Fermionization and Hubbard Models

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Abstract

We introduce a transformation which allows the fermionization of operators of any one-dimensional spin-chain. This fermionization procedure is independent of any eventual integrable structure and is compatible with it. We illustrate this method on various integrable and non-integrable chains, and deduce some general results. In particular, we fermionize XXC spin-chains and study their symmetries. Fermionic realizations of certain Lie algebras and superalgebras appear naturally as symmetries of some models. We also fermionize recently obtained Hubbard models, and obtain for the first time multispecies analogues of the Hubbard model, in their fermionic form. We comment on the conflict between symmetry enhancement and integrability of these models. Finally, the fermionic versions of the non integrable spin-1 and spin-$3/2$ Heisenberg chains are obtained.

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1 Introduction

Fermionization and its reverse procedure, bosonization, consist of a mapping between a set of bosonic variables and one of fermionic variables. Such mappings have long been essential tools of theoretical physics. They provide a way of expressing the same theory in two different languages and open the door to the use of a large number of techniques to study a particular system. They also reveal the interplay between symmetries and boundary conditions, and provide new models, in particular when the dimension of space or the underlying lattice is changed. Fermionization schemes exist in many dimensions. They take an added importance in two-dimensional conformal field theories; see for instance [1].

For the spin-$\frac{1}{2}$ one-dimensional quantum spin-chain a fermionization procedure exists which allows the mapping between spin operators and fermionic creation-annihilation operators. This Jordan-Wigner transformation [2] has been used in particular to study the polaron model [3] and the Hubbard model [4]. The fermionic forms appear as models of itinerant electrons used to model conduction properties, while the bosonic ones appear as spin-chains. More recently the Jordan-Wigner transformation was used in the study of the Bariev model of correlated-hopping electrons [5]. However the Jordan-Wigner transformation is limited in scope and has long been standing alone as a way to fermionize simple one-dimensional spin-chains. It is one of the aims of this paper to provide a general fermionization method.

We start by recalling the Jordan-Wigner transformation and then give its generalization. We show how several variants can be implemented to fermionize operators corresponding to any spin-chain, integrable or not. We then review how to fermionize, with the help of the Jordan-Wigner mapping, an integrable spin-chain model of considerable theoretical and experimental interest, the spin-$\frac{1}{2}$ Heisenberg spin-chain Hamiltonian. As boundary conditions are not preserved by the Jordan-Wigner transformation, we clarify this issue on this simple example. The fermionic version possesses non-local boundary terms. It is possible to replace these terms with periodic ones without spoiling integrability. This example serves also other purposes. The general features of integrability, both in the bosonic and fermionic settings, are introduced and quantities and equations which will reappear for other models are defined for this simple model. We also obtain some general results by studying this system. We then fermionize three of the recently found integrable XXC models and the $su(4)$ XXZ model, i.e. the model corresponding to the $R$-matrix of the fundamental representation of $su(4)$. The XXC models are hybrid trigonometric ones, with an underlying $su(2)$ structure and some $su(n)$ features. They possess symmetries much larger than the trigonometric models built from the fundamental representations of $su(n)$. We apply the fermionization at the Hamiltonian level and at the integrable structure level, by fermionizing the $L$- and $R$-matrices. While different versions of a given XXC model exist, they are known to be equivalent at the original bosonic level. This turns out to be false for most fermionized versions. We also show that fermionic realizations of algebras and superalgebras appear naturally as the symmetries of the fermionized models. Some symmetries are broken only to be replaced by new ones. From the examples studied we infer some general results.

The XXC models at their ‘free-point’ are known to be the building blocks of multistates generalizations of the Hubbard model [6, 7]. These generalizations inherit the large symmetries of the component models. This is another motivation to study fermionized versions of the XXC models. We recall the fermionization of the usual Hubbard model and do the same for two generalized Hubbard Hamiltonians and their $L$, $R$-matrices. This provides for the first time fermionic multiple-species generalizations of the Hubbard models. The Hamiltonian density can then be written on a lattice of any dimension and provides possible candidates for superconducting models in two dimensions.
The usual Hubbard model has a $U(1) \times U(1)$ symmetry in both its bosonic and fermionic forms. In its original fermionic definition, this symmetry is enhanced to an $SO(4)$ group symmetry. We investigate whether a similar enhancement takes place for our generalized Hubbard models.

We then derive the fermionic versions of two models of considerable physical importance, the spin-$\frac{1}{2}$ XXZ Heisenberg chains. We propose for the Hamiltonians slightly modified versions which do not contain non-local factors.

We conclude with general considerations and remarks about the foregoing fermionization scheme. Finally we provide a preliminary comparison with a different, recently introduced fermionization scheme for integrable models [8].

2 Fermionization

The mapping from purely bosonic operators to fermionic ones may seem to require the Hilbert space, at a given site, to have as dimension a power of two. This is obvious on general grounds. The Fock space of a set of anticommuting Fermi operators has dimension $2^d$ where $d$ is the number of species of spinless fermions, i.e. the number of pairs of mutually anticommuting creation-annihilation operators. However, a local space with arbitrary dimension can be embedded in the nearest larger space of dimension a power of two. The fermionization is then done in the latter space, with an eventual projection on the original space. We shall illustrate this procedure on various examples. Before going any farther we recall the Jordan-Wigner transformation for a chain of spin-$\frac{1}{2}$ variables [2].

2.1 Jordan-Wigner transformation

Let $\sigma^\pm$ and $\sigma^z$ be the three Pauli matrices:

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1)$$

The transformation at the site $m$ is non-local and given by

$$a_m^\dagger = \pi_m^{-1} \sigma_m^+ \sigma_m^{-1}, \quad a_m = \pi_{m-1}^{-1} \sigma^+_m \sigma^-_m \quad (2)$$

with

$$\pi_0^0 \equiv 1 \quad \text{and} \quad \pi^\sigma_{m-1} \equiv \sigma^z_1 \cdots \sigma^z_{m-1} \quad (3)$$

where $1$ is the identity operator. It is straightforward to check that the new operators are fermionic creation-annihilation operators,

$$\{a_m, a_n\} = \{a_m^\dagger, a_n^\dagger\} = 0, \quad \{a_m^\dagger, a_n\} = \delta_{mn} \quad (4)$$

where $\{,\}$ denotes the anticommutator, $\{A, B\} = AB + BA$. One may also have more than a single copy of the Pauli matrices at every site. The spin ‘up’ and ‘down’ fermions result in two copies of the Pauli matrices. The fermionization results in $p$ species of spinless fermions. The fermionization formulae for an $N$-sites chain are given by:

$$a_1^\dagger = \pi_{m-1}^{-1} \sigma^+_m \sigma^-_m, \quad a_1 = \pi_{m-1}^{-1} \sigma^-_m \sigma^+_m$$

$$a_2^\dagger = \pi_{N+1}^{-1} \sigma^+_m \sigma^-_m, \quad a_2 = \pi_{N+1}^{-1} \sigma^-_m \sigma^+_m$$

$$\cdots \cdots$$

$$a_{pm}^\dagger = \pi_{N+1}^{-1} \sigma^+_m \sigma^-_m, \quad a_{pm} = \pi_{N+1}^{-1} \sigma^-_m \sigma^+_m \quad (5)$$
where \( \pi^{(s)}_{m-1} \equiv \sigma^{(s)}_1 \cdots \sigma^{(s)}_{m-1} \), \( s = 1, ..., p \). The \( \pi^{(s)}_{N+1} \) ensure anticommutation between different species:

\[
\{ a_{sm}, a_{tn} \} = \{ a_{sm}^\dagger, a_{tn}^\dagger \} = 0 \quad , \quad \{ a_{sm}^\dagger, a_{tn} \} = \delta_{mn}\delta_{st}
\]

(6)

The Pauli matrices satisfy the \( su(2) \) algebra. However, \( \sigma^x \) and \( \sigma^y \) also satisfy the Clifford algebra and this is the relevant remark which allowed us to generalize the Jordan-Wigner transformation.

### 2.2 Generalization

Let \( d \) be a positive integer. The dimension \( 2d \) Clifford algebra, \( C_{2d} \), is generated by an even number of mutually anticommuting generators

\[
\{ \gamma_i, \gamma_j \} = +2\delta_{ij} \quad , \quad i, j = 1, ..., 2d
\]

(7)

It is a classical exercise to show that this algebra contains the following \( 2^d \) linearly independent elements:

\[
1 \quad ; \quad \gamma_i \quad ; \quad \gamma_{i_1}\gamma_{i_2} \quad , \quad i_1 < i_2 \quad ; \quad \ldots \ldots \quad ; \gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_{2d}} \quad , \quad i_1 < \ldots < i_{2d}
\]

(8)

The \( 2d \) generators of this algebra can be realized as \( 2^d \times 2^d \) hermitian matrices. Thus all \( 2^d \times 2^d \) matrices can be written in a unique way as linear combinations of products of \( \gamma \) matrices. For \( d = 1 \) we recover the Pauli matrices while for \( d = 2 \) we get the Dirac \( \gamma \)-matrices. There is a special element of \( C_{2d} \) which anticommutes with all elements \( \gamma \), and squares to one. It is the product of the \( 2d \) \( \gamma \)-matrices; we take a particular normalization for convenience:

\[
\gamma \equiv i^{-d} \gamma_1 \cdots \gamma_{2d}
\]

(9)

Let

\[
\gamma_j^\pm \equiv \frac{1}{2}(\gamma_{2j-1} \pm i\gamma_{2j}) \quad , \quad j = 1, ..., d
\]

(10)

We define a generalized Jordan-Wigner transformation by the following mapping between bosonic matrices and fermionic creation-annihilation operators of \( d \) species of spinless fermions:

\[
a_{jm}^\dagger = \pi_{m-1}^\gamma \gamma_{jm}^+, \quad a_{jm} = \pi_{m-1}^\gamma \gamma_{jm}^- \quad , \quad j = 1, ..., d
\]

