NEB in Analysis of Optical Flow 4 × 4 and 6 × 6-Patches

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Abstract. We apply the nudged elastic band technique to non-linear high-dimensional datasets, we analyze spaces of 4×4 and 6×6 optical flow patches and detect their topological properties. We experimentally prove that subsets of 4×4 and 6×6 optical flow patches can be modeled a circle, which confirm some results of 4×4 and 6×6 optical flow patches by using a new method-NEB, and expend Adams et al’s result to larger patches of optical flow.

1. Introduction
Optical flow is the pattern of clear motion of image objects between two continuous frames caused by the movement of objects [1]. Optical flow has many applications, for example, in object segmentation, motion estimation and video compression. Because the difficulty of collecting data of optical flow statistics, the spatial statistics of optical flow are comparatively unexploited. To analyze non-linear high-dimensional data is a difficult problem, computational topology is an efficient method for analyzing non-linear high-dimensional data, recently another tool (NEB) of analyzing nonlinear high-dimensional data appeared. In 2015, Adams, Atanasov, and Carlsson [2] used the nudged elastic band (NEB) technique to analyze data of optical flow, they found a new topological properties of an optical flow patches, that is, optical flow patches have a circle behavior, which initially demonstrate that the NEB method is an effective tool for high dimensional nonlinear data analysis.

In this paper, we use the nudged elastic band method [2] to high-contrast patches of optical flow, analyze , and optical flow patches, and identify topological structures in these data sets, the datasets used here comes from a database built in [3]. We use NEB to prove that there exist subsets of , and optical flow patches which has the homology of a circle, the results have been proven in [4] by using computational topology. For the same data sets, we used a completely different method from paper [4] to analyze them, and obtain the same result as in [4], therefore, these topological properties of the sets are their inherent nature. The NEB method is simpler than the method of computational topology, in NEB, we only use several cell complexes to identify topological properties of spaces, but in the computational topology method, we need several thousands even tens of thousands of complexes to get same results. This proves once again that NEB is an effective tool for analyzing high-dimensional nonlinear data sets and identifying their topological properties.
2. The spaces of optical flow patches
Our data are picked from the Roth and Black optical flow database [3], a sample is shown in Fig.1. For each flow field sequence in the database, we utilize the second optical flow frame of a sequence, and randomly select data sets of high contrast 4×4, and 6×6 patches from the optical flow database. The spaces $M_4$, and $M_6$ are sets of 4×4, and 6×6 patches of high contrast as created in [4], which is similar as [5], [6], [7]. And we use the set symbols of the paper [4]. Undefined concepts and symbols in this paper can be found in [2], [4~7].

We should note that one 4×4 patch is arranged as
\[
\begin{pmatrix}
(u_1,v_1) & (u_1,v_2) & (u_2,v_2) & (u_2,v_1) \\
(u_3,v_1) & (u_3,v_2) & (u_4,v_2) & (u_4,v_1)
\end{pmatrix},
\]
where $u$ denotes optical flow in the horizontal direction and $v$ denotes the vertical direction. We consider each 4×4 patch as a vector $x \in \mathbb{R}^{32}$, and each 6×6 patch as a vector $x \in \mathbb{R}^{72}$. For $y \in X$ and a positive integer $k$, we let $\rho_k(y) = |y - y_k|$, here $y_k$ is the $k$-th nearest neighbor of $y$. For a fixed $k$, we sort points of $X$ by descending density, we choose the points indicated by $X(k, p)$ whose densities are in the top $p$ percent. In this paper, we consider subsets $MS^4(200,30)$ and $MS^6(200,30)$.

![Figure 1. A sample from the optical flow database. Horizontal motion is on the left and vertical motion is on the right.](image)

3. Computing method
The outline of the computing method from the paper [2] is listed as following, for more details, please refer to [2].

3.1. Nudged elastic band
The nudged elastic band (NEB) comes from computational chemistry, that is an efficient method for finding a minimum energy path between a given two states. A NEB is computed by an optimization algorithm from an initial path according to the total force, the initial path is usually built from linear interpolation between the initial and final states. For more details about NEB, please refer to [8], [9], [10].

