THRESHOLD TWO–PION PHOTO– AND ELECTROPRODUCTION:
More neutrals than expected

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ABSTRACT

We present an exploratory study of two pion photo– and electroproduction off the nucleon in the threshold region. To calculate the pertinent amplitudes, we make use of heavy baryon chiral perturbation theory. We show that due to finite chiral loops the production cross section for final states with two neutral pions is considerably enhanced. The experimental implications are briefly discussed.
I. INTRODUCTION

Over the last few years, much interest has been focused on pion photo- and electro-production off nucleons. In particular, new accurate data for the processes $\gamma p \rightarrow \pi^0 p$ and $\gamma^* p \rightarrow \pi^0 p$ close to production threshold have become available [1,2].* These have led to many theoretical investigations. In particular, in refs.[3] baryon chiral perturbation theory was used to give a model-independent description of the pertinent differential cross sections, multipoles and so on. While the study of these reactions, also with charged pions in the final state, continues on the experimental as well as on the theoretical side, complementary information can be gained from the two pion production process $\gamma N \rightarrow \pi\pi N$, where $N$ denotes the nucleon and $\gamma$ the real or virtual photons. The two pions in the final state can both be charged, both neutral or one charged and one neutral. If the four–momentum of the photon is denoted by $k$, in case of $k^2 = 0$ one speaks of photoproduction and for $k^2 < 0$ of electroproduction. Here, we will be concerned with the threshold region, i.e. the photon in the initial state has just enough energy to produce the two pions (and the outgoing nucleon) at rest. This energy is very close to the first strong resonance excitation of the nucleon, the $\Delta(1232)$ resonance. In fact, presently available data focus on the resonance region and above. In that case, a two–step reaction mechanism of the form $\gamma N \rightarrow \pi\Delta \rightarrow \pi\pi N$ is appropriate to describe these data as detailed in refs.[4,5]. As we will show later, there is, however, a narrow window above threshold which is particularly sensitive to chiral loops, i.e. to the strictures of the spontaneously broken chiral symmetry. In the limit of vanishing current quark masses, which is a good first approximation for the two light flavors $u$ and $d$ relevant to the $\pi N$ system, the gauge theory of the strong interactions, QCD, exhibits this symmetry. While QCD is formulated in terms of (confined) quark and gluon fields, its low–energy behaviour is dominated by the almost massless Goldstone bosons related to the spontaneous breakdown of the chiral symmetry. This feature allows to formulate an effective field theory in terms of the asymptotically observed fields (here, the pions and the nucleons) which is amenable to a systematic energy expansion, baryon chiral perturbation theory. This framework has been applied to a wide variety of hadronic and nuclear processes as detailed in the reviews [6,7]. In the present context, CHPT amounts to a calculation of tree and pion one–loop diagrams. The latter are mandated to perturbatively restore unitarity and they are also demanded by the power counting scheme [8] underlying the effective field theory. In the baryon sector, a consistent one–to–one

* Here, $\gamma$ and $\gamma^*$ stand for the real and virtual photon, respectively. In what follows, we will exclusively use the symbol $\gamma$ but always spell out the pertinent four–momentum squared which is zero for real and less than zero for virtual photons, in order.
mapping between the energy and loop expansion is only possible if one transforms the nucleon mass term into a string of vertices of increasing powers in the inverse nucleon mass. This can be achieved in the extreme non–relativistic limit [9]. Evidently, such a procedure is nothing but a series of Foldy–Wouthuysen transformations known from the treatment of the Dirac equation for a very heavy fermion.

What makes the process $\gamma N \rightarrow \pi \pi N$ so interesting is that, as we will demonstrate in detail later on, the one–loop corrections lead to a dramatic increase in the production rates for final states containing neutral pions. This is a counter–intuitive result since in the case of single pion production the cross sections for production of charged pions are considerably larger than the ones with neutral pions in the final state. First measurements of two–pion production at low energies have been performed at MAMI and we expect that the theoretical predictions discussed below will give additional motivation to perform yet more detailed measurements of this particular reaction. Our study extends the one of Dahm and Drechsel [10] who discussed certain aspects of two–pion photoproduction in the framework of Weinberg’s chiral pion–nucleon Lagrangian [11].

The paper is organized as follows. In section 2, we discuss the formal aspects of threshold two–pion production, such as kinematics, threshold multipoles and the three–body phase space. We also give formulae for the total cross sections for the various final states, in particular we write down some very handy approximations which allow the interested reader to get a quick and fairly accurate first estimate. We extend this discussion to the amplitudes close to but not at threshold since these are important in the chiral expansion with two charged pions in the final state (see appendix A). Section 3 is devoted to a short discussion of the effective chiral pion–nucleon Lagrangian. We also discuss briefly the $\gamma \pi \Delta N$ system. The latter has to be included due to the closeness of the $\Delta (1232)$ resonance. In section 4, the chiral expansion of the threshold multipole amplitudes is given and the various contributions are discussed. While the tree and one–loop diagrams in the pion–nucleon system lead to unambiguous results, the tree diagrams including the $\Delta$ have potentially large uncertainties due to certain badly known off–shell parameters. However, to the first two orders in the chiral expansion only one of these enters and the final results are found to be weakly dependent on it. Section 5 contains the numerical results. We discuss the threshold multipoles and total cross sections making the approximation that the amplitude exactly at threshold represents the amplitude in the threshold region. For the particular channels $\gamma p \rightarrow \pi^+\pi^- p$ and $\gamma p \rightarrow \pi^0\pi^0 p$, we extend this calculation to the first non–vanishing corrections away from threshold. The experimental implications are also discussed. Most of the results presented are for the photoproduction case since the longitudinal multipoles appearing in electroproduction do not carry any relevant additional information to the first two orders in the chiral expansion. Section 6 contains the summary and outlook. Some technicalities are relegated to appendix B.
II. TWO-PION PRODUCTION: FORMAL ASPECTS

In this section, we will give the formalism necessary to treat two-pion photo- and electroproduction in the threshold region. We will only be concerned with the kinematics close or at threshold, the corresponding multipole decomposition and the total cross sections. For a more general discussion we refer the reader to [10].

II.1. GENERAL CONSIDERATIONS

Consider the process \( \gamma(k) + N(p_1) \rightarrow \pi^a(q_1) + \pi^b(q_2) + N(p_2) \), with \( N \) denoting the nucleon (proton or neutron), \( \gamma \) a real \( (k^2 = 0) \) or virtual \( (k^2 < 0) \) photon and pions of isospin \( a, b \). The pertinent four-momenta are given (cf. fig.1) and the polarization vector of the photon is denoted by \( \epsilon_\mu \). The corresponding current transition matrix element is

\[
T \cdot \epsilon = \quad \text{out} < \pi^a(q_1), \pi^b(q_2), N(p_2) | J_{\mu}^{em}(0) \epsilon^{\mu} | N(p_1) > \text{in} \tag{2.1}
\]

with \( J_{\mu}^{em} \) the electromagnetic current operator. From the two initial states \( \gamma p \) and \( \gamma n \) we can form in total six final states

\[
\begin{align*}
\gamma p &\rightarrow \pi^+ \pi^- p, \\
\gamma p &\rightarrow \pi^+ \pi^0 n, \\
\gamma p &\rightarrow \pi^0 \pi^0 p, \\
\gamma n &\rightarrow \pi^+ \pi^- n, \\
\gamma n &\rightarrow \pi^0 \pi^- p, \\
\gamma n &\rightarrow \pi^0 \pi^0 n. \tag{2.2}
\end{align*}
\]

In what follows we will mostly concentrate on the first three of these channels. To first order in the electromagnetic coupling \( e \) the threshold amplitudes for \( \gamma p \rightarrow \pi^+ \pi^0 n \) and \( \gamma n \rightarrow \pi^- \pi^0 p \) are equal as will become clear soon.

