NUMBER OF CLIQUES IN GRAPHS WITH A FORBIDDEN SUBDIVISION∗

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Abstract. We prove that for all positive integers \( t \), every \( n \)-vertex graph with no \( K_t \)-subdivision has at most \( 2^{50} t n \) cliques. We also prove that asymptotically, such graphs contain at most \( 2^t n \) cliques, where \( o(1) \) tends to zero as \( t \) tends to infinity. This strongly answers a question of Wood that asks whether the number of cliques in \( n \)-vertex graphs with no \( K_t \)-minor is at most \( 2^c t n \) for some constant \( c \).

Key words. minor, topological minor, subdivision, clique

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1. Introduction. A clique of a graph is a set of pairwise adjacent vertices. A graph \( H \) is a minor of a graph \( G \) if \( H \) can be formed from \( G \) by deleting edges and vertices and by contracting edges. An \( H \)-subdivision of a graph \( G \) is a subgraph of \( G \) that can be formed from an isomorphic copy of \( H \) by replacing edges with vertex-disjoint (nontrivial) paths. Trivially, if a graph has an \( H \)-subdivision, then it has an \( H \)-minor. But the converse is not true in general.

The problem of determining the maximum number of edges in graphs with no \( K_t \)-minor or no \( K_t \)-subdivision is a well-studied problem in extremal graph theory: Kostochka [8] and Thomason [14] proved that graphs with no \( K_t \)-minor have average degree at most \( c t \sqrt{\ln t} \), and Bollobás and Thomason [1] and, independently, Komlós and Szemerédi [7], proved that graphs with no \( K_t \)-subdivision have average degree at most \( c t^2 \), where \( c \) and \( c' \) are some absolute constants not depending on \( t \) (in fact, a theorem of Thomas and Wollan [13] can be used to show that \( c' \leq 10 \); see [2, Theorem 7.2.1]). A graph is \( d \)-degenerate if all its induced subgraphs contain a vertex of degree at most \( d \). The results mentioned above straightforwardly imply that graphs with no \( K_t \)-minor are \( c t \sqrt{\ln t} \)-degenerate, and graphs with no \( K_t \)-subdivision are \( c t^2 \)-degenerate.

We study a related problem of determining the maximum number of cliques in graphs with no \( K_t \)-minor or no \( K_t \)-subdivision. Our work can be viewed as an extension of Zykov’s theorem [16] that establishes a bound on the number of cliques in graphs with no \( K_t \) subgraphs. For planar graphs, Papadimitriou and Yannakakis [10] and Storch [12] proved a linear upper bound and, finally, Wood [15] determined the exact upper bound \( 8n - 16 \) for \( n \)-vertex planar graphs. Dujmović et al. [3] generalized this result to graphs on surfaces.

For graph with no \( K_t \)-minors, Reed and Wood [11] and Norine et al. [9] obtained an upper bound on the number of cliques by using the fact that an \( n \)-vertex \( d \)-degenerate graph with \( n \geq d \) has at most \( 2^d (n - d + 1) \) cliques. By the results

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mentioned above, this implies that graphs with no $K_t$-minor have at most $2^{c t \sqrt{\ln t}} n$ cliques and that graphs with no $K_t$-subdivision have at most $2^{10t^2} n$ cliques. Wood [15] then asked whether there exists a constant $c$ for which every $n$-vertex graph with no $K_t$-minor has at most $2^c n$ cliques. If true, then the bound would be best possible up to the constant $c$ in the exponent, since the $(t - 2)$th power of a path on $n$ vertices has no $K_t$-minor and contains $2^{t^2}(n - t + 3)$ cliques (including the empty set). See section 3 for an alternative construction.

The results of Wood were later improved to $2^{ct \ln \ln \ln t} n$ (for graphs with no $K_t$-minor) and $2^{ct \ln t}$ (for graphs with no $K_t$-subdivision) by Fomin, Oum, and Thilikos [4]. In this paper, we settle Wood’s question by proving the bound not only for graphs with no $K_t$-minor, but also for graphs with no $K_t$-subdivision.

**Theorem 1.1.** For all positive integers $t$, every $n$-vertex graph with no $K_t$-subdivision has at most $2^{500t} n$ cliques.

Our proof also implies that such graphs have at most $2^{(5+o(1))t} n$ cliques.

