A CRITICAL FRACTIONAL \( p \)-KIRCHHOFF TYPE PROBLEM INVOLVING DISCONTINUOUS NONLINEARITY

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Dedicated to Vicentiu D. R˘ adulescu on the occasion of his 60th birthday

Abstract. The aim of this paper is to discuss the existence and multiplicity of solutions for the following fractional \( p \)-Kirchhoff type problem involving the critical Sobolev exponent and discontinuous nonlinearity:

\[
M \left( \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right) (-\Delta)^s_p u = \lambda |u|^{p^*_s - 2} u + f(x, u) \quad \text{in} \quad \mathbb{R}^N,
\]

where \( M(t) = a + bt^{\theta-1} \) for \( t \geq 0, \ a \geq 0, b > 0, \theta > 1 \), \( (-\Delta)^s_p \) is the fractional \( p \)-Laplacian with \( 0 < s < 1 \) and \( 1 < p < N/s \), \( p^*_s = Np/(N - ps) \) is the critical Sobolev exponent, \( \lambda > 0 \) is a parameter, and \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is a function.

Under suitable assumptions on \( f \), we show that there exists \( \lambda_0 > 0 \) such that the above equation admits at least one nontrivial nonnegative solution provided \( \lambda < \lambda_0 \) by using the nonsmooth critical point theory for locally Lipschitz functionals. Furthermore, for any \( k \in \mathbb{N} \), there exists \( \Lambda_k > 0 \) such that the above equation has \( k \) pairs of nontrivial solutions if \( \lambda < \Lambda_k \). The main feature is that our paper covers the degenerate case, that is the coefficient of \( (-\Delta)^s_p \) may be zero at zero. Moreover, the existence results are obtained when \( f \) is discontinuous. Thus, our results are new even in the semilinear case.

1. Introduction and main results. In this paper we are concerned with the existence and multiplicity of solutions for a critical elliptic equations involving the fractional \( p \)-Laplacian. More precisely, we consider

\[
M \left( \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right) (-\Delta)^s_p u = \lambda |u|^{p^*_s - 2} u + f(x, u) \quad \text{in} \quad \mathbb{R}^N,
\]

where the Kirchhoff term \( M(t) = a + bt^{\theta-1} \) for \( t \geq 0, \ a \geq 0, b > 0, \theta > 1 \), \( N > sp \) with \( s \in (0, 1) \), and \( (-\Delta)^s_p \) is the fractional \( p \)-Laplacian which (up to normalization

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factors) may be defined for any \( x \in \mathbb{R}^N \) as
\[
(-\Delta)^s \varphi(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}} dy
\]
along any \( \varphi \in C_0^\infty(\mathbb{R}^N) \), where \( B_\varepsilon(x) \) denotes the ball in \( \mathbb{R}^N \) centered at \( x \) with radius \( \varepsilon \). In particular, \((-\Delta)^s_p\) becomes the fractional Laplacian \((-\Delta)^s\) as \( p = 2 \), and \((-\Delta)^s_p\) reduces to the standard \( p \)-Laplacian as \( s \uparrow 1 \) in the limit sense of Bourgain–Brézis–Mironescu, as shown in [9]. For more details about the fractional Laplacian, we refer to [17, 33, 41, 49] and the references therein.

In the last years, a great attention has been paid to the study of fractional and non-local problems involving critical nonlinearities. For example, a Brézis-Nirenberg type result for fractional Laplacian in bounded domain with homogeneous Dirichlet boundary datum is given in [44], see also [36] for related results for fractional \( p \)-Laplacian and the references cited there for further results. Nonexistence results for nonlocal equations involving critical and supercritical nonlinearities can be found in [43]. The multiplicity of solutions for critical fractional Laplacian equations is an interesting and difficult problem. For example, a multiplicity result for fractional Laplacian problems in \( \mathbb{R}^N \) is obtained in [5] by using the mountain pass theorem and the direct method in variational methods, where one of two superlinear nonlinearities could be critical or even supercritical. Infinitely many solutions for a critical Kirchhoff type problem involving a fractional operator are obtained in [22] by using a suitable truncation argument combined with Krasnoselskii’s genus theory, see also [32, 42] for more results in this direction. In [46], Wang and Xiang considered a superlinear fractional Choquard equation with fractional magnetic operators and critical exponent and obtained infinitely many solutions by critical point theory. It is worth mentioning that the interest in nonlocal fractional problems goes beyond the mathematical curiosity. Indeed, this type of operators arises in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, see for example [3, 11, 17, 27] and the references therein. The literature on non-local operators and their applications is quite large, here we just quote a few, see [34, 35, 49] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the readers to [17, 33].

Recently, Fiscella and Valdinoci in [20] proposed a stationary Kirchhoff variational equation which models the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. More precisely, they established a model given by the following formulation:
\[
\begin{cases}
M \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy \right)(-\Delta)^s u = \lambda f(x, u) + |u|^{2^*_s - 2} u & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
where \( M(t) = a + bt \) for all \( t \geq 0 \), here \( a > 0, b \geq 0 \). Note that \( M \) is this type, problem (2) is called non-degenerate if \( a > 0 \) and \( b \geq 0 \), while it is named degenerate if \( a = 0 \) and \( b > 0 \), see [42] for some physical explanations about non-degenerate Kirchhoff problems. Hence problem (2) is non-degenerate. For some motivation in the physical background for the fractional Kirchhoff model, we refer to [20, Appendix A]. In [47], Xiang et al. investigated the existence of solutions for Kirchhoff type problem involving the fractional \( p \)-Laplacian via variational methods, where
the nonlinearity is subcritical and the Kirchhoff function is non-degenerate. By using the mountain pass theorem and Ekeland’s variational principle, the authors in [48] studied the multiplicity of solutions to a nonhomogeneous Kirchhoff type problem driven by the fractional \( p \)-Laplacian, where the nonlinearity is convex-concave and the Kirchhoff function is degenerate. Using the same methods as in [48], Pucci et al. in [41] obtained the existence of multiple solutions for the nonhomogeneous fractional \( p \)-Laplacian equations of Schrödinger-Kirchhoff type in the whole space. Indeed, the fractional Kirchhoff problems have been extensively studied in recent years, for instance, we also refer to [12, 40] about non-degenerate Kirchhoff type problems and to [4, 21, 42] about degenerate Kirchhoff type problems for the recent advances in this direction.

When \( s = 1, a = 1 \) and \( b = 0 \), problem (1) reduces to the following equation
\[
-\Delta_p u = \lambda |u|^{p^* - 2} u + f(x, u) \quad \text{in } \mathbb{R}^N,
\]
where \(-\Delta_p u = -\text{div}(|\nabla u|^{p-2} \nabla u)\) is the \( p \)-Laplacian operator and \( f \) is a discontinuous nonlinearity. In [45], Shang studied problem (3) by using variational methods and obtained the existence and multiplicity of solutions. In [1], Alves and Bertone got two nonnegative solutions for the following equation
\[
-\Delta_p u = H(u - a)|u|^{p^* - 2} u + \lambda h(x) \quad \text{in } \mathbb{R}^N,
\]
where \( H \) is the Heaviside function, i.e. \( H(t) = 0 \) if \( t \leq 0 \), \( H(t) = 1 \) if \( t > 0 \). For more results on such kinds of discontinuous nonlinearity problems, we refer the readers to [2] and the references cited there.

Motivated by above cited papers, we will study the existence and multiplicity of solutions for problem (1) involving the fractional \( p \)-Laplacian in \( \mathbb{R}^N \) and discontinuous nonlinearity. To the best of our knowledge, there is no result in the literature on problem (1). Certainly, such a setting raises extra difficulties due to the lack of compactness and the nonlocal nature of the \( p \)-fractional Laplacian, as well as the presence of discontinuous nonlinearity. For this purpose, we shall use the principle of concentration compactness of P.L. Lions in fractional Sobolev spaces.