II.2. KINEMATICS IN THE THRESHOLD REGION

In general, one can form five/six Mandelstam variables for the two-pion photo/electroproduction process from the independent four-momenta. For our purpose, it is most convenient to work in the photon-nucleon center-of-mass frame. At threshold, the real or virtual photon has just enough energy to produce the two pions at rest. Let us denote by \( M_\pi \) and \( m \) the masses of the pion and nucleon, respectively.* The threshold center-of-mass energy squared is

\[
s_{\text{thr}} = (p_1 + k)_{\text{thr}}^2 = (m + 2M_\pi)^2 = m^2(1 + 4\mu + 4\mu^2) \tag{2.3}
\]

* In section II.5 we briefly discuss the relevance of differentiating the various masses in the six processes (2.2).
where we have introduced the small parameter

$$\mu = \frac{M_\pi}{m} \simeq \frac{1}{17}. \tag{2.4}$$

The photon center-of-mass energy can be expressed in terms of $s$ and the photon virtuality $k^2$ as

$$k_0 = \frac{s - m^2 + k^2}{2\sqrt{s}}, \quad k_0^{\text{thr}} = \frac{2m}{1 + 2\mu} \left[ \mu + \mu^2 + \frac{\nu}{4} \right] \tag{2.5}$$

introducing a second small parameter

$$\nu = \frac{k^2}{m^2} \tag{2.6}$$

since we will consider only small photon four-momenta squared $|k^2| \ll m^2$. When discussing experimental quantities like total cross sections, it is more common to consider the laboratory system in which $N(p_1)$ is at rest. The photon lab energy denoted by $E_\gamma$ is then

$$E_\gamma = \frac{s - m^2 - k^2}{2m} \tag{2.7}$$

and the threshold value for two pion-photoproduction is given by

$$E_{\gamma}^{\text{thr}} = 2M_\pi(1 + \mu). \tag{2.8}$$

For real photons only, we will calculate two-pion production amplitudes above threshold. In that case the pertinent four-momenta in the center-of-mass frame read

$$k^\mu = (k_0, \vec{k}), \quad p_1^\mu = (E_1, -\vec{k}), \quad p_2^\mu = (E_2, -\vec{q}_1 - \vec{q}_2), \quad q_1^\mu = (\omega_1, \vec{q}_1), \quad q_2^\mu = (\omega_2, \vec{q}_2) \tag{2.9}$$

together with

$$\omega_i^2 = q_i^2 + M_\pi^2, \quad E_1^2 = m^2 + k_0^2, \quad E_2^2 = m^2 + q_1^2 + q_2^2 + 2q_1q_2z, \quad q_1 \cdot k = k_0(\omega_1 - xq_1), \quad q_2 \cdot k = k_0(\omega_2 - yq_2) \tag{2.10}$$

where $x, y, z$ denote the corresponding cosines of the angles between the three-vectors and $k_0 = |\vec{k}| = (s - m^2)/2\sqrt{s}$, $q_i = |\vec{q}_i|$ give their lengths. From spherical geometry one can deduce the standard relation

$$y = xz + \sqrt{(1 - x^2)(1 - z^2)}\cos\varphi \tag{2.11}$$

with $\varphi$ the azimuthal angle between the planes spanned by $\vec{q}_1$ and $\vec{k}$ as well as $\vec{q}_2$ and $\vec{q}_2$. Finally, real photons satisfy $\epsilon \cdot k = 0$. In the following, we will work in the Coulomb gauge $\epsilon_0 = 0$, so that we have in photoproduction in addition $\vec{\epsilon} \cdot \vec{k} = 0$. 

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II.3. THRESHOLD MULTipoles

At threshold in the center-of-mass frame (i.e. \( \vec{q}_1 = \vec{q}_2 = 0 \)), the two-pion electroproduction current matrix element can be decomposed into multipole amplitudes as follows if we work to first order in the electromagnetic coupling \( e \),

\[
T \cdot \epsilon = \chi_f^\dagger \{ i \vec{\sigma} \cdot (\vec{\epsilon} \times \vec{k}) [M_1 \delta^{ab} + M_2 \delta^{ab} \tau^3 + M_3 (\delta^{a3} \tau^b + \delta^{b3} \tau^a)] \\
+ \vec{\epsilon} \cdot \vec{k} [N_1 \delta^{ab} + N_2 \delta^{ab} \tau^3 + N_3 (\delta^{a3} \tau^b + \delta^{b3} \tau^a)] \} \chi_i
\]  
(2.12)

with \( \chi_{i,f} \) two-component Pauli-spinors and isospinors and we used the gauge \( \epsilon_0 = 0 \).

Clearly, for real photons only the \( M_{1,2,3} \) can contribute. For virtual photons, gauge invariance \( T \cdot k = 0 \) allows to reconstruct \( T_0 = \vec{T} \cdot \vec{k}/k_0 \). The multipole amplitudes \( M_{1,2,3} \) and \( N_{1,2,3} \) encode the information about the structure of the nucleon as probed in threshold two pion photo- and electroproduction. The physical channels listed in eq.(2.2) give rise to the following linear combination of \( M_{1,2,3} \) (and \( N_{1,2,3} \) for \( k^2 < 0 \)).

\[
\begin{align*}
\gamma p \to \pi^+\pi^- p & : M_1 + M_2, \\
\gamma p \to \pi^+\pi^0 n & : \sqrt{2}M_3, \\
\gamma p \to \pi^0\pi^0 p & : M_1 + M_2 + 2M_3, \\
\gamma n \to \pi^+\pi^- n & : M_1 - M_2, \\
\gamma n \to \pi^0\pi^- p & : \sqrt{2}M_3, \\
\gamma n \to \pi^0\pi^0 n & : M_1 - M_2 - 2M_3
\end{align*}
\]  
(2.13)

which shows the abovementioned equality for the second and fifth channel.

II.4. TOTAL CROSS SECTIONS

Let us consider the two-pion photoproduction close to threshold. The invariant matrix element squared averaged over nucleon spins and photon polarizations takes the form

\[
|\mathcal{M}_{fi}|^2 = \vec{k}^2 |\eta_1 M_1 + \eta_2 M_2 + \eta_3 M_3|^2
\]  
(2.14)

with the isospin factors \( \eta_{1,2,3} \) given in eq.(2.13). The main dynamical assumption in this relation is that the two-pion photoproduction amplitude in the threshold region can be approximated by the amplitude at threshold. Expressing \( \vec{k}^2 \) in terms of \( s \) and supplementing \( |\mathcal{M}_{fi}|^2 \) by the photon flux factor \( m^2/p_1 \cdot k = 2m^2/(s - m^2) \), we find for the unpolarized total cross section

\[
\sigma_{\gamma N \to \pi\pi N}^{\text{tot}}(s) = \frac{m^2}{2s} (s - m^2) \Gamma_3(s) |\eta_1 M_1 + \eta_2 M_2 + \eta_3 M_3|^2 S. 
\]  
(2.15)
Here, $\Gamma_3(s)$ is the integrated three-body phase space and $S$ a Bose symmetry factor, $S = 1/2$ for the $\pi^0\pi^0$ final state and $S = 1$ otherwise. The integrated three-body phase space can be expressed as

$$\Gamma_3(s) = \frac{1}{32\pi^3} \int_0^{T_1} dT \sqrt{T(T + 2m)(T_1 - T)(T_2 - T)} \frac{1}{T_3 - T},$$

where $M_{\pi 1} \text{ and } M_{\pi 2}$ stand for the masses of the final state pions and one has the inequality $0 \leq T_1 \leq T_{2,3}$. For equal pion masses an excellent approximation to eq.(2.16) is given by

$$\Gamma_3(s) \approx \frac{M_{\pi} m^{5/2}}{64\pi^2(m + 2M_{\pi})^{7/2}} [E_\gamma - 2M_{\pi}(1 + \mu)]^2. \quad (2.17)$$