2. Proof of the theorem. One can enumerate all cliques of a given graph by choosing vertices one at a time and recursively exploring its neighbors. To be more precise, first choose a vertex $v_1$ of minimum degree and explore all cliques that contain $v_1$ by recursively applying the algorithm to the graph induced on the set $N(v_1)$. Once all cliques containing $v_1$ have been explored, remove $v_1$ from the graph, choose a vertex $v_2$ of minimum degree in the remaining graph, and repeat the algorithm. The algorithm enumerates each clique of the graph exactly once, since the $i$th step of the algorithm enumerates all cliques that contain $v_i$ but do not contain any vertex from $\{v_1, \ldots, v_{i-1}\}$ (where $v_j$ is the vertex chosen at the $j$th step). We emphasize that we always choose the vertex of minimum degree within the remaining graph since this choice blends particularly well with sparse graphs. This algorithm has been used in various previous works (see, e.g., [6]).

This simple algorithm immediately implies a reasonable result. Since $K_t$-minor-free graphs are $c t \sqrt{\ln t}$-degenerate, the vertex $v_i$ chosen at the $i$th step of the algorithm above will have degree at most $c t \sqrt{\ln t}$ in the remaining graph at that time. Since the neighborhood of $v_i$ is $K_{t-1}$-minor free, the number of cliques added at the $i$th step is at most

$$
\sum_{j=0}^{t-2} \left( \binom{c t \sqrt{\ln t}}{j} \right) \leq t (c t \sqrt{\ln t})^j \leq 2^{c t \ln \ln t},
$$

proving that $n$-vertex graphs with no $K_t$-minor have at most $2^{c t \ln \ln t} n$ cliques. One can similarly show that $n$-vertex graphs with no $K_t$-subdivision have at most $2^{c t \ln \ln t} n$ cliques by using the following theorem, which was mentioned in the introduction. Both of these bounds on the number of cliques were first proved in [4] using a different argument.  

**Theorem 2.1 (see [2, Theorem 7.2.1]).** For all $t \geq 1$, every graph of average degree at least $10t^2$ contains a $K_t$-subdivision.

In this section, we show how a more detailed analysis of the algorithm gives an improved bound on the number of cliques for graphs with no $K_t$-subdivision.

2.1. Enumerating cliques. The algorithm introduced above provides a natural tree structure, called the *clique search tree*, to the cliques of a given graph $G = (V, E)$,
where each node of the tree corresponds to one step of exploration in the algorithm and, at the same time, one clique of the graph. Formally, the clique search tree is a labeled tree defined as follows (since we are simultaneously considering two graphs, we denote the vertices of $G$ by $v, w, \ldots$, while we denote the vertices of the tree by $a, b, \ldots$ and refer to them as nodes):

1. Start with a tree having a single node $a_0$ as a root node with label $L_{a_0} = V$.
2. Choose a leaf node $a$ of the current tree with $L_a \neq \emptyset$ and let $L := L_a$.
   1. Choose a vertex $v \in L$ of minimum degree in $G[L]$.
   2. Add a child node $b$ to $a$ in the tree and label it by the set $L_b = L \cap N(v)$.
   3. Define $L \leftarrow L - \{v\}$.
   4. Repeat steps 2-1, 2-2, and 2-3 until $L = \emptyset$.
3. Repeat step 2 until all leaves have label $\emptyset$.

Denote this tree as $T_G$. Thus $T_G$ is a rooted labeled tree, where each node $a$ is labeled by some set $L_a \subseteq V(G)$ (distinct nodes might receive the same label). Note that the number of cliques in $G$ is exactly $|V(T_G)|$, since there exists a one-to-one correspondence between nodes of $T_G$ and cliques of $G$. (The root node of $T_G$ corresponds to the empty set, which is also a clique by definition.) Hence, to count cliques of $G$, it suffices to count nodes of $T_G$.

The following proposition lists some useful properties of the tree $T_G$. A subtree $T'$ of $T_G$ is a rooted subtree if $T'$ contains the root node of $T_G$. The boundary nodes of a rooted subtree $T'$ is the set of nodes of $T'$ that are adjacent in $T_G$ to a node not in $T'$.

**Proposition 2.2.** If $G$ is a graph with no clique of size $t$, then the clique search tree $T_G$ has the following properties.

