ON SYMBOL CORRESPONDENCES FOR QUARK SYSTEMS

P. A. S. ALCÂNTARA AND P. DE M. RIOS

Abstract. We present the characterization of symbol correspondences for mechanical systems that are symmetric under $SU(3)$, which we refer to as quark systems. The quantum systems are the unitary irreducible representations of $SU(3)$, denoted by $Q(p,q)$, $p,q \in \mathbb{N}_0$, together with their operator algebras. We study the cases when the classical phase space is a coadjoint orbit: either the complex projective plane $\mathbb{C}P^2$ or the flag manifold that is the total space of fiber bundle $\mathbb{C}P^1 \to E \to \mathbb{C}P^2$. In the first case, we refer to pure-quark systems and the characterization of their correspondences is given in terms of characteristic numbers, similarly to the case of spin systems, cf. [24]. In the second case, we refer to generic quark systems and the characterization of their correspondences is given in terms of characteristic matrices, which introduces various novel features. Furthermore, we present the $SU(3)$ decomposition of the product of quantum operators and their corresponding twisted products of classical functions, for both pure and generic quark systems. In preparation for asymptotic analysis of these twisted products, we also present the $SU(3)$ decomposition of the pointwise product of classical functions.

CONTENTS

1. Introduction 2
2. On the representations of $SU(3)$ 5
   2.1. Irreducible unitary representations and their GT basis 6
   2.2. Clebsch-Gordan series and the space of operators 9
   2.3. Mixed Casimir operators and symmetric CG coefficients 12
   2.4. Invariant decomposition of the operator product 14
   2.5. (Co)Adjoint orbits as invariant phase spaces 23
3. Pure-quark systems 25
   3.1. Classical pure-quark system 25
   3.2. Quantum pure-quark systems 27
   3.3. Symbol correspondences for pure-quark systems 28
   3.4. Twisted product for pure-quark system 37
4. Generic quark systems 40
   4.1. Classical generic quark system 40
   4.2. Quantum generic quark system 43
   4.3. Symbol correspondences for generic quark systems 44
   4.4. Twisted products for generic quark systems 53
5. Concluding remarks 55

2020 Mathematics Subject Classification. 17B08, 20C35, 22E46, 22E70, 43A85, 53D99, 81Q99, 81S10, 81S30.

Key words and phrases. Dequantization, Quantization, Symmetric mechanical systems, Symbol correspondences, Quark systems ($SU(3)$-symmetric systems).

The authors thank CAPES (finance code 001) for support.
1. Introduction

Inspired by the treatment of symbol correspondences between quantum and classical mechanical systems which are symmetric under SU(2), the so-called spin systems, cf. [24], in this paper we start a generalization of this treatment to the study of symbol correspondences between quantum and classical mechanical systems which are symmetric under a compact Lie group \( G \), by focusing exclusively on the case \( G = SU(3) \). Since \( SU(3) \) is the symmetry group of the strong force, in Physics, here we call such systems quark systems.

The first remarkable difference between the \( SU(2) \) and \( SU(3) \) cases is that in the former case there is just one classical system, namely the Poisson algebra of smooth functions on \( \mathbb{C}P^1 \simeq S^2 \), whereas in the latter case there are two types of symplectic phase space: the complex projective plane \( \mathbb{C}P^2 \) and the flag manifold that is a fiber bundle \( E \) over \( \mathbb{C}P^2 \) with fibers \( \mathbb{C}P^1 \), denoted \( \mathbb{C}P^1 \hookrightarrow E \to \mathbb{C}P^2 \). The quantum systems of interest here are the Hilbert spaces \( \mathcal{H}_{p,q} \) with an irreducible representation \( Q(\mathbb{N}) \) of \( SU(3) \), for \( p, q \in \mathbb{N}_0 \), together with their operator algebras.

Then the main question posed here can be addressed in the following way: when/how is it possible to injectively map the space of operators on \( \mathcal{H}_{p,q} \) to the space of smooth functions on \( \mathbb{C}P^2 \) or \( E \), in an \( SU(3) \)-equivariant way which also ensures that quantum observables give rise to classical observables?

To answer the question above we define symbol correspondences\(^2\) in the spirit of what is already done in literature, especially in [24], as linear injective maps
\[
W : \mathcal{B}(\mathcal{H}_{p,q}) \to C_c(\mathcal{O}) , \quad A \mapsto W_A ,
\]
where \( \mathcal{B}(\mathcal{H}_{p,q}) \) is the space of operators and \( \mathcal{O} \) is either \( \mathbb{C}P^2 \) or \( E \), satisfying a few extra properties: (i) any such map \( W \) is \( SU(3) \)-equivariant, (ii) the image of any Hermitian operator is a real function and (iii) the normalization condition
\[
\int_{\mathcal{O}} W_A(x)dx = \frac{1}{\dim Q(p,q)} \text{tr}(A)
\]
applies to every operator \( A \in \mathcal{B}(\mathcal{H}_{p,q}) \), with respect to a normalized left invariant integral on \( \mathcal{O} \). Condition (ii) encodes that \( W \) maps observables to observables and condition (iii) means it preserves expected values.

It turns out that for \( \mathcal{O} = \mathbb{C}P^2 \) we can only define symbol correspondences for irreducible representations of type \( Q(p,0) \) or \( Q(0,q) \). We refer to the classical and quantum systems associated to \( \mathbb{C}P^2 \) and Hilbert spaces \( \mathcal{H}_{p,0} \) or \( \mathcal{H}_{0,q} \) as pure-quark systems.\(^1\)

---

\(^1\)Initial efforts in this direction can be found in [17, 18, 20].

\(^2\)It may be fair to call them symplectic symbol correspondences because we are working only with symplectic manifolds. For \( SU(3) \), one could also try to work with a Poisson manifold irregularly foliated by symplectic leaves, trying to define Poissonian symbol correspondences.
systems, since the pertinent irreducible representations of $SU(3)$ emerge from systems of $p$ quarks only, or $q$ antiquarks only. Characterization of symbol correspondences for pure-quark systems is very similar to what is known for spin systems. In particular, the correspondences for $\mathcal{H}_{p,0}$, or $\mathcal{H}_{0,p}$, are unequivocally determined by an ordered set of nonzero real numbers

$$c_n \in \mathbb{R}^*, \quad 1 \leq n \leq p,$$

called characteristic numbers, so that the moduli space of symbol correspondences for pure-quark systems is $(\mathbb{R}^*)^p$, cf. Theorem 3.11 and Corollary 3.12.

However, when $O$ is the flag manifold $E$, we can define symbol correspondences for any $\mathcal{H}_{p,q}$, i.e. any irreducible representation $Q(p,q)$ of $SU(3)$, so we refer to the classical and quantum systems associated to $E$ and $\mathcal{H}_{p,q}$ as generic quark systems. Here we see some novel features and the characterization of symbol correspondences for generic quark systems is now presented via full-rank complex matrices, called characteristic matrices, cf. Theorem 4.11, so that the moduli space of symbol correspondences for a generic quark system is a product of noncompact Stiefel manifolds, cf. Corollary 4.12.

Just as it happens for spin systems, for both pure-quark and generic quark systems, Theorems 3.13 and 4.13 show that a symbol correspondence $W$ can be realized in terms of expectations over a Hermitian operator with unitary trace $K$, called the operator kernel, via

$$W_{A}(gx_0) = \text{tr}(AK^g),$$

for any $A \in \mathcal{B}(\mathcal{H}_{p,q})$ and $g \in SU(3)$, where $x_0 \in O$ is a suitable point related to a choice of basis for $\mathcal{H}_{p,q}$, with $gx_0$ denoting the (co)adjoint action of $g$ on $x_0 \in O$ and $K^g$ denoting the action of $g$ on $K \in \mathcal{B}(\mathcal{H}_{p,q})$ by conjugation,

$$SU(3) \ni g : K \mapsto \rho(g)K\rho(g)^{-1} = K^g,$$

where $\rho$ is the irreducible $SU(3)$-representation on the respective quantum system.

Thus, one can also interpret symbol correspondences for quark systems as expectation values over pseudo-states (Hermitian operators with unit trace which are not necessarily positive). When the operator kernel is also a positive operator, i.e. when $K$ is also a state, the correspondence maps positive-(definite) operators to (strictly-)positive functions and is called a mapping-positive correspondence.

On the other hand, if the correspondence $W$ induces an isometry between $\mathcal{B}(\mathcal{H}_{p,q})$ and the image set of $W$, for appropriately normalized invariant inner products in $\mathcal{B}(\mathcal{H}_{p,q})$ and $C^\infty_c(O)$, then $W$ is called a Stratonovich-Weyl correspondence. As for spin systems, Theorem 3.24 shows that no Stratonovich-Weyl correspondence is mapping-positive, for the case of pure-quark systems.

Adaptations of proofs from [24] show that, for pure-quark systems, the projector onto the highest weight space of $\mathcal{H}_{p,0}$ is an operator kernel, as well as the projector onto lowest weight space of $\mathcal{H}_{0,p}$, cf. Propositions 3.25 and 3.26. Then, Theorem 4.26 states that, for any $\mathcal{H}_{p,q}$, the projector onto the highest weight space as well as

---

3One can also interpret such pure-quark systems as solution spaces for the three dimensional isotropic oscillator, cf. Remark C.1.
4Note that quantum pure-quark systems are special cases of quantum generic quark systems.
5It must be emphasized that, whether $K$ is a state or just a pseudo-state, $K$ also has other defining properties for being the operator kernel of a symbol correspondence, in other words, not every state, or pseudo-state, is an operator kernel of a symbol correspondence.
the projector onto the lowest weight space are both operator kernels for a generic quark system. The failure of one of these projectors to be an operator kernel for a pure-quark system being due to a lack of greater symmetry required of an operator kernel for the map $\mathcal{O} = \mathbb{C}P^2$. The symbol correspondences that these projectors define are called Berezin correspondences, cf. Definitions 3.29 and 4.27. Besides being examples of mapping-positive correspondences, Berezin correspondences also exemplify a natural relation between correspondences for dual quark systems, which are defined as antipodal correspondences, cf. Definitions 3.33 and 4.30. Propositions 3.35 and 4.32.

In this paper, we also start the study of twisted products of symbols, that is, non-commutative products of functions in some finite dimensional subspaces of $C^\infty(\mathcal{O})$, which are induced by the operator product in $\mathcal{B}(\mathcal{H}_{p,q})$ via symbol correspondences. To do so, first we develop the $SU(3)$-invariant decomposition of the operator product in $\mathcal{B}(\mathcal{H}_{p,q})$, cf. Lemma 2.25 and Corollary 2.31, defining the Wigner product symbol that is a product of Wigner coupling and recoupling symbols, cf. Definitions 2.26, 2.27 and 2.29, which expresses all symmetries of the operator product, cf. Theorem 2.30. From this, in Theorems 3.40, 3.41, 4.40 and 4.41 we are able to present some explicit expressions for twisted products. And in Propositions 3.47 and 4.46 we show that antipodal correspondences induce a “reverse symbolic dynamics” via twisted product, cf. Remark 3.49.

This paper is organized as follows.

In Section 2 we first present some general facts concerning the Lie group $SU(3)$ and proceed in subsection 2.1 with the characterization of its irreducible unitary representations using the Gelfand-Tsetlin method for defining a standard basis of $\mathcal{H}_{p,q}$. Then, in subsection 2.2 we present the Clebsch-Gordan series of $SU(3)$ and in subsection 2.3 we use the method of mixed Casimir operators to distinguish subrepresentations with multiplicities in the $CG$ series. We then proceed in subsection 2.4 with the $SU(3)$-invariant decomposition of the operator product, using various Wigner symbols in order to highlight the symmetries of this invariant decomposition. Finally, in subsection 2.5 we present the description of $\mathbb{C}P^2$ and $\mathcal{E}$ as (co)adjoint orbits of $SU(3)$, along with some of their properties.

In Sections 3 and 4 we work out the description of symbol correspondences for pure-quark systems and generic quark systems, respectively. The first subsection of each section presents the construction of harmonic functions on each respective symplectic manifold that constitutes the classical phase space, from which the pertinent irreducible representations of $SU(3)$ are identified as possible quantum quark systems, and then proceeds with the $SU(3)$-invariant decomposition of the pointwise product of each of these harmonic functions. The last two subsections of each section are devoted to symbol correspondences and their twisted products.

Then, in Section 5 we briefly discuss some results obtained in this paper, with indication of some topics for future investigations, particularly the main topic still to be worked out: a general asymptotic analysis of twisted products of symbols.

Finally, in Appendix A we explain the Gelfand-Tsetlin method for $SU(3)$ used in Definition 2.1, in Appendix B we explain and exemplify the proof of Theorem 2.19 from [8], in Appendix C we justify the name in Definition 3.7 and in Appendix D we reproduce in greater detail the proof of Theorem 4.26 from [11, 30].
Acknowledgements: We thank Eldar Straume for stimulating initial discussions and interesting later comments. We also thank Igor Mencattini and Luiz San Martin for some pertinent comments.

2. On the representations of $SU(3)$

Let $SU(3)$ denote the special unitary subgroup of $GL_3(\mathbb{C})$, satisfying $\det g = 1$ and $gg^\dagger = g^\dagger g = e$, for all $g \in SU(3)$. As a manifold, $SU(3)$ can be seen as a fiber bundle over $S^5$ whose fibers are $S^3 \simeq SU(2)$ (see discussion in subsection 2.5), hence it is a compact Lie group of real dimension 8. The Lie algebra of $SU(3)$, denoted by $\mathfrak{su}(3)$, can be generated by $i\lambda_k$, for $k = 1, \ldots, 8$, where

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

are Hermitian matrices, known as Gell-Mann matrices, satisfying

$$\text{tr}(\lambda_a \lambda_b) = 2\delta_{ab}, \quad [\lambda_a, \lambda_b] = 2i \sum_{c=1}^8 f^{abc} \lambda_c,$$

with $f^{abc}$ totally antisymmetric and determined by Table 1 – see e.g. [13].

| $abc$ | 123 | 147 | 156 | 246 | 257 | 345 | 367 | 458 | 678 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $f^{abc}$ | 1   | 1/2 | -1/2 | 1/2 | 1/2 | -1/2 | 1/2 | -2/3 | 2/3 |

Table 1. Structure constants for Gell-Mann matrices.

Thus, $SU(3)$ is also a simple Lie group. In order to describe its irreducible representations, we take the complexification of $\mathfrak{su}(3)$ and define

$$F_k = \frac{\lambda_k}{2},$$

$$T_\pm = F_1 \pm IF_2, \quad V_\pm = F_4 \pm IF_5, \quad U_\pm = F_6 \pm IF_7, \quad T_3 = F_3, \quad Y = \frac{2}{\sqrt{3}} F_8.$$

Then, one can easily verify that

$$T_\pm^\dagger = T_\mp, \quad U_\pm^\dagger = U_\mp, \quad V_\pm^\dagger = V_\mp, \quad T_3^\dagger = T_3, \quad Y^\dagger = Y,$$

$[T_3, Y] = [T_\pm, Y] = 0,$

and furthermore,

$$\text{tr}(T_3 T_3) = \frac{1}{2}, \quad \text{tr}(Y Y) = \frac{2}{3}, \quad \text{tr}(T_3 Y) = 0,$$

thus $\mathfrak{su}(3)$ is of rank 2 and the set $\{iT_3, iY\}$ forms an orthogonal, but not normal basis of the Cartan subalgebra of $\mathfrak{su}(3)$. Then, by defining

$$U_3 = \frac{3}{4} Y - \frac{1}{2} T_3, \quad V_3 = \frac{3}{4} Y + \frac{1}{2} T_3,$$
among the various commutation relations that follow from (2.2)-(2.4), we have

\[
\begin{align*}
[T_3, T_{\pm}] &= \pm T_{\pm}, & [U_3, U_{\pm}] &= \pm U_{\pm}, & [V_3, V_{\pm}] &= \pm V_{\pm}, \\
[T_{\pm}, T_{\mp}] &= 2T_3, & [U_{\pm}, U_{\mp}] &= 2U_3, & [V_{\pm}, V_{\mp}] &= 2V_3,
\end{align*}
\]

(2.9)

\[
\begin{align*}
[U_3, T_{\pm}] &= \pm \frac{1}{2} U_{\pm}, & [V_3, T_{\pm}] &= \pm \frac{1}{2} V_{\pm}, \\
[V_3, U_{\pm}] &= \pm \frac{1}{2} U_{\pm}, & [T_3, V_{\pm}] &= \pm \frac{1}{2} V_{\pm}, & [T_3, U_{\pm}] &= \pm \frac{1}{2} U_{\pm},
\end{align*}
\]

(2.10)

\[
[T_3, U_3] = [U_3, V_3] = [V_3, T_3] = 0,
\]

hence the root system of $SU(3)$ is composed by three root systems of $SU(2)$, with the same length $6$, framing a regular hexagon as in Figure 1.

![Figure 1. Root diagram of $su(3)$.](image)

The roots $\alpha_1$, $\alpha_2$, $\alpha_3$ are associated to the ladder operators $T_+$, $U_+$, $V_+$, respectively. We choose the fundamental Weyl chamber as the blue hatched one, enclosed by the dashed lines, so that $\{\alpha_1, \alpha_2, \alpha_3\}$ is the set of positive roots and $\{\alpha_1, \alpha_2\}$ is the set of simple roots. Let $\omega_1$ and $\omega_2$ be the fundamental weights satisfying

\[
2 \frac{\langle \omega_j | \alpha_k \rangle}{\| \alpha_k \|^2} = \delta_{j,k}, \quad j, k \in \{1, 2\},
\]

(2.11)

where $\langle | \rangle$ is the canonical Euclidean inner product on the root space. Writing the fundamental weights as linear combination of the simple roots $\{\alpha_1, \alpha_2\}$, and using

\[
\langle \alpha_1 | \alpha_2 \rangle = -\| \alpha_1 \| \| \alpha_2 \| / 2, \quad \| \alpha_1 \| = \| \alpha_2 \|,
\]

(2.12)

the relations given by (2.11) imply

\[
\omega_1 = \frac{1}{3} (2\alpha_1 + \alpha_2), \quad \omega_2 = \frac{1}{3} (\alpha_1 + 2\alpha_2).
\]

(2.13)

2.1. Irreducible unitary representations and their GT basis. We label the classes of irreducible unitary representations of $SU(3)$ by two nonnegative integers $p$ and $q$, denoting each class by $Q(p, q)$, where $(p, q) \equiv p\omega_1 + q\omega_2$ is the highest weight of the representation. We shall often refer to an unitary irreducible representation of class $Q(p, q)$ in a less specific way simply as an irreducible representation $Q(p, q)$, or just as a representation $Q(p, q)$. Accordingly, for a representation $Q(p, q)$, $p$ and $q$ are the maximal integers such that $(T_{-})^p$ and $(U_{-})^q$ can be applied to the highest

\footnote{Among the infinitely many $SU(2)$ subgroups of $SU(3)$, the ones associated to $\{T_3, T_{\pm}\}$, $\{U_3, U_{\pm}\}$ and $\{V_3, V_{\pm}\}$ are singled out as the three standard $SU(2)$ subgroups of $SU(3)$.}
weight vector \( e_\gamma \), before vanishing, and an orthonormal basis of weight vectors for the representation \( Q(p,q) \) on a complex Hilbert space \( H \) of dimension
\[
\text{(2.14)} \quad \text{dim } Q(p,q) = \frac{(p+1)(q+1)(p+q+2)}{2},
\]
is obtained via linear combinations of the action of \((T_-)^a(U_-)^b(T_-)^c\) on \( e_\gamma \).

Then, any weight \( \omega \) of \( Q(p,q) \) can be expressed as a linear combination of the fundamental weights, \( \omega = m\alpha_1 + n\alpha_2 \), for some \( m, n \in \mathbb{Z} \) such that \( m/2, n/2 \) are the eigenvalues of \( T_3, U_3 \), respectively, according to the actions \( T_\pm : \omega \mapsto \omega \pm \alpha_1 \), \( U_\pm : \omega \mapsto \omega \pm \alpha_2 \). With this, one can define a partial order of weights [4] as follows:
\[
\text{(2.15)} \quad \omega > \tau \quad \text{if} \quad \omega - \tau = c_1\alpha_1 + c_2\alpha_2 \quad \text{for nonnegative integers} \quad (c_1, c_2) \neq (0,0).
\]
Although this is not a total order, we have \((p,q) > (m,n)\) for every weight \((m,n) \neq (p,q)\) of \( Q(p,q) \), hence the name highest. However, some other weights \( \omega = (m,n) \) may have multiplicity, as illustrated in Figure 2(B).

![Figure 2](image)

**Figure 2.** Examples of weight diagrams for \( SU(3) \). Each highest weight is highlighted as a square dot and the multiplicities of a weight are represented by rings around the weight.

Since for what follows we need to resolve the multiplicities of weights, we resort to labeling each weight \( \omega \) of \( Q(p,q) \) by a triple \((\nu_1, \nu_2, \nu_3)\) of nonnegative integers, using an extra nonnegative half-integer index \( J \) to distinguish weights with multiplicity, such that \( \nu_i, J \) satisfy the Gelfand-Tsetlin pattern
\[
\text{(2.16)} \quad 0 \leq r_- \leq q \leq r_+ \leq p + q, \quad r_- \leq \nu_3 \leq r_+,
\]
for \( r_+ \) and \( r_- \) integers. In Appendix [A] we present an explanation of the Gelfand-Tsetlin method for \( SU(3) \). In this Gelfand-Tsetlin labeling of weights,
\[
\text{(2.17)} \quad t = (\nu_1 - \nu_2)/2, \quad u = (\nu_2 - \nu_3)/2, \quad v = (\nu_1 - \nu_3)/2
\]
are the eigenvalues\(^8\) of the operators \( T_3, U_3, V_3 \), respectively, so that
\[
\text{(2.18)} \quad \begin{cases} T_\pm : (\nu_1, \nu_2, \nu_3) \mapsto (\nu_1 \pm 1, \nu_2 \mp 1, \nu_3), \\ U_\pm : (\nu_1, \nu_2, \nu_3) \mapsto (\nu_1, \nu_2 \pm 1, \nu_3 \mp 1), \\ V_\pm : (\nu_1, \nu_2, \nu_3) \mapsto (\nu_1 \pm 1, \nu_2, \nu_3 \mp 1), \end{cases}
\]
\(^7\text{Cf. Weyl dimensionality formula [15].}\)
\(^8\text{We shall explore this common representation of weights with its partial order in Appendix [D].}\)
\(^9\text{Henceforth the weights from which we are taking the eigenvalues will be specified or will be clear from the presence or absence of subscript and superscript in } t, u \text{ and } v.\)
and the index $J$ is the total spin number of the subrepresentation of the standard $SU(2)$ generated by $\{U_3, U_\pm\}$\textsuperscript{14} In particular, the highest weight is labelled by
\begin{equation}
(2.19) \quad \nu_+ = q , \quad \nu_- = \nu_3 = 0 \implies (\nu_1, \nu_2, \nu_3) = (p + q, q, 0) , \quad J = q/2 .
\end{equation}

Because the Gelfand-Tsetlin labeling distinguishes multiplicities, we use it to define a standard basis for any irreducible representation of class $Q(p, q)$ as follows.

**Definition 2.1** (cf. e.g. [2]). A Gelfand-Tsetlin basis, or simply a GT basis of an irreducible representation $\rho$ of class $Q(p, q)$ on $\mathcal{H}$, denoted $\{e((p, q); \nu, J)\}$, is an orthonormal basis indexed by $J$, the total spin number of $\{U_3, U_\pm\}$, and the triple
\begin{equation}
(2.20) \quad \nu = (\nu_1, \nu_2, \nu_3) ,
\end{equation}
as specified above, cf. (2.16)-(2.18), constructed by fixing a highest weight vector
\begin{equation}
(2.21) \quad e_+ = e((p, q); (p + q, q, 0), q/2) \in \mathcal{H} ,
\end{equation}
cf. (2.19), and determining the other $\text{dim } Q(p, q) - 1$ basis vectors of $\mathcal{H}$ by
\begin{equation}
(2.22) \quad U_- e((p, q); (\nu_1, \nu_2, \nu_3), J) = \sqrt{(J + u)(J - u + 1)} e((p, q); (\nu_1, \nu_2 - 1, \nu_3 + 1), J) ,
\end{equation}
\begin{equation}
T_- e((p, q); (\nu_1, \nu_2, \nu_3), J) =
\frac{\sqrt{(J + u)(J + u + 1)}e((p, q); (\nu_1 - 1, \nu_2 + 1, \nu_3), J - 1/2)}{2J(2J + 1)}
+ \frac{\sqrt{(J + u + 1)(J + u + 1 + \nu_3 - q)(J + u + 2 + \nu_3)}e((p, q); (\nu_1 - 1, \nu_2 + 1, \nu_3), J + 1/2)}{2(J + 1)(2J + 1)} .
\end{equation}

**Remark 2.2.** Thus, for any irreducible representation $\rho$ of class $Q(p, q)$, its GT basis for $\mathcal{H}$ is uniquely determined up to a choice of phase for $e_+$ as in (2.21). This “minimal indeterminacy” in the definition of a standard orthonormal basis for any irreducible representation of class $Q(p, q)$ is fundamental for all that follows.

Considering the anti-isomorphism $\mathcal{H}^* \leftrightarrow \mathcal{H}$ via inner product, for the dual representation $\bar{\rho} \leftrightarrow \rho$, we get that $\bar{T}_3 \leftrightarrow -T_3 , \bar{U}_3 \leftrightarrow -U_3 , \bar{V}_3 \leftrightarrow -V_3 , \bar{T}_\pm \leftrightarrow -T_\mp , \bar{U}_\pm \leftrightarrow -U_\mp$ and $\bar{V}_\pm \leftrightarrow -V_\mp$. Thus, the states of $\bar{\rho}$ are related to the states of $\rho$ by
\begin{equation}
(2.23) \quad \bar{J} = J ; \quad \bar{t} = -t , \quad \bar{u} = -u , \quad \bar{v} = -v ,
\end{equation}
which implies
\begin{equation}
(2.24) \quad \bar{\nu}_1 = p + q - \nu_1 , \quad \bar{\nu}_2 = p + q - \nu_2 , \quad \bar{\nu}_3 = p + q - \nu_3 .
\end{equation}
Therefore, from the above and (2.19)-(2.18), we have that $(q, p) = q\omega_1 + p\omega_2$ is the highest weight of $\bar{\rho}$, that is, the dual representation $\bar{\rho}$ is of class $Q(q, p)$, so that
\begin{equation}
(2.25) \quad Q(p, q)^* = Q(q, p) .
\end{equation}

\textsuperscript{14}For our purposes, this choice of standard $SU(2)$ subgroup is more convenient, but in the physics literature, it is more common to use the total spin number of the standard $SU(2)$ subgroup generated by $\{T_3, T_\pm\}$, which is often called the isospin [14].
Notation 1. In light of this dualization symmetry, we introduce the notation\footnote{We use the convention which identifies the set of natural numbers as \( \mathbb{N} = \{1, 2, 3, \ldots\} \) and denote \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \).}.

\begin{equation}
\rho = (p,q) \in \mathbb{N}_0 \times \mathbb{N}_0 \leftrightarrow \bar{\rho} = (q,p) , \quad \text{with} \quad |\rho| = |\bar{\rho}| = p + q .
\end{equation}

Then, to better state the relations in \eqref{eq:2.24}, we define

\begin{equation}
\Delta_{\nu,\mu}^{p+q} := \begin{cases} 1 , & \text{if} \quad \nu + \mu = (p + q, p + q, p + q) = (|\rho|, |\rho|, |\rho|) \\ 0 , & \text{otherwise} \end{cases}
\end{equation}

Definition 2.3. For a Gelfand-Tsetlin basis \( \{e(p; \nu, J)\} \) of an \( SU(3) \)-representation of class \( Q(p) \equiv Q(p,q) \), the induced Gelfand-Tsetlin basis of the dual representation of class \( Q(p,q)^* = Q(q,p) \equiv Q(\bar{p}) \) is the basis comprised by the vectors

\begin{equation}
\bar{e}(\bar{p}; \bar{\nu}, J) = (-1)^{2t_{\nu} + u_{\nu}} e^*(p; \nu, J) ,
\end{equation}

where \( e^*(p; \nu, J) \in Q(\bar{p}) \) is the Hermitian dual of \( e(p; \nu, J) \), that is, the dual of \( e(p; \nu, J) \) via Hermitian inner product on \( Q(p) \), and where \( t_{\nu} \) and \( u_{\nu} \) stand as in \eqref{eq:2.17}, with \( \nu \) and \( \bar{\nu} \) satisfying the duality relations \eqref{eq:2.27}, that is, using \eqref{eq:2.27},

\begin{equation}
\text{duality:} \quad \nu \leftrightarrow \bar{\nu} \iff \Delta_{\nu,\bar{\nu}}^{p+q} = 1 .
\end{equation}

Remark 2.4. A GT basis \( \{e(p; \nu, J)\} \) and the induced GT basis \( \bar{e}(\bar{p}; \bar{\nu}, J) \) are related by an involution. Considering the natural isomorphism between a finite dimensional vector space and its double dual, the dual GT basis induced by \( \bar{e}(\bar{p}; \bar{\nu}, J) \) is precisely \( \{e(p; \nu, J)\} \), that is,

\begin{equation}
e(p; \nu, J) = (-1)^{2(t_{\nu} + u_{\nu})} e^*(p; \nu, J) .
\end{equation}

This contrasts with standard convention for irreducible representations of \( SU(2) \), since for an \( SU(2) \)-representation with spin number \( j \), there is a phase \((-1)^{2j}\) between a standard basis and the basis of the double dual space induced by the isomorphism of the dual space, c.f. \cite{24}.

Definition 2.5. The Wigner D-functions (in the GT basis) of an irreducible unitary \( SU(3) \)-representation \( \rho \) of class \( Q(p) \equiv Q(p,q) \) are the functions

\begin{equation}
D_{\nu,\mu,L}^p(g) = \langle e(p; \nu, J) | \rho(g)e(p; \mu, L) \rangle .
\end{equation}

Using the conjugate symmetry of the inner product and the relation \( \langle v|w \rangle = \langle w^*|v^* \rangle \) between inner products of \( \mathcal{H} \) and \( \mathcal{H}^* \), we get, for \( \Delta_{\nu,\bar{\nu}}^{p+q} = \Delta_{\mu,\bar{\mu}}^{p+q} = 1 \),

\begin{equation}
\overline{D_{\nu,\mu,L}^p(g)} = \langle \rho(g)e(p; \mu, L) | e(p; \nu, J) \rangle = \langle e^*(p; \nu, J)|\bar{\rho}(g)\rangle e^*(p; \mu, L) \\
= (-1)^{2(t_{\nu} + u_{\nu} + t_{\mu} + u_{\mu})} \langle \bar{e}(\bar{p}; \bar{\nu}, J) | \bar{\rho}(g)\bar{e}(\bar{p}; \bar{\mu}, L) \rangle \\
= (-1)^{2(t_{\nu} + u_{\nu} + t_{\mu} + u_{\mu})} D_{\nu,\mu,L}^{\bar{p}}(g) .
\end{equation}

2.2. Clebsch-Gordan series and the space of operators. An irreducible unitary representation \( \rho \) of class \( Q(p) \) on \( \mathcal{H} \) extends to an unitary representation (with respect to the trace inner product) on \( \mathcal{B}(\mathcal{H}) \) via the \( \rho(g) \)-action by conjugation

\begin{equation}
\rho(g) : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) , \quad A \mapsto A^g := \rho(g)A\rho(g)^{-1} , \quad \forall g \in SU(3) .
\end{equation}

Now, we recall that \( \mathcal{B}(\mathcal{H}) \) is naturally isomorphic to \( \mathcal{H} \otimes \mathcal{H}^* \) in a manner that \eqref{eq:2.33} matches the representation \( \rho \otimes \bar{\rho} \) of class \( Q(p) \otimes Q(\bar{p}) \) on \( \mathcal{H} \otimes \mathcal{H}^* \).
The decomposition of a tensor product of irreducible unitary \(SU(3)\)-representations into a direct sum of irreducible unitary \(SU(3)\)-representations is known as the Clebsch-Gordan series of \(SU(3)\).

