
\]

Thanks to the condition \( q(1 + |w|^2) \in L^1_w \) and \( h(1 + |y| |w|^2 + |w|^3) \in L^1(\mathbb{R}^6) \) and Lemma 3.2 for the collision term, we can pass to the limit \( R \to +\infty \) using the dominated convergence theorem as above and we get

\[
- \frac{2}{3} \int_{\mathbb{R}^6} (q + h) \varphi_{y_0} |w|^2 \, dw \, dy - 4 \int_{\mathbb{R}^6} h \varphi_{y_0} |w|^2 \, dw \, dy + \frac{4}{3} \int_{\mathbb{R}^6} |w|^2 h y \cdot \nabla_{y} \varphi_{y_0} \, dy \, dw - \int_{\mathbb{R}^6} |w|^2 h w \cdot \nabla_{y} \varphi_{y_0} \, dw \, dy = 0.
\]

The limit \( y_0 \to +\infty \), using again \( h(1 + |y| |w|^2 + |w|^3) \in L^1_{y,w} \), gives

\[
- \frac{2}{3} \int_{\mathbb{R}^3} q |w|^2 \, dw = 0,
\]
which implies \( q \equiv 0 \). To show that also \( h \equiv 0 \), we multiply (3.13) with \( q = 0 \) by \( \chi_{R_1}(w)\chi_{R_2}(y)|w|^2 \) and integrate in \( \mathbb{R}^6 \). After taking the limit \( R_1 \to +\infty \) we obtain

\[
-\frac{2}{3} \int_{\mathbb{R}^6} h\chi_{R_2}|w|^2 \, dw \, dy - 4 \int_{\mathbb{R}^6} h\chi_{R_2}|w|^2 \, dw \, dy + \\
-\frac{4}{3} \int_{\mathbb{R}^6} |w|^2 h y \cdot \nabla y \chi_{R_2} \, dy \, dw - \int_{\mathbb{R}^6} |w|^2 h w \cdot \nabla y \chi_{R_2} \, dy \, dw = 0.
\]

The limit \( R_2 \to +\infty \) yields

\[
-\frac{14}{3} \int_{\mathbb{R}^6} h|w|^2 \, dw \, dy = 0,
\]

which implies, since \( h \geq 0 \), that \( h \equiv 0 \).

- Let \( \theta = -1/3 \). Mimicking the same calculation of the case \( \theta = 1/3 \), we multiply (3.13) by \( \chi_R(w)|w|^2 \varphi_{y_0} \), integrate over \( \mathbb{R}^6 \) and perform the limit \( R \to +\infty \). We get

\[
\frac{8}{3} \int_{\mathbb{R}^6} (q + h)\varphi_{y_0}|w|^2 \, dw \, dy - 2 \int_{\mathbb{R}^6} h\varphi_{y_0}|w|^2 \, dw \, dy + \\
-\frac{2}{3} \int_{\mathbb{R}^6} |w|^2 h y \cdot \nabla y \varphi_{y_0} \, dy \, dw - \int_{\mathbb{R}^6} |w|^2 h w \cdot \nabla y \varphi_{y_0} \, dy \, dw = 0.
\]

The limit \( y_0 \to +\infty \), using again \( h(1 + |w|^3) \in L^1(\mathbb{R}^6) \), gives

\[
\frac{8}{3} \int_{\mathbb{R}^3} q|w|^2 \, dw = 0,
\]

which implies \( q \equiv 0 \). To show that also \( h \equiv 0 \), we multiply (3.13) with \( q = 0 \) by \( \chi_{R_1}(w)\chi_{R_2}(y)|w|^2 \) and integrate in \( \mathbb{R}^6 \). The limits \( R_1, R_2 \to +\infty \) yield

\[
\frac{2}{3} \int_{\mathbb{R}^6} h|w|^2 \, dw \, dy = 0,
\]

which implies, since \( h \geq 0 \), that \( h \equiv 0 \).

This finishes the proof of the theorem.

\[
\square
\]

4. The Vlasov-Poisson-Landau System

In this section we analyze the following system:

\[
\partial_t f + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = Q_L(f, f),
\]

with

\[
F[f] = C \int_{\mathbb{R}^3} \frac{x - z}{|x - z|^3} \left[ \int_{\mathbb{R}^3} f(z, v) \, dv - n_0(z) \right] \, dz,
\]

where \( n_0(x) \geq 0 \) is a fixed function that models a neutralizing background. If \( C \geq 0 \) we are in the repulsive interaction case, if \( C < 0 \) we are in the attractive case.

Unlike Landau and Boltzmann, the Vlasov-Poisson-Landau equation only has a one-parameter scaling symmetry, and hence there is only one case to consider: \( \gamma = -3 \) and \( \theta = -\frac{1}{3} \). Therefore, our ansatz becomes

\[
f(t, x, v) = \phi(x, t, v) + \frac{1}{(t)^{1/3}} g \left( \frac{x}{(t)^{2/3}}, (t)^{1/3} v \right).
\]

For the analysis of the Vlasov-Landau-Poisson system, our proof requires the use of \( \ln g \) as a test function, which requires the following additional assumptions on the profile

\[
(1 + |w|)(\ln g - g) \in L^1_{y,w}, \quad \nabla \sqrt{g} \in L^\infty_y L^2_w, \quad g \in L^1_{y,w}.
\]
One new detail must be addressed: due to the non-locality in $x$ introduced by the interaction term, we must be more specific about the global structure of the solution. Our methods can handle any of the following three cases, each of which is physically relevant:

(a) $n_0 = 0$ and $f \in L^1(\mathbb{R}^6)$. This case is natural for studying gravitational interactions (where $f$ models the density of stars or galaxies and hence in the attractive case).

(b) The physical domain is $T^3_x$ and we take $n_0(x) = n_0 = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^6} f(x,v) dv$. This case is most natural for studying periodic perturbations arising in the kinetic theory of plasmas (where $f$ will model the density of electrons in a plasma and the $n_0$ models a homogeneous background of ions, hence the interactions are repulsive).

(c) The solution is given by $f(t,x,v) = \mu(v) + h(t,x,v)$ where $\mu$ is a Maxwellian with fixed density, momentum, and temperature, $h \in L^1(\mathbb{R}^6)$ with average zero, and $n_0 = \int_{\mathbb{R}^3} \mu(v) dv$. This case is most natural for studying localized perturbations of a homogeneous plasma (here $f$ models the density of electrons in a plasma and the $n_0$ models a homogeneous background of ions, hence the interactions are repulsive).

