DERIVED SUPEREQUIVALENCE FOR SPIN SYMMETRIC GROUPS AND ODD $\mathfrak{sl}_2$-CATEGORIFICATIONS

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Abstract. We show that actions of the odd categorification of $\mathfrak{sl}_2$ induce derived superequivalence analogous to those introduced by Chuang and Rouquier. Using Kang, Kashiwara and Oh’s action of the odd 2-category on blocks of the cyclotomic affine Hecke-Clifford algebra, our equivalences imply that blocks related by a certain affine Weyl group action are derived equivalent. By recent results of Kleshchev and Livesey, we show this implies Broué’s abelian defect conjecture for the modular representations of the spin symmetric group.

1. Introduction

Let $\mathbb{k}$ be an algebraically closed field of odd characteristic $p > 0$. Let $\mathbb{Z}_2 = \{1, z\}$ and consider the double cover $\widehat{S}_n$ of the symmetric group $S_n$.

Then there is a canonical central element $z \in \widehat{S}_n$ and central idempotent $e_z := (1 - z)/2 \in \mathbb{k}\widehat{S}_n$ giving an ideal decomposition

$$\mathbb{k}\widehat{S}_n = \mathbb{k}\widehat{S}_n e_z \oplus \mathbb{k}\widehat{S}_n (1 - e_z),$$

where the block $\mathbb{k}\widehat{S}_n (1 - e_z) \cong \mathbb{k}S_n$ correspond to the usual group algebra of the symmetric group. The other block of $\mathbb{k}\widehat{S}_n$ given by $\mathbb{k}\widehat{S}_n e_z$ are called spin blocks of the symmetric group that we denote by $\mathbb{k}\widehat{S}_n$. These blocks control the spin (or projective) representation theory of the symmetric group $S_n$, see [52, 32] and references therein for an excellent introduction to the spin theory of $S_n$.

The superblocks $B^{(\rho,d)}$ of $\mathbb{k}\widehat{S}_n$ are labelled by pairs $(\rho, d)$, where $\rho$ is a $\bar{p}$-core partition (see section [33, Section 2.3] and [34]) and $d$ is a non-negative integer such that $|\rho| + dp = n$. It is known that the defect group $D^{(\rho,d)}$ of $B^{(\rho,d)}$ is abelian when $d < p$. Let $b^{(\rho,d)}$ be the Brauer correspondent of the block $D^{(\rho,d)}$. Assume then that $d < p$, then Broué’s defect conjecture for the spin symmetric group states that the block $B^{(\rho,d)}$ with abelian defect group $D^{(\rho,d)}$ is derived equivalent to its Brauer correspondent $b^{(\rho,d)}$.

As explained in [33], unlike the blocks $\mathbb{k}\widehat{S}_n (1 - e_z) \cong \mathbb{k}S_n$ that are relatively well understood, very little is known about spin blocks of symmetric groups, and in particular, Broué’s conjecture for them is wide open. In this article, we prove the spin defect conjecture by utilizing so-called odd categorical $\mathfrak{sl}_2$-actions.

1.1. Chuang-Rouquier equivalences. Since Chuang and Rouquier’s pioneering work [11], the construction of highly nontrivial derived equivalences has been one of the most powerful applications arising from the theory of categorified quantum groups. Their work showed that categorical $\mathfrak{sl}_2$-actions gave rise to derived equivalences coming from a lift of the Weyl group action. Here we will call these CR-equivalences, or Chuang-Rouquier derived equivalences. They used these CR-equivalence to prove Broué’s abelian defect conjecture for the modular representation theory of the symmetric group.
Since Chuang and Rouquier’s work, a full theory of categorified quantum groups has emerged [35, 30, 31, 47, 48] including the discovery of 2-categories $\mathcal{U}(\mathfrak{g})$ categorifying integral forms of symmetrizable quantum Kac-Moody algebras. CR-equivalences were then extended by Cautis, Kamnitzer, and Licata [8] to more general Lie type $\mathfrak{g}$, where they correspond to simple transpositions giving rise to categorical braid group actions of type $\mathfrak{g}$. Such actions can be understood as giving the braidings in categorifications of the Jones and HOMFLYPT polynomials by Khovanov and Khovanov-Rozansky homology [8, 37, 45, 39] and even the braiding in bordered approaches to Knot Floer homology [40]. These equivalences have also been used to construct derived equivalences related to stratified flops in birational geometry [10, 9, 7].

1.2. Odd CR-equivalences. In this article, we construct new derived superequivalences coming from the ‘odd’ categorification of $\mathfrak{sl}_2$. This odd theory arose as an attempt to provide a higher representation theoretic explanation for a phenomena discovered in link homology theory where Khovanov homology exhibited two distinct even and odd variants [43, 44]. Ellis, Khovanov, and the second author initiated a program [18] to define odd analogs of quantum $\mathfrak{sl}_2$ and related structures. The result was the discovery of odd, noncommutative, analogs of many of the structures appearing in connection with $\mathfrak{sl}_2$ categorification including odd analogs of the Hopf algebra of symmetric functions [16, 18], the nilHecke algebra, and cohomologies of Grassmannians [18] and Springer varieties [38]. Subsequent work has shown these odd categorifications extend to arc algebras and constructions of odd Khovanov homology for tangles [42, 41, 21].

These investigations into odd categorification turned out to be closely connected with independent parallel investigations into Kac-Moody superalgebra categorifications [25, 23, 24], with the odd categorification of $\mathfrak{sl}_2$ lifting the rank one Kac-Moody superalgebra. These odd categorifications give categorifications of the theory of covering Kac-Moody algebras [20, 14, 12, 13]. Covering algebras $U_{q,\pi}(\mathfrak{g})$ generalize quantum enveloping algebras, depending on an additional parameter $\pi$ with $\pi^2 = 1$. When $\pi = 1$, it reduces to the usual quantum enveloping algebra $U_q(\mathfrak{g})$, while the $\pi = -1$ specialization recovers the quantum group of a super Kac-Moody algebra.

In the rank one case, the $\pi = 1$ specialization is $U_q(\mathfrak{sl}_2)$, while for $\pi = -1$ it gives the quantum group $U_q(\mathfrak{osp}(1|2))$ associated with the superalgebra $\mathfrak{osp}(1|2)$. Following a categorification of the positive parts of these algebras in [20], Ellis and the second author categorized the full rank one covering algebra proving a conjecture from [14]. In doing so, a 2-supercategory $\mathcal{U} := \mathcal{U}(\mathfrak{sl}_2)$ was defined [19] for the rank one covering algebra whose Grothendieck group recovers $U_{q,\pi}(\mathfrak{sl}_2)$. This categorification was later greatly simplified by Ellis and Brundan [4], where the 2-supercategory formalism was better developed, building off of their work [3].

Despite the discovery of an ‘odd categorification of $\mathfrak{sl}_2$’ over a decade ago, the analog of CR-equivalences has remained elusive. While it is relatively straightforward to define analogous complexes to the CR-equivalences, proving invertibility has proven more challenging. By utilizing new strategies for proving invertibility developed by the third author, we are able achieve the culmination of the odd categorification program and define odd analogs of the CR-complexes $\Theta$. In doing so, we achieve stronger results than those developed by Chuang and Rouquier. We show that the odd CR-complexes are invertible already in the homotopy category of complexes over the odd categorized quantum group $\mathcal{U}$. This implies that the odd CR-complexes will give equivalences in any 2-representation of $\mathcal{U}$.

1.3. Applications to spin defect conjecture and affine Hecke-Clifford algebras. Brundan and Kleshchev discovered a link between blocks of the affine Clifford-Hecke superalgebra (or affine Sergeev algebra) and quantum Kac-Moody superalgebras [2], see also [50, 1]. In much the same way that KLR-algebras (quiver Hecke algebras) in type $A$ can be viewed as a graded analog of the affine Hecke algebra via the Brundan-Kleshchev-Rouquier isomorphism [5, 47], Kang, Kashiwara, and Tsuchioka...
introduced quiver Hecke superalgebras that serve as graded analogs of the affine Hecke-Clifford and affine Sergeev superalgebras.

Let \( I = I_{\text{even}} \cup I_{\text{odd}} \) and \( A = (a_{ij})_{i,j \in I} \) a symmetrizable Cartan matrix satisfying various compatibility conditions to define a symmetrizable Kac-Moody superalgebra \( \mathfrak{g} \), see Section 6.1. To this Cartan data and an additional choice of certain skew polynomials \( \Omega \), Kang, Kashiwara, Tsuchioka introduced [25] quiver Hecke superalgebras \( R_n = R_n(\Omega) \) that categorify positive halves of quantum Kac-Moody superalgebras \( \mathfrak{g} \) associated to the data [23, 24]. To a dominant integral weight \( \Lambda \in P^+ \), these superalgebras admit cyclotomic quotients \( R_n^\Lambda \) giving categorifications of the highest weight representations \( V(\Lambda) \) of the quantum Kac-Moody superalgebra \( \mathfrak{g} \).

The quiver Hecke superalgebras can be extended to 2-supercategories \( \mathfrak{U}(\mathfrak{g}) \) introduced by Brundan and Ellis [4] that act via 2-representations on categories of modules over cyclotomic quiver Hecke superalgebras \( R_n^\Lambda \). For each \( i \in I_{\text{odd}} \), the 2-category contains a copy of the odd \( \mathfrak{sl}_2 \) 2-supercategory \( \mathfrak{U}(\mathfrak{sl}_2) \) and for \( i \in I_{\text{even}} \) it contains the usual even 2-category \( \mathfrak{U}(\mathfrak{sl}_2) \) from [35]. Hence, in any 2-representation of \( \mathfrak{U}(\mathfrak{g}) \) and any \( i \in I \), there is a derived superequivalence \( \Theta_i \) coming from either our newly defined odd CR-complexes if \( i \) is odd, or the original CR-complexes if \( i \) is even. In particular, our results imply derived superequivalences between blocks \( R_n^\Lambda \) of cyclotomic quiver Hecke algebras related by an action of corresponding Weyl group \( W = W_\mathfrak{g} \) (see Corollary 6.2).

There is a weak Morita superequivalence relating \( R_n \) and its cyclotomic quotients \( R_n^\Lambda \) to quiver Hecke-Clifford superalgebras \( RC_n \) and its cyclotomic quotients \( RC_n^\Lambda \), also introduced in [25]. Kang, Kashiwara, and Tsuchioka showed that quiver Hecke-Clifford superalgebras admit an analog of the Brundan-Kleshchev-Rouquier isomorphism, giving isomorphisms between blocks of cyclotomic affine Hecke-Clifford superalgebras \( ACH^\Lambda_{ij} \) (or cyclotomic affine Sergeev superalgebras \( \overline{ACH}^\Lambda_{ij} \) in the degenerate case) with blocks of their newly introduced quiver Hecke-Clifford algebras \( RC_{ij}^\Lambda \). Hence, our derived equivalences for blocks of \( R_n^\Lambda \) give rise to related equivalences for blocks of affine Hecke-Clifford and affine Sergeev superalgebras.

The spin symmetric group \( \mathfrak{k}_{\mathfrak{g}}^- \) superalgebra of order \( n \) appears in connection with the level one cyclotomic Sergeev superalgebra \( \overline{ACH}^\Lambda_n \). In particular, there is an isomorphism [49, 54]

\[
\overline{ACH}^\Lambda_n \cong \mathfrak{k}_{\mathfrak{g}}^- \otimes \mathfrak{c}_n
\]

(1.1)

where \( \mathfrak{c}_n \) is the Clifford superalgebra. Building off of foundational work of Kleshchev and Livesey [33], and categorical actions introduced in [23, 24], our derived equivalences for blocks of cyclotomic quiver Hecke superalgebras for \( \Lambda = \Lambda_0 \) implies the abelian defect conjecture for spin representations of the symmetric group (Theorem 6.3).

1.4. **Specifics.** After recalling some basic notions in super theory in Section 2, Section 3 introduces deformed cyclotomic quotients of the odd nilHecke algebras and shows they are Morita equivalent to a ring that can be thought of as an odd analog of the equivariant cohomology ring of a Grassmannian (see Theorem 3.11). Indeed, when coefficients are reduced modulo 2, this noncommutative ring reduces to the usual \( GL(N) \)-equivariant cohomology ring of a Grassmannian.

While writing this article, we became aware that Brundan and Kleshchev were independently working on the same problem with significant overlap to the results obtained here. In particular, they have independently constructed the Morita equivalence between deformed cyclotomic quotients and the odd equivariant cohomologies of Grassmannians [6]. They also construct the odd CR-complexes \( \Theta \) and give an independent proof of the invertibility of the complex \( \Theta \) acting on abelian 2-representations [6]. Our proofs of invertibility are entirely complementary. They have defined 2-representations of \( \mathfrak{U} \) directly on these deformed Grassmannian superalgebras, analogous to the equivariant 2-representations from [36] defined in the even case. They have significantly expanded the odd theory of symmetric functions and introduced bimodules that can be interpreted as odd equivariant cohomologies of two step flag
varieties. Their proof of invertibility of $\Theta$ then follows Chuang and Rouquier developing odd analogs of the theory of locally finite abelian 2-representations.

Here we work with the deformed cyclotomic quotients directly and extend to work of Kang, Kashiwara, and Oh to show that these deformed cyclotomic quotients admit 2-representations of $\mathcal{U}$, see Section 4.6. This extension enables us to avoid quotienting $\mathcal{U}$ by the ‘odd bubble’ (see (4.8)), whereas in [6] the odd bubble is set to zero. While our setting is more general, this has no material effect in constructing derived equivalences for quiver Hecke superalgebras. (These two representations appear to factor through the quotient by the odd bubble.)

By relating the 2-representation on deformed cyclotomic quotients to a universal 2-representations constructed as a quotient of the 2-category $\mathcal{U}$ itself, we are able to define odd analogs of the simple 2-representations analogous to those studied by Rouquier in the even case [47]. These 2-representations are often more convenient than the minimal categorifications developed in [11] as they are additive rather than abelian. Building off of the techniques developed by the third author, we can then study natural odd analogs of the Chuang-Rouquier complexes in Section 5.1 and prove in Theorem 5.11 that these are invertible complexes in the homotopy category of $\mathcal{U}$.

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2. Super theory

2.1. Superspaces. Let $k$ be a field with characteristic not equal to 2. A superspace is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. For a homogeneous element $v \in V$, write $|v|$ for the parity of $v$. An even linear map is a parity preserving $k$-module map. The usual tensor product of $k$-vector spaces is again a superspace with $(V \otimes W)_0 = V_0 \otimes W_0 \oplus V_1 \otimes W_1$ and $(V \otimes W)_1 = V_0 \otimes W_1 \oplus V_1 \otimes W_0$. Likewise, the tensor product $f \otimes g$ of two linear maps between superspaces is defined by

$$ (f \otimes g)(v \otimes w) := (-1)^{|g||v|} f(v) \otimes g(w). $$

We write $\text{SVect}$ for the category of superspaces and all linear maps. The underlying category $\text{SVect}$ consisting of super spaces and even linear maps is a symmetric monoidal category with symmetric braiding $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$. On a superspace, we have an involution

$$ \varphi(y) = (-1)^{|y|} y. $$

2.2. Supercategories. There are several variants of the definition of super monoidal category and 2-supercategory. Here we follow the notation of [3], where a comparison to the notation from [25, 23] is provided at the end of the introduction in [3].

Supercategories, superfunctors, and supernatural transformations are all defined via enriched category theory over the symmetric monoidal category $\text{SVect}$, see [3, Definition 1.1] for an unpacking of these definitions. In particular, the Hom spaces $\text{Hom}_\mathcal{A}(X, Y)$ of a supercategory $\mathcal{A}$ are equipped with superspace structures and composition and identities are given by even linear maps; a superfunctor $F: \mathcal{A} \to \mathcal{B}$ gives an even map of superspaces $\text{Hom}_\mathcal{A}(X, Y) \to \text{Hom}_\mathcal{B}(FX, FY)$ for all objects $X, Y$ of $\mathcal{A}$.

