A New Fixed Point Theorem for Non-expansive Mappings and Its Application

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Abstract. We use KKM theorem to prove the existence of a new fixed point theorem for non-expansive mapping; Let M be a bounded closed convex subset of Hilbert space H, and $A : M \rightarrow M$ be a non-expansive mapping, then exists a fixed point of $A$ in M, we also apply this Theorem to study the solution for an integral equation, we can weaken some conditions comparing with Banach’s contraction mapping principle.

Key words: Bounded closed convex subset, non-expansive mapping, fixed point, integral equation

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1 Introduction and Main Results

It is well-known that Banach, Schauder, Brouwer presented three kinds of fixed point theorems in 1910's-1930's. Banach’s fixed point theorem is also called contraction mapping principle, and Schauder's fixed point theorem is a generalization of Brouwer's from finite dimension into infinite dimension. Specifically, for a compact operator A, if its domain is a bounded non-empty closed convex subset, then there is at least one fixed point $u : A \cdot u = u$. All the above three fixed point theorems need good operators, for example, the compactness for the operator can make the unit closed sphere in infinite dimensional space be a compact set.

Here, a nature question is whether the fixed point theorem of other good operator or mapping could be established.

A problem is whether there is a fixed point theorem for $A$ when $k = 1$ (where A is called the non-expansive mapping). For solving this problem, the main tool we will use is KKM theorem. It’s well known that the Polish mathematician Knaster, Kuratowski, Mazurkiewicz got KKM theorem in 1929, they also applied it to the proof of Brouwer fixed point theorem.
**Theorem 1.1.** (Banach fixed point theorem([1])): Let \((X, d)\) be a complete metric space, and \(A: X \to X\) be a contraction mapping, that is, for any \(x, y \in X\), there holds \(d(Ax, Ay) \leq kd(x, y)\), \((0 \leq k < 1)\). Then there is a unique fixed point \(x^*\) for \(A\), i.e. \(Ax^* = x^*\).

Banach’s fixed point theorem can be applied to many fields in mathematics, especially to integral equation:

\[
    u(x) = \lambda \int_a^b F(x, y, u(y))dy + f(x), \quad a \leq x \leq b \tag{1.1}
\]

**Theorem 1.2.** ([2]) Assume

(a) \(f: [a, b] \to R\) is continuous
(b) \(F(x, y, u(y)) = K(x, y)u(y)\), and \(K: [a, b] \times [a, b] \to R\) is continuous.

Let \(\Gamma = \max_{a \leq x, y \leq b} |K(x, y)|\), and

(c) real number \(\lambda\) be given such that \((b - a) \cdot |\lambda| \cdot \Gamma < 1\).

Then the problem (1.1) has a unique solution \(u \in X = C[a, b]\).

In this paper, we get:

**Theorem 1.3.** Let \(M\) be a bounded closed convex subset of Hilbert space \(H\), and \(A: M \to M\) be a non-expansive mapping, then exists a fixed point of \(A\) in \(M\).

Notice that our fixed point theorem is in Hilbert Space, since any linear continuous functional in Hilbert space can be represented by the inner product, which is called Riesz representation theorem, while in the Banach space no such theorem.

We apply the above theorem to linear integral equation, we obtain

**Theorem 1.4.** For the integral equation (1.1) with \(F(x, y, u(y)) = K(x, y)u(y)\), assume

(1) The function \(f: [a, b] \to R\) is in \(L^2[a, b]\).
(2) \(K(x, y) \in L^2([a, b] \times [a, b]), K(x, \cdot) \in L^2[a, b]\), for any given \(x \in [a, b]\)
(3) Let the real number \(\lambda\) be given such that, \(|\lambda| \int_a^b \int_a^b |K(x, y)|^2dxdy \leq 1, if f \equiv 0\); and,

\[
    |\lambda| \int_a^b \int_a^b |K(x, y)|^2dxdy < 1, if f \neq 0.
\]

Then (1.1) has at least one solution \(u \in L^2[a, b]\).

