Compactness of Riesz transform commutator associated with Bessel operators

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Abstract: Let $\lambda > 0$ and $\Delta_\lambda := -\frac{d^2}{dx^2} - \frac{2\lambda}{x} \frac{d}{dx}$ be the Bessel operator on $\mathbb{R}_+ := (0, \infty)$. We first introduce and obtain an equivalent characterization of $\text{CMO}(\mathbb{R}_+, x^{2\lambda} dx)$. By this equivalent characterization and establishing a new version of the Fréchet-Kolmogorov theorem in the Bessel setting, we further prove that a function $b \in \text{BMO}(\mathbb{R}_+, x^{2\lambda} dx)$ is in $\text{CMO}(\mathbb{R}_+, x^{2\lambda} dx)$ if and only if the Riesz transform commutator $[b, R_{\Delta_\lambda}]$ is compact on $L^p(\mathbb{R}_+, x^{2\lambda} dx)$ for any $p \in (1, \infty)$.

Keywords: $\text{CMO}(\mathbb{R}_+, dm_\lambda)$; commutator; Bessel operator; Riesz transform.

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1 Introduction and Statement of Main Results

Let $\lambda$ be a positive constant and $\Delta_\lambda$ be the Bessel operator defined by by setting, for suitable functions $f$ and $x \in \mathbb{R}_+ := (0, \infty)$,

$$\Delta_\lambda f(x) := -\frac{d^2}{dx^2} f(x) - \frac{2\lambda}{x} \frac{d}{dx} f(x),$$

see [6, 29]. An early work concerning the Bessel operator is from Muckenhoupt and Stein [29]. They aimed to develop a theory associated to $\Delta_\lambda$ which is parallel to the classical one associated to the Laplace operator $\Delta$. After that, a lot of work concerning the Bessel operators was carried out. See, for example [1, 5, 7, 8, 9, 15, 24, 34, 35] and the references therein. Among the study of $\Delta_\lambda$, the properties of Riesz transforms associated to $\Delta_\lambda$ defined by

$$R_{\Delta_\lambda} f := \partial_x (\Delta_\lambda)^{-1/2} f,$$

have been studied extensively, see for example [1, 5, 7, 29, 34]. Characterizations of function spaces associated to the Bessel operator $\Delta_\lambda$ were also studied by many authors. Among these,
we point out that the Lebesgue space associated to the Bessel operator $\Delta_{\lambda}$ is of the form $L^p(\mathbb{R}_+, dm_{\lambda})$, where $1 < p < \infty$, $dm_{\lambda}(x) := x^{2\lambda} \, dx$, and $dx$ is the standard Lebesgue measure on $\mathbb{R}$ (see for example [6]). Moreover, in [6], Betancor et al. characterized the Hardy space $H^1(\mathbb{R}_+, dm_{\lambda})$ associated to $\Delta_{\lambda}$ in terms of the Riesz transform and the radial maximal function associated with the Hankel convolution of a class of suitable functions. More recently, Duong et al. [15] established a factorisation of the Hardy space associated to $\Delta_{\lambda}$ through commutators $[b, R_{\Delta_{\lambda}}]$, which is defined as follows:

$$[b, R_{\Delta_{\lambda}}]f(x) := b(x)R_{\Delta_{\lambda}}f(x) - R_{\Delta_{\lambda}}(bf)(x),$$

where $b \in L^1_{\text{loc}}(\mathbb{R}_+, dm_{\lambda})$ and $f \in L^p(\mathbb{R}_+, dm_{\lambda})$.

The aim of this paper is to provide a characterization of the compactness of the Riesz commutator $[b, R_{\Delta_{\lambda}}]$, based on the characterization in [15].

We recall that the first result on characterization of compactness of commutators of singular integrals is due to Uchiyama [33]. He refined the $L^p$-boundedness results of Coifman et al. [12] on the commutator with the symbol $b$ in the space BMO to compactness. This is achieved by requiring the symbol $b$ to be not just in BMO, but rather in CMO, which is the closure in BMO of the space of $C^\infty$ functions with compact supports. Since then, many authors focused on the compactness of commutators with certain singular integrals, including linear, nonlinear and bilinear operators on variant function spaces. See for example [3, 4, 11, 10, 18, 20, 25, 26, 27] and the references therein.

We further note that the compactness of the commutator has extensive applications in partial differential equations, see for example the application to $\bar{\partial}$–Neumann problem on forms [32, Chapter 12, Section 8]. Moreover, to study the $L^p$-theory of quasiregular mappings, Iwaniec [20] considered the linear complex Beltrami equation and derived the $L^p(\mathbb{C})$-invertibility of Beltrami operator $I - \mu T$, via the compactness of the commutator $[\mu, T]$ on $L^p(\mathbb{C})$ and the index theory of Fredholm operators on Banach spaces, where the Beltrami coefficient $\mu \in L^\infty(\mathbb{C}) \cap \text{CMO}(\mathbb{C})$ has compact support and $T$ is the Beurling-Ahlfors singular integral operator. In their remarkable work [2], Astala et al. further extended the result in [20] by removing the restrictive assumption on the compact support of $\mu$. Recently, based on the result of Iwaniec [20], Clop and Cruz [11] obtained a priori estimate in $L^p(\omega)$ for the generalized Beltrami equation and regularity for the Jacobian of certain quasiconformal mappings, where the weight $\omega$ belongs to the Muckenhoupt class $A_p$. See also [28, 14] for the application of the compactness of commutator generated by Beurling-Ahlfors transform and CMO functions to Beltrami equations.

Before stating our main result, we first recall the definition of the BMO space associated with the Bessel operator which is known to coincide with the standard BMO space on $(\mathbb{R}_+, dm_{\lambda})$. For every $x, r \in \mathbb{R}_+$, we define $I(x, r) := (x - r, x + r) \cap \mathbb{R}_+$.

**Definition 1.1** ([35]). A function $f \in L^1_{\text{loc}}(\mathbb{R}_+, dm_{\lambda})$ belongs to the space $\text{BMO}(\mathbb{R}_+, dm_{\lambda})$ if

$$\sup_{x, r \in (0, \infty)} M_{\lambda}(f, I(x, r)) := \sup_{x, r \in (0, \infty)} \frac{1}{m_{\lambda}(I(x, r))} \int_{I(x, r)} |f(y) - f_{I(x,r), \lambda}| y^{2\lambda} \, dy < \infty,$$

where

$$f_{I(x, r), \lambda} := \frac{1}{m_{\lambda}(I(x, \lambda))} \int_{I(x, r)} f(y)y^{2\lambda} \, dy. \tag{1.1}$$

We further denote by $\text{CMO}(\mathbb{R}_+, dm_{\lambda})$ the $\text{BMO}(\mathbb{R}_+, dm_{\lambda})$-closure of $\mathcal{D}$, the set of $C^\infty(\mathbb{R}_+)$ functions with compact supports.

The main result of this paper is stated as follows:
Theorem 1.2. Let $b \in \text{BMO}(\mathbb{R}_+, dm_\lambda)$. Then $b \in \text{CMO}(\mathbb{R}_+, dm_\lambda)$ if and only if the Riesz transform commutator $[b, R_{\Delta_\lambda}]$ is compact on $L^p(\mathbb{R}_+, dm_\lambda)$ for any $p \in (1, \infty)$.

The proof of Theorem 1.2 is carried out from Section 2 to 5, and contains the following ingredients:

(i) The doubling and reverse doubling properties of the space $(\mathbb{R}_+, x^{2\lambda}dx)$. More specifically, in Section 2, we first prove that the measure $dm_\lambda$ satisfies a doubling property with constant $2^{2\lambda+1}$ and reverse doubling property with constant $\min(2, 2^{2\lambda})$ (see Proposition 2.1 below). We remark that the constants are almost sharp in the sense that Proposition 2.1 is false if $\min(2, 2^{2\lambda})$ is replaced by 2 or $2^{2\lambda+1}$ replaced by $\max(2, 2^{2\lambda})$; see Remark 2.2 below.

(ii) Kernel bound estimates of the Riesz transforms (Lemma 2.3 and Proposition 2.5). We recall some known upper and lower bounds as well as the Hölder’s regularity of the kernel $R_{\Delta_\lambda}(x, y)$ of Riesz transform $R_{\Delta_\lambda}$ and establish a new estimate of the lower bound of $R_{\Delta_\lambda}(x, y)$, which plays a key role in the proof of the main result.

(iii) A new characterisation of the space $\text{CMO}(\mathbb{R}_+, x^{2\lambda}dx)$ which is also of independent interest (Theorem 3.1 in Section 3). We employ the idea of Uchiyama [33]. However, since the space $L^p(\mathbb{R}_+, dm_\lambda)$ is not invariant under translations, we need some new techniques and adapt the proof in [33] to our setting.

(iv) A new version of the Fréchet-Kolmogorov theorem in the Bessel setting (Theorem 4.2 in Section 4). We remark that Clop and Cruz [11] obtained a partial result of Fréchet-Kolmogorov theorem when $\omega$ belongs to $A_p(\mathbb{R}^n)$ with the Lebesgue measure and $p \in (1, \infty)$. However, in current setting, the weight $x^{2\lambda}$ for general $\lambda \in (0, \infty)$ might not belong to $A_p(\mathbb{R}_+)$. Estimates on the commutator $[b, R_{\Delta_\lambda}]$ are carried out in Section 5. By the upper and lower bounds and the Hölder’s regularity of $R_{\Delta_\lambda}(x, y)$ in Section 2, we first obtain a lemma for the upper and lower bounds of integrals of $[b, R_{\Delta_\lambda}]f_j$ on certain intervals, for $b \in \text{BMO}(\mathbb{R}_+, dm_\lambda)$ and proper function $f_j$. Using this and a contradiction argument in terms of the aforementioned equivalent characterization of $\text{CMO}(\mathbb{R}_+, dm_\lambda)$ in Section 3, we show that if $[b, R_{\Delta_\lambda}]$ is compact on $L^p(\mathbb{R}_+, dm_\lambda)$, then $b \in \text{CMO}(\mathbb{R}_+, dm_\lambda)$.

(vi) By the upper and lower bound and the Hölder’s regularity of $R_{\Delta_\lambda}$, together with the Fréchet-Kolmogorov theorem in Section 4, we show via a density argument that if $b \in \text{CMO}(\mathbb{R}_+, dm_\lambda)$, then $[b, R_{\Delta_\lambda}]$ is compact on $L^p(\mathbb{R}_+, dm_\lambda)$.

Here, for the necessity, we also note that Krantz and Li [26] showed that if $T$ is a singular integral operator bounded on $L^2(\mathcal{X})$ and $b \in \text{CMO}(\mathcal{X})$, then $[b, T]$ is compact on $L^p(\mathcal{X})$ for all $p \in (1, \infty)$. However, the underlying space $(\mathcal{X}, d, \mu)$ studied in [26] is a space of homogeneous type which satisfies the following condition: there exist positive constants $C$ and $\epsilon_0 \in (0, 1)$ such that for all $x, y \in \mathcal{X}$ and $d(x, y) \leq r \leq 1$,

$$
\mu(B(x, r) \setminus B(y, r)) + \mu(B(y, r) \setminus B(x, r)) \leq C \left( \frac{d(x, y)}{r} \right)^{\epsilon_0}.
$$

We point out that the underlying space $(\mathbb{R}_+, |\cdot|, dm_\lambda)$ in the Bessel setting does not fall into the scope of the space $(\mathcal{X}, d, \mu)$ studied by [26]. In fact, let $x := N + 1$, $y := N$, $r := 1$. Then we see that

$$
m_\lambda(I(N + 1, 1) \setminus I(N, 1)) = \int_{N+1}^{N+2} x^{2\lambda} dx \geq (N + 1)^{2\lambda} \to \infty,
$$
as $N \to \infty$, and that

$$
\frac{d(x, y)}{r} = 1.
$$
Hence (1.2) is not true for our $(\mathbb{R}_+, | \cdot |, dm_\lambda)$.

Throughout the paper, we denote by $C$ and $\tilde{C}$ positive constants which are independent of the main parameters, but they may vary from line to line. For every $p \in (1, \infty)$, $p'$ means the conjugate of $p$, i.e., $1/p' + 1/p = 1$. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we write $f \sim g$. For any $k \in \mathbb{R}_+$ and interval $I := I(x, r)$ for some $x, r \in (0, \infty)$, $kI := I(x, kr)$ and $I + y := \{x + y : x \in I\}$. For any $x, r \in (0, \infty)$, if $x < r$, then

$$I(x, r) = (0, x + r) = I\left(\frac{x + r}{2}, \frac{x + r}{2}\right).$$

Thus, for a given interval $I(x, r)$, without any specific condition, we may always assume that

$$x \geq r. \tag{1.3}$$

Moreover, for any $i \in \mathbb{Z}$, let

$$R_i := \{x \in \mathbb{R}_+ : x \leq 2^i\}. \tag{1.4}$$

If $I$ is a dyadic interval in $\mathbb{R}_+$, then $I = (k2^i, (k + 1)2^i]$, where $k \in \mathbb{Z}_+$, and $j \in \mathbb{Z}$.

## 2 Preliminaries: Reverse doubling property and bounds of Riesz transforms

In this section, we present some preliminary results, including the reverse doubling property of $dm_\lambda$ and upper and lower bound of Riesz transform $R_{\Delta \lambda}$. We begin with the following proposition, which implies that $(\mathbb{R}_+, \rho, dm_\lambda)$ is an RD space in [16], where $\rho(x, y) := |x - y|$ for all $x, y \in \mathbb{R}_+$.

