Abstract

A review of the appearance of integrable structures in the matrix model description of 2d-gravity is presented. Most of the ideas are demonstrated at the technically simple but ideologically important examples. Matrix models are considered as a sort of "effective" description of continuum 2d field theory formulation. The main physical role in such description is played by the Virasoro-W conditions, which can be interpreted as certain unitarity or factorization constraints. Both discrete and continuum (Generalized Kontsevich) models are formulated as the solutions to those discrete (continuous) Virasoro-W constraints. Their integrability properties are proved, using mostly the determinant technique highly related to the representation in terms of free fields. The paper also contains some new observations connected to formulation of more general than GKM solutions and deeper understanding of their relation to 2d gravity.
1 Introduction

The most geometrical way to derive the 2d gravity or string theory partition function (as well as the generating function for the correlators) is given by the Polyakov path integral which for the partition function reads

\[ F(\lambda) = \sum_{p-genus} \lambda^p F_p \]

\[ F_p = \int_{\Sigma_p} \mathcal{D}g \exp \gamma \int R \Delta^{-1} R \]  

(1)

Matrix models historically appeared \[\text{[1]}\] when one considers the discretization of (1) which in a sense turns to be exact at least for the simplest cases of ”empty” string theories, when the target space is low-dimensional (or in the limit of pure gravity does not exist at all). A well-known example of such theories is given by minimal \((p, q)\) models coupled to 2d gravity.

The difficulties with the continuum formulation \([1]\) as usual arise from the fact that it possesses additional complicated information (like the ”Verma-module” structure of the underlying 2d CFT) which might not be essential for description of final ”effective” theory. We have exactly here the case of a gauge-theory, when after coupling to gravity all conformal descendants are ”gauged away” and one might search for a sort of ”frame” description, which luckily appears in the form of matrix models.

Matrix models are usually defined by the integrals like

\[ Z_N = \int \mathcal{D}M_{N \times N} \exp[-TrV(M)] \]  

(2)

which in the continuum limit, requiring in part \(N \rightarrow \infty\)

\[ \log Z \xrightarrow{\scriptstyle{N \rightarrow \infty}} F \]  

(3)

can give the whole nonperturbative solution to \([1]\) \([2]\).

However, below we will mostly advocate the point of view different from the original definition \([2]\). Indeed, the main difference between the continuum \([1]\) and the matrix model formulation is that the first one requires some sort of the unitarity or factorization relations between different terms in the sum over topologies in \((1)\), while in the matrix
model formulation (2) these relations usually appear automatically. Moreover, for known solutions they usually appear in the well-known form of the Virasoro-W constraints, which in a sense may be considered as a definition of matrix model.

Below, we will usually start the description of various matrix model and matrix model-like theories as being particular solutions to these Virasoro-W recursion relations. It turns out that these relations lead to the integrability property of matrix models, namely the solutions to these constrains turn to be $\tau$-functions of the hierarchies of well-known integrable equations [3, 4, 5].

In terms of the partition function (1), (3)

$$F(T) = \log \tau(T)$$

(4)

where $T \equiv \{T_k\}$ is the set of times or the coupling constants in the theory of 2d gravity.

The formula (4) or the appearence of the integrability is exactly what allows one to make more progress in studying "frame" formulation (2) instead of the original one (1).

In what follows, we will first consider an example of discrete matrix models (finite $N$ in (2)) as being solution to the most simple discrete Virasoro constraints and then pass to the continuum case. Both kinds of solutions to the recursion relations are $\tau$-functions of well-known hierarchies and both particular solutions have representation in integral form.

We will also stress the moment that the solutions to discrete constraints correspond to the discretization of the world-sheet of string, while, as it will be seen below the matrix solutions to the continuum relations have rather interpretation of target-space theory or of an effective string field theory.

---

This should have an interpretation as a sort of world-sheet – target-space duality between the Virasoro symmetry as gauge symmetry for 2d gravity and Virasoro relations in the target-space (or better in the space of coupling constants).
2 Discrete hermitean 1-matrix model as a solution to the discrete Virasoro constraints

In this section we are going to consider the first (and simplest) example, demonstrating the above ideas, namely the solution to the discrete Virasoro constraints [6]:

\[ L_n Z[t] = 0, \quad n \geq -1 \]

\[ L_n \equiv \sum_{k=0}^{\infty} k t_k \partial \partial t_{k+n} + \sum_{a+b=n} \partial^2 \partial a \partial b \]

(5)

with an additional requirement (concerning \( t_0 \)-variable)

\[ \partial Z_N / \partial t_0 = -NZ_N \]

(later on \( N \) will be identified with the size of matrices in the formulas like (2)). The key idea how to solve the constraints (5) appears after one notices that the Virasoro generators (5) actually have the well-known form of the Virasoro operators in the theory of one free scalar field [2]. If we try to look for such solution in terms of holomorphic components of the scalar field

\[ \phi(z) = \hat{q} + \hat{p} \log z + \sum_{k \neq 0} \frac{J_{-k}}{k} z^{-k} \]

\[ [J_n, J_m] = n \delta_{n+m,0}, \quad [{\hat{q}}, {\hat{p}}] = 1 \]

(6)

the procedure is as follows. First, define the vacuum states

\[ J_k |0\rangle = 0, \quad \langle N | J_{-k} = 0, \quad k > 0 \]

\[ \hat{p} |0\rangle = 0, \quad \langle N | \hat{p} = N \langle N \]

(7)

"Half" of the stress-tensor components

\[ T(z) = \frac{1}{2} [\partial \phi(z)]^2 = \sum T_n z^{-n-2}, \quad T_n = \frac{1}{2} \sum_{k>0} J_{-k} J_{k+n} + \frac{1}{2} \sum_{a+b=n \atop a,b \geq 0} J_a J_b, \]

(8)

\[ ^2\text{In this sense we have } c = 1 \text{ for (6) though it is not too sensible to speak of the central charge, having only the half } (n \geq -1) \text{ of the Virasoro algebra, i.e.} \]

\[ [L_n, L_m] = (n - m) L_{n+m}, \quad n, m \geq -1 \]

always without the central element.

\[ ^3\text{For the sake of brevity, we omit the sign of normal ordering in the evident places, say, in the expression} \]


obviously vanish the $SL(2)$-invariant vacuum

$$T_n|0\rangle = 0, \quad n \geq -1$$

(9)

Second, we define the Hamiltonian by

$$H(t) = \frac{1}{\sqrt{2}} \sum_{k>0} t_k J_k = \oint_{C_0} V(z) j(z)$$

$$V(z) = \sum_{k>0} t_k z^k, \quad j(z) = \frac{1}{\sqrt{2}} \partial \phi(z).$$

(10)

Now one can easily construct a “conformal field theory” solution to (5) in two steps. The basic ”transformation”

$$L_n \langle N|e^{H(t)} \cdots = \langle N|e^{H(t)} T_n \cdots$$

(11)

can be checked explicitly. As an immediate consequence, any correlator of the form

$$\langle N|e^{H(t)} G|0\rangle$$

(12)

($N$ counts the number of zero modes, ”included” in $G$ – that is the role of the size of matrix in (2)) gives a solution to (5) provided by

$$[T_n, G] = 0, \quad n \geq -1$$

(13)

The conformal solution to (13) (and therefore to (5)) immediately comes from the basic properties of $2d$ conformal algebra. Indeed, any solution to

$$[T(z), G] = 0$$

(14)

is a solution to (13), and it is well-known that the solution to (14) is (by definition of the chiral algebra) a function of screening charges in the free scalar field theory given by

$$Q_\pm = \oint J_\pm = \oint e^{\pm \sqrt{2} \phi}.$$  

(15)

With a selection rule on zero mode it gives

$$G = \exp Q_+ \rightarrow \frac{1}{N!} Q_+^N$$

(16)

(Of course, the general case might be $G \sim Q_+^{N+M} Q_-^M$ but the special prescription for integration contours, proposed in [6], implies that the dependence of $M$ can be irrelevant and one can just put $M = 0$.) In this case the solution
after computation of the free theory correlator, analytic continuation of the integration contour gives the result

\[
Z_{2,N} = (N!)^{-1} \int \prod_{i=1}^{N} dz_i \exp \left( - \sum t_k z_i^k \right) \Delta_N^2(z) = (N! \text{Vol } U(N))^{-1} \int DM \exp \left( - \sum t_k M^k \right)
\]

\[
\Delta_N = \prod_{i<j}^{N} (z_i - z_j)
\]

in the form of \textit{multiple} integral over the "spectral parameter" or, in this particular case, in the form of the integral over Hermitian matrices of the type of eq. (18).

This point of view actually could be considered as a constructive one. Namely, instead of considering a special direct multi-matrix generalization of (18) one can use powerful tools of 2\textit{d} conformal field theories, where it is well known how to generalize almost all the steps of above construction: first, instead of looking for a solution to Virasoro constraints one can impose \textit{extended Virasoro} or \textit{W}-constraints on the partition function. In such case one would get Hamiltonians in terms of \textit{multi}-scalar field theory, and the second step is generalized directly using \textit{screening charges} for \textit{W}-algebras. The general scheme of solving discrete \textit{W}-constraints looks as follows [7]:

(i) Consider Hamiltonian as a linear combination of the Cartan currents of a level one Kac-Moody algebra \( G \)

\[
H(t^{(1)}, \ldots, t^{(\text{rank } G)}) = \sum_{\lambda, k>0} t_k^{(\lambda)} \mu_\lambda J_k,
\]

where \( \{\mu_i\} \) are basis vectors in Cartan hyperplane, which for \( SL(p) \) case are chosen to satisfy

\[
\mu_i \cdot \mu_j = \delta_{ij} - \frac{1}{p}, \quad \sum_{j=1}^{p} \mu_j = 0.
\]

(ii) The action of differential operators \( W_i^{(a)} \) with respect to times \( \{t_k^{(\lambda)}\} \) can be now defined from the relation

\[
W_i^{(a)} \langle N | e^{H(t)} \rangle \ldots = \langle N | e^{H(t)} W_i^{(a)} \rangle \ldots, \quad a = 2, \ldots, p; \quad i \geq 1 - a,
\]

where

\[
W_i^{(a)}(z) = \oint z^{a+i-1} W^{(a)}(z) dz
\]

\[
W^{(a)}(z) = \sum_{\lambda} \lambda \partial_\lambda \phi(z)^{\lambda}.
\]
are spin-$a$ W-generators of $W_p$-algebra written in terms of rank $G$-component scalar fields.

(iii) The conformal solution to the discrete $W$-constraints arises in the form \[ Z_{p,N}^{\text{CM}}[\{t\}] = \langle N|e^{H(t)}G\{Q^{(\alpha)}\}|0 \rangle \]

where $G$ is again an exponential function of screenings of level one Kac-Moody algebra (see \[9\] and references therein)

\[ Q^{(\alpha)} = \oint J^{(\alpha)} = \oint e^{\alpha \phi} \]

\{\alpha\} being roots of finite-dimensional simply laced Lie algebra $G$. The correlator (22) is still a free-field correlator and the computation gives it again in a multiple integral form

\[ Z_{p,N}^{\text{CM}}[\{t\}] \sim \prod_{\alpha} \prod_{i=1}^{N_{\alpha}} dz^{(\alpha)}_i \exp \left( - \sum_{\lambda,k>0} t^{(\lambda)}_k (\mu_{\lambda} \alpha)(z^{(\alpha)}_i)^k \right) \times \]

\[ \times \prod_{(\alpha,\beta)} \prod_{i=1}^{N_{\alpha}} \prod_{j=1}^{N_{\beta}} (z^{(\alpha)}_i - z^{(\beta)}_j)^{\alpha \beta} \] \hspace{1cm} (24)

The only difference with the one-matrix case (18) is that the expressions (24) have rather complicated representation in terms of multi-matrix integrals, the following objects will necessarily appear

\[ \prod_{i=1}^{N_{\alpha}} \prod_{j=1}^{N_{\beta}} (z^{(\alpha)}_i - z^{(\beta)}_j)^{\alpha \beta} = \left[ \det \{ M^{(\alpha)} \otimes I - I \otimes M^{(\beta)} \} \right]^{\alpha \beta} , \] \hspace{1cm} (25)

However, this is still a model with a chain of matrices and with closest neighbour interactions only (in the case of $SL(p)$).

Actually, it can be shown that CMM, defined by (22) as a solution to the $W$-constraints has a very rich integrable structure and possesses a natural continuum limit \[7, 10\]. To pay for these advantages one should accept a slightly less elegant matrix integral with the entries like (25).

The first non-trivial example is the $p = 3$ solution to $W_3$-algebra: an alternative to the conventional 2-matrix model. In this case one has six screening charges $Q^{(\pm \alpha_i)} (i = 1, 2, 3)$ which commute with

\[ W^{(\pm \alpha_i)}(z) = T(z) = \frac{1}{2} (\partial \phi(z))^2 \] \hspace{1cm} (26)
and

\[ W^{(3)}(z) = \sum_{\lambda=1}^{3} (\mu_{\lambda} \partial \phi(z))^3, \quad (27) \]

where \( \mu_{\lambda} \) are vectors of one of the fundamental representations (3 or \( \overline{3} \)) of \( SL(3) \).

The particular form of integral representation (24) depends on particular screening insertions to the correlator (22). We will concentrate on the solutions which have no denominators. One of the reasons of such choice is that these solutions possess the most simple integrable structure, though the other ones can still be analyzed in the same manner.

The simplest solutions which have no denominators correspond to specific correlators

\[ Z_{CMM}^{(3)}[N] = \langle N|e^{H(t)} \prod \exp Q_{\alpha_i}|0 \rangle \quad (28) \]

when we take \( \alpha_i \) to be “neighbour” (not simple!) roots: \( (\alpha_i, \alpha_j) = 1 \). In the case of \( SL(3) \) this corresponds, say, to insertions of only \( Q_{\alpha_1} \) and \( Q_{\alpha_2} \) (again, we use the notations of [9]) and gives

\[ Z_{CMM}^{(3)}[N][t, \overline{t}] = Z_{CMM}^{(3)}[N] = \frac{1}{N! M!} \langle N, M|e^{H(t, \overline{t})}(Q^{(\alpha_1)})^N(Q^{(\alpha_2)})^M|0 \rangle = \]

\[ = \frac{1}{N! M!} \int \prod dx_i dy_i \exp \left(-\sum[V(x_i) + \overline{V}(y_i)]\right) \Delta^2_N(x) \Delta^2_M(y) \prod_{i,j}(x_i - y_j) \quad (29) \]

Possible generalizations of the above scheme can include the ”supersymmetric matrix models” [11], where the authors looked for a solution to the system of equations \( L_n Z = 0 \) and \( G_m Z = 0 \), with generators \( \{L_n, G_m\} \) forming the \( N = 1 \) superconformal algebra, which in our language is nothing but a trivial generalization of the one-field case which has to be substituted by a scalar superfield. Then the insertion of screenings of \( N = 1 \) superconformal algebra immediately leads to the result of [11]. From this point of view the real problem with supersymmetric generalization can arise only in the \( N = 2 \) case because of the lack of appropriate screening operators.

2.1 Determinant representation and integrability of the solutions to Virasoro- \( W \) constraints

In the simplest case of the Hermitean one-matrix model the \( N \times N \) matrix integral \( (18) \)
\[ \langle i|j \rangle \equiv \langle P_i, P_j \rangle = \int P_i(m)P_j(m)e^{-V(m)}dm = \delta_{ij}e^{\phi_i(t)} \]  

(30)

and equals

\[ Z_N = \prod_{i=0}^{N-1} e^{\phi_i(t)} \]  

(31)

It follows from (30) and the definition of orthogonal polynomials

\[ P_i(m) = \sum_{j \leq i} a_{ij}m^j \]  

(32)

that

\[ \text{diag}(e^{\phi_i(t)}) = AHA^T, \]  

(33)

where \( A = \|a_{ij}\| \), \( A^T \) – transposed matrix, and \( H \) is so called matrix of moments

\[ H_{ij} = \int m^{i+j}e^{-V(m)}dm \]  

(34)

Thus,

\[ Z_N[t] = \text{det}[\text{diag}(e^{\phi_i(t)})] = \text{det} AHA^T = \text{det} H = \tau_N[t] \]  

(35)

Without going into all the details, which could be found in [12, 13] we will only point out that (33) is a kind of a Riemann-Hilbert problem and the determinant formula (35) is one of the basic definitions of \( \tau \)-function of Toda theory.

The relation (31) and (35) is actually based only on the fact that in the theory of Toda-chain there exists a relation between the potentials of the Toda-chain equations and the \( \tau \)-function having the form of a difference operator:

\[ e^{\phi_i(t)} = \frac{\tau_{i+1}(t)}{\tau_i(t)} \]  

(36)

Now to check that we have really got Toda chain hierarchy, let us check the first equation - flow in the direction \( \frac{\partial}{\partial \tau_i} \). This might be done after we introduce the Lax
operator for the Toda chain (see [12] for details), which in the basis of the orthogonal polynomials (30) acts by:

\[ mP_i(m) = P_{i+1}(m) - p_i(t)P_i(m) + R_i(t)P_{i-1}(m) \]  (37)

i.e. is determined by a trilinear matrix, what follows from a sort of ”unitarity condition” or just the properties of the basis (30). Now, from (30), (37) one can easily establish the relations among the ”potentials” \( \{\varphi_i(t)\} \) and the matrix elements \( \{R_i(t), p_i(t)\} \), first:

\[ \langle i|m|i-1 \rangle = e^{\varphi_i(t)} = R_i(t)e^{\varphi_{i-1}(t)} \]  (38)

gives

\[ R_i(t) = e^{\varphi_i(t)}e^{-\varphi_{i-1}} \]  (39)

Now, differentiating (30) for \( i=j \) one gets

\[ \frac{\partial}{\partial t_1}(i|j) = e^{\varphi_i(t)}\frac{\partial \varphi_i}{\partial t_1} = \int dme^{-\sum t_km_k}(-mP_i^2 + 2P_i\frac{\partial P_i}{\partial t_1}) = p_i(t)e^{\varphi_i(t)} \]  (40)

where the second term in brackets dissappear again from the orthogonality condition and property (32). From (40) it follows that

\[ p_i(t) = \frac{\partial \varphi_i(t)}{\partial t_1} \]  (41)

is just the momentum for the \( \varphi_i(t) \)-coordinate.