**Theorem 2.6** (cf. e.g. [9]). The Clebsch-Gordan series of \(SU(3)\) is given by

\[
(2.34) \quad Q(p_1, q_1) \otimes Q(p_2, q_2) = \bigoplus_{n=0}^{\min(p_1, q_2)} \bigoplus_{m=0}^{\min(p_2, q_1)} Q(p_1 - n, p_2 - m; q_1 - m, q_2 - n),
\]

where

\[
(2.35) \quad Q(r_1, r_2; s_1, s_2) = Q(r_1 + r_2, s_1 + s_2) \oplus \left( \bigoplus_{k=1}^{\min(r_1, r_2)} Q(r_1 + r_2 - 2k, s_1 + s_2 + k) \right) \oplus \left( \bigoplus_{k=1}^{\min(s_1, s_2)} Q(r_1 + r_2 + k, s_1 + s_2 - 2k) \right).
\]

**Corollary 2.7.** For \(p_1 = q_2 = p\) and \(q_1 = p_2 = q\), the Clebsch-Gordan series assumes the form

\[
(2.36) \quad Q(p, q) \otimes Q(q, p) = \bigoplus_{n=0}^{p} \bigoplus_{m=0}^{q} \left\{ Q(p + q - n - m, p + q - n - m) \right. \left. \oplus \bigoplus_{k=1}^{\min(p-n,q-m)} \left( Q(p + q - n - m - 2k, p + q - n - m + k) \right) \right. \left. \oplus Q(p + q - n - m + k, p + q - n - m - 2k) \right\}.
\]

Note that an irreducible unitary representation of class \(Q(a)\) may appear more than once in the CG series of \(Q(p_1) \otimes Q(p_2)\) and also of \(Q(p) \otimes Q(\tilde{p})\), in general.

**Notation 2.** We shall denote the multiplicity of \(Q(a) = Q(a, b)\) in the Clebsch-Gordan series of \(Q(p_1) \otimes Q(p_2)\) by

\[
(2.37) \quad m(p_1, p_2; a).
\]

To distinguish multiple appearances of the same class of representation in a Clebsch-Gordan series, we will write

\[
(2.38) \quad (a; \sigma) = (a, b; \sigma), \quad Q(a; \sigma) \equiv Q(a, b; \sigma),
\]

where the index \(\sigma\) counts the multiplicity starting from 1 to \(m(p_1, p_2; a)\).

We thus provide two basis for \(Q(p_1) \otimes Q(p_2)\).

**Definition 2.8.** An uncoupled GT basis of the tensor product representation \(Q(p_1) \otimes Q(p_2)\) is a basis comprised by the tensor product of GT basis \(\{e(p_1; \nu_1, J_1)\}\) of \(Q(p_1)\) and \(\{e(p_2; \nu_2, J_2)\}\) of \(Q(p_2)\).

**Definition 2.9.** A coupled GT basis of the tensor product representation \(Q(p_1) \otimes Q(p_2)\) is the union of GT basis \(\{e_{p_1, p_2}((a; \sigma); \nu, J)\}\) of each \(Q(a; \sigma) \equiv Q(a, b; \sigma)\) in the Clebsch-Gordan series of \(Q(p_1) \otimes Q(p_2)\).
Remark 2.10. To simplify notation, whenever clear from context, we write
\[(2.39) \quad e((a;\sigma);\nu,J) \equiv e_{p_1,p_2;}(a;\sigma);\nu,J).\]

Also, unless specified otherwise, from now on we shall always refer to the uncoupled and coupled basis of the tensor product as meaning their respective GT basis, and likewise for the Clebsch-Gordan coefficients defined below.

Both basis are orthonormal, so they are related by an unitary transformation.

Definition 2.11. The Clebsch-Gordan coefficients (in the GT basis) are the entries of the unitary transformation that relates a coupled and an uncoupled GT basis of \(Q(p_1) \otimes Q(p_2)\):
\[(2.40) \quad C_{\nu_{1j_1},\nu_{2j_2}}^{p_1,p_2;}(a;\sigma) = \langle e((a;\sigma);\nu,J) | e(p_1;\nu_1,J_1) \otimes e(p_2;\nu_2,J_2) \rangle,\]
where \(\langle \cdot | \cdot \rangle\) is the \(SU(3)\)-invariant inner product induced by the ones on each representation of the tensor product.

We are able to fix a relative phase between a coupled and an uncoupled basis so that all Clebsch-Gordan coefficients are real. Usually, one chooses some set of Clebsch-Gordan coefficients to be positive and the remaining coefficients are completely determined via the action of the step operators on the basis vectors, cf. e.g. [26]. In the next section, we shall return to this problem. What is important now is that we take Clebsch-Gordan coefficients as real numbers, so that we have
\[(2.41) \quad e(p_1;\nu_1,J_1) \otimes e(p_2;\nu_2,J_2) = \sum_{\nu_{1j_1},\nu_{2j_2}} C_{\nu_{1j_1},\nu_{2j_2}}^{p_1,p_2;}(a;\sigma) e((a;\sigma);\nu,J),\]
\[(2.42) \quad e((a;\sigma);\nu,J) = \sum_{\nu_{1j_1},\nu_{2j_2}} C_{\nu_{1j_1},\nu_{2j_2}}^{p_1,p_2;}(a;\sigma) e(p_1;\nu_1,J_1) \otimes e(p_2;\nu_2,J_2),\]
\[(2.43) \quad \sum_{\nu_{1j_1},\nu_{2j_2}} C_{\nu_{1j_1},\nu_{2j_2}}^{p_1,p_2;}(a;\sigma) C_{\nu_{1j_1},\nu_{2j_2}}^{p_1,p_2;}(a;\sigma) = \delta_{\nu_1,\nu_1'} \delta_{\nu_2,\nu_2'} \delta_{J_1,J_1'} \delta_{J_2,J_2'};\]
\[(2.44) \quad \sum_{\nu_{1j_1},\nu_{2j_2}} C_{\nu_{1j_1},\nu_{2j_2}}^{p_1,p_2;}(a;\sigma) C_{\nu_{1j_1},\nu_{2j_2}}^{p_1,p_2;}(b;\sigma) = \delta_{a,b} \delta_{\sigma_1,\sigma_2} \delta_{\nu,\nu'} \delta_{J,J'}.\]

Remark 2.12. From the way the GT basis for \(SU(3)\)-representations were constructed using \(SU(2)\)-subrepresentations, the \(SU(3)\) Clebsch-Gordan coefficients in the GT basis are related to the \(SU(2)\) Clebsch-Gordan coefficients by
\[(2.44) \quad C_{\nu_{1j_1},\nu_{2j_2}}^{p_1,p_2;}(a;\sigma) = C_{u_{1u_1},u_{2u_2},u_{1u_1}u_{2u_2}}^{J_1,J_2,J} C_{i_{1i_1},i_{2i_2}}^{p_1,p_2;}(a;\sigma),\]
where the second coefficient on the r.h.s. is called isoscalar factor, and this provides explicit equations for the \(SU(3)\) Clebsh-Gordan coefficients in the GT basis in terms of explicit equations for the \(SU(2)\) Clebsh-Gordan coefficients, as found in [26] and [27], for instance.

From decompositions \((2.41)-(2.42)\), we obtain some sufficient conditions for the Clebsch-Gordan coefficients to be zero. Since \(e(p_1;\nu_1,J_1)\) and \(e(p_2;\nu_2,J_2)\) are basis vectors of \(SU(2)\)-representations with spin numbers \(J_1\) and \(J_2\), their tensor product is a vector of the tensor product of the \(SU(2)\)-representations they belong to, that is, the Clebsch-Gordan coefficients are zero if \(J_1\), \(J_2\) and \(J\) do not satisfy the
triangle inequality. Also, using superscripts to identify the \(SU(3)\)-representations, the operators \(T_3\) and \(U_3\) in \(Q(p_1) \otimes Q(p_2)\) have the form

\[
\bigoplus_{(a;\sigma)} T_3^{(a;\sigma)} = T_3^{p_1} \otimes \mathbb{1} + \mathbb{1} \otimes T_3^{p_2}, \quad \bigoplus_{(a;\sigma)} U_3^{(a;\sigma)} = U_3^{p_1} \otimes \mathbb{1} + \mathbb{1} \otimes U_3^{p_2},
\]

where \(\mathbb{1}\) is the identity operator. Thus, the Clebsch-Gordan coefficients are zero if \(t \neq t_1 + t_2\) or \(u \neq u_1 + u_2\), for \(t, t_1, t_2, u, u_1\) and \(u_2\) being the eigenvalues of \(T_3\) and \(U_3\) related to the weights \(\nu, \nu_1\) and \(\nu_2\).

To summarize, let \(\delta(x, y, z)\) be equal to 1 if \(x, y\) and \(z\) satisfy the triangle inequality, or 0 otherwise, and let

\[
\nabla_{\nu, \mu} := \delta_{\nu, t_{\mu}} \delta_{u_{\nu}, u_{\mu}},
\]

where \(\delta_{m, n}\) is the Kronecker delta. Then,

\[
(2.45) \quad C_{\nu_1, t, \nu_2}^{p_1, p_2} (a;\sigma) \neq 0 \quad \implies \quad \begin{cases} \nabla_{\nu_1 + \nu_2, \nu} = 1 \\
\delta(J_1, J_2, J) = 1 \end{cases}
\]

2.3. Mixed Casimir operators and symmetric CG coefficients. We have avoided until now the problem of specifying a decomposition for degenerate representations in general Clebsch-Gordan series \(Q(p_1) \otimes Q(p_2)\). If \(Q(a)\) is such that \(m(p_1, p_2; a) > 1\) (cf. Notation 2), there is no canonically unique way to decompose

\[
(2.47) \quad \bigoplus_{\sigma = 1} Q(a; \sigma) \subset Q(p_1) \otimes Q(p_2)
\]

into irreducible representations of class \(Q(a)\). To fix a unique convention, we shall use the method of mixed Casimir operators, based on [8, 7, 22]. The envisaged decomposition provides Clebsch-Gordan coefficients with extra symmetries.

Let \(\mathcal{H}\) be a Hilbert space carrying an irreducible representation of class \(Q(p) = Q(p, q)\) and let \(A_{jk}\), for \(j, k \in \{1, 2, 3\}\), be the generators satisfying:

\[
(2.48) \quad A_{12} = T_+, \quad A_{23} = U_+, \quad T_3 = \frac{1}{2}(A_{11} - A_{22}), \quad U_3 = \frac{1}{2}(A_{22} - A_{33}),
\]

\[
(2.49) \quad A_{jk} = A_{kj}, \quad A_{11} + A_{22} + A_{33} = 0, \quad [A_{jk}, A_{lm}] = \delta_{j, l} A_{jm} - \delta_{j, m} A_{lk}.
\]

Then, \(Q(\tilde{p}) = Q(q, p)\) is generated by the operators

\[
(2.50) \quad \tilde{A}_{jk} = -A_{kj}.
\]

For \(Q(p, q)\), the quadratic and cubic Casimir operators are

\[
(2.51) \quad C_2 := \frac{1}{2} \sum_{j,k=1}^{3} A_{jk} A_{kj}, \quad C_3 := \sum_{j,k,l=1}^{3} A_{jk} A_{kl} A_{lj},
\]

so that (cf. 22)

\[
(2.52) \quad C_2 = \frac{1}{3} [(p + q)(p + q + 3) - pq] \mathbb{1},
\]

\[
C_3 - 3C_2 = \frac{1}{9} [(p - q)(p + 2q + 3)(2p + q + 3)] \mathbb{1}.
\]

Now, for \(x \in \{1, 2, 3\}\), consider \(\mathcal{H}_x\) carrying the representation \(Q(p_x) = Q(p_x, q_x)\) and the triple tensor product \(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3\). Let \(A_{jk}^{(x)}, C_2^{(x)}\) and \(C_3^{(x)}\) be operators...
relative to $Q(p_z)$ and, for simplicity, given an operator $A^{(x)}$ on $H_x$, we just write $A^{(x)}$ to denote its tensor product with the identity operator. Thus, for $x \neq y$,

$$A_{jk}^{(x)} A_{lm}^{(y)} = A_{lm}^{(y)} A_{jk}^{(x)} .$$

**Definition 2.13.** The quadratic and cubic mixed Casimir operators are defined by

$$C_2^{xy} := \sum_{j,k} A_j^{(x)} A_k^{(y)} = \sum_{j,k} A_j^{(y)} A_k^{(x)} := C_2^{yx} ,$$

$$C_3^{xyz} := \sum_{j,k,l} A_j^{(x)} A_k^{(y)} A_l^{(z)} ,$$

for superindices not all equal.

If $x, y, z$ are all distinct, equation (2.53) implies

$$C_3^{xyz} = C_3^{yxx} = C_3^{zyx} .$$

Furthermore,

$$C_3^{xyy} = \sum_{j,k,l} A_j^{(x)} A_k^{(y)} A_l^{(y)} = \sum_{j,k,l} A_k^{(y)} A_l^{(y)} A_j^{(x)} = C_3^{yyx} ,$$

but, on the other hand,

$$C_3^{xxy} = \sum_{j,k,l} A_j^{(x)} A_k^{(y)} A_l^{(x)} = \sum_{j,k,l} A_k^{(y)} A_l^{(x)} A_j^{(x)} = \sum_{j,k,l} A_k^{(y)} \left( \delta_{l,k} A_j^{(x)} - A_l^{(x)} A_j^{(y)} + A_j^{(x)} A_l^{(y)} \right) = C_3^{yxy} - C_3^{xxy} .$$

**Remark 2.14.** Equation (2.53) shows that, although formally we could write equation (2.56) in shorthand notation as $C_3^{xyz} \equiv \text{tr}(A^{(x)} A^{(y)} A^{(z)})$, this notation is actually misleading and must be used with care. We also note that such mixed Casimir operators in Definition 2.13 arise, for example, from the Casimir operators on $Q(p_1) \otimes Q(p_2) \otimes Q(p_3)$. For the quadratic, we have

$$\frac{1}{2} \sum_{j,k} \left( A_{jk}^{(1)} + A_{jk}^{(2)} + A_{jk}^{(3)} \right) \left( A_{kj}^{(1)} + A_{kj}^{(2)} + A_{kj}^{(3)} \right) = C_2^{(x)} + C_2^{(y)} + C_2^{(z)} + C_2^{(y)} + C_2^{(z)} + C_2^{(z)} ,$$

with $C_3^{xyz}$ in the decomposition of the cubic Casimir on $Q(p_1) \otimes Q(p_2) \otimes Q(p_3)$.

For representations $Q(p_z)$ generated by $A_{jk}^{(z)} = -A_{kj}^{(z)}$, cf. (2.50), we have

$$\tilde{C}_2^{xy} = \sum_{j,k} A_j^{(x)} A_k^{(y)} = \sum_{j,k} A_j^{(y)} A_k^{(x)} = C_2^{xy} ,$$

$$\tilde{C}_3^{xyz} = \sum_{j,k,l} A_j^{(x)} A_k^{(y)} A_l^{(z)} = -\sum_{j,k,l} A_k^{(y)} A_l^{(z)} A_j^{(x)} .$$

In particular, it is straightforward to see that

$$\tilde{C}_3^{xyy} = C_3^{yy} - C_3^{xyy} .$$

Now, from the mixed cubic Casimir operators, we define the operators

$$S_{xy} := \frac{1}{2} \left( C_3^{xy} - C_3^{yxx} \right) = \frac{1}{2} \left( C_3^{yxy} - C_3^{xyy} \right) ,$$

$$S_{xyz} := \frac{1}{3} \left( S_{xy} + S_{yz} + S_{zx} \right) ,$$
and likewise for \( \tilde{S}_{xy} \) and \( \tilde{S}_{xyz} \). Then, from (2.54)-(2.58) and (2.59)-(2.61) we have

**Lemma 2.15.** The operators \( S_{xy} \) and \( S_{xyz} \) are anti-symmetric under odd permutation of the indices and under dualization, that is,

\[
S_{xy} = -S_{yx} = -\tilde{S}_{xy} ,
\]

\[
S_{xyz} = -S_{yxz} = -S_{xzy} = -\tilde{S}_{xyz} .
\]

Now, let \( \mathcal{H}^0 = \mathcal{H}_{123}^0 \) be a maximal subspace of \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \) where \( SU(3) \) acts trivially. That is, for all \( j, k \in \{1, 2, 3\} \),

\[
A_{jk}^{(1)} + A_{jk}^{(2)} + A_{jk}^{(3)} = 0 \quad \text{on} \quad \mathcal{H}^0 .
\]

**Lemma 2.16.** \( \mathcal{H}^0 \) is not null if and only if there is a representation of class \( Q(p_3) \) in the Clebsch-Gordan series of \( Q(p_1) \otimes Q(p_2) \).

**Proof.** From the Clebsch-Gordan series

\[
Q(p_1) \otimes Q(p_2) = \bigoplus_{(a; \sigma)} Q(a; \sigma) ,
\]

\[
Q(p_1) \otimes Q(p_2) \otimes Q(p_3) = \bigoplus_{(a; \sigma)} Q(a; \sigma) \otimes Q(p_3) ,
\]

note that \( k \geq 1 \) in (2.35), so there exists a factor of class \( Q(0, 0) \) in the CG series of \( Q(a, b) \otimes Q(p_3, q_3) \) iff there is \( 0 \leq n \leq \min(a, q_3) \) and \( 0 \leq m \leq \min(p_3, b) \) satisfying

\[
a - n + p_3 - m = b - m + q_3 - n = 0 .
\]

The only possible solution is \( n = a = q_3 \) and \( m = p_3 = b \). Thus, \( \mathcal{H}^0 \) is not null iff there is a representation of class \( Q(a) = Q(p_3) \) in the r.h.s. of (2.67). \( \square \)

**Lemma 2.17.** On \( \mathcal{H}^0 \), the following holds:

\[
S_{12} := \frac{1}{3} (S_{12} + S_{23} + S_{31}) = S_{12} - \frac{1}{3} C_3^{(1)} + \frac{1}{3} C_3^{(2)} .
\]

**Proof.** Using (2.57)-(2.58), (2.62) and (2.66), by straightforward computation,

\[
S_{23} = \frac{1}{2} (C_3^{121} + 3 C_3^{122} + 2 C_3^{(2)} - C_2^{12}) .
\]

And by formal identification of the formulas,

\[
S_{31} = -S_{13} = -\frac{1}{2} (C_3^{122} + 3 C_3^{211} + 2 C_3^{(1)} - C_2^{21}) .
\]

Thus, from (2.69)-(2.70),

\[
S_{23} + S_{31} = C_3^{122} - C_3^{211} - C_3^{(1)} + C_3^{(2)} = 2 S_{12} - C_3^{(1)} + C_3^{(2)} ,
\]

where we used \( C_2^{12} = C_2^{21} \). Therefore, from (2.71) we obtain (2.68), on \( \mathcal{H}^0 \). \( \square \)

**Notation 3.** In view of the previous lemma, for \( \mathcal{H}^0 \neq 0 \), we shall denote

\[
S^0_{123} := S_{123} |_{\mathcal{H}^0} .
\]

**Lemma 2.18.** The operators \( S_{12} \) and \( S^0_{123} \) are Hermitian and \( SU(3) \)-invariant.
Proof. For hermiticity, by straightforward calculation, we have
\[
(C_3^{xyy})^\dagger = \sum_{j,k,l} (A_j^{(x)} A_k^{(y)} A_l^{(y)})^\dagger = \sum_{j,k,l} A_j^{(y)} A_k^{(y)} A_l^{(y)} = C_3^{xyy} = C_3^{xyy}
\]
\[
\implies (S_{12})^\dagger = \frac{1}{2}(C_3^{122} - C_3^{211})^\dagger = \frac{1}{2}(C_3^{122} - C_3^{211}) = S_{12} .
\]
For the SU(3)-invariance, by straightforward computation, we obtain
\[
\begin{align*}
A_{cd}^{(x)} C_3^{xyy} &= \sum_{k,l} A_{ck}^{(x)} A_{kl}^{(y)} A_{ld}^{(y)} - \sum_{j,l} A_{jd}^{(y)} A_{dl}^{(y)} A_{lj}^{(y)} + C_3^{xyy} A_{cd}^{(x)} \\
A_{cd}^{(x)} C_3^{yxx} &= \sum_{j,l} A_{j(cl)}^{(y)} A_{dl}^{(y)} A_{lj}^{(y)} - \sum_{k,l} A_{ck}^{(x)} A_{kl}^{(y)} A_{ld}^{(y)} + C_3^{yxx} A_{cd}^{(x)} \\
\implies [A_{cd}^{(1)}, C_3^{122} - C_3^{211}] &= [A_{cd}^{(2)}, C_3^{211} - C_3^{122}] \implies [A_{cd}^{(1)} + A_{cd}^{(2)}, S_{12}] = 0 .
\end{align*}
\]
Hence, $S_{12}$ is Hermitian and SU(3)-invariant. The result for $S_{123}^0$ follows immediately from Lemma 2.17.

In this way, we decompose degenerate irreducible representations in the Clebsch-Gordan series of $Q(p_1) \otimes Q(p_2)$ via diagonalization of the operator
\[
S := S_{12} - \frac{1}{3} C_3^{(1)} + \frac{1}{3} C_3^{(2)} ,
\]
satisfying
\[
S_{|H_{123}^0} = S_{123}^0 ,
\]
cf. (2.72). Because $S$ is built from Casimir operators, the eigenvalues $s_{123}$ of $S_{123}^0$ depend only on the subrepresentations comprising $H_{123}^0 \subset H_1 \otimes H_2 \otimes H_3$.

**Theorem 2.19.** (8.22). The eigenvalues of $S_{123}^0$ are distinct, for distinct irreducible subrepresentations in $H_{123}^0$.

**Remark 2.20.** In [3], the authors use a polynomial basis for SU(3)-representations, as constructed e.g. in [23], to compute the matrix entries of an operator equivalent to $S$ in order to prove Theorem 2.19. In the Appendix B, we introduce the approach used in [3] and illustrate the explicit (in general long) computations for a simple case: the degenerate representation of class $Q(1,1)$ within $Q(1,1) \otimes Q(1,1)$.

Thus, for
\[
m(p_1, p_2; a) = \bigoplus_{\sigma=1} Q(a; \sigma) \subset Q(p_1) \otimes Q(p_2) ,
\]
with $Q(\tilde{a}) = Q(p_3)$, cf. Lemma 2.16, we can define a function
\[
s_{p_1, p_2; a} : \{1, \ldots, m(p_1, p_2; a)\} \to \mathbb{R} , \ \sigma \mapsto s_{p_1, p_2; a}(\sigma) ,
\]
which indexes the eigenvalues of $S_{123}^0$ in this case. For simplicity, we shall denote
\[
s_{p_1, p_2; a} \equiv s_a
\]
and, henceforth, we shall adopt the convention that $s_a$ is an increasing function of the multiplicity counting index, that is,
\[
\{1, \ldots, m(p_1, p_2; a)\} \ni \sigma \mapsto s_a(\sigma) \in \mathbb{R} , \ s_a(\sigma) < s_a(\sigma + 1) .
\]
In other words, the eigenvalues of $S_{123}^0$ are indexed by increasing order.
Notation 4. For \( \bigoplus \alpha Q(\alpha; \sigma) \) in the CG series of \( Q(p_1) \otimes Q(p_2) \), the following involution in the set of multiplicity indices will be relevant:

\[
(2.79) \quad \{1, \ldots, m(p_1, p_2; a)\} \ni \sigma \mapsto \bar{\sigma} = m(p_1, p_2; a) - \sigma + 1.
\]

Remark 2.21. We emphasize that the involution (2.79) is at par with the involution \( s \mapsto -s \) in the set of eigenvalues for \( S \) under \( S \mapsto \bar{S} \), which is a consequence of (2.64)-(2.65), taking into account the convention (2.78).

From (2.93) and Lemmas 2.15-2.18, considering the convention in (2.78), we can choose coupled basis \( \{e_12((\bar{p}_3; \sigma); \nu, J)\}, \{e_21((\bar{p}_3; \sigma); \nu, J)\}, \{e_13((\bar{p}_3; \sigma); \nu, J)\} \) and \( \{\bar{e}_{12}((\bar{p}_3; \sigma); \nu, J)\} \) for \( Q(p_1) \otimes Q(p_2) \), \( Q(p_2) \otimes Q(p_1) \), \( Q(p_1) \otimes Q(p_3) \) and \( Q(\bar{p}_3) \otimes Q(\bar{p}_3) \), respectively, such that

\[
(2.80) \quad 1 \leq \sigma \leq m(p_1, p_2; \bar{p}_3) = m(p_2, p_1; \bar{p}_3) = m(p_1, p_3; \bar{p}_2) = m(p_1, p_2; p_3)
\]

and \( H^0 \) is spanned by

\[
\sum_{\nu, J} \left( -1 \right)^{2(t + u)} e_{12}((\bar{p}_3; \sigma); \nu, J) \otimes e(p_3; \nu, J)
\]

\[
= (-1)^{|p_1| + |p_2| + |\alpha|} \sum_{\nu, J} \left( -1 \right)^{2(t + u)} e_{21}((\bar{p}_3; \sigma); \nu, J) \otimes e(p_3; \nu, J)
\]

\[
= (-1)^{|p_1| + |p_2| + |\alpha|} \sum_{\nu, J} \left( -1 \right)^{2(t + u)} e_{13}((\bar{p}_3; \sigma); \nu, J) \otimes e(p_2; \nu, J)
\]

\[
= (-1)^{|p_1| + |p_2| + |\alpha|} \sum_{\nu, J} \left( -1 \right)^{2(t + u)} \bar{e}_{12}((\bar{p}_3; \sigma); \nu, J) \otimes \bar{e}(\bar{p}_3; \nu, J),
\]

where we have made use of (2.79). As consequence, now the Clebsch-Gordan coefficients satisfy a bigger set of symmetry relations, as follows.

Theorem 2.22. For representations of class \( Q(p_1) \) and \( Q(p_2) \), the Clebsch-Gordan coefficients for the Clebsch-Gordan series \( Q(p_1) \otimes Q(p_2) = \bigoplus Q(\alpha; \sigma) \) satisfy

\[
C_{\nu_1, J_1, \nu_2, J_2; \nu, J}^{p_1, p_2, (\alpha, \sigma)} = (-1)^{|p_1| + |p_2| + |\alpha|} C_{\nu_1, J_1, \nu_2, J_2; \nu, J}^{p_2, p_1, (\sigma, \alpha)}
\]

\[
= (-1)^{|p_1| - 2(t + u)} \sqrt{\frac{\dim Q(\alpha)}{\dim Q(p_2)}} C_{\nu_1, J_1, \nu_2, J_2; \nu, J}^{p_1, (\alpha, \sigma)}
\]

\[
= (-1)^{|p_1| + |p_2| + |\alpha|} C_{\nu_1, J_1, \nu_2, J_2; \nu, J}^{p_1, \bar{p}_2, (\sigma, \alpha)}
\]

\[
= (-1)^{|p_1| + |p_2| + |\alpha|} C_{\nu_1, J_1, \nu_2, J_2; \nu, J}^{\bar{p}_1, \bar{p}_2, (\sigma, \alpha)},
\]

Proof. Writing (2.81) with \( \alpha = \bar{p}_3 \) and taking the inner product with suitable uncoupled basis, the result follows straightforwardly from the symmetries of \( S \) satisfying Lemma 2.17, cf. (2.63)-(2.65), using (2.78) and (2.79). \( \square \)

We highlight that each generator \( A_{ijk} \) satisfying (2.48)-(2.49) can be realized as a real matrix on a GT basis, so \( S \) can be seen as a symmetric real matrix acting on

\[
\text{Span}_R \{ e(p_1; \nu, J) \otimes e(p_2; \mu, L) \},
\]

hence the elements of any basis of the previous paragraph can be constructed as real linear combinations of the respective uncoupled basis. With this convention, all Clebsch-Gordan coefficients are still real, and (2.43) and (2.46) still hold.
In particular, the Hermitian conjugate $\dagger$ of operators in $Q(p) \otimes Q(\bar{p})$ satisfies
\begin{equation}
\eta^\dagger((a;\sigma);\nu,J) = (-1)^{2(l+u)}\eta((\bar{a};\sigma);\bar{\nu},J) \tag{2.88}
\end{equation}
and the phases are chosen such that
\begin{equation}
\eta((0,0);(0,0,0),0) = \frac{(-1)^{|p|}}{\sqrt{\dim Q(p)}} \mathbb{I}, \tag{2.89}
\end{equation}
this being the only element with non-vanishing trace.

Besides Hermitian conjugate, the adjoint
\begin{equation}
\ast : Q(p) \otimes Q(\bar{p}) \rightarrow Q(\bar{p}) \otimes Q(p) : A \mapsto A^*, \tag{2.85}
\end{equation}
is given in the uncoupled basis by
\begin{equation}
\ast : e(p,\nu,J) \otimes \bar{e}(\bar{p},\mu,L) = (-1)^{2(l+u)}e(p,\nu,J) \otimes e^*(p,\bar{\mu},L) \tag{2.86}
\end{equation}
\begin{equation}
\rightarrow (-1)^{2(l+u)}e^*(p,\bar{\mu},L) \otimes e(p,\nu,J) = \bar{e}(\bar{p},\mu,L) \otimes e(p,\nu,J), \tag{2.87}
\end{equation}
cf. (2.28). Thus, for the coupled basis of $Q(p) \otimes Q(\bar{p})$, cf. (2.42), the adjoint is
\begin{equation}
\ast : e((a;\sigma);\nu,J) = \sum_{\nu_1,\nu_2,\nu_2} C^{p}_{\nu_1,\nu_2,\nu_2} p_{\nu_1,\nu_2} (a,\sigma) e(p,\nu_1,J_1) \otimes \bar{e}(\bar{p},\nu_2,J_2) \tag{2.88}
\end{equation}
\begin{equation}
\rightarrow \sum_{\nu_1,\nu_2,\nu_2} C^{\bar{p}}_{\nu_1,\nu_2,\nu_2} p_{\nu_1,\nu_2} (a,\bar{\sigma}) \bar{e}(\bar{p},\nu_2,J_2) \otimes e(p,\nu_1,J_1) \tag{2.89}
\end{equation}
\begin{equation}
= (-1)^{|a|} \sum_{\nu_1,\nu_2,\nu_2} C^{\bar{p}}_{\nu_1,\nu_2,\nu_2} p_{\nu_1,\nu_2} (a,\bar{\sigma}) \bar{e}(\bar{p},\nu_2,J_2) \otimes e(p,\nu_1,J_1). \tag{2.88}
\end{equation}
In the light of Remark 2.4 from the above calculation, we identify
\begin{equation}
\bar{e}((a;\bar{\sigma});\nu,J) := \sum_{\nu_1,\nu_2,\nu_2} C^{\bar{p}}_{\nu_1,\nu_2,\nu_2} p_{\nu_1,\nu_2} (a,\bar{\sigma}) \bar{e}(\bar{p},\nu_2,J_2) \otimes e(p,\nu_1,J_1), \tag{2.88}
\end{equation}
\begin{equation}
e^*((a;\sigma);\nu,J) = (-1)^{|a|} \bar{e}((a;\bar{\sigma});\nu,J). \tag{2.89}
\end{equation}

**Definition 2.23.** Given a coupled basis $\{e((a;\sigma);\nu,J)\}$ of $Q(p) \otimes Q(\bar{p})$, the induced coupled basis of $Q(\bar{p}) \otimes Q(p)$ is the basis $\{\bar{e}((a;\bar{\sigma});\nu,J)\}$ satisfying (2.88)-(2.89).

**2.4. Invariant decomposition of the operator product.** We now use the CG coefficients, introduced and studied in the previous sections, in order to establish the decomposition of the product of operators in the coupled basis.

But before, we establish the following relevant result, which leads to the decomposition of the product of harmonic functions, cf. Theorems 3.5 and 4.5 below, and will be important for asymptotic analysis of twisted products, in the future.

