On the pulsating strings in $\text{AdS}_5 \times T^{1,1}$

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Abstract
We study the class of pulsating strings in $\text{AdS}_5 \times T^{1,1}$. Using a generalized ansatz for pulsating string configurations, we find new solutions of this class. Further we semiclassically quantize the theory and obtain the first correction to the energy. The latter, due to AdS/CFT correspondence, is supposed to give the anomalous dimensions of operators in the dual $\mathcal{N} = 1$ superconformal gauge field theory.

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1. Introduction

The attempt to establish a correspondence between the large-$N$ limit of gauge theories and string theory has more than 30 years of history and over the years it has shown different faces. Recently an explicit realization of this correspondence was provided by the Maldacena conjecture about AdS/CFT correspondence [1]. The convincing results from the duality between type IIB string theory on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory [1–3] made this subject a major research area, and many fascinating new features have been established.

After the impressive achievements in the most supersymmetric example of AdS/CFT correspondence, namely $\text{AdS}_5 \times S^5$, it is important to extend the considerations to less supersymmetric gauge theories, moreover that the latter are more interesting from a physical point of view. There are several ways to find a theory with less supersymmetry. The experience from AdS/CFT correspondence suggests that one method is to take a stack of $N$ D3 branes and place them not in flat space, but at the apex of a conifold [4]. This model possesses many interesting features and allows one to build gauge theory operators of great physical importance, although there are hints that superstring theory on the resulting ten-dimensional spacetime, which takes form as the direct product $\text{AdS}_5 \times T^{1,1}$, is not integrable [5]. Since [4], infinite families of five-dimensional Sasaki–Einstein spaces, complementing the AdS space, have been constructed [6, 7], and also their gauge theory duals were identified [8–11]. Further developments can be traced in [6–20].
Semiclassical strings have played, and still play, an important role in studying various aspects of AdS5/SYM$_4$ correspondence [21–42]. The development in this subject gives strong hints about how the new emergent duality can be investigated. An important class of semiclassical string solutions is the class of pulsating strings introduced first in [43], and generalized further in [44–46]. In the case of an AdS$_5 \times T^{1,1}$ background, the pulsating strings are also expected to play an essential role, but thorough analysis and semiclassical quantization are still missing. The purpose of this paper is to analyse and semiclassically quantize the class of pulsating strings on the $T^{1,1}$ part of the AdS$_5 \times T^{1,1}$ background. The first correction to the energy, which according to the AdS/CFT conjecture gives the anomalous dimensions of gauge theory operators, will be the main subject of our considerations.

The paper is organized as follows. In the introduction, we provide some details about the dual gauge theory. In the next section, we present the pulsating strings and their semiclassical quantization for the case of the AdS$_5 \times T^{1,1}$ background, restricting the string dynamics to the $T^{1,1}$ part. The third section is devoted to the derivation of the correction to the energy. First we find the wavefunction associated with the Laplace–Beltrami operator on $T^{1,1}$, and then we compute the leading correction to the energy. We conclude with a brief discussion on the results.

**Dual field theory**

The respective $\mathcal{N} = 1$ superconformal gauge theory dual to string theory in AdS$_5 \times T^{1,1}$ is known as the Klebanov–Witten theory and was originally described in [4]. The theory has flavour symmetry $SU(2) \times SU(2)$. The elementary degrees of freedom are denoted by the fields $A$ and $B$, each a doublet of the factor $SU(2)$ groups and with conformal anomalous dimensions $\Delta_{A,B} = 3/4$. The gauge group is $SU(N) \times SU(N)$, and the two chiral multiplets $A$ and $B$ are correspondingly in the $(N, \bar{N})$ and $(\bar{N}, N)$ representations. The superpotential is

$$ W = \frac{\lambda}{2} e^{i/2} \frac{i}{e^{i/2}} \text{Tr} [A, B_i A, B_i], $$

where $i = 1, 2$. The chiral operators analogue of the $(X, Y, Z)$ operators in $\mathcal{N} = 4$ SYM are given by $\text{Tr}(AB)^k$ with R-charge $k$ and in the $(k/2, k/2)$ representation of the flavour group $SU(2) \times SU(2)$.