(11)

where

\[
\pi_0^\gamma \equiv 1 \quad \text{and} \quad \pi_{m-1}^\gamma = \gamma_1 \cdots \gamma_{m-1}
\]

(12)

It is straightforward to verify that the following anticommutation relations hold:

\[
\{ a_{jm}, a_{kn} \} = \{ a_{jm}^\dagger, a_{kn}^\dagger \} = 0 \quad , \quad \{ a_{jm}^\dagger, a_{kn} \} = \delta_{jk}\delta_{mn}
\]

(13)

Since \( \gamma \) squares to one, it is invertible and the above mapping is one-to-one. Let

\[
n_{jm} \equiv a_{jm}^\dagger a_{jm}
\]

(14)

be the number density operator for the species \( j \). One can then express the matrix \( \gamma \) at site \( m \) in terms of fermions operators

\[
\gamma_m = \prod_{i=1}^{d}(2n_{im} - 1)
\]

(15)

Since \( n_{jm} \) squares to one \( \gamma_m \) also squares one as needed, and the inverse equations are given by

\[
\gamma_{jm}^+ = \prod_{k=1}^{m-1} \left( \prod_{l=1}^{d}(2n_{lk} - 1) \right) a_{jm}^\dagger \quad , \quad \gamma_{jm}^- = \prod_{k=1}^{m-1} \left( \prod_{l=1}^{d}(2n_{lk} - 1) \right) a_{jm} \quad , \quad j = 1, ..., d
\]

(16)
Some remarks are in order. One could have chosen to group the $\gamma_i$ matrices two by two in an arbitrary order, with an eventual simple change in the definition of the $\gamma$ matrix. However there is no loss of generality in the foregoing choice, as no specific choice of representation of the algebra $C_{2d}$ has yet been made.

As we noted earlier, one should not be left with the impression that the Hilbert space at every site must have dimension $2^d$ to be able to fermionize. Indeed any space can be embedded in the nearest higher dimensional space of dimension $2^d$. More precisely, the operators at this site can be written as linear combinations of $E^{ij}$ matrices whose only non-vanishing elements is a one at row $i$ and column $j$, with $i, j$ varying from one to the dimension of the space. One first embeds the matrices $E^{ij}$ in the nearest $2^d$-dimensional Hilbert space by adding the required rows and columns of zero, and then proceeds with the fermionization. The transformed quantities act in the larger Hilbert space but also act in a stable way when restricted to the original space, where one has identified which states of the larger Fock space span the smaller one. One obtains in this way fermionized quantities for the chain. We shall illustrate this procedure on two simple examples.

The above transformations admit several variants. The spaces at different sites of a chain need not be equal or of equal dimension. Indeed, all that is required for all that was said above to still hold is the one-to-one mapping at one site and the anticommutation relations. And this can be ensured by replacing the string $\gamma_1 \cdots \gamma_{m-1}$ in equations (11,12) with

$$\pi^\gamma_{m-1} = \gamma^{(1)}_1 \cdots \gamma^{(m-1)}_{m-1}$$

(17)

where $\gamma^{(i)}_i$ is the special gamma matrix acting on the local space of dimension $2^{d_i}$ at site $i$. The dimension $2^{d_i}$ depends on the site. The $\gamma^\pm_j$ also correspond to the dimension $2^{d_i}$. Thus one can fermionize for instance a spin-chain with different representations at every site.

One can also choose to fermionize selectively, leaving some sites unfermionized while fermionizing others. This is achieved through another change in $\pi^\gamma_{m-1}$. For the sites where one keeps the bosonic variables, the bosonic operators are left unchanged by the mapping: $B_i \rightarrow B_i$. For the sites fermionized, one uses another version of (17) where the string has been replaced by one where the identity replaces the $\gamma$-matrix for all the sites not fermionized. One thus has ensured that the fermionic operators anticommutate among themselves as needed, while they commute with the bosonic operators for sites where no fermionization has taken place. In this way one gets a chain of mixed bosonic and fermionic local spaces.

There is also the multiple-copies fermionization, when a local space is a tensor product of smaller spaces. The fermionization formulae are just the $\gamma$-generalizations of formulae (5). One replaces the $p$ copies of Pauli matrices by $p$ copies of gamma matrices. Note that copies of gamma matrices at the same site need not correspond to the same dimension, as the space at a given site may be the tensor product of spaces of different dimensions. Again, all the above variants can still be implemented within the different copies, with copies of gamma matrices at different sites possibly having different dimensions. The selective fermionization is still possible. We shall give examples of fermionizations with two copies of $\gamma$ matrices. Finally, reverse fermionization i.e. bosonization, with all its variants, can be implemented by use of the inversion formulae (16).

For the specific calculations to follow we have adopted the definitions of $\gamma$-matrices found in [3]. We now fermionize operators of various spin chains.
3 Integrable spin-chains

We start with a well-studied model, the spin-$\frac{1}{2}$ XXZ spin-chain. Its Hamiltonian as well as its integrable structure have already been fermionized. This simple example is a warm-up exercise which allows us to introduce the general framework of integrability and general aspects of the fermionization procedure. We shall then fermionize the simplest XXC models. Unless otherwise indicated, roman symbols correspond to the original bosonic quantities, while calligraphic ones correspond to the fermionized ones.

3.1 The spin-$\frac{1}{2}$ XXZ chain

We first consider the Hamiltonian and the $(L, R)$-system, and then boundary and symmetry issues.

3.1.1 Hamiltonian and $L$-matrix

The Hamiltonian is given by

$$H_2 = \sum_m \left( x^{-1} \sigma_m^+ \sigma_{m+1}^+ - x \sigma_m^- \sigma_{m+1}^- + \Delta \sigma_m^z \sigma_{m+1}^z \right)$$

for $x = \pm 1$

Here we have introduced a twist parameter $x$. In the notation of [10] one is considering the model $(n_1 = 1, n_2 = 1)$. The parameter $x$ arises from threading the ring-chain by a flux $N\phi$. One then have $x = e^{i\phi(\vec{x}_m)}$ where

$$\phi(\vec{x}_m) = \int_{\vec{x}_m}^{\vec{x}_{m+1}} \vec{A}(\vec{x})d\vec{x}$$

is the Peierls phase.

The Jordan-Wigner transformation yields

$$\mathcal{H}_2 = -\sum_m \left( x a_{m+1}^\dagger a_m + x^{-1} a_m^\dagger a_{m+1} - \Delta (2n_m - 1)(2n_{m+1} - 1) \right)$$

where $\Delta \equiv -\cos\gamma/2$, and $n_m \equiv a_m^\dagger a_m$ is the number density operator. Let us also introduce another number density operator,

$$\bar{n}_m \equiv a_m a_m^\dagger = 1 - n_m \quad n_m \bar{n}_m = \bar{n}_m n_m = 0$$

as this quantity appears naturally farther on. Note that the transformation $a_m \leftrightarrow a_m^\dagger$ interchanges $n_m$ and $\bar{n}_m$. This is the particle-hole symmetry which reflects the fact that the roles of the creation and annihilation operators can be reversed. Since operators are independent of the choice of a fermionic vacuum $a$ and $a^\dagger$ play symmetrical roles, and it is only natural that $\bar{n}_m$ as well as $n_m$ should appear.

The bosonic and fermionic models, both with their respective periodic boundary conditions, are known to be integrable. The fermionization of the $L$-$R$ system was done in [3]. The fermionization of the integrable structure consists in applying a ‘gauge transformation’ on the Lax matrix at every site. The $R$-matrix intertwining two such matrices is then obtained by plugging the fermionic $L$-matrix into the $RLL$ equation and by cancelling out the transformation matrices.
We adopt the notations of [10] and consider a transformation for $L$ which differs slightly from the one found in [3, 11]; the difference is however not essential. Let $a = \sin(\gamma - \lambda)$, $b = \sin \lambda$, $c = \sin \gamma$, and $E^{\alpha \beta}$ be the matrix whose sole non-vanishing entry is a one at row $\alpha$ and column $\beta$. The $L = R$ matrix is given by:

$$L_m = \begin{pmatrix} a E_{11} + x b E_{22} & c E_{21} \\ c E_{12} & x^{-1} b E_{11} + a E_{22} \end{pmatrix}$$

(22)

Define the operators

$$V_m = \text{diag}(v_m, v_m^{-1}) = \begin{pmatrix} v_m & 0 \\ 0 & v_m^{-1} \end{pmatrix}, \quad v_m = \exp \left( \frac{i \pi}{2} \sum_{j=1}^{m-1} (n_j - 1) \right)$$

(23)

The fermionic $L$-matrix is obtained as:

$$L_m = V_{m+1} L_m V_m^{-1}$$

(24)

and one easily derives:

$$L_m = \begin{pmatrix} a n_m - i b x \bar{n}_m & -i c a_m \\ c a_m^\dagger & i a \bar{n}_m + b x^{-1} n_m \end{pmatrix}$$

(25)

The $\hat{RLL}$ equation is given by

$$\hat{R}(\lambda_1 - \lambda_2) L(\lambda_1) \otimes L(\lambda_2) = L(\lambda_2) \otimes L(\lambda_1) \hat{R}(\lambda_1 - \lambda_2)$$

(26)

and $\hat{R}$ satisfies the Yang-Baxter equation

$$\hat{R}_{12}(\lambda_1 - \lambda_2) \hat{R}_{23}(\lambda_1) \hat{R}_{12}(\lambda_2) = \hat{R}_{23}(\lambda_2) \hat{R}_{12}(\lambda_1) \hat{R}_{23}(\lambda_1 - \lambda_2)$$

(27)

