Inequalities concerning \( s^{th} \) derivative of a polynomial

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Abstract. If \( p(z) \) is a polynomial of degree \( n \) having no zero in \( |z| < k, k \geq 1 \), then for \( 0 \leq s < n \) and \( 1 \leq R \leq k \), Jain [2007 Turk. J. Math. 31 89-94] proved

\[
\max_{|z|=R} |p^{(s)}(z)| \leq \frac{1}{R^s + k^s} \left[ \left\{ \frac{d^s}{dx^s}(1 + x^n) \right\}_{x=1} \right] \left( \frac{R + k}{1 + k} \right)^n \max_{|z|=1} |p(z)|.
\]

In this paper, we improve as well as extend this inequality by involving certain coefficients of the polynomial. Further, our result improves and generalizes some well-known inequalities.

1. Introduction and Statement of Results

Let \( \mathbb{P}_n \) be the class of polynomials \( p(z) = \sum_{j=0}^{n} a_j z^j \) of degree \( n \). For a polynomial \( p \in \mathbb{P}_n \), we denote

\[
M(p, R) = \max_{|z|=R} |p(z)| \quad \text{and} \quad p'(z) \text{ the derivative of } p(z).
\]

Bernstein [2, 12] proved that if \( p \in \mathbb{P}_n \), then

\[
M(p', 1) \leq n M(p, 1).
\]

Although inequality (1.1) first appeared in a paper of Riesz [11], it is known as Bernstein’s inequality.

If we consider \( p \in \mathbb{P}_n \) such that \( p(z) \neq 0 \) inside the disk \( |z| < 1 \), Erdös conjectured that inequality (1.1) can be sharpened and replaced by

\[
M(p', 1) \leq \frac{n}{2} M(p, 1).
\]

Inequality (1.2) was later proved by Lax [9]. Several refinements and extensions of (1.2) have been added to literature over the years (see Malik [10], Bidkham and Dewan [3], Jain [8]).
Malik [10] generalized (1.2) by considering \( p \in \mathbb{P}_n \) which does not vanish in \(|z| < k, k \geq 1 \) and proved

\[
M(p', 1) \leq \frac{n}{1 + k} M(p, 1). \tag{1.3}
\]

Further, Bidkham and Dewan [3] generalized (1.3) and proved that if \( p \in \mathbb{P}_n \) and \( p(z) \neq 0 \) in \(|z| < k, k \geq 1 \), then for \( 1 \leq R \leq k \),

\[
M(p', R) \leq \frac{n(R + k)^{n-1}}{(1 + k)^n} M(p, 1). \tag{1.4}
\]

Jain [8] further extended (1.4) by considering the \( s^{th} \) derivative of the polynomial. In fact, he proved

**Theorem A.** If \( p \in \mathbb{P}_n \) and \( p(z) \neq 0 \) in \(|z| < k, k \geq 1 \), then for \( 0 \leq s < n \) and \( 1 \leq R \leq k \),

\[
M \left( p^{(s)}, R \right) \leq \frac{1}{R^s + k^s} \left[ \left\{ \frac{d^s}{dx^s} (1 + x^n) \right\} \right]_{x=1} \left( \frac{R + k}{1 + k} \right)^n M(p, 1). \tag{1.5}
\]

The result is sharp and equality holds with \( s = 1 \) for \( p(z) = (z + k)^n \).

Theorem A was further generalized and improved by Barchand and Dewan [5] by proving

**Theorem B.** If \( p \in \mathbb{P}_n \) and \( p(z) \neq 0 \) in \(|z| < k, k > 0 \), then for \( 0 < r \leq R \leq k \), and \( 1 \leq s < n \),

\[
M \left( p^{(s)}, R \right) \leq \frac{n(n-1) \ldots (n-s+1)}{R^s + k^s} \left( \frac{R + k}{r + k} \right)^n (M(p, r) - m). \tag{1.6}
\]

In this paper, by involving certain coefficients of the polynomial, we obtain an extension and improvement of (1.5) and an improvement of (1.6). More precisely, we prove