**2. Pulsating strings in AdS$_5 \times T^{1,1}$**

In this section, we consider a circular pulsating string expanding and contracting only on the $T^{1,1}$ part of AdS$_5 \times T^{1,1}$ ($\mathbb{R} \times T^{1,1}$). Then, the relevant metric we will work with is given by

$$ ds^2_{\text{p}^{1,1}} = R^2 (-dr^2 + ds^2_{T^{1,1}}), $$

where the metric of $T^{1,1}$ is

$$ ds^2_{T^{1,1}} = \frac{b}{4} \left[ \sum_{i=1}^2 (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + b \left( d\psi + \sum_{i=1}^2 \cos \theta_i d\phi_i \right)^2 \right], $$

where $0 \leq \psi < 4\pi, 0 \leq \theta_i \leq \pi, 0 \leq \phi_i < 2\pi$ and $b = 3/2$ (for $b = 1$ we recover $S^3$). Having in mind the explicit form of the $T^{1,1}$ metric (3), one can write it as

$$ ds^2_{T^{1,1}} = G_{ij} dx^i dx^j + \tilde{G}_{pq} dy^p dy^q, $$

where $G_{ij}$ is defined by

$$ G_{ij} = \text{diag} \left( \frac{b}{4}, \frac{b}{4} \right), \quad i, j = 1, 2, \quad x^1 = \theta_1, \quad x^2 = \theta_2, $$

and

$$ \tilde{G}_{pq} = \frac{b}{4} \text{diag} \left( \frac{b}{4}, \frac{b}{4} \right), \quad p, q = 1, 2, \quad y^1 = \phi_1, \quad y^2 = \phi_2. $$
Straightforward calculations yield

\[
(\hat{G}_{pq}) = \frac{b}{4} \begin{pmatrix}
  b \cos^2 \theta_1 + \sin^2 \theta_1 & b \cos \theta_1 \cos \theta_2 & -b \cos \theta_1 \\
  b \cos \theta_1 \cos \theta_2 & b \cos^2 \theta_2 + \sin^2 \theta_2 & -b \cos \theta_2 \\
  -b \cos \theta_1 & -b \cos \theta_2 & b
\end{pmatrix}.
\]

The residual worldsheet symmetry allows us to identify \( t \) with \( \tau \), and to obtain a classical pulsating string solution, we use the following ansatz:

\[
x^1 = x^1(\tau) = \theta_1(\tau), \quad x^2 = x^2(\tau) = \theta_2(\tau),
\]

\[
y^1 = \phi_1 = m_1 \sigma + h^1(\tau), \quad y^2 = \phi_2 = m_2 \sigma + h^2(\tau), \quad y^3 = \psi = m_3 \sigma + h^3(\tau).
\]

We are interested in the induced worldsheet metric, which in our case has the form

\[
d\sigma^2 = R^2 \left[ (1 - G_{ij} \dot{x}^i \dot{x}^j - \hat{G}_{pq} \dot{h}^p \dot{h}^q) \right] d\tau^2 + (\hat{G}_{pq} p_m p_q) d\sigma^2 + 2 (\hat{G}_{pq} p_m \dot{h}^q) d\tau d\sigma.
\]

The Nambu–Goto action

\[
S_{NG} = -T \int d\tau d\sigma \sqrt{-\det(\hat{G}_{\mu\nu} \partial_\mu X^\nu \partial_\beta X^\beta)}
\]

in this ansatz then reduces to the expression

\[
S_{NG} = -TR^2 \int d\tau d\sigma \sqrt{(1 - G_{ij} \dot{x}^i \dot{x}^j - \hat{G}_{pq} \dot{h}^p \dot{h}^q) (\hat{G}_{pq} p_m p_q) + (\hat{G}_{pq} p_m \dot{h}^q)^2},
\]