**Proposition 2.1.** For any $I \subset \mathbb{R}_+$,

$$\min \{2, 2^{2\lambda}\} m_\lambda(I) \leq m_\lambda(2I) \leq 2^{2\lambda+1}m_\lambda(I). \tag{2.1}$$

**Proof.** Let $I := I(x, r) \subset \mathbb{R}_+$ be an interval. As in the argument of (1.3), we can assume that $x \geq r$. Moreover, since when $x = r$, it is easy to see that $m_\lambda(2I) = (3/2)^{2\lambda+1}m_\lambda(I)$ and hence Proposition 2.1 holds. Thus, we further assume that $x > r$. Observe that

$$m_\lambda(I) := \int_{x-r}^{x+r} y^{2\lambda} dy = \frac{(x + r)^{2\lambda} - (x - r)^{2\lambda+1}}{2\lambda + 1}$$

and

$$m_\lambda(2I) = \begin{cases} 
\frac{(x + 2r)^{2\lambda+1} - (x - 2r)^{2\lambda+1}}{2\lambda + 1}, & x \geq 2r; \\
\frac{(x + 2r)^{2\lambda+1}}{2\lambda + 1}, & r < x < 2r.
\end{cases}$$

Let $t := r/x$,

$$f_\lambda(t) := \begin{cases} 
(1 + 2t)^{2\lambda+1} - (1 - 2t)^{2\lambda+1} - \min \{2, 2^{2\lambda}\} \left[(1 + t)^{2\lambda+1} - (1 - t)^{2\lambda+1}\right], & t \in [0, 1/2], \\
(1 + 2t)^{2\lambda+1} - \min \{2, 2^{2\lambda}\} \left[(1 + t)^{2\lambda+1} - (1 - t)^{2\lambda+1}\right], & t \in (1/2, 1); 
\end{cases}$$
and
\[\tilde{f}_\lambda(t) := \begin{cases} (1 + 2t)^{2\lambda+1} - (1 - 2t)^{2\lambda+1} - 2^{2\lambda+1}[(1 + t)^{2\lambda+1} - (1 - t)^{2\lambda+1}], & t \in [0, 1/2], \\ (1 + 2t)^{2\lambda+1} - 2^{2\lambda+1}[(1 + t)^{2\lambda+1} - (1 - t)^{2\lambda+1}], & t \in (1/2, 1). \end{cases}\]

To show (2.1), it suffices to prove that for all \(t \in (0, 1)\) and \(\lambda \in (0, \infty)\),
\[f_\lambda(t) \geq 0\] and \(\tilde{f}_\lambda(t) \leq 0\).

We first prove that \(f_\lambda(t) \geq 0\) by considering the following four cases:

Case (i) \(t \in (0, 1/2)\) and \(\lambda \in (0, 1/2)\). In this case,
\[f'_\lambda(t) = (2\lambda + 1) \left\{ 2(1 + 2t)^{2\lambda} + 2(1 - 2t)^{2\lambda} - 2^{2\lambda} \left[ (1 + t)^{2\lambda} + (1 - t)^{2\lambda} \right] \right\}.
\]

Observe that the function \(g(t) := t^{2\lambda}\) is a concave function of \(t\) for given \(\lambda \in (0, 1/2)\). By the fact that for any \(a, b \in (0, \infty)\),
\[(a + b)^{2\lambda} \leq a^{2\lambda} + b^{2\lambda} \leq 2^{1-2\lambda}(a + b)^{2\lambda},\]
we see that
\[f'_\lambda(t) \geq (2\lambda + 1) \left\{ 2(1 + 2t)^{2\lambda} + 2(1 - 2t)^{2\lambda} - 2^{2\lambda} \left[ (1 + t)^{2\lambda} + (1 - t)^{2\lambda} \right] \right\} = 0,
\]

which further implies that \(f_\lambda(t) \geq f_\lambda(0) = 0\) for any \(t \in (0, 1/2)\).

Case (ii) \(t \in (0, 1/2)\) and \(\lambda \in (1/2, \infty)\). In this case,
\[f'_\lambda(t) = 2(2\lambda + 1) \left\{ (1 + 2t)^{2\lambda} + (1 - 2t)^{2\lambda} - \left[ (1 + t)^{2\lambda} + (1 - t)^{2\lambda} \right] \right\}.
\]

By the fact that \(g(t) := t^{2\lambda}\) is a convex function for \(\lambda \in (1/2, \infty)\), we get that for any \(t \in (0, 1/2)\),
\[(1 + t)^{2\lambda} = \left[ \frac{1 - 2t}{4} + \frac{3(1 + 2t)}{4} \right]^{2\lambda} \leq \frac{(1 - 2t)^{2\lambda}}{4} + \frac{3(1 + 2t)^{2\lambda}}{4},\]

and
\[(1 - t)^{2\lambda} = \left[ \frac{3(1 - 2t)}{4} + \frac{1 + 2t}{4} \right]^{2\lambda} \leq \frac{3(1 - 2t)^{2\lambda}}{4} + \frac{(1 + 2t)^{2\lambda}}{4}.
\]

Combining these inequalities above, we see that \(f'_\lambda(t) \geq 0\) and hence \(f_\lambda(t) \geq f_\lambda(0) = 0\) for any \(t \in (0, 1/2)\).

Case (iii) \(t \in (1/2, 1)\) and \(\lambda \in (0, 1/2)\). In this case,
\[f_\lambda(t) = (1 + 2t)^{2\lambda+1} - 2^{2\lambda} \left[ (1 + t)^{2\lambda+1} - (1 - t)^{2\lambda+1} \right].\]

To show \(f_\lambda(t) \geq 0\) for all \(t \in (1/2, 1)\) and \(\lambda \in (0, 1/2)\), it suffices to prove that
\[g_\alpha(t) := 2^{-\alpha} - \left[ \left( \frac{1 + t}{1 + 2t} \right)^{\alpha+1} - \left( \frac{1 - t}{1 + 2t} \right)^{\alpha+1} \right] \geq 0.
\]
for all $t \in (1/2, 1)$ and $\alpha := 2\lambda \in (0, 1]$. On the other hand, observe that for fixed $t \in (1/2, 1)$, $g_\alpha(t)$ is decreasing in $\alpha \in (0, 1]$, which implies that for any $t \in (1/2, 1)$,

$$g_\alpha(t) \geq g_1(t) = \frac{(1 - 2t)^2}{2(1 + 2t)^2} \geq 0.$$ 

Thus, we conclude that $f_\lambda(t) \geq 0$ for all $t \in (1/2, 1)$.

Case (iv) $t \in (1/2, 1)$ and $\lambda \in (1/2, \infty)$. In this case, we also have $f_\lambda(t) \geq 0$ for all $t \in (1/2, 1)$. Therefore $f_\lambda(t)$ is increasing in $t \in (1/2, 1)$, and

$$f_\lambda(t) \geq f_\lambda(1/2) = 2^{2\lambda+1} - 2\left[\frac{3}{2}\right]^{2\lambda+1} - \frac{1}{2}^{2\lambda+1} \geq 0$$

for all $\lambda \in (1/2, \infty)$.

Combining the four cases above, we conclude that $f_\lambda(t) \geq 0$ for all $\lambda \in (0, \infty)$ and $t \in (0, 1)$.

Now we show $\tilde{f}_\lambda(t) \leq 0$ for all $t \in (0, 1)$ and $\lambda \in (0, \infty)$. Similarly, when $t \in (0, 1/2]$, we have

$$\tilde{f}_\lambda(t) = (2\lambda + 1) \left\{2(1 + 2t)^{2\lambda} + 2(1 - 2t)^{2\lambda} - 2^{2\lambda+1}\left[(1 + t)^{2\lambda} + (1 - t)^{2\lambda}\right]\right\} < 0,$$ 

and $\tilde{f}_\lambda(t) \leq \tilde{f}_\lambda(0) = 0$ for all $t \in (0, 1/2]$. When $t \in [1/2, 1)$,

$$\tilde{f}_\lambda(t) = (2\lambda + 1) \left\{2(1 + 2t)^{2\lambda} - 2^{2\lambda+1}\left[(1 + t)^{2\lambda} + (1 - t)^{2\lambda}\right]\right\} < 0,$$ 

and $\tilde{f}_\lambda(t) \leq \tilde{f}_\lambda(1/2) = 0$ for all $t \in [1/2, 1)$. This finishes the proof of Proposition 2.1.

\begin{remark}
We remark that the constants in (2.1) are almost sharp in the sense that (2.1) is false if $2^{2\lambda+1}$ is replaced by $\max(2, 2^{2\lambda})$ or $\min(2, 2^{2\lambda})$ replaced by 2. In fact, to see (2.1) is false if $2^{2\lambda+1}$ is replaced by $\max(2, 2^{2\lambda})$, it suffices to take $\lambda := 1/2$ and $r := x$ for any $x \in \mathbb{R}_+$. On the other hand, to see (2.1) is false if $\min(2, 2^{2\lambda})$ is replaced by 2, consider the case $\lambda \in (0, 1/2]$ and $r := x/2$ for any $x \in \mathbb{R}_+$. Let

$$h_{1/2}(\lambda) := 2^{2\lambda+1} - 2\left[\left(\frac{3}{2}\right)^{2\lambda+1} - \left(\frac{1}{2}\right)^{2\lambda+1}\right], \lambda \in [0, 1/2].$$

Then $h_{1/2}(0) = 0 = h_{1/2}(1/2)$ and $h_{1/2}(\lambda) = 2^{2\lambda+1}\tilde{h}_{1/2}(t)$, where

$$\tilde{h}_{1/2}(\lambda) := 1 - 2\left[\left(\frac{3}{4}\right)^{2\lambda+1} - \left(\frac{1}{4}\right)^{2\lambda+1}\right].$$

Observe that $\tilde{h}_{1/2}(\lambda) \leq 0$ on $[0, 1/2]$. This implies that $h_{1/2}(\lambda) \leq 0$ for all $\lambda \in (0, 1/2]$ and so

$$m_\lambda(I(x, x)) - 2m_\lambda(I(x, x/2)) = \frac{2^{2\lambda+1}}{2\lambda + 1}h_{1/2}(\lambda) \leq 0.$$ 

We now recall some known upper and lower bounds of the kernel $R_{\Delta_\lambda}(y, z)$ of $R_{\Delta_\lambda}$. The following estimates can be found in, for example, [7, 15].

\begin{lemma}
The kernel $R_{\Delta_\lambda}(y, z)$ satisfies the following estimates:

\end{lemma}
i) There exists a positive constant $C$ such that for any $y, z \in \mathbb{R}_+$ with $y \neq z$,

$$|R_{\Delta}(y, z)| \leq C \frac{1}{m_{\lambda}(I(y, |y - z|))}.$$  \hfill (2.2)

ii) There exists a positive constant $\tilde{C}$ such that for any $y, y_0, z \in \mathbb{R}_+$ with $|y_0 - z| < |y_0 - y|/2$,

$$|R_{\Delta}(y, y_0) - R_{\Delta}(y, z)| + |R_{\Delta}(y_0, y) - R_{\Delta}(z, y)| \leq \tilde{C} \frac{|y_0 - z|}{|y_0 - y|} \frac{1}{m_{\lambda}(I(y, |y_0 - y|))}.$$  \hfill (2.3)

iii) There exist $K_1 \in (0, 1)$ small enough and a positive constant $C_{K_1, \lambda}$ such that for any $y, z \in \mathbb{R}_+$ with $z < K_1 y$,

$$R_{\Delta}(y, z) \leq -C_{K_1, \lambda} \frac{1}{y^{2\lambda+1}}.$$  \hfill (2.4)

iv) There exist $K_2 \in (1/2, 1)$ such that $1 - K_2$ small enough and a positive constant $C_{K_2, \lambda}$ such that for any $y, z \in \mathbb{R}_+$ with $z/y \in (K_2, 1)$,

$$|R_{\Delta}(y, z) + \frac{1}{\pi} \frac{1}{y^{\lambda}z^\lambda} \frac{1}{y - z}| \leq C_{K_2, \lambda} \frac{1}{y^{2\lambda+1}} \left( \log_+ \frac{\sqrt{yz}}{|y - z|} + 1 \right).$$

**Remark 2.4.** We mention that by Lemma 2.3 iv), there exists $\tilde{K}_2 \in (K_2, 1)$ such that for any $y, z \in \mathbb{R}_+$ with $\tilde{K}_2 < z/y < 1$,

$$-R_{\Delta}(y, z) \geq \frac{1}{2\pi y^{\lambda}z^\lambda(y - z)}.$$  \hfill (2.4)

In fact, from Lemma 2.3 iv), it follows that for any $y, z \in \mathbb{R}_+$ with $K_2 < z/y < 1$,

$$-R_{\Delta}(y, z) \geq \frac{1}{\pi} \frac{1}{y^{\lambda}z^\lambda} \frac{1}{y - z} - C_{K_2, \lambda} \frac{1}{y^{2\lambda+1}} \left( \log_+ \frac{\sqrt{yz}}{|y - z|} + 1 \right).$$

To show (2.4), it suffices to prove that there exists $\tilde{K}_2 \in (K_2, 1)$ such that for all $y, z \in \mathbb{R}_+$ with $z/y \in (\tilde{K}_2, 1)$,

$$C_{K_2, \lambda} \frac{1}{y^{2\lambda+1}} \left( \log_+ \frac{\sqrt{yz}}{|y - z|} + 1 \right) \leq \frac{1}{2\pi y^{\lambda}z^\lambda} \frac{1}{y - z}.$$

Equivalently, we only need to show that

$$\left( \frac{z}{y} \right)^\lambda \frac{y - z}{y} \left( \log_+ \frac{\sqrt{yz}}{|y - z|} + 1 \right) \leq \frac{1}{2\pi C_{K_2, \lambda}}.$$

Note that

$$\left( \frac{z}{y} \right)^\lambda \frac{y - z}{y} \left( \log_+ \frac{\sqrt{yz}}{|y - z|} + 1 \right) = \left( \frac{z}{y} \right)^\lambda \left( 1 - \frac{z}{y} \right) \left( \log_+ \frac{\sqrt{z/y}}{|1 - z/y|} + 1 \right) \to 0,$$

as $z/y \to 1^-$. This implies the existence of $\tilde{K}_2$, which shows (2.4).
As a consequence of Lemma 2.3 iii) and Remark 2.4 above, we further establish a new version of lower bound for $R_{\Delta_\lambda}(y, z)$ for all $z < y$, which plays a key role in the proof of Theorem 1.2.

