MULTIPLE SIGN-CHANGING SOLUTIONS OF AN ELLIPTIC EIGENVALUE PROBLEM

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Abstract. We prove the existences of multiple sign-changing solutions for a semilinear elliptic eigenvalue problem with constraint by using variational methods under weaker conditions.

1. Introduction Multiple solutions for elliptic eigenvalue problem have been studied in some papers such as [7] [9], but they did not locate whether those solutions were sign-changing or not. [5] Theorem 4.4 got multiple sign-changing solutions under the assumption that the functional was of $C^2$. Both [6] and [8, Theorem 2] gave a result on existence of three solutions with one sign-changing for (4.1) with $f$ being not odd and obtained multiple sign-changing solutions under further condition. In this paper, we will obtain multiple sign-changing solutions and a pair of non sign-changing solutions of (4.1) under weaker conditions. Our result improves that of [5,6,7,8,9] and so on.

The paper is organized as follows. Section 2 contains some preliminary technical results. We construct a new pseudo-gradient flow of a $C^1$ functional, and prove the deformation lemma for their own sake. Section 3 is devoted to prove an abstract theory, which asserts multiplicity of sign-changing solutions. In fact, our theory is not confined to situations of cone-structures, we may consider the locations of critical points in relation to some given invariant sets of the flow. In section 4, we apply our theory to a nonlinear elliptic eigenvalue problem with constraint, establishing multiple sign-changing solutions under weaker conditions than those of correlated references.

2. Preliminaries

Lemma 2.1 Assume that $E$ is a Hilbert space, given $r > 0$, $S_r = \{u \in E : ||u||^2 = r^2\}$, $J \in C^1(S_r, R)$, $J'(u) = -a(u)u - A(u)$ for $u \in S_r$, where $a(u) = < -A(u), u >$, with $<, >$ the inner product of $E$, and $M'$ is a closed convex set of $E$, let $M = \ldots$
$M' \cap S_r$. Assume $A$ is bounded in $S_r$ and $A(M) \subset M$. Then there exists a pseudogradient vector field $W$ for $J$ on $S_r \setminus K$. Furthermore, if $J$ is even, $M = -M$, then $W$ is odd.

**Proof.** Let $E_0 = S_r \setminus K$, where $K = \{u \in S_r : J'(u) = 0\}$. For $u \in E_0$, define

$$N(u) = \{v \in E_0 : \|Av - Au\| < \frac{1}{16}\|J'(u)\|, \|J'(v)\| > \frac{1}{2}\|J'(u)\|\}.$$ 

Take

$$\tilde{N}_u = \begin{cases} N(u), & \text{if } u \in M, \\ N(u) \cap E_0 \setminus M, & \text{if } u \in E_0 \setminus M. \end{cases}$$

Now, $\{\tilde{N}_u : u \in E_0\}$ is an open covering of $E_0$, then $\tilde{N}_u$ possesses a locally finite refinement which will be denoted by $\{U_\lambda : \lambda \in \Lambda\}$. There exists a locally finite $C^{1,0}$ partition of unity $\{\beta_\lambda : \lambda \in \Lambda\}$ subordinated to $U_\lambda$, where $\Lambda$ is an index set. Set

$$B(u) = \sum_{\lambda \in \Lambda} \beta_\lambda(u)A(a_\lambda), \quad u \in E_0,$$

where, for any $\lambda \in \Lambda$, if $U_\lambda \cap M = \emptyset$, we choose $a_\lambda \in U_\lambda$ arbitrarily; if $U_\lambda \cap M \neq \emptyset$, fix a point $a_\lambda \in U_\lambda \cap M$.

Then $B : E_0 \to S_r$ is locally Lipschitz continuous as a consequence of the Lipschitz continuity of $\beta_\lambda$, and the locally finiteness of $U_\lambda$, so does $W(u) := -b(u)u - B(u)$, where $b(u) = \frac{\langle -B(u), u \rangle}{\|u\|^2}$. We claim $W$ is also Lipschitz continuous. Indeed,

$$|b(u_1) - b(u_2)| = \frac{1}{r^2} < -B(u_1), u_1 > - < -B(u_2), u_2 > | \leq \frac{1}{r} \|B(u_1) - B(u_2)\| + \frac{1}{r^2} \|B(u_2)\||u_1 - u_2|$$

$$\|b(u_1)u_1 - b(u_2)u_2\| \leq |b(u_1) - b(u_2)||u_1||u_1| + |b(u_2)||u_1 - u_2|$$

Since $A(u)$ is bounded in $S_r$, so are $B(u)$ and $b(u)$. Thus $W$ is also Lipschitz continuous.

