Blind SNR Estimation and Nonparametric Channel Denoising in Multi-Antenna mmWave Systems

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Abstract—We propose blind estimators for the average noise power, receive signal power, signal-to-noise ratio (SNR), and mean-square error (MSE) suitable for multi-antenna millimeter wave (mmWave) wireless systems. The proposed estimators can be computed at low complexity and solely rely on beamspace sparsity, i.e., the fact that only a small number of dominant propagation paths exist in typical mmWave channels. Our estimators can be used (i) to quickly track some of the key quantities in multi-antenna mmWave systems while avoiding additional pilot overhead and (ii) to design efficient nonparametric algorithms that require such quantities. We provide a theoretical analysis of the proposed estimators, and we demonstrate their efficacy via synthetic experiments and using a nonparametric channel-vector denoising task with realistic multi-antenna mmWave channels.

I. INTRODUCTION

Accurate knowledge of system parameters, such as average noise power, signal power, or the signal-to-noise ratio (SNR) is critical in communication systems, as many baseband processing tasks require knowledge of such quantities [1]. Most conventional communication systems dedicate a training phase to estimate such system parameters. However, since the propagation conditions can change at fast rates, especially at millimeter-wave (mmWave) frequencies where blockers or interferers may appear quickly [2], it is important to develop low-complexity solutions that accurately track such parameters.

Fortunately, modern wireless communication systems deal with high-dimensional data. For example, massive multiple-input multiple-output (MIMO) basestations are expected to be equipped with hundreds of antennas, or orthogonal frequency-division multiplexing (OFDM) systems have thousands of subcarriers. Since many of the signals in such systems are structured (e.g., exhibit sparsity or are taken from a discrete set), one can design statistical methods that blindly estimate parameters, without the need of a dedicated training stage.

A. Prior Art in Blind and Nonparametric Estimation

Blind estimators rely on the signal statistics rather than pilot sequences. Many of the existing blind noise variance and SNR estimators exploit modulation-specific structure, such as the cyclic prefix redundancy in OFDM [3] or periodicity of synchronization sequences [4]. Other methods use expectation maximization (EM) with sophisticated statistical models. For example, EM has been successfully used for blind SNR estimation [5] or to recover sparse signals [6]. However, the iterative nature of EM and its relatively high per-iteration complexity renders such methods unsuitable for real-time estimation in multi-antenna mmWave wireless systems that operate with high-dimensional data at gigabit per second sampling rates. In contrast, we propose a range of low-complexity blind estimators whose average-case complexity is only $O(D)$, where $D$ is the dimension of the processed data.

In order to deal with additional algorithm parameters, nonparametric methods have been proposed recently. The nonparametric equalizer (NOPE) [7], for example, performs linear minimum mean-square error (MSE) estimation in massive MIMO systems without knowledge of the SNR. NOPE combines approximate message passing [8], [9] with Stein’s unbiased risk estimate (SURE) [10] to automatically tune the algorithm parameters. The concept of estimating a subset of the algorithm parameters directly from the noisy measurements, has been used recently for adaptive denoising of mmWave [11]—[13] or OFDM [14] channel vectors. Such algorithms typically require two parameters (the noise power and a thresholding parameter), but since the threshold can be estimated using SURE from the noisy observations, such methods only need knowledge of the noise power. In this paper, we propose low-complexity blind estimators, which enable the design of nonparametric (i.e., parameter free) channel-vector denoisers.

B. Contributions

Our main contributions are as follows. We propose low-complexity blind estimators for the average noise power, signal power, SNR, and MSE after applying an element-wise function to the noisy observation. We provide a theoretical analysis by developing bounds on the accuracy in the large-dimension limit that depend on the SNR and sparsity. We provide simulation results with synthetic data to demonstrate the efficacy and limits of our estimators in finite dimensions. We showcase an application example of our blind estimators, which leads to a novel nonparametric channel-vector denoising algorithm for mmWave massive MIMO systems.

