Charged Dual String Vacua from Interacting Rotating Black Holes Via Discrete and Nonlinear Symmetries

Alfredo Herrera–Aguilar

Instituto de Física y Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Edificio C–3, Ciudad Universitaria, Morelia, Mich., CP 58040, México
e-mail: herrera@zeus.umich.mx

and

Marek Nowakowski

Departamento de Física, Universidad de los Andes,
Cra. 1 No. 18A–10, Santa Fe de Bogotá, Colombia
e-mail: mnowakos@uniandes.edu.co

Abstract

Using the stationary formulation of the toroidally compactified heterotic string theory in terms of a pair of matrix Ernst potentials we consider the four–dimensional truncation of this theory with no $U(1)$ vector fields excited. Imposing one time–like Killing vector permits us to express the stationary effective action as a model in which gravity is coupled to a matrix Ernst potential which, under certain parametrization, allows us to interpret the matter sector of this theory as a double Ernst system. We generate a web of string vacua which are related to each other via a set of discrete symmetries of the effective action (some of them involve $S$–duality transformations and possess non–perturbative character). Some physical implications of these discrete symmetries are analyzed and we find that, in some particular cases, they relate rotating black holes coupled to a dilaton with no Kalb–Ramond field, static black holes with non–trivial dilaton and antisymmetric tensor fields, and rotating and static naked singularities. Further, by applying a nonlinear symmetry, namely, the so–called normalized Harrison transformation, on the seed field configurations corresponding to these neutral backgrounds, we recover the $U(1)^n$ Abelian vector sector of the four–dimensional action of the heterotic string, charging in this way the double Ernst system which corresponds to each one of the neutral string vacua, i.e., the stationary and the static black holes and the naked singularities.

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1 Introduction

During the last years the concept of duality has played an important role in the understanding of non-perturbative aspects of string/M–theory [1]. Duality symmetries were originally discovered for toroidal compactifications of closed string theories (for a review see [2] and references therein). Subsequently it has been realized that they are a property of all string vacua for which the metric and the world sheet have Abelian isometries [3]. Afterward, another symmetry emerged from theories compactified to four and lower dimensions: the $S$–duality [4]; it involves an exchange of the electric and magnetic components of the vector fields and relates the strong and weak coupling regimes of the same/different string theories. It was precisely the appearance of the $T$– and $S$– dualities that provided the scenario for the correct understanding of the so-called $U$–duality, which contains the $T$– and $S$– dualities as subgroups [5].

The existence of these dualities represents another step in the classification of physically inequivalent string vacua [6]. In this context it is interesting to study solutions of Einstein’s equations in the vacuum since they are also solutions of the leading order string background equations with constant dilaton and antisymmetric tensor fields [7]. By applying duality to these solutions one generates new physically different string vacua. This approach has been used in the framework of the four–dimensional effective Einstein–Maxwell Dilaton–Axion theory (and some related models; see for example [8]–[11]) as well as in the context of the world sheet action for the bosonic string in a background with $N$ commuting isometries [12]), further generalized to the case of non–abelian isometries (see [13]–[14] and references therein).

In the framework of heterotic string theory toroidally compactified from $D$ to three dimensions, the use of the symmetry groups of these dualities led to the formulation of the effective action in terms of a pair of matrix Ernst potentials (MEP) coupled to gravity [15]. Afterward, in [16] it was shown that, under certain parametrization, the matter sector of the stationary effective action of the truncated $D = 4$ heterotic string theory with one time–like Killing vector and with no vector fields excited (electromagnetic fields set to zero) can be expressed in terms of a double Ernst system. We briefly review the MEP formalism and this parametrization in Section 2.

It turns out that, in the language of the Ernst potentials, this effective action has some discrete symmetries that can be used to relate a web of physically different string vacua which occasionally define distinct effective theories in the sense that they possess different field spectra. Some of these discrete symmetries involves transformations in which the four–dimensional dilaton field inverts its sign, corresponding in this way to $S$–dualities that possess non–perturbative character. Thus, the string vacua mentioned above are dual to each other since they have the same mathematical description, but describe distinct physical configurations. We present this web of dual theories and field configurations in Section 3 and analyze its physical meaning. We study the axisymmetric case when the double Ernst system corresponds to a double Kerr black hole in Section 4. The field configurations representing interacting black holes have been recently studied in the framework of both General Relativity and string theory [17].

By applying a nonlinear symmetry, namely, the normalized Harrison transformation (NHT) on a neutral field configuration which corresponds to the double Ernst system, one recovers the
\(U(1)^n\) Abelian sector that was previously set to zero \([16]\); this fact corresponds physically to charging the double Ernst system and, in particular, the double Kerr black hole in the axisymmetric case. In Section 5 we recall this result in order to further apply it to the whole web of dual field configurations related via the discrete symmetries mentioned above and interpret the obtained charged exact solutions. We summarize our results and discuss their physical implications in Section 6. Finally, in the Appendices A and B we clarify the \(S\)–duality character of some of the discrete symmetries that relate the different string vacua mentioned above.

2 The effective action and matrix Ernst potentials

We start with the \(D\)–dimensional effective action of the heterotic string at tree level (see \([18]\), for instance)

\[
S^{(D)} = \int d^{(D)}x \left| G^{(D)} \right| \frac{1}{2} e^{-\phi^{(D)}} \left( R^{(D)} + \phi^{(D)}_{;M} \phi^{(D);M} - \frac{1}{12} H^{(D)MNP} H^{(D)MNP} - \frac{1}{4} F^{(D)I}_M F^{(D)IM} \right),
\]

where

\[
F^{(D)I}_{MN} = \partial_M A^{(D)I}_N - \partial_N A^{(D)I}_M, \quad H^{(D)MNP} = \partial_M B^{(D)NP}_N - \frac{1}{2} A^{(D)I}_M F^{(D)IM}_N + \text{cycl perms of } M, N, P.
\]

Here \(G^{(D)}_{MN}\) is the metric, \(B^{(D)}_{MN}\) is the anti–symmetric Kalb-Ramond field, \(\phi^{(D)}\) is the dilaton, \(A^{(D)I}_M\) is a set of \(U(1)\) vector fields \((I = 1, 2, \ldots, n)\) and \(M, N, P = 1, 2, \ldots, D\). In the consistent critical case \(D = 10\) and \(n = 16\), but we shall leave these parameters arbitrary for the time being and will consider the \(D = 4\) theory later on, in Subsec. 2.1. In \([19]–[20]\) it was shown that after the compactification of this model on a \(D - 3 = d\)–torus, the resulting stationary theory possesses the \(SO(d + 1, d + 1 + n)\) symmetry group (\(U\)–duality); more precisely, the scheme of the dimensional reduction contains two steps: one first compactifies the original theory down to four dimensions on a torus (compact manifold) and then imposes a time–like Killing vector, i.e., stationarity. It is worth noticing that for stationary solutions, the equations of motion of this theory are the same as those of the \(D\)–dimensional heterotic string theory for field configurations independent of \(D - 3\) dimensions (the time and \(D - 4\) internal coordinates), since the character of the analysis is local (see, for instance, \([21]\)).

