ONE EXAMPLE ABOUT THE RELATIONSHIP BETWEEN THE CD INEQUALITY AND CDE’ INEQUALITY

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Abstract. In this paper, we will give an easy example to satisfy that we can not conclude CDE’ Inequality just from the CD Inequality.

1. INTRODUCTION

The curvature-dimension inequality (CD-inequality) was firstly introduced by Bakry-Emery as a substitute of the lower Ricci curvature bound of the underlying space. It was first studies on finite graphs by Lin-Yau[9][11].

Then some special cases CD(0,∞) are being studid by Liu-Peyerimhoff. They prove an optimal eigenvalue ratio estimate for finite weighted graphs satisfying the CD inequality. Also, they show taking Cartesian products to be an efficient way for construcing new weighted graphs satisfying CD(0,∞).[6][7].

In 2015, Paul-Lin-Liu-Yau prove Li-Yau type estimates for bounded and positives solutions of the heat equation on graphs, under the assumption of the curvature-dimension inequality CDE'(0,n), which can be consider as a notion of curvature for graphs. So the relation between CD and CDE’ is becoming more and more important for the study of the ricci estimate on finite graphs.[2]

In 2014, 2015, Munch introduced a new version of a curvature-dimension inequality for non-negative curvature. He used this inequality to prove a logarithmic Li-Yau inequality on finite graphs. The new calculus and the new curvature-dimension inequality coinciden with the common ones. In the case of graphs, they coincide in a limit. In that case, the new curvature-dimension inequality gives a more general concept of curvature on graphs. Then he showed the connection between the CDE’ and the CDψ inequality. Also, he showed that the CDE’ inequality implies the CD inequality.[3][4]

So can CD inequality implies CDE’? It is a problem. In this paper, the author gives a special case to show that the hypothesis is not right in some special cases.

The paper is organized into three parts:

Chapter 1 is the introduction of the graph, the Laplacians and CD, CDE’ inequalities on it.

Chapter 2 introduces some basic conclusions and lemmas in order to get the main result. These conclusions include some definitions such as local finite graph, weighted graph and the calculation of the Γ operator.
Chapter 3 is the main conclusion of this thesis.

2. GRAPHS, LAPLACIANS CD INEQUALITIES AND CDE’ INEQUALITIES

Given a graph $G = (V, E)$, for an $x \in V$, if there exists another $y \in V$ that satisfies $(x, y) \in E$, we call them are neighbors, and written as $x \sim y$. If there exists an $x \in V$ satisfying $(x, x) \in E$, we call it a self-loop. For a graph $G = (V, E)$. The neighborhood and the degree of a vertex $x \in V$ are defined, respectively, as $N_x = \{y \in V : xy \in E\}$ and $d_x = |N_x|$. For notational simplicity we work with $\mu_x = 1/d_x$.

Now we will introduce some basic definitions and theorems before we get the main results.

**Definition 2.1.** (difference in valuations) For a function $f \in R^V$ and two vertices $x, y \in V$ denote by $f(x, y) = f(y) - f(x)$ the difference in valuations.

**Definition 2.2.** (locally finite graph) We call a graph $G$ is a locally finite graph if for any $x \in V$, it satisfies $\#\{y \in V | y \sim x\} < \infty$. Moreover, it is called connected if there exists a sequence $\{x_i\}_{i=0}^n$ satisfying: $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$.

**Definition 2.3.** (Laplacians on locally finite graphs) On a locally finite graph $G = (V, E, \mu)$ the Laplacian has a form as follows:

$$\triangle f(x) = \mu_x \sum_{y \in N_x} (f(y) - f(x)), \quad \forall f \in C_0(V).$$

**Definition 2.4.** (gradient operator $\Gamma$) The operator $\Gamma$ is defined as follows:

$$\Gamma(f, g)(x) = \frac{1}{2}((\triangle f)g - f\triangle g - g\triangle f)(x).$$

Always we write $\Gamma(f, f)$ as $\Gamma(f)$.

**Definition 2.5.** (gradient operator $\Gamma_i$) The operator $\Gamma_i$ is defined as follows:

$$\Gamma_0(f, g) = fg$$

$$\Gamma_{i+1}(f, g) = \frac{1}{2}(\triangle(\Gamma_i(f, g)) - \Gamma_i(f, \triangle g) - \Gamma_i(\triangle f, g))$$

Also we have $\Gamma_2(f) = \Gamma_2(f, f) = \frac{1}{2}\triangle \Gamma(f) - \Gamma(f, \triangle f)$.