**Theorem.** If \( p \in \mathbb{P}_n \) and \( p(z) \neq 0 \) in \(|z| < k, k > 0 \), then for \( 0 \leq s < n \), and for \( 0 < r \leq R \leq k \),

\[
M \left( p^{(s)}, R \right) \leq \left[ \left\{ \frac{d^s}{dx^s} (1 + x^n) \right\} \right]_{x=1} \left\{ \frac{c(n, s)R + \frac{|a_s|}{|a_0| - m} k^{s+1}}{c(n, s)(k^{s+1} + R^{s+1}) + \frac{|a_s|}{|a_0| - m} (k^{s+1}R^s + Rk^2s)} \right\} \times B \{ M(p, r) - m \} \quad \text{for} \quad 1 \leq s < n \tag{1.7}
\]

and

\[
M(p, R) \leq M(p, r) B - (B - 1)m \quad \text{for} \quad s = 0. \tag{1.8}
\]

where here and throughout the paper

\[
c(n, s) = \frac{n!}{s!(n-s)!}, \quad m = \min_{|z|=k} |p(z)|
\]

and

\[
B = \left( \frac{R^2 + k^2 + \frac{2}{n} \left( \frac{|a_1|}{|a_0| - m} \right) k^2 R}{r^2 + k^2 + \frac{2}{n} \left( \frac{|a_1|}{|a_0| - m} \right) k^2 r} \right)^{\frac{1}{2}}. \tag{1.9}
\]
Remark 1. In the theorem, since \( p(z) \neq 0 \) in \( |z| < k \), \( k > 0 \), then for \( 0 < t \leq k \), \( p(tz) \neq 0 \) in \( |z| < \frac{k}{t} \), where \( \frac{k}{t} \geq 1 \). It follows by Rouche’s theorem that, for any real or complex number \( \lambda \) with \( |\lambda| < 1 \), the polynomial \( p(tz) - \lambda m' \), where \( m' = \min |p(tz)| \), does not vanish in \( |z| < \frac{k}{t} \), \( \frac{k}{t} \geq 1 \). Thus, applying (2.4) of Lemma 2.2 to \( p(tz) - \lambda m' \), we have

\[
\frac{1}{c(n, s)|a_0 - \lambda m'|} \left( \frac{k}{t} \right)^s \leq 1,
\]

which simplifies to

\[
\frac{1}{c(n, s)|a_0 - \lambda m'|} k^s \leq 1. \tag{1.10}
\]

Now, \( m' = \min |p(tz)| = \min |p(z)| = m \) and \( |a_0| \geq m \) (by Lemma 2.6). Choosing the argument of \( \lambda \) suitably such that \( |a_0 - \lambda m| = |a_0| - |\lambda|m \), inequality (1.10) becomes

\[
\frac{1}{c(n, s)|a_0| - |\lambda|m} k^s \leq 1. \tag{1.11}
\]

Taking \( |\lambda| \to 1 \), (1.11) reduces to

\[
\frac{1}{c(n, s)|a_0| - m} k^s \leq 1, \tag{1.12}
\]

which leads to

\[
\frac{c(n, s)t + \frac{|a_s|}{|a_0| - m} k^{s+1}}{c(n, s)(k^{s+1} + t^{s+1}) + \frac{|a_s|}{|a_0| - m} (k^{s+1}t^s + tk^{2s})} \leq \frac{1}{t^s + k^s} \text{ for } 0 < t \leq k. \tag{1.13}
\]

Since \( R \leq k \), taking \( t = R \) in (1.13), we get

\[
\frac{c(n, s)R + \frac{|a_s|}{|a_0| - m} k^{s+1}}{c(n, s)(k^{s+1} + R^{s+1}) + \frac{|a_s|}{|a_0| - m} (k^{s+1}R^s + Rk^{2s})} \leq \frac{1}{R^s + k^s}. \tag{1.14}
\]

It is evident from (1.14) of Remark 1 and Lemma 2.8 that our theorem is an improvement of Theorem B due to Barchand and Dewan [4].

Remark 2. Putting \( r = 1 \) in the theorem, we have the following improvement of Theorem A.

Corollary 1. If \( p \in P_n \) and \( p(z) \neq 0 \) in \( |z| < k \), \( k \geq 1 \), then for \( 0 \leq s < n \), and for \( 1 \leq R \leq k \),

\[
M \left( p^{(s)}, R \right) \leq \left\{ \int d^n x^n (1 + x^n) \right\}_{x=1} \left\{ \frac{c(n, s)R + \frac{|a_s|}{|a_0| - m} k^{s+1}}{c(n, s)(k^{s+1} + R^{s+1}) + \frac{|a_s|}{|a_0| - m} (k^{s+1}R^s + Rk^{2s})} \times C \{ M(p, 1) - m \} \right. \text{ for } 1 \leq s < n \right. \tag{1.15}
\]
and
\[ M(p, R) \leq M(p, 1)C - (C - 1)m, \quad \text{for } s = 0, \tag{1.16} \]
where
\[ C = \left( \frac{R^2 + k^2 + \frac{2}{n} \left( \frac{|a_1|}{|a_0| - m} \right) k^2 R}{1 + k^2 + \frac{2}{n} \left( \frac{|a_1|}{|a_0| - m} \right) k^2} \right)^{\frac{2}{n}}. \tag{1.17} \]

It is clear from inequality (1.14) of Remark 1 in conjunction with Lemma 2.8 for \( r = 1 \) that Corollary 1 is an improvement of Theorem A due to Jain [8].