Differentiating (30) with \( i > j \), we obtain

\[ 0 = -\int dme^{-\sum t_km_k}\left(mP_iP_j + P_j\frac{\partial P_i}{\partial t_1}\right) \]  (42)

comparing which with (37) and using (30) one gets

\[ \frac{\partial P_i}{\partial t_1} = R_iP_{i-1} \]  (43)

Now we are ready to differentiate (37)

\[ \frac{\partial P_i}{\partial t_1} \quad \frac{\partial P_{i+1}}{\partial t_1} \quad \frac{\partial p_i}{\partial t_1} \]  (44)
Multiplying (44) by $P_i$ integrating and using (30) and (43) one finally gets

$$
\frac{\partial p_i}{\partial t_1} = R_{i+1} - R_i \tag{45}
$$

or using (11), (39)

$$
\frac{\partial^2 \varphi_i}{\partial t_1^2} = e^{\varphi_{i+1} - \varphi_i} - e^{\varphi_i - \varphi_{i-1}} \tag{46}
$$

which is nothing but the first Toda-chain equation. In terms of the $\tau$-function (46) can be rewritten in the Hirota bilinear form:

$$
\tau_N(t) \frac{\partial^2}{\partial t_1^2} \tau_N(t) - \left( \frac{\partial \tau_N(t)}{\partial t_1} \right)^2 = \tau_{N+1}(t) \tau_{N-1}(t) \tag{47}
$$

The $\tau$-function of $p = 2$ case (35) can be also written in the form

$$
Z_{2,N}(t) = \text{det}_{N \times N} [\partial^j z^k - 2 C(t)] = \tau_N(t) \tag{48}
$$

with

$$
\partial_z C(t) = \partial_{\bar{t}} \bar{C}(\bar{t}) \tag{49}
$$

Eq.(49) means that $C(t)$ just has an integral representation

$$
C(t) = \int d\mu(z) \exp \sum t_k z^k, \tag{50}
$$

where $d\mu$ is some measure; the Virasoro constraints (5) fix the concrete measure ($d\mu = dz$) and the contour of integration in (50). The determinant form (48) is an explicit manifestation of the fact that the partition function does satisfy the Hirota bilinear relations, the simplest one of which in this particular case takes the form (17).

Now one can generalize (48) and (49) [7]. In the case of the $p = 3$ model (22) we have to introduce two functions instead of (50):

$$
C(t) = \int dz \exp[-V(z)], \quad \bar{C}(\bar{t}) = \int d\bar{z} \exp[-\bar{V}(\bar{z})] \tag{51}
$$

where

$$
V(z) = \sum_{k>0} t_k z^k, \quad \bar{V}(\bar{z}) = \sum_{k>0} \bar{t}_k \bar{z}^k
$$

and
Again, we can get the determinant representation, now having the form ($\partial \equiv \partial/\partial t_1$, $\bar{\partial} \equiv \partial/\partial \bar{t}_1$)

$$Z_{N,M}(t,\bar{t}) = \det \left[ \begin{array}{cccccc} C & \partial C & \ldots & \partial^{N-1}C & \bar{C} & \bar{\partial}C & \ldots & \bar{\partial}^{M-1}\bar{C} \\ \partial C & \partial^2 C & \ldots & \partial^{N}C & \bar{\partial}C & \bar{\partial}^2\bar{C} & \ldots & \bar{\partial}^{M}\bar{C} \\ \partial^{N+M-1}C & \partial^{N+M}C & \ldots & \partial^{2N+M-2}C & \bar{\partial}^{N+M-1}\bar{C} & \bar{\partial}^{N+M}\bar{C} & \ldots & \bar{\partial}^{2N+M-2}\bar{C} \end{array} \right] = \tau_{N,M}(t,\bar{t})$$

which is exactly the double-Wronskian representation of a $\tau$-function \cite{9}. From representation (53) it is easy to derive the analogs of the Hirota relation (47)

$$\frac{\partial^2}{\partial t_1\partial \bar{t}_1} \log \tau_{N,M}(t,\bar{t}) = \frac{\tau_{N+1,M-1}(t,\bar{t})\tau_{N-1,M+1}(t,\bar{t})}{\tau_{N,M}^2(t,\bar{t})}$$

"Higher-times" Hirota relations have more complicated form.

2.2 Fermionic representation

Now we shall proceed to the representation of the solutions to the Virasoro-W constraints in terms of the fermionic correlation functions \cite{11} developed for generic integrable systems in \cite{12}.

Indeed, the $\tau$-function of 2-component KP hierarchy defined by the fermionic correlator

$$\tau_{N,M}^{(2)}(x,y) = \langle N, M | e^{H(x,y)} G | N+M, 0 \rangle$$

(55)

with

$$H(x, y) = \sum_{k>0} (x_k J_k^{(1)} + y_k J_k^{(2)})$$

(56)

$$J_k^{(i)}(z) = \sum J_k^{(i)} z^{-k-1} = : \psi^{(i)}(z) \psi^{(i)*}(z) :$$

(57)

$$\psi^{(i)}(z) \psi^{(j)*}(z') = \frac{\delta_{ij}}{z-z'} + \ldots$$

(58)

Indeed, the two types of technique we are using are practically equivalent: the symmetry (better antisymmetry) of determinants under permutations is what reflects the anticommuting nature of the fermions.
is equivalent to (17) for certain G when (55) depends only on the differences $x_k - y_k$. To prove this we have to make use of the free-fermion representation of $SL(2)_{k=1}$ Kac-Moody algebra:

$$J_0 = \frac{1}{2} (\psi^{(1)} \psi^{(1)*} - \psi^{(2)} \psi^{(2)*}) = \frac{1}{2} (J^{(1)} - J^{(2)})$$

$$J_+ = \psi^{(2)} \psi^{(1)*} \quad J_- = \psi^{(1)} \psi^{(2)*}$$

(59)

Now let us take $G$ to be the following exponent of a quadratic form

$$G \equiv \exp \left( \int \psi^{(2)} \psi^{(1)*} \right)$$

(60)

The only term which contributes into the correlator (55) due to the charge conservation rule is:

$$G_{N,M} \equiv G_{N,-N} \delta_{M,-N} = \frac{1}{N!} \left( \int \psi^{(2)} \psi^{(1)*} \right)^N : \delta_{M,-N}$$

(61)

Now one can bosonize the fermions

$$\psi^{(i)*} = e^{\phi_i}, \quad \psi^{(i)} = e^{-\phi_i}$$

$$J^{(1)} = \partial \phi_1, \quad J^{(2)} = \partial \phi_2$$

(62)

and compute the correlator

$$\tau^{(2)}_N(x, y) \equiv \tau^{(2)}_{N,-N}(x, y) = \frac{1}{N!} \langle N, -N | \exp \left( \sum_{k>0} (x_k J^{(1)}_k + y_k J^{(2)}_k) \right) \left( \int : \psi^{(2)} \psi^{(1)*} : \right)^N | 0 \rangle =$$

$$= \frac{1}{N!} \langle N, -N | \exp \left( \oint [X(z) J^{(1)}(z) + Y(z) J^{(2)}(z)] \right) \left( \int : \exp(\phi_1 - \phi_2) : \right)^N | 0 \rangle$$

Introducing the linear combinations $\sqrt{2} \phi = \phi_1 - \phi_2, \sqrt{2} \tilde{\phi} = \phi_1 + \phi_2$ we finally get

$$\tau^{(2)}_N(x, y) = \frac{1}{N!} \langle \exp \left( \frac{1}{\sqrt{2}} \oint [X(z) + Y(z)] \partial \phi(z) \right) \times$$

$$\times \langle N | \exp \left( \frac{1}{\sqrt{2}} \oint [X(z) - Y(z)] \partial \tilde{\phi}(z) \right) \left( \int : \exp \sqrt{2} \phi : \right)^N | 0 \rangle = \tau^{(2)}_N(x - y)$$

(63)

since the first correlator is in fact independent of $x$ and $y$. Thus, we proved that the $\tau$-function (55) indeed depends only on the difference of two sets of times $\{x_k - y_k\}$, and coincides with (17).

The above simple example already contains all the basic features of at least all the
situation. In other words, the diagonal $U(1)$ $GL(2)$-current \( \tilde{J} = \frac{1}{2}(J^{(1)} + J^{(2)}) = \frac{1}{\sqrt{2}} \partial \tilde{\phi} \) decouples. This is an invariant statement which can be easily generalized to higher \( p \) cases.

In the case of $SL(p)$ we have to deal with the \( p \)-component hierarchy and instead of (55) for generic $\tau$-function one has

\[
\tau_N^{(p)}(x) = \langle N| e^{H(x)} G |0\rangle
\]

and now we have \( p \) sets of fermions \( \{\psi^{(i)}\}, \psi^{(i)} \} i = 1,\ldots,p \). The Hamiltonian is given by the Cartan currents of $GL(p)$

\[
H(t) = \sum_{i=1}^{p} \sum_{k>0} x_k^{(i)} J_k^{(i)}
\]

and the element of the Grassmannian in the particular case of CMM is given by an exponents of the other currents

\[
J^{(ij)} = \psi^{(i)}\psi^{(j)}, \quad \tilde{J}^{(ij)} = \psi^{(i)}\psi^{(j)*}, \quad J^{(ij)*} = \psi^{(i)*}\psi^{(j)*}, \quad i \neq j
\]

i.e.

\[
G \equiv \prod \exp(Q^{(ij)}) \exp(\tilde{Q}^{(ij)}) \exp(Q^{(ij)*})
\]

Since (68) are the $GL(p)_1$ Kac-Moody currents, (64) play the role of screening operators in the theory. It deserves mentioning that they are exactly the $SL(p)$ (not $GL(p)$) -screenings and thus the $\tau$-function (64) does not depend on \( \{\sum_{i=1}^{p} x_k^{(i)}\} \).

In the case of $SL(3)$ this looks as follows. The screenings are

\[
Q^{(\alpha)} = \oint J^{(\alpha)}
\]

where \( \{\alpha\} \) is the set of the six roots of $SL(3)$. In terms of fermions or bosons the screening currents look like
\[ J^{(\alpha_2)} = \psi^{(2)*} \psi^{(3)*} = \exp(\phi_2 + \phi_3) \]
\[ J^{(\alpha_3)} = \psi^{(1)} \psi^{(3)*} = \exp(\phi_3 - \phi_1) \]
\[ J^{(-\alpha_1)} = \psi^{(1)} \psi^{(2)} = \exp(-\phi_1 - \phi_2) \]
\[ J^{(-\alpha_2)} = \psi^{(2)} \psi^{(3)} = \exp(-\phi_2 - \phi_3) \]
\[ J^{(-\alpha_3)} = \psi^{(3)} \psi^{(1)*} = \exp(\phi_1 - \phi_3) \] 

The particular \( \tau \)-function is now described in terms of the correlator

\[ \tau^{(3)}_{N}(x) = \langle N| e^{H(x)} G |0\rangle \] 

with

\[ G \sim \prod_{\alpha} \exp Q^{(\alpha)} \] 

The condition of Cartan neutrality is preserved by compensation of charges between the operator (71) and left vacuum \( \langle N | \) in (70). It is obvious that in such case due to the condition of Cartan neutrality of the correlator (like in Wess-Zumino models) the mode \( \tilde{J} = \partial \tilde{\phi} = \frac{1}{\sqrt{p}} \sum_{i=1}^{p} \partial \phi_i \) decouples from the correlator, and

\[ \tau^{(3)}_{N}(x) = \langle N| e^{H(x)} G |0\rangle = \]
\[ = \langle 0| \exp \left( \sum_{k>0} \tilde{J}_k \sum_{i=1}^{3} x^{(i)}_k \right) |0\rangle \phi_{\tilde{N}| e^{H(t,\bar{t})} G |0\rangle \right|_{\sum_i \phi_i = 0} \] 

where the first correlator in the second row is trivially equal to unity. For the specific choice of the operator \( G \) in (72)

\[ G = G_{1,2} = \exp \left( \int J^{(\alpha_1)} \right) \exp \left( \int J^{(\alpha_2)} \right) \] 

we reproduce the formula (29).

Finally, let us only stress two main ideas we have demonstrated above: first, we proved that the solutions to the discrete Virasoro-\( W \) constraints can be rewritten from (free) bosonic to (free) fermionic language, which means automatically that they are solutions to integrable systems in the sense of [15]. Second, in general case the solutions to discrete constraints are presented in the form of conformal multimatrix models, being particular solutions to multicomponent hierarchies.
3 A solution to the continuum Virasoro - W constraints

This section is devoted to the derivation of the solution to the continuous Virasoro and W-constraints. To be more precise, we shall investigate them as being the direct consequence of the ”matrix” equations, which could be treated later on as the Ward identities for certain matrix integrals.

This is however not exactly the same what we had before for the case of discrete constraints. The reason is that the continuum case differs from the discrete one roughly speaking by replacement of ordinary free scalar fields by the same fields but with antiperiodic boundary conditions. This is the sense of so-called double scaling limit (see below for details) which can be done successfully for all conformal multimatrix models [7, 10], discussed above. The exact solution in terms of conformal correlators is much more complicated for the antiperiodic fields 5, thus instead here we are going to reformulate the problem.

Fortunately, it turns out that the continuum Virasoro constraints can be summed up into certain matrix differential operators. Namely, for the W(p)-algebra these operators are related to the Laplacians (or better Casimirs) for corresponding algebras, having the form of

$$\frac{\partial^p}{\partial \Lambda^p} + ...$$

(74)

where $\Lambda$ is $N \times N$ hermitean matrix (for the $SU(N)$-case). These equations might be identified with the Ward identities for ”continuum” matrix theories.

So, we start with an equation of the type of (74) operator vanish certain function, and prove that it is equivalent to the continuum Virasoro (or W) constraints.

$$\{ Tr \, \epsilon(\Lambda)[V \, ' (\partial/\partial \Lambda_{tr}) - \Lambda] \} F[\Lambda] = 0. \quad (75)$$

(75) can be certainly interpreted as a ward identity satisfied by a matrix integral which after the proper normalization gives a solution to continuum 2d gravity. The exact formula for corresponding partition function reads

$$Z^{(N)}[V|M] \equiv C^{(N)}[V|M]e^{TrV(M) - TrMV'M(M)} \int DX \, e^{-TrV(X) + TrV'M(M)X} \quad (76)$$
where the integral is taken over \( N \times N \) “Hermitean” matrices, with the normalizing factor given by Gaussian integral

\[
C^{(N)}[V|M]^{-1} \equiv \int \text{D}Y \ e^{-TrU_2[M,Y]},
\]

\[
U_2 \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} Tr[V(M + \epsilon Y) - V(M) - \epsilon YV'(M)]
\]  

and discuss only specific potentials, \( V(X) = \text{const} \cdot X^{p+1} \), giving rise (while substituted in (75)) to (74).

In the simplest example of \( p = 2 \) we have quadratic operator (or just Laplacian) and will prove the identity

\[
\frac{1}{\mathcal{F}} tr(\epsilon \frac{\partial^2}{\partial \Lambda_{tr}^2} - \epsilon \Lambda) \mathcal{F} = \frac{1}{Z} \sum_{n \geq -1} \mathcal{L}_n Z \ tr(\epsilon \Lambda^{-n-2})
\]  

\[
\mathcal{F}^{(2)} \{ \Lambda \} \equiv \int \text{D}X \ \exp(-trX^3/3 + tr\Lambda X) = C[\sqrt{\Lambda}] \exp(\frac{2}{3} tr\Lambda^{3/2}) Z^{(2)}(T_m)
\]

\[
T_m = \frac{1}{m} TrM^{-m} = \frac{1}{m} Tr\Lambda^{-m/2}, \quad m = \text{odd}
\]

with

\[
C[\sqrt{\Lambda}] = \det(\sqrt{\Lambda} \otimes I + I \otimes \sqrt{\Lambda})^{-\frac{1}{2}}
\]

and

\[
\mathcal{L}_n = \frac{1}{2} \sum_{k \geq \delta_{n+1,0}, k \text{ odd}} kT_k \frac{\partial}{\partial T_{k+2n}} + \frac{1}{4} \sum_{a+b=2n, a,b > 0; a,b \text{ odd}} \frac{\partial^2}{\partial T_a \partial T_b} +
\]

\[
+ \delta_{n+1,0} \cdot \frac{T_{n+1}^2}{4} + \delta_{n,0} \cdot \frac{1}{16} - \frac{\partial}{\partial T_{2n+3}}
\]

While (78) is valid for \( \text{any size} \) of the matrix \( \Lambda \), in the limit of infinitely large \( \Lambda \) \((N \to \infty)\) we can insist that all the quantities

\[\text{6The proof of the Virasoro constraints for generic potential is based on integrability and will be presented below.}\]
\[ tr(\epsilon \Lambda^{-n-2}) \]  
\[ (e.g. \ tr\Lambda^{p-n-2} \ for \ \epsilon = \Lambda^p) \] become algebraically independent, so that eq. (78) implies that

\[ \mathcal{L}_n Z\{T\} = 0, \ n \geq -1. \]  
(83)

Note that \( F\{\Lambda\} \) in (79), which we have to differentiate in order to prove (78), depends only upon eigenvalues \( \{\lambda_k\} \) of the matrix \( \Lambda \). Therefore, it is natural to consider eq. (78) at the diagonal point \( \Lambda_{ij} = 0, \ i \neq j \). The only “non-diagonal” piece of (78) which survives at this point is proportional to

\[ \frac{\partial^2 \lambda_k}{\partial \Lambda_{ij} \partial \Lambda_{ji}} \bigg|_{\Lambda_{mn} = 0, \ m \neq n} = \frac{\delta_{ki} - \delta_{kj}}{\lambda_i - \lambda_j} \]  
for \( i \neq j \).  
(84)

Eq. (84) is nothing but a familiar formula for the second order correction to the Hamiltonian eigenvalues in ordinary quantum-mechanical perturbation theory. It can be easily derived from the variation of determinant formula:

\[ \delta log(\det \Lambda) = tr \frac{1}{\Lambda} \delta \Lambda - \frac{1}{2} tr \left( \frac{1}{\Lambda} \delta \Lambda \frac{1}{\Lambda} \delta \Lambda \right) + \ldots \]  
(85)

For diagonal \( \Lambda_{ij} = \lambda_i \delta_{ij} \), but, generically, non-diagonal \( \delta \Lambda_{ij} \), this equation gives

\[ \sum_k \frac{\delta \lambda_k}{\lambda_k} = -\frac{1}{2} \sum_{i \neq j} \frac{\delta \Lambda_{ij} \delta \Lambda_{ji}}{\lambda_i \lambda_j} = \frac{1}{2} \sum_{i \neq j} \left( \frac{1}{\lambda_i} - \frac{1}{\lambda_j} \right) \frac{\delta \Lambda_{ij} \delta \Lambda_{ji}}{\lambda_i - \lambda_j} + \ldots , \]  
which proves (84).

