String Model for Analytic Nonlinear Regge Trajectories

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Abstract

We present a new generalized string model for Regge trajectories $J = J(E^2)$, where $J$ and $E$ are the orbital momentum and energy of the string, respectively. We demonstrate that this model is not to produce linear Regge trajectories, in contrast to the standard Nambu-Goto string, but generally nonlinear trajectories, which in many cases can be given in analytic form. As an example, we show how the model generates square-root, logarithmic and hyperbolic trajectories that have been discussed in the literature.

Key words: string models, nonrelativistic quark models, Regge trajectories, color screening

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1 Introduction

In QCD, the characteristic feature of the gluon mediators of the color force is their strong self-interaction, because the gluons themselves carry color charges. In analogy with the electric lines of force between two electric charges, one usually assumes that color charges (quarks) are held together by color lines of force, but the gluon-gluon interaction pulls these together into the form of a tube (string). Therefore, the use of the string picture for the phenomenological description of strong interactions seems to be completely justified.

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In fact, the formation of chromoelectric flux tubes between static quarks is by now a well established feature of lattice QCD simulations [1].

The action of the standard relativistic Nambu-Goto string with massive ends (quarks) in the parametrization \( \tau = t = x^0 \) is written as (\( \gamma \) is the string tension) [2]

\[
S = -\gamma \int_{t_1}^{t_2} dt \int_0^\pi d\sigma \sqrt{x'^2(1 - \dot{x}'^2) + (\dot{x}x')^2} - \sum_{i=1,2} m_i \int_{t_1}^{t_2} dt \sqrt{1 - \dot{x}_i'^2},
\]

where from now on the dot and the prime stand for the derivative with respect to \( t \) and \( \sigma \), respectively, unless otherwise specified. The action (1) is invariant under arbitrary transformations of \( \sigma \), therefore the string Lagrangian in Eq. (1) is degenerate, The corresponding constraint among the canonical variables \( x(t, \sigma) \) and \( p(t, \sigma) \) (\( p(t, \sigma) \equiv p(t, \sigma) \)) is

\[
p\dot{x}' = 0,
\]

for

\[
p = p(t, \sigma) = \frac{\partial L_{str}}{\partial \dot{x}} = \frac{\gamma^2}{L_{str}} \left[ x'(\dot{x}x')^2 - \dot{x}(x')^2 \right],
\]

\[
L_{str} = -\gamma \sqrt{x'^2(1 - \dot{x}'^2) + (\dot{x}x')^2}.
\]

With the equalities

\[
p\dot{x} - L_{str} = -\frac{\gamma^2}{L_{str}} x'^2, \quad p^2 + \gamma^2 x'^2 = \left( -\frac{\gamma^2}{L_{str}} x'^2 \right)^2,
\]

one obtains the expression for the canonical Hamiltonian, as follows,

\[
H = \int_0^\pi d\sigma \sqrt{p^2 + \gamma^2 x'^2} + \sum_{i=1,2} \sqrt{p_i^2 + m_i^2} = \int_0^\pi d\sigma |p + \gamma x'| + \sum_{i=1,2} \sqrt{p_i^2 + m_i^2},
\]

where \( p_i = m_i \dot{x}_i / \sqrt{1 - \dot{x}_i'^2} \).

In the nonrelativistic limit, \( |\dot{x}(t, \sigma)| \ll 1, |\dot{x}_i| \ll 1 \), Eq. (1) transforms into [3]

\[
S = -\gamma \int_{t_1}^{t_2} dt \int_0^\pi d\sigma \sqrt{x'^2} - \sum_{i=1,2} m_i \int_{t_1}^{t_2} dt + \sum_{i=1,2} m_i \frac{1}{2} \int_{t_1}^{t_2} dt \dot{x}_i^2.
\]

