MHD model of an incompressible polymeric fluid. Linear instability of the resting state

Alexander Blokhin$^{1,2}$ and Dmitry Tkachev$^{1,2}$
$^1$Novosibirsk State University, Novosibirsk, Russia, 630090
$^2$Sobolev Institute of Mathematics, Novosibirsk, Russia, 630090
E-mail: tkachev@math.nsc.ru

Abstract. We study the linear stability of a resting state for a generalization of the basic rheological Pokrovski–Vinogradov model for flows of solutions and melts of an incompressible viscoelastic polymeric medium to the nonisothermal case under the influence of magnetic field. We prove that the corresponding linearized problem describing magnetohydrodynamic flows of polymers in an infinite plane channel has the following property: for certain values of the conduction current which is given on the electrodes, i.e. on the channel boundaries, the problem has solutions whose amplitude grows exponentially (in the class of functions periodic along the channel).

In this work we study the generalization of the base structural-phenomenological Pokrovski–Vinogradov model that describes the flows of solutions and melts of incompressible viscoelastic polymeric mediums for the case when additionally the influence of heat and external magnetic field are taken into account. Fundamental for the model is the following representation of the medium: polymeric system is a suspension of polymer macromolecules, that move in anisotropic fluid, consisting, in case of solution for example, of the solvent and other macromolecules. The influence of the environment on the real macromolecule (in terms of the so called "monomolecular approximation") is approximated as the influence on the linear chain of Brownian particles, each of which represents a large enough part of the macromolecule. Brownian particles, often called "beads", are connected to each other by the sequence of elastic forces called "springs". In case of slow relaxation processes macromolecule is often considered as a chain of two particles called "dumbbell".

The physical representation of linear polymer flows described above results in the formulation of the Pokrovski–Vinogradov rheological model [1]:

\[
\rho \left( \frac{\partial}{\partial t} v_i + v_k \frac{\partial}{\partial x_k} v_i \right) = \frac{\partial}{\partial x_k} \sigma_{ik}, \quad \frac{\partial v_i}{\partial x_i} = 0,
\]

\[
\sigma_{ik} = -p \delta_{ik} + \frac{3}{\tau_0} \eta_0 a_{ik},
\]

\[
\frac{d}{dt} a_{ik} - v_{ij} a_{jk} - v_{kj} a_{ji} + \frac{1}{\tau_0} (k - \beta I) a_{ik} = \frac{2}{3} \gamma_{ik} - \frac{3}{\tau_0} a_{ij} a_{jk},
\]

\[
I = a_{11} + a_{22} + a_{33}, \quad \gamma_{ik} = \frac{v_{jk} + v_{ki}}{2},
\]
where $\rho$ is the polymer density, $v_i$ is the $i$-th velocity component, $\sigma_{ik}$ is the stress tensor, $p$ is the hydrostatic pressure, $\eta_0$, $\tau_0$ are the initial values of the shear viscosity and the relaxation time respectively for the viscoelastic component, $v_{ij}$ is the tensor of the velocity gradient, $a_{ik}$ is the symmetrical tensor of additional stresses of second rank, $I = a_{11} + a_{22} + a_{33}$ is the first invariant of the tensor of additional stresses, $\gamma_{ik} = \frac{\omega_{ik} + \omega_{ki}}{2}$ is the symmetrized tensor of the velocity gradient, $k$ and $\beta$ are the phenomenological parameters taking into account the shape and the size of the coiled molecule in the dynamics equations of the polymer macromolecule.

Structurally, the model consists of the incompressibility and motion equations (1) as well as the rheological relations (2), (3) connecting kinematic characteristics of the flow and its inner thermodynamic parameters.

Acquired for the viscosimetric flows results testify to the applicability of the defining rheological relation (2) for the description of the stationary and nonstationary shear flows of melts and solutions of linear polymers in a rather wide range of shear velocities. It is worth noting that model parameters $\beta$ and $k$ don’t depend on the molecular weight of the polymer and its concentration. The dynamic characteristics of the linear polymers (shear modulus and loss modulus) in the regime of overlaying small oscillating perturbations in parallel and orthogonal to the shear directions on the stationary shear flows were found. These results on the qualitative level match the results of experimental data.

The generalization of the base rheological Pokrovski–Vinogradov model is detailed in the first paragraph and describes the influence of the heat and magnetic field on the flow of the polymeric fluid in an infinite plane channel. We should note that for some parameters these factors significantly change the character of the flows: for example if we heat the boundaries of the channel for the polymeric analogue of the viscous Poiseuille flow (Navier-Stokes model), then it changes the symmetry of the velocity profile and the external magnetic field can even change the direction of the velocity profile to the opposite of the pressure force [7].

