Sparse Block–Jacobi Matrices with Exact Hausdorff Dimension

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Abstract

We show that the Hausdorff dimension of the spectral measure of a class of deterministic, i. e. nonrandom, block–Jacobi matrices may be determined exactly, improving a result of Zlatoš (J. Funct. Anal. 207, 216-252 (2004)).

1 Introduction

In [MWGA] two of the authors addressed the possibility that a spectral transition takes place in a deterministic model. The model is represented by a class of Jacobi matrices with a sparse potential in the sense that the perturbation of the free Jacobi matrix (the 0–Dirichlet Laplacean on $l^2(\mathbb{Z}_+)$) is a (direct) sum of a fixed $2 \times 2$ off–diagonal matrix placed at sites whose distances from one another grow exponentially. In the present work we improve and complement results of [MWGA] in two directions. The model is now represented by block–Jacobi matrices and we are able to compute, for sparse perturbations satisfying transversal homogeneity, the exact Hausdorff dimension of their spectral measures.

Denoting the set of non–negative integers by $\mathbb{Z}_+$, let $\Lambda = \mathbb{Z}_+ \times \{0, 1, \ldots, L - 1\}$ be a strip of width $L \geq 1$ on the $\mathbb{Z}_+^2$ plane and define, on the separable Hilbert space $l^2(\Lambda, \mathbb{C})$, an operator $\Delta_{P, \phi}$ for each sequence $P = (p_n)_{n \geq -1}$ of numbers $p_n \in (0, 1]$ and angle $\phi \in [0, \pi)$:

$$(\Delta_{P, \phi} u)(k, m) := p_k u(k + 1, m) + p_{k-1} u(k - 1, m) + u(k, m + 1) + u(k, m - 1), \quad (1.1)$$

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for all \((k,m) \in \Lambda\) with phase boundary conditions at \(k = -1:\)

\[ u(-1, m) \cos \phi - u(0, m) \sin \phi = 0 \] (1.2)

for each \(m \in \{0, \ldots, L - 1\}\), and periodic boundary conditions on the vertical direction: \(u(k, L) = u(k, 0)\) for each \(k \in \mathbb{Z}_+\).

The operator \(\Delta_{P,0}\) with phase boundary 0 is, in particular, defined on a cylinder with the 0–Dirichlet boundary condition on \(k = -1: u(-1, m) = 0\) for every \(m \in \{0, \ldots, L - 1\}\) and the operator \(\Delta_{0,\phi}\), defined by setting \(p_n = 1, \forall n\), reduces to the usual discrete Laplacean in \(\Lambda\) with phase boundary \(\phi\). We note that \(p_k\) lives on the horizontal edges \(\ll (k, m), (k + 1, m) \gg, m \in \{0, \ldots, L - 1\}\), and (1.1) is defined with the same value \(p_k\) to each \(m\). Such property is referred to in the present text as transversal homogeneity.

The sparse perturbation considered here is a natural extension of the perturbation employed on the one-dimensional problem developed on [MWGA]. By sparse perturbation we mean a perturbation about the Laplacean: \(\Delta_{P,\phi} = \Delta_{0,\phi} + V_P\) where the potential \(V_P\) is composed of infinitely many vertical ‘barriers’ whose distances from one another grow exponentially. The sequence \(P = (p_n)_{n \geq -1}\) of ‘barriers’ is of the form

\[ p_n = \begin{cases} 
1 - \delta & \text{if } n = a_j \in \mathcal{A}, \\
1 & \text{if } n \notin \mathcal{A},
\end{cases} \] (1.3)

for \(\delta \in (0, 1)\) and a set of positive integers \(\mathcal{A} = \{a_j\}_{j \geq 1}\) such that

\[ a_j - a_{j-1} \geq 2, \quad j = 2, 3, \ldots \] (1.4)

and

\[ \lim_{j \to \infty} \frac{a_{j+1}}{a_j} = \beta > 1. \]

Condition (1.4) makes each ‘barrier’ to be located in an isolated single column of horizontal edges and \(\beta\) is the so called “sparseness parameter”. As in [MWGA], the separations between the barriers are fixed as

\[ a_j - a_{j-1} = \beta^j, \quad j = 2, 3, \ldots \] (1.5)

with \(a_1 = \beta \geq 2\) an integer, in order to simplify our analysis. From now on, (1.3) with \(\mathcal{A}\) given by (1.5) will be the only sequence considered and we shall denote by \(\Delta_{\delta,\phi}\) the corresponding operator with \(P = (p_n)_{n \geq -1}\) of this form.

The operator \(\Delta_{P,\phi}\) with \(\phi = 0\) may be written in the block–Jacobi matrix form

\[ J_P = J_P \otimes I_L + I \otimes A_L, \] (1.6)
with $I$ the identity operator on $l^2(\mathbb{Z}_+)$, $J_P$ is defined by

$$J_P = \begin{pmatrix}
0 & p_0 & 0 & 0 & \cdots \\
p_0 & 0 & p_1 & 0 & \cdots \\
0 & p_1 & 0 & p_2 & \cdots \\
& & & & \ddots \\
& & & & 
\end{pmatrix},$$  

(1.7)

the $(p_n)_{n \geq 0}$ as in (1.3), $A_L$ and $I_L$ denoting, respectively, the $L \times L$ matrix

$$A_L = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 1 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
& & & & \ddots & \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix},$$

(1.8)

and the $L \times L$ identity. The matrix class above reduces to the one studied on [MWGA] by setting the strip width $L = 1$ ($I_L = 1$ and $A_L = 0$ in this case).

It is interesting to note that as $\delta$ varies in the interval $(0, 1)$, $J_\delta$ (given by $J_P$ with $P$ satisfying (1.3)) interpolates continuously two distinct situations: a dense pure point spectrum at $\delta = 1$ and an absolutely continuous spectrum at $\delta = 0$. For a more detailed discussion which includes a wider class of perturbations see Section 1 of [MWGA].

**Remark 1.1** The results presented in this paper are not restricted to the operator $\Delta_{\delta, \phi}$. We could extend our methods to any sparse perturbation that is block-diagonalizable, i.e., that can be decomposed into its one-dimensional constituents by a matrix conjugation. The transversal homogeneity condition allows us to use the discrete Fourier transform to reduce to this form. The results also hold if $\phi$ in (1.2) is different for each $m$.

**Remark 1.2** The operator $\Delta_{\delta, \phi}$ with $\phi$–phase boundary condition at $k = -1$ (1.3) may also be written in the block-Jacobi matrix form (1.6). If $J_{\delta, \phi}$ denotes the corresponding matrix, we have

$$J_{\delta, \phi} = J_{\delta} + E_0 \otimes \tan \phi I_L,$$

(1.8)

where $E_0$ is an operator on $l^2(\mathbb{Z}_+)$ with all elements zero except $(E_0)_{00} = 1$. If the $\phi$–phase condition varies for each $m$, $\tan \phi I_L$ in (1.8) is replaced by $\text{diag}\{\tan \phi_m\}_{m=0}^{L-1}$.

The very basic method employed to study the spectrum of sparse Schrödinger operators is given by Pearson [P]. Let $\Delta_{\delta, 0}$ be the sparse operator $\Delta_{\delta, 0}$ ($\Delta_{\delta, 0} = J_\delta$ for $L = 1$) with $(p_n)_n$ given by (1.3) if $n < a_k$ and $p_n = 1$ for all $n \geq a_k$, and let $\rho_k(\varphi)$ denote the corresponding spectral measure. Note that $d\rho_k/d\varphi$ exists for almost every $\varphi \in [0, \pi]$ and $\rho_k$ is absolutely continuous with respect
to the Lebesgue measure. The spectral measure $\rho$ of $\Delta_{\delta,0}$, which may be derived from the limit as $k \to \infty$ of $\rho_k$, is determined by the asymptotic behavior, as $n \to \infty$, of the solution $\psi_n = \psi_n(\varphi)$ of the equation

$$\left(\Delta_{\delta,0}^k \psi\right)_n = \lambda \psi_n , \quad \lambda = 2 \cos \varphi$$

in the following sense. If $R_k(\varphi)$ and $\theta_k(\varphi)$ are the radius and angle of Prüfer associated with $\psi_{a_k}(\varphi)$, it can be shown (see [P, P1, KR])

