Analysis of Two-Dimensional Nonlinear Transient Heat Conduction in Anisotropic Solids by Boundary Element Method Using Homotopy and Dual Reciprocity*

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This paper is concerned with an application of the homotopy boundary element method originally proposed by Liao and Chwang to the analysis of nonlinear transient heat conduction in anisotropic solids. Usually, domain integrals arise in the boundary integral equation of this formulation. Some ideas are needed to keep the boundary-only feature of BEM. In this paper, the resulting domain integrals are transformed into boundary integrals by the dual reciprocity method using a new set of radial basis functions. The mathematical formulations of this approach for two-dimensional problems are presented in detail. Two schemes are discussed in this paper: the “isotropic” scheme, in which the state before mapping is considered as steady-state heat conduction in isotropic solids; and the “anisotropic” scheme, where the state before mapping is considered as steady-state heat conduction in anisotropic solids. The proposed solution is applied to some typical examples, and the accuracy and other numerical properties of the proposed BEM are demonstrated through discussions of the results obtained.

Key Words: Boundary Element Method, Computational Mechanics, Numerical Analysis, Transient Heat Conduction, Anisotropic Solid, Homotopy, Dual Reciprocity

1. Introduction

The homotopy boundary element method proposed by Liao and Chwang (1) has been used to analyze nonlinear transient heat conduction problems in orthotropic solids. They formulated nonlinear transient heat conduction problems in orthotropic solids that can approximate steady-state heat conduction problems in isotropic solids. In their work, the state before mapping in homotopy can be considered as steady-state heat conduction in isotropic solids.

We have reported in our previous paper (2) that the homotopy boundary element method can be applied to nonlinear transient heat conduction problems in anisotropic solids. Furthermore, we have extended Liao and Chwang’s concept on the homotopy boundary element method; thus, the state before mapping can be considered as steady-state heat conduction in anisotropic solids. The computation has been performed by their method and our proposed method. For the computation, internal cells are required as the boundary integral equation includes domain integrals in the formulation. The computation cannot be performed using only boundary elements.

Good results were obtained in investigations (3) – (5) of applying the dual reciprocity method (6) without internal cells.

In this paper, domain integrals are transformed into boundary integrals using the dual reciprocity method (DRM) with RBF (7). Two schemes are discussed: the “isotropic” scheme proposed by Liao and Chwang, in which the state before mapping in homotopy is considered as steady-state heat conduction in isotropic solids; and the “anisotropic” scheme, where the state before mapping is considered as steady-state heat conduction in anisotropic solids. The proposed solution procedure is applied to some examples, and the usefulness of the proposed BEM is demonstrated through discussions of the results obtained.
2. Formulation

2.1 Boundary integral equation by homotopy

We shall consider two-dimensional nonlinear transient heat conduction in anisotropic solids. The governing differential equation is

\[
\frac{\partial}{\partial t} \left[ \lambda_{11}(\theta) \frac{\partial \theta}{\partial x_1} \right] + \frac{\partial}{\partial x_1} \left[ \lambda_{12}(\theta) \frac{\partial \theta}{\partial x_1} \right] + \frac{\partial}{\partial x_2} \left[ \lambda_{22}(\theta) \frac{\partial \theta}{\partial x_2} \right] + \rho c S = \frac{\partial \theta}{\partial t},
\]

where \( t \) is an arbitrary time, \( \theta \) is the temperature at the time \( t \) and \( S \) is the heat source. The mass density \( \rho \) and the specific heat \( c \) are independent of temperature. The thermal conductivities \( \lambda_{11}, \lambda_{12}, \lambda_{11}, \lambda_{22} \) are dependent on temperature. Using the homotopy boundary element method proposed by Liao and Chwang in which the time scale is divided into infinitesimal time intervals \( \Delta t \), Eq. (1) becomes

\[
\frac{\partial}{\partial t} \left[ \lambda_{11}(\theta_{n+1}) \frac{\partial \theta_{n+1}}{\partial x_1} \right] + \frac{\partial}{\partial x_1} \left[ \lambda_{12}(\theta_{n+1}) \frac{\partial \theta_{n+1}}{\partial x_1} \right] + \frac{\partial}{\partial x_2} \left[ \lambda_{22}(\theta_{n+1}) \frac{\partial \theta_{n+1}}{\partial x_2} \right] + \rho c S_{n+1} = \rho c \frac{\theta_{n+1} - \theta_{n}}{\Delta t},
\]