Integral over \( \sigma \) gives the length of the string (with the assumption that there are no singularities on the string). The variation of the first term in Eq. (6) with respect of the string coordinates leads to the requirement on the string to have the form of a linear rod connecting the massive ends. The effective action that leads to the equations of motion of the massive ends is therefore

\[
S_{eff} = \int_{t_1}^{t_2} dt \left( -\gamma |x_1(t) - x_2(t)| + \sum_{i=1,2} \frac{m_i \dot{x}_i^2}{2} \right).
\]
Hence, in the nonrelativistic limit, the string generates a linear potential between its massive ends: \( V(|x_1 - x_2|) = \gamma |x_1 - x_2| \). The same result holds for the string with massive ends in two-dimensional space-time [2] where \( x(t, \sigma) \) has only one component; therefore \( p(t, \sigma) = 0 \), via (3), and, in view of (5),

\[
H = \gamma \int_0^\pi d\sigma |x'| + \sum_{i=1,2} \sqrt{p_i^2 + m_i^2} = \gamma |x(0) - x(\pi)| + \sum_{i=1,2} \sqrt{p_i^2 + m_i^2}.
\]

The string model with constant tension is known to predict linearly rising Regge trajectories \( J = J(E^2) \) (\( J \) and \( E \) are the orbital momentum and energy of the string, respectively): \( J = E^2/(2\pi\gamma) \) [3]. The string trajectories are exactly linear in the case of the massless ends, and asymptotically linear in the case of the massive ends having some curvature in the region \( E \approx m_1 + m_2 \). The same picture of linear trajectories arises from a linear confining potential [3]. However, the realistic Regge trajectories extracted from data are nonlinear. Indeed, the straight line which crosses the \( \rho \) and \( \rho_3 \) squared masses corresponds to an intercept \( \alpha_\rho(0) = 0.48 \), whereas the physical intercept is located at 0.55, as extracted by Donnachie and Landshoff from the analysis of \( pp \) and \( \bar{p}p \) scattering data in a simple pole exchange model [4]. The nucleon Regge trajectory as extracted from the \( \pi^+p \) backward scattering data is [5]

\[
\alpha_N(t) = -0.4 + 0.9t + \frac{1}{2}0.25t^2,
\]

and contains positive curvature. Recent UA8 analysis of the inclusive differential cross sections for the single-diffractive reactions \( p\bar{p} \to pX, p\bar{p} \to X\bar{p} \) at \( \sqrt{s} = 630 \text{ GeV} \) reveals a similar curvature of the Pomeron trajectory [6]:

\[
\alpha_{\Pi}(t) = 1.10 + 0.25t + \frac{1}{2}(0.16 \pm 0.02)t^2.
\]

An essentially nonlinear \( a_2 \) trajectory was extracted in ref. [7] for the process \( \pi^-p \to \eta n \). Note that the nonlinearity of Regge trajectories was also proven on quite general grounds [8].

Once the nonlinearity of Regge trajectories is an established fact, a number of important issues immediately suggest themselves; e.g.,

(i) the understanding of the reasons for the nonlinearity of trajectories,

(ii) the development of the model(s) leading to analytic nonlinear Regge trajectories.

Possible reasons for the nonlinearity of trajectories may be related to the string breaking in QCD. In the absence of dynamical fermions (e.g., in zero temperature quenched QCD simulations) does not allow the screening of the potential between the heavy (static) \( QQ \) by virtual color-singlet light \( q\bar{q} \) pairs, and thus the interquark potential is expected to grow linearly with the separation \( R \) for arbitrarily large \( R \). With dynamical fermions, the static \( QQ \) meson can decay into two heavy-light mesons. Ignoring meson-meson interactions, one can expect the QCD string to break as soon as the potential exceeds twice the heavy-light mass, i.e., at about 1.5 fm. (Neglecting quark mass effects on the dynamics of the binding problem, which is a reasonable assumption once this mass is small compared to a typical binding energy of \( \sim 500 \text{ MeV} \), the string breaking distance should be shifted
by \( \Delta r \approx 2\Delta m/\sigma_{\text{eff}} \) when changing the quark mass by \( \Delta m \), where \( \sigma_{\text{eff}} \) is the (effective) string tension, to be discussed explicitly below.) The expectation of the string breaking due to the color screening described above has indeed been confirmed by some of the data on QCD lattice simulations [9].

