Runge-Kutta Characterization of the Generalized Summation-by-Parts Approach in Time

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Abstract

This article extends the theory of dual-consistent time-marching methods based on generalized summation-by-parts (GSBP) operators and simultaneous approximation terms by showing that they are Runge-Kutta (RK) schemes. The connection to RK schemes provides the analysis tools required to facilitate the construction of efficient, high-order, and stable GSBP time-marching methods, especially for those based on dense-norm operators. The minimum order of a GSBP time-marching method is shown to be, by definition, equivalent to the minimum guaranteed rate of superconvergence previously shown for initial value problems that are linear with respect to the solution. In addition, both the simplifying and full RK order conditions can be used to construct even higher-order GSBP time-marching methods. The RK connection also provides the conditions under which dense-norm GSBP time-marching methods are nonlinearly stable. GSBP time-marching methods are in general fully implicit; however, the RK characterization of these methods can be used to guide the construction of diagonally-implicit schemes, which are often more efficient, especially in terms of memory usage. The article concludes by presenting a few examples of known and novel RK time-marching methods which are based on GSBP operators.

Keywords:
Generalized Summation by Parts, Simultaneous Approximation Terms, Initial-Value Problems, Runge-Kutta Methods, Superconvergence,

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1. Introduction

Many dynamical systems in science and engineering are modelled by stiff initial value problems (IVPs) either directly or through semi-discretization of partial differential equations. The most stringent constraint on the numerical solution of stiff IVPs is often stability, motivating the use of unconditionally-stable implicit methods. The computational cost and complexity of such methods is offset by the freedom to choose much larger time steps, and therefore fewer time steps. To obtain the same level of accuracy with the larger time steps requires the use of higher-order methods. It is well-known that unconditionally-stable linear-multistep methods are at most second order [18], while unconditionally-stable multistage methods like implicit Runge-Kutta (RK) methods can be derived for arbitrary orders of accuracy. Hence, multistage methods are a common choice for the numerical simulation of stiff IVPs.

Recently, the work of [41, 42] was extended to show that all operators which satisfy the generalized summation-by-parts (GSBP) property [6, 19] can be used to construct high-order, unconditionally-stable, implicit, multistage time-marching methods [6]. The traditional SBP approach was first used with finite-difference (FD) schemes [39] as a means to derive stable and efficient boundary closures. The generalized framework extends both the definition of SBP and the permitted form of the operators. In doing so, the generalized framework unifies several known discretizations, including: classical FD-SBP operators [39, 45], collocated pseudo-spectral methods [12, 13, 31, 32, 33], certain discontinuous-Galerkin schemes, depending on the boundary implementation [24], and others. For practical problems, these operators are often augmented with simultaneous approximation terms (SATs) to weakly enforce boundary conditions and multiblock interface coupling [6, 14, 15, 19, 22, 23].

Dual-consistent time-marching methods based on GSBP operators and SATs have some desirable properties for stiff IVPs, including linear stability, non-linear stability for time-marching methods based on diagonal-norm GSBP operators, superconvergence of linear functionals of the numerical solution,
and superconvergence of the numerical solution itself at the final time $T_{f[i]}$ for each instance of a GSBP time-marching method [6, 41, 42]. Note that this value may be extrapolated if a solution point does not exist at the final time. In addition, high-order GSBP operators require substantially fewer nodes than classical FD-SBP operators with the same properties. This is significant as the resulting time-marching methods are in general fully implicit: the solution at any point in time within a block is fully coupled to all other solution points within that block. Thus, the GSBP time-marching methods can be significantly more efficient.

The objective of this paper is to facilitate the construction of more efficient high-order and stable GSBP time-marching methods by showing that they are RK schemes. Specifically, the RK connection enables the minimum superconvergence of the solution previously shown for IVPs that are linear with respect to the solution to be extended for general nonlinear problems, the use of full and simplifying RK order conditions to construct even higher-order GSBP time-marching methods, and the derivation of conditions under which dense-norm GSBP time-marching methods are nonlinearly stable. In addition, the RK connection is also used to derive the conditions under which GSBP time-marching methods are diagonally-implicit.

This paper is organized as follows: Section 3 gives a brief review of the GSBP framework presented in [6, 19]. The characterization of GSBP time-marching methods as Runge-Kutta schemes is developed in Section 4 and their properties are presented in Section 5. This includes reference to many previously obtained results as well as several novel developments of the properties of GSBP time-marching methods derived from the RK characterization. The theory is exemplified with the construction of some known and novel RK schemes based on GSBP operators shown in Section 6. A summary concludes the paper in Section 7.

2. Notation and Definitions

This article considers discrete approximations of IVPs for nonlinear non-autonomous systems of ordinary differential equations:

$$\mathcal{Y}' = \mathcal{F}(\mathcal{Y}, t), \quad \mathcal{Y}(T_{\text{initial}}) = f_{T_{\text{initial}}}, \quad \text{with} \quad T_{\text{initial}} \leq t \leq T_{\text{final}}, \quad (1)$$
where \( Y \in \mathbb{C}^M, Y' = \frac{dY}{dt}, F(Y, t) : \{ \mathbb{C}^M, \mathbb{R} \} \rightarrow \mathbb{C}^M, \) and \( f_{\text{initial}} \) is the vector of initial data. Here, continuous functions are represented by script upper-case letters, for example \( U(t) \in C^\infty[\text{\textit{T}}_{\text{Initial}}, \text{\textit{T}}_{\text{Final}}] \) represents a function which is infinitely differentiable over \( t \in [\text{\textit{T}}_{\text{Initial}}, \text{\textit{T}}_{\text{Final}}] \). The restriction of such functions onto a nodal distribution, \( \text{\textit{t}} = [t_1, \ldots, t_n] \), is represented by bold lower-case letters, for example \( \textbf{u} = [U(t_1), \ldots, U(t_n)]^T \) for the function given above. Furthermore, vectors with the subscript d, for example \( \textbf{u}_d \), denote a numerical solution to a system of discrete equations. Powers of vectors are computed element-wise, for example \( \text{\textit{t}}^j = [t_{1j}, \ldots, t_{nj}]^T \) forms a monomial with the convention that \( t^{-1} = 0 \).

