On the cobordism groups of cooriented, codimension one Morin maps

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Part I

Cobordism of fold maps and the Kahn–Priddy map

I.1 Formulation of the result

Let us denote by \( \text{Cob} \Sigma^{1,0}(n) \) the cobordism group of cooriented, codimension 1 fold maps of closed, smooth, \( n \)-dimensional manifolds in \( \mathbb{R}^{n+1} \) (see [Sz3]).

(A fold map may have only \( \Sigma^{1,0} \)-type (or \( A_1 \)-) singular points, see [AGV].)

**Theorem A**

a) \( \text{Cob} \Sigma^{1,0}(n) \) is a finite Abelian group.

b) Its odd torsion part is isomorphic to that of the \( n^{\text{th}} \) stable homotopy group of spheres, i.e. for any odd prime \( p \) \( \text{Cob} \Sigma^{1,0}(n)_p \approx \pi^s(n)_p \), where the lower index \( p \) denotes the \( p \)-primary part.

c) Its 2–primary part is isomorphic to the kernel of the Kahn–Priddy homomorphism [KP]:

\[
\lambda_*: \pi^s_{n-1}(RP^\infty) \longrightarrow \pi^s(n-1).
\]

**Remark** The group \( \pi^s_{n-1}(RP^\infty) \) is a 2–primary group and \( \lambda_* \) is onto the 2–primary part of \( \pi^s(n-1) \), see [KP].

Hence: \( \text{Cob} \Sigma^{1,0}(n) \approx \pi^s(n)_{\text{odd torsion part}} \oplus \ker (\lambda_*: \pi^s_{n-1}(RP^\infty) \longrightarrow \pi^s(n-1)) \).
I.2 The Kahn–Priddy map ([KP], [K])

Let us consider the composition of the following maps:

a) $RP^{q-1} \hookrightarrow O(q)$. A line $L \subset R^q$, $[L] \in RP^{q-1}$ is mapped into the reflection in its orthogonal hyperplane.

b) $O(q) \hookrightarrow \Omega^q S^q$ maps $A \in O(q)$ to the map $S^q = R^2 \cup \infty \rightarrow S^q = R^q \cup \infty$ defined as

$$
S^q = R^2 \cup \infty \rightarrow A(x) \quad \text{for } x \in R^q \text{ and }
$$

$$
\infty \rightarrow \infty.
$$

Take the adjoint of the composition map $RP^{q-1} \longrightarrow \Omega^q S^q$. It is a map $\lambda: \Sigma^q RP^{q-1} \longrightarrow \Omega^q S^q$.

If $n < q$, then the homotopy groups $\pi_{q+n}(\Sigma^q RP^{q-1})$ and $\pi_{q+n}(S^q)$ are stable, and

\[ \pi_{q+n}^\Sigma(RP^{q-1}) \approx \pi_{q+n}^\Sigma(RP^\infty). \]

The Kahn–Priddy homomorphism $\lambda_*: \pi_{q+n}^\Sigma(RP^\infty) \longrightarrow \pi_{q+n}^\Sigma(n)$ is the homomorphism induced by $\lambda$ in the stable homotopy groups (precomposed with the isomorphism (*)).

**Theorem 1** (Kahn–Priddy [KP]) \( \lambda_* \) is onto the 2–primary component of $\pi^*(n)$.

I.3 Koschorke’s interpretation of $\lambda_*$

Ulrich Koschorke gave a very geometric description of the Kahn–Priddy homomorphism through the so-called “figure 8 construction”. Given an immersion of an \((n-1)\)–dimensional (unoriented) manifold $N^{n-1}$ into $R^n$ the figure 8 construction associates with it an immersion of an oriented \(n\)–dimensional manifold $M^n$ into $R^{n+1}$ as follows:

Let us consider the composition $N^{n-1} \hookrightarrow R^n \hookrightarrow R^{n+1}$.

This has normal bundle of the form $\varepsilon^1 \oplus \zeta^1$, where $\varepsilon^1$ is the trivial line bundle (the $(n+1)^{\text{th}}$ coordinate direction in $R^{n+1}$) and $\zeta^1$ is the normal line bundle of $N^{n-1}$ in $R^n$.

Let us put a figure 8 in each fiber of $\varepsilon^1 \oplus \zeta^1$ symmetrically with respect to the reflection in the fiber $\zeta^1$. Choosing these figures 8 smoothly their union gives the image of an immersion of an oriented \(n\)–dimensional manifold $M^n$ into $R^{n+1}$. (Clearly $M^n$ is the total space of the circle bundle $S(\varepsilon^1 \oplus \zeta^1)$ over $N^{n-1}$.)
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This construction gives a map $\mathbf{8}_* : \pi_{n}^s(\mathbb{RP}^\infty) \longrightarrow \pi_{n}^s(n)$. Indeed, the cobordism group of immersion of unoriented $(n - 1)$–dimensional manifolds in $\mathbb{R}^n$ is isomorphic to $\pi_{n}^s(\mathbb{RP}^\infty)$, and that of oriented $n$–dimensional manifolds in $\mathbb{R}^{n+1}$ is $\pi^s(n)$.

