Renormalization of the noncommutative $\phi^3$ model through the Kontsevich model

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Abstract
We point out that the noncommutative selfdual $\phi^3$ model can be mapped to the Kontsevich model, for a suitable choice of the eigenvalues in the latter. This allows to apply known results for the Kontsevich model to the quantization of the field theory, in particular the KdV flows and Virasoro constraints. The 2-dimensional case is worked out explicitly. We obtain nonperturbative expressions for the genus expansion of the free energy and some $n$-point functions. The full renormalization for finite coupling is found, which is determined by the genus 0 sector only. All contributions in a genus expansion of any $n$-point function are finite after renormalization. A critical coupling is determined beyond which the model is unstable. The model is free of UV/IR diseases.

1 Introduction
One of the motivation for noncommutative field theory is the hope to achieve a better understanding of the UV divergences and renormalization in quantum field theory (QFT), by formulating QFT on a quantized or noncommutative (NC) space. The most popular example of such a space is the quantum plane, where the coordinate “functions” satisfy the canonical commutation relations $[x_i, x_j] = i\theta_{ij}$. This introduces a length scale, and divides space essentially into “Planck cells” of finite area. At first sight, one might then guess that this length scale corresponds to a cutoff $\Lambda_{NC} = \frac{1}{\sqrt{\theta}}$ in QFT. However, it turns out that $\Lambda_{NC}$ does not play the role supported by the FWF project P16779-N02.
of a UV cutoff, but more properly serves as a reflection point between scales in the UV and the IR on both sides of $\Lambda_{NC}$. Moreover, this generically leads to the so-called UV/IR mixing in divergent QFT’s, which is a serious obstacle to perturbative renormalization [1].

A way to overcome these problems has been found recently in [2–4] for the scalar $\phi^4$ model in 2 and 4 dimensions. This is achieved by adding a confining potential (a “box”) to the field theoretical models, i.e. an additional relevant term in the Lagrangian of the form $\Omega(\tilde{x}_i\phi)(\tilde{x}_i\phi)$ which makes them covariant under a duality [5]. This additional term essentially suppresses or controls the divergencies in the IR, and introduces discreteness into the models which can be viewed as generalized matrix models. Perturbative renormalizability was then proved using a renormalization group approach. In particular, there is a special point $\Omega = 1$ where the models become selfdual in a certain sense [5]. This is expected to be preserved under renormalization, and is therefore of particular interest.

The close relationship between noncommutative field theory and (generalized) matrix models can be seen in many ways. The most striking similarity is that Feynman diagrams are drawn on a Riemann surface, leading to a genus expansion. For generic momenta, only planar diagrams are divergent just like in ordinary matrix models. However, for exceptional momenta the nonplanar (higher-genus) diagrams become divergent as well, which leads to UV/IR mixing i.e. new divergences in the IR. This similarity to matrix models suggests to apply or adapt some of the powerful techniques which have been developed in the context of matrix models. This idea was applied in [6], where the selfdual complex $\phi^4$ model was formulated as matrix model, which in the degenerate $U(N)$-invariant case was solved exactly but turned out to be trivial (i.e. free). Later, in [7, 8] a strategy to analyze more general scalar models using matrix model techniques was proposed. This is based on the hypothesis that the eigenvalue distribution is localized as in ordinary matrix models, and strongly hints at nontriviality of e.g. the real $\phi^4$ model.

In the present paper, we show that the noncommutative Euclidean selfdual $\phi^3$ model can be solved (at least in 2 dimensions, but probably more generally) using matrix model techniques, and is related to the KdV hierarchy. This is achieved by rewriting the field theory as Kontsevich matrix model, for a suitable choice of the eigenvalues in the latter. The relation holds for any even dimension, and allows to apply some of the known, remarkable results for the Kontsevich model to the quantization of the $\phi^3$ model. We work out the 2-dimensional case explicitely. This allows to write down closed expressions for the genus expansion of the free energy, and also for some $n$-point functions by taking derivatives and using the equations of motion. It turns out that the required renormalization is determined by the genus 0 sector only, and can be computed explicitly. We show that using this renormalization, all contributions in a genus expansion of any $n$-point function correlation function are finite and well-defined for finite coupling. This implies but
is stronger than perturbative renormalization. Here we draw heavily on results of [9,10] for the Kontsevich model.

We thus obtain fully renormalized models with nontrivial interaction which are free of UV/IR diseases. Only the linear tadpole term in the action requires (logarithmic) renormalization, while mass and coupling constant do not run as expected. We also extract the leading perturbative contribution for certain correlators using the nonperturbative results, and verify that they coincide with the diagrammatic computations. All this shows that even though the $\phi^3$ may appear ill-defined at first, it is in fact much better under control than other models.

These results are obtained starting with purely imaginary coupling constants, but allow analytic continuation to real coupling. This turns out to be well-defined for finite coupling below some critical value, but becomes unstable for strong coupling. We derive the corresponding critical coupling, which is interpreted as instability induced by the finite potential barrier.

In view of the well-known relation between NC field theory and string theory in certain backgrounds [11], our results are relevant also in that context. In fact, the noncommutative $\phi^3$ model has been related to open strings in a $B$-field background in [12]. More recently, the relevance of the Konsevich model to open string field theory [13] and to branes on non-compact Calabi-Yau spaces [14] has been pointed out. This suggests that the language and techniques of NC field theory might be useful also in this string-theoretical context.

This paper is organized as follows: In section 2 we define the $\phi^3$ model under consideration, and rewrite it as Kontsevich model. Some useful identities for correlators are also given. We then recall the most important facts about the Kontsevich model in section 3 and specialize it to the field theoretical model under consideration in section 4. Renormalization and finiteness are established in section 4.1, which is the main result of this paper. We then perform some perturbative checks, and provide some approximation formulas valid for finite coupling. The critical point for real coupling is studied in section 4.3, and some perturbative computations are given in section 5. We conclude with a discussion and outlook.

## 2 The noncommutative $\phi^3$ model

We assume some familiarity with the formulation of noncommutative field theory, as reviewed e.g. in [15,16]. Consider the action

$$
\tilde{S} = \int_{\mathbb{R}^{2n}} \frac{1}{2} \partial_i \phi \partial_i \phi + \frac{\mu^2}{2} \phi^2 + \Omega^2 (\tilde{x}_i \phi) (\tilde{x}_i \phi) + \frac{i \tilde{\lambda}}{3!} \phi^3
$$

(1)
on the $2n$-dimensional quantum plane, which is generated by self-adjoint operators $x_i$ satisfying the canonical commutation relations

$$[x_i, x_j] = i\theta_{ij}. \quad (2)$$

We also introduce

$$\tilde{x}_i = \theta^{-1}_{ij} x_j, \quad [\tilde{x}_i, \tilde{x}_j] = i\theta^{-1}_{ij} \quad (3)$$

assuming that $\theta_{ij}$ is nondegenerate. The dynamical object is the scalar field $\phi = \phi^\dagger$, which is a self-adjoint operator acting on the representation space $\mathcal{H}$ of the algebra $\mathcal{A}$. The term $\Omega^2(\tilde{x}_i\phi)(\tilde{x}_i\phi)$ is included following [2, 5], making the model covariant under Langmann-Szabo duality, and taking care of the UV/IR mixing. We choose to write the action with an imaginary coupling $i\tilde{\lambda}$, assuming $\tilde{\lambda}$ to be real. The reason is that for real coupling $\tilde{\lambda}' = i\tilde{\lambda}$, the potential would be unbounded from above and below, and the quantization would seem ill-defined. We will see however that the quantization is completely well-defined for imaginary $i\tilde{\lambda}$, and allows an analytic continuation to real $\lambda' = i\lambda$ in a certain sense which will be made precise below. Therefore we accept for now that the action $\tilde{S}$ is not real.

Using the commutation relations (2), the derivatives $\partial_i$ can be written as inner derivatives $\partial_i f = -i[\tilde{x}_i, f]$. Therefore the action can be written as

$$\tilde{S} = \int -\frac{1}{2}[\tilde{x}_i, \phi][\tilde{x}_i, \phi] + \Omega^2\tilde{x}_i\phi\tilde{x}_i\phi + \frac{\mu^2}{2}\phi^2 + i\tilde{\lambda}\phi^3 \quad (4)$$

using the cyclic property of the integral. For the “self-dual” point $\phi = 1$, this action simplifies further to

$$\tilde{S} = \int (\tilde{x}_i\tilde{x}_i)\phi\phi + \frac{\mu^2}{2}\phi^2 + \frac{i\tilde{\lambda}}{3!}\phi^3. \quad (5)$$

In order to quantize the theory, we need to introduce a counterterm which is linear in $\phi$:

$$\tilde{S} = \int (\tilde{x}_i\tilde{x}_i)\phi\phi - (i\tilde{\lambda}\tilde{a})\phi + \frac{\mu^2}{2}\phi^2 + \frac{i\tilde{\lambda}}{3!}\phi^3. \quad (6)$$

We will include this additional term from now on, which is written as $(i\tilde{\lambda}\tilde{a})$ since then only the coupling $\lambda' = (i\tilde{\lambda})$ needs analytic continuation, while $\tilde{a}$ will always be real. The additional term implies that the potential $V(\phi) = \lambda'\tilde{a}\phi + \frac{\mu^2}{2}\phi^2 + \frac{\lambda'}{3!}\phi^3$ has a local minimum at $\phi_0 = \frac{1}{\lambda'}(-\mu^2 + \sqrt{\mu^2 - 2(\lambda')^2\tilde{a}})$ for real $\lambda'$.

