Local classical solutions of a three excitations kinetic system for a homogeneous condensed gas of bosons.

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M. Escobedo
Departamento de Matemáticas,
Universidad del País Vasco,
Apartado 644, E–48080 Bilbao, Spain.
E-mail: miguel.escobedo@ehu.es

Abstract: Short time existence of classical solutions is proved for a system of equations that involves a three excitations kinetic operator. The system is related to the description of a gas of bosons below but close to the critical temperature, where the three excitations integral aims at describing the interaction between the particles in the condensate and the excitations in the normal gas. Some qualitative properties of the solutions are obtained. Subject classification: 45K05, 45A05, 45M05, 82C40, 82C05, 82C22.

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1 Introduction

The goal of this article is to prove the local existence of classical solutions to the system,
\[ \frac{\partial F(\tau, \xi)}{\partial \tau} = n_c(\tau)Q(F(\tau))(\xi), \quad \tau \in (0, \tau^*), \xi > 0, \]
\[ \frac{dn_c(\tau)}{d\tau} = -n_c(\tau) \int_0^\infty Q(F(\tau))(\xi)\sqrt{\xi}d\xi, \quad \tau \in (0, \tau^*), \]
for some \( \tau^* > 0 \), with \( n_c(0) > 0 \),
\[ Q(F)(\xi) = \frac{1}{\sqrt{\xi}} \int_0^\xi \left[ F(\xi_1)(F(\xi - \xi_1) - F(\xi)) - (1 + F(\xi - \xi_1))F(\xi) \right]d\xi_1 - \]
\[ - \frac{2}{\sqrt{\xi}} \int_\xi^\infty \left[ F(\xi - \xi)(F(\xi) - F(\xi_1)) - F(\xi_1)(1 + F(\xi)) \right]d\xi_1, \]
and such that for some function \( A \in C(0, \tau_*), \)
\[ F(\tau, \xi) \sim A(\tau)\xi^{-1}. \]

Although there is a large number of seemingly very different approaches to the theoretical description of Bose–Einstein condensation at finite temperatures, it is accepted
that, under suitable conditions, in a condensed gas of bosons, the time evolution of the density functions $F$ and $n_c$ of respectively the excitations in the normal gas and of the condensate’s particles may be described by a Boltzmann equation for quantum particles coupled with a nonlinear Schrödinger equation (cf. for example [2], [9], [13], [14], [16] and [10], [17]).

In some regime of physical parameters, that system may be simplified by considering only the interactions between the excitations in the normal gas and particles in the condensate, and taking the free particle law as dispersion relation (cf. [5], [11] where other regimes are also described that are considered in [18]). A further simplification, assuming the system of particles to be spatially homogeneous and its momentum distribution to be isotropic, leads then to system (1.1)–(1.3) (cf. [14], [16]), where the collision integral $Q(F)$ describes $1 \leftrightarrow 2$ splitting of an excitation into two others in the presence of the condensate. In order to reduce numerical coefficients, dimensionless variables in units which minimize the number of prefactors are used throughout.

It was proved in [3] that for all non negative measure $F_0$ with a finite first moment, and for every constant $n_{c,0} > 0$, system (1.1)–(1.3) has a global weak solution $(F, n_c)$ with initial data $(F_0, n_{c,0})$. For all $\tau > 0$, $F(\tau)$ is a non negative measure that does not charge the origin with finite the first moment, and $n_c(\tau) > 0$.

However, one basic aspect of the non equilibrium behavior of the system condensate–normal fluid is the evolution of the condensate after its formation (cf. [2], [12]). It turns out that the evolution of $n_c(\tau)$ crucially depends on the behavior of $F(\tau, \xi)$ as $\xi \to 0$ (cf. for example Proposition 2 in [15] and Theorem 1.7 in [3]). If a regular behavior of $F(t)$ as $\xi \to 0$ is assumed, and Fubini’s Theorem may be applied in the right hand side of (1.2), it would follow,

$$\frac{dn_c(\tau)}{d\tau} = -n_c(\tau) \int_0^\infty Q(F)(\tau, \xi) \sqrt{\xi} d\xi = -n_c(\tau) \int_0^\infty F(\tau, \xi) \xi d\xi < 0$$

and $n_c$ could never grow. On the other hand, if the measure $F(\tau)$ is written as $F(\tau, \xi) = \xi^{-1/2} g(\tau, \xi)$, and $g(\tau)$ has no atomic part and has an algebraic behavior as $\xi \to 0$ then $F$ satisfies (1.4). It is well known, that equation (1.1) has a one parameter ($\beta > 0$) family of non trivial equilibria,

$$n_0(\xi) = \left( e^{\beta \xi} - 1 \right)^{-1}, \forall \xi > 0 \tag{1.5}$$

that satisfy (1.4) with $A(\tau) = \beta^{-1}$, where $\beta = (k_B T_e)^{-1}$, $k_B$ is the Boltzmann’s constant and $T_e$ is the temperature of the gas at equilibrium. Under condition (1.4) the following was proved in [15]

$$\lim_{\delta \to 0} \int_0^\infty Q(F(\tau))(\xi) \sqrt{\xi} d\xi = -\frac{\pi^2}{6} A(\tau)^2 + \int_0^\infty F(\tau, \xi) \xi d\xi, \tag{1.6}$$

assuming some regularity on $F$ for $\xi > 0$. However up to now, no solution of (1.1)–(1.3) is known satisfying (1.4) or these regularity properties.
1.1 The Main Result.

For a precise statement of the main result denote, for $\theta \in [0, 1/2)$

$$Y_\theta = \{ w \in L^{\infty}_{\text{loc}}(0, \infty); |w|_{Y_\theta} < \infty \}, \quad Y \equiv Y_0$$

(1.7)

$$||w||_{Y_\theta} = \sup_{\xi > 0} \left( \xi^\theta + \xi \right)^{1/2} |w(\xi)|$$

(1.8)

$$P_{\delta_1}(w_\xi) = \sup_{\xi > 0} \xi^{1-2\delta_1} \left| \frac{\partial w(\xi)}{\partial \xi} \right|, \quad \delta_1 = 1 - 2\varepsilon_1$$

(1.9)

$$Z_{\varepsilon_1, T} = \{ w \in C((0, T) \times [0, \infty)); ||w||_{\varepsilon_1, T} < \infty \}$$

(1.10)

$$||w||_{\varepsilon_1, T} = \sup_{t \in (0, T)} \left( t^\theta |w(t)|_Y + t^{2\delta_1} P_{\delta_1}(w_\xi(t)) \right)$$

(1.11)

for $\varepsilon_1 \in (0, 1/2), \delta_2 \in (\theta + \delta_1, 1 - \theta)$. For all $\Omega_0 \in Y_\theta$ define

$$F_0(\xi) = n_0(\xi) + n_0(\xi)(1 + n_0(\xi))\xi \Omega_0(\xi), \quad \forall \xi > 0.$$ 

Theorem 1.1. For all $\theta \in [0, 1/2)$, consider $\varepsilon_1 \in (\theta, 1/2), \delta_1 = 1 - 2\varepsilon_1$ and $\delta_2 \in (\theta + \delta_1, 1 - \theta)$. For all $\Omega_0 \in Y_\theta$ and $n_c(0) > 0$ there exists $\tau^* > 0$ and functions $F, n_c$, such that

(i) $n_c \in C([0, \tau^*)) \cap C^1((0, \tau^*))$, \hspace{1cm} (1.12)

(ii) $F(\tau, \xi) = n_0(\xi) + n_0(\xi)(1 + n_0(\xi))\xi \Omega(\tau, \xi)$, \hspace{1cm} (1.13)

$$\forall \tau_\ast \in (0, \tau^*): \Omega \in Z_{\varepsilon_1, \tau_\ast},$$

(1.14)

(iii) $\forall \tau_\ast \in (0, \tau^*), \varepsilon > 0$ as small as desired, $\exists C > 0$ such that

$$|Q(F(\tau, \xi))| \leq C n_0(\xi)(1 + n_0(\xi))\xi \times$$

$$\left( ||\Omega_0||_{Y_\theta}(1 + \Xi_\varepsilon(t(\tau), \xi^{1/2})) + ||\Omega||_{\varepsilon_1, \tau_\ast}(\Xi_\varepsilon(t(\tau), \xi^{1/2})) \right), \forall \tau \in (0, \tau_\ast), \forall \xi > 0$$

(1.15)

where $\Xi_\varepsilon$ and $\Xi_\theta$ are defined in \[4,56], \[0.32] respectively and

$$t(\tau) = \int_0^\tau n_c(\sigma) d\sigma, \tau \in (0, \tau^*)$$

(1.16)

(iv) $(F, n_c)$ satisfy \[1.1, \ref{1.2} for almost every $\tau \in (0, \tau^*), \xi > 0$ and

$$\lim_{\tau \to 0} \int_0^\infty F(\tau, \xi) \varphi \left( \xi^{1/2} \right) \frac{d\xi}{\sqrt{\xi}} = \int_0^\infty F_0(\xi) \varphi \left( \xi^{1/2} \right) \frac{d\xi}{\sqrt{\xi}}, \forall \varphi \in C(0, \infty) \cap L^\infty(0, \infty).$$

(1.17)

(v) $n_c(\tau) + \int_0^\infty F(\tau, \xi) \sqrt{\xi} d\xi = n_c(0) + \int_0^\infty F_0(\xi) \sqrt{\xi} d\xi, \forall \tau \in (0, \tau^*)$

(1.18)

$$\int_0^\infty F(\tau, \xi) \xi^{3/2} d\xi = \int_0^\infty F(0, \xi) \xi^{3/2} d\xi, \forall \tau \in (0, \tau^*).$$

(1.19)

There also exists a function $A \in C((0, \tau^*))$ such that, for $\delta > 0$ as small as desired and some constant $C > 0$, depending on $\delta$ and $\tau_\ast$, it holds for all $\sigma \in (0, \tau_\ast)$,

$$|\xi F(\sigma, \xi) - A(\sigma)| \leq C ||w||_{\varepsilon_1, \tau_\ast} \left( \frac{\sqrt{\xi}}{t(\sigma)} \right)^{1-\delta} t(\sigma)^{-\theta} + \frac{1-\delta-\theta}{2}, \forall \xi \in (0, \sigma^2/2).$$

(1.20)
Property (1.6) holds and for \( \theta \in (0, 1/2) \) there is a constant \( c_1 > 0 \) such that if \( w_0 \) satisfies,

\[
\exists \alpha > 0, \exists \varepsilon > 0; \quad w_0(\xi) = \alpha \xi^{-\theta} (1 + O(\xi^\varepsilon)) \quad (1.21)
\]

then,

\[
n'_{c}(\tau) = n_{c}(\tau) \left( c_1 \tau^{-2\theta} + O(\tau)^{-\theta} \right). \quad (1.22)
\]

Property (1.22) shows that for a singular initial data satisfying (1.21) with \( \theta \in (0, 1/2) \), the function \( n_c \) is initially strictly increasing with an exponential growth. An algebraic growth requires \( \theta = 1/2 \) but that critical value is not admissible in Theorem 1.1.

Forthcoming work may address several natural questions like the positivity of the solution \( F \) in Theorem 1.1, the possible extension to more general initial data, or the possible extension of the local solution to global classical solutions (where \( F \) belongs to some space of functions), and their long time asymptotic behavior. Notice that if \( (F, n_c) \) is the local solution given by Theorem 1.1 then, for \( \tau \in (0, \tau^*) \), the pair \( (F(\tau), n_c(\tau)) \) satisfies the hypothesis of Theorem 1.3 in [3] and \( (F, n_c) \) may then be extended to a global weak solution of (1.1), (1.2) in the sense of measures.

Similar questions to those treated in this work may arise in the context of the wave turbulence theory as exposed in [4], where the collision integral (1.3) has no linear terms.

2 Some arguments of the proof.

If the function \( n_c(\tau) \) in the right hand side of (1.1) is absorbed by the change of variable in (1.16), and we denote \( G(t, \xi) = F(\tau, \xi) \), the solution of system (1.1)–(1.3) is obtained as follows. First solve the equation

\[
\frac{\partial G(t, \xi)}{\partial t} = Q(G(t))(\xi), \quad (2.1)
\]

then invert the change of variable (1.16), deduce the function \( n_c \) and obtain \( F \).

We now look for a solution \( G(t, \xi) \) to (2.1) of the form,

\[
G(t, \xi) = n_0(\xi) + n_0(\xi)(1 + n_0(\xi))\xi w(t, \xi), \quad (2.2)
\]

where \( n_0 \) is one of the equilibria defined in (1.5). Without any loss of generality we may chose \( \beta = 2 \), and this simplifies somewhat the notations since then,

\[
G(t, \xi) = n_0(\xi) + \frac{\xi w(t, \xi)}{4 \sinh^2 \xi}. \quad (2.3)
\]

If the expression (2.3) is plugged into (2.1) it follows that \( w \) must satisfy

\[
\frac{\partial w(t, \xi)}{\partial t} = \mathcal{L}(w(t))(\xi) + \Pi(w(t), w(t))(\xi) \quad (2.4)
\]

where, for two positive numerical constants \( C_1 \) and \( C_2 \),

\[
\mathcal{L}(w(t))(\xi) = C_1 \int_0^\infty (w(t, \zeta) - w(t, \xi))\mathcal{M}(\xi, \zeta) \frac{d\zeta}{\zeta^{3/2}} \quad (2.5)
\]

\[
\mathcal{M}(\xi, \zeta) = \left( \frac{1}{\sinh |\xi - \zeta|} - \frac{1}{\sinh(\xi + \zeta)} \right) \frac{\zeta^{3/2} \sinh \xi}{\xi^{3/2} \sinh \zeta}. \quad (2.6)
\]
and
\[ \Pi(w, w)(\xi) = C_2 \left( N_1[w, w](\xi) - N_2[w, w](\xi) - N_3[w, w](\xi) + N_4[w, w](\xi) \right) \] (2.7)

where,
\[ N_1[w, \tilde{w}](\xi) = \frac{\sinh^2 \xi}{\xi^3/2} \int_0^{\xi/2} \tilde{H}(t, \xi) (H(t, \xi - \xi_1) - H(t, \xi)) \, d\xi_1, \]
\[ N_2[w, \tilde{w}](\xi) = \frac{\sinh^2 \xi}{\xi^3/2} \tilde{H}(t, \xi) \int_0^{\xi/2} H(t, \xi - \xi_1) \, d\xi_1, \]
\[ N_3[w, \tilde{w}](\xi) = \frac{\sinh^2 \xi}{\xi^3/2} \int_{\infty}^{\xi} \tilde{H}(t, \xi_1 - \xi) (H(t, \xi) - H(t, \xi_1)) \, d\xi_1, \]
\[ N_4[w, \tilde{w}](\xi) = \frac{\sinh^2 \xi}{\xi^3/2} H(t, \xi) \int_{\xi}^{\infty} \tilde{H}(t, \xi_1) \, d\xi_1, \]

and
\[ H(t, \xi) = \frac{\xi w(t, \xi)}{\sinh^2 \xi}, \quad \tilde{H}(t, \xi) = \frac{\xi \tilde{w}(t, \xi)}{\sinh^2 \xi}. \]

The linear operator \( \mathcal{L} \) was deduced in [8] (and is also recalled in [6, 7]). The nonlinear term \( \Pi(w, w) \) is obtained in the Appendix. A solution to equation (2.4) with initial data \( w_0 \) may now be obtained by means of the Duhamel’s formula,
\[ w(t, \xi) = (\Sigma(t)w_0)(\xi) + \int_0^t \Sigma(t - s)\Pi(w(s), w(s)) (\xi) \, ds. \] (2.8)

where \( v(t, \xi) = \Sigma(t)w_0 \) denotes a solution to the Cauchy problem,
\[ \frac{\partial v(t)}{\partial t} = \mathcal{L}(v(t)) \]
\[ v(0) = w_0. \] (2.9) (2.10)

The construction of the operator \( \Sigma \) is based on the approximation of (2.9, 2.10) by
\[ \frac{\partial u(t)}{\partial t} = L(u(t)), \quad t > 0, x > 0, \] (2.11)
\[ u(0) = u_0. \] (2.12)

where,
\[ L(u)(x) = \int_0^{\infty} (u(y) - u(x)) M(x, y) \frac{y \, dy}{x}, \] (2.13)
\[ M(x, y) = \left( \frac{1}{|x^2 - y^2|} - \frac{1}{x^2 + y^2} \right) \frac{y}{x}. \] (2.14)

The function \( M \) is obtained keeping only in the kernel \( \mathcal{M} \) of (2.6) the leading terms of the hyperbolic sine functions for small values of their arguments \( \xi \) and \( \zeta \) and expressing them in the new variables \( x^2 = \xi \) and \( y^2 = \zeta \).

The Cauchy problem (2.11), (2.12) is solved in the Section 3 and the Cauchy problem (2.9, 2.10) is solved in Section 4. The Section 5 is devoted to estimate the nonlinear term \( \Pi(w, w) \). The nonlinear equation (2.4) is then solved in Section 6 and Theorem 1.1 is proved in Section 7. The Appendix contains two short technical remarks.
3 The Cauchy problem (2.11), (2.12).

The Cauchy problem (2.11), (2.12) for initial data $u_0 \in L^\infty(0, \infty) \cap L^1(0, \infty)$ is solved in [6], Theorem 1.4, where it was proved that the solution denoted $S(t)u_0$ is given by

$$S(t)u_0(x) = \int_0^\infty u_0(y)\Lambda \left(\frac{t}{y}, \frac{x}{y}\right) \frac{dy}{y}, \forall t > 0, \forall x > 0,$$

(3.1)

with $\Lambda$ the fundamental solution of (2.11), (2.12) obtained in [6], Theorem 1.2. This result is extended in this Section to the following space of initial data,

$$X_\theta = \{u \in L^\infty(0, \infty); ||u||_{X_\theta} < \infty\}, \quad X \equiv X_0.,$$

(3.2)

$$||u||_{X_\theta} = \sup_{x > 0} (1 + x)|u(x)|.$$

(3.3)

The arguments and calculations are essentially the same as in [7] but they are presented here in some detail for the sake of completeness. The following auxiliary function is needed,

$$\Xi_\theta(t, x) = \begin{cases} \left(xt^{-1-\theta}, 0 < x < t < 1 \right) \\ \left(x^{-\theta}, 0 < t < x < 1 \right) \\ \left(x^{-1}t^{1-\theta}, 0 < t < 1 < x \right) \\ (t^2 + \log t), \forall t > \max(1, x) \\ (x^{-1}\log t + x^{-3+\varepsilon}t^{2-\varepsilon}), 1 < t < x, \end{cases}$$

(3.4)

with $\Xi \equiv \Xi_0$, that satisfies,

$$\forall T > 0, \exists C > 0; \quad \Xi(t, x) \leq Ct^{-\theta}\left(\frac{x}{t} + \frac{t}{x}\right)^{-1}, \forall x > 0, \forall t < (0, T).$$

(3.5)

**Proposition 3.1.** For all $g \in X_\theta$, the function $S(t)g$ in (3.1) is well defined for all $t > 0$, $x > 0$ and such that

$$S(\cdot)g \in L^\infty_{loc}((0, \infty); L^\infty(0, \infty)) \cap C((0, \infty) \times [0, \infty)).$$

(3.6)

Moreover, there exists a constant $C > 0$ that does not depend on $g$ such that for all $g \in X_\theta$ and for all $x > 0$,

$$|(S(t)g)(x)| \leq C||g||_{X_\theta}\Xi_\theta(t, x)$$

(3.7)

and then for all $t > 0$, $S(t)$ is a linear and bounded operator from $X_\theta$ into itself. Moreover,

(i) $\forall \varphi \in C(0, \infty) \cap L^\infty(0, \infty)$, $\lim_{t \to 0} \int_0^\infty (S(t)g)(x)\varphi(x)dx = \int_0^\infty g(x)\varphi(x)dx.$

(3.8)

(ii) $\forall g \in X \cap C(0, \infty)$, $\lim_{t \to 0} (S(t)g)(x) = g(x), \forall x > 0,$

(3.9)

(iii) $\forall g \in X_\theta \cap C^1(0, \infty)$: $\sup_{x \geq 1}|x^2|g'(x)| + x|g(x)| < \infty; \lim_{t \to 0} ||S(t)g - g||_{X_\theta} = 0.$

(3.10)
Proof. It was proved in [6] that for all $g \in L^\infty(0, \infty)$ the function $S(t)g(x)$ given by
\[
S(t)g(x) = \int_0^\infty \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y}
\]  
(3.12)
is well defined, and for all $t > 0$, $\|S(t)g\|_\infty \leq C\|g\|_\infty$ for some constant $C > 0$ independent on $g$. In order to prove (3.14) let us write now,
\[
S(t)g(x) = \int_0^t \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y} + \int_t^\infty \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y} = I_1 + I_2.
\]  
(3.13)

Properties (3.10) follow by classical Lebesgue integrals properties from estimates of the function under the integral signs in (3.13). Since these estimates are uniform for $(t, x)$ in compact subsets of $(0, \infty) \times (0, \infty)$ the continuity property $S(t \cdot)g \in C((0, \infty) \times (0, \infty))$ also follows. The first term $I_1$ is easily estimated as,
\[
|I_1| \leq \sup_{0 \leq y \leq t} \frac{1}{y} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \int_0^t |g(y)|dy.
\]  
(3.14)

If $t \in (0, 1],
\[
|I_1| \leq \sup_{0 \leq y \leq t} \frac{1}{y} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \|g\|_{X_\theta} \int_0^t y^{-\theta}dy = \sup_{0 \leq y \leq t} \frac{1}{y} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \|g\|_{X_\theta} \frac{t^{1-\theta}}{1-\theta}
\]  
(3.15)

and if $t > 1$
\[
\left| \int_0^t |g(y)|dy \right| \leq \|g\|_{X_\theta} \int_0^1 y^{-\theta}dy + \|g\|_{X_\theta} \int_1^t y^{-1}dy
\]
\[
\left| \int_0^t \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y} \right| \leq \sup_{0 \leq y \leq t} \frac{1}{y} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \left( \frac{\|g\|_{X_\theta}}{1-\theta} + \|g\|_{X_\theta} \log t \right).
\]  
(3.16)

Since $t/y > 1$ for $y \in (0, t)$, if $x/t \in (0, 1)$ it follows from Propositions 3.1, 3.2, 3.3 in [6],
\[
\left| \frac{1}{y} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \right| \leq C \left( t^{-3}y^2 + t^{-4}y^3 \right) = Ct^{-1} \left( (y/t)^2 + (y/t)^3 \right) \leq Ct^{-1}.
\]  
(3.17)

If on the contrary $x/t > 1$, then by the same Propositions in [6],
\[
\left| \frac{1}{y} \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \right| \leq C(x^{-3}y^2 + x^{-5}ty^3) = Cx^{-1} \left( (y/x)^2 + tx^{-2}(y/x)^3 \right) \leq Cx^{-1}.
\]  
(3.18)

Estimates (3.12), (3.15), (3.16), (3.17) and (3.18) yield,
\[
|I_1| \leq C \min \left( t^{-1}, x^{-1} \right) \left( \|g\|_{X_\theta} \mathbb{1}_{t<1} t^{1-\theta} + \mathbb{1}_{t>1} \|g\|_{X_\theta} \log t \right).
\]  
(3.19)

Consider now the term $I_2$ where $y > t$. If $x < t$ then by Proposition 3.5 in [6] and (3.24)
\[
\left| \Lambda \left( \frac{t}{y}, \frac{x}{y} \right) \right| \leq Ctx^5 y^{-6},
\]
\[
|I_2| \leq Ctx^5 \int_t^\infty |g(y)|y^{-7}dy.
\]
Therefore, when \( t \in (0, 1) \),
\[
|I_2| \leq C x t^{-1-\theta} M_\theta(g) + C x t^5 N_1(g) \leq C ||g|| x_o t^{-1-\theta}
\] 
(3.20)
and if \( t > 1 \),
\[
|I_2| \leq x t^{-2} ||g|| x_o.
\] 
(3.21)
On the other hand, when \( y > t \) if \( x > t \) there exists \( \delta > 0 \) small enough such that \( x > (1 + \delta)t \). The term \( I_2 \) must then be split,
\[
I_2 = \int_t^\infty \cdots \frac{dy}{y} + \int_{t}^{\infty} \cdots \frac{dy}{y} + \int_{t}^{\infty} \cdots \frac{dy}{y} = I_{2,1} + I_{2,2} + I_{2,3}
\] 
(3.22)
Since \( \frac{\delta}{y} > 1 + \delta \) in \( I_{2,1} \) and \( \frac{\delta}{y} < 1 - \delta \) in \( I_{2,3} \), by Proposition 3.5 in [6] for all \( \varepsilon > 0 \) as small as desired there exists a constant \( C_\varepsilon > 0 \) such that, for \( y \in (t, x/(1 + \delta)) \) and \( y > x/(1 - \delta) \),
\[
\left| \Lambda \left( \frac{t}{y} \right) \right| \leq C_\varepsilon \left( x^{-3+\varepsilon} t^{9-\varepsilon} y^{-6} + x^{-5} t^7 y^{-2} \right)
\]
and then
\[
|I_{2,1}| + |I_{2,3}| \leq C x^{-3+\varepsilon} t^{9-\varepsilon} \int_t^\infty y^{-7} |g(y)| dy + C x^{-5} t^7 \int_t^\infty y^{-3} |g(y)| dy.
\] 
(3.23)
When \( t > 1 \),
\[
\int_t^\infty y^{-7} |g(y)| dy \leq N_1(g) \int_t^\infty y^{-8} dy \leq \||g|| x_o t^{-7},
\]
\[
\int_t^\infty y^{-3} |g(y)| dy \leq N_1(g) \int_t^\infty y^{-4} dy \leq \||g|| x_o t^{-3},
\]
but, when \( t \in (0, 1) \),
\[
\int_t^\infty y^{-7} |g(y)| dy \leq \||g|| x_o \int_1^t y^{-7-\theta} dy + \||g|| x_o \int_1^\infty y^{-8} dy
\]
\[
\leq C \left( \||g|| x_o t^{-6-\theta} + \||g|| x_o \right)
\]
and,
\[
\int_t^\infty y^{-3} |g(y)| dy \leq M_\theta(g) \int_1^t y^{-3-\theta} dy + N_1(g) \int_1^\infty y^{-4} dy
\]
\[
\leq C \left( \||g|| x_o (t^{-2-\theta} - 1) + \||g|| x_o t^{-3} \right).
\]
All these estimates for \( t > 0 \) may be written,
\[
\int_t^\infty y^{-7} |g(y)| dy \leq C \left( \||g|| x_o t^{-6-\theta} - 1 \right) + \||g|| x_o \mathbb{1}_{t > t^{-7}}
\] 
(3.24)
\[
\int_t^\infty y^{-3} |g(y)| dy \leq \left( \||g|| x_o t^{-2-\theta} - 1 \right) + \||g|| x_o \mathbb{1}_{t > t^{-3}}.
\] 
(3.25)

8
It follows from (3.23)–(3.25) for \( t \in (0, 1) \) and \( x > t \),
\[
|I_{2,1}| + |I_{2,3}| \leq Cx^{-3+\varepsilon}\left(t^{3-\varepsilon-\theta}||g||_{X_{\theta}} + t^{9-\varepsilon}||g||_{X_{\theta}}\right) + Cx^{-5}\left(t^{5-\theta}||g||_{X_{\theta}} + t^7||g||_{X_{\theta}}\right) \leq C||g||_{X_{\theta}}x^{-3+\varepsilon}t^{3-\theta-\varepsilon} \tag{3.26}
\]

And for \( t > 1 \), \( x > t \),
\[
|I_{2,1}| + |I_{2,3}| \leq C\left(x^{-3+\varepsilon}t^{2-\varepsilon} + Cx^{-5}t^4\right) ||g||_{X_{\theta}} \leq C||g||_{X_{\theta}}x^{-3+\varepsilon}t^{2-\varepsilon} \tag{3.27}
\]

It follows that,
\[
|I_1| + |I_{1,1}| + |I_{2,3}| \leq \begin{cases} 
C||g||_{X_{\theta}}t^{-1-\theta}, & \forall 0 < x < t < 1 \\
C||g||_{X_{\theta}}t^2 + \log t, & \forall t > \max(1, x) \\
C||g||_{X_{\theta}}x^{-1}t^{1-\theta}, & 0 < t < 1, x > t \\
C||g||_{X_{\theta}}(x^{-1}\log t + x^{-3+\varepsilon}t^{2-\varepsilon}), & 1 < t < x.
\end{cases} \tag{3.28}
\]

In the second term \( I_{2,2} \) at the right hand side of (3.22) the regularising effect of \( S(t) \) must be used. With the change of variables \( z = x/y \),
\[
I_{2,2} = \int_{1-\delta}^{1+\delta} \frac{t_z}{x} |1 - z|^{2\varepsilon-1} \psi(t, z; x)g(x/z) \frac{dz}{z},
\]

In order to use Corollary 3.13 in [6] write now \( I_{2,2} \) as follows,
\[
I_{2,2} = \int_{1-\delta}^{1+\delta} \frac{t_z}{x} |1 - z|^{2\varepsilon-1} \psi(t, z; x)g(x/z) \frac{dz}{z},
\]

where, for all \( x > 0 \),
\[
\psi(t, z; x) = \left( A\left(t_z, z\right) \frac{x}{t_z} |1 - z|^{1-2\varepsilon}\right).
\]

By Corollary 3.13 and Corollary 3.14 in [6]
\[
(t, z) \mapsto \psi(t, z; x) \text{ belongs to } C([0, 1] \times [1 - \delta, 1 + \delta]).
\]
\[
C_{\varepsilon}(\psi, \delta) = \sup \left\{ |\psi(t, z; x)|; x > t, \forall z \in (1 - \delta, 1 + \delta) \right\} < \infty
\]

and then,
\[
|I_{2,2}| \leq C_{\varepsilon}(\psi, \delta) \int_{1-\delta}^{1+\delta} \frac{t_z}{x} |1 - z|^{2\varepsilon-1} |g(x/z)| \frac{dz}{z}.
\]