1. The number of nodes of $T_G$ is equal to the number of cliques of $G$. Moreover, for all nonnegative integers $\ell$, the number of nodes of $T_G$ that are at distance exactly $\ell$ from the root node is equal to the number of cliques of $G$ of size $\ell$.
2. For each node $a$ of $T_G$, the tree $T_G[L_a]$ is isomorphic (as a rooted labeled tree) to the subtree of $T_G$ induced on $a$ and its descendants.
3. If $b$ is a descendant of $a$, then $N_b \subseteq L_a$.
4. Let $T'$ be a rooted subtree of $T_G$ whose boundary nodes are all labeled by sets of size at most $m$. Then

$$|V(T_G)| \leq |V(T')| \cdot \sum_{i=0}^{t-1} \binom{m}{i} \leq |V(T')|2^m.$$  

**Proof.** Properties (i), (ii), and (iii) follow from the definition and the discussions given above. To prove property (iv), suppose that we are given a tree $T' \subseteq T_G$. Since $T'$ is a rooted subtree, each node in $T_G$ either is in $T'$ or is a descendant of a boundary node of $T'$. Furthermore, by properties (i) and (ii), each boundary node of $T'$ has at most $\sum_{i=1}^{t-1} \binom{m}{i}$ descendants in $T_G$.

$$|V(T_G)| \leq |V(T')| + \sum_{a: \text{boundary of } T'} \sum_{i=1}^{t-1} \binom{m}{i} \leq |V(T')| \cdot \sum_{i=0}^{t-1} \binom{m}{i}. \quad \Box$$

**2.2. Graphs of large minimum degree.** The simple argument given in the beginning of this section that proves the bound $2^{c't \ln \ln t \ln n}$ for $K_t$-minor-free graphs is equivalent to applying Proposition 2.2(iv) to the subtree induced on the root of $T_G$.
and its children. Hence, to improve on this bound, it would be useful to find a small rooted subtree $T'$ of $T_G$ whose boundary nodes are all labeled by small sets. When does such a subtree exist?

A graph $G$ is called $(\beta, N)$-locally sparse if every set $X$ of at least $N$ vertices has a vertex $v \in X$ of degree at most $\beta |X|$ in $G[X]$. This concept was first introduced by Kleitman and Winston [6] in their study of the number of $C_4$-free graphs on $n$ vertices, and has been successfully applied to several problems in extremal combinatorics.

In the following two lemmas, we utilize the concept of locally sparse graphs to handle a subcase of our theorem when the graph is small and dense. This subcase turns out to be an important ingredient in the proof of general cases.

**Lemma 2.3.** Let $G$ be an $m$-vertex graph with no $K_t$-subdivision. If $G$ has minimum degree at least $\frac{9}{10}m$, then $m \leq \max\{\frac{20}{11}t^2, \frac{20}{11}t\}$ and $G$ is $(1 - \frac{m}{20}, \frac{20}{11}t)$-locally sparse.

**Proof.** We may assume that $m \geq \frac{20}{11}t$, since otherwise the lemma is vacuously true. Let $X$ be a subset of vertices of size $|X| \geq \frac{20}{11}t$ and suppose that $G[X]$ has minimum degree at least $(1 - \frac{m}{20})|X|$ (note that this quantity may be negative). If we sum $e(Y)$, the number of edges in $Y$, over all $t$-element subsets $Y$ of $X$, then each edge in $X$ is counted exactly $\binom{|X| - 2}{t - 2}$ times. Therefore, there exists a $t$-element subset $Y$ of $X$ such that

$$e(Y) \geq \frac{\binom{|X|-2}{t-2}e(X)}{\binom{|X|}{t}} = \left(\frac{t}{2}\right)^2 \frac{e(X)}{\binom{|X|}{t}} > \left(\frac{t}{2}\right)^2 \left(1 - \frac{m}{21^2}\right) > \left(\frac{t}{2}\right)^2 - \frac{m}{4}.$$

Since every vertex of $G$ has degree at least $\frac{9}{10}m$, every pair of vertices of $G$ has at least $\frac{4}{5}m$ common neighbors. For each nonedge $e = \{v, v'\}$ in $Y$, we can greedily find a common neighbor $w_e \in V(G) \setminus Y$ of $v$ and $v'$ such that all chosen $w_e$ for all nonedges $e$ are distinct, because $Y$ has at most $\frac{4}{5}m$ nonedges in $Y$ and

