From Hubble diagrams to scale factors

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Abstract

We present a lower bound on the radius of the universe today \(a_0\) and a monotonicity constraint on the Hubble diagram. Our theoretical input is Einstein’s kinematics and maximally symmetric universes. Present supernova data yield \(a_0 > 1.2 \cdot 10^{26}\) m. A first attempt to quantify the monotonicity constraint is described. We do not see any indication of non-monotonicity.

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1 Introduction

In a homogeneous, isotropic, expanding universe, the apparent luminosity of a standard candle is a monotonically decreasing function of the time of flight of emitted photons if the universe is open. This is also true in spherical universes if the time of flight is small enough with respect to the radius divided by the speed of light. In principle the (apparent) luminosity $\ell(t)$ as a function of time can be used to measure the scale factor $a(t)$. In reality, arriving photons do not tell us their time of flight, but only their spectral deformation, $z := \nu_{\text{table}}/\nu_{\text{observed}} - 1$. In an expanding universe $z$ is positive, 'red shift'. If we pretend to know the scale factor $a(t)$ we can compute the luminosity $\ell(z)$ and confront it to the Hubble diagram. The transform $a(t) \rightarrow \ell(z)$ reminds us of the Fourier transform and of course we are interested in the inverse transform $\ell(z) \rightarrow a(t)$. Therefore we must ask three questions: What is the domain of definition of the initial transform, what is its image and is the transform injective? Should the measured luminosity $\ell(z)$ be ‘far away’ from the image our working hypotheses are put to test.

2 The hypotheses

We assume the kinematics of general relativity: (i) The gravitational field is coded in a time-space metric of signature $+ - - -$, the configuration space is the set of all such metrics. (ii) Massive and massless, pointlike test particles, subject only to gravity, follow timelike and lightlike geodesics. (iii) Pointlike clocks, e.g. atomic clocks, are necessarily massive. They move on timelike curves (not necessarily geodesics) and indicate proper time $\tau$.

This kinematics is covariant under general coordinate transformations. In 1983 the meter was officially abandoned as fundamental unit in favor of an absolute speed of light and we would like to stress that at least since then any non-covariant kinematics is void of any physical meaning.

We also assume the hypotheses of spatially maximally symmetric cosmology: (iv) The metric is Robertson-Walker,

$$c^2d\tau^2 = c^2dt^2 - a(t)^2\left[dx^2 + k\frac{(\vec{x} \cdot d\vec{x})^2}{1 - k|\vec{x}|^2}\right].$$

The scale factor $a(t)$ is a strictly positive function of time, $k = -1, 0, 1$ for the pseu-
dosphere, the Euclidean space and the sphere. We take the coordinates $\vec{x}$ dimensionless and call them ‘co-moving position’, while the scale factor is measured in meters. For closed universes, $k = 1$, we must restrict the spatial coordinates to the unit ball, $|\vec{x}| < 1$, they describe the northern hemisphere. For open universes, $\vec{x}$ varies in $\mathbb{R}^3$. (v) The test particles are (superclusters of) galaxies and photons. The former follow the timelike geodesics $t = \tau, \vec{x} = \text{constant}$. Note that due to the high degree of symmetry the proper time is universal for all these timelike geodesics and is taken as time coordinate. The photons are emitted from the a galaxy at time $t$ and arrive at our position at time $t_0$, today.
The symmetry hypothesis and the choice of the coordinates \((t, \vec{x})\) have reduced the configuration space to the set of positive functions \(a(t)\) on the real line and to the parameter \(k = 0, \pm 1\).

3 The Hubble diagram

From these hypotheses we can compute the (apparent) luminosity \(\ell(t)\) of a standard candle in Watt/m\(^2\) as a function of emission time \(t\)

\[
\ell(t) = \frac{L}{4\pi a_0^2} \frac{a(t)^2}{a_0^2 s^2(\chi(t))}
\]  

(2)

where \(L\) is the absolute luminosity of the standard candle in Watt, \(a_0 := a(t_0)\) is the scale factor today,

\[
\chi(t) := \int_t^{t_0} \frac{c\,d\tilde{t}}{a(\tilde{t})}
\]  

