On the zone of a circle in an arrangement of lines

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Abstract

Let \( L \) be a set of \( n \) lines in the plane, and let \( C \) be a convex curve in the plane, like a circle or a parabola. The zone of \( C \) in \( L \), denoted \( Z(C, L) \), is defined as the set of all faces in the arrangement \( A(L) \) that are intersected by \( C \). Edelsbrunner et al. (1992) showed that the complexity (total number of edges or vertices) of \( Z(C, L) \) is at most \( O(n\alpha(n)) \), where \( \alpha \) is the inverse Ackermann function, by translating the sequence of edges of \( Z(C, L) \) into a Davenport–Schinzel sequence \( S \) of order 3. Whether the worst-case complexity of \( Z(C, L) \) is only linear is a longstanding open problem.

Here we show that if \( C \) is a parabola, then \( S \) avoids not only the pattern \( ababa \), but another pattern \( u \) as well. Hence, if \( \text{Ex}(\{ababa, u, u^R\}, n) \) (the maximum length of a sequence with \( n \) distinct symbols that avoids the subsequences \( ababa \), \( u \), and the reversal of \( u \), and contains no adjacent repetitions) could be shown to be \( O(n) \), that would settle the problem for a parabola (and almost certainly also for a circle).

1 Introduction

Let \( L \) be a set of \( n \) lines in the plane. The arrangement of \( L \), denoted \( A(L) \), is the partition of the plane into vertices, edges, and faces induced by \( L \). Let \( C \) be another object in the plane. The zone of \( C \) in \( L \), denoted \( Z(C, L) \), is defined as the set of all faces in \( A(L) \) that are intersected by \( C \). The complexity of \( Z(C, L) \) is defined as the total number of edges, or vertices, in it.

The celebrated zone theorem states that, if \( C \) is another line, then \( Z(C, L) \) has complexity \( O(n) \) (Chazelle et al. [3]; see also Edelsbrunner et al. [5], Matoušek [12]).

If \( C \) is a convex curve, like a circle or a parabola, then \( Z(C, L) \) is known to have complexity \( O(n\alpha(n)) \), where \( \alpha \) is the very-slow-growing inverse Ackermann function (Edelsbrunner et al. [5]; see also Bern et al. [2], Sharir and Agarwal [21]). More specifically, the outer zone of \( Z(C, L) \) (the part that lies outside the convex hull of \( C \)) is known to have complexity \( O(n) \), whereas the complexity of the inner zone is only known to be \( O(n\alpha(n)) \). Whether the complexity of the inner zone is linear as well is a longstanding open problem [2, 21].

In this paper we make progress towards proving that the inner zone of a circle, or a parabola, in an arrangement of lines has linear complexity. The problem is more naturally formulated with a circle, but a parabola is easier to work with. Hence, throughout this paper we take for concreteness \( C \) to be the parabola \( y = x^2 \). The case where \( C \) is a circle is almost certainly equivalent, since a tiny portion of a circle is, at the limit, affinely identical to a parabola.

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\textsuperscript{1}If we allow \( L \) to be a set of pseudolines then the bound \( \Theta(n\alpha(n)) \) is worst-case tight [2].
1.1 Davenport–Schinzel sequences and their generalizations

Let $S$ be a finite sequence of symbols, and let $s \geq 1$ be a parameter. Then $S$ is called a Davenport–Schinzel sequence of order $s$ if every two adjacent symbols in $S$ are distinct, and if $S$ does not contain any alternation $\cdot \cdot \cdot a \cdot b \cdot \cdot \cdot a \cdot b \cdot \cdot \cdot$ of length $s + 2$ for two distinct symbols $a \neq b$. Hence, for $s = 1$ the “forbidden pattern” is $aba$, for $s = 2$ it is $abab$, for $s = 3$ it is $ababa$, and so on.

The maximum length of a Davenport–Schinzel sequence of order $s$ that contains only $n$ distinct symbols is denoted $\lambda_s(n)$. For $s \leq 2$ we have $\lambda_1(n) = n$ and $\lambda_2(n) = 2n - 1$. However, for fixed $s \geq 3$, $\lambda_s(n)$ is slightly superlinear in $n$: We have $\lambda_3(n) = 2\alpha(n) + O(n)$, $\lambda_4(n) = \Theta(n \cdot 2^{\alpha(n)})$, and in general, $\lambda_s(n) = \Theta(n \cdot 2^{\text{poly}(\alpha(n))})$, where the polynomial in the exponent is of degree roughly $s/2$. See Sharir and Agarwal [21], and subsequent improvements by Klazar [10], Nivasch [13], and Pettie [18].

A generalized Davenport–Schinzel sequence is one where the forbidden pattern is not restricted to be $abab\ldots$, but it can be any fixed subsequence $u$. In order for the problem to be nontrivial we must require $S$ to be $k$-sparse—meaning, every $k$ adjacent symbols in $S$ must be pairwise distinct—where $k = \|u\|$ is the number of distinct symbols in $u$. For example, if we take $u = abacacbc$, then $S$ must not contain any subsequence of the form $a \cdot \cdot \cdot b \cdot \cdot \cdot c \cdot a \cdot \cdot \cdot c \cdot a \cdot \cdot \cdot b \cdot \cdot \cdot c$ for $\|\{a, b, c\}\| = 3$, and every three adjacent symbols in $S$ must be pairwise distinct.

