Quantum phases of a Feshbach-resonant atomic Bose gas in one dimension

Yu-Wen Lee
Physics Department, Tunghai University, Taichung, Taiwan, R.O.C.

Yu-Li Lee
Physics Department, National Changhua University of Education, Changhua, Taiwan, R.O.C.

We study an atomic Bose gas with an s-wave Feshbach resonance in a one-dimensional optical lattice. It can be described by the Hamiltonian

\[
H = \sum_i \left( t_{m} b_{i+1}^\dagger b_i + t_{a} a_{i+1}^\dagger a_i + H.c. \right) - \mu \sum_i [n_{a,i} + (2\mu - \nu)n_{b,i}] + \frac{U_a}{2} n_{a,i} (n_{a,i} - 1) + \frac{U_m}{2} n_{b,i} (n_{b,i} - 1) + \sum_i U_b (b_i^\dagger a_i + H.c.) + U_{ab} \sum_i n_{a,i} n_{b,i} \tag{1}
\]

Here \(n_{a(b),i}\) is the atom (molecule) density operator and \(\mu\) is the chemical potential. This system can also be considered a kind of binary mixtures with two types of bosons. However, a main distinction between the present case and the binary mixture studied previously is that a Josephson coupling between atoms and molecules, the \(U\) term in Eq. (1), is absent in the latter. As we shall see, this will change the global phase diagram dramatically.

Our main results can be summarized in Fig. 1. For on-site interactions only, i.e. the Hamiltonian given by Eq. (1), the Josephson coupling dominates the low-energy physics in most of the parameter region, and the resulting phase exhibits strong superfluid fluctuations in both atomic and molecular sectors. This is because the phase fluctuations of both sectors are locked by the Josephson coupling. This phase is, in fact, a one-dimensional (1d) analog of the ASF phase in three dimensions (3d). (But we do not have real condensates here.) Therefore, it will be referred to as 1d ASF. The 1d ASF will collapse for sufficiently strong attractions between atoms and molecules \((U_{ab} < 0)\). When the density difference between atoms and molecules is close to some rational number, depending on the values of parameters as...
shown in Fig. 1(b), two additional phases may emerge by including nearest-neighbor repulsions between atoms (molecules): the two component Luttinger liquid (2-LL) where both the atomic and molecular sectors are gapless, and the inter-channel charge density wave (I-CDW) where the relative density fluctuations between atoms and molecules are frozen at low energy.\(^7\)

The continuum theory We now outline below the derivation of our results. For simplicity, we shall consider the case where both the densities of atoms and molecules, \(\rho_a = (n_{a,1})/a_0\) and \(\rho_b = (n_{b,1})/a_0\), respectively, are incommensurate with the lattice. (Here \(a_0\) is the lattice spacing. \(\rho_a\) and \(\rho_b\) satisfy the constraint: \(\rho_a + 2\rho_b = \rho_0\) where \(\rho_0\) is the density of bare atoms.) That is, we do not consider the possibility of the SF-Mott insulator transition. Then, in terms of the “bosonization” formula\(^3\):

\[
a_i \sim e^{-i\sqrt{\pi} \theta_i(x)} \int_{-\infty}^\infty e^{i\theta_i(x')} e^{i\sqrt{\pi} \phi_i(x')} dx' + \text{h.c.}
\]

and

\[
b_i \sim e^{-i\sqrt{\pi} \phi_i(x)} \int_{-\infty}^\infty e^{i\phi_i(x')} e^{i\sqrt{\pi} \theta_i(x')} dx' + \text{h.c.}
\]

the low-energy physics of \(H\) [Eq. (1)] can be described by the following effective Hamiltonian:

\[
H_{\text{eff}} = \frac{v_a}{2} \int dx \left[ K_a (\partial_x \theta_a)^2 + \frac{1}{K_a} (\partial_x \phi_a)^2 \right] + \frac{v_m}{2} \int dx \left[ K_m (\partial_x \theta_m)^2 + \frac{1}{K_m} (\partial_x \phi_m)^2 \right] + g_1 \int dx \cos [\sqrt{\pi}(\theta_m - 2\theta_a)] + g_2 \int dx \partial_x \phi_a \partial_x \phi_m + g_3 \int dx \cos [\sqrt{4\pi}(\phi_a - \phi_m) + 2\pi \delta x] ,
\]

where \(\delta = \rho_a - \rho_b\), \(v_{a/m}\) and \(K_{a/m}\) are sound velocities and Luttinger liquid (LL) parameters, respectively. In Eq. (2), only those terms which may become the most relevant in the renormalization group (RG) sense are retained. The values of \(v_{a/m}, K_{a/m}, g_1, g_2, g_3\), depend on the short-distance physics. In general, they must be extracted from numerics or experiments. \(K_{a/m} \gg 1\) in the weak coupling regime. On the other hand, \(K_{a/m} = 1\) in the Tonks limit, i.e. \(U_{a(m)}/t_{a(m)} \to +\infty\). Therefore, for on-site interactions only, \(1 \leq K_{a/m} < +\infty\). The value of \(K_a (K_m)\) can be further decreased by including nearest-neighbor repulsions between atoms (molecules).

