AN INTERSECTION FUNCTIONAL ON THE SPACE OF SUBSET CURRENTS ON A FREE GROUP

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Abstract. Kapovich and Nagnibeda introduced the space $\mathcal{SCurr}(F_N)$ of subset currents on a free group $F_N$ of rank $N \geq 2$, which can be thought of as a measure-theoretic completion of the set of all conjugacy classes of finitely generated subgroups of $F_N$. We define a product $\mathcal{N}(H, K)$ of two finitely generated subgroups $H$ and $K$ of $F_N$ by the sum of the reduced rank $\text{rk}(H \cap gKg^{-1})$ over all double cosets $HgK$ ($g \in F_N$), and extend the product $\mathcal{N}$ to a continuous symmetric $\mathbb{R}_{\geq 0}$-bilinear functional $\mathcal{N} : \mathcal{SCurr}(F_N) \times \mathcal{SCurr}(F_N) \to \mathbb{R}_{\geq 0}$. We also give an answer to a question presented by Kapovich and Nagnibeda. The definition of $\mathcal{N}$ originates in the Strengthened Hanna Neumann Conjecture, which has been proven by Mineyev and can be stated as follows: $\mathcal{N}(H, K) \leq \text{rk}(H)\text{rk}(K)$ holds for any finitely generated subgroups $H$ and $K$ of $F_N$. As a corollary to our theorem, this inequality is generalized to the inequality for subset currents.

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1. INTRODUCTION

In \cite{KN13} Kapovich and Nagnibeda introduced the space $\mathcal{SCurr}(F_N)$ of subset currents on a free group $F_N$ of rank $N \geq 2$ as an analogy of the space of geodesic currents on $F_N$ (see \cite{Kap06}). A subset current on $F_N$ is a positive $F_N$-invariant locally finite Borel measure on the space $\mathcal{C}_N$ of all closed subsets of the hyperbolic boundary $\partial F_N$ consisting of at least two points, where we endow $\mathcal{C}_N$ with the subspace topology of the Vietoris topology on the hyperspace of $\partial F_N$. The space $\mathcal{SCurr}(F_N)$ is equipped with the weak-* topology and with the $\mathbb{R}_{\geq 0}$-linear structure.

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Kapovich and Nagnibeda defined the counting subset current \( \eta_H \in \mathcal{SCurr}(F_N) \) for a nontrivial finitely generated subgroup \( H \leq F_N \) and proved that
\[
\{ c\eta_H \mid c \geq 0, H \text{ is a nontrivial finitely generated subgroup of } F_N \}
\]
is a dense subset of \( \mathcal{SCurr}(F_N) \). Counting subset currents have the following properties. For a nontrivial finitely generated subgroup \( H \leq F_N \) and a finite index subgroup \( H' \) of \( H \), we have \( \eta_{H'} = [H : H']\eta_H \). If two nontrivial finitely generated subgroups \( H, H' \leq F_N \) are conjugate, then \( \eta_H = \eta_{H'} \). For the trivial subgroup \( H = \{ \text{id} \} \) of \( F_N \) we set \( \eta_H = 0 \in \mathcal{SCurr}(F_N) \).

For a finitely generated free group \( F \) the reduced rank \( \overline{rk}(F) \) is defined as
\[
\overline{rk}(F) := \max\{ \text{rank}(F) - 1, 0 \},
\]
where rank\((F)\) is the cardinality of a free basis of \( F \). Let \( \Delta \) be a finite connected graph whose fundamental group is isomorphic to \( F \). Then we have
\[
\overline{rk}(F) = \max\{-\chi(\Delta), 0\},
\]
where \( \chi(\Delta) \) is the Euler characteristic of \( \Delta \). Note that for a finite index subgroup \( H \leq F \) we can obtain an equation
\[
\overline{rk}(H) = [F : H] \overline{rk}(F),
\]
which follows from a covering space argument. Kapovich and Nagnibeda [KN13] proved that there exists a unique continuous \( \mathbb{R}_{\geq 0} \)-linear functional
\[
\overline{rk}: \mathcal{SCurr}(F_N) \to \mathbb{R}_{\geq 0}
\]
such that for every finitely generated subgroup \( H \leq F_N \) we have
\[
\overline{rk}(\eta_H) = \overline{rk}(H).
\]
The map \( \overline{rk} \) is called the reduced rank functional.

The action of the automorphism group \( \text{Aut}(F_N) \) of \( F_N \) on \( \partial F_N \) induces a continuous and \( \mathbb{R}_{\geq 0} \)-linear action on \( \mathcal{SCurr}(F_N) \), and for \( \varphi \in \text{Aut}(F_N) \) and a finitely generated subgroup \( H \leq F_N \) we have \( \varphi\eta_H = \eta_{\varphi(H)} \). Since subset currents are \( F_N \)-invariant, the action of \( \text{Aut}(F_N) \) factors through the action of the outer automorphism group \( \text{Out}(F_N) \) on \( \mathcal{SCurr}(F_N) \). We see that \( \overline{rk} \) is \( \text{Out}(F_N) \)-invariant.

Define a product \( \mathcal{N}(H, K) \) of two finitely generated subgroups \( H, K \leq F_N \) as
\[
\mathcal{N}(H, K) := \sum_{HgK \in H \setminus F_N/K} \overline{rk}(H \cap gKg^{-1}),
\]
where \( H \setminus F_N/K \) is the set of all double cosets \( HgK \) (\( g \in F_N \)). Note that \( H \cap gKg^{-1} \neq \{ \text{id} \} \) for only finitely many double cosets \( HgK \). This definition originates in the Strengthened Hanna Neumann Conjecture, which has been proven by Mineyev [Min12]. By using the product \( \mathcal{N} \) it can be stated as follows:
\[
\mathcal{N}(H, K) \leq \overline{rk}(H)\overline{rk}(K)
\]
is satisfied for any finitely generated subgroups \( H, K \leq F_N \). The product \( \mathcal{N}(H, K) \) is closely related to a fiber product graph corresponding to \( H, K \), that is, each non-zero term of the sum in \( \mathcal{N}(H, K) \) is corresponding to a non-contractible connected component of the fiber product graph (see Section 3). Using the description by the fiber product graph, we can easily see that \( \mathcal{N} \) has the following property: if \( H' \) and \( K' \) are finite index subgroups of \( H \) and \( K \) respectively, then we have
\[
\mathcal{N}(H', K') = [H : H'][K : K']\mathcal{N}(H, K).
\]
Therefore it is natural to ask whether $N$ extends to a continuous $\mathbb{R}_{\geq 0}$-bilinear functional on $S\text{Curr}(F_N)$.

From the $\mathbb{R}_{\geq 0}$-linearity of the reduced rank functional, we have

$$N(H, K) = \sum_{HgK\in H\setminus F_N/K} \text{rk}(H \cap gKg^{-1})$$

$$= \text{rk} \left( \sum_{HgK\in H\setminus F_N/K} \eta_{H\cap gKg^{-1}} \right).$$

Kapovich and Nagnibeda [KN13] asked whether there exists a continuous $\mathbb{R}_{\geq 0}$-bilinear map

$$\cdot: \mathbb{S} \text{Curr}(F_N) \times \mathbb{S} \text{Curr}(F_N) \to \mathbb{S} \text{Curr}(F_N)$$

such that for any finitely generated subgroups $H, K \leq F_N$ we have

$$\cdot(\eta_H, \eta_K) = \sum_{HgK\in H\setminus F_N/K} \eta_{H\cap gKg^{-1}}.$$ 

If such a map $\cdot$ exists, then we immediately see that the product $N$ is extended to a continuous $\mathbb{R}_{\geq 0}$-bilinear functional $N: \mathbb{S} \text{Curr}(F_N) \times \mathbb{S} \text{Curr}(F_N) \to \mathbb{R}_{\geq 0}$. Moreover, the map $\cdot$ can be considered as a measure theoretical generalization of the fiber product graph. However, we prove that the map $\cdot$ cannot be continuous due to the requirements on $\cdot$ (see Proposition 3.3). Nevertheless, we can establish the following theorem by a different approach.

**Theorem 3.2.** There exists a unique continuous symmetric $\mathbb{R}_{\geq 0}$-bilinear functional $N: \mathbb{S} \text{Curr}(F_N) \times \mathbb{S} \text{Curr}(F_N) \to \mathbb{R}_{\geq 0}$ such that for any finitely generated subgroups $H, K \leq F_N$ we have

$$N(\eta_H, \eta_K) = N(H, K).$$

Moreover, $N$ is $\text{Out}(F_N)$-invariant.

We call $N$ the **intersection functional**. Our strategy of proving Theorem 3.2 is based on the proof of the existence of the reduced rank functional $\text{rk}$. We construct $N$ by a careful usage of occurrences, which were introduced in [KN13].

As a corollary to our theorem, the inequality in the Strengthened Hanna Neumann Conjecture is generalized to the inequality for subset currents:

**Corollary 3.12.** Let $\mu, \nu \in \mathbb{S} \text{Curr}(F_N)$. The following inequality holds:

$$N(\mu, \nu) \leq \text{rk}(\mu) \text{rk}(\nu).$$

Since $N(F_N, H) = \text{rk}(H)$ for every finitely generated subgroup $H \leq F_N$, the intersection functional $N$ is an extension of the reduced rank functional $\text{rk}$.

Inspired by the question presented by Kapovich and Nagnibeda, we also prove the following theorems.

**Theorems 4.1 and 4.2.** Let $I$ be the intersection map:

$$C_N \times C_N \to \{\text{closed subsets of } \partial F_N\}; \ (S_1, S_2) \mapsto S_1 \cap S_2.$$ 

For $\mu, \nu \in \mathbb{S} \text{Curr}(F_N)$ we can obtain a subset current $\widehat{I}(\mu, \nu)$ by defining

$$\widehat{I}(\mu, \nu)(U) := \mu \times \nu(I^{-1}(U)).$$
for any Borel subset $U \subset \mathcal{C}_N$. Then the map $\hat{I}$ is a non-continuous $\mathbb{R}_{\geq 0}$-bilinear map

$$\hat{I} : \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \text{SCurr}(F_N)$$

and $\hat{I}$ satisfies the conditions that

$$\hat{I}(\eta_H, \eta_K) = \sum_{HgK \in H \setminus F_N / K} \eta_{H \cap gKg^{-1}}$$

for any finitely generated subgroups $H, K \leq F_N$, and that

$$\text{rk} \circ \hat{I} = \mathcal{N}.$$

**Organization of this paper.** In Section 2 we set up notation and summarize without proofs some important properties on subset currents in [13]. We also recall some tools and methods used in the proof of the existence of the reduced rank functional $\text{rk}$. In Section 3, first, we give an answer to the question posed in [13], and construct the intersection functional $\mathcal{N}$. In Section 4 we represent $\mathcal{N}$ by using the intersection map $I$ and the reduced rank functional $\text{rk}$. 

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2. Preliminaries

2.1. Conventions regarding graphs and free groups. (cf. [13], Subsection 2.1 and 2.2) A graph is a 0 or 1-dimensional CW complex. The set of 0-cells of a graph $\Delta$ is denoted by $V(\Delta)$ and its elements are called vertices of $\Delta$. The set of (closed) 1-cells of a graph $\Delta$ is denoted by $E_{\text{top}}(\Delta)$ and its elements are called topological edges. The interior of every topological edge is homeomorphic to the interval $(0, 1) \subset \mathbb{R}$ and thus admits exactly two orientations. A topological edge endowed with an orientation on its interior is called an oriented edge of $\Delta$. The set of all oriented edges of $\Delta$ is denoted by $E(\Delta)$. For every oriented edge $e$ of $\Delta$ there are naturally defined (and not necessarily distinct) vertices $o(e) \in V(\Delta)$, called the origin of $e$, and $t(e) \in V(\Delta)$, called the terminal of $e$. Then the boundary of $e$ is $\{o(e), t(e)\}$. For an oriented edge $e \in E(\Delta)$ changing its orientation to the opposite one produces another oriented edge of $\Delta$ denoted by $e^{-1}$ and called the inverse of $e$. For a graph $\Delta$ giving an orientation to $\Delta$ is fixing an orientation of every topological edge of $\Delta$.

Let $\Delta$ be a graph and $v \in V(\Delta)$. The degree of $v$ in $\Delta$ is the number of oriented edges whose origin is $v$.