**Proposition 2.5.** There exists a positive constant $C_0$ such that for any $y, z \in \mathbb{R}_+$ with $z < y$,

$$R_{\Delta_\lambda}(y, z) \leq -C_0 \frac{1}{m_\lambda(I(y, y - z))}.$$  

**Proof.** Since $y > z$ and $y > y - z$, we first see that $m_\lambda(I(y, y - z)) \sim y^{2\lambda}(y - z)$, thus we only need to show

$$-R_{\Delta_\lambda}(y, z) \gtrsim \frac{1}{y^{2\lambda+1}} \gtrsim \frac{1}{y^{2\lambda}(y - z)}.$$

Recall that

$$R_{\Delta_\lambda}(y, z) = -\frac{2\lambda}{\pi} \int_0^\pi \frac{(y - z \cos \theta)(\sin \theta)^{2\lambda-1}}{(y^2 + z^2 - 2yz \cos \theta)^{\lambda+1}} \, d\theta; \quad (2.5)$$

see, for example, [6]. For any fixed $y, z \in \mathbb{R}_+$ with $y > z$, write $z = sy$. Then $s \in (0, 1)$. If $s < K_1$, where $K_1$ is as in iii) of Lemma 2.3, then by iii) of Lemma 2.3, we see that

$$-R_{\Delta_\lambda}(y, z) \gtrsim \frac{1}{y^{2\lambda+1}} \gtrsim \frac{1}{y^{2\lambda}(y - z)}.$$

On the other hand, for $\tilde{K}_2$ as in Remark 2.4, and any $y, z \in \mathbb{R}_+$ with $\tilde{K}_2 < y/z < 1$,

$$-R_{\Delta_\lambda}(y, z) \gtrsim \frac{1}{y^{\lambda}z^{\lambda}(y - z)}.$$

If $z = sy$ and $s \in (\tilde{K}_2, 1)$, then

$$-R_{\Delta_\lambda}(y, z) \gtrsim \frac{1}{y^{2\lambda}(y - z)}.$$

Thus, it remains to consider $s \in [K_1, \tilde{K}_2]$. By (2.5), we write

$$R_{\Delta_\lambda}(y, sy) = -\frac{2\lambda}{\pi} \frac{1}{y^{2\lambda+1}} \int_0^\pi \frac{(1 - s \cos \theta)(\sin \theta)^{2\lambda-1}}{(1 + s^2 - 2s \cos \theta)^{\lambda+1}} \, d\theta =: -\frac{2\lambda}{\pi} \frac{1}{y^{2\lambda+1}} I.$$

Since $s \in [K_1, \tilde{K}_2]$ and $\frac{2}{\pi} \theta \leq \sin \theta \leq \theta$ for any $\theta \in (0, \pi/2)$, we see that $1 - \tilde{K}_2 \leq 1 - s < 1$ and

$$I = \int_0^\pi \frac{[(1 - s) + s(1 - \cos \theta)]}{(1 + s^2 - 2s \cos \theta)^{\lambda+1}} (\sin \theta)^{2\lambda-1} \, d\theta$$

$$\geq \int_0^{\frac{\pi}{2}} \frac{1 - s}{[(1 - s)^2 + 2s(1 - \cos \theta)]^{\lambda+1}} \theta^{2\lambda-1} (\sin \theta)^{2\lambda-1} \, d\theta$$

$$\gtrsim (1 - s) \int_0^{\frac{\pi}{2}} \frac{\theta^{2\lambda-1}}{[(1 - s)^2 + 4s(\sin \frac{s \theta}{2})^2]^{\lambda+1}} \, d\theta$$

$$\gtrsim (1 - s) \int_0^{\frac{\pi}{2}} \frac{\theta^{2\lambda-1}}{[(1 - s)^2 + s\theta^2]^{\lambda+1}} \, d\theta$$

$$\gtrsim \frac{1}{(1 - s)^{2\lambda+1}} \int_0^{\frac{\pi}{2}} \frac{\theta^{2\lambda-1}}{[1 + (\sqrt{2}s \theta)^2]^{\lambda+1}} \, d\theta$$
\[ \int_0^{\pi \sqrt{1 - \kappa_1^2}} \frac{\beta^{2\lambda - 1}}{(1 + \beta^2)^{\lambda + 1}} d\beta \geq 1. \]

Thus, by the inequality above and the fact that \((1 - \tilde{K}_2)y \leq y - z < y\), we conclude that

\[-R_{\Delta y_1}(y, z) \geq \frac{1}{y^{2\lambda + 1}} \geq \frac{1}{y^{2\lambda}(y - z)},\]

and finish the proof of Proposition 2.5.

\section{An equivalent characterization of \textit{CMO}(\mathbb{R}_+, dm_\lambda)}

In this section, we establish an equivalent characterization of \textit{CMO}(\mathbb{R}_+, dm_\lambda), which is of independent interest. See also [33].

\textbf{Theorem 3.1.} Let \(f \in \text{BMO}(\mathbb{R}_+, dm_\lambda)\). Then \(f \in \text{CMO}(\mathbb{R}_+, dm_\lambda)\) if and only if \(f\) satisfies the following three conditions:

\begin{enumerate}[(i)]
    \item \(\lim_{a \to 0^+} \sup_{m_\lambda(I) = a} M_\lambda(f, I) = 0\),
    \item \(\lim_{a \to \infty} \sup_{m_\lambda(I) = a} M_\lambda(f, I) = 0\),
    \item \(\lim_{R \to \infty} \sup_{I \subset [R, \infty)} M_\lambda(f, I) = 0\).
\end{enumerate}

\textbf{Proof.} Assume that \(f \in \text{CMO}(\mathbb{R}_+, dm_\lambda)\). If \(f \in \mathcal{D}\), then (i)-(iii) hold. In fact, (i) holds for \(f\) since \(f\) is uniformly continuous, (ii) holds since \(f \in L^1(\mathbb{R}_+, dm_\lambda)\), and (iii) holds by the fact that \(f\) is compactly supported. If \(f \in \text{CMO}(\mathbb{R}_+, dm_\lambda) \setminus \mathcal{D}\), then for any given \(\epsilon > 0\), there exists \(f_\epsilon \in \mathcal{D}\) satisfying (i)-(iii) and \(\|f - f_\epsilon\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} < \epsilon\). By the triangle inequality of \textit{BMO}(\mathbb{R}_+, dm_\lambda) norm, we see that (i)-(iii) hold for \(f\).

Now we prove the converse. To this end, we assume that \(f\) satisfies (i)-(iii). To prove that \(f \in \text{CMO}(\mathbb{R}_+, dm_\lambda)\), it suffices to show that there exists a positive constant \(C_1\) depending only on \(\lambda\) such that, for any \(\epsilon \in (0, 1)\), there exists \(g_\epsilon \in \text{BMO}(\mathbb{R}_+, dm_\lambda)\) satisfying that

\[ \inf_{h \in \mathcal{D}} \|g_\epsilon - h\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} < C_1 \epsilon \quad \text{(3.1)} \]

and

\[ \|g_\epsilon - f\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} < C_1 \epsilon. \quad \text{(3.2)} \]

We prove (3.1) and (3.2) by the following two steps.

\textbf{Step I} We define an auxiliary function \(g_\epsilon\) via a set of dyadic intervals \(\mathcal{I}\) of \(\mathbb{R}_+\). In fact, by (i) and (ii), there exist \(i_\epsilon, k_\epsilon \in \mathbb{N}\) such that

\[ \sup \left\{ M_\lambda(f, I) : m_\lambda(I) \leq 2^{-i_\epsilon + 1} \right\} < \epsilon \quad \text{(3.3)} \]
From this and the mean value theorem, it follows that there exists an integer $j_\epsilon > k_\epsilon$ such that
\[ \sup \left\{ M_\lambda(f, I) : m_\lambda(I) \geq 2^{k_\epsilon} \right\} < \epsilon. \] (3.4)

By (iii), there exists an integer $j_\epsilon > k_\epsilon$ such that
\[ \sup \{ M_\lambda(f, I) : I \cap R_{j_\epsilon} = \emptyset \} < \epsilon, \] (3.5)
where $R_{j_\epsilon}$ is as in (1.4).

For the above $j_\epsilon$, we consider the dyadic intervals $R_{j_\epsilon} := (0, 2^{j_\epsilon}), R_m \setminus R_{m-1} := (2^{m-1}, 2^m), m > j_\epsilon$. Next for $k = 1, \ldots, 2^{j_\epsilon+i_\epsilon+2+2\lambda(j_\epsilon+1)}$, we denote by
\[ I_k^j := (k-1)2^{-i_\epsilon-2-(2\lambda(j_\epsilon+1)), k2^{-i_\epsilon-2-(2\lambda(j_\epsilon+1))}] \]
the descendants of $R_{j_\epsilon}$. Here for any $\alpha \in \mathbb{R}$, $[\alpha]$ means the largest integer $k$ such that $k \leq \alpha$. And similarly, for each $m > j_\epsilon$ and $k = 1, \ldots, 2^{j_\epsilon+i_\epsilon+1+2\lambda(j_\epsilon+1)}$, denote by
\[ I_k^m := (2^{m-1} + (k-1)2^{-i_\epsilon-2-(2\lambda(j_\epsilon+1))+m-j_\epsilon), 2^{m-1} + k2^{-i_\epsilon-2-(2\lambda(j_\epsilon+1))+m-j_\epsilon}] \]
the descendants of $R_m \setminus R_{m-1}$.

Then we list these dyadic descendants in order as follows:

\[ \mathcal{I} := \left\{ I_1^j, \ldots, I_1^{j_\epsilon}, \ldots, I_k^{j_\epsilon+i_\epsilon+2+2\lambda(j_\epsilon+1)}, I_k^{j_\epsilon+1}, \ldots, I_k^{j_\epsilon+i_\epsilon+1+2\lambda(j_\epsilon+1)}, I_{k+1}, \ldots \right\}. \]

For each $x \in \mathbb{R}_+$, we define $I_x$ as follows: if $I \in \mathcal{I}$ and $x \in I$, then $I_x := I$. Observe that for each $x \in \mathbb{R}_+$, such $I_x$ exists and is unique.

We claim that

(a) Every dyadic interval $I$ in
\[ \left\{ I_1^j, \ldots, I_k^{j_\epsilon+i_\epsilon+2+2\lambda(j_\epsilon+1)}, I_k^{j_\epsilon+1}, I_{k+1} \right\} \]
satisfies that $m_\lambda(I) \leq 2^{-i_\epsilon}$.

(b) For any $m_\epsilon > j_\epsilon$ and $x \in R_{m_\epsilon} \setminus R_{m_\epsilon-1},$
\[ 2^{j_\epsilon+2\lambda(j_\epsilon+1)+2+i_\epsilon} I_x = 2^{m_\epsilon} = |R_{m_\epsilon}|, \] (3.6)
\[ 2^{j_\epsilon+2\lambda(j_\epsilon+1)+2+i_\epsilon} I_x \subset R_{m_\epsilon+1}, \] (3.7)
moreover, if $2\lambda(m_\epsilon - j_\epsilon - 2) \geq 2$, then
\[ m_\lambda(I_x) \geq 2^{m_\epsilon-i_\epsilon-j_\epsilon}. \] (3.8)

In fact, since $m_\lambda(I_x)$ is non-decreasing with respect to $x$, to show (a), we only need to show that $m_\lambda(I_{2^{j_\epsilon+1}}) \leq 2^{-i_\epsilon}$. Observe that
\[ I_{2^{j_\epsilon+1}} := \left( 2^{j_\epsilon} + 2^{-i_\epsilon-1-(2\lambda(j_\epsilon+1)), 2^{j_\epsilon} + 2^{-i_\epsilon-2-(2\lambda(j_\epsilon+1))} \right]. \]

From this and the mean value theorem, it follows that there exists $\xi \in I_{2^{j_\epsilon+1}}$ such that
\[ m_\lambda(I_{2^{j_\epsilon+1}}) = \xi^{2\lambda}2^{-i_\epsilon-1-(2\lambda(j_\epsilon+1))} \]
Similarly, observe that

\[
\mathcal{I}_{2x+1+\lfloor 2\lambda(j+1)\rfloor+\epsilon} := \left(2^{m_\epsilon} - 2^{-i_\epsilon-2\lambda(j+1)} + m_\epsilon - j_\epsilon, 2^{m_\epsilon}\right),
\]

the last interval of \(\mathcal{I}\) included in \(R_{m_\epsilon} \setminus R_{m_\epsilon-1}\), we also have that (3.7) holds for any \(I_x\) with \(x \in R_{m_\epsilon} \setminus R_{m_\epsilon-1}\).

Finally, by the fact that \(m(I_x)\) is non-decreasing in \(x\), it suffices to show (3.8) holds for \(I_{m_\epsilon}^m\), the first dyadic interval of \(\mathcal{I}\) included in \(R_{m_\epsilon} \setminus R_{m_\epsilon-1}\). Observe that

\[
I_{m_\epsilon}^m := \left(2^{m_\epsilon-1}, 2^{m_\epsilon} + 2^{-i_\epsilon-2\lambda(j+1)} + m_\epsilon - j_\epsilon\right)
\]

and there exists \(\xi \in I_{m_\epsilon}^m\) such that

\[
m_\lambda(I_{m_\epsilon}^m) = \xi^{2\lambda(j_\epsilon + 2\lambda(j+1)) + m_\epsilon - j_\epsilon} \geq 2^{2\lambda(m_\epsilon - j_\epsilon - 2)} + 2^{m_\epsilon - j_\epsilon} \geq 2^{m_\epsilon - j_\epsilon}
\]

provided \(2\lambda(m_\epsilon - j_\epsilon - 2) - 2 \geq 0\). This implies (3.8).

Now for each \(x \in \mathbb{R}_+\), let \(\tilde{g}_\epsilon(x) := f_{I_x, \lambda}\), where \(f_{I_x, \lambda}\) is defined as in (1.1). Then from (ii), there exists an integer \(m_\epsilon > j_\epsilon\) such that

\[
\sup \{|\tilde{g}_\epsilon(x) - \tilde{g}_\epsilon(y)| : x, y \in R_{m_\epsilon} \setminus R_{m_\epsilon-1}\} < \epsilon.
\]  

(3.10)

To see this, by (ii), let \(m_\epsilon > j_\epsilon + k_\epsilon + i_\epsilon\) be large enough such that when \(m_\lambda(I) \geq 2^{m_\epsilon - i_\epsilon - j_\epsilon}\),

\[
M_\lambda(f, I) < \{C_2[j_\epsilon + 2\lambda(j_\epsilon + 1) + 2 + i_\epsilon]\}^{-1}\epsilon
\]  

(3.11)

for some positive constant \(C_2 > 2^{3(2\lambda+1)+2}\).

