Comparison Results for Poisson Equation with Mixed Boundary Condition on Manifolds

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Abstract. In this article, we establish a $L^1$ estimate for solutions to Poisson equation with mixed boundary condition, on complete noncompact manifolds with nonnegative Ricci curvature and compact manifolds with positive Ricci curvature respectively. On Riemann surfaces we obtain a Talenti-type comparison. Our results generalize main theorems in Alvino et al. (J Math Pures Appl 9(152):251–261, 2021) to Riemannian setting, and Chen–Li’s result (Talenti’s comparison theorem for poisson equation and applications on Riemannian manifold with nonnegative Ricci curvature, 2021. arXiv:2104.05568) to the case of variable Robin parameter.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with nonempty smooth boundary, $\Omega^\sharp \subset \mathbb{R}^n$ be a round ball with the same volume as $\Omega$, $f(x)$ be a nonnegative function on $\Omega$ and $f^\sharp$ be the Schwarz decreasing rearrangement of $f$ (see Definition 2.1). Assume $u(x)$ and $v(x)$ are solutions to

$$-\Delta u(x) = f(x), \quad x \in \Omega,$$

and

$$-\Delta v(x) = f^\sharp(x), \quad x \in \Omega^\sharp$$

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Moreover the equality occurs if and only if $f(x)$ is a radial function and $\Omega$ is a round ball, see [16]. The key tools of the proof are Schwarz symmetrization and isoperimetric inequalities on manifolds. Talenti’s comparison (1.1) plays important roles in both partial differential equations and geometry problems. There provide $L^\infty$ estimates for solutions to PDEs and Faber–Krahn type inequality for the first Dirichlet eigenvalue, see [8,14,16]. Talenti’s comparison was generalized to nonlinear elliptic and parabolic equations with Dirichlet boundary condition (cf. [4–6,20] and references therein), to compact Riemannian manifolds with positive Ricci curvature [11], and to noncompact manifolds with nonnegative Ricci curvature and positive asymptotic volume ratio [8]. We also refer the reader to the excellent books [15,17] for related topics.

Recently, Alvino et al. [3] studied the Poisson equation with Robin boundary condition when Robin parameter is a positive constant. They proved estimate (1.1) on planer domains, and a sharp $L^1$ estimate for higher dimensions. These results were generalized to Riemannian manifolds [9], to Robin boundary with variable Robin parameter [2], and to the torsion problem for the Hermite operator [10,12].

The purpose of the present paper is to study Talenti’s comparison and related estimates for solutions to Poisson equation with mixed boundary condition on manifolds. In particular, we generalize Alvino-Chiacchio-Nitsch-Trombetti’s result [2] to compact manifolds with positive Ricci curvature, and to noncompact manifolds with nonnegative Ricci curvature.

To state our results, we give some notations. Let $(M,g)$ be an $n$-dimensional complete Riemannian manifold, which is either compact with $\text{Ric} \geq (n-1)\kappa$ for $\kappa > 0,$ or noncompact with nonnegative Ricci curvature and positive asymptotic volume ratio. Denote by

$$\theta = \begin{cases} 
\lim_{r \to \infty} \frac{|B(r)|}{\omega_n r^n}, & \kappa = 0, \\
\frac{|M|}{|M_\kappa|}, & \kappa > 0,
\end{cases}$$

where $B(r)$ is a round geodesic ball of radius $r$ in $M,$ $M_\kappa$ is the $n$ dimensional space form of sectional curvature $\kappa,$ $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n,$ and $|M|$ denotes the volume of $M.$ It follows from the Bishop–Gromov volume comparison that $\theta \leq 1.$ Let $\Omega \subset M$ be a bounded domain with nonempty smooth boundary, $f(x)$ be a nonnegative smooth function not identically zero on $\Omega,$ and $\beta(x)$ be a positive smooth function on $\partial \Omega.$ We consider the following Poisson equation with mixed boundary condition

$$\begin{cases} 
-\Delta u(x) = f(x), & x \in \Omega, \\
\frac{\partial u}{\partial \nu} + \beta(x)u = 0, & x \in \partial \Omega,
\end{cases}$$

