Galois quotients of metric graphs and invariant linear systems

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Abstract

For a map \( \varphi : \Gamma \to \Gamma' \) between metric graphs and an isometric action on \( \Gamma \) by finite group \( K \), \( \varphi \) is a \( K \)-Galois covering on \( \Gamma' \) if \( \varphi \) is a morphism, the degree of \( \varphi \) coincides with the order of \( K \) and \( K \) induces a transitive action on every fibre. We prove that for a metric graph \( \Gamma \) with an isometric action by finite group \( K \), there exists a rational map, from \( \Gamma \) to a tropical projective space, which induces a \( K \)-Galois covering on the image. By using this fact, we also prove that for a hyperelliptic metric graph without one valent points and with genus at least two, the invariant linear system of the hyperelliptic involution \( \iota \) of the canonical linear system, the complete linear system associated to the canonical divisor, induces an \( \langle \iota \rangle \)-Galois covering on a tree. This is an analogy of the fact that a compact Riemann surface is hyperelliptic if and only if the canonical map, the rational map induced by the canonical linear system, is a double covering on a projective line \( P^1 \).

keywords: metric graph, invariant linear subsystem, rational map, Galois covering, hyperelliptic metric graph, canonical map

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1 Introduction

Tropical geometry is an algebraic geometry over tropical semifield $T = (\mathbb{R} \cup \{ -\infty \}, \max, +)$. A tropical curve is a one-dimensional object obtained from a compact Riemann surface by a limit operation called tropicalization and realized as a metric graph. In this paper, a metric graph means a finite connected multigraph where each edge is identified with a closed segment of $T$. Exactly as a compact Riemann surface, concepts of a divisor, a rational function and a complete linear system etc. are defined on a metric graph. A morphism of metric graphs is a (finite) harmonic map.

In tropical geometry, a hyperelliptic metric graph, i.e. a metric graph with a special action by two element group is investigated in detail in [8]. In this paper, we study a metric graph with an action by a finite group and develop a quotient metric graph in a tropical projective space as the image of a rational map. Note that in this paper, we always suppose that an action by a finite group on a metric graph is isometric.

**Theorem 1.0.1** (Theorem 3.3.21). Let $\Gamma$ be a metric graph and $K$ a finite group acting on $\Gamma$. Then, there exists a rational map, from $\Gamma$ to a tropical projective space, which induces a $K$-Galois covering on the image.

Here, the definition of a $K$-Galois covering on a metric graph is given as follows.

**Definition 1.0.2** (Definition 3.2.3). Assume that a map $\varphi : \Gamma \to \Gamma'$ between metric graphs and an action on $\Gamma$ by $K$ are given. Then, $\varphi$ is a $K$-Galois covering on $\Gamma'$ if $\varphi$ is a morphism of metric graphs, the degree of $\varphi$ coincides with the order of $K$ and the action on $\Gamma$ by $K$ induces a transitive action on every fibre by $K$.

For the proof of theorem 1.0.1, we use the following various results.

For a divisor $D$ on a metric graph $\Gamma$, $R(D)$ denotes the set of rational functions corresponding to the complete linear system $|D|$ together with a constant function of $-\infty$ on $\Gamma$, i.e. $R(D) := \{ f \mid f$ is a rational function other than $-\infty$ and $D + \text{div}(f)$ is effective.$\} \cup \{-\infty\}$. $R(D)$ becomes a tropical semimodule over $T$ ([8 Lemma 4]).

**Theorem 1.0.3** ([8, Theorem 6]). $R(D)$ is finitely generated.
$|D|$ is also finitely generated since $|D|$ is identified with the projection of $R(D)$. In this paper, we show the following theorem such that we consider a finite group on Theorem 1.0.3.

**Theorem 1.0.4 (Remark 3.1.3, Theorem 3.1.6 and Theorem 3.1.7).** Let $\Gamma$ be a metric graph, $K$ a finite group acting on $\Gamma$ and $D$ a $K$-invariant effective divisor on $\Gamma$. Then, the set $R(D)^K$ consisting of all $K$-invariant rational functions in $R(D)$ becomes a tropical semimodule and is finitely generated.

We can show that the set $|D|^K$ consisting of all $K$-invariant divisors in $|D|$ is identified with the projection of $R(D)^K$. Thus, the $K$-invariant linear system $|D|^K$ is also finitely generated. Let $\phi_{|D|^K}$ be the rational map, from $\Gamma$ to a tropical projective space, associated to $|D|^K$. Then, the following holds.

**Theorem 1.0.5 (Theorem 3.3.14).** $\phi_{|D|^K}$ induces a $K$-Galois covering on $\text{Im}(\phi_{|D|^K})$ if and only if $\phi_{|D|^K}$ maps distinct $K$-orbits to distinct points.

For each edge of $\text{Im}(\phi_{|D|^K})$, a natural measure defined by the $\mathbb{Z}$-affine structure of the tropical projective space is induced. In the proof of theorem 1.0.5, it is essential to show that $\phi_{|D|^K}$ is a local isometry for the edge length defined from this measure. For the fact that generally, the rational map defined by a finite number of rational functions on a metric graph may not induce a morphism of metric graphs since the rational map may not be harmonic, Theorem 1.0.5 states that the rational map induced by a $K$-invariant linear system induces a morphism of metric graphs if it satisfies the condition in Theorem 1.0.5. Moreover, we show the following theorem.

**Theorem 1.0.6 (Theorem 3.3.20).** There exists a $K$-invariant effective divisor $D$ on a metric graph with an action of a finite group $K$ such that $\phi_{|D|^K}$ maps distinct $K$-orbits to distinct points.

In conclusion, we obtain Theorem 1.0.1. Especially when the group $K$ is trivial, we have the following corollary.

**Corollary 1.0.7 (Corollary 3.3.22).** A metric graph is embedded in a tropical projective space by a rational map.

For a canonical map, which is the rational map induced by a canonical linear system, Haase–Musiker–Yu\cite{8} showed the following theorem.

**Theorem 1.0.8 (\cite{8} Theorem 49).** A metric graph whose canonical map is not injective is hyperelliptic.

In the proof of Theorem 1.0.8 Haase–Musiker–Yu\cite{8} gave all hyperelliptic metric graphs satisfying the condition concretely. Moreover, Haase–Musiker–Yu\cite{8} showed that the inverse of Theorem 1.0.8 does not hold and posed the problem of other characterizing metric graphs with non-injective canonical maps. As answers of this problem, we give Theorem 1.0.9 and Corollary 1.0.10 as follows.
Theorem 1.0.9 (Theorem 3.4.3). Let $\Gamma$ be a metric graph without one valent points. Then, the canonical map induces a morphism which is a double covering on the image if and only if the genus of $\Gamma$ is two.

By Theorem 1.0.5, Theorem 1.0.8 and its proof and Theorem 1.0.9, we have the following.

Corollary 1.0.10 (Corollary 3.4.4). Let $\Gamma$ be a metric graph of genus at least three without one valent points. Then, the canonical map of $\Gamma$ is not injective if and only if the map induced by the canonical map is not harmonic.

By using Theorem 1.0.1 and the fact that for the rational map induced by the canonical linear system associated with a divisor whose degree is two and whose rank is one on a metric graph, the image is a tree and the order of every fibre is one or two ([$8$, Proposition 48]), we have the following.

Theorem 1.0.11 (Theorem 3.4.7). For a hyperelliptic metric graph with genus at least two without one valent points, the invariant linear system of the hyperelliptic involution $\iota$ of the canonical linear system induces a rational map whose image is a tree and which is a $\langle \iota \rangle$-Galois covering on the image.

Theorem 1.0.11 means that an analogy of the fact the canonical map of a classical hyperelliptic compact Riemann surface is a double covering holds by the rational map induced not by the canonical linear system but by an invariant linear subsystem of the hyperelliptic involution of the canonical system for a hyperelliptic metric graph.

The length of each edge of $\text{Im}(\phi|_{D^K})$ in Theorem 1.0.5 have not given in [$8$]. We show this edge length is defined naturally and then we become to be able to argue whether a rational map is harmonic or not.

In this paper, we recall some basic facts corresponding to metric graphs in Section 2. We prove Theorem 1.0.1 and corresponding statements in Section 3. Metric graph with edge-multiplicities and harmonic morphisms between them, we need in Section 3, are defined in Section 4.

2 Preliminaries

In this section, we briefly recall some basic facts of tropical algebra ([$1$, $11$]), metric graphs ([$12$]), divisors on metric graphs ([$2$, $6$, $7$, $12$, $13$]), harmonic morphisms of metric graphs ([$6$, $8$, $10$]), and chip-firing moves on metric graphs ([$8$]), which we need later.

2.1 Tropical algebra

The set of $T := \mathbb{R} \cup \{-\infty\}$ with two tropical operations:

$$a \oplus b := \max\{a, b\} \quad \text{and} \quad a \odot b := a + b,$$
where both \( a \) and \( b \) are in \( T \), becomes a semifield. \( T = (T, \oplus, \odot) \) is called the tropical semifield and \( \oplus \) (resp. \( \odot \)) is called tropical sum (resp. tropical multiplication). We frequently write \( a \oplus b \) and \( a \odot b \) as “\( a + b \)” and “\( ab \)”, respectively.

A vector \( v \in T^n \) is primitive if all coefficients of \( v \) are integers and their greatest common divisor is one. For a vector \( u \in Q^n \), its length is defined as \( \lambda \) such that \( u = \lambda v \), where \( v \in Z^n \) is the primitive vector with the same direction as \( u \). For a vector \( u = (u_1, \ldots, u_n) \in T^n \), we define the length of \( u \) as \( \infty \) if each \( u_i \in Q \cup \{-\infty\} \) and some \( u_j = -\infty \). In each case, we call \( \lambda \) or \( \infty \) the lattice length of \( u \).

For \( (x_1, \ldots, x_n) \in T^n \) and \( a \in T \), we define a scalar operation in \( T^n \) as follows:

\[
\begin{align*}
\text{a}(x_1, \ldots, x_n) &:= (\text{a}x_1, \ldots, \text{a}x_n).
\end{align*}
\]

For \( x, y \in T^{m+1} \setminus \{ (-\infty, \ldots, -\infty) \} \), we define the following relation \( \sim \):

\[
x \sim y \iff \text{there exists a real number } \lambda \text{ such that } x = \lambda y.
\]

The relation \( \sim \) becomes an equivalence relation. \( TP^n := T^{n+1}/ \sim \) is called the \( n \)-dimensional tropical projective space.

Let \( u = (u_1 : \cdots : u_{n+1}) \) and \( v = (v_1 : \cdots : v_{n+1}) \) be distinct two points on \( TP^n (n \geq 2) \). A distance between \( u = (u_1 : \cdots : u_{n+1}) \) and \( v = (v_1 : \cdots : v_{n+1}) \) is defined as “the lattice length of \((u_1 - u_i) - (v_1 - v_i), \ldots, (u_n - u_i) - (v_n - v_i)\)” = \( l \cdot \gcd((u_1 - u_i) - (v_1 - v_i), \ldots, (u_n - u_i) - (v_n - v_i)) \) for some \( i \) if all \( u_j - u_i, v_j - v_i \) are rational numbers, where \( l \) is a positive rational number such that all \( \frac{(u_i - v_i)}{l} \) and \( \frac{(v_j - v_i)}{l} \) are integers. A distance between a point and itself on \( TP^n \) is defined by zero.

**Lemma 2.1.1.** Let \( u = (u_1 : \cdots : u_{n+1}) \) and \( v = (v_1 : \cdots : v_{n+1}) \) be distinct two points on \( TP^n (n \geq 2) \) such that for some \( i \), all \( u_j - u_i, v_j - v_i \) are integers. Then,

\[
\gcd((u_1 - u_i) - (v_1 - v_i), \ldots, (u_n - u_i) - (v_n - v_i)) = \gcd((u_1 - u_k) - (v_1 - v_k), \ldots, (u_n - u_k) - (v_n - v_k))
\]

holds for any \( k \).