\footnote{One can of course also start with the Moyal product on a classical space of functions. We will skip this point of view here, since for our purpose the matrix representation is crucial.}
The crucial observation which we will use in this paper is the fact that the self-dual model (6) can be written as a matrix model coupled to an external “source” matrix. This was already noted and applied in [6] in the context of the complex $\phi^4$ model. To see this, consider the operator

$$J = 4\pi(\theta \sum_i \bar{x}_i x_i + \mu^2 \theta).$$

(7)

We assume that $\theta_{ij}$ has the canonical form $\theta_{ij} = -\theta_{ji}$: $\theta$ (= $\theta_{34} = -\theta_{43}$ etc. for higher dimensions), focusing on the 2-dimensional case in this paper. Then $J$ is essentially the Hamiltonian of quantum mechanical harmonic oscillator $J \propto \sum \hat{x}^2 + \hat{p}^2 + \text{const}$, which in the usual basis of eigenstates diagonalizes with evenly spaced eigenvalues,

$$J|n\rangle = 4\pi(n + 1 + \frac{\mu^2 \theta}{2})|n\rangle, \quad n \in \{0, 1, 2, \ldots\};$$

(8)

we will only consider the 2-dimensional case from now on. Replacing $\int\lambda = (2\pi \theta)$, the action can be written as

$$\tilde{S} = 2\pi Tr \left( \frac{1}{4\pi} J \phi^2 + \frac{i\lambda \theta}{3!} \phi^3 - i\tilde{a} \theta \phi \right).$$

(9)

It is convenient to define the dimensionless constants

$$\lambda = 2\pi \theta \tilde{\lambda}, \quad a = \lambda^2 \tilde{a} = -(i\lambda)^2 \tilde{a},$$

(10)

and to subtract an irrelevant constant term $Tr(\frac{1}{3\lambda^2} J^3 + \frac{1}{\lambda^2} J a)$ from the action, in order to simplify some formulas below. Then we obtain the following action

$$S := Tr \left( \frac{1}{2} J \phi^2 + \frac{i\lambda}{3!} \phi^3 - \frac{a}{i\lambda} \phi - \frac{1}{3(i\lambda)^2} J^3 - \frac{1}{(i\lambda)^2} J a \right).$$

(11)

One can eliminate the quadratic term by shifting$^3$ the variable in (11) as

$$\tilde{\phi} = \phi + \frac{1}{i\lambda} J,$$

(12)

so that

$$S = Tr \left( - \frac{1}{2} \left( \frac{1}{i\lambda} J^2 + \frac{2a}{i\lambda} \tilde{\phi} + \frac{i\lambda}{3!} \tilde{\phi}^3 \right) \right) = Tr \left( - \frac{1}{2i\lambda} M^2 \tilde{\phi} + \frac{i\lambda}{3!} \tilde{\phi}^3 \right)$$

(13)

where

$$M = \sqrt{J^2 + 2a}.$$  

(14)

$^3$for the quantization, the integral for the diagonal elements is then defined via analytical continuation, and the off-diagonal elements remain hermitian since $J$ is diagonal.
Now the field $\tilde{\phi}$ couples linearly rather than quadratically to the source, which turns out to be very useful. Alternatively, the linear term can be eliminated by setting
\[
\tilde{\phi} = X + \frac{1}{i\lambda} M, \quad X = \tilde{\phi} - \frac{1}{i\lambda} M = \phi + \frac{J - M}{i\lambda}.
\] (15)

Then the action becomes
\[
S = Tr\left(\frac{1}{2} M X^2 + \frac{i\lambda}{3!} X^3 - \frac{1}{3(i\lambda)^2} M^3\right),
\] (16)
which has the form of the Kontsevich model [10].

2.1 Quantization and equations of motion

The quantization of the model (6) resp. (13) is defined by an integral over all Hermitian $N \times N$ matrices $\phi$, where $N$ serves as a UV cutoff. The partition function is defined as
\[
Z(M) = \int D\tilde{\phi} \exp(-S(M))
\] (17)
for $S = S(M)$ given by (13), and correlators or “$n$-point functions” are defined through
\[
\langle \phi_{i_1j_1}...\phi_{i_nj_n} \rangle = \frac{1}{Z} \int D\tilde{\phi} \exp(-S) \tilde{\phi}_{i_1j_1}...\tilde{\phi}_{i_nj_n}
\] (18)

The eigenvalues of $J$ resp. $M$ are then given by (8) resp. (14) for $n = 0, 1, ..., N - 1$. The nontrivial task is to show that all correlation functions have a well-defined and hopefully nontrivial limit $N \to \infty$, i.e. that the “low-energy physics” is well-defined and independent of the cutoff.

Using the symmetry $Z(M) = Z(U^{-1}MU)$ for $U \in U(N)$, we can assume that $M$ is diagonalized with (ordered) eigenvalues $m_i$. Then there is a residual $U(1)^N$ invariance $\phi_{ij} \to u_i^{-1}\phi_{ij}u_j$ with $u_i \in U(1)$. This implies certain obvious “index conservation laws”, e.g. $\langle \phi_{kl} \rangle = \delta_{kl} \langle \phi_{ll} \rangle$ etc.

In order to have a well-defined limit $N \to \infty$, we should require in particular that the 2-point function $\langle \phi_{ij}\phi_{kl} \rangle$ and also the one-point function $\langle \phi_{kl} \rangle$ have a well-defined limit. We therefore impose the renormalization conditions
\[
\langle \phi_{00}\phi_{00} \rangle = \frac{1}{2\pi} \frac{1}{\mu^2 \theta + 1},
\] (19)
\[
\langle \phi_{00} \rangle = 0
\] (20)
which hold in the free case $\lambda = 0$. This will uniquely determine the renormalization of $a$ (and $\mu^2$, which receives only finite quantum corrections and which will not be computed here).
Quantum equations of motion and correlators. We first derive some simple relations for the basic correlators. Using the identity

$$0 = \int d\tilde{\phi} \frac{d}{d\tilde{\phi}_{kk}} \exp \left( \frac{1}{2i\lambda} \tilde{\phi} M^2 - \frac{i\lambda}{3!} \tilde{\phi}^3 \right)$$

(21)

we obtain

$$0 = \left< \sum_{l,l\neq k} \tilde{\phi}_{kl} \tilde{\phi}_{lk} + \tilde{\phi}_{kk}^2 - \frac{1}{\lambda^2 m_k^2} \right>$$

(22)

for each matrix index $k$. Insertions of a diagonal factor $\tilde{\phi}_{kk}$ in the correlators can be achieved\(^5\) by acting with the derivative operator $\frac{\lambda}{m_k} \frac{\partial}{\partial m_k}$ on $Z$. More general non-diagonal insertions $\tilde{\phi}_{kl}$ will be discussed in section 4.5.

In the Kontsevich model resp. the $\phi^3$ model, there is a simple way to obtain also a certain class of non-diagonal insertions, by expressing the invariance of the integral $Z$ under an infinitesimal change of variable of the form \([9]\)

$$\tilde{\phi} \rightarrow \tilde{\phi} + i\varepsilon [X_{(kl)}, \tilde{\phi}], \quad \text{with} \quad (X_{(kl)})_{ab} = \delta_{ak}\delta_{bl}\tilde{\phi}_{kl}$$

(23)

which gives explicitly

$$\delta_{(kl)}\tilde{\phi}_{ij} = i\varepsilon \left( \delta_{ik}\tilde{\phi}_{il}\tilde{\phi}_{lj} - \delta_{jl}\tilde{\phi}_{ik}\tilde{\phi}_{kj} \right).$$

(24)

The Jacobian is $1 + i\varepsilon(\tilde{\phi}_{kl} - \tilde{\phi}_{kk})$, while the term $Tr\tilde{\phi}^3$ is invariant. Thus

$$0 = \left< \tilde{\phi}_{kl} \tilde{\phi}_{ik} + \frac{1}{2i\lambda} (m_k^2 - m_l^2) \tilde{\phi}_{lk} \tilde{\phi}_{ik} \right>, \quad \text{with no summation implied.}$$

(25)

In particular, we find for the propagator

$$\left< \tilde{\phi}_{kl} \tilde{\phi}_{ik} \right> = \frac{2i\lambda}{m_k^2 - m_l^2} \left< \tilde{\phi}_{kk} - \tilde{\phi}_{ll} \right>$$

(26)

for $k \neq l$ (no sum). Recalling that $\tilde{\phi} = \phi + \frac{1}{i\lambda} J$, this gives

$$\left< \phi_{kl}\phi_{ik} \right> = \frac{2i\lambda}{m_k^2 - m_l^2} \left( \left< \phi_{kk} - \phi_{ll} \right> + \frac{1}{i\lambda} (J_k - J_l) \right)$$

$$= \frac{2}{J_k + J_l} + \frac{2i\lambda}{m_k^2 - m_l^2} \left< \phi_{kk} - \phi_{ll} \right>$$

(27)

\(^4\)this is justified by analytic continuation using \([10]\).

\(^5\)Alternatively one could also promote $\phi$ to a matrix (commuting with $J$), and take derivatives w.r.t. $\alpha_i$. Since this presents no particular advantages, we will not do this in the following.
noting that \( J^2_k - J^2_l = m^2_k - m^2_l \). The first term is the free contribution, and the second the quantum correction. Thus we “only” need the 1-point functions

\[
\left< \tilde{\phi}_{kk} \right> = \frac{i\lambda}{m_k \partial m_k} \ln \tilde{Z}(m) = \frac{1}{i\lambda} J_k + \left< \phi_{kk} \right>.
\] (28)

They can be obtained from the Kontsevich model, as we will show in detail.

