Algebraic analysis of a model of two-dimensional gravity

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Abstract

An algebraic analysis of the Hamiltonian formulation of the model two-dimensional gravity is performed. The crucial fact is an exact coincidence of the Poisson brackets algebra of the secondary constraints of this Hamiltonian formulation with the SO(2,1)-algebra. The eigenvectors of the canonical Hamiltonian $H_c$ are obtained and explicitly written in closed form.

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I. INTRODUCTION

In this letter we consider the model which arises from the Einstein-Hilbert (EH) action
\[ S_D = \int d^Dx \sqrt{(-1)^{D-1}} gg^{\mu\nu} R_{\mu\nu} (g) \] (1)
where \( R_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\lambda_{\sigma\lambda} \Gamma^{\sigma\nu}_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} \Gamma^{\lambda\nu}_{\sigma\nu} \) is the Ricci tensor, \( \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \) is the affine connection and the action (1) is written in terms of a metric tensor \( g^{\mu\nu} \) and its second and first derivatives (comma “,” indicates the differentiation and \( D \) is a spacetime dimension). This is a “second-order” formalism. If we treat \( g_{\alpha\beta} \) and \( \Gamma^\lambda_{\mu\nu} \) as independent variables, then we have a “first-order” formulation
\[ S_D = \int d^Dx \sqrt{(-1)^{D-1}} gg^{\mu\nu} (\Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\lambda_{\sigma\lambda} \Gamma^{\sigma\nu}_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} \Gamma^{\lambda\nu}_{\sigma\nu}) \] (2)
which was originally introduced by Einstein \[ ] (not by Palatini, as it is generally believed \[ ]). By solving equations of motion for \( \Gamma^\lambda_{\mu\nu} \) in terms of \( g_{\mu\nu} \) and substituting the solutions to \( 2 \), it is easy to show the equivalence of these second- \( 1 \) and first-order \( 2 \) formulations of EH action for the dimensions of spacetime \( D \) higher than two \( (D > 2) \) \[ 5 \].

In \( D = 2 \) the field equations cannot be solved for \( \Gamma^\lambda_{\mu\nu} \) in terms of \( g_{\mu\nu} \) \[ 5 – 7 \], which is why equation \( 2 \) does not provide an equivalent first-order formulation of EH action in 2\( D \). For the Hamiltonian treatment of real two-dimensional gravity in second-order form see \[ 8, 9 \]. Although the action \( 2 \) does not reproduce the real 2\( D \) gravity, it can nevertheless be treated as a model which we call “two-dimensional gravity” (or 2DG for short), remembering that it is not equivalent to the second-order EH action when \( D = 2 \). This model is no worse than any other two-dimensional model arising from modifications of the EH action (e.g., dilaton 2\( D \) gravity, “string-inspired” 2\( D \) gravity, etc.).

In addition, and what is more important, this model can provide a deep insight into the first-order, affine-metric, formulation of the EH action in higher dimensions \[ 5, 10 \]. First of all, the action \( 2 \) is indeed equivalent to the original second-order EH action \( 1 \) in dimensions \( D > 2 \) \[ 5 \]. Second, the structure of constraints in the 2DG model is much closer to the higher dimensional first-order gravity \( 2 \) (see \[ 5, 10, 12 \]) than the structure of constraints of the real 2\( D \) gravity (see \[ 8, 9 \]). As it was shown in \[ 8 \], the Hamiltonian formulation of the second-order EH action (or real 2\( D \) gravity) in two dimensions leads
to three primary first class constraints which generate the gauge transformations consistent with zero degrees of freedom and triviality of the Einstein equations in 2D. If the constraints structure of the real 2D gravity imitated those of the higher dimensional first-order EH action, then there would be at least two primary and two secondary first class constraints.\footnote{In \[5, 10\] we showed that in the Hamiltonian formulation of the first-order EH action in dimensions $D > 2$ tertiary constraints should also appear which is consistent with counting of degrees of freedom. The explicit form of tertiary constraints as well as the closure of the Dirac procedure was demonstrated in \[11, 12\] for all dimensions $D > 2$.}

This would produce in 2D minus one degree of freedom \[13\] meaning that the system (in such a formulation) is overconstrained and non-physical. In 2DG model three primary and three secondary first class constraints appear which is also consistent with being zero degrees of freedom, but in contrast with the real 2D gravity \[8, 9\], two of the secondary constraints of the 2DG model, \[12, 13\], as well as the Poisson brackets among them \[15\], are exactly the same as in higher dimensional first-order EH action if we replace in \[12, 13\] the index “1” by “$k$” or “$n$” ($k, n = 1, 2, \ldots, D - 1$). For details see \[5, 10–12\].

The canonical analysis of the two-dimensional gravity model can be found in \[5, 10, 14, 15\]. We will briefly outline here the Hamiltonian formulation of this model. The Lagrangian density is

$$L_2 = h^{\mu\nu} \left( \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\lambda_{\sigma\lambda} \Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} \Gamma^\sigma_{\lambda\nu} \right)$$

(3)

where $h^{\mu\nu}$ is the metric density: $h^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ ($\mu, \nu = 0, 1$ are the spacetime indices). Note that in 2D we cannot express $g^{\mu\nu}$ in terms of $h^{\mu\nu}$ because $h = \det(h^{\mu\nu}) = -(\sqrt{-g})^{D-2}$, so that in two dimensions $h = -1$; however, the metric tensor appears in the Lagrangian in the combination $\sqrt{-g} g^{\mu\nu}$.

The analysis is simplified if we use instead of $\Gamma^\lambda_{\mu\nu}$ the linear combination

$$\xi^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} - \frac{1}{2} (\delta^\lambda_{\alpha} \Gamma^\sigma_{\beta\sigma} + \delta^\lambda_{\beta} \Gamma^\sigma_{\alpha\sigma}).$$

(4)

This covariant change of variables ($\Gamma^\lambda_{\alpha\beta} \rightarrow \xi^\lambda_{\alpha\beta}$) provides an alternative first-order formulation of the EH action in dimensions $D > 2$ and for $D = 2$ it gives the alternative two-dimensional gravity model with the Lagrangian density

$$L_2 = h^{\mu\nu} \left( \xi^\lambda_{\mu\nu,\lambda} - \xi^\lambda_{\mu\sigma} \xi^\sigma_{\nu\lambda} + \xi^\lambda_{\mu\lambda} \xi^\sigma_{\nu\sigma} \right).$$

(5)
We consider (5) as a model, treating $h^{\mu\nu}$ and $\xi^\alpha_\lambda$ as independent variables and the Lagrangian density $\tilde{L} = L - (h^{\alpha\beta}\xi^\lambda_\alpha)_\lambda$. Using this Lagrangian density, $\tilde{L}$, as a starting point of the Hamiltonian formulation allows completely avoid any integration as constraints (see below) follow directly from the Hamiltonian, contrary to the usual case when an additional spatial integration is often needed to single out the common field that appears in front of a constraint. The general discussion of the role of boundary terms can be found in [16].

