Globally positive solutions of linear parabolic PDEs of second order with Robin boundary conditions

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Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with boundary \( \partial \Omega \) of class \( C^2 \).

We consider a linear parabolic differential equation of second order

\[
(1a) \quad u_t = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + a_0(t,x)u, \quad t \in \mathbb{R}, x \in \Omega,
\]

where \( a_{ij} = a_{ji} \in C^1(\overline{\Omega}), \) \( \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0 \) for each \( x \in \overline{\Omega} \) and each nonzero \((\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\), and \( a_0 \in L^\infty(\mathbb{R} \times \Omega) \).

Equation (1a) is complemented with regular oblique (Robin) homogeneous boundary conditions

\[
(1b) \quad \frac{\partial u}{\partial \nu}(t,x) + c(x)u(t,x) = 0, \quad t \in \mathbb{R}, x \in \partial \Omega,
\]

where \( \nu \in C^1(\partial \Omega, \mathbb{R}^n) \) is a vector field pointing out of \( \Omega \) and \( c \in C^1(\partial \Omega) \) is a nonnegative function.

Motivated by deep results on one-dimensional parabolic PDE’s contained in a series of papers [4], [5] by S.-N. Chow, K. Lu and J. Mallet-Paret, I investigate the set of solutions to (1a)+(1b) that are globally positive.

Our main result is the following (see Corollary 2.4).

**Theorem.** Let \( u_1, u_2 \) be solutions of (1a)+(1b) defined and positive for all \( t \in \mathbb{R} \) and all \( x \in \Omega \). Then there is a positive constant \( \kappa \) such that \( u_1(t,x) = \kappa u_2(t,x) \) for all \( t \in \mathbb{R} \) and all \( x \in \Omega \).

A key idea in the proofs is to consider a family of linear parabolic PDEs obtained by letting the coefficient \( a_0(t,x) \) vary in some metrizable compact subset of the vector space \( L^\infty(\mathbb{R} \times \Omega) \) endowed with the weak* topology. That family generates in a natural way a compact linear skew-product semiflow on a (product) vector bundle with generic fiber \( L^1(\Omega) \).

For \( n \) arbitrary, the results obtained seem to be new even in the case of autonomous equations (compare the remarks on p. 288 in [5]).
I would like to mention that it was John Mallet-Paret who asked me about an “easy” way to establish uniqueness (up to multiplication) of globally positive solutions. I am grateful for his hospitality during my short stay at the Lefschetz Center for Dynamical Systems, Brown University.

0. Preliminaries

Throughout the paper, for a Banach space $X$ the symbol $\mathcal{L}(X)$ stands for the Banach space of bounded linear maps from $X$ into itself, endowed with the uniform operator topology.

Let $\mathcal{A}$ be the differential operator

\begin{equation}
\mathcal{A}u(x) := - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right)
\end{equation}

Denote by $\tilde{\mathcal{A}}_0$ the linear operator $\tilde{\mathcal{A}}_0 u := A\mathcal{A}$ acting on the Banach space of functions from $C^2(\overline{\Omega})$ satisfying the boundary condition (1b). For each $1 \leq p \leq \infty$ the operator $\tilde{\mathcal{A}}_0$ is closable and densely defined in $L^p(\Omega)$. Let $\tilde{\mathcal{A}}_p$ denote the closure of $\tilde{\mathcal{A}}_0$ in $L^p(\Omega)$. We write $\| \cdot \|_p$ for the norm on the Banach space $L^p(\Omega), 1 \leq p \leq \infty$.

It is well known (see e. g. Pazy [10]) that for each $1 \leq p < \infty$ the (unbounded) linear operator $\tilde{\mathcal{A}}_p$ is the infinitesimal generator of an analytic semigroup $\{e^{-\tilde{\mathcal{A}}_p t}\}_{t \geq 0}$ of compact linear operators from $\mathcal{L}(L^p(\Omega))$. Put $A_p := \tilde{\mathcal{A}}_p + \text{Id}$.

Lemma 0.1. For each $1 \leq p \leq \infty$ the spectrum $\sigma(A_p)$ of $A_p$ lies in the halfspace \{\(\zeta \in \mathbb{C} : \text{Re} \zeta \geq 1\}\}.

Proof. As $-A_p$ is the infinitesimal generator of an analytic semigroup $e^{-A_p t}$ of compact linear operators, for each $t > 0$ the spectrum $\sigma(e^{-A_p t})$ consists of eigenvalues and \{0\}. By the standard regularity theory the solutions of the abstract parabolic equation $u_t + A_p u = 0$ are classical ones, hence we can consider $e^{-A_p^{-1}} \in \mathcal{L}(C(\overline{\Omega}))$. The Krein–Rutman theorem (see e. g. Hess [8]) implies that the spectral radius of $e^{-A_p^{-1}}$ is a positive eigenvalue corresponding to a positive eigenfunction $v_0$. Multiplying $v_0$ by a positive constant, if necessary, we can assume that $v_0(x) \leq 1$ for each $x \in \overline{\Omega}$.

