Spectral properties of the $t-J$ model in the presence of hole-phonon interaction

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Abstract

We examine the effects of electron-phonon interaction on the dynamics of the charge carriers doped in two-dimensional (2D) Heisenberg antiferromagnet. The $t$-$J$ model Hamiltonian with a Fröhlich term which couples the holes to a dispersionless (optical) phonon mode is considered for low doping concentration. The evolution of the spectral density function, the density of states, and the momentum distribution function of the holes with an increase of the hole-phonon coupling constant $g$ is studied numerically. As the coupling to a phonon mode increases the quasiparticle spectral weight decreases and a “phonon satellite” feature close to the quasi-particle peak becomes more pronounced. Furthermore, strong electron-phonon coupling smears the multi-magnon resonances (“string states”) in the incoherent part of the spectral function. The jump in the momentum distribution function at the Fermi surface is reduced without changing the hole pocket volume, thereby providing a numerical verification of Luttinger theorem for this strongly interacting system. The vertex corrections due to electron-phonon interaction are negli-
gible in spite of the fact that the ratio of the phonon frequency to the effective bandwidth is not small.
I. INTRODUCTION

Since the discovery of high-temperature superconductivity in copper-oxide based materials, experimental studies have revealed a lot of evidence that interaction between electrons and lattice vibrations plays an important role in these compounds. The changes in position and width of the phonon peaks below superconducting transition temperature measured by Raman spectroscopy and by neutron scattering techniques, the existence of the isotope effect which varies as a function of both doping and rare earth ion substitution, clearly demonstrate that coupling between charge carriers and phonon modes in high-$T_c$ superconductors is not negligible. Nevertheless, theoretical studies of the effects of electron-phonon interaction on the dynamics of the charge carriers in electron-correlated systems are far from complete. The main problem for proper treatment of electron-phonon interaction in high-$T_c$ superconductors is that charge carrier motion in an antiferromagnetic background (or spin-fluctuating background at higher doping concentration) is strongly affected by electron-magnon interactions itself. Including the electron-phonon interaction deals with quasiparticles renormalized by interaction with magnons. As a consequence the renormalized bandwidth becomes comparable to phonon frequencies and the ‘classical’ Migdal-Eliashberg approach to the electron-phonon problem in metallic systems seems beyond the region of application.

In the present paper, we add a Fröhlich term to the $t-J$ model Hamiltonian to study the quasiparticle properties in the presence of hole-phonon interaction. This is particularly interesting, since it is widely believed that the $t-J$ model incorporates essential features of electron systems with strong local repulsion between electrons, characteristic of high $T_c$ copper oxides. We examine the properties of this model at low doping concentrations (near half-filling) where analytical and numerical results for the $t-J$ model are well-known. The optical phonon mode in the copper-oxide plane is assumed to have a substantial influence on the dynamics of doped holes in cuprate compounds. In the past, quasiparticle properties in the presence of electron-phonon interaction were studied by Engelsberg and Schrieffer for
a weakly correlated model of conventional metal. The authors examined the spectral density function \( A(\vec{k}, \omega) \) of an electron for phonon spectra of the Einstein and Debye forms. They found that spectral density function exhibits several branches of excitations rather than a single branch of a dressed electron. Our calculations of the quasiparticle energy dispersion show the same for the present model of a strongly correlated system with hole-phonon interaction. Recently the mass renormalization due to a coupling to optical phonons was studied for strongly correlated electrons by Ramšak et al. The authors found that the phonon-induced mass renormalization of a single hole that propagates in the \( t-J \) model on the scale \( 2J \) is much larger than that in the corresponding uncorrelated model for \( J \gtrsim t \). The mass enhancement, increasing with \( J/t \) ratio, is due to the slow motion of a spin polaron, which makes hole-phonon interaction more effective. The hole excursions from the center of the spin polaron are restricted to a few lattice sites and the bandwidth is on scale \( t^2/J \), as long as \( J/t \) is large. On the other hand, when \( J \ll t \) the confining antiferromagnetic potential becomes weak and a hole performs large radius incoherent excursions on scale \( t \). Therefore, the mass enhancement induced by the hole-phonon interaction becomes weaker and the coherent part of the spectral weight tends to zero.