where the temperature and heat source at \( t = t_{n+1} \) are expressed as \( \theta_{n+1} \) and \( S_{n+1} \), respectively. The nonlinear operator in Eq. (2) is defined as

\[
A(\theta_{n+1}) = \frac{\partial}{\partial x_1} \left[ \lambda_{11}(\theta_{n+1}) \frac{\partial \theta_{n+1}}{\partial x_1} \right] + \frac{\partial}{\partial x_1} \left[ \lambda_{12}(\theta_{n+1}) \frac{\partial \theta_{n+1}}{\partial x_1} \right] + \frac{\partial}{\partial x_2} \left[ \lambda_{22}(\theta_{n+1}) \frac{\partial \theta_{n+1}}{\partial x_2} \right] + \rho c S_{n+1} - \rho c \frac{\theta_{n+1} - \theta_{n}}{\Delta t}.
\]

The following differential equation is defined using an embedding parameter \( p (0 \leq p \leq 1) \).

\[
(1 - p) A(\theta(x;p)) - \theta_0(x) = -p A(\theta(x;p))
\]

The left-hand side of Eq. (4) is the state before mapping in homotopy and the right-hand side is the state after mapping. \( \Theta \) is the approximated temperature in homotopy at \( t = t_{n+1} \) and \( \theta_0(x) \) is the initial temperature of the approximated temperature \( \Theta \). The \( B \) operators on the left-hand side which denote steady-state heat conduction in isotropic solids and anisotropic solids are

\[
B = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2},
\]

\[
B = \lambda_{11}(\theta_0) \frac{\partial^2}{\partial x_1^2} + \lambda_{12}(\theta_0) \frac{\partial^2}{\partial x_1 \partial x_2} + \lambda_{22}(\theta_0) \frac{\partial^2}{\partial x_2^2}.
\]

where \( \theta_0 \) is a constant temperature, and \( \lambda_{11}, \lambda_{12}, \lambda_{21} \) and \( \lambda_{22} \) are dependent on \( \theta_0 \).

When Eq. (4) is differentiated with respect to \( p \), we can obtain Eq. (7) in isotropic solids and Eq. (8) in anisotropic solids, as the \( B \) operators are different from each other.

\[
\left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right] B^m \theta_{0m} = R_m \quad (m \geq 1)
\]

\[
\left[ \lambda_{11}(\theta_0) \frac{\partial}{\partial x_1} + \lambda_{12}(\theta_0) \frac{\partial}{\partial x_1 \partial x_2} + \lambda_{22}(\theta_0) \frac{\partial}{\partial x_2} \right] B^m \theta_{0m} = R_m \quad (m \geq 1)
\]

In addition, there are two relations as follows.

\[
\bar{\theta}_{0m}(x) = \frac{\partial^m \Theta(x;p)}{\partial p^m} \bigg|_{p=0} (m \geq 1)
\]

\[
\Theta(x;p) = \theta_0(x) + \int_0^\infty \left[ \delta_{0m}(x) \right] p^m \quad (m \geq 1)
\]

Considering Eqs. (7) and (8), it is revealed that they are similar to the governing differential equations of two-dimensional steady-state heat conduction in isotropic solids and anisotropic solids, respectively. Applying the standard boundary element method to these equations, we can obtain the following equation with respect to \( \bar{\theta}_{0m} \).

\[
a \bar{\theta}_{0m}(y) = \int_y T(x,y) \bar{\theta}_{0m}(x) d\Omega - \int_y q(x,y) \bar{\theta}_{0m}(x) d\Omega
\]

Here, \( a \) denotes the shape coefficient depending on the geometry in which the source point \( y \) is located on the boundary. \( \bar{\theta}_{0m} \) is the \( m \)th derivative with respect to \( p \) of the approximated heat flux which corresponds to the approximated temperature. \( T^* \) and \( q^* \) are the fundamental solutions of two-dimensional steady-state heat conduction problems concerning temperature and heat flux in isotropic solids or anisotropic solids.