For $x \in \overline{\Omega}, t > 0$, put $\tilde{v}(t, x) := e^{-t}$. The function $\tilde{v}$ is a supersolution (for the definition see e. g. Hess [8] or Smith [14]) for the initial boundary value problem

\begin{align*}
\frac{d u}{d t} + Au & = -u, \\
\frac{\partial u}{\partial \nu} + c(x)u & = 0, \\
u(0, x) & = 1 \text{ for } x \in \overline{\Omega}.
\end{align*}

An application of the parabolic maximum principle to $\tilde{v}$ and to the solution $v(t, x)$ to the initial boundary value problem

\begin{align*}
\frac{d v}{d t} + Av & = -v, \\
\frac{\partial v}{\partial \nu} + c(x)v & = 0, \\
v(0, x) & = v_0(x) \text{ for } x \in \overline{\Omega}.
\end{align*}

yields that \( v(t,x) \) converges to zero as \( t \to \infty \), uniformly in \( x \in \Omega \), with the exponential rate of decay not larger than \(-1\). So the spectral radius of \( e^{-A_{p}^{-1}} \in \mathcal{L}(C(\Omega)) \) (hence of \( e^{-A_{p}^{-1}} \in \mathcal{L}(L^p(\Omega)) \)) is not larger than \( e^{-1} \). As the spectral mapping theorem (see e. g. [10]) asserts that \( \sigma(e^{-A_{p}^{-1}}) = \exp(\sigma(-A_{p})) \cup \{0\} \), the desired result follows. □

For \( \alpha \geq 0 \) denote by \( A_{p}^{\alpha} \) the fractional power of \( A_{p} \), and by \( L^p(\Omega)^{\alpha} \) its domain endowed with the graph norm \( \| \cdot \|_{p,\alpha} := \|A_{p}^{\alpha} \cdot \|_p \). Obviously, \( A_{p}^{1} = A_{p} \) and \( L^p(\Omega)^{1} \) is the domain of \( A_{p} \).

**Theorem 0.2.** For each \( 1 \leq p < \infty \) the following holds:

(a) there is a positive constant \( C = C(p) \) such that

\[
\|e^{-A_{p}t}u\|_p \leq C\|u\|_p \quad \text{for all } u \in L^p(\Omega) \text{ and } t \geq 0,
\]

(b) for each \( u \in L^p(\Omega) \), \( t > 0 \) and \( \alpha \geq 0 \), \( e^{-A_{p}t}u \in L^p(\Omega)^{\alpha} \),

(c) for each \( 0 \leq \alpha \leq 1 \) there is a positive constant \( C_1 = C_1(p,\alpha) \) such that

\[
\|e^{-A_{p}t}u\|_{p,\alpha} \leq \frac{C_1}{t^\alpha}\|u\|_p \quad \text{for all } u \in L^p(\Omega) \text{ and } t > 0,
\]

(d) for each \( 0 < \alpha \leq 1 \) there is a positive constant \( C_2 = C_2(p,\alpha) \) such that

\[
\|(\text{Id} - e^{-A_{p}t})u\|_p \leq C_2t^\alpha\|u\|_{p,\alpha} \quad \text{for all } u \in L^p(\Omega)^{\alpha} \text{ and } t > 0.
\]

**Proof.** All the facts stated above are standard (see e. g. Henry [7] or Pazy [10]) □

For \( t \in \mathbb{R} \) and \( b \in L^\infty(\mathbb{R} \times \Omega) \), denote by \( b \cdot t \) the translate of \( b \), \((b \cdot t)(s,x) := b(t+s,x)\). Put

\[
\mathbb{B} := \text{w}^*\text{-cl}\{a_0 \cdot t : t \in \mathbb{R}\},
\]

where \( \text{w}^*\text{-cl} \) stands for the closure in the weak* topology of \( L^\infty(\mathbb{R} \times \Omega) \). Since \( \mathbb{B} \) is bounded, it is a compact metrizable space in the weak* topology of \( L^\infty(\mathbb{R} \times \Omega) \).

For \( b \in \mathbb{B} \) denote by \( M_p(b \cdot t) \) the multiplication operator by \( b(t, \cdot) \):

\[
(M_p(b \cdot t)\phi)(x) := b(t,x)\phi(x), \quad \phi \in L^p(\Omega), x \in \Omega.
\]

The following result is straightforward.

