ON THE DISCREPANCY FUNCTION IN ARBITRARY DIMENSION,
CLOSE TO $L^1$

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Abstract. Let $\mathcal{A}_N$ be $N$ points in the unit cube in dimension $d$, and consider the Discrepancy function

$$D_N(\vec{x}) := \#(\mathcal{A}_N \cap [\vec{0}, \vec{x}]) - N|[\vec{0}, \vec{x})|$$

Here, $\vec{x} = (x_1, \ldots, x_d)$, $[0, \vec{x}) = \prod_{i=1}^d [0, x_i)$, and $|[\vec{0}, \vec{x})|$ denotes the Lebesgue measure of the rectangle. We show that necessarily

$$\|D_N\|_{L^1(\log L)^{(d-1)/2}} \gtrsim (\log N)^{(d-1)/2}.$$ 

In dimension $d = 2$, the ‘$\log L$’ term has power zero, which corresponds to a Theorem due to (Halász, 1981). The power on $\log L$ in dimension $d \geq 3$ appears to be new, and supports a well-known conjecture on the $L^1$ norm of $D_N$. Comments on the Discrepancy function in Hardy space also support the conjecture.

1. Main Theorem

Our subject is irregularities of distribution of points with respect to rectangles in the unit cube. It is a familiar theme of the subject is to show that no matter how $N$ points are selected, they must be far from uniform. We give a new proof of a well-known theorem in the subject (Halász, 1981), concerning the $L^1$ norm of the Discrepancy function, and show that this result admits an extension to arbitrary dimension. We also make some remarks on the Discrepancy function and Hardy space.

Let $\mathcal{A}_N \subset [0, 1]^d$ be a set of cardinality $N$. Define the Discrepancy Function associated to $\mathcal{A}_N$ as a function on the unit square as follows.

$$D_N(\vec{x}) := \#(\mathcal{A}_N \cap [0, \vec{x})) - N|[0, \vec{x})|,$$

where $[0, \vec{x})$ is the rectangle in the unit cube with one vertex at the origin and the other at $\vec{x} = (x_1, \ldots, x_d)$, and $|[0, \vec{x})|$ denotes the Lebesgue measure of the rectangle. This is the difference between the number of points in the rectangles $[0, \vec{x})$ and the expected number
of points in the rectangle. The relative size of this function, in many senses, necessarily must increase with \( N \). The principal result here is that of (Roth, 1954).

**K. Roth's Theorem.** We have the estimate below, valid in all dimensions \( d \geq 2 \)
\[
\|D_N\|_2 \gtrsim (\log N)^{(d-1)/2}
\]
where the implied constant is only a function of dimension \( d \).

The same bound holds for the \( L^p \) norm, for \( 1 < p < \infty \), (Schmidt, 1977b), and is known to be sharp as to the order of magnitude. The endpoint cases of \( p = 1 \) and \( p = \infty \) are much harder. We concentrate on the case of \( p = 1 \) in this note, and refer the reader to (Beck, 1989; Bilyk et al.; Bilyk and Lacey; Halász, 1981) for more information about the case of \( p = \infty \).

In the case of \( d = 2 \), we have definitive information about the \( L^1 \) norm, namely that the Roth lower bound holds. See (Halász, 1981).

**Halász' Theorem.** In dimension \( d = 2 \) we have the uniform estimate
\[
\|D_N\|_1 \gtrsim \sqrt{\log N}
\]

A principal conjecture in the subject is that the Roth bound holds for the \( L^1 \) norm of the Discrepancy function in all dimensions.

**Conjecture on the \( L^1 \) norm of \( D_N \).** In dimension \( d \geq 3 \) we have the estimate
\[
\|D_N\|_1 \gtrsim (\log N)^{(d-1)/2}.
\]

The best known result for the \( L^1 \) norm directly is the Halász bound below.
\[
\|D_N\|_1 \gtrsim \sqrt{\log N}, \quad d \geq 3.
\]
This is a simple consequence of his argument in (Halász, 1981).

Our main result is a partial extension of Halász' Theorem to arbitrary dimension.

1.1. **Theorem.** In dimension \( d \geq 2 \) we have
\[
\|D_N\|_{L^1(\log L)^{(d-2)/2}} \gtrsim (\log N)^{(d-1)/2}.
\]

Here, we use an Orlicz norm which is ‘close’ to \( L^1 \). Its definition, the well-known one, is made precise in the next section. We remark that the orthogonal function method of (Roth, 1954), especially as modified by the observations in (Schmidt, 1977b), can be used to prove the estimate
\[
\|D_N\|_{L^1(\log L)^{(d-1)/2}} \gtrsim (\log N)^{(d-1)/2}.
\]
This does not contain Halász’ Theorem, as the power of the log $L$ is too large. Our proof is an elaboration of that of Halász, using appropriate version of the Chang-Wilson-Wolff inequality (Chang et al., 1985), and the variant in (Fefferman and Pipher, 1997).

