Renormalized QCD-inspired model for the pion and mesons

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1. Introduction

We address to the effective mass operator equation of the $\uparrow\downarrow$-model for the $q\bar{q}$ Light-Front Fock state component of the meson with mass $M$:

$$[M^2 - 4m^2 - 4k^2] \varphi(\vec{k}) = \int d\vec{p} U(\vec{k}, \vec{p}) \varphi(\vec{p}), \quad (1)$$

with the kernel

$$U(\vec{k}, \vec{p}) = -\frac{4\alpha}{3\pi^2m} \left[ \frac{2m^2}{(\vec{k} - \vec{p})^2} + 1 \right].$$

Equation (1) is mathematically not defined.

The aim of this paper is to give Eq.(1) a physical meaning by renormalization, i.e., by applying to it the “subtraction method”, a renormalization scheme for nonrelativistic quantum mechanics of singular interactions developed earlier [1].

This is an interesting problem because Eq.(1) as proposed in Ref.[2], the $\uparrow\downarrow$-model, is an effective Hamiltonian derived from Quantum Chromodynamics, meant to describe the lowest Fock-state component of the Light-Front meson wavefunction. It has been applied with reasonable success to the low-lying pseudo-scalar and vector mesons, by using a different renormalization scheme namely by regularization and subsequent renormalization [2]. The Coulomb-like effective potential is $V$ and the hyperfine singular interaction is $V^A$, which in the non-relativistic limit is the Dirac-delta. The matrix elements of these operators in momentum representation are given by:

$$<\vec{k}|V|\vec{p}> = -\frac{4m_s}{3\pi^2} <\vec{k}|\chi> \frac{\alpha}{Q^2} <\chi|\vec{p}>,$$

$$<\vec{k}|V^A|\vec{p}> = <\vec{k}|\chi> \frac{\lambda}{m_r} <\chi|\vec{p}>.$$

We are thus in the unique position to compare two drastically different schemes, both conceptually and numerically, and verify that they agree. This strong statement stays at the very basis of renormalization ideas, that no matter the intermediate steps one performs to mathematically define the initial equation (1), after renormalization all them produce the same physics.

2. Notation

For the purpose of presenting the subtraction method of Ref.[1], we introduce the notation below, and allow as well different quark masses and the relativistic phase space.

$$(M_0^2 + V + V^A)|\varphi> = M^2|\varphi>,$$

where the free mass operator of the quarks with masses $m_1$ and $m_2$ is $M_0 = E_1 + E_2$. The individual energies are $E_i = \sqrt{m_i^2 + k^2}$ ($i=1,2$) and $k \equiv |\vec{k}|$. The Coulomb-like effective potential is $V$ and the hyperfine singular interaction is $V^A$, which in the non-relativistic limit is the Dirac-delta. The matrix elements of these operators in the relativistic phase space are given by:

$$<\vec{k}|V|\vec{p}> = -\frac{4m_s}{3\pi^2} <\vec{k}|\chi> \frac{\alpha}{Q^2} <\chi|\vec{p}>.$$

where the total and reduced mass are denoted by $m_s = m_1 + m_2$ and $m_r = m_1m_2/m_s$, respectively.
\( \lambda \) is the bare strength of the Dirac-delta hyperfine interaction. The mean four-momentum transfer is taken as \( Q^2 = (\vec{k} - \vec{p})^2 \). The phase-space dimensionless function is
\[
\frac{1}{A(k)} = m_r \frac{E_1 + E_2}{E_1 E_2}.
\]
The form-factor is \( \chi(k) = \langle \vec{k} | \chi >= 1/\sqrt{A(k)} \).

### 3. Example: Dirac-Delta potential

Many authors in the past have renormalized the Schrödinger equation with contact interaction (see e.g. [1]). Here we want just to supply the essence of the “subtraction method” of Ref.[1]. Let us solve Eq.(1) with only
\[
U(\vec{k}, \vec{p}) = \lambda,
\]
which, as said, makes Eq.(1) not well defined.

The scattering amplitude for the potential (1) is the geometrical series
\[
\tau(M^2) = [\lambda^{-1} - I(M^2)]^{-1},
\]
solution of the scattering equation
\[
T(M^2) = V + V G_0^{(+)}(M^2) T(M^2),
\]
for the scattering state of mass \( M \). The Green’s function of the free mass operator equation with outgoing wave boundary condition is
\[
G_0^{(+)}(M^2) = [M^2 - M_0^2 + i\varepsilon]^{-1}.
\]
The function
\[
I(M^2) = \int d\vec{k} \frac{1}{M^2 - 4m^2 - 4k^2 + i\varepsilon}
\]
diverges linearly! This is the mathematical problem in Eq.(1).