In our paper, we study various quasiparticle properties as well as the mass renormalization due to hole-phonon interaction in the limit of a finite but small concentration of doped holes. The experimentally measured optical phonon frequencies are close to 50 meV and typical values of \( J \) are of the order of 100 meV. We assume the optical phonon frequency, \( \Omega \), to be \( \Omega = 0.5J \). The \( t-J \) model Hamiltonian appended with Fröhlich term will be denoted as the \( t-J-g \) model. Our paper is organized as follows. In Sec. II, the quasiparticle residue and the mass renormalization constant are calculated for \( t-J-g \) model. The incoherent part of the spectrum and Luttinger’s theorem are studied in Sec. III. Vertex corrections due to the hole-phonon interaction are analyzed in Sec. IV. Section V presents the optical conductivity in the \( t-J-g \) model. A summary is given in Sec. VI.
II. FORMULATION AND QUASI-PARTICLE SPECTRAL WEIGHT

To analyze the interaction between optical phonons and charge carriers in the copper oxide planes, we consider a two-dimensional $t-J$ model Hamiltonian appended by a Fröhlich term\textsuperscript{17} with the dispersionless (optical) phonon mode $\Omega$ and the coupling constant $g$. Using linear spin-wave approximation through Holstein-Primakoff transformation\textsuperscript{18} and spinless-fermion/ Schwinger-boson representation\textsuperscript{5} of the electron operators on a site $i$ with spin $\sigma$, $c_{i\sigma} \rightarrow h_i S_i^{-}$ and $c_{i\uparrow} \rightarrow h_i^{\dagger}$ the Hamiltonian can be written:

$$H_{tJg} = -t \sum_{\langle i,j \rangle} \left[ h_i h_j^{\dagger} S_j^{-} + S_i^{-} h_i h_j^{\dagger} + h.c. \right] + J \sum_{\langle i,j \rangle} \rho_i \cdot \tilde{S}_i \cdot \tilde{S}_j \rho_j^{e} - \mu \sum_i \rho_i^{h} + g \sum_i h_i^{\dagger} h_i (b_i + b_i^{\dagger}) + \Omega \sum_i b_i^{\dagger} b_i ,$$

where $h_i$ is a hole creation operator on site $i$ and $\tilde{S}_i$ is the on-site electron spin operator. ($S_i^{-}$ turns down the spin of the electron on site $i$.) The summation is over the nearest neighbor sites $\langle i, j \rangle$ of the square lattice, $b_i^{\dagger}$ is an optical phonon creation operator at site $i$. $\rho_i^{h} \equiv h_i^{\dagger} h_i$ is the local density operator of the spinless hole, while $\rho_i^{e} = 1 - \rho_i^{h}$ is the local density operator of electron under the single occupancy constraint. Further, $\mu$ is the chemical potential related to the concentration of the doped holes: $\langle \rho_i^{h} \rangle = x$ (at half-filling $x = 0$).

Since $t > J$ would correspond to a situation in cuprates, the usual perturbation approximation does not work in the strongly coupled $t-J$ model. However, it has been for some time known that the self-consistent noncrossing approximation for the hole self-energy gives a fairly accurate result because of the small vertex corrections. Namely, it was explained in Ref.\textsuperscript{19}, that the leading correction to the hole-spin wave vertex vanishes as a consequence of the electron (hole) spin conservation in the $t-J$ model. Mathematically, this fact is reflected in the special symmetry of the hole-spin wave interaction vertices $M_{k,q}$ and $N_{k,q}$, which leads to the identical vanishing of non-maximally crossing hole-spin wave vertex corrections\textsuperscript{19}. However, it is uncertain whether or not the vertex corrections due to the hole-phonon interaction are negligible in a strongly correlated electron system. Thus,
in the first place we assume that they are so, and later in Sec. IV this assumption will be supported by calculating the lowest order vertex corrections. To study the influence of hole-phonon coupling on the quasiparticle properties we treat the Hamiltonian within the noncrossing approximation for spin wave and phonon interactions. Fig. I shows four possible noncrossing diagrams contributing to the self-energy: spin wave emitting and absorbing diagrams, and phonon emitting and absorbing ones. The thick solid line is the dressed hole Green’s function, while the unperturbed hole propagator is denoted by the thin solid line. The dashed and wavy lines stand for the spin wave and phonon propagators, respectively. Here \( M, N \) are the hole-spin wave interaction vertices, while \( g \) is the hole-phonon vertex. After summing over the intermediate frequency, the hole self-energy at zero temperature becomes