The \(g_3\) term is a Umklapp process. It can be neglected when \(\delta\) is not close to zero (incommensurate filling). On the other hand, when \(\delta\) is close to zero (commensurate filling), the \(g_3\) term can affect low-energy physics and one may no longer neglect it in Eq. (2). When \(a_0|\delta|\) is close to some rational number \(k/l\) where \(k = 1, 2, \cdots, l = 2, 3, \cdots\), and \(k\) and \(l\) are co-prime with one another, other Umklapp processes must be taken into account, and one must include the following term in Eq. (2):

\[
\tilde{g}_3 \int dx \cos [\sqrt{4\pi l}(\phi_a - \phi_m) + 2\pi \delta x] . \tag{3}
\]

We shall focus on Eq. (2) in the following and discuss the effects of the \(\tilde{g}_3\) term later.

**Incommensurate filling** When \(\delta\) is incommensurate with the lattice, one may neglect the \(g_3\) term in Eq. (4). To analyze the effects of the \(g_1\) and \(g_2\) terms on low-energy physics, we resort to the RG method. A perturbative calculation up to the one-loop order already shows that the \(g_1\) and \(g_2\) terms alone do not form a closed operator algebra in the sense of operator product expansion (OPE). One must include the term \(\partial_x \theta_a \partial_x \theta_m\) in \(H_{\text{eff}}\) [Eq. (2)] . To simplify the analysis, we consider the case \(v_a = v_m = v_0\). (The effects of velocity anisotropy will be discussed later.) Furthermore, we rescale \(g_1\) by \(a_0^2 g_1 \to g_1\). The one-loop RG equations can be obtained by setting \(\lambda_3(t) = 0\) in Eqs. (10) — (15).

By solving these scaling equations, one may obtain the following results: Within the weak-coupling region, i.e. \(\pi |g_1|/(2v_0), |g_2|/(2v_0) \ll 1\), the \(g_1\) term is relevant in the regime \(D_1 < 2 + \sqrt{2|\pi|v_0}\), while it becomes irrelevant for \(D_1 > 2 + \sqrt{2|\pi|v_0}\), where \(D_1\) is the scaling dimension of the \(g_1\) term, defined by

\[
D_1 = \frac{1}{K_a} + \frac{1}{4K_m} . \tag{4}
\]

Therefore, there are two zero-temperature phases. It turns out that the phase transition between these two phases is of the KT type. We note that for on-site interactions only, i.e. \(1 \leq K_{a/m} < +\infty\), the \(g_1\) term is always relevant from Eq. (4).

In the regime where the \(g_1\) term is relevant, it is convenient to define new bosonic fields: \(\theta_{\pm} \equiv \theta_a \mp \frac{1}{2} \theta_m\) and \(\phi_{\pm} \equiv \frac{1}{2} \phi_a \pm \phi_m\). Then, the value of \(\theta_-\) is pinned and a gap \(\Delta_0 \sim |\pi |g_1|/(2v_0)|^{2-D_1}\) is opened for the \(\theta_-\) sector. That is, the phase fluctuations of atoms and molecules are locked by the Josephson coupling. By integrating out the gapped sector, the low-energy effective Hamiltonian describing the gapless \(\theta_+\) sector takes the form of LLs:

\[
H_+ = \frac{v}{2} \int dx \left[ K (\partial_x \theta_+)^2 + \frac{1}{K} (\partial_x \phi_+)^2 \right] . \tag{5}
\]