**Remark 3.** Putting \( s = 1 \), it is seen that inequality (1.15) of Corollary 1 is an improvement of inequality (1.4) due to Bidkham and Dewan [3].

**Remark 4.** For \( R = 1 \) and \( k = 1 \), inequality (1.15) of Corollary 1 gives an extended version of inequality (1.2) due to Lax [9] for the \( s \)th derivative.

**Remark 5.** For \( s = 0 \), inequality (1.16) of Corollary 1 gives an improvement of inequality (1.5) for \( s = 0 \).

### 2. Lemmas

We need the following lemmas to prove our result.

**The first lemma is due to Aziz and Rather [1].**

**Lemma 2.1.** If \( p \in \mathbb{P}_n \) and \( p(z) \neq 0 \) in \( |z| < k, \ k \geq 1 \), then for \( 1 \leq s < n \),
\[ M\left(p^{(s)}, 1\right) \leq n(n-1)\ldots(n-s+1) \left\{ \frac{c(n, s)|a_0| + |a_s|k^{s+1}}{c(n, s)|a_0|(1 + k^{s+1}) + |a_s|(k^{s+1} + k^{2s})} \right\} M(p, 1) \tag{2.1} \]
and
\[ \frac{1}{c(n, s)} \left| \frac{a_s}{a_0} \right| k^s \leq 1. \tag{2.2} \]

From Lemma 2.1, we can easily obtain the following lemma.

**Lemma 2.2.** If \( p \in \mathbb{P}_n \) and \( p(z) \neq 0 \) in \( |z| < k, \ k \geq 1 \), then for \( 0 \leq s < n \),
\[ M\left(p^{(s)}, 1\right) \leq \left\{ \frac{c(n, s)|a_0| + |a_s|k^{s+1}}{c(n, s)|a_0|(1 + k^{s+1}) + |a_s|(k^{s+1} + k^{2s})} \right\} \left\{ \left( \frac{d^s}{dx^s}(1 + x^n) \right)_{x=1} \right\} M(p, 1) \tag{2.3} \]
and
\[ \frac{1}{c(n, s)} \left| \frac{a_s}{a_0} \right| k^s \leq 1. \tag{2.4} \]

The following lemma was proved by Barchand and Dewan [5].

**Lemma 2.3.** If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, \ 1 \leq \mu \leq n \), is a polynomial of degree \( n \) such that \( p(z) \neq 0 \) in \( |z| < k, \ k > 0 \), then for \( 0 < r \leq R \leq k \),
\[ M(p, R) \leq M(p, r)B' - (B' - 1)m, \tag{2.5} \]
where \( m = \min_{|z|=k} |p(z)| \)
and
\[
B' = \exp \left\{ \int_{r}^{R} \frac{\mu}{n} \frac{|a_{\mu}|}{|a_{0}| - m} k^{\mu+1} t^{\mu-1} + \mu \left( k^{\mu+1} t^{\mu} + k^{2\mu} t \right) \right\}. \tag{2.6}
\]

Remark 6. If we take \( \mu = 1 \) in Lemma 2.3, then from (2.6), \( B' \) becomes
\[
\exp \left\{ \int_{r}^{R} \frac{1}{n} \frac{|a_{1}|}{|a_{0}| - m} k^2 + t \right\}, \tag{2.7}
\]

which, from the proof of Lemma 2.8 equals \( B \) given by (1.9) in the theorem.
Thus, for \( \mu = 1 \), we obtain from (2.5)
\[
M(p, R) \leq M(p, r)B - (B - 1)m. \tag{2.8}
\]

Lemma 2.4. The real-valued function \( f \) defined by
\[
f(x) = \frac{c(n, s) + |a_{s}| x}{c(n, s)(1 + k^{s+1}) + \frac{|a_{s}|}{x} (k^{s+1} + k^{2s})},
\]
where \( a_{s} \) is any complex number and \( k \geq 1 \), is non-increasing for all non-zero real \( x \).