The notion of 2-supercategory can be defined in a similar way, by enriching over the category $\text{SCat}$ of small supercategories and superfunctors with monoidal structure as in [4, Definition 1.2]. In particular, a 2-supercategory $\mathcal{U}$ has for each pair of objects $\lambda, \mu$ a supercategory of morphisms $\text{Hom}_\mathcal{U}(\lambda, \mu)$ and compositions and identities are given by monoidal superfunctors. For our purposes, the key feature of
2-supercategories is that the interchange law must be replaced by a superinterchange law so that given
2-morphisms $u: p \Rightarrow q: \lambda \to \mu, v': p' \Rightarrow q': \mu \to \nu, v: q \Rightarrow r: \lambda \to \mu, v': r' \Rightarrow r': \mu \to \nu$, we have
\begin{equation}
(uv') \circ (v'r) = (-1)^{|u||v|}(u \circ v)(u' \circ v'),
\end{equation}
where we denote horizontal composition by juxtaposition and vertical composition with $\circ$.

A graded $(Q, \Pi)$-supercategory is a graded supercategory $A$ together with superfunctors
\[ Q, Q^{-1}, \Pi: A \to A, \]
an odd supernatural isomorphism $\zeta: \Pi \to I$ that is homogeneous of degree 0, and even supernatural
isomorphisms $\sigma: Q \Rightarrow I, \tilde{\sigma}: Q^{-1} \Rightarrow I$ that are homogeneous of degrees 1 and -1, respectively. This data
makes $Q$ and $Q^{-1}$ mutually inverse graded superequivalences and $\Pi$ a self-inverse graded superequivalence. A graded $(Q, \Pi)$-2-supercategory can be defined similarly, where each hom category has the structure of a graded $(Q, \Pi)$-supercategory, see [3, Definition 6.5] for a precise
definition.

2.3. Superalgebras and supermodules. A superalgebra $A$ over the field $k$ is a superpace together with a structure of $k$-algebra such that $xy \in A_{i+j}$ for all $x \in A_i$ and $y \in A_j$. A superalgebra $A$ is said to be supercommutative if $xy = (-1)^{|y|x}yx$ for all $x \in A_i$ and $y \in A_j$. Given superalgebras $A$ and $B$, the multiplication on the tensor product $A \otimes B$ is given on $\mathbb{Z}_2$ homogeneous elements by
\[(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|}(a_1a_2) \otimes (b_1b_2).\]

A left (resp. right) supermodule over the superalgebra $A$ is a superspace $M$ together with a structure of left (resp. right) $A$-module such that $rm \in M_{i+j}$ (resp. $mr \in M_{i+j}$) for all $r \in A_i$ and $m \in M_j$. We denote the supercategory of left $A$-supermodules by $sMod(A)$. The category $Mod(A)$ of ordinary $A$-modules can be endowed with the structure of a $\Pi$-category, see [25, Section 2.2].

The supercategory $sMod(A)$ can be further equipped with the structure of a $\Pi$-supercategory. If $M$ is a left (resp. right) supermodule, we define a left (resp. right) supermodule
\[ \Pi M = \{ \pi(m), m \in M \} \]
by $(\Pi M)_i = \{ \pi(m), m \in M_{i+1} \}$ and $a\pi(m) = \pi(\varphi(a)m)$ (resp. $\pi(m)a = \pi(ama)$). An even morphism of
left (resp. right) supermodules is an even linear map $f$ such that $f(ay) = af(y)$. An odd morphism is an odd linear map $f$ such that $f(ay) = \varphi(a)f(y)$ (resp. $f(ya) = f(y)a$). Note that an odd morphism $M \to N$ is the same thing as an even morphism $M \to \Pi N$. Then the identity map on the underlying vector space defines an odd supermodule isomorphism $\zeta_M: \Pi M \to M$ giving the category $sMod(A)$
of $A$-supermodules and even $A$-linear homomorphisms the structure of a supercategory in the sense of
[25, Definition 2.1], or a $\Pi$-supercategory in the sense of [3, Definition 1.7] (see for example [3, Example
1.8]).

If in addition, $A$ is a $\mathbb{Z}_2$-graded superalgebra $A = \bigoplus_{i \in \mathbb{Z}} A_{i,0} \oplus \bigoplus_{i \in \mathbb{Z}} A_{i,1}$, we can consider the super
category of graded supermodules. This supercategory acquires a $(Q, \Pi)$-supercategory structure with $\Pi$
as above and $Q$ the grading shift functor defined on a $\mathbb{Z}_2$-graded supermodule $M = \bigoplus_{i \in \mathbb{Z}, j \in \mathbb{Z}_2} M_{i,j}$
by $(QM)_{i,j} = M_{i-1,j}$. Define the super opposite $A^{\text{op}}$ of the superalgebra $A$ as $A^{\text{op}} := \{ a^{\text{op}} | a \in A_i \}$ for $i \in \mathbb{Z}_2$ with
multiplication
\[ a^{\text{op}}b^{\text{op}} = (-1)^{|a||b|}(ba)^{\text{op}}. \]
Any left $A$-module can be regarded as a right $A^{\text{op}}$-module with $xa^{\text{op}} := (-1)^{|a||x|}ax$ for $\mathbb{Z}_2$-homogenous elements $a \in A$ and $x \in M$.

Given superalgebras $A$ and $B$, a superbimodule $M$ is a superspace $M$ and an even linear map
$m_{V}: A \otimes V \otimes B \to M$ making $M$ into an $(A, B)$-bimodule in the usual sense. A superbimodule homomorphism $f: M \to N$ is a linear map such that $f(abv) = (-1)^{|f||a|}af(v)b$. When $A$ and $B$ are graded superalgebras, the supercategory of graded $(A, B)$-superbimodules is also a $(Q, \Pi)$-supercategory [3, Example 1.8]. Given a graded $(A, B)$-superbimodule $M$, we have a functor $F_M: sMod(B) \to sMod(A)$
given by $N \mapsto M \otimes_B N$. This functor can be equipped with the structure of a $(Q, \Pi)$-superfunctor [24, Section 7.5].

3. Odd nilHecke algebra and deformed cyclotomic quotients

3.1. The odd nilHecke algebra. We recall here the odd nilHecke algebra introduced in [18, 25]. This algebra is closely related to the spin Hecke superalgebra appeared in earlier work of Wang [53]; many of the essential features of the odd nilHecke algebra including skew-polynomials appears much earlier in this and related works on spin symmetric groups [27, 28, 29].

Definition 3.1. The odd nilHecke algebra $\text{ONH}_n$ is the $\mathbb{Z}$-graded unital associative superalgebra generated by elements $x_1, \ldots, x_n$ of degree 2 and parity 1 and elements $\tau_1, \ldots, \tau_{n-1}$ of degree $-2$ and parity 1, subject to the relations

\begin{align}
(3.1) & \quad \tau_i^2 = 0, \quad \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \\
(3.2) & \quad x_i \tau_i + \tau_i x_{i+1} = 1, \quad \tau_i x_i + x_{i+1} \tau_i = 1, \\
(3.3) & \quad x_i x_j + x_j x_i = 0 \quad (i \neq j), \quad \tau_i \tau_j + \tau_j \tau_i = 0 \quad (|i - j| > 1), \\
(3.4) & \quad x_i \tau_j + \tau_j x_i = 0 \quad (i \neq j, j + 1).
\end{align}

For $w \in \mathfrak{S}_n$ and a choice of a reduced expression $w = s_{i_1} \cdots s_{i_k}$ in terms of simple transpositions $s_i = (i \ i + 1)$, define $\tau_w = \tau_{i_1} \cdots \tau_{i_k}$. Note that $\tau_w$ only depends on the reduced expression up to an overall sign. For $w_0$ the longest word in $\mathfrak{S}_n$ we fix a preferred choice of reduced expression

\begin{equation}
(3.5) \quad \tau_{w_0} = \tau_1 (\tau_2 \tau_1) \cdots (\tau_{n-1} \cdots \tau_1).
\end{equation}

One can show that $\tau_{w_0} = \tau_{n-1} (\tau_{n-2} \tau_{n-1}) \cdots (\tau_1 \cdots \tau_{n-1})$, see [18, (3.51)]. The elements

\begin{align}
(3.6) & \quad e_n := (-1)^{\binom{n}{2}} x_1 x_2^{n-1} x_2 \cdots x_n \tau_{w_0}, \\
(3.7) & \quad e'_n := (-1)^{\binom{n}{2}} \tau_{w_0} x_n x_{n-1}^{n-2} \cdots x_1,
\end{align}

are idempotents of $\text{ONH}_n$ and the left module $\text{ONH}_n e_n$ is, up to grading shifts, the unique indecomposable projective $\text{ONH}_n$-module, see [18].

Notation 3.2. Given a Laurent polynomial $f = \sum_j f_j q^j$ and object $M$ of a $\mathbb{Z}$-graded category, we write $f M$ to denote the direct sum $\bigoplus_j (Q^{-j} M)^{\oplus f_j}$.

The regular representation of $\text{ONH}_n$ decomposes as

\begin{equation}
(3.8) \quad \text{ONH}_n \cong \bigoplus_{j=0}^{\lfloor \frac{n}{2} \rfloor} \bigoplus_{i=0}^j [n]!(\text{ONH}_n e_n)
\end{equation}

Let $k$ be a commutative ring (usually we take $k = \mathbb{Z}$ or a field) and let

$$\text{OPol}_n = k \langle x_1, \ldots, x_n \rangle / (x_i x_j + x_j x_i \text{ if } i \neq j)$$

be the $\mathbb{Z}$-graded superalgebra of skew polynomials in $n$ variables. Its generators are given degree $\text{deg}(x_i) = 2$, and parity $|x_i| = 1$. The symmetric group $\mathfrak{S}_n$ acts on $\text{OPol}_n$ by

$$w(x_i) = (-1)^{\ell(w)} x_{w(i)}, \quad w(f g) = w(f) w(g)$$

where $\ell(w)$ denotes the length of $w \in \mathfrak{S}_n$. For $i = 1, \ldots, n - 1$, define the $i$-th odd divided difference operator $\partial_i$ to be the map $\text{OPol}_n \to \text{OPol}_n$ defined by

\begin{equation}
(3.9) \quad \partial_i(x_j) = \begin{cases} 1 & j = i, i + 1 \\ 0 & \text{otherwise} \end{cases}, \quad \partial_i(f g) = \partial_i(f) g + s_i(f) \partial_i(g).
\end{equation}
For any \( f \in \text{OPol}_n \) the action of the odd divided difference operator is given by the formula
\[
\partial_i(f) = \frac{(x_{i+1} - x_i)f - s_i(f)(x_{i+1} - x_i)}{x_{i+1}^2 - x_i^2},
\]
see [23, equation 4.19]. These operators play a role analogous to that of the divided difference operators of Kostant-Kumar.

Fix a reduced expression \( w = i_1 \cdots i_r \) for each \( w \in \mathfrak{S}_n \). If \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is an \( n \)-tuple, write \( x^\alpha \) for \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). The basic properties of \( \text{ONH}_n \) are as follows.

**Proposition 3.3 ([18]).**

1. Changing the choice of reduced expression only changes \( \tau_w \) by a possible factor of \(-1\), and
\[
\tau_{w\tau_{w'}} = \begin{cases} \pm \tau_{ww'} & \ell(w) + \ell(w') = \ell(ww'), \\ 0 & \text{otherwise.} \end{cases}
\]
2. The algebra \( \text{ONH}_n \) is a free \( k \)-module. Either of the sets \( \{\tau_w x^\alpha : w \in \mathfrak{S}_n, \alpha \in \mathbb{Z}_{\geq 0}\} \), \( \{x^\alpha \tau_w : w \in \mathfrak{S}_n, \alpha \in \mathbb{Z}_{\geq 0}\} \) is a basis.
3. The action on of the odd nilHecke algebra on odd polynomials given by sending \( x_i \) to the map of left multiplication by \( x_i \) and sending \( \tau_i \) to the odd divided difference operator \( \partial_i \) defines a superalgebra isomorphism
\[
\phi: \text{ONH}_n \to \text{End}_{\text{OL}_n}(\text{OPol}_n),
\]
where \( \text{OL}_n \) is the superalgebra of odd symmetric polynomials defined in the next section.

### 3.2. Odd symmetric functions.

Define the ring of **odd symmetric polynomials** to be the subring
\[
\text{OL}_n = \bigcap_{i=1}^{n-1} \ker(\partial_i) = \bigcap_{i=1}^{n-1} \text{im}(\partial_i)
\]
of \( \text{OPol}_n \). In the even case, with the usual divided difference operators, this definition would agree with the usual notion of symmetric polynomials \( \Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n} \). Define elements of \( \text{OPol}_n \) for each \( k \geq 1 \)
\[
\epsilon_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \bar{x}_{i_1} \cdots \bar{x}_{i_k}, \quad \text{where } \bar{x}_i = (-1)^{i-1}x_i,
\]
\[
h_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \bar{x}_{i_1} \cdots \bar{x}_{i_k}, \quad \text{where } \bar{x}_i = (-1)^{i-1}x_i.
\]
Set \( \epsilon_0 = h_0 = 1 \) and \( \epsilon_j = h_j = 0 \) for \( j < 0 \). If \( j > n \) then \( \epsilon_j = 0 \). Both of these families of skew polynomials are odd symmetric. We call the \( \epsilon_i \) the \( i \)th **odd elementary symmetric polynomial** and the \( h_i \) the \( i \)th **odd complete symmetric polynomial**. The odd elementary and odd complete symmetric functions are related by
\[
\sum_{j=0}^{\ell} (-1)^{j(j+1)} \epsilon_j h_{\ell-j} = 0 \quad \text{if } \ell \geq 1.
\]

We also have the following relations for any \( m \) and \( 0 \leq a \leq n \)
\[
\epsilon_m(x_1, \ldots, x_n) = \sum_{j=0}^{m} (-1)^a \epsilon_{m-j}(x_1, \ldots, x_a) \epsilon_j(x_{a+1}, \ldots, x_n).
\]
It was shown in [18] that the following relations hold in the ring $O\Lambda_n$:

\begin{align}
\varepsilon_i \varepsilon_{2m-i} &= \varepsilon_{2m-i} \varepsilon_i \quad (1 \leq i, 2m - i \leq n) \\
\varepsilon_i \varepsilon_{2m+1-i} + (-1)^i \varepsilon_{2m+1-i} \varepsilon_i &= (-1)^i \varepsilon_{i+1} \varepsilon_{2m-i} + \varepsilon_{2m-i} \varepsilon_{i+1} \quad (1 \leq i, 2m - i \leq n - 1) \\
\varepsilon_1 \varepsilon_{2m} + \varepsilon_{2m} \varepsilon_1 &= 2 \varepsilon_{2m+1} \quad (1 < 2m \leq n - 1).
\end{align}

(3.16)

Note that the third is the $i = 0$ case of the second. In particular, the ring $O\Lambda_n$ of odd symmetric functions is noncommutative.

Odd symmetric functions also have natural $\mathbb{Z}$ bases of odd Schur polynomials [16, 18]. There is also an odd analog of the Littlewood-Richardson rule

\[ s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda \]

for odd Schur polynomials developed by Ellis [17], where the mod 2 reduction of $c_{\mu\nu}^\lambda$ agrees with the mod 2 reduction of the usual Littlewood-Richardson coefficients. In particular, if $|\mu| + |\nu| \neq |\lambda|$, then $c_{\mu\nu}^\lambda = 0$.

**Proposition 3.4.** [18, Section 2.1] The superalgebra $O\Lambda_n$ has a presentation by generators $\varepsilon_1, \ldots, \varepsilon_n$ and relations (3.16). A basis of $O\Lambda_n$ in $\mathbb{Z}$-degree $2j$ is given by all products

\[ \varepsilon_\lambda := \varepsilon_{\lambda_1} \cdots \varepsilon_{\lambda_r} \]

with $n \geq \lambda_1 \geq \lambda_2 \geq \ldots \lambda_r \geq 1$ and $\lambda_1 + \cdots + \lambda_r = j$. The same result holds if all $\varepsilon$’s are replaced by complete symmetric polynomials $h$’s.

Combining the results of [16] and the identification of various definitions of odd Schur functions from [17], it follows that

\[ s_\lambda = h_\lambda + \sum_{\mu > \lambda} a_\mu s_\mu = \varepsilon_\bar{\lambda} + \sum_{\mu > \bar{\lambda}} b_\mu s_\mu \]

for integers $a_\mu$ and $b_\mu$, where the order is the usual lexicographic order on partitions and $\bar{\lambda}$ denotes the dual partition. We write $\ell(\lambda)$ for the number of nonzero parts of $\lambda$.

