Neyman-Pearson Detection of a Gaussian Source using Dumb Wireless Sensors

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Abstract

We investigate the performance of the Neyman-Pearson detection of a stationary Gaussian process in noise, using a large wireless sensor network (WSN). In our model, each sensor compresses its observation sequence using a linear precoder. The final decision is taken by a fusion center (FC) based on the compressed information. Two families of precoders are studied: random iid precoders and orthogonal precoders. We analyse their performance in the regime where both the number of sensors $k$ and the number of samples $n$ per sensor tend to infinity at the same rate, that is, $k/n \to c \in (0, 1)$. Contributions are as follows. 1) Using results of random matrix theory and on large Toeplitz matrices, it is proved that the miss probability of the Neyman-Pearson detector converges exponentially to zero, when the above families of precoders are used. Closed form expressions of the corresponding error exponents are provided. 2) In particular, we propose a practical orthogonal precoding strategy, the Principal Frequencies Strategy (PFS), which achieves the best error exponent among all orthogonal strategies, and which requires very few signaling overhead between the central processor and the nodes of the network. 3) Moreover, when the PFS is used, a simplified low-complexity testing procedure can be implemented at the FC. We show that the proposed suboptimal test enjoys the same error exponent as the Neyman-Pearson test, which indicates a similar asymptotic behaviour of the performance. We illustrate our findings by numerical experiments on some examples.

I. INTRODUCTION

The design of powerful tests allowing to detect the presence of a stochastic signal using large WSN’s is a crucial issue in a wide range of applications. We investigate the Neyman-Pearson detection of a Gaussian signal using a wireless network of $k$ sensors. Each sensor...
observes a finite sample of the signal of interest, corrupted by additive noise, and then forwards some information towards the FC which takes the final decision. Neyman-Pearson detection of Gaussian signals using large sensor networks has been thoroughly investigated in the literature (see for instance [1], [2] and references therein). In such works, the FC is assumed to have a perfect knowledge of the observation sequence of each sensor. Unfortunately, in a WSN, the amount of information forwarded by each sensor node to the FC is usually limited, due to channel capacity constraints. Thus, in practice, each sensor node must compress its information in some way before transmission to the FC. This compression step of course degrades the performance of the detection. A large number of works has been devoted to the determination of relevant compression strategies, essentially within the framework of distributed detection [3], [4]. In these works, the data is locally processed by each sensor: Typically, a local Neyman-Pearson test is made by each node, based on the knowledge of the probabilistic law of the source to be detected. Unfortunately, such approaches require at the same time that each sensor possesses a significant computational ability allowing involved processing of its data, and that each sensor has a full knowledge of the source statistics. On the opposite, this paper investigates the case of dumb WSN. By this term, we refer to the case where:

- Individual sensor nodes are not aware of their mission and their environment. They process the observed data with no or few instructions from the central processor.
- The processing abilities of each sensor node are limited due either to hardware or energy constraints.

Dumb WSN are of practical interest because they are simple, flexible (i.e., easily reconfigurable as a function of the sensor network’s mission) and avoid an excess of signaling overhead in the network.

The aim of this paper is to propose and to study different compressing strategies which satisfy the above constraints and which are attractive in terms of detection performance. The paper is organized as follows. Section II introduces the signal model. Each sensor is assumed to observe \( n \) noisy samples of a stationary (correlated) Gaussian source. The spectral density \( f \) of the source is known at the FC but is unknown at the sensor nodes. The aim is to detect the presence of the source. To that end, each node forwards a compressed version of its observed sequence to the FC. In our model, the latter compression is achieved through simple (linear) processing of the
data, allowing this way for low cost implementation. We refer to this step as linear precoding. Section III introduces the problem of the detection of the presence of the source (hypothesis $H_1$) versus the hypothesis that only thermal noise is observed (hypothesis $H_0$). It is well known that a uniformly most powerful (UMP) test is obtained by the celebrated Neyman-Pearson procedure. The corresponding test is derived in Subsection III-A. Intuitively, the good detection performance of the Neyman-Pearson test fundamentally relies on the relevant selection of the linear precoders used at the sensor nodes. Useful families of linear precoders are introduced, namely random iid precoders and orthogonal precoders. The detection performance associated with each of these families is studied in the asymptotic regime where both the number $k$ of sensors and the number $n$ of observations per sensor tend to infinity at the same rate ($k, n \to \infty$, $k/n \to c$ where $c \in (0, 1)$). More precisely, we show in Section IV that for any fixed $\alpha \in (0, 1)$, the miss probability of the NP test of level $\alpha$ converges exponentially to zero. Error exponents are characterized and compared for the precoding strategies of interest. In particular, it is proved that the so-called Principal Frequencies Strategy (PFS) achieves the best error exponent among all orthogonal strategies. Numerical computations of all the obtained error exponents on some examples conclude this section. In the case where PFS is used, a suboptimal (non UMP) test is proposed in Section V. Based on the proof of a Large Deviation Principle governing the proposed test statistics, it is shown that our suboptimal test achieves the same error exponent as the Neyman-Pearson test. Finally, Section VI is devoted to the simulations.

Notations

Column vectors are represented by bold symbols. Notation $\|y\|$ denotes the Euclidean norm of vector $y$. We denote by Leb the Lebesgue measure restricted to $[-\pi, \pi]$. For any function $f : [-\pi, \pi] \to \mathbb{R}$, we use notation $f^{-1}(A) = \{\omega \in [-\pi, \pi] : f(\omega) \in A\}$ for the inverse image of $A$, and we denote by Leb $\circ f^{-1}$ the image measure of Leb by $f$, i.e. which composes Leb with $f^{-1}$. For any square matrix $M$, $\rho(M)$ denotes its spectral radius. Finally, $I_k$ denotes the $k \times k$ identity matrix.
II. THE FRAMEWORK

A. Observation model at the sensor nodes

Consider a set of \( k \) sensors whose aim is to detect the presence of a certain source signal \( x(0), x(1), x(2), \ldots \). Each sensor \( i = 1 \ldots k \) collects \( n \) noisy samples of the source signal. We assume that \( n \geq k \). Denote by \( y_i = [y_i(0), \ldots, y_i(n-1)]^T \) the \( n \times 1 \) data vector observed by sensor \( i \). For each \( i = 1, \ldots, k \), we consider the following signal model:

\[
y_i = x + w_i,
\]

where \( x = [x(0), \ldots, x(n-1)]^T \) contains the time samples extracted from a zero mean stationary Gaussian process \( x \) with known spectral density function \( f(\omega), \omega \in [-\pi, \pi] \). Vector \( w_i = [w_i(0), \ldots, w_i(n-1)]^T \) is a zero mean white Gaussian process which stands for the thermal noise of sensor \( i \). We denote by \( \sigma^2 \) the variance of \( w_i(0) \) which is assumed to be the same for all \( i \). Random vectors \( x, w_1, \ldots, w_k \) are supposed to be independent. In the usual framework of Gaussian source detection, the aim is to detect whether the signal \( x \) of interest is present. Formally, this reduces to the following hypothesis testing problem:

\[
H_1: \quad y_i = x + w_i, \quad \forall i = 1 \ldots k
\]

\[
H_0: \quad y_i = w_i, \quad \forall i = 1 \ldots k.
\]

In this paper, we make the following technical assumptions on the spectral density \( f \):

A1. The spectral density \( f \) is continuous on \([-\pi, \pi]\).

A2. Measure \( \text{Leb} \circ f^{-1} \) does not put mass on points.

Assumption A2 says that \( f \) cannot be constant over a set of positive Lebesgue measure (say, an interval of positive length). This e.g. rules out a white noise for \( x \). On the other hand any ARMA process \( x \) that is not a white noise satisfies Assumptions A1 and A2.

B. Assumptions and constraints on the network

We assume that the decision is taken by a distant node (the fusion center). The latter is supposed to have a perfect knowledge of the noise variance \( \sigma^2 \) and of the spectral density \( f \) of the signal \( x \) to be detected. In this paper, we are interested in WSN satisfying the following constraints.
1) **Communication constraint:** In an ideal WSN architecture, each sensor \( i = 1, \ldots, k \) would transmit all available observations \( y_i(0), \ldots, y_i(n-1) \) to the FC. Unfortunately, perfect forwarding of the whole information sequence \( y_i \) by each sensor \( i \) is impractical in a large number of situations, the amount of information transmitted by each sensor node to the fusion center being usually limited. In this paper, we consider the case where only a compressed version of \( y_i \) is likely to be forwarded. More precisely, we assume that each sensor \( i \) forwards a single scalar \( z_i \) to the fusion center, where \( z_i \) is a certain mapping of the sequence \( y_i \) received by sensor \( i \).

2) **Signaling overhead constraint:** Depending on the particular mission of the network or on the particular spectral density \( f \) to be detected, the network should be easily reconfigurable using a limited number of feedback bits from the fusion center to the sensors. In the sequel we assume that the spectral density \( f \) is known at the fusion center but is unknown (or at most partially known) at the sensor nodes.