\[
\Sigma^R(\mathbf{k}, \omega) = \sum_q \left\{ \frac{M^2}{\mathbf{k}, \mathbf{q}} \int_0^\infty dy \frac{A(\mathbf{k} - \mathbf{q}, y)}{\omega - y - E_{\mathbf{q}} + i\delta} + \frac{N^2}{\mathbf{k}, \mathbf{q}} \int_{-\infty}^0 dy \frac{A(\mathbf{k} - \mathbf{q}, y)}{\omega - y + E_{\mathbf{q}} + i\delta} \right. \\
+ \left. g^2 \int_0^\infty dy \frac{A(\mathbf{k} - \mathbf{q}, y)}{\omega - y - \Omega + i\delta} + g^2 \int_{-\infty}^0 dy \frac{A(\mathbf{k} - \mathbf{q}, y)}{\omega - y + \Omega + i\delta} \right\},
\]

(2)

where the superscript \( R \) indicates retarded functions which are analytic in the upper half-plane of \( \omega \), \( E_{\mathbf{q}} \) is the spin wave energy, and \( A(\mathbf{k}, \omega) \) is the spectral density function of the interacting hole propagator given as

\[
A(\mathbf{k}, \omega) = -\frac{1}{\pi} \text{Im} G^R(\mathbf{k}, \omega).
\]

(3)

The hole Green’s function is then obtained from the self-energy

\[
G^R(\mathbf{k}, \omega) = \frac{1}{\omega - \Sigma^R(\mathbf{k}, \omega) + \mu + i\delta}.
\]

(4)

The self-consistent integral equation, Eq. 2, is solved numerically (See Ref. 15 for detail). Table II shows the quasiparticle residues at three different \( \mathbf{k} \) points and the energy shift due to both the hole-spin wave and hole-phonon interactions. \( \alpha_{\text{calc}}^{(\pi/2, \pi/2)} \) in the fifth column is the quasiparticle residue at the point \( (\pi/2, \pi/2) \) calculated by perturbation theory. The numerically computed mass renormalization factor due to the hole-phonon interaction is \( \lambda_{\text{num}} \). As expected, the quasiparticle residue decreases as the coupling constant increases.
At the $(0,0)$ point the hole loses its quasiparticle property more rapidly, as the vanishingly small spectral weight indicates. This is because the state at the $(0,0)$ point is located at a high hole energy region and hence it is more vulnerable to additional decay from the hole-phonon interaction. The energy shift due to the hole-phonon interaction is generally small ($-0.975$ for $g = 2.0J$), compared with that from the pure hole-spin wave interaction ($-5.378$). This suggests that a perturbative treatment for the hole-phonon interaction is possible. According to Eq. [2], the hole self-energy is composed of two terms, one from the hole-spin wave interaction and the other from the hole-phonon interaction. Thus the self-energy can be written as $\Sigma(\vec{k}, \omega) = \Sigma_m(\vec{k}, \omega) + \Sigma_g(\vec{k}, \omega)$, where the former comes from the hole-spin wave interaction and the latter from the hole-phonon interaction. If we define $X$ and $Y$ as follows

$$X = 1 - \frac{\partial}{\partial \omega} \text{Re} \Sigma_m(\vec{k}, \omega)|_{g=0},$$

$$Y = -\frac{\partial}{\partial \omega} \text{Re} \left( \Sigma_g(\vec{k}, \omega) + \Sigma_m(\vec{k}, \omega) - \Sigma_m(\vec{k}, \omega)|_{g=0} \right),$$

the quasiparticle residue can be approximated by

$$a_k = \frac{1}{X + Y} = \frac{1}{X} \left[ 1 - \left( \frac{Y}{X} \right) + \left( \frac{Y}{X} \right)^2 + \cdots \right]$$

$$\approx \frac{1}{X} \left[ 1 - \left( \frac{Y}{X} \right) + \left( \frac{Y}{X} \right)^2 \right]. \quad (5)$$

