Weighing the Black Hole via Quasi-local Energy

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Abstract
We set to weigh the black holes at their event horizons in various spacetimes and obtain masses which are substantially higher than their asymptotic values. In each case, the horizon mass of a Schwarzschild, Reissner-Nordström, or Kerr black hole is found to be twice the irreducible mass observed at infinity. The irreducible mass does not contain electrostatic or rotational energy, leading to the inescapable conclusion that particles with electric charges and spins cannot exist inside a black hole. This is proposed as the External Energy Paradigm. A higher mass at the event horizon and its neighborhood is obligatory for the release of gravitational waves in binary black hole merging. We describe how these mass values are obtained in the quasi-local energy approach and applied to the black holes of the first gravitational waves GW150914.

Keywords: Gravitational waves; black holes; quasi-local energy; horizon mass
Black holes are natural outcomes of solutions to Einstein’s equation. Since the discovery of the first gravitational waves from binary black hole merging in 2015 [1], black holes are now real astrophysical bodies. They are as legitimate as the elementary particles whose existence is confirmed indirectly. They may be abundant in the Universe and their properties can be investigated from the gravitational waves emitted in binary black hole merging.

A number of important theorems on black holes have been established between 1965 and 2005. They provide the conceptual framework and predict the properties of classical black holes in terms of temperature, entropy, irreversibility, thermodynamics, as well as energy conditions. They are known as

(1) Singularity Theorem (1965) [2],
(2) Area Non-decrease Theorem (1972) [3],
(3) Uniqueness Theorem (1975) [4],
(4) Positive Energy Theorem (1979) [5],
(5) Horizon Mass Theorem (2005) [6].

The first four theorems listed above have been extensively discussed in general relativity for many years and we take for granted that their contents are well known to general relativists. It is the last theorem, the Horizon Mass Theorem, which we shall discuss in this report and apply it to the black holes of the first gravitational waves GW150914.

2. Quasi-local Energy

The Horizon Mass Theorem is the final outcome of the quasi-local energy approach [7] applied to black holes. The quasi-local energy gives the total energy within a spatially bounded region instead of defining locally the energy
density for a gravitational system. It is obtained from a Hamiltonian-Jacobi analysis of the Hilbert action in general relativity and it is uniquely suited for investigating the dynamics of the gravitational field [8]. The mass of a black hole can be found anywhere by calculating the total energy contained in a Gaussian surface enclosing the black hole at a given coordinate distance. The usual mass of a black hole is the asymptotic mass seen by an observer at infinity.

A black hole has the strongest gravitational potential energy of any gravitational system. This energy exists outside the black hole. An observer at a distance sees the total of the constituent mass contained at the horizon and the intermediary potential energy. Since gravitational potential energy is always negative and extends throughout all space, the closer an observer gets to the black hole, the less gravitational potential energy the observer will see. Thus, the mass of the black hole increases as the observer gets near the horizon. This is a unique situation for black holes since for any other physical object, the gravitational potential energy is far insignificant compared to its mass and therefore the mass appears to be the same at all distances of observation.

The Brown and York expression for quasi-local energy is given in terms of the total mean curvature of a surface bounding a volume for a gravitational system in four-dimensional spacetime. It is given in the form of an integral

\[ E = \frac{c^4}{8\pi G} \int_B d^2 x \sqrt{\sigma} (k - k^0), \]  

(1)

where \( \sigma \) is the determinant of the metric defined on the two-dimensional surface \( B \); \( k \) is the trace of extrinsic curvature of the surface and \( k^0 \), the trace of curvature of a reference space. For asymptotically flat reference spacetime, \( k^0 \) is zero.

3. Horizon Mass Theorem

The Horizon Mass Theorem can be stated as follows,
**Theorem.** For all black holes; neutral, charged or rotating, the horizon mass is always twice the irreducible mass observed at infinity.

In notation, it has the simple form,

\[ M_{\text{horizon}} = 2M_{\text{irr}}, \]  \hspace{1cm} (2)

where \( M_{\text{irr}} \) is the irreducible mass. The derivation of this theorem is given fully in Ref. 6. The Horizon Mass Theorem relates the mass of a black hole at the event horizon to its irreducible mass. It is an exact result obtained only with the knowledge of the spacetime metrics of Schwarzschild, Reissner-Nordström, and Kerr without further assumption. It is a new addition to the previous theorems on classical black holes.

