A unified framework of continuous and discontinuous Galerkin methods for solving the incompressible Navier–Stokes equation

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Abstract

In this paper, we propose a unified numerical framework for the time-dependent incompressible Navier–Stokes equation which yields the $H^1$, $H(\text{div})$-conforming, and discontinuous Galerkin methods with the use of different viscous stress tensors and penalty terms for pressure robustness. Under minimum assumption on Galerkin spaces, the semi- and fully-discrete stability is proved when a family of implicit Runge–Kutta methods are used for time discretization. Furthermore, we present a unified discussion on the penalty term. Numerical experiments are presented to compare our schemes with classical schemes in the literature in both unsteady and steady situations. It turns out that our scheme is competitive when applied to well-known benchmark problems such as Taylor–Green vortex, Kovasznay flow, potential flow, lid driven cavity flow, and the flow around a cylinder.

Keywords: incompressible Navier–Stokes equation, discontinuous Galerkin method, mixed finite element method, energy stability, implicit Runge–Kutta methods, pressure robustness

1. Introduction

Continuous and discontinuous Galerkin methods for the incompressible Navier–Stokes (NS) equation have been an active research area and extensively studied, see, e.g., \textsuperscript{15} \textsuperscript{19} \textsuperscript{26} \textsuperscript{31} \textsuperscript{46} and references therein. Most of the classical $H^1$-conforming finite element methods weakly enforce the divergence free constraint and suffer from a loss of velocity accuracy due to the influence of pressure approximation and small viscosity, see, e.g., \textsuperscript{28}. To remedy the situation, one popular approach by Franca and Hughes \textsuperscript{17} is to add the grad-div stabilization term. Many works can be found in this direction, from both theoretical and computational point of view, see, e.g., \textsuperscript{36} \textsuperscript{37} \textsuperscript{38} \textsuperscript{39}. Recent research has shown that the grad-div stabilization is a penalization procedure \textsuperscript{3} \textsuperscript{27} \textsuperscript{34}, and large grad-div stabilization parameters might lead to Poisson locking phenomena if the finite element method is not inf-sup stable in the limiting case \textsuperscript{27}. To completely decouple the pressure and velocity, one may use the $H(\text{div})$-conforming methods. With the help of a carefully designed velocity and pressure finite element pair \textsuperscript{3} \textsuperscript{16} \textsuperscript{41}, the numerical velocity is actually pointwise divergence-free and

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Finally for two-dimensional incompressible flows, one may use the vorticity-stream formulation to automatically enforce the divergence-free constraint. For pressure robustness, discontinuous Galerkin (DG) methods usually penalize the jump of the velocity normal component. In , a new inf-sup condition involving the jump of pressure is constructed for steady incompressible NS equation and optimal convergence is observed when $P_{k+1} \times P_k$ DG space for velocity and pressure is used. In , a element-wise grad-div penalization has been used on tensor product meshes for non-isothermal flow, and an improvement of mass conservation is observed for both inf-sup stable $P_{k+1} \times P_k$ and $P_k \times P_k$ pairs. Readers are also referred to for DG methods with more than two variables. In particular, achieves pointwise divergence-free velocity by $H(\text{div})$-conforming finite element subspace, while a postprocessed divergence-free numerical velocity is obtained in .

In this paper, we present a unified framework for the spatial discretization of the time-dependent incompressible NS equation that covers the $H^1$-conforming, $H(\text{div})$-conforming, and DG methods including penalty term for pressure robustness and upwinding term for convection. With carefully designed numerical fluxes and consistent terms in the unified scheme, the semi-discrete stability for the first time is proved in Theorem for the time-dependent incompressible NS equation under minimal assumption on Galerkin spaces. Furthermore, a unified discussion on the penalty term for pressure robustness is presented, and thus, the motivation of penalization in $H^1$-conforming, $H(\text{div})$-conforming, and DG methods is quite transparent, see Section . Another distinct feature of this paper is the use of a family of implicit Runge–Kutta methods for time discretization of the NS equation, which is shown to guarantee fully-discrete kinetic energy stability. To the best of our knowledge, such stability analysis could not be found in existing literature, see Theorem for details.

In contrast to previous works, our numerical scheme incorporates the classical stress tensor $\tau_h = \nu \nabla h u_h$ or $\nu(\nabla h u_h + \nabla h u_h^T)$ as well as the full viscous stress tensor $\tau_h = \nu \big( \nabla h u_h + \nabla h u_h^T - \frac{2}{3} (\nabla h \cdot u_h) I \big)$. Due to the divergence-free constraint, the variational formulation based on $\tau = \nu \nabla u$ or $\tau = \nu (\nabla u + \nabla u^T)$ could be recovered from the corresponding formulation based on $\tau = \nu \big( \nabla u + \nabla u^T - \frac{2}{3} (\nabla \cdot u) I \big)$ at the continuous level. However, the equivalence breaks down at the discrete level because of insufficient regularity. Therefore, it is meaningful to check the numerical performance of those numerical methods based on the full viscous stress tensor. In Section , we shall test the performance of our schemes with full viscous stress tensor applied to a number of steady and unsteady benchmark problems.

Preliminary notations for numerical methods are introduced in the rest of this section. We use $T_h$ to denote a conforming and shape-regular simplex mesh on a bounded Lipschitz domain $\Omega$ in $\mathbb{R}^d$ where $d \in \{2, 3\}$. For each element $K \in T_h$, let $h_K$ denote the diameter of $K$. Let $F_h$ be the collection of faces of $T_h$ with $F_h^i$ the set of interior faces and $F_h^b$ the set of boundary faces. For any $(d-1)$-dimensional set $\Sigma$, we
use \( \langle \cdot, \cdot \rangle_{\Sigma} \) to denote the \( L^2 \) inner product on \( \Sigma \), and

\[
\langle \cdot, \cdot \rangle_{\partial T_h} := \sum_{K \in T_h} \langle \cdot, \cdot \rangle_{\partial K}, \quad \langle \cdot, \cdot \rangle_{\partial T_h^+} := \sum_{K \in T_h} \langle \cdot, \cdot \rangle_{\partial K \setminus \partial \Omega},
\]

\[
\langle \cdot, \cdot \rangle_{F_h} := \sum_{F \in F_h} \langle \cdot, \cdot \rangle_{F}, \quad \langle \cdot, \cdot \rangle_{F_h^+} := \sum_{F \in F_h^+} \langle \cdot, \cdot \rangle_{F}.
\]

For each \( F \in F_h^0 \), we fix a unit normal \( n_F \) to \( F \), which points from one element \( K^+ \) to the other element \( K^- \) on the other side. The jump and average operators are defined as:

\[
[\phi]_F = \phi|_{K^+} - \phi|_{K^-}, \quad [\phi n]_F = \phi|_{K^+} n_F - \phi|_{K^-} n_F, \quad \{\phi\} = \frac{1}{2} (\phi|_{K^+} + \phi|_{K^-}),
\]

\[
[v]_F = v|_{K^+} - v|_{K^-}, \quad [v \otimes n]_F = v|_{K^+} \otimes n_F - v|_{K^-} \otimes n_F, \quad \{v\} = \frac{1}{2} (v|_{K^+} + v|_{K^-}),
\]

where \( \phi \) and \( v \) are arbitrary scalar- and vector-valued functions, respectively. For a boundary face \( F \in F_h^\partial \) which is contained in a single element \( K \in T_h \), we further assume that \( n_F \) is the outward pointing normal to \( \partial \Omega \) and define

\[
[\phi]_F = \phi|_K, \quad [\phi n]_F = \phi|_K n_F, \quad \{\phi\} = \phi|_K,
\]

\[
[v]_F = v|_K, \quad [v \otimes n]_F = v|_K \otimes n_F, \quad \{v\} = v|_K.
\]

Throughout the rest of this paper, we use \( n \in \prod_{F \in F_h} \mathbb{R}^d \) to denote the piecewise constant vector defined on the skeleton \( F_h \) such that \( n_F := n_F \) for all \( F \in F_h \). Let \( P_j(K) \) denote the space of polynomials of degree at most \( j \). We shall make use of the following function spaces

\[
[H^m(T_h)]^d = \{ v \in [L^2(\Omega)]^d : v|_K \in [H^m(K)]^d, \ \forall K \in T_h \},
\]

\[
L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_{\Omega} q = 0 \right\},
\]

\[
V_h \subseteq \{ v_h \in [L^2(\Omega)]^d : v_h|_K \in [P_r(K)]^d, \ \forall K \in T_h \},
\]

\[
Q_h \subseteq \{ q_h \in L^2_0(\Omega) : q_h|_K \in P_k(K), \ \forall K \in T_h \},
\]

where \( V_h \) and \( Q_h \) will be given in Section \ref{sec:3}. Here we do not require a specific relationship between \( r \) and \( k \).

The rest of this paper is organized as follows. In Section \ref{sec:2} we first present the unified framework and then prove the semi- and fully-discrete stability of the general scheme. In Section \ref{sec:3} we derive \( H^1 \), \( H(\text{div}) \)-conforming, and DG methods from the unified scheme, and discuss the expression of the penalty term for each of the three methods. In Section \ref{sec:4} we test our schemes in both unsteady and steady situations, and compare the simulation results with classical schemes and data in the literature. Finally we conclude our paper in Section \ref{sec:5}.
2. General formulation

In this section, we present a general framework covering the $H^1$-conforming, $H(\text{div})$-conforming, and DG methods for the following incompressible Navier–Stokes equation

$$
\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u} + p \mathbf{I}) - \nu \nabla \cdot \boldsymbol{\tau} = \mathbf{f}, \quad \text{in} \quad (0, T] \times \Omega,
$$

$$
\nabla \cdot \mathbf{u} = 0, \quad \text{in} \quad (0, T] \times \Omega,
$$

$$
\mathbf{u} = 0, \quad \text{on} \quad (0, T] \times \partial \Omega,
$$

$$
\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad \text{in} \quad \Omega,
$$

where $\nu > 0$ is the viscosity constant, and $\boldsymbol{\tau}(\mathbf{u})$ is the viscous strain tensor that could be

$$
\boldsymbol{\tau}(\mathbf{u}) := \nabla \mathbf{u}, \text{ or } \boldsymbol{\tau}(\mathbf{u}) := \nabla \mathbf{u} + \nabla \mathbf{u}^T, \text{ or } \boldsymbol{\tau}(\mathbf{u}) := \nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} (\nabla \cdot \mathbf{u}) \mathbf{I}. \tag{1}
$$