2 Related Definitions and Well-known Theorems

**Definition 2.1.** (KKM mapping([3])): The mapping \(G: X \to 2^{R^n}\), where \(X\) is a non-empty subset of the vector space \(R^n\), is called KKM mapping. if \(\forall \{x_0, \ldots, x_n\} \subset X\), we have \(Co(\{x_0, \ldots, x_n\}) \subset \cup_{i=1}^n G(x_i)\), where \(Co(\{x_0, \ldots, x_n\})\) is the closed convex hull of \(x_0, \ldots, x_n\).
Theorem 2.1. \( (KKM \text{ Theorem (4)}) \) Let \( \Delta_n \) be \( n \) dimensional simplex which has vertices \( \{e_0, \ldots, e_n\} \) and is in \( R^{n+1} \). Let \( M_0, \ldots, M_n \) be \( n \) dimensional closed subsets in \( R^{n+1} \), which satisfy \( \forall \{e_{i0}, \ldots, e_{ik}\} \subset \{e_0, \ldots, e_n\} \), \( \text{Co}(\{e_{i0}, \ldots, e_{ik}\}) \subset \cup_{j=0}^k M_{ij} \). Then \( \cap M_i \) is not empty.

Theorem 2.2. \( (FKKM (5)) \): Let \( F^n \) be Hausdorff linear topological space, \( X \) be a non-empty subset of \( F^n \). Let \( G : X \to 2^{F^n} \) be a \( KKM \) mapping, and \( G(x) \) is weakly compact for any \( x \in X \). Then \( \cap_{x \in X} G(x) \) is non-empty.

Definition 2.2. \( (\text{Semi-continuous (6)}) \): Let \( X \) be a Banach space, \( X^* \) be its dual space. Let \( T : X \to X^* \). If \( \forall y \in X, \forall t_n \geq 0 \) and \( x_0 + t_n \cdot y \in X \), we have \( \lim_{x_0+t_n \to x_0} T(x_0 + t_n \cdot y) \to T(x_0) \), we call \( T \) is semi-continuous at point \( x_0 \). Furthermore if \( T \) is semi-continuous at every point \( x \) in \( X \), \( T \) is called semi-continuous on \( X \).

Definition 2.3. \( (\text{Monotonicity (7)}) \): Let \( X \) be a Banach space, \( X^* \) be its dual space. We call \( T : \to 2^{X^*} \) monotonous, if \( < u - v, x - y > \geq 0 \), for \( \forall x, y \in X, u \in T(x), v \in T(y) \).

3 Some Lemmas for the Proof of Our Theorem 1.3.

We require the following Lemmas for proving the fixed point theorem of the non-expansive mapping using the \( KKM \) theorem.

Lemma 3.1: Let \( M \) be a bounded closed convex subset of Hilbert space \( H \), \( L : M \to H \) be monotonous semi-continuous mapping. If \( x_0 \in M \), then

\[
< L(x_0), y - x_0 > \geq 0, \forall y \in M,
\]

if and only if

\[
< L(y), y - x_0 > \geq 0, \forall y \in M.
\]

Proof Let \( x_0 \in M \) such that : \( < L(x_0), y - x_0 > \geq 0, \forall y \in M \). Since the monotonicity of \( L \), we have

\[
< L(y) - L(x_0), y - x_0 > \geq 0 \iff < L(y), y - x_0 > - < L(x_0), y - x_0 > \geq 0, \forall y \in M.
\]

Thus ,

\[
< L(y), y - x_0 > \geq < L(x_0), y - x_0 > \geq 0, \forall y \in M.
\]

On the other hand, if \( x_0 \in M \) then

\[
< L(y), y - x_0 > \geq 0, \forall y \in M.
\]

We let

\[
y = h \cdot \mu + (1 - h)x_0, \forall h \in (0, 1], \forall \mu \in M,
\]
since $M$ is convex, then $y \in M$. Thus,

$$< L(y), y - x_0 > = < L(x_0 + h \cdot (\mu - x_0)), h \cdot (\mu - x_0) >= 0(h > 0).$$

We let $h \to 0$, then $x_0 + h \cdot (\mu - x_0) \to x_0$, since $L$ is semi-continuous, so

$$< L(x_0), (\mu - x_0) > = 0, \forall \mu \in M.$$

**Lemma 3.2**: Let $M$ be a bounded closed convex subset of Hilbert space $H$, and $L : M \to H, P : M \to 2^M$, and $P(y) = \{ x \in M, < L(x), x - y > \leq 0 \}, \forall y \in M$, then $P$ is a $KKM$ mapping. 