with Dirichlet boundary, respectively. Talenti [19] proved that

$$u^\sharp(x) \leq v(x), \quad x \in \Omega^\sharp. \quad (1.1)$$
where $\Delta$ denotes the Laplace–Beltrami operator and $\nu$ denotes the outward unit normal to $\partial \Omega$. Since $\beta(x) > 0$ and $f \geq 0$, it then follows easily from maximum principle that the solution to (1.3) is positive on $\Omega$, see Proposition 2.1. Let $\Omega^\sharp$ be a round geodesic ball in $M_\kappa$ satisfying $\theta|\partial \Omega^\sharp| = |\Omega|$, $f^\sharp$ defined on $\Omega^\sharp$ be the Schwarz decreasing rearrangement of $f$, and $v(x)$ be the solution to the Schwarz decreasing rearrangement equation of (1.3), namely

$$
\begin{align*}
-\Delta v(x) &= f^\sharp(x), & x \in \Omega^\sharp, \\
\frac{\partial v}{\partial \nu} + \bar{\beta}v &= 0, & x \in \partial \Omega^\sharp,
\end{align*}
$$

(1.4)

where $\bar{\beta}$ is a positive constant defined by

$$
\bar{\beta} = \frac{\theta|\partial \Omega^\sharp|}{\int_{\partial \Omega} \frac{1}{\beta(x)} dA}.
$$

(1.5)

Throughout the paper, $dA$ denotes the induced measure on boundary $\partial \Omega$ or $\partial \Omega^\sharp$. Since the boundary value problem (1.4) is radially symmetric, then the solution $v(x)$ is radially symmetric as well. Moreover $v(x)$ is monotone decreasing along the radial direction, see Proposition 2.2 below.

Our first result is concerning a $L^1$ comparison between $u(x)$ and $v(x)$. Precisely we prove

**Theorem 1.1.** Let $(M,g)$ be an n-dimensional complete Riemannian manifold, which is either compact with $\text{Ric} \geq (n-1)\kappa$ for $\kappa > 0$, or noncompact with nonnegative Ricci curvature and positive asymptotic volume ratio. Let $\Omega \subset M$ be a bounded domain with nonempty smooth boundary, $f(x)$ be a nonnegative smooth function not identically zero on $\Omega$, and $\beta(x)$ be a positive smooth function on $\partial \Omega$. Let $u(x)$ and $v(x)$ be the solutions to Eqs. (1.3) and (1.4) respectively. If $n \geq 3$, we assume further that for all measurable $E \subset \Omega$, it holds

$$
\int_E f(x) \, dx \leq \frac{|E|^{\frac{n-2}{n}}}{|\Omega|^{\frac{n-2}{n}}} \int_\Omega f(x) \, dx.
$$

(1.6)

Then

$$
||u||_{L^1(\Omega)} \leq \theta ||v||_{L^1(\Omega^\sharp)}.
$$

(1.7)

**Remark 1.2.** In this paper, we mainly focus on $L^1$ estimate (1.7) on manifolds and we then set up Theorem 1.1 in smooth case. In fact, Theorem 1.1 remains valid under the assumptions that $\Omega$ is a bounded domain with Lipschitz boundary, $f(x) \in L^2(\Omega)$ and $\beta(x)$ is a positive measurable function defined on $\partial \Omega$, see [2].

When $n = 2$ and $f$ is a constant, we prove a Talenti-type comparison, similarly as in [2].

**Theorem 1.3.** Let $(M,g)$ be a complete Riemann surface, which is either compact with $\text{Ric} \geq \kappa > 0$, or noncompact with $\text{Ric} \geq 0$ and positive asymptotic
volume ratio. Let \( u(x) \) and \( v(x) \) be the solutions to Eqs. (1.3) and (1.4) respectively. Suppose \( f(x) \equiv 1 \) and \( \beta(x) > 0 \). Then

\[
u^\sharp(x) \leq v(x), \quad x \in \Omega^\sharp.
\] (1.8)

Moreover, the equality case of (1.8) occurs if and only if \( M \) and \( \Omega \) are isometric to \( M_\kappa \) and \( \Omega^\sharp \) respectively, and \( \beta(x) \) is a constant on \( \partial \Omega \).