The three choices of viscous strain tensor yield the same problem in the smooth level. Let $(\cdot, \cdot)$ denote the usual $L^2$ inner product on $\Omega$ and $\nabla_h$ the broken gradient with respect to $T_h$. Our general semi-discrete scheme seeks unknowns $(\mathbf{u}_h(t), p_h(t)) \in V_h \times Q_h$ for each time $t \in (0, T]$ such that

$$
(\partial_t \mathbf{u}_h, v_h) - (\mathbf{u}_h \otimes \mathbf{u}_h, \nabla_h v_h) - (p_h, \nabla_h \cdot v_h) + \langle \tilde{\sigma}_h n, v_h \rangle_{\partial T_h}
$$

$$
- \frac{1}{2} \langle (\nabla_h \cdot u_h) u_h, v_h \rangle + \frac{1}{2} \langle u_h, n \{ u_h \cdot v_h \} \rangle_{\partial T_h} + d_h(u_h, v_h)
$$

$$
+ \nu \left[ (\tau_h(u_h), \nabla_h v_h) - \langle \tilde{\tau}_h n, v_h \rangle_{\partial T_h} + \langle \tilde{u}_h - u_h, \tau_h(v_h) n \rangle_{\partial T_h} \right] = (f, v_h),
$$

$$
(\nabla_h \cdot u_h, q_h) - 
\langle \{ u_h \} \cdot n, \{ q_h \} \rangle_{\partial T_h} = 0, \tag{2b}
$$

for all $(v_h, q_h) \in V_h \times Q_h$ subject to the initial condition $u_h(0) = I_h u_0$, where $I_h$ is a suitable interpolation onto $V_h$, and $\tau_h(u_h)$ is the discrete viscous strain tensor, which could be

$$
\tau_h(u_h) := \nabla_h u_h, \text{ or } \tau_h(u_h) := \nabla_h u_h + \nabla_h u_h^T, \text{ or } \tau_h(u_h) := \nabla_h u_h + (\nabla_h u_h)^T - \frac{2}{3} (\nabla \cdot u_h) \mathbf{I}. \tag{3}
$$

Note that $\frac{1}{2} \langle u_h, n \{ u_h \cdot v_h \} \rangle_{\partial T_h}$ is a consistent term added for convenience of analysis. In order to improve pressure robustness, we use the penalty term $d_h(u_h, v_h)$ which is consistent and positive semi-definite, namely, for all $v_h \in V_h$,

$$
d_h(v_h, v_h) \geq 0, \tag{4}
$$

$$
d_h(u, v_h) = 0. \tag{5}
$$

In principle, $d_h$ could also depend on the pressure although we have not found such examples in practice. The particular expression of $d_h$ will be specified later, see Section 3 for details. Let $h_F$ denote the diameter
of $F \in \mathcal{F}_h$ and $h = \{h_F\}_{F \in \mathcal{F}_h}$ the face size function. We recommend the following numerical fluxes

\[ \hat{\sigma}_h = \llbracket u_h \rrbracket \otimes \llbracket u_h \rrbracket + \llbracket p_h \rrbracket I + \zeta \llbracket u_h \rrbracket \cdot n \llbracket u_h \otimes n \rrbracket, \]

\[ \hat{\tau}_h = \llbracket \tau_h(u_h) \rrbracket - \eta h^{-1} \llbracket u_h \otimes n \rrbracket, \]

\[ \hat{u}_h = \llbracket u_h \rrbracket \text{ on } \mathcal{F}_h^i, \quad \hat{u}_h = 0 \text{ on } \mathcal{F}_h^o. \]

where $\zeta = \{z_F\}_{F \in \mathcal{F}_h}$ and $\eta = \{\eta_F\}_{F \in \mathcal{F}_h}$ are user specified piecewise non-negative constants for controlling the amount of numerical dissipation.

Now we introduce the viscous bilinear form $a_h$, the convective bilinear form $b_h$, and the convective form $c_h$ in the following.

\[ a_h(v_h, w_h) := (\tau_h(v_h), \nabla_h w_h) - \langle \llbracket \tau_h(v_h) \rrbracket - \eta h^{-1} \llbracket v_h \otimes n \rrbracket, n, w_h \rangle_{\partial F_h}, \]

\[ + \langle \llbracket v_h \rrbracket - v_h, \tau_h(w_h)n \rangle_{\partial F_h} - \langle v_h, \tau_h(w_h)n \rangle_{\partial \Omega}, \]

\[ b_h(v_h, q_h) := (\nabla_h \cdot v_h, q_h) - \langle \llbracket v_h \rrbracket \cdot n, q_h \rangle_{\mathcal{F}_h}, \]

\[ c_h(\beta_h, v_h, w_h) := -(v_h \otimes \beta_h, \nabla_h w_h) - \frac{1}{2}((\nabla_h \cdot \beta_h) v_h, w_h) \]

\[ + \langle (\zeta \llbracket \beta_h \rrbracket \cdot n) \llbracket v_h \otimes n \rrbracket, n, w_h \rangle_{\partial F_h} + \langle (\llbracket v_h \rrbracket \otimes \llbracket \beta_h \rrbracket) n, w_h \rangle_{\partial F_h} + \frac{1}{2} \langle \beta_h, n \llbracket v_h \cdot w_h \rrbracket \rangle_{\partial F_h}. \]

One can then rewrite (2) in the following compact form

\[ (\partial_t u_h, v_h) + N_h(u_h; v_h) - b_h(v_h, p_h) = (f, v_h), \quad \forall v_h \in V_h, \quad (7a) \]

\[ b_h(u_h, q_h) = 0, \quad \forall q_h \in Q_h, \quad (7b) \]

where

\[ N_h(u_h; v_h) := c_h(u_h; u_h, v_h) + \nu a_h(u_h, v_h) + d_h(u_h, v_h). \]

Note that $N_h(u_h; v_h)$ is nonlinear in $u_h$ but linear in $v_h$.

**Remark 1.** From the derivation given above, it can be observed that the scheme (7) is consistent if $u(t) \in [H^1(\Omega)]^d \cap [H^{\frac{3}{2}+\varepsilon}(\mathcal{T}_h)]^d$ and $p(t) \in L^2(\Omega) \cap H^{\frac{1}{2}+\varepsilon}(\mathcal{T}_h)$ with $\varepsilon > 0$.

Throughout the rest of this paper, we use $C$ to denote any positive absolute constant that is independent of $h$. We shall also make use of the following mesh-dependent norms

\[ \|v_h\|_{L^2(\mathcal{T}_h)} := \langle v_h, v_h \rangle_{\mathcal{T}_h}^{\frac{1}{2}}, \]

\[ \|v_h\|_{1,h} := \left(\|\tau_h(v_h)\|^2_{L^2(\Omega)} + \eta h^{-1} \|v_h\|^2_{L^2(\mathcal{T}_h)}\right)^{\frac{1}{2}}. \]

It is noted that $\|\cdot\|_{1,h}$ is a well-defined norm on $V_h$, see [4] [5] [12] for details. The next theorem shows that $a_h$ is coercive with respect to the norm $\|\cdot\|_{1,h}$. 


Lemma 1 (Positivity of $a_h$). For all $F \in \mathcal{F}_h$, assume that $\eta_F \geq \eta_0$, where $\eta_0$ is a sufficiently large constant independent of $h$. Then for $v_h \in V_h$ it holds that

$$a_h(v_h, v_h) \geq C \|v_h\|_{1,h}^2.$$  

Proof. Combining terms on $\mathcal{F}_h$, one can rewrite $a_h$ in the following symmetric form

$$a_h(v_h, w_h) = (\tau_h(v_h), \nabla_h w_h) - (\|v_h\|_{\mathcal{F}_h}, \|\tau_h(w_h)\|_{\mathcal{F}_h}) + (\eta h^{-1} \|v_h\|, \|w_h\|)_{\mathcal{F}_h}. \quad (8)$$

First we assume $\tau_h(v_h) = \nabla_h v_h + \nabla_h v_h^T - \frac{2}{3} (\nabla_h \cdot v_h) I$. It follows from (5) and the algebraic identity

$$(\tau_h(v_h), \nabla_h w_h) = \frac{1}{2} (\tau_h(v_h), \tau_h(w_h)) + \left( \frac{2}{3} - \frac{2d}{9} \right) (\nabla_h \cdot v_h, \nabla_h \cdot w_h)$$

that

$$a_h(v_h, v_h) \geq \frac{1}{2} \|\tau_h(v_h)\|^2_{L^2(\Omega)} - 2 (\|v_h\|, \|\tau_h(v_h)\|_{\mathcal{F}_h}) + (\eta h^{-1} \|v_h\|, \|v_h\|)_{\mathcal{F}_h}. \quad (8)$$

Assuming $\eta$ is sufficiently large, we conclude the proof from the trace and Cauchy–Schwarz inequalities, which is standard in the analysis of interior penalty DG methods, see, e.g., [2, 3] for details. The other two cases $\tau_h(u_h) = \nabla_h u_h$ and $\tau_h(u_h) = \nabla_h u_h + \nabla_h u_h^T$ can be proved in a similar way.

Lemma 2 (Positivity of $c_h$). Assume $\zeta_F \geq 0.5$ for all $F \in \mathcal{F}_h^0$. Then for $\beta_h, v_h \in V_h$, we have

$$c_h(\beta_h; v_h, v_h) \geq 0.$$  

Proof. Using integration by parts, $c_h$ could be rewritten as

$$c_h(\beta_h; v_h, w_h) = (\beta_h \cdot \nabla_h v_h, w_h) + \frac{1}{2} ((\nabla_h \cdot \beta_h) v_h, w_h) - ((\|\beta_h\| \cdot n) \|v_h\|, \|w_h\|)_{\mathcal{F}_h}$$

$$- \frac{1}{2} (\|\beta_h\| \cdot n, \|v_h \cdot w_h\|)_{\mathcal{F}_h} + (\zeta \|\beta_h\| \cdot n \|v_h\|, \|w_h\|)_{\mathcal{F}_h}. \quad (9)$$

It then follows from the following identities

$$\frac{1}{2} (\beta_h \cdot \nabla_h v_h, v_h) = \frac{1}{2} (v_h, v_h)_{\mathcal{F}_h} + (\|\beta_h\| \cdot n \|v_h\|, \|v_h\|)_{\mathcal{F}_h} + \frac{1}{2} (\beta_h \cdot n, v_h \cdot v_h)_{\mathcal{F}_h}$$

and (5) with $w_h = v_h$ that

$$c_h(\beta_h; v_h, v_h) = \zeta (\|\beta_h\| \cdot n \|v_h\|, \|v_h\|)_{\mathcal{F}_h} + (\zeta \|\beta_h\| \cdot n + 0.5 \beta_h \cdot n \|v_h \cdot v_h\|)_{\mathcal{F}_h}. \quad (10)$$

Finally, we conclude the proof by using $\zeta_F \geq 0.5$ for $F \in \mathcal{F}_h^0$.

Remark 2. The term $\zeta(\|\beta_h\| \cdot n, v_h \cdot v_h)_{\mathcal{F}_h}$ with $\zeta \geq 0.5$ plays an important role in guaranteeing the positive semi-definiteness of the convective form, which is crucial for proving the stability of the semi-discrete time-dependent incompressible NS equation. In contrast, such stability is not considered in DG schemes for steady-state NS equation. This new modification is one of the key differences of our formulation from the classical formulations in e.g., [11, 13].
With the help of Lemmata 1 and 2 we obtain the following stability.

