A Corbino disk viscometer for 2D quantum electron liquids

Andrea Tomadin,1,∗ Giovanni Vignale,2 and Marco Polini1

1NEST, Istituto Nanoscienze-CNR and Scuola Normale Superiore, I-56126 Pisa, Italy
2Department of Physics and Astronomy, University of Missouri, Columbia, Missouri 65211, USA

The shear viscosity of a variety of strongly interacting quantum fluids, ranging from ultracold atomic Fermi gases to quark-gluon plasmas, can be accurately measured. On the contrary, no experimental data exist, to the best of our knowledge, on the shear viscosity of two-dimensional quantum electron liquids hosted in a solid-state matrix. In this Letter we propose a Corbino disk device, which allows a determination of the viscosity of a quantum electron liquid from the dc potential difference that arises between the inner and the outer edge of the disk in response to an oscillating magnetic flux.

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Introduction.—The shear viscosity \( \eta \), which describes the diffusion of the average momentum density orthogonally to its direction, is one of the cornerstones of the hydrodynamic theory of fluids [1, 2]. The first estimate of the shear viscosity of a dilute gas as a function of its density and temperature was given by Maxwell in his celebrated article on the “Dynamical Theory of Gases”. He found that the shear viscosity of a dilute gas is independent of its density, a counterintuitive result that he felt needed immediate experimental testing [3]. Recent years have witnessed a surge of interest in the viscous flow of strongly interacting quantum fluids, for which hydrodynamics provides a powerful non-perturbative description [4]. Experimentally, the shear viscosity of quantum liquids like \(^3\)He and \(^4\)He can be measured by a variety of tools including capillary, rotation, and vibration viscometers [5]. The shear viscosity of cold atomic gases can be inferred from measurements of collective modes or by looking at the expansion of the gas in a deformed trap after the trapping potential is turned off [6–8]. The shear viscosity of quark-gluon plasmas can be extracted from elliptic flow measurements at relativistic heavy-ion colliders [9]. To the best of our knowledge, however, no protocols exist for measuring the shear viscosity of two-dimensional (2D) quantum electron liquids (QELs) in solid-state matrices [10, 11]. This gap is truly surprising in view of the large body of theoretical work [12] that has been carried out in connection with the shear viscosity of these systems. In this paper we try and fill the gap by proposing a method to measure the viscosity of electrons in a realistic experimental setup.

The concept of hydrodynamic viscosity \( \eta \) [1, 2, 4] becomes relevant in a regime of parameters in which the electron liquid is well described by a quasi-equilibrium distribution function characterized by slowly time-dependent density and drift velocity—the local counterparts of globally conserved particle number and momentum. In a solid-state device with linear dimension \( L \) this is ensured by the following chains of inequalities: \( \ell_{ee} \ll L \ll \ell_p \), where \( \ell_{ee} \) is the mean free path between quasiparticle collisions [13] and the length scale over which local thermodynamic equilibrium is achieved, while \( \ell_p \) is the length scale over which electron-impurity and electron-phonon scattering break momentum conservation. It is well known [14–16] that the above inequalities are satisfied in highly-pure 2DQELs e.g. in modulation-doped GaAs/AlGaAs semiconductor heterojunctions [10] for typical electron densities, in the temperature range 5 K \( \lesssim T \lesssim 35 \) K, and for devices with linear size 10 \( \mu m \lesssim L \lesssim 50 \) \( \mu m \). In this range of param-
eters momentum-non-conserving collisions can be safely
neglected, while electron-electron interactions establish
local equilibrium with a slowly-varying density \( n(r,t) \)
and drift velocity \( \mathbf{v}(r,t) \). Hydrodynamic electron flow
has indeed been experimentally generated and hydrody-
namic effects have been measured [17]. We now describe
our method for determining the viscosity.

Electrical measurement of the viscosity.—We consider a
2DQEL shaped in a Corbino disk (CD) geometry—see
Fig. 1. The CD lies in the \( z = 0 \) plane and has an
inner radius \( r_{\text{in}} \) and an outer radius \( r_{\text{out}} \). It is sepa-
rated from a back gate by a dielectric layer of thickness
\( d \) and dielectric constant \( \epsilon \). An oscillating magnetic flux
\( \Phi(t) = \Phi_0 \cos(\Omega t) \) oriented along the \( \hat{z} \) axis threads the
inner hole of the CD and induces, by Faraday’s law, an
azimuthal electric field, which is given by

\[
E_\theta(r,t) = -\frac{1}{2\pi cr} \partial_t \Phi(t) \hat{\theta},
\]

where \( c \) is the speed of light and \( \hat{\theta} \) is the unit vector in the
azimuthal direction. Fluctuations of the circularly-

capacitance approximation” is valid as long as
\( \epsilon/r \ll 1 \) [14]. The electron
drift velocity—in units of \( v_0 \) as defined in Eq. (4)—at \( t = 0 \),

\[
\Delta U_{\eta=0} = \frac{e^2}{mc} \frac{1}{2\pi r_{\text{in}}^2} \frac{1}{4} \Phi_0^2.
\]

