Achieving Global Optimality for Weighted Sum-Rate Maximization in the $K$-User Gaussian Interference Channel with Multiple Antennas

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Abstract

Characterizing the global maximum of weighted sum-rate (WSR) for the $K$-user Gaussian interference channel (GIC), with the interference treated as Gaussian noise, is a key problem in wireless communication. However, due to the users’ mutual interference, this problem is in general non-convex and thus cannot be solved directly by conventional convex optimization techniques. In this paper, by jointly utilizing the monotonic optimization and rate profile techniques, we develop a new framework to obtain the globally optimal power control and/or beamforming solutions to the WSR maximization problems for the GICs with single-antenna transmitters and single-antenna receivers (SISO), single-antenna transmitters and multi-antenna receivers (SIMO), or multi-antenna transmitters and single-antenna receivers (MISO). It is assumed that the transmitted signals have circularly symmetric complex Gaussian distributions and are independent over time. Different from prior work, this paper proposes to maximize the WSR in the achievable rate region of the GIC directly by exploiting the facts that the achievable rate region is a “normal” set and the users’ WSR is a “strictly increasing” function over the rate region. Consequently, the WSR maximization is shown to be in the form of monotonic optimization over a normal set and thus can be solved globally optimally by the existing outer polyblock approximation algorithm. However, an essential step in the algorithm hinges on how to efficiently characterize the intersection point on the Pareto boundary of the achievable rate region with any prescribed “rate profile” vector. This paper shows that such a problem can be transformed into a sequence of signal-to-interference-plus-noise ratio (SINR) feasibility problems, which can be solved efficiently by existing techniques. Numerical results validate that the proposed algorithms can achieve the global WSR maximum for the SISO, SIMO or MISO GIC, which serves as a performance benchmark for existing heuristic algorithms.

Index Terms

Beamforming, power control, interference channel, multi-antenna system, non-linear optimization, weighted sum-rate maximization.

I. INTRODUCTION

Gaussian interference channel (GIC) is a basic mathematical model that characterizes many real-life interference-limited communication systems. The information-theoretic study on the GIC has a long history, but the capacity...
region of the GIC still remains unknown in general, even for the two-user case. The best achievable rate region for the two-user GIC to date was established by Han and Kobayashi in [1], which utilizes rate splitting at transmitters, joint decoding at receivers, and time sharing among codebooks. This achievable rate region was recently proven to be within 1-bit of the capacity region of the GIC in [2]. However, capacity-approaching techniques in general require non-linear multi-user encoding and decoding, which may not be suitable for practical systems. A more pragmatic approach that leads to suboptimal achievable rates is to allow only single-user encoding and decoding by treating the interference from all other unintended users as additive Gaussian noise. For this approach, the key design challenge lies in how to optimally allocate transmit resources such as power, bandwidth, and antenna beam among different users to minimize the network performance loss due to their mutual interference. Recently, [3] showed that the circularly symmetric complex Gaussian (CSCG) distribution for the transmitted signals is in general non-optimal for the rate maximization in GIC with the interference treated as noise. By means of symbol extensions over time and/or asymmetric complex signaling, the weighted sum-rate (WSR) of GIC can be further improved. However, to our best knowledge, applying such techniques will result in more complicated WSR maximization problems, for which how to obtain the globally optimal solutions still remains an open problem, even for the case of 2-user GIC. Thus, for simplicity, in this paper we adopt the conventional assumption for the GIC that the transmitted signals have an independent CSCG distribution over time.

The research on the GIC with interference treated as noise has recently drawn significant attention due to the advance in cooperative inter-cell interference (ICI) management for cellular networks. Traditionally, most of the studies on resource allocation for cellular networks focus on the single-cell setup, while the ICI experienced by a receiver in one cell caused by the transmitters in other cells is minimized by means of frequency reuse, which avoids the same frequency band to be used by adjacent cells. However, most beyond-3G wireless systems advocate to increase the frequency reuse factor and even allow it to be one or so-called “universal frequency reuse”, due to which the issue of ICI becomes more crucial. Consequently, joint resource allocation across neighboring cells becomes a practically appealing approach for managing the ICI. If the mobile stations (MSs) in each cell are separated for transmission in frequency via orthogonal frequency-division multiple-access (OFDMA) or in time via time-division multiple-access (TDMA), then the active links in different cells transmitting at the same frequency tone or in the same time slot will interfere with each other, which can be modeled by a GIC. More specifically, if the base stations (BS) and MSs are each equipped with one single antenna, the system can be modeled as the single-input single-output (SISO) GIC, termed as SISO-IC. If the BSs are each equipped with multiple antennas while MSs are each equipped with one single antenna, then in the uplink the system can be modeled as the single-input multiple-output (SIMO) GIC, termed as SIMO-IC, and in the downlink as the multiple-input single-input (MISO) GIC, termed as MISO-IC.

The achievable rate region of SISO-IC, SIMO-IC or MISO-IC, with the single-user detection (SUD) by treating the interference as Gaussian noise, is in general a non-convex set due to the coupled interference among users. As

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1It is worth noting that even for the traditional single-cell setup with space-division multiple-access (SDMA), i.e., the multi-antenna BS simultaneously communicating with more than one single-antenna MSs, the MISO-IC and SIMO-IC models are also applicable if the linear transmit/receive precoding/equalization is implemented at the BS.
a result, how to efficiently find the optimal power control and/or beamforming solutions to achieve the maximum WSR for different types of GICs is a challenging problem. It is worth noting that a great deal of valuable scholarly work [4]-[14] has contributed to resolving this problem. For SISO-IC, various efficient power control schemes have been studied. The authors in [4] showed that in the two-user case the optimal power allocation to the sum-rate maximization problem is “binary”, i.e., either one user transmits with full power and the other user shuts down, or both users transmit with full power. However, this result does not hold in general when the number of users is greater than two. Based on game theory, an “asynchronous distributed pricing (ADP)” algorithm was proposed in [5] whereby locally optimal solutions can be obtained for WSR maximization. In [6], the WSR maximization problem was transformed into a signomial programming (SP) problem, which was efficiently solved by constructing a series of geometric programming (GP) problems through the approach of successive convex approximation. Similar to ADP, this algorithm only guarantees locally optimal solutions. As for the case of parallel SISO-IC, the authors in [7], [8] showed that the duality gap for the WSR maximization problem is zero when the number of parallel GICs becomes asymptotically large. As a result, the Lagrange duality method can be applied to decouple the problem into parallel sub-problems in the dual domain. However, the power optimization in each sub-problem for a given GIC is still non-convex. For an extensive survey of power control algorithms for SISO-IC, please refer to [9]. Furthermore, for MISO-IC, the optimality of transmit beamforming for achieving the maximum WSR with SUD has been proven in [10], [11]. In [10], [12], the complete characterization of all Pareto optimal rates for MISO-IC was studied. To maximize the WSR, an iterative algorithm was proposed in [13] from an egotistic versus altruistic viewpoint, and other “price-based” algorithms (see, e.g., [14] and references therein) were also developed. However, these algorithms in general cannot achieve the global WSR maximum for MISO-IC.

Different from the above prior work in which the power and/or beamforming vectors were optimized directly for WSR maximization in the GIC, in this paper we propose a new approach that maximizes the WSR in the achievable rate region of the GIC directly. This approach is based on the following two key observations: (1) the WSR is a strictly increasing function with respect to individual user rates; and (2) the achievable rate region is a “normal” set [15]. Accordingly, the WSR maximization problem for the GIC belongs to the class of optimization problems so-called monotonic optimization over a normal set, for which the global optimality can be achieved by an iterative “outer polyblock approximation” algorithm [15]. However, one challenging requirement of this algorithm is a unique characterization of the Pareto boundary of the achievable rate region since at each iteration of the algorithm one particular point on the Pareto boundary that corresponds to the maximum achievable sum-rate in a prescribed direction needs to be determined. This problem is efficiently solved in this paper by utilizing a so-called “rate profile” approach [16], which transforms the original problem into a sequence of signal-to-interference-plus-noise ratio (SINR) feasibility problems. It is also shown in this paper that such feasibility problems can be efficiently solved by existing techniques for various types of GICs.

It is worth noting that rate profile was first proposed in [16] as an alternative method to WSR maximization for characterizing the Pareto boundary of the capacity region for the multi-antenna Gaussian multiple-access channel (MAC), which is a convex set. This method was later applied to characterize the Pareto boundary of non-convex rate regions for the MISO-IC in [10] and the two-way multi-antenna relay channel in [17], for which the WSR
maximization approach is not directly applicable. A very similar idea to rate profile was also proposed in [18], where the proportional rate fairness is imposed as a constraint for WSR maximization in multi-user OFDM systems. As for the outer polyblock approximation algorithm, it was first proposed in [15], and later applied in [19] and [20] to solve the WSR maximization problems for the GIC. In [19], this algorithm was applied for SISO-IC together with the generalized linear fractional programming, which, however, cannot be extended to SIMO-IC or MISO-IC. In [20], this algorithm was applied to the two-user MISO-IC by exploiting a prior result in [12] that the optimal transmit beamforming vector to achieve any Pareto boundary rate-pair can be expressed as a linear combination of the zero-forcing (ZF) and maximum-ratio transmission (MRT) beamformers. However, this result only holds for the two-user MISO-IC and thus how to extend the algorithm in [20] to MISO-IC with more than two users remains unknown. In comparison, in this paper we show that by jointly utilizing the outer polyblock approximation algorithm and rate profile approach, the global optimality of the WSR maximization problem can be achieved for all SISO-IC, SIMO-IC and MISO-IC, with arbitrary number of users.

It is also worth noting that for the WSR maximization in SISO-IC, besides [19] that applies the outer polyblock approximation algorithm, there have been other algorithms developed based on the branch and bound method. For example, in [21] and [22], branch and bound methods combined with difference of convex functions (DC) programming have been proposed. A generalized branch and bound method applicable to problems in which the objective function cannot be expressed in the form of DC, has also been proposed in [23]. In this paper, we propose an alternative approach to that in the above prior work, whereby the WSR maximization problems for SISO-IC, SIMO-IC, and MISO-IC are all solvable.