Since the figure 8 construction respects the cobordism relation (i.e. it associates to cobordant immersions such ones) we obtain a map of the cobordism groups.

**Theorem 2** (Koschorke, [K, Theorem 2.1]) The maps $\lambda_*$ and $\mathbf{8}_*$ coincide.

This theorem of Koschorke will be the main tool in the computation of the cobordism groups of fold maps.

### I.4 Generalities on the cobordisms of singular maps

In [Sz3] we considered cobordism groups of singular maps with a given set $\tau$ of allowed local forms. (Such a map was called a $\tau$–map.) The cobordism group of (cooriented) $\tau$–maps of $n$–dimensional manifolds in Euclidean space was denoted by $\text{Cob}_\tau(n)$. A classifying space $X_\tau$ has been constructed for $\tau$–maps with the property that its homotopy groups are isomorphic to the groups $\text{Cob}_\tau(n)$.

An ancestor of the spaces $X_\tau$ was the classifying space for the cobordism groups of immersions. Namely given a vector bundle $\xi^k$ we denote by $\text{Imm}^\xi(n)$ the cobordism group of immersions of $n$–manifolds in $\mathbb{R}^{n+k}$ such that the normal bundle is induced from $\xi$. There is a classifying space $Y(\xi)$ such that

$$\pi_{n+k}(Y(\xi)) \approx \text{Imm}^\xi(n).$$

Namely $Y(\xi) = \Gamma(T\xi)$, where $T\xi$ denotes the Thom space of the bundle $\xi$, and $\Gamma = \Omega^\infty S^\infty$. (This follows by a slight modification from [W].)

Next we recall the so-called “key bundle”, that is the main tool in handling cobordism groups of singular maps.

Let $\tau$ be a list of allowed local forms, and let $\eta$ be a maximal element in it. (The set of local forms has a natural partial ordering, $\eta$ is greater than $\eta'$ if an isolated $\eta$–germ (at the origin) has an $\eta'$–point arbitrarily close to the origin.)

Let $\tau'$ be $\tau \setminus \{\eta\}$ (i.e. we omit the maximal element $\eta$).

Note that the stratum of $\eta$ points is immersed. We have established in [Sz3] that there is a universal bundle – denoted by $\tilde{\xi}_\eta$ – for the normal bundles of $\eta$–strata from
which these normal bundles always can be induced (with the smallest possible structure group).

In particular to the cobordism class \([f]\) of a \(\tau\)-map \(f: M^n \to \mathbb{R}^{n+k}\) we can associate the element in \(\text{Imm}^\tilde{\xi}_\eta(m)\) represented by the restriction of \(f\) to its \(\eta\)-stratum. (Here \(m\) is the dimension of the \(\eta\)-stratum.) Hence a homomorphism \(\text{Cob}_{\tau}(n) \to \text{Imm}^\tilde{\xi}_\eta(m)\) arises. Both these groups are homotopy groups (of \(X_{\tau}\) and \(\Gamma T\tilde{\xi}_\eta\) respectively). It turns out that this map is induced by a map of the classifying spaces \(X_{\tau} \to \Gamma T\tilde{\xi}_\eta\). Moreover the latter is a Serre fibration with (homotopy) fiber \(X_{\tau'}\). (This was shown in [Sz3] using some nontrivial homotopy theory. Terpai in [T] gave an elementary proof for it. This fibration is called the “key bundle”.)

I.5 Computation of the groups \(\text{Cob} \Sigma^{1,0}(n)\)

In the case of fold maps \(\tau = \{\Sigma^0, \Sigma^{1,0}\}\) where \(\Sigma^0\) denotes the germ of maximal rank and \(\Sigma^{1,0}\) denotes that of a Whitney umbrella \((\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)\) (multiplied by the germ of identity \((\mathbb{R}^{n-2}, 0) \to (\mathbb{R}^{n-2}, 0))\).

Hence here \(\eta = \Sigma^{1,0}\) and \(\tau' = \Sigma^0\). Note that a \(\tau'\)-map is nothing else but an immersion (cooriented and of codimension 1). Hence \(X_{\tau'} = \Gamma S^1\). Now the key bundle looks as follows:

\[
X \Sigma^{1,0} \xrightarrow{\Gamma S^1} \Gamma T\tilde{\xi}_{\Sigma^{1,0}}.
\]

It is not hard to see (see also [RSz]) that the bundle \(\tilde{\xi}_{\Sigma^{1,0}}\) is \(2\varepsilon^1 \oplus \gamma^1\), and so \(T\tilde{\xi}_{\Sigma^{1,0}} = S^2\mathbb{R}P^\infty\).