Of course, an analogous approximation can be derived for unequal pion masses. Consequently, the unpolarized total cross section can be approximated within a few percent by the handy formula

$$\sigma_{\text{tot}}^{\gamma N \to \pi^0 \pi^0 N}(E_\gamma) \approx \frac{M_{\pi}^2 (1 + \mu)}{32\pi^2(1 + 2\mu)^{11/2}} |\eta_1 M_1 + \eta_2 M_2 + \eta_3 M_3|^2 S (E_\gamma - E_{\gamma}^{\text{thr}})^2. \quad (2.18)$$

For electroproduction, the prefactor in eq.(2.18) has to be modified slightly to account for the virtual photon flux normalization and then it gives the transverse total electroproduction cross section. For the reaction $\gamma p \to \pi^+ \pi^- p$, we will also discuss results where the amplitude away from threshold is taken into account. In that case the total cross section has to be worked out numerically. A most efficient integration over the final state three-body phase space has the form

$$\sigma_{\text{tot}}^{\gamma p \to \pi^+ \pi^- p}(E_\gamma) = \frac{m}{64\pi^4 E_\gamma} \int \int_{z^2 < 1} d\omega_1 d\omega_2 \int_{-1}^{+1} dx \int_0^\pi d\varphi |\mathcal{M}_{f_1}|^2 (k_0, \omega_1, \omega_2, q_1, q_2, x, y, z)$$

where we have chosen as the four independent variables which characterize the final state configuration the pion energies $\omega_1$ and $\omega_2$, $x$ the cosine of the angle between $q_1$ and $k$ and the azimuthal angle $\varphi$ introduced in eq.(2.11). The cosine of the angle between $q_1$ and $q_2$ is already fixed by energy conservation to the value

$$z = z(\omega_1, \omega_2, s) = \frac{\omega_1 \omega_2 - \sqrt{s}(\omega_1 + \omega_2) + M_{\pi}^2 + \frac{1}{2}(s - m^2)}{\sqrt{(\omega_1^2 - M_{\pi}^2)(\omega_2^2 - M_{\pi}^2)}}. \quad (2.20)$$
Evidently, the quantity $z$ has to lie between $-1$ and $+1$ and this condition determines the allowed region in the $\omega_1\omega_2$ energy plane. We have also considered the first above threshold correction for $\gamma p \rightarrow \pi^0\pi^0 p$ as detailed in appendix A. This completes the necessary formalism.

II.5. REMARKS ON ISOSPIN-BREAKING

In the calculations to be performed later on, it is legitimate to work with one nucleon and one pion mass, which we will choose to be $m = m_p = 938.27$ MeV and $M_{\pi} = M_{\pi\pm} = 139.57$ MeV. However, the mass difference $M_{\pi\pm} - M_{\pi^0} = 4.6$ MeV in reality leaves a $11.9$ MeV gap between the production threshold of two neutral versus two charged pions. While we are not in position of performing a calculation including all possible isospin-breaking effects, a minimal procedure to account for the mass difference of the physical particles is to put in these by hand in the pertinent kinematics, such that the thresholds open indeed at the correct energy value. To be specific, for the $\pi^+\pi^- p$ final state the threshold photon energy is

$$E_{\gamma_{\text{thr}}}^{\text{thr}}(\pi^+\pi^- p) = 320.66 \text{ MeV}$$

(2.21)

whereas for $\pi^0\pi^0 p$ it is

$$E_{\gamma_{\text{thr}}}^{\text{thr}}(\pi^0\pi^0 p) = 308.77 \text{ MeV}.$$  

(2.22)

Also, in the pertinent three-body phase space integrals we will differentiate between neutral and charged pion mass when we present results incorporating the correct opening of the thresholds.

III. EFFECTIVE LAGRANGIAN

In this section, we will briefly discuss the chiral effective Lagrangian underlying our calculation. We will also present an extension to incorporate the $\Delta(1232)$ resonance. This is mandated by the closeness of the two-pion production threshold and the location of the $\Delta$-resonance, $m_\Delta - m - 2M_{\pi^\pm} = 14.6$ MeV. Concerning the chiral interactions of the $\pi N$ system our discussion will be brief. Many additional details are spelled out in refs.[6,12].

III.1. CHIRAL PION-NUCLEON LAGRANGIAN

To explore in a systematic fashion the consequences of spontaneous and explicit chiral symmetry breaking of QCD, we make use of baryon chiral perturbation theory (in the heavy mass formulation) [9] (HBCHPT). The nucleons are considered as extremely
heavy. This allows to decompose the nucleon Dirac spinor into "large" \((H)\) and "small" \((h)\) components

\[
\Psi(x) = e^{-imv \cdot x} \{H(x) + h(x)\}
\]

with \(v_\mu\) the nucleon four-velocity, \(v^2 = 1\), and the velocity eigenfields are defined via \(vH = H\) and \(vh = -h\). Eliminating the "small" component field \(h\) (which generates \(1/m\) corrections), the leading order chiral \(\pi N\) Lagrangian reads

\[
\mathcal{L}^{(1)}_{\pi N} = \bar{H} (iv \cdot D + g_A S \cdot u) H
\]

Here the pions are collected in a SU(2) matrix-valued field \(U(x)\)

\[
U(x) = \frac{1}{F} \left[ \sqrt{F^2 - \bar{\pi}(x)^2} + i \tau \cdot \bar{\pi}(x) \right]
\]

with \(F\) the pion decay constant in the chiral limit and the so-called \(\sigma\)-model gauge has been chosen which is of particular convenience for our calculations in the nucleon sector. In eq.(3.2) \(D_\mu = \partial_\mu + \Gamma_\mu\) denotes the nucleon chiral covariant derivative, \(S_\mu\) is a covariant generalization of the Pauli spin vector, \(g_A \approx 1.26\) the nucleon axial vector coupling constant (formally the one in the chiral limit) and

\[
u_\mu = i u_\mu \nabla_\mu U u_\mu^\dagger
\]

with \(u = \sqrt{U}\) and \(\nabla_\mu\) the covariant derivative acting on the pion fields. To leading order, \(\mathcal{O}(q)\) one has to calculate tree diagrams from

\[
\mathcal{L}^{(1)}_{\pi N} + \frac{F^2}{4} \text{Tr} \left\{ \nabla^\mu U \nabla_\mu U^\dagger + \chi_+ \right\}
\]

where the second term is the lowest order mesonic chiral effective Lagrangian, the non-linear \(\sigma\)-model coupled to external sources. At next-to-leading order \(\mathcal{O}(q^2)\) one has to consider tree graphs from

\[
\mathcal{L}^{(2)}_{\pi N} = \bar{H} \left\{ -\frac{1}{2m} D \cdot D + \frac{1}{2m} (v \cdot D)^2 + c_1 \text{Tr} \chi_+ + \left( c_2 - \frac{g_A^2}{8m} \right) v \cdot u v \cdot u + c_3 u \cdot u \\
+ c_4 [S^\mu, S^\nu] u_\mu u_\nu - \frac{ig_A}{2m} [S \cdot D, v \cdot u] - \frac{i}{4m} [S^\mu, S^\nu] (1 + \hat{k}_v) f_{\mu\nu}^+ + \frac{\hat{k}_s - \hat{k}_k}{2} \text{Tr} f_{\mu\nu}^+ \right\} H
\]

Some of the terms in eq.(3.6) are the \(1/m\) corrections from the original Dirac Lagrangian. But there are new terms proportional to the isoscalar and isovector anomalous magnetic moments \(\hat{k}_{s,v}\) in the chiral limit or the low energy constants \(c_1, c_2, c_3, c_4\). The latter are related to the \(\pi N\) \(\sigma\)-term and \(\pi N\) scattering lengths. In order to restore unitarity in
a perturbative fashion, one has to include (pion) loop diagrams. In HBCHPT, there exists a strict one-to-one correspondence between the expansion of any observable in small external momenta and quark masses and the expansion in the number of loops. In what follows we will work within the one-loop approximation corresponding to chiral power $O(q^3)$.