The article makes a distinction between diagonal and nondiagonal norm matrices. The latter will be referred to as dense norms following [19], but encompasses all nondiagonal norms whether they are strictly dense matrices or not. For example, classical FD-SBP operators with full, or restricted-full norms are referred to as dense-norm operators.

3. The Generalized SBP-SAT Approach

This section presents a brief review of the GSBP framework developed in [6, 19].

3.1. Generalized summation-by-parts operators

The motivation for operators which satisfy the generalized summation-by-parts property is the ability to use the energy method to determine the numerical stability of discrete approximations to IVPs. In the continuous case, the energy method is one means of determining the well-posedness of the IVP (1), given that a unique solution exists. The goal is to derive an energy estimate which bounds the norm of the solution at a given point in time with respect to the initial data [25, 26, 38]. The energy estimate is obtained by taking an inner product between the continuous solution and the IVP, and relating the integrals to the initial data through the use of integration-by-parts.

In the discrete case, the inner product between the numerical solution and the discrete IVP is defined by a symmetric positive-definite (SPD) matrix.
called the norm, which satisfies
\[ (u, v)_H = u^* H v \approx (\mathcal{U}, \mathcal{V}) = \int_{T_{\text{initial}}}^{T_{\text{final}}} \tilde{U} \tilde{V} dt, \quad ||u||^2_H = u^* H u \approx (\mathcal{U}, \mathcal{U}), \quad (2) \]

where \( \tilde{U} \) is the complex conjugate of \( U \), and \( u^* \) is the conjugate transpose of \( u \). The generalized summation-by-parts (GSBP) property [19]:
\[ u^* H D v + u^* D^T H v = u^* \tilde{E} v \approx \tilde{U} \tilde{V} \big|_{T_{\text{initial}}}^{T_{\text{final}}}, \quad (3) \]
is then used to relate the numerical quadrature to the initial data, where \( D \) is a linear first-derivative GSBP operator defined by:

**Definition 1. Generalized summation-by-parts operator [19]:** An operator \( D \) is an approximation to the first derivative of order \( q \) with the GSBP property if:
\[ D t^j = j t^{j-1}, \quad j \in [0, q \geq 1], \quad (4) \]

where \( t = [t_1, \ldots, t_n] \) is the nodal distribution with the only requirement that the \( t_i \) be distinct (non-confluent),
\[ H D + (HD)^T = \tilde{E}, \quad (5) \]

where \( H \) is an SPD matrix defining a discrete inner product (3), and
\[ (t^i)^T \tilde{E} t^j = (i + j) \int_{T_{\text{initial}}}^{T_{\text{final}}} \tilde{t}^{i+j-1} d\tilde{t} = T^{i+j}_{\text{final}} - T^{i+j}_{\text{initial}}, \quad i, j \in [0, r \geq q]. \quad (6) \]

In this article, the product \( HD \) will be given the symbol \( \Theta \). Additionally, note that often the nodal distribution of a GSBP operator is ordered, \( t_{i+1} > t_i \), and is within the acting domain of the operator, \( t_i \in [T_{\text{initial}}, T_{\text{final}}] \); however, neither is required here.

GSBP operators which satisfy Definition 1 exist if and only if a nodal quadrature rule exists, and the order of the GSBP operator and nodal quadrature
rule are strongly related [6, 19]. This was first observed for classical FD-SBP operators in [35] and used implicitly with several discretization strategies such as collocated pseudo-spectral methods. This makes the construction of novel GSBP operators fairly straightforward given a nodal quadrature rule, which was first demonstrated in [19]. The existence of a quadrature rule associated with a GSBP operator also presents a consistent and high-order method of evaluating linear functionals, which with careful choice of SAT coefficient are superconvergent [6, 34, 36, 41].

In this article, a nodal quadrature rule associated with a GSBP operator is denoted by \( 1^T H = w^T = [w_1, \ldots, w_n]^T \) such that
\[
\int_{T_{\text{initial}}}^{T_{\text{final}}} F dt \approx \sum_{k=1}^{n} w_k F(t_k)
\]
with order \( \tau \).

3.2. Generalized simultaneous approximation terms

A common approach to impose the initial data is through the use of simultaneous approximation terms (SATs) which weakly impose the initial data via penalty terms [6, 41, 42]. Choosing this approach, the fully-discrete form of the IVP (1) is written as:
\[
(D \otimes I_M) y_d = F(y_d, t) + \sigma(H^{-1} s_{T_{\text{initial}}}^T \otimes I_M)((s_{T_{\text{initial}}}^T \otimes I_M)y_d - (I_N \otimes f_0)),
\]
with
\[
y_d = \begin{bmatrix} y_{d,1} \\ \vdots \\ y_{d,n} \end{bmatrix}, \quad F(y_d, t) = \begin{bmatrix} F(y_{d,1}, t_1) \\ \vdots \\ F(y_{d,n}, t_n) \end{bmatrix}, \quad y_{d,i} = \begin{bmatrix} y_{d,k,1} \\ \vdots \\ y_{d,k,M} \end{bmatrix},
\]
where \( n \) is the number of nodes in the GSBP operator \( D \), \( \sigma \) is the SAT penalty parameter, and \( s_{T_{\text{initial}}} \) is an extrapolation operator defined by:
\[
s_{T_{\text{initial}}}^T \mathbf{t}^j = T_{\text{initial}}^j, \quad j \in [0, r \geq q].
\]
Note that if the nodal distribution includes the initial time \( T_{\text{initial}} \), then \( s_{T_{\text{initial}}}^T = [1, 0, \ldots, 0]^T \) becomes exact. This approach is also used to couple the interface between GSBP operators. To relate the definition of the
GSBP operators and the form of the SAT terms, a convenient decomposition of \( \tilde{E} = \Theta + \Theta^T \) using the extrapolation operators is:

\[
\tilde{E} = \Theta + \Theta^T = s_{T_{\text{final}}}^T s_{T_{\text{final}}} - s_{T_{\text{initial}}}^T s_{T_{\text{initial}}}.
\]  

This idea is not unique, as extrapolation operators have previously been used with FD and FD-SBP schemes in [3, 4, 20, 44], and \( \tilde{E} \) which are not equal to the classical definition of \( \text{Diag}(-1,0,\ldots,0,1) \) have also been considered in [1, 2]; however, to the authors’ knowledge this precise formulation was first proposed in [19]. This formulation of both \( \Theta \), i.e. that it have the property \( \Theta + \Theta^T = \tilde{E} = s_{T_{\text{final}}}^T s_{T_{\text{final}}} - s_{T_{\text{initial}}}^T s_{T_{\text{initial}}} \), and of the SAT terms is assumed for the rest of this article and is implicit in many of the theorems referred to from [6, 19].