Under the change of variables \( y = 2\varepsilon \log(1 - z), z = 1 - e^{-\frac{y}{2\varepsilon}}, dy = -2dz/(x(1 - z)) \)
\[
\int_{1-\delta}^{1} \frac{t_z}{x} |1 - z|^{2\varepsilon-1} |g(x/z)| \frac{dz}{z} =
\]
\[
= \frac{1}{2} \int_{-\infty}^{\frac{1}{1-e^{-\frac{y}{2\varepsilon}}}} e^{y(1-e^{-\frac{y}{2\varepsilon}})} \left|g\left(\frac{x}{1-e^{-\frac{y}{2\varepsilon}}}\right)\right| dy = J_1 + J_2
\]
where,

\[ J_1 = \frac{1}{2} \int_{-\infty}^{\frac{2t \log \delta}{x}} e^{y\left(\frac{1-e^{\frac{y}{2t \log \delta}}}{e^{\frac{y}{2t \log \delta}}} \right)} \left| g_<(\frac{x}{1-e^{\frac{y}{2t \log \delta}}}) \right| dy \]

\[ J_2 = \frac{1}{2} \int_{-\infty}^{\frac{2t \log \delta}{x}} e^{y\left(\frac{1-e^{\frac{y}{2t \log \delta}}}{e^{\frac{y}{2t \log \delta}}} \right)} \left| g_>(\frac{x}{1-e^{\frac{y}{2t \log \delta}}}) \right| dy \]

\[ g_<(z) = g(z)\mathbb{1}_{z<1}, \quad g_>(z) = g(z)\mathbb{1}_{z>1}. \]

Since \( g \in X_{\theta,1} \),

\[ |g_<(z)| \leq ||g||X_\theta z^{-\theta}, \quad |g_>(z)| \leq ||g||X_\theta z^{-1}. \]

Because condition \( x < 1 - e^{\frac{y}{2t \log \delta}} \) for some \( y \in (\infty, 2t \log \delta/x) \) requires \( x < 1 \), it follows that \( J_1(t,x) = 0 \) for \( x > 1 \). On the other hand, if \( y \in (\infty, 2t \log \delta/x) \) then \( e^{\frac{y}{2t \log \delta}} < \delta \) and so condition \( x > 1 - e^{\frac{y}{2t \log \delta}} \) requires \( x > 1 - \delta > 1/2 \) from where \( J_2(t,x) = 0 \) for all \( x \in (0,1/2) \). For \( 0 < t < x < 1 \),

\[ |J_1| \leq \frac{||g||X_\theta}{2x^\theta} \int_{-\infty}^{\frac{2t \log \delta}{x}} e^{y\left(\frac{1-e^{\frac{y}{2t \log \delta}}}{e^{\frac{y}{2t \log \delta}}} \right)} \left(1 - e^{\frac{y}{2t \log \delta}}\right)^\theta dy \]

\[ \leq C||g||X_\theta t^{-\theta} \int_{-\infty}^{\frac{2t \log \delta}{x}} e^{y\left(\frac{1-e^{\frac{y}{2t \log \delta}}}{e^{\frac{y}{2t \log \delta}}} \right)} dy \leq C||g||X_\theta t^{-\theta} \int_{-\infty}^{\frac{2t \log \delta}{x}} e^{y(1-\delta)} dy \]

\[ \leq C||g||X_\theta \leq C||g||X_\theta x^{-\theta}. \quad (3.29) \]

When \( x > 1/2 \),

\[ |J_2| \leq \frac{||g||X_\theta}{2x} \int_{-\infty}^{\frac{2t \log \delta}{x}} e^{y\left(\frac{1-e^{\frac{y}{2t \log \delta}}}{e^{\frac{y}{2t \log \delta}}} \right)} dy \]

\[ \leq C||g||X_\theta t^{-1} \int_{-\infty}^{\frac{2t \log \delta}{x}} e^{y\left(\frac{1-e^{\frac{y}{2t \log \delta}}}{e^{\frac{y}{2t \log \delta}}} \right)} ydy \leq C N_1(g) t^{-1} \int_{-\infty}^{\frac{2t \log \delta}{x}} e^{y(1-\delta)} ydy \]

\[ \leq C||g||X_\theta t^{-1} e^{-\frac{2(1-\delta t) \log \delta}{x}} \left(\frac{x}{t \log \delta}\right)^{-1} \leq C||g||X_\theta x^{-1}. \quad (3.30) \]

Similar estimates are obtained for the integral where \( z \in (1,1+\delta) \). Suppose now that \( \varphi \in C(0,\infty) \cap L^\infty(0,\infty) \)

\[ \int_0^\infty S(t)g(x)\varphi(x)dx = \int_0^\infty g(y) \int_0^\infty \Lambda \left(\frac{t}{y+y} \right) \varphi(x)dx \frac{dy}{y}. \]

By Proposition 8.2 in the Appendix, for all \( y > 0 \) fixed,

\[ \lim_{t \to 0} \int_0^\infty \Lambda \left(\frac{t}{y+y} \right) dx = \lim_{t \to 0} \int_0^\infty \Lambda \left(\frac{t}{y+\gamma} \right) \varphi(yz)dydz = y\varphi(y). \]
Let us find a function $H(y) \in L^1(0, \infty)$ such that for all $t > 0$ in a neighborhood of zero,

$$\left| g(y) \int_0^\infty \Lambda \left( \frac{t}{y}, z \right) \varphi(yz) dz \right| \leq H(y), \text{ a. e. } y > 0.$$  

By Corollary 3.15 of [6], there exists positive constant $C$, such that

$$\left| \int_0^\infty \Lambda \left( \frac{t}{y}, z \right) \varphi(yz) dz \right| \leq C \| \varphi \|_\infty, \text{ for all } \forall t > 0, \forall y > 0.$$  

The function $H(y)$ may then be equal to $C \| \varphi \|_\infty |u_0(y)|$ for bounded values of $y$. Suppose now that for some $\delta > 0$ small $t/y < 1 - \delta$. In that case,

$$\int_0^\infty \left| \Lambda \left( \frac{t}{y}, z \right) \varphi(yz) \right| dz \leq C \| \varphi \|_\infty \int_0^\infty \left| \Lambda \left( \frac{t}{y}, z \right) \right| dz$$

$$\leq \int_0^{t/y} [\cdots] dz + \int_{t/y}^{1-\delta} [\cdots] dz + \int_{1-\delta}^{1+\delta} [\cdots] dz + \int_{1+\delta}^\infty [\cdots] dz. \quad (3.31)$$

It follows from Proposition 3.5 of [6]

$$\int_0^{t/y} [\cdots] dz + \int_{t/y}^{1-\delta} [\cdots] dz + \int_{1-\delta}^{1+\delta} [\cdots] dz \leq C(t/y)^3. \quad (3.32)$$

For the third term at the right hand side of (3.31), by Proposition 3.6 of [6],

$$\int_{1-\delta}^{1+\delta} \left| \Lambda \left( \frac{t}{y}, z \right) \right| dz \leq \frac{Ct}{y} \int_{1-\delta}^{1+\delta} e^{\frac{t}{y} \log(1-|z|)} \frac{dz}{|1-z|}$$

$$\leq C \left( 1 - e^{-\left( \frac{t}{y} \log(1-\delta) \right)} \right) \leq C \left( 1 - e^{-\frac{t}{y}} \right)$$

and since $t \log |\delta| < 1$,

$$\int_{1-\delta}^{1+\delta} \left| \Lambda \left( \frac{t}{y}, z \right) \right| dz \leq C \left( 1 - e^{-\frac{1}{y}} \right).$$

The function $H$ may then be taken as follows,

$$H(y) = \begin{cases} 
C \| \varphi \|_\infty |g(y)|, \forall y \in (0, 2) \\
C \| \varphi \|_\infty |g(y)| y^{-1}, \forall y > 2.
\end{cases}$$

for a sufficiently large numerical constant $C > 0$, and (3.8) follows.

The proofs of (3.9) and (3.11) are based on very similar arguments and then only the proof of (3.11) is presented in detail. Then, suppose now that $g$ satisfies (3.10) and consider the expression (3.12) for $S(t)g$. By the properties of the fundamental solution obtained in [6], the region of integral at the right hand side of (3.12) where $z$ is away from $z = 1$ tends to zero as $t \to 0$ and it is the region where $z$ is around $z = 1$ that gives $g(x)$. In order to see this in some detail write (3.12) as follows, using the change of variables $x/y = z$,

$$S(t)g(x) = \int_0^{x/a} \Lambda \left( \frac{t}{x}, z \right) g(x/z) \frac{dz}{z} + \int_{x/a}^\infty \Lambda \left( \frac{t}{x}, z \right) g(x/z) \frac{dz}{z} = P_1 + P_2. \quad (3.33)$$
Both terms are treated with similar arguments. Consider first $P_1$. For $x < a$ then $z$ is away from $z = 1$ and two cases are possible. If $t > a$ then $x/t < x/a$ and

$$|P_1| \leq C ||g|| x_a \left( \int_0^{x/t} \left| \Lambda \left( \frac{t z}{x}, z \right) \rho_\theta(x/z) \frac{dz}{z} \right| + \int_{x/t}^{x/a} \left| \Lambda \left( \frac{t z}{x}, z \right) \rho_\theta(x/z) \frac{dz}{z} \right| \right) \tag{3.34}$$

In the first term at the right hand side of (3.38), $t z/x < 1$ and in the second $t z/x < 1$ while $x/t > 1$ in both. By Proposition 3.5 of [6], using the change of variables $x/z = y$, for $\varepsilon > 0$ as small as desired there is a constant $C_\varepsilon > 0$ such that,

$$\int_0^{x/t} \left| \Lambda \left( \frac{t z}{x}, z \right) \rho_\theta(x/z) \frac{dz}{z} \right| \leq C_\varepsilon \int_0^{x/t} \left( z^{-3+\varepsilon} \left( \frac{t z}{x} \right)^{9-\varepsilon} + z^{-5} \left( \frac{t z}{x} \right)^7 \right) \frac{dz}{z(x/z + (x/z)^\theta)} \leq C \left( \frac{t^{3-\varepsilon-\theta}}{x^{3-\varepsilon}} + \frac{t^{5-\theta}}{x^5} \right) \leq C t^{3-\varepsilon-\theta} \tag{3.35}$$

and,

$$\int_{x/t}^{x/a} \left| \Lambda \left( \frac{t z}{x}, z \right) \rho_\theta(x/z) \frac{dz}{z} \right| \leq C \int_{x/t}^{x/a} \frac{z^{-4} dz}{x/z + (x/z)^\theta} \leq C x^{-3} t^{3-\varepsilon-\theta}. \tag{3.36}$$

Then by (3.34) – (3.36),

$$|P_1| \leq C_\varepsilon ||g|| x_a t^{3-\varepsilon-\theta} x^{-3+\varepsilon}, \forall (t, x); 0 < x < a < t. \tag{3.37}$$

If $t \in (0, a)$, then $t z/x < 1$ for all $z \in (0, x/a)$ and by Proposition 3.5, if $x > t$,

$$|P_1| \leq C \int_0^{x/a} \left( z^{-3+\varepsilon} \left( \frac{t z}{x} \right)^{9-\varepsilon} + z^{-5} \left( \frac{t z}{x} \right)^7 \right) \frac{dz}{z(x/z + (x/z)^\theta)} \leq C t^{3-\varepsilon-\theta}, \tag{3.38}$$

and,

$$|P_1| \leq C \int_0^{x/a} \frac{z^{-5} dz}{x/z + (x/z)^\theta} \leq C a^{-6-\theta} t^5 x, \forall x \in (0, t). \tag{3.39}$$

From (3.38), (3.39),

$$|P_1| \leq C \left( t^{3-\varepsilon-\theta} x^{3-\varepsilon} \mathbf{1}_{a > x > t} + a^{-6-\theta} t^5 x \mathbf{1}_{x < t < a} \right). \tag{3.40}$$

Similar arguments show for $x > a$,

$$|P_2| \leq C \left( t^{3-\varepsilon-\theta} x^{-5} + x^{-3+\varepsilon} t^{3+\theta} \mathbf{1}_{t < a < x} \right). \tag{3.41}$$
By Proposition 3.5 of \([6]\), arguing as for \((3.35)\) or \((3.38)\),
\[
\int_0^{1-\delta} \cdots \, dz + \int_{1+\delta}^{x/a} \cdots \, dz \leq C_{\varepsilon} \|g\|_{X_{v\delta}} \left( t^2 x^{-\gamma} + t^2 x^{-\gamma} a^{-2-\theta} \right) \leq C_{\varepsilon} \|g\|_{X_{v\delta}} t^2 \tag{3.42}
\]

Consider next,
\[
\int_{1-\delta}^{1+\delta} A \left( \frac{t z}{x}, z \right) g(x/z) \frac{dz}{z} = \int_{1-\delta}^{1+\delta} \frac{t z}{x} \left| 1 - z |^{2 \alpha} \right|^{-1} \psi(t, z; x) g(x/z) \frac{dz}{z}.
\]
with, for all \(x > 0\),
\[
\psi(t, z; x) = \left( A \left( \frac{t z}{x}, z \right) \frac{x}{t z} \left| z \right|^{-2 \alpha} \right)
(t, z) \mapsto \psi(t, z; x) \text{ belongs to } C((0, 1) \times (1 - \delta, 1 + \delta))
\]

Under the change of variables \(y = 2t \log(1 - z)\), \(z = 1 - e^{y/2t}\)
\[
\int_{1-\delta}^{1} \frac{t z}{x} \left| 1 - z |^{2 \alpha} \right|^{-1} \psi(t, z; x) g(x/z) \frac{dz}{z} =
\]
\[
= \frac{1}{2x} \int_{-\infty}^{2t \log \delta} e^{\frac{y}{2t}} (1 - e^{y/2t}) \psi(t, 1 - e^{y/2t}; x) g \left( \frac{x}{1 - e^{y/2t}} \right) dy.
\]

Similarly, under the change \(y = 2t \log(z - 1)\), \(z = 1 + e^{y/2t}\)
\[
\int_{1}^{1+\delta} \frac{t z}{x} \left| 1 - z |^{2 \alpha} \right|^{-1} \psi(t, z; x) g(x/z) \frac{dz}{z} =
\]
\[
= \frac{1}{2x} \int_{-\infty}^{2t \log \delta} e^{\frac{y}{2t}} (1 + e^{y/2t}) \psi(t, 1 + e^{y/2t}; x) g \left( \frac{x}{1 + e^{y/2t}} \right) dy.
\]

Let us write,
\[
\frac{1}{2x} \int_{-\infty}^{2t \log \delta} e^{\frac{y}{2t}} (1 - e^{y/2t}) \psi(t, 1 - e^{y/2t}; x) g \left( \frac{x}{1 - e^{y/2t}} \right) dy - \frac{g(x)}{2} =
\]
\[
= \frac{1}{2x} \int_{-\infty}^{0} e^{\frac{y}{2t}} (1 - e^{y/2t}) \psi(t, 1 - e^{y/2t}; x) g \left( \frac{x}{1 - e^{y/2t}} \right) dy - \frac{g(x)}{2} +
\]
\[
+ \frac{1}{2x} \int_{0}^{2t \log \delta} e^{\frac{y}{2t}} (1 - e^{y/2t}) \psi(t, 1 - e^{y/2t}; x) g \left( \frac{x}{1 - e^{y/2t}} \right) dy
\]
\[
= \frac{1}{2} \left( \int_{-\infty}^{0} \frac{1}{x} e^{\frac{y}{2t}} (1 - e^{y/2t}) dy - 1 \right) g(x) +
\]
\[
+ \frac{1}{2} \int_{-\infty}^{0} \frac{1}{x} e^{\frac{y}{2t}} (1 - e^{y/2t}) \psi(t, 1 - e^{y/2t}; x) g \left( \frac{x}{1 - e^{y/2t}} \right) - g(x) \right) dy +
\]
\[
+ \frac{1}{2x} \int_{2t \log \delta}^{2t \log \delta} e^{\frac{y}{2t}} (1 - e^{y/2t}) \psi(t, 1 - e^{y/2t}; x) g \left( \frac{x}{1 - e^{y/2t}} \right) dy
\]
and finally,

\[
\frac{1}{2x} \int_{-\infty}^{2\log \delta} e^{\frac{y}{x}} \left( 1 - e^{\frac{x}{y}} \right) g(t, 1 - e^{\frac{x}{y}}; x) \left( \frac{x}{1 - e^{\frac{x}{y}}} \right) dy - \frac{g(x)}{2} = I_1 + I_2 + I_3 + I_4
\]

\[
I_1 = \frac{1}{2} \left( \int_{-\infty}^{0} \frac{1}{x} e^{\frac{y}{x}} \left( 1 - e^{\frac{x}{y}} \right) dy - 1 \right) g(x)
\]

\[
I_2 = \frac{1}{2} \int_{-\infty}^{0} \frac{1}{x} e^{\frac{y}{x}} \left( 1 - e^{\frac{x}{y}} \right) \left( \psi(t, 1 - e^{\frac{x}{y}}; x) - 1 \right) g \left( \frac{x}{1 - e^{\frac{x}{y}}} \right) dy
\]

\[
I_3 = \frac{1}{2} \int_{-\infty}^{0} \frac{1}{x} e^{\frac{y}{x}} \left( 1 - e^{\frac{x}{y}} \right) g \left( \frac{x}{1 - e^{\frac{x}{y}}} \right) \left( g \left( \frac{x}{1 - e^{\frac{x}{y}}} \right) - g(x) \right) dy
\]

\[
I_4 = \frac{1}{2x} \int_{2\log \delta}^{0} e^{\frac{y}{x}} \left( 1 - e^{\frac{x}{y}} \right) \psi(t, 1 - e^{\frac{x}{y}}; x) g \left( \frac{x}{1 - e^{\frac{x}{y}}} \right) dy.
\]

In the first term, the change of variables \( y = xz \) gives

\[
x|I_1(t, x)| \leq C||g||_{X_\theta} \left| \int_{-\infty}^{0} e^{z \left( 1 - e^{\frac{x}{y}} \right)} dz - 1 \right|.
\]

Since

\[
e^{z \left( 1 - e^{\frac{x}{y}} \right)} > e^z, \ \forall z < 0 \implies \int_{-\infty}^{0} e^{z \left( 1 - e^{\frac{x}{y}} \right)} dz > \int_{-\infty}^{0} e^z dz = 1,
\]

it follows,

\[
x|I_1(t, x)| \leq C||g||_{X_\theta} \left( \int_{-\infty}^{0} e^{z \left( 1 - e^{\frac{x}{y}} \right)} dz - 1 \right).
\]

Since \( x > a \), it follows,

\[
e^{-ze^{\frac{x}{y}}} < e^{-ze^{\frac{x}{y}}}
\]

and,

\[
x|I_1(t, x)| \leq C||g||_{X_\theta} \left( \int_{-\infty}^{0} e^{z \left( 1 - e^{\frac{x}{y}} \right)} dz - 1 \right).
\]

In the second term \( I_2 \), using that \( \rho \) is a decreasing function,

\[
\left| \frac{1}{2} \int_{-\infty}^{0} \frac{1}{x} e^{\frac{y}{x}} \left( 1 - e^{\frac{x}{y}} \right) \left( \psi(t, 1 - e^{\frac{x}{y}}; x) - 1 \right) g \left( \frac{x}{1 - e^{\frac{x}{y}}} \right) dy \right| \leq
\]

\[
\leq C||g||_{X_\theta} \rho(x) \int_{-\infty}^{0} \frac{1}{x} e^{\frac{y}{x}} \left( 1 - e^{\frac{x}{y}} \right) \left| \psi(t, 1 - e^{\frac{x}{y}}; x) - 1 \right| \rho \left( \frac{x}{e^{\frac{x}{y}}} \right) dy
\]

\[
\leq C||g||_{X_\theta} \rho(x) \int_{-\infty}^{0} e^{z \left( 1 - e^{\frac{x}{y}} \right)} \left| \psi(t, 1 - e^{\frac{x}{y}}; x) - 1 \right| dz
\]

\[
\leq C||g||_{X_\theta} \rho(x) \int_{-\infty}^{0} e^{z \left( 1 - e^{\frac{x}{y}} \right)} \left| \psi(t, 1 - e^{\frac{x}{y}}; x) - 1 \right| dz
\]

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By Corollary 3.13 in [6]

\[
\lim_{t \to 0} t^{-1} \left| e^{-1/t} Y \right|^{1-2t} \Lambda \left( t, 1 + e^{-1/t} Y \right) = 1
\]  

(3.43)

uniformly for \( Y \) on bounded subsets of \( \mathbb{R} \). Since \( x > a \), when \( t \to 0 \), \( 0 < t z / x t z / a \to 0 \) uniformly for \( x > a \), and \( z \in (1 - \delta, 1) \) and

\[
\lim_{t \to 0} \sup_{z \in (1-\delta(t), 0)} \left| \Lambda \left( \frac{t z}{x}, z \right) \frac{x}{t z} |1 - z|^{1-2z} - 1 \right| = 0.
\]

In the term \( I_3 \), by the mean value Theorem, for some \( r \in \left( \frac{x}{1 - e^{x t}}, \right) \),

\[
\left( g \left( \frac{x}{1 - e^{x t}} \right) - g(x) \right) = \left( \frac{x}{1 - e^{x t}} - x \right) g'(r).
\]

Then,

\[
\left| g \left( \frac{x}{1 - e^{x t}} \right) - g(x) \right| = x \left| \frac{1}{1 - e^{x t}} - 1 \right| |g'(r)|
\]

\[
\leq r \left| \frac{1}{1 - e^{x t}} - 1 \right| |g'(r)| \leq \sup_{r > a} r |g'(r)| \left| \frac{1}{1 - e^{x t}} - 1 \right|.
\]

Since by hypothesis \( r |g'(r)| \leq C r^{-1} \) for \( r > 1 \),

\[
\frac{1}{2} \int_{-\infty}^{0} \frac{1}{e^{x t}} \left| g \left( \frac{x}{1 - e^{x t}} \right) - g(x) \right| dy
\]

\[
\leq C \sup_{r > a} r |g'(r)| \int_{-\infty}^{0} \frac{1}{e^{x t}} \left| \frac{1}{1 - e^{x t}} - 1 \right| dy
\]

\[
\leq C a^{-1} \int_{-\infty}^{0} e^{z (1 - e^{x t})} \left( \frac{1}{1 - e^{x t}} - 1 \right) dz.
\]

(3.44)

Since \( z (1 - e^{x t}) < z (1 - e^{x t}) \) and \( \frac{1}{1 - e^{x t}} < \frac{1}{1 - e^{x t}}, \)

\[
\int_{-\infty}^{0} \frac{1}{e^{x t}} \left| \frac{1}{1 - e^{x t}} - 1 \right| dy = \int_{-\infty}^{0} e^{z (1 - e^{x t})} \left( \frac{1}{1 - e^{x t}} - 1 \right) dz
\]

\[
\leq \int_{-\infty}^{0} e^{z (1 - e^{x t})} \left( \frac{1}{1 - e^{x t}} - 1 \right) dz.
\]

It follows from (3.44), for \( x > a \) and \( 0 < t < a \),

\[
|I_3(t, x)| \leq C x^{-1} \int_{-\infty}^{0} e^{z (1 - e^{x t})} \left( \frac{1}{1 - e^{x t}} - 1 \right) dz.
\]

In the last term \( I_4 \), using again that \( \rho \) is decreasing

\[
|I_4| \leq \frac{1}{2x} \int_{2t \log \delta}^{0} e^{z (1 - e^{x t})} \psi(t, 1 - e^{x t}; x) \left| g \left( \frac{x}{1 - e^{x t}} \right) \right| dy.
\]

\[
\leq C \|g\|_{X_a} \frac{1}{x} \int_{2t \log \delta}^{0} e^{z (1 - e^{x t})} |\psi(t, 1 - e^{x t}; x)| \rho \left( \frac{x}{e^{x t}} \right) dy
\]

\[
\leq C \|g\|_{X_a} \rho(x) \frac{1}{x} \int_{2t \log \delta}^{0} e^{z (1 - e^{x t})} \rho \left( \frac{x}{e^{x t}} \right) dy = C \|g\|_{X_a} \rho(x) \frac{1}{x} \int_{2t \log \delta}^{0} e^{z (1 - e^{x t})} dz
\]

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and since \( \frac{2t \log \delta}{a} > \frac{2t \log \delta}{a} \), \(|I_4| \leq \|g\|_{X_\theta} \int_{\frac{2t \log \delta}{a}}^0 e^{\left(1 - e^{\frac{2t \log \delta}{a}}\right)} dz \). All this shows,

\[
\limsup_{t \to 0} x \left| \int_{1-\delta}^{1+\delta} \frac{tz}{x} |1 - z|^{2t} x^{-1} \psi(t, z; x) g(x/z) \frac{dz}{z} - \frac{g(x)}{2} \right| = 0. \tag{3.45}
\]

A similar argument shows,

\[
\limsup_{t \to 0} x \left| \int_{1-\delta}^{1+\delta} \frac{tz}{x} |1 - z|^{2t} x^{-1} \psi(t, z; x) g(x/z) \frac{dz}{z} - \frac{g(x)}{2} \right| = 0 \tag{3.46}
\]

and then,

\[
\limsup_{t \to 0} x |S(t)g(x) - g(x)| = 0. \tag{3.47}
\]

The term \( P_2 \) when \( x < a \) is treated in the same way as \( P_1 \) for \( x > a \), writing first,

\[
|P_2| \leq \int_{x/a}^{\infty} \Lambda \left( \frac{tz}{x} \right) g(x/z) \frac{dz}{z} = \int_{x/a}^{1-\delta} [\cdots] dz + \int_{1-\delta}^{1+\delta} [\cdots] dz + \int_{1+\delta}^{\infty} [\cdots] dz. \tag{3.48}
\]

In the first and third integrals, two cases arise depending on whether \( x > t \) or \( x < t \). In both cases, use of Proposition 3.1, Proposition 3.2 and Proposition 3.5 of [6] yields the existence of a constant \( C > 0 \) such that,

\[
\int_{x/a}^{1-\delta} [\cdots] dz + \int_{1+\delta}^{\infty} [\cdots] dz \leq C\|g\|_{X_\theta} \left( t^{-\theta} \mathbb{1}_{0<t<x} + x^{-3} t^{3-\theta} \mathbb{1}_{t<x<a} \right), 0 < x < a. \tag{3.49}
\]

The estimate of the second integral in the right hand side of (3.48) follows from the same arguments that lead to (3.45), (3.46) and give,

\[
\limsup_{t \to 0} x^{\theta'} |S(t)g(x) - g(x)| = 0. \tag{3.50}
\]

It follows from (3.49)–(3.50),

\[
\limsup_{t \to 0} x^{\theta'} |S(t)g(x) - g(x)| = 0, \tag{3.51}
\]

and then (3.11), from (3.47) and (3.51). Similar arguments prove (3.9).