The rest of this paper is organized as follows. Section II introduces the system models for various GICs including SISO-IC, SIMO-IC and MISO-IC, and formulates their WSR maximization problems. Section III presents a new framework to solve the formulated problems based on monotonic optimization and rate profile techniques. Section IV completes the proposed algorithms by addressing the solutions to various SINR feasibility problems. Section V provides numerical examples to validate the proposed results. Finally, Section VI concludes the paper.

Notation: Scalars are denoted by lower-case letters, vectors denoted by bold-face lower-case letters, and matrices denoted by bold-face upper-case letters. $I$ and $0$ denote an identity matrix and an all-zero matrix, respectively, with appropriate dimensions. For a square matrix $S$, $S^{-1}$ denotes its inverse (if $S$ is full-rank). For a matrix $M$ of arbitrary size, $M^H$ and $M^T$ denote the conjugate transpose and transpose of $M$, respectively. $\text{Diag}(X_1, \cdots, X_K)$ denotes a block diagonal matrix with the diagonal matrices given by $X_1, \cdots, X_K$. The distribution of a CSCG random vector with mean vector $\mathbf{x}$ and covariance matrix $\Sigma$ is denoted by $CN(\mathbf{x}, \Sigma)$; and $\sim$ stands for “distributed as”. $\mathbb{C}^{x \times y}$ denotes the space of $x \times y$ complex matrices. $\mathbb{R}$ denotes the real number space, while $\mathbb{R}_x$ denotes the $x \times 1$ real vector space and $\mathbb{R}_{x+}$ denotes its non-negative orthants. $||x||$ denotes the Euclidean norm of a complex vector $x$. $e_k$ denotes a vector with its $k$th component being 1, and all other components being 0. For two real vectors $x$ and $y$, $x \geq y$ means that $x$ is greater than or equal to $y$ in a component-wise manner.

II. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a $K$-user GIC, in which $K$ mutually interfering wireless links communicate simultaneously over a common bandwidth, as shown in Fig. 1. Firstly, consider the case where all transmitters and receivers are each
equipped with one single antenna. The system is thus modeled as SISO-IC, for which the discrete-time baseband signal received at the $k$th receiver is given by
\[
y_k = h_{k,k} \sqrt{p_k} x_k + \sum_{j \neq k} h_{k,j} \sqrt{p_j} x_j + z_k, \quad k = 1, \cdots, K, \tag{1}
\]
where $h_{k,j}$ denotes the complex channel gain from the $j$th transmitter to the $k$th receiver, $p_k$ denotes the transmit power of the $k$th transmitter, $x_k$ denotes the transmitted signal from the $k$th transmitter, and $z_k$ denotes the background noise at the $k$th receiver. It is assumed that $z_k \sim \mathcal{CN}(0, \sigma_k^2)$, $\forall k$, and all $z_k$'s are independent.

We assume independent encoding across different transmitters and thus $x_k$'s are independent over $k$. It is also assumed that the Gaussian codebook is used and thus $x_k$'s are independent over $k$. Accordingly, the SINR of the $k$th receiver is expressed as
\[
\gamma_{SISO-IC}^k = \frac{\| h_{k,k} \|^2 p_k}{\sum_{j \neq k} \| h_{k,j} \|^2 p_j + \sigma_k^2}.
\tag{2}
\]

**Remark 2.1:** It is worth noting that in the above signal model, we have made the following two assumptions:

A1. The interference is treated as additive Gaussian noise.

A2. The Gaussian input $x_k$ for user $k$ is assumed to be CSCG distributed and independent over time, i.e., asymmetric Gaussian signalling with time-domain symbol expansion in [3] is not used.

Note that for the subsequent studies on SIMO-IC and MISO-IC, the above two assumptions are similarly made.

Secondly, consider the case where all transmitters are each equipped with one single antenna but each receiver is equipped with multiple antennas, i.e., SIMO-IC. Assuming that the $k$th receiver is equipped with $M_k > 1$ antennas, its discrete-time baseband received signal is given by
\[
y_k = w_k^H (h_{k,k} \sqrt{p_k} x_k + \sum_{j \neq k} h_{k,j} \sqrt{p_j} x_j + z_k), \quad k = 1, \cdots, K, \tag{3}
\]

![System model for the K-user MISO-IC (SISO-IC if each transmitter has one single antenna, or SIMO-IC in the reverse link transmission).](image-url)
where \( \mathbf{w}_k^H \in \mathbb{C}^{1 \times M_k} \) is the receive beamforming vector for the \( k \)th receiver, \( \mathbf{h}_{k,j} \in \mathbb{C}^{M_k \times 1} \) is the channel vector from the \( j \)th transmitter to the \( k \)th receiver, and \( z_k \in \mathbb{C}^{M_k \times 1} \) is the noise vector at the \( k \)th receiver. It is assumed that \( z_k \sim \mathcal{CN}(0, \sigma_k^2 \mathbf{I}) \). Thus, the SINR of the \( k \)th receiver can be expressed as

\[
\gamma_k^{\text{SIMO-IC}} = \frac{p_k \| \mathbf{w}_k^H \mathbf{h}_{k,k} \|^2}{\mathbf{w}_k^H (\sum_{j \neq k} p_j \mathbf{h}_{k,j} \mathbf{h}_{k,j}^H + \sigma_k^2 \mathbf{I}) \mathbf{w}_k}.
\]

(4)

Thirdly, consider the MISO-IC case in which all transmitters are each equipped with multiple antennas while each receiver is equipped with one single antenna. Assume that the \( k \)th transmitter is equipped with \( N_k > 1 \) antennas. The discrete-time baseband signal at the \( k \)th receiver is then given by

\[
y_k = \mathbf{h}_{k,k}^H \mathbf{v}_k x_k + \sum_{j \neq k} \mathbf{h}_{k,j}^H \mathbf{v}_j x_j + z_k, \quad k = 1, \ldots, K,
\]

(5)

where \( \mathbf{v}_k \in \mathbb{C}^{N_k \times 1} \) is the transmit beamforming vector at the \( k \)th transmitter, and \( \mathbf{h}_{k,j}^H \in \mathbb{C}^{1 \times N_j} \) denotes the channel vector from the \( j \)th transmitter to the \( k \)th receiver. Accordingly, the SINR of the \( k \)th receiver can be expressed as

\[
\gamma_k^{\text{MISO-IC}} = \frac{\| \mathbf{h}_{k,k}^H \mathbf{v}_k \|^2}{\sum_{j \neq k} \| \mathbf{h}_{k,j}^H \mathbf{v}_j \|^2 + \sigma_k^2}.
\]

(6)

**Remark 2.2:** It is worth noting that for the above signal model for MISO-IC, we assume that all transmitters employ rank-one beamforming. This is because it has been shown in [10] and [11] that under Assumptions A1 and A2, beamforming achieves all the points on the Pareto boundary of the achievable rate region for MISO-IC, i.e., beamforming is optimal for WSR maximization in MISO-IC.

With \( \gamma_k \) defined in (2), (4) or (6), the achievable rate of the \( k \)th receiver can be formulated as

\[
R_k(\gamma_k) = \log_2(1 + \gamma_k), \quad k = 1, \ldots, K.
\]

(7)

Next, we define the achievable rate region for each type of GIC, which constitutes all the rate-tuples simultaneously achievable by all the users under a given set of transmit-power constraints denoted by \( P_1^{\text{max}}, \ldots, P_K^{\text{max}} \):

\[
\mathcal{R}^{\text{SISO-IC}} \triangleq \bigcup_{\{p_k\} : 0 \leq p_k \leq P_k^{\text{max}}, \forall k} \left\{ (r_1, \ldots, r_K) : 0 \leq r_k \leq R_k(\gamma_k^{\text{SISO-IC}}), k = 1, \ldots, K \right\},
\]

(8)

\[
\mathcal{R}^{\text{SIMO-IC}} \triangleq \bigcup_{\{p_k, \{\mathbf{w}_k\} : 0 \leq p_k \leq P_k^{\text{max}}, \forall k} \left\{ (r_1, \ldots, r_K) : 0 \leq r_k \leq R_k(\gamma_k^{\text{SIMO-IC}}), k = 1, \ldots, K \right\},
\]

(9)

\[
\mathcal{R}^{\text{MISO-IC}} \triangleq \bigcup_{\{\mathbf{v}_k\} : 0 \leq \| \mathbf{v}_k \| \leq P_k^{\text{max}}, \forall k} \left\{ (r_1, \ldots, r_K) : 0 \leq r_k \leq R_k(\gamma_k^{\text{MISO-IC}}), k = 1, \ldots, K \right\}.
\]

(10)

The upper-right boundary of each defined rate region is called the **Pareto boundary**, constituted by rate-tuples for each of which it is impossible to improve one particular user’s rate without decreasing the rate of at least one of the other users.

The WSR maximization problems for SISO-IC, SIMO-IC and MISO-IC are then formulated as (P1.1)-(P1.3) as follows.

\[
(P1.1): \quad \text{Maximize} \quad U(p) := \sum_{k=1}^{K} \mu_k R_k(\gamma_k^{\text{SISO-IC}})
\]

Subject to \( 0 \leq p_k \leq P_k^{\text{max}}, \forall k \),
\[ (P1.2): \text{Maximize} \quad U(W, p) := \sum_{k=1}^{K} \mu_k R_k(\gamma_k^{\text{SIMO-IC}}) \]

Subject to \[ 0 \leq p_k \leq P_k^{\text{max}}, \forall k, \]

\[ (P1.3): \text{Maximize} \quad U(V) := \sum_{k=1}^{K} \mu_k R_k(\gamma_k^{\text{MISO-IC}}) \]

Subject to \[ \|v_k\|^2 \leq P_k^{\text{max}}, \forall k, \]

where \( p = (p_1, \cdots, p_K) \) denotes the transmit power vector, \( W = (w_1, \cdots, w_K) \) and \( V = (v_1, \cdots, v_K) \) constitute the receive and transmit beamforming vectors, respectively, and \( \mu_k \) is the non-negative rate weight for user \( k \). Since the objective functions are all non-concave with respect to the power values or beamforming vectors due to the coupled interference, all the WSR maximization problems in \( (P1.1)-(P1.3) \) are non-convex and thus cannot be solved globally optimally by conventional convex optimization techniques.

