Modification of the fictitious absorption method for solving integral equations of mixed problems for arbitrary simply connected regions

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Abstract. We presented a modification of the fictitious absorption method (FAM), which is used to solve integral equations (IE) of mixed dynamic problems. The method was generalized for the case of the deflector stamp occupying a non-convex in plane area. For regions of complex configuration, we assumed the possibility of their representation as a union of convex bounded closed regions, possibly with common boundary sets. We proposed to modify the method in terms of the selection of basis functions that are present in the solution only under the signs of integral operators. The derivatives of Dirac delta functions with supports in the boundary sets of the considered regions were chosen as such. A more complex form of the functions used provided a more convenient form of the solution. As an illustration of the modified MFA use in the basis functions we present solutions of integral equations of the first kind for an axisymmetric problem of steady-state vibrations of an elastic layer with a clamped bottom edge under the action of a surface load and a horizontally oriented internal source.

1. Introduction

The study of the processes of contact interaction between bodies subject to vibration is an important task in mechanical engineering, construction and other industries. The study of contact phenomena in the system of geological formations also leads to mixed dynamic problems, which can be reduced to integral equations (IE) and IE systems (IES). There are various known methods that are used to solve the IE of these problems: variational, factorization and asymptotic methods, the method of matched asymptotic expansions, the method of orthogonal functions, etc. [1–3, etc.] Comparison of the most frequently used approaches in solving various problems of solid mechanics and an analysis of their effectiveness were carried out in [4].

In the present work we developed a method for solving IE and SIE of mixed contact problems, called the fictitious absorption method (FAM) [5–7] in relation to problems for simply connected areas of complex shape.

The method is based on the transformation of a strongly oscillating and slowly decreasing symbol of the IE kernel (the IES matrix symbol), which makes it possible to go over to solving the IE with the
kernel symbol exponentially decreasing with increasing argument. In this case, the oscillating components of the solution are singled out so that the non-oscillating function acts as a new unknown.

The aim of this work is to develop an effective method for solving integral equations with an oscillating kernel symbol, which are characteristic for mixed problems in the theory of elasticity with steady-state mode of vibration. A modification of the FAM in terms of the basis functions choice is described, including a generalized one for the case when a stamp or defect has a non-convex area in the plane. In contrast to the commonly used scheme presented in [5–7], for the introduced auxiliary component of the solution, we use the derivatives of Dirac $\delta$-functions as basis functions, where the boundary set of the contact area is chosen as a support.

Section 2 presents a FAM algorithm for solving the integral equation of a contact problem for a simply connected arbitrary (possibly non-convex) contact area. Section 3 contains an example of method’s application for solving an IE of an axisymmetric problem using the proposed modification of basis functions. Here, we present the analytical form of the solution of the IE for the right-hand side in the form of the Bessel function of the first kind as well as the results of computational experiments and the distribution analysis of the contact stresses. Section 4 presents conclusions about the scope of the FAM application.

2. Application of the fictitious absorption method for an arbitrary simply connected region

When studying the contact phenomena, a special place is occupied by the analysis of mixed initial-boundary value problems and the choice of effective methods for their solution. Problems without initial conditions for steady-state oscillations (with frequency $\omega$) of objects on the surface of an elastic medium, arising during the study of the conditions for the interaction of an oscillation source with a deformable foundation, are reduced to solving an IE or IES with oscillating kernel. Further, we consider the method of fictitious absorption, which makes it possible to arrive to the solution of the IE with exponentially decreasing with the growth of the argument kernel symbols, the solution of which is used by the indicated method as the basic one when solving the initial IE. The FAM allows us to consider as auxiliary the corresponding problems for media with absorption or static problems, for the solution of which there are many approximate methods that have proven themselves well.

Let us consider the general scheme of the FAM for solving the IE in the area of arbitrary configuration. To do this, we introduce a Cartesian coordinate system $x_1Ox_2$ in the plane of contact between the stamp and elastic medium. When solving problems concerning vibration of internal defects – in the plane of the defect location. We write the IE of the mixed problem in the form

$$\mathcal{K} q = \int_{\Omega} k \left( x_1 - \xi_1, x_2 - \xi_2 \right) q \left( \xi_1, \xi_2 \right) d\xi_1 d\xi_2 = f \left( x_1, x_2 \right), \ (x_1, x_2) \in \Omega. \quad (1)$$

Here, the unknown function $q \left( x_1, x_2 \right)$ describes the distribution of contact stresses under the stamp, and the function $f \left( x_1, x_2 \right)$ describes the given amplitude of displacements of the punch foot.

