NEW OSTROWSKI TYPE INEQUALITIES FOR $m$–CONVEX FUNCTIONS AND APPLICATIONS

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Abstract. In this paper, we establish new inequalities of Ostrowski type for functions whose derivatives in absolute value are $m$–convex. We also give some applications to special means of positive real numbers. Finally, we obtain some error estimates for the midpoint formula.

1. INTRODUCTION

Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on $I$, the interior of the interval $I$, such that $f' \in L([a,b])$ where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality holds (see [2]):

$$
\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq \frac{M}{b-a} \left[ \frac{(x-a)^2 + (b-x)^2}{2} \right].
$$

This inequality is well known in the literature as the Ostrowski inequality. For some results which generalize, improve, and extend the above inequality, see [2],[5],[6],[10] and [11], the references therein.

In [12], G. Toader defined $m$–convexity, an intermediate between the usual convexity and starshaped property, as the following:

**Definition 1.** The function $f : [0,b] \to \mathbb{R}, b > 0$, is said to be $m$–convex, where $m \in [0,1]$, if we have

$$
f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)
$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Denote by $K_m(b)$ the set of the $m$–convex functions on $[0,b]$ for which $f(0) \leq 0$.

**Definition 2.** The function $f : [0,b] \to \mathbb{R}, b > 0$ is said to be starshaped if for every $x \in [0,b]$ and $t \in [0,1]$ we have:

$$
f(tx) \leq tf(x).
$$

For $m = 1$, we recapture the concept of convex functions defined on $[0,b]$ and $m = 0$ we get the concept of starshaped functions on $[0,b]$.

The following theorem contains the Hermite-Hadamard type integral inequality (see [8]).

**Theorem 1.** Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an $M$–Lipschitzian mapping on $I$ and $a, b \in I$ with $a < b$. Then we have the inequality:

$$
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq M \frac{(b-a)}{4}.
$$

In [13] E. Set, M.E. Özdemir, M.Z. Sarıkaya established the following theorem.

**Theorem 2.** Let $f : I^o \subset [0,b^*] \to \mathbb{R}, b^* > 0$, be a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$. If $|f'|^q$ is $m$–convex on $[a,b]$, $q > 1$ and $m \in (0,1)$, then the following inequality holds:

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(2) \[ \left| \frac{f(a+b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq (b-a) \left( \frac{3^{1-\left(\frac{1}{q}\right)}}{8} \right) \left( |f'(a)| + m^{\frac{1}{q}} |f'(b)| \right). \]

where \( \frac{b}{m} < b^* \).

In [14] U. Kirmaci proved the following theorem.

**Theorem 3.** Let \( f : I^0 \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^0 \), \( a, b \in I^0 \) with \( a < b \). If the mapping \( |f'| \) is convex on \( [a,b] \), then we have

(3) \[ \left| \frac{f(a+b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{8} \left( |f'(a)| + |f'(b)| \right). \]

In [9] S.S. Dragomir and G. Toader proved the following Hermite-Hadamard type inequality for \( m \)-convex functions.

**Theorem 4.** Let \( f : [0,\infty) \to \mathbb{R} \) be an \( m \)-convex function with \( m \in (0,1] \). If \( 0 \leq a < b < \infty \) and \( f \in L^1([a,b]) \) then

(4) \[ \frac{1}{b-a} \int_a^b f(x)dx \leq \min \left\{ \frac{f(a) + mf(b)}{2}, \frac{f(b) + mf(a)}{2} \right\}. \]

Some generalizations of this result can be found in [4].

In [3] M.K. Bakula, M.E. Özdemir and J. Pečarić proved the following theorems.

**Theorem 5.** Let \( I \) be an open real interval such that \([0,\infty) \subset I \). Let \( f : I \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L([a,b]) \), where \( 0 \leq a < b < \infty \). If \( |f'|^q \) is \( m \)-convex on \( [a,b] \) for some fixed \( m \in (0,1] \) and \( q \in [1,\infty) \), then

(5) \[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right), \]

where

\[
\mu_1 = \min \left\{ \frac{|f'(a)|^q + m |f'(\frac{a+b}{2m})|^q}{2}, \frac{|f'(\frac{a+b}{2m})|^q + m |f'(\frac{b}{m})|^q}{2} \right\},
\]

\[
\mu_2 = \min \left\{ \frac{|f'(b)|^q + m |f'(\frac{a+b}{2m})|^q}{2}, \frac{|f'(\frac{a+b}{2m})|^q + m |f'(\frac{b}{m})|^q}{2} \right\}.
\]