**Theorem 1.** [Semi-discrete Stability Estimate] Let the assumptions in Lemmata 1 and 2 hold and
\[
\|f\|_{L^1(0,t;L^2(\Omega))} := \int_0^t \|f(s)\|_{L^2(\Omega)} ds < \infty, \quad \forall t \in [0,T].
\]
Then for all \(0 \leq t \leq T\), the scheme \(7\) admits the following semi-discrete stability
\[
\|u_h(t)\|_{L^2(\Omega)} \leq \|u_h(0)\|_{L^2(\Omega)} + \|f\|_{L^1(0,t;L^2(\Omega))}.
\]

**Proof.** Taking \(v_h = u_h\) in (7a) and \(q_h = p_h\) in (7b), we have
\[
\frac{1}{2} \frac{d}{dt} \|u_h\|_{L^2(\Omega)}^2 + N_h(u_h; u_h) = (f, u_h).
\]
It then follows from the identity given above, the positivity of \(a_h, c_h, d_h\) (see (4) and Lemmata 1 and 2) and the Cauchy–Schwarz inequality that
\[
\|u_h\|_{L^2(\Omega)} \frac{d}{dt} \|u_h\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)},
\]
which implies
\[
\frac{d}{dt} \|u_h\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.
\]
Integrating \(12\) over \([0,t]\) yields
\[
\|u_h(t)\|_{L^2(\Omega)} \leq \|u_h(0)\|_{L^2(\Omega)} + \|f\|_{L^1(0,t;L^2(\Omega))}.
\]
The proof is complete.

2.1. Fully discrete stable scheme

Let the time interval \([0,T]\) be partitioned into \(0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\). For each \(n\), let \(\tau_n := t_{n+1} - t_n\). We use the Runge–Kutta (RK) method (see, e.g., [24]) to discretize the semi-discrete finite-dimensional system (7). In particular, an \(m\)-stage RK method is determined by parameters \(\{a_{ij}\}_{i,j=1}^m, \{b_i\}_{i=1}^m, \{c_i\}_{i=1}^m\). Applying this RK method to the time direction in (7), we obtain the following fully discrete RK-DG method
\[
(u_{h}^{n+1}, v_h) = (u^n_h, v_h) + \tau_n \sum_{i=1}^m b_i \left\{ (f^i, v_h) - N_h(U^i_h; v_h) + b_h(v_h, P^i_h) \right\}, \quad \forall v_h \in V_h,
\]
where \(f^i = f(t_n + c_i \tau_n)\), and the internal stages \(U^i_h \in V_h\) and \(P^i_h \in Q_h\) with \(1 \leq i \leq m\) are determined by
\[
(U^i_h, v_h) = (u^n_h, v_h) + \tau_n \sum_{j=1}^m a_{ij} \left\{ (f^j, v_h) - N_h(U^j_h; v_h) + b_h(v_h, P^j_h) \right\}, \quad \forall v_h \in V_h,
\]
\[
b_h(U^i_h, q_h) = 0, \quad \forall q_h \in Q_h.
\]
Note that $0 \leq c_i \leq 1$ for all RK methods. The internal stages $U^i_h$ and $P^i_h$ have useful approximation property. In fact, $U^i_h \approx u_h(t_n + c_i \tau_n)$ and $P^i_h \approx p_h(t_n + c_i \tau_n)$.

Although the semi-discrete stability is proved in Theorem 1, a traditional time discretization such as the family of Backward Differentiation Formulas (BDF) methods would usually destroy such nice dynamic structure. In general, it is quite delicate to design a stability preserving time integration technique for complex dynamical systems, see, e.g., [20] for stability preserving RK schemes for hyperbolic conservation laws and [31 40] for stable low order time difference schemes for incompressible NS equations.

In this subsection, we consider a family of Gauss–Legendre collocation Runge–Kutta (GLRK) methods [7 25] that achieve arbitrarily high order accuracy. The parameters $\{c_i\}_{i=1}^m$ are zeros of the Gauss–Legendre polynomial $\frac{d^m}{ds^m} (s^m (1-s)^m)$. Then $\{a_{ij}\}_{i, j=1}^m$ and $\{b_i\}_{i=1}^m$ are uniquely determined by $\{c_i\}_{i=1}^m$. For instance, if $m = 1$, then $c_1 = \frac{1}{2}$, $a_{11} = \frac{1}{2}$, $b_1 = 1$, which is equivalent to the Crank–Nicolson scheme. If $m = 2$, then $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}, \quad b_1 = b_2 = \frac{1}{2}, \quad a_{11} = \frac{1}{4}, \quad a_{12} = \frac{1}{4} - \frac{\sqrt{3}}{6}, \quad a_{21} = \frac{1}{4} + \frac{\sqrt{3}}{6}, \quad a_{22} = \frac{1}{4}.$

It is well-known that any GLRK method satisfies (see [25])

$$b_i b_j - b_i a_{ij} - b_j a_{ji} = 0 \quad \forall 1 \leq i, j \leq m,$$

$$\sum_{i=1}^m b_i = 1, \quad b_i > 0, \quad \forall 1 \leq i \leq m.$$  \hspace{1cm} (15)

The next theorem shows that the GLRK method preserves the semi-discrete stability given in Theorem 1.

**Theorem 2 (Fully Discrete Kinetic Energy Estimate).** Let the assumptions in Theorem 1 hold. In addition, we assume one of the three following conditions holds: (a) $\tau_h(u_h) = \nabla_h u_h$; (b) $\tau_q(u_h) = \nabla_h u_h + \nabla_h u_h^T$; (c) $\|v_h\|_{L^2(\Omega)} \leq C\|v_h\|_{1, h}$ when $\tau_h(u_h) = \nabla_h u_h + \nabla_h u_h^T - \frac{2}{3} (\nabla_h \cdot u_h) I$. Then we have the following fully discrete kinetic energy estimate

$$\|u^n_h\|_{L^2(\Omega)}^2 \leq \|u^0_h\|_{L^2(\Omega)}^2 + C \sum_{j=0}^{n-1} \tau_j \sum_{i=1}^m b_i \|f(t_j + c_i \tau_j)\|_{L^2(\Omega)}^2, \quad \forall n \geq 1.$$  \hspace{1cm} (17c)

**Proof.** Since $N_h(u_h; v_h)$ is linear in $v_h$, there exists a unique $R_h(u_h) \in V_h$ such that

$$(R_h(u_h), v_h) = N_h(u_h; v_h) \quad \text{for all } v_h \in V_h.$$  \hspace{1cm} (13)

Let $B_h : V_h \to Q_h$ denote the linear operator associated with $b_h$, i.e.,

$$(B_h v_h, q_h) = b_h(v_h, q_h) \quad \text{for all } q_h \in Q_h.$$  \hspace{1cm} (14)

Therefore, (13) and (14) translate into

$$u^{n+1}_h = u^n_h + \tau_n \sum_{i=1}^m b_i F^i_h,$$  \hspace{1cm} (17a)

$$U^i_h = u^n_h + \tau_n \sum_{j=1}^m a_{ij} F^j_h,$$  \hspace{1cm} (17b)

$$(U^n_h, B^T_h q_h) = (B_h U^T_h, q_h) = 0, \quad \forall q_h \in Q_h.$$  \hspace{1cm} (17c)
where $F_h^i := f^i - R_h(U_h^i) + B_h^i P_h^i$. It follows from (17a) that
\[
\|u_h^{n+1}\|_{L^2(\Omega)}^2 = \|u_h^n\|_{L^2(\Omega)}^2 + \tau_n \sum_{i=1}^m b_i \left( F_h^i, u_h^n \right) + \tau_n \sum_{j=1}^m b_j \left( u_h^n, F_j^i \right) + \tau_n^2 \sum_{i,j=1}^m b_i b_j \left( F_h^i, F_j^i \right).
\] (18)

Using (17b), the second term on the right hand side becomes
\[
\tau_n \sum_{i=1}^m b_i \left( F_h^i, u_h^n \right) = \tau_n \sum_{i=1}^m b_i \left( F_h^i, U_h^i - \tau_n \sum_{j=1}^m a_{ij} F_j^i \right)
= \tau_n \sum_{i=1}^m b_i \left( f^i - R_h(U_h^i), U_h^i \right) - \tau_n^2 \sum_{i,j=1}^m b_i a_{ij} (F_h^i, F_j^i),
\] (19)

where (17c) is used in the last equality. Similarly, it holds that
\[
\tau_n \sum_{j=1}^m b_j \left( u_h^n, F_j^i \right) = \tau_n \sum_{j=1}^m b_j \left( f^j - R_h(U_h^j), U_h^j \right) - \tau_n^2 \sum_{i,j=1}^m b_j a_{ji} (F_i^j, F_j^i).
\] (20)

Collecting (18), (19), (20) and using (15), we obtain
\[
\|u_h^{n+1}\|_{L^2(\Omega)}^2 = \|u_h^n\|_{L^2(\Omega)}^2 + 2 \tau_n \sum_{i=1}^m b_i \left( f^i - R_h(U_h^i), U_h^i \right) + \tau_n^2 \sum_{i,j=1}^m b_i b_j - \tau_n \sum_{i,j=1}^m b_i a_{ij} (F_h^i, F_j^i)
= \|u_h^n\|_{L^2(\Omega)}^2 + 2 \tau_n \sum_{i=1}^m b_i \left( f^i, U_h^i \right) - \tau_n \sum_{i,j=1}^m \left( f^i, U_h^j \right) - \left( R_h(U_h^i), U_h^j \right).
\] (21)

For $v_h \in V_h$, recall the discrete Poincaré inequality (cf. [5] [13])
\[
\|v_h\|_{L^2(\Omega)} \leq C \left( \|\nabla_h v_h\|_{L^2(\Omega)} + \|h^{-\frac{1}{2}} [v_h]\|_{L^2(\Omega)} \right),
\] (22)

and the discrete Korn’s inequality (see Eq. (1.19) in [3])
\[
\|\nabla_h v_h\|_{L^2(\Omega)} \leq C \left( \|\nabla_h v_h + \nabla_h v_h^T\|_{L^2(\Omega)} + \|h^{-\frac{1}{2}} [v_h]\|_{L^2(\Omega)} \right).
\] (23)

Using (22), (23) and the definition of $\| \cdot \|_{1,h}$, it holds that
\[
\|v_h\|_{L^2(\Omega)} \leq C \|v_h\|_{1,h}, \quad \forall v_h \in V_h,
\] (24)

when $\tau_h(v_h) = \nabla_h v_h$ or $\tau_h(v_h) = \nabla_h v_h + \nabla_h v_h^T$. Otherwise, the previous inequality follows from the assumption (c). Now combining Lemmata [1] [2] Equation (4) and using (24), (16), we have
\[
\sum_{i=1}^m b_i \left( f^i, U_h^i \right) - \left( R_h(U_h^i), U_h^i \right) \leq \sum_{i=1}^m b_i \left( f^i, U_h^i \right) - \nu a_h(U_h^i, U_h^i)
\leq \sum_{i=1}^m b_i \left( f^i, \frac{1}{\nu} U_h^i \right) - \nu \frac{1}{\nu} \left( U_h^i, U_h^i \right)
\leq \sum_{i=1}^m b_i \left( \frac{1}{\nu} \|f^i\|_{L^2(\Omega)}^2 + \frac{1}{\nu} \left( U_h^i, U_h^i \right) - \nu \frac{1}{\nu} \left( U_h^i, U_h^i \right) \right)
\leq \sum_{i=1}^m b_i \left( \frac{1}{\nu} \|f^i\|_{L^2(\Omega)}^2 - \left( \nu - \frac{1}{2} C \right) \left( U_h^i, U_h^i \right) \right),
\] (25)
where $C$ is in [24] and $\varepsilon > 0$. It then follows from the estimate given above with $\varepsilon = 2C^{-1} \nu$ that

$$\|u_h^{n+1}\|^2_{L^2(\Omega)} \leq \|u_h^n\|^2_{L^2(\Omega)} + \tau_n \sum_{i=1}^m \frac{\varepsilon^{-1}}{2} b_i \|f^i\|^2_{L^2(\Omega)}.$$ 

The proof is complete.