**Lemma 0.3.** For each \( b \in \mathbb{B} \), each \( 1 \leq p < \infty \) and each \( \phi \in L^p(\Omega) \) we have

\[
\|M_p(b \cdot t)\phi\|_p \leq R\|\phi\|_p, \quad \text{where } R \text{ is the essential supremum of } a_0.
\]
1. Linear skew-product (semi)dynamical systems

We consider the following abstract parabolic equation

\begin{equation}
(1.1) \quad u_t + A_p u = M_p(b \cdot t)u,
\end{equation}

together with an initial condition

\begin{equation}
(1.2) \quad u(0) = u_0.
\end{equation}

In the present section we investigate, for $1 \leq p < \infty$, the dependence of solutions to (1.1) satisfying (1.2) on the coefficient $b \in \mathbb{B}$ and the initial condition $u_0 \in L^p(\Omega)$.

The abstract initial value problem (1.1)+(1.2) can be written in the form of the following integral equation

\begin{equation}
(1.3) \quad u(t) = e^{-A_p t}u_0 + \int_0^t e^{-A_p (t-s)}M_p(b \cdot s)u(s) \, ds.
\end{equation}

We refer to solutions of (1.3), that is, functions $u \in C([0, \infty), L^p(\Omega))$ satisfying (1.3) for each $t \geq 0$ as mild solutions of (1.1)+(1.2). If a (mild) solution is defined for all $t \in \mathbb{R}$, we say it is a global solution.

**Proposition 1.1.** For each $b \in L^\infty(\mathbb{R} \times \Omega)$, each $1 \leq p < \infty$ and each $u_0 \in L^p(\Omega)$ there exists precisely one solution $u(\cdot; u_0, b) : [0, \infty) \to L^p(\Omega)$ to (1.3).

**Proof.** Applying the contraction mapping principle yields the desired solution. For a similar result, see Problem 7.1.5 in Henry’s book [7]. □

The proof of the following proposition rests on the estimates occurring in the proofs of Lemma 3.2 and Thm. 3.3 in [5], and we did not repeat it here.

**Proposition 1.2.** a) For $T > 0$ and $1 \leq p < \infty$ the mapping $L^p(\Omega) \times \mathbb{B} \ni (u_0, b) \mapsto u(\cdot; u_0, b) \in C([0, T], L^p(\Omega))$ is continuous.

b) For $0 < t_1 \leq t_2$, $1 \leq p < \infty$ and $0 \leq \alpha < 1$ the mapping $L^p(\Omega) \times \mathbb{B} \ni (u_0, b) \mapsto u(\cdot; u_0, b) \in C([t_1, t_2], (L^p(\Omega))^\alpha)$ is continuous.

**Theorem 1.3.** a) For $t > 0$, $u_0 \in L^1(\Omega)$ and $b \in \mathbb{B}$ we have $u(t; u_0, b) \in C^1(\overline{\Omega})$.

b) For $0 < t_1 \leq t_2$ the mapping $L^1(\Omega) \times \mathbb{B} \ni (u_0, b) \mapsto u(\cdot; u_0, b) \in C([t_1, t_2], C^1(\Omega))$ is continuous.

**Proof.** By Amann [2] or Pazy [10], the domain of $A_1$ embeds continuously in the Sobolev space $W^{1,1}(\Omega)$. A corollary of the Nirenberg–Gagliardo inequality (see e. g. [7]) states that

$$L^1(\Omega)^{1/2} \subset L^q(\Omega) \quad \text{if} \quad -n/q \leq 1/2 - n,$$

that is, $L^1(\Omega)^{1/2} \subset L^q(\Omega)$ for each $1 < q < 2n/(2n - 1)$. From Proposition 1.2 we deduce that $u(t; u_0, b) \in L^q(\Omega)$ for each $t > 0$, each $u_0 \in L^1(\Omega)$ and each $b \in \mathbb{B}$.

By [10], for $1 < p < \infty$ the domain of $A_p$ embeds continuously in the Sobolev space $W^{2,p}(\Omega)$. A corollary of the Nirenberg–Gagliardo inequality states that

$$L^p(\Omega)^{1/2} \subset L^q(\Omega) \quad \text{if} \quad -n/q \leq 1 - n/p.$$
Therefore, $L^p(\Omega)^{1/2} \subset L^q(\Omega)$ for each $p < q < np/(n - p)$. Consequently, $L^p(\Omega)^{1/2} \subset L^r(\Omega)$, where $r := n/(n - 1)$. Repeating the reasoning sufficiently many times, we prove that $u(t; u_0, b) \in L^p(\Omega)$ for each $t > 0$, each $u_0 \in L^1(\Omega)$ and each $b \in \mathbb{B}$ and $1 \leq p < \infty$.

For $p > 3n$ a corollary of the Nirenberg–Gagliardo inequality states that

$$L^p(\Omega)^{3/4} \subset C^\mu(\overline{\Omega}) \quad \text{if } 0 \leq \mu < 3/2 - 1/3 = 7/6.$$ 

It follows that $u(t; u_0, b) \in C^1(\overline{\Omega})$ for each $t > 0$, each $u_0 \in L^1(\Omega)$ and each $b \in \mathbb{B}$.

