Arithmetic Brownian motion subordinated by tempered stable and inverse tempered stable processes

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Abstract
In the last decade the subordinated processes have become popular and found many practical applications. Therefore in this paper we examine two processes related to time-changed (subordinated) classical Brownian motion with drift (called arithmetic Brownian motion). The first one, so called normal tempered stable, is related to the tempered stable subordinator, while the second one - to the inverse tempered stable process. We compare the main properties (such as probability density functions, Laplace transforms, ensemble averaged mean squared displacements) of such two subordinated processes and propose the parameters’ estimation procedures. Moreover we calibrate the analyzed systems to real data related to indoor air quality.

Keywords: subordination, Brownian motion, tempered stable, diffusion, anomalous diffusion, calibration

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1. Introduction

Processes based on the Brownian motion were considered in many aspects and have found various practical applications [1–8]. But the assumption of normality for the observations seems not to be reasonable in the number of examined phenomenon. Therefore in many Gaussian models the Brownian motion is replaced by its various modifications. One of the simples modification is the extension of Gaussian by another distribution, for example the Lévy-stable one. Processes based on the stable distribution are very useful in modeling data that exhibit fat tails. For example, the classical Ornstein-Uhlenbeck process was extended to the stable case and analyzed in [9, 10] as a suitable model for financial data description, see also [11]. The another possibility of modification for Brownian-type processes is introduction a time-changed Brownian models. This extension is related to replacement of real time in Brownian systems by non-decreasing Lévy process (called subordinator), that in this case plays a role of random (operational) time. The new process is called subordinate. The idea of subordination was introduced in 1949 by Bochner [12] and expounded in his book in [13]. The theory of subordinated processes is also explored in details in [14].

The subordinated processes based on the diffusive Brownian motion were considered in many disciplines. The Brownian motion subordinated by gamma process, so called variance gamma, is analyzed in [15] in the context of option price modeling. The other applications and main characteristics of such system one can find in [16]. Subordination of Brownian motion by the inverse Gaussian process is called the normal inverse Gaussian (NIG) and was considered for example in [17] and proposed to modeling turbulence and financial data. Applications of NIG processes to asset returns are also shown in [18] while analysis of real environmental data by using NIG distribution is presented in [19]. Let us mention that the Brownian motion driven by strictly increasing Lévy subordinator is also a Lévy process, moreover when the subordinator is temporally homogeneous Markov then the subordinate has also this property, [14].

Another possibility of subordination is replacement the real time in Brownian diffusion by inverse subordinators and processes that arise after this transformation are called anomalous diffusion. In the domain of anomalous diffusion the typical approach is based on continuous time random walk (CTRW), [20, 21], and subordinated Lévy processes can be treat as a limit in distribution of CTRW, [22]. The key issue in the framework of CTRW as well as in subordination
technique is the waiting-times distribution corresponding to observed constant time periods [23]. In the last decade
the anomalous diffusion processes were analyzed by various number of authors in many disciplines. For example
the subordinated Brownian motion driven by inverse Lévy-stable subordinator was considered in [11, 22, 24, 25], the
inverse tempered stable subordinator was examined in [23, 26–28] while inverse gamma process was mentioned in
[29]. The general case of Lévy processes that can play a role of inverse subordinators were explored for example in
[30–32].

In this paper we examine two processes related to subordinated Brownian motion. The first one, so called normal
tempered stable [33, 34], is a Brownian motion with drift (called arithmetic Brownian motion, ABM) driven by
tempered stable subordinator, while the second one is an ABM subordinated by inverse tempered stable subordinator.
The tempered stable processes are extension of the \( \alpha \)--stable Lévy systems but possess also the properties of Gaussian
models, therefore in the last few years they have become popular and very useful in description of many real data,
[10, 34, 35]. We compare the main statistical properties of the ABM driven by tempered stable and inverse tempered
stable subordinator. Moreover in two considered cases we propose the parameters’ estimation procedures and validate
them. In order to illustrate the theoretical results we calibrate the examined processes to real data related to indoor air
quality.