**Theorem 6.** Let \( I \) be an open real interval such that \([0,\infty) \subset I \). Let \( f : I \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L([a,b]) \), where \( 0 \leq a < b < \infty \). If \( |f'|^q \) is \( m \)-convex on \( [a,b] \) for some fixed \( m \in (0,1] \) and \( q \in [1,\infty) \), then

(6) \[ \left| \frac{f(a+b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \min \left\{ \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right), \left( \frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right\} \left( \frac{m |f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}. \]

The main purpose of this paper is to establish new Ostrowski type inequalities for functions whose derivatives in absolute value are \( m \)-convex. Using these results we give some applications to special means of positive real numbers and we obtain some error estimates for the midpoint formula.
In [1], in order to prove some inequalities related to Ostrowski inequality, M. Alomari and M. Darus used the following lemma with the constant \((b - a)\), but we changed it with the constant \((a - b)\) to obtain an equality in Lemma 7.

**Lemma 7.** Let \(f : I \subset \mathbb{R} \rightarrow \mathbb{R}\) be a differentiable mapping on \(I^0\) where \(a, b \in I\) with \(a < b\). If \(f' \in L([a, b])\), then the following equality holds:

\[
(7) \quad f(x) - \frac{1}{b - a} \int_a^b f(u)du = (a - b) \int_0^1 p(t)f'(ta + (1 - t)b)dt
\]

for each \(t \in [0, 1]\), where

\[
p(t) = \begin{cases} 
    t & , \quad t \in [0, \frac{b - x}{b - a}] \\
    t - 1 & , \quad t \in \left(\frac{b - x}{b - a}, 1\right]
\end{cases},
\]

for all \(x \in [a, b]\).

**Theorem 8.** Let \(I\) be an open real interval such that \([0, \infty) \subset I\). Let \(f : I \rightarrow \mathbb{R}\) be a differentiable function on \(I\) such that \(f' \in L([a, b])\), where \(0 \leq a < b < \infty\). If \(|f'|\) is \(m\)-convex on \([a, b]\) for some fixed \(m \in (0, 1]\), then the following inequality holds:

\[
(8) \quad \left| f(x) - \frac{1}{b - a} \int_a^b f(u)du \right| \leq (b - a) \min \left\{ \begin{array}{l}
\left[ \frac{1}{6} - \frac{1}{2} \left( \frac{b - x}{b - a} \right)^2 + \frac{2}{3} \left( \frac{b - x}{b - a} \right)^3 \right] |f'(a)| \\
+ m \left[ \frac{1}{6} - \frac{1}{2} \left( \frac{b - x}{b - a} \right)^2 - \frac{1}{3} \left( \frac{b - x}{b - a} \right)^3 + \frac{1}{3} \left( \frac{a - x}{b - a} \right)^3 \right] |f'(b)| \\
+ m \left[ \frac{1}{3} \left( \frac{b - x}{b - a} \right)^2 - \frac{1}{6} \left( \frac{b - x}{b - a} \right)^3 + \frac{1}{6} \left( \frac{a - x}{b - a} \right)^3 \right] |f'(\frac{a}{m})| \end{array} \right\}
\]

for each \(x \in [a, b]\).

**Proof.** By Lemma 7 we have

\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(u)du \right| \leq (b - a) \int_0^1 t |f'(ta + (1 - t)b)| dt
\]

\[
= (b - a) \int_{\frac{b - x}{b - a}}^{1} (1 - t) |f'(ta + (1 - t)b)| dt
\]