### 3.3. Modular reduction.

For a graded abelian group $V$ of finite rank in each degree define the graded rank of $V$ as

\[ \text{rk}_q(V) = \sum_{i \in \mathbb{Z}} \text{rk}(V_i)q^i. \]

Likewise, for a graded vector space $W$ over a field $k$ of finite dimension in each degree, define the graded dimension of $W$ as

\[ \text{dim}_q(k)(W) = \sum_{i \in \mathbb{Z}} \text{dim}_k(W_i)q^i. \]

In what follows we make use of results from [18, Section 2] about various reductions mod 2. Consider the reduction map $\Lambda_n \rightarrow \Lambda_n \otimes_{\mathbb{Z}} \mathbb{Z}/2$. The images of the usual elementary symmetric functions under this map are nonzero in $\Lambda_n \otimes_{\mathbb{Z}} \mathbb{Z}/2$. In particular, $\text{rk}_q(\Lambda_n) = \text{dim}_q(\mathbb{Z}/2)(\Lambda_n)$.

Likewise, in the odd setting we also have

\[ \text{rk}_q(O\Lambda_n) = \text{dim}_q(\mathbb{Z}/2)(O\Lambda_n) \quad \text{and} \quad \text{rk}_q(O\text{Pol}_n) = \text{dim}_q(\mathbb{Z}/2)(O\text{Pol}_n). \]

Products of odd elementary symmetric functions provide a $\mathbb{Z}/2$-basis for $O\Lambda_n \otimes_{\mathbb{Z}} \mathbb{Z}/2$. Since the definitions of odd divided difference operators, odd elementary symmetric functions, and odd polynomials agree with their even counterparts, when reduced modulo 2, we have isomorphisms

\[ O\text{Pol}_n \otimes_{\mathbb{Z}} (\mathbb{Z}/2) \cong P_n \otimes_{\mathbb{Z}} (\mathbb{Z}/2) \quad \text{and} \quad O\Lambda_n \otimes_{\mathbb{Z}} (\mathbb{Z}/2) \cong \Lambda_n \otimes_{\mathbb{Z}} (\mathbb{Z}/2). \]
The first one fixes the generators $x_1, \ldots, x_n$, and the second one is the restriction of the first.

**Lemma 3.5.** Let $A$ be an ring and let $M$ be an $A$-module such that $A, M$ are free as a $\mathbb{Z}$-modules. Let $S$ be a subset of $M$. Assume that $S \otimes 1$ is a linearly independent subset of $M \otimes \mathbb{Z}_2$ as an $A \otimes \mathbb{Z}_2$-module. Then $S$ is linearly independent over $A$.

**Proof.** We proceed by contradiction. Assume that we have a non-trivial relation

$$\sum_{s \in S} a_s s = 0. \tag{3.23}$$

If all the coefficients in (3.23) are in $2A$, then we can divide the relation by 2 to produce a new non-trivial relation by freeness of $M$. By freeness of $A$, this procedure terminates and we may assume that at least one $a_s$ is not in $2A$. In $M \otimes \mathbb{Z}_2$, we obtain a relation

$$\sum_{s \in S} (a_s \otimes 1)(s \otimes 1) = 0$$

with at least one coefficient $a_s \otimes 1$ non-zero, a contradiction. \qed

### 3.4. Deformed cyclotomic quotient

Let $A_n$ be the graded supercommutative superalgebra generated by $\chi_1, \ldots, \chi_{\lfloor n/2 \rfloor}$ of parity $\bar{0}$ and $d$ of parity $\bar{1}$, subject to the relations

$$d^2 = 0, \quad d\chi_{\lfloor n/2 \rfloor} = 0 \quad \text{if } n \text{ is even}. \tag{3.24}$$

The $\mathbb{Z}$-grading on $A_n$ is defined by declaring $d$ to have degree 1 and $\chi_i$ to have degree $2i$. Note that for all odd elements $x, y$ of $A_n$, $xy = 0$. Consequently, $A_n$ is also commutative. Put

$$c_i = \begin{cases} 
\chi_i & \text{if } i \text{ is even,} \\
\frac{d \chi_{i+1}}{2} & \text{if } i \text{ is odd.}
\end{cases}$$

The element $c_i$ is of degree $i$ and parity $\bar{i}$. Furthermore, we have the relation $c_{2i+1} = c_1 c_{2i} = c_2 c_1$.

We can realize $A_n$ as a quotient of $O\Lambda_n$, via the morphism

$$\begin{cases} 
O\Lambda_n & \rightarrow A_n, \\
\varepsilon_i & \mapsto (-1)^{\bar{i}} c_i.
\end{cases}$$

**Notation 3.6.** If $M$ is a right-module over $O\Lambda_n$, we put

$$\overline{M} = M \otimes_{O\Lambda_n} A_n. \tag{3.25}$$

Consider the polynomial $a^n(t) \in A_n[t]$ defined by

$$a^n(t) = \sum_{t=0}^{n} (-1)^{\bar{t}} t^t c_{n-t}. \tag{3.26}$$

An important point is that when $t$ is a variable of parity $\bar{1}$, the polynomial $a^n(t)$ neither commutes nor supercommutes with $t$. Instead, we have the relations

$$\begin{align*}
ta^n(t) &= a^n(t)(t - 2d), \\
a^n(t)t &= (t + (-1)^{n}d)a^n(t).
\end{align*} \tag{3.27}$$

**Definition 3.7.** The deformed cyclotomic quotient is the graded superalgebra $\text{ONH}_n^k$ defined as

$$\text{ONH}_n^k = (\text{ONH}_k \otimes A_n)/a^n(x_1).$$
The deformed cyclotomic quotients of Kang-Kashiwara-Oh [23, 24] are specializations of \( \text{ONH}^n_k \).
More precisely, we have an isomorphism of graded superalgebras
\[
\text{ONH}^n_k \otimes_{\Lambda_n} (\Lambda_n/d) \simeq R^n(k),
\]
where \( R^n(k) \) is the notation used in [23].

3.5. Odd equivariant cohomology of Grassmannians. Recall that in the even case, the deformed cyclotomic quotient \( \text{NH}^n_k \) is isomorphic to a matrix ring over the \( GL(n) \)-equivariant cohomology ring \( H^s_{\GL(n)}(Gr(k;n)) \) of Grassmannian \( Gr(k;n) \) of \( k \)-planes in \( \mathbb{C}^n \),
\[
(H^s_{\GL(n)}(Gr(k;n)) \cong \Lambda^k \otimes \Lambda^\Lambda_{n-k}
\]
See [46, Proposition 9]. There is an isomorphism
\[
H^s_{\GL(n)}(Gr(k;n)) \cong \Lambda_k \otimes \Lambda_{n-k}
\]
making \( H^s_{\GL(n)}(Gr(k;n)) \) a free graded module of rank \( \binom{n}{k} \) over \( \Lambda_n \), with an integral basis given by the set of Schur polynomials \( s^\lambda_{\nu} \in \Lambda_k \) with \( \lambda = (\lambda_1, \ldots, \lambda_k) \) and \( \lambda_i \leq (n - k) \) for \( 1 \leq i \leq k \). In these statements, we view \( \Lambda_k \otimes \Lambda_{n-k} \) as a subalgebra of \( \text{P}_n \).

By analogy with the even case that will become apparent in this section, we define a superring using the notation from (3.25) as
\[
\text{OH}^s_{\GL(n)}(Gr(k;n)) := \overline{\Omega \Lambda_k \otimes \Omega \Lambda_{n-k}},
\]
that will refer to as the odd equivariant cohomology\(^1\) of the Grassmannian. The bimodule structure of \( \Omega \Lambda_k \otimes \Omega \Lambda_{n-k} \) as an \( \Omega \Lambda_n \)-bimodule is given by
\[
\varepsilon_r \cdot (a \otimes b) := \left( \sum_{j=0}^{r} \varepsilon_{r-j} \otimes \varepsilon_j \right)(a \otimes b) \quad (a \otimes b) \cdot \varepsilon_r := (a \otimes b) \left( \sum_{j=0}^{r} (-1)^{(r-j)} \varepsilon_{r-j} \otimes \varepsilon_j \right)
\]
Note that the left action is obtained by viewing \( \Omega \Lambda_k \otimes \Omega \Lambda_{n-k} \) as a subalgebra of \( \text{OPol}_n \), whereas the right action is twisted by an involution coming from the symmetric braiding.

**Lemma 3.8.** The odd equivariant cohomology superring \( \text{OH}^s_{\GL(n)}(Gr(k;n)) \) is a free graded \( \Lambda_n \)-supermodule of rank \( \binom{n}{k} \) with basis given by the set
\[
\{s^\lambda_{\nu} \otimes 1 \mid \lambda = (\lambda_1, \ldots, \lambda_k), \lambda_i \leq (n - k) \text{ for } 1 \leq i \leq k \}.
\]

**Proof.** By base change, it suffices to prove that the set
\[
S = \{s^\lambda_{\nu} \mid \lambda = (\lambda_1, \ldots, \lambda_k), \lambda_i \leq (n - k) \text{ for } 1 \leq i \leq k \}
\]
is a basis of \( \Omega \Lambda_k \otimes \Omega \Lambda_{n-k} \) as a right \( \Omega \Lambda_n \)-module. Let \( X = \{x_1, \ldots, x_k\} \) and \( Y = \{x_{k+1}, \ldots, x_{n-k}\} \).
By (3.15), (3.19), and (3.30) we have that the right action of \( s^\lambda_{\nu} \) on \( \Omega \Lambda_k \otimes \Omega \Lambda_{n-k} \) is given by right multiplication
\[
(a \otimes b) \cdot s^\lambda_{\nu} = (a \otimes b) \cdot \sum_{\mu, \nu} \alpha^\lambda_{\mu \nu} s^\mu_{\nu} \otimes s^\nu_{\nu}
\]
for some coefficients \( \alpha^\lambda_{\mu \nu} \in \mathbb{Z} \) equal to the odd Littlewood-Richardson coefficients up to a sign. Hence, arguing inductively as in the even case, we deduce that \( S \) spans \( \Omega \Lambda_k \otimes \Omega \Lambda_{n-k} \) as a right \( \Omega \Lambda_n \)-module.

---

\(^1\)We emphasize that ‘odd cohomology’ is not an actual cohomology theory that can be applied to any manifold. In general, ‘odd cohomology’ is a noncommutative analog of usual singular cohomology that becomes isomorphic to the usual singular cohomology when coefficients are reduced modulo two. See [19] for a discussion of odd cohomologies of Grassmannians and [36] for a discussion on odd cohomologies of Springer varieties and a broader ‘oddification’ program.
We now prove that \( S \) is linearly independent over \( \mathcal{O}_n \). The isomorphisms (3.22) yield an isomorphism of algebras \( \mathcal{O}_n \otimes \mathbb{Z} \mathcal{Z}_n \simeq \Lambda_n \otimes \mathbb{Z} \mathcal{Z}_2 \) and an isomorphism of right \( \Lambda_n \otimes \mathbb{Z} \mathcal{Z}_2 \)-modules
\[
(3.33) \quad (\mathcal{O}_n \otimes \mathcal{O}_{n-k}) \otimes \mathbb{Z} \mathcal{Z}_2 \simeq (\Lambda_k \otimes \Lambda_{n-k}) \otimes \mathbb{Z} \mathcal{Z}_2.
\]
Denote by \( S^\text{ev} \) the even counterpart of \( S \). Since even and odd Schur functions coincide modulo 2, the isomorphism in (3.33) sends \( S \otimes 1 \) to \( S^\text{ev} \otimes 1 \). Furthermore, \( S^\text{ev} \) is a basis of \( \Lambda_k \otimes \Lambda_{n-k} \) as a right \( \Lambda_n \)-module, so \( S^\text{ev} \otimes 1 \) forms a basis of \( (\Lambda_k \otimes \Lambda_{n-k}) \otimes \mathbb{Z} \mathcal{Z}_2 \) as a right \( \Lambda_n \otimes \mathbb{Z} \mathcal{Z}_2 \)-module. It follows that \( S \otimes 1 \) is a basis of \( (\mathcal{O}_n \otimes \mathcal{O}_{n-k}) \otimes \mathbb{Z} \mathcal{Z}_2 \) as a right \( \mathcal{O}_n \otimes \mathcal{Z}_2 \)-module. The conclusion now follows from Lemma 3.5.

\[ \square \]

3.6. Deformed quotient as a matrix ring over the odd cohomology of a Grassmannian. In this section we give a characterization of the deformed cyclotomic quotient as matrix ring over the odd equivariant cohomology of a Grassmannian. This description will be useful to study the braiding complexes in Section 5.

Let \( \phi : \mathsf{ON}_{\mathfrak{k}} \to \mathsf{End}_{\mathcal{O}_n}(\mathsf{OPol}_k) \) be the isomorphism (3.10). Under this isomorphism, a polynomial \( f \) acts by multiplication. To compute a set of defining relations for the odd cyclotomic quotient we must compute the image of the polynomial \( a^\alpha(x_1) \) from (3.26), and in particular, the matrix representing various powers of \( \phi(x_1) \). It is most convenient to work in the basis
\[ \mathcal{H}_\alpha := \{ \bar{x}^\alpha | \alpha \leq (k-1, k-2, \ldots, 1, 0) \} \]
of \( \mathsf{OPol}_k \) as a free module over \( \mathcal{O}_n \). For each multi-index \( \beta \) obtained by replacing \( \alpha_1 \) by zero in some \( \alpha \) appearing in \( \mathcal{H}_\alpha \), consider the \( \mathcal{O}_k \)-submodule of \( \mathsf{OPol}_k \) with basis
\[ B_\beta = \{ \bar{x}^{k-1} \bar{x}^\beta, \bar{x}^{k-2} \bar{x}^\beta, \ldots, \bar{x}^\beta \} \]
with \( \bar{x}^\beta = x_2^{\alpha_2} x_3^{\alpha_3} \cdots x_k^{\alpha_k} \). Then \( \phi(x_1) \) sends the span of each \( B_\beta \) to itself and is given in this basis by the \( k \times k \)-matrices \( X \) [18, Lemma 5.1], where
\[
X := \phi(x_1)_\beta = \begin{pmatrix}
\varepsilon_1 & 1 & 0 & \cdots & 0 \\
\varepsilon_2 & 0 & 1 & \cdots & 0 \\
-\varepsilon_3 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^{(k-3)} \varepsilon_{k-1} & 0 & 0 & \cdots & 1 \\
(-1)^{(k-2)} \varepsilon_k & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

Lemma 3.9. Denote the \((i, j)\) entry of \( X^m \) by \( b_{i,j}^m \). Then \( b_{i,j}^m \) are completely determined by the relations
\[
(3.34) \quad b_{i,j}^m = \begin{cases}
h_{m+i-j} - \sum_{l=1}^{i-1} h_l b_{i-l,j}^m & \text{if } j \leq m, \\
\delta_{i+m,j} & \text{if } j > m.
\end{cases}
\]

Proof. We prove this by induction on \( m \). For \( m = 1 \), the only non-trivial cases we need to check are for \( j = 1 \) which follows from (3.14)
\[ b_{i,1}^1 := (-1)^{(i+1)} \varepsilon_i = h_i + \sum_{l=1}^{i-1} (-1)^{(i+1)} h_{i-l} \varepsilon_l = h_{1+i-1} - \sum_{l=1}^{i-1} (-1)^{(i-l-1)} h_l \varepsilon_{i-l} \]
For the inductive step, observe by the exact same reasoning as [46, Lemma 10], that \( b_{i,j+1}^m = b_{i,j}^{m+1} \). This implies that \( b_{i,j}^{m+1} \) satisfy the required relations for \( j \geq 2 \), so all that remains to be checked are the \( j = 1 \) entries.
We prove that the $b^{m+1}_{i,j}$ entries satisfy (3.34) by induction on $i$. The induction hypothesis on $m$ implies $b^m_{1,j} = h_{m+1-j}$. Thus, the $(1,1)$ entry in $X^{m+1}$ is

$$b^{m+1}_{1,1} = \sum_{j=1}^{k} (-1)^{(j-1)} h_{m+1-j} \varepsilon_j = \sum_{j=1}^{m} (-1)^{(j-1)} h_{m+1-j} \varepsilon_j = h_{m+1}.$$  

This proves the $i = 1$ base case. For the $i$-induction step, assume $b^{m+1}_{p,j}$ satisfies the relations for all $p < i + 1$. Then

$$b^{m+1}_{i+1,1} = \sum_{r=1}^{k} (-1)^{(r-1)} b^{m+1}_{i+1,r} \varepsilon_r = \sum_{r=1}^{k} (-1)^{(r-1)} \left( h_{m+i+1-r} - \sum_{l=1}^{i} h_l b^{m}_{i+l-l,r} \right) \varepsilon_r$$

$$= h_{m+(i+1)} - \sum_{l=1}^{i} h_l \sum_{r=1}^{k} (-1)^{(r-1)} b^{m}_{i+1-l,r} \varepsilon_r = h_{m+(i+1)+1} - \sum_{l=1}^{i} h_l b^{m+1}_{i+1-l,r}$$

Thus, $b^{m+1}_{i+1,j}$ satisfies the relations.