**Definition 2.6.** ($CD(K, n)$ condition) We call a graph satisfies $CD(K, n)$ condition if for any $x \in V$, we have

$$\Gamma_2(f)(x) \geq \frac{1}{n}(\triangle f)^2(x) + K\Gamma(f)(x). \quad K \in \mathbb{R}.$$
Definition 2.7. ($CDE(K,n)$ condition) Let $f : V \to \mathbb{R}^+$ satisfy $f(x) > 0$, $\Delta f(x) < 0$. We call a graph satisfies $CDE(x,K,n)$ condition if for any $x \in V$, we have

$$\Gamma_2(f)(x) - \Gamma \left( f, \frac{\Gamma(f)}{f} \right)(x) \geq \frac{1}{n}(\Delta f)(x)^2 + K \Gamma(f)(x). \quad K \in \mathbb{R}.$$ 

Definition 2.8. ($CDE'(K,n)$ condition) We say that a graph $G$ satisfies the exponential curvature dimension inequality $CDE(K,n)$ if for any positive function $f : V \to \mathbb{R}^+$ such that $\Delta f(x) < 0$, we have

$$\Gamma_2(f)(x) - \Gamma(f, \frac{\Gamma(f)}{f}(x)) \geq \frac{1}{n}f(x)^2(\Delta log f)(x)^2 + k \Gamma(f)(x).$$

Lemma 2.9.

$$\Gamma(f)(x) = \frac{1}{2} \mu_x \sum_{y \in N_x} f(x,y)^2.$$ 

Here we define $N_x = \{y \in V : xy \in E\}$ and $d_x = |N_x|$. For notational simplicity we work with $\mu_x = \frac{1}{d_x}$.

Proof

We have

$$\Gamma(f)(x) = \frac{1}{2} \Delta (f^2)(x) - f(x)(\Delta f)(x)$$

$$= \frac{1}{2} \mu_x \sum_{y \in N_x} (f^2)(x,y) - f(x) \mu_x \sum_{y \in N_x} f(x,y)$$

$$= \frac{1}{2} \mu_x \sum_{y \in N_x} (f(x,y)(f(y) + f(x)) - 2f(x,y)f(x))$$

$$= \frac{1}{2} \mu_x \sum_{y \in N_x} f(x,y)^2.$$ 

Lemma 2.10.

$$\Gamma_2(f)(x) = \frac{1}{2}(\Delta f)^2(x) + \mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} (f(y,z)^2 - \frac{1}{2}f(x,z)^2)).$$

Proof

We have

$$\Delta(\Gamma(f))(x) = \mu_x \sum_{y \in N_x} \Gamma(f)(x,y) = \mu_x \sum_{y \in N_x} \frac{1}{2} \mu_y \sum_{z \in N_y} (f(y,z)^2 - f(x,y)^2).$$
\begin{align*}
\Gamma(f, \triangle f)(x) &= \frac{1}{2}(\Delta (f \cdot \triangle f)(x) - f(x) \cdot (\triangle^2 f)(x) - (\triangle f)^2(x)) \\
&= \frac{1}{2}(\triangle^2 f)(x) + \frac{1}{2} \mu_x \sum_{y \in N_x} ((f \triangle f)(x, y) - f(x)(\triangle f)(x, y)) \\
&= \frac{1}{2}(\triangle^2 f)(x) + \frac{1}{2} \mu_x \sum_{y \in N_x} f(x, y)(\triangle f)(y) \\
&= \frac{1}{2}(\triangle^2 f)(x) + \frac{1}{2} \mu_x \sum_{y \in N_x} f(x, y)\mu_y \sum_{z \in N_y} f(y, z)
\end{align*}

thus
\begin{align*}
\Gamma_2(f)(x) &= \frac{1}{2} \Delta (\Gamma(f))(x) - \Gamma(f, \triangle f)(x) \\
&= \frac{1}{2}(\triangle^2 f)(x) + \frac{1}{2} \mu_x \sum_{y \in N_x} \mu_y \sum_{z \in N_y} \left(\frac{1}{2} f(y, z)^2 - \frac{1}{2} f(x, y)^2 - f(x, y) f(y, z)\right) \\
&= \frac{1}{2}(\triangle f)^2 + \frac{1}{2} \mu_y \sum_{y \in N_x} \mu_y \sum_{z \in N_y} \left(f(y, z)^2 - \frac{1}{2} f(x, z)^2\right)
\end{align*}

Lemma 2.11. If $\Delta f(x) < 0$ in $x \in V, CDE'(K, n)$ implies $CDE(K, n)$.