Halász’ proof is by way duality,\(^1\) namely one constructs an appropriate bounded function $\Psi$, and obtains a uniform lower bound on the inner product $\langle \Psi, D_N \rangle$. And in particular, the proof uses a Bernoulli product construction. Our construction is not Bernoulli product, though once the function is constructed, many details are variants of the arguments in (Halász, 1981).

The concluding section of the paper includes some remarks about the Discrepancy function and Hardy spaces, and proves a result which can be thought of supporting evidence for the $L^1$ conjecture above.

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2. Preliminary Facts

We suppress many constants which do not affect the arguments in essential ways. $A \lesssim B$ means that there is an absolute constant $K > 0$ so that $A \leq KB$. Thus $A \lesssim 1$ means that $A$ is bounded by an absolute constant. And if $A \lesssim B \lesssim A$, we write $A \cong B$.

Inequalities. We collect some standard estimates from the Probability and Harmonic Analysis literature. Let $\{r_j \mid j \geq 1\}$ be a sequence of Rademacher random variables, thus independent identically distributed random variables on a probability space $(\Omega, \mathcal{P})$ such that $\mathcal{P}(r_j = 1) = \mathcal{P}(r_j = -1) = \frac{1}{2}$. We have the moment function inequality

$$
\mathbb{E} \exp \left( \lambda \sum_j c_j r_j \right) \leq \exp \left( \frac{1}{2} \lambda^2 \sum_j c_j^2 \right), \quad \lambda \in \mathbb{R}.
$$

This holds for all sequences of coefficients $\{c_j \mid j \in \mathbb{N}\}$ such that the right hand side above is finite. This implies the distributional inequality

$$
\mathcal{P} \left( \sum_j c_j r_j > t \right) \leq \exp \left( -\frac{t^2}{2 \sum_j c_j^2} \right), \quad t > 0.
$$

An equivalent formulation is in terms of the Khintchine inequalities.

\(^1\)Extremal choices of the point set will result in a Distribution Function supported whose $L^1$ norm is determined on a set which is nearly the whole square. A proof by duality is natural.
Khintchine Inequalities for Rademacher Random Variables. We have the inequalities
\[ \left\| \sum_j c_j r_j \right\|_p \leq C \sqrt{p} \left[ \sum_j c_j^2 \right]^{1/2}, \quad 0 < p < \infty. \]

The best constants in these inequalities are of significant interest. For the range \( p \geq 2 \), of interest to us, see (Young, 1976).

There is a powerful extension to these inequalities to the setting of Haar series, or more generally, conditionally symmetric martingales. We state the results in a convenient form.

In one dimension, the class of dyadic intervals in the unit interval are \( D := \{[j2^{-k}, (j+1)2^{-k}) | j, k \in \mathbb{N}, 0 \leq j < 2^k \} \). Let \( D_n \) denote the dyadic intervals of length \( 2^{-n} \), and by abuse of notation, also the sigma field generated by these intervals. For an integrable function \( f \) on \([0, 1]\), the conditional expectation is
\[ f_n = \mathbb{E}(f \mid D_n) := \sum_{I \in D_n} 1_I \cdot |I|^{-1} \int_I f(y) \, dy. \]
The sequence of functions \( \{f_n | n \geq 0\} \) is a martingale. The martingale difference sequence is \( d_0 = f_0 \), and \( d_n = f_n - f_{n-1} \) for \( n \geq 1 \). The sequence of functions \( \{d_n | n \geq 0\} \) are pairwise orthogonal. The square function is
\[ S(f) := \left[ \sum_{n=0}^{\infty} |d_n|^2 \right]^{1/2}. \]
We have the following extension of the Khintchine inequalities.

2.1. Theorem. The inequalities below hold, for some absolute choice of constant \( C > 0 \).
\[ \|f\|_p \leq C \sqrt{p} \|S(f)\|_p, \quad 2 \leq p < \infty. \]
In particular, this inequality holds for Hilbert space valued functions \( f \).

For real-valued martingales, this was observed by (Chang et al., 1985). The extension to Hilbert space valued martingales is useful for us, and is proved in (Fefferman and Pipher, 1997). Indeed, the best constants in these inequalities are known for \( p \geq 3 \) (Wang, 1991).

Orlicz Spaces. For background on Orlicz Spaces, we refer the reader to (Lindenstrauss and Tzafriri, 1977). Consider a symmetric convex function \( \psi \), which is zero at the origin, and is otherwise non-zero. Let \((\Omega, \mathcal{F}, P)\) be a probability space, on which our functions are defined, and let \( \mathbb{E} \) denote expectation over the probability space. We can define
\[ \|f\|_{L^\psi} = \inf\{K > 0 \mid \mathbb{E}\psi(f \cdot K^{-1}) \leq 1\}, \]
where we define the infimum over the empty set to be $\infty$. The set of functions $L^\psi = \{ f \mid \| f \|_{L^\psi} < \infty \}$ is a normed linear space, with norm as above. It is the Orlicz space associated with $\psi$.