We denote by $\text{Ex}(u, n)$ the maximum length of a $k$-sparse, $u$-avoiding sequence $S$ on $n$ distinct symbols, where $k = \|u\|$. For every fixed forbidden pattern $u$, $\text{Ex}(u, n)$ is at most slightly superlinear in $n$: $\text{Ex}(u, n) = O(n \cdot 2^{\text{poly}(\alpha(n))})$, where the polynomial in the exponent depends on $u$ (Klazar [8], Nivasch [13], Pettie [19]).

Similarly, $\text{Ex}(\{u_1, u_2, \ldots, u_j\}, n)$ denotes the maximum length of a sequence that avoids all the patterns $u_1, \ldots, u_j$, is $k$-sparse for $k = \min\{\|u_1\|, \ldots, \|u_j\|\}$, and contains only $n$ distinct symbols.

Here we recall the following known facts:

- $\text{Ex}\{ababa, abacacbc\}, n) = \Theta(n\alpha(n))$ (Pettie [16]). Indeed, the “standard” superlinear-length, $ababa$-free sequences of Hart and Sharir [7] avoid $abacacbc$ as well. See Appendix A

- $\text{Ex}(abacacbc, n) = \Theta(n\alpha(n))$ (Pettie [17]). The lower bound is achieved by a modification of the Hart–Sharir construction, which does not avoid $ababa$ anymore.

- It is unknown whether $\text{Ex}\{ababa, abacacbc\}, n) \text{ or } \text{Ex}\{ababa, abacacbc, (abacacbc)^R\}, n)$ are superlinear in $n$ (where $u^R$ denotes the reversal of $u$). We conjecture that they are both $O(n)$.

Applications of generalized DS sequences Generalized Davenport–Schinzel sequences have found a few applications. Cibulka and Kyncl [4] used them to bound the size of sets of permutations with bounded VC-dimension. Valtr [22] and Fox et al. [6] used the “N-shaped” forbidden pattern $a_1 \cdot \cdot \cdot a_\ell \cdot \cdot \cdot a_1 \cdot \cdot \cdot a_\ell$, and [17] also used the forbidden pattern $(a_1 \cdot \cdot \cdot a_\ell)^m$, to bound the number of edges in graphs with no $k$ pairwise crossing edges.

\footnote{Spaces are just for clarity. The Hart–Sharir construction also avoids other patterns, such as $abcbadacbd$ (Klazar [9]; see Pettie [16]).}
Pettie considered $\text{Ex}(\{abababa, abaabba\}, n)$ for analyzing the deque conjecture for splay trees [14], and $\text{Ex}(\{ababab, abbaabba\}, n)$ for analyzing the union of fat triangles in the plane [15].

1.2 Transcribing the zone into a Davenport–Schinzel sequence

Here we recall the argument of Edelsbrunner et al. [5] showing that the inner complexity of $\mathcal{Z}(C, \mathcal{L})$ is $O(n^{\alpha(n)})$.

Let $\mathcal{L}$ be a set of $n$ lines in the plane, and let $C$ be the parabola $y = x^2$. Assume general position for simplicity: No line is vertical, no two lines are parallel, no three lines are concurrent, no line is tangent to $C$, and no two lines intersect $C$ at the same point. (Perturbing $\mathcal{L}$ into general position can only increase the complexity of $\mathcal{Z}(C, \mathcal{L})$.) We can also assume that every line of $\mathcal{L}$ intersects $C$, since otherwise the line would not contribute to the complexity of the inner zone of $C$.

The lines $\mathcal{L}$ partition the convex hull of $C$ into faces, only one of which is unbounded.

Let $\mathcal{L}'$ be the set of $n$ segments obtained by intersecting each line of $\mathcal{L}$ with the convex hull of $C$. Let $G$ be the intersection graph of $\mathcal{L}'$, i.e. the graph having $\mathcal{L}'$ as vertex set, and having an edge connecting two elements of $\mathcal{L}'$ if and only if they intersect. Then, all the bounded faces in $\text{conv}(C)$ are simple (touch $C$ in a single interval) if and only if $G$ has a single connected component. We can assume without loss of generality that this is the case: If $G$ has several connected components, then we can separately bound the complexity produced by each one and add them up; this works because our desired bound is superlinear in $n$.

The complexity of the unbounded face is at most $n$ (as is the complexity of any single face). To bound the complexity of the remaining faces, we traverse the boundary of the inner zone by starting at the leftmost endpoint of $\mathcal{L}'$, and walking around the boundary of the faces, as if the segments were walls which we touch with the left hand at all times, until we reach the rightmost endpoint of $\mathcal{L}'$. See Figure 1.

Let $S'$ be the sequence resulting from the tour. For each segment $a$, denote its left endpoint by $L_a$ and its right endpoint by $R_a$.

Let $a', a, a'', b', b, b''$ be two intersecting segments, with $L_a$ left of $L_{b'}$. Then the restriction of $S'$ to $\{a', a, a'', b', b, b''\}$ is of the form

$$(a')^* a^* (b')^* b^* a^* (a'')^* b^* (b'')^* (a'')^* \quad \text{or} \quad (b')^* (a')^* a^* (b')^* b^* a^* (a'')^* b^* (b'')^*,$$

where * denotes zero or more repetitions.
Hence, the restriction of $S'$ to first-type symbols contains no alternation $abab$, and it contains no adjacent repetitions either, as can be easily seen. Hence, it is an order-2 DS-sequence and so it has linear length. The same is true for the restriction of $S'$ to third-type symbols.