Our proof easily adapts to any of these three cases, so we focus on the simplest one, which is case (a). It is straightforward to extend the argument to cases (b) and (c). As in the previous section, we will assume $\phi$ satisfies (2.7), which, for $i = 0$ and $\theta = \frac{1}{\gamma} = -\frac{1}{3}$, reads

\[
\lim_{t \to 0} \left( -t \right)^{1 + \frac{1}{p} + \frac{2}{3} \ell - \frac{1}{3} j} \sup_{|y| \leq R} \left\| D_x^j D_v^\ell \phi(t, (-t)^{2/3} y, \cdot) \right\|_{L^p_x} = 0,
\]

for all $1 \leq p \leq \infty$, $0 \leq j \leq 1$, $0 \leq \ell \leq 2$, $R > 0$. Due to the nonlocality in $x$ of the interaction force, we also enforce the condition that the density $\rho_\phi = \int_{\mathbb{R}^3} \phi(t,x,v) dv$ satisfies

\[
\lim_{t \to 0} \left( -t \right)^{1 + \frac{1}{p}} \| \rho_\phi(t) \|_{L^p_x} = 0,
\]

for some $p < 3$ and also for $p = \infty$ (and hence everything in between by interpolation).

**Theorem 4.1.** Let $f$ satisfy (2.3) and (2.4), $\phi$ satisfy (2.7) and (4.4), and $g$ satisfy (2.5) and (4.2). Then for any solution to the Vlasov Landau Poisson system of the form

\[
f(t,x,v) = \phi(x,t,v) + \frac{1}{(-t)} g \left( \frac{x}{(-t)^{2/3}}, (-t)^{1/3} v \right),
\]

we must have $g \equiv 0$.

**Proof.** Define the self similar variables

\[
y := \frac{x}{(-t)^{2/3}}, \quad w := v(-t)^{1/3}.
\]

A brief computation shows that $F[f]$ transforms as

\[
F \left[ \phi + \frac{1}{(-t)} g \right] \left( (-t)^{2/3} y \right) = F[\phi] \left( (-t)^{2/3} y \right) + \frac{1}{(-t)^{4/3}} F[g](y).
\]
We now plug (4.1) into the system (1.5). The resulting equation, after multiplying by \((-t)^2\), reads as

\[
(-t)^2[\partial_t \phi + v \cdot \nabla_x \phi + F[\phi] \cdot \nabla_v \phi - Q_L(\phi, \phi)]
\]

\[
+ (-t)^{4/3} F[\phi] \cdot \nabla_w g + (-t)^{2/3} F[g] \cdot \nabla_v \phi
\]

\[
- (t)[Q_{L,w}(\phi, g) + Q_{L,w}(g, \phi)]
\]

\[
g + \frac{2}{3} y \cdot \nabla_y g - \frac{1}{3} w \cdot \nabla_w g + w \cdot \nabla_y g + F[g] \cdot \nabla_w g = Q_{L,w}(g, g) = 0.
\]

We now define the error as

\[
E(\phi, g) := (-t)^2[\partial_t \phi + v \cdot \nabla_x \phi + F[\phi] \cdot \nabla_v \phi - Q_L(\phi, \phi)] + (-t)^{4/3} F[\phi] \cdot \nabla_w g
\]

\[
+ (-t)^{2/3} F[g] \cdot \nabla_v \phi - (t)[Q_{L,w}(\phi, g) + Q_{L,w}(g, \phi)].
\]

We claim \(E(\phi, g) \to 0\) as \(t \to 0\), uniformly on compact sets of \(R^3_y \times R^3_w\). All the terms, except the ones with \(F[\cdot]\), appeared already in \(E_1\) and \(E_2\) and converge to zero, as proven in Lemma 3.3 and 3.4. We start by analyzing

\[
(-t)^2 F[\phi] \cdot \nabla_v \phi.
\]

We have

\[
\lim_{t \to 0} \sup_{|y| \leq R} (-t)^2 F[\phi] \cdot \nabla_v \phi \leq \lim_{t \to 0} \sup_{|y| \leq R} (-t)^{4/3} F[\phi] \leq \lim_{t \to 0} \sup_{|y| \leq R} (-t)^{2/3} \|\nabla_v \phi\|_{L^\infty} = 0,
\]

thanks to (4.3) with \(p = \infty, \ell = 0, j = 1\). Note that \(\sup_{|y| \leq R} \|F[g]\|\) is bounded thanks to our assumption \(g \in L_{y,loc}^\infty F_w^1\) and \(g \in L_y^1\), and therefore we similarly have

\[
\lim_{t \to 0} \sup_{|y| \leq R} \|(-t)^{2/3} F[g] \cdot \nabla_v \phi\| \leq \lim_{t \to 0} \sup_{|y| \leq R} \|F[g] \| (-t)^{2/3} \|\nabla_v \phi\|_{L^\infty} = 0.
\]

We turn next to the term \((-t)^{4/3} F[\phi] \cdot \nabla_w g\). For this, we use the interpolation (3.5) with \(s = -2, r = \infty\), and \(1 \leq p < 3\) to obtain

\[
(-t)^{4/3} \|F[\phi]\| \leq (-t)^{4/3} \|\rho_\phi\|_{L_p}^{p/3} \|\rho_\phi\|_{L^\infty}^{1-p/3}
\]

\[
\lesssim \left((-t)^{1+\frac{1}{p}} \|\rho_\phi\|_{L_p}\right)^{p/3} \left((-t) \|\rho_\phi\|_{L^\infty}\right)^{1-p/3},
\]

and, hence, the associated term vanishes by (4.4).

Thus, in the limit as \(t \to 0\), we obtain

\[
g + \frac{2}{3} y \cdot \nabla_y g - \frac{1}{3} w \cdot \nabla_w g + w \cdot \nabla_y g + F[g] \cdot \nabla_w g = Q(g, g).
\]

From where, we multiply by \(\chi_{R_1}(w) \chi_{R_2}(y) \log g\) and integrate in both variables; after integration by parts we get

\[
\int_{R^6} g \chi_{R_2}(y) \chi_{R_1}(w) \, dw \, dy - \frac{2}{3} \int_{R^6} (g \ln g - g) \chi_{R_1}(w) y \cdot \nabla_y \chi_{R_2}(y) \, dw \, dy
\]

\[
+ \frac{1}{3} \int_{R^6} (g \ln g - g) \chi_{R_2}(y) w \cdot \nabla_w \chi_{R_1}(w) \, dw \, dy - \int_{R^6} (g \ln g - g) \chi_{R_1}(w) w \cdot \nabla_y \chi_{R_2}(y) \, dw \, dy
\]

\[
- \int_{R^6} (g \ln g - g) \chi_{R_2}(y) F[g] \cdot \nabla_w \chi_{R_1}(w) \, dw \, dy
\]

\[
= \int_{R^6} \chi_{R_2}(y) \chi_{R_1}(w) Q_{L,w}(g, g) \, dw \, dy.
\]
With the assumptions on $g$, we can pass to the limit $R_1 \to +\infty$ by the dominated convergence theorem and get
\[
\int_{\mathbb{R}^6} g \chi_{R_2}(y) \, dw \, dy - \frac{2}{3} \int_{\mathbb{R}^6} (g \ln g - g) y \cdot \nabla_y \chi_{R_2}(y) \, dw \, dy
- \int_{\mathbb{R}^6} (g \ln g - g) w \cdot \nabla_y \chi_{R_2}(y) \, dw \, dy \leq 0,
\]
using Lemma 3.2 for the right hand side. Thanks to the assumption
\[(1 + w)(g \ln g - g) \in L^1_{w,y},\]
we take the limit $R_2 \to +\infty$ and obtain
\[
\int_{\mathbb{R}^6} g \, dw \, dy \leq 0,
\]
which implies $g \equiv 0$. □