2.2 Application of DRM

2.2.1 Isotropic scheme

The right-hand side of Eq. (7) is approximated as

\[
R_m = \sum_{j=1}^{N+L} a_j^m f(x,j).
\]

where \( N \) is the number of nodes, \( L \) is the number of inner points, \( a_j^m \) are unknown coefficients, \( f(x,j) \) is an approximated function and \( j \) are collocation points of DRM. Applying Eq. (14), Eq. (7) is written as

\[
\left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right] \theta_0^m = \sum_{j=1}^{N+L} a_j^m f(x,j).
\]
We define a particular solution \( \hat{q}_0^{[m]}(x, z_j) \) as
\[
\left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right] \hat{q}_0^{[m]}(x, z_j) = f(x, z_j).
\tag{16}
\]
Substituting Eq. (16) into Eq. (15),
\[
\left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right] \hat{q}_0^{[m]}(x, z_j) = \sum_{j=1}^{N_{\lambda}L} \alpha_j^m \left[ \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right] \hat{q}_0^{[m]}(x, z_j).
\tag{17}
\]
The following equation can be obtained by integrating Eq. (17) over the whole domain multiplied by the fundamental solution of steady-state heat conduction problems in isotropic solids, and by applying integration by parts.
\[
a \hat{q}_0^{[m]}(y) = \int_{\Gamma} T^r(x, y) \hat{q}_0^{[m]}(x) d\Gamma - \frac{N_{\lambda}}{2} \int_{\Gamma} q(x, y) \hat{q}_0^{[m]}(x) d\Gamma - \int_{\Gamma} T^r(x, y) \hat{q}_0^{[m]}(x, z_j) d\Gamma + \sum_{j=1}^{N_{\lambda}L} \alpha_j^m \left[ \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right] \hat{q}_0^{[m]}(x, z_j) d\Gamma.
\tag{18}
\]
Here, \( q_0^{[m]}(x, z_j) \) is the normal derivative of the particular solution \( \hat{q}_0^{[m]}(x, z_j) \).

In this study, the following equation is used as an approximated function.
\[
f(x, z_j) = \begin{cases} (1 - r_j)^3(3r_j + 1), & r_j \leq 1 \\ 0, & r_j > 1 \end{cases}
\tag{19}
\]
Equation (19) is the compactly supported radial basis function. \( r_j \) is the dimensionless distance between an observation point \( x \) and a DRM collocation point \( z_j \), divided by the support radius 1 of this function. Applying Eq. (19), the particular solution \( \hat{q}_0^{[m]}(x, z_j) \) and the normal derivative \( \hat{q}_0^{[m]}(x, z_j) \) are
\[
\hat{q}_0^{[m]} = \frac{1}{4} (r_j)^3 - \frac{3}{8} (r_j)^2 + \frac{8}{25} (r_j)^5 - \frac{1}{12} (r_j)^6
\tag{20}
\]
\[
\hat{q}_0^{[m]} = \left( n_1(r_j) + n_2(r_j) \right) \left\{ \frac{1}{2} (r_j)^2 + \frac{8}{5} (r_j)^3 - \frac{1}{2} (r_j)^4 \right\}
\tag{21}
\]
where \( (r_j)_1 \) and \( (r_j)_2 \) are components in the \( x_1 \) and \( x_2 \) directions, respectively, and \( n_1 \) and \( n_2 \) are components of the normal unit vector at the boundary \( \Gamma \).