Let
\[
S = \inf_{u \in D^{s,p}((\mathbb{R}^N) \setminus \{0\})} \frac{\|u\|_{p^*}^p}{\|u\|_{p^*}^{p^*}},
\]
which is positive by the fractional Sobolev inequality. Here and throughout this paper, we shortly denote by \( \| \cdot \|_q \) the norm of \( L^q(\mathbb{R}^N) \) for any \( q \in (1, \infty) \). Recently, Brasco et al. in [8] obtained that there exists a radially symmetric nonnegative decreasing minimizer \( U = U(r) \) for \( S \). The authors also showed that \( U \) is a weak
solution of (4) and satisfies
\[\|U\|^p = |U|_{p*}^{p*} = S^{N/p}.\]

Set
\[f(x,t) = \lim \text{ess inf}_{\delta \to 0^+} f(x,s), \quad \overline{f}(x,t) = \lim \text{ess sup}_{\delta \to 0^+} f(x,s).\]

In this paper, we assume \(f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}\) is a measurable function satisfying:

1. \(f(x,t) = 0\) if \(t \leq 0\) for all \(x \in \mathbb{R}^N\) and for all \(t \in \mathbb{R}\), there exist the limits:
\[f(x,t+\delta) = \lim_{\delta \to 0^+} f(x,t+\delta), \quad f(x,t-\delta) = \lim_{\delta \to 0^+} f(x,t-\delta);\]

2. \(|f(x,t)| \leq c_1(x) + c_2(x)|t|^{q-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \quad c_1(x), c_2(x) \geq 0,\]
\[c_1 \in L^{\frac{p^*}{2}}(\mathbb{R}^N), \quad c_2 \in L^{\frac{p^*}{q-1}}(\mathbb{R}^N), \quad \theta_p < q < p^*_s;\]

3. There exists a positive constant \(\sigma \in (p,p^*_s)\) such that \(\min \{f(x,t) : (x,t) \in \Omega\} \geq \sigma F(x,t) > 0\) for all \(t \in \mathbb{R}\setminus \{0\}\) and a.e. \(x \in \mathbb{R}^N\), where \(F(x,t) = \int_0^t f(x,\tau)\,d\tau;\)

4. There exist an open non-empty subset \(\Omega \subset \mathbb{R}^N\), \(C_0 > 0\) and \(q_1 \in (\theta_p,p^*_s)\) such that
\[F(x,t) \geq C_0|t|^{q_1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.\]

A simple example of \(f\) which satisfies (1) is given by
\[f(x,t) = C(x)\text{sgn}(t-a_0)|t|^{q-1},\]
where \(C(x) \in C^\infty_0(\Omega)\) with positive minimum and \(\Omega \subset \mathbb{R}^N\) is bounded, \(\text{sgn}\) denotes the sign function, \(a_0 > 0, \theta_p < q < p^*_s\) and \(t^+ = \max\{0,t\}\).

Define \(g(t) := \frac{\theta_p}{\theta_p - S^{-\theta_p/p}p^{\theta_p} - c_1} S^{-q/p} - c_2\) for all \(t \geq 0\). Since \(p^*_s > \theta_p\) by \(\theta < N/(N+sp)\) and \(q > \theta_p\), we deduce that there exists \(\rho > 0\) such that \(g(\rho) > 0\). Based on this fact, we further assume that
\[|c_1| \frac{p^*}{r^*_p - 1} \leq g(\rho) \left(\frac{\theta_p}{\theta_p - S^{-\theta_p/p}p^{\theta_p}}\right)^{-\frac{r^*_p - 1}{\theta_p}} S^p \left(\frac{h}{2}\right)^{-\frac{\theta_p}{\theta_p - s},} \quad (5)\]
which will be used later.

Note that by assumption (f1), it is easy to see that
\[\overline{f}(x,t) = \max\{f(x,t-0), f(x,t+0)\}, \quad f(x,t) = \min\{f(x,t-0), f(x,t+0)\}.\]

We give the definition of weak solutions for problem (1).

**Definition 1.1.** We say that \(u \in D^{s,p}(\mathbb{R}^N)\) is a (weak) solution of problem (1) if there exists \(\tilde{w}(x) \in [f(x,u),\overline{f}(x,u)]\) a.e. in \(\mathbb{R}^N\) such that
\[(a + b||u||^{(q-1)p}) \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}}
\[= \int_{\mathbb{R}^N} |u|^{p^* - 2} u\varphi\,dx + \int_{\mathbb{R}^N} \tilde{w}\varphi\,dx, \quad \forall \varphi \in D^{s,p}(\mathbb{R}^N).\]

Now we can state the first result as follows.
Theorem 1.2. Assume that $f$ satisfies $(f_1)-(f_4)$ and $c_1$ satisfies (5). Then there exists $\lambda_0 > 0$ such that problem (1) admits at least one nontrivial nonnegative solution for all $\lambda \in (0, \lambda_0)$.

One of the main motivations is to consider a particular and relevant case associated to problem (1) given by

$$(a + b\|u\|^{(\theta - 1)p})(-\Delta)^s_p u = \lambda |u|^{p^*_s - 2} u + \tilde{f}(x,u)$$

(6)

where $\tilde{f}(x,u) = \begin{cases} f(x,u) & \text{if } u \neq \Theta, \\ [0, h(x)|u|^{q - 2}u] & \text{if } u = \Theta, \end{cases}$

and $f(x,u) = h(x)H(u - \Theta)|u|^{q - 2}u$, $H$ is the Heaviside function, $\lambda$ and $\Theta$ are positive real parameters, $\theta p < q < p^*_s$ and $h : \mathbb{R}^N \to [0, \infty)$ is a nonnegative measurable function.

We are in a position to state the second result of our paper as follows:

Theorem 1.3. Let $\theta p < q < p^*_s$, $h \geq 0$ and $h \in L^{\frac{p^*_s}{p^*_s - q}}(\mathbb{R}^N)$. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ and all $\Theta > 0$ problem (6) possesses a nontrivial nonnegative solution in $D^{s,p}(\mathbb{R}^N)$.

To study the multiplicity result, we need the following assumption:

(\text{f}'_1) \quad f(x,0) = 0$ and there exist the limits:

$$f(x,t + 0) = \lim_{\delta \to 0^+} f(x,t + \delta), \quad f(x,t - 0) = \lim_{\delta \to 0^+} f(x,t - \delta).$$

Theorem 1.4. Assume that $f$ satisfies $(f_1)$ and $(f_2)-(f_5)$ and $c_1$ satisfies (5). Then for each $k > 0$ there is $\lambda_k > 0$ such that problem (1) has at least $k$ pairs of nontrivial solutions for all $\lambda \in (0, \lambda_k)$.

Finally, let us simply describe the main approach to obtain Theorem 1.2. The proof relies on some results of convex analysis since the functional $I_\lambda$ associated to problem (1) is locally Lipschitz. To get critical points for $I_\lambda$, we use a version of the mountain pass theorem for locally Lipschitz functional. It is worth stressing that the major difficulties in this paper are as follows: first of all, because we are working with the fractional $p$-Laplacian, which is not linear and local, the growth of the nonlinear part is critical. Secondly, arguments involved the case that $f(x,t)$ is continuous with respect to $t$ (the classical case) could not be used easily in our context. In particular, we establish a new estimate which is a crucial ingredient to prove that the energy functional verifies the nonsmooth Palais-Smale condition at some levels.

This paper is organized as follows. In Section 2, we first recall some basic properties of generalized gradient of locally Lipschitz functionals and the concentration compactness lemma in fractional Sobolev spaces $D^{s,p}(\mathbb{R}^N)$. In Section 3, using the mountain pass theorem combined with the principle of concentration compactness in $D^{s,p}(\mathbb{R}^N)$, we first establish the existence of one nontrivial and nonnegative solutions for problem (1) with a suitable range of positive parameter $\lambda$. Then using the symmetric mountain pass theorem, we obtain the multiplicity of solutions for problem (1).
2. Preliminaries. Throughout this paper, for any set $A \subset \mathbb{R}^N$, $|A|$ is the Lebesgue measure of $A$. Set $u^\pm = \max\{\pm u, 0\}$. Denote by $o(1)$ a real vanishing sequence, and let $C$ be various positive constants.

Let us now recall some definitions and properties of generalized gradient of locally Lipschitz functionals, which will be used later and can be found in [15]. Let $X$ be a Banach space, $X^*$ be its topological dual and $\langle \cdot, \cdot \rangle$ be the duality. A functional $I : X \to \mathbb{R}$ is said to be locally Lipschitz if for all $u \in X$ there exists a neighborhood $U$ of $u$ and a constant $K > 0$ depending on $U$ such that

$$|I(u) - I(v)| \leq K\|u - v\|, \quad \forall u, v \in U.$$  

For a locally Lipschitz functional $I$, the directional derivative at $u \in X$ in the direction $v \in X$ is defined by

$$I^0(u; v) = \limsup_{\gamma \to 0, \delta \to 0} \frac{I(u + \gamma + \delta v) - I(u + \gamma)}{\delta}.$$  

It is easy to know that $I^0(u; v)$ is subadditive and positively homogeneous. Moreover, there is a $k > 0$ such that $|I^0(u; v)| \leq k\|v\|$ for all $v \in X$, that is, for each $u \in X$, the functional $I^0(u; \cdot)$ is a continuous on $X$.