5. The Boltzmann equation

We recall the Boltzmann equation
\[
\partial_t f + v \cdot \nabla_x f = Q_B(f, f) := \int_{\mathbb{R}^3} \int_{S^2} B(v - v_s, \sigma) [f(v')f(v') - f(v_s)f(v)] \, d\sigma \, dv_s.
\]
The velocities are related by the formulas
\[
v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2} \sigma, \tag{5.1}
\]
\[
v'_s = \frac{v + v_s}{2} - \frac{|v - v_s|}{2} \sigma, \tag{5.2}
\]
and the pre-post collisional angle $\eta$ (usually denoted $\theta$ in the literature) is defined by
\[
\cos \eta = \left\langle \frac{v - v_s}{|v - v_s|}, \sigma \right\rangle.
\]
We take the standard non-cutoff collision kernel described by
\[
B(v - v_s, \sigma) := |v - v_s|^\gamma b(\cos \eta),
\]
for some $\gamma \in (-3, 1]$, with the angular cross-section $b$ satisfying the asymptotics
\[
b(\cos \eta) \approx \eta^{-2 - 2s} \quad \text{as } \eta \to 0, \tag{5.3}
\]
for some $s \in (0, 1)$. We assume $\gamma + 2s < 0$ for ease of presentation. Results similar to Theorem 5.1 should also be available when $\gamma + 2s \geq 0$.

As mentioned above, the Boltzmann equation obeys the same family of scaling laws as the Landau equation, so the approximately self-similar ansatz (1.4) takes the same form.

In our main result for the Boltzmann equation, we derive the same conclusion as Theorem 2.2 under similar hypotheses:

**Theorem 5.1.** Let $\gamma > -3$ and $s \in (0, 1)$ be such that $\gamma + 2s < 0$, and assume $-1 < \theta < 1/(2s)$. Let $f$ be a smooth solution of the Boltzmann equation (1.6) that satisfies (2.3) and (2.4). Assume that $\phi$ satisfies (2.7). For $g$, assume it satisfies (2.5) as well as $(1 + |w|^{2+\gamma})g(y, \cdot) \in L^1_w(\mathbb{R}^3)$ for all $y \in \mathbb{R}^3$, and that there exist $h$ and $q$ such that
\[
g(y, w) = q(w) + h(y, w),
\]
with
\[(1 + |y| + |w|) h \in L^1_{y,w}(\mathbb{R}^6) \quad \text{and} \quad q \in L^1_w(\mathbb{R}^3).\]

Finally, if $\theta = \pm 1/3$ we additionally assume that
\[(1 + |y| |w|^2 + |w|^3) h \in L^1_{y,w}(\mathbb{R}^6) \quad \text{and} \quad (1 + |w|^2) q \in L^1_w(\mathbb{R}^3).\]

Then, for any solution to the Boltzmann equation (1.6) of the form
\[f(t,x,v) = \phi(t,x,v) + \frac{1}{(-t)^{1+\theta(3+\gamma)}} g \left( \frac{x}{(-t)^{1+\theta}}, \frac{v}{(-t)^{1+\theta}} \right),\]
we must have $g \equiv 0$.

**Remark 5.2.** As for Landau, if $g \in L^1_{y,w}$, then it suffices to assume $(1 + |w|)g \in L^1_{y,w}$ (and $(1 + |w|^3)g \in L^1_{y,w}(\mathbb{R}^6)$).

Specializing to the homogeneous case as above, we have the following result:

**Theorem 5.3.** With $\gamma$ and $s$ as in Theorem 5.1 and $\frac{1}{|a|} < \theta < \frac{1}{2\gamma}$, assume that $f = f(v,t)$ has finite mass and second moment and satisfies
\[f \in C^\infty((-T,0) \times \mathbb{R}^3), \quad f \in C^\infty((-T,0] \times \mathbb{R}^3).\]

Assume that $\phi$ satisfies (2.7), and that $g = g(v)$ is such that
\[g \in C^\infty(\mathbb{R}^3), \quad (1 + |w|^{2+\gamma})g \in L^1_w.\]

If $\theta = 1/3$, then additionally assume that $g$ satisfies $(1 + |w|^2)g \in L^1_w$.

Then, for any solution to the homogeneous Boltzmann equation of the form
\[f(t,v) = \phi(t,v) + \frac{1}{(-t)^{1+\theta(3+\gamma)}} g \left( \frac{v}{(-t)^{1+\theta}} \right),\]
we must have $g \equiv 0$, and hence no approximate self-similar singularity of this type can occur.

To prove Theorem 5.1, we need the following decomposition of the collision operator $Q_B(f_1, f_2)$ into two terms: by adding and subtracting $f_1(v'_s)f_2(v)$ inside the integral, we write $Q_B(f_1, f_2) = Q_1(f_1, f_2) + Q_2(f_1, f_2)$, with
\[
Q_1(f_1, f_2) = \text{p.v.} \int_{\mathbb{R}^3} \int_{S^2} b(\cos \eta) |v - v_s|^\gamma (f_2(v') - f_2(v)) f_1(v'_s) \, d\sigma \, dv_s,
\]
\[
Q_2(f_1, f_2) = f_2(v) \int_{\mathbb{R}^3} \int_{S^2} b(\cos \eta) |v - v_s|^\gamma (f_1(v'_s) - f_1(v_s)) \, d\sigma \, dv_s,
\]
for functions $f_1$ and $f_2$ defined on $\mathbb{R}^3$. The term $Q_1(f_1, f_2)$ acts as a fractional differential operator of order $2s$, and can roughly be thought of as analogous to the term $\text{tr}(\bar{a}D_v^2 f_2)$ from the Landau collision operator. The following lemma, quoted from [70], makes this point of view clearer:

**Lemma 5.4.** ([70], Section 4) There holds
\[Q_1(f_1, f_2) = \int_{\mathbb{R}^3} [f_2(v + h) - f_2(v)] K_{f_1}(v, h) \, dh,
\]
where
\[K_{f_1}(v, h) \approx |h|^{-3-2s} \int_{\{z: z \cdot h = 0\}} f_2(v + z)|z|^{\gamma + 2s + 1} \, dz,
\]
with implied constants depending only on $\gamma, s$, and the angular cross-section $b$. The kernel $K_{f_1}$ is symmetric ($K_{f_1}(v, -h) = K_{f_1}(v, h)$) and satisfies the following bounds: for any $r > 0$,

$$
\int_{B_r \setminus B_r} K_{f_1}(v, h) \, dh \leq C \left( \int_{\mathbb{R}^3} |z|^{\gamma+2s} f_1(v + z) \, dz \right) r^{-2s},
$$

whenever the right-hand side is finite.