**Proof**: If $P$ is not a $KKM$ mapping. By the definition of $KKM$ mapping, $\exists \{ x_0 \ldots x_n \} = w$, $w \subset M$, let $\bar{x} = \sum h_i \cdot x_i \in Co(w), (\sum h_i = 1, h_i \geq 0)$, then $\bar{x} \notin \cup_{i=0}^n P(x_i)$, that is to say, $< L(\bar{x}), \bar{x} - y >> 0$. As the arbitrariness of $y$, let $y = \sum h_i \cdot x_i$, we have $< L(\bar{x}), \bar{x} - x_i >> 0$, then $< L(\bar{x}), h_i \cdot (\bar{x} - x_i) >> 0(\exists h_i > 0)$, thus,

$$< L(\bar{x}), \sum h_i \cdot (\bar{x} - x_i) >= < L(\bar{x}), \sum h_i \cdot \bar{x} - \sum h_i \cdot x_i >= < L(\bar{x}), \bar{x} - y >= 0.$$

This is a contradiction with $< L(\bar{x}), \bar{x} - y >> 0, \forall y \in M$. Hence $P$ is a $KKM$ mapping.

**Lemma 3.3**: Let $M$ be a bounded closed convex subset of Hilbert space $H$, $L : M \to H$ be a semi-continuous monotonous mapping, then $\exists x_0 \in M$ s.t.

$$< L(x_0), x_0 - y > \leq 0, \forall y \in M$$

**Proof**: Let $P : M \to 2^M$ and $P(y) = \{ x \in M, < L(x), x - y > \leq 0 \}, \forall y \in M$. According to Lemma 3.2, $P$ is a $KKM$ mapping. Mean-while, let mapping $J : M \to 2^M$ and

$$J(y) = \{ x \in M, < L(y), x - y > \leq 0 \}, \forall y \in M.$$

By the monotonicity of the operator $L$ and Lemma 3.1,

$$P(y) \subset J(y), \forall y \in M.$$ 

Since $P$ is a $KKM$ mapping, $Co(\{ x_0 \ldots x_n \}) \subset \cup_{i=0}^n P(x_i) \subset \cup_{i=0}^n J(x_i)$. hence $J$ is $KKM$ mapping.

Furthermore, it can be easy to prove that $J(y)$ is weakly compact. In fact, Since $M$ is a bounded closed convex subset, it can be convinced that $M$ is weakly compact. But $J(y) \subset M$, thus we only need to prove $J(y)$ is weakly closed. In order to do that let $y_n \in J(y) \subset M$, s.t.

$$y_n \to y_0, \text{so}, y_n - y \to y_0 - y, \forall y \in M.$$

According to the semi-continuity of $L, < L(y), y_n - y > \to < L(y), y_0 - y >$. By $y_n \in J(y)$, we have $< L(y), y_n - y > \leq 0$, so $< L(y), y_0 - y > \leq 0$. Hence $y_0 \in J(y)$, that is to say, $J(y)$ is weakly closed.

From the above proof, $J$ is a $KKM$ mapping, and $\forall y, J(y)$ is weakly compact. According to $FKKM$ theorem, $\cap_{y \in M} J(y) \neq \emptyset$. According to Lemma 3.1, $\cap_{y \in M} J(y) = \cap_{y \in M} P(y) \neq \emptyset$, thus

$$\exists x_0 \in \cap_{y \in M} P(y), \text{s.t., } < L(x_0), x_0 - y > \leq 0, \forall y \in M.$$
4 The Proof of Fixed Point Theorem 1.3 for the Non-Expansive Mapping

**Proof:** Let $L(x) = x - A(x), \forall x \in M$, then $L : M \to H$. By the non-expansive property for the operator $A$, we will prove $L = I - A$ is a monotonous operator:

$$< L(x) - L(y), x - y > \geq 0, \forall x, y \in M.$$ 

In fact,

$$< L(x) - L(y), x - y > = < x - y - (Ax - Ay), x - y >$$

$$= ||x - y||^2 - < Ax - Ay, x - y >$$

$$\geq ||x - y||^2 - ||Ax - Ay||^2 \cdot ||x - y||^2$$

$$\geq ||x - y||^2 - ||x - y||^2 = 0.$$ 

Since $A$ is non-expansive mapping : $\forall x, y \in M, ||Ax - Ay|| \leq ||x - y||$, it can be derived that $L$ is semi-continuous mapping. According to Lemma 3.3, $\exists x_0 \in M$, s.t.

$$< L(x_0), x_0 - x > \leq 0, \forall x \in M \quad (4.1)$$

Specially we let $x = A(x_0) \in M$ and substitute it into (4.1), we get:

$$< L(x_0), x_0 - A(x_0) > \leq 0 \quad (4.2)$$

Since $L(x_0) = x_0 - A(x_0)$, so from (4.2) we have $||x_0 - A(x_0)|| \leq 0$. It’s clearly that norm $||x_0 - A(x_0)|| \geq 0$, hence $||x_0 - A(x_0)|| = 0$, that is to say, $A(x_0) = x_0$.