### 2.2.2 Anisotropic scheme

The right-hand side of Eq. (8) is approximated as in the isotropic scheme as
\[
R_m = \sum_{j=1}^{N_{\lambda}L} \alpha_j^m f(x, z_j).
\tag{22}
\]
Applying Eq. (22), Eq. (8) is written as
\[
\begin{aligned}
&\lambda_{11}(\theta_0) \frac{\partial^2}{\partial x_1^2} + \lambda_{12}(\theta_0) \frac{\partial^2}{\partial x_1 \partial x_2} + \lambda_{21}(\theta_0) \frac{\partial^2}{\partial x_2^2} + \lambda_{22}(\theta_0) \frac{\partial^2}{\partial x_1 \partial x_2} \\
&+ \lambda_{22}(\theta_0) \frac{\partial^2}{\partial x_2^2} \right] \hat{q}_0^{[m]}(x, z_j) = \sum_{j=1}^{N_{\lambda}L} \alpha_j^m f(x, z_j).
\tag{23}
\end{aligned}
\]
A particular solution \( \hat{q}_0^{[m]}(x, z_j) \) is defined as
\[
\begin{aligned}
&\lambda_{11}(\theta_0) \frac{\partial^2}{\partial x_1^2} + \lambda_{12}(\theta_0) \frac{\partial^2}{\partial x_1 \partial x_2} + \lambda_{21}(\theta_0) \frac{\partial^2}{\partial x_2^2} + \lambda_{22}(\theta_0) \frac{\partial^2}{\partial x_1 \partial x_2} \\
&+ \lambda_{22}(\theta_0) \frac{\partial^2}{\partial x_2^2} \right] \hat{q}_0^{[m]}(x, z_j) = f(x, z_j).
\tag{24}
\end{aligned}
\]
Substitution of Eq. (24) into Eq. (23) yields
\[
\begin{aligned}
&\lambda_{11}(\theta_0) \frac{\partial^2}{\partial x_1^2} + \lambda_{12}(\theta_0) \frac{\partial^2}{\partial x_1 \partial x_2} + \lambda_{21}(\theta_0) \frac{\partial^2}{\partial x_2^2} + \lambda_{22}(\theta_0) \frac{\partial^2}{\partial x_1 \partial x_2} \\
&+ \lambda_{22}(\theta_0) \frac{\partial^2}{\partial x_2^2} \right] \hat{q}_0^{[m]}(x, z_j) = f(x, z_j).
\tag{25}
\end{aligned}
\]
Equation (25) is multiplied by the fundamental solution of steady-state heat conduction problems in anisotropic solids, and then integrated over the whole domain. Integrating by parts, we finally obtain an equation similar to that in the isotropic scheme as
\[
\begin{aligned}
&\lambda_{11}(\theta_0) \frac{\partial^2}{\partial x_1^2} + \lambda_{12}(\theta_0) \frac{\partial^2}{\partial x_1 \partial x_2} + \lambda_{21}(\theta_0) \frac{\partial^2}{\partial x_2^2} + \lambda_{22}(\theta_0) \frac{\partial^2}{\partial x_1 \partial x_2} \\
&+ \lambda_{22}(\theta_0) \frac{\partial^2}{\partial x_2^2} \right] \hat{q}_0^{[m]}(x, z_j) = f(x, z_j).
\tag{26}
\end{aligned}
\]
The approximated function is treated as a function of distance in the isotropic scheme. In the anisotropic scheme, the following variable is defined in consideration of anisotropy as well as distance.
\[
w_{\lambda} = \sqrt{\frac{\lambda_{22}(\theta_0)}{\lambda_{11}(\theta_0) \lambda_{22}(\theta_0) - \lambda_{12}(\theta_0)^2}} \left( x_1 - \frac{\lambda_{12}(\theta_0)}{\lambda_{22}(\theta_0)} x_2 \right)^2
\tag{27}
\]
Let us consider the coordinate transformation
\[
\begin{aligned}
&\xi = \sqrt{\frac{\lambda_{22}(\theta_0)}{\lambda_{11}(\theta_0) \lambda_{22}(\theta_0) - \lambda_{12}(\theta_0)^2}} \left( x_1 - \frac{\lambda_{12}(\theta_0)}{\lambda_{22}(\theta_0)} x_2 \right)^2 \\
&\eta = \frac{1}{\sqrt{\lambda_{22}(\theta_0)}} x_2.
\end{aligned}
\tag{28}
\]
where
\[
\begin{aligned}
w_{\lambda} = \sqrt{\xi^2 + \eta^2}.
\end{aligned}
\tag{29}
\]
Applying Eq. (28), we obtain, instead of Eq. (24),
\[
\left( \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \hat{q}_0^{[m]}(x, z_j) = f(x, z_j).
\tag{30}
\]
If Eqs. (27) and (28) are used, the approximated function can be described as in the isotropic scheme as
\[
f(x, z_j) = \begin{cases} (1 - w_{\lambda})^3(3w_{\lambda} + 1), & w_{\lambda} \leq 1 \\ 0, & w_{\lambda} \geq 1 \end{cases}
\tag{31}
\]