The $X, Y$ are determined by

$$X = \frac{1}{a_k(g = 0)},$$

$$Y = \frac{1}{a_k} - \frac{1}{a_k(g = 0)}.$$

Substituting $X$ and $Y$ into Eq. [3] leads to the $a_{\text{calc}}^{(\pi/2, \pi/2)}$ in the fifth column of Table [I]. For coupling constants less than $1.5J$, agreement with the numerical results is quite satisfactory. However, the perturbative treatment based on the two term expansion breaks down for $g > 1.5J$, since $(Y/X)$ increases significantly for the strong hole-phonon interaction. In fact, $Y$ is the approximate hole-phonon mass renormalization constant $\lambda$ evaluated numerically.
on the basis of the $t - J$ model Hamiltonian. This is listed as $\lambda_{num}$ in the sixth column. The mass renormalization constant for a small coupling constant can also be computed from perturbation theory. We used a similar method employed by Ramšak et al.\textsuperscript{11}. Since the hole-phonon coupling constant is small compared with the hole-spin wave interaction strength, the effective mass of the hole can be computed using the lowest-order perturbation correction to the hole self-energy,

$$\omega_{\vec{k}}(g) - \omega_{\vec{k}} = \frac{1}{(2\pi)^2} \int d^2q \frac{a_{\vec{k} - \vec{q}}^2}{\omega_{\vec{k}} - \omega_{\vec{k} - \vec{q}} - \Omega},$$

where $\omega_{\vec{k}}(g)$ and $\omega_{\vec{k}}$ are the quasiparticle dispersion functions in the presence of the hole-phonon interaction and in its absence, respectively. Approximately, $\omega_{\vec{k}}$ is given by

$$\omega_{\vec{k}} = \frac{(\vec{k} - (\pm \pi/2, \pm \pi/2))^2}{2m_{eff}},$$

where $m_{eff}$ is the averaged effective mass of the hole near the bottom of the energy dispersion function, which is taken as $3.36/t$ according to Martinez et al.\textsuperscript{8}. From the second derivative around the points $\vec{k} = (\pm \pi/2, \pm \pi/2)$, we arrive at

$$\frac{1}{m_{eff}(g)} - \frac{1}{m_{eff}} \approx -\frac{16ag^2m_{eff}}{\pi} \int_0^{\pi} dq \frac{q^3}{(q^2 + 2m_{eff}\Omega)^3} = -\frac{1}{m_{eff}} \frac{2ag^2m_{eff}}{\pi\Omega} \left[4 \int_0^{y_m} dy \frac{y^3}{(y^2 + 1)^2} \right],$$

where $y_m$ is given by $\pi/\sqrt{2m_{eff}\Omega}$ and $\bar{a}$ is the averaged quasiparticle residue near the point $(\pi/2, \pi/2)$. Hence,

$$\lambda_{eff} = \frac{m_{eff}(g)}{m_{eff}} - 1 \approx \frac{1}{1 - 1.5ag^2m_{eff}/(\pi\Omega)} - 1 \approx 1.5ag^2m_{eff}/(\pi\Omega),$$

where a small $\lambda_{eff}$ is assumed. For $g/J = 0.5$ and 1.0, the substitution of $\bar{a} \approx 0.34$ gives 0.127 and 0.822, respectively. The former value is favorably compared with 0.102, the numerically calculated mass renormalization constant for $g/J = 0.5$. Clearly $g/J = 1.0$ is too strong to apply perturbation theory, because of the too large effective mass, $m_{eff} = 3.36/t$. Besides which, the large value of $\sqrt{2m_{eff}\Omega}$, namely, 1.16 makes the calculation very
sensitive to the upper bound, $y_m$, for the integration, leading to an additional difficulty. A straightforward calculation of $\lambda_{\text{eff}}$ for noninteracting electrons yields 0.016, 0.068, 0.167 and 0.342 for $g/J = 0.5, 1.0, 1.5$ and 2.0, respectively. In this case, the above mentioned difficulties do not occur because of the small effective mass, $m_{\text{eff}} = 1/2t$. The calculation shows that the mass renormalization factor is more enhanced for the strongly correlated electrons than the factor for the noninteracting electron system. This enhancement factor which is 6.4 for $g/J = 0.5$ is substantially large, compared with 3.5 which Ramšak et al. reported. The discrepancy between these two values originates from the use of different definitions for the mass enhancement parameter. The authors of this paper obtained the enhancement factor by explicitly computing the change in the curvature of the quasiparticle dispersion along the line $(\pi/2, \pi/2) - (0, 0)$, namely, $\lambda_{\parallel} = m_{\parallel}(g)/m_{\parallel} - 1$, while the effective mass in the present calculation, $\lambda_{\text{eff}} = \sqrt{m_{\parallel}(g)m_{\perp}(g)/m_{\parallel}m_{\perp} - 1}$, is averaged out around the point $(\pi/2, \pi/2)$. This indicates even larger mass enhancement along the line $(0, \pi) - (0, 0)$, which is consistent with the observation that the hole moves slowest along this direction.