Now the bundle \((\ast\ast)\) gives the following exact sequence of homotopy groups:

\[
\pi_{n+1}(\Gamma S^1) \to \pi_{n+1}(X \Sigma^{1,0}) \to \pi_{n+1}(\Gamma S^2\mathbb{R}P^\infty) \xrightarrow{\partial} \pi_n(\Gamma S^1) \text{ i.e.}
\]

\[
\pi^s(n) \to \text{Cob} \Sigma^{1,0}(n) \to \pi^s_n(\mathbb{R}P^\infty) \xrightarrow{\partial} \pi^s(n-1) \to
\]

Lemma 3 The boundary map \(\partial\) coincides with the map \(8_s\) and hence with the Kahn–Priddy homomorphism \(\lambda_s\).

Proof of Theorem A is immediate from Theorem 1, Theorem 2 and Lemma 3.

Proof of Lemma 3 The boundary map \(\partial: \pi^s_{n+1}(S^2\mathbb{R}P^\infty) \approx \pi_{n+1}(X \Sigma^{1,0}, \Gamma S^1) \to \pi_n(\Gamma S^1)\) can be interpreted geometrically as follows:
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The source group \( \pi_{n+1}(X \Sigma^{1,0}, \Gamma S^1) \) is isomorphic to the cobordism group of fold maps

\[
f: (M^n, \partial M^n) \to (D^{n+1}, S^n)
\]

and the map \( \partial f = f \bigg|_{\partial M^n} \) is an immersion of \( \partial M^n \) into \( S^n \). (Here \( M^n \) is an oriented compact smooth \( n \)-dimensional manifold with boundary \( \partial M^n \).)

Let \([f]\) denote the (relative) cobordism class of \( f \), and let \([\partial f]\) be that of the immersion \( \partial f: \partial M^n \to S^n \). Then \( \partial [f] = [\partial f] \).

Now let \( V \) denote the set of singular points of \( f \). This is a submanifold of \( M^n \) of codimension 2. The restriction of \( f \) to \( V \) is an immersion, its image we denote by \( \overline{V} (= f(V)) \). Let \( T \) be the (immersed) tubular neighbourhood of \( \overline{V} \). More precisely there exist a \( D^3 \)-bundle \( T' \to V \) over \( V \), a submersion \( F \) of \( T' \) into \( D^n \), \( (F(T') = \overline{T}) \) and \( F \) extends the immersion \( f \mid V: V \to D^n \). The bundle \( T' \to V \) has the form \( \overline{2\varepsilon^1} \oplus \zeta^1 \), where \( \zeta^1 \) is a line bundle.

Let \( T \) be the tubular neighbourhood of \( V \) in \( M \) such that \( f(T) \subset \overline{T} \). The map \( f \mid T: T \to \overline{T} \) can be decomposed into a map \( \tilde{f}: T \to T' \) and the submersion \( F: T' \to \overline{T} \), where \( \tilde{f} \) maps each fiber \( D^2 \) of the bundle \( T \to V \) into a fiber \( D^3 \) of \( T' \to \overline{T} \) as a Whitney umbrella and \( \tilde{f}^{-1}(\partial T') = \partial T \). On the boundary of each fiber \( D^3 \) we obtain a “curved figure 8” as image of \( \tilde{f} \). The manifold with boundary \( M \setminus T \) will be denoted by \( W \). Note that its boundary is \( \partial W = \partial_1 W \sqcup \partial_2 W \), where \( \partial_1 W = \partial M \), and \( \partial_2 W = \partial T \).

The image of \( \partial_2 W \) at \( f \) is the union of the (\( F \)-images of the) above mentioned curved figures 8. This is a codimension 2 framed immersed submanifold in \( D^{n+1} \), we will denote it by \( \overline{V} \). (The first framing is the \( F \)-image of the inside normal vector of \( \partial T' \) in \( T' \). The second framing is the normal vector of the curved figure 8 in \( S^2 = \partial D^3 \).)

It remained to show the following two claims.

**Claim a)** \( \overline{V} \) with the given 2-framing is framed cobordant to the immersion \( \partial f: \partial M \to S^n \) (compared with the framed embedding \( S^n \subset D^{n+1} \)).

**Claim b)** \( \overline{V} \) is obtained from the immersion \( f \mid V: V^{n-2} \to D^{n+1} \) by the figure 8 construction.

We have to make some remarks in order to clarify the above statements a) and b).

