We discuss $B \rightarrow \rho$ form factors within the framework of perturbative QCD, including the higher twist contributions and study the validity of such an approach in calculating quantities such as form factors, which in principle and quite generally are thought to be completely non-perturbative objects and are expected to receive large contributions from the non-perturbative regime in the calculations. It is shown that including the Sudakov and threshold resummation effects, the general expectations of the pQCD approach are clearly met and the form factors do indeed receive most of the contribution from the perturbative region. We do not make an attempt to precisely evaluate the form factors but rather try to study the gross features and behaviour of the same. We also find that use of single wave function for the B-meson may actually underestimate various quantities. The results clearly indicate the validity and reliability of pQCD calculations, at least in this particular case.

PACS numbers: 13.20.He, 12.38.Bx, 12.38.Cy

I. INTRODUCTION

B-decays offer a very fertile soil for understanding, analysing and testing the basic structure of particle interactions, widely ranging from electroweak aspects, including CP violation, to the mysterious world of QCD (for a quick review of various issues see [1]). The study of B-decays and related observables has greatly enhanced and affected our understanding of the underlying principles that guide and govern these phenomena. The Standard Model (SM) of particle physics, including the QCD corrections, seems to be a highly successful candidate in explaining almost all the experimental data, including a variety of issues concerning B-decays themselves. However, as we enter the precision era in B-physics, there is a compelling need to uncover more and more that goes as input while evaluating and analysing these decays, and as can be expected, the need to unearth the relative importance of QCD corrections is overwhelming. In particular, the experience with $b \rightarrow s \gamma$ shows that QCD can, in fact, alter the results by a large amount and therefore it becomes imperative to include higher order QCD corrections to get more and more sensible and accurate results.

Semileptonic decays of the B-mesons are of particular interest owing to their cleanliness and relatively simpler

*Electronic address: nmahajan@mri.ernet.in
calculational aspects. It also means that these decays provide us with the opportunity to test QCD corrections more reliably and much more easily, compared to pure hadronic decay modes. This has been widely recognized and a lot of work has been done in this direction. Owing to the large mass of B-mesons, it is expected that the heavy quark effective theory (HQET) description is a good one and the leading term is given simply by the quark level process (see [2]). The sub-leading terms are suppressed by the B-meson mass and in most of the applications, are therefore neglected. Another simplifying assumption that is employed is the idea of factorization [3]. Quite simply, it means that in an energetic process, like the decay of a B-meson, since the energy released is large, the outgoing hadrons move out of the interaction region quite quickly, thereby implying that there are no soft interactions between the various hadronic subsystems. The full meson level amplitude for any process is thus written as a convolution integral of the hard scattering kernel (which can be reliably computed using standard perturbative formalism) and the non-perturbative meson wave functions that are universal and are obtained from lattice studies, fits to the experimental data or sum rules of one kind or the other. This assumption forms the backbone of most of the calculations involving B-mesons. The issue of the extent of reliability of such an approximation, which seems to work quite well for most of the cases, is still an open question. It is however found and argued that not in all cases this is a good approximation and one must look for the sub-leading terms to this approximation scheme. For example it suffers from the problem of scale dependence [4] and in case of $B \rightarrow J/\Psi$ it seems difficult to match the branching ratios using the approximation [5]. Furthermore, it is now accepted that though the leading term in the HQET gives results that are quite close to the observed numbers, the reality that the mass of a B-meson is not infinite but close to 5 GeV, implies that the sub-leading terms can not be naively neglected anymore. One can try and estimate these sub-leading terms in HQET itself or employ some other methods. It is worthwhile to try and explore the idea of estimating the above mentioned sub-leading terms suppressed by the B-meson mass. To do such a calculation, one can follow either of the following two approaches. Both the approaches treat the hard scattering kernels perturbatively and the mesonic wave functions are the non-perturbative ingredients. The main difference lies in the treatment of quantities like from factors.

(a) **QCD factorization**: The form factors are also thought to be purely non-perturbative objects, not calculable perturbatively [6].

(b) **pQCD**: It is believed that the form factors can be reliably and satisfactorily computed in perturbation theory. This approach was developed by Brodsky and Lepage and others [7]. The factorization theorem for relevant for exclusive B-meson has been proved [8].

There is no clear consensus on either of these two methods and both seem to have some advantageous features while both suffer from lack of very sound theoretical footing on certain issues. Both the approaches have invited intense activity in the recent past. In some cases, the results of the two approaches tend to be consistent while in some other cases, they seem to be distance apart.

In this note, we investigate $B \rightarrow \rho$ form factors in the second of the approaches listed above, namely the pQCD method. We have chosen semileptonic process for the merit of cleanliness and relatively simplified calculations and
the fact that with just two hadrons in the process, we do not have to bother about the subtleties and complications involved in a purely hadronic process and the reliability can be checked more clearly. The same process has been considered by [9] and [10]. We extend their study and elaborate on the differences in the next sections. The aim of this study is not to precisely pin down the value of the form factors from the calculations but to explore and study the behaviour of the same with respect to the various parameters that enter the calculation and check the reliability of such a calculation. In particular, we keep in mind the objections and criticisms that are generally raised regarding the validity of such a scheme and try to see for ourselves, whether, following a more consistent treatment compared to whatever exists in literature till now, we get a clue to some of the unresolved and mysterious issues that we are forced to live with in such calculations. We would like to emphasize again that the goal is not a very accurate numerical study of the form factors but to study the general behaviour for some suitable choices of various parameters like the shape variables appearing in the meson wave functions etc. Furthermore, we would like to remind ourselves that in such a situation, we may finally end up over- or under-estimating some of the quantities but we hope to clarify certain issues in the end. In this spirit, this study aims at extending the ongoing debate between the two approaches in order to get a clearer picture of what exactly is happening and can we understand and explain the same. The article is organized as follows: in the next section we briefly review the chief ingredients of the pQCD approach. We summarize some of the main objections/criticisms that this method faces. Next we discuss the form factor calculation including the higher twist contributions to the meson wave functions. We then qualitatively study the behaviour of the various form factors without attempting to determine the values very precisely. The last section discusses and summarizes our results and conclusions.