III. INCOHERENT SPECTRUM AND LUTTINGER’S THEOREM

Because of the additional scattering channel, we expect that there are more spectra associated with the optical phonon excitations above the quasiparticle pole in the spectral density function. Fig. 2 shows the spectral density function for four different hole-phonon coupling constants $g$. As $g$ increases, the spin wave peaks become more suppressed, whereas the phonon peaks become stronger. Since the coupling strength for the hole-spin wave interaction is much larger than that for the hole-phonon interaction, the weak phonon features appear on top of the spin wave peaks as a satellite structure. For $g \geq 1.5J$, the phonon induced peaks are even sharper and larger in height than the spin wave peaks. Especially the appearance of the multiple phonon peaks below the chemical potential for $g = 2.0J$, compared with the spectral density function for $g = 0$, is clearly noticeable. The spikes due to finite size effects are gradually reduced, since those artificial peaks get smears out due
to the additional decay induced by the hole-phonon interaction. Fig. 3 presents the hole density of states for various coupling constants. The general behavior for the hole density of states is similar to that for the spectral density function, since in most of the Brillouin zone the spectral density function is very similar to that at the point \((\pi/2, \pi/2)\). As the hole-phonon coupling constant increases, the position of the strongest phonon peak right above the Fermi energy \((\omega = 0)\) is shifted upward, although this is not well pronounced due to finite size effects. This is understood based on a (lattice) polaronic formation. For the strong hole-phonon coupling constant, the hole is surrounded by an increasing number of phonons. Hence, approximately 2 phonons are involved in the formation of the polaron for \(g = 2.0J\), since the position of the peak is close to \(1.0J\) and the optical phonon frequency is \(0.5J\). The figure for \(g = 0\) shows, however, 2.5 spin waves with energy \(2J\) (i.e. from the top of the spin wave band where the density of spin wave states is sharply peaked) participate in a magnetic polaron. The incoherent spectrum below the Fermi energy in the density of states is crucial in satisfying Luttinger’s theorem, as far as the quasiparticle (coherent) contribution to the density of the occupied states (at \(T = 0\)) is substantially suppressed due to the strong hole-spin wave (and phonon) coupling. The momentum distribution function is shown in Fig. 4. We chose a much larger cluster (240 × 240) for \(n(\vec{k})\) calculation, than the original cluster (24 × 24) used for the self-consistent calculation of the self-energy in the \(\vec{k}\) points along the line \((\pi/2, \pi/2) - (0, \pi)\) or \(S - Y\). This provides detailed information about the behavior of \(n(\vec{k})\) in the vicinity of the Fermi surface. The four very elongated ellipses in the inset denote the Fermi surface. The distribution function shows a sharp drop at the same \(\vec{k}\) point for all the coupling constants we have studied, as seen in the figure. We also compared the doping concentration \(x\) computed from the spectral density function, with the ratio of the number of \(\vec{k}\) states inside the Fermi surface to the number in the entire (antiferromagnetic) Brillouin zone. The former yields \(x = 0.030, 0.031, 0.032, 0.030, 0.030\) for \(g = 2.0J, 1.5J, 1.0J, 0.5J, 0\), respectively, while the latter shows 0.032. They agree with each other within less than 4% on the average. These two features numerically verify Luttinger’s theorem in the \(t - J - g\) model for a small doping concentration. As
the coupling constant increases, the distribution function inside the Fermi surface decreases gradually from 0.36 to 0.23. This is due to the reduced quasiparticle residue for the strong hole-phonon interaction, as seen in Table I. At the same time, however, some density of the occupied states also appears outside the Fermi surface. This is because in the $t - J - g$ model the spectral density function $A(\vec{k}, \omega)$ at the $\vec{k}$ points outside the Fermi surface possesses a strong incoherent tail below the chemical potential.

IV. VERTEX CORRECTIONS

Since quasiparticles move coherently on a reduced energy scale $2J$, the Fermi energy is quite small for a small doping case, i.e. almost on the order of the phonon frequency or even less, which signals a possible breakdown of the standard strong (phonon) coupling theory. In the present section, the Migdal-type vertex corrections are studied in the $t - J - g$ model. Below $k$ is defined as $(\vec{k}, ik_n)$ where $k_n$ is a Matsubara frequency. Hence a summation over $k$ means the summation over both momenta $\vec{k}$ and Matsubara frequencies $k_n$. The lowest order vertex corrections to the hole-phonon interaction in Fig. 5 can be written as $g \Gamma(k, k + q)$, where

$$
\Gamma(k, k + q) = -\frac{1}{\beta} \sum_{k'} G(k')G(k' + q)B(k - k') . 
$$

(9)