For \( g \in X_\theta \), the function \( S(t)g \) defined by (3.1) satisfies the following regularity property,

**Proposition 3.2.** There exists a constant \( C > 0 \) such that if \( g \in X \) and \( f = S(t)g \),

\[
\begin{align*}
\left| \frac{\partial f}{\partial x}(t, x) \right| &\leq C\|g\|_{X_\theta} t^{1-\theta}, \text{ if } 0 < x < t < 1. \tag{3.52} \\
\left| \frac{\partial f}{\partial x}(t, x) \right| &\leq C\|g\|_{X_\theta} t^{-2}, \text{ if } 0 < x < t, t > 1 \tag{3.53} \\
\left| \frac{\partial f}{\partial x}(t, x) \right| &\leq C\|g\|_{X_\theta} x^{-2}, \text{ if } x > t > 1. \tag{3.54}
\end{align*}
\]

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For all $\delta \in (0, 1)$ there exists a constant $C > 0$ depending on $\delta$, such that

\[
\left| \frac{\partial f}{\partial x}(t, x) \right| \leq C ||g||_{X_\sigma} x^{-1+\delta} t^{-\delta-\theta}, \text{ if } 0 < t < x < 1
\]

(3.55)

\[
\left| \frac{\partial f}{\partial x}(t, x) \right| \leq C ||g||_{X_\sigma} \left( x^{-1+\delta} t^{-\delta-\theta} + x^{-2} \right), \text{ if } 0 < t < 1 < x
\]

(3.56)

**Proof.** Suppose first that $x \in (0, t)$ and write

\[
\frac{\partial f}{\partial x}(t, x) = I_1 + I_2
\]

\[
I_1 = \int_0^t \frac{\partial \Lambda}{\partial z} \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y^2}, \quad I_2 = \int_t^\infty \frac{\partial \Lambda}{\partial z} \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y^2}.
\]

In the term $I_1$, $y \in (0, t)$ and then, Proposition 3.4 in [6] may be applied to obtain,

\[
|I_1| \leq C \int_0^t \left( \frac{t}{y} \right)^{-4} \left( A + O \left( \frac{x}{t}^{\delta} \right) \right) g(y) \frac{dy}{y^2} \leq Ct^{-4} \int_0^t g(y) y^2 dy
\]

\[
\leq \begin{cases} 
C ||g||_{X_\sigma} t^{-4} \int_0^1 y^{2-\theta} dy = C ||g||_{X_\sigma} t^{1-\theta}, & t < 1, \\
Ct^{-4} \int_0^1 g(y) y^2 dy + Ct^{-4} \int_1^t g(y) y^2 dy & t > 1
\end{cases}
\]

(3.57)

In the term $I_2$, $t/y < 1$ and then, by (3.32) in Proposition 3.5 of [6], there exists constant $C$ such that,

\[
\left| \frac{\partial \Lambda}{\partial z} \left( \frac{t}{y}, \frac{x}{y} \right) \right| \leq C \left( tx^{-1-\sigma_2^*} y^{\sigma_1^*} + tx^{-1-\sigma_2^*} y^{\sigma_1^*} + t^2 y^{-2} \right), \forall x \in (0, t).
\]

where $\sigma_j^*$ are given real numbers given by (8.1) in the Appendix. Since $y > t > x$ in the integration’s domain of $I_2$ and $\sigma_2^* < \sigma_1^* < 0$ it follows that

\[
tx^{-1-\sigma_2^*} y^{\sigma_1^*} \leq tx^{-1-\sigma_2^*} y^{\sigma_0^*}.
\]

Then,

\[
|I_2| \leq C tx^{-1-\sigma_1^*} \int_t^\infty y^{\sigma_1^* - 2} |g(y)| dy + Ct^2 \int_t^\infty |g(y)| y^{-4} dy
\]

where the first term in the right hand side may estimated as follows. If $t \in (0, 1),$

\[
\int_t^\infty y^{\sigma_1^* - 2} |g(y)| dy \leq \int_1^t y^{\sigma_1^* - 2} |g(y)| dy + \int_1^\infty y^{\sigma_1^* - 2} |g(y)| dy
\]

\[
\leq C ||g||_{X_\sigma} t^{-1+\sigma_1^* - \theta} + ||g||_{X_\sigma} \int_1^\infty y^{\sigma_1^* - 3} dy
\]

\[
\leq C ||g||_{X_\sigma} \left( t^{-1+\sigma_1^* - \theta} + 1 \right),
\]

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and then
\[ tx^{-1-\sigma_1^*} \int_t^\infty y^{\sigma_1^*-2} |g(y)| dy \leq C \| g \|_{X_\sigma} \left( t^{\sigma_1^*-\theta} x^{-1-\sigma_1^*} + tx^{-1-\sigma_1^*} \right). \]

On the other hand,
\[ \int_t^\infty |g(y)| y^{-4} dy \leq \| g \|_{X_\sigma} \int_t^1 y^{-4-\theta} dy + \| g \|_{X_\sigma} \int_1^\infty y^{-4} dy \leq C \| g \|_{X_\sigma} t^{-3-\theta}. \]

Since \( x \in (0, t) \) and \( \sigma_1^* < 0 \), it follows that,
\[ t^{\sigma_1^*-\theta} x^{-1-\sigma_1^*} = t^{-\theta} x^{-1} \left( \frac{x}{t} \right)^{-\sigma_1^*} < t^{-\theta} x^{-1}, \]
\[ tx^{-1-\sigma_1^*} = t^{-\theta} x^{-1} \left( x^{-\sigma_1^*} t^{1+\theta} \right) \leq t^{-\theta} x^{-1} t^{-\sigma_1^*+1+\theta} \leq t^{-\theta} x^{-1}, \]
\[ t^2 t^{-3-\theta} = t^{-1-\theta} \leq t^{-\theta} x^{-1}. \]

Since \( 0 < x < t \leq 1 \) it follows \( tx^{-1-\sigma_1^*} < t^{\sigma_1^*-\theta} x^{-1-\sigma_1^*} < t^{-1-\theta} \), and then,
\[ |I_2| \leq C \| g \|_{X_\sigma} t^{-1-\theta}. \quad (3.58) \]

If on the contrary \( t > 1 \),
\[ |I_2| \leq C t x^{-1-\sigma_1^*} \int_t^\infty y^{\sigma_1^*-2} |g(y)| dy + C t^2 \int_t^\infty |g(y)| y^{-4} dy \]
\[ \leq t x^{-1-\sigma_1^*} \| g \|_{X_\sigma} \int_t^1 y^{\sigma_1^*-3} dy + C t^2 \| g \|_{X_\sigma} \int_1^\infty y^{-5} dy \]
\[ \leq C \| g \|_{X_\sigma} \left( t^{\sigma_1^*-1} x^{-1-\sigma_1^*} + t^{-2} \right). \]

Since \( x < t \) and \( \sigma_1^* < -1 \), it follows that \( t^{\sigma_1^*-1} x^{-1-\sigma_1^*} \leq t^{-2} \) and,
\[ |I_2| \leq C \| g \|_{X_\sigma} t^{-2}. \quad (3.59) \]

This shows (3.52) and (3.53).

If \( x > t > y \), by Proposition 3.4, for \( \varepsilon > 0 \) as small as desired there exists \( C_\varepsilon > 0 \) such that,
\[ \left| \frac{\partial \Lambda}{\partial z} \left( \frac{t}{y} \cdot \frac{x}{y} \right) \right| \leq C_\varepsilon x^{-4} + y^4 x^{-4-\varepsilon} \leq C_\varepsilon x^{-4} y^4, \quad \forall x > t. \]

Then,
\[ |I_1| \leq C x^{-4} \int_0^t y^2 |g(y)| dy \]
\[ \leq \begin{cases} C \| g \|_{X_\sigma} x^{-4} \int_0^t y^{-\theta} dy = C \| g \|_{X_\sigma} x^{-4} t^{3-\theta}, & t < 1, \\ C x^{-4} \int_0^1 g(y) y^2 dy + C x^{-4} \int_1^t g(y) y^2 dy & t > 1 \end{cases} \]
\[ \leq \begin{cases} C v \| g \|_{X_\sigma} x^{-4} t^{3-\theta}, & t < 1, \\ C \| g \|_{X_\sigma} x^{-4} t^2 & t > 1. \end{cases} \quad (3.60) \]
We wish to estimate \( I_2(t, x) \) for \( y > t \) and \( x > t \) and suppose first that \( t \in (0, 1) \). Then
\[
I_2(t, x) = \int_t^1 \frac{\partial \Delta}{\partial z} \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y^2} + \int_1^\infty \frac{\partial \Delta}{\partial z} \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y^2} = K_1(t, x) + K_2(t, x).
\]
By Lemma 6.5 in [7], for all \( \delta_1 \in (0, 1) \) there exists a constant \( C > 0 \) such that
\[
\left| \frac{\partial \Delta}{\partial z} \left( \frac{t}{y}, \frac{x}{y} \right) \right| \leq C \left( \frac{x}{y} \right)^{-1+\delta_1}. \tag{3.61}
\]
Therefore
\[
|K_1(t, x)| \leq C x^{1-\delta_1} \int_t^1 y^{-1-\delta_1} |g(y)|dy \leq C' x^{1-\delta_1} t^{1-\theta} ||g||_{X_\theta}. \tag{3.61}
\]
The way to estimate \( K_2 \) actually depends on whether \( x \in (t, 1) \) or \( x > 1 \). In the first case \( x/y \in (0, 1) \) and since \( t/y \in (0, 1) \), by Lemma 6.1 for \( c' > 1 + \sigma^*_1 \) but as close to \( 1 + \sigma^*_1 \) as desired, there exists a constant \( C > 0 \) such that,
\[
\left| \frac{\partial \Delta}{\partial z} \left( \frac{t}{y}, \frac{x}{y} \right) \right| \leq C \left( \frac{x}{y} \right)^{-c'} + \frac{t}{y}. \tag{3.62}
\]
Then, if \( r < -\sigma^*_1 \), there exists \( c' > 1 + \sigma^*_1 \) such that \( r < 1 - c' \) and
\[
|K_2(t, x)| \leq C x^{-c'} \int_1^\infty |g(y)|y^{c'-2}dy + Ct \int_1^\infty |g(y)|y^{-3}dy \leq C x^{-c'} ||g||_{X_\theta} \int_1^\infty y^{-3}dy + Ct ||g||_{X_\theta} \int_1^\infty y^{-4}dy \leq C ||g||_{X_\theta} \left( x^{-c'} + t \right). \tag{3.63}
\]
It follows from (3.60), (3.61), (3.63), for \( 0 < t < x < 1 \),
\[
|I_1(t, x)| + |I_2(t, x)| \leq C ||g||_{X_\theta} \left( x^{-4\delta_3 - \theta} + x^{-1+\delta_1 t^{-\delta_1 - \theta}} + x^{-c'} + t \right). \tag{3.64}
\]
Since \( x^{-4\delta_3 - \theta} < x^{-1+\delta_1 t^{-\delta_1 - \theta}} \) (because \( x > t \)) and \( x^{-c'} + t \leq \kappa x^{-1+\delta_1 t^{-\delta_1 - \theta}} \) for some numerical constant \( \kappa > 0 \) (since \( 0 < x < 1 \) and \( 0 < t < 1 \), (3.55) follows from (3.61).

If we suppose on the contrary that \( x > 1 \), the \( K_2 \) must again be split in two integrals,
\[
K_2(t, x) = \int_1^x \frac{\partial \Delta}{\partial z} \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y^2} + \int_x^\infty \frac{\partial \Delta}{\partial z} \left( \frac{t}{y}, \frac{x}{y} \right) g(y) \frac{dy}{y^2} = K_{2,1}(t, x) + K_{2,2}(t, x).
\]
In \( K_{2,1}(t, x) \) Lemma 6.5 of [7] is used again to obtain,
\[
|K_{2,1}(t, x)| \leq C x^{1-\delta_1} \int_1^x y^{-1-\delta_1} |g(y)|dy \leq C x^{1-\delta_1} ||g||_{X_\theta} \quad \text{for some } \kappa > 0 \tag{3.65}
\]

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In $K_{2,2}(t, x)$, $x/y \in (0, 1)$ and then again Lemma 8.1 gives, if $r < 1 - c'$,

$$|K_{2,2}(t, x)| \leq Cx^{-c'} \int_x^\infty |g(y)|y^{c'-2}dy + Ct \int_x^\infty |g(y)|y^{-3}dy$$

$$\leq Cx^{-c'} \|g\|_{X_0} \int_x^\infty y^{c'-3}dy + Ct\|g\|_{X_0} \int_x^\infty y^{-4}dy$$

$$\leq C\|g\|_{X_0} (x^{-2} + tx^{-3}) \leq C\|g\|_{X_0} x^{-2}.$$  (3.66)

By (3.65) and (3.66), if $y > t$ and $x > t$, $t \in (0, 1)$ and $x > 1$

$$|K_2(t, x)| \leq C\|g\|_{X_0} (x^{-1+\delta_1} + x^{-2})$$  (3.67)

and,

$$|I_1(t, x)| + |I_2(t, x)| \leq C\|g\|_{X_0} \left( (x^{-4} t^{3-\theta} + x^{-1+\delta_1} t^{-\delta_1-\theta} + x^{-1+\delta_1} + x^{-2} t^{-\delta_2}) \right).$$  (3.68)

Using again that $x > t$, and $t \in (0, 1)$, (3.66) follows from (3.68).

Suppose now that $y > t$, $x > t$ and $t > 1$. Then,

$$I_2(t, x) = \int_t^x \frac{\partial}{\partial z} \left( \frac{t}{y} - \frac{x}{y} \right) g(y) \frac{dy}{y^2} + \int_x^\infty \frac{\partial}{\partial z} \left( \frac{t}{y} - \frac{x}{y} \right) g(y) \frac{dy}{y^2} = I_{2,1}(t, x) + I_{2,2}(t, x).$$

In the first term $I_{2,1}(t, x)$, Lemma 6.5 in [7] gives,

$$|I_{2,1}(t, x)| \leq C x^{-1+\delta_1} \int_t^x y^{-1-\delta_1} |g(y)| \frac{dy}{y}$$

$$\leq C x^{-1+\delta_1} \|g\|_{X_0} \int_t^x y^{-1-\delta_1-\theta} \frac{dy}{y} \leq C\|g\|_{X_0} x^{-1+\delta_1} t^{-\delta_1-\theta}.$$  (3.69)

Estimate (3.62) may be used in $I_{2,2}(t, x)$ to obtain,

$$|I_{2,2}(t, x)| \leq C x^{-c'} \int_x^\infty |g(y)|y^{c'-2}dy + Ct \int_x^\infty |g(y)|y^{-3}dy$$

$$\leq C x^{-c'} \|g\|_{X_0} \int_x^\infty y^{c'-2+r}dy + Ct\|g\|_{X_0} \int_x^\infty y^{-4}dy \leq C\|g\|_{X_0} x^{-2}.$$  (3.70)

By (3.69) and (3.70), for $y > t$ and $x > t > 1$,

$$|I_2(t, x)| \leq C x^{-1+\delta_1} t^{-\delta_1-\theta} M_0(g) + C\|g\|_{X_0} x^{r-1} \leq C\|g\|_{X_0} x^{r-1}.$$  (3.71)

Combination of (3.60) and (3.71) gives

$$|I_1(t, x)| + |I_2(t, x)| \leq C\|g\|_{X_0} (x^{-4} t^2 + x^{-2})$$  (3.72)

and (3.64) follows because $x > t$ and then $x^{-4} t^2 < x^{-2}$. □

**Proposition 3.3.** For all $g \in X_0$, the function $f = S(t)g \in L^\infty_{\text{loc}}((0, \infty); X)$ is such that

$$L(f) \in L^\infty_{\text{loc}}((0, \infty); L^\infty(0, \infty)).$$  (3.73)
For all $\varepsilon > 0$ there exists a constant $C_{\varepsilon,T} > 0$, depending on $\varepsilon$ such that, for all $g \in X_{\theta}$, for all $t \in (0,T)$ and all $x > 0$,

$$\left| \frac{\partial f}{\partial t}(t,x) \right| \leq C_{\varepsilon,T} \|g\|_{X_{\theta}} \zeta_{\varepsilon,\theta}(t,x)$$

(3.74)

where

$$\zeta_{\varepsilon,\theta}(t,x) = \begin{cases} 
  x^{-4}t^{3-\theta} + x^{-3+\varepsilon}t^{4-\theta}, & \text{if } x > 3t \\
  xt^{-2-\theta}, & \text{if } t > \frac{3x}{2} \\
  x^{-1}t^{-\theta}, & \text{if } \frac{2t}{3} < x < 3t
\end{cases}$$

(3.75)

and for almost every $t > 0$ and $x > 0$,

$$\frac{\partial f(t,x)}{\partial t} = L(f(t))(x).$$

Proof. The proof of Proposition 4.3 exactly follows the proof of points (1.34), (1.35) and the property that $u$ satisfies equation (1.1) in Theorem 1.4 of [6]. It is only necessary to replace $\|g\|_1$ by $\|g\|_{X_{\theta}}$ at each step when needed. \qed

4 The Cauchy Problem (2.9), (2.10).

The operator $\Sigma$ is defined in this Section, via the following Proposition.

Proposition 4.1. For all $w_0 \in Y_\theta$, there exists a function $v \in L_{\text{loc}}((0,\infty);Y_\theta) \cap C((0,\infty) \times [0,\infty))$ satisfying (2.9) for almost every $t > 0, \xi > 0$ and the following properties.

1.- For all $T > 0$ there exists a constant $C > 0$ depending on $T$ but not of $w_0$ nor $v$ such that

$$\sup_{0 \leq t \leq T} t^\theta \|v(t)\|_V \leq C, \quad \forall t \in (0,T), \xi > 0.$$

(4.1)

2.- For all $\delta \in (0,1)$ and $T > 0$, there exists a constant $C > 0$ not depending on $w_0$ nor $v$ such that, for all $t \in (0,T)$,

$$\left| \frac{\partial v}{\partial \xi}(t,\xi) \right| \leq C \|w_0\|_{Y_\theta} t^{-\delta-\theta} \xi^{-1+\frac{\delta}{2}} \forall \xi \in (t^2,1),$$

(4.2)

$$\left| \frac{\partial v}{\partial \xi}(t,\xi) \right| \leq C \|w_0\|_{Y_\theta} t^{-\theta} \left( 1 - \log \left( \frac{\xi^{1/2}}{t} \right) \right) \xi^{-1/2}, \forall \xi \in (0,t^2),$$

(4.3)

$$\left| \frac{\partial v}{\partial \xi}(t,\xi) \right| \leq C \|w_0\|_{Y_\theta} \left( t^{-\delta-\theta} \xi^{-1+\frac{\delta}{2}} + \xi^{-\frac{3}{2}} \right), \forall \xi > 1.$$  

(4.4)

4.- For all $\varphi \in C((0,\infty) \cap L^\infty(0,\infty)$

$$\lim_{t \to 0} \int_0^\infty v(t,\xi) \frac{\varphi(\sqrt{\xi})}{\sqrt{\xi}} d\xi = \int_0^\infty w_0(\xi) \frac{\varphi(\sqrt{\xi})}{\sqrt{\xi}} d\xi.$$
If \( w_0 \in Y_\theta \cap C(0, \infty) \),
\[
\lim_{t \to 0} v(t, \xi) = w_0(\xi), \ \forall \xi > 0 \tag{4.5}
\]

If \( w_0 \in C^1(0, \infty) \) is such that
\[
\sup_{\xi > 1} \left( \xi^{3/2} |w_0| + \xi^{1/2} |w_0(\xi)| \right) < \infty, \tag{4.6}
\]
then,
\[
\lim_{t \to 0} ||w(t) - w_0||_{Y_\theta} = 0, \ \forall \theta' > \theta. \tag{4.7}
\]

The problem (2.9), (2.10) is written as a perturbation of (2.11), (2.12) by means of the change of variables,
\[
x = \xi^{1/2}, \ u(t, x) = w(t, \xi), \ \forall t > 0, \xi > 0. \tag{4.8}
\]

In these new variables,
\[
\mathcal{L}(w)(\xi) = 2C_1 \int_0^\infty (u(y) - u(x)) \left( \frac{1}{\sinh |x^2 - y^2|} - \frac{1}{x^2 + y^2} \right) \frac{y^3 \sinh x^2}{x^3 \sinh y^2} dy
\]
\[
= 2C_1 \left( L(u)(x) + \Psi(u)(x) \right), \tag{4.9}
\]
where
\[
\Psi(u)(x) = -u(x) \int_0^\infty T(x, y) dy + \int_0^\infty T(x, y) u(y) dy \tag{4.10}
\]
\[
T(x, y) = \frac{y^3 \sinh x^2}{x^3 \sinh y^2} \left( \frac{1}{\sinh |x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)} \right) - \frac{y}{x} \left( \frac{1}{|x^2 - y^2|} - \frac{1}{x^2 + y^2} \right). \tag{4.11}
\]

The function \( T \) may be split as follows,
\[
T(x, y) = T_1(x, y) + T_2(x, y)
\]
\[
T_1(x, y) = \frac{y}{x} \left( \frac{1}{\sinh |x^2 - y^2|} - \frac{1}{|x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)} + \frac{1}{|x^2 - y^2|} \right)
\]
\[
T_2(x, y) = \frac{y^3}{x^3} \left( \frac{\sinh x^2}{\sinh y^2} - \frac{x^2}{y^2} \right) \left( \frac{1}{\sinh |x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)} \right).
\]

After a simple scaling of the time variable to absorb the constant \( C_1 \), the Cauchy problem (2.9), (2.11) reads in the new variables,
\[
\frac{\partial u(t, x)}{\partial t} = (L(u)(x) + \Psi(u)(x)), \ t > 0, x > 0 \tag{4.12}
\]
\[
u(0, x) = u_0(x), \ x > 0. \tag{4.13}
\]

Problem (4.12), (4.13) was considered in [7] for initial data \( u_0 \in L^1(0, \infty) \). It is solved in this Section for \( u_0 \in X \), and some of its properties are obtained.
Proposition 4.1. For all $\theta \in [0, 1)$, $\Psi$ is a linear bounded operator from $X_\theta$ into itself.

Proof. By definition,
\[
\Psi(g)(x) = -g(x) \int_0^\infty T(x, y)dy + \int_0^\infty T(x, y)g(y)dy \equiv \Psi_1(g)(x) + \Psi_2(g)(x). \tag{4.14}
\]

By Corollary 2.4 of [7], there exists a constant $M > 0$ such that $\int_0^\infty |T(x, y)|dy \leq M$ for all $x \geq 0$. Then $\Psi_1(g) \in X_\theta$ and
\[
||\Psi_1(g)||_{X_\theta} \leq M ||g||_{X_\theta}. \tag{4.15}
\]

On the other hand
\[
\int_0^\infty T(x, y)g(y)dy = \int_1^1 T(x, y)g(y)dy + \int_1^\infty T(x, y)g(y)dy.
\]

with
\[
\left| \int_0^1 T(x, y)g(y)dy \right| \leq ||g||_{X_\theta} \int_0^1 |T(x, y)|y^{-\theta}dy \tag{4.16}
\]
\[
\left| \int_1^\infty T(x, y)g(y)dy \right| \leq ||g||_{X_\theta} \int_1^\infty |T(x, y)|y^{-1}dy \tag{4.17}
\]

The two integrals in the right hand sides of (4.16) and (4.17) must be estimated for $x \in (0, 1)$ and $x > 1$. Consider first the integral in the right hand side of (4.16). If $x \in (0, 1),
\[
\int_0^1 |T(x, y)|y^{-\theta}dy \leq C \int_0^1 (1 + xy)y^{-\theta}dy \leq C. \tag{4.18}
\]

When $x > 1$ and $y \in (0, 1),
\[
|T(x, y)| \leq C \left( \frac{y}{x} \left( \frac{1}{\sinh \frac{3\theta}{4}} + \frac{y^2}{x^4} \right) + \frac{1}{\sinh \frac{3\theta}{4}} \right) \leq C \left( \frac{1}{\sinh \frac{3\theta}{4}} + \frac{y^2}{x^4} \right)
\]

and then,
\[
\int_0^1 |T(x, y)|y^{-\theta}dy \leq C \frac{1}{x^4}. \tag{4.19}
\]

It follows from (4.16), (4.18) and (4.19),
\[
\int_0^1 |T(x, y)|y^{-\theta}dy \leq \frac{C}{1 + x^4}, \forall x > 0, \tag{4.20}
\]
\[
\left| \int_0^1 T(x, y)g(y)dy \right| \leq C ||g||_{X_\theta} \frac{1}{1 + x^4}, \forall x > 0. \tag{4.21}
\]

We must now consider the integral in the right hand side of (4.16). If $x \in (0, 1),
\[
|T(x, y)| \leq Cx \left( \frac{1}{y^3} + \frac{y}{\sinh y^2} \right)
\]
\[
\int_1^\infty |T(x, y)|y^{-1}dy \leq Cx \int_1^\infty \left( \frac{1}{y^3} + \frac{y}{\sinh y^2} \right)y^{-1}dy = Cx. \tag{4.22}
\]
Suppose now that \( x > 1 \). The integral may then be split in three different regions, where \( T(x, y) \) satisfies different estimates.

\[
\int_1^\infty |T(x, y)|y^{-1}dy = \int_{|x-y|<\delta} |T(x, y)|y^{-1}dy + \int_{\delta<|x-y|<x/2} |T(x, y)|y^{-1}dy + \int_{|x-y|>x/2} |T(x, y)|y^{-1}dy = J_1 + J_2 + J_3.
\]

In \( J_1 \), \( T(x, y) \leq C \) for some constant \( C > 0 \) and then,

\[
J_1 \leq Cx^{-1}, \ x > 1. \tag{4.23}
\]

In \( J_2 \),

\[
|T_1(x, y)| \leq \frac{Cy}{x} \left( \frac{2 \min(x, y)^2}{(x^2 - y^2)(x^2 + y^2)} + \frac{1}{\sinh|x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)} \right)
\leq C \left( \frac{1}{x} + \left| \frac{1}{\sinh\frac{|x+y|}{\delta}} - \frac{1}{\sinh(x^2 + y^2)} \right| \right),
\]

\[
|T_2(x, y)| \leq C \left| \frac{1}{\sinh|x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)} \right| \leq C \left( \left| \frac{1}{\sinh\frac{|x+y|}{\delta}} - \frac{1}{\sinh(x^2 + y^2)} \right| \right)
\]

from where,

\[
J_2 \leq \frac{C}{x} \int_{|x-y|<\delta} y^{-1}dy = O(x)^{-1}, \ x > 1. \tag{4.24}
\]

In \( J_3 \), if \( y > 3x/2 \),

\[
|T_1| \leq \frac{Cy}{x} \left( \frac{e^{-C_1 \max(x, y)^2}}{(1 - e^{-\beta(x^2+y^2)})(1 - e^{-C_1 \max(y, x)^2})} \right) + \frac{\min(y, x)^2}{\max(y, x)^4}
\leq C \left( \frac{ye^{-C_1 y^2}}{x} + \frac{x}{y^3} \right) \leq \frac{Cy e^{-C_1 y^2}}{x} + \frac{Cx}{y^3}
\]

\[
|T_2| \leq \frac{Cy^3}{x^3} e^{-(y^2-x^2)} \left( e^{-(y^2-x^2)} + \frac{x^2}{y^2} \right) \leq \frac{Cy^3}{x^3} e^{-\frac{10y^2}{9}} + \frac{Cy}{x} e^{-\frac{5y^2}{9}}
\]

and then,

\[
\int_{3x/2}^\infty |T(x, y)|y^{-1}dy \leq Cx^{-2}, \ x > 1. \tag{4.25}
\]

If on the other hand \( y \in (1, x/2) \),

\[
|T_1(x, y)| \leq \frac{Cy}{x} \left( \frac{1}{\sinh\frac{3x^2}{4}} + \frac{\min(y, x)^2}{\max(y, x)^4} \right) \leq \frac{Cy}{x \sinh\frac{3x^2}{4}} + \frac{Cy^3}{x^3}
\]

\[
|T_2(x, y)| \leq \frac{C}{\sinh\frac{3x^2}{4}}.
\]
from where
\[ \int_0^{x/2} |T(x,y)|y^{-1}dy \leq Cx^{-1}, \quad x > 1 \]
and then \( J_3 \leq Cx^{-2} \). \( \text{(4.26)} \)

It follows from \( \text{(4.23)}, \text{(4.24)} \) and \( \text{(4.26)}, \)
\[ \int_1^\infty |T(x,y)|y^{-1}dy = O(x^{-1}), \quad x > 1 \] \( \text{(4.27)} \)
and from \( \text{(4.17)}, \text{(4.22)} \) and \( \text{(4.27)}, \)
\[ \int_1^\infty |T(x,y)|y^{-1}dy \leq Cx^{-1} + x^{-2}, \quad \forall x > 0 \] \( \text{(4.28)} \)
\[ \int_1^\infty |T(x,y)g(y)|dy \leq C||g||_{X_\theta} e^{Ct^2}, \quad \forall t \in (0,T) \] \( \text{(4.29)} \)

This concludes the proof of Proposition 4.1.

In the next Proposition a mild solution of \( \text{(4.12)}, \text{(4.13)} \) is obtained.

**Proposition 4.2.** For any \( u_0 \in X_\theta \) there exists a unique function \( u \in L^\infty((0,\infty;X_\theta)) \cap C((0,\infty) \times (0,\infty)) \) such that
\[ \forall T > 0, \sup_{0 \leq t \leq T} t^\theta||u(t)||_X < \infty \] \( \text{(4.30)} \)
\[ \forall t > 0, \quad u(t) = S(t)u_0 + \int_0^t S(t-s)\Psi(u(s))ds \in X \] \( \text{(4.31)} \)

that we denote \( u(t) = \mathcal{S}(t)(u_0) \). The function \( u \) also satisfies,
\[ u \in C((0,\infty) \times [0,\infty)) \]
\[ \forall T > 0, \exists C > 0; \]
\[ t^\theta||u(t)||_X \leq C||u_0||_{X_\theta} e^{Ct^2}, \quad \forall t \in (0, T). \] \( \text{(4.32)} \)
\[ ||u(t)||_{X_\theta} \leq C||u_0||_{X_\theta} e^{Ct}, \quad \forall t \in (0, T) \] \( \text{(4.33)} \)

For all \( \varphi \in C(0,\infty) \cap L^\infty(0,\infty) \),
\[ \lim_{t \to 0} \int_0^\infty u(t,x)\varphi(x)dx = \int_0^\infty u_0(x)\varphi(x)dx. \] \( \text{(4.34)} \)

If \( u_0 \in X \cap C(0,\infty) \),
\[ \lim_{t \to 0} u(t,x) = u_0(x), \quad \forall x > 0 \] \( \text{(4.35)} \)

If
\[ u_0 \in C^1(0,\infty), \quad \sup_{x \geq 1} (x^2|u_{0x}(x)| + x|u_0(x)|) < \infty, \] \( \text{(4.36)} \)
then,
\[ \lim_{t \to 0} ||u(t) - u_0||_{X_\theta} = 0. \] \( \text{(4.37)} \)
Proof. Notice first that if \( u \) and \( v \) are two functions satisfying (4.30) and (4.31),

\[
\|u(t) - v(t)\|_{X_\theta} \leq \int_0^t \|S(t-s)\Psi(u(s) - v(s))\|_{X_\theta} ds.
\]

By Proposition 3.1 and Proposition 4.1,

\[
\|u(t) - v(t)\|_{X_\theta} \leq C \int_0^t \|(u(s) - v(s))\|_{X_\theta} ds
\]

and by Gronwall’s Lemma, \( u = v \).