Since matrix \( \epsilon \) is assumed to be a function of \( \Lambda \), it can be, in fact, treated as a function of eigenvalues \( \lambda_i \). Then, we use actually only

a) the concrete form of the normalization (77)

b) the fact that \( Z[T(\lambda_i)] \) is a complicated function, i.e. we should differentiate it as depending on \( \{\lambda_i\} \) only through time variables. After that, (78) can be rewritten in the following way:
$$\frac{e^{-\frac{2}{3}tr \Lambda^{3/2}}}{C(\sqrt{\Lambda})Z\{T\}} \left[ \text{tr} \epsilon \left\{ \frac{\partial^2}{\partial \Lambda^2} - \Lambda \right\} C(\sqrt{\Lambda})e^{\frac{2}{3}tr \Lambda^{3/2}} Z\{T\} = \right.$$ (86)

$$= \frac{1}{Z} \sum_{a,b>0} \frac{\partial^2 Z}{\partial T_a \partial T_b} \sum_i \epsilon(\lambda_i) \frac{\partial T_a}{\partial \lambda_i} \cdot \frac{\partial T_b}{\partial \lambda_i} +$$ (87)

$$+ \frac{1}{Z} \sum_{n \geq 0} \frac{\partial Z}{\partial T_n} \sum_{i,j} \epsilon(\lambda_i) \frac{\partial^2 T_n}{\partial \Lambda_{ij} \partial \Lambda_{ji}} + 2 \sum_i \epsilon(\lambda_i) \frac{\partial T_n}{\partial \lambda_i} \frac{\partial \log C}{\partial \lambda_i} +$$ (88)

$$+ 2 \sum_i \epsilon(\lambda_i) \frac{\partial T_n}{\partial \lambda_i} \left( \frac{2}{3} \frac{\partial}{\partial \lambda_i} tr \Lambda^{3/2} \right) +$$

$$+ \left[ \sum_i \epsilon(\lambda_i) \left( \frac{\partial}{\partial \lambda_i} \left( \frac{2}{3} tr \Lambda^{3/2} \right) \right)^2 - \sum_i \lambda_i \epsilon(\lambda_i) +$$ (89)

$$+ \sum_{i,j} \epsilon(\lambda_i) \left( \frac{\partial^2}{\partial \Lambda_{ij} \partial \Lambda_{ji}} \left( \frac{2}{3} tr \Lambda^{3/2} \right) \right) +$$ (90)

$$+ 2 \sum_i \epsilon(\lambda_i) \left( \frac{2}{3} \frac{\partial tr \Lambda^{3/2}}{\partial \lambda_i} \frac{\partial \log C}{\partial \lambda_i} +$$ (91)

$$+ \frac{1}{C} \sum_{i,j} \epsilon(\lambda_i) \frac{\partial^2 C}{\partial \Lambda_{ij} \partial \Lambda_{ji}} \right] \right.$$ (92)

with $tr \Lambda^{3/2} = \sum_k \lambda_k^{3/2}$ and

$$C = \prod_{i,j} (\sqrt{\lambda_i} + \sqrt{\lambda_j})^{-1/2}. \tag{93}$$

The calculation of all the quantities in (87) - (92) is just an exercise of taking derivatives, using (84), all necessary details can be found in [16, 18]. Careful calculation shows that all the terms after taking the derivatives contain only negative powers of $\sqrt{\lambda_i}$ and can be "arbsorbed" in times. The result is:

$$\frac{e^{-\frac{2}{3}tr \Lambda^{3/2}}}{C(\sqrt{\Lambda})Z\{T\}} \left[ \text{tr} \epsilon \left\{ \frac{\partial^2}{\partial \Lambda^2} - \Lambda \right\} C(\sqrt{\Lambda})e^{\frac{2}{3}tr \Lambda^{3/2}} Z\{T\} = \right.$$ (86)

$$= \frac{1}{Z} \sum_{n \geq -1} tr (\epsilon_{n,\Lambda^{-n-2}}) \left\{ \frac{1}{2} \sum_{k=0}^n \frac{kT_k}{\partial T_{2n+k}} \frac{\partial}{\partial T_{2n+k}} + \frac{1}{4} \sum_{a+b=2n \atop a,b>0} \frac{\partial^2}{\partial T_a \partial T_b} +$$ (94)

$$+ \frac{1}{8} \sum_{n \geq 0} \frac{\partial}{\partial T_n} + \frac{1}{8} \sum_{n \geq 0} T_n \frac{\partial}{\partial T_n} \right\} Z(T) = 0 \right.$$
or just the set of Virasoro constraints for the case $p = 2$.

In the case of generic $p$ the analog of the derivation actually involves the same steps:

– Represent $\mathcal{F}[\Lambda]$ as

$$
\mathcal{F}^{(p)}[\Lambda] = g_p[\Lambda]Z^{(p)}(T_n),
$$

with

$$
g_p[\Lambda] = \frac{\Delta(M)}{\Delta(\Lambda)} \prod_i [V''(\mu_i)^{1/2} e^{(\mu_i V'(\mu_i) - V(\mu_i))}] = \frac{\Delta(\Lambda^{1/p})}{\Delta(\Lambda)} \prod_i [\lambda_i^{\frac{p}{2(p+1)} e^{p+1} \lambda_i^{1+1/p}}].
$$

– Substitute this $\mathcal{F}^{(p)}[\Lambda]$ into (75), which in the particular case of $V_p(X) = \frac{X^{n+1}}{p+1}$, looks like

$$
\{\text{Tr } \epsilon(\Lambda)[(\frac{\partial}{\partial \Lambda_{tr}})^p - \Lambda] \} g_p[\Lambda]Z^{(p)}(T_n) = 0.
$$

Higher-order derivatives, $\frac{\partial^i Z}{\partial \Lambda_{tr}^i}$, they are defined with the help of relations like (84).

– Perform the shift of variables

$$
T_n \rightarrow \hat{T}_n = T_n - \frac{p}{n} \delta_{n,p+1}
$$

(this procedure doesn’t change the derivatives).

– After all these substitutions the l.h.s. of eq.(77) acquires the form of an infinite series where every item is a product of $\text{Tr}[\tilde{\epsilon}(M)M^{-k}]$ and a linear combination of generators of $\mathcal{W}_p$-algebra, acting on $Z^{(p)}(T_n)$. In the case of $p = 3$ this equation looks like

$$
\frac{1}{27} \text{Tr} \left[ \tilde{\epsilon}(M)M^{-3} \left\{ \sum M^{-3n} \mathcal{W}_{3n}^{(3)} + 
+ 9 \sum M^{-3n-1/3} \left\{ \sum (3k - 2) \hat{T}_{3k-2} \mathcal{W}_{3n+3k}^{(2)} + \sum \frac{\partial}{\partial T_{3a+1}} \mathcal{W}_{3b-3}^{(2)} \right\} + 
+ 9 \sum M^{-3n-2/3} \left\{ \sum (3k - 2) \hat{T}_{3k-2} \mathcal{W}_{3n+3k}^{(2)} + \sum \frac{\partial}{\partial T_{3a+1}} \mathcal{W}_{3b-3}^{(2)} \right\} \right\} \right] Z^{(3)} = 0,
$$

– If $N = \infty$ all the quantities $\text{Tr}\tilde{\epsilon}(M)M^{-k}$ with given $k$ but varying $\tilde{\epsilon}(M)$ become independent, and (97) may be said to give $W$-constraints. The exact proof for the case
4 Integrability of GKM

The purpose of this section is to prove that the solution to the continuum Virasoro- and \( W \)-constraints found above is nothing but a particular solution to the integrable KP system. Namely:

(A) The partition function \( Z_N^{\{V\}}[M] \) (76), if considered as a function of time-variables \( T_k = \frac{1}{k} Tr M^{-k}, k \geq 1 \); (100) is a KP \( \tau \)-function for \textit{any} value of \( N \) and \textit{any} potential \( V[X] \).

(B) As soon as \( V[X] \) is homogeneous polynomial of degree \( p + 1 \), \( Z_N^{\{V\}}[M] = Z_N^{\{p\}}[M] \) is in fact a \( \tau \)-function of \( p \)-reduced KP hierarchy [20].

In order to prove these statements, first, we rewrite (76) in terms of determinant formula

\[
Z_N^{\{V\}}[M] = \det \left( \frac{\phi_i(\mu_j)}{\Delta(\mu)} \right) \quad i, j = 1, \ldots, N. \tag{101}
\]

Then, we show that \textit{any} KP \( \tau \)-function in the Miwa parameterization does have the same determinant form. 

The main thing which distinguishes matrix models from the point of view of solutions to the KP-hierarchy is that the set of functions \( \{\phi_i(\mu)\} \) in (101) is not arbitrary. Moreover, this whole \textit{infinite} set of functions is expressed in terms of a \textit{single} potential \( V[X] \) (i.e. instead of arbitrary matrix \( A_{ij} \) in \( \phi_i(\mu) = \sum A_{ij} \mu^j \) we have here only a \textit{vector} \( V_i \) or \( V[\mu] = \sum V_i \mu^i \)). This is the origin of \( L_{-1} \) and other \( W \)-constraints (which in the context of KP-hierarchy may be considered as implications of \( L_{-1} \)). All these constraints are in fact contained in the Ward identity (75).

\[\text{Moreover, actually, } \frac{\partial Z^{\{V\}}}{\partial T_{np}} = 0. \]

\[\text{As a check of self-consistency, it can be proven (see for example [18]) that any determinant formula (101) with \textit{any} set of functions \( \{\phi_i(\mu)\} \) satisfies the Hirota difference bilinear equation.}\]
4.1 Integrability from the determinant formula

We begin from evaluation of the integral:

$$\mathcal{F}^{(V)}_N[\Lambda] \equiv \int DX \ e^{-Tr[V(X) - Tr\Lambda X]}.$$  \hspace{1cm} (102)

The integral over the "angle" \(U(N)\)-matrices can be easily taken \[21, 22\] and if eigenvalues of \(X\) and \(\Lambda\) are denoted by \(\{x_i\}\) and \(\{\lambda_i\}\) respectively, this integral can be rewritten as

$$\frac{1}{\Delta(\Lambda)} \left[ \prod_{i=1}^{N} \int dx_i e^{-V(x_i)+\lambda_i x_i} \right] \Delta(X).$$ \hspace{1cm} (103)

\(\Delta(X)\) and \(\Delta(\Lambda)\) are Van-der-Monde determinants, \(e.g.\) \(\Delta(X) = \prod_{i>j}(x_i - x_j)\).

The r.h.s. of (103) can be rewritten as

$$\Delta^{-1}(\Lambda) \Delta \left( \frac{\partial}{\partial \lambda} \right) \prod \int dx_i e^{-V(x_i)+\lambda_i x_i} = \Delta^{-1}(\Lambda) \det_{(ij)} F_i(\lambda_j)$$ \hspace{1cm} (104)

with

$$F_{i+1}(\lambda) \equiv \int dx \ x^i e^{-V(x)+\lambda x} = \left( \frac{\partial}{\partial \lambda} \right)^i F_1(\lambda).$$ \hspace{1cm} (105)

Note that

$$F_1(\lambda) = \mathcal{F}^{(V)}_{N=1}[\lambda].$$ \hspace{1cm} (106)

If we recall that

$$\Lambda = V'(M)$$ \hspace{1cm} (107)

and denote the eigenvalues of \(M\) through \(\{\mu_i\}\), then:

$$\mathcal{F}^{(V)}_N[V'(M)] = \frac{\det \tilde{\Phi}_i(\mu_j)}{\prod_{i>j}(V'(\mu_i) - V'(\mu_j))},$$ \hspace{1cm} (108)

with
Proceed now to the normalization (77). Indeed, it is given by the Gaussian integral:

\[ C^{(N)}[V|M]^{-1} \equiv \int DX \ e^{-U_2(M,X)}. \] (110)

Making use of \( U(N) \)-invariance of Haar measure \( dX \) one can easily diagonalize \( M \). Of course, this does not imply any integration over angular variables and provide no factors like \( \Delta(X) \). Then for evaluation of (110) it remains to use the obvious rule of Gaussian integration,

\[ \int DX \ e^{-\sum_{i,j}^N U_{ij}X_{ij}X_{ji}} \sim \prod_{i,j}^N U_{ij}^{-1/2} \] (111)

and substitute the explicit expression for \( U_{ij}(M) \). If potential is represented as a formal series,

\[ V(X) = \sum \frac{v_n}{n} X^n, \] (112)

we have

\[ U_2(M,X) = \sum_{n=0}^\infty v_{n+1} \left\{ \sum_{a+b=n-1} \text{Tr} M^a X M^b X \right\}, \]

and

\[ U_{ij} = \sum_{n=0}^\infty v_{n+1} \left\{ \sum_{a+b=n-1} \mu_i^a \mu_j^b \right\} = \sum_{n=0}^\infty v_{n+1} \frac{\mu_i^n - \mu_j^n}{\mu_i - \mu_j} = \frac{V''(\mu_i) - V''(\mu_j)}{\mu_i - \mu_j}. \]

Coming back to (76), we conclude that

\[ Z^{(V)}_N[M] = e^{Tr[V(M)-M V'(M)]} C^{(N)}[V|M] \mathcal{F}_N[V'(M)] \sim \]

\[ \sim [\det \Phi_1(\mu_j)] \prod_{i>j}^N \frac{U_{ij}}{(V''(\mu_i) - V''(\mu_j))} \prod_{i=1}^N s(\mu_i) = \frac{[\det \Phi_1(\mu_j)]}{\Delta(M)} \prod_{i=1}^N s(\mu_i). \] (113)

\[ s(\mu) = [V''(\mu)]^{1/2} e^{V(\mu) - \mu V'(\mu)} \] (114)

The product of \( s \)-factors at the \( r.h.s. \) of (113) can be absorbed into \( \Phi \)-functions:

\[ Z^{(V)}_N[M] \Delta(M) \mathcal{F}_N[V'(M)] \sim \]

\[ \prod_{i=1}^N s(\mu_i) \] (115)
where
\[
\Phi_i(\mu) = s(\mu)\tilde{\Phi}_i(\mu) \xrightarrow{\mu \to \infty} \mu^{i-1}(1 + O(\frac{1}{\mu})).
\] (116)

where the asymptotic is crucial for the determinant (115) to be a solution to the KP hierarchy in the sense of [20].

