An excision scheme for black holes in constrained evolution formulations: spherically symmetric case

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I. INTRODUCTION

Relativistic simulations of astrophysical phenomena involving one or several black holes (BHs) have undergone significant improvements in the last decade, in particular with the first successful studies of binary BH systems [1–3]. One of the major difficulties in performing such simulations is the handling of the physical singularity of the BH, where some physical fields may diverge. In order to cope with this problem, essentially two types of methods have been proposed in the literature: i) Excision, where the singularity, together with its neighborhood, is removed from the computational domain and eventually replaced by boundary conditions (see e.g. [4,5]). ii) Punctures, where the BH is set in such initial data that the physical singularity is not included, but instead the spatial hypersurface containing the initial data follows a wormhole through to another copy of spacetime, which is compactified and its infinity is reduced to a point, the “puncture” (see e.g. [6,7]). The wormhole topology is prescribed analytically in the conformal factor (see Eq. (2.2) hereafter), which diverges at the puncture location.

Both of these approaches have been successfully applied in simulations of binary BH systems, but with different formulations of Einstein equations and gauge choices: excision has been used in conjunction with the generalized harmonic gauge [4,8], and punctures have usually been associated with the so-called BSSN (from Baumgarte-Shapiro [9] Shibata-Nakamura [10]) formulation [2,3]. All these studies use free evolution schemes, in which the constraint equations arising in the 3+1 decomposition of Einstein equations are not enforced during the evolution. If the constraint equations are satisfied initially, they are also satisfied during the evolution theoretically, but this is not necessarily the case in numerical simulations. In these formulations, the constraint equations are satisfied by the initial data and then monitored during the evolution to check the validity of the numerical solution.

In such free evolution schemes, most of the resulting partial differential equations (PDE), coming from Einstein and matter equations (in the case of non-vacuum spacetimes), are of hyperbolic type. In particular their characteristics, computed inside the BH (apparent) horizon, are all directed towards the singularity. This means that in these schemes, within the excision approach and adopting excision surfaces lying inside the apparent horizon (AH) such that their evolution worldtubes are of spacelike character, there is no need for imposing any inner boundary condition.

However, when solving constraints arising in the 3+1 formulation of Einstein equations, the elliptic nature of these PDEs requires that correct corresponding boundary conditions at the excision surface have to be defined and tested. Otherwise, wrong boundary conditions will give wrong physical content to the numerical solution and, therefore invalidate the whole simulation. This is particularly true in the case of a Fully Constrained Formulation (FCF), as the one devised by Bonazzola et al. [11], where the constraint equations are regularly solved and enforced during the numerical simulation.

A geometric approach to define proper boundary conditions for the elliptic part of Einstein equations has been undertaken on the basis of the isolated horizon paradigm [12,13] or the dynamical “trapping horizon” concept [18]. They have been successfully applied to stationary spacetimes [19,20], which can then be used as initial data for further dynamical evolutions.

In this paper, we propose a different approach for the
definition of boundary conditions in the FCF formulation using the excision technique in dynamical spacetimes, with the motivation of simulating astrophysical scenarios like a star collapsing to a BH. In this context, initial data are usually regular and no BH is present yet. During the simulation, a BH forms and, particularly when using singularity avoiding time coordinates (e.g. maximal slicing), an AH is found before the appearance of the physical singularity [21]. When no symmetry is assumed, the AH does not have a simple shape in general and, therefore, it is not easy to use it numerically as an excision surface to impose boundary conditions. Here, we suggest to use an arbitrary but nearby sphere inside the AH to define similar ideas with the motivation of simulating astrophysical scenarios in spherical symmetry, too. Scheel et al. [22] set the excision boundary at the AH, at a fixed radial coordinate; the main difference with this approach is that we here set the excision boundary to be an arbitrary sphere located strictly inside the AH, and let this AH evolve in time. This approach allows in particular for a very straightforward extension to spacetimes without symmetries, where the AH can form with a shape deviating from a coordinate sphere. The latter approach is also followed in the work of Rinne & Moncrief [23], where the value of the conformal lapse function is frozen and evolution equations are used to update the values of the remaining variables at the excision surface. Here we propose and analyse a different prescription for the boundary conditions at the excision surface, that underline the geometric features of the system.

We will describe how the excision technique is used in dynamical evolutions and show the practical applicability of this approach in the case of a Schwarzschild BH spacetime and the accretion of a scalar field into a spherical BH. Previous works by Scheel et al. [22] and Rinne & Moncrief [23] have presented similar approaches in constrained formulations in spherical symmetry, too. Scheel et al. [22] have considered the situation of dust collapse in Brans-Dicke theory of gravity, whereas Rinne & Moncrief [23] have studied scalar and Yang-Mills fields coupled to gravity in a constant mean curvature slicing. Scheel et al. [22] set the excision boundary at the AH, at a fixed radial coordinate; the main difference with this work is that here we set the excision boundary to be an arbitrary sphere located strictly inside the AH, and let this AH evolve in time. This approach allows in particular for a very straightforward extension to spacetimes without symmetries, where the AH can form with a shape deviating from a coordinate sphere. The latter approach is also followed in the work of Rinne & Moncrief [23], where the value of the conformal lapse function is frozen and evolution equations are used to update the values of the remaining variables at the excision surface. Here we propose and analyse a different prescription for the boundary conditions at the excision surface, that underline the geometric features of the system.

The paper is organized as follows. The FCF of Einstein equations is briefly overviewed in Sec. II, which also contains some remarks in the spherically symmetric case. In Sec. III we describe the excision method, including our excision region and the boundary conditions imposed on each slice. Sec. IV discusses numerical results. A summary of our conclusions is given in Sec. V. We use units in which $c = G = M_\odot = 1$. Greek indices run from 0 to 3, Latin indices from 1 to 3, and we adopt the standard convention for the summation over repeated indices. $\partial_\alpha$ denotes partial derivatives.