C. Notation

Lowercase boldface letters denote column vectors. The $d$th entry of the vector $\mathbf{a} \in \mathbb{C}^D$ is $a_d$, the real and imaginary parts are $\Re\{\mathbf{a}\}$ and $\Im\{\mathbf{a}\}$, respectively, and we use $b = |a|^2$ to refer to $b_d = |a_d|^2$ for $d = 1, \ldots, D$. Statistical quantities are denoted by plain symbols, e.g., the variance $E_x = \frac{1}{D} \mathbb{E}[|x|^2]$ of the random vector $\mathbf{x} \in \mathbb{C}^D$, where $\mathbb{E}[\cdot]$ denotes expectation; sample estimates are denoted by a bar, e.g., the sample variance
is \( E_x = \frac{1}{T} ||x||^2 \); corresponding blind estimators are denoted by a hat, e.g., \( \hat{E}_x \). For \( x \in \mathbb{R} \), rounding towards plus and minus infinity is denoted by \( [x] \) and \( [x] \), respectively, and \( [x]_+ = \max\{x, 0\} \). The big-O notation is \( \mathcal{O}(\cdot) \).

II. LOW-COMPLEXITY BLIND ESTIMATORS

A. System Models

We say that a vector \( s \in \mathbb{C}^D \) is sparse if the number of nonzero entries \( K \) is smaller than the dimension of the vector \( D \). As a sparsity measure, one can use the \( \ell_0 \) pseudo norm \( ||s||_0 = K \), which counts the number of nonzero entries of \( s \). As we will show in Section III, sparsity is one of the key ingredients to our blind estimators. In the rest of the paper, we focus on the following two system models.

System Model 1. Let \( s \in \mathbb{C}^D \) be a sparse signal with average energy \( E_s = \frac{1}{D} \mathbb{E}[||s||^2] \). We model a noisy observation of the sparse signal using the following input-output relation:

\[
y = s + n.
\]

(1)

Here, \( y \in \mathbb{C}^D \) is the noisy observation and \( n \in \mathbb{C}^D \) models noise with circularly-symmetric i.i.d. complex Gaussian entries with variance \( N_0 \). We assume that the sparse signal vector \( s \) and noise vector \( n \) are statistically independent.

In what follows, we do not make assumptions on the signal sparsity \( ||s||_0 = K \), even though the performance of the proposed estimators depends on this parameter; see Section III for the details. Furthermore, unlike pilot-based estimators, we assume that the sparse vector \( s \) is unknown, which makes parameter estimation nontrivial in our scenario.

System Model 1 finds numerous applications in wireless communication systems. Prime examples are modeling the estimated channel vectors in multi-antenna mmWave systems [11]–[13] or in OFDM systems [14], where the beamspace representation or the delay-domain representation of these channel vectors are typically sparse, respectively.

System Model 2. Let \( y \) be a noisy observation as in System Model 1. Fix a weakly differentiable function \( \eta(\cdot) \) that operates element-wise on vectors. We model the input-output relation

\[
\eta(y) = s + e,
\]

(2)

where \( e \in \mathbb{C}^D \) contains residual distortion.

We emphasize that in (2), the residual distortion \( e \) is not necessarily independent of the sparse signal \( s \). System Model 2 is relevant when applying (entry-wise) estimation or denoising functions, which finds use for channel-vector denosing [11], [12], or to model nonlinearities at the output of System Model 1 when modeling the impact of hardware impairments [15].

B. Low-Complexity Blind Estimators

The blind estimators proposed next make frequent use of the sample median, which we define as follows.

Definition 1 (Sample Median). Let \( s \in \mathbb{R}^D \) be a vector and \( s^{\text{sort}} \in \mathbb{R}^D \) be its sorted version (entries sorted in ascending order). Then, the sample median is defined as

\[
\hat{m}(s) = \frac{1}{2} \left( s^{\text{sort}}[\lceil(D+1)/2 \rceil] + s^{\text{sort}}[\lceil(D+1)/2 \rceil] \right).
\]

(3)

The sample median is robust to outliers [16], [17], which makes it amenable to System Models 1 and 2 as the nonzero entries of sparse vectors can be considered outliers. We emphasize that the sample median can be computed at low complexity in \( \mathcal{O}(D) \) average time using quickselect [18] and in \( \mathcal{O}(D \log(D)) \) worst-case time by performing merge sort.