Since \(|f'|\) is \(m\)-convex on \([a, b]\) we know that for any \(t \in [0, 1]\)

\[
|f'(ta + (1 - t)b)| = \left| f'(ta + m(1 - t) \frac{b}{m}) \right| \leq t |f'(a)| + m(1 - t) \left| f'(\frac{b}{m}) \right|,
\]

for some fixed \(m \in (0, 1]\).
Hence
\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right|
\leq (b-a) \int_0^1 t \left[ t |f'(a)| + m(1-t) \left| f'(\frac{b}{m}) \right| \right] dt
\]
\[
+ (b-a) \int_{b-x}^{b-a} (1-t) \left[ t |f'(a)| + m(1-t) \left| f'(\frac{b}{m}) \right| \right] dt
\]
\[
= (b-a) \left\{ \left[ \frac{1}{6} - \frac{1}{2} \left( \frac{b-x}{b-a} \right)^2 + \frac{2}{3} \left( \frac{b-x}{b-a} \right)^3 \right] |f'(a)| \right\}
\]
\[
+ m \left[ \frac{1}{3} \left( \frac{b-x}{b-a} \right)^2 - \frac{1}{3} \left( \frac{b-x}{b-a} \right)^3 + \frac{1}{3} \left( \frac{x-a}{b-a} \right)^3 \right] |f'(\frac{b}{m})| \right\}
\]
where we use the facts that
\[
\int_0^{b-x} t \left[ t |f'(a)| + m(1-t) \left| f'(\frac{b}{m}) \right| \right] dt
\]
\[
= \frac{1}{3} \left( \frac{b-x}{b-a} \right)^3 |f'(a)| + m \left[ \frac{1}{2} \left( \frac{b-x}{b-a} \right)^2 - \frac{1}{3} \left( \frac{b-x}{b-a} \right)^3 \right] |f'(\frac{b}{m})| \right, \]
and
\[
\int_{b-x}^{b-a} (1-t) \left[ t |f'(a)| + m(1-t) \left| f'(\frac{b}{m}) \right| \right] dt
\]
\[
= \left[ \frac{1}{6} - \frac{1}{2} \left( \frac{b-x}{b-a} \right)^2 + \frac{1}{3} \left( \frac{b-x}{b-a} \right)^3 \right] |f'(a)| + m\left[ \frac{1}{3} \left( \frac{x-a}{b-a} \right)^3 \right] |f'(\frac{b}{m})| \right, \]
and analogously
\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right|
\leq (b-a) \left\{ \left[ \frac{1}{6} - \frac{1}{2} \left( \frac{b-x}{b-a} \right)^2 + \frac{2}{3} \left( \frac{b-x}{b-a} \right)^3 \right] |f'(b)| \right\}
\]
\[
+ m \left[ \frac{1}{2} \left( \frac{b-x}{b-a} \right)^2 - \frac{1}{3} \left( \frac{b-x}{b-a} \right)^3 + \frac{1}{3} \left( \frac{x-a}{b-a} \right)^3 \right] |f'(\frac{b}{m})| \right\}
\]
The proof is completed. □

**Remark 1.** Suppose that all the assumptions of Theorem 8 are satisfied. If we choose \( x = \frac{a+b}{2} \), then we have
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(u)du \right|
\leq \frac{b-a}{8} \min \left\{ |f'(a)| + m \left| f' \left( \frac{b}{m} \right) \right|, |f'(b)| + m \left| f' \left( \frac{a}{m} \right) \right| \right\}
\]
which is \([\Theta], q=1\).

**Remark 2.** Suppose that all the assumptions of Theorem 8 are satisfied. Then

(A) If we choose \( m = 1 \) and \( x = \frac{a+b}{2} \), we obtain
\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{b-a}{8} \left( |f'(a)| + |f'(b)| \right),
\]
which is \((3)\).

(B) In (A). Additionally, if we choose \( |f'(x)| \leq M, M > 0 \)
Theorem 9. Let $I$ be an open real interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on $I$ such that $f' \in L([a, b])$, where $0 \leq a < b < \infty$. If $|f'| p \cdot q$ is $m$-convex on $[a, b]$ for some fixed $m \in (0, 1)$ and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

\[
\left| f \left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq M \frac{(b-a)}{4}
\]

which is (1).

Proof. From Lemma 7 and using the Hölder inequality, we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{1}{(p+1)^{\frac{1}{2}}} \left\{ \begin{array}{l}
\frac{(b-x)^2}{b-a} \left[ \min \left\{ \frac{|f'(b)|^q + m |f'(\frac{a+b}{2})|^q}{2}, \frac{|f'(x)|^q + m |f'(\frac{a+b}{2})|^q}{2} \right\} \right]^{\frac{1}{q}} \\
\frac{(x-a)^2}{b-a} \left[ \min \left\{ \frac{|f'(a)|^q + m |f'(\frac{a+b}{2})|^q}{2}, \frac{|f'(x)|^q + m |f'(\frac{a+b}{2})|^q}{2} \right\} \right]^{\frac{1}{q}}
\end{array} \right\}
\]

for each $x \in [a, b]$.

