Virial theorem in clusters of galaxies with MOND

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ABSTRACT
A specific modification of Newtonian dynamics known as MOND has been shown to reproduce the dynamics of most astrophysical systems at different scales without invoking non-baryonic dark matter (DM). There is, however, a long-standing unsolved problem when MOND is applied to rich clusters of galaxies in the form of a deficit (by a factor around two) of predicted dynamical mass derived from the virial theorem with respect to observations. In this article we approach the virial theorem using the velocity dispersion of cluster members along the line of sight rather than using the cluster temperature from X-ray data and hydrostatic equilibrium. Analytical calculations of the virial theorem in clusters for Newtonian gravity+DM and MOND are developed, applying pressure (surface) corrections for non-closed systems. Recent calibrations of DM profiles, baryonic ratio and baryonic equilibrium. Analytical calculations of the virial theorem in clusters for Newtonian gravity+DM and MOND are developed, applying pressure (surface) corrections for non-closed systems. Recent calibrations of DM profiles, baryonic ratio and baryonic (β model or others) profiles are used, while allowing free parameters to range within the observational constraints. It is shown that solutions exist for MOND in clusters that give similar results to Newton+DM—particularly in the case of an isothermal β model for β = 0.55 – 0.70 and core radii r_c between 0.1 and 0.3 times r_500 (in agreement with the known data). The disagreements found in previous studies seem to be due to the lack of pressure corrections (based on inappropriate hydrostatic equilibrium assumptions) and/or inappropriate parameters for the baryonic matter profiles.

Key words: gravitation — dark matter — galaxies: clusters

1 INTRODUCTION
In present-day astrophysics, many lines of investigation support the existence of large amounts of non-baryonic dark matter (DM) elsewhere in galaxies and in the Universe at large, the most obvious example being the asymptotically flat rotation curve of galaxies, which indicates the existence of massive DM haloes. Considerable fine tuning is required, however, to justify their observed properties, the most striking example possibly being the baryonic Tully–Fisher relation (see e.g. McGaugh (2012) and reference therein). Because of this, over the years more than one proposal has been made to find alternative explanations not involving DM. In particular, it has been shown that a specific modification of Newtonian dynamics, known as MOND (Milgrom 1983a,b,c), is able to describe many kinds of behaviour of galaxies and other cosmic structures generally ascribed to the presence of DM. The basic idea of MOND is that an acceleration (a_0) exists, below which Newtonian dynamics is no longer valid.

The MOND hypothesis has profound and far-reaching implications. Since the seminal papers by Milgrom (1983a,b,c), MOND has been applied to several astrophysical objects including (in increasing order of size) wide binary stars (Hernández, Jiménez & Allen 2012; Hernández, Cookson & Cortes 2022), globular clusters (Scarpa et al. 2003; Scarpa & Falomo 2010; Scarpa et al. 2011; Hernández & Lara-D I 2020), dwarf galaxies (Milgrom 1995; McGaugh & Milgrom 2013; Sanders 2021), gas dominated galaxies (McGaugh 2012; Sanders 2019), spiral galaxies (Sanders 1996; Gentile, Famaya & de Blok 2011; Milgrom & Sanders 2007) including our Milky Way (Chrobáková et al. 2020), elliptical galaxies (Milgrom & Sanders 2003; Durarao et al. 2018; Tian & Ko 2016), satellites around galaxies (Angus et al. 2008; Klypin & Prada 2009), pairs of galaxies (Milgrom 1983c; Scarpa, Falomo & Treves 2022), groups of galaxies (Milgrom 2019; McGaugh et al. 2021), gravitational lenses (Sanders 2014), and cluster of galaxies (Sanders 1999, 2003). In all cases except one MOND may describe the observations without the need for DM. The problematic case being rich clusters of galaxies, which are a longstanding problem, thus far unsolved by MOND, and on which we try to shed some light here.

We know the virial theorem works in clusters of galaxies for standard Newtonian gravity within the usual assumption of the existence of non-baryonic dark matter as predicted by ΛCDM models (e.g., Evrard et al. 2008; Zhang et al. 2011; Munari et al. 2013), but it has not worked for MOND so far. Using a hydrostatic isothermal model with temperatures derived from X-ray data, the MOND mass prediction falls short by a factor ~ 2 (Sanders 1999; Pointecouteau & Silk 2005). A more recent analysis by Ettori et al. (2019) finds that MOND scenarios underestimate hydrostatic masses of cluster by 40% at r_1000 (r_x being the radius of the sphere for which the average density inside it is x times the critical density ρ_c), but with a decreasing tension as the radius increases, and reaches ~15% at r_200. However, this hydrostatic model has certain drawbacks which, according to some authors, may lead to important systematic errors of up to a factor 2 for the mass ((Bartelmann & Steinmetz 1996; Ballard & Blanchard 1997)(Sadat 1997, §4.2)).