Estimating the convolution as in the proof of Lemma 3.1, we see that Lemma 5.4 implies

$$
\int_{B_r \setminus B_r} K_{f_1}(v, h) \, dh \leq C ||f_1||_{L^p_v}^2 ||f_1||_{L^\infty_v}^{2s},
$$

for all $r > 0$ and $1 \leq p < 3/(\gamma + 3 + 2s)$. For $f_1$ depending on $(x, v)$ or $(t, x, v)$, we will write $K_{f_1}(x, v, h)$ or $K_{f_1}(t, x, v, h)$. Note that the “p.v.” in $Q_1$ is only necessary when $s > 1/2$. We omit the “p.v.” from now on, since our functions are smooth enough ($C^2$ in $v$) that the value of the integral is well-defined.

For $Q_2(f_1, f_2)$, symmetry effects for grazing collisions (see [1]) imply the following representation:

**Lemma 5.5.** [70, Lemmas 5.1 and 5.2] The integral in $Q_2(f_1, f_2)$ satisfies

$$
Q_2(f_1, f_2) = f_2(v) \int_{\mathbb{R}^3} f_1(v + z)(C|z|\gamma) \, dz,
$$

where the constant $C$ depends only on $\gamma$ and $s$.

In other words, surprisingly, $Q_2(f_1, f_2)$ is equal up to a constant to $\tilde{c} f_1 f_2$ in the notation of the Landau equation.

Lemmas 5.4 and 5.5 imply in particular that $Q_B(f_1, f_2)$ is well-defined in a pointwise sense whenever $f_1 \in L^1_v \cap L^\infty_w$ and $f_2 \in L^\infty_v \cap C^2_w$. As in the previous sections (see Lemma 3.2), we need to use a form of the identities $\int Q_B(g, g) \, dw = \int |w|^2 Q_B(g, g) \, dw = 0$:

**Lemma 5.6.** With $\chi \in C^\infty(B(0, 2))$ a smooth-cutoff with $\chi(|x|) = 1$ for $|x| \leq 1$, and with $g \in L^2_v \cap L^\infty_w(\mathbb{R}^3)$ satisfying (2.3) as well as $(1 + |w|^{2+\gamma})g \in L^1$, we have

$$
\lim_{R \to \infty} \int_{\mathbb{R}^3} \chi \left( \frac{|w|}{R} \right) Q_B(g, g) \, dw = 0,
$$

and

$$
\lim_{R \to \infty} \int_{\mathbb{R}^3} \chi \left( \frac{|w|}{R} \right) |w|^2 Q_B(g, g) \, dw = 0.
$$

This lemma is more or less understood in the literature on the Boltzmann equation. We give a proof for the convenience of the reader, and because we could not find an easy reference to apply in our setting.

**Proof.** The well-known weak formulation of the Boltzmann collision operator allows one to make sense of integrals of the form $\int_{\mathbb{R}^3} \varphi Q_B(g, g) \, dw$ using smoothness of $\varphi$. For any function $f$, let us introduce the abbreviations $f = f(w)$, $f_\ast = f(w_\ast)$, $f' = f(w')$, and $f'_\ast = f(w'_\ast)$. Applying the pre-post collisional change of variables $(\sigma, w, w_\ast) \leftrightarrow (\sigma, w', w'_\ast)$ (with unit Jacobian) one has

$$
\int_{\mathbb{R}^3} \varphi Q_B(g, g) \, dw = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(w - w_\ast, \sigma) g g_\ast [\varphi' - \varphi] \, d\sigma \, dw_\ast \, dw.
$$
Symmetrizing further with the change of variables $w \leftrightarrow w_*$, which also exchanges $w'$ and $w'_*$, one has

\begin{equation}
\int_{\mathbb{R}^3} \varphi Q_B(g, g) \, dw = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} B(w - w_*, \sigma) g g_*(\varphi'_* + \varphi' - \varphi_ - \varphi) \, d\sigma \, dw_* \, dw.
\end{equation}

These formal calculations can be justified rigorously under our assumption that $g$ is smooth, provided that $\varphi$ is (say) $C^2$ and compactly supported.

The expression $\varphi'_* + \varphi' - \varphi_ - \varphi$ is equal to zero for the following three cases: $\varphi = 1$, $\varphi = w$, and $\varphi = |w|^2$. This reflects the conservation of mass, momentum, and energy during collisions.

Taylor expanding $\varphi$ and using $w'_* + w' = w_* + w$ and $|w'_* - w_*| = |w' - w|$, we have

\begin{align*}
\varphi'_* + \varphi' - \varphi_ - \varphi &= \nabla \varphi_* \cdot (w'_* - w_*) + \nabla \varphi \cdot (w' - w) + O(\|D^2 \varphi\|_{L^\infty}|w' - w|^2) \\
&= (\nabla \varphi - \nabla \varphi_*) \cdot (w' - w) + O(\|D^2 \varphi\|_{L^\infty}|w' - w|^2).
\end{align*}

It follows from the geometry of collisions that $|w' - w| \approx |w_*|_\eta$. Therefore, the second term in the last expression is proportional to $\eta^2 |w_*|^2$, which is good enough to cancel the angular singularity $\eta^{-2-2s}$, but the first term is only proportional to $\eta |w - w_*|^2$. We get around this problem in a standard way, by parametrizing $S^2$ in spherical coordinates $(\phi, \eta) \in [0, 2\pi] \times [0, \pi]$ (where $\eta = 0$ corresponds to $w = w'$) and realizing that $\left| \int_0^{2\pi} (w' - w) \, d\phi \right| \lesssim |w - w_*| \eta^2$. We now have

\begin{equation}
\left| \int_0^{2\pi} [\varphi'_* + \varphi' - \varphi_ - \varphi] \, d\phi \right| \lesssim \|D^2 \varphi\|_{L^\infty} \eta^2 |w - w_*|^2,
\end{equation}

and

\begin{equation}
\left| \int_{\mathbb{R}^3} \varphi Q_B(g, g) \, dw \right| \lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} g g_* |w - w_*|^{-2-2s} \eta^2 \sin \eta \, d\eta \, dw_* \, dw \\
\lesssim \|D^2 \varphi\|_{L^\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g g_* |w - w_*|^{-2} \, dw_* \, dw.
\end{equation}

The last integral is convergent by our assumption that $(1 + |w|^{\gamma+2}) g \in L^1$.

Now, with the choice $\varphi(w) = \chi(|w|/R)$, since $\|D^2 \varphi\|_{L^\infty} \lesssim R^{-2}$, we see directly that $\int \chi(|w|/R) Q_B(g, g) \, dw \to 0$ as $R \to \infty$.

If we choose $\varphi(w) = |w|^2 \chi(|w|/R)$, then $\|D^2 \varphi\|_{L^\infty}$ is bounded independently of $R$. Writing (5.6) as the integral over $\mathbb{R}^3 \times \mathbb{R}^3 \times [0, \pi]$ of $F_R(w, w_*, \eta) := |w - w_*|^{-2} \chi(|w|/R) g g_* \int_0^{2\pi} [\varphi'_* + \varphi' - \varphi_ - \varphi] \, d\phi,$

then $F_R$ converges to 0 pointwise as $R \to \infty$, and by the above integrability estimates, we may apply Dominated Convergence to conclude $\int |w|^2 \chi(|w|/R) Q_B(g, g) \, dw \to 0$.