3) **Complexity constraint:** Only low complexity data processing is likely to be implemented at the sensors’ side. More precisely, we assume that each sensor node \( i = 1 \ldots k \) forwards a linear combination

\[
z_i = a_i^T y_i
\]

of its observation sequence \( y_i \) to the fusion center, where \( a_i \) is a \( n \times 1 \) vector to be determined. Figure 1 provides an illustration of the sensing scheme. Such a set of vectors \( a_1, \ldots, a_k \) will be referred to as a *linear precoder*. The \( n \times k \) matrix \( A_n = [a_1, \ldots, a_k] \) will be referred to as the
precoding matrix.

III. LIKELIHOOD RATIO TEST

A. Expression of the Likelihood Ratio

We denote by $P_0$ and $P_1$ the probability under $H_0$ and $H_1$ and by $E_0$ and $E_1$ the corresponding expectations. Denote by $z = [z_1, \ldots, z_k]^T$ the available $k \times 1$ observation vector at the fusion center, where for each $i$, $z_i$ is defined by (2). We denote by $p_0 : \mathbb{R}^k \rightarrow \mathbb{R}_+$ and $p_1 : \mathbb{R}^k \rightarrow \mathbb{R}_+$ the joint probability density function of $z_1, \ldots, z_k$ under hypotheses $H_0$ and $H_1$ respectively. Due to the celebrated Neyman-Pearson’s Lemma, the Likelihood Ratio Test (LRT) is uniformly most powerful. The LRT rejects the null hypothesis for large values of the log-likelihood ratio (LLR) defined by:

$$L_{A_n} = \log \frac{p_1(z)}{p_0(z)}.$$  \hfill (3)

In the above definition, the lowerscript $A_n$ has been introduced to recall that the distribution of the random variable $L_{A_n}$ depends on the particular choice of the precoding matrix $A_n = [a_1, \ldots, a_k]$. We now derive a closed form expression of the LLR $L_{A_n}$. It is worth noting that multiplying each $a_i$ by a non-zero constant does not modify the performance of the likelihood ratio test. Hence we may normalize $A_n$ so that $\|a_i\| = 1$ for each $i$ in the following. In this case, $z$ is a zero mean Gaussian random vector with covariance matrix

$$\mathbb{E}_1(zz^T) = A_n^T \Gamma_n A_n + \sigma^2 I_k$$

under hypothesis $H_1$, where $\Gamma_n = \mathbb{E}_1(xx^T)$ represents the $n \times n$ covariance matrix of vector $x$. Matrix $\Gamma_n$ is the $n \times n$ Toeplitz matrix associated to the spectral density $f$ of process $x$, namely,

$$\Gamma_n = T_n(f) = \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega) e^{i\omega(k-l)} d\omega \right]_{1 \leq k, l \leq n}.$$ \hfill (4)

Under $H_0$, the covariance matrix of vector $z$ simply coincides with $\mathbb{E}_0(zz^T) = \sigma^2 I_k$. Using these remarks, it is straightforward to show that

$$2 L_{A_n} = k \log \frac{\|z\|^2}{\sigma^2} - \log \det(A_n^T \Gamma_n A_n + \sigma^2 I_k) - z^T (A_n^T \Gamma_n A_n + \sigma^2 I_k)^{-1} z.$$ \hfill (5)

In the sequel, we assume as usual that the threshold of the test, say $\gamma_n$, is fixed in such a way that the probability of false alarm (PFA) does not exceed a level $\alpha$ ($0 < \alpha < 1$), which reads

$$\mathbb{P}_0(L_{A_n} > \gamma_n) \leq \alpha.$$ \hfill (6)

We now analyze the miss probability of the above LLR test as a function of $A_n$. 

DRAFT
B. Introduction to error exponents

Let $P_M(\alpha; A_n)$ denote the miss probability of the LLR test with level $\alpha$ based on the observation $z_1, \ldots, z_k$:

$$P_M^n(\alpha; A_n) = \inf P_1(\mathcal{L}_{A_n} \leq \gamma_n) ,$$

where the inf is taken over all threshold values $\gamma_n$ verifying the PFA constraint (6). The miss probability is generally the key metric to characterize the performance of hypothesis tests. Unfortunately, an exact expression of the miss probability as a function of $A_n$ is difficult to obtain in the general case. Following [5], we thus analyze the asymptotic behaviour of the miss probability as the number of available observations tends to infinity. More precisely, we study the asymptotic regime where both the number of sensors $k$ and the number of observations $n$ per sensor tend to infinity at the same rate:

$$n \to \infty, \quad k \to \infty, \quad \frac{k}{n} \to c$$

(7)

where $c \in (0, 1)$. Any sequence of $n \times k$ precoding matrices $\mathcal{A} = (A_n)_{n \geq 0}$ will be referred to as a linear strategy. Loosely speaking, we will prove that, at least for certain linear strategies of interest, the miss probability behaves as

$$P_M^n(\alpha; A_n) \simeq e^{-nK_\alpha(A)}$$

in the asymptotic regime (7), where $K_\alpha(A)$ is a certain constant which depends on the linear strategy but, as a matter of fact, does not depend on the level $\alpha$. Such a constant is called the error exponent. It is a key indicator of the way the power of the test is influenced by the chosen linear strategy. More formally, we define for each $\mathcal{A}$,

$$K_\alpha(\mathcal{A}) = \liminf_{k \to \infty} -\frac{1}{n} \log P_M^n(\alpha; A_n) ,$$

(8)

$$\overline{K}_\alpha(\mathcal{A}) = \limsup_{k \to \infty} -\frac{1}{n} \log P_M^n(\alpha; A_n) ,$$

(9)

and we define the error exponent of $\mathcal{A}$ as $K_\alpha(\mathcal{A}) = K_\alpha(\mathcal{A}) = \overline{K}_\alpha(\mathcal{A})$ as soon as (8) and (9) coincide. In the sequel, our aim is therefore to determine linear precoding strategies $\mathcal{A}$ having a large error exponent $K_\alpha(\mathcal{A})$ (and for which $K_\alpha(\mathcal{A})$ is well-defined, of course). The following Lemma (see [5]) provides a practical way to evaluate error exponents.
Lemma 1 ([5]) The following inequalities hold:

\[
K_\alpha(A) \geq \sup \left\{ t : \liminf_{n \to \infty} P_0 \left[ \frac{1}{n} \log \frac{p_0(z)}{p_1(z)} \leq t \right] < \alpha \right\}
\]

\[
\overline{K}_\alpha(A) \leq \sup \left\{ t : \limsup_{n \to \infty} P_0 \left[ \frac{1}{n} \log \frac{p_0(z)}{p_1(z)} \leq t \right] \leq \alpha \right\}.
\]

In particular if, under hypothesis \( H_0 \), \( -n^{-1} \mathcal{L}_{A_n} \) converges in probability to a deterministic constant \( \xi \), then \( \overline{K}_\alpha(A) = K_\alpha(A) = K_\alpha(A) = \xi \) is necessary equal to this limit.

According to the above lemma, the asymptotic performance analysis of the LLR test reduces to the characterization of the limit in probability of the normalized LLR as \( n \to \infty \), as soon as this limit exists. Moreover, in this case, the error exponent \( K_\alpha \) is independent from level \( \alpha \).

C. Some families of precoders

A natural approach to design relevant precoders would be to characterize the linear strategies \( \mathcal{A} \) which maximize the limit in probability (if it exists) of the LLR \( \mathcal{L}_{A_n} \) as \( n, k \to \infty \). Ideally, this would lead to the strategies with maximal error exponent. Unfortunately, such a characterization is difficult and would moreover lead to linear strategies which would deeply depend on the spectral density \( f \) of the signal to be detected. The practical implementation of such optimal linear strategies would typically require to communicate the whole function \( f \) to each sensor via a feedback link from the central processor. In this paper, we focus on the opposite on the case of “dumb” sensors \( i.e. \), sensors which are able to process information with few or no knowledge of their mission or their environment. More precisely, we separately study the following linear strategies.

1) **Random iid precoders**: A natural way to design dumb sensor networks is to select each sensor’s precoder at random, independently from the network’s mission. Motivated by first by the simplicity of the approach and second by its widespread use in compressive sensing applications [6], we assume that matrix \( A_n \) is one realization of a \( n \times k \) random matrix with zero mean iid entries. In the case of random iid precoders, sensors are able to precode their information without any instructions from the fusion center.

2) **Orthogonal precoders**: In this case, matrix \( A_n \) is such that \( A_n^T A_n = I_k \) \( i.e. \), the precoders \( a_1, \ldots, a_k \) are orthogonal. Orthogonal precoders will reveal useful for the design of dumb but nevertheless efficient sensor networks. Indeed, under this constraint, we are able to
exhibit strategies that achieve the best error exponent. In addition, when such precoders are used, we will show that a low complexity testing procedure can be implemented as an alternative to the costly Likelihood Ratio Test, without decreasing the error exponent.