### III. Proposed Solution Based on Outer Polyblock Approximation and Rate Profile

In this section, we solve the formulated WSR maximization problems in \( (P1.1)-(P1.3) \) globally optimally by a new approach based on the outer polyblock approximation and rate profile techniques.

#### A. A New Look at the Problem: Optimizing WSR Directly in Rate Region

Traditionally, Problems \( (P1.1)-(P1.3) \) are solved in the power allocation and/or beamforming domain, which results in non-convex optimization problems. In this subsection, we study the WSR maximization problem utilizing a new formulation, which maximizes the WSR directly in the achievable rate region.

If the achievable rate vector \( r = (R_1, \cdots, R_K) \) is treated as the design variable, where \( R_k \) is the achievable rate of user \( k \) defined in (7), the WSR maximization problems \( (P1.1)-(P1.3) \) can be unified in the following form.

\[ (P2): \text{Maximize} \quad U(r) := \sum_{k=1}^{K} \mu_k R_k \]

Subject to \[ r \in \mathcal{R}, \]

where the rate region \( \mathcal{R} \) is defined in (8), (9) or (10) for SISO-IC, SIMO-IC or MISO-IC.

Next, we will show that Problem \( (P2) \) belongs to one special class of optimization problems: monotonic optimization over a “normal” set. Two useful definitions are given first as follows.

**Definition 3.1:** A function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be strictly increasing on \( \mathbb{R}_+^n \) if for any \( x', x \in \mathbb{R}_+^n, x' \geq x \) and \( x' \neq x \) imply that \( f(x') > f(x) \).

**Definition 3.2:** A set \( D \subseteq \mathbb{R}_+^n \) is called normal if given any point \( x \in D \), all the points \( x' \) with \( 0 \leq x' \leq x \) satisfy that \( x' \in D \).

Based on the above definitions, we declare the following two facts regarding Problem \( (P2) \), which can be easily verified to be true.

**Fact 1:** The objective function of Problem \( (P2) \) is a strictly increasing function with respect to \( r \).
Fact 2: The achievable rate region defined in (8), (9) or (10) is a normal set.

Facts 1 and 2 imply that Problem (P2) maximizes a strictly increasing function over a normal set. In [15], the “outer polyblock approximation” algorithm was proposed to achieve the global optimality for this type of problems. In the following, we will apply this algorithm to solve Problem (P2).

B. Outer Polyblock Approximation Algorithm

In this subsection, we introduce the outer polyblock approximation algorithm to solve Problem (P2). First, two definitions are given as follows.

Definition 3.3: Given any vector \( v \in \mathbb{R}^n_+ \), the hyper rectangle \([0, v] = \{x | 0 \leq x \leq v\}\) is referred to as a box with vertex \( v \).

Definition 3.4: A set is called a polyblock if it is the union of a finite number of boxes.

Next, we show one important property of the polyblock in the following proposition.

Proposition 3.1: The maximum of a strictly increasing function \( f(x) \) over a polyblock is achieved at one of the vertices of the polyblock.

Proof: Suppose that \( x^* \) is the globally optimal solution over the polyblock, and it is not a vertex. Then, there exists at least one vertex \( x' \) satisfying \( x' \geq x^* \) but \( x' \neq x^* \). Since \( f(x) \) is a strictly increasing function, \( f(x^*) < f(x') \) must hold, which contradicts to the presumption. The proof is thus completed.

According to Proposition 3.1, the maximum of an increasing function over a polyblock can be obtained efficiently by enumeration of the vertices of that polyblock. Consequently, we can construct a sequence of polyblocks to approximate the rate region \( \mathcal{R} \) with the increasing accuracy for Problem (P2). In other words, we need to find an iterative method to generate a sequence of polyblocks of shrinking sizes such that

\[
P^{(1)} \supset P^{(2)} \supset \cdots \supset \mathcal{R},
\]

\[
\lim_{n \to \infty} \left[ \max_{r \in P^{(n)}} U(r) \right] = \max_{r \in \mathcal{R}} U(r),
\]

where \( P^{(n)} \) denotes the polyblock generated at the \( n \)th iteration.

Next, we present one method to generate the polyblocks satisfying (11) and (12). Let \( \mathcal{Z}^{(n)} \) denote the set containing all the vertices of the polyblock \( P^{(n)} \). The vertex that achieves the maximum WSR in polyblock \( P^{(n)} \) can be formulated as

\[
\hat{z}^{(n)} = \arg \max_{z \in \mathcal{Z}^{(n)}} U(z).
\]

Define \( \delta \hat{z}^{(n)} \) as the line that connects the two points \( 0 \) and \( \hat{z}^{(n)} \), and \( r^{(n)} \) as the intersection point on the Pareto boundary with the line \( \delta \hat{z}^{(n)} \). The following method can be used to generate \( K \) new vertices adjacent to \( \hat{z}^{(n)} \):

\[
z^{(n), i} = \hat{z}^{(n)} - (\hat{z}^{(n)}_i - r^{(n)}_i) e_i, \quad i = 1, \ldots, K,
\]

where \( z^{(n), i} \) denotes the \( i \)th new vertex generated at the \( n \)th iteration; \( \hat{z}^{(n)}_i \) and \( r^{(n)}_i \) denote the \( i \)th element of vectors \( \hat{z}^{(n)} \) and \( r^{(n)} \), respectively. Then, the new vertex set can be expressed as

\[
\mathcal{Z}^{(n+1)} = \mathcal{Z}^{(n)} \backslash \hat{z}^{(n)} \bigcup \{z^{(n),1}, \ldots, z^{(n),K}\}.
\]
The polyblock $P^{(1)}$ with vertex $z^{(1)}$

The polyblock $P^{(2)}$ with vertices $z^{(1),1}, z^{(1),2}$

The polyblock $P^{(n)}$ which approaches the rate region with increasing $n$

Fig. 2. Illustration of the procedure for constructing new polyblocks.

Each vertex in the set $Z^{(n+1)}$ defines a box, and thus the new polyblock $P^{(n+1)}$ is the union of all these boxes. An illustration about the above procedure to generate polyblocks for the case of two-user rate region is given in Fig. 2. In the following proposition, we show the feasibility of the above polyblock generation method.

**Proposition 3.2:** If the rate region $\mathcal{R}$ is a normal set (as we have already shown), the polyblocks generated by (15) satisfy (11).

**Proof:** Please refer to [15].

Proposition 3.2 ensures that the above polyblock generation method can be used to approximate the rate region from the outside with increasing accuracy. Let $r^* = (R_1^*, \ldots, R_K^*)$ denote the optimal solution to Problem (P2). Based on the above method, in the following we present an algorithm to find $r^*$ in the rate region $\mathcal{R}$. It is worth noting that $r^*$ must be on the Pareto boundary of the rate region; thus, we only need to search over the Pareto boundary to find $r^*$.

The outer polyblock approximation algorithm works iteratively as follows. In the $n$th iteration, the optimal vertex $\tilde{z}^{(n)}$ is first obtained by (13). According to Proposition 3.1, in the polyblock $P^{(n)}$ the maximum WSR is $U(\tilde{z}^{(n)})$. Since Proposition 3.2 implies that $P^{(n)}$ always contains the rate region $\mathcal{R}$, $U(\tilde{z}^{(n)})$ is an upper bound of $U(r^*)$. Then, the intersection point $r^{(n)}$ on the Pareto boundary with the line $\delta \tilde{z}^{(n)}$ is obtained. Define the best intersection point up to the $n$th iteration as

$$\tilde{r}^{(n)} = \operatorname{arg\,max}\{U(r^{(n)}), U(\tilde{r}^{(n-1)})\}. \quad (16)$$
Consequently, $U(\tilde{r}^{(n)})$ is the tightest lower bound of $U(r^\star)$ by the $n$th iteration. Next, we can compute the value of $U(\tilde{z}^{(n)}) - U(\tilde{r}^{(n)})$, which is the difference between the upper and lower bounds of the optimal value of Problem (P2) at the $n$th iteration. If this difference is less than $\eta$ (a small positive number), the algorithm can terminate and $\tilde{r}^{(n)}$ is at least an $\eta$-optimal solution to Problem (P2) because

$$U(r^\star) - U(\tilde{r}^{(n)}) < U(\tilde{z}^{(n)}) - U(\tilde{r}^{(n)}) < \eta, \quad (17)$$

Otherwise, we construct a new polyblock $P^{(n+1)}$ by the above polyblock generation method. We repeat the above procedure until an $\eta$-optimal solution is found.