The rest of the paper is organized as follows: In section 2 we introduce the ABM and tempered stable subordi-
nator. We present the main properties of those processes and define the time-changed ABM driven by tempered stable
process. For this system we also examine the main statistical characteristics, such as probability density function,
Laplace transform and ensemble averaged mean squared displacement that can be an useful tool to distinction be-
tween diffusion and anomalous diffusion models. In this section we propose also the estimation procedure based on
the distance between theoretical and empirical Laplace transforms and validate it. In section 3 we examine inverse
tempered stable subordinator and its main statistical properties as well as define the ABM driven by inverse tempered
stable process. In this section we also present the estimation procedure for unknown parameters. In section 4 we
model real data sets related to indoor air quality by using the mentioned subordinated processes. Last section contains
conclusions.

2. Arithmetic Brownian motion with tempered stable subordinator

2.1. Arithmetic Brownian motion

The arithmetic Brownian motion (ABM) is a process \( \{X(t), t \geq 0\} \) defined by [23, 36]:

\[
dX(t) = \beta \, dt + dB(t),
\]

(1)

where \( \{B(t), t \geq 0\} \) is a classical Brownian motion and \( \beta > 0 \). The solution of equation (1) takes the form

\[
X(t) = X(0) + \beta t + B(t).
\]

In the further analysis we assume \( X(0) = 0 \) with probability one and in this case the process defined above has
Gaussian distribution with mean \( \beta t \) and variance \( t \). It is called also Brownian motion with drift, [23] and first of
all was used to description of stock prices [1, 37]. It is an extension of the classical Brownian motion therefore its
modifications have found many other practical applications, like diffusion in liquids modeling, [38] and description
of hydrology time series [39].

2.2. Tempered stable subordinator

The tempered stable subordinator \( \{T(t), t \geq 0\} \) is a strictly increasing Lévy process with tempered stable incre-
ments, i.e. with the following Laplace transform, [27, 28]:

\[
< e^{-\lambda T(t)} > = e^{\lambda^{\alpha} - (\lambda + \lambda^{\alpha}) t}, \lambda > 0, 0 < \alpha < 1.
\]

(2)

When \( \lambda = 0 \), then \( \{T(t)\} \) becomes totally skewed \( \alpha \)--stable Lévy process. The probability density function (pdf) of
tempered stable subordinator can be express in the following form:

\[
f_{T(t)}(x) = e^{-\lambda x^{\alpha}} f_{U(t)}(x),
\]
where \(\{U(t), \ t \geq 0\}\) is a totally skewed \(\alpha\)–stable Lévy motion with the stability index \(\alpha\), [23, 40]. By using tail approximation of stable density, [41], we obtain the following:

\[
f_{T(t)}(x) \sim 2\alpha c \rho e^{-\lambda x + \rho^2 x^{-\alpha + 1}}, \ x \to \infty
\]

for some constant \(c\). Therefore the right tail can be approximated by:

\[
1 - F_{T(t)}(x) \sim e^{-\lambda x + \rho^2 x^{-\alpha}}, \ x \to \infty,
\]

where \(F_{T(t)}(\cdot)\) is the distribution function of \(T(t)\). Let us mention that for \(\lambda = 0\) (the \(\alpha\)–stable case), the above formula reduces to \(1 - F_{U(t)}(x) \sim x^{-\alpha}\) for fixed \(t\).

2.3. ABM with tempered stable subordinator

The ABM driven by tempered stable subordinator is the process \(\{Y_t(t), \ t \geq 0\}\) defined as follows:

\[
y_t(t) = X(T(t)),
\]

where \(\{X(t)\}\) is ABM defined in (1) and \(\{T(t)\}\) is strictly increasing Lévy process with tempered stable increments with the Laplace transform given in (2). We assume those two processes are independent.

Using the form of the solution of equation (1) we can write the exact formula for the process \(\{Y_t(t)\}\), namely:

\[
y_t(t) = B(T(t)) + \beta T(t). \tag{3}
\]

The process \(\{Y_t(t)\}\) is known in the literature as normal tempered stable, [33] and the algorithm for generating its sample trajectories in points \(t_1, t_2, \ldots, t_n\) proceeds follows:

1. Simulate the increments of subordinator \(dT_i = T(t_i) - T(t_{i-1})\) with the initial value \(T_0 = 0\). In order to do this use the simple algorithm presented for example in [27].
2. Simulate \(n\) standard Gaussian random variables \(N_1, N_2, \ldots, N_n\)
3. Define \(dY_t(i) = N_i \sqrt{T_i} + \beta dT_i\) and put \(Y_t(t_i) = \sum_{k=1}^{i} \Delta Y_t\).