The rest of this paper is organized as follows. In Sect. 2, we recall the Schwarz decreasing rearrangement and isoperimetric inequalities on manifolds, and give some properties on solutions to Poisson equation with Robin boundary condition. In Sect. 3, we prove Theorem 1.1 and Theorem 1.2.

2. Preliminaries

2.1. Schwarz Decreasing Rearrangements

Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold with \( \text{Ric} \geq (n - 1)\kappa_\ast \), which is either noncompact with \( \kappa = 0 \) and positive asymptotic volume ratio, or compact with \( \kappa > 0 \). Let \( M_\kappa \) be the \(n\)-dimensional space form of constant sectional curvature \( \kappa \). Let \( \Omega \) be a bounded domain in \( M \) and \( \Omega^\sharp \) be a round geodesic ball in \( M_\kappa \) with volume \(|\Omega|/\theta\), where \( \theta \in (0, 1) \) is a constant defined by (1.2).

We recall the definitions and properties of the Schwarz decreasing rearrangement of non-negative functions on manifolds, see also [8, Section 2] and [11, Section 2].

**Definition 2.1.** Let \( h(x) \) be a nonnegative measurable function on \( \Omega \). Denote by \( \Omega_{h,t} = \{x \in \Omega : h(x) > t\} \) and \( \mu_h(t) = |\Omega_{h,t}| \), the decreasing rearrangement \( h^\ast \) of \( h \) is defined by

\[
h^\ast(s) = \begin{cases} \text{ess sup}_{x \in \Omega} h(x), & s = 0, \\ \inf\{t \geq 0 : \mu_h(t) < s\}, & s > 0, \end{cases}
\] (2.1)

for \( s \in [0, |\Omega|] \). The Schwarz decreasing rearrangement of \( h(x) \) is defined by

\[
h^\sharp(x) = h^\ast(\theta \omega_n r(x)^n), \quad x \in \Omega^\sharp,
\] (2.2)

where \( r(x) \) is the distance function from the center of \( \Omega^\sharp \) in \( M_\kappa \), and \( \omega_n \) is the volume of unit ball in \( \mathbb{R}^n \).

It follows from (2.1) and (2.2) that

\[
mu_h(t) = \theta \mu_{h^\sharp}(t)
\] (2.3)

for \( t \geq 0 \). The Fubini’s theorem gives that for \( h \in L^p(\Omega) \) and \( p \geq 1 \) it holds

\[
\int_\Omega h^p(x) \, dx = \int_0^{[\Omega]} (h^\ast)^p(s) \, ds = \theta \int_{\Omega^\sharp} (h^\sharp)^p(x) \, dx.
\]
For any nonnegative functions \( f(x) \) and \( g(x) \), the Hardy-Littlewood inequality gives
\[
\int_{\Omega} f(x)g(x) \, dx \leq \int_{0}^{\Omega} f^*(s)g^*(s) \, ds,
\]
and taking \( g(x) \) as the characteristic function of \( \Omega_{h,t} \) in above inequality yields
\[
\int_{\Omega_{h,t}} f(x) \, dx \leq \int_{0}^{\mu_{h,t}} f^*(s) \, ds.
\]

2.2. Isoperimetric Inequalities on Manifolds
To prove Theorems 1.1 and 1.3, we require the following isoperimetric inequality on manifolds with Ricci curvature bounded from below.

**Theorem 2.2.** Let \( M, \theta, \Omega \) and \( \Omega^2 \) be the same as in Theorem 1.1. It holds
\[
|\partial \Omega| \geq \theta|\partial \Omega^2|,
\]
where \( |\partial \Omega| \) denotes the \((n-1)\)-dimensional area of \( \partial \Omega \). Moreover the equality case of (2.6) occurs if and only if \( \Omega \) is isometric to \( \Omega^2 \).

When \( \kappa > 0 \), inequality (2.6) was known as Lévy-Gromov isoperimetric inequality [13], see also [18, Theorem 2.1]. When \( M \) is noncompact and \( \kappa = 0 \), inequality (2.6) was proved by Agostiniani, Fogagnolo and Mazzieri [1] when \( n = 3 \), and by Brendle [7] for all dimensions. Brendle proved in [7] that the isoperimetric inequality (2.6) also holds true when \( \Omega \) is an \( n \)-dimensional compact minimal submanifold of \( M \) of dimension \( n + 2 \) with nonnegative sectional curvature as well.