**Proof.** Let \( l_i := \gcd((u_1 - u_i) - (v_1 - v_i), \ldots, (u_n - u_i) - (v_n - v_i)) \) for any \( i \). For any \( k \), we have integers \( m_k \) and \( t_k \) such that \((u_k - u_i) - (v_k - v_i) = l_i \cdot m_k, \gcd(m_1, \ldots, m_n) = 1, \) \((u_k - u_j) - (v_k - v_j) = l_j \cdot t_k \) and \( \gcd(t_1, \ldots, t_n) = 1 \). Since \((u_k - u_i) - (v_k - v_i) = (u_k - v_k) - (u_i - v_i), \)

\[
\begin{align*}
l_j &= \gcd((u_1 - u_j) - (v_1 - v_j), \ldots, (u_n - u_j) - (v_n - v_j)) \\
    &= \gcd((u_1 - v_1) - (u_j - v_j), \ldots, (u_n - v_n) - (u_j - v_j)) \\
    &= \gcd((u_i - v_i) + l_i \cdot m_1 - (u_j - v_j), \ldots, (u_i - v_i) + l_i \cdot m_1 - (u_j - v_j)) \\
    &= \gcd((u_i - u_j) - (v_i - v_j) + l_i \cdot m_1, \ldots, (u_i - u_j) - (v_i - v_j) + l_i \cdot m_1) \\
    &= \gcd(l_j \cdot t_i + l_i \cdot m_1, \ldots, l_j \cdot t_i + l_i \cdot m_n).
\end{align*}
\]

Then \( l_j \) must divide \( l_i \cdot m_1, \ldots, l_i \cdot m_n \). As \( \gcd(m_1, \ldots, m_n) = 1 \), \( l_j \) divides \( l_i \). Thus \( l_j \leq l_i \). The inverse inequality holds since \( i \) and \( j \) are arbitrary. \( \square \)
By Lemma 2.1.1, the above distance between two points of $TP^n$ satisfying the condition is well-defined.

A tropical semimodule on $T$ is defined like a classical module on a ring. Note that a tropical semimodule on $T$ has two tropical operations: tropical sum $\oplus$ and tropical scalar multiplication $\odot$. Let $R$ and $R'$ be tropical semimodules on $T$, respectively. A map $f : R \to R'$ is said a homomorphism if for any $a, b \in R$ and $\lambda \in T$, $f(a \oplus b) = f(a) \oplus f(b)$ and $f(\lambda \odot a) = \lambda \odot f(a)$ hold. For a homomorphism $f : R \to R'$ of tropical semimodules, $f$ is an isomorphism if there exists a homomorphism $f' : R' \to R$ of tropical semimodules such that $f' \circ f = id_R$ and $f \circ f' = id_{R'}$. Then, $f'$ is also an isomorphism. Two tropical semimodules $R$ and $R'$ are isomorphic if there exists an isomorphism of tropical semimodules between them.

2.2 Metric graphs

In this paper, a graph means an unweighted, finite connected nonempty multigraph. Note that we allow the existence of loops. For a graph $G$, the sets of vertices and edges are denoted by $V(G)$ and $E(G)$, respectively. The genus of $G$ is defined by $g(G) := |E(G)| - |V(G)| + 1$. The valence $\text{val}(v)$ of a vertex $v$ of $G$ is the number of edges emanating from $v$, where we count each loop as two. A vertex $v$ of $G$ is a leaf end if $v$ has valence one. A leaf edge is an edge of $G$ adjacent to a leaf end.

An edge-weighted graph $(G, l)$ is the pair of a graph $G$ and a function $l : E(G) \to R_{>0} \cup \{\infty\}$ called a length function, where $l$ can take the value $\infty$ on only leaf edges. A metric graph is the underlying $\infty$-metric space of an edge-weighted graph $(G, l)$, where each edge of $G$ is identified with the closed interval $[0, l(e)]$ and if $l(e) = \infty$, then the leaf end of $e$ must be identified with $\infty$. Such a leaf end identified with $\infty$ is called a point at infinity and any other point is said to be a finite point. For the above metric graph $\Gamma$, $(G, l)$ is said to be its model. There are many possible models for $\Gamma$. We construct a model $(G_0, l_0)$ called the canonical model of $\Gamma$ as follows. Generally, we determine $V(G_0) := \{x \in \Gamma \mid \text{val}(x) \neq 2\}$, where the valence $\text{val}(x)$ of $x$ is the number of connected components of $U \setminus \{x\}$ with any sufficiently small connected neighborhood $U$ of $x$ in $\Gamma$ except following two cases. When $\Gamma$ is a circle, we determine $V(G_0)$ as the set consisting of one arbitrary point on $\Gamma$. When $\Gamma$ is the $\infty$-metric space obtained from the graph consisting only of two edges with length of $\infty$ and three vertices adjacent to these edges, $V(G_0)$ consists of the two endpoints of $\Gamma$ (those are points at infinity) and an any point on $\Gamma$ as the origin, Since connected components of $\Gamma \setminus V(G_0)$ consist of open intervals, whose lengths determine the length function $l_0$. If a model $(G, l)$ of $\Gamma$ has no loops, then $(G, l)$ is said to be a loopless model of $\Gamma$. For a model $(G, l)$ of $\Gamma$, the loopless model for $(G, l)$ is obtained by regarding all midpoints of loops of $G$ as vertices and by adding them to the set of vertices of $G$. The loopless model for the canonical model of a metric graph is called the canonical loopless model.

For terminology, in a metric graph $\Gamma$, an edge of $\Gamma$ means an edge of the underlying graph $G_0$ of the canonical model $(G_0, l_0)$. Let $e$ be an edge of $\Gamma$ which is not a loop. We regard $e$ as a closed subset of $\Gamma$, i.e., including the endpoints $v_1, v_2$ of $e$. The relative interior of $e$ is $e^0 = e \setminus \{v_1, v_2\}$. For a point $x$ on $\Gamma$, a connected component of $U \setminus \{x\}$ with any sufficiently small connected neighborhood $U$ of $x$ is a half-edge of $x$. 


For a model \((G, l)\) of a metric graph \(\Gamma\), we frequently identify a vertex \(v\) (resp. an edge \(e\)) of \(G\) with the point corresponding to \(v\) on \(\Gamma\) (resp. the closed subset corresponding to \(e\) of \(\Gamma\)).

The genus \(g(\Gamma)\) of a metric graph \(\Gamma\) is defined to be its first Betti number, where one can check that it is equal to \(g(G)\) of any model \((G, l)\) of \(\Gamma\). A metric graph of genus zero is called a tree.

### 2.3 Divisors on metric graphs

Let \(\Gamma\) be a metric graph. An element of the free abelian group \(\text{Div}(\Gamma)\) generated by points on \(\Gamma\) is called a divisor on \(\Gamma\). For a divisor \(D\) on \(\Gamma\), its degree \(\deg(D)\) is defined by the sum of the coefficients over all points on \(\Gamma\). We write the coefficient at \(x\) as \(D(x)\). A divisor \(D\) on \(\Gamma\) is said to be effective if \(D(x) \geq 0\) for any \(x\) in \(\Gamma\). If \(D\) is effective, we write simply \(D \geq 0\).

For an effective divisor \(D\) on \(\Gamma\), the set of points on \(\Gamma\) where the coefficient(s) of \(D\) is not zero is called the support of \(D\) and written as \(\text{supp}(D)\). The canonical divisor \(K_\Gamma\) of \(\Gamma\) is defined as \(K_\Gamma := \sum_{x \in \Gamma} (\text{val}(x) - 2) \cdot x\).

A rational function on \(\Gamma\) is a constant function of \(-\infty\) or a piecewise linear function with integer slopes and with a finite number of pieces, taking the value \(\pm \infty\) only at points at infinity. \(\text{Rat}(\Gamma)\) denotes the set of rational functions on \(\Gamma\). For a point \(x\) on \(\Gamma\) and \(f, g\) in \(\text{Rat}(\Gamma)\) which is not constant \(-\infty\), the sum of the outgoing slopes of \(f\) at \(x\) is denoted by \(\text{ord}_x(f)\). If \(x\) is a point at infinity and \(f\) is infinite there, we define \(\text{ord}_x(f)\) as the outgoing slope from any sufficiently small connected neighborhood of \(x\). Note when \(\Gamma\) is a singleton, for any \(f\) in \(\text{Rat}(\Gamma)\), we define \(\text{ord}_x(f) := 0\). This sum is 0 for all but finite number of points on \(\Gamma\), and thus

\[\text{div}(f) := \sum_{x \in \Gamma} \text{ord}_x(f) \cdot x\]

is a divisor on \(\Gamma\), which is called the principal divisor defined by \(f\). Two divisors \(D\) and \(E\) on \(\Gamma\) are said to be linearly equivalent if \(D - E\) is a principal divisor. We handle the values \(\infty\) and \(-\infty\) as follows. Let \(f, g\) in \(\text{Rat}(\Gamma)\) take the value \(\infty\) and \(-\infty\) at a point \(x\) at infinity on \(\Gamma\) respectively, and \(y\) be any point in any sufficiently small neighborhood of \(x\). When \(\text{ord}_x(f) + \text{ord}_x(g)\) is negative, then \((f \circ g)(x) := \infty\). When \(\text{ord}_x(f) + \text{ord}_x(g)\) is positive, then \((f \circ g)(x) := -\infty\). Remark that the constant function of \(-\infty\) on \(\Gamma\) does not determine a principal divisor. For a divisor \(D\) on \(\Gamma\), the complete linear system \(|D|\) is defined by the set of effective divisors on \(\Gamma\) being linearly equivalent to \(D\).

For a divisor \(D\) on a metric graph, let \(R(D)\) be the set of rational functions \(f \neq -\infty\) such that \(D + \text{div}(f)\) is effective together with \(-\infty\). When \(\deg(D)\) is negative, \(|D|\) is empty, so is \(R(D)\). Otherwise, from the argument in Section 3 of \([8]\), \(|D|\) is not empty and consequently so is \(R(D)\). Hereafter, we treat only divisors of nonnegative degree.

**Remark 2.3.1** ([9] and cf. [8, Lemma 4]). \(R(D)\) becomes a tropical semimodule on \(T\) by extending above tropical operations onto functions, giving pointwise sum and product.

For a tropical subsemimodule \(M\) of \((R \cup \{\pm \infty\})^\Gamma\) (or of \(R^\Gamma\)), \(f\) in \(M\) is called an extremal of \(M\) when it implies \(f = g_1\) or \(f = g_2\) that any \(g_1\) and \(g_2\) in \(M\) satisfies \(f = g_1 \oplus g_2\).
Remark 2.3.2 ([9]). Any finitely generated tropical subsemimodule $M$ of $R(D) \subset (\mathbb{R} \cup \{\pm\infty\})^\Gamma$ is generated by the extremals of $M$.

For a divisor $D$ on a metric graph $\Gamma$, we set $r(D)$, called the rank of $D$, as the minimum integer $s$ such that for some effective divisor $E$ with degree $s - 1$, the complete linear system associated to $D - E$ is empty set.

A Riemann–Roch theorem for finite loopless graphs was established by Baker–Norine ([4]). A Riemann–Roch theorem for metric graphs was proven independently by Gathmann–Kerber ([7]) and by Mikhalkin–Zharkov ([13]).

Remark 2.3.3 (Riemann–Roch theorem for metric graphs). Let $\Gamma$ be a metric graph and $D$ a divisor on $\Gamma$. Then, $r(D) - r(K_\Gamma - D) = \deg(D) + 1 - g(\Gamma)$ holds.

Let $\Gamma$ be a metric graph of genus at least two. $\Gamma$ is hyperelliptic if there exists a divisor on $\Gamma$ whose degree is two and whose rank is one. A binary group action on $\Gamma$ with a tree quotient is called a hyperelliptic involution of $\Gamma$. Chan ([6]), Amini–Baker–Brugallé–Rabinoff ([2]) and Kawaguchi–Yamaki ([12]) investigated hyperelliptic metric graphs.

Remark 2.3.4 ([12, Theorem 5]). Let $\Gamma$ be a metric graph of genus at least two without one valent points. Then, the following are equivalent:

(1) $\Gamma$ is hyperelliptic;

(2) $\Gamma$ has a hyperelliptic involution.

Furthermore, a hyperelliptic involution is unique.

2.4 Harmonic morphisms

Let $\Gamma, \Gamma'$ be metric graphs, respectively, and $\varphi : \Gamma \to \Gamma'$ be a continuous map. The map $\varphi$ is called a morphism if there exist a model $(G,l)$ of $\Gamma$ and a model $(G',l')$ of $\Gamma'$ such that the image of the set of vertices of $G$ by $\varphi$ is a subset of the set of vertices of $G'$, the inverse image of the relative interior of any edge of $G'$ by $\varphi$ is the union of the relative interiors of a finite number of edges of $G$ and the restriction of $\varphi$ to any edge $e$ of $G$ is a dilation by some nonnegative integer factor $\deg_e(\varphi)$. Note that the dilation factor on $e$ with $\deg_e(\varphi) \neq 0$ represents the ratio of the distance of the images of any two points $x$ and $y$ except points at infinity on $e$ to that of original $x$ and $y$. If an edge $e$ is mapped to a vertex of $G'$ by $\varphi$, then $\deg_e(\varphi) = 0$. The morphism $\varphi$ is said to be finite if $\deg_e(\varphi) > 0$ for any edge $e$ of $G$. For any half-edge $h$ of any point on $\Gamma$, we define $\deg_h(\varphi)$ as $\deg_e(\varphi)$, where $e$ is the edge of $G$ containing $h$.