IV. ERROR EXPONENTS

A. Case of random iid precoders

Before stating the main result of this subsection, remark that the performance of the test is of course expected to depend on the covariance matrix $\Gamma_n$ of the signal to be detected. In particular, it is useful to recall some well known results on the behaviour of the eigenvalues of $\Gamma_n$. From classical results on large Toeplitz matrices [7], it is known that $\Gamma_n$ can be approximated by a circulant matrix with eigenvalues $f(0), f(\frac{2\pi}{n}), \ldots, f(\frac{2\pi(n-1)}{n})$. More precisely, for any Hermitian $n \times n$ matrix $Q$, we denote by $F_Q(t) = \frac{\# \{i, \lambda_i(Q) \leq t \}}{n}$ the distribution function associated with the empirical distribution of the eigenvalues $\lambda_1(Q), \ldots, \lambda_n(Q)$ of $Q$ (the corresponding probability measure is often referred to as the spectral measure of $Q$). Szegő’s Theorem ([7], p.64) states that, provided that Assumption A1 holds, $F_{\Gamma_n}$ converges weakly to the distribution function $\Phi$ defined by:

$$\Phi(t) = \frac{1}{2\pi} \text{Leb} o f^{-1}((\infty, t]) , \quad (10)$$

where we recall that Leb $o f^{-1}$ is the measure which composes the Lebesgue measure Leb on $[-\pi, \pi]$ with $f^{-1}$ (the inverse image under $f$). The error exponent merely depends on the latter limiting spectral measure $\Phi$, as stated by the following Theorem.

**Theorem 1** Suppose that [7] holds for some $c \in (0, 1)$ and assume A1. For each $n$, let $A_n = (A_{ij}^n)$ be a $n \times k$ real random matrix such that $A_{ij}^n$ for all $n, i, j$ are iid zero mean random variables with finite second order moment. Consider any fixed level $\alpha \in (0, 1)$. Then the linear strategy $A = (A_n)_n$ admits an error exponent $K_\alpha(A) = K_{\text{rnd}}(c)$ given by:

$$K_{\text{rnd}}(c) = -c + \sigma^2 c^2 \beta - \frac{c}{2} \log(\sigma^2 \beta) + \frac{1}{2} \int \log(1 + ct \beta) d\Phi(t) , \quad (11)$$

where $\beta$ is the unique solution to the following equation:

$$\sigma^2 = \frac{1}{\beta} - \int \frac{t}{1 + ct \beta} d\Phi(t) . \quad (12)$$

The proof is provided in Appendix [A-A]
B. Case of orthogonal precoders

We now focus on the case where $A_n^T A_n = I_k$. Our aim is first to prove that among all orthogonal strategies, we may determine some that achieve the maximum error exponent and second, to determine this maximum error exponent. Results are provided below in Theorem 2.

We first provide some definitions along with some insights on the results.

Loosely speaking, it is easy to think of a relevant orthogonal strategy as follows. Focus on one given sensor $i = 1 \ldots k$ for the sake of simplicity. Under $H_0$, the received sequence $y_i = w_i$ corresponds to a white Gaussian noise of variance $\sigma^2$. Therefore the law of $z_i = a_i^T y_i$ is $\mathcal{N}(0, \sigma^2)$. Under $H_1$, it is straightforward to show that $z_i \sim \mathcal{N}(0, a_i^T \Gamma_n a_i + \sigma^2)$, where we recall that $\Gamma_n = \mathbb{E}(xx^T)$ is the signal covariance matrix. Clearly, the best way for the sensor $i$ to discriminate $H_1$ versus $H_0$ is to chose the precoder $a_i$ which maximizes the variance $a_i^T \Gamma_n a_i + \sigma^2$. This is achieved when $a_i$ coincides with the eigenvector of $\Gamma_n$ associated with the largest eigenvalue. Generalizing this remark to $k$ sensors, it is natural to introduce the strategy for which the $k$ precoders $a_1 \ldots a_k$ coincide with the $k$ eigenvectors of $\Gamma_n$ associated with the largest eigenvalues. We shall refer to this strategy as the Principal Component Strategy (PCS).

**Definition 1 (principal components strategy (PCS))** Let $(v^n_i)_{1 \leq i \leq n}$ be the eigenvectors of $\Gamma_n$ and $(\lambda^n_i)_{1 \leq i \leq n}$ be the corresponding eigenvalues, ordered in such a way that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$. The principal component strategy $V$ is defined as the sequence of $n \times k$ matrices $V_n$ given by:

$$V_n = [v^n_1, \ldots, v^n_k].$$

As will be stated by Theorem 2 below, PCS achieves the maximum error exponent among all orthogonal strategies. Unfortunately, exact PCS might be difficult to implement in a dumb sensor network, as each node needs to be informed of a whole eigenvector of the covariance matrix $\Gamma_n$. This requires involved cooperation between the nodes and the fusion center. In order to reduce the amount of overhead in the network, we propose an alternative strategy which turns out to achieve the same error exponent as PCS. Let $F_n = [F_n(i, j)]_{0 \leq i, j \leq n-1}$ denote the $n \times n$ real-valued orthogonal Fourier basis matrix, that is $F_n = [e^n_0, \ldots, e^n_{n-1}]$, where the columns $e^n_j$
are defined, up to a normalizing constant by
\[
\begin{align*}
e_0^n & \propto [1, \ldots, 1]^T \\
e_j^n & \propto [\cos(2\pi ij/n)]_{i=0,\ldots,n-1} \quad \text{for} \quad j = 1, \ldots, [n/2] \\
e_{n-j}^n & \propto [\sin(2\pi ij/n)]_{i=0,\ldots,n-1} \quad \text{for} \quad j = 1, \ldots, [(n-1)/2].
\end{align*}
\]

The main idea is to remark that for large \( n \), the covariance matrix \( \Gamma_n \) can be approximated by the matrix
\[
F_n \text{diag}(f(0) \ldots f(2\pi(n-1)/n)) F_n^T,
\]
see [7], [8] for more details. As a consequence, it seems reasonable to propose a strategy inspired of PCS, only substituting the above matrix (13) with the true covariance matrix \( \Gamma_n \). This leads to the following definition.

**Definition 2 (principal frequencies strategy (PFS))** For each \( n \), denote by \( (j_n^1, \ldots, j_n^n) \) any permutation of \( \{0, 1, \ldots, n-1\} \) such that \( f(2\pi j_1^n/n) \geq \cdots \geq f(2\pi j_n^n/n) \). The principal frequencies strategy \( W \) is defined as the sequence of \( n \times k \) matrices \( W_n \) given by:
\[
W_n = \left[ e_{j_n^1}^n, \ldots, e_{j_n^k}^n \right],
\]
where \( e_1^n, \ldots, e_n^n \) are the columns of matrix \( F_n \).

Note that PFS only requires to transmit one of the \( k \) indices \( j_n^1 \ldots j_n^k \) corresponding to the principal frequencies of \( f \) to each sensor. In return, each sensor \( i \) computes the scalar product between the \( j_i^n \)th column of Fourier matrix \( F_n \) and its received sequence \( y_i \). In other words, it computes the value of the (real) periodogram of \( y_i \) at frequency \( 2\pi j_i^n/n \). The following result proves furthermore that both PCS and PFS achieve the best error exponent among all orthogonal strategies.

For any \( c \in (0, 1) \) denote by \( \Delta_c \) the following set of frequencies:
\[
\Delta_c = \{ \omega \in (-\pi, \pi) : \Phi \circ f(\omega) \geq 1 - c \}.
\]

It is worth noting that the Lebesgue measure of \( \Delta_c \) is equal to \( 2\pi c \) (see Lemma 5).

**Theorem 2** Suppose that (7) holds for some \( c \in (0, 1) \), and assume A1 and A2. Let \( V \) and \( W \) respectively denote the PCS and PFS as defined above. For any \( \alpha \in (0, 1) \), the error exponents
\( K_\alpha(\mathcal{V}) \) and \( K_\alpha(\mathcal{W}) \) associated with \( \mathcal{V} \) and \( \mathcal{W} \) exist, and are such that \( K_\alpha(\mathcal{V}) = K_\alpha(\mathcal{W}) = K_{\text{orth}}(c) \) where

\[
K_{\text{orth}}(c) = \frac{1}{2\pi} \int_{\Delta_c} D \left( \mathcal{N}(0, \sigma^2 \| \mathcal{N}(0, f(\omega) + \sigma^2) \right) d\omega ,
\]

where \( D \) denotes the Kullback-Leibler contrast. Moreover, for any orthogonal strategy \( \mathcal{A} \),

\[
\overline{K}_\alpha(\mathcal{A}) \leq K_{\text{orth}}(c) .
\]

The proof is provided in Appendix A-B. Let us briefly comment the best error exponent formula (16). First we recall that for any \( \sigma_1^2, \sigma_2^2 > 0 \),

\[
D \left( \mathcal{N}(0, \sigma_1^2 \| \mathcal{N}(0, \sigma_2^2) \right) = -\frac{1}{2} \left[ \log \frac{\sigma_1^2}{\sigma_2^2} + 1 - \frac{\sigma_1^2}{\sigma_2^2} \right] ,
\]

which is increasing as \( \sigma_1/\sigma_2 \) gets away from 1 from above or below. Since \( \Phi \) is nondecreasing, we see that the frequencies \( \omega \) lying in \( \Delta_c \) are those that maximize \( D \left( \mathcal{N}(0, \sigma_1^2 \| \mathcal{N}(0, f(\omega) + \sigma_2^2) \right) \) in \([\pi, \pi]\). Thus \( K_{\text{orth}}(c) \) can be interpreted as some distance between the two spectral densities \( \sigma_1^2 \) (corresponding to \( H_0 \)) and \( f + \sigma_2^2 \) (corresponding to \( H_1 \)) restricted to a set of frequencies where these two spectral densities are the furthest apart.