Thus, this stationary effective field theory describes gravity with the metric tensor

\[
g_{\mu\nu} = e^{-2\phi} \left( G^{(D)}_{\mu\nu} - G^{(D)}_{m+3,\mu} G^{(D)}_{n+3,\nu} G^{mn} \right),
\]

coupled to the following set of three–fields:

a) scalar fields

\[
G \equiv G = G^{(D)}_{mn} = G^{(D)}_{m+3,n+3}, \quad B \equiv B = B^{(D)}_{mn} = B^{(D)}_{m+3,n+3}, \quad A \equiv A = A^{(D)I}_m = A^{(D)I}_{m+3}, \quad \phi = \phi^{(D)} - \frac{1}{2} \ln |\det G|.
\]

b) antisymmetric tensor field

\[
B_{\mu\nu} = B^{(D)}_{\mu\nu} - 4 B_{mn} A^m_\mu A^n_\nu - 2 \left( A^m_\mu A^{m+d}_\nu - A^m_\nu A^{m+d}_\mu \right),
\]
c) vector fields $A_\mu^{(a)} = ((A_1)_\mu^m, (A_2)_\mu^{m+d}, (A_3)_\mu^{2d+1})$

$$(A_1)_\mu^m = \frac{1}{2} G^{mn} G_{n+3,\mu}, (A_3)_\mu^{I+2d} = -\frac{1}{2} A^{(D)_I}_\mu + A'_n A^m_\mu, (A_2)_\mu^{m+d} = \frac{1}{2} B^{(D)_I}_{mn} - B_{mn} A^m_n + \frac{1}{2} A'_I A^{I+2d}_\mu$$

(5)

where the subscripts $m, n = 1, 2, ..., d; \mu, \nu = 1, 2, 3$; and $a = 1, ..., 2d + n$. In this paper we set $B_{\mu\nu} = 0$ to remove the effective cosmological constant from our consideration.

All vector fields in three dimensions can be dualized on–shell [15],[20]:

$$\nabla \times \vec{A}_1 = \frac{1}{2} e^{-2\phi} G^{-1} \left( \nabla u + (B + \frac{1}{2} A A^T) \nabla v + A \nabla s \right),$$

$$\nabla \times \vec{A}_3 = \frac{1}{2} e^{2\phi} (\nabla s + A^T \nabla v) + A^T \nabla \times \vec{A}_1,$$

$$\nabla \times \vec{A}_2 = \frac{1}{2} e^{2\phi} G \nabla v - (B + \frac{1}{2} A A^T) \nabla \times \vec{A}_1 + A \nabla \times \vec{A}_3.$$  (6)

Thus, the effective stationary theory describes gravity $g_{\mu\nu}$ coupled to the scalars $G, B, A, \phi$ and pseudoscalars $u, v, s$. These matter fields can be arranged in the following pair of matrix Ernst potentials:

$$\mathcal{X} = \begin{pmatrix} -e^{-2\phi} + v^T X v + v^T As + \frac{1}{2} s^T s & v^T X - u^T X v + u + As \\ X v + u + As & X \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} s^T + v^T A \\ A \end{pmatrix},$$

(7)

where $X = G + B + \frac{1}{2} A A^T$. These matrices have dimensions $(d + 1) \times (d + 1)$ and $(d + 1) \times n$, respectively. In terms of the MEP the effective stationary theory adopts the form

$$\mathcal{S} = \int d^3 x \sqrt{g} \left\{ \mathcal{L}_R + \frac{1}{4} \left( \nabla \mathcal{X} - \nabla \mathcal{A} A^T \right) G^{-1} \left( \nabla \mathcal{X}^T - \mathcal{A} \nabla \mathcal{A}^T \right) G^{-1} + \frac{1}{2} \nabla \mathcal{A}^T G^{-1} \nabla \mathcal{A} \right\},$$

(8)

where $\mathcal{X} = \mathcal{G} + B + \frac{1}{2} A A^T$, then $\mathcal{G} = \frac{1}{2} \left( \mathcal{X} + \mathcal{X}^T - A A^T \right)$ and

$$\mathcal{G} = \begin{pmatrix} -e^{-2\phi} + v^T G v & v^T G \\ G v & G \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & v^T B - u^T \\ B v + u & B \end{pmatrix}. $$

(9)

Interestingly there exist a map between the stationary actions of the heterotic string and Einstein–Maxwell theories [15]:

$$\mathcal{X} \longleftrightarrow -E, \quad \mathcal{A} \longleftrightarrow F,$$

matrix transposition $\longleftrightarrow$ complex conjugation,  (10)

where $E$ and $F$ are the complex Ernst potentials (gravitational and electromagnetic, respectively) of the stationary Einstein–Maxwell theory [22]. This map allows us to extrapolate the results obtained in the EM theory to the heterotic string one using the MEP formulation.
2.1 Bosonic string truncation and double Ernst system

In [16] the stationary bosonic string sector of the whole $D = 4$ effective field theory of the heterotic string was formulated in terms of a double Ernst system coupled to three–dimensional gravity. This truncated action arises by dropping the matrix $A$ in the action (8), i.e. by setting to zero all the $U(1)$ Abelian vector fields which correspond to the winding modes of the reduced theory (we will recover these Abelian fields later on in the paper by using the so-called normalized Harrison transformation). Right now we have

$$3S = \int d^3 x | g \left[ \frac{1}{2} \left\{ -3R + \frac{1}{4} \text{Tr} \left[ \nabla \mathcal{X} \mathcal{G}^{-1} \nabla \mathcal{X}^T \mathcal{G}^{-1} \right] \right\} \right] = \int d^3 x | g \left[ \frac{1}{2} \left\{ -3R + \frac{1}{4} \text{Tr} \left( J^X J^X^T \right) \right\} \right]$$

(11)

where now $\mathcal{X} = \mathcal{G} + \mathcal{B}$, $\mathcal{G} = \frac{1}{2} (\mathcal{X} + \mathcal{X}^T)$ and $J^X = \nabla \mathcal{X} \mathcal{G}^{-1}$. For this theory, since $D = d + 3$, then $d = 1$ and $\mathcal{X}$ is a $2 \times 2$–matrix which can be fully parameterized in the following form

$$\mathcal{X} = \frac{1}{p_2} \begin{pmatrix} p_1 & p_1 q_2 - p_2 q_1 \\ p_1 q_2 + p_2 q_1 & p_1(p_2^2 + q_2^2) \end{pmatrix}.$$  

(12)

By substituting (12) into (11), the action of the matter fields takes the form

$$3S_m = \frac{1}{2} \int d^3 x | g \left[ \frac{1}{2} \left\{ p_1^{-2} \left[ (\nabla p_1)^2 + (\nabla q_1)^2 \right] + p_2^{-2} \left[ (\nabla p_2)^2 + (\nabla q_2)^2 \right] \right\} \right],$$

(13)

which allows us to introduce two independent Ernst–like potentials

$$\epsilon_1 = p_1 + i q_1, \quad \epsilon_2 = p_2 + i q_2.$$  

(14)

Thus, in terms of these field variables, the action of the field system can be rewritten in the form

$$3S = \int d^3 x | g \left[ \frac{1}{2} \left\{ -3R + 2 \left( J^{\epsilon_1} J^{\epsilon_1^T} + J^{\epsilon_2} J^{\epsilon_2^T} \right) \right\} \right],$$

(15)

where $J^{\epsilon_1} = \nabla \epsilon_1 (\epsilon_1 + \tau_1)^{-1}$ and $J^{\epsilon_2} = \nabla \epsilon_2 (\epsilon_2 + \tau_2)^{-1}$, which precisely coincides with the action of a double Ernst system expressed in the Kähler form in the framework of General Relativity. In the particular axisymmetric case, the double Ernst system describes a pair of interacting rotating black holes located on the symmetry axis. However, the action (15) can be obtained for any stationary theory of the form (11) with the potential $\mathcal{X}$ parameterized in the form (12).