**Remark 3.** In order to prove [24] with $\tau_h(v_h) = \nabla_h v_h + \nabla_h v_h^T - \frac{2}{3}(\nabla_h \cdot v_h) I$, one needs the corresponding discrete Korn’s inequality [23], which is not known in the literature. A possible proof should rely on the characterization of the kernel $\{v \in [H^1(K)]^d : \tau_h(v) = 0\}$ on each element $K \in T_h$ and estimation of suitable semi-norm associated with that kernel, see [4].

For each $n \geq 0$, let

$$\delta_t u_h^n := \frac{u_h^{n+1} - u_h^n}{\tau_n}, \quad u_h^{n+\frac{1}{2}} := \frac{u_h^{n+1} + u_h^n}{2}, \quad f^{n+\frac{1}{2}} := f \left(t_n + \frac{1}{2} \tau_n\right).$$

The Crank–Nicolson time discretization to [4] can be written as

$$(\delta_t u_h^n, v_h) + N_h \left(u_h^{n+\frac{1}{2}} ; v_h\right) - b_h \left(v_h, p_h^{n+\frac{1}{2}}\right) = (f^{n+\frac{1}{2}}, v_h), \quad \forall v_h \in V_h,$$ (26a)$$b_h \left(u_h^{n+\frac{1}{2}}, q_h\right) = 0, \quad \forall q_h \in Q_h.$$ (26b)

Here $p_h^{n+\frac{1}{2}}$ approximates $p_h(t_n + \frac{1}{2} \tau_n)$. The popular Crank–Nicolson scheme can be written as the 1-stage GLRK ($m = 1, a_{11} = \frac{1}{2}, b_1 = 1, c_1 = \frac{1}{2}$) as mentioned before. Therefore, we obtain the unconditional stability of the fully discrete scheme [26] from Theorem [2].

### 3. Three methods from the unified formulation

In this section, we derive three pressure-robust methods from the unified scheme [7] proposed in Section 2. For a positive integer $k$, we introduce the $H^1$-conforming Taylor–Hood finite element spaces

$$V_h^C := \left\{v_h \in [C^0(\Omega)]^d : v_h|_K \in [P_{k+1}(K)]^d, \forall K \in T_h \text{ and } v_h|_{\partial \Omega} = 0\right\},$$

$$Q_h^C := \left\{q_h \in C^0(\Omega) \cap L^2(\Omega) : q_h|_K \in P_k(K), \forall K \in T_h\right\}.\quad (27)$$

We shall also make use of the $P_{k+1} \times P_k$ discontinuous Galerkin spaces

$$V_h^{DG} := \left\{v_h \in [L^2(\Omega)]^d : v_h|_K \in [P_{k+1}(K)]^d, \forall K \in T_h\right\},$$

$$Q_h^{DG} := \left\{q_h \in L^2(\Omega) : q_h|_K \in P_k(K), \forall K \in T_h\right\},$$ (28)

where $k$ could be any nonnegative integer in [28]. Let $H(\text{div}; \Omega) := \{v \in [L^2(\Omega)]^d : \nabla \cdot v \in L^2(\Omega)\}$. Let

$$Q_k(K) := [P_{k+1}(K)]^d \quad \text{or} \quad Q_k(K) := [P_k(K)]^d + P_k(K)x,$$

which is the Raviart–Thomas [44] or Brezzi–Douglas–Marini [5] shape function space, respectively. The $H(\text{div})$-conforming finite element space is

$$V_h^{\text{DIV}} := \{v_h \in H(\text{div}; \Omega) : v_h|_K \in Q_k(K), \forall K \in T_h \text{ and } v_h \cdot n|_{\partial \Omega} = 0\}.\quad (29)$$
3.1. Pressure robustness

Consider the following discrete divergence-free space

\[ Z_h := \{ v_h \in V_h : b_h(v_h, q_h) = 0, \forall q_h \in Q_h \}. \]

For error estimation of velocity in (7), the term \(|b_h(v_h, p - p_h)|\) with \(v_h \in Z_h\) measures the inconsistency of convective bilinear form and serves as a guide to design \(d_h(u_h, v_h)\). Note that this inconsistency is directly related to the concept of pressure robustness [28], that is, the error in pressure induces a velocity error. The goal of \(d_h\) is to reduce the influence of pressure approximation on velocity approximation, which in this paper is said to improve pressure robustness. For any \(v_h \in Z_h\), we have \(b_h(v_h, p_h) = 0\) and thus

\[ |b_h(v_h, p - p_h)| = |b_h(v_h, p)| = \| (\nabla_h \cdot v_h, p - p_h) \|_{H^1(T)} + \| (\nabla_h \cdot v_h) \cdot n \|_{L^2(F_h)} \leq \| \nabla_h \cdot v_h \|_{L^2(\Omega)} \| p \|_{L^2(\Omega)} + \frac{\varepsilon}{2} \| \nabla_h \cdot v_h \|_{L^2(F_h)}^2, \]

where \(0 < \varepsilon \ll 1\) is a small number. In view of \(\| \nabla_h \cdot v_h \|_{L^2(\Omega)}\) and \(\| [v_h] \cdot n \|_{L^2(F_h)}\) in the previous estimate, it is reasonable to add the penalization term

\[ d_h(u_h, v_h) := \gamma_{gd}(\nabla_h \cdot u_h, \nabla_h \cdot v_h) + \langle \gamma_F([u_h] \cdot n), [v_h] \cdot n \rangle_{F_h}, \quad (30) \]

where \(\gamma_{gd}, \{ \gamma_F \}_{F \in F_h} \geq 0\) are sufficiently large (piecewise) constants.

3.2. \(H^1\)-conforming method

Let \(u_h, v_h \in V_h^C\) and \(p_h, q_h \in Q_h^C\). Then the form [8] simplifies to

\[ a_h(u_h, v_h) = \begin{cases} 
(\nabla u_h, \nabla v_h) & \text{when } \tau_h(u_h) := \nabla u_h, \\
(\nabla u_h, \nabla v_h) + (\nabla u_h, \nabla v_h) & \text{when } \tau_h(u_h) := \nabla u_h + \nabla u_h^T, \\
(\nabla u_h, \nabla v_h) + \frac{1}{3}(\nabla u_h, \nabla v_h) & \text{when } \tau_h(u_h) := \nabla u_h + \nabla u_h^T - \frac{2}{3}(\nabla \cdot u_h)I, 
\end{cases} \quad (31) \]

where the identity \((\nabla u_h^T, \nabla v_h) = (\nabla \cdot u_h, \nabla \cdot v_h)\) under \(u_h|_{\partial \Omega} = 0\) is used. The forms [6], [9], [30] simplify to

\[ b_h(u_h, v_h) = (\nabla \cdot v_h, q_h), \]

\[ c_h(u_h; u_h, v_h) = (u_h \cdot \nabla u_h, v_h) + \frac{1}{2}(\langle \nabla \cdot u_h \rangle u_h, v_h), \]

\[ d_h(u_h, v_h) = \gamma_{gd}(\nabla_h \cdot v_h, \nabla_h \cdot u_h). \]

Therefore, the corresponding scheme [7] with \(V_h = V_h^C\), \(Q_h = Q_h^C\) recovers the skew symmetric formulation [11, 31] with grad-div stabilization [17, 28]. The inf-sup condition is guaranteed by

\[ C\|q_h\|_{L^2(\Omega)} \leq \sup_{v_h \in V_h^C(\Omega)} \frac{\langle q_h, \nabla \cdot v_h \rangle}{\|v_h\|_{H^1(\Omega)}}, \quad \forall q_h \in Q_h^C. \]
Remark 4. The $H^1$-conforming method has a minimum number of degrees of freedom, hence significantly reduces the computational cost. However, the $H^1$-conforming method is not able to handle convection dominated flows.

3.3. $H(\text{div})$-conforming method

Let $u_h, v_h \in V_h^{\text{DIV}}$ and $p_h, q_h \in Q_h^{\text{DG}}$. It follows from $[u_h \cdot n] = 0$ on $\mathcal{F}_h$, the inclusion $\nabla \cdot V_h^{\text{DIV}} \subseteq Q_h^{\text{DG}}$, and (7b) that

$$\nabla \cdot u_h = 0 \quad \text{on } \Omega.$$

Therefore the full viscous strain tensor $\tau_h(u_h) = \nabla_h u_h + \nabla_h u_h^T - \frac{2}{3}(\nabla \cdot u_h)I$ and the symmetric gradient strain tensor $\tau_h(u_h) = \nabla_h u_h + \nabla_h u_h^T$ coincide. Then $a_h$ and $b_h$ reduce to

$$a_h(u_h, v_h) = \begin{cases} (\nabla_h u_h, \nabla_h v_h) - \langle [v_h], [\nabla_h u_h] n \rangle_{\mathcal{F}_h} \\
\quad - \langle [u_h], [\nabla_h v_h] n \rangle_{\mathcal{F}_h} + \langle \eta h^{-1} [u_h], [v_h] \rangle_{\mathcal{F}_h}, \quad \text{when } \tau_h(u_h) := \nabla_h u_h, \\
(\nabla_h u_h + \nabla_h u_h^T, \nabla_h v_h) - \langle [v_h], [\nabla_h u_h + \nabla_h u_h^T] n \rangle_{\mathcal{F}_h} \\
\quad - \langle [u_h], [\nabla_h v_h + \nabla_h v_h^T] n \rangle_{\mathcal{F}_h} + \langle \eta h^{-1} [u_h], [v_h] \rangle_{\mathcal{F}_h} \end{cases}$$

and

$$b_h(u_h, v_h) = (\nabla_h \cdot v_h, q_h),$$

respectively. Similarly, using $\nabla \cdot u_h = 0$ and $[u_h \cdot n] = 0$, we obtain the simplified convective term

$$c_h(u_h; u_h, v_h) = (u_h \cdot \nabla_h u_h, v_h) - \langle [u_h \cdot n], [u_h] \rangle_{\mathcal{F}_h} + \langle \zeta |u_h \cdot n|, [u_h] \rangle_{\mathcal{F}_h},$$

and the vanishing penalty term $d_h$ [30], i.e.,

$$d_h(u_h, v_h) = 0.$$

In this case, the scheme (7) with $V_h = V_h^{\text{DIV}}$, $Q_h = Q_h^{\text{DG}}$ reduces to the classical $H(\text{div})$-conforming method [23, 43], but with symmetrical gradient formulation for the viscous bilinear form. Finally the inf-sup condition is guaranteed by

$$C\|q_h\|_{L^2(\Omega)} \leq \sup_{v_h \in V_h^{\text{DIV}} \backslash \{0\}} \frac{(q_h, \nabla \cdot v_h)}{\|v_h\|_{H(\text{div};\Omega)}}, \quad \forall q_h \in Q_h^{\text{DG}}.$$