II. PQCD APPROACH - A QUICK LOOK

Exclusive processes enjoy the simplicity and elegance they derive from factorization theorems invoked in some form or the other (for a general review of factorization theorem and its related issues see [11]). It has been demonstrated that for an exclusive process involving a large scale $Q$, the amplitude can be neatly written as

$$A \sim C(t) \otimes H(t) \otimes \left( \prod_i \Phi_i(x_i) \right) \otimes \exp[-S]$$

(2.1)

where $C(t)$ denotes the Wilson coefficients relevant to the problem, $H(t)$ is the hard scattering kernel that is perturbatively evaluated, $\Phi_i(x)$ describes the distribution of the partons in the $i$-th hadron (here $x$ is the momentum fraction carried by the parton) and we have collectively put all the resummed quantities in the factor $S$ for simplicity. The factor $S$ therefore contains the Sudakov logarithms and the relevant evolution factor (these are discussed below). The factorization theorem implies that in a hard exclusive process, the non-perturbative dynamics can be separated from the perturbative pieces and the final result is simply the convolution of these. We now concentrate on a specific process involving B-meson decay, namely $B \rightarrow \rho \ell \nu$, as our prototype hard exclusive process. The large mass of the B-meson acts as the natural large scale in the problem. Here and in the following we do not differentiate between a $B$ or a $\bar{B}$-meson. We therefore have the following scales in the problem: mass of the $b$-quark (we do not differentiate between the $b$-quark mass and the B-meson mass) $m_b \sim 5$ GeV, the W-boson mass $M_W \sim 100$ GeV,
the renormalization scale $\mu$ (which is of the order of $m_b$) and a scale of the order of $\Lambda_{QCD} \equiv \Lambda \sim 250$ GeV. The Wilson coefficients appearing in the effective Hamiltonian contain the information of the high scales ($> m_b$) and have the resummed logarithms of the form $\ln(M_W^2/\mu^2)$. The scale $\Lambda$ characterizes the non-perturbative scale and crucially enters the meson wave functions. The precise shape and behaviour of the meson wave function are crucial inputs for any calculation to make physical sense. For example, if the wave function does not vanish at the end points ($x \to 0,1$), then it can be shown that the amplitude is infra-red divergent \cite{12}. Also, it has been pointed out that if the small transverse momentum of the parton, generically denoted as $k_\perp \sim O(\Lambda)$ is ignored, the dominant contribution arises from the end point regions. The way out is to retain the $k_\perp$ components, which in turn regulate the infra-red divergence that has just been mentioned \cite{13}. If the variable conjugate to $k_\perp$ is denoted by $b$, then apart from the large logarithms that are reorganised using standard renormalization group techniques into Wilson coefficients, we have another potential source of large logarithms - $\ln(\mu b)$. The inclusion of the transverse momentum for the partons in turn results in double logarithms of the form $\ln^2(P_b)$, where $P$ is the momentum of the meson (generally the larger of the light-cone momentum components). These large logarithms are resummed and lead to the so called Sudakov form factor $\exp[-S(p,b)]$. This exponentially damping factor suppresses the long-distance contributions from the large $b$ regions and the factor $S(P,b)$ vanishes as $b \to 1/\Lambda$. Since the infra-red divergences in a theory are manifestations of the non-perturbative dynamics, such effects are absorbed in the wave functions. These wave functions are universal in character and do not depend on the specific process. After, reorganising all the large logarithms and absorbing the infra-red divergences in the meson wave functions, we are left with finite quantities that are believed to be perturbatively calculable. The main philosophy behind the pQCD method is that all the factors except the hadronic wave functions can be perturbatively calculated such that the dominant contribution to various quantities, like form factors, comes from the hard gluon exchanges. It is important that this is true because otherwise the dominant contribution would arise from the non-perturbative regime - a region where the perturbative calculations do not make sense and there is absolutely no way of determining them in this way.

The $B \to \rho$ form factors are parameterized as follows \cite{14}:

$$
\langle \rho(P_\rho,\lambda) | (V - A)_{\mu} | B(P_B) \rangle = -i(m_B + m_\rho)A_1(q^2)\epsilon_{\mu}^{*(\lambda)} + \frac{IA_2(q^2)}{(m_B + m_\rho)}(\epsilon^{*} \cdot P_B)(P_B + P_\rho)_{\mu} + \frac{IA_3(q^2)}{(m_B + m_\rho)}(\epsilon^{*} \cdot P_B)(P_B - P_\rho)_{\mu} + \frac{2V(q^2)}{(m_B + m_\rho)}\epsilon_{\mu\nu\alpha\beta}P_B^\nu P_\rho^\alpha \epsilon_\perp^{*}\beta
$$

where $q^2 = (P_B - P_\rho)^2$ is the momentum transferred. For the process under consideration, we have the following two diagrams contributing at the one gluon exchange level in pQCD:

![Diagram](image-url)

FIG. 1: One gluon exchange diagrams contributing to $B \to \rho$ form factors. The cross denotes the weak interaction vertex.

where $\phi_{B,\rho}$ represents the corresponding meson wave function, $\xi$ is the momentum fraction carried by the lighter
(anti)quark and $l_\perp$ is the transverse momentum while $x$ and $k_\perp$ denote the analogous quantities for the $\rho$-meson. As mentioned before, we do not distinguish between $b$-quark mass and $B$-meson mass and set them equal in the calculations. The light quarks - $u, d, s$ are taken to be massless. Our convention for the light-cone variables is as follows. For any four momentum $a^\mu = (a^+, a^-, a_\perp)$ we define $a^\pm = a^0 \pm a^3$ and $a_\perp = (a^1, a^2)$. The scalar product is defined as: $a \cdot b = \frac{a^+ b^- + a^- b^+}{2} - a_\perp \cdot b_\perp$. With these conventions, we have for various momenta:

$$P_B = m_B(1, 1, 0_\perp) \quad P_\rho = \frac{m_B}{\eta} (r^2, \eta^2, 0_\perp)$$