$$\frac{4}{5}m - \left(t + \frac{m}{4}\right) = \frac{11}{20}m - t \geq 0.$$

Then $Y$, together with all chosen $w_e$, induces a $K_t$-subdivision in $G$, contradicting our assumption. Therefore, $G$ is $(1 - \frac{m}{20}, \frac{20}{11}t)$-locally sparse. Since $G$ has minimum degree at least $\frac{9}{10}m$, if $m \geq \frac{20}{11}t$, then this implies

$$\frac{9}{10}m \leq \left(1 - \frac{m}{21^2}\right)m,$$

from which $m \leq \frac{\beta}{\alpha}$ follows. □

**Lemma 2.4.** Let $G$ be an $m$-vertex graph with no $K_t$-subdivision with $m \leq \frac{t^2}{4}$. If $G$ is $(1 - \frac{m}{20}, \frac{20}{11}t)$-locally sparse, then $G$ contains less than $2^t$ cliques.

**Proof.** If $m < 5t$, then trivially $G$ contains less than $2^t$ cliques, and therefore we may assume that $m \geq 5t$. Let $T_G$ be the clique search tree of $G$, and let $T'$ be the

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2It is more common to define a $(\beta, N)$-locally sparse graph as a graph satisfying the following slightly stronger property: each subset $X$ of size at least $N$ contains at most $\beta |X|^2$ edges.
As has cardinality at most $10^{2\ln \frac{m}{t}}$ from the root. Then by the local sparsity condition, the label set of each boundary node of $T'$ has cardinality less than $\max\{20t, \frac{102t}{9}\} = \frac{20t}{11}$, because

$$
(1 - \frac{m}{2t^2})^{2\ln \frac{m}{t}} m < e^{-\ln \frac{m}{t}m} \frac{10t}{9} = \frac{10t}{9},
$$

where the last inequality follows from $m \leq \frac{t^2}{9}$. By Proposition 2.2(i), the number of nodes of $T'$ is at most the number of cliques of size at most $2\frac{2\ln \frac{m}{t}}{m}$, and so we have the following inequality:

$$
|V(T')| \leq \sum_{i=0}^{\lfloor \frac{2\ln \frac{m}{t}}{m} \rfloor} \binom{m}{i}.
$$

As $\ln \frac{m}{t} \leq \frac{1}{2}$ for all $x > 0$, $2\frac{2\ln \frac{m}{t}}{m} \leq \frac{2e}{3} < m$. Since $\sum_{i=0}^{\lfloor \frac{m}{i} \rfloor} \binom{m}{i} \leq \sum_{i=0}^{\lfloor \frac{m}{i} \rfloor} \binom{m}{i} \left(\frac{m}{i}\right)^{i-k} \leq \left(\frac{e}{2}\right)^{\frac{m}{i}}$ for all $k \leq m$, we have

$$
|V(T')| \leq \left(\frac{em}{2^2m \ln \frac{m}{t}}\right)^{2\ln \frac{m}{t}} \leq \left(\frac{m^2}{t^2}\right)^{2\ln \frac{m}{t}} e^{-\frac{m^2}{t^2}} < e^{-\frac{m^2}{t^2}},
$$

because $2 \ln \frac{m}{t} \geq 2 \ln 5 > e$. As $\ln^2 \frac{m}{t} \leq \frac{4}{e^2}$ for all $x > 1$,

$$
|V(T')| \leq e^{-\frac{m^2}{t^2}} < 2^{\frac{16t}{t^2}}.
$$

Since the label set of each boundary node of $T'$ has cardinality less than $\frac{20t}{11}$, by Proposition 2.2(iv),

$$
|V(T_G)| \leq |V(T')| \cdot 2^{\frac{20t}{11}} < |V(T')|2^{1.82t}.
$$

It follows that $G$ has at most $2^{(3.13+1.82)t} < 2^{5t}$ cliques.  \( \square \)

### 2.3. Finishing the proof.

In this subsection, we prove Theorem 1.1.

We may assume that $t \geq 4$, because otherwise $G$ is a forest and contains at most $2n$ cliques. Given a graph $G$ with no $K_t$-subdivision, let $T_G$ be its clique search tree. By Theorem 2.1, $G$ is $10t^2$-degenerate. Therefore every nonroot node has a label set of cardinality at most $10^2t^2$, and thus has at most $10^2t^2$ children.