To a): The framed immersion \( \partial f: \partial M \to S^n \) and its composition with \( i: S^n \subset D^{n+1} \subset R^{n+1} \) (with the added second framing, the inside normal vectors of \( S^n \) in \( D^{n+1} \))
represent the same element in $\pi^i(n-1)$. Indeed, the composition with $i$ corresponds to applying the suspension homomorphism in homotopy groups of spheres. But the cobordism group of framed \textit{immersions} is isomorphic to the corresponding \textit{stable} homotopy group of spheres, so the suspension homomorphism gives the identity map of these groups.

To b): The figure 8 construction was defined for a codimension one immersed submanifold in a Euclidean space. Here we apply it to the codimension 3 immersed submanifold $\widetilde{V}^{n-2}$ in $D_{n+1}$. But $\widetilde{V}^{n-2}$ has two linearly independent normal vector fields in $D_{n+1}$ as was described above. Identify $D_{n+1}$ with $R^n_{q+1}$ and apply the so-called multicompression theorem by Rourke–Sanderson [RS], thus one can make the two normal vectors parallel to the last two coordinate axes in $R^n_{q+1}$, we can project the immersion to $R^n_{q+1}$ and then we have a codimension 1–immersion, so claim b) makes sense. (This needs some more clarification since the multicompression theorem deals with embeddings. See below.)

**Proof of Claim a)** It can be supposed that the center $c$ of $D_{n+1}$ does not belong to $\partial \tilde{T} \cup f(M)$. Let us omit from $D_{n+1}$ a small ball centered around $c$ and still disjoint from $\partial \tilde{T} \cup f(M)$. In the remaining manifold $S^n \times I$ the direction of $I$ will be called vertical. Take the product with an $R^q$ for big enough $q$, so that the immersions $f|_W: W^n \hookrightarrow S^n \times I$ and $\partial T \hookrightarrow S^n \times I$ become embeddings after small perturbations.

Now $W$ is embedded in $S^n \times I \times R^q$, it is framed with $q+1$ normal vectors ($q$ are parallel to the coordinate axes of $R^q$).

One can suppose that the two boundary components of $W$ are embedded as follows

1) $\partial_1 W = \partial M$ is embedded into $S^n \times \{0\} \times R^q$.

2) $\partial_2 W = \partial T$ is embedded into the interior part $\text{int}(S^n \times I) \times R^q$.

Both have $(q+2)$-framings.

Now applying the multicompression theorem we make by an isotopy the first framing vector (the one coming from the normal vector of $\partial \tilde{T}$ in $\tilde{T}$) vertical, i.e. parallel to the direction of $I$ in $S^n \times I \times R^q$, while the $q$ last framing vectors (coming from $R^q$) we keep parallel to themselves. The other boundary component $\partial_1 W \subset S^n \times \{0\} \times R^q$ is kept fixed.

We arrive at such an embedding of $W$ in $S^n \times I \times R^q$ for which the outward normal vector along $\partial_2 W$ in $W$ is vertical (i.e. parallel to the direction of $I$). Now by a vertical shift we can deform $\partial_2 W$ into $S^n \times \{1\} \times R^q$. 

This deformation can be extended to $W$. Projecting into $S^n \times I$ this new position of $W$ in $S^n \times I \times R^q$ we obtain an immersion cobordism between the immersions of the two boundary components.

On the first component $(\partial_1 W)$ we obtain $\partial[f]$. On the second component we obtain the same framed cobordism class as was that of $\tilde{V}$ in $\partial\tilde{T} \subset D^{n+1}$ (the union of curved figures 8). Claim a) is proved.

**Proof of Claim b)** Deform the immersed manifold $\tilde{V}^{n-1}$ (formed by the union of curved figures 8) as follows. Contract each curved figure 8 by an isotopy along the corresponding sphere $S^2$ into a small neighbourhood of its double point obtaining an almost flat (very small) figure 8. As we have noticed the normal bundle of $f(V)$ in $D^{n+1}$ has the form $2\varepsilon_1 \oplus \zeta^1 = \varepsilon_1^1 \oplus \varepsilon_1^2 \oplus \zeta^1$. The first trivial normal line bundle $\varepsilon_1^1$ can be identified with the direction of the double line of the umbrella, the second one $\varepsilon_1^2$ can be the symmetry axes of the figures 8 (both of the curved and the flattened ones). $\zeta^1$ is the direction orthogonal to the symmetry axes.

Now considering the maps of $V$ and $\tilde{V}$ in $R^{n+1}$ (instead of $D^{n+1}$) applying again the multicompression theorem we make the two trivial normal directions $\varepsilon_1^2$ and $\varepsilon_1^1$ parallel to the last two coordinate axes and then project $V$ to $R^{n-1}$. In this way we obtain a new immersion $g: V^{n-2} \rightarrow R^{n-1}$ and $\varepsilon_1^2$ (the symmetry axes of the figures 8) will be parallel to the normal vector of $R^{n-1}$ in $R^n$. Now the (flattened) figures 8 (of $\tilde{V}$) are placed exactly as by the original figure 8 construction applied to $g$.