Introducing momenta $\pi^{\alpha\beta}$ and $\Pi^\mu\nu_\lambda$ conjugate to the variables $h^{\alpha\beta}$ and $\xi^\alpha_\mu\nu$, respectively, with the fundamental Poisson brackets (PB) among them $[h^{\alpha\beta}, \pi_{\mu\nu}] = \Delta^{\alpha\beta}_{\mu\nu} = \frac{1}{2}(\delta^\alpha_{\mu}\delta^\beta_{\nu} + \delta^\beta_{\mu}\delta^\alpha_{\nu})$ and $[\xi^\alpha_\mu\nu, \Pi^\mu\nu_\sigma] = \delta^\alpha_\sigma\Delta^{\alpha\beta}_{\mu\nu}$, we obtain the primary constraints

\[ \Phi^{\alpha\beta} = \pi^{\alpha\beta} + \xi^0_\alpha\beta \approx 0, \quad \Phi^\mu\nu_\lambda = \Pi^\mu\nu_\lambda \approx 0. \] (6)

The total Hamiltonian density is defined as

\[ H_T = H_c + h^{\alpha\beta}_0 \Phi^{\alpha\beta} + \xi^\lambda_\mu\nu,0 \Phi^\mu\nu_\lambda \] (7)

where

\[ H_c = h^{\alpha\beta}_0 \pi^{\alpha\beta} + \xi^\lambda_\mu\nu,0 \Pi^\mu\nu_\lambda - \tilde{L}_2. \] (8)

There is the second class subset of constraints among those in equation (6) which is of a special form [17]. Because these constraints are of a special form, we can eliminate the canonical pair of variables

\[ \Pi^\mu\nu_0 = 0, \quad \xi^0_\alpha\beta = -\pi^{\alpha\beta} \] (9)

from the total Hamiltonian and constraints. We then obtain the reduced total Hamiltonian density

\[ H_T = H_c + \xi^{1\mu\nu}_\mu\nu. \] (10)

The conservation in time of the primary constraints $\Phi^{\mu\nu}_1 = \Pi^{\mu\nu}_1$ leads to the secondary constraints:

\[ \chi^{11}_1 = -(h^{11}_{11} + 2h^{11}_{10}\pi^{11}_{01} + 2h^{01}_{21}\pi^{01}_{00}), \] (11)

\[ \chi^{01}_1 = -(h^{01}_{11} - h^{11}_{11}\pi^{11}_{01} + h^{00}_{00}\pi^{00}_{00}). \] (12)
\[
\chi_1^{00} = -(h_1^{00} - 2h_1^{01}\pi_{11} - 2h_1^{00}\pi_{01}).
\]

The canonical Hamiltonian density \( H_c \)

\[
H_c = -\xi_{11}^1\chi_1^{11} - 2\xi_{01}^1\chi_1^{01} - \xi_{00}^1\chi_1^{00}
\]

is just a linear combination of the secondary first class constraints (14). The secondary constraints \( \chi_1^{\alpha\beta} \) have zero PB with the primary constraints \( \Phi_1^{\alpha\beta} \) and among themselves have the following PBs

\[
[\chi_1^{01}(x,t),\chi_1^{00}(y,t)] = \chi_1^{00}(x,t)\delta(x-y),
\]

(15)

\[
[\chi_1^{01}(x,t),\chi_1^{11}(y,t)] = -\chi_1^{11}(x,t)\delta(x-y),
\]

(16)

\[
[\chi_1^{11}(x,t),\chi_1^{00}(y,t)] = 2\chi_1^{01}(x,t)\delta(x-y).
\]

(17)

The complete analysis of this model can be found in [5]. In next sections we present the analysis of the model based on local coincidence of the constraint algebra of (17) with the Lie algebra of SO(2,1).

II. ALGEBRAIC ANALYSIS OF THE MODEL 2DG

Introducing the three operators \( K_- = \chi_1^{11}, K_0 = \chi_1^{01} \) and \( K_+ = \chi_1^{00} \), equation (17) takes the form

\[
[K_0, K_+] = K_+ , \quad [K_0, K_-] = -K_- , \quad [K_-, K_+] = 2K_0.
\]

(18)

Equations (18) coincide with the commutation relations for the three generators of the SO(2,1)-algebra (see, e.g., Eq. (5.14) in [18]). This coincidence of the PB between the secondary constraints in the 2DG model and the generators of the SO(2,1)-algebra means that there is a uniform relation between the corresponding representations of these two algebras. The Hamiltonian \( H_c \), equation (14), can now be written in the form

\[
H_c = -2\xi_{01}^1K_0 - \xi_{11}^1K_- - \xi_{00}^1K_+ = -2\xi_{01}^1K_0 - i(\xi_{11}^1 - \xi_{00}^1)K_1 + (\xi_{11}^1 + \xi_{00}^1)K_2,
\]

(19)
i.e. as a linear combination of the three generators of the SO(2,1) algebra. In this equation we take
\[ K_1 = \frac{i}{2}(K_+ - K_-), \quad K_0 = K_0, \quad K_2 = -\frac{1}{2}(K_+ + K_-), \] (20)
so that \( K_{\pm} = \pm i(K_1 \pm iK_2) \). The coefficients in this linear form, (19), are some \( \xi \)-numbers which ensure the correct relation with General Relativity (GR) (see below). Note that equations (14) and (19) are the simplest linear forms which are acceptable for the \( H_c \) Hamiltonian in General Relativity.

By studying the relation between the algebra of secondary constraints of this 2DG model and the SO(2,1)-algebra we can come to some conclusions about properties and spectra of the Hamiltonian of 2DG. Furthermore, for any operator represented as a linear combination of generators of the SO(2,1)-algebra one can apply a simple procedure which allows one to determine all eigenvalues and the corresponding eigenvectors. In the present case this procedure is based on the use of coherent states constructed from the SO(2,1)-algebra [18].

The eigenvectors of \( H_c \) are the classical eigenstates. However, they are closely related to the corresponding quantum states. Indeed, for any dynamical system with a classical analogue, a state for which quantum description is valid is represented in quantum mechanics by a wave packet [19]. In the Schrödinger representation such a wave function is of the form \( \Psi(q,t) = A(q,t) \cdot \exp(i \frac{S(q,t)}{\hbar}) \), where \( A \) and \( S \) are the amplitude and phase of the total wave function \( \Psi \). It can be shown [19] that the phase function \( S(q,t) \) satisfies the following equation (to the lowest order in \( \hbar \))
\[ \frac{\partial S}{\partial t} = -H_c \left( q, \frac{\partial S}{\partial q} \right) \] (21)
which is known as the Hamilton-Jacobi equation (compare with equation (46) below). Note that this equation involves the classical Hamiltonian \( H_c \), in which all momenta are replaced by the corresponding partial derivatives of the Jacobi function \( S \).