**TABLE I**

**OUTER POLYBLOCK APPROXIMATION ALGORITHM FOR SOLVING PROBLEM (P2)**

| Step | Description |
|------|-------------|
| a)   | Initialize: Set $n = 1$, $Z^{(1)} = \{z^{(1)}\}$; |
| b)   | **While ($\epsilon, \eta$)-accuracy is not reached, do** |
| 1)   | Find the optimal vertex $\tilde{z}^{(n)}$ that maximizes the WSR in the set $Z^{(n)}$ based on $\tilde{z}^{(n)} = \arg\max_{z \in Z^{(n)}} U(z)$, $\epsilon$ is a small positive number and $Z^{(n)} = \{z \in Z^{(n)} | z_k \geq \epsilon, \forall k\}$; |
| 2)   | Compute the intersection point $\tilde{r}^{(n)}$ on the Pareto boundary of the rate region $R$ with the line $\delta \tilde{z}^{(n)}$; |
| 3)   | Update the best intersection point until the $n$th iteration $\tilde{r}^{(n)}$ according to (16); |
| 4)   | If $U(\tilde{z}^{(n)}) - U(\tilde{r}^{(n)}) \leq \eta$, then Stop and $\tilde{r}^{(n)}$ is an ($\epsilon, \eta$)-optimal solution to Problem (P2); |
| 5)   | **else** Compute $K$ new vertices that are adjacent to $\tilde{z}^{(n)}$ by (14) and update the vertex set $Z^{(n+1)}$ by (15); |
| 6)   | **end** |
| 7)   | $n = n + 1$; |
| c)   | **end** |

The above algorithm, denoted as Algorithm I, is summarized in Table I. It is worth noting that in Algorithm I $\tilde{z}^{(n)}$ is obtained by enumeration in the set $Z^{(n)}$ rather than $Z^{(n)}$. This is because in [15] it was shown that if the optimal solution lies in a strip defined by $\{r^\star | 0 \leq R_k^\star \leq \epsilon\}$ with arbitrary $k$ and a small value $\epsilon > 0$, then as $\tilde{z}^{(n)}$ approaches this strip, the algorithm converges very slowly. Consequently, $\epsilon$ is chosen to balance the tradeoff between the accuracy and complexity of Algorithm I. With $\epsilon$, Algorithm I solves the following problem

$$(P2 - A): \text{Maximize } U(r) := \sum_{k=1}^{K} \mu_k R_k$$

Subject to $r \in R^\epsilon$, where $R^\epsilon$ is defined as

$$R^\epsilon = R \cap \{(r_1, \ldots, r_K) : r_k \geq \epsilon, \forall k\}. \quad (19)$$

Thus, the corresponding solution is called an ($\epsilon, \eta$)-optimal solution to Problem P2.
Next, we address the convergence issue of Algorithm I. According to Proposition 3.2, \( P^{(n)} \supset P^{(n+1)} \) always holds. Moreover, the optimal vertex \( \tilde{z}^{(n)} \) is removed from \( Z^{(n+1)} \) after each iteration. Thus, \( U(\tilde{z}^{(n+1)}) < U(\tilde{z}^{(n)}) \) also holds. Furthermore, the lower bound \( U(\tilde{r}^{(n)}) \) is non-decreasing. Consequently, the value of \( U(\tilde{z}^{(n)}) - U(\tilde{r}^{(n)}) \) will decrease after each iteration. It was shown in [15] that as \( n \) increases, the difference between the upper and lower bounds can be reduced to an arbitrary small positive number in a finite number of iterations. Thus, Algorithm I converges given small positive values \( \epsilon \) and \( \eta \). More details about the selection of the values of \( \epsilon \) and \( \eta \) will be given later in Section V-A.

Last, we explain how to obtain an initial vertex \( z^{(1)} = (z^{(1)}_1, \ldots, z^{(1)}_K) \) for the first iteration of Algorithm I. Since the box \([0, z^{(1)}] \) needs to contain the rate region \( \mathcal{R} \), for any user \( k \), \( z^{(1)}_k \) can be obtained when all other users switch off their transmission (thus no interference exists for user \( k \)), and user \( k \) transmits its maximum power \( P_{k}^{\text{max}} \). More specifically, for SISO-IC,

\[
z^{(1)}_k = \log_2 \left( 1 + \frac{P_{k}^{\text{max}} \| h_{k,k} \|^2}{\sigma_k^2} \right), \quad \forall k.
\]  

(20)

Since for MISO-IC,

\[
\gamma_{k}^{\text{MISO-IC}} = \frac{\| h_{k,k}^H w_k \|^2}{\sum_{j \neq k} \| h_{k,j}^H w_j \|^2 + \sigma_k^2} \leq \frac{P_{k}^{\text{max}} \| h_{k,k} \|^2}{\sigma_k^2}, \quad \forall k,
\]  

(21)

where (a) is due to the Cauchy-Schwarz inequality, \( z^{(1)}_k \) can thus be set as

\[
z^{(1)}_k = \log_2 \left( 1 + \frac{P_{k}^{\text{max}} \| h_{k,k} \|^2}{\sigma_k^2} \right), \quad \forall k.
\]  

(22)

The initial vertex for SIMO-IC can be obtained similarly to (22), and is thus omitted for brevity.

To summarize, the only challenge that remains unaddressed in Algorithm I is on how to compute the intersection point \( r^{(n)} \) on the Pareto rate boundary with the line \( \delta \tilde{z}^{(n)} \) at the \( n \)th iteration, which will be addressed next.

C. Finding Intersection Points by the “Rate Profile” Approach

In this subsection, we show how to obtain the intersection point on the Pareto boundary of the rate region with the line \( \delta \tilde{z}^{(n)} \), to complete Algorithm I. Let \( R_{\text{sum}} = \sum_{k=1}^{K} R_k \) denote the sum-rate of all the users, \( \alpha = \tilde{z}^{(n)}/ \sum_{k=1}^{K} \tilde{z}_k^{(n)} \) denote the slope of the line \( \delta \tilde{z}^{(n)} \). Consequently, the intersection point at the \( n \)th iteration can be expressed as \( r^{(n)} = R_{\text{sum}}^* \alpha \), where \( R_{\text{sum}}^* \) is the optimal value of the following problem:

Maximize \( R_{\text{sum}} \)
Subject to \( R_{\text{sum}} \alpha \in \mathcal{R} \).  

(23)

The above approach to find the intersection point on the Pareto boundary of the rate region is known as rate profile [10], [16], [17]. In the following, we solve Problem (23) to obtain the intersection point \( r^{(n)} \) on the Pareto boundary with a given \( \delta \tilde{z}^{(n)} \).
Problem (23) is solvable via solving a sequence of feasibility problems shown as follows. Given a target sum-rate \( \bar{R}_{\text{sum}} \), the feasibility problems for SISO-IC, SIMO-IC and MISO-IC can be expressed in the following problems (P3.1)-(P3.3), respectively.

(P3.1): Find \( \{ p_k \} \)

Subject to \[
\log_2 (1 + \gamma_k^{\text{SISO-IC}}) \geq \alpha_k \bar{R}_{\text{sum}}, \quad \forall k
\]
\[
p_k \leq P_k^{\text{max}}, \quad \forall k.
\]

(P3.2): Find \( \{ w_k \}, \{ p_k \} \)

Subject to \[
\log_2 (1 + \gamma_k^{\text{SIMO-IC}}) \geq \alpha_k \bar{R}_{\text{sum}}, \quad \forall k
\]
\[
p_k \leq P_k^{\text{max}}, \quad \forall k.
\]

(P3.3): Find \( \{ v_k \} \)

Subject to \[
\log_2 (1 + \gamma_k^{\text{MISO-IC}}) \geq \alpha_k \bar{R}_{\text{sum}}, \quad \forall k
\]
\[
\| v_k \|^2 \leq P_k^{\text{max}}, \quad \forall k.
\]

If any of Problems (P3.1), (P3.2) and (P3.3) is feasible, it follows that \( R^*_{\text{sum}} \geq \bar{R}_{\text{sum}} \); otherwise, \( R^*_{\text{sum}} < \bar{R}_{\text{sum}} \). Hence, \( R^*_{\text{sum}} \) can be obtained for Problem (23) by applying a simple bisection method \cite{24}, for which the detail is omitted for brevity.

The remaining challenge is on solving the feasibility problems (P3.1)-(P3.3), which is addressed next. Let \( \bar{\gamma}_k = 2^{\alpha_k \bar{R}_{\text{sum}}} - 1 \), \( \forall k \). Then, the first constraint of each feasibility problem can be re-expressed as

\[
\gamma_k \geq \bar{\gamma}_k, \quad \forall k.
\]

Therefore, given any sum-rate target \( \bar{R}_{\text{sum}} \), the feasibility problems (P3.1)-(P3.3) are equivalent to finding whether a corresponding SINR target vector \( \bar{\gamma} = (\bar{\gamma}_1, \cdots, \bar{\gamma}_K) \) is achievable. In the next section, we will propose efficient algorithms to solve these SINR feasibility problems.

Remark 3.1: In the case where a set of minimum rate constraints \( R_k \geq R_k^{\text{min}}, \quad \forall k \), are added to the WSR maximization problem (P2), where \( R_k^{\text{min}} \) is the minimum rate required for user \( k \), we can solve this new problem by modifying Algorithm I as follows. Since the new rate region \( \mathcal{R}' \) is the intersection of the original rate region with the set \( \{(r_1, \cdots, r_K) : r_k \geq R_k^{\text{min}}, \quad \forall k\} \), we should change the initial point from \( 0 \) to \( r^{\text{min}} \) in Algorithm I where \( r^{\text{min}} = (R_1^{\text{min}}, \cdots, R_K^{\text{min}}) \) is the rate constraint vector. Thus, at each iteration we need to compute the intersection point on the Pareto boundary with the line passing through the optimal vertex \( \tilde{z}^{(n)} \) and the point \( r^{\text{min}} \) (instead of \( 0 \) in Algorithm I). In addition, any point \( r \) on this line with \( R_{\text{sum}} = \sum_{k=1}^{K} R_k \) can be rewritten as

\[
r = r^{\text{min}} + \alpha (R_{\text{sum}} - \sum_{k=1}^{K} R_k^{\text{min}}),
\]

where the rate profile \( \alpha \) is obtained by

\[
\alpha = \frac{\tilde{z}^{(n)} - r^{\text{min}} \cdot \sum_{k=1}^{K} R_k^{\text{min}}}{\sum_{k=1}^{K} \tilde{z}_k^{(n)} - \sum_{k=1}^{K} R_k^{\text{min}}}.
\]
IV. SOLUTIONS TO SINR FEASIBILITY PROBLEMS

In this section, we solve Problems (P3.1)-(P3.3) subject to the equivalent SINR constraints given in (24) for SISO-IC, SIMO-IC and MISO-IC, respectively.