C. Illustration and comparisons

Error exponents \( K_{\text{orth}} \) and \( K_{\text{iid}} \) defined in sections IV-B and IV-A depends on the following parameters: the spectral density \( f \), the noise level \( \sigma \), along with the sensors growth ratio \( c \). When using the orthogonal strategy, one can expect that the more peaky \( f \) is, the more efficient the compression will be. That is, by using only a few sensors configured at the peak frequencies, one will get an attractive exponent error. This should also lead to a sharp increase of the error exponent curve \( K_{\text{orth}}(c) \) for small \( c \). On the contrary, when \( f \) is nearly flat (with a small range of values), there are no privileged frequencies for the sensors to forward and the error exponent should increase slowly as \( c \) gets larger. Let us illustrate these intuitive arguments with numerical experiments. We consider two spectral densities corresponding to ARMA processes. The corresponding plots are depicted in Fig. 2.

\[
f_1(\omega) = s_1^2 \left| \frac{1 - \frac{1}{2} \exp(i\omega) + \frac{1}{2} \exp(2i\omega)}{1 - \frac{1}{2} \exp(i\omega) - \frac{1}{2} \exp(2i\omega) - \frac{1}{10} \exp(3i\omega)} \right|^2 ;
\]

\[
f_2(\omega) = s_2^2 \left| \frac{1 + \frac{1}{2} \exp(i\omega) - \frac{1}{2} \exp(2i\omega)}{1 + \frac{1}{2} \exp(i\omega) - \frac{1}{10} \exp(2i\omega)} \right|^2 .
\]
Fig. 2. Left: Spectral density $f_1$ with $s_1$ adjusted such that $\frac{1}{2\pi} \int_{-\pi}^{\pi} f_1 = 1$. Right: Spectral density $f_2$ with $s_2$ adjusted such that $\frac{1}{2\pi} \int_{-\pi}^{\pi} f_2 = 1$. One can notice that $f_1$ has a sharp peak while $f_2$ takes its values in much smaller range.

Fig. 3. Error Exponents $K_{\text{iid}}(c)$, $K_{\text{orth}}(c)$ and $K_{\text{fst}}(c)$ as functions of the growth ratio $c = \lim k/n$, for spectral density functions $f_1$ (left) and $f_2$ (right).

Fig. 3 represents $K_{\text{iid}}(c)$ and $K_{\text{orth}}(c)$ for $\sigma = 1$. For comparison, we also plotted another error exponent curve, corresponding to an orthogonal, yet suboptimal strategy which uses precoding matrices $A_n = [I_k \ 0]$. This strategy amounts to keep only the first $k$ values of the signal, independently of $f$. It is straightforward to prove that the corresponding error exponent writes $K_{\text{fst}}(c) = c \cdot K_{\text{orth}}(1)$.

One can notice several numerical facts on Fig. 3. First, as expected from section IV-B, $K_{\text{fst}}$ is always below $K_{\text{orth}}$. Remark that, as expected, $K_{\text{orth}}$ has a sharper increase near $c = 0$ when used with $f_1$ than when used with $f_2$. The fact that the random iid strategy seems to behave better for $c$ close to 1 is more surprising but it reveals the following interesting fact: in some circumstances, a non-orthogonal strategy may outperform an optimal orthogonal strategy.

Let us try to interpret this result. When setting up the sensor network design, one faces a tradeoff.
Fig. 4. Error Exponent curves for spectral density functions $f_1$ (left) and $f_2$ (right) with $\sigma^2 = 1/2$ (top) and $\sigma^2 = 4$ (bottom).

Either use an extra sensor over the same frequency that the previous sensor in order to denoise their common measurement, or use this extra sensor over a new frequency in order to discover another part of the spectral density $f$. At small levels of noise, it is always more interesting to discover $f$ at new frequencies than to denoise ones already used by other sensors; indicating that the orthogonal strategy is always the best. But for high levels of noise, it may become more efficient to repeat (and thus denoise) key frequencies than to discover less important ones. To support this claim, we refer to Fig. 4 where we chose two levels of noise, one that is larger ($\sigma^2 = 4$) than the one used in Fig. 3 and one that is smaller ($\sigma^2 = .5$). One can see that when $\sigma^2 = .5$, the best orthogonal strategy outperforms the two others, whereas for $\sigma^2 = 4$ the upcrossing of $K_{iid}$ over $K_{orth}$ near $c = 1$ is more important than on Fig. 3. The same conclusions hold for both spectral densities $f_1$ and $f_2$.

V. A PFS-BASED LOW COMPLEXITY TEST

Results of the previous section indicate that the principal frequencies strategy is a good candidate for implementation in dumb sensor networks. Indeed it requires only few cooperation
between the nodes and the fusion center, and is attractive from an error exponent perspective. In this section, we prove furthermore that when PFS is used, then a test procedure can be proposed which is much less complex than the LRT, and which achieves nevertheless the same error exponent.

We assume throughout this section that PFS is used, i.e., each precoding matrix is given by $A_n = W_n$ where $W_n$ is defined by (14).

A. A low complexity test

Recall that the LRT rejects the null hypothesis when the LLR (5) is above a threshold. As the terms $k \log \sigma^2$ and $\log \det(A_n^T \Gamma_n A_n + \sigma^2 I_k)$ are constant w.r.t. the observation $z$, it is clear that the LRT reduces to the test which rejects the null hypothesis for large values of the statistics:

$$\frac{\|z\|^2}{\sigma^2} - z^T (A_n^T \Gamma_n A_n + \sigma^2 I_k)^{-1} z.$$  \hspace{1cm} (18)

Unfortunately, the evaluation of the above statistics is computationally demanding as $k$ gets larger, since it requires the inversion of the $k \times k$ matrix $A_n^T \Gamma_n A_n + \sigma^2 I_k$. In order to avoid this, we propose to replace matrix $\Gamma_n$ in (18) with its circulant approximation given by (13). In other words, product $A_n^T \Gamma_n A_n$ is replaced by:

$$A_n^T F_n \text{diag}(f(0) \ldots f(2\pi(n-1)/n)) F_n^T A_n = \text{diag}(f(2\pi j^n_1/n) \ldots f(2\pi j^n_k/n)).$$

This leads directly to the following procedure.

**PFS low complexity (PFSLC) Test:** Reject hypothesis $H_0$ when the statistics $T_n$ defined by:

$$T_n = \sum_{\ell=1}^{k} |z_{\ell}|^2 \left( \frac{1}{\sigma^2} - \frac{1}{\sigma^2 + f(2\pi j^n_\ell/n)} \right),$$  \hspace{1cm} (19)

is larger than a threshold.

Although this statistics cannot give rise to a better test than the LRT, its numerical simplicity makes it worth to be considered. In the next paragraph, we study the performance of the test and we prove that it performs as well as the LRT in terms of error exponent.

B. Asymptotic optimality of the PFSLC test

As the statistics (19) is no longer a likelihood ratio, Lemma 1 cannot be used to evaluate the error exponent associated with the test (19). Instead, we must resort to arguments of large
deviations theory. Specifically, we shall study the large deviation behaviour of the test associated to this statistic, that is the limit of $-n^{-1} \log \mathbb{P}_1(\mathcal{T}_n \leq \eta_n(\alpha))$ where $\eta_n(\alpha)$ is the $(1-\alpha)$-quantile of the statistic $\mathcal{T}_n$ under $H_0$, $\mathbb{P}_0(\mathcal{T}_n > \eta_n(\alpha)) = \alpha$. Under mild assumptions, we show below that this limit is given by the error exponent of the PFS. Hence, as far as error exponents are considered, there is no loss in the performance in using the statistic $\mathcal{T}_n$ defined in (19) rather than the likelihood ratio.

**Theorem 3** Assume that A1 and A2 hold true. For any level $\alpha \in (0, 1)$, the statistics $\mathcal{T}_n$ defined in (19) satisfies the following property. For $\eta_n(\alpha)$ such that $\mathbb{P}_0(\mathcal{T}_n > \eta_n(\alpha)) = \alpha$, the miss probability $\mathbb{P}_1(\mathcal{T}_n \leq \eta_n(\alpha))$ satisfies

$$\lim_{n \to \infty} -\frac{1}{n} \log \mathbb{P}_1(\mathcal{T}_n \leq \eta_n(\alpha)) = K_{\text{orth}}.$$ 

The proof of this result is provided in Appendix A-C.