We are interested in, for instance, $\psi(x) = e^{x^2} - 1$, in which case we indicate the Orlicz space as $\exp(L^2)$. More generally, for $0 < \alpha < 1$, we let $\psi_\alpha(x)$ be a symmetric convex function which equals $e^{|x|^\alpha} - 1$ for $|x|$ sufficiently large, depending upon $\alpha$. And we write $L^{\psi_\alpha} = \exp(L^\alpha)$.

The following proposition is well-known, and follows from elementary methods.

2.4. Proposition. We have the following equivalence of norms valid for all $\alpha > 0$:

$$\|f\|_{\exp(L^\alpha)} \simeq \sup_{p>1} p^{-1/\alpha} \| f \|_p.$$  

Comparing this Proposition to the results of the previous section, we see that an equivalent form of the Khintchine Inequalities for Rademacher random variables is as follows.

2.5. Theorem. For all square summable coefficients $c_j$ we have the following estimate for the Rademacher sequence of random variables \( \{r_j \mid j\} \).

$$\left\| \sum_j c_j r_j \right\|_{\exp(L^2)} \lesssim \|c_j\|_{\ell^2}.$$  

Likewise, the following reformulation of Theorem 2.1 is frequently referred to as the Chang-Wilson-Wolff inequality.

2.6. Chang-Wilson-Wolff Inequality. We have this inequality valid for all Hilbert space valued martingales.

$$\|f\|_{\exp(L^2)} \lesssim \|S(f)\|_\infty.$$  

For $\alpha > 0$, let $\varphi_\alpha(x)$ be a symmetric convex function which equals $|x|(\log 3 + |x|)^\alpha$ for $|x|$ sufficiently large, depending upon $\alpha$. The Orlicz space $L^{\varphi_\alpha}$ is denoted as $L^{\varphi_\alpha} = L(\log L)^\alpha$. These are the spaces that we used in the statement of our Main Theorems.

The point of interest here is following Proposition, which is in the standard references we have cited.

2.7. Proposition. For $0 < \alpha < \infty$, the two Orlicz spaces $L(\log L)^\alpha$ and $\exp(L^{1/\alpha})$ are Banach spaces which are dual to one-another.

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2. We are only interested in measuring the behavior of functions for large values of $f$, so this requirement is sufficient. For $\alpha > 1$, we can insist upon this equality for all $x$.

3. For $\alpha \geq 1$, we can take this as the definition for all $|x| \geq 0$. 
In particular, the dual to $L(\log L)$ is $\exp(L)$, and the dual to $\exp(L^2)$ is $L \sqrt{\log L}$.

**Discrepancy.** We recall some definitions and facts about Discrepancy which are well represented in the literature, (Roth, 1954; Schmidt, 1977a; Beck and Chen, 1987).

Each dyadic interval has a left and right half, $I_{\text{left}}, I_{\text{right}}$ respectively, which are also dyadic. Define the Haar function associated with $I$ by
\[ h_I \equiv -1_{I_{\text{left}}} + 1_{I_{\text{right}}} \]
Note that this is an $L_\infty$ normalization of these functions.

In dimension $d$, a **dyadic rectangle** is a product of dyadic intervals, thus an element of $D^d$. A Haar function associated to $R$ we take to be the product of the Haar functions associated with each side of $R$, namely for $R_1 \times \cdots \times R_d$,
\[ h_{R_1 \times \cdots \times R_d}(x_1, \ldots, x_d) \equiv \prod_{t=1}^d h_{R_t}(x_t). \]

We will concentrate on rectangles of a fixed volume, contained in in $[0,1]^d$.

We call a function $f$ an $r$ **function with parameter** $\vec{r} = (r_1, \ldots, r_d)$ if $\vec{r} \in \mathbb{N}^d$, and
\[ f = \sum_{R \in \mathcal{R}_r} \varepsilon_R h_R, \quad \varepsilon_R \in \{\pm 1\}. \]

In this sum, we let
\[ \mathcal{R}_r \equiv \{R = (R_1, \ldots, R_d) \mid R \text{ dyadic}, |R_t| = 2^{-r_t}, 1 \leq t \leq d\}. \]
We will use $f_{\vec{r}}$ to denote a generic $r$ function. A fact used without further comment is that $f_{\vec{r}}^2 \equiv 1$.

Let $|\vec{r}| = \sum_{t=1}^d r_t = n$, which we refer to the index of the $r$ function. And let
\[ \mathbb{H}_n^d \equiv \{\vec{r} \in \{1, \ldots, n\}^d \mid |\vec{r}| = n\}. \]
It is fundamental to the subject that $\mathbb{H}_n^d \simeq n^{d-1}$. We refer to $\{f_{\vec{r}} \mid r \in \mathbb{H}_n^d\}$ as hyperbolic $r$ functions.

The next four Propositions are standard.

