Optical soliton solutions to a (2+1) dimensional Schrödinger equation using a couple of integration architectures

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Abstract

In this work, we consider a (2+1) dimensional nonlinear Schrödinger system which appears in the theory of nonlinear optics and describe transmission of the optical pulses in optical fibers. We attain certain special type traveling wave solutions of the under investigated model by help of finite series expansion and auxiliary differential equations. In this manner, we exploit $\exp(-\phi(\epsilon))$ and modified Kudryashov approaches as solution procedures. Moreover, we make tanh ansatz because of the being even order of the reduced ordinary differential equation. The obtained solutions are in the form of dark soliton, combined soliton, symmetrical Lucas sine, Lucas cosine functions, and periodic wave solutions. We present also some graphical simulations of the solutions corresponding to values of parameters which leads to a better understanding the phenomena.

Keywords: (2+1) dimensional nonlinear Schrödinger equation, traveling wave solutions, soliton solutions.

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1 Introduction

As well-known, many physical phenomena occurred in the nonlinear science are governed by the nonlinear evolution equations (NLEEs). We observe these phenomena in many models of applied sciences such as nonlinear optics, nuclear physics, shallow water wave theory, plasma physics, biology, chemistry, etc. Therefore, solving these models via analytic mathematical methods is quite important for revealing the physical explanations of the considered model. In the last three decades, we have witnessed many powerful analytical methods such as the theory of Lie groups, Hirota’s bilinear method, Painleve property, homogeneous balance method, inverse scattering method, Backlund transformation, Darboux transformation, Lax pairs, He’s the exp-function method, Kudryashov’s simplest equation method, sine-cosine method, etc. (see, [1]-[13] and references therein).

The one-dimensional nonlinear Schrödinger equation (NLSE) is the basic equation of physics for describing quantum mechanical behavior. It is also frequently called the Schrödinger wave equation, and is a partial
differential equation (PDE),
\[ iq_t + 2|q|^2 q + q_{xx} = 0 \]  
for complex field \( q(x,t) \). The Eq.(1) models many nonlinearity effects in a fiber, including but not limited to self-phase modulation, four-wave mixing, second harmonic generation, stimulated Raman scattering, optical solitons, ultrashort pulses, etc. It is a classical field equation whose principal applications are to the propagation of light in nonlinear optical fibers and planar waveguides [15] and to Bose-Einstein condensates confined to highly anisotropic cigar-shaped traps, in the mean-field regime.

It has been noted [16] that NLEEs in higher order spatial dimensions correspond to certain holomorphic vector bundles over the twistor space, the connection being provided by the Penrose correspondence. The self-dual Yang Mills (SDYM) equation is the best known example admitting such an interpretation and interestingly demonstrated that the integrable many models have been found to be embedded within this equation [17]. By dimensionally reducing the SDYM equation, Strachan [18] introduced a (2+1) dimensional spatio-temporal non-local system [19], [20]
\[ kq_t = \frac{1}{2} q_{xy} - q \int \partial_x [p.q] d_x, \]
\[ -kp_t = \frac{1}{2} p_{xy} - p \int \partial_y [p.q] d_x. \]  
which is a generalization of the NLS equation and showed it to be integrable from the point of view of geometrical consideration. Under potential function
\[ v_x(p,q) = 2\partial_y [p.q], \]  
with the ansatz \( p = -q^* = u \) and \( k = -i/2 \), Eq.(2) reduces to
\[ iu_t = u_{xy} + uv, \]
\[ v_x = 2(|u|^2)_y. \]  
Eq.(4) converts to classical NLSE while \( \partial_x = \partial_y \).

In [21], the authors constructed the possible explicit parametric representations of the bounded travelling wave solutions and all kinds of phase portraits in the parametric space of Eq.(4) by using the approach of dynamical systems and the theory of bifurcations. In [22] the authors obtained different nature exact solutions of Eq.(4) such as triangular-type, soliton-type, doubly periodic-like, single and combined non-degenerate Jacobi elliptic function like solutions by using the Fan sub-equation method.

The main aim of this study is to reveal the exact traveling solutions of the Eq.(2) via three distinct algorithms. In this regard, in the second section, we briefly present the model equation and converts it into the ordinary differential equation (ODE) system. In section 3, we present the exact solution schemas of \( \exp(-\phi(\varepsilon)) \), modified Kudryashov, and tanh function method. Section 4 is devoted to the application of the methods to our model separately. In the last section, we give numerical simulations and graphics of the obtained solutions for better understanding the physical phenomena. Moreover, in this section we give some concluding remarks on the presented methods and future works.