The eigenvectors of \( H_c \) can be used to obtain further information about the Hamiltonian formulation of the 2DG model. Furthermore, using the conclusions drawn from the 2DG model we can predict some useful properties of the Hamiltonian \( H_c \) in four-dimensional and \( N \)-dimensional GR. This is the main goal of our work.
III. SELF-ADJOINT IRREDUCIBLE REPRESENTATIONS OF THE SO(2,1)-
ALGEBRA

First of all, let us describe the self-adjoint irreducible representations of the SO(2,1)-
algebra \cite{20}. By using the operators $K_-, K_+$ and $K_0$ we can construct the Casimir operator

$$
\hat{C}_2 = K_0^2 - \frac{1}{2}(K_+K_- + K_-K_+) = K_0^2 - K_0 - K_+K_-.
$$

(22)

This operator commutes with the $K_-, K_+, K_0$. Moreover, $\hat{C}_2$ commutes with an arbitrary
function of these three operators. In particular, it commutes with the $H_c$ operator defined
above. As follows from Schur’s lemma, the operator $\hat{C}_2$ is diagonal, i.e., $\hat{C}_2 = \lambda \hat{I}$, where $\hat{I}$
is the unit operator. In the case of SO(2,1)-algebra the numerical constant $\lambda$ is designated
as $k(k-1)$, where $k$ is some number (see below).

All self-adjoint and irreducible representations of the SO(2,1)-algebra can be constructed
with the use of $\hat{C}_2$ and $K_0$ operators. In general, for the SO(2,1)-algebra one finds the
two discrete and two continuous series of representations (see, e.g., \cite{20}). There are the
positive and negative discrete series, and principal and supplementary series of continuous
representations. First, consider the positive discrete series. Analysis of the negative discrete
series is almost identical and essentially based on the use of the same derivation. In the case
of positive discrete series the two following conditions must be obeyed for each of the $| k, m \rangle$
basis vectors

$$
\hat{C}_2 | k, m \rangle = k(k-1) | k, m \rangle, \quad K_0 | k, m \rangle = m | k, m \rangle,
$$

(23)

where $k$ is semi-integer: $k = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$, while $m = k, k+1, \ldots, k+n, \ldots$ ($n$ is a nonnegative
integer). In other words, we need to determine these vectors from the following eigenvalue
equations

$$
[(\chi^{01}_1)^2 - \chi^{01}_1 - \chi^{11}_1 \chi^{00}_1] | k, m \rangle = k(k-1) | k, m \rangle, \quad \chi^{01}_1 | k, m \rangle = m | k, m \rangle.
$$

(24)

For continuous series of representations the last two equations are traditionally written in
the form

$$
[(\chi^{01}_1)^2 - \chi^{01}_1 - \chi^{11}_1 \chi^{00}_1] | \lambda, \mu \rangle = k(k-1) | \lambda, \mu \rangle, \quad \chi^{01}_1 | \lambda, \mu \rangle = \mu | \lambda, \mu \rangle,
$$

(25)

where $k = \frac{1}{2} + i \lambda, \mu = 0, \pm 1, \pm 2, \ldots$, while $\lambda$ is an arbitrary real number (see below).
The theory of representations of the \(SO(2,1)\)-algebra is a very well developed area of theoretical physics (see, e.g., [18] and references therein). The non-compact \(SO(2,1)\)-algebra and its representations were introduced for the first time by Bargmann [21]. The most detailed and complete analysis of the \(SO(2,1)\)-algebra and its representations can be found in [20]. Nevertheless, in applications to the problems of 2DG the Casimir operator \(\hat{C}_2\) of the \(SO(2,1)\)-algebra plays a very restricted, secondary role. The actual Hamiltonian of the problem \(H_c\) has a different structure. Below, the structure of \(H_c\) is considered in detail.

One needs to develop an approach which allows one to construct the eigenvectors of the \(H_c\) operator from the \(|k,m\rangle\) basis vectors (or \(|\lambda,\mu\rangle\) basis vectors).

IV. PROPERTIES OF THE \(H_c\) HAMILTONIAN

The most important feature of the \(H_c\) operator (or \(H_c\) Hamiltonian) follows from its explicit form, equations (14) and (19). Briefly, the \(H_c\) operator is the linear and homogeneous combination of the three operators \(\chi_{11}^{11}, \chi_{01}^{01}, \chi_{10}^{10}\) (or \(K_-, K_0, K_+\)) and three values \(\xi_{00}, \xi_{01}, \xi_{11}\) which depend upon the components of the metric tensor \(g_{\mu\nu}\) and their derivatives. (Note that the values \(\xi_{00}, \xi_{01}, \xi_{11}\) are linearly related to the affine connection \(\Gamma^\lambda_{\alpha\beta}\) as

\[
\Gamma^\lambda_{\alpha\beta} = \xi^\lambda_{\alpha\beta} - \frac{1}{D-1} (\delta^\lambda_\alpha \xi_{\beta\sigma} + \delta^\lambda_\beta \xi_{\alpha\sigma})
\]

which can be found by inverting equation (11).

The linear invertible relation between \(\xi^\lambda_{\alpha\beta}\) and \(\Gamma^\lambda_{\alpha\beta}\) means that \(\xi_{11}, \xi_{01}\) and \(\xi_{00}\) do not form a two-dimensional tensor as \(\Gamma^\lambda_{\alpha\beta}\) is not a tensor. Furthermore, all these three equal zero identically in the case of a Galilean two-dimensional system. It is easy to understand that only such values can be used in General Relativity in order to obey the fundamental principle of equivalence. This explains why only the Hamiltonian, (14), is acceptable in GR, but all alternative ‘Hamiltonians’ which can be constructed from the \(SO(2,1)\)-algebra, e.g., taking the Hamiltonian to be \(H = K_0^2 - K_0 - K_+ K_-\), make no sense in GR. Below, to emphasize the non-tensorial nature of \(\xi_{11}, \xi_{01}\) and \(\xi_{00}\) we shall call them the \(\xi-\)symbols. Note that in this notation the affine connections \(\Gamma^\lambda_{\alpha\beta}\) can be also recognized as the \(\xi-\)symbols.