$$\rho(\Sigma) = \lim_{k \to \infty} \rho_k(\Sigma) = \lim_{k \to \infty} \frac{2}{\pi} \int_{\Sigma} \sin^2 \varphi R_k^2(\varphi) d\varphi$$

for any Borel set $\Sigma \subset (0, \pi)$. Pearson’s idea is that sparse ‘barriers’ lead to ‘independence’ of certain (deterministic) functions which behave as functions of an uniformly distributed random variable. As a consequence, we have

$$\left(\frac{1}{R_k^2(\varphi)}\right)^{1/k} = \prod_{m=1}^{k} \left(\frac{R_{m-1}^2(\varphi)}{R_m^2(\varphi)}\right)^{1/k}$$

$$= \exp \left(\frac{1}{k} \sum_{m=1}^{k} \ln \frac{R_{m-1}^2(\varphi)}{R_m^2(\varphi)}\right)$$

$$\equiv \exp \left(\frac{1}{k} \sum_{m=1}^{k} \ln f(\varphi, (a_m - a_{m-1})\varphi, \theta_{m-1}(\varphi))\right)$$

$$\longrightarrow \exp \left(\frac{1}{\pi} \int_0^\pi \ln f(\varphi, u, \theta) \, du\right) \equiv 1/r$$

(1.9)

with probability one with respect to that uniform distribution, by the weak law of large numbers. His method was modified in [MWGA] by exploiting the uniform distribution of a sequence $(\zeta_m(\varphi))_{m \geq 1}$, for almost every $\varphi$, defined by a linear interpolation of Prüfer angles:

$$\theta_m(\varphi) = g(\theta_{m-1}(\varphi)) - (a_m - a_{m-1})\varphi , \quad m \geq 2$$

(1.10)

with $\theta_1 = \theta_0 - a_1\varphi$, for the monotone increasing function

$$g(\theta) = \tan^{-1} \left((\tan \theta + \cot \varphi)/(1 - \delta)^2 - \cot \varphi\right)$$

that maps the interval $(-\pi/2, \pi/2]$ into itself. The crucial observation here is that $f(\varphi, (a_m - a_{m-1})\varphi, \theta_{m-1}(\varphi))$ in (1.9) can be rewritten as $f(\varphi, \theta_m(\varphi))$ for a different, although similar, function $f$. Equation (1.9) thus gives an exact decay rate $1/r$ of $\psi_n(\varphi)$ without evoking ‘independence’ of

1Our definition of Prüfer angles differs slightly from that of [P] and other authors. By $\theta_j$ we mean the Prüfer angle at the site $a_j$ immediately before the $j$–th barrier takes place. Pearson’s definition is at the point $b_j$ right after the barrier.
the Bernoulli shift sequence \( u_m = (a_m - a_{m-1}) \varphi \mod \pi \), which would require an extremely sparse condition.

The Hausdorff dimension of the spectral measure \( \rho \) can be determined using an extension due to Jitomiskaya–Last [JL] of the Gilbert–Pearson theory of subordinance [GP], which relates the spectral property of \( \rho \) to the growth rate of solution \( \psi_n^{a}(\varphi) \) of the Schrödinger equation \( \Delta_{\varphi}^{k} \psi_n = \lambda \psi_n \). Note that \( \Delta_{\varphi}^{k} \psi_n = \Delta_{\varphi}^{k} + E_0 \tan \varphi \) and the phase boundary is important since the exact Hausdorff dimension holds only for almost every \( \varphi \) w.r.t. the Lebesgue measure. It is worth mentioning that Zlatoš [Z] has applied the Jitomiskaya–Last method to a sparse model very similar to the one considered in [MWGA] (whose ‘barriers’ locate at sites, not at edges). He has obtained the exact Hausdorff measure for a sparse random model in which the distances from one to another ‘barrier’ are given by \( a_j - a_j-1 + \omega_j \) with \( (\omega_j)_{j \geq 1} \) independent random variables uniformly distributed in the interval \([-j, -j+1, \ldots, j]\). The improvement of Pearson’s method (1.9) given in [MWGA] allows the Hausdorff dimension of the spectral measure to be determined without adding a random variable to the sparse condition. This our main result (Theorem 3.11).

The present paper is organized as follows. We present some preliminary facts on the spectrum of \( J_{\delta} \) on Section 2 and on Section 3 we establish the exact Hausdorff dimension of the spectral matrix measure of \( \Delta_{\delta,\varphi} \). Our main result, Theorem 3.11, is stated and proved in this section, after we have extended to the block–Jacobi matrix \( J_{\delta,\varphi} \) several preliminary results of [JL, Z].

## 2 The Spectrum of \( J_{\delta} \) and Notation

In order to introduce the spectral measure of block–Jacobi matrices considered and to fix notation we shall first consider the 0–Dirichlet Laplacean operator \( \Delta_{0,0} \). For convenience, we always change the order of the tensor product in (1.6): \( I_L \otimes J_{\delta} + A_L \otimes I = \Pi (J_{\delta} \otimes I_L + I \otimes A_L) \Pi^{-1} \) by an appropriate permutation matrix \( \Pi \) and we call it by \( J_{\delta} \) as well. The Kronecker sum

\[
J_0 = I_L \otimes J_0 + A_L \otimes I ,
\]

with \( J_0 \) the free Jacobi matrix

\[
J_0 = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & 1 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix},
\]

is thus unitarily similar to (1.6) for (1.3) with \( \delta = 0 \); consequently, its spectrum remains unchanged.

The structure of (2.1) permits to give a simple answer to the spectrum of this operator. It is well known (see e.g. [La]) that if \( \{\eta_k\}_{k=1}^n \) and \( \{\lambda_j\}_{j=1}^m \) are the eigenvalues of the matrices \( A \) and \( B \), respectively, then \( \{\eta_k + \lambda_j\}_{k,j=1}^{n,m} \) are the eigenvalues of the Kronecker sum \( I_m \otimes A + B \otimes I_n \). Since
the interval $[-2, 2]$ is the essential spectrum of $J_0$, the essential spectrum of $J_0$ is given by

$$
\sigma_{\text{ess}}(J_0) = \bigcup_{j=0}^{L-1} (\eta_j + \sigma_{\text{ess}}(J_0)) = \bigcup_{j=0}^{L-1} I_j ,
$$

(2.3)

with $\{\eta_j\}_{j=0}^{L-1}$, $\eta_j = 2 \cos (2\pi j / L)$, the eigenvalues of $A_L$. Thus,

$$
\sigma_{\text{ess}}(J_0) = \begin{cases} 
[-2 + 2 \cos (\pi (L - 1) / L), 4] & \text{if } L \text{ is odd} \\
[-4, 4] & \text{if } L \text{ is even} 
\end{cases}
$$

(2.4)

holds for $L \geq 2$.

It is also well known that the essential spectrum of the free Jacobi matrix $J_0$, defined by (2.2), is purely absolutely continuous. As $J_0$ is in some sense a free matrix, we have

**Proposition 2.1** The essential spectrum of $J_0$, given by (2.4), is purely absolutely continuous.

Proof. Let

$$
M(z) = \int \frac{d\rho(x)}{x - z}
$$

be the $L \times L$ $M$–matrix defined by the Borel transform of the spectral matrix $\rho$. By the spectral theorem, the $M$–matrix of $J$ is related to the resolvent matrix $(J - zI_L \otimes I)^{-1}$ as follows. If $J$ is the matrix representation of a self–adjoint operator $H$ in the separable space $\mathcal{H}$ with an orthonormal basis $\{\varphi_{(k,m)}\}_{(k,m) \in \Lambda}$, we have

$$
(J - zI_L \otimes I)^{-1} = (\varphi_{(0,m)}, (H - zI)^{-1} \varphi_{(0,m')}) = \int \frac{d\rho_{mm'}(x)}{x - z} = M_{mm'}(x)
$$

By the fact that $\Delta_{0,0}$ has periodic condition on the vertical direction, $A_L$ is cyclic and the resolvent can be block–diagonalized by the Fourier matrix:

$$
(F_L^{-1} \otimes I)(J - zI_L \otimes I)^{-1}(F_L \otimes I) = \text{diag} \left\{ (J_0 - z_j I)^{-1} \right\}_{j=0}^{L-1},
$$

(2.5)

with $z_j = z - 2 \cos(2\pi j / L)$, $j = 0, \ldots, L - 1$, and $F_L := [v_1 \cdots v_L]$ the matrix built up with the eigenvectors $v_k = (1, \xi^k, \ldots, \xi^{(L-1)k}) / \sqrt{L}$, $\xi = \exp\{2\pi i / L\}$ of the shift matrix $S : (x_0, \ldots, x_{L-1}) \mapsto (x_1, \ldots, x_{L-1}, x_0)$ on its columns.