2 The derivation of reduced ODE of Eq.(4)

In order to reveal the exact solutions of Eq.(4), the following transformation
\[ u(x,y,t) = U(\varepsilon) e^{i\psi}, v(x,y,t) = V(\varepsilon) \]
Optical soliton solutions to a (2+1) dimensional Schrödinger equation

is chosen in Eq.(4) where

\[ \psi = \gamma_1 x + \gamma_2 y + \gamma_3 t \quad \text{and} \quad \epsilon = \eta_1 x + \eta_2 y + \eta_3 t. \]  

In Eq.(6) \( \gamma_i \) and \( \eta_i \) \((i = 1, 2, 3)\) are arbitrary constants which to be determined later [22]. By substitution of Eq.(5) into Eq.(4) and then decomposing the real and imaginary parts of it, we get respectively

\[ \eta_1 \eta_2 U''(\epsilon) + U(\epsilon)V(\epsilon) + (\gamma_3 - \gamma_1 \eta_2)U(\epsilon) = 0, \]  

\[ (\eta_1 \eta_2 + \gamma_1 \eta_2 + \eta_3)U'(\epsilon) = 0, \]  

and

\[ \eta_1 V'(\epsilon) = 2\eta_2(U^2(\epsilon))'. \]  

It is readily seen that the Eq.(8) should holds the following constraint:

\[ (\eta_1 \eta_2 + \gamma_1 \eta_2 + \eta_3) = 0. \]  

After the integration of Eq.(9) with respect to \( \epsilon \), we yield

\[ V(\epsilon) = \frac{2\eta_2}{\eta_1} U^2(\epsilon). \]  

Plugging the Eq.(11) into Eq.(7), we get second order nonlinear ordinary differential equation (NLODE) as follows [22]:

\[ U''(\epsilon) + \frac{2}{\eta_1^2} U^3(\epsilon) + \left( \frac{\gamma_3 - \gamma_1 \eta_2}{\eta_1 \eta_2} \right) U(\epsilon) = 0. \]  

3 Methods

3.1 \( Exp(-\phi(\epsilon)) \) Method

The main properties of the method can be found in [23–25]. Let us consider (2+1) dimensional NLEEs (involving polynomial derivatives)

\[ F(U, U_x, U_y, U_t, U_{xy}, U_{xt}, U_{yt}, ...) = 0 \]  

where \( U \) is dependent variable and \( x, y \) and \( t \) are independent variables. Using the wave variable transformation \( u(x, y, t) = U(\epsilon), \) where \( \epsilon = \eta_1 x + \eta_2 y + \eta_3 t \), Eq.(13) transforms to the following NLODE of integer order

\[ G(U, U', U'', ...) = 0. \]  

We are looking for the following finite series expansion solution

\[ U(\epsilon) = \sum_{n=0}^{N} a_n \exp(-N\phi(\epsilon)) = a_0 + a_1 \exp(-\phi(\epsilon)) + \ldots + a_N \exp(-N\phi(\epsilon)), \quad a_N \neq 0 \]  

for Eq.(14). In Eq.(15), the coefficients \( a_n \) \( n = 1, 2, \ldots, N \) will be determined later. The positif integer \( N \) is computed by the homogenous balance between highest order nonlinear terms and linear terms. The \( \phi(\epsilon) \) analytic function holds the following first order NLODE
\[ \phi'(\varepsilon) = \exp(-\phi(\varepsilon)) + \mu \exp(\phi(\varepsilon)) + \lambda. \] (16)

We now present some solution sets of Eq.(16) with respect to coefficient classifications.

**Case 1** If \( \lambda^2 - 4\mu > 0 \) and \( \mu \neq 0 \), then
\[ \phi_1(\varepsilon) = \ln \left( -\sqrt{\lambda^2 - 4\mu} \tanh((\sqrt{\lambda^2 - 4\mu}/2)(\varepsilon + C)) - \lambda \right). \] (17)

**Case 2** If \( \lambda^2 - 4\mu > 0 \), \( \mu = 0 \) and \( \lambda \neq 0 \), then
\[ \phi_2(\varepsilon) = -\ln \left( \frac{\lambda}{\cosh(\lambda(\varepsilon + C)) + \sinh(\lambda(\varepsilon + C)) - 1} \right). \] (18)

**Case 3** If \( \lambda^2 - 4\mu < 0 \) and \( \mu \neq 0 \), then
\[ \phi_3(\varepsilon) = \ln \left( \sqrt{4\mu - \lambda^2} \tan((\sqrt{4\mu - \lambda^2}/2)(\varepsilon + C)) - \lambda \right). \] (19)

By inserting Eq.(15) into Eq.(14) with the assistance of Maple and equating the coefficients of same terms of \( \exp(-\phi(\varepsilon)) \), we obtain an algebraic system which involves \( a_n \)'s, \( \eta_1 \), \( \eta_2 \) and \( \eta_3 \). Having solved this system we capture these coefficients easily. If necessary arrangements are made, Eq.(15) gives the traveling wave solutions of Eq.(14).