(3)

is the dimensionless co-moving geodesic distance covered by the photon and

\[
s(\chi) := \begin{cases} 
\sin \chi, & k = 1 \\
\chi, & k = 0 \\
\sinh \chi, & k = -1
\end{cases}
\]  

(4)

The luminosity has a singularity at \(t = t_0\), 'short distance divergency', for any value of \(k\) and any scale factor. For closed universes, \(k = 1\) we may have additional singularities for \(\chi = \pi\), 'antipode divergencies', for \(\chi = 2\pi\) we are back at a second short distance divergency and so forth if \(\chi\) continues to grow. Of course there might be a horizon, i.e. an upper bound for \(\chi\), masking all or some of the singularities.

From the above hypotheses we can also compute the spectral deformation as function of emission time,

\[
z(t) = \frac{a_0}{a(t)} - 1.
\]  

(5)

The theoretical Hubble diagram is the parametric plot in the \(z - \ell\) plane as \(t\) varies.

If we suppose that the scale factor is strictly increasing with \(\dot{a} := da/dt > 0\) then \(z\) is positive, 'red shift', and we can invert the function \(z(t)\). By abuse of notations we write \(t(z)\) for its inverse. Then the Hubble diagram is a function \(\ell(t(z))\) that still by abuse is written \(\ell(z)\),

\[
\ell(z) = \frac{L}{4\pi a_0^2} \frac{1}{(z + 1)^2 s^2(\chi(z))}
\]  

(6)

where

\[
\chi(z) := \chi(t(z)) = \frac{c}{a_0} \int_0^z \frac{d\tilde{z}}{H(\tilde{z})}
\]  

(7)
and $H(z) := \dot{a}(t(z))/a(t(z))$ is the Hubble rate.

The short distance divergency now is at $z = 0$ and easy to get rid of: let us define the regularized luminosity

$$f(z) := z^2 \ell(z) = \frac{L}{4 \pi a_0^2} \frac{z^2}{(z + 1)^2 s^2(\chi(z))}. \tag{8}$$

Indeed,

$$f(0) = \frac{L}{4 \pi c^2} H_0^2, \quad f'(0) = \frac{L}{4 \pi c^2} H_0^2 (q_0 - 1), \tag{9}$$

where the prime is differentiating with respect to $z$, $q(z)$ is the deceleration parameter,

$$q(z) = -\frac{\ddot{a}}{\dot{a} a}(t(z)), \tag{10}$$

and $H_0 := H(0)$, $q_0 = q(0)$.

### 4 The transform $a(t) \rightarrow \ell(z)$

Let us first try to describe the image, that is all luminosity functions $\ell(z)$ which can be obtained from strictly increasing scale factors $a(t)$ with $\dot{a} > 0$ and with $k = 0, \pm 1$.

We already know that $\ell(z)$ comes with the short distance divergency at $z = 0$ which is such that $z^2 \ell(z) =: f(z)$ is regular there. For open universes there are no other singularities. Indeed, $(z + 1)^2 \ell(z) =: g(z)$ is a decreasing function. For closed universes on the other hand, $g(z)$ goes through a minimum as the photons pass the equator, $\chi = \pi/2$. From there on the luminosity increases again and goes to the antipode divergency. It might of course happen that the equator is masked by the horizon in which case $g(z)$ remains decreasing for ever, even though the universe is closed.

Let us now ask whether the transform is injective in the domain of increasing scale factors.

If we pretend to know $k$ the answer is affirmative for open universes. Indeed, solving equation (8) for $s(\chi)$ and differentiating with respect to $z$ yields

$$\frac{a_0}{c} \sqrt{4 \pi a_0^2 L^{-1} (z + 1)^2 f(z) - k z^2} \left/ \left(1 - \frac{z}{2} (z + 1) f'(z)/f(z)\right)\right. = \frac{z(z + 1)}{1 - \frac{z}{2} (z + 1) f'(z)/f(z)}. \tag{11}$$

Therefore we can reconstruct the Hubble rate from the luminosity.