The $M$–matrix can thus be written as

$$
M(z) = F_L \text{diag} \left\{ (J_0 - z_j I)_{00}^{-1} \right\}_{j=1}^{L} F_L^{-1},
$$

(2.6)

where the $00$–element $(J_0 - z_j I)_{00}^{-1}$ of the resolvent matrix $(J_0 - z_j I)^{-1}$ is the Weyl–Titchmarsh $m$–function of the free Jacobi matrix $J_0$ evaluated at $z_j$. It is a simple exercise to calculate the $m$–function for the one-dimensional free problem. If $u_1$, $u_2$ are the linear independent solutions
of the Schrödinger equation $J_0 u = zu$ satisfying, respectively, Dirichlet (3.8) and Neumann (3.9) boundary conditions at $n = -1$, $m(z)$ is uniquely defined by imposing that $u = u_2 - m(z)u_1$ is $l^2(Z_+, \mathbb{C})$. Explicitly

$$m(z_j) = \frac{z_j}{2} + \sqrt{\frac{z_j^2}{4} - 1} , \quad j = 0, 1, \ldots, L - 1 .$$

(2.7)

for $\Re z_j > 0$.

Now, let $m(z) = \int d\mu(x)/(z - x)$,

$$\Im m(\zeta) = \limsup_{\xi \downarrow 0} \Im m(z) ,$$

$z = \zeta + i\xi$, and let $L(\rho)$ be the set of all $\zeta \in \mathbb{R}$ for which this limit exists. It is known (see Appendix B from [T]) that the minimal (or essential) supports $M$, $M_{ac}$ and $M_s$ of $\mu$, the absolutely continuous part $\mu_{ac}$ and the singular part $\mu_s$ of $\mu$, with respect to the Lebesgue measure in $\mathbb{R}$, are, respectively, given by $\zeta \in L(\rho)$ such that $0 < \Im m(\zeta) \leq \infty$, $0 < \Im m(\zeta) < \infty$ and $\Im m(\zeta) = \infty$. These criteria can be obtained using de la Vallée-Poussin’s decomposition theorem [S], the Lebesgue-Radon-Nikodym theorem and the following Lemma (see e.g. [GP]):

**Lemma 2.2** If $(d\mu/d\nu)(\zeta)$ (the Radon-Nikodym derivative) exists finitely or infinitely, then $\Im m(\zeta)$ also exists and $(d\mu/d\nu)(\zeta) = (1/\pi)\Im m(\zeta)$ ($\nu$ is some Lebesgue measure on $\mathbb{R}$).

Returning to the $M$–matrix (2.6), its diagonal elements are given by

$$M_{mm}(z) = \sum_{j,k=0}^{L-1} (F_L)_{mj} \left( \text{diag} \ (J_0 - z_i I)^{-1}_{00} \right)_{jk} (F_L^{-1})_{km}$$

$$= \sum_{j=1}^{L} m(z_j) |(F_L)_{mj}|^2 = \frac{1}{L} \sum_{j=1}^{L} m(z_j) ,$$

(2.8)

with $m(z)$ given by (2.7). This equation, together with

$$\lim_{\xi \downarrow 0} \Im m(\zeta_j + i\xi) = \begin{cases} 0 & \text{if } |\zeta_j| \geq 2 \\ \sqrt{1 - \zeta_j^2/4} & \text{if } |\zeta_j| < 2 \end{cases} ,$$

$\zeta_j = \zeta - 2 \cos(2\pi j/L)$, and Lemma 2.2 leads to

$$\lim_{\xi \downarrow 0} \frac{d\rho_{mm}}{d\zeta}(\zeta + i\xi) = \frac{1}{\pi L} \sum_{j=1}^{L} \lim_{\xi \downarrow 0} \Im m(\zeta_j + i\xi) ,$$

(2.9)

which is strictly positive for almost every $\zeta$ with respect to the Lebesgue measure on the essential support (2.4) of $J_0$ and zero on its complement. The proof of Proposition 2.1 is thus concluded evoking the above criteria.
A natural question to ask is whether the essential spectrum of the matrix $J_\delta$ is, regardless of $\delta \in (0, 1)$, the same of $J_0$. This question is settled by the following

**Theorem 2.3** Let $J_\delta$ be the block-Jacobi matrix defined by (1.6) with $P$ given by (1.3) and (1.5). The essential spectrum of $J_\delta$ is the set (2.3) and, consequently,

$$\sigma_{\text{ess}}(J_\delta) = \sigma_{\text{ess}}(J_0)$$

holds for any $\delta \in (0, 1)$.

**Remark 2.4** Theorem 2.3 is an extension of Theorem 2.1 from [MWGA]. We follow its proof step by step.

**Remark 2.5** The operator $\Delta_{\delta, \phi}$ with $\phi$-phase boundary condition at $k = -1$ (1.2) may also be written in the block-Jacobi matrix form (1.4) (see equation (1.8)). Clearly $E_0 \otimes \tan \phi I_L$ is a rank-$L$ perturbation of $J_\delta$ and $\sigma_{\text{ess}}(J_{\delta, \phi}) = \sigma_{\text{ess}}(J_\delta)$, by Weyl’s invariance principle (see e.g. [RS]). Thus, it is sufficient to deal with $J_\delta$ to determine the essential spectrum of $J_{\delta, \phi}$.

**Proof.** Firstly, let us show that $\sigma_{\text{ess}}(J_\delta) \subseteq \sigma_{\text{ess}}(J_0)$. Define for $u = (u(k, m))_{(k, m) \in \Lambda} \in l_2(\Lambda)$ the $2L$-dimensional column vectors

$$u_k = (u(k, 0), u(k + 1, 0), \ldots, u(k, L - 1), u(k + 1, L - 1))$$

and the $2L \times 2L$ matrices

$$h_k^L = p_k I_L \otimes A_2 + \frac{1}{2}A_L \otimes I_2 . \tag{2.10}$$

Then, the quadratic form associated with $J_\delta$ can be written as

$$(u, J_\delta u) = \sum_{k=1}^{\infty} u_k \cdot h_k^L u_{k+1} + \sum_{k=1}^{L} u(0, k) u(0, k + 1) . \tag{2.11}$$

The factor $1/2$ present in (2.10) avoids double counting of terms in (2.11); the second sum present in (2.11) corrects the counting of the interacting terms between the elements of the first column.

We follow the strategy used in Proposition 2.1 to calculate the eigenvalues of $h_n^L$. The characteristic polynomial of $h_k^L$ reads

$$\det [h_k^L - \lambda I_L \otimes I_2] = \det [(F_L^{-1} \otimes I_2) (h_k^L - \lambda I_L \otimes I_2) (F_L \otimes I_2)]$$

$$= \det \left[ \text{diag} (S_m - \lambda I_2)^{L-1} \right]$$

$$= \prod_{m=0}^{L-1} \det [S_m - \lambda I_2] ,$$

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So, the eigenvalues of $h^L_k$ are $\lambda_{k,m}^\pm = \pm p_k + \cos(2\pi m/L)$, $m = 0, \ldots, L - 1$. Inserting the spectral decomposition of $h^L_k$

$$h^L_k = \sum_{m=0}^{L-1} \left( \lambda_{k,m}^+ P_{k,m}^+ + \lambda_{k,m}^- P_{k,m}^- \right)$$

into (2.11), where $P_{k,m}^\pm$ are the projectors in the direction of the eigenvectors associated with $\lambda_{k,m}^\pm$, we have

$$2\lambda^- \leq \frac{(u, \mathcal{J}_\delta u)}{(u, u)} \leq 2\lambda^+, \quad \text{with} \quad \lambda^+ = \sup_{k,m} \lambda_{k,m}^+ = 2$$

and

$$\lambda^- = \inf_{k,m} \lambda_{k,m}^- = \begin{cases} -2 & \text{if} \ L \text{ is even} \\ -1 + \cos(\pi L/2) & \text{if} \ L \text{ is odd} \end{cases}$$

concluding, together with (2.12), that $\sigma_{\text{ess}}(\mathcal{J}_\delta) \subseteq \sigma_{\text{ess}}(\mathcal{J}_0)$.