### 3.2 Modified Kudryashov Method

The preliminary steps of the modified Kudryashov method [25–27] are as previous method. With the same traveling wave transformation, the original NLEE can be converted to a NLODE. Supposed solution form is
\[ U(\varepsilon) = a_0 + a_1 Q(\varepsilon) + \ldots + a_N Q^N(\varepsilon), \quad a_N \neq 0, \] (20)
where the constants \( a_n, n = 0, 1, 2, \ldots, N \) shall be determined later, \( N \) is a positive number which is assessed by the technique of homogeneous balance, and
\[ Q(\varepsilon) = 1/(1 + da^\varepsilon) \] (21)
is an explicit function that satisfies the following NLODE
\[ Q'(\varepsilon) = Q(\varepsilon)(Q(\varepsilon) - 1) \ln a. \] (22)

By substituting Eq.(22) into Eq.(14) with the help of Maple and equating the coefficients of like terms of \( Q(\varepsilon) \), we will derive an algebraic system for obtaining \( a_n \)'s, \( \eta_1 \), \( \eta_2 \) and \( \eta_3 \). If necessary arrangements are made, Eq.(20) furnishes the exact traveling wave solutions of Eq.(14).

As stressed out in the works of Stakhov and Ruzin in [28] and Sayed and Alurrfi in [29] that the Eq.(21) could be described in terms of Lucas symmetric hyperbolic sine and cosine functions. From those works, we recognize that the Lucas symmetric hyperbolic sine and cosine functions are defined respectively as follows:
\[ s_{\text{Ls}}(\varepsilon) = a^\varepsilon - a^{-\varepsilon}, \] (23)
\[ c_{\text{Ls}}(\varepsilon) = a^\varepsilon + a^{-\varepsilon}. \] (24)
Similarly, Lucas symmetric hyperbolic tangent and cotangent functions are read as
\[ \text{tLs}(\varepsilon) = \frac{a^\varepsilon - a^{-\varepsilon}}{a^\varepsilon + a^{-\varepsilon}}, \quad \text{ctLs}(\varepsilon) = \frac{a^\varepsilon + a^{-\varepsilon}}{a^\varepsilon - a^{-\varepsilon}}. \]

Moreover, it is demonstrated that there exist an identity \[ [28], [29]\) between Lucas symmetric hyperbolic sine and cosine functions as the follows:
\[ [\text{cLs}(\varepsilon)]^2 - [\text{sLs}(\varepsilon)]^2 = 4 \]

If one chooses \( a = e \) in Eq. (22) then classical Kudryashov method is obtained \[26\].

### 3.3 Tanh Method

In this subsection, we give a detailed description of the tanh method \[30–32\]. The preliminary steps of the tanh method are again as \exp(-\phi(\varepsilon)) \) method. In order to integrate Eq.(13) and to deduce \( U(x,y,t) \) explicitly, one can pursue the following steps:

**Step 1:** Use the traveling wave transformation:

\[ U = U(\varepsilon), \varepsilon = \eta_1 x + \eta_2 y + \eta_3 t \]

where, \( \eta_1, \eta_2 \) and \( \eta_3 \) are constants to be fixed latter. Then, the NLEE Eq.(13) is reduced to a NLODE for \( U = U(\varepsilon) \) as follows:

\[ G(U, U', U'',...) = 0. \]

**Step 2:** Suppose that the NLODE Eq.(26) has the following solution in the form of finite series expansion:

\[ U(\varepsilon) = \sum_{n=0}^{N} a_n \tan(\varepsilon)^n = a_0 + a_1 \tan(\varepsilon) + ... + a_N \tan(\varepsilon)^N, \quad a_N \neq 0, \]

where, \( a_n(n = 0,1,...,N) \) are constants to be fixed later and \( N \) is a positive integer to be determined in step 3.

**Step 3:** Determine the positive integer \( N \) by balancing the highest order derivatives of linear terms and nonlinear terms appearing in Eq.(26).

**Step 4:** Inserting Eq.(27) into Eq.(26) we get an algebraic equations involving \( a_n \) and \( \eta_i \) \( (i = 1,2,3) \). In this stage, we equate the expressions of different power of \( (\tan(\varepsilon))^n \) to zero. Solving those equations sequently by Maple the coefficients \( a_n \) and the parameters \( \eta_1, \eta_2 \) and \( \eta_3 \) are easily determined.

**Step 5:** Plugging \( a_n, \eta_1, \eta_2 \) and \( \eta_3 \) into Eq.(27), we can yield the traveling wave solutions of Eq.(13).