The purpose of the present letter is to present a model for nonlinear Regge trajectories. This model is the generalization of the standard string model. Indeed, if the string breaking really happens in QCD, as should become clear with the forthcoming sophisticated lattice measurements, there is a necessity to reconsider the canonical string model and modify the notion of a constant string tension, in order to include the color screening effects. Before we proceed with the presentation of such a modified string model, let us note that potential models, in general, lead to nonlinear trajectories (for potentials that are different from a linearly rising one), but these trajectories cannot generally be cast into analytic form [10]. The model we shall present in what follows in many cases leads to analytic nonlinear Regge trajectories.

2 Generalized string model

Here we wish to generalize the standard string formulation reviewed above on the case of arbitrary potential between the string massive ends. Such generalization is done by the modification of the standard string tension into the effective string tension which is a function of \(|x|\), as follows:

\[
S_{\text{gen}} = -\int_{t_1}^{t_2} dt \int_0^\pi d\sigma \gamma(|x|) \sqrt{x'^2(1 - \dot{x}^2) + (\dot{x}x')^2} - \sum_{i=1,2} m_i \int_{t_1}^{t_2} dt \sqrt{1 - \dot{x}_i^2}. \tag{8}
\]

As we explain below, in the nonrelativistic limit the effective string tension is the derivative of the interaction potential between the massive ends of the string. Therefore, different choices of the effective string tension would be related to different potentials, which makes it possible to deal, among the others, with color-screened potentials, i.e., potentials that approach constant values at large separations; e.g., \( V(\rho) = \gamma/\mu (1 - \exp(-\mu \rho)) \) which is used to fit the lattice QCD data (e.g., first paper of [9]), and two examples in Section 3. As is clear from the above discussion, such color-screened potentials may be relevant to QCD.

Obviously, the new action (8) is again invariant under arbitrary transformations of \( \sigma \), and therefore the constraint (2), and Eqs. (3)-(5) hold in this case as well, with \( \gamma = \gamma(|x|) \). In the nonrelativistic limit, however, in place of (7) one will now obtain

\[
S_{\text{gen,eff}} = \int_{t_1}^{t_2} dt \left( -\int_0^\pi d\sigma \gamma(|x|)|x'| + \sum_{i=1,2} m_i \dot{x}_i^2 \right) \tag{9}
\]

In contrast to the previous case of the standard string, it is seen in the above relations that now

\[
\gamma(|x|) = \frac{1}{|x'|} \frac{dV(|x|)}{d\sigma} = \frac{dV(|x|)}{d|x|}, \tag{10}
\]
i.e., the effective string tension is the derivative of the potential with respect to the distance. Obviously, in the case of a linear potential, the effective string tension reduces to the standard (constant) one.

It should be noted that the other generalizations of the standard string have also been discussed in the literature \[11, 12\]. Specifically, refs. \[11\] deal with the linearized string Lagrangian that generally has different coefficients of \(\dot{x}^2\) and \(x'^2\), which are to be associated with the string mass density and tension, respectively. Both of these coefficients are functions of \(\sigma\). Refs. \[12\] deal with the so-called wiggly string (membrane) for which the string mass density and tension are generally different also. With \(\gamma\) being a function of \(\sigma\) not \(x\), as in \[11\], (i) the invariance of the action (8) under arbitrary transformations of \(\sigma\) disappears, and (ii) the interpretation of \(\gamma\) as an effective string tension, via Eq. (10), is lost, as seen in (9). Also, there are claims in the literature that the choice of the string mass density and tension being different from each other contradicts the principles of relativity which require that the both coincide \[13\]. In our case (upon the linearization of the Lagrangian in Eq. (8)) both the string mass density and tension would be equal to \(\gamma(|x|)\).