The Hamiltonian, \(H_c\), can be presented in the equivalent form as

\[
H_c = -2\xi_{01}^1 K_0 - i(\xi_{11}^1 - \xi_{00}^1)K_1 + (\xi_{11}^1 + \xi_{00}^1)K_2 = 2\sqrt{-G}(n_0 \cdot K_0 - n_1 \cdot K_1 - n_2 \cdot K_2),
\]
where $G = \xi_{00}^{1} \xi_{11}^{1} - (\xi_{01}^{1})^{2}$ is the determinant of the symmetric $2 \times 2$ matrix formed from three $\xi-$numbers: $\xi_{00}^{1}$, $\xi_{01}^{1}$, and $\xi_{11}^{1}$. We have introduced the three-dimensional unit-norm vector $\mathbf{n} = (n_{0}, n_{1}, n_{2})$ (in pseudo-Euclidean $(2,1)$-space) whose components are

$$n_{0} = -\frac{\xi_{01}^{1}}{\sqrt{-G}} , \quad n_{1} = \frac{\xi_{11}^{1} - \xi_{00}^{1}}{2\sqrt{-G}} , \quad n_{2} = -\frac{\xi_{11}^{1} + \xi_{00}^{1}}{2\sqrt{-G}}.$$  \hspace{1cm} (28)

For this vector $\mathbf{n}$ we always have $n_{0}^{2} - n_{1}^{2} - n_{2}^{2} = 1$. All vectors used below are in the pseudo-euclidean $(2,1)$-space only. These vectors in the $(2,1)$ pseudo-euclidean space have nothing to do with the real vectors and/or tensors in the original two-dimensional Riemannian spacetime with the metric $g_{\mu\nu}$. The actual vectors and/or tensors transform according the rules dictated by the metric tensor which has three independent components $g_{00}, g_{01}(= g_{10})$ and $g_{11}$ in two-dimensional spacetime. In contrast, vectors in the $(2,1)$ pseudo-euclidean space are only formal constructions. They are needed to describe self-adjoint, irreducible representations of the non-compact SO$(2,1)$-algebra.

The general form of the $H_{c}$ Hamiltonian of (19), $H_{c} \simeq (\mathbf{n} \cdot \mathbf{K})$, is very similar to the chirality operator $(\mathbf{n} \cdot \mathbf{S})$ for moving particles where $\mathbf{S}$ is the spin and $\mathbf{n}$ is the direction of motion. Analogous chirality operators are defined for various fields. In the case of the 2DG model the Hamiltonian, $H_{c}$ in equation (19), is formally written as the scalar product of the two $(2,1)$-vectors $\mathbf{K} = (K_{0}, K_{1}, K_{2})$ and $\mathbf{n} = (n_{0}, n_{1}, n_{2})$. The vector $\mathbf{n}$ can be considered as a ‘direction of propagation’ of the free field defined in the $(2,1)$ pseudo-euclidean space. All components of this vector are $\xi-$symbols. Analogically the vector $\mathbf{K}$ represents some internal property of this field. The components of this $(2,1)$-vector, $\mathbf{K}$, are the three secondary constraints of 2DG. All secondary constrains do not change with time (see, e.g., [22]), so in this sense the Hamiltonian, $H_{c}$, can be considered as the ‘chirality operator’ of the 2DG model.

V. EIGENVECTORS OF THE $H_{c}$ HAMILTONIAN

The Hamiltonian, $H_{c}$, in (27) is the linear form of the three $\xi-$numbers and the three generators of the SO$(2,1)$-algebra. In general, for an arbitrary operator which is represented as a linear combination of three generators of this algebra, there is a well developed procedure which can be used to obtain the eigenvalues and eigenvectors of this operator based on the use of coherent states of the SO$(2,1)$-algebra.
The system of coherent states for the discrete series of $SO(2,1)$-algebra representations has been constructed in [18]. In our paper we shall follow the procedure described in [18]. At the first step one chooses an arbitrary vector $|\psi_0\rangle$. It is shown in [18] that there are some advantages to choosing such a vector to be in the form $|\psi_0\rangle = |k,k\rangle$, i.e. the vector $|k,k+m\rangle$ for which $m = 0$. The coherent states derived from the $|k,k\rangle$ vector have properties which are similar to the properties of the corresponding ‘classical’ states. At the second stage we represent the unit-norm pseudo-euclidean vector $n = (n_0, n_1, n_2)$ in the following two-parameter form

$$
(n_0, n_1, n_2) = (\cosh \tau, \sinh \tau \cos \phi, \sinh \tau \sin \phi).
$$

(29)

These vectors can be used to designate a corresponding coherent state $|n\rangle$. Moreover, $|n\rangle = D(n) |\psi_0\rangle$, where $D(n)$ is some operator which is represented in the following three-parameter form

$$
D(n) = \exp(\alpha K_-) \exp(\beta K_0) \exp(\gamma K_+),
$$

(30)

where $\beta = -\ln(1 - |\alpha|^2)$ and $\gamma = -\overline{\alpha}$ ( $\overline{\alpha}$ is the complex conjugate to $\alpha$). The coherent state $|n\rangle(= D(n) |\psi_0\rangle)$ can be designated with the use of this one parameter ($\alpha$) only (see, equation (31) below). There is a relation between the parameter $\alpha$ and three components of the vector $n$ (29), given by $\alpha = \tanh(\xi) \exp(i\phi)$. The transition from variables $n$ (or $\tau, \phi$) to the variable $\alpha$ corresponds to the stereographic projection from south pole of hyperboloid, i.e., $n_0 = (-1, 0, 0)$, on the complex $\alpha$-plane.

The operator $D(n)$ defined in equation (30) is similar to the $D$-matrix known for the compact $SO(3)$-algebra. Here we do not discuss this analogy in detail (such a discussion can be found in [18], see also references therein). For our present analysis it is important to write the coherent states as infinite expansions upon the basis set of unit-norm vectors $|k,m\rangle$ defined above for the positive series of representations (see equations (23) and (24))

$$
|\alpha\rangle = |\alpha, \beta(\alpha), \gamma(\alpha)\rangle = (1 - |\alpha|^2) \sum_{m=0}^{\infty} \sqrt{(m + 2k - 1)! / m!(2k - 1)!} \cdot \alpha^m |k, k + m\rangle,
$$