The sample trajectory of the process \(\{Y_t(t)\}\) with parameters \(\alpha = 0.8\), \(\lambda = 1\) and \(\beta = 1\) is presented on the top panel of Fig.1.

On the basis of the formula for Laplace transform of general subordinated processes given in [14] we can calculate this characteristic of the process \(\{Y_t(t)\}\), namely:

\[
\exp \left( \left( x^\alpha - \left( \lambda + \beta z - \frac{1}{2} z^2 \right) \right)^\alpha \right).
\]

The pdf of the process \(\{Y_t(t)\}\) defined in (3) can be calculated by using the following formula:

\[
f_{Y_t(t)}(x) = \int_0^\infty f_{X(t)}(x)f_{T(t)}(z)dz,
\]

where \(f_{X(t)}(\cdot)\) and \(f_{T(t)}(\cdot)\) denote the pdfs of the ABM \(\{X(t)\}\) and tempered stable subordinator \(\{T(t)\}\), respectively. Therefore we obtain:

\[
f_{Y_t(t)}(x) = \int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{-\frac{(x-\beta z)^2}{2z} - \lambda z + \rho^2 z^{-\alpha + 1}}f_{U(t)}(z)dz,
\]

where \(f_{U(t)}(\cdot)\) is the pdf of the totally skewed \(\alpha\)–stable Lévy motion \(\{U(t), \ t \geq 0\}\) mentioned above.

On the basis of formulas (3) and (4) we can also calculate the main characteristics such as mean and autocovariance function:

\[
\langle Y_t(t) \rangle = \beta t a x^{\alpha - 1}
\]
cov(t, s) = \langle Y_T(t), Y_T(s) \rangle - \langle Y_T(t) \rangle \langle Y_T(s) \rangle = \min(s, t) (\alpha \lambda^{\alpha-1} + \beta^2 \alpha (1 - \alpha) \lambda^{\alpha-2}).

We consider also one of the most popular characteristic of the process that is especially important in real data analysis because it can be an useful tool for recognition between diffusion and anomalous diffusion models. This characteristic is called mean squared displacement (MSD). Here we consider the ensemble averaged MSD that is defined as a second moment of a given process. Therefore for \{Y_T(t)\} it takes the form, [29]:

< Y_T^2(t) > = \int_0^\infty x^2 f_{Y_T(t)}(x) dx,

where \(f_{Y_T(t)}(x)\) is a pdf of the process \(Y_T(t)\). Let us emphasize, that the ensemble averaged MSD for real data can be calculated only in case when there are available more than one trajectory. For the process \(Y_T(t)\) defined in (3) the ensemble averaged MSD is a polynomial of the second order with respect to \(t\):

< Y_T^2(t) > = t^2 (\beta^2 \alpha \lambda^{\alpha-1})^2 + t (\alpha \lambda^{\alpha-1} + \beta^2 \alpha (1 - \alpha) \lambda^{\alpha-2}).

As we observe, in case of \(\beta = 0\), the ensemble averaged MSD scales as \(t\). In Fig. 2 (top panel) we present this characteristic calculated on the basis of 1000 trajectories for the ABM with tempered stable subordinator with \(\alpha = 0.8\), \(\lambda = 1\) and \(\beta = 0.01\). Moreover we also show the fitted second-order polynomial.
2.4. Estimation procedure

We propose to estimate the parameters $\alpha$, $\lambda$ and $\beta$ by using the formula of the Laplace transform given in (4). Because $(Y_T(t))$ is a Lévy process, [14], therefore its increments are independent and have the same distribution with the following characteristic:

$$< e^{-z(Y_T(t+d) - Y_T(t))} >= \exp\left( d \left( \lambda^\alpha - \left( \lambda + \beta z - \frac{1}{2} z^2 \right)^\alpha \right) \right).$$

The estimation scheme for random sample $y_1, y_2, ..., y_n$ proceeds as follows:

1. Calculate increments of the observations: $dy_i = y_{i+1} - y_i$ for $i = 1, 2, ..., n - 1$.
2. For the increments find the empirical Laplace transform by using the following formula:

$$\phi(z) = \frac{1}{n-1} \sum_{i=1}^{n-1} e^{-z\Delta y_i}.$$
3. By using the least squares method find the estimators $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\beta}$ that satisfy:

$$(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \min_{\alpha, \lambda, \beta} \left( \phi(z) - \exp \left( \lambda^\nu - \left( \lambda + \beta z - \frac{1}{2} z^2 \right)^\nu \right) \right)^2.$$

In order to show the efficiency of the described method, in Fig. 3 we present the values of the estimated parameters for simulated process $\{Y_T(t)\}$. To the analysis we take 1000 trajectories of length 1000. The theoretical values are: $\alpha = 0.26$, $\lambda = 6$, $\beta = 0.11$. As we observe the theoretical values are close to the estimated values (are between appropriate quantiles) that indicates the procedure works properly.

3. **ABM with inverse tempered stable subordinator**

In this section we examine the main characteristics of the ABM with driven by the inverse tempered stable subordinator. Such kind of systems were considered in the literature in many aspects, [23, 26–29], therefore some properties we present without proofs.
3.1. Inverse tempered stable subordinator

The inverse tempered stable subordinator is the process \( S(\tau), \tau \geq 0 \) defined as follows:

\[
S(\tau) = \inf\{t > 0 : T(t) > \tau\},
\]

where \( [T(t), \ t \geq 0] \) is the tempered stable Lévy process defined via its Laplace transform in (2). In Fig.4 we present the trajectories of the tempered stable subordinator \( \{T(t)\} \) and corresponding tempered stable inverse subordinator \( \{S(\tau)\} \) for parameters \( \alpha = 0.26 \) and \( \lambda = 6 \). As we observe, the processes are non-decreasing but exhibit completely different behavior.

Using the relation \( P(S(\tau) \leq x) = P(T(x) > \tau) \) we can calculate the pdf of the process \( \{S(\tau)\} \), namely:

\[
f_{S(\tau)}(x) = -\frac{d}{dx} \int_0^\tau f_{T(x)}(u)\,du.
\]

Therefore we get:

\[
f_{S(\tau)}(x) = -\frac{d}{dx} \int_0^\tau e^{-\lambda u+\lambda x} f_{U(x)}(u)\,du,
\]
where \([U(t)]\) is a totally skewed \(\alpha\)-stable Lévy process described in the previous section. Because \(f_{U(t)}(u) = \frac{1}{x^{\alpha}} f_{U(1)}(u/x^{1/\alpha})\), [42], therefore we get:

\[
\begin{align*}
fs(t)(x) &= -\frac{d}{dx} \int_0^t e^{-\mu x} \int_0^x \frac{1}{x^{1/\alpha}} f_{U(1)}(u/x^{1/\alpha}) du \\
&= -\frac{1}{\alpha x} \int_0^t e^{-\mu x} \int_0^x \frac{1}{x^{1/\alpha}} f_{U(1)}(u/x^{1/\alpha}) du \\
&= \left(\frac{1}{\alpha x} - \lambda^\alpha\right) \int_0^t e^{-\mu x} \int_0^x \frac{1}{x^{1/\alpha}} f_{U(1)}(u/x^{1/\alpha}) du + \frac{1}{\alpha x} \int_0^t e^{-\mu x} \int_0^x \frac{1}{x^{1/\alpha}} f_{U(1)}(u/x^{1/\alpha}) du \\
&= \frac{1}{\alpha x} \int_0^t e^{-\mu x} \left(\int_0^x f_{U(1)}(u) du + \int_0^x e^{-\mu x} \frac{1}{x^{1/\alpha}} f_{U(1)}(u/x^{1/\alpha}) du \right) dx \\
&= \frac{1}{\alpha x} \int_0^t e^{-\mu x} \left(\int_0^x f_{U(1)}(u) du \right) dx.
\end{align*}
\]