In order to understand the Horizon Mass Theorem, it is necessary to introduce the various mass terms involved.

1. The **asymptotic mass** is the mass of a neutral, charged or rotating black hole including electrostatic and rotational energy. It is the mass observed at infinity used in the various spacetime metrics.

2. The **horizon mass** is the mass which cannot escape from the horizon of a neutral, charged or rotating black hole. It is the mass of the black hole observed at the horizon.

3. The **irreducible mass** is the final mass of a charged or rotating black hole when its charge or angular momentum is removed by adding external particles to the black hole. It is the mass observed at infinity.

The Horizon Mass Theorem is remarkable in that the mass contained at the event horizon depends only on the irreducible mass of the black hole. The irreducible mass does not contain electrostatic or rotational energy. This leads to the surprising conclusion that the electrostatic and rotational energy exist only outside the black hole. They are all external quantities. An asymptotic observer investigating a charged or rotating black hole is in fact exploring a Schwarzschild black hole with external energies in between.
4. External Energy Paradigm

There are profound implications which follow from the Horizon Mass Theorem. Since all electric field lines terminate at electric charges and electrostatic energy is external, this indicates that electrical particles cannot exist inside a black hole. They can only stay at the surface. Similarly, since rotational energy is external, any particle with angular momentum also cannot exist inside the black hole and must stay outside, as required by the Horizon Mass Theorem. Together, this implies that all elementary particles possessing charges and spins can only stay outside the horizon. If a black hole is formed from the collapse of a star made of ordinary matter, the result will be a hollow and thin spherical shell with all constituent mass at the horizon. This is a radical view of the black hole and it follows inescapably from the property of the irreducible mass. We are thus led to introduce a new paradigm for black holes to be called the External Energy Paradigm. All energies of a black hole are external quantities. Matter particles are forbidden inside a black hole and can only stay outside or at the horizon. These energies include constituent mass, gravitational energy, heat energy, electrostatic energy and rotational energy. It explains naturally why the entropy of a black hole is proportional to the area and not to the volume because matter particles are all at the surface.

5. Schwarzschild Black Hole

The total energy contained in a sphere enclosing the black hole at a coordinate distance $r$ is given by the expression \[ E(r) = \frac{r c^4}{G} \left[ 1 - \sqrt{1 - \frac{2GM}{rc^2}} \right], \quad (3) \]

where $M$ is the mass of the black hole observed at infinity, $c$ is the speed of light and $G$ is the gravitational constant. At the horizon, the Schwarzschild radius is $r = R_S = 2GM/c^2$. Evaluating the expression in Eq.(3), we find that the metric coefficient $g_{00} = (1 - 2GM/rc^2)^{1/2}$ vanishes identically and
the horizon energy is therefore

\[ E(r) = \left( \frac{2GM}{c^2} \right) \frac{c^4}{G} = 2Mc^2. \]  

(4)

The horizon mass of the Schwarzschild black hole is simply twice the asymptotic mass \( M \) observed at infinity. The negative gravitational energy outside the black hole is as great as the asymptotic mass.

Equation (3) can be used to evaluate the mass seen by an observer at any distance \( r \). We show some particular values in Table 1.

| Coordinate \( r \) in \( R_S \) | Mass in \( M_\infty = M \) |
|-------------------------------|--------------------------|
| 1                             | 2.000                    |
| 2                             | 1.172                    |
| 3                             | 1.101                    |
| 4                             | 1.072                    |
| 5                             | 1.056                    |
| 6                             | 1.046                    |
| 7                             | 1.039                    |
| 8                             | 1.033                    |
| 9                             | 1.029                    |
| 10                            | 1.026                    |
| 100                           | 1.003                    |
| \( \infty \)                 | 1.000                    |

From the listed values, it is seen that 90% of the negative potential energy lies within a distance of two Schwarzschild radii outside the horizon, i.e. \( R_S < r < 3R_S \). At a distance of \( r = 100R_S \), the mass is seen to be only 0.3% higher than the asymptotic value \( M \). An observer at that location is approaching a near flat spacetime.
6. Reissner-Nordstrøm Black Hole