To obtain all contributions at order $q^3$ one has to supplement the chiral effective Lagrangian by additional terms $\mathcal{L}^{(4)}_{\pi\pi} + \mathcal{L}^{(3)}_{\pi N}$ which will also serve to cancel the divergences of certain loop diagrams. In our case of two-pion electroproduction at threshold, however, all individual contributions at order $q^3$ are finite.

The corrections at order $q^3$ to the two-pion threshold photo/electroproduction amplitudes can be grouped into three classes, (i) the one-loop diagrams with insertion from the leading order Lagrangian eq.(3.5), (ii) kinematical corrections with inverse powers of $m$ arising from the (chiral) expansion of the relativistic nucleon pole graphs and (iii) contact graphs from $\mathcal{L}^{(3)}_{\pi N}$ with a priori unknown coefficients related to dynamics at higher mass scales and tree graphs with one vertex from $\mathcal{L}^{(4)}_{\pi\pi}$ (e.g. the Wess-Zumino term which takes care about the anomalous Ward identities of QCD). In estimating the a priori unknown coefficients of the contact graphs we will make use of the resonance saturation principle and relate them to the $\Delta(1232)$ exchange contributions. This is motivated by the closeness of the $\Delta(1232)$ resonance and its relatively large couplings to the $\gamma\pi N$ system. The effects of heavier baryon resonances are neglected and vector meson exchange is suppressed by higher powers of $q$ at threshold. Therefore we have now to give the chiral $\pi N\Delta$ Lagrangian to lowest order,

$$\mathcal{L}^{(1)}_{\pi N\Delta} = \frac{3g_A}{2\sqrt{2}} \overline{\Delta}^a_{\mu} [g^{\mu\nu} - (Z + \frac{1}{2})\gamma^\mu \gamma^\nu] u^a_{\nu} \Psi + \text{h.c.}$$

(3.7)

with $u^a_{\nu} = \frac{1}{2} \text{Tr}(\tau^a u_{\nu})$ and we used the well known SU(4) relation $g_{\pi N\Delta} = 3g_{\pi N}/\sqrt{2}$ together with the Goldberger-Treiman relation. The relativistic formulation using a Rarita-Schwinger spinor for the $\Delta$ introduces an off-shell parameter $Z$ into the $\pi N\Delta$ and $\gamma\pi N\Delta$ vertices which would be lost in the nonrelativistic isobar formulation. It allows us to keep track of the uncertainties in the $\Delta$ resonance contribution to certain observables. Furthermore one has to consider the $\gamma N \rightarrow \Delta$ transition vertex which is proportional to the electromagnetic field strength tensor. Since the couplings of this vertex do not enter our final result for the two-pion production amplitudes at threshold calculated up to order $q^3$, we do not explicitely give it here. The interested reader can find a discussion on this topic in ref.[14]. We have now assembled all tools to perform a complete calculation up to and including $O(q^3)$ of two-pion photo/electroproduction.

* For more accurate calculations, it seems to be necessary to include all terms of order $q^4$ as demonstrated for a specific example in ref.[13].
IV. CHIRAL EXPANSION OF THE THRESHOLD AMPLITUDES

In this section, we will be concerned with the chiral expansion of the threshold amplitudes $M_{1,2,3}$ and $N_{1,2,3}$. In each case we will give two complete chiral powers, the leading and next-to-leading term. Before giving the explicit expressions, a few general remarks are in order. Directly at threshold the calculation simplifies enormously. In the center-of-mass frame the final state nucleon and the two pions are at rest, i.e. $\vec{q}_1 = \vec{q}_2 = 0$. With $v^\mu = (1, 0, 0, 0)$ the spin-operator is the usual one $S^\mu = (0, \vec{\sigma}/2)$ and the two pion momenta are equal, $q_1^\mu = q_2^\mu = (M_\pi, 0)$. We therefore have in the Coulomb gauge $\epsilon_0 = 0$, $\epsilon \cdot v = \epsilon \cdot q_i = 0$, $S \cdot q_i = 0$, $v \cdot (q_1 - q_2) = 0$. (4.1)

In photoproduction, we have in addition $\epsilon \cdot k = 0$. Here, the advantage of the heavy mass formulation (HBCHPT) clearly shows since the conditions eq.(4.1) make most diagrams vanishing at threshold and only very few diagrams are left to contribute. In words eq.(4.1) says, that the photon nucleon vertex, the (out-going) pion nucleon vertex and the Weinberg vertex for the two out-going pions all vanish at threshold. In photoproduction even the photon coupling to an out-going pion becomes zero. Furthermore, the $\gamma \pi \pi NN$ contact term in $L^{(1)}_{\pi N}$ is vanishing since it is proportional to $\epsilon \cdot v$. Such selection rules at threshold are extremely useful to fish out the non-vanishing one-loop graphs from a huge number of diagrams. See also ref.[12] where such methods were applied to nucleon Compton scattering and $\pi^0$ photoproduction at threshold demonstrating their power.

IV.1. THE TRANSVERSE THRESHOLD AMPLITUDES

Let us first discuss the transverse amplitudes $M_{1,2,3}$. In this case we can still exploit the condition $\epsilon \cdot k = 0$ but we will not restrict $k^2$ to be zero in order to get also the result for electroproduction. The various contributions to $M_{1,2,3}$ are ordered by their chiral power. To leading order $O(q)$ we have the tree graphs from $L^{(1)}_{\pi N}$ and one finds quite easily that due to the abovementioned selection rules all these graphs are exactly zero at threshold. The leading nonzero contributions therefore comes from tree graphs with one insertion from $L^{(2)}_{\pi N}$. The two non-vanishing diagrams are shown in fig.2. It is interesting to note that the first non-vanishing term for two-pion photoproduction at threshold start at order $q^2$ in all channels. This is very different from single pion photoproduction where the charged channels are strongly enhanced.

Next, we have to study all contributions at chiral power $O(q^3)$. Among these are the one-loop contributions. Again the selection rules eq.(4.1) and $\epsilon \cdot k = 0$ reduce drastically the number of graphs which are nonzero at threshold to four. These are depicted in fig.3. Finally we have to work out all polynomial contributions at order $q^3$. Part of
these (the $1/m^2$ corrections) can be most easily obtained if we calculate the nucleon pole graphs relativistically and then expand in powers of $M_\pi$ and $k^2$. Further terms are given by tree graphs involving two vertices from $\mathcal{L}^{(2)}_{\pi N}$ where at least one carries a constant $c_1, c_2, c_3, c_4$ or $\vec{K}_a, \vec{K}_v$. The two possible graphs are shown in fig.4a, their sum, however, vanishes since the nucleon propagators in both have opposite sign. Then we have to study tree graphs with one vertex from $\mathcal{L}^{(4)}_{\pi \pi}$. Most of these insertions just lead to a multiplicative renormalization of the $\mathcal{O}(q)$ graphs, which are, however, all equal to zero. $\mathcal{L}^{(4)}_{\pi \pi}$ contains the Wess-Zumino term incorporating the anomalous (natural parity violating) vertex $\gamma \to 3\pi$. Nevertheless, the diagram shown in fig.4b vanishes at threshold, simply because it is proportional to $\epsilon_{\mu \nu \alpha \beta} q_1^\alpha q_2^\beta = 0$ with $q_1 = q_2$. Finally, in order to complete the list of all possible contributions at $\mathcal{O}(q^3)$ we have to consider the (genuine) contact terms from $\mathcal{L}^{(3)}_{\pi N}$. These carry new low energy constants which are not known a priori and have to be determined from phenomenology. We will invoke here the resonance saturation principle (which has been tested in the meson sector to work very well) and estimate these constants from the $\Delta(1232)$-resonance contribution to two-pion photo/electroproduction at threshold. One expects sizeable effects from the $\Delta(1232)$ since first it is quite close to threshold and second its couplings to the $\gamma\pi N$ system are very large (about twice the nucleon couplings). On first sight the distance of only 14.6 MeV of the $\Delta(1232)$ from threshold seems to give rise to overwhelming contributions since one naively expects that the very small denominator

$$\frac{1}{m_\Delta^2 - s_{\text{thr}}} = \frac{1}{m_\Delta - m - 2M_\pi} \frac{1}{m_\Delta + m + 2M_\pi}$$