3 Dual String Vacua Via Discrete Symmetries

A mathematically equivalent $2 \times 2$–matrix representation of the three–dimensional effective theory arises from (13) by making use of the discrete symmetry $p_1 \leftrightarrow p_2, \ q_1 \leftrightarrow q_2$, a fact that allows us to define the new matrix potential

$$\mathcal{X}' = \frac{1}{p_1} \begin{pmatrix} p_2 & p_2 q_1 - p_1 q_2 \\ p_1 q_2 + p_2 q_1 & p_2(p_2^2 + q_2^2) \end{pmatrix},$$

(16)

The dual nature of this discrete symmetry is clarified in the Appendix A. In the next Sec. we shall consider such a duality within the framework of two interacting rotating black holes of Kerr type.
and to write down the action corresponding to these quantities

\[ S = \int d^3x \left| g \right|^{\frac{1}{2}} \left\{ -\frac{3R}{4} + \frac{1}{4} \text{Tr} \left( J^{\sigma'} J^{\sigma} \right) \right\} = \int d^3x \left| g \right|^{\frac{1}{2}} \left\{ -\frac{3R}{4} + 2 \left( J^{i'i} J^{j'j} + J^{i'j} J^{j'i} \right) \right\} , \]

where similarly \( J^{\sigma'} = \nabla \sigma' \mathcal{G}^{-1} \), \( J^{i'i} = \nabla e_1' (e_1' + e_1')^{-1} \), \( J^{i'j} = \nabla e_2' (e_2' + e_2')^{-1} \), \( e_1' = p_2 + iq_2 \) and \( e_2' = p_1 + iq_1 \).

In terms of the MEP and complex Ernst potentials, the above-mentioned discrete transformation reads

\[ \mathcal{X}[\epsilon_1, \epsilon_2] \leftrightarrow \mathcal{X}'[\epsilon_2, \epsilon_1], \quad \epsilon_1 \leftrightarrow \epsilon_1' = \epsilon_2, \quad \epsilon_2 \leftrightarrow \epsilon_2' = \epsilon_1. \] (18)

From the point of view of the three-dimensional quantities, this map relates the unprimed

\[ G_{tt} = \frac{Re \epsilon_1}{Re \epsilon_2} \left| \epsilon_2 \right|^2, \quad B \equiv 0, \quad e^{-2\phi} = -\frac{Re \epsilon_1 Re \epsilon_2}{\left| \epsilon_2 \right|^2}, \quad u = Im \epsilon_1, \quad v = \frac{Im \epsilon_2}{\left| \epsilon_2 \right|^2} \] (19)

and primed

\[ G'_{tt} = \frac{Re \epsilon_2}{Re \epsilon_1} \left| \epsilon_1 \right|^2, \quad B' \equiv 0, \quad e^{-2\phi'} = -\frac{Re \epsilon_1 Re \epsilon_2}{\left| \epsilon_1 \right|^2}, \quad u' = Im \epsilon_2, \quad v' = \frac{Im \epsilon_1}{\left| \epsilon_1 \right|^2} \] (20)

field configurations parameterized in terms of the stationary double Ernst system. By looking carefully at this symmetry one realizes that it mixes the gravitational and matter degrees of freedom of both theories, since the potentials \( p_k \) and \( q_k \) \((k = 1, 2)\) enter the matrix \( \mathcal{X} \) in a non-symmetric way, even if they appear in the action in a completely symmetric form. Namely, under this correspondence, the \( G_{t\varphi} \)-component of the four-dimensional metric is directly related to the \( B_{t\varphi}^{(4)} \)-component of the Kalb–Ramond field, since from one side, the pseudoscalar field \( u \) \((u')\) is directly related to the component \( G_{t\varphi} \) \((G_{t\varphi}'\)) of the metric (which defines the rotation of the gravitational field) and, from the other, \( v \) \((v')\), defines the component \( B_{t\varphi}^{(4)} \) \((B_{t\varphi}'^{(4)}\)) of the antisymmetric tensor field, (this identification is established through the dualization relations (6)).

The relationship between the \( G_{t\varphi} \) and \( B_{t\varphi}^{(4)} \) tensor components also takes place when applying a Buscher transformation \((T\text{-duality})\) on a given stationary solution of the low-energy string equations [3]. However, in the framework of that discrete transformation the dilaton field is just shifted and cannot change its sign. Thus, the strong and weak coupling regimes of the theory cannot be related each other as in the framework of the symmetry (18) (see Appendix A for details).

A similar effect was found by Bakas in [9] as well, where the implemented \( Z_2 \) discrete symmetry interchanges the field content of an \( SL(2, \mathbb{R})/U(1) \) Ernst \( \sigma \)-model which describes the gravitational sector of the theory with another \( SL(2, \mathbb{R})/U(1) \) Ernst \( \sigma \)-model which parameterizes the axidilaton one. However, the approach of Bakas is quite different from the one we present here since the four-dimensional action and field equations considered in [9] were dimensionally reduced down to two dimensions in the presence of two commuting space-like
Killing symmetries with an ansatz in which $G_{iA} = 0$ \((i=1,0; A=2,3)\). Thus, the symmetry group which arises in the effective two–dimensional theory turns out to be infinite (it corresponds to the string Geroch group) and the theory itself, integrable (as it happens within the effective two–dimensional theory of General Relativity with a similar ansatz).

In the framework of our approach, the discrete symmetry \((18)\) arises after imposing stationarity (a single time–like Killing symmetry) and lives in an effective three–dimensional world, where the vector fields $G_{\tau}\varphi$ and $B_{\tau}\varphi^{(4)}$ are non–trivial (they generate the pseudoscalar fields $u$ and $v$, respectively). Certainly, the discrete map \((18)\) also relates two non–linear $\sigma$–models where the gravitational and matter degrees of freedom are already mixed due to the three–dimensional character of the effective theory. However, it is remarkable that both approaches are quite similar since both discrete transformations relate two non–linear $\sigma$–models, even if they are defined in different two– and three–dimensional effective theories. In this context, it is interesting to consider the further reduction of the theory down to two dimensions in order to study the relation of the discrete symmetry \((18)\) to the infinite dimensional string Geroch group found by Bakas [9]. However, this topic requires a separate investigation.

Finally, it should be noticed that, under the transformation \((18)\), the dilaton fields $\phi^{(4)}_{1}$ and $\phi^{(4)}_{2}$ are just interchanged. This means that, for example, rotating solutions of the Kaluza–Klein–Dilaton theory are dual to static configurations of the bosonic string theory with nontrivial Kalb–Ramond field of dipole type. This fact will be illustrated in the next Section.

Another discrete symmetry which is present in the effective three–dimensional action \((11)\) relates the matrix potential $X$ to its inverse $X^{-1}$, giving rise to the double–primed matrix potential

$$X'' = X^{-1} = \frac{1}{p_2(p_1^2 + q_1^2)} \begin{pmatrix} p_1(p_2^2 + q_2^2) & p_2q_1 - p_1q_2 \\ -(p_1q_2 + p_2q_1) & p_1 \end{pmatrix},$$

which defines an action identical to \((11)\) and \((17)\), but in terms of the matrix variable $X''$ (the dual character of this symmetry is pointed out in Appendix B).

In terms of the MEP and the complex Ernst potentials this symmetry reads

$$X \longleftrightarrow X'' = X^{-1}, \quad \epsilon_1 \longleftrightarrow \epsilon''_1 = \epsilon^{-1}_1, \quad \epsilon_2 \longleftrightarrow \epsilon''_2 = \epsilon^{-1}_2,$$

whereas the three–dimensional double–primed fields written in the language of the Ernst potentials adopt the form

$$G''_{tt} = \frac{Re\epsilon_1}{Re\epsilon_2} \frac{1}{|\epsilon_1|^2}, \quad B'' \equiv 0, \quad e^{-2\phi''} = -\frac{Re\epsilon_1 Re\epsilon_2}{|\epsilon_1|^2}, \quad u'' = -Im\epsilon_1, \quad v'' = -Im\epsilon_2.$$  \(23\)

Thus, the transformation \((22)\) relates $\epsilon_1$ and $\epsilon_2$ to their inverse quantities as well. In the particular case when the imaginary parts of these potentials vanish (static configurations with no antisymmetric Kalb–Ramond field since $u$, $u''$, $v$ and $v''$ are set to zero), this map establishes a correspondence between black holes and naked singularities since it relates the $G_{tt}$–component to its inverse $G''_{tt} = G_{tt}^{-1}$. Such a relationship also takes place when comparing
three–dimensional primed and double–primed fields since the $G'_{tt}$–component also transforms into its inverse $G''_{tt} = G'_{tt}^{-1}$. This effect is well known in the framework of General Relativity since it translates horizons into naked singularities and was pointed out for the first time in the framework of string theory in [10], when applying $T$–duality on a given solution (see also [11]–[13]). It will be illustrated in the next Section with an explicit example.