Other applications of the virial theorem within the framework of MOND are discussed in several works (Milgrom 1994, 2010, 2014; Sanders 1999, 2003).
2 APPLICATION OF THE VIRIAL THEOREM

2.1 Virial theorem

We assume spherical symmetry in a rich cluster with mass density distribution \( \rho(x) \) and mass interior to each radius \( r \)

\[
M(r) = 4\pi \int_0^r dx x^2 \rho(x).
\]

The potential energy with MOND or Newtonian gravity within a radius \( r_{\text{max}} \) is (Fabris & Velten 2009)

\[
V(r_{\text{max}}) = -4\pi G \int_0^{r_{\text{max}}} dr r \sqrt{1 + \left( \frac{r}{r_{cM}(r)} \right)^2 \rho(r) M(r)},
\]

with \( r_{cM}(r) \rightarrow \infty \) for Newtonian gravity. The distance \( r_{cM}(r) \) is related to the usual parameter \( a_0 \) by means of

\[
r_{cM}(r) = \sqrt[3]{\frac{G M(r)}{a_0}}
\]

and \( a_0 = 1.2 \times 10^{-10} \text{ m}^2 \text{s}^{-2} \) for MOND or \( a_0 = 0 \) for Newton.

The kinetic energy is

\[
K(r_{\text{max}}) = 3\pi \int_0^{r_{\text{max}}} dR \Sigma(R) \sigma_v^2(R),
\]

\[
\Sigma(R) = 2 \int_0^{\sqrt{\frac{R^2}{R^2 + z^2}}} dz \rho(\sqrt{R^2 + z^2}),
\]

where \( \sigma_v(R) \) is the line-of-sight velocity dispersion (in the rest-frame of the cluster) as a function of the projected distance \( R \) and \( \Sigma(R) \) is the surface mass density. This can also be expressed as:

\[
K(r_{\text{max}}) = \frac{3}{2} M(r_{\text{max}}) \sigma_{v,r < r_{\text{max}}}^2.
\]

In a virialized cluster, in the limit of \( r_{\text{max}} \rightarrow \infty \), the following condition should be followed:

\[
2K(r_{\text{max}}) + V(r_{\text{max}}) = 0.
\]

Hence,

\[
\sigma_{v,r < r_{\text{max}}} = \sqrt{-\frac{V(r_{\text{max}})}{3M(r_{\text{max}})}}.
\]

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\[ \ln \left( \frac{1 + C \frac{r_{\text{max}}}{r_{200}}}{1 + C} \right) \]

For the virial radius \( r_{\text{max}} = r_{200} \), and applying the virial theorem Eq. (7), with the pressure correction referred at Eq. (10) (\( F_P \) is independent of the mass), we get

\[ \sigma_{v,r,200,\text{Newton+DM}} = (382 \text{ km s}^{-1}) \left( \frac{M_{200}}{10^{14} \text{ M}_\odot} \right)^{1/3} \left( \frac{C}{1 + C} \right)^{1/2} \times F_P(C), \]

\[ F_P = \left[ 1 - \frac{0.400}{(1 + C)^2} \left( \frac{C}{1 + C} \right)^{1/2} \right]^{-1/2}. \]

This dependence with the mass to the power of 1/3 is also well known from simulations (Evans et al. 2008; Munari et al. 2013).

Throughout this paper, we shall calculate the dispersion of velocities as a function of \( M_{500} \equiv M(r_{500}) \). This amount might also be related to other parameters or measurements; for instance, the Sunyaev-Zel’dovich effect amplitude (\( Y_{SZ} \)) (Arnaud et al. 2010; López-Corredoira, Gutiérrez & Génova-Santos 2017; Aguado-Barahona et al. 2022). In terms of \( M_{500} \) (with \( M_{200}/M_{500} \) derived from Eq. (B1)), and including all the dependence of \( C \) in a single factor,

\[ \sigma_{v,r,200,\text{Newton+DM}} = A_{\text{Newton,NFW}}(C) \left( \frac{M_{500}}{10^{14} \text{ M}_\odot} \right)^{1/3}, \]