It remained to note that the described deformations do not change the cobordism class of a framed immersion. Claim b) is proved.

Thus Theorem A is also proved.

**Remark** The stable homotopy groups of $RP^\infty$ were computed by Liulevicius [Liu] in dimensions not greater than 9.

Below in the first line we show his result, in the second one the stable homotopy groups of spheres. These two lines by Theorem A give the groups $\text{Cob} \Sigma^{1,0}(n)$ for $n \leq 10$ given in the third line. (Here for example $(Z_2)^3$ stands for $Z_2 \oplus Z_2 \oplus Z_2$.)

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----|---|---|---|---|---|---|---|---|---|----|
| $\pi^s_0(RP^\infty)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_8$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$ | $(\mathbb{Z}_2)^3$ | $(\mathbb{Z}_2)^3$ | ? |
| $\pi^s(n)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_{240}$ | $(\mathbb{Z}_2)^2$ | $(\mathbb{Z}_2)^2$ | $\mathbb{Z}_6$ |
| $\text{Cob} \Sigma^{1,0}(n)$ | 0 | 0 | $\mathbb{Z}_3$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_6$ |
Part II

Cusp maps

II.1 Formulation of the results

Here we consider cusp maps, i.e. maps having at most cusp singularities. (In the previous terms these are \( \tau \)-maps for \( \tau = \{ \Sigma^0, \Sigma^{1,0}, \Sigma^{1,1} \} \).) The cobordism group of cusp maps of oriented \( n \)-dimensional manifolds in \( R^{n+1} \) will be denoted by \( \text{Cob} \Sigma^{1,1}(n) \).

We shall compute these groups modulo their 2–primary and 3–primary parts. Let \( C_{\{2,3\}} \) be the minimal class of groups containing all 2–primary and 3–primary groups.

**Theorem B**

\[
\text{Cob} \Sigma^{1,1}(n) \approx C_{\{2,3\}} \pi^4(n) \oplus \pi^3(n-4)
\]

where \( \approx \) means isomorphism modulo the class \( C_{\{2,3\}} \), and \( \pi^m \) denotes the \( m \)th stable homotopy group of spheres.

II.2 Preliminaries on Morin maps

Morin maps are those of types \( \Sigma^{1,0}, \Sigma^{1,1,0}, \ldots, \Sigma^{r,0}, \ldots, r = 1, 2, \ldots \) (See [AGV].)

For \( \eta = \Sigma^{1,0} \) the universal normal bundle \( \tilde{\xi}_r \) will be denoted by \( \tilde{\xi}_r \). It was established in [RSz] and [R] that the structure group of \( \xi_r \) is \( Z_2 \) and the bundle \( \tilde{\xi}_r \) is associated to a representation \( \lambda_2 : Z_2 \longrightarrow O(2r+1) \) with the property that \( \lambda_2(Z_2) \subset SO(2r + 1) \) precisely when \( r \) is even. It follows that \( \tilde{\xi}_r \) is the direct sum \( i \cdot \gamma^1 \oplus j \cdot \varepsilon^1 \), where \( i + j = 2r+1 \), and \( i \equiv r \mod 2 \). Here \( \varepsilon^1, \gamma^1 \) are the trivial and the universal line bundles respectively. Hence the Thom space \( T\tilde{\xi}_r \) is \( S'(RP^{\infty}/RP^{r-1}) \). It is easy to see that for any odd \( p \) the reduced mod \( p \) cohomology \( \overline{H}^i(\Gamma \tilde{\xi}_r; \mathbb{Z}_p) \) vanishes if \( r \) is odd, and the natural inclusion \( S^{2r+1} \subset T\tilde{\xi}_r \) (as a “fiber”) induces isomorphism of the cohomology groups with \( \mathbb{Z}_p \)–coefficients for \( r \) even. Consequently by Serre’s generalization of the Whitehead theorem [S2] – the inclusion \( \Gamma S^{2r+1} \subset \Gamma T\tilde{\xi}_r \) (recall \( \Gamma = \Omega^\infty S^\infty \)) induces isomorphism of the odd torsion parts of the homotopy groups for \( r \) even, while for \( r \) odd \( \pi_*(\Gamma T\tilde{\xi}_r) \) are finite 2–primary groups.
II.3 Computation of the cusp cobordism groups

In the case of cusps $r = 2$ and we have that the inclusion $\Gamma S^5 \subset \Gamma T\tilde{\xi}_2$ is a mod $C_2$ homotopy equivalence ($C_2$ is the class of 2–primary groups).