Let $P$ be the permutation operator on the tensor product of two spaces: $P(x \otimes y) = y \otimes x$, where $x$ and $y$ are two vectors. Define $R(\lambda) \equiv P \hat{R}(\lambda)$, then an equivalent form of equation (26) is:

$$R(\lambda_1 - \lambda_2) \frac{1}{L} (\lambda_1) \frac{2}{\hat{L}} (\lambda_2) = \frac{2}{\hat{L}} (\lambda_2) \frac{1}{L} (\lambda_1) R(\lambda_1 - \lambda_2)$$

(28)

where $\frac{1}{L} (\lambda_1) = L(\lambda_1) \otimes I$ and $\frac{2}{\hat{L}} (\lambda_2) = I \otimes L(\lambda_2)$. The Yang-Baxter equation for $R$ is given by

$$R_{12}(\lambda_1 - \lambda_2) R_{13}(\lambda_1) R_{23}(\lambda_2) = R_{23}(\lambda_2) R_{13}(\lambda_1) R_{12}(\lambda_1 - \lambda_2)$$

(29)

To find the fermionic version of (26) one inverts (24) to obtain $L(\lambda_1)$ and $L(\lambda_2)$, and replaces them into equation (24). Upon commuting the $V$’s through, one finds that they cancel each other out after leaving phases which modify the matrix $\hat{R}$. Equation (26) becomes

$$\hat{R}(\lambda_1 - \lambda_2) \mathcal{L}(\lambda_1) \otimes \mathcal{L}(\lambda_2) = \mathcal{L}(\lambda_2) \otimes \mathcal{L}(\lambda_1) \hat{R}(\lambda_1 - \lambda_2)$$

(30)

where the tensor product of two operators $A$ and $B$ is graded according to

$$(A \otimes B)_{ij,kl} = (-1)^{(P(i)+P(k))P(j)} A_{ik} B_{jl}, \quad P(i) \in \mathbb{Z}_2$$

(31)

and

$$\hat{R}(\lambda) = F^{-1} \hat{R}(\lambda) F, \quad F = \text{diag}(1, 1, i, i) = (\text{diag}(1, i)) \otimes \mathbb{I}_2$$

(32)
For (23) one has: $P(1) = 0$ and $P(2) = 1$. Note that $F$ is a diagonal matrix defined up to an overall normalization. We shall choose throughout this paper $F_{11} = 1$. This gives the above $F$, and $F^4 = I_4$.

One can then define a graded permutation operator $P$ whose action on two vectors of well-defined parity is given by: $P(x \otimes y) = (-1)^{P(x)P(y)} y \otimes x$. In components one has: $P_{ijkl} = (-1)^{P(i)P(j)} \delta_{ij} \delta_{jk}$. Let $R = P \tilde{R}$, and equation (25) becomes

$$\mathcal{R}(\lambda_1 - \lambda_2) = \frac{1}{L(\lambda_1)} \mathcal{L}(\lambda_2) = \frac{2}{L(\lambda_2)} \mathcal{L}(\lambda_1) \mathcal{R}(\lambda_1 - \lambda_2)$$

(33)

where $\frac{1}{L(\lambda_1)} = \mathcal{L}(\lambda_1) \otimes I$ and $\frac{2}{L(\lambda_2)} = I \otimes \mathcal{L}(\lambda_2)$. The matrices $\tilde{R}$ and $\mathcal{R}$ satisfy the Yang-Baxter equations (27) and (29) respectively. However, while (27) written in components for $\tilde{R}$ is unchanged, equation (29) for $\mathcal{R}$ has grading signs due to the graded permutation operator.

We shall obtain similar results for the fermionization of all the integrable systems considered here. The above equations will still hold, with the corresponding matrices $R$, $L$, $F$, and the appropriate grading.

### 3.1.2 Boundary terms and integrability

It is well-known that the Jordan-Wigner transformation does not conserve periodic boundary conditions. One finds

$$\sigma_N^+ \sigma_1^- = -(2n_2 - 1) \cdots (2n_{N-1} - 1)a_N^+ a_1^- , \quad \sigma_N^- \sigma_1^+ = -(2n_2 - 1) \cdots (2n_{N-1} - 1)a_N^- a_1^+$$

(34)

Thus the fermionic Hamiltonian (20) with the JW-twisted fermionic boundary conditions

$$a_{N+1} \equiv +(2n_2 - 1) \cdots (2n_{N-1} - 1)a_1 , \quad a_{N+1}^\dagger \equiv +(2n_2 - 1) \cdots (2n_{N-1} - 1)a_1^\dagger$$

(35)

is the Hamiltonian equivalent to the bosonic one (18) with periodic boundary conditions. If instead one takes periodic fermionic boundary conditions for (20), with $a_{N+1} \equiv a_1$ and $a_{N+1}^\dagger \equiv a_1^\dagger$, the resulting Hamiltonian is not anymore equivalent to the bosonic one with periodic boundary conditions. It is however still integrable. This follows from the fermionic relations (30).

To this end we recall the general tenets of integrability. The transfer matrix, $\tau(\lambda)$, is the generating functional of the infinite set of conserved quantities. Its construction in the framework of the Quantum Inverse Scattering Method (QISM) is well known [12, 13, 14]. Given an $(L, R)$ pair, the trace over the auxiliary space of the monodromy matrix $T(\lambda)$ yields the transfer matrix:

$$\tau(\lambda) \equiv \text{Tr}_0 \left[ T(\lambda) \right] \equiv \text{Tr}_0 \left[ M_0 \ L_{0N}(\lambda) \cdots L_{01}(\lambda) \right]$$

(36)

where $N$ is the number of sites on the chain and 0 is the auxiliary space. The introduction of the numerical matrix $M$ corresponds to integrable periodic $M$-twisted boundary conditions, with $M = I$ corresponding to periodic conditions. This twisting of the periodic boundary conditions is different from the one arising from the Jordan-Wigner transformation. For matrices $M$ such that $[M \otimes M, \tilde{R}(\lambda)] = 0$ equation (24) implies the RTT relations:

$$\tilde{R}(\lambda_1 - \lambda_2) T(\lambda_1) \otimes T(\lambda_2) = T(\lambda_2) \otimes T(\lambda_1) \tilde{R}(\lambda_1 - \lambda_2)$$

(37)

Taking the trace over the auxiliary spaces one obtains $[\tau(\lambda_1), \tau(\lambda_2)] = 0$. A set of local conserved quantities is given by

$$H_{p+1} = \left( \frac{d^p \ln \tau(\lambda)}{d\lambda^p} \right)_{\lambda=0} , \quad p \geq 0$$

(38)

The Hamiltonians $H_{p+1}$ mutually commute and the system is said to be integrable.
In the fermionic setting the monodromy matrix is given by

\[ T(\lambda) \equiv M_0 L_{0N}(\lambda) \cdots L_{01}(\lambda) \]  

This is so because the Lax matrix is a homogeneous even-parity matrix. The generalization of the QISM framework to the graded context was initiated in [15, 16]. Equation (30) replaces equation (26), with the appropriate grading, and equation (37) is replaced by

\[ \mathcal{R}(\lambda_1 - \lambda_2) T(\lambda_1) \otimes T(\lambda_2) = T(\lambda_2) \otimes T(\lambda_1) \mathcal{R}(\lambda_1 - \lambda_2) \]  

It can be shown in all generality that periodic fermionic boundary conditions correspond to a matrix \( M \) which implements a supertrace:

\[ M_{ij} = \delta_{ij}(-1)^{P(i)} \quad , \quad \text{Tr}(MA) = \text{Str}(A) = \sum_i (-1)^{P(i)} A_{ii} \]  

This shows that Hamiltonian (29) with fermionic periodic boundary conditions is also integrable.

We restrict ourselves to \( M = I \) for the bosonic case, and to a matrix implementing the supertrace in the fermionic setting. The latter matrix corresponds to the grading arising in equation (30). For the latter case we omit \( M \) and replace the trace by a supertrace in (36).

From now on, unless otherwise indicated, ‘bosonic system’ means we consider bosonic periodic boundary conditions, while ‘fermionic system’ means we consider periodic fermionic boundary conditions. Thus we are considering the bosonic and fermionic systems with their ‘natural’ boundary conditions; as we have seen, these systems are inequivalent.

An immediate and general consequence of all this is the existence of at least two integrable boundary conditions for any integrable system: Periodic and possibly many periodic-twisted conditions through fermionization or bosonization. This applies to the general fermionization scheme we introduced.

### 3.1.3 Symmetries

Let us now consider the symmetries of the bosonic spin-1/2 XXZ model. For arbitrary \( \Delta \) the operator \( \sigma^z = \sum_{i=1}^N \sigma_i^z \) commutes with the transfer matrix and therefore with all the conserved quantities. At the rational point where \( \Delta = \frac{1}{2} \), the full symmetry is \( su(2) \), with the two remaining generators being \( \sigma^+ = \sum_{i=1}^N \sigma_i^+ \) and \( \sigma^- = \sum_{i=1}^N \sigma_i^- \). Their fermionic counterparts are given by:

\[ \sigma^+ = \sum_{i=1}^N \left( \prod_{j=1}^{i-1} (2n_j - 1) \right) a_i^\dagger, \quad \sigma^- = \sum_{i=1}^N \left( \prod_{j=1}^{i-1} (2n_j - 1) \right) a_i, \quad \sigma^z = \sum_{i=1}^N (2n_i - 1) \]  

To show that \( \sigma^z \) commutes with the fermionic transfer matrix, we showed and used

\[ [n_m, \mathcal{L}_m(\lambda)] = -\frac{1}{2}[\sigma_0^z, \mathcal{L}_m(\lambda)] \]  

At \( \Delta = \frac{1}{2} \), are \( \sigma^\pm \) still symmetries of the periodic fermionic system? We have verified that the fermionic Hamiltonian does not commute with \( \sigma^\pm \). Thus, although the rational limit of \( \mathcal{L} \) and \( \mathcal{R} \) is well-defined, the symmetry enhancement at the rational point does not take place for the fermionic system. This is in fact a general (negative) result for all models fermionized, whenever the rational limit provides an enlarged symmetry for the bosonic system.