Similarly to the standard case of a constant string tension which represents the relativization à la Poincaré of the nonrelativistic two-body problem with linear potential \[14\], the generalized string that we are discussing here can be considered as the relativization of a nonrelativistic two-body problem with arbitrary potential.

### 2.1 The dynamics of the generalized string model

By varying the action of the generalized string with massive ends (here the dot stands for the derivative with respect to \(\tau\)),

\[
S_{gen} = \int \int d\tau d\sigma L(x, \dot{x}, x') + \sum_{i=1,2} L^{(m)}(\dot{x}_i),
\]

one obtains the equations of motion of the generalized string,

\[
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}} + \frac{d}{d\sigma} \frac{\partial L}{\partial x'} = \frac{\partial L}{\partial x}, \quad (12)
\]

and the boundary conditions which represent the equations of motion of the massive ends:

\[
\frac{d}{d\tau} \frac{\partial L^{(m)}}{\partial \dot{x}_i} = \frac{\partial L}{\partial x'}, \quad x = x_i. \quad (13)
\]

In the gauge \(\tau = t\) discussed above, the equation of motion of the generalized string reduce to

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d}{d\sigma} \frac{\partial L}{\partial x'} = \frac{\partial L}{\partial x'}, \quad x \equiv x, \quad (14)
\]

and the boundary conditions are

\[
m_1 \frac{d}{dt} \frac{\dot{x}_1}{\sqrt{1 - \dot{x}_1^2}} = \frac{\partial L}{\partial x'}, \quad \sigma = 0, \quad (15)
\]

\[
m_2 \frac{d}{dt} \frac{\dot{x}_2}{\sqrt{1 - \dot{x}_2^2}} = \frac{\partial L}{\partial x'}, \quad \sigma = \pi.
\]
Let us show that, similarly to the standard case of the string with constant tension, there are solutions to the equations of motion of the generalized string in the form of a rigid rod connecting the massive ends and rotating with frequency $\omega$ about its center of mass:

$$x(t, \sigma) = \rho(\sigma) \left( \cos(\omega t), \sin(\omega t), 0 \right).$$

(16)

Indeed, $\gamma = \gamma(|x|) = \gamma(\rho)$, since $x^2 = \rho^2$; therefore $d\rho/dx = x/\rho = (\cos(\omega t), \sin(\omega t), 0)$, and $d\gamma/dx = d\gamma/d\rho \cdot d\rho/dx = d\gamma/d\rho \cdot (\cos(\omega t), \sin(\omega t), 0)$. Hence

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial \gamma} \frac{d\gamma}{dx} = \frac{d\gamma}{d\rho} \rho' \sqrt{1 - \omega^2 \rho^2} \left( \cos(\omega t), \sin(\omega t), 0 \right).$$

(17)

Since also

$$\frac{d}{dt} \frac{\partial L}{\partial x} = -\frac{\gamma \omega^2 \rho \rho'}{\sqrt{1 - \omega^2 \rho^2}} \left( \cos(\omega t), \sin(\omega t), 0 \right),$$

(18)

$$\frac{d}{d\sigma} \frac{\partial L}{\partial x} \left|_{x' x} \right. = \left( -\frac{d\gamma}{d\rho} \rho' \sqrt{1 - \omega^2 \rho^2} + \frac{\gamma \omega^2 \rho \rho'}{\sqrt{1 - \omega^2 \rho^2}} \right) \left( \cos(\omega t), \sin(\omega t), 0 \right),$$

(19)

(the last relation is obtained via $d\gamma/d\sigma = d\gamma/d\rho \rho'$), one sees that the equations of motion (14) are satisfied.