In this work we study the Lyapunov linear stability of the resting state for the model describing the flow of the incompressible viscoelastic polymeric fluid in an infinite plane channel.

Linearized about the resting state model and formulation of the main results, namely statements 1 and 2, are done in second paragraph.

In the third and last paragraph while proving statements 1 and 2 we also get that if the quantity of a conductivity current on the channel walls is sufficiently small, then the linearized about the resting state problem (we consider periodic along the channel walls functions as a perturbation class) has exponentially growing with time solutions which guarantees linear instability by Lyapunov of the chosen solution.

Results of this work are closely connected to the results of papers [2–6], where we got asymptotic representations for eigenvalues of mixed problems that arise while describing polymeric flows in an infinite plane channel. There we used different generalizations of the Pokrovski-Vinogradov model as mathematical models and Poiseuille-type flows as base solutions.

Let us formulate a mathematical model describing magnetohydrodynamic flows of an incompressible polymeric fluid for which, in the equation for the inner energy we introduce some dissipative terms. In a dimensionless form this model reads (we keep the notations from [7]):

$$\text{div}\vec{u} = u_x + v_y = 0, \quad \text{(4)}$$

$$\text{div}\vec{H} = L_x + M_y = 0, \quad \text{(5)}$$

$$\frac{d\vec{u}}{dt} + \nabla P = \text{div}(Z\Pi) + \sigma_m(\vec{H}, \nabla)\vec{H} + Gr(Z - 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{(6)}$$

$$\frac{da_{11}}{dt} - 2A_1u_x - 2a_{12}u_y + L_{11} = 0, \quad \text{(7)}$$
\[
\begin{align*}
\frac{da_{22}}{dt} - 2A_2v_y - 2a_{12}v_x + L_{22} &= 0, \\
\frac{da_{12}}{dt} - A_1v_x - A_2u_y + \frac{K_1a_{12}}{\tau_0(Z)} &= 0, \\
\frac{dZ}{dt} &= \frac{1}{Pr} \Delta_{x,y}Z + \frac{A_r}{Pr} ZD_l + \frac{A_m}{Pr} \sigma_m D_m, \\
\frac{d\bar{H}}{dt} &= (\bar{H}, \nabla)\bar{u} - b_m \Delta_{x,y}\bar{H} = 0,
\end{align*}
\]

where \( t \) is the time, \( u, v \) and \( L, 1 + M \) are the components of the velocity vector \( \bar{u} \) and the magnetic field \( \bar{H} \) respectively in the Cartesian coordinate system \( x, y \);

\[
P = p + \sigma_m \frac{L^2 + (1 + M)^2}{2},
\]

\( p \) is the pressure; \( a_{11}, a_{22}, a_{12} \) are the components of the symmetrical anisotropy tensor of second rank;

\[
\Pi = \frac{1}{Re}(a_{ij}), \quad i, j = 1, 2; \quad L_{ii} = \frac{K_1a_{ii} + \beta(a_{i1}^2 + a_{12}^2)}{\tau_0(Z)}, \quad i = 1, 2;
\]

\[
K_I = \frac{W^{-1} + \bar{k}}{3} I, \quad \bar{k} = k - \beta;
\]

\( I = a_{11} + a_{22} \) is the first invariant of the anisotropy tensor; \( k, \beta \) \((0 < \beta < 1)\) are the phenomenological parameters of the rheological model; \( A_i = Wi^{-1} + a_{ii}, i = 1, 2; Z = \frac{T_0}{T} \); \( T \) is the temperature, \( T_0 \) is an average temperature (room temperature; we will further assume that \( T_0 = 300 \) K);

\[
\bar{K}_I = K_I + \beta I; \quad \bar{\tau}_0(Z) = \frac{1}{Z(Z)} J(Z) = \exp\{E_A \frac{Z - 1}{Z}\},
\]

\( E_A = \frac{E_A}{T_0} \), constants \( E_A \) (activation energy), \( Re \) (Reynolds number), \( Wi \) (Weissenberg number), \( Gr \) (Grashoff number), \( Pr \) (Prandtl number), \( Ra \) (Rayleigh number), \( A_r, A_m \) (dissipative coefficients),

\[
D_\Gamma = a_{11}u_x + (v_x + u_y)a_{12} + a_{22}v_y, \\
D_m = L^2u_x + L(1 + M)(v_x + u_y) + (1 + M)^2v_y, \\
\frac{d}{dt} \frac{\partial}{\partial t} + (\bar{u}, \nabla) = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}, \\
\Delta_{x,y} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \text{ is the Laplace operator.}
\]