This simple fermionization already illustrate features that will also arise for most if not all integrable systems.
3.2 The (1, 2)-XXC model

This model naturally appears when one takes the infinite coupling limit of the Hubbard model; see for instance [17]. The dimension of the the Hilbert space at one site is three. We therefore utilize the Clifford algebra to fermionize this model.

The Hamiltonian is given by

\[
H_2 = \sum \left( \sum_{\beta=2}^3 (x_{1\beta} E_m^{1\beta} E_{m+1}^{1\beta} + x_{1\beta}^{-1} E_m^{1\beta} E_{m+1}^{1\beta}) - \cos \gamma (E_m^{11} E_{m+1}^{11} + \sum_{\beta, \beta' = 2}^3 E_m^{\beta\beta'} E_{m+1}^{\beta\beta'}) \right)
\]

We took \(A = \{1\}\) and \(B = \{2, 3\}\) in the notation of [10]. In the bosonic setting the two other choices are unitarily equivalent to this one [18].

There are four possible equivalent embeddings of the \(3 \times 3\) matrices into \(4 \times 4\) matrices:

\[
(1, 2, 3) \rightarrow (1, 2, 3, 4), \quad (1, 2, 3) \rightarrow (1, 2, 4, 3),
\]
\[
(1, 2, 3) \rightarrow (1, 3, 4, 2), \quad (1, 2, 3) \rightarrow (2, 3, 4, 1).
\]

For instance, the second choice corresponds to letting

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a_{11} & a_{12} & 0 & a_{13} \\
a_{21} & a_{22} & 0 & a_{23} \\
0 & 0 & 0 & 0 \\
a_{31} & a_{32} & 0 & a_{33}
\end{pmatrix}
\]

while the fourth choice corresponds to

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & a_{11} & a_{12} & a_{13} \\
0 & a_{21} & a_{22} & a_{23} \\
0 & a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

In the specific representation we adopted for the \(\gamma\)-matrices, the canonical basis \((e_1, e_2, e_3, e_4)\) is sent to the four fermionic states as follows:

\[
e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow a_1^\dagger a_2^\dagger |0\rangle \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow |0\rangle
\]

\[
e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow a_1^\dagger |0\rangle \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow a_2^\dagger |0\rangle
\]

We choose, for obvious reasons but again without loss of generality, the fourth embedding which means that the Hamiltonian and the higher conserved quantities, \(H_p\), automatically annihilate all states of the chain which have a double occupancy on at least one site. The third choice corresponds to the annihilation of all states which have a vacuum on at least one site.

The fermionized Hamiltonian density is given by

\[
H_{mm+1} = (x_{23} a_{1m}^\dagger a_{1m+1} + x_{23}^{-1} a_{1m+1}^\dagger a_{1m}) \bar{n}_{2m} \bar{n}_{2m+1} + (x_{24} a_{2m}^\dagger a_{2m+1} + x_{24}^{-1} a_{2m+1}^\dagger a_{2m}) \bar{n}_{1m} \bar{n}_{1m+1}
\]

\[
- \frac{1}{2} \cos \gamma (1 + C_m C_{m+1})
\]

(49)
where $C = (2n_1 - 1)(2n_2 - 1) - n_1n_2$ is the ‘conjugation operator’. It is obvious that the action of $H_2$ is stable on the subspace of dimension $3^N$. Define the projection operator which annihilates any state with double occupancy on at least one site:

$$p = \prod_{m=1}^N (1 - n_{1m}n_{2m})$$  

(50)

$H_2$ can be interpreted as acting on the reduced Hilbert space or as $pH_2p$ acting on the full $4^N$-dimensional space.

The Lax matrix is fermionized with the help of the matrix

$$V_m = \text{diag}(v_m, v_m^{-1}, v_m^{-1}) \quad \text{where} \quad v_m = \exp\left(\frac{i\pi}{2} \sum_{i=1}^{m-1} \sum_{j=1}^2 (n_{ji} - 1)\right)$$  

(51)

One easily finds

$$L_m = \begin{pmatrix} L_{22} & -ica_{1m}^\dagger \tilde{n}_{2m} & ic a_{2m}^\dagger \tilde{n}_{1m} \\ -ca_{1m}\tilde{n}_{2m} & L_{33} & -ia a_{2m}^\dagger a_{1m} \\ ca_{2m}\tilde{n}_{1m} & -ia a_{1m}^\dagger a_{2m} & L_{44} \end{pmatrix}$$  

(52)

$$L_{22} = -a \tilde{n}_{1m}\tilde{n}_{2m} - ib(x_{23} n_{1m}\tilde{n}_{2m} + x_{24} \tilde{n}_{1m}n_{2m})$$

$$L_{33} = ia n_{1m}\tilde{n}_{2m} - bx_{23} \tilde{n}_{1m}\tilde{n}_{2m}$$

$$L_{44} = ia \tilde{n}_{1m}n_{2m} - bx_{24} \tilde{n}_{1m}\tilde{n}_{2m}$$

Note that $(1 - n_{1m}n_{2m})L_m = L_m(1 - n_{1m}n_{2m}) = L_m$. The grading is given by: $P(2) = 0$, $P(3) = P(4) = 1$. We obtain for the matrix $F$

$$F = \text{diag}(1, 1, 1, -i, -i, -i, -i, -i) = \text{diag}(1, -i, -i) \otimes I_3$$  

(53)

The symmetry generators for the conserved quantities are given by

$$X^{34} = \sum_m a_{2m}^\dagger a_{1m} \quad X^{43} = \sum_m a_{1m}^\dagger a_{2m} \quad X^1 = \sum_m n_{1m} \quad X^2 = \sum_m n_{2m}$$  

(54)

Recall that the off-diagonal generators are symmetries provided the following constraint hold:

$$x_{23} = x_{24}$$  

(50)

We see in the next section how to show similar commutation relations with the fermionic transfer matrix. The symmetries are the generators of $su(2) \times u(1)$, and reflect a (partial) symmetry between the two species of spinless fermions. Because of no-double-occupancy, $X = \sum m n_{1m}n_{2m}$ is a trivial symmetry. Note that, here and below, these fermions can be seen as one type of fermion with spin $\frac{1}{2}$. We however reserve the terms ‘up’ and ‘down’ for the two XX copies used in constructing Hubbard models.

### 3.3 The (2, 2)-XXC model: first avatar

We first write this XXC model with $A = \{1, 2\}$ and $B = \{3, 4\}$. The fermionic Hamiltonian density is given by

$$H^{(1)}_{mm+1} = a_{1m}^\dagger a_{1m+1} \left[ x_{23} \tilde{n}_{2m}\tilde{n}_{2m+1} + x_{24} \tilde{n}_{1m}\tilde{n}_{1m+1} \right]$$  

(55)

$$+ a_{1m+1}^\dagger a_{1m} \left[ x_{23} \tilde{n}_{2m}\tilde{n}_{2m+1} + x_{24} \tilde{n}_{1m}\tilde{n}_{1m+1} \right]$$

$$+ a_{2m}^\dagger a_{2m+1} \left[ x_{24} \tilde{n}_{1m}\tilde{n}_{1m+1} - x_{13} n_{1m}n_{1m+1} \right]$$

$$+ a_{2m+1}^\dagger a_{2m} \left[ x_{24} \tilde{n}_{1m}\tilde{n}_{1m+1} - x_{13} n_{1m}n_{1m+1} \right]$$

$$- \frac{1}{2} \cos \gamma \left( 1 + C_m^1 C_m^{(-1)} \right)$$
where $\mathcal{C}^{(1)} = (2n_1 - 1)(2n_2 - 1)$.

The Lax matrix is fermionized with the help of matrix

$$V_m = \text{diag}(v_m, v_m, v_m^{-1}, v_m^{-1}) \quad \text{where} \quad v_m = \exp\left(\frac{i\pi}{2} \sum_{i=1}^{m-1} \sum_{j=1}^2 (n_{ji} - 1)\right) \quad (56)$$

We find

$$L_m^{(1)} = \begin{pmatrix}
L_{11}^{(1)} & -a a_{1m} a_{2m} & -i c a_{2m} n_{1m} & -i c a_{1m} n_{2m} \\
-a a_{1m} a_{2m} & L_{22}^{(1)} & -i a a_{1m} n_{2m} & i c a_{2m} n_{1m} \\
c a_{2m} n_{1m} & -c a_{1m} n_{2m} & L_{33}^{(1)} & -i a a_{1m} a_{2m} \\
c a_{1m} n_{2m} & c a_{2m} n_{1m} & -i a a_{2m} a_{1m} & L_{44}^{(1)}
\end{pmatrix} \quad (57)$$

The tensor product of equation $[30]$ is now graded according to

$$P(1) = P(2) = 0, \quad P(3) = P(4) = 1 \quad (58)$$

For the matrix $F$ we found

$$F = \text{diag}(1, 1, 1, 1, -1, -1, -1, i, i, i, i, i, i, i, i) = \text{diag}(1, -1, i, i) \otimes \mathbb{I}_4 \quad (59)$$

Again one has $F^4 = \mathbb{I}_{16}$.