2.3. Poisson equation with Robin boundary
In this subsection, we collect some known facts about solutions to Poisson equation with Robin boundary condition.

**Proposition 2.1.** Let \( u(x) \) be the solution to (1.3). Assume \( f(x) \) is nonnegative and not identically zero on \( \Omega \), and \( \beta(x) > 0 \) on \( \partial \Omega \). Then we have
\[
u(x) > 0
\]
for all \( x \in \Omega \).

**Proof.** Letting \( \bar{u}(x) = \frac{1}{2}(u(x) - |u(x)|) \) and using integration by parts, we estimate that
\[
\int_{\Omega} \langle \nabla \bar{u}, \nabla u \rangle \, dx \leq \int_{\partial \Omega} \bar{u} \frac{\partial u}{\partial \nu} \, dA - \int_{\Omega} \bar{u} \Delta u \, dx
\]
\[
= -\int_{\partial \Omega} \beta u \bar{u} \, dA + \int_{\Omega} \bar{u} f \, dx
\]
\[
\leq 0,
\]
\( \Box \)
where we used Eq. (1.3) in the second equality and nonnegativity of \( f \) in the inequality. Observing that
\[
\int_{\Omega} \langle \nabla \bar{u}, \nabla u \rangle \, dx = \int_{\Omega} |\nabla \bar{u}|^2 \, dx,
\]
then we have
\[
\int_{\Omega} |\nabla \bar{u}|^2 \, dx = 0.
\]
So \( \bar{u}(x) \) is a constant on \( \Omega \) and inequality (2.7) holds as an equality, hence
\[
\bar{u}(x) = 0,
\]
equivalently \( u(x) \geq 0 \). Since \( u(x) \) is a supharmonic function by Eq. (1.3) and \( f \) is not identically zero, then we conclude from strong maximum principle that \( u(x) > 0 \) on \( \Omega \). \( \square \)

**Proposition 2.2.** Let \( v(x) \) be the solution to (1.4), rewritten as \( \varphi(r(x)) \). If \( f(x) \) is non-negative and not identically zero on \( \Omega \), and \( \beta > 0 \), then
\[
\varphi'(r) < 0 \tag{2.8}
\]
for \( r \in (0, R_0) \). Where \( \Omega^s = B(R_0) \), and \( R_0 \leq \frac{\pi}{\sqrt{\kappa}} \) if \( \kappa > 0 \) in view of the Myers theorem.

**Proof.** We rewrite \( f^s(x) \) as \( h(r(x)) \) and denote by
\[
\sin_{\kappa}(r) = \begin{cases} \sin \frac{\sqrt{\kappa}r}{\sqrt{\kappa}}, & \kappa > 0, \\ r, & \kappa = 0, \end{cases}
\]
then Eq. (1.4) is equivalent to
\[
\varphi''(r) + (n - 1) \frac{\sin'_{\kappa}(r)}{\sin_{\kappa}(r)} \varphi'(r) = -h(r).
\]
Since \( h(r) \geq 0 \), we then have
\[
(\sin^{n-1}_{\kappa}(r)\varphi'(r))' = -\sin^{n-1}_{\kappa}(r)h(r) \leq 0
\]
for \( r \in (0, R_0) \). Since \( h(r) \) is monotone nonincreasing in \( (0, R_0) \) and not identically zero, then there exists a \( \varepsilon_0 < R_0 \) such that \( h(r) > 0 \) for \( r < \varepsilon_0 \). Hence \( \varphi'(r) < 0 \) for all \( r \in (0, R_0) \). \( \square \)

### 3. Proofs of Theorems 1.1 and 1.3

In this section, we will prove the main theorems. For simplicity, we rewrite \( \Omega_{u,t} \) as \( \Omega_t \), and \( \Omega_{v,t}^s \) as \( \Omega_t^s \) for short. For \( s \in (0, |\Omega^s|) \), we denote
\[
a(s) = \frac{s^{\frac{n-1}{n}}}{|\partial B_s|},
\]
where \( B_s \) is a round geodesic ball in \( M_\kappa \) with volume \( s \). It is easily checked that \( a(s) = n^{-1} \omega^{-1/n}_n \) if \( \kappa = 0 \), and \( a(s) \) is monotone increasing in \( s \) if \( \kappa > 0 \).
Using isoperimetric inequality (2.6), we prove the following lemma, which will be used later.