3.2. CW complexes
A $k$-dimensional closed ball $\{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$ is called a $k$-cell. A CW complex $X$ is defined by follow inductive steps. The 0-skeleton $X^0$ of $X$ is a set of 0-cell. The 1-skeleton $X^1$ is created by gluing the endpoints of 1-cells to the 0-skeleton. Inductively, the $k$-skeleton $X^k$ are built by gluing the boundaries of $k$-cells to the $(k-1)$-skeleton $X^{k-1}$.  

3.3. Density estimator

Given a set \( X \subseteq \mathbb{R}^n \), let \( \Phi_{x,\sigma} : \mathbb{R}^n \rightarrow [0, \infty) \) be the probability density of a normal distribution centered at \( x \in X \), we apply a differentiable density estimator \( g(y) = |X|^{-1} \sum_{x \in X} \Phi_{x,\sigma}(y) \) to approach the unknown density.

3.4. 0-cells

To find 0-cells, we randomly choose an initial point \( y_0 \in X \), and iteratively generate a sequence \( \{y_0, y_1, \ldots\} \) with \( y_{n+1} = m(y_n) \), where \( m(y) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the mean shift function given by the formula

\[
m(y) = \frac{\sum_{x \in X} \Phi_{y-x}(y) x}{\sum_{x \in X} \Phi_{y-x}(y)}.
\]

The sequence \( \{y_n\} \) converges to a local maxima of \( g \) [11]. In order to identify different 0-cells, we use single-linkage clustering to cluster the convergent points, and choose the densest member from each cluster as a 0-cell.

3.5. 1-cells

For two 0-cells, there is a 1-cell between them if we find a convergent band between them by using NEB. For an initial band \( \{U_0, U_1, \ldots, U_N\} \), where \( U_0 \) and \( U_N \) are 0-cells. The total force on each node \( U_i \) is computed by

\[
F_i = \left( \|U_i - U_{i-1}\| - \|U_i - U_{i+1}\| \right) r_i - c \nabla g(U_i) \|F_{sm}\text{,}
\]

where \( c = (\sigma \sqrt{2\pi})^2 \sqrt{e} \) is the gradient constant. The smoothing force \( F_{sm} \) is given by the similar formula as in [2].

Given a set \( X \subseteq \mathbb{R}^n \) from unknown probability density function \( f : \mathbb{R}^n \rightarrow [0, \infty) \). For super-level sets \( X^\alpha = f^{-1}([0, \alpha]) \), we construct CW complex models \( Z^\alpha \) to approximate the super-level sets \( X^\alpha \).

We construct only one-dimensional cell complexes. First step, we establish a differentiable density estimator to approach the unknown probability density function by using the given set. Then, we find local maxima of the density estimate to get 0-cells. Lastly, we randomly produce initial bands, then find the convergent bands by NEB, thus we get 1-cells. Hence we can detect the topology properties of the data set \( X \) by constructing cell complexes \( Z^\alpha \).

4. Experimental results

The author of the paper [4] use persistent homology to discuss the topological structure of spaces \( M_4 \) and \( M_6 \) of 4×4 and 6×6 Optical flow patches, and discover that the homology of the subsets changes from a circle to a 3-circle space. Specially, he experimentally show that there exist core subsets \( MS^\alpha(200,30) \) of \( M_n \) (\( n = 4, 6 \)), their homology is that of a circle, and \( M_4, M_6 \) have subset \( CC^\alpha(400), CC^\alpha(400) \) respectively with the topology of the three circle space \( C_3 \).

In this paper we select three kinds subsets of \( M_n \) to analyze (\( n=4, 6 \)): (1) random subsets \( MR^\alpha(10000) \) of \( M_n \), with size 10000; (2) core subsets \( MS^\alpha(200,30) \); (3) selected subset \( CC^\alpha(400) \). For a tidy projected graph, we make DCT transformation for our considered sets. For example, we transform u-component and v-component of \( MS^\alpha(200,30) \) respectively, then points of the set \( MS^\alpha(200,30) \) become 30-dimensional points. In fact, DCT transformation does not change the topology of a space.

4.1. 4 × 4 optical flow patches
Our data set information are displayed in Table 1. For the set $MR^4(10000)$, we take standard deviation $\sigma = 0.35$, we obtain four 0-cells with densities in $[1.285, 3.844]$, and four 1-cells with densities in $[1.133, 2.052]$, all these cells form a loop. Thus for $\alpha = 1.133$, the $Z^\alpha$ is a circle (Fig.2).