Applying Eqs. (27), (28), (29) and (30), we obtain the particular solution $\hat{\theta}_{0}^{m}(x, z)$ and the normal derivative $\hat{\theta}_{0}^{m}(x, z)$, respectively, as

$$
\hat{\theta}_{0}^{m} = \frac{1}{4} w_{i}^{2} - \frac{3}{8} w_{j} w_{i}^{2} + \frac{25}{4} w_{j}^{2} - \frac{1}{12} (w_{i})^{6}
$$

and

$$
\hat{\theta}_{0}^{m} = \lambda_{11}(\dot{\theta}_{0}) w_{i}(w_{i} + \lambda_{12}(\dot{\theta}_{0}) w_{j} + \lambda_{12}(\dot{\theta}_{0}) n_{1}(w_{i}) + n_{2}(w_{j} + \lambda_{12}(\dot{\theta}_{0}) n_{1}(w_{i})) + \lambda_{12}(\dot{\theta}_{0}) w_{j})
$$

+ \lambda_{22}(\dot{\theta}_{0}) n_{1}(w_{i}) n_{2}(w_{j})

+ \lambda_{22}(\dot{\theta}_{0}) n_{1}(w_{i}) n_{2}(w_{j}) \sqrt{1 + \frac{1}{2}(w_{i})^{2} + \frac{3}{5}(w_{j})^{2} - \frac{1}{2}(w_{i}^{2})^{4}}.
$$

where

$$
(w_{i})_{1} = \xi = \sqrt{\frac{\lambda_{22}(\dot{\theta}_{0})}{\lambda_{22}(\dot{\theta}_{0}) - \lambda_{12}(\dot{\theta}_{0})^{2}} (x_{1} - \lambda_{12}(\dot{\theta}_{0}) \lambda_{22}(\dot{\theta}_{0}) x_{2}),
$$

$$
(w_{j})_{2} = \eta = \frac{1}{\sqrt{\lambda_{22}(\dot{\theta}_{0})}} x_{2}.
$$

### 2.3 Derivation of unknown coefficients $\alpha_{j}^{m}$

In this paper, the formulation is described using the approximated function and the unknown coefficients to transform the domain integrals into the boundary integrals. We need to determine the unknown coefficients to obtain temperature and heat flux at the boundary. When $m = 1$ in Eqs. (14) and (22), the following equation is obtained.

$$
- \frac{\partial \lambda_{11}}{\partial \theta_{0}} \left( \frac{\partial \theta_{0}}{\partial x_{1}} \right)^{2} - 2 \frac{\partial \lambda_{12}}{\partial \theta_{0}} \left( \frac{\partial \theta_{0}}{\partial x_{1}} \right) \left( \frac{\partial \theta_{0}}{\partial x_{2}} \right) - 2 \lambda_{12} \left( \frac{\partial^{2} \theta_{0}}{\partial x_{1} \partial x_{2}} \right)
$$

$$
- \frac{\partial \lambda_{22}}{\partial \theta_{0}} \left( \frac{\partial \theta_{0}}{\partial x_{2}} \right)^{2} - \lambda_{22} \left( \frac{\partial^{2} \theta_{0}}{\partial x_{2} \partial x_{2}} \right) + \frac{\rho c}{\Delta t} \theta_{0} - \frac{\theta_{i}}{\Delta t} = \sum_{j=1}^{N} \alpha_{i} \hat{f}(x, z_{j})
$$

We can determine the unknown coefficient $\alpha_{j}^{m}$ if the left-hand side of Eq. (35) is known at points not less than $N + L$. Therefore, the first and second derivatives of temperature with respect to the coordinates are estimated as follows. The temperature $\hat{\theta}_{0}$ can be approximated as the following polynomial, which is dependent on $x_{1}$ and $x_{2}$ coordinates.

$$
\hat{\theta}_{0} = \sum_{j=1}^{J} \beta_{i-j+1} x_{1}^{i-1} x_{2}^{j-1}
$$

Here, $\beta_{i-j+1}$ are unknown coefficients. The temperature $\hat{\theta}_{0}$, and the coordinates $x_{1}$ and $x_{2}$ are known. If the product $(I \cdot J)$ of the approximated degrees $I$ and $J$ is not more than the sum $(N + L)$ of the nodes and collocation points of DRM, the unknown coefficients are obtained by the least square method. Hence, the first and second derivatives of temperature can be obtained when Eq. (36) is differentiated.