To prove the inclusion $\sigma_{\text{ess}}(\mathcal{J}_\delta) \supseteq \sigma_{\text{ess}}(\mathcal{J})$, we use the Weyl criterion (Theorem VII.12 of [RS]): if $B$ is a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$, $\lambda$ belongs to the spectrum $\sigma(B)$ of $B$ if and only if there exists a sequence $(\psi_n)_{n \in \mathbb{N}}$ in $\mathcal{H}$, with $\|\psi_n\| = 1$, such that

$$\lim_{n \to \infty} \|(B - \lambda)\psi_n\| = 0.$$ 

Let $\lambda_m(\varphi) = 2(\cos \varphi - \cos(2\pi m/L))$, $\varphi \in [0, \pi]$, $m = 0, \ldots, L - 1$, and define

$$\psi_{n,m} = \psi_n \otimes v_m,$$

with $\psi_n = (1/\sqrt{n}) (e^{i\varphi_1}, \ldots, e^{i\varphi_j}, 0, \ldots)$ and $v_m$ the $m$-th eigenvector of the shift operator $S$ (see equation (2.5)). Clearly $\psi_{n,m} \in l_2(\Lambda)$ and $\{\lambda_m(\varphi)\}$ is in one-to-one correspondence with (2.3). We claim that, for each $m = 0, \ldots, L - 1$,

$$\|\mathcal{J}_\delta \psi_{n,m} - \lambda_m \psi_{n,m}\| \leq c \frac{\ln n}{\sqrt{n}} \quad (2.12)$$

holds with $c = c(\beta)$ independent of $n$. To prove (2.12), we just have to note that $\mathcal{J}_\delta \psi_{n,m} - \lambda_m \psi_{n,m}$ consists of the action on $\psi_{n,m}$ of a sum of local matrices, bounded in norm by one; 2 of them involve the extreme points $e^{i\varphi}$ and $e^{in\varphi}$, and there are $O(\ln n)$ nondiagonal matrices. The $O(\ln n)$ is due to the fact that the sequence $(a_j)_{j \geq 1}$ satisfies the sparseness condition (1.5), with at most $r$ points $a_j$ within $[1, n]$; $r$ is such that

$$r \leq \frac{\ln n}{\ln \beta}.$$
Note that \( A_L v_m = 2 \cos(2\pi m / L) v_m \) and this part of the tensor product in \( J_\delta \) has no effect to the limit process. This proves the inclusion \( \sigma_{\text{ess}}(J_\delta) \supseteq \sigma_{\text{ess}}(J) \) and completes the proof of Theorem 2.3.

\[ □ \]

### 3 Exact Hausdorff dimension

This section is devoted to the determination of the Hausdorff dimension of the spectral measure of (1.6).

#### 3.1 Basic Definitions and Subordinacy

We start by some useful definitions. A more complete description is found in [L].

Given a Borel set \( S \subset \mathbb{R} \) and \( \alpha \in [0,1] \), we define the number

\[
Q_{\alpha,\delta}(S) = \inf \left\{ \sum_{\nu=1}^{\infty} |b_\nu|^\alpha : |b_\nu| < \delta; S \subset \bigcup_{\nu=1}^{\infty} b_\nu \right\},
\]

(3.1)

the infimum taken over all \( \delta \)-covers by intervals of size at most \( \delta \). The limit \( \delta \to 0 \),

\[
h_{\alpha}(S) = \lim_{\delta \to 0} Q_{\alpha,\delta}(S),
\]

(3.2)

is called \( \alpha \)-dimensional Hausdorff measure. This measure can be viewed as a continuous interpolation of the counting measure at \( \alpha = 0 \) (which assigns to each set \( S \) the number of points in it) and the Lebesgue measure at \( \alpha = 1 \). It is clear by the definitions (3.1) and (3.2) that \( h_{\alpha}(S) \) is an outer measure on \( \mathbb{R} \), and its restriction to Borel sets is a Borel measure (see e.g. [F]). For \( \beta < \alpha < \gamma \),

\[
\delta^{\alpha-\gamma} Q_{\gamma,\delta}(S) \leq Q_{\alpha,\delta}(S) \leq \delta^{\alpha-\beta} Q_{\beta,\delta}(S),
\]

holds for any \( \delta > 0 \) and \( S \subset \mathbb{R} \). So, if \( h_{\alpha}(S) < \infty \), then \( h_{\gamma}(S) = 0 \) for \( \gamma > \alpha \); if \( h_{\alpha}(S) > 0 \), then \( h_{\beta}(S) = \infty \) for \( \beta < \alpha \). Thus, for every Borel set \( S \), there is an unique \( \alpha_S \) such that \( h_{\alpha}(S) = 0 \) if \( \alpha > \alpha_S \) and \( h_{\alpha}(S) = \infty \) if \( \alpha_S < \alpha \). The number \( \alpha_S \) is called the Hausdorff dimension of the set \( S \).

Another useful concept is the exact dimension of a measure, due to Rodgers-Taylor [RT]:

**Definition 3.1** A measure \( \mu \) defined on \( \mathbb{R} \) is said to be of exact dimension \( \alpha \), \( \alpha \in [0,1] \), if and only if two requirements hold: (1) for every \( \beta \in [0,1] \) with \( \beta < \alpha \) and \( S \) a set of dimension \( \beta \), \( \mu(S) = 0 \) (which means that \( \mu(S) \) gives zero weight to any set \( S \) with \( h_{\alpha}(S) = 0 \)); (2) there is a set \( S_0 \) of dimension \( \alpha \) which supports \( \mu \) in the sense that \( \mu(\mathbb{R} \setminus S_0) = 0 \).
Given a positive, finite measure \( \mu \) and \( \alpha \in [0,1] \), we define the Hausdorff upper derivative by the limit

\[
D_\mu^\alpha(x) \equiv \limsup_{\epsilon \downarrow 0} \frac{\mu((x-\epsilon,x+\epsilon))}{(2\epsilon)^\alpha}.
\]  
(3.3)

Definition (3.3) is the generalization of the Radon-Nikodym derivative for Hausdorff measures. Note that the limit \( \epsilon \downarrow 0 \) does not need to be defined. Clearly, if \( D_\mu^\alpha(x_0) < \infty \) for some \( x_0 \) then, for all \( \beta < \alpha \),

\[
D_\mu^\beta(x_0) = \limsup_{\epsilon \downarrow 0} (2\epsilon)^{\alpha-\beta} \frac{\mu((x-\epsilon,x+\epsilon))}{(2\epsilon)^\alpha} = \limsup_{\epsilon \downarrow 0} (2\epsilon)^{\alpha-\beta} D_\mu^\alpha(x) = 0.
\]

In a similar fashion, if \( D_\mu^\alpha(x_0) > 0 \) for some \( x_0 \), then \( D_\mu^\beta(x_0) = \infty \) for all \( \beta > \alpha \). Thus, we can define for each \( x_0 \) the local Hausdorff dimension \( \alpha(x_0) \), given by

\[
\alpha_\mu(x_0) \equiv \liminf_{\epsilon \downarrow 0} \frac{\ln \mu((x-\epsilon,x+\epsilon))}{\ln(2\epsilon)}.
\]  
(3.4)

Finally, we introduce the notion of continuity and singularity of a measure with respect to the Hausdorff measure. Given \( \alpha \in [0,1] \), a measure \( \mu \) is called \( \alpha \)-continuous if \( \mu(S) = 0 \) for every set \( S \) with \( h^\alpha(S) = 0 \); it is called \( \alpha \)-singular if it is supported on some set \( S \) with \( h^\alpha(S) = 0 \). We can reformulate Definition 3.1 in this context: a measure \( \mu \) is said to have exact dimension \( \alpha \) if, for every \( \epsilon > 0 \), it is simultaneously \((\alpha - \epsilon)\)-continuous and \((\alpha + \epsilon)\)-singular.