We construct a rooted subtree $T'$ of the clique search tree $T_G$ according to the following recursive rule. First, take the root node. Then for a node $a$ in $T'$, take its child $a'$ to be in $T'$ if $\sqrt{10}t \leq |L_{a'}| < \frac{5}{10^2}|L_a|$. Since the label set of every nonroot node has cardinality at most $10^2t^2$ and the cardinality of the label sets decreases by a factor of at least $\frac{9}{10^2}$ at each level, we see that $T'$ is a tree of height at most $1 + \frac{\ln(10t^2)}{2\ln(10^{10})}$. Since the root of $T_G$ has exactly $n$ children, the number of nodes of $T'$ satisfies

$$
|V(T')| \leq n \cdot (10t^2)^{\frac{\ln(10t^2)}{2\ln(10^{10})}} = n \cdot 2^{\frac{\ln(10t^2)^2}{6\ln(10^{10})}}
$$

$$
\leq n \cdot 2^{\frac{\ln^2(160)}{6\ln(10^{10})}} < 2^{44.1t}n,
$$

where the second-to-last inequality follows from the fact that $t \geq 4$ and $\frac{\ln^2(160)}{x}$ is decreasing for $x > \frac{e^2}{\sqrt{10}}$.

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Further note that for each boundary node \( a \) of \( T' \), we have either that \( |L_a| \leq \sqrt{10t} \) or that there exists a child \( a' \) of \( a \) for which \( |L_{a'}| \geq \frac{1}{\beta} |L_a| \). In the first case, the number of descendants of \( a \) in \( T_G \) is clearly at most \( 2\sqrt{10t} \), which is less than \( 2^{5t} \). In the latter case, let \( v_1, v_2, \ldots, v_{|L_a|} \) be the vertices in \( L_a \) listed in the order that they were chosen by the algorithm, and let \( a_1, a_2, \ldots, a_{|L_a|} \) be the corresponding nodes of \( T_G \). Suppose that \( i \) is the minimum index for which \( |L_{a_i}| \geq \frac{1}{\beta} |L_a| \). Define \( X_a = \{ v_i, v_{i+1}, \ldots, v_{|L_a|} \} \), and let \( G_a = G[X_a] \). Notice that the clique search tree \( T_{G_a} \) is isomorphic to the subtree of \( T_G \) induced on \( a, a_1, \ldots, a_{|L_a|} \), and the descendants of \( a, a_1, a_2, \ldots, a_{|L_a|} \) in \( T_G \). Hence, the total number of nodes of \( T_G \) is at most

\[
|V(T_G)| \leq |V(T')| + \max_{a: \text{boundary of } T'} |V(T_{G_a})|.
\]

By the definition of our algorithm, the vertex \( v_i \) is a vertex of minimum degree in the graph \( G_a \), and hence \( G_a \) has minimum degree at least \( |L_a| \geq \frac{1}{\beta}|L_a| \geq \frac{1}{\beta} |X_a| \).

By Lemma 2.3, \( G_a \) is \((1 - |X_a|/\beta |t|)\)-locally sparse and \( |X_a| \leq \max\{\frac{\beta}{11}, \frac{1}{\beta^2}\} \). If \( |X_a| \leq \frac{1}{\beta^2} \), then \( G_a \) satisfies the conditions of Lemma 2.4, and therefore the tree \( T_{G_a} \) has at most \( 2^{5t} \) nodes. Otherwise, \( |X_a| \leq \frac{\beta}{11} |t| \), and by Proposition 2.2(i), the tree \( T_{G_a} \) has at most \( 2^{10t} \) nodes. In either case, we have

\[
|V(T_{G_a})| \leq 2^{5t}.
\]

By substituting this bound and (2.1) into (2.2), we obtain the desired inequality

\[
|V(T_G)| \leq 2^{5t}|V(T')| < 2^{(5+44.1)t}n < 2^{50t}n.
\]

3. Remarks. In this paper, we proved Theorem 1.1, asserting that every \( n \)-vertex graph with no \( K_t \)-subdivision has at least \( 2^{50t}n \) cliques. In fact, our proof shows that such graphs have at most \( 2^{(5+o(1))t}n \) cliques, since (2.1) could have been replaced by the inequality \( |V(T')| \leq 2^{o(t)}n \).