By (3.6) and (3.7), we see that

\[
2^{j_\epsilon + \lfloor 2\lambda(j+1)\rfloor + 2 + i_\epsilon} I_x \subset R_{m_\epsilon+1} \subset 4 \cdot 2^{j_\epsilon + \lfloor 2\lambda(j+1)\rfloor + 2 + i_\epsilon} I_x.
\]

This together with (2.1) and (3.11) implies that

\[
\left|f_{2^{j_\epsilon + \lfloor 2\lambda(j+1)\rfloor + 2 + i_\epsilon} I_x, \lambda} - f_{R_{m_\epsilon+1}, \lambda}\right| \leq \frac{m_\lambda(R_{m_\epsilon+1})}{m_\lambda(2^{j_\epsilon + \lfloor 2\lambda(j+1)\rfloor + 2 + i_\epsilon} I_x)} M_\lambda(f, R_{m_\epsilon+1})
\]  

\[
< 2^{2(2\lambda+1)} \frac{\epsilon}{C_2[j_\epsilon + 2\lambda(j_\epsilon + 1) + 2 + i_\epsilon]}
\]  

(3.12)

\[
< \epsilon/8.
\]

Similarly, observe that \(R_{m_\epsilon+1} \subset 8(R_{m_\epsilon} \setminus R_{m_\epsilon-1})\). Thus by (2.1),

\[
|f_{R_{m_\epsilon+1}} - f_{R_{m_\epsilon} \setminus R_{m_\epsilon-1}}| \leq \frac{m_\lambda(R_{m_\epsilon+1})}{m_\lambda(R_{m_\epsilon} \setminus R_{m_\epsilon-1})} M_\lambda(f, R_{m_\epsilon+1})
\]  

(3.13)
By (3.11), (3.12), (3.13) and (3.8), we conclude that for any $x \in R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}$,

\[
\left| f_{I_x, \lambda} - f_{R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}, \lambda} \right|
\leq \left| f_{2^{j_x}+2\lambda(j_x+1)+1+i_\epsilon, \lambda} - f_{R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}, \lambda} \right| + \sum_{j=0}^{j_x+2\lambda(j_x+1)+1+i_\epsilon} \left| f_{2^{j+1}I_x, \lambda} - f_{2^{j+1}I_y, \lambda} \right|
\leq 2^{2\lambda+1} \frac{\epsilon}{C_2[j_\epsilon + 2\lambda(j_\epsilon + 1) + 2 + i_\epsilon]}
\leq \epsilon/8 + \epsilon/8 + \frac{2^{2\lambda+1}}{C_2} \epsilon
< \epsilon/2.
\] (3.14)

So for any $I_x, I_y$ with $x, y \in R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}$,

\[
\left| f_{I_x, \lambda} - f_{I_y, \lambda} \right| \leq \left| f_{I_x, \lambda} - f_{R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}, \lambda} \right| + \left| f_{R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}, \lambda} - f_{I_y, \lambda} \right| < \epsilon.
\]

This shows (3.10).

**Step II** Define $g_\epsilon(x) := \tilde{g}_\epsilon(x)$ when $x \in R_{m_\epsilon}$ and $g_\epsilon(x) := f_{R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}, \lambda}$ when $x \in \mathbb{R}_+ \setminus R_{m_\epsilon}$.

Before proving (3.1) and (3.2), we first claim that there exists a positive constant $C_3$ such that if $I_x \cap I_y \neq \emptyset$ or $x, y \in \mathbb{R}_+ \setminus R_{m_{\epsilon-1}}$, then

\[
\left| g_\epsilon(x) - g_\epsilon(y) \right| < C_3 \epsilon.
\] (3.15)

In fact, assume that $x < y$. We first show that if $x, y \in \mathbb{R}_+ \setminus R_{m_{\epsilon-1}}$, then (3.15) holds.

Firstly, if $x, y \in \mathbb{R}_+ \setminus R_{m_\epsilon}$, then

\[
g_\epsilon(x) = g_\epsilon(y) = f_{R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}, \lambda}
\]

and (3.15) holds. Secondly, if $x, y \in R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}$, then from (3.10), we deduce that

\[
\left| g_\epsilon(x) - g_\epsilon(y) \right| = \left| \tilde{g}_\epsilon(x) - \tilde{g}_\epsilon(y) \right| < \epsilon.
\]

Thirdly, if $x \in R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}$ and $y \in \mathbb{R}_+ \setminus R_{m_\epsilon}$, then from (3.14), it follows that

\[
\left| g_\epsilon(x) - g_\epsilon(y) \right| = \left| \tilde{g}_\epsilon(x) - f_{R_{m_\epsilon} \setminus R_{m_{\epsilon-1}}, \lambda} \right| < \epsilon/2.
\]

Now we show if $I_x \cap I_y \neq \emptyset$, then (3.15) holds. In fact, assume that $I_x \neq I_y$ and define $I := I_x \cup I_y$. Observe that by the choice of $I_x, |I_x|/2 \leq |I_x| \leq 2|I_y|$ if $I_x \cap I_y \neq \emptyset$. If $x, y \in R_{j_x}$ and $I_x, I_y \in \{I_1^+, \ldots, I_{2^{j_x+i_\epsilon+2\lambda+1+2\lambda(j_x+1)}}^+ \}$, by (3.9), we have that $m_\lambda(I) \leq m_\lambda(I_1^+) \leq 2^{-i_\epsilon+1}$. And from (3.3) and (2.1), it follows that

\[
\left| g_\epsilon(x) - g_\epsilon(y) \right| \leq \left| \tilde{g}_\epsilon(x) - f_{I, \lambda} \right| + \left| f_{I, \lambda} - \tilde{g}_\epsilon(y) \right| \lesssim M_\lambda(f, I) \lesssim \epsilon.
\]
Similarly, if \( I_x = I^{\hat{j}_+}_{2\hat{j}_+ + 2 + 2\lambda(j_++1)} \), \( I_y = I^{\hat{j}_+}_{1} + 1 \) and \( I_x = I^{\hat{j}_+}_{1} \) and \( I_y = I^{\hat{j}_+}_{2} + 1 \), then arguing as in (3.9), we see that \( m_\lambda(I) \leq 2^{-i} + 1 \). By (3.3) and (2.1) again,

\[
|g_\epsilon(x) - g_\epsilon(y)| \leq |\tilde{g}_\epsilon(x) - f_I, \lambda| + |f_I, \lambda - \tilde{g}_\epsilon(y)| \lesssim M_\lambda(f, I) \lesssim \epsilon.
\]

Finally, if \( I_x \notin \{ I^{\hat{j}_+}_{1}, \cdots, I^{\hat{j}_+}_{2\hat{j}_++2 + 2\lambda(j_++1)} \}, I^{\hat{j}_+}_{1} \) and \( y > x \), then \( I \cap R_{j_+} = \emptyset \). It follows from (3.5) and (2.1) that

\[
|g_\epsilon(x) - g_\epsilon(y)| \leq |\tilde{g}_\epsilon(x) - f_I, \lambda| + |f_I, \lambda - \tilde{g}_\epsilon(y)| \lesssim M_\lambda(f, I) \lesssim \epsilon.
\]

Combining these cases, (3.15) holds.

The function \( g_\epsilon \) satisfies (3.1). In fact, let

\[
\tilde{h}_\epsilon(x) := g_\epsilon(x) - f_{R_{m_\epsilon}} \setminus R_{m_\epsilon - 1}, \lambda.
\]

Then by the definition of \( g_\epsilon \), we see that

\[
\tilde{h}_\epsilon(x) = 0 \text{ for any } x \in \mathbb{R}_+ \setminus R_{m_\epsilon}, \quad \|\tilde{h}_\epsilon - g_\epsilon\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} = 0.
\]

Moreover, if \( I_x \cap I_y \neq \emptyset \) or \( x, y \in \mathbb{R}_+ \setminus R_{m_\epsilon - 1} \), then from (3.15), it follows that

\[
|\tilde{h}_\epsilon(x) - \tilde{h}_\epsilon(y)| = |g_\epsilon(x) - g_\epsilon(y)| < C_3 \epsilon.
\]

Observe that \( \text{supp}(\tilde{h}_\epsilon) \subset R_{m_\epsilon} \) and there exists a function \( h_\epsilon \in C_c(\mathbb{R}_+) \) such that for any \( x \in \mathbb{R}_+ \),

\[
|\tilde{h}_\epsilon(x) - h_\epsilon(x)| < C_3 \epsilon.
\]

Then let \( \omega \in C_c(\mathbb{R}) \) be a positive valued function with \( \int_\mathbb{R} \omega(x) \, dx = 1 \) and \( \omega_t(x) := \frac{1}{t} \omega\left(\frac{x}{t}\right) \) for any \( t \in \mathbb{R}_+ \) and \( x \in \mathbb{R} \). Then we see that \( \omega_t * h_\epsilon(x) \to h_\epsilon(x) \) uniformly for \( x \in \mathbb{R}_+ \) as \( t \to 0^+ \), which yields the following inequality:

\[
\|\omega_t * h_\epsilon - g_\epsilon\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} \leq \|\omega_t * h_\epsilon - h_\epsilon\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} + \|h_\epsilon - \tilde{h}_\epsilon\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)}
+ \|\tilde{h}_\epsilon - g_\epsilon\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} 
\lesssim \|\omega_t * h_\epsilon - h_\epsilon\|_{L^\infty(\mathbb{R}_+, dm_\lambda)} + \epsilon.
\]

Hence, by letting \( t \to 0^+ \) we get that (3.1) holds.

Now we show (3.2). From the definitions of \( i_\epsilon \) and \( j_\epsilon \), we deduce that for any \( x \in R_{m_\epsilon} \),

\[
\int_{I_x} |f(y) - g_\epsilon(y)| \, y^{2\lambda} \, dy \leq \epsilon m_\lambda(I_x).
\]

In fact,

\[
\int_{I_x} |f(y) - g_\epsilon(y)| \, y^{2\lambda} \, dy = \int_{I_x} |f(y) - \tilde{g}_\epsilon(y)| \, y^{2\lambda} \, dy = \int_{I_x} |f(y) - f_{I_x, \lambda}| \, y^{2\lambda} \, dy.
\]

If \( I_x \cap R_{j_+} = \emptyset \), then by (3.5), (3.16) holds. If \( I_x \cap R_{j_+} \neq \emptyset \), i. e., \( I_x \in \{ I^{\hat{j}_+}_{1}, \cdots, I^{\hat{j}_+}_{2\hat{j}_++2 + 2\lambda(j_++1)} \} \), then \( m_\lambda(I_x) \leq 2^{-i_\epsilon} \). From this fact and (3.3), (3.16) follows.
Let $I$ be an arbitrary interval in $\mathbb{R}_+$. To show (3.2), we only need to prove that

$$M_\lambda(f - g_\epsilon, I) \lesssim \epsilon. \quad (3.17)$$

To this end, we consider the following four cases:

Case i) $I \subset R_{m_\epsilon}$ and $\max\{|I_x| : I_x \cap I \neq \emptyset\} > 4|I|$. In this case, the cardinality of the set $\{I_x : I_x \cap I \neq \emptyset\}$ is at most 2 and hence, $I_{x_i} \cap I_{x_j} \neq \emptyset$ if $I_{x_i} \cap I \neq \emptyset$ and $I_{x_j} \cap I \neq \emptyset$. By (3.15), we have

$$M_\lambda(g_\epsilon, I) \leq \frac{1}{m_\lambda(I)} \sum_{i : I_{x_i} \cap I \neq \emptyset} \int_{I_{x_i} \cap I} \frac{1}{m_\lambda(I)} \sum_{j : I_{x_j} \cap I \neq \emptyset} \int_{I_{x_j} \cap I} |g_\epsilon(x) - g_\epsilon(y)| y^{2\lambda} dy x^{2\lambda} dx \lesssim \epsilon.$$

Moreover, if $I \cap R_{j_\epsilon} \neq \emptyset$, then $m_\lambda(I) \leq m_\lambda(I_{j_\epsilon+1}) \leq 2^{-i_\epsilon}$. By (3.3), we see that $M_\lambda(f, I) < \epsilon$ and so

$$M_\lambda(f - g_\epsilon, I) \leq M_\lambda(f, I) + M_\lambda(g_\epsilon, I) < \epsilon + M_\lambda(g_\epsilon, I) \lesssim \epsilon.$$

If $I \cap R_{j_\epsilon} = \emptyset$, then by (3.5), we also see that $M_\lambda(f, I) < \epsilon$ and $M_\lambda(f - g_\epsilon, I) \lesssim \epsilon$.

Case ii) $I \subset R_{m_\epsilon}$ and $\max\{|I_x| : I_x \cap I \neq \emptyset\} \leq 4|I|$. In this case, from (2.1), it follows that

$$\sum_{i : I_{x_i} \cap I \neq \emptyset} m_\lambda(I_{x_i}) \sim m_\lambda(I).$$

Since $I \subset R_{m_\epsilon}$, then $x \in R_{m_\epsilon}$ if $I_x \cap I \neq \emptyset$. By this and (3.16), we see that

$$M_\lambda(f - g_\epsilon, I) \lesssim \frac{1}{m_\lambda(I)} \sum_{i : I_{x_i} \cap I \neq \emptyset} \int_{I_{x_i}} |f(y) - g_\epsilon(y)| y^{2\lambda} dy \lesssim \frac{1}{m_\lambda(I)} \sum_{i : I_{x_i} \cap I \neq \emptyset} m_\lambda(I_{x_i}) \epsilon \lesssim \epsilon.$$

Thus, (3.17) holds in this case.

Case iii) $I \subset (\mathbb{R}_+ \setminus R_{m_\epsilon-1})$. In this case, $I \cap R_{j_\epsilon} = \emptyset$. By (3.5), we see that $M_\lambda(f, I) < \epsilon$. Similar to Case i), it then suffices to estimate $M_\lambda(g_\epsilon, I)$. However, by (3.15), $M_\lambda(g_\epsilon, I) \lesssim \epsilon$. Thus, (3.17) holds.