Now consider the symmetries of the transfer matrix. It is easy to check that the following commutation relations hold provided all the parameter $x_{a\beta}$ are equal to each other, but not necessarily to $\pm 1$:

$$\{a_{2m}^{\dagger} a_{1m}^{\dagger}, L_m^{(1)}\} = \{E_{01}^{(1)}, L_m^{(1)}\} \quad \{a_{1m} a_{2m}, L_m^{(1)}\} = \{E_{01}^{21}, L_m^{(1)}\}$$

$$\{a_{2m}^{\dagger} a_{1m}^{\dagger}, L_m^{(1)}\} = -\{E_{01}^{(1)}, L_m^{(1)}\} \quad \{a_{1m} a_{2m}^{\dagger}, L_m^{(1)}\} = -\{E_{01}^{21}, L_m^{(1)}\}$$

$$\{n_{1m} n_{2m}, L_m^{(1)}\} = -\{E_{01}^{(1)}, L_m^{(1)}\} \quad \{\bar{n}_{1m} \bar{n}_{2m}, L_m^{(1)}\} = -\{E_{01}^{21}, L_m^{(1)}\}$$

Let

$$X^{12} = \sum_m a_{2m}^{\dagger} a_{1m}^{\dagger} \quad X^{21} = \sum_m a_{1m} a_{2m}$$

$$X^{34} = \sum_m a_{2m} a_{1m}^{\dagger} \quad X^{43} = \sum_m a_{1m} a_{2m}^{\dagger}$$

$$X^{11} = \sum_m n_{1m} n_{2m} \quad X^{22} = \sum_m \bar{n}_{1m} \bar{n}_{2m}$$

$$X^{33} = \sum_m n_{1m} \bar{n}_{2m} \quad X^{44} = \sum_m \bar{n}_{1m} n_{2m} \quad (61)$$

Relations $[60]$ imply the following commutation relations with the transfer matrix of the fermionic system:

$$\{X^{12}, \tau^{(1)}(\lambda)\} = 0 \quad \{X^{21}, \tau^{(1)}(\lambda)\} = 0$$

$$\{X^{34}, \tau^{(1)}(\lambda)\} = 0 \quad \{X^{43}, \tau^{(1)}(\lambda)\} = 0$$

$$\{X^{11}, \tau^{(1)}(\lambda)\} = 0 \quad \{X^{22}, \tau^{(1)}(\lambda)\} = 0$$

$$\{X^{33}, \tau^{(1)}(\lambda)\} = 0 \quad \{X^{44}, \tau^{(1)}(\lambda)\} = 0 \quad (62)$$

We come to the unusual fact that the zero parity operators $X^{12}$ and $X^{21}$ anticommute with the zero parity operator $\tau(\lambda)$! However, a simple proof by induction shows that the logarithmic
derivatives of the transfer matrix are made up of sums of product of an even number of the transfer matrix and its derivatives. This shows that the symmetries \( X_{12} \) and \( X_{21} \) still commute with the conserved quantities \( \mathcal{H}_p \), for \( p \geq 1 \). The anticommutation with \( \exp(\mathcal{H}_1) \) does not of course prevent the simultaneous diagonalization of \( X_{12} \) or \( X_{21} \) with all the other conserved quantities, as the spectral spaces are stable under anticommutation and/or commutation relations. The symmetry operators clearly are the generators of \( su(2) \times su(2) \times u(1) \), with a fermionic realization of the generators of this algebra.

### 3.4 The (2, 2)-XXC model: second avatar

We consider here the (2, 2)-XXC model in another realization, where \( A = \{1, 3\} \) and \( B = \{2, 4\} \). As first pointed out in [18], this bosonic model is unitarily equivalent to the bosonic model of the preceding section. The orthogonal operator linking the second avatar to the first one is constructed as follows. Consider the orthogonal matrix

\[
U = U^{-1} = U^t = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The \( N \)-fold tensor product \( U^{(N)} = U \otimes \cdots \otimes U \) transforms the models into each other, modulo redefinitions of the parameters \( x_{a\beta} \). This can be seen at the Hamiltonian level or generally at the transfer matrix level

\[
U^{(N)} \tau^{(1)}(\lambda) U^{(N)} = \tau^{(2)}(\lambda) \implies U^{(N)} H_p^{(1)} U^{(N)} = H_p^{(2)}
\]

At the level of the \( L \)-matrices one has:

\[
L_m^{(2)} = U_0 U_m L_m^{(1)} U_m U_0
\]

Note that \( U^{(N)} \) is an orthogonal and therefore unitary operator.

The fermionic Hamiltonian density is now given by:

\[
\mathcal{H}^{(2)}_{m m+1} = a_{1m} \dagger a_{1m+1} \left[ x_{32}^{-1} \bar{n}_{2m} \bar{n}_{2m+1} - x_{14}^{-1} n_{2m} n_{2m+1} \right] \\
+ a_{1m} \dagger a_{1m+1} \left[ x_{12} a_{2m} a_{2m+1} - x_{34} a_{2m+1} a_{2m} \right] \\
+ a_{1m+1} \dagger a_{1m} \left[ x_{32} \bar{n}_{2m} \bar{n}_{2m+1} - x_{14} n_{2m} \bar{n}_{2m+1} \right] \\
+ a_{1m+1} \dagger a_{1m} \left[ x_{12} a_{2m+1} a_{2m} - x_{34} a_{2m+1} a_{2m} \right] \\
- \frac{1}{2} \cos \gamma \left( 1 + \mathcal{C}_m^{(2)} \mathcal{C}_{m+1}^{(2)} \right)
\]

where \( \mathcal{C}^{(2)} = 2n_1 - 1 \). The \( V \)-matrix, the induced grading and matrix \( F \) are the same as in the previous section. We find:

\[
\mathcal{L}^{(2)}_m = \begin{pmatrix}
\mathcal{L}^{(2)}_{11} & -c a_{1m} a_{2m} & -i a a_{2m} n_{1m} & -i c a_{1m} n_{2m} \\
-c a_{1m} a_{2m} & \mathcal{L}^{(2)}_{22} & -i c a_{1m} n_{2m} & i a a_{2m} n_{1m} \\
-a a_{2m} n_{1m} & -c a_{1m} \bar{n}_{2m} & \mathcal{L}^{(2)}_{33} & -i c a_{2m} n_{1m} \\
c a_{1m} n_{2m} & a a_{2m} \bar{n}_{1m} & -i c a_{1m} a_{2m} & \mathcal{L}^{(2)}_{44}
\end{pmatrix}
\]
\( \mathcal{L}^{(2)}_{11} = a n_{1m}n_{2m} - b \{ x_{12} \bar{n}_{1m}\bar{n}_{2m} + i x_{14} \bar{n}_{1m}n_{2m} \} \)

\( \mathcal{L}^{(2)}_{22} = -a \bar{n}_{1m}\bar{n}_{2m} + b \{ x_{12}^{-1}n_{1m}n_{2m} - i x_{32}^{-1}n_{1m}\bar{n}_{2m} \} \)

\( \mathcal{L}^{(2)}_{33} = i a n_{1m}\bar{n}_{2m} - b \{ x_{32} \bar{n}_{1m}\bar{n}_{2m} - i x_{34} \bar{n}_{1m}n_{2m} \} \)

\( \mathcal{L}^{(2)}_{44} = i a \bar{n}_{1m}n_{2m} + b \{ x_{14}^{-1}n_{1m}n_{2m} + i x_{34}^{-1}n_{1m}\bar{n}_{2m} \} \)

This matrix is not fundamentally different from the one in the preceding section.

Unlike the fermionic version of the first realization, the symmetries which survive fermionization are only the diagonal ones: the \( X^i, i = 1, \ldots, 4 \). They generate \( u(1) \times u(1) \times u(1) \). There are no constraints on the parameters \( x_{\alpha\beta} \). The off-diagonal bosonic symmetry generators become non-local in the fermionic setting and it is easy to check that they do not commute with the fermionic Hamiltonian. This breaks the bosonic \( su(2) \oplus su(2) \oplus u(1) \) to the three \( u(1) \)’s.

The obstruction to symmetry comes from the periodic fermionic boundary conditions. Locally the two fermionic versions are however equivalent. At the level of the \( \mathcal{L} \) matrices one has:

\[
\mathcal{L}^{(2)}_m = (V_{m+1}U_0 \mathcal{U}_m V_{m}^{-1}) \mathcal{L}^{(1)}_m (V_m U_0 \mathcal{U}_m V_m^{-1})
\]

The operator \( \mathcal{U}_m \) is non-local:

\[
\mathcal{U}_m = n_{2m} + \left( \prod_{j=1}^{m-1} (2n_{1j} - 1)(2n_{2j} - 1) \right) \bar{n}_{2m}(a_{1m}^\dagger + a_{1m})
\]

At the Hamiltonian level the correspondence is:

\[
\mathcal{H}^{(2)}_{mm+1} = \mathcal{U}_N \cdots \mathcal{U}_1 \mathcal{H}^{(1)}_{mm+1} \mathcal{U}_1 \cdots \mathcal{U}_N, \quad m = 1, \ldots, N - 1
\]

However this equivalence breaks down for the last term: \( \mathcal{H}_{N1} \). Thus the two fermionic models, despite sharing a common bosonic antecedent, are *not* equivalent. This is due to the periodic fermionic boundary conditions.

This is not the end of the story however. By fine tuning the twist parameters one can recover a large superalgebra symmetry. Integrability is preserved since it does not depend on the values of these parameters. We take

\[
x \equiv x_{12} = x_{32} = -x_{34} = x_{14}
\]

The following operators then are symmetries of all the conserved quantities:

\[
Y^{11} = \sum_m n_{1m}n_{2m} \quad Y^{22} = \sum_m \bar{n}_{1m}\bar{n}_{2m} \\
Y^{33} = \sum_m n_{1m}\bar{n}_{2m} \quad Y^{44} = \sum_m \bar{n}_{1m}n_{2m} \\
Y^{13} = \sum_m n_{1m}a_{2m}^\dagger \quad Y^{31} = \sum_m n_{1m}a_{2m} \\
Y^{24} = -\sum_m \bar{n}_{1m}a_{2m}^\dagger \quad Y^{42} = -\sum_m \bar{n}_{1m}a_{2m}
\]

They generate a fermionic realization of two copies of a Lie superalgebra: \( gl(1|1) \times gl(1|1) \). The off-diagonal operators \( Y \) appear as the ‘localized’ counterparts of the bosonic operators.