II. FULLY CONSTRAINED FORMULATION

Given an asymptotically flat spacetime $(\mathcal{M}, g_{\mu\nu})$, we consider a 3+1 splitting by spacelike hypersurfaces $\Sigma_t$, denoting time like unit normals to $\Sigma_t$ by $n^\mu$. The spacetime on each spacelike hypersurface $\Sigma_t$ is described by the pair $(\gamma_{ij}, K^{ij})$, where $\gamma_{ij} = g_{ij} + n_i n_j$ is the Riemannian metric induced on $\Sigma_t$. We choose the convention $K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}$ for the extrinsic curvature. With the lapse function $N$ and the shift vector $\beta^i$, the Lorentzian metric $g_{\mu\nu}$ in the 3+1 formalism can be expressed in coordinates $(x^\mu)$ as

$$g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt).$$  \hspace{1cm} (2.1)

As in [11], we introduce a time independent flat metric $f_{ij}$, which satisfies $\mathcal{L}_f f_{ij} = \partial_t f_{ij} = 0$ and coincides with $\gamma_{ij}$ at spatial infinity. With the definitions $\gamma := \det \gamma_{ij}$ and $f := \det f_{ij}$, we introduce the following conformal decomposition of the spatial metric,

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}, \quad \psi = (\gamma/f)^{1/12}. \hspace{1cm} (2.2)$$

The difference between the conformal metric and the flat fiducial one is denoted by $h^{ij}$, $h^{ij} := \tilde{\gamma}^{ij} - f^{ij}$. The chosen prescriptions for the gauge variables in [11] are the maximal slicing,

$$K = 0, \hspace{1cm} (2.3)$$

and the so-called generalized Dirac gauge,

$$\mathcal{D}_i \tilde{\gamma}^{ij} = \mathcal{D}_i h^{ij} = 0, \hspace{1cm} (2.4)$$

where $K = \gamma^{ij} K_{ij}$ denotes the trace of the extrinsic curvature and $\mathcal{D}_k$ stands for the Levi-Civita connection associated with the flat metric $f_{ij}$. Finally, we introduce the conformal decomposition

$$\hat{A}^{ij} := \psi^{10} K^{ij}. \hspace{1cm} (2.5)$$

In this formulation, Einstein equations result in a coupled elliptic-hyperbolic system: the elliptic sector acts on the variables $\psi$, $N$ and $\beta^i$, while the hyperbolic sector acts on $h^{ij}$ and $\hat{A}^{ij}$. More details of the analysis of both sectors can be found in [22, 23].

The decomposition of $\hat{A}^{ij}$ in longitudinal and transverse-traceless (TT) parts

$$\hat{A}^{ij} = (LX)^{ij} + \hat{A}_TT^{ij}, \hspace{1cm} (2.6)$$

where $(LX)^{ij} := \mathcal{D}^i X^j + \mathcal{D}^j X^i - \frac{2}{\psi} f^{ij} \mathcal{D}_k X^k$ and $\mathcal{D}_i \hat{A}^{ij}_TT = 0$, can be considered motivated by the local uniqueness properties of the elliptic sector shown in [23].

We decompose classically the energy-momentum tensor, $T^{\mu\nu}$, measured by the observer of 4-velocity $n^\nu$ (Eulerian observer), in terms of the energy density $E := T_{\mu\nu} n^\mu n^\nu$, the momentum density $S_i := -\gamma^{ij} T_{\mu\nu} n_\mu n^j$, and the stress tensor $S_{ij} := T_{\mu\nu} \gamma^{\mu i} \gamma^{\nu j}$, with $S := \gamma^{ij} S_{ij}$ being its trace.
The resulting elliptic equations in the FCF are
\[ \tilde{\Delta} \psi = -2\pi \psi^{-1}E^* - \frac{\tilde{\gamma}_i \tilde{\gamma}_m \tilde{A}^m A_{ij}}{8\psi^7} + \psi \tilde{R} \],
(2.7)
where the operator \( \tilde{\Delta} \) is defined as
\[ \tilde{\Delta} \psi = \tilde{\gamma}^{kl} D_k D_l \psi \] (2.10)
(analogously for \( N \psi \)) and
\[ \tilde{\Delta} \beta^i = \tilde{\gamma}^{kl} D_k D_l \beta^i + \frac{1}{3} \tilde{\gamma}^{ik} D_k D_l \beta^l, \]
(2.11)
where \( \tilde{\gamma}^{kl} = \tilde{\gamma}^{lk} \) is the difference between Christoffel symbols of the conformal and flat metrics.

The resulting hyperbolic equations are evolution equations for \( h^{ij} \) and \( A^{ij} \),
\[ \partial_t h^{ij} = 2N \psi^{-6} \tilde{A}^{ij} + \beta^k D_k h^{ij} - \tilde{\gamma}^{ik} D_k \beta^j - \frac{2}{3} \tilde{\gamma}^{ij} D_k \beta^k, \] (2.14)
\[ \partial_t \tilde{A}^{ij} = (S_\tilde{A})^{ij}, \]
(2.15)
where the explicit expression for the source \( (S_\tilde{A})^{ij} \) can be found in [20].

If a TT decomposition is performed for \( \tilde{A}^{ij} \), an extra elliptic equation for the vector \( X^t \) is added and Eq. (2.15) can be viewed as an evolution equation for \( \tilde{A}^{ij}_{TT} \).

**A. Spherical symmetry**

It has been proved in [27] that a spherically symmetric spacetime can be locally foliated by a maximal slicing and using isotropic coordinates for the spatial metric onto the spatial hypersurfaces \( \Sigma_t \). This statement refers to neighborhoods, and does not involve global prescritions or boundaries. The FCF in spherical symmetry and with topologically \( \mathbb{R}^3 \) spatial hypersurfaces \( \Sigma_t \) reduces to the isotropic gauge, where \( \tilde{\gamma}^{ij} = f^{ij} \). This is not true anymore for more general topologies, like the \( \mathbb{R}^3 - B \) case, where \( B \) is a ball, when general boundary conditions on the boundary of \( B \) are given. In a non-convex topology such as \( \mathbb{R}^3 - B \), the expression for \( \tilde{\gamma}^{ij} \) evolving in time in the FCF in spherical symmetry is instead given by
\[
\tilde{\gamma}^{ij}(t) = \left( \begin{array}{ccc}
1 + \frac{\omega(t)}{r^2} & 0 & 0 \\
0 & 1 + \frac{\omega(t)}{r^2} & 0 \\
0 & 0 & 1 + \frac{\omega(t)}{r^2}
\end{array} \right),
\]
(2.16)
where \( \omega(t) \) is a real and derivable twice function of time \( t \). A proof of this statement is presented in Appendix [X]. Note that on \( \mathbb{R}^3, \omega = 0 \) is required at the origin for metric regularity. If a ball containing the origin is excised and the value for \( \omega \) is not zero at the excision boundary, the spatial metric is not necessarily expressed as a conformally flat one.