Case iv) $I \cap (\mathbb{R}_+ \setminus R_{m_\epsilon}) \neq \emptyset$ and $I \cap R_{m_\epsilon-1} \neq \emptyset$. Let $p_I$ be the smallest integer such that $I \subset R_{p_I}$. Then by (2.1),

$$M_\lambda(f - g_\epsilon, I) \lesssim M_\lambda(f - g_\epsilon, R_{p_I}).$$

Moreover,

$$M_\lambda(f - g_\epsilon, R_{p_I}) m_\lambda(R_{p_I}) \lesssim \int_{R_{p_I}} \left| (f - g_\epsilon)(x) - (f - g_\epsilon)_{R_{p_I} \setminus R_{m_\epsilon}} \right| x^{2\lambda} dx$$

$$\lesssim \int_{R_{p_I}} \left| f(x) - f_{R_{p_I} \setminus R_{m_\epsilon}} \right| x^{2\lambda} dx + \int_{R_{p_I}} \left| g_\epsilon(x) - (g_\epsilon)_{R_{p_I} \setminus R_{m_\epsilon}} \right| x^{2\lambda} dx.$$

On the one hand, observe that $m_\lambda(R_{p_I}) \geq m_\lambda(R_{m_\epsilon}) \geq 2^{k_\epsilon}$. By this, (2.1) and (3.4), we have that

$$\int_{R_{p_I}} \left| f(x) - f_{R_{p_I} \setminus R_{m_\epsilon}} \right| x^{2\lambda} dx$$

$$\leq \int_{R_{p_I}} \left| f(x) - f_{R_{p_I}} \right| x^{2\lambda} dx + \int_{R_{p_I}} \left| f_{R_{p_I}} - f_{R_{p_I} \setminus R_{m_\epsilon}} \right| m_\lambda(R_{p_I})$$
\begin{equation*}
\lesssim \int_{R_{P_1}} |f(x) - f_{R_{P_1}, \lambda}| x^{2\lambda} \, dx \lesssim \epsilon m_{\lambda}(R_{P_1}).
\end{equation*}

On the other hand, it is obvious that
\begin{equation*}
\sum_{i : I_{x_i} \in \mathcal{I}, I_{x_i} \subseteq R_{m_c}} m_{\lambda}(I_{x_i}) = m_{\lambda}(R_{m_c}).
\end{equation*}

From this, the fact that $g_{\epsilon}(x) = g_{\epsilon}(y)$ for any $x, y \in (\mathbb{R}_+ \setminus R_{m_c})$, (3.4) and (3.16), we deduce that
\begin{equation*}
\int_{R_{P_1}} |g_{\epsilon}(x) - (g_{\epsilon})_{R_{P_1} \setminus R_{m_c}, \lambda}| x^{2\lambda} \, dx
\end{equation*}
\begin{equation*}
\leq \frac{1}{m_{\lambda}(R_{P_1} \setminus R_{m_c})} \int_{R_{P_1}} \int_{R_{P_1} \setminus R_{m_c}} |g_{\epsilon}(x) - g_{\epsilon}(y)| y^{2\lambda} dy x^{2\lambda} dx
\end{equation*}
\begin{equation*}
= \frac{1}{m_{\lambda}(R_{P_1} \setminus R_{m_c})} \int_{R_{m_c}} \int_{R_{P_1} \setminus R_{m_c}} |g_{\epsilon}(x) - g_{\epsilon}(y)| y^{2\lambda} dy x^{2\lambda} dx
\end{equation*}
\begin{equation*}
\leq \frac{1}{m_{\lambda}(R_{P_1} \setminus R_{m_c})} \int_{R_{P_1} \setminus R_{m_c}} \sum_{i : I_{x_i} \in \mathcal{I}, I_{x_i} \subseteq R_{m_c}} \int_{I_{x_i}} |g_{\epsilon}(x) - f(x)| x^{2\lambda} dx y^{2\lambda} dy
\end{equation*}
\begin{equation*}
+ \int_{R_{m_c}} \left[ |f(x) - f_{R_{m_c}, \lambda}| + |f_{R_{m_c}, \lambda} - f_{R_{m_c} \setminus R_{m_c-1, \lambda}}| \right] x^{2\lambda} dx
\end{equation*}
\begin{equation*}
\lesssim \frac{1}{m_{\lambda}(R_{P_1} \setminus R_{m_c})} \int_{R_{P_1} \setminus R_{m_c}} \epsilon \sum_{i : I_{x_i} \in \mathcal{I}, I_{x_i} \subseteq R_{m_c}} m_{\lambda}(I_{x_i}) y^{2\lambda} dy + \int_{R_{m_c}} |f(x) - f_{R_{m_c}, \lambda}| x^{2\lambda} dx
\end{equation*}
\begin{equation*}
\lesssim \epsilon m_{\lambda}(R_{m_c})
\end{equation*}
\begin{equation*}
\lesssim \epsilon m_{\lambda}(R_{P_1}).
\end{equation*}

This implies (3.2) and finishes the proof of Theorem 3.1. \hfill \Box

4 The Fréchet-Kolmogorov theorem in the Bessel setting

In this section, we provide a version of Fréchet-Kolmogorov theorem in the Bessel setting, stating a necessary and sufficient condition for a subset of $L^p$ to be relatively compact, which is useful in the proof of Theorem 1.2. For the original Fréchet-Kolmogorov theorem, we refer the readers to Yosida [36]. See also [11, 17].

We first recall that a metric space $(\mathcal{X}, \delta)$ is totally bounded if for every $\epsilon > 0$, there exists a finite number of open balls of radius $\epsilon$ whose union is the space $\mathcal{X}$, and a metric space $(\mathcal{X}, \delta)$ is compact if and only if it is complete and totally bounded; see, for example, [11].

Lemma 4.1. ([17]) Let $(\mathcal{X}, \delta)$ be a metric space. Suppose that for every $\epsilon > 0$, there exists some $\delta > 0$, a metric space $(\mathcal{W}, \tilde{d})$ and a mapping $\Phi : \mathcal{X} \rightarrow \mathcal{W}$ such that $\Phi(\mathcal{X})$ is totally bounded, and whenever $x, y \in \mathcal{X}$ are such that $\tilde{d}(\Phi(x), \Phi(y)) < \delta$, then $d(x, y) < \epsilon$. Then $X$ is totally bounded.

The main result of this section is as follows.
Theorem 4.2. For $1 < p < \infty$, a subset $F$ of $L^p(\mathbb{R}_+, dm_\lambda)$ is totally bounded (or relatively compact) if and only if the following statements hold:

(a) $F$ is uniformly bounded, i.e., $\sup_{f \in F} \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)} < \infty$;

(b) $F$ uniformly vanishes at infinity, i.e., for every $\epsilon > 0$, there exists some positive constant $M$ such that for every $f \in F$,

$$
\int_M^\infty |f(x)|^p x^{2\lambda} \, dx < \epsilon^p;
$$

(c) $F$ is uniformly equicontinuous, i.e., for every $\epsilon > 0$, there exists some positive constant $\rho$, such that for every $f \in F$ and $y \in \mathbb{R}_+$ with $y < \rho$,

$$
\int_0^\infty |f(x + y) - f(x)|^p x^{2\lambda} \, dx < \epsilon^p.
$$

Proof. Assume that $F \subset L^p(\mathbb{R}_+, dm_\lambda)$ satisfies the three conditions. By Lemma 4.1, to show $F$ is totally bounded, it suffices to prove that for any $\epsilon > 0$, there exists a mapping $\Phi$ on $L^p(\mathbb{R}_+, dm_\lambda)$ such that $\Phi(F)$ is totally bounded and that

$$
\|f - g\|_{L^p(\mathbb{R}_+, dm_\lambda)} < \epsilon \quad (4.1)
$$

for any $f, g \in F$ such that

$$
\|\Phi(f) - \Phi(g)\|_{L^p(\mathbb{R}_+, dm_\lambda)} < \epsilon/2. \quad (4.2)
$$

To this end, given $\epsilon > 0$, pick $M$ as in the condition (b), such that

$$
\sup_{f \in F} \|f - f\chi_{(0, M)}\|_{L^p(\mathbb{R}_+, dm_\lambda)} < \epsilon/12. \quad (4.3)
$$

Let $\rho$ be as in condition (c) such that

$$
\sup_{y \in (0, \rho)} \left( \sup_{f \in F} \|f(\cdot + y) - f(\cdot)\|_{L^p(\mathbb{R}_+, dm_\lambda)} \right) < \frac{\epsilon}{\left(2(2\lambda + 1)\right)^{1/p}12}. \quad (4.4)
$$

Let $N := [M/\rho] + 1$, $\tilde{I}_1 := (0, \rho]$ and $\tilde{I}_j := \tilde{I}_1 + (j - 1)\rho$, $j = 2, \ldots, N$. Then $\{\tilde{I}_j\}_{j=1}^N$ are mutually non-overlapping intervals and

$$(0, M) \subset \bigcup_{j=1}^N \tilde{I}_j.
$$

Now define the mapping $\Phi$ by setting for any $f \in F$ and $x \in \mathbb{R}_+$,

$$
\Phi(f)(x) := f_{\tilde{I}_1, \lambda} x_{\tilde{I}_1}(x) + \sum_{j=2}^N \frac{1}{|\tilde{I}_j|} \int_{\tilde{I}_j} f(z) \, dz \chi_{\tilde{I}_j}(x).
$$

We first see that for $f \in F$, $\Phi(f)$ is well defined. In fact, if $x \in \tilde{I}_1$, then it follows from the Hölder inequality that

$$
|\Phi(f)(x)| = \frac{1}{m_\lambda(\tilde{I}_1)} \int_{\tilde{I}_1} |f(y)| y^{2\lambda} \, dy \leq \frac{1}{[m_\lambda(\tilde{I}_1)]^{1/p}} \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)} < \infty; \quad (4.5)
$$
while if \( x \in \tilde{I}_j \), \( j = 2, \ldots, N \), by another application of the Hölder inequality, we also have

\[
|\Phi(f)(x)| = \frac{1}{|I_j|} \int_{I_j} f(z) \, dz \leq \frac{1}{|I_j|} \|f\|_{L^p(\mathbb{R}^+, dm_\lambda)} \left( \int_{I_j} z^{-\frac{2\lambda}{p}'} \, dz \right)^{1/p'} < \infty. \tag{4.6}
\]

Let \( \mathcal{B}_N \) be the linear space spanned by \( \{\chi_{\tilde{I}_j}\}_{j=1}^N \). Then \( \mathcal{B}_N \) is a finite dimensional Banach space endowed with the norm \( \| \cdot \|_{L^p(\mathbb{R}^+, dm_\lambda)} \) for \( p \in (1, \infty) \). Observe that \( \Phi(\mathcal{F}) \) is a subset of \( \mathcal{B}_N \). Moreover, by (4.5) and (4.6), for any \( N \),

\[
\|\Phi(f)\|_{L^p(\mathbb{R}^+, dm_\lambda)} \leq \sum_{j=2}^N \int_{I_j} \left\| f \chi_{\tilde{I}_j} - \Phi(f)(x) \right\|_{L^p(\mathbb{R}^+, dm_\lambda)} \leq \epsilon/12 + \|f \chi_{(0, M)} - \Phi(f)\|_{L^p(\mathbb{R}^+, dm_\lambda)}
\]

\[
\leq \epsilon/12 + \left( \sum_{j=1}^N \int_{I_j} |f(x) - \Phi(f)(x)|^p x^{2\lambda} \, dx \right)^{1/p}.
\]

By the Hölder inequality, a change of variable and (4.4), we see that

\[
\sum_{j=2}^N \int_{I_j} |f(x) - \Phi(f)(x)|^p x^{2\lambda} \, dx
\]

\[
\leq \sum_{j=2}^N \frac{1}{|I_j|} \left( \int_{I_j} \int_{I_j} |f(x) - f(z)|^p \, dz \, x^{2\lambda} \, dx \right)
\]

\[
= \sum_{j=2}^N \frac{1}{|I|} \int_{I_j} \left[ \int_{I_j: z < x} + \int_{I_j: z \geq x} \right] |f(x) - f(z)|^p \, dz \, x^{2\lambda} \, dx
\]

\[
\leq \sum_{j=2}^N \frac{1}{|I|} \int_{I_j} \int_{I_j} \left[ |f(x) - f(x - y)|^p + |f(x) - f(x + y)|^p \right] \, dy \, x^{2\lambda} \, dx
\]

\[
= \frac{1}{|I|} \int_{I} \sum_{j=2}^N \int_{I_j} \left[ |f(x) - f(x - y)|^p + |f(x) - f(x + y)|^p \right] \, dy \, x^{2\lambda} \, dx
\]

\[
\leq \frac{2}{|I|} \int_{I} \int_{0}^\infty |f(x) - f(x + y)|^p x^{2\lambda} \, dy \, dx
\]

< (\epsilon/12)^p.

On the other hand, by the Hölder inequality, we see that

\[
\int_{I_1} |f(x) - \Phi(f)(x)|^p x^{2\lambda} \, dx = \int_{I_1} \left| f(x) - f_{\tilde{I}_1, \lambda} \right|^p x^{2\lambda} \, dx
\]
Thus (b) holds.

Every $g,$ there exists $\rho > 0$ such that

$$\| f - f(x) \|_{L^p(R_+, dm_\lambda)} < \epsilon/4.$$ (4.7)

By (4.7) and the linearity of $\Phi,$ we further deduce that for any $f, g \in \mathcal{F}$ satisfying (4.2),

$$\| f - g \|_{L^p(R_+, dm_\lambda)} \leq \| f - \Phi(f) \|_{L^p(R_+, dm_\lambda)} + \| \Phi(f) - \Phi(g) \|_{L^p(R_+, dm_\lambda)} + \| \Phi(g) - g \|_{L^p(R_+, dm_\lambda)} < \epsilon.$$

Thus (4.1) holds and $\mathcal{F}$ is totally bounded by Lemma 4.1.

For the converse, assume that $\mathcal{F}$ is totally bounded. For every $\epsilon > 0,$ the existence of a finite $\epsilon-$cover of $\mathcal{F}$ implies the boundedness of $\mathcal{F},$ thus the condition (a) holds.