For any $u \in E_0$, there are only a finite number of $U_\lambda$ denoted by $U_{\lambda_i}$ ($i = 1, 2, \cdots, n(u)$), which contain $u$. For each $\lambda_i$, there exists $U_{\lambda_i} \in E_0$ such that $U_{\lambda_i} \subset \tilde{N}_{u_{\lambda_i}}$. Then we have $a_{\lambda_i} \in U_{\lambda_i} \subset \tilde{N}_{u_{\lambda_i}}, u \in U_{\lambda_i} \subset \tilde{N}_{u_{\lambda_i}}$. Therefore, for $i = 1, 2, \cdots, n(u)$,

$$\|A(u) - A(a_{\lambda_i})\| \leq \|A(u) - A(u_{\lambda_i})\| + \|A(u_{\lambda_i}) - A(a_{\lambda_i})\| < \frac{1}{8}\|J'(u_{\lambda_i})\| < \frac{1}{4}\|J'(u)\|.$$ 

From this inequality, we easily get
\[ \|W(u)\| = \sum_{\lambda \in \Lambda} \beta_\lambda(u) - \frac{-A(a_\lambda), u >}{r^2} u - A(a_\lambda) \]
\[ = \sum_{\lambda \in \Lambda} \beta_\lambda(u) - \frac{-A(u), u >}{r^2} u - A(u) + \frac{-A(u), u >}{r^2} u \]
\[ - \frac{-A(a_\lambda), u >}{r^2} u + A(u) - A(a_\lambda) \]
\[ \leq \sum_{\lambda \in \Lambda} \beta_\lambda(u) \{ \| - a(u)u - A(u)\| + 2\|A(u) - A(a_\lambda)\| \} \]
\[ \leq \sum_{\lambda \in \Lambda} \beta_\lambda(u)(\|J'(u)\| + \frac{1}{2}\|J'(u)\|) \]
\[ \leq 2\|J'(u)\|. \quad (2.1) \]
\[ < J'(u), W(u) > \]
\[ = \sum_{\lambda \in \Lambda} \beta_\lambda(u) < J'(u), -\frac{-A(a_\lambda), u >}{r^2} u - A(a_\lambda) > \]
\[ = \sum_{\lambda \in \Lambda} \beta_\lambda(u) \{ < J'(u), -a(u)u - A(u) > + < J'(u), \frac{-A(u), u >}{r^2} u - \frac{-A(a_\lambda), u >}{r^2} u > \}
\[ + < J'(u), A(u) - A(a_\lambda) > \}
\[ \geq \sum_{\lambda \in \Lambda} \beta_\lambda(u)(\|J'(u)\|^2 - \frac{1}{2}\|J'(u)\|^2) \]
\[ = \frac{1}{2}\|J'(u)\|^2. \quad (2.2) \]

which imply W is a pseudo-gradient vector field for J on \( S_p \setminus K \).

Finally, if J is even and \( M = -M \), we enlarge the covering \{ \( U_\lambda : \lambda \in \Lambda \) \} by adding the sets \( \{-U_\lambda : \lambda \in \Lambda \} \). Then the vector \( V(u) = \frac{1}{2}(W(u) - W(-u)) \) is odd, locally Lipschitz continuous and satisfies (2.1)/(2.2).

**Remark 2.1** Since subspace \( W_0^{2, q}(\Omega) \) \((\Omega \subset \mathbb{R}^N \text{ is a bounded domain}) \) is dense in \( H^1_0(\Omega) \) for \( q \geq 2 \), the \( a_\lambda \) of \( B(u) = \sum_{\lambda \in \Lambda} \beta_\lambda(u)A(a_\lambda) \) can be chosen in \( W_0^{2, q}(\Omega) \)

for \( E = H^1_0(\Omega) \).

**Remark 2.2** In Lemma 2.1, we can denote \( \tilde{N}_u : u \in E_0 \) to be \( \{B(u, r_u)\} \) with \( r_u \in (0, 1) \). In applications, we may assume \( \{U_\lambda : \lambda \in \Lambda \} := \{B(u, r_u) : u \in E_0, r_u \in (0, 1)\} \).