A. The SISO-IC Case

We first study the feasibility problem (P3.1) for SISO-IC. Given a SINR target vector $\bar{\gamma} = (\bar{\gamma}_1, \cdots, \bar{\gamma}_K)$ with $\bar{\gamma}_k = 2^{\alpha_k R_{\text{sum}}} - 1$, we will check whether it is achievable under users’ individual power constraints.

Let $G$ denote the $K \times K$ normalized channel gain matrix given by

$$ G_{k,j} = \begin{cases} \frac{\bar{\gamma}_k \|h_{k,j}\|^2}{\|h_{k,k}\|^2}, & k \neq j \\ 0, & k = j, \end{cases} $$

(26)

and $\eta$ denote the $K \times 1$ normalized noise vector given by

$$ \eta_k = \frac{\bar{\gamma}_k \sigma_k^2}{\|h_{k,k}\|^2}, \quad \forall k. $$

(27)

To achieve the SINR target, the transmit power vector for users is given by

$$ p = (I - G)^{-1} \eta. $$

(28)

Let $\rho(B)$ denote the spectral radius (defined as the maximum eigenvalue in absolute value) of the non-negative matrix $B$. The following propositions were shown in [25], which play important roles in solving Problem (P3.1).

**Proposition 4.1:** The power allocation $p$ given by (28) satisfies $p \geq 0$ if and only if $\rho(G) < 1$.

**Proposition 4.2:** If $\rho(G) < 1$, the power allocation $p$ given by (28) is component-wise minimum in the sense that any other power allocation $p'$ that satisfies (24) needs to satisfy $p' \geq p$.

Propositions 4.1 and 4.2 imply that a SINR target vector $\bar{\gamma}$ is feasible if and only if: (1) $\rho(G) < 1$, and (2) the power solution obtained by (28) satisfies $p_k \leq P_{\text{max}}^k, \forall k$. Consequently, we propose Algorithm II in Table II to solve Problem (P3.1).

| **Algorithm II** Algorithm for Solving Problem (P3.1) |
|-----------------------------------------------------|
| a) Given any SINR target vector $\bar{\gamma} = (\bar{\gamma}_1, \cdots, \bar{\gamma}_K)$, compute the spectrum radius of matrix $G$. If it is larger than 1, conclude that there is no feasible power allocation to meet the SINR target and exit the algorithm; otherwise, go to step b); |
| b) Compute the power allocation $p$ by (28), and check for any user $k$, whether the power constraint $p_k \leq P_{\text{max}}^k$ is satisfied. If so, conclude that the SINR target is feasible; otherwise, the SINR target is not feasible. |

**Remark 4.1:** It is worth comparing Algorithm II with the MAPEL algorithm proposed in [19]. MAPEL solves Problem (P1.1) for SISO-IC in the SINR region (as opposed to the rate region in our approach) due to the fact that the problem to characterize the Pareto boundary of the SINR region for SISO-IC can be transformed into a generalized linear fractional programming problem and thus efficiently solved by Dinkelbach-type algorithm [26]. However, this transformation does not work for SIMO-IC or MISO-IC if the beamforming vectors are involved. Consequently, MAPEL cannot be extended to the GIC with multiple antennas. As comparison, in this paper we
solve the WSR maximization problem in the rate region directly because the Pareto boundary can be characterized completely by the rate profile approach, as along as the associated SINR feasibility problem can be solved. Thus, our proposed algorithm is more applicable than MAPEL in solving the WSR maximization problems for SIMO-IC and MISO-IC, as shown next.

B. The SIMO-IC Case

The feasibility of Problem (P3.2) can be checked by using the optimal value of the following SINR balancing problem:

\[
\text{Maximize} \quad \min_{1 \leq k \leq K} \frac{\gamma_k}{\bar{\gamma}_k} \\
\text{Subject to} \quad p_k \leq P_{\text{max}}^k, \quad \forall k.
\]  

(29)

If the optimal value of Problem (29) is no smaller than 1, then the SINR target vector \(\bar{\gamma} = (\bar{\gamma}_1, \cdots, \bar{\gamma}_K)\) is achievable; otherwise, this SINR target cannot be achieved.

In [27], an efficient algorithm was proposed to solve a SINR balancing problem similar to Problem (29), where only one sum-power constraint is imposed. However, the algorithm in [27] cannot be directly applied to solve Problem (29) due to multiple users’ individual power constraints. To utilize the algorithm proposed in [27], we decouple Problem (29) into \(K\) sub-problems, with the \(i\)th sub-problem formulated as:

\[
\text{Maximize} \quad \min_{1 \leq k \leq K} \frac{\gamma_k}{\bar{\gamma}_k} \\
\text{Subject to} \quad p_i \leq P_{\text{max}}^i, \quad \forall k.
\]

(30)

Therefore, for the \(i\)th sub-problem only the \(i\)th user’s power constraint is considered. Next, we show how to solve Problem (30) by extending the algorithm in [27], and then reveal an important relationship between Problems (29) and (30), based upon which we further propose an efficient algorithm to solve Problem (29).

1) Solution to Problem (30):

In this part, we extend the algorithm proposed in [27] to solve Problem (30) for a given \(i\).

One important property of the SINR balancing problem in (30) is that given any receive beamforming vectors \(\bar{W} = (\bar{w}_1, \cdots, \bar{w}_K)\), the corresponding optimal power allocation \(\bar{p}\) must satisfy the following two conditions:

\[
\frac{\gamma_k(\bar{W}, \bar{p})}{\bar{\gamma}_k} = C(\bar{W}), \quad \forall k,
\]

(31)

\[
\bar{p}_i = P_{\text{max}}^i,
\]

(32)

where \(C(\bar{W})\) is the maximum SINR balancing value for all users given \(\bar{W}\).

We justify the above conditions as follows. (31) can be shown by contradiction. Supposing that the SINR balancing values are not the same for all the users, then we select the user with the highest SINR balancing value and decrease its transmit power by a small amount such that its new SINR balancing value is still above \(\min \frac{\gamma_k}{\bar{\gamma}_k}\). Since the other users’ SINR balancing values will increase, the minimum SINR balancing value among all the users will increase accordingly. Thus, whenever the SINR balancing values are not the same for all users, we can proceed as above to improve the optimal value. Hence, (31) must hold. Similarly to show (32) by contradiction,
suppose $\bar{p}_i < P^\text{max}_i$. With $\alpha = \frac{P^\text{max}_i}{\bar{p}_i} > 1$, we can multiply the transmit power values of each user by $\alpha$, and the SINRs of all users will be increased accordingly. Hence, (32) must hold.

We can express (31) for all $k$’s in the following matrix form:

\[
\bar{p} \frac{1}{C(W)} = D\Psi(W)\bar{p} + D\sigma, \tag{33}
\]

where $D = \text{Diag}\{\|\bar{w}_1^H h_{1,1}\|^2, \ldots, \|\bar{w}_K^H h_{K,K}\|^2\}$, $\sigma = [\sigma_1^2\|\bar{w}_1\|^2, \ldots, \sigma_K^2\|\bar{w}_K\|^2]^T$, and the $K \times K$ non-negative matrix $\Psi(W)$ is a function of $\bar{W}$ defined as

\[
[\Psi(W)]_{k,j} = \begin{cases} 
\|\bar{w}_k^H h_{k,j}\|^2, & k \neq j \\
0, & k = j.
\end{cases} \tag{34}
\]

By multiplying both sides of (33) by $e_i^T$, we obtain

\[
e_i^T \bar{p} \frac{1}{C(W)} = P^\text{max}_i C(W) = e_i^T D\Psi(W)\bar{p} + e_i^T D\sigma. \tag{35}
\]

Therefore, by combining (33) and (35), it follows that

\[
\frac{1}{C(W)}\bar{p}_{\text{ext}} = A_i(W)\bar{p}_{\text{ext}}, \tag{36}
\]

where $\bar{p}_{\text{ext}} = \begin{pmatrix} \bar{p} \\ 1 \end{pmatrix}$ and

\[
A_i(W) = \begin{pmatrix} D\Psi(W) & D\sigma \\
\frac{1}{P^\text{max}_i} e_i^T D\Psi(W) & \frac{1}{P^\text{max}_i} e_i^T D\sigma \end{pmatrix}. \tag{37}
\]

Next, we show one important property for (36) in the following lemma.

**Lemma 4.1:** Given any fixed $\bar{W}$, there exists a unique solution $(\bar{p}, C(W))$ to the equation in (36).

**Proof:** Please refer to Appendix A. \hfill \blacksquare

According to Perron-Frobenius theory [29], for any nonnegative matrix, there is at least one positive eigenvalue and the spectral radius of the matrix is equal to the largest positive eigenvalue. Furthermore, according to Lemma 4.1, there is only one strictly positive eigenvalue to matrix $A_i(\bar{W})$. Accordingly, it follows from (36) that given $\bar{W}$, the inverse of the optimal SINR balancing value $1/C(W)$ is the spectral radius of $A_i(\bar{W})$. Consequently, the maximum SINR balancing solution to Problem (30) is obtained as

\[
C^* = \frac{1}{\min_{\bar{W}} \rho(A_i(\bar{W}))}. \tag{38}
\]

Next, by defining a cost function as

\[
\Upsilon(W, p_{\text{ext}}) = \max_{x \geq 0} \frac{x^T A_i(W)p_{\text{ext}}}{x^T p_{\text{ext}}}, \tag{39}
\]

then the min-max characterization of the spectral radius of $A_i(W)$ can be expressed as [27], [30]

\[
\rho(A_i(W)) = \min_{p_{\text{ext}}} \Upsilon(W, p_{\text{ext}}). \tag{40}
\]

Taking (40) into (38), it follows that

\[
\frac{1}{C^*} = \min_{W} \min_{p_{\text{ext}}} \Upsilon(W, p_{\text{ext}}). \tag{41}
\]
Similar to [27], we can solve Problem (41) via the alternating optimization shown as follows. First, given $\bar{W}$, we find the optimal power allocation for $p_{\text{ext}}$. Let $\bar{p}_{\text{ext}}$ denote the dominant eigenvector corresponding to the spectral radius of $A_i(\bar{W})$. It then follows that

$$\frac{x^TA_i(\bar{W})\bar{p}_{\text{ext}}}{x^T\bar{p}_{\text{ext}}} = \rho(A_i(\bar{W})) = \min_{p_{\text{ext}}} \Upsilon(W, p_{\text{ext}}).$$

(42)

Thus, $\bar{p}_{\text{ext}}$ is the optimal power allocation given $\bar{W}$.