Integrating the Hubble rate with respect to $t$ gives us the scale factor with the ambiguity of the initial condition $a_0$. But for flat universes this initial condition is unphysical, by a coordinate transformation of $\vec{x}$, we can set $a_0 = 1$ m. This is different for curved universes where $a_0$ is related to a local observable, curvature. In the closed case, $a_0$ is also related to a global observable, the radius of the universe today. However, unless $g(z)$ already exhibits an increase, only a lower bound on the radius today can be reconstructed from the luminosity,

$$a_0 \geq \sqrt{\frac{L}{4 \pi g_{\min}}}, \quad g_{\min} := \min_{z > 0}\{(z + 1)^2 \ell(z)\}. \tag{12}$$
Note that this lower bound does not depend on the absolute luminosity \( L \).

If we admit that we do not know \( k \) and if the luminosity function \( \ell(z) \) satisfies: (i) \( z^2 \ell(z) \) is regular at \( z = 0 \), (ii) \((z + 1)^2 \ell(z) \) is decreasing, then there are three positive functions \( a_{\pm}(t) \) and \( a(t) \), such that the universes with scale factor \( a_{-}(t) \), \( k = -1 \), with scale factor \( a(t) \), \( k = 0 \) and with scale factors \( a_{+}(t) \), \( k = 1 \) have the same luminosity function \( \ell(z) \). These three scale factors satisfy

\[
a_{-}(t_0) \sinh \chi_{-} = a(t_0) \chi = a_{+}(t_0) \sin \chi_{+}. \tag{13}
\]

Note that in the flat case the ‘initial condition’ \( a(t_0) \) carries no information while in the closed case \( a_{+}(t_0) \) must satisfy the inequality \( (12) \).

**Example** (constant deceleration parameter):
Take the scale factor,

\[
a(t) := a_0(pH_0 t)^{1/p}, \quad p > 0, \quad 0 < t < t_0 = \frac{1}{pH_0}. \tag{14}
\]

Then the Hubble rate is \( H = H_0(z + 1)^p \) and the deceleration parameter is constant, \( q \equiv p - 1 \). The co-moving distance is

\[
\chi = \frac{c}{a_0 H_0} \ln(z + 1) = - \frac{c}{a_0 H_0} \ln(H_0 t), \quad p = 1, \tag{15}
\]

\[
\chi = \frac{c}{(p - 1)a_0 H_0} (1 - (z + 1)^{1-p}) = \frac{c}{(p - 1)a_0 H_0} (1 - (pH_0 t)^{1-1/p}), \quad p \neq 1. \tag{16}
\]

For \( p > 1 \), there is a horizon at \( \chi = c/((p - 1)a_0 H_0) \). In particular for \( p = 2, k = 0 \) the regularized luminosity \( f(z) \) is constant.

**Example** (constant regularized luminosity):
Suppose we have measured a constant regularized luminosity \( f(z) \equiv LH_0^2/(4\pi c^2) \). Then the Hubble rate is

\[
H = (z + 1) \sqrt{H_0^2(z + 1)^2 - kc^2z^2/a_0^2}. \tag{17}
\]

The three solutions of these three differential equations, \( k = \pm 1, 0 \), in terms of the scale factors are obtained from equations \( (13) \) and \( (14) \):

\[
a(t) = a_0(2H_0 t)^{1/2} \sqrt{1 - k \left[ \frac{1 - (2H_0 t)^{1/2}}{a_0 H_0/c} \right]^2}, \quad 0 < t < t_0 = \frac{1}{2H_0}. \tag{18}
\]

To alleviate notations we have suppressed the subscripts \( \pm \) from \( a(t) \) and \( a_0 \). In the closed case, \( a_0 \) must satisfy the inequality \( (12) \):

\[
a_0 \geq \frac{c}{H_0} \frac{z_{\text{max}}}{z_{\text{max}} + 1} \tag{19}
\]
and the inflection point of $g(z)$ is hidden behind the horizon at $\chi = \arcsin[c/(a_0 H_0)]$.

**Example** (constant Hubble rate):
To have an example without horizon consider the scale factor

$$a(t) = a_0 \exp[H_0(t - t_0)].$$  \hspace{1cm} (20)

This is a limiting case of the first example with $p \to 1$. It has constant Hubble rate and constant deceleration parameter, $H(z) \equiv H_0$, $q(z) \equiv -1$, and $\chi = cz/(a_0 H_0)$. In the closed case, $g(z)$ has an infinite number of inflection points alternating with short distance divergencies. The inflection points are all minima, the first is located at $z_{\text{infl}} = \pi a_0 H_0/(2c)$.