Let us consider the following pull-back diagram defining the space $X^{fr}\Sigma^{1,1}$

$$
\begin{array}{ccc}
X^{fr}\Sigma^{1,1} & \longrightarrow & X\Sigma^{1,1} \\
\downarrow XSigma^{1,0} & & \downarrow XSigma^{1,0} \\
\Gamma S^5 & \longrightarrow & \Gamma T\tilde{\xi}_2
\end{array}
$$

(Note that $X^{fr}\Sigma^{1,1}$ is the classifying space of those cusp maps for which the normal bundle of the $\Sigma^{1,1}$–stratum in the target is trivialized. Equivalently these are the cusp maps for which the kernel of the differential is trivialized over the cusp-stratum.)

The horizontal maps of the diagram induce isomorphisms of the odd-torsion parts of the homotopy groups. Now we show that the homotopy exact sequence of the left-hand side fibration “almost has a splitting”.

**Definition** Let $p: E \longrightarrow B$ be a fibration and let $t$ be a natural number. We say that this fibration has an algebraic $t$–splitting if for each $i$ there is homomorphism $S_i: \pi_i(B) \longrightarrow \pi_i(E)$ such that the composition of $S_i$ with the map $p_*$ induced by $p$ is a multiplication by $t$. We say that the fibration $p$ has a geometric $t$–splitting if it has an algebraic one such that all $S_i$ are induced by a map $s: B \longrightarrow E$ (the same map $s$ for each $i$).

**Lemma 4** The fibration $X^{fr}\Sigma^{1,1} \longrightarrow \Gamma S^5$ has a 6–splitting.

**Remark** We shall prove this only in algebraic sense, since we will need only that. For the existence of geometric splitting we give only a hint.

**Proof of Theorem B** is immediate from Lemma 4. Indeed,

$$
\text{Cob } \Sigma^{1,1}(n) \approx \pi_{n+1}(X\Sigma^{1,1}) \approx \pi_{n+1}(X^{fr}\Sigma^{1,1}).
$$

Now the homotopy exact sequence of the fibration $p^{fr}: X^{fr}\Sigma^{1,1} \longrightarrow X\Sigma^{1,0} \longrightarrow \Gamma S^5$ has a 6–splitting, hence modulo the class $C_{(2,3)}$ we have

$$
\pi_{n+1}(X^{fr}\Sigma^{1,1}) \approx \pi_{n+1}\left(S^5 \oplus \pi_{n+1}(X\Sigma^{1,0})\right) \approx \pi^f(n-4) \oplus \pi^f(n).
$$

(In the last mod $C_2$ isomorphism we used Theorem A.)

Theorem B is proved except Lemma 4. 

$\square$
Proof of Lemma 4 will follow from the following two claims.

Claim 1 If there is a map of an oriented 4–dimensional manifold into $\mathbb{R}^5$ with $t$ cusp points (algebraically counting them), then the fibration $p^{fr}$ has a $t$–splitting (algebraically).

Claim 2 There is a cusp-map $f: M^4 \rightarrow \mathbb{R}^5$ with $t$–cusp points (counting algebraically).

Proof of Claim 1 Let $f: M^4 \rightarrow \mathbb{R}^5$ be a cusp-map with $t$ cusp-points. Let $x$ be an element in $\pi_m(\Gamma S^5) \approx \pi^s(m-5)$. It can be represented by a framed, immersed $(m-5)$–dimensional manifold $A^{m-5}$ in $\mathbb{R}^m$, let us denote its immersion by $\alpha$.

Take the product $A^{m-5} \times M^4$ and its map into the direct product $A^{m-5} \times D^5$ by id$_A \times f$. Now the target $A^{m-5} \times D^5$ can be mapped by a submersion $F$ into $\mathbb{R}^m$ onto the immersed tubular neighbourhood of $\alpha(A)$ using the framing to map the $D^5$–fibers). The composition $A \times M^4 \rightarrow \mathbb{R}^m$ is clearly a cusp map and its cusp-singularity stratum represents the element $t \cdot x$ in $\pi^s(m-5)$.

Claim 1 is proved (at least its algebraic version. The geometric one follows from the fact that we use the same element $[f: M^4 \rightarrow D^5]$ for any element $x \in \pi^s(i)$ and for any dimension $i$ to construct the element $S_i(x)$. The classifying space $\Gamma S^5$ can be obtained as the limit of target spaces of codimension 5 framed immersions.)

Proof of Claim 2 is a compilation of the following two theorems.