Notice that \( \Delta U_{\eta=0} \) is independent of the frequency \( \Omega \)
of the driving flux. To estimate the magnitude of the dc
response, we use the following parameters [18]:

\[
r_{\text{in}} = 2.0 \, \mu \text{m}, \quad r_{\text{out}} = 20.0 \, \mu \text{m},
\]

\[
\Phi_0/(2\pi r_{\text{in}}) = 10 \, \text{mT}, \quad m = 0.067 \, m_e,
\]

where \( m_e \) is the bare electron mass in vacuum and the
value of \( m \) given above is appropriate for electrons in
GaAs [10]. From Eq. (5), we find \( \Delta U_{\eta=0} = 63 \, \mu \text{eV} \).
The analytical result (4) is plotted as a solid line in Fig. 2.

Including the shear viscosity has three main effects on
which we further elaborate below: (i) the spatial vari-
ation of the velocity field is considerably reduced (see
Fig. 2), (ii) the flow acquires a non-curl-free dependence
on the radial position \( r \), i.e., a non-zero vorticity \( \Omega \)
\( \omega = \nabla \times \mathbf{v} \) appears near the inner and outer edges
(see Fig. 4), and (iii) the dc potential drop \( \Delta U \) becomes
strongly frequency-dependent (see Fig. 3). Indeed, \( \Delta U \)
decreases by a factor 20 as the frequency decreases from

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**Fig. 2.** Radial profile of the azimuthal component of the
drift velocity—see Eq. (4) at \( t = 0 \), obtained by solving Eq. (15)
with boundary conditions (12) (main panel) and (13) (inset). The results are obtained with
\( r_{\text{out}}/r_{\text{in}} = 10.0 \), and for several values of the dimensionless parameter \( \xi \)
defined in Eq. (16): \( \xi = 0.2 \) (dashed line), 0.5
(dotted line), and 1.0 (dash-dotted line). The value of the
viscosity in the different solutions increases as shown by the
arrow. The solid line corresponds to the analytical result (4),
which holds at \( \xi = 0 \).
500 MHz (where the effect the viscosity is practically negligible) to 1 MHz. The profile of \( \Delta U \) depends only on the ratio of the viscosity to the average mass density \( \bar{m} \), i.e. the kinematic viscosity \[ \nu = \frac{\eta}{\bar{m}}. \]

By measuring the frequency dependence of \( \Delta U \), subtracting the frequency-independent background (5), and fitting the theoretical curve to the experimental result one can determine \( \nu \). In the remainder of this Letter we supply the main steps of the calculation of \( \Delta U \).

**Hydrodynamic equations and their solution.**—In the hydrodynamic regime, the response of the 2DQEL is governed by the Navier-Stokes equation \[ 1, 2 \]

\[ \rho(r, t) \{ \partial_t \mathbf{v}(r, t) + \mathbf{v}(r, t) \cdot \nabla \mathbf{v}(r, t) \} = - \varepsilon \mathbf{E}(r, t) n(r, t) - \nabla P(r, t) + \eta \nabla^2 \mathbf{v}(r, t) \]  

\[ + \zeta \nabla \cdot \mathbf{v}(r, t) \]  

combined with the continuity equation

\[ \partial_t n(r, t) + \nabla \cdot [n(r, t) \mathbf{v}(r, t)] = 0. \]  

Here, \( \rho(r, t) = mn(r, t) \) is the mass density, \( P(r, t) \) is the pressure, and \( \zeta \) is the bulk viscosity, which we can neglect here since it vanishes at long wavelengths \[ 1, 13 \]. In Eq. (8) we have also included the contribution of the electric field \( \mathbf{E}(r, t) \). The azimuthal symmetry of the system implies that all quantities depend on the radial coordinate \( r \) only and that the derivatives with respect to the azimuthal angle \( \theta \) vanish. The radial component \( E_r(r, t) \) of the electric field is given by \( E_r(r, t) = -\hat{r} \partial_r U(r, t)/\varepsilon \)

where the electric potential energy \( U(r, t) \) is obtained by solving the Poisson equation in the CD geometry with a constant boundary condition at the gate position \( z = -d \). If the typical wavelength of density fluctuations is larger than \( d \), it is easy to see that

\[ U(r, t) \approx -\varepsilon^2 n(r, t)/\varepsilon, \]  

which immediately leads to Eq. (2). Finally, the pressure gradient in Eq. (8) can be neglected when \( \partial P(n)/\partial n \ll \varepsilon^2 n/\varepsilon \), i.e. when \( d \gg a_B^2/\varepsilon^2 \). This inequality is always well satisfied since \( a_B^2 \equiv e^2/(\mu m^2) \) is the material Bohr radius, which is \( \sim 10 \) nm for GaAs.