**Lemma 3.1.** Under the hypotheses of Theorem 1.1, we have

\[
\theta^2 \left( \frac{\mu u(t)}{\theta} \right)^{\frac{2n-2}{n}} \leq a^2 \left( \frac{\mu u(t)}{\theta} \right) \left( -\mu'_u(t) + \int_{\partial \Omega_t \cap \partial \Omega} \frac{1}{\beta(x) u(x)} \, dA \right) \int_0^{\mu u(t)} f^*(s) \, ds,
\]

(3.1)

and

\[
\mu_v^n(t) = a^2 (\mu_v(t)) \left( -\mu'_v(t) + \int_{\partial \Omega_t^v \cap \partial \Omega^v} \frac{1}{\beta v(x)} \, dA \right) \int_0^{\mu_v(t)} (f^v)^*(s) \, ds
\]

(3.2)

for a.e. \( t > 0 \).

**Proof.** By the Morse–Sard theorem, we have

\[
\partial \Omega_t = \{ x \in \Omega : u(x) = t \} \cup \{ x \in \partial \Omega : u(x) \geq t \}
\]

for a.e. \( t > 0 \). Let

\[
g(x) = \begin{cases} |\nabla u|, & x \in \partial \Omega_t \cap \Omega, \\ -\frac{\partial u}{\partial \nu}, & x \in \partial \Omega_t \cap \partial \Omega. \end{cases}
\]

Applying the divergence theorem and Eq. (1.3), we derive that

\[
\int_{\partial \Omega_t} g(x) \, dA = \int_{\partial \Omega_t} -\frac{\partial u}{\partial \nu} \, dA = -\int_{\Omega_t} \Delta u \, dx = \int_{\Omega_t} f(x) \, dx \leq \int_0^{\mu u(t)} f^*(s) \, ds,
\]

(3.3)

where we have used (2.5) in the inequality. Using the Hölder inequality, we estimate

\[
|\partial \Omega_t|^2 \leq \int_{\partial \Omega_t} g(x) \, dA \int_{\partial \Omega_t} \frac{1}{g(x)} \, dA \\
\leq \int_0^{\mu u(t)} f^*(s) \, ds \left( \int_{\partial \Omega_t \cap \Omega} \frac{1}{|\nabla u|} \, dA + \int_{\partial \Omega_t \cap \partial \Omega} \frac{1}{\beta(x) u(x)} \, dA \right) \\
= \int_0^{\mu u(t)} f^*(s) \, ds \left( -\mu'_u(t) + \int_{\partial \Omega_t \cap \partial \Omega} \frac{1}{\beta(x) u(x)} \, dA \right),
\]

(3.4)

where we have used inequality (3.3) in the second inequality and the coarea formula in the equality.

On the other hand, the isoperimetric inequality (2.6) yields

\[
|\partial \Omega_t| \geq \theta |\partial (\Omega_t^u)| = \frac{\theta}{a(\mu_u(t)/\theta)} (\frac{\mu u(t)}{\theta}) \frac{n-1}{n}.
\]

(3.5)

Assembling inequalities (3.4) and (3.5), we get

\[
\frac{\theta^2}{a^2(\mu_u(t)/\theta)} \mu_v^n(t) \leq \int_0^{\mu u(t)} f^*(s) \, ds \left( -\mu'_u(t) + \int_{\partial \Omega_t \cap \partial \Omega} \frac{1}{\beta(x) u(x)} \, dA \right),
\]

(3.6)
which proves inequality (3.1).