Theorem (Sz1, Sz2, L) Given a generic immersion $g: M \rightarrow Q \times \mathbb{R}^1$ and a natural number $r$, let us denote by $\Delta_{r+1}(g)$ the manifold of (at least) $(r+1)$–tuple points in $M^n$. Let $f: M \rightarrow Q$ be the composition of $g$ with the projection $Q \times \mathbb{R}^1 \rightarrow Q$. Let us denote by $\Sigma_{1r}(f)$ the closure of the set of $\Sigma_{1r}$ singular points of $f$. Then the manifolds $\Delta_{r+1}(g)$ and $\Sigma_{1r}(f)$ are cobordant. If $M$ and $Q$ are oriented and dim $Q - \dim M$ is odd, then these manifolds are oriented and they are oriented-cobordant.

Theorem (Eccles–Mitchell EM) There is an oriented closed 4–dimensional manifold $M^4$ and an immersion $g: M^4 \rightarrow \mathbb{R}^6$ with (algebraically) 2 triple points.

Theorem B is proved.
Part III

Higher Morin maps

Most of the previous arguments can be applied in the computation of cobordism groups of Morin maps having at most $\Sigma^1_i$ singular points for any $r$ (the codimensions of the considered maps are still equal to one, and the maps are cooriented). The only problem is that we need a generalization of the Theorem of Eccles–Mitchell.

Below we shall give a weak form of such a generalization. This will allow us to compute the groups $\text{Cob} \Sigma^1_i(n)$ modulo the $p$–primary part for $p \leq r + 1$.

Notation: Let $C\{p \leq 2r + 1\}$ denote the minimal class of groups containing all $p$–primary groups for any prime $p \leq 2r + 1$. The main result of this Part III is the following.

**Theorem C**  Let us denote by $\text{Cob} \Sigma^1_i(n)$ the cobordism group of $\Sigma^1_i$–map of oriented $n$–manifolds in $R^{n+1}$ (i.e. $\tau$–maps for $\tau = \{\Sigma^0, \Sigma^1, \ldots, \Sigma^1, 0, \ldots, \Sigma^1, 1, \ldots, 1\}$, $i$ digits $1$). Then for any $r$

\[
\text{Cob} \Sigma^1_{2r+1}(n) \cong \frac{\text{Cob} \Sigma^1_{2r}(n)}{C\{p \leq 2r+1\} \bigoplus \pi^s(n-4i)}.
\]

The proof is very similar to that given in Parts I and II. It goes by induction on $r$. First we give a weak analogue of the Theorem of Eccles and Mitchell.

**Lemma 1**  a) For any natural number $k$ there is a positive integer $t(k)$ such that for any immersion of an oriented, closed, smooth $4k$–dimensional manifold in $R^{4k+2}$ the algebraic number of $(2k+1)$–tuple points is divisible by $t(k)$, and there is a case when this number is precisely $t(k)$.

b) The number $t(k)$ coincides with the order of the cokernel of the stable Hurewicz homomorphism

\[
\pi^s_{4k+2}(CP^\infty) \rightarrow H_{4k+2}(CP^\infty).
\]

Before proving Lemma 1 it will be useful to recall a result on the cokernel of the stable Hurewicz map.

**Theorem**  (Arlettaz [A])  Let $X$ be a $(b-1)$–connected space and let $\varrho_j$ be the exponent of the stable homotopy group of spheres $\pi^s(j)$. Let $h_m: \pi^s_m(X) \rightarrow H_m(X)$ be the stable Hurewicz homomorphism. Then $(\varrho_1 \ldots \varrho_{m-b-1})(\text{coker } h_m) = 0$. 
Next we recall a theorem of Serre on the prime divisors of the numbers \( g_j \).

**Theorem (Serre [S1])** \( \pi^i(i) \otimes \mathbb{Z}_p = 0 \) if \( i < 2p - 3 \) and \( \pi^i(2p - 3) \otimes \mathbb{Z}_p = \mathbb{Z}_p \).

Hence \( g_j \) is not divisible by a prime \( p \) if \( p > \frac{j + 3}{2} \), in other words, \( g_j \) may have a prime \( p \) as a divisor only if \( p \leq \frac{j + 3}{2} \).

Applying Arlettaz’ theorem to \( X = \mathbb{C}P^\infty, b = 2, m = 4r + 2 \) we obtain that \( t(r) \) has no prime divisor greater than \( 2r + 1 \).