Proof of Lemma 2.4. The proof follows from the first derivative test of \( f(x) \) for any non-zero real \( x \) and \( k \geq 1 \), that is,
\[
f'(x) = \frac{c(n, s)k^{2s}(1 - k^{2})|a_{s}| x^2}{c(n, s)(1 + k^{s+1}) + \frac{|a_{s}|}{x} (k^{s+1} + k^{2s})^2} \leq 0.
\]

Lemma 2.5. If \( p \in \mathbb{P}_n \) such that \( p(z) \neq 0 \) in \( |z| < k, k \geq 1, \) then for \( 1 \leq s < n \),
\[
M \left( p^{(s)}, 1 \right) \leq \left\{ \left. \frac{d^{s}}{dx^{s}}(1 + x^n) \right|_{x=1} \right\} c(n, s) + \left( \frac{|a_{s}|}{|a_{0}| - m} \right) k^{s+1} \frac{c(n, s)(1 + k^{s+1}) + \frac{|a_{s}|}{x} (k^{s+1} + k^{2s})}{\{ M(p, 1) - m \}}, \tag{2.9}
\]
where \( m = \min_{|z|=k} |p(z)| \).
Lemma 2.2 to

Thus, taking $|z| = k$. Hence, it follows by Rouche’s theorem that for every real or complex number $\lambda$ such that $|\lambda| < 1$, the polynomial $p(z) - \lambda m$ also has no zero in $|z| < k, k \geq 1$. Thus, applying inequality (2.3) of Lemma 2.2 to $p(z) - \lambda m$, we have for $1 \leq s < n$,

$$M\left(p(z) - \lambda m, 1\right) \leq \left\{ \frac{d^s}{dx^s}(1 + x^n) \right\}_{x=1} \times \left\{ \frac{c(n, s)|a_0 - \lambda m| + |a_s|k^{s+1}}{c(n, s)|a_0 - \lambda m|(1 + k^{s+1}) + |a_s|(k^{s+1} + k^{2s})} \right\} M\left(p - \lambda m, 1\right).$$

(2.10)

Inequality (2.10) implies that

$$M\left(p^{(s)}, 1\right) \leq \left\{ \frac{d^s}{dx^s}(1 + x^n) \right\}_{x=1} \times \left\{ \frac{c(n, s) + |a_s|k^{s+1}}{c(n, s)(1 + k^{s+1}) + |a_s|(k^{s+1} + k^{2s})} \right\} M\left(p - \lambda m, 1\right).$$

(2.11)

Using Lemma 2.6, $|p(z)| \geq m$ for $|z| \leq k$, i.e., in particular $|a_0| \geq m$, therefore $|a_0| - |\lambda|m \leq |a_0 - \lambda m|$, then it follows by Lemma 2.4 that

$$f(|a_0| - |\lambda|m) \geq f(|a_0 - \lambda m|).$$

(2.12)

Further,

$$M(p - \lambda m, 1) = \max_{|z|=1} |p(z) - \lambda m|.$$

Let $z_0$ on $|z| = 1$ be such that

$$\max_{|z|=1} |p(z) - \lambda m| = |p(z_0) - \lambda m|.$$

We choose the argument of $\lambda$ such that

$$|p(z_0) - \lambda m| = |p(z_0)| - |\lambda|m \leq M(p, 1) - |\lambda|m.$$

(2.13)

Hence, using the facts of (2.12) and (2.13), inequality (2.11) gives

$$M\left(p^{(s)}, 1\right) \leq \left\{ \frac{d^s}{dx^s}(1 + x^n) \right\}_{x=1} \times \left\{ \frac{c(n, s) + |a_s|k^{s+1}}{c(n, s)(1 + k^{s+1}) + |a_s|(k^{s+1} + k^{2s})} \right\} \times \left( M(p, 1) - |\lambda|m \right).$$

(2.14)

Thus, taking $|\lambda| \to 1$ in (2.14), we obtain the desired conclusion of the lemma.
Lemma 2.6. If $p(z)$ is a polynomial of degree $n$ such that $p(z) \neq 0$ in $|z| < k$, $k > 0$, then

$$|p(z)| \geq m \quad \text{for} \quad |z| \leq k,$$

(2.15)

and in particular

$$|a_0| \geq m,$$

(2.16)

where $m = \min_{|z|=k}|p(z)|$.

This lemma is due to Gardner et al. [6, see Lemma 2.6].