(2.3)

where $r = m_\rho/m_B$ and $\eta = 1 - q^2/m_B^2$. It is convenient to define two light-like momenta $n_+ = (\sqrt{2}, 0, 0_\perp)$ and $n_- = (0, \sqrt{2}, 0_\perp)$ such that $n_+ \cdot n_- = 1$. The transverse momenta $k_\perp$ and $l_\perp$ as introduced above should be thought of as four-vectors with only the transverse components non vanishing. Schematically, therefore, the various form factors (generically called $F$), after using the factorization theorem, can be written as

$$F = \int_0^1 dx \int_0^1 d\xi \; \phi_B(\xi, q^2) H(x, \xi, q^2) \phi_\rho(x, q^2)$$

(2.4)

where as before $\phi$’s denote the meson wave functions and $H(x, \xi, q^2)$ represents the hard kernel evaluated perturbatively. The basic theme behind such a factorization is that the long-distance or the soft interactions occur before and after the hard decay process and therefore the two effects simply decouple. The hard contribution takes place at short-distance scales and therefore we only need to specify the distribution of quarks (partons) inside the mesons apart from explicitly computing the perturbative piece. The information regarding the parton distribution, along with all the infra-red dynamics, is contained in the wave functions. In the asymptotic region, the factorization theorem has been shown to be valid. However, it has been pointed out \[12\] that in the case of pion’s electromagnetic form factor at lower energy scales (few GeV’s), large contributions come from the end point regions $(x, \xi \to 0)$. The perturbative analysis is not valid in such a region and this casts serious doubts about such a method. This problem, even when the total convolution integral does not contain any divergences, is called the *endpoint problem*. The way out is to introduce Sudakov factors that regulate this undesirable endpoint behaviour and thus make the pQCD calculations sensible at these lower energy scales \[15\]. The Sudakov factors typically suppress the long-distance effects at large transverse distances (denoted by variable $b$ earlier) i.e. small transverse momenta. Therefore, the endpoint behaviour is a crucial aspect to get any meaningful results.

A further issue of concern is that as the momentum fraction carried by the spectator tends to zero, the form factors become divergent due to divergent behaviour of the meson wave functions related to the light-cone distribution amplitudes (LCDAs). There is a need to resum the contributions of the form $\ln^2 x$. This is achieved by threshold resummation into a jet function $S_t(x)$ such that $S_t(x) \to 0$ as $x \to 0, 1$ \[16\]. Therefore, the threshold resummation modifies the endpoint behaviour of the distribution amplitudes (DAs).

After including all these factors in the DAs, one expects that the pQCD calculations are reliable and can be carried out without much trouble. However, the method still faces some more objections/criticisms which we now briefly discuss. We summarize the main issues as discussed in \[17\]

- Are Sudakov suppression factors strong enough to regulate the large transverse separation contributions?
particular, it is quite possible that for intermediate values of the variables, away from the endpoints, there can be a slight enhancement. How effective and efficient is this effect?

- How small are the contributions from the non-perturbative regime?

- The biggest uncertainties presumably arise from the precise lack of knowledge about the meson wave functions. In particular, the use of a single wave function for the B-meson has been questioned.

All these and related issues cast a serious cloud of doubt on the applicability and reliability of pQCD calculations. We try to address and clarify some of these issues below and also elaborate on various issues.

III. MESON DISTRIBUTION AMPLITUDES

The DAs refer to the distribution of partons inside a hadron (here mesons). A DA thus, carries the information regarding the momentum fraction carried by partons inside the meson in a specific Fock state. The DA is related to the the Bethe-Salpeter wave function by the following relation

$$\phi(x) \sim \int |k_\perp|^{<\mu} d^2k_\perp \phi_{BS}(x, k_\perp)$$

(3.1)

where $\mu$ is the ultra-violet cut-off (for a general discussion see 18). In the discussions below, we do not remain very careful and do not distinguish between the wave functions and DAs and freely interchange one for the other in the discussions.

We closely follow the approach and results for the $\rho$-meson DAs as outlined in 19. Accordingly, the the LCDAs are defined as meson-to-vacuum transition matrix elements of the non-local gauge invariant operators in the light-cone picture. For the light vector mesons, the DAs are split into chiral even and chiral odd contributions. In terms of $n_\pm$, the vector meson momentum is written as $P_\rho = E_n - m_\rho n_\perp/(4E)$, such that the transverse plane is defined with respect to the vectors $n_\pm$. Fourier transforming the DAs, we obtain the momentum space representation of the $\rho$-meson light-cone projection 20 (keeping terms to twist-3 of the two particle quark-antiquark distribution only and not writing the path ordered integral and also remembering a relative $-i$ factor between the definitions here and those in 19)

$$\langle \rho(P_\rho, \lambda)|\tilde{d}_\alpha(z)u_\delta(0)|0\rangle \equiv M^P_\delta\alpha = M^P_{\delta\alpha\|} + M^P_{\delta\alpha\perp}$$

(3.2)

where

$$M^P_{\delta\alpha\|} = \left( -\frac{i\mu}{4} \frac{m_\rho (e^* \cdot n_\perp)}{2E} E_{\rho} N_{\perp} \phi_{\parallel}(x) - \frac{i\mu}{4} \frac{m_\rho (e^* \cdot n_\perp)}{2E} \left[ -\frac{i}{2} \sigma_{\mu\nu} n_\mu^\alpha n_\nu^\beta h^{(t)}(x) \sigma_{\alpha\beta} \right] h^{(t)}(x) \right)_{\delta\alpha}$$

(3.3)
\[ M_{\delta \perp}^\rho = \left( -\frac{i f_1}{4} E \kappa \cdot \mathbf{\dot{h}}_\perp \phi_\perp(x) - \frac{i f_\rho \alpha}{4} \left[ \kappa \cdot \mathbf{\dot{g}}_\perp^{(v)}(x) - E \int_0^x dv \phi_\perp(v) - g_\perp^{(v)}(v) \right] \right) \delta_{\alpha} \]  

(3.4)

By \( \partial^{k\perp} \) we mean \( \partial_{\perp k} \) and we have

\[ \epsilon^{\mu}_\perp \equiv \epsilon^{\mu} - \frac{\epsilon^* \cdot n_+}{2} n_- - \frac{\epsilon^* \cdot n_-}{2} n_+ \]  

