A slightly compressible hyperelastic material model with the Mullins effect

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Abstract. The paper presents a hyperelastic material model which is intended to describe slightly compressible materials with encountered the Mullins effect. It is assumed the volumetric-isochoric stored energy function split regarding the primary response of a material. Constitutive equations based on the concept of pseudo-elasticity are derived. The problems of a simple uniaxial compression/tension involving one finite element and a compression of a ball are solved using ABAQUS. The material model is defined with the help of user subroutine UHYPER in terms of the primary material response. Its main advantage is a physically correct description of volumetric changes while the polynomial model in ABAQUS does not meet basic growth conditions. In order to include the Mullins effect, it is combined with available in the software library the Mullins effect material model which is based on Ogden-Roxburgh proposition of stress softening function. Experimental data on synthetic rubber neoprene published in the literature is utilized.

1. Introduction
Rubber-like materials show a significant change in their mechanical response resulting from the first deformation [1]. The stress softening phenomenon, observed in filled and non-filled rubber, has been investigated intensively by Mullins and his co-workers [2] and consequently is called the Mullins effect. More precisely, the Mullins effect refers to a strain-induced stress softening of the material that takes place almost exclusively after the first loading path of loading cycles. It is often assumed that a damage causing the stress softening only depends on the maximum strain energy attained previously by the material.

There are a large number of works concerning the Mullins effect [3]. Many authors use the thermodynamical framework of continuum damage mechanics to develop constitutive equations. Others derive a large strain viscoelastic constitutive model with damage for rubber-like materials. A different approach is based on the network theory which states that parts of the network are reformed during loading.

Ogden and Roxburgh [4,5] proposed a pseudo-elastic model which differs from the mentioned theories because the stress softening function - that depends on the maximum strain energy - is activated only on unloading paths. A single continuous damage parameter is incorporated into the model in order to modify the stored energy function so that the material responses differently on unloading paths. The approach greatly simplifies the identification of material parameters. Its main advantage is a convenient description of the stress-strain relationship in a cyclic loading as well as their numerical implementation for the finite element method [6,7]. Based on the concept of pseudo-elasticity, Naumann and Ihlemann...
proposed an integral form of the specific energy function to derive the constitutive equations. However, the Mullins effect described using the approach should be treated as a quasi-static phenomenon.

In the paper we derive constitutive equations based on the concept of pseudo-elasticity for a slightly compressible hyperelastic material model [10-12]. Example boundary value problems, which concern a simple compression/tension and a compression of a ball, are solved using ABAQUS/Standard [6,7]. The material model is defined via user subroutine UHYPER in terms of the primary material response. It is combined with available in the software library the Mullins effect material model which is based on the Ogden-Roxburgh proposition.

2. Pseudo-hyperelasticity

2.1. An internal variable model

The local equations of thermodynamics yield the Clausius-Duhem [9] inequality for purely mechanical material models

\[ D = \frac{1}{2} \dot{\mathbf{C}} \cdot \dot{\mathbf{C}} - \Psi \geq 0 \]  

where \( D \) denotes the mechanical dissipation, \( \mathbf{T} \) the second Piola-Kirchhoff stress tensor, \( \mathbf{C} = \mathbf{F}^T \mathbf{F} \) the right Cauchy-Green tensor, and \( \Psi \) the specific free energy. Here \( \mathbf{F} \) denotes the deformation gradient with \( J = \det \mathbf{F} > 0 \). We assume that \( \Psi \) is a function of \( \mathbf{C} \) and an internal variable \( g \), i.e. \( \Psi = \Psi(\mathbf{C}, g) \).

Hence, the inequality (1) can be written in the form

\[ D = \left( \frac{1}{2} \mathbf{T} - \frac{\partial \Psi}{\partial \mathbf{C}} \right)_{\mathbf{c}=\mathbf{c}'} \cdot \dot{\mathbf{C}} + P_0 g \geq 0, \quad P = -\frac{\partial \Psi}{\partial g} \]  

where \( P \) a thermodynamic force associated with a damage variable \( g \). In the case of pure hyperelasticity, we have \( g = 0 \) and \( D = 0 \) (process is fully reversible). A constitutive relation can be established as

\[ \mathbf{T} = 2 \frac{\partial \Psi}{\partial \mathbf{C}} \]  