Both symmetries (18) and (22) can be combined in order to determine another discrete symmetry, namely

$$X \longleftrightarrow X''' = X'^{-1}, \quad \epsilon_1 \longleftrightarrow \epsilon_1''' = \epsilon_1'^{-1} = \epsilon_2^{-1}, \quad \epsilon_2 \longleftrightarrow \epsilon_2''' = \epsilon_2'^{-1} = \epsilon_1^{-1}, \quad (24)$$

where the role of the matrix potential is now played by the matrix

$$X''' = X'^{-1} = \frac{1}{p_1(p_2^2 + q_2^2)} \begin{pmatrix} p_2(p_1^2 + q_1^2) & p_1q_2 - p_2q_1 \\ -(p_2q_1 + p_1q_2) & p_2 \end{pmatrix} \quad (25)$$

which also defines an action of the form (11) in the language of $X'''$. The three–dimensional triple–primed fields written with the aid of the Ernst potentials are

$$G'''_{tt} = \frac{Re\epsilon_2}{Re\epsilon_1} |\epsilon_2|^2, \quad B''' \equiv 0, \quad e^{-2\phi''' = -\frac{Re\epsilon_1 Re\epsilon_2}{|\epsilon_2|^2}, \quad u''' = -\frac{Im\epsilon_2}{|\epsilon_2|^2}, \quad v''' = -Im\epsilon_1. \quad (26)$$

In particular, when applied on an original string background, the transformation (24) combines the effects of the symmetries (18) and (22) since it mixes the gravitational and matter degrees of freedom of the dual configurations (as before, this fact can be seen by switching off one of the Ernst potentials, let us say $\epsilon_2 = 1$) and interchanges the $G'_{tt}$–component with its inverse $G''_{tt} = G'_{tt}^{-1}$, i.e., it relates rotating black holes with no Kalb–Ramond field to static naked singularities coupled to non–trivial antisymmetric tensor field. However, this discrete symmetry does not constitute a $S$–duality transformation since the four–dimensional dilaton field does not change its sign under it (see Appendix B for details).

We have established in this way a web of discrete symmetries between physically different string vacua which sometimes correspond to distinct theories (in the sense that they contain distinct field spectra) sharing the same mathematical description in the language of their effective three–dimensional actions.

4 Double Kerr Solution and Explicit String Vacua

In this Section we shall give some simple explicit examples of the dual string vacua related by the discrete symmetries established above. For concreteness let us consider the axisymmetric double Kerr black hole system, where the line element in the Lewis–Papapetrou form reads

$$ds^2 = G_{tt} (dt + 2(A_1)\phi d\phi)^2 + e^{2\phi} \left[ e^{2\gamma} \left( d\rho^2 + dz^2 \right) + \rho^2 d\phi^2 \right], \quad (27)$$

where $\gamma$, $\phi$, $G_{mn}$ and $(A_1)_\phi$ are $\phi$–independent; moreover, the function $\gamma$, that accounts for the general relativistic interaction between the black holes, is $\gamma \equiv \gamma^\ell_1 + \gamma^\ell_2$, whereas the Ernst
potentials are

$$\epsilon_k = 1 - \frac{2m_k}{r_k + ia_k \cos \theta_k},$$

where $m_k$ and $a_k$ are constant parameters which define the masses and rotations of the sources of the Kerr field configurations, respectively. The Weyl and Boyer–Lindquist coordinates are related through the relations

$$\rho = \sqrt{(r_k - m_k)^2 - \sigma_k^2 \sin \theta_k}, \quad z = z_k + (r_k - m_k) \cos \theta_k,$$

where $z_k$ stands for the positions of the sources, $\sigma_k^2 = m_k^2 - a_k^2$ and, finally, for the function $\gamma_k$ we have

$$e^{2\gamma_k} = \frac{P_k}{Q_k},$$

with

$$P_k = \Delta_k - a_k^2 \sin^2 \theta_k,$$

$$Q_k = \Delta_k + \sigma_k^2 \sin^2 \theta_k,$$

$$\Delta_k = r_k^2 - 2m_k r_k + a_k^2.$$

The relationship (18) can be illustrated by switching off one of the potentials, let us say $\epsilon_2 = 1$; therefore, we get a four–dimensional field configuration that corresponds to a single rotating black hole located at the point $z_1$ with vanishing dilaton and Kalb–Ramond fields (see Eqs. (2)–(6)), namely:

$$ds^2_4 = G_{tt} (dt + \omega_\varphi d\varphi)^2 + \frac{e^{2\phi^{(4)}}}{|G_{tt}|} \left[ P_1 \left( \frac{dr_1^2}{\Delta_1} + d\theta_1^2 \right) + \Delta_1 \sin^2 \theta_1 d\varphi^2 \right],$$

$$G_{tt} = \frac{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}{(r_1^2 + \alpha_1^2 \cos^2 \theta_1)}, \quad \omega_\varphi \equiv 2(\vec{A}_1)_{\varphi} = \frac{2m_1 \alpha_1 r_1 \sin^2 \theta_1}{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}, \quad B \equiv 0,$$

$$e^{2\gamma} = \frac{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1 + m_1^2 \sin^2 \theta_1)}, \quad \phi^{(4)} = 0, \quad B^{(4)}_{t\varphi} \equiv 2(\vec{A}_2)_{\varphi} = 0.$$

Under the discrete symmetry (18) this solution is mapped into a static black hole configuration endowed with non–trivial dilaton and Kalb–Ramond field of dipole type:

$$ds^2_4 = G'_{tt} dt^2 + \frac{e^{2\phi'(4)}}{|G'_{tt}|} \left[ P_1 \left( \frac{dr_1^2}{\Delta_1} + d\theta_1^2 \right) + \Delta_1 \sin^2 \theta_1 d\varphi^2 \right], \quad \omega'_{\varphi} \equiv 2(\vec{A}_1)_{\varphi} = 0, \quad B' \equiv 0,$$

$$G'_{tt} = \frac{(r_1^2 - 4m_1 r_1 + 4m_2^2 + \alpha_1^2 \cos^2 \theta_1)}{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}, \quad e^{2\gamma} = \frac{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1 + m_1^2 \sin^2 \theta_1)}.$$
However, this topic deserves a separate investigation. The origin of these naked singularities in general and their relation to comparing the three–dimensional primed and double–primed fields. It is interesting to study the horizon into a naked singularity [13] (see as well [10]–[11], for instance). Thus, under the string in a background with non–abelian isometries, it was shown that such a relationship maps with naked singularities. For example, in the context of the worldshe et action for the bosonic the framework of both General Relativity and string theory and connects black hole geometries metric also transforms into its inverse. As pointed out above, this relationship is well known in the known duality between the rotational sector of the metric \( \omega_\varphi \) and the \( B_{t\varphi}^{(4)} \)–component of the Kalb–Ramond field (see [3] and [9]) also takes place under the discrete symmetry (18). A similar effect takes place when we switch off the potential \( \epsilon_1 = -1 \). However, it is worth noticing that if we do not switch off any potential \( \epsilon_k \), the sources of the four–dimensional dilaton fields just exchange their positions \( z_1 \leftrightarrow z_2 \), in other words, \( \phi^{(4)}(z_1) \leftrightarrow \phi^{(4)}(z_2) \).