(31)

where $\alpha$ is one of the three parameters of the coherent state $|\alpha, \beta, \gamma\rangle$. The properties of these coherent states are discussed in [18]. The most important of these properties is: this state is an eigenstate of the $(n_0 \cdot K_0 - n_1 \cdot K_1 - n_2 \cdot K_2)$ operator. This immediately follows
from our choice of the unit vector \(| \mathbf{n} \rangle\) and from the identity \(D(\mathbf{n})K_0D^{-1}(\mathbf{n}) = (\mathbf{n} \cdot \mathbf{K})\), where \(\mathbf{K} = (K_0, K_1, K_2)\). Indeed, from the definition \(| \mathbf{n} \rangle = D(\mathbf{n}) | \psi_0 \rangle\) we can write \(D^{-1}(\mathbf{n}) | \mathbf{n} \rangle = | \psi_0 \rangle\); and therefore

\[
K_0(D^{-1}(\mathbf{n}) | \mathbf{n} \rangle) = K_0 | \psi_0 \rangle = k | \psi_0 \rangle, \tag{32}
\]

since \(| \psi_0 \rangle\) was chosen to be an eigenvector of \(K_0\). From here one finds

\[
D(n)K_0D^{-1}(n) | n \rangle = k | \psi_0 \rangle. \tag{33}
\]

On the other hand, we have the identity \(D(\mathbf{n})K_0D^{-1}(\mathbf{n}) = (\mathbf{n} \cdot \mathbf{K})\). By combining equation \([33]\) and this identity we obtain

\[
(n \cdot K) | n \rangle = k | n \rangle. \tag{34}
\]

In other words, the vector \(| \mathbf{n} \rangle\) is an eigenvector of the \((\mathbf{n} \cdot \mathbf{K})\) operator and \(k\) is its eigenvalue.

In our notation this eigenvalue equation can also be written in the form \((\mathbf{n} \cdot \mathbf{K}) | \alpha \rangle = k | \alpha \rangle\), where \(\alpha\) is the complex parameter which determines the coherent state for the discrete series of representations of the \(SO(2,1)\)-algebra. It follows from here that for the \(| \alpha \rangle\) vector the following equation is also obeyed

\[
H_c | \alpha \rangle = 2\sqrt{-G}(n_0 \cdot K_0 - n_1 \cdot K_1 - n_2 \cdot K_2) | \alpha \rangle = 2k\sqrt{-G} | \alpha \rangle. \tag{35}
\]

This means that the coherent state \(| \alpha \rangle\) (= \(| \alpha, \beta, \gamma \rangle\) is the eigenvector of the \(H_c\) operator with the eigenvalue \(\lambda = 2k\sqrt{-G}\). This eigenvalue equals zero in any flat two-dimensional spacetime. It should be mentioned that there is another condition which is always obeyed for the \(| \alpha \rangle\) vector [18],

\[
(K_- - 2\alpha K_0 + \alpha^2 K_+) | \alpha \rangle = 0. \tag{36}
\]

The role of this condition for the 2DG model is not quite clear, since it contains a mixture of the regular numbers (unity) and \(\xi\)-values (\(\alpha\) and \(\alpha^2\)).

The discrete series of self-adjoint irreducible representations of the \(SO(2,1)\)-algebra constructed above (see equations \([31]\) and \([34]\)) is quite restricted when considering actual problems of GR. It is clear that we need to construct analogous coherent states for the main (or continuous) series of self-adjoint irreducible representations of the \(SO(2,1)\)-algebra. Let \(| m \rangle \equiv k, m \rangle\) be the unit-norm basis in some Hilbert space \(\mathcal{H}\). All these vectors are the eigenvectors of the operator \(K_0\), i.e. \(K_0 | \mu \rangle = \mu | \mu \rangle\). Also, they are eigenvectors of the
Casimir operator $\hat{C}_2$ defined above. The coherent states can be expanded in terms of these basis vectors. Below, we shall designate the corresponding coherent state by $| \alpha \rangle$, while the notation $| m \rangle$ always means the eigenvectors of the $K_0$ and $\hat{C}_2$ operators.

The coherent state $| \alpha \rangle$ can be represented as an infinite sum of eigenstates $| m \rangle$, i.e.

$$| \alpha \rangle = \sum_{n=-\infty}^{\infty} u_n(\alpha) | n \rangle$$  \hspace{1cm} (37)

where the coefficients $u_n(\alpha)$ are

$$u_n(\alpha) = \langle n | \alpha \rangle = u^\lambda_n(\tau, \phi).$$  \hspace{1cm} (38)

Here $\tau$ and $\phi$ are the parameters which define the unit-norm pseudo-euclidean vector $\mathbf{n} = (n_0, n_1, n_2)$ \hspace{1cm} (29). For continuous series of the irreducible representation of the SO(2,1)-algebra the relation between parameters $\alpha$ and $\tau, \phi$ is $\alpha = -\tanh(\frac{\tau}{2}) \exp(i\phi)$; i.e. it differs by sign from the analogous relation used above for the discrete series.

The coefficients $u^\lambda_n(\tau, \phi)$ defined in equation (38) have three following properties \hspace{1cm} [18]

$$u^\lambda_n(0, \phi) = \delta_{0n}, \hspace{0.5cm} u^\lambda_n(\tau, \phi) = \exp(-i\phi) R^\lambda_n(\tau)$$  \hspace{1cm} (39)

and

$$\sum_{n=-\infty}^{\infty} | u^\lambda_n(\tau, \phi) |^2 = 1$$  \hspace{1cm} (40)

for arbitrary $\tau, \phi$ and $\lambda$. The last equality follows from the fact that all coherent states $| \alpha \rangle$ have unit norm.

It can also be shown that these coefficients $u^\lambda_n(\tau, \phi)$ coincide with the corresponding eigenfunctions of the Laplace-Beltrami operator $\tilde{\Delta}$ (see, e.g., \hspace{1cm} [23]) constructed for the Lobachevskii plane; i.e.

$$\tilde{\Delta} u^\lambda_n(\tau, \phi) = \left[ \frac{\partial^2}{\partial \tau^2} + \cosh \tau \frac{\partial}{\partial \tau} + \frac{1}{\sinh^2 \tau} \frac{\partial^2}{\partial \phi^2} \right] u^\lambda_n(\tau, \phi) = \Lambda u^\lambda_n(\tau, \phi) = -\left( \frac{1}{4} + \lambda^2 \right) u^\lambda_n(\tau, \phi)$$  \hspace{1cm} (41)

for principal series of SO(2,1)-representations $\lambda$ is an arbitrary real number, while for the supplementary series: $\lambda = i\sigma$, where $\sigma$ is also real, but $| \sigma | \leq \frac{1}{2}$. Below, we shall consider only the principal series.