The following remarkable result is due to Rodgers-Taylor [RT] and was extracted from Del Rio-Jitomirskaya-Last-Simon [DJLS]:

**Theorem 3.2 (Rodgers-Taylor)** Let \( \mu \) be any measure and \( \alpha \in [0,1] \). Let

\[
T_\infty = \{ x : D_\mu^\alpha(x) = \infty \}
\]

and let \( \chi_\alpha \) denote its characteristic function. Let \( d\mu_{\alpha s} = \chi_\alpha d\mu \) and \( d\mu_{\alpha c} = (1-\chi_\alpha) d\mu \). Then \( d\mu_{\alpha s} \) and \( d\mu_{\alpha c} \) are, respectively, singular and continuous with respect to \( h^\alpha \).

**Remark 3.3** The restriction \( \mu(T_+ \cap \cdot) \) to the set \( T_+ = \{ x : 0 < D_\mu^\alpha(x) < \infty \} \) is absolutely continuous with respect to \( h^\alpha \), in the sense that it is given by \( f(x)dh^\alpha(x) \) for some \( f \in L^1(\mathbb{R},dh^\alpha) \).

**Remark 3.4** Theorem 3.2 permits an extension of the standard Lebesgue decomposition of a Borel measure into continuous and singular parts, with respect to the Hausdorff measure. The decomposition into absolutely continuous, singular-continuous and pure point parts can also be extended (see [L] for a complete study). All these measure decompositions lead to a corresponding spectral decomposition of the Hilbert space.
Let $J$ be an essentially self–adjoint operator on $l^2(\mathbb{Z}_+)$ given by a Jacobi matrix and let
\[ Ju = \lambda u , \tag{3.5} \]
be the corresponding Schrödinger equation. Jitomirskaya-Last [JL] extended, for Hausdorff measures, the Gilbert–Pearson theory of subordinacy [GP], for Lebesgue measures, which relates the spectral property of $\rho$ to the rate of growth of the solutions of the Schrödinger equation. A solution $u$ of (3.5) is said to be subordinate if
\[ \lim_{l \to \infty} \frac{\|u\|_l}{\|v\|_l} = 0 \]
holds for any linearly independent solution $v$ of (3.5), where $\|\cdot\|_l$ denotes the $l^2(\mathbb{Z}_+)$–norm truncated at the length $l \in \mathbb{R}$, i.e.,
\[ \|u\|_l^2 = \sum_{n=0}^{[l]} |u(n)|^2 + (l-[l])|u([l]+1)|^2 , \]
$[l]$ the integer part of $l$.

We shall see that the theory in [MWGA] permits to distinguish different kinds of singular–continuous spectra, suitable for the study of the spectral measure $\rho_j(\lambda)$ associated to each one–dimensional component of $J_\delta$, since their singularity becomes more pronounced when $\lambda$ varies from the center to the border of the spectrum (see Theorem 4.4 of [MWGA]).

To extend the block-diagonalization ideas used in Section 2 to study the spectral measure of $\Delta_{\delta,\phi}$, given by (1.1), we define operators
\[ (H^j_{\delta,\phi} \psi)(n) = p_n \psi(n+1) + p_{n-1} \psi(n-1) + V_j \psi(n) , \]
on $l^2(\mathbb{Z}_+, \mathbb{C})$ subjected to a $\phi$–boundary condition at $n = -1$:
\[ \psi(-1) \cos \phi - \psi(0) \sin \phi = 0 , \tag{3.6} \]
for each $j \in \{0, \ldots, L-1\}$. The “potential” $V_j = 2 \cos(2\pi j/L)$ arises from the block-diagonalization of $\Delta_{\delta,\phi}$ by the Fourier matrix $F_L \otimes I$. Note that each $H^j_{\delta,\phi}$ is the projection of $\Delta_{\delta,\phi}$ into its $j$–th one-dimensional constituent.

To each $H^j_{\delta,\phi}$ there corresponds a Schrödinger equation
\[ J_\delta u_j = \lambda_j u_j , \tag{3.7} \]
with $J_\delta$ given by (1.7); we incorporate the factor $V_j$ to the spectral parameter $\lambda$ and define $\lambda_j = \lambda - 2 \cos(2\pi j/L)$.

Now, let $\lambda \in \mathbb{R}$ and $u_{1,j}$ be the solution of (3.7) which satisfies the Dirichlet boundary condition at $-1$, namely
\[ u_{1,j}(-1) = 0, \quad u_{1,j}(0) = 1 , \tag{3.8} \]
and let $u_{2,j}$ be the solution which satisfies the Neumann boundary condition

$$u_{2,j}(-1) = 1, \quad u_{2,j}(0) = 0. \quad (3.9)$$

Following Jitomirskaya-Last [JL], we define for any given $\epsilon > 0$ and for each $j \in \{1, \ldots, l\}$ a length $l_j(\epsilon) \in (0, \infty)$ by the equality

$$\|u_{1,j}\|_{l_j(\epsilon)} \|u_{2,j}\|_{l_j(\epsilon)} = \frac{1}{2\epsilon} \quad (3.10)$$

(see equation (1.12) from [JL]).

Since at most one of the solutions $\{u_{1,j}, u_{2,j}\}$ of (3.7) is $l_2$ (thanks to the Wronskian constancy), the left-hand side of (3.10) is a monotone increasing function of $l$ which vanishes at $l = 0$ and diverges as $l \to \infty$. On the other hand, the right-hand side of (3.10) is a monotone decreasing function of $\epsilon$ which diverges as $\epsilon \to 0$. We conclude that the function $l(\epsilon)$ is a well defined monotone decreasing and continuous function of $\epsilon$ which diverges as $\epsilon \to 0$.

$l_j(\epsilon)$ being defined, we can apply Theorem 1.1 of [JL] for each Weyl-Titchmarsh $m_j$-function related to each pair of solutions $u_{1,j}$ and $u_{2,j}$, $j \in \{1, \ldots, l\}$: for fixed $\epsilon > 0$

$$\frac{5 - \sqrt{24}}{m_j(\lambda + i\epsilon)} \leq \frac{\|u_{1,j}\|_{l_j(\epsilon)}}{\|u_{2,j}\|_{l_j(\epsilon)}} \leq \frac{5 + \sqrt{24}}{m_j(\lambda + i\epsilon)}.$$

Note that Theorem 1.2 of [JL] and its corollaries also holds: if $\mu_j$ denotes the spectral measure of $H^j_{\delta,\phi}$, then, with $b = \alpha/(2 - \alpha)$,

$$\limsup_{\epsilon \to 0} \frac{\mu_j((\lambda - \epsilon, \lambda + \epsilon))}{(2\epsilon)\alpha} = \infty \quad (3.11)$$

if and only if

$$\liminf_{l \to \infty} \frac{\|u_{1,j}\|_{l}}{\|u_{2,j}\|_{l}^b} = 0. \quad (3.12)$$

3.2 Extension to Block–Jacobi Matrices

We may ask whether these results can be extended to the diagonal elements $\rho_{nn}$ of the spectral matrix $\rho$ of $\Delta_{\delta,\phi}$. The generalization of Theorem 1.2 from [JL] is as follows. Since all diagonal elements of $M$ are equal, it is enough to consider $\rho_{00}$.

**Theorem 3.5** Let $\Delta_{\delta,\phi}$ be given by (1.1), $\lambda \in \mathbb{R}$ and $\alpha \in (0, 1)$. Then

$$D^\alpha_{\rho_{00}}(\lambda) = \limsup_{\epsilon \to 0} \frac{\rho_{00}(\lambda - \epsilon, \lambda + \epsilon))}{(2\epsilon)\alpha} = \infty \quad (3.13)$$

if and only if

$$\liminf_{l \to \infty} \frac{\|u_{1,j}\|_{l}}{\|u_{2,j}\|_{l}^b} = 0 \quad (3.14)$$
for at least one \( j \in \mathcal{I}(\lambda) \), where
\[
\mathcal{I}(\lambda) := \{ m \in \{0, \ldots, L-1 \} : I_m \ni \lambda \} ,
\] (3.15)

\( I_m \) is defined in (2.3) and \( b = \frac{\alpha}{2 - \alpha} \).