For a graph $\Delta$, we always give the path metric $d_\Delta$ to $\Delta$, where the length of each topological edge of $\Delta$ is 1.

A graph morphism $f : \Delta \to \Delta'$ is a continuous map from a graph $\Delta$ to a graph $\Delta'$ that maps the vertices of $\Delta$ to vertices of $\Delta'$ and such that each interior of an edge of $\Delta$ is mapped isometrically to an interior of an edge of $\Delta'$. A graph isomorphism is a bijective graph morphism.

Let $N \geq 2$ be an integer. We fix a free basis $A = \{a_1, \ldots, a_N\}$ of the free group $F_N$. We denote by $\text{id}$ the identity element of $F_N$. We denote by $\text{Sub}(F_N)$ the set of all non-trivial finitely generated subgroups of $F_N$. 

Let $X$ be the Cayley graph of $(F_N, A)$, where $V(X) := F_N$, $E_{\text{top}}(X) := F_N \times A$, and for every topological edge $(g, a) \in E_{\text{top}}(X)$ the boundary is $\{g, ga\}$. The Cayley graph $X$ is a tree. The free group $F_N$ naturally acts on $X$ from the left by graph isomorphisms. Let $R_N$ be the quotient graph $F_N \backslash X$ and denote by $q : X \to R_N$ the canonical projection, which is a universal covering map. The quotient graph $R_N$ is isomorphic. Let $A$ statement depends on the basis $A \N$ and denote by $\partial X \N$ corresponding to an element of $A \N$. The space of subset currents on $F \N$.

2.2. The space of subset currents on $F_N$. In this subsection, we summarize necessary definitions and properties of subset currents on $F_N$ (see [KMR] for details).

We denote by $\mathcal{E}_N$ the set of all closed subsets $S \subset \partial X$ such that the cardinality $\#S \geq 2$. We endow $\mathcal{E}_N$ with the subspace topology from the Vietoris topology on the hyperspace of $\partial X$, which consists of all closed subsets of $\partial X$. Then, $\mathcal{E}_N$ is a locally compact totally disconnected metrizable space. If we give a distance on $\partial X$ which is compatible with the topology on $\partial X$, then the topology on $\mathcal{E}_N$ induced by the Hausdorff distance coincides with the topology which we defined above.

For an oriented edge $e \in E(X)$, we define the cylinder $\text{Cyl}(e)$ to be the subset of $\partial X$ consisting of equivalence classes of all geodesic rays in $X$ emanating from the oriented edge $e$. A cylinder $\text{Cyl}(e)$ is an open and compact subset of $\partial X$ for any $e \in E(X)$, and the collection of all $\text{Cyl}(e) \ (e \in E(X))$ is a basis of $\partial X$.

We denote by $\text{Sub}(X)$ the set of all non-degenerate finite subtrees of $X$, which are finite subtrees with at least two distinct vertices.

Let $T \in \text{Sub}(X)$ and let $e_1, \ldots, e_m$ be all the terminal edges of $T$, which are oriented edges whose terminal vertices are precisely the vertices of $T$ of degree 1. Define the sub-
set cylinder $S \text{Cyl}(T)$ to be the subset of $\mathcal{E}_N$ consisting of $S \in \mathcal{E}_N$ satisfying the condition that

$$S \subset \bigcup_{i=1}^{m} \text{Cyl}(e_i) \text{ and } S \cap \text{Cyl}(e_i) \neq \emptyset \ (\forall i = 1, 2, \ldots m).$$

For $T \in \text{Sub}(X)$ the subset $S \text{Cyl}(T) \subset \mathcal{E}_N$ is compact and open, and the collection of all $S \text{Cyl}(T) \ (T \in \text{Sub}(X))$ forms a basis for the topology on $\mathcal{E}_N$.

Note that the left continuous action of $F_N$ on $\partial X$ naturally extends to a left continuous action on $\mathcal{E}_N$.

A subset current on $F_N$ is a Borel measure on $\mathcal{E}_N$ that is $F_N$-invariant and locally finite (i.e., finite on all compact subsets of $\mathcal{E}_N$).

The set of all subset currents on $F_N$ is denoted by $S \text{Curr}(F_N)$. The space $S \text{Curr}(F_N)$ has the $\mathbb{R}_{\geq 0}$-linear structure, and the space $S \text{Curr}(F_N)$ is endowed with the natural weak-* topology.

**Proposition 2.1** (See [KMR], Proposition 3.7).

1. Let $\mu, \mu_n \in S \text{Curr}(F_N)$, where $n = 1, 2, \ldots$. Then $\lim_{n \to \infty} \mu_n = \mu$ in $S \text{Curr}(F_N)$ if and only if for every $T \in \text{Sub}(X)$ we have

$$\lim_{n \to \infty} \mu_n(\text{Cyl}(T)) = \mu(\text{Cyl}(T)).$$
Therefore Φ is bijective, and this implies

\[ \text{SCur}(F_N) \to \mathbb{R}_{\geq 0}; \ \mu \mapsto \mu(\text{Syl}(T)) \]

is continuous and \( \mathbb{R}_{\geq 0} \)-linear.

Recall that for a subgroup \( H \) of a group \( G \) the commensurator or virtual normalizer \( \text{Comm}_G(H) \) of \( H \) in \( G \) is defined as

\[ \text{Comm}_G(H) := \{ g \in G \mid [H : H \cap gHg^{-1}] < \infty \text{ and } [gHg^{-1} : H \cap gHg^{-1}] < \infty \}. \]

Let \( H \in \text{Sub}(F_N) \). The limit set \( \Lambda(H) \) of \( H \) in \( \partial X(= \partial F_N) \) is the set of all \( \xi \in \partial X \) such that there exists a sequence of \( h_n \in H \) (\( n = 1, 2, \ldots \)) satisfying \( \lim_{n \to \infty} h_n = \xi \) in \( X \cup \partial X \). See [KN13, Proposition 4.1 and Proposition 4.2] for elementary properties of limit sets.

Note that for every \( H \in \text{Sub}(F_N) \) we have \( \Lambda(H) \in \mathcal{C}_N \), and

\[ \text{Comm}_{F_N}(H) = \text{Stab}_{F_N}(\Lambda(H)), \]

where \( \text{Stab}_{F_N}(\Lambda(H)) := \{ g \in F_N \mid g\Lambda(H) = \Lambda(H) \} \) is the stabilizer of \( \Lambda(H) \).

For \( H \in \text{Sub}(F_N) \) a subset current \( \eta_H \), which is called a counting subset current, is defined as follows. Set \( \hat{H} = \text{Comm}_{F_N}(H) \).

Suppose first that \( H = \hat{H} \). Define a Borel measure \( \eta_H \) on \( \mathcal{C}_N \) as

\[ \eta_H := \sum_{H' \in [H]} \delta_{\Lambda(H')}, \]

where \([H]\) is the conjugacy class of \( H \) in \( F_N \) and \( \delta_{\Lambda(H')} \) is the Dirac measure on \( \mathcal{C}_N \) which means that for a Borel subset \( U \subset \mathcal{C}_N \), if \( \Lambda(H') \in U \), then \( \delta_{\Lambda(H')}(U) = 1 \); if \( \Lambda(H') \not\in U \), then \( \delta_{\Lambda(H')}(U) = 0 \).

Now let \( H \) be an arbitrary nontrivial finitely generated subgroup of \( F_N \). Define \( \eta_H \) as \( \eta_H := [\hat{H} : H]|\eta_{\hat{H}} \). Note that \( \text{Comm}_{F_N}(\hat{H}) = \hat{H} \) and \([\hat{H} : H]\) is finite.

We can see that \( \eta_H \) is a subset current, especially, locally finite (see [KN13, Lemma 4.4]). For the trivial subgroup \( H = \{ \text{id} \} \) we set \( \eta_H = 0 \in \text{SCurr}(F_N) \). A subset current \( \mu \in \text{SCurr}(F_N) \) is said to be rational if \( \mu = r\eta_H \) for some \( r \geq 0 \) and \( H \in \text{Sub}(F_N) \). The set of all rational subset currents is a dense subset of \( \text{SCurr}(F_N) \) (see [KN13, Theorem 5.8], and Kap13).

We observe the following proposition.

**Proposition 2.2.** Let \( H \in \text{Sub}(F_N) \). Then we have

\[ \eta_H = \sum_{gH \in F_N/H} \delta_{g\Lambda(H)}. \]

**Proof.** First, we assume \( H = \text{Comm}_{F_N}(H)(= \text{Stab}_{F_N}(\Lambda(H))) \). Then we can define the following map:

\[ \Phi : F_N/H \to [H]; \ gH \mapsto \text{Stab}_{F_N}(g\Lambda(H)) = gHg^{-1}. \]

The map \( \Phi \) is clearly surjective and we can prove that \( \Phi \) is injective as follows. Let \( g_1, g_2 \in F_N/H \), and suppose \( g_1Hg_1^{-1} = g_2Hg_2^{-1} \). Then we have \( g_2^{-1}g_1Hg_1^{-1}g_2 = H \), and so \( g_2^{-1}g_1 \in H \) by the definition of the commensurator. Hence \( g_1H = g_2H \). Therefore \( \Phi \) is bijective, and this implies

\[ \eta_H = \sum_{H' \in [H]} \delta_{\Lambda(H')} = \sum_{gH \in F_N/H} \delta_{\Lambda(\Phi(gH))} = \sum_{gH \in F_N/H} \delta_{g\Lambda(H)}. \]
In general, put \( \hat{H} := \text{Comm}_{F_N} H \) and \( m := [\hat{H} : H] \). Then we can choose \( h_1, \ldots, h_m \in \hat{H} \) such that \( \{h_i H\}_{i=1}^m \) is a complete system of representatives of \( \hat{H}/H \). If \( \{g_j \hat{H}\}_{j \in J} \) is a complete system of representatives of \( F_N/\hat{H} \), then \( \{g_j h_i H\}_{i=1}^m, j \in J \) is a complete system of representatives of \( F_N/\hat{H} \). Since \( h_i \in \hat{H} = \text{Stab}_{F_N}(\Delta(H)) \) and \( \Delta(H) = \Delta(\hat{H}) \), we have
\[
\sum_{gH \in F_N/H} \delta_{g\Delta(H)} = \sum_{i,j} \delta_{g_j h_i \Delta(H)} = m \sum_{j \in J} \delta_{g_j \hat{H}} = m \eta_{\hat{H}} = \eta_H,
\]
as required. \( \square \)

If \( \varphi \in \text{Aut}(F_N) \) is an automorphism of \( F_N \), then \( \varphi \) induces a quasi-isometry of \( X \), and moreover, the quasi-isometry extends to a homeomorphism \( \varphi: \partial X \to \partial X \), where we still denote it by \( \varphi \). Thus \( \text{Aut}(F_N) \) has a natural action on \( \mathcal{E}_N \). Moreover, \( \text{Aut}(F_N) \) acts on \( \text{SCurr}(F_N) \) \( \mathbb{R}_{\geq 0} \)-linearly and continuously by pushing forward. Explicitly,
\[
(\varphi \mu)(U) := \mu(\varphi^{-1}(U))
\]
for \( \varphi \in \text{Aut}(F_N), \mu \in \text{SCurr}(F_N) \) and every Borel subset \( U \subset \mathcal{E}_N \). Then for \( \varphi \in \text{Aut}(F_N) \) and \( H \in \text{Sub}(F_N) \) we have \( \varphi \eta_H = \eta_{\varphi(H)} \). Since subset currents are \( F_N \)-invariant, the action of \( \text{Aut}(F_N) \) on \( \text{SCurr}(F_N) \) factors through the action of the outer automorphism group \( \text{Out}(F_N) \) on \( \text{SCurr}(F_N) \). The action of \( \text{Out}(F_N) \) on \( \text{SCurr}(F_N) \) is effective.

2.3. \( R_N \)-graphs and occurrences. In this subsection, first, following [KN13 Subsection 4.2] we define \( R_N \)-graphs and also occurrences for an \( R_N \)-graph and for \( T \in \text{Sub}(X) \), the set of all non-degenerate finite subtrees of \( X \). Occurrences play an essential role in studying rational subset currents on \( F_N \) (Lemma 2.4).