Finally, an assumption is stated here for future reference, which is required for a well functioning procedure [41, 42], to guarantee a unique solution for linear IVPs, and for the RK characterization presented in subsequent sections:

**Assumption 1.** For \( \sigma < -\frac{1}{2} \), all eigenvalues of \( \Theta - \sigma s_{T_{\text{initial}}}^T s_{T_{\text{initial}}} \) have strictly positive real parts.

A proof of this assumption for the classical second-order FD-SBP operator is presented in [42] with numerical demonstration of the assumption for higher-order diagonal-norm classical FD-SBP operators. Furthermore, when all operators in a discretization are identical and of fixed size, this can easily be verified numerically.

### 3.3. The dual problem

In addition to the GSBP property itself, a key tool in the proof of many accuracy results obtained for GSBP time-marching methods is the Lagrangian dual problem. This section briefly introduces the continuous and discrete dual problems along with the concept of dual consistency. The latter is required for the RK characterization, as well as many accuracy and stability results referred to in this article. Consider an IVP linear with respect to the
solution:
\[
\mathcal{Y}' = \lambda \mathcal{Y} + \mathcal{G}(t), \quad \mathcal{Y}(0) = f_0, \quad \text{with} \quad T_{\text{initial}} \leq t \leq T_{\text{final}},
\]
which is referred to as the primal problem. A linear functional of the solution to (11) has the form:
\[
\mathcal{J}(\mathcal{Y}(t)) = (\mathcal{H}(t), \mathcal{Y}(t)) + \alpha \mathcal{Y}(T_{\text{final}}).
\]
which is used along with the primal problem to derive the dual problem (See e.g. [34, 36]):
\[
- \Phi' = \bar{\lambda} \Phi + \mathcal{H}(t), \quad \Phi(T_{\text{final}}) = \bar{\alpha}, \quad \text{with} \quad T_{\text{initial}} \leq t \leq T_{\text{final}}.
\]
Applying a GSBP time-marching method to each of these equations results in the discrete primal problem:
\[
D \mathbf{y}_d = \lambda \mathbf{y}_d + \mathcal{G}(t) + \sigma H^{-1} s_{T_{\text{initial}}} (s_{T_{\text{initial}}}^T \mathbf{y}_d - f_{T_{\text{initial}}}),
\]
discrete linear functional:
\[
J_H(\mathbf{y}_d) = (\mathcal{H}(t), \mathbf{y}_d)_H + \alpha s_{T_{\text{final}}}^T \mathbf{y}_d,
\]
and discrete dual problem (See e.g. [6, 41]):
\[
- D \phi_d = \bar{\lambda} \phi_d + \mathcal{H}(t) - (1 + \sigma) H^{-1} s_{T_{\text{initial}}} s_{T_{\text{initial}}}^T \phi_d - H^{-1} s_{T_{\text{final}}} (s_{T_{\text{final}}}^T \phi_d - \bar{\alpha}).
\]
The discrete dual problem (16) becomes a consistent approximation of the continuous dual problem (13) if the SAT penalty parameter is \(\sigma = -1\). This is known as dual consistency (DC) [40]. Dual consistency is required for many of the accuracy and stability results referred to in subsequent sections, but also decouples the solution within a given block from the solution of subsequent blocks in time. Here a block is defined as an instance of a GSBP operator, therefore the solution within that block is the nodal solution values associated with that instance. This means that instead of having a global discretization
of the IVP, each block can be solved sequentially in time, though in general
the solution within a block is fully-implicit. This decoupling is what permits
the characterization as an RK scheme.

4. Runge-Kutta Representation

In this section it is shown that DC GSBP time-marching methods are
implicit Runge-Kutta (RK) schemes. An RK method can be written as

\[ \tilde{y}^{[i]} = \tilde{y}^{[i-1]} + h \sum_{j=1}^{s} b_j \mathbf{F}(y_{d,j}, t^{[i-1]} + c_j h), \]  

(17)

with \( n \) internal stage approximations:

\[ y_k = \tilde{y}^{[i-1]} + h \sum_{j=1}^{s} A_{kj} \mathbf{F}(y_{d,j}, t^{[i-1]} + c_j h) \]  

for \( k = 1, \ldots, n \),

(18)

where \( A_{kj} \) and \( b_j \) are the coefficients of the method with abscissa \( c \), and \( h \) is
the step size. This can also be written as a partitioned matrix system of the
form

\[ \begin{bmatrix} \mathbf{y}_d \\ \tilde{y}^{[i]} \end{bmatrix} =  
\begin{bmatrix} A & \mathbf{1} \\ b^{T} & 1 \end{bmatrix} \begin{bmatrix} h \mathbf{F}(\mathbf{y}, t^{[i-1]} \mathbf{1} + h \mathbf{c}) \\ \tilde{y}^{[i-1]} \end{bmatrix}, \]

(19)

where \( \mathbf{1} = [1, \ldots, 1]^T \). The notation chosen here foreshadows the relationship
between the GSBP time-marching methods and RK schemes.