VI. SIMULATIONS

The error exponent theory is inherently asymptotic. In this section we provide numerical experiments to analyze the performance of the PFS on simulated data for finite $n$ since we have already proved that the error exponent curve is the same. The point here is to test how well the error exponent theory is relevant for finite $n$.

We use the same spectral density functions $f_1$ and $f_2$ as in section IV-C whose error exponents are displayed in Fig. 3. We now compare, for a couple of values for $c$, the finite sample performances of the LRT with the iid, PFS and PCS Strategies by using their empirical Receiver Operating Characteristic (ROC) curves. When the PFS is used, we also consider the PFSLC test of section V. We have shown that the LRT with the PFS or the PCS and the PFSLC test share the same error exponent curve. How well this measure of the performance impacts the whole ROC curves at finite samples is displayed in Fig. 5. It turns out that the PCS, the PFS and the PFSLC have similar ROC curves, as indicated by the error exponent analysis. One can also notice the good performance of the PFS, PFSLC and PCS when $c = .1$, $\sigma = 1$ and $n = 100$ for $f_1$, which confirms the conclusions drawn from the error exponent curves in Fig. 3. For $c = .9$, $\sigma = 2$ and $n = 100$, one can notice that error exponent curves also provide a good prediction: the iid strategy slightly outperforms the PFS, PCS and PFSLC.
VII. CONCLUSION

In this paper, we studied the performance of the Neyman-Pearson detection of a stationary Gaussian process in noise, using a large wireless sensor network (WSN). Our results are relevant for the design of sensor networks which are constrained by limited signaling and communication overhead between the fusion center and the sensor nodes. We studied the case where each sensor compresses its observation sequence using either a random iid linear precoder or an orthogonal precoder. In the random precoder case, we determined the error exponent governing the asymptotic behaviour of the miss probability, when $k, n \to \infty$ and $k/n \to c \in (0,1)$. In the orthogonal precoder case, we exhibit strategies (PCS and PFS) that achieve the best error exponent among all orthogonal strategies. The PFS has moreover the attractive property of being well suited for WSN with signaling overhead constraints. In addition, we proved that when the PFS is used, a low complexity test can be implemented at the FC as an alternative to the Likelihood Ratio (Neyman-Pearson) test. Interestingly, the proposed test performs as well as the
LRT in terms of error exponents.

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APPENDIX A

PROOFS OF MAIN RESULTS

Observe that we may set $\sigma^2 = 1$ without loss of generality, since it amounts to divide $f$ by $\sigma^2$ and the data $y_i$ by $\sigma$. Hence in the following proof sections, we assume $\sigma = 1$. In particular the LLR in (5) for a precoding matrix $A_n$ with normalized precoders $a_i, i = 1, \ldots, k$ is given by

$$2\mathcal{L}_{A_n} = \|z\|^2 - \log \det(A_n^T \Gamma_n A_n + I_k) - z^T (A_n^T \Gamma_n A_n + I_k) z.$$  \hspace{1cm} (20)

A. Proof of Theorem 1

We assume without restriction that $\mathbb{E}((A_{11}^2)) = 1$. Due to Lemma 1, it is sufficient to prove that the normalized LLR associated to strategy $A$ converges in probability to the rhs of equation (11) under $H_0$. Expression (20) of the LLR relies on the assumption that each precoder $a_i$ has unit norm, which is generally not the case for $A_n$ defined as in Theorem 1. Since the false alarm and miss probabilities of this LRT do not depend on the norms $\|a_i\|, i = 1, \ldots, k$, it is equivalent to consider precoders defined by the matrix

$$\tilde{A}_n = A_n P_n^{-1/2},$$  \hspace{1cm} (21)

where $P_n = \text{diag} (\sum_{i=1}^n (A_{ij}^n)^2 : j = 1, \ldots, k)$. With this definition, we may use expression (20) which is valid for normalized precoders. In order to prove Theorem 1 it is now sufficient to show that $-(1/n)\mathcal{L}_{\tilde{A}_n}$ converges to the constant $K_{\text{rnd}}$ defined in (11).

The main issue lies in the asymptotic study of the two terms $\frac{1}{n} \log \det(\tilde{A}_n^T \Gamma_n \tilde{A}_n + I_k)$ and $\frac{1}{n} z^T (\tilde{A}_n^T \Gamma_n \tilde{A}_n + I_k)^{-1} z$. This can be done by successively using the results of [9], [10] and [11]. The crucial point is to characterize the limiting spectral measure of matrix $\tilde{A}_n^T \Gamma_n \tilde{A}_n$. Define:

$$R_n = \tilde{A}_n^T \Gamma_n \tilde{A}_n,$$

$$S_n = \frac{1}{n} A_n^T \Gamma_n A_n.$$
First, we prove (see Lemma 2 below) that the spectral measure of $R_n$ is asymptotically close to the one of $S_n$ in a sense which is made clear below. Second, we apply the results of [12], [10] along with [7] to determine the limiting spectral measure of $S_n$. Finally, closed form expressions of the desired quantities follow from the results of [11].

Denote by $d$ the Lévy distance on the set of distribution functions. We recall that $F_Q$ denotes the distribution function of the spectral measure of $Q$ (see Section IV-A).

**Lemma 2** As $n, k \to \infty$, $d(F_{R_n}, F_{S_n})$ converges to zero in probability.

**Proof:** The proof relies on Bai’s formula (see [13] and [9]) which provides the following bound on the Lévy distance:

$$d^4(F_{A^T A}, F_{B^T B}) \leq \frac{2}{nk} |A - B|^2 (|A|^2 + |B|^2)$$  \hspace{1cm} (22)

for any $n \times k$ real matrices $A, B$, where $|A| = \sqrt{\text{Tr}[A^T A]}$ denotes the Frobenius norm of $A$. We now use equation (22) with $A \leftarrow \Gamma_n^{1/2} \tilde{A}_n$ and $B \leftarrow \frac{1}{\sqrt{n}} \Gamma_n^{1/2} A_n$. Using (21) and introducing $\Delta_n = (P_n^{-1/2} - \frac{1}{\sqrt{n}} I_n)^2$, it is straightforward to show that:

$$d^4(F_{R_n}, F_{S_n}) \leq \frac{2n}{k} \text{Tr}[\Delta_n S_n] \left( \text{Tr}[P_n^{-1} S_n] + \frac{1}{n} \text{Tr} S_n \right).$$  \hspace{1cm} (23)

Note that $\frac{1}{n} \text{Tr} S_n \leq \frac{\rho(\Gamma_n)}{n^2} \text{Tr}[A_n^T A_n]$, where $\rho(\Gamma_n)$ denotes the spectral radius of $\Gamma_n$. Similarly, $\text{Tr} P_n^{-1} S_n = \text{Tr} R_n \leq \rho(\Gamma_n) \frac{1}{n} \text{Tr} \tilde{A}_n^T \tilde{A}_n = \rho(\Gamma_n) \frac{k}{n}$. Finally, $\text{Tr} \Delta_n S_n \leq \rho(\Gamma_n) \rho(\frac{1}{n} A_n^T A_n) \text{Tr} \Delta_n$. Putting all pieces together,

$$d^4(F_{R_n}, F_{S_n}) \leq \kappa_n \text{Tr} \Delta_n$$  \hspace{1cm} (24)

where

$$\kappa_n = \frac{2n}{k} \rho(\Gamma_n)^2 \left( \frac{k}{n} + \frac{1}{n^2} \text{Tr} A_n^T A_n \right) \rho \left( \frac{1}{n} A_n^T A_n \right).$$

From [7], $\rho(\Gamma_n)$ converges to $M_f = \sup(f)$. By the law of large numbers, $\frac{1}{n^2} \text{Tr} A_n^T A_n$ converges almost surely (a.s.) to $c$. From [14], $\rho(\frac{1}{n} A_n^T A_n)$ converges a.s. to $(1 + \sqrt{c})^2$. Therefore, $\kappa_n$ converges a.s. to $4M_f^2 c (1 + (1 + \sqrt{c})^2)$. In order to prove that $d(F_{R_n}, F_{S_n})$ converges in probability to zero, it is sufficient, by equation (24), to prove that $\text{Tr} \Delta_n$ converges in probability to zero.

We write $\text{Tr} \Delta_n$ as follows:

$$\text{Tr} \Delta_n = \frac{1}{n} \sum_{j=1}^k \xi_{n,j}$$
where $\xi_{n,j} = \left(\frac{1}{n} \sum_{i=1}^{n} (A_{ij}^n)^2\right)^{-1/2} - 1)^2$. Note that for a fixed $n$, $\xi_{n,j}$ are iid for all $j$. Let $\epsilon > 0$. By Markov inequality,

$$
\mathbb{P}(\text{Tr } \Delta_n > \epsilon) \leq \frac{\mathbb{E} \left( \sum_{j=1}^{k} \xi_{n,j} \right)}{n \epsilon} = \frac{k \mathbb{E} (\xi_{n,1})}{n \epsilon}.
$$

(25)

As $\xi_{n,1}$ converges a.s. to zero, we conclude that $\mathbb{P}(\text{Tr } \Delta_n > \epsilon)$ tends to zero. This completes the proof of Lemma 2.