We assert that \( B(M) \subset M \). Indeed, if \( u \in M \cap E_0 \) and \( \beta_\lambda(u) \neq 0 \) for some \( \lambda \in \Lambda \), then \( u \in U_\lambda \cap M \). This implies that \( U_\lambda \cap M \neq \emptyset \) and \( a_\lambda \in U_\lambda \cap M \). By the condition \( A(M) \subset M \), \( A(a_\lambda) \in M \). Then \( B(u) \in M \), since \( \sum_{\lambda \in \Lambda} \beta_\lambda(u) = 1 \) and \( M \) is convex.

Consider the pseudo-gradient flow \( \sigma \) on \( E \) associated to the vector field \( W := -b - B, \)

\[
\begin{align*}
&\frac{d}{dt} \sigma(t, u) = -W(\sigma(t, u)) = b(\sigma(t, u))\sigma(t, u) + B(\sigma(t, u)), \quad t \geq 0, \\
&\sigma(0, u) = u.
\end{align*}
\]

(2.3)
We see that \( \sigma \) is odd in \( u \), if \( W \) is odd in \( u \). For any \( u \in M \),

\[
    u - hW(u) = u + hh(u)u + hB(u) = (1 + hh(u))u + hB(u).
\]

Obviously, \( 1 + hh(u) > 0 \) with \( h \) small enough. Since \( B(u) \in M \), for small \( h \), we have

\[
    \text{dist}_E(u - hW(u), M) = 0.
\]

The Brezis-Martin Theorem (see [2, Theorem 1.6.3]) implies that \( \sigma(t, M) \subset M \) for

\( \sigma(t) = 0 \).

**Remark 2.3** Since \( W : S_r \setminus K \to S_r \) is local Lipschitz continuous, [1] Theorem 2.2 implies \( \sigma(t, u_0) \subset S_r \) with \( u_0 \in S_r \).

**Definition 2.1** With the flow \( \sigma \), we call a subset \( M \subset S_r \) an invariant set if \( \sigma(t, M) \subset M \) for \( t \geq 0 \).

Let \( E \) be a Hilbert space and \( X \subset E \) a Banach space densely embedded in \( E \). Assume that \( E \) has a closed convex cone \( P_E \) and that \( P := P_E \cap X \) has interior points in \( X \), i.e., \( P = \text{int}P \cup \partial P \), with \( \text{int}P \) the interior and \( \partial P \) the boundary of \( P \) in \( X \).

We use the following notation: \( K = K(J) = \{ x \in S_r : J'(x) = 0 \}, J^b = \{ x \in S_r : J(x) \leq b \}, K_c = \{ x \in K : J(x) = c \}, K([a, b]) = \{ x \in K : J(x) \in [a, b] \}, \) for \( a, b, c \in R \). Let \( \| \cdot \| \) and \( \| \cdot \|_X \) denote the norms in \( E \) and \( X \) respectively. We use \( d_E(\cdot, \cdot) \) and \( d_X(\cdot, \cdot) \) to denote the distance in \( E \) and \( X \).

The first assumption we make is:

(\( \Phi \) ) \( K(J) \subset \tilde{S}_r, J'(u) = -a(u)u - A(u) \) for \( u \in S_r, A : \tilde{S}_r \to \tilde{S}_r \) is continuous and bounded in \( S_r \).

Under condition (\( \Phi \)), we have \( \sigma(t, x) \in \tilde{S}_r \) for \( x \in \tilde{S}_r \), and \( \sigma \) is continuous in \( (t, x) \in R \times \tilde{S}_r \).

**Lemma 2.2**[5] Let \( \Psi : E \to R \) be a locally Lipschitz continuous function. Then \( \Psi_X : X \to R \) is also locally Lipschitz continuous in the topology of \( X \).

**Definition 2.2** Let \( M \subset S_r \) be an invariant set under \( \sigma \). We say \( M \) is an admissible invariant set for \( J \), if (a) \( M \) is the closure of an open set in \( \tilde{S}_r \), i.e., \( M = \text{int}M \cup \partial M \); (b) if \( u_n = \sigma(t_n, v) \) for some \( v \not\in M \) and \( u_n \to u \) in \( S_r \) as \( t_n \to \infty \) for some \( u \in K \), then \( u_n \to u \) in \( \tilde{S}_r \); (c) if \( u_n \in K \cap M \) such that \( u_n \to u \) in \( S_r \), then \( u_n \to u \) in \( \tilde{S}_r \); (d) for any \( u \in \partial M \setminus K, \sigma(t, u) \in \text{int}M \) for \( t > 0 \).