**Example** (closed universe):
Suppose we have measured the luminosity:

$$\ell(z) = \frac{LH_0^2}{4\pi c^2} \frac{z + 2}{z^2}.$$ \hspace{1cm} (21)

It is strictly decreasing and has the correct short distance divergency. Its $g(z)$ has one and only one inflection point at $z_{\text{infl}} = (1 + \sqrt{17})/2 \sim 2.56$ and suggests a closed universe with $a_0 \sim 0.337 c/H_0$.

**Counter-example** (wiggling $g(z)$):
Suppose we have measured the luminosity, figure 1:

$$\ell(z) = \frac{LH_0^2}{4\pi c^2} \frac{(\sin z/z)^2 + 0.1}{z^2}.$$ \hspace{1cm} (22)

Again it is strictly decreasing and has the correct short distance divergency. Now $g(z)$ has a maximum, figure 2. Therefore no Robertson-Walker universe, neither open nor closed, exists with this luminosity.

The last two examples illustrate that our constraint of monotonic $g(z)$ is stronger than the constraint of monotonic luminosity $\ell(z)$. 

![Figure 1: The monotonic luminosity](22)
5 Non-monotonic scale factors

If the scale factor is strictly decreasing we get similar results with a negative spectral deformation: $-1 < z < 0$, ’blue shift’.

One might think that one can produce non-monotonic functions $g(z)$ by starting from non-monotonic scale factors $a(t)$. This is not true. In fact any non-monotonic scale factor produces multivalued luminosities in terms of the spectral deformation $z$. The first example is of course the constant scale factor with no spectral deformation, $z \equiv 0$, but varying luminosity. A more generic example is, see figure 3:

$$a(t) = \frac{1}{2}gt^2 + \alpha, \quad g, \alpha > 0.$$ \hspace{1cm} (23)

For positive $t_0$, we have a maximal redshift of $z_{\text{max}} = a_0/\alpha - 1$. We have $z = 0$ for $t = \pm t_0$ and a blue shift for $t < -t_0$:

$$-1 < z_{\text{min}} = \frac{a_0}{\frac{1}{2}gt^2 + \alpha} - 1 < 0.$$ \hspace{1cm} (24)
The dimensionless co-moving distance is

\[
\chi(t) = c \sqrt{\frac{2}{g\alpha}} \left[ \arctan \left( \sqrt{\frac{g}{2\alpha} t_0} \right) - \arctan \left( \sqrt{\frac{g}{2\alpha} t} \right) \right]
\]  

(25)

and the relation between emission time \( t \) and spectral deformation \( z \) is

\[
t(z) = \pm \sqrt{\frac{2}{g}} \sqrt{\frac{a_0}{z + 1} - \alpha}.
\]  

(26)

Note that the Hubble rate vanishes at the inflection time \( t = 0, \ z = z_{\text{max}} \) while the deceleration parameter diverges there. Note also that the regularized luminosity vanishes at \( t = -t_0, \ z = 0 \). As \( t \) tends to minus infinity \( z \) tends to \(-1\) and the luminosity, regularized or not, diverges, the ‘ultra-violet divergence’. In the closed case, we have in addition antipode and short distance divergencies. Figure 4 shows only one antipode singularity.

![Figure 4: The regularized luminosity of the non-monotonic scale factor](image)

6 Data and conclusion

We use the ‘Gold’ sample data compiled by Riess et al. (2004) [3], with 157 SN’s including a few at \( z > 1.3 \) from the Hubble Space Telescope (HST GOODS ACS Treasury survey). For convenience we normalize the luminosity to the maximum absolute SN luminosity estimated by Jha et al. (1999) [4], Saha et al. (2001) [5] and Gibson & Stetson (2001) [6], \( L = L_{\text{max}} = (1 \pm 0.1) \cdot 10^{35} \text{ W} \). The Hubble rate today \( H_0 \) is taken as \((70 \pm 5) \text{ km s}^{-1} \text{ Mpc}^{-1}\) [7,8].
Figure 5: The regularized luminosity as measured today [3], with a binning of 0.02 in red shift. The full line at low redshift corresponds to a second order polynomial extrapolation fit.