The variables \( t, x, y, u, v, p, a_{11}, a_{22}, a_{12}, L, M \) in system (4)–(11) correspond to the following values:

\[
\frac{1}{u_0}, l, \theta, pu_0, \frac{\bar{u}^2}{2}, Z, T_0, H_0, \text{ where } H_0 \text{ is the characteristic magnitude of the magnetic field (see figure 1).}
\]

Our main problem is the problem of finding solutions to the mathematical model (4)–(11) describing magnetohydrodynamic flows of an incompressible polymeric fluid in a plane channel with the depth \( 1(l) \) and bounded by horizontal walls which are the electrodes \( C^+ \) and \( C^- \) along which we have electric currents with the current strength of \( J^+ \) of \( J^- \) respectively (see figure 1).

The channel is placed in a uniform external magnetic field with the components \( L = 0 \) and \( M = 0 \), i.e., \( 1 + M = 1(H_0) \).

The domains \( S^+_1 \) and \( S^-_1 \) external for the channel are magnets with the magnetic susceptibilities \( \chi^-_1 \) and \( \chi^+_1 \). On the walls of the channel the following boundary conditions hold:

\[
\begin{aligned}
y &= \pm \frac{1}{2} : \quad \bar{u} = 0 \quad \text{(no-slip condition)}, \\
y &= \frac{1}{2} : \quad Z = 1 \quad (T = T_0), \\
y &= -\frac{1}{2} : \quad Z = 1 + \bar{\theta} \quad (\bar{\theta} = \frac{\theta}{T_0}, \theta = T - T_0).
\end{aligned}
\]
The temperature $T = T_0$ in the domain $S_1^+$ and on the electrode $C^+$;

$$y = -\frac{1}{2} : Z = 1 + \bar{\theta}, \bar{\theta} = \frac{\theta}{T_0}, \theta = T - T_0,$$

i.e., for $\bar{\theta} > 0$ there is heating from below ($T$ is the temperature in the domain $S_1^-$ and on the electrode $C^-$), and for $\bar{\theta} < 0$ there is heating from above.

**Remark 1** We will consider the electrodes $C^+$ and $C^-$ as the boundaries between two uniform isotropic magnetics. Therefore, on the boundaries $C^+$ and $C^-$ the following known conditions hold:

$$\begin{align*}
y &= \frac{1}{2}(C^+) : & L &= -J^+, & M_y &= 0, \\
y &= -\frac{1}{2}(C^-) : & L &= -J^-, & M_y &= 0.
\end{align*}$$

(13)

We get the boundary condition $M_y = 0$ at $y = \pm \frac{1}{2}$ by assuming that relation (5) holds for $y = \pm \frac{1}{2}$ and by taking into account the conditions $L = -J^+$ ($y = \frac{1}{2}$) and $L = -J^-$ ($y = -\frac{1}{2}$) (see (13)) that gives us $M_y = 0$ ($y = \pm \frac{1}{2}$).

Let's choose the resting state $\hat{u} \equiv 0$, $\hat{\alpha}_{11} = \hat{\alpha}_{12} = \hat{\alpha}_{22} \equiv 0$, $p(t, x, y) = \hat{p}_0 = \text{const}$, $\hat{Z} \equiv 1$ (here and below $\hat{\theta} = 0$), $\hat{L} = J_0$ (here and below $J^\pm = J_0$), $\hat{M} = \lambda(= \chi_0^+ = \chi_0^-)$ as a base solution for problem (4)–(11), (12), (13).

Linearization of the system (4)–(11) with respect to the chosen solution gives us the linear
system:
\[
\begin{align*}
&u_t - (\alpha_{11} u_x + (\alpha_{22})_x - (\alpha_{12}) y + \Omega_x + \sigma_m (1 + \hat{\lambda}) \omega_m = 0, \\
v_t - (\alpha_{12})_x + \Omega_y - Gr Z + \sigma_m J_0 \omega_m = 0, \\
(\alpha_{11})_t - 2 \varepsilon^2 u_{xx} + W^{-1} \alpha_{11} = 0, \\
(\alpha_{12})_t - \varepsilon^2 v_{xx} - \varepsilon^2 u_{yy} + W^{-1} \alpha_{12} = 0, \\
(\alpha_{22})_t - 2 \varepsilon^2 v_{yy} + W^{-1} \alpha_{22} = 0,
\end{align*}
\]  
(14)