Part b) follows similarly by Proposition 1.2. \(\square\)

We recall now the definition of a skew-product dynamical system (with discrete time).

Let $B \times X$ be a product Banach bundle, where the base space $B$ is a compact metrizable space and the fiber $X$ is a Banach space. For a homeomorphism $\phi : B \to B$ the iterates $\phi^k$, $k \in \mathbb{Z}$, form a (discrete time) dynamical system on $B$. A compact linear skew-product dynamical system \(\{\psi^k\}_{k=1}^{\infty}\) on $B \times X$, covering $\phi$, is given by a family of compact linear operators \(\{\psi(b) : b \in B\}\) depending continuously on $b \in B$ in the uniform operator topology, in the following way

$$\psi(b, u) = (\phi(b), \psi(b)u), \quad b \in B, u \in X.$$ 

In other words, $\psi$ is a vector bundle endomorphism. The iterates $\psi^k$, $k \in \mathbb{N}$, are given by

$$\psi^k(b, u) = (\phi^k(b), \psi(k)(b)u),$$

where we denote

$$\psi(k)(b) := \psi(\phi^{k-1}(b)) \circ \psi(\phi^{k-2}(b)) \circ \cdots \circ \psi(\phi(b)) \circ \psi(b).$$

(the cocycle identity).

Let $X^*$ stand for the Banach space dual to $X$. For a compact linear skew-product dynamical system \(\{\psi^k\}_{k=1}^{\infty}\) on $B \times X$ we define its dual system \(\{\psi^k\}_{k=1}^{\infty}\) on $B \times X^*$

by

$$\psi^k(b, u^*) := (\phi^{-1}(b), \psi^k(b)u^*), \quad b \in B, u^* \in X^*,$$

where $\psi^*(b) \in L(X^*)$ is the dual operator to $\psi(b)$. It is straightforward that \(\{\psi^k\}_{k=1}^{\infty}\) is a compact linear skew-product dynamical system covering $\phi^{-1}$.

**Theorem 1.4.** Equation (1a)+(1b) generates a compact linear skew-product dynamical system \(\{\psi^k\}_{k=1}^{\infty}\) (resp. \(\{\tilde{\psi}^k\}_{k=1}^{\infty}\)) on the Banach bundle $\mathcal{B} := \mathbb{B} \times L^1(\Omega)$ (resp. $\tilde{\mathcal{B}} := \mathbb{B} \times C^1(\overline{\Omega})$), where $\phi(b) := b \cdot 1$ for $b \in \mathbb{B}$, and $\psi(b)u_0 := u(1; b, u_0)$ for $(b, u_0) \in \mathcal{B}$.

The system \(\{\psi^k\}_{k=1}^{\infty}\) factorizes through \(\{\tilde{\psi}^k\}_{k=1}^{\infty}\) in the sense that $\psi(b)u \in C^1(\Omega)$ for each $(b, u) \in \mathcal{B}$ and

$$\tilde{\psi}(b) = \psi(b) \circ i \quad \text{for each } b \in \mathbb{B},$$
where \( i \) stands for the natural embedding \( C^1(\overline{\Omega}) \subset L^1(\Omega) \).

Proof. The continuity of the assignment \( b \mapsto \psi(b) \) or \( b \mapsto \hat{\psi}(b) \) as a mapping from \( B \) into the space of bounded linear operators on \( L^1(\Omega) \) or \( C^1(\overline{\Omega}) \) with the strong operator topology follows by Theorem 1.3. The compactness follows by Proposition 1.2 and by the fact that for \( p \) sufficiently large the space \( L^p(\Omega)^{3/4} \) embeds continuously in \( C^p(\overline{\Omega}) \). We derive the continuity in the uniform operator topology along the lines of Thm. 2.3.2. in Pazy [10]. □

An immediate consequence is that for the dual skew-product dynamical system, \( \psi^*(b) : C^1(\overline{\Omega})^* \to L^\infty(\Omega) \) for each \( b \in B \). However, by the Green’s formula it follows that \( \psi^*(b)u^*, u^* \in C^1(\overline{\Omega})^* \), equals the value at time 1 of a solution of the adjoint equation. Since the adjoint equation satisfies all the assumptions, we have the following

**Theorem 1.5.** The linear skew-product dynamical system \( \{ (\Psi^*)^k \}_{k=1}^\infty \) dual to the system \( \{ \Psi^k \} \) generated by equation (1a)+(1b) has the property that \( \psi^*(b) : C^1(\overline{\Omega})^* \to C^1(\overline{\Omega}) \) for each \( b \in B \).

## 2. ORDER AND MONOTONICITY

For a Banach space \( X \) consisting of functions defined on \( \Omega \), by \( X_+ \) we denote the cone of nonnegative functions. It is straightforward that \( X_+ \) is a closed convex set such that

a) For each \( u \in X_+ \) and \( \alpha \geq 0 \) one has \( \alpha u \in X_+ \), and

b) \( X_+ \) contains no one-dimensional subspace.