Lemma 2.7. If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu}z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree $n$, $p(z) \neq 0$ for $|z| < k$, $k \geq 1$, and if $m = \min_{|z|=k}|p(z)|$, then

$$\frac{|a_\mu k^\mu|}{|a_0| - m} \leq \frac{n}{\mu},$$

(2.17)

This lemma is due to Gardner et al. [7, Proof of Lemma 3].

Lemma 2.8. If $p \in \mathbb{P}_n$ has no zero in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$,

$$\left( \frac{R^2 + \frac{1}{n} \frac{|a_1|}{|a_0| - m} 2k^2 R + k^2}{r^2 + \frac{1}{n} \frac{|a_1|}{|a_0| - m} 2k^2 r + k^2} \right)^{\frac{n}{2}} \leq \left( \frac{R + k}{r + k} \right)^n,$$

(2.18)

where $m = \min_{|z|=k}|p(z)|$.

Proof of Lemma 2.8. Since $p(z) \neq 0$ in $|z| < k$, $k > 0$, the polynomial $P(z) = p(tz) \neq 0$ in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$, where $0 < t \leq k$. Applying Lemma 2.7 for $\mu = 1$, we have

$$\frac{|a_1| t}{|a_0| - m} \left( \frac{k}{t} \right) \leq n,$$

which is equivalent to

$$n \frac{1}{n} \frac{|a_1|}{|a_0| - m} k^2 + t \leq \frac{1}{t + k}.$$

(2.19)

Integrating both sides of (2.19) with respect to $t$ from $r$ to $R$ where $0 < r \leq R \leq k$, we have

$$\frac{n}{2} \left[ \log \left( r^2 + \frac{1}{n} \frac{|a_1|}{|a_0| - m} 2k^2 r + k^2 \right) \right]_r^R \leq n \left[ \log(t + k) \right]_r^R.$$

i.e.

$$\left( \frac{R^2 + \frac{1}{n} \frac{|a_1|}{|a_0| - m} 2k^2 R + k^2}{r^2 + \frac{1}{n} \frac{|a_1|}{|a_0| - m} 2k^2 r + k^2} \right)^{\frac{n}{2}} \leq \left( \frac{R + k}{r + k} \right)^n.$$

$\square$
3. Proof of the Theorem
Proof. We first prove inequality (1.7).

Since $p(z)$ has no zero in $|z| < k$, $k > 0$, then the polynomial $P(Rz)$ has no zero in $|z| < \frac{k}{R}$.

\[
\frac{k}{R} \geq 1.
\]

Hence, applying Lemma 2.5 to $p(Rz)$, we have for $0 < m = \min \{|p(z)| : |z| = \frac{k}{R}\}$, which is equivalent to

\[
R^s M \left( p^{(s)}, R \right) \leq \left\{ \frac{d^s}{dx^s} (1 + x^n) \right\}_{x=1} \left( \frac{c(n, s) + \frac{|a_s|}{|a_0| - m'} \left( \frac{k}{R} \right)^{s+1}}{c(n, s) + \left( \frac{k}{R} \right)^{s+1}} + \frac{|a_s|}{|a_0| - m'} \left( \frac{k^{s+1} + R^{s+1}}{R^{s+1} + k^{2s}} \right) \right) \times (M(p, R) - m'),
\]

where, $m' = \min \{|p(Rz)| : |z| = \frac{k}{R}\} = \min \{|p(z)| : |z| = k\}$,

which is equivalent to

\[
M \left( p^{(s)}, R \right) \leq \left\{ \frac{d^s}{dx^s} (1 + x^n) \right\}_{x=1} \left( \frac{c(n, s)R + \frac{|a_s|}{|a_0| - m} k^{s+1}}{c(n, s)(k^{s+1} + R^{s+1}) + \frac{|a_s|}{|a_0| - m} (k^{s+1}R^{s} + Rk^{2s})} \right) \times (M(p, R) - m),
\]

Using inequality (2.8) to the right hand side of (3.1), we have for $0 < r \leq R \leq k$,

\[
M \left( p^{(s)}, 1 \right) \leq \left\{ \frac{d^s}{dx^s} (1 + x^n) \right\}_{x=1} \left( \frac{c(n, s) + \frac{|a_s|}{|a_0| - m} k^{s+1}}{c(n, s)(k^{s+1} + 1^{s+1}) + \frac{|a_s|}{|a_0| - m} (k^{s+1}1^{s} + Rk^{2s})} \right) \times B \{M(p, r) - m\},
\]

which is inequality (1.7).

Next, for $s = 0$, inequality (1.8) follows simply from inequality (2.8) of Remark 6.

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