Recall that \( R_N \) is the quotient graph \( F_N/\Delta \), which is an \( N \)-rose. An \( R_N \)-graph is a graph \( \Delta \) with a graph morphism \( \tau: \Delta \to R_N \). We call \( \tau \) an \( R_N \)-graph structure. Let \( (\Delta_1, \tau_1) \) and \( (\Delta_2, \tau_2) \) be \( R_N \)-graphs. A graph morphism \( f: \Delta_1 \to \Delta_2 \) is called an \( R_N \)-graph morphism if \( \tau_1 = \tau_2 \circ f \). For an \( R_N \)-graph \( (\Delta, \tau) \) and \( v \in V(\Delta) \) we call a pair \( ((\Delta, \tau), v) \) (or simply \( (\Delta, v) \)) a based \( R_N \)-graph with a base point \( v \).

An \( R_N \)-graph \( (\Delta, \tau) \) is said to be folded if \( \tau \) is locally injective (immersion). We say that a finite \( R_N \)-graph \( (\Delta, \tau) \) is an \( R_N \)-core graph if \( (\Delta, \tau) \) is folded and has no degree-one and degree-zero vertices. We do not assume that \( R_N \)-core graphs are connected.

For a graph \( \Delta \) giving an \( R_N \)-structure \( \tau: \Delta \to R_N \) to \( \Delta \) can be regarded as giving a label structure \( E_{\text{top}}(\Delta) \to A \) and giving an orientation to \( \Delta \). We say a topological or oriented edge \( e \) of \( \Delta \) has a label \( a \in A \) when the graph morphism \( \tau \) maps the edge \( e \) to the loop of \( R_N \) corresponding to \( a \in A \).

**Definition 2.3.** Let \( \Delta \) be a graph. Let \( T \in \text{Sub}(X) \). The **interior** of \( T \) is \( T \setminus \{ \text{degree-one vertices of } T \} \). A graph morphism \( f: T \to \Delta \) is said to be **locally homeomorphic in the interior** if the restriction of \( f \) to the interior of \( T \) is locally homeomorphic, in other words, the degree of \( v \) in \( T \) equals to the degree of \( f(v) \) in \( \Delta \) for every \( v \in V(T) \) with degree more than one.
Let \( T \in \text{Sub}(X) \), and \( Y \) be a (not necessarily finite) subtree of \( X \). We say that \( Y \) is an extension of \( T \) and denote by \( T \subset Y \), if \( T \subset Y \) and if the inclusion map is locally homeomorphic in the interior.

**Proposition 2.4.** Let \( S \in \mathcal{N} \) and \( T \subset \text{Sub}(X) \). Then \( S \in \text{SCyl}(T) \) if and only if \( T \subset \text{Conv}(S) \), where \( \text{Conv}(S) \) is the convex hull of \( S \) in \( X \).

**Proof.** Let \( e_1, \ldots, e_m \) be all the terminal edges of \( T \) and \( v_i \) be the terminal of \( e_i \). Then we have
\[
T = \bigcup_{i,j=1,\ldots,m} [v_i, v_j],
\]
where \([v_i, v_j]\) is the geodesic from \( v_i \) to \( v_j \) in \( X \). Similarly,
\[
\text{Conv}(S) = \bigcup_{\xi, \zeta \in S} ([\xi, \zeta])
\]
where \((\xi, \zeta)\) is the bi-infinite geodesic from \( \xi \) to \( \zeta \) in \( X \).

Suppose \( S \in \text{SCyl}(T) \). By the definition \( S \subset \bigcup_i \text{Cyl}(e_i) \) and \( S \cap \text{Cyl}(e_i) \neq \emptyset \) for some \( i \). If \( \xi, \zeta \in S \cap \text{Cyl}(e_i) \), then \((\xi, \zeta)\) does not contain any edges of \( T \). Thus \( S \subset \bigcup_i \text{Cyl}(e_i) \) implies that there exists \( v_j \) such that \((\xi, \zeta)\) is an extension of \([v_i, v_j]\). Therefore we have \( T \subset \text{Conv}(S) \).

Next, we assume \( T \subset \text{Conv}(S) \). For every \( v_i, v_j (i \neq j) \) there exist \( \xi, \zeta \in S \) such that \((\xi, \zeta)\) is an extension of \([v_i, v_j]\) and then we have \( \xi \in \text{Cyl}(e_i) \) and \( \zeta \in \text{Cyl}(e_j) \). Hence \( S \cap \text{Cyl}(e_i) \neq \emptyset \) for some \( i \). If \( \xi, \zeta \) contains some edges of \( T \), then \( T \subset \text{Conv}(S) \) implies that \( v_j \) such that \((\xi, \zeta)\) is an extension of \([v_i, v_j]\). Thus \( S \subset \bigcup_i \text{Cyl}(e_i) \). If \((\xi, \zeta)\) does not contain any edges of \( T \), we choose \( \zeta' \in S \cap \text{Cyl}(e_j) \) if \( j \neq i \). Then \((\zeta, \zeta')\) is an extension of \([v_i, v_j]\). Since \((\xi, \zeta) \cup (\zeta, \zeta') \supset (\zeta, \zeta')\), the geodesic \((\zeta, \zeta')\) contains some edges of \( T \). Now, we can apply the above argument to \((\xi, \zeta')\).

**Definition 2.5.** Let \( T \subset \text{Sub}(X) \). We can regard \( T \) as an \( R_N \)-graph with the \( R_N \)-graph structure inherited from the canonical projection \( q \): \( X \rightarrow R_N \). Let \( \Delta \) be an \( R_N \)-core graph. An occurrence of \( T \) in \( \Delta \) is an \( R_N \)-graph morphism \( f: T \rightarrow \Delta \) which is locally homeomorphic in the interior. Let \( \text{Occ}(T, \Delta) \) be the set of all occurrences of \( T \) in \( \Delta \).

Consider a based \( R_N \)-graph \((T, x)\) such that \( T \subset \text{Sub}(X) \). For a based and folded \( R_N \)-graph \((\Delta, v)\) there exists at most one based occurrence \( f: (T, x) \rightarrow (\Delta, v) \), where \( f(x) = v \). In order to compute \( \#\text{Occ}(T, \Delta) \), it is sufficient to see whether there exists a based \( R_N \)-graph morphism \( f: (T, x) \rightarrow (\Delta, v) \) for each \( v \in V(\Delta) \). Hence
\[
\#\text{Occ}(T, \Delta) = \# \{ v \in V(\Delta) \mid \exists f: (T, x) \rightarrow (\Delta, v) \text{ a based occurrence} \}.
\]

**Notation 2.6.** Let \( H \subset \text{Sub}(F_N) \). Set \( X_H := \text{Conv}(\Lambda(H)) \), which is the unique minimal \( H \)-invariant subtree of \( X \). We define \( \Delta_H \) to be the quotient graph \( H \backslash X_H \) and denote by \( q_H: X_H \rightarrow \Delta_H \) the canonical projection. Then \( \Delta_H \) becomes an \( R_N \)-core graph by the induced graph morphism \( \tau_H: \Delta_H \rightarrow R_N \) from \( q: X \rightarrow R_N \).

We denote by \( \text{Sub}(X, \text{id}) \) the set of all nontrivial finite based subtrees of \( X \) with the base point id. For \((T, \text{id}) \in \text{Sub}(X, \text{id}) \) we denote it briefly by \( T \).
The following lemma is a direct corollary from Section 4.2 and Section 4.3 in [KN13], and plays an essential role in studying rational subset currents on $F_N$.

**Lemma 2.7.** Let $H \in \text{Sub}(F)$ and $T \in \text{Sub}(X, \text{id})$. Then we have

$$\eta_H(\text{SCyl}(T)) = \#\text{Occ}(T, \Delta_H) \quad (= \#\{v \in V(\Delta_H) \mid \exists f: (T, \text{id}) \to (\Delta_H, v) \text{ a based occurrence}\}).$$

### 2.4. The reduced rank functional $\overline{\text{rk}}$.

Recall that the rank $\text{rank}(F)$ of a finitely generated free group $F$ is the cardinality of a free basis of $F$, and the reduced rank $\overline{\text{rk}}(F)$ is defined as

$$\overline{\text{rk}}(F) := \max\{\text{rank}(F) - 1, 0\}.$$  
If $\text{rank}(F) \geq 1$, then $\overline{\text{rk}}(F) = \text{rk}(F) - 1$, and for a finite connected graph $\Delta$ whose fundamental group is isomorphic to $F$ we have $\overline{\text{rk}}(F) = -\chi(\Delta)$, where $\chi(\Delta) := \#V(\Delta) - \#E_{\text{top}}(\Delta)$ is the Euler characteristic of $\Delta$.

**Theorem 2.8** (cf. [KN13], Theorem 8.1). There exists a unique continuous $\mathbb{R}_{\geq 0}$-linear functional

$$\overline{\text{rk}}: \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}$$

such that for every $H \in \text{Sub}(F_N)$ we have

$$\overline{\text{rk}}(\eta_H) = \overline{\text{rk}}(H).$$

Moreover, $\overline{\text{rk}}$ is $\text{Out}(F_N)$-invariant.

The functional $\overline{\text{rk}}$ is called the reduced rank functional. In order to prove Theorem 2.8 we will use a method used in the proof of [KN13] Theorem 8.1. We give a proof of Theorem 2.8 following the argument in [KN13] almost step by step in the remaining part of this subsection.

Note that for $H \in \text{Sub}(F_N)$ we have $\text{rank}(H) \geq 1$, and so

$$\text{rank}(H) = \#E(\Delta_H) - \#V(\Delta_H).$$

We extend each term of the right hand side of this equation to a continuous $\mathbb{R}_{\geq 0}$-linear functional on $\text{SCurr}(F_N)$, namely construct continuous $\mathbb{R}_{\geq 0}$-linear functionals

$$E, V: \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}$$

such that $E(\eta_H) = \#E(\Delta_H)$ and $V(\eta_H) = \#V(\Delta_H)$ for $H \in \text{Sub}(F_N)$.

Let $e_a$ be the topological edge in $X$ with the boundary $\{\text{id}, a\} \ (a \in A)$. Let $H \in \text{Sub}(F_N)$. By considering $e_a$ as a subtree of $X$ we have $(e_a, \text{id}) \in \text{Sub}(X, \text{id})$. Then $\eta_H(\text{SCyl}(e_a)) = \#\text{Occ}(e_a, \Delta_H)$ coincides with the number of topological edges of $\Delta_H$ with the label $a$. From this we define the map

$$E: \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}; \ \mu \mapsto \sum_{a \in A} \mu(\text{SCyl}(e_a)).$$

Then we have $E(\eta_H) = \#E_{\text{top}}(\Delta_H)$ for $H \in \text{Sub}(F_N)$, and $E$ is continuous and $\mathbb{R}_{\geq 0}$-linear from Proposition 2.1.

In the case of $V$, we need the following definition.

**Definition 2.9** (Round graphs. cf. [Kap13] Definition 3.6). For an integer $r \geq 1$, we say that $T \in \text{Sub}(X)$ is a round graph of grade $r$ in $X$ if there exists a (necessarily unique) vertex $v$ of $T$ such that for every degree-one vertex $u$ of $T$ we have $d(v, u) = r$. We call a pair $(T, v)$ a based round graph of grade $r$ if $v$ satisfies
the above condition, where the degree of $v$ have to be more than one. Let $R_r$ denote the set of all based round graphs of grade $r$ with the base point id. Thus $R_r$ is a subset of $\text{Sub}(X, \text{id})$.

**Remark 2.10.** Fix a positive integer $r$. For any $R_N$-graph $\Delta$ and $v \in V(\Delta)$ there exists a unique $T_r(v) \in R_r$ such that there exists a based occurrence $(T_r(v), \text{id}) \rightarrow (\Delta, v)$. We can think of $T_r(v)$ as an $r$-neighborhood of $v$ in $\Delta$.

Let $v \in V(X)$ and $\rho \in \mathbb{R}_{\geq 0}$. Set $B(v, \rho) := \{ x \in X \mid d_X(v, x) \leq \rho \}$ the closed ball with radius $\rho$ and center $v$ in $X$. For $T \in R_r$ and every $S \in \text{SCyl}(T)$ we have $\text{Conv}(S) \cap B(\text{id}, r) = T$ from Proposition 2.4. Therefore, if $T_1 \neq T_2$ for $T_1, T_2 \in R_r$, then $\text{SCyl}(T_1) \cap \text{SCyl}(T_2) = \emptyset$.