Now consider the \( i^{th} \) block within a multi-block GSBP-SAT discretization
of (1) with \( M = 1 \) and DC SAT coefficient \( \sigma = -1 \):

\[ D\mathbf{y}_d^{[i]} = H^{-1} \Theta \mathbf{y}_d^{[i]} = \mathbf{F} \left( \mathbf{y}_d^{[i]}, t \right) - H^{-1} s_0 \left( s_0^T \mathbf{y}_d^{[i]} - \bar{s}_1^T \mathbf{y}_d^{[i-1]} \right), \]

(20)

where the components from the preceding block in time with index \( i - 1 \) are
marked with an overbar \( \bar{\Box} \), and the subscripts denote the boundaries of the
block: 0 for the initial boundary at \( t^{[i-1]} \) and 1 for the final boundary at \( t^{[i]} \). Recall that in addition to dual consistency, the choice of \( \sigma = -1 \) forces
the numerical solution within each block to be independent of the solution
within subsequent blocks in time. Next, rearranging (20) using Assumption
1 gives

\[ y_d^{[i]} = (\Theta + s_0 s_0^T)^{-1} H F \left( y_d^{[i]}, t \right) + (\Theta + s_0 s_0^T)^{-1} s_0 s_1^T y_d^{[i-1]}, \tag{21} \]

and extrapolating the solution to the block boundary yields

\[ s_1^T y_d^{[i]} = \tilde{y}_1^{[i]} = s_1^T (\Theta + s_0 s_0^T)^{-1} H F \left( y_d^{[i]}, t \right) + s_1^T (\Theta + s_0 s_0^T)^{-1} s_0 \tilde{s}_1^T y_d^{[i-1]}. \tag{22} \]

Note again that if the nodal distribution includes the final time \( T_{\text{final}} \), then the solution at the block boundary is the final nodal solution value, equivalent to stiff-accuracy for RK schemes. These equations describe a set of intermediate values (21) which are used to generate a solution one step forward in time (22), analogously to an RK scheme. Formally, we let the nodal solution values \( y_d \) of the GSBP time-marching method be the internal stage approximations of the RK scheme, and the solution at block boundaries, \( s_1^T y_d^{[i]} = \tilde{y}_1^{[i]} \) and \( s_1^T y_d^{[i-1]} = \tilde{y}_1^{[i-1]} \), be the solution points \( \tilde{y}_1^{[i]} \) and \( \tilde{y}_1^{[i-1]} \), respectively. For the analogy to be valid requires that the identities

\[ (\Theta + s_0 s_0^T)^{-1} s_0 = 1, \tag{23} \]

and

\[ s_1^T (\Theta + s_0 s_0^T)^{-1} s_0 = 1, \tag{24} \]

hold. The first identity is proved by multiplying (23) through by \( H^{-1}(\Theta + s_0 s_0^T) \) and rearranging:

\[ H^{-1} \Theta 1 = H^{-1} s_0 (1 - s_0^T 1). \tag{25} \]

By Definition 1, a consistent GSBP operator must be associated with an extrapolation operator of order \( r \geq q \geq 1 \) or be exact. This implies that \( s_0^T \mathbf{1} = 1 \), and (25) simplifies to

\[ D \mathbf{1} = 0, \tag{26} \]
which is condition (4) with \( j = 0 \), a prerequisite imposed by Definition 1. The second identity (24) follows immediately for a consistent GSBP operator which satisfies Definition 1, which implies that \( s_1^T \mathbf{1} = 1 \). These two identities are often referred to as preconsistency conditions for more general time-marching methods.

Finally, the coefficient matrices of the RK scheme defined in terms of the components of a GSBP operator are:

\[
A = \frac{1}{h} (\Theta + s_0 s_0^T)^{-1} H,
\]

\[
b^T = s_1^T A = \frac{1}{h} s_1^T (\Theta + s_0 s_0^T)^{-1} H,
\]

with abscissa

\[
c = \frac{t - 1}{T_0} h,
\]

where \( h = T_1 - T_0 \) is the size of the time interval associated with the \( i^{th} \) block, \( T_0 \leq t^{[i]} \leq T_1 \). Note that all the coefficient matrices are solely a function of the \( i^{th} \) block and are independent of the components of the preceding block. The application of RK methods, (17) and (18), sees the coefficient matrices \( A \) and \( b^T \) multiplied by \( h \), which for GSBP-SAT discretizations is included in \( H \); hence the factor of \( \frac{1}{h} \) in (27). Similarly, the abscissa indicates the location of the intermediate solution point between time levels \([i + 0]\) and \([i + 1]\) respectively; hence the form in (28) which rescales and translates the nodal distribution from \([T_0, T_1]\) to \([0, 1]\).

5. Properties

This section examines common properties of RK methods and how they relate to the conditions imposed on time-marching methods based on GSBP operators. It further explores concepts which are not imposed by the GSBP theory and how they can be exploited to generate more favourable GSBP time-marching methods in terms of accuracy, stability, and efficiency.
5.1. The abscissa and nodal distribution

For simplicity, the relationship between the abscissa and nodal distribution of GSBP operators is examined first. In particular, the interest is in the relationship formed by the application of GSBP operators to the abscissa with respect to the nodal distribution and the conditions enforced by Definition 1.

To begin, recall the definition of the abscissa with respect to the nodal distribution (28) and that powers are computed element-wise
\[
c^p = [c^p_1, \ldots, c^p_n]^T.
\] (29)

Substituting (28) into (29) and expanding yields
\[
c^p = \frac{1}{h^p} \sum_{i=0}^{p} \binom{p}{i} t^{p-i}(-T_0)^i,
\] (30)

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) is the binomial coefficient. Applying a GSBP operator of order at least \( p \) yields
\[
Dc^p = \frac{1}{h^p} \sum_{i=0}^{p} \binom{p}{i} (p-i)t^{p-1-i}(-T_0)^i.
\] (31)

Substituting the relationship that \( (p-k)\binom{p}{k} = p\binom{p-1}{k} \) and recalling that \( t^{-1} = 0 \), gives
\[
Dc^p = \frac{p}{h^p} \sum_{i=0}^{p-1} \binom{p-1}{i} t^{p-1-i}(-T_0)^i = \frac{p}{h} c^{p-1}.
\] (32)

The additional factor of \( \frac{1}{r} \) comes from the norm \( H \) of the GSBP operator \( D = H^{-1} \Theta \), which is defined for the interval \([T_0, T_1]\) of size \( h \), rather than \([0, 1]\) on which the abscissa is defined. In summary, the identity \( Dc^p = \frac{p}{h} c^{p-1} \) holds for GSBP operators of order greater than or equal to \( p \).