We investigate next the Reissner-Nordstrøm black hole in the quasi-local energy approach. The total energy of a charged black hole contained within a radius at coordinate \( r \) is now given by

\[
E(r) = \frac{r c^4}{G} \left[ 1 - \sqrt{1 - \frac{2GM}{r c^2} + \frac{GQ^2}{r^2 c^4}} \right].
\]  

(5)

Here, \( M \) is the mass of the black hole including electrostatic energy observed at infinity and \( Q \) is the electric charge. At the horizon radius

\[
r_+ = \frac{GM}{c^2} + \frac{GM}{c^2} \sqrt{1 - \frac{Q^2}{GM^2}},
\]  

(6)

the metric coefficient \( g_{00} \) given by the square root in Eq.(5) also vanishes and the horizon energy becomes

\[
E(r_+) = \frac{r_+ c^4}{G} = Mc^2 + Mc^2 \sqrt{1 - \frac{Q^2}{GM^2}}.
\]  

(7)

For the Reissner-Nordstrøm black hole, the irreducible mass which is obtained when the charge is removed by adding oppositely charged particles has the expression

\[
M_{irr} = \frac{M}{2} + \frac{M}{2} \sqrt{1 - \frac{Q^2}{GM^2}}.
\]  

(8)

Combining Eqs.(7) and (8), we find the horizon energy to be exactly twice the irreducible energy

\[
E(r_+) = 2M_{irr}c^2,
\]  

(9)

which depends only on the mass of the black hole when the charge is neutralized.

7. Kerr Black Hole

The rotating black hole is considerably more complicated to handle in the quasi-local energy approach because one is comparing a rotating spacetime
with a fixed spacetime. It is therefore not possible to give an exact analytical expression as in the previous two cases. An approximate energy expression [10] is available for a slowly rotating black hole with angular momentum $J$ and angular momentum parameter $\alpha = J/Mc$, where $0 < \alpha \ll GM/c^2$,

$$
E(r) = \frac{rc^4}{G} \left[ 1 - \sqrt{1 - \frac{2GM}{rc^2} + \frac{\alpha^2}{r^2}} \right]
+ \frac{\alpha^2c^4}{6rG} \left[ 2 + \frac{2GM}{rc^2} \left( 1 + \frac{2GM}{rc^2} \right) \sqrt{1 - \frac{2GM}{rc^2} + \frac{\alpha^2}{r^2}} \right]
+ \cdots \tag{10}
$$

Again, with the horizon radius of the Kerr black hole

$$
r_+ = \frac{GM}{c^2} + \sqrt{\frac{G^2M^2}{c^4} - \frac{J^2}{M^2c^2}} \tag{11}
$$

and the definition of the irreducible mass

$$
M_{irr}^2 = \frac{M^2}{2} + \frac{M^2}{2} \sqrt{1 - \frac{J^2c^2}{G^2M^4}}, \tag{12}
$$

we arrive at a very good approximate relation for the horizon energy

$$
E(r_+) \simeq 2M_{irr}c^2 + O(\alpha^2). \tag{13}
$$

For general and fast rotations, the energy can be accurately obtained by numerical evaluation in the teleparallel equivalent of general relativity [11]. The result shows almost perfectly that the horizon mass is twice the irreducible mass. For an exact and impeccable relationship, we have to employ a formula known for the area of a Kerr black hole valid for all rotations [12], i.e.

$$
A = 4\pi(r_+^2 + \alpha^2) = \frac{16\pi G^2 M_{irr}^2}{c^4}. \tag{14}
$$

This area is exactly the same as that of a Schwarzschild black hole with an asymptotic mass $M_{irr}$,

$$
A = 4\pi R_S^2 = 4\pi \left( \frac{2GM_{irr}}{c^2} \right)^2. \tag{15}
$$
As shown earlier, the horizon mass of such a Schwarzschild black hole is twice the irreducible mass. We have therefore established the Horizon Mass Theorem for all black hole cases. The profound consequence of this theorem is that elementary particles with charges or spins cannot exist inside a black hole.

8. Black Holes of GW150914

The discovery of gravitational waves GW150914 by LIGO confirmed the existence of two black holes in a binary system. They merged to form a single black hole with the release of gravitational energy. We realize that the energy of the gravitational waves comes from outside the black holes and not from their interiors. The waves are generated predominately from near the horizon and they are gravitationally redshifted as they propagate to infinity. The horizon energy therefore becomes important. Without a higher mass at the event horizon and its neighborhood, there can be no gravitational waves emitted in black hole merging.