Now we are ready to prove our main result for the Boltzmann equation.

Proof of Theorem 5.1. Proceeding as in the proof of Theorem 2.2 we plug the ansatz (1.4) into the Boltzmann equation (1.6), and change variables to $y$ and $w$. The left-hand side transforms in the same way as before. For the right-hand side,

\begin{equation*}
Q_B(f, f) = Q_B(\phi, \phi) + \frac{1}{(-t)^{1+\theta(3+\gamma)}} [Q_B(\phi, g) + Q_B(g, \phi)] + \frac{1}{(-t)^{2+2\theta(3+\gamma)}} Q_B(g, g),
\end{equation*}


Now we are ready to prove our main result for the Boltzmann equation.
where \( g \) is evaluated at \((x/(-t)^{1+\theta}, v/(-t)\theta)\). Applying the decomposition \( Q_B = Q_1 + Q_2 \) and changing variables appropriately, we have

\[
Q_1(\phi, g) \approx \int_{\mathbb{R}^3} [g((v + h)/(-t)\theta) - g(v/(-t)\theta)]|h|^{-3-2s} \int_{z \perp h} \phi(v + z)|z|^\gamma dz dh
\]

\[
= (-t)^{-2s\theta} \int_{\mathbb{R}^3} [g(w + \tilde{h}) - g(w)]|\tilde{h}|^{-3-2s} \int_{z \perp h} \phi((-t)\theta w + z)|z|^\gamma dz d\tilde{h}
\]

\[
=: (-t)^{-2s\theta} \hat{Q}_1(\phi, g),
\]

and

\[
Q_1(g, \phi) \approx \int_{\mathbb{R}^3} [\phi(v + h) - \phi(v)]|h|^{-3-2s} \int_{z \perp h} g((v + z)/(-t)^\theta)|z|^\gamma dz dh
\]

\[
= (-t)^{(\gamma+2s+3)\theta} \int_{\mathbb{R}^3} [\phi((-t)^\theta w + h) - \phi((-t)^\theta w)]|\tilde{h}|^{-3-2s} \int_{z \perp h} g(w + \tilde{z})|\tilde{z}|^\gamma dz d\tilde{h}
\]

\[
=: (-t)^{(\gamma+2s+3)\theta} \hat{Q}_1(g, \phi).
\]

(Note that \( \{z \perp h\} \) is a two-dimensional subspace.) By similar calculations, we have

\[
Q_1(g, g) = (-t)^{(\gamma+3)\theta} \int_{\mathbb{R}^3} [g(w + h) - g(w)]|\tilde{h}|^{-3-2s} \int_{z \perp h} g(w + \tilde{z})|\tilde{z}|^\gamma dz d\tilde{h}
\]

\[
=: (-t)^{(\gamma+3)\theta} \hat{Q}_1(g, g).
\]

Since \( Q_2(h_1, h_2) \approx c_h h_2 \), calculations from the proof of Theorem 2.2 imply

\[
Q_2(\phi, g) \approx c_\phi g, \quad Q_2(g, \phi) \approx (-t)^{\theta(\gamma+3)}c_\theta \phi,
\]

and we abuse notation by writing \( = \) instead of \( \approx \) (which amounts to a change of constants). Making these substitutions in the right-hand side of (1.6), multiplying through by \((-t)^{2+\theta(3+\gamma)}\), and grouping terms, we have

\[
0 = g + (1 + \theta)g \cdot \nabla_y g + \theta w \cdot \nabla_w g + w \cdot \nabla_y g - Q_B, g, g \quad - (t)^{1-2s\theta} \hat{Q}_1(\phi, g)
\]

\[
+ (t)^{1+\theta(3+\gamma)}(\partial_t \phi + v \cdot \nabla_x \phi - Q_B(\phi, \phi)),
\]

where, as above, \( \phi \) is understood to stand for \( \phi(t, (-t)^{1+\theta}y, (-t)^\theta w) \), and \( g = g(y, w) \). We remark that, as \( s \to 1 \), all exponents in this expansion converge to the exponents of the corresponding terms in (3.8) in the proof of Theorem 2.2.

The error is defined as

\[
\mathcal{E}(\phi, g) = -(-t)^{-1-2s\theta} \hat{Q}_1(\phi, g) - (-t)^{1+\theta(3+\gamma)} \hat{Q}_1(g, \phi)
\]

\[
+ (-t)^{1+\theta(3+\gamma)}(\partial_t \phi + v \cdot \nabla_x \phi - Q_B(\phi, \phi))
\]

\[
=: (-t)^{1+\theta(3+\gamma)}(\partial_t \phi - Q_1(\phi, \phi) - (\partial_t \phi + v \cdot \nabla_x \phi - Q_B(\phi, \phi)).
\]

We claim that for all \( R_1, R_2 > 0 \),

\[
\lim_{t \to 0} \sup_{|y| \leq R_1} \sup_{|w| \leq R_2} |\mathcal{E}(\phi, g)| = 0.
\]

First, all terms in \( \mathcal{E}(\phi, g) \) that do not involve \( Q_1 \) or \( Q \) are equal to corresponding terms in the proof of Theorem 2.2 so the same arguments (which do not require any restriction on \( \theta \) from above) imply convergence to zero in the sense of (5.9) for those terms.
Now we address the singular integral terms. For any integer \( k \), let \( A_k \) denote the annulus \( \{2^k \leq |v| < 2^{k+1}\} \). Splitting \( \tilde{Q}_1(\phi, g) \) into integrals over \( |\tilde{h}| < 1 \) and \( |\tilde{h}| \geq 1 \), we have, using (5.5),

\[
(5.10) \quad (-t)^{1-2\theta} \int_{|\tilde{h}| \geq 1} |g(w + \tilde{h}) - g(w)| K_1(t, (-t)^{1+\theta} y, (-t)^{\theta} w, \tilde{h}) \, d\tilde{h}
\]

\[
= (-t)^{1-2\theta} \sum_{k \geq 0} \int_{A_k} |g(w + \tilde{h}) - g(w)| K_1(t, (-t)^{1+\theta} y, (-t)^{\theta} w, \tilde{h}) \, d\tilde{h}
\]

\[
\lesssim (-t)^{1-2\theta} \|g\| \|w\|^{2}\sum_{k \geq 0} 2^{-2k} \| \phi(t, (-t)^{1+\theta} y, \cdot) \| L^p_{\gamma} \| \phi(t, (-t)^{1+\theta} y, \cdot) \| L^\infty_{\gamma}^{1-(\gamma+2s+3)p/3}
\]

\[
\leq \left[ (-t)^{1+\theta(3+\gamma)-3\theta/p} \| \phi(t, (-t)^{1+\theta} y, \cdot) \| L^p_{\gamma} \right]^{(\gamma+2s+3)p/3}
\]

\[
\cdot \left[ (-t)^{1+\theta(3+\gamma)} \| \phi(t, (-t)^{1+\theta} y, \cdot) \| L^\infty_{\gamma} \right]^{1-(\gamma+2s+3)p/3}
\]

with \( 1 \leq p < 3/(\gamma + 2s + 3) \). The second factor converges to 0 by (2.7). For the first factor, we use (2.7) again, which requires \( p > \theta/\left( 1 + \theta(3 + \gamma) \right) \). Therefore, an admissible \( p \) satisfies

\[
\frac{3\theta}{1 + \theta(3 + \gamma)} < p < \frac{3}{\gamma + 2s + 3},
\]

which is possible since \( \theta < 1/(2\gamma) \).