The generalized gradient of $I$ at $u$ is defined as

$$\partial I(u) = \{\omega \in X^* : I^0(u; v) \geq \langle \omega, v \rangle, \forall v \in X\}.$$  

Then, for each $v \in X$, $I^0(u; v) = \sup\{\langle \omega, v \rangle : \omega \in \partial I(u)\}$. A point $u \in X$ is a critical point of $I$ if $0 \in \partial I(u)$. If $u \in X$ is a local minimum or maximum, then $0 \in \partial I(u)$.

The solutions of (1) are exactly the critical points of the functional given by

$$I_\lambda(u) = \frac{a}{p} \|u\|^p + \frac{b}{\theta p} \|u\|^\theta p - \frac{\lambda}{p^*_s} \|u\|^{p^*_s} - \int_{\mathbb{R}^N} F(x, u) dx, \quad \forall u \in D^{s,p}(\mathbb{R}^N).$$  

We say that $I_\lambda$ satisfies the nonsmooth $(PS)_c$ condition on $X$, if any sequence $\{u_n\}_n \subset X$ such that $I_\lambda(u_n) \to c$ and $m(u_n) = \min\{\|\omega\|_{(D^{s,p}(\mathbb{R}^N))_c} : \omega \in \partial I_\lambda(u_n)\}$ → 0 as $n \to \infty$, possesses a convergent subsequence. To prove our main results, we need the generalizations of the mountain pass theorem [14] and of the symmetric mountain pass theorem [23].

**Theorem 2.1.** (see [14]) Let $X$ be a reflexive Banach space, $I : X \to \mathbb{R}$ is locally Lipschitz functional which satisfies the nonsmooth $(PS)_c$ condition, $I(0) = 0$ and there are $\rho, \alpha > 0$ and $\epsilon \in X$ with $\|\epsilon\| > \rho$, such that

$$I(u) \geq \alpha \quad \text{if} \quad \|u\| = \rho \quad \text{and} \quad I(\epsilon) \leq 0.$$  

Then there exists $u_0 \in X$ such that $0 \in \partial I(u_0)$ and $I(u_0) = c$, where

$$c = \inf_{\gamma \in \Gamma} \max_{\tau \in [0, 1]} I(\gamma(\tau))$$  

and

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = \epsilon\}.$$  

**Theorem 2.2.** (see [23]) Let $X$ be a reflexive Banach space, $I : X \to \mathbb{R}$ is even locally Lipschitz functional satisfying the nonsmooth $(PS)_c$ condition. Let $X^+, X^- \subset X$ be closed subspace of $X$ with codim$X^+ \leq \dim X^- < \infty$ and suppose $X = X^+ + X^-$. Also suppose that there holds:

1. $I(0) = 0$,
2. there exist $\rho, \alpha > 0$ such that $I(u) \geq \alpha$ if $\|u\| = \rho$ for all $u \in X^+$. 


(3) there is $R > 0$ such that $\forall u \in X^-, I(u) \leq 0$ if $\|u\| \geq R$.

Then for each $j$, $1 \leq j \leq k = \dim X^- - \text{codim} X^+$, the numbers
\[ c_j = \inf_{h \in \Gamma} \sup_{u \in X_j} I(h(u)), \]
are critical. Moreover, $c_k \geq c_{k-1} \geq \cdots \geq c_1 \geq \alpha$, where
\[ \Gamma = \{ h \in C(\mathbb{X}, \mathbb{X}) : h \text{ is odd, } h(u) = u \text{ if } u \in X^- \text{ and } \|u\| \geq R \}, \]
and $X_1 \subset X_2 \subset \cdots \subset X_k = X^-$ are fixed subspaces of dimension $\dim X_j = \text{codim} X^+ + j$.

In [50], Xiang et al. established the principle of concentration compactness in $D^{s,p}(\mathbb{R}^N)$, which can be regarded as a fractional counterpart of the principle of concentration compactness in classical Sobolev space $W^{1,p}(\mathbb{R}^N)$. Now, we recall the fractional concentration-compactness principle, which will be the keystone that enables us to verify that $I_\lambda$ satisfies the nonsmooth $(PS)_c$ condition.

Let $C_c(\mathbb{R}^N) = \{ u \in C(\mathbb{R}^N) : \text{supp } u \text{ is a compact subset of } \mathbb{R}^N \}$ and denote by $C_0(\mathbb{R}^N)$ the closure of $C_c(\mathbb{R}^N)$ with respect to the norm $|\eta|_\infty = \sup_{x \in \mathbb{R}^N} |\eta(x)|$. As well known, a finite measure on $\mathbb{R}^N$ is a continuous linear functional on $C_0(\mathbb{R}^N)$. Now we give a norm for measure $\mu$
\[ \|\mu\| = \sup_{\mu \in C_0(\mathbb{R}^N), |\eta|_\infty = 1} |(\mu, \eta)|, \]
where $(\mu, \eta) = \int_{\mathbb{R}^N}\eta d\mu$.

**Definition 2.3.** Let $\mathcal{M}(\mathbb{R}^N)$ denote the finite nonnegative Borel measure space on $\mathbb{R}^N$. For any $\mu \in \mathcal{M}(\mathbb{R}^N)$, $\mu(\mathbb{R}^N) = \|\mu\|$ holds. We say that $\mu_n \rightharpoonup \mu$ weakly* in $\mathcal{M}(\mathbb{R}^N)$, if $(\mu_n, \eta) \rightarrow (\mu, \eta)$ holds for all $\eta \in C_0(\mathbb{R}^N)$ as $n \rightarrow \infty$.

**Theorem 2.4.** (see [50]) Let $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ with upper bound $C > 0$ for all $n \geq 1$ and
\[ u_n \rightarrow u \text{ weakly in } D^{s,p}(\mathbb{R}^N), \]
\[ \int_{\mathbb{R}^N} \frac{|u_n(x) - u_0(y)|^p}{|x - y|^{N+sp}} dy \rightharpoonup \mu \text{ weakly* in } \mathcal{M}(\mathbb{R}^N), \]
\[ |u_n(x)|^{p^*_s} \rightharpoonup \nu \text{ weakly* in } \mathcal{M}(\mathbb{R}^N). \]

Then
\[ \mu = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dy + \sum_{j \in J} \mu_j \delta_{x_j} + \check{\mu}, \quad \mu(\mathbb{R}^N) \leq C^p, \]
\[ \nu = |u|^{p^*_s} + \sum_{j \in J} v_j \delta_{x_j}, \quad \nu(\mathbb{R}^N) \leq S^{p^*_s}C^p, \]
where $J$ is at most countable, sequences $\{\mu_j\}_j, \{v_j\}_j \subset [0, \infty), \{x_j\}_j \subset \mathbb{R}^N$, $\delta_{x_j}$ is the Dirac mass centered at $x_j$, $\mu$ is a non-atomic measure,
\[ \nu(\mathbb{R}^N) \leq S^{-p^*_s/p} \mu(\mathbb{R}^N)^{p^*_s/p}, \]
\[ v_j \leq S^{-p^*_s/p} \mu_j^{p^*_s/p} \quad \forall j \in J, \]
and $S > 0$ is the best constant of $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p^*_s}(\mathbb{R}^N)$.

The following lemma is fundamental to prove Theorem 2.4.
Lemma 2.5. (see [50]) Assume \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) is the sequence given by Theorem 2.4 and let \( x_0 \in \mathbb{R}^N \) fixed and let \( \varphi \in C_0^\infty(\mathbb{R}^N) \) such that \( 0 \leq \varphi \leq 1; \varphi \equiv 1 \) in \( B(0,1), \varphi \equiv 0 \) in \( \mathbb{R}^N \setminus B(0,2) \) and \( |\nabla \varphi| \leq 2 \). Set \( \varphi_{\varepsilon,0}(x) = \varphi((x-x_0)/\varepsilon) \) for all \( x \in \mathbb{R}^N \). Then

\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p|\varphi_{\varepsilon,0}(x) - \varphi_{\varepsilon,0}(y)|^{p}}{|x-y|^{N+ps}} \ dx \ dy = 0.
\]

Evidently, Theorem 2.4 does not provide any information about the possible loss of mass at infinity for a weakly convergent sequence. The following theorem expresses this fact in quantitative terms.