For \(\pi |g_1|/(2v_0), |g_2|/(2v_0) \ll 1\), \(v\) and \(K\) can be related to the short distance variables with the help of the one-loop RG equations, yielding

\[
\frac{v}{v_0} = \sqrt{\left(\frac{K^*_a}{4} + K^*_m + 2\lambda'_1\right)\left(\frac{1}{K^*_a} + \frac{1}{4K^*_m} + 2\lambda'_2\right)} , \tag{6}
\]

\[
\frac{K}{K_0} = \sqrt{\left(\frac{K^*_a}{4} + K^*_m + 2\lambda'_1\right)\left(\frac{1}{K^*_a} + \frac{1}{4K^*_m} + 2\lambda'_2\right)} , \tag{6}
\]

where \(K^*_a = K_a + \frac{1}{2D_1}\), \(K^*_m = K_m + \frac{1}{4(2-D_2)}\), \(\lambda'_1 = \frac{g_2}{2v_0} + \frac{1}{2K_a K_m (2-D_2)}\), and \(\lambda'_2 = -\frac{1}{2(2-D_2)}\). Equation (6) indicates that the 1d ASF will become unstable provided that the inequality is satisfied:

\[
\frac{1}{K^*_a} + \frac{1}{4K^*_m} < -2\lambda'_2 . \tag{This}
\]
Green functions of atoms and molecules: In other words, the system will collapse for sufficiently
v
is because v ≤ 0, indicating an instability of this system. In other words, the system will collapse for sufficiently strong attraction between atoms and molecules.

The 1d ASF can be characterized by the single-particle Green functions of atoms and molecules:

\[
\langle \Psi_a(x)\Psi_a(0) \rangle = A_1 \left(\frac{a_0}{r}\right)^{\alpha_1} + \cdots,
\]

\[
\langle \Psi_m(x)\Psi_m(0) \rangle = A_2 \left(\frac{a_0}{r}\right)^{\alpha_2} + \cdots, \tag{7}
\]

where \(x = (\tau, x)\), \(r = \sqrt{(v_\tau)^2 + x^2}\), \(\Psi_a(x) = a_i/\sqrt{a_0}\), \(\Psi_m(x) = b_i/\sqrt{a_0}\), \(A_{1,2}\) are nonuniversal constants, and

\[
\alpha_2 = 4\alpha_1 = \frac{1}{2K}. \tag{8}
\]

We would like to stress that this exact relation between \(\alpha_1\) and \(\alpha_2\) [Eq. (8)] results from the fact that the phase fluctuations of atoms and molecules are locked by the Josephson coupling, and is a characteristic of the 1d ASF. On the other hand, the density correlation functions of atoms and molecules at nonzero momenta decay exponentially due to the presence of the gap \(\Delta_0\). Equation (8) implies that both the atomic and molecular sectors exhibit the behavior of a 1d SF as long as \(K > 1/4\). For \(1 \leq K_{a/m} < +\infty\), this condition is satisfied, which can be verified from Eq. (8). It gives further support to our claim that this phase is a 1d analog of the ASF in 3d.

In the regime where the \(g_1\) term is irrelevant, the excitation spectrum consists of two branches of gapless excitations with linear dispersion relations. This phase, which has been thoroughly discussed in Ref. [8], will be referred to as the 2-LLs. We just mention that in the present situation this phase can exist only by including sufficiently strong repulsions between atoms or molecules. This is because \(D_1 < 2\) for on-site interactions only.

**Commensurate filling** When \(\delta\) is close to zero, the \(g_3\) term must be retained. For simplicity, we shall consider \(\delta = 0\). Let us first define the scaling dimension of the \(g_3\) term:

\[
D_2 = K_a + K_m. \tag{9}
\]

The \(g_1\) \((g_3)\) term will be relevant if \(D_1 < 2\) \((D_2 < 2)\). Accordingly, a tree-level scaling analysis suggests the phase diagram in the \(K_a-K_m\) space as shown in Fig. 2(a). \(D_1 < 2\) and \(D_2 > 2\) corresponds to the 1d ASF, \(D_1 > 2\) and \(D_2 < 2\) will be referred to as the I-CDW, and \(D_1, D_2 > 2\) corresponds to the 2-LL. However, a competition between the two relevant operators, the \(g_1\) and \(g_3\) terms, occur when \(D_1, D_2 < 2\). To determine whether Region IV in Fig. 2(a) corresponds to a new phase or not, we employ the one-loop RG analysis.