Estimator 1 (Average Noise Power). Consider System Model 1. We propose the following blind estimator

\[
\hat{N}_0 = \frac{\mathbb{E}||y||^2}{\log(2)},
\]

(4)

to estimate the average noise power defined as

\[
N_0 = \frac{1}{D} \mathbb{E}[||n||^2].
\]

(5)

Estimator 1 only requires the absolute square entries of the noisy observation \( y \) in (1) and can be computed efficiently. The estimator exploits sparsity in the signal \( s \), but is independent of the sparsity level \( K \), the signal power, or the statistical sparsity model. It is, however, important to understand that the accuracy of this estimator depends on all of these factors as it relies on the fact that the nonzero entries of the sparse vector \( s \) can be treated as outliers. The derivation of this blind estimator and its analysis are provided in Section III-C.

Estimator 2 (Average Signal Power). Consider System Model 2. We propose the following blind estimator

\[
\hat{E}_s = \left[ \frac{||y||^2}{D} - \hat{N}_0 \right]_+.
\]

(6)

for the average signal power defined as

\[
E_s = \frac{1}{D} \mathbb{E}[||s||^2].
\]

(7)

Estimator 2 only uses the sample estimate of the average receive power \( \hat{E}_y = \frac{1}{D} ||y||^2 \) and the blind noise estimate \( \hat{N}_0 \) from Estimator 1. The derivation of this estimator and an analysis of its key properties are provided in Section III-D.

Estimator 3 (Signal-to-Noise Ratio). Consider System Model 2. We propose the following blind estimator

\[
\hat{\text{SNR}} = \left[ \frac{||y||^2}{D\hat{N}_0} - 1 \right]_+.
\]

(8)

for the SNR defined as

\[
\text{SNR} = \frac{\mathbb{E}[||s||^2]}{\mathbb{E}[||n||^2]}.
\]

(9)

Estimator 3 is blind as it combines the sample estimate of the average receive power \( \hat{E}_y = \frac{1}{D} ||y||^2 \) and the blind noise estimate \( \hat{N}_0 \) from Estimator 1. The derivation of this estimator and an analysis of its key properties are provided in Section III-E.

Estimator 4 (Mean-Square Error). Consider System Model 2 with function \( \eta(\cdot) \). We propose the following blind estimator

\[
\hat{E}_0 = \frac{1}{D} \mathbb{E}[\eta(y) - y]^2 + \hat{N}_0 + \frac{\hat{N}_0}{D} \sum_{d=1}^{D} \left( \frac{\partial \mathbb{R}[\eta(y_d)]}{\partial y_d} + \frac{\partial \mathbb{I}[\eta(y_d)]}{\partial y_d} \right)_+.
\]

(10)

for the MSE defined as

\[
E_0 = \frac{1}{D} \mathbb{E}[||\eta(y) - s||^2] = \frac{1}{D} \mathbb{E}[||e||^2].
\]

(11)
Estimator \( \hat{N}_0 \) only uses the receive signal \( y \), the estimate \( \hat{N}_0 \) from Estimator 1 and the function \( \eta(\cdot) \). Note that if we use the identity function \( \eta(y) = y \), the MSE corresponds to \( E_0 = N_0 \) while the estimated MSE corresponds to \( \hat{E}_0 = \hat{N}_0 \), as expected. The derivation of this estimator and an analysis of its properties is provided in Section III-F.

## III. THEORY

### A. Convergence of the Sample Median for \( D \to \infty \)

We will use the following definition of the median.

**Definition 2** (Median). Let \( X \) be an absolutely continuous random variable (RV) with cumulative distribution function (CDF) \( F_X(x) \). Then, the median \( m_X \) of \( X \) is defined as

\[
F_X(m_X) = \frac{1}{2}.
\]

(12)

We will frequently make use of the following result.