For areas of complex configuration, it is assumed that they can be represented as a union of a finite number of convex closed bounded regions $\Omega = \bigcup_{m=1}^{M} D_m$, $M < \infty$. Split areas can share common border sections. Then (1) can be represented by the system

$$\sum_{m=1}^{M} \mathcal{K}_m q_m = f_s \left( x_1, x_2 \right), \ (x_1, x_2) \in D_s, \ s = 1, M, \ f_s = \int_{D_s} f \left( x_1, x_2 \right), \quad (2)$$

$$\mathcal{K}_m q_m = \int_{D_m} k \left( x_1 - \xi_1, x_2 - \xi_2 \right) q_m \left( \xi_1, \xi_2 \right) d\xi_1 d\xi_2, \ (x_1, x_2) \in D_m,$$

$$k \left( x_1, x_2 \right) = \frac{1}{4\pi} \int_{\Gamma_1, \Gamma_2} \left( k(x_1 + \alpha_2 x_2) \exp(-i(x_1 + \alpha_2 x_2)) \right) d\alpha d\alpha_2, \quad \alpha_1^2 + \alpha_2^2 = \alpha^2. \quad (3)$$
Here \( P_{D_s} \) is a projector for the area \( D_s \), \( i – \) imaginary unit. The properties of the kernel symbol \( K \) are determined by models of an elastic medium and are described in detail in [5, 7]. In what follows, we will assume that the symbol of the Green's function \( K(\alpha,\alpha_\perp) \equiv K(\alpha) \) and \( K(\alpha) = K(-\alpha) \) and can have a finite number (depending on the vibration frequency) of real poles \( p_k \) and zeros \( z_k \) \((k = 1, N)\) as well as a countable set of complex \( p_k \), \( z_k \) \((k = N + 1, \infty)\) with condensation points in sectors of small angles at the imaginary axis [5, 7]. \( K(\alpha) = \mathcal{C}|\alpha|^{-1}\left[1 + O(|\alpha|^{-1})\right] \), \( |\alpha| \to \infty \). We choose contours \( \Gamma_1, \Gamma_2 \) in accordance with the principle of limiting absorption [7].

Let’s consider further local coordinate systems, where \( O_m(a_m,b_m) \in D_m \), with axes parallel to the axes \( x_Ox_2, R = R_m(\psi) \) are the boundary \( S_m \) equations. According to [9], for some ratios of the IE parameters, the problems of vibration for the system of stamps may have more than one solution. This is also true for the environments with multiple defects. Further, it is believed that for the boundaries \( D_s \) IE have unique solutions [5]. This is true for many problems encountered in practice.

\[
\int_0^{2\pi} f_{i\ell}(\psi) \exp\left(i\psi R(x,\psi)\cos(\psi - \gamma)\right) d\psi = b_{i\ell}(\gamma), 0 \leq \gamma \leq 2\pi, k = 1, N, s = 1, M, b_{i\ell} \in \mathcal{C}(0,2\pi). \tag{4}
\]

The scheme of the method consists of the following. Let us introduce a function \( \Pi(\alpha) = E_{\gamma}(\alpha^2) Q_N(\alpha^2) \) according to the FAM algorithm [5, 7]. Here \( E_{\gamma}(\alpha^2) = \prod_{k=1}^N(\alpha^2 - p_k^2) \), \( Q_N(\alpha^2) = \prod_{k=1}^N(\alpha^2 - z_k^2) \) – respectively, poles and zeros of \( K(\alpha) \), numbered in the order of their modules ascendance. It is easy to see that \( \Pi(\alpha) = 1 + O(\alpha^{-1}) \), \( |\alpha| \to \infty \). Kernel symbol \( K(\alpha) \) (3) we can write as \( K(\alpha) = K(\alpha)p(\xi,\alpha) \Pi(\alpha) \), where \( K(\alpha) \) – kernel symbol of IE for the problem of a medium with strong absorption (static problem) is a regular function on the real axis. The asymptotic behaviors of \( K(\alpha) \) and \( K(\alpha) \) are identical for \( |\alpha| \to \infty \). In the integral equation (1), a new function \( p(\xi,\alpha) \) we introduce by the relation \( q(\xi,\alpha) = p(\xi,\alpha) + \varphi(\xi,\alpha) \) [5], where \( \varphi(\xi,\alpha) = \sum_{k=1}^M \varphi_k(\xi,\alpha) \) has the form

