Missing solution in a Cornell potential

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1To appear in Annals of Physics
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Abstract

Missing bound-state solutions for fermions in the background of a Cornell potential consisting of a mixed scalar-vector-pseudoscalar coupling is examined. Charge-conjugation operation, degeneracy and localization are discussed.

Key-words: Dirac equation; scalar-vector-pseudoscalar potential; isolated bound-state solution; Cornell potential; effective Compton wavelength

PACS: 03.65.Ge; 03.65.Pm

Highlights:

- The Dirac equation with scalar-vector-pseudoscalar Cornell potential is investigated.
- The isolated solution from the Sturm-Liouville problem is found.
- Charge-conjugation operation, degeneracy and localization are discussed.
In a recent article, Hamzavi and Rajabi [1] investigated the Dirac equation in a 1+1 dimension with the Cornell potential. They mapped the Dirac equation into a Sturm-Liouville problem of a Schrödinger-like equation and obtained a set of bound-state solutions by recurring to the properties of the biconfluent Heun equation. Nevertheless, an isolated solution from the Sturm-Liouville scheme was not taken into account. The purpose of this work is to report on this missing bound-state solution.

In the presence of time-independent interactions the most general 1+1 dimensional time-independent Dirac equation for a fermion of rest mass $m$ and momentum $p$ reads

$$\left( c\sigma_1 \hat{p} + \sigma_3 m + \frac{I + \sigma_3 V_\Sigma}{2} + \frac{I - \sigma_3 V_\Delta}{2} + \sigma_2 V_p \right) \psi = E\psi$$

(1)

where $E$ is the energy of the fermion, $V_\Sigma = V_v + V_s$ and $V_\Delta = V_v - V_s$. The subscripts for the terms of potential denote their properties under a Lorentz transformation: $v$ for the time component of the two-vector potential, $s$ for the scalar potential and $p$ for the pseudoscalar potential. $\sigma_i$ are the Pauli matrices and $I$ is the $2 \times 2$ unit matrix. From now on we restrict our attention to the case $V_\Delta = 0$. On that ground, Eq. (1) can also be written as

$$-i \frac{d\psi_-}{dx} + m\psi_+ + V_\Sigma \psi_+ - i V_p \psi_- = E\psi_+$$

$$-i \frac{d\psi_+}{dx} - m\psi_- + i V_p \psi_+ = E\psi_-$$

(2)

where $\psi_+$ and $\psi_-$ are the upper and lower components of the Dirac spinor respectively. It is clear from the pair of coupled first-order differential equations (2) that both $\psi_+$ and $\psi_-$ have opposite parities if the Dirac equation is covariant under $x \rightarrow -x$. In this sense, $V_p$ changes sign whereas $V_\Sigma$ remains the same. Because the invariance under reflection through the origin, components of the Dirac spinor with well-defined parities can be built. Thus, it suffices to concentrate attention on the half line and impose boundary conditions on $\psi$ at the origin and at infinity. Components of the Dirac spinor on the whole line with well-defined parities can be constructed by taking symmetric and antisymmetric linear combinations of $\psi$ defined on the half line:

$$\psi^{(\lambda)}_{\pm}(x) = [\theta (+x) \pm \lambda \theta (-x)] \psi_{\pm}(|x|)$$

(3)

where $\lambda (-\lambda)$ denotes the parity of $\psi_+$ ($\psi_-$) and $\theta (x)$ the Heaviside step function. For a normalized spinor, the expectation value of any observable $O$ may be given by

$$\langle O \rangle = \int dx \psi^\dagger O \psi$$

(4)

where the matrix $O$ must be Hermitian for insuring that $\langle O \rangle$ is a real quantity. In particular, $\psi^\dagger V \psi$ is an integrable quantity. Here $V$ is the potential matrix

$$V = \frac{I + \sigma_3}{2} V_\Sigma + \sigma_2 V_p$$

(5)
For $E \neq -m$, the searching for solutions can be formulated as a Sturm-Liouville problem for the upper component of the Dirac spinor, as done in Ref. [1] for bound states.