Remark 5. The $H(\text{div})$-conforming method is naturally pressure robust, since the pressure approximation is completely decoupled from the velocity approximation [23, 24]. With the help of upwind flux ($\zeta \geq 0$), the $H(\text{div})$-conforming method could deal with convection dominated flow.
3.4. Discontinuous Galerkin method

The scheme \( \mathbf{V}_h = \mathbf{V}_h^{DG} \) and \( Q_h = Q_h^{DG} \) yields our DG scheme. If the normal component of the velocity is penalized sufficiently, then we obtain that for \( \mathbf{v}_h \in Z_h, \)

\[
0 = b_h (\mathbf{v}_h, q_h) = (\nabla_h \cdot \mathbf{v}_h, q_h) - (\langle \| \mathbf{v}_h \| \cdot n_F, \| q_h \| \rangle)_{\mathcal{F}_h} \approx (\nabla_h \cdot \mathbf{v}_h, q_h), \quad \forall q_h \in Q_h^{DG}.
\]

By \( \nabla_h \cdot \mathbf{V}_h^{DG} \subseteq Q_h^{DG} \) and the previous reasoning, we may further conclude that

\[
\nabla_h \cdot \mathbf{v}_h \equiv 0.
\]

Hence \( d_h \) should be of the form

\[
d_h (\mathbf{u}_h, \mathbf{v}_h) = \sum_{F \in \mathcal{F}_h} \gamma_F (\mathbf{v}_h \cdot n_F, [\mathbf{u}_h] \cdot n_F)_{\mathcal{F}_h}.
\]

Assuming \( \gamma_F = \gamma h^{-1} \), (cf. [12][23]), the penalty term \( d_h \) further simplifies to

\[
d_h (\mathbf{u}_h, \mathbf{v}_h) = \gamma \sum_{F \in \mathcal{F}_h} h^{-1} (\| \mathbf{v}_h \| \cdot \| \mathbf{u}_h \| \cdot n_F)_{\mathcal{F}_h},
\]

where \( \gamma \) is a sufficiently large parameter. The symmetric form in (8) is given as

\[
a_h (\mathbf{u}_h, \mathbf{v}_h) = \begin{cases}
(\nabla_h \mathbf{u}_h, \nabla_h \mathbf{v}_h) - (\| \mathbf{v}_h \| \cdot \| \nabla_h \mathbf{u}_h \| n)_{\mathcal{F}_h}, & \text{when } \tau_h (\mathbf{u}_h) := \nabla_h \mathbf{u}_h, \\
- (\| \mathbf{u}_h \| \cdot \| \nabla_h \mathbf{v}_h \| n)_{\mathcal{F}_h} + \langle \eta h^{-1} [\mathbf{u}_h], [\mathbf{v}_h] \rangle_{\mathcal{F}_h}, & \text{when } \tau_h (\mathbf{u}_h) := \nabla_h \mathbf{u}_h + \nabla_h \mathbf{u}_h^T, \\
(\nabla_h \mathbf{u}_h + \nabla_h \mathbf{u}_h^T) \cdot \nabla_h \mathbf{v}_h - (\| \mathbf{v}_h \| \cdot \| \nabla_h \mathbf{u}_h \| n)_{\mathcal{F}_h}, & \text{when } \tau_h (\mathbf{u}_h) := \nabla_h \mathbf{u}_h + \nabla_h \mathbf{u}_h^T - \frac{2}{3} (\nabla_h \cdot \mathbf{u}_h) I.
\end{cases}
\]

For \( b_h (\mathbf{u}_h, \mathbf{v}_h) \) and \( c_h (\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \), we use the same form as in (4) and (5), respectively. Finally the pressure stability is guaranteed by observing the following inf-sup condition [12]

\[
C \| q_h \|_{L^2(\Omega)} \leq \sup_{\mathbf{v}_h \in \mathbf{V}_h^{DG} \setminus \{ 0 \}} \frac{b_h (\mathbf{v}_h, q_h)}{\| q_h \|_{\text{lip}}}, \quad \forall q_h \in Q_h^{DG},
\]

where \( \| \mathbf{v}_h \|_{\text{lip}} := (\| \mathbf{v}_h \|^2_{L^2(\Omega)} + \| h^{-\frac{1}{2}} \| \mathbf{v}_h \|_{L^2(\mathcal{F}_h)}^2)^{\frac{1}{2}}. \)

**Remark 6.** It is clear that \( \mathbf{u}|_{\partial \Omega} = 0 \) holds point-wise on the boundary for the \( H^1 \)-conforming method. However, for \( H(\text{div}) \)-conforming and DG methods, \( \mathbf{u}|_{\partial \Omega} = 0 \) is weakly imposed in (7). In fact, any non-homogeneous Dirichlet boundary condition could be weakly enforced via modifying the right hand side of (26).

In particular, the scheme \( \mathbf{V}_h^{DG} \) under the boundary condition \( \mathbf{u}|_{\partial \Omega} = \mathbf{g} \) is modified as

\[
(\delta_i \mathbf{u}_h^n, \mathbf{v}_h) + N_h \left( \mathbf{u}_h^{n+\frac{1}{2}}; \mathbf{v}_h \right) - b_h \left( \mathbf{v}_h, p_h^{n+\frac{1}{2}} \right) = \left( f^{n+\frac{1}{2}}, \mathbf{v}_h \right) + \nu f_h^{n+\frac{1}{2}} (\mathbf{v}_h) + s f_h^{n+\frac{1}{2}} (\mathbf{v}_h) + s f_h^{n+\frac{1}{2}} (\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h,
\]

\[
b_h \left( \mathbf{u}_h^{n+\frac{1}{2}}, q_h \right) = s f_h^{n+\frac{1}{2}} (q_h), \quad \forall q_h \in Q_h,
\]
where \( s = 0 \) for the \( H(\text{div}) \) scheme \((V_h \times Q_h = V_h^{\text{DIV}} \times Q_h^{\text{DG}})\) and \( s = 1 \) for the DG scheme \((V_h \times Q_h = V_h^{\text{DG}} \times Q_h^{\text{DG}})\). The newly introduced \( f_{h,a}^{n+\frac{1}{2}}, f_{h,b}^{n+\frac{1}{2}}, f_{h,c}^{n+\frac{1}{2}}, f_{h,d}^{n+\frac{1}{2}} \) are defined as

\[
\begin{align*}
&f_{h,a}^{n+\frac{1}{2}}(v_h) := \sum_{F \in \mathcal{F}_h^0} \frac{n}{h} \left\langle g^{n+\frac{1}{2}} , v_h \right\rangle_F - \sum_{F \in \mathcal{F}_h^0} \left\langle g^{n+\frac{1}{2}} , \tau_h(v_h)n \right\rangle_F , \\
&f_{h,b}^{n+\frac{1}{2}}(q_h) := \sum_{F \in \mathcal{F}_h^0} \left\langle g^{n+\frac{1}{2}} , q_h n \right\rangle_F , \\
&f_{h,c}^{n+\frac{1}{2}}(v_h) := \sum_{F \in \mathcal{F}_h^0} \left\langle \zeta g^{n+\frac{1}{2}} \cdot n , g^{n+\frac{1}{2}} , v_h \right\rangle_F , \\
&f_{h,d}^{n+\frac{1}{2}}(v_h) := \sum_{F \in \mathcal{F}_h^0} \frac{\gamma}{h} \left\langle g^{n+\frac{1}{2}} , n , v_h \cdot n \right\rangle_F .
\end{align*}
\]

Remark 7. Similarly to the \( H(\text{div}) \)-conforming method, the DG method is able to handle convection dominated flows when upwind flux is introduced. In addition, the DG scheme allows non-conforming and polygonal meshes. However, the DG scheme may lack pressure robustness, which could be cured by increasing the parameter \( \gamma \).

It is worth mentioning that energy-stable and convergent \( H(\text{div}) \) and DG schemes in [23] are designed for the Euler equation modelling incompressible and inviscid flows. Our \( H(\text{div}) \)-conforming scheme shares the same convective form \( c_h \) with the \( H(\text{div}) \) scheme in [23]. However, in contrast to our DG scheme, the convective form \( c_h \) of the DG scheme in [23] relies on a postprocessed velocity. We also point out that, only semi-discrete stability is shown in [23] and the BDF1 time integrator used in the fully discrete scheme there might not yield decaying numerical energy.

4. Numerical Experiments

In this section, we test the performance of several methods in the form (26) with

\[ V_h \times Q_h = V_h^{\text{C}} \times Q_h^{\text{C}} , \text{ or } V_h^{\text{DIV}} \times Q_h^{\text{DG}} , \text{ or } V_h^{\text{DG}} \times Q_h^{\text{DG}} . \]

The corresponding scheme is denoted as Scheme \( H^1 \), \( H(\text{div}) \), or DG-N, respectively. The viscous strain tensor in (26) is chosen as \( \tau_h(u_h) = \nabla u_h + \nabla^T u_h - \frac{2}{3} (\nabla u_h \cdot u_h) I \). Schemes \( H^1 \) and \( H(\text{div}) \) are considered in the first and second experiments, while the DG-N scheme from our framework are tested in all experiments. Recall that the incompressibility condition \( \nabla \cdot u = 0 \) is weakly enforced via the condition \( b_h(u_h, q_h) = 0 \ \forall q_h \in Q_h \), where the bilinear form \( b_h \) is introduced in Section 3. When implementing our schemes, that condition yields the linear system of equations \( B_h U_h = 0 \), where \( U_h \) is the vector representation of \( u_h \) and \( B_h \) is a matrix representing \( b_h \). For nonhomogeneous boundary condition, the right hand side of \( b_h(u_h, q_h) = 0 \) (and \( B_h U_h = 0 \)) is modified as discussed in Remark 6. Although our framework is designed for unsteady problems, we compare our DG-N spatial discretization with the scheme proposed in [11, 15], which we will
denote as DG-C and is of the form (7) with the following bilinear and convective forms

\[ a_h(u_h, v_h) = (\nabla_h u_h, \nabla_h v_h) - (\langle v_h \rangle, \langle \nabla_h u_h \rangle n)_{F_h} - (\langle u_h \rangle, \langle \nabla_h v_h \rangle n)_{F_h} + (\eta h^{-1} \langle u_h \rangle, \langle v_h \rangle)_{F_h}, \]

\[ b_h(v_h, q_h) = (\nabla_h \cdot v_h, q_h) - (\langle v_h \rangle \cdot n, \langle q_h \rangle)_{F_h}, \]

\[ c_h(u_h; v_h, q_h) = (u_h \cdot \nabla_h u_h, v_h) - ((\langle u_h \rangle \cdot n) \langle u_h \rangle, \langle v_h \rangle)_{F_h} + \left( \frac{1}{2} |\langle u_h \rangle \cdot n| \langle u_h \rangle, \langle v_h \rangle \right)_{F_h}, \]

\[ d_h(u_h, v_h) = \gamma (\nabla \cdot u_h, \nabla \cdot v_h) + \sum_{F \in \mathcal{F}_h} \eta h^{-1} (\langle v_h \rangle \cdot n_F, \langle u_h \rangle \cdot n_F)_F. \]

In contrast to DG-N, the scheme DG-C in [1, 15] is designed only for steady incompressible flow and not proved to be energy stable for unsteady flow.