where \( TR^2 = \sqrt{\lambda} \). For our considerations, it is useful to pass to Hamiltonian formulation. For this purpose, we have to find first the canonical momenta of our dynamical system. Straightforward calculations yield

\[
\Pi_i = \sqrt{\lambda} \frac{(\hat{G}_{pq} p_m p_q) G_{ij} \dot{x}^j}{\sqrt{(1 - G_{ij} \dot{x}^i \dot{x}^j - \hat{G}_{pq} \dot{h}^p \dot{h}^q) (\hat{G}_{pq} p_m p_q) + (\hat{G}_{pq} p_m \dot{h}^q)^2}}, \quad i = 1, 2,
\]

\[
\hat{p}_p = \sqrt{\lambda} \frac{(\hat{G}_{pq} p_m p_q) \hat{G}_{pq} \dot{h}^q}{\sqrt{(1 - G_{ij} \dot{x}^i \dot{x}^j - \hat{G}_{pq} \dot{h}^p \dot{h}^q) (\hat{G}_{pq} p_m p_q) + (\hat{G}_{pq} p_m \dot{h}^q)^2}}, \quad p = 1, 2, 3,
\]

which also implies the constraint

\[
m_p \hat{p}_p = 0.
\]

Solving for the derivatives in terms of the canonical momenta and substituting back into the Legendre transform of the Lagrangian, we find the Hamiltonian

\[
H^2 = G^{ij} \Pi_i \Pi_j + \hat{G}^{pq} \hat{p}_p \hat{p}_q + \lambda (\hat{G}_{pq} p_m p_q).
\]

The interpretation of this relation is as in the case of \( \text{AdS}_5 \times S^5 \) [45]. Namely, the first two terms represent kinetic energy, while the last one is considered as a potential \( V \), which in our case has the form

\[
V(\theta_1, \theta_2) = \lambda \hat{G}_{pq}(\theta_1, \theta_2) p_m p_q.
\]

The approximation where our considerations are valid assumes high energies, which suggests that one can think of this potential term as a perturbation. For later use, we write down the explicit form of the potential

\[
V(\theta_1, \theta_2) = \frac{b}{4} \left[ (b \cos^2 \theta_1 + \sin^2 \theta_1) m_1^2 + (b \cos^2 \theta_2 + \sin^2 \theta_2) m_2^2 + bm_3^2 + 2b \cos \theta_1 \cos \theta_2 m_1 m_2 - 2b \cos \theta_1 m_1 m_3 - 2b \cos \theta_2 m_2 m_3 \right].
\]
or
\[ V(\theta_1, \theta_2) = \frac{b}{4} \left[ \sum_{i=1}^{2} m_i^2 \sin^2 \theta_i + b \left( m_3 - \sum_{i=1}^{2} m_i \cos \theta_i \right)^2 \right] . \]  

The above perturbation to the free action will produce the correction to the energy and therefore the anomalous dimension. In order to calculate the correction to the energy as a perturbation due to the above potential, however, we need the normalized wavefunction associated with the \( T^{1,1} \) space. All these issues are addressed in the following section.

3. Semiclassical correction to the energy

In this section, we compute the semiclassical correction to the energy of the pulsating string on \( T^{1,1} \). As we discussed in the previous section, the Hamiltonian of the pulsating string is interpreted as a dynamical system with high energy described by the free theory Schrödinger equation and perturbed by potential (17).

First, we have to find the wavefunction associated with the Laplace–Beltrami operator on \( T^{1,1} \) and then to obtain the correction to the energy due to the induced potential.