The deformed cyclotomic quotient has defining relations in $\text{End}_{\Lambda_k} (\text{OPol}_k)$ given by setting the following element to zero

$$\phi(a^n(x_1)) = a^n(\phi(x_1)) = a^n(X) = \sum_{r=0}^{n} (-1)^{n-r} X^{n-r} \otimes c_r \in \text{Mat}_{k!}(\Lambda_k \otimes A_n).$$

Denoting this matrix by $C$, we have that $C_{i,j} = (-1)^n \sum_{r=0}^{n} (-1)^r b^{n-r}_{i,j} \otimes c_r$. Then, under the isomorphism $\phi$ from (3.10)

$$\text{ONH}_k^n \cong \text{Mat}_{k!}(\Lambda_k \otimes A_n/\langle C_{i,j} \mid 1 \leq i, j \leq k \rangle).$$

In particular, $(-1)^n C_{1,j} = \sum_{r=0}^{n} (-1)^r h_{n+1-j-r} \otimes c_r = h'_{n-j+1} \in \Lambda_k \otimes A_n$, where we define

$$h'_{m} = \sum_{r=0}^{m} (-1)^r h_{m-r} \otimes c_r \in \Lambda_k \otimes A_n.$$  

**Lemma 3.10.** Let $I'$ denote the ideal generated by the entries of the first row of $C$,

$$I' := \langle C_{1,j} \mid 1 \leq j \leq k \rangle = \langle h'_{m} \mid n-k+1 \leq m \leq n \rangle.$$  

Then $C_{i,j} \in I'$ for all $1 \leq i,j \leq k$.

**Proof.** We prove this by induction on $i$. The statement is immediate by definition for $i = 1$, so now assume that $C_{p,j} \in I'$ for all $1 \leq p < i$. Then we have

$$(-1)^n C_{i,j} = \sum_{r=0}^{n} (-1)^r \left( h_{n+1-j-r} - \sum_{l=1}^{i} h_l b^{n-r}_{i-l,j} \right) \otimes c_r = h'_{n-i-j} - \sum_{l=1}^{i} h_l C_{l-j}$$

Each $h_l C_{l-j}$ is in $I'$ by the induction hypothesis, so we only need to show that $h'_{n+1-j} \in I'$ for all $1 \leq i,j \leq k$. We know that $h'_{n+1-j}$ is a generator of $I'$ for $i < j$, so it suffices to prove that $h'_{n+s} \in I'$ for all $s \geq 0$. We prove this with a simple induction argument on $s$. The case $s = 0$ is clearly true, so now assume that $h'_{n+r} \in I'$ for all $0 \leq r < s$. It follows that

$$h'_{n+s} = \sum_{r \geq 0} (-1)^r h_{n+s-r} \otimes c_r = -\sum_{r \geq 0} (-1)^r \left( \sum_{i \geq 1} (-1)^{(i+1)} \varepsilon_i h_{n+s-r-i} \otimes c_r \right)$$
\[
= - \sum_{i \geq 1} (-1)^{\binom{i+1}{2}} \varepsilon_i h'_{n+i-1} \in I'
\]

Consider the \(A_n\)-superalgebra defined by

\[
M^n_k := (\mathcal{OA}_k \otimes A_n) / \langle h'_m \mid n - k + 1 \leq m \leq n \rangle.
\]

**Theorem 3.11.** There is an isomorphism of graded superrings

\[
\text{ONH}_k^n \cong \text{Mat}_k(M^n_k)
\]

**Proof.** The deformed cyclotomic quotient is given by (3.35). By Lemma 3.10 the quotient is generated by the first row. The result follows. \(\Box\)

We will show in Proposition 3.14 that \(M^n_k\) is isomorphic to \(\text{OH}_{\text{Gl}(n)}^n(Gr(k; n))\).

**Lemma 3.12.** Let \(\mu = (\mu_1, \ldots, \mu_k)\) be a composition satisfying \(\mu_i \leq n - k\) for all \(1 \leq i \leq k\). Then the product \(h_\mu := h_{\mu_1} \cdots h_{\mu_k}\) of odd complete symmetric functions in \(M^n_k\) is in the span of the set \(\langle h_\lambda \mid \lambda = (\lambda_1, \ldots, \lambda_k), \text{ with } 0 \leq \lambda_k \leq \cdots \leq \lambda_1 \leq (n-k) \rangle\).

**Proof.** We prove this by induction on \(|\mu|\). For \(|\mu| = 0\), this is trivial as \(h_0 = 1 \otimes 1\). Assume then that \(h_\mu\) is in the span of the set (3.39) for all compositions \(\mu = (\mu_1, \ldots, \mu_k)\) with \(\mu_i \leq n - k\) for all \(1 \leq i \leq k\) and \(|\mu| < r\). We establish the \(|\mu| = r\) case.

Let \(h_\nu\) satisfy the hypothesis of the Lemma with \(|\nu| = r\). We use [18, Remark 2.4] with \(\varepsilon\) replaced by \(h\) to sort a product of odd complete symmetric functions into a non-increasing order. If \(p < l\) and \(p + l\) is odd, then

\[
h_{\nu} h_{\lambda} = \begin{cases} h_{\nu} h_{\lambda} + 2 \sum_{i=1}^{p} \varepsilon_i h_{\nu+i} h_{\lambda-p-i} & \text{if } p \text{ even}, \\ -h_{\nu} h_{\lambda} + 2 \sum_{i=1}^{p} (-1)^{\binom{i+1}{2}} h_{\nu+i} h_{\lambda-p-i} & \text{if } p \text{ odd}. \end{cases}
\]

Hence, \(h_\nu\) can be written \(h_\nu = \sum \gamma c_\gamma h_\gamma\) for integers \(c_\gamma\) where \(\gamma = (\gamma_1, \ldots, \gamma_k)\) varies over the set of partitions of \(|\nu| = r\) with \(\ell(\gamma) \leq \ell(\nu)\). It suffices to show that all of the \(h_\gamma\) lie in the span (3.39).

For any \(a \in A_n\) and composition \(a = (a_1, \ldots, a_k)\) with \(a_i > n - k\), the relations of \(M^n_k\) imply

\[
h_\alpha \otimes a = \sum_{j=1}^{\alpha} (-1)^{a_i+1+j+ j(a_i+1+\cdots+a_k)} (h_{(a_1,\ldots,a_{i-1},a_i-j,a_{i+1},\ldots,a_k)} \otimes c_j a).
\]

Therefore, we can write each \(h_\gamma\) as a linear combination \(h_\gamma = \sum_\beta b_\beta h_\beta \otimes d_\beta\) for some \(d_\beta \in A_n\), and \(b_\beta \in \mathbb{Z}\) where \(\beta = (\beta_1, \ldots, \beta_k)\) varies over compositions with \(\beta_i \leq n - k\) for all \(1 \leq i \leq k\) and \(|\beta| < |\nu| = r\). Each \(h_\beta\) lies in the span of (3.39) by the induction hypothesis, and the result follows. \(\Box\)

**Lemma 3.13.** The \(A_n\)-supersmodule \(M^n_k\) is generated by the set

\[
\{ h_\lambda \mid \lambda = (\lambda_1, \ldots, \lambda_k), \text{ with } 0 \leq \lambda_k \leq \cdots \leq \lambda_1 \leq (n-k) \}\}

**Proof.** The superalgebra \(\mathcal{OA}_k\) has a basis given by \(\{ s_\lambda \mid \lambda = (\lambda_1, \ldots, \lambda_k) \}\). It follows from (3.19) and \(\varepsilon_m = 0 \in \mathcal{OA}_k\) for \(m > k\) that any Schur function \(s_\lambda\) with \(\ell(\lambda) > k\) vanishes in \(\mathcal{OA}_k\). We argue that any \(h_\mu\) with \(\ell(\mu) > k\) can be written as a linear combination of \(h_{\mu'}\) with \(\ell(\mu') \leq k\) in \(\mathcal{OA}_k\). To see this, use (3.19) to write \(h_\mu\) in terms of Schur functions

\[
h_\mu = s_\mu - \sum_{\mu' > \mu} a_{\mu} s_{\mu'}.
\]
The $s_\mu$ term vanishes in $\mathcal{O} \Lambda_k$ and if $\mu'$ has $\ell(\mu') > k$ this term also vanishes in the summation. Otherwise, since $\mu' > \mu$ it suffices to assume that all $\mu'$ appearing in the summation have $\ell(\mu') \leq k$. Now we use (3.19) so that (3.42) can be written

$$h_\mu = - \sum_{\mu' > \mu} a_\mu s_{\mu'} = - \sum_{\mu' > \mu} a_\mu h_{\mu'} - \sum_{\mu'' > \mu} \sum_{\mu' > \mu'' > \mu'} a_{\mu''} h_{\mu'}. $$

Since $\mu' < \mu''$ are both partitions of the same size and $\ell(\mu') \leq k$, it follows that all $\mu''$ appearing in the sum have $\ell(\mu'') \leq k$.

Therefore, to show that the set (3.41) spans, it suffices to show that for any $k$-partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we can write $h_\lambda$ as an $A_n$-linear combination of elements from (3.41). Observe that in the quotient module $M_k^n$, any $h_j = h_j \otimes 1$ with $j > n - k$ can be written as an $A_n$-linear combination of $h_\ell$ with $\ell \leq n - k$. However, because of the noncommutativity of the odd complete symmetric functions, this rewriting process may produce terms $h_\mu \otimes a \in \mathcal{O} \Lambda_k \otimes A_n$ where $\mu$ is a composition $\mu = (\mu_1, \ldots, \mu_k)$ satisfying $\mu_i \leq n - k$ for all $1 \leq i \leq k$. The claim then follows from Lemma 3.12. Hence, the set (3.41) is a spanning set for $M_k^n$.

**Proposition 3.14.** There is an isomorphism of $A_n$-superalgebras

(3.43) \[ M_k^n \to OH_{GL(n)}^*(Gr(k; n)) \]

\[ f \otimes a \mapsto f \otimes 1 \otimes a. \]

**Proof.** To see that this map is a well-defined morphism of $A_n$-superalgebras, we need to check that the elements $h'_m$, $n-k+1 \leq m \leq n$, from (3.36) vanish in $OH_{GL(n)}^*(Gr(k; n))$. This follows from (3.14). Now observe that for a partition $\lambda$, the image of $h_\lambda \otimes 1$ is

$$\left( s_\lambda - \sum_{\mu > \lambda} a_\mu s_\mu \right) \otimes 1 \otimes 1$$

where the coefficients $a_\mu$ are as in (3.19). Therefore, the image of the generating set (3.41) from Lemma 3.13 is a basis of $OH_{GL(n)}^*(Gr(k; n))$ as an $A_n$-module by Lemma 3.8. The conclusion follows. \qed

**Corollary 3.15.** The map

\[ g : \text{ONH}_k^n \to \text{ONH}_k \otimes \text{ONH}_n - k = \text{End}_{\mathcal{O} \Lambda_n}(\mathcal{O} \Lambda_k \otimes \mathcal{O} \Lambda_{n-k}), \]

\[ h \mapsto (h \otimes 1) \otimes 1 \] if $h \in \text{ONH}_k$,

\[ z \mapsto (1 \otimes 1) \otimes z \] if $z \in A_n$,

is an isomorphism (using the notation from (3.25)). Thus, $\text{ONH}_k^n$ is a matrix algebra of size $k!$ over the superverring $OH_{GL(n)}^*(Gr(k; n))$.

**Proof.** This is immediate from Theorem 3.11 and the isomorphism of Proposition 3.14. \qed

4. The odd 2-category and faithful universal quotients

4.1. **Definition of the odd 2-category.** We recall here the definition of the rank one super Kac-Moody 2-category from [4]. This presentation greatly simplifies the presentation from [19] where it was referred to as the odd categorification of $\mathfrak{sl}_2$.

**Definition 4.1.** The odd 2-supercategory $\mathfrak{U} = \mathfrak{U}(\mathfrak{sl}_2)$ is the 2-supercategory consisting of

- objects $\lambda$ for $\lambda \in \mathbb{Z}$,
- generating 1-morphisms $E \mathbb{1}_\lambda : \lambda \to \lambda + 2$ and $F \mathbb{1}_\lambda : \lambda \to \lambda - 2$ for each $\lambda \in \mathbb{Z}$,
• \( \mathbb{Z} \times \mathbb{Z}_2 \)-graded generating 2-morphisms \( x : \mathcal{E} \mathbb{I}_\lambda \to \mathcal{E} \mathbb{I}_\lambda \) of parity \( \bar{1} \) and degree 2, \( \tau : \mathcal{E} \mathcal{E} \mathbb{I}_\lambda \to \mathcal{E} \mathcal{E} \mathbb{I}_\lambda \) of parity \( \bar{1} \) and degree -2, and \( \eta : \mathbb{I}_\lambda \to \mathcal{F} \mathcal{E} \mathbb{I}_\lambda \) and \( \varepsilon : \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \to \mathbb{I}_\lambda \) both of parity 0 and degree \( 1 + \lambda \) and \( 1 - \lambda \), respectively. These generators are subject to the relations given below, which are most conveniently expressed in the string diagrammatics of supercategories from [3].

The identity 2-morphism of the 1-morphism \( \mathcal{E} \mathbb{I}_n \) is represented by an upward oriented line (likewise, the identity 2-morphism of \( \mathcal{F} \mathbb{I}_n \) is represented by a downward oriented line).

Horizontal and vertical composites of the above diagrams are interpreted using the conventions for supercategories explained in [3]. The 2-supercategory structure implies that diagrams with odd parity skew commute. The 2-morphisms satisfy the following relations (see [4] for more details).

1. Odd nilHecke: The odd nilHecke relations from Definition 3.1 are satisfied for upward oriented strands and any \( \lambda \in \mathbb{Z} \).
2. Right adjunction axioms: \( \mathcal{F} \mathbb{I}_{\lambda+2} \) is a right dual of \( \mathcal{E} \mathbb{I}_\lambda \) with unit \( \eta \) and counit \( \varepsilon \).
3. Odd \( \mathfrak{sl}_2 \) isomorphisms: we define a new 2-morphisms \( \lambda : \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \to \mathcal{F} \mathcal{E} \mathbb{I}_\lambda \) of parity \( \bar{1} \) and degree 0. Then there are (non homogeneous) isomorphisms

\[
\begin{align*}
\lambda_{\lambda} & : \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \to \mathcal{F} \mathcal{E} \mathbb{I}_\lambda \\
\lambda_{\lambda} & : \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \to \mathcal{F} \mathcal{E} \mathbb{I}_\lambda
\end{align*}
\]

for \( \lambda \geq 0 \) and

\[
\begin{align*}
\lambda_{\lambda} & : \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \to \mathcal{F} \mathcal{E} \mathbb{I}_\lambda \\
\lambda_{\lambda} & : \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \to \mathcal{F} \mathcal{E} \mathbb{I}_\lambda
\end{align*}
\]

for \( \lambda \leq 0 \) in the 2-category \( \mathfrak{U} \).