(3.5)

Authors in [9] and [10] did not include the \( \partial^{k\perp} \) terms in their calculations. We retain these terms as well and finally show that they contribute significantly to some of the form factors. All the individual distribution amplitudes are normalized to unity i.e. \( \int_0^1 dx \phi_\parallel(x) = 1 \) etc. For the sake of completeness, we list the individual distribution amplitudes [19]:

\[ \phi_\parallel(x) = 6x(1-x)[1 + 0.27(5[2x - 1]^2 - 1)] \]

\[ h^{(t)}_\parallel(x) = 3(2x - 1)^2 + 0.3(2x - 1)^2[5(2x - 1)^2 - 3] + 0.21[3 - 30(2x - 1)^2 + 35(2x - 1)^4] \]

\[ h^{(s)}_\parallel(x) \equiv \frac{\partial h^{(t)}_\parallel(x)}{\partial x} = 6(2x - 1)[1 + 0.76(10x^2 - 10x + 1)] \]

\[ \phi_\perp(x) = 6x(1-x)[1 + 0.3(5[2x - 1]^2 - 1)] \]

\[ g^{(v)}_\perp(x) = \frac{3}{4}[1 + (2x - 1)^2] + 0.24[3(2x - 1)^2 - 1] + 0.12[3 - 30(2x - 1)^2 + 35(2x - 1)^4] \]

\[ g^{(a)}_\perp(x) = 6x(1-x)[1 + 0.23(5[2x - 1]^2 - 1)] \]

\[ g^{(a)}_\perp(x) \equiv \frac{\partial g^{(a)}_\perp(x)}{\partial x} \]

Fourier transforming the wave functions with respect to the transverse momentum leads us to the wave functions expressed in the transverse separation \( b \)-space. The variable \( b \) defines the transverse separation between the quark and the anti-quark inside the meson. In principle, the various DAs listed above can have different transverse momentum dependence. But for simplicity, we assume same dependence for all of them. This dependence is not known from first principles and therefore, it is generally assumed that the full wave function, including the transverse momentum dependence, is of the form

\[ \Psi(x, k_\perp) = \phi(x) \Sigma(k_\perp) \]  

(3.7)

such that apart from the normalization condition for \( \phi \) listed above, we have for the transverse part, \( \int d^2 k_\perp \Sigma(k_\perp) = 1 \).

The functional form of \( \Sigma(k_\perp) \) is assumed to be a simple Gaussian distribution. In the \( b \)-space, we therefore have

\[ \Sigma_\rho(b_\rho) = \exp \left( -\frac{\omega_\rho^2 b_\rho^2}{2} \right) \]  

(3.8)
For the heavy B-meson, following [20] and [21], we have the following momentum space projection operator
\[ M_B^{\delta \alpha} = -i f_B \left( (P_B + m_B) \left[ \Psi_B(\xi) + \frac{1}{2} \Psi_B(\xi) \right] - \frac{1}{2} \triangle(l) \gamma^\mu \partial_\mu \right)_5 \delta_\alpha \] (3.9)
where \( \Psi_B(\xi) \) and \( \bar{\Psi}_B(\xi) \) are the B-meson wave functions corresponding to the DAs \( \phi_B \) and \( \bar{\phi}_B \) defined as
\[ \phi_B = \frac{\phi_B^+ + \phi_B^-}{2} \quad \bar{\phi}_B = \frac{\phi_B^+ - \phi_B^-}{2} \] (3.10)
The term with coefficient \( \triangle(l) \) works out to be proportional to \( l^+ / m_B \) and is therefore generally dropped. We retain this as well in the analysis and comment on its contribution later. Also, it has been argued that the \( \bar{\Psi}_B(\xi) \) term is sub-leading compared to the first term in the projection operator and is also dropped. This is one of the major issues of debate concerning the pQCD calculations. In the present study, we retain this sub-leading term also. Another reason of retaining this term is the confusion whether the pseudoscalar and axial matrix elements can both be described by the same DA in the heavy quark limit. It has been pointed out that the two DAs considered by the authors of [22] (and of subsequent works based on their proposal), do not satisfy the equations of motion and further raise more doubts. In this study, we follow [17] and use, as a model, the following form for the two DAs which satisfy the equations of motion
\[ \phi_B^+(\xi) = \sqrt{\frac{2}{\pi}} \frac{\xi^2 m_B^2}{\omega_B^2} \exp \left[ -\frac{\xi^2 m_B^2}{2\omega_B^2} \right] \] (3.11)
and
\[ \phi_B^-(-\xi) = \sqrt{\frac{2}{\pi}} \frac{1}{\omega_B} \exp \left[ -\frac{\xi^2 m_B^2}{2\omega_B^2} \right] \] (3.12)
The quantity \( \triangle(l) \) introduced above is related to the DAs as follows
\[ \triangle(l) = \int_0^{l^+} dl (\phi_B - \phi_B^+) \] (3.13)
We next include the transverse momentum dependence for the B-meson through the following
\[ \Sigma_B(b_B) = \exp \left[ -\frac{\omega_B^2 b_B^2}{4} \right] \] (3.14)
The question whether the above DAs are actually realistic and whether they correctly reproduce all the features is an intriguing one and has to be dealt separately. We therefore proceed with the assumption that they do or at least capture the important features.

**IV. SUDAKOV FORM FACTORS AND THRESHOLD RESUMMATION**

The resummation of large transverse separation regions leads to Sudakov form factors while the resummation over small fractional momenta leads to threshold resummation.