3.1. Laplace–Beltrami operator and wavefunction

The line element of \( T^{1,1} \) in global coordinates is explicitly given by (3)
\[ ds^2_{T^{1,1}} = \frac{b}{4} \left[ \sum_{i=1}^{2} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + b \left( d\psi + \sum_{i=1}^{2} \cos \theta_i d\phi_i \right)^2 \right] . \]  

3.1.1. Laplace–Beltrami operator on \( T^{1,1} \).

Using the standard definition of the Laplace–Beltrami operator in global coordinates, we find (see section 3 of [47] for the general case of \( T^{p,q} \))
\[ \Delta_{T^{1,1}} = \frac{4}{b^2} \left[ \frac{b}{\sin \theta_1} \frac{\partial}{\partial \theta_1} \left( \frac{\sin \theta_1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \right) + \frac{b}{\sin^2 \theta_1} \frac{\partial^2}{\partial \phi_1^2} + \frac{2b \cos \theta_1}{\sin^2 \theta_1} \frac{\partial^2}{\partial \phi_1 \partial \psi^2} + \frac{b}{\sin^2 \theta_1} \frac{\partial^2}{\partial \phi_1 \partial \theta_2} + \left( 1 + \frac{b \cos \theta_1}{\sin^2 \theta_1} + \frac{b \cos \theta_2}{\sin^2 \theta_2} \right) \frac{\partial^2}{\partial \psi^2} \right] , \]  

or
\[ \Delta_{T^{1,1}} = \frac{4}{b^2} \left[ b \left( \frac{1}{\sin \theta_1} \frac{\partial}{\partial \theta_1} \left( \frac{\sin \theta_1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \right) + \frac{1}{\sin^2 \theta_1} \left( \frac{\partial}{\partial \phi_1} + \cos \theta_1 \frac{\partial}{\partial \psi} \right)^2 \right] + \right. \]
\[ + \left. b \left( \frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left( \frac{\sin \theta_1}{\sin \theta_2} \frac{\partial}{\partial \theta_1} \right) + \frac{1}{\sin^2 \theta_2} \left( \frac{\partial}{\partial \phi_2} + \cos \theta_2 \frac{\partial}{\partial \psi} \right)^2 \right] + \frac{\partial^2}{\partial \psi^2} \right] . \]  

The full measure on \( T^{1,1} \) is
\[ d\Omega(\theta_1, \theta_2) = \sqrt{\det(G_{\mu\nu})} d\theta_1 d\theta_2 = 2 \left( \frac{b}{4} \right)^3 \sin \theta_1 \sin \theta_2 d\theta_1 d\theta_2 . \]
3.1.2. Wavefunction. The Schrödinger equation for the wavefunction is

$$\Delta_{T^1} \Psi(\theta_1, \theta_2, \phi_1, \phi_2, \psi) = -E^2 \Psi(\theta_1, \theta_2, \phi_1, \phi_2, \psi).$$  \hspace{1cm} (23)$$

To separate the variables, we define $\Psi$ as

$$\Psi(\theta_1, \theta_2, \phi_1, \phi_2, \psi) = f_1(\theta_1) f_2(\theta_2) f(\phi_1, \phi_2, \psi),$$  \hspace{1cm} (24)$$

where

$$f(\phi_1, \phi_2, \psi) = \exp(i l_1 \phi_1) \exp(i l_2 \phi_2) \exp(i l_3 \psi), \quad l_1, l_2, l_3 \in \mathbb{Z}.$$  \hspace{1cm} (25)$$

With this choice we can solve for the eigenfunctions, replacing the derivatives along Killing directions $(\partial_{\theta_1}, \partial_{\theta_2}, \partial_{\phi})$ by $(l_1, l_2, l_3)$, correspondingly. Note that (14) implies the following relation among the parameters $l_1, l_2, l_3$:

$$m_1 l_1 + m_2 l_2 + m_3 l_3 = 0.$$  \hspace{1cm} (26)$$

Substituting (24) into (23), together with (21), we arrive at

$$E^2 = \frac{4}{b^2} \left( (b E_1^2 + b E_2^2 + l_3^2) \right),$$  \hspace{1cm} (27)$$

where $E_1$ and $E_2$ are determined by the ordinary differential equations

$$\left[ \frac{1}{\sin \theta_1} \frac{d}{d \theta_1} \left( \sin \theta_1 \frac{d}{d \theta_1} \right) \right] \left( l_i + \cos \theta_i l_3 \right) f_i(\theta_i) = -E_i^2 f_i(\theta_i), \quad i = 1, 2.$$  \hspace{1cm} (28)$$