$B(k - k')$ is the Green’s function of the optical phonon. Using the spectral representation for the hole Green’s function and converting the summation over Matsubara frequencies into a contour integration leads to

$$
\Gamma(k, k + q) = g^2 \sum_{\vec{k}, \vec{q}} \int \int d\omega d\omega' A(\vec{k}', \omega)A(\vec{k}' + \vec{q}, \omega') \frac{1}{\omega - \omega'} 
\times \left\{ \frac{1}{-ik_n + \omega + \Omega} [F(\omega) - N(\Omega) - 1] - \frac{1}{-ik_n + \omega - \Omega} [F(\omega) + N(\Omega)] 
- \frac{1}{-ik_n + \omega' + \Omega} [F(\omega') - N(\Omega) - 1] + \frac{1}{-ik_n + \omega' - \Omega} [F(\omega') + N(\Omega)] \right\} ,
$$

(10)

where $F(\omega)$ and $N(\omega)$ are the Fermi and Bose distribution functions respectively, and the $iq_n \to 0$ limit is taken first for numerical simplicity. Since the numerical computation for
$q \neq 0$ shows a similar result to $q = 0$ case, we restrict ourselves to the latter case. By taking the $T = 0$ and $\vec{q} \to 0$ limits and the analytic continuation $ik_n \to k_0 + i\delta$ as well as by noting that $\Theta(x) = 1 - \Theta(-x)$ and that (an integral is in the principal value sense)

$$
\int d\omega' A(\vec{k}', \omega') \frac{\omega}{\omega - \omega'} = \text{Re} G(\vec{k}, \omega),
$$

we arrive at

$$
\Gamma(k_0) = 2g^2 \sum_{\vec{k}} \int d\omega A(\vec{k}, \omega) \text{Re} G(\vec{k}, \omega)
\times \left\{ \frac{\Theta(\omega)}{-k_0 + \omega + \Omega - i\delta} + \frac{\Theta(-\omega)}{-k_0 + \omega - \Omega - i\delta} \right\}
= 2g^2 \sum_{\vec{k}} \int d\omega A(\vec{k}, \omega) \text{Re} G(\vec{k}, \omega)
\times \frac{1}{-k_0 + \omega + \Omega \text{sign}(\omega) - i\delta}.
$$

Therefore, the real and imaginary parts of the lowest order vertex correction are found as

$$
\text{Re} \Gamma(k_0) = 2g^2 \sum_{\vec{k}} \int_{-\infty}^{\infty} d\omega A(\vec{k}, \omega) \text{Re} G(\vec{k}, \omega)
\times \frac{1}{-k_0 + \omega + \Omega \text{sign}(\omega)}.
$$

and

$$
\text{Im} \Gamma(k_0) = \begin{cases} 
2\pi g^2 \sum_{\vec{k}} A(\vec{k}, k_0 + \Omega) \text{Re} G(\vec{k}, k_0 + \Omega) & \text{if } k_0 < -\Omega \\
0 & \text{if } -\Omega < k_0 < \Omega \\
2\pi g^2 \sum_{\vec{k}} A(\vec{k}, k_0 - \Omega) \text{Re} G(\vec{k}, k_0 - \Omega) & \text{if } k_0 > \Omega.
\end{cases}
$$

Fig. 6 and Fig. 7 show the real and imaginary parts of the lowest order vertex correction for several hole-phonon coupling constants, respectively. As the coupling strength increases, the real and imaginary parts of the vertex correction grow. In spite of the expectation that the Migdal approximation may break down due to the small hole band width $2J$ determined from the $t - J$ model, the vertex correction is much smaller than unity for up to $g \sim 2J$. For a noninteracting electron system, first order vertex correction has been known to be of the order of $\omega_D/E_F$, where $\omega_D$ is the Debye frequency and $E_F$ is the
Fermi energy. But, the adiabatic argument valid for a weakly interacting system breaks down for strongly correlated electrons, since the Fermi velocity is comparable to or even less than the phonon phase velocity in a considerable part of the Brillouin zone. According to Eq. 12, first order vertex correction is roughly proportional to the square of the quasiparticle residue. This indicates a possibility that the significant renormalization \((0.2 - 0.3)\) of the quasiparticle residue for a strongly correlated electron system makes the vertex correction much reduced, thereby accounting for the small vertex correction from the hole-phonon interaction in the \(t - J - g\) model. Hence, the present calculation numerically corroborates using the noncrossing approximation for the self-energy both in the hole-magnon and hole-phonon interactions.