**Lemma 2.3** Assume \( J \in C^1(S_r, R) \) satisfies (\( \Phi \)) and (PS) condition. Let \( M \subset \tilde{S}_r \) be an admissible invariant set to the pseudo-gradient flow \( \sigma \) of \( J \). Define \( K^*_c = K_c \cap \text{int}M, K^*_c = K_c \cap \tilde{S}_r \) for some \( c \). Assume \( K_c \cap \partial M = \emptyset \), there exits \( \delta > 0 \) such that \( (K^*_c)_{4\delta} \cap (K^*_c)_{4\delta} = \emptyset \), where \( (K^*_c)_{4\delta} = \{ u \in S_r : d_E(u, K^*_c) < 4\delta \} \) for \( i = 1, 2 \). Then there is \( \epsilon_0 \) such that, for any \( \epsilon > 0 \) and any compact subset \( A \subset J^{ε−}\tilde{S}_r \cup M \), there is \( \eta \in C([0, 1] \times \tilde{S}_r, \tilde{S}_r) \) such that

(i) \( \eta(t, u) = u \), if \( t = 0 \) or \( u \not\in J^{-1}(c - 3\epsilon_0, c + 3\epsilon_0) \setminus (K^*_c)_3 \);

(ii) \( \eta(1, A \setminus (K^*_c)_3) \subset J^{-\epsilon} \cup M \), and \( \eta(1, A) \subset J^{-\epsilon} \cup M \) if \( K^*_c = \emptyset \);

(iii) \( \eta(t, \cdot) \) is a homeomorphism of \( \tilde{S}_r \) for \( t \in [0, 1] \);

(iv) \( J(\eta(\cdot, u)) \) is nonincreasing for any \( u \in \tilde{S}_r \);

(v) \( \eta(t, M) \subset M \) for any \( t \in [0, 1] \);
\((vi)\) \(\eta(t, \cdot)\) is odd, if \(J\) is even and \(M\) is symmetric about the origin.

**Proof.** Due to the \((PS)\) condition, we can choose \(\varepsilon_0 > 0\), such that

\[
\frac{\|J'(u)\|^2}{1 + 2\|J'(u)\|} \geq \frac{6\varepsilon_0}{\delta}, \quad \forall u \in J^{-1}([c - 3\varepsilon_0, c + 3\varepsilon_0]) \setminus (K_c)_\delta, \tag{2.4}
\]

and

\[
K([c - 3\varepsilon_0, c + 3\varepsilon_0]) \cap \partial M = \emptyset, \quad K([c - 3\varepsilon_0, c + 3\varepsilon_0]) \subset (K_c)_\delta. \tag{2.5}
\]

The followings are similar to the proof of Lemma 2.4 \([5]\), in which \(X\) is replaced by \(\tilde{S}_r\), and the gradient vector \(\nabla \phi(u)\) is replaced by the pseudo gradient vector \(W(u)\). We omit them.

**Remark 2.4** It is easy to check that both the union and intersection of a finite number of admissible invariant sets for \(J\) are still admissible invariant sets for \(J\).

3. An abstract theory and its proof

First, we need the notion of genus and its properties (see \([7]\)). Let

\[
\Sigma_X = \{ A \subset \tilde{S}_r : A \text{ compact}, \ A = -A \},
\]

\[
\Sigma_E = \{ A \subset S_r : A \text{ compact}, \ A = -A \}.
\]

For preciseness, we denote \(i_X(A)\) and \(i_E(A)\) to be the genus of \(A\) in \(\tilde{S}_r\) and \(S_r\) respectively.

**Proposition 3.1** Assume that \(A, B \in \Sigma_X, h \in C(\tilde{S}_r, \tilde{S}_r)\) is an odd homeomorphism. Then:

(i) \(A \subset B \Rightarrow i_X(A) \leq i_X(B)\);

(ii) \(i_X(A \cup B) \leq i_X(A) + i_X(B)\);

(iii) \(i_X(A) \leq i_X(h(A))\);

(iv) If \(A\) is compact, then \(i_X(A) < +\infty\) and there exists a neighborhood \(N \in \Sigma_X\), such that \(A \subset intN \subset N\) and \(i_X(N) = i_X(A)\).