where \( A_{\text{Newton,NFW}}(C) \) is plotted in Figure 1. The dependence on \( C \) is quite small for \( C \) between 2 and 8. Several values are given in the literature: from \( C = 2.9 \pm 0.2 \) (Lin, Mohr & Stanford 2004; Macciò, Dutton & van den Bosch 2008) derived from analyses of observational X-ray data; or \( C = 4.6^{+1.8}_{-1.4} \) from purely theoretical dynamical models in Prada et al. (2012) (the error bars represent here the r.m.s., not the error of the average; assuming that the 10, 90 per cent percentiles of fig. 13 in Prada et al. (2012) are 1.28 times the r.m.s., as it corresponds to a Gaussian distribution). The concentration index \( C \) has a modest dependence on mass (Lin, Mohr & Stanford 2004; Macciò, Dutton & van den Bosch 2008; Prada et al. 2012; Ettori et al. 2019). The above range 2.7 < \( C < 6.4 \) gives a variation of only ~ 25% of \( A_{\text{Newton,NFW}}(C) \), which is negligible compared to other sources of errors. We take in the following as default \( C = 3 \), for which \( A_{\text{Newton,NFW}}(C) = 522 \text{ km s}^{-1} \) (\( F_P = 1.244 \)).

### 2.4 MOND with only baryonic matter

We set \( a_0 = 1.2 \times 10^{-10} \text{ m s}^{-2} \). For MOND, there is only baryonic mass. To calculate the amount of baryonic mass, we use the relationship obtained by González et al. (2013):

\[ M_{\text{bar,500}}(M_{500}) = (0.117 \pm 0.004) \times 10^{14} \left( \frac{M_{500}}{10^{14} \text{ M}_\odot} \right)^{0.16 \pm 0.04} \text{ M}_\odot. \]

For a baryonic density distribution of the type \( \rho_{\text{bar}}(r) = \rho_0 J \left( \frac{r}{r_c} \right) \), with \( J \) a generic function and cluster core radius \( r_c \) proportional to \( r_{500} \) (we define the parameter independent of the mass \( x_{500} \equiv r_c/r_{500} \)); (Pacaud et al. 2016)), the potential and kinetic energies are

\[ V(r_{\text{max}}) = -(6.366 \times 10^{55}) \frac{1}{x_{500}^2} \left( \frac{M_{\text{bar,500}}}{10^{14} \text{ M}_\odot} \right)^{5/3} \]

\[ K(r_{\text{max}}) = (2.980 \times 10^{50}) \left[ \sigma_{v,r_{\text{max}}} \right]^2 \left( \frac{M_{\text{bar,500}}}{10^{14} \text{ M}_\odot} \right)^{1/3} \left( \frac{r_{\text{max}}}{r_{500}} \right)^{1/2}. \]

Note that the radii \( r_{500} \) or \( r_{200} \) are here quite similar to the one obtained from Newton+dark matter(DM)+NFW. For instance, with Eqs. (16), (C1), we get

\[ r_{500} = 1.345 \text{ Mpc} \times \left( \frac{M_{\text{bar,500}}}{10^{14} \text{ M}_\odot} \right)^{0.386}, \]

which is similar although slightly lower than the \( r_{500} \) from Eq. (B1) of Newton+DM. In Fig. 2, we offer a plot with a numerical example for \( M_{\text{500}} = 5 \times 10^{14} \text{ M}_\odot \). This approximate coincidence is expected because the ratio of baryonic/total matter in the cluster is similar to the ratio of baryonic/total matter in the Universe (\( \Omega_b/\Omega_m \), with \( \Omega_m = 1 \) respectively for Newton+MOND). The fact that this ratio is the same one in clusters and in the Universe implies that the central cluster density in MOND is similar to the cluster central density in standard gravity times \( \Omega_b/\Omega_m \), thus leading to similar \( r_{500} \)'s in the two models. The fact that the similarity is tighter for \( r_{500} \) than for \( r_{200} \) is a coincidence.
The amplitude $A$ is however quite dependent on the parameters of the $\beta$ model.

For the usual values of $\beta = 2/3$, $x_{500} = 0.15$ (Pacaud et al. 2016), $A = 493$ km s$^{-1}$, $B = 0.295$. The pressure factor correction is $F_P = 1.357$. For comparison, the prediction of the velocity dispersion with only baryonic matter following $\beta$-model profile and Newtonian gravitation [equivalent to substituting $\beta = 0$ within the root square inside the integral of Eq. (17)], for the same parameters $\beta = 2/3$, $x_{500} = 0.15$ and the same range of masses, is $A = 206$ km s$^{-1}$, $B = 0.387$. Clearly, the effect of MOND is quite significant. It is almost enough to compensate for the absence of non-baryonic dark matter: it is $5-15\%$ (depending on the mass) lower than $\sigma_v$ for Newton+DM for $C = 3.0$.