Proof. From Lemma 7 and using the Hölder inequality, we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq (b-a) \left( \int_0^{\frac{b-x}{b-a}} t^p dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{x-a}{b-a}} |f'(ta + (1-t)b)|^q dt \right)^{\frac{1}{q}}
\]

\[
\leq (b-a) \left( \frac{b-x}{b-a} \right)^{\frac{p+1}{p}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{b-x}{b-a} \right)^{\frac{1}{q}}
\]

\[
\times \left( \min \left\{ \frac{|f'(b)|^q + m |f'(\frac{a+b}{2})|^q}{2}, \frac{|f'(x)|^q + m |f'(\frac{a+b}{2})|^q}{2} \right\} \right)^{\frac{1}{q}}
\]

\[
\leq (b-a) \left( \frac{x-a}{b-a} \right)^{\frac{p+1}{p}} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \frac{x-a}{b-a} \right)^{\frac{1}{q}}
\]

\[
\times \left( \min \left\{ \frac{|f'(a)|^q + m |f'(\frac{a+b}{2})|^q}{2}, \frac{|f'(x)|^q + m |f'(\frac{a+b}{2})|^q}{2} \right\} \right)^{\frac{1}{q}}
\]

\[
= \frac{1}{(p+1)^{\frac{1}{2}}} \frac{1}{b-a} \left\{ \begin{array}{l}
\frac{(b-x)^2}{b-a} \left[ \min \left\{ \frac{|f'(b)|^q + m |f'(\frac{a+b}{2})|^q}{2}, \frac{|f'(x)|^q + m |f'(\frac{a+b}{2})|^q}{2} \right\} \right]^{\frac{1}{q}} \\
\frac{(x-a)^2}{b-a} \left[ \min \left\{ \frac{|f'(a)|^q + m |f'(\frac{a+b}{2})|^q}{2}, \frac{|f'(x)|^q + m |f'(\frac{a+b}{2})|^q}{2} \right\} \right]^{\frac{1}{q}}
\end{array} \right\}
\]

where we use the facts that

\[
\int_0^{\frac{b-x}{b-a}} t^p dt = \left( \frac{b-x}{b-a} \right)^{p+1} \frac{1}{p+1},
\]

\[
\int_0^{1} (1-t)^p dt = \left( \frac{x-a}{b-a} \right)^{p+1} \frac{1}{p+1},
\]
and by Theorem 4 we get
\[
\frac{b - a}{b - x} \int_0^{b - x} \left| f'(ta + (1 - t)b) \right|^q dt \\
\leq \min \left\{ \frac{|f'(b)|^q + m |f'(\frac{x}{m})|^q}{2}, \frac{|f'(x)|^q + m |f'(\frac{b}{m})|^q}{2} \right\},
\]
\[
\frac{b - a}{x - a} \int_{b - x}^{1} \left| f'(ta + (1 - t)b) \right|^q dt \\
\leq \min \left\{ \frac{|f'(a)|^q + m |f'(\frac{x}{m})|^q}{2}, \frac{|f'(x)|^q + m |f'(\frac{b}{m})|^q}{2} \right\}.
\]
The proof is completed.

\[\square\]

**Corollary 10.** Suppose that all the assumptions of Theorem 9 are satisfied, if we choose $|f'(x)| \leq M$, $M > 0$, then we have
\[
\left| f(x) - \frac{1}{b - a} \int_a^b f(u)du \right| \\
\leq \left( \frac{1}{(p + 1)^{\frac{1}{p}}} \right) \left( \frac{1 + m}{2} \right)^{\frac{1}{q}} M \left[ \frac{(b - x)^2 + (x - a)^2}{b - a} \right].
\]