For $E = -m$, though, one can write

$$\frac{d\psi_+}{dx} - V_p \psi_+ = 0$$  \hspace{1cm} (6)$$

$$\frac{d\psi_-}{dx} + V_p \psi_- = -i (V_\Sigma + 2m) \psi_+$$

whose solution is

$$\psi_+ (x) = N_+ e^{v(x)}$$

$$\psi_- (x) = [N_- - iN_+ I (x)] e^{-v(x)}$$  \hspace{1cm} (7)$$

where $N_+$ and $N_-$ are normalization constants, and

$$v (x) = \int^x dy V_p (y)$$

$$I(x) = \int^x dy [V_\Sigma (y) + 2m] e^{+2v(y)}$$  \hspace{1cm} (8)$$

It is worthwhile to note that this sort of isolated solution cannot describe scattering states and is subject to the normalization condition $\int dx (|\psi_+|^2 + |\psi_-|^2) = 1$. Because $\psi_+$ and $\psi_-$ are normalizable functions, the possible isolated solution presupposes $V_p \neq 0$. More precisely, the singularities of $V_p$ and its asymptotic behaviour determines if the solution exists or does not exist [2]-[5].

For the mixed Cornell potential

$$V_\Sigma = -\frac{a_\Sigma}{|x|} + b_\Sigma |x|$$

$$V_p = -\frac{a_p}{x} + b_px$$  \hspace{1cm} (9)$$

one finds $v (|x|) = -a_p \ln |x| + b_px^2/2$ so that

$$\psi_+ (|x|) = N_+ |x|^{-a_p} e^{+b_px^2/2}$$

$$\psi_- (|x|) = [N_- - iN_+ I (|x|)] |x|^{+a_p} e^{-b_px^2/2}$$  \hspace{1cm} (10)$$

A normalizable solution for $b_p > 0$ is possible if $N_+ = 0$. Thus,

$$\psi (|x|) = N|x|^{a_p} e^{-b_px^2/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad b_p > 0$$  \hspace{1cm} (11)$$
regardless of $a_\Sigma$, $b_\Sigma$ and $m$. Note that (11) is square-integrable even if it is singular at the origin for $-1/2 < a_p < 0$. Nevertheless, near the origin

$$\hat{p}^2 \psi \propto a_p (a_p - 1) |x|^{a_p-2} \tag{12}$$

so that the operator $\hat{p}^2$ may not be a legitimate linear transformation, because it may carry functions out of Hilbert space. This observation allows us to impose additional restrictions on the eigenfunction. $\hat{p}^2 \psi$ is in Hilbert space only if

$$a_p = 0 \text{ or } a_p = 1 \text{ or } a_p > 3/2 \tag{13}$$

As for $b_p < 0$, a normalizable solution requires $N_- = 0$, and a good behaviour of $\psi_+$ near the origin, in the sense of normalization, demands $a_p < 1/2$. We must, however, calculate $I(|x|)$ to make explicit the behaviour of $\psi_-$. For the Cornell potential given by (9), $I(|x|)$ can be expressed in terms of the incomplete gamma function \[6\]

$$\gamma(a, z) = \int_0^z dt e^{-t} t^{a-1}, \quad \text{Re} a > 0 \tag{14}$$

As a matter of fact,

$$I(|x|) = \frac{|b_p|^{a_p-1/2}}{2} \left[ 2m \gamma(-a_p + 1/2, |b_p|x^2) + \frac{b_\Sigma}{|b_p|} \gamma(-a_p + 1, |b_p|x^2) - a_\Sigma \sqrt{|b_p|} \gamma(-a_p, |b_p|x^2) \right] \tag{15}$$

It follows immediately from the condition imposed on the definition of the incomplete gamma function that $a_p < 1/2$ if $m \neq 0$, $a_p < 1$ if $b_\Sigma \neq 0$ and $a_p < 0$ if $a_\Sigma \neq 0$. Now, because $\gamma(a, z)$ tends to $\Gamma(a)$ as $z$ tends to infinity, $\psi_-$ is not, in general, a square-integrable function. An exception, though, occurs when $m = a_\Sigma = b_\Sigma = 0$ ($I(|x|) = 0$) just for the reason that $\psi_-$ vanishes identically. Therefore,

$$\psi(|x|) = N|x|^{-a_p} e^{-|b_p|x^2/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b_p < 0, \quad m = a_\Sigma = b_\Sigma = 0 \tag{16}$$