Proceeding as in [9], we can insert (25) into (22) which leads to

\[
\frac{m^2_k}{\lambda^2} = -\left< \tilde{\phi}_{kk}^2 \right> - (2i\lambda) \sum_{l, l \neq k} \frac{\left< \tilde{\phi}_{kk} - \tilde{\phi}_{ll} \right>}{m^2_k - m^2_l}.
\] (29)

These manipulations can be generalized: consider

\[
0 = \delta_{(kl)} Z \left< \tilde{\phi}_{kk} \right> = \delta_{(kl)} \int D\tilde{\phi} \exp(-S(J)) \tilde{\phi}_{kk}
\]
\[
= Z \left< \left( \tilde{\phi}_{ll} - \tilde{\phi}_{kk} \right) + \frac{1}{2i\lambda} (m^2_k - m^2_l) \tilde{\phi}_{kl} \tilde{\phi}_{lk} \left( \tilde{\phi}_{kk} + \delta_{(kl)} \tilde{\phi}_{kk} \right) \right>
\]
\[
= Z \left< \left( \tilde{\phi}_{ll} - \tilde{\phi}_{kk} \right) + \frac{1}{2i\lambda} (m^2_k - m^2_l) \tilde{\phi}_{kl} \tilde{\phi}_{lk} \left( \tilde{\phi}_{kk} + \left( \tilde{\phi}_{kl} \tilde{\phi}_{lk} - \delta_{kl} \tilde{\phi}_{kk} \tilde{\phi}_{kk} \right) \right) \right>.
\] (30)

using

\[
\delta_{(kl)} \tilde{\phi}_{kk} = i\varepsilon \left( \tilde{\phi}_{kl} \tilde{\phi}_{lk} - \delta_{kl} \tilde{\phi}_{kk} \tilde{\phi}_{kk} \right).
\] (31)

This implies

\[
\left< \tilde{\phi}_{kl} \tilde{\phi}_{lk} \tilde{\phi}_{kk} \right> = \frac{2i\lambda}{m^2_k - m^2_l} \left< \left( \tilde{\phi}_{kk} - \tilde{\phi}_{ll} \right) \tilde{\phi}_{kk} - \left( \tilde{\phi}_{kl} \tilde{\phi}_{lk} - \delta_{kl} \tilde{\phi}_{kk} \tilde{\phi}_{kk} \right) \right>
\]
\[
= \frac{2i\lambda}{m^2_k - m^2_l} \left< \left( 1 + \delta_{kl} \right) \tilde{\phi}_{kk} \tilde{\phi}_{kk} - \tilde{\phi}_{ll} \tilde{\phi}_{kk} - \frac{2i\lambda}{m^2_k - m^2_l} \left( \tilde{\phi}_{kk} - \tilde{\phi}_{ll} \right) \right>
\] (32)

(no sum). Therefore \( \left< \tilde{\phi}_{kl} \tilde{\phi}_{lk} \tilde{\phi}_{kk} \right> \) is finite and known exactly provided the 1- and 2-point functions are known, which indeed will be obtained from the Kontsevich model.

Clearly these manipulations can be generalized. However, we will present a different argument in section 4.5 which establishes finiteness of general correlation functions more directly, using the renormalization procedure explained below. The relation (32) could also be used to demonstrate that the model is not free, in contrast to [6]. We will not bother to elaborate this, since the lowest nontrivial term in a Taylor expansion in \( \lambda \) is manifestly finite and nonzero, and the model will be renormalized for finite coupling.
It is worth pointing out that (29) yields the matrix Airy equations

\[
\left[ m_k^2 - \left( \frac{1}{m_k} \frac{\partial}{\partial m_k} \right)^2 - 2 \sum_{l \neq k} \frac{1}{m_k^2 - m_l^2} \left( \frac{1}{m_k} \frac{\partial}{\partial m_k} - \frac{1}{m_l} \frac{\partial}{\partial m_l} \right) \right] Z = 0
\]  

(33)
as shown in [9], which was called “Master equation” in [17]. This in turn can be translated into the Virasoro constraints,

\[ L_m Z = 0, \quad m \geq -1 \]  

(34)
for suitable operators \( L_m \). It is also useful to define as in [6, 17]

\[ W(m_k) := \frac{1}{m_k} \frac{\partial}{\partial m_k} \ln Z(m) = \frac{1}{i\lambda} \langle \tilde{\phi}_{ii} \rangle. \]  

(35)

Then (33) can be written as

\[ W(m_k)^2 + \frac{1}{m_k} \frac{\partial}{\partial m_k} W(m_k) + 2 \sum_{l \neq k} \frac{W(m_k) - W(m_l)}{m_k^2 - m_l^2} - m_k^2 = 0. \]  

(36)

Before proceeding, we briefly recall the usual treatment [17] of the genus 0 solution of the Kontsevich model. In that case one can neglect the term \( \frac{1}{m_k} \frac{\partial}{\partial m_k} W(m_k) \) in (36), which would be suppressed by \( \frac{1}{N} \). Then (36) constitutes \( N \) equations in \( N \) unknowns \( W(m_k), k = 1, \ldots, N \). This can be solved in the large \( N \) limit, by assuming that \( W(m_k) \) becomes an analytic function with cuts in the large \( N \) limit, where \( m_k, k = 1, \ldots, N \) becomes a continuous variable (recall that \( m_k \) are the ordered eigenvalues of \( M \)).

In the present case, we are not allowed to neglect the term \( \frac{1}{m_k} \frac{\partial}{\partial m_k} W(m_k) \) since there is no factor \( N \) involved in (35). Correspondingly \( \langle \tilde{\phi}_{kk}^2 \rangle \neq \langle \tilde{\phi}_{kk} \rangle^2 \), as can be seen explicitly in perturbation theory (see section 5). Therefore we have to use the complete genus expansion, which is indeed available more-or-less explicitly. It will turn out that only the genus 0 contribution requires renormalization. We start by recalling some more facts about the Kontsevich model.

3 Some useful facts for the Kontsevich model

The Kontsevich model is defined by

\[
Z^{Kont}(\tilde{M}) = e^{F^{Kont}} = \frac{\int dX \exp \left\{ Tr \left( -\frac{\tilde{M}X^2}{2} + i\frac{X^3}{6} \right) \right\}}{\int dX \exp \left\{ -Tr \left( \tilde{M}X^2 \right) \right\}}
\]  

(37)
where $\tilde{M}$ is a given hermitian $N \times N$ matrix, and the integral is over Hermitian $N \times N$ matrices $X$. This model has been introduced by Kontsevich [10] as a combinatorial way of computing certain topological quantities (intersection numbers) on moduli spaces of Riemann surfaces with punctures, which in turn were related to the partition function of the general one-matrix model by Witten [18]. It turns out to have an extremely rich structure related to integrable models (KdV flows) and Virasoro constraints, and was studied using a variety of techniques. For our purpose, the most important results are those of [9,10,17] which provide explicit expressions for the genus expansion of the free energy of (37). Note that $\lambda$ can be introduced via

$$\int dX \exp \left\{ -\lambda \frac{2}{3} \tilde{M}X^2 + \frac{i}{6} \tilde{X}^3 \right\} = \int d\tilde{X} \exp \left\{ -\frac{1}{2} Tr M\tilde{X}^2 \right\},$$

(38)

where $X = -\lambda^{1/3} \tilde{X}$, $M = \lambda^{-2/3} \tilde{M}$, which allows to obtain the analytic continuation in $\lambda$.

The matrix integral in (37) and its large $N$ limit can be defined rigorously in terms of its asymptotic series. Defining the normalized measure

$$d\mu_M(X) = \frac{dX \exp(-\frac{1}{2} Tr \tilde{M}X^2)}{\int dX \exp(-\frac{1}{2} Tr \tilde{M}X^2)},$$

(39)

for Hermitian $N \times N$ matrices, one considers the matrix Airy function

$$Z^{Kont}(N) = \int d\mu_M(X) \exp \left( \frac{i}{6} Tr X^3 \right) = \sum_{k \geq 0} Z^{Kont}(N)_k(\tilde{M})$$

$$Z^{Kont}(N)_k(\tilde{M}) = \frac{(-1)^k}{(2k)!} \int d\mu_M(X) \left( \frac{Tr X^3}{6} \right)^{2k}.$$  

(40)

A crucial fact [10] is that the terms $Z^{Kont}(N)_k(\tilde{M})$ can be expressed as polynomials with rational coefficients in the variables

$$\theta_r = \frac{1}{r} Tr \tilde{M}^{-r}.$$  

(41)

Then $Z^{Kont}(N)_k$ is homogeneous of degree $3k$, if we set

$$\deg \theta_r = r.$$  

(42)

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Only the first \( N \) of the \( \theta_r \) are algebraically independent for an \( N \times N \) matrix \( \tilde{M} \). However, it is known [9, 10] that

Considered as a function of \( \theta \equiv \{ \theta_1, \theta_2, \ldots \} \), \( Z_{k}^{Kont(N)}(\tilde{M}) \) is independent of \( N \) for \( 3k \leq N \) and depends only on \( \theta_r \), \( 1 \leq r \leq 3k \).

This allows one to define unambiguously the series \( Z^{Kont}(\theta_i) = \sum_{k \geq 0} Z_{k}^{Kont}(\theta_i) \) where \( Z_{k}^{Kont}(\theta_i) \) without any further reference to \( N \). Even though we formally work with an infinite number of variables \( \theta_r \), each \( Z_{k}^{Kont}(Z_0 = 1) \) depends on finitely many of them.

Furthermore, the following remarkable facts hold [10]:

1. \[
\frac{\partial Z^{Kont}}{\partial \theta_{2r}} = 0 \tag{43}
\]

2. \( Z^{Kont}(\theta_r) \) is a \( \tau \)-function for the Korteweg-de Vries equation.

Namely if

\[
t_r = -(2r + 1)!! \theta_{2r+1} = -(2r - 1)!! tr\tilde{M}^{-2r-1}
\]

\[
u := \frac{\partial^2}{\partial t_0^2} \ln Z^{Kont} \tag{44}
\]

then

\[
\frac{\partial u}{\partial t_1} = \frac{\partial}{\partial t_0} \left( \frac{1}{12} \frac{\partial^2 u}{\partial t_0^2} + \frac{1}{2} u^2 \right) \tag{45}
\]

and more generally

\[
\frac{\partial}{\partial t_n} u = \frac{\partial}{\partial t_0} R_{n+1}. \tag{46}
\]

Here the \( R_n \) denote the Gelfand-Dikii differential polynomials (derivatives are taken with respect to \( t_0 \))

\[
R_2 = \frac{u^2}{2} + \frac{u''}{12}
\]

\[
R_3 = \frac{u^3}{6} + \frac{u u''}{12} + \frac{u'^2}{24} + \frac{u^{(4)}}{240}
\]

\[
R_4 = \frac{u^4}{24} + \frac{u u'^2}{24} + \frac{u^2 u''}{24} + \frac{u u^{(4)}}{240} + \frac{u' u'''}{120} + \frac{(u'')^2}{160} + \frac{u^{(6)}}{6720}
\]

\[
\ldots
\]

\[
R_n = \frac{u^n}{n!} + \ldots \tag{47}
\]
computed from

\[(2n + 1)R'_{n+1} = \frac{1}{4} R''_n + 2u R'_n + u' R_n. \quad (48)\]

For the present case, we have

\[\theta_r = \frac{\lambda^{2r/3}}{r} \sum_{n \geq 0} \frac{1}{(J_n^2 + 2a)^{r/2}} . \quad (49)\]

Without renormalization (i.e. for finite or zero \(a\)), \(\theta_r\) is logarithmically divergent for \(r = 1\), and finite for \(r \geq 2\). This is a first indication that the model requires renormalization, and we will see that \(a \propto \lambda^2 \ln N\). However, it will turn out that even in the properly renormalized case a different set of variables is more suitable.