How to give meaning to \( \tau(M^2) \)? We use the renormalization idea. Suppose \( \tau(\mu^2) \) is known from experiment, then we rewrite \( \tau(M^2) \) using this piece of data:
\[
\tau(M^2) = \left[ \tau^{-1}(\mu^2) + I(\mu^2) - I(M^2) \right]^{-1}
\]
and know the subtraction of the divergence appears! A closer look to
\[
I(\mu^2) - I(M^2) = (M^2 - \mu^2) \int d\vec{k}
\]
\[
\frac{1}{\lambda} \left( \mu^2 - 4m^2 - 4k^2 + i\varepsilon \right) (M^2 - 4m^2 - 4k^2 + i\varepsilon)
\]
shows that it is finite with \( \mu \) being the subtraction point! This is the essence of the “subtraction method” of Ref.[1].

The renormalized form of the potential \( V_\lambda^{\delta} \), has the bare strength written as a function of the renormalized one \( \lambda_R(\mu^2) = \tau(\mu^2) \)
\[
\lambda = \frac{1}{1 + \lambda_R(\mu^2) I(\mu^2)} \lambda_R(\mu^2),
\]
in which the physical input and the counter terms that subtract all the infinities in the scattering matrix at the mass scale \( \mu \) are present [1].

### 4. Renormalized Model

The “subtraction method” exemplified in sec.3 is applied to the effective model defined by the mass operator of Eq.(2). The scattering matrix comes from the solution of the scattering equation with the renormalized potential [1, 4]
\[
T_R(M^2) = V_R + V_R G_0^{(+)}(M^2) T_R(M^2),
\]
where \( V_R = V + V_\lambda^{\delta} \). In finding the bound state, one could as well diagonalize the mass operator \( M_0^2 + V_R \).

The renormalized Dirac-delta interaction is written formally as below [4]:
\[
V_\lambda^{\delta} = \left[ 1 + T_R^{\delta}(\mu^2) G_0^{(+)}(\mu^2) \right]^{-1} T_R^{\delta}(\mu^2),
\]
where \( T_R^{\delta}(\mu^2) \) is the renormalized T-matrix of the Dirac-delta interaction, with matrix elements given by
\[
< \vec{p} | T_R^{\delta}(\mu^2) | \vec{q} > = \chi(\vec{p}) \lambda_R(\mu^2) \chi(\vec{q}).
\]

The solution of Eq. (11) is [1]:
\[
T_R(M^2) = T^V(M^2) + |F > t_R(M^2) < F|,
\]
where
\[
|F > = \left( 1 + T^V(M^2) G_0^{(+)}(M^2) \right) |\chi >,
\]
\[
< F| = < \chi | \left( G_0^{(+)}(M^2) T^V(M^2) + 1 \right),
\]
\[
t_R^{-1}(M^2) = \lambda_R^{-1}(\mu^2) - G(M) + G_0(\mu),
\]
\[ G(M) = \langle \chi | [M^2 - M_0^2 - V + i\varepsilon]^{-1} |\chi > , \]

and

\[ G_0(M) = \langle \chi | [M^2 - M_0^2 + i\varepsilon]^{-1} |\chi > . \]

The regular potential T-matrix, \( T^V(M^2) \), is the solution of the scattering equation (11) for \( V \).

The scattering equation with the renormalized interaction appears in a subtracted form (12), in which all the divergent momentum integrals are explicitly removed:

\[
T_R^R(M^2) = T_R^R(\mu^2) \left[ 1 + \left( G_0^{(+)}(M^2) - G_0^{(+)}(\mu^2) \right) T_R(M^2) \right]. \tag{12}
\]

For a regular potential Eq.(12) is completely equivalent to the traditional Lippman-Schwinger scattering equation.

We use the renormalization condition that at the pion mass, \( M = m_\pi \), the T-matrix, for \( m_1 = m_2 = m_u = m_d \) has a bound-state pole, where \( t_1^{-1}(m_\pi^2) = 0 \). The choice \( \mu = m_\pi \) implies that \( \lambda_R^{-1}(m_\pi^2) = G(m_\pi) - G_0(m_\pi) \).