For other values of the parameters $\beta, x_{500}$ can also approximately reproduce the dynamics of Newton+DM. Values of $A$ similar to the amplitude of Newton+DM ($= 522$ km s$^{-1}$) are obtained in the yellow–violet colour area of the left plot of Fig. 3: $\beta$ between 0.55 and 0.80. For reasonable values of $0.1 < x_{500} < 0.3$ [which lead, through Eq. (19) to $0.12 < r_c < 0.36$ Mpc for an average mass of $M_{500} = 5 \times 10^{14}$ M$_\odot$; of the order of $r_c = 0.25$ Mpc given by Jones & Forman (1984)], the values of $\beta$ are constrained between 0.55 and 0.70 in order to match the Newtonian amplitude. In Fig. 5, we plot the dispersion of velocities for the parameters $\beta = 0.65$, $x_{500} = 0.3$ which are very close to the Newton+DM results, giving $A = 553$ km s$^{-1}$, $B = 0.294$, $F_P = 1.448$ (note that the pressure factor here is 16% higher than with Newton). The values of $\beta$ in the literature (e.g., Bahcall & Lubin 1994; Henning et al. 2009) are of the same order, between 0.50 and 0.65 for rich clusters.

### 2.4.2 The Patej & Loeb (2015) model

The isothermal $\beta$-model is known to be insufficient for characterizing the range of cluster gas distributions (Vikhlinin et al. 2006; Patej & Loeb 2015). Other profiles could be used that give a better fit to the gas distribution. Here we use the one given by Patej & Loeb (2015):

$$\rho_{\text{bar}}(r) = \frac{\Gamma f_g \left( \frac{r}{r_s} \right)^{3\Gamma-3}}{r^{2\Gamma}} \rho_{\text{DM}} s \left( \frac{r}{r_s} \right)^\Gamma,$$

where $f_g$ is the fraction of gas with respect the total (in Newton+DM), i.e., $f_g = M_{500,\text{gas}} / M_{500}$; $\rho_{\text{DM}}(r)$ is the profile of the total mass including dark matter, in our case given by the NFW profile (see §8B) with scale $r_s = \frac{r_{200}}{C}$, with concentration index $C$, and $\Gamma, s$ are two extra free parameters. For $\Gamma = 1$, we would have that the baryonic mass traces the dark matter $\rho_{\text{bar}}(r) = f_g \rho_{\text{DM}}(r)$. Like the $\beta$ model, the above expression is also motivated on a theoretical basis within standard Newtonian gravity. Here, with MOND, we use it because it simply fits the observational profile of gas in clusters of galaxies, as a function that describes baryonic matter, and the theoretical derivation would have no sense.

This gives

$$J(x) = \frac{x^{2\Gamma-3}}{(1 + f_s x)^2},$$

and applying the virial theorem from Eqs. (7), with $V$ and $K$ of Eqs. (17), (18), we get a dependence that is fitted in the range $M_{500} = (1 - 10) \times 10^{14}$ M$_\odot$ with high accuracy by

$$\sigma_{v,r_{200},\text{MOND}} \approx A(\beta, x_{500}) \left( \frac{M_{500}}{10^{14} \text{M}_\odot} \right)^{B(\beta, x_{500})},$$

where $A(\beta, x_{500})$ and $B(\beta, x_{500})$ are plotted at Fig. 3. The exponent $B$ is almost constant, between 0.29 and 0.30 for most of the cases. The amplitude $A$ is however quite dependent on the parameters of the $\beta$ model.

For the range $1 < \Gamma \leq 2$, $0 < x_{500} < 1$, $0 < f_s \leq 0.4$, the exponent $D(\Gamma, x_{500}, f_s)$ falls always in the range between 0.29
and 0.34 in approximate agreement with Newton+DM. The amplitude $E(\Gamma, x_{500}, f_5)$ is plotted at Fig. 4 for $\Gamma = 1.1, 1.3, 1.5$, which are within the constraints obtained by Patej & Loeb (2015) of $1 < \Gamma \leq 1.5$. There is a wide range of possible values compatible with Newton+DM. For instance, if we assume an average value of $\Gamma = 1.5, x_{500} = 0.5$ (hence, $s = 0.5r_{500}$), $E = 646$ km $s^{-1}$ implies $f_5 \approx 0.6$ (hence, $r_5 \approx 0.8r_{500}$, equivalent to a concentration index $C \approx 2$).

2.5 Comparison with observations

Estimates of masses and velocity dispersions carried out by other teams for some clusters are shown in Figure 5.

