The validity of the Schönberg-Chandrasekhar limit in describing stellar evolution

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ABSTRACT
We discuss the Schönberg-Chandrasekhar limit in the light of the modern theory of stellar evolution. We find it remains a valid concept with the numerical value of the limiting fractional mass of the He core lying in a relatively narrow range of 0.09–0.11 for stars in the mass range of 1.4–7M⊙. Crossing this limit is associated with some restructuring of the star, which manifests itself by a characteristic kink (first a decrease then an increase) in the evolution of the radius vs. time. This also coincides with the moment the star moves from the subgiant branch to the red giant branch. The crossing is associated with some acceleration of the rate of the growth of the stellar radius and luminosity, but this acceleration is not substantial and the evolution does not change qualitatively. Therefore, the Schönberg-Chandrasekhar limit, while remaining a valid concept, is not a particularly useful tool in constraining the evolutionary status of a specific object.

Key words: stars: general – stars: evolution – stars: interiors – Hertzsprung-Russell and colour-magnitude diagrams

1 INTRODUCTION
The concept of the Schönberg-Chandrasekhar (hereafter S-C) limit is one of the classic results of the early stellar evolution theory. Its history started with a suggestion by Gamov (1938) that a He core left after the hydrogen exhaustion in the central part of a star should become isothermal due to the lack of the energy sources within the core. Next, Henrich & Chandrasekhar (1941) calculated simplified models containing such isothermal cores but not consisting of helium; they constructed homogeneous models in which chemical composition was the same for the core and the envelope. They found that it was not possible to construct a model if the relative mass of the isothermal core, \( q_c \equiv M_c / M \), was \( \lesssim 0.35 \). In a later paper, Schönberg & Chandrasekhar (1942) constructed more physically realistic models in which the configuration was composed of a He core and a H-rich envelope. Such configurations are a natural outcome of the core H-burning phase. They found the limiting value of \( q_c \approx 0.10 \) in this case, which has become known as the S-C limit.

Soon, it became clear that the S-C limit cannot be universal. Models of Schönberg & Chandrasekhar (1942) were very simplified. In particular, they were using the ideal gas equation of state. As noted by Hayashi, Hoshi & Sugimoto (1962), the He cores of stars with the total mass of \( M \leq 1.3M_\odot \) become quickly degenerate, and then they remain isothermal even above the S-C limit. On the other hand, Cox & Giuli (1968) claimed that the cores of stars with \( M \geq 6M_\odot \) are heavier than the S-C limit already at the moment of the central hydrogen exhaustion (which cannot be true literally since at that moment the mass of a core must be zero; this statement should rather be ‘soon after the central hydrogen exhaustion’). To summarize the above: according to the state of knowledge in 1968, the S-C limit should be valid only for stars in the mass range of about 1.3–6M⊙ (Cox & Giuli 1968).

The S-C limit was later analysed by some other authors, though not much progress has been achieved with respect to the above results. Some authors also discussed similar limits, called ‘S-C-like’, in different astrophysical situations. Eggelton & Faulkner (1981) and Eggleton, Faulkner & Cannon (1998) discussed whether the existence of the S-C limit could by itself explain the transformation of a star into a red giant after the central hydrogen exhaustion. The answer was negative; the S-C limit can explain why the core has to contract but it cannot explain why the envelope has to simultaneously expand. Beech (1988) considered configurations similar to that discussed in the original Henrich & Chandrasekhar (1941) and Schönberg & Chandrasekhar (1942) papers. He demonstrated that it is possible to derive the S-C limit by approximating the star with a simplified composite model with an isothermal core surrounded by an \( n = 1 \) polytrope envelope. Beech (1988) found that the limiting value of the relative mass of the isothermal core is 0.27 for a chemically homogeneous configuration and 0.10 for the configuration composed of a He core and an H-rich envelope; the latter fully agrees with Schönberg & Chandrasekhar (1942). The former differs from the result of Henrich & Chandrasekhar (1941), with 0.27 instead of 0.38. This is probably because an \( n = 1 \) polytrope (chosen for computational convenience) is a worse approximation of the envelope than one with a more realistic \( n = 3 \) used by