One can easily check that for the rotation (16), the energy of the generalized string is given, in view of (5), by

$$H = \int d\sigma \sqrt{\rho^2 + \gamma^2 x'^2} = \int d\sigma \frac{\gamma \rho'}{\sqrt{1 - \omega^2 \rho^2}} = \int \frac{d\rho \gamma(\rho)}{\sqrt{1 - \omega^2 \rho^2}}.$$  

(20)

Similarly, the orbital momentum of the generalized string is

$$J = J_z = \int d\sigma (xp_y - yp_x) = \int d\sigma \frac{\gamma \omega \rho^2 \rho'}{\sqrt{1 - \omega^2 \rho^2}} = \int \frac{d\rho \gamma(\rho) \omega \rho^2}{\sqrt{1 - \omega^2 \rho^2}}.$$  

(21)

Interestingly enough, in his book [15] Perkins also presents the above relations for the energy and orbital momentum of the generalized string. He does not however derive these relations from the first principles Lagrangian, as in Eq. (8).

By adding the contribution of the massive ends, one finally has the expressions for the total energy and orbital momentum of the generalized string with massive ends:

$$E = \int_0^{r_1} \frac{d\rho \gamma(\rho)}{\sqrt{1 - \omega^2 \rho^2}} + \int_0^{r_2} \frac{d\rho \gamma(\rho)}{\sqrt{1 - \omega^2 \rho^2}} + \frac{m_1}{\sqrt{1 - \omega^2 r_1^2}} + \frac{m_2}{\sqrt{1 - \omega^2 r_2^2}},$$

(22)

$$J = \int_0^{r_1} \frac{d\rho \gamma(\rho) \omega \rho^2}{\sqrt{1 - \omega^2 \rho^2}} + \int_0^{r_2} \frac{d\rho \gamma(\rho) \omega \rho^2}{\sqrt{1 - \omega^2 \rho^2}} + \frac{m_1 \omega r_1^2}{\sqrt{1 - \omega^2 r_1^2}} + \frac{m_2 \omega r_2^2}{\sqrt{1 - \omega^2 r_2^2}}.$$  

(23)

Note that the boundary conditions (15) define the separations of the massive ends from the center of mass through the following nonlinear equations:

$$\frac{m_i \omega^2 r_i}{\sqrt{1 - \omega^2 r_i^2}} = \gamma(r_i) \sqrt{1 - \omega^2 r_i^2}, \quad i = 1, 2.$$  

(24)
2.2 Generalized massless string

The energy and orbital momentum of the generalized massless string, \( m_1 = m_2 = 0 \), are given by

\[
E = 2 \int_0^R \frac{d\rho \gamma(\rho)}{\sqrt{1 - \omega^2 \rho^2}}, \quad J = 2 \int_0^R d\rho \gamma(\rho) \omega \rho^2 \sqrt{1 - \omega^2 \rho^2},
\]

where \( R = 1/\omega \) is half of the string length for a given \( \omega \). The condition \( \omega R = 1 \) follows from, e.g., Eqs. (24) with \( m_i \to 0 \).

By eliminating \( \omega \) from Eqs. (25) one can obtain \( J \) as a function of \( E^2 \), the Regge trajectory. As we demonstrate below, the trajectory is generally nonlinear, and in many cases it is given in an analytic form. It will be shown elsewhere \([16]\) that it is possible to uniquely recover the potential \( (V(\rho) \sim \int d\rho \gamma(\rho)) \) from the known analytic form of Regge trajectory, for both generalized massless and massive strings, and the techniques of the corresponding inverse problem will be presented in detail. It should be noticed that potential corresponding to an analytic Regge trajectory cannot always be recovered itself in an analytic form, but only as some power series in \( \rho \). However, in the most important cases of analytic nonlinear Regge trajectories the corresponding potentials happen to be recovered analytically. Below we present such potentials for three examples of analytic nonlinear Regge trajectories that have been discussed in the literature. For our present purposes, here we present only the final results.