2.9. **Proposition.** For any selection $\mathcal{A}_N$ of $N$ points in the unit cube the following holds. Fix $n$ with $2N < 2^n \leq 4N$. For each $\vec{r} \in \mathbb{H}_n^d$, there is an $r$ function $f_{\vec{r}}$ with
\[ \langle D_N, f_{\vec{r}} \rangle \gtrsim 1. \]
Proof. There is a very elementary one dimensional fact: For all dyadic intervals $I$,

\[(2.10) \quad \int_{0}^{1} x \cdot h_{I}(x) \, dx = \frac{1}{4} |I|^2.\]

This immediately implies that in any dimension

\[(2.11) \quad \langle \| [0, \bar{x}] \|, h_{R}(\bar{x}) \rangle = 4^{-d} |R|^2.\]

Recall that $\mathcal{A}_{N}$, the distribution of $N$ points in the unit cube, is fixed. Call a cube $R \in R_{r}$ good if $R$ does not intersect $\mathcal{A}_{N}$, otherwise call it bad. Set

\[(2.12) \quad f_{R} := \sum_{R \in R_{r}} \text{sgn}(\langle D_{N}, h_{R} \rangle) h_{R}.\]

Each bad rectangle contains at least one point in $\mathcal{A}_{N}$, and $2^n \geq 2N$, so there are at least $N$ good rectangles. Moreover, one should observe that the counting function $\#(\mathcal{A}_{N} \cap [0, \bar{x}])$ is orthogonal to $h_{R}$, for each good rectangle $R$. Thus, by (2.11),

$$\langle D_{N}, h_{R} \rangle = -N \langle \| [0, \bar{x}] \|, h_{R}(\bar{x}) \rangle = N 2^{-2n-2d} \lesssim -2^{-n}.$$ 

Hence, we can estimate

$$\langle D_{N}, f_{R} \rangle \geq \sum_{\substack{R \in R_{r} \\ R \text{ is good}}} \langle D_{N}, h_{R} \rangle \gtrsim 2^{-n} \#(R \in R_{r} \mid R \text{ is good}) \geq 1.$$ 

And so our proof is complete.

\[\square\]

2.13. Proposition. Let $f_{\bar{x}}$ be any $r$ function with $|\bar{s}| > n$. We have

$$|\langle D_{N}, f_{\bar{x}} \rangle| \lesssim N 2^{-|s|}.$$ 

Proof. This is a brute force proof. Consider the linear part of the Discrepancy function. By (2.10), we have

$$|\langle N \prod_{j=1}^{d} x_{ij}, f_{\bar{s}} \rangle| \lesssim N 2^{-|s|},$$

as claimed.

Consider the part of the Discrepancy function that arises from the point set. Observe that for any point $\bar{x}_{0}$ in the point set, we have

$$|\langle 1_{[\bar{x}_{0}, \bar{x}]}, f_{\bar{s}} \rangle| \lesssim 2^{-|s|}.$$
Indeed, of the different Haar functions that contribute to $f_{\vec{s}}$, there is at most one with non-zero inner product with the function $1_{[\vec{s},\vec{s}]}(\vec{x}_0)$ as a function of $\vec{x}$. It is the one rectangle which contains $x_0$ in its interior. Thus the inequality above follows. Summing it over the $N$ points in the point set finish the proof of the Proposition. 

In two dimensions, the decisive product rule holds. It is as follows, and we omit the proof.

2.14. **Proposition.** In dimension $d = 2$ the following holds. Let $\vec{r}_1, \ldots, \vec{r}_k$ be elements of $\mathbb{H}^2_n$ where one of the vectors occurs an odd number of times. Then, the product $\prod_{j=1}^k f_{\vec{r}_j}$ is also a $r$-function. If the $\vec{r}_j$ are distinct and $k \geq 2$, the product has index larger than $n$.

2.15. **Proposition.** In dimension $d = 2$ the following holds. Fix a collection of $r$ functions $\{f_{\vec{r}} \mid \vec{r} \in \mathbb{H}^2_n\}$. Fix an integer $2 \leq v \leq n$ and $\vec{s}$ with $|\vec{s}| \geq n + v - 1$. Let $\text{Count}(\vec{s}; v)$ be the number of ways to choose distinct $r_1, \ldots, r_v \in \mathbb{H}^2_n$ so that $\prod_{w=1}^v f_{\vec{r}_w}$ is an $\vec{s}$ function. We have

$$\text{(2.16)} \quad \text{Count}(\vec{s}; v) = \binom{|\vec{s}| - n - 1}{v - 2}.$$

**Proof.** Fix a vector $\vec{s}$ with $|\vec{s}| > n$, and suppose that

$$\prod_{w=1}^v f_{\vec{r}_w}$$

is an $\vec{s}$ function. Then, the maximum of the first coordinates of the $\vec{r}_w$ must be $s_1$, and similarly for the second coordinate. Thus, the vector $s$ completely specifies two of the $\vec{r}_w$.

The remaining $v - 2$ vectors must be distinct, and take values in the first coordinate that are less than $s_1$, and in the second less than $s_2$. The hyperbolic assumption then says that there are at most $|\vec{s}| - n - 1$ possible choices for these vectors, and from them we select $v - 2$. This completes the proof. 