\[
\varphi_k(\xi,\alpha) = \sum_{i=1}^N \varphi(x_i - a_i, x_2 - b_i, G_k, R_k, f_{ik}) = \\
= \sum_{k=1}^N G_k(-\Delta)^{2k} \int_0^{2\pi} \delta[x_i - a_i, x_2 - b_i, R_k(\phi)] \delta[x_2 - b_i, R_k(\phi)] g_{ik}(\phi) d\phi, \tag{5}
\]

Here \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \), \( g_{ik} \) – single-valued functions that will be determined during the construction of the solution. The choice of basis functions can be different; in this work, the FAM for \( \varphi(\xi,\alpha) \) in the form (5) is presented. For the introduced auxiliary function in the case of single poles [5], the following conditions are satisfied:

\[
V_{2}\varphi(\xi,\alpha) = V_{2}q(\xi,\alpha), V_{2}\varphi_k(\xi,\alpha) = V_{2}q(x_i,\alpha, D_k), \alpha^2 = p_k^2, k = 1, N, \]

where \( V_{2} \) – operator of two-dimensional Fourier transform. Then \( V_{2}p(\xi,\alpha) = 0, \alpha^2 = p_k^2, k = 1, N \).
Further, we will assume that the function $K(\alpha)$ has no multiple real zeros $z_k$ and poles $p_k$. For the case of multiple poles, the conditions are formulated in [5].

By substituting the representation $q(x_1,x_2)$ in (1), taking into account the relation for $K(\alpha)$

$$\mathcal{K}_0 t = f(x_1,x_2) - \mathcal{K}_0 V_2^{-1} \Pi (\alpha, N) V_2 \varphi,$$

(6)

where the new unknown $t(x_1,x_2) = V_2^{-1} \Pi (\alpha, N) V_2 p(x_1,x_2)$, is determined with the arbitrariness introduced by the function $\varphi(x_1,x_2)$, $V_2^{-1}$ is the inverse Fourier operator.

The integral operator $\mathcal{K}_0$ has a strong damping, and there are many approximate methods for constructing the inverse $\mathcal{K}_0^{-1}$ which are effectively used in solving static problems [8].

Let’s write equation (6) in the form

$$\mathcal{K}_0 f = f(x_1,x_2) - \mathcal{K}_0 \varphi - S \varphi,$$

$$S \varphi = \frac{1}{4 \pi^2} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty K_0(\alpha) \left[ \Pi(\alpha) - 1 \right] \exp(-i(\alpha_1 x_1 - \xi_1) + \alpha_2(x_2 - \xi_2)) d\alpha_1 d\alpha_2 \varphi(\xi_1,\xi_2) d\xi_1 d\xi_2.$$

From here $t = \mathcal{K}_0^{-1} f(x_1,x_2) - \varphi - \mathcal{K}_0^{-1} S \varphi$ and the solution of the original IE can be represented as

$$q = p + \varphi = V_2^{-1} \Pi^{-1} V_2 t + \varphi = V_2^{-1} \Pi^{-1} V_2 \left[ \mathcal{K}_0^{-1} f - \mathcal{K}_0^{-1} S \varphi \right] + \varphi =$$

$$= \mathcal{K}_0^{-1} f - \mathcal{K}_0^{-1} S \varphi + V_2^{-1} \left[ \Pi^{-1} - 1 \right] V_2 \left[ \mathcal{K}_0^{-1} f - \mathcal{K}_0^{-1} S \varphi \right].$$

According to [5], it is necessary to require the fulfillment of the condition

$$V_2 \left[ \mathcal{K}_0^{-1} f - \mathcal{K}_0^{-1} S \varphi \right] = 0, \quad \alpha^2 = z_k^2, \quad k = 1, N.$$

(7)