The inequality (2) contains two parts describing the dissipation such that \( D = D_1 + D_2 \), which are functions of \( \mathbf{C} \) and \( \dot{g} \). In the article we assume that the specific internal energy [9] is of the form

\[ \Psi = \Psi(\mathbf{C}, g) = \phi_1(g) W_0(\mathbf{C}) + \phi_2(g) \]  

where \( W_0(\mathbf{C}) \) denotes the stored energy function of the virgin material model (it is assumed that in the undeformed configuration \( W_0(\mathbf{I}) = 0 \)). Therefore, the inequality (2.2) yields

\[ D = \left( \frac{1}{2} \mathbf{T} - \phi_1 \frac{\partial W_0}{\partial \mathbf{C}} \right)_{\mathbf{c}=\mathbf{c}'} \cdot \dot{\mathbf{C}} + P_0 g = D_1 + D_2 \geq 0 \]

\[ P = -\frac{\partial \Psi(\mathbf{C}, g)}{\partial g} = \left( \phi_1 \frac{\partial W_0}{\partial g} + \phi_2 \frac{\partial \phi_2}{\partial g} \right) = \dot{g} \alpha \]

Here, both \( \dot{g} \geq 0 \) and \( \dot{g} \leq 0 \) are possible. If \( \alpha > 0 \) and \( D_1 = 0 \), equations (5) give the constitutive equation for stress tensor and the evolution equation for internal variable
\[
T = 2\phi_1 \frac{\partial W_0}{\partial C} \bigg|_{C \to C'}, \quad g = \frac{P}{\alpha} = -\frac{1}{\alpha} \left( \frac{\partial \phi_1}{\partial g} W_0 + \frac{\partial \phi_2}{\partial g} \right) \tag{6}
\]

Equations (6) also state basic continuum damage mechanics models [3]. In the case of a typical damage modelling, the internal variable \( g \) often describes only an irreversible process, so that only \( \dot{g} \geq 0 \) is possible.

### 2.2. The Ogden-Roxburgh model

Assumption \( \dot{g} = 0 \) yields the pseudo-elastic Ogden–Roxburgh model [4]

\[
\frac{\partial \phi_1}{\partial g} W_0 + \frac{\partial \phi_2}{\partial g} = 0 \tag{7}
\]

In such case it holds \( D_2 = 0 \). Therefore, the function \( \Psi \), which meets condition (7), is called a pseudo-elastic potential (it is not a stored elastic energy potential) and \( \phi_1 \) is a damage (softening) function. In the work [4] it is assumed that \( \phi_1 = \eta \) and \( \phi_2 = \phi(\eta) \), so that \( \partial \phi / \partial \eta = -W_0 \). The damage variable \( \eta \) is given by

\[
\eta = 1 - \frac{1}{c_2} \operatorname{Erf} \left[ \frac{1}{c_1} (W_0 - W_{max}) \right] \tag{8}
\]

where \( c_1, c_2 \) are positive constants and \( W_{max}(t) = \max \{W_0(\tau), \tau \leq t\} \). Thus, the damage of material is controlled by the maximum energy state \( W_{max} \) attained.

### 2.3. An integral form of pseudo-hyperelastic potential

In the approach, the internal energy dissipation occurs during loading and unloading cycles of the material. It is convenient to define a pseudo-hyperelastic potential in an integral form [8]

\[
\Psi(W_0, W_{max}) = \int_0^{W_{max}} \eta(\xi, W_{max}) d\xi \tag{9}
\]

where \( \eta = \eta(\xi, W_{max}) \) is the softening function. The definition (9) directly leads to

\[
\Psi(W_0(\mathbf{I}) = 0, W_{max}) = 0 \quad \text{Material derivative of } \Psi \text{ yields}
\]

\[
\Psi = \frac{\partial \Psi(W_0, W_{max})}{\partial W_0} W_0 + \frac{\partial \Psi(W_0, W_{max})}{\partial W_{max}} W_{max} = \eta(W_0, W_{max}) W_0 + \frac{\partial \Psi(W_0, W_{max})}{\partial W_{max}} W_{max} \tag{10}
\]

so that from (1) we have

\[
D = \left( \frac{1}{2} \mathbf{T} - \eta \frac{\partial W_0}{\partial C} \right) \mathbf{C} - \frac{\partial \Psi(W_0, W_{max})}{\partial W_{max}} W_{max} \geq 0 \tag{11}
\]

Similarly to the previous case, it holds

\[
\mathbf{T} = 2\eta \frac{\partial W_0}{\partial C} \mathbf{C} - \frac{\partial \Psi(W_0, W_{max})}{\partial W_{max}} W_{max} \leq 0 \tag{12}
\]

By definition \( W_{max} \) is always non-negative. Then, a sufficient condition for thermomechanical consistency is
\[
\frac{\partial \eta(W_0, W_{\text{max}})}{\partial W_{\text{max}}} \leq 0
\]  