The cones in \( L^p(\Omega), \ 1 \leq p \leq \infty \), as well as in \( C^1(\overline{\Omega}) \) are generating, that is, \( X_+ + X_+ = X \).

We write \( v \leq u \) if \( u - v \in X_+ \), and \( v < u \) if \( v \leq u \) and \( v \neq u \).

A cone \( X_+ \) is called solid if its interior \( X_{++} \) is nonempty. We write \( v \ll u \) if \( u - v \in X_{++} \). The standard cone in \( L^p(\Omega) \) is solid if and only if \( p = \infty \). The standard cone in \( C^1(\overline{\Omega}) \) is solid.

We say that a bounded linear functional \( u^* \in X^* \) is nonnegative if \( \langle u^*, u \rangle \geq 0 \) for each \( u \in X_+ \), where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing. When the cone \( X_+ \) is generating, the set of all nonnegative functionals forms a cone \( (X^*)_+ \), called the dual cone.

A functional \( u^* \in (X^*)_+ \) is called uniformly positive if there is a constant \( K > 0 \) such that \( \langle u^*, u \rangle \geq K\|u\| \) for each nonzero \( u \in X_+ \), where \( \| \cdot \| \) stands for the norm in \( X \). Among the spaces \( L^p(\Omega) \), only \( L^1(\Omega) \) admits uniformly positive functionals: They are represented by functions in \( L^\infty(\Omega) \) whose values are a.e. larger than some \( K > 0 \) (in other words, elements of \( L^\infty(\Omega)_{++} \)). For more on cones the reader is referred to Amann’s paper [1].

**Theorem 2.1.** Let \( \{ \Psi^k \}_{k=1}^\infty \) be the linear skew-product dynamical system generated on \( B \) by (1a)+(1b). There exists an invariant decomposition \( B = B_1 \oplus B_2 \), such that

(i) \( \text{codim} B_1 = 1 \), and for each \( b \in B \) the fiber \( B_1(b) \) is represented as the nullspace of a uniformly positive functional \( v^*(b) \in L^1(\Omega)^* \).

(ii) \( B_2 \setminus \mathcal{Z} \subset B \times (L^1(\Omega)_+ \cup -L^1(\Omega)_+) \), where \( \mathcal{Z} \) denotes the null section of \( B \).

(iii) The mapping \( \Psi|B_2 \) is a bundle automorphism.
(iv) There are constants \( D \geq 1 \) and \( 0 < \lambda < 1 \) such that
\[
\frac{\|\hat{\psi}^{(k)}(b)v_1\|_{L^1(\Omega)}}{\|\hat{\psi}^{(k)}(b)v_2\|_{L^1(\Omega)}} \leq D\lambda^k \frac{\|v_1\|_{L^1(\Omega)}}{\|v_2\|_{L^1(\Omega)}}
\]
for each \((b, v_1) \in B_1\), \((b, v_2) \in B_2 \setminus Z\), and each \( k \in \mathbb{N} \).

Proof. By Theorem 1.3, for each \( b \in \mathbb{B} \), each \( u_0 \in L^1(\Omega) \) and each \( t > 0 \) we have \( u(t; u_0, b) \in C^1(\overline{\Omega}) \). Assume that a nonzero \( u_0 \in L^1(\Omega)_+ \). As \( u(t; u_0, b) \) may fail to be a classical solution, we cannot apply the standard parabolic strong maximum principle as put forward e. g. in Protter and Weinberger [12]. However, since \( u \) as well as its spatial derivatives are continuous, we can establish the parabolic strong maximum principle by reasoning along the lines of Chapters 8 and 9 in Gilbarg and Trudinger [6], which enables us to show after the pattern of Thm. 4.1 in Hirsch [9] or Cor. 7.2.3 in Smith [14] that \( u(t; u_0, b) \gg 0 \) for \( t > 0 \).

Applying Thm. 1 in Poláčik and Tereščák [11] to the linear skew-product dynamical system \( \{\hat{\Psi}^k\}_{k=1}^\infty \), we obtain the existence of two \( \hat{\Psi} \)-invariant subbundles \( \mathcal{S} \) and \( \mathcal{T} \) of \( \hat{\mathcal{B}} \) such that:

(A1) \( \hat{\mathcal{B}} = \mathbb{B} \times C^1(\overline{\Omega}) \) is the direct sum of \( \mathcal{S} \) and \( \mathcal{T} \).

(A2) \( \dim \mathcal{S} = 1 \), and \( \mathcal{S} \setminus \hat{\mathcal{Z}} \subset \mathbb{B} \times (C^1(\overline{\Omega})_+ + \cup -C^1(\overline{\Omega})_+ +) \), where \( \hat{\mathcal{Z}} \) stands for the null section of \( \hat{\mathcal{B}} \).