It remains to determine the best possible constants \( c \) and \( C \) for which the number of cliques in an \( n \)-vertex graph with no \( K_t \)-subdivision is at most \( 2^{(c+o(1))t}n \) and at most \( 2^{ct} n \). We showed that \( c \leq 5 \) and \( C \leq 50 \), while, as mentioned in the introduction, the \((t - 2)\)th power of a path shows that \( c \geq 1 \). Lemma 2.3 can be written as follows: if \( G \) is an \( m \)-vertex \( K_t \)-subdivision-free graph of minimum degree at least \( (1 - \alpha)m \), then \( m \leq \max\{\frac{t}{1 - 2\alpha - \beta/2}, \frac{\beta}{\beta^2}\} \) and \( G \) is \((1 - \frac{m}{t^2}, 1 - (2\alpha - \beta/2))\)-locally sparse. By taking \( \alpha = 0.01 \) and \( \beta = 0.65 \) and following an almost identical proof, we can obtain \( c < 4 \). Similarly, by taking \( \alpha = 0.35 \) and \( \beta = 0.4 \), we can obtain \( C < 20 \).

(In the modified proof, when we compute an upper bound on the number of cliques in a graph on \( \gamma t \) vertices, we may use the inequality \( \sum_{i=0}^{\gamma t} (\gamma t)^i \leq (\gamma e)^t \) instead of \( 2^{\gamma t} \) to achieve a better bound depending on \( \gamma \).)

Wood [15] showed that \( c \geq \frac{4}{3} \log_3 3 \approx 1.057 \) because the complete \( k \)-partite graph \( K_{2,2,\ldots,2} \) contains \( 3^k \) cliques and has no \( K_t \)-subdivision for \( t > \lfloor 3k/2 \rfloor \).

We remark that Kawarabayashi and Wood [5] proved that \( n \)-vertex graphs with no odd-\( K_t \)-minor have at most \( O(n^2) \) cliques and unlike the case of graph minors, \( n^2 \) cannot be improved because \( K_{n,n} \) has no odd-\( K_3 \)-minor.

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REFERENCES

[1] B. Bollobás and A. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs, European J. Combin., 19 (1998), pp. 883–887.
[2] R. Diestel, Graph Theory, 4th ed., Grad. Texts Math. 173, Springer, Heidelberg, 2010.
[3] V. Dujmović, G. Fijavž, G. Joret, T. Sulanke, and D. R. Wood, On the maximum number of cliques in a graph embedded in a surface, European J. Combin., 32 (2011), pp. 1244–1252.
[4] F. V. Fomin, S. Oum, and D. M. Thilikos, Rank-width and tree-width of \( H \)-minor-free graphs, European J. Combin., 31 (2010), pp. 1617–1628.
[5] K.-I. Kawarabayashi and D. R. Wood, Cliques in odd-minor-free graphs, in Proceedings of the Eighteenth Computing: The Australasian Theory Symposium - Volume 128 (CATS '12), Australian Computer Society, Inc., Darlinghurst, Australia, 2012, pp. 133–138.
[6] D. J. Kleitman and K. J. Winston, On the number of graphs without 4-cycles, Discrete Math., 41 (1982), pp. 167–172.
[7] J. Komlós and E. Szemerédi, Topological cliques in graphs II, Combin. Probab. Comput., 5 (1996), pp. 79–90.
[8] A. V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, Metody Diskret. Analiz., no. 38, 1982, pp. 37–58 (in Russian).
[9] S. Norine, P. Seymour, R. Thomas, and P. Wollan, Proper minor-closed families are small, J. Combin. Theory Ser. B, 96 (2006), pp. 754–757.
[10] C. H. Papadimitriou and M. Yannakakis, The clique problem for planar graphs, Inform. Process. Lett., 13 (1981), pp. 131–133.
[11] B. Reed and D. R. Wood, A linear-time algorithm to find a separator in a graph excluding a minor, ACM Trans. Algorithms, 5 (2009), 39.
[12] T. Storch, How randomized search heuristics find maximum cliques in planar graphs, in Proceedings of the 8th Annual Conference on Genetic and Evolutionary Computation, ACM, New York, 2006, pp. 567–574.
[13] R. Thomas and P. Wollan, An improved linear edge bound for graph linkages, European J. Combin., 26 (2005), pp. 309–324.
[14] A. Thomason, An extremal function for contractions of graphs, Math. Proc. Cambridge Philos. Soc., 95 (1984), pp. 261–265.
[15] D. R. Wood, On the maximum number of cliques in a graph, Graphs Combin., 23 (2007), pp. 337–352.
[16] A. A. Zykov, On some properties of linear complexes, Mat. Sbornik N.S., 24 (1949), pp. 163–188 (in Russian); Amer. Math. Soc. Translation, no. 79, 1952 (in English).