Let $\Gamma'$ be not a singleton and $x$ a point on $\Gamma$. The morphism $\varphi$ is harmonic at $x$ if the number

$$\deg_x(\varphi) := \sum_{x \in h \to h'} \deg_h(\varphi)$$

is independent of the choice of half-edge $h'$ emanating from $\varphi(x)$, where $h$ is a connected component of the inverse image of $h'$ by $\varphi$ containing $x$. The morphism $\varphi$ is harmonic if it
is harmonic at all points on \( \Gamma \). One can check that if \( \varphi \) is a finite harmonic morphism, then the number 
\[
\deg(\varphi) := \sum_{x \to x'} \deg_x(\varphi)
\]
is independent of the choice of a point \( x' \) on \( \Gamma' \), and is said the degree of \( \varphi \), where \( x \) is an element of the inverse image of \( x' \) by \( \varphi \). If \( \Gamma' \) is a singleton and \( \Gamma \) is not a singleton, for any point \( x \) on \( \Gamma \), we define \( \deg_x(\varphi) \) as zero so that we regard \( \varphi \) as a harmonic morphism of degree zero. If both \( \Gamma \) and \( \Gamma' \) are singletons, we regard \( \varphi \) as a harmonic morphism which can have any number of degree.

The collection of metric graphs together with harmonic morphisms between them forms a category.

Let \( \varphi : \Gamma \to \Gamma' \) be a finite harmonic morphism between metric graphs. The pull-back of \( f' \) in \( \text{Rat}(\Gamma') \) is the function \( \varphi^*f' : \Gamma \to \mathbb{R} \cup \{ \pm \infty \} \) defined by \( \varphi^*f' := f' \circ \varphi \). We define the push-forward homomorphism on divisors \( \varphi_* : \text{Div}(\Gamma) \to \text{Div}(\Gamma') \) by
\[
\varphi_*(D) := \sum_{x \in \Gamma} D(x) \cdot \varphi(x).
\]
The pull-back homomorphism on divisors \( \varphi^* : \text{Div}(\Gamma') \to \text{Div}(\Gamma) \) is defined to be
\[
\varphi^*(D') := \sum_{x \in \Gamma} \deg_x(\varphi) \cdot D'(\varphi(x)) \cdot x.
\]
One can check that \( \deg(\varphi_*(D)) = \deg(D) \) and \( \varphi^*(\text{div}(f')) = \text{div}(\varphi^*f') \) for any divisor \( D \) on \( \Gamma \) and any \( f' \) in \( \text{Rat}(\Gamma')^\times \), respectively (cf. \cite[Proposition 4.2]{[5]}).

### 2.5 Chip-firing moves

In \cite{[8]}, Haase, Musiker and Yu used the term subgraph of a metric graph as a compact subset of the metric graph with a finite number of connected components and defined the chip firing move \( \text{CF}(\tilde{\Gamma}_1, l) \) by a subgraph \( \tilde{\Gamma}_1 \) of a metric graph \( \tilde{\Gamma} \) and a positive real number \( l \) as the rational function \( \text{CF}(\tilde{\Gamma}_1, l)(x) := -\min(l, \text{dist}(x, \tilde{\Gamma}_1)) \), where \( \text{dist}(x, \tilde{\Gamma}_1) \) is the infimum of the lengths of the shortest path to arbitrary points on \( \tilde{\Gamma}_1 \) from \( x \). They proved that every rational function on a metric graph is an (ordinary) sum of chip firing moves (plus a constant) (\cite[Lemma 2]{[8]}) with the concept of a weighted chip firing move. This is a rational function on a metric graph having two disjoint proper subgraphs \( \tilde{\Gamma}_1 \) and \( \tilde{\Gamma}_2 \) such that the complement of the union of \( \tilde{\Gamma}_1 \) and \( \tilde{\Gamma}_2 \) in \( \tilde{\Gamma} \) consists only of open line segments and such that the rational function is constant on \( \tilde{\Gamma}_1 \) and \( \tilde{\Gamma}_2 \) and linear (smooth) with integer slopes on the complement. A weighted chip firing move is an (ordinary) sum of chip firing moves (plus a constant) (\cite[Lemma 1]{[8]}).

With unbounded edges, their definition of chip firing moves needs a little correction. Let \( \Gamma_1 \) be a subgraph of a metric graph \( \Gamma \) which does not have any connected components consisting only of points at infinity and \( l \) a positive real number or infinity. The chip firing move by \( \Gamma_1 \) and \( l \) is defined as the rational function \( \text{CF}(\Gamma_1, l)(x) := -\min(l, \text{dist}(x, \Gamma_1)) \).
Remark 2.5.1 ([9]). A weighted chip firing move on a metric graph is a linear combination of chip firing moves having integer coefficients (plus a constant).

Remark 2.5.2 ([9]). Every rational function on a metric graph is a linear combination of chip firing moves having integer coefficients (plus a constant).

A point on $\Gamma$ with valence two is said to be a smooth point. We sometimes refer to an effective divisor $D$ on $\Gamma$ as a chip configuration. We say that a subgraph $\Gamma_1$ of $\Gamma$ can fire on $D$ if for each boundary point of $\Gamma_1$ there are at least as many chips as the number of edges pointing out of $\Gamma_1$. A set of points on a metric graph $\Gamma$ is said to be cut set of $\Gamma$ if the complement of that set in $\Gamma$ is disconnected.

3 Rational maps induced by $|D|^K$

In this section, our main concern is the rational map induced by an invariant linear system on a metric graph with an action by a finite group $K$. We find a condition that the rational map induces a $K$-Galois covering on the image.

3.1 Generators of $R(D)^K$

In this subsection, for an effective divisor $D$ on a metric graph and a finite group $K$ acting on the metric graph, we give two proofs, other than that in [9], of the statement that the $K$-invariant set $R(D)^K$ of $R(D)$ is finitely generated as a tropical semimodule. When $D$ is $K$-invariant, $R(D)/R$ is identified with the subset $|D|^K$ of $|D|$ consisting of all $K$-invariant elements of $|D|$, so the $K$-invariant linear system $|D|^K$ is finitely generated by the generating set of $R(D)^K$ modulo tropical scaling (except by $-\infty$).

Remark 3.1.1 ([8, Lemma 6]). Let $\tilde{\Gamma}$ be a metric graph, $\tilde{D}$ be a divisor on $\tilde{\Gamma}$ and $S$ be the set of rational functions $f$ in $R(\tilde{D})$ such that the support of $\tilde{D} + \text{div}(f)$ does not contain any cut set of $\tilde{\Gamma}$ consisting only of smooth points. Then

1. $S$ contains all the extremals of $R(\tilde{D})$,
2. $S$ is finite modulo tropical scaling (except by $-\infty$), and
3. $S$ generates $R(\tilde{D})$ as a tropical semimodule.

Remark 3.1.2 ([8, Theorem 14]). Let $G$ be a model of $\tilde{\Gamma}$ and let $S_G$ be the set of functions $f \in R(D)$ such that the support of $D + \text{div}(f)$ does not contain an interior cut set (i.e. a cut set consisting of points in interior of edges in the model $G$). Then

1. $S_G$ contains the set $S$ from Remark 3.1.1 and
2. $S_G$ is finite modulo tropical scaling (except by $-\infty$).
Though in the above remarks they assume that \( R(\mathcal{D}) \) is a subset of \( R^{\mathcal{F}} \), the proof is applied even in the case that \( R(\mathcal{D}) \) is a subset of \( (R \cup \{ \pm \infty \})^{\mathcal{F}} \) with preparations in Section 2. Also, the above remarks throws the relation between \( S \) (resp. \( S_G \)) and \( \mathcal{D} \) into relief, hence hereafter we write \( S \) (resp. \( S_G \)) for \( \mathcal{D} \) as \( S(\mathcal{D}) \) (resp. \( S_G(\mathcal{D}) \)).

Next, for \( R(D)^K \), the following holds.

**Remark 3.1.3** \([9]\). \( R(D)^K \) is a tropical semimodule.

Note that \( R(D + \text{div}(f))^K = R(D)^K \circ (-f) \) for any \( K \)-invariant rational function \( f \).

The following lemma is an extension of \([8\, \text{Lemma 5}]\).

**Remark 3.1.4** \([9]\). Let \( f \) be in \( \text{Rat}(\mathcal{G}) \). Then, \( f \) is an extremal of \( R(D)^K \) if and only if there are not two proper \( K \)-invariant subgraphs \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) covering \( \mathcal{G} \) such that each can fire on \( D + \text{div}(f) \).

If \( D \) is \( K \)-invariant, \( R(D)^K / R \) is naturally identified with the subset \( |D|^K \) of \( |D| \) consisting of all \( K \)-invariant elements of \( |D| \). In fact, let \( D \) be a \( K \)-invariant effective divisor on \( \mathcal{G} \). For any \( D' \in |D|^K \), there exists \( f \in \text{Rat}(D)^K \) such that \( D' = D + \text{div}(f) \). Since both \( D \) and \( D' \) are \( K \)-invariant, \( D' = D + \text{div}(f \circ \sigma) \) for any \( \sigma \in K \). Thus \( 0 = \text{div}(f) - \text{div}(f \circ \sigma) = \text{div}(f - f \circ \sigma) \) and there exists \( c \in R \) such that \( f - f \circ \sigma = c \), i.e. \( f = c \circ f \circ \sigma \) by Liouville’s theorem. Since the order \( k \) of \( \sigma \) is finite, \( f = c \circ \cdots \circ c \circ f \) holds, where \( c \) is multiplied \( k \) times. As \( k \) is not zero, \( c \) must be zero. Therefore \( f \) is \( K \)-invariant and then \( f \in \text{Rat}(D)^K \). Conversely, \([g] \in \text{Rat}(D)^K / R \) corresponds to an element \( D + \text{div}(g) \) in \( |D|^K \).

Let \( \mathcal{G} \) be a metric graph, \( K \) a finite group acting on \( \mathcal{G} \) and \( D \) an effective divisor on \( \mathcal{G} \). In this subsection, we prove that \( R(D)^K \) is finitely generated as a tropical semimodule in two different ways from that in Subsection 4.1. In one of them, we return arguments about effective divisors to ones about \( K \)-invariant divisors to use the condition of generators of \( R(D) \) found by Haase, Musiker and Yu. In the other, we find generators of \( R(D)^K \) by an algebraic way. By later proof, we know that the number of generators of \( R(D)^K \) is not greater than that of \( R(D) \).

Let \( D_1 \) be the maximum \( K \)-invariant part of \( D \), i.e. \( D_1 := \sum_{x \in \mathcal{G}} \min_{x' \in Kx} \{ D(x') \} \cdot x \).

By the definition, both \( D_1 \) and the \( K \)-variant part \( D_2 := D - D_1 \) are effective.

**Lemma 3.1.5.** \( R(D)^K = R(D_1)^K \) holds.

**Proof.** \( D_1 + \text{div}(f_1) \geq 0 \) holds for any element \( f_1 \) of \( R(D_1)^K \). Therefore \( D + \text{div}(f_1) = D_2 + (D_1 + \text{div}(f_1)) \geq 0 \), i.e. \( f_1 \in R(D_1)^K \).

For arbitrary element \( f \) of \( R(D)^K \), the set of poles of \( f \) is \( K \)-invariant as \( f \) is \( K \)-invariant and the set is contained in the support of \( D_1 \). This means \( D_1 + \text{div}(f) \geq 0 \). In fact, if \( D_1 + \text{div}(f) < 0 \) holds, then there exists a point \( x \) on \( \mathcal{G} \) whose orbit by \( K \) is a subset of \( \text{supp}(D_1 + \text{div}(f)) \). Therefore \( 0 > D_2 + (D_1 + \text{div}(f)) = D + \text{div}(f) \) holds and this contradicts to \( f \in R(D)^K \).

By Lemma 3.1.5, we can prove (1), (3) of the following theorem in the same way of the proof of \([8\, \text{Lemma 6}]\) and (2) clearly holds by \([8\, \text{Lemma 6}]\).
Theorem 3.1.6. In the above situation, the following hold:

1. $S_G(D_1)^K$ contains all the extremals of $R(D_1)^K$,
2. $S_G(D_1)^K$ is finite modulo tropical scaling (except by $-\infty$), and
3. $S_G(D_1)^K$ generates $R(D_1)^K$ as a tropical semimodule.