Next, consider the application of an extrapolation operator \( s_0 \) of order greater than or equal to \( p \) to a monomial of the abscissa:
\[
s_0^T c^p = \frac{1}{h^p} \sum_{i=0}^{p} \binom{p}{i} T_0^{p-i}(-T_0)^i.
\] (33)
Pulling out a factor of $T_0^p$ from the summation and given that $\sum_{i=0}^{p} \binom{p}{i} (-1)^i = 0$ for $p > 0$, (33) simplifies to

$$s_0^T c^p = 0 \text{ for } p > 0. \quad (34)$$

If $p = 0$ then $s_0^T c^p = s_0^T 1 = 1$ by Definition 1. Finally, consider the application of an extrapolation operator $s_1$ of order greater than or equal to $p$ to a monomial of the abscissa:

$$s_1^T c^p = \frac{1}{h^p} \sum_{i=0}^{p} \binom{p}{i} T_1^{p-i} (-T_0)^i. \quad (35)$$

Note that $h^p = (T_1 - T_0)^p$ can be expanded as $\sum_{i=0}^{p} \binom{p}{i} T_1^{p-i} (-T_0)^i$. Therefore, simplifying (35) gives

$$s_1^T c^p = 1 \text{ for } p \geq 0. \quad (36)$$

These relationships greatly simplify the analysis in the subsequent sections as the components of GSBP time-marching methods can be applied directly to the abscissae found in the RK conditions.

### 5.2. Consistency

In this section, it is shown that all time-marching methods based on GSBP operators are consistent and therefore convergent. The conditions for consistency and stage consistency of an RK method are that they be preconsistent and satisfy:

$$b^T 1 = 1, \quad \text{and} \quad A 1 = c. \quad (37)$$

Beginning with the stage-consistency condition and substituting the expression for $A$, gives

$$A 1 = \frac{1}{h} (\Theta + s_0 s_0^T)^{-1} H 1 = c. \quad (38)$$
Multiplying through by $hH^{-1}(\Theta + s_0s_0^T)$ and simplifying yields

$$hDc + hH^{-1}s_0s_0^Tc = \mathbb{1}.$$  \hspace{1cm} (39)

Using the relationships derived in Section 5.1 and that by Definition 1, $r \geq q \geq 1$, the first term on the left-hand side simplifies to $\mathbb{1}$ and the second term to $0$. Therefore, (39) becomes an identity. From this result, the condition for consistency immediately follows:

$$b^T\mathbb{1} = s_1^TAd = s_1^Tc = 1$$  \hspace{1cm} (40)

which is also an identity. Therefore, all DC time-marching methods based on GSBP operators and SATs are consistent and stage-consistent RK methods and therefore convergent.

5.3. Stage-order conditions

The stage order of an RK scheme is the order to which the intermediate stage values (18) approximate $Y(T_0 + c_i)$. This property is particularly important for problems with stiff source terms as it influences the rate of convergence [6, 41, 43]. For time-marching methods based on GSBP operators cast as RK schemes, these are the nodal solution values $y_d$. In [6] the pointwise order of accuracy of these solution values was shown to be order $q$ in general. This section formally relates these two properties by substituting the RK characterization of time-marching methods based on GSBP operators into the stage-order conditions of RK schemes.

The stage-order conditions for an RK method can be written as

$$c^i = iAc^{i-1}, \text{ for } i \in [1, \hat{q}],$$  \hspace{1cm} (41)

where $\hat{q}$ is the stage order to be satisfied. Substituting the GSBP expression for $A$ (27) gives:

$$c^i = iAc^{i-1} = i\frac{1}{h}(\Theta + s_0s_0^T)^{-1}Hc^{i-1}, \text{ for } i \in [1, \hat{q}].$$  \hspace{1cm} (42)
Multiplying through by $hH^{-1}(\Theta + s_0s^T_0)$ and simplifying yields

$$hDc^i - ic^{i-1} = -hH^{-1}s_0s^T_0c^i, \text{ for } i \in [1, \hat{q}; \quad (43)$$

which is guaranteed to be an identity for $\hat{q} \leq q$. Therefore, all time-marching methods based on GSBP operators that have a minimum uniform order $q$, also have stage order $q$.

5.4. Order conditions

Often the primary result of an RK scheme is the solution update (17), rather than the intermediate stage approximations (18). Therefore, it is of interest to consider the order of the solution update in addition to what has just been shown for the stage approximations. In [6] it was shown that the numerical solution at the final time $T[i]$ of each instance of a GSBP time-marching method applied to IVP linear with respect to the solution is superconvergent to the order of $\min(2q + 1, \rho)$, where $\rho$ is defined by:

**Definition 2. Accuracy of a GSBP norm:** The accuracy $\rho$ of a norm $H$ associated with a GSBP operator $D = H^{-1}\Theta$ is the minimum order to which:

- $(y, z)_H$ approximates $(Y, Z)$; and
- $(\phi, D y + H^{-1}s_T^{initial} (s_T^{initial} y - f_T^{initial}))_H$ approximates $(\Phi, Y')$,

where $\Phi$ the solution to dual problem (13), and $f_T^{initial}$ is the initial condition of the primal problem (11).

For diagonal norms associated with GSBP operators, $\rho = \min(2q + 1, \tau)$ and for dense norms associated with GSBP operators, $\rho = \min(2q + 1, s)$, where $2[q/2] \leq s \leq \tau$. The derivation of these values, along with a more precise definition of $s$ is presented in [6]. Note also that $\rho$ is always greater than or equal to the order of the GSBP operator itself.

In this section, the order conditions of RK solution updates (17) are examined in the context of time-marching methods based on GSBP operators, extending the accuracy results previously obtained to both nonlinear IVPs and for higher-order updates relative to the stage-order.
5.4.1. Simplifying assumptions

The full RK order conditions for general nonlinear IVPs (see Section 5.4.2) become increasingly difficult to solve as the order increases. To ease the construction of higher-order RK schemes, a simpler set of sufficient conditions were derived in [8]. This simplified set of equations are referred to as simplifying order conditions and are summarized with the following theorem:

**Theorem 1.** If the coefficients $A$, $b^T$, and $c$ of an RK method satisfy the following conditions:

\begin{align}
B(p) : \quad & b^T c^{j-1} = \frac{1}{j}, \quad j \in [1, p] \\
C(\hat{q}) : \quad & A c^{j-1} = \frac{c^j}{j}, \quad j \in [1, \hat{q}] \\
D(\xi) : \quad & A^T B_d c^{j-1} = \frac{1}{j} B_d (1 - c^j), \quad j \in [1, \xi]
\end{align}

with $p \leq 2\hat{q} + 2$ and $p \leq \hat{q} + \xi + 1$, where $B_d$ is a diagonal matrix formed by the entries of $b$, then the RK method will be of order $p$.

