Abstract
Let \( G = (V,E) \) be a connected graph. A subset \( H \) of \( V \) is called a hub set of \( G \) if for any two distinct vertices \( u,v \in V - H \), there exists a \( u-v \) path \( P \) in \( G \) such that all the internal vertices of \( P \) are in \( H \). A hub set \( H \) of \( V \) is called an open hub set if the induced subgraph \( <H> \) has no isolated vertices. The minimum cardinality of an open hub set of \( G \) is called the open hub number of \( G \) and is denoted by \( h_o(G) \). In this paper, we present several basic results on the open hub number.

Keywords
Open hub set, Open hub number.

AMS Subject Classification
05C40, 05C99.

1 Department of Mathematics, PRNSS College, Mattanur-670702, Kerala, India.
2 Department of Mathematics, Kannur University, Kannur-670002, Kerala, India.
*Corresponding author: 1 ragiputhanveettil@gmail.com; 2ramakrishnantvknr@gmail.com
Article History: Received 29 April 2020; Accepted 30 August 2020

Contents
1 Introduction ................................................. 1375
2 Main Results ............................................... 1375

References ...................................................... 1377

1. Introduction

By a graph \( G = (V,E) \) we mean a finite ordered graph with no loops and no multiple edges. For graph theoretic terminology we refer [1]. Let \( G = (V,E) \) be a connected graph. The concept of hub set is introduced by M. Walsh [3]. A subset \( H \) of \( V \) is called a hub set of \( G \) if for any two distinct vertices \( u, v \in V - H \), there exists a \( u-v \) path \( P \) in \( G \) such that all the internal vertices of \( P \) are in \( H \). The minimum cardinality of a hub set of \( G \) is called the hub number of \( G \) and is denoted by \( h_c(G) \). A dominating set of a graph \( G \) is a sub set \( D \) of \( V \) such that every vertex not in \( D \) is adjacent to at least one member of \( D \). The domination number \( \gamma(G) \) is the number of vertices in a smallest dominating set for \( G \). A dominating set \( D \) is said to be a connected dominating set if the subgraph \( <D> \), induced by \( D \) is connected in \( G \). The minimum of the cardinalities of the connected dominating sets of \( G \) is called the connected domination number and is denoted by \( \gamma_c(G) \). In this paper we introduce the Open hub number of a graph \( G \). A hub set \( H \) of \( V \) is called an open hub set if the induced subgraph \( <H> \) has no isolated vertices. The minimum cardinality of an open hub set of \( G \) is called the open hub number of \( G \) and is denoted by \( h_o(G) \). Since an open hub set has at least two elements we have \( h_o(G) \geq 2 \).

We use the following results to prove our main results.

Lemma 1.1. [3] For any graph \( G \), \( \gamma(G) \leq h(G) + 1 \).

Lemma 1.2. [3] Let \( d(G) \) denote the diameter of \( G \). Then \( h(G) \geq d(G) - 1 \), and the inequality is sharp.

Theorem 1.3. [3] If \( C_n \) is the cycle with \( n \) vertices then, \( h(C_n) = n - 3 \).

Theorem 1.4. [3] If \( P_n \) is the path with \( n \) vertices then, \( h(P_n) = n - 2 \).

Theorem 1.5. [4] If \( G \) is a connected graph and \( n \geq 3 \), then \( \gamma_c(G) = n - e_f(G) \leq n - 2 \), where \( e_f(G) \) is the maximum number of pendant vertices of underlying spanning tree of \( G \).

Theorem 1.6. [5] For any connected graph \( G \), \( h(G) \leq h_c(G) \leq \gamma_c(G) \leq h(G) + 1 \).

2. Main Results

Theorem 2.1. For every connected graph \( G \), \( h(G) \leq h_o(G) \leq 2h(G) \).

Proof. Since every open hub set is a hub set we have \( h(G) \leq h_o(G) \). Let \( H = \{v_1, v_2, ..., v_k\} \) be a minimum hub
Theorem 2.5. If $\Delta (G)$ is a minimum open hub set, the open hub number of tree $G$. Hence $h_{O}(G) \leq |H \cup H'| \leq 2|H| = 2h(G)$.

Proposition 2.2. For a connected graph $G$, if $h_{c}(G) \geq 2$ then every connected hub set is an open hub set.

Proof. Suppose $h_{c}(G) \geq 2$ and assume $H$ is a connected hub set of $G$. Then $<H>$ is connected and contains more than one vertex. Hence $H$ is an open hub set.

Proposition 2.3. For any connected graph $G$, if $\Delta (G) = n - 1$, then $h_{O}(G) = 2$.

Proof. Suppose $\Delta (G) = n - 1$. Let $u$ be a vertex of $G$ having degree $n - 1$. Then $\{u, v\}$ where $v \in N(u)$, the neighborhood of $u$, forms an open hub set of $G$. Hence the result.