**Lemma 2.24.** The pointwise product of Wigner $D$-functions of $SU(3)$, cf. Definition 2.2, can be decomposed into a sum of the form
\begin{equation}
D^p_{\nu_1,J_1,\nu_1,J_1} D^p_{\nu_2,J_2,\nu_2,J_2} = \sum_{(a,\sigma)} \sum_{\nu_1,\nu_1,\nu_1,\nu_1} C^{p_1}_{\nu_1,\nu_1,\nu_1,\nu_1} p_{\nu_1,\nu_1} (a,\sigma) C^{p_2}_{\nu_1,\nu_1,\nu_1,\nu_1} p_{\nu_1,\nu_1} (a,\sigma) D_a^{\nu_1,J_1,\nu_1,J_1} \tag{2.90}
\end{equation}
where the summations are restricted to $\nabla_{\nu_1+\nu_2,\nu_1+\nu_2} = \nabla_{\nu_1'+\nu_2',\nu_1'+\nu_2'} = 1$, $\delta(J_1, J_2, J) = \delta(J_1, J_2, J') = 1$ and $(a;\sigma)$ such that $Q(a;\sigma)$ is in the CG series of $Q(p_1) \otimes Q(p_2)$.\]
Proof. Let $g \in SU(3)$. From (2.41) and (2.46), we have
\begin{equation}
\sum_{\nu_1, J_1, \nu_2, J_2} D_{\nu_1 J_1, \nu_2 J_2} (g) D_{\nu_2 J_2, \nu_1 J_1} (g) e(p_1; \nu_1, J_1) \otimes e(p_2; \nu_2, J_2) =
\end{equation}
(2.91)
where the sum over $(a; \sigma)$, $\nu'$ and $J'$ satisfies the statement. From (2.42) and (2.46),
\begin{equation}
\sum_{\nu_1, J_1, \nu_2, J_2} D_{\nu_1 J_1, \nu_2 J_2} (g) D_{\nu_2 J_2, \nu_1 J_1} (g) e(p_1; \nu_1, J_1) \otimes e(p_2; \nu_2, J_2) =
\end{equation}
(2.92)
where $\nu, \nu_1, \nu_2, J, J_1$ and $J_2$ are related as in the statement. The decomposition in a basis is unique, so this finishes the proof. □

Now, for the operator product. Again, let $\mathcal{H}$ be a Hilbert space with an irreducible $SU(3)$-representation of class $Q(p)$. Also, let $\{e(p; \nu, J)\}$ be a GT basis of such space and $\{\tilde{e}(\tilde{p}; \tilde{\nu}, \tilde{J})\}$ be the induced GT basis of its dual space. The trivial representation within $\mathcal{B}(\mathcal{H})$ is spanned by the normalized operator
\begin{equation}
\frac{1}{\sqrt{\dim Q(p)}} 1 = \frac{1}{\sqrt{\dim Q(p)}} \sum_{\nu, J} e(p; \nu, J) \otimes e^*(p; \nu, J)
\end{equation}
(2.93)
For operators $A, R \in \mathcal{B}(\mathcal{H})$,
\begin{equation}
A = e(p; \nu, J) \otimes \tilde{e}(\tilde{p}; \nu', J') , \quad R = e(p; \mu, L) \otimes \tilde{e}(\tilde{p}; \mu, L) ,
\end{equation}
their product is given by
\begin{equation}
AR = \delta_{J', L'} \Delta_{\nu', \nu}^{[p]} \delta_{\nu' - \nu, \nu'} (-1)^{2(t_{\nu' + u_{\nu'})} e(p; \nu, J) \otimes \tilde{e}(\tilde{p}; \mu, L) ,
\end{equation}
where $t_{\nu'} = (\nu'_1 - \nu'_2)/2$, $u_{\nu'} = (\nu'_3 - \nu'_3)/2$, for $\nu' = (\nu'_1, \nu'_2, \nu'_3)$, cf. Definition 2.33

Lemma 2.25. Let $(a_1; \sigma_1)$ and $(a_2; \sigma_2)$ label $SU(3)$-representations in the Clebsch-Gordan series of $Q(p) \otimes Q(\tilde{p})$. The operator product of elements of a coupled basis $\{e[(a; \sigma); \nu, J] \equiv e_{p, p'}[(a; \sigma); \nu, J]\}$ of $Q(p) \otimes Q(\tilde{p})$ can be decomposed as
\begin{equation}
e((a_1; \sigma_1); \nu_1, J_1) e((a_2; \sigma_2); \nu_2, J_2) = \sum_{(a; \sigma) \nu, J} \mathcal{M}[p]_{a; \sigma, \nu, J} e((a; \sigma); \nu, J) ,
\end{equation}
with summation over $(a; \sigma)$ restricted to $Q(a; \sigma)$ in the Clebsch-Gordan series of $Q(p) \otimes Q(\tilde{p})$, and
\begin{equation}
\mathcal{M}[p]_{a_1, a_2; \sigma_1, \sigma_2, \nu_1, J_1, \nu_2, J_2} = \sum_{\mu_1, \mu_2, \mu_3, L_1, L_2, L_3} (-1)^{2(t_{\mu_2 + u_{\mu_2})} C_{\mu_1, \mu_2, \mu_3, L_1, L_2, L_3} e_{p, p'}[(a_1; \sigma_1); \nu_1, J_1] \mathcal{C}_{\mu_2, L_2, \mu_3, L_3} \mathcal{C}_{\mu_1, L_1, \mu_3, L_3} e_{p, p'}[(a_2; \sigma_2); \nu_2, J_2] ,
\end{equation}
(2.96)
In particular,
\begin{equation}
\mathcal{M}[p; \nu_1, J_1, \nu_2, J_2, \nu J] \neq 0 \quad \Rightarrow \quad \begin{cases}
\nabla_{\nu_1 + \nu_2, \nu} = 1 \\
J \leq J_1 + J_2 + |p|
\end{cases}
\end{equation}

Proof. Using (2.14), we write
\[ e((a_1; \sigma_1); \nu_1, J_1) = \sum_{\mu_1, \mu_2, \nu_1, J_1} C_{\mu_1, \mu_2, \nu_1, J_1}^{p, \nu} \cdot \bar{\mu}_1, \bar{\mu}_2, \bar{\nu}_1, \bar{\nu}_2, \bar{J}_1, \bar{J}_2 \times \bar{e}(\bar{\mu}_1, \bar{\nu}_1, J_1) \]
\[ e((a_2; \sigma_2); \nu_2, J_2) = \sum_{\mu_3, \mu_4, \nu_2, J_2} C_{\mu_3, \mu_4, \nu_2, J_2}^{p, \nu} \cdot \bar{\mu}_3, \bar{\mu}_4, \bar{\nu}_2, \bar{J}_2 \times \bar{e}(\bar{\mu}_3, \bar{\nu}_2, J_2) \]

From (2.95), we have
\[ e((a_1; \sigma_1); \nu_1, J_1)e((a_2; \sigma_2); \nu_2, J_2) = \sum_{\mu_1, \mu_2, \mu_3, \mu_4, \nu_1, J_1, \nu_2, J_2} (-1)^{2(|\mu_1|+|\mu_2|)} C_{\mu_1, \mu_2, \mu_3, \mu_4, \nu_1, J_1, \nu_2, J_2}^{p, \nu} \cdot \bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4, \bar{\nu}_1, \bar{\nu}_2, \bar{J}_1, \bar{J}_2 \times \bar{e}(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4, \bar{\nu}_1, \bar{\nu}_2, J_1, J_2) \]
then, using (2.41) and (2.46),
\begin{equation}
(2.99)
\sum_{(a; \sigma), \nu_1, J_1, \nu_2, J_2} (-1)^{2(|\mu_1|+|\mu_2|)} C_{\mu_1, \mu_2, \mu_3, \mu_4, \nu_1, J_1, \nu_2, J_2}^{p, \nu} \cdot \bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4, \bar{\nu}_1, \bar{\nu}_2, J_1, J_2 \times \bar{e}(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4, \bar{\nu}_1, \bar{\nu}_2, J_1, J_2) \]
where \(\nabla_{\mu_1, \mu_2, \nu_1} = \nabla_{\mu_3, \mu_4, \nu_2} = \nabla_{\mu_1, \mu_2, \nu_1} = 1, L_1 \leq J_1 + L_2, L_3 \leq J_2 + L_2 \) and \(J \leq L_1 + L_3\). It follows that \(\nabla_{\mu_1 + \mu_2, \nu_1} = 1\) and, considering that \(L_2 \leq |p|/2\), which can be inferred from (2.10), we have \(J \leq J_1 + J_2 + |p|\). \(\square\)

For SU(2), CG coefficients can be substituted by other coefficients with neater symmetry properties, the so-called Wigner 3jm-symbols (c.f. e.g. [24, 27, 29]).

We shall use such variations of the CG coefficients and the recoupling coefficients (defined below), in order to rewrite the decomposition of the operator product in the coupled basis in a way that shows explicitly all symmetries of the product.

**Definition 2.26.** The Wigner coupling symbol is the coefficient denoted by the round brackets below:
\begin{equation}
(2.100) \ \left( \begin{array}{ccc} p_1 & p_2 & (a; \sigma) \\ \nu_1, J_1 & \nu_2, J_2 & \nu, J \end{array} \right) = \frac{(-1)^{|a|+2(|\mu_1|+|\mu_2|)}}{\sqrt{\dim Q(a)}} C_{p_1, p_2, (a; \sigma)}^{p_1, \nu_1, \nu_2, \nu, J}.
\end{equation}

Thus, from Theorem 2.22, we have the symmetries
\begin{equation}
(2.101) \ \left( \begin{array}{ccc} p_1 & p_2 & (a; \sigma) \\ \nu_1, J_1 & \nu_2, J_2 & \nu, J \end{array} \right) = (-1)^{|p_1|+|p_2|+|a|} \left( \begin{array}{ccc} p_2 & p_1 & (a; \sigma) \\ \nu_2, J_2 & \nu_1, J_1 & \nu, J \end{array} \right) \]
\[ = (-1)^{|p_1|+|p_2|+|a|} \left( \begin{array}{ccc} p_1 & a & (p_2; \tilde{\sigma}) \\ \nu_1, J_1 & \nu, J & \nu_2, J_2 \end{array} \right) \]
\[ = (-1)^{|p_1|+|p_2|+|a|} \left( \begin{array}{ccc} \tilde{p}_1 & \tilde{p}_2 & (a; \tilde{\sigma}) \\ \bar{\nu}_1, J_1 & \bar{\nu}_2, J_2 & \bar{\nu}, J \end{array} \right).
\]

\[\text{References [4, 10] define the generalized Wigner 3jm-symbols for general compact groups. Here, we shall follow the conventions in [23] for SU(3).}\]
Now, consider the two Clebsch-Gordan sub-series for \(Q(p_1) \otimes Q(p_2) \otimes Q(p_3)\),
\[
Q(p_1) \otimes Q(p_2) = \bigoplus_{(a_{12}; \sigma_{12})} Q(a_{12}; \sigma_{12}), \quad Q(a_{12}) \otimes Q(p_3) = \bigoplus_{(a; \sigma)} Q(a; \sigma),
\]
\[
Q(p_2) \otimes Q(p_3) = \bigoplus_{(a_{23}; \sigma_{23})} Q(a_{23}; \sigma_{23}), \quad Q(p_1) \otimes Q(a_{23}) = \bigoplus_{(\alpha'; \sigma')} Q(\alpha'; \sigma'),
\]
defining two bases, \(\{e((a_{12}; \sigma_{12}), (a; \sigma); \mu, L)\}\) and \(\{e((a_{23}; \sigma_{23}), (\alpha'; \sigma'); \mu', L')\}\),
for \(Q(p_1) \otimes Q(p_2) \otimes Q(p_3)\) satisfying (cf. (2.42) and (2.100)):
\[
\langle e((a_{12}; \sigma_{12}), (a; \sigma); \mu, L) | e((a_{23}; \sigma_{23}), (\alpha'; \sigma'); \mu', L') \rangle \neq 0 \iff \begin{cases} a = a' \\ (\mu, L) = (\mu', L') \end{cases}
\]
Also, the coefficients \(\langle e((a_{12}; \sigma_{12}), (a; \sigma); \mu, L) | e((a_{23}; \sigma_{23}), (\alpha'; \sigma'); \mu', L') \rangle\) don’t depend on weight and spin number \(L\) since these vectors can be generated from the highest weight vectors of their respective representations by applying ladder operators and we can write a highest weight vector of one basis as a linear combination of highest weight vectors of the other basis for equivalent representations.

**Definition 2.27.** The Wigner recoupling symbol is the coefficient denoted by the curly brackets below:
\[
\begin{pmatrix} p_1 & p_2 & p_3 \\ a_{12}; \sigma_{12} & (a; \sigma, \sigma') & (a_{23}; \sigma_{23}) \end{pmatrix} = \frac{(-1)^{|a_{23}|+|p_2|+|p_3|}}{\sqrt{\dim Q(a_{12}) \dim Q(a_{23})}} \sum_{\nu_1, J_1} \sum_{\nu_2, J_2} \sum_{\nu_3, J_3} C^{a_{12}}_{\mu_{12}; L_{12}, \nu_{12}, J_{12}} C^{a_{23}}_{\mu_{23}; L_{23}, \nu_{23}, J_{23}} C^{p_1}_{\mu_1, J_1} C^{p_2}_{\mu_2, J_2} C^{p_3}_{\mu_3, J_3} \prod_{\nu, J} C^{(a; \sigma)}_{\nu, J} C^{(\alpha'; \sigma')}_{\nu, J}
\times \prod_{\nu, J} C^{(a_{12}; \sigma_{12})}_{\nu, J} C^{(a_{23}; \sigma_{23})}_{\nu, J}
\]
\[
= \sum_{\nu_1, J_1} \sum_{\nu_2, J_2} \sum_{\nu_3, J_3} (-1)^{|p_2|+|p_3|+|a_{12}|+2(\mu_{12}+\mu_{23}+\mu_{23})} \times \begin{pmatrix} a_{12} & p_3 & (\tilde{\alpha}; \sigma) \\ \mu_{12}, L_{12} & \nu_3, J_3 & \tilde{\mu}, L \end{pmatrix} \begin{pmatrix} p_1 & p_2 & (\tilde{\alpha}_{12}; \sigma_{12}) \\ \nu_1, J_1 & \nu_2, J_2 & \tilde{\mu}_{12}, L_{12} \end{pmatrix}
\times \begin{pmatrix} a_{23} & p_1 & (\tilde{\alpha}; \sigma') \\ \mu_{23}, L_{23} & \nu_1, J_1 & \tilde{\mu}, L \end{pmatrix} \begin{pmatrix} p_2 & p_3 & (\tilde{\alpha}_{23}; \sigma_{23}) \\ \nu_2, J_2 & \nu_3, J_3 & \tilde{\mu}_{23}, L_{23} \end{pmatrix}.
\]
The name *recombination symbol* being justified by the following equation:

\begin{equation}
(2.105)
\begin{aligned}
e((a_{23}; \sigma_{23}), (a; \sigma'); \mu, L) &= \sum_{(a_{12}; \sigma_{12})} (-1)^{|a_{23}|+|p_2|+|p_3|} \sqrt{\dim Q(a_{12}) \dim Q(a_{23})} \\
& \quad \times \left\{ p_1 \begin{pmatrix} a_{12} \\ p_2 \end{pmatrix} (a; \sigma, \sigma') (a_{23}; \sigma_{23}) e((a_{12}; \sigma_{12}), (a; \sigma); \mu, L) .
\right.
\end{aligned}
\end{equation}

Then, from (2.102) and (2.103), one obtains Wigner’s identity\(^{13}\):

\begin{equation}
(2.106)
\begin{aligned}
\sum_{\mu_{23}, L_{23}} (-1)^{2(t_{\nu_1}+u_{\nu_1})} (a_{23} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} (a; \sigma, \sigma') (a_{23}; \sigma_{23}) e((a_{23}; \sigma_{23}), (a; \sigma); \mu, L)) .
\end{aligned}
\end{equation}

And using (2.101), we obtain the symmetries

\begin{equation}
(2.107)
\begin{aligned}
&\begin{cases}
\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} (a; \sigma, \sigma') (a_{23}; \sigma_{23}) \\
\begin{pmatrix} p_3 \\ a \end{pmatrix} (\sigma, \sigma_{23}) (a_{23}; \sigma')
\end{cases} = \begin{cases}
\begin{pmatrix} a_{12} \\ p_2 \end{pmatrix} (\sigma_{12}; \sigma) (a_{23}; \sigma_{23}) \\
\begin{pmatrix} p_3 \\ a_{12} \end{pmatrix} (\sigma_{12}; \sigma) (a_{23}; \sigma_{23})
\end{cases} \\
\begin{cases}
\begin{pmatrix} p_3 \\ a \end{pmatrix} (\sigma, \sigma_{23}) (a_{23}; \sigma') = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (\sigma_{12}; \sigma') (a_{23}; \sigma_{23})
\end{cases}
\end{aligned}
\end{equation}

**Proposition 2.28.** The coefficients \(\mathcal{M}[p]^{(a_{1}; \sigma_{1}), (a_{2}; \sigma_{2}), (a; \sigma); \nu, L_{1}, \nu_{2}, L_{2}, \nu; J]\) in the decomposition of the operator product, cf. (2.96)-(2.97), are given by

\begin{equation}
(2.108)
\mathcal{M}[p]^{(a_{1}; \sigma_{1}), (a_{2}; \sigma_{2}), (a; \sigma); \nu, L_{1}, \nu_{2}, L_{2}, \nu; J] = \sqrt{\dim Q(a_{1}) \dim Q(a_{2}) \dim Q(a)} \sum_{\sigma'} (-1)^{|p|+2(t_{\nu}+u_{\nu})} \begin{pmatrix} a_1 \\ p \end{pmatrix} (\sigma_{1}; \sigma) (a_{2}; \sigma) (a; \sigma) \begin{pmatrix} \nu_1, J_1 \\ \nu_2, J_2, \nu; J \end{pmatrix} .
\end{equation}

**Proof.** From (2.106), we have

\begin{equation}
(2.109)
\sum_{\mu_{1}, L_{1}} (-1)^{2(t_{\nu_1}+u_{\nu_1})} (p \begin{pmatrix} a_{1} \\ \nu_{1}, L_{1} \end{pmatrix} (\sigma_{1}; \nu_{2}, L_{2}; \mu_{2}, L_{2}) (\sigma_{2}; \nu; J) (p; \sigma_{2}) .
\end{equation}

\[\sum_{\nu', \nu''} (-1)^{|a'|+|a_2|+2(t_{\nu_1}+u_{\nu_1})} \dim Q(a') \begin{pmatrix} a_1 \\ p \end{pmatrix} (\sigma_{1}; \sigma_{2}) (a; \sigma) \begin{pmatrix} \nu_1, J_1 \\ \nu_2, J_2, \nu; J \end{pmatrix} .
\end{equation}

\[^{13}\text{An equivalent formula is deduced for } SU(2) \text{ in } [22] \text{ and for a general compact group in } [8].\]
Multiplying both sides by \((-1)^{2(l_u + u_v + l_{\mu_2} + u_{\mu_2})}\) and summing over \(\mu_1, L_1, \mu_2, L_2\), using (2.101) and (2.107), we obtain (2.108).

From (2.108), we identify

**Definition 2.29.** The Wigner product symbol is the coefficient denoted by the square brackets below:

\[
(a_1; \sigma_1) \otimes (a_2; \sigma_2) \otimes (a; \sigma) \left[ \begin{array}{c} \nu_1, J_1 \\ \nu_2, J_2 \\ \nu, J \end{array} \right] [p] = \sqrt{\dim Q(a_1) \dim Q(a_2) \dim Q(a)}
\]

\[
\times \sum_{\sigma'} \left\{ a_1 a_2 \right\} \left( \begin{array}{c} a_1 a_2 \end{array} \right) \left( \begin{array}{c} a_1 a_2 \end{array} \right) \left( \begin{array}{c} a_1 a_2 \end{array} \right) \left( \begin{array}{c} a_1 a_2 \end{array} \right)
\]

Then, from (2.106), plus symmetries (2.101) and (2.107), recalling (2.79), we have

**Theorem 2.30.** The Wigner product symbol satisfies:

\[
(a_1; \sigma_1) \otimes (a_2; \sigma_2) \otimes (a; \sigma) \left[ \begin{array}{c} \nu_1, J_1 \\ \nu_2, J_2 \\ \nu, J \end{array} \right] [p] = \begin{cases} 0 & \text{if } \nu_{1+2}, \nu' = 1 \\
\delta(J_1, J_2, J) = 1 \\
\end{cases}
\]

\[
= (-1)^{\sum_{k=1}^{3} |a_k|} \left[ \begin{array}{c} a_1; \sigma_1 \\ a_2; \sigma_2 \\ a; \sigma \end{array} \right] \left[ \begin{array}{c} \nu_1, J_1 \\ \nu_2, J_2 \\ \nu, J \end{array} \right] [p] = \begin{cases} 0 & \text{if } \nu_{1+2}, \nu' = 1 \\
\delta(J_1, J_2, J) = 1 \\
\end{cases}
\]

Thus, Lemma 2.25 can be rewritten as

**Corollary 2.31.** The operator product of elements of a coupled basis \(e((a; \sigma); \nu, J) \equiv e_{p, \tilde{p}}((a; \sigma); \nu, J)\) of \(Q(p) \otimes Q(\tilde{p})\) can be decomposed as

\[
e((a_1; \sigma_1); \nu_1, J_1) e((a_2; \sigma_2); \nu_2, J_2) = \sum_{(a; \sigma)} (-1)^{|p| + 2(l_u + u_v)} \left[ \begin{array}{c} a_1; \sigma_1 \\ a_2; \sigma_2 \\ a; \sigma \end{array} \right] \left[ \begin{array}{c} \nu_1, J_1 \\ \nu_2, J_2 \\ \nu, J \end{array} \right] [p] e((a; \sigma); \nu, J)
\]

where summation over \((a; \sigma)\) is restricted to \(Q(a; \sigma)\) in the Clebsch-Gordan series of \(Q(p) \otimes Q(\tilde{p})\), and summations over \(\nu\) and \(J\) effectively restricted by (2.111).

**Remark 2.32.** In particular, if we calculate the product of any \(e((a; \sigma); \nu, J)\) by the identity operator, recalling (2.84), we obtain

\[
e((0; 0); \nu, J) = \left[ \begin{array}{c} (0; 0) \\ (0, 0), 0 \\ \nu, J \end{array} \right] \left[ \begin{array}{c} a; \sigma \\ a'; \sigma' \end{array} \right] \left[ \begin{array}{c} \mu, L \\ \mu, L \end{array} \right] [p] = (-1)^{2(l_u + u_v)} \delta_{\nu, \mu} \delta_{J, J} \delta_{a, a'} \delta_{\sigma, \sigma'}
\]
Theorem 2.30 and Corollary 2.31 conclude our treatment of the $SU(3)$-invariant decomposition of the product of operators acting on the complex Hilbert space $H_p$ of an irreducible representation of $SU(3)$ of class $Q(p)$. As we shall see further below, equation (2.113) shall be used for the twisted product of functions on a symplectic manifold which are related to operators on $H_p$ via a symbol correspondence, as defined in Sections 3 and 4 below. But for that, first we have to define the symplectic manifolds that are invariant under $SU(3)$.

2.5. (Co)Adjoint orbits as invariant phase spaces. $SU(3)$ being a simple compact Lie group, the coadjoint and adjoint orbits of $SU(3)$ are isomorphic\(^{14}\), so here we focus on the adjoint action of $SU(3)$ on its Lie algebra, which provides a real representation whose complexification is of class $Q(1,1)$. We identify the root diagram of $su(3)$ with the Cartan subalgebra generated by $iT_3$ and $iU_3$ by making $\alpha_1 \equiv 2iT_3$ and $\alpha_2 \equiv 2iU_3$. Then, we obtain

\[
\omega_1 \equiv \frac{i}{2} \lambda_3 + \frac{i}{2\sqrt{3}} \lambda_8 = \frac{i}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \omega_2 = \frac{i}{\sqrt{3}} \lambda_8 = \frac{i}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\]

It is well known that each orbit intersects the closed positive Weyl chamber

(2.116) \( C = \{x\omega_1 + y\omega_2 : x, y \geq 0\} \)

in precisely one single point (see e.g. [5]). Let $O_{x,y}$ be the orbit passing through

(2.117) \( \xi_{x,y} = x\omega_1 + y\omega_2 \in C \setminus \{0\} \).

It is clear from (2.115) that, for $x > 0$, the isotropy subgroup of $\xi_{x,0}$ is

\[
H := \left\{ \begin{pmatrix} \det(U)^{-1} & 0 \\ 0 & U \end{pmatrix} : U \in U(2) \right\} \simeq S(U(2) \times U(1)) \simeq U(2),
\]

whereas, for $y > 0$, the isotropy subgroup of $\xi_{0,y}$ is

\[
\tilde{H} := \tilde{\delta}H\tilde{\delta}^{-1} = \tilde{\delta}H\tilde{\delta} = \left\{ \begin{pmatrix} U & 0 \\ 0 & \det(U)^{-1} \end{pmatrix} : U \in U(2) \right\}
\]

\( \simeq S(U(1) \times U(2)) \simeq U(2), \)

where

(2.120) \( \tilde{\delta} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in SU(3). \)

On the other hand, the isotropy subgroup of $\xi_{x,y}$ is the maximal torus\(^{15}\)

\[
T := \left\{ \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} : \theta_1 + \theta_2 + \theta_3 = 0 \right\}
\]

\( \simeq S(U(1) \times U(1) \times U(1)) \simeq U(1) \times U(1) \).

Therefore, we have two types of non trivial (co)adjoint orbits:

\( O_{x,0} \simeq SU(3)/H \simeq SU(3)/\tilde{H} \simeq O_{0,y} \)

and

\( O_{x,y} \simeq SU(3)/T \simeq O_{y,x}, \quad \text{for} \quad x, y > 0. \)

\(^{14}\)A general discussion of coadjoint orbits of semisimple Lie groups can be found in [3].

\(^{15}\)It is a matter of simple calculation to verify that $T = \delta T \delta = T$. 
For a better realization of such orbits, we recall the complex projective space \( \mathbb{C}P^2 \): the quotient of \( \mathbb{C}^3 \setminus \{0\} \) by the equivalence relation
\[
\begin{align*}
\sim & \iff z = a z', \quad a \in \mathbb{C}^*.
\end{align*}
\]
To construct \( \mathbb{C}P^2 \) using this equivalence relation, we can look only to the unitary vectors of \( \mathbb{C}^3 \), reducing our analysis to the \( SU(3) \)-homogeneous space \( S^5 = \{ z \in \mathbb{C}^3 : ||z|| = 1 \} \). Since the point \((1, 0, 0) \in S^5\) has
\[
(2.122) \quad \left\{ \begin{array}{l}
(1 \quad 0 \\
0 \quad 0 \\
\end{array} \right] U \in SU(2) \right\} \subset SU(3)
\]
as isotropy subgroup, we have \( S^5 \cong SU(3)/SU(2) \). Also note that, \( \forall z \in S^5, e^{i \theta} z \sim z \). So \( \mathbb{C}P^2 \cong S^5/S^1 \) and the isotropy subgroup of the equivalence class \( 1 : 0 : 0 \in \mathbb{C}P^2 \) is \( H \), i.e., \( SU(3)/H \cong \mathbb{C}P^2 \). By similar argument we get \( U(2)/(U(1) \times U(1)) \cong CP^1 \), so \( SU(3)/T \cong \mathcal{E} \), where \( \mathcal{E} \) is the total space of a fiber bundle \( \pi : \mathcal{E} \to \mathbb{C}P^2 \) with fiber \( CP^1 \), which we denote by \( E(\mathbb{C}P^2, CP^1, \pi) \), that is,
\[
(2.123) \quad \mathcal{E} = E(\mathbb{C}P^2, CP^1, \pi) := \mathbb{C}P^1 \hookrightarrow \mathcal{E} \xrightarrow{\pi} \mathbb{C}P^2.
\]
Thus,
\[
(2.125) \quad \mathcal{O}_{x,y} \simeq \left\{ \begin{array}{l}
\mathbb{C}P^2, \text{ if } x = 0 \text{ or } y = 0 \\
\mathcal{E}, \text{ if } x, y > 0
\end{array} \right.
\]

The orbits \( \mathcal{O}_{x,y} \) and \( \mathcal{O}_{y,x} \) are related by the involution \( \iota = -id \) on \( \mathfrak{su}(3) \). Indeed,
\[
(2.126) \quad \iota(x, y) = (-x, -y) = Ad_g(y, x, y) \equiv g x_0, 
\]
so \( \iota(O_{x,y}) = O_{y,x} \). Thus, \( \iota \) is an involution on \( O_{x,y} \).

Let \( x_0 = [1 : 0 : 0] \in \mathbb{C}P^2 \), whose isotropy subgroup is \( H \), so that the isotropy subgroup of \( \delta x_0 = [0 : 0 : 1] \) is \( \tilde{H} \), and let \( z_0 \in \pi^{-1}(x_0) \subset \mathcal{E} \) be a point with \( T \) as isotropy subgroup. Then consider the equivariant diffeomorphisms
\[
(2.127) \quad \psi_{x,0} : \mathcal{O}_{x,0} \to \mathbb{C}P^2 : \psi_{x,0} = \Ad_g \xi_{x,0} \to g x_0,
\]
\[
(2.128) \quad \psi_{0,y} : \mathcal{O}_{0,y} \to \mathbb{C}P^2 : \psi_{0,y} = Ad_g \xi_{0,y} \to g z_0,
\]
\[
(2.129) \quad \psi_{x,y} : \mathcal{O}_{x,y} \to \mathcal{E} : \psi_{x,y} = \Ad_g \xi_{x,y} \to g z_0,
\]
for \( x, y > 0 \) still holding. Therefore,
\[
(2.130) \quad \psi_{x,0} \circ \iota \circ \psi_{0,x}^{-1} : \mathbb{C}P^2 \to \mathbb{C}P^2
\]
is the identity map, and
\[
(2.131) \quad \alpha := \psi_{x,y} \circ \iota \circ \psi_{y,x}^{-1} : \mathcal{E} \to \mathcal{E}
\]
is an \( SU(3) \)-equivariant involution. Of course, there is an equivalent involution on each \( \mathcal{O}_{x,y} \), namely
\[
(2.132) \quad \alpha_{x,y} := \psi_{x,y}^{-1} \circ \psi_{y,x} \circ \iota : \mathcal{O}_{x,y} \to \mathcal{O}_{x,y}
\]
which reduces to \( \iota \) for \( x = y \).

Every \( SU(3) \)-coadjoint orbit is a \( SU(3) \)-invariant symplectic manifold, that is, every \( \mathcal{O}_{x,y} \) carries an \( SU(3) \)-invariant symplectic form, which is the restriction to each symplectic leaf of the so-called Kirillov-Arnold-Kostant-Souriau bracket on

\textsuperscript{16} This construction is presented in \( \mathbb{R} \), for instance.

\textsuperscript{17} \( SU(2) \cong S^3 \) hence \( SU(3) \) is a 3-sphere bundle over \( S^5 \).
Further, the $SU(3)$-invariant symplectic form on $\mathcal{O}_{x,y}$ induces a normalized left invariant integral on the orbit $\mathcal{O}_{x,y}$ such that any other left invariant integral differs from it by a scalar factor. Then, we can fix this factor for $\mathcal{O}_{x,y}$ so that the lift $\tilde{f} \in C(SU(3))$ of any $f \in C(\mathcal{O}_{x,y})$ satisfies
\begin{equation}
\int_{SU(3)} \tilde{f}(g) \, dg = \int_{\mathcal{O}_{x,y}} f(x) \, dx
\end{equation}
for the Haar integral on $SU(3)$ (cf. eg. [12]). With no danger of causing confusion, we may denote the $SU(3)$-invariant inner product in $L^2(\mathcal{O}_{x,y})$ with respect to such integral simply by $\langle | \rangle$, that is, for any $f_1, f_2 \in L^2(\mathcal{O}_{x,y})$,
\begin{equation}
\langle f_1 | f_2 \rangle = \int_{\mathcal{O}_{x,y}} f_1(x) f_2(x) \, dx.
\end{equation}

Throughout this paper, we consider $\mathbb{C}P^2$ and $\mathcal{E}$ as homogeneous spaces (by the adjoint action of $SU(3)$) equipped with the mentioned symplectic forms and normalized left invariant integrals. Thus, $\mathbb{C}P^2$ and $\mathcal{E}$ are $SU(3)$-invariant phase spaces for classical quark systems. That is, in what follows, we shall identify as a classical quark system, the Poisson algebra of smooth functions on $\mathcal{O}$. When $\mathcal{O} = \mathbb{C}P^2$, we shall refer to a classical pure-quark system. When $\mathcal{O} = \mathcal{E}$, we shall refer to a generic classical quark system.