Thanks to Lemma 2, it is sufficient to study the asymptotic behaviour of $F_{S_n}$. The latter is provided by [12], [10]. In order to introduce this result, we need to recall some definitions. For any distribution function $F$, the Stieltjes transform $b_F$ of $F$ is given by:

$$
b_F(z) = \int \frac{dF(t)}{t - z}
$$

for each $z \in \mathbb{C}^+$, where $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ with $\Im(z)$ denoting the imaginary part of $z$. Recall that, from the results of [7], the spectral distribution function $F_{\Gamma_n}$ of $\Gamma_n$ converges weakly to $\Phi$ given by (10). By straightforward application of the results of [12], [10], we obtain that, with probability one, $F_{S_n}$ converges weakly to a deterministic measure $F$ whose Stieltjes transform $b = b(z)$ is the unique solution in $\mathbb{C}^+$ of:

$$
z = -\frac{1}{b} + \int \frac{t}{1 + ctb} d\Phi(t),
$$

(26)

for each $z \in \mathbb{C}^+$. The above result along with Lemma 2 implies that

$$
\forall \epsilon > 0, \quad \mathbb{P}(d(F_{R_n}, F) > \epsilon) \to 0.
$$

(27)

We are now in a position to study the limit of the LLR. We obtain immediately from (20):

$$
-\frac{1}{n} \mathcal{L}_{\tilde{A}_n} = \frac{k}{2n} \left\{ -\frac{\|z\|^2}{k} + \beta_n + \gamma_n + \delta_n \right\}
$$

(28)

where

$$
\beta_n = \frac{1}{k} \text{Tr} (I_k + R_n)^{-1} = \int \frac{1}{1 + t} dF_{R_n}(t)
$$

$$
\gamma_n = \frac{1}{k} \log \det (I_k + R_n) = \int \log (1 + t) dF_{R_n}(t)
$$

$$
\delta_n = \frac{1}{k} z^T (I_k + R_n)^{-1} z - \beta_n.
$$

Using (27), $\beta_n$ and $\gamma_n$ respectively converge in probability to the constants $\beta$ and $\gamma$ defined by:

$$
\beta = \int \frac{1}{1 + t} dF(t) \quad \text{and} \quad \gamma = \int \log (1 + t) dF(t).
$$
Recalling that \( z \sim \mathcal{N}(0, I_k) \) under \( H_0 \), the term \( \frac{1}{k} \|z\|^2 \) in the rhs of (28) converges a.s. to one. Since \( z \) is independent of \( \tilde{A}_n \) and since the spectral radius of \((R_n + I_k)^{-1}\) is bounded, it is straightforward to show that \( \delta_n \xrightarrow{a.s.} 0 \) (use for instance Lemma 2.7 in [15]). Finally, \(-\frac{1}{n} \mathcal{L}_{\tilde{A}_n}\) converges in probability to:

\[
K_n(A) = \frac{c}{2}(-1 + \beta + \gamma) .
\]

(29)

Constant \( \beta \) coincides with the Stieltjes transform of \( F \) at point \(-1\), that is \( \beta = b(-1) \) where we defined for each \( x < 0 \), \( b(x) = \lim_{z \to C^+ x} b(z) \). Constant \( \beta \) is thus the unique solution to (12).

A closed form expression for \( \gamma \) can as well be obtained using (for instance) [11]. Using the fact that the limiting spectral measure associated with \( F \) has a bounded support, the dominated convergence Theorem applies to the function \( x \mapsto \int \log(x + t) dF(t) \). One easily obtains after some algebra:

\[
\gamma = \int_1^\infty \left( \frac{1}{t} - b(-t) \right) dt .
\]

Following [11], we conclude that \( \gamma = C(1) \) where \( C \) is the function defined for each \( x > 0 \) by:

\[
C(x) = -1 + xb(-x) - \log(xb(-x)) + \frac{1}{c} \int \log(1 + ct b(-x)) d\Phi(t) .
\]

This statement can simply be proved by noting that \( C'(t) = \frac{1}{t} - b(-t) \) (where \( C' \) is the derivative of \( C \)) and \( C(\infty) = 0 \). Plugging the above expression of \( \gamma \) into (29), we obtain the claimed error exponent \( K_{\text{rnd}} \).

\[B.\ Proof\ of\ Theorem\ \[2\]

We start with some useful definitions and technical preliminaries. Let \( c \in (0, 1) \). Denote \( m_f = \inf(f) \) \( M_f = \sup(f) \) so that, by definition of \( \Phi \) in (10), \( \Phi(t) = 0 \) for all \( t < m_f \) and \( \Phi(t) = 1 \) for all \( t \geq M_f \). We define the set

\[
\Lambda_c = \{ \lambda \in (m_f, M_f) : \Phi(\lambda) \geq (1 - c) \} .
\]

By assumptions \( A1 \) and \( A2 \), \( \Phi \) is continuously strictly increasing from \([m_f, M_f]\) to \([0, 1]\), we denote by \( \Phi^{-1} \) its inverse continuous function defined from \([0, 1]\) to \([m_f, M_f]\). Hence \( \Lambda_c = [\Phi^{-1}(1 - c), M_f] \). Moreover, using again \( A2 \), we obtain that \( 1_{\Lambda_c} \) is almost surely continuous with respect to \( \text{Leb} \circ f^{-1} \). By the uniform mapping theorem, this implies that, for any sequence
of probability measures \((\mu_n)\) weakly converging to \((2\pi)^{-1}\text{Leb} \circ f^{-1}\), we have, for all continuous function \(g: \mathbb{R} \to \mathbb{R}\), as \(n \to \infty\),

\[
\int_{\Lambda_c} g(\lambda) d\mu_n(\lambda) \to \frac{1}{2\pi} \int_{\Delta_c} g(\lambda) d\{\text{Leb} \circ f^{-1}\}(\lambda) = \frac{1}{2\pi} \int_{\Delta_c} g \circ f(\omega) d\omega
\]

(30)

where the last equality follows from the definition of \(\Delta_c\) in (15) by setting \(\lambda = f(\omega)\).

1) The PCS case: The outline of the proof is the following.

Step 1. Assume that

\[
k = \max\{i \in \{1, \ldots, n\} : \Phi(\lambda_i^n) \geq 1 - c\},
\]

(31)

where \((\lambda_i^n)_{1 \leq i \leq n}\) is given in Definition 1. Then \(k\) satisfies (7) and strategy \(V\) has error exponent \(K_{\text{orth}}(c)\).

Step 2. Strategy \(V\) with any sequence \(k\) satisfying (7) also has the error exponent \(K_{\text{orth}}(c)\).

Step 3. Under Condition (7) strategy \(V\) is optimal among all orthogonal strategies, that is, (17) holds for any \(A\).

Step[7] Let \(\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i^n}\) denote the empirical spectral measure of \(\Gamma_n\) defined in (4). Szegö’s Theorem states that \(\mu_n\) converges weakly to \(\frac{1}{2\pi} \text{Leb} \circ f^{-1}\) ([7], p.64). Applying (30) and then Lemma 5 then gives

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\Lambda_c}(\lambda_i^n) = \frac{1}{2\pi} \int_{\Delta_c} d\omega = c.
\]

That is, \(k\) defined by (31) satisfies (7). Recall that here \(V_n\) is given in Definition 1 with \(k\) given by in (31). The empirical spectral measure of \(V_n^T \Gamma_n V_n + I_k\) is thus given by \(\frac{1}{n} \sum_{i=1}^{n} \delta_{1+\lambda_i^n} \mathbb{1}_{\Lambda_c}(\lambda_i^n)\).

Hence we have as above that

\[
\lim_{n \to \infty} \frac{1}{n} \log \det(V_n^T \Gamma_n V_n + I_k) = \frac{1}{2\pi} \int_{\Delta_c} \log(1 + f(\omega)) d\omega,
\]

(32)

\[
\lim_{n \to \infty} \text{Tr} [(V_n^T \Gamma_n V_n + I_k)^{-1}] = \frac{1}{2\pi} \int_{\Delta_c} \frac{1}{1 + f(\omega)} d\omega.
\]

(33)

The spectral radius \(\rho[(V_n^T \Gamma_n V_n + I_k)^{-1}]\) is bounded by \(1/(1 + \Phi^{-1}(1 - c))\). Using eqs (20), (32)–(33), Lemma 3 and (16), we obtain that, under \(H_0, -\frac{1}{n} \mathcal{L}_V \xrightarrow{p} K_{\text{orth}}(c)\). As a consequence of Lemma 1, we obtain the assertion of Step 1.

Step[2] Observe that the error exponent associated to a strategy \(V\) is increasing with \(k\). Now let \(k\) be a sequence satisfying (7). For any \(c'\) and \(c''\) such that \(c' < c < c''\), define \(k'\) and \(k''\) by (31) with \(c\) replaced by \(c'\) and \(c''\) respectively. Then, as seen in Step[1] \(k'\) and \(k''\) also satisfy (7) with \(c\) replaced by \(c'\) and \(c''\) respectively. Thus, eventually, \(k' \leq k \leq k''\), and, applying Step[1] the
error exponent of $\mathcal{V}$ belongs to $[K_{\text{orth}}(c'), K_{\text{orth}}(c'')]$. This, with the continuity of $K_{\text{orth}}$, yields the assertion of Step 2.