(4.2)

enters the result. We have worked out all possible graphs with a single or double $\Delta$ excitation (cf. fig.5) and found that at threshold the small and dangerous denominator $m_\Delta - m - 2M_\pi$ always gets cancelled by exactly the same term in the numerator. This is a very important feature since otherwise the $M_\pi$ expansion would be in serious trouble. $M_\pi/(m_\Delta - m - 2M_\pi)$ formally counts as small, $\mathcal{O}(q)$, but numerically it is of course very large. After this detailed discussion of the chiral expansion of the transverse threshold

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amplitude, let us now give the final result,

\[ M_1 = \frac{e g_A^2 M_\pi}{4 m^2 F_\pi^2} + O(q^2), \]

\[ M_2 = \frac{e}{4 m F_\pi^2} (2g_A^2 - 1 - \kappa_v) + \frac{e M_\pi}{4 m^2 F_\pi^2} (g_A^2 - \kappa_v) - \frac{eg_A^2 M_\pi}{8 m m^2 F_\pi^2} B_{\Delta} \]

\[ + \frac{e g_A^2 M_\pi}{64\pi F_\pi^4} \left\{ \frac{8 + 4r}{\sqrt{1+r}} \arctan \sqrt{1+r} - \frac{r}{1+r} - \frac{1 + r + r^2}{(1+r)^{3/2}} \frac{\pi}{2} + \arctan \frac{r}{\sqrt{1+r}} \right\} + \frac{e g_A^2 M_\pi}{16 m m^2 F_\pi^2} B_{\Delta} \]

\[ + i \left\{ \frac{\sqrt{3}(2 + r)}{1 + r} - \frac{1 + r + r^2}{(1+r)^{3/2}} \ln \frac{2 + r + \sqrt{3(1+r)}}{\sqrt{1+r + r^2}} \right\} + O(q^2), \]

\[ M_3 = \frac{e}{8 m F_\pi^2} (1 + \kappa_v - 2g_A^2) + \frac{e M_\pi \kappa_v}{8 m^2 F_\pi^2} + \frac{e g_A^2 M_\pi}{16 m m^2 F_\pi^2} B_{\Delta} \]

\[ + \frac{e g_A^2 M_\pi}{256\pi F_\pi^4} \left\{ 6 - \frac{4 + 2r}{\sqrt{1+r}} \arctan \sqrt{1+r} - \frac{r}{1+r} - \frac{1 + r + r^2}{(1+r)^{3/2}} \frac{\pi}{2} + \arctan \frac{r}{\sqrt{1+r}} \right\} \]

\[ + i \left\{ \frac{\sqrt{3}(2 + r)}{1 + r} - \frac{1 + r + r^2}{(1+r)^{3/2}} \ln \frac{2 + r + \sqrt{3(1+r)}}{\sqrt{1+r + r^2}} \right\} + O(q^2) \]

(4.3)

with the ratio \( r = -k^2/4M_\pi^2 \) and

\[ B_{\Delta} = \frac{2m_\Delta^2 + m_\Delta m - m^2}{m_\Delta - m} + 4Z[m_\Delta (1 + 2Z) + m(1 + Z)] \]

(4.4)

which involves the off-shell parameter \( Z \) of the \( \pi N\Delta \) vertex. In fact, taking the allowed range of \( Z \) given in ref.[14], \textit{i.e.} \(-0.8 < Z < 0.3\) we find a weak \( Z \)-dependence, \textit{i.e.} \( 9.9 \text{ GeV} < B_{\Delta} < 15.1 \text{ GeV} \). Furthermore, from the isospin factors of eq.(2.13) we see that to order \( M_\pi \) the \( \Delta \) contributions are absent in the \( \pi^0\pi^0 \) channels.

It is interesting to note that \( M_1 \) is zero to \( O(q^0) \) and \( O(q) \). At next order \( q^2 \) it only receives a contribution from the relativistic \( N \)-pole graphs, but no loop or counter term contribution. The result eq.(4.3) has been written completely in terms of physical parameters, not the chiral limit values which enter the effective Lagrangian. For \( F_\pi, M_\pi, g_A \) and \( m \) these differences only would show up at the next order \( O(q^4) \). The isovector anomalous magnetic moment, however, has been renormalized and includes a non-analytic piece \( \sim \sqrt{m} \) [12],

\[ \kappa_v = \hat{\kappa}_v - \frac{g_A^2 m M_\pi}{4\pi F_\pi^2}. \]

(4.5)

Note that the first terms in the expansion of \( M_{2,3} \) arising at order \( q^2 \) differ from the corresponding expressions given in ref.[10] by the term proportional to \( \kappa_v \). In that
reference, the Weinberg Lagrangian was used and the photon was coupled in via minimal substitution. Such a procedure can, however, not generate the photon nucleon coupling proportional to the anomalous magnetic moment. It is furthermore important to write down such an anomalous photon nucleon vertex in a manifestly chiral invariant fashion using in eq. (3.6) the quantity $f_{\mu\nu}^+$ and not just the photon field strength tensor. When expanding $f_{\mu\nu}^+$ in powers of the pion field the pertinent $\gamma\pi\pi NN$ vertex (proportional to $\tilde{\kappa}_v$) based on chiral symmetry is automatically generated.

Another point worth mentioning is that the transverse multipoles $M_{2,3}$ are $k^2$-dependent only through their loop contribution. This can be understood from the fact that the tree graphs have to be polynomial in both $M_\pi$ and $k^2$ and that a term linear in $k^2$ is already of higher order in the chiral expansion. It is also not possible to further expand the $r$-dependent functions since $r = -k^2/4M_\pi^2$ counts as order one and all terms have to be kept.

At first sight the multipoles $M_{2,3}$ seem to behave singular in the chiral limit $M_\pi = 0$ since then $r$ becomes infinite. This is, however, not the case and the chiral limit is perfectly smooth (as it should be) with

$$M_2^{\text{loop}} = -2M_3^{\text{loop}} = \frac{e\tilde{g}_A^2}{128F_\pi^4}\sqrt{-k^2} = \frac{e}{4\tilde{r}F^4}[\tilde{R}_v - \tilde{F}_2(k^2)]$$

(4.6)

where $\tilde{F}_2(k^2)$ is the nucleon isovector magnetic form factor in the chiral limit calculated up to order $q^3$ (see ref. [12]).

The loop contribution to the transverse multipoles of two pion production as given in eq. (4.3) have a nonzero imaginary part even at threshold. This comes from the rescattering type graphs. Due to unitarity the pertinent loop functions have a right hand cut starting at $s = (m + M_\pi)^2$ (the single pion production threshold) and these functions are here to be evaluated at $s = (m + 2M_\pi)^2$ (the two-pion production threshold). For photoproduction the complicated $k^2$-dependence of the loop terms disappears and we have the complex constants

$$M_2^{\text{loop}} = \frac{eg_A^2M_\pi}{64\pi F_\pi^4}\left\{\frac{3\pi}{2} + i[2\sqrt{3} - \ln(2 + \sqrt{3})]\right\}$$

$$M_3^{\text{loop}} = \frac{eg_A^2M_\pi}{256\pi F_\pi^4}\left\{6 - \frac{3\pi}{2} + i[2\sqrt{3} - \ln(2 + \sqrt{3})]\right\}.$$  

(4.7)

This completes the discussion of the transverse threshold multipoles up-to-and-including order $q^3$. 