3. Pure-quark systems

Here we focus on the simpler possible phase space for a classical quark system: $\mathbb{C}P^2$. First, we describe the set of harmonic functions on $\mathbb{C}P^2$, which imposes a restriction on the classes of irreducible representations of $SU(3)$ with possible correspondences to smooth functions on $\mathbb{C}P^2$. Then, we proceed to describe the relevant $SU(3)$-representations for this case as quantum quark systems. Finally, we work out the characterization of all symbol correspondences from such quantum quark systems to the classical quark system of interest and describe the induced twisted products of symbols on $\mathbb{C}P^2$. The construction and characterization of symbol correspondences in this section is very close to what is done for spin systems in [24]. Accordingly, proofs of some propositions are identical to the $SU(2)$ case. The quantum and classical systems in correspondence, in this chapter, are both called “pure-quark system” and this name is explained in Appendix C.

3.1. Classical pure-quark system.

Definition 3.1. The classical pure-quark system consists of $\mathbb{C}P^2$ equipped with its $SU(3)$-invariant symplectic form, together with its Poisson algebra on $C^\infty(\mathbb{C}P^2)$.

Since $\mathbb{C}P^2 \simeq SU(3)/H$, where $H \simeq U(2)$, cf. (2.118), we look for representations $Q(p,q)$ with weights satisfying $t = u = J = 0$ (cf. (2.16)-(2.17)) to determine the harmonic functions on $\mathbb{C}P^2$.

Proposition 3.2. The representations of $SU(3)$ with non null vectors fixed by $H \simeq U(2)$ are the representations $Q(n,n)$ with $t = u = J = 0$. The space fixed by $H$ is spanned by $e((n,n);(n,n,n),0)$.

This bracket was actually first identified for general Lie groups by Sophus Lie [28].
Proof. Let \( e((p,q);(\nu_1,\nu_2,\nu_3), J) \) be such that \( t = u = J = 0 \), so \( \nu_1 = \nu_2 = \nu_3 = \nu \). From the constraints (2.10), we get \( r_+ = r_- = \nu = q \). Thus, putting \( n = p = q \), we finish the proof. \( \square \)

**Definition 3.3.** The \( \mathbb{C}P^2 \) harmonics are the functions \( X_{\nu,j}^n : \mathbb{C}P^2 \to \mathbb{C} \), such that

\[
X_{\nu,j}^n(gx_0) = (n + 1)^{3/2}D_{\nu,j,(n,n,n)}^{(n,n)}(g),
\]

for \( x_0 = [1 : 0 : 0] \in \mathbb{C}P^2 \), \( g \in SU(3) \) and \( D_{\nu,j,(n,n,n)}^{(n,n)} \) a Wigner D-function as in Definition 2.6.

The factor \((n + 1)^{3/2}\) in the definition of \( \mathbb{C}P^2 \) harmonics is the square root of the dimension of the representation \( Q(n,n) \) and is used to ensure normalization according to Schur’s Orthogonality Relations, so that we have

\[
\langle X_{\nu,j}^n | X_{\mu,L}^m \rangle = \delta_{n,m} \delta_{\nu,\mu} \delta_{j,L}
\]

with respect to the inner product described in section 2.5, cf. (2.131 - 2.132).

We note that

\[
X_{(0,0,0),0}^0 = 1
\]

and, cf. (2.32),

\[
X_{\nu,j}^n = (-1)^{2(\ell+u)} X_{\nu,j}^n, \text{ for } \Delta_{\nu,\nu}^n = 1.
\]

**Remark 3.4.** Fixed \( x > 0 \), the diffeomorphism \( \psi_{x,0} \) can be used to carry \( \mathbb{C}P^2 \) harmonics to \( \mathcal{O}_{x,0} \) by means of the composition \( X_{\nu,j}^n \circ \psi_{x,0} \), cf. (2.127). Equivalently, \( X_{\nu,j}^n \circ \psi_{x,0} \) are the \( \mathbb{C}P^2 \) harmonics carried to the orbit \( \mathcal{O}_{x,0} \). Consequently, we have a set of harmonic functions on \( \mathcal{O}_{x,0} \) related to a set of harmonic functions on \( \mathcal{O}_{0,x} \) by the map \( t \), cf. (2.128).

From Lemma 2.24 we have the following.

**Theorem 3.5.** The pointwise product of \( \mathbb{C}P^2 \) harmonics decomposes as

\[
X_{\nu_1,j_1}^{n_1} X_{\nu_2,j_2}^{n_2} = \sum_{(n,\nu,\sigma)} \left( \frac{(n_1 + 1)(n_2 + 1)}{n + 1} \right)^{3/2} C_{\nu_1,j_1,\nu_2,j_2}^{(n_1,n_1),(n_2,n_2),(n,\nu,\sigma)}
\]

\[
\times C_{\nu_1,j_1,\nu_2,j_2}^{(n_1,n_1),(n_2,n_2),(n,\nu,\sigma)} X_{\nu,j}^n,
\]

where \( n = (n_k,n_k,n_k) \) for \( k = 1,2 \) and \( n = (n,n,n) \), and summation is restricted to \( \nabla_{\nu_1,\nu_2,\nu} = 1 \), \( \delta(J_1,J_2,J) = 1 \) and \( Q(n,n;\sigma) \) in the Clebsch-Gordan series of \( Q(n_1,n_1) \otimes Q(n_2,n_2) \); in particular, \( |n_1 - n_2| \leq n \leq n_1 + n_2 \).

**Proof.** With a little abuse of notation, we write

\[
X_{\nu_k,j_k}^{n_k} = (n_k + 1)^{3/2} D_{\nu_k,j_k,n_k}^{(n_k,n_k)} (g);
\]

and apply Lemma 2.24 to get

\[
X_{\nu_1,j_1}^{n_1} X_{\nu_2,j_2}^{n_2} = \sum_{(\alpha,\sigma)} \sum_{\mu,0} \left( \frac{(n_1 + 1)(n_2 + 1)}{n + 1} \right)^{3/2} C_{\nu_1,j_1,\nu_2,j_2}^{(n_1,n_1),(n_2,n_2),(\alpha,\sigma)}
\]

\[
\times C_{\nu_1,j_1,\nu_2,j_2}^{(n_1,n_1),(n_2,n_2),(\alpha,\sigma)} \rho_{\nu,j} D_{\nu,j,\mu,0}^{(n,n)} (g),
\]
where $\nabla_{\nu_1+\nu_2,\nu} = \nabla_{n_1+n_2,\mu} = 1$ and $\delta(J_1, J_2, J) = 1$, so $\mu = (\mu, \mu, \mu)$. But $e(a; (\mu, \mu, \mu), 0)$ only exists if $a = (\mu, \mu)$. Thus, we set $a = (n, n)$ and $\mu = (n, n, n)$. The restriction over $n$ follows from Theorem 2.4.

Remark 3.6. The fact that $\mathbb{C}P^2$ is a symplectic manifold plays no part in the decomposition of the pointwise product of $\mathbb{C}P^2$ harmonics. Accordingly, the next step in the study of the classical pure-quark system, with the purpose of studying symbol correspondences, amounts to decomposing the Poisson bracket of $\mathbb{C}P^2$ harmonics. However, this problem is considerably harder and is deferred to a later study.

3.2. Quantum pure-quark systems. From Proposition 3.2, we look for representations $Q(p, q)$ such that the tensor product $Q(p, q) \otimes Q(q, p)$ splits into representations of the form $Q(n, n)$, without multiplicities. From Corollary 2.4 we have that $Q(p, 0)$ and $Q(0, p)$ are the only ones that satisfy these requirements. These are special cases of quantum quark systems.

Definition 3.7. Let $p \in \mathbb{N} \times \{0\} \cup \{0\} \times \mathbb{N}$ with $|p| = p$. A quantum pure-quark system is a complex Hilbert space $H_p \simeq \mathbb{C}^d$, where

$$d = \frac{(p+1)(p+2)}{2},$$

cf. (2.14), with an irreducible unitary $SU(3)$-representation of class $Q(p)$ together with its operator algebra $B(H_p)$.

The reason for the name pure-quark systems is explained in Appendix C.

In the pure-quark case, the Gelfand-Tsetlin pattern (2.10) is reduced to

$$Q(p) = Q(p, 0) \implies \begin{cases} 0 \leq r \leq p, \\ \nu_1 = p - r, \nu_2 = r - \nu_3, 0 \leq \nu_3 \leq r, \\ J = \frac{r}{2}. \end{cases}$$

(3.8)

$$Q(p) = Q(0, p) \implies \begin{cases} 0 \leq r \leq p, \\ \nu_1 = p - r, \nu_2 = p + r - \nu_3, r \leq \nu_3 \leq p, \\ J = \frac{p-r}{2}. \end{cases}$$

(3.9)

In both cases, $J$ is determined by $\nu$, so we can simplify the notation for $p \in (\mathbb{N} \times \{0\}) \cup \{0\} \times \mathbb{N}$ as

$$e(p; \nu) := e(p, \nu, J), \quad \tilde{e}(p; \nu) := \tilde{e}(p, \nu, J).$$

To clear even more the notation, we will denote the elements of a coupled basis of $B(H_p)$ that lies in the $Q(n, n)$-invariant subspace by

$$e(n; \nu, J) := e((n, n); \nu, J) \equiv e_{p, \tilde{p}}((n, n); \nu, J),$$

cf. (2.39). Thus, notation for the Clebsch-Gordan coefficients can be simplified to

$$C_{\nu_1, \nu_2, \nu}^p := C_{\nu_1, \nu_2, \nu}^p, \quad \tilde{C}_{\nu_1, \nu_2, \nu}^p := \tilde{C}_{\nu_1, \nu_2, \nu}^p, \quad C_{\nu_1, \nu_2, \nu}^{(n, n)}.$$  

19We are ignoring the trivial representation $Q(0, 0)$.  


And applying the same simplification to the Wigner product symbol, expressions \((2.111)\) and \((2.112)\) in Theorem 2.30 now read as
\[
(3.13) \quad \begin{bmatrix} n_1 & n_2 & n_3 \\ \nu_1, J_1 & \nu_2, J_2 & \nu_3, J_3 \end{bmatrix} \neq 0 \implies \begin{cases} \nabla_{\nu_1 + \nu_2, \nu_3} = 1 \\ \delta(J_1, J_2, J_3) = 1 \end{cases},
\]
\[
(3.14) \quad = \begin{bmatrix} n_1 & n_2 & n_3 \\ \nu_1, J_1 & \nu_2, J_2 & \nu_3, J_3 \end{bmatrix} = \begin{bmatrix} n_3 & n_1 & n_2 \\ \nu_3, J_2 & \nu_1, J_2 & \nu_2, J_2 \end{bmatrix},
\]
Therefore, Corollary 2.51 takes the special form:

**Theorem 3.8.** For a quantum pure-quark system \(\mathcal{H}_p\), \(|p| = p\), the product of elements of a coupled basis of the space of operators \(\mathcal{B}(\mathcal{H}_p)\) decomposes in the form
\[
eq |n_1, \nu_1, J_1 \rangle \langle n_2, \nu_2, J_2 | = \sum_{n=0}^{p} \sum_{\nu, J} (-1)^{\nu_1 + \nu_2 - \nu} \begin{bmatrix} n_1 & n_2 & n_3 \\ \nu_1, J_1 & \nu_2, J_2 & \nu_3, J_3 \end{bmatrix} \mathcal{C}^{p, p, p}(\nu, \nu, \nu) \big| (n, n, n) \big> \big< (n, n, n), J \big|
\]
for \(0 \leq n_1, n_2 \leq p\), with summations over \(\nu\) and \(J\) effectively restricted by \((3.13)\). We also identify the operator algebra \(\mathcal{B}(\mathcal{H}_p)\) with the matrix algebra \(\mathcal{M}_z(d)\) by means of an uncoupled basis of \(Q(p) \otimes Q(\tilde{p})\). So let \(\nu\) and \(\tilde{\nu}\) be such that \(\Delta |p, \nu\rangle = 1\), the operator \(e(p, \nu) \otimes \mathcal{C}(\tilde{p}, \tilde{\nu})\) is a diagonal matrix and its decomposition in the coupled basis can be written as
\[
eqn.(3.16) \quad e(p, \nu) \otimes \mathcal{C}(\tilde{p}, \tilde{\nu}) = \sum_{n=0}^{p} \sum_{\nu, J} C^{p, p, p}(\nu, \nu, \nu) \big| (n, n, n) \big> \big< (n, n, n), J \big|
\]
That is, any diagonal matrix is a linear combination of \(\{e(n, (n, n, n), J)\}\). Since the cardinality of this set is \((p + 1)(p + 2) / 2\), it is the set of diagonal matrices of a coupled basis. CG coefficients being real implies that such matrices are also real.

3.3. Symbol correspondences for pure-quark systems. Let \(p \in \{\mathbb{N} \times \{0\}\} \cup \{(0) \times \mathbb{N}\}\) with \(|p| = p\).

**Definition 3.9.** A symbol correspondence for a pure-quark system \((\mathcal{H}_p, Q(p))\), referred to simply as a symbol correspondence or just as a correspondence, is an injective linear map \(W : \mathcal{B}(\mathcal{H}_p) \rightarrow C^\infty_c(\mathbb{C}P^2) : A \mapsto W_A\), satisfying, \(\forall A \in \mathcal{B}(\mathcal{H}_p)\),
\[
\text{i) Equivariance: } \forall g \in SU(3), W_A^g = (W_A)^g ;
\]
\[
\text{ii) Reality: } W_{A^\dagger} = \overline{W_A} ;
\]
\[
\text{iii) Normalization: } \int_{\mathbb{C}P^2} W_A(x)dx = \frac{2}{(p + 1)(p + 2)} \text{tr}(A) .
\]

**Remark 3.10.** If one replace \(\mathbb{C}P^2\) in Definition 3.7 by the orbit \(O_{x,0}\) or \(O_{0,x}\), for \(x > 0\), the definition remains essentially the same. Given any symbol correspondence \(W : \mathcal{B}(\mathcal{H}_p) \rightarrow C^\infty_c(\mathbb{C}P^2)\), the map \(W' : \mathcal{B}(\mathcal{H}_p) \rightarrow C^\infty_c(O_{x,0}), W'_A = W_A \circ \psi_{x,0}\), satisfies the desired properties and \(W_A = W'_A \circ \psi_{x,0}^{-1}\). Using \(\psi_{x,0}\), we get symbol correspondences with codomain \(C^\infty_c(O_{x,0})\). Conveniently, one can define such symbol correspondences as maps \(W' : \mathcal{B}(\mathcal{H}_p) \rightarrow C^\infty_c(O_{p})\).
Theorem 3.11. A linear map \( W : \mathcal{B}(\mathcal{H}_p) \to C^\infty_\mathbb{R}(\mathbb{C}P^2) \) is a symbol correspondence if and only if it maps (cf. (3.11))

\[
W : \sqrt{\frac{(p+1)(p+2)}{2}} e(n; \nu, J) \mapsto c_n X_{\nu,J}^n
\]

for \((c_1, \ldots, c_p) \in (\mathbb{R}^*)^p\) and \(c_0 = (-1)^p\).

Proof. By Schur’s Lemma, \( W \) is equivariant and injective if and only if it provides a mapping of the form (3.11) with \( c_n \neq 0 \) for every \( n \in \{0, \ldots, p\} \). Since \( e^\dagger(n, \nu, J) = (-1)^{2(t+u)} e(n, \nu, J) \) and \( X_{\nu,J}^n = (-1)^{2(t+u)} X_{\nu,J}^n \), reality holds if and only if the constants \( c_n \) are all real numbers. To finish, we have

\[
(-1)^p \sqrt{\frac{(p+1)(p+2)}{2}} e(0; (0,0,0),0) = 1
\]

and \( X_{0,0}^0 = 1 \), then the normalization condition is satisfied iff \( c_0 = (-1)^p \). \( \square \)

Corollary 3.12. The moduli space of symbol correspondences for a pure-quark system of class \( Q(p) \) is \((\mathbb{R}^*)^p\).

But there is another way to construct symbol correspondences. Again, let

\[
x_0 = [1 : 0 : 0] \in \mathbb{C}P^2.
\]

Given an operator \( K \in \mathcal{B}(\mathcal{H}_p) \) fixed by \( H \), consider the operator-valued function \( \mathbb{C}P^2 \to \mathcal{B}(\mathcal{H}_p) : x \mapsto K(x) = K^g \), where \( g \in SU(3) \) is such that \( x = gx_0 \).

Theorem 3.13. A map \( W : \mathcal{B}(\mathcal{H}_p) \to C^\infty_\mathbb{R}(\mathbb{C}P^2) : A \mapsto W_A \) is a symbol correspondence satisfying (3.11) if and only if

\[
W_A(gx) = \text{tr}(AK(x)) \iff W_A(gx_0) = \text{tr}(AK^g),
\]

\( \forall g \in SU(3), \) for \( K \in \mathcal{B}(\mathcal{H}_p) \) of the form

\[
K = \frac{2}{(p+1)(p+2)} + \sum_{n=1}^{p} c_n \sqrt{\frac{2(n+1)^3}{(p+1)(p+2)}} e(n; (n,n,n),0).
\]

In particular, \( K \) is a diagonal matrix with real entries and unitary trace.

Proof. Suppose \( W \) is a symbol correspondence given by (3.11). The map \( A \mapsto W_A(x_0) \) is a linear functional, then \( \exists K \in \mathcal{B}(\mathcal{H}_p) \) s.t. (3.20) holds for the identity of \( SU(3) \). From equivariance, \( W_A(g^{-1}x_0) = (W_A)^g(x_0) = W_{A^g}(x_0) = \text{tr}(A^gK) = \text{tr}(AK^{g^{-1}}), \forall g \in SU(3) \). Since \( x_0 \) is fixed by \( H \), we have \( \text{tr}(AK^g) = \text{tr}(AK) \) for every \( g \in H \) and every operator \( A \). Thus, we can write (cf. Proposition 3.2)

\[
K = \sum_{n=0}^{p} k_n e(n; (n,n,n),0),
\]

so \( K \) is a diagonal matrix. Taking \( A = e(n; \nu, J) = (-1)^{2(t+u)} e^\dagger(n; \nu, J) \), we have

\[
W_A(gx_0) = \text{tr}(AK^g) = k_n (-1)^{2(t+u)} D_{\nu,J,n,n,n}^{(n,n)}(g) = k_n D_{\nu,J,n,n,n}^{(n,n)},
\]

cf (2.82). Then,

\[
W : e(n; \nu, J) \mapsto \frac{k_n}{(n+1)^{3/2}} X_{\nu,J}^n.
\]
It follows from Theorem 3.11 that
\[
k_n = c_n \sqrt{\frac{2(n+1)^3}{(p+1)(p+2)}}.
\]
On the other hand, for \(K\) given by (3.21), equations (3.22)-(3.24) imply that (3.20) defines a symbol correspondence given by (3.17).

Remark 3.14. In a nutshell: from \(\text{tr}(A^\dagger) = \text{tr}(A)\), the reality condition is equivalent to \(K\) being Hermitian, and, since diagonal operators span the space of \(H\)-invariant operators, reality plus invariance is equivalent to \(K\) being real diagonal, then, normalization condition is equivalent to \(\text{tr}(K) = 1\), as shown in [24], and, finally, injectivity is equivalent to all \(c_n\)'s being nonzero, in decomposition (3.21).

The results from Theorems 3.11 and 3.13 and Corollary 3.12 are completely analogous to the case of spin systems, cf. [24], so we come with the next definition.

Definition 3.15. An operator kernel \(K \in \mathcal{B}(\mathcal{H}_p)\) is an operator that induces a symbol correspondence via (3.19)-(3.20). That is, \(K\) is given by (3.21) with nonzero real numbers \((c_n)\) which are called characteristic numbers of both the operator kernel and the symbol correspondence.

If \(K \in \mathcal{B}(\mathcal{H}_p)\) is an operator kernel, it is diagonal with real entries, thus it is a linear combination of projections of the form
\[
K = \sum_{\nu} a_{\nu} \Pi_{\nu},
\]
for real coefficients \(a_{\nu}\), where \(\Pi_{\nu}\) is an orthogonal projector onto the weight space \(\text{span}\{e(p;\nu)\}\). We can separate the summation as
\[
K = \sum_{j=0}^{p} K_j, \quad K_j = \sum_{\nu \in j/2} a_{\nu} \Pi_{\nu},
\]
where \(\nu \in j/2\) means \(e(p;\nu) = e(p;\nu, j/2)\), cf. (3.8)-(3.10).

Proposition 3.16. If \(K \in \mathcal{B}(\mathcal{H}_p)\) is an operator kernel, then
\[
K = \sum_{j=0}^{p} a_j \sum_{\nu \in j/2} \Pi_{\nu},
\]
where the coefficients \(a_j\) are real numbers satisfying
\[
\sum_{j=0}^{p} a_j (j+1) = 1.
\]

Proof. Every operator kernel is fixed by \(H\) of (2.118), so \(K\) must be fixed also by the \(SU(2)\) of (2.123), the standard \(SU(2)\) related to \(\{U_3, U_\pm\}\). Decomposing \(K\) as in (3.24), we have that each component \(K_j\) is an operator on the subrepresentation \(j/2\) of that standard \(SU(2)\) and it must commute with \(\{U_3, U_\pm\}\), which implies \(K_j = a_j \sum_{\nu \in j/2} \Pi_{\nu}\), where \(a_j\) is real. To finish, \(\text{tr}(K) = 1\) implies (3.29). □

We can also use an operator kernel to construct a symbol correspondence in an implicit way, as follows.
Proposition 3.17. Let $K$ be an operator kernel with characteristic numbers $(c_n)$. The equation
\begin{equation}
A = \frac{(p+1)(p+2)}{2} \int_{\mathbb{C}P^2} \overline{W}_A(x)K(x)dx.
\end{equation}

defines a symbol correspondence $\overline{W}$ with characteristic numbers $(\overline{c}_n)$ given by
\begin{equation}
\overline{c}_n = 1/c_n.
\end{equation}

Proof. We have
\begin{equation}
\int_{\mathbb{C}P^2} X_{\nu,J}^n(x)K(x)dx = \int_{SU(3)} X_{\nu,J}^n(gx_0)K^g dg
\end{equation}
\begin{equation}
= \sum_{n',\mu,L} k_{n'} \int_{SU(3)} X_{\nu,J}^n(gx_0)D_{\mu,L}(n',n')D_{g}dx e(n';\mu,L)
\end{equation}
\begin{equation}
= \sum_{n',\mu,L} k_{n'} \left( X_{\nu,J}^n \right)^* \left( X_{\nu,J}^n \right) e(n;\mu,L) = \frac{k_n}{(n+1)3/2} e(n;\nu,J),
\end{equation}
where $k_n$ is given by (3.25). Thus,
\begin{equation}
\frac{(p+1)(p+2)}{2} \int_{\mathbb{C}P^2} \frac{1}{c_n} X_{\nu,J}^n(x)K(x)dx = \sqrt{\frac{(p+1)(p+2)}{2}} e(n;\nu,J).
\end{equation}

By Theorem 3.11, $\overline{W}$ is a symbol correspondence with characteristic numbers $(\overline{c}_n)$ satisfying (3.31).

Actually, there is a duality relation between the symbol correspondences defined by the same operator kernel via (3.20) and via (3.30) considering the normalized inner product $\langle \cdot \rangle_p$ on $\mathcal{B}(\mathcal{H}_p)$ given by
\begin{equation}
\langle A|R \rangle_p = \frac{2}{(p+1)(p+2)} \langle A|R \rangle = \frac{2}{(p+1)(p+2)} \text{tr}(A^\dagger R).
\end{equation}

Definition 3.18. Given a symbol correspondence $W: \mathcal{B}(\mathcal{H}_p) \to C^\infty_\mathcal{C}(\mathbb{C}P^2)$, its dual correspondence is the symbol correspondence $\overline{W}: \mathcal{B}(\mathcal{H}_p) \to C^\infty_\mathcal{C}(\mathbb{C}P^2)$ that satisfies, for all $A,R \in \mathcal{B}(\mathcal{H}_p)$,
\begin{equation}
\langle A|R \rangle_p = \left\langle \overline{W}_A | W_R \right\rangle = \left\langle W_A | \overline{W}_R \right\rangle.
\end{equation}
The operator kernel of $\overline{W}$ is also said to be dual to the operator kernel of $W$.

Proposition 3.19. Let $K$ be an operator kernel. The symbol correspondences defined by $K$ via (3.20) and via (3.30) are dual to each other, that is, for any symbol correspondence with characteristic numbers $(c_n)$, its dual correspondence has characteristic numbers $(1/c_n)$.

Proof. Given any two operators $A$ and $R$, if we write $A^\dagger$ as in (3.30) and use the reality property, we get
\begin{equation}
\frac{2}{(p+1)(p+2)} \text{tr}(A^\dagger R) = \int_{\mathbb{C}P^2} \overline{W}_A(x)\text{tr}(RK(x))dx = \int_{\mathbb{C}P^2} \overline{W}_A(x)W_R(x)dx,
\end{equation}
that is, $\langle A|R \rangle_p = \left\langle \overline{W}_A | W_R \right\rangle$. For $R$ as in (3.30), we get $\langle A|R \rangle_p = \left\langle W_A | \overline{W}_R \right\rangle$. \qed
**Definition 3.20.** A symbol correspondence \( W : \mathcal{B}(\mathcal{H}_p) \to C^\infty_{\mathbb{R}}(\mathbb{C}P^2) \) is a Stratonovich-Weyl correspondence if it is an isometry, that is,
\[
(A|R)_p = \langle W_A|W_R \rangle
\]
for all \( A, R \in \mathcal{B}(\mathcal{H}_p) \).

That is, a symbol correspondence \( W \) is an isometry if and only if it is self-dual, and it follows immediately from Proposition 3.19.

**Corollary 3.21.** A symbol correspondence is a Stratonovich-Weyl correspondence if and only if its characteristic numbers \( (c_n) \) satisfy \( |c_n| = 1 \).

Although isometry is a nice property for a symbol correspondence, there is another property for symbol correspondences that is very reasonable, from a physical perspective, and very good, from analytical considerations. Recall that a Hermitian operator with only nonnegative eigenvalues is called positive, or is positive-definite if all of its eigenvalues are positive, and a real function that takes only nonnegative values is called positive, or strictly-positive if it only takes positive values.

**Definition 3.22.** A symbol correspondence for a pure-quark system is mapping-positive if it maps positive-definite operators to (strictly-)positive functions. A symbol correspondence for a pure-quark system which is dual to a mapping-positive correspondence is a positive-dual correspondence.

In Theorem 3.13 we characterize all symbol correspondences as expected values with respect to \( K^\alpha \), where \( K \) is an \( H \)-invariant Hermitian operator with unitary trace satisfying equation (3.21) with \( c_n \in \mathbb{R^*} \). From Physics, an operator on a complex Hilbert space is a state if it is a positive operator with unitary trace. Since a general operator kernel might have negative eigenvalues, \( K \) is identified with a *pseudo-state* and we can see it as providing pseudo-probabilities just as a state provide actual probabilities. In fact, we have:

**Proposition 3.23.** A symbol correspondence \( W : \mathcal{B}(\mathcal{H}_p) \to C^\infty_{\mathbb{R}}(\mathbb{C}P^2) \) with operator kernel \( K \) is mapping-positive if and only if \( K \) is also a state, that is, \( K \) given by (3.21), with \( c_n \in \mathbb{R^*} \), is in the convex hull of \( \{\Pi_\nu\} \), that is, \( K \in \text{Conv}\{\Pi_\nu\} \).

**Proof.** We assume that \( K \) is an operator kernel, thus \( K \) is given by (3.21) with \( c_n \in \mathbb{R^*} \). Now, let \( K \) be decomposed as in (3.23). Suppose \( K \in \text{Conv}\{\Pi_\nu\} \), so that the coefficients \( a_j \) in the summation are all nonnegative. An operator \( A \in \mathcal{B}(\mathcal{H}_p) \) is positive if and only if \( A = R^\dagger R \) for some \( R \in \mathcal{B}(\mathcal{H}_p) \), and \( A \) is positive-definite if and only if \( R \) is an automorphism. Thus, for any \( g \in SU(3) \) and \( \bar{R} = R\rho(g) \),
\[
W_A(gx_0) = \sum_{j=0}^p a_j \sum_{\nu \in j/2} \text{tr}(R^\dagger R\rho(g)\Pi_\nu\rho(g)^\dagger) = \sum_{j=0}^p a_j \sum_{\nu \in j/2} \text{tr}(R\rho(g)\Pi_\nu\rho(g)^\dagger R^\dagger)
\]
\[
= \sum_{j=0}^p a_j \sum_{\nu \in j/2} \text{tr}(\bar{R}\Pi_\nu\bar{R}^\dagger) = \sum_{j=0}^p a_j \sum_{\nu \in j/2} \left|\bar{R}(e(p, \nu))\right|^2 \geq 0 ,
\]
where the inequality is strict if \( \bar{R} \) is an automorphism, which is true if \( R \) is an automorphism, that is, if \( A \) is positive-definite.

Now, if \( K \not\in \text{Conv}\{\Pi_\nu\} \), then there is a coefficient \( a_j < 0 \) and any projector \( \Pi_\nu \) with \( \nu \in j/2 \) is a positive operator satisfying \( W_{\Pi_\nu}(x_0) = \text{tr}(K\Pi_\nu) < 0 \).

\[\text{The real numbers } a_j \text{ in (3.23) can be negative.}\]
For a pure-quark system $\mathcal{H}_p$, let $S^p_\leq$, $S^p_\geq$ and $S^p_\geq$ be the sets of Stratonovich-Weyl, mapping-positive and positive-dual correspondences, respectively.

**Theorem 3.24.** The sets $S^p_\leq$, $S^p_\geq$ and $S^p_\geq$ are mutually disjoint.

**Proof.** If $W \in S^p_\leq$, then, from Proposition 3.23, its operator kernel is given by

$$K = \sum_{\nu} (-1)^{2(t+u)} a_{\nu} e(p;\nu) \otimes \tilde{e}(\tilde{p};\tilde{\nu}),$$

where $a_{\nu}$ are coefficients of a convex combination, that is, they are non negative and sum up to 1. From (3.21), its characteristic numbers are

$$c_n = \sqrt{(p+1)(p+2)} \sum_{\nu} (-1)^{2(t+u)} a_{\nu} C_{p,\nu}^p \tilde{p}.n,$$

Since CG coefficients are coefficients of a unitary transformation, their absolute values are bounded above by 1, so

$$|c_n| \leq \sqrt{(p+1)(p+2)} a_{\nu} \sum_{\nu} = \sqrt{(p+1)(p+2)} 2(n+1)^3.$$

Thus,

$$|c_p| < 1$$

and the characteristic numbers $(\tilde{c}_n)$ of $\tilde{W} \in S^p_\geq$ dual to $W \in S^p_\leq$ satisfy

$$|\tilde{c}_p| = \frac{1}{|c_p|} > 1,$$

cf. Proposition 3.19. From Proposition 3.21 and inequalities (3.40)-(3.41), we conclude the statement. □

To verify the existence of a mapping-positive correspondence, we consider the defining representation $\rho_1$ of SU$(3)$ on $C^3$, and invoke the canonical projection

$$\tilde{\pi} : S^5 \rightarrow \mathbb{C}P^2,$$

inside $C^3 \rightarrow \mathbb{C}P^2$, and

$$\Phi_p : C^3 \rightarrow C^{(p+1)(p+2)/2}, (z_1, z_2, z_3) \mapsto (z_1^p, \ldots, \sqrt{\left(\sum_{j,k,l} p_{j,k,l} z_j^k z_3^l \right)}, z_3^p),$$

where $C^{(p+1)(p+2)/2} \simeq \mathcal{H}_{p,0}$ for a representation $\rho_p$ of class $Q(p,0)$, cf. (C.3).

**Proposition 3.25.** The map $B : B(\mathcal{H}_{p,0}) \rightarrow C_{C^p}^\infty(\mathbb{C}P^2) : A \mapsto B_A$, with

$$B_A(x) = \langle \Phi_p(n), A \Phi_p(n) \rangle$$

for $x \in \mathbb{C}P^2$ and $n \in S^5$ related by $\tilde{\pi}(n) = x$, is a mapping-positive symbol correspondence whose operator kernel is the projection $\Pi_{(p,0,0)}$ onto the highest weight space of $Q(p,0)$ and whose characteristic numbers are

$$b_n = (-1)^p \sqrt{(p+1)(p+2)} C_{(p,0), (0,p), n}^{(p,0,0), (0,p,p), (n,n,n), 0}.$$
Proof. First of all, for \( n, n' \in S^5 \), we have \( \tilde{\pi}(n) = \tilde{\pi}(n') \) if and only if \( n' = e^{i\theta} n \), but \( \Phi_p(e^{i\theta} n) = e^{i\theta} \Phi_p(n) \), so

\[
\langle \Phi_p(e^{i\theta} n) | A \Phi_p(e^{i\theta} n) \rangle = \langle e^{i\theta} \Phi_p(n) | e^{i\theta} A \Phi_p(n) \rangle = \langle \Phi_p(n) | A \Phi_p(n) \rangle,
\]

hence \( B_A \) is a well defined function on \( \mathbb{C}P^2, \forall A \in B(\mathcal{H}_{p,0}) \). It is also smooth, since \( \tilde{\pi} \) is a surjective submersion and \( B_A \circ \tilde{\pi} \) is a composition of smooth functions of \( S^5 \).