**Lemma 1** (Lemma C.1 from [19]). Suppose that \( f_X(x) \) is a differentiable probability density function (PDF) in some neighborhood of the median \( m_X \), and vector \( x \) contains i.i.d. samples of \( X \). Then, for any \( c > 0 \) the sample median \( \hat{m}(x) \) satisfies

\[
\lim_{D \to \infty} \Pr[|\hat{m}(x) - m_X| \geq c] = 0.
\]

(13)

In words, Lemma 1 implies that in the large-dimension limit, i.e., when \( D \to \infty \), the sample median \( \hat{m}(x) \) converges to the median \( m_X \). Hence, by observing a large number of samples, which is possible in modern multi-antenna mmWave or OFDM systems, we can accurately estimate the true median.

### B. Statistical Model for Complex-Valued Sparse Vectors

In order to derive and analyze the blind estimators proposed in Section II we need a suitable statistical model for the sparse signal \( s \). In what follows, we consider Bernoulli complex Gaussian (BCG) random vectors \([9], [20]\).

**Definition 3** (BCG Random Vector). Each entry in the sparse vector \( s \in \mathbb{C}^D \) is nonzero with activity rate \( p \in (0, 1] \), and the nonzero entries are i.i.d. circularly-symmetric complex Gaussian with variance \( E_s/p \). The PDF of each entry \( s_d \), \( d = 1, \ldots, D \), is therefore given by

\[
f_s(s_d) = (1 - p)\delta(s_d) + p\frac{1}{\pi E_s/p} e^{-\frac{|s_d|^2}{E_s/p}},
\]

(14)

where \( \delta(\cdot) \) is the Dirac delta function.

With this statistical model, the expected number of nonzero entries (the sparsity) is \( K = Dp \) and the average power of the sparse signal vector \( s \) corresponds to \( E_s = \frac{1}{D} \mathbb{E}[||s||^2] \).

In System Model 1, we assumed that the noise vector \( n \) is i.i.d. circularly-symmetric complex Gaussian with variance \( N_0 \) per complex entry. Hence, the PDF of each entry is given by \( f_n(n_d) = \frac{1}{\pi N_0} e^{-|n_d|^2/N_0} \). Consequently, the PDF of the noisy observation vector \( y = n + s \) in (1) is as follows.

**Definition 4** (Noisy BCG Random Vector). The PDF of the entries \( y_d \), \( d = 1, \ldots, D \), of a BCG random vector per Definition 3 observed in System Model 1 is given by

\[
f_Y(y_d) = (1 - p)\frac{1}{\pi N_0} e^{-\frac{|y_d|^2}{N_0}} + p\frac{1}{\pi (N_0 + E_s/p)} e^{-\frac{|y_d|^2}{N_0 + E_s/p}}.
\]

(15)

For this signal and observation model, we are now able to derive and analyze the proposed blind estimators.

### C. Analysis of Estimator 1

We start with the blind noise variance estimator defined in Estimator 1. We have the following key result. The proof is given in Appendix A.

**Theorem 1.** Let \( y \) be a noisy BCG random vector with PDF as in Definition 3 and with activity rate

\[
p \leq \frac{1}{2}e^{-2} \approx 0.421.
\]

Then, the average noise variance \( N_0 \) satisfies

\[
\min\left\{ \log(\frac{1}{1-p}), \log(2)(1 + \text{SNR}) \right\} \leq N_0 \leq \frac{m_z}{\log(2)} \left( 1 - p + \frac{p^2}{p + \text{SNR}} \right),
\]

(17)

where \( m_z \) is the median of an entry \( z_d \) of \( z = |y|^2 \).

First, we reiterate that in the large-dimension limit, i.e., when \( D \to \infty \), the sample median \( \hat{m}(z) \) converges to the true median \( m_z \) as established by Lemma 1. Hence, by assuming the large-dimension limit, Theorem 1 has the following implications on Estimator 1 (i) Since

\[
N_0 \leq \frac{m_z}{\log(2)} \left( 1 - p + \frac{p^2}{p + \text{SNR}} \right) \leq \frac{m_z}{\log(2)} = \hat{N}_0,
\]