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Henrich & Chandrasekhar (1941). Eggleton et al. (1998) considered simple analytic two-polytrope models in which the core and envelope were approximated by an $n = 5$ and 1 polytropes, respectively. They assumed a jump in the chemical composition between the core and the envelope. The size of this jump (or, exactly, the ratio $\alpha$ of the mean molecular weight of the gas in the core and in the envelope) was a free parameter of the model. They found that the S-C phenomenon is not present (i.e., the fractional core mass can be arbitrarily large) for $\alpha < 3$. For $\alpha = 3$, the S-C phenomenon abruptly appears and the value of the limit is $2/\pi \approx 0.64$, while the classical value of 0.10. The phenomenon is also present for all values of $\alpha > 3$. However, $\alpha \geq 3$ does not correspond to a typical case of a He core and a Population I envelope, where $\alpha = 2.2$. Only in the case of carbon-oxygen core there is $\alpha = 3.3$. Their unexpected result, namely the lack of the S-C phenomenon in the most typical case appears to be due to the use of the $n = 5$ polytrope as an approximation for the core of the star, while the isothermal core formally corresponds to a polytrope with $n \to \infty$. Indeed, when using $n > 5$, Eggleton et al. (1998) found the S-C phenomenon at any $\alpha$.

Ball et al. (2011); Ball, Tout & Zytkow (2012) and Ball (2012) carried out an extensive analysis of different astrophysical situations in which limits analogous to the S-C one appear (they called them generally ‘S-C-like’ limits). The first class of objects they discuss are so-called quasi-stars. Quasi-stars are objects consisting of a stellar-mass black hole embedded in a massive, hydrostatic, giant-like envelope. The evolution of such object is driven by the accretion of matter from the envelope onto the central black hole. The authors constructed models following the evolution of quasi-stars of different masses using the Cambridge STARS stellar evolution code. One of the results was the existence of a robust upper limit on the ratio of the inner black hole mass to the total mass, equal to 0.119, independent of the quasi-star mass. This is certainly an S-C-like limit. The authors carried out also simplified polytropic analysis modelling the envelopes of quasi-stars with the $n = 2$, 3 and 4 polytropes. They got similar values for the S-C-like limit: 0.127, 0.102 and 0.089 for $n = 2$, 3 and 4 respectively.

The next group of objects discussed by Ball et al. (2011, 2012) and Ball (2012) are intermediate mass (3–15$M_{\odot}$) Population I stars near the end of the core H burning (the central H content is $\approx 0.05$). With help of their evolutionary models (using the STARS stellar evolution code) they found S-C-like limits for all investigated stars. The numerical values of the limit were in the range from 0.216 for a 3$M_{\odot}$ star to 0.367 for a 15$M_{\odot}$ star. The models showing S-C-like phenomenon were close to (but not exactly at) the leftward turn performed by stars in the Hertzsprung-Russell (H-R) diagram. These particular S-C-like limits have little in common with the classical S-C limit: they occur in a different evolutionary phase, they are strongly mass-dependent and the numerical values of the limits are much higher than the classical S-C value $\approx 0.10$. The authors carried out a similar analysis for low mass, 0.5–1.0$M_{\odot}$, Population I stars soon after the end of the core H burning. Again, they found the S-C-like limits with the numerical values in the range from 0.394 for 0.5$M_{\odot}$ star to 0.205 for 1.0$M_{\odot}$ star. These S-C-like limits are closer to the classical S-C limit as they occur in the same evolutionary phase. However, they are also strongly mass-dependent, the numerical values of the limits are much higher than the classical S-C limit and they are applied to less massive stars than typically considered in the context of S-C limit. Finally, those authors considered pure He stars with the masses of 0.5–1.0$M_{\odot}$ and again find some S-C-like phenomena exist under a wider range of circumstances than previously thought.