By integrating out the fast modes, the one-loop RG equations are given by

\[
\frac{dK_a(l)}{dl} = 2 \left[\lambda_1^2(l) - K_a^2(l)\lambda_3^2(l)\right], \tag{10}
\]

\[
\frac{dK_m(l)}{dl} = \frac{1}{2} \left[\lambda_1^2(l) - 4K_m^2(l)\lambda_2^2(l)\right], \tag{11}
\]

\[
\frac{d\lambda_1(l)}{dl} = \left[2 - \frac{1}{K_a(l)} - \frac{1}{4K_m(l)}\right]\lambda_1(l), \tag{12}
\]

\[
\frac{d\lambda_2(l)}{dl} = \frac{\lambda_1^2(l)}{K_a(l)K_m(l)} - 2\lambda_2^2(l), \tag{13}
\]

\[
\frac{d\lambda_3(l)}{dl} = \left[2 - K_a(l) - K_m(l)\right]\lambda_3(l), \tag{14}
\]

\[
\frac{d\lambda_4(l)}{dl} = 2K_a(l)K_m(l)\lambda_2(l)\lambda_3(l) - \lambda_3^2(l), \tag{15}
\]

with the initial values: \(K_a(0) = K_a\), \(K_m(0) = K_m\), \(\lambda_{1(3)}(0) = \pi g_{1(3)}/(2v_0)\), \(\lambda_2(0) = g_2/(2v_0)\), and \(\lambda_4(0) = 0\). By solving Eqs. (10) — (15), one may find three kinds of behaviors of the RG flow of \(\lambda_1(l)\) and \(\lambda_3(l)\): (i) \(\lambda_1(l)\) flows to strong coupling while \(\lambda_3(l)\) flows to zero. This is the 1d ASF. (ii) \(\lambda_3(l)\) flows to strong coupling while \(\lambda_1(l)\) flows to zero. This is the I-CDW. (iii) Both \(\lambda_1(l)\) and \(\lambda_3(l)\) flow to zero. This is the 2-LL. Thus, Region IV in Fig. 2(a) shrinks to a transition line between the 1d ASF and I-CDW in the \(K_a-K_m\) space as shown in Fig. 2(b). Both the phase transition between the 1d ASF and 2 LL and that between the I-CDW and 2-LL belong to the KT type. The transition between the 1d ASF and I-CDW is of second order. Further, all these transition lines coincide at two tricritical points, the point \(A\) and \(B\) in Fig. 2(b). The very reason why there is no way for both \(\lambda_1(l)\) and \(\lambda_3(l)\) flowing to strong coupling simultaneously is that the operators \(\cos \sqrt{\pi}(\theta_m - \theta_a)\) and \(\cos \sqrt{4\pi}(\phi_a - \phi_m)\) are exclusive to one another, that is, the field configurations which minimize one perturbation term do not minimize the other. The interplay between these two competing relevant operators then produces a novel quantum phase transition.

In the I-CDW, it is convenient to define new bosonic fields: \(\phi_+ = \frac{1}{2}(\phi_a + \phi_m)\), \(\phi_- = \phi_a - \phi_m\), \(\theta_+ = \theta_a + \theta_m\), and \(\theta_- = \frac{1}{2}(\theta_a - \theta_m)\). The \(\phi_+\) and \(\phi_-\) fields describe the in-phase and out-of-phase density fluctuations, respectively.
Due to the relevant perturbation \( \cos \sqrt{4\pi} \tilde{\phi}_- \), the value of \( \tilde{\phi}_- \) is pinned and a gap is opened for the \( \tilde{\phi}_- \) sector, while the \( \tilde{\phi}_+ \) sector is still gapless. On account of this, both the single-particle Green functions of atoms and molecules decay exponentially. On the other hand, the \( 2\pi \rho_{a/b} \) parts of the density fluctuations for atoms and molecules are enhanced:

\[
\langle \rho_{a(b)}(x) \rho_{a(b)}(0) \rangle \bigg|_{2\pi \rho_{a(b)}} \sim \left( \frac{\alpha_0}{|x|} \right)^\gamma,
\]

with \( \gamma < 2 \) for \( \pi |g_1|/(2v_0), |g_2|/(2v_0) \ll 1 \). Here \( \rho_{a/b}(x) = n_{a/b}/a_0 \).

To understand the nature of the ground state of the I-CDW, a simple picture can be obtained in the limit of strong atom-molecule interactions \( (g_3) \), where the potential energy (the \( g_1 \) term) dominates over quantum fluctuations. In this case, atoms and molecules form a regular lattice (Wigner crystal of hard-core bosons). For \( g_3 > 0 \), the energy of the repulsion between atoms and molecules

\[
g_3 \cos \sqrt{4\pi} \tilde{\phi}_- = -g_3 \cos \sqrt{4\pi} (\phi_a - \phi_m + \sqrt{\pi/2} ],
\]

is minimized by a relative phase shift of \( \sqrt{\pi}/2 \) between atoms and molecules, which corresponds to a shift of the atom (or molecule) lattice by half-a-period.\(^{3}\) Thus, the I-CDW respects the symmetry of translation by one site, \( a_i \to a_{i+1} \) and \( b_i \to b_{i+1} \), but spontaneously breaks the reflection symmetry about the origin, \( a_i \to -a_{-i} \) and \( b_i \to -b_{-i} \) (or \( a_{i/m} \to -a_{-i/m} \) and \( \theta_{a/m} \to \theta_{a/m} \)).