We define the map

$$V : \text{SCurr}(F_N) \rightarrow \mathbb{R}_{\geq 0}; \mu \mapsto \sum_{T \in R_1} \mu(\text{SCyl}(T)) = \mu \left( \bigcup_{T \in R_1} \text{SCyl}(T) \right).$$

Then

$$V(\eta_H) = \sum_{T \in R_1} \eta_H(\text{SCyl}(T))$$

$$= \sum_{T \in R_1} \# \{ v \in V(\Delta_H) \mid T_1(v) = T \} = \# V(\Delta_H)$$

for $H \in \text{Sub}(F_N)$, and $V$ is continuous and $\mathbb{R}_{\geq 0}$-linear. Note that for any positive integer $r$, we have

$$\bigcup_{T \in R_r} \text{SCyl}(T) = \{ S \in \mathbb{C}_N \mid \text{Conv}(S) \ni \text{id} \}.$$

**Proof of Theorem 2.8.** Set $r_k = E - V$. Then for $H \in \text{Sub}(F_N)$,

$$\overline{r_k}(\eta_H) = E(\eta_H) - V(\eta_H) = \# E(\Delta_H) - \# V(\Delta_H) = r_k(H).$$

Since $\overline{r_k}(r\eta_H) = r\overline{r_k}(H) \geq 0$ for any rational subset current $r\eta_H (r \geq 0, H \in \text{Sub}(F_N))$, we have $r_k(\mu) \geq 0$ for any $\mu \in \text{SCurr}(F_N)$. The uniqueness and Out($F_N$)-invariance of $r_k$ is obvious from the denseness of the rational subset currents in $\text{SCurr}(F_N)$.

### 3. The intersection functional $\mathcal{N}$

Define a product $\mathcal{N}(H, K)$ of two finitely generated subgroups $H$ and $K$ of $F_N$ as

$$\mathcal{N}(H, K) := \sum_{HgK \in H\backslash F_N/K} \overline{r_k}(H \cap gKg^{-1}),$$

where $H\backslash F_N/K$ is the set of all double cosets $HgK$ ($g \in F_N$). By this definition the Strengthened Hanna Neumann Conjecture (SHNC), which has been proven by Mineyev, can be stated as follows:

**Theorem 3.1 (SHNC, see [Min12]).** For any finitely generated subgroups $H, K \leq F_N$, the following inequality follows:

$$\mathcal{N}(H, K) \leq \overline{r_k}(H) \overline{r_k}(K).$$

In this section we give a proof of the following theorem, which is our main result.
Theorem 3.2. There exists a unique continuous symmetric $\mathbb{R}_{\geq 0}$-bilinear functional

$$\mathcal{N}: \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}$$

such that for $H, K \in \text{Sub}(F_N)$ we have

$$\mathcal{N}(\eta_H, \eta_K) = \mathcal{N}(H, K).$$

Moreover, $\mathcal{N}$ is $\text{Out}(F_N)$-invariant, that is, for any $\varphi \in \text{Out}(F_N)$ and $\mu, \nu \in \text{SCurr}(F_N)$ we have $\mathcal{N}(\varphi \mu, \varphi \nu) = \mathcal{N}(\mu, \nu)$.

We call $\mathcal{N}$ the intersection functional. By the definition of the product $\mathcal{N}$ and the $\mathbb{R}_{\geq 0}$-linearity of the reduced rank functional $\underline{\text{rk}}$, we have

$$\mathcal{N}(H, K) = \sum_{HgK \in H \setminus F_N / K} \underline{\text{rk}}(H \cap gKg^{-1}).$$

Kapovich and Nagnibeda asked whether there exists a continuous $\mathbb{R}_{\geq 0}$-bilinear map

$$\hat{\mathcal{H}}: \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \text{SCurr}(F_N)$$

such that

$$\hat{\mathcal{H}}(\eta_H, \eta_K) = \sum_{HgK \in H \setminus F_N / K} \eta_{H \cap gKg^{-1}}$$

for any finitely generated subgroups $H, K \leq F_N$ (see [KN13, Subsection 10.4]). If such a map $\hat{\mathcal{H}}$ exists, then Theorem 3.2 follows immediately, and moreover, $\hat{\mathcal{H}}$ can be considered as a measure theoretical generalization of the construction of the fiber product graph $\Delta_H \times_{R_N} \Delta_K$ (see Definition 3.4 and the following argument). However, by the following proposition we answer that there does not exist such a map $\hat{\mathcal{H}}$. Nevertheless we construct a non-continuous $\mathbb{R}_{\geq 0}$-bilinear map $\hat{\mathcal{I}}: \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \text{SCurr}(F_N)$ with reasonable properties (see Section 4).

Proposition 3.3. If there exists an $\mathbb{R}_{\geq 0}$-bilinear map

$$\hat{\mathcal{H}}: \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \text{SCurr}(F_N)$$

such that

$$\hat{\mathcal{H}}(\eta_H, \eta_K) = \sum_{HgK \in H \setminus F_N / K} \eta_{H \cap gKg^{-1}}$$

for any finitely generated subgroups $H, K \leq F_N$, then $\hat{\mathcal{H}}$ is not continuous.

Proof. Recall that $A = \{a_1, \ldots, a_N\}$ is a free basis of $F_N$. Set subgroups $H_n := \langle a_1^n a_2 \rangle$ ($n = 1, 2, \ldots$) and $H := \langle a_1 \rangle$. Then from Proposition 2.1 and Lemma 2.7 we can see

$$\frac{1}{n} \eta_{H_n} \to \eta_H \ (n \to \infty).$$

We also have

$$\hat{\mathcal{H}}(\eta_{H_n}, \eta_H) = 0 \ (n = 1, 2, \ldots), \ \hat{\mathcal{H}}(\eta_H, \eta_H) = \eta_H,$$

which implies that $\frac{1}{n} \hat{\mathcal{H}}(\eta_{H_n}, \eta_H)$ does not converges to $\hat{\mathcal{H}}(\eta_H, \eta_H)$. Therefore, $\hat{\mathcal{H}}$ is not continuous. \qed
To prove Theorem 3.2 we use the following graph theoretical description of $\mathcal{N}$. First, we recall the definition of the fiber product graph in [Sta83]. Since by a graph we mean a 0 or 1-dimensional CW complex, we rearrange the definition.

**Definition 3.4** (cf. [Sta83]). Let $(\Delta_1, \tau_1), (\Delta_2, \tau_2)$ be $R_N$-graphs. The fiber product graph $\Delta_1 \times_{R_N} \Delta_2$ corresponding to $(\Delta_1, \tau_1)$ and $(\Delta_2, \tau_2)$ is the fiber product of $(\Delta_1, \tau_1)$ and $(\Delta_2, \tau_2)$ in the category of topological spaces with a graph structure induced by $(\Delta_1, \tau_1)$ and $(\Delta_2, \tau_2)$. Explicitly,

\[
\Delta_1 \times_{R_N} \Delta_2 = \{(x_1, x_2) \in \Delta_1 \times \Delta_2 \mid \tau_1(x_1) = \tau_2(x_2)\};
\]

\[
V(\Delta_1 \times_{R_N} \Delta_2) = \{(v_1, v_2) \in V(\Delta_1) \times V(\Delta_2)\};
\]

\[
E_{\text{top}}(\Delta_1 \times_{R_N} \Delta_2) = \{(e_1, e_2) \in E_{\text{top}}(\Delta_1) \times E_{\text{top}}(\Delta_2) \mid \tau_1(e_1) = \tau_2(e_2)\}.
\]

Here, $V(\Delta_1 \times_{R_N} \Delta_2)$ is given as above because $R_N$ has only one vertex. For $(e_1, e_2) \in E_{\text{top}}(\Delta_1 \times_{R_N} \Delta_2)$, fix orientations of $e_1, e_2$ such that $\tau_1(e_1) = \tau_2(e_2)$ in $E(R_N)$. Then the boundary of $(e_1, e_2)$ is $\{(o(e_1), o(e_2)), (t(e_1), t(e_2))\}$, which does not depend on the choice of the orientations of $e_1$ and $e_2$.

There are natural graph morphisms $\phi_i : \Delta_1 \times_{R_N} \Delta_2 \rightarrow \Delta_i, (x_1, x_2) \mapsto x_i$ ($i = 1, 2$), and the graph morphism $\phi_i$ induces

\[
\phi_i : V(\Delta_1 \times_{R_N} \Delta_2) \rightarrow V(\Delta_i); \ (v_1, v_2) \mapsto v_i,
\]

\[
\phi_i : E_{\text{top}}(\Delta_1 \times_{R_N} \Delta_2) \rightarrow E_{\text{top}}(\Delta_i); \ (e_1, e_2) \mapsto e_i.
\]

The fiber product graph $\Delta_1 \times_{R_N} \Delta_2$ becomes an $R_N$-graph by the map $\tau_1 \circ \phi_1 (= \tau_2 \circ \phi_2)$. If $(\Delta_1, \tau_1)$ and $(\Delta_2, \tau_2)$ are folded $R_N$-graphs, then so is $(\Delta_1 \times_{R_N} \Delta_2, \tau_1 \circ \phi_1)$. However, $\Delta_1 \times_{R_N} \Delta_2$ may not be connected and may have degree-zero and degree-one vertices even if $\Delta_1$ and $\Delta_2$ are connected $R_N$-core graphs.

Let $H, K \in \text{Sub}(F_N)$. In the context of the SHNC it has been proven that every non-zero term in the sum of $\mathcal{N}(H, K)$ corresponds to a non-contractible component of $\Delta_H \times_{R_N} \Delta_K$, and the following equality holds (see [Neu90]):

\[
\mathcal{N}(H, K) = - \sum_{i=1}^{k} \chi(\Gamma_i),
\]

where $\Gamma_1, \ldots, \Gamma_k$ are all the non-contractible components of $\Delta_H \times_{R_N} \Delta_K$. Using this description of $\mathcal{N}$ our strategy of proving Theorem 3.2 is the same as that of Theorem 2.8. We denote by $c(\Delta_H \times_{R_N} \Delta_K)$ the number of contractible components of $\Delta_H \times_{R_N} \Delta_K$. Since the Euler characteristic of a contractible component is 1, we have

\[
\mathcal{N}(H, K) = \#E_{\text{top}}(\Delta_H \times_{R_N} \Delta_K) - \#V(\Delta_H \times_{R_N} \Delta_K) + c(\Delta_H \times_{R_N} \Delta_K).
\]

We extend each term of the right hand side of this equation to a continuous symmetric $\mathbb{R}_{\geq 0}$-bilinear functional on $SCur(F_N)$.

Note that for $T_1, T_2 \in \text{Sub}(X)$ we have a continuous $\mathbb{R}_{\geq 0}$-bilinear functional

\[
SCur(F_N) \times SCur(F_N) \rightarrow \mathbb{R}_{\geq 0}; \ (\mu, \nu) \mapsto \mu(S\text{Cyl}(T_1))\nu(S\text{Cyl}(T_2)),
\]

which plays a fundamental role in constructing the intersection functional $\mathcal{N}$.

Since $\#V(\Delta_H \times_{R_N} \Delta_K) = \#V(\Delta_H)\#V(\Delta_K)$, we define the continuous $\mathbb{R}_{\geq 0}$-bilinear map

\[
\tilde{V} : SCur(F_N) \times SCur(F_N) \rightarrow \mathbb{R}_{\geq 0}
\]
We prove the map $f$ given by Lemma 3.5. Since $1$, we denote by $R$ then there exists an of $\Delta$ $f$ vertex of $\Delta T$ $\top$ topological edges of $\Delta H$ edges of $\Delta$ we have $\Delta H$ with the label $a$ equals to the product of the number of topological edges of $\Delta H$ with the label $a$ and that of $\Delta K$ with the label $a$. Therefore, we have $\#E_{\text{top}}(\Delta H \times R_N \Delta K) = \sum_{a \in A} \eta_{H}(SCyl(e_a)) \eta_{H}(SCyl(e_a))$.

Now, we define the continuous $\mathbb{R}_{\geq 0}$-bilinear map

$$\tilde{E}: \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}$$

by

$$\tilde{E}(\mu, \nu) := \sum_{a \in A} \mu(\text{Cyl}(e_a)) \nu(\text{Cyl}(e_a)).$$

Then we have $\tilde{E}(\eta_H, \eta_K) = \#E_{\text{top}}(\Delta H \times R_N \Delta K)$.