In this case, near the origin

$$\hat{p}^2 \psi \propto a_p (a_p + 1) |x|^{-a_p-2} \tag{17}$$

so that $\hat{p}^2 \psi$ is in Hilbert space only if

$$a_p = 0 \text{ or } a_p = -1 \text{ or } a_p < -3/2 \tag{18}$$

It is instructive to note that

$$\psi^\dagger V \psi = V_\Sigma |\psi_+|^2 - 2V_p \text{Im} \psi_-^* \psi_+ \tag{19}$$

so that all the isolated solutions make $\langle V \rangle = 0$. Notice from (6) that when $m = 0$ and $V_\Sigma = 0$, $\psi_+$ turns into $\psi_-$ and vice versa as $V_p$ changes its sign. As a matter of fact, these zero-energy solutions are related by the charge-conjugate operation [3].
The extension of $\psi$ compatible with (6) for the whole line demands that $\psi_+$ and $\psi_-$ be continuous at the origin, otherwise (6) would contain the $\delta$-functions. On the whole line they are even functions if $a_p = 0$. If $|a_p| = 1$ or $|a_p| \geq 3/2$, though, the eigenvalue $E = -m$ is two-fold degenerate due to the existence of symmetric and asymmetric extensions of $\psi$. It is worthwhile to note that this sort of degeneracy for bound-state solutions in a one-dimensional system contrasts with a well-known nondegeneracy theorem in the nonrelativistic theory [7]. However, it has been shown that the theorem is not necessarily valid for singular potentials [8]. Our results are summarized in Table 1.

$$
\psi(|x|) = SN|x|^{|a_p|}e^{-|b_p|x^2/2} \\
sgn(b_p) = sgn(a_p), |a_p| = 0 \text{ or } 1 \text{ or } > 3/2
$$

| $S$ | Formula |
|---|---|
| $|a_p| = 0$ | Only symmetrical extension |
| $|a_p| > 3/2$ | Symmetrical and antisymmetrical extensions |

Table 1: Summary of the results for the solution with $E = -m$ ($\langle V \rangle = 0$) defined on the whole line. sgn($z$) stands for the sign function of $z$. For massless fermions, the zero energy solutions with $b_p > 0$ and $b_p < 0$ are related by charge-conjugate operation when $a_{\Sigma} = b_{\Sigma} = 0$.

The uncertainties in the position and momentum for the solution on the whole line can be written as

$$
\Delta x = \frac{1}{\sqrt{|b_p|}}\sqrt{|a_p| + 1/2} \quad (20) \\
\Delta p = \sqrt{|b_p|}\sqrt{|a_p| - 1/4} \quad (21)
$$

If $\Delta x$ shrinks then $\Delta p$ will must swell, in consonance with the Heisenberg uncertainty principle. Notice that $\Delta x \Delta p$ is independent of $|b_p|$, and when $a_p = 0$ one obtains the minimum-uncertainty solution ($\Delta x \Delta p = 1/2$) due to the Gaussian form of the eigenspinor, as it happens in the ground-state solution for the nonrelativistic...
harmonic oscillator. Nevertheless, the maximum uncertainty in the momentum is comparable with $m$ requiring that is impossible to localize a fermion in a region of space less than or comparable with half of its Compton wavelength. This impasse can be broken by resorting to the concepts of effective mass and effective Compton wavelength [2], [4]. Indeed, if one defines an effective mass as $m_{\text{eff}} = \sqrt{|b_p|}$ and an effective Compton wavelength as $\lambda_{\text{eff}} = 1/m_{\text{eff}}$ one will find that the high localization of fermions, related to high values of $|b_p|$ never menaces the single-particle interpretation of the Dirac theory. In fact, $(\Delta x)_{\text{min}} = \sqrt{\frac{2}{m_{\text{eff}}}}$ occurs for $a_p = 0$, and $(\Delta p)_{\text{max}} = \sqrt{6m_{\text{eff}}}/2$ for $a_p = 1$.

Acknowledgments
This work was supported in part by means of funds provided by CAPES and CNPq. This work was partially done during a visit (L.B. Castro) to UNESP-Campus de Guaratinguetá.

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