The discrete symmetry (22) also acts in a non–trivial way on a given string background. This fact can be illustrated by considering, for example, static configurations with no antisymmetric Kalb–Ramond field (the pseudoscalar fields \( u, u'' \), \( v \) and \( v'' \) are set to zero, therefore, the imaginary parts of the Ernst potentials vanish). Thus, the metric of an original four–dimensional Schwarzschild solution in the Einstein frame\(^2\) with the source located at the position \( z_1 \) and coupled to the dilaton field \( e^\phi^{(4)} = R e \epsilon_2 = 1 - 2 m_2/r_2 \) with its source located at \( z_2 \) reads

\[
\begin{align*}
  ds^2_E &= -\left(\frac{r_1 - 2 m_E}{r_1}\right) dt^2 + \left(\frac{r_1}{r_1 - 2 m_E}\right) \left[ \frac{r_1^2 - 2 m_2 r_2}{r_1^2 - 2 m_2 r_2 + m_2^2 \sin^2 \theta_2} \right] \times \\
  &\quad \left( dr_1^2 + \left( r_1^2 - 2 m_E r_1 \right) d\theta_1^2 \right) + \left( r_1^2 - 2 m_E r_1 \right) \sin^2 \theta_1 d\psi^2; \quad (34)
\end{align*}
\]

its under (22) this solution defines a geometry represented by the following metric

\[
\begin{align*}
  ds''^2_E &= -\left(\frac{r_1}{r_1 - 2 m_E}\right) dt^2 + \left(\frac{r_1 - 2 m_E}{r_1}\right) \left[ \frac{r_1^2 - 2 m_2 r_2}{r_1^2 - 2 m_2 r_2 + m_2^2 \sin^2 \theta_2} \right] \times \\
  &\quad \left( dr_1^2 + \left( r_1^2 - 2 m_E r_1 \right) d\theta_1^2 \right) + \left( r_1^2 - 2 m_E r_1 \right) \sin^2 \theta_1 d\psi^2, \quad (35)
\end{align*}
\]

with the dilaton field inverted \( \phi''^{(4)} = 1/R \epsilon_2 = (1 - 2 m_2/r_2)^{-1} \). The \( G_{tt} \)–component of the metric also transforms into its inverse. As pointed out above, this relationship is well known in the framework of both General Relativity and string theory and connects black hole geometries with naked singularities. For example, in the context of the worldsheet action for the bosonic string in a background with non–abelian isometries, it was shown that such a relationship maps the horizon into a naked singularity [13] (see as well [10]–[11], for instance). Thus, under the discrete duality symmetry (22), the black hole geometry is translated into a naked singularity, whereas the four dimensional dilaton field inverts its sign. A similar effect can be established by comparing the three–dimensional primed and double–primed fields. It is interesting to study the origin of these naked singularities in general and their relation to \( S \)–duality transformations. However, this topic deserves a separate investigation.

\(^2\)The relationship between the metrics in the Einstein and string frames in four dimensions reads \( ds^2_E = e^{-\phi^{(4)}} ds^2_{st} \).
As it was pointed out above, the discrete map (24) combines the physical effects of the previous symmetries. In order to illustrate this result we shall switch off the Ernst potential $\epsilon_2 = 1$. Thus, under this symmetry, the string background (32) representing a rotating black hole configuration with vanishing dilaton and Kalb–Ramond fields transforms into the following field configuration

$$
\text{ds}_{4''}^2 = G_{tt}'' dt^2 + \frac{e^{2\phi''(4)}}{|G_{tt}''|} \left[ P_1 \left( \frac{dr}{\Delta_1^2} + d\theta_1^2 \right) + \Delta_1 \sin^2 \theta_1 d\varphi^2 \right], \quad \omega'' \equiv 2(A_1)_{\varphi} = 0, \quad B'' \equiv 0,
$$

$$
G_{tt}'' = -\frac{(r_1^2 + \alpha_1^2 \cos^2 \theta_1)}{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}, \quad e^{2\gamma} = \frac{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}{(r_1^2 - m_1 r_1 + \alpha_1^2 \cos^2 \theta_1 + m_2^2 \sin^2 \theta_1)},
$$

$$
B_{t\varphi}'' \equiv 2(A_2)_{\varphi} = -\frac{2m_1 r_1 \alpha_1 \sin^2 \theta_1}{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}, \quad \phi''(4) = \ln \frac{(r_1^2 + \alpha_1^2 \cos^2 \theta_1)}{(r_1^2 - 2m_1 r_1 + \alpha_1^2 \cos^2 \theta_1)}, \quad (36)
$$

which describes a static metric coupled to non–trivial dilaton and antisymmetric tensor field of dipole type. By looking at the behaviour of the $G_{tt}$–component of the metric tensor under (24), one realizes that it again gets inverted. Hence, the transformed geometry constitutes a naked singularity. Therefore, the discrete transformation (24) establishes a relationship between four–dimensional rotating black holes with no matter fields and static naked singularities coupled to non–trivial dilaton and Kalb–Ramond fields. However, since the dilaton field does change its sign under the discrete symmetry (24), the transformed string background does not involve or contain a $S$–duality transformation.

Similar relationships can be established between the four families of string vacua connected via the discrete symmetries (18), (22) and (24). We emphasize that each symmetry acts in a non–trivial way on the starting four–dimensional field configurations.

## 5 Charged Field Configurations

In the language of the MEP the stationary action (8) possesses a set of symmetries which has been classified according to their charging properties in [23]. Among them, only the nonlinear Ehlers and Harrison transformations act in a non–trivial way on the spacetime [24]. For instance, the so–called normalized Harrison transformation allows us to construct charged string vacua from neutral ones preserving the asymptotic values of the seed fields. Namely, the matrix transformation

$$
\mathcal{A} \to \left( 1 + \frac{1}{2} \Sigma \lambda \lambda^T \right) \left( 1 - A_0 \lambda^T + \frac{1}{2} \lambda \lambda^T \right)^{-1} \left( A_0 - \lambda \right) + \Sigma, \quad (37)
$$

$$
\mathcal{X} \to \left( 1 + \frac{1}{2} \Sigma \lambda \lambda^T \right) \left( 1 - A_0 \lambda^T + \frac{1}{2} \lambda \lambda^T \right)^{-1} \left( A_0 + \left( A_0 - \frac{1}{2} \lambda \right) \lambda^T \Sigma \right) + \frac{1}{2} \Sigma \lambda \lambda^T \Sigma,
$$

11
where $\Sigma = \text{diag}(-1, -1; 1, 1, \ldots, 1)$ and $\lambda$ is an arbitrary constant $(d + 1) \times n$–matrix parameter, generates charged string solutions (with non–zero electromagnetic potential $A$) from neutral ones if we start from the seed potentials $X_0 \neq 0$ and $A_0 = 0$. Thus, this solution generation procedure allows us to generate the $U(1)^n$ electromagnetic spectrum of the effective heterotic string theory starting with just the bosonic string spectrum. In other words, if we apply the NHT on uncharged seed solutions (like (19) (20), (23) or (26), for instance), we recover the $U(1)^n$ vector field sector of the heterotic string theory, charging, by this means, the double Ernst system and, in particular, interacting Kerr black hole configurations. It is interesting to see what kind of transformation undergo the unprimed and primed seed solutions after making use of the NHT.