Corollary 2.4. $h_{O}(W_{n}) = 2$

Proof. The result is obvious from above proposition since $\Delta (W_{n}) = n - 1$.

Now we characterise a class of graphs having open hub number $n - 3$

Theorem 2.5. Suppose $G$ is a connected graph of order $n$ such that $\Delta (G) \neq n - 1$, then $h_{O}(G) = n - 3$ if and only if $G$ isomorphic to one of the following graphs

1. The cycle $C_{n}$
2. A subdivision of $K_{1,3}$
3. $C_{k}$ with a path attached for any $k$.
4. $C_{3}$ with two paths attached
5. $C_{3}$ with three paths attached
6. A graph with exactly two cycles $C_{3}$ and $C_{k}$ for any $k \geq 3$, with one edge common if $G$ has no pendent vertices.
7. A graph with exactly two cycles $C_{3}$ and $C_{k}$ for any $k \geq 3$, with one edge common and a path attached to a vertex of degree 2 in $C_{3}$

Proof. Suppose $h_{O}(G) = n - 3$. Since $h_{O}(G) \leq n - \Delta (G)$ we have $\Delta (G) \leq 3$.

If $\Delta (G) = 1$, $G \cong K_{2,2}$, a contradiction.

If $\Delta (G) = 2$, $G$ is either a cycle or a path. But open hub number of the path $P_{2}$ is $n - 2$. Hence $G$ is a cycle.

If $\Delta (G) = 3$

Case 1 Suppose $G$ is a tree having $l$ leaves.

Since the set of all non leaf vertices of a tree (which is not a star) is a minimum open hub set, the open hub number of tree is $n - l$, we have $l = 3$. That is $G$ is a tree having 3 leaves and $\Delta = 3$.

Therefore $G$ must be isomorphic to a subdivision of $K_{1,3}$.

Case 2 $G$ is not a tree.

Then $G$ contains cycles. If $G$ contains two disjoint cycles $C_{l} = (u_{1}, u_{2}, ... , u_{l})$ and $C_{k} = (v_{1}, v_{2}, ... , v_{k})$. Let $H_{1}$ and $H_{2}$ are minimum open hub sets of $C_{l}$ and $C_{k}$ respectively such that $|H_{1}| = |V(C_{l})| - 3$ and $|H_{2}| = |V(C_{k})| - 3$.

Then $H = H_{1} \cup H_{2} \cup T$ where $T = V(G) - (V(C_{l}) \cup V(C_{k}))$ is an open hub set of $G$, a contradiction to $h_{O}(G) = n - 3$. Also since $\Delta = 3$, no cycle has only single vertex in common. Hence any two cycles have common edge. Thus $G$ is either unicyclic or exactly two cycles with a common edge.

Subcase I: $G$ is unicyclic.

Suppose $G$ contains a cycle $C_{k} = (v_{1}, v_{2}, ... , v_{k})$.

Let $S = \{v \in V(C_{k}) | \delta (v) = 3\}$. Then $|S| \leq 3$.

If $|S| = 0$ Then $G$ is isomorphic to $C_{k}, k = n$.

If $|S| = 1$, then $G$ is isomorphic to the graph $C_{k}$ with a path attached to one vertex.

If $|S| = 2$, then $G$ is isomorphic to $C_{2}$ with two paths attached.

If $|S| = 3$, $G$ is isomorphic to $C_{l}$ with 3 paths attached.

Subcase II: $G$ is not Unicyclic

Then $G$ contains exactly 2 cycles and at least one cycle should be $C_{3}$. In this case if $G$ has no pendent vertices, then it is isomorphic to a graph with 2 cycles $C_{3}$ and $C_{k}, k \geq 3$ with one common edge. If $G$ has pendent vertices then it is isomorphic to a graph with 2 cycles $C_{3}$ and $C_{k}, k \geq 4$ with one edge common and a path attached to vertex of degree 2 in $C_{3}$.

Converse is trivial.

Figure 1. Class of graphs in Theorem 2.5

Theorem 2.6. Given two integers $k$ and $n$ with $2 \leq k \leq \left[ \frac{n}{2} \right]$, there exist a connected graph $G$ of order $n$ with $h_{O}(G) = k$.

Proof. Let $K_{n-k}$ be the complete graph with $V(K_{n-k}) = \{v_{1}, v_{2}, ... , v_{n-k}\}$. Let $G$ be the graph obtained from $K_{n-k}$ by adding $k$ new vertices $u_{1}, u_{2}, ... u_{k}$ and $k$ new edges $u_{i}v_{j}, 1 \leq i \leq k$.

Then $G$ is a connected graph of order $n$.

The domination number $\gamma(G) = k$. Hence $h(G) \geq k - 1$.