A further crucial fact is the existence of a

**Genus expansion.** As usual for matrix models, one can consider the genus expansion

\[\ln Z^{Kont} = F^{Kont} = \sum_{g \geq 0} F^{Kont}_g \quad (50)\]

by drawing the Feynman diagrams on a suitable Riemann surface. In principle, this genus expansion can be obtained as a \(\frac{1}{N}\) expansion by introducing an explicit factor \(N\) in the action, so that the action takes the form

\[S' = Tr N(-\frac{1}{2} M'^2 \phi' + \frac{1}{3!} \phi'^3). \quad (51)\]

However, it was shown in [9] that the \(F^{Kont}_g\) can also be computed using the KdV equations and the Virasoro constraints (34), which allows to find closed expressions for small \(g\). It is useful to use the following set of variables:

\[I_k(u_0, t_i) = \sum_{p \geq 0} t_{k+p} \frac{u^p_0}{p!} \quad (52)\]

where \(u_0\) is given by the solution of the implicit equation

\[u_0 = I_0(u_0, t_i). \quad (53)\]

We note that using the definition (51), \(I_k\) can be resummed as

\[I_k(u_0, t_i) = -(2k - 1)!! \sum_{i \geq 0} \frac{1}{(m_i^2 - 2u_0)^{k+\frac{1}{2}}}, \quad (54)\]

in particular

\[u_0 = -\sum_{i \geq 0} \frac{1}{\sqrt{m_i^2 - 2u_0}} = I_0. \quad (55)\]
These variables turn out to be more useful for our purpose than the \( t_r \), since the \( \tilde{m}_i^2 - 2u_0 \) will be finite in the renormalized model, while the \( t_r \) are not. Using the KdV equations, [9] found the following explicit formulas:

\[
F_0^{\text{Kont}} = \frac{u_0^3}{6} - \sum_{k \geq 0} \frac{u_0^{k+2} t_k}{k + 2! k!} + \frac{1}{2} \sum_{k \geq 0} \frac{u_0^{k+1}}{k + 1} \sum_{a+b=k} \frac{t_a t_b}{a! b!}
\]  

(56)

\[
F_1^{\text{Kont}} = \frac{1}{24} \ln \frac{1}{1 - I_1},
\]  

(57)

\[
F_2^{\text{Kont}} = \frac{1}{5760} \left[ \frac{5}{(1 - I_1)^3} + 29 \frac{I_3 I_2}{(1 - I_1)^4} + 28 \frac{I_3^2}{(1 - I_1)^5} \right],
\]  

(58)

etc. All \( F_g^{\text{Kont}} \) with \( g \geq 2 \) are given by finite sums of polynomials in \( I_k/(1 - I_1)^{2k+1} \), the number of which is \( p(3g - 3) \) with \( p(n) \) being the number of partitions of \( n \). Expanded in the original variables \( t_r \), this becomes [9]

\[
F_0^{\text{Kont}} = \frac{t_0^3}{3!} + t_1^3 + \left( \frac{t_2 t_0}{4!} + 2 \frac{t_1^2 t_0}{2! 3!} \right) + \left( \frac{t_1^5}{5!} + 3t_1 t_2 \frac{t_0}{4!} + 6 \frac{t_0^3 t_0}{3! 3!} \right)
\]

(59)

\[
\quad + \left[ \frac{t_4 t_0}{6!} + \left( \frac{t_2^2 t_0}{2!} + 4t_1 t_3 \right) \frac{t_0^2}{5!} + 24 \frac{t_0^3 t_1}{3! 4!} + 12t_1 t_2 \frac{t_0^4}{2! 4!} \right]
\]

\[
\quad + \left[ \frac{t_5 t_0}{7!} + (5t_1 t_4 + 10t_2 t_3) \frac{t_0^6}{6!} + 120 \frac{t_0^3 t_1^5}{3! 5!} + \left( 30t_1 \frac{t_2^2}{2!} + 20t_3 \frac{t_0^4}{2!} \right) \frac{t_0^5}{5!} + 60t_2 \frac{t_1^3 t_0^3}{3! 4!} \right]
\]

\[
\quad + \ldots
\]

and

\[
24 F_1^{\text{Kont}} = t_1 + \left( \frac{t_2^2}{2!} + t_0 t_2 \right) + \left( \frac{2t_1^3}{3!} + t_3 \frac{t_0^2}{2!} + 2t_0 t_1 t_2 \right)
\]

(60)

\[
\quad + \left( \frac{6t_4 t_0}{4!} + t_4 \frac{t_0^2}{3!} + 4 \frac{t_2^2 t_0}{2! 2!} + 6t_0 t_2 \frac{t_0^2}{2!} + 3t_1 t_3 \right)
\]

\[
\quad + \left( 24 \frac{t_4^3}{5!} + t_5 \frac{t_0^4}{4!} + 24 t_0 t_2 \frac{t_0^3}{3!} + (4t_1 t_4 + 7t_2 t_3) \frac{t_0^6}{3!} + 16 \frac{t_0^3 t_0^2}{2! 2!} + 12t_1 \frac{t_0^4 t_0^2}{2! 2!} \right)
\]

\[
\quad + \ldots
\]

An alternative, more useful form of \( F_0^{\text{Kont}} \) can be obtained by solving directly the “master-equation” (36) at genus 0, as discussed in section 2.1. This leads to [17]

\[
F_0^{\text{Kont}} = \frac{1}{3} \sum_{i=0}^{N} \tilde{m}_i^3 - \frac{1}{3} \sum_{i=0}^{N} (\tilde{m}_i^2 - 2u_0)^{3/2} - u_0 \sum_{i=0}^{N} (\tilde{m}_i^2 - 2u_0)^{1/2}
\]

\[
\quad + \frac{u_0^3}{6} - \frac{1}{2} \sum_{i,k=0}^{N} \ln \left\{ \frac{(\tilde{m}_i^2 - 2u_0)^{1/2} + (\tilde{m}_k^2 - 2u_0)^{1/2}}{\tilde{m}_i + \tilde{m}_k} \right\}
\]  

(61)
which is equivalent to (56) but more useful in our context because it gives explicitly the analytic continuation. The parameter $u_0$ is again given by the implicit equation (55). This constraint can alternatively be obtained by considering $F^{Kont}(\tilde{m}_i; u_0)$ with $u_0$ as an independent variable, since its equation of motion
\[
\frac{\partial}{\partial u_0} F^{Kont}(\tilde{m}_i; u_0) = \frac{1}{2}(u_0 - I_0)^2 = 0
\]
reproduces the constraint. All sums now range from 0 to $N$, and will be convergent after renormalization as $N \to \infty$ for the physical observables.

4 Applying Kontsevich to the $\phi^3$ model

We need
\[
Z = Z^{Kont}[\tilde{M}]Z^{free}[\tilde{M}] \exp(\frac{1}{3(i\lambda)^2}TrM^3)
\]
where
\[
Z^{free}[\tilde{M}] = e^{F_{free}} = \int dX \exp \left(-Tr \left(\frac{\tilde{M}X^2}{2}\right)\right) = \prod_i \frac{1}{\sqrt{\tilde{m}_i}} \prod_{i<j} \frac{2}{\tilde{m}_i + \tilde{m}_j}
\]
up to irrelevant constants, so that
\[
F_{free} = -\frac{1}{2} \sum_{i,j=1}^N \ln(\tilde{m}_i + \tilde{m}_j) \ (+\text{const}).
\]

Therefore
\[
F_0 := F^{Kont}_0 + F_{free} + \frac{1}{3(i\lambda)^2}TrM^3
\]
\[
= -\frac{1}{3} \sum_{i=0}^N \sqrt{\tilde{m}_i^2 - 2u_0} - u_0 \sum_{i=0}^N \sqrt{\tilde{m}_i^2 - 2u_0}
\]
\[
+ \frac{u_0^3}{6} - \frac{1}{2} \sum_{i,k=0}^N \ln(\sqrt{\tilde{m}_i^2 - 2u_0} + \sqrt{\tilde{m}_k^2 - 2u_0}).
\]

In the present case, the eigenvalues $\tilde{m}_i$ are given by (38), (14)
\[
\tilde{m}_i = \lambda^{-2/3} \sqrt{J_i^2 + 2a},
\]
and the model will be ill-defined without renormalization since \( u_0 \) is logarithmically divergent. However, we note that only the combinations \( \sqrt{\bar{m}_i^2 - 2u_0} \) enter in (54) and (60), which can be rewritten as

\[
\sqrt{\bar{m}_i^2 - 2u_0} = \lambda^{-2/3} \sqrt{J_i^2 + 2(a - \lambda^{4/3}u_0)} = \lambda^{-2/3} \sqrt{J_i^2 + 2b} \tag{68}
\]

where

\[
b = a - \lambda^{4/3}u_0 = a - \lambda^{4/3}I_0. \tag{69}
\]

The point is that \( b \) will be finite after renormalization, which makes the model well-defined.