The next Lemma is elementary, and has probably been observed before, but is not a standard fact. For integers $1 \leq v \leq n$, let

$$\text{(2.17)} \quad G_v := \sum_{\{\vec{r}_1, \ldots, \vec{r}_v\} \subset \mathbb{H}^2_n} \prod_{w=1}^v f_{\vec{r}_w}.$$

This sum is over all subsets of $\mathbb{H}^2_n$ of cardinality $v$, that is the vectors $\{\vec{r}_1, \ldots, \vec{r}_v\}$ are all distinct.
2.18. Lemma. For odd integers \( k \) we have

\[
(2.19) \quad \left[ \sum_{\vec{r} \in H_2^n} f_{\vec{r}} \right]^k = \sum_{v=1}^{\min(k, n)} \gamma(k, n) \cdot n^{(k-v)/2} \cdot G_v,
\]

\[
(2.20) \quad 0 < \gamma(k, n) \leq \frac{k!}{(k-v)!} [C_0(k-v)]^{(k-v)/2} \leq k! C^{k-v}.
\]

where \( C_0 \) and \( C \) are absolute constants.

Proof. The point is that \( f_{\vec{r}}^2 \equiv 1 \), and that \( k \) is odd. Thus, even products of distinct \( r \) functions cannot arise in this product. (There would be a different formulation for \( k \) even.)

For a fixed subset \( \{\vec{r}_v^1, \ldots, \vec{r}_v^v\} \subset H_2^n \), with \( v \) odd and at most \( k \), we consider the number of ways that the product

\[
\prod_{u=1}^{v} f_{\vec{r}_u^v}
\]

can arise in the left hand side of (2.19). From the \( k \) terms on the left in (2.19), we choose \( v \) from which we take one of the pre-specified \( r \) functions \( f_{\vec{r}_1^v}, \ldots, f_{\vec{r}_v^v} \) in some order. There are \( k!/(k-v)! \) ways to do this.

In the remaining \( k-v \) terms, an even number of terms, we need to estimate \( \gamma'(k-v) \), the number of ways to assign \( r \) functions to these, so that they can be divided into distinct pairs of equal \( r \) functions. This is a somewhat tricky combinatorial problem. Let us observe that this problem arises in a concrete way in one approach to the Khintchine inequalities.

Let \( \phi_r \) be the \( r \) function as in (2.8), when all the choices of signs \( \epsilon_R \) are identically one. Thus, \( \phi_r \) are Rademacher functions. Then, the term we are seeking to bound is

\[
\gamma'(k-v) = \int_{[0,1]^P} \left[ \sum_{|j|=n} \phi_{r_j} \right]^{k-v} \, dx
\]

Indeed, expanding the \((k-v)\)th power on the right, the integral of the product below

\[
\int_{[0,1]^P} \prod_{u=1}^{k-v} \phi_{\vec{r}_u^v} \, dx
\]

will be non-zero iff the \( \{\vec{r}_1^v, \ldots, \vec{r}_{k-v}^v\} \) can be written as a disjoint union of pairs of equal vectors.
Classical proofs of the Khintchine inequality, see (Lindenstrauss and Tzafriri, 1977), estimate this norm directly. We can use the well-known best constants to estimate as below. The best constants in the range we are interested were established in (Young, 1976). The only information that we use here is the asymptotic order, which follows from other sources, such as (Chang et al., 1985; Wang, 1991; Pipher, 1986), as well as references therein. For an absolute constant $C$ we have

$$\int_{[0,1]^2} \left[ \sum_{|r|=n} \varphi_r \right]^{k-v} dx = \left\| \sum_{|r|=n} \varphi_r \right\|^{k-v}_{k-v} \leq C \sqrt{k-v} \cdot \sqrt{n}^{k-v}.$$  

This completes our proof.

\[\square\]

3. The Proof of Halász’ Theorem

The point of this section is to provide a new proof of Halász’ Theorem, on the $L^1$ norm of the Discrepancy function in two dimensions. It suffices to prove this for sufficiently large $N$. The proof is by way of duality. Fix the point distribution $\mathcal{A}_N \subset [0, 1]^2$. Set $2N < 2^n \leq 4N$, so that $n \approx \log N$. Proposition 2.9 provides us with $r$ functions $f_r$ for $r \in \mathbb{H}_n^2$. We show that

$$\langle D_N, \Psi \rangle \gtrsim \sqrt{n},$$

where

$$\Psi := \sin(\epsilon \cdot n^{-1/2} \sum_{r \in \mathbb{H}_n} f_r).$$

Here $\epsilon > 0$ is a small positive constant. It needs to be smaller than the $1/C_1$ for $C_1$ as in (3.3). Of course $\Psi$ is a bounded function, so this will prove the Theorem.