Regarding the classical S-C limit, we note that at present we have tools to construct precise stellar models. There is no necessity to invoke the S-C limit concept. On the other hand, this concept might still be valuable as a simplified tool in analysis of the evolutionary status of specific objects, in particular to distinguish between the subgiant and giant branches. However, it is sometimes used in a not fully correct way.

In particular, King (1993) placed a limit on the mass of the donor in the binary V404 Cygni by arguing that for $M_{\star}/M_{\odot}$ below the S-C limit the star would not have left the main sequence. However, as we discuss below, that limit is achieved relatively long after leaving the main sequence, around the time of the transition from a subgiant to giant, or from the luminosity class IV to III, with the difference between those two stellar types being relatively mild. The result of King (1993) was then used by Múnoz-Darias, Casares & Martínez-Pais (2008) to constrain the donor mass in the binary GX 339–4. Subsequently, Heida et al. (2017) used the constraint of Múnoz-Darias et al. (2008) to place a strict upper limit on the mass of the black hole and a lower limit on the inclination of that binary (while their two stellar templates giving best fit to the donor spectrum were one of the class III and one of the class IV), which constraints have then been widely used in papers on that binary. Moreover, the numerical value of the limit used by King (1993), 0.17, is arbitrary and significantly above the correct value of $\approx 0.10$. Additionally, the claimed maximum donor masses for V404 Cyg and GX 339–4 are $\approx 1.3M_{\odot}$ and $\approx 1.1M_{\odot}$, respectively, where the S-C limit does not apply.

In order to clarify the existing uncertainty about the meaning of the C-S limit, we have decided to construct detailed evolutionary models to check its validity, and, wherever it is valid, to determine the numerical value of the limit. For the purpose of this paper we define the S-C limit as the critical value of the relative mass of the He core for which its crossing is associated with a restructuring of the star manifesting as a characteristic kink (first a decrease then an increase) in the evolution of the radius as a function of time. We use this definition instead of that of the departure from isothermality since the latter happens quite gradually, starting at relatively low values of $q_{\star}$, see Section 2. The S-C limit is reached during the post-main sequence evolutionary phase (after the central H exhaustion) when a sufficiently large He core develops.

2 THE EVOLUTIONARY MODELS

We use the Warsaw stellar evolution code of Paczyński (1969, 1970), also developed by M. Kozłowski and R. Sienkiewicz. Its main current features and updates (e.g., the used opacities, nuclear reaction rates, equation of state) are described in Pamyatnykh et al. (1998) and Ziolkowski (2005). In Zdziarski et al. (2016), the code was calibrated to reproduce the Sun at the solar age. This resulted in the H mass fraction of $X = 0.74$, the metallicity of $Z = 0.014$, and the mixing length parameter of 1.55. Here, we use these values to follow evolution of stars in the mass range of 1–7$M_{\odot}$ up to advanced (luminosity of $\sim 10^5L_{\odot}$) red giant branch. The evolutionary tracks in the H-R diagram are shown in Fig. 1.

The crossings of the S-C limit marked in this figure by black crosses are defined as the points at which the radius as a function

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1 We have considered the evolution of an 8$M_{\odot}$ star (where we did find an S-C type behaviour), but for this mass the central He burning already takes place, and our numerical code is not accurate in that regime. Thus, the results presented in this work concern the mass range of $M \leq 7M_{\odot}$. 