**Corollary 11.** Suppose that all the assumptions of Theorem 9 are satisfied, if we choose $x = \frac{a + b}{2}$ and \( \frac{1}{2} < \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} < 1 \), then we have
\[
\left| f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(u)du \right| \leq \frac{b - a}{4} \left( \mu_1^{\frac{1}{2}} + \mu_2^{\frac{1}{2}} \right),
\]
where
\[
\mu_1 = \min \left\{ \frac{|f'(b)|^q + m |f'(\frac{a + b}{2m})|^q}{2}, \frac{|f'(\frac{a + b}{2})|^q + m |f'(\frac{b}{m})|^q}{2} \right\},
\]
\[
\mu_2 = \min \left\{ \frac{|f'(a)|^q + m |f'(\frac{a + b}{2m})|^q}{2}, \frac{|f'(\frac{a + b}{2})|^q + m |f'(\frac{a}{m})|^q}{2} \right\}.
\]

**Remark 3.** Corollary 11 is similar to inequality, but for the left-hand side of Hermite-Hadamard inequality.

**Remark 4.** Suppose that all the assumptions of Theorem 9 are satisfied. Then in Corollary 11

(D) $|f'|$ is increasing and $m = 1$ then we have
\[
\left| f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(u)du \right| \leq \frac{b - a}{2} |f'(b)|,
\]

(E) $|f'|$ is decreasing and $m = 1$ then we have
\[
\left| f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(u)du \right| \leq \frac{b - a}{2} |f'(a)|,
\]

(F) $|f'(b)| = |f'(a)| = |f'(\frac{a + b}{2})|$ and $m = 1$ then we have
\[
\left| f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(u)du \right| \leq \frac{b - a}{2} \left| f'\left( \frac{a + b}{2} \right) \right|.
Theorem 12. Let $I$ be an open real interval such that $(0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on $I$ such that $f' \in L([a, b])$, where $0 \leq a < b < \infty$. If $|f'|^q$ is $m$–convex on $[a, b]$ for some fixed $m \in (0, 1]$ and $q \in [1, \infty)$, $x \in [a, b]$, then the following inequality holds:

\begin{equation}
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(u) du \right| \\
\leq (b-a) \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \\
\left\{ \begin{array}{l}
\left( \frac{b-x}{b-a} \right)^{2\left(1-\frac{1}{q}\right)} \left[ \left( \frac{1}{3} \left( \frac{b-x}{b-a} \right)^{3} |f'(a)|^{q} + m \frac{(b-x)^{2}(b-3a+2x)}{6(b-a)^{3}} |f' \left( \frac{b}{m} \right) |^{q} \right]^{\frac{1}{q}} \right. \\
\left. + \left( \frac{x-a}{b-a} \right)^{2\left(1-\frac{1}{q}\right)} \left[ \left( \frac{1}{6} + \frac{(b-x)^{2}(3a-b-2x)}{6(b-a)^{3}} \right) |f'(a)|^{q} + m \frac{1}{3} \left( \frac{x-a}{b-a} \right)^{3} |f' \left( \frac{b}{m} \right) |^{q} \right]^{\frac{1}{q}} \right\}
\end{array} \right.
\end{equation}

for each $x \in [a, b]$.

Proof. By Lemma 7 and using the well known power mean inequality we have

\[ \left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(u) du \right| \]

\[ \leq (b-a) \int_{0}^{1} t |f'(ta + (1 - t)b)| dt \\
+ (b-a) \int_{\frac{b-x}{b-a}}^{1} (1 - t) |f'(ta + (1 - t)b)| dt \\
\leq (b-a) \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left( \int_{0}^{\frac{b-x}{b-a}} t |f'(ta + (1 - t)b)|^{q} dt \right) \\
+ (b-a) \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left( \int_{\frac{b-x}{b-a}}^{1} (1 - t) |f'(ta + (1 - t)b)|^{q} dt \right) \\
\leq (b-a) \left( \frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ \left( \frac{b-x}{b-a} \right)^{2\left(1-\frac{1}{q}\right)} \left[ \left( \frac{1}{3} \left( \frac{b-x}{b-a} \right)^{3} \right) |f'(a)|^{q} + m \frac{(b-x)^{2}(b-3a+2x)}{6(b-a)^{3}} |f' \left( \frac{b}{m} \right) |^{q} \right]^{\frac{1}{q}} \right. \\
\left. + \left( \frac{x-a}{b-a} \right)^{2\left(1-\frac{1}{q}\right)} \left[ \left( \frac{1}{6} + \frac{(b-x)^{2}(3a-b-2x)}{6(b-a)^{3}} \right) |f'(a)|^{q} + m \frac{1}{3} \left( \frac{x-a}{b-a} \right)^{3} |f' \left( \frac{b}{m} \right) |^{q} \right]^{\frac{1}{q}} \right\}
\]