The two black holes of GW150914 are rotating black holes. For a Kerr black hole, rotation necessarily contributes to the overall mass observed at infinity. To find the irreducible mass of the Kerr black hole, we need to know the dimensionless spin parameter \( a \), which is the ratio of the angular momentum \( J \) to the maximum possible angular momentum, i.e.

\[
a = \frac{J}{\left(\frac{GM^2}{c}\right)} = \frac{Jc}{GM^2}.
\]

The irreducible mass, from Eq.(12), is then given by

\[
M_{\text{irr}} = \left[ \frac{M^2}{2} + \frac{M^2}{2} \sqrt{1 - a^2} \right]^\frac{1}{2}
\]
and the horizon mass can be found as twice the irreducible mass,

\[
M(r_+) = 2M_{\text{irr}} = \left[ 2M^2 + 2M^2 \sqrt{1 - a^2} \right]^\frac{1}{2}.
\]

For the black holes of GW150914 [13], the primary black hole has a mass of 36\( M_{\odot} \) and an average model spin parameter \( a = 0.32 \). The secondary
black hole has a mass of $29M_{\text{Sun}}$ and average model spin parameter $a = 0.44$. The final black hole has a mass of $62M_{\text{Sun}}$ and a spin parameter $a = 0.67$. Accordingly, $3M_{\text{Sun}}$ of energy is released as gravitational waves to infinity as report by LIGO, i.e.

$$36M_{\text{Sun}} + 29M_{\text{Sun}} = 62M_{\text{Sun}} + 3M_{\text{Sun}}.$$  \hspace{1cm} (19)

However, this $3M_{\text{Sun}}$ wave energy has been significantly redshifted and Eq.(19) does not account for the missing energy. Without additional source of energy, the waves cannot propagate away from the deep potential of the black hole. To understand the energy of the waves at the source, we need to know the mass of the black holes at the event horizon. For the primary black hole, the horizon mass is found to be $71M_{\text{Sun}}$ and for the secondary black hole, a horizon mass of $57M_{\text{Sun}}$. The final black hole has a horizon mass of $116M_{\text{Sun}}$. In an ideal merging, the energy at the horizon would follow the equation

$$71M_{\text{Sun}} + 57M_{\text{Sun}} = 116M_{\text{Sun}} + (3M_{\text{Sun}} + 3M_{\text{Sun}}) + 6M_{\text{Sun}}.$$  \hspace{1cm} (20)

In this account, the energy for the redshift is now available. Analysis of a mass removed from the surface of a black hole shows that the energy required has the same magnitude as the energy of the waves observed at infinity [14]. The total energy required to release $3M_{\text{Sun}}$ of wave energy to infinity is therefore $3M_{\text{Sun}} + 3M_{\text{Sun}} = 6M_{\text{Sun}}$. The remaining $6M_{\text{Sun}}$ of the energy is for uncertainties in LIGO data. These mass values are additional properties for the binary black hole merger of GW150914.

We may further provide the rotational energy of the black holes by comparing the asymptotic mass with the irreducible mass. The rotational mass of the primary black hole is $36M_{\text{Sun}} - 35.5M_{\text{Sun}} = 0.5M_{\text{Sun}}$, while that of the secondary black hole is $29M_{\text{Sun}} - 28.25M_{\text{Sun}} = 0.75M_{\text{Sun}}$. The final black hole has a higher rotational mass that is $62M_{\text{Sun}} - 58M_{\text{Sun}} = 4M_{\text{Sun}}$. The initial black holes in a binary system generally have different spin orientations. Thus most of the rotational energy of the final black hole comes from the orbiting energy of the binary system.
9. Conclusion

The detection of the first gravitational waves GW150914 shows the need to understand the energy of the black hole at the event horizon. This was first emphasized by the author in 2003 in the paper “The Gravitational Energy of a Black Hole” [14] before the success of LIGO was certain. At the end, there was the remark:

*Therefore, in detecting any gravitational signals from a black hole collision such as that proposed in the LIGO project, any conclusion about the strength of the signals near its source should be based on the black hole energy formula.*

The quasi-local energy approach and the Horizon Mass Theorem are indispensable tools in the latest development of general relativity.
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