Since in the general case the support of the function $V_2^{-1} \Pi (\alpha, N) V_2 \varphi$ is the entire plane, we will select in its representation the generalized component $\varphi_0$ with the support in $\Omega$. The rest $\varphi_2$ will be a classic function that has a support in $\mathbb{R}^2$. For this purpose, we’ll consider the relation

$$V_2^{-1} \Pi (\alpha, N) V_2 \varphi = V_2^{-1} \Pi (\alpha, N) \sum_{k=1}^N \sum_{l=1}^M G_k(\alpha^2) \exp(i(a_1 \alpha_1 + b_1 \alpha_2)) \int_0^{2\pi} g_{kl}(\phi) \exp(i \alpha R_1(\phi) \cos(\phi - \gamma)) d\phi$$

and select the polynomial components of the rational functions

$$\Pi(\alpha, N) G_k(\alpha^2) = P_k(\alpha^2) + R_k(\alpha^2),$$

where

$$P_k(\alpha^2) = E_N(\alpha^2) - E_N(p_k^2)(\alpha^2 - p_k^2)^{-1} \approx \alpha^{2N-2}, \quad R_k(\alpha^2) = E_N(p_k^2)(\alpha^2 - p_k^2)^{-1} \approx \alpha^{-2}, \quad |\alpha| \to \infty.$$

In this way,

$$\varphi_2 = V_2^{-1} \Pi (\alpha, N) V_2 \varphi_0 + \varphi_2(x_1,x_2),$$

$$\varphi_2 = V_2^{-1} \sum_{k=1}^N \sum_{l=1}^M R_k(\alpha^2) \exp(i(a_1 \alpha_1 + b_1 \alpha_2)) \int_0^{2\pi} g_{kl}(\phi) \exp(i \alpha R_1(\phi) \cos(\phi - \gamma)) d\phi.$$

(8)

As a result, we have

$$t = \mathcal{K}_0^{-1} f(x_1,x_2) - \varphi_0(x_1,x_2) - \mathcal{K}_0^{-1} P_{[\alpha]} \mathcal{K}_0 \varphi_2.$$

Since the support of the function $\varphi_2$ is the entire plane, in the last term on the right side, we will apply a projector $P_{[\alpha]}$ to the $\Omega$, then

$$p(x_1,x_2) = V_2^{-1} \Pi^{-1} V_2 \left[ \mathcal{K}_0^{-1} f(x_1,x_2) - \varphi_0(x_1,x_2) - \mathcal{K}_0^{-1} P_{[\alpha]} \mathcal{K}_0 \varphi_2 \right].$$

We can write conditions (7) as

$$V_2^{-1} P_{[\alpha]} \left[ \mathcal{K}_0^{-1} f(x_1,x_2) - \varphi_0 - \mathcal{K}_0^{-1} P_{[\alpha]} \mathcal{K}_0 \varphi_2 \right] = 0, \quad \alpha^2 = z_n^2, \quad n = 1, N.$$

The last relation is reduced to the form
\[
\sum_{k=1}^{N} P_k(z_k) \int_{0}^{2\pi} g_{u_k}(\phi) \exp(i z_k R_1(\phi) \cos(\phi - \gamma)) \, d\phi = \\
= \exp(-i(a, \alpha_1 + b, \alpha_2)) V_1 P_{[\phi]} \left[ \mathcal{X}_0^{-1} f(x_1, x_2) - \mathcal{X}_0^{-1} P_{[\phi]} \mathcal{X}_0 \phi_2 \right], \quad \alpha^2 = z^2, \quad n = 1, N, \quad s = 1, M.
\]

From (4), the left side of the system is solvable in relation to \( g_{u_k}(\phi) \), and they also enter the right-hand side, as seen from (8). After inverting the left side, we arrive at a system of linear IE of the second kind with respect to \( g_{u_k}(\phi) \). This system can be solved through discretization [9].