**The \( (2, 2) - XXC \) model: third avatar**

There is also a 3rd avatar for this model where one takes \( A = \{1, 4\} \) and \( B = \{2, 3\} \). We found the quantities \( \mathcal{H}^{(3)}, \mathcal{L}^{(3)} \) and verified that \( V, F \) and the grading are unchanged. The symmetry breaks down to \( u(1) \times u(1) \times u(1) \). However, upon localization of the fermionic generators and fine tuning of the twist parameters, the symmetry is enlarged to \( gl(1|1) \times gl(1|1) \). Again, due to the boundary terms, this fermionic system is not equivalent to the previous two versions. Going back to bosonic variables, we find three *integrable* periodic-twisted boundary conditions, in addition to the starting bosonic periodic one. We say more about this in section \( 3.7 \).
3.5 The (1, 3)-XXC model

We now consider the second and last XXC model with four states per site. This model is not equivalent to the above two models. It has $su(3) \oplus u(1)$ symmetry. As we have seen on the preceding two models, a particular choice of states for $A$ and $B$ is not restrictive, at the bosonic level.

We take here $A = \{1, 2, 3\}$ and $B = \{4\}$. The fermionized Hamiltonian density then reads:

$$
\mathcal{H}_{mm+1} = -x_{14} a_{1m}^\dagger a_{1m+1} n_{2m} n_{2m+1} - x_{14} a_{1m+1} a_{1m} n_{2m} n_{2m+1} + x_{24} a_{2m}^\dagger a_{2m+1} n_1 n_{1m+1} + x_{24} a_{2m+1} a_{2m} n_1 n_{1m+1} - x_{34} a_{1m}^\dagger a_{1m+1} a_{2m}^\dagger a_{2m+1} a_{2m}\n - \frac{1}{2} \cos \gamma (1 + C_m C_{m+1})
$$

where $C = 2n_1 - 1$. The matrices $V$ and $F$, and the grading are again unchanged. The $L$-matrix reads:

$$
L_m = \begin{pmatrix}
L_{11} & -a a_{1m} a_{2m} & -i a a_{1m} n_{1m} & -i c a_{1m} n_{2m} \\
-a a_{1m} a_{2m}^\dagger & L_{22} & -i a a_{1m} n_{2m} & i c a_{1m} n_{1m} \\
-a a_{2m}^\dagger n_{1m} & -a a_{1m} n_{2m} & L_{33} & -i c a_{2m} a_{1m} \\
ca_{1m}^\dagger n_{2m} & c a_{2m} n_{1m} & -i c a_{1m} a_{2m} & L_{44}
\end{pmatrix}
$$

$$
L_{11} = a n_{1m} n_{2m} - i b x_{14} n_1 n_{1m} \\
L_{22} = -a n_{1m} n_{2m} - i b x_{24} n_1 n_{1m} \\
L_{33} = i a n_{1m} n_{2m} + i b x_{34} n_1 n_{1m} \\
L_{44} = i a n_{1m} n_{2m} + b \left( x_{14} n_{1m} n_{2m} - x_{24}^{-1} n_1 n_{2m} + x_{34}^{-1} n_1 n_{2m} \right)
$$

For $x_{14} = x_{24}$ and the other $x$'s unconstrained, the surviving symmetry is $su(2) \times u(1) \times u(1)$. The generators are $X^{12}$ and $X^{21}$ and the four diagonal ones in (51). If in additions one takes $x_{14} = x_{24} = -x_{34}$, one gets a fermionic realization of the Lie superalgebra $gl(2|1)$, and an enlarged symmetry: $gl(2|1) \times u(1)$. The complete set of generators is: $X^{12}$, $X^{21}$, $X^{11}$, $X^{22}$, $X^{33}$, $X^{44}$ of (51), $Y^{13}$, $Y^{31}$ of (52), and $Y^{23} = \sum_m n_{2m} a_{1m}$, $Y^{32} = \sum_m n_{2m} a_{1m}^\dagger$.

There are three other avatars of this models and the analysis made for the (2, 2) model can be done again here. In particular, three (not four!) fermionic versions are inequivalent and we obtain 1 + 3 integrable periodic and periodic-twisted boundary conditions.

3.6 The $su(4)$-XXZ model

This model, which is not an XXC one, is based on the fundamental representation of $su(4)$. In the notation of (43) where the XXC models were further generalized, one has the (1, 1, 1; 1, 4, 4) system. We could fermionize the $su(3)$ XXZ as was done for (1, 2)-XXC model. We remain however at four states per site and study the $su(4)$ model.

The Hamiltonian density is given by:

$$
\mathcal{H}_{mm+1} = a_{1m}^\dagger a_{1m+1} \sum x_{23} n_{2m} n_{2m+1} - x_{14}^{-1} n_{2m} n_{2m+1} + a_{1m+1} a_{1m} x_{23} n_{2m} n_{2m+1} - x_{14} n_{2m} n_{2m+1} + a_{2m}^\dagger a_{2m+1} x_{24} n_{1m} n_{1m+1} - x_{13}^{-1} n_{1m} n_{1m+1}
$$

14
The grading, $V$ of the parameters tries, these survive at the fermionic level and there are no other symmetries, whatever the values parity operators are symmetries provided the twist parameters are chosen in a certain way.

The XXC model has. This generalizes to the models of [19]. The non-locality of the generalized each XXC system is associated a number of integrable boundary conditions equal to 1, for the

Let us pause and recapitulate what we have learned so far. Several conclusions arrived at for the foregoing examples are of a general nature. The first concerns boundary conditions. Going

\[
\begin{align*}
L_{11} &= a' n_1 n_2 - b' \left\{ x_{12} n_1 n_2 + i x_{13} n_1 n_2 + i x_{14} n_1 n_2 \right\} \\
L_{22} &= -a' \tilde{n}_1 n_2 + b' \left\{ x_{12}^\dagger n_1 n_2 - i x_{23} n_1 \tilde{n}_2 - i x_{24} \tilde{n}_1 n_2 \right\} \\
L_{33} &= i a' n_1 \tilde{n}_2 + b' \left\{ x_{13}^\dagger n_1 n_2 - x_{23} \tilde{n}_1 n_2 + i x_{34} \tilde{n}_1 n_2 \right\} \\
L_{44} &= i a' \tilde{n}_1 n_2 + b' \left\{ x_{14}^\dagger n_1 n_2 - x_{24} \tilde{n}_1 n_2 + i x_{34} \tilde{n}_1 n_2 \right\}
\end{align*}
\]

The grading, $V$ and $F$ matrices are unchanged. As the bosonic model has only diagonal symmetries, these survive at the fermionic level and there are no other symmetries, whatever the values of the parameters $x$. The symmetries are the $X^{ii}$ of (\[15\]), and they generate $u(1) \times u(1) \times u(1)$.

### 3.7 General remarks

Let us pause and recapitulate what we have learned so far. Several conclusions arrived at for the foregoing examples are of a general nature. The first concerns boundary conditions. Going back to bosonic variables generates many integrable systems. The non-local boundary terms are generically of the type $E_1^{\alpha \beta} (U \gamma U) \cdots (U \gamma U)_{N-1} E_N^{\beta \alpha}$, with their hermitian conjugate. Thus, to each XXC system is associated a number of integrable boundary conditions equal to 1, for the usual periodic bosonic conditions, plus a number less than or equal to the number of versions the XXC model has. This generalizes to the models of [19]. The non-locality of the generalized Jordan-Wigner transformation appears as a strength rather than a handicap.

Another important result concerns the symmetries. We have seen that symmetry generators of the bosonic model, which are local in fermionic variables, are also symmetries of the fermionic system with periodic fermionic boundary conditions. The generators which become non-local are not symmetries of the fermionic model. However we can consider their ‘localized’ versions, obtained by a simple deletion of the non-local string stemming for the fermionization. These odd parity operators are symmetries provided the twist parameters are chosen in a certain way. This also generalizes to the models of [19].
All trigonometric models admit a rational limit. The rational limit of $L$-$R$ is obtained by letting $\lambda \to \gamma \lambda$, dividing by $\sin \gamma$ and taking the limit $\gamma \to 0$. These manipulations conserve all the properties of the $L$, $R$-matrices and can be applied directly to the fermionic form, without any other change. However the eventual extended symmetries which some bosonic models may enjoy breaks down for their fermionic versions.

The general form of $v_m$ is obvious:

$$v_m = \exp \left[ \frac{i\pi}{2} \sum_{k=1}^{m-1} \sum_{j=1}^{d} (n_{jk} - 1) \right]$$ \hspace{1cm} (79)$$

The four phases $\pm 1$ and $\pm i$ arise from the braiding relations between $v_m$ and the other fermionic operators. The particular structure of $V_m$ is fixed by the choice of representation of $C_{2d}$. This is true also for the grading and the matrix $F$. The matrix elements of the diagonal matrix $F$ are found by solving a redundant system of very simple linear equations. It is in fact enough to solve part of this system, although we have solved it completely for the systems studied here.

Most of these general results extend straightforwardly to the generalized Hubbard models considered in the following section.

## 4 Hubbard models

The Jordan-Wigner transformation turns the Hubbard model into a spin-chain of two coupled XX chains [4]. In [6, 7] the construction of bosonic Hubbard models was generalized to all XX models. The latter are XXC models at their ‘free-fermions’ point: $\gamma = \frac{\pi}{2}$. The fermionic integrable structure of the original Hubbard model was obtained by generalizing the method used for the XXZ model. We first briefly recall this method [11] and then generalize it to fermionize generalized Hubbard models by fusing some of the models obtained in the preceding section. The left and right copies are labeled $\uparrow$ and $\downarrow$, respectively.