The seed MEP that correspond to the neutral stationary double Ernst system are

$$X_0 = \begin{pmatrix}
\frac{p_1}{p_2} & \frac{p_1q_2 - q_1p_2}{p_2} \\
\frac{p_1q_2 + q_1p_2}{p_2} & \frac{p_2(p_2^2 + q_2^2)}{p_2}
\end{pmatrix}, \quad A_0 = 0. \quad (38)$$

For the simplest case the charge matrix $\lambda$ that parameterizes the normalized Harrison transformation has the form

$$\lambda = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \ldots & \lambda_{1n} \\
\lambda_{21} & \lambda_{22} & \ldots & \lambda_{2n}
\end{pmatrix}, \quad (39)$$

where $n \geq 2$ for consistency; these parameters $\lambda_{1i}$ and $\lambda_{2i}$ ($i = 1, 2, \ldots, n$) can be interpreted as the electromagnetic charges of the generated field configuration. When $n = 6$ the generated field spectrum corresponds to the bosonic sector of $\mathcal{N} = 4, D = 4$ supergravity; here we shall leave it arbitrary for the sake of generality. After applying the normalized Harrison transformation on a generic double Ernst seed solution with this matrix $\lambda$, the transformed MEP read

$$X_{11} = \frac{1}{\Xi} \left[ \left( 4 + \Lambda^2 |\epsilon_2|^2 \right) \text{Re} \epsilon_1 + 2 \left( \lambda_{11}^2 + \lambda_{21}^2 |\epsilon_1|^2 \right) \text{Re} \epsilon_2 - 4 \lambda_{1i} \lambda_{2i} \text{Re} \epsilon_1 \text{Re} \epsilon_2 \text{Im} \epsilon_1 \right], \quad (40)$$

$$X_{12} = \frac{1}{\Xi} \left\{ \Gamma_+ \left( \text{Re} \epsilon_1 \text{Im} \epsilon_2 - \text{Re} \epsilon_2 \text{Im} \epsilon_1 \right) + 2 \lambda_{11} \lambda_{2i} \left[ (1 - |\epsilon_2|^2) \text{Re} \epsilon_1 + (1 - |\epsilon_1|^2) \text{Re} \epsilon_2 \right] \right\}, \quad (41)$$

$$X_{21} = \frac{1}{\Xi} \left\{ \Gamma_- \left( \text{Re} \epsilon_1 \text{Im} \epsilon_2 + \text{Re} \epsilon_2 \text{Im} \epsilon_1 \right) + 2 \lambda_{1i} \lambda_{21} \left[ (1 - |\epsilon_1|^2) \text{Re} \epsilon_2 - (1 - |\epsilon_2|^2) \text{Re} \epsilon_1 \right] \right\}, \quad (42)$$

$$X_{22} = \frac{1}{\Xi} \left[ \left( \Lambda^2 + 4 |\epsilon_2|^2 \right) \text{Re} \epsilon_1 + 2 \left( \lambda_{2i}^2 + \lambda_{1i}^2 |\epsilon_1|^2 \right) \text{Re} \epsilon_2 + 4 \lambda_{1i} \lambda_{2i} \text{Re} \epsilon_1 \text{Re} \epsilon_2 \text{Im} \epsilon_1 \right], \quad (43)$$

$$A_{1j} = \frac{-2}{\Xi} \left\{ \left[ 2 + \lambda_{2i}^2 |\epsilon_2|^2 \right] \text{Re} \epsilon_1 + \left( 2 + \lambda_{2i}^2 |\epsilon_1|^2 \right) \text{Re} \epsilon_2 + \lambda_{1i} \lambda_{2i} \left( \text{Re} \epsilon_1 \text{Im} \epsilon_2 - \text{Re} \epsilon_2 \text{Im} \epsilon_1 \right) \text{Im} \epsilon_1 \right\} \lambda_{1j} + \text{const}.$$
\[
\left\{ (2 - \lambda_{1i}^2) \left( Re\epsilon_1 Im \epsilon_2 - Re\epsilon_2 Im \epsilon_1 \right) - \lambda_{1i} \lambda_{2i} \left( |\epsilon_2|^2 Re \epsilon_1 + |\epsilon_1|^2 Re \epsilon_2 \right) \right\} \lambda_{2j},
\]

\[
A_{2j} = -\frac{2}{\Xi} \left\{ \left( (2 - \lambda_{2i}^2) \left( Re\epsilon_1 Im \epsilon_2 + Re\epsilon_2 Im \epsilon_1 \right) - \lambda_{1i} \lambda_{2i} \left( Re \epsilon_1 + |\epsilon_1|^2 Re \epsilon_2 \right) \right) \lambda_{1j} + \left( \lambda_{1i}^2 + 2|\epsilon_2|^2 \right) Re \epsilon_1 + \left( 2 + \lambda_{1i}^2 |\epsilon_1|^2 \right) Re \epsilon_2 + \lambda_{1i} \lambda_{2i} \left( Re\epsilon_1 Im \epsilon_2 + Re\epsilon_2 Im \epsilon_1 \right) \lambda_{2j} \right\},
\]

\[
\Xi = 2 \left( \lambda_{1i}^2 + \lambda_{2i}^2 |\epsilon_2|^2 \right) Re \epsilon_1 + \left( 4 + \Lambda^2 |\epsilon_1|^2 \right) Re \epsilon_2 + 4 \lambda_{1i} \lambda_{2i} Re \epsilon_1 Im \epsilon_2,
\]

where \( \Gamma_- = 4 - 2\lambda_{1i}^2 + 2\lambda_{2i}^2 - \Lambda^2 \), \( \Gamma_+ = 4 + 2\lambda_{1i}^2 - 2\lambda_{2i}^2 - \Lambda^2 \) and \( \Lambda^2 = \lambda_{1i}^2 \lambda_{2j}^2 - (\lambda_{1i} \lambda_{2j})^2 \).

Now we can consider the explicit seed solutions corresponding to the field configurations (19), (20), (23) and (26), substitute the expressions of \( \epsilon_1 \) and \( \epsilon_2 \) in the transformed MEP formulae, compute the original fields with the aid of (2)–(6), and give an interpretation of the generated families of solutions.

After applying the NHT on the seed solution (19) we get the following unprimed three–dimensional field configurations:

\[
G \equiv G_{tt} = \chi_{22} - \frac{1}{2} A_{2j}^2, \quad B \equiv 0, \quad A = A_{2j}, \quad v = \frac{1}{2G} (\chi_{12} + \chi_{21} - A_{1j} A_{2j}),
\]

\[
u = v \chi_{22} - \chi_{12}, \quad s^T = A_{1j} - v A_{2j}, \quad e^{-2\phi} = \frac{1}{2} [v(\chi_{12} + \chi_{21}) + A_{1j} s] - \chi_{11},
\]

where the appearance of the electromagnetic potential is obvious. Similar relations hold for the primed, double– and triple–primed field systems that arise under the interchanges (18), (22) and (24), respectively.

With the aid of the dualization relations (6) we get explicit expressions for the non–trivial components of the vector fields

\[
\omega_\varphi \equiv 2(A_{1})_\varphi = \left[ -4\lambda_{11} + 2\lambda_{1i} \lambda_{2i} + \left( \lambda_{1i}^2 \lambda_{2j}^2 - (\lambda_{1i} \lambda_{2i})^2 \right) \chi_{12} - 2\lambda_{2i} \lambda_{2i} + 4\lambda_{1i} \lambda_{2i} \chi_{23} \right] / DQ,
\]

\[
B_{t \varphi}^{(4)} \equiv 2(A_{2})_\varphi = \left[ -2\lambda_{2i} \chi_{11} + \left( \lambda_{2i}^2 \lambda_{2j}^2 - (\lambda_{1i} \lambda_{2i})^2 \right) \chi_{21} + 2\lambda_{2i} \lambda_{2i} \chi_{12} - 4\chi_{22} - 4\lambda_{1i} \lambda_{2i} \chi_{13} \right] / DQ,
\]

\[
2(A_{3})^t_\varphi = 2(DQ)^{-2} \left\{ \lambda_{1i} \lambda_{2j} \left[ DQ \left( \chi_{12} + \chi_{21} \right) - \left( 4 - 2\lambda_{2i}^2 \right) \chi_{22} + \left( \lambda_{1i}^2 \lambda_{2j}^2 - 2\lambda_{2i}^2 - (\lambda_{1i} \lambda_{2i})^2 \right) \chi_{21} - 4\lambda_{1i} \lambda_{2i} \chi_{13} \right] + 2 \left[ \left( 1 - \lambda_{1i}^2 \right) \left( 2 - \lambda_{2j}^2 \right) - (1 - \lambda_{1i}^2) (1 - \lambda_{2j}^2) \right] \left( \chi_{13} - \chi_{23} \right) \right\} \lambda_{1i}^t +
\]