In the remaining part of this section we construct a continuous $\mathbb{R}_{\geq 0}$-bilinear functional

$$\tilde{c}: \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}$$

such that $\tilde{c}(\eta_H, \eta_K) = c(\Delta H \times R_N \Delta K)$.

Set $\text{Sub}(X, id) := \text{Sub}(X, id) \cup \{\{id\}\}$, where $\{id\}$ is regarded as a subtree of $X$ consisting of one vertex id.

To construct $\tilde{c}$ we use the following lemmas. First one is obvious from the definition of the fiber product graph.

**Lemma 3.5.** Let $(\Delta_1, \tau_1), (\Delta_2, \tau_2)$ be $R_N$-core graphs and $v_i \in V(\Delta_i) \ (i = 1, 2)$. Let $\Gamma$ be the connected component of $\Delta_1 \times_{R_N} \Delta_2$ containing the vertex $(v_1, v_2)$. Let $f_i: (T_i, id) \to (\Delta_i, v_i) \ (T_i \in \text{Sub}(X, id))$ be a based occurrence $(i = 1, 2)$, and set $T = T_1 \cap T_2$. Then we have an $R_N$-graph morphism

$$f: T \to \Delta_1 \times_{R_N} \Delta_2; \ x \mapsto (f_1(x), f_2(x)),$$

and $f$ is locally homeomorphic in the interior.

**Lemma 3.6.** Let $T \in \widehat{\text{Sub}}(X, id)$ such that $T \subset B(id, r)$ for an integer $r \geq 0$. Let $T_1, T_2 \in \mathcal{R}_{r+1}$ with $T_1 \cap T_2 = T$. Let $(\Delta_1, \tau_1), (\Delta_2, \tau_2)$ be $R_N$-core graphs and $v_i$ a vertex of $\Delta_i \ (i = 1, 2)$. If there are based occurrences $f_i: (T_i, id) \to (\Delta_i, v_i) \ (i = 1, 2)$, then there exists an $R_N$-graph isomorphism from $T$ to the connected component $\Gamma$ of $\Delta_1 \times_{R_N} \Delta_2$ containing the vertex $(v_1, v_2)$. In particular, $\Gamma$ is contractible.

**Proof.** We denote by $\phi_i$ the natural $R_N$-graph morphism from $\Delta_1 \times_{R_N} \Delta_2$ to $\Delta_i \ (i = 1, 2)$, and consider the $R_N$-graph morphism $f: T \to \Delta_1 \times_{R_N} \Delta_2; \ x \mapsto (f_1(x), f_2(x))$ given by Lemma 3.5. Since $f(id) = (f_1(id), f_2(id)) = (v_1, v_2)$, we have $f(T) \subset \Gamma$. We prove the map $f: T \to \Gamma$ is an $R_N$-graph isomorphism.
First, we prove the surjectivity of \( f \). Take any \( x \in \Gamma \) and a locally isometric path \( p : [0, l] \to \Gamma \) such that \( p(0) = (v_1, v_2) \) and \( p(l) = x \). Assume that \( l \leq r + 1 \). Since \( f_i \) is locally homeomorphic in the interior, for the path \( \phi_i \circ p \) in \( \Delta_i \) we can take the lift \( \tilde{p}_i : [0, l] \to T_i \) such that \( f_i \circ \tilde{p}_i = \phi_i \circ p \) and \( \tilde{p}_i(0) = id \) (\( i = 1, 2 \)). Then \( \tilde{p}_i \) can be regarded as the lift of \( \tau_1 \circ \phi_i \circ p : ([0, l], 0) \to (R_N, x_0) \) with respect to the universal covering \( q : (X, id) \to (R_N, x_0) \).

Since \( \tau_1 \circ \phi_1 \circ p = \tau_2 \circ \phi_2 \circ p \), we have \( \tilde{p}_1 = \tilde{p}_2 \) from the uniqueness of the lift. Therefore \( \tilde{p}_1(l) = \tilde{p}_2(l) \in T_1 \cap T_2 = T \), and \( f(\tilde{p}_1(l)) = (\phi_1 \circ p(l), \phi_2 \circ p(l)) = x \). Hence \( f \) is surjective, and moreover, it follows that \( l \leq r \). If the length \( l \) of \( p \) is greater than \( r + 1 \), we can do the same argument for \( p_{[0,r+1]} \) and this leads to a contradiction, which concludes that there is no locally isometric path starting from \((v_1, v_2)\) in \( \Gamma \) with length greater than \( r \).

In particular, we can see that \( \Gamma \) is a tree.

Since \( f \) is an \( R_N \)-graph morphism from the tree \( T \) to the tree \( \Gamma \), the injectivity of \( f \) follows. Therefore, \( f \) is an \( R_N \)-graph isomorphism from \( T \) to \( \Gamma \).

\section*{Lemma 3.7.}
Let \( T \in \overline{\text{Sub}}(X, \text{id}) \) with \( T \subset B(\text{id}, r) \). Let \((\Delta_1, \tau_1)\) and \((\Delta_2, \tau_2)\) be \( R_N \)-core graphs and \( \Gamma \) the connected component of \( \Delta_1 \times_{R_N} \Delta_2 \) containing a vertex \((v_1, v_2) \in V(\Delta_1) \times V(\Delta_2) \). Then there exists a based \( R_N \)-graph isomorphism from \((T, \text{id})\) to \((\Gamma, (v_1, v_2))\) if and only if \( T_{r+1}(v_1) \cap T_{r+1}(v_2) = T \), where \( T_{r+1}(v_i) \) is \((r + 1)\)-neighborhood of \( v_i \) (\( i = 1, 2 \)).

\begin{proof}
The "if" part follows from Lemma 3.6. To prove the "only if" part, we suppose there exists an \( R_N \)-graph isomorphism \( \varphi : (T, \text{id}) \to (\Gamma, (v_1, v_2)) \). Let \( f_i : (T_{r+1}(v_i), \text{id}) \to (\Delta_i, v_i) \) be the occurrence (\( i = 1, 2 \)). From Lemma 3.5 we have the \( R_N \)-graph morphism

\[ f : T_{r+1}(v_1) \cap T_{r+1}(v_2) \to \Gamma \subset \Delta_1 \times_{R_N} \Delta_2 ; \quad x \to (f_1(x), f_2(x)) \]

which is locally homeomorphic in the interior. Then the \( R_N \)-graph morphism \( \varphi^{-1} \circ f : T_{r+1}(v_1) \cap T_{r+1}(v_2) \to T \) is locally homeomorphic in the interior and \( \varphi^{-1} \circ f(\text{id}) = id \), which implies that \( T_{r+1}(v_1) \cap T_{r+1}(v_2) \subset T \). Since \( T \subset B(\text{id}, r) \), for the natural \( R_N \)-graph morphism \( \phi_i : \Delta_1 \times_{R_N} \Delta_2 \to \Delta_i \), we have the lift \( \tilde{\phi}_i : T \to T_{r+1}(v_i) \) such that \( f_i \circ \phi_i = \phi_i \circ \varphi \) and \( \tilde{\phi}_i(\text{id}) = id \) (\( i = 1, 2 \)). Therefore, \( T \subset T_{r+1}(v_1) \cap T_{r+1}(v_2) \), which concludes that \( T_{r+1}(v_1) \cap T_{r+1}(v_2) = T \).

\end{proof}

\section*{Notation 3.8.}
Let \( T \in \overline{\text{Sub}}(X, \text{id}) \) and let \((\Delta_1, \tau_1), (\Delta_2, \tau_2)\) be \( R_N \)-core graphs. We denote by \( c(\Delta_1 \times_{R_N} \Delta_2, T) \) the number of contractible components of \( \Delta_1 \times_{R_N} \Delta_2 \) that are \( R_N \)-graph isomorphic to \( T \).

We define \( \overline{\text{Sub}}(X, \text{id})/F_N \) to be the set of all the equivalence classes of the following equivalence relation: \( T_1 \in \overline{\text{Sub}}(X, \text{id}) \) is equivalent to \( T_2 \in \overline{\text{Sub}}(X, \text{id}) \) if there exists \( g \in F_N \) such that \( gT_1 = T_2 \). We denote by \([T]\) the equivalence class containing \( T \in \overline{\text{Sub}}(X, \text{id}) \).

For any contractible component \( \Gamma \) of \( \Delta_1 \times_{R_N} \Delta_2 \) we have a lift \( \Gamma \to X \) of \( \tau_1 \circ \phi_i : \Gamma \to R_N \) with respect to the universal covering \( q : X \to R_N \), which implies that there exists a unique equivalence class \([T]\) in \( \overline{\text{Sub}}(X, \text{id})/F_N \) such that \( \Gamma \) is \( R_N \)-graph isomorphic to \( T \). Therefore we have

\[ c(\Delta_1 \times_{R_N} \Delta_2) = \sum_{[T] \in \overline{\text{Sub}}(X, \text{id})/F_N} c(\Delta_1 \times_{R_N} \Delta_2, T). \]
Let $H, K \in \text{Sub}(F_N)$ and $T \in \widehat{\text{Sub}}(X, \text{id})$ with $T \subset B(\text{id}, r)$ for an integer $r \geq 0$. From Lemma 3.7 and Lemma 2.7 we obtain

\[
c(\Delta_H \times_{R_N} \Delta_K, T) = \sum_{T_1, T_2 \in R_{r+1}} \#\{v \in V(\Delta_H) : T_{r+1}(v) = T_1\} \cdot \#\{v \in V(\Delta_K) : T_{r+1}(v) = T_2\}
\]

\[
= \sum_{T_1, T_2 \in R_{r+1}} \eta_H(\text{SCyl}(T_1)) \eta_K(\text{SCyl}(T_2)).
\]

**Notation 3.9.** Let $T \in \widehat{\text{Sub}}(X, \text{id})$. Set

\[R(T) := \{(S_1, S_2) \in \mathcal{C}_N \times \mathcal{C}_N \mid \text{Conv}(S_1) \cap \text{Conv}(S_2) = T\}.\]

Let $r$ be a positive integer which satisfies $T \subset B(\text{id}, r)$. Then we have the following equality:

\[
(*) \quad R(T) = \bigsqcup_{T_1, T_2 \in R_{r+1}} \text{SCyl}(T_1) \times \text{SCyl}(T_2),
\]

and so

\[
c(\Delta_H \times_{R_N} \Delta_K, T) = \eta_H \times \eta_K(R(T)),
\]

where $\eta_H \times \eta_K$ is the product measure of $\eta_H$ and $\eta_K$.

From the definition of $R(T)$ and $(*)$, we immediately have the following proposition.

**Proposition 3.10.** Let $T, T' \in \widehat{\text{Sub}}(X, \text{id})$.

1. If $T \neq T'$, then we have $R(T) \cap R(T') = \emptyset$.

2. If there exists $g \in F_N$ such that $gT = T'$, then for any $\mu, \nu \in S\text{Curr}(F_N)$

\[
\mu \times \nu(R(T)) = \mu \times \nu(R(T')).
\]

3. The map

\[
S\text{Curr}(F_N) \times S\text{Curr}(F_N) \to \mathbb{R}_{\geq 0}; \ (\mu, \nu) \mapsto \mu \times \nu(R(T))
\]

is a continuous $\mathbb{R}_{\geq 0}$-bilinear map.