Note that the $u^\lambda_n(\tau, \phi)$ functions are the eigenfunctions of the two commuting and self-adjoint operators $\tilde{\Delta}$ and $-i \frac{\partial}{\partial \phi}$. This means that these functions of the $\tau$ and $\phi$ variables form
a complete system of orthogonal (basis) functions on the Lobachevskii plane. In other words, an arbitrary function of the \( \tau \) and \( \phi \) variables can be approximated by linear combinations of the \( u^\lambda_n(\tau, \phi) \) functions. For the principal series of representations of the \( \text{SO}(2,1) \)-algebra the orthogonality relation for the \( u^\lambda_n(\tau, \phi) \) functions takes the form

\[
\frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} T_n^\lambda(\tau, \phi) u^\lambda_{n_1}(\tau, \phi) \sinh \tau d\tau d\phi = N_n(\lambda) \delta_{n_{n_1}} \delta(\lambda - \lambda_1). \tag{42}
\]

To conclude this section we want to note that there is an obvious analogy between the \( u^\lambda_n(\tau, \phi) \) functions in the Lobachevskii plane and the plane waves \( \exp(\mathbf{i} k \mathbf{r}) \) in the Euclidean plane (for more details, see [18]). The second comment is related to the fact that the functions \( \sqrt{\sinh \tau} R^\lambda_n(\tau) \), where \( R^\lambda_n(\tau) \) are defined in equation (39), obey the non-relativistic Schrödinger equation with the (scattering) potential \( V(\tau) = (n^2 - \frac{1}{4}) \sinh^{-2} \tau \). This analogy allows one to obtain many additional properties of the \( u^\lambda_n(\tau, \phi) \) functions.

VI. THE HAMILTON-JACOBI METHOD

It is crucial for our analysis that the canonical Hamiltonian density of the problem, \( H_c \) (14), contains only secondary constraints which do not change with time \( t \) [22]. The only variables which are included in the \( H_c \) Hamiltonian density are \( \xi_{00}, \xi_{01} \) and \( \xi_{11} \). The variables \( (t, \xi_{00}, \xi_{01} \text{ and } \xi_{11}) \) can be considered as the four actual variables of the problem. In this case it follows from (14) that

\[
H_c \delta t = -\chi_{11}^{11} \delta \xi_{11} - 2\chi_{10}^{01} \delta \xi_{01} - \chi_{00}^{00} \delta \xi_{00} \tag{43}
\]

or, in other words,

\[
\frac{\partial H_c}{\partial \xi_{00}} = -\chi_{11}^{11}, \quad \frac{\partial H_c}{\partial \xi_{01}} = -2\chi_{10}^{01}, \quad \frac{\partial H_c}{\partial \xi_{11}} = -\chi_{00}^{00}. \tag{44}
\]

This means that we can write

\[
H_c = \xi_{11}^{11} \frac{\partial H_c}{\partial \xi_{11}} + \xi_{01}^{01} \frac{\partial H_c}{\partial \xi_{01}} + \xi_{00}^{00} \frac{\partial H_c}{\partial \xi_{00}} \tag{45}
\]

where all partial derivatives on the right-hand side do not depend upon \( t \). Bearing this equation in mind, let us try to find the function \( S(t, \xi_{00}, \xi_{01}, \xi_{11}) \) for which the following equation is always obeyed

\[
\frac{\partial S}{\partial t} = -\xi_{11}^{11} \frac{\partial S}{\partial \xi_{11}} - \xi_{01}^{01} \frac{\partial S}{\partial \xi_{01}} - \xi_{00}^{00} \frac{\partial S}{\partial \xi_{00}}. \tag{46}
\]
In particular, we can try to represent the function \( S(t, \xi_{00}, \xi_{01}, \xi_{11}) \) in the form \( S(t, \xi_{00}, \xi_{01}, \xi_{11}) = f(t) \cdot S_0(\xi_{00}, \xi_{01}, \xi_{11}) \), where \( S_0(x, y, z) \) is a homogeneous function of power \( b \), i.e.
\[
x \frac{\partial S_0}{\partial x} + y \frac{\partial S_0}{\partial y} + z \frac{\partial S_0}{\partial z} = b S_0, \tag{47}
\]
where \( b \) is a real number. In this case from (46) one finds
\[
\frac{df(t)}{dt} = -bf(t), \tag{48}
\]
or \( f(t) = A \exp(-bt) \). Therefore, the function \( S(t, \xi_{00}, \xi_{01}, \xi_{11}) \) does exist and can be found.
Moreover, it can be presented in the form
\[
S(t, \xi_{00}, \xi_{01}, \xi_{11}) = A \exp(-bt) \cdot S_0(\xi_{00}, \xi_{01}, \xi_{11}), \tag{49}
\]
where \( S_0(x, y, z) \) is an arbitrary homogeneous function of power \( b \). In the last equation the variables \( \xi_{00}, \xi_{01} \) and \( \xi_{11} \) do not depend upon \( t \). The function \( S(t, \xi_{00}, \xi_{01}, \xi_{11}) \) depends upon time \( t \) only by the exponential factor (decaying factor for \( b > 0 \)). In general, the function \( S(t, \xi_{00}, \xi_{01}, \xi_{11}) \) is the Jacobi function, while \( S_0 \) is the so-called short Jacobi function.

VII. SECOND QUANTIZED FORM OF THE \( H_c \) HAMILTONIAN

The Hamiltonian, \( H_c \), can be represented in a second quantized form. In fact, such a form immediately follows from equation (27) and the following theorem about unitary representations of the SO(2,1)-algebra. Let \( a_1, a_2, a_1^+ \) and \( a_2^+ \) be the four bosonic operators for which the following commutation relations are obeyed \([a_i, a_j^+] = \delta_{ij}, [a_i^+, a_j^+] = 0 \) and \([a_i, a_j] = 0 \), where \( i = 1, 2 \) and \( j = 1, 2 \). In this case the three following operators
\[
X_1 = \frac{i}{2}(a_1^+ a_2 + a_2^+ a_1) \quad X_2 = -\frac{1}{2}(a_1^+ a_2 - a_2^+ a_1) \quad X_0 = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2) \tag{50}
\]
form the SO(2,1)-algebra [24]. Furthermore, let \(| \phi, m \rangle = N_{\phi, m}(a_1^+)^{\phi+m}(a_2^+)^{-m} | 0 \rangle \) be the basis vectors, where \( a_1 | 0 \rangle = 0 \) and \( a_2 | 0 \rangle = 0 \). On these vectors one finds for the operators \( X_0 \) and \( \hat{C}_2 = X_0^2 - X_1^2 - X_2^2 \)
\[
X_0 | \phi, m \rangle = m | \phi, m \rangle \quad \hat{C}_2 | \phi, m \rangle = \phi(\phi + 1) | \phi, m \rangle. \tag{51}
\]
The operator \( \hat{C}_2 \) is the Casimir operator of the SO(2,1)-algebra.
Let us discuss the unitary representations of the SO(2,1)-algebra. In this case \( \phi(\phi + 1) \) must be real and all three operators \( X_0, X_1, X_2 \) must be self-adjoint. The positive series of unitary representations of the SO(2,1)-algebra is obtained by applying the condition \( m = -\phi, -\phi + 1, \ldots, -\phi + n, \ldots \). The negative series corresponds to the choice \( m = \phi, \phi - 1, \ldots, \phi - n, \ldots \). The choice \( \phi = -\frac{1}{2} + i\rho \), where \( \rho \) is real, represents the principal series of unitary representations of the SO(2,1)-algebra. Here we do not want to discuss applications of this theorem to various problems. Note, however, that this theorem allows one to represent the Hamiltonian, \( H_c \), in the second quantized form. Indeed, from equation (27) and equation (50) one finds

\[
H_c = \sqrt{-G}\left[n_0 \cdot (a_1^+ a_1 - a_2^+ a_2) - i \cdot n_1 \cdot (a_1^+ a_2 + a_2^+ a_1) - n_2 \cdot (a_1^+ a_2 - a_2^+ a_1)\right]
\]