The regularized luminosity allows to extract $LH_0^2$ from the Hubble diagram with small redshift, figure 5. The value $f(0)$ is extracted by a second order polynomial extrapolation fit on the SN data up to a red shift of 0.1. By construction, the fitted value $f(0)$ is equal to $LH_0^2/(4\pi c^2) = (4.3 \pm 0.4) \cdot 10^{26}$ W Mpc$^{-2}$ where the error is only coming from the fit itself.

### 6.1 Lower bound on the radius of the universe today

The first of equations (9) and equation (12) give the lower bound on $a_0$ as

$$a_0 \geq \sqrt{\frac{f(0)}{g_{\text{min}}} \frac{c}{H_0}}.$$  \hspace{1cm} (27)

The minimal value of $g(z)$ is obtained from the SN recorded at $z = 1.3$ and is equal to $g_{\text{min}} = (3.83 \pm 0.72) \cdot 10^{26}$ W Mpc$^{-2}$. At 95% confidence level (CL) we have the upper bound on the minimal value of $g(z)$: $g_{\text{min}} < 5.6 \cdot 10^{26}$ W Mpc$^{-2}$. Combining the errors from $f(0)$ and $g_{\text{min}}$ gives the lower bound on $a_0$ at 95% CL:

$$a_0 > 0.88 \frac{c}{H_0} \sim 3.8 \text{ Gpc} \sim 1.2 \cdot 10^{26} \text{ m} \sim 1.3 \cdot 10^{10} \text{ light years.}$$  \hspace{1cm} (28)
This lower bound on $a_0$ translates into a lower bound on the curvature density,

$$
\Omega_k := -k \left( \frac{c}{a_0 H_0} \right)^2 > -1.29 \quad \text{at 95\% CL.} \quad (29)
$$

Note that this limit is independent of the values of the absolute luminosity $L$ and of the Hubble rate today. It is also independent of any dynamical hypothesis.

Let us compare this bound with the one obtained from the SN data but now adding the dynamics of the $\Lambda CDM$ cosmology fitting the matter density $\Omega_m$, the cosmological constant density $\Omega_\Lambda$ and the nuisance normalisation parameter and without any other input constraint: $\Omega_k > -1$ at 95\% CL.

### 6.2 Wiggles and non-monotonicity

We must now ask the question whether the data is compatible with a monotonic luminosity $\ell(z)$. We will also ask the finer question whether $g(z)$ is monotonic.

To detect non-monotonicity in the SN data set we assume that the luminosity $\ell(z)$ and its $g(z)$ can be described by monotonic functions to which we add a simple Gaussian:

$$
L(z) = \ell(z, LH_0^2, p) + a_w \exp -\frac{(z - z_w)^2}{2\delta z_w^2}
$$

and

$$
G(z) = g(z, LH_0^2, p) + a_w \exp -\frac{(z - z_w)^2}{2\delta z_w^2}.
$$

The monotonic functions $\ell(z, LH_0^2, p)$ and $g(z, LH_0^2, p)$ are derived from a power law parameterization of the scale factor $a(t) := a_0 (pH_0 t)^{1/p}$ (constant deceleration parameter), $1/5 < p < 1$, $k = 0$. It yields the monotonic luminosity

$$
\ell(z, LH_0^2, p) = \frac{LH_0^2}{4\pi c^2} \left( \frac{p - 1}{(z + 1)[1 - (z + 1)^{1-p}]} \right)^2,
$$

and with $p = 0.69$ it describes well the $\Lambda CDM$ cosmology up to a redshift of 1.8 [9]. Its acceleration is positive with $q \equiv -0.31$.

Our wiggle detection procedure consists of scanning the plane in wiggle position $z_w$ and wiggle width $\delta z_w$ in steps of 0.01 in both directions. In each point $(z_w, \delta z_w)$ of this plane we fit the normalization $LH_0^2$, the power $p$ and the wiggle amplitude $a_w$.

Warning: if the wiggle amplitude $a_w$ is smaller than a critical amplitude $a_w^c$ the modified functions (30) and (31) will still be monotonic.