Hence, every softening function which monotonically decreases with respect to \( W_{\text{max}} \) leads to a positive dissipation. Taking into consideration (8), the specific energy function may be explicitly stated as

\[
\Psi(W_0, W_{\text{max}}) = \frac{c_1}{c_2 \sqrt{\pi}} \exp \left[-\frac{1}{c_1^2} (\xi - W_{\text{max}})^2 \right] + \frac{1}{c_2} (\xi - W_{\text{max}}) \text{Erf} \left[ \frac{1}{c_1} (\xi - W_{\text{max}}) \right]^{W_0}
\]

3. A slightly compressible pseudo-hyperelastic material model

3.1. Slightly compressible hyperelastic models

Models of isotropic slightly compressible materials are a generalisation of models describing incompressible materials. In the case we assume that no coupling between the stored energy function of isochoric \( \tilde{W} \) and volumetric \( W_{\text{vol}} \) deformations occurs [10-12]. Therefore, it is convenient to express stored energy function in the form

\[
\tilde{W}(\tilde{I}_1, \tilde{I}_2, J) = \tilde{W}(\tilde{I}_1, \tilde{I}_2) + W_{\text{vol}}(J)
\]

where the modified invariants are given by \( \tilde{I}_1 = \text{tr} \tilde{B}, \tilde{I}_2 = \text{tr} \tilde{B}^{-1} \) and \( \tilde{B} = FF^T, \tilde{F} = J^{-1/3} \tilde{F} \). These can be explicitly written as

\[
\tilde{I}_1 = J^{-3/2} I_1, \quad \tilde{I}_2 = J^{-4} I_2
\]

Due to the form of stored energy function (15), the constitutive relation in the current configuration yields

\[
\sigma = \frac{1}{J} F F^T = \frac{\partial W_{\text{vol}}}{\partial J} I + \frac{\partial \tilde{W}}{\partial \tilde{I}_1} \tilde{B}_D - \frac{\partial \tilde{W}}{\partial \tilde{I}_2} \tilde{B}_D^{-1}
\]

where \( \sigma \) is the Cauchy stress tensor and \( \tilde{B}_D \) stands for deviatoric part of the modified left Cauchy-Green tensor \( \tilde{B} \) as follows

\[
\tilde{B}_D = \tilde{B} - \frac{1}{3} \tilde{I}_1 I, \quad \tilde{B}_D^{-1} = \tilde{B}^{-1} - \frac{1}{3} \tilde{I}_2 I
\]

3.2. The pseudo-hyperelastic model

As mentioned in the previous section, Ogden and Roxburgh proposed a phenomenological pseudo-elastic model for the Mullins effect in filled rubber which concerns an incompressible material [4,11]. In the case of slightly compressible material, the pseudo-hyperelastic potential is modified such that

\[
\Psi = \Psi(\tilde{I}_1, \tilde{I}_2, \eta) = \eta \tilde{W}(\tilde{I}_1, \tilde{I}_2) + \phi(\eta) + W_{\text{vol}}(J)
\]

The consequence of the above form of the function is that the Mullins effect is associated only with the deviatoric part of the deformation. The pressure stress of the augmented response is the same as that of the primary response [6]. Here, the inequality (1)
\[ D = \left[ \frac{1}{2} \mathbf{T} - \eta \frac{\partial W(\mathbf{T}, \mathbf{I})}{\partial \mathbf{C}} \right]_{\mathbf{C} = \mathbf{C}'} - \frac{J}{2} \frac{\partial W_{vol}(J)}{\partial J} \mathbf{C}^{-1} \cdot \mathbf{C} + P \mathbf{g} \geq 0 \]  

(20)

yields the constitutive relation of the form

\[ \mathbf{T} = J \frac{\partial W_{vol}(J)}{\partial J} \mathbf{C}^{-1} + 2\eta \frac{\partial W(\mathbf{T}, \mathbf{I})}{\partial \mathbf{C}} \bigg|_{\mathbf{C} = \mathbf{C}'} \]  

(21)

or in the spatial description

\[ \mathbf{\sigma} = \frac{1}{J} \mathbf{F} \mathbf{T} \mathbf{F}^T = \frac{\partial W_{vol}}{\partial J} \mathbf{I} + \frac{2}{J} \eta \left( \mathbf{F} \frac{\partial \mathbf{W}}{\partial \mathbf{T}_1} \mathbf{B}_D - \frac{\partial \mathbf{W}}{\partial \mathbf{T}_2} \mathbf{B}_0^{-1} \right) \]  

(22)

In the article we consider so called MCMV five-parameter model which deviatoric part is defined by the stored energy function of the form

\[ W(\mathbf{T}, \mathbf{I}) = \frac{1}{2} \left[ a_1 (\mathbf{I}_1 - 3) + \frac{1}{2} a_2 (\mathbf{I}_1^3 - 9) + \frac{1}{3} a_3 (\mathbf{I}_2^3 - 27) + a_4 (\mathbf{I}_2 - 3) + a_5 (\mathbf{I}_1 \mathbf{I}_2 - 9) \right] \]  

(23)

It is worth mentioning that the function (23) is a consistent third order approximation in terms of deformation tensor \( \mathbf{C} \) norm such that

\[ \left| W(\mathbf{T}, \mathbf{I}) - \mathbf{W}(\mathbf{I}, \mathbf{I}) \right| = O \left( \left\| \mathbf{C} \right\|^3 \right) \]  

(24)

For more details regarding the model we refer the reader to [10,12].