\[
2(DQ)^{-2} \left\{ 2DQ (\chi_{11} + \chi_{22}) - DQ \lambda_{1i}^2 (\chi_{12} + 2 \chi_{21}) + 2DQ \lambda_{1i} \lambda_{2i} (\chi_{13} - \chi_{23}) -
\]
\[2\lambda_{i_1}\lambda_{i_2} \left[ \lambda_{i_1}\lambda_{i_2} \left( \chi_{21} + \chi_{22} \right) + \left( 4 - \lambda_{i_1}^2 - \lambda_{i_2}^2 \right) \chi_{13} + \left( \lambda_{i_2}^2 - \lambda_{i_1}^2 \right) \chi_{23} \right] \right] \lambda_{1_1'}, \quad (48)\]

where \(i' = 1, 2, \ldots n\) and the functions \(\chi_{kl} \ (l = 1, 2, 3;)\) and \(DQ\) read

\[
\chi_{k1} = \frac{\alpha_{k} m_{k} r_{k} \sin^{2} \theta_{k}}{(r_{k}^{2} - 2 m_{k} r_{k} + \alpha_{k}^{2} \cos^{2} \theta_{k})}, \quad \chi_{k2} = \frac{\alpha_{k} m_{k} (r_{k} - 2 m_{k}) \sin^{2} \theta_{k}}{(r_{k}^{2} - 2 m_{k} r_{k} + \alpha_{k}^{2} \cos^{2} \theta_{k})},
\]

\[
\chi_{k3} = \frac{m_{k} (r_{k}^{2} - 2 m_{k} r_{k} + \alpha_{k}^{2}) \cos \theta_{k}}{(r_{k}^{2} - 2 m_{k} r_{k} + \alpha_{k}^{2} \cos^{2} \theta_{k})}, \quad DQ = 4 - 2 \lambda_{i_1}^2 - 2 \lambda_{i_2}^2 + \lambda_{i_1}^2 \lambda_{i_2}^2 - (\lambda_{i_1} \lambda_{i_2})^2. \quad (49)\]

It is worth noticing that since the \(\chi_{k3}\) functions do not vanish at spatial infinity (they involve the so-called NUT parameters of the gravitational field), in order to get an asymptotically flat gravitational field configuration, i.e., to deal with charged black holes, we should impose the orthogonality condition on the pair of charge vectors \(\lambda_{i_1}\) and \(\lambda_{i_2}\):

\[
\lambda_{i_1} \lambda_{i_2} = 0. \quad (50)\]

However, even with this restriction on the \((A_3)_{\phi}'\) fields the Dirac string singularity is still present; in order to remove it we could, in principle, normalize the charge vector \(\lambda_{i_1}\):

\[
\lambda_{i_1}^2 = 1, \quad (51)\]

but we shall keep it unnormalized for the sake of generality (in fact, this singularities could correspond to the Dirac strings of monopole–type solutions).

Thus, after imposing the conditions (50), the transformed metric preserves its form (27) with the following field configurations in the string frame

\[
G_{tt} = \frac{(2 - \lambda_{i_2}^2) P_{1} P_{2} \left[ 4 R_{2} \tilde{r}_{1} + \lambda_{i_1}^2 R_{1} \tilde{r}_{2} - 4 \lambda_{i_1}^4 (P_{1} P_{2} - 4 m_{1} m_{2} \alpha_{1} \alpha_{2} \cos \theta_{1} \cos \theta_{2}) \right]}{\left[ -2 P_{1} \left( \lambda_{i_1}^2 \tilde{r}_{2} + \lambda_{i_2}^2 R_{2} \right) + P_{2} \left( 4 \tilde{r}_{1} + \lambda_{i_1}^2 \lambda_{i_2}^2 R_{1} \right) \right]^2},
\]

\[
e^{\phi(4)} = \frac{(2 - \lambda_{i_2}^2) \left[ 4 R_{2} \tilde{r}_{1} + \lambda_{i_1}^4 R_{1} \tilde{r}_{2} - 4 \lambda_{i_1}^4 (P_{1} P_{2} - 4 m_{1} m_{2} \alpha_{1} \alpha_{2} \cos \theta_{1} \cos \theta_{2}) \right]}{(2 - \lambda_{i_1}^2) \left[ -2 P_{1} \left( \lambda_{i_1}^2 \tilde{r}_{2} + \lambda_{i_2}^2 R_{2} \right) + P_{2} \left( 4 \tilde{r}_{1} + \lambda_{i_1}^2 \lambda_{i_2}^2 R_{1} \right) \right]^2},
\]

\[
\omega_{\phi} \equiv 2(A_{1})_{\phi} = \left( -4 \chi_{11} + 2 \lambda_{i_1}^2 \chi_{21} + \lambda_{i_1}^2 \lambda_{i_2}^2 \chi_{12} - 2 \lambda_{i_2}^2 \chi_{22} \right) / \left[ (2 - \lambda_{i_1}^2) \left( 2 - \lambda_{i_2}^2 \right) \right],
\]

\[
B_{t\phi}^{(4)} \equiv 2(A_{2})_{\phi} = \left( -2 \lambda_{i_2}^2 \chi_{11} + \lambda_{i_1}^2 \lambda_{i_2}^2 \chi_{21} + 2 \lambda_{i_1}^2 \chi_{12} - 4 \chi_{22} \right) / \left[ (2 - \lambda_{i_1}^2) \left( 2 - \lambda_{i_2}^2 \right) \right],
\]

\[
2(A_{3})_{\phi} = 4 \left( 1 - \lambda_{i_1}^2 \right) \left( \chi_{13} - \chi_{23} \right) \lambda_{1_1'} \left( 2 - \lambda_{i_1}^2 \right)^2 +
\]

\[
\left[ 4 \left( \chi_{11} + \chi_{22} \right) - 2 \lambda_{i_1}^2 \left( \chi_{12} + 2 \chi_{21} \right) \right] \lambda_{2_1'} / \left[ (2 - \lambda_{i_1}^2) \left( 2 - \lambda_{i_2}^2 \right) \right], \quad (52)\]

where \(R_{k} = (r_{k} - 2 m_{k})^{2} + \alpha_{k}^2 \cos^{2} \theta_{k}, \ \tilde{r}_{k} = r_{k}^{2} + \alpha_{k}^2 \cos^{2} \theta_{k}\) and \(e^{2\phi}\) remains the same.
6 Conclusion and Discussion

We have presented a web of dual four-dimensional bosonic string vacua that have the same mathematical description in terms of a matrix Ernst potential. These backgrounds are related by discrete symmetries and all of them describe distinct physical configurations. In particular, it turns out that the discrete symmetry (18) mixes the gravitational and matter degrees of freedom of the dual backgrounds, i.e., the $G_{tt}$ component of the gravitational metric is interchanged with the $B_{t^4}$ component of the Kalb–Ramond antisymmetric tensor field, establishing in this way a duality between rotating solutions of the Kaluza–Klein–Dilaton theory and static configurations of the bosonic string theory with nontrivial antisymmetric tensor field of dipole type.

Another discrete symmetry (22) establishes a correspondence between black holes and naked singularities since it relates the $G_{tt}$-component to its inverse $G''_{tt} = G_{tt}^{-1}$ (at least for static field configurations). This interesting non–trivial effect on the singularities of the dual theories is accompanied by the inversion if the sign of the four–dimensional dilaton field. Thus, in this particular case, the discrete symmetry (22) corresponds to a $S$–duality transformation and possesses non–perturbative character. We obtain the same physical result by comparing the three–dimensional primed and double–primed fields, namely, the $G'_{tt}$–component is also related to its inverse $G'''_{tt} = G_{tt}^{-1}$ under this symmetry.

