Approximate 3-Dimensional Electrical Impedance Imaging

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Abstract
We discuss a new approach to three-dimensional electrical impedance imaging based on a reduction of the information to be demanded from a reconstruction algorithm. Images are obtained from a single measurement by suitably simplifying the geometry of the measuring chamber and by restricting the nature of the object to be imaged and the information required from the image. In particular we seek to establish the existence or non-existence of a single object (or a small number of objects) in a homogeneous background and the location of the former in the \((x,y)\)-plane defined by the measuring electrodes. Given in addition the conductivity of the object rough estimates of its position along the \(z\)-axis may be obtained. The approach may have practical applications.

1 Introduction
The aim of electrical impedance tomography (EIT) is to reconstruct the conductivity distribution \(\sigma(x)\) in the interior of an object \(\Omega \subset \mathbb{R}^3\) from electrical...
measurements on the boundary $\partial \Omega$. For this purpose a number of different current distributions are applied to the surface of the object via electrodes and the resulting potentials on the surface are recorded. Applications can be envisaged both in medicine and industry [1].

Conservation of the current $\mathbf{j}(\mathbf{x})$ and Maxwell’s equations in the quasi-static limit lead to the following differential equation for the potential $\Phi(\mathbf{x})$:

$$\nabla \cdot [\sigma(\mathbf{x}) \nabla \Phi(\mathbf{x})] = 0. \quad (1)$$

In the following we take as the object a rectangular box and investigate whether statements on the conductivity distribution can be made if the surface potential can only be measured on one side of the box. Such a model relates to typical situations in geological and medical imaging.

The general inverse conductivity problem for the box requires current- and potential-measurements for a large number (in principle infinite) of applied current configurations on the surface of the box. For the reconstruction of the conductivity distribution in this and related problems the boundary conditions must be known precisely and all calculations of potentials be performed with high accuracy. All these conditions are difficult to be achieved in practice, which explains the comparative lack of success of the impedance method in medical applications. In many cases, specifically breast cancer screening, it is actually not absolutely necessary to have a complete image of the region. If we restrict the reconstruction to a shadow on a plane and require only rough information on size and location of the cancerous region, the reconstruction can be done analytically using a single measurement. This problem has also been discussed from different points of view [2].

2 Description of the problem

We are interested in the conductivity distribution $\sigma(x)$ inside a rectangular box with sides $a$, $b$, $c$, as pictured in figure (1).

The region of interest $\Omega$ is therefore of the form

$$\Omega = \{(x, y, z)| 0 < x < a, 0 < y < b, 0 < z < c\}.$$

The boundary is made up of six rectangles

$$\partial \Omega = \partial \Omega_{x=0} \cup \partial \Omega_{x=a} \cup \partial \Omega_{y=0} \cup \partial \Omega_{y=b} \cup \partial \Omega_{z=0} \cup \partial \Omega_{z=c},$$
where, for instance, \( \partial \Omega_{x=0} \) means
\[
\partial \Omega_{x=0} = \{(x, y, z) | x = 0, 0 \leq y \leq b, 0 \leq z \leq c\}.
\]
Similar definitions hold for the other rectangular regions.

The following discussion assumes that a fixed external current enters on one of the side surfaces and leaves on the opposite surface. The current is taken to be constant on the two surfaces. i.e.
\[
\sigma \frac{\partial \phi}{\partial n} = -I; \quad (x, y, z) \in \partial \Omega_{x=0} \quad (2)
\]
\[
\sigma \frac{\partial \phi}{\partial n} = I; \quad (x, y, z) \in \partial \Omega_{x=a} \quad (3)
\]
\[
\sigma \frac{\partial \phi}{\partial n} = 0; \quad otherwise, \quad (4)
\]
where \( \frac{\partial}{\partial n} \) denotes the normal derivative. For simplicity we set \( I = 1 \), which can always be achieved by a suitable choice of units for the conductivity and the potential.

We further assume that conditions are such that the resulting potential can only be measured on the plane \( \partial \Omega_{z=0} \) (see fig.(1)).

Given \( \sigma(x) \) the resulting potential \( \phi(x) \) can be obtained by solving the differential equation (1) with the Neumann boundary condition Eq.(2,3,4).