(A3) For each \( b \in \mathbb{B} \) the fiber \( \mathcal{T}(b) \) can be (uniquely) represented by a normalized functional \( \hat{\psi}^*(b) \in C^1(\overline{\Omega})^*_+ \). In particular, \( \mathcal{T} \cap (\mathbb{B} \times C^1(\overline{\Omega})_+) = \hat{\mathcal{Z}} \).

(A4) \( \hat{\Psi} \mid \mathcal{S} \) is a bundle automorphism.

(A5) There are constants \( d \geq 1 \) and \( 0 < \lambda < 1 \) such that
\[
\frac{\|\hat{\psi}^{(k)}(b)v_1\|_{C^1(\overline{\Omega})}}{\|\hat{\psi}^{(k)}(b)v_2\|_{C^1(\overline{\Omega})}} \leq d\lambda^k \frac{\|v_1\|_{C^1(\overline{\Omega})}}{\|v_2\|_{C^1(\overline{\Omega})}}
\]
for each \((b, v_1) \in \mathcal{T}\), \((b, v_2) \in \mathcal{S}\), \( v_2 \neq 0 \) and each \( k \in \mathbb{N} \).

Now, the desired bundle \( B_2 \) equals simply \( \mathcal{S} \) considered a subbundle of the bundle \( \mathcal{B} = \mathbb{B} \times L^1(\Omega) \).

In order to construct \( B_1 \), notice first that the mapping \( \mathbb{B} \ni b \mapsto \hat{\psi}^*(b) \in C^1(\overline{\Omega})^* \) is continuous. Moreover, the \( \hat{\Psi} \)-invariance of \( \mathcal{T} \) means that \( \hat{\psi}^*(\hat{\phi}^{-1}(b))\hat{\psi}^*(\hat{\phi}^{-1}(b)) = \gamma \hat{\psi}^*(b) \) for some \( \gamma = \gamma(b) > 0 \). From Theorem 1.5 we derive that for each \( b \in \mathbb{B} \) the functional \( \hat{\psi}^*(b) \) is represented as a function from \( C^1(\overline{\Omega}) \) depending continuously on \( b \). Denote by \( v^*(b) \) the functional \( \hat{\psi}^*(b) \) viewed as an element of \( C^1(\overline{\Omega}) \). Applying the parabolic strong maximum principle to the adjoint equation we get that \( v^*(b) \in C^1(\overline{\Omega})_+ \), consequently it is a uniformly positive functional from \( L^1(\Omega)^* \).

To prove (iii) and (iv), use (A4), (A5) and the fact that since \( B_1 \) has finite dimension, both the \( C^1(\overline{\Omega}) \)-norm and the \( L^1(\Omega) \)-norm on it are equivalent.

The property described in (iv) is usually referred to as exponential separation (continuous separation in [11]). The direct sum decomposition \( \mathcal{B} = B_1 \oplus B_2 \) uniquely defines the bundle projection \( P \) with image \( B_2 \) and kernel \( B_1 \). The exponential separation can be formulated in the following way:
There are constants $D \geq 1$ and $0 < \lambda < 1$ such that

$$\frac{\|\text{Id} - P(\phi^k(b)))\psi^k(b)v\|_{L^1(\Omega)}}{\|P(\phi^k(b)))\psi^k(b)v\|_{L^1(\Omega)}} \leq D\lambda^k \frac{\|\text{Id} - P(b)\psi\|_{L^1(\Omega)}}{\|P(b)\|_{L^1(\Omega)}}$$

for each $(b, v) \in B \setminus B_1$ and each $k \in \mathbb{N}$.

For $t \in \mathbb{R}$ and $b \in B$ put $\phi_t b$ to be $b \cdot t$. The family $\{\phi_t\}_{t \in \mathbb{R}}$ forms a flow (= continuous-time dynamical system) on the compact metrizable space $B$. Define for $t \geq 0$ and $b \in B$ a linear operator $\psi(t, b) \in \mathcal{L}(L^1(\Omega))$ by the formula

$$\psi(t, b)u_0 := u(t; b, u_0) \quad \text{for } u_0 \in L^1(\Omega).$$

For each $t \geq 0$ the mapping $\Psi_t$ defined as

$$\Psi_t b, u := (\phi_t b, \psi(t, b)u)$$

is a bundle endomorphism of $B$. The family $\{\Psi_t\}_{t \geq 0}$ forms a linear skew-product semiflow on the product bundle $B$. A consequence of the semiflow axioms is the following cocycle identity

$$(2.1) \quad \psi(t_1 + t_2, b) = \psi(t_1, \phi_{t_1} b) = \psi(t_2, \phi_{t_2} b) \quad \text{for } t_1, t_2 \geq 0.$$

For basic properties of linear skew-products semiflows the reader is referred to Sacker and Sell [13] or to Chow and Leiva [3].