V. OPTICAL CONDUCTIVITY IN THE \(t - J - g\) MODEL

The current operator after Bogoliubov transformations of the spin wave operators becomes\(^{12}\)

\[
J_x(\vec{q}) = \frac{2et}{N} \sum_{\vec{k}, \vec{p}} h_{\vec{k}} h_{\vec{p}}^* \left[ C_{\vec{k}, \vec{p}} \alpha_{\vec{k} - \vec{p} - \vec{q}}^+ + D_{\vec{k}, \vec{p}} \alpha_{\vec{k} + \vec{p} + \vec{q}}^- \right],
\]

(13)

where the bare current vertices \(C_{\vec{k}, \vec{p}}, D_{\vec{k}, \vec{p}}\) are defined as

\[
C_{\vec{k}, \vec{p}} = u_{\vec{k} - \vec{p}} \sin p_x + v_{\vec{k} - \vec{p}} \sin k_x,
\]

\[
D_{\vec{k}, \vec{p}} = v_{\vec{k} - \vec{p}} \sin p_x + u_{\vec{k} - \vec{p}} \sin k_x.
\]

(14)

Due to the special nature of the interaction vertices, it was established that the lowest order contribution to the optical conductivity dominates at \(\omega \neq 0\). Hence, we consider only the lowest order diagrams in the present study. The lowest order contribution comes from two diagrams in Fig. 8 owing to the structure of the current operator Eq. 13. In the combined limits of zero temperature \((T \to 0)\) and long wavelength electromagnetic radiation \((\vec{q} \to 0)\), the optical conductivity becomes

\[
\sigma_1(q_0) = \frac{\pi}{q_0} \left( \frac{2t}{N} \right)^2 \int_0^{q_0} d\omega \sum_k A(\vec{k}, \omega - q_0)
\]

on page 13.
\[ \times \sum_{\vec{p}} C_{k,p}^2 A(p, \omega - E_{\vec{k},p}). \]  

(15)

Fig. 9 presents the optical conductivity for the \( t - J - g \) model, as the hole-phonon coupling constant varies. As expected from the corresponding spectral density function, the contribution to the conductivity from the multi-magnon excitations (“string structure”) decreases, while strong absorption appears right above the \( 2J \) peak. This new peak in the absorption comes from the hole-phonon interaction, as can be seen in the corresponding spectral density function. Generally, the peak at \( 2J \) peak and higher energy incoherent spin wave peaks are suppressed and broadened in the presence of the strong hole-phonon interaction. This may be associated with the growth of featureless spectral weight at the mid-infrared region, when the CuO\(_2\) plane is doped with charge carriers\( ^{20} \).

VI. CONCLUSION

The influence of the hole-phonon interaction on various physical quantities was studied within the noncrossing approximation for the spin wave and optical phonon interactions on the same footing. As the hole-phonon coupling constant \( g \) increases, the quasiparticle residue is further reduced and spin wave peaks in the spectral density function and optical conductivity are more suppressed. Phonon peaks in the spectral density function \( A(\vec{k}, \omega) \), instead, grow more pronounced around the quasiparticle pole at a low energy on the scale of \( \Omega \). A sharp drop in the hole momentum distribution function is found for all the hole-phonon coupling constants we have studied. The invariance of the volume enclosed by the Fermi surface for all the chosen hole-phonon coupling constants numerically verifies Luttinger’s theorem for doped holes in the \( t - J - g \) model. Our numerical estimate of the lowest order vertex corrections to the hole-phonon coupling vertex \( g \) due to the hole-phonon interaction, gives relatively small values \( \ll 1 \). This means that electron-phonon vertex corrections are not important and Migdal’s approximation can be used in calculating the hole self-energy. The smallness of the effective hole bandwidth of order \( 2J \), is compensated by suppressed quasiparticle residues. Due to the presence of additional phonon induced absorption, \( 2J \) and
higher hole-multi-spin wave peaks in optical conductivity are suppressed and broadened.