**Analytic continuation and elimination of \( u_0 \).** Replacing \( u_0 \) by (69), the genus 0 contribution to the partition function (66) takes the form

\[
F_0 = \ln Z_{g=0} = -\frac{\lambda^{-2}}{3} \sum_{i=0}^{N} \sqrt{J_i^2 + 2b}^3 - \lambda^{-2}(a - b) \sum_{i=0}^{N} \sqrt{J_i^2 + 2b} + \frac{\lambda^{-4}}{6}(a - b)^3 - \frac{1}{2} \sum_{i,k=0}^{N} \ln \left( \lambda^{-2/3} \sqrt{J_i^2 + 2b} + \lambda^{-2/3} \sqrt{J_k^2 + 2b} \right). \tag{70}
\]

We consider \( F = F(J) \) as a function of (the eigenvalues of) \( J \) from now on. \( b \) satisfies the implicit constraint (69), which becomes

\[
(b - a) = \lambda^2 \sum_{i=0}^{N} \frac{1}{\sqrt{J_i^2 + 2b}}. \tag{71}
\]

In particular, the \( \bar{m}_k \) can be analytically continued as long as \( \sqrt{J_i^2 + 2b} \) is well-defined. Imposing the constraint explicitly we have

\[
F_0 = -\frac{\lambda^{-2}}{3} \sum_{i=0}^{N} \sqrt{J_i^2 + 2b}^3 - \left( -\sum_{k=0}^{N} \frac{1}{\sqrt{J_k^2 + 2b}} \right) \sum_{i=0}^{N} \sqrt{J_i^2 + 2b} + \frac{1}{6} \lambda^2 \left( -\sum_{k=0}^{N} \frac{1}{\sqrt{J_k^2 + 2b}} \right)^3 - \frac{1}{2} \sum_{i,k=0}^{N} \ln \left( \lambda^{-2/3} \sqrt{J_i^2 + 2b} + \lambda^{-2/3} \sqrt{J_k^2 + 2b} \right). \tag{72}
\]

Now the model depends only on the \( J_k \), and \( b \) will be fixed together with the counterterm \( a \) by the renormalization condition (20) and the constraint (71). Also, note that the unexpected \( \lambda^{-2} \) dependence is spurious due to the term \( -\frac{\lambda^2}{3} Tr.J^3 \) in (11), which has no physical significance an could be subtracted.
For some computations it is useful to consider \( b \) in (70) as an independent auxiliary variable, which then satisfies the constraint through the e.o.m as in (62). The essential observation is

\[
\frac{\partial}{\partial J_i} F_0(J_i) = \frac{\partial}{\partial J_i} F_0(J_i; b) + \frac{\partial}{\partial b} F_0(J_i; b) \frac{\partial}{\partial J_i} b = \frac{\partial}{\partial J_i} F_0(J_i; b)
\]

(73)

using

\[
\frac{\partial}{\partial b} F_0(J_i; b) = -\frac{1}{2} \left( \lambda^{-2}(b - a) - \sum_{i=0}^{N} \frac{1}{\sqrt{J_i^2 + 2b}} \right)^2 = 0,
\]

(74)
due to the constraint (71).

We can now compute various \( n \)-point functions, by taking derivatives of \( F = \sum_g F_g \) (where \( F_g = F_g^{\text{Kont}} \) for \( g \geq 1 \)) w.r.t. the \( J_k \). In doing so, we must keep in mind that \( b \) depends implicitly on the \( J_k \) through the constraint (71). On the other hand, the model should be fixed, i.e. \( a \) is fixed for given \( N \) (this will be done below). However, for derivatives up to second order we can as well consider \( u_0 \) resp. \( b \) as an independent variable as discussed above, taking advantage of (74). This simplifies some of the computations below.

### 4.1 Renormalization and finiteness

We can now determine the required renormalization of \( a \), by considering the one-point function. Using (70), (74) and (71), the genus zero contribution is

\[
\langle \tilde{\phi}_{kk} \rangle_{g=0} = \frac{i\lambda}{J_k} \frac{\partial}{\partial J_k} F_0
\]

\[
= \frac{1}{i\lambda} \sqrt{J_k^2 + 2b} + \frac{1}{i\lambda} \frac{a - b}{\sqrt{J_k^2 + 2b}} - (i\lambda) \sum_{j=0}^{N} \frac{(J_j^2 + 2b)^{-1/2}}{\sqrt{J_j^2 + 2b} + \sqrt{J_j^2 + 2b}}
\]

\[
+ \frac{\partial F_0}{\partial b} \frac{i\lambda}{J_k} \frac{\partial b}{\partial J_k}
\]

\[
= \frac{1}{i\lambda} \sqrt{J_k^2 + 2b} + (i\lambda) \sum_{j=0}^{N} \frac{1}{\sqrt{J_j^2 + 2b} \sqrt{J_j^2 + 2b} + (J_j^2 + 2b)}.
\]

(75)

This is manifestly finite for the values (8) of \( J_j \) provided \( b \) is finite, which strongly suggests that a renormalization of the model is achieved if \( b \) is finite. To establish this, we have to check that the higher genus contributions are then also finite. Indeed,

\[
I_p = -(2p - 1)!! \sum_{i=0}^{N} \frac{1}{(\bar{m}_i^2 - 2u_0)^{p+1/2}} = -(2p - 1)!! \lambda^{2(2p+1)/3} \sum_{i=0}^{N} \frac{1}{(J_i^2 + 2b)^{p+1/2}}
\]

(76)
is finite for \( p \geq 1 \). We also obtain from (71) that
\[
\frac{1}{J_k} \frac{\partial}{\partial J_k} b = -\frac{\lambda^2}{(J_k^2 + 2b)^{3/2}} \sum_{i=0}^{N} \frac{\lambda^2}{(J_i^2 + 2b)^{3/2}} \frac{1}{J_k} \frac{\partial}{\partial J_k} b,
\]
(77)

hence
\[
\frac{1}{J_k} \frac{\partial}{\partial J_k} b = -\frac{\lambda^2}{(J_k^2 + 2b)^{3/2}} \frac{1}{1 - I_1}
\]
(78)
is finite since \( 1 - I_1 \neq 0 \) for small \( |i\lambda| \). Therefore taking into account the implicit dependence of \( b \) on \( J_k \), we see that
\[
\frac{1}{J_k} \frac{\partial}{\partial J_k} I_p = (2p + 1)!! \frac{\lambda^2(2p+1)^{1/3}}{(J_k^2 + 2b)^{p+1/2}} (1 + \frac{1}{J_k} \frac{\partial}{\partial J_k} b)
\]
(79)
is also finite. Together with the structure of the higher genus contributions \( F_g \) stated below (58) as found by [9], this implies that

All derivatives of \( F_g \) w.r.t. \( J_k \) for \( g \geq 0 \) as well as all \( F_g \) for \( g \geq 1 \) are finite and have a well-defined limit \( N \to \infty \), provided \( b \) is finite.

Since the connected \( n \)-point functions are given by the derivatives of \( F = \sum_{g \geq 0} F_g \) w.r.t. \( J_k \), this implies that all contributions in a genus expansion of the correlation functions for diagonal entries \( \langle \phi_{kk}...\phi_{ll} \rangle \) are finite and well-defined. The general non-diagonal correlation functions are discussed in section 4.5 and also turn out to be finite for arbitrary genus \( g \) provided \( b \) is finite. Putting these results together we have established renormalizability of the model to all orders in a genus expansion, i.e.

The (connected) genus \( g \) contribution to any given \( n \)-point function is finite and has a well-defined limit \( N \to \infty \) for all \( g \), provided \( b \) is finite.

Moreover, they can in principle be computed explicitly using the above formulas. In particular, since any contribution to \( F_g \) has order at least \( \lambda^{4g-2} \), this implies renormalizability of the perturbative expansion to any order in \( \lambda \).

Next we show that the renormalization conditions\(^6\) \( \langle \phi_{00} \rangle = \langle \tilde{\phi}_{00} \rangle - \frac{b}{\lambda} = 0 \) has indeed a solution with finite \( b \). At genus 0, this amounts to
\[
J_0(\sqrt{1 + 2\frac{b}{J_0} - 1}) = -(i\lambda)^2 \sum_{j=0}^{N} \frac{1}{(J_0^2 + 2b)^{1/2}(J_j^2 + 2b)^{1/2} + (J_0^2 + 2b)}
\]
\[
= -(i\lambda)^2 \left( \sum_{j=0}^{N} \frac{1}{J_0 J_j + J_j^2} - O(b) \right)
\]
\(^6\)we will not bother to impose \( \langle \phi_{00} \rangle = \langle \tilde{\phi}_{00} \rangle - \frac{b}{\lambda} = 0 \) exactly, since \( \mu^2 \) does not require renormalization

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\begin{align*}
\approx & - (i\lambda)^2 \left( \frac{1}{4\pi} \int_{J_0}^{\infty} dJ \frac{1}{J_0 J + J^2} - O(b) \right) \\
= & - (i\lambda)^2 \left( \frac{\ln 2}{4\pi} \frac{1}{J_0} - O(b) \right).
\end{align*}

approximating the sum by an integral. The lhs is an unbounded increasing function of $b$ starting at 0, while the rhs is decreasing and positive for real $\lambda$. Therefore (80) has a unique solution

$$b = b(\lambda) > 0.$$  

(81)

The higher-genus terms contribute only to higher order in $\lambda$, and therefore cannot change this conclusion for small enough $|\lambda|$. In particular, $b \to 0$ as $\lambda \to 0$. This determines $b$ to be finite in the properly renormalized model, which represents together with the mass and coupling constant one of the free physical, properly renormalized parameters of the model. This in turn determines the counterterm $a$ through (69),

$$a = b + \lambda^{4/3} I_0$$  

(82)

which is therefore logarithmically divergent in $N$, as is $I_0$. This will be worked out in more detail below.