**Step 3** Assume $A = (A_n)$ now denotes any orthogonal, that is $A_n$ is a $n \times k$ orthogonal matrix for all $n$, where $k$ satisfies (7). Let us prove that the bound (17) holds. By Lemma 1, for all real $t$, we have

$$\lim_{n \to \infty} \mathbb{P}_0 \left[ -\frac{1}{n} \mathcal{L}_{A_n} > t \right] = 0 \Rightarrow K_\alpha(A) \leq t.$$  

(34)

Let $t > \limsup_{n \to \infty} \mathbb{E}_0 [-\mathcal{L}_{A_n}/n]$. Using Markov inequality, we have for $n$ large enough,

$$\mathbb{P}_0 \left[ -\frac{1}{n} \mathcal{L}_{A_n} > t \right] \leq \frac{\text{Var}_0 \left[ \frac{1}{n} \mathcal{L}_{A_n} \right]}{(t + \mathbb{E}_0 \left[ \frac{1}{n} \mathcal{L}_{A_n} \right])^2}.$$  

Using (20), we have

$$\text{Var}_0 \left[ \frac{1}{n} \mathcal{L}_{A_n} \right] = \frac{1}{4} \text{Var}_0 \left[ z^T \{I_k - (I_k + A_n^T \Gamma_n A_n)^{-1}\} z \right] \xrightarrow{n \to \infty} 0,$$

where the convergence follows from Lemma 3 by noticing that $I_k - (I_k + A_n^T \Gamma_n A_n)$ has eigenvalues in $[0, 1]$ and, under $H_0$ (recall that we set $\sigma^2 = 1$), $z \sim \mathcal{N}(0, I_k)$. The last two displays show that $\mathbb{P}_0 \left[ -\frac{1}{n} \mathcal{L}_{A_n} > t \right] \to 0$ as $n \to \infty$ for all $t > \limsup_{n \to \infty} \mathbb{E}_0 [-\mathcal{L}_{A_n}/n].$

With (34), we get that

$$K_\alpha(A) \leq \limsup_{n \to \infty} \mathbb{E}_0 [-\mathcal{L}_{A_n}/n].$$

To conclude the proof, it thus only remains to show that $\limsup_{n \to \infty} \mathbb{E}_0 [-\mathcal{L}_{A_n}/n] \leq K_{\text{orth}}(c)$. We have, by (20),

$$\mathbb{E}_0 [-\mathcal{L}_{A_n}] = -k + \log \det(A_n^T \Gamma_n A_n + I_k) + \text{Tr} \left( (A_n^T \Gamma_n A_n + I_k)^{-1} \right).$$

Since $x \mapsto \log x + 1/x$ is nondecreasing on $[1, +\infty]$, Lemma 4 thus implies that $\mathbb{E}_0 [-\mathcal{L}_{A_n}] \leq \mathbb{E}_0 [-\mathcal{L}_{V_n}]$. We proved in Step 1 that $\mathbb{E}_0 [-\mathcal{L}_{V_n}/n] \to K_{\text{orth}}(c)$. Hence the proof is achieved.

2) The PFS case: We now prove that the PFS strategy also achieves the error exponent $K_{\text{orth}}(c)$ under the condition (7). Using the same argument as in Step 2 of the PCS case, we can in fact take $k$ as defined by

$$k = \max\{i \in \{1, \ldots, n\} : \Phi \circ f(2\pi j_i/n) \geq 1 - c\},$$  

(35)

where $(j_i^n)_{0 \leq i < n}$ is given in Definition 2 which we assume in the following.

It is known ([16], Lemma 4.6) that $\Gamma_n$ defined in (4) is asymptotically equivalent to $F_n^T D_n F_n$ where $D_n$ denotes the $n \times n$ diagonal matrix with entries $f(2\pi k/n)$, $k = 0, \ldots, n - 1$. As in
we denote asymptotic equivalence between matrices $A_n$ and $B_n$ by $A_n \sim B_n$. Asymptotic equivalence is preserved by elementary matrix operations ([17], Proposition 2.1). Hence, $F_n$ being unitary,

$$D_n \sim F_n \Gamma_n F_n^T.$$ 

Also, from the definition of $W_n$,

$$W_n = F_n^T S_n,$$

where $S_n$ is a $n \times k$ selection matrix the columns of which belong to the canonical basis. Hence, $S_n$ being unitary,

$$S_n^T D_n S_n \sim W_n^T \Gamma_n W_n .$$

Eq (36) implies

$$\lim_{n \to \infty} \frac{1}{n} \log \det (W_n^T \Gamma_n W_n + I_k) = \lim_{n \to \infty} \frac{1}{n} \sum_{\text{PF}^n(f;\varepsilon)} \log(1 + f(2\pi k/n))$$

$$= \frac{1}{\pi} \int_{\Delta} \log(1 + f(\omega))d\omega ,$$

And

$$\lim_{n \to \infty} \text{Tr} \left[ (W_n^T \Gamma_n W_n + I_k)^{-1} \right] = \lim_{n \to \infty} \frac{1}{n} \sum_{\text{PF}^n(f;\varepsilon)} \frac{1}{1 + f(2\pi k/n)}$$

$$= \frac{1}{2\pi} \int_{\Delta} \frac{1}{1 + f(\omega)}d\omega .$$

We then conclude as in Step [1] of the PCS case.

C. Proof of Theorem 3

Again we can take $k$ defined by (31) without loss of generality.

Let $D_n$ denote the $n \times n$ diagonal matrix with entries $D_n(\ell, \ell) = \tilde{f}(2\pi \ell/n)$, $\ell = 0, \ldots, n - 1$,

where we defined the function

$$\tilde{f}(\omega) = \left( \frac{1}{1 + \sigma^2} - \frac{1}{1 + \sigma^2 + f(\omega)} \right) 1(\Phi \circ f(\omega) \geq 1 - c) .$$

Then by Definition [2] and [19], we have

$$\mathcal{T}_n \overset{d}{=} u_n^T M_n u_n ,$$
where $M_n = F_n^T D_n F_n$ and $u_n$ is a $n$-sample of a centered stationary Gaussian process with spectral density $g_1(\omega) = 1 + \sigma^2 + f(\omega)$ under $H_1$ and $g_0(\omega) = 1 + \sigma^2$ under $H_0$. We shall apply [18, Proposition 2] which provides a large deviation principle (LDP) for quadratic forms of stationary Gaussian processes. Recall that we denote by $T_n(g)$ the $n \times n$ covariance matrix associated to the spectral density $g$ (see (4)). Let $S_n = \text{Sp}(T_n(g)^{1/2} M_n T_n(g)^{1/2})$ denote the set of eigenvalues of $T_n(g)^{1/2} M_n T_n(g)^{1/2}$. Since $M_n$ is non-negative, to apply this result, we successively show that for $g = g_0$ or $g = g_1$,

(i) $\bar{a}_n = \max(S_n)$ is bounded above by $M_f M_g$,

(ii) the following weak convergence holds $n^{-1} \sum_{\lambda \in S_n} \delta_\lambda \Rightarrow \frac{1}{2\pi} \text{Leb} \circ [\tilde{f} g]^{-1}$,

(iii) $\bar{a}_n \rightarrow M_{fg}$ as $n \rightarrow \infty$.

Observe that the eigenvalues of $D_n$ are given by $\tilde{f}(2\pi \ell/n)$, with $\ell = 0, \ldots, n - 1$, and those of $T_n(g)$ are bounded by $M_g$. Hence we have (i). Assertion (ii) is a consequence of Lemma 5 in [16] and Theorem 2.1 in [17]. By (i) and (ii), we have

$$\limsup \bar{a}_n \leq M_f M_g \quad \text{and} \quad M_{fg} \leq \liminf \bar{a}_n.$$ 

Thus Assertion (iii) follows by observing that $f$, $g_0$ and $g_1$ achieve their maxima at the same points, thus $M_{fg} = M_f M_g$ for $g = g_0$ or $g_1$. Since Assertions (i)–(iii) hold, Propostion 3 and Corollary 2 in [18] give that for $i = 0, 1$, under $H_i$, $n^{-1} T_n$ satisfies a LDP with good rate function

$$I_i(x) = \sup_{y \in \mathbb{R}} \left( y x + \frac{1}{4\pi} \int_{-\pi}^{\pi} \log(1 - 2y \tilde{f} g_i(\omega)) \, d\omega \right),$$

(37)

with $g = g_0$ or $g = g_1$ under $H_0$ or $H_1$, respectively. As in [18], we assume for convenience that $\log(x) = -\infty$ when $x \leq 0$.