Suppose now that a function \( u \) is such that \( u(t) \in X_\theta \) and, \( \sup_{0\leq t\leq T} t^\theta \|u(t)\|_X < \infty \) for some \( T > 0 \). Then, for all \( t \in (0,T) \) and \( s \in (0,t) \), by Proposition 4.1, \( \Psi(u(s)) \in X_\theta \). By Proposition 3.1, \( S(\cdot)u_0 \in C((0,\infty) \times [0,\infty)) \) and \( S(t-s)\Psi(u(s)) \in C(0,\infty) \) for all \( t > 0 \) and \( s \in (0,t) \). By Proposition 3.1 and Proposition 4.1 given \( t > 0 \) fixed, for all \( x \geq 0 \),

\[
|\langle S(t-s)\Psi(u(s))\rangle(x)| \leq C\|\Psi(u(s))\|_X \Xi(t-s, x)
\]

\[
\leq C\left( \sup_{s \in (0,T)} s^\theta \|u(s)\|_X \right) s^{-\theta} \Xi(t-s, x) \in L^1((0,t), ds)
\]

and then, for all \( t > 0 \),

\[
\int_0^t (S(t-s)\Psi(u(s)))(x)ds \in C([0,\infty)).
\]

On the other hand for , for \( x > 0 \), \( t > t' > 0 \),

\[
\left| \int_0^t S(t-s)\Psi(u(s))(x)ds - \int_0^{t'} S(t'-s)\Psi(u(s))(x)ds \right| \leq \int_0^{t'} |S(t-s)\Psi(u(s))(x)|ds + \int_0^t |S(t-s)\Psi(u(s))(x) - S(t'-s)\Psi(u(s))(x)|ds
\]

By Proposition 3.1 and Proposition 4.1, for all \( \delta > 0 \), \( x \geq \delta \) and \( T > t \),

\[
\left| 1_{(t', t)}(S(t-s)\Psi(u(s)))(x) \right| \leq C\|u(s)\|_X \in L^1(0, T)
\]

\[
1_{(0,t)} \left| S(t-s)\Psi(u(s))(x) - S(t'-s)\Psi(u(s))(x) \right| \leq C\|u(s)\|_X \in L^1(0, T)
\]

By Proposition 3.1 for all \( s \in (0,t') \) and \( x > 0 \) fixed

\[
\lim_{t \to t'} |S(t-s)\Psi(u(s))(x) - S(t'-s)\Psi(u(s))(x)| = 0.
\]

It follows that the map

\[
(t, x) \to \int_0^t (S(t-s)\Psi(u(s)))(x)ds
\]

belongs to \( C((0,\infty) \times (0,\infty)) \).
Consider now the operator
\[
\sigma(u(t,x)) = S(t)u_0(x) + \int_0^t S(t-s)\Psi(u(s))(x)ds
\]
on the space,
\[
E_{R,T} = \left\{ u \in L^\infty_{loc}((0,\infty); X); ||u||_{X_{\sigma},T} < R \right\}
\]
\[
||u||_{X_{\sigma},T} = \sup_{0 \leq t \leq T} t^\sigma ||u(t)||_X.
\]
By Proposition 4.1 and Proposition 3.1, for all \( u \in L^\infty_{loc}((0,\infty); X) \), all \( t > 0 \) and \( x > 0 \),
\[
|\sigma(u)(t)(x)| \leq C||u_0||_{X_\sigma}\Xi(t,x) + C\int_0^t ||u(s)||_{X_{\sigma}}\Xi(t-s,x)ds,
\]
and by (3.5),
\[
x|\sigma(u)(t)(x)| \leq Ct^{-\theta}||u_0||_{X_\sigma} + C||u||_{X_{\sigma},T}t^{2(1-\theta)}, \quad \forall x > 1
\]
\[
x^{\theta}|\sigma(u)(t)(x)| \leq Ct^{-\theta}||u_0||_{X_\sigma} + C||u||_{X_{\sigma},T}t^{1-\theta}, \quad \forall x \in (0,1)
\]
and, then,
\[
||\sigma(u(t))||_{X,T} \leq C||u_0||_{X_\sigma} + C||u||_{X_{\sigma},T}T^{1-\theta}
\]
and, for \( R > 2C||u_0||_{X_\sigma} \), and \( T \in (0,1) \) such that \( CT^{1-\theta} < 1/2 \) it follows \( \sigma(u) \in E_{R,T} \). In order for \( \sigma \) to be a contraction from \( E_{R,T} \) into itself notice,
\[
|\sigma(u)(t)(x) - \sigma(v)(t)(x)| \leq \int_0^t |S(t-s)(u(s) - v(s))(x)|ds \leq C||u - v||_{X_{\sigma},T}T^{1-\theta},
\]
and the contraction property follows if \( CT^{1-\theta} < 1 \). The map \( \sigma \) is then a strict contraction from \( E_{R,T} \) into itself for \( T \) small enough and has a fixed point \( u \in E_{R,T} \) satisfying (4.31).
Since there exists \( C > 0 \) such that for all \( t \in (0,T), x > 0 \), \( \Xi(t,x) \leq Ct^{-\theta} \)
\[
||u(t)(x)|| \leq C||u_0||_{X_\sigma}\Xi(t,x) + C\int_0^t ||u(s)||_{X_\sigma}\Xi(t-s,x)ds
\]
\[
\leq Ct^{-\theta}||u_0||_{X_\sigma} + C\int_0^t ||u(s)||_{X_{\sigma}}(t-s)^{1-\theta}ds, \quad \forall t \in (0,T), \forall x > 0
\]
and, for all \( t \in (0,T) \),
\[
t^{\theta}||u(t)||_{X_\sigma} \leq C||u_0||_{X_\sigma} + Ct^{\theta}\int_0^t ||u(s)||_{X_{\sigma}}(t-s)^{1-\theta}ds.
\]
That may be written, in order to apply Gronwall’s Lemma applied to the function \( \varphi(t) = t^{\theta}||u(t)||_{X_\sigma} \)
\[
\varphi(t) \leq C||u_0||_{X_\sigma} + Ct^{\theta}\int_0^t \varphi(s)(t-s)^{1-\theta}ds
\]
It follows, for some constant $C > 0$,

$$t^\theta ||u(t)||_{X_\theta} \leq C ||u_0||_{X_\theta} e^{Ct^\theta}, \forall t \in (0, T).$$  \hfill (4.42)

Similarly, for some constant $C > 0$,

$$||u(t)||_{X_\theta} \leq C ||u_0||_{X_\theta} + C \int_0^t ||u(s)||_{X_\theta} ds, \forall s \in (0, T)$$

and then,

$$||u(t)||_{X_\theta} \leq C ||u_0||_{X_\theta} e^{Ct}.$$  \hfill (4.43)

By the uniqueness property shown at the beginning of the proof of this Proposition, this solution may be extended to a maximal time interval $T_\ast$. The time to which the solution defined on an interval $(0, T]$ may be extended only depends on $||u(T)||_{X_\theta}$ and then a classical arguments shows that either $T_\ast = \infty$ or $||u(t)||_{X_\theta} \to \infty$ as $t \to T_\ast$. Since $u \in C((0, T) \times [0, \infty))$, estimate (4.43) holds for all $t \in [0, T_\ast)$ from where,

$$\limsup_{t \to T_\ast} ||u(t)||_{X_\theta} \leq C ||u_0||_{X_\theta} e^{CT_\ast}$$  \hfill (4.44)

and since it follows that $T_\ast = \infty$, and $u \in C((0, \infty) \times [0, \infty)) \cap L^\infty((0, \infty); L^\infty(0, \infty))$ is a global solution of (3.31). Estimate (4.4) follows applying Gronwall’s as above. By property (3.3),

$$|u(t, x) - S(t)u_0(x)| \leq C \int_0^t ||u(s)||_{X_\theta} \Xi(t - s, x) ds \leq C \sup_{0 \leq s \leq t} ||u(s)||_{X_\theta} (1 + x)^{-1} t^{1 - \theta}$$

and then,

$$||u(t) - S(t)u_0||_{X_\theta} \leq C \sup_{0 \leq s \leq t} ||u(s)||_{X_\theta} t^{1 - \theta}. $$  \hfill (4.45)

Properties (4.34), (4.35) follow from (3.8), (3.9) respectively. Property (4.37) follows from (3.11) using (4.45).

The function $u$ obtained in Proposition 4.2 satisfies the following regularity estimates.

**Proposition 4.3.** For all $\delta \in (0, 1)$, $\theta \in (0, 1)$, $T > 0$, there exists a positive constant $C > 0$ such that, if $u$ is the function that satisfies (4.31) given by Corollary 4.2 with initial data $u_0 \in X_\theta$ then, for all $t \in (0, T)$ and $x > 0$,

$$\left| \frac{\partial u}{\partial x} (t, x) \right| \leq C ||u_0||_{X_\theta} t^{-\delta - \theta} x^{-1 + \delta}, \forall x \in (t, 1), \hfill (4.46)$$

$$\left| \frac{\partial u}{\partial x} (t, x) \right| \leq C ||u_0||_{X_\theta} t^{-\theta} \left( 1 - \log \left( \frac{x}{t} \right) \right), \forall t \in (0, T), \forall x \in (0, t), \hfill (4.47)$$

$$\left| \frac{\partial u}{\partial x} (t, x) \right| \leq C ||u_0||_{X_\theta} \left( t^{-\delta - \theta} x^{-1 + \delta} + x^{-2} \right) , \forall t \in (0, T), \forall x > 1. \hfill (4.48)$$

**Proof.**

$$\frac{\partial u}{\partial x} (t, x) = \frac{\partial(S(t)u_0)}{\partial x} (x) + \int_0^t \frac{\partial}{\partial x} (S(t - s)F(u(s)))(x) ds$$  \hfill (4.48)
Suppose first \( \theta \in [0, 1) \). Then, by Lemma 3.2 when \( x \in (0, 1) \),

\[
\frac{\partial S(t)u_0}{\partial x}(t, x) \leq C||u_0||_X x^{-\delta} t^{-\delta} x^{-1+\delta}, \forall x \in (t, 1), \quad (4.50)
\]

\[
\frac{\partial S(t)u_0}{\partial x}(t, x) \leq C||u_0||_X x^{-1-\theta}, \forall x \in (0, t), \quad (4.51)
\]

On the other hand, since \( u(s) \in X \) for \( s > 0 \), it follows by Lemma 4.1 that \( F(u(s)) \in X \) too and then, if \( 0 < t < x < 1 \) by Lemma 3.2 with \( \theta = 0 \),

\[
\int_0^t \left| \frac{\partial S(t-s)u(s)}{\partial x}(x) \right| ds \leq C x^{-1+\delta} \int_0^t ||u(s)||_X (t-s)^{-\delta} ds
\]

and by 4.11,

\[
\int_0^t \left| \frac{\partial S(t-s)u(s)}{\partial x}(x) \right| ds \leq C ||u_0||_X x^{-1+\delta} \int_0^t (t-s)^{-\delta} s^{-\theta} ds
\]

\[
\leq C ||u_0||_X x^{-1+\delta} t^{1-\delta-\theta}.
\]

Estimate (4.49) follows.

When \( x \in (0, t) \) with \( t < 1 \), if \( s \in (0, t-x) \) then \( x < t-s \) and then,

\[
\int_0^{t-x} \left| \frac{\partial S(t-s)u(s)}{\partial x}(x) \right| ds \leq C ||u_0||_X x^{-1+\delta} \int_0^{t-x} (t-s)^{-1-\theta} ds
\]

\[
= C ||u_0||_X x^{-\theta} B \left[ 1 - \frac{x}{t}, 1 - \theta, 0 \right].
\]

But if \( s \in (t-x, t) \) then, \( x > t-s \) and

\[
\int_{t-x}^t \left| \frac{\partial S(t-s)u(s)}{\partial x}(x) \right| ds \leq C ||u_0||_X x^{-1+\delta} \int_{t-x}^t (t-s)^{-\delta} s^{-\theta} ds
\]

\[
= C ||u_0||_X t^{1-\delta-\theta} x^{-1+\delta} \left( \frac{\Gamma(1-\delta)\Gamma(1-\theta)}{\Gamma(2-\delta-\theta)} - B \left[ \frac{1}{2}, 1 - \theta, 1 - \delta \right] \right).
\]

By definition, if \( z \in (1/2, 1) \),

\[
B \left[ z, 1 - \theta, 0 \right] = \int_0^{1/2} \sigma^{-\theta}(1-\sigma)^{-1} d\sigma + \int_{1/2}^z \sigma^{-\theta}(1-\sigma)^{-1} d\sigma
\]

\[
\leq \int_0^{1/2} \sigma^{-\theta}(1-\sigma)^{-1} d\sigma + 2^\theta \int_{1/2}^z (1-\sigma)^{-1} d\sigma
\]

\[
= \left( B \left[ \frac{1}{2}, 1 - \theta, 0 \right] - 2^\theta \log 2 \right) - 2^\theta \log(1-z)
\]

but if \( z \in (0, 1/2) \), then \( 1 - z > 1/2 \) and

\[
B \left[ z, 1 - \theta, 0 \right] \leq 2 \int_0^z \sigma^{-\theta} d\sigma = \frac{2z^{1-\theta}}{1-\theta}
\]

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Therefore,
\[
\int_0^{t-x} \left| \frac{\partial S(t-s)u(s)}{\partial x} (x) \right| ds \leq C \|u_0\| X_\theta \left( 1 - \log \left( \frac{x}{t} \right) \right).
\]

On the other hand, if \( z \in (0, 1/2) \),
\[
\Gamma(1-\delta_1)\Gamma(1-\theta) \leq B [1-z, 1-\theta, 1-\delta] = \int_1^{1-z} (1-\sigma)^{-\delta} d\sigma \\
\leq 2^\theta \int_1^{1-z} (1-\sigma)^{-\delta} d\sigma = C z^{1-\delta}.
\]

For \( z \in (1/2, 1) \),
\[
\int_1^{1-z} (1-\sigma)^{-\delta} d\sigma = \int_1^{1/2} (1-\sigma)^{-\delta} d\sigma + \int_1^{1/2} (1+\sigma)^{-\delta} d\sigma \\
\leq 2^\delta \int_1^{1/2} (1-\sigma)^{-\delta} d\sigma + \int_1^{1/2} (1+\sigma)^{-\delta} d\sigma.
\]

and,
\[
\int_0^1 \left| \frac{\partial S(t-s)u(s)}{\partial x} (x) \right| ds \leq C \|u_0\| X_\theta \left( 1 - \theta \right) x^{-1+\delta} \left( \frac{x}{t} \right)^{1-\delta} = C \|u_0\| X_\theta t^{-\theta}.
\]

Then
\[
\left| \frac{\partial u}{\partial x} (t, x) \right| \leq C \|u_0\| X_\theta \left( t^{-1} + \left( 1 - \log \left( \frac{x}{t} \right) \right) + 1 \right) \\
\leq C \|u_0\| X_\theta t^{-\theta} \left( t^{-1} - \log \left( \frac{X}{t} \right) \right)
\]
i. e. (4.47).

Suppose now that \( t \in (0, 1) \), and \( x > 1 \). Then,
\[
\left| \frac{\partial S(t)u_0}{\partial x} (t, x) \right| \leq C \left( t^{-\theta} x^{-1+\delta} + x^{-2} \right) \|u_0\| X_\theta, \quad (4.52)
\]

When \( x > 1 \) and \( t \in (0, 1) \), then \((t-s) < 1 < x\) for all \( s \in (0, t)\) and, there exists a constant \( C > 0 \) such that,
\[
\left| \frac{\partial u}{\partial x} (t, x) \right| \leq C \left( t^{-\theta} x^{-1+\delta} + x^{-2} \right) \|u_0\| X_\theta + \\
\quad \quad + C \int_0^t \left( (t-s)^{-\theta} x^{-1+\delta} + x^{-2} \right) \|F(u(s))\| X ds \\
\leq C \left( t^{-\theta} x^{-1+\delta} + x^{-2} \right) \|u_0\| X_\theta + \\
\quad \quad + C \|u_0\| X_\theta \int_0^t \left( (t-s)^{-\theta} x^{-1+\delta} + x^{-2} \right) s^{-\theta} ds \\
= C \|u_0\| X_\theta \left( t^{-\theta} x^{-1+\delta} + x^{-2} \right) + t^{-\theta} \left( t^{-\theta} x^{-1+\delta} + x^{-2} \right) + t x^{-2} \\
\leq C \|u_0\| X_\theta \left( t^{-\theta} x^{-1+\delta} + x^{-2} \right)
\]

(4.53)
Proposition 4.4. Suppose that $u_0 \in X$ and let $u$ be the function that satisfies (4.31) given by Proposition (4.2). Then, for almost every $t > 0$, $x > 0$,

$$\frac{\partial u(t, x)}{\partial t} = L(u(t))(x).$$  \hspace{1cm} (4.54)

For all $T > 0$ and $\varepsilon > 0$ as small as desired, there exists a constant $C_{\varepsilon,T} > 0$ such that, for all $u_0 \in X$, all $t \in (0, T)$ and $x > 0$,

$$\left| \frac{\partial u(t, x)}{\partial t} \right| \leq C_{\varepsilon,T} \|u_0\|_X (1 + \Xi_{\varepsilon}(t, x)),$$  \hspace{1cm} (4.55)

with

$$\Xi_{\varepsilon}(t, x) = \begin{cases} 
(x^{-4}t^3 + x^{-3+\varepsilon}t^{6-\varepsilon}), & \text{if } x > 3t \\
(t^{-1} + |\log(x/t)| + x^4), & \text{if } t > \frac{3x}{2} \\
(x^{-1} + x^4), & \text{if } 2t^3 < x < 3t.
\end{cases}$$  \hspace{1cm} (4.56)

Proof. Since $u$ satisfies (4.31), for all $t > 0$,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\Psi(u(s))ds$$

By Proposition 3.3, for almost every $t > 0$ and $x > 0$,

$$\frac{\partial}{\partial t} (S(t)u_0(x)) = L(S(t)u_0(x)).$$  \hspace{1cm} (4.57)

Since for all $s > 0$, $\Psi(u(s)) \in X$, for almost every $x > 0$, $t \in (0, T)$ and $s \in (0, t)$, by the same argument,

$$\frac{\partial}{\partial t} [(S(t-s)\Psi(u(s)))(x)] = L(S(t-s)\Psi(u(s)))(x).$$  \hspace{1cm} (4.58)

Let us prove now that the function

$$\psi(t, x) = \int_0^t S(t-s)\Psi(u(s))(x)ds$$

is derivable with respect to $t$ for almost every $x > 0$ and $t > 0$ and

$$\frac{\partial}{\partial t} \psi(t, x) = \Psi(u(t)) + \int_0^t \frac{\partial}{\partial t} (S(t-s)\Psi(u(s))(x))ds.$$  \hspace{1cm} (4.59)

Notice first, for $t > 0$ and $h \in [0, t-s]$,

$$\frac{\psi(t+h, x) - \psi(t, x)}{h} = I_1 + I_2$$

$$I_1 = \int_0^t \left( \frac{(S(t+h-s)\Psi(u(s)))(x) - (S(t-s)\Psi(u(s)))(x)}{h} \right) ds$$

$$I_2 = \frac{1}{h} \int_t^{t+h} (S(t+h-s)\Psi(u(s))(x)ds.$$
By (3.75), (3.74) of Proposition 3.3 above, and Proposition 4.1 for all $T > 0$, $t \in (0,T)$ and $s \in (0,t)$,

$$\left| \frac{\partial}{\partial t}(S(t-s)\Psi(u(s)))(x) \right| \leq C||u(s)||_{X} (x^{-1} + T^{1+\varepsilon-\theta}).$$  

(4.60)

Therefore, by (4.58) and (4.60), for all $x > 0$,

$$\lim_{h \to 0} I_{1} = \int_{0}^{t} \lim_{h \to 0} \left( \frac{S(t+h-s)\Psi(u(s)))(x) - (S(t-s)\Psi(u(s)))(x)}{h} \right) \, ds$$

$$= \int_{0}^{t} L(\Psi(u(s))) \, ds.$$

On the hand,

$$I_{2} - \Psi(u(t))(x) = \frac{1}{h} \int_{t}^{t+h} \left( (S(t+h-s)\Psi(u(s)))(x) - \Psi(u(t))(x) \right) \, ds.$$

By the continuity of $u$ with respect to $t$ and property (3.9), for all $\varepsilon > 0$ there exists $h_{\varepsilon} > 0$ sufficiently small such that for all $h \in (0, h_{\varepsilon})$,

$$\left| (S(t+h-s)\Psi(u(s))(x) - \Psi(u(t))(x) \right| < \varepsilon$$

and then $|I_{2} - \Psi(u(t))(x)| < \varepsilon$ too. This ends the proof of (4.59). By (4.57) and (4.59), (4.54) follows.

Moreover, Proposition 3.3 also yields, for some numerical constant $C_{\varepsilon} > 0$,

$$\left| \frac{\partial(S(t)u_{0})(x)}{\partial t} \right| \leq C_{\varepsilon}||u_{0}||_{X_{\varepsilon,0}}(t,x)$$

(4.61)

$$\left| \frac{\partial(S(t-s)\Psi(u(s))}{\partial t}(x) \right| \leq C_{\varepsilon}||\Psi(u(s))||_{X_{\varepsilon,0}}(t-s,x).$$

(4.62)

with $\zeta_{\varepsilon}$ defined in (3.75), from where it follows by (4.61), (4.62), Proposition 4.1 and 4.1

$$\left| \frac{\partial}{\partial t}(S(t)u_{0})(x) \right| + \int_{0}^{t} \left| \frac{\partial(S(t-s)\Psi(u(s))}{\partial t}(x) \right| \, ds \leq$$

$$\leq C||u_{0}||_{X_{\varepsilon}}(\zeta,0,t,x) + \int_{0}^{t} \zeta,0(t-s,x) \, ds = C||u_{0}||_{X_{\varepsilon}}(t,x).$$

(4.63)

The estimate (4.55), follows now from (4.63), Proposition 4.1 and 4.1.

**Corollary 4.5.** Suppose $u_{0} \in X_{\theta}$ and $u$ is the function given by Proposition 4.2. Then, for all $\varphi \in C_{b}^{1}([0,\infty))$, the map $t \mapsto \int_{0}^{\infty} \varphi(x)u(t,x)dx$ belongs to $W_{loc}^{1,1}(0,\infty)$ and

$$\frac{d}{dt} \int_{0}^{\infty} \varphi(x)u(t,x)dx = \int_{0}^{\infty} \mathcal{L}(u(t))(x)\varphi(x)dx, \forall t > 0.$$

(4.64)

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Proof. If we multiply both sides of (4.31) by $\varphi \in C^1_b([0, \infty))$ and integrate,
\[
\int_0^\infty u(t, x)\varphi(x)dx = \int_0^\infty \varphi(x)(S(t)u_0)(x)dx + \int_0^\infty \varphi(x)f(t)S(t)\Psi(u(s))(x)dsdx.
\]
By (4.55),
\[
\frac{d}{dt} \int_0^\infty u(t, x)\varphi(x)dx \leq \int_0^\infty \left| \frac{\partial u(t, x)}{\partial t} \varphi(x) \right| dx \leq C_c||u_0||_{X_\theta}\int_0^\infty (1 + \zeta, \theta(t, x))|\varphi(x)|dx \leq C||u_0||_{X_\theta}||\varphi||_\infty (1 + t^{-\theta})
\]
It follows from this and (4.11) that the function $t \to \int_0^\infty u(t, x)\varphi(x)dx$ belongs to $W^{1,1}_{loc}(0, \infty)$. By (4.57), (4.58),
\[
\frac{d}{dt} \int_0^\infty u(t, x)\varphi(x)dx = \int_0^\infty \varphi(x)\frac{\partial}{\partial t}(S(t)u_0)(x)dx + \int_0^\infty \varphi(x)\Psi(u(t, x))dx + \int_0^\infty \varphi(x)\int_0^t \frac{\partial}{\partial t}(S(t-s)\Psi(u(s)))(x)dsdx
\]
\[
= \int_0^\infty \varphi(x)L(S(t)u_0)(x)dx + \int_0^\infty \varphi(x)L(S(t)\Psi(u(s)))(x)dx + \int_0^\infty \varphi(x)\int_0^t L(S(t-s)\Psi(u(s)))(x)dsdx
\]
\[
\leq \int_0^\infty \varphi(x)L'(u(s))(x)dx.
\]
\[
\text{Proof of Proposition 4.4.} \text{ Proposition 4.4 follows from Proposition 4.2, Proposition 4.3 and Proposition 4.4 using the change of variables } \xi = x^2, \ u(t, x) = w(t, \xi).
\]

4.1 The behavior of $(\Sigma(t)w_0)(\xi)$ as $\xi \to 0$.

Let us denote the incomplete Beta function as $B[ \cdot, \cdot, \cdot]$. 

Proposition 4.6. Let $\delta > 0$ as small as desired, $T > 0$ be fixed, and denote $v(t) = \mathcal{I}(t)w_0$. Then, there exists $C > 0$ such that, for all $t \in (0, T)$
\[
|v(t, \xi) - b(t)| \leq C||w_0||_{Y_\theta} \left( \frac{\sqrt{\xi}}{t} \right)^{1-\delta} t^{-\theta} + \xi^{1+b}\), \quad \forall \xi \in (0, t^2/2), \quad (4.65)
\]
\[
b(t) = \lambda(w_0; t) + \int_0^t \lambda(\Psi(v(s)); t - s)ds \quad (4.66)
\]
where the function $\lambda$ is defined in Proposition 8.5 of the Appendix, and there exists a constant $C > 0$ such that
\[
|b(t)| \leq C||w_0||_{Y_\theta} t^{-\theta}, \quad \forall t \in (0, T). \quad (4.67)
\]
By definition, \( v(t, \xi) = u(t, \sqrt{\xi}) \) where \( u = u(t, x) \) satisfies,

\[
    u(t, x) = \mathcal{F}(t)u_0(x) \equiv S(t)u_0(x) + \int_0^t S(t-s)(\Psi(u(s))(x)ds, \ t > 0, \ x > 0.
\]

By Proposition 1.5 in [6], for \( \ell(u_0; t) = \lambda(u_0, t) \) with \( \lambda(u_0, t) \) defined in Proposition 8.3 of the Appendix,

\[
|S(t)u_0(x) - \ell(u_0; t)| \leq C \left( t^{-2+\delta} \int_0^t |u_0(y)|dy + t^{5+\delta} \int_t^\infty \frac{|u_0(y)|dy}{y^{\delta}} \right) x^{1-\delta}
\]

\[
\leq C||u_0||_{X_s} t^{-1+\delta-\theta} x^{1-\delta}, \ \forall x \in (0, t/2). \quad (4.68)
\]

On the other hand, for \( x \in (0, t/2) \) and \( s \in (0, t) \),

\[
\int_0^t S(t-s)\Psi(u(s))(x)ds = \int_0^t \int_0^\infty \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds = I_1 + I_2,
\]

\[
I_1 = \int_0^t \int_0^{t-s} \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds,
\]

\[
I_2 = \int_0^t \int_{t-s}^\infty \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds,
\]

Each of the integrals \( I_1 \) and \( I_2 \) are split again as follows,

\[
I_i = I_{i,1} + I_{i,2}, \quad i = 1, 2,
\]

\[
I_{1,1} = \int_0^{t-2x} \int_0^{t-s} \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds,
\]

\[
I_{1,2} = \int_0^t \int_0^{t-s} \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds,
\]

\[
I_{2,1} = \int_0^{t-2x} \int_{t-s}^\infty \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds,
\]

\[
I_{2,2} = \int_0^t \int_{t-s}^\infty \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds,
\]

\[
I_{2,3} = \int_0^{t-2x} \int_{t-s}^\infty \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds.
\]

In the first integral, \( I_{1,1} \), \((t-s)/y > 1\) and \( x/(t-s) < 1/2\) and then, using Proposition 3.1, Proposition 3.2 and Proposition 3.3 of [6], and (4.42),

\[
\left| I_{1,1} - \int_0^t \int_0^{t-s} \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds \right| \leq C \int_{t-x}^t \int_0^{t-s} \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \left| \frac{dy}{y} ds \right|
\]

\[
\leq C||u_0||_{X_s} \int_0^t \int_{t-x}^{t-s} \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \left| \frac{dy}{y} ds \right|
\]

\[
\leq C||u_0||_{X_s} \int_0^t \int_{t-x}^{t-s} (x^{-3}y^2 + x^{-5}(t-s)y^3) dyds \leq \frac{Ct^{4-2\theta}||u_0||_{X_s}}{1 + (t/x)^{4-\theta}}
\]
Arguing as in the Proof of (1.38) of [8]

\[
\left| \int_t^1 \int_0^{t-s} \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} ds - \int_t^1 \ell(\Psi(u(s)); t-s) ds \right| \leq C \sup_{0 \leq s \leq t} ||u_0||_{X_\theta} t^{\delta-\theta} x^{1-\delta},
\]

and then, for all \( T > 0 \) there exists \( C > 0 \) such that, for all \( t \in (0, T) \) and \( x \in (0, t/2) \),

\[
|I_{1,1} - \int_0^t \ell(\Psi(u(s)); t-s) ds| \leq C ||u_0||_{X_\theta} \left( t^{\delta-\theta} x^{1-\delta} + \frac{C t^{1-2\theta}}{1 + (t/x)^{4-\theta}} \right). 
\]

(4.69)

In the second integral \( I_{1,2} \) one still has \((t-s)/y > 1\) but now \(x/(t-s) > 1/2\). Then, Proposition 3.1, Proposition 3.2 and Proposition 3.3 of [8] yield, for all \( T > 0 \) the existence of a constant such that for all \( t \in (0, T) \) and \( x \in (0, t/2) \),

\[
|I_{1,2}| \leq ||u_0||_{X_\theta} \int_{t-2x}^t \int_0^{t-s} \left( x^{-3} y^2 + (t-s)x^{-5} y^3 \right) dy ds \leq \frac{C t^{4-2\theta} ||u_0||_{X_\theta}}{1 + (t/x)^{4-\theta}}. 
\]

(4.70)

Similar arguments in \( I_{2,1} \) and \( I_{2,3} \), and Proposition 3.5 of [8] give,

\[
|I_{2,1}| + |I_{2,3}| \leq C ||u_0||_{X_\theta} x^{1-\theta}. 
\]

(4.71)

In the term \( I_{2,2} \), the change of variables \( x/y = z \) gives

\[
\int_{t-s}^\infty \Lambda \left( \frac{t-s}{y}, \frac{x}{y} \right) \Psi(u(s))(y) \frac{dy}{y} = \int_0^{\frac{t-s}{x}} \Lambda \left( \frac{(t-s)z}{x}, z \right) \Psi(u(s))(x/z) \frac{dz}{z}. 
\]

Since \( x/(t-s) \in (1/2, 2) \) when \( s \in (t-2x, t-(x/2)) \), for \( \delta > 0 \) small,

\[
I_{2,2} = \int_{t-2x}^{t-x} \int_0^{1-\delta} \Lambda \left( \frac{(t-s)z}{x}, z \right) \Psi(u(s))(x/z) \frac{dz}{z} ds + \int_{t-2x}^{t-x} \int_0^{1+\delta} \left[ \cdots \right] \frac{dz}{z} ds + \int_{t-2x}^{t-x} \int_0^{1+\delta} \left[ \cdots \right] \frac{dz}{z} ds. 
\]

Using Proposition 3.5 as in the previous arguments,

\[
\int_{t-2x}^{t-x} \int_0^{1-\delta} \Lambda \left( \frac{(t-s)z}{x}, z \right) \Psi(u(s))(x/z) \frac{dz}{z} ds + \int_0^{\frac{t-s}{x}} \left[ \cdots \right] ds \leq C ||u_0||_{X_\theta} t^{\delta-\theta} x. 
\]

(4.72)