Furthermore, we know that given any power allocation $p_{\text{ext}}$, the optimal receive beamformer in $W$ to maximize the SINR is minimum-mean-squared-error (MMSE) based for each of the users. Therefore, we propose an iterative algorithm in Table III, denoted as Algorithm III, to solve Problem (30).

**TABLE III**

**Algorithm III: Algorithm for Solving Problem (30)**

a) Initialize: $n = 0, p^{(0)} = [0, \ldots, 0]^T$ and $\rho^{(0)} = \infty$;

b) repeat

1) $n = n + 1$;

2) Update $W^{(n)}$ by $w_k^{(n)} = (\sum_{j \neq k} p_j^{(n-1)} h_{k,j} h_{k,j}^H + \sigma_k^2 I)^{-1} h_{k,k}, \forall k$;

3) Update $p_{\text{ext}}^{(n)}$ as the dominant eigenvector of the matrix $A_i(W^{(n)})$;

4) $\rho^{(n)} = \rho(A_i(W^{(n)}))$ and $C^{(n)} = \frac{1}{\rho^{(n)}}$;

c) until $\rho^{(n-1)} - \rho^{(n)} < \epsilon$.

The convergence of Algorithm III can be shown in the following way. Since given any power allocation $p_{\text{ext}}$ for the $n$th iteration, $W^{(n+1)}$ minimizes $\Upsilon(W, p_{\text{ext}}^{(n)})$, i.e.,

$$\Upsilon(W^{(n+1)}, p_{\text{ext}}^{(n)}) \leq \Upsilon(W^{(n)}, p_{\text{ext}}^{(n)}) = \rho^{(n)}.$$  (43)

Moreover, given $W^{(n+1)}$, $p_{\text{ext}}^{(n+1)}$ minimizes $\Upsilon(W^{(n+1)}, p_{\text{ext}})$ as

$$\rho^{(n+1)} = \Upsilon(W^{(n+1)}, p_{\text{ext}}^{(n+1)}) \leq \Upsilon(W^{(n+1)}, p_{\text{ext}}^{(n)}).$$  (44)

Hence, we can guarantee $\rho^{(n+1)} \leq \rho^{(n)}$ after each iteration. Since $\rho$ is lower-bounded by 0, Algorithm III thus converges.

Finally, the convergence of Algorithm III to the global optimality of Problem (30) can be proven similarly as Section IV.A in [27], and the proof is thus omitted for brevity. After convergence, $C^{(n)}\bar{\gamma}$ is the maximum achievable SINR vector and $p^{(n)}$, $W^{(n)}$ are the optimal power and receive beamforming vectors to achieve this SINR vector, respectively.

2) **Solution to Problem (29):**

Next, we show that Problem (29) can be efficiently solved via solving Problem (30) for all $i$’s. Let $W^*$ and $p^*$ denote the optimal beamforming vectors and power allocation for Problem (29), respectively. Let $W_i^*$ and $p_i^*$ denote the optimal beamforming vectors and power allocation for the $i$th sub-problem in (30), respectively. Next, we provide a theorem to reveal the relationship between the optimal solutions to Problems (29) and (30).
Theorem 4.1: For all sub-problems in (30) with \( i = 1, \cdots, K \), there exists one and only one sub-problem for which the optimal power solution satisfies all users’ individual power constraints of Problem (29). Furthermore, let \( i^* \) denote the index of the corresponding sub-problem in (30), then it holds that \( W^* = W^*_{i^*} \) and \( p^* = p^*_{i^*} \).

Proof: Please refer to Section IV.B in [31].

Theorem 4.1 reveals that Problem (29) can be solved as follows. First, we apply Algorithm III to solve Problem (30) in the order of \( i = 1, \cdots, K \). If the optimal power solution to any of these problems satisfies all users’ individual power constraints, the algorithm terminates, and the obtained optimal power and beamforming solutions to Problem (30) are also those to Problem (29). The above algorithm, denoted by Algorithm IV, is summarized in Table IV.

| TABLE IV |
| ALGORITHM IV: ALGORITHM FOR SOLVING PROBLEM (29) |

a) Initialize: \( i = 0; \)

b) repeat

1) \( i = i + 1; \)

2) Solve the \( i \)th sub-problem in (30) by Algorithm III and find the optimal beamforming solution \( W^*_{i} \) and power solution \( p^*_{i} \);

3) Check whether \( p^*_{i} \) satisfies all power constraints of Problem (29). If so, exit the algorithm and set \( W^*_{i} \) and \( p^*_{i} \) as the optimal solution to Problem (29); otherwise, continue the algorithm;

c) until \( i = K \).

C. The MISO-IC Case

In this subsection, we show how to solve the feasibility problem (P3.3) for MISO-IC under the equivalent SINR constraints given by (24). It was shown in [10] that this problem can be transformed into a second-order cone programming (SOCP) problem, which is briefly described as follows for the sake of completeness. The SINR constraints in Problem (P3.3) can be rewritten as

\[
(1 + \frac{1}{\gamma_k}) ||h^H_{k,k}v_k||^2 \geq \sum_{j=1}^{K} ||h^H_{k,j}v_j||^2 + \sigma_k^2, \quad \forall k. \tag{45}
\]

Without loss of generality, we can assume that \( h^H_{k,k}w_k \) is a positive real number, \( \forall k \). Thus we can reformulate the above SINR constraints as

\[
\sqrt{1 + \frac{1}{\gamma_k}} h^H_{k,k}v_k \geq \sqrt{\sum_{j=1}^{K} ||h^H_{k,j}v_j||^2 + \sigma_k^2}, \quad \forall k. \tag{46}
\]

Denote \( \mathbf{x} = [v_1^T, \cdots, v_K^T, 0]^T \) of dimension \((K^2 + 1) \times 1\), \( \mathbf{n}_k = [0, \cdots, 0, \sigma_k]^T \) of dimension \((K + 1) \times 1\), and \( \mathbf{E}_k = \text{Diag}(h^H_{k,1}, \cdots, h^H_{k,K}, 0) \) of dimension \((K + 1) \times (K^2 + 1)\), \( \forall k \). We further define \( \mathbf{L}_k \) as

\[
\mathbf{L}_k = \begin{bmatrix}
\mathbf{0}_{K \times K} & \cdots & \mathbf{0}_{K \times K} \\
\mathbf{I}_{K \times K} & \cdots & \mathbf{0}_{K \times K} \\
\mathbf{0}_{K \times K} & \cdots & \mathbf{0}_{K \times 1}
\end{bmatrix}_{(K - k) \times (K - k)}. \tag{47}
\]
where $0^{K \times K}$ and $0^{K \times 1}$ denote the $K \times K$ all-zero matrix and $K \times 1$ all-zero vector, respectively, and $I^{K \times K}$ denotes the $K \times K$ identity matrix. Thus, (46) can be reformulated as

$$
\| E_k x + n_k \| \leq \sqrt{1 + \frac{1}{\gamma_k} h_{k,k}^H L_k x}, \quad \forall k. 
$$

(48)

Moreover, we can reformulate the power constraints as

$$
\| L_k x \| \leq \sqrt{P_{\text{max}}^k}, \quad \forall k. 
$$

(49)

Using (48) and (49), Problem (P3.3) can be transformed into a SOCP feasibility problem over $x$ and efficiently solvable by existing software [32].

**Remark 4.2:** It is worth comparing our proposed algorithm with that in [20] for solving the WSR maximization problem (P1.3) for MISO-IC. The algorithm in [20] is based on a prior result in [12] that for the special case of two-user MISO-IC, any point on the Pareto boundary of the rate region can be achieved by transmit beamforming vectors that are obtained by linearly combining the ZF and MRT beamformers. In [20], the outer polyblock approximation algorithm was applied to find the optimal beamformer combining coefficients. However, since this result does not hold for MISO-IC with more than two users, the algorithm in [20] cannot be extended to the general $K$-user MISO-IC with $K > 2$. In contrast, our proposed algorithm can be applied to MISO-IC with arbitrary number of users.

**V. Numerical Results**

In this section, we provide numerical results to validate the proposed algorithms in this paper. We assume that $\mu_k = 1$, $\forall k$, i.e., the sum-rate maximization problem is considered. We also assume that $P_{\text{max}}^k = 3$, $\forall k$. For SIMO-IC and MISO-IC, we further assume that $M_k = 2$ and $N_k = 2$, respectively, $\forall k$. The numerical results with related discussions are presented in the following subsections.

**A. Convergence Performance**

Firstly, we study the convergence performance of Algorithm [I] for SISO-IC. We assume that there are 4 users, i.e., $K = 4$, and there is a minimum rate constraint for each user with $R_{k}^\text{min} = 0.5$, $\forall k$. We set the parameters to control the accuracy of Algorithm [I] as $\epsilon = 0.01$ and $\eta = 0.5$. We consider the following matrix:

$$
H = \begin{bmatrix}
0.4310 & 0.0022 & 0.0105 & 0.0042 \\
0.0200 & 0.4102 & 0.0180 & 0.0035 \\
0.0210 & 0.0200 & 0.5162 & 0.0112 \\
0.0210 & 0.0021 & 0.0063 & 0.3634
\end{bmatrix},
$$

(50)

with each element denoting the power of the corresponding channel gain, i.e., $H_{k,j} = \| h_{k,j} \|^2$.