For any fixed $n$, $n^{-1/2} \sum_{r \in \mathbb{H}_n} f_r$ is a bounded function, hence the Taylor series expansion of $\Psi$ is absolutely convergent. Lemma 2.18 shows that the Taylor series for $\Psi$ satisfies

$$\Psi = \sum_{k \text{ odd}} \frac{(-1)^{k+1/2}}{k!} n^{-k/2} e^k \left[ \sum_{r \in \mathbb{H}_n} f_r \right] k$$

$$= \sum_{k \text{ odd}} (-1)^{(k+1)/2} e^{k} \sum_{v=1}^{n} \gamma(k, v) n^{-v/2} G_v$$

$$= \sum_{v=1}^{n} \delta(v) n^{-v/2} G_v.$$

(3.2)
In this last display, we can take \( \delta(v) \) to be a sequence of constants with \( \delta(1) = C_0 \varepsilon \), for a choice of constant \( C_0 \), which depends only on the constant \( C \) in (2.20). For \( v > 1 \),

\[(3.3) \quad |\delta(v)| \leq (C_1 \varepsilon)^v, \quad v > 1.\]

for an absolute constant \( C_1 \).

We turn our attention to the terms in (3.2). Now, by construction, we have

\[(3.4) \quad \langle D_N, \delta_0 n^{-1/2} G_1 \rangle \gtrsim \delta_0 n^{-1/2} \sum_{\vec{r} \in \mathcal{H}_v^2} \langle D_N, f_{\vec{r}} \rangle \gtrsim \delta_0 n^{1/2} \approx \varepsilon \sqrt{\log N}.\]

As for the terms \( 3 \leq v \leq n \), note that by Proposition 2.13, Proposition 2.15 and the definition of \( G_v \), we have

\[
|\langle D_N, G_v \rangle| \leq N \sum_{s=n+v-1}^{2n} 2^{-s} \left( s - n - 1 \right) v - 2.
\]

And so we estimate as follows, using (3.2) and (3.3). Here is convenient that the sum is only over odd \( v \geq 3 \).

\[
\sum_{\substack{v=3 \\text{odd} \\ \ \ \ \ \ n \\ v \ odd \ \ n}} (C_1 \varepsilon)^v n^{-v/2} |\langle D_N, G_v \rangle| \leq N \sum_{\substack{v=3 \\text{odd} \\ \ \ \ \ n \\ v \ odd \ \ n}} \sum_{s=n+v-1}^{2n} (C_1 \varepsilon)^v 2^{-s} n^{-v/2} \left( s - n - 1 \right) v - 2.
\]

\[
\leq N n^{-1} \sum_{s=n+3}^{2n} 2^{-s} \sum_{v=0}^{n} (C_1 \varepsilon)^v n^{-v/2} \left( s - n - 1 \right) v.
\]

\[
\leq n^{-1} \sum_{s=0}^{n} 2^{-s} (1 + (C_1 \varepsilon)^2 n^{-1/2})^s.
\]

This estimate holds for \( n \) sufficiently large, and tends to zero with \( n \), so we can combine it with (3.4) to prove the Halász Theorem for sufficiently large \( N \).

4. The Proof of Theorem 1.1

The proof is by duality. The Orlicz space in which we seek an estimate of \( D_N \) is \( L^1(\log L)^{(d-2)/2} \). The dual to this space, by Proposition 2.7, is the exponential Orlicz space \( \exp(L^{2/(d-1)}) \). Namely we define an \( \Phi \) so that

\[\langle D_N, \Phi \rangle \gtrsim (\log N)^{(d-1)/2}, \quad \|\Phi\|_{\exp(L^{2/(d-1)})} \leq 1.\]
Given the point set $\mathcal{A}_N \subset [0, 1]^d$, we fix $2N \leq 2^n < 4N$, and take the $r$ functions as in Proposition 2.9. We construct our test function as follows.

For $\vec{s} \in \mathbb{N}^{d-2}$, let

$$F_s := \sum_{r \in H^d \mid r_j = s_j, \ 1 \leq j \leq d-2} f_r$$

That is, we sum over vectors $\vec{r}$ which equal $\vec{s}$ in the first $d-2$ coordinates. Clearly, we need to require at a minimum that $|\vec{s}| < n$, so that there are only $\leq n^{d-2}$ possible values $\vec{s}$ for which the definition above is non-zero. In dimension $d = 2$ we interpret this as no restriction on the coordinates. In dimension $d = 3$ only the first coordinate is restricted.

Our test function is

$$\Phi := n^{-(d-2)/2} \sum_{|s| \leq \frac{3}{4} n} \sin(\epsilon n^{-1/2} F_s).$$

Here $\epsilon > 0$ is the small positive constant of the previous section. The argument in the two dimensional case can be modified to show that

$$\langle D_N, \sin(n^{-1/2} F) \rangle \gtrsim n^{1/2}, \quad |s| \leq \frac{3}{4} n.$$ 

Therefore, it follows that $\langle D_N, \Phi \rangle \gtrsim n^{(d-1)/2}$.