If \( v(x) \) is the solution to Eq. (1.4), \( v(x) \) is a radial function on \( \Omega^2 \) and decreasing along the radial direction by inequality (2.8), and \( \Omega^2_t \) is a round ball. Therefore all previous inequalities hold as equalities if we replace \( u(x) \) by \( v(x) \), hence

\[
\mu_v^{\frac{2n-2}{n}}(t) = a^2(\mu_v(t)) \left( \mu_v'(t) + \int_{\partial\Omega^2_t \cap \partial\Omega^2} \frac{1}{\beta v(x)} \ dA \right) \int_0^t (f^2)^*(s) \ ds
\]

for all \( t > 0 \). Thus we complete the proof of the lemma. \( \square \)

**Lemma 3.2.** Suppose \( u \) and \( v \) are solutions to (1.3) and (1.4). Then both \( u \) and \( v \) attain their minima on \( \partial \Omega \) and \( \partial \Omega^2 \). Moreover if we denote by \( u_0 \) and \( v_0 \) the minima of \( u \) and \( v \) respectively, then

\[
u_0 \leq v_0. \tag{3.6}
\]

**Proof.** Recall that \( f \) is nonnegative, then \( -\Delta u \geq 0 \) and \( -\Delta v \geq 0 \). Therefore \( u \) and \( v \) attain their minima on \( \partial \Omega \) and \( \partial \Omega^2 \). Moreover \( u(x) > u_0 \) for \( x \in \Omega \), \( v(x) > v_0 \) for \( x \in \Omega^2 \) unless \( f \) is identically zero.

Noting that \( v(x) = v_0 \) on \( \partial \Omega^2 \) and using integration by parts we compute that

\[
v_0 |\partial \Omega^2| = \int_{\partial \Omega^2} \frac{1}{\beta} \ dA \int_{\partial \Omega^2} \bar{v}(x) \ dA
\]

\[
= \int_{\partial \Omega^2} \frac{1}{\beta} \ dA \int_{\Omega^2} -\Delta v \ dx
\]

\[
= \frac{1}{\theta^2} \int_{\partial \Omega} \frac{1}{\beta(x)} \ dA \int_{\Omega^2} f^2(x) \ dx
\]

\[
= \frac{1}{\theta^2} \int_{\partial \Omega} \frac{1}{\beta(x)} \ dA \int_{\Omega} -\Delta u(x) \ dx
\]

\[
= \frac{1}{\theta^2} \int_{\partial \Omega} \frac{1}{\beta(x)} \ dA \int_{\Omega} \beta(x) u(x) \ dA,
\]

where we have used Eqs. (1.3), (1.4), and equality (1.5). By the Hölder inequality, it holds

\[
\left( \int_{\partial \Omega} \sqrt{u(x)} \ dA \right)^2 \leq \int_{\partial \Omega} \frac{1}{\beta(x)} \ dA \int_{\partial \Omega} \beta(x) u(x) \ dA,
\]

then we have

\[
v_0 |\partial \Omega^2|^2 \geq \frac{1}{\theta^2} \left( \int_{\partial \Omega} \sqrt{u(x)} \ dA \right)^2 \geq \frac{u_0}{\theta^2} |\partial \Omega|^2 \geq u_0 |\partial \Omega^2|^2,
\]

where we used isoperimetric inequality (2.6) in the last inequality. Thus we have \( u_0 \leq v_0 \), proving (3.6). \( \square \)

Now we turn to prove Theorem 1.1.
Proof of Theorem 1.1. Set $\tilde{\mu}_u(t) = \frac{\mu_u(t)}{\bar{\sigma}}$. Dividing inequality (3.1) by $\tilde{\mu}_u(t)\tilde{\tau}^{-1}$ and integrating over $[0, \tau]$, we obtain

$$
\theta^2 \int_0^\tau \tilde{\mu}_u(t) \, dt \leq \int_0^\tau \left( a^2(\tilde{\mu}_u(t))\tilde{\mu}_u(t)\tilde{\tau}^{-1} \left( \int_0^{\tilde{\mu}_u(t)} f^*(s) \, ds \right) \left( -\tilde{\mu}_u(t) \right) \right) \, dt
$$

$$
+ \int_0^\tau a^2(\tilde{\mu}_u(t))\tilde{\mu}_u(t)\tilde{\tau}^{-1} \int_0^{\tilde{\mu}_u(t)} f^*(s) \, ds \int_0^{\tilde{\mu}_u(t)} \frac{1}{\beta(x)} \frac{1}{u(x)} \, dA \, dt
$$