The aim is to obtain an image of \( \sigma(x) \) from the measurement potential on the boundary \( \partial \Omega_{z=0} \). If the conductivity does not differ much from a constant distribution \( \sigma_0 \), we can write
\[
\sigma(x) = \sigma_0 + \delta \sigma(x). \quad (5)
\]
Without loss of generality we can set $\sigma_0 = 1$. For $\sigma \equiv \sigma_0$ the solution of the boundary value problem is obviously

$$\phi_0(x, y, z) = x + \text{const.} \quad (6)$$

In the following section we try to answer the question to what extent $\delta \sigma(x)$ can be reconstructed by measuring the potential only on the lower surface of the box, i.e. on the boundary surface $\partial \Omega_{z=0}$.

3 Reconstruction

As the potential distribution is only defined up to a constant, it is convenient to require that the average of the potential distribution vanishes on the boundary surface $\partial \Omega$

$$\int_{\partial \Omega} \phi = 0 \quad (7)$$

If we assume that the current on the surface $\partial \Omega$ is square integrable, it is in the space

$$\mathcal{L}_2^2(\partial \Omega) := \left\{ f \in \mathcal{L}_2^2(\partial \Omega), \int_{\partial \Omega} f = 0 \right\}. \quad (8)$$

Any change $\delta \sigma$ of a homogeneous conductivity distribution $\sigma_0$ produces a corresponding change $\delta \phi$ in the potential distribution $\phi_0$. Then, for any function $g \in \mathcal{L}_2^2(\partial \Omega)$, it can be shown (see appendix), that in linear approximation

$$< \delta \phi, g >_{\mathcal{L}_2^2(\partial \Omega)} = \int_{\partial \Omega} \delta \phi \ g = - \int_{\Omega} \nabla \phi_0 \cdot \nabla \phi \ g \ \delta \sigma \ , \quad (9)$$

where $\phi_g$ represents the solution of the potential problem for constant conductivity $\sigma_0 \equiv 1$ and external current distribution $g \in \mathcal{L}_2^2(\partial \Omega)$. We have checked in model calculations that the linearization yields good qualitative images even for objects with large conductivity. This refers only to the geometrical appearance of the objects and not to the actual numerical value of the reconstructed conductivity. A small spherical metallic object of infinite conductivity in a homogeneous background of conductivity 1 unit, for example, will be imaged as an object of conductivity 3 units. We will therefore not attempt to determine numerical values of the conductivity of the hidden objects but only existence and location of such objects. This restriction of the
scope of the reconstruction will still yield useful results in applications such as mammography.

For the given experimental set-up we measure a change in potential $\delta \phi_{\text{exp}}$, which we normalize so that it is in $L^2_0(\partial \Omega_{z=0})$, which is defined in analogy to Eq. (8). We consider a base $\{u_n\}$, which is complete and orthonormal in $L^2_0(\partial \Omega_{z=0})$.

It turns out to be useful to introduce in addition a set of functions $\tilde{u}_n \in L^2_0(\partial \Omega)$ (not complete), which are defined on the full surface $\partial \Omega$ of the box,

$$\tilde{u}_n(x) := \begin{cases} u_n(x) &; x \in \partial \Omega_{z=0} \\ 0 &; \text{else} \end{cases}. \quad (10)$$

Then, by Eq. (11), the moments $\langle \delta \phi_{\text{exp}}, u_n \rangle_{L^2_0(\partial \Omega_{z=0})}$ satisfy in linear approximation

$$\langle \delta \phi_{\text{exp}}, u_n \rangle_{L^2_0(\partial \Omega_{z=0})} = \langle \delta \phi, \tilde{u}_n \rangle_{L^2_0(\partial \Omega)} = -\int_{\Omega} \nabla \phi_0 \cdot \nabla \phi_{\tilde{u}_n} \, \delta \sigma. \quad (11)$$

We introduce a linear operator $A$ acting on the change in conductivity $\delta \sigma$ through

$$A \delta \sigma := -\sum_n \left( \int_{\Omega} \nabla \phi_0 \cdot \nabla \phi_{\tilde{u}_n} \, \delta \sigma \right) u_n. \quad (12)$$