Since the value for \( \omega \) can be chosen arbitrarily taking into account the previous general expression for \( \tilde{\gamma}^{ij} \) (it represents just a gauge freedom, as the Dirac gauge is a differential one), the spatial metric can be expressed as a conformally flat one, i.e., \( \tilde{\gamma}^{ij} = f^{ij} \), or equivalently, \( h^{ij} = 0 \). Note that in this case Eq. (2.14) is a time-independent prescription for \( \tilde{A}^{ij} \),
\[
\tilde{A}^{ij} = \frac{\psi^6}{2N} \left( \tilde{\gamma}^{ik} D_k \beta^j + \tilde{\gamma}^{kj} D_k \beta^i - \frac{2}{3} \tilde{\gamma}^{ij} D_k \beta^k \right),
\]
(2.17)
Eqs. (2.7), (2.9) can be solved to obtain \( N, \psi \) and \( \beta^i \), and Eq. (2.15) is a redundant condition in the bulk. This redundant condition will be used as a compatibility condition for the prescription of boundary conditions in the constrained system resolution.

In a general spacetime with no spherical symmetry \( h^{ij} = 0 \) cannot be imposed, and \( h^{ij} \) and \( \tilde{A}^{ij} \) have to be evolved in time. Boundary conditions on the excision boundary for hyperbolic equations may or may not be imposed, depending on the characteristic structure of the system. This is not the case for elliptic equations, for which wrong boundary conditions invalidate the solution in the whole domain. The general case (no spherically symmetric spacetimes) is beyond the scope of this work and will be analyzed in future ones.

**III. EXCISION METHOD**

Due to the singular character of BH interior solutions, measures have to be taken in numerical simulations of BH spacetimes. Quite a number of codes relying on hyperbolic formulations of Einstein equations are based on an adaptive slicing which is typically designed to avoid the BH singularity by coordinate stretching and proper shift vector. This is the case of the BSSN formulation in combination with the puncture method, very popular in binary BH simulations. An alternative approach is known as stuffed BHs, where one fills BH interiors with unimportant (but regular) junk data in a hyperbolic formulation, and then evolves the regularized spacetime [28, 29].
In this work, we want to present how the excision technique can be used in the FCF in spherically symmetric spacetimes, where the presence of elliptic PDEs has to be taken into account. This technique consists in removing from each spatial hypersurface $\Sigma_t$ the open interior of a topological sphere $S^2$, and solving Einstein equations in the remaining hypersurface. The sphere is assigned both physical and geometrical characteristics, tailored so that it is located strictly inside the apparent horizon of the modelled BH region, and so that it encloses the gravitational singularity. Those properties are then encoded as boundary conditions of the available elliptic equations to be solved.

In BH initial data problems, a natural approach consists in placing the excision sphere at the outermost marginally outer trapped surface in the initial slice, namely the AH. Much work has been done in this field, including how to impose this prescription and transcribe it in terms of 3+1 spacetime metric quantities (see, e.g., [12–17]). In the evolution case, the set of excision spheres at every $\Sigma_t$ can be prescribed to describe a hyperpunct of marginally outer trapped surfaces, leading to a trapping horizon as outlined in [18]. Here we will rather follow a more generic approach in which the excision surface is not enforced at the BH AH, but rather at the interior of the AH worldtube.

### A. Excision surface geometry

We first define the geometrical setting. Let $S_t$ be a topological 2-sphere embedded in $\Sigma_t$, and its induced 2-metric $g_{ab}$. Let $s^\mu$ be the unit outward directed spacelike vector normal to $S_t$, that is also tangent to $\Sigma_t$. Let $H$ be the hyperpunct formed by the set of excision spheres at every $\Sigma_t$. At the excision surface, three other vector fields are defined: the outward and inward future-directed null vectors

$$l^\mu = (n^\mu + s^\mu)/\sqrt{2}, \quad k^\mu = (n^\mu - s^\mu)/\sqrt{2}, \quad (3.1)$$

respectively, and the evolution vector on $H$ normal to sections $S_t$ and carrying $S_t$ onto $S_{t+\delta t}$

$$h^\mu = N n^\mu + b s^\mu, \quad (3.2)$$

that we adapt to the 3+1 evolution vector $t^\mu = N n^\mu + \beta^\mu$, so that $b = \beta^s s^i$ (note that $b$ is only defined at $H$). This entails that the excision surface is kept at the same spatial coordinate location along the evolution. Two additional geometric quantities are the scalar outward expansion $\theta^{(l)}$ and outward shear $\sigma_{ab}$ along $l^\mu$, defined as

$$\mathcal{L}_{l} e^S_{ab} = \theta^{(l)} e^S_{ab}, \quad (3.3)$$

$$\sigma^{(l)}_{ab} = \frac{1}{2} \left[ \mathcal{L}_{l} g_{ab} - \theta^{(l)} g_{ab} \right], \quad (3.4)$$

where $e^S_{ab}$ is the area element on $S_t$. Analogously, the inward expansion $\theta^{(k)}$ and the corresponding shear can be defined.