At low ($z < 0.10$) and intermediate ($0.10 < z < 0.30$) redshifts, we use velocity dispersion data within $r_{200}$ from clusters of Sohn et al. (2020, Table 2), including the mass $M_{500}$ within this table estimated from X-ray observations by Piffaretti et al. (2011). We use only the clusters with $M_{500} \gtrsim 10^{14} M_\odot$. These comprise 74 clusters with $z < 0.10$ and 96 clusters with $0.10 < z < 0.30$. The $M_{500}$ errors are not provided; here we assume they have a 20% of error, which is typical of other estimates of X-ray masses (Vikhlinin et al. 2006; Walker et al. 2012; Martino et al. 2014; Haines et al. 2018; Whelan et al. 2022).

X-ray data for $M_{500}$, $\sigma_Y$, Error($\sigma_Y$) are obtained for eight high-redshift clusters ($0.50 < z < 0.65$) from the NIKA2 cluster survey (Mayet et al. 2020): see Table 1. Rest-frame velocity dispersions were calculate using public Sloan Digital Sky Survey (SDSS) data of galaxies’ velocities and applying a biweight technique (Beers, Flynn & Gebhardt 1990). X-ray masses were derived from REFLEX (Böhringer et al. 2004) and REXCESS (Böhringer et al. 2007) cluster surveys applying the method by Arnaud et al. (2010), whose error bars are estimated with the relative error bar of the X-ray luminosities (when available, or the average value of similar clusters of this sample otherwise).

Note that the values of $M_{500}$ from X-ray data correspond always to the estimations using the standard model Newton+DM. As it was remarked throughout the paper, in a MOND model, this would be approximately related to the total (baryonic) mass within $r_{500}$ through $M_{\text{bar},500} = 0.117 \times 10^{14} \left( \frac{M_{500}}{10^{14} M_\odot} \right)^{1.16} M_\odot$.

In cases with small numbers of galaxies, there may be some important biases in the galaxy cluster velocity dispersion (Ferragamo et al. 2020). Here we do not introduce any correction to take them into account, since the number of galaxies per cluster is high enough and the corrections of the statistics for small numbers are negligible. For the comparison with the theoretical predictions, we also assume that the r.m.s. of $\sigma_Y$ is much smaller than its average value within $r_{200}$, as is usually the case (Ferragamo et al. 2020). The observed velocities may be slightly different from the average because the average line-of-sight velocities were measured within a radius smaller than $r_{200}$ and our approximation of almost constant dispersion of velocities with radius might introduce some higher values of dispersion in the observations than in the theory. We assume that these differences are lower than the error bars.

Other effects could produce a few small systematics (Krízek, Krízek & Somer 2014): relativistic effects of high velocities, gravitational redshift, and gravitational lensing in a curved space, which would decrease the Hubble–Lemaître parameter, intergalactic baryonic matter, gravitational aberration, etc.

In Fig. 5, we see that the points for 178 clusters are close to the predictions of virial theorem within a virial radius of $r_{200}$ for Newtonian gravity, or for MOND with some parameters. The best power-law weighted fit (taking into account both the errors of masses and velocities) is:

$$\sigma_Y, \text{best fit data} = (613 \pm 22) \left( \frac{M_{500}}{10^{14} M_\odot} \right)^{0.230 \pm 0.027} \text{km s}^{-1}$$

The data present a correlation between $X = \ln \left( \frac{M_{500}}{10^{14} M_\odot} \right)$ and $Y = \ln \left( \frac{\sigma_Y}{\text{km s}^{-1}} \right)$, including the same weighting factors:

$$C \pm \Delta C = \left( \frac{X}{Y} \right) - 1 \pm \left( \frac{\sigma_X \sigma_Y}{\sqrt{\langle X \rangle \langle Y \rangle}} \right) = (9.3 \pm 1.3) \times 10^{-3}$$

a correlation at 7.2$\sigma$ level. This sigma-level does not strictly correspond to a Gaussian distribution, but practically indistinguishable from a Gaussian one. Perhaps the complement to one of the confidence level might be somewhat larger than the seven sigmas Gaussian one ($\sim 10^{-12}$), but it may certainly be said that in the present case the null hypotheses may be reject with at a confidence level larger than 99.99%. The estimator of the correlation coefficient is a sum of some 178 terms. Each of these terms is the product of two Gaussian variables, assuming that the errors of both mass and velocity dispersion conform to Gaussian statistics. Therefore, each of these
Figure 4. Values of $E$ in Eq. (25) as a function of $\Gamma$, $x_{500}$ and $f_s$ in the Patej & Loeb (2015) model of the baryonic matter+MOND.