3 Analytic nonlinear Regge trajectories

In what follows, we consider only the generalized massless string.

3.1 Square-root trajectory

The square-root Regge trajectory, \( J \propto E_{th} - \sqrt{E_{th}^2 - E^2} \), where \( E_{th} \) is the trajectory energy threshold, has been widely discussed by Kobylinsky et al. \([17]\) as the simplest choice of a trajectory for dual amplitude with Mandelstam analyticity (DAMA) \([18]\). It also provides for the Orear (small-\( t \)) regime of the amplitude. The corresponding potential is \( (\gamma, \mu = \text{const}, V(\rho) \to \gamma/2\mu \text{ as } \rho \to \infty) \), and hence \( E \to E_{th} = \gamma/\mu \)

\[
V(\rho) = \frac{\gamma}{\pi \mu} \arctan(\pi \mu \rho), \quad \gamma(\rho) = \frac{dV(\rho)}{d\rho} = \frac{\gamma}{1 + (\pi \mu \rho)^2},
\]

for which

\[
E = \frac{\pi \gamma}{\omega^2 + \pi^2 \mu^2}, \quad J = \frac{\gamma}{\pi \mu^2} \left( 1 - \frac{\omega}{\sqrt{\omega^2 + \pi^2 \mu^2}} \right).
\]

\[\]

7
Eliminating $\omega$ from the above relations gives

$$J = \frac{1}{\pi \mu} \left( \frac{\gamma}{\mu} - \sqrt{\left(\frac{\gamma}{\mu}\right)^2 - E^2} \right),$$

(29)
i.e., the square-root Regge trajectory. For $E \ll \gamma/\mu$, it reduces to an (approximate) linear trajectory, $J \simeq E^2/(2\pi\gamma)$. Note that in the corresponding quantum case where $J$ takes on integer values only and the trajectory would develop a (nonzero) intercept (the quantum defect), $J(0)$, there would be a finite number of states lying on the trajectory, with orbital momenta $0 \leq J \leq J_{max} \leq J(0) + \gamma/(\pi \mu^2)$.

### 3.2 Logarithmic trajectory

The logarithmic Regge trajectory, $J \propto -\log(1 - E^2/E_{th}^2)$, is the ingredient of dual amplitude with logarithmic trajectories (DALT) [19]. There are certain theoretical reasons [20] to consider logarithmic trajectories that, for large $-t (= -E^2)$, are compatible with fixed angle scaling behavior of the amplitude. The corresponding potential is (again $\gamma, \mu = \text{const}$, $V(\rho) \rightarrow \gamma/2\mu$ as $\rho \rightarrow \infty$, and hence $E \rightarrow E_{th} = \gamma/\mu$)

$$V(\rho) = \frac{\gamma}{2\pi \mu} \left( 2 \arctan(2\pi \mu \rho) - \frac{\log[1 + (2\pi \mu \rho)^2]}{2\pi \mu \rho} \right),$$

(30)

for which

$$\gamma(\rho) = \frac{\gamma \log[1 + (2\pi \mu \rho)^2]}{(2\pi \mu \rho)^2},$$

(31)

and leads to

$$E = \frac{\gamma}{2\pi \mu^2} \left( \sqrt{\omega^2 + 4\pi^2 \mu^2} - \omega \right), \quad J = \frac{\gamma}{2\pi \mu^2} \log \frac{\omega + \sqrt{\omega^2 + 4\pi^2 \mu^2}}{2\omega},$$

(32)

from which eliminating $\omega$ (viz., $\omega = \pi(\gamma^2 - \mu^2 E^2)/\gamma E$) gives

$$J = -\frac{\gamma}{2\pi \mu^2} \log \left( 1 - \frac{E^2}{(\gamma/\mu)^2} \right),$$

(33)
i.e., the logarithmic Regge trajectory. For $E \ll \gamma/\mu$, it again reduces to an (approximate) linear form, $J \simeq E^2/(2\pi\gamma)$. Note that, in contrast to the previous example of square-root trajectory, although the trajectory has an energy threshold, in the corresponding quantum case the number of states lying on the trajectory would be infinite.