Thus, the important part of the sequence $S'$ is its restriction to second-type symbols. From now on we denote this subsequence $S$, and we call it the lower inner-zone sequence of $Z(C, L)$. The sequence $S$ contains no alternation $ababa$, and it contains no adjacent repetitions, as can be easily seen. Hence, $S$ is an order-3 DS-sequence, and hence its length is at most $O(n^{\alpha(n)})$.

1.3 Our results

In this paper we offer some evidence for the following conjecture, and make some progress towards proving it:

**Conjecture 1.** If $\mathcal{L}$ is a set of $n$ lines and $C$ is the parabola $y = x^2$, then the lower inner-zone sequence $S$ of $Z(C, \mathcal{L})$ has length $O(n)$, and hence, $Z(C, \mathcal{L})$ has at most linear complexity.

We first show in Section 3 that $S$ avoids a certain pattern $u'$ of length $|u'| = 37$ and alphabet size $\|u'\| = 11$. This, result, however, is useless for establishing Conjecture 1, since $u'$ contains both $abacacbc$ and its reversal. Therefore, by the above-mentioned result of Pettie, the Hart–Sharir construction avoids both $u'$ and $(u')^R$ (which is actually the same as $u'$), and so $\text{Ex}(\{ababa, u', (u')^R\}, n) = \Theta(n^{\alpha(n)})$.

Section 3 is just a warmup for Section 4 in which we construct another forbidden pattern $u$. Unfortunately, $|u| = 2157$ and $\|u\| = 665$. However, the Hart–Sharir construction does not avoid $u$—in fact, that is where we took $u$ from. (The Hart–Sharir construction does avoid $u^R$.) Therefore, as far as we know, it might be that $\text{Ex}(\{ababa, u, u^R\}, n)$ is linear in $n$, which would imply Conjecture 1.

Our hope seems to hang from a very thin thread: How hard can it be to make a tiny modification in Hart–Sharir construction so that it avoids $ababa$ as well as the humongous
pattern $u$? However, in Section 5 we discuss why there seems to be a fundamental barrier to achieving a lower inner-zone sequence $S$ of superlinear length. Hence, in our opinion, Conjecture [1] rests on reasonably solid ground.

1.4 Relation to lower envelopes

If $\mathcal{F} = \{f_1, \ldots, f_n\}$ is a collection of $x$-monotone curves in the plane (continuous functions $\mathbb{R} \to \mathbb{R}$), then the \textit{lower envelope} of $\mathcal{F}$ is their pointwise minimum (or the part that can be seen from the point $(0, -\infty)$), and the lower-envelope \textit{sequence} is the sequence of functions that appear in the lower envelope, from left to right. If the $f_i$’s are partially defined functions (say, each one is defined only on an interval of $\mathbb{R}$), then the definition is the same, except that the symbol “$\infty$” might also appear in the lower-envelope sequence.

In our case, the lower-envelope sequence of the set of segments $\mathcal{L}'$ is a subsequence of $S$: It contains only those parts that can be seen from $-\infty$. We shall denote this sequence by $N = N(\mathcal{L}')$.

In general, the lower-envelope sequence of a set of $n$ line segments can have length as large as $na(n) - O(n)$ (Wiernik and Sharir [23]; see also [12, 21]). However, it is unknown whether superlinear length can be achieved when all the endpoints lie on a circle/parabola (like our set $\mathcal{L}'$), or more generally on a convex curve. Sharir and Agarwal raise this question in [21, p. 112]. Proving a linear upper bound for the length of $N$ might be easier than for the length of $S$.

2 Geometric preliminaries

\textbf{Observation 2.} Let $a, b \in \mathbb{R}$ be fixed. Then the affine transformation $m : \mathbb{R}^2 \to \mathbb{R}^2$ given by $m(x, y) = (ax + b, 2abx + a^2y + b^2)$ maps the parabola $C$ to itself and keeps vertical lines vertical. Therefore, we are free to horizontally translate and scale the $x$-coordinates of the endpoints of the segments, without affecting the resulting lower inner-zone sequence $S$ or the lower-envelope sequence $N$.

\textbf{Definition 1.} Let $a_1, a_2, \ldots, a_n$ be pairwise intersecting nonvertical segments, listed by increasing slope. These segments are said to \textit{intersect concavely} if the points of intersection $a_1 \cap a_2, a_2 \cap a_3, \ldots, a_{n-1} \cap a_n$, in this order, are ordered from right to left. If they are ordered from left to right, then the segments are said to \textit{intersect convexly}. See Figure 2.

\footnote{See [3] on how to avoid losing a factor of 2 in the interpolation step.}
**Observation 3.** Suppose segments $a_1, a_2, \ldots, a_n$ intersect concavely (or convexly), and let $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ be increasing indices. Then $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ also intersect concavely (or convexly).

**Observation 4.** Let $S$ be lower inner-zone sequence of the parabola $C$ in an arrangement of lines $A(L)$. Let $L'$ be the set of segments corresponding to $L$. Then,

1. If $S$ contains the subsequence $abab$, then segments $a, b \in L'$ cross, and their endpoints are ordered $L_a, L_b, R_a, R_b$ from left to right.

2. If $S$ contains the subsequence $abcba$, then $R_c$ lies left of $L_a$. Similarly, if $S$ contains $cbcba$, then $R_c$ lies left of $R_a$.

3. If $S$ contains $axaxbx$ or $axaxyby$, then $R_a$ lies left of $L_b$.

**Lemma 5.** Let $a, b, c, d$ be four points on the parabola $C$, having increasing $x$-coordinates $a_x < b_x < c_x < d_x$. Let $z = ac \cap bd$. Define the horizontal distances $p = b_x - a_x$, $q = d_x - c_x$, $r = z_x - b_x$, $s = c_x - z_x$. Then $p/q = r/s$. See Figure 3 left.