The Kac-Schwarz operator [23, 24]. From eqs. (109), (114) and (116) one can deduce that \( \Phi_i(\mu) \) can be derived from the basic function \( \Phi_1(\mu) \) by the relation
\[
\Phi_i(\mu) = \left[ V''(\mu) \right]^{1/2} \int x^{i-1}e^{-V(x)+xV'(\mu)}dx = A_{\{V\}}^{-1}(\mu)\Phi_1(\mu),
\] (117)
where \( A_{\{V\}}(\mu) \) is the first-order differential operator
\[
A_{\{V\}}(\mu) = s \frac{\partial}{\partial \lambda} s^{-1} = \frac{e^{V(\mu)-\mu V''(\mu)} \partial e^{-V(\mu)+\mu V'(\mu)}}{[V''(\mu)]^{1/2} \partial \mu [V''(\mu)]^{1/2}} = \frac{1}{V''(\mu)} \frac{\partial}{\partial \mu} + \mu - \frac{V''(\mu)}{2[V''(\mu)]^2}.
\] (118)

In the particular case of \( V(x) = \frac{x^{p+1}}{p+1} \)
\[
A_{\{p\}}(\mu) = \frac{1}{p \mu^{p-1}} \frac{\partial}{\partial \mu} + \mu - \frac{p-1}{2p \mu^p}
\]
coinsides (up to the scale transformation of \( \mu \) and \( A_{\{p\}}(\mu) \)) with the operator which determines the finite dimensional subspace of the Grassmannian in ref. [23]. We emphasize that the property
\[
\Phi_{i+1}(\mu) = A_{\{V\}}(\mu)\Phi_i(\mu) \quad (F_{i+1}(\lambda) = \frac{\partial}{\partial \lambda} F_i(\lambda))
\] (119)
is exactly the thing which distinguishes partition functions of GKM from the expression for generic \( \tau \)-function in Miwa’s coordinates,
\[
\tau^{\{\phi_i\}}_N[M] = \frac{[\det \phi_i(\mu_j)]}{\Delta(M)},
\] (120)
with arbitrary sets of functions \( \phi_i(\mu) \). In the next section we demonstrate that the quantity (120) is exactly a KP \( \tau \)-function in Miwa coordinates.
4.2 KP $\tau$-function in Miwa parameterization

A generic KP $\tau$-function is a correlator of a special form \[^{13}\]:

$$\tau^G\{T_n\} = \langle 0 | : e^{\sum T_n J_n} : G | 0 \rangle \quad (121)$$

with

$$J(z) = \tilde{\psi}(z)\psi(z); \quad G = : \exp \sum \tilde{\psi}_m \psi_n : \quad (122)$$

in the theory of free 2-dimensional fermionic fields $\psi(z), \tilde{\psi}(z)$ with the action $\int \bar{\psi} \partial \psi$. The vacuum states are defined by conditions

$$\psi_n | 0 \rangle = 0 \quad n < 0 , \quad \tilde{\psi}_n | 0 \rangle = 0 \quad n \geq 0 \quad (123)$$

where $\psi(z) = \sum Z \psi_n z^n \, dz^{1/2}, \quad \tilde{\psi}(z) = \sum Z \tilde{\psi}_n z^{-n-1} \, dz^{1/2}$.

The crucial restriction on the form of the correlator, implied by (122) is that the operator $: e^{\sum T_n J_n} : G$ is Gaussian exponential, so that the insertion of this operator may be considered just as a modification of $\langle \tilde{\psi}\psi \rangle$ propagator, and the Wick theorem is applicable. Namely, the correlators

$$\langle 0 | \prod i \tilde{\psi}(\mu_i)\psi(\lambda_i)G | 0 \rangle \quad (124)$$

for any relevant $G$ are expressed through the pair correlators of the same form:

$$(124) = \det_{(ij)} \langle 0 | \tilde{\psi}(\mu_i)\psi(\lambda_j)G | 0 \rangle \quad (125)$$

The simplest way to understand what happens to the operator $e^{\sum T_n J_n}$ after the substitution of (100) is to use the free-boson representation of the current $J(z) = \partial \varphi(z)$. Then $\sum T_n J_n = \sum_i \left\{ \sum_n \frac{1}{n} \varphi_{n} \right\} = \sum_i \varphi(\mu_i)$, and

$$: e^{\sum_i \varphi(\mu_i)} := \frac{1}{\prod_{i<j}(\mu_i - \mu_j)} \prod_i : e^{\varphi(\mu_i)} : \quad (126)$$

In fermionic representation it is better to start from

$$T_n = \frac{1}{2} \sum \left( \begin{array}{c} 1 \ 1 \end{array} \right) \quad (127)$$
instead of (100). Then

\[ \sum_{T_nJ_n} : e^{\sum_{T_nJ_n}} := \prod_{i,j} (\tilde{\mu}_i - \mu_j) \prod_{i > j} (\tilde{\mu}_i - \tilde{\mu}_j) \prod_i \tilde{\psi}(\tilde{\mu}_i) \psi(\mu_i) . \]  

(128)

In order to come back to (100) it is necessary to shift all \( \tilde{\mu}_i \)'s to infinity. This may be expressed by saying that the left vacuum is substituted by

\[ \langle N | \sim \langle 0 | \tilde{\psi}(\infty)\tilde{\psi}'(\infty)\ldots\tilde{\psi}(N-1)(\infty). \]

The \( \tau \)-function now can be represented in the form:

\[ \tau^G_N[M] = \langle 0 | : e^{\sum_{T_nJ_n}} : G | 0 \rangle = \Delta(M)^{-1} \langle N | \prod_i e^{\varphi(\mu_i)} : G | 0 \rangle = \lim_{\tilde{\mu}_i \to \infty} \frac{\prod_{i,j} (\tilde{\mu}_i - \mu_j)}{\prod_{i > j} (\mu_i - \mu_j) \prod_{i > j} (\tilde{\mu}_i - \tilde{\mu}_j)} \langle 0 | \prod_i \tilde{\psi}(\tilde{\mu}_i) \psi(\mu_i) G | 0 \rangle \]

applying the Wick’s theorem (124), (125) and taking the limit \( \tilde{\mu}_i \to \infty \) we obtain:

\[ \tau^G_N[M] = \frac{\det \phi_i(\mu_j)}{\Delta(M)} \]

(130)

with functions

\[ \phi_i(\mu) \sim \langle 0 | \tilde{\psi}^{(i-1)}(\infty) \psi(\mu) G | 0 \rangle \xrightarrow{\mu \to \infty} \mu^{i-1}(1 + O(\frac{1}{\mu})). \]

(131)

Thus, we proved that KP \( \tau \)-function in Miwa coordinates (100) has exactly the determinant form (101), or, put differently, (101) is a \( \tau \)-function of KP hierarchy. Below we will discuss how from generic point of Grassmannian described by \( G = \exp \sum A_{ij} \tilde{\psi}_i \psi_j \) or infinite matrix with two indices (\( \infty^2 \)) one can restrict it to a solution, determined only by one function (\( \infty \)) or two functions (\( 2 \times \infty \))..

5 Universal \( L_{-1} \)-constraint and string equation

Let us return to the question of specifying particular ”stringy” solutions to the KP hierarchy. It turns out that using integrability it is enough to prove only the so-called string equation or \( L_{1} \)-constraint, all other recursion relation follow from these two statements [4, 26].
It is well known that $\mathcal{L}_{-1}$-constraint is closely related to the action of operator

$$
Tr \frac{\partial}{\partial \Lambda_{tr}} = Tr \frac{1}{V''(M)} \frac{\partial}{\partial M_{tr}}.
$$

(132)

Therefore it is natural to examine, how this operator acts on

$$
Z^{(V)}[M] = \frac{\det \tilde{\Phi}_i(\mu_j)}{\Delta(M)} \prod_i s(\mu_i),
$$

(133)

$$
s(\mu) = (V''(\mu))^{1/2} e^{V'(\mu)} - \mu V'(\mu),
$$

(134)

$$
\tilde{\Phi}_i(\mu) = F_i(\lambda) = (\partial/\partial \lambda)^i F_1(\lambda), \; \lambda = V'(\mu).
$$

First of all, if $Z^{(V)}$ is considered as a function of $T$-variables,

$$
\frac{1}{Z^{(V)}} Tr \frac{\partial}{\partial \Lambda_{tr}} Z^{(V)} = - \sum_{n \geq 1} Tr \frac{1}{V''(M) M_{n+1}} \frac{\partial \log Z^{(V)}}{\partial T_n}.
$$

(135)

On the other hand, if we apply (132) to explicit formula (133), we obtain:

$$
\frac{1}{Z^{(V)}} Tr \frac{\partial}{\partial \Lambda_{tr}} Z^{(V)}
= - Tr M + \frac{1}{2} \sum_{i,j} \frac{1}{V''(\mu_i)V''(\mu_j)} \frac{V''(\mu_i) - V''(\mu_j)}{\mu_i - \mu_j} + Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j),
$$

(136)

Below, we will argue that

$$
\frac{1}{Z^{(V)}} \mathcal{L}_{-1}^{(V)} Z^{(V)} = - \frac{\partial}{\partial T_1} \log Z^{(V)} + Tr M - Tr \frac{\partial}{\partial \Lambda_{tr}} \log \det F_i(\lambda_j).
$$

(137)

can be used in order to suggest the formula for the universal operator $\mathcal{L}_{-1}^{(V)}$.

Here

$$
\mathcal{L}_{-1}^{(V)} = \sum_{n \geq 1} Tr \left[ \frac{1}{V''(M) M_{n+1}} \right] \frac{\partial}{\partial T_n} + \\
+ \frac{1}{2} \sum_{i,j} \frac{1}{V''(\mu_i)V''(\mu_j)} \frac{V''(\mu_i) - V''(\mu_j)}{\mu_i - \mu_j} - \frac{\partial}{\partial T_1}.
$$

(138)
and this turns into more common expression when $V(X) = X^{p+1}/(p+1)$ (note that the items with $i = j$ are included into the sum at the r.h.s. in (138)).

So, in order to prove the $\mathcal{L}_{-1}^{(V)}$-constraint, one should prove that the r.h.s. of (137) vanishes, i.e.

$$
\frac{\partial}{\partial T_1} \log Z_N^{(V)} = Tr M - Tr \frac{\partial}{\partial \Lambda tr} \log \det F(\lambda_j),
$$

This is possible to prove only if we remember that $Z_N^{(V)} = \tau_N^{(V)}$. In this case the l.h.s. may be represented as residue of the ratio

$$
\text{res}_\mu \frac{\tau_N^{(V)}(T_n + \mu^{-n}/n)}{\tau_N^{(V)}(T_n)} = \frac{\partial}{\partial T_1} \log \tau_N^{(V)}(T_n).
$$

However, if expressed through Miwa coordinates, the $\tau$-function in the numerator is given by the same formula with one extra parameter $\mu$, i.e. is in fact equal to $\tau_{N+1}^{(V)}$. This idea is almost enough to deduce (139). Let us begin with an illustrative example of $N = 1$. Then ($\lambda = V'(\mu)$)

$$
\tau_1^{(V)}(T_n) = \tau_1^{(V)}[\mu_1] = e^{V(\mu_1) - \mu_1 V'(\mu_1)}[V''(\mu_1)]^{1/2}F(\lambda_1),
$$

$$
\tau_1^{(V)}(T_n + \mu^{-n}/n) = \tau_2^{(V)}[\mu_1, \mu] =
$$

$$
e^{V(\mu_1) - \mu_1 V'(\mu_1)} e^{V(\mu) - \mu V'(\mu)} \frac{[V''(\mu_1)V''(\mu)]^{1/2}}{\mu - \mu_1} \left[ F(\lambda_1) \frac{\partial F(\lambda)}{\partial \lambda} - F(\lambda) \frac{\partial F(\lambda_1)}{\partial \lambda_1} \right] =
$$

$$
e^{V(\mu) - \mu V'(\mu)} [V''(\mu)]^{1/2} F(\lambda) \frac{\partial F(\lambda)}{\partial \lambda_1} \tau_1^{(V)}[\mu_1] \cdot [-\partial \log F(\lambda_1)/\partial \lambda_1 + \partial \log F(\lambda)/\partial \lambda].
$$

The function

$$
F(\lambda) = \int dx \ e^{-V(x)} + \lambda x \sim e^{V(\mu) - \mu V'(\mu)}[V''(\mu)]^{-1/2}\{1 + O\left( \frac{V'''}{V''} \right) \}.
$$

If $V(\mu)$ grows as $\mu^n$ when $\mu \to \infty$, then $V'''/(V'')^2 \sim \mu^{-n}$, and for our purposes it is enough to have $n = p + 1 > 1$, so that in the braces at the r.h.s. stands $\{1 + o(1/\mu)\}(\mu \cdot o(\mu) \to 0$ as $\mu \to \infty$). Then numerator at the r.h.s. of (141) is $\sim 1 + o(1/\mu)$, while the second item in square brackets behaves as $\partial \log F(\lambda)/\partial \lambda \sim \mu (1 + o(1/\mu))$. Combining all
\[
\frac{\partial}{\partial T_1} \log \tau_1^{\{V\}} = \text{res}_\mu \left\{ \frac{1 + o(1/\mu)}{\mu - \mu_1} \left[ -\partial \log F(\lambda_1)/\partial \lambda_1 + \mu (1 + o(1/\mu)) \right] \right\} = \\
= \mu_1 - \partial \log F(\lambda_1)/\partial \lambda_1. \tag{143}
\]
i.e. (139) is proved for the particular case of \( N = 1 \).

The proof is literally the same for any \( N \), we will omit here the details which can be found in [18]. After a simple but cumbersome calculation one gets

\[
\frac{\partial}{\partial T_1} \log \tau_N^{\{V\}} = \\
= \text{res}_\mu \left\{ \frac{1 + o(1/\mu)}{\prod_{j=1}^N (\mu - \mu_j)} \mu^N \left\{ [1 + o(1/\mu)] - \\
- \frac{1}{\mu} \left[ \text{Tr} \frac{\partial}{\partial \Lambda_{tr}} \log \det F(\lambda_j) \cdot [1 + O(1/\mu)] \right] \right\} \right\} = \\
= \sum_{j=1}^N \mu_j - \text{Tr} \frac{\partial}{\partial \Lambda_{tr}} \log \det F(\lambda_j). \tag{144}
\]

which completes the proof of eq.(3.13) and thus of the universal \( \mathcal{L}_{-1}^{\{V\}} \)-constraint.

In the particular case of monomial potential \( V \equiv V_p = \sum_{p=0}^{p+1} a_{p+1} \) (138) turns into more common form [4, 5]:

\[
\mathcal{L}_{-1}^{\{p\}} = \frac{1}{p} \sum_{n \geq 1} (n + p) T_n^{p+1} \frac{\partial}{\partial T_n} + \\
+ \frac{1}{2p} \sum_{a+b=p, a,b \geq 0} aT_a bT_b - \frac{\partial}{\partial T_1}, \tag{145}
\]

6 GKM versus Toda theory and discrete models

Now, first, without any special reference to GKM there exists an explicit relation between KP-like (in Miwa variables),

\[
\tau_{KP}[T_k] = \frac{\det_{ij} \phi_i(\mu_j)}{\Delta(\mu)}, \tag{146}
\]
\[ \tau_N[T_{-k}, T_k] = \det_{ij} H_{i+N,j+N}[T_{-k}, T_k], \]  
\text{(147)}

representations of \( \tau \)-functions, where

\[ \Delta(\mu) = \prod_{i>j}(\mu_i - \mu_j), \]  
\text{(148)}

\[ \phi_i(\mu) = \mu^{i-1}(1 + O(\frac{1}{\mu})), \]  
\text{(149)}

\[ T_k = \frac{1}{k} \sum_i \mu_i^{-k}, \quad k > 0, \]  
\text{(150)}

\[ \partial H_{ij}/\partial T_k = H_{i,j-k}, \quad j > k > 0, \]  
\text{(151)}

and

\[ \partial H_{ij}/\partial T_{-k} = H_{i-k,j}, \quad i > k > 0. \]  
\text{(152)}

Relation between (146) and (147) is formulated in terms of the Schur polynomials, which are defined by:

\[ P[z|T_k] \equiv \exp\{\sum_{k>0} T_k z^k\} = \sum z^k P_k[T], \]  
\text{(153)}

e.g. \( P_{-n} = 0 \) for any \( n > 0; \) \( P_0[T] = 1; \) \( P_1[T] = T_1; \) \( P_2[T] = T_2 + \frac{1}{2} T_1^2; \) \( P_3[T] = T_3 + T_2 T_1 + \frac{1}{6} T_1^3 \) etc. The crucial property of Schur polynomials is:

\[ \partial P_k/\partial T_n = P_{k-n} \]  
\text{(154)}

(this is just because \( \partial P/\partial T_k = z^k P \)). This feature allows one to express all the dependence on time-variables of \( H_{ij}[T] \), which satisfies eqs. (151) and (152), through the Schur polynomials:

\[ H_{ij}[T_{-p}, T_p] = \sum_{\substack{k\leq i \\ell\geq j}} P_{i-k}[T_{-p}] H_{kl} P_{l+j}[T_p], \]  
\text{(155)}
Let us begin our consideration from the case, when all \( N = T_{-k} = 0 \), then look what happens if \( N > 0 \) and introduce \( T_{-k} \)-variables.

Given the system of basic vectors \( \phi_i(\mu) \) for \( i > 0 \), we put by definition

\[
H_{ij}[T_{-k} = 0, T_k] = \oint_{z \to 0} \phi_i(z) z^{-j} \mathcal{P}[z|T_k] dz, \quad i > 0.
\]

The integration contour is around zero and it can be deformed to encircle infinity and the singularities of \( \mathcal{P}[z] \), if any. If we just substitute the definition (153) of \( \mathcal{P}[z] \) into (156), we get (155) with \( P_{k-i}[T_{-m} = 0] = \delta_{ki} \) and

\[
H_{kl} = \oint_{z \to 0} \phi_k(z) z^l dz.
\]

In order to prove the identity between (146) and (147) let us substitute the Miwa transformation (150) to (153)

\[
\mathcal{P}[z|T_k] = \frac{\det M}{\det(M - I z)} = M \prod_i \frac{\mu_i}{(\mu_i - z)} = \prod_i \mu_i \sum_k \frac{(-)^k \Delta_k(\mu)}{(z - \mu_k) \Delta(\mu)},
\]

where \( \Delta_k(\mu) \equiv \prod_{i > j, i \neq j} (\mu_i - \mu_j) \). Now the integral (156) picks up contributions only from the poles of \( \mathcal{P}[z|T_k] \) at the points \( \mu_k \):

\[
H_{ij}[T_{-k} = 0, T_k] = \oint_{z \to 0} \phi_i(z) z^{-j} \mathcal{P}[z|T_k] dz = \prod_k \frac{\mu_i}{\Delta(\mu)} \sum_k (-)^k \phi_i(\mu_k) \frac{\Delta_k(\mu)}{\mu_k^j}.
\]

The sum at the r.h.s. has a form of a matrix product and we conclude that

\[
\det H_{ij} = \det \phi_i(\mu_k) \cdot \prod_k \left[ \frac{\prod_i \mu_i}{\Delta(\mu)} (-)^k \Delta_k(\mu) \right] \cdot \det \frac{1}{\mu_k^j}.
\]

The last determinant at the r.h.s. is equal to \( \Delta(1/\mu) \sim \Delta(\mu) \cdot \left[ \prod_k \mu_k^N \right]^{-1} \). Note also that

\[
\prod_k \left[ \frac{\Delta_k(\mu)}{\Delta(\mu)} \right] = \Delta(\mu)^{-2},
\]

and taking all this together, we see that there exists the equality

\[
\det H_{ij} = \frac{\det \phi_i(\mu_j)}{\Delta(\mu)},
\]

as required.