Ellis and Brundan argue that this compact definition implies the existence of 2-morphisms \( \eta' : \mathbb{I}_\lambda \to \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \) and \( \varepsilon' : \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \to \mathbb{I}_\lambda \) satisfying left adjunction axioms up to a sign. We depict these maps as

\[
\begin{align*}
\eta' & : \mathbb{I}_\lambda \to \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \\
\varepsilon' & : \mathcal{E} \mathcal{F} \mathbb{I}_\lambda \to \mathbb{I}_\lambda
\end{align*}
\]

where we have indicated a \( Q \)-grading and parity as an ordered tuple \( (x, \bar{y}) \). One can also deduce that \( \text{ONH}_{\alpha}^{sop} \) acts on \( \mathcal{F}^\alpha \mathbb{I}_\lambda \) for all \( \lambda \) and \( \alpha \geq 0 \) where the generators \( \tau_{i}^{sop} \) and \( x_{i}^{sop} \) act by

\[
\begin{align*}
x_{i}^{sop} & \mapsto \lambda + 2 \\
\tau_{i}^{sop} & \mapsto \lambda + 2
\end{align*}
\]

4.2. Bubble Relations. The isomorphisms (4.3) – (4.4) imply a number of other diagrammatic relations, see [19, 4]. In particular, these isomorphisms imply important relations for ‘dotted bubbles’ which are various composites of the 2-morphisms \( \eta, \eta', \varepsilon, \varepsilon' \) and \( x \) giving endomorphisms of \( \mathbb{I}_\lambda \). The
relations imply that dotted bubbles of negative degree are zero and degree zero bubbles are equal to the identity, so that for all \( m \geq 0 \)

\[
    (4.6) \quad \bigcirc_{m}^{\lambda} = \delta_{m,\lambda-1}\text{Id}_{1,\lambda} \quad \text{if } m \leq \lambda - 1,
    \quad \bigcirc_{m}^{-\lambda} = \delta_{m,-\lambda-1}\text{Id}_{1,\lambda} \quad \text{if } m \leq -\lambda - 1.
\]

We will sometimes make use of the shorthand notation that highlights the degree of the bubble which are \( 2n \) in both cases below

\[
    (4.7) \quad n^{+} \bigcirc_{m}^{\lambda} := \lambda - 1 + n \bigcirc_{m}^{\lambda}
    \quad \lambda \bigcirc_{m}^{n^{+}} := \lambda - \lambda + n
\]

The degree two bubble is given a special notation as in \((4.8)\) and squares to zero by the superinterchange law. We call this map the odd bubble and denote it by

\[
    (4.8) \quad \bigotimes := \begin{cases} 
        \bigcirc^{\lambda} & \text{if } \lambda \geq 0 \\
        \bigcirc^{\lambda - \lambda} & \text{if } \lambda \leq 0
    \end{cases}
\]

We call a clockwise (resp. counterclockwise) bubble fake if \( m + \lambda - 1 < 0 \) and (resp. if \( m - \lambda - 1 < 0 \)). These correspond to positive degree bubbles that are labeled by a negative number of dots. These are to be interpreted as formal symbols recursively defined by the odd infinite Grassmannian relations

\[
    (4.9) \quad \bigcirc_{m}^{2n^{+}} := - \sum_{i=1}^{n} \bigcirc_{2n^{+}-\ell}^{2n^{+}} \quad \text{for } 0 \leq 2n < -\lambda,
    \quad \bigcirc_{m}^{2n^{+}} := - \sum_{i=1}^{n} \bigcirc_{2n^{+}}^{2n^{+}} \quad \text{for } 0 \leq 2n < \lambda,
\]

4.3. \((Q, \Pi)\)-envelopes and graded \((Q, \Pi)\)-2-supercategories.

**Definition 4.2** ([4] Definition 1.6). Given a graded 2-supercategory \( \mathcal{U} \), its \((Q, \Pi)\)-envelope \( \mathcal{U}_{q,\pi} \) is the graded 2-supercategory with the same objects as \( \mathcal{U} \), 1-morphisms defined from

\[
    \text{Hom}_{\mathcal{U}_{q,\pi}}(\lambda, u) := \{ Q^{m}\Pi^{a}F \mid \text{for all } F \in \text{Hom}_{\mathcal{U}}(\lambda, \mu) \text{ with } m \in \mathbb{Z} \text{ and } a \in \mathbb{Z}/2\mathbb{Z} \}
\]

with composition law \((Q^{m}\Pi^{a}G)(Q^{m}\Pi^{b}F) := Q^{m+n}\Pi^{a+b}(GF)\). The 2-morphisms are defined by

\[
    \text{Hom}_{\mathcal{U}_{q,\pi}}(Q^{m}\Pi^{a}F, Q^{n}\Pi^{b}G) := \{ x_{m,a}^{n,b} \mid \text{for all } x \in \text{Hom}_{\mathcal{U}}(F, G) \}
\]

viewed as a superspace with addition given by \( x_{m,a}^{n,b} + y_{m,a}^{n,b} := (x + y)_{m,a}^{n,b} \) and scalar multiplication given by \( c(x_{m,a}^{n,b}) := (cx)_{m,a}^{n,b} \). The degrees are given by \( \deg(x_{m,a}^{n,b}) = \deg(x) + n - m, \ |x_{m,a}^{n,b}| = |x| + a + b \). The
Theorem 4.4. The 2-supercategory $\mathcal{U}_{q,\pi}$ is Krull-Schmidt. A complete set of pairwise non-isomorphic indecomposable 1-morphisms of $\mathcal{U}_{q,\pi}$ is given by

$$Q^t\Pi^s\mathcal{E}(a)\mathcal{F}(b)_{\mathcal{U}} = \bigcup_{\lambda\leq b-a, s \in \mathbb{Z}, t \in \{0,1\}} \{Q^s\Pi^t\mathcal{E}(a)\mathcal{F}(b)_{\mathcal{U}}_{\mathcal{U}}, \lambda > b-a, s \in \mathbb{Z}, t \in \{0,1\}\}
$$

Furthermore, if $X$ is an indecomposable 1-morphism of $\mathcal{U}_{q,\pi}$ then

$$\dim_{q,\pi} \text{Hom}_{\mathcal{U}_{q,\pi}}(X, X) \in 1 + q\mathbb{N}[g].$$

Proof. See [19] Section 3.6.1, Proposition 8.2, Proposition 8.3. □
4.5. 2-superfunctors and 2-representations. Recall the notion of a 2-superfunctor from [3, Definition 2.2 (ii)] and graded \((Q, \Pi)\)-2-superfunctor [4, Definition 5.2 (ii)], see also the equivalent notation of superbifunctor from [24, Definition 7.2]. The key point, is that a 2-superfunctor \(\Phi: A \to B\) gives superfunctors \(\Phi_{X,Y}: \text{Hom}_A(X, Y) \to \text{Hom}_B(\Phi X, \Phi Y)\) on each Hom supercategory (compatible with the coherence data when \(A\) and \(B\) are not strict 2-supercategories). In addition, if \(\Phi\) is a \((Q, \Pi)\)-2-superfunctor, then \(\Phi_{X,Y}\) is a \((Q, \Pi)\)-superfunctor.

For any graded 2-supercategory \(A\) there is a canonical strict 2-superfunctor \(\mathbb{J}: K \to K_{q,\pi}\) mapping \(A\) to its \((Q, \Pi)\)-envelope \(A_{q,\pi}\). If the graded 2-supercategory \(A\) is already \((Q, \Pi)\) complete, so that there are 1-morphisms \(\pi_{\lambda}, q_{\lambda}: \lambda \to \lambda\), even 2-isomorphisms \(q_{\lambda} \cong 1_{\lambda}\), and odd 2-isomorphisms \(\pi_{\lambda} \cong 1_{\lambda}\) for all objects \(\lambda\) of \(A\), then \(\mathbb{J}\) is a 2-equivalence [3, Lemma 4.6].

For a \((Q, \Pi)\)-2-supercategory \(B\), let \(\nu\) denote the underlying 2-supercategory. If \(A\) is a graded 2-supercategory and \(B\) is a \((Q, \Pi)\)-2-supercategory, then given any 2-superfunctor \(\Phi: A \to B\), there is a canonical graded 2-superfunctor \(\widetilde{\Phi}: A_{q,\pi} \to B\) such that \(\Phi = \widetilde{\Phi}\mathbb{J}\). By [3, Theorem 4.9], there is a functorial equivalence between the 2-category of graded 2-superfunctors \(\text{Hom}(A, B)\) and the category \(\text{Hom}(A_{q,\pi}, B)\) of \((Q, \Pi)\)-2-superfunctors that sends \(\Phi\) to \(\widetilde{\Phi}\). In particular, \(\widetilde{\Phi}\) has the structure of a \((Q, \Pi)\)-2-superfunctor.

**Definition 4.5.** Let \(\mathcal{U}\) be a graded 2-supercategory. A 2-representation of \(\mathcal{U}\) is a 2-superfunctor \(\Phi: \mathcal{U} \to K\) for some \((Q, \Pi)\)-complete graded 2-supercategory \(K\). By the above remarks, such a 2-representation extends to a canonical \((Q, \Pi)\)-2-superfunctor \(\Phi: \mathcal{U}_{q,\pi} \to K\). In addition, \(K\) is idempotent complete, then \(\Phi\) extends uniquely to a graded \((Q, \Pi)\)-2-superfunctor \(\Phi: \mathcal{U}_{q,\pi} \to K\) by the universal property of idempotent completions.

Breaking down the above definition, given a 2-representation \(\Phi: \mathcal{U} \to K\), the \((Q, \Pi)\)-2-superfunctor \(\Phi\) sends each object \(\lambda\) in \(\mathcal{U}_{q,\pi}\) to a \((Q, \Pi)\)-supercategory \(\Phi(\lambda)\). Each morphism morphism of \(\mathcal{U}_{q,\pi}\) maps to \((Q, \Pi)\)-superfunctors, and 2-morphisms are sent by \(\Phi\) to supernatural transformations.

4.6. 2-representations on deformed cyclotomic quotients. In this section we extend the 2-representations on cyclotomic quotients defined by Kang, Kashiwara, and Oh [23, 24]. The primary difference in our presentation is that the deformed quotients they consider only contain even elements as coefficients, while ours contain odd elements. More precisely, the \(c_i\) are zero in (3.26) when \(i\) is odd. Some of the arguments in [23] extend in a straightforward manner, while others are complicated by the additional odd elements. In this section we focus mainly on those arguments that do not immediately extend.

We start by defining the structure of 2-representation on deformed cyclotomic quotients. The injective morphism

\[
\begin{cases}
\text{ONH}_k & \to \text{ONH}_{k+1}, \\
x_a & \mapsto x_{a+1}, \\
\tau_a & \mapsto \tau_{a+1},
\end{cases}
\]

descends to a morphism \(\text{ONH}_k^n \to \text{ONH}_{k+1}^n\). We denote by \(E_n\) (resp. \(F_n\)) the induction (resp. restriction) superbimodule along these morphisms. We will think of \(E_n\) as the \((\text{ONH}_{k+1}^n, \text{ONH}_k^n)\)-superbimodule \(\text{ONH}_{k+1}^n\). There is an endomorphism \(x\) of \(E_n\) of degree 2 and parity \(\bar{1}\) given by right multiplication by \(x_{k+1}\). More precisely, \(x\) is the \((\text{ONH}_{k+1}^n, \text{ONH}_k^n)\)-superbimodule endomorphism defined by

\[
\begin{cases}
\text{ONH}_{k+1}^n & \to \text{ONH}_{k+1}^n, \\
x & \mapsto x_{k+1},
\end{cases}
\]

with \(\varphi\) as in (2.2). Similarly, there is an endomorphism \(\tau\) of \(F_n\) of degree \(-2\) and parity \(\bar{1}\) given by right multiplication by \(\tau_{k+1}\). More precisely, \(\tau\) is the \((\text{ONH}_{k+2}^n, \text{ONH}_k^n)\)-superbimodule endomorphism
defined by
\[
\begin{cases}
\text{ONH}_{k+2}^n & \rightarrow \text{ONH}_{k+2}^n, \\
h & \mapsto -\varphi(h)\tau_{k+1}.
\end{cases}
\]

Consider the \((Q, \Pi)\)-supercategory
\[\mathcal{L}(n) = \bigoplus_{k=0}^{n} \text{sProj}(\text{ONH}_k^n)\]

**Theorem 4.6.** The functors \((E_n, F_n)\) restrict to an adjoint pair of superendofunctors of \(\mathcal{L}(n)\). Together with the data of the natural transformations \(x: E_n \rightarrow \Pi Q^{-2} E_n\) and \(\tau: E_n E_n \rightarrow \Pi Q^2 E_n E_n\), this defines a 2-representation of \(\mathcal{U}_{q, \pi}\) on \(\mathcal{L}(n)\).

In the case where \(d\) is specialized to 0, this theorem is the type \(A_1\) case of [23, Theorems 8.9 and 9.6]. In the rest of this subsection, we explain how to generalize the proof to our case.

The first main argument of the proof is to realize \(E_n\) as a cokernel. Consider the following \((\text{ONH}_{k+1}, \text{ONH}_k^n)\)-superbimodules:
\[
E = \text{ONH}_{k+1} \otimes_{\text{ONH}_k} \text{ONH}_k^n,
\]
\[
\overline{E} = \text{ONH}_{k+1, \xi} \otimes_{\text{ONH}_k} \Pi^{n+k} \text{ONH}_k^n,
\]
where \(\text{ONH}_{k+1, \xi} = \text{ONH}_{k+1}\) as a left superbimodule, with the right action of \(\text{ONH}_k\) given by the morphism
\[
\xi: \begin{cases}
\text{ONH}_k & \rightarrow \text{ONH}_{k+1}, \\
x_a & \mapsto x_{a+1}, \\
\tau_a & \mapsto \tau_{a+1}.
\end{cases}
\]

Then, there is a short exact sequence of \((\text{ONH}_{k+1}, \text{ONH}_k^n)\)-superbimodules
\[(4.16)\]
\[0 \rightarrow \overline{E} \xrightarrow{P} E \rightarrow E_n \rightarrow 0,
\]
where the map \(E \rightarrow E_n\) is the canonical quotient map, and \(P\) is the morphism of degree \(2(n-k)\) and parity 0 given by
\[
\begin{cases}
\overline{E} & \rightarrow E, \\
h \otimes \pi^{n+k} y & \mapsto h a^n(x_1)\tau_1 \ldots \tau_k \otimes y.
\end{cases}
\]

Establishing this exact sequence can be done as in [23, Section 8]. As a consequence, we can deduce that \(\text{ONH}_{k+1}^n\) is projective as a right \(\text{ONH}_k^n\)-supermodule [23, Theorem 8.7]. This proves that \((E_n, F_n)\) send projective modules to projective modules, and thus restrict to superendofunctors of \(\mathcal{L}(n)\).

A key difference with [23], however, is that the map \(P\) is not \(A_n[t]\)-linear, in a sense that we now explain. The superbimodules \(\overline{E}, E, E_n\) can be endowed with a structure of right \(A_n[t]\)-modules given by the following formulas.

- **On** \(E_n, z \cdot t = zx_{k+1}\).
- **On** \(E, (h \otimes y) \cdot t = hx_{k+1} \otimes \varphi(y)\).
- **On** \(\overline{E}, (h \otimes \pi^{n+k} y) \cdot t = (-1)^k hx_1 \otimes \pi^{n+k} \varphi(y)\).

Then, the canonical quotient map \(E \rightarrow E_n\) is \(A_n[t]\)-linear. But because of relations \((3.27)\), \(P\) is not \(A_n[t]\)-linear. Instead, it is \(A_n[t]\)-linear *up to an automorphism* of \(A_n[t]\). More precisely, we have the relation
\[(4.17)\]
\[P(yt) = P(y)(t - 2d).
\]

Together with Mackey formulas for odd nilHecke algebras (see [23, Section 5]), the short exact sequence \((4.16)\) gives rise to an exact sequence of \((\text{ONH}_{k+1}^n, \text{ONH}_k^n)\)-superbimodules
\[0 \rightarrow \ker(A) \rightarrow E_n \text{Proj}_n \xrightarrow{\sigma} F_n E_n \rightarrow \text{coker}(A) \rightarrow 0,
\]
where $A : A_n[t] \otimes_{A_n} \Pi^n \text{ONH}_{k} \to A_n[t] \otimes_{A_n} \text{ONH}_{k}$ is a certain morphism of $(\text{ONH}_{k}, \text{ONH}_{k})$-superbimodules induced by $P$ (see [23, Section 9]). In this sequence, the morphism $\sigma$ is the image in $\mathcal{L}(n)$ of the 2-morphism $\bigotimes$ of the 2-supercategory $\mathcal{U}$.