When $m \geq 2$, the first and second derivatives of $\hat{\theta}_{0}^{m-1}$ appear. The unknown coefficients $\alpha_{j}^{m}$ can be obtained by the same procedure.

### 2.4 Notes on numerical computation

For numerical computation, Eq. (18) is discretized in the isotropic scheme and Eq. (26) is discretized in the anisotropic scheme. The first and second terms of the equations can be discretized by the standard method. The third and other terms of the equations contain the unknown coefficients $\alpha_{j}^{m}$. Because we have described the derivation in section 2.3, whole terms can be derived. Then $\hat{\theta}_{0}^{m}$ at the boundary and internal collocation points are determined, and temperature can be obtained using Eq. (12). The convergence radius $\gamma$ of Eq. (12) is dependent on material properties, initial conditions, boundary conditions, $\Delta t$ and the number of nodes. If $\gamma < 1$, Eq. (12) can approximate the following equation.

$$
\theta_{k+1}(x, (n+1)\Delta t) = \theta_{k}(x, (n+1)\Delta t) + \sum_{m=1}^{M} \frac{\alpha_{j}^{m} \hat{\theta}_{0}^{m}(x)}{m!}
$$

$$(k = 0, 1, 2, 3, \ldots)
$$

Here, $\kappa(0 < \kappa < \gamma)$ is an iteration parameter and $M$ is the order of the approximation. The root mean square of the residuals concerning Eq. (2) is obtained using Eq. (38). Computation is repeated until the root mean square is the minimum or the rate of the change in the root mean square is less than the criterion $\varepsilon_{ref}$, as shown by Eq. (39). When the procedure is advanced subsequently for $t = t_{2} + \Delta t$, $t_{3}(= t_{2} + \Delta t)$ and $t_{4}$, the solution at each time can be obtained.

$$
\Delta_{k+1}(= \text{RMS}) = \sqrt{\sum_{i=1}^{L} \left( A_{i} \dot{\theta}_{k+1}(i) \right)^{2}}
$$

$$
\frac{\Delta_{k+1} - \Delta_{k}}{\Delta_{k}} \leq \varepsilon_{ref}
$$

### 3. Numerical Results

Examples are computed by the proposed boundary element method to investigate its usefulness. The present solutions are compared with FEML solutions. We use MSC. visualNastran for Windows for FEM computation. Dimensionless numbers are used in these computations.

#### 3.1 Analysis case I

Figure 1 shows the analysis model of a square plate. The whole boundary is divided into 20 quadratic elements and 81 internal collocation points are located in the domain. The boundary condition and the initial conditions are as follows.

**Boundary condition:**

$$
\Theta = 0 \quad (x_{1} = 0, 0.5; \ x_{2} = 0, 0.5)
$$

**Initial conditions:**

$$
\Theta(x_{1}, x_{2}, 0) = \sin(2\pi x_{1}) \sin(2\pi x_{2}), \ S_{1} = 0
$$

The material properties are

Series A, Vol. 49, No. 2, 2006

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\[ \lambda_{11} = 0.75e^{\theta} + 0.5e^{2\theta}, \]
\[ \lambda_{12} = \lambda_{21} = 0.43301e^{\theta} - 0.86603e^{2\theta}, \]
\[ \lambda_{22} = 0.25e^{\theta} + 1.5e^{2\theta}, \rho_c = 1.0. \]

(42)

The time interval \( \Delta t \) is
\[ \Delta t = 0.00001. \]

(43)

In the anisotropic scheme, we need to set the constant temperatures \( \dot{\theta}_0 \). These are
\[ \dot{\theta}_0 = 0, 0.5, 1.0. \]

(44)

The approximated degrees \( I \) and \( J \) of polynomial (36) are 7. The order of the approximation \( M \) is 1 while temperature is computed using Eq. (37). In this computation, the minimum root mean square is used.