Proof. Let $f : V \to R^+$ be a positive function for which $\Delta f(x) < 0$. Since $\log s \leq s - 1$ for all positive $s$, we can write
\begin{equation*}
\Delta \log f(x) = \sum_{y \sim x} (\log f(y) - \log f(x)) \leq \sum_{y \sim x} \frac{f(y) - f(x)}{f(x)} = \frac{\Delta f(x)}{f(x)} < 0.
\end{equation*}

Hence squaring everything reverses the above inequality and we get
\begin{equation*}
(\Delta f(x))^2 \leq f(x)^2 (\Delta \log f(x))^2,
\end{equation*}
and thus $CDE(K, n)$ is satisfied
\begin{equation*}
\Gamma_2(f)(x) \geq \frac{1}{n} f(x)^2 (\Delta \log f(x))^2 + k \Gamma(f)(x) > \frac{1}{n} (\Delta f(x))^2 + k \Gamma(f)(x).
\end{equation*}

Lemma 2.12. The $CDE'$ inequality implies $CD$ inequality. It was the work in [3]

3. BASIC CONCLUSION

Remark 3.1. In this section, we just concern the easiest graph: $x$ is the initial point and its neighborhood is $y$, also $y$ has another neighborhood $z$. So the graph consists three points which created from $x$, $x$ and $z$ are not connected. We give the special graph a name: $EG$. Also, we concern the special case in $CD$ and $CDE'$ inequality: $CD(0, n)$ and $CDE'(0, n)$.
Lemma 3.2. If the EG satisfies $CD(0, m)$, then we have $m \geq 2$.

Proof For the simplicity, we rewrite $f(x) = x, f(y) = y$ and $f(z) = z$. Then according to the lemma before, we can have the following: $\Delta f(x) = y - x, \Gamma(f)(x) = \frac{1}{2}(y - x)^2$. Also, we need to calculate the $\Gamma_2(f)$,

\[
\Gamma_2(f)(x) = \frac{1}{2}(y - x)^2 + \frac{1}{4}((z - y)^2 - \frac{1}{2}(z - x)^2 + (x - y)^2)
\]

\[
= \frac{1}{2}(y - x)^2 + \frac{1}{4}(z, y)^2 - \frac{1}{8}(z - x)^2 + \frac{1}{4}(x - y)^2
\]

\[
= \frac{1}{8}(z^2 + z(2x - 4y) + 8y^2 + 5x^2 - 12xy)
\]

Absolutely, we will use the knowledge of quadratic function. The minimum of $z^2 + z(2x - 4y)$ can be obtained when $z = 2y - x$. Then we take $z = 2y - x$ in the equality.

\[
8y^2 + 5x^2 - 12xy + (2y - x)^2 + (2y - x)(2x - 4y)
\]

\[
= 8y^2 + 5x^2 - 12xy + 4y^2 + x^2 - 4xy + 4xy - 8y^2 - 2x^2 + 4xy
\]

\[
= 4x^2 + 4y^2 - 8xy
\]

So we have the estimate of $\Gamma_2(f)$

\[
\Gamma_2(f) \geq \frac{1}{2}(x^2 + y^2 - 2xy) = \frac{1}{2}(x - y)^2 \geq \frac{1}{m}(y - x)^2
\]

So we have $m \geq 2$.

Theorem 3.3. If the EG satisfies $CD(0, m)$, we can take the special value in the point $y$, then it may not be satisfied in $CDE'(0, m)$.

Proof Without loss of generality, we assume $x = 1$. According the theorem before, we need to find some special $y$ that make sure the EG does not satisfy $CDE'(0, 2)$.

As before, the value of $\Gamma_2(f)$ is given

\[
\Gamma_2(f) = \frac{3}{4}(x - y)^2 + \frac{1}{4}(z, y)^2 - \frac{1}{8}(z, x)^2
\]

So the next work is to calculate the value of $\Gamma(f, \frac{\Gamma(f)}{f})$

\[
\Gamma(f, \frac{\Gamma(f)}{f}) = \frac{1}{2}\Delta(\Gamma(f)) - \frac{1}{2}f(x)\Delta(\frac{\Gamma(f)}{f(x)}) - \frac{1}{2}\Delta(f)\frac{\Gamma(f)}{f(x)}
\]

\[
= \frac{1}{2}\Delta(\Gamma(f)) - \frac{1}{2}\Delta(\frac{\Gamma(f)}{f}) - \frac{1}{2}\Delta f \Gamma(f)
\]

\[
= I_1 - I_2 - I_3.
\]
Then we get the value of $I_1$.

