Fin stabilized projectiles point mass model

Tomasz Meda

Military Institute of Armament Technology, Wyszyńskiego 7 str., 05-220 Zielonka, Poland:

Email: merdat@witu.mil.pl.

Abstract. To calculate flight of unguided projectiles several types of the flight models are used. The application mostly depends on demand of the model accuracy, complexity and computational cost. As a good compromise of these three features for flight calculation of spin stabilized projectiles Modified Point Mass Trajectory Model was created. The MPMTM is commonly used for firing tables creation. In the paper the author shown conception of flight model designed for fin (aerodynamic) stabilized projectiles. At In the first part of the paper the motivation of the model designing was explained. Additionally analysis of the phenomena that should be used in the model was made. In the next part of the paper main equations of the model were described. Some of them were empirical. In the third part of the paper the accuracy of the designed model and two other model types was compared. Calculations results were compared with experimental data and the conclusion were drawn.

1. Introduction

One of the main parts of the exterior ballistic is modelling flight of the projectiles. Across the ages, during evolution of the exterior ballistic new types of the flight models were created [1]. At the second part of the nineteen century Siacci and Mayewski develop first modern drag shapes of the projectiles used at these times [2]. That allow to calculate the projectile flight using material point model with these days sufficient accuracy. At the first part of the twenty century in many countries the researchers try to develop the most complexity model (including forces and moments) of the projectile flight – the 6 degree of freedom (6-dof) flight model. The first incomplete 6-dof model was published in 1920 by Fowler, Gallop, Lock and Richmond [3]. It was completed by Nielsen and Synge in 1943 [4] and in that form and its modification is used for today. 6-dof models are much more accurate that point mass models [2]. Main disadvantages of the 6-dof model are much higher computational cost (comparing with point mass model) and necessity of determine more aerodynamic coefficients. As a compromise between 6-dof and point mass models, the Modified Point Mass Trajectory Model (MPMTM) was develop by Leskie and Reiter in 1966 [5]. The MPMTM allow to determine flight trajectories of spin stabilised projectiles with better accuracy without significant computational cost of calculation than standard point mass models. That is especially necessary for Fire Control Systems where time of the calculation is strictly limited. The necessity is intensifies by increasing ranges of new types of guns and ammunition which demands higher accuracy of computations and increasing their time. The range increasing concerns also mortar ammunition. Nowadays self-propelled mortars have range above 10 km (RAK-Mortar) which make point mass model much less sufficient. Instead of spin stabilized projectiles, the aerodynamically (fin) stabilized projectiles didn’t have specialized compromised trajectory model. The main objective of this paper is to present a the compromised trajectory model.
for long range mortar ammunition. To avoid confusion with MPMTM the author named it Fin Stabilized Projectiles Point Mass Model (FSPPMM).

2. Phenomenon’s analysis

Modified Point Mass Trajectory Model was created at first stage by analysis of swerving motion of the projectile using results from 6-dof model. The second stage was linearization of the aerodynamic parameters. That allowed to neglect factors with small influence of trajectory and simplify equations. FSPPMM should be built by the same way. Main difference between spin and fin stabilized projectiles is rotation velocity (around longitudinal axis). For fin stabilized projectiles the spin velocity is much lower or even close to zero. The rotation is used only to compensate aerodynamic asymmetry. Therefore, it can be assumed that spin velocity do not have influence on the trajectory. That simplification neglect calculation of spin velocity, Magnus force and moment. Taking into account also that pitching moment behaviour of pitch moment and lift force coefficients will be assumed these angles can be analysed independent. When wind changes during flight will be enough slow to not change yaw angle rapidly it will depends on its initial value, cross wind and initial yaw velocity. At the beginning of the flight will harmonically oscillate with damping to 0 value. Pitch angle behaviour is more complicated. To illustrate it pitch values of two different trajectories with the same initial disturbances of pitch were shown on figure 1.

![Figure 1. Pitch versus time for two different trajectories, where: QE – quadrant elevation [°], V – velocity [m/s].](image)

Pitch of first trajectory (with higher muzzle velocity and lower quadrant elevation) after initial distribution have damped oscillation, but convergence value is different than 0. This value also change during flight. The second trajectory pitch (with lower muzzle velocity and high quadrant elevation) initial damped oscillation increase greatly close to trajectory apogee. The cause of that peak is too low value of pitching moment close to apogee which cannot allow projectile to follow the trajectory curvature which is the biggest close to apogee. It is the pitch visible also after oscillation convergence is different than for the first trajectory and its changes are greatly. Described behaviour of pitch and yaw should be included in the model. Due that lift force, pitch and pitch damping moments have to be
include. However the model is developing for long range shooting calculations, the pitch peak close to apogee problem will not be consider. Analysis presented in next chapter shown that pitch calculation method is sufficient accurate even for 70 degrees elevation angle. Analysis about modelling of very sharp trajectories will be made in further work. To increase the accuracy also Coriolis effect and gravity vector change.