As a final result we obtain:

\[
\begin{align*}
fs(t)(x) &= \frac{1}{\alpha x} \int_0^t f_{U(1)}(x) dx + \frac{1}{\alpha x} \int_0^t e^{-\lambda^\alpha x} f_{U(1)}(x) dx \\
&= \frac{1}{\alpha x} \int_0^t f_{U(1)}(x) dx + \frac{1}{\alpha x} \int_0^t \lambda^\alpha x f_{U(1)}(x) dx.
\end{align*}
\]

In case of \(\lambda = 0\) we have:

\[
fs(t)(x) = \frac{1}{\alpha x} \int_0^t f_{U(1)}(x) dx,
\]

that coincides with the result presented in [42] for \(\alpha\)-stable case. For large \(t\) the density given in (5) tends to:

\[
fs(t)(x) \sim \frac{1}{\alpha x} \int_0^t f_{U(1)}(x) dx.
\]

Similar, for small \(t\) we get:

\[
fs(t)(x) \sim \frac{1}{\alpha x} \int_0^t f_{U(1)}(x) dx.
\]

On the basis of equation (5) we can calculate also the Laplace transform of \([S(t)]\), namely:

\[
< e^{-\lambda S(t)} > = \int_0^\infty e^{-\lambda x} fs(t)(x) dx.
\]

However such derivation requires numerical approximations.

### 3.2. ABM driven by inverse tempered stable subordinator

The ABM driven by the inverse tempered stable subordinator is defined as follows:

\[
Y_S(t) = X(S(t)) = \beta S(t) + N_S(t),
\]

where \([X(t)]\) is the classical ABM explored in section 2. The simulation procedure of the process \([Y_S(t)]\), \(t \geq 0\) is completely described in [27]. The sample trajectory of the process is presented in Fig. 1 (bottom panel). To the simulation we take the following values of the parameters: \(\alpha = 0.8, \lambda = 1\) and \(\beta = 1\). Let us remind that on the top panel we present the sample trajectory of the ABM with tempered stable subordinator \([T(t)]\). For \([Y_S(t)]\) we observe constant time periods that is typical for subdiffusive processes. This behavior is not visible for the \([Y_T(t)]\).

The Laplace transform of the ABM driven by inverse tempered stable subordinator \([S(t)]\) can be calculated by using the following formula:

\[
< e^{-\lambda S(t)} > = \int_0^\infty e^{-\beta x} \frac{1}{\alpha x} \int_0^\infty e^{-\lambda x} fs(t)(x) dx,
\]
where $f_{S(t)}(\cdot)$ is given in (5). The pdf $f_{Y(t)}(x)$ of the process $\{ Y_s(\tau) \}$ satisfies the generalized Fokker-Planck equation:

$$
\frac{d f_{Y(t)}(x)}{d \tau} = \left[ -\beta \frac{d}{dx} + \frac{1}{2} \frac{d^2}{d^2x} \right] \Phi(f_{Y(t)}(x)).
$$

where $\Phi$ is an operator defined in [23]. On the other hand we can use the formula for pdf of subordinated processes given in [14]:

$$
f_{Y(t)}(x) = \int_0^\infty f_{X(z)}(x) f_{S(t)}(z) dz.
$$

Therefore we obtain the following:

$$
f_{Y(t)}(x) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-(x-\beta z)^2/2} f_{S(t)}(z) dz,
$$

The mean of the process $\{ Y_s(\tau) \}$ is given by, [29]:

$$
< Y_s(\tau) > = \beta < S(\tau) > = \beta \int_0^\infty e^{-\lambda t} u^{\alpha-1} E_{\alpha,\beta}(\lambda u) du,
$$

where $E_{\alpha,\beta}(z)$ is a generalized Mittag-Leffler function defined as follows, [43]:

$$
E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(ak+\beta)}.
$$

For small $\tau$, $< Y_s(\tau) > \sim \tau^\alpha$, while for large values of $\tau$ the mean tends to $\tau$, [26]. Moreover the ensemble averaged MSD takes the form, [45]:

$$
< Y_s^2(\tau) > = \beta^2 < S^2(\tau) > + < B^2(S(\tau)) > = \beta^2 < S^2(\tau) > + < S(\tau) >
$$

where

$$
< S^2(\tau) > = \int_0^\infty x^2 f_{S(t)}(x) dx.
$$

For small $\tau$ the last characteristic behaves:

$$
< S^2(\tau) > = \int_0^\infty x^2 \frac{1}{\alpha x} f_{Y(t)}(\tau) dx = \int_0^\infty \frac{x}{\alpha x^{1/\alpha}} \tau e^{-\tau x + x^{\alpha}} f_{U(1)}(\tau/\lambda^{1/\alpha}) dx = \int_0^\infty \tau^{2\alpha} x^{-2\alpha} e^{-\tau x + x^{\alpha}} f_{U(1)}(\lambda x^{1/\alpha}) dx \sim \tau^{2\alpha}.
$$