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TABLE I. Quasiparticle residue, energy shift and mass renormalization factor due to the hole-phonon interaction. The first three columns after the column corresponding to the value of $g/J$ are quasiparticle residues at three different $\vec{k}$ points and the fourth column is the energy shift due to the hole-magnon and hole-phonon interactions at the $(\pi/2, \pi/2)$ point. $a_{\text{calc}}^{(\pi/2, \pi/2)}$ is the quasiparticle pole strength at the $(\pi/2, \pi/2)$ point obtained by perturbation theory. $\lambda_{\text{num}}$ is the numerically calculated renormalization factor due to the hole-phonon interaction.

| $g/J$ | $a_{(0,0)}$ | $a_{(0,\pi/2)}$ | $a_{(\pi/2,\pi/2)}$ | $a_{\text{calc}}^{(\pi/2, \pi/2)}$ | $\Delta \mu / J$ | $\lambda_{\text{num}}$ |
|-------|-------------|-----------------|------------------|-----------------------------|-----------------|------------------|
| 0.0   | 0.103       | 0.408           | 0.349            | 0.349                       | -5.378          | 0.000            |
| 0.5   | 0.001       | 0.376           | 0.337            | 0.337                       | -5.425          | 0.102            |
| 1.0   | 0.002       | 0.322           | 0.304            | 0.305                       | -5.603          | 0.424            |
| 1.5   | 0.003       | 0.258           | 0.266            | 0.274                       | -5.934          | 0.894            |
| 2.0   | 0.007       | 0.231           | 0.216            | 0.266                       | -6.353          | 1.765            |
FIGURES

FIG. 1. Four different diagrams contributing to the self-energy, magnon emitting and absorbing processes, and phonon emitting and absorbing ones.

FIG. 2. Spectral density function for four different coupling constants $g$ at $\vec{k} = (\pi/2, \pi/2)$ for a $24 \times 24$ momentum cluster. The figures for $g \neq 0$ are shifted upward by 0.2. The doping concentration is 3.2%. The phonon frequency $\Omega$ is equal to $0.5J$.

FIG. 3. Density of states for the dressed holes for four different coupling constants $g$ for a $24 \times 24$ momentum cluster. The figures for $g \neq 0$ are shifted upward by 0.2. The doping concentration is 3.2%.

FIG. 4. Momentum distribution function for four different coupling constants $g$ for a $24 \times 24$ momentum cluster. It is scanned from $S = (\pi/2, \pi/2)$ to $M = (\pi, 0)$. The ellipse-like Fermi surface pockets are calculated numerically for the doping concentration 3.2%.

FIG. 5. Lowest order vertex correction due to hole-phonon interaction.

FIG. 6. Real part of the lowest order vertex correction due to hole-phonon interaction for a $24 \times 24$ momentum cluster. The figures for $g/J \geq 1.0$ are shifted upward by 0.1. The doping concentration is 3.2%.

FIG. 7. Imaginary part of the lowest order vertex correction due to hole-phonon interaction for a $24 \times 24$ momentum cluster. The figures for $g/J \geq 1.0$ are shifted upward by 0.1. The doping concentration is 3.2%.

FIG. 8. (a) Lowest order diagram for the optical conductivity, corresponding to an intermediate magnon emitting process. (b) Other lowest order diagram for the optical conductivity. At zero temperature, this contribution vanishes, since it involves a magnon absorption.
FIG. 9. Optical conductivity calculated by the analytic expressions given in the text for four different hole-phonon coupling constants $g$ for a $24 \times 24$ momentum cluster. The units are $2\pi e^2 \times 10^2$ chosen for convenience (compare Ref. 9). The figures for $g \neq 0$ are shifted upward by 20. The doping concentration is 3.2%.
$J/t = 0.4$, $\delta = 0.03$, $k = (\pi/2, \pi/2)$

$A(k, \omega)$

$\omega / J$

$g = 2.0$

$g = 1.0$

$g = 0.5$

$g = 0.0$
$J/t = 0.4, \delta = 0.03$

- \text{solid line: } g = 0.0
- \text{dashed line: } g = 0.5
- \text{dotted line: } g = 1.0
- \text{dashed-dotted line: } g = 2.0
\( k + q \quad k' + q \quad k' \quad k \)
$t/J = 2.5, \delta = 0.03$

$g = 2.0$

$g = 1.5$

$g = 1.0$

$g = 0.5$
$t/J = 2.5$, $\delta = 0.03$

$g = 2.0$

$g = 1.5$

$g = 1.0$

$g = 0.5$
$J/t = 0.4, \delta = 0.03$

$g = 2.0$

$g = 1.0$

$g = 0.5$

$g = 0.0$