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of time, $R(t)$, executing a characteristic kink in the $R-t$ plane starts to increase again after a temporary decrease, as shown in Fig. 2. We found that the value of the S-C limit slowly increases with $M$ growing from 0.090 for $M = 1.4M_\odot$ to 0.098 for $4M_\odot$, 0.109 for 6$M_\odot$ and 0.114 for 7$M_\odot$. Summarizing, the value of the S-C limit lies in a rather narrow range of $M/M_\odot \approx 0.09–0.11$. This narrowness indeed expresses the universality of this limit, taking into account the large spans of other parameters of the stars at the moment of the crossing of the S-C limit. For example, we have the radius $R = 2.76R_\odot$ and the central temperature of $3.0 \times 10^7$ K for $M = 1.4M_\odot$ and $54.8R_\odot$, $1.0 \times 10^8$ K for $7M_\odot$.

By inspecting Fig. 1, we can state with a reasonable accuracy that the crossing of the S-C limit coincides with the star moving from the subgiant branch of into red giant branch. We use here the terms subgiant and red giant in the sense of the location in the H-R diagram. However, we must remember that the classification of a specific object as either subgiant or giant is usually based on the spectroscopic criteria and may not exactly coincide with the location in the H-R diagram.

Conclusions concerning the nature of the S-C limit can then be drawn from the rates of the evolutionary change in the vicinity of the S-C limit crossing. We have chosen the radius of the star as a relevant parameter, and show $R(t)$ in Fig. 2. We see that at $1M_\odot$ the evolution proceeds through the relevant region very smoothly, without showing any sign of the S-C limit, as expected. We see an acceleration of the evolution along the red giant branch but this is due to the rate of the growth of the He core accelerating with the increasing luminosity. At $1.4M_\odot$, a weak disturbance in $R(t)$ appears in the form of a slight flattening, which is still a weak manifestation of the S-C limit. We see that this kink becomes first stronger with the increasing $M$, and it becomes again weaker for $M \geq 4M_\odot$. The S-C limit occurs in the range of $q_c \approx 0.09–0.11$, as noted above. The crossing is accompanied by an acceleration of the rate of the radius increase, but it is not dramatic; the evolution does not change qualitatively. On both sides of the S-C limit there are long-lived configurations that may correspond to observed stars.

We notice that Zdziarski, Ziolkowski & Mikołajewska (2019), while studying the evolutionary status of the donor in GX 339–4, considered models with $1.4M_\odot$, $q_c = 0.11$, and $2M_\odot$, $q_c = 0.05$. The latter was rejected because it was too luminous but from the point of view of the stability it was a valid solution, in spite of having its mass above the maximum and $q_c$ below the minimum implied by the analysis of King (1993), as discussed in Section 1. Then, Ziolkowski & Zdziarski (2017) in studying the the evolutionary status of GRS 1915+105 considered the donor parameters in the range of $0.28M_\odot$, $q_c = 0.90$ to $1.4M_\odot$, $q_c = 0.20$. In those cases, models of stripped giants were considered, and their He cores were highly degenerate. Thus, the S-C limit did not apply there. These examples show that valid models for accreting binaries can be found on both sides of the formal S-C limit.

The fact that the evolution does not change qualitatively at the crossing of the S-C limit means that, while remaining a valid concept, that limit not a practically useful tool in analysing the evolutionary status of a specific object; valid solutions could be found on both sides of the limit. This contrasts, e.g., the case of crossing of the Chandrasekhar mass limit by an accreting white dwarf, where obviously stable solutions exist only on one side, or of the Hertzsprung gap, where the evolution is much faster than either before or after it.

One might wonder what is the reason that a star passes the S-C limit relatively smoothly. The answer lies in the fact that even before approaching the limit, the core begins to contract, but the contraction is not rapid and permits solutions stable on long time scales. The contracting cores are no longer isothermal even before reaching that limit. Thus, the basic assumption in deriving the S-C limit is not satisfied. Therefore, the existence of stable cores more massive than the S-C limit is permitted. We found that when the He core is relatively small, e.g., $q_c = 0.06$, it remains with a reasonable approximation isothermal. This is true for the stars of all masses.
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**Figure 2.** The stellar radius vs. the evolutionary time (from the zero-age main sequence state) for $M = 1–7M_\odot$. The red dots show the points where $q_c = 0.09$, 0.095 and 0.1 (from left to right), and the red crosses show the points where $q_c = 0.06$ (left) and 0.12 (right). The single red dot for $M = 5M_\odot$ star corresponds to $q_c = 0.1$. The kinks in the shape of $R(t)$, i.e., at $dR/dt = 0$, defined the S-C limit. They are present for $M = 1.4–7M_\odot$. The lower kink in $R(t)$ for $2M_\odot$ corresponds to the time of exhaustion of the central hydrogen (the Hertzsprung gap), i.e., $q_c = 0$.