**Proof.** Suppose that (3.13) holds. Then, by (2.8) and (2.9), there exists at least one \( j \in \mathcal{I}(\lambda) \) such that
\[
\limsup_{\epsilon \downarrow 0} \mu_j(\lambda - \epsilon, \lambda + \epsilon)) \frac{(2\epsilon)^\alpha}{(2\epsilon)^\alpha} = \infty
\] (3.16)
and by Theorem 1.2 in [JL] applied to the operator \( H^j_{\delta,\phi} \) (equations (3.11) and (3.12)), this holds if and only if (3.14) holds.

Suppose now that (3.14) holds for some \( j \in \mathcal{I}(\lambda) \). The same Theorem 1.2 of [JL] leads to (3.16). But we know from (2.8) and (2.9) that this implies (3.13), concluding the proof of Lemma 3.5.

\[\square\]

The resulting corollaries of Theorem 1.2 in [JL] can be extended on a similar fashion. Of particular interest are Corollaries 4.4 and 4.5 of [JL]. The new version of the first is given by

**Corollary 3.6** Suppose that for some \( \alpha \in [0,1) \) and every \( \lambda \) in some Borel set \( A \), every solution \( v_j \) of (3.5) obeys
\[
\limsup_{l \to \infty} \frac{\|v_j\|^2}{l^{2-\alpha}} < \infty
\]
for all \( j \in \mathcal{I}(\lambda) \neq \emptyset \). Then the restriction \( \rho_{00}(A \cap \cdot) \) is \( \alpha \)-continuous.

**Proof.** The proof follows the same structure as the proof of Corollary 4.4 in [JL]. Let \( \lambda \in A \). From the constancy of the Wronskian, \( \|u_{1,j}\|, \|u_{2,j}\| \geq l \) holds for every \( j \), and since, by hypothesis, \( \|u_{2,j}\|^2 < Cl^{2-\alpha} \) for some constant \( C \), it follows that \( \|u_{1,j}\| > C^{-1/2}l^{\alpha/2} \) for every \( j \in \mathcal{I}(\lambda) \). Thus, we have
\[
\frac{\|u_{1,j}\|}{\|u_{2,j}\|} > C^{-(1+b)/2}l^{\alpha/2 - b(2-\alpha)/2} = C^{-(1+b)/2} > 0 ,
\]
since \( b = \alpha/(2 - \alpha) \). It follows from Theorem 3.5 that \( \rho_{00}(A \cap \cdot) \) is \( \alpha \)-continuous.

\[\square\]

Corollary 3.6 can be rewritten in terms of the one-dimensional \( 2 \times 2 \) transfer matrices
\[
T_j(n; \lambda) = T_j(n, n-1; \lambda)T_j(n-1, n-2; \lambda) \cdots T_j(0, -1; \lambda) ,
\] (3.17)
where
\[
T_j(n, n-1; \lambda) = \begin{pmatrix}
\lambda_j & -p_{n-1} \\
p_n & p_n \\
1 & 0
\end{pmatrix} = T(n, n-1; \lambda_j)
\] (3.18)
is related to the equation $(3.7)$ for every $j \in \{0, \cdots, L-1\}$. Note that $T(n, n-1; \lambda)$ is precisely the transfer matrix considered in [MWGA] (see equation (2.2) therein). Moreover, for the sequence $(p_n)_{n \geq -1}$ of the form $(1.3)$, only three different $2 \times 2$ matrices appear in the r.h.s. of $(3.17)$:

$$
T_- = \begin{pmatrix}
\lambda_j & -1 \\
1-\delta & 1-\delta
\end{pmatrix}, \quad T_+ = \begin{pmatrix}
\lambda_j & -1+\delta \\
1 & 0
\end{pmatrix} \quad \text{and} \quad T_0 = \begin{pmatrix}
\lambda_j & -1 \\
1 & 0
\end{pmatrix}
$$

(3.19)
depending on whether the left, the right or none of the two entries $n$ and $n-1$ in $(3.18)$ belong to $A$, respectively. As

$$
\left( \begin{array}{c}
u_j(n+1) \\
u_j(n)
\end{array} \right) = T_j(n; \lambda) \left( \begin{array}{c}
u_j(0) \\
u_j(-1)
\end{array} \right),
$$

$T_j(n; \lambda)$ is also the fundamental matrix of $(3.7)$

$$
T_j(n; \lambda) = \begin{pmatrix}
u_{1,j}(n+1) & \nu_{2,j}(n+1) \\
u_{1,j}(n) & \nu_{2,j}(n)
\end{pmatrix}.
$$

(3.20)

Marchetti et al. [MWGA] have determined precisely the growth of the norm of $T(n; \lambda)$ given by the product of $(3.18)$ with $\lambda_j = \lambda$ and $P$ given by $(1.3)$ and $(1.5)$. This together with a result due to Zlatoš [Z] permits the determination of the Hausdorff dimension of $\rho_{00}$.

Given $(3.20)$, we have

**Corollary 3.7** Suppose that for some $\alpha \in [0,1)$ and every $\lambda$ in some Borel set $A$,

$$
\limsup_{t \to \infty} \frac{1}{t^{1-\alpha}} \sum_{n=0}^{t} \|T_j(n; \lambda)\|^2 < \infty,
$$

(3.21)

for all $j \in I(\lambda)$, with $\|\cdot\|$ some matrix norm. Then the restriction $\rho_{00}(A \cap \cdot)$ is $\alpha$–continuous.

Proof. Theorem 2.3 from [KLS] states that there are two positive constants $c_1, c_2$, such that

$$
c_1 \max \{ |u_{1,j}(n+1)|^2, |u_{2,j}(n+1)|^2 \} \leq \|T_j(n; \lambda)\|^2 \leq c_2 \max \{ |u_{1,j}(n+1)|^2, |u_{2,j}(n+1)|^2 \}
$$

(3.22)

This leads to

$$
\sum_{n=0}^{t} \|T_j(n; \lambda)\|^2 \geq c \max \{ \|u_{1,j}\|^2_{l+1}, \|u_{2,j}\|^2_{l+1} \}
$$

(3.23)

for every $j \in I(\lambda)$. Hypothesis (3.21), together with (3.23), implies Corollary 3.7.

□

It is interesting to note that the growth of the norm of the transfer matrix gives exactly the growth of the increasing solution. This fact will be of great importance later.

The new version of Corollary 4.5 is
Corollary 3.8 Suppose that for at least one \( j \in I(\lambda) \neq \emptyset \)

\[
\liminf_{l \to \infty} \frac{\|u_{1,j}\|_{l}^2}{l^\alpha} = 0 \tag{3.24}
\]

for every \( \lambda \) in some Borel set \( A \). Then the restriction \( \rho_{00}(A \cap \cdot) \) is \( \alpha \)-singular.

Proof. Let \( \lambda \in A \) and \( b = \alpha/(2 - \alpha) \). By hypothesis, there is at least one \( j \in I(\lambda) \) that satisfies (3.24). Again, by the constancy of the Wronskian, \( \|u_{1,j}\|_l \|u_{2,j}\|_l \geq l \), and so \( \|u_{2,j}\|_l^b \geq (l/\|u_{1,j}\|_l)^b \). This implies

\[
\liminf_{l \to \infty} \frac{\|u_{1,j}\|_l^p}{\|u_{2,j}\|_l^b} = \liminf_{l \to \infty} \left( \frac{\|u_{1,j}\|_l^2}{l^\alpha} \right)^{1/(2-\alpha)} = 0.
\]

It follows from Theorem 3.5 that \( \rho_{00}(A \cap \cdot) \) is \( \alpha \)-singular.

\[
\square
\]

3.3 Main Result

In order to state the result concerning the Hausdorff dimension of the measure \( \rho_{00} \), we need a result due to Zlatoš [Z] on the growth and decay of the solutions of (3.7) in the span \{\( u_{1,j}, u_{2,j} \}\}. We shall give an improved version of Lemma 2.1 of [Z].