where we use the facts that

\[ \int_{0}^{\frac{b-x}{b-a}} t dt = \frac{1}{2} \left( \frac{b-x}{b-a} \right)^{2}, \]

\[ \int_{0}^{\frac{b-x}{b-a}} t |f'(ta + (1 - t)b)|^{q} dt \]

\[ \leq \frac{1}{3} \left( \frac{b-x}{b-a} \right)^{3} \left| f'(a) \right|^{q} + m \frac{(b-x)^{2}(b-3a+2x)}{6(b-a)^{3}} \left| f' \left( \frac{b}{m} \right) \right|^{q}. \]

\[ \int_{\frac{b-x}{b-a}}^{1} (1 - t) dt = \frac{1}{2} \left( \frac{x-a}{b-a} \right)^{2}, \]
\[
\int_{\frac{a+b}{b-a}}^1 (1-t) \left| f'(ta + (1-t)b) \right|^q dt \\
\leq \left[ \frac{1}{6} + \frac{(b-x)^2(3a-2x-b)}{6(b-a)^3} \right] |f'(a)|^q + m \frac{1}{3} \left( \frac{x-a}{b-a} \right)^3 \left| f' \left( \frac{b}{m} \right) \right|^q.
\]

The proof is completed.

**Remark 5.** Suppose that all the assumptions of Theorem 12 are satisfied. If we choose \( x = \frac{a+b}{2} \), we obtain

\[
\left| f \left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq (b-a) \left( 3^{1-\frac{1}{q}} \right) \left( |f'(a)| + m^{\frac{1}{q}} \left| f' \left( \frac{b}{m} \right) \right| \right)
\]

which is (2).

### 3. APPLICATIONS TO SPECIAL MEANS

Let us recall the following means for two positive numbers.

**(AM)** The Arithmetic mean

\[
A = A(a,b) = \frac{a+b}{2}; \quad a, b > 0,
\]

**(p−LM)** The p-Logarithmic mean

\[
L_p = L_p(a,b) = \begin{cases} 
\frac{a}{b^{p+1}-a^{p+1}} & \text{if } a = b \\
\frac{1}{p} & \text{if } a \neq b; \quad a, b > 0,
\end{cases}
\]

**(IM)** The Identric mean

\[
I = I(a,b) = \begin{cases} 
\frac{a}{\left( \frac{b^p}{a^q} \right)^{\frac{1}{p-q}}} & \text{if } a = b \\
\frac{1}{2} \left( \frac{b}{a} \right)^{\frac{1}{p-q}} & \text{if } a \neq b; \quad a, b > 0.
\end{cases}
\]

The following propositions hold:

**Proposition 13.** Let \( a, b \in [0, \infty) \), and \( a < b \), \( n \geq 2 \) with \( m \in (0,1] \). Then we have

\[
|A^n(a,b) - L^n_p(a,b)| \leq \frac{b-a}{8} \min \left\{ 2A \left( a^{n-1}, m \left( \frac{b}{m} \right)^{n-1} \right), 2A \left( b^{n-1}, m \left( \frac{a}{m} \right)^{n-1} \right) \right\}.
\]

**Proof.** The proof follows by Remark 1 on choosing \( f : [0, \infty) \to [0, \infty), \quad f(x) = x^n, \quad n \in \mathbb{Z}, \quad n \geq 2 \) which is \( m \)−convex on \([0, \infty)\).

**Proposition 14.** Let \( a, b \in [0, \infty), \) and \( a < b \), with \( m \in (0,1] \). Then we have

\[
\left| \ln \frac{I(a+1,b+1)}{A(a,b)+1} \right| \leq \frac{b-a}{4} \left( \eta_1^\frac{1}{q} + \eta_2^\frac{1}{q} \right),
\]

where

\[
\eta_1^\frac{1}{q} = \min \left\{ \left( \frac{1}{b+1} \right)^\frac{1}{q} + m \left( \frac{2m}{a+b+2m} \right)^\frac{1}{q} , \left( \frac{2}{a+b+2} \right)^\frac{1}{q} + m \left( \frac{m}{b+m} \right)^\frac{1}{q} \right\},
\]

\[
\eta_2^\frac{1}{q} = \min \left\{ \left( \frac{1}{a+1} \right)^\frac{1}{q} + m \left( \frac{2m}{a+b+2m} \right)^\frac{1}{q} , \left( \frac{2}{a+b+2} \right)^\frac{1}{q} + m \left( \frac{m}{a+m} \right)^\frac{1}{q} \right\}.
\]