To show (b) holds, given $\epsilon > 0,$ let $\{U_1, \ldots, U_m\}$ be an $\epsilon-$cover of $\mathcal{F},$ and choose $g_j \in U_j$ for $j = 1, \ldots, m.$ Let $M > 0$ such that

$$\int_M^\infty |g_j(x)|^p x^{2\lambda} dx < \epsilon^p, \quad j = 1, \ldots, m.$$ 

If $f \in U_j,$ then $\| f - g_j \|_{L^p(R_+, dm_\lambda)} < \epsilon,$ and so

$$\left( \int_M^\infty |f(x)|^p x^{2\lambda} dx \right)^{1/p} \leq \left( \int_M^\infty |f(x) - g_j(x)|^p x^{2\lambda} dx \right)^{1/p} + \left( \int_M^\infty |g_j(x)|^p x^{2\lambda} dx \right)^{1/p} < 2\epsilon.$$

Thus (b) holds.

For condition (c), given $\epsilon > 0,$ we pick an $\epsilon-$cover $\{U_1, \ldots, U_m\}$ of $\mathcal{F}.$ Since $\mathcal{D}$ is dense in $L^p(R_+, dm_\lambda),$ there exists $g_j \in U_j \cap \mathcal{D},$ for each $j = 1, \ldots, m.$ It is not hard to see that, for every $g \in \mathcal{D},$

$$\lim_{y \to 0^+} \int_0^\infty |g(x + y) - g(x)|^p x^{2\lambda} dx = 0.$$ 

Then there exists $\rho > 0$ such that

$$\int_0^\infty |g_j(x + y) - g_j(x)|^p x^{2\lambda} dx < \epsilon^p, \quad y \in (0, \rho), \quad j = 1, \ldots, m.$$
Moreover, for any \( f \in \mathcal{F} \), we see that \( f \in U_j \) for certain \( j = 1, \ldots, m \) and hence,

\[
\left( \int_0^\infty |f(x + y) - f(x)|^p x^{2\lambda} \, dx \right)^{1/p} \\
\leq \left( \int_0^\infty |f(x + y) - g_j(x + y)|^p x^{2\lambda} \, dx \right)^{1/p} + \left( \int_0^\infty |g_j(x + y) - g_j(x)|^p x^{2\lambda} \, dx \right)^{1/p} \\
+ \left( \int_0^\infty |g_j(x) - f(x)|^p x^{2\lambda} \, dx \right)^{1/p} \\
\leq \left( \int_0^\infty |f(x + y) - g_j(x + y)|^p (x + y)^{2\lambda} \, dx \right)^{1/p} + \left( \int_0^\infty |g_j(x + y) - g_j(x)|^p x^{2\lambda} \, dx \right)^{1/p} \\
+ \left( \int_0^\infty |g_j(x) - f(x)|^p x^{2\lambda} \, dx \right)^{1/p} \\
\leq \left( \int_0^\infty |g_j(x + y) - g_j(x)|^p x^{2\lambda} \, dx \right)^{1/p} + 2 \left( \int_0^\infty |g_j(x) - f(x)|^p x^{2\lambda} \, dx \right)^{1/p} \\
< 5\varepsilon.
\]

This finishes the proof of Theorem 4.2. \( \square \)

## 5 The proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. To begin with, we first recall the following boundedness of \([b, R_{\Delta\lambda}]\) established in [15].

**Lemma 5.1.** ([15]) Let \( b \in \bigcup_{q>1} L^q_{\text{loc}}(\mathbb{R}_+, dm_\lambda) \) and \( p \in (1, \infty) \). Then \( b \in \text{BMO}(\mathbb{R}_+, dm_\lambda) \) if and only if \([b, R_{\Delta\lambda}]\) is bounded on \( L^p(\mathbb{R}_+, dm_\lambda) \). Moreover, there exists a positive constant \( C \in (1, \infty) \) such that

\[
C^{-1} \|b\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} \leq \|[b, R_{\Delta\lambda}]\|_{L^p(\mathbb{R}_+, dm_\lambda) \to L^p(\mathbb{R}_+, dm_\lambda)} \leq C \|b\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)}.
\]

Before giving the proof of Theorem 1.2, we first obtain a lemma for the upper and lower bounds of integrals of \([b, R_{\Delta\lambda}]f\) on certain intervals. To this end, we recall the median value in [30, 19], see also [22, 31, 23, 21]. For \( f \in L^1_{\text{loc}}(\mathbb{R}_+, dm_\lambda) \) and \( I \subseteq \mathbb{R}_+ \), let \( \alpha_I(f) \) be a real number such that

\[
\inf_c \frac{1}{m_\lambda(I)} \int_I |f(x) - c| \, dm_\lambda(x)
\]

is attained. Note that \( \frac{1}{m_\lambda(I)} \int_I |f - c| \, dm_\lambda \) is uniformly continuous in \( c \), so such \( \alpha_I(f) \) exists and may not be unique. Moreover, as in [23, p. 30] where the setting of \((\mathbb{R}_+, |\cdot|, dx)\) was considered, \( \alpha_I(f) \) satisfies that

\[
m_\lambda(\{x \in I : f(x) > \alpha_I(f)\}) \leq m_\lambda(I)/2 \tag{5.1}
\]

and

\[
m_\lambda(\{x \in I : f(x) < \alpha_I(f)\}) \leq m_\lambda(I)/2. \tag{5.2}
\]

In fact, if \( \alpha_I(f) \) does not satisfy (5.1), then

\[
m_\lambda(\{x \in I : f(x) > \alpha_I(f)\}) > m_\lambda(I)/2.
\]
Take $\varepsilon > 0$ small enough such that
\[ m_\lambda(\{x \in I : f(x) > \alpha_I(f) + \varepsilon\}) > m_\lambda(I)/2. \]
We define $I_1 := \{x \in I : f(x) > \alpha_I(f) + \varepsilon\}$ and $I_2 := I \setminus I_1$. Then
\[
\begin{align*}
\int_I |f(x) - \alpha_I(f)| \, dm_\lambda(x) &= \int_{I_1} |f(x) - \alpha_I(f)| \, dm_\lambda(x) + \int_{I_2} |f(x) - \alpha_I(f)| \, dm_\lambda(x) \\
&= \int_{I_1} |f(x) - \alpha_I(f)| \, dm_\lambda(x) + \int_{I_2} |f(x) - \alpha_I(f) - \varepsilon| \, dm_\lambda(x) \\
&\quad - \int_{I_1} (f(x) - \alpha_I(f) - \varepsilon) \, dm_\lambda(x) - \int_{I_2} (\alpha_I(f) + \varepsilon - f(x)) \, dm_\lambda(x) \\
&\geq \varepsilon (m_\lambda(I_1) - m_\lambda(I_2)) > 0.
\end{align*}
\]
This violates the choice of $\alpha_I(f)$. The proof of (5.2) is similar and omitted.

Moreover, by the choice of $\alpha_I(f)$ and Definition 1.1, it is easy to see that for any interval $I \subset \mathbb{R}_+$,
\[
M_\lambda(f, I) \sim \frac{1}{m_\lambda(I)} \int_I |f(y) - \alpha_I(f)| \, y^{2\lambda} \, dy. \tag{5.3}
\]

**Lemma 5.2.** Assume that $b \in BMO(\mathbb{R}_+, dm_\lambda)$ with $\|b\|_{BMO(\mathbb{R}_+, dm_\lambda)} = 1$ and there exist $\delta \in (0, \infty)$ and a sequence $\{I_j\}_{j=1}^\infty := \{I(x_j, r_j)\}_j$ of intervals such that for each $j$,
\[
M_\lambda(b, I_j) > \delta. \tag{5.4}
\]
Then there exist functions $\{f_j\}_j \subset L^p(\mathbb{R}_+, dm_\lambda)$, positive constants $A_1 > 4$, $\widetilde{C}_0$, $\bar{C}_1$ and $\bar{C}_2$ such that for any integers $j$ and $k \geq \lfloor \log_2 A_1 \rfloor$, $\|f_j\|_{L^p(\mathbb{R}_+, dm_\lambda)} \leq \bar{C}_0$.
\[
\int_{I_j^k} |[b, R_{\Delta_k}] f_j(y)|^p \, y^{2\lambda} \, dy \geq \bar{C}_1 \delta^p \frac{[m_\lambda(I_j)]^{p-1}}{[m_\lambda(2kI_j)]^{p-1}}, \tag{5.5}
\]
where $I_j^k := (x_j + 2^k r_j, x_j + 2^{k+1} r_j)$; and
\[
\int_{2^{k+1}I_j \setminus 2^kI_j} |[b, R_{\Delta_k}] f_j(y)|^p \, y^{2\lambda} \, dy \leq \bar{C}_2 \frac{[m_\lambda(I_j)]^{p-1}}{[m_\lambda(2^kI_j)]^{p-1}}. \tag{5.6}
\]

**Proof.** For each $j$, define the function $f_j$ as follows:
\[
f_j^1 := \chi_{I_{j,1}} - \chi_{I_{j,2}} := \chi_{\{x \in I_j : b(x) > \alpha_{I_j}(b)\}} - \chi_{\{x \in I_j : b(x) < \alpha_{I_j}(b)\}},
\]
\[
f_j^2 := a_j \chi_{I_j} \quad \text{and} \quad f_j := [m_\lambda(I_j)]^{-1/p} \left( f_j^1 - f_j^2 \right),
\]
where $a_j$ is a constant such that
\[
\int_0^\infty f_j(x) x^{2\lambda} \, dx = 0. \tag{5.7}
\]
Then by the definition of $a_j$, (5.1) and (5.2), we see that $|a_j| \leq 1/2$. Moreover, we also have that $\text{supp}(f_j) \subset I_j$, and that for any $y \in I_j$, $$f_j(y) [b(y) - \alpha_I(b)] \geq 0. \quad (5.8)$$

On the other hand, since $|a_j| \leq 1/2$, we see that for any $y \in \langle I_{j,1} \cup I_{j,2} \rangle$, $$|f_j(y)| \sim [m_\lambda (I_j)]^{-1/p} . \quad (5.9)$$

Moreover, since $\text{supp}(f_j) \subset I_j$, we have that $\|f_j\|_{L^p(\mathbb{R}^+, dm_\lambda)} \leq 1$. Observe that $$[b, R_\lambda]f = R_\lambda \left( [b - \alpha_I(b)]f \right) - [b - \alpha_I(b)] R_\lambda(f). \quad (5.10)$$

Let $A_1 > 4$ large enough. Then for any integer $k \geq [\log_2 A_1]$, $$2^{k+1}I_j \subset 8I_j = \left(x_j - \frac{5}{2} \cdot 2^k r_j, x_j + \frac{11}{2} \cdot 2^k r_j\right) \cap \mathbb{R}^+ \subset 2^{k+3}I_j, \quad (5.11)$$

and by (2.1), $$m_\lambda(I_j^k) \sim m_\lambda(2^k I_j). \quad (5.12)$$

We first prove the inequality (5.5). By the fact that $|y - x_j| > 2|z - x_j|$ for any $y \in \mathbb{R}^+ \setminus 2I_j$ and $z \in I_j$, (5.9), (5.7), (2.3), we see that, $$\left| [b(y) - \alpha_I(b)] R_\lambda(f_j)(y) \right| = |b(y) - \alpha_I(b)| \left| \int_{I_j} [R_\lambda(y, z) - R_\lambda(y, x_j)] f_j(z)z^{2\lambda} \, dz \right|$$

$$\leq |b(y) - \alpha_I(b)| \int_{I_j} |R_\lambda(y, z) - R_\lambda(y, x_j)||f_j(z)||z^{2\lambda} \, dz$$

$$\lesssim r_j \left[ m_\lambda(I_j) \right]^{-1/p} \frac{|b(y) - \alpha_I(b)|}{|x_j - y| m_\lambda(I(y, |y - x_j|))}. \quad (5.13)$$

Moreover, by the well known John-Nirenberg inequality ([13, p. 594]) and (2.1), we conclude that for each $k \in \mathbb{N}$ and $I \subset \mathbb{R}^+$, $$\int_{2^{k+1}I} |b(y) - \alpha_I(b)|^p y^{2\lambda} \, dy$$

$$\lesssim \int_{2^{k+1}I} |b(y) - \alpha_{2^{k+1}I}(b)|^p y^{2\lambda} \, dy + m_\lambda \left(2^{k+1}I \right) |\alpha_{2^{k+1}I}(b) - \alpha_I(b)|^p$$

$$\lesssim k^p m_\lambda \left(2^k I \right). \quad (5.14)$$

By this fact, the fact that for any $x$ and $y$,

$$m_\lambda(I(x, |x - y|)) \sim m_\lambda(I(y, |y - x|)), \quad (5.15)$$

(5.13), (5.11) and (2.1), we see that there exists a positive constant $C_4$, such that for any $k \in \mathbb{N}$,

$$\int_{I_j} \left| [b(y) - \alpha_I(b)] R_\lambda(f_j)(y) \right|^p y^{2\lambda} \, dy \lesssim \int_{I_j} \left[ m_\lambda(I_j) \right]^{p-1} |b(y) - \alpha_I(b)|^p y^{2\lambda} \, dy.$$
\[
\begin{align*}
\lesssim & \frac{1}{2^{kp}} \frac{[m_{\lambda}(I_j)]^{p-1}}{[m_{\lambda}(2^{k}I_j)]^{p}} \int_{2^{k+1}I_j} |b(y) - \alpha I_j(b)|^p y^{2\lambda} dy \\
\lesssim & \frac{k^p}{2^{kp}} \frac{[m_{\lambda}(I_j)]^{p-1}}{[m_{\lambda}(2^{k}I_j)]^{p}} m_{\lambda}(2^{k+1}I_j) \\
\leq & C_4 \frac{k^p}{2^{kp}} \frac{[m_{\lambda}(I_j)]^{p-1}}{[m_{\lambda}(2^{k}I_j)]^{p-1}}. \quad (5.16)
\end{align*}
\]

Next, observe that \(y > z\) for any \(y \in I_j^k\) and \(z \in I_j\). By Proposition 2.5, (5.8), (5.9), (5.3) and (5.4), we have that
\[
|R_{\Delta\lambda} \left[ (b - \alpha I_j(b))f_j \right](y) | = \int_{(I_j \cup I_j, 2)} |R_{\Delta\lambda}(y, z)| |[b(z) - \alpha I_j(b)] f_j(z)| z^{2\lambda} dz \\
\geq & \left[ m_{\lambda}(I_j) \right]^{-1/p} \int_{I_j} \frac{|b(z) - \alpha I_j(b)| z^{2\lambda}}{m_{\lambda}(I(y, |y - z|))} dz \\
\geq & \delta \left[ m_{\lambda}(I_j) \right]^{1/p'} \frac{1}{m_{\lambda}(I(y, |y - x_j|))}.
\]