Proposition 3.9 Let \( A = (a_n)_{n \geq 1} \) be given by (1.3), \( \lambda \in \mathbb{R} \) and let us assume that, for \( j \in I(\lambda) \), the sequence \( (\theta^j_n)_{n \geq 0} \) of Prüfer angles, defined by (1.10) with \( \varphi \) replaced by \( \varphi_j \), is uniformly distributed mod \( \pi \) for every \( \theta^j_0 \in [0, \pi] \) and almost every \( \varphi_j \in [0, \pi] \) (w.r.t. Lebesgue measure) where \( 2 \cos \varphi_j = \lambda_j = \lambda - 2 \cos(2\pi j/L) \). Then, there is a generalized eigenfunction \( u_j \) (i.e., \( u_j \) satisfies (3.7) and the phase boundary condition (3.6)) for energy \( \lambda \) such that

\[
C_n^{-1} r_j^{n/2} \leq |u_j(a_n + 1)| \leq C_n r_j^{n/2}, \tag{3.25}
\]

holds for a constants \( r_j > 1 \) given by (with \( p = 1 - \delta \))

\[
r_j = r(p, \lambda_j) = 1 + \frac{(1 - p^2)^2}{p^2(4 - \lambda_j^2)} \tag{3.26}
\]

and \( C_n^{1/n} \searrow 1 \) as \( n \to \infty \). In addition, there exists a subordinate solution \( v_j \) for energy \( \lambda \) such that, for all sufficiently large \( n \),

\[
|v_j(a_n + 1)| \leq \tilde{C}_n r_j^{-n/2} \tag{3.27}
\]

holds with \( \tilde{C}_n^{1/n} \searrow 1 \) as \( n \to \infty \).
Proof. We shall combine ideas of [Z] with Theorem 8.1 of [LS] and estimates of [MWGA]. Let us denote the spectral norm of the transfer matrix $\|T_j(a_n + 1; \lambda)\|$ by $t_{j,n}$. Equation (3.25), together with (3.22), implies that $t_{j,n}$ satisfies the same upper and lower bounds. Under the hypotheses of Proposition 3.9, it follows from (3.8) and (4.19) of [MWGA] that

$$C^{-1} r_j^{n/2} \leq t_{j,n} \leq C r_j^{n/2}, \quad (3.28)$$

with $r_j$ given by (3.26).

By (3.18) and (1.3),

$$\|T_j(k, k-1; \lambda)\|^2 \leq \|T_j(k, k-1; \lambda)\|_E \leq 1 + \frac{1 + \lambda_j^2}{(1-\delta)^2} < \infty$$

if $\delta \in (0, 1)$, where $\|\cdot\|_E$ is the Euclidean matrix norm, for $k, k-1 \in \mathcal{A}$; otherwise $T_j(k, k-1; \lambda)$ is similar to a clockwise rotation $R(\varphi_j)$ by $\varphi_j = (1/2) \arccos \lambda_j$: $R(\varphi_j) = U T_0 U^{-1}$ (see (2.8) of [MWGA]). We write

$$T_j(a_n + 1; \lambda) = A_n \cdots A_1$$

where, for each $m$

$$A_m = T_j(a_m + 1, a_m; \lambda) \cdots T_j(a_m + 2, a_m + 1; \lambda) = T^m_+ T^m_-$$

by (3.19). Denoting $s_{j,n} = \|A_n\|$, we thus have

$$s_{j,n} \leq C \left(1 + \frac{1 + \lambda_j^2}{(1-\delta)^2}\right) \equiv B_j$$

$$C = (1 + |\cos \varphi_j|)/(1 - |\cos \varphi_j|),$$

uniformly in $n$. As a consequence,

$$\sum_{n=1}^{\infty} \frac{s_{j,n+1}^2}{t_{j,n}^2} < \infty \quad (3.30)$$

verifies the assumption of Theorem 8.1 of [LS] and provides the existence of a subordinate solution $v_j$ for energy $\lambda$. The idea of Zlatoš is to use the proof of Last–Simon to establish the decay of the subordinate solution. We shall reproduce the main steps, for convenience.

Since $T_0^+, T_-^+ := T_+ T_-^+$ given by (3.19) and, consequently, $T_j(a_n + 1; \lambda)$ and $T_j^+(a_n + 1; \lambda)$ are $2 \times 2$ unimodular real matrices, $T_j^+(a_n + 1; \lambda) T_j(a_n + 1; \lambda)$ is a $2 \times 2$ unimodular symmetric real matrix whose eigenvalues are $t_{j,n}^2$ and $t_{j,n}^2$, with corresponding orthonormal eigenvectors $v_{j,n}^+$ and $v_{j,n}^-$. $(v_{j,n}^+, v_{j,n}^-) = 0$. We write $v_\alpha = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$ and define $\alpha_n$ by

$$v_{\alpha_n} = v_{j,n}^-.$$
Clearly, $v^+_{j,n} = v_{\alpha_n + \pi/2}$ and by the spectral theorem, we have
\[
\|T_j(a_n + 1; \lambda)v_{\alpha}\|^2 = (v_{\alpha}, T_j^*(a_n + 1; \lambda)T_j(a_n + 1; \lambda)v_{\alpha})
\]
\[
= t^2_{j,n} |(v_{\alpha}, v_+)|^2 + t^{-2}_{j,n} |(v_{\alpha}, v_-)|^2
\]
\[
= t^2_{j,n} \sin^2(\alpha - \alpha_n) + t^{-2}_{j,n} \cos^2(\alpha - \alpha_n) . \tag{3.32}
\]

By the properties of a matrix norm together with (3.32) for $n + 1$ and definition (3.31), it can be shown (see proof of Theorem 8.1 of [LS])

\[
|\alpha_n - \alpha_{n+1}| \leq \frac{\pi s^2_{j,n+1}}{2 t^2_{j,n}} .
\]

Condition (3.30) implies that the sequence $(\alpha_n)_{n \geq 1}$ has a limit $\alpha^* = \lim_{n \to \infty} \alpha_n$. Hence, equation (3.32) and the telescope estimate

\[
|\alpha_n - \alpha^*| \leq \sum_{m=n}^{\infty} |\alpha_m - \alpha_{m+1}| \leq \frac{\pi}{2} \sum_{m=n}^{\infty} s^2_{j,m+1} t^2_{j,m}
\]

yields

\[
\|T(a_n + 1; \lambda_j)v_{\alpha^*}\|^2 \leq t^2_{j,n} (\alpha^* - \alpha_n)^2 + t^{-2}_{j,n}
\]

\[
\leq \frac{\pi}{2} B_j t^2_{j,n} \left( \sum_{m=n}^{\infty} \frac{1}{t^2_{j,m}} \right)^2 + t^{-2}_{j,n}
\]

which, together with (3.28), gives (3.27) concluding the proof of Proposition 3.9. Note that, by definition of transfer matrix, $v_j(a_n + 1) = (T_j(a_n + 1; \lambda)v_{\alpha^*})_2$ is a subordinate solution evaluated at $a_j + 1$ since $u_j(a_n + 1) \equiv (T_j(a_n + 1; \lambda)v_{\alpha^* + \pi/2})_2$ satisfies

\[
\lim_{n \to \infty} \frac{|v_j(a_n + 1)|}{|u_j(a_n + 1)|} = 0
\]

in view of $\|T(a_n + 1; \lambda_j)v_{\alpha^* + \pi/2}\| \geq t^2_{j,n}/2$ for sufficiently large $n$.

\[\square\]

**Remark 3.10** Equation (3.28), where $t_{j,n} \equiv \|T_j(a_n + 1; \lambda)\|$, holds for every $j \in I(\lambda)$ for $\lambda_j = \lambda - 2 \cos(2\pi j/L) \in (-2, 2) \setminus A_{\theta_0}$, $A_{\theta_0}$ a set of zero Lebesgue measure possibly depending on the initial Pr"ufer angle $\theta_0$, which depends on $\phi$–condition and $\varphi_j$ (see eq. (3.7) and Theorem 4.4 of [MWGA]).

We are now ready to present our main result.
Theorem 3.11 Let \( \Delta_{\delta, \phi} \) be given by (1.1) with \( \delta \in (0, 1) \) and \( \phi \)-boundary condition (1.2). Let \( \rho \) be its spectral matrix measure. For any closed interval of energies \( I \subset \bigcup_j I_j \), where

\[
I_j = \left( -2 + 2 \cos \left( \frac{2\pi j}{L} \right), 2 + 2 \cos \left( \frac{2\pi j}{L} \right) \right),
\]

and for almost every boundary condition \( \phi \), the element \( \rho_{00} \) of the spectral measure \( \rho \) restricted to \( I \) has, for every \( \varepsilon > 0 \), the Hausdorff dimension

\[
\alpha_{\rho_{00}}(\lambda) \in \left( \alpha_{\rho_\ast}(\lambda_{j^\ast}) - \varepsilon, \alpha_{\rho_\ast}(\lambda_{j^\ast}) + \varepsilon \right)
\]

where

\[
\alpha_{\rho_\ast}(\lambda_{j^\ast}) = \min_{j \in I(\lambda)} \alpha_{\rho_j}(\lambda_j) = \min_{j \in I(\lambda)} \left( 1 - \frac{\ln r_j}{\ln \beta} \right),
\]

with \( r_j = r(p, \lambda_j) \) given by (3.26) \( (p = 1 - \delta) \), if the sparseness parameter satisfies \( \beta > \beta_0 \) for some \( \beta_0 = \beta_0(\delta, \lambda_{j^\ast}, \varepsilon) \) large enough.