The following theorem is proved with a purely algebraic way from stacked point of view.

Theorem 3.1.7. Let $\Gamma$ be a metric graph, $K$ be a finite group acting on $\Gamma$ and $D$ a $K$-invariant effective divisor on $\Gamma$. For a minimal generating set $\{f_1, \ldots, f_n\}$ of $R(D)$, $\{g_1, \ldots, g_n\}$ is a generating set of $R(D)^K$, where $g_i := \sum_{\sigma \in K} f_i \circ \sigma$.

Proof. For any $i$ and $\tau \in K$,

$$g_i \circ \tau = \left( \sum_{\sigma \in K} f_i \circ \sigma \right) \circ \tau = \sum_{\sigma \in K} f_i \circ (\sigma \circ \tau) = \sum_{\sigma \in K} f_i \circ \sigma = g_i$$

and

$$0 \leq \tau(D + \text{div}(f_i)) = \tau(D) + \text{div}(f_i \circ \tau) = D + \text{div}(f_i \circ \tau)$$

hold. Thus, each $g_i$ is in $R(D)^K$. For any element $g = \sum_{i=1}^{n} a_i f_i \in R(D)^K$,

$$g = \sum_{\sigma \in K} g \circ \sigma = \sum_{\sigma \in K} \left( \sum_{i=1}^{n} a_i f_i \right) \circ \sigma = \sum_{i=1}^{n} a_i \left( \sum_{\sigma \in K} f_i \circ \sigma \right) = \sum_{i=1}^{n} a_i g_i.$$ 

Hence $\{g_1, \ldots, g_n\}$ generates $R(D)^K$.

Corollary 3.1.8. Let $\Gamma$ be a metric graph, $K$ be a finite group acting on $\Gamma$ and $D$ an effective divisor on $\Gamma$. For a minimal generating set $\{f_1, \ldots, f_n\}$ of $R(D_1)$, $\{g_1, \ldots, g_n\}$ is a generating set of $R(D)^K = R(D_1)^K$, where $g_i := \sum_{\sigma \in K} f_i \circ \sigma$.

Remark 3.1.9. $\{g_1, \ldots, g_n\}$ is not always minimal. Using Lemma 3.1.4, we can obtain a minimal generating set by omitting elements not being extremal from $\{g_1, \ldots, g_n\}$.

3.2 Galois covering on metric graphs

Let $\Gamma$ be a metric graph and $K$ a finite group acting on $\Gamma$. We define $V_1(\Gamma)$ as the set of points $x$ on $\Gamma$ such that there exists a point $y$ in any neighborhood of $x$ whose stabilizer is not equal to that of $x$.

Remark 3.2.1. $V_1(\Gamma)$ is a finite set.
We set \((G_0, l_0)\) as the canonical loopless model of \(\Gamma\). By Lemma 3.2.1, we obtain the model \((\tilde{G}_1, l_1)\) of \(\Gamma\) by setting the \(K\)-orbit of the union of \(V(G_0)\) and \(V_1(\Gamma)\) as the set of vertices \(V(\tilde{G}_1)\). Naturally, we can regard that \(K\) acts on \(V(\tilde{G}_1)\) and also on \(E(\tilde{G}_1)\). Thus, the sets \(V(\tilde{G}')\) and \(E(\tilde{G}')\) are defined as the quotient sets of \(V(\tilde{G}_1)\) and \(E(\tilde{G}_1)\) by \(K\), respectively. Let \(\tilde{G}'\) be the graph obtained by setting \(V(\tilde{G}')\) as the set of vertices and \(E(\tilde{G}')\) as the set of edges. Since \(G_1\) is connected, \(\tilde{G}'\) is also connected. We obtain the loopless graph \(G'\) from \(\tilde{G}'\) and the loopless model \((G_1, l_1)\) of \(\Gamma\) from the inverse image of \(V(G')\) by the map defined by \(K\). Note that \(V(G_1)\) contains \(V(\tilde{G}_1)\). Since \(K\) is a finite group acting on \(\Gamma\), the length function \(l' : E(G') \to \mathbb{R}_{\geq 0} \cup \{\infty\}, [e] \mapsto |K_e| \cdot l_1(e)\) is well-defined, where \([e]\) and \(K_e\) mean the equivalence class of \(e\) and the stabilizer of \(e\), respectively. Let \(\Gamma'\) be the metric graph obtained from \((G', l')\). Then, \(\Gamma'\) is the quotient metric graph of \(\Gamma\) by \(K\).

For any edge \(e\) of \(G_1\), by the Orbit-Stabilizer formula, \(|K_e|\) is a positive integer. Thus, for \((G_1, l_1)\) and \((G', l')\), there exists only one morphism \(\pi : \Gamma \to \Gamma'\) that satisfies \(\deg_e(\pi) = |K_e|\) for any edge \(e\) of \(G_1\).

**Remark 3.2.2 ([9]).** \(\pi\) is a finite harmonic morphism of degree \(|K|\).

Note that whether \(\Gamma\) is a singleton or not agrees with whether \(\Gamma'\) is a singleton.

Let \(\varphi : \Gamma \to \Gamma'\) be a finite harmonic morphism of metric graphs. We write the isometry transformation group of \(\Gamma\) as \(\text{Isom}(\Gamma)\), i.e. \(\text{Isom}(\Gamma) := \{\sigma : \Gamma \to \Gamma | \sigma \text{ is an isometry}\}\).

**Definition 3.2.3.** Assume that a map \(\varphi : \Gamma \to \Gamma'\) between metric graphs and an action on \(\Gamma\) by \(K\) are given. Then, \(\varphi\) is a \(K\)-Galois covering on \(\Gamma'\) if \(\varphi\) is a harmonic morphism of metric graphs, the degree of \(\varphi\) is coincident with the order of \(K\) and the action on \(\Gamma\) by \(K\) induces a transitive action on every fibre by \(K\). \(K\) is called the Galois group of \(\varphi\).

If \(\varphi\) is a \(K\)-Galois covering, then \(\varphi\) is finite since \(K\) transitively acts on every fibre.

**Remark 3.2.4.** A \(K\)-Galois covering can be \(K'\)-Galois for a finite group \(K'\) which is not conjugate to \(K\).

**Lemma 3.2.5.** There exists a finite harmonic morphism of degree one (i.e. an isomorphism) \(\psi\) from the quotient metric graph \(\Gamma/K\) to \(\Gamma'\) which satisfies \(\varphi = \psi \circ \pi\), where \(\pi : \Gamma \to \Gamma/K\) is the natural surjection.

**Proof.** Let \((G, l), (G', l')\) and \((G'', l'')\) be models of \(\Gamma, \Gamma'\) and \(\Gamma/K\) corresponding to \(\varphi\) and \(\pi\), respectively. Let \(\psi : \Gamma/K \to \Gamma'\) be a map defined by \([x] \mapsto \varphi(x)\). Since \([x] = Kx \subset \Gamma\) and \(\varphi\) is \(K\)-Galois, \(\varphi(Kx) = \varphi(x)\) holds. Thus \(\psi\) is well-defined. By the definition of \(\psi\), \(\varphi = \psi \circ \pi\) holds. As \(\varphi\) is continuous, so is \(\psi\). For any edge \(e \in E(G)\), since \(\varphi\) is \(K\)-Galois,

\[
\deg(\varphi) = \sum_{e_1 \in K e} \deg_{e_1}(\varphi) = \sum_{e_1 \in K e} \deg_{e_1}(\varphi) = |K e| \cdot \deg_e(\varphi).
\]

Therefore,

\[
\deg_e(\varphi) = \frac{\deg(\varphi)}{|K e|} = \frac{|K|}{|K e|} = |K e|.
\]
Remark 3.3.2. For any edge $[e] \in E(G'')$, 
\[ \frac{l''([e])}{l''(\varphi(e))} = \frac{l(e) \cdot \deg_e(\pi)}{l(e) \cdot \deg_e(\varphi)} = \frac{l(e) \cdot |K_e|}{l(e) \cdot |K_e|} = 1. \]
Therefore, $\psi$ is a finite harmonic morphism of degree one. \qed

3.3 Rational maps induced by $|D|^K$

Several concepts and statements appearing in this subsection are based on Section 7 of [5].

Let $\Gamma$ be a metric graph, $K$ a finite group acting on $\Gamma$ and let $D$ be a $K$-invariant effective divisor on $\Gamma$. For a finite generating set $F = \{f_1, \ldots, f_n\}$ of $R(D)^K$, let $(G, l)$ be the model of $\Gamma$ such that $V(G) := V(G_0) \cup \bigcup_{i=1}^{n} (\text{supp}(D + \text{div}(f_i)))$, where $G_0$ is the underlying graph structure of the canonical loopless model $(G_0, l_0)$ of $\Gamma$. $\phi_F : \Gamma \to \mathbb{TP}^{n-1}, x \mapsto (f_1(x) : \cdots : f_n(x))$ denotes the rational map induced by $F$.

**Proposition 3.3.1.** $\text{Im}(\phi_F)$ is a metric graph in $\mathbb{TP}^{n-1}$.

**Proof.** If $n = 1$, then $\phi_F$ is a constant map from $\Gamma$ to $\mathbb{TP}^0$ and $\text{Im}(\phi_F)$ is a metric graph.

Let us assume that $n \geq 2$. As $R(D)^K$ contains all constant functions, $\phi_F$ is well-defined. Since all $f_i$s are $\mathbb{Z}$-affine function, the image of $\phi_F$ is a one-dimensional polyhedral complex.

Let $e = v_1v_2$ be an edge of $G$. As each $f_i$ is constant on $e$, $\phi_F(e)$ is a segment or a point in $\mathbb{TP}^{n-1}$ and the distance between $\phi_F(v_1)$ and $\phi_F(v_2)$ can be measured by the definition of rational functions on metric graphs. Hence $\text{Im}(\phi_F)$ becomes a metric graph. \qed

When does $\phi_F$ induce a finite harmonic morphism from $\Gamma$ to $\text{Im}(\phi_F)$? Moreover, when does $\phi_F$ induce a $K$-Galois covering on $\text{Im}(\phi_F)$? We consider an answer for these questions.

**Remark 3.3.2.** $R(D)^K$ is isomorphic to $R(D')^K$ as a tropical semimodule for any element $D'$ of $|D|^K$. In fact, since $D'$ is linearly equivalent to $D$ and both are $K$-invariant, there exists $f \in R(D)^K$ such that $D' = D + \text{div}(f)$ (see Subsection 3.1). $\psi : R(D)^K \to R(D')^K$, $g \mapsto g - f$ and $\psi' : R(D')^K \to R(D)^K$, $g \mapsto g + f$ are homomorphisms of tropical semimodules and are inverses of each other.

**Definition 3.3.3.** Let $\Gamma$ be a metric graph, $K$ a finite group acting on $\Gamma$ and let $D$ be a $K$-invariant effective divisor on $\Gamma$. $D$ is $K$-very ample if for any elements $x$ and $x'$ of $\Gamma$ whose orbits by $K$ differ from each other, there exist $f$ and $f'$ in $R(D)^K$ such that $f(x) - f(x') \neq f'(x) - f'(x')$. We call $D$ $K$-ample if some positive multiple $kD$ is $K$-very ample. When $K$ is trivial, we use words “very ample” or “ample” simply.

**Remark 3.3.4.** $D$ is $K$-very ample if and only if $D'$ is $K$-very ample for any element $D'$ of $|D|^K$. In fact, if $D$ is $K$-very ample, for any points $x$ and $x'$ on $\Gamma$ whose orbits by $K$ differ from each other, there exist $g$ and $g'$ in $R(D)^K$ such that $g(x) - g(x') \neq g'(x) - g'(x')$. Using $\psi$ given in Remark 3.3.2, $(g - f)(x) - (g - f)(x') = (g(x) - g(x')) - (f(x) - f(x')) \neq (g(x) - g(x')) - (f'(x) - f'(x')) = (g - f')(x) - (g - f')(x')$. Thus, $D'$ is $K$-very ample. The converse is shown in the same way.
Definition 3.3.5. Let $F = \{f_1, \ldots, f_n\}$ be a finite generating set of $R(D)^K$. $\phi_F$ is $K$-injective if $\phi_F$ separates different $K$-orbits on $\Gamma$, i.e. for any $x$ and $x'$ in $\Gamma$ whose $K$-orbits differ from each other, $\phi_F(x) \neq \phi_F(x')$ holds.