Contrary to the map $P$, the map $A$ is not $A_n[t]$-linear, even up to an automorphism. However, the default of $A_n[t]$-linearity is controlled. Namely, for all $g(t) \in A_n[t] \otimes A_n \mathcal{Z}(\text{ONH}_{k})$ of degree $\ell$ in $t$, the polynomial $A(\nu g(t)) - A(g)g((t - 2d))$ has degree at most $\ell - 1$ in $t$. This can be proved as in [23, Lemma 9.5], keeping in mind that the $A_n[t]$-linearity of $P$ is replaced by formula (4.17) in our case.

Once this is established, we deduce that $A(t^m \otimes \pi^n 1)$ is a monic polynomial in $t$ of degree $m + n - 2k$, up to a sign. From this, it follows that

$$\ker(A) = \left( \bigoplus_{i=0}^{n-2k-1} t^i \otimes \text{ONH}_{k} \right) \quad \text{and} \quad \coker(A) = 0 \quad \text{if} \quad n - 2k < 0,$$

$$\ker(A) = 0 \quad \text{and} \quad \coker(A) = \left( \bigoplus_{k=0}^{n-2k-1} t^i \otimes \text{ONH}_{k} \right) \quad \text{if} \quad n - 2k \geq 0.$$ 

The commutation relations (4.3) then follow, completing the proof of Theorem 4.6.

### 4.7. Universal quotients and their properties

In this section it is convenient to work first with the graded 2-supercategories $\mathcal{U}$ and its 2-representations and then later pass to its $(Q, \Pi)$-envelope $\mathcal{U}_{q, \pi}$.

Consider the 2-representation $\text{Hom}_{\mathcal{U}}(-n, -)$ of $\mathcal{U}$. Let $R(n)$ be the 2-subrepresentation generated by $\mathcal{F}_{-n}$. The universal quotient is the 2-representation $\mathcal{U}^n$ of $\mathcal{U}$ defined as

$$\mathcal{U}^n = \text{Hom}_{\mathcal{U}}(-n, -)/R(n).$$

In particular, for all weights $m < -n$, the 1-morphisms $\mathbb{I}_m$ vanish in this quotient. This implies that $\mathbb{I}_m = 0$ for $m > n$, but we do not need this fact in what follows.

**Proposition 4.7.** [47, Lemma 5.4] Let $\mathcal{V}$ be a 2-representation of $\mathcal{U}$, and let $M \in \mathcal{V}_{-n}$ be such that $\mathcal{F}M = 0$. Then there is a canonical morphism of superalgebras $\text{End}_{\mathcal{U}^n}(\mathbb{I}_{-n}) \to \text{End}_{\mathcal{V}}(M)$ and a fully-faithful morphism of 2-representations

$$\Phi_M : \mathcal{U}^n \otimes_{\text{End}_{\mathcal{U}^n}(\mathbb{I}_{-n})} \text{End}_{\mathcal{V}}(M) \to \mathcal{V}$$

such that $\Phi_M(\mathbb{I}_{-n}) = M$.

Assume furthermore that every object of $\mathcal{V}$ is a direct sum of direct summands of objects of the form $\mathcal{E}^i M$ for $i \in \mathbb{N}$. Then $\Phi_M$ is an equivalence of 2-representations.

**Proof.** This proof is analogous to the even case. $\square$

**Proposition 4.8.** We have an isomorphism of superalgebras $\text{End}_{\mathcal{U}^n}(\mathbb{I}_{-n}) \simeq A_n$ and an equivalence of 2-representations

$$\mathcal{U}^n \simeq \mathcal{L}(n)$$

of $\mathcal{U}$. In particular, $\mathcal{L}(n)$ satisfies the universal property of Proposition 4.7. By the discussion in Section 4.5, this equivalence extends to an equivalence $(\mathcal{U}^n)_{q, \pi} \cong \mathcal{L}(n)$ of 2-representations of $\mathcal{U}_{q, \pi}$.

**Proof.** Put $A'_n = \text{End}_{\mathcal{U}^n}(\mathbb{I}_{-n})$. The object $A_n \in \mathcal{L}(n)_{-n}$ satisfies $F_{\mathcal{U}}(A_n) = 0$. Furthermore, every object of $\mathcal{L}(n)$ is a direct sum of direct summands of objects of the form $E^i_n(A_n)$. Thus, by the universal property, there is a canonical morphism of superalgebras $A'_n \to A_n$ and an equivalence of 2-representations

$$\mathcal{U}^n \otimes_{A'_n} A_n \simeq \mathcal{L}(n)$$

sending $\mathbb{I}_{-n}$ to $A_n$. To prove the proposition, it suffices to show that the canonical morphism $A'_n \to A_n$ is an isomorphism. We do this by constructing an inverse.
We know from the proof of the nondegeneracy conjecture [15, Theorem 6.2] that \text{End}_U(1_n) has a basis given by certain products of bubbles defined in Section 4.2. Since \( A'_n \) is a quotient of \text{End}_U(1_n), we deduce that (classes of) monomials in bubbles span \( A'_n \). However, by definition of \( \Omega^n \), real bubbles vanish in \( A'_n \) since they factor through the weight \(-n - 2\). So \( A'_n \) is spanned by monomials in fake bubbles (see (4.9)). Using relations (4.9), we deduce that there is a surjective morphism

\[
\begin{cases}
A_n & \rightarrow A'_n, \\
c_i & \mapsto \circ
\end{cases}
\]

Let us now prove that the canonical morphism \( A'_n \rightarrow A_n \) sends the fake bubble of degree \( i \) to \( c_i \). The structure of 2-representation of \( L(n) \) in weights \(-n\) and \(-n + 2\) is given by

\[
\begin{align*}
\circ \quad \text{Proj}(A_n) & \quad \text{Proj}(A_n[x_1]/a^n(x_1)) \\
E_n = \text{Ind} & \quad F_n = \text{Res}
\end{align*}
\]

The \( \mathfrak{sl}_2 \)-relations of \( \Omega \) give rise to an isomorphism of \((A_n, A_n)\)-superbimodules in \( L(n) \):

\[
(\beta_0, \ldots, \beta_{n-1}) \mapsto \sum_{i=0}^{n-1} x_i^i a_i.
\]

Let \([\alpha_0 \ldots \alpha_{n-1}]\) denote this direct sum of maps and define

\[
\begin{bmatrix}
\beta_0 \\
\vdots \\
\beta_{n-1}
\end{bmatrix} = [\alpha_0 \ldots \alpha_{n-1}]^{-1}
\]

where \( \eta = \alpha_0 \) is the unit of the adjunction. Then by [4, Definition 2.3], for all \( i \in \{1, \ldots, n\} \) we have

\[
(-1)^{i+1} \beta_{n-i} \circ x^n \circ \eta.
\]

Now, it suffices to observe that

\[
(\beta_{n-i} \circ x^n \circ \eta)(1) = \beta_{n-i}(x^n_1) = \beta_{n-i} \left( \sum_{\ell=0}^{n-1} (-1)^{n-\ell-1} x_1^\ell c_{n-\ell} \right) = (-1)^{i-1} c_i.
\]

Hence, the image of the fake bubble of degree \( i \) in \( A_n \) is \( c_i \). It follows that the canonical morphism \( A'_n \rightarrow A_n \) is an isomorphism. \[ \square \]

4.8. Faithfulness of universal quotients. Define a \((Q, \Pi)\)-supercategory

\[
L(n)\text{-sbim} = \bigoplus_{0 \leq k, \ell \leq n} (\text{ONH}_k^n, \text{ONH}_\ell^n)\text{-sbim}.
\]

The structure of 2-representation on \( L(n) \) induces \((Q, \Pi)\)-superfunctors \( \Phi_{n, \lambda} : \Omega_{n, \lambda} \rightarrow L(n)\text{-sbim} \) for all \( \lambda \in \mathbb{Z} \). To simplify the notation, we drop the dependence on \( \lambda \) and simply denote this superfunctor by \( \Phi_n \).
Theorem 4.9.

(1) Let $C$ be a complex of 1-morphisms of $\Omega$. If $\Phi_n(C)$ is null-homotopic for all $n \in \mathbb{N}$, then $C$ is null-homotopic.

(2) Let $f$ be a morphism between complexes of 1-morphisms of $\Omega$. If $\Phi_n(f)$ is a homotopy equivalence for all $n \in \mathbb{N}$, then $f$ is a homotopy equivalence.

Under mild boundedness assumptions for the complexes, the theorem can be improved so that it suffices to verify the assumptions on $\Phi_n(C)$ and $\Phi_n(f)$ in the derived category of superbimodules.

Theorem 4.10.

(1) Let $C$ be a complex of 1-morphisms of $\Omega$ such that for every integrable 2-representation $\mathcal{V}$ and every object $M$ of $\mathcal{V}$, the complex $C(M)$ is bounded. If $\Phi_n(C)$ is acyclic for all $n \in \mathbb{N}$, then $C$ is null-homotopic.

(2) Let $f$ be a morphism between complexes of 1-morphisms of $\Omega$ that both satisfy the boundedness condition in (1). If $\Phi_n(f)$ is a quasi-isomorphism for all $n \in \mathbb{N}$, then $f$ is a homotopy equivalence.

These theorems are odd versions of [51, Theorems 4.3, 4.14]. The proofs are similar, and we recall the main arguments. Since $\Omega$ is Krull-Schmidt (see [19, Section 3.6.1]), then a general result about representations of Krull-Schmidt categories ([51, Theorem 4.4]), reduces the proof of these theorems to the following lemma.

Lemma 4.11. Let $G$ be an indecomposable 1-morphism of $\Omega\mathbb{A}$. If $n$ is large enough and of the same parity as $\lambda$, then $\Phi_n(G)$ is indecomposable and the morphism $\text{End}_{\mathcal{L}(n)}(G) \to \text{End}_{\mathcal{L}(n)-\text{sbim}}(\Phi_n(G))$ is local.

Using [19, Proposition 8.3], which essentially follows from the corresponding arguments in the even case [35, Proposition 9.8] using adjunction, it suffices to prove this lemma for indecomposable 1-morphisms of the form $G = E(a)\mathbb{1}_\lambda$ for $a \in \mathbb{N}$ and $\lambda \in \mathbb{Z}$. To do so, we must study the bimodules $\Phi_n(E(a)1_{n+2(k-a)})$, where $a \leq k$. For this task, it is more convenient to work with the description $\text{ONH}_k^\mathbb{A} \cong \text{ONH}_k \otimes \text{OA}_{n-k}$ established in Corollary 3.15. Unraveling the definitions, we see that as an $(\text{ONH}_k \otimes \text{OA}_{n-k}, \text{ONH}_{k-a} \otimes \text{OA}_{n-k+a})$-bimodule, we have

$$\Phi_n(E(a)1_{n+2(k-a)}) = \text{ONH}_k e_{[k-a+1,k]} \otimes \text{OA}_{n-k},$$

where the right action of $\text{OA}_{n-k+a}$ is defined in (3.30), and for $\ell \leq m \leq k$, $e_{[\ell,m]}$ is the idempotent of $\text{ONH}_k$ defined by

$$e_{[\ell,m]} = (-1)^{\ell+1}(\tau_{\ell} \ldots \tau_{m-1})(\tau_{\ell} \ldots \tau_{m-2}) \ldots \tau_{\ell} x_{m}^{0} x_{m-1}^{1} \ldots x_{\ell}^{m-\ell}.$$ 

We put $x_{[\ell,m]} = (-1)^{\ell+1} x_{m}^{0} x_{m-1}^{1} \ldots x_{\ell}^{m-\ell}$, and $\tau_{[\ell,m]} = (\tau_{\ell} \ldots \tau_{m-1})(\tau_{\ell} \ldots \tau_{m-2}) \ldots \tau_{\ell}$, so that $e_{[\ell,m]} = \tau_{[\ell,m]} x_{[\ell,m]}$. Note that if $\ell \leq \ell' \leq m \leq k$, then $e_{[\ell,m]} e_{[\ell',m']} = e_{[\ell',m]} e_{[\ell,m]} = e_{[\ell,m]}$. Similarly, we have

$$\Phi_n(F(a)1_{n+2k}) = e_{[k-a+1,k]} \text{ONH}_k \otimes \text{OA}_{n-k},$$

where, for $\ell \leq m \leq k$, $e_{[\ell,m]}$ is the idempotent of $\text{ONH}_k$ defined by

$$e_{[\ell,m]} = (-1)^{m+1}(\tau_{m-1} \tau_{m-2} \ldots \tau_{\ell})(\tau_{m-1} \tau_{m-2} \ldots \tau_{\ell}).$$

We put $x_{[\ell,m]} = (-1)^{m+1} x_{m}^{0} x_{m-1}^{1} \ldots x_{\ell}^{m-\ell}$. Note that $(\tau_{m-1} \tau_{m-2} \ldots \tau_{\ell})(\tau_{m-1} \tau_{m-2} \ldots \tau_{\ell}) = \tau_{[\ell,m]}$ by [18, (3.51)]. Thus, $e_{[\ell,m]} = x_{[\ell,m]} \tau_{[\ell,m]}$. Note that if $\ell \leq \ell' \leq m \leq k$, then $e_{[\ell,m]} e_{[\ell',m']} = e_{[\ell',m']} e_{[\ell,m]} = e_{[\ell,m]}$. 
To study the indecomposability of the bimodule $\text{ONH}_{k\varepsilon[k-a+1,k]} \otimes \Lambda_n$, it is simpler to pass to the coefficient ring using Morita equivalences.

**Definition 4.12.** For any disjoint subsets $U_1, \ldots, U_k \subset \{1, \ldots, n\}$, we write $\Lambda_n^{U_1, \ldots, U_k} \subset \text{OPol}_n$ for the set of odd polynomials that are odd symmetric separately in each of the subsets of variables $\{x_j, j \in U_i\}$ for $1 \leq i \leq k$. For example, $\Lambda_n^0 = \text{OPol}_n$, and $\Lambda_n^{[1,n]} = \Lambda_n$.

Put $X = \{1, \ldots, k-a\}$, $Y = \{k-a+1, \ldots, k\}$ and $Z = \{k+1, \ldots, n\}$. With this notation, using Morita equivalences, we see that the endomorphism algebra of the bimodule $\text{ONH}_{k\varepsilon[k-a+1,k]} \otimes \Lambda_n$ is isomorphic to the endomorphism algebra of $\Lambda_n^{X,Y} \otimes \Lambda_n$ as an $(\Lambda_n \otimes \Lambda_n, \Lambda_n \otimes \Lambda_n \otimes \Lambda_n)$-bimodule, where again the right action of $\Lambda_n$ is defined in (3.30). Identifying $\Lambda_n^{X,Y} \otimes \Lambda_n$ with $\Lambda_n^{X,Y,Z}$ via the multiplication map, we can further identify this endomorphism algebra as the endomorphism algebra of $\Lambda_n^{X,Y,Z}$ as a $(\Lambda_n^{X,Y,Z}, \Lambda_n^{X,Y,Z})$-bimodule, where the isomorphism involves the involution used to define the right action in equation (3.30). Hence, to prove the result, it suffices to prove that $\Lambda_n^{X,Y,Z}$ is indecomposable and that it has $k$ as a bimodule endomorphism algebra in degree 0.

In Theorem 4.13, we prove that $\Lambda_n^{X,Y,Z}$ is generated as an $(\Lambda_n^{X,Y,Z}, \Lambda_n^{X,Y,Z})$-bimodule by the element 1. This implies that $\Lambda_n^{X,Y,Z}$ is indecomposable as a bimodule, and that there is an injective homogeneous map

$$\text{End}(\Lambda_n^{X,Y,Z}, \Lambda_n^{X,Y,Z}) \hookrightarrow \Lambda_n^{X,Y,Z},$$

given by $f \mapsto f(1)$. In particular, the degree zero part of the endomorphism algebra is $k$. Once this is established, Theorem 4.9 follows.

In the even case, proving the indecomposability is elementary (see [51, Lemma 4.11]). The proof in the odd case is much more difficult due to the fact that (symmetric) odd polynomials do not commute.

**Theorem 4.13.** The multiplication map $\Gamma: \Lambda_n^{X,Y,Z} \otimes \Lambda_n^{X,Y,Z} \to \Lambda_n^{X,Y,Z}$ is surjective. Thus, $\Lambda_n^{X,Y,Z}$ is cyclic and generated by 1 as a $(\Lambda_n^{X,Y,Z}, \Lambda_n^{X,Y,Z})$-bimodule.