Theorem 2.6. (see [50]) Let \( \{u_n\}_n \subset D^{s,p}(\mathbb{R}^N) \) be a bounded sequence such that

\[
\int_{\mathbb{R}^N} \frac{|u_n(x)|^{p} - u_n(y)|^p}{|x - y|^{N+ps}} \ dy \to \mu \quad \text{weakly * in } \mathcal{M}(\mathbb{R}^N),
\]

\[
|u_n|^p \to \nu \quad \text{weakly * in } \mathcal{M}(\mathbb{R}^N),
\]

and define

\[
\mu_{\infty} = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\{x \in \mathbb{R}^N: |x| > R\}} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} \ dy \ dx,
\]

and

\[
\nu_{\infty} = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\{x \in \mathbb{R}^N: |x| > R\}} |u_n|^p \ dx.
\]

Then the quantities \( \mu_{\infty} \) and \( \nu_{\infty} \) are well defined and satisfy

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} \ dy \ dx = \int_{\mathbb{R}^N} d\mu + \mu_{\infty},
\]

and

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p \ dx = \int_{\mathbb{R}^N} d\nu + \nu_{\infty}.
\]

Moreover, the following inequality holds

\[
S_{\infty}^{p/p^*} \leq \mu_{\infty}.
\]

3. Proofs of Theorems 1.2–1.4. Before giving the proof of main results, let us first show that the functional \( \mathcal{I}_\lambda \) is a locally Lipschitz functional. Set

\[
\Phi(u) = \frac{a}{p} \|u\|^p + \frac{b}{\theta p} \|u\|^\theta p - \frac{\lambda}{p^*_s} \int_{\mathbb{R}^N} |u|^{p^*_s} \ dx
\]

and

\[
\Psi(u) = \int_{\mathbb{R}^N} F(x,u) \ dx
\]

for all \( u \in D^{s,p}(\mathbb{R}^N) \). Then \( \mathcal{I}_\lambda(u) = \Phi(u) - \Psi(u) \).

Lemma 3.1. Assume \( f \) satisfies (f2). Then \( \Psi(u) \) is locally Lipschitz in \( D^{s,p}(\mathbb{R}^N) \). Moreover, if \( \omega \in \partial\Psi(u) \), then \( \omega \in [F(x,u), F(x,u)] \) a.e. in \( \mathbb{R}^N \).
Proof. Let \( u, v \in D^{s,p}(\mathbb{R}^N) \). It follows from (f_2), the mean value theorem and Hölder’s inequality that

\[
|\Psi(u) - \Psi(v)| \leq \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} f(x,t)dt \right| dx \\
\leq \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} |c_1(x) + c_2(x)|t|^{q-1}|dt \right| dx \\
\leq \left( |c_1| \frac{p^*_v}{p^*_v-1} + \max_{w \in U} |w(x)|^{\frac{q-1}{p^*_v-1}} |c_2| \frac{p^*_v}{p^*_v-q} \right) |u-v|_{p^*_v},
\]

where \( w(x) = \tau v(x) + u(x) \) with \( \tau \in (0, 1) \). Using the embedding \( D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p^*_v}(\mathbb{R}^N) \), we get that there is a neighborhood \( U \subset D^{s,p}(\mathbb{R}^N) \) of \( u, v \) such that

\[
|\Psi(u) - \Psi(v)| \leq S^{-\frac{1}{q}} \left( |c_1| \frac{p^*_v}{p^*_v-1} + \max_{w \in U} |w(x)|^{\frac{q-1}{p^*_v-1}} |c_2| \frac{p^*_v}{p^*_v-q} \right) \|u-v\|,
\]

Hence, \( \Psi(u) \) is locally Lipschitz in \( D^{s,p}(\mathbb{R}^N) \).

By the definition of the directional derivative, there exist functions \( v \in C_0^\infty(\mathbb{R}^N) \), \( h_n \to 0 \) in \( D^{s,p}(\mathbb{R}^N) \) and \( \delta_n \downarrow 0 \) such that

\[
\Psi^0(u) = \lim_{n \to \infty} \frac{\Psi(u + h_n + \delta_n v) - \Psi(u + h_n)}{\delta_n} \\
= \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{F(x,u+h_n+\delta_nv) - F(x,u+h_n)}{\delta_n} dx \\
= \lim_{n \to \infty} \left( \int_{\{v>0\}} G_n(v(x))dx + \int_{\{v\leq0\}} G_n(v(x))dx \right),
\]

where

\[
G_n(v(x)) = \frac{F(x,u+h_n+\delta_nv) - F(x,u+h_n)}{\delta_n},
\]

and \( \{ v > 0 \} := \{ x \in \mathbb{R}^N : v(x) > 0 \} \). Since \( h_n \to 0 \) in \( D^{s,p}(\mathbb{R}^N) \), up to a subsequence, we may assume that \( h_n \to 0 \) a.e. in \( \mathbb{R}^N \). By \( h_n \to 0 \) in \( D^{s,p}(\mathbb{R}^N) \), we have \( h_n \to 0 \) in \( L^{p^*_v}(\mathbb{R}^N) \). Thus, there is \( h \in L^{p^*_v}(\mathbb{R}^N) \) such that \( |h_n(x)| \leq h(x) \) a.e. in \( \mathbb{R}^N \).

First, we consider \( v(x) > 0 \) for all \( x \in \mathbb{R}^N \). Then

\[
\limsup_{n \to \infty} G_n(v(x)) \leq \overline{f}(x,u(x)+0)v(x) \quad \text{a.e. in } \mathbb{R}^N.
\]

Observe that the mean value theorem yields

\[
|G_n(v(x))| \leq \frac{1}{\delta_n} \int_{u+h_n}^{u+h_n+\delta_nv} (c_1(x) + c_2(x)|t|^{q-1})dt \\
\leq C(c_1(x) + c_2(x)|u|^{q-1} + c_2|h_n|^{q-1} + |\delta_nv|^{q-1})|v| \\
\leq C(c_1(x) + c_2(x)|u|^{q-1} + c_2|h|^{q-1} + |v|^{q-1})|v| \leq L^1(\mathbb{R}^N).
\]

Thus, by Fatou’s lemma, one can deduce that

\[
\limsup_{n \to \infty} \int_{\{v>0\}} G_n(v(x))dx \leq \int_{\{v>0\}} \overline{f}(x,u(x)+0)v(x)dx.
\]

Similarly,

\[
\limsup_{n \to \infty} \int_{\{v\leq0\}} G_n(v(x))dx \leq \int_{\{v\leq0\}} \overline{f}(x,u(x)-0)v(x)dx.
\]
Therefore, we arrive at the following inequality
\begin{equation}
\Psi^0(u) \leq \int_{\{v \geq 0\}} v\bar{f}(x,u)dx + \int_{\{v < 0\}} v\bar{f}(x,u)dx.
\end{equation}

Let $\omega \in \partial \Psi(u) \subset L^{p^*_\infty}(\mathbb{R}^N)$. Next we prove that
\[
\bar{f}(x,u) \leq \omega \leq \underline{f}(x,u) \quad \text{a.e. in } \mathbb{R}^N.
\]

Arguing by contradiction, we assume that there is a set $A \subset \mathbb{R}^N$ with $|A| > 0$ such that
\[
\omega(x) < \underline{f}(x,u) \text{ in } A.
\]
Taking $v = -\chi_A(x)$ in (7) and using the definition of the generalized gradient, we obtain
\[
-\int_A \omega(x)dx = \int_{\mathbb{R}^N} \omega(x)v(x)dx \leq -\int_A \underline{f}(x,u)dx,
\]
which contradicts (8). Thus, $\underline{f}(x,u) \leq \omega(x)$ a.e. in $\mathbb{R}^N$. Similarly, we can obtain $\omega(x) \leq \bar{f}(x,u)$ a.e. in $\mathbb{R}^N$. Therefore, the proof is complete. \hfill \Box

It is easy to see that $\Phi(u) \in C^1(D^{s,p}(\mathbb{R}^N), \mathbb{R})$, this, together with Lemma 3.1, implies that $\mathcal{I}_\lambda$ is a locally Lipschitz functional. Thus, we have $\omega \in \partial \mathcal{I}_\lambda(u)$ if and only if there exists $\varpi(x) \in [\underline{f}(x,u), \bar{f}(x,u)]$ a.e. in $\mathbb{R}^N$ such that
\begin{equation}
\langle \omega, \varphi \rangle = (a + b\|u\|^{(\theta-1)p}) \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(|\varphi(x) - \varphi(y)|^{(\theta-1)p}) dx dy - \lambda \int_{\mathbb{R}^N} |u|^{p^*_s} u \varphi dx - \int_{\mathbb{R}^N} \varpi \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N)
\end{equation}

By the definition of $D^{s,p}(\mathbb{R}^N)$, we know that (9) holds for any $\varphi \in D^{s,p}(\mathbb{R}^N)$.