As mentioned earlier, the Sudakov factor suppresses the double logarithms of the form \( \ln^2 P \) arising due to the
overlap of soft and collinear divergences. The parameter $b$ regulates these kind of infra-red contributions. In terms of the variables

$$\hat{q} \equiv \ln \left[ \frac{xQ}{\sqrt{2} \Lambda} \right], \quad \hat{b} \equiv \ln \left[ \frac{1}{b\Lambda} \right]$$

(4.1)

the exponent of the Sudakov form factor, to NLO, reads

$$s(x, b, Q) = \frac{A^{(1)}}{2\beta_1} \hat{q} \ln \left( \frac{\hat{q}}{b} \right) - \frac{A^{(1)}}{2\beta_1} (\hat{q} - \hat{b}) + \frac{A^{(2)}}{4\beta_1^2} \left( \frac{\hat{q}}{b} - 1 \right) - \left[ \frac{A^{(2)}}{4\beta_1^2} - \frac{A^{(1)}}{4\beta_1} \ln \left( \frac{e^{2\gamma_E} - 1}{2} \right) \right] \ln \left( \frac{\hat{q}}{b} \right)$$

(4.2)

$$+ \frac{A^{(1)}}{4\beta_1^2} \hat{q} \ln \left( \frac{\ln(2\hat{q}) + 1}{\hat{q}} \right) - \frac{\ln(2\hat{b}) + 1}{b} + \frac{A^{(2)}}{8\beta_1^3} \ln^2(2\hat{q} - \ln^2(2\hat{b}))$$

$$- \frac{A^{(2)}}{16\beta_1^4} \left[ \frac{2\ln(2\hat{q}) + 3}{\hat{q}} - \frac{2\ln(2\hat{b}) + 3}{\hat{b}} \right] - \frac{A^{(2)}}{16\beta_1} \frac{\hat{q} - \hat{b}}{b^2} \left[ 2\ln(2\hat{b}) + 1 \right]$$

$$+ \frac{A^{(2)}}{432\beta_1^6} \frac{\hat{q} - \hat{b}}{b^3} [9\ln^2(2\hat{b}) + 6\ln(2\hat{b}) + 2]$$

$$+ \frac{A^{(2)}}{1728\beta_1^8} \left[ 18\ln^2(2\hat{q}) + 30\ln(2\hat{q}) + 19 - 18\ln^2(2\hat{b}) + 30\ln(2\hat{b}) + 19 \right]$$

where $\gamma_E$ is the Euler’s constant and the various coefficients are given as follows:

$$\beta_1 = \frac{33 - 2n_f}{12}, \quad \beta_2 = \frac{153 - 19n_f}{24}$$

(4.3)

$$A^{(1)} = \frac{4}{3}, \quad A^{(2)} = \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{27}n_f + \frac{8}{3}\beta_1 \ln \left( \frac{e^{\gamma_E}}{2} \right)$$

(4.4)

with $n_f$ being the number of flavours. The strong coupling constant to NLO is given by

$$\frac{\alpha_s(\mu)}{\pi} = \frac{1}{\beta_1 \ln(\mu^2/\Lambda^2)} - \frac{\beta_2}{\beta_1^2} \frac{\ln(\mu^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)}$$

(4.5)

The Sudakov factor therefore falls for large $b$ regions and vanishes as $b > 1/\Lambda$. Also, the wave functions (or equivalently DAs) defined above are valid only for scales $1/b$. To have the correct results at an arbitrary $\mu$ a RG-evolution is required which generates an evolution factor in the exponential. We club this factor along with the Sudakov exponent and define the Sudakov as $\exp(-S)$ with

$$S(x, b, Q, \mu) = s(x, b, Q) + s(1-x, b, Q) - \frac{1}{\beta_1} \ln \left( \frac{\ln(\mu/\Lambda)}{\ln(1/(b\Lambda))} \right)$$

(4.6)

In practical calculations, $\mu$ is identified with the factorization scale of the hard kernel. It is to be noted that for the B-meson, the Sudakov factor is considered only for the lighter quark. Therefore, the exponents of the individual Sudakov factors (including the evolution function) are given as follows:

$$S_p = s(x, b_p, m_B) + s(1-x, b_p, m_B) - \frac{1}{\beta_1} \ln \left( \frac{\ln(\mu/\Lambda)}{\ln(1/(b_p\Lambda))} \right)$$

(4.7)
\[ S_B = s(\xi, b_B, m_B) - \frac{1}{\beta_1} \ln \frac{\ln(\mu/\Lambda)}{\ln(1/(b_B \Lambda))} \]  

The last major ingredient to be introduced is the threshold resummation. The double logarithms of the form \( \ln^2 x \) diverge at the endpoints and therefore they are also resummed. In order to achieve this goal, a jet function, \( S_i(x) \) is introduced which vanishes at the endpoints. It has been proposed \[10\] that for phenomenological studies, a simple parameterization can be used. The proposed parameterization is

\[ S_i(x) = \frac{2^{1+2c} \Gamma(3/2 + c)}{\sqrt{\pi} \Gamma(1 + c)} [x(1 - x)]^c \]

with \( c \sim 0.3 \). We use this parameterization in our numerical study but agree that this may not be the accurate form. Finally, we remark that it is not very clear whether the power suppressed corrections arising due to working in different gauges for different quantities are going to play an important role \[17\]. This is one feature that has to be checked but we leave it for a separate study. The DAs discussed in the previous section are assumed to be finally multiplied by the corresponding Sudakov factors, the transverse dependence carrying functions - the \( \Sigma \)'s and threshold functions.

V. EVALUATING \( B \to \rho \) FORM FACTORS

The \( B \to \rho \) form factors as introduced above can also be written in a slightly different form. We make the following identifications. The coefficients of \( P_B + P_\rho \) and \( P_B - P_\rho \) define the form factors \( A_2 \) and \( A_3 \), as discussed above. Instead, we define \( \tilde{A}_2 \) and \( \tilde{A}_3 \) as coefficients of \( P_B \) and \( P_\rho \) such that \( A_2 \) and \( A_3 \) are simply sum and difference of the new form factors. The remaining two have the same definition as above. This has been done for convenience only. We evaluate the “parallel” and “transverse” contributions from both the diagrams separately. The individual expressions read (again as mentioned in the last section, we assume the DAs to be multiplied by suitable Sudakov, threshold and momentum dependence factors and do not write them explicitly here in the expressions)

\[ \mathcal{M}_{\nu}^{(1)} = (-4\pi N_c C_F) \left( \frac{i f_B m_B^2 m_B^2}{16 \sqrt{2} E} \right) (\epsilon^* \cdot n_\perp) \int dx \, d\xi \, d^2 k_\perp \alpha_s \]