For the integral over \( |\tilde{h}| < 1 \), we write

\[
g(w + \tilde{h}) - g(w) = \nabla_w g(w) \cdot \tilde{h} + E(w, \tilde{h})|\tilde{h}|^2,
\]

with \( |E(w, \tilde{h})| \lesssim \| D^2 w g \| L^\infty \). By the symmetry of the kernel \( K_1 \), the term with \( \nabla_w g(w) \) vanishes, and we have, with \( p \) as in the previous paragraph,

\[
(-t)^{1-2\theta} \int_{|\tilde{h}| < 1} |g(w + \tilde{h}) - g(w)| K_1(t, (-t)^{1+\theta} y, (-t)^{\theta} w, \tilde{h}) \, d\tilde{h}
\]

\[
= (-t)^{1-2\theta} \sum_{k < 0} \int_{A_k} E(w, \tilde{h})|\tilde{h}|^2 K_1(t, (-t)^{1+\theta} y, (-t)^{\theta} w, \tilde{h}) \, d\tilde{h}
\]

\[
\lesssim (-t)^{1-2\theta} \| D^2 w g \| L^\infty \sum_{k < 0} 2^{(2-2s)k} \| \phi(t, (-t)^{1+\theta} y, \cdot) \| L^p_{\gamma} \| \phi(t, (-t)^{1+\theta} y, \cdot) \| L^\infty_{\gamma}^{1-(\gamma+2s+3)p/3},
\]

which converges to zero using (2.7), as above. For \( \tilde{Q}_1(\phi, g) \), we divide the \( h \) integral into the annuli \( \tilde{A}_k = \{ (-t)^{\theta} 2^k < |v| \leq (-t)^{\theta} 2^{k+1} \} \) (which are the same as \( A_k \), read in \( h \) variables rather than \( v \) variables).
than $\hat{h}$). By a similar Taylor expansion for $|h| < 1$, we have, with $p$ as above,

$$
(-t)^{1+\theta(\gamma+2s+3)} \left( \sum_{k\geq 0} \int \mathcal{A}_k \left[ \phi((-t)^\theta w + h) - \phi((-t)^\theta w) \right] K_g(y, w, h) \, dh \right.
$$

$$
+ \sum_{k<0} \int \mathcal{A}_k \left[ \phi((-t)^\theta w + h) - \phi((-t)^\theta w) \right] K_g(y, w, h) \, dh \right)
$$

\begin{equation}
\lesssim (-t)^{1+\theta(\gamma+2s+3)} \left( \left\| \phi(t, (-t)^{1+\theta} y, \cdot) \right\|_{L^\infty} \sum_{k\geq 0} (-t)^{-2s\theta} 2^{-2sk} \right.
$$

$$
\left. + \left\| D_v^2 \phi(t, (-t)^{1+\theta} y, \cdot) \right\|_{L^\infty} \sum_{k<0} (-t)^{(2-2s)\theta} 2^{(2-2s)k} \right) \left\| g \right\|_{L^\infty_{w}}^{(\gamma+2s+3)/3} \left\| g \right\|_{L^\infty_{w}}^{-(\gamma+2s)/3} \right)
$$

\begin{equation}
\lesssim (-t)^{1+\theta(\gamma+3)} \left\| \phi(t, (-t)^{1+\theta} y, \cdot) \right\|_{L^\infty} + (-t)^{1+\theta(\gamma+5)} \left\| D_v^2 \phi(t, (-t)^{1+\theta} y, \cdot) \right\|_{L^\infty} ,
\end{equation}

which converges to 0 by \[(2.7)\].

For the term $Q_1(\phi, \phi)$, we apply \[51\] Lemma 2.3 directly to obtain, with $p$ as above and $\phi = \phi(t, (-t)^{1+\theta} y, (-t)^\theta w)$,

$$
(-t)^{2+\theta(3+\gamma)} Q_1(\phi, \phi) \lesssim (-t)^{2+\theta(3+\gamma)} \left\| D_v^2 \phi \right\|_s^{s} \left\| \phi \right\|_{L^\infty_w}^{1-s} \int_{\mathbb{R}^3} \phi(t, (-t)^{1+\theta} y, (-t)^\theta w - z) |z|^{\gamma+2s} \, dz
$$

\begin{equation}
\lesssim (-t)^{2+\theta(3+\gamma)} \left\| D_v^2 \phi \right\|_{L^\infty_w}^{s} \left\| \phi \right\|_{L^\infty_w}^{2-s-(\gamma+2s+3)p/3} \left\| \phi \right\|_{L^\infty_w}^{(\gamma+2s+3)p/3}
$$

\begin{equation}
= \left( (-t)^{1+\theta(5+\gamma)} \left\| D_v^2 \phi \right\|_{L^\infty_w} \right)^s \left( (-t)^{1+\theta(3+\gamma)} \left\| \phi \right\|_{L^\infty_w} \right)^{2-s-(\gamma+2s+3)p/3}
$$

\begin{equation}
\cdot \left( (-t)^{1+\theta(3+\gamma)-3\theta/p} \left\| \phi \right\|_{L^\infty_w} \right)^{(\gamma+2s+3)p/3},
\end{equation}

which also converges to 0 by \[(2.7)\]. In the second line, we performed a convolution estimate as in \[(5.3)\]. We could apply \[(2.7)\] because $1 + \theta(3 + \gamma) - 3\theta/p \geq 0$.

Multiplying \[(5.8)\] by any smooth, compactly supported test function and sending $t \to 0$, we conclude

\begin{equation}
(5.11)
\end{equation}

$$
g + (1 + \theta) y \cdot \nabla_y g + \theta w \cdot \nabla_w g + w \cdot \nabla_y g - Q_{B,w}(g, g) = 0,
$$

in the sense of distributions. As above, the regularity assumptions \[(2.5)\] for $g$ imply \[(5.11)\] holds pointwise.