### 4 Applications

#### 4.1 Application of the \exp(-\phi(\varepsilon)) \) method to Eq.(4)

In this section, we are looking for the exact solutions of Eq.(4) using the \exp(-\phi(\varepsilon)) \) method. As noted above the Eq.(4) is transformed to the second order NLODE Eq.(12) by the wave variables Eq.(5). Firstly, balancing \( U'' \) with \( U^3 \) in Eq.(12), one gets the finite series order as \( N = 1 \). Therefore our solution form is as follows

\[ U(\varepsilon) = a_0 + a_1 \exp(-\phi(\varepsilon)). \]

By substituting Eq.(28) into Eq.(12) and equating the coefficients of same terms of \exp(-\phi(\varepsilon)),we yield an algebraic system

\[ \exp(-3\phi(\varepsilon)) \text{ Coeff.:} 2a_1 \eta_2 (a_1^2 + \eta_1^2) = 0, \]
exp(−2φ(ε)) Coeff.: 
\[3a_1\eta_2(\lambda \eta_1^2 + 2a_0a_1) = 0,\]

exp(−φ(ε)) Coeff.: 
\[2a_1((\frac{1}{2}\lambda^2 + \mu)\eta_1^2 - \frac{1}{2}\eta_1\gamma_1 + 3a_0^2)\eta_2 + \frac{1}{2}\eta_1\gamma_3) = 0,\]

Const: 
\[(\lambda \mu a_1\eta_1^2 + 2a_0^3 - a_0\eta_1\gamma_1)\eta_2 + a_0\eta_1\gamma_3 = 0.\]

which contain \(a_n\)'s, \(\eta_1\), \(\eta_2\) and \(\eta_3\). Solving these equations by help of Maple the unknown aforementioned coefficients are fixed.

Four different coefficient sets are derived after solving the above system which are given as follows:

Set 1. \(a_0 = \frac{1}{2}i\lambda \eta_1, a_1 = i\eta_1, \eta_1 = \eta_1, \eta_2 = \frac{2\gamma_3}{\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1}, \eta_3 = \eta_3.\)

Set 2. \(a_0 = \frac{1}{2}i\lambda \eta_1, a_1 = -i\eta_1, \eta_1 = \eta_1, \eta_2 = \frac{2\gamma_3}{\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1}, \eta_3 = \eta_3.\)

Set 3. \(a_0 = -\frac{1}{2}i\lambda \eta_1, a_1 = i\eta_1, \eta_1 = \eta_1, \eta_2 = \frac{2\gamma_3}{\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1}, \eta_3 = \eta_3.\)

Set 4. \(a_0 = -\frac{1}{2}i\lambda \eta_1, a_1 = -i\eta_1, \eta_1 = \eta_1, \eta_2 = \frac{2\gamma_3}{\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1}, \eta_3 = \eta_3.\)

Therefore, the exact solution forms Eq.(5) of Eq.(4) (together with Eq.(11) and Eq.(28)) corresponding to above sets can be given as follows:

\[u_1(x,y,t) = \left(\frac{1}{2}i\lambda \eta_1 + i\eta_1 e^{-\phi(\epsilon)}\right) e^{i\psi},\]

\[v_1(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left(\frac{1}{2}i\lambda \eta_1 + i\eta_1 e^{-\phi(\epsilon)}\right)^2.\]  

\[u_2(x,y,t) = \left(\frac{1}{2}i\lambda \eta_1 - i\eta_1 e^{-\phi(\epsilon)}\right) e^{i\psi},\]

\[v_2(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left(\frac{1}{2}i\lambda \eta_1 - i\eta_1 e^{-\phi(\epsilon)}\right)^2.\]

\[u_3(x,y,t) = \left(-\frac{1}{2}i\lambda \eta_1 + i\eta_1 e^{-\phi(\epsilon)}\right) e^{i\psi},\]

\[v_3(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left(-\frac{1}{2}i\lambda \eta_1 + i\eta_1 e^{-\phi(\epsilon)}\right)^2.\]

\[u_4(x,y,t) = \left(-\frac{1}{2}i\lambda \eta_1 - i\eta_1 e^{-\phi(\epsilon)}\right) e^{i\psi},\]

\[v_4(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left(-\frac{1}{2}i\lambda \eta_1 - i\eta_1 e^{-\phi(\epsilon)}\right)^2.\]
We now insert the $\phi(\varepsilon)$ which is classified in Eqs.(17)-(19) into the Eqs.(29)-(32).