Remark 3.3.6. Let $F_1 = \{f_1, \ldots, f_n\}$ and $F_2 = \{g_1, \ldots, g_n\}$ be minimal generating sets of $R(D)^K$, respectively. Since both $F_1$ and $F_2$ are minimal, each $g_i$ is written as $a_i \circ f_i$ with some real number $a_i$ by changing numbers if we need. Thus, we can move $\text{Im}(\phi_{F_1})$ to $\text{Im}(\phi_{F_2})$ by the translation $(x_1 : \cdots : x_n) \mapsto (x_1 + a_1 : \cdots : x_n + a_n)$. Hence $\phi_{F_1}$ is $K$-injective if and only if $\phi_{F_2}$ is $K$-injective.

Lemma 3.3.7. $D$ is $K$-very ample if and only if the rational map associated to any finite generating set is $K$-injective.

Proof. (“if” part) Let $F = \{f_1, \ldots, f_n\}$ be a generating set of $R(D)^K$. Assume that $\phi_F$ is $K$-injective, i.e. for any $Kx \neq Kx'$, $\phi_F(x) \neq \phi_F(x')$. If for any $i$ and $j$, $f_i(x) - f_i(x') = f_j(x) - f_j(x') =: c$, then

$$\phi_F(x) = (f_1(x) : \cdots : f_n(x)) = (f_1(x') + c : \cdots : f_n(x') + c) = (f_1(x') : \cdots : f_n(x')) = \phi_F(x').$$

This is a contradiction. Thus there exist $i \neq j$ such that $f_i(x) - f_i(x') \neq f_j(x) - f_j(x')$.

(“only if” part) Suppose that there exists a finite generating set $F = \{f_1, \ldots, f_n\} \subset R(D)^K$ such that $\phi_F$ is not $K$-injective. There exist distinct points $x$ and $x'$ on $\Gamma$ whose $K$-orbits are different from each other and whose images by $\phi_F$ are same. Therefore, there exists a real number $c$ such that $f_i(x') + c = f_i(x)$ for any $i$. This means that $f_i(x) - f_i(x') = f_j(x) - f_j(x')$ for any $i$ and $j$. Hence for any $f$ and $f'$ in $R(D)^K$, $f(x) - f(x') = f'(x) - f'(x')$ since $F$ generates $R(D)^K$ as a tropical semimodule.

If the induced rational map associated to a minimal generating set of $R(D)^K$ is $K$-injective, then $D$ is $K$-very ample since all generating sets of $R(D)^K$ contain a minimal generating set of $R(D)^K$ consisting only of extremals of $R(D)^K$ (see [9]). Therefore, we obtain the following corollary.

Corollary 3.3.8. $D$ is $K$-very ample if and only if the rational map associated to a minimal generating set of $R(D)^K$ is $K$-injective.

Lemma 3.3.9. If $\Gamma$ dose not consist only of one point and for any point $x$ on $\Gamma$, there exists $f$ in $R(D)^K$ such that the support of $D + \text{div}(f)$ contains the orbit of $x$ by $K$, then for any edge $e$ of $G$, there exists $f_i$ which has slope one on $e$.

Proof. Suppose that there exists an edge $e = v_1v_2 \in E(G)$ such that any $f_i$ does not have slope one on $e$. By assumptions, there exists $f_j$ has slope at least two on $e$. By changing numbers if we need, we may assume that $f_j(v_1) > f_j(v_2)$. Let $(f_j)_{\geq t} := \max\{f_j(x), f_j(t)\}$ for any $t \in \Gamma$. Since both $f_j$ and the constant $f_j(t)$ function are in $R(D)^K$ and $(f_j)_{\geq t}$ is the tropical sum of them, $(f_j)_{\geq t} \in R(D)^K$ holds. $\Gamma_{v_1} := \{x \in \Gamma | f_j(x) \geq f_j(v_1)\}$ is $K$-invariant and can fire on $D + \text{div}((f_j)_{\geq t(v_1)})$. In fact, for any $x \in \Gamma_{v_1}$ and $\sigma \in K$, since
$f_j(\sigma(x)) = f_j(x) \geq f_j(v_1)$ holds, $\Gamma_{v_1}$ is $K$-invariant. For any sufficiently close point $t$ to $v_1$ on $e$ and $\Gamma_t := \{ x \in \Gamma \mid f_j(x) \geq f_j(t) \}$, $g := (f_j)_{\geq f_j(t)} - (f_j)_{\geq f_j(v_1)}$ has a constant integer slope which is different from zero on any closure of connected component of $\Gamma_t \setminus \Gamma_{v_1}$. Therefore for any point $x$ on the boundary set of $\Gamma_{v_1}$ and any positive number $l$ less than the minimum of lengths of these closures,

$$(D + \text{div}((f_j)_{\geq f_j(v_1)} + (\text{CF}(\Gamma_{v_1}, l))))(x) \geq (D + \text{div}((f_j)_{\geq f_j(v_1)} + g))(x) = (D + \text{div}(f_j))(x) \geq 0.$$ 

Thus $(f_j)_{\geq f_j(v_1)} + \text{CF}(\Gamma_{v_1}, l)$ is in $R(D)^K$ and has slope one on $[t, v_1] \subset e$. This is a contradiction.

\begin{remark}
When $K$ is trivial, the condition “for any point $x$ on $\Gamma$, there exists $f$ in $R(D)^K$ such that the support of $D + \text{div}(f)$ contains the orbit of $x$ by $K$” means that the rank $r(D)$ of $D$ is greater than or equal to one.
\end{remark}

\begin{lemma}
If $\phi_F$ is $K$-injective, then for any $x \in \Gamma$, there exists $f \in R(D)^K$ such that $\text{supp}(D + \text{div}(f)) \supseteq Kx$.
\end{lemma}

\begin{proof}
If $\Gamma$ consists only of one point $p$, then $D$ must be of the form $kp$ with a positive integer $k \in \mathbb{Z}_{>0}$ and $R(D)^K = R(D)$ consists only of constant functions on $\Gamma$. Therefore for any $f \in R(D)^K$, the support of $D + \text{div}(f)$ coincides with the support of $D$ and it is $\{p\}$.

Let us assume that $\Gamma$ does not consist only of one point. Since $\Gamma$ is connected, $\Gamma$ contains a closed segment. We show the contraposition. Suppose that there exists a point $x$ on $\Gamma$ such that for any $f \in R(D)^K$, the support of $D + \text{div}(f)$ does not contain $Kx$. In particular, for any $i$, the support of $D + \text{div}((f_i)_{\geq f_i(x)})$ (resp. the support of $D + \text{div}(f_i)$) does not contain $Kx$, where $(f_i)_{\geq f_i(x)}(t) := \max\{f_i(x), f_i(t)\}$ and it is in $R(D)^K$ as it is the tropical sum of $f_i$ and the constant $f_i(t)$ function for any $t \in \Gamma$. Therefore $(D + \text{div}((f_i)_{\geq f_i(x)}))(x) = 0$ (resp. $(D + \text{div}(f_i))(x) = 0$) holds. By the definition of $(f_i)_{\geq f_i(x)}$, $D(x) \geq 0$ and $(\text{div}((f_i)_{\geq f_i(x)}))(x) \geq 0$. Thus $D(x) = (\text{div}((f_i)_{\geq f_i(x)}))(x) = 0$, and then $(\text{div}(f_i))(x) = 0$. If $f_i$ is nonconstant around $x$, then there must exist a direction on which $f_i$ has positive slope and another direction on which $f_i$ has negative slope at $x$. This means that $(\text{div}((f_i)_{\geq f_i(x)}))(x) \geq 1$ and this is a contradiction. Consequently, $f_i$ is locally constant at $x$. As $F$ is finite, we can choose a connected neighborhood $U_x$ of $x$ such that $\phi_F(U_x) = \phi_F(x)$. Since $K$ is finite, $\phi_F$ is not $K$-injective.
\end{proof}

\begin{remark}
Since $D$ is effective, $R(D)^K$ contains all constant functions on $\Gamma$. Therefore for any edge $e$ of $G$, there exists $f_i$ which has slope zero on $e$.
\end{remark}

\begin{remark}
In Section 6, we define metric graphs with edge-multiplicities and harmonic morphisms between them. Hereafter, we use these concepts and so we recommend seeing Section 6.
\end{remark}

\begin{theorem}
If $\phi_F$ is $K$-injective, then $\phi_F$ induces a $K$-Galois covering on $\text{Im}(\phi_F)$ with some edge-multiplicities.
\end{theorem}
Lemma 3.3.16. If \( \Gamma \) is a singleton, then the image of \( \phi_\Gamma \) is also a singleton. Since \( \phi_\Gamma \) induces a finite harmonic morphism between singletons, then it is \( K \)-Galois.

Assume that \( \Gamma \) is not a singleton. By Proposition 3.3.1, Lemma 3.3.9, Lemma 3.3.11 and Remark 3.3.12, \( \phi_\Gamma \) is a local isometry. In fact, for any edge \( e = v_1v_2 \) of \( G \),

\[
\phi_\Gamma(v_2) = (f_1(v_2) : \cdots : f_n(v_2)) = (f_1(v_1) + s_1 \cdot l(e) : \cdots : f_n(v_1) + s_n \cdot l(e)),
\]

where each \( s_i \) is the slope of \( f_i \) on \( e \) from \( v_1 \) to \( v_2 \). Let \( j \) be a number such that \( f_j \) has slope zero on \( e \), i.e. \( s_j = 0 \). Then, the distance between \( \phi_\Gamma(v_1) \) and \( \phi_\Gamma(v_2) \) is

\[
\text{"the lattice length of } (f_1(v_2) - f_j(v_2)) - (f_1(v_1) - f_j(v_1)), \ldots, (f_n(v_2) - f_j(v_2)) - (f_n(v_1) - f_j(v_1))\text{"} = \text{"the lattice length of } (s_1 \cdot l(e), \ldots, s_n \cdot l(e))\text{"} = l(e) \cdot \gcd(s_1, \ldots, s_n) = l(e).
\]

Let \((G_o, l_o)\) (resp. \((G'_o, l'_o)\)) be the canonical model of \( \Gamma \) (resp. \( \operatorname{Im}(\phi_\Gamma) \)). We show that we can choose loopless models \((G_1, l_1)\) and \((G', l')\) of \( \Gamma \) and \( \operatorname{Im}(\phi_\Gamma) \) respectively for that \( \phi_\Gamma \) induces a \( K \)-Galois covering on \( \operatorname{Im}(\phi_\Gamma) \) with the edge-multiplicities \(1 : E(G_1) \to \mathbb{Z}_{\geq 0}, e \mapsto 1\), and \( m' : E(G') \to \mathbb{Z}_{\geq 0}, e' \mapsto |K_e|\), where \( e \) is an edge of \( G_1 \) whose image by \( \phi_\Gamma \) is \( e' \). For any \( x' \in V(G'_o) \setminus \phi_\Gamma(V(G_o)) \), since \( \phi_\Gamma \) is \( K \)-injective, there exists a unique orbit \( Kx \) in \( \Gamma \) whose image by \( \phi_\Gamma \) is \( x' \). \( x \) is smooth. Let \( e \) be the edge of \( G_o \) containing \( x \). If there exist no elements of \( K \) which inverse \( e \), then \( \phi_\Gamma(x) \) is smooth and this is a contradiction. Hence there exists an element of \( K \) which inverse \( e \). As \( x' \) is not smooth, by the proof of Lemma 3.2.1, \( x \) is the midpoint of \( e \) and \( x' \) has valence one. Thus, let \( V(G_1) := V(G) \cup \bigcup_{x' \in V(G'_o) \setminus V(G_o)} \phi_\Gamma^{-1}(x') \) and \( V(G') := \phi_\Gamma(V(G_1)) \). Then \( \phi_\Gamma \) induces a finite harmonic morphism from \( \Gamma \) to \( \operatorname{Im}(\phi_\Gamma) \) of degree \(|K|\) with the edge-multiplicities \(1\) and \( m' \). By the definition of the action of \( K \) on \( \Gamma \) and by the assumption, the induced finite harmonic morphism is a \( K \)-Galois covering on \( \operatorname{Im}(\phi_\Gamma) \).

\[\square\]

Corollary 3.3.15. If \( \phi_\Gamma \) is \( K \)-very ample, then \( \phi_\Gamma \) induces a \( K \)-Galois covering on \( \operatorname{Im}(\phi) \).

Lemma 3.3.16. If \( \phi_\Gamma \) induces a \( K \)-Galois covering (with the edge-multiplicities in Theorem 3.3.14), then \( \phi_\Gamma \) is \( K \)-injective.