Fig. 3 shows the convergence of Algorithm [I] under the above channel setup. It is observed that this algorithm takes about 300 iterations to converge. The converged sum-rate is 11.4605 with users’ individual rates given by $[3.1982, 2.6297, 2.8441, 2.7884]$. To verify that the global sum-rate maximum is achieved, we compare the obtained maximum sum-rate with that by an exhaustive search, which is equal to 11.5349. Thus, Algorithm [I] does achieve
Fig. 3. Convergence performance of Algorithm I for SISO-IC with weak interference channel gains.

Fig. 4. Convergence performance of Algorithm I for SISO-IC with strong interference channel gains.
the global optimality of sum-rate maximization within a guaranteed error $11.5349 - 11.4605 = 0.0744$, which is smaller than the set threshold $\eta = 0.5$.

Next, we consider a SISO-IC with stronger cross-user interference channel gains than those given in (50) by keeping all diagonal elements of $H$ unchanged, but scaling all off-diagonal elements by 10 times. As shown in Fig. 4 for this new channel setup, Algorithm II takes about 2900 iterations to converge. The converged sum-rate in this case is 5.1184 with users’ individual rates given by $[0.5408, 1.9119, 0.5060, 2.1597]$, while that obtained by the exhaustive search is 5.1392. Thus, as compared to the previous case with weaker interference channel gains, the global optimality for sum-rate maximization is achieved in this case with a much slower convergence. The reason is as follows. With stronger interference channel gains, the optimal power allocation for sum-rate maximization is more likely to render some users transmit at their minimum required rates (e.g., user 1 and user 3 in this example). Hence, the corresponding optimal rate values will lie in the strip defined by $\{r^* | R_k^{\min} \leq R_k^* \leq R_k^{\min} + \epsilon\}$ for some $k$’s. Since in Algorithm II each new polyblock is generated from the previous one by cutting off some unfit portions, the cuts become shallower and shallower as $\tilde{z}^{(n)}$ approaches the above strip. This can be observed from Fig. 4 that after the 1300th iteration, the best intersection point $\tilde{r}^{(n)}$ has never changed. However, to make $U(\tilde{z}^{(n)}) - U(\tilde{r}^{(n)}) \leq \eta$ hold, another 1700 iterations are taken just to reduce the value of $U(\tilde{z}^{(n)})$. Since this reduction becomes very inefficient near the strip, the algorithm converges much more slowly to the desired accuracy with the increasing of interference channel gains. From this observation, we infer that the values of $\epsilon$ and $\eta$ need to be properly set to balance between the accuracy and convergence speed of our proposed algorithm.

Next, we give another example to illustrate the important role of parameter $\epsilon$ in balancing between the accuracy and convergence speed of our proposed algorithm. We assume that $K = 3$, and there are no minimum rate requirements for the users. We consider the following channel matrix:

$$
H = \begin{bmatrix}
0.4310 & 0.0187 & 0.0893 \\
0.1700 & 0.4102 & 0.1530 \\
0.1785 & 0.1700 & 0.5162
\end{bmatrix},
$$

(51)

with $H_{k,j} = \|h_{k,j}\|^2$. By an exhaustive search, the optimal sum-rate is obtained as 4.8079 with users’ individual rates given by $[3.2146, 1.5933, 0]$.

Table V shows the convergence speed and the converged sum-rate of our proposed algorithm for different values of $\epsilon$ with $\eta = 0.2$. We observe that as $\epsilon$ increases, the algorithm convergence speed improves rapidly, but the converged sum-rate decreases. When $\epsilon = 0.45$, the difference between the optimal sum-rate and converged sum-rate is $4.8079 - 4.5880 = 0.2199$, which is even larger than $\eta = 0.2$. This is because that as we show in Section II-B with non-zero $\epsilon$, we are in fact solving Problem P2-A instead of the original problem P2. Consequently, the proposed algorithm can only guarantee that the difference between the maximum sum-rate of Problem P2-A and the converged sum-rate is less than $\eta$, but not necessarily for Problem P2. Thus, if the value of $\epsilon$ is selected to be too large such that all the $\eta$-optimal solutions lie in the excluded strips, the difference of the converged sum-rate and the maximum sum-rate of Problem (P2) will be larger than $\eta$. Therefore, the value of $\epsilon$ should be carefully selected based on the value of $\eta$. In this numerical example, we can select $\epsilon = 0.40$ such that the $\eta$-optimal solution is still guaranteed and also the converged speed is reasonably fast.
TABLE V
SELECTION OF $\epsilon$ ON THE PERFORMANCE OF THE PROPOSED ALGORITHM

| Value of $\epsilon$ | Number of iterations | Converged WSR |
|--------------------|----------------------|---------------|
| 0.05               | 8183                 | 4.7625        |
| 0.10               | 3498                 | 4.7438        |
| 0.15               | 2212                 | 4.7275        |
| 0.20               | 1642                 | 4.6942        |
| 0.25               | 1396                 | 4.6825        |
| 0.30               | 1148                 | 4.6620        |
| 0.35               | 1029                 | 4.6350        |
| 0.40               | 866                  | 4.6165        |
| 0.45               | 651                  | 4.5880        |

B. Providing Performance Benchmark for Other Heuristic Algorithms

A key application of our proposed algorithm is to provide performance benchmarks for other heuristic algorithms for achieving the maximum WSR in the GIC, especially in cases of MISO-IC and SIMO-IC where the globally optimal solution by exhaustive search is hardly possible. In the following, we provide an example to show how to utilize our proposed algorithm to evaluate the performance of other suboptimal algorithms for WSR maximization in MISO-IC and SIMO-IC.

We consider the “price-based” suboptimal algorithm, e.g., the ADP algorithm, which was proposed in [5] as an efficient distributed algorithm for WSR maximization in SISO-IC. Since to our best knowledge, extensions of ADP to the multi-antenna GIC are not yet available in the literature, we provide the details for such extensions.
Fig. 6. Performance comparison for Algorithm I versus the price-based algorithm in MISO-IC.

for SIMO-IC and MISO-IC in Appendix B

Figs. 5 and 6 show the achievable sum-rates by the price-based algorithm versus Algorithm I for 4-user SIMO-IC and MISO-IC, respectively, without the minimum rate constraints. Each element in all channel vectors involved is randomly generated by the CSCG distribution with zero mean and unit variance. We set the parameters to control the accuracy of Algorithm I as $\epsilon = 0.01$ and $\eta = 0.5$. In Fig. 5, the price-based algorithm converges to the sum-rate of 10.6989 in SIMO-IC, while the maximum sum-rate achieved by Algorithm I is 11.9182. In Fig. 6, the price-based algorithm converges to the sum-rate of 4.8216 (although it has reached almost 6 before convergence) in MISO-IC, while Algorithm I achieves the maximum sum-rate of 10.6193. Based on these results as well as other numerical examples (not shown in this paper due to the space limitation), we infer that in general the price-based algorithm for SIMO-IC performs better than MISO-IC, as both compared with our proposed algorithm that achieves the global sum-rate maximum. Moreover, the price-based algorithm for MISO-IC does not converge under certain channel setups, while even when the algorithm converges, the resulted sum-rate can be far from the global maximum. In contrast, for SIMO-IC, the price-based algorithm usually achieves the sum-rate very close to the global maximum, and even converges to it under certain channel setups.

VI. CONCLUDING REMARKS

In this paper, we propose a new framework to achieve the global optimality of WSR maximization problems in SISO-IC, SIMO-IC, and MISO-IC, with the interference treated as Gaussian noise. Although the studied problems are non-convex with respect to the power allocation and/or beamforming vectors, we show that they belong to the monotonic optimization over a normal set by reformulating them as maximizing the WSR in the achievable rate
regions directly. Therefore, the outer polyblock approximation algorithm can be applied to achieve the global WSR maximum. Furthermore, by utilizing the approach of rate profile, at each iteration of the proposed algorithm, the updated intersection point on the Pareto boundary of the achievable rate region is efficiently obtained via solving a sequence of SINR feasibility problems. It is worth noting that although the developed framework in this paper is aimed to solve the WSR maximization problem for the GIC, it can be similarly applied to other multiuser communication systems with non-convex rate regions provided that the problem of characterizing the intersection Pareto boundary point with an arbitrary rate-profile vector can be efficiently solved.

It is worth pointing out that based on our numerical experiments, the proposed algorithm in this paper is found to converge very slowly when the number of users becomes large (e.g., $K \geq 6$), and thus may not be suitable for real-time implementation. Nevertheless, the proposed algorithm can be applied to provide performance benchmarks for other real-time algorithms that usually guarantee only suboptimal solutions. It is our hope that this paper will motivate future work to improve the convergence speed of the proposed algorithm and thus make it more applicable in practical systems, even with large number of users. For example, in [33], the original point for the algorithm is shifted from the origin to a point in the negative plane, which is shown to speed up the convergence to some extent.

After the submission of this manuscript, we become aware of one interesting related work [34] that is worth mentioning. In [34], a similar framework is proposed to optimize the system performance for multi-cell downlink MISO beamforming (similar to MISO-IC in nature), e.g., sum-rate performance and proportional fairness, by making use of monotonic optimization and rate profile techniques. One difference between [34] and our work is that for the monotonic optimization part, a so-called “branch-reduce-and-bound” algorithm is used in [34] as compared to the outer polyblock approximation algorithm in our paper.

APPENDIX

A. Proof of Lemma 4.1

Note that under the sum power constraint, a similar result to this lemma has been shown in [28]. However, the proof in [28] is not directly applicable in our case since in (36), there is an individual power constraint rather than the sum power constraint. Thus, we need to provide a new proof for this lemma shown as follows.