**Case 1:** If $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, then the combined soliton solution as follows:

$$u_1(x,y,t) = \left( \frac{1}{2} i\lambda \eta_1 + \frac{2\mu i\eta_1 \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))}{-\sqrt{\lambda^2 - 4\mu} \sinh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C)) - \lambda \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))} \right) e^{iy},$$

$$v_1(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \times \left( \frac{1}{2} i\lambda \eta_1 + \frac{2\mu i\eta_1 \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))}{-\sqrt{\lambda^2 - 4\mu} \sinh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C)) - \lambda \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))} \right)^2,$$

$$u_2(x,y,t) = \left( \frac{1}{2} i\lambda \eta_1 - \frac{2\mu i\eta_1 \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))}{-\sqrt{\lambda^2 - 4\mu} \sinh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C)) - \lambda \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))} \right) e^{iy},$$

$$v_2(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \times \left( \frac{1}{2} i\lambda \eta_1 - \frac{2\mu i\eta_1 \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))}{-\sqrt{\lambda^2 - 4\mu} \sinh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C)) - \lambda \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))} \right)^2,$$

$$u_3(x,y,t) = \left( -\frac{1}{2} i\lambda \eta_1 + \frac{2\mu i\eta_1 \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))}{-\sqrt{\lambda^2 - 4\mu} \sinh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C)) - \lambda \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))} \right) e^{iy},$$

$$v_3(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \times \left( -\frac{1}{2} i\lambda \eta_1 + \frac{2\mu i\eta_1 \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))}{-\sqrt{\lambda^2 - 4\mu} \sinh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C)) - \lambda \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))} \right)^2,$$

$$u_4(x,y,t) = \left( -\frac{1}{2} i\lambda \eta_1 + \frac{2\mu i\eta_1 \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))}{\sqrt{\lambda^2 - 4\mu} \sinh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C)) + \lambda \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))} \right) e^{iy},$$

$$v_4(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \times \left( -\frac{1}{2} i\lambda \eta_1 + \frac{2\mu i\eta_1 \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))}{\sqrt{\lambda^2 - 4\mu} \sinh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C)) + \lambda \cosh((\sqrt{\lambda^2 - 4\mu/2})(\varepsilon + C))} \right)^2.$$

**Case 2:** If $\lambda^2 - 4\mu > 0$, $\mu = 0$ and $\lambda \neq 0$, then the combined soliton solution:
If \( v_u(x,y,t) = \left( \frac{1}{2} i \lambda \eta_1 + \frac{2mi\eta_1}{\sqrt{4\mu - \lambda^2 \tan((\sqrt{4\mu - \lambda^2/2})(\epsilon + C)) - \lambda}} \right) e^{i\psi} \),

\[ u_1(x,y,t) = \left( \frac{1}{2} i \lambda \eta_1 + \frac{2mi\eta_1}{\sqrt{4\mu - \lambda^2 \tan((\sqrt{4\mu - \lambda^2/2})(\epsilon + C)) - \lambda}} \right) e^{i\psi}, \tag{37} \]

\[ v_1(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left( \frac{1}{2} i \lambda \eta_1 + \frac{2mi\eta_1}{\sqrt{4\mu - \lambda^2 \tan((\sqrt{4\mu - \lambda^2/2})(\epsilon + C)) - \lambda}} \right)^2. \]

\[ u_2(x,y,t) = \left( \frac{1}{2} i \lambda \eta_1 - \frac{i\eta_1\lambda}{\cosh(\lambda(\epsilon + C)) + \sinh(\lambda(\epsilon + C)) - 1} \right) e^{i\psi}, \tag{38} \]

\[ v_2(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left( \frac{1}{2} i \lambda \eta_1 - \frac{i\eta_1\lambda}{\cosh(\lambda(\epsilon + C)) + \sinh(\lambda(\epsilon + C)) - 1} \right)^2. \]

\[ u_3(x,y,t) = \left( \frac{1}{2} i \lambda \eta_1 + \frac{i\eta_1\lambda}{\cosh(\lambda(\epsilon + C)) + \sinh(\lambda(\epsilon + C)) - 1} \right) e^{i\psi}, \tag{39} \]

\[ v_3(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left( \frac{1}{2} i \lambda \eta_1 + \frac{i\eta_1\lambda}{\cosh(\lambda(\epsilon + C)) + \sinh(\lambda(\epsilon + C)) - 1} \right)^2. \]

\[ u_4(x,y,t) = \left( \frac{1}{2} i \lambda \eta_1 - \frac{i\eta_1\lambda}{\cosh(\lambda(\epsilon + C)) + \sinh(\lambda(\epsilon + C)) - 1} \right) e^{i\psi}, \tag{40} \]

\[ v_4(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left( \frac{1}{2} i \lambda \eta_1 - \frac{i\eta_1\lambda}{\cosh(\lambda(\epsilon + C)) + \sinh(\lambda(\epsilon + C)) - 1} \right)^2. \]