**Proof.** By a straightforward algebraic calculation.

**Lemma 6.** Let $a, b, c, d, e, f$ be six points on the parabola $C$, listed by increasing $x$-coordinate. Suppose the segments $ad, be, cf$ intersect concavely. Define $\alpha_1 = b_x - a_x$, $\alpha_2 = c_x - b_x$, $\gamma = d_x - c_x$, $\beta_1 = e_x - d_x$, $\beta_2 = f_x - e_x$. Then:

1. $\alpha_1/\beta_1 > \alpha_2/\gamma$ and $\beta_2/\alpha_2 > \beta_1/\gamma$;

2. $\alpha_1/\beta_1 > \alpha_2/\beta_2$;

3. Either $\beta_1 < \beta_2$ or $\alpha_2 < \gamma + \beta_1 + \beta_2$ (or both).

See Figure 3 right.

**Proof.** Let $g = bc \cap cf$, $h = ad \cap be$. Subdivide $\gamma$ into $\gamma_1 = g_x - c_x$, $\gamma_2 = h_x - g_x$, $\gamma_3 = d_x - h_x$. By Lemma 5 we have

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2 + \gamma_1 + \gamma_2}{\gamma_3}, \quad \frac{\alpha_2}{\beta_2} = \frac{\gamma_1}{\gamma_2 + \gamma_3 + \beta_1};$$
from which the first two claims follow.

By the first claim we have

\[
\frac{\beta_1}{\beta_2} < \frac{\gamma}{\alpha_2} < \frac{\gamma + \beta_1 + \beta_2}{\alpha_2},
\]

hence, if \( \beta_1/\beta_2 \) is larger than 1, then so is \((\gamma + \beta_1 + \beta_2)/\alpha_2\), implying the third claim.

We shall call a set of concavely intersecting segments with endpoints on \( C \) a fan.\footnote{Matoušek in \cite{12} gives the word \textit{fan} a different, but closely related, meaning.} Let \( s_1, s_2, \ldots, s_n \) be a fan, with the segments listed by increasing slope. Denote the left and right endpoint of each \( s_i \) by \( a_i \) and \( b_i \), respectively. If \( b_{kx} - a_{1x} > 2(b_{(k-1)x} - a_{1x}) \) for each \( 2 \leq k \leq n \), then we shall call the fan \textit{wide}.

\textbf{Lemma 7.} Let \( s_1 = a_1b_1, s_2 = a_2b_2, \ldots, s_n = a_nb_n \) be a wide fan, and let \( \alpha_k = a_{(k+1)x} - a_{kx} \) for \( 1 \leq k \leq n - 1 \). Then \( \alpha_k > \alpha_{k+1} + \cdots + \alpha_{n-1} \) for each \( 1 \leq k \leq n - 2 \).

\textbf{Proof.} Let \( \gamma_k = b_{kx} - a_{1x} \) for \( 1 \leq k \leq n \). See Figure 4. We are given that \( \gamma_k > 2\gamma_{k-1} \) for each \( 2 \leq k \leq n \). Applying the first claim of Lemma 6 to segments \( s_k, s_{k+1}, s_n \), we get

\[
\frac{\alpha_k}{\alpha_{k+1} + \cdots + \alpha_{n-1}} > \frac{\gamma_{k+1} - \gamma_k}{\gamma_k - \sum \alpha_i} > \frac{\gamma_{k+1} - \gamma_k}{\gamma_k} > 1.
\]

The claim follows.

\section{Warmup: A simple but useless forbidden pattern}

\textbf{Theorem 8.} Let \( S \) be the lower inner-zone sequence of the parabola \( C \) in an arrangement of lines. Then \( S \) cannot contain a subsequence isomorphic to

\[
u' = 81a1b12181c1d12dedc3ab34b4c49434d4e49.
\]
Figure 5: Impossible realization of the lower inner-zone sequence $u'$. 

Note that $u'$ has length $|u'| = 37$ and alphabet size $\|u'\| = 11$.

**Proof.** By repeated application of Observation 4, the endpoints of our segments must appear in the following order along the parabola:

$L_8, L_1, L_{a}, L_{b}, L_{2}, R_8, L_c, L_d, R_1, R_2, L_e, R_a, L_{3}, L_4, R_b, R_c, L_9, R_3, R_d, R_e, R_4, R_9.$

(For example, the order $L_8, L_1$ follows from the subsequence 8181; the order $L_b, L_2$ follows from $b1212$; the order $R_8, L_c$ follows from $8181c1c$; and so on.)

Furthermore, the intersection point of segments 1 and 2, which we shall call $A$, must lie left of $R_8$, and the intersection point of segments 3 and 4, which we shall call $B$, must lie right of $L_9$. See Figure 5.