Table 1. Clusters at high redshift ($z > 0.5$) used for our analysis, ordered by increasing redshift. The third column indicates the number of galaxies used for the measurement of the rest-frame velocity dispersion (column 5). Column 4 gives the estimated masses using either X-rays.

| Name               | Redshift | Nr. of galaxies | $M_{500}$ ($10^{14} M_\odot$) | $\sigma_v$     |
|--------------------|----------|-----------------|-------------------------------|----------------|
| PSZ2 G211.21+38.66 | 0.503    | 25              | $7.0 \pm 1.6$                | $760 \pm 150$ |
| PSZ2 G212.44+63.19 | 0.532    | 15              | $4.2 \pm 1.6$                | $840 \pm 270$ |
| PSZ2 G201.50-27.31 | 0.534    | 47              | $9.3 \pm 1.4$                | $1430 \pm 240$|
| PSZ2 G094.56+51.03 | 0.541    | 55              | $6.6 \pm 1.6$                | $1180 \pm 180$|
| PSZ2 G228.16+75.20 | 0.542    | 26              | $11.0 \pm 1.9$               | $1130 \pm 250$|
| PSZ2 G111.61-45.71 | 0.547    | 30              | $9.6 \pm 1.4$                | $700 \pm 140$ |
| PSZ2 G183.90+42.99 | 0.559    | 21              | $6.6 \pm 1.6$                | $1000 \pm 260$|
| PSZ2 G099.86+58.45 | 0.618    | 13              | $7.1 \pm 1.6$                | $1000 \pm 300$|

Terms are random variables following a Rayleigh distribution. The Rayleigh distribution is somewhat more extended than a Gaussian, but the central limit theorem assures that the sum of many variable following that distribution, or any distribution with well defined mean and variance, tends to a Gaussian. In fact, the sum of only four of them is already quite close to a Gaussian, although not in the farthest positions of the wings.

For the respective redshift ranges of $z$ (low: $< 0.10$; intermediate: $0.10 \leq z < 0.30$; high: $\geq 0.30$), we get

$$\sigma_v, \text{best fit data, z}<0.10 = (636 \pm 28) \left( \frac{M_{500}}{10^{14} M_\odot} \right)^{0.203\pm0.045} \text{ km s}^{-1}$$

(28)

$$\sigma_v, \text{best fit data, 0.10} \leq z < 0.30 = (550\pm43) \left( \frac{M_{500}}{10^{14} M_\odot} \right)^{0.294\pm0.050} \text{ km s}^{-1}$$

(29)
Virial theorem in clusters of galaxies with MOND

Figure 5. Upper panel: Rest-frame dispersion of velocities along the line of sight as a function of the $M_{500}$ mass in 178 clusters of galaxies. $M_{500}$ represents the total mass within the radius $r_{500}$ for Newtonian gravity+dark matter, whereas in MOND it is related to the total (baryonic) mass within $r_{500}$ through $M_{\text{bar},500} = 0.117 \times 10^{14} \times \left( \frac{M_{500}}{10^{14} \, M_\odot} \right)^{1.16} \, M_\odot$. The lines represent the power-law fit (the shaded area covers the range within a 1-$\sigma$ error of the fit), or the predictions with the standard Newtonian gravity+non-baryonic dark matter following a NFW profile. Bottom panel: predictions with the standard Newtonian gravity+non-baryonic dark matter following a NFW profile or MOND for different baryonic matter profiles, or Newtonian gravity with only baryonic matter.
\[ \sigma_v, \text{best fit data,} z > 0.50 = (690 \pm 560) \left( \frac{M_{500}}{10^{14} \text{ M}_\odot} \right)^{0.19 \pm 0.39} \text{ km s}^{-1} \]

(30)

There is no significant difference in the trend between low and high redshift clusters.

3 DISCUSSION AND CONCLUSIONS

The relationship between velocity dispersion and masses in clusters was known to work properly within Newton+DM (e.g., Evrard et al. 2008; Zhang et al. 2011; Munari et al. 2013), and also in the case of some modifications of gravity different from MOND without including DM (e.g., Brownstein & Moffat 2006). However, they did not work in MOND (Sanders 1999; Pointecouteau & Silk 2005; Ettori et al. 2019). We explored here the reason for this inconsistency and make major improvements in the application of the virial theorem. In particular our virial theory analytical relationship of velocity dispersion in galaxies with given mass profiles includes a pressure (surface) term, which, although its relevance is recognized in some literature (e.g., The & White 1986; Carlberg et al. 1996; Carlberg, Yee & Ellingson 1997; Girardi et al. 1998), is not usually considered in analytical calculations, although it is implicitly taken into account when carrying out numerical simulations. We also applied an updated calibration of non-baryonic mass in the NFW profile, baryonic ratio in analytical calculations, although it is implicitly taken into account when using our MOND model or a Patej & Loeb (2015) model.