Proof. If there exist two \( K \)-orbits in \( \Gamma \) whose images by \( \phi_\Gamma \) consistent with each other, then the inverse image by \( \phi_\Gamma \) contains at least two \( K \)-orbits. Thus \( K \) does not act transitively on the fibre.

\[\square\]

Corollary 3.3.17. \( \phi_\Gamma \) induces a \( K \)-Galois covering with the edge-multiplicities in Theorem 3.3.14 if and only if \( \phi_\Gamma \) is \( K \)-injective.

Remark 3.3.18. By the same proof of Lemma 3.3.16, we have the statement \(	ext{"Every } K \text{-Galois covering on a metric graph (with edge-multiplicities) maps distinct } K \text{-orbits to distinct points."\text{"}}, which is more general than Lemma 3.3.16.
We then have an answer for the question “when does $\phi_F$ induce a $K$-Galois covering on $\text{Im}(\phi_F)$”.

Next, we pose a question “whether there exists a divisor which induces a $K$-Galois covering induced by $K$-invariant linear system or not”.

**Remark 3.3.19** ([S Corollary 46]). Every divisor of positive degree is ample.

**Theorem 3.3.20.** Every effective $K$-invariant divisor of positive degree is $K$-ample.

**Proof.** Let $\pi : \Gamma \to \Gamma' := \Gamma/K$ be the natural surjection. By the construction, $\pi$ is $K$-Galois. Thus $\pi$ is $K$-injective. Let $x$ and $y$ be points on $\Gamma$ whose $K$-orbits are different from each other and let $x' := \pi(x)$ and $y' := \pi(y)$. Let $D$ be an effective $K$-invariant divisor on $\Gamma$ of positive degree. $\pi_*(D)$ is ample since $\deg(\pi_*(D)) = \deg(D) \geq 1$. Therefore there exists a positive integer $k$ such that $k\pi_*(D)$ is very ample. Let $f_1'$ and $f_2'$ be in $R(k\pi_*(D))$ such that $f_1'(x') - f_1'(y') \neq f_2'(x') - f_2'(y')$. As $D$ is $K$-invariant and $\pi$ is $K$-injective,

$$
\pi^*(\pi_*(D)) = \pi^* \left( \sum_{x \in \Gamma} D(x) \cdot \pi(x) \right)
= \sum_{x \in \Gamma} \deg_x(\pi) \cdot \left( \sum_{y \in \Gamma} D(y) \cdot \pi(y) \right) \cdot \pi(x) 
= \sum_{x \in \Gamma} \deg_x(\pi) \cdot \left( \sum_{y \in \Gamma} D(y) \right) \cdot x 
= \sum_{x \in \Gamma} \deg_x(\pi) \cdot \sum_{y \in \Gamma} (|Kx| \cdot D(x)) \cdot x 
= \sum_{x \in \Gamma} |K| D(x) \cdot x = |K| D.
$$

Since $k\pi_*(D) + \text{div}(f_1')$ is effective,

$$
\pi^*(k\pi_*(D) + \text{div}(f_1')) = k\pi^*(\pi_*(D)) + \pi^*(\text{div}(f_1')) = k|K|D + \text{div}(\pi^*f_1')
$$

is also effective. This means $\pi^*f_1' \in R(k|K|D)$. As

$$
\pi^*f_1'(x) - \pi^*f_1'(y) = f_1'(\pi(x)) - f_1'(\pi(y)) = f_1'(x') - f_1'(y') 
\neq f_2'(x') - f_2'(y') = f_2'(\pi(x)) - f_2'(\pi(y)) 
= \pi^*f_2'(x) - \pi^*f_2'(y),
$$

$k|K|D$ is $K$-very ample. 

Therefore, the answer is “always”.

In conclusion, we have the following theorem.
Theorem 3.3.21. Let $\Gamma$ be a metric graph and $K$ a finite group acting on $\Gamma$. Then, there exists a rational map, from $\Gamma$ to a tropical projective space, which induces a $K$-Galois covering on the image with edge-multiplicities.

Especially when the group $K$ is trivial, we have the following corollary.

Corollary 3.3.22. A metric graph is embedded in a tropical projective space by a rational map.

Proposition 3.3.23. If $\phi_F$ induces a $K$-Galois covering $\phi$, then $\phi^*(\text{Rat}(\text{Im}(\phi_F))) = \text{Rat}(\Gamma)^K$ holds.

Proof. For any $f' \in \text{Rat}(\text{Im}(\phi_F))$, obviously $\phi^*(f') = f' \circ \phi \in \text{Rat}(\Gamma)^K$ holds.

Let $(G, l)$ (resp. $(G', l')$) be a model of $\Gamma$ (resp. $\text{Im}(\phi_F)$) corresponding to $\phi$. Let $f$ be an element of $\text{Rat}(\Gamma)^K$. Since $\varphi$ is $K$-injective, there exists a one-to-one mapping between $K$-orbits of $\Gamma$ and $\text{Im}(\phi_F)$. Let $g(x') := f(\varphi^{-1}(x'))$, $x' \in \text{Im}(\phi_F)$ and $g$ is well-defined. By the definition of $g$, for any $x \in \Gamma$, $\phi^*(g)(x) = g \circ \phi(x) = g(\phi(x)) = f(x)$ holds. Thus, $f = \phi^*(g) \in \phi^*(\text{Rat}(\text{Im}(\phi_F)))$. \qed

Remark 3.3.24. Let $\Gamma$ be a metric graph, $K$ a finite group acting on $\Gamma$ and $\varphi : \Gamma \to \Gamma' := \Gamma/K$ the natural surjection. Let $(G_1, l_1)$ (resp. $(G', l')$) be the model of $\Gamma$ (resp. $\Gamma'$) in Section 3. $\text{Rat}(\Gamma)_K$ denotes the set consisting of $K$-invariant rational functions $f$ on Gamma whose each slope on $e$ is a multiple of $|K_e|$, where $e$ is a connected component of $\Gamma \setminus (\text{supp}(\text{div}(f)) \cup V(G_1))$. Then, $\varphi^*(\text{Rat}(\Gamma')) = \text{Rat}(\Gamma)_K$.

Proof. Let $f' \in \text{Rat}(\Gamma')$. By the definition of pull-back of a function, $\varphi^*(f') = f' \circ \varphi \in \text{Rat}(\Gamma)^K$. Let $e$ be a connected component of $\Gamma \setminus (\text{supp}(\text{div}(\varphi^*(f'))) \cup V(G_1))$. By the construction of $\Gamma'$, $l'(\varphi(e)) = l'([e]) = |K_e|l(e)$. $\varphi^*(f')$ has the slope which is a multiple by $K_e$ of the one of $f'$ on $\varphi(e)$. Therefore, $\varphi^*(f')$ is in $\text{Rat}(\Gamma)_K$.

Let $f$ be an element of $\text{Rat}(\Gamma)_K$. Let $g$ be the rational function on $\Gamma'$ defined by the following (1) and (2).

(1) Fix a point $x_0$ on $\Gamma$. $g(\varphi(x_0)) := f(x_0)$.

(2) For a connected component $e$ of $\Gamma \setminus (\text{supp}(\text{div}(f)) \cup V(G_1))$, $g$ has the slope $\frac{\text{(the slope of $f$ on $e$)}}{|K_e|}$.

Then, $g$ is well-defined and $f = \varphi^*(g) \in \varphi^*(\text{Rat}(\Gamma'))$. In fact, the following fold. Let $x_1, x_2$ be any two point on $\Gamma$ and $P_1 = e_{11} \cdots e_{1n_1}$ and $P_2 = e_{21} \cdots e_{2n_2}$ any two paths from $x_1$ to $x_2$. Let $s_{ij}$ be the slope of $f$ on $e_{ij}$. As

$$f(x_2) = f(x_1) + \sum_{j=1}^{n_1} l_1(e_{1j}) s_{1j} = f(x_1) + \sum_{j=1}^{n_2} l_1(e_{2j}) s_{2j}$$

holds, then we have

$$l_1(e_{1j}) s_{1j} = \sum_{j=1}^{n_2} l_1(e_{2j}) s_{2j}.$$
Therefore,
\[
\sum_{j=1}^{n_1} \frac{|K_{e_{1j}}| \cdot l_1(e_{1j}) \cdot s_{1j}}{|K_{e_{1j}}|} = \sum_{j=1}^{n_1} \frac{|K_{e_{2j}}| \cdot l_1(e_{1j}) \cdot s_{2j}}{|K_{e_{2j}}|}
\]
and then \( g \) is well-defined. Let \( x_1 \) be \( x_0 \). For any \( x_2 \),
\[
f(x_2) = f(x_0) + \sum_{j=1}^{n_1} l_1(e_{1j}) s_{1j} = g(\varphi(x_0)) + \sum_{j=1}^{n_1} \frac{|K_{e_{1j}}| \cdot l_1(e_{1j}) \cdot s_{1j}}{|K_{e_{1j}}|} = g(\varphi(x_2)).
\]
Then, \( f = \varphi^*(g) \).

\[\square\]

### 3.4 Applications

In [8], Haase, Musiker and Yu give a problem “give a characterization of metric graphs whose canonical divisors are not very ample” (see [8] Problem 51). In this subsection, we give an answer to this problem and at the same time, we consider an analogy of the fact the canonical map of a hyperelliptic compact Riemann surface is a double covering.

Let \( \Gamma \) be a metric graph and \( D \) a divisor on \( \Gamma \). \( \phi_{|D|} \) denotes the rational map induced by \( |D| \), i.e. for a minimal generating set \( \{f_1, \ldots, f_n\} \) of \( R(D) \), \( \phi_{|D|} := (f_1 : \cdots : f_n) : \Gamma \to TP^{n-1}, x \mapsto (f_1(x) : \cdots : f_n(x)) \).

**Remark 3.4.1** ([8] Proposition 48]). If \( \deg(D) = 2 \), then \( \phi_{|D|}(\Gamma) \) is a tree. If in addition \( r(D) = 1 \), then the fibre \( \phi_{|D|}^{-1}(x) = \{ y \in \Gamma \mid \phi_{|D|}(y) = x \} \) has size one or two for all \( x \) in the image.

By Remark 3.4.1 we have the following lemma.

**Lemma 3.4.2.** Let \( \Gamma \) be a hyperelliptic metric graph without one valent points and \( D \) a divisor on \( \Gamma \) whose degree is two and whose rank is one. Then, the complete linear system \( |D| \) is invariant by the hyperelliptic involution \( \iota \) and the rational map associated to \( |D| \) induces a \( \langle \iota \rangle \)-Galois covering on a tree.

**Proof.** Obviously \( |D| \) is invariant by \( \langle \iota \rangle \). By Remark 3.4.1 \( \text{Im}(\phi_{|D|}) \) is a tree. By the proof of Remark 3.4.1, for any point \( x \) on a bridge of \( \Gamma \), \(|\phi_{|D|}^{-1}(\phi_{|D|}(x))| = 1 \) and any point \( y \) not on a bridge but on a cycle of \( \Gamma \), \(|\phi_{|D|}^{-1}(\phi_{|D|}(y))| = 2 \) and \( \phi_{|D|}^{-1}(\phi_{|D|}(y)) = \{ y, \iota(y) \} \). Therefore \( \phi_{|D|} \) is \( \langle \iota \rangle \)-injective. Thus \( \phi_{|D|} \) induces a \( \langle \iota \rangle \)-Galois covering. \( \square \)

The *canonical map* is the rational map induced by the canonical linear system \( |K_{\Gamma}| \) on a metric graph \( \Gamma \).

**Theorem 3.4.3.** Let \( \Gamma \) be a metric graph without one valent points and \( \phi_{|K_{\Gamma}|} \) the canonical map of \( \Gamma \). Then \( \phi_{|K_{\Gamma}|} \) induces a \( \mathbb{Z}/2\mathbb{Z} \)-Galois covering on the image of \( \phi_{|K_{\Gamma}|} \) if and only if the genus \( g \) of \( \Gamma \) is two.
Proof. Since $\text{deg}(K_{\Gamma}) = 2g - 2$ and $r(K_{\Gamma}) = g - 1$ by Riemann–Roch theorem, when $g = 0$, $\phi|_{K_{\Gamma}}$ is not induced and when $g = 1$, $\phi|_{K_{\Gamma}}$ is a constant map. When $g = 2$, $K_{\Gamma}$ has degree two and rank one and then by Lemma 3.4.2, $\phi|_{K_{\Gamma}}$ is a $\mathbb{Z}/2\mathbb{Z}$-Galois covering on a tree. When $g \geq 3$, for $K_{\Gamma}$ is not very ample, $\Gamma$ must be one of the following two type of hyperelliptic metric graphs by [8, Theorem 49].