Define:

$$
\alpha_1 = L_{bx} - L_{ax}, \quad \beta_1 = R_{bx} - R_{ax}, \\
\alpha_2 = L_{cx} - L_{bx}, \quad \beta_2 = R_{cx} - R_{bx}, \\
\alpha_3 = L_{dx} - L_{cx}, \quad \beta_3 = R_{dx} - R_{cx}, \\
\alpha_4 = L_{ex} - L_{dx}, \quad \beta_4 = R_{ex} - R_{dx}.
$$

Segments $a, b, c, d, e$ must intersect concavely, so by the second claim of Lemma 6 we must have

$$
\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2} > \frac{\alpha_3}{\beta_3} > \frac{\alpha_4}{\beta_4}.
$$

(1)
We will show, however, that this is impossible. Define:

\[ p = L_{2x} - L_{1x}, \]
\[ p' = L_{4x} - L_{3x}, \]
\[ r = A_x - L_{2x}, \]
\[ r' = B_x - L_{4x}, \]
\[ s = R_{1x} - A_x, \]
\[ s' = R_{3x} - B_x, \]
\[ q = R_{2x} - R_{1x}, \]
\[ q' = R_{4x} - R_{3x}. \]

Then,

\[ \alpha_1 < p, \quad \beta_1 > p', \]
\[ \alpha_2 > r, \quad \beta_2 < r', \]
\[ \alpha_3 < s, \quad \beta_3 > s', \]
\[ \alpha_4 > q, \quad \beta_4 < q'. \]

Furthermore, by Lemma 5 we have \( p/q = r/s, \) \( p'/q' = r'/s'. \) Hence,

\[ \frac{\alpha_1 \alpha_3}{\beta_1 \beta_3} < \frac{ps}{p's'} = \frac{qr}{q'r'} < \frac{\alpha_2 \alpha_4}{\beta_2 \beta_4}, \]

contradicting (1).

If we are only interested in a pattern avoided by \( N, \) the lower-envelope sequence, then we can omit the symbols 8 and 9 from \( u'. \) Their only role is preventing the intersection points \( A \) and \( B \) from “hiding” above the segment \( c. \)

Unfortunately, as we said in the Introduction, the forbidden pattern \( u' \) is useless for establishing Conjecture 1, since \( u' \) contains both \( abacakbc \) and its reversal (e.g., \( be4b44e4, 1a11d1d1ad). \) Furthermore, there does not seem to be a simple way to “fix” \( u'. \)

4 A more promising forbidden pattern

We now construct a forbidden pattern \( u \) which is quite long, though, to the best of our knowledge, might still satisfy \( \text{Ex}(\{ababa, u, u^R\}, n) = O(n). \) At the very least, the Hart–Sharir construction does eventually contain \( u; \) indeed, that is where we took \( u \) from; see Appendix A.

To define \( u \) we shall consider sequences \( A \) that are partitioned into contiguous blocks. Some of these blocks in \( A \) are \textit{special}, and they satisfy the following two properties:

- All special blocks in \( A \) have the same length.
- Special blocks entirely consist of \textit{first} occurrences of symbols (hence, every symbol appears in at most one special block).

We denote special blocks by enclosing them in parentheses.

\textbf{Definition 2.} Let \( A \) be a sequence that has \( k \) special blocks of length \( m, \) and let \( B \) be a sequence that has \( \ell \) special blocks of length \( k. \) Then the \textit{shuffle} of \( A \) and \( B, \) denoted \( A \circ B, \) is a new sequence having \( k\ell \) special blocks of length \( m + 1, \) formed as follows: We make \( \ell \) copies of
of \( A \) (one for each special block of \( B \)), each one having “fresh” symbols that do not occur in \( B \) or in any other copy of \( A \).

For each special block \( \Gamma_i = a_1 a_2 \cdots a_k \) in \( B \), \( 1 \leq i \leq \ell \), let \( A_i \) be the \( i \)-th copy of \( A \). For each special block \( \Delta_j \) in \( A_i \), \( 1 \leq j \leq k \), we insert the symbol \( a_j \) at the end of \( \Delta_j \) (so its length grows from \( m \) to \( m+1 \)) and we duplicate the \( m \)-th symbol of \( \Delta_j \) immediately after \( \Delta_j \). Then we place another copy of \( a_k \) immediately after \( A_i \). Call the resulting sequence \( A_i' \).

Then \( A \circ B \) is obtained from \( B \) by replacing each special block \( \Gamma_i \) in it by \( A_i' \).

In the construction of \( A \circ B \), the copies of \( A \) are called \textit{local}, and the copy of \( B \) is called \textit{global}.

For example, let \( A = (a)(b)(c)\) and \( B = (123)21(456)5414525636 \). Then,

\[
A \circ B = (a1)\textit{a}(b2)b(c3)\textit{cb}abc \quad 21 \quad (a'4)a'(b'5)b'(c'6)c'b'a'b'c'6 \quad 5414525636.
\]

Figure 6 shows how \( A \), \( B \), and \( A \circ B \) must be realized, by Observation \[4\].

If \( |v| \) and \( \|v\| \) denote, respectively, the length and alphabet size of a sequence \( v \), then we have

\[
|A \circ B| = |B| + \ell(|A| + k + 1),
\]

\[
\|A \circ B\| = \|B\| + \ell\|A\|.
\]

Let us define the sequences

\[
K = 12(3)(4)56543212(7)273(8)38456(9)69
\]

and, for \( m \geq 2 \),

\[
L_m = abc(x_1 \cdots x_m) \ d x_m \cdots x_1 \ c x_1 \cdots x_m \ d b e b a f (y_1 \cdots y_m) \ g y_m \cdots y_1 \ f y_1 \cdots y_m \ g b e.
\]

In a realization of \( K \), the relative order of all the endpoints is determined, except for the right endpoints \( R_7 \), \( R_8 \), and \( R_9 \). In a realization of \( L_m \), the relative order of all the endpoints is determined, except for \( L_c \), \( R_d \), \( L_f \), and \( R_g \). See Figure 7.
Theorem 9. The lower inner-zone sequence of the parabola C in an arrangement of lines cannot contain a sequence isomorphic to

\[ u = K \circ (((L_1 \circ L_2) \circ L_4) \circ L_8) \circ L_{16} \]  

(nor to \(u^R\)).