We would claim that $\ell(z)$ and a fortiori the luminosity $g(z)$ is not monotonic if the ratio between the fitted wiggle amplitude and the associated error is greater than 5 ($5\sigma$ level detection) and if the wiggle amplitude is greater than the critical one. The sensitivity of the method is computed by Monte Carlo simulation. The same SN sample than the Riess data set with the same statistical power is simulated assuming the $\Lambda CDM$ cosmology and a wiggle of positive or negative amplitude is added to $\ell(z)$ and $g(z)$ for each point in the $(z_w, \delta z_w)$ plane. We apply the wiggle detection procedure on each simulation, restricted
to a small grid of points around the simulated one to speed up the processing. The significance on the wiggle amplitude ($|a_{w,\text{fitted}}|/\sigma_{a_w}$) is computed in each point and the smaller value from positive or negative wiggle amplitude is retained. The sensitivity is computed at a 2σ level corresponding to a 95% confidence level exclusion limit on the wiggle magnitude $m_w$ defined by

$$a_w(z_w) = \pm (10^{-m_w/2.5} - 1)\ell(z_w). \quad (33)$$

Figure 6 shows the significance of the wiggle fit performed on the Riess data sample (color contours) for the luminosity $\ell(z)$ and $g(z)$ with $z_w$ varying from 0.01 to 1.8 and $\delta z_w$ from 0.01 to 2 in steps of 0.01 in both directions. The maximum significance for both $\ell(z)$ and $g(z)$ is 2.4 for a wiggle at the position $z_w = 0.45$, with width of 0.07. The wiggle magnitude is $m_w = 0.16$ for the luminosity $\ell(z)$ and $m_w = 0.25$ for $g(z)$. The dashed lines indicate the location of the critical wiggle magnitude $m^c_w$ of a positive wiggle that breaks the monotonicity.

No wiggle greater than this value is observed and we conclude that no wiggle is detected at a 5σ level using the actual SN data set. On the same figures, the sensitivity for different values of the wiggle magnitude is shown (plain line). Wiggles of magnitude greater than 2 are excluded at 95% CL up to a redshift of 1.6. Up to a redshift of 1, the 95% CL exclusion limit on the wiggle magnitude is 0.6. These two magnitudes are below the critical magnitudes and therefore these wiggles do not upset the monotonicity.

References

[1] see for example M. Berry, Principles of Cosmology and Gravitation, Cambridge University Press (1976)

[2] G. Esposito-Farèse & D. Polarski, Scalar tensor gravity in an accelerated Universe, [gr-qc/0009034], Phys. Rev. D63 (2001) 063504

[3] A. G. Riess et al. Astroph. J. 607 (2004) 665

[4] S. Jha , P. M. Garnavich, R. P. Kirshner et al., The Type Ia Supernova 1998 BU in M96 and the Hubble Constant, ApJS 125 (1999) 73

[5] A. Saha, A. Sandage, G. A. Tamman et al., Cepheid calibration of the peak brightness of Type Ia supernovae XI. Sn 1998ap in NGC 3982, ApJ 562 (2001) 314

[6] B. K. Gibson & P. B. Stetson, Supernova 1991T and the value of the Hubble Constant, ApJ 547 (2001) L103

[7] L. M. Krauss, Space, time and matter: Cosmological Parameters 2001, in Identification of Dark Matter (2001)

[8] J. Raux, Photométrie différentielle de supernovae de type Ia lointaines (0.5 < z < 1.2) mesurées avec le télescope spatial Hubble et estimation des paramètres cosmologiques, PhD Université Paris 11 (2003)
[9] F. Henry-Couannier et al., *Negative Energies and a Constantly Accelerating Flat Universe* to be published
Figure 6: Color contours: Significance of wiggle detection (vertical colour scale) as function of the wiggle position $z_w$ and width $\delta z_w$, logarithmic scale. Full lines: Expected $2\sigma$ sensitivity of the wiggle detection as a function of wiggle magnitude $m_w$. The dashed lines indicate the location of the critical wiggle magnitude $m_w^c$ of a positive wiggle that breaks the monotonicity. The upper panel is the exclusion plot for a non-monotonic luminosity $\ell(z)$, the lower panel for non-monotonic $g(z)$. 