Suppose that there are two solutions to (36), denoted by $(\bar{p}, C(\bar{W}))$ and $(\bar{p}', C'(\bar{W}))$. Define a sequence of $\theta_k$’s as $\theta_k = \frac{P^i_k}{\bar{p}^i_k}, \forall k$. We can re-arrange $\theta_k$’s in a decreasing order by

$$\theta_{t1} \geq \theta_{t2} \geq \cdots \geq \theta_{tK}. \quad (52)$$

Since according to (32) we have $\bar{p}_i = \bar{p}'_i = P^i_{\max}$, it follows that $\theta_i = 1$ must hold. Hence, $\theta_{t1} \geq \theta_i = 1$. Moreover, in (52), at least one strict inequality must hold because otherwise $\theta_k = 1, \forall k$, which then implies that only one unique solution to (36) exists.
Next, we derive the SINR balancing value of user $t_1$ as follows:

$$C'_{t_1}(\bar{W}) = \frac{\bar{p}_{t_1}' ||\bar{w}_{t_1}^H h_{t_1,t_1}||^2}{\bar{w}_{t_1}^H (\sum_{j \neq t_1} \bar{p}_j h_{t_1,j} h_{t_1,j}^H + \sigma_{t_1}^2 I) \bar{w}_{t_1} \bar{\gamma}_{t_1}}$$

$$= \frac{\bar{p}_{t_1}' ||\bar{w}_{t_1}^H h_{t_1,t_1}||^2}{\bar{w}_{t_1}^H (\sum_{j \neq t_1} \bar{p}_j h_{t_1,j} h_{t_1,j}^H + \sigma_{t_1}^2 I \frac{1}{\bar{p}_{t_1}'}) \bar{w}_{t_1} \bar{\gamma}_{t_1}}$$

$$> \frac{\bar{p}_{t_1}' ||\bar{w}_{t_1}^H h_{t_1,t_1}||^2}{\bar{w}_{t_1}^H (\sum_{j \neq t_1} \bar{p}_j h_{t_1,j} h_{t_1,j}^H + \sigma_{t_1}^2 I) \bar{w}_{t_1} \bar{\gamma}_{t_1}}$$

$$= C_{t_1}(\bar{W}). \quad (53)$$

Based on (31), we have

$$C'(\bar{W}) = C'_{t_1}(\bar{W}) > C_{t_1}(\bar{W}) = C(\bar{W}). \quad (54)$$

Similarly, we can show that $C'_{t_K}(\bar{W}) < C_{t_K}(\bar{W})$, which yields

$$C'(\bar{W}) = C'_{t_K}(\bar{W}) < C_{t_K}(\bar{W}) = C(\bar{W}). \quad (55)$$

Since (54) and (55) contradict to each other, there must be one unique solution to (36). Lemma 4.1 is thus proven.

### B. Price-Based Algorithm for SIMO-IC and MISO-IC

In this part, we provide the details of the suboptimal price-based algorithms for Problems (P1.2) in SIMO-IC and (P1.3) in MISO-IC, which can be viewed as extensions of the ADP algorithm proposed in [5] for SISO-IC. In ADP, each user announces a price that reflects its sensitivity to the interference from all other users, and then updates its transmit power by maximizing its own utility offset by the sum interference price received from all the other users. It was shown in [5] that ADP can converge to the solution that has the same Karush-Kuhn-Tucker (KKT) conditions as that of the WSR maximization problem, and is thus guaranteed to achieve at least a locally optimal solution. In the following, we extend the ADP algorithm in [5] to SIMO-IC and MISO-IC, but without the proof of convergence.

1) **Price-Based Algorithm for SIMO-IC:**

In this part, we extend the ADP or price-based algorithm to SIMO-IC. First, without loss of generality, we substitute the optimal MMSE-based receive beamforming vectors for $w_k$’s into the SINR expression given in (4). Then, given any transmit power vector $p$, the achievable rate for user $k$ can be expressed as

$$R_k(p) = \log_2(1 + \gamma_k^{SIMO-IC}) = \log_2 \left(1 + p_k h_{k,k}^H (\sum_{j \neq k} p_j h_{k,j} h_{k,j}^H + \sigma_k^2 I)^{-1} h_{k,k}\right). \quad (56)$$

Thus in Problem (P1.2), we only need to find the optimal transmit power solution, without the need to consider the receive beamforming optimization.

Next, we present the KKT optimality conditions of Problem (P1.2) with the objective function specified in (56). For any locally optimal power solution $p^*$, there exist unique Lagrangian multipliers $\lambda = (\lambda_1, \cdots, \lambda_K)$ such that
for any \( k = 1, \cdots, K \),
\[
\mu_k \frac{\partial R_k(p^*)}{\partial p_k} + \sum_{j \neq k} \mu_j \frac{\partial R_j(p^*)}{\partial p_k} = \lambda_k,
\]
(57)
\[
\lambda_k (p_{k}^{\text{max}} - p_k) = 0,
\]
(58)
\[
\lambda_k \geq 0.
\]
(59)

Now, for the price-based algorithm, define the price charged by receiver \( j \) to transmitter \( k \), which indicates the sensitivity of the achievable rate of receiver \( j \) subject to the power change of transmitter \( k \), as
\[
\pi_{j,k} = -\frac{\partial R_j(p)}{\partial p_k} = \frac{p_j \| h_{j,j}^H \left( \sum_{i \neq j} p_i h_{j,i} h_{j,i}^H + \sigma_j^2 I \right)^{-1} h_{j,k} \|^2}{\ln 2 \left( 1 + p_j h_{j,j}^H \left( \sum_{i \neq j} p_i h_{j,i} h_{j,i}^H + \sigma_j^2 I \right)^{-1} h_{j,j} \right)}.
\]
(60)

Consequently, we see that the KKT conditions in (57), (58) and (59) are both necessary and sufficient for the optimal solution to the following problem for user \( k \), \( k = 1, \cdots, K \):
\[
\begin{align*}
\text{Maximize} & \quad \mu_k \log_2 \left( 1 + p_k h_{k,k}^H (\sum_{j \neq k} p_j h_{k,j} h_{k,j}^H + \sigma_k^2 I)^{-1} h_{k,k} \right) - p_k \sum_{j \neq k} \mu_j \pi_{j,k} \\
\text{Subject to} & \quad p_k \leq P_k^{\text{max}},
\end{align*}
\]
(61)

where \( p_j \) and \( \pi_{j,k} \) are fixed, \( \forall j \neq k \).

Similar to the ADP algorithm in [5], we propose the following algorithm to update the price and transmit power iteratively for all users in SIMO-IC. Specifically, at each iteration the algorithm does the following:

1. Each user announces its price obtained using (60) to all the other users;
2. Each user updates its transmit power by solving Problem (61), i.e.,
\[
p_k = \left[ \ln 2 \sum_{j \neq k} \mu_j \pi_{j,k} - \frac{1}{h_{k,k}^H \left( \sum_{j \neq k} p_j h_{k,j} h_{k,j}^H + \sigma_k^2 I \right)^{-1} h_{k,k}} \right] \frac{p_k^{\text{max}}}{0}, \quad \forall k,
\]
(62)

where \([x]_a^b = \max(\min(x, b), a)\).

Because Problems (P1.2) and (61) possess the same KKT optimality conditions, when the above algorithm converges to a set of optimal solutions to problems in (61) for all \( k \)’s, this set of solutions will be at least a locally optimal solution to Problem (P1.2).

2) Price-Based Algorithm for MISO-IC:

Next, we extend the ADP algorithm to MISO-IC. For any given transmit beamforming vectors \( V \), we first define the price for user \( k \) as
\[
\pi_k = -\frac{\partial R_k}{\partial \Gamma_k} = \frac{\| h_{k,k}^H v_k \|^2}{\ln 2(\| h_{k,k}^H v_k \|^2 + \Gamma_k + \sigma_k^2))},
\]
(63)

where \( \Gamma_k = \sum_{j \neq k} \| h_{k,j}^H v_j \|^2 \) is the total interference power at the \( k \)th receiver. Let \( S_k = v_k v_k^H, \forall k \). Given fixed interference prices and beamforming vectors of all other users, the following problem is to be solved by any user
for its own transmit beamforming update:

\[
\text{Maximize } \quad \frac{\mu_k \log_2(1 + \frac{h_{k,k}^H S_k h_{k,k}}{\Gamma_k + \sigma_k^2})}{\sum_{j \neq k} \mu_j \pi_j h_{k,j}^H S_k h_{j,k}} - \sum_{j \neq k} \mu_j \pi_j h_{k,j}^H S_k h_{j,k}
\]

Subject to \n\[
\text{Tr}(S_k) \leq P_k^{\text{max}}
\]

\[
S_k \succeq 0,
\]

where \(S_k \succeq 0\) means that \(S_k\) is a positive semi-definite matrix. Similar to the previous case of SIMO-IC, we can show that the KKT conditions of Problem (64) with \(k = 1, \cdots, K\) are also those of Problem (P1.3) by replacing \(v_k v_k^H\) with \(S_k\), \(\forall k\). However, Problem (P1.3) requires that the optimal solution \(S_k^*\) in Problem (64) to be rank-one, which is not guaranteed a priori. Thus, Problem (64) is a relaxation of the original WSR maximization problem (P1.3) without considering the rank-one constraint.

Interestingly, it was recently shown in [10] that the optimal solution to Problem (64) is always of rank-one, i.e., \(S_k^* = v_k v_k^H\). Hence, we propose a price-based algorithm for MISO-IC in a similar way to that for SIMO-IC. When this algorithm converges to a set of optimal solutions to problems in (64) with \(k = 1, \cdots, K\), this set of solutions are all rank-one and thus corresponds to at least a locally optimal solution to Problem (P1.3).

For this price-based algorithm for MISO-IC, the interference price can be iteratively updated according to (63). As for the update of beamforming vectors, we need to solve Problem (64) for each user \(k\). It can be verified that Problem (64) is convex with strictly feasible points, and thus it can be solved by the standard Lagrangian duality method [24] with a zero duality gap. The details of solving Problem (64) can be found in Appendix I of [10], and are thus omitted here.

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