For a Schwarzschild BH, in the case of adapting the excision surface to the AH, the excision worldtube $H$ is a null hypersurface, meaning that the time evolution vector $h^\mu$ at the excision surface is null. This provides the following relationship between metric quantities at the AH: $b = N$. The outward expansion $\theta^{(l)}$ can be expressed as follows (see e.g. Eq. (11.8) in [18]):

$$\psi^2 \theta^{(l)} = 4\tilde{S} \tilde{D}_i \ln \psi + \tilde{D}_i \tilde{s}^i + K_{ij} \tilde{s}^i \tilde{s}^j, \quad (3.5)$$

where $\tilde{s}^i := \psi^2 s^i$ and $\tilde{D}$ is the Levi-Civita connection associated with the conformal metric $\tilde{g}_{ij}$. This relation could be used as a (non-linear) Robin boundary condition on the conformal factor in order to compute initial data. In particular, the outward expansion $\theta^{(l)}$ vanishes if the excision surface is placed at the AH, or can be prescribed to be negative to place the excision surface inside the AH, as we shall do here. One can set $h^{\mu} = 0$ at the boundary and on the whole spacetime, due to the particular form that the Dirac gauge takes in spherical symmetry (see Appendix A).

The value of the lapse subsists at the excision boundary as a free condition for initial data: since the maximal slicing gauge provides us only with an elliptic constraint on the bulk, one can still choose freely the value at the inner excision surface. In the case of a dynamical evolution of a spherically symmetric BH, the value of the lapse at the excision surface has to be consistent with the assumption $h^{\mu} = 0$, as it is described in next subsection.

### B. Dynamical approach in the spherically symmetric case

We consider now the time-dependent case in spherical symmetry involving matter content. Such situations include pure gauge evolutions (as illustrated in Sec. [IV B] and matter evolution (here, with the particular case of a scalar field, see Sec. [IV C]). As an astrophysical application, we have in mind the BH formation in stellar gravitational collapse simulations. Starting from a simulation for regular data evolving in $\mathbb{R}^3 \times [0, t_0]$, a trapped region forms at a given time $t = t_0$. To pursue the simulation long enough to study the subsequent evolution of the BH, one performs excision inside the trapped region, switching to a simulation in $(\mathbb{R}^3 - \mathcal{B}) \times [t_0, +\infty)$. The algorithm has the three following conditions to fulfill: i) for numerical stability reasons, transition between the two topologies has to occur smoothly, meaning that all the metric quantities solved for must be continuous and derivable in time at $t = t_0$; ii) dynamical excision has to avoid coordinate stretching and high gradient fields that would cause high and increasing inaccuracies in the computation; iii) the Schwarzschild solution should be recovered in the stationary limit.

The outermost trapped surface corresponding to the adopted spherically symmetric slicing can be located with
an AH finder. Since there is no previous control on the geometry of this trapped region, the outermost trapped surface might be stretched or deformed in 3-dimensional models, and thus is not in general an optimal candidate for the excision surface, unless we make an adaptation of the coordinates to the horizon that would imply a remapping of all data on the slice and likely introduce a copious amount of noise.

At time \( t = t_0 \), we choose the excision surface \( S_{t_0} \) to be located strictly inside the trapped region. The quantities \( N, \psi \) and \( \beta \) are determined at \( S_{t_0} \) by the previous evolution and are employed as initial values for the subsequent evolution. The outgoing scalar expansion is generically (and in average) negative, \( \theta^{(t)}_{l=0} \leq 0 \).

Once the initial excised surface has been chosen, one needs to determine a geometrical prescription for the evolution of the excision surface in time, i.e. to characterize the excision hypertube. If the initial surface were the AH, one could prescribe it to span an AH worldtube in time, by imposing the vanishing of the outward expansion \( \theta^{(l)} \) at all times on the sphere of constant radius \( R \) [13]. Contrary to the stationary case, we do not have in general \( b = N \) at the horizon. In particular, in the spherically symmetric case one has (see e.g. Eq. (38) in [32], with \( 2C = b^2 - N^2 \) and vanishing angular derivatives)

\[
\frac{b^2 - N^2}{2} = -\frac{\sigma^{(l)}_\mu \sigma^{(l)} \nu + T_{\mu \nu} \ell^\mu \ell^\nu}{L_k \theta^{(l)}}. \tag{3.6}
\]

Under the null condition the numerator is non-positive, so that the fulfillment of an outer trapping horizon condition [30], namely \( L_k \theta^{(l)} < 0 \), implies \( b \geq N \). In this case the horizon is either null (stationary case), or spacelike (dynamical case), depending of the vanishing or not of the energy flux across the BH horizon. Unfortunately, spherically symmetric trapping horizons do not necessarily fulfill the (stability) outer condition [51], so that \( a \) \textbf{a priori} we cannot guarantee in general the spacelike of \( H \) in the dynamical case. This, together with the wish of avoiding a coordinate adaption of the excision surface to the AH, leads us to choose a excursion sphere strictly inside the AH and look for an appropriate characterization of the excision worldtube. For instance, one could also impose a (non-positive) value of the expansion throughout the evolution, which would also determine a hypertube geometry.

Here we will rather follow an effective approach in which we control the radial component \( b \) of the evolution vector \( h^\mu \) on an excised coordinate sphere located strictly inside the AH. From \( b = \beta \psi s_i = \psi^2 \beta \psi s_i \), it follows in spherical symmetry \( b = \beta \psi^2 \). The imposition of a constant value in time of \( b \) at the excised surface, given by the data at \( t = t_0 \), provides us with a simple boundary condition for the shift vector through time. We want the excision hypertube to be spacelike, so that the quantity \( b - N \) should remain positive. Although we do not impose this condition directly, we monitor \( b - N \) along the evolution so that \( b \) could be dynamically adapted if needed to guarantee the spacelike character of \( H \).