Finally, for the solution of (1) we get the following representation:
\[
q(x_1, x_2) = V_1^{-1} \Phi^{-1}(\alpha, N) V_2 \left[ \phi_2 + \mathcal{X}_0^{-1} f(x_1, x_2) - \mathcal{X}_0^{-1} P_{[\phi]} \mathcal{X}_0 \phi_2 \right],
\]
\[
\phi_2 = V_2 \sum_{k=1}^{N} \sum_{s=1}^{M} E_N \left( p^2 \right) \left( \alpha^2 - p^2 \right)^{-1} \tilde{g}_{s} (\alpha) \exp(i(a, \alpha_1 + b, \alpha_2)),
\]
\[
\tilde{g}_{s} (\alpha) = \sum_{k=1}^{2\pi} g_{u_k}(\phi) \exp(i \alpha R_1(\phi) \cos(\phi - \gamma)) \, d\phi.
\]

Let \( \varphi^s_2 = \varphi_2(x_1, x_2), \quad (x_1, x_2) \in D_s \) be the components of \( \varphi_2 \) with support in \( D_s \) and \( \varphi^{s-} = \varphi_2(x_1, x_2) - \sum_{s=1}^{M} \varphi^{s+}_2, \quad s = 1, M \). We’ll introduce \( q_s(x_1, x_2) = q(x_1, x_2, D_s) = q(x_1, x_2) \) for \( (x_1, x_2) \in D_s \), then for the solutions of (2) the relation (9) will have the form
\[
q(x_1, x_2, D_s) = V_2^{-1} \Phi^{-1}(\alpha, N) F(\alpha_1, \alpha_2, D_s), \quad F(\alpha_1, \alpha_2, D_s) = V_2 P_{[\phi]} \left[ \mathcal{X}_0^{-1} f - \mathcal{X}_0^{-1} P_{[\phi]} \mathcal{X}_0 \phi_2 \right] + V_2 \varphi^{s-}_2.
\]
The \( g_{u_k} \) included in the solution are found from the system [5]: \( F(\alpha_1, \alpha_2, D_s) = 0, \quad \alpha^2 = z^2, \quad n = 1, N \).

The final expression for \( q(x_1, x_2) \) gives an idea of its structure. It should be noted that the introduced function \( \phi \) that contains generalized functions appears in the solution only under the signs of integral operators, the unknowns introduced at the joining boundaries \( D_s \), are mutually annihilated.

3. An example of the application of the modified method

We consider an axisymmetric formulation of the problem for vibration (with frequency \( \omega \)) of a round stamp \( \{r \leq a, z = 0\} \) on an elastic layer \( \{r \geq 0; -h \leq z \leq 0\} \), rigidly connected to a non-deformable foundation and containing a horizontally oriented internal source \( \{r < a, z = -h_0\} \). We will assume that there is no friction in the area of contact between the stamp and the medium and a load is applied in the center of the stamp \( P = \{0, P \exp(-i\omega t)\} \). The load on the inclusion \( X = \{X_r, X_z\} \), \( X_i = \Re[f_1(r) \delta(z + h_0) \exp(-i\omega t)], \quad l = r, z \), is modeled by a localized body force. In cylindrical coordinates \( \{r, z\} \) the displacements are described by the Lame equations [10, 11]. Further, the time factor \( \exp(-i\omega t) \) is omitted everywhere. To find the amplitudes of displacements \( u = \{u_r, u_z\} \) for the elastic layer and the distribution of stresses under the stamp base, one can apply an approach using the Hankel (Fourier – Bessel) integral transform, presented in [12, 13].

To find the amplitude characteristic of the stresses under the stamp, we arrive at an integral equation of the Fredholm type of the first kind
\[
\int_{r_0}^{a} k(r, \tau) q(\tau) r d\tau = u_z(r, 0) - A(r), \quad 0 \leq r \leq a, \quad 0 \leq r \leq a,
\]
\[
k(r, \tau) = \int_{a_0}^{\sigma} K(\alpha) J_0(\alpha r) J_0(\alpha \tau) d\alpha, \quad A(r) = \int_{a_0}^{\sigma} U(\alpha) J_0(\alpha r) d\alpha,
\]
\[ K(\alpha) = \frac{k_2^2}{4\rho c_0} \left[ \sigma_i \left( \alpha \chi_{ch}(\sigma,h) \chi_{sh}(\sigma,h) - \sigma_i \chi_{sh}(\sigma,h) \chi_{ch}(\sigma,h) \right) \right] \], \quad \sigma_i = \sqrt{\alpha^2 - k^2}, \quad k^2 = (\alpha/c_i)^2, \]