This proposition is still true when we replace \(\Sigma_X\) by \(\Sigma_E\) with obvious modification. We shall use the notion

\[
S = \tilde{S}_r \setminus (P \cup -P).
\]

**Lemma 3.1[5]** If \(A \in \Sigma_X\) with \(2 \leq i_X(A) < \infty\), then \(A \cap S \neq \emptyset\).

**Lemma 3.2[5]** Let \(A \in \Sigma_E\), then \(A \cap X \in \Sigma_X\) and \(i_E(A) \geq i_X(A \cap X)\).

**Theorem 3.1** Let \(J \in C^1(S_r, R)\) and \((\Phi)\) hold. Assume that \(J\) is even, bounded from below and satisfies \((PS)_c\) condition for all \(c \neq 0\). Let \(P \cap S_r \subset X\) be an admissible invariant set for \(J\). Assume \(K(J) \cap \partial P = \emptyset\). Then \(J\) has infinitely many distinct pairs of critical points in \(\tilde{S}_r \setminus (P \cup -P)\).

**Proof.** That \(J\) possesses infinitely many pairs of critical points follow from e.g.\([7,\ Theorem 8.10]\). The question is whether they belong to \(P\) and \(-P\). We now rule this out.
For $m \geq 1$ define
$$\Sigma_m = \{ A \in \Sigma_X : i_X(A) \geq m \}.$$ and for $m \geq 2$ define
$$c_m = \inf_{A \in \Sigma_m} \sup_{A \cap S} J(x), \quad \text{with } S = \tilde{S}_r \setminus (P \cup -P).$$

Since $J$ is bounded from below, Lemma 3.1 implies
$$-\infty < c_2 \leq c_3 \leq \cdots \leq c_m.$$ We claim that for $m = 2, 3, \cdots$,
$$K_{c_m} \bigcap (\tilde{S}_r \setminus (P \cup -P)) \neq \emptyset. \quad (3.1)$$ Moreover (PS) condition implies $K_{c_m} \bigcap (\tilde{S}_r \setminus (P \cup -P))$ is a compact set. We also claim that if, for $l \geq 1$ and $k \geq 2$,
$$c := c_{l+1} = c_{l+2} = \cdots = c_{l+k},$$
then
$$i_E (K_c \bigcap (\tilde{S}_r \setminus (P \cup -P))) \geq k. \quad (3.2)$$

Please see the proof of Theorem 2.1 [5] for those of (3.1) (3.2) in detail. Then there are infinitely many pairs of critical points in $\tilde{S}_r \setminus (P \cup -P)$.

**Remark 3.2** In the theorem above, the positive and negative cones $P \cup (-P)$ can be changed to a symmetric admissible invariant set $M$, such that for any $A$ with $i_E(A) \in [2, \infty)$, we get $A \bigcap (\tilde{S}_r \setminus M) \neq \emptyset$. Then the theorem is still valid.

4. Application As application of our abstract theory, we consider a nonlinear eigenvalue problem: for $r > 0$ fixed,
\[
\begin{cases}
-\Delta u = \lambda f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega, \\
\int_{\Omega} |\nabla u|^2 dx = r^2.
\end{cases}
\] (4.1)

where, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. We want to find solution of the form $(\lambda, u)$. We say $(\lambda, u)$ a positive solution if $u$ is positive, a negative solution if $u$ negative, and a sign-changing solution if $u$ sign-changing.

We refer the following assumptions.

\begin{itemize}
\item[(f_1)] $f \in C(\overline{\Omega} \times R)$, and there are $c > 0, 0 < \alpha < 2^* - 1$ (for $N = 1, 2$, take $2^* = \infty$) such that
$$|f(x, u)| \leq c(1 + |u|^\alpha), \quad (x, u) \in \Omega \times R.$$
\item[(f_2)] $f(x, t)t \geq 0$ and for any $\delta > 0$, $f(x, t)t \neq 0$ in both $\Omega \times (-\delta, 0)$ and $\Omega \times (\delta, 0)$.
\item[(f_3)] $f(x, u)$ is odd in $u$.
\end{itemize}
Let $E = H^1_0(\Omega)$, $X = C^1_0(\Omega)$ which is a Banach space and embedded densely in $E$. Let $S_r = \{ u \in H^1_0(\Omega) : \int_{\Omega} \nabla u^2 \, dx = r^2 \}$, $\tilde{S}_r = S_r \cap C^1_0(\Omega)$, $F(x, u) = \int_0^u f(x,t) \, dt$ for any $u \in R$, $\Psi(u) = -\int_0^u F(x,u(x)) \, dx$ for any $u \in H^1_0(\Omega)$, $J = \Psi|_{\tilde{S}_r}$. Note that

$$J'(u) = \Psi'(u) - \frac{\langle \Psi'(u), u \rangle}{\|u\|^2} u := -a(u)u - A(u),$$

where $\langle . , . \rangle$ is the inner product in $E$ given by $\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v + u v \, dx$, $A(u) = (-\triangle)^{-1} f(x,u)$, $a(u) = \frac{\langle \Psi'(u), u \rangle}{\|u\|^2}$.