The values for \( \psi, N \) and \( h^{ij} \) have still to be determined. The trace part of the evolution equations gives a consistent time evolution for \( \psi \), valid everywhere and at all time (see, e.g., Eq. (42) of [11]),

\[
\partial_t \psi = \beta^k D_k \psi + \frac{\psi}{6} D_k \beta^k. \tag{3.7}
\]

This equation, following from the kinematic definition of the extrinsic curvature, provides an additional coherent boundary condition for the conformal factor, which is obtained by solving the corresponding (elliptic) Eq. (2.7) with this boundary condition at the excised surface. The value of the lapse at the excised surface is the last (gauge) freedom left in the algorithm. We address this issue making use of the form for \( \tilde{\gamma}^{ij} \) in a \((\mathbb{R}^3 - \mathbb{B}) \) topology in Eq. (2.10). In particular, adopting the gauge \( \omega(t) = 0 \) fixes the remaining degree of freedom in the system and, in particular, fixes the value of the lapse on the excised surface. Indeed \( \tilde{\gamma}^{ij} \) adopts then a conformally flat form (or \( h^{ij} = 0 \) equivalently) at all times, and using Eq. (2.15) to update the extrinsic curvature at the excised surface we can fix the boundary condition for the lapse from any non-degenerate component of the Eq. (2.14), for instance

\[
N = \frac{\psi^6 (L \beta)^{ij} s_i s_j}{A^{ij} s_i s_j}. \tag{3.8}
\]

Our strategy can be summarized as follows: updated values for \( \psi \) and \( A^{ij} \) at the excised surface are obtained by solving Eqs. (3.7) and (2.15), respectively; imposition of constant \( b \) and Eq. (3.8) are used to obtain updated values of \( N \) and \( \beta \) at the excised surface; finally, \( h^{ij} \) is vanishing throughout the evolution. During the numerical simulation, we check that \( b - N \geq 0 \) (see Sec. [14]; under the choice of an excursion worldtube closely tracking the AH from its interior, this quantity is indeed expected to be non-negative for matter satisfying standard energy conditions in stellar collapses (note however that in scenarios not considered here that involve much larger BHs, the denominator in Eq. (3.9) can actually change sign, cf. [31]). With these boundary conditions, all elliptic equations can be solved in the numerical domain for all times, and no evolution equations are solved in the bulk.

In this approach for using the excision technique we have not considered the TT decomposition of the conformal extrinsic curvature given by Eq. (2.15) motivated by a uniqueness pathology of the elliptic sector. We find in our numerical simulations of spherically symmetric spacetimes that the given boundary conditions at the excised surface for solving the elliptic equations are enough to avoid any convergence problem in the numerical resolution of the elliptic sector. However, this question is open for more general spacetimes. In any case, the value of the \( X^i \) vector of the TT decomposition of the conformal extrinsic curvature at the excised surface is actually a degree of freedom [32].
C. Convergence to a stationary solution

Let us comment here about the convergence of metric fields to stationary values in our approach. Indeed, we find in our numerical simulations that the metric exponentially converges to a stationary solution. This fact means that the foliation induced by the boundary conditions described in Sec. III B is such that the coordinates adapt to the stationarity of the spacetime. Since no evolution equations for $h^{ij}$ and $A^i$ are computed in the bulk in this approach, and solutions to elliptic equations are determined by the boundary conditions at the excised surface, we should focus on the analysis of the values of the metric variables at the excised surface. The value for $b$ is fixed to be constant at the excised surface, so a convergence of the conformal factor at the excised surface to a stationary value will imply also a convergence of the shift vector at the excised surface to a stationary value. The evolution of the lapse $N$ at the excised surface should be such that it is compatible with the setting $b^{ij} = 0$ in the whole spacetime. This setting should in turn be compatible with coordinates adapted to stationarity.

We therefore focus on the value of the conformal factor $\psi$ at the excised surface, whose evolution is governed by Eq. (3.7). Let us define $\bar{\beta} = \beta^2 \psi^2$, which coincides with $b$ at the excised surface. Eq. (3.7) can be rewritten in terms of $\psi$ and $\bar{\beta}$ as

$$\partial_t \psi = \frac{\partial_t \bar{\beta}}{6 \psi} + \frac{2 \bar{\beta}}{3} \frac{\partial_t \psi}{\psi^2} + \frac{\bar{\beta}}{3 r^2}.$$  \hspace{1cm} (3.9)

We have imposed $b$ to be constant to its initial given value at the excised surface, located at a fixed coordinate radius, say $R > 0$, during the evolution: $b_R = \bar{\beta}_R = \beta_0 > 0$ (since in the initial data $N$ is strictly positive and $(b - N)$ is positive, $b_0$ is strictly positive, too).

Motivated by the results we observe in our numerical simulations, let us assume in the present consistency analysis that $(\partial_t \bar{\beta})|_R$ is negative and does not change significantly during the evolution, so $(\partial_t \bar{\beta})|_R \approx b_1 < 0$, $b_1$ being a constant. These assumptions are compatible with the ones found in our numerical simulations with an excised surface strictly inside the AH. In particular, such hypotheses (constant and negative value for $b_1$) are checked in the different numerical simulations of Sec. IV.

Let us assume a profile for $\psi$ of the form

$$\psi \approx 1 + \frac{c(t)}{r^p}, \quad p \geq 1.$$  \hspace{1cm} (3.10)

This profile can be considered as the one containing the leading term for $r$, taking into account that $\psi \to 1$ when $r \to \infty$ (first term) and that the conformal factor should diverge at the center of the BH ($p \geq 1$). Therefore, $\partial_t \psi \approx -p c(t)/r^{p+1}$ and $\partial_t \bar{\beta} \approx c'(t)/r^p$, where the prime denotes the derivative.

The hypotheses assumed here are not imposed during the numerical evolution of the system; they are used only to analyze the behavior of the metric variables in stationary spacetimes.