(52)

where all parameters in this formula have been defined in the main text. This form of \( H_c \) is of interest in Quantum Gravity.

Note that the commutation relations between operators \( a_i \) and \( a_j^+ \) \((i,j = 1, 2)\) mentioned above exactly coincide with the Poisson brackets between a set of canonical variables known from the Hamilton Classical Mechanics (see, e.g., [25]). Therefore, we can consider, Eq.(14), as the Hamiltonian \( H_c \) which is already written in canonical variables. Let us obtain the canonical equations for four operators \( a_i \) and \( a_j^+ \) \((i,j = 1, 2)\), i.e. for these canonical variables. By using the explicit form of \( H_c \), Eq.(14), one finds

\[
\frac{da_1}{dt} = [a_1, H_c] = -\xi_{01}^1 a_1 + \xi_{11}^1 a_2,
\]

(53)

\[
\frac{da_2}{dt} = [a_2, H_c] = -\xi_{00}^1 a_1 + \xi_{01}^1 a_2.
\]

(54)

Analogous equations for the \( a_j^+ \) \((j = 1, 2)\) are

\[
\frac{da_1^+}{dt} = [a_1^+, H_c] = \xi_{01}^1 a_1^+ + \xi_{00}^1 a_2^+,
\]

(55)

\[
\frac{da_2^+}{dt} = [a_2^+, H_c] = -\xi_{11}^1 a_1^+ - \xi_{01}^1 a_2^+.
\]

(56)

To determine the actual time-dependence of these operators, let us assume that \( a_i(t) = a_i(0) \exp(i\omega t) \), and therefore, \( a_i^+(t) = a_i^+(0) \exp(-i\omega t) \). In this case from equations of motion...
Eqs. (53) - (54) one obtains

\[ \omega a_1 = -\xi_{01}^1 a_1 + \xi_{11}^1 a_2, \]  
\[ \omega a_2 = -\xi_{00}^1 a_1 + \xi_{01}^1 a_2. \]  

(57)  

(58)

This system of equations has a non-trivial solution if the determinant of the following $2 \times 2$ matrix

\[
\begin{pmatrix}
\xi_{01}^1 + i\omega & -\xi_{11}^1 \\
\xi_{00}^1 & -\xi_{01}^1 + i\omega
\end{pmatrix}
\]

equals zero. In this case the solutions are

\[ \omega_{1,2} = \pm \sqrt{(\xi_{01}^1)^2 - \xi_{00}^1 \xi_{11}^1}. \]  

(59)

Without loss of generality, we shall choose the positive root, i.e. $\omega = \omega_1 = \sqrt{(\xi_{01}^1)^2 - \xi_{00}^1 \xi_{11}^1}$. In this case the evolution in time of the $a_1(t)$ and $a_2(t)$ operators is represented in the form $a_i(t) = a_i(0) \exp(i\omega t)$. Other possible forms of time dependence for these two operators will not be discussed in this study. Now, we can introduce the two pairs of conjugate operators $Q_i$ and $P_i$ which are simply and canonically related to the $a_i$ and $a_i^+$ operators

\[ a_i = \frac{1}{\sqrt{2\omega}} (\omega Q_i + iP_i) \]  

(60)

and

\[ a_i^+ = \frac{1}{\sqrt{2\omega}} (\omega Q_i - iP_i). \]  

(61)

The inverse relations take the form

\[ Q_i = \frac{1}{\sqrt{2\omega}} (a_i + a_i^+) \]  

(62)

and

\[ P_i = i\sqrt{\frac{\omega}{2}} (a_i^+ - a_i). \]  

(63)

Note that the operators $P_i$ are Hermitian. From (62) and (63) one finds

\[ [Q_i, Q_j] = 0, \quad [Q_i, P_j] = i\delta_{ij} \quad (\text{or} \quad [P_i, Q_j] = -i\delta_{ij}), \quad [P_i, P_j] = 0 \]  

(64)

where $i = 1, 2$ and $j = 1, 2$. In other words, the two ‘coordinates’ $Q_1, Q_2$ and two momenta $P_1, P_2$ can be considered as the new canonical variables, which are related to the old canonical
variables $a_1, a_2, a_1^+, a_2^+$ by a canonical transformation, Eqs.(60) - (63). The Hamiltonian $H_c$, Eq.(52), written in these new canonical variables takes the form

$$H_c = -\frac{\xi_{01}}{2\sqrt{(\xi_{01})^2 - \xi_{00}\xi_{11}}} \left[ \left( P_1^2 + \omega^2 Q_1^2 \right) - \left( P_2^2 + \omega^2 Q_2^2 \right) \right]$$

$$+ \frac{(\xi_{11} - \xi_{01})}{2\sqrt{(\xi_{01})^2 - \xi_{00}\xi_{11}}} \left( P_1 P_2 + \omega^2 Q_1 Q_2 \right)$$

$$- \frac{i(\xi_{11} + \xi_{00})}{2\sqrt{(\xi_{01})^2 - \xi_{00}\xi_{11}}} (Q_1 P_2 - Q_2 P_1).$$

(65)

By using this form of $H_c$ written in the canonical variables $P_1, P_2, Q_1, Q_2$ it is straightforward to derive the corresponding canonical equations for these variables. In general, the Hamiltonian density, Eq.(55), corresponds to the case of two coupled harmonic oscillators. Formally, the quantization of the canonical equations for the $P_1, P_2, Q_1, Q_2$ variables does not present any difficulty. Note that the coupling of the two classical oscillators is described by the two last terms in Eq.(55). Briefly, we can say that such a coupling cannot be found in any vibrational system known in classical mechanics. The next step of our procedure is to perform the explicit quantization of the Hamiltonian $H_c$ and a related system of canonical equations which represent the two-dimensional gravity.