Our results show that we can reconcile MOND with the virial theorem in clusters. This agreement is obtained when: 1) the pressure term is taken into account in the virial theorem, which gives a 10–15% higher velocity dispersion for MOND than for Newton+DM; 2) we explore a range of possible parameters in the baryonic matter profile rather than adopting a fixed one. In particular for MOND we predict velocity dispersions equivalent to Newton+DM by adopting a \( \beta \) model with \( \beta = 0.55 – 0.70 \), and core radii \( r_c < 0.30 S_{500} \), as in agreement with the known data. Lower concentration favours a higher MOND effect, so \( \sigma_{500} \sim r_{500} \sim 0.3 \) increases the dispersion of velocities by a factor 10–15% with respect to \( \sigma_{500} = 0.15 \) for the same \( \beta \sim 2/3 \); decreasing \( \beta \) with \( \sigma_{500} = 0.15 \) also decreases the concentration and produces similar results. This last effect is easy to understand in MOND since lower concentrations enhance the MOND effect because the galaxies spend a longer time during their orbits in the MOND regime of low \( (< a_0) \) accelerations. Also the greater pressure term for MOND is due to a lower concentration of baryons than DM. Calculations without pressure and with default parameters \( (\sigma_{500} \text{ much lower than} 0.3 \text{ and } \beta = 2/3) \) would give a \( \sigma_v \) 15–25% lower than Newton+DM. Given that the dynamical mass is proportional to \( \sigma_v^{3/2} \), this means dynamical masses 40–60% lower, and this would explain the discrepancies found in previous studies.

MOND in the regime of very low accelerations forms a field [‘phantom mass’; (Milgrom 1986, 2009; Wu & Kroupa 2015; López-Corredoira & Betancort-Rijo 2021)] which has an effect dynamically similar to the presence of non-baryonic dark matter in Newtonian gravity. Here we observe that MOND fits the predictions of the virial theorem in rich clusters of galaxies, which should not be surprising, given that the MOND phantom mass effect is equivalent to the non-baryonic dark matter. If some inconsistency arises, a revision of our knowledge of the distribution of baryons would be needed because, with appropriate profiles and calibrations of \( M_{\text{bary}} / M_{\text{DM}} \), there is always a mathematical solution able to mimic non-baryonic dark matter.

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APPENDIX A: PRESSURE (SURFACE) TERM IN THE VIRIAL THEOREM

When the virial theorem is applied to a portion of a stable gravitating system, it takes a different form from when it is applied to the whole system. In the latter case, the pressure term, which is present in general, cancels (asymptotically) and the familiar result of $2K + V = 0$ [Eq. (7)] holds, where $K$ is the kinetic energy and $V$ is the potential energy. However, the theorem is usually applied to the inner parts of a more extended system. This is the case, for instance, when the entities forming the objects (i.e., galaxies in the case of cluster of galaxies) are increasingly more difficult to discriminate from the interlopers. The relevance of this pressure (surface) term in standard gravity is known (e.g., The & White 1986; Carlberg et al. 1996; Carlberg, Yee & Ellingson 1997; Girardi et al. 1998).

For the general form, we must use the general scalar virial theorem, which takes the form

$$\sum_i \vec{F}_i \cdot \vec{r}_i = \sum_i m_i |\vec{v_i}|^2,$$

where $\vec{F}_i$ is the force acting on $i$-th particle and $\vec{r}_i$ its position vector. The sum $i$ extends to all particles. The right hand side of the equation is $2K$, while the left hand side is equal to $-V$ plus the ‘pressure term’. This last term appears because the force $\vec{F}_i$ acting on the $i$-th particle is due not only to the gravitational fields but also to pressure for particles at the boundary of the system.

The contribution of this term is simply $3P_r V_r$, where $P_r$ is the radial pressure at the boundary of the virial radius $r_{\text{max}}$, assumed to be constant over it; and $V_r$ is the volume within it. Assuming a stationary system with null average velocity, for which

$$\langle v^2 \rangle = 3\sigma_v^2,$$  \hspace{1cm} (A2)

where $\sigma_v$ is the rms of the velocities along the line of sight. Hence, after dividing by the total mass, the virial theorem reads

$$\sigma_{v,r < r_{\text{max}}}^2 = \sigma_{v,r < r_{\text{max}}, P=0}^2 + \frac{P_r (r_{\text{max}}) \frac{4}{3} \pi r_{\text{max}}^3}{M (r_{\text{max}})},$$  \hspace{1cm} (A3)

For isotropic pressure,

$$P_r (r_{\text{max}}) = \sigma_{v,r = r_{\text{max}}}^2 \rho (r_{\text{max}}),$$  \hspace{1cm} (A4)

For a more general case, when the pressure is not isotropic,

$$P_r (r_{\text{max}}) = \frac{3}{3 - 2\beta_\theta} \sigma_{v,r = r_{\text{max}}}^2 \rho (r_{\text{max}}),$$  \hspace{1cm} (A5)

where $\beta_\theta$ is the anisotropy parameter:

$$\beta_\theta = 1 - \frac{\sigma_\theta^2}{\sigma_r^2},$$  \hspace{1cm} (A6)

$$3 \sigma_\theta^2 = \sigma_r^2 + 2 \sigma_\phi^2,$$

and where we have used the fact that for spherical systems $\sigma_\theta^2 = \sigma_\phi^2$ ($r, \theta, \phi$ denote radial, declination and azimuthal directions).