Taking into account previous assumptions, Eq. (3.7) at the excised radius $R$ is rewritten as

$$c'(t) = \frac{c_1}{[R^p + c(t)]} + \frac{c_2}{[R^p + c(t)]^2},$$  \hspace{1cm} (3.11)

where

$$c_1 = \frac{R^{2p-1}}{3} \left( \frac{b_1 R}{2} - b_0 (2p - 1) \right) < 0,$$  \hspace{1cm} (3.12)

$$c_2 = \frac{2 b_0 p R^{2p-1}}{3} > 0,$$  \hspace{1cm} (3.13)

are constants. We can integrate previous differential equation, expressed in an implicit way,

$$t = f(R^p + c(t)) = c_0 + \frac{[R^p + c(t)]^2}{2 c_1} - \frac{c_2 [R^p + c(t)]}{c_1^2} + \frac{c_2^2}{c_1^2} \log |c_1 [R^p + c(t)] + c_2|,$$  \hspace{1cm} (3.14)

where $c_0$ is an integration constant. The implicit equation for $[R^p + c(t)]$ is well defined in both branches $c_1 [R^p + c(t)] + c_2 < 0$ and $c_1 [R^p + c(t)] + c_2 > 0$. The function $f$ has a monotonic behaviour on each branch (increasing or decreasing depending of the specific one). In both branches, the range of $t$ is $\mathbb{R}$. For the initial value $[R^p + c(t = t_0)] = -c_2/c_1$, the solution of Eq. (3.14) is simply the constant value $[R^p + c(t)] = -c_2/c_1$. For close by initial values, an exponential convergence of $[R^p + c(t)]$ to the value $-c_2/c_1$ is found when $t \to +\infty$.

This discussion shows, under the assumed conditions, that we shall obtain exponential convergence to stationary values for the conformal factor $\psi$, and therefore to all metric variables, if the initial data are close enough to the stationary solution. This fact is checked numerically in Sec. IV.

IV. NUMERICAL RESULTS

In order to illustrate that the excision method presented in Sec. III works well in practice in a numerical code, we have implemented and tested it on two toy-models, which are detailed hereafter.

A. Setup

We consider spherically symmetric spacetimes in the two following physical scenarios: the evolution of a slicing of Schwarzschild BH (vacuum spacetime) in Sec. IV B and the spherical accretion of a massless scalar field onto an existing BH in Sec. IV C. In both cases, we use spherical (polar) coordinates and start from an existing BH, with the excision sphere located at the coordinate radius $r = 1$. As stated in Sec. III A, the hypothesis of a spherically symmetric spacetime in which adapted boundary
conditions are chosen, such that $\omega(t) = 0$ implies that, as far as the gravitational field is concerned, we only need to solve for $N$, $\beta$ and $\psi$.

We do not study here the full collapse process involving in particular the formation of the AH. We rather start from initial data which already contain a BH and then evolve the system in the above described excision scheme. Therefore we need to construct initial data at $t = t_0 = 0$ by solving the elliptic system (3.7)-(3.9), with the stress-energy tensor being either zero (vacuum) in the Schwarzschild BH evolution, or given by expressions (3.5)-(3.7) of Appendix B in the case of the scalar field accretion. The initial excision surface is chosen as a sphere with a given (arbitrary) value of the outward expansion (3.3) $\theta^i(t = 0, r = 1) = \theta_0^i$, prescribed to be negative. This guarantee that the initial excision surface is inside the AH, that then is located using an AH finder. Initial data boundary conditions are needed for the elliptic system and are taken as follows:

- Setting $\theta_0^i$ and using Eq. (3.3), a (non-linear) Robin boundary condition is obtained for the conformal factor $\psi$.
- The boundary value of the lapse $N$ is fixed yielding a Dirichlet condition.
- A value is prescribed for the quantity $b - N$ (see Eq. (3.2)), from which one can get a Dirichlet boundary condition for the radial component of the shift $\beta^r$, the other two components ($\beta^\theta, \beta^\varphi$) being zero in spherical symmetry.

The elliptic system giving the metric functions is solved iteratively, starting from a first guess (flat metric) and inverting linear Laplace operators with the library LORENE [34], using multi-domain spectral methods, with a coordinate transform $u = 1/r$ in the last domain, extending to infinity and allowing for the imposition of boundary conditions at $r \to \infty$ (see, e.g. [35]).

During the evolution, metric quantities are obtained by the resolution of the same elliptic system, but with different boundary conditions. As described in Sec. III B the following boundary conditions are used during the dynamical evolution:

- A Dirichlet boundary condition for the conformal factor $\psi$ is obtained from the time integration of Eq. (3.3) at the excision surface. As this surface is a sphere, there is no need for a boundary condition to integrate Eq. (3.3) in time.
- The value of the lapse at the boundary is set by the Dirichlet condition (3.5). To compute it, we need to integrate in time Eq. (2.15) at the excision surface.
- The radial component of the shift is imposed at the excision boundary by keeping the value of $b(t)$ constant in time.

Time integration of Eqs. (3.7)-(3.9) is done with a second-order (explicit) Adams-Bashforth scheme.

B. Evolution of a Schwarzschild black hole

A spherically symmetric spacetime is computed for $r \geq 1$, with boundary conditions defined hereabove, and the specific values $\theta_0^i = -0.01$, $N(t=0, r=1) = 0.55$ and $(b - N)(t=0) = 0.01$ for these initial data. Once having solved the elliptic system (2.7)-(2.9), one verifies that this setting induces a non-zero value for the Arnowitt-Deser-Misner (ADM) mass, more precisely $M_{\text{ADM}} \approx 1.09$. Numerically, it is obtained in isotropic gauge from the asymptotic behavior of the conformal factor (see e.g. [11]),

$$M_{\text{ADM}} = -\frac{1}{2\pi} \int_\infty^\infty D_i \psi dS^i,$$

where the integral is taken over a sphere of radius $r \to +\infty$. This makes the excision sphere to be located at $r \approx 0.916 M_{\text{ADM}}$. Using the numerical AH finder described in [36], we have found in the initial data an AH located at the coordinate radius $r_s \approx 0.94 M_{\text{ADM}}$. These are evidences that initial data represent a BH in spherical symmetry.

This BH spacetime is then numerically evolved for $t \geq 0$, through the time integration of boundary conditions, while solving the elliptic system (2.7)-(2.9) at every given number of timesteps, as described in Sections III B and IV A. This is obviously only a gauge evolution, since the spacetime is Schwarzschild by construction. The time evolution of the metric variables at the excision surface ($r = 1 = 0.916 M_{\text{ADM}}$), namely $N$, $\psi$ and $b - N$, as well as the coordinate radius $r_s$ of the AH, in the interval $0 \leq t \leq 50$, are displayed in Fig. 1. An exponential convergence toward stationary values is observed for all metric quantities, with an explicit behavior shown for the lapse $N$, as expected from the analysis carried out in Sec. III C $r_s$ increases with time (right bottom panel of Fig. 1), but not the overall mass of the black hole (see hereafter).