\[ \Lambda(\alpha) = \sigma_j \sigma_s \left( s^2 + \lambda^2 \right) \chi_{ch}(\sigma,h) \chi_{sh}(\sigma,h) - \alpha^2 \left( s^2 + \sigma_i^2 \right) \chi_{sh}(\sigma,h) \chi_{sh}(\sigma,h) - 2\alpha^2 \sigma_i \sigma_j, \quad s = \alpha^2 - 0.5k_2^2, \]

\[ c_1 = \sqrt{(\lambda + 2\mu)/\rho}, \quad c_2 = \sqrt{\mu/\rho} \] respectively, the velocities of the longitudinal and transverse waves in the layer (\( \lambda, \mu \) – Lamé characteristics, \( \rho \) – density), \( J_n(ar) \) – Bessel function, \( U(\alpha) \) – the Hankel transform for the amplitude of vertical displacements caused by the action of a buried source, the form of which is given in [13]. Contour \( \sigma_o \) is located in the complex plane in accordance with the principle of limiting absorption [5–7].

To construct a solution for IE (10), we use the auxiliary equation

\[ \int_0^a k(r,\tau)q_\eta(\tau)\,d\tau = J_0(\eta r), \quad 0 \leq r \leq a. \] (11)

Here kernel symbols can be presented in the form \( K(\alpha) = K_0(\alpha)\Pi(\alpha) \), as in [5–7], for reasons of simplicity for the factorization, we choose \( K_0(\alpha) = C(\alpha^2 + B^2)^{-\lambda/2}, \ B \) – given approximation parameter, as described above, \( \Pi(\alpha, N) = \prod_{k=1}^{N} (\alpha^2 - z_k^2)(\alpha^2 - p_k^2)^{-1} \). When introducing a new unknown, the following function was used \( \varphi(r) \):

\[ \varphi(r) = \sum_{k=1}^{N} C_k G_k(L) \delta(r-a), \quad L = \left( \frac{1}{r} \frac{d}{dr} \right), \]

\[ G_k(\alpha^2) = (\alpha^2 - p_k^2) \cdots (\alpha^2 - p_k^2) \cdots (\alpha^2 - p_k^2), \ C_k \) – unknowns determined in the future.

Thus, we have chosen the derivatives of the Dirac \( \delta \)-functions with support on the boundary of the stamp \( a \) as a system of linearly independent functions. Next, we introduce

\[ t(r) = \int_0^a \Pi(\alpha) P(\alpha) J_0(\alpha r) \, a \, d\alpha \]

and write (6) for this case

\[ \int_0^a k_0(r,\tau)t(\tau)\,d\tau = J_0(\eta r) - \sum_{k=1}^{N} \int_0^a k(r,\tau)C_k G_k(L) \delta(r-a) \, d\tau, \] (12)

By using the approximate solution (12) for the right side of the \( J_0(\eta r) \) IE of the static problem with the chosen kernel symbol \( K_0(\alpha) \), we obtain solution (11) in the form

\[ q_\eta(r) = J_0(\eta r) K^{-1}(\eta) + \frac{i\pi a}{2} K_0^{-1} \sum_{i=1}^{N} \beta_i J_0(z_i r) G_i(\eta, z_i) - \frac{a\pi^2}{4} \sum_{i=1}^{N} g_i \sum_{i=1}^{N} \beta_i J_0(z_i r) G_2(z_i, p_i) + \]

\[ + b(\eta) \left( \frac{e^{-B(a-r)}}{\sqrt{a^2 - r^2}} \right) + \frac{a\pi^2}{2} \sum_{i=1}^{N} \beta_i H_0^1(z_i a) J_0(z_i r) \right) \]

\[ + \left[ \frac{a\pi^2}{2} \frac{e^{-B(a-r)}}{\sqrt{a^2 - r^2}} - \frac{a\pi^2}{2} \right] \sum_{i=1}^{N} \beta_i J_0(z_i r) H_0^1(z_i a) \sum_{i=1}^{N} \frac{g_i}{B - \nu_k} H_0^1(p_k a) \]

up to a constant in the asymptotic \( K(\alpha) \) when \( |\alpha| \rightarrow \infty, \ \varepsilon = 1/\sqrt{B} \),

\[ G_i(\eta, z) = \left[ \eta J_{\alpha i}^1(\eta a) H_0^1(z a) - z J_{\alpha}^1(\eta a) H_{\alpha i}^1(z a) \right] (\eta^2 - z^2)^{-1}, \]

\[ G_2(z, p) = \left[ p H_{\alpha i}^1(pa) H_0^1(z a) - z H_{\alpha i}^1(pa) H_0^1(z a) \right] (z^2 - p^2)^{-1}, \]