**Theorem 2.2.** Let $\{\Psi_t\}_{t \geq 0}$ be the linear skew-product semiflow generated on the product bundle $B = B \times L^1(\Omega)$ by Equation (1a)+(1b). Then

(i) The subbundles $B_1$ and $B_2$ are $\Psi_t$-invariant in the sense that if $(b, u) \in B_i$ then $(\phi_t b, \psi(t, b)u) \in B_i$ for $t \geq 0$, where $i = 1, 2$.

(ii) $\{\Psi_t|B_2\}$ extends uniquely to a linear skew-product flow on $B_2$.

(iii) There are constants $D' > 1$ and $\mu > 0$ ($\mu = -\log \lambda$) such that

$$\frac{\|\psi(t, b)v_1\|_{L^1(\Omega)}}{\|\psi(t, b)v_2\|_{L^1(\Omega)}} \leq D'e^{-\mu t}\frac{\|v_1\|_{L^1(\Omega)}}{\|v_2\|_{L^1(\Omega)}}$$

for each $(b, v_1) \in B_1$, $(b, v_2) \in B_2 \setminus Z$, and each $t \geq 0$.

**Proof.** We start by proving (iii). Indeed, by the cocycle identity we have $\psi(t, b) = \psi(t - \lfloor t \rfloor, \phi(\lfloor t \rfloor, b)) \circ \psi(\lfloor t \rfloor, b)$, where $\lfloor t \rfloor$ stands for the integer part of $t$. By the standard argument (compare e.g., the proof of Lemma 3.3 in Sacker and Sell [13]) there is a positive constant $C'$ such that $\|\psi(s, b)u\|_{L^1(\Omega)} \leq C'\|u\|_{L^1(\Omega)}$ for $s \in [0, 1]$, $b \in B$ and $u \in L^1(\Omega)$. Further, as $B_2$ is one-dimensional, there is a positive constant $C''$ such that $\|\psi(s, b)u\|_{L^2(\Omega)} \geq C''\|u\|_{L^1(\Omega)}$ for $s \in [0, 1]$ and $(b, u) \in B_2$. From this it follows that for $(b, v_1) \in B_1$, $(b, v_2) \in B_2 \setminus Z$, $t \geq 0$, we have

$$\frac{\|\psi(t, b)v_1\|_{L^1(\Omega)}}{\|\psi(t, b)v_2\|_{L^1(\Omega)}} \leq \frac{C''}{C'}\|\psi(t, b)v_1\|_{L^1(\Omega)} \leq \frac{C'D}{C''\lambda^t}\frac{\|v_1\|_{L^1(\Omega)}}{\|v_2\|_{L^1(\Omega)}} \leq \frac{C'D}{C''\lambda^t}\frac{\|v_1\|_{L^1(\Omega)}}{\|v_2\|_{L^1(\Omega)}}$$

Putting $D' := C''/C'\lambda^{C''}$ and $\mu := -\log \lambda$ gives the desired result.

Part (ii) is a consequence of the fact that $\Psi|B_2$ is a bundle automorphism and the cocycle property.
The proof of the $\Psi_t$-invariance of $\mathcal{B}_2$ is straightforward: $(b, u) \in \mathcal{B}_2$ is equivalent to $(\Id - P(b))u = 0$, which yields, by part (iii), $(\Id - P(\phi_t b))\psi(t, b)u$ for all $t \geq 0$. In order to establish the $\Psi_t$-invariance of $\mathcal{B}_1$, suppose by way of contradiction that for some $(b', v) \in \mathcal{B}_1$ and some $t' > 0$, $t' \notin \mathbb{N}$, we have $\psi(t', b')v \notin \mathcal{B}_1$. This means that $P(\phi_{t'} b')v \neq 0$. As $\psi([t'] + 1, b') = \psi([t'] + 1 - t', \phi_{t'} b')v$, from (iii) we deduce that the ratio

$$\frac{\| (\Id - P(\phi_{[t'] + 1} b'))\psi([t'] + 1, b')v \|_{L^1(\Omega)}}{\| P(\phi_{[t'] + 1} b')\psi([t'] + 1, b')v \|_{L^1(\Omega)}}$$

is finite, which contradicts the fact that $(\phi_{[t'] + 1} b', \psi([t'] + 1, b')v) \in \mathcal{B}_1$. \hfill \Box

**Theorem 2.3.** Let $u : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ be a nonzero global solution to (1a)+(1b) such that $u(t, x) \geq 0$ for each $t \in \mathbb{R}$ and each $x \in \overline{\Omega}$. Then for each $t \in \mathbb{R}$ the pair $(a_0 \cdot t, u(t, \cdot))$ belongs to the one-dimensional subbundle $\mathcal{B}_2$.

**Proof.** An application of the parabolic strong maximum principle yields $u(t, \cdot) \in C^1(\overline{\Omega})_{++}$ for each $t \in \mathbb{R}$.