The linearity of \( B \) is trivial. The equivariance follows straightforwardly from the equivariance of \( \Phi_p \). For any \( g \in SU(3) \),

\[
B_A(g)(x) = \langle \Phi_p(n) | A \Phi_p(n) \rangle = \langle \Phi_p(n) | \rho_p(g) A \rho_p(g)^{-1} \Phi_p(n) \rangle
\]

\[
= \langle \rho_p(g)^{-1} \Phi_p(n) | A \rho_p(g)^{-1} \Phi_p(n) \rangle = \langle \Phi_p(\rho_1(g)^{-1} n) | A \Phi_p(\rho_1(g)^{-1} n) \rangle
\]

\[
= B_A(g^{-1} x) = (B_A)^g(x),
\]

cf. (2.33). Equivariance implies that \( \ker B \) is an invariant subspace, and we use that to prove \( B \) is injective by means of contradiction. Suppose \( B \) is not injective, then \( \ker B \) contains an irreducible representation of the form \( Q(n, n) \), so the diagonal matrix \( e(n, n, J) \) lies in \( \ker B \), that is, there exists a non-zero diagonal matrix \( D = \text{diag}(d_{p,0,0}, ..., d_{j,k,l}, ..., d_{0,0,p}) \in \ker B \). Thus, we have

\[
B_D(x) = \langle \Phi_p(n) | D \Phi_p(n) \rangle = \sum_{j+k+l=1} \binom{p}{j, k, l} d_{j,k,l} |z_j|^2 |z_k|^2 |z_l|^2 = 0
\]

for every \( n = (z_1, z_2, z_3) \in S^5 \), so \( D \) must be a zero matrix, and this is the desired contradiction, therefore \( B \) is injective. Now, we know that

\[
\Pi_{(p,0,0)} = e((p, 0); (p, 0, 0)) \otimes e^*((p, 0); (p, 0, 0))
\]

\[
= (-1)^p e((p, 0); (p, 0, 0)) \otimes e((0,0); (0, p, p))
\]

\[
= (-1)^p \sum_{n=0}^{p} C_{(p,0), (0,p), (n,n,n), 0} e(n; (n, n, n), 0)
\]

\[
= \frac{2}{(p + 1)(p + 2)} \mathbb{1} + (-1)^p \sum_{n=1}^{p} C_{(p,0), (0,p), (n,n,n), 0} e(n; (n, n, n), 0),
\]

where the last equality follows from \( \text{tr} \langle \Pi_{(p,0,0)} \rangle = 1 \). Since the CG coefficients are real\(^{24} \) and \( B \) is an injective map, it is sufficient to show \( B_A(g x_0) = \text{tr} \left( A \Pi_{(p,0,0)} \right) \).

We have that \( n_0 = (1, 0, 0) \in S^5 \) satisfies \( \pi(n_0) = x_0 \) and \( \Phi_p(n_0) = (1, 0, ..., 0) \), so

\[
B_A(x_0) = \langle \Phi_p(n_0) | A \Phi_p(n_0) \rangle = \text{tr} \langle A \Pi_{(p,0,0)} \rangle
\]

and therefore

\[
B_A(g x_0) = (B_A)^g(x_0) = B_{A_{g^{-1}}}(x_0) = \text{tr} \left( A^{g^{-1}} \Pi_{(p,0,0)} \right) = \text{tr} \left( A \Pi_{(p,0,0)}^g \right).
\]

This concludes the proof that \( B \) is a symbol correspondence with operator kernel \( \Pi_{(p,0,0)} \). Equation (3.44) for characteristic numbers follows from (3.34). Finally, Proposition 3.23 implies that \( B \) is a mapping-positive correspondence. \( \square \)

24For a more natural argument, cf. Remark 6.14.
Consider \(\tilde{\rho}_1\) and \(\tilde{\rho}_p\) the dual representations of \(\rho_1\) and \(\rho_p\), respectively, so that
\[
\sigma : \mathbb{C}^3 \rightarrow \mathbb{C}^3 : (z_1, z_2, z_3) \mapsto (-\overline{z}_3, \overline{z}_2, -\overline{z}_1)
\]
and \(\Phi_p\) are both equivariant maps in the following sense (cf. (3.44)): \(\tilde{\rho}_p(g) \circ \sigma = \sigma \circ \rho_p(g)\).

\[
\Phi_p(g) \circ \tilde{\rho}_p(g) = \Phi_p \circ \tilde{\rho}_p(g).
\]

**Proposition 3.26.** The map \(B^- : \mathcal{B}(\mathcal{H}_{0,p}) \rightarrow C_c^\infty(\mathbb{C}P^2) : A \mapsto B^-_A\), with
\[
B^-_A(x) = \langle \Phi_p \circ \sigma(n) | A \Phi_p \circ \sigma(n) \rangle
\]
for \(x \in \mathbb{C}P^2\) and \(n \in S^5 \subset \mathbb{C}^3\) related by \(\pi(n) = x\), is a mapping-positive symbol correspondence whose operator kernel is the projection \(\Pi_{(0,p,p)}\) onto the lowest weight space of \(Q(0,p)\) and whose characteristic numbers are
\[
b_{n_0} = (-1)^p \sqrt{\frac{(p+1)(p+2)}{2(n+1)^3}} \sigma_{C_c^\infty(\mathbb{C}P^2)}(0,p,n) \cdot
\]

**Proof.** The proof goes just as the proof of Proposition 3.25, we just highlight that the following hold: \(\Phi_p \circ \sigma(e^{i\theta}n) = e^{-i\theta} \Phi_p \circ \sigma(n)\),
\[
\Pi_{(0,p,p)} = \tilde{e}((0,p);(0,p,p)) \otimes e^*((0,p);(0,p,p))
\]
\[
= (-1)^p \tilde{e}((0,p);(0,p,p)) \otimes e((0,p);(0,p,0))
\]
\[
= \frac{2}{(p+1)(p+2)} e_{0}^{(p,0),(p,0,0),(0,0,0)} e_{0}^{(0,0),(0,0,0),(0,0,0)}
\]
and \(n_0 = (1,0,0) \in S^5\) satisfies \(\Phi_p \circ \sigma(n_0) = (0,...,0,(-1)^p)\). \(\square\)

**Remark 3.27.** One could expect that \(\Pi_{(0,0,p)} \in \mathcal{B}(\mathcal{H}_{p,0})\), the projection onto the lowest weight space of \(Q(p,0)\), also is an operator kernel according to (3.19)-(3.20). This is not the case since \(\Pi_{(0,0,p)}\) is not \(H\)-invariant, cf. Propositions 3.20 and 3.21. However, this is just a matter of convention, a choice of \(U(2)\) subgroup by which we impose invariance. Recalling (C.3), we have \(\rho_p(\delta) e_{j,k,l} = (-1)^p e_{l,k,j}\) for \(\delta\) as in (2.120), so that \(\Pi_{(0,0,p)} = \Pi_{(0,0,0)}\) and \(\Pi_{(0,0,p)}\) is fixed by \(\hat{H} = \delta \hat{H} \delta\), cf. (2.119). If we set
\[
\tilde{x}_0 = \tilde{\delta} x_0 = [0 : 0 : 1] \in \mathbb{C}P^2,
\]
then we can construct a symbol correspondence \(A \mapsto B'_A\) using \(\Pi_{(0,0,p)}\) as operator kernel via the modified rule (compare with (3.25)) given by
\[
B'_A(g \tilde{x}_0) = \text{tr} \left( A \Pi^g_{(0,0,p)} \right).
\]

But in fact, \(B'\) and \(B\) are the same map, that is,
\[
B'_A(g \tilde{x}_0) = \text{tr} \left( A \Pi^g_{(0,0,p)} \right) = \text{tr} \left( A \Pi^{g^*}_{(0,0,0)} \right) = B_A^* g \tilde{x}_0 = B_A(g \tilde{x}_0).
\]

In the same vein, using the modified rule
\[
B^-_A(\tilde{x}_0) = \text{tr} \left( A \Pi^{g^*}_{(0,p,0)} \right),
\]
we can identify the highest weight projector of \(Q(0,p)\) as the operator kernel of the symbol correspondence \(A \mapsto B'^{-}_A\). But again, \(B'^{-}_A \equiv B^{-}_A\).
So far, practically all results obtained on symbol correspondences for pure-quark systems have analogous results for spin systems, as well, cf. [24] and [1]. However, the next proposition, extending Remark 3.27, sets an important distinction between symbol correspondences for pure-quark systems and for spin systems.

**Proposition 3.28.** Among all projectors onto weight spaces of \( Q(p, 0) \), projector \( \Pi_{(p, 0, 0)} \in \mathcal{B}(\mathcal{H}_{p, 0}) \) is the only one that is an operator kernel in the sense of (3.19)-(3.20), cf. Definition 3.15. Likewise, \( \Pi_{(0, p, p)} \in \mathcal{B}(\mathcal{H}_{0, p}) \) is the unique projector onto a weight space of \( Q(0, p) \) that is an operator kernel, in the above sense.

**Proof.** If \( \Pi_{\nu} \) is an operator kernel for \( Q(p) \) according to (3.19)-(3.20), from (3.28) we get \( j = 0 \), that is, \( e(p; \nu) = e(p; 0, 0) \), cf. (3.11). Using (3.8)-(3.9), we get that \( \nu = (p, 0, 0) \) for \( p = (p, 0) \) and \( \nu = (0, p, p) \) for \( p = (0, p) \). \( \square \)

**Definition 3.29.** The symbol correspondences \( B : \mathcal{B}(\mathcal{H}_{p, 0}) \rightarrow C_{\mathbb{C}}^\infty(\mathbb{C}P^2) \), with operator kernel \( \Pi_{(p, 0, 0)} \), and \( B^- : \mathcal{B}(\mathcal{H}_{0, p}) \rightarrow C_{\mathbb{C}}^\infty(\mathbb{C}P^2) \), with operator kernel \( \Pi_{(0, p, p)} \), are the symmetric Berezin correspondences of \( Q(p, 0) \) and \( Q(0, p) \), respectively, with each unique dual being the respective symmetric Toeplitz correspondence.

**Proposition 3.30.** The symmetric Berezin (or Toeplitz) correspondences of \( Q(p, 0) \) and \( Q(0, p) \) have the same characteristic numbers.

**Proof.** This follows from (3.44) and (3.48) using Theorem 2.22. \( \square \)

By Corollary 3.28, the moduli space of Stratonovich-Weyl correspondences for a pure-quark system \( Q(p) \) is \( \mathbb{Z}_2 \)\(^p \), with different Stratonovich-Weyl correspondences lying in different connected components of the moduli space \( \mathbb{R}^+ \)\(^p \) of all correspondences. Thus, there is an unique Stratonovich-Weyl correspondence that can be continuously deformed from the symmetric Berezin correspondence for \( Q(p) \).

**Definition 3.31.** The symbol correspondence for a pure-quark system \( Q(p) \) with characteristic numbers given by (cf. (3.44) and (3.48) and Proposition 3.30)

\[
(3.53) \quad c_n = b_n / |b_n| = b_n^- / |b_n^-|
\]

is called the symmetric Stratonovich-Weyl correspondence.

**Remark 3.32.** The impossibility of both projectors onto lowest and highest weight spaces of the same representation defining symbol correspondences for pure-quark systems via (3.19)-(3.20) is a direct consequence of Proposition 3.28, which follows from imposing \( H \simeq U(2) \) invariance. If one relaxes this invariance condition, both the lowest projector for \( Q(p, 0) \) and the highest projector for \( Q(0, p) \) can define less symmetric Berezin correspondences via (3.19)-(3.20), but the symbols are now functions on the generic orbit \( \mathbb{C}P^1 \hookrightarrow \mathcal{E} \hookrightarrow \mathbb{C}P^2 \), cf. Theorem 4.26.

Now, we introduce the following:

**Definition 3.33.** For a correspondence \( W : \mathcal{B}(\mathcal{H}_p) \rightarrow C_{\mathbb{C}}^\infty(\mathbb{C}P^2) \), its antipodal correspondence \( \tilde{W} : \mathcal{B}(\mathcal{H}_p) \rightarrow C_{\mathbb{C}}^\infty(\mathbb{C}P^2) \) is the one given by (cf. (2.85)-(2.87)):

\[
(3.54) \quad \tilde{W}_A^* = W_A.
\]

**Remark 3.34.** Recalling Remark 3.10, if one defines a symbol correspondence as a map \( W' : \mathcal{B}(\mathcal{H}_p) \rightarrow C_{\mathbb{C}}^\infty(\mathcal{O}_p) \), its antipodal correspondence \( \tilde{W}' : \mathcal{B}(\mathcal{H}_p) \rightarrow C_{\mathbb{C}}^\infty(\mathcal{O}_p) \) is related to \( W' \) by

\[
(3.55) \quad \tilde{W}'_A^* = W_A^* \circ \iota,
\]
where \( i = -\text{id} \) on \( \text{su}(3) \), thus the name antipodal.

**Proposition 3.35.** Two symbol correspondences \( W_1 : \mathcal{B}(\mathcal{H}_p) \to C^\infty_c(\mathbb{C}P^2) \) and \( W_2 : \mathcal{B}(\mathcal{H}_p) \to C^\infty_c(\mathbb{C}P^2) \) are antipodal to each other if and only if their characteristic numbers are equal.

**Proof.** The result follows from (2.87) and Theorem 3.11 \( \square \)

**Corollary 3.36.** The Berezin correspondences for \( \mathcal{B}(\mathcal{H}_p) \) and \( \mathcal{B}(\overline{\mathcal{H}}_p) \) are antipodal to each other. Also, a correspondence for a pure-quark system is Stratonovich-Weyl correspondence if and only if its antipodal correspondence is Stratonovich-Weyl.

3.4. **Twisted product for pure-quark system.** Again, \( p \in (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N}) \) with \( |p| = p \). It is obvious from Theorem 3.11 that the images of all symbol correspondences for \( \mathcal{H}_p \) and \( \mathcal{H}_\overline{p} \) are the same space, namely, the space spanned by the \( \mathbb{C}P^2 \) harmonics \( X_{\nu,j}^n \) with \( 0 \leq n \leq p \). This space shall be denoted by

\[
X_p = \text{Span}_\mathbb{C}\{X_{\nu,j}^n\}_{0 \leq n \leq p}.
\]

Now, we translate the operator algebra \( \mathcal{B}(\mathcal{H}_p) \) to \( X_p \) using a symbol correspondence.

**Definition 3.37.** Given a symbol correspondence \( W : \mathcal{B}(\mathcal{H}_p) \to C^\infty_c(\mathbb{C}P^2) \), the twisted product of symbols induced by \( W \) is the binary operation \( * \) on \( X_p \) given by

\[
(W_A * W_R)(p) = W_{AR}(p),
\]

for any \( A, R \in \mathcal{B}(\mathcal{H}_p) \). The algebra \( (X_p, *) \) is called a twisted \( p \)-algebra.

**Proposition 3.38.** Any twisted \( p \)-algebra \( (X_p, *) \) is

\[
i) \quad \text{SU}(3)\text{-equivariant}: (f_1 \ast f_2)g = f_1^g \ast f_2^g; \\
ii) \quad \text{Associative}: (f_1 \ast f_2) \ast f_3 = f_1 \ast (f_2 \ast f_3); \\
iii) \quad \text{Unital}: 1 \ast f = f \ast 1 = f; \\
iv) \quad \text{A \ast algebra}: f_1 \ast f_2 = f_2 \ast f_1;
\]

where \( f_1, f_2, f_3, f \in X_p, g \in \text{SU}(3) \) and \( 1 \in X_p \) is the constant function equal to 1 on \( \mathbb{C}P^2 \), cf. (3.3).

**Proof.** The operator space \( \mathcal{B}(\mathcal{H}_p) \) is an \( \text{SU}(3) \)-equivariant unital associative \( \ast \)-algebra with respect to Hermitian conjugate, where \( \mathbb{I} \) is the identity. Since any symbol correspondence \( W \) for \( \mathcal{H}_p \) is an \( \text{SU}(3) \)-equivariant linear isomorphism between \( \mathcal{B}(\mathcal{H}_p) \) and \( X_p \) satisfying reality and \( W_1 = 1 \), the statement is true. \( \square \)

**Proposition 3.39.** Fixed \( p \in (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N}) \), any two twisted \( p \)-algebras are naturally isomorphic, and any twisted \( p \)-algebra is naturally anti-isomorphic to any twisted \( \overline{p} \)-algebra.

**Proof.** Let \( W_1, W_2 : \mathcal{B}(\mathcal{H}_p) \to C^\infty_c(\mathbb{C}P^2) \) be symbol correspondences. Then \( W_1 \circ W_2^{-1} : X_p \to X_p \) is an isomorphism because each \( W_k \) is an isomorphism onto \( X_p \).

If, now, we suppose \( W_2 : \mathcal{B}(\mathcal{H}_\overline{p}) \to C^\infty_c(\mathbb{C}P^2) \), then \( W_1 \circ \ast \circ W_2^{-1} : X_p \to X_p \) is an anti-isomorphism since the adjoint map \( \ast \) is an anti-isomorphism and, again, each \( W_k \) is an isomorphism onto \( X_p \). \( \square \)

Twisted products of \( \mathbb{C}P^2 \) harmonics can be easily computed and determine the twisted product for all functions in \( X_p \) by bilinearity of the product.
Theorem 3.40. If \( W : \mathcal{B}(\mathcal{H}_p) \rightarrow \mathcal{C}_\mathbb{C}^\infty(\mathbb{C}P^2) \) is a symbol correspondence with characteristic numbers \( (c_n) \), then the induced twisted product is given by (3.57)

\[
X^{n_1}_{\nu_1, J_1} \ast X^{n_2}_{\nu_2, J_2} = \frac{(p+1)(p+2)}{2} \sum_{n=0}^{p} \sum_{\nu, J} (-1)^{p+2(n+u_\nu)} \left[ \begin{array}{ccc} n_1 & n_2 & n \\ \nu_1, J_1 & \nu_2, J_2 & \nu, J \end{array} \right] [p] \times \frac{c_n}{c_{n_1}c_{n_2}} X^n_{\nu, J}
\]

for \( 0 \leq n_1, n_2 \leq p \), where summations over \( \nu \) and \( J \) are effectively restricted to \( \nabla_{\nu_1+\nu_2, \nu} = 1 \) and \( \delta(J_1, J_2, J) = 1 \) due to (3.13).

Proof. The result follows directly from Theorems 3.8 and 3.11. \( \Box \)

Any twisted product on \( X_p \) admits an integral formulation, supposedly allowing one to compute it without decomposing functions in the basis of \( \mathbb{C}P^2 \) harmonics.

Theorem 3.41. For \( W : \mathcal{B}(\mathcal{H}_p) \rightarrow \mathcal{C}_\mathbb{C}^\infty(\mathbb{C}P^2) \) a correspondence with operator kernel \( K \) and characteristic numbers \( (c_n) \), its induced twister product is given by (3.58)

\[
f_1 \ast f_2(x) = \int_{\mathbb{C}P^2 \times \mathbb{C}P^2} f_1(x_1)f_2(x_2) L(x_1, x_2, x) \, dx_1 \, dx_2
\]

for any \( f_1, f_2 \in X_p \), where (3.59)

\[
L(x_1, x_2, x_3) = \left( \frac{(p+1)(p+2)}{2} \right)^2 \text{tr} (\tilde{K}(x_1) \tilde{K}(x_2) K(x_3))
\]

\[
= (-1)^p \sqrt{\frac{(p+1)(p+2)}{2}} \sum_{n=0}^{p} \sum_{\nu_1, J_1, \nu_2, J_2} \left[ \begin{array}{ccc} n_1 & n_2 & n_3 \\ \nu_1, J_1 & \nu_2, J_2 & \nu_3, J_3 \end{array} \right] [p] \times \frac{c_n}{c_{n_1}c_{n_2}} X^n_{\nu_1, J_1} X^{n_2}_{\nu_2, J_2} X^{n_3}_{\nu_3, J_3}(x_3)
\]

where \( \tilde{K} \) is the operator kernel dual to \( K \), with characteristic numbers \( \tilde{c}_n = (1/c^n) \).

Proof. Let \( A_1, A_2 \in \mathcal{B}(\mathcal{H}_p) \) so that \( f_1 = W A_1 \), \( f_2 = W A_2 \). By Proposition 3.19

\[
\left( \frac{(p+1)(p+2)}{2} \right)^2 \int_{\mathbb{C}P^2 \times \mathbb{C}P^2} f_1(x_1)f_2(x_2) \text{tr} (\tilde{K}(x_1) \tilde{K}(x_2) K(x)) \, dx_1 \, dx_2
\]

\[
= \text{tr} \left( \left( \frac{(p+1)(p+2)}{2} \right)^2 \int_{\mathbb{C}P^2 \times \mathbb{C}P^2} f_1(x_1)f_2(x_2) \tilde{K}(x_1) \tilde{K}(x_2) K(x) \, dx_1 \, dx_2 \right)
\]

\[
= \text{tr} (A_1 A_2 K(x)) = W A_1 A_2(x) = f_1 \ast f_2(x).
\]

Now, \( \tilde{K}(x_1) \), \( \tilde{K}(x_2) \) and \( K(x_3) \) can be expanded using Wigner D-functions so that \( L(x_1, x_2, x_3) \) is a linear combinations of \( X^{n_1}_{\nu_1, J_1}(x_1) X^{n_2}_{\nu_2, J_2}(x_2) X^{n_3}_{\nu_3, J_3}(x_3) \). Then, Theorem 3.40 implies the second equality in (3.59). \( \Box \)

Definition 3.42. The integral trikernel \( L \in \mathcal{C}_\mathbb{C}^\infty(\mathbb{C}P^2 \times \mathbb{C}P^2 \times \mathbb{C}P^2) \) of a twisted product induced by a symbol correspondence \( W : \mathcal{B}(\mathcal{H}_p) \rightarrow \mathcal{C}_\mathbb{C}^\infty(\mathbb{C}P^2) \) is the function given by (3.59) so that the twisted product is given by (3.58).

Obviously, the integral in (3.59) is well defined for any pair of smooth functions on \( \mathbb{C}P^2 \), so it leads to a product on \( \mathcal{C}_\mathbb{C}^\infty(\mathbb{C}P^2) \).
Proposition 3.43. Let $L$ be the integral trikernel of a twisted product $\ast$ induced by a symbol correspondence $W : \mathcal{B}(\mathcal{H}_p) \rightarrow \mathcal{C}_c^\infty(\mathbb{C}P^2)$. Then the binary operation $\ast$ on $\mathcal{C}_c^\infty(\mathbb{C}P^2)$,

$$(3.60) \quad f_1 \ast f_2(x) = \int_{\mathbb{C}P^2 \times \mathbb{C}P^2} f_1(x_1)f_2(x_2)L(x_1, x_2, x)\,dx_1\,dx_2$$

for any $f_1, f_2 \in \mathcal{C}_c^\infty(\mathbb{C}P^2)$, defines an $SU(3)$-equivariant associative $*$-algebra with respect to complex conjugation. In particular, if $f_1, f_2 \in X_p$, we have $f_1 \ast f_2 = f_1 \star f_2$. But, if either $f_1$ or $f_2$ is orthogonal to $X_p$, we have $f_1 \ast f_2 = 0$.

Proof. Linearity of integral implies the product is bilinear, hence it defines an algebra. By definition, it is clear that $f_1 \ast f_2 = f_1 \ast f_2$ if $f_1, f_2 \in X_p$. Now, suppose $f_k$ is orthogonal to $X_p$. Thus, it is orthogonal to every $X_{p, j}$ with $n \leq p$, which implies the integral over $x_k$ in (3.60) results in 0, so $f_1 \ast f_2 = 0$. Since any $f \in \mathcal{C}_c^\infty(\mathbb{C}P^2)$ can be decomposed into $f = f_\parallel + f_\perp$, where $f_\parallel \in X_p$ and $f_\perp$ is orthogonal to $X_p$, the $SU(3)$-equivariant, associative and $*$-algebra properties of $\ast$ extends to $\ast$. □

Remark 3.44. For a product $\ast$ as in (3.60), the constant function 1 is no longer the identity, now it gives an orthogonal projection $\mathcal{C}_c^\infty(\mathbb{C}P^2) \rightarrow X_p : f \mapsto 1 \ast f = f = f \ast 1$.

Notation 5. Before stating general properties of integral trikernels, we denote the reproducing kernel on $X_p$ by

$$(3.61) \quad R_p(x_1, x_2) = \sum_{n=0}^p \sum_{\nu, j} X_{\nu, j}^\ast(x_1)X_{\nu, j}(x_2) = R_p(x_2, x_1),$$

satisfying

$$(3.62) \quad \int_{\mathbb{C}P^2} f(x_1)R_p(x_1, x_2)\,dx_1 = f(x_2), \quad \forall f \in X_p.$$ 

Proposition 3.45. Let $L$ be an integral trikernel of a twisted product $\ast$ on $X_p$. Then, for every $g \in SU(3)$ and every $x_1, x_2, x_3, x_4 \in \mathbb{C}P^2$,

i) $L(x_1, x_2, x_3) = L(gx_1, gx_2, gx_3);$

ii) $\int_{\mathbb{C}P^2} L(x_1, x_2, x)L(x, x_3, x_4)\,dx = \int_{\mathbb{C}P^2} L(x_1, x, x_4)L(x_2, x_3, x)\,dx;$

iii) $\int_{\mathbb{C}P^2} L(x, x_1, x_2)\,dx = \int_{\mathbb{C}P^2} L(x_1, x, x_2)\,dx = R_p(x_1, x_2);$

iv) $L(x_1, x_2, x_3) = L(x_2, x_1, x_3).$

Proof. Let $f_1, f_2 \in X_p$. Writing the equality $(f_1)^\ast(f_2)^\ast = (f_1 \star f_2)^\ast$ in the integral form, we get that $SU(3)$-equivariance of $\ast$ is equivalent to property (i). In the same vein, we conclude that each property of this statement is equivalent to the property of Proposition 3.45 with same number. □

Remark 3.46. Although the integral formulation of a twisted product on $X_p$ is supposed to circumvent the necessity of decomposing symbols (elements of $X_p$) in the basis of $\mathbb{C}P^2$ harmonics, the formula (3.59) for an integral trikernel uses these harmonics explicitly. In [24], new formulas for integral trikernels of spin systems were obtained using $SU(2)$-invariant 2-point and 3-point functions on $\mathbb{C}P^1$, but a similar exercise for pure-quark systems is much harder and is deferred for later.

We finish this section with a relation between twisted algebras induced by antipodal correspondences.
Proposition 3.47. The twisted products $\star$ and $\bar{\star}$ induced by a symbol correspondence and its antipodal correspondence satisfy

\[ f_1 \star f_2 = f_2 \bar{\star} f_1. \]

Proof. For $f_1 = W_{F_1} = \tilde{W}_{F_1'}$ and $f_2 = W_{F_2} = \tilde{W}_{F_2'}$,

\[ f_1 \star f_2 = W_{F_1 F_2} = \tilde{W}_{(F_1 F_2)'} = \tilde{W}_{F_2' F_1'} = f_2 \bar{\star} f_1. \]

\[ \square \]

Corollary 3.48. For $\star$ and $\bar{\star}$ as in the previous proposition, their integral trikernels $L$ and $\tilde{L}$ satisfy

\[ L(x_1, x_2, x_3) = \tilde{L}(x_2, x_1, x_3). \]

Remark 3.49. We already mentioned that the notion of antipodal correspondences for quark systems is analogous to alternation for spin systems considering the appropriate characterization. In addition to the previous discussion, we present a related phenomenon encoded in Proposition 3.47 which also happens for spin systems. The commutator $[\cdot, \cdot]$, of a twisted product $\star$ satisfies

\[ [f_1, f_2] \star = [f_2, f_1] \bar{\star}, \]

where $\bar{\star}$ is the twisted product induced by the antipodal correspondence. In this way, $\bar{\star}$ can be seen as defining the reverse symbolic dynamics of the one defined by $\star$. In fact, recalling Heisenberg’s equation for an operator $F$ subject to a Hamiltonian $H$,

\[ \frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t}, \]

if $F$ has no explicit temporal dependence, then under a symbol correspondence $W$ its symbol $f$ satisfies

\[ \frac{df}{dt} = [f, h] \star, \]

where $h = W_H$. It follows that if we set $H^*$ as the Hamiltonian of the dual space, the symbolic dynamics of $F^*$ under $\tilde{W}$ is given by

\[ \frac{df}{dt} = [f, h] \bar{\star} = -[f, h] \star. \]

### 4. Generic quark systems

In this section, we begin a study of correspondences for generic quark systems, that is, representations of generic class $Q(p, q)$ and generic coadjoint orbit $\mathcal{E}(\mathbb{C}P^2, \mathbb{C}P^1, \pi)$. Although we proceed by basically reproducing what we have done for $Q(p, 0)$ (or $Q(0, q)$) and $\mathbb{C}P^2$, some new phenomena shall appear.

#### 4.1. Classical generic quark system.

**Definition 4.1.** The generic classical quark system is the symplectic total space $\mathcal{E}$ of the fiber bundle $\mathcal{E}(\mathbb{C}P^2, \mathbb{C}P^1, \pi)$, with base $\mathbb{C}P^2$, fiber $\mathbb{C}P^1$ and projection $\pi$,

\[ \mathbb{C}P^1 \hookrightarrow \mathcal{E} \xrightarrow{\pi} \mathbb{C}P^2, \]

together with its Poisson algebra on $C_c^\infty(\mathcal{E})$.

We have $\mathcal{E} \simeq SU(3)/T$, where $T$ is the maximal torus $\mathbb{T}$ of $SU(3)$. So we look for representations with weights satisfying $t = u = 0$, cf. (2.10)-(2.17).
**Proposition 4.2.** The representations of SU(3) with non null vectors fixed by $T$ are of the form $Q(a, b)$ for $a \equiv b \pmod{3}$. For 

$$k = |a - b|/3,$$

the space fixed by $T$ is spanned by the set

$$\{e((a, b); J_{\gamma}, \nu): \gamma = 1, ..., \min\{a, b\} + 1\},$$

where

$$\nu_{(a, b)} = \begin{cases} 
(a + 2k, a + 2k, a + 2k) & \text{if} \quad \min\{a, b\} = a \\
(b + k, b + k, b + k) & \text{if} \quad \min\{a, b\} = b 
\end{cases}$$

and

$$J_{\gamma} = \gamma - 1 + k.$$

**Proof.** Let $e((a, b); (\nu_1, \nu_2, \nu_3), J)$ be such that $t = u = 0$. From (2.16)-(2.17), we get that $\nu_1 = \nu_2 = \nu_3 = \nu \in \mathbb{N}_0$, with

$$2\nu = r_+ + r_- \quad \text{and} \quad 3\nu = a + 2b,$$

for

$$0 \leq r_- \leq b \leq r_+ \leq a + b, \quad r_- \leq \nu \leq r_+.$$

From (4.5), $a + 2b \equiv 0 \pmod{3}$, which implies $a \equiv b \pmod{3}$.