**Analytic continuation to real $(i\lambda)$**. It is easy to see that $b(\lambda)$ is analytic in $\lambda$ near the origin, and allows an analytic continuation to real $(i\lambda)$. The analytic property of $b(\lambda)$ can be understood by considering the inverse function

$$(i\lambda)^2 = -J_0(\sqrt{1 + 2\frac{b}{J_0^2}} - 1) \left( \sum_{j=0}^{N} \frac{1}{(J_0^2 + 2b)^{1/2}(J_j^2 + 2b)^{1/2} + (J_j^2 + 2b)} \right)^{-1}$$

$$\approx -J_0^2(\sqrt{1 + 2\frac{b}{J_0^2}} - 1) \left( \frac{4\pi}{\ln 2} + O(b) \right)$$

$$= - \frac{4\pi}{\ln 2} b(1 + O(b)).$$  

(83)

as shown above. This means that

$$b = b(\lambda) = (i\lambda)^2 f((i\lambda)^2)$$  

(84)

where $f$ is an analytic function in $(i\lambda)^2$ near the origin. The higher genus contributions are of higher order in $(i\lambda)^2$ and do not change this conclusion. This implies that $b(\lambda)$ can be analytically continued to to real $\lambda' = (i\lambda)$ in some neighborhood of the origin.

This can be seen more explicity. Consider again the first line of (83) for any real $\lambda' = (i\lambda)$. For $2b \in [-J_0^2,0]$, the function $-J_0\left( \sqrt{1 + 2\frac{b}{J_0^2}} - 1 \right)$ covers the interval $[J_0,0]$, while the term $(\cdot)^{-1}$ covers the interval $[0,\frac{\ln 2}{4\pi J_0}]$. Therefore there exists a

\footnote{The lower bound is approximate}
solution $b(\lambda')$ for real $\lambda = (i\lambda)$ provided $|\lambda'|$ is small enough.

If $J_0^2 + 2b < 0$, some of the eigenvalues $m_i$ of the Kontsevich model become imaginary. It seems unlikely that there is a meaningful solution of (83) in that case. We will not pursue this any further in this paper.

### 4.2 Small coupling expansion and checks

We want to verify that the results derived from the Kontsevich integrals coincide with the leading non-trivial perturbative computations. Only genus 0 contributes here.

Consider first the one-point function $\langle \phi_{kk} \rangle$. Using the fact that $b = O(\lambda^2)$ as shown above, we have using (75)

$$
\langle \phi_{kk} \rangle = \frac{J_k}{i\lambda} \left( \sqrt{1 + 2 \frac{b}{J_k^2}} - 1 \right) + (i\lambda) \sum_{j=0}^{N} \frac{1}{(J_k^2 + 2b)^{1/2} + (J_j^2 + 2b)^{1/2} + (J_k^2 + 2b)}
$$

$$
= \frac{1}{i\lambda} \frac{b}{J_k} + (i\lambda) \sum_{j=0}^{N} \frac{1}{J_k J_j + J_j^2} + O(\lambda^3)
$$

$$
= \frac{1}{J_k} \left( \frac{b}{i\lambda} + (i\lambda) \sum_{j=0}^{N} \left( \frac{1}{J_j} - \frac{1}{J_k + J_j} \right) \right) + O(\lambda^3). \quad (85)
$$

On the other hand, the constraint (71) is to lowest order\(^8\)

$$
(b - a) = \lambda^2 \sum_{i=0}^{N} \frac{1}{J_i} + O(\lambda^4) \quad (86)
$$

and combined with the above this gives

$$
\langle \phi_{kk} \rangle = \frac{1}{J_k} \left( \frac{b}{i\lambda} \sum_{j=0}^{N} \left( \frac{i\lambda}{J_j} - \frac{i\lambda}{J_k + J_j} \right) \right) + O(\lambda^3)
$$

$$
= \frac{1}{J_k} \left( \frac{a}{i\lambda} - \sum_{j=0}^{N} \frac{i\lambda}{J_k + J_j} \right) + O(\lambda^3)
$$

$$
\approx \frac{1}{J_k} \left( \frac{a}{i\lambda} - \frac{i\lambda}{4\pi} \ln \frac{J_k + J_N}{J_k + J_0} \right) + O(\lambda^3). \quad (87)
$$

This agrees precisely with a perturbative computation, as shown in section [5]. We note in particular that the dependence on $k$ of $\langle \phi_{kk} \rangle$ is nontrivial, and we cannot

\(^8\)note that all higher order corrections are finite
require that $\langle \phi_{kk} \rangle = 0$ for all $k$. Setting $\langle \phi_{00} \rangle = 0$ determines the required one-loop counterterm

$$a = (i\lambda)^2 \sum_{j=0}^{N} \frac{1}{J_0 + J_j} + O(\lambda^4) \approx \frac{(i\lambda)^2}{4\pi} \ln \frac{J_0 + J_N}{2J_0} + O(\lambda^4)$$

(88)

which gives

$$b = (i\lambda)^2 \sum_{j=0}^{N} \frac{1}{J_0 + J_j} - \frac{1}{J_j} \approx \frac{(i\lambda)^2}{4\pi} \left( \ln \frac{J_0 + J_N}{J_0 + J_0} - \ln \frac{J_N}{J_0} \right) + O(\lambda^4)$$

$$\approx \frac{(i\lambda)^2}{4\pi} \ln 2 + O(\lambda^4)$$

(89)

for large $N$.

**The propagator** $\langle \phi_{kl} \phi_{lk} \rangle$. We can use (27) for the lowest-order correction to the 2-point function $\langle \phi_{kl} \phi_{lk} \rangle$ for $k \neq l$. First we observe that

$$\langle \phi_{kk} - \phi_{ll} \rangle J_k^2 - J_l^2 + \langle \phi_{kk} + \phi_{ll} \rangle (J_k + J_l)^2 = 2 \left( J_k \langle \phi_{kk} \rangle - J_l \langle \phi_{ll} \rangle \right)$$

$$= \frac{2i\lambda}{(J_k + J_l)^2} \sum_{j=0}^{N} \frac{1}{J_k + J_j} \frac{1}{J_l + J_j} + O(\lambda^3)$$

(90)

using (87). Therefore (27) gives

$$\langle \phi_{kl} \phi_{lk} \rangle = \frac{2}{J_k + J_l} - 2(i\lambda) \langle \phi_{kk} + \phi_{ll} \rangle (J_k + J_l)^2 + \frac{4(i\lambda)^2}{(J_k + J_l)^2} \sum_{j=0}^{N} \frac{1}{J_k + J_j} \frac{1}{J_l + J_j} + O(\lambda^4).$$

(91)

This again coincides with the diagrammatic result (113).

**The propagator** $\langle \phi_{ll} \phi_{kk} \rangle$. As a further example, consider the 2-point function $\langle \phi_{ll} \phi_{kk} \rangle$ for $k \neq l$, which vanishes in the free case. To compute it from the effective action, we need in principle

$$\langle \tilde{\phi}_{ll} \tilde{\phi}_{kk} \rangle - \langle \tilde{\phi}_{kk} \rangle \langle \tilde{\phi}_{ll} \rangle = \frac{i\lambda}{J_l} \frac{i\lambda}{J_k} \left( \frac{\partial}{\partial J_l} \frac{\partial}{\partial J_k} \right) (F_0 + F_1 + ...)$$

(92)

Even though this corresponds to a nonplanar diagram with external legs, it is obtained by taking derivatives of a closed genus 0 ring diagram. Therefore we expect
that only \( F_0 \) will contribute, and indeed the derivatives of \( F_1 \) contribute only to order \( \lambda^4 \). We need
\[
\frac{i\lambda}{J_i} \frac{\partial}{\partial J_i} J_k \frac{\partial}{\partial J_k} F_0 =
\]
\[
= -\frac{i\lambda}{J_i} \frac{\partial}{\partial J_i} \frac{1}{\sqrt{J_i^2 + 2b}} \left( \lambda^{-2}(J_k^2 + 2b) + \lambda^{-2}(a-b) + \sum_{j=0}^{N} \frac{1}{\sqrt{J_j^2 + 2b + \sqrt{J_j^2 + 2b}}} \right)
\]
\[
= -\frac{(i\lambda)^2}{\sqrt{J_k^2 + 2b}} J_i \frac{\partial}{\partial J_i} \left( \sum_{j=0}^{N} \frac{1}{\sqrt{J_j^2 + 2b + \sqrt{J_j^2 + 2b}}} \right)
\]
\[
= (i\lambda)^2 \frac{1}{\sqrt{J_k^2 + 2b}} \frac{1}{\sqrt{J_i^2 + 2b}} \left( \frac{1}{\sqrt{J_j^2 + 2b + \sqrt{J_j^2 + 2b}}} \right)^2.
\]
(93)

Therefore\(^9\) to lowest order we obtain
\[
\langle \phi_u \phi_{kk} \rangle = \langle \phi_{kk} \rangle \langle \phi_u \rangle + (i\lambda)^2 \left( \frac{1}{J_k J_i} \left( \frac{1}{J_k + J_i} \right) \right)^2,
\]
(94)
in complete agreement with the perturbative computation (112).

### 4.3 Approximation formulas for finite coupling

In this section we derive some closed formulas which are appropriate for finite coupling \( \lambda \), in the large \( N \) limit. This is done by approximating the various sums by integrals. Using (74), we have
\[
I_0 = -\lambda^{2/3} \sum_{i=0}^{N} \frac{1}{(J_i^2 + 2b)^{2}} \approx -\frac{\lambda^{2/3}}{\sqrt{2b}} \frac{\sqrt{2b}}{4\pi} \int_{x_0}^{x_N} dx \frac{1}{(x^2 + 1)^{1/2}}
\]
\[
= -\frac{\lambda^{2/3}}{4\pi} \ln \left( \frac{x_N + \sqrt{1 + x_N^2}}{x_0 + \sqrt{1 + x_0^2}} \right)
\]
(95)

where
\[
x_n = \frac{4\pi}{\sqrt{2b}} \left( n + \frac{1 + \mu^2}{2} \right), \quad dx = \frac{4\pi}{\sqrt{2b}} \, dn.
\]
(96)

Furthermore, we will need
\[
I_1 = -\lambda^2 \sum_{i=0}^{N} \frac{1}{(J_i^2 + 2b)^{2}}
\]

\(^9\)This computation was again simplified by ignoring the implicit dependence of \( b \) on the \( J_i \) taking advantage of (62). This is no longer possible for higher derivatives.
\[
\left. \begin{array}{r}
\approx - \frac{\lambda^2}{2b} \frac{\sqrt{2b}}{4\pi} \int_{x_0}^{x_N} dx \frac{1}{(x^2 + 1)^{3/2}} \\
= - \frac{\lambda^2}{2b} \frac{1}{4\pi} \left( \frac{1}{\sqrt{1 + x_N^{-2}}} - \frac{1}{\sqrt{1 + x_0^{-2}}} \right) \\
\approx - \frac{\lambda^2}{2b} \frac{1}{4\pi} \left( 1 - \frac{1}{\sqrt{1 + x_0^{-2}}} \right). 
\end{array} \right. 
\tag{97}
\]

This is valid also for \( b < 0 \) (by analytic continuation) as long as \( 1 + x_0^{-2} > 0 \). Similarly, all \( I_p \) can be approximated by elementary, convergent integrals.