Figure 2 shows the temperature at \( x_2 = 0.25 \) with \( \kappa = 0.0001 \). The symbols in Fig. 2 are the same as those in the other figures described below. Figure 3 shows the relation between the iteration parameter \( \kappa \) and the error of the temperature. The error \( Er \) can be defined as
\[ Er[\%] = 100 \frac{K}{\kappa} \left( \frac{1}{N_\Omega} \sum_{j=1}^{N_\Omega} \left| \theta_i(j) - \theta_i^{FEM}(j) \right| \right) \]
where \( K \) is the total number of stepping times (\( K = 50 \)) and \( \theta_i^{FEM} \) is the FEM solution in which the domain is equally divided into 100 quadratic elements. Then, we can preset the convergence tolerance for temperature, the convergence criterion(10) is 0.001 for the computation. Therefore, the FEM solution converges sufficiently, so that we think it is adequate for the error estimation. The figure shows that the accuracy is stable until \( \kappa = 0.0014 \) in the isotropic scheme and \( \kappa = 0.0020 \) at \( \dot{\theta}_0 = 0, \kappa = 0.0048 \) at \( \dot{\theta}_0 = 0.5 \) and \( \kappa = 0.0110 \) at \( \dot{\theta}_0 = 1.0 \) in the anisotropic scheme. In this example, the accuracy for the isotropic scheme is better than that for the anisotropic scheme.

Figure 4 shows the relation between the iteration parameter \( \kappa \) and the total number of iterations. The total
number of iterations decreases when the iteration parameter $\kappa$ becomes large in a stable zone of accuracy. The total number of iterations for the anisotropic scheme is larger than that for the isotropic scheme.

3.2 Analysis case 2

Figure 5 shows the analysis model of a square plate which is divided identically as shown in Fig. 1 of analysis case 1. The boundary conditions and the initial conditions are as follows.

Boundary conditions:
\[ \Theta(0, x_2) = 0, \quad Q(0, x_2) = 0 \]

Initial conditions:
\[ \Theta(x_1, x_2, 0) = \sin(\pi x_1), \quad S_1 = 0 \]

The material properties are
\[ \lambda_{11} = 1.134 + 2.067\Theta, \]
\[ \lambda_{12} = \lambda_{21} = -0.5 - 0.25\Theta, \]
\[ \lambda_{22} = 2.866 + 2.933\Theta, \quad \rho c = 1.0. \]

The quantities for the computation, the time interval $\Delta t$ and other parameters are the same quantities as those in analysis model 1.

Figure 6 shows the temperature at $x_2 = 0.25$ with $\kappa = 0.00005$.

Figure 7 shows the relation between the iteration parameter $\kappa$ and the error $E_r$. The error $E_r$ is defined by Eq.(45) as in analysis model 1. The accuracy is stable until $\kappa = 0.00075$ in the isotropic scheme and $\kappa = 0.00085$ at $\dot{\theta} = 0$, $\kappa = 0.00165$ at $\dot{\theta} = 0.5$ and $\kappa = 0.00245$ at $\dot{\theta} = 1.0$ in the anisotropic scheme. The stable zone for the accuracy in the anisotropic scheme is larger than that in the isotropic scheme. A different result in which the accuracy for the anisotropic scheme is better than that for the isotropic scheme appears in this case compared with that in analysis model 1.

4. Conclusion

Two-dimensional nonlinear transient heat conduction problems in anisotropic solids is analyzed using the ho-
motopy boundary element method originally proposed by Liao and Chwang using the dual reciprocity method with RBF. Through this method, integral equation can be expressed in terms of only boundary integral terms. This means that we can analyze the above problems by the discretization of the boundary using conventional boundary elements and the arrangement of internal collocation points. Through computations of some examples based on these formulations, we compared the solution in the isotropic scheme, the solution in the anisotropic scheme and the FEM solution. It is revealed that the present method can provide good solutions.

In this paper, we have studied two-dimensional problems. As a future work, we may recommend this work to be extended to three-dimensional problems.

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