5 The Proof of Theorem 1.4

**Proof:** Let $(Au)(x) = \lambda \int_a^b F(x, y, u(y))dy + f(x),$

$$||Au - Av||^2 = \int_a^b |\lambda \int_a^b K(x, y)((u(y) - v(y))dy|^2dx$$

$$\leq \lambda^2 \int_a^b [\int_a^b |K|^2dy \int_a^b |u(y) - v(y)|^2dy]dx$$

$$= [\lambda^2 \int_a^b \int_a^b |K(x, y)|^2dxdy] \cdot ||u - v||_{L^2}.$$ 

If $[\lambda^2 \int_a^b \int_a^b |K(x, y)|^2dxdy] \leq 1$, then $A$ is non-expansive mapping.
Furthermore, we restrict A on a bounded closed convex subset M of $L^2$ such that $AM \subseteq M$. Given $r > 0$, let $M = \{ u \in L^2 \| u \|_{L^2} \leq r \}$,

$$
\|Au(x)\|_{L^2} \leq \int_a^b F(x, y, u(y))dy \|_{L^2} + \|f\|_{L^2}
$$

$$
\leq \|\lambda\| \int_a^b [\int_a^b Fdy|^{2dx}]^{1/2} + \|f\|_{L^2}
$$

$$
\leq \|\lambda\| \{\int_a^b [\int_a^b |K(x, y)u(y)|dy|^{2dx}]^{1/2} + \|f\|_{L^2}
$$

$$
\leq \|\lambda\| \{\int_a^b [\int_a^b |K|^{2dy} \int_a^b |u(y)|^{2dy}]^{1/2}
$$

$$
= \|\lambda\| \{\int_a^b \int_a^b |K(x, y)|^{2dx}dy\}^{1/2} \cdot \|u\|_{L^2} + \|f\|_{L^2}
$$

$$
\leq \|\lambda\| \{\int_a^b \int_a^b |K(x, y)|^{2dx}dy\}^{1/2} \cdot r + \|f\|_{L^2}
$$

$$
\leq r.
$$

(1). If $f \equiv 0$, and

$$
|\lambda| \{\int_a^b \int_a^b |K(x, y)|^{2dx}dy\}^{1/2} \leq 1.
$$

Then we can choose any given $r > 0$.

(2). If $f \neq 0$, and

$$
|\lambda| \{\int_a^b \int_a^b |K(x, y)|^{2dx}dy\}^{1/2} < 1.
$$

Then choose r such that,

$$
r \geq \frac{\|f\|_{L^2}}{1 - |\lambda| \{\int_a^b \int_a^b |K(x, y)|^{2dx}dy\}^{1/2}}.
$$

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References

[1] E. Zeidler, Applied functional analysis: main principles and their applications, Springer, 1995.

[2] E. Zeidler, Applied functional analysis: application to mathematical physics, Springer, 1995, P. 22-23.

[3] K. C. Border, Fixed point theorems with applications to economics and game theory, Cambridge University Press, 2009, P. 26.
[4] B.Knaster, C.Kuratowski and S.Mazurkiewicz, 'Ein Beweis des Fixepunktensatzes fur n-dimensional simplexe', Fund. Math. 14 (1929), P.132-137.

[5] K.Fan, 'A generalization of Tychonoff’s fixed point theorem', Ann. Math. 142 (1961), P.303-310.

[6] W.Hang, X.Cheng, 'An introduction to variational inequalities: elementary theory, numerical analysis and applications', Academic Press, 2003, P.92.

[7] B.Meng, X.Cao, 'A course in operator and game theory', The MIT Press, 1998, P.36.

[8] S.Itoh, W.Takahashi, K.Yanagi, 'Variational Inequalities and complementarity problems', J. Math. Soc. Japan, 1978, P.17-55.