Moreover, $R(T)$ $(T \in \widehat{\text{Sub}}(X, \text{id}))$ has the following properties:

\[
\bigsqcup_{T \in \text{Sub}(X, \text{id})} R(T) = \{(S_1, S_2) \in \mathcal{C}_N \times \mathcal{C}_N \mid \exists T \in \widehat{\text{Sub}}(X, \text{id}), \text{Conv}(S_1) \cap \text{Conv}(S_2) = T\}
\]

\[
\subset \{(S_1, S_2) \in \mathcal{C}_N \times \mathcal{C}_N \mid \text{Conv}(S_1), \text{Conv}(S_2) \ni \text{id}\}
\]

\[
= \bigsqcup_{T_1, T_2 \in R_{r+1}} \text{SCyl}(T_1) \times \text{SCyl}(T_2).
\]
Therefore, for $\mu, \nu \in \mathcal{SCurr}(F_N)$

$$
\sum_{T \in \text{Sub}(X, \text{id})} \mu \times \nu(\mathcal{R}(T)) = \mu \times \nu \left( \bigcup_{T \in \text{Sub}(X, \text{id})} \mathcal{R}(T) \right)
\leq \mu \times \nu \left( \bigcup_{T_1, T_2 \in \mathcal{R}_1} \mathcal{SCyl}(T_1) \times \mathcal{SCyl}(T_2) \right)
= \mu \left( \bigcup_{T_1 \in \mathcal{R}_1} \mathcal{SCyl}(T_1) \right) \nu \left( \bigcup_{T_2 \in \mathcal{R}_1} \mathcal{SCyl}(T_2) \right)
= V(\mu)V(\nu).
$$

Hence the infinite sum

$$
\sum_{T \in \text{Sub}(X, \text{id})} \mu \times \nu(\mathcal{R}(T))
$$
always converges. Since $\#[T] = \#V(T)$ for $T \in \text{Sub}(X, \text{id})$, we have

$$
\sum_{T \in \text{Sub}(X, \text{id})} \mu \times \nu(\mathcal{R}(T)) = \sum_{[T] \in \text{Sub}(X, \text{id})/F_N} \#V(T)\mu \times \nu(\mathcal{R}(T)).
$$

Let $H, K \in \text{Sub}(F_N)$. Then

$$
c(\Delta_H \times_{R_N} \Delta_K) = \sum_{[T] \in \text{Sub}(X, \text{id})/F_N} c(\Delta_H \times_{R_N} \Delta_K, T)
= \sum_{[T] \in \text{Sub}(X, \text{id})/F_N} \eta_H \times \eta_K(\mathcal{R}(T)).
$$

We define the $\mathbb{R}_{\geq 0}$-bilinear map

$$
\hat{c}: \mathcal{SCurr}(F_N) \times \mathcal{SCurr}(F_N) \to \mathbb{R}_{\geq 0}
$$
as

$$
\hat{c}(\mu, \nu) := \sum_{[T] \in \text{Sub}(X, \text{id})/F_N} \mu \times \nu(\mathcal{R}(T)).
$$

Then we have $\hat{c}(\eta_H, \eta_K) = c(\Delta_H \times_{R_N} \Delta_K)$ for $H, K \in \text{Sub}(F_N)$.

Note that $\hat{c}(\mu, \nu)$ can be represented by

$$
\hat{c}(\mu, \nu) = \sum_{m=1}^{\infty} \sum_{[T] \in \text{Sub}(X, \text{id})/F_N} \mu \times \nu(\mathcal{R}(T))_{\#V(T)=m}.
$$

**Theorem 3.11.** The $\mathbb{R}_{\geq 0}$-bilinear map $\hat{c}$ is continuous.

**Proof.** Let $\mu_1, \mu_2 \in \mathcal{SCurr}(F_N)$ and $\mu_1^n, \mu_2^n \in \mathcal{SCurr}(F_N)$ $(n = 1, 2, \ldots)$ such that $\mu_1^n \to \mu_1$ $(n \to \infty)$ $(i = 1, 2)$. We prove that $\hat{c}(\mu_1^n, \mu_2^n) \to \hat{c}(\mu_1, \mu_2)$ $(n \to \infty)$.

Fix $\varepsilon > 0$. Since $\mathcal{V}: \mathcal{SCurr}(F_N) \to \mathbb{R}_{\geq 0}$ is continuous, $\mathcal{V}(\mu_1^n) \to \mathcal{V}(\mu_1)$ $(n \to \infty)$, and so we set

$$
M = \frac{3}{\varepsilon} \sup\{\mathcal{V}(\mu_1^n)\mathcal{V}(\mu_2^n) \mid n = 1, 2, \ldots\}(< \infty).
$$
Take a positive integer $L \geq M$. Then,

$$L \sum_{m=L}^{\infty} \sum_{[T] \in \text{Sub}(X, \text{id}) / F_N \#V(T) = m} \mu_1^n \times \mu_2^n(\Re(T))$$

$$\leq \sum_{m=L}^{\infty} \sum_{[T] \in \text{Sub}(X, \text{id}) / F_N \#V(T) = m} m \mu_1^n \times \mu_2^n(\Re(T))$$

$$\leq \sum_{m=1}^{\infty} \sum_{[T] \in \text{Sub}(X, \text{id}) / F_N \#V(T) = m} \mu_1^n \times \mu_2^n(\Re(T))$$

$$= \sum_{T \in \text{Sub}(X, \text{id})} \#V(T) \mu_1^n \times \mu_2^n(\Re(T))$$

$$\leq \sum_{T \in \text{Sub}(X, \text{id})} \#V(T) \mu_1^n \times \mu_2^n(\Re(T))$$

$$\leq V(\mu_1^n) V(\mu_2^n).$$

Consequently, we have

$$\sum_{m=L}^{\infty} \sum_{[T] \in \text{Sub}(X, \text{id}) / F_N \#V(T) = m} \mu_1^n \times \mu_2^n(\Re(T))$$

$$\leq \frac{1}{L} V(\mu_1^n) V(\mu_2^n) \leq \frac{1}{M} V(\mu_1^n) V(\mu_2^n) \leq \frac{\varepsilon}{3} \quad (n = 1, 2, \ldots).$$

In the same way, we have

$$\sum_{m=L}^{\infty} \sum_{[T] \in \text{Sub}(X, \text{id}) / F_N \#V(T) = m} \mu_1 \times \mu_2(\Re(T)) \leq \frac{\varepsilon}{3},$$

Since

$$\sum_{m=1}^{L-1} \sum_{[T] \in \text{Sub}(X, \text{id}) / F_N \#V(T) = m} \mu_1^n \times \mu_2^n(\Re(T))$$

is a finite sum, this converges to

$$\sum_{m=1}^{L-1} \sum_{[T] \in \text{Sub}(X, \text{id}) / F_N \#V(T) = m} \mu_1 \times \mu_2(\Re(T))$$

when $n \to \infty$. If $n$ is large enough, then the absolute value of the difference of the above two sums is smaller than $\varepsilon / 3$. Hence,

$$|\tilde{\mathcal{c}}(\mu_1^n, \mu_2^n) - \tilde{\mathcal{c}}(\mu_1, \mu_2)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof. \hfill \Box

**Proof of Theorem 3.2** We define the $\mathbb{R}_{\geq 0}$-bilinear functional

$$\mathcal{N} : \text{SCurr}(F_N) \times \text{SCurr}(F_N) \to \mathbb{R}_{\geq 0}$$
as
\[ N(\mu, \nu) := \tilde{E}(\mu, \nu) - \tilde{V}(\mu, \nu) + \tilde{c}(\mu, \nu) \]
for \( \mu, \nu \in SCurr(F_N) \). Then \( N \) is a continuous symmetric \( \mathbb{R}_{\geq 0} \)-bilinear functional, and we have \( N(\eta_H, \eta_K) = N(H, K) \) for \( H, K \in \text{Sub}(F_N) \). The uniqueness and \( \text{Out}(F_N) \)-invariance of \( N \) follows immediately from the denseness of the rational subset currents in \( SCurr(F_N) \). \( \square \)

Now we recall the inequality in the SHNC (Theorem \ref{thm:shnc}). Since the rational subset currents are dense in \( SCurr(F_N) \), and
\[ \text{rk} : SCurr(F_N) \times SCurr(F_N) \rightarrow \mathbb{R}_{\geq 0}; (\mu, \nu) \mapsto \text{rk}(\mu) \text{rk}(\nu) \]
is a continuous \( \mathbb{R}_{\geq 0} \)-bilinear map, we have the following corollary.

**Corollary 3.12.** Let \( \mu, \nu \in SCurr(F_N) \). The following inequality holds:
\[ N(\mu, \nu) \leq \text{rk}(\mu) \text{rk}(\nu). \]

4. THE INTERSECTION MAP AND THE INTERSECTION FUNCTIONAL

Let \( I \) be the intersection map:
\[ I : \mathfrak{C}_N \times \mathfrak{C}_N \rightarrow \{ \text{closed subsets of } \partial X \}; (S_1, S_2) \mapsto S_1 \cap S_2. \]
For \( \mu, \nu \in SCurr(F_N) \), the intersection map \( I \) induces a Borel measure \( I_*(\mu \times \nu) \) on \( \mathfrak{C}_N \) by pushing forward: for a Borel subset \( U \subset \mathfrak{C}_N \), we define
\[ I_*(\mu \times \nu)(U) := \mu \times \nu(I^{-1}(U)). \]
Note that it is not trivial that \( I^{-1}(U) \) is a measurable set of \( \mu \times \nu \) (see Appendix \ref{app:measure}). The Borel measure \( I_*(\mu \times \nu) \) is a subset current on \( F_N \), that is, an \( F_N \)-invariant locally finite Borel measure on \( \mathfrak{C}_N \). When we consider the diagonal action of \( F_N \) on \( \mathfrak{C}_N \times \mathfrak{C}_N \), the product measure \( \mu \times \nu \) is \( F_N \)-invariant. Therefore, \( I_*(\mu \times \nu) \) is also an \( F_N \)-invariant positive Borel measure.

Next, we check the local finiteness of \( I_*(\mu \times \nu) \). For every \( T \in \text{Sub}(X, \text{id}) \)
\[ I^{-1}(SCyl(T)) \subset \{(S_1, S_2) \in \mathfrak{C} \times \mathfrak{C} \mid \text{Conv}(S_1), \text{Conv}(S_2) \supset \text{id}\} = \bigsqcup_{T_1, T_2 \in \mathfrak{R}_1} SCyl(T_1) \times SCyl(T_2). \]
Thus we have
\[ I_*(\mu \times \nu)(SCyl(T)) = \mu \times \nu(I^{-1}(SCyl(T))) \leq \mu \times \nu \left( \bigsqcup_{T_1, T_2 \in \mathfrak{R}_1} SCyl(T_1) \times SCyl(T_2) \right) \]
\[ = V(\mu)V(\nu) < \infty. \]
This implies that \( I_*(\mu \times \nu) \) is locally finite.

We define the \( \mathbb{R}_{\geq 0} \)-bilinear map
\[ \widehat{I} : SCurr(F_N) \times SCurr(F_N) \rightarrow SCurr(F_N), \]
as
\[ \widehat{I}(\mu, \nu) := I_*(\mu \times \eta). \]
For $H, K \subseteq \text{Sub}(F_N)$ and any Borel subset $U \subseteq \mathcal{C}_N$, we have

\[
\widehat{I}(\eta_H, \eta_K)(U) = \eta_H \times \eta_K(I^{-1}(U))
\]

\[
= \sum_{(g_1 H, g_2 K) \in F_N / H \times F_N / K} \delta_{g_1 \Lambda(H)} \times \delta_{g_2 \Lambda(K)}(I^{-1}(U))
\]

\[
= \sum_{(g_1 H, g_2 K) \in F_N / H \times F_N / K} \delta_{g_1 \Lambda(H) \cap g_2 \Lambda(K)}(U).
\]

Therefore, we have the following explicit description of $\widehat{I}(\eta_H, \eta_K)$:

\[
\widehat{I}(\eta_H, \eta_K) = \sum_{(g_1 H, g_2 K) \in F_N / H \times F_N / K} \delta_{g_1 \Lambda(H) \cap g_2 \Lambda(K)}.
\]

Since $\Lambda(H' \cap K') = \Lambda(H') \cap \Lambda(K')$ for any $H', K' \subseteq \text{Sub}(F_N)$, we have

\[
\widehat{I}(\eta_H, \eta_K) = \sum_{(g_1 H, g_2 K) \in F_N / H \times F_N / K} \delta_{\Lambda(g_1 H g_1^{-1} \cap g_2 K g_2^{-1})}.
\]

**Theorem 4.1.** For $H, K \subseteq \text{Sub}(F_N)$ we have

\[
\widehat{I}(\eta_H, \eta_K) = \sum_{H g K \subseteq H \setminus F_N / K} \eta_{H \cap g K g^{-1}}.
\]

Hence,

\[
\overline{\text{rk}} \circ \widehat{I}(\eta_H, \eta_K) = \overline{\mathcal{N}}(\eta_H, \eta_K) (H, K, K \subseteq \text{Sub}(F_N)).
\]