APPENDIX B: THE NFW PROFILE

Assuming a critical density of $\rho_c = 8.5 \times 10^{-27}$ kg m$^{-3}$ (for $H_0 = 67.4$ km s$^{-1}$ Mpc$^{-1}$), a virial radius equal to $r_{200}$ and concentration index $C$, the Navarro–Frenk–White (NFW, Navarro, Frenk & White (1997)) profile follows these relationships for the mass density $\rho(r)$ and the mass within the sphere of radius $r$ $M(r)$:

$$M(r) = M_{200} \times \frac{\ln \left(1 + \frac{C r}{r_{200}}\right) - \frac{C r}{r_{200} + C r}}{\ln (1 + C) - \frac{C}{1 + C}},$$  \hspace{1cm} (B1)

$$r_{200} = 0.9834 \text{ Mpc} \times \left(\frac{M_{200}}{10^{14} \text{ M}_\odot}\right)^{1/3},$$

$$r_{500} = 0.7246 \text{ Mpc} \times \left(\frac{M_{500}}{10^{14} \text{ M}_\odot}\right)^{1/3},$$
\[
\rho(r) = \frac{\rho_0}{\left(1 + \frac{C}{r_{200}}\right)^2},
\]

\[
\rho_0 = 8.368 \times 10^{12} \frac{C^3}{\ln(1+C) - \frac{C}{1+C}} \text{M}_\odot \text{Mpc}^{-3},
\]

where \(M_x \equiv M(r_x)\) and \(r_x\) is the radius of the sphere for which the average density inside it is \(x\) times the critical density \(\rho_c\). That is, 
\[x\rho_c = \frac{M(r_x)}{\frac{x}{\pi} r_x^3}.
\]

**APPENDIX C: PROFILE FOR BARYONIC MATTER**

We assume a critical density of baryonic matter \(\rho_{cb} = \rho_c \frac{\Omega_b}{\Omega_m}\) with 
\[\rho_c = 8.5 \times 10^{-27} \text{ kg m}^{-3} \text{ (for } H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}); \Omega_m = 0.315, \Omega_b = 0.0493 \text{ (Planck Collaboration 2020).}
\]

For a mass density profile 
\[\rho_{\text{bar}}(r) = \rho_0 J\left(\frac{r}{r_c}\right),\]

the mass within the sphere of radius \(r\) is:
\[M_{\text{bar}}(r) = \frac{M_{\text{bar,500}}}{I(r_{500})} I(r), \quad (C1)\]

\[
I(r) = \int_0^{r/r_c} dx x^2 J(x),
\]

\[
r_{200} = 1.357 r_{500} \left(\frac{I(r_{200})}{I(r_{500})}\right)^{1/3},
\]

\[
r_{500} = 1.345 \text{ Mpc} \times \left(\frac{M_{\text{bar,500}}}{10^{14} \text{ M}_\odot}\right)^{1/3}
\]

\[
\rho_0 = \frac{M_{\text{bar,500}}}{4\pi r_c^3 I(r_{500})}.
\]

where \(M_{\text{bar,500}} \equiv M_{\text{bar}}(r_{500})\) and \(r_x\) is the radius of the sphere for which the average density inside it is \(x\) times the critical baryonic density \(\rho_{cb}\). The value of \(r_{200}\) is solved iteratively.

An isothermal \(\beta\) model is usual in the description of gas in clusters of galaxies (Cavaliere & Fusco-Femiano 1997), where 
\[J(x) = \frac{1}{(1 + x^2)^{3/2}}.\]

For \(\beta = 2/3\), which is usual the assumed value (Arnaud 2009), \(I(r)\) has an analytical solution: 
\[I(r)\beta = 2/3] = \frac{r_c}{r_c - \tan^{-1}\left(\frac{r}{r_c}\right)}.\]

A value of a cluster core radius scale equal to \(r_c \sim 0.25 \text{ Mpc}\) is expected (Jones & Forman 1984); although a dependence on the mass is also expected.