Consider again equation (80), which determines \( b \) as a function of the coupling constant at genus 0. It can be written for finite \( b \) using the above approximation as

\[
J_0(\sqrt{1 + 2b J_0^2} - 1) = - (i\lambda)^2 \sum_{j=0}^{N} \frac{1}{(J_0^2 + 2b)^{1/2}(J_j^2 + 2b)^{1/2} + (J_j^2 + 2b)} \\
\approx - \frac{(i\lambda)^2}{2b} \frac{\sqrt{2b}}{4\pi} \int_{x_0}^{x_N} dx \frac{1}{\sqrt{x_0^2 + 1 \sqrt{x^2 + 1} + x^2 + 1}} \\
\approx - \frac{(i\lambda)^2}{4\pi} \frac{1}{\sqrt{2b} x_0} \ln \left( 1 + \frac{1}{\sqrt{1 + x_0^{-2}}} \right) 
\tag{98}
\]

using

\[
\int_{x_0}^{\infty} dx \frac{x_0}{\sqrt{x_0^2 + 1 \sqrt{x^2 + 1} + (x^2 + 1)}} = \ln \left( 1 + \frac{1}{\sqrt{1 + x_0^{-2}}} \right). 
\tag{99}
\]

Therefore (88) becomes

\[
J_0^2(\sqrt{1 + 2b J_0^2} - 1) = - \frac{(i\lambda)^2}{4\pi} \ln \left( 1 + \frac{1}{\sqrt{1 + \frac{2b}{J_0^2}}} \right). 
\tag{100}
\]

This implies again that there exists a solution \( b = b(\lambda) \) not only for real \( \lambda \), but also for real \( (i\lambda) \) at least in some neighborhood of the origin. We can also recover the leading perturbative result (89),

\[
2b = - \frac{(i\lambda)^2}{4\pi} \frac{1}{x_0^2} \frac{1}{\sqrt{1 + x_0^{-2}}} - 1 \ln(1 + \frac{1}{\sqrt{1 + x_0^{-2}}}) = - \frac{(i\lambda)^2}{4\pi} \ln 2 + O(x_0^{-2}). 
\tag{101}
\]

Using these and similar expressions, one can obtain explicit formulas for the genus expansion of \( F \) and the correlators considered in the previous section, which are appropriate for finite coupling. Since there is no particular difficulty we will not write them down explicitly.
4.4 Critical line and instability.

We have seen that for small enough coupling $|\lambda|$, the free energy $F = F_0 + F_1 + ...$ is regular and finite for any given genus in the renormalized model (i.e. for finite $b$), since all $I_k$ with $k \geq 1$ are finite provided $I_1 \neq 1$.

However, as is manifest in the explicit formulas for $F_g$ at higher genus \[57\] ff., there is a singularity at $I_1 = 1$. Using \[97\], this critical point is given by

$$
1 = I_1 = \frac{(i\lambda)^2}{2b} \frac{1}{4\pi} \left( 1 - \frac{1}{\sqrt{1 + x_0^{-2}}} \right)
$$

for large $N$. The lhs is negative for both real $\lambda$ and real $i\lambda$, as can be seen e.g. from \[83\]. Since $x_0^2 = \frac{2b}{4\pi(1 + \mu^2)}$, \[60\], \[102\] has a solution only for negative $b$ i.e. real $(i\lambda)$. Combining this with \[100\] which is valid for finite $b$, we get

$$
\left( 1 - \frac{1}{\sqrt{1 + x_0^{-2}}} \right) = -\frac{\ln(1 + \frac{1}{\sqrt{1 + x_0^{-2}}})}{x_0^2(\sqrt{1 + x_0^{-2}} - 1)}
$$

which has a unique solution $x_0^{-2} \approx -0.873759$. Inserting this into \[102\] gives

$$
\frac{4\pi^2(1 + \mu^2)^2}{2b} \frac{8\pi b}{(i\lambda)^2} = x_0^2 \frac{8\pi b}{(i\lambda)^2} = x_0^2 \left( 1 - \frac{1}{\sqrt{1 + x_0^{-2}}} \right) \approx 2.07665
$$

$$
\frac{1 + \mu^2}{i\lambda} \approx \pm 0.0646989
$$

This\[10\] indicates that for the $\phi^3$ model with real coupling constant $\lambda' = i\lambda$ stronger than this critical coupling, the model becomes unstable. This is very reasonable, since the potential is unbounded, and the potential barrier around the local minimum becomes weaker for stronger coupling. Therefore this critical line could be interpreted as the point where the quantum fluctuations of $\phi$ become large enough to see the global instability, so that the field “spills over” the potential barrier. Similar transitions for a cubic potential are known e.g. for the ordinary matrix models, but may also be relevant in the context of string field theory and tachyon condensation [13, 19]. In particular, it is interesting to note that this singularity occurs simultaneously for each given genus, which suggest that some double-scaling limit near this critical point can be taken, again in analogy with the usual matrix models (for a review, see e.g. [20]). Again, such a scaling limit for the Kontsevich model is discussed in [9]. We leave this issue for future work.

\[10\] Recall that this is obtained imposing the renormalization conditions $\langle \phi_{00} \rangle = 0$ at genus 0.
4.5 General $n$-point functions

Finally we show that all contributions in the genus expansion (and therefore perturbative expansion) of the expectation values of any $n$-point functions of the form

$$\langle \phi_{i_1j_1}\ldots\phi_{i_nj_n} \rangle$$

(105)

have a well-defined and finite limit as $N \to \infty$ provided $b$ is finite, which means that the model is fully renormalized.

In view of (13), the insertion of a factor $\tilde{\phi}_{ij}$ can be obtained by acting with the derivative operator $2(i\lambda)\frac{\partial}{\partial J_{ij}}$ on $Z(J)$ resp. $F_g(J)$ for fixed genus $g$. We use $J^2$ rather than $J$ to simplify the notation, which is not a problem defining the inverse by the local diffeomorphism given by the positive square-root of a Hermitian matrix with distinct positive eigenvalues (alternatively one could act with $(-i\lambda\frac{\partial}{\partial a_{ij}} + \frac{1}{i\lambda}J_{ji})$ promoting $a$ to a matrix, see (11)). Since $Z(J)$ and $F_g(J)$ depend only on the eigenvalues of $J$, we can diagonalize $J$ as $J = U^{-1}\text{diag}(J_a)U$, and rewrite these derivatives as

$$\frac{\partial}{\partial J_{ij}^2} = \sum_a \frac{\partial J_a}{\partial J_{ij}^2} \frac{\partial}{\partial J_a} + \sum_\alpha \frac{\partial V_\alpha}{\partial J_{ij}^2} \frac{\partial}{\partial V_\alpha}$$

(106)

where $V_\alpha$ denotes suitable coordinates on the coset space $U(N)/(\prod U(1))$ near the origin. Now for any given correlation function of type (105), let $k$ denote the highest index involved. Then taking derivatives of the type (106) amounts to considering matrices $J$ of the form

$$J = \begin{pmatrix} J_{k\times k} & 0 \\ 0 & \text{diag}(J_{k+1},\ldots,J_N) \end{pmatrix}$$

(107)

We can assume that $J_{k\times k}$ is a small perturbation around $\text{diag}(J_1,\ldots,J_k)$ with distinct increasing eigenvalues. Therefore only the first $k$ eigenvalues of $J$ are deformed, while the higher eigenvalues $J_{k+1},\ldots,J_N$ are fixed and given by (8). This means that the diagonalization and the transformation (106) takes place only in the upper-left $k \times k$ block, and is independent of $N$ for $N > k$. In particular, only the coset coordinates $V_\alpha$ for $U(k)/(\prod U(1))$ enter. Therefore acting with multiple operators of type (106) on $F_g(J)$ (resp. $Z(J)$) produces a result which is finite and independent of $N$ except possibly through the terms $\frac{\partial}{\partial J_a}\ldots\frac{\partial}{\partial J_a} F_g(J)$ (resp. $\frac{\partial}{\partial V_\alpha}\ldots\frac{\partial}{\partial V_\alpha} Z(J)$), which in turn were shown to be finite and convergent for any $g$ in section 4.1. This completes the proof that each genus $g$ contribution to the general (connected) correlators $\langle \phi_{i_1j_1}\ldots\phi_{i_nj_n} \rangle$ is finite and convergent as $N \to \infty$. This implies in particular (but is stronger than) renormalizability of the perturbative expansion to any order in $\lambda$. 