We claim that there is a positive constant $L$ such that

$$\|P(b)w\|_{L^1(\Omega)} \geq L \|w\|_{L^1(\Omega)} \quad \text{for each } b \in \mathcal{B} \text{ and each } w \in L^1(\Omega)_+. \quad (2.2)$$

Indeed, as the fiber $\mathcal{B}_1(b)$ is the nullspace of the functional $\hat{\psi}^*(b)$, one has $\langle \hat{\psi}^*(b), w \rangle = \langle \hat{\psi}^*(b), P(b)w \rangle$ for all $w \in L^1(\Omega)$. Further, since $\mathcal{B}_2$ has dimension one and the functional $\hat{\psi}^*(b)$ is uniformly positive, there is a positive number $h(b)$ such that $\|w\|_{L^1(\Omega)} = h(b)\langle \hat{\psi}^*(b), w \rangle$ for all $w \in L^1(\Omega)_+$ with $(b, w) \in \mathcal{B}_2$. Consequently $\|P(b)w\|_{L^1(\Omega)} = h(b)\langle \hat{\psi}^*(b), P(b)w \rangle$ for all $w \in L^1(\Omega)_+$. The positive function $h(\cdot)$ is easily seen to be continuous, so there is $h > 0$, $h = \min \{h(b) : b \in \mathcal{B}\}$, such that $\|P(b)w\|_{L^1(\Omega)} \geq h\langle \hat{\psi}^*(b), w \rangle$ for all $b \in \mathcal{B}$ and $w \in \mathcal{B}$. Now it remains to notice that by the continuity of the mapping $\mathbb{B} \ni b \mapsto \hat{\psi}^*(b) \in L^1(\Omega)^*$ and the fact that uniformly positive functionals form an open set in $L^1(\Omega)^*$, the positive constant in the definition of uniform positivity can be chosen independent of $b$. Formula (2.2) follows immediately.

As a consequence, for each nonzero $w \in L^1(\Omega)_+$ and each $b \in \mathcal{B}$ one has

$$\frac{\| (\Id - P(b))w \|_{L^1(\Omega)}}{\| P(b)w \|_{L^1(\Omega)}} \leq \frac{1 + N}{L},$$

where $N := \sup \{ \|P(b)w\|_{L^1(\Omega)} : b \in \mathcal{B}, \|w\|_{L^1(\Omega)} = 1 \}$.

For each $t \in \mathbb{R}$ denote $\hat{u}(t) = u(t, \cdot)$ regarded as an element of $L^1(\Omega)$. Suppose to the contrary that there is $t'$ such that $(a_0 \cdot t', \hat{u}(t'))$ does not belong to $\mathcal{B}_2$. Put

$$M := \frac{\| (\Id - P(a_0 \cdot t'))\hat{u}(t') \|_{L^1(\Omega)}}{\| P(a_0 \cdot t')\hat{u}(t') \|_{L^1(\Omega)}} > 0,$$

and

$$t'' := t' + \frac{1}{\mu} \log \frac{ML}{2D'(1 + N)},$$

where $\mu$ and $D'$ are constants from Theorem 2.2(iii). As $M \leq (1 + N)/L$, we have $t'' < t'$. 

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An application of Theorem 2.2(iii) yields
\[
\frac{\| (\text{Id} - P(a_0 \cdot t')) \bar{u}(t') \|_{L^1(\Omega)}}{\| P(a_0 \cdot t') \bar{u}(t') \|_{L^1(\Omega)}} = \frac{\| \psi(t' - t'', a_0 \cdot t'') (\text{Id} - P(a_0 \cdot t'')) \bar{u}(t'') \|_{L^1(\Omega)}}{\| \psi(t' - t'', a_0 \cdot t'') P(a_0 \cdot t'') \bar{u}(t'') \|_{L^1(\Omega)}} \\
\leq D'e^{-\mu (t'' - t')} \frac{\| \psi(t' - t'', a_0 \cdot t'') (\text{Id} - P(a_0 \cdot t'')) \bar{u}(t'') \|_{L^1(\Omega)}}{\| \psi(t' - t'', a_0 \cdot t'') P(a_0 \cdot t'') \bar{u}(t'') \|_{L^1(\Omega)}} \\
\leq D'e^{-\mu (t'' - t')} \frac{1 + N}{L} \\
\leq \frac{M}{2},
\]
a contradiction. \(\square\)

Finally, we formulate now our main result.

**Corollary 2.4.** Assume that \(u_1, u_2\) are nonzero global solutions of (1a)+(1b) such that \(u_1(t, x) \geq 0\) and \(u_2(t, x) \geq 0\) for all \(t \in \mathbb{R}\) and all \(x \in \Omega\). Then there is a positive constant \(\kappa\) such that \(u_1(t, x) = \kappa u_2(t, x)\) for all \(t \in \mathbb{R}\) and all \(x \in \Omega\).

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