**Figure 3.** The interior temperature as a function of the ratio of the running mass (i.e., that inside the sphere of a given radius), $M_r$, for $M = 1.4$ and $6M_\odot$ and for different values of the fractional mass of the core, $q_c$. Notice that when the He core is relatively small, $q_c = 0.06$, the core remains with a reasonable approximation isothermal. The core departs from the isothermality with the growing $q_c$, as illustrated, e.g., by the case with $M = 6M_\odot$ and $q_c = 0.09$. When the core reaches the S-C limit, which happens at 0.09 and 0.11 for 1.4 and $6M_\odot$, respectively, it is already quite far from being isothermal. This is also true for the higher values of $q_c$. However, if the core becomes degenerate, as happens at $1.4M_\odot$ and $q_c = 0.30$, it returns to isothermality.

We should also point out that the He core is defined as the sphere filled with He and with zero hydrogen content. However, the realistic border between the core and the envelope is quite fuzzy, and above the He core there is a region containing mostly He but still with a some H content, especially for $q_c$ below the S-C limit. While formally it is no longer the He core, it behaves similarly to it up to $X \approx 0.005$. This can seen in Fig. 3 for the cases with $q_c = 0.06$ and 0.09. We see that at the formal boundary of the core, i.e., at the running mass of $M_r/M = q_c$, the temperature profile does not

in the considered range of $1–7M_\odot$. Examples for $M = 1.4$ and $6M_\odot$ are shown in Fig. 3, which shows the interior temperature as a function of the mass content with the sphere of a given radius. As the fractional mass of the He core grows, it departs more and more from the isothermality (unless becoming degenerate, see below).
show any feature and remains very smooth. Only at higher values of $M_r/M$, at $\approx 0.11–0.12$ in these cases, we see the temperature drop associated with the H-burning shell.

We should also remember that the cores of low mass stars will eventually recover the isothermality when strong degeneracy sets in. This is illustrated in Fig. 3 for $1.4M_\odot$ and $q_c = 0.30$. In this case, another assumption used in deriving the S-C limit is not satisfied, namely the equation of state is not that of the ideal gas.

3 CONCLUSIONS

Our main results are listed below.

1. The S-C limit remains a valid concept in the description of the stellar evolution, and it takes place in the mass range of $1.4–7M_\odot$, within which the limiting value of fractional mass in the He core lies in the relatively narrow range of $0.09–0.11$.

2. The crossing coincides with the beginning of the ascending to the red giant branch in the H-R diagram. Equivalently, it approximately marks the border between the subgiant and the red giant branches.

3. The crossing of the S-C limit is associated with some restructuring of the star, manifesting itself by a characteristic kink (first a decrease then increase) in the evolution of the radius as a function of time.

4. While this crossing is associated with some acceleration of the stellar evolution, specifically the rates of growth of the stellar radius and luminosity, this acceleration is not substantial and the rate of the evolution does not change qualitatively between the subgiant and giant branches.

5. Soon after the central hydrogen exhaustion when the He cores are still relatively small, $q_c \approx 0.06$, the core remains approximately isothermal. However, already before approaching the S-C limit, the core begins to contract and develops a strong temperature gradient.

6. The fact that the acceleration of the evolution at the crossing of the S-C limit is not substantial means that on both sides of the border line there exist long-lived configurations that may correspond to observed stars.

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