**Proof of Theorem C** should be clear now, since it is completely analogous to that of Theorem B.

First we consider the “key bundle”

\[
X \Sigma^{12r+1} \longrightarrow \Gamma T \xi_{2r+1} \text{ with fiber } X \Sigma^{12r}.
\]

Remember that \( H^*(\Gamma T \xi_{2r+1}; \mathbb{Z}_p) = 0 \) for any odd \( p \), so – by the mod \( C \) Whitehead theorem [S2] we obtain the first (mod \( C_2 \)) isomorphism in the Theorem

\[
\text{Cob} \Sigma^{12r+1}(n) \approx \text{Cob} \Sigma^{12r}(n).
\]

In order to prove the second (mod \( C\{p \leq 2r + 1\} \)) isomorphism recall that modulo the class \( C_2 \) the key bundle

\[
X \Sigma^{12r} \longrightarrow \Gamma T \xi_{2r} \text{ with fibre } X \Sigma^{12r-1}
\]

can be replaced by the bundle

\[
X^{fr} \Sigma^{12r} \longrightarrow \Gamma S^{4r+1} \text{ with the same fibre.}
\]

The later bundle has a(n algebraic) \( t(r) \)--splitting.

By Lemma 1 and the theorems of Arlettaz and Serre \( t(r) \) has no prime divisor greater than \( 2r + 1 \), hence by induction on \( r \) we obtain the second isomorphism in Theorem C.

**Proof of Lemma 1** By Herbert’s theorem the algebraic number of the \( (2k + 1)\)--tuple points of an immersion \( f : M^{4k} \hookrightarrow R^{4k+2} \) is \( \langle \overline{\pi}_1 \rangle, [M^{4k}] \rangle \). The immersion \( f \) represents an element \([f]\) of the corresponding cobordism group of immersions of oriented \( 4k \)--manifolds in \( R^{4k+2} \). This cobordism group is isomorphic to the group \( \pi^{4k+2}(MSO(2)) \), the element of the later group corresponding to \([f]\) will be denoted by \( \{\alpha_f\} \). Here \( \alpha_f \) is the Pontrjagin–Thom map \( S^{q+4k+2} \longrightarrow S^{q}MSO(2) \), for \( q \) big enough.
Let us consider the following composition of maps
\[ \pi^s_{4k+2}(MSO(2)) \to H_{4k+2+q}(S^qMSO(2)) \xrightarrow{\approx} H_{4k}(BSO(2)) \xrightarrow{\approx} \mathbb{Z}. \]
Here \( \circlearrowright_1 \) is the stable Hurewicz homomorphism, \( \circlearrowright_2 \) is the Thom isomorphism in the homologies
\[ x \to S^qU \cap x \]
where \( U \) is the Thom class of \( MSO(2) \) and \( S^qU \) its \( q \)th suspension, and also the Thom class of \( S^qMSO(2) \).
\( \circlearrowright_3 \) is the evaluation on the class \( p_1^k \)
\[ y \to \langle y, p_1^k \rangle. \]
Since the maps \( \circlearrowright_2 \) and \( \circlearrowright_3 \) are isomorphisms, the cokernel of this composition is the same as the cokernel of \( \circlearrowright_1 \), i.e. of the stable Hurewicz homomorphism.

On the other hand, we show that the image of this composition map is \( t(k) \mathbb{Z} \), and that will prove part b) of Lemma 1. (Part a) follows then as well, since the rational stable Hurewicz homomorphism
\[ \pi^s_m(X) \otimes Q \to H_m(X; Q) \] is an isomorphism.

Claim The composition of the maps \( \circlearrowright_1, \circlearrowright_2, \circlearrowright_3 \) has image \( t(k) \cdot \mathbb{Z} \).

Proof It is enough to show that the image of \( [\alpha_f] \in \pi^s_{4k+2}(MSO(2)) \) is \( \langle \overline{p}_1^k, [M^{4k}] \rangle \).

[\alpha_f] goes by the map \( \circlearrowright_1 \) to \( (\alpha_f)_*[S^q+4k+2] \), that is mapped by \( \circlearrowright_2 \) to \( (\alpha_f)_*[S^q+4k+2] \cap S^qU \).

Let \( \nu \) be the normal bundle of \( f \), let us denote by \( T\nu \) its Thom space, let \( pr: S^q+4k+2 \to S^qT\nu \) be the Pontrjagin–Thom map, \( \beta_f: S^qT\nu \to S^qMSO(2) \) the fiberwise map of Thom spaces that on the base spaces is the map \( \nu_f: M \to BSO(2) \) inducing the normal bundle \( \nu \).

Now \( (\alpha_f)_*[S^q+4k+2] = (\beta_f)_* pr_*[S^q+4k+2] = (\beta_f)_*[S^qT\nu] \). Here \( [S^qT\nu] \) is the fundamental homology class of \( S^qT\nu \). Therefore
\[ \langle p_1^k, (\alpha_f)_*[S^q+4k+2] \cap S^qU \rangle = \langle p_1^k, (\beta_f)_*[S^qT\nu] \cap S^qU \rangle \]
\[ = \langle p_1^k, (\nu_f)_*[M] \rangle = \langle \nu_f^* p_1^k, [M] \rangle = \langle \overline{p}_1^k, [M] \rangle. \]
Q.E.D.
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