Remark 3.12 Theorem 3.11 generalizes (from the one-dimensional case to the finite strip problem) and improves (it establishes the Hausdorff dimension of the spectral measure) Theorem 4.1 of Zlatos \([Z]\).

Proof. Let \( I \) be given by (3.33) and let us, provisionally, assume that for \( \lambda \in I \) the sequence \((\theta_n^j)_{n \geq 0}\) of Prüfer angles is uniformly distributed mod \( \pi \) for every \( \theta_0^j \in [0, \pi] \) and almost every \( \varphi_j = (\cos^{-1} \lambda_j) / 2 \in [0, \pi] \), for every \( j \in I(\lambda) \). It follows from (3.28) and Theorem 4.4 of \([MWGA]\) that, there is an \( A_{\theta_0} \) with zero Lebesgue measure such that for any \( \lambda \in I \setminus A_{\theta_0} \) and any \( k \in \mathbb{Z}_+ \) such that \( a_n \leq k < a_{n+1} \), we have

\[
\|T_j(k; \lambda)\| \leq C_n r_j^{n/2} \leq C'_n a_{\gamma_j}^{n/2} \leq C''_n k^{\gamma_j/2},
\]

with \( \gamma_j \equiv \ln r_j / \ln \beta \) and \( \lim_{n \to \infty} (C''_n)^{1/n} = 1 \), by the sparseness condition (1.5).

It follows from the constancy of \( \|T_j(k; \lambda)\| \) on \([a_n + 1, a_{n+1}]\) (see Section 4 of \([MWGA]\)), together with the above equation,

\[
\sum_{k=0}^{l} \|T_j(k; \lambda)\|^2 \leq cl^{1+\gamma_j}
\]

holds for some \( c > 0 \) and every \( \lambda \in I \setminus A_{\theta_0} \).

The application of Proposition 3.9 for these values of \( \lambda \) guarantees the existence of a subordinate solution \( v_j \) which satisfies

\[
|v_j(a_n + 1)|^2 \leq C''_n a_n^{-\gamma_j}
\]

for every \( j \in I(\lambda) \). Since every solution of (3.5) has constant modulus on the interval \([a_n + 1, a_{n+1}]\), we have

\[
\|v_j\|_t^2 \leq c'l^{1-\gamma_j},
\]

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for some \( c' > 0 \).

Since the measure \( \rho_{00} \) restricted to \( I \) is supported on the set of those \( \lambda \) for which each \( u_j^{\text{sub}} \) satisfies the boundary condition \( \phi \) (due to the fact that each constituent of \( \rho_{00} \) has no absolutely continuous part; see Theorem 1 of \([GP]\)) we have \( u_{1,j} = v_j \).

Thus, by (3.36) and (3.37)

\[
\limsup_{l \to \infty} \frac{1}{l^{2-\alpha}} \sum_{k=0}^{l} ||T_j(k; \lambda)||^2 < \infty \quad (3.38)
\]

and

\[
\liminf_{l \to \infty} \frac{||u_{1,j}||^2}{l^\alpha} = 0 \quad (3.39)
\]

hold for each \( j \in \mathcal{I}(\lambda) \), provided \( 2 - \alpha \geq 1 + \gamma_j \) and \( \alpha' > 1 - \gamma_j \).

Corollary 3.7 says that if (3.38) is satisfied for all \( j \in \mathcal{I}(\lambda) \), the restriction \( \rho_{00}((I \cup \cup_j A_{\theta_j}) \cap \cdot) \) is \( \alpha \)-continuous. Clearly, \( \alpha = \min_j (1 - \gamma_j) \) satisfies the requirement:

\[
\limsup_{l \to \infty} \frac{1}{l^{2-\alpha}} \sum_{n=1}^{l} ||T_j(n; \lambda)||^2 \leq \limsup_{l \to \infty} \frac{1}{l^{1+\gamma_j}} \sum_{n=1}^{l} ||T_j(n; \lambda)||^2 < \infty ,
\]

which implies that (3.38) holds simultaneously for every \( j \in \mathcal{I}(\lambda) \), provided \( \lambda \in I \setminus \cup_j A_{\theta_j} \). Thus \( \rho_{00}((I \setminus \cup_j A_{\theta_j}) \cap \cdot) \) is at most \( \alpha \)-continuous.

We affirm that \( \rho_{00}((I \setminus \cup_j A_{\theta_j}) \cap \cdot) \) is at least \( \alpha \)-singular with \( \alpha = \min_j (1 - \gamma_j) \). We have from Corollary 3.8 that the restriction above is \( \eta \)-singular for every \( \eta > \alpha \) (since (3.24) is satisfied for at least one \( j \)). However, (3.39) is satisfied for every \( j \); this proves our assertion.

Thus, by the definition of Hausdorff dimension to measures, \( \rho_{00}((I \setminus \cup_j A_{\theta_j}) \cap \cdot) \) has exact dimension

\[
\alpha = \min_j (1 - \gamma_j) \quad (3.40)
\]

which, together with the definition of \( \gamma_j \), is exactly (3.35).

We now replace the Prüfer angles \((\theta_n^{j})_{n \geq 0}\) by a sequence \((\zeta_n^{j})_{n \geq 0}\) of continuous piecewise linear functions \( \zeta_n^{j} = \zeta_n^{j}(\phi_j) \) which can be shown to be uniformly distributed mod \( \pi \) by the general metric criterion (see Section 5 of \([MWGA]\)) and whose difference of their respective Birkhoff average

\[
E = \frac{1}{N} \sum_{n=1}^{N} (f(\theta_n^{j}) - f(\zeta_n^{j})) ,
\]

for any uniformly continuous function \( f \) defined in \([0, \pi]\), can be made arbitrarily small by taking the sparseness parameter \( \beta \) sufficiently large (see Theorem 5.6 of \([MWGA]\)). As a consequence, (3.28) is replaced by

\[
C_n^{-1} e^{n(\ln r_j - 2|E|)/2} \leq t_{j,n} \leq C_n e^{n(\ln r_j + 2|E|)/2}
\]
and equations (3.38) and (3.39) are affected only by an $\varepsilon$ uncertainty, leading to (3.34).

Finally, by the theory of rank one perturbations, we know that $\rho_{00}(\cup_j A_{\theta_j}) = 0$ holds for almost every $\phi$, and so for almost every $\phi$ the restriction $\rho_{00}(I \cap \cdot)$ has (3.35) as its Hausdorff dimension. This concludes the proof of Theorem 3.11.

□

An interesting conclusion drawn from Theorem 3.11 is that the spectral measure $\rho_{00}$ always inherits the most singular behavior between its components. Let us explain what this assertion means.

Let $B$ be a Borel set, $B \subset I$ (I given by (3.33)). If $\alpha_{\rho_{00}}(\lambda) > 0$ for every $\lambda \in B$, then $\rho_{00}(B \cap \cdot)$ is purely singular-continuous. We see from (3.35) and (3.26) that this holds if, and only if,

$$(4 - \lambda_j^2)(\beta - 1) > \left(\frac{1 - p^2}{p}\right)^2$$

(3.41)

is satisfied for every $j \in I(\lambda)$. This is exactly the expression (4.30) of [MWGA], which gives a necessary condition for the existence of singular-continuous spectrum (the result follows directly from Theorem 2.1 of [SS] and Theorem 3.2 of [LS]).

Thus, if condition (3.41) fails to be satisfied for at least one $j$ in some Borel set $B$, then the spectrum of $\rho_{00}(B \cap \cdot)$ if singular-continuous, it has 0 Hausdorff dimension. This result is a direct consequence of Corollary 3.7.

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