And arguing as before, for \( s \in (t-2x, t-(x/2)) \), and using the change of variables \( y = xz \),

\[
\int_{1-\delta}^{1+\delta} \Lambda \left( \frac{(t-s)z}{x}, z \right) \Psi(u(s))(x/z) \frac{dz}{z} \leq C ||u_0||_{X_\theta} \frac{2x}{y} \int_0^0 e^z \left(1 - e^{\frac{t-s}{2x(z)}} \right) dy \leq C ||u_0||_{X_\theta} \frac{2x}{y} \int_0^0 e^z \left(1 - e^{\frac{t-s}{2x(z)}} \right) dz \leq C ||u_0||_{X_\theta} \rho o(x) \int_0^0 e^{z(1-e^{xz})} dz \leq C ||u_0||_{X_\theta} \rho o(x),
\]

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and then,
\[ \int_{t-2x}^{t-x} \int_{1-\delta}^{1+\delta} \left| \Lambda \left( \frac{(t-s)z}{x}, z \right) \Psi(u(s))(x/z) \right| \frac{dz}{z} ds \leq C ||u_0||_{X_\theta} x^{1-\theta}. \]  

(4.73)

By (4.69) – (4.73),
\[ \left| \int_0^t S(t-s)\Psi(u(s))(x)ds - \int_0^t \ell(\Psi(u(s)); t-s)ds \right| \leq ||u_0||_{X_\theta} \left( t^{\delta-\theta} x^{1-\delta} + \frac{C t^{4-2\theta}}{1 + (t/x)^{\delta-\theta}} + x^{1-\theta} + xt^{-\theta} \right), \]

from where, using also (4.68), for some constant $C > 0$ and all $t \in (0, T)$,
\[ |u(t, x) - b(t)| \leq C ||u_0||_X \left( \left( \frac{x}{t} \right)^{1-\delta} t^{-\theta} + x^{1-\theta} \right), \forall \xi \in (0, t^2/2). \]  

(4.74)

Properties (4.65) and (4.66) follow from (4.74). On the other hand, by Proposition (4.1) and (4.11) in Proposition 4.2, for every $T > 0$ there exists $C > 0$ such that,
\[ ||\Psi(u(s))(x)||_{X_\theta} \leq ||\Psi(u(s))||_{X_\theta} (x^\theta + x)^{-1} \leq C ||u_0||_{X_\theta} (x^\theta + x)^{-1}, \]

and property (4.67) follows from (4.11).

\[ \Box \]

5 Estimate of the non linear term $\Pi(w, w)$.

Proposition 5.1. Suppose that $w$ and $\tilde{w}$ are such that, for some $T > 0$ and $R > 0$,
\[ \sup_{\xi > R} \xi^{1/2} |w(t, \xi)| = r_{\infty}(t) < \infty, \sup_{\xi > R} \xi^{1/2} |\tilde{w}(t, \xi)| = \tilde{r}_{\infty}(t) < \infty, \]  

(5.1)

for $t \in (0, T)$ and for some $\alpha > 1/2$,
\[ \sup_{\xi > R} \xi^\alpha \left| \frac{\partial w(t, \xi)}{\partial \xi} \right| = \rho_{\infty}(t) < \infty, \sup_{\xi > R} \xi^\alpha \left| \frac{\partial \tilde{w}(t, \xi)}{\partial \xi} \right| = \tilde{\rho}_{\infty}(t) < \infty. \]  

(5.2)

for $t \in (0, T)$. Then, there exists a positive constant $C > 0$ such that, for $t \in (0, T)$,
\[ |\Pi(w(t, \tilde{w})(t))| \leq C (r_{\infty}(t) + \rho_{\infty}(t)) \tilde{r}_{\infty}(t) \xi^{-1} \log \xi, \forall \xi > 1. \]  

(5.3)

\[ \text{Proof.} \]

In order to simplify the notation let us denote $N_1(t, \xi)$ for $N_1[w(t), \tilde{w}(t)](\xi)$. The estimates of $N_2(t, \xi)$ and $N_3(t, \xi)$ are straightforward.

\[ |N_1(t, \xi)| \leq \frac{|w(t, \xi)|}{\xi^{1/2}} \tilde{r}_{\infty}(t) \int_\xi^{\infty} \frac{\xi_1^{1/2} d\xi_1}{\sinh^2(\xi_1)} \leq Cr_{\infty}(t) \tilde{r}_{\infty}(t) e^{-2\xi} \xi^{-1}, \xi \to \infty, \]

and

\[ |N_2(t, \xi)| \leq \frac{|w(t, \xi)|}{\xi^{1/2}} \tilde{r}_{\infty}(t) \int_{\xi/2}^{\xi} \frac{\xi^{1/2} d\xi'}{\sinh^2(\xi')}, \xi \to \infty. \]  

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In the estimate for $N_3$ Taylor’s formula gives,
\begin{equation}
H(\xi) - H(\xi_1) = (\xi - \xi_1) \int_0^1 H'(\sigma \xi + (1 - \sigma)\xi_1) \, d\sigma.
\end{equation}
where the $t$ dependence has been dropped to simplify the notation and,
\begin{equation}
H'(\xi) = \frac{w(\xi) + \xi w_\xi(\xi)}{\sinh^2 \xi} - \frac{2\xi w(\xi) \cosh \xi}{\sinh^3 \xi}.
\end{equation}
By hypothesis, for $\xi$ large enough,
\begin{equation}
|w(\xi) + \xi w_\xi(\xi) + 2\xi w(\xi)| \leq r_\infty(t)\xi^{-1/2} + \rho_\infty(t)\xi^{1 - \alpha} + 2r_\infty(t)\xi^{1/2} \leq R_\infty(t)\xi^{1/2}
\end{equation}
for $R_\infty(t) = r_\infty(t) + \rho_\infty(t)$, and then,
\begin{equation}
|H'(t, \sigma \xi + (1 - \sigma)\xi_1)| \leq \frac{CR_\infty(t)(\sigma \xi + (1 - \sigma)\xi_1)^{1/2}}{\sinh^2(\sigma \xi + (1 - \sigma)\xi_1)}.
\end{equation}
Therefore, the integral $I_3$ in the term $N_3$ is estimated as follows.
\begin{equation}
I_3 = \int_\xi^{\infty} \tilde{H}(\xi_1 - \xi) |H(\xi) - H(\xi_1)| \, d\xi_1 \leq \leq CR_\infty(t)\tilde{r}_\infty(t) \int_0^1 \int_\xi^{\infty} \frac{|\xi - \xi_1|^2 (\sigma \xi + (1 - \sigma)\xi_1)^{1/2} d\xi_1 \, d\sigma}{\sinh^2(\xi_1 - \xi) \sinh^2(\sigma \xi + (1 - \sigma)\xi_1)}
\end{equation}
\begin{equation}
= CR_\infty(t)\tilde{r}_\infty(t)\xi^2 \int_0^1 \int_\xi^{\infty} \frac{(\rho - 1)^2 (\sigma + (1 - \sigma)\rho)^{1/2} \rho \, d\rho \, d\sigma}{\sinh^2((\rho - 1)\xi) \sinh^2(\sigma \xi + (1 - \sigma)\rho \xi)}.
\end{equation}
Since $|\sigma + (1 - \sigma)\rho| = (\rho(1 - \sigma) + \sigma)$, if $\sigma \in (1/2, 1)$ then $\rho(1 - \sigma) + \sigma > 1/2$ and, if $\sigma \in (0, 1/2)$, then $\rho(1 - \sigma) + \sigma > \rho/2$. In both cases, $|\sigma + (1 - \sigma)\rho| \geq \xi/2$ as $\xi \to \infty$ and $\sinh^2(\sigma \xi + (1 - \sigma)\rho \xi) > \frac{1}{2} e^{2(\sigma(1 - \rho)\xi + \rho \xi)} / 2$ as $\xi \to \infty$. Then,
\begin{equation}
\int_0^1 \int_\xi^{\infty} \frac{(\rho - 1)^2 (\sigma + (1 - \sigma)\rho)^{1/2} \rho \, d\rho \, d\sigma}{\sinh^2((\rho - 1)\xi) \sinh^2(\sigma \xi + (1 - \sigma)\rho \xi)} \leq C \int_0^1 \int_1^{\infty} \frac{(\rho - 1)^2 (\sigma + (1 - \sigma)\rho)^{1/2} e^{-2(\sigma \xi + (1 - \sigma)\rho \xi)}}{\sinh^2((\rho - 1)\xi)} \, d\rho \, d\sigma
\end{equation}
The integral with respect to $\rho$ may be split in two parts depending on whether $\rho(1 - \xi) > 1$ or $\rho - 1(1 - \xi) \leq 1$. In the first $\sinh^2((\rho - 1)\xi) \geq \frac{1}{2} e^{-2(\rho - 1)}$ for $\xi$ large enough and then,
\begin{equation}
\int_0^1 \int_{1 - \frac{1}{\xi - 1}}^{\infty} (\rho - 1)^2 (\sigma + (1 - \sigma)\rho)^{1/2} e^{-2(\sigma \xi + (1 - \sigma)\rho \xi)} \, d\rho \, d\sigma \leq C \int_1^{\infty} \int_0^{1 - \frac{1}{\xi - 1}} (\rho - 1)^2 (\sigma + (1 - \sigma)\rho)^{1/2} e^{-2(\sigma \xi + (1 - \sigma)\rho \xi)} e^{-2(\rho - 1)} \, d\sigma \, d\rho.
\end{equation}
The integral in the right hand side may be calculated using Mathematica and its behavior as $\xi \to \infty$, is bounded by $Ce^{-2\xi}/3$. from where,
\begin{equation}
\xi^2 \int_0^1 \int_{1 - \frac{1}{\xi - 1}}^{\infty} (\rho - 1)^2 (\sigma + (1 - \sigma)\rho)^{1/2} d\rho d\sigma \leq C \xi^{1/2} e^{-2\xi}, \forall \xi > R.
\end{equation}
The second part of the integral at the right hand side of (5.5) is for $(\rho - 1)\xi < 1$, where \( \sinh 2((\rho - 1)\xi) \geq \frac{1}{2}(\rho - 1)\xi \),

\[
\xi^{7/2} \int_0^1 \int_1^{1+\frac{\xi}{\rho}} \frac{(\rho - 1)^2(\sigma + (1 - \sigma)\rho)^{1/2}e^{-2(\sigma\xi + (1 - \sigma)\rho\xi)}}{\sinh^2((\rho - 1)\xi)} \, d\rho \, d\sigma \leq \\
\leq \xi^{3/2} \int_0^1 \int_1^{1+\frac{\xi}{\rho}} (\sigma + (1 - \sigma)\rho)^{1/2}e^{-2(\sigma\xi + (1 - \sigma)\rho\xi)} \, d\rho \, d\sigma.
\]

with,

\[
\int_1^{1+\frac{\xi}{\rho}} (\sigma + (1 - \sigma)\rho)^{1/2}e^{-2(\sigma\xi + (1 - \sigma)\rho\xi)} \, d\rho = \frac{1}{1 - \sigma} \left( E[-1/2, 2]\xi - \left( \frac{1 - \sigma + \xi}{\xi} E[-1/2, 2 - 2\sigma + 2\xi] \right) \right) \frac{1}{1 - \sigma} \leq \\
\leq \frac{e^{-2\xi}}{\xi} \left( 1 + \frac{4\xi - 1}{4\xi} (1 - \sigma) + C(1 - \sigma)^2 \right).
\]

For all \( \delta \in (0, 1/2) \) there exists \( R_1 > 1 \) and a constant \( C > 0 \) depending on \( \delta \) and \( R_1 \) such that for \( \xi > R_1 \) and \( \sigma \in (0, \delta) \),

\[
\left( E[-1/2, 2]\xi - \left( \frac{1 - \sigma + \xi}{\xi} E[-1/2, 2 - 2\sigma + 2\xi] \right) \right) \frac{1}{1 - \sigma} \leq \\
\leq \frac{e^{-2\xi}}{2(1 - \sigma)}(1 - e^{2(\delta - 1)}) + C\xi^{-2}
\]

On the other hand, for \( \sigma \in (\delta, 1) \) there exists \( R_2 > 0 \) and a constant \( C > 0 \) depending on \( R_2 \) and \( \delta \) such that for \( \xi > R_2 \) and \( \sigma \in (\delta, 1) \),

\[
\left( E[-1/2, 2]\xi - \left( \frac{1 - \sigma + \xi}{\xi} E[-1/2, 2 - 2\sigma + 2\xi] \right) \right) \frac{1}{1 - \sigma} \leq \\
\leq \frac{e^{-2\xi}}{2(1 - \sigma)}(1 - e^{2(\delta - 1)}) + C\xi^{-2}
\]

It follows that, for \( \xi \) large enough,

\[
\left| \int_0^1 \left( E[-1/2, 2]\xi - \left( \frac{1 - \sigma + \xi}{\xi} E[-1/2, 2 - 2\sigma + 2\xi] \right) \frac{d\sigma}{1 - \sigma} \right) \leq Ce^{-2\xi}\xi^{-1},
\]

and then

\[
\xi^{7/2} \int_0^1 \int_1^{1+\frac{\xi}{\rho}} \frac{(\rho - 1)^2(\sigma + (1 - \sigma)\rho)^{1/2}e^{-2(\sigma\xi + (1 - \sigma)\rho\xi)}}{\sinh^2((\rho - 1)\xi)} \, d\rho \, d\sigma \leq C\xi^{3/2}e^{-2\xi}
\]

It follows that

\[
|N_3(t, \xi)| \leq CR_\infty(t)\tilde{\tau}_\infty(t)\xi^{-1}, \quad \xi > R.
\]
In order to estimate $N_1$ consider the integral,

$$I_1(t, \xi) = \int_{\xi/2}^{\xi} \bar{H}(\xi_1) (H(\xi - \xi_1) - H(\xi)) \, d\xi_1$$

$$= \int_{\xi/2}^{\xi} \bar{H}(\xi - \xi_1) (H(\xi) - H(\xi)) \, d\xi_1$$

and, then, as in $I_1$,

$$|I_1(t, \xi)| \leq CR_\infty(t)\bar{r}_\infty(t)\xi^2 \int_{1/2}^{1} \int_{0}^{1} (\rho - 1)^2 (\sigma + (1 - \sigma)\rho)^{1/2} \times$$

$$\times \sinh^2((\rho - 1)\xi) \times e^{-2(\sigma\xi + (1 - \sigma)\rho\xi)} \, d\sigma d\rho.$$ 

and using again $\sinh((\rho - 1)\xi) \geq \frac{1}{2}(\rho - 1)\xi$,

$$|I_1(t, \xi)| \leq CR_\infty(t)\bar{r}_\infty(t)\xi^2 \int_{1/2}^{1} \int_{0}^{1} (\sigma + (1 - \sigma)\rho)^{1/2} \times$$

$$\times e^{-2(\sigma\xi + (1 - \sigma)\rho\xi)} \, d\sigma d\rho$$

$$= CR_\infty(t)\bar{r}_\infty(t)\xi^3 \int_{1/2}^{1} \left( \rho^{3/2} E[-1/2, 2\rho\xi] - E[-1/2, 2\xi] \right) (1 - \rho)^{-1} \, d\rho$$

(5.6)

The integral in the right hand side of (5.6) is now split in two parts where $\rho \in (1/2, 1 - 1/\xi)$ and $\rho \in (1 - 1/\xi, 1)$. In the first, we use that for all $R > 2$ there exists $C(R) > 0$ such that for all $\xi > R$ and $\rho \in (1/2, 1 - 1/\xi)$,

$$|\rho^{3/2} E[-1/2, 2\rho\xi] - E[-1/2, 2\xi]| (1 - \rho)^{-1} \leq \frac{e^{-2\xi}}{2\xi(1 - \rho)} \left( 1 + \rho^{1/2} + C\xi^{-1/2} \right)$$

and then, for $\xi$ large enough,

$$\left| \int_{1/2}^{1} \left( \rho^{3/2} E[-1/2, 2\rho\xi] - E[-1/2, 2\xi] \right) (1 - \rho)^{-1} \, d\rho \right| \leq C \frac{e^{-2\xi} \log \xi}{2\xi}.$$

(5.7)

In the second part of the integral at the right hand side of (5.6) we use that since $\sigma \in (0, 1)$ and $\rho \in (1/2, 1)$, $(\sigma + (1 - \sigma)\rho)^{1/2} \leq \sqrt{2}$ and,

$$CR_\infty(t)\bar{r}_\infty(t)\xi^3 \int_{1/2}^{1} \int_{0}^{1} (\sigma + (1 - \sigma)\rho)^{1/2} e^{-2(\sigma\xi + (1 - \sigma)\rho\xi)} \, d\sigma d\rho$$

(5.8)

$$\leq CR_\infty(t)\bar{r}_\infty(t)\xi^3 \int_{1/2}^{1} \int_{0}^{1} e^{-2(\sigma\xi + (1 - \sigma)\rho\xi)} \, d\sigma d\rho$$

$$= CR_\infty(t)\bar{r}_\infty(t)\xi^3 \int_{1/2}^{1} \frac{e^{-2\rho\xi} - e^{-2\xi}}{2\xi - 2\xi\rho} \, d\rho \leq CR_\infty(t)\bar{r}_\infty(t) e^{-2\xi^{1/2}}.$$ 

(5.9)

It follows from (5.6), (5.7) and (5.9) for $\xi$ large enough,

$$|I_1(t, \xi)| \leq CR_\infty(t)\bar{r}_\infty(t) \frac{\log \xi}{\xi}.$$ 

□

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Proposition 5.2. Suppose that \( w \) and \( \tilde{w} \) satisfy (5.1), (5.2) and \( w(t) \in C([0, 1]), \tilde{w}(t) \in C([0, 1]) \). Suppose also that, for some constant \( \beta < 1 \) and \( T > 0 \),

\[
\sup_{\xi \in (0, 1)} \left| \frac{\partial w(t, \xi)}{\partial \xi} \right| = \rho_0(t) < \infty, \ t \in (0, T) \tag{5.10}
\]

\[
\sup_{\xi \in (0, 1)} \left| \frac{\partial \tilde{w}(t, \xi)}{\partial \xi} \right| = \tilde{\rho}_0(t) < \infty, \ t \in (0, T) \tag{5.11}
\]

Then there exists a constant \( C > 0 \) such that, for all \( \xi \in (0, 1) \) and \( t \in (0, T) \),

\[
|\Pi(w(t), \tilde{w}(t))(\xi)| \leq C |w(t)||\tilde{w}(t)||\xi^{\frac{1}{2}}| \log \xi| + \left( |w(t)|^\infty (\rho_0(t) + \tilde{\rho}_0(t)) + |\tilde{w}(t)|^\infty (\rho_0(t) + \rho_\infty(t)) \right) \xi^{\frac{1}{2} - \beta}.
\]

In order to prove Proposition 5.10 we denote \( a(t) = w(t, 0) = \lim_{\xi \to 0} w(t, \xi) \), and \( \kappa(t, \xi) \) be the function such that,

\[
w(t, \xi) = a(t) + \kappa(t, \xi) \ \forall \xi \in (0, 1). \tag{5.12}
\]

Then,

\[
|w(t, \xi) - a(t)| \leq \int_0^\xi |w_\xi(t, \zeta)|d\zeta,
\]

by hypothesis,

\[
|w(t, \xi) - a(t)| \leq \int_0^\xi |w_\xi(t, \zeta)|d\zeta \leq \frac{\rho_0(t)\xi^{1 - \beta}}{1 - \beta}, \ \forall \xi \in (0, 1)
\]

and then,

\[
w(t, \xi) = a(t) + \kappa(t, \xi), \ |\kappa(t, \xi)| \leq C \rho_0(t)\xi^{1 - \beta}, \ \forall t \in (0, 1), \ \forall \xi \in (0, 1) \tag{5.13}
\]

A similar expression holds with \( \tilde{a} = \tilde{w}(t, 0) \) and \( \tilde{\kappa}(t, \xi) = \tilde{w}(t, \xi) - \tilde{a} \). Proposition 5.2 follows from the estimates of \( N_1(t, \xi) - N_2(t, \xi) \) and \( N_3(t, \xi) - N_4(t, \xi) \) obtained in the two next Lemmas.

Lemma 5.3.

\[
|N_1(t, \xi) - N_2(t, \xi)| \leq C \sup_{0 \leq \xi \leq 1} \left( |\tilde{w}(t, \xi)| \right) \rho_0(t)\xi^{\frac{1}{2} - \beta}.
\]

**Proof.** In the term \( N_2 \) write

\[
J_2 = \int_0^{\xi/2} \frac{(\xi - \xi_1)w(t, \xi - \xi_1)}{\sinh^2(\xi - \xi_1)} d\xi_1 =
\]

\[
= \int_0^{\xi/2} \left( \frac{(\xi - \xi_1)a(t)}{\sinh^2(\xi - \xi_1)} + \frac{(\xi - \xi_1)(w(t, \xi - \xi_1) - a(t))}{\sinh^2(\xi - \xi_1)} \right) d\xi_1
\]

\[
= a(t) \int_0^{\xi/2} \frac{(\xi - \xi_1)d\xi_1}{\sinh^2(\xi - \xi_1)} + J_{2,1}(t, \xi)
\]

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with

$$|J_{2,1}(t, \xi)| \leq \int_{0}^{\xi/2} \frac{|\xi - \xi_1||w(t, \xi - \xi_1) - a(t)|}{\sinh^2(\xi - \xi_1)} d\xi_1.$$ 

By hypothesis

$$|w(t, \xi - \xi_1) - a(t)| \leq C \rho_0(t) \int_{0}^{\xi - \xi_1} s^{-\beta} d\zeta = C \rho_0(t)|\xi - \xi_1|^{1-\beta}.$$ 

and, as \(\xi \to 0\),

$$|J_{2,1}(t, \xi)| \leq C \rho_0(t) \int_{0}^{\xi/2} \frac{|\xi - \xi_1|^{2-\beta}}{\sinh^2(\xi - \xi_1)} d\xi_1 = C \rho_0(t) \left(\xi^{1-\beta} + O(\xi)^{3-\beta}\right).$$

Then,

$$J_2(t, \xi) = \frac{a(t) \tilde{w}(t, \xi)}{\xi^{1/2}} \int_{\xi/2}^{\xi} \frac{\xi_1 d\xi_1}{\sinh^2 \xi_1} + \frac{\tilde{w}(t, \xi)}{\xi^{1/2}} J_{2,1}(t, \xi)$$

$$= \frac{a(t) \tilde{a}(t)}{\xi^{1/2}} \int_{\xi/2}^{\xi} \frac{\xi_1 d\xi_1}{\sinh^2 \xi_1} + \frac{a(t) \tilde{\kappa}(t, \xi)}{\xi^{1/2}} \int_{\xi/2}^{\xi} \frac{\xi_1 d\xi_1}{\sinh^2 \xi_1} + \frac{\tilde{w}(t, \xi)}{\xi^{1/2}} J_{2,1}(t, \xi)$$

and

$$\left|J_2(t, \xi) - \frac{a(t) \tilde{a}(t) \log 2}{\sqrt{\xi}}\right| \leq C \sup_{0 \leq \xi \leq 1} |\tilde{w}(t, \xi)| \rho_0(t) \xi^{\frac{3}{2}-\beta}. \quad (5.14)$$

Similar arguments hold for \(N_1\). First,

$$J_1(t, \xi) = \int_{0}^{\xi/2} \frac{\xi_1 w(t, \xi_1)}{\sinh^2 \xi_1} \left(\frac{(\xi - \xi_1)w(t, \xi - \xi_1)}{\sinh^2(\xi - \xi_1)} - \frac{\xi w(t, \xi)}{\sinh^2 \xi}\right) d\xi_1 =$$

$$= \int_{\xi/2}^{\xi} \frac{(\xi - \xi_1)(\tilde{a}(t) + (w(t, \xi - \xi_1) - \tilde{a}(t)))}{\sinh^2(\xi - \xi_1)} \times$$

$$\times \left(\frac{\xi_1 w(t, \xi_1)}{\sinh^2 \xi_1} - \frac{\xi w(t, \xi)}{\sinh^2 \xi}\right) d\xi_1$$

$$= \int_{\xi/2}^{\xi} \frac{(\xi - \xi_1)}{\sinh^2(\xi - \xi_1)} \left(\frac{\xi_1 w(t, \xi_1)}{\sinh^2 \xi_1} - \frac{\xi w(t, \xi)}{\sinh^2 \xi}\right) (\tilde{a}(t) + \tilde{\kappa}(t, \xi - \xi_1)) d\xi_1$$

Since,

$$\frac{\xi_1 w(t, \xi_1)}{\sinh^2 \xi_1} - \frac{\xi w(t, \xi)}{\sinh^2 \xi} = \left(\frac{\xi_1}{\sinh^2 \xi} - \frac{\xi}{\sinh^2 \xi}\right) w(t, \xi_1) + \frac{\xi (w(t, \xi_1) - w(t, \xi))}{\sinh^2 \xi}$$

it follows,

$$J_1(t, \xi) = \int_{\xi/2}^{\xi} \frac{(\xi - \xi_1)}{\sinh^2(\xi - \xi_1)} \left(\frac{\xi_1}{\sinh^2 \xi} - \frac{\xi}{\sinh^2 \xi}\right) (a(t) + \kappa(t, \xi_1)) +$$

$$+ \frac{\xi (w(t, \xi_1) - w(t, \xi))}{\sinh^2 \xi} (\tilde{a}(t) + \tilde{\kappa}(t, \xi - \xi_1)) d\xi_1$$

$$= J_{1,1}(t, \xi) + J_{1,2}(t, \xi) + J_{1,3}(t, \xi).$$
Use of (5.13) yields,

\[ J_{1,1}(t, \xi) = a(t) \tilde{a}(t) \int_{\xi/2}^{\xi} \left( \frac{\xi - \xi_1}{\sinh^2(\xi - \xi_1)} \right) \left( \frac{\xi_1}{\sinh^2 \xi_1} - \frac{\xi}{\sinh^2 \xi} \right) d\xi \]

\[ J_{1,2}(t, \xi) = \tilde{a}(t) \int_{\xi/2}^{\xi} \left( \frac{\xi - \xi_1}{\sinh^2(\xi - \xi_1)} \right) \left( \frac{\xi_1}{\sinh^2 \xi_1} - \frac{\xi}{\sinh^2 \xi} \right) \kappa(t, \xi_1) d\xi \]

\[ J_{1,3}(t, \xi) = \frac{\xi}{\sinh^2 \xi} \int_{\xi/2}^{\xi} \left( \frac{\xi - \xi_1}{\sinh^2(\xi - \xi_1)} \right) \frac{1}{\sinh^2(\xi - \xi_1)} \tilde{w}(t, \xi - \xi_1) d\xi. \]

The first term \( J_{1,1}(t, \xi) \) may be calculated by Mathematica and its asymptotic behavior as \( \xi \to 0 \) is,

\[ J_{1,1}(t, \xi) = a(t) \tilde{a}(t) \left( \log \frac{2}{\xi} + \mathcal{O}(\xi) \right), \quad \xi \to 0. \]

In \( J_{1,2}(t, \xi) \) the following is used,

\[ \frac{\xi_1}{\sinh^2 \xi_1} - \frac{\xi}{\sinh^2 \xi} = \left( \frac{1}{\xi_1} - \frac{1}{\xi} \right) + \left( \frac{1}{\xi_1^2} + \frac{2 \xi \coth \xi}{\sinh^2 \xi} \right)(\xi_1 - \xi) + \]

\[ + (\xi_1 - \xi)^2 \left( \frac{\xi}{5} + \mathcal{O}(\xi)^2 \right), \quad \xi \to 0 \]

to obtain

\[ |J_{1,2}(t, \xi)| \leq |\tilde{a}(t)| \left( \int_{\xi/2}^{\xi} (\xi - \xi_1)^{-1} \left( \frac{1}{\xi_1} - \frac{1}{\xi} \right) \kappa(t, \xi_1) d\xi_1 + \right. \]

\[ + \left( \frac{1}{\xi_1^2} + \frac{1}{\sinh^2 \xi} - \frac{2 \xi \coth \xi}{\sinh^2 \xi} \right) \int_{\xi/2}^{\xi} \kappa(t, \xi_1) d\xi_1 + C \xi \int_{\xi/2}^{\xi} (\xi_1 - \xi) \kappa(t, \xi_1) d\xi_1 \]

where,

\[ \frac{1}{\xi_1^2} + \frac{1}{\sinh^2 \xi} - \frac{2 \xi \coth \xi}{\sinh^2 \xi} = -\frac{1}{3} + \mathcal{O}(\xi)^2, \quad \xi \to 0. \]

Use of (5.13) yields,

\[ |J_{1,2}(t, \xi)| \leq C \rho_0(t)| \tilde{a}(t)| \left( \int_{\xi/2}^{\xi} (\xi - \xi_1)^{-1} \left( \frac{1}{\xi_1} - \frac{1}{\xi} \right) \xi^{1-\beta} d\xi_1 + \right. \]

\[ + C \rho_0(t) \left( \frac{1}{\xi_1^2} + \frac{1}{\sinh^2 \xi} - \frac{2 \xi \coth \xi}{\sinh^2 \xi} \right) \int_{\xi/2}^{\xi} \xi^{1-\beta} d\xi_1 + C \rho_0(t) \xi \int_{\xi/2}^{\xi} (\xi_1 - \xi) \xi^{1-\beta} d\xi_1 \]

\[ \leq C \rho_0(t)| \tilde{a}(t)| \xi^{-\beta}. \]

In \( J_{1,3}(t, \xi) \)

\[ |w(t, \xi) - w(t, \xi_1)| \leq \int_{\xi_1}^{\xi} \left| \frac{\partial w(t, \xi)}{\partial \xi} \right| d\xi \leq C \rho_0(t) \left| \xi^{1-\beta} - \xi_1^{1-\beta} \right| \]

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then
\[ |J_{1,3}(t, \xi)| \leq \frac{C \rho_0(t) ||\tilde{w}(t)||_{L^\infty(0,1)}}{\sinh^2 \xi} \int_{\xi/2}^\xi \frac{|\xi - \xi_1|^{2-\beta}}{\sinh^2(\xi - \xi_1)} d\xi_1 \leq C \rho_0(t) ||\tilde{w}(t)||_{L^\infty(0,1)} \xi^{-\beta} \]

It follows,
\[ |N_1(t, \xi) - \frac{a(t)\tilde{w}(t) \log \frac{2}{\xi}}{\sqrt{\xi}}| \leq C \sup_{0 \leq \xi \leq 1} |\tilde{w}(t, \xi)| \rho_0(t) \xi^{\frac{1}{2} - \beta}. \tag{5.15} \]

and Lemma 5.3 follows from (5.14), (5.15). □

The estimate of \( N_3(t, \xi) - N_4(t, \xi) \) is obtained from the same arguments, although slightly more involved. In the next Lemma, the shorthand notation \( w = w(t, \xi), w_1 = w(t, \xi_1), w_2 = w(t, \xi_1 - \xi) \) is used.