3. Mathematical formulation of the model
Main base of the model should be the point mass model. The basic equations, its explanation and linearization are widely decribed in [2] and other works. Equation (1) present change of projectile position coordinates.

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
V_x \\
V_y \\
V_z
\end{bmatrix}
\]  

(1)

Where: x,y,z – projectile position coordinates, Vx, Vy, Vz – projectile velocity components;

Change of projectile velocity including lift force is presented in equation 2.

\[
\begin{bmatrix}
\dot{V}_x \\
\dot{V}_y \\
\dot{V}_z
\end{bmatrix} =
\begin{bmatrix}
-\nu_x C_d - (\nu_y \alpha + \nu_z \beta) C_l \\
-\nu_y C_d + \nu_x C_l \alpha \\
-\nu_z C_d - (\nu_x^2 + \nu_y^2) 0.5 C_l \beta
\end{bmatrix} + G + \text{Cor}
\]

(2)

Where: f – factor described below Cd – drag coefficient, Cl – linearized lift coefficient, \(\nu_x, \nu_y, \nu_z\) – components of velocity relative to atmosphere described below \(\alpha\) – pitch angle, \(\beta\) – yaw angle, G – acceleration due gravity described below, Cor – Coriolis effect factor described below

Value of f and \(\nu_x, \nu_y, \nu_z\) are given by:

\[
f = \frac{\rho \bar{V} S}{2m}
\]

(3)

Where: \(\rho\) – atmosphere density, S – projectile reference area, m – projectile mass, \(\bar{V}\) – velocity relative to atmosphere

\[
\begin{bmatrix}
\dot{\nu}_x \\
\dot{\nu}_y \\
\dot{\nu}_z
\end{bmatrix} =
\begin{bmatrix}
\nu_x \\
\nu_y \\
\nu_z
\end{bmatrix} - 
\begin{bmatrix}
W_x \\
W_y \\
W_z
\end{bmatrix}
\]

(4)

Where: Wx, Wy, Wz – components of atmosphere velocity (wind)

\[
\bar{V} = \left(\begin{array}{c}
\bar{V}_x \\
\bar{V}_y \\
\bar{V}_z
\end{array}\right) 0.5
\]

(5)

\[
G = g \begin{bmatrix}
x/R \\
1 - 2y/R \\
0
\end{bmatrix}
\]

(6)

Where: g – gravity acceleration value, R – radius of Earth
Cor = \[ 2\Omega \begin{vmatrix} -V_y \cos L \sin AZ - V_z \sin L \\ V_x \cos L \sin AZ + V_z \cos L \cos AZ \\ V_x \cos L - V_y \cos L \cos AZ \end{vmatrix} \] (7)

Where: \( \Omega \) – rotation velocity of Earth, \( L \) – geographical latitude of firing site, \( AZ \) – azimuth of firing

Pitch and yaw angles have to be calculated. Without initial disturbances yaw will have 0 value but pitch will have non 0 value. In the paper [6] author deals with the issue. Key to determine the value of the pitch is analyze equilibrium state of pitch during flight. The analysis show that equilibrium state of pitch is when pitching and pitch damping moments are equal. That conclusion allowed to write equation of mean pitch:

\[ \alpha_m = \frac{C_{Ma}}{C_{Mq}} g \cos(\theta) \] (8)

Where: \( \alpha_m \) – mean pitch, \( C_{Ma} \) – linearized pitch damping moment coefficient, \( C_{Mq} \) – linearized pitching moment coefficient, \( \theta \) – angle between velocity vector and ground plane

Comparison results from equation 8 for trajectories showed in the figure 1 is presented in figure 2.

Received values have good correlation between results from equation 8 and 6-dof model even for 70° elevation angle. To improve accuracy modelling of oscillation after initial perturbation have to be added. To avoid significant increasing of computational cost author decide to use damped harmonic motion theory to predict projectile yaw and pitch oscillations from initial conditions. To not remind the whole theory in the paper will be shown only basic equation. The main formula is presented below (variable names are changed from standard to not be duplicate in the paper).

\[ \beta = A_0 e^{-bt} \cos(\omega t + \phi_0) \] (9)

Where: \( A_0 \) – maximum amplitude of oscillation, \( b \) – damping coefficient, \( \omega \) – frequency of the oscillations, \( \phi_0 \) – initial phase

The biggest difference between results from 6-dof calculations and damped harmonic oscillations is frequency behaviour. In the theory frequency is constant, but 6-dof analysis shown that is significantly
changing. To solve that virtual time is added to equation 9. After analysis to complete pitch formula mean pitch with reverse damping coefficient have to be considered. Complete pitch calculation equation looks:

\[ \alpha = A_0 e^{-bt} \cos(\omega t_v + \varphi_0) + \alpha_m (1 - e^{-bt})^2 \]  \hspace{1cm} (10)

Where: \( t_v \) – virtual time

Yaw calculation formula is simpler.