We have \(|K| = 25, \|K\| = 9, |L_m| = 17 + 6m, \|L_m\| = 7 + 2m\). Therefore \(|u| = 2157\) and it contains \(\|u\| = 665\) distinct symbols.

Definition 3. Let \(\mathcal{L}'\) be a set of segments with endpoints in \(C\), let \(z\) be the rightmost endpoint of \(\mathcal{L}'\), and let \(s \in \mathcal{L}'\) be a segment. We say \(s\) is short if \(z_x - R_{sx} > R_{sx} - L_{sx}\).

Lemma 10. In a realization of \(L_m\), either all the segments \(x_1, \ldots, x_m\) or all the segments \(y_1, \ldots, y_m\) must be short.

Proof. By the third claim of Lemma \(5\) considering the segments \(a, b, e\).

Remark 1. If we could find a sequence in which a specific segment must certainly be short (unlike the sequence \(L_1\), in which one of two segments must be short), we could avoid the explosive growth in the length of \(u\).

Corollary 11. The sequence \(\cdots (((L_1 \circ L_2) \circ L_4) \circ L_8) \circ L_{2^n}\) has \(2^{n+1}\) special blocks of length \(n+1\), composed of \(x\)'s and \(y\)'s. In a realization of the sequence, each special block corresponds to a fan, and at least one of the fans must be wide.

Proof of Theorem 9. The sequence \(((L_1 \circ L_2) \circ L_4) \circ L_{16}\) has 32 special blocks of length 5. One of them, denote it \(z_1 z_2 z_3 z_4 z_5\), must correspond to a wide fan, and therefore Lemma \(7\) applies to it. Let \(p_1, \ldots, p_5\) denote the left endpoints of \(z_1, \ldots, z_5\), respectively. When this special block is shuffled with a copy of \(K\), the endpoints of the segments \(z_1, \ldots, z_5, 1, \ldots, 6\) must appear in the following order:

\[ L_1, L_2, L_3, p_1, L_4, p_2, L_5, L_6, R_1, p_3, R_2, p_4, R_3, R_4, R_5, p_5, R_6. \]
Hence, we have both

\[ R_{3x} - R_{1x} > p_{4x} - p_{3x} > p_{5x} - p_{4x} > R_{5x} - R_{3x} \]

and

\[ L_{5x} - L_{3x} > p_{2x} - p_{1x} > p_{5x} - p_{2x} > R_{5x} - L_{5x}, \]

contradicting the third claim of Lemma 6 on the segments 1, 3, 5.

5 Discussion

Our argument supporting Conjecture 1 is as follows:

The only known way to construct a set of line segments whose lower-envelope sequence has superlinear length is to realize the Hart–Sharir sequences (described in Appendix A below). (There are two constructions that do this: the original one by Wiernik and Sharir [23], described in Chapter 4.2 of [21]; and a simpler one by Shor, described in Chapter 4.3 of [21] and in Chapter 7 of [12].) However, as we have proven, the Hart–Sharir sequences cannot be realized as lower inner-zone sequences of a parabola in an arrangement of lines.

Specifically, what goes wrong in Shor’s construction is the following: In the construction, very wide fans are created, and they are then shuffled into other fans. In order to do this shuffling, the segments of the global wide fan are given slopes \(1 + \varepsilon_1, 1 + \varepsilon_2, \ldots\) for very small values of \(\varepsilon_1, \varepsilon_2, \ldots\). Then the distances between the left endpoints of the global fan have a lot of freedom, which allows us to do the shuffling properly while still making sure the global fan intersects concavely.

However, if we want all endpoints to lie on a parabola, then the slopes in the global wide fan must increase very rapidly, which leads to the absurd requirement that the distances between the left endpoints decrease very rapidly. Then it is impossible to properly shuffle the global fan into some local structure.

A possible line of attack Perhaps the Hart–Sharir sequences are the only way to achieve superlinear-length ababa-free sequences. Meaning, perhaps for every Hart–Sharir sequence \(S_k(m)\) we have \(\text{Ex}\{\{ababa, S_k(m), (S_k(m))^R\}, n\} = O(n)\). That would imply Conjecture 1.

This is known to be true for \(k = 1\), since \(S_1(m)\) are N-shaped sequences [11, 15]. The first open case is for \(S_2(2) = abacdacb\) (which is the same as \((S_2(2))^R\)). (However, as we mentioned in the Introduction, even the weaker conjecture, that \(\text{Ex}\{\{ababa, abacbc\}, n\} = O(n)\), is still open.)

Related open problems

• What if we do not require \(C\) to be a parabola, but only a convex curve? It still seems impossible to implement the above-mentioned construction.

• The Hart–Sharir sequences have length \(n\alpha(n) - O(n)\). However, Nivasch [13] constructed Davenport–Schinzel sequences of order 3 of length \(2n\alpha(n) - O(n)\). Can they be realized as lower envelopes of line segments? We can perhaps attack this question by finding some forbidden patterns.
• The longest Davenport–Schinzel sequences of order 4 (ababab-free) have length \( \Theta(n \cdot 2^{\alpha(n)}) \). However, no one knows how to realize them as lower-envelope sequences of parabolic segments. Perhaps it is impossible. One could start by finding forbidden patterns here as well.