Assertion (ii) above also implies that $n^{-1} T_n \xrightarrow{P} \frac{1}{2\pi} \int_{-\pi}^{\pi} [\tilde{f} g](\omega) \, d\omega$ with the same convention for $g$. Hence the sequence $(\eta_n(\alpha))$ in Theorem 3 satisfies $n^{-1} \eta_n(\alpha) \rightarrow x_0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} [\tilde{f} g_0](\omega) \, d\omega$. Thus the LDP under $H_1$ gives

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_1(T_n \leq \eta_n(\alpha)) \leq \inf_{c > x_0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_1(n^{-1} T_n \leq c) \leq \inf_{c > x_0} \sup_{x \leq c} -I_1(x),$$

and

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_1(T_n \leq \eta_n(\alpha)) \geq \sup_{c < x_0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_1(n^{-1} T_n < c) \geq \sup_{c < x_0} \sup_{x < c} -I_1(x),$$

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By Lemma 6, we conclude that
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}_1(T_n \leq \eta_n(\alpha)) = \inf_{x \leq x_0} I_1(x) .
\]
To conclude the proof, it only remains to show that \(I_1(x_0) = K_\alpha(\mathcal{W}^c)\) and \(I_1(x)\) is nonincreasing on \((\infty, x_0]\. \)

By definition of \(\overline{f}\) and \(I_1\), we have \(I_1(x) = \sup_y F_x(y)\), where
\[
F_x(y) = yx + \frac{1}{4\pi} \int_{\Delta_c} \log(1 - 2y[g_1/g_0 - 1](\omega)) \, d\omega.
\]

Using the definition of \(x_0\), we further have \(yx_0 = \int_{\Delta_c} y[1 - g_0/g_1](\omega) d\omega\) and hence \(F_{x_0}(y) = \frac{1}{4\pi} \int_{\Delta_c} f_\omega(2y) \, d\omega\) with
\[
f_\omega(y) = y[1 - g_0/g_1](\omega) + \log(1 - y[g_1/g_0 - 1](\omega)) .
\]

For any \(\omega\), it is straightforward to show that \(f_\omega(y)\) is maximized at \(y = -1\) at which it takes value \(f_\omega(-1) = 2D(\mathcal{N}(0, g_0(\omega))) || \mathcal{N}(0, g_1(\omega))\). Since the maximizing \(f_\omega(y)\) does not depend on \(\omega\) we obtain
\[
I_1(x_0) = \sup_{y \in \mathbb{R}} \frac{1}{4\pi} \int_{\Delta_c} f_\omega(2y) \, d\omega = \frac{1}{4\pi} \int_{\Delta_c} \sup_{y \in \mathbb{R}} f_\omega(2y) \, d\omega = K_\alpha(\mathcal{W}^c) .
\]

We now consider \(x \leq x_0\). By differentiating \(F_x\), the \(y\) maximizing \(F_x(y)\) satisfies
\[
x = \int_{\Delta_c} \frac{[g_1/g_0 - 1](\omega)}{1 - 2y[g_1/g_0 - 1](\omega)} d\omega .
\]

Note that \(g_1/g_0 - 1\) is non-negative and has a positive integral on \(\Delta_c\) hence the right-hand side of the previous display has a strictly positive derivative w.r.t. \(y\). It follows that \(y(x)\), defined as the \(y\) maximizing \(F_x(y)\), is strictly increasing with \(x\). On the other hand, we know from above that \(y(x_0) = -1/2\). Thus, for all \(x \leq x_0\), we have \(I_1(x) = \sup_{y \leq -1/2} F_x(y)\). Now observe that for all \(x' \leq x\) and all \(y \leq 0\) we have \(F_{x'}(y) - F_x(y) = (x' - x)y \geq 0\). It follows that \(I_1\) is nonincreasing on \((\infty, x_0]\), which achieves the proof.

**APPENDIX B**

**TECHNICAL LEMMAS**

**Lemma 3** Assume that for each \(n > 0\), \(x_n \sim \mathcal{N}(0, \Sigma_n)\) where \(\Sigma_n\) has bounded spectral radius \(\rho(\Sigma_n)\); and assume \(Q_n\) is a family of quadratic forms with bounded spectral radius \(\rho(Q_n)\). Then,
\[
\var\left[\frac{1}{n} x_n^T Q_n x_n\right] \xrightarrow{n \to \infty} 0 .
\]
If, moreover,
\[
\lim_{n \to \infty} \frac{1}{n} \text{Tr}(Q_n \Sigma_n) \to c,
\]
Then \(\frac{1}{n} (x_n^T Q_n x_n)\) converges in the \(L_2\) sense towards \(c\).

**Proof:** One has \(E[x_n^T Q_n x_n] = \text{Tr}[Q_n \Sigma_n]\). Let us estimate \(\text{Var}[x_n^T Q_n x_n]\).

\[
x_n^T Q_n x_n = y_n^T \Delta_n y_n
\]
with \(y_n\) a standard centered gaussian vector and \(\Delta_n\) diagonal and congruent to \(\Sigma_n^\frac{1}{2} Q_n \Sigma_n^\frac{1}{2}\).

\[
\text{Var}[x_n^T Q_n x_n] = 2 \text{Tr}[\Delta_n^2] \leq 2n \rho(\Delta_n^2) \leq 2n \rho(\Sigma_n)^2 \rho(Q_n)^2 \leq C \cdot n
\]
where \(C\) is a constant. Thus we have, as sought,

\[
\text{Var}\left[\frac{1}{n} x_n^T Q_n x_n\right] \to_{n \to \infty} 0.
\]

\[\square\]

**Lemma 4** ([19], p. 189) Let \(Q\) be a symmetric \(n \times n\) matrix, and \(V\) be a \(r\)-dimensional subspace of \(\mathbb{R}^n\). Denote by \(Q_V\) the restriction of \(Q\) to \(V\), \(\lambda_i, i \in \{1, \ldots, n\}\), the eigenvalues of \(Q\) in increasing order and \(\mu_j, j \in \{1, \ldots, r\}\), the eigenvalues of \(Q_V\) in increasing order. Then, for all \(i = 1, \ldots, r\), we have \(\lambda_i \leq \mu_i \leq \lambda_{n+i-r}\).

**Lemma 5** Under A1-A2, we have for any \(c \in [0, 1]\), \(\text{Leb}(\Delta_c) = 2\pi c\), where \(\Delta_c\) is defined by (15).

**Proof:** We have \(\Delta_c = (\Phi \circ f)^{-1}((1-c, \infty)) \cap (-\pi, \pi)\), where \((\Phi \circ f)^{-1}\) denotes the inverse image under \(\Phi \circ f\). Observe that \((\Phi \circ f)^{-1} = f^{-1} \circ \Phi^{-1}\). Moreover as we have seen in the preamble of Appendix A-B, \(\Phi\) is continuously and strictly increasing from \([m_f, M_f]\) to \([0, 1]\) and constant on \([M_f, \infty)\), hence \(\Phi^{-1}((1-c, \infty)) = [\Phi^{-1}(1-c), \infty)\), where \(\Phi^{-1}\) here denotes the inverse function from \([0, 1]\) to \([0, M_f]\). Hence \(\Delta_c = f^{-1}([\Phi^{-1}(1-c), \infty)\). Now since \(\Phi\) is the distribution function of the probability measure \((2\pi)^{-1}\text{Leb} \circ f^{-1}\) and using again that it is continuously and strictly increasing from \([0, M_f]\) to \([0, 1]\), we get that \((2\pi)^{-1}\text{Leb}(\Delta_c) = 1 - (1-c) = c\), which concludes the proof.

\[\square\]

**Lemma 6** Let \(I(x)\) be defined for \(x \in \mathbb{R}\) by (37) with values in \(\mathbb{R} \cup \{\infty\}\) for some non-negative bounded function \(h = [\tilde{f} g]\). Then \(I(x)\) is finite and continuous for \(x > 0\).
Proof: Let \( J_x(y) = yx + \frac{1}{4\pi} \int_{-\pi}^{\pi} \log(1 - 2y[f(u)](\omega)) \, d\omega \) so that \( I(x) = \sup_y J_x(y) \). Let \( M_h \) denote the essential sup of \( h \). Then \( J_x(y) = -\infty \) for all \( y > 1/(2M_h) \). Let \( \epsilon > 0 \). Note that \( J_x(0) = 0 \) and for all \( x \geq \epsilon \) and \( y \leq 0 \), \( J_x(y) \leq yx + \log(1 - 2yM_h)/2 \to -\infty \) as \( y \to -\infty \). Thus there exists \( y_\epsilon \) only depending on \( \epsilon \) such that \( J_x(y) \leq 0 \) for all \( x \geq \epsilon \) and \( y \leq y_\epsilon \). From these facts, it follows that for all \( x \geq \epsilon \), \( I(x) = \sup_{y \in [y_\epsilon, 1/(2M_h)]} |J_x(y)| \). Finally we observe that for all \( x, x' \geq \epsilon \), \( \sup_{y \in [y_\epsilon, 1/(2M_h)]} |J_x(y) - J_{x'}(y)| \leq (y_\epsilon \vee 1/(2M_h)) |x - x'| \) which now yields the result. 

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