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5 Perturbative computations

We write the action (9) as
\[
\tilde{S} = Tr\left( \frac{1}{4} (J\phi^2 + \phi^2 J) + \frac{i \lambda}{3!} \phi^3 - \frac{a}{i \lambda} \phi \right)
\]
\[
= Tr\left( \frac{1}{2} \phi_j^i G_{i;k}^{j;l} \phi_k^l + \frac{i \lambda}{3!} \phi^3 - \frac{a}{i \lambda} \phi \right)
\]  \hspace{1cm} (108)

where the kinetic term is
\[
G_{i;k}^{j;l} = \frac{1}{2} \delta^i_j \delta^k_l \left( J_i + J_j \right),
\]
and the propagator is
\[
\Delta_{j;i}^{i;k} = \langle \phi_j^i \phi_k^l \rangle = \delta^i_j \delta^k_l \frac{2}{J_i + J_j} = \delta^i_j \delta^k_l \frac{1/(2\pi)}{i + j + (\mu^2 \theta + 1)}.
\]  \hspace{1cm} (109)

In 2 dimensions, the only divergent primitive graph is the tadpole, which requires the counterterm \( a \). A one-loop computation gives
\[
\langle \phi_{ii} \rangle = \frac{a}{i \lambda} \left[ \frac{1}{2} \sum_{k=0}^{N} \frac{2}{J_i + J_k} - \frac{i \lambda}{J_i} \left( \frac{a}{\lambda^2} + \frac{1}{4\pi} \ln \frac{J_i + J_N}{J_i + J_0} \right) \right] 
\]
\[
= -\frac{1}{2\pi} \frac{i \lambda}{2i + \mu^2 + 1} \left( \frac{a}{\lambda^2} + \frac{1}{4\pi} \ln \frac{N + i + \mu^2 + 1}{i + \mu^2 + 1} \right)
\]  \hspace{1cm} (110)

In particular, \( \langle \phi_{00} \rangle = 0 \) implies
\[
a + \frac{\chi^2}{4\pi} \ln \frac{N + \mu^2 + 1}{\mu^2 + 1} = 0.
\]  \hspace{1cm} (111)

Next we compute the leading contribution to the 2-point function \( \langle \phi_{ll} \phi_{kk} \rangle \) for \( l \neq k \), which vanishes at tree level. The leading contribution comes from the nonplanar graph in figure 1 which gives

\[
\langle \phi_{ll} \phi_{kk} \rangle = \langle \phi_{kk} \rangle \langle \phi_{ll} \rangle + \frac{1}{4} \frac{(i \lambda)^2}{J_k J_l} \left( \frac{2}{J_k + J_l} \right)^2
\]  \hspace{1cm} (112)

Figure 1: one-loop contribution to \( \langle \phi_{ll} \phi_{kk} \rangle \)
(for \( l \neq k \)) indicating the symmetry factors, where the disconnected contributions are given by (110). This is in complete agreement with the result (94) obtained from the Kontsevich model approach.

Similarly, the leading contribution to the 2-point function \( \langle \phi_{kl} \phi_{lk} \rangle \) for \( l \neq k \), has the contribution indicated in figure 2 which gives the result

\[
\langle \phi_{kl} \phi_{lk} \rangle = \frac{2}{J_k + J_l} - 2(i\lambda) \langle \phi_{kk} + \phi_{ll} \rangle \left( \frac{2}{J_k + J_l} \right)^2 + \frac{4(i\lambda)^2}{(J_k + J_l)^2} \sum_{j=0}^{N} \frac{1}{J_k + J_j} \frac{1}{J_l + J_j} + O(\lambda^4).
\]

The first term is the free propagator, the second term the tadpole contributions including counterterms, and the last term the one-loop contribution in figure 2. This is in complete agreement with the result (91) obtained from the Kontsevich model approach.

One can also check the leading terms for \( F_0, F_1 \) in (59), (60) explicitly. The lowest order contribution to \( F_0 \) is given by the planar diagrams in figure 3 which are given by

\[
F_0^{(1),a} = (i\lambda)^2 \sum_{i,j,k} \frac{2}{J_i + J_j} \frac{2}{J_i + J_k} \frac{2}{J_i + J_l},
\]

\[
F_0^{(1),b} = (i\lambda)^2 \sum_{i,j,k} \frac{2}{J_i + J_j} \frac{2}{2J_i} \frac{2}{J_j + J_k}.
\]

It is not difficult to see that

\[
F_0^{(1)} := \frac{(i\lambda)^2}{2} \left( \frac{1}{6} \text{Tr} \phi^3 \right)_0 = \frac{1}{72} \left( 3F_0^{(1),a} + 9F_0^{(1),b} \right) = \frac{(i\lambda)^2}{6} \left( \sum \frac{1}{J_i} \right)^3 = \frac{\lambda^2 t_0^3}{6}.
\]
(setting $a = 0$ to this lowest order), which indeed coincides with the first term in (59). Similarly, the lowest order contribution to $F_1$ is given by the nonplanar diagrams in figure 3, which sum up to

$$F_1^{(1)} := \left(\frac{i\lambda}{2}\right)^2 \frac{1}{6} Tr\phi^3 \frac{1}{6} Tr\phi^3 \sum_i \left(\frac{1}{J_i}\right)^3 = \frac{\lambda^2}{24} t_1$$

again in agreement with the first term in (60). This illustrates the remarkable role of the variables $t_r$ (44) in the Kontsevich model.

We also would like to point out that the perturbative expressions obtained in this model are very similar to those considered by Connes and Kreimer in [21]. It might be interesting to study their Hopf-algebraic formulation of renormalization in this nonperturbative framework.

6 Discussion and conclusion

In this paper, we have shown that the selfdual NC $\phi^3$ model can be mapped to the Kontsevich (matrix) model, for a suitable external source resp. eigenvalues of the latter. This map works for any even dimensions. We concentrate on the 2dimensional case in this paper, leaving the case of 4 dimensions for a future publication. We showed how to use known results for the Kontsevich model to quantize and renormalize the noncommutative $\phi^3$ model. This provides closed expressions for each given genus $g$ in a genus expansion, which are valid for finite nonzero coupling. The appropriate renormalization is found, and we have shown that the resulting
contributions for each genus are finite and well-defined for nonzero coupling. This implies but is stronger than renormalization order-by-order in perturbation theory. An instability is found if the real coupling constant reaches a critical coupling, as expected for the $\phi^3$ model.

The techniques used in this paper are very powerful, but more-or-less restricted to the $\phi^3$ interaction (note however the possible generalizations of the Kontsevich model pointed out in [9,10,22]). However, one important message is the fact that the required renormalization is determined by the genus 0 contribution only. This can be expected to hold more generally for scalar NC models. Since the genus 0 contribution should accessible more easily and does include essential nonperturbative information, this supports the strategy of applying matrix-methods such as those in [7,8] more generally in the NC case.

Perhaps the main gap in our treatment is the lack of control over the sum over all genera $g$. While the contributions for each genus are manifestly analytic in the coupling constant $\lambda$, we have not shown that the sum over $g$ converges in a suitable sense. However, it seems very plausible that this is the case, and the sum defines an analytic function in $\lambda$ near the origin. This would amount to a full construction of the model. It should be possible to establish this using the relation with the KdV hierarchy or the relation with topological gravity, which is however beyond the scope of this paper.

In view of the well-known relation between NC field theory and string theory in certain backgrounds [11], our results are quite relevant also in that context. Renewed interest arises through the recent discussions of the relevance of the Konsevich model e.g. for open string field theory and open/closed string duality [13], and for branes on Calabi-Yau spaces [14].

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**References**

[1] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **02**, 020 (2000), hep-th/9912072.

[2] H. Grosse and R. Wulkenhaar, *Renormalisation of $\phi^4$ theory on noncommutative $\mathbb{R}^2$ in the matrix base*, JHEP **12**, 019 (2003), hep-th/0307017.

[3] H. Grosse and R. Wulkenhaar, *Renormalisation of $\phi^4$ theory on noncommutative $\mathbb{R}^4$ in the matrix base*, Commun. Math. Phys. **256**, 305 (2005), hep-th/0401128.
[4] V. Rivasseau, F. Vignes-Tourneret, and R. Wulkenhaar, *Renormalization of noncommutative phi**4-theory by multi-scale analysis*, (2005), hep-th/0501036.

[5] E. Langmann and R. J. Szabo, *Duality in scalar field theory on noncommutative phase spaces*, Phys. Lett. **B533**, 168 (2002), hep-th/0202039.

[6] E. Langmann, R. J. Szabo, and K. Zarembo, *Exact solution of quantum field theory on noncommutative phase spaces*, JHEP **01**, 017 (2004), hep-th/0308043.

[7] H. Steinacker, *A non-perturbative approach to non-commutative scalar field theory*, JHEP **03**, 075 (2005), hep-th/0501174.

[8] H. Steinacker, *Quantization and eigenvalue distribution of noncommutative scalar field theory*, (2005), hep-th/0511076.

[9] C. Itzykson and J. B. Zuber, *Combinatorics of the modular group. 2. The Kontsevich integrals*, Int. J. Mod. Phys. **A7**, 5661 (1992), hep-th/9201001.

[10] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Commun. Math. Phys. **147**, 1 (1992).

[11] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **09**, 032 (1999), hep-th/9908142.

[12] O. Andreev and H. Dorn, *Diagrams of noncommutative Phi**3 theory from string theory*, Nucl. Phys. **B583**, 145 (2000), hep-th/0003113.

[13] D. Gaiotto and L. Rastelli, *A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model*, JHEP **07**, 053 (2005), hep-th/0312196.

[14] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa, *Topological strings and integrable hierarchies*, Commun. Math. Phys. **261**, 451 (2006), hep-th/0312085.

[15] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. **73**, 977 (2001), hep-th/0106048.

[16] R. J. Szabo, *Quantum field theory on noncommutative spaces*, Phys. Rept. **378**, 207 (2003), hep-th/0109162.

[17] Y. Makeenko and G. W. Semenoff, *Properties of Hermitean matrix models in an external field*, Mod. Phys. Lett. **A6**, 3455 (1991).

[18] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys Diff. Geom. **1**, 243 (1991).
[19] A. Sen and B. Zwiebach, *Tachyon condensation in string field theory*, JHEP 03, 002 (2000), hep-th/9912249.

[20] P. Di Francesco, P. H. Ginsparg, and J. Zinn-Justin, *2-D Gravity and random matrices*, Phys. Rept. 254, 1 (1995), hep-th/9306153.

[21] A. Connes and D. Kreimer, *Hopf algebras, renormalization and noncommutative geometry*, Commun. Math. Phys. 199, 203 (1998), hep-th/9808042.

[22] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, and A. Zabrodin, *Unification of all string models with c < 1*, Phys. Lett. B275, 311 (1992), hep-th/9111037.