It is convenient to define new variables $z_i = \cos \theta_i$. Then the equations can be written as

$$\left( 1 - z_i^2 \right) \frac{d^2}{dz_i^2} - 2z_i \frac{d}{dz_i} - \frac{1}{1 - z_i^2} (l_i + z_i l_3)^2 + E_i^2 \right) f_i(z_i) = 0.$$  \hspace{1cm} (29)$$

The solutions to these equations are

$$f_i(z_i) = \left( 1 - z_i \right) \frac{1}{\left( 1 + z_i \right)^{1/2}} \left( 1 + z_i \right) \frac{1}{\left( 1 - z_i \right)^{1/2}} 2F_1 \left[ \frac{1}{2} \left( |l_i + l_3| + |l_i - l_3| + 1 - \sqrt{1 + 4(l_3^2 + E_i^2)} \right), \right.$$

$$\left. \frac{1}{2} \left( |l_i + l_3| + |l_i - l_3| + 1 + \sqrt{1 + 4(l_3^2 + E_i^2)} \right); 1 + |l_i - l_3| : \frac{1 + z_i}{2} \right].$$  \hspace{1cm} (30)$$

In addition, we have to ensure that the solutions $f_i(\theta_i)$ are square integrable with respect to the measure for $\theta_i$, which leads to the following restrictions on the parameters:

$$\sqrt{1 + 4(l_3^2 + E_i^2)} - |l_i + l_3| - |l_i - l_3| - 1 = 2n_i, \quad n_i \in \mathbb{N}.$$  \hspace{1cm} (31)$$

From (27) and (31) it follows that the squares of the bare dimensions of dual operators are

$$\Delta^2 \equiv E^2 = \frac{4}{b^2} \left( \frac{b}{2} \sum_{i=1}^{3} (2n_i + |l_i + l_3| + |l_i - l_3| + 1)^2 + (1 - 2b)l_3^2 - \frac{b}{2} \right).$$  \hspace{1cm} (32)$$

Introducing new parameters $\alpha_i \equiv |l_i - l_3|$ and $\beta_i \equiv |l_i + l_3|$, the solution can be written in terms of Jacobi polynomials,

$$f_i(z_i) = \left( 1 - z_i \right)^{\alpha_i/2} \left( 1 + z_i \right)^{\beta_i/2} \frac{n_i! \Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + 1 + n_i)} P_{n_i}^{(\alpha_i, \beta_i)}(z_i).$$  \hspace{1cm} (33)$$

The normalized wavefunctions look like

$$\Psi_{n_i}^{\alpha_i, \beta_i}(z_i) = \left( \frac{(\alpha_i + \beta_i + 1 + 2n_i) n_i! \Gamma(\alpha_i + \beta_i + 1 + n_i)}{2^{\alpha_i+\beta_i+1} \Gamma(\alpha_i + 1 + n_i) \Gamma(\beta_i + 1 + n_i)} \right)^{1/2} \times \left( 1 - z_i \right)^{\alpha_i/2} \left( 1 + z_i \right)^{\beta_i/2} P_{n_i}^{(\alpha_i, \beta_i)}(z_i), \quad i = 1, 2.$$  \hspace{1cm} (34)$$
3.2. Leading correction to the energy

It is convenient to write potential (17) in the following form:

\[
V(\theta_1, \theta_2) = \lambda \frac{b}{4} \left[ b \sum_{i=1}^{3} m_i^2 + (1 - b) \sum_{i=1}^{2} m_i^2 \sin^2 \theta_i + 2bm_1m_2 \cos \theta_1 \cos \theta_2 - 2bm_3 \sum_{i=1}^{2} m_i \cos \theta_i \right].
\]