Lemma 5.4.
\[ |N_3 - N_4| \leq C ||\tilde{w}(t)||_{L^\infty} \xi^{\frac{1}{2} - \beta} \left(||w(t)||_{L^\infty} \log \xi + \rho_0(t) + \rho_\infty(t)\right), \quad \forall \xi \in (0,1) \tag{5.16} \]

Proof. Define,
\[
J(t, \xi, \xi_1) = \frac{\sinh^2 \xi}{\xi^{3/2}} \frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) - \frac{w}{\xi^{1/2}} \frac{\xi_1 \tilde{w}_1}{\sinh^2 \xi_1} = K_1(t, \xi, \xi_1) + K_2(t, \xi, \xi_1) \tag{5.17}
\]
\[
K_1(t, \xi, \xi_1) = \sqrt{\xi} \frac{\xi_2 \tilde{w}_2}{\sinh^4 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) - \frac{w}{\xi^{1/2}} \frac{\xi_1 \tilde{w}_1}{\sinh^2 \xi_1} \tag{5.18}
\]
\[
K_2(t, \xi, \xi_1) = \frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) \mathcal{O}(\xi^{3/2}, \xi \to 0). \tag{5.19}
\]

and \( N_3(t, \xi) - N_4(t, \xi) \) is then written as,
\[
N_3(t, \xi) - N_4(t, \xi) = \int_\xi^\infty K_1(t, \xi, \xi_1) d\xi_1 + \int_\xi^\infty K_2(t, \xi, \xi_1) d\xi_1. \tag{5.20}
\]

Consider first the second term in the right hand side of (5.20). For all \( \rho > 0 \),
\[
\int_\xi^\infty \frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) d\xi_1 \leq \int_\xi^\rho [\cdots] d\xi_1 + \int_\rho^\infty [\cdots] d\xi_1. \tag{5.21}
\]

In the second term at the right hand side of (5.21), \( \xi_1 > \rho \) and \( \xi \in (0, \rho/2), \xi_2 > \xi_1/2 \) then
\[
\left| \int_\rho^\infty \frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) d\xi_1 \right| \leq C ||w(t)||_{L^\infty} ||\tilde{w}(t)||_{L^\infty} \times
\]
\[
\times \int_\rho^\infty \frac{\xi_1}{\sinh^2(\xi_1/2)} \left( \frac{\xi w}{\sinh^2 \xi} + \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) d\xi_1 \leq C ||w(t)||_{L^\infty} ||\tilde{w}(t)||_{L^\infty} \xi^{-1}, \xi \to 0,
\]
from where,
\[
\xi^{3/2} \int_\rho^\infty \frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) d\xi_1 \leq C ||w(t)||_{L^\infty} ||\tilde{w}(t)||_{L^\infty} \xi, \xi \to 0. \tag{5.22}
\]
In the first term at the right hand side of (5.21),
\[
\frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} = \left( \frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1} \right) w + (w - w_1) \frac{\xi_1}{\sinh^2 \xi_1}
\]
A “brute force argument” (Mathematica) gives,
\[
\int_\xi^\rho \frac{\xi_2}{\sinh^2 \xi_2} \left( \frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1} \right) d\xi_1 = O \left( \frac{|\log \xi|}{\xi} \right), \quad \xi \to 0.
\]
By hypothesis,
\[
|w - w_1| \leq \int_\xi^{\xi_1} |\partial_\xi w(t, \zeta)| d\zeta \leq C \rho_0(t) \left( \xi_1^{1-\beta} - \xi^{1-\beta} \right) \leq C \rho_0(t)(\xi_1 - \xi)^{1-\beta}
\]
and then,
\[
\int_\xi^\rho \frac{\xi_2}{\sinh^2 \xi_2} \frac{\xi_1}{\sinh^2 \xi_1} d\xi_1 \leq \frac{C t^{-\alpha}}{1 - \beta} \int_\xi^\rho (\xi_1 - \xi)^{-\beta} \xi_1^{-1} d\xi_1 \leq \frac{C \rho_0(t)}{(1 - \beta) \xi^\beta} \int_\xi^\rho (\xi_1 - \xi)^{-\beta} d\xi_1 \leq \frac{C \rho_0(t)}{(1 - \beta) \xi^\beta}.
\]
It follows,
\[
\left| \int_\xi^\rho \frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) d\xi_1 \right| \leq C ||w(t)||_\infty ||\tilde{w}(t)||_\infty \left( \frac{|\log \xi|}{\xi} \right) + \frac{C ||\tilde{w}(t)||_\infty \rho_0(t)}{(1 - \beta) \xi^\beta}.
\]
Then,
\[
\xi^{3/2} \left| \int_\xi^\rho \frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) d\xi_1 \right| \leq C ||w(t)||_\infty ||\tilde{w}(t)||_\infty \xi^{1/2} |\log \xi| + C ||\tilde{w}(t)||_\infty \rho_0(t) \xi^{3/2 - \beta}.
\]
and, from (5.22), (5.23), there exists a constant $C > 0$ such that for $\xi \in (0, 1)$
\[
\int_\xi^\infty K_2(t, \xi, \xi_1) d\xi_1 \leq C \left( ||w(t)||_\infty ||\tilde{w}(t)||_\infty \xi^{1/2} |\log \xi| + ||\tilde{w}(t)||_\infty \rho_0(t) \xi^{3/2 - \beta} \right).
\]
We are then left with the first term in the right hand side of (5.20). The integral is here again split in two parts,
\[
\int_\xi^\infty K_1(t, \xi, \xi_1) d\xi_1 = \int_\xi^R K_1(t, \xi, \xi_1) d\xi_1 + \int_\xi^\infty K_1(t, \xi, \xi_1) d\xi_1.
\]
In the second term of (5.26),
\[
K_1(t, \xi, \xi_1) = \frac{\sqrt{\xi} \xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi_2}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) - w_1 \frac{\xi_1 \tilde{w}_1}{\xi^{1/2} \sinh^2 \xi_1}
\]
\[
= w \left( \frac{\xi^{3/2}}{\sinh^2 \xi} - \frac{1}{\sqrt{\xi}} \right) \frac{\xi_1 \tilde{w}_1}{\sinh^2 \xi_1} + \frac{\xi^{3/2}}{\sinh^2 \xi} \left( \frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} - \frac{\xi_1 \tilde{w}_1}{\sinh^2 \xi_1} \right) - \frac{\sqrt{\xi} \xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \frac{\xi_1 w_1}{\sinh^2 \xi_1}.
\]
\[
\text{By Taylor's formula,}
\]
\[
\frac{\xi_2}{\sinh^2 \xi_2} - \frac{\xi_1}{\sinh^2 \xi_1} = -\xi \int_0^1 h(\xi - \sigma \xi) d\sigma,
\]
\[
h(\xi) = \frac{\partial}{\partial \xi} \left( \frac{\xi}{\sinh^2 \xi} \right) = \frac{1 - 2 \xi \coth \xi}{\sinh^2 \xi},
\]
and by hypothesis,
\[
|\tilde{w}_2 - \tilde{w}_1| \leq \int_{\xi_1 - \xi}^{\xi_1} \left| \frac{\partial \tilde{w}(t, \xi)}{\partial \xi} \right| d\xi \leq (\bar{\rho}_0(t) + \bar{\rho}_\infty(t))\xi,
\]
from where,
\[
\left| \int_{R}^{\infty} K_1(t, \xi, \xi_1) d\xi \right| \leq C \xi^{3/2} ||w(t)||_\infty ||\tilde{w}(t)||_\infty + \frac{C \xi^{3/2} ||w(t)||_\infty}{\sinh^2 \xi} \times
\]
\[
\times \left( \xi \left| \int_0^1 h(\xi - \sigma \xi) d\sigma \right| ||\tilde{w}||_\infty + (\bar{\rho}_0(t) + \bar{\rho}_\infty(t))\xi \right) \quad (5.27)
\]
Use of Mathematica gives,
\[
\int_R^{\infty} h(\xi - \sigma \xi) d\xi = \int_R^{\infty} \frac{1 - 2(\xi_1 - \sigma \xi) \coth(\xi_1 - \sigma \xi)}{\sinh^2(\xi_1 - \sigma \xi)} d\xi_1
\]
\[
= M \left( \frac{-R + \sigma \xi}{\sinh^2(R - \sigma \xi)} \right)^2
\]
\[
\int_0^1 \left( \frac{-R + \sigma \xi}{\sinh^2(R - \sigma \xi)} \right)^2 d\sigma = \xi^{-1} \left( \xi \coth(R - \xi) + \log \left( \frac{\sinh(R - \xi)}{\sinh R} \right) \right) -
\]
\[
- \frac{R \sinh \xi}{\sinh R \sinh (R - \xi)} = - \frac{R}{\sinh^2 R} + \mathcal{O}(\xi), \quad \xi \to 0.
\]
By (5.27), for all $\varepsilon \in (0, 1)$ there exists a positive constant $C$ such that for all $\xi \in (0, \varepsilon)$,
\[
\left| \int_R K_1(t, \xi, \xi_1) d\xi_1 \right| \leq C\|w(t)\|_{\infty} (||\tilde{w}(t)||_{\infty} + (\tilde{\rho}_0(t) + \tilde{\rho}_\infty(t))) \xi^{1/2}, \forall \xi \in (0, \infty). \quad (5.28)
\]

In order to estimate the first term at the right hand side of (5.25) the integrand of $N_3$ is written in terms of $a(t)$ and $\kappa$ as follows,
\[
\begin{align*}
\xi_2 &= \xi_1 - \xi, \quad w_2 = w(t, \xi_2), \quad w_1 = w(t, \xi_1), \quad w = w(t, \xi), \\
\kappa_2 &= \kappa_2(t, \xi_2), \quad \kappa_1 = \kappa_1(t, \xi_1), \quad \kappa = \kappa(t, \xi), \quad a = a(t)
\end{align*}
\]

\[
\begin{align*}
\frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) &= \frac{\xi_2(a + \kappa_2)}{\sinh^2 \xi_2} \left( \frac{\xi(a + \kappa)}{\sinh^2 \xi} - \frac{\xi_1(a + \kappa_1)}{\sinh^2 \xi_1} \right) + \\
&\quad + \frac{\tilde{a} \xi_2}{\sinh^2 \xi_2} \left( \frac{\xi \kappa}{\sinh^2 \xi} - \frac{\xi_1 \kappa_1}{\sinh^2 \xi_1} \right)
\end{align*}
\]

The second term in the right hand side may now be split to give,
\[
\begin{align*}
\frac{\xi \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) &= \frac{\xi \tilde{a} \xi_2}{\sinh^2 \xi_2} \left( \frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1} \right) + \\
&\quad + \frac{\xi_2 \tilde{a} \kappa_2}{\sinh^2 \xi_2} \left( \frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1} \right) \left( \frac{\xi \kappa}{\sinh^2 \xi} - \frac{\xi_1 \kappa_1}{\sinh^2 \xi_1} \right) + \\
&\quad + \frac{\tilde{a} \xi_2}{\sinh^2 \xi_2} \left( \frac{\xi \kappa}{\sinh^2 \xi} - \frac{\xi_1 \kappa_1}{\sinh^2 \xi_1} \right)
\end{align*}
\]

The two last terms in the right hand side may now be put together to give,
\[
\begin{align*}
\xi_2 \tilde{w}_2 \left( \frac{\xi w}{\sinh^2 \xi} - \frac{\xi_1 w_1}{\sinh^2 \xi_1} \right) &= \xi_2 \tilde{a} \xi_2 \left( \frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1} \right) + \\
&\quad + \xi_2 \tilde{a} \kappa_2 \left( \frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1} \right) \left( \frac{\xi \kappa}{\sinh^2 \xi} - \frac{\xi_1 \kappa_1}{\sinh^2 \xi_1} \right) + \\
&\quad + \tilde{a} \xi_2 \left( \frac{\xi \kappa}{\sinh^2 \xi} - \frac{\xi_1 \kappa_1}{\sinh^2 \xi_1} \right). \quad (5.29)
\end{align*}
\]

Similarly, in $N_4(t, \xi),$
\[
\begin{align*}
\frac{\xi_1 w_1}{\sinh^2 \xi_1} &= (a + \kappa) \frac{\xi_1(a + \kappa_1)}{\sinh^2 \xi_1} = \frac{a^2 \xi_1}{\sinh^2 \xi_1} + \frac{a \xi_1 \kappa_1}{\sinh^2 \xi_1} + \frac{\kappa_1 w_1}{\sinh^2 \xi_1}. \quad (5.30)
\end{align*}
\]

Use of (5.29) and (5.30) permits to write $K_1(t, \xi, \xi_1)$ as,
\[
\begin{align*}
K_1(t, \xi, \xi_1) &= \frac{a \sqrt{\xi} \xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left( \frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1} \right) + \frac{a \sqrt{\xi} \xi_2 \tilde{a} \kappa_2}{\sinh^2 \xi_2} \left( \frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1} \right) + \\
&\quad + \frac{\sqrt{\xi} \xi_2 \tilde{a} \kappa_1}{\sinh^2 \xi_2} \left( \frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1} \right) - \frac{\tilde{a} \xi_2}{\sqrt{\xi} \sinh^2 \xi_1} - \frac{a \xi_1 \kappa_1}{\sqrt{\xi} \sinh^2 \xi_1} - \frac{\kappa \xi_1 w_1}{\sqrt{\xi} \sinh^2 \xi_1},
\end{align*}
\]
and then,

\[ K_1(t, \xi, \xi_1) = a\tilde{a}\left(\frac{\sqrt{\xi} \xi_2}{\sinh^2 \xi_2} \left(\frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1}{\sqrt{\xi} \sinh^2 \xi_1}\right) + \]

\[ + \left(\frac{a\sqrt{\xi} \xi_2 \tilde{\kappa}_2}{\sinh^2 \xi_2} \left(\frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1}\right) - \frac{a\xi_1 \tilde{\kappa}_1}{\sqrt{\xi} \sinh^2 \xi_1}\right) + \]

\[ + \frac{\sqrt{\xi} \xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left(\frac{\xi \kappa}{\sinh^2 \xi} - \frac{\xi_1 \kappa_1}{\sinh^2 \xi_1}\right) - \frac{\kappa_1 \tilde{w}_1}{\sqrt{\xi} \sinh^2 \xi_1} \right], \quad (5.31) \]

When the third term in the right hand side of (5.31) is written as:

\[ \frac{\xi \kappa}{\sinh^2 \xi} - \frac{\xi_1 \kappa_1}{\sinh^2 \xi_1} = \left(\frac{\xi}{\sinh^2 \xi} - \frac{\xi_1}{\sinh^2 \xi_1}\right) \kappa + \frac{\xi_1}{\sinh^2 \xi_1} (\kappa - \kappa_1) \]

it follows,

\[ K_1(t, \xi, \xi_1) = a\tilde{a}\left(\frac{\xi_2}{\sinh^2 \xi_2} \left(\frac{\xi^2}{\sinh^2 \xi} - \frac{\xi \xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1}{\sinh^2 \xi_1}\right) + \]

\[ + \frac{a}{\sqrt{\xi}} \left(\frac{\xi_2 \tilde{\kappa}_2}{\sinh^2 \xi_2} \left(\frac{\xi^2}{\sinh^2 \xi} - \frac{\xi \xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1 \tilde{\kappa}_1}{\sinh^2 \xi_1}\right) + \]

\[ + \frac{\kappa}{\sqrt{\xi}} \left(\frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left(\frac{\xi^2}{\sinh^2 \xi} - \frac{\xi \xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1 \tilde{w}_1}{\sinh^2 \xi_1}\right) + \frac{\sqrt{\xi} \xi_2 \tilde{w}_2 (\kappa - \kappa_1)}{\sinh^2 \xi_2 \sinh^2 \xi_1}. \quad (5.32) \]

The way second and third terms are written still need to be slightly modified. In the second, adding and subtracting \(\frac{a\xi\tilde{\kappa}_1}{\sqrt{\xi} \sinh^2 \xi_1}\),

\[ \frac{a}{\sqrt{\xi}} \left(\frac{\xi_2 \tilde{\kappa}_2}{\sinh^2 \xi_2} \left(\frac{\xi^2}{\sinh^2 \xi} - \frac{\xi \xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1 \tilde{\kappa}_1}{\sinh^2 \xi_1}\right) = \frac{a\xi_2 \tilde{\kappa}_1}{\sqrt{\xi} \sinh^2 \xi_1} - \frac{a\xi_1 \tilde{\kappa}_1}{\sqrt{\xi} \sinh^2 \xi_1} + \]

\[ + \frac{a}{\sqrt{\xi}} \left(\frac{\xi_2 \tilde{\kappa}_2}{\sinh^2 \xi_2} \left(\frac{\xi^2}{\sinh^2 \xi} - \frac{\xi \xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1 \tilde{\kappa}_1}{\sinh^2 \xi_1}\right) \]

\[ = \frac{\kappa_1 (\tilde{\kappa}_2 - \tilde{\kappa}_1)}{\sqrt{\xi} \sinh^2 \xi_1} \left(\frac{\xi_2}{\sinh^2 \xi_2} \left(\frac{\xi^2}{\sinh^2 \xi} - \frac{\xi \xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1}{\sinh^2 \xi_1}\right). \]

In the third term, adding and subtracting \(\frac{\xi_1 \tilde{w}_2 \xi}{\sqrt{\xi} \sinh^2 \xi_1}\)

\[ \frac{\kappa}{\sqrt{\xi}} \left(\frac{\xi_2 \tilde{w}_2}{\sinh^2 \xi_2} \left(\frac{\xi^2}{\sinh^2 \xi} - \frac{\xi \xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1 \tilde{w}_1}{\sinh^2 \xi_1}\right) = \]

\[ = \frac{\kappa}{\sqrt{\xi}} \left(\frac{\xi_2}{\sinh^2 \xi_2} \left(\frac{\xi^2}{\sinh^2 \xi} - \frac{\xi \xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1}{\sinh^2 \xi_1}\right) \tilde{w}_2 + \frac{\xi_1 (\tilde{w}_2 - \tilde{w}_1) \kappa}{\sqrt{\xi} \sinh^2 \xi_1}. \]

Therefore, after introduction of the function,

\[ \Phi(\xi, \xi_1) = \frac{\xi_2}{\sinh^2 \xi_2} \left(\frac{\xi^2}{\sinh^2 \xi} - \frac{\xi \xi_1}{\sinh^2 \xi_1}\right) - \frac{\xi_1}{\sinh^2 \xi_1} \]
Then, every $\varepsilon > 0$ small there exists a constant $C > 0$ such that the integral of the different terms in the right hand side of (5.33) give the following. The first parenthesis:

$$\int_R \frac{1}{\sqrt{\xi}} \left| \int_{\xi}^{R} (a^{2} \Phi(\xi, \xi_{1}) + a\kappa_{2} \Phi(\xi, \xi_{1}) + \kappa \xi_{2} \Phi(\xi, \xi_{1})) \xi_{1} d\xi_{1} \right| \leq \xi \leq \xi_{1}(\xi_{1} + \beta) \int_{\xi}^{R} \frac{\xi_{1} d\xi_{1}}{\sinh^{2} \xi_{1}}$$

Since by hypothesis,

$$\int_{\xi}^{R} \frac{\xi_{1}(a + \beta)(\bar{w}_{2} - \bar{w}_{1})}{\sqrt{\xi} \sinh^{2} \xi_{1}} \leq \xi_{1}(a + \beta) \int_{\xi}^{R} \frac{\xi_{1} d\xi_{1}}{\sinh^{2} \xi_{1}}$$

the integral of the second term in the right hand side of (5.33),

$$\int_{\xi}^{R} \frac{\xi_{1}(a + \beta)(\bar{w}_{2} - \bar{w}_{1})}{\sqrt{\xi} \sinh^{2} \xi_{1}} \leq \xi_{1}(a + \beta) \int_{\xi}^{R} \frac{\xi_{1} d\xi_{1}}{\sinh^{2} \xi_{1}}$$

The integral of the last term in the right hand side of (5.33) gives

$$\int_{\xi}^{R} \frac{\sqrt{\xi} \xi_{2} \xi_{1}(w - w_{1})}{\sinh^{2} \xi_{2} \sinh^{2} \xi_{1}} \leq \int_{\xi}^{R} \frac{\xi_{1} d\xi_{1}}{\sinh^{2} \xi_{1}} \leq \int_{\xi}^{R} \frac{\xi_{1} d\xi_{1}}{(1 - \xi) \beta \xi_{1}}$$

It follows from (5.33), (5.34), (5.35), (5.36)

$$\int_{\xi}^{R} K_{1}(\xi, \xi_{1}) d\xi \leq C \|ar{w}(\xi)\|_{\infty} \xi^{\frac{1}{2} - \beta} \left( |w(\xi)| \log |\xi| + (\rho_{0}(t) + \rho_{\infty}(t)) \right)$$
and, from \(5.20, 5.24, 5.28, 5.37\)

\[|N_3 - N_4| \leq C||\tilde{w}(t)||_{\infty}\xi^{\frac{1}{2} - \beta} (||u(t)||_{\infty} |\log \xi| + \rho_0(t) + \rho_\infty), \forall \xi \in (0, \varepsilon).\] (5.38)

Proposition follows from \(5.24, 5.38\) and \(5.20\).

6 The equation (2.4).

**Proposition 6.1.** For all \(\theta \in [0, 1/2]\), \(\varepsilon_1 \in (\theta, 1/2)\), \(\delta_1 = 1 - 2\varepsilon_1\) and \(\delta_2 \in (\theta + \delta_1, 1 - \theta)\) and all \(w_0 \in Y_\theta\) there exists \(T_{\max} > 0\) and a unique function \(w \in Z_{\epsilon_1,T}\) for all \(T \in (0, T_{\max})\) such that

\[w(t, \xi) = (\Sigma(t)w_0)(\xi) + \int_0^t (\Sigma(t-s)\Pi(w(s), w(s))) \xi ds, \forall \xi > 0.\] (6.1)

and either \(T_{\max} = \infty\) or \(||w(t)||_{Y_\theta} \to \infty\). Moreover, for all \(\varphi \in C(0, \infty) \cap L^\infty(0, \infty)\),

\[\lim_{t \to T_{\max}} \int_0^\infty w(t, \xi)\varphi(\xi^{1/2})\xi^{-1/2}d\xi = \int_0^\infty w_0(\xi)\varphi(\xi^{1/2})\xi^{-1/2}d\xi.\] (6.2)

If \(w_0 \in Y \cap C(0, \infty)\),

\[\lim_{t \to 0} w(t, \xi) = u_0(\xi), \forall \xi > 0\] (6.3)

If

\[w_0 \in C^1(0, \infty) \text{ and } \sup_{\xi > 1} \left(\xi^{3/2}|w'_0(\xi)| + \xi^{1/2}|w_0(\xi)|\right) < \infty,\] (6.4)

then,

\[\lim_{t \to 0} ||w(t) - w_0||_{Y_\theta} = 0, \forall \theta' > \theta.\] (6.5)

**Proof.** Consider the operator \(T\) defined as

\[T(w)(t, \xi) = (\Sigma(t)w_0)(\xi) + \int_0^t \left(\Sigma(t-s)\Pi(w(s), w(s))\right) \xi ds.\]

If \(w_0 \in Y\), the function \(u_0\) defined as \(u_0(x) = w_0(x^2)\) is such that \(u_0 \in X\), by Corollary 4.2 \(T(t)u_0\) is well defined and \(||T(t)u_0||_X \leq Ct^{-\theta}||u_0||_{X_\theta}\). By definition,

\[(\Sigma(t)w_0)(\xi) = (T(t)w_0)(\sqrt{\xi})\] (6.6)

\[|\Sigma(t-s)\Pi(w(s))|(\xi) = (T(t-s)h(s))(\sqrt{\xi})\] (6.7)

where \(h(t, x) = \Pi(w(t), w(t))(x^2)\). If \(w_0 \in Y_\theta\), by Proposition 4.2,

\[||(\Sigma(t)w_0)||_Y = ||(T(t)u_0)||_X \leq Ct^{-\theta}||u_0||_{X_\theta}, \forall t \in (0, 1).\]
For $\varepsilon \in (\theta, 1/2)$ consider
\[
P_{\delta_1}(w_\xi) = \sup_{\xi > 0} \varepsilon^{1-\frac{\delta_1}{2}} \left| \frac{\partial w(t, \xi)}{\partial \xi} \right|, \quad \delta_1 = 1 - 2\varepsilon_1. \tag{6.8}
\]
and, for $R > 0$, $T \in (0, 1)$ define for $\delta_2 \in (\delta_1, 1 - \theta)$,
\[
B_{R,T,\varepsilon_1} = \{ w \in C((0, T), Y); ||w||_{1,T} < R \}
\]
\[
||w||_{1,T} = \sup_{0 < t < T} t^\theta ||w(t)||_Y + t^{\delta_2} P_{\delta_1}(w_\xi(t)).
\]
Suppose that $w \in Z_{\varepsilon_1,T}$. Since $w(t) \in Y$ it follows by definition that $|w(t, \xi)| \leq ||w(t)||_Y \xi^{-1/2}$ for $\xi > 1$ and then, by Proposition [5.1] with $\alpha = 1 - \frac{\delta_2}{2} > 1/2$, for all $\xi > 1$,
\[
||\Pi(w(t), w(t))|| \leq C \left( ||w(t)||_Y t^{-\theta} + P_{\delta_1}(w_\xi(t)) t^{-\delta_2} \right) t^{-\theta} ||w(t)||_Y \xi^{-1} \log \xi. \tag{6.9}
\]
On the other hand, by Proposition [5.2] with $\beta = 1 - \frac{\delta_2}{2} = \frac{1}{2} - \varepsilon_1$, for all $\xi \in (0, 1)$,
\[
||\Pi(w(t), w(t))|| \leq C \left( ||w(t)||_Y^2 t^{-\theta} \xi^{1/2 - \beta} \log \xi + ||w(t)||_{1, T} P_{\delta_1}(w_\xi(t)) t^{-\delta_2} \xi^{1/2 - \beta} \right) \tag{6.10}
\]
and it follows from [6.9], [6.11] that $\Pi(w(t)) \in Y$ with
\[
||\Pi(w(t), w(t))||_Y \leq C ||w(t)||_Y t^{-\theta} \left( t^{-\theta} ||w(t)||_Y + P_{\delta_1}(w_\xi(t)) t^{-\delta_2} \right). \tag{6.11}
\]
The function $h \in X$ defined as
\[
h(t, x) = \Pi(w(t), w(t))(x^2) \tag{6.12}
\]
satisfies then
\[
||h(t)||_X = \sup_{x > 0} (x^\theta + x) ||\Pi(w(t), w(t))(x^2)|| = ||\Pi(w(t), w(t))||_Y
\]
and by [6.11], for all $w \in B_{R,T,\varepsilon_1}$
\[
||h(t)||_X \leq C ||w(t)||_Y t^{-\theta} \left( ||w(t)||_Y t^{-\theta} + P_{\delta_1}(w_\xi(t)) t^{-\delta_2} \right) \leq CR^2 \left( t^{2\theta} + t^{-\delta_2 - \theta} \right).
\]
On the other hand, by Proposition [4.2], for all $t > 0$ and $s \in (0, t)$, $\Sigma(t-s)\Pi(w(s)) = \mathcal{S}(t-s)h(s) \in X$ and,
\[
||\Sigma(t-s)\Pi(w(s), w(s))||_Y = ||\mathcal{S}(t-s)h(s)||_X \leq CR^2 \left( s^{-2\theta} + s^{-\delta_2 - \theta} \right).
\]
Both terms of this inequality may be integrated on $s \in (0, t)$
\[
\int_0^t ||\Sigma(t-s)\Pi(w(s), w(s))||_Y ds = \int_0^t ||\mathcal{S}(t-s)h(s)||_X ds
\]
\[
\leq CR^2 \int_0^t \left( s^{-2\theta} + s^{-\delta_2 - \theta} \right) ds \leq CR^2 \left( t^{1-2\theta} + t^{1-\delta_2 - \theta} \right). \tag{6.13}
\]
Therefore,
\[ ||\mathcal{T}(w)(t)||_Y \leq ||\mathcal{T}(w_0)||_Y + \int_0^t ||\mathcal{T}(t-w(s))\Pi(w(s),w(s))||_Y ds \]
\[ \leq ||(\mathcal{T}(t)w_0)||_X + \int_0^t ||\mathcal{T}(t-w(s))h(s)||_X ds \]
\[ \leq Ct^{-\theta}\|u_0\|_{X_0} + CR^2\int_0^t \left( s^{-2\theta} + s^{-\delta_2-\theta} \right) ds \]
\[ \leq Ct^{-\theta}\|u_0\|_{X_0} + CR^2 \left( t^{1-2\theta} + t^{1-\delta_2-\theta} \right) \]
\[ = Ct^{-\theta}\|u_0\|_{X_0} + CR^2(t^{1-2\theta} + t^{1-\delta_2-\theta}). \]

Consider now \( \left( \sum(t-s)\Pi(w(s)) \right)_\xi \) for \( t \in (0,1) \).

Suppose first that \( \xi \in (0,1) \).

Two cases are possible: \( \xi \in (0,t^2) \) and \( \xi \in (t^2,1) \).