**Proof.** Refer to Theorem 7 in [8].

The first simplifying condition $B(p)$ (44) is the requirement that $b^T$ be a quadrature rule of order $p$. In Section 3.1 the relationship between a GSBP operator and an associated quadrature rule of order $\tau$ was discussed. It turns out that the elements of the coefficient matrix $b^T$ are the weights $w$ of this quadrature rule scaled for the domain $[0, 1]$. Consider the coefficient matrix $b^T$:

\[ b^T = \frac{1}{h} s_1^T (\Theta + s_1 s_1^T)^{-1} H, \]

(47)

By Lemma 1 from [6] $s_1^T (\Theta + s_1 s_1^T)^{-1} = \mathbb{1}^T$, and therefore (44) simplifies to:

\[ b^T = \frac{1}{h} \mathbb{1}^T H = \frac{1}{h} w^T. \]

(48)
Hence, the first simplifying condition \( B(p) \) (44) is satisfied for \( p = \tau \). The second simplifying condition \( C(\tilde{q}) \) (45) is equivalent to the stage-order conditions discussed in Section 5.3 which were been shown to be satisfied up to \( \tilde{q} = q \) for all time-marching methods based on GSBP operators.

At this point, it has been shown that time-marching methods based on GSBP operators are at least order \( p = \min(\tau, q + 1) \). For the final simplifying condition \( D(\xi) \) (46), time-marching methods associated with diagonal and dense norms are investigated separately. Beginning with diagonal norms and inserting the GSBP expressions for the RK coefficient matrices, the third simplifying condition \( D(\xi) \) becomes:

\[
\left( \frac{1}{h} \left( \Theta + s_0 s_0^T \right)^{-1} H \right)^T B_d c^{j-1} = \frac{1}{j} B_d (\mathbf{1} - c^j), \quad j \in [1, \xi].
\]

(49)

For diagonal norms \( B_d \) is equivalent to the norm \( H \). Making this substitution, multiplying through by \( h H^{-1} (\Theta + s_0 s_0^T)^T H^{-1} \), and simplifying, yields:

\[
\frac{1}{h} c^{j-1} = \frac{1}{j} H^{-1} (\Theta^T + s_0 s_0^T) (\mathbf{1} - c^j), \quad j \in [1, \xi].
\]

(50)

Substituting the definition of \( \Theta \) that it satisfy \( \Theta + \Theta^T = \tilde{E} = s_1 s_1^T - s_0 s_0^T \) and expanding gives:

\[
\frac{1}{h} c^{j-1} = \frac{1}{j} \left( -H^{-1} \Theta (\mathbf{1} - c^j) + H^{-1} s_1 s_1^T (\mathbf{1} - c^j) \right), \quad j \in [1, \xi].
\]

(51)

Using the relationships derived in Section 5.1, for \( \xi \leq q \) the first term on the left-hand side reduces to \( \frac{1}{h} c^{j-1} \) and the second term to \( \mathbf{0} \), thus recovering an identity. Therefore, all DC time-marching methods based on diagonal-norm GSBP operators are RK methods of order \( p = \min(\tau, 2q + 1) \). This is equivalent to the superconvergence shown previously in [6], but for general nonlinear IVPs. Furthermore, the third simplifying order condition (49) can be used to guide the construction of diagonal-norm GSBP time-marching methods one order higher.

In the case of time-marching methods based on dense-norm GSBP operators the same simplifications do not follow and the resulting method is only
guaranteed to be order \( p = \min(\tau, q + 1) \). This is equivalent to the minimum guaranteed rate of superconvergence shown previously in [6], but for general nonlinear IVPs. In the case of dense-norm GSBP time-marching methods, the third simplifying order condition can be used to guide the construction of significantly higher-order methods.

5.4.2. Full order conditions

The full order conditions for an RK scheme can be found in several references, for example [11, 28]. For reference, the conditions for orders one through four are:

\[
\begin{align*}
    b^T 1 &= 1 \\
    b^T c &= \frac{1}{2} \\
    b^T c^2 &= \frac{1}{3} \\
    b^T Ac &= \frac{1}{6} \\
    b^T c^3 &= \frac{1}{4} \\
    b^T Ac^2 &= \frac{1}{8} \\
    b^T CAc &= \frac{1}{12} \\
    b^T AAc &= \frac{1}{24} \\
\end{align*}
\]

where \( C = \text{diag}(c) \) and it is assumed that \( A1 = c \). Given the relationship between the GSBP operators and the RK coefficient matrices derived above, one can simply insert these relationships into the algebraic order conditions and solve for the coefficients. These systems of equations are not necessarily easy to solve, but can be exploited, for example when constructing diagonally-implicit GSBP-RK schemes which are limited to stage-order \( \hat{q} = 1 \), since \( A \) must be invertible. This implies that the pointwise accuracy of the GSBP operator is order 1, but does not limit the superconvergence of the solution at the final time, or in other words the solution update. Without using the full or simplifying RK order conditions, the maximum order diagonally-implicit RK scheme is only guaranteed to be third-order; however, it is known that higher-order diagonally-implicit RK methods exist. An example of a diagonally implicit GSBP-RK scheme with stage order 1 and solution update order 4 is presented in Section 6 using these full order conditions.
5.5. Stability

In [6] it was shown that all dual-consistent time-marching methods based on GSBP operators are unconditionally stable for linear problems. This includes A-stability, L-stability, and stability for certain systems of linear IVPs introduced in [41], termed linear stability. In addition, [6] showed that dual-consistent time-marching methods based on diagonal-norm GSBP operators are unconditionally stable for nonlinear problems. This included both BN-stability\(^2\) and energy-stability, the latter being another stability definition introduced in [41]. In this section, the nonlinear stability results are extended for time-marching methods based on dense-norm GSBP operators.