**Proof.** To prove this, we only need to show that $s_\mu(X)s_\lambda(Y)s_\nu(Z)$ lies in the image. This follows from Lemma 4.14 below. \hfill \Box

Equation (3.15) implies the following equation.

$$-1)^{m(k-a)}\varepsilon_m(Y) = \varepsilon_m(X \cup Y) - \sum_{j=0}^{m-1} (-1)^{j(k-a)}\varepsilon_{m-j}(X)\varepsilon_j(Y)$$

**Lemma 4.14.** For any partitions $\mu, \lambda, \nu$, the product $s_\mu(X)s_\lambda(Y)s_\nu(Z) \in \Gamma(\Lambda_n^{X,Y,Z} \otimes \Lambda_n^X)$.

**Proof.** Since homogeneous odd symmetric polynomials in disjoint sets of variables commute up to a sign, it suffices to prove that $s_\lambda(Y) \in \Gamma(\Lambda_n^{X,Y,Z} \otimes \Lambda_n^X)$. The decomposition of odd Schur functions in terms of odd elementary symmetric functions (3.19) implies that this is true if $\varepsilon_m(Y) \in \Gamma(\Lambda_n^{X,Y,Z} \otimes \Lambda_n^X)$ for all $m$. This follows immediately by induction on $m$ from (4.19) and the fact that $\varepsilon_{m-j}(X)$ and $\varepsilon_j(Y)$ commute up to a sign. \hfill \Box

5. Braiding complexes

5.1. Definition of super equivalences.

**Definition 5.1.** Given $\lambda \in \mathbb{Z}$, we define the odd Chuang-Rouquier complex $\Theta_{\mathbb{Z},\lambda}$ of objects of $\text{Hom}_\mathfrak{sl}(\lambda, -\lambda)$ as follows.
The $r$th component of $ΘI_λ$ is $Q^rF^{(λ+r)}E^{(r)}$.

- The differential $d$ is given by the composition of $1_F^{(λ+r)}η_1E^{(r)} : F^{(λ+r)}E^{(r)} → F^{(λ+r)}FIE^{(r)}$ with the projection on $F^{(λ+r+1)}E^{(r+1)}$ given by $e'_{λr+1}$. The fact that the differential squares to zero follows from the fact that $e_2e'_{2} = 0$.

Let $n ∈ N$. For all $λ ∈ Z$, the complex $Φ_n(ΘI_λ)$ is bounded. Let us describe this explicitly. Let $k ∈ \{0, \ldots, n\}$. Then

$$Φ_n(ΘI_{n-k+2}) =$$

$$0 → ΩH_n^e[k+1,n-k] → \cdots → e'_{n-k+1,n-r}ΩH_n^e[k+1,n-r] → \cdots e'_{n-k+1,n}ΩH_n^e[k+1,n] → 0$$

This complex has $k + 1$ non-zero terms, the last one being in cohomological degree $n - k$.

Now let $k ∈ \{\frac{n}{2}, \ldots, n\}$. Then as complexes of graded $(ΩH_n^e, ΩH_n^f)$-bimodules we have

$$Φ_n(ΘI_{n-k+2}) =$$

$$0 → e'_{n-k+1,k}ΩH_n^f[k] → \cdots → e'_{n-k+1,n-r}ΩH_n^f[k+1,n-r] → \cdots e'_{n-k+1,n}ΩH_n^f[k+1,n] → 0$$

This complex has $n - k + 1$ non-zero terms, the last one being in cohomological degree $n - k$.

### 5.2. Invertibility

#### 5.2.1. Bases.

We fix an integer $n$ and study the action of the odd Chuang-Rouquier complex $Θ$ in the 2-representation $L(n)$. We start by constructing a basis for $e'_{[t,m]}ΩH_n^e[k,m]$, which is the form of a general term of $Φ_n(Θ)$, as a left $Ω\Lambda_n^{[t,n]}$-module. Define the following sets:

$$X_{t,m} = \{a = (a_1, \ldots, a_m), a_i ≤ n - i\},$$

$$Y_{t,m} = \{a ∈ X_{t,m}, a_1 > \cdots > a_m\}.$$  

We do not indicate the dependence on $n$ (which is fixed for this section) in the notation for simplicity.

**Lemma 5.2.** The set $\{∂_{ω[\ell,m]}(x^n), a ∈ Y_{t,m}\}$ generates $Ω\Lambda_n^{[t,m],[m+1,n]}$ as a left $Ω\Lambda_n^{[t,n]}$-module.

**Proof.** We know that $\{x^a, a ∈ X_{t,m}\}$ is a basis of $Ω\Lambda_n^{[m+1,n]}$ as an $Ω\Lambda_n^{[t,n]}$-module. Furthermore, the map $∂_{ω[\ell,m]} : Ω\Lambda_n^{[m+1,n]} → Ω\Lambda_n^{[t,m],[m+1,n]}$ is surjective and $Ω\Lambda_n^{[t,n]}$-linear, up to an automorphism. It follows that $\{∂_{ω[\ell,m]}(x^n), a ∈ X_{t,m}\}$ generates $Ω\Lambda_n^{[t,m],[m+1,n]}$ as a left $Ω\Lambda_n^{[t,n]}$-module.

Put $b(a) = ∂_{ω[\ell,m]}(x^n)$. To finish proving the lemma, it suffices to prove that

$$\text{Span}_{ℤ} \{b(a), a ∈ X_{t,m}\} = \text{Span}_{ℤ} \{b(a), a ∈ Y_{t,m}\}.$$  

To do so, we prove by induction on $r$ the following property $P(r)$:

∀ $a ∈ X_{t,m}$, $b(a) ∈ \text{Span}_{ℤ} \{b(a), a_1 < \ldots < a_{m-r} ≤ \max\{a_m, \ldots, a_{m-r}\}, a_i = a_i$ if $i < m-r\}$.

The property $P(0)$ clearly holds. Assume that $P(r-1)$ holds and let $a ∈ X_{t,m}$. To prove $P(r)$, we may assume that $a_m < \ldots < a_{m-r+1}$ by $P(r-1)$. If we also have $a_{m-r+1} < a_{m-r}$, then there is nothing to prove. Otherwise, let $k ≤ 0$ be such that $a_{m-r} = a_{m-r+1} - k$. We proceed by induction on $k$. If $k = 0$, then $b(a) = 0$, so $P(r)$ holds. If $k ≥ 1$, by the Shuffle Lemma [18, Lemma 4.4] we have $b(a) ∈ \text{Span}_{ℤ} \{b(s_{m-r}(a)), b(s_{m-r}(a) + (\ldots, -1, 1, \ldots)), b(a + (\ldots, -1, 1, \ldots))\},$  

where $s_{m-r}$ denotes the usual action of the transposition $s_{m-r}$ on tuples, and $(\ldots, -1, 1, \ldots)$ is the tuple whose entry $m - r - 1$ is $-1$, entry $m - r$ is $1$, and all other entries are $0$. The property $P(r-1)$ applies directly to $b(s_{m-r}(a))$ and $b(s_{m-r}(a) + (\ldots, -1, 1, \ldots))$, and the case $k - 2$ of $P(r)$ applies to $b(a + (\ldots, -1, 1, \ldots))$. It follows that the property $P(r)$ holds for $b(a)$. Thus, the induction is complete. The lemma now follows from $P(m-ℓ)$.

**Corollary 5.3.** The set $\{∂_{ω[\ell,m]}(x^n), a ∈ Y_{t,m}\}$ is a basis of $Ω\Lambda_n^{[t,m],[m+1,n]}$ as a left $Ω\Lambda_n^{[t,n]}$-module.
Proof. By the previous lemma, the set \( \{ \partial_{\omega_0[t,m]}(x^a), a \in Y_{t,m} \} \) generates \( \mathcal{O}\Lambda^{[t,m],[m+1,n]}_n \) as a left \( \mathcal{O}\Lambda^{[t,n]}_n \)-module. We can conclude using graded dimensions as in [51, Lemma 5.5]. \( \square \)

**Corollary 5.4.** The set \( \{ \partial_{\omega_0[t,m]}(x^a) \otimes 1, a \in Y_{t,m} \} \) is a basis of \( \mathcal{O}\Lambda^{[t,m],[m+1,n]}_n \) as a left \( \mathcal{O}\Lambda^{[t,n]}_n \)-module.

We denote by \( \mathcal{S}_{[k,m]} \) the subgroup of \( \mathcal{S}_m \) generated by the transpositions \( s_k, \ldots, s_{m-1} \). For each coset of \( \omega \in \mathcal{S}_m/\mathcal{S}_{[k,m]} \), we fix a reduced expression for the longest element in \( \omega \) and denote by \( \tau_\omega \) the corresponding element of \( \mathcal{ONH}_n \). The element \( \tau_\omega \) depends on the choice of the reduced expression up to an overall sign only.

**Definition 5.5.** For \( a \in Y_{t,m} \) and \( \omega \in \mathcal{S}_m/\mathcal{S}_{[k,m]} \), define an element \( b_m(a, \omega) \in e'[t,m] \mathcal{ONH}^n_m e'[k,m] \) by

\[
b_m(a, \omega) = e'[t,m] x^a \tau_\omega e'[k,m].
\]

**Theorem 5.6.** The set \( \{ b_m(a, \omega), a \in Y_{t,m}, \omega \in \mathcal{S}_m/\mathcal{S}_{[k,m]} \} \) is a basis of \( e'[t,m] \mathcal{ONH}^n_m e'[k,m] \) as a \( \mathcal{O}\Lambda^{[t,n]}_n \)-module.

**Proof.** Observe that

\[
\tau_\omega(t,m)x^a \in \partial_{\omega_0(t,m)}(x^a) + \sum_{z > 1} \mathcal{O}\Lambda^{[m+1,n]}_n \tau_z.
\]

Hence, we have

\[
b_m(a, \omega) \in x'[t,m] \partial_{\omega_0(t,m)}(x^a) \tau_\omega e'[k,m] + \sum_{z > \omega} \mathcal{O}\Lambda^{[m+1,n]}_n \tau_z e'[k,m].
\]

We also know that

- \( \{ \tau_\omega e'[k,m], \omega \in \mathcal{S}_m/\mathcal{S}_{[k,m]} \} \) is free as a \( \mathcal{OPol}_n \)-module,
- \( \{ \partial_{\omega_0(t,m)}(x^a), a \in Y_{t,m} \} \) is free as a \( \mathcal{O}\Lambda^{[t,n]}_n \)-module by **Corollary 5.3**.

We deduce that the set \( \{ b_m(a, \omega), a \in Y_{t,m}, \omega \in \mathcal{S}_m/\mathcal{S}_{[k,m]} \} \) is free over \( \mathcal{O}\Lambda^{[t,n]}_n \). To conclude that the set is a basis, we compute graded dimensions as in [51, Theorem 5.3]. \( \square \)

**5.2.2. Differential.** In the general form considered for the terms of \( \Phi_n(\Theta) \), the differential of \( \Phi_n(\Theta) \) takes the form

\[
d_m = \begin{cases} 
  e'[t,m] \mathcal{ONH}^n_m e'[k,m] & \rightarrow e'[t,m+1] \mathcal{ONH}^n_{m+1} e'[k,m+1], \\
  h & \rightarrow e'[t,m+1] h e'[k,m+1]. 
\end{cases}
\]

**Theorem 5.7.** We have \( \ker(d_{m+1}) = \text{im}(d_m) \).

To prove this theorem, we merely need adapt the proof of [51, Proposition 5.8] to the odd case. The proof is based on computing \( d_m(b_m(a, \omega)) = e'[t,m+1] x^a \tau_\omega e'[k,m+1] \).

In the case \( a_m > 0 \), then we simply have \( d_m(b_m(a, \omega)) = b_{m+1}((a, 0), \omega) \).

In the case \( a_m = 0 \), let \( r \) be the largest integer such that \( a_m = 0 < a_{m-1} = 1 < \ldots < a_{m-r} = r \). In particular, we have \( a_{m-r-1} > r + 1 \). The key point of the computation is to observe that we have the following equality in \( \mathcal{ONH}_{m+1} \):

\[
\tau_{m-r} x^{m-r+1} x_m \tau_{m-r} = x_{m-r} x_m x_{m-r+1}.
\]

It follows that

\[
\tau_{\omega_0(t,m+1)} x_a^{\ell} \cdots x_{m-r+1}^{a_{m-r+1}} x_{m-r} x_m \tau_{m-r} = (-1)^{\frac{r(r-1)}{2}} \tau_{\omega_0(t,m+1)} x_a^{\ell} \cdots x_{m-r+1}^{a_{m-r+1}} (x_{m-r} x_{m-r+1} \tau_{m-r}) x_{m-r+2} \cdots x_m
\]

\[
= (-1)^{\frac{r(r-1)}{2}} \tau_{\omega_0(t,m+1)} x_a^{\ell} \cdots x_{m-r+1}^{a_{m-r+1}} (x_{m-r} x_{m-r+1}) x_{m-r+2} \cdots x_m.
\]
This last term is 0 since \( \tau_{\omega_l}m_{l+1} \tau_m^r = 0 \). Hence, we have
\[
(1) \quad \tau_{\omega_l}m_{l+1}x^\ell r_{m-r+1} x^r_{m-r+1} \cdots \tau_m \tau_m^r m_{m-r+2} \cdots m
\]
We can repeat this argument for \( \tau_{\omega_l}m_{l+1} \) and obtain
\[
\tau_{\omega_l}m_{l+1}x^\ell r_{m-r+1} x^r_{m-r+1} \cdots \tau_m \tau_m^r m_{m-r+1} \cdots m_{m-r+2} \cdots m_{m-1}.
\]
It follows that
\[
d_m(b_m(a, \omega)) = (-1)^{e[0]} e'[m_{l+1}]x^\ell r_{m-r+1} x^r_{m-r+1} \cdots \tau_m \tau_m^r m_{m-r+1} \cdots m_{m-r+2} \cdots m_{m-1}.
\]
Hence, the computation of \( d_m(b_m(a, \omega)) \) is the same as in the even case, up to a sign. Thus, the results from [51] apply and Theorem 5.7 follows.

5.2.3. Top Cohomology. As a consequence of Theorem 5.7, we deduce the following corollary.

**Corollary 5.8.** For all \( \lambda \in \mathbb{Z} \), the cohomology of \( \Phi_n(\Theta \mathbb{L}) \) is concentrated in top cohomological degree. We now prove that the top cohomology of \( \Phi_n(\Theta \mathbb{L}) \) is invertible as a bimodule. Put
\[
C_{n,k} = H^\bullet(\Phi_n(\Theta \mathbb{L}^{n-2k})).
\]
To prove that \( C_{n,k} \) is an invertible \((\text{ONH}^n_{n-k}, \text{ONH}^n_{n-k})\)-bimodule, it is simpler to pass to the coefficient rings first and establish the following result.

**Theorem 5.9.** We have an isomorphism of \((\text{ONH}^n_{n-k}, \text{ONH}^n_{n-k}), (\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n})\)-bimodules
\[
e'[1-n-k]C_{n,k}e[1-k] \simeq (\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n}),
\]
where, on \( (\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n}) \), the left action of \( (\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n}) \) is given by multiplication, and the right action of \( (\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n}) \) is given by multiplication pre-composed by \( \omega_0 \). In particular, \( e'[1-n-k]C_{n,k}e[1-k] \) is an invertible \((\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n}), (\text{ONH}^n_{n-k}, (\text{ONH}^n_{n-k})))\)-bimodule.