In \( a_h \) and \( c_h \), the penalization parameters \( \eta \) and \( \zeta \) are empirically set to be \( \eta = 3(k+1)(k+2) \) (cf. [15]) and 0.5 respectively, where \( k \) is the degree of polynomials in \( (28) \). The penalty parameters \( \gamma \) (for DG-N and DG-C) and \( \gamma_{gd} \) (for \( H^1 \)) will be specified in each numerical example.

The numerical simulations are performed in FEniCS [30] on a laptop with Intel Core i5 CPU (2.7 GHz) and 8 GB RAM. We use the Newton nonlinear solver with the MUMPS linear solver inside FEniCS to solve the nonlinear systems of equations arising from fully discrete schemes. We set absolute and relative error tolerances used in the Newton solver to be \( 10^{-8} \) for dynamic problems and \( 10^{-10} \) for stationary problems.

### 4.1. Taylor–Green Vortex

The analytical solutions of Taylor–Green vortex [23] in \( \mathbb{R}^2 \) are given by

\[ u(t, x) = \left( \sin(x_1) \cos(x_2)e^{-2\nu t}, -\cos(x_1) \sin(x_2)e^{-2\nu t} \right), \]

\[ p(t, x) = \frac{1}{4} \left( \cos(2x_1) + \cos(2x_2) \right) e^{-4\nu t} \]

with \( \nu = 0.01 \). The space domain and time interval are set to be \( \Omega := [0, 2\pi]^2 \) and \( [0, T] \) with \( T = 1s \), respectively. All schemes are based on the Crank–Nicolson time discretization with uniform time step \( \tau = 0.01s \).

The space domain is partitioned by uniform meshes with mesh sizes \( h_{\text{max}} \in \{0.8886, 0.4443, 0.2221, 0.1777\} \), see Figure 1 for sample meshes. For the \( H^1 \) scheme, we choose \( k \in \{1, 2\} \) in (27) and \( \gamma_{gd} = 0 \) in (30). Note that the Taylor–Hood space (27) with \( k = 0 \) is not inf-sup stable. For \( H(\text{div}) \) and DG schemes, we set \( k \in \{0, 1, 2\} \) in (28) and \( Q_k(K) = [P_{k+1}(K)]^d \) (Brezzi-Douglas-Marini element) in (29). In addition, the DG scheme uses the penalty parameter \( \gamma \in \{0, 10\} \). Numerical results are presented in Figure 2 and Tables 1 to 3.

From Tables 2 and 3 we observe that both DG schemes achieve the expected convergence rates (when \( \gamma = 10 \)), and achieve roughly the same level of accuracy for both velocity and pressure with the same order of runtime. In addition, we observe a decreasing of errors in both velocity and pressure when we increase \( \gamma \) from 0 to 10. In order to ensure stability, and to test the behaviors of the full viscous strain tensor, our DG scheme has more terms (in both \( a_h \) and \( c_h \)) to be updated at each time step compared with DG-C.
Therefore, the running time of DG-N scheme is slightly longer. We do not observe a clear trend of runtime when we increase $\gamma$ from 0 to 10, for both DG schemes. Figure 2 shows that the approximation becomes better when increasing polynomial degree and/or decreasing mesh sizes for our DG-N scheme.

Numerical results on $H^1$ and $H(\text{div})$ schemes are presented in Table 1. Due to smaller numbers of degrees of freedom, the runtime of the $H^1$ scheme is less than $H(\text{div})$ and DG schemes. An interesting phenomenon is the apparent superconvergence of the $H^1$ scheme when $k = 1$. It can be observed from Tables 1 and 2 that errors of $H(\text{div})$, DG-N, and DG-C schemes are of the same magnitude, while the $H^1$ scheme is much less accurate. It is noted that the $H(\text{div})$ scheme has a longer running time than DG schemes although it has less number of degrees of freedom and a simpler expression. We will not pursue a rigorous explanation on this and conjecture that the ‘unreasonable’ runtime of $H(\text{div})$ schemes might be due to the inefficiency of assembling process for Brezzi–Douglas–Marini elements in FEniCS.

4.2. Kovasznay Flow

In this experiment, we consider the steady Kovasznay flow [15] with the analytical solutions given by

$$u(t, x) = \left(1 - e^{\lambda x_1} \cos(2\pi x_2), \frac{\lambda}{2\pi} e^{\lambda x_1} \sin(2\pi x_2)\right),$$

$$p(t, x) = -\frac{1}{2} e^{2\lambda x_1} - \frac{1}{8\lambda} \left(e^{-\lambda} - e^{3\lambda}\right)$$

with $\lambda = \frac{1}{2\nu} - (\frac{1}{4\nu^2} + 4\pi^2)^{\frac{1}{2}}$ and the simulation domain $\Omega := [-0.5, 0] \times [1.5, 2]$. All schemes with $\nu = 0.025$ are tested on uniform meshes with mesh sizes $h_{\text{max}} \in \{0.1768, 0.0884, 0.0442, 0.0354\}$. Other parameters are identical to those given in Experiment 4.1.

The overall performance of $H^1$, $H(\text{div})$, and DG schemes are similar to those in Experiment 4.1, see Tables 4, 5, and 6. The $H^1$ scheme is the best among all schemes when $k = 2$. Figure 3 shows that the approximation becomes better when increasing polynomial degree and/or decreasing the mesh size for our DG scheme.
Figure 2: Taylor–Green Vortex: Contours of vorticity $\nabla_h \times u_h$ from DG-N with $h_{\text{max}} = 0.8886$ (left) and $h_{\text{max}} = 0.1777$ (right) at $t = 1.0$ s when $k = 0$ (upper row) and $k = 2$ (bottom row), $\nu = 0.01$, $\gamma = 10$. 
Table 1: Taylor–Green vortex: Velocity (at $t = 1.0s$) and pressure (at $t = 0.995s$) of $H^1$ and upwind $H(\text{div})$ schemes, $\nu = 0.01$

| $k$ | $h_{\text{max}}$ | $H^1$ | $H(\text{div})$ |
|-----|------------------|-------|------------------|
|     | d.o.f | $\|u - u_h\|_{L^2(\Omega)}$ error order | $\|p - p_h\|_{L^2(\Omega)}$ error order | Runtime | d.o.f | $\|u - u_h\|_{L^2(\Omega)}$ error order | $\|p - p_h\|_{L^2(\Omega)}$ error order | Runtime |
| 0   |       |       |       |       |       |       |       |       |
| 0.8886 | N/A | N/A | N/A | 841 | 2.26e-1 | 4.55e-1 | 1.97s |
| 0.4443 | N/A | N/A | N/A | 3281 | 5.21e-2 | 2.25e-1 | 5.71s |
| 0.2221 | N/A | N/A | N/A | 12991 | 1.20e-2 | 1.12e-1 | 20.78s |
| 0.1777 | N/A | N/A | N/A | 20201 | 7.57e-3 | 2.00e-1 | 35.62s |
| 1   |       |       |       |       |       |       |       |       |
| 0.8886 | 1004 | 2.86e-1 | 1.54e-1 | 1.83es | 2161 | 2.01e-2 | 6.80e-2 | 5.17s |
| 0.4443 | 3804 | 2.55e-2 | 2.70 | 4.66s | 8521 | 2.44e-3 | 1.72e-1 | 26.31s |
| 0.2221 | 14804 | 5.62e-3 | 2.08 | 16.18s | 33841 | 2.03e-3 | 4.31e-3 | 104.10s |
| 0.1777 | 23004 | 3.58e-3 | 2.02 | 27.13s | 52801 | 1.49e-3 | 2.76e-2 | 170.04s |
| 2   |       |       |       |       |       |       |       |       |
| 0.8886 | 2364 | 2.56e-2 | 4.26s | 4081 | 1.29e-3 | 7.05e-2 | 14.42s |
| 0.4443 | 9124 | 5.67e-3 | 2.07 | 13.23s | 16161 | 1.68e-4 | 8.99e-2 | 38.00s |
| 0.2221 | 35844 | 3.48e-4 | 2.23 | 20.78s | 64321 | 4.57e-6 | 1.11e-4 | 360.00s |
| 0.1777 | 55804 | 1.68e-4 | 3.38 | 88.89s | 100401 | 1.88e-6 | 5.71e-5 | 622.18s |

Table 2: Taylor–Green vortex: Comparison of DG-N and DG-C schemes for the velocity at $t = 1.0s$ and pressure at $t = 0.995s$ when $\nu = 0.01$ and $\gamma = 10$.

| $k$ | $h_{\text{max}}$ | d.o.f | DG-N | DG-C | Runtime | DG-N | DG-C | Runtime |
|-----|------------------|-------|-------|-------|---------|-------|-------|---------|
|     |                  |       |       |       |         |       |       |         |
| 0   |       |       |       |       |         |       |       |         |
| 0.8886 | 1401 | 2.35e-1 | 4.55e-1 | 2.30s | 2.27e-1 | 4.51e-1 | 1.69s |
| 0.4443 | 5601 | 2.11 | 2.26e-1 | 1.01 | 5.73s | 5.28e-2 | 2.26e-1 | 5.64s |
| 0.2221 | 22401 | 2.11 | 1.12e-1 | 1.01 | 29.65s | 1.24e-2 | 1.13e-1 | 22.50s |
| 0.1777 | 35001 | 2.08 | 8.97e-2 | 1.00 | 54.50s | 7.78e-3 | 8.99e-2 | 38.00s |
| 1   |       |       |       |       |         |       |       |         |
| 0.8886 | 3001 | 2.11 | 6.80e-2 | 4.05s | 2.00e-2 | 8.68e-2 | 3.78s |
| 0.4443 | 12001 | 2.19 | 1.72e-2 | 1.99 | 17.32s | 2.42e-3 | 2.23e-2 | 15.62s |
| 0.2221 | 48001 | 2.00 | 4.31e-3 | 2.00 | 98.24s | 2.83e-3 | 5.60e-3 | 87.66s |
| 0.1777 | 75001 | 3.06 | 2.76e-3 | 2.00 | 175.95s | 1.42e-3 | 3.58e-3 | 158.56s |
| 2   |       |       |       |       |         |       |       |         |
| 0.8886 | 5201 | 2.00 | 7.04e-3 | 9.93s | 1.37e-3 | 8.00e-3 | 9.38s |
| 0.4443 | 20801 | 4.15 | 8.90e-4 | 55.93s | 7.80e-5 | 9.72e-4 | 52.93s |
| 0.2221 | 83201 | 4.05 | 1.11e-3 | 3.00 | 316.97s | 4.65e-6 | 1.20e-4 | 308.49s |
| 0.1777 | 130001 | 4.02 | 5.71e-5 | 3.00 | 569.20s | 1.90e-6 | 6.15e-5 | 562.17s |
Table 3: Taylor–Green vortex: Comparison of DG-N and DG-C schemes for the velocity at $t = 1.0s$ and pressure at $t = 0.995s$, when $\nu = 0.01$, $\gamma = 0$.