For representations of class $Q(a, a + 3k)$, with $a, k \in \mathbb{N}_0$, we have that

$$\nu = a + 2k$$

and the GT states fixed by $T$ are given by $r_+$ and $r_-$ satisfying

$$\begin{cases} 
2r_+ + r_- = 2a + 4k \\
0 \leq r_- \leq a + 3k \leq r_+ \leq 2a + 3k 
\end{cases}.$$

The system has $a + 1$ solutions:

$$\begin{cases} 
r_+ = 2a + 3k, \quad r_- = k \\
\vdots \\
r_+ = a + 3k, \quad r_- = a + k 
\end{cases},$$

so that the subspace fixed by $T$ is spanned by

$$\{e((a, a + 3k); (a + 2k, a + 2k, a + 2k), J): J = k, ..., a + k\}.$$

For a representation of class $Q(b, b + 3k)$, since it is dual to $Q(b, b + 3k)$, the subspace fixed by $T$ is spanned by

$$\{e((b + 3k, b); (b + k, b + k, b + k), J): J = k, ..., b + k\}.$$

To finish, we order the $J$-multiplicities by crescent $J$ in both cases (4.9)-(4.10) by setting $J_{\gamma} = \gamma + k - 1$, where $1 \leq \gamma \leq \min\{a, b\} + 1$. \hfill $\square$

**Definition 4.3.** The $E$ harmonics are the functions on $E$ given by

$$Z_{\nu, J, \gamma}^{(a,b)}(gz_0) = \sqrt{\frac{(a + 1)(b + 1)(a + b + 2)}{2}} D_{\nu, J, \gamma}^{(a,b)}(g),$$

for $z_0 \in \pi^{-1}([1 : 0 : 0]) \subset E$ with $T$ as isotropy subgroup, $g \in SU(3)$, with $a \equiv b \pmod{3}$ and $(\nu_{(a,b)}, J_{\gamma})$ as in Proposition 4.2 and where $D_{\nu, J, \gamma}^{(a,b)}$ is a Wigner $D$-function as in Definition 2.5.
Just as in definition of CP² harmonics, the factor $\sqrt{(a+1)(b+1)(a+b+2)/2}$ is the square root of the dimension of the representation $Q(a,b)$ and is used to ensure normalization according to Schur’s Orthogonality Relations, so that

\[(4.12) \quad \left\langle Z_{\nu,J}^{(a,b,\gamma)} | Z_{\mu,L}^{(c,d,\zeta)} \right\rangle = \delta_{a,c} \delta_{b,d} \delta_{\gamma,\zeta} \delta_{J,L} .\]

**Remark 4.4.** Analogously to Remark 3.4, for any $x, y > 0$, we can take the generic harmonics as functions on $O_{x,y}$ via the compositions $Z_{\nu,J}^{(a,b,\gamma)} \circ \psi_{x,y}$ so that the harmonic functions on $O_{x,y}$ are related to the ones on $O_{y,x}$ by $\alpha_{x,y} \circ \iota$, cf. (2.130). Besides that, the involution $\alpha_{x,y}$ generates another, but somewhat equivalent, set of harmonic functions on $O_{x,y}$, just as $\alpha$ does to $\mathcal{E}$ (cf. (2.124)):

\[(4.13) \quad \tilde{Z}_{\nu,J}^{(a,b,\gamma)}(g\mathbf{z}_0) = Z_{\nu,J}^{(a,b,\gamma)}(g\tilde{\mathbf{z}}_0) .\]

As expected, we have

\[(4.14) \quad Z^{(0,0)}_{(0,0,0),0} = 1\]

and, cf. (2.32),

\[(4.15) \quad Z_{\nu,J}^{(a,b,\gamma)} = (-1)^{2(t+u)} Z_{\nu,J}^{(b,a,\gamma)}, \text{ for } \Delta_{\nu,\nu}^{a+b} = 1 .\]

**Theorem 4.5.** The decomposition of pointwise product of $\mathcal{E}$ harmonics is given by

\[(4.16) \quad Z_{\nu_1,J_1}^{(a_1,\gamma_1)} Z_{\nu_2,J_2}^{(a_2,\gamma_2)} = \sum_{(a,\sigma)} \sqrt{\frac{\dim Q(a_1) \dim Q(a_2) \dim Q(a)}{\dim Q(a)}} C_{a_1, a_2, (a,\sigma)}^{a, \nu_1, J_1, \nu_2, J_2, \nu, J} \times C_{a_1, a_2, (a,\sigma)}^{a, \nu_1, J_1, \nu_2, J_2, \nu, J}_{\nu_1, J_1, \nu_2, J_2},\]

for $(\nu_{a_1}, J_{\gamma_1})$ and $(\nu_{a_2}, J_{\gamma_2})$ as in Proposition 4.4 and summation restricted to $\nabla_{\nu_1 + \nu_2, \nu} = 1$, $\delta(J_1, J_2, J) = \delta(J_{\gamma_1}, J_{\gamma_2}, J) = 1$ and $Q(a; \sigma)$ in the Clebsch-Gordan series of $Q(a_1) \otimes Q(a_2)$.

**Proof.** With a little abuse of notation, again,

\[(4.17) \quad Z_{\nu,J}^{(a,\gamma)} = \sqrt{\dim Q(a)} \overline{D}_{\nu,J}^{a, \nu_a, \nu_J, J_k},\]

and Lemma 2.23 give us

\[(4.18) \quad Z_{\nu_1, J_1}^{(a_1,\gamma_1)} Z_{\nu_2, J_2}^{(a_2,\gamma_2)} = \sum_{(a,\sigma)} \sum_{\nu, J} \sqrt{\dim Q(a_1) \dim Q(a_2) C_{a_1, a_2, \nu_J}^{a, \nu_1, J_1, \nu_2, J_2, \nu, J}} \times C_{a_1, a_2, \nu_J}^{a, \nu_1, J_1, \nu_2, J_2, \nu, J}_{\nu_1, J_1, \nu_2, J_2, \mu, L},\]

where $\nabla_{\nu_1 + \nu_2, \nu} = \nabla_{\nu_1 + \nu_2, \nu} = 1$ and $\delta(J_1, J_2, J) = \delta(J_{\gamma_1}, J_{\gamma_2}, L) = 1$, so $\mu = (\mu, \mu, \mu)$. But $e(a; (\mu, \mu, \mu), L)$ only exists if $a$ and $(\mu, L)$ are as in Proposition 4.2. Thus, we set $\mu = \nu_a$ and $L = J_{\gamma}$. \(\square\)

**Remark 4.6.** As in the decomposition of the pointwise product of CP² harmonics, the decomposition of the pointwise product of $\mathcal{E}$ harmonics follows directly as a special case of Lemma 2.23 and does not “see” the symplectic structure on $\mathcal{E}$. Thus, as in the pure-quark case, the next step is to decompose the Poisson bracket of $\mathcal{E}$ harmonics, but this is a much harder problem that is deferred to a later study.
4.2. Quantum generic quark system. Now, we want representations \( Q(p, q) \) such that \( Q(p, q) \otimes Q(q, p) \) splits only into representations of the form \( Q(a, b) \), \( a \equiv b \) (mod 3), with multiplicity less than or equal to \( \min\{a, b\} + 1 \). From Corollary 2.7 if we suppose, without loss of generality, that \( \min\{a, b\} = a \), then the occurrences of \( Q(a, b) \oplus Q(b, a) \) are given by the solutions of

\[
(4.18) \quad a = p + q - n - m - 2k, \quad 0 \leq n \leq p - k, \quad 0 \leq m \leq q - k,
\]

where \( b = a + 3k \). Of course, we can also assume without loss of generality that \( p \geq q \). If \( a + k \leq q \), then we have \( a + 1 \) solutions:

\[
(4.19) \quad \begin{cases} n = p - k - a, & m = q - k \\ n = p - k - a + 1, & m = q - k - 1 \\ \vdots \\ n = p - k, & m = q - a - k \end{cases}
\]

Otherwise, we need to eliminate some lines of the above solutions, which means \( Q(a, b) \oplus Q(b, a) \) have multiplicity less than \( \min\{a, b\} + 1 \). Then, we have:

**Definition 4.7.** Let \((p, q) \in (\mathbb{N} \times \mathbb{N}_0) \cup (\mathbb{N}_0 \times \mathbb{N})\). A quantum generic quark system is a complex Hilbert space \( \mathcal{H}_{p,q} \simeq \mathbb{C}^d \), where \( d = \dim Q(p, q) \) is given by (2.14), with an irreducible unitary \( SU(3) \)-representation of class \( Q(p, q) \) together with its operator algebra \( \mathcal{B}(\mathcal{H}_{p,q}) \). If \( p \geq q \), the pair \((p, q)\) and the system \( \mathcal{H}_{p,q} \) are called material. If \( p > q \), they are called baryonic and if \( p = q \) they are called mesonic. Alternatively, if \( p < q \), they are called antibaryonic.

**Remark 4.8.** The nomenclature in Definition 4.7 is related to the number of quarks vs. antiquarks for a generic quark system, cf. Appendix C. From Theorem 2.6

\[
(4.20) \quad Q(p, 0) \otimes Q(0, q) = \bigoplus_{n=0}^{\min\{p,q\}} Q(p-n, q-n),
\]

so a generic representation \( Q(p, q) \) is the invariant space of \( Q(p, 0) \otimes Q(0, q) \) where the product of the highest weight vectors lives in. That means material quark systems can be constructed from systems with number \( p \) of quarks greater than or equal to number \( q \) of antiquarks, cf. Appendix C. The names baryonic and mesonic refer to systems with positive and null baryon number, respectively, recalling that a system of \( p \) quarks and \( q \) antiquarks has baryon number \( B = (p - q)/3 \). In Physics, if \( B > 0 \), the system is a baryon (an antibaryon if \( B < 0 \)); if \( B = 0 \), it is a meson.

In particular, quantum generic quark systems encompass quantum pure-quark systems as special cases. But now, all forms of (2.111) and (2.112) are relevant to us, and we cannot further simplify Corollary 2.31 as we did for pure-quark systems. Also, since quantum mesonic systems are self-dual, it is possible to identify

\[
\mathcal{B}(\mathcal{H}_{p,p}^*) \ni \mathcal{e}((a; \tilde{\sigma}); \nu, J) \longleftrightarrow \mathcal{e}((a; \sigma); \nu, J) \in \mathcal{B}(\mathcal{H}_{p,p}),
\]

which is an \( SU(3) \)-invariant isomorphism, analogously to what is implicitly done for spin systems in [24]. But we will not use such identification to avoid confusion.

---

22 Again, we are ignoring the trivial representation \( Q(0, 0) \).
4.3. Symbol correspondences for generic quark systems. Let $p \in (\mathbb{N} \times \mathbb{N}) \cup (\mathbb{N}_0 \times \mathbb{N})$. The following is completely analogous to Definition 3.9.

**Definition 4.9.** A symbol correspondence for a generic quark system $(\mathcal{H}_p, Q(p))$, referred to simply as a symbol correspondence or just as a correspondence, is an injective linear map $W : B(\mathcal{H}_p) \to C^\infty_c(\mathcal{E}) : P \mapsto W_P$ satisfying, $\forall A \in B(\mathcal{H}_p)$,

1. **Equivariance:** $\forall g \in SU(3)$, $W_{A^g} = W_A$.
2. **Reality:** $W_{A^\dagger} = \overline{W_A}$.
3. **Normalization:** $\int_\mathcal{E} W_A(z)dz = \frac{1}{\dim Q(p)} \tr(A)$.

**Remark 4.10.** In the spirit of Remark 3.17 here one could replace $\mathcal{E}$ by $\mathcal{O}_{x,y}$ for any $x, y > 0$, using the diffeomorphism $\psi_{x,y}$ in (2.127), so that one could define symbol correspondences as linear injective maps $W^i : B(\mathcal{H}_p) \to C^\infty_c(\mathcal{O}_p)$, satisfying equivariance, reality and normalization, as in Definition 4.9.

**Notation 6.** Recalling the notations

$$a = (a,b), \quad p = (p,q) \iff \tilde{p} = (q,p),$$

from now on, we shall use the notation

$$m(a) = m(a,b) = \min\{a, b\} + 1.$$

and simplify the notation $m(p, \tilde{p}, a)$ for the multiplicity of $Q(a) = Q(a,b)$ in the Clebsch-Gordan series of $Q(p) \otimes Q(\tilde{p}) = Q(p,q) \otimes Q(q,p)$ by setting

$$m(p,a) = m(p, \tilde{p}, a).$$

Finally, we set

$$B(p,a) = \bigoplus_{\sigma=1}^{m(p,a)} Q(a;\sigma) \subset B(\mathcal{H}_p).$$

**Theorem 4.11.** A linear map $W : B(\mathcal{H}_p) \to C^\infty_c(\mathcal{E}) : A \mapsto W_A$ is a symbol correspondence if and only if, for each $Q(a;\sigma) \in B(\mathcal{H}_p)$, it maps

$$W : \sqrt{\dim Q(p)} e((a;\sigma); \nu, J) \mapsto \sum_{\gamma=1}^{m(a)} c^\sigma_{\gamma}(a) Z^{(a;\gamma)}_{\nu,\gamma},$$

where $e((a;\sigma); \nu, J) \equiv e_p,\tilde{p}, a; \nu, J$, cf. (2.39), and $c^\sigma_{\gamma}(a)$ is the $\gamma \times \sigma$ entry of a complex full rank matrix of order $m(a) \times m(p;a)$ denoted by $C(a)$, that is,

$$C(a) = [c^\sigma_{\gamma}(a)],$$

with $C(a)$ satisfying $\overline{C(a)} = C(\tilde{a})$ and $C(0,0) = (-1)^{|p|}$.

**Proof.** Since $W$ is injective and equivariant, the image of $Q(a;\sigma)$ is a representation isomorphic to $Q(a)$. For $\{f((a;\sigma); \nu, J)\}$ a GT basis of the image of $Q(a;\sigma)$,

$$W : \sqrt{\dim Q(p)} e((a;\sigma); \nu, J) \mapsto \alpha_{(a,\sigma)} f((a;\sigma); \nu, J), \quad \alpha_{(a,\sigma)} \neq 0.$$  

Because the multiplicity of $Q(a)$ in $C^\infty_c(\mathcal{E})$ is $m(a)$, cf. (4.22) we must have

$$f((a;\sigma); \nu, J) = \sum_{\gamma=1}^{m(a)} \beta^{(a;\gamma)}_{\nu,\gamma} Z^{(a;\gamma)}_{\nu,\gamma},$$

where $Z^{(a;\gamma)}_{\nu,\gamma}$ are the $\mathcal{E}$ harmonics, cf. Definition 4.3. Let

$$c^\sigma_{\gamma}(a) = \alpha_{(a,\sigma)} \beta^{(a;\sigma)}_{\gamma}.$$
The injection hypothesis implies that the union of basis \( \bigcup_{\sigma=1}^{m(p,a)} \{ f((a; \sigma); \nu, J) \} \) is a linearly independent set, hence \( \{ (\beta^{\sigma(a; \sigma)}, ..., \beta^{\sigma(a; \sigma)}) : \sigma = 1, ..., m(p,a) \} \) is a linearly independent set in \( \mathbb{C}^{m(a)} \), cf. (4.27). This means that \( \{ (c^{\sigma(a)}(a), ..., c^{\sigma(a)}(a)) : \sigma = 1, ..., m(p,a) \} \) is a linearly independent set too, cf. (4.28), so the complex matrix \( \mathcal{E}(a) \) whose \( \gamma \times \sigma \) entry is \( c^{\gamma}(a) \) is of full rank.

We have that \( e^\dagger((a; \sigma); \nu, J) = (-1)^{2(t+u)} e((\tilde{a}; \sigma); \tilde{\nu}, J) \), cf. (2.84) and (4.14), hence \( Z^{\langle a, \gamma \rangle}_{\nu, J} = (-1)^{2(t+u)} Z^{\langle \tilde{a}, \gamma \rangle}_{\tilde{\nu}, J} \), cf. (4.15), so the reality condition implies

\[
(4.29) \quad c^{\gamma}(a) = c^{\gamma}(\tilde{a}),
\]

or in a concise form, the matrices \( \mathcal{E}(a) \) satisfy \( \overline{\mathcal{E}(a)} = \mathcal{E}(\tilde{a}) \).

The normalization property implies

\[
(4.30) \quad W : (-1)^{|p|} \sqrt{\dim Q(p, q)} e((0, 0); (0, 0, 0), 0) \mapsto Z^{(0,0)}_{(0,0),0},
\]

cf. (2.81) and (1.14), hence \( \mathcal{E}(0, 0) = (-1)^{|p|} \).

It is more straightforward to prove the converse, that is, to check that a map as described by Theorem 4.11 satisfies the properties for a symbol correspondences expressed in Definition 4.9 so we leave this to the reader. \( \square \)

**Corollary 4.12.** The moduli space \( \mathcal{S}_p \) of correspondences for a generic quark system \( \mathcal{H}_p \) can be described as

\[
(4.31) \quad \mathcal{S}_p = \left( \prod_{a=0}^{|p|} V_{m(p,a,a)}(\mathbb{R}^{a+1}) \right) \times \left( \prod_{a<b} V_{m(p,a,b)}(\mathbb{C}^{a+1}) \right),
\]

where \( V_k(\mathbb{K}^n) = GL_n(\mathbb{K})/GL_n(\mathbb{K}) \), for \( GL_n(\mathbb{K}) \subset GL_n(\mathbb{K}) \) a maximal subgroup that fixes a \( k \)-dimensional subspace, is a non compact Stiefel manifold.

In particular, for a mesonic quark system \( \mathcal{H}_{p,p} \),

\[
(4.32) \quad \mathcal{S}_{p,p} = \left( \prod_{a=0}^p V_{a+1}(\mathbb{R}^{a+1}) \right) \times \left( \prod_{a=p+1}^{2p} V_{2p-a+1}(\mathbb{R}^{a+1}) \right)
\]
\[
\times \left( \prod_{a=0}^{p+1} V_a(\mathbb{C}^{a+1}) \right) \times \left( \prod_{k=1}^{p-1} V_{2p-a-2k+1}(\mathbb{C}^{a+1}) \right).
\]

**Proof.** The description of a generic \( \mathcal{S}_p \) in (4.31) follows directly from the characterization of symbol correspondences in Theorem 4.11. In the mesonic case, from Corollary 2.7, the multiplicity of a representation \( Q(a, a + 3k) \) or \( Q(a + 3k, a) \) in the CG series of \( Q(p, p) \otimes Q(p, p) \) is given by the number of solutions \((n, m)\) of

\[
(4.33) \quad n + m = 2p - a - 2k
\]
for $0 \leq n, m \leq p - k$. We have:

$$a + k \leq p \implies \begin{cases} n = p - a - k, \ m = p - k; \\
\vdots \\
n = p - k, \ m = p - a - k. \end{cases}$$

(4.34)

$$a + k > p \implies \begin{cases} n = 0, \ m = 2p - a - 2k; \\
\vdots \\
n = 2p - a - 2k, \ m = 0. \end{cases}$$

(4.37)

The bounds of the products in (4.32) follow from $a = 2p - n - m - 2k$. □

The matrices $\mathcal{E}(a)$ are matrix representations of the map $W$ restricted to a weight space of $\mathcal{B}(p; a)$ with respect to a coupled basis of $\mathcal{B}(\mathcal{H}_p)$ and the $\mathcal{E}$ harmonics. They are analogous to characteristic numbers of symbol correspondences for pure-quark system: in the latter case, the domain and codomain of a symbol correspondence are multiplicity free and have only representations $Q(n, n)$, so it provides a $1 \times 1$ real matrix indexed by $n$. The moduli space in that case is a product of $V_1(\mathbb{R}) = \mathbb{R}^*$. We now prove a theorem analogous to Theorem 3.13. As usual, let $z_0 \in \pi^{-1}(x_0) = \pi^{-1}([1 : 0 : 0]) \subset \mathcal{E}$ be a point with $T$ as isotropy subgroup. Now, for an operator $K \in \mathcal{B}(\mathcal{H}_p)$ fixed by $T$, we have $\mathcal{E} \to \mathcal{B}(\mathcal{H}_p) : z \mapsto K(z) = K^g$, where $g \in SU(3)$ is such that $z = gz_0$.

**Theorem 4.13.** A map $W : \mathcal{B}(\mathcal{H}_p) \to \mathcal{C}^\infty(\mathcal{E}) : A \mapsto W_A$ is a symbol correspondence satisfying (4.24) if and only if

$$W_A(z) = \text{tr}(AK(z)) \iff W_A(gz_0) = \text{tr}(AK^g),$$

for $K \in \mathcal{B}(\mathcal{H}_p)$ of the form

$$K = \frac{1}{\dim Q(p)}1 + \sum_{(a; \sigma)} \sum_{\gamma=1}^{m(a)} c_{\gamma}(a) \sqrt{\frac{\dim Q(a)}{\dim Q(p)}} e((a; \sigma); \nu_\gamma, J),$$

with $c_{\gamma}(a) = [\mathcal{E}(a)]_{\gamma}^\sigma$ as in Theorem 4.13, where the summation is over all $(a; \sigma)$ in the CG series of $Q(p) \otimes Q(p)$. In particular, $K$ is Hermitian with unitary trace.

**Proof.** Assuming $W$ is a symbol correspondence, we can reproduce the proof of Theorem 3.13 to conclude $W_A(gz_0) = \text{tr}(AK^g)$, where $K$ is a linear combination of the vectors fixed by $T$, so

$$K = \sum_{(a; \sigma)} \sum_{\gamma} k_{\gamma}(a; \sigma)e((a; \sigma); \nu_\gamma, J),$$

cf. Proposition 4.2. For $A = e((a; \sigma); \nu, J) = (-1)^{2(t + u)}e^1((a; \sigma); \nu, J)$ we get

$$W_A(gz_0) = \text{tr}(AK^g) = \sum_{\gamma=1}^{m(a)} k_{\gamma}(a; \sigma)(-1)^{2(t + u)}D_{\nu; J}(g) = \sum_{\gamma=1}^{m(a)} k_{\gamma}(a; \sigma)D_{\nu; J}(g),$$

cf (2.32). Then, from the above and Theorem 4.11 we have

$$W_A = \sum_{\gamma=1}^{m(a)} k_{\gamma}(a; \sigma)Z_{\nu, J}(a, \gamma),$$

(4.40)
ON SYMBOL CORRESPONDENCES FOR QUARK SYSTEMS

\[ k_\gamma^{(\tilde{a},\sigma)} = c_\gamma^\sigma(a) \sqrt{\frac{\dim Q(a)}{\dim Q(p)}} \iff k_\gamma^{(a,\sigma)} = c_\gamma^\sigma(\tilde{a}) \sqrt{\frac{\dim Q(\tilde{a})}{\dim Q(p)}}. \]

Using (4.20) and \( \dim Q(a) = \dim Q(\tilde{a}) \), we obtain (4.37).

Hermitean property of \( K \) follows from (4.20) and (4.83) plus the fact that, if \( Q(a;\sigma) \) is in the CG series of \( Q(p) \otimes Q(\tilde{p}) \), then so is \( Q(\tilde{a};\sigma) \). Unitary trace for \( K \) is immediate from every \( e((a;\sigma);\nu_a,J_\gamma) \) being traceless (orthogonal to 1).

The converse is, again, analogous to Theorem 3.13, and it is rather straightforward to verify that, for \( K \) given by (4.37), equations (4.38)-(4.40) imply that (4.36) defines a symbol correspondence given by (4.24).

**Definition 4.14.** Any \( K \in \mathcal{B}(\mathcal{H}_p) \) that induces a symbol correspondence via (4.36)-(4.39) is an operator kernel. Thus, \( K \) is given by (4.37), where the numbers \( (c_\gamma^\sigma(a)) \) are called characteristic parameters and the matrices \( \mathcal{C}(a) \) with \( c_\gamma^\sigma(a) \) in the \( \gamma \times \sigma \) entry, cf. (4.27), are the characteristic matrices of both the operator kernel and the symbol correspondence.

**Remark 4.15.** Note that an operator kernel \( K \) of a correspondence for a pure-quark system \( \mathcal{H}_{p,0} \) (or, equivalently, \( \mathcal{H}_{0,p} \)) can also be seen as an operator kernel of a correspondence for \( \mathcal{H}_{p,0} \) taken as a generic quark system. If \( K \) has characteristic numbers \( (c_\gamma) \), it has characteristic parameters \( (c_\gamma^\sigma(n,n)) \) given by \( c_\gamma^\sigma(n,n) = c_\gamma \delta_{\gamma,1} \).

**Proposition 4.16.** Let \( K \in \mathcal{B}(\mathcal{H}_p) \) be an operator kernel with characteristic matrices \( \mathcal{C}(a) \). A symbol correspondence \( \tilde{W} : \mathcal{B}(\mathcal{H}_p) \to C^\infty(\mathcal{E}) \) satisfies

\[ A = \dim Q(p) \int_\mathcal{E} \tilde{W}_a(z) K(z) dz \]

if and only if it has characteristic matrices \( \tilde{\mathcal{C}}(a) \) such that \( (\tilde{\mathcal{C}}(a))^\dagger \tilde{\mathcal{C}}(a) = 1 \).

**Proof.** By straightforward calculation, we get

\[
\int_\mathcal{E} Z^{(a',\gamma)}_{\nu,J}(z) K(z) dz = \sum_{(a',\gamma')} k_{\gamma\gamma'}^{(a',\sigma')} \int_{SU(3)} Z^{(a',\gamma')}_{\nu,J}(g z_0) D_{\mu L,\nu_a,J_\gamma}(g) d e((a';\sigma'); \mu, L)
\]

\[
= \sum_{(a',\gamma')} \frac{k_{\gamma\gamma'}^{(a',\sigma')}}{\sqrt{\dim Q(a)}} \left< Z_{\mu L}^{(a',\gamma')} | Z_{\nu,J}^{(a',\gamma')} \right> e((a';\sigma'); \mu, L) = \sum_{\sigma'=1}^{m(p,a)} c_{\gamma\gamma'}^{(a,\sigma')} \left< Z_{\nu,J}^{(a,\sigma')} \right> e((a;\sigma'); \nu, J).
\]

where \( k_{\gamma\gamma'}^{(a',\sigma')} \) is given by (4.20) and (4.11). So, for \( c_\gamma^\sigma(a) \) and \( \tilde{c}_\gamma^\sigma(a) \) being the characteristic parameters of \( \mathcal{C}(a) \) and \( \tilde{\mathcal{C}}(a) \), respectively, we have, cf. (4.24),

\[
\dim Q(p) \int_\mathcal{E} Z \tilde{\mathcal{C}}(a)_{\nu,J}(z) K(z) dz = \sqrt{\dim Q(p)} \sum_{\gamma,\sigma'} \tilde{c}_\gamma^\sigma(a) c_{\gamma\gamma'}^{(a')} \left< Z_{\nu,J}^{(a')} \right> e((a;\sigma'); \nu, J).
\]

Hence, (4.42) holds for \( A = e((a;\sigma); \nu, J) \) if and only if

\[ \sum_{\gamma=1}^{m(a)} \tilde{c}_\gamma^\sigma(a) c_{\gamma\gamma'}^{(a)} = \delta_{\sigma,\sigma'}, \]

which means \( (\tilde{\mathcal{C}}(a))^\dagger \tilde{\mathcal{C}}(a) = 1 \), or equivalently \( (\tilde{\mathcal{C}}(a))^\dagger \mathcal{C}(a) = 1 \). \( \square \)
Now, let

\[(4.44) \quad \langle A|R \rangle_p = \frac{1}{\dim Q(p)} \langle A|R \rangle = \frac{1}{\dim Q(p)} \tr(A^\dagger R)\]

be the normalized inner product in \(B(H_p)\) and let \(\|\cdot\|_p\) be its induced norm.

**Definition 4.17.** Two symbol correspondences \(W, \widetilde{W} : B(H_p) \rightarrow C^\infty(\mathcal{E})\) satisfying

\[(4.45) \quad \langle A|R \rangle_p = \left\langle \widetilde{W}_A \big| W_R \right\rangle = \left\langle W_A \big| \widetilde{W}_R \right\rangle\]

for every \(A, R \in B(H_p)\) are said to be dual correspondences. In this case, the operator kernel of \(\widetilde{W}\) is also said to be dual to the operator kernel of \(W\).

**Proposition 4.18.** Two symbol correspondences \(W, \widetilde{W} : B(H_p) \rightarrow C^\infty(\mathcal{E})\) with characteristic matrices \(C(a)\) and \(\widetilde{C}(a)\) are dual to each other if and only if

\[(4.46) \quad \widetilde{C}(a)^\dagger C(a) = 1.\]

**Proof.** The proof follows analogous to the proof of Proposition 3.19 by writing the operators using \(4.42\) and symbols using \(1.36\). \(\square\)

**Remark 4.19.** For symbol correspondences of generic quark systems, duality is no longer \(1 \leftrightarrow 1\) because characteristic matrices may have more than one left inverse. Consider, for instance, the correspondences \(W, \widetilde{W} : B(H_p) \rightarrow C^\infty(\mathcal{E})\) defined respectively by the characteristic parameters \(c_1(a) = \delta_{\gamma,\sigma}\) and \(\tilde{c}_1(a) = \delta_{\gamma,\sigma} + \delta_{\gamma,m(a)}\delta_{\sigma,1}\). Then, \(\widetilde{W}\) and \(W\) itself are both dual to \(W\).

The correspondence \(W\) of the previous remark is obviously an isometry. In addition to such special cases of correspondences which are isometric, now we also have correspondences given by a direct sum of conformal maps.

**Definition 4.20.** A symbol correspondence \(W : B(H_p) \rightarrow C^\infty(\mathcal{E})\) is a Stratonovich-Weyl correspondence if it is an isometry, that is,

\[(4.47) \quad \langle A|R \rangle_p = \langle W_A|W_R \rangle\]

for all \(A, R \in B(H_p)\). If \(W\) preserves angles for each \(B(p; a)\), that is,

\[(4.48) \quad \frac{\|A\|_p \|R\|_p}{\langle A|R \rangle_p} = \frac{\langle W_A|W_R \rangle}{\|W_A\| \|W_R\|}\]

for all non null \(A, R \in B(p; a)\) and every \(B(p; a) \subset B(H_p)\), cf. \((4.23)\), then \(W\) shall be called a semi-conformal correspondence.

**Proposition 4.21.** A correspondence \(W : B(H_p) \rightarrow C^\infty(\mathcal{E})\) is a Stratonovich-Weyl correspondence if and only if its characteristic matrices are semi-unitary matrices, that is, they satisfy \((C(a))^\dagger C(a) = 1\). Furthermore, \(W\) is a semi-conformal correspondence if and only if its characteristic matrices are semi-conformal matrices, that is, \((C(a))^\dagger C(a) = \alpha(a) 1\) for \(\alpha(a) > 0\), where \(\alpha(a) = \alpha(\tilde{a})\) and \(\alpha(0,0) = 1\).

**Proof.** From Proposition 4.18, \(W\) is its own dual if and only if \((C(a))^\dagger C(a) = 1\) holds, which proves the first part of the statement. For the second part, we use that a linear map is conformal iff it is a positive real multiple of an unitary map, thus \(W\) is a semi-conformal correspondence if and only if there is \(\alpha(a) > 0\) for each \(B(p; a)\) such that \(\alpha(a)^{-1/2} W_{|B(p; a)}\) is an unitary map, and this is true if and only
if the characteristic matrices of $W$ satisfy $(\mathcal{E}(a))^\dagger \mathcal{E}(a) = \alpha(a) \mathbf{1}$. The equations $\alpha(0,0) = 1$ and $\alpha(a) = \alpha(\tilde{a})$ follows from $\mathcal{E}(a) = \mathcal{E}(\tilde{a})$ and $\mathcal{E}(0,0) = (-1)^{|p|}$. □

**Remark 4.22.** A symbol correspondence $W$ is an actual conformal map if and only if $W = \sqrt{\alpha} W'$ for $\alpha > 0$ and some Stratonovich-Weyl correspondence $W'$. Since $W_1 = W_1'$, we must have $\alpha = 1$, so the only actual conformal correspondences are the isometric ones. For pure-quark systems (likewise for spin systems), every symbol correspondence is semi-conformal, with $\alpha(a) = \alpha(n,n) = c_n^2$.

Propositions 4.16, 4.21 illustrate how characteristic matrices encode all the information about symbol correspondences for generic quark systems in the same vein of characteristic numbers for pure-quark system. The existence of multiple correspondences in duality can be explained by the existence of invariant subspaces $Q(a)$ with higher degeneracy within $C_\infty^\infty(\mathcal{E})$ than within $\mathcal{B}(\mathcal{H}_p)$, or equivalently by the existence of multiple left inverses of characteristic matrices.

In general, there is no natural way to choose a unique dual correspondence. For semi-conformal symbol correspondences, however, we make the following definition:

**Definition 4.23.** Let $W : \mathcal{B}(\mathcal{H}_p) \to C_\infty^\infty(\mathcal{E})$ be a semi-conformal correspondence with characteristic matrices $\mathcal{E}(a)$ satisfying $(\mathcal{E}(a))^\dagger \mathcal{E}(a) = \alpha(a) \mathbf{1}$. Its canonical dual correspondence $\tilde{W} : \mathcal{B}(\mathcal{H}_p) \to C_\infty^\infty(\mathcal{E})$ is the one with characteristic matrices

\[
\tilde{\mathcal{E}}(a) = \frac{1}{\alpha(a)} \mathcal{E}(a) .
\]

Thus, Stratonovich-Weyl correspondences for generic quark systems are their own canonical dual correspondences.