The Navier-Stokes and continuity equations (8)-(9) must be complemented by suitable boundary conditions (BCs) expressed in terms of the flux density tensor \[ 1, 2 \]

\[ \Pi_{r,k}(r, t) = P(r, t) \delta_{r,k} + \rho \mathbf{v}(r, t) \mathbf{v}(r, t) - \sigma'_{r,k}(r, t), \]  

where \( \sigma'_{r,k}(r, t) \) is the viscous stress tensor. We require the radial diffusion of azimuthal momentum, which is proportional to the viscosity \( \eta \), to vanish at the outer and inner rims of the CD, i.e.

\[ \sigma'_{r,\theta}(r, t) \Big|_{r=r_{in}} = \sigma'_{r,\theta}(r, t) \Big|_{r=r_{out}} = 0. \]  

The off-diagonal component of the viscous stress tensor reads \( \sigma'_{r,\theta} = \eta (\partial_r v \delta_{r,\theta} + \partial_\theta v_r/r - v_\theta/r) \) \[ 1, 2 \]. Because of circular symmetry, the BCs (11) reduce to

\[ \partial_r v_\theta(r, t) \mid_{r=r_i} = v_\theta(r, t) \mid_{r=r_i} \]  

where \( r_i = r_{in}, r_{out} \). Moreover, for the setup in Fig. 1, two further BCs should be imposed. First, the radial component of the current \( j(r, t) = n(r, t) \mathbf{v}(r, t) \) must vanish at the outer rim, where the CD is isolated. Second, the electric potential at the inner rim is fixed, hence the electron density \( n(r_{in}, t) \) at the inner rim of the CD must coincide with the average value \( \bar{n} = eV_G/C \) fixed by the gate voltage \( V_G \).

We notice that the BCs (12) differ from the standard “no-slip” BCs

\[ v_\theta(r, t) \mid_{r=r_{in}} = v_\theta(r, t) \mid_{r=r_{out}} = 0, \]  

which are commonly employed \[ 1, 2 \] to describe fluid adhesion to the walls of a container. While the use of the BCs in Eq. (13) is not immediately justified in our case, we have checked that the results in Fig. 3 do not change qualitatively if the BCs in Eq. (13) are used instead of those in Eq. (12). The agreement between the results obtained with two different sets of BCs gives us confidence in the robustness of the effect illustrated in Fig. 3.

We solve Eqs. (8)-(9) by expanding the hydrodynamic variables in powers of the amplitude \( \Phi_0 \) of the magnetic flux \[ 19 \]:
where \( \mathbf{u}^{(k)}(r,t), n^{(k)}(r,t) \sim (\Phi_0)^k \cos(k \Omega t) \). Since the hydrodynamic equations of motion are nonlinear, in the expansion (14) we include constant contributions \( \delta \mathbf{v}(r), \delta n(r) \sim O(\Phi_0) \) arising from the self-mixing of the signal at frequency \( \Omega \) [19]. At first order, we find the following differential equation for the radial profile of the azimuthal component of the velocity:

\[
- i \Omega \omega_\theta^{(1)}(r) \bigg[ \frac{\partial^2}{\partial r^2} \psi_\theta^{(1)}(r) + \frac{1}{r} \frac{\partial}{\partial r} \psi_\theta^{(1)}(r) - \frac{1}{r^2} \psi_\theta^{(1)}(r) \bigg] = - i \Omega \frac{\Phi_0}{mc} \frac{e \psi_\theta^{(1)}(r)}{2 \pi r} .
\]

(15)

In the general \( \nu \neq 0 \) case, Eq. (15) must be solved numerically [20]. It is convenient to introduce dimensionless variables by rescaling \( r \) with \( r_{in} \) and \( \psi_\theta^{(1)}(r) \) with \( \psi_{in} \), which has been defined in Eq. (4). Then, Eq. (15) depends only on the dimensionless parameter

\[
\xi \equiv \left( \frac{L_\eta}{r_{in}} \right)^2 ,
\]

(16)

where \( L_\eta = \sqrt{\nu / \Omega} \) is the vorticity penetration depth [1, 2] during a time \( 1/\Omega \). The solution of Eq. (15) is shown in Fig. 2 for several values of \( \xi \) and both sets of BCs. We note that with increasing viscosity the amplitude of the velocity flow diminishes. Since according to Eq. (3) \( \Delta U \) depends on the integral of the square of the velocity profile, increasing the viscosity suppresses the dc response \( \Delta U \), thereby explaining the results in Fig. 3.