VIII. CONCLUSION

We have analyzed the algebraic structure of the model of two-dimensional gravity. It is shown that the algebraic structure of this model is locally isomorphic to the SO(2,1)-algebra. The canonical Hamiltonian of 2DG is expressed as a linear combination of the three generators of this algebra. These generators coincide with the three secondary first class constraints defined in the model. These secondary constraints do not change with time. The coefficients included in the Hamiltonian are the $\xi -$numbers which are uniformly related to the affine connections $\Gamma^\lambda_{\alpha\beta}$. The linear form of the Hamiltonian allows us to apply a special procedure which has been developed earlier to determine its eigenvalues. This procedure is based on the use of coherent states for the SO(2,1)-algebra. We consider the coherent states constructed for the discrete and principal series of representations of the SO(2,1)-algebra.

Finally, the analysis of the 2DG model can be presented in the following way. If an arbitrary point in the actual two-dimensional metric space $g_{00}, g_{01}, g_{11}$ is given, then we
can define the tensor density $h^{\alpha \beta} = \sqrt{-g} g^{\alpha \beta}$. By using the components of this tensor we can determine the momenta, $\pi_{\alpha \beta}$, conjugate to each of the components. One finds an explicit formula for the three secondary constraints $\chi_{11}^{11}, \chi_{11}^{01}$ and $\chi_{11}^{00}$. The PB algebra of the constraints is an $SO(2,1)$-algebra, while the canonical Hamiltonian of 2DG is written as a linear combination of these three secondary first class constraints.

It should be mentioned that coherent states for non-Abelian $SO(3)$-algebra have been constructed for the first time by Radcliffe in [26]. Perelomov [27] considered a more general case of arbitrary non-Abelian algebras. The method developed in [27] (see also [18]) allows one to construct various systems of coherent states for many non-Abelian algebras, including $SO(2,1)$, $SO(3,1)$, $SO(N,1)$ algebras and others. The importance of coherent states in our analysis is based on the fact that such states essentially coincide with the corresponding eigenvectors of the canonical Hamiltonian $H_c$. In turn, the $H_c$ Hamiltonian plays a central role in the 2DG model.

As we mentioned above, the model of two-dimensional gravity that we have considered has a number of advantages. In particular, the methods developed for our model can be applied to formulations of both metric and tetrad gravity. For instance, let us consider the Hamiltonian of tetrad gravity which is derived from its first-order formulation. It was shown in [28] that up to a total spatial derivative, the canonical Hamiltonian density of three-dimensional tetrad gravity is a linear combination of the secondary first class constraints (“rotational” ($\chi^0_{0(\rho)}$) and “translational” ($\chi^0_{0(\alpha\beta)}$) constraints), i.e.

$$H_c = -\epsilon^0_{0(\rho)}\chi^0_{0(\rho)} - \omega^0_{0(\alpha\beta)}\chi^0_{0(\alpha\beta)}$$  \hspace{0.5cm} (66)

where $\epsilon^0_{0(\rho)}$ are tetrads, while $\omega^0_{0(\alpha\beta)}$ are the spin connections. (For notation and the explicit form of the constraints see [28].) The result of [28] allows us to infer that the form of (66) is common for all dimensions $D > 2$. Further developments along this line can be found in [29, 30] where (66) is obtained in higher dimensions. Translational and rotational invariance in the tangent space is the general property of the first-order tetrad (or N-bein) gravity in all dimensions $D > 2$. In higher dimensions the only possible modification is the Poisson bracket among translational constraints which might differ from zero but proportional to the constraints and in 3$D$ limit gives zero [29,30]. This work is in progress and the results will be reported elsewhere.

The $D$ translational $\chi^0_{0(\rho)}$ and $\frac{D(D-1)}{2}$ rotational $\chi^0_{0(\alpha\beta)}$ constraints of the Poincaré algebra
form a closed and local algebraic structure. To classify this algebra, let us redefine the above
constraints \( M^{\mu \nu} = \frac{1}{2} \chi^{0(\mu \nu)} \) and \( P^{\mu} = \frac{1}{2} \chi^{0(\mu)} \). In this notation, the Poisson brackets obtained in [28] for \( D = 3 \) have a \( D \)-dimensional form
\[
\{ M^{\mu \nu}, M^{\rho \sigma} \} = \eta^{\nu \sigma} M^{\mu \rho} + \eta^{\rho \sigma} M^{\mu \nu} - \eta^{\mu \rho} M^{\nu \sigma} - \eta^{\mu \sigma} M^{\nu \rho},
\]
where \( M^{\mu \nu} = -M^{\nu \mu}, \mu, \nu, \ldots = 0, 1, 2, \ldots, \) and \( \eta^{\mu \nu} = \text{diag}(1, -1, -1, \ldots) \) is the Minkowski tensor. In three-dimensional spacetime the maximal value of indices in these definitions equals \( 2 \). With this notation we see that the PBs, equations (67) - (68), coincide with the commutation relations known for the generators of Poincaré algebra \( \text{ISO}(D - 1, 1) = \text{P}(1, D - 1) \). In \( D \) dimensions the Poincaré algebra \( \text{ISO}(D - 1, 1) \) is represented as a semi-direct sum of its ideal \( t^D \) (which contains \( D \) translations only) and corresponding Lorentz group \( \text{SO}(D - 1, 1) \) (which contains the \( \frac{D(D - 1)}{2} \) rotations). So that \( \text{ISO}(D - 1, 1) = t^D \rtimes \text{SO}(D - 1, 1) \).

With this notation, equation (66) gives a proportional to the secondary first class constraints part of the canonical Hamiltonian density which takes the form
\[
H_c = -2e_{0(\rho)} P^\rho - 2\omega_{0(\alpha \beta)} M^{\alpha \beta}.
\]
As follows from this equation, this part is linear in the secondary first class constraints \( P^\rho \) and \( M^{\alpha \beta} \) and also linear in the tetrads \( e_{0(\rho)} \) and spin connections \( \omega_{0(\alpha \beta)} \). This form of the Hamiltonian \( H_c \) is common in all problems related to tetrad gravity. Since \( e_{0(\rho)} \) and \( \omega_{0(\alpha \beta)} \) in equation (69) are real, while all generators \( P^\rho \) and \( M^{\alpha \beta} \) are self-adjoint, all eigenvectors and corresponding eigenvalues of the Hamiltonian \( H_c \) can be found with the use of the procedure which is generalization of our method for the case of Poincaré algebra. Formally, all these eigenvectors and eigenvalues will contain \( e_{0(\rho)} \) and \( \omega_{0(\alpha \beta)} \) as parameters. This means that there is a remarkable analogy of our simple model of two-dimensional gravity and tetrad gravity for an arbitrary \( D \)-dimensional spacetime.

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