\footnote{discussion of \( N < 0 \) can be found in [23]}
Proceed now to introducing of zero- and negative-time variables. The zero-time $n$ arises just as the simultaneous shifts of indices $i$ and $j$ of $H_{ij} : H_{ij} \to H_{i+N,j+N}$. We can write:

$$H_{i+N,j+N}[0,T_k] = \oint_{z \to 0} \phi^{(N)}_i(z) z^{-j} \mathcal{P}[z] dz$$

with

$$\phi^{(N)}_i(z) = z^{-N} \phi_{i+N}(z)$$

This exhausts the problem of restoring the $N$-dependence for positive integer values of $N$.

As for negative-times, as soon as $H_{kl}$ is defined, they are introduced with the help of (155) and

$$H_{i+N,j+N}[T_{-k}, T_k] \equiv \sum_{l \leq i} P_{i-k}[T_{-l}] H_{k+N,j+N}[0,T_l] =$$

$$= \oint_{z \to 0} \phi^{(T_{-k},N)}_i(z) z^{-j} \mathcal{P}\left[\frac{1}{z} |T_{-k}\right] \mathcal{P}[z |T_k] dz,$$

with

$$\phi^{(T_{-k},N)}_i(z) \equiv \left\{ \mathcal{P}\left[\frac{1}{z} |T_{-k}\right]\right\}^{-1} \sum_{l \geq k} P_{i-k}[T_{-l}] \phi^{(N)}_k =$$

$$= z^{-N} \exp\left\{ - \sum_{k>0} T_{-k} z^{-k} \right\} \sum_{l \geq k} P_{k}[T_{-l}] \phi_{i+N-k}(z).$$

The role of the exponential prefactor in (162) is to guarantee the proper asymptotic behaviour

$$\phi^{(T_{-k},N)}_i(z) = z^i \mathcal{O}\left(\frac{1}{z}\right).$$

The important reduction from Toda lattice is Toda chain (see, for example, [50]). It can be easily written both in terms of element $G$ in the fermionic language ($[G, J_k + J_{-k}] = 0$ – see also the comments above, and in determinant form. Latter one merely implies the symmetry property:
where $\Lambda$ is shift matrix $\Lambda_{ij} \equiv \delta_{i,j-1}$. This condition leads to $\tau$-function of Toda chain hierarchy (proper rescaled by exponential of bilinear form of times) which depends only on the sum of positive and negative times $t_k = \frac{1}{2}(T_k + T_{-k})$, but not on their difference (one can consider this as defining property of Toda chain hierarchy). Let us remark that one possible solution to constraint (164) is matrix $H_{i,j} = H_{i} - H_{j}$. We can combine both reductions to reproduce forced Toda chain hierarchy. In this case one can easily transform $H_{i} - H_{j}$ to matrix $\tilde{H}_{i+j}$ by permutations of columns what does not effect to the determinant. This matrix just corresponds to one-matrix model case [12, 13]. Thus, we consider again the determinant of size $N \times N$, which now can be represent in the form:

$$\tau_N = \det_{N \times N} \frac{\partial x \partial t_1, \partial H/\partial t_k = \partial^k H.}$$

Like the Toda lattice case, the forced Toda chain is unambiguously continuable to negative values of zero-time.

Now, we can easily introduce zero- and negative-time variables into GKM in such a way that its partition function becomes a $\tau$-function of the Toda lattice hierarchy. The relevant set of functions $\{\phi_i(\mu)\}$ – the point in Grassmannian for GKM – is given by the following integral formula:

$$\phi^V_i(\mu) = e^{V(\mu) - \mu V'(\mu)} \sqrt{V'(\mu)} \int dx \ x^{i-1} e^{-V(x) + x V'(\mu)} \equiv$$

$$\equiv s(\mu) \int dx \ x^{i-1} e^{-V(x) + x V'(\mu)} \equiv \langle x^{i-1} \rangle_{\mu}$$

Dependence of $N$ and $T_{-k}$ is now introduced by the rule $^\text{10}$

$$\phi^{V,N,T_{-k}}_i(\mu) \equiv \langle x^{i-1} \left[ \frac{x}{\mu} \right]^N \exp \left\{ \sum_{l>0} T_{-l}(x^{-l} - \mu^{-l}) \right\} \rangle_{\mu} =$$

$$= \frac{\sqrt{V'(\mu)e^{V(\mu) - \mu V'(\mu)}}}{\mu^N} \int dx \ x^{N+i-1} e^{-V(x) + x V'(\mu)} \exp \left\{ \sum_{l>0} T_{-l}(x^{-l} - \mu^{-l}) \right\} \equiv \langle x^{i-1} \rangle_{\mu}$$

\text{10}Let us point out that the exponential of negative powers in normalization does not essentially effect to the KP $\tau$-function as it reduces to trivial exponential of bilinear form of times in front of $\tau$-function and corresponds to the freedom in its definition. Indeed, $\tau \sim \det \{\exp[i \sum_k a_k z^{-k}_j \phi_i(z_j)]\} \sim \det \{\exp[i \sum_k a_k z^{-k}_j \phi_i(z_j)]\}$.
\[
\hat{V}(X) \equiv V(X) - N \log X - \sum_{k>0} t_{-k} X^{-k} \tag{168}
\]

with

\[
\hat{Z}_{\{\hat{V}\}}[M] = e^{\text{Tr} \hat{V}(M) - \text{Tr} M \hat{V}^*_0(M)} \frac{\int DX \ e^{-\text{Tr} \hat{V}(X) + \text{Tr} \hat{V}^*_0(M)X}}{\int dX \ e^{-\text{Tr} U_+(X,M)}} \tag{169}
\]

Since we devote this section to discussion of Toda lattice hierarchies in the context of matrix models, we can not avoid touching the main conclusion of \([12]\) that all the discrete matrix models do correspond to particular cases of Toda hierarchies. In the simplest case of Hermitean one-matrix model one gets a Toda chain, other multi-matrix models correspond to other reductions of the Toda lattice hierarchy. Moreover, all discrete matrix models fall into the class of forced hierarchies \([13]\).

The idea is to perform a Miwa transformation of times \(T_k\) with \(T_{-k}\) fixed, so that \(H_{ij}\)'s become averages of polynomial functions of \(X\). Then \(\det H_{ij}\) may be transformed with the help of orthogonal polynomials technique.

The main result of all these calculations is that partition functions arise in the form \([146]\), with \(\phi_i(z)\) being proportional to orthogonal polynomials.

Indeed, using

\[
T_k = \frac{1}{k} \text{Tr} \Lambda^{-k} = \frac{1}{k} \sum_{i=1}^{\tilde{N}} \lambda_i^{-k} + \tilde{t}_k \tag{170}
\]

\((k > 0)\) \([11]\) (note that \(\tilde{N}\) – the size of matrix \(\Lambda\) has nothing to do with \(N\) – the size of matrices \(M\), being integrated over in \([18]\) ) the partition functions of discrete models \(\{\tau_N\}\) acquire the form of KP \(\tau\)-function \([146]\). E.g., for the Hermitean one-matrix model one has

\[
T_{-k} \equiv \tilde{T}_k = \frac{1}{k} \text{Tr} M^{-k} = \frac{1}{k} \sum_{i=1}^{\tilde{N}} \mu_i^{-k} + \tilde{T}_{-k}
\]
\[
\tau_N(t) = (N!)^{-1} \int \prod_i dm_i \Delta^2(m) \exp\left\{-\sum_{i,k} t_k m_i^k\right\} = \\
= (N!)^{-1} \int \prod_i dm_i \Delta^2(m) e^{-\tilde{V}(m_i)} \prod_{i,a} \left(1 - \frac{m_i}{\lambda_a}\right) = \\
= (N!)^{-1} \prod_a \lambda_{a}^{-N} \int \prod_i dm_i e^{-\tilde{V}(m_i)} \Delta(m) \frac{\Delta(m, \lambda)}{\Delta(\lambda)} = \\
= (N!)^{-1} \prod_a \lambda_{a}^{-N} \Delta^{-1}(\lambda) \int \prod_i dm_i e^{-\tilde{V}(m_i)} \times
\]

\[
\times \det_{N \times N} \tilde{P}_{i-1}(m_j) \quad \text{det}_{(N+\tilde{N}) \times (N+\tilde{N})} \left[\begin{array}{cccc}
\tilde{P}_{i-1}(m_j) & \tilde{P}_{N+b-i}(m_j) \\
\ldots & \ldots \\
\tilde{P}_{i-1}(\lambda_a) & \tilde{P}_{N+b-i}(\lambda_a)
\end{array}\right],
\]

where \(i, j = 1, \ldots, N; a, b = 1, \ldots, \tilde{N};\) and \(\{\tilde{P}_i(m)\}\) are corresponding orthogonal polynomials with respect to deformed measure \(e^{-\tilde{V}} dm:\)

\[
< \tilde{P}_i, \tilde{P}_j > = \int \tilde{P}_i(m) \tilde{P}_j(m) e^{-\tilde{V}(m)} dm = \delta_{ij} e^{\tilde{\phi}_i(s)},
\]

so that

\[
\tilde{P}_i(m) = m^i + O(m^{i-1}).
\]

Computing determinants in \((171)\) and using orthogonality condition \((172)\) one obtains

\[
\tau_N[\lambda|\tilde{t}] = \prod_a \lambda_{a}^{-N} \Delta^{-1}(\lambda) \det_{N \times \tilde{N}} \tilde{P}_{N+a-1}(\lambda_b) \prod_i e^{\tilde{\phi}_i(\tilde{t})} = \\
= \left[\prod_i e^{\tilde{\phi}_i(\tilde{t})}\right] \frac{\det_{(ab)} \phi^{(N)}(\lambda_b)}{\Delta(\lambda)} = \tau_N[\tilde{t}] \times \frac{\det_{(ab)} \phi^{(N)}(\lambda_b)}{\Delta(\lambda)},
\]

i.e. the \(\tau\)-function of the discrete Hermitean one-matrix model acquires the form of eq.\((146)\) with

\[
\phi^{(N)}(\lambda) = \lambda^{-N} \tilde{P}_{N+a-1}(\lambda)
\]

\((173)\) is natural representation for \textit{all} discrete matrix models.

Now, in order to be representable as GKM the above formulas still need to arise in a somewhat specific form. Namely, components of the vector \(\{\phi_i(\mu)\}\) should possess a
\[ \phi_i(\mu) = \left< x^{i-1} \right>_\mu \]

Since in the study of discrete matrix models \( \phi_i(\mu) \) arise as the orthogonal polynomials \( P_{n+1}(\mu) \), what is necessary is a kind of integral representation of these polynomials, with \( i \)-dependence coming only from the \( x^{i-1} \)-factor in the integrand. It is an interesting problem to find out such kind of representation for various discrete models, but it is easily available only whenever orthogonal polynomials are associated with the Gaussian measure: the relevant Hermit polynomials are known to possess integral representation, which is exactly of the form which we need.

The final statement we have got here is that Hermitean one-matrix model with the matrix size \( N \) is equivalent to GKM with \( \hat{V}(X) = X^2/2 - N \log X \).

Indeed, we have proven the remarkable explicit identity between Gaussian matrix integrals,

\[
\frac{\int DM_{N\times N} \det(I - M/\Lambda) e^{-TrM^2/2}}{\int DM_{N\times N} e^{-TrM^2/2}} = \frac{\int DX_{\tilde{N}\times \tilde{N}} \det(I - iX/\Lambda)^N e^{-TrX^2/2}}{\int DX_{\tilde{N}\times \tilde{N}} e^{-TrX^2/2}} \quad (175)
\]

Note that the size of matrix in the l.h.s. is \( N \times N \) and in the r.h.s. is \( \tilde{N} \times \tilde{N} \), and these parameters are absolutely independent. This identity is indeed true for any \( N \) and \( \tilde{N} \), as follows from the proof given above and integral representation of Hermit polynomials \( \phi_j(\mu) = He_j(i\mu) \).

7 Double-scaling limit

Now, we are going directly to the discussion of the connection between discrete and continuum theories. Indeed, it turns out \([27]\) that this connection at the language of free scalar fields is nothing but a change of spectral parameter

\[
u^2 = 1 + az \quad (176)
\]

Then the continuous Virasoro constraints \([3]\) which are modes of the stress tensor

\[
\left[ \frac{1}{2} \partial_z \Phi^2 - \frac{1}{16} z^2 \right] = \sum_{L} L_n z^n + 2 \quad (177)
\]
can be deduced [27] from analogous constraints in Hermitian one-matrix model by taking
the continuum limit. The procedure is as follows.

The partition function of Hermitian one-matrix model [2] satisfies the discrete Virasoro
constraints (5). In order to obtain the continuum constraints (7, 8) one has to consider a
reduction of model (18) to the pure even potential $t_{2k+1} = 0$.

Let us denote by the $\tau_{\text{red}}^{\text{red}}(t_{2k})$ the partition function of the reduced matrix model
\begin{equation}
\tau_{\text{red}}^{\text{red}}\{t_{2k}\} = \int D\mathcal{M} \exp \text{Tr} \sum_{k=0} t_{2k} M^{2k}
\end{equation}
and consider the following change of the time variables
\begin{equation}
g_{m} = \sum_{n \geq m} \frac{(-)^{n-m}}{(n-m)!} \frac{\Gamma(n + \frac{3}{2}) a^{-n-\frac{3}{2}}}{\Gamma(m + \frac{1}{2})} T_{2n+1},
\end{equation}
($g_{m} \equiv m t_{2m}$ and this expression can be used also for the zero discrete time $g_{0} = N$ that
plays the role of the dimension of matrices in the one-matrix model), deduced from the
following prescription. Take the free scalar field with periodic boundary conditions for
\begin{equation}
\partial \phi(u) = \sum_{k \geq 0} g_{k} u^{2k-1} + \sum_{k \geq 1} \frac{\partial}{\partial t_{2k}} u^{-2k-1},
\end{equation}
and analogous scalar field with antiperiodic boundary conditions:
\begin{equation}
\partial \Phi(z) = \sum_{k \geq 0} \left( \left( k + \frac{1}{2} \right) T_{2k+1} z^{k-\frac{3}{2}} + \frac{\partial}{\partial T_{2k+1}} z^{-k-\frac{3}{2}} \right).
\end{equation}
Then the equation
\begin{equation}
\frac{1}{\tau} \partial \Phi(z) = a \frac{1}{\tau_{\text{red}}} \partial \phi(u) \tau_{\text{red}}, \quad u^{2} = 1 + a z
\end{equation}
generates the correct transformation rules (179), (184) and gives rise to the expression for $A_{nm}$ (187). Taking the square of the both sides of the identity (182),
\begin{equation}
\frac{1}{\tau} T(z) = \frac{1}{\tau_{\text{red}}} T(u) \tau_{\text{red}},
\end{equation}
one can obtain after simple calculations that the relation (188) is valid.

Derivatives with respect to $t_{2k}$ transform as
\begin{equation}
\frac{\partial}{\partial t_{2k}} = \sum_{n \geq 0} \frac{\Gamma(k + \frac{1}{2}) a^{n+\frac{3}{2}}}{\Gamma(n + \frac{3}{2})} \frac{\partial}{\partial T_{2n+1}},
\end{equation}
where the auxiliary continuum times $\tilde{T}_{2n+1}$ are connected with “true” Kazakov continuum times $T_{2n+1}$ via
\begin{equation}
T_{2k+1} = \tilde{T}_{2k+1} + a \frac{k}{k + 1/2} \tilde{T}_{2(k-1)+1},
\end{equation}
and coincide with $T_{2n+1}$ in the double-scaling limit when $a \to 0$.

Let us rescale the partition function of the reduced one-matrix model by exponent of quadratic form of the auxiliary times $\tilde{T}_{2n+1}$
\begin{equation}
\tilde{\tau} = \exp \left( -\frac{1}{2} \sum_{m,n \geq 0} A_{mn} \tilde{T}_{2m+1} \tilde{T}_{2n+1} \right) \tau^\text{red}_N.
\end{equation}
with
\begin{equation}
A_{mn} = \frac{\Gamma \left( n + \frac{3}{2} \right) \Gamma \left( m + \frac{3}{2} \right)}{2 \Gamma^2 \left( \frac{1}{2} \right)} \frac{(-)^{n+m} a^{-n-m-1}}{n! m! (n + m + 1)(n + m + 2)}.
\end{equation}

Then a direct though tedious calculation \cite{27} demonstrates that the relation
\begin{equation}
\tilde{\mathcal{L}}_n \tilde{\tau} = a^{-n} \sum_{p=0}^{n-1} C^p_{n+1} (-1)^{n+1-p} \frac{L^\text{red}_{2p} \tau^\text{red}}{\tau^\text{red}},
\end{equation}
is valid, where
\begin{equation}
L^\text{red}_{2n} \equiv \sum_{k=0}^{\infty} k t_{2k} \frac{\partial}{\partial t_{2(k+n)}} + \sum_{0 \leq k \leq n} \frac{\partial^2}{\partial t_{2k} \partial t_{2(n-k)}}
\end{equation}
and
\begin{align*}
\tilde{\mathcal{L}}_{-1} &= \sum_{k \geq 1} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial \tilde{T}_{2(k-1)+1}} + T_1^2 \frac{1}{16G}, \\
\tilde{\mathcal{L}}_0 &= \sum_{k \geq 0} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial \tilde{T}_{2k+1}}, \\
\tilde{\mathcal{L}}_n &= \sum_{k \geq 0} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial \tilde{T}_{2(k+n)+1}} \\
&\quad + \sum_{0 \leq k \leq n-1} \frac{\partial}{\partial \tilde{T}_{2k+1}} \frac{\partial}{\partial \tilde{T}_{2(n-k)-1}} - \frac{(-)^n}{16a^n}, \quad n \geq 1.
\end{align*}

Here $C^p_n = \frac{n!}{p!(n-p)!}$ are binomial coefficients.