### 4.1 The Hubbard model

The Hubbard Hamiltonian is that of a spin-$\frac{1}{2}$ electron hopping on a lattice with an on-site Coulomb interaction. There are four possible states per site and:

$$H_2 = -\sum_{m} \sum_{s=\uparrow, \downarrow} \left( x_s a_{sm+1}^{\dagger} a_{sm} + x_s^{-1} a_{sm}^{\dagger} a_{sm+1} \right) + U \sum_m (2n_{\uparrow m} - 1)(2n_{\downarrow m} - 1)$$ \hspace{1cm} (80)$$

Its bosonic version [4] is given by

$$H_2 = \sum_m \left( x_{\uparrow}^{-1} \sigma_m^{\uparrow} \sigma_{m+1}^{-} + x_{\uparrow} \sigma_m^{-} \sigma_{m+1}^{\uparrow} + x_{\downarrow}^{-1} \tau_m^{\uparrow} \tau_{m+1}^{-} + x_{\downarrow} \tau_m^{-} \tau_{m+1}^{\uparrow} \right) + U \sum_m \sigma_m^{z} \tau_m^{z}$$ \hspace{1cm} (81)$$

where $\sigma^i$ and $\tau^i$ are two commuting copies of Pauli matrices. Note that this model is integrable without any constraint on the parameters $x_s$. Its $L$-matrix is given by

$$L(\lambda) = I(h) \ L_{\uparrow}(\lambda) \otimes \ L_{\downarrow}(\lambda) \ I(h)$$ \hspace{1cm} (82)$$

where

$$h = h(\lambda) = \frac{1}{2} \text{Arctanh} |U \sin(2\lambda)| \quad \text{and} \quad I(h) = \exp \left( \frac{h}{2} \sigma_{10}^{z} \sigma_{10}^{z} \right)$$ \hspace{1cm} (83)$$

The relation between $h$ and $\lambda$ is the same for all models of this section. The $V$-matrix is defined as follows:

$$V_m = V_{\uparrow m} \otimes V_{\downarrow m}$$ \hspace{1cm} (84)$$
Two spectral parameters appear instead of just a difference because the Lie algebra \( \text{so}_4 \) has fermionic constituents. For group symmetry \([20, 21, 22, 23, 24, 25]\). The two model is not additive; this is an unusual feature of the Hubbard models considered here.

We give below the results for the fusion of some of the models studied in the preceding section. This extended symmetry arises after coupling of the two XX models. It exponentiates to the \( SO(4) \) group symmetry and characterizes many physical features of the one-dimensional Hubbard model (see for instance \([26]\)).

The integrable hierarchy defined by the Hubbard Hamiltonian is known to have an \( \text{SO}(4) \) group symmetry and \( \text{SO}(4) \) is given by: \( P(1) = P(4) = 0 \) and \( P(2) = P(3) = 1 \). Two spectral parameters appear instead of just a difference because the \( R \)-matrix of the Hubbard model is not additive; this is an unusual feature of all the Hubbard models considered here.

The integrable hierarchy defined by the Hubbard Hamiltonian is known to have an \( SO(4) \) group symmetry \([20, 21, 22, 23, 24, 25]\). The two \( u(1) \) generators are inherited from the two fermionic constituents. For \( x = x' = \pm 1 \), the symmetry further extends and the generators of the Lie algebra \( \text{so}(4) = \text{su}(2) \times \text{su}(2) \) are given by:

\[
S^+ = \sum_m a_{\uparrow m}^\dagger a_{\uparrow m}, \quad S^- = \sum_m a_{\downarrow m}^\dagger a_{\downarrow m}, \quad S^z = \frac{1}{2} \sum_m (n_{\uparrow m} - n_{\downarrow m})
\]

\[
\eta^+ = \sum_m (-1)^m a_{\uparrow m}^\dagger a_{\downarrow m}^\dagger, \quad \eta^- = \sum_m (-1)^m a_{\downarrow m} a_{\uparrow m}, \quad \eta^z = \frac{1}{2} \sum_m (n_{\uparrow m} + n_{\downarrow m} - 1)
\]

This extended symmetry arises after coupling of the two XX models. It exponentiates to the \( SO(4) \) group symmetry and characterizes many physical features of the one-dimensional Hubbard model (see for instance \([26]\)).

The fermionization of the bosonic Hubbard models of \([7]\) is carried out in a similar manner. We give below the results for the fusion of some of the models studied in the preceding section.

### 4.2 The \((1, 1) \times (2, 2)\) model

We use as right copy the first version of the \((2, 2)\)-XXC model. The resulting model has a local space of dimension eight. The Hamiltonian reads:

\[
\mathcal{H}_2 = \mathcal{H}_2^\dagger + \mathcal{H}_2 + U \sum_m (2n_{\uparrow m} - 1)(2n_{\downarrow 1m} - 1)(2n_{\downarrow 2m} - 1)
\]
where $\mathcal{H}^\uparrow_2$ corresponds to (24) and $\mathcal{H}^\downarrow_2$ to (23), both at $\gamma = \pi/2$. The $V$-matrix is a tensor product of $V_{\uparrow m}$ of the preceding section on the left, and of

$$V_{\downarrow m} = \begin{pmatrix} v_{\uparrow N+1}v_{\downarrow m} & 0 & 0 & 0 \\ 0 & v_{\uparrow N+1}v_{\downarrow m}^{-1} & 0 & 0 \\ 0 & 0 & v_{\uparrow N+1}v_{\downarrow m}^{-1} & 0 \\ 0 & 0 & 0 & v_{\uparrow N+1}v_{\downarrow m}^{-1} \end{pmatrix}$$

(93)

$$v_{\downarrow m} = \exp \left( \frac{i \pi}{2} \sum_{i=1}^{m-2} \sum_{j=1}^{2} (n_{sji} - 1) \right)$$

(94)

on the right. This yields for the Lax matrix $\mathcal{L}$:

$$\mathcal{L}_m = I(h) \tilde{\mathcal{L}}_{\uparrow m} \otimes \mathcal{L}_{\downarrow m} I(h)$$

(96)

where

$$I(h) = \exp \left( \frac{h}{2} \gamma_5 \gamma_0 \right) \quad \text{and} \quad \gamma_5 = \text{diag}(1, 1, -1, -1)$$

(97)

$\tilde{\mathcal{L}}_{\uparrow m}$ has been defined in the preceding section while $\mathcal{L}_{\downarrow m}$ is a copy of (57). The grading above is:

$$P_{\uparrow}(1) = 0, \quad P_{\uparrow}(2) = 1, \quad P_{\downarrow}(1) = P_{\downarrow}(2) = 0, \quad P_{\downarrow}(3) = P_{\downarrow}(4) = 1$$

(98)

The $64 \times 64$ matrix $F$ is given by

$$F = \sigma^z \otimes \text{diag}(1, i, -1, -i, i, i, i, i) \otimes \mathbb{I}_4$$

(99)

and the grading here is:

$$P(1) = P(2) = P(7) = P(8) = 0, \quad P(3) = P(4) = P(5) = P(6) = 1$$

(100)

This grading is used in the supertrace.

For all $x$’s on the right equal to each other, the symmetry of this Hubbard model is $u(1)_\uparrow \times su(2)_\uparrow \times su(2)_\downarrow \times u(1)_\downarrow$. Note that is is possible to take the second (or third) copy of the (2,2) model. The Hamiltonian (22) is modified accordingly. By choosing the $x$’s on the right as indicated in section (5.3), one restores the symmetry to: $u(1)_\uparrow \times gl(1|1)_\downarrow \times gl(1|1)_\downarrow$. It is also possible to choose the model (1,3) instead of (2,2), and the appropriate choice of $x$’s, with a resulting symmetry of: $u(1)_\uparrow \times gl(2|1)_\downarrow \times u(1)_\downarrow$. The $S, V, F$ matrices and the gradings are unchanged.

### 4.3 The $(2,2) \times (2,2)$ model

Again we use two copies of the first version of the $(2,2)$-XXC model. The dimension of the local Hilbert space is sixteen. The Hamiltonian is now given by:

$$\mathcal{H}_2 = \mathcal{H}^\uparrow_2 + \mathcal{H}^\downarrow_2 + U \sum_m (2n_{\uparrow 1m} - 1)(2n_{\uparrow 2m} - 1)(2n_{\downarrow 1m} - 1)(2n_{\downarrow 2m} - 1)$$

(101)

where $\mathcal{H}^\uparrow_2$ and $\mathcal{H}^\downarrow_2$ corresponds to two copies of (53) at $\gamma = \pi/2$. The numerical coupling matrix is given by: $I(h) = \exp(\frac{h}{2} C_{\uparrow 0} C_{\downarrow 0})$ where $C = \gamma_5 = \text{diag}(1, 1, -1, -1)$. The $V$-matrices are:

$$v_{sm} = \exp \left( \frac{i \pi}{2} \sum_{i=1}^{m-1} \sum_{j=1}^{2} (n_{sji} - 1) \right) \quad , \quad s = \uparrow, \downarrow$$

(102)

$$V_m = \text{diag}(v_{\uparrow m}, v_{\downarrow m}, v_{\uparrow m}^{-1}, v_{\downarrow m}^{-1}) \otimes \text{diag}(v_{\uparrow N+1} v_{\downarrow m}, v_{\uparrow N+1} v_{\downarrow m}^{-1}, v_{\uparrow N+1} v_{\downarrow m}^{-1}, v_{\uparrow N+1} v_{\downarrow m}^{-1})$$
The $L_{sm}$ matrices are copies of matrix (57) at $\gamma = \pi/2$. The grading in (86–87) is given by:

$P(1) = P(2) = 0$, $P(3) = P(4) = 1$, and $S = \text{diag}(1, -1, i, i)$ for the 'up' copy. For (30) and the supertrace the grading is:

$P(1) = P(2) = P(3) = P(4) = 1$ (103)

The $256 \times 256$ matrix $F$ is given by

$F = \sigma^z \otimes I_2 \otimes \text{diag}(1, -1, i, -i, 1, 1, -i, i, -1, -1, 1, 1) \otimes I_4$ (104)

For all $x$’s, on both left and right independently, equal to each other, the symmetry of this Hubbard model is $\text{su}(2)^\uparrow \times \text{su}(2)^\uparrow \times \text{u}(1)^\uparrow \times \text{su}(2)^\downarrow \times \text{u}(1)^\downarrow$. As indicated at the end of section (4.2), it is possible to fuse any two copies of the other versions of (2, 2) or (1, 3). The resulting symmetry changes accordingly, with the appropriate choices of $x$’s. The gradings and $V, S, F$ are unchanged. It is possible to couple also (1, 2) models, or in general any two XX models in their fermionic form. A fermionic Hubbard Hamiltonian is simply:

$H_2 = H_2^\uparrow + H_2^\downarrow + \sum_m c_m^\dagger c_m$ (105)

### 4.4 Symmetry and Integrability

There are many models of itinerant electrons in the literature which qualify as such for the name Hubbard model. However, most of these models have an additive $R$-matrix, or do not share the same algebraic structure of the original Hubbard model (the Bariev model). Here we chose to use the term Hubbard models for a class of models which share the same integrable and algebraic structure as the original model along with a similar structure for the non-additive $R$-matrix. These bosonic multistates Hubbard models were constructed and studied in [6, 7]. (Lax pairs were derived in [27].) They correspond to arbitrary up and down copies of the generalized XX models, with and on-site coupling. This structure, which is the natural generalization of the Hubbard model one, has been fermionized here, and we obtained the first multispecies generalizations written in fermionic form.