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\[ b(\eta) = \sqrt{\frac{\alpha}{2\pi}} \left[ H_0^j(\eta a)\sqrt{B+i\eta} + H_0^j(\eta a)\sqrt{B-i\eta} \right], \quad \beta_l = \prod_{k=1}^{N} \left( z_{j_k}^2 - p_i^2 \right) \prod_{k=1, l \neq j_k}^{N} \left( z_{j_k}^2 - z_{<k}^2 \right), \]

\( H_0^j \) – Hankel function \((j = 1, 2)\). Unknowns \( C_k \) that are: \( g_k = aC_J \psi_0(\alpha \eta) \prod_{k=1}^{N} \left( \alpha^2 - z_{k}^2 \right) \). And \( g_k \) are determined from the following algebraic system:

\[ \sum_{k=1}^{N} g_k \left[ \left( p_i H_0^j(\alpha \eta) J_0(\alpha \eta) - \alpha H_0^j(\alpha \eta) J_0(\alpha \eta) \right)\left( \alpha^2 - p_i^2 \right)^{-1} - 2 \sqrt{\frac{\varepsilon}{\pi}} \left( B - \alpha \eta \right) H_0^j(\alpha \eta) J_0(\alpha \eta) \right] = \begin{cases} \frac{2i}{\pi} \sqrt{B^2 + \eta^2} \left[ \alpha J_0(\alpha \eta) J_0(\alpha \eta) - \eta J_0(\alpha \eta) J_0(\alpha \eta) \right] + \frac{2i}{\pi} \sqrt{\frac{2\varepsilon}{\alpha}} J_0(\alpha \eta) b(\eta), & \alpha = z_l, \quad l = 1, N. \end{cases} \]

The IE solution \((1)\) can be represented as \( q(r) = q_1(r) + q_2(r) \), where \( q_1(r) \) – is the corresponding amplitude of contact stresses caused by surface \((j=1)\) and buried \((j=2)\) loads. When carrying out computational experiments, the stress distribution on a horizontally oriented inclusion was specified in the form \([13]\): \( f(r) = kr + d \), \( r \geq k = d(\varepsilon_1 - 1)r_0^1, \quad d = 2(\varepsilon_1 + 1)r_0^1, \quad r \in [0, r_0] \). A frequency range of \( \nu = 0 \div 10 \) Hz (dimensionless frequency \( \omega = 2\pi \nu \varepsilon_0 / \epsilon_0 \)) was selected. The figure illustrates the results of calculating the amplitude values of the stresses under the stamp base for the following problem parameters: \( c_1 = 0.2 \cdot 10^3 \) m/c, \( c_2 = 0.12 \cdot 10^3 \) m/s, \( \rho = 1.4 \cdot 10^3 \) kg/m\(^3\), \( h = 20 \) m, \( \varepsilon_1 = 1 \), \( a = 2 \) m, \( r_0 = 2 \) m, \( h_0 = 10 \) m. Here \( q_1(r) \) (surface load) corresponds the dotted line and the buried load \( q_2(r) \) – the solid line.

![Figure 1](image-url)  
Figure 1. Dependence of the vertical stress component amplitude from the vibration frequency

As can be seen from the figure, the presence of a buried load on a horizontal inclusion insignificantly affects the magnitude, distribution and nature of the stresses in the area of contact between the stamp and the elastic layer at low frequencies.

4. Conclusion
The presented modification of the FAM can be used to solve the IE problems in various areas of mechanics, including linked fields mechanics \([7]\), tribology \([14]\), fracture mechanics \([15]\), defectoscopy, acoustoelectronics, etc. The area of application for the obtained formulas, which give us approximate solutions to dynamic problems, is determined by the area of application for the solutions of the corresponding problems for media with dampening. The method can be used to study the
composition of regional lithospheric structures, to calculate the characteristics of the stress-strain state of soil foundations, vibration effects on foundations, etc.

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Appendices
IE – integral equation;
IES – system of integral equations;
FAM – fictitious absorption method.

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