Using $\delta \phi_{\text{exp}} = \sum_n < \delta \phi_{\text{exp}}, u_n >_{L^2_0(\partial \Omega_{z=0})} u_n$ the relation between $\delta \sigma$ and the associated change in potential $\delta \phi_{\text{exp}}$ reads

$$A \delta \sigma = \delta \phi_{\text{exp}}. \quad (13)$$

A natural choice for the base $\{u_n\}$ associated to the upper surface is

$$u_{i,j}(x, y) = C_{i,j} \cos \frac{i \pi x}{a} \cos \frac{j \pi y}{b}, \quad i, j = 0, 1, \ldots, (i, j) \neq (0, 0), \quad (14)$$

$$C_{i,j} = \begin{cases} 2/\sqrt{ab} &; i, j \neq 0 \\ \sqrt{2/(ab)} &; \text{else} \end{cases}. \quad (15)$$

where the index $n$ is replaced by two indices $i, j$. The set of functions $\tilde{u}_n \rightarrow \tilde{u}_{i,j}$ referring to the whole surface is then defined in accordance with Eq. (11). To make use of Eq. (11) to calculate $\delta \sigma$ we need the potential $\phi_{\tilde{u}_{i,j}}$ resulting from an external current distribution $\tilde{u}_{i,j} \in L^2_0(\partial \Omega)$ and conductivity $\sigma_0 \equiv 1$. It is a simple exercise to show that

$$\phi_{\tilde{u}_{i,j}}(x, y, z) = \frac{C_{i,j}}{\delta_{i,j}(1 - e^{-2\delta_{i,j}c})} \cos \frac{i \pi x}{a} \cos \frac{j \pi y}{b} \left\{ e^{-\delta_{i,j}z} + e^{\delta_{i,j}z - 2\delta_{i,j}c} \right\},$$

(16)
with the abbreviation
\[ \delta_{i,j} = \pi \sqrt{(i/a)^2 + (j/b)^2}. \]

If we define
\[ \sigma_{i,j} = \| \nabla \phi_0 \cdot \nabla \tilde{u}_{i,j} \| \]
\[ = \frac{i \pi}{a \delta_{i,j} (1 - e^{-2\delta_{i,j}c})} \left( \frac{1}{2\delta_{i,j}} (1 - e^{-4\delta_{i,j}c}) + 2c e^{-2\delta_{i,j}c} \right)^{1/2}, \tag{17} \]
\[ v_{i,j} = \frac{-\nabla \phi_0 \cdot \nabla \tilde{u}_{i,j}}{\| \nabla \phi_0 \cdot \nabla \tilde{u}_{i,j} \|_{L^2(\Omega)}} \]
\[ = C_{i,j} \sin \frac{i \pi x}{a} \cos \frac{j \pi y}{b} \left\{ e^{-\delta_{i,j}z} + e^{\delta_{i,j}z - 2\delta_{i,j}c} \right\} \]
\[ \times \left( \frac{1}{2\delta_{i,j}} (1 - e^{-4\delta_{i,j}c}) + 2c e^{-2\delta_{i,j}c} \right)^{-1/2}, \tag{18} \]
then Eq.(12) can be written in the form
\[ A \delta \sigma = \sum_{i=1, j=0}^{\infty} \sigma_{i,j} < \delta \sigma, v_{i,j} >_{L^2(\Omega)} u_{ij}. \tag{19} \]

This is our main result. It is obvious from Eq.(19) that the set \{v_{i,j}\} is a complete orthogonal system in \(N(A)^\perp\). We have thus explicitly constructed the singular system \{v_{i,j}, u_{i,j}; \sigma_{i,j}\} of the operator \(A\) and its generalized inverse can be written down explicitly. The generalized or least square solution of Eq.(19) is then simply given by
\[ \delta \sigma = \sum_{i=1, j=0}^{\infty} \sigma_{i,j}^{-1} < \delta \phi_{exp}, u_{i,j} > v_{i,j}. \tag{20} \]

This generalized solution is still not continuous in the data and must be regularized in a suitable manner. We employed for convenience mainly the method of truncating the singular values or truncating the indices \(i, j\) in Eq.(20). The latter procedure turns out to produce better images. The cut off values of the indices is determined by a version of the discrepancy principle, i.e. by requiring that the resolution implied by the Fourier series Eq.(14) should not exceed the distance between the electrodes which measure the potential on the surface \(\partial \Omega_{z=0}\). We assume the latter constitutes the main source of the experimental error.
It should be pointed out that we only reconstruct a three dimensional picture which is a projection on the set of functions \( \{v_{i,j}\} \). It is sufficient to view the image at \( z = 0 \), because the images for \( z \neq 0 \) follow uniquely from the \( z = 0 \) one and contain no additional information.