The physics described by the theory does not dependent on the arbitrary renormalization point, this imposes \( \frac{d}{d\mu^2} V_R^2 = 0 \) and qualifies the interaction as the fixed-point of this equation. The inhomogeneous term of Eq.(12) runs as the Callan-Symanzik equation

\[
\frac{d}{d\mu^2} T_R(\mu^2) = -T_R(\mu^2) G_0^{(+)}(\mu^2)^2 T_R(\mu^2). \tag{13}
\]

The renormalized T-matrix (11) is invariant under the change of \( \mu \) to \( \mu' \), and thus \( \frac{d}{d\mu^2} t_R(M^2) = 0 \), and the renormalized strength runs as according to \( \lambda_R^{-1}(\mu^2) = \lambda_R^{-1}(\mu^2) + G(\mu') - G(\mu) + G_0(\mu') \).

The bare strength is obtained by equating Eq.(4) to Eq.(10), and using \( \lambda_R(m_\pi^2) \), one finds \( \lambda_{bare} = \frac{m_\pi}{m_\pi} G(m_\pi)^{-1} \), with the reduced mass \( m_\pi = 1/2 \) and \( G(m_\pi) \) calculated for \( m_1 = m_2 = m_u = m_d \). With this, the pole of the T-matrix (11) at the bound-state mass, \( M_b \), is given by

\[
t_1^{-1}(M_b^2) = \frac{m_\pi}{m_\pi} G(m_\pi) - G(M_b) = 0 , \tag{14}
\]

for s-wave states with any quark mass.

5. Comparing Renormalization Schemes

Here we compare the results obtained with the Yukawa form of the smeared Dirac-delta interaction (11), Eq.(15), and the “subtraction method”, Eq.(16), with \( A(k) = 1 \). Using Eq.(17), the pion mass, \( m_\pi \), of 140 MeV and the first excited s-wave state mass, \( m_{\pi}^* \), of 768 MeV were fitted with the parameters \( \alpha = 0.763 \) and \( \eta = 1148\text{MeV} \) see (12), for \( m = 406 \text{MeV} \). The solution of Eq.(14), for

\( m_\pi = 140 \text{MeV} \), shows that \( \alpha = 0.763 \) is \( m_{\pi}^* \) = 766 MeV, in remarkable agreement with the previous result. Both renormalization methods have the same physical inputs. In one set of calculations, \( \alpha \) was varied, with a fixed \( m_\pi = 140 \text{MeV} \). In the other set of calculations, \( m_{\pi}^* \) = 768 MeV was kept fixed. In the model of Eq.(17) the value of \( \eta \) was fitted to the \( m_\pi \) or \( m_{\pi}^* \) for a given \( \alpha \).

In figure 1, the results of \( m_{\pi}^* \) as a function of \( \alpha \) for the two renormalization methods are shown. The agreement between the “subtraction method” and the smeared delta renormalization method is within few percent, which we relate to the rather drastically different methods. The values of \( \eta \) for \( \alpha \) going to zero increase towards infinite, to keep the ground state at the pion mass, while \( m_{\pi}^* \) tends to the scattering threshold at 812 MeV. For \( \alpha \) increasing the values of \( \eta \) decreases to keep \( m_\pi \) fixed, and the \( m_{\pi}^* \) which is Coulomb dominated, has to decrease, as we observe in figure 1. The effect of the relativistic phase-space, Eq.(3), has been studied in Ref.(11) and it is of the order of only few percents.

The results for \( m_{\pi} \) as a function of \( \alpha \) for \( m_{\pi}^* = 768 \text{MeV} \), are presented in figure 2. The threshold for zero pion mass occurs for \( \alpha \) with the value about 0.75. The value of \( m_{\pi} \) increases with \( \alpha \), corresponding to a decreasing binding energy, which means that the intensity of the short-range interaction, that dominates the ground state, diminishes. In fact to keep constant \( m_{\pi}^* \), as the effective Coulomb interaction increases it demands a weaken short-range interaction. The calculation of \( m_\pi \) with Eq.(17) does not go beyond \( \alpha = 0.97 \) because \( \eta \) vanishes and the mass of 768 MeV of the excited state is reproduced with the effective Coulomb interaction. The “subtraction method” does not present the same limitation.
6. Effective Meson Model

Now, we use the “subtraction method” applied to the ↑↓-model to calculate the mass gap between the ground states of the pseudo-scalar and vector mesons, corresponding to (π, ρ)[139,768], (K±, K∗)[494,892], (D0, D∗0)[1865,2007] and (B±, B∗)[5279,5325] (experimental values of the meson masses within the square brackets in MeV). The ground state masses of the pseudoscalar mesons comes from the solution of Eq.(14) with A(k) given by Eq.(5). The pion mass is fixed to its experimental value and the mass \( m_1 \) of one of the constituents quarks are varied with \( m_1 = 406 \) MeV and \( \alpha \) constant. We choosed the values for \( \alpha = 0.18, 0.4 \) and 0.5.