The conditions under which dense-norm GSBP time-marching methods are nonlinearly stable are derived from algebraic criteria of B-stability derived in [10, 17]:

\begin{enumerate}
  \item \(b_i \geq 0\) for \(i = 1, \ldots, s\)
  \item \(A\) is invertible
  \item \(\widehat{M} = B_d A^{-1} + (A^{-1})^T B_d - (A^{-1})^T b b^T A^{-1}\) is non-negative definite,
\end{enumerate}

where \(B_d\) is a diagonal matrix formed by the elements of \(b\). This result was later superseded in 1979 by algebraic criteria for BN-stability [7]:

\begin{enumerate}
  \item \(b_i \geq 0\) for \(i = 1, \ldots, s\)
  \item \(M = B_d A + A^T B_d - b b^T\) is non-negative definite,
\end{enumerate}

eliminating the need for \(A\) to be invertible. If \(A\) is invertible, then the two sets of criteria are equivalent: \(M = A^T \widehat{M} A\) is non-negative definite if \(A\) is invertible and \(\widehat{M}\) is non-negative definite; however, only the criteria for

\(^2\)BN-stability is sometimes referred to as B-stability when the distinction between autonomous and non-autonomous ODEs is not made (Compare Definitions 2.9.2 and 2.9.3 of [37] and Definition 12.2 in [29]).
BN-stability are given the name algebraic-stability. It is known that for non-confluent RK schemes, which all GSBP time-marching methods are, the concepts of algebraic-stability, BN-stability, B-stability, and AN-stability are all equivalent [29]. These algebraic criteria for nonlinear stability provide a tool for deriving the conditions under which time-marching methods based on dense-norm GSBP operators are nonlinearly stable as well.

The first criterion is that the elements of the coefficient matrix $b$ be non-negative. This is equivalent to requiring that the weights of the quadrature rule associated with the norm of the GSBP time-marching method be non-negative. By Definition 1 and Assumption 1, the coefficient matrix $A$ of a GSBP time-marching method is invertible. Finally, substituting the GSBP expressions for the RK coefficient matrices into the last criterion and simplifying yields:

$$
\widehat{M} = B_d H^{-1}(\Theta + s_0 s_0^T) + (\Theta + s_0 s_0^T)^T H^{-1} B_d - s_1^T s_1.
$$

(53)

If the norm is diagonal, then $B_d$ is equivalent to $H$, and $\widehat{M}$ simplifies to $s_0 s_0^T$. The eigenvalues of $\widehat{M}$ are then zero and $s_0^T s_0$; the former having multiplicity $n - 1$ and the latter being non-negative. Hence, DC diagonal-norm GSBP time-marching methods are always algebraically-stable, as shown previously.

If the norm is dense, the criteria for nonlinear stability require that the associated quadrature rule have positive weights and the coefficients of the GSBP operator satisfy (53) such that $\widehat{M}$ is non-negative definite.

6. Sample Implicit GSBP-RK Methods

This section applies the theory developed in this article to construct some known and novel RK schemes which are based on GSBP operators. These methods are presented here as examples and are not necessarily optimal in any sense. For numerical examples of the RK analogy in practice, refer to the results presented in [6] which were obtained with an RK implementation of the GSBP operators.

6.1. Lobatto IIIC discontinuous-collocation RK methods

The diagonal-norm GSBP operators with Gauss-Lobatto nodal distributions derived in [24] lead to DC time-marching methods which are equivalent
to Lobatto IIIC discontinuous-collocation RK methods [5, 9, 16, 21, 30]. As an example, consider the four-node GSBP operator defined for the interval $[-1, 1]$ based on Gauss-Lobatto points:

$$t = \left[ -1 \quad -\frac{1}{6}\sqrt{5} \quad \frac{1}{6}\sqrt{5} \quad 1 \right]^T.$$  \hspace{1cm} (54)

The corresponding norm, whose entries are the Gauss-Lobatto quadrature weights, and resulting GSBP operator are:

$$H = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} & \frac{5}{6} & \frac{1}{6} \\ \frac{5}{6} & 0 & -\frac{\sqrt{5}}{2} & -\frac{5\sqrt{5}}{\sqrt{5}+5} \end{bmatrix}, \quad D = \begin{bmatrix} -3 & -\frac{5\sqrt{5}}{\sqrt{5}+5} & -\frac{5\sqrt{5}}{\sqrt{5}+5} & \frac{1}{2} \\ -\frac{\sqrt{5}}{\sqrt{5}+5} & 0 & -\frac{\sqrt{5}}{2} & -\frac{5\sqrt{5}}{\sqrt{5}+5} \\ -\frac{5\sqrt{5}}{\sqrt{5}+5} & -\frac{\sqrt{5}}{2} & 0 & -\frac{5\sqrt{5}}{\sqrt{5}+5} \\ -\frac{1}{2} & \frac{5\sqrt{5}}{\sqrt{5}+5} & \frac{5\sqrt{5}}{\sqrt{5}+5} & 3 \end{bmatrix}, \hspace{1cm} (55)$$

with exact extrapolation operators $s_{-1} = [1, 0, \ldots, 0]^T$ and $s_1 = [0, \ldots, 0, 1]^T$. Applying the RK characterization derived in Section 4, the coefficients of the equivalent RK scheme are

$$A = \begin{bmatrix} \frac{1}{12} & -\frac{\sqrt{5}}{12} & -\frac{\sqrt{5}}{12} & -\frac{1}{12} \\ \frac{1}{12} & \frac{1}{4} & \frac{10-7\sqrt{5}}{60} & \frac{\sqrt{5}}{60} \\ \frac{1}{12} & \frac{10+7\sqrt{5}}{60} & \frac{1}{4} & -\frac{\sqrt{5}}{60} \\ \frac{1}{12} & \frac{5}{12} & \frac{5}{12} & \frac{1}{12} \end{bmatrix}, \hspace{1cm} (56)$$

and

$$b = \left[ \frac{1}{12} \quad \frac{5}{12} \quad \frac{5}{12} \quad \frac{1}{12} \right], \hspace{1cm} (57)$$

with abscissa:

$$c = \left[ 0 \quad \frac{1}{2} - \frac{1}{10}\sqrt{5} \quad \frac{1}{2} + \frac{1}{10}\sqrt{5} \quad 1 \right]^T. \hspace{1cm} (58)$$