In terms of the new variables \(z_i\), the potential and measure (22) become

\[
V(z_1, z_2) = \lambda \frac{b}{4} \left[ b \sum_{i=1}^{3} m_i^2 + (1 - b) \sum_{i=1}^{2} m_i^2 (1 - z_i^2) + 2b \left(m_1m_2z_1z_2 - m_3 \sum_{i=1}^{2} m_i z_i\right) \right].
\]

The first correction to the energy is given then by the expression

\[
\delta E^2 = \int_{-1}^{1} \int_{-1}^{1} d\Omega(z_1, z_2) V(z_1, z_2) \left[ \Psi_{n_1}^{a_1, \beta_1}(z_1) \right]^2 \left[ \Psi_{n_2}^{a_2, \beta_2}(z_2) \right]^2.
\]

The explicit form of the correction to the energy is obtained by plugging the various wavefunctions and the potential in (38)

\[
\delta E^2 = \lambda \left( \frac{b}{4} \right)^4 \left[ b \sum_{i=1}^{3} m_i^2 + (1 - b) \sum_{i=1}^{2} m_i^2 \int_{-1}^{1} dz_i (1 - z_i^2) \left[ \Psi_{n_i}^{a_i, \beta_i}(z_i) \right]^2 + 2bm_1m_2 \right.
\]

\[
\times \int_{-1}^{1} dz_1 z_1 \left[ \Psi_{n_1}^{a_1, \beta_1}(z_1) \right]^2 \int_{-1}^{1} dz_2 z_2 \left[ \Psi_{n_2}^{a_2, \beta_2}(z_2) \right]^2
\]

\[
- 2bm_3 \sum_{i=1}^{2} m_i \int_{-1}^{1} dz_i z_i \left[ \Psi_{n_i}^{a_i, \beta_i}(z_i) \right]^2 \right).
\]

In short notations, it looks like

\[
\delta E^2 = \lambda \left( \frac{b}{4} \right)^4 \left[ b \sum_{i=1}^{3} m_i^2 + (1 - b) \sum_{i=1}^{2} m_i^2 l_1^i + 2bm_1m_2l_1^2 - 2bm_3 \sum_{i=1}^{2} m_i l_2^i \right],
\]

where the integrals \(l_1^i\) and \(l_2^i\) are explicitly calculated as follows:

\[
l_1^i = \int_{-1}^{1} dz_i (1 - z_i^2) \left[ \Psi_{n_i}^{a_i, \beta_i}(z_i) \right]^2 = \left(n_i + \alpha_i + \beta_i + 1\right) \left(n_i + \alpha_i + \beta_i + 2\right) \left(n_i + \alpha_i + \beta_i + 3\right)
\]

\[
\times \left(2n_i + \alpha_i + \beta_i + 1\right) \left(2n_i + \alpha_i + \beta_i + 2\right) \left(2n_i + \alpha_i + \beta_i + 3\right)
\]

\[
+ n_i \left(n_i + \alpha_i + \beta_i + 1\right) \left(2n_i + \alpha_i + \beta_i + 2\right) ^2
\]

\[
- \frac{n_i + \alpha_i + 1}{n_i (n_i - 1)(n_i + \alpha_i)(n_i + \beta_i)} \left(2n_i + \alpha_i + \beta_i + 2\right) ^2
\]

\[
+ \frac{n_i + \alpha_i + 1}{n_i (n_i - 1)(n_i + \alpha_i)(n_i + \beta_i)} \left(2n_i + \alpha_i + \beta_i + 1\right) \left(2n_i + \alpha_i + \beta_i + 2\right) ^2.
\]

\[
l_2^i = \int_{-1}^{1} dz_i z_i \left[ \Psi_{n_i}^{a_i, \beta_i}(z_i) \right]^2 = \frac{2(n_i + \beta_i) (n_i + \alpha_i + \beta_i)}{\left(2n_i + \alpha_i + \beta_i + 1\right) \left(2n_i + \alpha_i + \beta_i + 2\right)} + \frac{2(n_i + 1)(n_i + \alpha_i + 1)}{\left(2n_i + \alpha_i + \beta_i + 1\right) \left(2n_i + \alpha_i + \beta_i + 2\right)}.
\]
The expression for the correction to the energy looks very complicated. Therefore, we use the fact that the approximation we work in is for large quantum numbers, say $n_{1,2} \gg \alpha_{1,2}(\beta_{1,2}) \gg 1$. Within this approximation, the integrals behave like