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IV.3. LONGITUDINAL MULTipoles AT THRESHOLD

In the electroproduction case, we also have the longitudinal threshold amplitudes $N_{1,2,3}$. Since we can no more exploit the condition $\epsilon \cdot k = 0$ the photon coupling to an out-going pion line is non-vanishing and therefore we obtain a nonzero contribution already at leading order $\mathcal{O}(q)$ involving a pion propagator (cf. fig.6a). Adding up all terms which arise at order $q$ and $q^2$ we find the following results:

$$N_1 = \mathcal{O}(q),$$

$$N_2 = \frac{eM}{F^2}(1 + \mu) + \frac{e(2g^2 - 1)}{4mF^2} + \mathcal{O}(q),$$

$$N_3 = -\frac{1}{2}N_2 + \mathcal{O}(q).$$

(4.8)

It is interesting to note that none of the low energy constants $c_1, c_2, c_3, c_4$ and the anomalous magnetic moments which enter $\mathcal{L}_{\pi N}^{(2)}$ show up in the final result. The graph of fig.6b contains the higher derivative $\pi\pi NN$ vertex proportional to $c_1, c_2, c_3$ but the isospin factor $\delta^{ac}e^{3b} + (a \leftrightarrow b) = 0$ annihilates this diagram. With eq.(4.8) we have given two full chiral powers (leading and next-to-leading) for the expansion of the longitudinal threshold amplitudes $N_{1,2,3}$. We do not work out here the third chiral power, which comes from loops and polynomial counter terms. We will now return to photoproduction and present our results for observables like total cross sections.

V. RESULTS AND DISCUSSION

In this section, we will first present numerical results for the threshold multipoles and total cross sections in the isospin limit. For the experimental implications, we also consider the kinematical shifts for the various channels as discussed in section 2.5.

V.1. RESULTS IN THE ISOSPIN LIMIT

First, we must fix parameters. We work with $m = m_p, M = M_{\pi^\pm}, F = 93$ MeV, $g_{\pi N}^2/4\pi = 14.28$ and $e^2/4\pi = 1/137.036$. In fig.7, we show the invariant matrix–elements $|M_{ij}|$ for the $\gamma p$ initial state versus the off–shell parameter $Z$ defined in eq.(3.7) ($k^2 = 0$). As stressed after eq.(4.4), the $\Delta$ does not contribute to the $\pi^0\pi^0$ final state. For the allowed range of $Z$ taken from ref.[14], the $Z$–dependence is weak. The surprising result is the dominance of the $\pi^0\pi^0$ final state for the threshold multipoles chirally expanded to $\mathcal{O}(M_{\pi})$. In table 1, we give the $\mathcal{O}(1)$ and $\mathcal{O}(1) + \mathcal{O}(M_{\pi})$ contributions with and without the $\Delta$ term. To lowest order, the $\pi^0\pi^0$ final states are suppressed, as already
noted in ref. [10]. It is interesting to compare these results to the case of single pion photoproduction. In that case, the Kroll–Ruderman term dominates charged production and the loop effects are small. For $\gamma p \rightarrow \pi^0 p$, however, the loop corrections are quite large [3]. The difference to the two pion production case is that the total cross sections for photoproducing a charged pion are nevertheless much larger than the corresponding ones for neutral pions (in the threshold region). Notice also that for the $\pi^+\pi^-$ final state the $O(M_\pi)$ loop and kinematical corrections are largely cancelled by the ones from the $\Delta$. In case of electroproduction, the $k^2$–dependence of the $M_i$ ($i = 1, 2, 3$) is weak. For $k^2$ ranging from 0 to $-0.1$ GeV$^2$, the $|M_{ij}|$ (in GeV$^{-3}$) change from 11.2 to 9.4, 20.1 to 15.1 and 35.2 to 29.7 for $\gamma p \rightarrow \pi^+\pi^- p$, $\pi^+\pi^0 n$ and $\pi^0\pi^0 p$, respectively. The corresponding longitudinal multipoles $N_i$ ($i = 1, 2, 3$) are completely dominated by the pion pole (compare eq.(4.8)) and are not shown here.

In fig.8, we show the total cross sections for the first three channels given in (2.2) by approximating the amplitudes in the threshold region through their threshold values. For $E_\gamma = 330$ MeV, we find

\begin{align*}
\sigma_{\text{tot}}(\gamma p \rightarrow \pi^0\pi^0 p) &= 0.36 \text{ nb}, \\
\sigma_{\text{tot}}(\gamma p \rightarrow \pi^+\pi^0 n) &= 0.22 \text{ nb}, \\
\sigma_{\text{tot}}(\gamma p \rightarrow \pi^+\pi^- p) &= 0.08 \text{ nb}.
\end{align*}

These are very small cross sections, but we refer to section 5.2. for further discussion. In fig.9, we compare the lowest order cross section for the production of two charged pions with the first correction from $O(q)$, cf. appendix A. While the former goes like $(E_\gamma - E_\gamma^{\text{thr}})^2$, the latter approximatively rises as $(E_\gamma - E_\gamma^{\text{thr}})^3$. At $E_\gamma \approx 323$ MeV, these two contributions are of the same size. This narrows the window in which the $\pi^0\pi^0$ final state is dominant to the first few MeV above threshold. The first correction to this particular channel is rather small as discussed below (cf. appendix A). Let us now, however, review these results in the light of the different thresholds already mentioned a few times.

V.2. EXPERIMENTAL IMPLICATIONS

To connect to the experimental situation, we now consider the three–body phase space with the physical masses for the corresponding pions. This automatically takes care of the various threshold energies. In the loops we work, however, with one pion

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* These numbers differ from the ones reported in ref.[15] since in that reference the $\Delta$–contribution was omitted.
mass since that effect is small as discussed in appendix B. Chiefly, the loop functions are sensitive to \((M_{\pi^+} - M_{\pi^0})/M_{\pi^+} = 0.03\), i.e. a few percent effect. Fig.10 shows the calculations with the correct phase–space and using the threshold matrix–elements. For \(\gamma p \to \pi^+\pi^- p\), we also show the first correction above threshold from \(L_{\pi N}^{(1)}\). At \(E_\gamma = 320\) MeV, the total cross section for \(\pi^0\pi^0\) production is 0.5 nb whereas the competing \(\pi^0\pi^+ n\) final state has \(\sigma_{\text{tot}} = 0.07\) nb. Double neutral pion production reaches \(\sigma_{\text{tot}} = 1.0\) nb at \(E_\gamma = 324.3\) MeV in comparison to \(\sigma_{\text{tot}}(\gamma p \to \pi^0\pi^+ n) = 0.26\) nb and \(\sigma_{\text{tot}}(\gamma p \to \pi^+\pi^- p) < 0.1\) nb. This means that for the first 10...12 MeV above \(\pi^0\pi^0\) threshold, one has a fairly clean signal and much more neutrals than expected. Remember that to leading order in the chiral expansion, the production of two neutral pions is extremely suppressed. Of course, the above threshold correction for this channel, which comes from \(L_{\pi N}^{(2)}\) (and higher orders) should be calculated systematically. The first correction, which vanishes proportional to \(|\vec{q}_i|\) \((i = 1, 2)\) at threshold, has been calculated (see appendix A) and found to be very small. The corresponding cross section at \(E_\gamma = 320, 325\) and 330 MeV is \(\sigma_{\text{tot}}^{\text{first corr}} = 0.009, 0.026\) and 0.056 nb, i.e a few percent of the leading order result. It is, therefore, conceivable that the qualitative features described above will not change if even higher order corrections are taken into account.

VI. BRIEF SUMMARY AND OUTLOOK

In this paper, we have performed an exploratory study of two–pion photo– and electroproduction off the nucleon. We have focused on the threshold region and used heavy baryon chiral perturbation theory to calculate the first two powers in the chiral expansion of the transversal \((M_{1,2,3})\) and longitudinal \((N_{1,2,3})\) multipoles defined in eq.(2.12). The pertinent results (we focus on the photoproduction case) are:

(i) To leading order in the chiral expansion, only the multipoles \(M_2\) and \(M_3\) are non–vanishing, with \(M_2 = -2M_3\). Therefore, the production of two neutral pions is strictly suppressed.