From the top-right panel of Fig. 1 one can check that the difference $b - N$ remains positive, as assumed in the discussion of Sec. III C. The hypothesis for $(\partial_t \beta)$ to be negative, assumed in the same analysis of Sec. III C is fulfilled too during the evolution, and its value does not change significantly (absolute value of the relative difference with respect to its initial value $\lesssim 4 \cdot 10^{-3}$). Moreover, from the formulas (4.2) and (4.3), and the computed limit in Sec. III C for the conformal factor at the excision boundary, one can check that the hypothesis (4.10) is valid; the numerically deduced value of $p$ from $\lim_{t \to \infty} \psi(t, r = 1)$, and values of $b_0$ and $b_1$, is $p \approx 1.01$, which gives the expected behavior for the conformal factor. In Fig. 2 is displayed the time evolution of the outward expansion $\theta^i(t)$, which is decreasing during the simulation and remains always negative, ensuring that the
excision surface remains always inside the apparent horizon.

The scheme is clearly stable (we have run it for $t \sim 1000$) and, in order to check its accuracy, we have monitored the variation of the AH irreducible mass, $M_{AH}$, defined as

$$M_{AH}(t) = \frac{1}{2} \sqrt{\frac{A_{AH}(t)}{4\pi}} = \frac{1}{2} \psi^2(t, r_*(t)) r_*(t),$$

and determined by the AH finder. Penrose inequality conjecture, in particular its rigidity part, provides a practical manner of characterising the Schwarzschild solution. It states

$$A \leq 16\pi M_{ADM}^2 \Rightarrow M_{AH} \leq M_{ADM},$$

with equality if and only for slices of the Schwarzschild spacetime. The conservation of the $M_{AH}$ is displayed in Fig. 3 in particular showing the numerical consistency with a Schwarzschild solution using the Penrose inequality test. The conservation of the ADM mass (4.1) has also been checked, with quantitatively very similar results to the conservation of the AH surface. Finally, a second-order convergence of these conserved quantities has been obtained numerically with decreasing the time-step, in agreement with the implemented second-order Adams-Bashforth scheme.

The exponential convergence obtained for the chosen initial data, given by the specific values of $\theta_0^{(l)}$, $N_{(t=0,r=1)} > 0$ and $(b - N)_{(t=0)} > 0$, is independent of these initial data. A similar behavior is found for different initial values.
C. Accretion of a massless scalar field

In this case we evolve a BH spacetime with energy content in the form of a minimally coupled massless scalar field in spherical symmetry; see Appendix B for details concerning the expressions of the projections of the energy-momentum tensor, the corresponding evolution equations for the scalar field and the way they are solved. Initial data are given by a Gaussian profile for the scalar field outside the excision surface, i.e. for \( r \geq 1 \),

\[
\phi(r, t = 0) = \frac{\phi_0 r^2}{1 + r^2} \left( e^{-(r-r_0)^2/\sigma^2} + e^{-(r+r_0)^2/\sigma^2} \right), \tag{4.4}
\]

with \( \phi_0, r_0 \) and \( \sigma \) three constants, following e.g. [37].

The scalar field is evolved on the numerical grid up to \( r = R_{\text{max}} = 120 \simeq 110 \, M_{\text{ADM}} \) (i.e. not in the compactified domain). Thus, the effect on the metric at the horizon of any scalar field wave reflected from the artificial boundary at \( r = R_{\text{max}} \) can be in principle neglected up to \( t \sim 200 \). For this simulation, we have used 24 numerical domains: one nucleus, 22 shells and a compactified domain (for details about the grid setting, see [35]). As said, the wave equation is not solved in the compactified domain, but instead the outgoing boundary conditions (B9)-(B10) are imposed at \( r = R_{\text{max}} \). 25 Chebyshev coefficients are used in each domain, in order to describe the wave accurately enough.
We evolve these initial data with $\phi_0 = 0.01$, $r_0 = 5$ and $\sigma = 1$. A fraction of the scalar field is radiated away, the other part is accreted onto the BH, its time evolution at the excision surface being given in Fig. 5. The metric, $N$, $\beta^r$ and $\psi$ follow a similar evolution with respect to the vacuum case and, once the scalar field has been accreted to the BH, they settle rapidly to stationary values. Fig. 6 gives the evolution of the AH mass as a function of the proper time of the observer that is located at the AH. As expected, the AH grows in time while accreting energy from the scalar field, before reaching a stationary limit. This limit does not represent all the ADM mass of the spacetime, as part of this asymptotic mass is still contained in the scalar field traveling to higher radii. We have checked the accuracy of the code by monitoring the variation of the ADM mass, computed by Eq. (4.1) in Fig. 6. The conservation of this quantity, up to a level $10^{-6}$, shows that the scalar field stress-energy enters the BH and makes it grow accordingly. The first two spikes for $t \lesssim 10$ can be attributed to the scalar field which is entering the black hole apparent horizon. Further oscillations that are seen in this figure for $t \gtrsim 20$ can be related to the passing of the scalar field wave from one spectral domain to another. Note that the overall level of this ADM mass violation is $10^{-6}$ and that it converges away with both time and spatial resolutions.

This simulation shows that the excision boundary conditions that have been designed here allow us to study the growth of a spherically symmetric accreting BH in a stable and accurate way.

V. CONCLUSIONS

In this work we have presented a new excision technique for the dynamical evolution of spherically symmetric spacetimes in the FCF in order to numerically simulate systems forming a BH.

FCF belongs to the so-called constrained formulations of Einstein equations, in which the constraints are solved for each time step. On the contrary, in free evolution formulations the evolution equations are in general of hyperbolic type and constraints are used to monitor the validity of the numerical solution (and/or as damping terms in the evolution scheme). The puncture method has been used in combination with the BSSN formulation for binary BH evolutions and the excision technique in combination with the generalized harmonic gauge. The difficulty of using the excision technique in the case of constrained formulations comes from the fact that constraints are elliptic-type PDEs and wrong boundary conditions at the excision surface invalidate the physical solution in the whole numerical domain.