$$
\leq \theta^2 \int_0^\tau \left( a^2(\tilde{\mu}_u(t))\tilde{\mu}_u(t)\tilde{\tau}^{-1} \int_0^{\tilde{\mu}_u(t)} f^*(s) \, ds \right) \left( -d\tilde{\mu}_u(t) \right)
$$

$$
+ \theta a^2(|\Omega^2|)|\Omega^2|\tilde{\tau}^{-1} \int_0^{\tilde{\tau}} (f^2)^*(s) \, ds \int_0^\tau \left( \int_{\partial\Omega^2_\tau \cap \partial\Omega} \frac{1}{\beta(x)} \frac{1}{u(x)} \, dA \right) \, dt,
$$

where we have used

$$
\mu_u(t)\tilde{\tau}^{-1} \int_0^{\mu_u(t)} f^*(s) \, ds \leq |\Omega|\tilde{\tau}^{-1} \int_\Omega f(x) \, dx = |\Omega|\tilde{\tau}^{-1} \int_0^{|\Omega|} f^*(s) \, ds,
$$

which follows from the assumption (1.6) and the fact $a(s)$ is monotone increasing in $s$. Applying Fubini’s theorem, we estimate

$$
\int_0^\tau \int_{\partial\Omega^2_\tau \cap \partial\Omega} \frac{1}{\beta(x) u(x)} \, dA \, dt \leq \int_0^\infty \int_{\partial\Omega^2_\tau \cap \partial\Omega} \frac{1}{\beta(x) u(x)} \, dA \, dt = \int_{\partial\Omega} \frac{1}{\beta(x)} \, dA
$$

(3.7)

then we conclude

$$
\theta^2 \int_0^\tau \tilde{\mu}_u(t) \, dt \leq -\theta^2 F(\tilde{\mu}_u(\tau)) + \theta^2 F(\tilde{\mu}_u(0))
$$

$$
+ \theta a^2(|\Omega^2|)|\Omega^2|\tilde{\tau}^{-1} \int_0^{\tilde{\tau}} (f^2)^*(s) \, ds \int_0^\tau \frac{1}{\beta(x)} \, dA,
$$

(3.8)

where $F(s)$ is defined by

$$
F(s) := \int_0^s a^2(\sigma)\sigma\tilde{\tau}^{-1} \int_0^\sigma (f^2)^*(r) \, dr \, d\sigma.
$$

Substituting (1.5) into inequality (3.8) yields

$$
\int_0^\tau \tilde{\mu}_u(t) \, dt \leq -F(\tilde{\mu}_u(\tau)) + F(\tilde{\mu}_u(0)) + a^2(|\Omega^2|)|\Omega^2|\tilde{\tau}^{-1}
$$

$$
\int_0^{\tilde{\tau}} (f^2)^*(s) \, ds \int_{\partial\Omega^2_\tau} \frac{1}{\beta} \, dA.
$$

(3.9)

For $\tau > v_0$, we have $\Omega^2_\tau \subset \Omega^2$, and then equality (3.2) becomes to

$$
\mu_v^{2n-2}(t) = a^2(\mu_v(t))(-\mu_v'(t)) \int_0^{\mu_v(t)} (f^2)^*(s) \, ds,
$$

implying

$$
\int_0^\tau \mu_v(t) \, dt + F(\mu_v(\tau)) = \int_0^{v_0} \mu_v(t) \, dt + F(\mu_v(v_0)).
$$

(3.10)
Dividing equation (3.2) by $\mu_v(t)^{\frac{2-n}{2n}}$ and integrating over $[0,v_0]$ yields

$$\int_0^{v_0} \mu_v(t) \, dt + F(\mu_v(v_0)) = F(\mu_v(0)) + a^2(|\Omega^2|)|\Omega^2|^\frac{2}{\beta} - \int_0^{[|\Omega^2|]} (f^\tau)^*(s) \, ds \int_{\partial\Omega^2} \frac{1}{\beta} \, dA,$$

and substituting above equality to (3.10), we obtain for all $\tau > v_0$ that

$$\int_0^{\tau} \mu_v(t) \, dt + F(\mu_v(\tau)) = F(\mu_v(0)) + a^2(|\Omega^2|)|\Omega^2|^\frac{2}{\beta} - \int_0^{[|\Omega^2|]} (f^\tau)^*(s) \, ds \int_{\partial\Omega^2} \frac{1}{\beta} \, dA.$$