We effectively see a two-dimensional image, which represents a kind of shadow of the object. For many purposes (such as in cancer screening), when one is only interested in the presence or absence of an object, this is sufficient information. As discussed above, the actual value of the object’s conductivity cannot be reconstructed quantitatively.

In Fig.(2) and (3) we present images obtained from synthetic data which were calculated for \( 10 \times 10 \) grid points on the plane \( \partial \Omega_{z=0} \). We also show how the image deteriorates when errors are assigned to the data.

![Image](image.png)

Figure 2: Images of a spherical object obtained from exact and error affected data.

Figure (2) shows images of a spherical metallic object of diameter \( d = 1 \) at a distance \( z = 2 \) (all in units of the grid spacing) from the surface of measurement, (a) with exact data, and (b) with data \( \delta \phi_{\text{exp}} \) corrupted with a 20% random uniform multiplicative error. It is amazing that even with errors of such a magnitude a reasonable image is produced.

Figure (3) shows images of two spherical objects obtained from exact data. Case (a) shows the image for two spheres of diameter 1.5 and 1 respectively both at a distance \( z = 2 \) from the surface and case (b) shows the image for two spheres of equal diameter at distance \( z = 1.5 \) and \( z = 2 \) respectively.
Figure 3: Images of two spherical objects

Qualitatively the image gets larger and flatter as the object is moved away from the measuring plate. In addition the image gets brighter (but not larger!) as the object gets larger. The same effect is observed when the conductivity is increased. It is not possible to distinguish volume- from conductivity effects. This is true as long as the objects are small or not too close to the surface, as they effectively behave as dipoles (see section 4). Given additional information, e.g. that the object’s conductivity is constant and of a given magnitude, it may be possible to quantify this observation and obtain a full three dimensional image of the object. This is exemplified in the next section.

4 Spherical Object

In the following we consider a single spherical object $K$ of conductivity $\kappa$ and radius $a$ immersed in the box $\Omega$ filled with a liquid of conductivity 1.

Let $n_K$ be the normal to the surface of $K$ and $n_\Omega$ the normal on $\partial \Omega$. The boundary value problem for a current distribution $f \in L^2(\partial \Omega)$ can be
defined as follows,
\[
\triangle \phi(x) = 0 \quad , x \in \Omega \setminus \partial K, \quad (21)
\]
\[
\frac{\partial \phi}{\partial n}(x) = f(x) \quad , x \in \partial \Omega, \quad (22)
\]
\[
\lim_{h \to 0^+} \left( \phi(x + h n_K) - \phi(x - h n_K) \right) = 0 \quad , x \in \partial K, \quad (23)
\]
\[
\lim_{h \to 0^+} \left( \frac{\partial \phi(x + h n_K)}{\partial n_K} - \kappa \frac{\partial \phi(x - h n_K)}{\partial n_K} \right) = 0 \quad , x \in \partial K, \quad (24)
\]
\[
\int_{\partial \Omega} \phi \, ds = 0. \quad (25)
\]

Equation (23) guarantees the continuity of the potential while (24) describes current conservation. These two equations determine the boundary conditions on the surface of the sphere. The other three equations represent the well-known boundary value problem of the Laplace equation. The Neumann boundary condition (22) is given in (23). For \(\kappa = 1\) one obtains \(\phi_0\), the solution of (3).

For the case of a small sphere of constant conductivity and not too close to the surface \(\Omega\), the change in potential \(\delta \phi\) is given by the dipole term,
\[
\delta \phi(\vec{x}) = -\frac{\alpha \alpha^3 \nabla \phi_0 \cdot (\vec{x} - \vec{x}_0)}{|\vec{x} - \vec{x}_0|^3}; \quad (26)
\]
where \(\vec{x}_0\) is the coordinate of the centre of the sphere, and
\[
\alpha = \frac{\kappa - 1}{\kappa + 2} \quad (27)
\]
This result can be derived by noting that our boundary value problem is equivalent to that of a dielectric sphere in a uniform electric field.

The variation of \(\alpha\) with \(\kappa\) shows quite clearly the limited sensitivity of EIT to changes of conductivity. In practical reconstructions \(\kappa = 10\) can hardly be distinguished from \(\kappa = \infty\).