On the other hand for large $\tau$ we have:

$$
< S^2(\tau) > = \tau^{2\alpha} e^{-\tau} \int_0^\infty x^{-2\alpha} e^{\tau^{1/\alpha} x} f_{U(1)}(\lambda x^{1/\alpha}) dx.
$$

Therefore as a final result we obtain that the ensemble averaged MSD for the process $\{ Y_s(\tau) \}$ satisfies:

$$
< Y_s^2(\tau) > \sim c_1 \tau^{2\alpha} + c_2 \tau^\alpha, \quad \text{when } \tau \to 0
$$

$$
< Y_s^2(\tau) > \sim d_1 \tau^{2\alpha} e^{-d_2 \tau}, \quad \text{when } \tau \to \infty
$$

for some constants $c_1$, $c_2$, $d_1$, $d_2$. Let us mention $c_1$ and $d_1$ depend on $\beta$ in such way that for $\beta = 0$, $c_1 = d_1 = 0$, therefore for $\beta = 0$ the ensemble averaged MSD behaves like $\tau^\alpha$ for small values of $\tau$ and like $\tau$ for large values of this parameter, [26].

The ensemble averaged MSD calculated on the basis of 1000 trajectories for the process $\{ Y_s(\tau), \tau \geq 0 \}$ we present on the bottom panel of Fig. 2. To the simulation we take $\alpha = 0.8$, $\lambda = 1$ and $\beta = 0.01$. We show also the fitted power function for small and large values of arguments. Let us mention that on the top panel we present the ensemble averaged MSD for the ABM with tempered stable subordinator $\{ Y_T(t), t \geq 0 \}$ with fitted polynomial of order 2.
3.3. Estimation procedure

The estimation procedure for parameters $\alpha$, $\beta$ and $\lambda$ of ABM with inverse tempered stable subordinator is partially described in [23]. We only mention here that it is based on the decomposition of the time series into two vectors. The first one is responsible for the lengths of constant time periods typical for the subdiffusive processes, while the second one appears after removing the constant periods. According to the theory, in our case the first vector constitutes independent identically distributed sample from tempered stable distribution. Because the right tail of the tempered stable distribution behaves like $e^{-\lambda x} x^{-\alpha}$ therefore we propose to estimate the parameters $\alpha$ and $\lambda$ by fitting this function to the empirical tail by using the least squares method. The second vector is related to the external process $\{X(t)\}$, therefore in the case of ABM the differenced series has Gaussian distribution with mean $\beta$. Therefore the parameter we estimate as a mean of the differenced series.

In order to present the efficiency of the presented method we simulate 1000 trajectories (each of length 1000) of ABM with inverse tempered stable subordinator $\{Y_S(\tau)\}$ with $\alpha = 0.4$, $\lambda = 0.2$ and $\beta = 0$. Next, we estimate the parameters by using the presented scheme and in Fig. 5 we show the results. As we observe the fitted parameters are close to the theoretical values.

Figure 5: The boxplots of the values of estimators for parameters of ABM with inverse tempered stable subordinator $\{Y_S(\tau)\}$. The values were calculated on the basis of 1000 trajectories of length 1000 each. The theoretical values are: $\alpha = 0.4$, $\lambda = 0.2$, $\beta = 0$. 