**Lemma 3.2.** Assume that $\theta < N/(N - sp)$, $a > 0$ and $b > 0$. Suppose $f$ satisfies $(f_2)$ and $(f_3)$. Then $\mathcal{I}_\lambda$ satisfies the nonsmooth $(PS)_c$ condition provided that $c < \lambda(1 - \frac{1}{\theta p} - \frac{1}{\theta p^*_s})^{p^*_s/(p^*_s - \theta p)}$.

**Proof.** Let $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ be such that $\mathcal{I}_\lambda(u_n) \to c$ and $m(u_n) = \min\{\|\omega\|_{X^*} : \omega \in \partial \mathcal{I}_\lambda(u_n)\} \to 0$ as $n \to \infty$. Here $X^* = (D^{s,p}(\mathbb{R}^N))^*$.

We first show that $\{u_n\}_n$ is bounded in $D^{s,p}(\mathbb{R}^N)$. Let $\omega_n \in \partial \mathcal{I}_\lambda(u_n)$ such that $\|\omega_n\|_{X^*} = m(u_n) = o(1)$. By (9), we have
\[
\langle \omega_n, u_n \rangle = a\|u_n\|^p + b\|u_n\|^{\theta p} - \lambda \int_{\mathbb{R}^N} |u_n|^{p^*_s} dx - \int_{\mathbb{R}^N} \varpi u_n dx,
\]

where $\varpi_n(x) \in [\underline{f}(x,u_n), \bar{f}(x,u_n)]$ a.e. in $\mathbb{R}^N$. From $(f_3)$, we get
\[
\varpi_n u_n \geq \sigma F(x,u_n).
\]

Hence, we deduce
\[
C(1 + \|u_n\|) \geq \mathcal{I}_\lambda(u_n) - \frac{1}{\sigma} \langle \omega_n, u_n \rangle \geq a \left( \frac{1}{p} - \frac{1}{\sigma} \right) \|u_n\|^p + b \left( \frac{1}{\theta p} - \frac{1}{\sigma} \right) \|u_n\|^{\theta p},
\]
which together with $\sigma > \theta p > p$ implies that $\{u_n\}_n$ is bounded in $D^{s,p}(\mathbb{R}^N)$. Thus, up to a subsequence, there exists a nonnegative function $u \in D^{s,p}(\mathbb{R}^N)$ such that $u_n \to u$ in $D^{s,p}(\mathbb{R}^N)$, $u_n \to u$ in $L^{p^*_s}_{\text{loc}}$ for $\sigma \in [1, p^*_s)$, and $u_n \to u$ a.e. in $\mathbb{R}^N$.

By Theorem 2.4, up to a subsequence, there exists a (at most) countable set $J$, a
Using the Hölder inequality and Lemma 2.5, we deduce
\[
\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dy \to \mu = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dy + \sum_{j \in J} \mu_j \delta_{x_j} + \tilde{\mu} \quad (10)
\]
and
\[
|u_n|^p \to \nu = |u|^p + \sum_{j \in J} \nu_j \delta_{x_j} \quad (11)
\]
in the measure sense, where \( \delta_{x_j} \) is the Dirac measure concentrated \( x_j \). Moreover,
\[
\nu_j \leq S^{-p^*_j/p} \mu_j^{p^*_j/p} \quad \forall j \in J,
\]
and \( S > 0 \) is the best constant of the embedding \( D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p^*_j}(\mathbb{R}^N) \).

Next we claim that \( J = \emptyset \). Suppose by contradiction that \( J \neq \emptyset \). Fix \( j \in J \). For \( \varepsilon > 0 \), choose \( \varphi_{\varepsilon,j} \in C_0^\infty(\mathbb{R}^N) \) such that
\[
\varphi_{\varepsilon,j} = 1 \text{ for } |x-x_j| \leq \varepsilon; \quad \varphi_{\varepsilon,j} = 0 \text{ for } |x-x_j| \geq 2\varepsilon,
\]
and \( |\nabla \varphi_{\varepsilon,j}| \leq 2/\varepsilon \). Obviously, \( \varphi_{\varepsilon,j} u_n \in D^{s,p}(\mathbb{R}^N) \). Replacing \( u_n \) with \( \varphi_{\varepsilon,j} u_n \) in \( \langle \omega_n, u_n \rangle \) we have
\[
\langle \omega_n, \varphi_{\varepsilon,j} u_n \rangle = \left( a + b \| u_n \|_{(\theta-1)p} \right) \langle \langle u_n, u_n \varphi_{\varepsilon,j} \rangle \rangle - \lambda \int_{\mathbb{R}^N} |u_n|^{p^*_j} \varphi_{\varepsilon,j} dx + \int_{\mathbb{R}^N} \nabla_n \varphi_{\varepsilon,j} u_n dx, \quad (13)
\]
where
\[
\langle \langle u_n, u_n \varphi_{\varepsilon,j} \rangle \rangle :=
\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p^*_j-2}(u_n(x) - u_n(y)) (\varphi_{\varepsilon,j}(x) u_n(x) - \varphi_{\varepsilon,j}(y) u_n(y))}{|x-y|^{N+ps}} dxdy.
\]
Using the Hölder inequality and Lemma 2.5, we deduce
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left| M \left( \| u_n \|^p \right) \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p^*_j-2}(u_n(x) - u_n(y)) (\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)) u_n(x)}{|x-y|^{N+ps}} dxdy \right| 
\leq C \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left( \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{(p-1)/p}
\times \left( \int_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)) u_n(x)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p}
\leq C \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left( \int_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y)) u_n(x)|^p}{|x-y|^{N+ps}} dxdy \right)^{1/p} = 0. \quad (14)
\]
By (10) and (11), we have
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \left( a + b \|u_n\|^{(\theta-1)p} \right) \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \varphi_{\varepsilon,j}(y) dy dx \\
\geq \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left( a \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \varphi_{\varepsilon,j}(y) dy dy \\
+ b \left( \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \varphi_{\varepsilon,j}(y) dy dx \right) \theta \right)
\geq b \mu^\theta_j,
\]
and
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{p^*_s} \varphi_{\varepsilon,j} dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |u|^{p^*_s} \varphi_{\varepsilon,j} dx + \nu_j = \nu_j.
\]
By (f_2) and \( u_n(x) \in \{f(x, u_n), f(x, u_n)\} \), we obtain
\[
|u_n(x)| \leq c_1(x) + c_2(x) |u_n|^{q-1} \text{ for a.e. } x \in \mathbb{R}^N.
\]
It follows from the boundedness of \( \{u_n\} \) that \( \{u_n\} \) is bounded in \( L^{p^*_s} (\mathbb{R}^N) \).
Hence there exists \( \varpi \in L^{p^*_s} (\mathbb{R}^N) \) such that
\[
\varpi_n \to \varpi \text{ weakly in } L^{p^*_s} (\mathbb{R}^N)
\]
and \( \varpi \in \{f(x, u), f(x, u)\} \). Thus,
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^N} \varpi \varphi_{\varepsilon,j} u_n dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \varpi \varphi_{\varepsilon,j} u dx = 0.
\]
It follows from (13)–(17) that
\[
\lambda \nu_j \geq b \mu^\theta_j.
\]
Combining this inequality with (12), we obtain
\[
\nu_j \geq \left( \frac{b \theta^\theta}{\lambda} \right)^{\frac{p^*_s}{p^*_s - p}}.
\]
For \( R > 0 \), assume \( \psi_R \in C^\infty_0 (\mathbb{R}^N) \) satisfies \( \psi_R \in [0, 1] \), \( \psi_R(x) = 1 \) for \( |x - x_j| \leq R \), \( \psi_R(x) = 0 \) for \( |x - x_j| > 2R \), and \( |\nabla \psi_j| \leq 2/R \). By (10) and (11), we obtain
\[
c = \lim_{n \to \infty} \left( I_\lambda (u_n) - \frac{1}{\theta p} \langle \omega_n, u_n \rangle \right) \\
\geq \lim_{R \to \infty} \lim_{n \to \infty} \left( \left( \frac{a}{p} - \frac{a}{\theta p} \right) \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} \psi_R(x) dy dx + \left( \frac{a}{p} - \frac{a}{\theta p} \right) \mu_j \\
+ \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \int_{\mathbb{R}^N} |u_n|^{p^*_s} \psi_R dx - \left( 1 - \frac{a}{p} \right) \int_{\mathbb{R}^N} F(x, u_n) dx \right) \\
\geq \lim_{R \to \infty} \left( \left( \frac{a}{p} - \frac{a}{\theta p} \right) \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \psi_R(x) dy dx + \left( \frac{a}{p} - \frac{a}{\theta p} \right) \mu_j \\
+ \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \int_{\mathbb{R}^N} |u|^{p^*_s} \psi_R dx + \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \nu_j \right) \\
\geq \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \nu_j.
\]
This together with (18) implies that
\[ c \geq \lambda \left( \frac{1}{\theta} - \frac{1}{p_*} \right) \left( \frac{bS^\theta}{\lambda} \right)^{\frac{1}{p_*-p}}. \]
which is a contradiction. Hence the claim holds.