(18)

where the right hand side (RHS) bound is obtained for \( \text{SNR} = 0 \), the proposed blind estimate \( \hat{N}_0 \) bounds the average noise variance \( N_0 \) from above, i.e., we developed a pessimistic estimator. (ii) By letting \( p \to 0 \), the left hand side (LHS) and RHS of (17) coincide and the inequalities in (18) hold with equality; this implies that the average noise variance will approach the value of the blind estimator for sparse signals (irrespective of the SNR). (iii) By letting \( \text{SNR} \to 0 \), the LHS and RHS of (17) coincide and the inequalities in (18) hold with equality; this implies that the noise variance will approach the value of the blind estimator at low SNR (irrespective of the signal’s sparsity). In summary, the proposed noise variance estimate is pessimistic but accurate for sparse vectors or in low-SNR scenarios for high-dimensional data. While the above observations are only true for our blind estimator in the large-dimension limit and for the noisy BCG model in Definition 3, we will use simulations in Section IV to demonstrate the accuracy of Estimator 1 for finite (and small) dimensions \( D \) and showcase an example application for channel denoising in multi-antenna mmWave systems.

### D. Analysis of Estimator 2

For the blind estimate \( \bar{E}_s \) of the average signal power \( E_s \), we use the following standard result, which follows from the fact that the entries of the vector \( z = |y|^2 \) are i.i.d. with expected value of \( \mu = \mathbb{E}[|z_d|^2] = E_s + N_0 \), \( d = 1, \ldots, D \).

**Lemma 2**. Let \( y \) be a noisy BCG random vector with PDF as in Definition 3. Then, we have that

\[
\lim_{D \to \infty} \frac{1}{D} \mathbb{E}[|y|^2] \overset{a.s.}{\to} E_s + N_0,
\]

(19)

where \( \overset{a.s.}{\to} \) implies almost sure convergence.
By defining the RV $W_D = \frac{1}{D}||y||^2 - N_0$, we have that $\lim_{D \to \infty} W_D \xrightarrow{a.s.} E_s$. By replacing the average noise power $N_0$ by the blind estimate $\hat{N}_0$ in (14) from Estimator 1 and by clipping the result, we obtain Estimator 2 in (16). Since, in the large-dimension limit, the noise power estimate $\hat{N}_0$ is overestimating the true average noise power, the blind estimate $\hat{E}_s$ in (6) tends to underestimate the signal power. From Theorem 2 it follows that for $p \to 0$ or $SNR \to 0$, the blind signal power estimate is exact. While the above observations only hold for $D \to \infty$, we showcase their accuracy for finite dimensions $D$ in Section IV.

E. Analysis of Estimator 2

The blind SNR estimator is obtained by simply taking the ratio of $\hat{E}_s$ in (6) and $\hat{N}_0$ in (16), followed by clipping to prevent negative estimates. For $D \to \infty$, the blind signal estimate overestimates the average signal power and the noise power estimate underestimates the average noise power, which means that the blind SNR estimate in (8) understimates the SNR. From Theorem 1 it follows that for $D \to \infty$ with either $p \to 0$ or $SNR \to 0$ the blind SNR estimate is exact. We provide simulation results for finite dimensions $D$ in Section IV.

F. Analysis of Estimator 3

In order to analyze Estimator 3 we first assume that the average noise power $N_0$ is known. For this scenario, we can borrow the following two theorems from [13].

Theorem 2 (Thm. 1 of [13]). Let $y$ be a noisy random vector of observations of $s$ as in System Model 1 and apply a weakly differentiable function $\eta(\cdot)$ to the entries of $y$ as in System Model 2. Then, Stein’s unbiased risk estimate given by

$$SURE = \frac{1}{D}||\eta(y) - \hat{y}\Vert_2^2 - N_0 + \frac{N_0}{D} \sum_{d=1}^{D} \left( \frac{\partial \eta(\tau y_d)}{\partial \eta(y_d)} \right)^2 + \frac{\partial^2 \eta(\tau y_d)}{\partial \eta(y_d)^2},$$

is an unbiased estimate of the MSE so that $\mathbb{E}[SURE] = E_0$.

Under the same assumptions, we have the following result which characterizes the behavior in the large-dimension limit.

Theorem 3 (Thm. 3 of [13]). In the large-dimension limit, i.e., when $D \to \infty$, SURE in (20) converges to the MSE in (11), i.e., we have $\lim_{D \to \infty} SURE = E_0$.