By (4.4) and (4.7) with \( \theta = 0 \),
\[ \left| \frac{\partial}{\partial x} \left( \sum(t-s)\Pi(w(s),w(s)) \right) (x) \right| \leq C\|h(s)\|_X x^{-1+\delta_1} (t-s)^{-\delta_1}, \forall s \in (t-x,t), \] (6.14)
\[ \left| \frac{\partial}{\partial x} \left( \sum(t-s)\Pi(w(s),w(s)) \right) (x) \right| \leq C\|h(s)\|_X \left( 1 - \log \left( \frac{x}{t-s} \right) \right), \forall s \in (0,t-x). \] (6.15)

Since,
\[ \frac{\partial}{\partial \xi} \left( \sum(t-s)\Pi(w(s),w(s)) \right)_\xi (x) = \frac{1}{2\sqrt{\xi}} \frac{\partial}{\partial x} \left( \mathcal{T}(t-s)h(s) \right)(x), \]
it follows, for \( t \in (0,1) \) and \( \xi \in (0,t^2) \),
\[ \left| \frac{\partial}{\partial \xi} \left( \sum(t-s)\Pi(w(s),w(s)) \right)_\xi (\xi) \right| \leq C\|h(s)\|_X \xi^{-1+\frac{\delta_1}{2}} (t-s)^{-\delta_1}, \forall s \in (t-\sqrt{\xi},t), \]
\[ \left| \frac{\partial}{\partial \xi} \left( \sum(t-s)\Pi(w(s),w(s)) \right)_\xi (\xi) \right| \leq C\|h(s)\|_X \xi^{-1/2} \times \]
\[ \left( 1 - \log \left( \frac{\sqrt{\xi}}{t-s} \right) \right), \forall s \in (0,t-\sqrt{\xi}). \]

Then, if \( 0 < \sqrt{\xi} < t \),
\[ \int_0^t \left| \frac{\partial}{\partial \xi} \left( \sum(t-s)\Pi(w(s),w(s)) \right)_\xi (\xi) \right| ds \leq \int_0^{t-\sqrt{\xi}} \left| \frac{\partial}{\partial \xi} \left( \sum(t-s)\Pi(w(s),w(s)) \right)_\xi (\xi) \right| ds + \]
\[ + \int_{t-\sqrt{\xi}}^t \left| \frac{\partial}{\partial \xi} \left( \sum(t-s)\Pi(w(s),w(s)) \right)_\xi (\xi) \right| ds = I_1 + I_2 \]

where,
\[ I_1 \leq C\xi^{-1/2} \int_0^{t-\sqrt{\xi}} \|h(s)\|_X \left( 1 - \log \left( \frac{\sqrt{\xi}}{t-s} \right) \right) ds \] (6.16)
\[ I_2 \leq C\xi^{-1+\frac{\delta_1}{2}} \int_{t-\sqrt{\xi}}^t \|h(s)\|_X (t-s)^{-\delta_1} ds. \] (6.17)

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The integral in (6.16) may be estimated for \( t \in (0, 1) \) as,

\[
\int_{t - \sqrt{\xi}}^{t} |h(s)|X \left( 1 - \log \left( \frac{\sqrt{\xi}}{t - s} \right) \right) ds \\
\leq C \int_{t - \sqrt{\xi}}^{t} |h(s)|X \left( 1 + |\log(t - s)| + |\log \xi| \right) ds \\
\leq CR^2 \int_{t - \sqrt{\xi}}^{t} \left( s^{-2\theta} + s^{-\delta_2 - \theta} \right) \left( 1 + |\log(t - s)| + |\log \xi| \right) ds \\
\leq CR^2 t^{1-\delta_2 - \theta} \left( 1 + |\log \xi| + |\log t| \right)
\]

It then follows, since \( \xi \leq t^2 < 1 \),

\[
I_1 \leq CR^2 t^{1-\delta_2 - \theta} \xi^{-1/2} (|\log \xi|)
\]

and since \( \theta \leq \delta_1, \sqrt{\xi} < t \leq 1 \), for all \( \varepsilon' \in (0, \varepsilon) \),

\[
I_1 \leq CR^2 t^{1-\delta_2 - \theta} \left( \xi^{-1/2 - \varepsilon'} |\log \xi| \right) \xi^{\varepsilon'} \\
\leq CR^2 t^{2\varepsilon' + 1-\delta_2 - \theta} \xi^{-\varepsilon - \varepsilon'}.
\]

In the integral in (6.17), using that \( \theta < \delta_2 \) and \( t \in (0, 1) \),

\[
\int_{t - \sqrt{\xi}}^{t} |h(s)||X(t - s)^{-\delta_1} ds \leq CR^2 \int_{t - \sqrt{\xi}}^{t} s^{-\delta_1 - \theta} (t - s)^{-\delta_1} ds \\
\leq CR^2 t^{1-2\theta - \delta_1} \left( B[1, 1 - \theta - \delta_2, 1 - \delta_1] - B \left[ 1 - \frac{\xi^{1/2}}{t}, 1 - \theta - \delta_2, 1 - \delta_1 \right] \right) \\
\leq CR^2 t^{1-\theta - \delta_1 - \delta_2} \left( \frac{\xi^{1/2}}{t} \right)^{1-\delta_1} \left( \left( 1 - \frac{\xi^{1/2}}{t} \right)^{1-2\theta} + 1 \right) \\
\leq CR^2 t^{1-\theta - \delta_2} \xi^{1/2} + 1/2
\]

and then, since \( 2\theta < 1 \) and \( \delta_1 + \theta < 1 \),

\[
|I_2| \leq CR^2 t^{-\theta - \delta_2} \xi^{-1/2 + \delta_1/2}
\]

and so, for \( \sqrt{\xi} < t \) and \( \theta \leq \delta_1 \),

\[
I_2 \leq CR^2 t^{-\theta - \delta_2 + 2\varepsilon + \delta_1} \xi^{-1/2 - \varepsilon},
\]

Then, for \( t \in (0, 1), \sqrt{\xi} < t, \) and \( \varepsilon' \in (0, \varepsilon_1) \),

\[
\int_{0}^{t} \left| \frac{\partial}{\partial \xi} \left( \sum(t - s)\Pi(w(s), w(s)) \right)(\xi) \right| ds \leq CR^2 t^{-\theta - \delta_2 + 2\varepsilon + \delta_1} \xi^{-1/2 - \varepsilon_1}.
\]
On the other hand, by (4.46), when $\xi \in (t^2, 1)$,
\[
\int_0^t \left| \frac{\partial}{\partial \xi} \left( \sum (t-s) \Pi(w(s), w(s)) \right)(\xi) \right| ds \leq CR^2 \xi^{-1+\frac{\theta}{2}} \times \\
\times \int_0^t (s^{-2\theta} + s^{-\delta_2-\theta})(t-s)^{-\delta_1} ds \leq CR^2 \xi^{-1+\frac{\theta}{2}} t^{1-\delta_1-\delta_2-\theta}.
\]
By Proposition 4.3,
\[
\left| \frac{\partial (\mathcal{S}(t)u_0)}{\partial x} (x) \right| \leq C||u_0||_x t^{-\theta-\delta_2} x^{-1+\delta_5}, \quad \forall x \in (t, 1)
\]
\[
\left| \frac{\partial (\mathcal{S}(t)u_0)}{\partial x} (x) \right| \leq C||u_0||_x t^{-\theta} \left( 1 + \left| \frac{\log x}{t} \right| \right) \leq C||u_0||_x t^{-\theta} (1 + |\log x|), \quad \forall x \in (0, t),
\]
Since, for all $\delta_5 \in (0, 1)$ there exists $C > 0$ such that for $0 < x < t < 1$
\[
1 + |\log x| \leq C t^{-\delta_5} x^{-1+\delta_5},
\]
it follows,
\[
\left| \frac{\partial (\mathcal{S}(t)u_0)}{\partial x} (x) \right| \leq C||u_0||_x t^{-\theta-\delta_2} x^{-1+\delta_5}, \quad \forall x \in (0, 1).
\]
Then, for $t \in (0, 1)$ and $\xi \in (0, 1)$
\[
\left| \frac{\partial \mathcal{S}(w)(t, \xi)}{\partial \xi} \right| \leq \left| \frac{\partial}{\partial \xi} \left( \sum (t-s) \Pi(w(s)) \right)(\xi) \right| ds
\]  
\[
\leq C||u_0||_{\mathcal{S}} t^{-\theta-\delta_2-\delta_5} \xi^{-1+\frac{\theta}{2}} + CR^2 \xi^{-1+\frac{\theta}{2}} t^{1-\delta_1-\delta_2-\theta}.
\]
Since $\delta_2 > \theta + \delta_1$, the choice $\delta_5 = \delta_1$ yields for $t \in (0, 1)$ and $\xi \in (0, 1)$
\[
t^{\delta_2} \xi^{1-\frac{\theta}{4}} \left| \frac{\partial \mathcal{S}(w)(t, \xi)}{\partial \xi} \right| \leq C||u_0||_{\mathcal{S}} t^{\delta_2-\theta-\delta_5} \xi^{\frac{\delta_5-\delta_1}{2}} + CR^2 t^{1-\delta_1-\theta}
\]  
\[
\leq C||u_0||_{\mathcal{S}} + CR^2 t^{1-\delta_1-\theta}. \quad (6.18)
\]
If $t \in (0, 1)$ but $\xi \geq 1$. By Proposition 4.3 for $\delta \in (0, 1)$, there exists $C > 0$,
\[
\left| \frac{\partial (\mathcal{S}(t)u_0)}{\partial x} (x) \right| \leq C||u_0||_{\mathcal{S}} t^{-\theta} x^{-1+\delta_1}, \quad \forall t \in (0, 1), \quad \forall x > 1
\]
and
\[
\left| \frac{\partial (\mathcal{S}(t-s)h(s))}{\partial x} (x) \right| \leq C||h(s)||_{\mathcal{S}} (t-s)^{-\delta_1} x^{-1+\delta_1} \forall t \in (0, 1), \quad \forall x > 1 \quad (6.19)
\]
from where it follows,
\[
\left| \frac{\partial}{\partial \xi} \left( \sum (t-s) \Pi(w(s))(\xi) \right) \right| \leq C||h(s)||_{\mathcal{S}} \xi^{-1+\frac{\theta}{4}} (t-s)^{-\delta_1},
\]  
\[
\leq CR^2 \left( s^{-2\theta} + s^{-\delta_2-\theta} \right) \xi^{-1+\frac{\theta}{4}} (t-s)^{-\delta_1}, \forall \xi > 1
\]
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and, since \( \theta < \delta_1 < 1 \),

\[
\left| \frac{\partial \mathcal{F}(w)(t, \xi)}{\partial \xi} \right| \leq C\|u_0\|_{X_\theta} t^{-\delta_2-\delta_1-\delta} \xi^{-1+\frac{\delta_1}{2}} + CR^2 \xi^{-1+\frac{\delta_1}{2}} \int_0^t (s^{-2\theta} + s^{-\delta_2-\theta})(t-s)^{-\delta_1} ds \\
\leq C\|u_0\|_{X_\theta} t^{-\delta_2-\delta_1-\theta} \xi^{-1+\frac{\delta_1}{2}} + CR^2 \xi^{-1+\frac{\delta_1}{2}} t^{1-\delta_1-\delta_2-\theta}.
\]

If \( \delta > 0 \) is chosen such that \( \delta \leq \delta_1, \delta + \theta \leq \delta_2 \)

\[
t^\delta_2 \xi^{1-\frac{\delta_1}{2}} \left| \frac{\partial \mathcal{F}(w)(t, \xi)}{\partial \xi} \right| \leq C\|u_0\|_{X_\theta} t^{\delta_2-\delta_1-\theta} \xi^{1-\frac{\delta_1}{2}} + CR^2 t^{1-\delta_1-\theta},
\]

and there exists two numerical constants \( C_1 > 0 \) and \( C_2 > 0 \) such that for all \( w \in B_{R,T,\varepsilon} \),

\[
|||\mathcal{F}(w)|||_{1,T} \leq C_1\|u_0\|_{X_\theta} + C_2 R^2 T^{1-\delta_1-\theta}.
\]

Then, for \( R > 0 \) large enough and \( T > 0 \) small enough such that

\[
C_1\|u_0\|_{X_\theta} \leq \frac{R}{2},
\]

\[
C_2 R^2 T^{1-\delta_1-\theta} < C_1\|u_0\|_{X_\theta}
\]

the operator \( \mathcal{F} \) sends \( B_{R,T,\varepsilon} \) into itself. On the other hand, for \( w \) and \( \tilde{w} \) in \( E_{R,T,\varepsilon} \),

\[
\mathcal{F}(w(t)) - \mathcal{F}(\tilde{w}(t)) = \int_0^t \sum(t-s) \left( \Pi(w(s),w(s)) - \Pi(\tilde{w}(s),\tilde{w}(s)) \right)(\xi) ds.
\]

Since,

\[
\Pi(w(s),w(s)) - \Pi(\tilde{w}(s),\tilde{w}(s)) = \Pi(w(s) - \tilde{w}(s),w(s)) + \Pi(\tilde{w}(s),w(s) - \tilde{w}(s))
\]

use of Proposition 5.2 and Proposition 5.1 as above gives,

\[
||\Pi(w(s),w(s)) - \Pi(\tilde{w}(s),\tilde{w}(s))||_Y \leq C(||w(s)||_{Y} + ||\tilde{w}(s)||_{Y}) s^{-\theta} \times \\
\times \left( ||w(s) - \tilde{w}(s)||_{Y} s^{-\theta} + s^{-\delta_2} P_{\delta_1}(w(s) - \tilde{w}(s)) + \\
+ ||w(s) - \tilde{w}(s)||_{Y} s^{-\theta-\delta_2} \left( P_{\delta_1}(\omega(s)) + P_{\delta_1}(\tilde{\omega}(s)) \right) \right)
\]

and similar arguments as in the proof of \( (6.21) \) show that, for some positive constant \( C_3 \),

\[
||\mathcal{F}(w) - \mathcal{F}(\tilde{w})||_{1,T} \leq C_3 R T^{1-\delta_1-\theta} ||w - \tilde{w}||_{1,T}
\]

Then, for \( T \) satisfying not only \( (6.22) \) but also

\[
C_3 R T^{1-\delta_1-\theta} < 1
\]

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The operator $\mathcal{F}$ has a fixed point $w$ in $B_{R,T,\varepsilon_1}$ such that
\[ w(t) = \Sigma(t)(w_0)(\xi) + \int_0^t \Sigma(t-s)(\Pi(w(s), w(s)))ds, \quad \text{in } X, \tag{6.27} \]
\[ |||w|||_{\varepsilon_1, T} \leq C_1 ||u_0||_{X_0} + C_2 R^2 T^{1-\delta_1-\theta}. \tag{6.28} \]
If $w$ and $\tilde{w}$ were two solutions of (6.27) such that, for some $\varepsilon > 0$ and $T > 0$,
\[ \sup_{0 < t < T} \theta^2(|||w(t)|||_Y + ||\tilde{w}(t)|||_Y) + \theta^2 (P_{\delta_1}(w_\xi(t)) + P_{\delta_1}(\tilde{w}_\xi(t))) = M < \infty. \]
Then, using (6.14), (6.15), (6.19) and (6.25),
\[ \int_0^t \left( s^{-2\theta} |||w(s) - \tilde{w}(s)|||_Y + P_{\delta_1}(w_\xi(s) - \tilde{w}_\xi(s)) s^{-\delta_2-\theta} \right) ds, \quad \forall t \in (0, T) \tag{6.29} \]
and by Gronwall’s Lemma, $w = \tilde{w}$ for $t \in (0, T)$.
Therefore, the solution $w \in B_{R,T,\varepsilon_1}$, obtained by the fixed point argument may be extended to a maximal time interval $(0, T_{\max})$. Since the time interval only depends on the norm of the initial data $||w_0||_{Y_0}$ it follows by a classical argument that either the solution is global or
\[ \lim_{t \to T_{\max}} ||w(t)|||_{Y_0} = \infty \]
By (3.3) and (4.1), for all $\xi > 0$ and $t \in (0, T_{\max})$ small enough,
\[ |w(t, \xi) - \Sigma(t) w_0(\xi)| \leq \int_0^t |\Sigma(t-s) \Pi(w(s), w(s))| ds \leq \int_0^t |||\mathcal{F}(t-s) h(s)|||_X ds \]
\[ \leq C(x^\theta + x)^{-1} \int_0^t ||h(s)||_X (s^{-2\theta} + s^{-2\theta-\delta_2}) ds \]
\[ \leq C |||w|||_{\varepsilon_1, t}^2 (x^\theta + x)^{-1} \int_0^t s^{-\delta_2-\theta} ds \leq C |||w|||_{\varepsilon_1, t}^2 (\theta^{\theta/2} + \xi^{1/2})^{-1} t^{1-\delta_2-\theta}. \]
Properties (6.2), (6.3), (6.5) follow then from (4.31), (4.35) and (4.37) respectively.

**Proposition 6.2.** Suppose that $w_0 \in Y_0$ and let $T_{\max}$ and $w$ be given by Proposition 6.1. Then, for almost every $t \in (0, T_{\max})$ and $\xi > 0$,
\[ \frac{\partial w(t, \xi)}{\partial t} = \mathcal{L}(w(t))(\xi) + \Pi(w(t), w(t))(\xi). \tag{6.30} \]
For all $\varepsilon > 0$, $T \in (0, T_{\max})$, there exists a constant $C > 0$, independent of $w_0$, such that,
\[ \left| \frac{\partial w(t, \xi)}{\partial t} \right| \leq C \left( ||w_0||_{Y_0} (1 + \Xi_\varepsilon(t, \xi^{1/2}) + \frac{|||w|||_{\varepsilon_1, T} \Xi_\theta(t, \xi)}{(\theta^\theta + \xi^{1/2})}) \right) \forall t \in (0, T), \forall \xi > 0, \tag{6.31} \]
with $\Xi_\varepsilon$ defined in (4.56) and
\[ \Xi_\theta(t, \xi) = \begin{cases} \xi^{-\theta} t^\theta + t^{-\theta-\delta_2}, & \xi \geq t^2 \\ t^{-\theta-\delta_2} + |\log \xi|, & \xi \in (0, t^2). \end{cases} \tag{6.32} \]
Proof. The proof closely follows that of Proposition (4.4) using (6.11) to control the term $\Pi(w(s), w(s))$. By Proposition (4.54) and the definition of $\Sigma$,

$$
\frac{\partial}{\partial t} (\Sigma(t)w_0(x)) = \mathcal{L}(\Sigma(t)w_0(x)) \tag{6.33}
$$

$$
\frac{\partial}{\partial t} \left[ (\Sigma(t-s)\Pi(w(s), w(s)))(\xi) \right] = \mathcal{L}(\Sigma(t-s)\Pi(w(s), w(s)))(\xi) \tag{6.34}
$$

As in the proof of Proposition (4.4) and with the same argument, the function

$$
\tilde{\psi}(t, \xi) = \int_0^t \Sigma(t-s)\Pi(w(s), w(s))(\xi)ds
$$

is derivable with respect to $t$ for almost every $x > 0$ and $t \in (0, T_{\text{max}})$,

$$
\frac{\partial}{\partial t} \tilde{\psi}(t, \xi) = \Pi(w(t), w(t))(\xi) + \int_0^t \frac{\partial}{\partial t} \left( \Sigma(t-s)\Pi(w(s), w(s)) \right)(\xi)ds. \tag{6.35}
$$

Property (6.30) follows from (6.33), (6.34) and (6.35). Estimate (6.31) is deduced as follows. By (4.55) and (6.11), for $\varepsilon > 0$ small there exists $C > 0$ such that,

$$
\left| \frac{\partial w(t, \xi)}{\partial t} \right| \leq \left| \frac{\Sigma(t)(w_0)(\xi)}{\partial t} \right| + |\Pi(w(t), w(t))(\xi)| + \int_0^t \left| \frac{\partial \Sigma(t-s)\gamma(s)}{\partial t} \right| (\xi)ds
$$

$$
\leq C||w_0||Y_\theta(1 + \Xi_\varepsilon(t, \xi^{1/2}) + C||w||_{e_1,T}t^{-\theta-\delta_2}(\xi^\theta + \xi)^{-1/2} + C||w||_{e_1,T}(\xi^\theta + \xi)^{-1/2} \int_0^t (1 + \Xi_\varepsilon(t-s, \xi^{1/2})s^{-\theta-\delta_2}ds
$$

and (6.31) follows. \hfill \square

6.1 Behavior of $w(t)$ as $\xi \to 0$.

Proposition 6.3. Suppose that $\theta, \delta_1, \delta_2, w$ are as in in Proposition (6.1). Then, for all $\delta \in (0, 1)$, $T \in (0, T_{\text{max}})$, and $t \in (0, T)$,

$$
|w(t, \xi) - B(t)| \leq C||w||_{e_1,T} \left( \frac{\sqrt{\xi}}{t} t^{-\theta} + \xi^{1-\theta-\delta_2} \right), \forall \xi \in (0, t^2/2) \tag{6.36}
$$

with,

$$
B(t) = \lambda(w_0, t) + \int_0^t \lambda \left( \Psi(\Sigma(s)w_0); t-s \right) ds + \int_0^t \lambda \left( \gamma(s); t-s \right) ds + \int_0^t \int_0^{t-s} \lambda \left( \Psi(\Sigma(r)\gamma(s)); t-s-r \right) dr ds, \tag{6.37}
$$

where, $\gamma(s) = \Pi(w(s), w(s)), \forall s \in (0, T)$. \tag{6.38}

Moreover, there exists $C > 0$ such that, for all $t \in (0, T)$,

$$
|B(t) - \lambda(w_0, t)| \leq C \left( ||w_0||_{Y_\theta} t + ||w||_{e_1,T}^2 t^{1-\theta-\delta_2} \right). \tag{6.39}
$$
Proof. By construction,

\[ w(t, \xi) - \Sigma(t)w_0(\xi) = \int_0^t (\Sigma(t-s)\gamma(s))(\xi)ds \]
\[ = \int_0^{t-2\sqrt{\xi}} (\Sigma(t-s)\gamma(s))(\xi)ds + \int_{t-2\sqrt{\xi}}^t (\Sigma(t-s)\gamma(s))(\xi)ds \quad (6.40) \]

By Proposition 4.6, for all \( \delta > 0 \) as small as desired and for all \( T \in (0, T_{\text{max}}) \) there exists \( C > 0 \) such that for all \( t \in (0, T) \), and all \( s \in (0, t) \),

\[
\left| \left( \Sigma(t-s)\gamma(s) \right)(\xi) - \lambda(\gamma(s); (t-s)) - \int_0^{t-s} \lambda \left( \Sigma(r)\gamma(s); t-s-r \right)dr \right| \leq C||\gamma(s)||_Y \left( \frac{\sqrt{\xi}}{t-s} \right)^{1-\delta}(t-s)^{-\theta} + \xi^{\frac{1-\delta}{2}}, \quad \forall \xi \in (0, (t-s)^2/2). \quad (6.41)
\]

It follows,

\[
\left| \int_0^{t-2\sqrt{\xi}} (\Sigma(t-s)\gamma(s))(\xi)ds - \int_0^{t-2\sqrt{\xi}} \lambda(\gamma(s); t-s)ds - \int_0^{t-2\sqrt{\xi}} \int_0^{t-s} \lambda \left( \Sigma(r)\gamma(s); t-s-r \right)drds \right| \leq \int_0^t C||\gamma(s)||_Y \left( \frac{\sqrt{\xi}}{t-s} \right)^{1-\delta}(t-s)^{-\theta} + \xi^{\frac{1-\delta}{2}} ds
\]
\[
\leq \int_0^t C(s^{-2\theta} + s^{-\delta_2-\theta}) \left( \left( \frac{\sqrt{\xi}}{t-s} \right)^{1-\delta}(t-s)^{-\theta} + \xi^{\frac{1-\delta}{2}} \right) ds
\]
\[
\leq C||w_0||_{Y_\theta} \left( t^{\delta-\delta_2-2\theta} + t^{1-\delta_2-\theta} \xi^{\frac{1-\delta}{2}} \right) \leq Ct^{-1+\delta-\theta} \xi^{\frac{1-\delta}{2}},
\]

since \( \delta > \theta \). In the second term at the right hand side of (6.40),

\[
\int_{t-2\sqrt{\xi}}^t |(\Sigma(t-s)\gamma(s))(\xi)| ds \leq C \int_{t-2\sqrt{\xi}}^t ||\gamma(s)||_Y ds
\]
\[
\leq C||w||_{\epsilon_1, T}^2 \int_{t-2\sqrt{\xi}}^t (s^{-2\theta} + s^{-\theta-\delta_2})ds \leq C||w||_{\epsilon_1, T}^2 \xi^{\frac{1-\delta-\theta}{2}}.
\]

Properties (6.36), (6.37) and (6.38) follow from Proposition 4.6. On the other hand by
and (6.39) follows.

Proposition 6.4. By Fubini’s and Lebesgue’s Theorem $G$ 15, although the function $G$ mainly for the sake of completeness since it is almost identical to that of Proposition 2 of 15, although the function $G$ does not satisfy the same hypothesis.

Property (1.6) is essentially shown in the next Proposition. Its proof is given here mainly for the sake of completeness since it is almost identical to that of Proposition 2 of 15, although the function $G$ does not satisfy exactly the same hypothesis.

Proposition 6.4.

$$\lim_{\delta \to 0} \int_{\delta}^{\infty} Q(G(t))((\xi))\sqrt{\xi}d\xi = -\frac{\pi^2}{6} A(t)^2 + \int_{0}^{\infty} G(t, \xi)\xi d\xi. \quad (6.42)$$

where, $A(t) = \frac{1}{2} + B(t). \quad (6.43)$

Proof. For all $\delta > 0$,

$$\int_{\delta}^{\infty} Q(G(t))((\xi))\sqrt{\xi}d\xi = A_{\delta}(G, G) - \int_{\delta}^{\infty} G(t, \xi)\xi d\xi + 2 \int_{\delta}^{\infty} \int_{\xi}^{\infty} G(t, \xi_1)d\xi_1 d\xi,$$

where, following the notations of 15,

$$A_{\delta}(G, \tilde{G}) = A_{\delta}^{1}(G, \tilde{G}) + A_{\delta}^{2}(G, \tilde{G}),$$

$$A_{\delta}^{1}(G, \tilde{G}) = \int_{\delta}^{\infty} \int_{0}^{\xi} \left( G(\xi - \xi')\tilde{G}(\xi') - G(\xi)\tilde{G}(\xi - \xi') - G(\xi)\tilde{G}(\xi') - G(\xi) \right) d\xi' d\xi,$$

$$A_{\delta}^{2}(G, \tilde{G}) = 2 \int_{\delta}^{\infty} \int_{\xi}^{\infty} \left( G(\xi') + G(\xi)\tilde{G}(\xi') + G(\xi' - \xi)\tilde{G}(\xi') - G(\xi)\tilde{G}(\xi' - \xi) \right) d\xi' d\xi.$$

By Fubini’s and Lebesgue’s Theorem

$$\lim_{\delta \to 0} \left( \int_{\delta}^{\infty} G(t, \xi)\xi d\xi - 2 \int_{\delta}^{\infty} \int_{\xi}^{\infty} G(t, \xi_1)d\xi_1 d\xi \right) = -\int_{0}^{\infty} G(\xi)\xi d\xi. \quad (6.44)$$

On the other hand, if we define, as in 15, for some $d > 0$ small fixed,

$$g_{>}(t, \xi) = G(t, \xi)1_{\xi >d},$$

$$g_{<}(t, \xi) = G(t, \xi)1_{\xi <d},$$

$$g_{<}(t, \xi) = h_0(t, \xi)1_{\xi<d} + H(t, \xi), \quad h_0(t, \xi) = \frac{A(t)}{\xi}.$$
The function $H(t, \xi)$ satisfies,

$$H(t, \xi) = \left(G(t, \xi) - \frac{A(t)}{\xi}\right) \mathbb{1}_{\xi < d} = \left(n_0 + n_0(1 + n_0)\xi w(t) - \frac{A(t)}{\omega}\right) \mathbb{1}_{\omega < d}$$

Then,

$$A_\delta(G, G) = A_\delta(g_\delta, g_\delta) + A_\delta(g_\delta, g_\delta) + A_\delta(g_\delta, g_\delta) + A_\delta(g_\delta, g_\delta)$$

As in [13] by Lebesgue’s convergence,

$$\lim_{\delta \to 0} A_\delta(g_\delta, g_\delta) = \lim_{\delta \to 0} A_\delta(g_\delta, g_\delta) = \lim_{\delta \to 0} A_\delta(g_\delta, g_\delta) = 0,$$

and we are then left with, $A_\delta(g_\delta, g_\delta)$ that may be written,

$$A_\delta(g_\delta, g_\delta) = A_\delta(H, H) + A_\delta(h_0, H) + A_\delta(H, h_0) + A_\delta(h_0, h_0).$$

The last term is explicit and gives $A_\delta(h_0, h_0) = -\pi^2 A(t)^2/6$.

On the other hand, by Proposition 6.3, for all $\delta > 0$ as small as desired, $T \in (0, T_{\text{max}})$, and $t \in (0, T)$, there exists $C > 0$ such that for all $\xi \in (0, t^2/2)$,

$$|\xi G(t, \xi) - A(t)| \leq C||w||_{\ell^1,T} \left(\frac{\sqrt{\xi}}{t} \right)^{1-\delta} t^{-\theta} + \xi^{1+\theta+\delta},$$

and then,

$$|H(t, \xi)| \leq C||w||_{\ell^1,T} \left(\xi^{-\frac{1-\delta}{1+\theta+\delta}} t^{-1+\delta-\theta} + \xi^{\frac{1-\theta-\delta}{2}}\right).$$

The function $H(t, \cdot)$ is then integrable for all $t > 0$ fixed and,

$$\lim_{\delta \to 0} A_\delta(H, H) = A_0(H, H) = 0.$$

We slightly rearrange now the term $A_\delta(h_0, H) + A_\delta(H, h_0)$ as

$$A_\delta(h_0, H) + A_\delta(H, h_0) = A_3^1(h_0, H) + A_3^2(h_0, H) + A_3^3(H, h_0) + A_3^4(H, h_0) = (A_3^1(h_0, H) + A_3^3(H, h_0)) + (A_3^2(h_0, H) + A_3^4(H, h_0)).$$

with

$$A_3^1(h_0, H) + A_3^3(H, h_0) = \int_0^\infty \int_0^\xi \left((H(\xi) - \xi) - H(\xi))h_0(\xi)\right) d\xi' d\xi$$

and

$$A_3^2(h_0, H) + A_3^4(H, h_0) = 2 \int_0^\infty \int_\xi^\infty \left((H(\xi) - H(\xi')) - H(\xi) - H(\xi'))h_0(\xi)\right) d\xi' d\xi.$$
The argument still follows as in [15], even if our function $H$ satisfies slightly different conditions than (A.13) and (A.14). Indeed we claim that here also, the two functions under the integral signs in (6.45) and (6.46) are integrable on $(0, \infty)$. The only delicate region is where both $\xi$ and $|\xi' - \xi|$ are arbitrarily small.