In order to present the efficiency of the presented method we simulate 1000 trajectories (each of length 1000) of ABM with inverse tempered stable subordinator $\{Y_S(\tau)\}$ with $\alpha = 0.4$, $\lambda = 0.2$ and $\beta = 0$. Next, we estimate the parameters by using the presented scheme and in Fig. 5 we show the results. As we observe the fitted parameters are close to the theoretical values.
4. Applications

In this section we examine real data sets that describe humidity (in %) and temperature (in °C) of the indoor air in some open space of huge company. For simplicity we denote humidity as DATA1 and temperature - as DATA2. Those two analyzed quantities were measured by three sensors placed in the open space. Therefore for DATA1 and DATA2 we have three paths (trajectories). The humidity and temperature was quoted per minute. To the analysis we take data from 9:48–23:59 on 12.08.2010 (851 observations of each trajectory). On Fig. 6 we present paths of humidity (top panel) and data related to temperature (bottom panel).

As we observe on Fig. 6, DATA2 exhibit behavior typical to subdiffusive process \{Y_S(\tau)\} described in previous section, namely the constant time periods. This behavior is not visible for DATA1 (top panel). In the first step we examine the empirical ensemble averaged MSD calculated on the basis of three trajectories (of DATA1 and DATA2), see Fig. 7. To the ensemble averaged MSD of DATA1 we fit the second-order polynomial (top panel), while for small and large values of the MSD of DATA2 - the power functions. In both cases we use the least squares method. As we observe, the ensemble averaged MSD for DATA1 behaves like second-order polynomial for all arguments that suggests behavior similar to this observed in Fig. 2 (top panel). The empirical MSD of the temperature that we observe on bottom panel of Fig. 7 clearly suggests subdiffusive behavior.

In the next step of our analysis we estimate the parameters of the suggested processes, i.e. ABM with tempered stable subordinator for DATA1 and ABM driven by inverse tempered stable subordinator for DATA2. In both cases
we use the procedures presented in sections 2 and 3, respectively. In the Table 1 we present the estimated parameters for DATA1.

| Trajectory | \( \hat{\alpha} \) | \( \lambda \) | \( \beta \) |
|------------|-----------------|-------------|-------------|
| trajectory 1 | 0.23            | 6.1         | 0.11        |
| trajectory 2 | 0.29            | 5.9         | 0.12        |
| trajectory 3 | 0.24            | 6.1         | 0.12        |

Table 1: Estimated parameters of the ABM with tempered stable subordinator for DATA1.

Moreover in Fig. 8 we present the empirical Laplace transform for each of three trajectories of DATA1 and theoretical one given in (4). In order to confirm that the tempered stable subordinator is better than the stable one we show also the Laplace transform for the process \( \{ Y_t(t) \} \) with \( \lambda = 0 \) (the other parameters we estimate by using the same method as this presented in section 2). Moreover we test also the Gaussian and \( \alpha \)-stable distributions for differenced series of DATA1 by using the methods based on the distance between empirical and theoretical distribution functions,\[44\]. Those tests reject the hypothesis of the stable or Gaussian behavior of the observed data.

After analysis of the humidity we start the examination of temperature. The ensemble averaged MSD indicates the
data can be considered as subdiffusive process. Therefore we propose to use the ABM with inverse tempered stable subordinator described in section 3. Let us mention that the \( \alpha \)-stable subordinator is not appropriate in this case because of the behavior of ensemble averaged MSD that in stable case behaves like \( \tau^\alpha \) for all values of arguments. The estimated values of \( \alpha \), \( \lambda \) and \( \beta \) parameters for three trajectories we present in Table 2.

| DATA2          | \( \hat{\alpha} \) | \( \hat{\lambda} \) | \( \hat{\beta} \) |
|----------------|---------------------|---------------------|------------------|
| trajectory 1   | 0.41                | 0.13                | -0.0085          |
| trajectory 2   | 0.3936              | 0.11                | -0.0039          |
| trajectory 3   | 0.4906              | 0.129               | -0.012           |

Table 2: Estimated parameters of the ABM with inverse tempered stable subordinator for DATA2.

5. Conclusions

In this paper we have examined two processes related to subordinated ABM driven by tempered stable type systems. The first one, so called normal tempered stable, arises after subordination of ABM by strictly increasing...
tempered stable process, while the second one is a result of subordination of ABM with inverse tempered stable model. We have compared the main characteristics of such systems, like Laplace transforms, asymptotic behavior of pdf as well as the ensemble averaged MSD, that can be an useful tool of recognition between considered models. We have described also the estimation procedures for the parameters of considered processes and validate them. Finally, we have analyzed the real data related to indoor air quality in context of presented methodology.

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