Let us check the integrability properties of $\Phi$. That is, using Proposition 2.4, we should verify that

$$(4.1) \quad ||\Phi||_p \lesssim p^{(d-2)/2}.$$ 

We shall find that $d - 2$ applications of the Littlewood–Paley are enough to prove this estimate.

Consider the function $\Phi$ with the variables $x_2, \ldots, x_d$ fixed. The terms below form a martingale difference sequence in $\sigma_1$:

$$\sum_{s_1 = \sigma_1} n^{-(d-2)/2} \sin(\epsilon n^{-1/2} F_s).$$

The square function of this martingale, which we denote by $S_1(\Phi)$ is

$$S_1(\Phi) = n^{-(d-2)/2} \left[ \sum_{\sigma_1} \left| \sum_{s_1 = \sigma_1} \sin(\epsilon n^{-1/2} F_s) \right|^2 \right]^{1/2}.$$
A useful (but not absolutely essential) point to observe is that the term on the right above is a Hilbert space valued function

\[ S_1(\Phi) = n^{-(d-2)/2} \left\| \sum_{\| \vec{s} \| \leq n^{-1/2} F_{\vec{s}}} \sin(\epsilon n^{-1/2} F_{\vec{s}}) \mid \sigma_1 \in \mathbb{N} \right\|_{F(\sigma_1)} \]

Continuing by induction, holding the variables \(x_3, \ldots, x_d\) constant above, we see that the expression above is a Hilbert space martingale, with martingale difference sequence

\[ \left\{ \sum_{\| \vec{s} \| \leq n^{-1/2} F_{\vec{s}}} \sin(\epsilon n^{-1/2} F_{\vec{s}}) \mid \sigma_1, \sigma_2 \in \mathbb{N} \right\}. \]

We accordingly define

\[ S_2(\Phi) = \left[ \sum_{\sigma_1} \sum_{\sigma_2} \left| \sum_{\| \vec{s} \| \leq n^{-1/2} F_{\vec{s}}} \sin(\epsilon n^{-1/2} F_{\vec{s}}) \right|^2 \right]^{1/2}. \]

This notation is extended to \(d - 2\), where

\[ S_{d-2}(\Phi) = n^{-(d-2)/2} \left( \sum_{\| \vec{s} \|} \sin^2(\epsilon n^{-1/2} F_{\vec{s}}) \right)^{1/2} \leq 1. \]

We now apply the Littlewood-Paley inequalities, as phrased in Theorem 2.1, to conclude that for \(2 \leq p < \infty\),

\[ \| \Phi \|_p \leq C \sqrt{p} \| S_1(\Phi) \|_p \leq C^2 p \| S_2(\Phi) \|_p \leq \ldots \leq C^{p-2} p^{(d-2)/2} \| S_{d-2}(\Phi) \|_p \lesssim p^{(d-2)/2}. \]

That is, (4.1) holds, finishing our proof. Our proof of the main result is complete.

We comment, that with the \(L^\infty\) bound on the iterated square function, especially in (4.2), we are employing a variant of the Chang-Wilson-Wolff inequality as described in (Pipher, 1986; Fefferman and Pipher, 1997).

In dimension \(d = 3\), an outstanding conjecture is that there is the universal estimate

\[ \| D_N \|_1 \gtrsim \log N, \]
valid for all point sets $\mathcal{A}_N$. To attempt to prove this estimate, it would be natural to consider the function

$$\Psi := \sin\left(n^{-1/2} \sum_a \sin\left(n^{-1/2} \sum_{r \in H_N} f_r\right)\right).$$

But there are difficulties to this line of approach that are similar to those that complicate the argument in J. Beck’s paper (Beck, 1989). Indeed, any improvement in our main Theorem, in dimension $d = 3$ appears to be of interest.

5. A Remark on Hardy Spaces and the Discrepancy Function

For functions $f$ on $\mathbb{R}^d$, and $1 \leq t \leq d$ we define the integration operator in coordinate $t$ by

$$\text{Int}_t f(x) := \int_0^1 f(x_1, \ldots, x_d) \, dx_t.$$  \((5.1)\)

For a point set $\mathcal{A}_N$ of cardinality $N$, with Discrepancy function $D_N$, we define

$$\tilde{D}_N = \left(1 - \sum_{T \subseteq \{1, \ldots, d\}} (-1)^{|T|} \prod_{i \in T} \text{Int}_i \right) D_N.$$  \((5.2)\)

Here Id is the Identity operator. We prove

5.3. **Theorem.** For $d \geq 2$ and all $0 < p \leq 1$, we have the estimate

$$\|\tilde{D}_N\|_{H^p} \gtrsim (\log N)^{(d-1)/2}.$$  

Here, $H^p$ is the dyadic real-valued Hardy space of $d$ parameters on $\mathbb{R}^d$.

The Hardy space we have in mind in this Theorem is given in the next Definition.