Table 1  Data set information

|                  | $MR^4(10000)$ | $MS^4(200,30)$ | $CC^4(400)$ |
|------------------|---------------|---------------|-------------|
| size of data set | 10000         | 15000         | 1200        |
| dimension $n$    | 30            | 30            | 30          |
| standard deviation $\sigma$ | 0.35          | 0.30          | 0.45        |

For the set $MS^4(200,30)$, let $\sigma = 0.30$, we get four 0-cells with densities in $[194.8, 769.4]$ and four 1-cells having densities in $[69.29, 214.6]$, these cells compose a circle (Fig.3).

For the set $CC^4(400)$, we take standard deviation $\sigma = 0.45$, we have four 0-cells with densities in $[0.003253, 0.00429]$ and four 1-cells having densities in $[0.002607, 0.003009]$, these cells have the topology of a circle (Fig.4). When we take $\sigma$ to be values of 0.25, 0.30, 0.35, 0.40, we can't conclude a definite conclusion. We can't find that $CC^4(400)$ has the topology of $C_3$ using the current method.

Figure 2. $MR^4(10000)$ and the circle $Z^{1.133}$, projected to a plane.

Figure 3. $MS^4(200,30)$ and the circle $Z^{69.29}$, projected to a plane.

Figure 4. $CC^4(400)$ complex $Z^{0.002607}$, projected to plane.

4.2. 6 × 6 optical flow patches

Table 2 gives information of data sets. For the set $MR^4(10000)$, taking standard deviation $\sigma = 0.33$, we find four 0-cells, and four 1-cells, which form a loop. The four 0-cells have densities 11400, 17210, 20670, 43200, and four 1-cells have densities 8425, 9866, 17210, 18250 respectively. Hence for $\alpha = 8425$, $Z^\alpha$ is a circle (Fig.5).

Table 2  Data set information

|                  | $MR^4(10000)$ | $MS^4(200,30)$ | $CC^4(400)$ |
|------------------|---------------|---------------|-------------|
| size of data set | 10000         | 15000         | 1200        |
| dimension $n$    | 70            | 70            | 70          |
| standard deviation $\sigma$ | 0.33          | 0.35          | 0.45        |

For the set $MS^4(200,30)$, supposed $\sigma = 0.35$, we get three 0-cells with densities 342.4, 623.6, 2315, and three 1-cells with densities 192.7, 209.8, 600.7 respectively, all these cells form a loop (Fig.6). If we take $\sigma = 0.30$, we get four 0-cells, and four 1-cells, which form a loop.
For the set $CC_j^\sigma(400)$, we take $\sigma = 0.45, 0.50$, for each case we find four and four 1-cells, these cells form a loop. For example, when $\sigma = 0.45$, we have four 0-cells with densities in $[0.0000279, 0.00003068]$ and four 1-cells with densities in $[0.00002147, 0.00002391]$ (Fig.7). When we take $\sigma = 0.25, 0.30, 0.35$ and $0.40$, we do not find a definite conclusion, and we can't discover that $CC_j^\sigma(400)$ has the topology of $C_j$ for other values of $\sigma$.

5. Conclusion
In this short note we utilize the nudged elastic band method to discuss topological qualitative analysis of optical flow small patches. We experimentally prove that the spaces of $4 \times 4$ and $6 \times 6$ high contrast patches have subsets modeled as a circle. The most important parameter of the NEB method is standard deviation $\sigma$, we should choose its proper values to get a stable result. The advantage of cell complexes for analyzing high-dimensional data is its simplicity. For example, we only use several cells to model $MS^\sigma(200,30)$ as a circle, for the same result, we need several tens of thousands witness complexes to model $MS^\sigma(200,30)$ as a circle. If we apply witness complexes to analyze subsets of $M_s$ and $M_\phi$, we detect that the topology of the subsets changes from a circle to a 3-circle space, but we can't find a 3-circle model for various subsets of $M_s$ and $M_\phi$ by cell complexes. Hence the NEB method may only identify coarse topology of a high-dimensional data set.

![Figure 5.](image5.png)  
**Figure 5.** $MR^8(10000)$ and the circle $Z^{8425}$, projected to a plane.

![Figure 6.](image6.png)  
**Figure 6.** $MS^8(200,30)$ and the circle, $Z^{192.7}$, projected to a plane.

![Figure 7.](image7.png)  
**Figure 7.** $CC_j^\sigma(400)$ and the complex $Z^{0.00004127}$, projected to a plane.

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