\[
\left\{ P_{\rho \mu} \left( \frac{2 i E f_\rho}{m_B} \right) \phi_\parallel(x) \left[ \frac{\sqrt{2} \eta}{m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) + \frac{4 \eta}{m_B(\eta^2 - r^2)} \Psi_B(\xi) \right] \\
+ \frac{\sqrt{2} \eta}{m_B(\eta^2 - r^2)} + \frac{2 x}{m_B} \bar{\Psi}_B(\xi) + \frac{\sqrt{2} x}{m_B} \bar{\Psi}_B(\xi) + \frac{2 x(\eta + r^2) \eta}{m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) \right] \\
+ f_\perp h_\parallel^{(1)}(x) \left[ \frac{2 \sqrt{2} i \eta}{m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) + \frac{2 i \eta}{m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) - \frac{4 \sqrt{2} i \eta^2 x}{m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) \right] \\
+ 4 i f_\perp E \left[ \int_0^x dv [\phi_\perp(v) - h_\parallel^{(1)}(v)] \right] (\xi) \frac{\eta}{m_B(\eta^2 - r^2)} (\partial^k \cdot \partial^{l_\perp}) \\
+ 2 \sqrt{2} i f_\perp h_\parallel^{(1)}(x) \left[ - \frac{\eta}{\sqrt{2} m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) + \frac{2 x}{m_B} \bar{\Psi}_B(\xi) \right] \right) \]

The expressions above are written in a slightly different form, making use of the threshold and momentum dependence factors. The coefficients of the form factors are defined in the previous section. The parameterization can be used for phenomenological studies, simplifying the expressions.
where the derivatives should be viewed as acting on the hard kernel and we have already set terms proportional to $k_\perp$ and $l_\perp$ appearing in the numerator to be zero. Similarly we have

\[
\mathcal{M}_{\mu\nu}^{(2)} = (-4\pi N_c C_F) \left( \frac{i f_B m_B^2 m_B^2}{8 E} \right) (\epsilon^* \cdot n_+) \int dx \, d\xi \, d^2 l_\perp \, d^2 k_\perp \alpha_s \tag{5.2}
\]

\[
\left\{ P_{\rho\mu} \left( 2i E f_B \phi_\parallel(x) \left[ \frac{\sqrt{2} \eta^2}{m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) - \frac{\sqrt{2} \eta^2 r^2}{m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) - \frac{\sqrt{2}}{m_B} \bar{\Psi}_B(\xi) - \frac{2r^2}{m_B} \Psi_B(\xi) \right] \right. \right.
\]

\[
+ f_{\perp h^{(\perp)}(x)} \left[ - \frac{\sqrt{2} \eta^2}{m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) + \frac{2 \eta^2}{m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) + \frac{4 \sqrt{2} \eta^2 r^2}{m_B(\eta^2 - r^2)} \Psi_B(\xi) \right]
\]

\[
+ 2 \sqrt{2} f_{\perp h^{(a)}(x)} \left[ \frac{\eta^2}{\sqrt{2} m_B(\eta^2 - r^2)} \bar{\Psi}_B(\xi) - \frac{1}{m_B} \Psi_B(\xi) \right] \left[ \frac{1}{[x\xi m_B^2 - (l_\perp - k_\perp)^2]} \right] \tag{5.3}
\]

\[
\mathcal{M}_{\mu\perp}^{(1)} = (4\pi N_c C_F) \left( - \frac{f_B m_B^2}{4} \right) \int dx \, d\xi \, d^2 l_\perp \, d^2 k_\perp \alpha_s \tag{5.3}
\]
\[
\mathcal{M}^{(2)}_{\mu\nu} = \left( -4\pi N_c C_F \right) \left( -\frac{f_B m_B^2}{4} \right) \int dx \, d\xi \, d^2 k_\perp \, d^2 k_\parallel \, \alpha_s
\]

\[
\left\{ \begin{array}{l}
\varepsilon_\mu \cdot P_{B\rho} \left( f_{\mu\nu} g^{(v)}_{\perp \perp} \right) \left( -\frac{2i}{m_B} \bar{\Psi}_B(\xi) + \frac{\sqrt{2} \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \Psi_B(\xi) \right) - \frac{\sqrt{2} \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \bar{\Psi}_B(\xi) \\
+ \frac{f_{\mu\nu} g^{(r(a))}_{\perp \perp}}{8} \left[ \frac{\sqrt{2} \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \bar{\Psi}_B(\xi) - \frac{4 \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \bar{\Psi}_B(\xi) + \frac{\sqrt{2} \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \bar{\Psi}_B(\xi) - \frac{2 \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \bar{\Psi}_B(\xi) \right] \\
- \frac{f_{\mu\nu} g^{(r(a))}_{\perp \perp}}{8} \left[ \frac{\sqrt{2} \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \bar{\Psi}_B(\xi) + \frac{4 \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \bar{\Psi}_B(\xi) - \frac{\sqrt{2} \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \bar{\Psi}_B(\xi) - \frac{2 \xi \eta}{m_B^2 (\eta^2 - r^2)^2} \bar{\Psi}_B(\xi) \right] \\
+ \frac{f_{\mu\nu} E g^{(r(a))}_{\perp \perp}}{4} \frac{1}{[x \xi m_B^2 - (l_\perp - k_\perp)^2][x \eta m_B^2 - k_\perp^2]} \end{array} \right\}
\]
the expressions, we introduce the following multiplied by the Sudakov, threshold resummation and momentum dependence factors. While Fourier transforming 