$$I_1 = \frac{1}{2}(\Gamma(f)(y) - \Gamma(f)(x))$$

$$= \frac{1}{2} \left[ \frac{1}{4}([y - x]^2 + (z, y)^2] - \frac{1}{2}(y - x)^2 \right]$$

$$= \frac{1}{2} \left[ \frac{1}{4}(y - x)^2 + \frac{1}{4}(y - z)^2 - \frac{1}{2}(y - x)^2 \right]$$

$$= \frac{1}{8}(y - z)^2 - \frac{1}{8}(y - x)^2.$$

Also the value of $I_2$.

$$I_2 = \frac{1}{2} \frac{\Gamma(f)(y) - \Gamma(f)(x)}{y} - \frac{1}{2} \frac{\Gamma(f)(y) - \Gamma(f)(x)}{x}$$

$$= \frac{1}{2} \frac{1/4([y - x]^2 + (z - y)^2]}{y} - \frac{1}{4}(y - x)^2$$

Then the value of $I_3$.

$$I_3 = \frac{1}{2} \Delta(f) \Gamma(f)(x)$$

$$= \frac{1}{2}(y - x)^2 \frac{1}{2}(y - x)^2$$

$$= \frac{1}{4}(y - x)^3$$

We get the following inequality according the definition of $CDE'(m, 0)$

$$\frac{3}{4}(y - x)^2 + \frac{1}{4}(y - z)^2 - \frac{1}{8}(z - x)^2 - \frac{1}{8}(y - z)^2 + \frac{1}{8}(y - x)^2 + \frac{1}{8}(y - z)^2 - \frac{1}{4}(y - z)^2 + \frac{1}{4}(y - x)^3 \geq \frac{1}{2}(log y)^2$$

Firstly, we deal with the polynomial with $z$ like the situation before.

$$\frac{1}{8}(y - z)^2 - \frac{1}{8}(z - x)^2 + \frac{1}{8y}(y - z)^2$$

$$= \frac{1}{8}(y^2 + z^2 - 2yz - z^2 - 1 + 2z) + \frac{1}{8y}(y^2 + z^2 - 2yz)$$

$$= \frac{1}{8}y^2 - \frac{1}{4}yz + \frac{1}{4}z - \frac{1}{8}y + \frac{1}{8}z^2 - \frac{1}{4}z$$

$$= \frac{1}{8y}z^2 - \frac{1}{4}yz + \frac{1}{8}y^2 + \frac{1}{8}y - \frac{1}{8}$$

Here, we take $z = \frac{1}{2y} = y^2$. Then we get

$$\frac{1}{8y}y^4 - \frac{1}{4}y^3 + \frac{1}{8}y^2 + \frac{1}{8}y - \frac{1}{8} = -\frac{1}{8}y^3 + \frac{1}{8}y^2 + \frac{1}{8}y - \frac{1}{8}.$$
Then the inequality becomes

\[
\frac{1}{4}(y - 1)^3 + \frac{5}{8}(y - 1)^2 + \frac{1}{8y}(y - 1)^2 - \frac{1}{8}y^3 + \frac{1}{8}y^2 + \frac{1}{8y} - \frac{1}{8}y - \frac{3}{8} + \frac{1}{8y} = \frac{1}{4}(y^3 - 3y^2 + 3y - 1) + \frac{5}{8}y^2 + \frac{5}{8} - \frac{10}{8}y - \frac{1}{8}y^3 + \frac{1}{8}y^2 + \frac{1}{4}y - \frac{3}{8} + \frac{1}{8y}
\]

\[
= \frac{1}{8}y^3 + y(\frac{3}{4} - \frac{10}{8} + \frac{1}{8} + \frac{1}{8}) + \frac{1}{8y} = \frac{1}{8}y^3 - \frac{1}{4}y + \frac{1}{8y}
\]

So the inequality becomes:

\[
y^3 - 2y + \frac{1}{y} \geq 4(\log y)^2
\]

Here we distinguish \(y\) into two parts: \(y > 1\) and \(0 < y < 1\)

I. First, we concern the situation \(y > 1\), because \(e^x \geq x + 1\), so we have \(\log y < y - 1\)

Then we need to prove

\[
y^3 - 2y + \frac{1}{y} \geq 4(y - 1)^2
\]

It equals

\[
y^3 - 4y^2 + 6y - 4 + \frac{1}{y} \geq 0
\]

we set the function \(h(y) = y^3 - 4y^2 + 6y - 4 + \frac{1}{y}\).