The vector meson mass is associated to the sum \( m_1 + m_2 \). In that sense, the vector meson mass does not have the contribution of the strong attractive hyperfine interaction, and the Coulomb effective attraction produces a binding energy too small compared to the mass, which we have desconsidered here. In figure 3, the results of the difference of the squared masses \( m_v^2 - m_{ps}^2 \) of the ground state of the vector mesons and pseudoscalar mesons as a function of the mass of the ground state of the pseudo-scalar mesons \( m_{ps} \) are shown. The experimental results are reasonably described with \( \alpha = 0.4 \). The value of \( \alpha \) used to describe the data depends on \( m_1 \). In our previous model calculation, the value of 386 MeV was used and \( \alpha \) was found to be 0.5 [4]. The linear raising behaviour of the difference \( m_v^2 - m_{ps}^2 \) with \( m_{ps} \) observed in figure 3, is due to the saturation of the binding energy when \( m_2 \to \infty \). It is not clear if a confining potential, not present in the model, can change such a trend.

7. Conclusion

The “subtraction method” [1] was applied to renormalize the ↑↓-model [2] which contains an effective Coulomb interaction and a hyperfine zero-ranged singular. We have compared with a different renormalization scheme that make use of regularization and subsequent renormalization [2]. The two drastically different schemes, both conceptually and numerically, agree. Here we provide one more simple example, that the physics of the renormalized theory does not recognize the intermediate steps one performs to mathematically define the initial undefined theory.

Acknowledgments: TF thanks to H. Leutwyler for suggesting the plot of figure 3. TF also thanks to H.C. Pauli for the warm hospitality.
at the Max-Planck Institute in Heidelberg, where this work has been written, and to CNPq and FAPESP for financial support.

A. Equations in Momentum Space

The effective model of \[1\] corresponds to use the non-relativistic phase-space \(A(k) = 1\) in Eq.\[1\] and a smeared delta-interaction of a Yukawa form:

\[
m_\pi^2 \varphi(\vec{k}) = \left[ 4m^2 + 4k^2 \right] \varphi(\vec{k}) - \frac{4}{3\pi^2} \alpha \int \frac{d\vec{p}}{m} \times \left( \frac{2m^2}{(\vec{k} - \vec{p})^2} + \frac{\eta^2}{\eta^2 + (\vec{k} - \vec{p})^2} \right) \varphi(\vec{p}). \tag{15}\]

In the renormalized model for the Coulomb plus Dirac-delta interaction the bound state masses of the pion ground and excited states in s-wave, are found numerically from the zeroes of Eq.\[14\]:

\[
\int_0^\infty dp \frac{4\pi p^2}{A(p)} \left[ \frac{1}{m_\pi^2 - M_0^2(p)} - \frac{1}{m_\pi^2 - M_0^2(p)} \right] = -\frac{t^V(p, q; m_\pi^2)}{(m_\pi^2 - M_0^2(p))(m_\pi^2 - M_0^2(q))} = 0, \tag{16}\]

with \(m_\pi^*\) the excited s-wave state mass. The free mass of the two quark system is \(M_0(k) = \sqrt{k^2 + m_1^2 + \sqrt{k^2 + m_2^2}}\). The s-wave projected T-matrix of the Coulomb potential in Eq.\[16\] is

\[
t^V(p, q; M^2) = \int_1^{-1} d\cos(\theta) \langle \vec{p} | T^V(M^2) | \vec{q} \rangle; \tag{17}\]

which is the solution of

\[
t^V(p, q; M^2) = \frac{4m}{3\pi^2} \frac{\alpha}{pq} \frac{\ln \left( \frac{(p+q)^2}{(p-q)^2} \right)}{\sqrt{A(p)A(q)}} \int_0^\infty dp \int_0^\infty dq \frac{\ln \left( \frac{(p+q)^2}{(p-q)^2} \right)}{\sqrt{A(p)A(q)}}, \tag{18}\]

the momentum space representation of s-wave projection of the scattering equation for the T-matrix \(T^V(M^2)\).

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