Consider for example the term $(H(t, \xi) - H(t, \xi'))h_0(t, \xi - \xi')$ for $\xi' \in (0, \xi)$ in (6.45).

With our notation’s convention,

$$H(t, \xi) - H(t, \xi') = n_0 + n_0(1 + n_0)\xi w(t) - n_0' - n_0'(1 + n_0')\xi' w'(t) - \frac{A(t)}{\xi} + \frac{A(t)}{\xi'} = \varphi(\xi) - \varphi(\xi') + \psi(\xi) - \psi(\xi'),$$

$$\varphi(\xi) = \left[n_0 - \frac{1}{\xi}\right]; \quad \psi(\xi) = \left[n_0(1 + n_0)\xi w(t, \xi) - \frac{A(t) - 1}{\xi}\right]$$

It is now immediate that the function $\varphi(\xi) - \psi(\xi')$ is written as

$$\psi(\xi) - \psi(\xi') = n_0(1 + n_0)\xi (w(t) - w'(t)) + \left(n_0(1 + n_0)\xi - n_0'(1 + n_0')\xi'\right) w'(t) - \frac{A(t) - 1}{\xi} + \frac{A(t) - 1}{\xi'}$$

the two terms in the right hand side of (6.49) are estimated as follows. In the first one, for $\xi$ and $\xi'$ small

$$|n_0(1 + n_0)\xi (w(t) - w'(t))| \leq \frac{C|w(t, \xi) - w(t, \xi')|}{\xi}$$

For each $t > 0$ fixed, if $\xi$ and $\xi'$ are small enough,

$$|w(t, \xi) - w(t, \xi')| \leq |\xi - \xi'| \int_0^1 |w'_{\xi}(t, (\xi r + (1 - r)\xi'))| dr$$

$$\leq C||w||_{1, T} t^{-\delta_2} |\xi - \xi'| \int_0^1 ((\xi - \xi')^r + (\xi')^r)^{-1 + \frac{\delta_1}{2}} dr$$

$$\leq C||w||_{1, T} t^{-\delta_2} (\xi - \xi')^{\frac{\delta_1}{2}}$$

$$\Longrightarrow |n_0(1 + n_0)\xi (w(t) - w'(t))| \leq \frac{C(\xi - \xi')^{\frac{\delta_1}{2}}}{\xi^{\delta_2}}$$

(6.50)

The second term in the right hand side of (6.49), is written,

$$\left(n_0(1 + n_0)\xi - n_0'(1 + n_0')\xi'\right) w'(t) - \frac{A(t)}{\xi} + \frac{A(t) - 1}{\xi'} = I_1 + I_2,$$

$$I_1 = \left(n_0(1 + n_0)\xi - n_0'(1 + n_0')\xi'\right) (w'(t) - A(t) + 1)$$

$$I_2 = \left(n_0(1 + n_0)\xi - \frac{1}{\xi} - n_0'(1 + n_0')\xi' + \frac{1}{\xi'}\right) (A(t) - 1).$$

(6.51)

Since the function $\xi \mapsto n_0(1 + n_0)\xi - \xi^{-1}$ is Lipschitz and then, the factor of $a(t) - 1$ in the right hand side of (6.52) yields,

$$\left[n_0(1 + n_0)\xi - \frac{1}{\xi} - n_0'(1 + n_0')\xi' + \frac{1}{\xi'}\right] \leq C(\xi - \xi').$$

(6.52)

(6.53)
For the right hand side of (6.51) we notice that,
\[ \frac{d}{d\xi} (n_0(1+n_0)\xi) = -\frac{1}{4} \left( 1 + \xi \coth(\xi/2) \right) (\csch(\xi/2))^2 = -\frac{1}{\xi^2} + O(1), \quad \xi \to 0. \]
from where, if we call \( g(\xi) = n_0(1+n_0)\xi, \) for \( \xi \) and \( \xi' \) small,
\[ |n_0(1+n_0)\xi - n_0'(1+n_0')\xi'| \leq (\xi - \xi') \int_0^1 |\frac{dg}{d\xi}(r\xi + (1-r)\xi')| \, dr \]
\[ \leq C(\xi - \xi') \int_0^1 (r\xi + (1-r)\xi')^{-2} \, dr \leq \frac{C|\xi - \xi'|}{\xi\xi'}. \]
and using also Proposition 6.3 for \( \delta \in (\theta, 1) \) and if \( \xi \in (0, t^2/2), \)
\[ |I_1| \leq C||w||_{e,T} \left( t^{-1+\delta-\theta} \frac{|\xi - \xi'|}{\xi^{1+\frac{\delta}{2}} \xi'} + \frac{|\xi - \xi'|}{\xi^{1+\frac{\delta}{2}} \xi'} \right). \quad (6.54) \]
It follows from (6.41), (6.49)–(6.54) that for \( d \) small, \( \xi \in (0, d) \) and \( \xi' \in (0, \xi) \) there exists a constant \( C > 0 \) that depends on \( T, d, \) and \( ||w||_{e,T}, \)
\[ |(H(t, \xi) - H(t, \xi'))h_0(t, \xi - \xi')| \leq \frac{C}{(\xi - \xi')} \left( (\xi - \xi') + \frac{\delta \xi}{t^{1+\delta-\theta}} + \frac{\delta \xi}{(\xi - \xi')^{1-\frac{\delta}{2}}} \right) \]
\[ + \frac{t^{-1+\delta-\theta}(\xi - \xi')}{\xi \xi^{1+\frac{\delta}{2}} \xi'} + \frac{|\xi - \xi'|}{\xi^{1+\frac{\delta}{2}} \xi'} \leq C \left( 1 + \frac{\delta t}{\xi(\xi - \xi')^{1-\frac{\delta}{2}}} + \frac{t^{1+\delta-\theta}}{\xi \xi^{1+\frac{\delta}{2}} \xi'} + \frac{\delta}{\xi^{1+\frac{\delta}{2}} \xi'} \right). \]
Moreover, if \( \xi' \in (0, \xi), \) were such that \( \xi > d + \xi' \) or \( \xi' > d \) it would follow that \( (H(t, \xi) - H(t, \xi'))h_0(t, \xi - \xi') = 0. \) Therefore,
\[ \int_0^\infty \int_0^\xi |(H(t, \xi) - H(t, \xi'))h_0(t, \xi - \xi')| \, d\xi' \, d\xi = \]
\[ = \int_0^{2d} \int_0^\xi |(H(t, \xi) - H(t, \xi'))h_0(t, \xi - \xi')| \, d\xi' \, d\xi \]
\[ \leq C \int_0^{2d} \int_0^\xi \left( 1 + \frac{\delta t}{\xi(\xi - \xi')^{1-\frac{\delta}{2}}} + \frac{t^{1+\delta-\theta}}{\xi \xi^{1+\frac{\delta}{2}} \xi'} + \frac{\delta}{\xi^{1+\frac{\delta}{2}} \xi'} \right) \, d\xi' \, d\xi' \to 0, \quad \text{as} \ \delta \to 0. \]
Arguing in the same way for all the other terms in (6.45) and (6.46) it follows that,
\[ \lim_{\delta \to 0} A_\delta(h_0, H) + A_\delta(H, h_0) = 0, \]
and Proposition 6.4 follows.

7 Proof of Theorem 1.1.

In order to define \( F \) and \( n_c \) the change of time variable (1.10), must be inverted.
Proposition 7.1. Suppose that $w_0 \in Y_\theta$, let $w$ be the function given by Proposition 6.1 on the time interval $(0, T_{\text{max}})$ and $G$ defined in (2.2). If

$$m(t) = \int_0^\infty Q(G(t))(\xi) \sqrt{\xi} d\xi$$

then,

$$|m(t)| < \infty, \forall t \in (0, T_{\text{max}}),$$

and

$$\int_0^t \left| \int_0^\infty Q(G(s))(\xi) \sqrt{\xi} d\xi \right| ds < \infty, \forall t \in (0, T_{\text{max}}).$$

Proof. By Proposition 6.4,

$$\left| \int_0^\infty Q(G(t,\xi))(\xi) \sqrt{\xi} d\xi \right| \leq \left| -\frac{\pi^2}{6} A(t)^2 + \int_0^\infty G(t,\xi) \xi d\xi \right|$$

By definition of $A(t)$ and $G(t)$, and by (6.39) and (8.11),

$$\left| \int_0^\infty n_0(1 + n_0)w(t,\xi) \xi d\xi \right| \leq C |||w|||_{\epsilon_1, T\theta}^{-\theta} \int_0^\infty \frac{n_0(1 + n_0)(\xi)}{\xi(\xi + \theta)^{1/2}} d\xi$$

Then, for all $T \in (0, T_{\text{max}})$ there exists a constant $C > 0$ such that,

$$|m(t)| \leq C t^{-2\theta}, \forall t \in (0, T),$$

from where (7.2) and (7.3) follow.

If one denotes,

$$q_c(t) = n_c(0) - \int_0^t m(s) ds, \forall r > 0.$$

by (7.3), $q_c \in C(0, T_{\text{max}})$ and, since $n_c(0) > 0$ there exists $\delta > 0$ such that $q_c(t) > 0$ for all $t \in (0, \delta)$ and then,

$$T_* = \sup \left\{ \delta \in (0, T_{\text{max}}); q_c(t) > 0, \forall t \in (0, \delta) \right\} > 0.$$

Proposition 7.2. Let $\tau_* > 0$ defined as

$$\tau_* = \int_0^{T_*} \frac{ds}{q_c(s)}$$

Then for all $\tau \in (0, \tau_*)$ there exists a unique $t \in (0, T)$ such that

$$\tau = \int_0^t \frac{ds}{q_c(s)}, \forall \tau > 0.$$
Proof. By construction $q_c(t) > 0$ for all $t \in (0, \infty)$ and the integral in the right hand sides of (7.7) and (7.8) are well defined. It follows that the function

$$t \mapsto \tau(t) = \int_0^t \frac{ds}{q_c(s)}$$

(7.9)
is well defined and monotone non decreasing on the interval $[0, \tau_*)$ with values in the whole interval $[0, \tau_*)$. It is then invertible an the inverse function is well defined and non decreasing on $[0, \tau_*)$.

7.1 The proof of Theorem 1.1

Let $w(t)$ be the function given by Proposition 6.1 on the time interval $(0, T)$ with initial data $w_0$ and $F(t)$ defined as in (1.13). For all $\tau \in (0, \tau_*)$ let $t \in [0, T_*)$ be given by Proposition 7.2 and define $\Omega(\tau, \xi) = w(t, \xi)$. Then,

$$F(\tau, \xi) = G(t, \xi), \forall \xi > 0,$$

(7.10)

$$n_c(\tau) = q_c(t),$$

(7.11)

$$\mu(\tau) = \int_0^\infty Q(F(\tau))(\xi) \sqrt{\xi} \, d\xi,$$

(7.12)

by (7.1), (7.10), (7.12), $\mu(\tau) = m(t)$ and is defined for all $\tau \in (0, \tau_*)$. By definition of $q_c$,

$$\frac{d\tau}{dt} = \frac{1}{q_c(t)} \implies \frac{dt}{d\tau} = q_c(t) = n_c(\tau).$$

By a straightforward application of the chain rule,

$$\frac{d\mu_c(\tau)}{d\tau} = \frac{d\tau}{dt} \frac{d\mu_c(t)}{dt} = -m(t)n_c(\tau) = -\mu(\tau)n_c(\tau), \quad \forall \tau \in (0, \tau_*).$$

Then $n_c$ satisfies (1.2) and $n_c \geq 0$. On the other hand, since for almost every $\tau \in (0, \tau_*)$ and $\xi > 0$ $\partial_\tau F(\tau, \xi) = \partial_t F(t, \xi) \frac{dt}{d\tau} = n_c(\tau)F(\tau, \xi)$, the function $F(\tau)$ satisfies (1.1).

If $T^* < T_{\text{max}}$,

$$\lim_{t \to (T^*)^-} q_c(t) = 0,$$

and by the mean value Theorem, for all $t \in (0, T_*)$ there exists $\zeta \in (t, T_*)$ such that

$$q_c(t) = (t - T_*) q_c'(\zeta) = (T_* - t) m(\zeta).$$

By (7.4), for all $t \in (0, T_*)$,

$$0 < q_c(t) \leq (T_* - t)|m(\zeta)| \leq C(T_* - t)^{1-2\theta}$$

and the integral in the right hand side of (7.7) is then divergent. The function (7.9) is well defined and monotone for all $t \in (0, T_{\text{max}})$ and $\tau$ varies on the whole interval

$$\tau \in (0, \tau^*), \quad \tau^* = \int_0^{T_{\text{max}}} \frac{ds}{q_c(s)}.$$
Property (1.6) follows from Proposition 6.4 and property (1.15) follows from (6.31). Property (1.17) follows from (6.2), since, by (7.8), $t \to 0$ when $\tau \to 0$.

Since the function $F$ satisfies (1.1) for almost $\tau \in (0, \tau^*)$ and $\xi > 0$, after multiplication of both sides by $\sqrt{\xi}$ and integration,

$$n_c(\tau) \int_0^\infty Q(F(\tau))(\xi) \sqrt{\xi} d\xi = \int_0^\infty \frac{\partial F(\tau, \xi)}{\partial \tau} \sqrt{\xi} d\xi,$$

where by (7.2) both integrals are convergent. The estimate (6.31) then gives,

$$\int_0^\infty Q(F(\tau))(\xi) \sqrt{\xi} d\xi = \frac{d}{dt} \int_0^\infty F(\tau, \xi) \sqrt{\xi} d\xi.$$

and since $n_c$ satisfies (1.2), it follows,

$$\frac{d}{d\tau} \int_0^\infty F(\tau, \xi) \sqrt{\xi} d\xi + n'_c(\tau) = 0.$$

In a similar way, if both sides of (1.1) are multiplied by $\xi^{3/2}$ it follows after application of Fubini’s Theorem,

$$\frac{d}{dt} \int_0^\infty F(\tau, \xi) \xi^{3/2} d\xi = \int_0^\infty \frac{\partial F(\tau, \xi)}{\partial \tau} \xi^{3/2} d\xi = 0, \forall \tau \in (0, \tau^*).$$

Property (1.20) is now a simple consequence of Proposition 6.3.

**Proposition 7.3.** For all $\delta > 0$ and $\tau \in (0, \tau^*)$ there exists a constant $C > 0$, that depends on $\delta$ and $\tau$, such that for all $\sigma \in (0, \tau)$ and $\xi \in (0, t(\sigma)^2/2)$,

$$|\xi F(\sigma, \xi) - A(t(\sigma))| \leq C |||\Omega|||_{\varepsilon, \tau} \left( \frac{\sqrt{\xi}}{t(\sigma)} \right)^{1-\delta} t(\sigma)^{-\theta} + \xi^{\frac{1-\theta-\delta}{2}}.$$  (7.13)

where, $t(\sigma) = \int_0^\sigma n_c(\rho) d\rho.$  (7.14)

**Proof.** By Proposition 6.3, for all $\sigma \in (0, \tau)$ and $\xi \in (0, t(\sigma)^2/2)$,

$$|F(\sigma, \xi) + \frac{1}{2} - \frac{1 + B(t)}{\xi}| \leq |n_0 + \frac{1}{2} - \frac{1}{\xi}| + |n_0(1 + n_0)\xi - \frac{1}{\xi}| |w(t, \xi)| + \left| \frac{w(t, \xi) - B(t)}{\xi} \right|$$

$$\leq C \xi (1 + |w(t, \xi)|) + C |||w||| ||w|||_{e, \tau} \xi^{-1} \left( \left( \frac{\sqrt{\xi}}{t(\sigma)} \right)^{1-\delta} t^{-\theta} + \xi^{\frac{1-\theta-\delta}{2}} \right)$$

from where (7.13) follows.

In order to prove (1.22) and finish the proof of Theorem 1.1 just notice that, by (1.13) and (6.33),

$$-\frac{\pi^2}{6} A(\tau)^2 + \int_0^\infty F(\tau, \xi) \xi d\xi = -\frac{\pi^2}{6} \left( \frac{1}{2} + B(\tau) \right)^2 +$$

$$+ \int_0^\infty \left( n_0(\xi) + n_0(\xi)(1 + n_0(\xi) \Omega(\tau, \xi)) \right) \xi d\xi. \quad (7.15)$$

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The constant $\pi^2/24$ in the right hand side of (7.15) cancels and then,
\[
-\frac{\pi^2}{6} A(\tau)^2 + \int_0^\infty F(\tau, \xi) d\xi = -\frac{\pi^2}{6} (B^2(\tau) + B(\tau)) + \int_0^\infty n_0(\xi)(1 + n_0(\xi)\Omega(\tau, \xi))\xi^2 d\xi.
\]

Property (1.22) follows now from (6.39), (8.12) with
\[
c_1 = \frac{24}{\pi^2} \left( \frac{\alpha B(1)\Gamma(\theta)}{B(\theta)} \right)^2.
\]

8 Appendix

We present in this Appendix some results about the function $\Lambda$ defined in [6]. In Proposition 2.1 the sequences of numbers $\sigma_j^+$ for $j \in \mathbb{N}$ were introduced, for whose values only the following estimates are given,
\[
\sigma_j^+ \in \left(-2(2j+1), -2(2j+1) + 1\right), j = 1, 2, \cdots
\]
(a misprint in [6] made us write $j = 0, 1, \cdots$. In particular $s_1^+ = -6$ and $\sigma_1^+ \in (-6, -5)$.

Lemma 8.1. For all $c' \in \left(1 + \sigma_1^+, 0\right)$ there exists a constant $C > 0$ such that
\[
\left| \frac{\partial \Lambda}{\partial x}(t, x) \right| \leq C \left( t + x^{-c'} \right), \forall t \in (0, 1), \forall x \in (0, 1).
\]

Proof. The proof is based on the following representation formula of the function $\partial_x \Lambda(t, x)$:
\[
\frac{\partial \Lambda}{\partial x}(t, x) = \left( x \frac{\partial}{\partial x} \right)^3 (J(t, x))
\]
\[
J(t, x) = -\frac{1}{4\pi^2} \int_{c-i\infty}^{c+i\infty} \int_{\Re(\sigma) = \beta} \frac{t^{-\sigma-1}B(s-1)\Gamma(\sigma - s + 1)}{B(\sigma)} (s-1)s^{-3} \left( \frac{x}{t} \right)^{-s} dsd\sigma.
\]
for all $\beta \in (0, 2)$ and $c \in (0, \beta)$. Because $x \in (0, 1)$ we first deform the $s$ integration contour in $J$ towards smaller values, and cross the value $s = 0$ that is the first pole of the function under the integration signs. Since, for some constants $C_i > 0$,
\[
Res \left( \frac{B(s-1)}{s^3} \left( \frac{x}{t} \right)^{-s}, s = 0 \right) = C_0 + C_1 (\log(x/t)) + C_2 (\log(x/t))^2 + C_3 (\log(x/t))^3
\]
\[
= R (\log(x/t))
\]
it is possible to write,
\[
J(t, x) = R (\log(x/t)) \int_{\Re(\sigma) = \beta} \frac{t^{-\sigma-1}\Gamma(\sigma + 1)}{B(\sigma)} + J_1(t, x)
\]
\[
J_1(t, x) = -\frac{1}{4\pi^2} \int_{\Re(\sigma) = \beta}^{c'+i\infty} \int_{c'-i\infty} t^{-\sigma-1}B(s-1)\Gamma(\sigma - s + 1)(s-1)s^{-3} \left( \frac{x}{t} \right)^{-s} dsd\sigma
\]
for \( c' \in (1 + \sigma_1, 0) \), where \( 1 + \sigma_1 \in (-5, -4) \). In the integral at the right hand side of (8.2), the integration contour may be deformed towards lower values of \( \Re \epsilon(\sigma) \). The first singularity to be crossed seems to be the pole of the function \( \Gamma(\sigma + 1) \) at \( \sigma = -1 \). But \( \sigma = -1 \) is also a pole of \( B(\sigma) \). Therefore the first singularity to be crossed is the pole of \( \Gamma(\sigma + 1) \) at \( \sigma = -2 \) and then,

\[
\int_{\Re \epsilon(\sigma)=\beta} \frac{t^{-\sigma-1}\Gamma(\sigma + 1)}{B(\sigma)} = Ct + O(t^2), \quad t \to 0.
\]

On the other hand, in the term \( J_1 \), for \( c' \in (1 + \sigma_1, 0) \) the \( \sigma \) integration contour may be deformed towards lower values of \( \beta \) and cross the pole \( \sigma = -1 + c' \) of the function \( \Gamma(\sigma - s + 1) \) when \( s = c' \) to deduce,

\[
J_1(t, x) = \frac{t^{-c'}}{2\pi B(c' - 1)} \int_{c'-i\infty}^{c'+i\infty} \frac{B(s-1)(s-1)}{s^3} \left( \frac{x}{t} \right)^{-s} ds - J_2
\]

(8.4)

\[
J_2 = \frac{1}{4\pi^2} \int_{\Re \epsilon(\sigma)=\beta'} \int_{c'-i\infty}^{c'+i\infty} \frac{t^{-\sigma-1}B(s-1)\Gamma(\sigma - s + 1)}{B(\sigma)} (s-1)s^{-3} \left( \frac{x}{t} \right)^{-s} dsd\sigma.
\]

(8.5)

for \( \beta' \in (-2 + c', -1 + c') \). The terms in the right hand side of (8.4) are estimated as follows

\[
\left| \frac{t^{-c'}}{2\pi B(c' - 1)} \int_{c'-i\infty}^{c'+i\infty} \frac{B(s-1)(s-1)}{s^3} \left( \frac{x}{t} \right)^{-s} ds \right| \leq Cx^{-c'}
\]

(8.6)

\[
\leq Cx^{-c'} \int_{\mathbb{R}} \left| \frac{B(c' - 1 + iv)(c' - 1 + iv)}{(c' + iv)^3} \right| dv \leq Cx^{-c'},
\]

(8.7)

\[
|J_2| \leq Cx^{-c'} t^{-\beta' - 1 + c'}.
\]

(8.8)

and since \(- \beta' - 1 + c' > 0\) and \( t \in (0, 1) \), the Lemma follows.

\[ \square \]

**Proposition 8.2.** Let \( \Lambda \) be the function defined in Corollary 2.11 of \([6]\). Then, for all \( \varphi \in C(0, \infty) \cap L^\infty(0, \infty) \),

\[
\lim_{t \to 0} \int_0^\infty \Lambda(t, x) \varphi(x) dx = \varphi(1)
\]

**Proof.** Let \( \psi \in C_c(0, \infty) \) such that \( 0 \leq \psi \leq 1 \), \( \psi(x) = 1 \) when \( |x - 1| \leq 1/4 \) and \( \psi(x) = 0 \) if \( |x - 1| \geq 1/2 \). Then

\[
\int_0^\infty \Lambda(t, x) \varphi(x) dx = \int_0^\infty \Lambda(t, x) \varphi(x) \psi(x) dx + \int_0^\infty \Lambda(t, x) \varphi(x)(1 - \psi(x)) dx
\]

By Corollary 2.14 of \([6]\),

\[
\lim_{t \to 0} \int_0^\infty \Lambda(t, x) \varphi(x) \psi(x) dx = \varphi(1).
\]

On the other hand,

\[
\left| \int_0^\infty \Lambda(t, x) \varphi(x)(1 - \psi(x)) dx \right| \leq ||\varphi||_\infty \left( \int_0^{3/4} |\Lambda(t, x)| dx + \int_{3/4}^\infty |\Lambda(t, x)| dx \right).
\]

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By Proposition 3.5 of [6], for \( \varepsilon > 0 \) as small as desired there exists positive constants \( C_{\varepsilon,1}, C_{\varepsilon,2} \) such that for all \( t > 0 \),
\[
\int_0^{3/4} |\Lambda(t,x)| \, dx \leq C_{\varepsilon,1} \left( t^5 \int_0^t x \, dx + \int_t^{5/4} (x^{-3+\varepsilon}t^{9-\varepsilon} + x^{-5}t) \, dx \right)
\]
\[
\int_{5/4}^{\infty} |\Lambda(t,x)| \, dx \leq C_{\varepsilon,2} \int_{5/4}^{\infty} (x^{-3+\varepsilon}t^{9-\varepsilon} + x^{-5}t) \, dx
\]
and Proposition follows.

**Proposition 8.3.** There exists a positive constant \( B(1) \) such that, for all \( \theta \in [0,1) \), \( w_0 \in Y_\theta \) and \( \delta \in (0,1) \),
\[
(\Sigma(t)w_0)(\xi) = \lambda(t;w_0) + O(\xi^{\theta/2}), \quad x \to 0,
\]
\[
\lambda(t;w_0) = \frac{6B(1)}{\pi^2} \frac{1}{2\pi i} \int_{\mathfrak{H}(\varepsilon) = \beta} \frac{\Gamma(\sigma)}{B(\sigma)} t^{-\sigma} \left( \int_0^{t^2} w_0(\zeta) \zeta^{-1+\frac{\sigma}{2}} \, d\zeta \right) \, d\sigma, \quad \beta \in (0,2).
\] (8.10)

For all \( T > 0 \) there exists \( C > 0 \) such that,
\[
|\lambda(t;w_0)| \leq C ||w_0||_{\mathcal{Y}_\theta} t^{-\theta}, \quad \forall t \in (0,T).
\] (8.11)

If \( w_0(\xi) = \alpha \xi^{-\theta/2} (1 + O(\xi^{\varepsilon})) \) as \( \xi \to 0 \) for some \( \varepsilon > 0 \), then,
\[
\lambda(t;w_0) = \frac{12B(1)\Gamma(\theta)}{\pi^2 B(\theta)} t^{-\theta} + O(t)^{-\theta+\varepsilon}, \quad t \to 0.
\] (8.12)

**Proof.** Consider the function \( g(x) = w_0(x^2) \) for \( x > 0 \). Then, as given in [6],
\[
(S(t)g)(x) = \int_0^t g(y)\Lambda \left( \frac{t}{y} \frac{x}{y} \right) \, dy + \int_t^{\infty} g(y)\Lambda \left( \frac{t}{y} \frac{x}{y} \right) \, dy
\]
\[
= \frac{1}{2\pi i} \int_{\mathfrak{H}(\varepsilon) = \varepsilon} x^{-s} \int_0^{t^2} g(y)U \left( \frac{t}{y}, s \right) y^{s-1} \, dy ds + \int_t^{\infty} g(y)\Lambda \left( \frac{t}{y} \frac{x}{y} \right) \, dy
\]
where \( c \in (0,2) \). By Proposition 3.5 in [6],
\[
\left| \int_t^{\infty} g(y)\Lambda \left( \frac{t}{y} \frac{x}{y} \right) \, dy \right| \leq C_xt^5 \int_t^{\infty} |g(y)| \, dy.
\]

On the other hand, for all \( t > 0 \), \( y > 0 \) the function \( U(t/y, \cdot) \) is analytic for \( \mathfrak{H}(s) \in (-1,0) \cup (0,2) \) and has simple poles at \( s = -1 \) and \( s = 0 \). Then for all \( \delta \in (0,1) \),
\[
\frac{1}{2\pi i} \int_{\mathfrak{H}(\varepsilon) = \varepsilon} x^{-s} \int_0^t g(y)U \left( \frac{t}{y}, s \right) y^{s-1} \, dy ds = \int_0^t g(y)y^{s-1} \, dy + \frac{1}{2\pi i} \int_{\mathfrak{H}(\varepsilon) = -\delta} x^{-s} \int_0^t g(y)U \left( \frac{t}{y}, s \right) y^{s-1} \, dy ds.
\]

By Proposition 2.10 in [6],
\[
\left| \frac{1}{2\pi i} \int_{\mathfrak{H}(\varepsilon) = -\delta} x^{-s} \int_0^t g(y)U \left( \frac{t}{y}, s \right) y^{s-1} \, dy ds \right| \leq
\]
\[
\leq C_x \delta \int_{-\infty}^{\infty} (\delta^2 + \zeta^2)^{-1} \int_0^t g(y)(\delta^2 + \zeta^2)^{-\frac{1}{2}} \, dy dv \leq C_x \delta^2 \int_0^1 g(tz)z^{-\frac{3}{2}} \, dz.
\]

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By (2.21) and (2.27) in [6],

\[
\text{Res} \left( U \left( \frac{t}{y}, s \right) \big| s = 0 \right) = \frac{12 B(1)}{\pi^2} \frac{1}{2i\pi} \int_{\Re \sigma = \beta} \left( \frac{t}{y} \right)^{-\sigma} \frac{\Gamma(\sigma)}{B(\sigma)} d\sigma
\]

where \( B(1) > 0 \) by (2.7) in [6]. Properties (8.9) and (8.10) follow using that \( g \in X_\theta \). On the other hand, by definition,

\[
\lambda(t, w_0) = \frac{12 B(1)}{\pi^2} \frac{1}{2i\pi} \int_{\Re \sigma = \beta} \frac{\Gamma(\sigma)}{B(\sigma)} t^{-\sigma} \left( \int_0^t u_0(x) x^{-1+\sigma} dx \right) d\sigma
\]

where, by hypothesis,

\[
u_0(x) x^{-1+\sigma} = \alpha x^{-1+\sigma-\theta} + s(x), \quad |s(x)| \leq C x^{-1+\sigma-\theta+\epsilon}, \quad \forall x \in (0, t).
\]

Then,

\[
\lambda(t, w_0) = \frac{12 \alpha B(1) t^{-\theta}}{\pi^2} \frac{1}{2i\pi} \int_{\Re \sigma = \beta} \frac{\Gamma(\sigma)}{B(\sigma)} \left( \sigma - \theta \right) d\sigma + O(t)^{-\theta+\epsilon}, \quad t \to 0
\]

from where (8.12) follows. \( \square \)

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