5.4. **Definition.** The dyadic real-valued Hardy space of $d$ parameters on $\mathbb{R}^d$ is the set of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that the following norm is finite.

$$\|f\|_{H^p} := \|Mf\|_p,$$  \((5.5)\)

$$Mf(x) := \sup_{\substack{x \in R \text{ dyadic} \atop R \text{ dyadic}}} \frac{1}{|R|} \int_R f(y) \, dy.$$  \((5.6)\)

Here, at a given point $x$, the supremum is over all dyadic rectangles in $\mathbb{R}^d$ that contain $x$. Note that we do not take the absolute value of the function $f$. 
The Hardy space in question is a ‘real-variable’ extension of the classical space of analytic functions introduced by G. H. Hardy. It is a subtle object, whose crucial properties have been identified by Alice Chang, Robert Fefferman, (Chang and Fefferman, 1980; 1985), Jean-Lin Journé (Journé, 1986) and Jill Pipher (Pipher, 1986), among others. The dyadic setting, of interest to us, is specifically addressed in the paper (Bernard, 1979). See especially the paper (Chang and Fefferman, 1985), as well as the references in these papers.

There is an alternative definition, which we recall here. Let us define the *Square Function* by

\[ S(f) := \left[ \sum_{R} \frac{|\langle f, h_R \rangle|^2}{|R|^2} 1_{R} \right]^{1/2}. \]

The sum is over dyadic rectangles in \( \mathbb{R}^d \).

**An Equivalent Definition of Real Valued Hardy Space.** The following equivalence of norms holds.

\[ \|M f\|_p \simeq \|S(f)\|_p , \quad 0 < p \leq 1. \]

It is well-known that Hardy spaces are an appropriate substitute for \( L^p \) spaces for a variety of issues concerning Harmonic Analysis. In particular, there is a third equivalent definition of the Hardy space norm in terms of Hilbert transforms, which we omit in this discussion. There is a fourth ‘atomic decomposition’ approach that we also omit (See (Fefferman, 1985)), referring all these issues to the cited references.

The point of the subtraction in (5.2) is that \( D_N \) is not a priori a member of Hardy space, but \( \tilde{D}_N \) is. Note that the conclusion of the Theorem provides partial support for the Conjecture below, which is an extension of the \( L^1 \) Conjecture mentioned in the introduction. There is nothing known about this Conjecture, even in dimension \( d = 2 \).

**Conjecture on the \( L^p \) norm of \( D_N \).** In dimension \( d \geq 2 \) we have the estimate

\[ \|D_N\|_p \gtrsim (\log N)^{(d-1)/2} , \quad 0 < p < 1. \]

**Proof of Theorem 5.3.** We use the following elementary fact. Let \( G_1, \ldots, G_j \) be measurable subsets of a probability space \( (\Omega, \mathcal{P}) \), with \( \mathcal{P}(G_j) \geq \frac{1}{2} \) for all \( 1 \leq j \leq J \). Then,

\[ \mathcal{P}\left( \sum_{j=1}^{J} 1_{G_j} > J/4 \right) \geq \frac{1}{4}. \]
Indeed, suppose this is not the case, then we have the contradiction

\[ \frac{J}{2} \leq \left\| \sum_{j=1}^{J} 1_{G_j} \right\|_1 \]

\[ \leq J/4 + J \cdot \mathbb{P} \left( \sum_{j=1}^{J} 1_{G_j} > J/4 \right) < \frac{J}{2}. \]

It follows that we have

\[ (5.7) \quad \left\| \sum_{j=1}^{J} 1_{G_j} \right\|_p \geq J, \quad 0 < p \leq \infty. \]

Fix a point set \( \mathcal{A}_N \), and let \( 2N \leq 2^n < 4N \). For a vector \(|\vec{r}| = n\), let

\[ \mathcal{G}_\vec{r} := \{ R : |R_t| = 2^n, 1 \leq t \leq d, R \cap \mathcal{A}_N = \emptyset \}. \]

These rectangles, which avoid the point set \( \mathcal{A}_N \) are the ‘good rectangles’ in the proof of Proposition 2.9. Set \( G_{\vec{r}} = \bigcup_{R \in \mathcal{G}_\vec{r}} R \). By the pigeonhole principle, \( |G_{\vec{r}}| \geq \frac{1}{2} \). Moreover, a standard computation, see (2.11), shows that

\[ \frac{|\langle \tilde{D}_N, h_R \rangle |^2}{|R|^2} \geq 1, \quad R \in \mathcal{G}_{\vec{r}}. \]

The implied constant only depends upon dimension.

We conclude a lower bound on the Square Function of \( \tilde{D}_N \):

\[ S(\tilde{D}_N) \geq \left[ \sum_{|\vec{r}|=n} 1_{G_{\vec{r}}} \right]^{1/2}. \]

As there are \( n^{d-1} \approx (\log N)^{d-1} \) choices of \( \vec{r} \), we conclude from (5.7) that we have

\[ \|S(\tilde{D}_N)\|_p \geq n^{(d-1)/2}, \quad 0 < p \leq 1. \]

By the definition of the Hardy space norm, this completes our proof. \( \square \)

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