**Proof.** Let $H, K \subseteq \text{Sub}(F_N)$. We denote by $F_N \setminus (F_N / H \times F_N / K)$ the quotient set of $F_N / H \times F_N / K$ by the natural diagonal action of $F_N$. Then, we have the bijective map:

\[
F_N \setminus (F_N / H \times F_N / K) \to H \setminus F_N / K: [g_1 H, g_2 K] \mapsto H g_1^{-1} g_2 K.
\]

Denote $H^g$ to be $g H g^{-1}$ for $g \in F_N$. Then for each $[g_1 H, g_2 K] \in F_N \setminus (F_N / H \times F_N / K)$ with fixed $g_1$ and $g_2$ we have a bijective map:

\[
[g_1 H, g_2 K] \to F_N / (H^{g_1} \cap K^{g_2}); (g_1 H, g_2 K) \mapsto g (H^{g_1} \cap K^{g_2}).
\]

For $(g_1^1 H, g_2^1 K) \in [g_1 H, g_2 K]$ we can see that $H^{g_1^1} \cap K^{g_2^1}$ is conjugate to $H^{g_1} \cap K^{g_2}$. Therefore $\eta_{H^{g_1} \cap K^{g_2}}$ does not depend on the choice of $g_1$ and $g_2$. Consequently,

\[
\widehat{I}(\eta_H, \eta_K)
\]

\[
= \sum_{(g_1 H, g_2 K) \in F_N / H \times F_N / K} \eta_{H \cap g K g^{-1}}
\]

\[
= \sum_{[g_1 H, g_2 K] \in F_N \setminus (F_N / H \times F_N / K)} \sum_{(g_1 H, g_2 K) \in [g_1 H, g_2 K]} \sum_{(g_1 H, g_2 K) \in F_N \setminus (F_N / H \times F_N / K)} \eta_{H^{g_1} \cap K^{g_2}}
\]

\[
= \sum_{H g K \subseteq H \setminus F_N / K} \eta_{H \cap g K g^{-1}},
\]

as required. \qed
Note that from Proposition 3.3 the map \( \hat{I} \) is not continuous. However, we can establish the following theorem. From Theorem 4.1 and Theorem 4.2 we can think of \( \hat{I} \) as a generalization of the construction of the fiber product graph, which we considered in the beginning of Section 3. One of the points of \( \hat{I} \) is that we do not use the Cayley graph \( X \) in the definition of \( \hat{I} \).

**Theorem 4.2.** The following equality holds:

\[
\overline{rk} \circ \hat{I} = \mathcal{N}.
\]

**Proof.** Let \( \mu, \nu \in \mathcal{S} \text{Curr}(F_N) \). We prove the above equality by representing \( \overline{rk} \circ \hat{I}(\mu, \nu) \) and \( \mathcal{N} \) explicitly. Most parts of this proof consist of technical calculations.

First, by the definition of \( \overline{rk} \) and \( \hat{I} \),

\[
\overline{rk} \circ \hat{I}(\mu, \nu) = \sum_{a \in A} \mu \times \nu \left(I^{-1}(\text{SCyl}(e_a)) - \mu \times \nu \left(I^{-1} \left( \bigcup_{T \in \mathcal{R}_1} \text{SCyl}(T) \right) \right) \right).
\]

For any \( (S_1, S_2) \in I^{-1}(\text{SCyl}(e_a)) \), we have

\[
e_a \subset \text{Conv}(S_1 \cap S_2) \subset \text{Conv}(S_1) \cap \text{Conv}(S_2),
\]

that is, \( e_a \subset \text{Conv}(S_1) \) and \( e_a \subset \text{Conv}(S_2) \). Consequently,

\[
I^{-1}(\text{SCyl}(e_a)) \subset \text{SCyl}(e_a) \times \text{SCyl}(e_a).
\]

Similarly, for any \( (S_1, S_2) \in I^{-1}(\bigcup_{T \in \mathcal{R}_1} \text{SCyl}(T)) \) we have \( \text{id} \in \text{Conv}(S_i) \) \( (i = 1, 2) \), which implies that

\[
I^{-1} \left( \bigcup_{T \in \mathcal{R}_1} \text{SCyl}(T) \right) \subset \bigcup_{T_1, T_2 \in \mathcal{R}_1} \text{SCyl}(T_1) \times \text{SCyl}(T_2).
\]

Therefore, we have

\[
\overline{rk} \circ \hat{I}(\mu, \nu)
\]

\[
= \sum_{a \in A} \mu \times \nu (\text{SCyl}(e_a) \times \text{SCyl}(e_a))
\]

\[
- \sum_{T_1, T_2 \in \mathcal{R}_1} \mu \times \nu (\text{SCyl}(T_1) \times \text{SCyl}(T_2))
\]

\[
- \sum_{a \in A} \mu \times \nu \left( \text{SCyl}(e_a) \times \text{SCyl}(e_a) \setminus I^{-1}(\text{SCyl}(e_a)) \right)
\]

\[
+ \mu \times \nu \left( \bigcup_{T_1, T_2 \in \mathcal{R}_1} \text{SCyl}(T_1) \times \text{SCyl}(T_2) \setminus I^{-1} \left( \bigcup_{T \in \mathcal{R}_1} \text{SCyl}(T) \right) \right).
\]

Here each number from (1) to (4) represents a term of the equation, and we have (1) = \( \hat{E}(\mu, \nu) \) and (2) = \( -\hat{V}(\mu, \nu) \). Hence it suffices to show that (3) + (4) = \( \hat{c}(\mu, \nu) \).

**Step 1:** First, we consider (4). Let \( (S_1, S_2) \in \text{SCyl}(e_a) \times \text{SCyl}(e_a) \). By the definition of subset cylinders, \( (S_1, S_2) \) does not belong to \( I^{-1}(\text{SCyl}(e_a)) \) if and only if either \( S_1 \cap S_2 = \emptyset \), or \( S_1 \cap S_2 \neq \emptyset \) and \( S_1 \cap S_2 \subset \text{Cyl}(e_a) \) or \( S_1 \cap S_2 \subset \text{Cyl}(e_a)^{-1} \), where we endow \( e_a \) with the orientation such that \( o(e_a) = \text{id}, t(e_a) = a \), and \( (e_a)^{-1} \) is the inverse of \( e_a \). Then, we have
\[
\left( \text{SCyl}(e_a) \times \text{SCyl}(e_a) \setminus I^{-1}(\text{SCyl}(e_a)) \right) \cap I^{-1}(\emptyset)
\]

\[
= \left( \text{SCyl}(e_a) \times \text{SCyl}(e_a) \right) \cap I^{-1}(\emptyset)
\]

\[
= \bigcup_{T \in \text{Sub}(X, \text{id})} R(T).
\]

For \( a \in A \cup A^{-1} \) put

\[
U(a) := \left( \text{SCyl}(e_a) \times \text{SCyl}(e_a) \setminus I^{-1}(\text{SCyl}(e_a)) \right) \setminus I^{-1}(\emptyset),
\]

where \( e_a \ (a \in A \cup A^{-1}) \) is the oriented edge with origin \( \text{id} \) and terminal \( a \) in \( X \).

We will use \( U(a) \) (\( a \in A^{-1} \)) later. It follows that

\[
-3 = \sum_{a \in A} \mu \times \nu \left( \text{SCyl}(e_a) \times \text{SCyl}(e_a) \setminus I^{-1}(\text{SCyl}(e_a)) \right)
\]

\[
= \sum_{a \in A} \mu \times \nu \left( \left( \text{SCyl}(e_a) \times \text{SCyl}(e_a) \setminus I^{-1}(\text{SCyl}(e_a)) \right) \cap I^{-1}(\emptyset) \right)
\]

\[
+ \sum_{a \in A} \mu \times \nu(U(a))
\]

\[
= \sum_{a \in A} \sum_{T \in \text{Sub}(X, \text{id})} \mu \times \nu(R(T)) + \sum_{a \in A} \mu \times \nu(U(a))
\]

\[
= \sum_{[T] \in \text{Sub}(X, \text{id})/F_N} \#E_{\text{top}}(T) \mu \times \nu(R(T)) + \sum_{a \in A} \mu \times \nu(U(a)).
\]

\[
(5)
\]

\[
\text{Step 2:} \quad \text{Next, we consider } 3 \text{ in a similar way. First, we have}
\]

\[
\left( \bigcup_{T_1, T_2 \in \mathcal{R}_1} \text{SCyl}(T_1) \times \text{SCyl}(T_2) \setminus I^{-1} \left( \bigcup_{T \in \mathcal{R}_1} \text{SCyl}(T) \right) \right) \cap I^{-1}(\emptyset)
\]

\[
= \left( \bigcup_{T_1, T_2 \in \mathcal{R}_1} \text{SCyl}(T_1) \times \text{SCyl}(T_2) \right) \cap I^{-1}(\emptyset)
\]

\[
= \bigcup_{T \in \text{Sub}(X, \text{id})} R(T).
\]

Put

\[
U := \left( \bigcup_{T_1, T_2 \in \mathcal{R}_1} \text{SCyl}(T_1) \times \text{SCyl}(T_2) \setminus I^{-1} \left( \bigcup_{T \in \mathcal{R}_1} \text{SCyl}(T) \right) \right) \setminus I^{-1}(\emptyset).
\]
Then,

\[ \mathcal{H} = \mu \times \nu \left( \bigcup_{T \in \text{Sub}(X, \text{id})} \mathcal{R}(T) \right) + \mu \times \nu(U) \]

(6) \[
\sum_{[T] \in \text{Sub}(X, \text{id})/F_N} \#V(T) \mu \times \nu(\mathcal{R}(T)) + \mu \times \nu(U).
\]

**Step 3:** From the equation (5) and (6), we obtain

\[ \overline{\text{rk}} \circ \hat{I}(\mu, \nu) = \hat{E}(\mu, \nu) - \hat{V}(\mu, \nu) \]

\[ - \sum_{[T] \in \text{Sub}(X, \text{id})/F_N} \#E_{\text{top}}(T) \mu \times \nu(\mathcal{R}(T)) - \sum_{a \in A} \mu \times \nu(U(a)) \]

\[ + \sum_{[T] \in \text{Sub}(X, \text{id})/F_N} \#V(T) \mu \times \nu(\mathcal{R}(T)) + \mu \times \nu(U). \]

Since for any \( T \in \widehat{\text{Sub}}(X, \text{id}) \) we have \#\( V(T) - \#E(T) = \chi(T) = 1 \),

\[ \overline{\text{rk}} \circ \hat{I}(\mu, \nu) = \hat{E}(\mu, \nu) - \hat{V}(\mu, \nu) \]

\[ + \sum_{[T] \in \text{Sub}(X, \text{id})/F_N} \mu \times \nu(\mathcal{R}(T)) + \mu \times \nu(U) - \sum_{a \in A} \mu \times \nu(U(a)) \]

\[ = E(\mu, \nu) - V(\mu, \nu) \]

\[ + \hat{c}(\mu, \nu) + \mu \times \nu(U) - \sum_{a \in A} \mu \times \nu(U(a)). \]

Now, we show that \( \mu \times \nu(U) = \sum_{a \in A} \mu \times \nu(U(a)) \). By the definition of \( U(a) \) \( (a \in A) \), for any \( (S_1, S_2) \in U(a) \) we have \( S_1 \cap S_2 \neq \emptyset \) and either \( S_1 \cap S_2 \subset \text{Cyl}(e_a) \) or \( S_1 \cap S_2 \subset \text{Cyl}((e_a)^{-1}) \). For \( e \in E(X) \) we set

\[ \mathcal{H}(e) := \{ S \subset \partial X \mid S \text{ is closed and } S \subset \text{Cyl}(e) \}. \]

Then

\[ U(a) = \left( U(a) \cap I^{-1}(\mathcal{H}(e_a)) \right) \cup \left( U(a) \cap I^{-1}(\mathcal{H}((e_a)^{-1})) \right) \]

and

\[ a^{-1}U(a) = a^{-1}\left( \text{SCyl}(e_a) \times \text{SCyl}(e_a) \right) \setminus I^{-1}(\{\emptyset\}) \]

\[ = \left( \text{SCyl}(e_a^{-1}) \times \text{SCyl}(e_a^{-1}) \right) \setminus I^{-1}(\{\emptyset\}) \]

\[ = U(a^{-1}). \]