Recall that (see section (4.1)) the Hubbard model has, after coupling of its two components, an extended symmetry not shared by its two independent components. This symmetry also exponentiates to a group symmetry. It is therefore natural to ask whether the new models we obtained here have such a property. We have looked for additional symmetries of the new Hubbard models, and found it is unlikely that they exist. This negative result can be explained by a conflict between higher symmetries and the strictures of integrability. The form of the fermionized XXC models shows that they are not constructed out of group invariants and their symmetries do not exponentiate. Coupling does not solve this problem. There is however a finite number of discrete symmetries which appear; they correspond to fermion species interchanges. In fact one could easily build models of itinerant electrons with larger symmetries. However integrability breaks down the symmetry group manifold to a discrete set of points, some corresponding to the symmetries of the XXC blocks and some appearing only for the Hubbard models.

### 5 Non integrable models

The fermionized Hamiltonians seen so far were obtained directly from their bosonic counterparts without use of any integrability criterion. The generalized fermionization procedure is independent of any integrable structure. We now fermionize two non integrable models of considerable physical interest.
5.1 The spin-1 XXZ Heisenberg chain

The bosonic Hamiltonian is given by:

$$H_2 = \sum_m (S^x_m S^x_{m+1} + S^y_m S^y_{m+1} + \Delta_z S^z_m S^z_{m+1})$$  \hspace{1cm} (106)$$

where the $S^i$ form a spin one representation of $su(2)$, at every site. For arbitrary $\Delta_z$, the diagonal operator $S^z = \sum_m S^z_m$ commutes with $H_2$. For $\Delta_z = 1$, $S^{x,y} = \sum_m S^{x,y}_m$ in addition commute with the Hamiltonian.

To fermionize this model one has four choices of embeddings. We chose the same one as in section (3.2). All states with double occupancy on at least one site are therefore excluded. The fermionization then yields:

$$H_{mm+1} = (a_{1m+1}^\dagger a_{1m} + a_{1m}^\dagger a_{1m+1} + \bar{n}_{2m}\bar{n}_{2m+1})$$  \hspace{1cm} (107)$$

where

$$\pi_0 \equiv 1, \quad \pi_{m-1} = (2n_{11} - 1)(2n_{21} - 1) \cdots (2n_{1m-1} - 1)(2n_{2m-1} - 1)$$  \hspace{1cm} (108)$$

Note that, contrary to all the integrable models we have studied so far, there are non-local contributions associated with odd-parity combinations in the Hamiltonian density. It is also easily seen that its action is stable on states without double occupancy. Let $p$ be the projection operator (13) defined in section (3.2). The action of $H_2 = \sum_m H_{mm+1}$ on the $3^N$-dimensional subspace is equivalent to the action of $pH_2 p$ on the $4^N$-dimensional space.

We may remove the $\pi_{m-1}$’s, leaving their factors in, wherever they appear in (107), and obtain a local density which combines bosonic and fermionic pieces. The local spin-one fermionic version the XXZ Heisenberg model may then be defined this way. The diagonal operator $X^z = \sum_m \bar{n}_{1m}(2n_{2m} - 1)$ still commutes with the Hamiltonian with periodic boundary conditions. However as seen for the spin-$\frac{1}{2}$ Heisenberg chain, the extended symmetry at $\Delta_z = 1$ does not survive the fermionization.

5.2 The spin-$\frac{3}{2}$ XXZ Heisenberg chain

The bosonic Hamiltonian is given by (106) where now the $S^i$ span a spin $\frac{3}{2}$ representation of $su(2)$. Here the Hilbert space of the chain is $4^N$-dimensional. We find for the fermionized Hamiltonian density:

$$H_{mm+1} = \frac{1}{2} a_{1m}^\dagger a_{1m+1} \left[ 3a_{2m}^\dagger a_{2m+1} + 3a_{2m} a_{2m+1}^\dagger + 3a_{2m} a_{2m+1} \right]$$  \hspace{1cm} (109)$$

+ $3a_{2m} a_{2m+1}^\dagger + 4 \bar{n}_{2m}\bar{n}_{2m+1}$

+ $\frac{1}{2} a_{1m+1}^\dagger a_{1m} \left[ 3a_{2m+1}^\dagger a_{2m} + 3a_{2m+1} a_{2m}^\dagger + 3a_{2m} a_{2m+1} \right]$

+ $3a_{2m+1} a_{2m}^\dagger + 4 \bar{n}_{2m}\bar{n}_{2m+1}$

+ $\frac{\sqrt{3}}{2} a_{1m}^\dagger a_{1m+1} \left[ a_{2m+1}^\dagger \bar{n}_{2m+1} - a_{2m}\bar{n}_{2m+1} - a_{2m+1}^\dagger \bar{n}_{2m} - a_{2m+1} \bar{n}_{2m} \right] \pi_{m-1}$

+ $\frac{\sqrt{3}}{2} a_{1m} a_{1m+1} \left[ a_{2m+1}^\dagger \bar{n}_{2m+1} - a_{2m}\bar{n}_{2m+1} + a_{2m+1}^\dagger \bar{n}_{2m} + a_{2m+1} \bar{n}_{2m} \right] \pi_{m-1}$

+ $\frac{\Delta_z}{4} (2n_{1m} - 1)(4n_{2m} - 1)(2n_{1m+1} - 1)(4n_{2m+1} - 1)$

+ $\sqrt{3} a_{1m+1}^\dagger a_{1m} \left[ a_{2m+1}^\dagger \bar{n}_{2m+1} - a_{2m}\bar{n}_{2m+1} - a_{2m+1}^\dagger \bar{n}_{2m} - a_{2m+1} \bar{n}_{2m} \right] \pi_{m-1}$
Remarks similar to those made for the spin-1 case hold here. In particular a local density may be defined by the deletion of $\pi_{m-1}$. The diagonal commuting operator is given by

$$X^z = \frac{1}{2} \sum_m (2n_{1m} - 1)(4n_{2m} - 1)$$  \hspace{1cm} (110)

6 Conclusion

We have introduced a generalized Jordan-Wigner transformation which allows the fermionization/bosonization of integrable and non-integrable spin chains. Several variants of this transformation were described. We have shown on various examples how the mapping works. We also obtained the first fermionic versions of generalized Hubbard models.

The study of particular models allowed to infer some general results. In particular we showed that the non-conservation of boundary conditions, far from being a ‘shortcoming’, is in fact an advantage and generates many new integrable boundary conditions of the periodic-twisted type. Another issue is the mapping of certain local operators to non-local ones. However we have shown that, the non-local fermionic operators can be made local and the symmetry of the model turned into supersymmetry, without any loss of integrability. Fermionic realizations of Lie algebras and superalgebras appeared naturally in this context.

Periodic boundary conditions were considered here. But the fermionization method can be directly applied to models with other types of boundary conditions, within or without the framework of integrability.

Fermionization and bosonization of one-dimensional systems are an expression of a local equivalence between a bosonic language and a fermionic one. This is clearly seen at the Hamiltonian level, for integrable and non integrable models. The fermionization of the $L$-matrix is a local procedure. The important point is to notice that a periodic boundary condition breaks this equivalence, while an open one, where $H_{N1}$ is dropped, does not. (Integrability may be lost but this is a secondary point.) The issue of boundary conditions is intrinsic to any fermionization scheme.

The present work was near completion when [8] appeared. The authors of [8] proposed a fermionization scheme which applies only to integrable models. Their main example corresponds to the one in section (3.2), at $\gamma = \pi/2$. The Hamiltonians are given by the same expressions while $L$-matrices have the same structure. The phases $\pm 1$ do not appear, and this is general, in the method of [8]. It would be interesting and instructive to find, for integrable systems, the exact relation between the two methods. However let us stress that boundary issues invariably arise for any fermionization method, and only the foregoing approach tackles this issue. The method of Göhmann and Murakami and the generalized Jordan-Wigner transformation are therefore complementary.

A given one-dimensional Hamiltonian density can be transposed on a higher-dimensional lattice. While in one-dimension bosons and fermions are closely related, the relationship breaks down in higher dimensions. This is one important motivation for finding the fermionic expression of a bosonic model in one-dimension, before going to higher dimensions. The physical properties of a model may change drastically with dimension. As noted in section (4.4), the study of one-dimensional models would shed some light on the interplay between integrability and symmetries. But it is only on a two-dimensional square lattice with nearest-neighbor interaction that the Hubbard model exhibits $d$-wave superconductivity. It would be worthwhile to study the new Hubbard models in one and two dimensions.

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