(ii) At next order in the chiral expansion, one has to consider kinematical \((1/m)\) corrections, one–loop contributions and tree graphs with insertion of the \(\Delta(1232)\) (chirally expanded). The loop contributions make the \(\pi^0\pi^0\) final state dominant in contrast to the expectation from single pion production.

(iii) We have calculated total cross sections for the various channels eq.(2.2) under the assumption that the amplitudes in the threshold region can be approximated by the exact threshold amplitude. Taking into account the different pion masses in the respective phase spaces, these cross sections are shown in fig.10. There is a window
of about 10 MeV above $\pi^0\pi^0$ threshold in which one can detect much more neutrals than expected.

(iv) We have considered the first above threshold corrections for the processes $\gamma p \rightarrow \pi^+\pi^- p$ and $\gamma p \rightarrow \pi^0\pi^0 p$. While the former are sizeable even very close to threshold, the latter are rather small. We conclude that ultimately one should consider the full amplitude above threshold (including also more dynamics from the resonance region) to draw more quantitative conclusions. We hope to report on this in the not too far future.

APPENDIX A: LEADING ORDER AMPLITUDE ABOVE THRESHOLD

In section IV, we considered the two-pion photoproduction amplitude exactly at threshold and we have seen that the selection rules make the $\mathcal{O}(q)$ contribution exactly vanishing at threshold in all channels. Here we will now work out the amplitude for the process $\gamma p \rightarrow \pi^+\pi^- p$ (with $k^2 = 0$) away from threshold to leading order $\mathcal{O}(q)$. For the $\pi^0\pi^0$ channels to this order the amplitude above threshold is still equal to zero everywhere. The five diagrams shown in fig.11 (plus their crossed partners with the pion lines interchanged) give rise to the following amplitude, with the kinematics taken over from eq.(2.9),

$$T \cdot \epsilon = \frac{e}{2F^2} \left\{ -\bar{\epsilon} \cdot \vec{q}_1 \frac{\omega_1}{q_1 \cdot k} - \bar{\epsilon} \cdot \vec{q}_2 \frac{\omega_1}{q_2 \cdot k} + \frac{g^2_A}{\omega_2} \bar{\epsilon} \cdot \vec{q}_2 \bar{\sigma} \cdot \bar{\epsilon} + \frac{g^2_A}{\omega_1} \bar{\epsilon} \cdot \sigma \cdot \bar{q}_1 + \frac{g^2_A}{\omega_2 q_1 \cdot k} \bar{\sigma} \cdot \delta \cdot (k - \vec{q}_1) + \frac{g^2_A}{\omega_1 q_2 \cdot k} \bar{\sigma} \cdot \delta \cdot (k - \vec{q}_2) \right\} \tag{A.1}$$

To calculate the unpolarized total cross section we have to average over initial spins and sum over final spins, i.e. we must compute the quantity

$$\frac{1}{2} \text{Tr}[T \cdot \epsilon (T \cdot \epsilon)^\dagger] \tag{A.2}$$

where the trace is taken over $2 \times 2$ spin-matrices. From (A.2) we get a quadratic form in the polarization vector $\bar{\epsilon}$ which we have to average over the two photon polarizations perpendicular to $k$. The pertinent polarization averages are done via the substitutions

$$(\bar{\epsilon} \cdot \vec{q}_1)^2 \rightarrow \frac{1}{2} q_1^2 (1 - x^2),$$

$$(\bar{\epsilon} \cdot \vec{q}_2)^2 \rightarrow \frac{1}{2} q_2^2 (1 - y^2),$$

$$\bar{\epsilon} \cdot \vec{q}_1 \bar{\epsilon} \cdot \vec{q}_2 \rightarrow \frac{1}{2} q_1 q_2 (z - xy),$$

$$(\bar{\epsilon})^2 \rightarrow 1.$$
with \(x, y, z\) the cosines of the inclined angles. Therefore, the invariant matrix element that enters the cross section integral eq.(2.19) takes the form

\[
|M_{fi}|^2 = \frac{e^2}{8 F_{\pi}^4} \left\{ \frac{q_1^2 \omega_2}{(q_1 \cdot k)^2} (1 - x^2) + 2 \frac{q_1 q_2 \omega_1 \omega_2}{q_1 \cdot k \cdot q_2 \cdot k} (z - xy) + \frac{q_2^2 \omega_1}{(q_2 \cdot k)^2} (1 - y^2) \right. \\
- 2 g_A^2 \left[ q_1^2 (1 - x^2) \omega_2 \left( \frac{1}{\omega_1} + \frac{q_2 (k_0 y - q_1 z)}{\omega_2 q_1 \cdot k} \right) + q_2^2 (1 - y^2) \omega_1 \left( \frac{1}{\omega_2} + \frac{q_1 (k_0 x - q_2 z)}{\omega_1 q_2 \cdot k} \right) \right] \\
+ q_1 q_2 (z - xy) \left( \frac{\omega_2}{q_1 \cdot k} + \frac{q_1 (k_0 x - q_2 z)}{\omega_1 q_2 \cdot k} \right) + \frac{\omega_1}{q_2 \cdot k} \left( \frac{1}{\omega_1} + \frac{q_2 (k_0 y - q_1 z)}{\omega_2 q_1 \cdot k} \right) \right] \\
+ g_A^4 \left[ \frac{2 q_1^2}{\omega_1^2} + \frac{2 q_2^2}{\omega_2^2} + \frac{q_1^2 q_2^2 (1 - x^2)}{\omega_2^2 (q_1 \cdot k)^2} (k_0^2 - 2 k_0 \omega_1 + q_1^2) + \frac{q_1^2 q_2^2 (1 - y^2)}{\omega_1^2 (q_2 \cdot k)^2} (k_0^2 - 2 k_0 \omega_2 + q_2^2) \right] \\
+ \frac{2 q_1 q_2^2}{\omega_1 \omega_2 q_2 \cdot k} (k_0 (x + z - 2 x y^2) + q_2 (x y - z)) + \frac{2 q_1 q_2}{\omega_1 \omega_2 q_1 \cdot k} (k_0 (y + z - 2 x^2 y) + q_1 (x y - z)) \\
- \frac{4 q_1 q_2}{\omega_1 \omega_2} x y + \frac{2 q_1^2 q_2^2 (z - xy)}{\omega_1 \omega_2 q_1 \cdot k q_2 \cdot k} (k_0^2 (2 x y - z) - k_0 q_1 y - k_0 q_2 x + q_1 q_2) \right\} \quad (A.4)
\]

The lengthy expression of the invariant matrix element from the rather simple amplitude (A.1) makes clear that calculations above threshold are rather voluminous. In fact, for the first 10 MeV above threshold the last two terms in eq.(A.1) could be safely neglected since they vanish with a higher power of \(|\vec{q}_i|\) \((i = 1, 2)\). In order to represent the most general amplitude above threshold more than a dozen of amplitude functions are needed to characterize a specific channel (see ref.[10]) and each of them depends on five independent variables (e.g. \(E_\gamma, \omega_1, \omega_2, x, y\)). In comparison to this enormous complexity single pion photoproduction is rather simple. The process is completely described in terms of four amplitude functions which depend on \(E_\gamma\) and e.g. the cms scattering angle.