In each of the two examples considered above, the potential belongs to the family of the color-screened potentials mentioned in Section 2. (The exact form of the color-screened potential, if it is realized in QCD, will be discussed elsewhere.) Note that with a different choice of signs in Eqs. (29),(33) (e.g., $J = \gamma/(2\pi \mu^2) \log[1 + E^2/(\gamma/\mu)^2]$) there would be a difficulty with the analytic continuation of the trajectory in the region $E^2 (= t) < 0$, since in this region the trajectory cannot have imaginary part. Note also that besides the trajectories (29),(33), a hybrid square-root–logarithmic trajectory, $J \propto -\log(1 + \beta \sqrt{E_{th}^2 - E^2})$, has been motivated and discussed in [20, 21]. We have not managed to obtain the analytic form of the corresponding potential.
3.3 Hyperbolic trajectory

The hyperbolic trajectory, \( J \propto \cosh(E) - 1 \), results in the \( \kappa \)-deformed Poincaré phenomenology \([22]\). Here the corresponding potential is

\[
V(\rho) = \frac{\gamma \rho \Phi(- (\pi \mu \rho)^2, 2, 1/2)}{4} = \gamma \rho \sum_{k=0}^{\infty} \frac{[-(\pi \mu \rho)^2]^k}{(2k+1)^2} = \gamma \rho \left( 1 - \frac{(\pi \mu \rho)^2}{9} + \frac{(\pi \mu \rho)^4}{25} - \ldots \right),
\]

where

\[
\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s}
\]

is the so-called Lerch’s transcendent. The corresponding \( \gamma(\rho) \),

\[
\gamma(\rho) = \gamma \frac{\arctan(\pi \mu \rho)}{\pi \mu \rho},
\]

leads to

\[
E = \frac{\gamma}{\mu} \log \left( \frac{\pi \mu/\omega + \sqrt{1 + (\pi \mu/\omega)^2}}{} \right),
\]

\[
J = \frac{\gamma}{\pi \mu^2} \left( \sqrt{1 + (\pi \mu/\omega)^2} - 1 \right),
\]

from which eliminating \( \omega \) (\( \pi \mu/\omega = \sinh(\mu E/\gamma) \)) gives

\[
J = \frac{\gamma}{\pi \mu^2} \left[ \cosh \left( \frac{E}{\gamma/\mu} \right) - 1 \right],
\]

i.e., the hyperbolic trajectory. For \( E \ll \gamma/\mu \), it reduces to the linear form, as well as in the above two cases: \( J \simeq E^2/(2\pi \gamma) \).

4 Concluding remarks

We have presented a new generalized string model that leads to nonlinear Regge trajectories which in many cases can be given in analytic form. We have considered three examples of how this model generates square-root, logarithmic and hyperbolic trajectories that have been discussed in the literature.

The main phenomenological implication of the model presented here is the possibility to obtain Regge trajectory for an arbitrary (nonrelativistic) potential in general, and for a color-screened potential in particular. Since trajectory for the latter is characterized by an energy threshold, and in some cases by a finite number of states, as in the example of square-root trajectory considered, it may be of relevance to QCD to predict the numbers of states lying on different trajectories, and the corresponding energy thresholds, if a color-screened potential of, e.g., the type (26) is indeed realized in QCD.

Of certain interest would be further exploration of the generalized string model discussed here, and the applications of this model to the hadron scattering (the form of the corresponding scattering amplitude) and the thermodynamics of hot hadronic matter.
(the equation of state of the ensemble of the generalized strings), as well as the quantization of this model. These, and related issues, e.g., the understanding of the form of the color-screened potential, if is realized in QCD, will be the subjects of further study, to be undertaken elsewhere.

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