| $k$ | $h_{\text{max}}$ | d.o.f | DG-N | DG-C |
|-----|-------------------|-------|-----|------|
|     |                   |       | $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ | $\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ | $\|\mathbf{p} - \mathbf{p}_h\|_{L^2(\Omega)}$ | $\|\mathbf{p} - \mathbf{p}_h\|_{L^2(\Omega)}$ | Runtime | Runtime |
| 0   | 0.8886            | 1401  | 8.11e-1 | 5.34e-1 | 2.99s | 8.28e-1 | 5.17e-1 | 2.57s |
|     | 0.4443            | 5601  | 3.04e-2 | 2.55e-1 | 9.36s | 3.05e-1 | 2.52e-1 | 1.04 |
|     | 0.2221            | 22401 | 9.92e-2 | 1.21e-1 | 28.35s | 1.04e-1 | 1.21e-1 | 1.06 |
|     | 0.1777            | 35001 | 6.89e-2 | 9.58e-2 | 42.47s | 7.26e-2 | 9.58e-2 | 1.06 |
| 1   | 0.8886            | 3001  | 1.66e-1 | 8.80e-2 | 6.98s | 1.50e-1 | 7.90e-2 | 4.14s |
|     | 0.4443            | 12001 | 2.60e-2 | 1.98e-2 | 20.10s | 2.30e-2 | 1.86e-2 | 2.09 |
|     | 0.2221            | 48001 | 3.18e-3 | 4.55e-3 | 92.16s | 3.24e-3 | 4.45e-3 | 2.06 |
|     | 0.1777            | 75001 | 1.62e-3 | 2.87e-3 | 163.94s | 1.70e-3 | 2.83e-3 | 2.03 |
| 2   | 0.8886            | 5201  | 6.99e-3 | 7.47e-3 | 11.95s | 5.95e-3 | 7.24e-3 | 9.57s |
|     | 0.4443            | 20801 | 3.30e-4 | 9.01e-4 | 55.58s | 3.19e-4 | 8.98e-4 | 3.01 |
|     | 0.2221            | 83201 | 1.90e-5 | 1.12e-4 | 315.91s | 1.99e-5 | 1.12e-4 | 3.00 |
|     | 0.1777            | 130001| 7.78e-6 | 5.74e-5 | 570.54s | 8.27e-6 | 5.74e-5 | 3.00 |

From Tables 5 and 6 we observe that both DG schemes achieve the expected convergence rates for velocity and pressure when $\gamma \in \{0, 10\}$. The errors and running time of DG-N are slightly smaller than DG-C when $\gamma = 0$. In contrast to the Taylor–Green vortex, the running time of both DG schemes for the stationary Kovasznay flow are similar because there is no dynamic update at each time step. We also observe that the runtime tends to decrease when $\gamma$ increases from 0 to 10, especially for DG-C. An interesting observation is that there is a trend of increasing of errors in both velocity and pressure when we increase $\gamma$ from 0 to 10, which indicates that the penalty term (33) may fail to reduce the errors in some cases when the convective term appears. It is shown in [1] that the solution will converge to BDM solution if $\gamma \to \infty$ for the Stokes problem, which indicates a decreasing of absolute errors when increasing $\gamma$ (at least for Stokes flow). Due to nonlinearity of Naiver–Stokes equations, theoretical analysis on the optimality of $\gamma$ seems not available. Hence we will investigate this influence numerically in the next experiment.

4.3. Influence of $\gamma$ and $\gamma_{gd}$

In this subsection, we go back to the original form of the penalty term (30) with $\gamma_F = \gamma h_F^{-1}$ which is

$$d_h(\mathbf{u}_h, \mathbf{v}_h) = \gamma_{gd}(\nabla_h \cdot \mathbf{u}_h, \nabla_h \cdot \mathbf{v}_h) + \gamma \sum_{F \in \mathcal{F}_h} h_F^{-1} \langle [\mathbf{v}_h], \mathbf{n}_F, [\mathbf{u}_h] \cdot \mathbf{n}_F \rangle_F$$
### Table 4: Kovasznay flow: Velocity and pressure of $H^1$ and upwind $H(\text{div})$ schemes, $\nu = 0.025$.

| $k$ | $h_{\text{max}}$ | d.o.f | $H^1$ | $H(\text{div})$ |
|-----|-----------------|-------|-------|-----------------|
|     | $\|u - u_h\|_{L^2(\Omega)}$ | $\|p - p_h\|_{L^2(\Omega)}$ | Runtime | $\|u - u_h\|_{L^2(\Omega)}$ | $\|p - p_h\|_{L^2(\Omega)}$ | Runtime |
| 0   | 0.1768          | N/A   | N/A   | 2113            | 4.45e-2 —— | 6.73e-2 —— | 1.49e-1s |
|     | 0.0884          | N/A   | N/A   | 8321            | 1.00e-2 2.15 | 3.00e-2 1.17 | 6.01e-1s |
|     | 0.0442          | N/A   | N/A   | 33025           | 2.46e-3 2.03 | 1.43e-2 1.06 | 3.06s    |
|     | 0.0354          | N/A   | N/A   | 51521           | 1.57e-3 1.99 | 1.14e-2 1.03 | 5.05s    |
| 1   | 0.1768          | 2468  | 3.37e-3 —— | 5473 | 2.69e-3 —— | 3.70e-3 —— | 5.79e-1s |
|     | 0.0884          | 9540  | 4.17e-4 3.01 | 21697 | 3.31e-4 3.02 | 8.11e-4 2.19 | 2.93s    |
|     | 0.0442          | 37508 | 5.20e-5 3.00 | 86401 | 4.13e-5 3.00 | 1.87e-4 2.12 | 15.56s   |
|     | 0.0354          | 58404 | 2.66e-5 3.00 | 134881 | 2.11e-5 3.00 | 1.17e-4 2.08 | 26.68s   |
| 2   | 0.1768          | 5892  | 1.61e-4 —— | 10369 | 1.68e-4 —— | 3.70e-3 —— | 5.79e-1s |
|     | 0.0884          | 23044 | 1.01e-5 4.00 | 41217 | 1.08e-5 3.96 | 3.12e-5 3.29 | 10.32s   |
|     | 0.0442          | 91140 | 6.32e-7 4.00 | 86401 | 6.87e-7 3.98 | 3.50e-6 3.16 | 58.00s   |
|     | 0.0354          | 142084| 2.59e-7 4.00 | 256641 | 2.82e-7 3.99 | 1.75e-6 3.10 | 105.40s  |

### Table 5: Kovasznay flow: Comparison of DG-N and DG-C schemes when $\nu = 0.025$ and $\gamma = 10$.

| $k$ | $h_{\text{max}}$ | d.o.f | DG-N | DG-C |
|-----|-----------------|-------|------|------|
|     | $\|u - u_h\|_{L^2(\Omega)}$ | $\|p - p_h\|_{L^2(\Omega)}$ | $\|u - u_h\|_{L^2(\Omega)}$ | $\|p - p_h\|_{L^2(\Omega)}$ | Runtime | Runtime |
| 0   | 0.1768          | 3585  | 3.77e-2 —— | 5.89e-2 —— | 1.27e-1s | 4.91e-2 —— | 6.52e-2 —— | 1.25e-1s |
|     | 0.0884          | 14337 | 9.62e-3 1.97 | 2.86e-2 1.04 | 5.55e-1s | 1.04e-2 2.24 | 3.01e-2 1.11 | 5.81e-1s |
|     | 0.0442          | 57345 | 2.44e-3 1.98 | 1.41e-2 1.02 | 2.91s | 2.46e-3 2.08 | 1.47e-2 1.04 | 2.96s    |
|     | 0.0354          | 89601 | 1.57e-3 1.98 | 1.13e-2 1.01 | 4.82s | 1.57e-3 2.02 | 1.17e-2 1.02 | 4.90s    |
| 1   | 0.1768          | 7681  | 2.59e-3 —— | 2.99e-3 —— | 4.65e-1s | 2.59e-3 —— | 2.55e-3 —— | 4.62e-1s |
|     | 0.0884          | 30721 | 3.25e-4 3.00 | 7.01e-4 2.09 | 2.61s | 3.23e-4 3.00 | 5.64e-4 2.17 | 2.63s    |
|     | 0.0442          | 122881 | 4.07e-5 3.00 | 1.69e-4 2.05 | 14.73s | 4.03e-5 3.00 | 1.31e-4 2.11 | 14.98s   |
|     | 0.0354          | 192001 | 2.08e-5 3.00 | 1.08e-4 2.03 | 26.13s | 2.06e-5 3.00 | 8.24e-5 2.07 | 26.25s   |
| 2   | 0.1768          | 13313 | 1.37e-4 —— | 2.02e-4 —— | 1.58s | 1.37e-4 —— | 1.83e-4 —— | 1.53s    |
|     | 0.0884          | 53249 | 8.86e-6 3.95 | 2.50e-5 3.01 | 9.17s | 8.87e-6 3.95 | 2.36e-5 2.96 | 9.16s    |
|     | 0.0442          | 212993 | 5.62e-7 3.98 | 3.08e-6 3.02 | 53.67s | 5.63e-7 3.98 | 2.96e-6 2.99 | 55.35s   |
|     | 0.0354          | 332801 | 2.31e-7 3.99 | 1.57e-6 3.01 | 101.30s | 2.31e-7 3.99 | 1.52e-6 3.00 | 95.62s   |
Figure 3: Kovasznay Flow: Contours of velocity magnitude from the DG-N scheme with $h_{\text{max}} = 0.1768$ (left) and $h_{\text{max}} = 0.0354$ (right) when $k = 0$ (upper row) and $k = 2$ (bottom row), $\nu = 0.025$ and $\gamma = 10$. 
Table 6: Kovasznay flow: Comparison of DG-N and DG-C schemes when \( \nu = 0.025 \) and \( \gamma = 0 \).