Just as for pure-quark systems, a positive operator kernel provides a special type of symbol correspondence for generic quark systems.

**Definition 4.24.** A symbol correspondence $W$ for a generic quark system is mapping-positive if it maps positive-(definite) operators to (strictly-)positive functions. If $\tilde{W}$ is dual to a mapping-positive correspondence, then $\tilde{W}$ is a positive-dual correspondence.

**Proposition 4.25.** A symbol correspondence $W : \mathcal{B}(\mathcal{H}_p) \to C_\infty^\infty(\mathcal{E})$ with operator kernel $K$ is mapping-positive if and only if $K$ is also a state, that is, $K$ is also a positive operator.

Proof. Suppose $K$ is a positive operator, so $K = R^\dagger R$ for some $R \in \mathcal{B}(\mathcal{H}_p)$, and let $A = M^\dagger M \in \mathcal{B}(\mathcal{H}_p)$ be a positive operator. Then, for any $g \in SU(3)$,

\[
W_A(gz_0) = \text{tr}(M^\dagger M \rho(g) K \rho(g)^\dagger) = \text{tr}(M \rho(g) K \rho(g)^\dagger M^\dagger) = \text{tr}\left(\tilde{M} R^\dagger \tilde{R} M^\dagger\right) \geq 0 ,
\]

where $\tilde{M} = M \rho(g)$ and $\tilde{M} R^\dagger \tilde{R} M^\dagger$ is a positive operator. Since $K$ is non null, $R$ is also non null, so there exists $w_0 \in \mathcal{H}_p$ s.t. $||R(w_0)||^2 > 0$. If $A$ is positive-definite, $\tilde{M}^\dagger$ is an automorphism and we can set $w = (\tilde{M}^\dagger)^{-1}(w_0)$ so that $||w|| > 0$ and

\[
W_A(z) = \text{tr}\left(\tilde{M} R^\dagger \tilde{R} M^\dagger\right) \geq \frac{\langle w | \tilde{M} R^\dagger \tilde{R} M^\dagger | w \rangle}{||w||^2} = \frac{\langle R M^\dagger | R M^\dagger \rangle}{||w||^2} = \frac{||R M^\dagger||^2}{||w||^2} = \frac{||R(w_0)||^2}{||w||^2} > 0 .
\]
Now, suppose $K$ is not positive. Then, $K$ has a negative eigenvalue. Let $\Pi$ be the projection onto an eigenspace of $K$ associated to a negative eigenvalue. We have that $\text{tr}(\Pi K) < 0$. 

Clearly, projector $\Pi_{(p,0,0)} \in \mathcal{B}(\mathcal{H}_p)$, or $\Pi_{(0,p,p)} \in \mathcal{B}(\mathcal{H}_p)$, is an operator kernel of correspondence $\mathcal{B}(\mathcal{H}_p) \to C^\infty(\mathcal{E})$, or $\mathcal{B}(\mathcal{H}_p) \to C^\infty(\mathcal{E})$, respectively, with corresponding symbols on $CP^1 \to \mathcal{E} \to CP^2$ being constant extensions of functions on $CP^2$, cf. Remark 4.15. From Remark 4.27 the only impediment to $\Pi_{(0,0,p)} \in \mathcal{B}(\mathcal{H}_p)$, or $\Pi_{(p,0,0)} \in \mathcal{B}(\mathcal{H}_p)$, being an operator kernel for a pure-quark system is the lack of $H \simeq U(2)$ invariance. But they are invariant by the torus $T$, so each of these projectors is also an operator kernel for a generic quark system, though the corresponding symbols will no longer be constant along the fibers of $\mathcal{E}$, in general.

However, in greater generality, this is true for every projector onto the highest or the lowest weight space of any irreducible representation $Q(p,q)$.

**Theorem 4.26.** For any $p \in (\mathbb{N} \times \mathbb{N}_0) \cup (\mathbb{N}_0 \times \mathbb{N})$, both the projector onto the highest weight space and onto the lowest weight space of $Q(p)$, $\Pi_\succ \in \mathcal{B}(\mathcal{H}_p)$ and $\Pi_\prec \in \mathcal{B}(\mathcal{H}_p)$, are operator kernels in the sense of Definition 4.27 and Theorem 4.13.

In Appendix I, we present a detailed proof of the above theorem, by specializing to $SU(3)$ the main argument in [11, 30] for general compact semisimple Lie groups.

**Definition 4.27.** A mapping-positive correspondence whose operator kernel is $\Pi_\succ$, shall be called the highest Berezin correspondence for a generic quark system. Likewise for the lowest Berezin correspondence in the case of $\Pi_\prec$.

For the case of pure-quark systems, the highest Berezin correspondence for $Q(p,0)$ and the lowest Berezin correspondence for $Q(0,p)$, both invariant under the full group $H \simeq U(2)$, are called symmetric Berezin correspondences and they have explicit constructions, cf. Propositions 3.25 and 3.26. For the case of generic quark systems, we do not know of such an explicit construction of the highest or lowest Berezin correspondence for $Q(p,q)$ in general, but we have the following:

**Corollary 4.28.** For any $p = (p,q) \in (\mathbb{N} \times \mathbb{N}_0) \cup (\mathbb{N}_0 \times \mathbb{N})$, the characteristic parameters of the highest and the lowest Berezin correspondences are, respectively,

\[
\begin{align*}
(b_\succ)_\gamma^\tau (\alpha) &= (-1)^{|\alpha|} \sqrt{\frac{\dim Q(p)}{\dim Q(\alpha)}} C_{p}^{p_{\alpha}} \tilde{p}_{\gamma}^{(\alpha, \tau)} \in \mathcal{B}(\mathcal{H}_p) \\
(b_\prec)_\gamma^\tau (\alpha) &= (-1)^{|\alpha|} \sqrt{\frac{\dim Q(p)}{\dim Q(\alpha)}} C_{p}^{p_{\alpha}} \tilde{p}_{\gamma}^{(\alpha, \tau)} \in \mathcal{B}(\mathcal{H}_p) 
\end{align*}
\]

**Proof.** The characteristic parameters of $\Pi_\succ$ and $\Pi_\prec$ can be obtained just as in Propositions 3.25 and 3.26 let $(\nu_\succ, J_\succ) = ((p + q, q, 0), q/2)$ and $(\nu_\prec, J_\prec) = ((0, q, p + q), p/2)$, so

\[
\Pi_\succ = (-1)^{|\alpha|} e(p; \nu_\succ, J_\succ) \otimes \tilde{e}(\tilde{p}; \nu_\succ, J_\succ) \\
\Pi_\prec = (-1)^{|\alpha|} \sum_{(\alpha, \sigma)} \sum_{\gamma=1}^{m(\alpha)} C_{\nu_\prec, J_\prec}^{p_{\alpha}} \tilde{p}_{\gamma}^{(\alpha, \sigma)} e((\alpha, \sigma); \nu_\prec, J_\prec) ,
\]

\[
\text{Proof.}
\]
\[ 
\Pi_\gamma = (-1)^{|p|} e(p; \nu <, J <) \otimes \tilde{e}(\tilde{p}; \tilde{\nu} <, J <) 
\]

(4.53)

\[ 
= (-1)^{|p|} \sum_{m(\sigma)} \sum_{(\alpha; \sigma)} \sum_{\gamma=1} C^{p, \tilde{p}, \nu <, J <} \tilde{C}^{\nu <, J <} e((\alpha; \sigma); \nu_a, J_\gamma) . 
\]

From (4.37), since the CG coefficients are real, we get (4.50)-(4.51).

Remark 4.29. For pure-quark systems, Proposition 3.28 asserts there is only one projector that defines a (symmetric) symbol correspondence and for generic quark systems Theorem 4.20 asserts that the highest and lowest projectors define symbol correspondences for every \(Q(p, q)\). However, given a representation of class \(Q(p, q)\), we don’t know which other projectors can define symbol correspondences. Also, we still don’t have explicit examples of mapping-positive correspondences for every \(Q(p, q)\), other than the highest and the lowest Berezin correspondences.

Following as in Definition 5.31 one could expect to define highest and lowest Stratonovich-Weyl correspondences via continuous deformations from the highest and lowest Berezin correspondences. There are, however, infinite Stratonovich-Weyl correspondences connected via continuous deformation from either one, so this can not be done unambiguously. To see this, consider a Berezin correspondence with characteristic matrices \(C(a)\). By continuous application of Gram-Schmidt process on the columns of \(C(a)\), we obtain a semi-unitary matrix. Then, if we apply any rotation to the columns of this semi-unitary matrix, it remains semi-unitary.

Remark 4.30. For a symbol correspondence \(W: B(H_p) \rightarrow C^\infty_c(E)\), its antipodal correspondence \(\tilde{W}: B(H_{\tilde{p}}) \rightarrow C^\infty_c(E)\) is given by (cf. (2.83)-(2.84))

(4.54)

\[ 
\tilde{W}_A^\alpha = W_A . 
\]

Remark 4.31. For generic quark systems, given a symbol correspondence \(W': B(H_p) \rightarrow C^\infty_c(O_p)\), defined in accordance to Remarks 3.10 and 4.10, for \(O_p = O_{p,q} \equiv O_{x,y}\) the (co)adjoint orbit as in section 2.2 its antipodal correspondence \(\tilde{W}': B(H_{\tilde{p}}) \rightarrow C^\infty_c(O_{\tilde{p}})\) is related to \(W'\) by

(4.55)

\[ 
\tilde{W}'_A^\alpha = W'_A^\alpha \circ \iota \circ \alpha_{p} ,
\]

cf. (2.130). In particular, for mesonic systems, \(O_p = O_{\tilde{p}} = O_{p,p}\) and \(\alpha_{p,p} = \iota\), so

(4.56)

\[ 
\tilde{W}'_A^\alpha = W_A^\alpha .
\]

Proposition 4.32. The symbol correspondences \(\tilde{W}: B(H_p) \rightarrow C^\infty_c(E)\) with characteristic parameters \((c^\tilde{p}_\gamma(a))\) is antipodal to the symbol correspondence \(W: B(H_p) \rightarrow C^\infty_c(E)\) with characteristic parameters \((c^p_\gamma(a))\) if and only if

(4.57)

\[ 
c^p_\gamma(a) = (-1)^{|a|} \tilde{c}^\tilde{p}_\gamma(a) .
\]
Proof. The result follows from (2.89) and Theorem 4.11. □

We now state another important distinction between correspondences for generic quark and pure-quark systems.

**Proposition 4.33.** For any representation of class \(Q(p)\), there may exist symbol correspondences \(W_1, W_2 : \mathcal{B}(\mathcal{H}_p) \to C^\infty_C(\mathcal{E})\) with different image sets, that is, such that \(W_1(\mathcal{B}(\mathcal{H}_p)) \neq W_2(\mathcal{B}(\mathcal{H}_p))\).

Proof. Consider \(W_1 \neq W_2 : \mathcal{B}(\mathcal{H}_p) \to C^\infty(\mathcal{E})\) determined by characteristic matrices \(C[1](a)\) and \(C[2](a)\) with respective characteristic parameters \(c[1]_\gamma(a)\) and \(c[2]_\gamma(a)\).

Since \(m(p, |p|, |p|) = 1\), for \(Q(a) = Q(|p|, |p|)\), we can drop the index \(\sigma\) for the characteristic parameters \(c[1]_\gamma(|p|, |p|)\) and \(c[2]_\gamma(|p|, |p|)\). Then, for \(C[1](|p|, |p|)\) and \(C[2](|p|, |p|)\) such that

\[
C[1](|p|, |p|) \cdot C[2](|p|, |p|) = \sum_{\gamma=1}^{|p|+1} c[1]_\gamma(|p|, |p|)c[2]_\gamma(|p|, |p|) = 0 ,
\]
we have that the following two subspaces of \(C^\infty(\mathcal{E})\),

\[
W_1(Q(|p|, |p|)) = \text{span}\left\{ ZC[1](|p|, |p|, \nu, J) \right\} ,
\]

\[
W_2(Q(|p|, |p|)) = \text{span}\left\{ ZC[2](|p|, |p|, \nu, J) \right\} ,
\]
are orthogonal to each other\(^{24}\), hence \(W_1(\mathcal{B}(\mathcal{H}_p)) \neq W_2(\mathcal{B}(\mathcal{H}_p))\). □

**Notation 7.** For any symbol correspondence \(W : \mathcal{B}(\mathcal{H}_p) \to C^\infty(\mathcal{E})\), we shall denote by \(S_p(W)\) the image of \(W\) in \(C^\infty_C(\mathcal{E})\), that is,

\[
(4.58) \quad S_p(W) = W(\mathcal{B}(\mathcal{H}_p)) \subset C^\infty_C(\mathcal{E}) .
\]

Recalling (2.79), we have from (4.57) that the characteristic matrices of a symbol correspondence and of its antipodal differ just by a constant \((-1)^{|a|}\) factor and the reverse ordering of columns. We then have the following:

**Corollary 4.34.** For generic quark systems, a symbol correspondence and its antipodal have the same image in \(C^\infty_C(\mathcal{E})\).

**Proposition 4.35.** For any \(p = (p, q) \in (\mathbb{N} \times \mathbb{N}) \cup (\mathbb{N}_0 \times \mathbb{N})\), the Berezin correspondence with operator kernel \(\Pi_\succ \in \mathcal{B}(\mathcal{H}_p)\) is antipodal to the Berezin correspondence with operator kernel \(\Pi_\prec \in \mathcal{B}(\mathcal{H}_\bar{p})\).

Proof. The result follows straightforwardly from Propositions 4.26 and 4.32 and the symmetry relations (2.82). □

**Proposition 4.36.** If \(W, \tilde{W} : \mathcal{B}(\mathcal{H}_p) \to C^\infty_C(\mathcal{E})\) are symbol correspondences dual to each other, then their respective antipodal correspondences are dual to each other.

Proof. The result follows from

\[
(4.59) \quad \sum_{\gamma=1}^{m(a)} \left( (-1)^{|a|} \frac{c[\gamma](a)}{\overline{c[\gamma](a)}} \right) \left( (-1)^{|a|} \frac{c'[\gamma](a)}{\overline{c'[\gamma](a)}} \right) = \sum_{\gamma=1}^{m(a)} \frac{c[\gamma](a)}{\overline{c[\gamma](a)}} \frac{c'[\gamma](a)}{\overline{c'[\gamma](a)}} = \delta_{\sigma, \sigma'} ,
\]
for \((c[\gamma](a))\) and \((\overline{c[\gamma](a)})\) characteristic parameters of \(W\) and \(\tilde{W}\), respectively. □

\(^{24}\)We recall that the constant function 1 on \(\mathcal{E}\) is in the image of \(Q(0, 0)\), thus \(1 \notin W(Q(|p|, |p|))\), for any \(p \in (\mathbb{N} \times \mathbb{N}) \cup (\mathbb{N}_0 \times \mathbb{N})\) and for any correspondence \(W\).
Corollary 4.37. A symbol correspondence for a generic quark system is a semi-conformal (resp. Stratonovich-Weyl) correspondence if and only if its antipodal correspondence is also a semi-conformal (resp. Stratonovich-Weyl) correspondence.

4.4. Twisted products for generic quark systems. Let \( p \in \mathbb{N} \times \mathbb{N} \) still. Recalling Proposition 4.33 and Notation 4.10 we have

**Definition 4.38.** For a symbol correspondence \( W : \mathcal{B}(\mathcal{H}_p) \to C_c^\infty(\mathcal{E}) \), the twisted product of symbols induced by \( W \) is the binary operation \( * \) on \( \mathcal{S}_p(W) \) given by

\[
W_A \ast W_R = W_{AR}
\]

for any \( A, R \in \mathcal{B}(\mathcal{H}_p) \). The algebra \( (\mathcal{S}_p(W), *) \) is called a twisted \( p \)-algebra.

**Proposition 4.39.** Any twisted \( p \)-algebra \( (\mathcal{S}_p(W), *) \) is

(i) \( SU(3) \)-equivariant: \( (f_1 \ast f_2)^q = f_1^q \ast f_2^q \); (ii) Associative: \( (f_1 \ast f_2) \ast f_3 = f_1 \ast (f_2 \ast f_3) \); (iii) Unital: \( 1 \ast f = f \ast 1 = f \); (iv) A \( * \)-algebra: \( \mathcal{S}_p(W) \) is the constant function equal to 1 on \( \mathcal{E} \).

Any two twisted \( p \)-algebras are naturally isomorphic, and any twisted \( p \)-algebra is naturally anti-isomorphic to any twisted \( \bar{p} \)-algebra.

**Proof.** The first part follows as in the proof of Proposition 3.38. For the second part, although we may have correspondences \( W_1, W_2 : \mathcal{B}(\mathcal{H}_p) \to C_c^\infty(\mathcal{E}) \) with different images, we still have that each \( W_k \) is an isomorphism onto its image, so \( W_1 \circ W_2^{-1} : \mathcal{S}_p(W_2) \to \mathcal{S}_p(W_1) \) is an isomorphism. Finally, if \( W_2 : \mathcal{B}(\mathcal{H}_p) \to C_c^\infty(\mathcal{E}) \), then \( W_1 \circ W_2^{-1} : \mathcal{S}_p(W_2) \to \mathcal{S}_p(W_1) \) is an anti-isomorphism because the adjoint map \( * \) is an anti-isomorphism and each \( W_k \) is an isomorphism onto its image. \( \square \)

From Corollary 2.31 and Theorem 4.11 we obtain

**Theorem 4.40.** If \( W : \mathcal{B}(\mathcal{H}_p) \to C^\infty(\mathcal{E}) \) is a symbol correspondence with characteristic matrices \( \mathcal{C}(\alpha) \), then the induced twisted product is given by

\[
(\mathcal{Z}_1 \mathcal{C}(\alpha_1))_{\nu_1, J_1} \ast (\mathcal{Z}_2 \mathcal{C}(\alpha_2))_{\nu_2, J_2} = \sqrt{\dim Q(p)} \sum_{(\sigma; \nu)} (-1)^{|\nu|} \left[ (a_1; \sigma_1)_{\nu_1, J_1} (a_2; \sigma_2)_{\nu_2, J_2} (a; \sigma)_{\nu, J} \right] [p] \mathcal{Z}(\alpha)^{\sigma}_{\nu, J} ,
\]

with summations over \( \nu \) and \( J \) effectively restricted by 2.111.

**Theorem 4.41.** If \( W : \mathcal{B}(\mathcal{H}_p) \to C^\infty(\mathcal{E}) \) is a correspondence with operator kernel \( K \) and characteristic matrices \( \mathcal{C}(\alpha) \), then the induced twisted product is given by

\[
f_1 \ast f_2(z) = \int_{\mathcal{E} \times \mathcal{E}} f_1(z_1) f_2(z_2) \mathcal{L}(z_1, z_2, z) \, dz_1 \, dz_2
\]

for any \( f_1, f_2 \in \mathcal{S}_p(W) \), where

\[
\mathcal{L}(z_1, z_2, z_3) = (\dim Q(p))^2 \text{tr} \left( \tilde{K}(z_1) \tilde{K}(z_2) K(z_3) \right)
\]

\[
= (-1)^{|\nu|} \sqrt{\dim Q(p)} \sum_{(\sigma; \nu_k)} \left[ (a_1; \sigma_1)_{\nu_1, J_1} (a_2; \sigma_2)_{\nu_2, J_2} (a_3; \sigma_3)_{\nu_3, J_3} \right] [p] \times Z \mathcal{C}(\alpha_1)^{\sigma_1}_{\nu_1, J_1}(z_1) Z \mathcal{C}(\alpha_2)^{\sigma_2}_{\nu_2, J_2}(z_2) Z \mathcal{C}(\alpha_3)^{\sigma_3}_{\nu_3, J_3}(z_3)
\]

for \( \mathcal{C}(\alpha) \) being the characteristic matrices of an operator kernel \( \tilde{K} \) dual to \( K \).
Proof. The proof follows analogously to Theorem 3.41 but now the second equality comes from Theorem 4.40. We emphasize that, although the expression (4.63) for integral trikernels depends explicitly on the choice of dual representation, the twisted product given by (4.62) does not have such dependence. By definition,

\[
\int_{\mathcal{E}} (Z\mathcal{E}(e_{\nu,j}^{a})(z)Z\mathcal{E}(e_{\nu',j'}^{a'})(z)) \, dz = \left( \int Z\mathcal{E}(e_{\nu,j}^{a})(z)Z\mathcal{E}(e_{\nu',j'}^{a'})(z) \, dz \right)
\]

\[\begin{align*}
= \dim \langle \phi((a; \sigma); \nu, J)|e((a; \sigma'); \nu', J') \rangle_p &= \langle \phi((a; \sigma); \nu, J)|e((a; \sigma'); \nu', J') \rangle_p = \langle \phi((a; \sigma); \nu, J)|e((a; \sigma'); \nu', J') \rangle_p ,
\end{align*}\]

no matter which dual correspondence is used.

\[\square\]

Definition 4.42. An integral trikernel \( \mathbb{L} \in C^\infty(\mathcal{E} \times \mathcal{E} \times \mathcal{E}) \) of a twisted product induced by a symbol correspondence \( W : \mathcal{B}(\mathcal{H}_p) \rightarrow C^\infty(\mathcal{E}) \) is a function of the form (4.63) so that the twisted product is given by (4.62). If \( W \) is a semi-conformal correspondence, the integral trikernel constructed using its canonical dual correspondence is the canonical integral trikernel.

Note that, in Theorems 4.40 and 4.41 we could not decompose twisted products in the harmonic basis, as done in Theorems 3.40 and 3.41 because in general \( \mathcal{S}_p(W) \) is not spanned by the generic harmonics \( Z_{\nu,j}^{(a, \gamma)} \), but by the linear combinations expressed in (4.24). Now, one may use (4.62) to expand a twisted product on \( \mathcal{S}_p(W) \) induced by a symbol correspondence \( W \) to a product \( \mathbb{\cdot} \) on all \( C^\infty(\mathcal{E}) \) in the same way we did for pure-quark systems in Proposition 3.43. But integral trikernels are not unique, so such expansions are not unique either. In addition, the product \( \mathbb{\cdot} \) in general fails to vanish for functions orthogonal to \( \mathcal{S}_p(W) \) since we may find a symbol correspondence \( W \) dual to \( W \) with \( \mathcal{S}_p(W) \neq \mathcal{S}_p(W) \), cf. Remark 4.19. But for semi-conformal correspondences and canonical integral trikernels, we have:

Proposition 4.43. Let \( \mathbb{L} \) be the canonical integral trikernel of a twisted product \( \star \) induced by a semi-conformal correspondence \( W : \mathcal{B}(\mathcal{H}_p) \rightarrow C^\infty(\mathcal{E}) \). The binary operation \( \mathbb{\cdot} \) given by

\[
\mathbb{L}(z_1, z_2, z_3) = \int_{\mathcal{E} \times \mathcal{E}} f_1(z_1) f_2(z_2) \mathbb{L}(z_1, z_2, z) \, dz_1 dz_2
\]

for any \( f_1, f_2 \in C^\infty(\mathcal{E}) \), defines an \( SU(3) \)-equivariant associative \( * \)-algebra on \( C^\infty(\mathcal{E}) \) with respect to complex conjugation. In particular, if \( f_1, f_2 \in \mathcal{S}_p(W) \), we have \( f_1 \mathbb{\cdot} f_2 = f_1 \star f_2 \). But, if either \( f_1 \) or \( f_2 \) is orthogonal to \( \mathcal{S}_p(W) \), then \( f_1 \mathbb{\cdot} f_2 = 0 \) (and thus \( C^\infty(\mathcal{E}) \rightarrow \mathcal{S}_p(W) : f \mapsto f \cdot f = f \) is an orthogonal projection).

Proof. The proof follows from the same arguments applied to Proposition 3.43 but now it is needed to point out that

\[
\mathbb{L}(z_1, z_2, z_3) = (-1)^{|p|} \sqrt{\dim Q(p)} \sum_{(a_1, \sigma_1), (a_2, \sigma_2), (a_3, \sigma_3)} \begin{bmatrix} \sigma_3 & \sigma_2 & \sigma_1 \\ \nu_1, J_1 & \nu_2, J_2 & \nu_3, J_3 \end{bmatrix} [p] \times \frac{1}{\alpha(a)} Z\mathcal{E}(a_1)^{\nu_1, J_1}(z_1) Z\mathcal{E}(a_2)^{\nu_2, J_2}(z_2) Z\mathcal{E}(a_3)^{\nu_3, J_3}(z_3) .
\]

Thus, if \( f_k \in C^\infty(\mathcal{E}) \) is orthogonal to \( \mathcal{S}_p(W) \), it is orthogonal to every \( Z\mathcal{E}(a)^{\nu, J} \) and this implies that the integral over \( z_k \) in (4.64) vanishes.

\[\square\]

Proposition 4.44. Let \( \mathbb{L} \) be an integral trikernel of a twisted product \( \star \) induced by \( W : \mathcal{B}(\mathcal{H}_p) \rightarrow C^\infty(\mathcal{E}) \) as in (4.63). Then, \( \forall g \in SU(3) \) and \( \forall z_1, z_2, z_3, z_4 \in \mathcal{E} \),
highlight two important differences. First, because representations distinguish to what is known for spin systems, being also determined by ordered operator kernel that is also an "actual state", thus being also a positive operator.

positive correspondence is a correspondence generated as expectation values over a unitary trace. Then, the more restricted case of a mapping-

\[ R_W^p (z_1, z_2) = \sum_{(a, \sigma) \nu, j} Z{\mathcal{C}(a)}_{\nu, j}^a (z_1) Z{\mathcal{C}(a)}_{\nu, j}^a (z_2) \]

satisfies \( \int_{E} f(z_1) R_W^p (z_1, z_2) \, dz_1 = f(z_2) \) for every \( f \in \mathcal{S}_p(W) \);

\[ \int_{E} R_W^p (z_1, z_2) \, dz_1 = R_W^p (z_2, z_1) \]

\( R_W^p \) is the reproducing kernel on \( \mathcal{S}_p(W) \).

Finally, the following proposition, whose proof is analogous to the proof of Proposition 3.45 implies the same kind of phenomenon described in Remark 3.49.

Remark 4.45. In general, \( R_W^p (z_1, z_2) \neq R_W^p (z_2, z_1) \). But if \( W \) is a semi-conformal correspondence and \( \mathcal{L} \) is its canonical integral trikernel, then (iii) is satisfied with

\[ R_W^p (z_1, z_2) = \sum_{(a, \sigma) \nu, j} \frac{1}{\alpha(a)} Z{\mathcal{C}(a)}_{\nu, j}^a (z_1) Z{\mathcal{C}(a)}_{\nu, j}^a (z_2) = R_W^p (z_2, z_1) \]

so that \( R_W^p \) is the reproducing kernel on \( \mathcal{S}_p(W) \).

Proposition 4.46. The twisted products \( \ast \) and \( \tilde{\ast} \) induced by a symbol correspondence and its antipodal correspondence satisfy

\[ f_1 \ast f_2 = f_2 \tilde{\ast} f_1 . \]

Corollary 4.47. For \( \ast \) and \( \tilde{\ast} \) as above, we can choose integral trikernels satisfying

\[ \mathcal{L}(z_1, z_2, z_3) = \tilde{\mathcal{L}}(z_2, z_1, z_3) \] .

5. CONCLUDING REMARKS

The main problem studied in this paper, the characterization of symbol correspondences between quantum and classical mechanical systems symmetric under \( SU(3) \), referred to as quark systems, is often settled on facts pertaining to systems symmetric under more general compact Lie groups, thus some of the features presented here are common to the case of spin systems (\( SU(2) \)-symmetric systems).

For example, the realization of any symbol correspondence as expectation values over an operator kernel, which is a special "pseudo-state", that is, a special Hermitian operator with unitary trace. Then, the more restricted case of a mapping-positive correspondence is a correspondence generated as expectation values over an operator kernel that is also an "actual state", thus being also a positive operator.

In particular, symbol correspondences for pure-quark systems show little formal distinction to what is known for spin systems, being also determined by ordered \( n \)-tuples of non zero real numbers, the characteristic numbers. Nonetheless, we highlight two important differences. First, because representations \( Q(p, 0) \) and...
Q(0, p) are not self-dual, antipodal correspondences are defined for pure-quark systems dual to each other and have the same characteristic numbers\(^2\). Second, for any pure-quark system, there exists only one Berezin (and thus only one Toeplitz) correspondence from quantum operators to functions on \(\mathbb{C}P^2\).

However, correspondences for generic quark systems present some new features originated from the degeneracy of representations within both the quantum operator space \(B(\mathcal{H}_{p,q})\) and the classical function space \(C^\infty_\C(\mathcal{E})\). Then, the characterization of correspondences for generic quark systems, in the same vein of what is done for pure-quark systems, are given not in terms of characteristic numbers, but in terms of characteristic matrices. As consequence, there are multiple correspondences linked by a dual relation and, in addition to isometric (Stratonovich-Weyl) correspondences, we have the more general definition of semi-conformal correspondences as special cases of symbol correspondences, alongside the special cases of mapping-positive and positive-dual correspondences.

Future work shall be dedicated to the problem of asymptotic behavior of symbol correspondences and verifying the conditions under which the Poisson algebra of a classical system emerges as an asymptotic limit of operator algebras of quantum systems, via twisted products. In this respect, the first problem at hand could be how to decompose the Poisson bracket of \(\mathbb{C}P^2\)-harmonics, or \(\mathcal{E}\)-harmonics, similarly to what has been done in the case of spin systems and spherical harmonics.

Due to the similarity with spin systems, it seems that the asymptotic analysis of pure-quark systems could be more feasible. On the other hand, generic quark systems seem to be quite more subtle for asymptotic analysis, since the relevant quantum systems are labeled by two indices, \((p, q) \in (\mathbb{N} \times \mathbb{N}_0) \cup (\mathbb{N}_0 \times \mathbb{N})\), so the construction and study of sequences of generic quark symbol correspondences, in line with what was done for spin systems (cf. \(\text{[24]}\) and also \(\text{[1]}\)), may involve some choices yet to be identified and understood (in greater generality, we could have to deal with bi-sequences of correspondences and study the asymptotic limit \(d \to \infty\), where \(d = d(p, q)\) is the dimension of \(Q(p, q)\)). Furthermore, different correspondences for the same quantum generic quark system may have different images, cf. Proposition \(\text{[1.33]}\), so it might be the case that we could generate sequences (or bi-sequences) of symbol correspondences whose images never reach some harmonic functions \(f \in C^\infty_\C(\mathcal{E})\). In that light, integral formulations of twisted products, as in Theorem \(\text{[4.41]}\), may turn out to be more useful in the generic case. Anyway, we expect that the factorization obtained in Corollary \(\text{[2.31]}\) together with the various symmetries presented in Theorem \(\text{[2.30]}\) shall be useful in some asymptotic approaches similar to the one performed for spin systems in \(\text{[24]}\).

Another related direction to be explored is the study of symbol correspondences from quantum quark systems to \(SU(3)\)-invariant Poisson manifolds, particularly \(S^7 \subset \mathbb{R}^8 \cong su(3)\). The 7-sphere can be split as \(S^7 \simeq M_+ \cup N \cup M_-\), where \(M_+\) and \(M_-\) are two copies of \(\mathbb{C}P^2\) and \(N\) is an uncountable disjoint union of \(\mathcal{E}\) copies. In other words, \(\mathcal{E}\) and \(\mathbb{C}P^2\) are isomorphic to the symplectic leaves of \(S^7\), with \(\mathcal{E}\) isomorphic to the regular leaves of this singular foliation of \(S^7\). In this respect, the first problem at hand could be understanding how to “glue” the harmonic functions of \(\mathbb{C}P^2\) and \(\mathcal{E}\) in order to obtain \(SU(3)\)-equivariant smooth functions on \(S^7\).

\(^2\)The antipodal relation stems from the action of the longest element of the Weyl group, which opens the question of other possible relations associated to the action of other elements of the Weyl group.
Appendix A. An explanation of Definition 2.1

In this appendix, we explain the Gelfand-Tsetlin method applied to the case of SU(3), which is used in Definition 2.1 cf. (2.16)-(2.19) and (2.22). For a general description of the Gelfand-Tsetlin method, see [19, 31].