These Virasoro generators differ from (78) by terms which are singular in the limit $a \to 0$. At the same time $L^\text{red}_{2p} \tau^\text{red}$ at the r.h.s. of (188) do not need to vanish, since
\begin{equation}
0 = L_{2p} \tau^\text{red} \bigg|_{\tau_{2k+1}=0} = L^\text{red}_{2p} \tau^\text{red} + \sum \frac{\partial^2 \tau^\text{red}}{\partial \tau^\text{red}},
\end{equation}
It was shown in [27] that these two origins of difference between (78) and (190) actually cancel each other, provided eq.(188) is rewritten in terms of the square root $\sqrt{\tilde{\tau}}$ rather than $\tilde{\tau}$ itself:

$$\frac{L^{\text{cont}}_n \sqrt{\tilde{\tau}}}{\sqrt{\tau}} = a^{-n} \sum_{p=0}^{n+1} C_p (-1)^{n+1-p} \frac{L_{2p} \tau}{\tau} \bigg|_{t_{2k+1}=0} (1 + O(a)).$$

(192)

The proof of this cancelation, as given in [27], is not too much simple and makes use of integrable equations for $\tau$.

Now we will use the demonstrated above fact, that the discrete Hermitean one-matrix model is equivalent to GKM with $\hat{V}(X) = X^2/2 - N \log X$, and also that its double-scaling continuum limit is described by GKM with $V(X) = X^3/3$. Thus, we should conclude that

$$\lim_{\text{d.s. } N \to \infty} Z\{\hat{V}\} = Z^2\{V\}.$$

(193)

This relation should certainly be understandable just in terms of GKM itself.

Let us recall that double-scaling continuum limit for the model of interest implies that only even times $t_{2k} = \frac{1}{2k} Tr \frac{1}{\Lambda^{2k}}$ should remain non-zero, while all odd times $t_{2k+1} = 0$. This obviously implies that the matrix $M$ should be of the block form:

$$\Lambda = \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}$$

(194)

and, therefore, the matrix integration variable is also naturally decomposed into block form:

$$X = \begin{pmatrix} X \\ Z \end{pmatrix}$$

(195)

Then

$$Z\{\hat{V}=X^2/2-N \log X\} = \int D\mathcal{X} D\mathcal{Y} D^2 Z \, \det(\mathcal{X}\mathcal{Y} - \frac{1}{\mathcal{Y}} Z\mathcal{Y})^N e^{-Tr\{|Z|^2 + X^2/2 + Y^2/2 - M X + M Y\}}.$$

(196)

To take the limit $N \to \infty$, one should assume certain scaling behaviour of $\mathcal{X}$, $\mathcal{Y}$ and $Z$. Moreover, the notion of double-scaling limit implies a specific fine tuning of this scaling
\[ X = \alpha(i\beta I + x), \]
\[ Y = \alpha(-i\beta I + y), \]
\[ Z = \alpha \zeta, \]

with some large real \( \alpha \) and \( \beta \).

As for behaviour of the matrix of Miwa’s parameters, it should be dictated by the change of the spectral parameter (176), which dictates the right Kazakov change of variables. So, we will take the anzatz

\[ M^2 = A + Bm \]

which under particular scaling behaviour \( A/B \to \infty \) could be linearized giving rise to

\[ M = \alpha^{-1}(i\gamma I + m) \]

with

\[
\begin{align*}
A^{1/2} &= \frac{\gamma}{\alpha} \\
B &\to 1 \\
2A^{1/2} &= \frac{1}{\alpha}
\end{align*}
\]

If expressed through these variables, the action becomes:

\[
Tr\{|Z|^2 + X^2/2 + Y^2/2 - M\mathcal{X} + M\mathcal{Y} - N \log(\mathcal{X}\mathcal{Y} - \bar{Z}^\dagger \bar{Y}^\dagger)\} = \\
= \frac{\gamma^2}{2} Tr\{(i\beta I + x)^2 + \frac{\gamma^2}{2} Tr(i\beta I - y)^2 + \gamma^2 |\zeta|^2\} - Tr(i\alpha I + m)(2i\beta I + x - y) - \\
- NTr \log \beta^2 \gamma^2 \left\{ 1 - i \frac{x - y}{\beta} + \frac{xy}{\beta^2} - \frac{|\zeta|^2}{\beta^2} (1 + o(1/\beta)) \right\} = \\
= [2\alpha \beta - \beta^2 \gamma^2 - 2N \log \beta \gamma] Tr I - 2i \beta Tr m + \\
+i(\beta \gamma^2 - \alpha + \frac{N}{\beta})(Tr x - Tr y) + \frac{1}{2}(\gamma^2 - \frac{N}{\beta^2})(Tr x^2 + Tr y^2) + \\
+(\gamma^2 - \frac{N}{\beta^2})Tr|\zeta|^2 - \\
- Tr mx + Tr my + iN \frac{1}{3\beta^2} Tr(x^3 - y^3) + \\
+ O(N(\beta^2) + 2N|\zeta|^2 N) \]
We want to adjust the scaling behaviour of $\alpha$, $\beta$ and $\gamma$ in such a way that only the terms in the line $(D)$ survive. This goal is achieved in several steps.

The line $(A)$ describes normalization of functional integral, it does not contain $x$ and $y$. Thus, it is not of interest for us at the moment.

Two terms in the line $(B)$ are eliminated by adjustment of $\alpha$ and $\gamma$:

$$\gamma^2 = \frac{N}{\beta^2}, \quad \alpha = \frac{2N}{\beta}. \quad (202)$$

As we shall see soon, $\gamma^2 = N/\beta^2$ is large in the limit of $n \to \infty$. Thus, the term $(C)$ implies that the fluctuations of $\zeta$-field are severely suppressed, and this is what makes the terms of the second type in the line $(E)$ negligible. More general, this is the reason for the integral $Z\{\hat{V}\}$ to split into a product of two independent integrals leading to the square of partition function in the limit $n \to \infty$ (this splitting is evident as, if $Z$ can be neglected, the only mixing term $\log \det \begin{pmatrix} \mathcal{X} & Z \\ Z & \mathcal{Y} \end{pmatrix}$ turns into $\log \mathcal{X} \mathcal{Y} = \log \mathcal{X} + \log \mathcal{Y}$).

Thus, we remain with a single free parameter $\beta$ which can be adjusted so that

$$\frac{\beta^3}{N} \to \text{const} \quad \text{as} \quad N \to \infty \quad (i.e. \ \beta \sim N^{1/3}), \quad (203)$$

making the terms in the last line $(E)$ vanishing and the third term in the line $(D)$ finite.

Let us finish the discussion of double-scaling limit, making some remarks on ref. [28]. It is claimed there that in order to get the Kontsevich model from a discrete Hermitean one, one should not necessarily care of the reduction to even times (194) and the particular Kazakov change of variables, inspired by (176), (198); instead it is just enough to take (199) and make the rescalings similar to what we done above. However, in such case it is not clear what should be instead of Kazakov change of variables, and what is indeed the integrable hierarchy we get in continuum limit. As for the first question, one might hope that the related change of variables in in the class of ”allowed redefinitions” of Kazakov times, similar to those we used in (188), though the second question, certainly deserves further investigation.
8 GKM as a solution to topological \((p, 1)\) models

For various choices of the potential \(V(X)\) the model \((76)\) formally reproduces various \((p,q)\)-series: the potential \(V(X) = \frac{X^{p+1}}{p+1}\) can be associated with the entire set of \((p,q)\)-minimal string models with all possible \(q\)'s. In order to specify \(q\) one needs to make a special choice of \(T\)-variables: all \(T_k = 0\), except for \(T_1\) and \(T_{p+q}\) (the symmetry between \(p\) and \(q\) is implicit in this formulation).

Let us briefly discuss two simple examples. First, we will fix \(p = 2\), \(i.e.\) the case of the KdV reduction to the KP hierarchy. The second number \(q\) should be coprime to \(p\), thus we have here \(q = 2m - 1\).

Now the string equation is of the form

\[
\frac{1}{\tau_{KdV}} L_{-1} \tau_{KdV} = \frac{1}{2} \sum_{k \geq 1, \ k \ odd} kT_k \frac{\partial}{\partial T_{k-2}} \log \tau_{KdV} + \frac{T_1^2}{4} = 0
\]

(204)

or taking the \(\frac{\partial}{\partial T_1}\)-derivative, one gets

\[
\sum_{k \geq 1, \ k \ odd} kT_k \frac{\partial^2}{\partial T_{k-2} \partial T_1} \log \tau_{KdV} + T_1 = 0
\]

(205)

or using the formula for the Gelfand-Dikii polynomials

\[
\frac{\partial^2}{\partial T_{k-2} \partial T_1} \log \tau_{KdV} = \left[ L^{2m-1} \right]_{-1} = R_{m}[u]
\]

(206)

we have

\[
\sum_{m \geq 0} (2m + 1)T_{2m+1}R_m[u] = 0
\]

(207)

Now, we should use the ”axiomatics” of [4] how to extract the concrete \((2, 2m - 1)\) solutions from (205), (207). The very simple example is \(m = 1\): where

\[
3T_3 \frac{\partial^2}{\partial T_1^2} \log \tau_{KdV} + T_1 = 0
\]

(208)

using that

\[
u \sim \frac{\partial^2}{\partial T_1^2} \log \tau_{KdV}
\]

the solution to the KdV-equation is

\[
u_{KdV} = \log \tau_{KdV} + \frac{T_1^2}{2}
\]
\[ u \sim \frac{T_1}{T_3} \quad \text{(209)} \]

or fixing \( T_3 \), one gets

\[ F = \log \tau \sim T_1^3 \quad \text{(210)} \]

which is a well known fact from the \( c = -2 \) theory coupled to gravity where

\[ \langle P^3 \rangle = 1 \]

with \( P \) being the puncture operator \( P = c\overline{c}e^\phi \). This is the case of topological gravity.

Less simple example is with \( m = 2 \), where \[ (207) \] and the fact that

\[ R_2 \sim u^2 + u'' \]

gives the Painleve equation

\[ u^2 + u'' = T_1 \quad \text{(211)} \]

This is the case of pure gravity where the solution is actually much more complicated than in the previous case.

From this point of view, the presence of all \((p, q)\) solutions in GKM is a rather formal consideration. For the potential \( V(X) = \frac{X^{p+1}}{p+1} \) the partition function \( Z[V|T_k] = \tau_V[T_k] \equiv \tau_p[T_k] \) satisfies the string equation which looks like

\[ \sum_{k=1}^{p-1} k(p-k)T_kT_{p-k} + \sum_{k=1}^{\infty} (p+k)(T_{p+k} - \frac{p}{p+1}\delta_{k,1}) \frac{\partial}{\partial T_k} \log \tau_p[T] = 0 \quad \text{(212)} \]

i.e. \( \tau \)-function is defined with all Miwa times \[ (\text{l00}) \] around zero values (in \( 1/M \) decomposition like in original Kontsevich model) with the only exception - \( T_{p+1} \) is shifted what corresponds obviously to \((p, 1)\) model. Thus, we see that the matrix integral gives an explicit solution to \((p, 1)\) string models which must be nothing but particular topological matter coupled to topological gravity.

Of course, we still have an opportunity for analytic continuation in string equation, using the definition of Miwa’s times \[ (\text{l00}) \]. We have to satisfy the following conditions:
\[ T_2 = 0 \]

... 

\[ T_{p+1} - \frac{p}{p+1} = 0 \]

\[ T_{p+q} = t_{p+q} = \text{fixed} \]

\[ T_{p+q+1} = 0 \]  

(213)

which is a system of equations on the Miwa parameters \( \{\mu_i\}, i = 1, ..., N \). So, to do this analytic continuation one has to decompose the whole set

\[ \{\mu_i\} = \{\xi_a\} \oplus \{\mu'_s\} \]

\[ T_k = \frac{1}{k} Tr M^{-k} = \frac{1}{k} \sum_{j=1}^{N} \mu_j^{-k} = \frac{1}{k} \sum \xi_a^{-k} + \frac{1}{k} \sum_{j=1}^{N'} \mu_j'^{-k} \equiv T_k^{(cl)} + T_k' \]  

(214)

into “classical” and “quantum” parts respectively. In principle it is clear that we have now to solve the equations

\[ T_k^{(cl)} = \frac{1}{k} \sum \xi_a^{-k} = t_{p+q} \delta_{k,p+q} - \frac{p}{p+1} \delta_{k,p+1} \]  

(215)

and this can be done adjusting a certain block form of the matrix \( M \) \cite{18, 30}. However, in such a way we can only vanish several first times, the rest ones can be vanished only adjusting correct behaviour in the limit \( N \to \infty \). The most elegant way to do this \cite{33} is to use the formula

\[ \exp(- \sum_{k=1}^{\infty} \lambda^k T_k^{(cl)}) = \lim_{K \to \infty} (1 - \frac{1}{K} \sum_{k=1}^{\infty} \lambda^k T_k^{(cl)})^K = \prod_a (1 - \frac{\lambda}{\xi_a}) \]  

(216)

and then the solution to (215) will be given by \( K \) sets of roots of the equation

\[ \sum_{k=1}^{\infty} \lambda^k T_k^{(cl)} - K = t_{p+q} \lambda^{p+q} - \frac{p}{p+1} \lambda^{p+1} - K = 0 \]  

(217)

Obviously, the eigenvalues \( \xi_a \) will now depend on the size of the matrix \( N = (p+q)K + N' \) through explicit \( K \)-dependence \( (\xi_a \sim K^{1/(p+q)}) \) and we lose one of the main features of \((p,1)\) theories mentioned above – trivial dependence of the size of the matrix. Now we

\(^{12}\)due to A.Zabrodin
can consider only matrices of infinite size and deal only with the infinite determinant formulas.

That is why this way to get higher critical points is a formal one. Below we will present an alternative way of thinking [29, 31, 32], connected with so-called $p$-times. Indeed, there exists a priori another integrable structure in the model (76), connected with time variables, related to the non-trivial coefficients of the potential $V$. As a results, the cases of monomial potential $V_p(X) = \frac{X^{p+1}}{p+1}$ and arbitrary polynomial of the same degree $(p+1)$ are closely connected with each other.

In order to demonstrate this, first, we return to the derivatives of $Z_{GKM}$ with respect to the time-variables $T_k$. Such derivatives define nonperturbative correlators in string models and are of their own interest for the theory of GKM. The derivatives with respect to $T_k$ with $k \geq p+1$ (responsible for the correlators of irrelevant operators) are not very easy to evaluate, things are simpler for $T_k$ with $1 \leq k \leq p$, where using the obvious notation of average so that $Z_{GKM} = \langle 1 \rangle$, we have

$$\left. \frac{\partial Z_{GKM}}{\partial T_k} \right|_V = \langle Tr M^k - Tr X^k \rangle, \quad 1 \leq k \leq p \tag{218}$$

It is implied that the derivative in the l.h.s. is taken preserving the form of the potential $V = \sum_{k}^{p+1} \frac{v_k}{k} X^k$.

The r.h.s. of (218) can be also represented as

$$\left. \frac{\partial Z_{GKM}}{\partial T_k} \right|_V = \left\langle Tr \frac{\partial V(M)}{\partial v_k} - Tr \frac{\partial V(X)}{\partial v_k} \right\rangle, \quad 1 \leq k \leq p \tag{219}$$

which looks similar but actually is different from $-\frac{\partial}{\partial v_k} Z_{GKM}$, as it would be if (219) does contain some corrections. The problem is that $\frac{\partial}{\partial v_k} Z_{GKM}$ gets contributions not only from differentiating $V(X) - V(M)$ in exponentials in (76) but also from the term $V'(M)(X-M) \equiv W(M)(X-M)$ as well as from the pre-exponential (77). The corrections are of the two different types

$$\mathcal{O}(\frac{\partial}{\partial v_k} W) + \text{"quantum corrections"} \tag{220}$$

First type should dissappear if one introduce a new "spectral parameter"
Thus, we are led to special time variables induced by a special transformation of the spectral parameter $\mu$

$$\tilde{T}_k = \frac{1}{k} Tr \tilde{M}^{-k}$$  \hspace{1cm} (222)

The formula (222) also demonstrates that the $\{v_k\}$ are not the true time variables for a given arbitrary potential. Indeed, it appears that the right variables are the parameters $\{t_k\}$ being certain linear combinations of the coefficients $\{v_k\}$ of the potential \[33, 36\]

$$t_k = -\frac{p}{k(p - k)} \text{Res} \frac{W^{1-k/p}(\mu)}{\mu} d\mu$$  \hspace{1cm} (223)

Using (223) one can get two important formulas:

$$\mu = \frac{1}{p} \sum_{-\infty}^{p+1} k t_k \tilde{\mu}^{k-p}, \hspace{1cm} (224)$$

and

$$V(\mu) - \mu V'(\mu) = -\sum_{-\infty}^{p+1} t_k \tilde{\mu}^k. \hspace{1cm} (225)$$

The second one implies the natural interpretation of the exponential pre-factor in eq.(76) as the standard essential singularity factor in the Baker-Akhiezer function of $p$-time variables.