**Case 3:** If \( \lambda^2 - 4\mu < 0 \) and \( \mu \neq 0 \), then the periodic wave solutions:

\[ u_1(x,y,t) = \left( \frac{1}{2} i \lambda \eta_1 + \frac{2mi\eta_1}{\sqrt{4\mu - \lambda^2 \tan((\sqrt{4\mu - \lambda^2/2})(\epsilon + C)) - \lambda}} \right) e^{i\psi}, \tag{41} \]

\[ v_1(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left( \frac{1}{2} i \lambda \eta_1 + \frac{2mi\eta_1}{\sqrt{4\mu - \lambda^2 \tan((\sqrt{4\mu - \lambda^2/2})(\epsilon + C)) - \lambda}} \right)^2. \]

\[ u_2(x,y,t) = \left( \frac{1}{2} i \lambda \eta_1 - \frac{2mi\eta_1}{\sqrt{4\mu - \lambda^2 \tan((\sqrt{4\mu - \lambda^2/2})(\epsilon + C)) - \lambda}} \right) e^{i\psi}, \tag{42} \]

\[ v_2(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left( \frac{1}{2} i \lambda \eta_1 - \frac{2mi\eta_1}{\sqrt{4\mu - \lambda^2 \tan((\sqrt{4\mu - \lambda^2/2})(\epsilon + C)) - \lambda}} \right)^2. \]

\[ u_3(x,y,t) = \left( \frac{1}{2} i \lambda \eta_1 + \frac{2mi\eta_1}{\sqrt{4\mu - \lambda^2 \tan((\sqrt{4\mu - \lambda^2/2})(\epsilon + C)) - \lambda}} \right) e^{i\psi}, \tag{43} \]

\[ v_3(x,y,t) = \frac{4\gamma_3}{\eta_1(\lambda^2 \eta_1 - 4\mu \eta_1 + 2\gamma_1)} \left( \frac{1}{2} i \lambda \eta_1 + \frac{2mi\eta_1}{\sqrt{4\mu - \lambda^2 \tan((\sqrt{4\mu - \lambda^2/2})(\epsilon + C)) - \lambda}} \right)^2. \]
4.2 Application of the modified Kudryashov method to Eq.(4)

We note that again the order of series expansion is \( N = 1 \). Therefore, the solution form of Eq.(20) can be given as

\[
U(\psi) = a_0 + a_1 Q(\psi).
\]  

(45)

By substituting Eq.(45) into Eq.(12) with the help of Maple and equating the coefficients of same terms of \( Q(\psi) \), we reach to an algebraic system involving \( a_i \)'s, \( \eta_1 \), \( \eta_2 \) and \( \eta_3 \) as

\[
\begin{align*}
Q^2(\psi) & \text{ Coeff.:} \\
-3a_1 \ln(a)^2 + \frac{6a_0 a_1^2}{\eta_1^2} = 0 \\
Q(\psi) & \text{ Coeff.:} \\
a_1 \ln(a)^2 + \frac{6a_0 a_1}{\eta_1^2} - \frac{a_1 \gamma_1}{\eta_1 \eta_2} + \frac{a_1 \gamma_3}{\eta_1 \eta_2} = 0 \\
\text{Const:} \\
\frac{2a_0^3}{\eta_1^2} - \frac{a_0 \gamma_1}{\eta_1} + \frac{a_0 \gamma_3}{\eta_1 \eta_2} = 0
\end{align*}
\]

Four different results are obtained after solving the above system.

Set 1. \( a_0 = -\frac{1}{2} \ln(a) \eta_1, a_1 = \ln(a) \eta_1, \eta_1 = \eta_1, \eta_2 = \frac{2\gamma_1 - \ln(a)^2 \eta_1}{2\gamma_3}, \eta_3 = \gamma_3, \)

Set 2. \( a_0 = \frac{1}{2} \ln(a) \eta_1, a_1 = -\ln(a) \eta_1, \eta_1 = \eta_1, \eta_2 = \frac{2\gamma_1 - \ln(a)^2 \eta_1}{2\gamma_3}, \eta_3 = \gamma_3. \)

Thus, the exact traveling wave solutions of Eq.(5) can be given as follows:

\[
\begin{align*}
u_4(x,y,t) &= \left( -\frac{1}{2} \ln(a) \eta_1 + \ln(a) \eta_1 Q(\psi) \right) e^{i\psi}, \\
u_4(x,y,t) &= \frac{2\gamma_1 - \ln(a)^2 \eta_1}{\eta_1 \gamma_3} \left( -\frac{1}{2} \ln(a) \eta_1 + \ln(a) \eta_1 Q(\psi) \right)^2, \\
u_2(x,y,t) &= \left( \frac{1}{2} \ln(a) \eta_1 - \ln(a) \eta_1 Q(\psi) \right) e^{i\psi}, \\
v_2(x,y,t) &= \frac{2\gamma_1 - \ln(a)^2 \eta_1}{\eta_1 \gamma_3} \left( \frac{1}{2} \ln(a) \eta_1 - \ln(a) \eta_1 Q(\psi) \right)^2.
\end{align*}
\]  

(46)  