Let \( R > 0 \), we define
\[
\mu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\{x \in \mathbb{R}^N : |x| > R\}} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} dy dx,
\]
and
\[
\nu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\{x \in \mathbb{R}^N : |x| > R\}} |u_n|^{p_*} dx.
\]
It follows from Theorem 2.6 that \( \mu_\infty \) and \( \nu_\infty \) are well defined and satisfy
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}} dy dx = \int_{\mathbb{R}^N} d\mu + \mu_\infty, \tag{19}
\]
and
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{p_*} dx = \int_{\mathbb{R}^N} d\nu + \nu_\infty. \tag{20}
\]
Assume that \( \chi_R \in C^\infty(\mathbb{R}^N) \) satisfies \( \chi_R \in [0, 1] \) and \( \chi_R(x) = 0 \) for \( |x| < R \), \( \chi_R(x) = 1 \) for \( |x| > 2R \), and \( |\nabla \chi_R| \leq 2/R \). By Theorem 2.6, we have
\[
\mu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p|\chi_R(x)p|}{|x - y|^{N+ps}} dy dx \tag{21}
\]
and
\[
\nu_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n(x)\chi_R(x)|^{p_*} dx. \tag{22}
\]
Moreover, we have
\[
S_{\mu_\infty}^{p/p_*} \leq \mu_\infty. \tag{23}
\]
Since \( ||u_n||^p \) and \( |u_n|^{p_*} \) are bounded, up to a subsequence, we can assume that \( ||u_n||^p \) and \( |u_n|^{p_*} \) are both convergent. Then by (19) and (20), we obtain
\[
\lim_{n \to \infty} ||u_n||^p = \int_{\mathbb{R}^N} d\mu + \mu_\infty \tag{24}
\]
and
\[
\lim_{n \to \infty} |u_n|^{p_*} = \int_{\mathbb{R}^N} d\nu + \nu_\infty. \tag{25}
\]
It follows from \( \langle \omega_n, \chi_R u_n \rangle \to 0 \) as \( n \to \infty \) that
\[
\begin{align*}
&\left( a + b||u_n||^{(p-1)p} \right) \left[ \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p|\chi_R(x)p|}{|x - y|^{N+ps}} dxdy \\
&+ \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))u_n(y)(\chi_R(x) - \chi_R(y))}{|x - y|^{N+ps}} dxdy \right] \\
&= \lambda \int_{\mathbb{R}^N} |u_n|^{p_*} \chi_R dx + \int_{\mathbb{R}^N} \omega_n u_n \chi_R dx + o(1). \tag{26}
\end{align*}
\]
Using a similar discussion as in [50], we get
\[
\lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))u_n(y)(\chi_R(x) - \chi_R(y))}{|x-y|^{N+ps}} \, dx \, dy = 0.
\]
Hence we deduce from (21), (24), (26) and (27) that
\[
\lim_{R \to \infty} \limsup_{n \to \infty} \left( a + b \|u_n\|^{(\theta-1)p} \right) \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p \chi_R(x)}{|x-y|^{N+ps}} \, dx \, dy
\leq \left( a + b \mu_\infty \right) \mu_\infty
\leq b \mu_\infty, \tag{28}
\]
thanks to the fact that \( \theta > 1 \). By (f2) and \( \varpi_n \in [\mathcal{F}(x, u_n), \mathcal{F}(x, u_n)] \), it is easy to see that
\[
\lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N} \varpi_n \chi_R u_n \, dx = \lim_{R \to \infty} \int_{\mathbb{R}^N} \varpi \chi_R u \, dx = 0. \tag{29}
\]
Therefore, we conclude from (26)–(29) and (22) that \( b \mu_\infty \leq \lambda \nu_\infty \), which together with (23) yields \( \nu_\infty = 0 \) or
\[
\nu_\infty \geq \frac{bS^\theta}{\lambda^{\frac{p^*_s}{p^*_s - p}}} \tag{30}
\]
Assume that (30) holds. Then
\[
c = \lim_{n \to \infty} \left( I_\lambda(u_n) - \frac{1}{\theta p} \langle \omega_n, u_n \rangle \right)
\geq \lim_{n \to \infty} \left( \left( \frac{a}{p} - \frac{a}{\theta p} \right) \|u_n\|^p + \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \int_{\mathbb{R}^N} |u_n|^{p^*_s} \, dx - \left( 1 - \frac{\sigma}{\theta p} \right) \int_{\mathbb{R}^N} F(x, u_n) \, dx \right)
\geq \left( \frac{a}{p} - \frac{a}{\theta p} \right) \int_{\mathbb{R}^N} d\mu + \left( \frac{a}{p} - \frac{a}{\theta p} \right) \mu_\infty + \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \int_{\mathbb{R}^N} d\nu + \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \nu_\infty
\geq \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \nu_\infty.
\]
It follows from (30) that
\[
c \geq \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \left( bS^\theta \right)^{\frac{p^*_s}{p^*_s - p}} \tag{31}
\]
which is a contradiction. Hence \( \nu_\infty = 0 \). In view of \( J = \emptyset \) and (25), we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{p^*_s} \, dx = \int_{\mathbb{R}^N} |u|^{p^*_s} \, dx.
\]
Now we show that $u_n \to u$ in $D^{s,p}(\mathbb{R}^N)$. To this aim, we first assume that $d := \inf_{n \geq 1} \|u_n\| > 0$.

For any $w, v \in D^{s,p}(\mathbb{R}^N)$, we define

$$[w, v] = \int_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+ps}} dxdy.$$ 

Since $(\omega_n, u_n - u) \to 0$ as $n \to \infty$, we have

$$
\left(a + b\|u_n\|^{(\theta-1)p}\right) [u_n, u_n - u] - \lambda \int_{\mathbb{R}^N} |u_n|^{p^*_s - 2} u_n (u_n - u) dx
- \int_{\mathbb{R}^N} \nabla_n (u_n - u) dx = o(1). \tag{32}
$$

Thus,

$$
\left(a + b\|u_n\|^{(\theta-1)p}\right) ([u_n, u_n - u] - [u, u_n - u]) + \left(a + b\|u_n\|^{(\theta-1)p}\right) [u, u_n - u]
= \int_{\mathbb{R}^N} \left(|u_n|^{p^*_s - 2} u_n - |u|^{p^*_s - 2} u\right) (u_n - u) dx + \int_{\mathbb{R}^N} \nabla_n (u_n - u) dx + o(1).
$$

Because $\{u_n\}$ is bounded and $u_n \to u$ in $D^{s,p}(\mathbb{R}^N)$, we deduce

$$
\lim_{n \to \infty} \left(a + b\|u_n\|^{(\theta-1)p}\right) [u, u_n - u] = 0. \tag{33}
$$

From $(f_2)$, we have

$$
\left|\int_{\mathbb{R}^N} \nabla_n (u_n - u) dx\right| \leq \int_{\mathbb{R}^N} c_1(x) |u_n - u| dx + c_2(x) |u_n - u|^q dx.
$$

Note that for each measurable subset $A \subset \mathbb{R}^N$ there holds

$$
\int_A c_1(x) |u_n - u| dx \leq \left(\int_A c_1(x) |u_n|^{p^*_s} dx\right)^{\frac{p^*_s - 1}{p^*_s}} \|u_n\|_{p^*_s} \leq C \left(\int_A c_1(x) |u_n|^{p^*_s} dx\right)^{\frac{p^*_s - 1}{p^*_s}},
$$

which yields that $\{c_1(x) |u_n - u|\}$ is equi-integrable in $\mathbb{R}^N$. By $u_n \to u$ a.e. in $\mathbb{R}^N$, one has $c_1(x) |u_n - u| \to 0$ a.e. in $\mathbb{R}^N$. Thus, Vitali’s convergence theorem yields that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} c_1(x) |u_n - u| dx = 0.
$$