In the literature, there are a large number of propositions of the volumetric stored energy function [12]. Such function needs to meet certain growth conditions. Here, we define

\[ W_{vol}(J) = K_0 \left[ \frac{1}{4} (J^2 - 1) - \frac{1}{2} \ln J \right] \]  

(25)

where is the \( K_0 \) initial bulk modulus, which is several orders of magnitude greater than the initial shear modulus in the case of rubber-like materials. We emphasize that available in the ABAQUS standard material library all hyperelastic models include the volumetric stored energy function which do not meet basic growth conditions. It may lead to unrealistic response of the material model.

4. Example boundary value problems

In this section we present solutions of two boundary value problems which illustrates application of the discussed model. The example problems are solved using ABAQUS/Standard which is a finite element program designed for general use in nonlinear problems [6,7]. The MCMV model is defined with the help of user subroutine UHYPER in terms of the primary material response. It is combined with available in the software library the Mullins effect material model which is based on the Ogden-Roxburgh model. The damage variable is given by

\[ \eta = 1 - \frac{1}{c_2} \text{Erf} \left[ \frac{W_{\text{max}} - W_0}{c_1 + \beta W_{\text{max}}} \right] \]  

(26)

which generalizes the function (8) by \( \beta \) parameter. We use values of parameters given in table 1 for the problems presented in the following part of the article. The initial bulk modulus is step up to be \( K_0 = 1000 \mu_0 = 340.74 \) MPa.
4.1. A simple compression/tension test
In order to illustrate basic properties of the model, a simple compression/tension test involving one finite element (C3D8H) is considered. The loading is realised by prescribed displacement over one face according to figure 1.

Results in the form of the Cauchy stress versus principal stretch are shown in figure 2. The first part of loading process concerns compression – path marked as ‘1’. Unloading path ‘2’ clearly shows predicted stress softening. One can notice that the path in the range of tension is not a smooth curve. After marked in the plot filled point a primary response of the material occurs. It means that the stored energy function $W(\mathbf{I}_1, \mathbf{I}_2)$ attains greater value than at maximum compression. Following unloading path ‘3’ is indeed a smooth one.

4.2. Compression of a ball
The next discussed problem is a boundary value problem of compression of a ball with radius $R = 1$, see figure 3. The force is applied by a contact with non-deformable membranes with prescribed traction on one of them. The computational model consists of 9572 finite elements: type C3D8H with linear interpolation functions for the ball and type R3D4 for the membranes.
In order to show the Mullins effect, the ball is compressed and then unloaded. Figure 4 presents two curves of the Cauchy stress versus displacement of the ball’s top point. Values of the stress are obtained from four top, central elements on the edge. On the other hand, values of internal energy are given in the figure 5 for the whole model (it does not concern membranes as they are fully rigid). In the example the damage energy (ALLDMD) is much smaller than the recoverable (elastic) one (ALLSE). In this case the internal energy denoted in ABAQUS as ALLIE is a sum of ALLSE and ALLDMD.

Figure 3. Finite element mesh of the ball.

Figure 4. The Cauchy stress versus normalized displacement of the ball’s top point.
5. Conclusions

The paper concerns a phenomenological, constitutive modelling of the Mullins effect. We derive constitutive equations based on the concept of pseudo-elasticity for a slightly compressible hyperelastic material model. The approach was firstly proposed by Ogden and Roxburgh [4] where an incompressible material model is discussed. In this case the Mullins effect should be treated as a quasi-static phenomenon as the theory does not predict any rate-dependent effects.

In order to demonstrate the Mullins effect occurring in a cycle loading, example boundary value problems are solved using ABAQUS/Standard. The proposed material model is defined via user subroutine UHYPER which is suitable for definition of isotropic material models. The primary hyperelastic response is combined with available in the software library the Mullins effect material model which is based on the Ogden-Roxburgh model with stress softening function. The main advantage of the proposed model is the physically correct description of volumetric changes while the polynomial model in ABAQUS does not meet basic growth conditions.
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