From this point on, the proof is the same as for the Landau equation (the proof of Theorem \[2.2\]), since $\lim_{R \to \infty} \int_{\mathbb{R}^3} x(|w|/R)Q_{B,w}(g, g) \, dw = 0$ holds thanks to Lemma \[5.6\] as well as $\lim_{R \to \infty} \int_{\mathbb{R}^3} x(|w|/R)|w|^2 Q_{B,w}(g, g) \, dw = 0$ in the distinguished cases $\theta = \pm 1/3$. (We can apply Lemma \[5.6\] because $|w|^{2+\gamma} g \in L^1_w$, by assumption.) Applying the same argument as above, we conclude $g \equiv 0$ in all cases. \qed

**References**

[1] R. Alexandre, L. Desvillettes, C. Villani, and B. Wennberg. Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal.*, 152(4):327–355, 2000.

[2] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. Regularizing effect and local existence for the non-cutoff Boltzmann equation. *Arch. Ration. Mech. Anal.*, 198(1):39–123, 2010.

[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. Bounded solutions of the Boltzmann equation in the whole space. *Kinet. Relat. Models*, 4(1):17–40, 2011.
[4] R. Alexandre and C. Villani. On the Landau approximation in plasma physics. *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, 21(1):61 – 95, 2004.

[5] A. A. Arsen’ev and N. V. Peskov. The existence of a generalized solution of Landau’s equation. *Z. Vyčisl. Mat. i Mat. Fiz.*, 17(4):1063–1068, 1996, 1977.

[6] G. Barenblatt and Y. B. Zel’Dovich. Self-similar solutions as intermediate asymptotics. *Annual Review of Fluid Mechanics*, 4(1):285–312, 1972.

[7] G. I. Barenblatt. *Scaling, self-similarity, and intermediate asymptotics: dimensional analysis and intermediate asymptotics*. Number 14. Cambridge University Press, 1996.

[8] A. Blanchet and P. Lauren¸ cot. Finite mass self-similar blowing-up solutions of a chemotaxis system with non-linear diffusion. *Communications on Pure & Applied Analysis*, 11(1):47, 2009.

[9] T. Buckmaster, S. Shkoller, and V. Vicol. Formation of point shocks for 3D compressible Euler. *arXiv preprint arXiv:1912.04429*, 2019.

[10] T. Buckmaster, S. Shkoller, and V. Vicol. Formation of shocks for 2D isentropic compressible Euler. *Communications on Pure and Applied Mathematics*, 2019.

[11] S. Cameron, L. Silvestre, and S. Snelson. Global a priori estimates for the inhomogeneous Landau equation with moderately soft potentials. *Annales de l’Institut Henri Poincaré (C) Analyse Non Linéare*, 35(3):625–642, 2018.

[12] K. Carrapatoso, L. Desvillettes, and L. He. Estimates for the large time behavior of the Landau equation in the Coulomb case. *Arch. Ration. Mech. Anal.*, (2):381–420, 2017.

[13] K. Carrapatoso and S. Mischler. Landau equation for very soft and Coulomb potentials near Maxwellians. *Annals of PDE*, 3(1):1, Jan 2017.

[14] D. Chae. Nonexistence of asymptotically self-similar singularities in the Euler and the Navier–Stokes equations. *Mathematische Annalen*, 338(2):435–449, 2007.

[15] D. Chae and R. Shvydkoy. On formation of a locally self-similar collapse in the incompressible Euler equations. *Archive for Rational Mechanics and Analysis*, 209(3):999–1017, 2013.

[16] D. Chae and T.-P. Tsai. Remark on Luo-Hou’s ansatz for a self-similar solution to the 3D Euler equations. *Journal of Nonlinear Science*, 25(1):193–202, 2015.

[17] D. Chae and J. Wolf. On the local type I conditions for the 3D Euler equations. *Archive for Rational Mechanics and Analysis*, 230(2):641–663, 2018.

[18] S. Chaturvedi. Stability of vacuum for the landau equation with hard potentials. *Preprint. arXiv:2001.07208*, 2029.

[19] S. Chaturvedi, J. Luk, and T. T. Nguyen. The vlasov–poisson–landau system in the weakly collisional regime. *arXiv preprint arXiv:2104.05692*, 2021.

[20] J. Chen, T. Y. Hou, and D. Huang. On the finite time blowup of the De Gregorio model for the 3D Euler equation. *arXiv preprint arXiv:1905.06387*, 2019.

[21] Y. Chen, L. Desvillettes, and L. He. Smoothing effects for classical solutions of the full Landau equation. *Archive for Rational Mechanics and Analysis*, 193(1):21–55, 2009.

[22] J. Chern and M. Guadagni. Uniqueness of higher integrable solution to the Landau equation with Coulomb interactions. *Math. Res. Let.*, 2021.

[23] D. Christodoulou. *The formation of shocks in 3-dimensional fluids*, volume 2. European Mathematical Society, 2007.

[24] D. Christodoulou and S. Miao. *Compressible flow and Euler’s equations*, volume 9. International Press Somerville, MA, 2014.

[25] C. Collot. Nonradial type II blow up for the energy-supercritical semilinear heat equation. *Analysis & PDE*, 10(1):127–252, 2017.

[26] C. Collot, T.-E. Ghoul, and N. Masmoudi. Singularity formation for Burgers equation with transverse viscosity. *arXiv preprint arXiv:1803.07826*, 2018.

[27] C. Collot, T.-E. Ghoul, N. Masmoudi, and V. T. Nguyen. Refined description and stability for singular solutions of the 2D keller-segel system. *arXiv preprint arXiv:1912.00721*, 2019.

[28] L. Desvillettes. Entropy dissipation estimates for the Landau equation in the Coulomb case and applications. *Journal of Functional Analysis*, 269(5):1359 – 1403, 2015.

[29] L. Desvillettes, L.-B. He, and J. J-C. A new monotonicity formula for the spatially homogeneous Landau equation with Coulomb potential and its applications. *arXiv Preprint*.

[30] R. Duan and H. Yu. The vlasov-poisson-landau system near a local maxwellian. *Advances in Mathematics*, 362:106956, 2020.
[31] J. Eggers and M. A. Fontelos. Singularities: formation, structure, and propagation, volume 53. Cambridge University Press, 2015.

[32] T. M. Elgindi. Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$. arXiv preprint arXiv:1904.04795, 2019.

[33] T. M. Elgindi and I.-J. Jeong. Finite-time singularity formation for strong solutions to the axi-symmetric 3D Euler equations. Annals of PDE, 5(2):1–51, 2019.

[34] N. Fournier. Uniqueness of bounded solutions for the homogeneous Landau equation with a Coulomb potential. Communications in Mathematical Physics, 299(3):765–782, 2010.

[35] T.-E. Ghoul and N. Masmoudi. Stability of infinite time blow up for the Patlak Keller Segel system. arXiv preprint arXiv:1610.00456, 2016.

[36] F. Golse, M. Gualdani, C. Imbert, and A. Vasseur. Partial regularity in time for the space homogeneous Landau equation with Coulomb potential. To appear in Annales scientifiques de l' ENS (2021), 2019.

[37] F. Golse, C. Imbert, C. Mouhot, and A. Vasseur. Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. Annali della Scuola Normale Superiore di Pisa, XIX(1):253–295, 2019.