Now, the direct calculation shows that

$$Z[V|T_k] = \tau_V[T_k] =$$

$$= \exp \left( -\frac{1}{2} \sum A_{ij}(t)(\tilde{T}_i - t_i)(\tilde{T}_j - t_j) \right) \tau_p[\tilde{T}_k - t_k], \hspace{1cm} (226)$$

where

$$A_{ij} = \text{Res}_\mu W^{i/p} dW^{j/p}_{+}, \hspace{1cm} (227)$$

and $f(\mu)_+$ denotes the positive part of the Laurent series $f(\mu) = \Sigma f_i \mu^i$. It is also easy to demonstrate, that

$$\tau_p[T] \equiv \tau_V[T]$$  \hspace{1cm} (228)

- is the $\tau$-function of $p$-reduction.

Formula (226) means that “shifted” by flows along $p$-times (223) $\tau$-function is easily expressed through the $\tau$-function of $p$-reduction, depending only on the difference of the
time-variables \( \tilde{T}_k \) and \( t_k \). The change of the spectral parameter in (213) \( M \to \tilde{M} = f(M) = W^{1/p}(M) \) (and corresponding transformation of times \( T_k \to \tilde{T}_k \)) is a natural step from the point of view of equivalent hierarchies.

Indeed, the relation between \( \tau \)-functions of the equivalent hierarchies can be easily derived from an identical transformation:

\[
\tau(T) = \frac{\Delta(\mu)}{\Delta(\tilde{\mu})} \prod_i [f'(\mu_i)]^{1/2} \tilde{\tau}(\tilde{T})
\]

where \( \tilde{\tau}(\tilde{T}) \) as function of times \( \tilde{T} \) has the determinant form (101) with the basic vectors

\[
\tilde{\phi}(\tilde{\mu}) = \left[f'(\mu(\tilde{\mu}))\right]^{1/2} \phi_i(\mu(\tilde{\mu}))
\]

By a direct calculation one can show that pre-factor in eq.(229) may be represented in the form

\[
\frac{\Delta(\mu)}{\Delta(\tilde{\mu})} \prod_i [f'(\mu_i)]^{1/2} = \exp \left(-\frac{1}{2} \sum_{i,j} A_{ij} \tilde{T}_i \tilde{T}_j \right)
\]

where

\[
A_{ij} = \text{Res } f^i(\lambda)d\lambda f^j_+(\lambda)
\]

From (229) we see that

\[
\tau(T(\tilde{T})) = \tilde{\tau}(\tilde{T}) \exp \left(-\frac{1}{2} \sum_{i,j} A_{ij} \tilde{T}_i \tilde{T}_j \right)
\]

Let us introduce the \( \tau \)-function \( \tilde{\tau}(\tilde{T}) \) of the \( p \)-reduced KP hierarchy defined by:

\[
\tilde{\tau}(\tilde{T}) \equiv \frac{\tilde{\tau}(\tilde{T})}{\tau_0(t)} \exp \left(\sum_j j t_j - j \tilde{T}_j \right)
\]

\[
\tau_0(t) = e^{-\frac{1}{2} \sum A_{ij} t_i t_j}
\]

for which instead of

\[
\tau_V[T] = \frac{\det \phi_i(\mu_j)}{\Delta(\mu)}
\]

we have

\[
\frac{\tau_p[\tilde{T} - t]}{\tau_p[t]} = \frac{\det \phi_i(\tilde{\mu}_j)}{\Delta(\tilde{\mu})}
\]

with the corresponding points of the Grassmannian determined by the basic vectors
\[ \hat{\phi}(\hat{\mu}) = [p\hat{\mu}^{p-1}]^{1/2} \exp \left( -\sum_{j=1}^{p+1} t_j \hat{\mu}_j \right) \int x^{i-1} e^{-V(x)+x\hat{\mu}} \, dx \]  

(238)

respectively. Then it is easy to show that \( \hat{\tau}_p(T) \) satisfies the \( L_{-1} \)-constraint with shifted KP-times in the following way

\[ \sum_{k=1}^{p-1} k(p-k)(\hat{T}_k - t_k)(\hat{T}_{p-k} - t_{p-k}) + \sum_{k=1}^{\infty} (p+k)(\hat{T}_{p+k} - t_{p+k}) \frac{\partial}{\partial \hat{T}_k} \log \hat{\tau}_p[\hat{T} - t] = 0 \]  

(239)

where \( t_i \) defined by (223) are identically equal to zero for \( i \geq p + 2 \).

The formulas (226, 239) demonstrate at least two things. First, the partition function in the case of deformed monomial potential (\( \equiv \text{polynomial} \) of the same degree) is expressed through the equivalent solution (in the sense \[39, 40\]) of the same \( p \)-reduced KP hierarchy, second – not only \( t_{p+1} \) but all \( t_k \) with \( k \leq p + 1 \) are not equal to zero in the deformed situation. We will call such theories as topologically deformed \( (p, 1) \) models (in contrast to pure \( (p, 1) \) models given by monomial potentials \( V_p(X) \)), the deformation is “topological” in the sense that it preserves all the features of topological models. Moreover, this “topological” deformation preserves almost all features of 2\( d \) Landau-Ginzburg theories and from the point of view of continuum theory they should be identified with the twisted Landau-Ginzburg topological matter interacting with topological gravity \[13\].

(Of course, the basic vectors of the pure \( (p, 1) \) model corresponding to monomial potential \( V_p \) can be obtained by setting \( t_1 \),..., \( t_{p+1} \) = 0, \( t_{p+1} = \frac{p}{p+1} \)). As a solution to string equation this deformed case differs only in analytic continuation along first \( p \) times.

These topologically deformed \( (p, 1) \) models as we already said preserve all topological properties of \( (p, 1) \) models. Indeed, according to [3] shifting of first times \( t_1, ..., t_{p+1} \) is certainly not enough to get higher critical points. To do this one has to obtain \( t_{p+q} \neq 0 \), but this cannot be done using above formulas naively, because it is easily seen from definition (223) of \( p \)-times, that \( t_k \equiv 0 \) for \( k \geq p + 2 \).

9 General scheme and \( pq \)-duality

The above scheme has a natural quasiclassical interpretation. Indeed, the solution to \( (p, 1) \) theories given by the partition function (76) can be considered as a “path integral”
representation of the solution to Douglas equations \[3\]

\[
[\hat{P}, \hat{Q}] = 1
\]  
(240)

where \(\hat{P}\) and \(\hat{Q}\) are certain differential operators (of order \(p\) and \(q\)) respectively and obviously \(p - th\) order of \(\hat{P}\) dictates \(p\)-reduction, while \(q\) stands for \(q - th\) critical point. Quasiclassically, (240) turns into Poisson brackets relation \([36, 37]\)

\[
\{P, Q\} = 1
\]  
(241)

where \(P(x)\) and \(Q(x)\) are now certain (polynomial) functions. It is easily seen that the above case corresponds to the first order polynomial \(Q(x) \equiv x\) and the \(p\)-th order polynomial \(P(x)\) should be identified with \(W(x) \equiv V'(x)\). Thus, the exponentials in (76), (237) and (238) acquire an obvious sense of action functionals

\[
S_{p,1}(x, \mu) = -V(x) + xW(\mu) = -\int_0^x dy \ W(y)Q'(y) + Q(x)W(\mu)
\]

\[
W(x) = V'(x) = x^p + \sum_{k=1}^{p} v_k x^{k-1}
\]

\[
Q(x) = x^{14}
\]  
(242)

and we claim that the generalization to arbitrary \((p, q)\) case must be

\[
S_{W,Q} = -\int_0^x dy \ W(y)Q'(y) + Q(x)W(\mu)
\]

\[
W(x) = V'(x) = x^p + \sum_{k=1}^{p} v_k x^{k-1}
\]

\[
Q(x) = x^q + \sum_{k=1}^{q} \bar{v}_k x^{k-1}
\]  
(243)

Now the “true” co-ordinate is \(Q\), therefore the extreme condition of action (243) is still

\[
W(x) = W(\mu)
\]  
(244)

\[\text{14For example: } W(z) = z^2 + t_1, \ Q(z) = z, \text{ then for (241)} \]

\[
\{W, Q\} = \frac{\partial W}{\partial t_1} \frac{\partial Q}{\partial z} - \frac{\partial Q}{\partial t_1} \frac{\partial W}{\partial z} = 1
\]
having $x = \mu$ as a solution, and for extreme value of the action one gets

$$S_{W,Q}|_{x=\mu} = \int_0^\mu dy \ W'(y)Q(y) =$$

$$= \sum_{k=-\infty}^{p+q} t_k \tilde{\mu}^k$$

(245)

where $\tilde{\mu}^p = W(\mu)$ and

$$t_k \equiv t_k^{(W,Q)} = -\frac{p}{k(p-k)} \text{Res} \ W^{1-k/p} dQ .$$

(246)

We should stress that the extreme value of the action (243), represented in the form (245), determines the quasiclassical (or dispersionless) limit of the $p$-reduced KP hierarchy [36, 37] with $p + q - 1$ independent flows. We have seen that in the case of topologically deformed $(p, 1)$ models the quasiclassical hierarchy is exact in the strict sense: topological solutions satisfy the full KP equations and the first basic vector is just the Baker-Akhiezer function of our model (76) restricted to the small phase space. Unfortunately, this is not the case for the general $(p, q)$ models: now the quasiclassics is not exact and in order to find the basic vectors in the explicit form one should solve the original problem and find the exact solutions of the full KP hierarchy along first $p + q - 1$ flows. Nevertheless, the presence of the “quasiclassical component” in the whole integrable structure of the given models can give, in principle, some useful information, for example, we can make a conjecture that the coefficients of the basic vectors are determined by the derivatives of the corresponding quasiclassical $\tau$-function.

Returning to eq.(246) we immediately see, that now only for $k \geq p + q + 1$ $p$-times are identically zero, while

$$t_{p+q} \equiv t_{p+q}^{(W,Q)} = \frac{p}{p+q}$$

(247)

and we should get a correct critical point adjusting all $\{t_k\}$ with $k < p + q$ to be zero. The exact formula for the Grassmannian basis vectors in general case acquires the form

$$\phi_i(\mu) = [W'(\mu)]^{1/2} \exp(-S_{W,Q}|_{x=\mu}) \int dM_Q(x) f_i(x) \exp S_{W,Q}(x, \mu)$$

(248)

where $dM_Q(x)$ is the integration measure. We are going to explain, that the integration measure for generic theory determined by two arbitrary polynomials $W$ and $Q$ has the form
by checking the string equation. For the choice (249) to insure the correct asymptotics of basis vectors \( \phi_i(\mu) \) we have to take \( f_i(x) \) being functions (not necessarily polynomials) with the asymptotics

\[
f_i(x) \sim x^{i-1}(1 + O(1/x))
\]  

To satisfy the string equation, one has to fulfill two requirements: the reduction condition

\[
W(\mu)\phi_i(\mu) = \sum_j C_{ij} \phi_j(\mu)
\]  

and the Kac-Schwarz (118) operator action

\[
A^{(W,Q)}\phi_i(\mu) = \sum A_{ij}\phi_j(\mu)
\]

with

\[
A^{(W,Q)} \equiv N^{(W,Q)}(\mu) \frac{1}{W'(\mu)} \frac{\partial}{\partial \mu} [N^{(W,Q)}(\mu)]^{-1} = 
\]

\[
= \frac{1}{W'(\mu)} \frac{\partial}{\partial \mu} - \frac{1}{2W'(\mu)^2} + Q(\mu)
\]

\[
N^{(W,Q)}(\mu) \equiv [W'(\mu)]^{1/2} \exp(-S_{W,Q}|_{x=\mu})
\]

These two requirements are enough to prove string equation. The structure of action immediately gives us that

\[
A^{(W,Q)}\phi_i(\mu) = N^{(W,Q)}(\mu) \int d\mathcal{M}_Q(z) Q(z) f_i(z) \exp S_{W,Q}(z,\mu)
\]  

and the condition (252) can be reformulated as a \( Q \)-reduction property of basis \( \{f_i(z)\} \)

\[
Q(z)f_i(z) = \sum A_{ij} f_i(z)
\]

Let us check now the reduction condition. Multiplying \( \phi_i(\mu) \) by \( W(\mu) \) and integrating by parts we obtain

\[
W(\mu)\phi_i(\mu) = 
\]

\[
= N^{(W,Q)}(\mu) \int d\mathcal{M}_Q(z) f_i(z) \frac{1}{Q'(z)} \frac{\partial}{\partial z} [\exp Q(z)W(\mu)] \exp[-\int_0^z dy W(y)Q'(y)] =
\]

\[
= -N^{(W,Q)}(\mu) \int d\mathcal{M}_Q(z) \exp[S_{W,Q}(z,\mu)] \left( \frac{1}{Q'(z)} \frac{\partial}{\partial z} - \frac{1}{2} \frac{Q''(z)}{Q'(z)^2} - W(z) \right) f_i(z) \equiv
\]

\[
= -N^{(W,Q)}(\mu) \int d\mathcal{M}_Q(z) \exp[S_{W,Q}(z,\mu)] A^{(Q,W)} f_i(z)
\]
Therefore, in the “dual” basis \( \{ f_i(z) \} \) the condition (31) turns to be

\[
A^{(Q,W)} f_i(z) = - \sum C_{ij} f_j(z) \tag{257}
\]

with \( A^{(Q,W)}(\neq A^{(W,Q)}) \) being the “dual” Kac-Schwarz operator

\[
A^{(Q,W)} = \frac{1}{Q''(z)} \frac{\partial}{\partial z} - \frac{1}{2 Q'(z)^2} - W(z) \tag{258}
\]

The representation (248), (249) is an exact integral formula for basis vectors solving the \( (p,q) \) string model. It has manifest property of \( p-q \) duality (in general \( W-Q \)), turning the \( (p,q) \)-string equation into the equivalent \( (q,p) \)-string equation.

Now let us transform (248), (249) into a little bit more explicit \( p-q \) form. As before for \( (p,1) \) models we have to make substitutions, leading to equivalent KP solutions:

\[
\tilde{\mu}^p = W(\mu), \tilde{z}^q = Q(z) \tag{259}
\]

Then we can rewrite (248) as

\[
\hat{\phi}_i(\tilde{\mu}) = [p\tilde{\mu}^{p-1}]^{1/2} \exp \left( - \sum_{k=1}^{p+q} t_k \tilde{\mu}^k \right) \int d\tilde{z}[q\tilde{z}^{q-1}]^{1/2} \hat{f}_i(\tilde{z}) \exp S_{W,Q}(\tilde{z},\tilde{\mu}) \tag{260}
\]

where action is given now by

\[
S_{W,Q}(\tilde{z},\tilde{\mu}) = - \left[ \int_{0}^{\tilde{z}} d\tilde{y} q\tilde{y}^{q-1} W(y(\tilde{\mu})) \right] + \tilde{z}^q \tilde{\mu}^p = \sum_{k=1}^{p+q} \tilde{t}_k \tilde{z}^k + \tilde{z}^q \tilde{\mu}^p \tag{261}
\]

In new coordinates the reduction conditions are

\[
\tilde{\mu}^p \hat{\phi}_i(\tilde{\mu}) = \sum_j \tilde{C}_{ij} \hat{\phi}_j(\tilde{\mu})
\]

\[
\tilde{z}^q \hat{f}_i(\tilde{z}) = \sum_j \tilde{A}_{ij} \hat{f}_j(\tilde{z}) \tag{262}
\]

and for the Kac-Schwarz operators one gets conventional formulas [23, 24, 18]

\[
\tilde{A}^{(p,q)} = \frac{1}{p\tilde{\mu}^{p-1}} \frac{\partial}{\partial \tilde{\mu}} - \frac{p-1}{2p} \frac{1}{\tilde{\mu}^p} + \frac{1}{p} \sum_{k=1}^{p+q} k \tilde{t}_k \tilde{\mu}^{k-p} - \frac{q-1}{2q} \frac{1}{\tilde{z}^q} + \frac{1}{q} \sum_{k=1}^{p+q} k \tilde{t}_k \tilde{z}^{k-q} \tag{262}
\]
where for \((q,p)\) models we have introduced the “dual” times:

\[
\bar{t}_k \equiv t_k^{(Q,W)} = \frac{q}{k(q-k)} \text{Res} \, Q^{1-k/q} \, dW
\]  

(264)

in particularly, \(\bar{t}_{p+q} = -\frac{q}{p} t_{p+q} = -\frac{q}{p+q}\). Now string equations give correspondingly

\[
\bar{A}^{(p,q)} \hat{\phi}_i(\bar{\mu}) = \sum \bar{A}_{ij} \hat{\phi}_j(\bar{\mu})
\]

\[
\bar{A}^{(q,p)} \hat{f}_i(\bar{z}) = -\sum \bar{C}_{ij} \hat{f}_j(\bar{z})
\]  

(265)

By these formulas we get a manifestation of \(p-q\) duality if solutions to \(2d\) gravity.

As a main result of formulas presented above, one may conclude that a generic solution to \(c \leq 1\) \(2d\) gravity should be described by two functions \(W(x)\) and \(Q(x)\).