From (f2), we have $\langle \Psi'(u), u \rangle \neq 0$ for $u \in S_r$ and know that there is a one-to-one correspondence between critical point of $C^1$ functional $J$ and weak solution of (4.1). Thus, if $u$ is a critical point of $J$, then $(\lambda, u)$ is a weak solution of (4.1) with

$$\lambda = -\frac{\langle \Psi'(u), u \rangle}{\|u\|^2} > 0.$$

On $E$, let use define $P_E = \{ u \in E : u(x) \geq 0, \text{ a.e. in } \Omega \}$, which is a closed convex cone. Set $P = P_E \cap X$, then $P$ is a closed convex cone in $X$. Furthermore, $P = \text{int} P \cup \partial P$ under the topology of $X$, i.e., there exist interior points in $X$. We may define a partial order relation: $u, v \in X, u > v \iff u - v \in P \setminus \{0\}$.

**Proposition 4.1** Under (f1), (f2) is satisfied.

**Proof.** By the standard elliptic regular theory, $K(J) \subset C^{1,\alpha}_0(\Omega) \cap S_r \subset X \cap S_r := \tilde{S}_r$. A direct computation gives $J'(u) = -a(u)u - A_E(u)$, where $A_E(u) = (-\triangle)^{-1} f(x,u)$, $A_E : E \to E$ is continuous and compact, $A = A_E|_X : \tilde{S}_r \to \tilde{S}_r$ is continuous.

**Remark 4.1** Proposition 4.1 implies Lemma 2.1. From (2.3), it follows that

$$\sigma(t, u) = e^{\int_0^t b(\sigma(s,u)) \, ds} (u + \int_0^t e^{-\int_0^s b(\sigma(t,u)) \, dt} B(\sigma(s,u)) \, ds), \quad t \geq 0.$$  

(4.2)

$$B(u) = \sum_{\lambda \in \Lambda} \beta_\lambda(u) A(a_\lambda), \quad u \in S_r.$$  

Let $K = \{ \sigma(t, u) : t \in [0, T(u))] \}$, where $T(u)$ is the maximal existence interval of $\sigma$.

**Proposition 4.2** Let $v \in K$ then $\|v\|_E = r$, where $r > 0$ is given. If $\|a_\lambda\|_E \geq r + 3$, then $\beta_\lambda(v) = 0$ for all $v \in K$.

**Proof.** If not, there exists a $\lambda \in \Lambda$ with $\|a_\lambda\|_E \geq r + 3$ such that $\beta_\lambda(v) \neq 0$ for some $v \in K$. Then $\text{supp} \beta_\lambda(v) \subset B(a_\lambda, r_\lambda + 1) \subset E \setminus B(0, r + 1)$, while $v \in S_r$, a contradiction.

**Remark 4.2** From Proposition 4.2, it follows that

$$B(\sigma(t,u)) = \sum_{\|a_\lambda\|_E \leq r + 3} \beta_\lambda(\sigma(t,u)) A(a_\lambda), \quad u \in S_r.$$  

(4.3)

**Proposition 4.3** Let $X_0 = W_0^{2,q}(\Omega)$ for some $q \geq 2$ large such that the embedding from $X_0$ into $X = C^1_0(\Omega)$ is compact. It is well known that there exist a finite sequence of Banach spaces $X_1, X_2, \ldots, X_m$, such that $X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_m$ and the embeddings from $X_k$ into $X_m$ are compact for $k = 0, 1, \ldots, m - 1$, $A(X_k) \subset X_{k-1}$ with $1 \leq k \leq m$ maps bounded sets to bounded sets. Then $B(X_k) \subset X_{k-1}$ maps bounded sets to bounded sets.
Increasing. Denote the inverse function of \( \aleph \). Since \( \aleph \) is bounded in \( X_m \), (4.3) implies that \( B(v) \) for any \( v \in \aleph \) is bounded in \( X_{m-1} \), so does \( \aleph \). Next, we prove \( \aleph \) is bounded in \( X_{m-2} \).