In the limit \( \xi \ll 1 \) (low viscosity or high frequency) the solution of Eq. (15) is given by the curl-free profile \( \psi_\theta^{(1)}(r) \propto r \) solves the problem, satisfying the BCs.

To obtain the dc response \( \Delta U \) in Eq. (3) we expand Eq. (9) and the radial component of Eq. (8) to second order and we average the resulting expressions over a period \( T = 2 \pi / \Omega \) of the oscillating flux \( \Phi(t) \). The solution of the time-averaged equations yields \( \delta \mathbf{v}_r(r) \equiv 0 \) and

\[
\delta n(r) = \frac{mC_e}{e^2} \int_{r_{in}}^{r_{out}} dr' \frac{1}{r'} \left[ \psi_\theta^{(1)}(r,t) \right]^2 ,
\]

(17)

where \( \langle g(t) \rangle_t \equiv T^{-1} \int_0^T dt' g(t') \) denotes the time-average of a function \( g(t) \) over one period of the oscillating magnetic flux. Finally, Eq. (3) can be easily obtained by setting \( r = r_{out} \) in Eq. (17) and making use of \( \left[ \psi_\theta^{(1)}(r,t) \right]^2_t = |\psi_\theta^{(1)}(r)|^2 / 2 \) and of the local capacitance formula, Eq. (10).

**Vorticity and dissipation.**—Further insights on the physical properties of the solution shown in Fig. 2 can be obtained by looking at the radial profile of the vorticity \( \omega_z^{(1)}(r,t) \) (in units of \( \psi_{in}/r_{in} \)) at time \( t = 0 \). The inset shows the position \( L_{max} \) of the maximum of \( \psi_\theta^{(1)}(r,t) = 0 \), as displayed in Fig. 2, for several values of the vorticity penetration depth \( L_\eta \) (in units of \( r_{in} \)). The dashed line corresponds to the expected relation \( L_{max} = L_\eta \).

FIG. 4. Same as in Fig. 2 but for the radial profile of the vorticity \( \omega_z^{(1)}(r,t) \) (in units of \( \psi_{in}/r_{in} \)) at time \( t = 0 \). The inset shows the position \( L_{max} \) of the maximum of \( \psi_\theta^{(1)}(r,t) = 0 \), as displayed in Fig. 2, for several values of the vorticity penetration depth \( L_\eta \) (in units of \( r_{in} \)). The dashed line corresponds to the expected relation \( L_{max} = L_\eta \).

Before concluding, we would like to quantify the power \( W \) dissipated in the fluid by viscosity [1, 2] in a period of the oscillating flux. This is given by the following expression:

\[
W = - \pi \eta \int_{r_{in}}^{r_{out}} dr \frac{1}{r} \left[ \psi_\theta^{(1)}(r) - r \partial_r \psi_\theta^{(1)}(r) \right]^2 .
\]

(18)

To leading order in \( \eta \) for \( \eta \to 0 \) we obtain \( W \to -8 \pi (\eta/m) \Delta U |_{\eta=0} \). As far as order-of-magnitudes are concerned, we find that \( W \) is of the order of microwatts for \( \nu \sim 10 \text{ cm}^2/\text{s} \) and \( \bar{n} \sim 10^{11} \text{ cm}^{-2} \). Away from the perturbative regime, we have verified that \( \eta \sim -a W/(8\pi \Delta U) \) where \( a \sim 0.5 \) is a numerical coefficient. This relation is valid in the range \( 0.25 \lesssim \xi \lesssim 1.0 \). Thus, an independent measure of \( W \) and \( \Delta U \) at a single frequency \( \Omega \) of the oscillating magnetic flux yields the value of \( \eta \). This provides a possible alternative to the method described after Eq. (7), which requires a measurement of \( \Delta U \) as a function of frequency.

In summary, we have demonstrated that the hydrodynamic shear viscosity of a two-dimensional quantum electron liquid can be obtained by studying the response of the system to an oscillating magnetic flux in a Corbino
disk geometry. We truly hope that this work will stimulate further studies of viscometers for two-dimensional quantum electron liquids and related experimental activities on the shear viscosity of these systems, which may pave the way for the discovery of solid-state nearly perfect fluids [5].

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* andrea.tomadin@sns.it

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[21] This can be seen as follows: ω(1)(r, t) obeys the 2D equation of motion [2] ∂ω(1)(r, t) / ∂t = ν∇rω(1)(r, t). Therefore, ν plays the role of diffusion constant for ω(1)(r, t). This happens because the 2D flow in our case is uniform and incompressible to O(Φ0).