## 10 String field theory and \(c \to 1\) limit

Now we will make some remarks why this model could be considered as an attempt of constructing a string field theory or effective theory of string models. It is necessary to point out from the beginning that by string field theory we would mean more than a conventional definition as a field theory of functionals defined on string loops - it must rather mean a sort of effective theory which gives all the solutions to classical string equations of motion (\(2d\) conformal field theories coupled to \(2d\) gravity) as its vacua and allows us to consider all of them on equal footing (within the same Lagrangian framework) and maybe even describe the flows between different string vacua. Of course, it should reproduce perturbation expansion around any of these vacua. In this sense, the conventional string field theory was not true effective model, because it contained a fixed set of variables which correspond to a concrete vacuum (say, 26 free scalar fields). Therefore, it doesn’t have even a priori a possibility to make a flow to another classical solution (maybe only except for some simple change of a background), i.e. conventional string field theory might describe only some small perturbation around given classical solution in terms of the coordinates equivalent to the matter variables in the Polyakov path integral [\(\square\)].

Moreover, in conventional approach even the perturbative expansion was ill-defined due to the presence of tachyon in the spectrum or in other words due to the instability of...
make sense to the non-perturbative description, presented above, and existing up to now only in the case of non-critical and moreover “non-tachyonic” strings. That means that what we know up to now is the only case of “highly-noncritical” string models where the total matter-gravity central charge

$$c_{\text{matter}} + c_{\text{gravity}} = 26$$

is “dominated” by the contribution of 2d gravity. This is far from the case of critical string ($c_{\text{matter}} = 26$) and close to the case of pure two-dimensional gravity. Unfortunately, up to now this is the only region where it is possible to formulate string theory consistently and at least put the question what is the internal principle which might allow one to choose dynamically a string vacuum.

Now, the above scheme is a string field theory in a sense it is highly related to the underlying module space \[33, 34\]. That is actually the main difference with ordinary field theory: the geometrical sense of construction above was motivated by cell-decomposition of module space \[46, 47, 33, 34\].

So, at the moment we have a sort of theory, describing various ($p, q$) models coupled to 2d gravity beyond the perturbation theory, and technically advocating to the equations in the space of coupling constants, which determine the generation function for all the correlators. Such function (being the logarithm of the $\tau$-function cannot be defined globally in this space, and around each “critical point” it has some sort of perturbative definition, which should reproduce the original ”first-quantized” theory \[1\]). However, moving in this space from one solution to another naively one will meet the divergences of such expansion, then it is necessary to continue analytically, and the piece we would get in such a way is exactly a nonperturbative correction. For simplest ($p, 1$) topological theories, this scheme can be described at the language of effective (“path”) or rather matrix integral with almost trivial integration measure, the exact integral formulas for generic case involve more complicated structures.

This scheme, in principle, should also be true for the ”barrier” $c = 1$ solutions. However, a generic $c = 1$ ”phase” is rather complicated theory, and naive one-matrix formulation leads only to some highly restricted $c = 1$ cases.
the determinant form of Penner model \[16, 17, 48\] partition function implies already that for fixed values of times it is a Toda lattice tau-function in the sense of and allows us to apply to this case the the Toda theory representation for Generalized Kontsevich models. Indeed, the solution to the Penner model

\[ Z \sim \det H_{ij}^{(\alpha)} \]  

(267)

with

\[ H_{ij}^{(\alpha)} = \Gamma(\alpha + i + j - 1) \]

(268)

is nothing but a specific case of GKM.

Now one can easily introduce positive- and negative-times dependence in (267) and then reconstruct \( \Phi_k^\{V\} (z) \) from (162) Indeed,

\[ h_{ij}^{(\alpha)} = H_{ij}^{(\alpha)} = \Gamma(\alpha - 1 + i + j) = \]

(269)

\[ = \int_0^\infty dy \, e^{-y} y^{\alpha-1+i+j} = \oint \phi_i^{(\alpha)}(z) z^j \]

immediately gives

\[ \phi_i^{(\alpha)}(z) = \int_0^\infty dy \, e^{zy} y^{\alpha-1+i} \]

(270)

which is a sort of GKM-like representation. The difference with more common situation for \( c < 1 \) is in the definition of the contour in (270) and also in the fact that \( z \)-dependence is trivial, because integral is easily taken with the result

\[ \phi_i^{(\alpha)}(z) = \frac{\Gamma(\alpha + i)}{(z - 1)^{\alpha+i}} \equiv \phi_{\alpha+i}(z) \]  

(271)

and

\[ \left( \frac{\partial}{\partial z} \right)^j \phi_i^{(\alpha)}(z) = (-)^j \phi_{\alpha+i+j}(z) = (-)^j \frac{\Gamma(\alpha + i + j)}{(z - 1)^{\alpha+i+j}} \]

(272)

Introducing negative times, one gets \[30\]

\[ \phi_i^{(\alpha)}(z | T_p) = z^{-\alpha} \exp \left( - \sum_{p>0} T_{-p} z^{-p} \right) \sum_k P_k[T_p] \phi_{i-k}^{(\alpha)}(z) \]

(273)

where \( P_k[T_p] \) are Schur polynomials \( (\exp \sum T_p z^p = \sum z^n P_n[T_p]) \), or simply

\[ Z_{\alpha=1} \sim \int \mathcal{D}Y \exp Tr ZY + \alpha Tr log Y + \sum T_{-k} Tr Y^{-k} \]

(274)
with
\[ T_{+k} = \frac{1}{k} Tr Z^k \]  
and amazingly this formula turns to be consistent with the calculations of the tachyonic amplitudes [45].

Let us now make more comments on \( c = 1 \) situation. From basic point of view we need in generic situation to get the most general (unreduced) KP or Toda-lattice tau-function satisfying some (unreduced) string equation. In a sense this is not a limiting case for \( c < 1 \) situation but rather a sort of “direct sum” for all \((p, q)\) models. This reflects that in conformal theory coupled to 2d gravity there is, in a sense, less difference between \( c < 1 \) and \( c = 1 \) situations than this coupling.

However, there are several particular cases when one can construct a sort of direct \( c \to 1 \) limit and which should correspond to certain highly “degenerate” \( c = 1 \) theories. From the general point of view presented above these are nothing but very specific cases of \((p, q)\) string equations, and they could correspond only to a certain very reduced subsector of \( c = 1 \) theory.

Indeed, it is easy to see, that for two special cases \( p = \pm q \) the equations \((251), (252)\) can be simplified drastically, actually giving rise to a single equation instead of a system of them. Of course, these two cases don’t correspond to minimal series where one needs \((p, q)\) being coprime numbers. However, we still can fulfill both reduction and Kac-Schwarz condition and these solutions to our equations using naively the formula for the central charge, one might identify with \( c = 1 \) for \( p = q \) and \( c = 25 \) for \( p = -q \).

Now, the simplest theories should be again with \( q = 1 \). For such case “\( c = 1 \)” turns to be equivalent to a discrete matrix model [25] while “\( c = 25 \)” is exactly what one would expect from generalization of the Penner approach [30, 45]. Indeed, taking in general non-polynomial functions, like
\[ W(x) = x^{-\beta} \]
\[ Q(x) = x^{\beta} \]  
the action would acquire a logarithmic term
\[ S_{-\beta,\beta} = -\beta \log x + \frac{x^{\beta}}{\beta} \]  
\[ (277) \]
while equations (251), (252) give rise just to rational solutions. It is very easy to see that 

\[ \beta = 1 \]

immediately gives the Penner model in the external field, which rather corresponds to “dual” to \( c = 1 \) situation with matter central charge being \( c_{\text{matter}} = 25 \) with a highly non-unitary realization of conformal matter.\(^{15}\)

On the other hand, \( p = q = 1 \) solution is nothing but a trivial theory, which however becomes a nontrivial discrete matrix model for unfrozen zero-time. Moreover, these particular \( p = \pm q \) solutions become nontrivial only if one considers the Toda-lattice picture with negative times being involved into dynamics of the effective theory. On the contrary, we know that \( c < 1 \) \((p, q)\)-solutions in a sense trivially depends on negative times with the last ones playing the role of symmetry of string equation [25]. It means, that we don’t yet understand enough the role of zero and negative times in the Toda-lattice formulation.

\[ \text{11 Conclusion} \]

Let us, finally make several concluding remarks. We tried to demonstrate above that using additional structure of integrability arised in the ”discretized” formulation of 2d gravity one might go far from what is known in string perturbation expansion. Namely, a generic \((p, q)\) solution to the \( c < 1 \) case can be described as a solution to the integrable system and its explicit form can be studied using various integral representations for these ”stringy” solutions. In the most simplest case of topological theories, these integral formulas can be combined into the matrix integral giving rise to a matrix ”zero-dimensional” field theory description.

Much more complicated is the case of \( d \geq 1 \) which at the moment seems to be almost unclear from the point of view be discussed below. One might only hope that there exists a possible generalization concerning non-Cartan flows and formed by them an integrable system. One might also hope that such integrable systems are related to the matrix-like integrals with a nontrivial integration over ”angle variables”, i.e. when in matrix integral the action is no longer the trace of function of a matrix, but is rather complicated objects. Such type of systems were discussed in the literature [53] and the technique

\(^{15}\)This \( c = 1 - c = 25 \) duality might be also connected with the known fact that there exists a Legendre transform between the Gross-Klebanov solution to \( c=1 \) matrix model and the Penner model.
was mostly based on the more complicated operations with the Itzykson-Zuber integrals \[21\]. This might be connected to a sort of “group-theoretical” $\tau$-functions, to be discussed elsewhere \[54\].

12 Acknowlegements

I would like to thank J.Petersen and J.Sidenius who initiated the course of lectures after which this paper has appeared. I am also indebted to D.Boulatov, L.Chekhov, P.Di Vecchia, K.Ito, C.Kristjansen H.Nillsen and especially to J.Ambørn for valuable discussions and remarks. And I am certainly grateful to A.Losev, S.Kharchev, A.Mironov, A.Morozov and A.Zabrodin for collaboration and permanent discussions on presented above and related problems. This work was financially supported by NORDITA.

References

[1] J.Ambjørn, B.Durhuus, J.Frolich, Nucl.Phys., B257[FS14] (1985) 433, F.David, Nucl.Phys., B257[FS14] (1985) 45
V.Kazakov, Phys.Lett., 150B (1985) 282
V.Kazakov, I.Kostov, A.Migdal, Phys.Lett., 157B (1985) 295.

[2] V.Kazakov, Mod.Phys.Lett., A4 (1989) 2125
E.Brézin and V.Kazakov, Phys.Lett., B236 (1990) 144
M.Douglas and S.Shenker, Nucl.Phys., B335 (1990) 635
D.Gross and A.Migdal, Phys.Rev.Lett., 64 (1990) 127.

[3] M.Douglas Phys.Lett., 238B (1990) 176

[4] M.Fukuma, H.Kawai, and R.Nakayama Int.J.Mod.Phys., A6 (1991) 1385

[5] R.Dijkgraaf, H.Verlinde, and E.Verlinde, Nucl.Phys., B348 (1991) 435.

[6] A.Marshakov, A.Mironov, and A.Morozov, Phys.Lett., 265B (1991) 99.
[7] S.Kharchev at al. *Conformal multimatrix models*, preprint FIAN/TD/9-92, ITEP-M-4/92 (July, 1992).

[8] V.Fateev and S.Lukyanov, *Int.J.Mod.Phys.*, A3 (1988) 507.

[9] A.Gerasimov et al. *Int.J.Mod.Phys.*, A5 (1990) 2495

[10] A.Mironov and S.Pakuliak, *Double scaling limit in the multi-matrix models of a new type*, preprint FIAN/TD/05-92 (May, 1992).

[11] L.Alvarez-Gaume and et al., preprint CERN (1991).

[12] A.Gerasimov et al. *Nucl.Phys.*, B357 (1991) 565.

[13] S.Kharchev et al., *Nucl.Phys.*, 366B (1991) 569.

[14] R.Hirot, Y.Ohta, J.Satsuma, *J.Phys.Soc.Japan*, 57 (1988) 1901.

[15] E.Date, M.Jimbo, M.Kashiwara, and T.Miwa, *Transformation group for soliton equation: III*, preprint RIMS-358 (1981), 
E.Date, M.Jimbo, M.Kashiwara, and T.Miwa. In: *Proc.RIMS symp.Nonlinear integrable systems — classical theory and quantum theory*, page 39, Kyoto, 1983.

[16] A.Marshakov, A.Mironov, and A.Morozov, *Phys.Lett.*, B274 (1992) 280.

[17] S.Kharchev et al. *Phys.Lett.*, 275B (1992) 311.

[18] S.Kharchev et al. *Nucl.Phys.*, B380 (1992) 181.

[19] A.Mikhailov *W-constraints in GKM*, preprint, (1992)

[20] G.Segal and G.Wilson, *Publ.I.H.E.S.*, 61 (1985) 1.

[21] C.Itzyzson and J.-B.Zuber, *J.Math.Phys.*, 21 (1980) 411.

[22] M.L.Mehta, *Commun.Math.Phys.*, 79 (1981) 327.

[23] V.Kac, A.Schwarz, *Phys.Lett.* B257 (1991) 329

[24] A.Schwarz, *Mod Phys Lett.*, A6 (1991) 611; 2713.
[25] S.Kharchev et al. *Generalized Kontsevich model versus Toda hierarchy and discrete matrix models*, preprint FIAN/TD/3-92, ITEP-M-3/92 (February, 1992), hepth/9203043.

[26] M.Fukuma, H.Kawai, and R.Nakayama, *Infinite dimensional Grassmannian structure of two-dimensional quantum gravity*, preprint UT-572, KEK-TH-272 (1990).

[27] Yu.Makeenko et al. *Nucl.Phys.*, **B356** (1991) 574.

[28] J.Ambjørn, L.Chekhov, C.Kristjansen, Yu.Makeenko, *Matrix model calculations beyond the spherical limit*, preprint NBI-HE-92-89 (1992).

[29] S.Kharchev et al. *Landau-Ginzburg topological theories in the framework of GKM and equivalent hierarchies*, preprint FIAN/TD/7-92, ITEP-M-5/92 (July, 1992), hepth/9208046.

[30] A.Marshakov, *On string field theory for c ≤ 1*, preprint FIAN/TD/08-92 (June, 1992), hepth/9208022.

[31] S.Kharchev, A.Marshakov, *Topological versus non-topological theories and p − q duality in matrix models*, preprint FIAN/TD/15-92 (September, 1992), hepth/9210072.

[32] S.Kharchev, A.Marshakov, *On p − q duality and explicit solutions in c ≤ 1 2d gravity models*, preprint NORDITA-93/20, FIAN/TD/04-93 (February, 1993), hepth/9303100.

[33] M.Kontsevich *Funk.Anal.&Prilozh.*, **25** (1991) 50 (in Russian)

[34] M.Kontsevich, preprint Max-Planck Inst., 1991; *Comm. Math. Phys.*

[35] R.Dijkgraaf, H.Verlinde, and E.Verlinde, *Nucl.Phys.*, **B352** (1991) 59.

[36] I.Krichever, *Comm.Math.Phys.*, **143** (1992) 415; *The tau-function of the universal Whitham hierarchy, matrix models and topological field theories*, preprint LPTENS-92/18.

[37] K.Takasaki and T.Takebe, *Sdiff(2) KP hierarchy*, preprint RIMS-814 (October,
[38] M.Fukuma, H.Kawai, and R.Nakayama, *Explicit solution for p – q duality in two-dimensional quantum gravity*, preprint UT-582, KEK-TH-289 (May 1991)

[39] T.Shiota, *Invent.Math.*, **83** (1986) 333.

[40] T.Takebe, *From general Zakharov-Shabat equations to the KP and the Toda lattice hierarchies*, preprint RIMS-779 (1991).

[41] A.Lossev, *Descendants constructed from matter fields and K.Saito higher residue pairing in Landau-Ginzburg theories coupled to topological gravity*, preprint ITEP/TPI-MINN (May, 1992).

A.Losev, I.Polyubin, *On connection between topological Landau-Ginzburg gravity and integrable systems*, preprint ITEP/Uppsala, (1993)

[42] W.Bailey, *Generalized hypergeometric series*. London, Cambridge Univ. press, 1935

[43] H.Bateman, A.Erdelyi, *Higher transcendental functions*, vol.1, NY, 1953

[44] E.D.Rainville, *Special functions*, Macmillan, NY, 1960

[45] R.Dijkgraaf, G.Moore, R.Plessner, *The partition function of the 2d string theory*, preprint IASSNS-HEP-92/48 (1992).

[46] J.Harer and D.Zagier, *Invent.Math.*, **85** (1986) 457.

[47] R.Penner, *J.Diff.Geom.*, **27** (1987) 35.

[48] J.Distler and C.Vafa, *Mod.Phys.Lett.*, **A6** (1991) 259.

[49] A.Marshakov, A.Mironov, and A.Morozov, *Mod.Phys.Lett.*, **A7** (1992) 1345.

[50] K.Ueno and K.Takasaki, *Adv.Studies in Pure Math.*, **4** (1984) 1.

[51] P.Grinevich and A.Orlov, *Flag spaces in KP theory and Virasoro action on det \( \bar{\partial}_j \) and Segal-Wilson \( \tau \)-function*, preprint Cornell Univ. (September, 1989).

[52] T.Miwa, *Proceedings of the Japan Academy*, **58** (1982) 9.
[53] V.Kazakov, A.Migdal *preprint* PUPT-1322, (1992)
    A.Migdal *preprint* PUPT-1322
    D.Gross *Some new/old approaches to QCD*, *preprint* PUPT-1355, (1992) and references therein
    S.Shatashvili, *IAS preprint*, (1992)

[54] S.Kharchev *et.al.*, to appear.