Combining inequality (3.9) and equality (3.11), we get

$$\int_0^{\tau} \tilde{\mu}_u(t) \, dt - \int_0^{\tau} \mu_v(t) \, dt \leq - F(\tilde{\mu}_u(t)) + F(\mu_v(t)) - F(\tilde{\mu}_u(0)) - F(\mu_v(0))$$

$$= - F(\tilde{\mu}_u(t)) + F(\mu_v(t)),$$

where in the equality we used that

$$\tilde{\mu}_u(0) = \frac{|\Omega|}{\theta} = |\Omega^2| = \mu_v(0).$$

Letting $\tau \to \infty$ in (3.12), we get

$$\int_0^{\infty} \mu_u(t) \, dt \leq \int_0^{\infty} \theta \mu_v(t) \, dt,$$

where we have used $\lim_{\tau \to \infty} \mu_u(\tau) = \lim_{\tau \to \infty} \mu_v(\tau) = 0$. Hence (1.7) holds true. \qed

Now we deal with the case $n = 2$ and $f(x) \equiv 1$.

**Proof of Theorem 1.3.** When $n = 2$ and $f(x) \equiv 1$, inequality (3.1) gives

$$\theta \leq \left( - \mu'_u(t) + \int_{\partial\Omega^2 \cap \partial\Omega} \frac{1}{\beta(x)u(x)} \, dA \right) a^2(\mu_u(t)/\theta)$$

$$\leq \left( - \mu'_u(t) + \int_{\partial\Omega^2 \cap \partial\Omega} \frac{1}{\beta(x)u(x)} \, dA \right) a^2(|\Omega^2|),$$

where we have used the fact that $a(s)$ is monotone increasing in $s$ in the last inequality. Integrating inequality (3.13) over $[0,\tau]$ yields

$$\theta\tau \leq \left( - \mu_u(\tau) + |\Omega| + \int_0^{\tau} \int_{\partial\Omega^2 \cap \partial\Omega} \frac{1}{\beta(x)u(x)} \, dA \, dt \right) a^2(|\Omega^2|)$$

$$\leq \left( - \mu_u(\tau) + |\Omega| + \int_{\partial\Omega} \frac{1}{\beta(x)} \, dA \right) a^2(|\Omega^2|),$$

where we used (3.7) in the last inequality.

Analogously, by using equality (3.2) we have

$$\tau = \left( - \mu_v(\tau) + |\Omega^2| + \int_{\partial\Omega^2} \frac{1}{\beta} \, dA \right) a^2(|\Omega^2|)$$

for $\tau \geq v_0$, then putting (3.14) and (3.15) together we get

$$\mu_u(\tau) - \theta \mu_v(\tau) \leq 0.$$
For \( \tau < v_0 \), it follows clearly that
\[
\mu_u(\tau) \leq |\Omega| = \theta|\Omega^\sharp| = \theta \mu_v(\tau),
\]
thus
\[
\mu_u(\tau) - \theta \mu_v(\tau) \leq 0
\]
holds for all \( \tau > 0 \), hence inequality (1.8) holds.

If inequality (1.8) holds as an equality, then inequality (3.14) holds as an equality as well, which implies \( \Omega_\tau \subset \subset \Omega \), and isoperimetric inequality (2.6) holds as an equality for \( \Omega_\tau \). Hence \( \Omega_\tau \) is a geodesic ball in \( M_\kappa \) for \( \tau \geq v_0 \). So \( \Omega \) is a round geodesic ball in \( M_\kappa \), \( \theta = 1 \), and \( u(x) \) is a radial function. Thus we conclude \( \beta(x) \) is a constant on \( \partial \Omega \) and \( M \) is isometric to \( M_\kappa \). We complete the proof of Theorem 1.3. \( \square \)

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**Declarations**

**Conflict of interest** The authors have not disclosed any competing interests.

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