(type 1) $\Gamma$ is a metric graph consisting two vertices $x, y$ and $g + 1$ multiple edges between them. See Figure 1.

The rational functions $f_{i1}, f_{i2}$ and $f_{i3}$ in Figure 2 are extremals of $R(K_{\Gamma})$ and the rational map $e_i^2 \to TP^2, t \mapsto (f_{i1}(t) : f_{i2}(t) : f_{i3}(t))$ is injective.

On the other hand, obviously all extremals of $R(K_{\Gamma})$ attain maximal only at $x$ and $y$ by Lemma 3.1.4 (in this case, $K$ is trivial). Hence $\phi|_{K_{\Gamma}\setminus \{x,y\}}$ is injective and $\phi|_{K_{\Gamma}}(x) = \phi|_{K_{\Gamma}}(y)$. Thus $\phi|_{K_{\Gamma}}$ is not a $\mathbb{Z}/2\mathbb{Z}$-Galois covering.

(type 2) $\Gamma$ is a metric graph of the form in Figure 3, $e_{g+2}$ and $e_{g+3}$ have a same length.

Since $K_{\Gamma}$ is linearly equivalent to $D := (g - 1)(x + y)$, $\phi|_{K_{\Gamma}} = \phi|_{D}$ holds. Similarly to the proof of type 1, we have three extremals $f_{i1}, f_{i2}$ and $f_{i3}$ of $R(D)$ which induce an injective rational map on $e_i^2, i = 1, \ldots, g$. The rational functions $h_1, \ldots, h_5$ and $h_6$ in Figure
Figure 3: type 2

are extremals of $R(D)$ and the map \((e_g \cup e_{g+1}) \setminus \{p, q\} \to TP^2, t \mapsto (h_1(t) : h_2(t) : h_3(t) : h_4(t) : h_5(t) : h_6(t))\) is injective. The rational functions $h_7$ and $h_8$ in Figure 5 are extremals of $R(D)$ and the map \((e_{g+2} \cup e_{g+3}) \setminus \{x, y\} \to TP^2, t \mapsto (f_{11}(t) : h_7(t) : h_8(t))\) is injective. In particular, when $g = 3$, see Figure 6. Hence $\phi|_{\Gamma\setminus\{x, y\}}$ is injective. On the other hand, by the same reason, $\phi|_{D|}\{x\} = \phi|_{D|}\{y\}$. In conclusion, $\phi|_{D|}$ is not a $\mathbb{Z}/2\mathbb{Z}$-Galois covering.

Corollary 3.4.4. Let $\Gamma$ be a metric graph of genus $\geq 3$ without one valent points. $K_\Gamma$ is not very ample if and only if the canonical map is not harmonic. In particular, $\Gamma$ is hyperelliptic and $g(\text{Im}(\phi|_{K_\Gamma})) = g(\Gamma) + 1$.

Proof. In the proof of Theorem 3.4.3, we can directly check that the degrees of $\phi|_{K_\Gamma}$ (resp. $\phi|_{D|}$) at $x$ and $y$ are different from the degree of $\phi|_{K_\Gamma}$ (resp. $\phi|_{D|}$) at any other point. 

Theorem 3.4.3 and Corollary 3.4.4 mean that an analogy of the fact the canonical map of a hyperelliptic compact Riemann surface is a double covering on a projective line $P^1(\mathbb{C})$ with non-zero degree does not hold for a metric graph and in stead of this, we have the following by Lemma 3.4.2.

Proposition 3.4.5. Let $\Gamma$ be a hyperelliptic metric graph with genus at least two without one valent points. Then, an invariant linear subsystem of the hyperelliptic involution $\iota$ of the canonical linear system induces a rational map whose image is a tree and which is a $\langle \iota \rangle$-Galois covering on the image.

Proof. As $\iota$ is an isometry, $K_\Gamma$ is of the form $D + E$, where both $D$ and $E$ are effective divisors on $\Gamma$ and $\deg(D) = 2$, $r(D) = 1$. Since $K_\Gamma$ and $D$ are invariant by $\iota$, so is $E$. Thus the canonical linear system $|K_\Gamma| = |D + E|$ contains the invariant linear subsystem $\Lambda$ of the hyperelliptic involution whose elements are of the form $D_1 + E$, where $D_1$ is effective and
Figure 4: $w_1$ (resp. $w_2$) is the midpoint of $e_g$ (resp. $e_{g+1}$). $w_3$ and $w_5$ (resp. $w_4$ and $w_6$) are the internally dividing points obtained by internally dividing $e_g$ (resp. $e_{g+1}$) by $(g-2) : (g-1)$, where $w_3$ (resp. $w_4$) is further than $w_5$ (resp. $w_6$) from $p$ (resp. $q$). $h_1, h_2, h_3, h_4, h_5$ and $h_6$ define principal divisors such that $D + \text{div}(h_1) = (2g-2)w_1$, $D + \text{div}(h_2) = (2g-2)w_2$, $D + \text{div}(h_3) = p + (2g-3)w_3$, $D + \text{div}(h_4) = p + (2g-3)w_4$, $D + \text{div}(h_5) = q + (2g-3)w_5$ and $D + \text{div}(h_6) = q + (2g-3)w_6$, respectively.
Figure 5: (case $g \geq 4$) $z_{11}$, $w_7$ and $w_8$ are the midpoints of $e_1$, $e_{g+2}$ and $e_{g+3}$, respectively. $h_7$ and $h_8$ define principal divisors such that $D + \text{div}(h_7) = (2g - 6)z_{11} + p + 2w_7$ and $D + \text{div}(h_8) = (2g - 6)z_{11} + q + 2w_8$, respectively.

Figure 6: (case $g = 3$) $w_7$ and $w_8$ are the midpoints of $e_{g+2}$ and $e_{g+3}$, respectively. $h_7$ and $h_8$ define principal divisors such that $D + \text{div}(h_7) = p + 2w_7$ and $D + \text{div}(h_8) = q + 2w_8$, respectively.
linearly equivalent to $D$. Let $R$ be the subsemimodule of $R(K_G) = R(D + E)$ corresponding to $\Lambda$. Then $R = R(D)$. In fact, for any $f \in R$, there exists $D_1 + E \in \Lambda$ such that $(D+E) + \text{div}(f) = D_1 + E \geq 0$ and $D_1$ is effective. Thus $D + \text{div}(f) = D_1 \geq 0$, i.e. $f \in R(D)$. Conversely, for any $g \in R(D)$, there exists $D_1 \in \Lambda'$ such that $D + \text{div}(g) = D_1 \geq 0$, where $\Lambda'$ is the linear system corresponding to $R(D)$. Hence $D + E + \text{div}(g) = D_1 + E \geq 0$ and then $g \in R$. Therefore, by Lemma 3.4.2 $\Lambda$ induces a rational map which is a $(\iota)$-Galois covering on a tree. \hfill\Box

Moreover, we have the following lemma.

**Lemma 3.4.6.** Let $\Gamma$ be a metric graph, $K$ a finite group and $D$ a divisor on $\Gamma$. For finitely generated $K$-invariant linear subsystems $\Lambda_1 \subset \Lambda_2 \subset |D|$, let $\phi_{\Lambda_1} = (f_1 : \cdots : f_n)$ (resp. $\phi_{\Lambda_2} = (g_1 : \cdots : g_m)$) be the rational map induced by $\Lambda_1$ (resp. $\Lambda_2$). If $\phi_{\Lambda_1}$ induces a $K$-Galois covering, then $\phi_{\Lambda_2}$ induces a $K$-Galois covering.

**Proof.** Let $f_i = "\sum_{j=1}^m a_{ij}g_j"$. Since $m$ is finite and each $g_i$ is a rational function on a metric graph, we can choose a model $(G, l)$ of $\Gamma$ satisfying the following condition: for any $i$ and edge $e$ of $G$, there exists a number $i_e$ such that $f_i|e = a_{ii_e} + g_{i_e}$. Let $f'_i := f_i - a_{ii_e}$. Then, $F' := \{f'_1, \ldots, f'_n\}$ is a minimal generating set of $R_1$, where $R_1$ is the tropical subsemimodule of $R(D)$ corresponding to $\Lambda_1$. As $\phi_{\Lambda_1}$ is $K$-Galois, by Theorem 3.3.14 it is $K$-injective and then $\phi_{F'}$ is also $K$-injective by Remark 3.3.6. By the definition of $F'$, $\phi_{\Lambda_2}|e = \phi_{F'}|e$ holds. Hence $\phi_{\Lambda_2}$ is $K$-injective on $e$. Since $e$ is arbitrary, $\phi_{\Lambda_2}$ is $K$-injective. In conclusion, by Theorem 3.3.14 again, $\phi_{\Lambda_2}$ induces a $K$-injective. \hfill\Box

Consequently, by Proposition 3.4.3 and Lemma 3.4.6 the following holds.

**Theorem 3.4.7.** For a hyperelliptic metric graph with genus at least two without one valent points, the invariant linear system of the hyperelliptic involution $\iota$ of the canonical linear system induces a rational map whose image is a tree and which is a $(\iota)$-Galois covering on the image.

### 4 Metric graphs with edge-multiplicities

In Section 3, we prove that the induced rational map by $|D|^K$ which is $K$-injective is a finite harmonic morphism (and then a $K$-Galois covering) of metric graphs with an edge-multiplicity. We define in this section metric graphs with edge-multiplicities and harmonic morphisms between them. Compare Subsections 2.2, 2.3 and 2.4. Note that all of them are original definitions of the author and we may need more improvements.

#### 4.1 Metric graphs with edge-multiplicities

**Definition 4.1.1.** Let $\Gamma$ be a metric graph, and $(G, l)$ a model of $\Gamma$. We call a function $m : E(G) \to \mathbb{Z}_{>0}$ an *edge-multiplicity function* on $G$. $1$ is the edge-multiplicity function assigning multiplicity one to all edges and called a *trivial* edge-multiplicity function. Two
triplets \((G, l, m)\) and \((G', l', m')\) are said to be isomorphic if there exists an isomorphism between \(G\) and \(G'\) keeping the length and the multiplicity of each edge. We define \(\text{Isom}(G,l)(\Gamma)\) as the subset of the isometry transformation group \(\text{Isom}(\Gamma)\) of \(\Gamma\) whose element keeps the length of each edge of \(G\). We set \(\text{Isom}(G,l,m)(\Gamma)\) as the subset of \(\text{Isom}(G,l)(\Gamma)\) whose each element keeps the multiplicity of each edge of \(G\).

**Definition 4.1.2** (Subdivision of models). Let \(\Gamma\) be a metric graph, and \((G, l)\), \((G', l')\) models of \(\Gamma\). \((G, l)\) is said to be a subdivision of \((G', l')\) and written as \((G, l) \succ (G', l')\) if \(V(G')\) is a subset of \(V(G)\).

**Definition 4.1.3.** Let \(\Gamma\) be a metric graph, and \((G, l) \succ (G', l')\) models of \(\Gamma\). A triplet \((G, l, m)\) is said to be a subdivision of a triplet \((G', l', m')\) and written as \((G, l, m) \succ (G', l', m')\) if for any \(e' \in E(G')\) and \(e_i \in E(G)\) such that \(e' = e_1 \sqcup \cdots \sqcup e_n\), \(m'(e')\) divides all \(m(e_i)\). In particular, if \(m'(e')\) and all \(m(e_i)\) equals, then \((G', l', m')\) is said to be a trivial subdivision of \((G, l, m)\) and then \((G', l', m')\) is denoted by \((G', l', m)\).

**Definition 4.1.4.** For a quadruplet \((\Gamma, G, l, m)\), the metric graph with an edge-multiplicity, denoted by \(\Gamma_m\), is defined by the pair of metric graph \(\Gamma\) and \(m\) such that we can choose only models \((G', l') \prec (G, l)\) of \(\Gamma\). The word “a point \(x\) on \(\Gamma_m\)” means that \(x \in \Gamma\). The genus of \(\Gamma_m\) is the genus of \(\Gamma\).

**Definition 4.1.5.** Let \(\Gamma_m\) be a metric graph with an edge-multiplicity. \(\text{Div}(\Gamma_m)\) is defined by \(\text{Div}(\Gamma)\) and an element of \(\text{Div}(\Gamma_m)\) is called a divisor on \(\Gamma_m\). The canonical divisor on \(\Gamma_m\) is the canonical divisor on \(\Gamma\). We define \(\text{Rat}(\Gamma_m)\) as \(\text{Rat}(\Gamma)\). We call an element of \(\text{Rat}(\Gamma_m)\) a rational function on \(\Gamma_m\).