$$I_1^i = \frac{1}{8} + \frac{1}{32} \left( 2\alpha_{i}^2 + 2\beta_{i}^2 - 1 \right) \frac{1}{n_{i}^2} + O \left( \frac{1}{n_{i}^3} \right) , \quad (43)$$

$$I_2^i = 1 + \frac{1}{4} \left( \beta_{i}^2 - \alpha_{i}^2 \right) \frac{1}{n_{i}^2} + O \left( \frac{1}{n_{i}^3} \right) . \quad (44)$$

Since $\alpha_i \equiv |l_i - l_3|$ and $\beta_i \equiv |l_i + l_3|$, the above integrals look like

$$I_1^i = \frac{1}{8} + \frac{1}{8} \left( l_i^2 + l_3^2 - \frac{1}{4} \right) \frac{1}{n_{i}^2} + O \left( \frac{1}{n_{i}^3} \right) , \quad (45)$$

$$I_2^i = 1 - \frac{l_3 l_i}{n_{i}^2} + O \left( \frac{1}{n_{i}^3} \right) . \quad (46)$$

Ignoring the terms of higher order, we obtain

$$\delta E^2 \approx \kappa \left( \frac{b}{4} \right)^4 \left[ b \left( m_3 - \sum_{i=1}^2 m_i \right)^2 + \frac{1 - b}{8} \sum_{i=1}^2 m_i^2 \left( 1 - \frac{1}{4n_i^2} + \frac{l_i^2 + l_3^2}{n_{i}^2} \right) \right. \right.$$  

$$+ \left. 2b \sum_{i=1}^2 \left( m_i m_3 - m_1 m_2 \right) \frac{l_3 l_i}{n_{i}^2} \right] . \quad (47)$$

### 4. Conclusion

Our study is motivated by the recently suggested duality between string theory in $\text{AdS}_5 \times T^{1,1}$ and $\mathcal{N} = 1$ superconformal field theory. The results obtained so far [4–19] provide important understanding of string/gauge theory dualities, particularly in the region of strong coupling. The purpose of this paper is to investigate the pulsating string solutions in the AdS$_5 \times T^{1,1}$ background. The class of pulsating strings has been used to study the AdS/CFT correspondence in the case of AdS$_5 \times S^5$ [43, 45, 46], and the leading correction to the string energy has been associated with anomalous dimensions of certain operators in the dual gauge theory.

Here we consider a generalized string ansatz for a pulsating string in the $T^{1,1}$ part of the geometry. Next we derive the correction to the classical energy. From the AdS/CFT point of view, the correction gives the anomalous dimensions of operators in SYM theory and therefore it is of primary interest. For this purpose, we consider the Nambu–Goto action and find the Hamiltonian. After that we quantize the resulting theory semiclassically and obtain the correction to the energy. Since we consider a highly excited system, the kinetic term is dominating. This means that we effectively perform summation over all classical solutions (not only those that have been explicitly found), while the effective potential term serves as a small perturbation. The obtained correction to the classical energy looks complicated, but in a certain limit, one can find a relatively simple expression. To identify the contributions of the different terms, it is instructive to look at the solutions for the $S^5$ case. Since they correspond to a subsector well known from AdS$_5 \times S^5$ considerations, one can identify the origin of the various contributions. One can see that the correction to the energy in $T^{1,1}$ has an analogous structure to the case of pulsating strings in $S^5$, for example. The mixing between quantum numbers of different isometry directions shows up in an analogous, but slightly more
complicated, way. This can be seen using the result from the $S^3$ subsector and its embedding in $T^1,1$.

As a final comment, we note that in order to complete the study from the AdS/CFT point of view, it is of great importance to perform an analysis comparing our result to that on the SYM side. We leave this problem for future research.

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