• Higher dimensions: Raz [20] recently proved that the combinatorial complexity of the outer zone of the boundary of a convex body in an arrangement of hyperplanes in \( \mathbb{R}^d \) is \( O(n^{d-1}) \). The complexity of the inner zone is only known to be \( O(n^{d-1} \log n) \) (Aronov et al. [1]). Whether the latter is also linear in \( n \) is an open question.

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A The Hart–Sharir superlinear-length ababa-free sequences

In this appendix we do three things:

• We recall the Hart–Sharir construction \[7\] of superlinear-length Davenport–Schinzel sequences of order 3.

• We recall Pettie’s proof \[16\] that these sequences avoid not only ababa but also abcacbc\[5\] so the forbidden pattern \(u'\) of Section \[3\] is useless.

• We show that these sequences do not avoid the forbidden pattern \(u\) of Setion \[4\] so there is a chance that this pattern is useful\[6\].

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\[5\] They do not avoid \((abcacbc)^R\), however.

\[6\] They do avoid \(u^R\), since \(u^R\) contains abcacbc.
We define a two-dimensional array of sequences $S_k(m)$, which satisfy the following properties:

- $S_k(m)$ contains a number of special blocks, each of length $m$.
- Each special block is composed of first occurrences of symbols.
- Each symbol in $S_k(m)$ makes its first appearance in a special block.
- $S_k(m)$ contains no adjacent repetitions and no alternation $ababa$.

The sequences are defined as follows. For $k = 1$ we let

$$S_1(m) = \left(12\cdots(m-1)m\right)(m-1)\cdots212\cdots m,$$

with a single special block of length $m$. (Actually, of all sequences $S_1(m)$, we only use $S_1(2) = (12)(12)$.)

For $m = 1$, $k \geq 2$, we let $S_k(1)$ be equal to $S_{k-1}(2)$, but with each special block of size 2 split into two special blocks of size 1. Finally, for $m, k \geq 2$, we let

$$S_k(m) = S_k(m-1) \circ S_{k-1}(N),$$

using the shuffle operation defined in Section 4, where $N$ is the number of special blocks in $S_k(m-1)$.

Thus, we have, up to a renaming of the symbols,

- $S_2(1) = (1)(2)12$,
- $S_2(2) = (12)1(34)313424$,
- $S_2(3) = (123)21(456)5414525636$,
- $S_2(4) = (1234)321(5678)76515626737848$,

$$
\vdots$$

- $S_3(1) = (1)(2)1(3)(4)313424$,
- $S_3(2) = (12)(34)31(56)5(78)75157378642$
  $\quad (9A)9(BC)B9(DE)D(FG)FD9DFBFGECA2AC4C6EG8G$,

$$
\vdots$$

- $S_4(1) = (1)(2)1(3)(4)31(5)(6)5(7)(8)75157378642$
  $\quad (9)(A)9(B)(C)B9(D)(E)D(F)(G)FD9DFBFGECA2AC4C6EG8G$,

$$
\vdots$$

Note that, in the construction of $S_k(m)$, the special blocks of $S_{k-1}(N)$ “dissolve”, and the only special blocks present in $S_k(m)$ are those that come from the copies of $S_k(m-1)$ (enlarged by one).

If $A$ and $B$ are two sequences with special blocks in them, we say that $B$ contains $A$ if $B$ contains a subsequence $A'$ that not only is isomorphic to $A$, but for every two symbols in $A$ that lie in a common special block, the corresponding symbols in $A'$ also lie in a common special
block, and for every two symbols in \( A \) that lie in different special blocks, the corresponding symbols in \( A' \) also lie in different special blocks. For example, \((12)(3)213\) contains \(ab(c)\), but not \(a(bc)\) nor \((a(b))(c)\).

**Lemma 12.** Each special block \((1 \cdots m)\) in \(S_k(m)\) is immediately followed by \((m-1) \cdots 1\).

**Proof.** By induction. \(\square\)

**Observation 13.** Let \( N \) be the number of special blocks in \(S_k(m)\). Then \(S_k(m+1)\) contains \(S_k(m)\) using the first \( N \) special blocks of \(S_k(m+1)\), and using the first \( m \) symbols of each special block. Hence, this property carries on to \(S_k(m+c)\) for all \( c \geq 1 \).

**Lemma 14.** Let \( N \) be the number of special blocks in \(S_k(m)\). Then \(S_{k+1}(m)\) contains \(S_k(m)\) using the first \( N \) special blocks of \(S_k(m)\).

**Proof.** By induction. The case \( m = 1 \) follows since \(S_{k+1}(1)\) is obtained from \(S_k(2)\), and \(S_k(1)\) is obtained from \(S_{k-1}(2)\), and the claim is true for \(S_k(2)\) by induction. (The case \( m = 1 \) does not follow from the fact that \(S_k(2)\) contains \(S_k(1)\), since the resulting copy of \(S_k(1)\) in \(S_{k+1}(1)\) would skip some special blocks.)