**Experimental signatures** We suggest a few methods to detect the above results experimentally. The most important distinction between the 2-LL and the 1d ASF is that the ground state is a coherent state formed by hybridizing the atoms and molecules. Therefore, by suddenly changing the detuning, some kind of Rabi oscillation will be observed between the atomic and molecular condensates. Such a change of detuning can be achieved by applying magnetic field pulses to the ASF.\(^{10} \) Although there is no true condensate in 1d due to strong phase fluctuations, a similar oscillation can also be observed between the densities of atoms and molecules, which is strongly damped by the phase fluctuations. Further support to the 1d ASF is the examination of Eq. \(^3\), which can be achieved by a measurement of single-particle Green functions of atoms and molecules through the absorption line shape.

**Discussions** Finally, two points should be addressed here. First of all, when \( a_0 |\delta| \) is close to some rational number, one must replace the \( g_3 \) term in Eq. \(^2\) by the \( g_3 \) term [Eq. \(^3\)], with the scaling dimension \( D_{2l} \). A tree-level scaling analysis suggests the phase diagram in the \( K_a-K_m \) space as shown in Fig. \(^3\). We note that there is no direct transition between the 1d ASF and I-CDW. Moreover, the I-CDW can be reached only with very small values of \( K_{a/m} \). Next, a RG study shows that small velocity anisotropy, i.e. \( v_a \neq v_m \), just shifts the phase boundary, and does not affect our conclusions. This is because the OPE’s between the additional operators arising from velocity anisotropy and the already existing operators do not generate themselves.

When this work was finished, we were aware of two recent papers dealing with a related system — a Feshbach-resonant atomic Fermi gas in 1d.\(^{11} \) Their results about the charge sector are similar to ours for the 1d ASF.

The work of Y.-W. Lee is supported by the National Science Council of Taiwan under grant NSC 93-2112-M-029-007. The work of Y.L. Lee is supported by the National Science Council of Taiwan under grant NSC 93-2112-M-018-009.

---

* Electronic address: ywlee@thu.edu.tw
† Electronic address: ylle@cc.ncue.edu.tw

[1] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, E.A. Cornell, Science 269, 198 (1995); K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
[2] S. Jochim, M. Bartenstein, A. Altmeyer, G. Hendl, S. Riedl, C. Chin, J. Hecker Denschlag, and R. Grimm, Science 302, 2101 (2003); M. Greiner, C.A. Regal, and D.S. Jin, Nature 426, 537 (2003); M.W. Zwierlein, C.A. Stan, C.H. Schunck, S.M.F. Raupach, S. Gupta, Z. Hadzibabic, and W. Ketterle, Phys. Rev. Lett. 91, 250401 (2003).
[3] Y. Ohashi and A. Griffin, Phys. Rev. Lett. 89, 130402 (2002).
[4] L. Radzihovsky, J. Park, and P.B. Weichman, Phys. Rev. Lett. 92, 160402 (2004); M.W.J. Romans, R.A. Duine, S. Sachdev, and H.T.C. Stoof, ibid. 93, 020405 (2004); Y.W. Lee and Y.L. Lee, Phys. Rev. B 70, 224506 (2004).
[5] K.K. Das, Phys. Rev. Lett. 90, 170403 (2003).
We emphasize that most of our results can also be applied to the case where the Feshbach-resonant atomic gas is put in a very elongated trap, such as the toroidal one.

For a review, see, for example, T. Giamarchi, *Quantum Physics in One Dimension*, (Oxford University Press, Oxford, 2004).

A similar situation also occurs in quantum wires with two nearly equivalent subbands. See, for example, O.A. Starykh, D.L. Maslov, W. Häusler, and L.I. Glazman in: *Low-Dimensional Systems: Interaction and Transport Properties*, (Springer-Verlag, Berlin, 2000).

S.J.J.M.F. Kokkelmans and M.J. Holland, Phys. Rev. Lett. 89, 180401 (2002).

D.E. Sheehy and L. Radzihovsky, cond-mat/0505681; R. Citro and E. Orignac, cond-mat/0505706.