From this, (5.15) and (5.12), we deduce that there exists a positive constant \(C_5\) such that
\[
\int_{I_j^k} |R_{\Delta\lambda} \left[ (b - \alpha I_j(b))f_j \right](y) |^p y^{2\lambda} dy \geq \frac{\delta^p}{2} \left[ m_{\lambda}(I_j) \right]^{p-1} \int_{I_j^k} \frac{1}{m_{\lambda}(I(y, |y - x_j|))^{p'}} y^{2\lambda} dy \\
\geq & \delta^p \left[ \frac{m_{\lambda}(I_j)}{[m_{\lambda}(2^{k}I_j)]^{p-1}} \right] m_{\lambda}(I_j) \\
\geq & \delta^p C_5 \left[ \frac{m_{\lambda}(I_j)}{[m_{\lambda}(2^{k}I_j)]^{p-1}} \right]. \quad (5.17)
\]

Take \(A_1\) large enough such that for any integer \(k \geq \lceil \log_2 A_1 \rceil\),
\[
C_5 \frac{\delta^p}{2^{p-1}} - C_4 \frac{k^p}{2^{kp}} \geq C_5 \frac{\delta^p}{2^p}.
\]

By (5.10), (5.17) and (5.16), we conclude that for any integer \(k \geq \lceil \log_2 A_1 \rceil\),
\[
\int_{I_j^k} |[b, R_{\Delta\lambda}]f_j(y) |^p y^{2\lambda} dy \\
\geq \left[ \frac{1}{2^{p-1}} \int_{I_j^k} |R_{\Delta\lambda} \left[ (b - \alpha I_j(b))f_j \right](y) |^p y^{2\lambda} dy \\
- \int_{I_j^k} |[b(y) - \alpha I_j(b)] R_{\Delta\lambda}(f_j)(y) |^p y^{2\lambda} dy \right] \\
\geq \left( C_5 \frac{\delta^p}{2^{p-1}} - C_4 \frac{k^p}{2^{kp}} \right) \left[ m_{\lambda}(I_j) \right]^{p-1} \geq C_5 \frac{\delta^p}{2p} \left[ m_{\lambda}(I_j) \right]^{p-1}.
\]

This shows the inequality (5.5).

Now we show the inequality (5.6). From \(\text{supp}(f_j) \subset I_j\), (2.2), (5.3) and (5.9), we deduce that for any \(y \in \mathbb{R}_+ \setminus 2I_j\),
\[
|R_{\Delta\lambda} \left[ (b - \alpha I_j(b))f_j \right](y) | \lesssim \left[ m_{\lambda}(I_j) \right]^{-1/p} \int_{I_j} \frac{|b(z) - \alpha I_j(b)| z^{2\lambda}}{m_{\lambda}(I(y, |y - z|))} dz
\]
Proof of Theorem 1.2. Sufficiency:

We use the idea in [33]. We first show that if \([b, R_{\Delta \lambda}]\) is a compact operator on \(L^p(\mathbb{R}_+, dm_\lambda)\), then \(b \in \text{CMO}(\mathbb{R}_+, dm_\lambda)\). Since \([b, R_{\Delta \lambda}]\) is compact on \(L^p(\mathbb{R}_+, dm_\lambda)\), \([b, R_{\Delta \lambda}]\) is bounded on \(L^p(\mathbb{R}_+, dm_\lambda)\). By Lemma 5.1, we see that \(b \in \text{BMO}(\mathbb{R}_+, dm_\lambda)\). Without loss of generality, we may assume that \(\|b\|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} = 1\). To show \(b \in \text{CMO}(\mathbb{R}_+, dm_\lambda)\), we use a contradiction argument via Theorem 3.1. Observe that if \(b \notin \text{CMO}(\mathbb{R}_+, dm_\lambda)\), \(b\) does not satisfy at least one of (i)-(iii) in Theorem 3.1.

We now consider the following three cases.

Case i), \(b\) does not satisfy (i) in Theorem 3.1. Then there exists \(\delta \in (0, \infty)\) and a sequence \(\{I_j\}_{j=1}^\infty\) of intervals satisfying (5.4) and that \(m_\lambda(I_j) \to 0\) as \(j \to \infty\). Let \(f_j, \tilde{C}_1, \tilde{C}_2, A_1\) be as in Lemma 5.2 and \(A_2 > A_1\) large enough such that

\[
A_3 := 8^{(1-p)(2\lambda+1)}\tilde{C}_1\delta^p A_1^{(1-p)(2\lambda+1)} > \frac{2\tilde{C}_2}{1 - \min(2^{2\lambda}, 2)^{1-p} \min(2^{2\lambda}, 2)^{\lfloor \log_2 A_2 \rfloor (p-1)}}.
\]

Since \(m_\lambda(I_j) \to 0\) as \(j \to \infty\), we may choose a subsequence \(\{I_j^{(1)}\}\) of \(\{I_j\}\) such that

\[
\frac{m_\lambda(I_j^{(1)})}{m_\lambda(I_j^{(1)})} < \frac{1}{(2 A_2)^{2\lambda+1}}.
\]  

(5.18)

For fixed \(\ell, m \in \mathbb{N}\), denote

\[
\mathcal{J} := \left( s_{j_\ell}^{(1)} + A_1 s_{j_\ell}^{(1)}, s_{j_\ell}^{(1)} + A_2 s_{j_\ell}^{(1)} \right),
\]
\[ \mathcal{J}_1 := \mathcal{J} \setminus \left\{ y \in \mathbb{R}_+ : \left| y - x_{j\ell+m}^{(1)} \right| \leq A_2 r_{j\ell+m}^{(1)} \right\} \]
and
\[ \mathcal{J}_2 := \left\{ y \in \mathbb{R}_+ : \left| y - x_{j\ell+m}^{(1)} \right| > A_2 r_{j\ell+m}^{(1)} \right\}. \]
Note that
\[ \mathcal{J}_1 \subset \left\{ y \in \mathbb{R}_+ : \left| y - x_{j\ell}^{(1)} \right| \leq A_2 r_{j\ell}^{(1)} \right\} \cap \mathcal{J}_2 \quad \text{and} \quad \mathcal{J}_1 = \mathcal{J} \setminus (\mathcal{J}_1 \cup \mathcal{J}_2). \]

We then have
\[
\begin{align*}
\| [b, R_{\Delta \lambda}] (f_{j\ell}) - [b, R_{\Delta \lambda}] (f_{j\ell+m}) \|_{L^p(\mathbb{R}_+, dm)} & \geq \left( \int_{\mathcal{J}_1} \left| [b, R_{\Delta \lambda}] (f_{j\ell})(y) - [b, R_{\Delta \lambda}] (f_{j\ell+m})(y) \right|^p y^{2\lambda} dy \right)^{1/p} \\
& \geq \left( \int_{\mathcal{J}_1} \left| [b, R_{\Delta \lambda}] (f_{j\ell})(y) \right|^p y^{2\lambda} dy \right)^{1/p} - \left( \int_{\mathcal{J}_2} \left| [b, R_{\Delta \lambda}] (f_{j\ell+m})(y) \right|^p y^{2\lambda} dy \right)^{1/p} \\
& = \left( \int_{\mathcal{J}_1 \setminus (\mathcal{J}_1 \cup \mathcal{J}_2)} \left| [b, R_{\Delta \lambda}] (f_{j\ell})(y) \right|^p y^{2\lambda} dy \right)^{1/p} - \left( \int_{\mathcal{J}_2} \left| [b, R_{\Delta \lambda}] (f_{j\ell+m})(y) \right|^p y^{2\lambda} dy \right)^{1/p} \\
& =: F_1 - F_2. \quad (5.19)
\end{align*}
\]

We first consider the term $F_1$. To begin with, we now estimate the measure of $\mathcal{J}_1 \setminus \mathcal{J}_2$. Assume that $E_{j\ell} := \mathcal{J}_1 \setminus \mathcal{J}_2 \neq \emptyset$. Then $E_{j\ell} \subset A_2 I_{j\ell+m}^{(1)}$. Hence, we have
\[
m_\lambda (E_{j\ell}) \leq m_\lambda \left( A_2 I_{j\ell+m}^{(1)} \right) \leq (2A_2)^{2\lambda+1} m_\lambda \left( I_{j\ell+m}^{(1)} \right) < m_\lambda \left( I_{j\ell}^{(1)} \right), \quad (5.20)
\]
where the second inequality follows from the doubling condition (2.1), and the last inequality follows from (5.18).

Now let
\[ I_{j\ell}^k := \left( x_{j\ell}^{(1)} + 2^k r_{j\ell}^{(1)}, x_{j\ell}^{(1)} + 2^{k+1} r_{j\ell}^{(1)} \right), \quad \text{with} \ k \geq 1. \]

Then by (5.12) and (2.1),
\[ m_\lambda \left( I_{j\ell}^k \right) \geq \left[ \min \left( 2^{2\lambda}, 2 \right) \right]^k m_\lambda \left( I_{j\ell}^{(1)} \right), \]
which, together with (5.20), implies that
\[ m_\lambda \left( I_{j\ell}^k \right) \geq m_\lambda (E_{j\ell}). \]

From this fact, it follows that there exist at most two intervals, $I_{j\ell}^{k_0}$ and $I_{j\ell}^{k_0+1}$, such that $E_{j\ell} \subset (I_{j\ell}^{k_0} \cup I_{j\ell}^{k_0+1})$. By (5.5) and (2.1),
\[
\begin{align*}
F_1^p & \geq \sum_{k=\lfloor \log_2 A_1 \rfloor + 1, k \neq k_0, k_0 + 1}^{\lfloor \log_2 A_2 \rfloor} \int_{I_{j\ell}^k} \| [b, R_{\Delta \lambda}] (f_{j\ell})(y) \|^p y^{2\lambda} dy \\
& \geq \tilde{C}_1 \delta_p \sum_{k=\lfloor \log_2 A_1 \rfloor + 1, k \neq k_0, k_0 + 1}^{\lfloor \log_2 A_2 \rfloor} \frac{m_\lambda (I_{j\ell}^{(1)}))^{p-1}}{[m_\lambda (2^k I_{j\ell}^{(1)}))^{p-1}}
\end{align*}
\]
Compactness of Riesz transform commutator associated with Bessel operators

By these two inequalities and (5.19), we get
\[ \geq C_1 \delta^p \sum_{k=\lfloor \log_2 A_2 \rfloor + 3}^{\lfloor \log_2 A_2 \rfloor} \frac{1}{2k(p-1)(2\lambda+1)} \geq C_1 \delta^p A_2 \lambda = A_3. \]

If \( E_{j_\ell} := J \setminus J_2 = \emptyset \), the inequality above still holds.

On the other hand, from (5.6) and (2.1), we deduce that
\[ F_2^p \leq \sum_{k=\lfloor \log_2 A_2 \rfloor}^{\infty} \int_{2k+1}^{2k+2} \left| [b, R_{\Delta \lambda}] (f_{j_{\ell+m}}) (y) \right|^p y^{2\lambda} dy \]
\[ \leq C_2 \sum_{k=\lfloor \log_2 A_2 \rfloor}^{\infty} \frac{\left[ m_\lambda (f_{j_{\ell+m}}) \right]^{p-1}}{\left[ m_\lambda (2k f_{j_{\ell+m}}) \right]^{p-1}} \]
\[ \leq C_2 \sum_{k=\lfloor \log_2 A_2 \rfloor}^{\infty} \frac{1}{\min(2^{2\lambda}, 2) |k(p-1)|} \leq \frac{C_2}{1 - \min(2^{2\lambda}, 2)^{1-p} \min(2^{2\lambda}, 2)^{1-\log_2 A_2 |p-1|}} < A_3/2. \]

By these two inequalities and (5.19), we get
\[ \| [b, R_{\Delta \lambda}] (f_{j_\ell}) - [b, R_{\Delta \lambda}] (f_{j_{\ell+m}}) \|_{L^p(\mathbb{R}_+, dm_\lambda)} \gtrsim (A_3)^{1/p}. \]

Thus, \( \{[b, R_{\Delta \lambda}] f_j\}_j \) is not relatively compact in \( L^p(\mathbb{R}_+, dm_\lambda) \), which implies that \( [b, R_{\Delta \lambda}] \) is not compact on \( L^p(\mathbb{R}_+, dm_\lambda) \). Therefore, \( b \) satisfies condition (i).