We can take the matrices $E_{jk}$, with $j, k \in \{1, 2, 3\}$, given by $(E_{jk})_{l,m} = \delta_{j,k} \delta_{k,m}$ as generators of the unitary group $U(3)$, so that $E_{jk}$ is a raising operator if $j < k$, it is a lower operator if $j > k$ and it is a Cartan operator if $j = k$. Those operators satisfy the commutation relations

\begin{equation}
[E_{jk}, E_{lm}] = \delta_{j,l} E_{jm} - \delta_{j,m} E_{lk}.
\end{equation}

The triples $(\nu_1, \nu_2, \nu_3)$ widely used in this work are weights of representations of $U(3)$, nonnegative eigenvalues of $E_{11}$, $E_{22}$ and $E_{33}$, respectively.\footnote{The operators $E_{jk}$ differ from the operators $A_{jk}$ outlined in subsection 2.2 only for indices $j = k$, cf. (2.16)-(2.19).}

One obtains generators of SU(3) by maintaining the ladder operators and taking $(E_{11} - E_{22})/2$ and $(E_{22} - E_{33})/2$ as Cartan operators for SU(3). Thus, an irreducible representation $Q(p, q)$ of SU(3) gives rise to an irreducible representation of U(3) with highest weight $(p + q + m, q + m, m)$, for any nonnegative integer $m$, and vice-versa.\footnote{An irreducible representation of U(3) with highest weight $(a_1, a_2, a_3)$ corresponds to the Young tableau with $a_j$ boxes in the $j$-th row.}

We conveniently choose $m = 0$, so that we can identify an irrep $Q(p, q)$ of SU(3) with the irrep of U(3) with highest weight $(p + q, q, 0)$.

Now, we want to unambiguously index an orthonormal basis of the representation consisting only of weight vectors. To do so, we consider the subrepresentations of the $U(2)$ related to the generators $E_{jk}$ with $j, k \in \{1, 2\}$. Then we decompose the subrepresentations of this $U(2)$ into irreducible subrepresentations of the $U(1)$ generated by $E_{33}$. Since $U(1)$ is abelian, its irreducible representations are unidimensional, so we can get an orthonormal basis for $Q(p, q)$ by the restriction of $(p + q, q, 0)$ to the chain of subgroups $U(1) \subset U(2) \subset U(3)$.

The classification of subrepresentations within a given representation for a chain of groups is usually called branching rule, and the branching rule for $U(n-1) \subset U(n)$ is well known: they are multiplicity free and given by the so called betweenness condition \footnote{The operators $E_{jk}$ differ from the operators $A_{jk}$ outlined in subsection 2.2 only for indices $j = k$, cf. (2.16)-(2.19).}.

In our case of interest, with the choice $m = 0$ as explained above, this means that the subrepresentations of $U(2)$ are determined by all pairs of integers $(r_+, r_-)$ such that $r_+$ is between $q$ and $p + q$ and $r_-$ is between 0 and $q$, that is,

\begin{equation}
0 \leq r_- \leq q \leq r_+ \leq p + q.
\end{equation}

Then, for each subrepresentation $(r_+, r_-)$ of U(2), the subrepresentations of U(1) associated to generator $E_{33}$ are given by integers $\nu_3$ satisfying

\begin{equation}
r_- \leq \nu_3 \leq r_+.
\end{equation}

It is straightforward to verify from (A.1) that $E_{11} + E_{22} + E_{33}$ is invariant by $U(3)$. By applying it to the vector with highest weight $(p + q, q, 0)$, we get

\begin{equation}
E_{11} + E_{22} + E_{33} = (p + 2q) \mathbb{1} \implies E_{11} = (p + 2q) \mathbb{1} - (E_{22} + E_{33}).
\end{equation}

Analogously, in a subrepresentation $(r_+, r_-)$ of U(2), we have

\begin{equation}
E_{22} + E_{33} = (r_+ + r_-) \mathbb{1} \implies E_{22} = (r_+ + r_-) \mathbb{1} - E_{33}.
\end{equation}
Therefore, $\nu_1$ and $\nu_2$ are given by
\begin{equation}
(A.6) \quad \nu_1 = p + 2q - (r_+ + r_-) \quad \text{and} \quad \nu_2 = r_+ + r_- - \nu_3.
\end{equation}

Just as for $SU(3) \subset U(3)$, the operators $\{E_{23}, E_{32}, (E_{22} - E_{33})/2\}$ are generators of $SU(2)$ among the generators of the chosen $U(2)$, so that we can identify the representation $(r_+, r_-)$ of $U(2)$ with the representation of $SU(2)$ with spin number
\begin{equation}
(A.7) \quad J = \frac{r_+ - r_-}{2}.
\end{equation}

Then, from $U_- = E_{32}$ and $T_- = E_{21}$, the coefficients in (2.22) can be explicitly carried out by straightforward calculations as done in [3].

**Appendix B. An example for Theorem 2.14**

In [25], the irreducible representation $Q(p, q)$ is constructed using complex polynomials in six variables, $x_k$ and $x_k^*$ for $j = 1, 2, 3$. For the matrices
\begin{equation}
(N) \quad N = \begin{pmatrix} x_1 \partial_1 & x_1 \partial_2 & x_1 \partial_3 \\ x_2 \partial_1 & x_2 \partial_2 & x_2 \partial_3 \\ x_3 \partial_1 & x_3 \partial_2 & x_3 \partial_3 \end{pmatrix}, \quad \bar{N} = \begin{pmatrix} x_1^* \partial_1^* & x_2^* \partial_2^* & x_3^* \partial_3^* \\ x_1^* \partial_2^* & x_2^* \partial_1^* & x_3^* \partial_3^* \\ x_1^* \partial_3^* & x_2^* \partial_3^* & x_3^* \partial_1^* \end{pmatrix},
\end{equation}
where $\partial_j$ and $\partial_j^*$ are, respectively, derivatives with respect to $x_j$ and $x_j^*$, the generators $A_{jk}$ satisfying (2.48)-(2.49) are given by the corresponding entry of
\begin{equation}
(B.2) \quad A = N - \bar{N} - \frac{p - q}{3} I.
\end{equation}

For $l \in \{1, 2\}$, let $N^{(l)}$, $\bar{N}^{(l)}$ and $A^{(l)}$ be the operators given as in (B.1)-(B.2) for the representation $Q(p_1, q_1)$ built over variables $x_k$ and $x_k^*$, $k = 1, 2, 3$. Then, from (B.2) and the defining equations (2.50) and (2.62), we get, for some $\lambda \in \mathbb{R}$,
\begin{equation}
(B.3) \quad S_{12} = \frac{1}{2} \text{tr} \left( A^{(1)} (A^{(1)} - A^{(2)}) A^{(2)} \right) = \lambda I - S',
\end{equation}
\begin{equation}
(B.4) \quad S' = \frac{1}{2} \text{tr} \left( (p_1 - p_2) N^{(1)} N^{(2)} - (q_1 - q_2) \bar{N}^{(1)} \bar{N}^{(2)} + (q_1 + p_2 + 1) N^{(1)} \bar{N}^{(2)} \\ - (p_1 + q_2 + 1) N^{(1)} \bar{N}^{(2)} - N^{(1)} \bar{N}^{(1)} N^{(2)} + N^{(2)} \bar{N}^{(2)} N^{(1)} \\ + N^{(1)} \bar{N}^{(1)} N^{(2)} - N^{(2)} \bar{N}^{(2)} N^{(1)} \right),
\end{equation}
cf. [8], (2.13)-(2.14)], where in (B.3)-(B.4) we are using the shorthand notation
\begin{equation}
\text{tr}(AB) = \sum_{j,k=1}^{3} A_{jk} B_{kj}, \quad \text{tr}(ABC) = \sum_{j,k,l=1}^{3} A_{jk} B_{kl} C_{lj},
\end{equation}
recalling that this shorthand notation must be used with care, cf. Remark 2.14.

Now, a subrepresentation $Q(a, b)$ in the CG series of $Q(p_1, q_1) \otimes Q(p_2, q_2)$ is generated by $\psi_{(a,b)}$ satisfying
\begin{equation}
(B.5) \quad A_{12} \psi_{(a,b)} = A_{32} \psi_{(a,b)} = A_{13} \psi_{(a,b)} = 0.
\end{equation}
From [25, eq. (4.4a)-(4.4b)], $\psi_{(a,b)}$ can be given by
\begin{align}
(B.6) \quad \psi_{(a,b)} &= P B_{12} B_{21} C^{*}(x_1^1)^{p_1-u-s}(x_2^2)^{p_2-u-s}(x_2^2)^{q_2-u}, \quad \text{or} \\
(B.7) \quad \psi_{(a,b)} &= P B_{12} B_{21} C^{*}(x_1^1)^{p_1-u-s}(x_2^2)^{p_2-v-s}(x_2^2)^{q_1-v-s}(x_2^2)^{q_2-u-s}.
\end{align}
where

$$B_{kl} = \sum_{j=1}^{3} x_j^k x_j^{l*}, \quad C = x_1^1 x_3^2 - x_3^1 x_1^2, \quad C^* = x_3^1 x_2^2 - x_2^1 x_3^2,$$

and $P$ is the projection onto the subspace of polynomials satisfying

$$\sum_{k=1}^{3} \partial_k^1 \partial_k^{l*} \psi = \sum_{k=1}^{3} \partial_k^2 \partial_k^{2*} \psi = 0,$$

and

$$a = p_1 + p_2 - u - v - 2s, \quad b = q_1 + q_2 - u + v + s, \quad \text{for } (B.6),$$

$$a = p_1 + p_2 - u - v + s, \quad b = q_1 + q_2 - u - v - 2s, \quad \text{for } (B.7),$$

with

$$0 \leq u \leq \min\{p_1, q_2\}, \quad 0 \leq v \leq \min\{p_2, q_1\}.$$

Then, from (2.73), in order to study the eigenvalues of $S'_{123}$ one studies the eigenvalues of $S_{12}$ on the subrepresentation $\bigoplus_{\alpha=1}^{m(p_1, p_2; \alpha)} Q(a, \sigma)$ in the CG series of $Q(p_1) \otimes Q(p_2)$. But because $S_{12}$ is a Casimir operator, its eigenvalues only depend on the highest weight vector of each irreducible subrepresentation $Q(a, \sigma)$, so we can restrict $S_{12}$ to the subspace spanned by these highest weight vectors $\psi^\sigma_{\alpha}, \sigma = 1, \cdots, m(p_1, p_2; \alpha)$, and, from (B.3), it is equivalent to study the eigenvalues of $S'$ restricted to this subspace. This is done in $\hat{S}$ via a brute force calculation outlined by using (B.1)-(B.4) and (B.5)-(B.11), where the final result of the calculation shows that this restricted $S'$ has a cyclic vector, hence has only distinct eigenvalues.

To illustrate in detail the general computation outlined in $\hat{S}$, we consider

$$Q(1, 1) \otimes Q(1, 1) = Q(2, 2) \oplus Q(0, 3) \oplus Q(3, 0) \oplus Q(1, 1) \oplus Q(1, 1) \oplus Q(0, 0),$$

cf. Corollary (2.7). The subrepresentation $Q(1, 1) \oplus Q(1, 1)$ in the CG series of $Q(1, 1) \otimes Q(1, 1)$ is generated by the polynomials

$$\psi_1 = PB_{21} x_1^1 x_2^{2*}, \quad \psi_2 = PB_{12} x_1^2 x_2^{1*}.$$ 

Furthermore, (B.4) in this case simplifies to

$$S' = \frac{1}{2} \text{tr} \left( 3\nabla_1 N_2 - 3\nabla_1 \nabla_2 - N_1 \nabla_1 N_2 + N_2 \nabla_2 N_1 + N_1 \nabla_1 N_2 - N_2 \nabla_2 N_1 \right),$$

where we are now using the simpler notation $N_l = N^{(l)}, \nabla_l = \nabla^{(l)}, l = 1, 2$.

From (2.73) and (B.3), it is enough to show that the eigenvalues of $S'$ are different, for each of the two irreducible subrepresentations $Q(1, 1)$ in the CG series. Thus, from now on, we consider $\hat{S}$ as the restriction of $S'$ on span$\{\psi_1, \psi_2\}$ and show that $\hat{S}$ has two distinct eigenvalues. To compute the entries $[\hat{S}]_{j,k}$,

$$\hat{S}\psi_k = \sum_{k=0}^{1} [\hat{S}]_{j,k} \psi_j,$$
we make the substitution \( \psi_k = \phi_k + \psi'_k \), where \( \phi_k \) is \( \psi_k \) without the projection \( P \) and \( \psi'_k \) contains terms with \( B_{11} \) and \( B_{22} \). In the r.h.s. of (B.14), however, we can ignore terms with \( B_{11} \) and \( B_{22} \). Thus, for \( \mathbf{S} \), we can consider only the terms

\[
\begin{align*}
\text{(B.15)} & \quad \text{tr}(N_1 N_2) = B_{21} \sum_{j=1}^{3} \delta_{ij}^2 \delta_{ij}^2, \\
\text{(B.16)} & \quad \text{tr}(N_k N_k N_l) = B_{kl} \sum_{i,j=1}^{3} x_i^k \delta_{ij}^k \delta_{ij}^k \delta_{ij}^k, \\
\text{(B.17)} & \quad \mathbf{S} = \frac{1}{2} \begin{pmatrix} 7 & -2 \\ 2 & -7 \end{pmatrix},
\end{align*}
\]

for \( k \neq l \in \{1, 2\} \). By a straightforward calculation using \( \text{(B.12)}-\text{(B.16)} \), we get\(^{27}\)

\[
\text{(B.17)} \quad \mathbf{S} = \frac{1}{2} \begin{pmatrix} 7 & -2 \\ 2 & -7 \end{pmatrix},
\]

and it is easy to see that \( \mathbf{S} \) has a cyclic vector, so its eigenvalues are distinct (more precisely, in this \( 2 \times 2 \) case the eigenvalues of \( \mathbf{S} \) are easily computed to be \( \pm 3\sqrt{5}/2 \)).

**Appendix C. A justification for Definition 3.7**

An explicit way to construct a representation \( Q(p, 0) \) is by the so-called tensor method. Consider the defining representation \( \rho \) of class \( Q(1, 0) \) on \( H_{1,0} \approx \mathbb{C}^3 \), and let the canonical basis \( \{e_1, e_2, e_3\} \) match a GT basis, with each vector associated to a weight on the diagram of Figure 2(A), \( e_1 \) with the highest weight and \( e_2, e_3 \) ordered counterclockwise. Then, the tensor product space \( H = H_{1,0} \otimes \ldots \otimes H_{1,0} \) with \( p \) copies of \( H_{1,0} \) carries the induced representation \( \rho = \rho_1 \otimes \ldots \otimes \rho_1 \).

Let \( H_{p,0} = \text{Sym}^p(H_{1,0}) \subset H \) be the subspace of totally symmetric tensors,

\[
\begin{equation}
\sum_{i_1, \ldots, i_p=1}^{3} c_{i_1, \ldots, i_p} e_{i_1} \otimes \ldots \otimes e_{i_p} \in H_{p,0} \iff c_{i_{f(1)}, \ldots, i_{f(p)}} = c_{i_1, \ldots, i_p}
\end{equation}
\]

for every permutation \( f \in S_p \). It is immediate that \( H_{p,0} \) is an invariant subspace. We can get a basis for \( H_{p,0} \) by means of symmetrization. For \( e_{i_1} \otimes \ldots \otimes e_{i_p} \in H \), let \( j, k \) and \( l \) be the numbers of occurrence of index 1, 2 and 3, respectively, and take

\[
\begin{equation}
e_{j,k,l} = \left( \frac{p!}{j!k!l!} \right)^{-1/2} \sum_{f \in S_p} e_{i_{f(1)}} \otimes \ldots \otimes e_{i_{f(p)}} \in H_{p,0}.
\end{equation}
\]

The set \( \{e_{j,k,l} : j + k + l = p\} \) is an orthonormal basis of \( H_{p,0} \) considering the inner product induced by \( H_{1,0} \) on \( H \). Starting with the element \( e_{0,0,0} \), we can obtain the basis \( \{e_{j,k,l} : j + k + l = p\} \) by recursively applying the ladder operators \( T_- \) and \( U_- \) and normalizing the result. As can be seen from the diagram of Figure 2(A), \( e_{j,k,l} = \mu_{j,k,l}(U_-)^k(T_-)^{k+l} e_{p,0,0} \), where \( \mu_{j,k,l} > 0 \). Since \( \dim H_{p,0} = (p+1)(p+2)/2 \) and \( e_{0,0,0} \) is a highest weight vector\(^{28}\) with eigenvalues \( p/2 \) for \( T_3 \) and 0 for \( U_3 \), we

\(^{27}\)The matrix is not equal to its conjugate transpose because the base \( \{\psi_1, \psi_2\} \) is not orthogonal.

\(^{28}\)Also, \( e_{0,0,0} \) is a lowest weight vector.
conclude that the $SU(3)$-representation on $\mathcal{H}_{p,0}$ is an irreducible representation of class $Q(p,0)$. In particular, the following map is equivariant

\begin{equation}
H_1 \to \mathcal{H}_{p,0}: w = (z_1, z_2, z_3) \mapsto w \otimes \ldots \otimes w = \sum_{j+k+l=1} p! \frac{p!}{j!k!l!} z_j^3 z_2^k z_3^l e_{j,k,l} \,.
\end{equation}

An equivalent procedure starting with $\mathcal{H}_{0,1} = \mathcal{H}_1^{*,0}$ gives us the space $\mathcal{H}_{0,p} = \mathcal{H}_{p,0}^*$ with a representation of class $Q(0,p)$ and the equivariant map

\begin{equation}
H_{0,1} \to \mathcal{H}_{0,p}: w^* = (z_1, z_2, z_3) \mapsto w^* \otimes \ldots \otimes w^* = \sum_{j+k+l=1} p! \frac{p!}{j!k!l!} z_j^3 z_2^k z_3^l \tilde{e}_{j,k,l} \,;
\end{equation}

where $\{\tilde{e}_{j,k,l}: j+k+l=p\}$ is the basis induced by $\{\tilde{e}_1 = -e_3, \tilde{e}_2 = e_2, \tilde{e}_3 = -e_1\}$ just like $\{e_{j,k,l}: j+k+l=p\}$ is induced by $\{e_1, e_2, e_3\}$, cf. Definition 2.3.

In Physics, the space of colors (resp. flavors) of a quark is precisely the representation $Q(1,0)$, with $e_1 \equiv \text{red}$ (resp. up quark), $e_2 \equiv \text{blue}$ (resp. down quark) and $e_3 \equiv \text{green}$ (resp. strange quark). Thus, $Q(p,0)$ is the totally symmetric part of a system of $p$ quarks. Analogously, $Q(0,q)$ is the totally symmetric part of a system of $q$ antiquarks since the representation $Q(0,1)$ describes an antiquark, $\tilde{e}_1 \equiv \text{antigreen}$ (strange antiquark), $\tilde{e}_2 \equiv \text{antiblue}$ (down antiquark) and $\tilde{e}_3 \equiv \text{antired}$ (up antiquark). Thus, in the context of quark systems, such spaces arise in description of systems of $p$ identical quarks only (or $p$ identical antiquarks only). Hence, we call them pure-quark systems because the number of antiquarks (or quarks) is zero.

On the other hand, because the highest or lowest weight space of $Q(p,0)$ or $Q(0,p)$ have the maximal isotropy subgroup $H \simeq U(2)$, these representations are also called symmetric representations, in the Mathematics literature. So, pure-quark systems could also be referred to as symmetric quark systems.

**Remark C.1.** There is another interpretation for the representation $Q(p,0)$ as a quantum system whose classical phase space is $\mathbb{C}P^2$. A quantum three-dimensional isotropic harmonic oscillator is an affine quantum system with Hamiltonian

\begin{equation}
H = \sum_{i=1}^3 \left( \frac{1}{2m} P_i^2 + \frac{1}{2} m \omega^2 X_i^2 \right) \,;
\end{equation}

where $P_i$ and $X_i$ are the component operators of momentum and position, for some positive parameters $m$ and $\omega$. It has degenerate energy levels\(^{29}\)

\begin{equation}
E = \left( n_1 + n_2 + n_3 + \frac{3}{2} \right) \omega \,;
\end{equation}

with $SU(3)$-symmetry given by representations $Q(p,0)$ for $p = n_1 + n_2 + n_3$, so that $E = E_p = p + 3/2$, by setting $\omega = 1$ \(^{29}\). For the classical three-dimensional isotropic harmonic oscillator the phase space is $T^*\mathbb{R}^3 \simeq \mathbb{R}^6$ and the Hamiltonian is

\begin{equation}
h(\vec{x}, \vec{p}) = \sum_{i=1}^3 \left( \frac{p_i^2}{2m} + \frac{1}{2} m \omega x_i^2 \right) \,;
\end{equation}

\(^{29}\)Again, $e_{p,0,0}$ is a highest weight vector and $\tilde{e}_{0,0,p}$ is a lowest weight vector.

\(^{30}\)We set $h = 1$ throughout this paper.
where $p_i$ and $x_i$ are the components of momentum and position, $\vec{p}, \vec{x} \in \mathbb{R}^3$, in analogy to (6.5). By rescaling, a region of fixed energy is identified with $S^5 \subset \mathbb{R}^6$. The solution passing through a point of $S^5$ is the orbit of the point under an $SO(2)$-action, where $SO(2)$ acts via rotations on each $\mathbb{R}^2$ of pairs $(x_i, p_i)$. But the action of $SO(2)$ on $\mathbb{R}^2$ is equivalent to the action of $U(1)$ on $\mathbb{C}$, so the set of solutions of a classical 3-d isotropic harmonic oscillator is identified with $S^5/S^1 = \mathbb{CP}^2$.

APPENDIX D. A PROOF OF THEOREM 4.26

We start by proving for $H_\geq$. Let $\rho$ be an irreducible representation of class $Q(\mathbf{p})$ in $\mathcal{H}_\rho$. Now, the really hard-to-check property in Definition 4.9 that the map
\[ B(\mathcal{H}_\rho) \ni A \mapsto \left( B_A : \mathcal{E} \to \mathbb{C} : z \mapsto B_A(z) = \text{tr}(A\Pi_\geq(z)) \right) \]
needs to satisfy in order to be a symbol correspondence is injectivity. Here we reproduce in greater detail, for the specific case of $SU(3)$, the argument for injectivity of the map (D.1) as presented by Wildberger for general compact semisimple Lie groups in [30], and reworked by Figueroa, Gracia-Bondía and Várilly in [11].

Because multiplicities of weights are not relevant for the argument, instead of the Gelfand-Tsetlin labeling used throughout the rest of the paper, here we use the representation of weights as linear combinations of the fundamental weights $\{\omega_1, \omega_2\}$, cf. (2.13), so that the highest weight of $Q(\mathbf{p})$ is $\mathbf{p}$. Then, the action of $T_\pm$ on a weight $\omega$ produces the weight $\omega \pm \alpha_1$, and the action of $U_\pm$ on $\omega$ produces the weight $\omega \pm \alpha_2$, where $\alpha_1$ and $\alpha_2$ are the simple roots of $SU(3)$, cf. Figure 1. We recall the partial order (2.15) on the set of weights, so that, although this is not a total order, we have $\mathbf{p} > \omega$ for every weight $\omega \neq \mathbf{p}$ of $Q(\mathbf{p})$. For pairs of weights, we consider the lexicographical order induced from this ordering of weights.

We denote by $\mathcal{H}_\rho^\omega$ the subspace of $\mathcal{H}_\rho$ spanned by vectors with weight $\omega$, and let $B_{\omega, \tau} = \text{Hom}(\mathcal{H}_\rho^\omega, \mathcal{H}_\rho^\omega)$ so that
\[ \mathcal{H}_\rho = \bigoplus_\omega \mathcal{H}_\rho^\omega, \quad B(\mathcal{H}_\rho) = \bigoplus_{\omega, \tau} B_{\omega, \tau}. \]

Given $A \in B(\mathcal{H}_\rho)$, we have a decomposition $A = \sum_{\omega, \tau} A_{\omega, \tau}$ such that $A_{\omega, \tau} \in B_{\omega, \tau}$.

We now introduce the sets
\[ \mathcal{A} = \{ A \in B(\mathcal{H}_\rho) : A \neq 0, B_A = 0 \}, \quad \mathcal{P} = \{ (\omega, \tau) : A_{\omega, \tau} \neq 0 \text{ for some } A \in \mathcal{A} \}. \]

Lemma D.1. If $\mathcal{A} \neq \emptyset$, then $\max \mathcal{P} = (\mathbf{p}, \mathbf{p})$.

Proof. Since the order on pair of weights is only a partial order, there might be more than one maximal element in $\mathcal{P}$. Let $(\omega, \tau)$ be a maximal element of $\mathcal{P}$ and take $A \in \mathcal{A}$ satisfying $A_{\omega, \tau} \neq 0$. Given basis $\{u_1, \ldots, u_n\}$ of $\mathcal{H}_\rho^\omega$ and $\{v_1, \ldots, v_m\}$ of $\mathcal{H}_\rho^\tau$, we have
\[ A_{\omega, \tau} = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{j,k} u_j \otimes v_k^* = \sum_{k=1}^{m} \left( \sum_{j=1}^{n} a_{j,k} u_j \right) \otimes v_k^* = \sum_{k=1}^{m} w_k \otimes v_k^*, \]
where $w_k = \sum_{j=1}^{n} a_{j,k} u_j$. Since $A_{\omega, \tau} \neq 0$, there is some $k_0 \in \{1, \ldots, m\}$ such that $w_{k_0} \neq 0$. If $\omega < \mathbf{p}$, there is $E_j \in \{E_1 = T_+, E_2 = U_+\}$ such that $E_j(w_{k_0}) \neq 0$. So $[E_j, A]$ has a non zero component $[E_j, A]_{\omega + \alpha_j, \tau}$. However, by equivariance of $B$, etc.
we have $B_A = 0 \implies B_{[E_j, A]} = 0$, which contradicts the maximality of $(\omega, \tau)$ in $P$. Thus, $\omega = p$, and

$$A_{p, \tau} = \sum_{k=1}^{n} a_k e_0 \otimes v_k^* = e_0 \otimes \left( \sum_{k=1}^{n} a_k v_k^* \right) = e_0 \otimes v^*,$$

where $e_0$ is some unit vector in $H^p_0$ and $v = \sum_{k=1}^{n} a_k v_k$ is non zero. Now, if $\tau < p$, there is, again, $E_k \in \{ E_1, E_2 \}$ such that $E_k(v) \neq 0$, so $B_{[E_k^\dagger, A]} = 0$. Thus, $\tau = p$ and $(\omega, \tau) = (p, p)$. □

From the above lemma, if $A \neq \emptyset$, then there exists $A \in A$ such that $A_{p, p} \neq 0$. But, given an unit vector $e_0 \in H^p_0$,

$$B_A(z_0) = \text{tr}(A\Pi_>) = \langle e_0 | A e_0 \rangle = \langle e_0 | A_{p,p} e_0 \rangle = A_{p,p} \neq 0,$$

a contradiction. Therefore, $A = \emptyset$, that is, $B_A = 0$ only if $A = 0$, hence the map (D.1) is injective. One can easily check that (D.1) also satisfies all other properties in Definition 4.9 thus the highest weight projector $\Pi_>$ is an operator kernel.

Finally, we recall Remarks 3.27 and 4.8 to conclude that $\Pi_\leq = \Pi_\geq_\sigma$, so projector onto the lowest weight space is an operator kernel as well.

References

[1] P. A. S. Alcántara and P. de M. Rios. Asymptotic localization of symbol correspondences for spin systems and sequential quantizations of $S^2$. To appear in Advances in Theoretical and Mathematical Physics. Preprint available at arXiv:2004.03929. 2022.

[2] G. Alexanian et al. Fuzzy $C\mathbb{P}^2$. J. Geom. Phys. 42 (2002), pp. 28–53.

[3] G. E. Baird and L. C. Biedenharn. On the representations of the semisimple Lie groups. II. J. Math. Phys. 4 (1963), pp. 1449–1466.

[4] J. Bernatska and P. Holod. “Geometry and topology of coadjoint orbits of semisimple Lie groups”. In: Proceedings of the Ninth International Conference on Geometry, Integrability and Quantization. Ed. by Ivailo M. Mladenov. Bulgarian Academy of Sciences, 2008, pp. 146–166.

[5] R. Bott et al. “The geometry and representation theory of compact Lie groups”. In: Representation Theory of Lie Groups. London Mathematical Society Lecture Note Series. Cambridge University Press, 1980, pp. 65–90.

[6] P. H. Butler. Coupling coefficients and tensor operators for chains of groups. Philos. Trans. R. Soc. Lond. 277 (1975), pp. 545–585.

[7] C. K. Chew and R. T. Sharp. $SU(3)$ isoscalar factors. Nucl. Phys. B. 2 (1967), pp. 697–712.

[8] C. K. Chew and R. T. Sharp. On the degeneracy problem in $SU(3)$. Can. J. Phys. 44 (1966), pp. 2789–2795.

[9] S. Coleman. The Clebsch-Gordan series for $SU(3)$. J. Math. Phys. 5 (1964), pp. 1343–1344.

[10] J.-R. Derome and W. T. Sharp. Racah Algebra for an Arbitrary Group. J. Math. Phys. 6 (1965), pp. 1584–1590.

[11] H. Figueroa, J. M. Gracia-Bondía, and J. C. Varilly. Moyal quantization with compact symmetry groups and noncommutative harmonic analysis. J. Math. Phys. 31 (1990).
REFERENCES

[12] G. B. Folland. *A Course in Abstract Harmonic Analysis*. CRC Press, Taylor & Francis, 2016.

[13] W. Greiner and B. Müller. *Quantum Mechanics*. Springer, 1994.

[14] W. Holman and L. Biedenharn. “The Representations and Tensor Operators of the Unitary Groups U(n)”. In: *Group Theory and its Applications*. Ed. by E. Loebl. Academic Press, 1971, pp. 1–73.

[15] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer New York, 1973.

[16] A. A. Kirillov. *Lectures on the Orbit Method*. American Mathematical Society, 2004.

[17] A. B. Klimov and H. de Guise. General approach to SU(n) quasi-distribution functions. *J. Phys. A* 43 (2010), p. 402001.

[18] A. B. Klimov, J. L. Romero, and H. de Guise. Generalized SU(2) covariant Wigner functions and some of their applications. *J. Phys. A* 50 (2017), p. 323001.

[19] J. D. Louck. Recent progress toward a theory of tensor operators in the unitary groups. *Am. J. Phys.* 38 (1970), pp. 3–42.

[20] A. C. N. Martins, A. B. Klimov, and H. de Guise. Correspondence rules for Wigner functions over SU(3)/U(2). *J. Phys. A* 52 (2019), p. 285202.

[21] R. Murgan and A. Zender. Energy eigenvalues of the three-dimensional quantum harmonic oscillator from SU(3) cubic Casimir operator. *Eur. J. Phys.* 40 (2018), p. 015405.

[22] Z. Pluhar, Y. F. Smirnov, and V. N. Tolstoy. Clebsch-Gordan coefficients of SU(3) with simple symmetry properties. *J. Phys. A* 19 (1986), pp. 21–28.

[23] Z. Pluhar, L. J. Weigert, and P. Holan. Symmetry properties of s-classified SU(3) 3j-, 6j- and 9j-symbols. *J. Phys. A* 19 (1986), pp. 29–34.

[24] P. de M. Rios and E. Straume. *Symbol Correspondences for Spin Systems*. Birkhäuser/Springer, 2014.

[25] R. T. Sharp and H. von Baeyer. Polynomial bases and isoscalar factors for SU(3). *J. Math. Phys.* 7 (1966), p. 1105.

[26] J. J. de Swart. The octet model and its Clebsch-Gordan coefficients. *Rev. Mod. Phys.* (1963), pp. 916–939.

[27] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii. *Quantum Theory of Angular Momentum*. WSPC, 1988.

[28] A. Weinstein. Sophus Lie and symplectic geometry. *Exposition. Math.* 1 (1983), pp. 95–96.

[29] E. Wigner. *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*. Academic Press, 1959.

[30] N. Wildberger. On the Fourier transform of a compact semisimple Lie group. *J. Austral. Math. Soc.* 56 (1994), pp. 64–116.

[31] D. Zhelobenko. *Compact Lie Groups and Their Representations*. American Mathematical Society, 1973.