| \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f | \( k \) | \( h_{\text{max}} \) | d.o.f |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.1768 | 3585 | 4.04e-2 | — | 6.05e-2 | — | 1.62e-1s | 8.00e-2 | — | 1.08e-1 | — | 1.28e-1s | 0.0884 | 14337 | 1.03e-2 | 1.97 | 2.84e-2 | 1.09 | 6.86e-1s | 2.65e-2 | 2.11 | 1.40e-2 | 1.08 | 3.59s | 0.0442 | 57345 | 1.58e-3 | 2.06 | 1.38e-2 | 1.04 | 4.86s | 1.16e-3 | 2.06 | 1.12e-2 | 1.02 | 5.67s | 0.0354 | 89601 | 2.50e-3 | 2.05 | 1.38e-2 | 1.04 | 3.21s | 2.65e-3 | 2.11 | 1.40e-2 | 1.08 | 3.59s | 0.0354 | 192001 | 1.58e-3 | 2.05 | 1.38e-2 | 1.04 | 3.21s | 2.65e-3 | 2.11 | 1.40e-2 | 1.08 | 3.59s |
| 1 | 0.1768 | 7681 | 2.34e-3 | — | 2.18e-3 | — | 4.54e-1s | 2.29e-3 | — | 2.12e-3 | — | 4.75e-1s | 0.0884 | 30721 | 2.87e-4 | 3.96 | 5.01e-4 | 2.12 | 2.60s | 7.78e-6 | 3.96 | 1.97e-5 | 2.94 | 9.74s | 0.0442 | 122881 | 4.90e-7 | 3.98 | 2.28e-6 | 2.99 | 2.49s | 4.93e-7 | 3.98 | 2.49e-6 | 2.99 | 55.67s | 0.0354 | 332801 | 2.01e-7 | 3.99 | 7.77e-5 | 2.02 | 25.49s | 2.02e-7 | 3.99 | 1.27e-6 | 3.00 | 26.52s |
| 2 | 0.1768 | 13313 | 1.21e-4 | — | 1.42e-4 | — | 1.61s | 1.21e-4 | — | 1.51e-4 | — | 1.56s | 0.0884 | 53249 | 7.75e-6 | 2.96 | 1.82e-5 | 2.97 | 9.39s | 7.78e-6 | 2.96 | 1.97e-5 | 2.94 | 9.74s | 0.0442 | 212993 | 4.90e-7 | 3.98 | 2.28e-6 | 2.99 | 54.09s | 4.93e-7 | 3.98 | 2.49e-6 | 2.99 | 55.67s | 0.0354 | 332801 | 2.01e-7 | 3.99 | 1.17e-6 | 3.00 | 94.04s | 2.02e-7 | 3.99 | 1.27e-6 | 3.00 | 94.78s |

and study the influence of \( \gamma_{gd} \) and \( \gamma \) on the velocity approximation of the DG schemes. The model problems is the potential flow (cf. [1])

\[
\mathbf{u}(t, \mathbf{x}) = \left( 5x_1^4 - 30x_1^2x_2^2 + 5x_2^4, -20x_1^3x_2 + 20x_1x_2^3 \right),
\]

\[
p(t, \mathbf{x}) = -\frac{1}{2} |\mathbf{u}(t, \mathbf{x})|^2
\]
on the domain \( \Omega := [-1, 1]^2 \) consisting of ten colliding jets which meets at the stagnation point \((0, 0)\) (see Figure 4) and the Kovasznay flow (in Subsection 4.2) with \( \nu = 0.025 \) and \( \gamma \) (respectively \( \gamma_{gd} \)) ranging from \( 0, 1, 5, 25, 125 \) when \( k = 2, 3 \) and \( h = 0.0884 \).

It can be observed from Tables 7 and 8 that larger \( \gamma \) (while \( \gamma_{gd} = 0 \)) decreases the velocity error tremendously in potential flow, but increases the errors in Kovasznay flow while keeping the same order of error magnitude. This observation indicates that the addition of penalty term \( \frac{\gamma}{\sum_{F \in \mathcal{F}}} h_F^{-1} \mathbf{F}^{-1} \langle [\mathbf{v}_h] \cdot \mathbf{n}_F, [\mathbf{u}_h] \cdot \mathbf{n}_F \rangle_{\mathcal{F}} \) may fail to decrease errors, but it may not affect the order of error too much. To the best of our knowledge, we have not seen a similar report for the DG schemes in the literature. Finally, we do not observe an obvious increasing or decreasing error when increasing \( \gamma_{gd} \) (while keeping \( \gamma = 0 \)) except for the case from \( \gamma_{gd} = 0 \) to \( \gamma_{gd} = 1 \).

4.4. Lid Driven Flow

In this section, we consider lid driven flow for the DG-N scheme with \( Re = 100, 400 \) (which corresponds to \( \nu = 0.01, 0.0025 \) respectively in this setting) [18] and the square domain \( \Omega := [0, 1]^2 \). There is a tangential
Table 7: Comparison between DG-N and DG-C schemes with $\nu = 0.025$ and $h_{\text{max}} = 0.0884$ for Kovasznay flow and potential flow when $\gamma_{gd} = 0$.

| $k$ | $\gamma$ | Kovasznay flow $\|u - u_h\|_{L^2(\Omega)}$ | Potential flow $\|u - u_h\|_{L^2(\Omega)}$ |
|-----|-----------|----------------------------------------|--------------------------------------|
|     |           | DG-N | DG-C | DG-N | DG-C |
| 2   | 0         | 7.75e-6 | 7.78e-6 | 3.26e-4 | 2.60e-4 |
|     | 1         | 8.25e-6 | 8.27e-6 | 2.09e-4 | 2.09e-4 |
|     | 5         | 8.72e-6 | 8.73e-6 | 9.76e-5 | 9.13e-5 |
|     | 25        | 8.96e-6 | 8.96e-6 | 2.86e-5 | 2.57e-5 |
|     | 125       | 9.02e-6 | 9.02e-6 | 7.48e-6 | 6.82e-6 |
| 3   | 0         | 1.41e-7 | 1.42e-7 | 6.43e-6 | 5.05e-6 |
|     | 1         | 1.47e-7 | 1.47e-7 | 4.39e-6 | 4.24e-6 |
|     | 5         | 1.55e-7 | 1.55e-7 | 2.05e-6 | 1.92e-6 |
|     | 25        | 1.60e-7 | 1.59e-7 | 5.86e-7 | 5.32e-7 |
|     | 125       | 1.61e-7 | 1.61e-7 | 1.30e-7 | 1.16e-7 |

Table 8: Comparison between DG-N and DG-C schemes with $\nu = 0.025$ and $h_{\text{max}} = 0.0884$ for Kovasznay flow and potential flow when $\gamma = 0$.

| $k$ | $\gamma_{gd}$ | Kovasznay flow $\|u - u_h\|_{L^2(\Omega)}$ | Potential flow $\|u - u_h\|_{L^2(\Omega)}$ |
|-----|----------------|----------------------------------------|--------------------------------------|
|     |                | DG-N | DG-C | DG-N | DG-C |
| 2   | 0              | 7.75e-6 | 7.78e-6 | 3.26e-4 | 2.60e-4 |
|     | 1              | 7.80e-6 | 7.82e-6 | 2.47e-4 | 2.63e-4 |
|     | 5              | 7.81e-6 | 7.82e-6 | 2.47e-4 | 2.61e-4 |
|     | 25             | 7.81e-6 | 7.82e-6 | 2.48e-4 | 2.61e-4 |
|     | 125            | 7.81e-6 | 7.82e-6 | 2.48e-4 | 2.61e-4 |
| 3   | 0              | 1.41e-7 | 1.42e-7 | 6.43e-6 | 5.05e-6 |
|     | 1              | 1.44e-7 | 1.44e-7 | 4.20e-6 | 4.17e-6 |
|     | 5              | 1.44e-7 | 1.45e-7 | 4.05e-6 | 4.04e-6 |
|     | 25             | 1.44e-7 | 1.45e-7 | 4.02e-6 | 4.01e-6 |
|     | 125            | 1.44e-7 | 1.45e-7 | 4.02e-6 | 4.01e-6 |
velocity $\mathbf{u} = (1, 0)$ on the top, while no-slip boundary conditions are applied on the other sides. In this test, we use polynomial $k = 3$ for the pressure, $\gamma = 10$ and $h = 0.0283$ (which corresponds to a mesh size of $50 \times 50$).

Figures 5 and 6 show the contour and streamlines of the lid driven cavity flow. The important aspect of Figure 6 is that a small corner vortex at the bottom right corner, which normally requires a very high mesh resolution, has been predicted by our scheme. In Figures 7 and 8 we compare our simulation results with the data reported in [18], we find our simulation results match the data perfectly except at the coordinate point $(0.9063, 0.5)$ for vertical velocity; we are not sure if there is a typo in the original data from [18] as even in classical fluid finite element book like [48] (Figure 4.5(b), page 134) this point is ignored when making comparisons.
Figure 6: Lid Driven Cavity: Stream trace from DG-N with $Re = 100$ (left) and $Re = 400$ (right) when $k = 3$, $h_{\text{max}} = 0.0283$ and $\gamma = 10$.

Figure 7: Lid Driven Cavity: Horizontal velocity from DG-N with $Re = 100$ (left) and $Re = 400$ (right) when $k = 3$, $h_{\text{max}} = 0.0283$, $\gamma = 10$ at $x_1 = 0.5$ (with $x_2$ varying)
Figure 8: Lid Driven Cavity: Vertical velocity from DG-N with $Re = 100$ (left) and $Re = 400$ (right) when $k = 3$, $h_{\text{max}} = 0.0283$, $\gamma = 10$ at $x_2 = 0.5$ (with $x_1$ varying)

Figure 9: Mesh for flow around a cylinder
Figure 10: Contour of velocity magnitude when $t = 2s, 3s, 5s, 6s$ from top to bottom—BDF2 time discretization
Figure 11: Contour of velocity magnitude when $t = 2\,\text{s}, 3\,\text{s}, 5\,\text{s}, 6\,\text{s}$ from top to bottom—Crank–Nicolson time discretization
4.5. Flow around a cylinder

In the last example, we follow [32, 42, 30] and consider the flow over a cylinder with our DG-N scheme when $\gamma = 10$ using both BDF2 and Crank–Nicolson time discretization. We follow the example in [30] and use the FEniCS mshr tool to generate a fixed mesh, see Figure 9. Note that the BDF2 method is not necessarily stability preserving. The simulation domain $\Omega := [(0, 2.2) \times [0, 0.41)] \setminus B$, where $B$ is the disk centered at $(0.2, 0.2)$ with radius 0.05. The simulation time interval is $[0, T]$ with $T = 8s$ and the time step $\tau = 0.01$. A primary feature of this benchmark is the formation of von Kármán vortex street. Our goal is to study the influence of time discretization on the formulation of the vortex. The inflow and outflow profile is given (cf. [32]) as

$$u(t, 0, x_2) = u(t, 2.2, x_2) = \frac{6}{0.41^2} \sin(\pi t/8)x_2(0.41 - x_2),$$

$$v(t, 0, x_2) = v(t, 2.2, x_2) = 0,$$

and the boundary condition on the rest of $\partial\Omega$ is set to be $u = 0$. The Reynolds number $Re$ corresponding to the mean velocity inflow ranges from 0 to 100. From Figures 10 and 11 we observe that the vortex forms gradually over time when Crank–Nicolson is used for the time discretization, while there is no apparent vortex formulation when using BDF2, which indicates the importance of using stability preserving time discretization.

5. Conclusion

We have developed a general framework including the $H^1$, $H(div)$ and DG methods with the use of different stress tensors. We proved the stability in general and discussed the expressions for penalty terms for each of the three cases. For Taylor–Green vortex and Kovasznay flow, our DG schemes are comparable to classical schemes in the literature, while the $H^1$ scheme from the framework is less accurate but with much less runtime with the Taylor-Green vortex. The $H(div)$ scheme has the longest runtime among four schemes for Taylor-Green vortex implemented in FEniCS. We also show through examples that penalizing normal component of the velocity in DG schemes may fail to decrease absolute errors. In general, we are not able to demonstrate it rigorously and the choice of $\gamma$ is empirical. In addition, we show that our DG scheme agrees very well with the features and data of lid driven flow. Finally, the importance of stability preserving time discretization has been shown by comparing the BDF2 and Crank–Nicolson scheme for the flow around a cylinder.

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