In addition, for \( a \in A \) we have

\[ a^{-1}I^{-1}(\mathcal{H}((e_a)^{-1})) = I^{-1}(\mathcal{H}(e_a^{-1})), \]

and so

\[ a^{-1}\left( U(a) \cap I^{-1}(\mathcal{H}((e_a)^{-1})) \right) = U(a^{-1}) \cap I^{-1}(\mathcal{H}(e_a^{-1})). \]
Since $\mu \times \nu$ is $F_N$-invariant with respect to the diagonal action of $F_N$ on $\mathcal{C}_N \times \mathcal{C}_N$, we obtain

$$\sum_{a \in A} \mu \times \nu(U(a))$$

$$= \sum_{a \in A} \left\{ \mu \times \nu(U(a) \cap I^{-1}(\mathcal{H}(e_a))) + \mu \times \nu(U(a) \cap I^{-1}(\mathcal{H}((e_a)^{-1}))) \right\}$$

$$= \sum_{a \in A} \left\{ \mu \times \nu(U(a) \cap I^{-1}(\mathcal{H}(e_a))) + \mu \times \nu(U(a^{-1}) \cap I^{-1}(\mathcal{H}(e_{a^{-1}}))) \right\}$$

$$= \sum_{a \in A \cup A^{-1}} \mu \times \nu(U(a) \cap I^{-1}(\mathcal{H}(e_a)))$$

$$= \mu \times \nu \left( \bigsqcup_{a \in A \cup A^{-1}} U(a) \cap I^{-1}(\mathcal{H}(e_a)) \right),$$

where we note that if $a, a' \in A \cup A^{-1}$ and $a \neq a'$, then $\mathcal{H}(e_a) \cap \mathcal{H}(e_{a'}) = \emptyset$. Now it suffices to show that the following equality holds:

$$\bigsqcup_{a \in A \cup A^{-1}} U(a) \cap I^{-1}(\mathcal{H}(e_a)) = U.$$

The key claim for this equality is that for a given $S \in \mathcal{C}_N$, $\text{Conv}(S) \neq \emptyset$ if and only if there exists $a \in A \cup A^{-1}$ such that $S \subset \text{Cyl}(e_a)$.

Take $(S_1, S_2) \in U(a) \cap I^{-1}(\mathcal{H}(e_a))$ $(a \in A \cup A^{-1})$. Since $\text{Conv}(S_i) \supset e_a$, there exists $T_i \in \mathcal{R}_1$ such that $S_i \in \text{SCyl}(T_i)$ $(i = 1, 2)$. In addition, $S_1 \cap S_2 \neq \emptyset$. Also, $S_1 \cap S_2 \subset \text{Cyl}(e_a)$ implies that $\text{Conv}(S_1 \cap S_2) \neq \emptyset$, namely $S_1 \cap S_2 \not\subset \bigsqcup_{T \in \mathcal{R}_1} \text{SCyl}(T)$. Therefore $S_1 \cap S_2 \in U$.

Take $(S_1, S_2) \in U$. Then $S_1 \cap S_2 \neq \emptyset$ and $S_1 \cap S_2 \not\subset \bigsqcup_{T \in \mathcal{R}_1} \text{SCyl}(T)$, and so there exists $a \in A \cup A^{-1}$ such that $S_1 \cap S_2 \subset \text{Cyl}(e_a)$. Since $\text{Conv}(S_i) \supset e_a$, we have $\text{Conv}(S_i) \supset e_a$. Therefore we obtain $(S_1, S_2) \in U(a) \cap I^{-1}(\mathcal{H}(e_a))$. \qed

**Appendix A.**

The purpose of this appendix is to show that for any Borel subset $U \subset \mathcal{C}_N$, the preimage $I^{-1}(U)$ of the intersection map $I$ is a measurable set of a product measure of two Borel measures on $\mathcal{C}_N$. We prove this in a general setting.

The notation in this appendix is different from that in the main text of this paper.

Let $X$ be a compact metrizable space. Then we see that $X$ is second countable. Fix a countable basis $\{U_n\}_{n=1}^\infty$ of $X$. Let $\mathcal{C}$ be the set of all closed (compact) subsets of $X$. We provide $\mathcal{C}$ with the Vietoris topology, which has the sub-basis consisting of all sets of the forms

$$[U]_\subset := \{K \in \mathcal{C} \mid K \subset U\}$$

and

$$[U]_{\neq \emptyset} := \{K \in \mathcal{C} \mid K \cap U \neq \emptyset\},$$

where $U$ is an open subset of $X$. See [Kec95] for details of the Vietoris topology. Since $\emptyset \subset \emptyset$, $\emptyset$ is an isolated point of $\mathcal{C}$. The topology of the subspace $\mathcal{C} \setminus \{\emptyset\}$ coincides with the topology induced by the Hausdorff metric when we give a distance on $X$ which is compatible with the topology of $X$. Moreover, we can see
that \( \mathcal{C} \) is compact. Therefore \( \mathcal{C} \setminus \{\emptyset\} \) is a compact metrizable space, which implies that \( \mathcal{C} \setminus \{\emptyset\} \) is second countable, and so is \( \mathcal{C} \).

Let \( O_X \) be the set of all open subsets of \( X \) and \( \mathcal{O} \) the set of all open subsets of \( \mathcal{C} \). Then the \( \sigma \)-algebra \( \sigma(\mathcal{O}) \) generated by \( \mathcal{O} \) is the set of all Borel subsets of \( \mathcal{C} \).

Now, we consider the intersection map

\[
I : \mathcal{C} \times \mathcal{C} \to \mathcal{C}; \ (K_1, K_2) \mapsto K_1 \cap K_2.
\]

The goal of this appendix is to prove the following proposition.

**Proposition A.1.** The intersection map \( I \) is a Borel map, which means that for any Borel subset \( S \subset \mathcal{C} \) the preimage \( I^{-1}(S) \) is a Borel subset of \( \mathcal{C} \times \mathcal{C} \).

Note that a measurable set of a product measure of two Borel measures on \( \mathcal{C} \) is an element of the \( \sigma \)-algebra generated by the set

\[
\sigma(\mathcal{O}) \times \sigma(\mathcal{O}) := \{U_1 \times U_2 \mid U_1, U_2 \in \sigma(\mathcal{O})\}.
\]

Since \( \mathcal{C} \) is a second countable space, the \( \sigma \)-algebra generated by \( \sigma(\mathcal{O}) \times \sigma(\mathcal{O}) \) coincides with the \( \sigma \)-algebra generated by the set of all open subsets of \( \mathcal{C} \times \mathcal{C} \).

To prove the above proposition we prepare a “good” generating set of \( \sigma(\mathcal{O}) \) as a \( \sigma \)-algebra, and it suffices to show that \( I^{-1}(U) \) is a Borel subset of \( \mathcal{C} \times \mathcal{C} \) for \( U \) belonging to the “good” generating set of \( \sigma(\mathcal{O}) \). First, since \( \mathcal{C} \) is a second countable space, the sub-basis

\[
\{[U]_C \mid U \in O_X\} \cup \{[U]_{\neq \emptyset} \mid U \in O_X\}
\]

is a generating set of \( \sigma(\mathcal{O}) \).

From now on, we assume that the countable basis \( \{U_n\}_{n=1}^\infty \) of \( X \) is closed under finite union, that is, \( \bigcup_{i=1}^n V_i \in \{U_n \mid n = 1, 2, \ldots\} \) for any \( V_1, \ldots, V_k \in \{U_n \mid n = 1, 2, \ldots\} \).

**Lemma A.2.** The set \( \{[U]_{\neq \emptyset} \mid U \in O_X\} \) is a generating set of \( \sigma(\mathcal{O}) \).

**Proof.** Take any \( U \in O_X \). It suffices to show that \( [U]_C \) belongs to the \( \sigma \)-algebra generated by the above set. For \( U \in O_X \) and \( K \in \mathcal{C} \) we can see that \( K \) belongs to \( [U]_C \) if and only if there exists \( U_n \) such that \( U^c \subset U_n \) and \( K \cap U_n = \emptyset \). The “if” part is obvious. We prove the “only if” part. For any \( p \in U^c \) there exists \( U_{n_p} \) such that \( p \in U_{n_p} \) and \( U_{n_p} \cap K = \emptyset \). Since \( U^c \) is compact, there exist \( p_1, \ldots, p_m \in U^c \) such that \( U \subset \bigcup_{i=1}^m U_{n_{p_i}} \) and \( (\bigcup_{i=1}^m U_{n_{p_i}}) \cap K = \emptyset \). Hence there exists \( U_n \) such that \( U^c \subset U_n \) and \( U_n \cap K = \emptyset \). From the above equivalence, we obtain

\[
[U]_C = \bigcup_{U^c \subset U_n} \{K \in \mathcal{C} \mid K \cap U_n = \emptyset\} = \bigcup_{U^c \subset U_n} ([U_n]_{\neq \emptyset})^c,
\]

as required. \( \square \)

For a compact subset \( A \subset X \), set \( [A]_{\neq \emptyset} := \{K \in \mathcal{C} \mid K \cap A \neq \emptyset\} \).

**Lemma A.3.** The set \( \{[A]_{\neq \emptyset} \mid A \subset X : \text{compact}\} \) is a generating set of \( \sigma(\mathcal{O}) \).

**Proof.** First, note that for any compact subset \( A \subset X \), we have

\[
[A]_{\neq \emptyset} = \bigcap_{A \subset U_n} [U_n]_{\neq \emptyset}.
\]
Here there is $U_n$ containing $A$ since $\{U_n\}$ is closed under a finite union. Hence $[A]_\neq \emptyset$ belongs to $\sigma(O)$. For any $U \in O_X$ by taking a sequence of compact subsets $\{K_n\}$ of $X$ such that $U = \bigcup_{n=1}^{\infty} K_n$, we have
\[
[U]_{\neq \emptyset} = \bigcup_{n=1}^{\infty} [K]_{\neq \emptyset},
\]
as required.

Proof of Proposition A.7. Take any compact subset $A$ of $X$. We show that the preimage $I^{-1}([A]_{\neq \emptyset})$ belongs to the $\sigma$-algebra generated by $\sigma(O) \times \sigma(O)$.

For each positive integer $n$, take $p_n \in U_n$. Since $X$ is a metrizable space, we can define a distance function $d$ on $X$ which is compatible with the topology of $X$. For $x \in X$ and $r \geq 0$ we set
\[
B(x, r) := \{y \in X \mid d(y, x) \leq r\}.
\]
Take $(K_1, K_2) \in \mathcal{C} \times \mathcal{C}$. We show that $(K_1, K_2)$ belongs to $I^{-1}([A]_{\neq \emptyset})$ if and only if for any positive integer $k$ there exists $p_n$ such that
\[
K_1 \cap A \cap B(p_n, \frac{1}{k}) \neq \emptyset \quad (i = 1, 2).
\]

First, we prove the “only if” part. Since $K_1 \cap K_2 \cap A \neq \emptyset$, take $p \in K_1 \cap K_2 \cap A$ and take a subsequence $\{p_{j_n}\}$ of $\{p_n\}$ such that $\{p_{j_n}\}$ converges to $p$. Then for any positive integer $k$ there exists $p_{j_n}$ satisfying the above condition.

Next, we prove the “if” part by contradiction. Assume that $K_1 \cap K_2 \cap A = \emptyset$. Then there exists a positive integer $k$ such that
\[
\frac{1}{k} < \frac{1}{2}d(K_1 \cap A, K_2 \cap A).
\]
From the assumption there exists $p_n$ such that
\[
K_1 \cap A \cap B(p_n, \frac{1}{k}) \neq \emptyset \quad (i = 1, 2).
\]
Hence, we can take $a_i \in K_i \cap A$ such that $d(a_i, p_n) \leq 1/k \ (i = 1, 2)$. Therefore,
\[
dl(a_1, a_2) \leq \frac{2}{k} < d(K_1 \cap A, K_2 \cap A),
\]
which leads to a contradiction.

From the above, we have
\[
I^{-1}([A]_{\neq \emptyset}) = \bigcap_{k=1}^{\infty} \left( \bigcup_{n=1}^{\infty} [A \cap B(p_n, \frac{1}{k})]_{\neq \emptyset} \times [A \cap B(p_n, \frac{1}{k})]_{\neq \emptyset} \right),
\]
as required.

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