(47)
We now replace the term $Q(\varepsilon) = \frac{1}{1+da^s}$ in Eqs.(46)-(47)

\begin{align*}
    u_1(x,y,t) &= \left( -\frac{1}{2} \ln(a) \eta_1 + \frac{\ln(a) \eta_1}{1 + da} \right) e^{i\psi'}, \\
    v_1(x,y,t) &= \frac{2\gamma_1 - \ln(a)^2 \eta_1}{\eta_1 \gamma_3} \left( -\frac{1}{2} \ln(a) \eta_1 + \frac{\ln(a) \eta_1}{1 + da} \right)^2, \\
    u_2(x,y,t) &= \left( \frac{1}{2} \ln(a) \eta_1 - \frac{\ln(a) \eta_1}{\eta_1 + \frac{1}{\gamma_1}} \right) e^{i\psi'}, \\
    v_2(x,y,t) &= \frac{2\gamma_1 - \ln(a)^2 \eta_1}{\eta_1 \gamma_3} \left( \frac{1}{2} \ln(a) \eta_1 - \frac{\ln(a) \eta_1}{\eta_1 + \frac{1}{\gamma_1}} \right)^2.
\end{align*}

where $\psi'$ and $\eta'$ are arbitrary constants. Eq. (48) can be described in terms of Lucas symmetric hyperbolic sine and cosine functions as

\begin{align*}
    u_1(x,y,t) &= \left( -\frac{1}{2} \ln(a) \eta_1 + \frac{2\ln(a) \eta_1}{2 + d(s\psi + c\psi)} \right) e^{i\psi'}, \\
    v_1(x,y,t) &= \frac{2\gamma_1 - \ln(a)^2 \eta_1}{\eta_1 \gamma_3} \left( -\frac{1}{2} \ln(a) \eta_1 + \frac{2\ln(a) \eta_1}{d(s\psi + c\psi)} \right)^2.
\end{align*}

where $\psi = \eta_1 x + \left( \frac{2\gamma_1 - \ln(a)^2 \eta_1}{2\gamma_3} \right) y + \gamma_1 t$.

### 4.3 Application of the $tanh$ method to Eq.(4)

In this subsection, we are looking for the exact solutions of Eq.(4) by using the $tanh$ method. Based on the order of finite series expansion $N = 1$, our solution form is as follows

\begin{equation}
    U(\varepsilon) = a_0 + a_1 \tanh(\varepsilon).
\end{equation}

By substituting Eq.(50) into Eq.(12) with the help of Maple and equating the coefficients of same terms of $\tanh(\varepsilon)$, we yield an algebraic system containing $a_n$’s, $\eta_1$, $\eta_2$ and $\eta_3$. The coefficients of some various powers of $\tanh(\varepsilon)$ are listed as follows:

$tanh^3(\varepsilon)$ Coeff.:

\begin{equation}
    2a_1^2 + \frac{2a_1^3}{\eta_1^2} = 0
\end{equation}

$tanh^2(\varepsilon)$ Coeff.:

\begin{equation}
    \frac{6a_0a_1^2}{\eta_1^2} = 0
\end{equation}

$tanh(\varepsilon)$ Coeff.:

\begin{equation}
    -2a_1^2 + \frac{6a_0a_1^2}{\eta_1^2} - \frac{a_1 \gamma_1}{\eta_1} + \frac{a_1 \gamma_3}{\eta_1 \eta_2} = 0
\end{equation}
Const:
\[ \frac{2a_0^3}{\eta_1^2} - \frac{a_0 \gamma_1}{\eta_1} + \frac{a_0 \gamma_3}{\eta_1 \eta_2} = 0 \]

Having solved above algebraic system by the help of Maple, we attain the following coefficient sets:

Set 1. \( a_0 = 0, a_1 = 1, \eta_1 = i, \eta_2 = -\frac{\gamma_3 + 2\gamma_1}{\gamma_1}, \eta_3 = \eta_3. \)

Set 2. \( a_0 = 0, a_1 = 1, \eta_1 = i, \eta_2 = \frac{\gamma_3 - 2\gamma_1}{\gamma_1}, \eta_3 = \eta_3. \)

Set 3. \( a_0 = 0, a_1 = 1, \eta_1 = -i, \eta_2 = -\frac{\gamma_3 + 2\gamma_1}{\gamma_1}, \eta_3 = \eta_3. \)

Set 4. \( a_0 = 0, a_1 = 1, \eta_1 = -i, \eta_2 = \frac{\gamma_3 - 2\gamma_1}{\gamma_1}, \eta_3 = \eta_3. \)