These results imply that if $N_0$ were known perfectly, one can estimate the MSE without knowing the sparse signal vector $s$ in the large-dimension limit when $D \to \infty$. To obtain the blind version in Estimator 4 we have replaced the true average noise power $N_0$ by its estimate $\hat{N}_0$ and clipped the result to prevent negative values. Consequently, for $D \to \infty$ and either $p \to 0$ or $SNR \to 0$, Theorem 1 implies that $\hat{N}_0$ will be exact, which implies that Estimator 2 will be exact in this scenario. To demonstrate the efficacy of this estimator in finite dimensions, we will show synthetic results and an application for mmWave channel-vector denoising in Section IV.

IV. NUMERICAL RESULTS

A. Synthetic Results

We perform Monte-Carlo simulations with 10000 trials to characterize the accuracy of the estimators proposed in Section II-B. We use the sparse signal model in Definition 4 and fix $N_0 = 1$, without loss of generality. We show the effect of the $SNR = E_0/N_0$ on the proposed estimators for an activity rate of $p = 0.1$ and a dimension of $D = 64$.

Fig. 1 shows the accuracy when varying the SNR. The results for our blind estimators are shown in green, where thick solid lines refer to the average performance and light green areas indicate the standard deviation. For the MSE $E_0$, we pick the soft-thresholding function $\eta(x; \tau) = x/|x| \max\{|x| - \tau, 0\}$ for $x \neq 0$ and $\eta(x; \tau) = 0$ for $x = 0$. Furthermore, for each received vector $y$, we adaptively select the denoising threshold $\tau \geq 0$ that minimizes the estimated MSE $\hat{E}_0$ as done in (11). As a comparison, we include EM for a zero-mean complex Gaussian mixture (average performance shown with a blue dashed line and standard deviation with a light blue area) with a maximum of 30 iterations and early stopping if the total parameter change is below 0.1%. We also show the accuracy of genie-aided estimators (average performance shown with a red dash-dotted line and standard deviation with a light red area) that have separate knowledge of $n$ and $s$. We compute $\hat{E}_s = \frac{1}{D}\|s\|^2$, $\hat{N}_0 = \frac{1}{D}\|n\|^2$, $SNR = \frac{E_0}{\hat{N}_0}$, $E_0 = \frac{1}{D}\|\eta(y; \tau) - s\|^2$. The reference parameters used in our simulations are shown with black dotted lines. Note that there is no reference for $E_0$, as the adaptive threshold prevents us from computing $E_0$ analytically.

From Fig. 1 we observe the following facts: (i) The standard deviation of the proposed blind estimators is comparable to that of the genie-aided methods that have separate access to $n$ and $s$. (ii) Even though the sample size is small ($D = 64$), our estimators are quite accurate with a standard deviation comparable to that of the genie-aided estimators; increasing $D$ would further reduce the standard deviation of all considered estimators. (iii) As predicted by our theory, the average noise power and $SNR$ are overestimated and the signal power is slightly underestimated. At low $SNR$, the three estimators become exact. (iv) For the considered scenario, the MSE is appreciably underestimated. However, the key requirement for adaptive parameter tuning is that the MSE $\hat{E}_0$ (which is the function to be minimized) has a similar shape as $E_0$. As we show next, the MSE estimate still performs well in practice.

In comparison with EM, our method provides a less-accurate estimate at higher $SNRs$, but requires significantly lower complexity. The complexity (in terms of the number of real-valued additions, real-valued multiplications, and exponentials) of EM is more than $N(16D + 12) + 3D$ operations, where $N$ is the number of EM iterations—the average number of iterations observed in our simulations ranges from 8 to 28 depending on the SNR. In contrast, our proposed median-based noise estimator has an average complexity of no more than $7.7D + 9$ operations, when computing the median using quickselect. Hence, our proposed blind estimator is more than $17\times$ less complex than EM (and avoids the evaluation of complex operations such as exponentials and divisions), which
renders our method suitable for (i) low-complexity parameter estimation and (ii) as a potential initializer for EM-based methods—the latter aspect is part of ongoing research.