In the context of constrained formulations, excision has been used to generate initial data. Dynamical evolutions using constrained formulations were presented in the work of Scheel et al. [22], in the case of dust collapse in Brans-Dicke theory of gravity; in their work, the excision boundary was considered to be the AH at a fixed radial coordinate. In non spherically symmetric spacetimes, the coordinate shape of the AH can be very complicated and this approach has strong limitations. In our case, as in Rinne & Moncrief [23], the excision boundary is a coordinate sphere located strictly inside the AH, and we let the coordinate location of the AH to freely evolve in time. This approach allows in particular for a very straightforward extension to spacetimes without symmetries, where the AH can form with a shape deviating from a coordinate sphere.

The proposed approach uses an arbitrary coordinate sphere located strictly inside the AH as excision surface and a set of simple boundary conditions for the elliptic equations to be solved. It permits more freedom in the choice of the excision surface, which could be set as a simple coordinate sphere in spacetimes without symmetries. We have checked the practical applicability of this approach in two cases: the numerical simulation of a Schwarzschild spacetime, and of a spherically symmetric accreting BH with energy content in the form of a massless scalar field. Our numerical results are stable and accurate. We have also theoretically analyzed the behavior of these boundary conditions in the proximity of stationary spacetimes and found an exponential adjustment of the coordinates to stationary values, independently of the chosen initial data. This behavior has also been checked numerically. Although we have restricted this work to spherical symmetric spacetimes, we plan to extend this approach in the future to more general spacetimes with less symmetries in forthcoming studies.

The proposed excision technique can be used in the
context of several astrophysical scenarios like a stellar collapse to a BH, the formation of an accretion disk and/or the launching of a jet. Notice that in these scenarios the initial data are usually regular, and during the simulation a BH forms and, particularly when using singularity avoiding time coordinates (e.g. maximal slicing), an AH is found before the appearance of the physical singularity [21]. Then the excision technique can be used to continue the numerical simulation, say in the accreting boundary (excision at $r=R_{\text{max}}$), before the compactified domain, and the consistency condition for the metric. Note that, in the case of maximal slicing condition ($K = 0$), the last term of Eq. (B7) vanishes.

Matching across different domains is done along characteristic fields, in an upwind manner. At the outer boundary ($r = R_{\text{max}}$), before the compactified domain, we impose a Sommerfeld-like condition,

$$\partial_t (\Pi - \Phi_r)_{r = R_{\text{max}}} = 0,$$

and the consistency condition for $\phi$,

$$\partial_t \phi |_{r = R_{\text{max}}} = (-N\Pi + \beta^r \Phi_r)_{r = R_{\text{max}}}. \quad (B10)$$

No boundary condition is needed for either field at the inner boundary (excision at $r = 0.916M_{\text{ADM}}$), all characteristics being directed out of the computational domain as long as the excision surface is spacelike, which is verified in our case with $b - N > 0$ in our numerical simulations. All matching and boundary conditions are implemented in a collocation approach to spectral variables.

\[ T_{\mu\nu} = \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{2} \gamma_{\mu\nu} \nabla_{\rho} \phi \nabla^{\rho} \phi. \] (B2)

Its projections are given by

$$E = \frac{1}{2N^2} ((\partial_{t} - L_{e})\phi)^2 + \frac{1}{2} D_{\mu} \phi D^{\mu} \phi,$$ (B3)

$$S_{i} = \frac{1}{N} ((\partial_{t} - L_{e})\phi) D_{i} \phi,$$ (B4)

$$S_{ij} = D_{i} \phi D_{j} \phi - \frac{1}{2} \gamma_{ij} \left[ D_{k} \phi D^{k} \phi - \frac{1}{N^2} ((\partial_{t} - L_{e})\phi)^2 \right],$$ (B5)

where $D$ is the Levi-Civita connection associated with the spatial metric $\gamma_{ij}$.

The wave equation [B1] is rewritten as a first order system in space and time, by introducing the auxiliary scalar $\Pi$, defined from Eq. (B6) below, and the vector $\Phi_i = D_i \phi$, considered also as a constraint of the system, as

$$\partial_t \phi = -N \Pi + \beta^i D_i \phi,$$ (B6)

$$\partial_t \Pi = -N \gamma^{ij} D_i \Phi_j + \beta^i D_i \Pi$$

$$- \frac{\Phi_i D_j (N \psi^{\delta}_{\gamma^{ij}})}{\psi^{\delta}} + \Pi N K,$$ (B7)

$$\partial_t \Phi_i = -D_i (N \Pi) + \Phi_k D_i \beta^k + \beta^k D_k \Phi_i.$$ (B8)

This system is solved using a second-order Adams-Bashforth scheme, the time-step being a submultiple of the one used for the evolution of boundary conditions for the metric. Note that, in the case of maximal slicing condition ($K = 0$), the last term of Eq. (B7) vanishes.

\[ \gamma_{ij} = \begin{pmatrix} A(r) & 0 & 0 \\ 0 & B(r) & 0 \\ 0 & 0 & C(r) \end{pmatrix}, \quad (A1) \]

with the additional determinant condition

$$A(r) B(r) C(r) = \det f^{ij} = 1. \quad (A2)$$

Conditions (2,3) and previous one for the determinant can be written as

$$\partial_r A + \frac{2A - B - C}{r} = 0, \quad B = C = \frac{1}{\sqrt{A}}$$

$$\Leftrightarrow \partial_r A + \frac{2(A - 1/\sqrt{A})}{r} = 0, \quad B = C = \frac{1}{\sqrt{A}} \quad (A3)$$

A general solution for $A$ is

$$A(r) = \left(1 + \frac{\omega}{r_0}\right)^{2/3}, \quad \omega \in \mathbb{R}. \quad (A4)$$

\[ \text{Appendix B: Massless scalar field in a curved spacetime} \]

The massless Klein-Gordon equation, or simply wave scalar equation, is given by

$$\nabla^\mu \nabla_\mu \phi = 0, \quad (B1)$$

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