Therefore, exact traveling wave solutions (dark soliton) of Eq.\((4)\) can be given as follows:

\[
\begin{align*}
\textbf{u}_1(x,y,t) &= \tanh \left( ix + \left( \frac{i^2 \gamma_3 - 2\gamma_1}{\gamma_1} \right) y + \eta_3 t \right) e^{iy}, \\
\textbf{v}_1(x,y,t) &= \frac{2i^2 \gamma_3 - 4\gamma_1}{\gamma_1 i} \tanh^2 \left( ix + \left( \frac{i^2 \gamma_3 - 2\gamma_1}{\gamma_1} \right) y + \eta_3 t \right). 
\end{align*}
\] (51)

\[
\begin{align*}
\textbf{u}_2(x,y,t) &= \tanh \left( ix + \left( -\frac{i^2 \gamma_3 + 2\gamma_1}{\gamma_1} \right) y + \eta_3 t \right) e^{iy}, \\
\textbf{v}_2(x,y,t) &= -\frac{2i^2 \gamma_3 + 4\gamma_1}{\gamma_1 i} \tanh^2 \left( ix + \left( -\frac{i^2 \gamma_3 + 2\gamma_1}{\gamma_1} \right) y + \eta_3 t \right). 
\end{align*}
\] (52)

\[
\begin{align*}
\textbf{u}_3(x,y,t) &= \tanh \left( -ix + \left( \frac{i^2 \gamma_3 - 2\gamma_1}{\gamma_1} \right) y + \eta_3 t \right) e^{iy}, \\
\textbf{v}_3(x,y,t) &= -\frac{2i^2 \gamma_3 + 4\gamma_1}{\gamma_1 i} \tanh^2 \left( -ix + \left( \frac{i^2 \gamma_3 - 2\gamma_1}{\gamma_1} \right) y + \eta_3 t \right). 
\end{align*}
\] (53)

\[
\begin{align*}
\textbf{u}_4(x,y,t) &= \tanh \left( -ix + \left( -\frac{i^2 \gamma_3 + 2\gamma_1}{\gamma_1} \right) y + \eta_3 t \right) e^{iy}, \\
\textbf{v}_4(x,y,t) &= \frac{2i^2 \gamma_3 + 4\gamma_1}{\gamma_1 i} \tanh^2 \left( -ix + \left( -\frac{i^2 \gamma_3 + 2\gamma_1}{\gamma_1} \right) y + \eta_3 t \right). 
\end{align*}
\] (54)

The solutions in (33) (see Figures 1 and 2), (48) (see Figures 3 and 4) and (51) (see Figures 5 and 6) of Eq.(4) are shown in Figs. 1-6 for \( y = 0 \).
Fig. 1 3D-plot for $|v(x,y,t)|^2$ when $y = 0$ where $\mu = -1, \lambda = 1, \gamma_1 = 1, \gamma_3 = 1, \eta_1 = 1, \eta_3 = 1, C = 0$. (combined soliton solution)

Fig. 2 3D-plot for $|v(x,y,t)|^2$ when $y = 0$ where $\mu = -1, \lambda = 1, \gamma_1 = 1, \gamma_3 = 1, \eta_1 = 1, \eta_3 = 1, C = 0$. (combined soliton solution)

Fig. 3 3D-plot for $|v(x,y,t)|^2$ when $y = 0$ where $a = 1.5$ and all other parameter values are 1 (Lucas combined hyperbolic function)
Optical soliton solutions to a (2+1) dimensional Schrödinger equation

Fig. 4 3D-plot for $|v(x,y,t)|^2$ when $y = 0$ where $a = 1.5$ and all other parameter values are 1 (Lucas combined hyperbolic function)

Fig. 5 3D-plot for $|v(x,y,t)|^2$ when $y = 0$ where all parameter values are 1 (dark soliton)
5 Conclusions and discussions

In this study, we have obtained various different structures of traveling wave solutions to a special (2+1)-dimensional Schrödinger equation. These solutions are in the form of dark soliton, combined soliton, symmetrical Lucas sine, Lucas cosine functions, and periodic waves. Some of the solutions are the same (dark solitons and periodic waves) with those obtained by [22] while the rest of solutions seems to be new. With the comparison of the performed methods, we conclude that $\exp(-\phi(\varepsilon))$ give us many different solution structures yet modified Kudryashov and tanh ansatz approaches lead to merely one type solution prototype (see Eq.(21) and Eq.(27)). Thus, $\exp(-\phi(\varepsilon))$ method has an apparent advantage over the other two methods.

The obtained exact (traveling wave) solutions might be used as an initial value in the initial/boundary value problems. In addition, these solutions can be utilized in numerical schemas as a benchmark and stability theories. Though omitted in the present research, there are several topics which will be addressed in our future works:

1) The group invariant solutions can be sought which are different than the wave solutions by the theory of Lie groups.
2) The different types of soliton interactions can be investigated depend upon Hirota’s bilinearization method.
3) The conservation laws of the model can be studied from the point of view of variational or Bell polynomials.
4) The Painleve analysis and interaction solutions can also be constructed.

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