Here \( \Omega = p - \alpha_{22}, \omega_m = M_x - L_y, \varepsilon^2 = \frac{1}{W \varepsilon}. \) Note that relation (5) is omitted due to the remark 1.4. Moreover, if \( (\alpha_{11} + \alpha_{22})|_{t=0} = 0, \) then from third, fifth and last equations of the system (14) it follows that

\[
\alpha_{11} = -\alpha_{22} \quad |y| < \frac{1}{2}, \quad t > 0, \quad x \in R^1.
\]  
(15)

Boundary conditions (12), (13) are transformed into the following for the problem (14):

\[
u = v = Z = L = M_y = 0, \quad y = \pm \frac{1}{2}, \quad t > 0, \quad x \in R^1.
\]  
(16)

Lets assume that areas \( S_1^+ \) contain nonconductive mediums. Then, due to the Maxwell equations [8], small perturbations of \( M_1^\pm \), independent of the perturbation time, satisfy the Laplace equation with additional condition at the infinity:

\[
\begin{align*}
\Delta_{x,y} M_1^+ &= 0, \quad S_1^+, \quad M_1^+ \to 0 \quad y \to \infty, \\
\Delta_{x,y} M_1^- &= 0, \quad S_1^-, \quad M_1^- \to 0 \quad y \to -\infty.
\end{align*}
\]  
(17)

If components \( M_1^\pm \) are periodic functions with respect to \( x \), then

\[
M_1^\pm (x, y) = \tilde{M}^\pm (y) e^{i\omega x}, \quad \omega \in R,
\]
and from (17) we get that

\[
\begin{align*}
\tilde{M}_1^+ (y) &= M_1^+|_{y=\frac{1}{2} + 0} e^{-|\omega|(y - \frac{1}{2})}, \quad y > \frac{1}{2}, \\
\tilde{M}_1^- (y) &= M_1^-|_{y=\frac{1}{2} - 0} e^{-|\omega|(y + \frac{1}{2})}, \quad y < -\frac{1}{2}.
\end{align*}
\]

Thus, due to the continuity of magnetic induction vector normal component on the channel boundaries, the components \( L_1^\pm, 1 + M_1^\pm \) of the tension vector \( \tilde{H} \) in areas \( S_1^\pm \) are defined.

Lets consider a particular case of the model (14), (16), where \( b_m = 0 \) (absolute conductivity) and additionally assume that \( A_m = 0. \) Then the problem (14), (16) is significantly simplified and describes connections only between four unknowns: \( u, v, \alpha_{12}, \Omega. \)

We choose functions periodic with respect to \( x \) functions as our perturbation class, specifically

\[
\begin{align*}
\tilde{U} = (u, v, \alpha_{12})^T &= \tilde{V}(y) \exp\{\lambda t + i\omega x\}, \\
\Omega &= \tilde{\Omega}(y) \exp\{\lambda t + i\omega x\},
\end{align*}
\]  
(18)

where \( \lambda = \eta + i\omega_0, \omega_0, \omega \in R \) (below symbol "\( \sim \)" for the unknown functions is omitted).
Using representation (18), we transform the boundary problem (14), (16) to the spectral problem

\[
\begin{cases}
\vec{G}' = A\vec{G}, & |y| < \frac{1}{2}, \\
L\vec{G}(\pm\frac{1}{2}) = 0,
\end{cases}
\]

(19)

where

\[
\vec{G} = (u, v, \alpha_{12}, \Omega)^T, \quad L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

\[
A = \begin{pmatrix}
0 & -i\omega & q_0 & 0 \\
-i\omega & 0 & 0 & 0 \\
\bar{\rho}_1 & a_1 & a_2 & \bar{\rho}_0 i\omega \\
\bar{\rho}_0 & b_1 & b_2 & b_3 & a_2
\end{pmatrix},
\]

Constants \(\bar{\rho}_1, \rho_1, \rho_0, a_1, b_1, b_2, b_3\) are defined through physical parameters of the problem (4)–(10), (12), (13). It is true that

**Statement 1** If \(J_0\) is sufficiently small, then the boundary problem (19) has nontrivial solutions for \(\text{Re}\lambda(=\eta) > 0\).

And consequently it is also true that

**Statement 2** The resting state of the system (4)–(11) with boundary conditions (12), (13) \((\bar{\theta} = 0, J^\pm = J_0, J_0\) is sufficiently small in case of the absolute conductivity \((b_m = 0)\), and if additionally \(A_m = 0\), then the resting state is linearly unstable by Lyapunov.

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