Similarly, since $c_2(x)$ is in $L^{\frac{p^*_s}{p^*_s - 1}}(\mathbb{R}^N)$, one can obtain

$$
\lim_{n \to \infty} \lim_{m \to \infty} \int_{\mathbb{R}^N} c_2(x) |u_n - u|^q dx = 0.
$$

Hence, we get

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} \nabla_n (u_n - u) dx = 0. \tag{34}
$$

A similar discussion gives that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{p^*_s - 2} u_n u dx = \int_{\mathbb{R}^N} |u|^{p^*_s} dx. \tag{35}
$$
Therefore, we conclude from (31) and (32)–(35) that
\[
\lim_{n \to \infty} \left( a + b\|u_n\|^{(\theta-1)p} \right) (\|u_n\| - \|u\| + \|u_n - u\|) = 0,
\]
this together with \( d := \inf_{n \geq 1} \|u_n\| > 0 \) and \( b > 0 \) implies that
\[
\lim_{n \to \infty} (\|u_n\| - \|u\| + \|u_n - u\|) = 0. \tag{36}
\]

Let us now recall the well-known Simon inequalities:
\[
|\xi - \eta|^p \leq \begin{cases} 
2^{p-2} \left( |\xi|^{p-2} - |\eta|^{p-2} \right) \cdot (\xi - \eta) & \text{for } p \geq 2 \\
C_p'' \left( |\xi|^{p-2} - |\eta|^{p-2} \right) \cdot (\xi - \eta)^{p/2} & \text{for } 1 < p < 2 
\end{cases}
\tag{37}
\]
for all \( \xi, \eta \in \mathbb{R}^N \), where \( C_p' \) and \( C_p'' \) are positive constants depending only on \( p \).

If \( p > 2 \), then it follows from (37) that
\[
\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y) - u(x) + u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \leq C_p' \left( \|u_n - u\| - |u_n - u| \right) \to 0,
\]
as \( n \to \infty \). Hence \( u_n \to u \) in \( D^{s,p}(\mathbb{R}^N) \).

It remains to consider the case \( 1 < p < 2 \). To this aim, from (37) we have
\[
\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y) - u(x) + u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \\
\leq C_p'' \left( \|u_n - u\| - |u_n - u| \right)^{p/2} \\
\times \left\{ \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y) + u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right\}^{(2-p)/2} \\
\leq C \left( \|u_n - u\| - |u_n - u| \right)^{p/2} \\n\rightarrow 0,
\]
as \( n \to \infty \). Hence \( u_n \to u \) in \( D^{s,p}(\mathbb{R}^N) \). In conclusion, we get \( u_n \to u \) strongly in \( D^{s,p}(\mathbb{R}^N) \) as \( n \to \infty \).

Finally, we consider the case \( \inf_n \|u_n\|_{D^{s,p}(\mathbb{R}^N)} = 0 \). Then either 0 is an accumulation point of the sequence \( \{u_n\}_n \) and so there exists a subsequence of \( \{u_n\}_n \) strongly converging to \( u = 0 \), or 0 is an isolated point of the sequence \( \{u_n\}_n \) and so there exists a subsequence, still denoted by \( \{u_n\}_n \), such that \( \inf_n \|u_n\| > 0 \). In the first case we are done, while in the latter case we can processed as above. \( \square \)

**Lemma 3.3.** Assume that \((f_2)\) and (5) hold. Then for any \( \lambda \in (0, 1) \) there exist \( \alpha \) and \( \rho > 0 \) such that \( \mathcal{I}_\lambda(u) \geq \alpha \) for all \( u \in D^{s,p}(\mathbb{R}^N) \) with \( \|u\| = \rho \).

**Proof.** For any \( u \in D^{s,p}(\mathbb{R}^N) \), by \((f_2)\) and the fractional Sobolev inequality, one has
\[
\mathcal{I}_\lambda(u) = \frac{a}{p} \|u\|^p + \frac{b}{\theta p} \|u\|^{\theta p} - \frac{\lambda}{p_s} \int_{\mathbb{R}^N} |u|^p \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx \\
\geq \frac{b}{\theta p} \|u\|^{\theta p} - \frac{\lambda}{p_s} S^{-\frac{p_s}{p_s - \theta}} \|u\|^{p_s} - \|c_1\| \frac{p_s}{p_s - \theta} \|u\|^{p_s} - \|c_2\| \frac{p_s}{p_s - \theta} \|u\|^{q_p} \\
\geq \frac{b}{\theta p} \|u\|^{\theta p} - \frac{1}{p_s} S^{-\frac{p_s}{p_s - \theta}} \|u\|^{p_s} - \|c_1\| \frac{p_s}{p_s - \theta} S^{-\frac{p_s}{p_s - \theta}} \|u\|^q - \|c_2\| \frac{p_s}{p_s - \theta} S^{-\frac{p_s}{p_s - \theta}} \|u\|^q, \tag{38}
\]
thanks to the fact that $\lambda < 1$. By Young’s inequality, for any $\varepsilon > 0$ we obtain

$$|c_1| \frac{p^*_s}{p^*_s - 1} S^{-\frac{q}{p}} \|u\| \leq \varepsilon \frac{\|u\|^\theta p}{\theta p} + \varepsilon^{-\frac{1}{\theta p - 1}} \left( S^{-\frac{q}{p}} |c_1| \frac{p^*_s}{p^*_s - 1} \right)^\frac{\theta p}{\theta p - 1}.$$ 

Putting this inequality with $\varepsilon = b/2$ into (38), we deduce

$$I_\lambda(u) \geq \frac{b}{2\theta p} \|u\|^\theta p - \frac{1}{p^*_s} S^{-\frac{q}{p}} \|u\|^p S^{-\frac{q}{p}} \|u\|^q - |c_2| \frac{p^*_s}{p^*_s - 1} \|\|u\|^q.$$ 

Define $g(t) := \frac{b}{2\theta p} t^{\theta p} - \frac{1}{p^*_s} S^{-\frac{q}{p}} t^{p} - |c_2| \frac{p^*_s}{p^*_s - 1} S^{-\frac{q}{p}} t^q$ for all $t \geq 0$. Since $p^*_s > \theta p$ by $\theta < N/(N - sp)$ and $q > \theta p$, we deduce that there exists $\rho > 0$ such that $\max_{t \geq 0} g(t) = g(\rho) > 0$. Then for any $\lambda \in (0, 1)$, by (5) we get

$$I_\lambda(u) \geq g(\rho) - \frac{b}{2} \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \left( S^{-\frac{q}{p}} |c_1| \frac{p^*_s}{p^*_s - 1} \right)^\frac{\theta p}{\theta p - 1}.$$ 

Thus we complete the proof.

Put $\varphi \in C_0^\infty(\Omega)$ such that $\|\varphi\| = 1$ and $\int_\Omega |\varphi|^q_1 dx > 0$. Then, we have

**Lemma 3.4.** If $(f_4)$ is satisfied. Then there exist $\lambda_0 \in (0, 1)$ and $t_0 > 0$ such that $I_\lambda(t_0 \varphi) < 0$ for all $\lambda \in (0, \lambda_0)$ and $\|t_0 \varphi\| > \rho$, where $\rho$ is the number given in Lemma 3.3. Moreover, for any $\lambda \in (0, \lambda_0)$, we have

$$\sup_{t \geq 0} I_\lambda(t \varphi) < \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \left( \frac{bs^\theta}{\lambda} \right)^{\frac{p^*_s}{p^*_s - 1} \theta p}.$$ 

**Proof.** By $(f_4)$, we get

$$F(x, t) \geq C_0 |t|^{q_1} \text{ for all } (x, t) \in \Omega \times \mathbb{R}.$$ 

Then, for any $t \geq 1$,

$$I_\lambda(t \varphi) = \frac{a}{p} t^p + \frac{b}{\theta p} t^{\theta p} - \lambda \frac{t^{p^*_s}}{p^*_s} \int_{\mathbb{R}^N} |\varphi|^{p^*_s} dx - \int_{\mathbb{R}^N} F(x, t \varphi) dx$$

$$\leq \frac{a}{p} t^p + \frac{b}{\theta p} t^{\theta p} - C_0 \int_{\Omega} |t \varphi|^{q_1} dx.$$ 

Since $q_1 > \theta p > p$ and $\int_\Omega |\varphi|^{q_1} dx > 0$, there exists $t_0 \geq 1$ sufficiently large such that $I_\lambda(t_0 \varphi) < 0$ and $\|t_0 \varphi\| > \rho$, where $\rho$ is given by Lemma 3.3.