Note that for an edge \(e\) of \(G\) and \(f \in \text{Rat}(\Gamma_m)\), \(f\) has different finite slopes on \(e\) since \(f\) may have plural pieces.

For a metric graph with an edge-multiplicity, we use same terms and notations for the underlying metric graph.

### 4.2 Harmonic morphisms with edge-multiplicities

**Definition 4.2.1.** Let \(\Gamma_m, \Gamma_m'\) be metric graphs with edge-multiplicities \(m, m'\), respectively, and \(\varphi_{m,m'} : \Gamma_m \to \Gamma_m'\) be a continuous map. The map \(\varphi_{m,m'}\) is called a morphism if \(\varphi_{m,m'}\) is a morphism as loopless models \((G, l)\) and \((G', l')\). For an edge \(e\) of \(G\), if \(\varphi_{m,m'}(e)\) is a vertex of \(G'\), let \(m'(\varphi_{m,m'}(e)) := 0\) formally. The morphism \(\varphi_{m,m'}\) is said to be finite if \(\varphi_{m,m'}\) is finite as a morphism of loopless models.

**Definition 4.2.2.** Let \(\varphi_{m,m'} : \Gamma_m \to \Gamma_m'\) be a morphism of metric graphs with edge-multiplicities.

Let \(\Gamma_m'\) be not a singleton and \(x\) a point on \(\Gamma_m\). The morphism \(\varphi_{m,m'}\) is harmonic at \(x\) if for any edge \(e_1\) of \(G\) adjacent to \(x\), \(m(e_1)\) divides \(m'(\varphi_{m,m'}(e_1))\) and the number

\[
\deg_x^{m,m'}(\varphi_{m,m'}) := \sum_{x \in h \to h'} \frac{m'(\varphi_{m,m'}(e_1))}{m(e)} \cdot \deg_h(\varphi_{m,m'})
\]

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is independent of the choice of half-edge $h'$ emanating from $\varphi_{m,m'}(x)$, where $h$ is a connected component of the inverse image of $h'$ by $\varphi_{m,m'}$ containing $x$ and $e$ is the edge of $G$ containing $h$. The morphism $\varphi_{m,m'}$ is harmonic if it is harmonic at all points on $\Gamma_m$. For a point $x'$ on $\Gamma'_m$, 

$$\deg_{m,m'}(\varphi_{m,m'}) := \sum_{x \to x'} \deg_{x}(\varphi_{m,m'})$$

is said the degree of $\varphi_{m,m'}$, where $x$ is an element of the inverse image of $x'$ by $\varphi_{m,m'}$. If $\Gamma'_m$ is a singleton and $\Gamma_m$ is not a singleton, for any point $x$ on $\Gamma_m$, we define $\deg_{x}(\varphi_{m,m'})$ as zero so that we regard $\varphi_{m,m'}$ as a harmonic morphism of degree zero. If both $\Gamma_m$ and $\Gamma'_m$ are singletons, we regard $\varphi_{m,m'}$ as a harmonic morphism which can have any number of degree.

**Lemma 4.2.3.** $\sum_{x \to x'} \deg_{x}(\varphi_{m,m'})$ is independent of the choice of a point $x'$ on $\Gamma'$.

**Proof.** It is sufficient to check that for any vertex of $G'$, the sum is same. Let $x'_1$ and $x'_2$ be vertices of $G'$ both adjacent to an edge $e'$ of $G'$. Let $h'_1$ be the half-edge of $x'_1$ contained in $e'$. Then

$$\sum_{x_1 \to x'_1} \deg_{x_1}(\varphi_{m,m'}) = \sum_{x_1 \to x'_1} \left( \sum_{x_1 \to h'_1} \deg_{h'_1}(\varphi_{m,m'}) \right)$$

$$= \sum_{x_1 \to x'_1} \left( \sum_{x_1 \to e'_{_1}} \deg_{e'_{_1}}(\varphi_{m,m'}) \right)$$

$$= \sum_{e \to e'} \deg_{e}(\varphi_{m,m'}).$$

Similarly,

$$\sum_{x_2 \to x'_2} \deg_{x_2}(\varphi_{m,m'}) = \sum_{e \to e'} \deg_{e}(\varphi_{m,m'}).$$

The collection of metric graphs with edge-multiplicities together with harmonic morphisms between them forms a category.

**Definition 4.2.4.** Let $\varphi_{m,m'} : \Gamma_m \to \Gamma'_m$ be a finite harmonic morphism of metric graphs with edge-multiplicities. For $f$ in $\text{Rat}(\Gamma_m)$, the push-forward of $f$ is the function $(\varphi_{m,m'})_* f : \Gamma'_m \to \mathbb{R} \cup \{\pm \infty\}$ defined by

$$(\varphi_{m,m'})_* f(x') := \sum_{\varphi_{m,m'}(x) = x'} \deg_{x}(\varphi_{m,m'}) \cdot f(x).$$

The pull-back of $f'$ in $\text{Rat}(\Gamma'_m)$ is the function $(\varphi_{m,m'})^* f' : \Gamma_m \to \mathbb{R} \cup \{\pm \infty\}$ defined by $(\varphi_{m,m'})^* f' := f' \circ \varphi_{m,m'}$. We define the push-forward homomorphism on divisors $(\varphi_{m,m'})_* : \text{Div}(\Gamma_m) \to \text{Div}(\Gamma'_m)$ by homomorphism

$$(\varphi_{m,m'})_*(D) := \sum_{x \in \Gamma_m} D(x) \cdot \varphi_{m,m'}(x).$$

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The pull-back homomorphism on divisors \((\varphi_{m,m'})^* : \text{Div}(\Gamma_m') \to \text{Div}(\Gamma_m)\) is defined to be

\[
(\varphi_{m,m'})^*(D') := \sum_{x \in \Gamma_m} \deg^m_{x}(\varphi_{m,m'}) \cdot D'(\varphi_{m,m'}(x)) \cdot x.
\]

**Remark 4.2.5.** We need not to assume that \(\varphi_{m,m'}\) is finite to define pull-backs of rational functions and divisors.

**Proposition 4.2.6.** For any divisors \(D\) on \(\Gamma_m\) and \(D'\) on \(\Gamma_{m'}\), \(\deg((\varphi_{m,m'})^*(D)) = \deg(D)\) and \(\deg((\varphi_{m,m'})^*(D')) = \deg^m_{m'}(\varphi_{m,m'})\) hold.

**Proof.** The first equation holds obviously.

Let \(x'\) be a point on \(\Gamma_{m'}\). Since \(\sum_{x \to x'}((\varphi_{m,m'})^*(D'))(x) = \sum_{x \to x'}\deg^m_{x}(\varphi_{m,m'}) \cdot D'(x') = \deg^m_{m'}(\varphi_{m,m'}) \cdot D'(x')\), we have the second equation. \(\square\)

**Definition 4.2.7.** Let \(\varphi_{m,m'} : \Gamma_m \to \Gamma_{m'}\) be a finite harmonic morphism of metric graphs with edge-multiplicities. For a rational function \(f\) on \(\Gamma_m\) other than \(-\infty\), we define the number

\[
\text{div}^m_{m'}(f) := \sum_{x \in \Gamma_m} \left( \sum_{e \in E(G)} \frac{m'(\varphi_{m,m'}(e))}{m(e)} \cdot (\text{the outgoing slope of } f \text{ on } e \text{ at } x) \cdot x \right)
\]

and call it the principal divisor with edge-multiplicities \(m\) and \(m'\) defined by \(f\).

**Proposition 4.2.8.** For any rational functions \(f\) on \(\Gamma_m\) and \(f'\) on \(\Gamma_{m'}\) both other than \(-\infty\), \((\varphi_{m,m'})^*(\text{div}^m_{m'}f) = \text{div}((\varphi_{m,m'})^*(f))\) and \((\varphi_{m,m'})(\text{div}(f')) = \text{div}^{m,m'}((\varphi_{m,m'})^*f')\) hold.

**Proof.** Let us write \(\varphi_{m,m'}\) as \(\varphi\) simply. We may break \(\Gamma_m\) and \(\Gamma_{m'}\) into sets \(S\) and \(S'\) of segments along which \(f\) and \(\varphi_* f\), respectively, are linear and such that each segment \(s \in S\) is mapped linearly to some \(s' \in S'\). Then at any point \(x'\) on \(\Gamma_{m'}\), we have

\[
\varphi_*(\text{div}^m_{m'}(f))(x') = \sum_{x \in \Gamma_m \atop x \to x'} \text{div}^m_{m'}(f)(x) = \sum_{x \in \varphi^{-1}(x')} \sum_{s = xy \in S} \frac{m'(\varphi(s))}{m(s)} \cdot \frac{f(y) - f(x)}{l(s)}
\]

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and

\[ \text{div}(\varphi_* f)(x') = \sum_{s'=x'y' \in S'} \left( \sum_{\varphi(y)=y'} \left( \frac{m'(s')}{m(s)} \cdot \frac{l'(s')}{l(s)} \right) f(y) - \sum_{x \in \varphi^{-1}(x')} \left( \frac{m'(s')}{m(s)} \cdot \frac{l'(s')}{l(s)} \right) f(x) \right) \cdot \frac{1}{l'(s')} \]

\[ = \sum_{s'=x'y' \in S'} \left( \sum_{\varphi(y)=y'} \left( \frac{m'(s')}{m(s)} \cdot \frac{1}{l(s)} \right) f(y) - \sum_{x \in \varphi^{-1}(x')} \left( \frac{m'(s')}{m(s)} \cdot \frac{1}{l(s)} \right) f(x) \right) \]

\[ = \sum_{s'=x'y' \in S'} \left( \sum_{s=xy \in S', \varphi(s)=s'} \left( \frac{m'(s')}{m(s)} \cdot \frac{f(y) - f(x)}{l(s)} \right) \right) \]

Let us assume that \( \Gamma_m \) and \( \Gamma_{m'} \) are broken into \( S_1 \) and \( S'_1 \) of segments along which \( \varphi^* f' \) and \( f' \), respectively, have the same conditions as that of \( S \) and \( S' \). Then for any point \( x \) on \( \Gamma_m \), we have

\[ (\varphi^*(\text{div}(f')))(x) = \deg_{x}^{m,m'}(\varphi_{m,m'}) \cdot (\text{div}(f')(\varphi(x))) \]

\[ = \deg_{x}^{m,m'}(\varphi_{m,m'}) \cdot \left( \sum_{s'=\varphi(x)y' \in S'} \frac{f'(y') - f'(\varphi(x))}{l'(s')} \right) \]

\[ = \sum_{s'=\varphi(x)y' \in S'} \deg_{x}^{m,m'}(\varphi_{m,m'}) \cdot \frac{f'(y') - f'(\varphi(x))}{l'(s')} \]

\[ = \sum_{s'=\varphi(x)y' \in S'} \frac{m'(s')}{m(s)} \cdot \frac{l'(s')}{l(s)} \cdot \frac{f'(y') - f'(\varphi(x))}{l'(s')} \]

\[ = \sum_{s'=\varphi(x)y' \in S'} \frac{m'(s')}{m(s)} \cdot \frac{f'(\varphi(y)) - f'(\varphi(x))}{l(s)} \]

\[ = \sum_{s=xy \in S} \frac{m'(s')}{m(s)} \cdot \frac{(\varphi^* f')(y) - (\varphi^* f')(x)}{l(s)} \]

\[ = (\text{div}_{x}^{m,m'}(\varphi^*(f')))(x). \]
**Definition 4.2.9.** Let \( \varphi_{m,m'} : \Gamma_m \to \Gamma_{m'} \) be a map between metric graphs with edge-multiplicities \( m \) and \( m' \) and let \( K \) be a finite group. \( \varphi_{m,m'} \) is a \( K \)-Galois covering on \( \Gamma_{m'} \) if \( \varphi_{m,m'} \) is a finite harmonic morphism of metric graphs with edge-multiplicities, \( |K| = \deg_{m,m'}(\varphi_{m,m'}) \) and \( K \) acts transitively on every fibre and \( K \) keeps edge-multiplicities.

**Remark 4.2.10.** If \( \varphi_{m,m'} : \Gamma_m \to \Gamma_{m'} \) is \( K \)-Galois, then there exists a group homomorphism \( K \to \text{Isom}_{(G,l,m)} \) for a model \( (G,l) \) of \( \Gamma_m \).

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