Let us briefly turn to the first above threshold correction for \(\gamma p \rightarrow \pi^0 \pi^0 p\). The corresponding tree diagrams must include at least one insertion from \(L_{\pi N}^{(2)}\) as noted before. From these, the ones with the \(\gamma \pi NN\) vertex from \(L_{\pi N}^{(2)}\) go to zero as \(|\vec{q}_i|\) \((i = 1, 2)\) as one approaches threshold whereas the ones with a photon–nucleon vertex from \(L_{\pi N}^{(2)}\) and two pion–nucleon vertices from \(L_{\pi N}^{(1)}\) go as \(|\vec{q}_1| \cdot |\vec{q}_2|\), i.e. they vanish faster. So to leading order in small momenta, we consider only the first type of diagrams and find

\[
T \cdot \epsilon = \frac{i e g_A^2}{2 m F_{\pi}^2} \left[ \frac{\omega_1}{\omega_2} \vec{\sigma} \cdot (\vec{\epsilon} \times \vec{q}_2) + \frac{\omega_2}{\omega_1} \vec{\sigma} \cdot (\vec{\epsilon} \times \vec{q}_1) \right] \quad (A.5)
\]

and from this the total cross section follows as

\[
\sigma_{\gamma p \rightarrow \pi^0 \pi^0 p} (E_\gamma) = \frac{e^2 g_A^4}{384 \pi^3 E_\gamma m F_{\pi}^4} \int \int_{z^2 < 1} d\omega_1 \ d\omega_2 \left( \frac{\omega_1 q_2^2}{\omega_2^2} + \frac{\omega_2 q_1^2}{\omega_1^2} + 2 q_1 q_2 z \right) \quad . \quad (A.6)
\]
In this case, one can perform the $x$ and $\varphi$ integrations in $|M_{fi}|^2$ analytically. We have, however, also performed the four-dimensional integration numerically as a check and found good agreement with the results based on (A.6).

**APPENDIX B: LOOP FUNCTIONS**

The calculation of the one-loop graphs shown in fig.3 leads to the following loop integrals (in $d$ space time dimensions):

$$\frac{1}{i} \int \frac{d^dl}{(2\pi)^d} \frac{l_\mu l_\nu}{v \cdot l (M^2_\pi - l^2)(M_\pi^2 - (l + k)^2)} = g_{\mu\nu} \gamma_3(\omega, k^2) + \ldots,$$

$$\frac{1}{i} \int \frac{d^dl}{(2\pi)^d} \frac{l_\mu l_\nu}{v \cdot l (M^2_\pi - l^2)(M_\pi^2 - (l + k)^2)(M_\pi^2 - (l + k - 2q)^2)} = g_{\mu\nu} Q_3(\omega, M_\pi, k^2) + \ldots$$

\[(B.1)\]

with $\omega = v \cdot k$ and $q = M_\pi v$. All propagators are understood to have an additional infinitesimal negative imaginary part in the denominator, which makes the Wick-rotation to euclidean space time unique. The ellipsis stand for terms proportional to $v_\mu v_\nu, k_\mu k_\nu$ etc., which are not needed. Using standard Feynman parameter representations we obtain for the symmetric sums at $d = 4$ which enter the threshold amplitudes $M_{2,3}$

$$\gamma_3(\omega, k^2) + \gamma_3(-\omega, k^2) = \frac{1}{8\pi} \int_0^1 dx \sqrt{M^2_\pi - x^2 \omega^2 + x(x - 1)k^2},$$

$$Q_3(\omega, M_\pi, k^2) + Q_3(-\omega, -M_\pi, k^2) = \frac{1}{32\pi M_\pi(\omega - M_\pi)} \times \int_0^1 dx \left\{ \sqrt{M^2_\pi - x^2 \omega^2 + x(x - 1)k^2} - \sqrt{M^2_\pi + 4xM_\pi(\omega - M_\pi) - x^2 \omega^2 + x(x - 1)k^2} \right\}.$$ \[(B.2)\]

These are then to be evaluated at $\omega = 2M_\pi + i0$. Let us finally comment on isospin breaking effects. If we consider $\pi^0 \pi^0$ production the pions in the loop are all charged, therefore $M_\pi = M_{\pi^+}$ in eq.(B.2) and the functions are to be evaluated at $\omega = 2M_{\pi^0} + i0$. Since both real and imaginary part of the functions in eq.(B.2) are smooth around $\omega = 2M_{\pi^+}$ with finite first derivative the isospin breaking effect from the different pion masses is of order $(M_{\pi^+} - M_{\pi^0})/M_{\pi^+} \simeq 3\%$ and therefore very small. This is quite different to the reaction $\gamma p \rightarrow \pi^0 p$ where the respective loop functions vary strongly around the branch point $\omega = M_{\pi^+}$ (of square root type) and the effective isospin breaking parameter is $\sqrt{(M_{\pi^+} - M_{\pi^0})/M_{\pi^+}} \simeq 18\%$. 

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FIGURE CAPTIONS

Fig.1 The process $\gamma(k) + N(p_1) \rightarrow \pi^a(q_1) + \pi^b(q_2) + N(p_2)$. Here, $\gamma$, $\pi$ and $N$ denote the photon, the pion and the nucleon, in order. $a, b$ are isospin indices, $\epsilon$ is the polarization vector of the photon and the corresponding four–momenta are indicated.

Fig.2 Lowest order diagrams which lead to the leading order result for the multipoles $M_{1,2,3}$. The circle–cross denotes an insertion from $\mathcal{L}^{(2)}_{\pi N}$.

Fig.3 One–loop diagrams contributing to $M_{1,2,3}$ at order $M_\pi$.

Fig.4 (a) Tree diagrams with two insertions from $\mathcal{L}^{(2)}_{\pi N}$. (b) Tree diagram sensitive to the anomalous $\gamma \rightarrow 3\pi$ vertex. These vanish as described in the text.

Fig.5 Classification of the tree diagrams with $\Delta(1232)$ insertions. (a) and (b) lead to the lowest order $\mathcal{O}(M_\pi)$ contribution. (c) e.g. contains the dangerous energy denominator as described in the text. Crossed diagrams are not shown.

Fig.6 (a) Lowest order diagrams contributing to the longitudinal multipoles $N_{1,2,3}$. (b) The circle–cross denotes an insertion from $\mathcal{L}^{(2)}_{\pi N}$. The terms proportional to $c_i$ ($i = 1, 2, 3, 4$) and $\kappa v, s$ do not contribute as described in the text.

Fig.7 Invariant matrix–elements (in GeV$^{-3}$) for the $\gamma p$ initial state ($k^2 = 0$) as a function of the off–shell parameter $Z$ (in the isospin limit).

Fig.8 Total cross sections (in nb) for the $\gamma p$ initial state ($k^2 = 0$) as a function of the photon lab energy $E_\gamma$ (in the isospin limit).

Fig.9 Total cross sections (in nb) for the $\gamma p \rightarrow \pi^+\pi^- p$. The dashed line gives the result from the exact threshold amplitude and the solid one the first above threshold correction (see appendix A).

Fig.10 Total cross sections (in nb) for the $\gamma p$ initial state ($k^2 = 0$) with the correct three–body phase space. The various thresholds are indicated.

Fig.11 Tree diagrams with insertions exclusively from $\mathcal{L}^{(1)}_{\pi N}$ which lead to the first above threshold correction for $\gamma p \rightarrow \pi^+\pi^- p$, eq.(A.1).
| process          | $\mathcal{O}(1)$ | $\mathcal{O}(1) + \mathcal{O}(M_\pi)$ | $\mathcal{O}(1) + \mathcal{O}(M_\pi)$ |
|------------------|------------------|---------------------------------|---------------------------------|
|                  | without $\Delta$ | with $\Delta$                   | with $\Delta$                   |
| $\gamma p \to \pi^+\pi^- p$ | 11.01            | 16.10                           | 11.22                           |
| $\gamma p \to \pi^+\pi^0 n$   | 7.78             | 14.18                           | 20.06                           |
| $\gamma p \to \pi^0\pi^0 p$   | 0.00             | 35.25                           | 35.25                           |
| $\gamma n \to \pi^+\pi^- n$   | 11.01            | 12.84                           | 10.72                           |
| $\gamma n \to \pi^-\pi^0 p$   | 7.78             | 14.18                           | 20.06                           |
| $\gamma n \to \pi^0\pi^0 n$   | 0.00             | 30.96                           | 30.96                           |

Table 1: Invariant matrix-elements $|\eta_1 M_1 + \eta_2 M_2 + \eta_3 M_3|$ for the various two pion production channels in GeV$^{-3}$ ($k^2 = 0$).
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