The potential \(\phi = \phi_0 + \delta \phi\) still does not satisfy the Neumann boundary condition (22) on \(\partial \Omega\). This problem can, in principle, be solved by an infinite number of image dipoles. We checked that the series converges rapidly. For the case \(a, b \to \infty\), i.e. the case of two infinite plates, the sum takes on the simple form:
\[ \delta \phi(x) = -\alpha r^3 \sum_{n=0}^{\infty} \left\{ \frac{\nabla \phi_0 \cdot (x - x_0, y - y_0, z - (2nc - z_0))^T}{|(x - x_0)^2 + (y - y_0)^2 + (z - (2nc - z_0))^2|^3} + \frac{\nabla \phi_0 \cdot (x - x_0, y - y_0, z - (-2nc + z_0))^T}{|(x - x_0)^2 + (y - y_0)^2 + (z - (-2nc + z_0))^2|^3} \right\} \] (28)

The knowledge of \( \delta \phi \) allows to calculate the generalized inverse according to Eq. (20). For rough estimates and when the sphere is not too close to the surface, the image dipoles can be neglected. As the conductivity of the sphere is assumed to be known, Eq. (26) can be used to obtain an estimate of the position and the volume of the sphere. As a test we measure the synthetic data on the surface \( \partial \Omega z=0 \) by \( 10 \times 10 \) electrodes. Given the rough knowledge of the coordinates \((x_0, y_0)\) of the centre of the sphere, we fit \( \delta \phi \) of Eq. (26) plus the background potential \( \phi_0 \) to data taken on neighboring electrodes. As a typical example, we find for data afflicted with 10% multiplicative uniform error, the following results: \( z_0 = 2.32 \) (instead of 2.0) and \( r = 0.56 \) (instead of 0.5).

In realistic applications the object to be detected will in general not be spherical. Nonetheless one may obtain rough information on size and depth of the location of the object by assuming a spherical shape and applying the analysis above.

5 Conclusion

We have presented in this note an electric impedance imaging system based on a specific simple geometry of the device which guarantees a uniform current distribution in the case of constant conductivity. If we further impose the condition that only a single object (or possible a small number of objects) is to be detected, then we show that an image can be obtained in a single measurement of the surface potential. To test of the effectiveness of the method, we create synthetic data which can be afflicted with errors. The image obtained by inverse problem techniques represents a projection or shadow on the surface where the potential is measured. This image is amazingly stable against data errors. We also indicate how rough estimates on the size and the depth of the object may be obtained. The actual construction of such an imaging system is planned.
A Appendix

We will give the sketch of a proof of Eq.(9). Let a potential, denoted by $u(x)$, satisfy the EIT differential equation

$$\nabla(\sigma \nabla u(x)) = 0$$

for a given conductivity $\sigma(x)$ and surface current $f(x)$

$$f = \sigma \left. \frac{\partial u}{\partial n} \right|_{\partial \Omega}.$$  

Let $v(x)$ be an arbitrary solution of the EIT differential equation Eq.29. Then we can define a functional

$$b_f[v] := \int_{\partial \Omega} f \cdot v$$

and a bilinear form.

$$a_\sigma[u, v] := \int_\Omega \sigma \nabla u \cdot \nabla v.$$

The EIT boundary value problem with Neumann boundary conditions is known [4] to be equivalent to the condition

$$b_f[v] = a_\sigma[u, v] \ \forall v$$

We now change $\sigma \rightarrow \sigma + \delta \sigma$, keeping the same Neumann boundary condition [30]. Then the potential will change as $u \rightarrow u + \delta u$ and the condition Eq.33 will read

$$b_f[v] = a_{\sigma + \delta \sigma}[u + \delta u, v] \ \forall v$$

Or

$$b_f[v] = a_\sigma[u, v] + a_\sigma[\delta u, v] + a_{\delta \sigma}[u, v] + a_{\delta \sigma}[\delta u, v]$$

Neglecting the last term, using Eq.33 and the symmetry of the bi-linear form, we obtain the relation

$$a_{\delta \sigma}[u, v] = -a_\sigma[v, \delta u] \ \forall v$$

As $v(x)$ is the solution of Eq.29 for some boundary current $g(x)$, it must satisfy

$$b_g[w] = a_\sigma[v, w]$$

for all $w(x)$, in particular for $w(x) = \delta u(x)$. We finally obtain therefore

$$\int_{\partial \Omega} g \cdot \delta u = - a_{\delta \sigma}[u, v],$$

which is just Eq.(9).
References

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