Let $H$ be a hub set of $G$. Then either $v_{j} \in H \forall j$ $1 \leq j \leq k$ or $u_{i} \in H \forall j, 1 \leq j \leq k$. Therefore $h(G) = k$ and $\{v_{1}, v_{2}, ... , v_{k}\}$ is a minimum hub set of $G$ and the induced subgraph $H >$ is the path graph $v_{1}v_{2}...v_{k}$ and hence it is an open hub set. Hence the result.

□
Given two integers \( k \) and \( n \) with \( 2 \leq k \leq n - 2 \), there exist a connected graph \( G \) of order \( n \) with \( h_0(G) = k \).

**Proof.** Let \( K_{n-k} \) be the complete graph with \( V(K_{n-k}) = \{v_1, v_2, \ldots, v_{n-k}\} \). Let \( P_{k+1} \) be the path \( w_1w_2\ldots w_{k+1} \). Let \( G \) be the graph obtained by identifying the vertices \( v_1 \) and \( w_1 \). We claim that \( h(G) = k \). It follows from Theorem 1.5 that \( \chi_c(G) = k \) and by Theorem 1.6 \( h(G) = k \) or \( k - 1 \). Now suppose \( h(G) = k - 1 \). Let \( H \) be a hub set of \( G \) with cardinality \( k - 1 \). If both \( v_1 \) and \( w_{k+1} \) are not in \( H \), then \( w_i \in H \) for \( 2 \leq i \leq k \). Since cardinality of \( H \) is \( k - 1 \), we have \( H = \{w_2, w_3, \ldots, w_k\} \). A contradiction to \( H \) is a hub set of \( G \). Suppose \( v_1 \in S \) and \( w_{k+1} \notin S \). Then if \( w_2 \notin H \) then \( w_i \in H \) for \( 3 \leq i \leq k \). In this case \( H = \{v_1, v_3, \ldots, w_k\} \), which is not a hub set, again a contradiction. We must have \( w_2 \in H \). By similar argument we have \( w_i \in H \) for \( 3 \leq i \leq k \), a contradiction. We get a similar contradiction if \( v_1 \notin H \) and \( w_{k+1} \in H \) or if both \( v_1 \) and \( w_{k+1} \in H \). Thus \( h(G) \neq k - 1 \). Hence \( h_0(G) \geq k \).

Now \( S = \{v_1, w_2, \ldots, w_k\} \) is a minimum hub set of \( G \) and the induced subgraph \( \langle S \rangle \) is the path graph \( v_1, w_2, w_3, \ldots, w_k \). Hence has no isolated vertices. Hence \( S \) is an open hub set of \( G \) so that \( h_0(G) \leq k \). Hence the result.

**Corollary 2.8.** For each positive integer \( n \), there exist a connected graph \( G \) of order \( n \) such that \( h_0(G) = n - \Delta(G) \).

**Definition 2.9.** Let \( G_1, G_2, \ldots, G_r \) be connected graphs, then the graph \( G \) obtained by joining each vertex of \( G_i \) with each vertex of \( G_{i+1} \), \( 1 \leq i \leq r - 1 \), is called the successive join of \( G_1, G_2, \ldots, G_r \) and is denoted by \( G_1 + G_2 + \ldots + G_r \).

**Theorem 2.10.** Let \( G_1, G_2, \ldots, G_r \) be connected graphs and let \( G = G_1 + G_2 + \ldots + G_r \), then

\[
 h_0(G) = \begin{cases} 
 2 & \text{if } r = 2, 3 \\
 r - 2 & \text{if } r \geq 4 
\end{cases}
\]

**Proof.** Case I \( r = 2, 3 \)

Here \( H = \{u, v\} \) where \( u \in V(G_1) \) and \( v \in V(G_2) \), is an open hub set of \( G \).

Case II \( r \geq 4 \)

In this case \( H = \{v_2, v_3, \ldots, v_{r-1}\} \), where \( v_i \in V(G_i) \) for \( 2 \leq i \leq r - 1 \), is an open hub set of \( G \).

Hence \( h_0(G) \leq r - 2 \). Now the diameter of \( G, d(G) = r - 1 \) and hence by Lemma 1.2, \( h_0(G) \geq r - 2 \). Hence \( h_0(G) = r - 2 \).

**References**

[1] F. Harary, *Graph Theory*, Addison-Wesley Pub House, 1963.

[2] C. Gary and Z. Ping, *Introduction to Graph Theory*, Tata McGraw-Hill, 2006.

[3] W. Matthew, The hub number graphs, *International Journal of Mathematics and Computer Science*, 1(2006), 117–124.

[4] W.H. Teresa, T.H. Stephan and J.S. Peter, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc, 2008.

[5] G. Tracy, A.H. Stephan and J. Adam, The hub number of a graph, *Information Processing Letters*, 108(2008), 226–228.