A third discrete symmetry arises when one combines the transformations (18) and (22). Thus, as one could expect, when applied on a generic string background, the transformation (24) combines the effects of the symmetries (18) and (22) and mixes the gravitational and matter degrees of freedom of the dual configurations and interchanges the $G_{tt}$–component with its inverse $G'''_{tt} = G_{tt}^{-1}$, i.e., relates pure rotating black hole configurations to static naked singularities coupled to non–trivial antisymmetric Kalb–Ramond and dilaton fields. However, this discrete symmetry does not involve any transformation which could correspond to $S$–duality since the four–dimensional dilaton field does not change its sign under it.

These are just some simple examples of the dual field configurations and theories connected via the discrete symmetries (18), (22) or (24). Some of the physical effects that take place under these transformations are of interest since they have non–perturbative character. It should be stressed that each symmetry acts in a non–trivial way on the starting four–dimensional string background.

Finally, by making use of the nonlinear charging symmetry called normalized Harrison transformation we endow all the generated string vacua with electromagnetic charges. We clarify the conditions under which the constructed charged gravitational configurations are asymptotically flat in order to interpret the obtained field configurations from (19) as charged black holes.

In the framework of these results, it is of interest as well to know whether the discrete symmetries presented in this work can map singular regions into regular regions of one string black hole geometry (as it happens in [25]) or of different black hole geometries [26], or relate charged black strings to boosted (uncharged) black strings as in [27] after applying the NHT.

In this context, it is interesting to consider the further reduction of the theory down to two dimensions in order to study the relation of the discrete symmetries (18) and (22) to the infinite
dimensional string Geroch group found by Bakas [9].

It would be interesting as well to extend this kind of discrete dualities to spaces compactified in Calabi–Yau manifold since they have no continuous isometries (the O(d+1,d+1) symmetry group in this case) but they are known to have duality–like symmetries [28]. Another interesting issue concerns the possibility of getting an Ernst system corresponding to $n > 2$ aligned rotating sources (black holes, in particular) by means of a suitable parametrization of the $(d+1) \times (d+1)$–matrix potential $X$.

**Appendix A**

Let us analyze the dual nature of the discrete symmetry (18). As pointed out in the Introduction, the $U$–duality group of the low–energy string theory contains the $S$– and $T$–duality groups as subgroups (see [5] and [20], for instance). The fact that the symmetry (18) relates the gravitational and matter degrees of freedom under the transformation found by Buscher [3] indicates that the discrete map (18) contains the $T$–duality symmetry. For this reason, we shall show below that the symmetry (18) also contains the $S$–duality transformation since it interchanges the sign of the four–dimensional dilaton field, and, thus, relates the strong and weak coupling regimes of the effective theory under consideration.

From relations (19) it follows that the Ernst potentials read

$$
\epsilon_1 = \frac{-\sqrt{-G_{tt}}}{e^{\phi}} + iu, \quad \epsilon_2 = \frac{\sqrt{-G_{tt}} e^{\phi} - i G_{tt} e^{2\phi} v}{1 - G_{tt} e^{2\phi} v^2}.
$$

Under the discrete symmetry (18) the expressions for $\epsilon_1$ and $\epsilon_2$ interchange and we get

$$
\epsilon_1 = \frac{-\sqrt{-G_{tt}'} e^{\phi'}}{1 - G_{tt}' e^{2\phi'} v'^2}, \quad \epsilon_2 = \frac{\sqrt{-G_{tt}'} e^{\phi'} - i G_{tt}' e^{2\phi'} v'}{1 - G_{tt}' e^{2\phi'} v'^2},
$$

Thus, the primed three–dimensional fields are related to the unprimed ones through the following equations

$$
G_{tt}' = -\frac{u^2 e^{2\phi} - G_{tt}}{1 - G_{tt} e^{2\phi} v^2}, \quad u' = \frac{-G_{tt} e^{2\phi} v}{1 - G_{tt} e^{2\phi} v^2},
$$

$$
e^{2\phi'} = -\frac{(u^2 e^{2\phi} - G_{tt})(1 - G_{tt} e^{2\phi} v^2)}{G_{tt} e^{2\phi}}, \quad v' = \frac{u e^{2\phi}}{u^2 e^{2\phi} - G_{tt}}.
$$

By looking at the behaviour of the four–dimensional dilaton field (see eq. (3)) under such a symmetry we observe that it changes its sign:

$$
e^{\phi(4)'} = e^{\phi'} \sqrt{-G_{tt}'} = \frac{u^2 e^{2\phi} - G_{tt}}{e^{\phi} \sqrt{-G_{tt}}} = \frac{u^2 e^{2\phi} - G_{tt}}{e^{\phi(4)}}.
$$

Thus, one can conclude that the discrete symmetry (18) involves transformations in which the dilaton changes its sign and, thus, contains the $S$–duality symmetry.
Appendix B

As we did in the Appendix A, here we shall study the effect of the discrete symmetry (22) on the four-dimensional dilaton and show that under certain conditions it also inverts the sign of the dilaton field. In Appendix A we pointed out that from relations (19) it follows that the Ernst potentials $\epsilon_k$ adopt the form (53) in terms of the three-dimensional fields. Under the discrete symmetry (22) the Ernst potentials transform into their inverse and, thus, we can express them in terms of the double-primed fields

$$
\epsilon_1 = -\frac{\sqrt{-G_{tt}''}e^{\phi''} + iv''e^{2\phi''}}{u'^{2}e^{2\phi''} - G_{tt}''}, \quad \epsilon_2 = \frac{1}{\sqrt{-G_{tt}''}e^{\phi''} - iv''}.
$$

(57)

From (53) and (57) we can extract the relationships that connect the double-primed and unprimed three-dimensional field variables

$$
G_{tt}''' = -\frac{e^{-2\phi} - G_{tt}v^2}{u^2 - G_{tt}e^{-2\phi}}, \quad u''' = \frac{-u}{u^2 - G_{tt}e^{-2\phi}},
$$

$$
e^{2\phi''} = -\frac{(u^2 - G_{tt}e^{-2\phi}) (e^{-2\phi} - G_{tt}v^2)}{G_{tt}e^{-2\phi}}, \quad v''' = \frac{G_{tt}v}{e^{-2\phi} - G_{tt}v^2}.
$$

(58)

By computing the transformed four-dimensional dilaton field we get

$$
e^{\phi(4)'''} = e^{\phi'''}\sqrt{-G_{tt}'''} = \frac{e^{-2\phi} - G_{tt}v^2}{e^{-\phi}\sqrt{-G_{tt}}} = e^{-\phi(4)} + v^2e^{\phi(4)}.
$$

(59)

This expression indicates us that when considering field configurations in which $v = 0$ we deal with a strong–weak coupling duality transformation. Thus, the discrete symmetry (22) contains the $S$–duality.

It should be pointed out as well that the discrete symmetry (24) does not correspond to a $S$–duality transformation since the implementation of the discrete symmetry (22) reverts the effect generated by the (18), i.e., (22) reverts the inversion of the sign that the four-dimensional dilaton field received under (18). Thus, as a result, the dilaton field does not change its sign under the discrete transformation (24). This fact can be seen as follows.

The expressions for the Ernst potentials in terms of the triple-primed fields are

$$
\epsilon_1 = \frac{-1}{\sqrt{-G_{tt}'''}e^{\phi'''}}, \quad \epsilon_2 = \frac{\sqrt{-G_{tt}'''e^{\phi'''}} - iv'''e^{2\phi'''}}{u'^{2}e^{2\phi'''} - G_{tt}'''}.
$$

(60)

Correspondingly, the expressions of the triple-primed field variables in terms of the unprimed ones read

$$
G_{tt}''' = 1/G_{tt}, \quad u''' = -v, \quad e^{\phi'''} = e^{\phi}, \quad v''' = -u.
$$

(61)
As before, we compute the expression for the transformed four–dimensional dilaton field

\[ e^{\phi'''} = \frac{e^\phi}{\sqrt{-G_{tt}}}. \]  

(62)

and observe that it does not correspond to the original four–dimensional dilaton field with the inverted sign. Thus, we can conclude that the discrete symmetry (24) does not involve a S–duality transformation.

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