**Proof.** Since the action of the differential on the bases constructed in 5.2.1 is the same as in the even case up to a sign, we know by [51, Theorem 5.11] that there is an isomorphism of left \( (\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n}) \)-modules
\[
\begin{align*}
(\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n}) & \rightarrow e'[1-n-k]C_{n,k}e[1-k], \\
f & \mapsto f x'y'_{[n-1-k]} x'_{[n+1-k,n]} \tau_{\omega_0(1-n-2k)} x'[1-k]x'[k+n].
\end{align*}
\]
To finish the proof, we must prove that this map is also compatible with the right-module structure. Put
\[
b' = x'y'_{[n-1-k]} x'_{[n+1-k,n]} \tau_{\omega_0(1-n-2k)} x'[1-k]x'[k+n].
\]
We need to check that for all \( f \in (\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n}) \), \( b' \cdot f - \omega_0(b') \cdot b' \) is a coboundary of \( e'[1-n-k]C_{n,k}e[1-k] \).
We have
\[
b' \cdot f = b'f e[1-k]e[1-k] + f e'[1-k] x'y'_{[n-1-k]} x'_{[n+1-k,n]} \tau_{\omega_0(1-n-2k)} x'[1-k]x'[k+n].
\]
We need to check that for all \( f \in (\text{ONH}^n_{n-k}, \text{ONH}^n_{k+n}) \), \( b' \cdot f - \omega_0(b') \cdot b' \) is a coboundary of \( e'[1-n-k]C_{n,k}e[1-k] \).
Let $u \in \{0, 1\}$ be such that $\tau_{\omega_0[1,n]} = (-1)^u \tau_x \tau_{\omega_0[n+1,n]} \tau_{\omega_0[1,n]}$, where

$$\sigma_k = (s_{n-k} \ldots s_1) \ldots (s_{n-2} \ldots s_k) (s_{n-1} \ldots s_k).$$

Observe that

$$\tau_{\omega_0[1,n]} \tau_{\omega_0[1,k]} \tau_{\omega_0[n+1,n]} \tau_{\omega_0[1,k]} = \tau_{\omega_0[1,k]} \tau_{\omega_0[n+1,n]} \tau_{\omega_0[n+1,n]} \tau_{\omega_0[1,k]}.$$

Hence,

$$b' \cdot f = (-1)^u x'_{[1,n-k]} x'_{[n-k+1,n]} \tau_x e_{1,k} e_{[k+1,n]} e_{[k+1,n]}$$

modulo coboundaries

$$= (-1)^u x'_{[1,n-k]} x'_{[n-k+1,n]} \sigma_k (f) \tau_x e_{1,k} e_{[k+1,n]} e_{[k+1,n]}$$

$$= (-1)^u x'_{[1,n-k]} x'_{[n-k+1,n]} \tau_{\omega_0[1,n-k]} \tau_{\omega_0[1,k]} w \tau_0(f) \tau_x e_{1,k} e_{[k+1,n]} e_{[k+1,n]}$$

$$= e'_{[1,n-k]} e'_{[n-k+1,n]} w \tau_0(f) b'$$

$$= \omega_0(f) \cdot b'.$$

\[\square\]

The following corollary follows immediately from the theorem.

**Corollary 5.10.** For all $k \leq n$, $H^\bullet(\Phi_n(\Theta_{-\lambda+n+2k}))$ is an invertible $(\text{ONH}^n_{n-k}, \text{ONH}^n_k)$-bimodule.

**Theorem 5.11.** For all $\lambda \in \mathbb{Z}$, the odd Chuang-Rouquier complex $\Theta_{\lambda}$ is invertible in the homotopy category of $\mathcal{U}$.

**Proof.** Denote by $\Theta^\vee_{-\lambda}$ the right adjoint of $\Theta_{\lambda}$, so that $\Theta^\vee_{-\lambda}$ is a complex of 1-morphisms $-\lambda \to \lambda$. Consider the unit of adjunction $\eta : 1_{-\lambda} \to \Theta^\vee \Theta_{1_{-\lambda}}$. By Corollary 5.10, $\Phi_n(\eta)$ is a quasi-isomorphism for all $n \in \mathbb{N}$. By Theorem 4.10, it follows that $\eta$ is a homotopy equivalence. The same argument can be applied to the counit of adjunction $\varepsilon : \Theta \Theta^\vee 1_{-\lambda} \to 1_{-\lambda}$, proving that it is a homotopy equivalence as well. The result follows.

\[\square\]

6. Applications

6.1. Quiver Hecke superalgebras. Let $I$ be an index set partitioned into $I = I_{\text{even}} \cup I_{\text{odd}}$ and $A = (a_{ij})_{i,j \in I}$ a symmetrizable Cartan matrix satisfying $a_{ij} \in 2\mathbb{Z}$ for all $i \in I_{\text{odd}}$ and $j \in I$. Define a parity function $p : I \to \{0, 1\}$ by $p(i) = 1$ if $i \in I_{\text{odd}}$ and $p(i) = 0$ if $i \in I_{\text{even}}$. Let $\mathcal{P}_{i,j} := k(\omega, z)/(\omega w - (-1)^{p(i)p(j)} w z)$ be the $\mathbb{Z} \times 2$-graded $k$-algebra where $w$ and $z$ where $w$ and $z$ have $\mathbb{Z} \times 2$-grading $((m, \alpha), (p, q))$ and $((\alpha_1, \alpha_j, p(j))$, respectively.

To a Cartan datum and a matrix $Q_{i,j} \in \mathcal{P}_{i,j}$ satisfying various conditions, Kang, Kashiwara, and Tsuchioka define a *quiver Hecke superalgebra* $R_n = R_n(\mathcal{Q})$. The odd nilHecke algebra studied in Section 3 is the rank one quiver Hecke superalgebra with a single odd $i \in I$. The quiver Hecke algebra $R_n$ decomposes into blocks $R_\beta$ for $\beta = \sum m_i \alpha_i \in \mathbb{Q}^+$ with $\sum m_i = n$. For dominant integral weight $\Lambda \in \mathbb{P}^+$, the quiver Hecke superalgebra admits a cyclotomic quotient $R_n^\Lambda$. The direct sum $R_n^\Lambda := \oplus_n R_n^\Lambda$ categorifies the representation $V(\Lambda)$ of the Kac-Moody superalgebra associated with the Cartan datum [23, 24].

The quiver Hecke superalgebra extends to a 2-supercategory $\mathcal{U}(g)$ introduced by Brundan and Ellis [4]. For a single odd vertex, $i \in I$, this 2-supercategory agrees with the 2-supercategory $\mathcal{U}(\mathfrak{sl}_2)$ from Definition 4.1.

6.2. 2-representations. Let $A$ be a $\mathbb{Z}$-graded superalgebra. To avoid repetition, let $X \in \{\emptyset, \text{super}\}$ so that $\text{Mod}_X(A)$ can be used to denote either the supercategory $\text{Mod}(A)$ of left $A$ modules if $X = \emptyset$, or the supercategory $\text{sMod}(A)$ of left $A$ supersuperalgebras with $\mathbb{Z}$-degree preserving homomorphisms if $X = \text{super}$. Likewise, we denote by $\text{Rep}_X(A)$ the supercategory of $\mathbb{Z}$-graded $A$-modules (resp. $A$-superalgebras) that are finite-dimensional over $k_0$ if $X = \emptyset$ (resp. $X = \text{super}$). Then $\text{Proj}_X(A)$ denotes
either the supercategory of finitely generated projective $A$-modules if $X = \emptyset$, or the supercategory of finitely generated projective $A$-supermodules if $X =$ super.

By the results in Section 9 of [23] for $X = \emptyset$, and by Theorem 8.13 [24] and the results in Section 8.3 for $X =$ super, we have the following result.

**Proposition 6.1.** For each $i \in I$ and $\beta \in Q^+$, there exist $(Q, \Pi)$-superfunctors

$$E_i^\Lambda : \operatorname{Mod}_X(R^\Lambda(\beta + \alpha_i)) \to \operatorname{Mod}_X(R^\Lambda(\beta))$$

$$F_i^\Lambda : \operatorname{Mod}_X(R^\Lambda(\beta)) \to \operatorname{Mod}_X(R^\Lambda(\beta + \alpha_i))$$

that are exact on $\operatorname{Rep}_X(R^\Lambda)$ and $\operatorname{Proj}_X(R^\Lambda)$. Furthermore, there exists natural isomorphisms of endofunctors on $\operatorname{Mod}_X(R^\Lambda(\beta))$ given by

$$E_i^\Lambda F_j^\Lambda \to q^{-(\alpha_i|\alpha_j)} \Pi_{r \in (\Lambda - \beta)} E_j^\Lambda F_i^\Lambda$$

if $i \neq j$,

$$\Pi_i q_i^{2k} F_i^\Lambda E_i^\Lambda \oplus \bigoplus_{k=0}^{-(\Lambda, \Lambda - \beta) - 1} \Pi_i q_i^{2k} \to E_i^\Lambda F_i^\Lambda$$

if $(h, \Lambda - \beta) \geq 0$,

$$\Pi_i q_i^{2k} F_i^\Lambda E_i^\Lambda \oplus \bigoplus_{k=0}^{-(\Lambda, \Lambda - \beta) - 1} \Pi_i q_i^{2k} \to E_i^\Lambda F_i^\Lambda$$

if $(h, \Lambda - \beta) < 0$.

Then the main result of Brundan and Ellis [4] implies that there is a 2-representation of $\mathcal{U}(\mathfrak{g})$ on the supercategories

$$\operatorname{Rep}_X(R^\Lambda) := \bigoplus_{\beta \in Q^+} \operatorname{Rep}_X(R^\Lambda(\beta)), \quad \operatorname{Proj}_X(R^\Lambda) := \bigoplus_{\beta \in Q^+} \operatorname{Proj}_X(R^\Lambda(\beta)).$$

**Corollary 6.2.** For any $\beta \in Q^+$, $w \in W$, and reduced expression $w = r_{i_1} \ldots r_{i_k}$ into simple transpositions, there is a superequivalence of supercategories

$$\Theta_w : K^b(R^\Lambda(\beta)) \cong K^b(\operatorname{Rep}_X(R^\Lambda_{\Lambda - w \alpha_i})).$$

**Proof.** If $i \in I$ is odd, then the 2-supercategory $\mathcal{U}(\mathfrak{g})$ contains a subcategory isomorphic to $\mathcal{U}(\mathfrak{sl}_2)$ from Definition 4.1, and if $i \in I$ is even, it contains a subcategory isomorphic to the usual categorification of $\mathfrak{sl}_2$. In both cases, there is a derived (super)equivalence $\Theta_i$ coming either from the usual Chuang-Rouquier complex in the even case, or the newly defined complexes from Definition 5.1. Hence, for any element $w$ in the Weyl group of type $\mathfrak{g}$ and reduced expression $w = r_{i_1} \ldots r_{i_k}$, there is a derived superequivalence $\Theta_w := \Theta_{i_1} \ldots \Theta_{i_k}$. \qed

### 6.3. Abelian defect conjecture for spin symmetric groups

In this section, let $k$ be a field of odd characteristic $p = 2\ell + 1$. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra of type $A^{(2)}_{2\ell}$ whose vertices of its Dynkin diagram are labelled $\{0, 1, \ldots, \ell\}$. Let $\delta$ be the null root, $\{\alpha_i \mid i \in I\}$ the simple roots, $\{\Lambda_i \mid i \in I\}$ the corresponding fundamental dominant weights, $Q_+$ the non-negative part of the root lattice, $P_+$ the set of dominant integral weights.

For $\beta = \sum_{i \in I} m_i \alpha_i \in Q_+$, let $R^\beta$ denote Kang-Kashiwara-Tsuchioka’s [25] quiver-Hecke superalgebra of type $\mathfrak{g}$. For $\Lambda \in P_+$, we let $R^\Lambda$ denote the cyclotomic quiver Hecke superalgebra. Likewise, let $RC^\beta$ and $RC^\Lambda$ denote the quiver Hecke-Clifford superalgebra. Kang, Kashiwara, Tsuchioka show that there are super Morita equivalences

$$RC^\beta \simeq_{\text{Mor}} R^\beta \otimes \mathfrak{c}_{m_0}, \quad RC^\Lambda \simeq_{\text{Mor}} R^\Lambda \otimes \mathfrak{c}_{m_0}$$

where $\mathfrak{c}_{m_0}$ is the Clifford superalgebra of size $m_0$. They relate these cyclotomic quiver Hecke-Clifford superalgebras with blocks of the affine Sergeev superalgebra. Our results then give new nontrivial
derived superequivalences between blocks of the affine Sergeev superalgebra related by an action of the Weyl group \( W \) of type \( A_{2l}^{(2)} \).

When \( \Lambda = \Lambda_0 \), The Kang, Kashiwara, Oh categorification theorem [23] implies that \( R_{\beta}^{\Lambda_0} \) is nonzero, if and only if \( \Lambda_0 - \beta \) is a nonzero weight of \( V(\Lambda_0) \). Nonzero weights of \( V(\Lambda_0) \) can be understood in terms of the set of \( p \)-strict partitions \( \lambda \) utilizing its residue content \( \cont(\lambda) \in \mathbb{Q}_+ \), see [33, Section 2.1d]. Expressed in terms of the \( \bar{p} \)-core \( \rho \) of the \( p \)-strict partition \( \lambda \), the nonzero weights of \( V(\Lambda_0) \) are of the form \( \cont(\rho) + d\delta \) for some \( \bar{p} \)-core partition \( \rho \) and \( d \in \mathbb{Z}_{\geq 0} \). See [33, Section 2.3a] and the references therein for details on the combinatorics of \( p \)-strict partitions and their cores.

Taking \( \Lambda = \Lambda_0 \), we have that, up to tensoring with a Clifford superalgebra, the cyclotomic quotient \( R_{\cont(\rho)+d\delta}^{\Lambda_0} \) is Morita superequivalent to a spin block \( \mathcal{B}^{\rho,d} \) of the symmetric group. Kleshchev and Livesey reduced the abelian defect conjecture for spin blocks to a conjecture \([33, Conjecture 2] \) which implies the Kessar-Schaps conjecture [26] and completes the program to prove Broué’s abelian defect conjecture for the spin symmetric groups. Our derived superequivalences give a proof of this conjecture.

**Theorem 6.3** ([33 Conjecture 2]). Let \( \rho \) and \( \rho' \) be \( \bar{p} \)-cores and \( d, d' \in \mathbb{Z}_{\geq 0} \). If \( d = d' \), then the algebras \( R_{\cont(\rho)+d\delta}^{\Lambda_0} \) and \( R_{\cont(\rho')+d'\delta}^{\Lambda_0} \) are derived equivalent and \( R_{\cont(\rho)+d\delta}^{\Lambda_0} \otimes \mathcal{C}_1 \) and \( R_{\cont(\rho')+d'\delta}^{\Lambda_0} \otimes \mathcal{C}_1 \) are derived equivalent.

**Proof.** Assume that \( d = d' \). By the KKO categorification theorem, \( R_{\beta}^{\Lambda_0} \) is nonzero if and only if \( \Lambda_0 - \beta \) is a nonzero weight of \( V(\Lambda_0) \). Furthermore, by [22, Section 12], the nonzero weights of \( V(\Lambda_0) \) are all of the form \( w\Lambda_0 - d\delta \) with \( w \in W \) and \( d \in \mathbb{Z}_{\geq 0} \). By [33, Lemma 3.1.39] there is a map from the set of \( p \)-strict partitions \( \mathcal{P}_p \) to the set of weights \( P \) given by

\[
(6.5) \quad \kappa: \mathcal{P}_p \to P \quad \lambda \mapsto \Lambda_0 - \cont(\lambda),
\]

whose image is the set \( \{w\Lambda_0 - d\delta \mid w \in W, d \in \mathbb{Z}_{\geq 0}\} \). Further, this map restricts to a bijection between the set of \( \bar{p} \)-cores \( \mathcal{C}_p \) and affine Weyl group orbits \( W\Lambda_0 \) of the highest weight \( \Lambda_0 \). In particular, for \( \bar{p} \)-cores \( \rho, \rho' \in \mathcal{C}_p \), we have

\[
\Lambda_0 - \cont(\rho) = w\Lambda_0, \quad \Lambda_0 - \cont(\rho') = w'\Lambda_0
\]

for some affine Weyl group elements \( w, w' \in W \). Let \( \tau = w'w^{-1} \in W \) so that \( \tau w = w' \). Write \( \tau = \tau_{i_1} \ldots \tau_{i_k} \) a reduced expression and let

\[
\Theta_\tau := \Theta_{i_k} \ldots \Theta_{i_1}
\]

be the corresponding sequences of derived superequivalences defined in Section 5.1. Then by Corollary 6.2, \( \Theta_\tau \) defines a derived superequivalence from \( \text{Rep}_X(R_{\cont(\rho)+d\delta}^{\Lambda_0}) \) to \( \text{Rep}_X(R_{\beta}^{\Lambda_0}) \) where \( \Lambda_0 - \beta \) is the weight

\[
\tau(\Lambda_0 - \cont(\rho) - d\delta) = \tau(\Lambda_0 - \cont(\rho)) - \tau(d\delta) = \tau(w\Lambda_0) - d\tau(\delta) = w'\Lambda_0 - d\delta = \Lambda_0 - \cont(\rho') - d\delta
\]

where we used that \( \delta \) is \( W \)-invariant. The theorem follows by [25, Lemma 2.7].

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