These expressions are Fourier transformed to the $b$-space. The wave functions/DAs are assumed to have been multiplied by the Sudakov, threshold resummation and momentum dependence factors. While Fourier transforming the expressions, we introduce the following $h_i$ functions:

$$h_1 = K_0(\sqrt{x\xi m_B b_B})\{\theta(b_B - b_p)I_0(\sqrt{x\xi m_B b_B})K_0(\sqrt{x\xi m_B b_B}) + \theta(b_p - b_B)I_0(\sqrt{x\xi m_B b_B})K_0(\sqrt{x\xi m_B b_B})\} \tag{5.5}$$

$$h_2 = K_0(\sqrt{x\xi m_B b_B})\{\theta(b_B - b_p)I_0(\sqrt{x\xi m_B b_B})K_0(\sqrt{x\xi m_B b_B}) + \theta(b_p - b_B)I_0(\sqrt{x\xi m_B b_B})K_0(\sqrt{x\xi m_B b_B})\} \tag{5.6}$$

$$h_3 = K_1(\sqrt{x\xi m_B b_B})\{\theta(b_B - b_p)I_0(\sqrt{x\xi m_B b_B})K_0(\sqrt{x\xi m_B b_B}) + \theta(b_p - b_B)I_0(\sqrt{x\xi m_B b_B})K_0(\sqrt{x\xi m_B b_B})\} \tag{5.7}$$

$$h_4 = K_1(\sqrt{x\xi m_B b_B})\{\theta(b_B - b_p)I_0(\sqrt{x\xi m_B b_B})K_0(\sqrt{x\xi m_B b_B}) + \theta(b_p - b_B)I_0(\sqrt{x\xi m_B b_B})K_0(\sqrt{x\xi m_B b_B})\} \tag{5.8}$$
and finally set $r^2 = 0$ to retain terms up to twist-3.

Before ending this section we would like to discuss the singular nature of the strong coupling constant as the scale $\Lambda$ is approached. Close to this scale, it is easy to convince oneself that the perturbation theory should break down and therefore the calculation outlined above makes no sense as this scale is approached. For the reliability of any perturbative calculation, it has to be ensured that most of the contribution to the calculated quantity comes from the perturbative regime. It has been observed for the pion form factor that this is not really true and a large contribution does come from the non-perturbative region, something which makes the predictions highly unreliable. Near this scale, the coupling constant acquires a large value, thereby, rendering the breakdown of the RG improved perturbation theory. The infra-red behaviour of the strong coupling constant is one of the major challenges of modern day physics. However, it has been observed that there is a good phenomenological evidence to believe that close to (and below) the scale $\Lambda$, the coupling constant gets frozen to a value which is not a big number literally (see [24] and references therein). The concept of a frozen coupling constant was proposed long ago by Cornwall [25]. Also, it has been proposed that the gluon propagator should be modified in order to circumvent the problem of singularity in the coupling constant and this has proved a successful phenomenological ansatz. If one adopts this point of view that as one approaches the scale $\Lambda$, the gluon propagator should be appropriately modified (using some regulator, say) and if one also assumes that the coupling constant gets frozen to some fixed value, then, one can try to carry out the analysis with the frozen value of the coupling constant for scales close to or below $\Lambda$. If the contributions are small, then this establishes the validity of the perturbative treatment.

VI. BEHAVIOUR OF FORM FACTORS AND ROUGH NUMERICAL ESTIMATES

The various form factors are read from the expressions obtained above. To numerically study the behaviour of the form factors, the choice of the scale $\mu$ has to be made. We take $\mu = \text{Max}(\sqrt{\alpha_s}m_B, 1/b_B, 1/b_\rho)$. This scale must be greater than $\Lambda$ in order to avoid the singular behaviour of the strong coupling constant. We would like to emphasize again that in this study we do not make any attempt of precisely calculating the form factors but focus on the behaviour of the same as functions of $b_B$ and $b_\rho$ and see whether the perturbative calculations make any sense. The choices for the other parameters are as follows:

$\Lambda = 0.25 \text{ GeV}$ \hspace{0.5cm} $m_B = 5.279 \text{ GeV}$ \hspace{0.5cm} $m_\rho = 0.77 \text{ GeV}$

$f_B = 0.18 \text{ GeV}$ \hspace{0.5cm} $f_\rho = 0.198 \text{ GeV}$ \hspace{0.5cm} $f_\perp = 0.152 \text{ GeV}$

$\omega_B = 0.35 \text{ GeV}$ \hspace{0.5cm} $\omega_\rho = 0.3 \text{ GeV}$

The parameters $\omega_B$ and $\omega_\rho$ should be $O(\Lambda)$ and therefore have been chosen as above. Figures 2 to 5 show the behaviour of various form factors for large recoil i.e. $q^2 \rightarrow 0$ or $\eta \rightarrow 1$. From the figures it is quite evident that the Sudakov and threshold resummation factors are successful in regulating the endpoint behaviour of the form factors. Furthermore, it is not hard to convince oneself by looking at the figures that the perturbative region supplies the
dominant contribution. Also, it can be seen that for intermediate values of $b_B$ and $b_\rho$, there is indeed an enhancement as can be expected and as discussed earlier. The Sudakov suppression is much weaker for the B-meson as is clear from the variation with $b_B$. The reason for the appearance of a turning point while studying the variation with $b_B$ is the fact that the Sudakov suppression is weaker and there is a tendency for the strong coupling constant to blow up as $1/b_B \to \Lambda$. However, if one assumes the phenomenological ansatz of frozen coupling constant as $\Lambda$ is approached, then explicit evaluations show that the contributions from the non-perturbative region are small.

We summarize the numerical values for various form factors with these choices of parameters:

\[
V = 0.43 \quad A_1 = 0.39
\]
\[
A_2 = 1.73 \quad A_3 = 1.92
\]

Very evidently, $V$ and $A_1$ are slightly overestimated while $A_2$ and $A_3$ deviate significantly. We would like to stress upon the fact that in this particular study we have not bothered to carry out a precise numerical analysis which would involve varying various other parameters like $\omega_B$ and $\omega_\rho$ and then finally concluding which choices give the best results. It is straightforward to note that slightly higher values for both these parameters would have given more suppression. This has been the case with \cite{10} and \cite{21}. The best value which the authors in \cite{10} use for $\omega_B$ was obtained by comparing the pQCD results to the pion form factor with the use of single wave function for the B-meson. However, when the other pieces are also included, this value is bound to change.