The first derivate of \(h(y)\) is

\[
h'(y) = 3y^2 - 8y + 6 - \frac{1}{y^2}
\]

The second derivate of \(h(y)\) is

\[
h''(y) = 6y - 8 + 2\frac{1}{y^3}
\]

Then we use the cauchy inequality to get that

\[
6y - 8 + \frac{1}{y^3} = 2y + 2y + 2y + 2\frac{1}{y^3} - 8 = 2(y + y + y + \frac{1}{y^3}) - 8 \geq 2 + 4(yyy \frac{1}{y^3})^{(1/4)} - 8 = 0
\]

So the function \(h'(y)\) is increasing when \(y > 1\).

\[
h'(y) > h'(1) = 3 - 8 + 6 - 1 = 0
\]

So the function \(h(y)\) is increasing too.

\[
h(y) \geq h(1) = 1 - 4 + 6 - 4 + 1 = 0
\]

When \(y > 1\), we can get the \(CDE'(0,m)\)
II: Second, we concern the situation $y < 1$ as our discussion. We need to prove the following

$$y^3 - 2y + \frac{1}{y} \geq 4(\log f)^2$$

Actually, it equals that

$$y^3 - 2y + \frac{1}{y} \geq 4(\log \frac{1}{y})^2$$

Just as discussed before, then we have to prove

$$y^3 - 2y + \frac{1}{y} \geq 4(\frac{1}{y} - 1)^2$$

Here we set $Q(y) = y^3 - 2y + \frac{1}{y} - 4(\frac{1}{y} - 1)^2 = y^3 - 2y + \frac{9}{y} - \frac{4}{y^2} - 4$. The next step is to analysis $Q(y)$

$$Q(y) = y^3 - 2y + \frac{9}{y} - \frac{4}{y^2} - 4$$

$$= y^3 - y^2 + y^2 - y - y + 1 - 5 + \frac{5}{y} + \frac{4}{y} - \frac{4}{y^2}$$

$$= (y - 1)(y^2 + y - 1 - \frac{5}{y} + \frac{4}{y^2})$$

$$= (y - 1)(y^2 - y + 2y - 2 + 1 - \frac{1}{y} - \frac{4}{y} + \frac{4}{y^2})$$

$$= (y - 1)^2(y + 2 + \frac{1}{y} - \frac{4}{y^2})$$

$$= (y - 1)^2(y - 1 + 3 - \frac{3}{y} + \frac{4}{y} - \frac{4}{y^2})$$

$$= (y - 1)^3(1 + \frac{3}{y} + \frac{4}{y^2})$$

Of course $1 + \frac{3}{y} + \frac{4}{y^2} > 0$, but $y - 1 < 0$, so $(y - 1)^3(1 + \frac{3}{y} + \frac{4}{y^2}) < 0$, which is contrary to what we need. Clearly, we can set $y = 0.1$, then the left is less than the right. So we get the conclusion that we can not conclude $CDE'$ just from $CD$ situation.

References

[1] Chao Gong and Yong Lin, Equivalent Properties for CD Inequalities On Graphs With Unbounded Laplacians, preprint.

[2] Paul Horn, Yong Lin, Shuang Liu, Shing-Tung Yau, Volume doubling, Poincare inequality and Gaussian heat kernel estimate for nonnegative curvature graphs, Mathematics, 2015

[3] Florentin Munch, Remarks on curvature dimension conditions on graphs, arXiv:1501.05839v1

[4] Florentin Munch, Li-Yau inequality on finite graphs via non-linear curvature dimension conditions, arXiv:1412.3840v1

[5] Yong Lin and Shuang Liu, Equivalent properties of CD inequality on graphs, preprint.
[6] David Cushing, Shiping Liu, Norbert Peyerimhoff, Bakry-Emery curvature functions of graphs, arXiv:1606.01496
[7] Shiping Liu, Florentin Munch, Norbert Peyerimhoff, Bakry-Emery curvature and diameter bounds on graphs, arXiv:1608.07778
[8] F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi and S.-T. Yau, Li-Yau inequality on graphs, Journal of Differential Geometry, 99, 359-405, 2015.
[9] Y. Lin and S. T. Yau, Ricci curvature and eigenvalue estimate on locally finite graphs, Math. Res. Lett., 17(2):343-356, 2010.
[10] B. Hua, Y. Lin, Stochastic completeness for graphs with curvature dimension conditions, preprint, 2014
[11] Y. Lin, L. Lu, S. T. Yau, Ricci curvature of graphs, Tohoku Math. J. 63, 605-627, 2011

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