Now assume \( m \geq 2 \). Let us see how the desired property carries on from \(S_{k+1}(m)\) to \(S_{k+1}(m+1)\): We have

\[
S_k(m + 1) = S_k(m) \circ S_{k-1}(N),
S_{k+1}(m + 1) = S_{k+1}(m) \circ S_k(N') \quad \text{for some } N' \geq N.
\]

Thus, \(S_k(m + 1)\) contains a number local copies of \(S_k(m)\); and \(S_{k+1}(m + 1)\) contains an even larger number of local copies of \(S_{k+1}(m)\), which by induction contains \(S_k(m)\) using its first \( N \) special blocks. The same operation that is applied on the global copy \(S_{k-1}(N)\) in constructing \(S_k(m+1)\), is applied on the global copy \(S_k(N')\) in constructing \(S_{k+1}(m+1)\), with one exception: The last \((N\text{-th})\) symbol in each special block of \(S_{k-1}(N)\) is duplicated just after the local copy, but this is not true for the \(N\text{-th} \) symbol of \(S_k(N')\), if \(N' > N\). But we do not need this duplication, since the \(N\text{-th} \) symbol immediately appears after the special block, by Lemma 12. \(\square\)

**Corollary 15.** \(S_{k'}(m')\) contains \(S_k(m)\) for all \(k' \geq k, m' \geq m\).

**Lemma 16.** \(S_k(m)\) does not contain \(ababa\) nor \(abcacbc\).

**Proof.** For the first claim, suppose for a contradiction that \(k \) and \(m \) are minimal such that \(S_k(m)\) contains \(ababa\). Recall that \(S_k(m) = S_k(m-1) \circ S_{k-1}(N)\). Each of the symbols \(a, b\) must have come either from a copy of \(S_k(m-1)\) (in which case we call the symbol local) or from \(S_{k-1}(N)\) (in which case we call it global). A case analysis shows that none of the possibilities work. For example, it cannot be that \(a\) is local and \(b\) is global, because then there would be at most one \(b\) between two \(a\)'s. It cannot be either that \(a\) is global and \(b\) is local, because then only the first \(a \) could appear between two \(b\)'s.

For the second claim, we first show by induction that \(S_k(m)\) does not contain \((bc)c\cb\), and if it contains \(bc)c\cb\ then \(c\) is not the last symbol in the special block. Therefore, if \(k,m\) are minimal such that \(S_k(m)\) contains \(abcacbc\), then it cannot be that \(a\) is local and \(b, c\) are global. The other possibilities do not work either. \(\square\)
Recall that

\[ K = 12(3)(4)\cdot 56543212(7)\cdot 273(8)\cdot 38456(9)\cdot 69, \]
\[ L_m = abc (x_1 \cdots x_m) d x_m \cdots x_1 c x_1 \cdots x_m \cdot d\text{beba}\text{f} (y_1 \cdots y_m) g y_m \cdots y_1 f y_1 \cdots y_m \cdot g \text{be}, \]
\[ u = K \circ (((L_1 \circ L_2) \circ L_4) \circ L_8) \circ L_{16}. \]

**Lemma 17.** \( L_m \) is contained in \( S_6(m + 3) \). Furthermore, if we want the symbols \( x_1, \ldots, x_m, y_1, \ldots, y_m \) to lie in specific positions \( 2 \leq i_1 < \cdots < i_m \) within their special blocks, then \( L_m \) is contained in this desired way in \( S_6(i_m + 3) \).

**Proof.** We build \( L_m \) piece by piece.

- \( S_2(3) \) contains \((?b)b()b\), and \( S_3(2) \) contains \((a())a\).
- Therefore \( S_3(3) \) contains \((a()b)ab()b\) (taking \( S_3(2) \) as local and \( S_2(8) \), which contains \( S_2(3) \), as global).
- Therefore, \( S_3(4) \) contains \((a()b?)bab()b\) (taking \( S_3(3) \) as local).
- Therefore \( S_4(2) \) contains \((a()b()bab)()b\), and so \( S_4(3) \) contains \((a()b?)bab()b\).
- \( S_5(3) \) contains \((??e)e\); therefore, \( S_4(4) \) contains \((a()b?)bab()b()be\), so \( S_5(2) \) contains \( ab()bebab()be\).
- \( S_1(m+2) \) contains \( c(x_1 \cdots x_m) d x_m \cdots x_1 c x_1 \cdots x_m d \), which is the same as \( f(y_1 \cdots y_m) g y_m \cdots y_1 f y_1 \cdots y_m g \); therefore, so does \( S_6(m + 2) \).
- Therefore, \( S_6(m + 3) \) contains \( L_m \) (taking \( S_6(m + 2) \) as local and \( S_5(N) \), which contains \( S_5(2) \), as global). Here, the symbols \( \{x_1, y_1\}, \{x_2, y_2\}, \ldots \) lie in consecutive positions \( 2, 3, \ldots, m + 1 \) within their special blocks.

For the second claim, find \( L_{m'} \) for \( m' \geq m \), and ignore the undesired \( x \)'s and \( y \)'s.

**Corollary 18.** \( S_7(6) \) contains \( L_1 \circ L_2 \).

**Proof.** We know that \( S_7(4) \) contains \( L_1 \), in which there are two special blocks \( (x_1) \) and \( (y_1) \). Therefore, \( S_7(5) \) contains a local copy of \( S_7(4) \) that is neither the first nor the last. Therefore, in \( S_7(6) = S_7(5) \circ S_6(N) \), there is a copy of \( L_2 \) for which its global version of the symbols \( x_1, x_2, y_1, y_2 \) get into the right places.

**Corollary 19.** \( S_7(n + 5) \) contains \( \cdots (((L_1 \circ L_2) \circ L_4) \circ \cdots) \circ L_{2n} \).

**Lemma 20.** \( K \) is contained in \( S_4(2) \).

**Proof.** We can verify that \( S_3(6) \) contains \((123456)\cdot 543212(7)\cdot 273(8)\cdot 38456(9)69 \). Therefore, \( S_4(2) \), which uses \( S_3(N) \) as global, contains this sequence but with the symbols \( 1, \ldots, 6 \) in separate special blocks.

**Corollary 21.** The sequence \( u \) is contained in \( S_8(2) \).