Since the embedding from \( X_{m-1} \) into \( X_m \) is compact, \( \aleph \) is compact in \( X_m \) and for all \( v \in \aleph \), there exists a neighborhood \( U_v \) of \( v \), such that \( U_v \) intersects with finitely many sets of \( \{ U_{\lambda} : \lambda \in \Lambda \} \). Then there exist finitely many \( \beta_\lambda \in \{ \beta_\lambda : \lambda \in \Lambda \} \) such that \( \text{supp} \beta_\lambda \cap U_v \neq \emptyset \). Let \( \mathcal{R} = \{ U_v : v \in \aleph \} \) be the open covering of \( \aleph \) under the topology of \( X_m \), then there are finitely many coverings \( U_{v_1}, \ldots, U_{v_n} \) of \( \aleph \). Set \( \{ \beta_{\lambda_1}, \ldots, \beta_{\lambda_n} \} = \{ \beta_\lambda : \text{supp} \beta_\lambda \cap U_v \neq \emptyset \} \) for \( i = 1, 2, \ldots, n \), and \( \mathcal{S} = \bigcup_{i=1}^n \{ \beta_{\lambda_1}, \ldots, \beta_{\lambda_n} \} \), then for all \( v \in \aleph \) and for all \( \beta_\lambda \in \{ \beta_\lambda : \lambda \in \Lambda \} \) \( \mathcal{S} \), \( \beta_\lambda(v) = 0 \). Otherwise, we get \( v_0 \in \aleph \) and \( \beta_{\lambda_0} \in \{ \beta_\lambda : \lambda \in \Lambda \} \) \( \mathcal{S} \), such that \( \beta_{\lambda_0}(v_0) \neq 0 \).

Then \( \text{supp} \beta_{\lambda_0} \cap \aleph \neq \emptyset \), which implies \( \text{supp} \beta_{\lambda_0} \cap \bigcup_{i=1}^n U_{v_i} \neq \emptyset \). It follows that there exists a \( U_{v_i}, i \in \{1, \cdots, n\} \) such that \( \text{supp} \beta_{\lambda_0} \cap U_{v_i} \neq \emptyset \), which implies \( \beta_{\lambda_0} \in \mathcal{S} \), a contraction. This implies

\[
\| B(v) \|_{X_{m-2}} = \left\| \sum_{i=1}^n \sum_{j=1}^k \beta_{\lambda_j}^i(v) A(a_{\lambda_j}^i) \right\|_{X_{m-2}} \\
\leq \max_{1 \leq j \leq k, 1 \leq t \leq n} \| A(a_{\lambda_j}^i) \|_{X_{m-2}}
\]

i.e., \( \{ B(v) : v \in \aleph \} \) is bounded in \( X_{m-2} \), so does \( \aleph \).

Similarly, we see that \( \aleph \) is bounded in \( X_j \), for \( j = 0, 1, \cdots, m-3 \).

**Proposition 4.4** Under the assumption \( f \in C(\overline{\Omega} \times \mathbb{R}) \) and \((f_2)\), the flow \( \sigma \) defined by (2.3) has the following properties:

\[ \sigma(t,u_0) \in \pm \text{int} P \text{ for all } u_0 \in \pm P \cap \overline{S_r} \text{ and } t \geq 0. \]

**Proof.** The proof is similar to that of Lemma 1.6 [6].

Firstly, by the strong maximum principle, we know that if \( u \in P \setminus \{0\} \), then \( A(u) \in \text{int} P \), so does \( B(u) \).

Secondly, \( \forall u \in P \), since \( B(u) \in P \), we can choose \( \delta > 0 \) small such that if \( \delta > h > 0 \), we have

\[ u + h(b(u)u + B(u)) = (1 + hb(u))u + hB(u) \in P. \]

Breizis-Martin theorem implies \( \sigma(t,u) \in P \) for all \( t \in [0,T(u)) \). It follows that

\[ B(\sigma(t,u_0)) \in \text{int} P, \quad \text{with } u_0 \in P \cap \overline{S_r}, \quad t \geq 0. \quad (4.4) \]