### B. Application to Nonparametric Channel-Vector Denoising

We now show an application of Estimator 4 for beamspace channel estimation. As in [11], we simulate a mmWave massive MU-MIMO system with $D = 128$ basestation antennas, 8 user equipments, uncoded 16-QAM transmission, and line-of-sight (LoS) mmMAGIC QuaDRiGa mmWave channels [21].

Fig. 2 shows simulation results for the uncoded bit-error rate (BER) and the MSE for different channel estimation methods. We simulate beamspace channel estimation (BEACHES) as in [11], which denoises the channel vector by applying soft-thresholding to the noisy observation. The thresholding parameter is adaptively selected for each noisy observation by minimizing SURE with perfect knowledge of the average noise power $N_0$ using an $O(D \log(D))$ algorithm. We compare the nonparametric BEACHES variant, where we use Estimator 4 to blindly learn the denoising threshold. We also include a variant that we call EM BEACHES, where we use Estimator 4 but replace $N_0$ by the EM noise estimate. As a reference, we show the performance of perfect channel state information (CSI) that uses the ground truth (noiseless) channel vector, and maximum likelihood (ML) estimation that simply takes the noisy observation as the channel estimate.

From Fig. 2, we observe that the nonparametric BEACHES algorithm achieves virtually the same performance as that of the original BEACHES algorithm which requires knowledge of $N_0$ (except at high SNR where Estimator 1 tends to over-estimate $N_0$). We reiterate that the nonparametric BEACHES algorithm requires no parameters and exhibits exactly the same complexity as the original algorithm as the latter already sorts the entries of $|y|^2$, which we can reuse to compute Estimator 4. EM BEACHES achieves higher (worse) MSE as realistic channels deviate from the BCG model in Definition 3 and exhibits higher complexity than our nonparametric method.

### V. Conclusions

We have proposed blind estimators for the average noise power, signal power, SNR, and MSE. Our estimators can be calculated in $O(D)$ average time and $O(D \log(D))$ worst-case time, and only require the noisy observation vector, which avoids the need for additional pilot signals. We have analyzed our estimators for a complex Bernoulli-Gaussian sparsity model and evaluated their accuracy via simulations. Using a channel vector denoising task in multi-antenna mmWave systems, we have demonstrated that our blind estimators lead to a novel nonparametric denoiser that achieves comparable performance and the same complexity as BEACHES in [11], which requires knowledge of the average noise power. We believe that the proposed blind estimators find potential use in a large number of other applications in wireless systems that contain sparse signals and require low complexity.

### Appendix A

#### Proof of Theorem 1

**A. Prerequisites**

In what follows, we need the distribution of $z = |y|^2$. Since the absolute-square entry of a circularly-symmetric complex Gaussian RV $Q$ with variance $E_0$ is exponentially distributed with CDF $F_Q(q) = 1 - e^{-q/E_0}$, $q \geq 0$, the CDF of each entry of the absolute-square noisy observation is as follows.

**Definition 5 (Noisy BCG Power RV).** Let $y$ be as in Definition 2 and let $z = |y|^2$ be defined by $z_d = |y_d|^2$, $d = 1, \ldots, D$. Then, for $z \geq 0$, the CDF of each entry of $z$ is given by

$$F_Z(z_d) = (1 - p)(1 - e^{-z_d/E_0}) + p(1 - e^{-z_d/N_0 + E_0/p}).$$

**B. Upper Bounds on the Median**

We start with the following upper bound on the median $m_z$ of a noisy BCG power RV $Z$ with CDF given in (21).