**Proof.** The proof follows by Corollary 11 on choosing \( f : [0, \infty) \to (-\infty, 0], \quad f(x) = -\ln(x+1) \) which is \( m \)−convex on \([0, \infty), \quad p > 1\).
4. APPLICATIONS TO THE MIDPOINT FORMULA FOR 1–CONVEX FUNCTIONS

Let $d$ be a division $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of the interval $[a, b]$ and consider the quadrature formula

$$\int_a^b f(x)dx = M(f, d) + E(f, d),$$

where

$$M(f, d) = \sum_{i=1}^{n-1} (x_{i+1} - x_i) f \left( \frac{x_{i+1} + x_i}{2} \right)$$

is the midpoint formula and $E(f, d)$ denotes the associated approximation error (see [7]).

Here, we obtain some error estimates for the midpoint formula.

**Proposition 15.** Let $I$ be an open real interval such that $(0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on $I$ such that $f' \in L([a, b])$, where $0 \leq a < b < \infty$. If $|f'|^q$ is 1–convex on $[a, b]$ for some fixed $m \in (0, 1]$ and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then in (11), for every division $d$ of $[a, b]$, the midpoint error satisfies

$$|E(f, d)| \leq \frac{1}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left( \mu_1^q + \mu_2^q \right),$$

where

$$\mu_1 = \min \left\{ \frac{|f'(x_i)|^q + |f'(\frac{x_i+x_{i+1}}{2})|^q}{2}, \frac{|f'(x_{i+1})|^q + |f'(x_i)|^q}{2} \right\} = \frac{|f'(\frac{x_i+x_{i+1}}{2})|^q + |f'(x_i)|^q}{2},$$

$$\mu_2 = \min \left\{ \frac{|f'(x_{i+1})|^q + |f'(\frac{x_i+x_{i+1}}{2})|^q}{2}, \frac{|f'(x_{i+1})|^q + |f'(x_i)|^q}{2} \right\} = \frac{|f'(\frac{x_i+x_{i+1}}{2})|^q + |f'(x_{i+1})|^q}{2}.$$

**Proof.** On applying Corollary [11] with $m = 1$ on the subinterval $[x_i, x_{i+1}]$ ($i = 0, 1, 2, \ldots, n-1$) of the division, we have

$$\left| f \left( \frac{x_{i+1} + x_i}{2} \right) - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)dx \right| \leq \frac{x_{i+1} - x_i}{4} \left( \mu_1^q + \mu_2^q \right),$$

where

$$\mu_1 = \frac{|f'(\frac{x_i+x_{i+1}}{2})|^q + |f'(x_i)|^q}{2},$$

$$\mu_2 = \frac{|f'(\frac{x_i+x_{i+1}}{2})|^q + |f'(x_{i+1})|^q}{2}.$$

Hence, in (11) we have

$$\left| \int_a^b f(x)dx - M(f, d) \right| = \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)dx - (x_{i+1} - x_i) f \left( \frac{x_{i+1} + x_i}{2} \right) \right|$$

$$\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)dx - (x_{i+1} - x_i) f \left( \frac{x_{i+1} + x_i}{2} \right) \right|$$

$$\leq \frac{1}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left( \mu_1^q + \mu_2^q \right),$$
which completes the proof. □

**Proposition 16.** Let $I$ be an open real interval such that $[0, \infty) \subset I$. Let $f : I \to \mathbb{R}$ be a differentiable function on $I$ such that $f' \in L([a, b])$, where $0 \leq a < b < \infty$. If $|f'|^q$ is 1–convex on $[a, b]$ for some fixed $m \in (0, 1]$ and $q \in [1, \infty)$, $x \in [a, b]$, then in (11), for every division $d$ of $[a, b]$, the midpoint error satisfies

$$|E(f, d)| \leq \left(\frac{3^{1-\frac{1}{q}}}{8}\right)^{n-1} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left(|f'(x_i)| + |f'(x_{i+1})|\right).$$

**Proof.** The proof is similar to that of Proposition 15 and using Remark 5 with $m = 1$. □

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