In \cite{21}, it has been remarked that the contribution from the $\Delta(l)$ term is roughly $20\%$. We agree with the general arguments and conclusions and reiterate that this term is crucial. Also, the $\bar{\Psi}_B(\xi)$ term, though is sub-leading, but can still change the results when incorporated by at least a few percent. Furthermore, setting the two B-meson wave functions same is equivalent to setting the $\bar{\Psi}_B(\xi)$ term to zero and also the $\Delta(l)$ term to zero. Therefore, the use of a single wave function for the B-meson does more than expected and therefore introduces more uncertainty than thought of. We attribute the reason of large deviations in $A_2$ and $A_3$ to the presence of not really insignificant $\bar{\Psi}_B(\xi)$ and $\Delta(l)$ terms. However, as mentioned earlier, the power suppressed terms that would arise due to different choices of gauges for defining different quantities have not been included and the relative sign of the same can be crucial in a complete and accurate numerical analysis.

\section{Conclusions and Summary}

We have studied the $B \to \rho$ form factors within the perturbative QCD approach including the twist-3 contributions to the DAs. For the B-meson, we employed a model for the wave functions that satisfies the relevant equations of motion and passes through this level of criticism. The use of single wave function has been criticised and we avoid this aspect by retaining the so called sub-leading pieces, which turn out to be not so sub-leading and insignificant. The appropriate Sudakov and threshold resummation functions have been included. With these ingredients the results in figures clearly are very encouraging and seem to suggest that the pQCD calculation of the various form factors is reliable and valid in the sense that dominant contribution clearly appears to stem from the perturbative regime.
However, since the aim was not a very accurate determination of the numerical values, we have overestimated, at least two of the form factors. But we believe that a careful choice of various parameters can lead to a consistent determination of the same. Also, there can be slight differences once a complete and precise jet function is employed in the calculations rather than the phenomenologically motivated function that has been used here.

Encouraged by the results obtained, and mainly the fact that the endpoint behaviour is actually regulated and the contribution from the perturbative regime is the dominant one, we would like to propose the following as a working hypothesis for such calculations:

* Use of two wave functions for the B-meson ie. retaining the generally thought sub-leading terms.
* Including proper Sudakov and threshold resummation
* To use, as a phenomenological ansatz, the notion of a frozen coupling constant for scales very near or below $\Lambda_{QCD}$.

The precise form of the B-meson wave functions is not known but we believe that the results will only improve with the advancement of our understanding of them. The results of this study, particularly the behaviour of the form factors with $b_B$ and $b_\rho$ are very encouraging and require a detailed and careful numerical scrutiny.

[1] A. Ali, R. Fleischer, H-n. Li, hep-ph/0312303; R. Fleischer, hep-ph/0405091; H-n. Li, hep-ph/0303116.
[2] M. Neubert, Phys. Rept. 245, 260 (1994).
[3] M. Bauer, B. Stech and M. Wirbel, Z. Phys. C34, 103 (1987); M. Bauer, B. Stech and M. Wirbel, Z. Phys C29, 637 (1985).
[4] H. Y. Cheng, H-n. Li and K. C. Yang, Phys. Rev. D60, 094005 (1999).
[5] T. W. Yeh and H-n. Li, Phys. Rev. D56, 1615 (1997).
[6] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Phys. Rev. Lett. 83, 1914 (1999); M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Nucl. Phys. B591, 313 (2000).
[7] G. P. Leepage and S. J. Brodsky, Phys. Rev. 22, 2157 (1980); A. V. Efremov and A. V. Radyushkin, Theor. Math. Phys. 42, 97 (1980); A. V. Efremov and A. V. Radyushkin, Phys. Lett. B94, 245 (1980); A. Duncan and A. H. Mueller, Phys. Lett. B90, 159 (1980).
[8] A. Szczepaniak, E. M. Henley and S. J. Brodsky, Phys. Lett. B243, 287 (1990).
[9] D. S. Hwang and B-H. Lee, Eur. Phys. J. C6, 663 (1999).
[10] T. Kurimoto, H-n. Li and A. I. Sanda, Phys. Rev. D65, 014007 (2001).
[11] J. C. Collins, D. E. Soper and G. Sterman in Perturbative Quantum Chromodynamics, ed. A. H. Mueller, World Scientific (1989).
[12] N. Isgur and C. H. Llewellyn Smith, Phys. Rev. Lett. 52, 1080 (1984); N. Isgur and C. H. Llewellyn Smith, Phys. Lett. B217, 535 (1989); N. Isgur and C. H. Llewellyn Smith, Nucl. Phys. B381, 129 (1992); A. V. Radyushkin, Nucl. Phys. A532, 141 (1991).
[13] H-n. Li and G. Sterman, Nucl. Phys. B381, 129 (1992).
[14] P. Ball and V. M. Braun, Phys. Rev. D55, 5561 (1997).
[15] J. Botts and G. Sterman, Nucl. Phys. B325, 62 (1989).
[16] H-n. Li, Phys. Rev. D66, 094010 (2000).
[17] S. Descotes-Genon and C. T. Sachrajda, Nucl. Phys. B625, 239 (2002).
[18] S. J. Brodsky and G. P. Lepage in Perturbative Quantum Chromodynamics, ed. A. H. Mueller, World Scientific (1989).
[19] P. Ball and V. M. Braun, hep-ph/9808229.
[20] M. Beneke and T. Feldmann, Nucl. Phys. B592, 3 (2001).
[21] Z.-T. Wei and M.-Z. Yang, Nucl. Phys. B624, 263 (2002).
[22] H-n. Li and H. Yu, Phys. Rev. D53, 2480 (1996).
[23] H-n. Li, Phys. Rev. D52, 3958 (1994).
[24] A. C. Aguilar, A. Mihara and A. A. Natale, Phys. Rev. D65, 054011 (2002).
[25] J. M. Cornwall, Phys. Rev. D26, 1453 (1982).
[26] P. Ball and V. M. Braun, Phys. Rev. D58, 094016 (1998).
FIG. 2: Variation of $V$ with $b_B$ (left) and $b_\rho$ (right).

FIG. 3: Variation of $A_1$ with $b_B$ (left) and $b_\rho$ (right).
FIG. 4: Variation of $\tilde{A}_2$ with $b_B$ (left) and $b_\rho$ (right).

FIG. 5: Variation of $\tilde{A}_3$ with $b_B$ (left) and $b_\rho$ (right).