[38] P. T. Gressman and R. M. Strain. Global classical solutions of the Boltzmann equation without angular cut-off. J. Amer. Math. Soc., 24(3):771–847, 2011.

[39] M. Gualdani and N. Guillen. Estimates for radial solutions of the homogeneous Landau equation with Coulomb potential. Anal. PDE, 9(8):1773–1810, 2016.

[40] M. Gualdani and N. Guillen. On $A_p$ weights and the Landau equation. Calculus of Variations and Partial Differential Equations, 58(1):17, 2018.

[41] Y. Guo. The Landau equation in a periodic box. Communications in Mathematical Physics, 231(3):391–434, 2002.

[42] Y. Guo. The vlasov-poisson-landau system in a periodic box. Journal of the American Mathematical Society, 23(3):759–812, 2010.

[43] L. He. Well-posedness of spatially homogeneous Boltzmann equation with full-range interaction. Comm. Math. Phys., 312(2):447–476, 2012.

[44] L. He and X. Yang. Well-posedness and asymptotics of grazing collisions limit of Boltzmann equation with Coulomb interaction. Ann. Inst. H. Poincaré Anal. Non Linéaire 37 (2020), no. 6, 1345–1377.

[45] C. Henderson and S. Snelson. $C^\infty$ smoothing for weak solutions of the inhomogeneous Landau equation. Arch. Ration. Mech. Anal. 236 (2020), no. 1, 113–143.

[46] C. Henderson, S. Snelson, and A. Tarfulea. Local solutions of the Landau equation with rough, slowly decaying initial data. Ann. Inst. H. Poincaré Anal. Non Linéaire 37 (2020), no. 6, 1345–1377.

[47] C. Henderson, S. Snelson, and A. Tarfulea. Local existence, lower mass bounds, and a new continuation criterion for the Landau equation. Journal of Differential Equations, 266(2-3):1536–1577, 2019.

[48] C. Henderson, S. Snelson, and A. Tarfulea. Local well-posedness of the Boltzmann equation with polynomially decaying initial data. Kinetic and Related Models, 13(4):837–867, 2020.

[49] T. Y. Hou and P. Liu. Self-similar singularity of a 1D model for the 3D axisymmetric Euler equations. Research in the Mathematical Sciences, 2(1):1–26, 2015.

[50] C. Imbert, C. Mouhot, and L. Silvestre. Decay estimates for large velocities in the Boltzmann equation without cut-off. J. Éc. polytech. Math. 7 (2020), 143–184, 2018.

[51] C. Imbert, C. Mouhot, and L. Silvestre. Gaussian lower bounds for the Boltzmann equation without cutoff. SIAM J. Math. Anal., 52(3):2930–2944, 2020.

[52] C. Imbert and L. Silvestre. The weak Harnack inequality for the Boltzmann equation without cut-off. J. Eur. Math. Soc. (JEMS) 22 (2020), no. 2, 507–592.

[53] C. Imbert and L. Silvestre. Global regularity estimates for the Boltzmann equation without cut-off. Preprint. arXiv:1909.12729, 2019.

[54] R. Kohn and Y. Giga. Asymptotically self-similar blow-up of semilinear heat equations. Communications on Pure and Applied Mathematics, 38:297–319, 1985.

[55] M. Lemou, F. Méhats, and P. Raphaël. The orbital stability of the ground states and the singularity formation for the gravitational Vlasov Poisson system. Archive for Rational Mechanics and Analysis, 189(3):425–468, 2008.

[56] M. Lemou, F. Méhats, and P. Raphaël. Stable self-similar blow up dynamics for the three dimensional relativistic gravitational Vlasov-Poisson system. Journal of the American Mathematical Society, 21(4):1019–1063, 2008.
[57] S. Liu and X. Ma. Regularizing effects for the classical solutions to the Landau equation in the whole space. *Journal of Mathematical Analysis and Applications*, 417(1):123 – 143, 2014.

[58] X. Lu and C. Mouhot. On measure solutions of the Boltzmann equation, part I: Moment production and stability estimates. *Journal of Differential Equations*, 252(4):3305 – 3363, 2012.

[59] J. Luk. Stability of vacuum for the Landau equation with moderately soft potentials. *Annals of PDE*, 5(1):11, 2019.

[60] G. Luo and T. Y. Hou. Potentially singular solutions of the 3D axisymmetric Euler equations. *Proceedings of the National Academy of Sciences*, 111(36):12968–12973, 2014.

[61] H. Matano and F. Merle. On nonexistence of type II blowup for a supercritical nonlinear heat equation. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 57(11):1494–1541, 2004.

[62] F. Merle and P. Raphael. The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Annals of mathematics*, pages 157–222, 2005.

[63] F. Merle, P. Raphael, I. Rodnianski, and J. Szeftel. On smooth self similar solutions to the compressible Euler equations. *arXiv preprint arXiv:1912.10998*, 2019.

[64] F. Merle, P. Raphael, I. Rodnianski, and J. Szeftel. On the implosion of a three dimensional compressible fluid. *arXiv preprint arXiv:1912.11009*, 2019.

[65] Y. Morimoto, S. Wang, and T. Yang. Measure valued solutions to the spatially homogeneous Boltzmann equation without angular cutoff. *J. Stat. Phys.*, 165(5):866–906, 2016.

[66] J. Nečas, M. Ružička, and V. Šverák. On Leray’s self-similar solutions of the Navier-Stokes equations. *Acta Mathematica*, 176(2):283–294, 1996.

[67] P. Quittner and P. Souplet. *Superlinear parabolic problems*. Springer, 2019.

[68] P. Raphaèl and R. Schweyer. On the stability of critical chemotactic aggregation. *Mathematische Annalen*, 359(1):267–377, 2014.

[69] G. Seregin and V. Šverák. On type I singularities of the local axi-symmetric solutions of the Navier–Stokes equations. *Communications in Partial Differential Equations*, 34(2):171–201, 2009.

[70] L. Silvestre. A new regularization mechanism for the Boltzmann equation without cut-off. *Comm. Math. Phys.*, 348(1):69–100, 2016.

[71] L. Silvestre. Upper bounds for parabolic equations and the Landau equation. *Journal of Differential Equations*, 262(3):3034 – 3055, 2017.

[72] R. M. Strain and K. Zhu. The vlasov–poisson–landau system in r3. *Archive for Rational Mechanics and Analysis*, 210(2):615–671, 2013.

[73] T.-P. Tsai. On Leray’s self-similar solutions of the Navier-Stokes equations satisfying local energy estimates. *Archive for Rational Mechanics and Analysis*, 143(1):29–51, 1998.

[74] C. Villani. On the Cauchy problem for Landau equation: sequential stability, global existence. *Adv. Differential Equations*, 1(5):793–816, 1996.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742 USA
Email address: jacob@cscamm.umd.edu

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TEXAS AT AUSTIN, 2515 SPEEDWAY, AUSTIN TX, 78712
Email address: gualdani@math.utexas.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, FLORIDA INSTITUTE OF TECHNOLOGY, MELBOURNE, FL 32901
Email address: ssnelson@fit.edu