Let \( w(t) = -\int_0^t b(\sigma(s,u_0)) ds \), we have \( w'(t) > 0, w(t) > 0 \) and \( w(t) \) is strictly increasing. Denote the inverse function of \( w(t) \) by \( w^{-1}(t) \). From (4.4), for \( u_0 \in P \cap \overline{S_r} \), we have

\[ \frac{1}{w'(t)} B(\sigma(t,u_0)) \in \text{int} P. \quad (4.5) \]
Let $C(t) = \frac{1}{w(t)} B(\sigma(t, u_0))$, $F_t = \{C(s) : 0 \leq s \leq t\}$. Note that $F_t$ is a compact set in $X$. It follows from (4.5) that $F_t \in \text{int} P$ and hence $\overline{C F_t} \in \text{int} P$, where $\overline{C F_t}$ is the closed convex hull of $F_t$ in $X$. Note that

$$
\frac{1}{e^{w(t)}} - 1 \int_0^t e^{w(s)} B(\sigma(s, u_0))ds = \frac{1}{e^{w(t)}} - 1 \int_0^{e^{w(t)}} B(\sigma(w^{-1}(\ln s), u_0)) \frac{ds}{w'(w^{-1}(\ln s))}
$$

$$
= \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} C(w^{-1}(\ln(1 + \frac{i}{m}(e^{w(t)} - 1)))).
$$

Therefore

$$
\frac{1}{e^{w(t)}} - 1 \int_0^t e^{w(s)} B(\sigma(s, u_0))ds \in \overline{C F_t} \in \text{int} P.
$$

Finally, it follows (4.2) that

$$
\sigma(t, u_0) = e^{-w(t)} u_0 + (1 - e^{-w(t)}) \frac{1}{e^{w(t)}} - 1 \int_0^t e^{w(s)} B(\sigma(s, u_0))ds \in \text{int} P.
$$

As the same proof above, we have

$$
\sigma(t, u_0) \in (-\text{int} P), \quad u_0 \in -P \bigcap \tilde{S}_r, \quad t \geq 0.
$$

Remark 4.3 Proposition 4.4 implies that $P \bigcap \tilde{S}_r$ and $-P \bigcap \tilde{S}_r$ are invariant sets of the negative gradient flow $\sigma$, and for any $u \in \partial(\pm P) \setminus K, \sigma(t, u_0) \in \pm \text{int} P$ for $t > 0$.

Proposition 4.5 Under $(f_1)(f_2)$, $M = P \bigcap \tilde{S}_r$ is an admissible invariant set for $J$.

Proof. [7, Theorem 8.10] implies that $J$ satisfies $(PS)$ condition. The requirement (a) is satisfied automatically. Remark 4.3 implies that $P \bigcap \tilde{S}_r$ is an invariant set of the negative gradient flow of $J$, and (d) is satisfied. To prove (b), let $u_n = \sigma(t_n, v)$ for some $v \in P \setminus \{P \cup (-P)\}$, and let $t_n \to \infty$ be a sequence such that $u_n \to u$ in $\tilde{S}_r$, for some $u \in K(J)$, then $\sigma(t, v)$ is bounded in $X_m$ for $t \geq 0$. Proposition 4.3 implies $\sigma(t, v)$ for $t \geq 0$ is bounded in $X$. Since the embedding from $X_0$ into $X$ is compact, $u_n = \sigma(t_n, v) \to u$ in $X$. For (c), if $u_n \in K(J) \bigcap (P \bigcup (-P)), u_n \to u$ in $S_r$, then $u_n$ is bounded in $E$. By condition $(f_1)$, the elliptic theory and the bootstrap argument, we get $u_n$ is bounded in $W^{2,q}_0(\Omega)$ for some $q$ large such that the embedding from $W^{2,q}_0(\Omega)$ into $X$ is compact, thus $u_n \to u$ in $\tilde{S}_r$.

Theorem 4.1 Under $(f_1, 2, 3)$, (4.1) has infinitely many sign-changing solutions.

Proof. Under these hypothesis, we know that $\Psi(u) < 0$ for any $u \in S_r$. As is proved in [7] that $J$ is bounded from below and satisfies $(PS)_c$ for all $c < 0$. By the maximum principle $K(J) \bigcap \partial P = \emptyset$. It follows proposition 4.5 that $P \bigcap \tilde{S}_r$ is an admissible invariant set for $J$. Then Theorem 3.1 implies the result.

Corollary 4.4 Under $(f_1, 2)$, (4.1) has at least a positive solution and a negative solution.

Proof. Please see Theorem 1 [8] for the proof in detail.
Remark 4.4 With much weaker conditions than Theorem 4.4 [5], we get more conclusions.

Remark 4.5 We refer the reader to the classical treatment of nonlinear eigenvalue problems in [4] [7], and recent work on sign-changing solution in [5] [6], on positive solutions for p-Laplacian systems problem in [3] and so on.

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