**Lemma 3.** For a noisy BCG power RV in Definition 5 with $p < 0.5$, the median is bounded from above by

$$N_0 \log \left( \frac{2 \cdot 2p}{1 - 2p} \right) \geq m_z.$$  \hspace{1cm} (22)

**Proof.** Using (21), we obtain the expression for the median of a RV $Z$ with CDF in (21) according to equation (12):

$$\left(1 - p\right)(1 - e^{-m_Z/E_0}) + p(1 - e^{-m_Z/N_0 + E_0/p}) = \frac{1}{2}.$$  \hspace{1cm} (23)
Since the second term is nonnegative, we can omit it to obtain the following inequality:

\[(1-p)(1-e^{-\frac{mg}{N_0}}) \leq \frac{1}{2}.\]  \tag{24}

Note that this bound will be useful for vectors \(s\) that are sparse, i.e., where \(p\) is small. We can simplify (24) as follows

\[\frac{mg}{N_0} \leq -\log \left(1 - 2p \right)\]  \tag{25}

which leads to an upper bound on the median \(m_Z\). In order to take the logarithm in (25), we require \(p \in (0, 1/2).\) ∎

**Lemma 4.** For a noisy BCG power RV \(Z\) in Definition 5 with \(p \leq \frac{1}{e^{1-p} - e^{-2}}\), the median is bounded from above by

\[
\log(2)(N_0 + E_s) \geq m_Z. \tag{26}
\]

**Proof.** From (21), we have that

\[
\frac{1}{2} = (1-p)e^{-\frac{N_0}{m_Z} + pe^{-\frac{N_0 + E_s}{m_Z}}}. \tag{27}
\]

Let us define the function \(g(r) = e^{-1/r} + r\) with \(r > 0\). We can now rewrite (27) as follows:

\[
\frac{1}{2} = (1-p)g\left(\frac{N_0}{m_Z} + \frac{p}{m_Z}N_0 + E_s\right). \tag{28}
\]

The function \(g(r)\) is concave for \(r \geq 1/2\), which holds when

\[2N_0 \geq m_Z.\]  \tag{29}

We first verify when (29) holds. From (27), we have that

\[
\frac{1}{2} \geq (1-p)e^{-\frac{2N_0}{m_Z} + pe^{-\frac{2N_0 + E_s}{m_Z}}}. \tag{30}
\]

\[
1 - e^{-2} \geq p \left(1 - e^{-2}\right), \tag{31}
\]

which implies that the condition (16) ensures concavity of \(g(r)\). By assuming that the condition (16) holds, we can use Jensen’s inequality on the expression in (28) to get

\[
\frac{1}{2} \leq g\left(\frac{1}{2}\right) \geq \frac{1}{2} \frac{N_0}{m_Z} + \frac{p}{m_Z}N_0 + E_s = e^{-m_Z} \frac{1}{1-e^{-2}}. \tag{32}
\]

We can now simplify this expression to

\[
\log(1/2) \leq -m_Z\frac{1}{N_0 + E_s}, \tag{33}
\]

\[
\log(2)(N_0 + E_s) \geq m_Z, \tag{34}
\]

which is what we show in Lemma 4. ∎

**C. Lower Bound on the Median**

We now establish the following lower bound on the median.

**Lemma 5.** For a noisy BCG power RV \(Z\) in Definition 5 with \(p \in (0, 1]\), the median is bounded from below by

\[
\log(2)N_0 \leq \frac{m_Z}{1 - e^{-\frac{m_Z}{N_0}}} \tag{35}
\]

**Proof.** Since the exponential CDF \(F_Q(q) = 1 - e^{-\frac{q}{\lambda}}\) for \(\lambda \geq 0\) is concave in \(q\), Jensen’s inequality leads to

\[
(1-p)(1-e^{-\frac{m_Z}{N_0}}) + p(1-e^{-\frac{m_Z}{N_0} + \frac{p}{m_Z}N_0}) = \frac{1}{2} \tag{36}
\]

\[
1 - e^{-1 - \frac{m_Z}{N_0} - \frac{p}{m_Z}N_0} \geq \frac{1}{2}. \tag{37}
\]

We can simplify this expression to obtain the following bound

\[
\frac{1}{2} \geq e^{-1 - \frac{m_Z}{N_0} - \frac{p}{m_Z}N_0} \geq \frac{1}{2} \tag{38}
\]

\[
\log(2)N_0 \leq m_Z \left(1 - p + \frac{p^2}{m_Z}\right), \tag{39}
\]

which is what we have in (35). ∎

**D. Combining the Results**

Finally, we can combine Lemma 3 with Lemma 4 and Lemma 5 to obtain the desired result in (17).

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