Define $J(t) := \frac{a}{p} t^p + \frac{b}{\theta p} t^{\theta p} - C_0 t^{q_1} \int_{\Omega} |\varphi|^{q_1} dx$. Then $I_\lambda(t \varphi) \leq J(t)$. Hence, we obtain

$$\sup_{t \geq 0} I_\lambda(t \varphi) \leq \sup_{t \geq 0} J(t).$$

It is easy to verify that there exists $t_1 > 0$ such that $\sup_{t \geq 0} J(t) = J(t_1) \in (0, \infty)$, since $q_1 > \theta p \geq p$. Thus, there exists $\lambda_0 \in (0, 1)$ such that

$$\sup_{t \geq 0} J(t) < \lambda_0 \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \left( \frac{bs^\theta}{\lambda_0} \right)^{\frac{p^*_s}{p^*_s - 1} \theta p}.$$
Therefore, for any \( \lambda \in (0, \lambda_0) \), we can deduce the desired conclusion. This completes the proof. \( \square \)

**Proof of Theorem 1.2.** Obviously, \( I_{\lambda}(0) = 0 \). By Lemmas 3.2–3.4 and Theorem 2.1, there exists \( \lambda_0 \in (0, 1) \), for all \( \lambda \in (0, \lambda_0) \), we can find a \( u \in D^{s,p}(\mathbb{R}^N) \) such that \( I_{\lambda}(u) = c > 0 \) and \( 0 \in \partial I_{\lambda}(u) \). Thus, \( u \) is a nontrivial solution of (1). That is, there exists \( \varpi(x) \in [f(x, u, \mathcal{F}(x, u)) \) such that

\[
(a + b\|u\|^{(q-1)p}) \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+ps}} dxdy = \lambda \int_{\mathbb{R}^N} |u|^{ps-2}u \varphi dx + \int_{\mathbb{R}^N} \varpi \varphi dx, \ \forall \varphi \in D^{s,p}(\mathbb{R}^N),
\]

for all \( \varphi \in D^{s,p}(\mathbb{R}^N) \).

Next we show that \( u \) is a nonnegative solution. Taking \( \varphi = -u^- \) in (39) and using assumption \((f_1)\), we have

\[
(a + b\|u\|^{(q-1)p}) \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) (-u^- (x) + u^- (y))}{|x - y|^{N+ps}} dxdy = \lambda \int_{\mathbb{R}^N} |u^-|^{ps} dx.
\]

From above identity together with the following facts:

\[
|\xi^- - \eta^-|^p \leq |\xi - \eta|^p - (\xi^- + \eta^-)
\]

and \( |\xi^- - \eta^-| \leq |\xi - \eta| \), for all \( \xi, \eta \in \mathbb{R} \), we obtain

\[
(a + b\|u^-\|^{(q-1)p}) \int_{\mathbb{R}^{2N}} \frac{|u^- (x) - u^- (y)|^p}{|x - y|^{N+ps}} dxdy \leq \lambda \int_{\mathbb{R}^N} |u^-|^{ps} dx. \tag{40}
\]

In view of the definition of \( S \), we deduce

\[
(a + b\|u^-\|^{(q-1)p}) \int_{\mathbb{R}^{2N}} \frac{|u^- (x) - u^- (y)|^p}{|x - y|^{N+ps}} dxdy \leq \lambda S^{-\frac{ps}{p'}} \|u^-\|^{ps}.
\]

Moreover, by \( a \geq 0 \) and \( b > 0 \), we get

\[
b\|u^-\|^p \leq \lambda S^{-\frac{ps}{p'}} \|u^-\|^{ps},
\]

from which it is easy to see that

\[
\|u^-\| \geq \left( \frac{b}{\lambda S^{-\frac{ps}{p'}}} \right)^{\frac{1}{1-rac{ps}{p}}}.
\]

On the other hand, by (39) one has

\[
a\|u\|^p + b\|u\|^p = \lambda \int_{\mathbb{R}^N} |u|^{ps} dx + \int_{\mathbb{R}^N} \varpi(x) u dx.
\]

It follows from this equality and \((f_3)\) that

\[
c = I_{\lambda}(u) = a \left( \frac{1}{p} - \frac{1}{\sigma} \right) \|u\|^p + b \left( \frac{1}{\theta p} - \frac{1}{\sigma} \right) \|u\|^{\theta p} + \lambda \left( \frac{1}{\sigma} - \frac{1}{p^*_s} \right) \int_{\mathbb{R}^N} |u|^{ps} dx + \int_{\mathbb{R}^N} \left[ \frac{1}{\theta} \varpi(x) u - F(x, u) \right] dx \geq b \left( \frac{1}{\theta p} - \frac{1}{\sigma} \right) \|u\|^{\theta p} + \lambda \left( \frac{1}{\sigma} - \frac{1}{p^*_s} \right) \int_{\mathbb{R}^N} |u|^{ps} dx. \tag{41}
\]
Notice that
\[ b\|u^−\|^{\theta p} \leq \lambda \int_{\mathbb{R}^N} |u^−|^{p^*_s} dx, \]
due to (40). Then by (41) we obtain
\[ c \geq b \left( \frac{1}{\theta p} - \frac{1}{\sigma} \right) \|u\|^{\theta p} + \left( \frac{1}{\sigma} - \frac{1}{p^*_s} \right) b\|u\|^{\theta p} \]
\[ \geq b \left( \frac{1}{\theta p} - \frac{1}{\sigma} \right) \|u^−\|^{\theta p} + \left( \frac{1}{\sigma} - \frac{1}{p^*_s} \right) b\|u^−\|^{\theta p} \]
\[ = \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \frac{bS^0}{\lambda} \theta p, \]
which contradicts the fact that \( c < \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \left( \frac{bS^0}{\lambda} \right)^{p^*_s/(p^*_s - \theta p)} \). Hence, \( u^− = 0 \).
This means that \( u \) is a nontrivial nonnegative solution of (1).

Proof of Theorem 1.3. Let \( f(x, u) = h(x)H(u - \Theta)|u|^{q-2}u \). Then \( f \) has only one discontinuity point \( \Theta \). Thus, by Theorem 1.2, we obtain that \( u \in D^{s,p}(\mathbb{R}^N) \) is a nontrivial nonnegative solution of problem (6).

Let \( X = D^{s,p}(\mathbb{R}^N) \). Since \( X \) is a separable and reflexive Banach space, there exist \( \{e_n\}_{n=1}^{\infty} \subset X \) and \( \{e^*_m\}_{m=1}^{\infty} \subset X^* \) such that
\[ \langle e^*_m, e_n \rangle = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m, \end{cases} \]
and
\[ X = \text{span}\{e_n : n = 1, 2, \cdots\}, \quad X^* = \text{span}\{e^*_m : m = 1, 2, \cdots\}. \]
For \( k = 1, 2, \cdots \), we set
\[ X_k = \text{span}\{e_k\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \bigoplus_{j=k}^{\infty} X_j. \]

Proof of Theorem 1.4. It follows from \( (f_5) \) and Lemma 3.1 that \( \mathcal{I}_\lambda \) is an even locally Lipschitz functional. By Lemma 3.2, we know that \( \mathcal{I}_\lambda \) satisfies the nonsmooth \((PS)_c\) condition for any \( c < \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \left( \frac{bS^0}{\lambda} \right)^{p^*_s/(p^*_s - \theta p)} \). We take \( X^- = Y_k \) and \( X^+ = X \), by Lemma 3.3, there exist \( \rho, \alpha > 0 \) such that \( \mathcal{I}_\lambda(u) \geq \alpha \) for all \( u \in X^+ \) with \( \|u\| = \rho \). Using a similar discussion as in Lemma 3.4, we obtain
\[ \mathcal{I}_\lambda(u) \to -\infty \quad \text{as } \|u\| \to \infty, \quad u \in X^-, \]
and there exists \( \Lambda_k \in (0, 1) \) such that for all \( \lambda \in (0, \Lambda_k) \)
\[ c_k = \inf_{h \in \Gamma} \sup_{u \in Y_k} \mathcal{I}_\lambda(h(u)) < \lambda \left( \frac{1}{\theta p} - \frac{1}{p^*_s} \right) \left( \frac{bS^0}{\lambda} \right)^{p^*_s/(p^*_s - \theta p)}, \]
where
\[ \Gamma = \{ h \in C(X, X) : h \text{ is odd}, h(u) = u \text{ if } u \in X^- \text{ and } \|u\| \geq R \}. \]
Obviously, \( \mathcal{I}_\lambda(0) = 0 \). Thus, by Theorem 2.6, we obtain that problem (1) has \( k \) pairs of nontrivial solutions. Therefore, the proof is complete.
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