Case ii), \( b \) violates (ii) in Theorem 3.1. In this case, we also have that there exist \( \delta \in (0, \infty) \) and a sequence \( \{I_j\} \) of intervals satisfying (5.4) and that \( m_\lambda (I_j) \to \infty \) as \( j \to \infty \). We take a subsequence \( \{I_{j_\ell}^{(2)}\} \) of \( \{I_j\} \) such that
\[ \frac{m_\lambda (I_{j_\ell})}{m_\lambda (I_{j_{\ell+1}})} < \frac{1}{(2A_2)^{2\lambda+1}}. \]  

We can use a similar method as in the previous case and redefine our sets in a reversed order. That is, for fixed \( \ell \) and \( m \), let
\[ \tilde{J} := \left( x_{j_{\ell+m}}^{(2)} + A_1 r_{j_{\ell+m}}^{(2)} , x_{j_{\ell+m}}^{(2)} + A_2 r_{j_{\ell+m}}^{(2)} \right), \]
\[ \tilde{J}_1 := \tilde{J} \setminus \left\{ y \in \mathbb{R}_+ : \left| y - x_{j_\ell}^{(2)} \right| \leq A_2 r_{j_\ell}^{(2)} \right\} \]
and
\[ \tilde{J}_2 := \left\{ y \in \mathbb{R}_+ : \left| y - x_{j_\ell}^{(2)} \right| > A_2 r_{j_\ell}^{(2)} \right\}. \]

Then we have that
\[ \tilde{J}_1 \subset \left\{ y \in \mathbb{R}_+ : \left| y - x_{j_{\ell+m}}^{(2)} \right| \leq A_2 r_{j_{\ell+m}}^{(2)} \right\} \cap \tilde{J}_2 \text{ and } \tilde{J}_1 = \tilde{J} \setminus \left( \tilde{J} \setminus \tilde{J}_2 \right). \]

As in Case i), by Lemma 5.2 and (5.21), we see that \( [b, R_{\Delta \lambda}] \) is not compact on \( L^p(\mathbb{R}_+, dm_\lambda) \). This contradiction implies that \( b \) satisfies (ii) of Theorem 3.1.
Case iii), condition (iii) in Theorem 3.1 does not hold for $b$. Then there exists $\delta > 0$ such that for any $R > 0$, there exists $I \subset [R, \infty)$ with $M_\lambda(b, I) > \delta$. We claim that for the $\delta$ above, there exists a sequence $\{I_j^{(3)}\}_j$ of intervals such that for any $j$,

$$M_\lambda(b, I_j^{(3)}) > \delta,$$

and that for any $\ell \neq m$,

$$A_2 I_\ell^{(3)} \cap A_2 I_m^{(3)} = \emptyset.$$  

(5.23)

In fact, let $C_\delta > 0$ to be determined later. Then for $R_1 > C_\delta$, there exists an interval $I_1^{(3)} := I(x_1, r_1) \subset [R_1, \infty)$ such that (5.22) holds. Similarly, for $R_j := x_{j-1} + 4A_2 C_\delta$, $j = 2, 3, \ldots$, there exists $I_j^{(3)} := I(x_j, r_j) \subset [R_j, \infty)$ satisfying (5.22). Repeating this procedure, we obtain $\{I_j^{(3)}\}_j$ satisfying (5.22) for each $j$. Moreover, as $b$ satisfies the condition (ii) in Theorem 3.1, for $\delta$ aforementioned, there exists a constant $\widetilde C_\delta$ such that

$$M_\lambda(b, I) < \delta$$

for any interval $I$ satisfying $m_\lambda(I) > \widetilde C_\delta$. This together with the choice of $\{I_j^{(3)}\}_j$ implies that $m_\lambda(I_j^{(3)}) \leq \widetilde C_\delta$ for all $j$. Since for any $j$, $m_\lambda(I_j^{(3)}) \sim x_j^{2\lambda} r_j > r_j^{2\lambda+1}$, it follows that $r_j < C\widetilde C_\delta^{\frac{1}{2\lambda+1}} =: C_\delta$. Therefore, by the choice of $\{I_j^{(3)}\}_j$, (5.23) holds. This implies the claim.

Now we define

$$\widetilde J_1 := (x_\ell + A_1 r_\ell, x_\ell + A_2 r_\ell),$$

and

$$\widetilde J_2 := \{y \in \mathbb{R}_+ : |y - x_{\ell+m}| > A_2 r_{\ell+m}\}.$$  

Note that $\widetilde J_1 \subset \widetilde J_2$. Thus, similar to the estimates of $F_1$ and $F_2$ in Case i), for any $\ell, m$, we get

$$\| [b, R_{\Delta \lambda}] (f_\ell) - [b, R_{\Delta \lambda}] (f_{\ell+m}) \|_{L^p(\mathbb{R}_+, dm_\lambda)}$$

$$\geq \left\{ \int_{\widetilde J_1} \| [b, R_{\Delta \lambda}] (f_\ell)(y) - [b, R_{\Delta \lambda}] (f_{\ell+m})(y) \|^p y^{2\lambda} dy \right\}^{1/p}$$

$$\geq \left\{ \int_{\widetilde J_1} \| [b, R_{\Delta \lambda}] (f_\ell)(y) \| y^{2\lambda} dy \right\}^{1/p} - \left\{ \int_{\widetilde J_2} \| [b, R_{\Delta \lambda}] (f_{\ell+m})(y) \| y^{2\lambda} dy \right\}^{1/p}$$

$$\geq (A_3)^{1/p}.$$

This contradicts to the compactness of $[b, R_{\Delta \lambda}]$ on $L^p(\mathbb{R}_+, dm_\lambda)$, so $b$ also satisfies condition (iii) in Theorem 3.1.

**Necessity:**

To see the converse, we show that when $b \in \text{CMO}(\mathbb{R}_+, dm_\lambda)$, the commutator $[b, R_{\Delta \lambda}]$ is compact on $L^p(\mathbb{R}_+, dm_\lambda)$. For any $\epsilon > 0$, there exists $b_\epsilon \in \mathcal{D}$ such that

$$\| b - b_\epsilon \|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} < \epsilon$$

and

$$\|[b, R_{\Delta \lambda}] - [b_\epsilon, R_{\Delta \lambda}]\|_{L^p(\mathbb{R}_+, dm_\lambda) \to L^p(\mathbb{R}_+, dm_\lambda)} \lesssim \| b - b_\epsilon \|_{\text{BMO}(\mathbb{R}_+, dm_\lambda)} \lesssim \epsilon.$$
Thus, it suffices to show that \([b, R_{\Delta_i}]\) is a compact operator for \(b \in \mathcal{D}\).

Let \(b \in \mathcal{D}\), to show \([b, R_{\Delta_i}]\) is compact on \(L^p(\mathbb{R}_+, dm_\lambda)\), it suffices to show that for every bounded subset \(F \subseteq L^p(\mathbb{R}_+, dm_\lambda)\), \([b, R_{\Delta_i}]F\) is relatively compact. Thus, we only need to show that \([b, R_{\Delta_i}]F\) satisfies the conditions (a)—(c) in Theorem 4.2. We first point out that by Lemma 5.1 and the fact that \(b \in \text{BMO}(\mathbb{R}_+, dm_\lambda)\), \([b, R_{\Delta_i}]\) is bounded on \(L^p(\mathbb{R}_+, dm_\lambda)\), which implies \([b, R_{\Delta_i}]F\) satisfies (a) in Theorem 4.2. Next, since \(b \in \mathcal{D}\), by (2.2) and the Hölder inequality, there exists \(M\) such that for any \(x > M\),

\[
\|b, R_{\Delta_i}\|_{L^p(\mathbb{R}_+, dm_\lambda)} \leq \frac{1}{m_\lambda(I(x, x))}. 
\]

Hence (b) in Theorem 4.2 holds for \([b, R_{\Delta_i}]F\). Therefore, it remains to prove \([b, R_{\Delta_i}]F\) also satisfies (c).

Let \(\epsilon\) be a fixed positive constant in \((0, \frac{1}{2})\) and \(z \in \mathbb{R}_+\) small enough. Then for any \(x \in \mathbb{R}_+\),

\[
[b, R_{\Delta_i}]f(x) - [b, R_{\Delta_i}]f(x + z) = \int_0^\infty R_{\Delta_i}(x, y)[b(x) - b(y)]f(y)y^{2\lambda}\,dy - \int_0^\infty R_{\Delta_i}(x + z, y)[b(x + z) - b(y)]f(y)y^{2\lambda}\,dy \\
= \int_{|x-y| > \epsilon^{-1}z} R_{\Delta_i}(x, y)[b(x) - b(x + z)]f(y)y^{2\lambda}\,dy \\
+ \int_{|x-y| > \epsilon^{-1}z} R_{\Delta_i}(x, y) - R_{\Delta_i}(x + z, y)[b(x + z) - b(y)]f(y)y^{2\lambda}\,dy \\
+ \int_{|x-y| < \epsilon^{-1}z} R_{\Delta_i}(x, y)[b(x) - b(y)]f(y)y^{2\lambda}\,dy \\
- \int_{|x-y| < \epsilon^{-1}z} R_{\Delta_i}(x + z, y)[b(x + z) - b(y)]f(y)y^{2\lambda}\,dy =: \sum_{j=1}^4 L_i.
\]

From (2.3) and \(\epsilon \in (0, 1/2)\), it follows that

\[
|L_2| \lesssim |z| \int_{|x-y| > \epsilon^{-1}z} \frac{|f(y)|}{m_\lambda(I(x, |x-y|))} y^{2\lambda}\,dy.
\]

By this and the Hölder inequality, we have that

\[
\int_0^\infty |L_2|^p x^{2\lambda\,dx} \lesssim z^p \int_0^\infty \left[ \int_0^\infty \left( \frac{1}{m_\lambda(I(x, |x-y|))} \right)^{1/p'+1/p} |f(y)| y^{2\lambda}\,dy \right]^p x^{2\lambda\,dx} \\
\lesssim z^p \int_0^\infty \left\{ \int_{|x-y| > \epsilon^{-1}z} \frac{y^{2\lambda}}{m_\lambda(I(x, |x-y|))} \,dy \right\}^{p/p'} \\
\times \int_{|x-y| < \epsilon^{-1}z} \frac{|f(y)| p}{m_\lambda(I(x, |x-y|))} y^{2\lambda}\,dy \right\} x^{2\lambda\,dx} \\
\lesssim z^p (\epsilon z^{-1})^{(p/p'+1)|f|_{L^p(\mathbb{R}_+, dm_\lambda)}} \lesssim \epsilon^p \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)}^p,
\]

where the last-to-second inequality follows from the fact that

\[
\int_{|x-y| > \epsilon^{-1}z} \frac{y^{2\lambda}}{m_\lambda(I(x, |x-y|))} \,dy
\]
\[ \sim \sum_{k=0}^{\infty} \frac{1}{2^k e^{z}} \int_{2^k e^{z} < |x-y| \leq 2^{k+1} e^{z}} \frac{y^{2\lambda}}{m_{\lambda}(I(x, 2^k e^{z}))} dy \]
\[ \lesssim \sum_{k=0}^{\infty} \frac{1}{2^k e^{z}} \frac{m_{\lambda}(I(x, 2^{k+1} e^{z}))}{m_{\lambda}(I(x, 2^k e^{z}))} \lesssim \epsilon z^{-1}. \]

By (2.2), the fact that \( b \in D \) and the mean value theorem, we conclude that

\[ |L_3| \lesssim \int_{|x-y| \leq \epsilon z} \frac{|x-y|}{m_{\lambda}(I(x, |x-y|))} |f(y)| y^{2\lambda} dy \]
and

\[ |L_4| \lesssim \int_{|x-y| \leq \epsilon z} \frac{|x+z-y|}{m_{\lambda}(I(x+z, |x+z-y|))} |f(y)| y^{2\lambda} dy. \]

Then by the fact that

\[ \int_{|x-y| \leq \epsilon z} \frac{|x-y|}{m_{\lambda}(I(x, |x-y|))} y^{2\lambda} dy \]
\[ \sim \sum_{k=-\infty}^{1} 2^k e^{-z} \int_{2^k e^{-z} < |x-y| \leq 2^{k+1} e^{-z}} \frac{y^{2\lambda}}{m_{\lambda}(I(x, 2^k e^{-z}))} dy \]
\[ \lesssim \sum_{k=-\infty}^{-1} 2^k e^{-z} \frac{m_{\lambda}(I(x, 2^{k+1} e^{-z}))}{m_{\lambda}(I(x, 2^k e^{-z}))} \lesssim \epsilon^{-1} z, \]
we see that

\[ \int_{0}^{\infty} |L_3|^p x^{2\lambda} dx \lesssim \int_{0}^{\infty} \left\{ \int_{|x-y| \leq \epsilon z} \frac{|x-y|}{m_{\lambda}(I(x, |x-y|))} y^{2\lambda} dy \right\}^{p/p'} \]
\[ \times \int_{|x-y| \leq \epsilon z} \frac{|x-y| |f(y)|^p}{m_{\lambda}(I(x, |x-y|))} y^{2\lambda} dy \] \[ x^{2\lambda} dx \]
\[ \lesssim (\epsilon^{-1} z)^p \| f \|_{L_p(\mathbb{R}^+, dm_{\lambda})}^p, \]

and

\[ \int_{0}^{\infty} |L_4|^p x^{2\lambda} dx \lesssim \int_{0}^{\infty} \left\{ \int_{|x+z-y| \leq \epsilon z} \frac{|x+z-y|}{m_{\lambda}(I(x+z, |x+z-y|))} y^{2\lambda} dy \right\}^{p/p'} \]
\[ \times \int_{|x+z-y| \leq \epsilon z} \frac{|x+z-y| |f(y)|^p}{m_{\lambda}(I(x+z, |x+z-y|))} y^{2\lambda} dy \] \[ x^{2\lambda} dx \]
\[ \lesssim (\epsilon^{-1} z + z)^p \| f \|_{L_p(\mathbb{R}^+, dm_{\lambda})}^p \]
\[ \lesssim (\epsilon^{-1} z)^p \| f \|_{L_p(\mathbb{R}^+, dm_{\lambda})}^p. \]

Moreover, observe that

\[ |L_1| \leq |b(x) - b(x+z)| \sup_{t>0} \left| \int_{|x-y|>t} R_{\Delta \lambda}(x,y) f(y) y^{2\lambda} dy \right| =: |b(x) - b(x+z)| R_{\Delta \lambda, f}(x). \]
Since by i) and ii) of Lemma 2.3, $R_{\Delta}x, y$ is a Calderón-Zygmund kernel in space of homogeneous type, we see that $R_{\Delta}x$ is bounded on $L^p(\mathbb{R}^+, dm_{\lambda})$ for any $p \in (1, \infty)$; see, for example, [19] and [7]. Then we have that

$$\int_0^{\infty} |L_1|^p x^{2\lambda} \, dx \lesssim \int_0^{\infty} [\{b(x) - b(x + z)|R_{\Delta}x f(x)\}]^p x^{2\lambda} \, dx.$$ 

As $b$ is uniformly continuous, by letting $z$ small enough depending on $\epsilon$, we have that

$$\int_0^{\infty} |L_1|^p x^{2\lambda} \, dx \lesssim \epsilon^p \|f\|_{L^p(\mathbb{R}^+, dm_{\lambda})}^p.$$ 

Combining the estimates of $L_i$, $i \in \{1, 2, 3, 4\}$, we conclude that

$$\left[ \int_0^{\infty} |\{b, R_{\Delta}x \} f(x) - \{b, R_{\Delta}x \} f(x + z)|^p x^{2\lambda} \, dx \right]^{1/p} \lesssim \epsilon \|f\|_{L^p(\mathbb{R}^+, dm_{\lambda})}.$$ 

This shows that $[b, R_{\Delta}x]^F$ satisfies the condition (c) in Theorem 4.2. Hence, $[b, R_{\Delta}x]$ is a compact operator. This finishes the proof of Theorem 1.2. \hfill \Box

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