\[ \beta = A_0 e^{-bt} \cos(\omega t_v + \varphi_0) \] \hspace{1cm} (11)

Virtual time formula is empirical. The author build it up using 6-dof results for best match of few first oscillations because they have the biggest influence to the trajectory. To better understanding of the equation it will be presented partially.

\[ t_v = t(e^{-bt})^{A_0^{0.5}} A_0^{0.25} (1 + |\dot{\alpha}|)^{0.5} \] \hspace{1cm} (12)

Where \( k \): 

\[ k = \frac{C_D 0}{C_m} \frac{100}{V_0^2} \left( \frac{\omega_0}{\omega} \right)^2 (1 - 0.25 \sin(\theta_0)) \] \hspace{1cm} (13)

Where: \( C_D 0 \) – initial drag coefficient, \( C_m \) – linearized pitching moment, \( V_0 \) – initial velocity, \( \omega_0 \) – oscillation frequency without damping, \( \omega \) – oscillation frequency with damping, \( \theta_0 \) – initial fire elevation.

Presented formulation of the model allow to predict yaw and pitch in as a function of time. That allow to reduce number of equation calculated by Fire Control System and computational time. Next step of the paper is checking the model accuracy.

4. Check model accuracy

At first step trajectories from three different models and experimental data will be compared. In the figure 3 were presented trajectories from 6-dof, SFPPMM and point mass models and trajectory measured by Doppler radar during shooting of supersonic mortar projectile.

**Figure 3.** Comparison of calculated and experimental trajectory.
Differences between calculated trajectories and experimental were less than 0.5% of experimental range. 6-dof and FSPPMM give a bit to high apogee, but point mass model with the same as experimental apogee give bigger range. Trajectories from 6-dof and FSPPMM model are almost equal. Disadvantage of these test is that initial perturbations are unknown. At the next stage behaviour of yaw and pitch disturbed by different factors will be checked. At the beginning response for strong crosswind will be compared with 6-dof model. Main initial parameters were: \( V_0 = 200 \) m/s, \( W_z = 15 \) m/s, \( \theta_0 = 15^\circ \). Initial 6 seconds of the results are presented in the figure 4.

![Figure 4. Comparison of yaw behaviour due strong cross wind.](image)

First period of the oscillations was simulated with very good accuracy. After that differences between results become increasing, but in that time amplitudes are much smaller and less significant than for the first period. Influence yaw calculation on the trajectory is shown in the figure 5.

![Figure 5. Trajectories influenced by strong crosswind calculated by different models.](image)
In case presented in the figure 5 differences between 6-dof and SFPPMM are much smaller than between 6-dof and point mass model. Shape of the curves show that FSPPMM is able to calculate aerodynamic jump due a crosswind.

Pitch test will include initial pitch and pitch velocity at high and low muzzle velocity. For lower muzzle velocity disturbances were bigger to achieve longer and greater oscillations. High velocity results are shown in the figure 6. Main initial parameters were: \( V_0 = 500 \text{ m/s} \), \( \alpha = 0.5^\circ \), \( \dot{\alpha} = 15^\circ/\text{s} \), \( \theta_0 = 45^\circ \).

![Figure 6. Comparison of the pitch for high muzzle velocity.](image)

It is necessary to keep in mind that for tested projectile at Mach number \( \approx 1.3 \) pitching moment coefficient value is higher two times more than for Mach number \( \approx 1.5 \) (muzzle velocity). That induce lower amplitudes of oscillations that can be calculated using initial conditions. For low velocity test parameters were set to: \( V_0 = 150 \text{ m/s} \), \( \alpha = 1.5^\circ \), \( \dot{\alpha} = 30^\circ/\text{s} \), \( \theta_0 = 70^\circ \). The results were presented in figure 7.

![Figure 7. Comparison of pitch for low muzzle velocity.](image)
For this case after first period results are not matched. One of the causes of that is high velocity reduction due the gravity acceleration ($\theta_0=70^\circ$). For flat trajectory ($\theta_0=15^\circ$) presented at figure 8 the match is better.

![Comparison of pitch for low muzzle velocity for flat trajectory.](image)

**Figure 8.** Comparison of pitch for low muzzle velocity for flat trajectory.

### 5. Conclusion

Presented in the paper model is designed for fast calculations of fin stabilized projectiles trajectories. Main difference between FSPPMM and standard point mass model is that FSPPMM include yaw, pitch and they oscillations. FSPPMM have good agreement with 6-dof model, but computational time is similar to standard point mass model. It have also possibility to calculate aerodynamic jump due crosswind, yaw and pitch and radial velocities of the projectile. That features makes the FSPPMM very good tool for Fire Control Systems, where calculation time and precision is necessary or for calculating of firing tables which require large amount of calculations.

The Author is aware that some of empirical equations used in the model may be not the best solution. Also testing process should use many cases including projectiles with precisely defined coefficients. Therefore the testing process is not complete and some modifications to the model can be added. It is also field to cooperation.

### References

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