6.2. Diagonally-implicit GSBP-RK methods

To demonstrate some of the knowledge gained from the extended theory developed in this article, diagonally-implicit RK schemes are constructed
which are based on GSBP operators. Diagonally-implicit methods are often more efficient than fully-implicit schemes, especially in terms of memory usage, and are therefore of particular interest. For these examples, coefficients of the GSBP operator are first constrained such that the resulting RK scheme is diagonally-implicit and satisfies the minimal requirements of Definition 1. The former is done by using the fact that the inverse of a lower triangular matrix is also lower triangular. Therefore, decomposing \( \Theta \) into symmetric \( \Theta_S = \frac{1}{2} \tilde{E} \) and anti-symmetric components \( \Theta_A \) (See e.g. [19]), the coefficients of \( \Theta_A \) are chosen such that \( H^{-1}(\Theta + s_0^T s_0^T) \), the inverse of the coefficient matrix \( A \), is lower triangular. The remaining coefficients in the GSBP operator and corresponding RK scheme, including the nodal distribution and weights of the associated quadrature, are solved for using the full-order conditions of RK schemes.

The first example is a novel three-stage GSBP-RK scheme. Note that the scheme is not necessarily optimal in any sense, but is shown here as an example of how the theory can be applied. The nodal distribution of the GSBP operator determined by solving the order conditions (52) is:

\[
\mathbf{c} = \begin{bmatrix}
\frac{13}{20} & -\frac{1791531357580309}{5132388141631403} & \frac{8865729326410787}{20641615062507019}
\end{bmatrix}^T,
\]

which is already chosen to be for the domain \([0, 1]\). Likewise, the norm is determined to be:

\[
H = \begin{bmatrix}
\frac{13}{20} & 0 & 0 \\
0 & \frac{452918275472559}{484196979970915} & 0 \\
0 & 0 & \frac{993416917985809}{3873575823896732}
\end{bmatrix},
\]

which are the weights of the associated quadrature rule, and the GSBP derivative operator is:

\[
D = \begin{bmatrix}
0 & -\frac{140}{109} & \frac{140}{109} \\
-\frac{453720308822687}{944696543100651} & -\frac{1362276155224454}{51495418365075} & \frac{520526231758587}{710649509866024} \\
-\frac{22415516029965940}{12787899975934583} & -\frac{1177280456698345}{14940332242497112} & \frac{530641460134326}{549953965844729}
\end{bmatrix},
\]

with extrapolation operators:
$$s_0 = \begin{bmatrix} 1 & \frac{91}{109} & -\frac{91}{109} \end{bmatrix}^T, \quad \text{and} \quad s_1 = \begin{bmatrix} 1 & -\frac{49}{109} & \frac{49}{109} \end{bmatrix}^T. \quad (62)$$

It is interesting to note that the nodal distribution is not ordered, $t_i \not> t_{i-1}$, nor are all $t_i$ within the acting domain of the operator, e.g. $T_{\text{final}} = -\frac{1791531357580399}{5132588141631403} \notin [0,1]$. This is not uncommon for time-marching methods, though it is often required for GSBP operators (See e.g. [19]). The additional flexibility enables a third-order method to be constructed with only three nodal points that is diagonally-implicit. The equivalent diagonally-implicit RK scheme has the following coefficient matrices:

$$A = \begin{bmatrix} \frac{13}{20} & 0 & 0 \\ -\frac{49}{70} & \frac{944096543160651}{4537203088822687} & 0 \\ -\frac{49}{70} & \frac{1888192086221302}{4537203088822687} & \frac{337338079946117}{591310322486307} \end{bmatrix}, \quad (63)$$

and

$$b = \begin{bmatrix} \frac{13}{20} & \frac{452918275472559}{484196977970915} & \frac{993416917985809}{98755782986532} \\ \frac{83162513996114}{1074657575374628} & \frac{2569786067480504}{2618583692794293} & \frac{4910818521271745}{1000000000000000} \end{bmatrix}, \quad (64)$$

with $c = t$. Note that even though the GSBP derivative operator is dense, the resulting RK scheme is diagonally-implicit. In addition, since the norm associated with the GSBP operator is diagonal, the scheme is by definition L-stable, linearly-stable, Algebraically-stable and energy-stable.

As a second example, a fourth-order diagonally-implicit GSBP-RK scheme is constructed. This goes beyond the order guaranteed by the GSBP theory alone. The full RK order conditions are used to derive all of the coefficients of the GSBP operators and hence the equivalent RK scheme. It is theoretically possible to do the same with the simplifying order conditions, but in this case the equivalent RK scheme does not satisfy the fourth-order simplifying conditions. Again, this scheme is not necessarily optimal in any sense, but is shown here as an example of how the theory can be applied. The nodal distribution of the scheme determined by solving the full RK order conditions (52) is:

$$t = \begin{bmatrix} 3996719253698849 \\ 668508890105010 \\ 7127423865815527 \\ 2618583692794293 \\ 4910818521271745 \end{bmatrix}, \quad (65)$$
already chosen to be for the domain \([0, 1]\). Likewise, the corresponding norm is:

\[
H = \begin{pmatrix}
15629811955606409 
& 0 & 0 & 0 \\
29693966436542602 
& 11453716769462519 
& 0 
& 0 \\
0 
& 5329863756145130 
& 3814633038509597 
& 0 \\
0 
& 0 
& 5283095949055420 
& 18464984304571329 \\
0 
& 0 
& 0 
& 0 \\
\end{pmatrix},
\]

which defines the weights of the associated quadrature rule, and the GSBP derivative operator is:

\[
D = \begin{pmatrix}
2271390838991971 
& 7752721717421943 
& 2367084780665803 
& 686571278778671 \\
1139079839212987 
& 4086898652837736 
& 2352920003148328 
& 1529889993395618 \\
2631310607557697 
& 3145596460071109 
& 1555816077753852 
& 672004562251629 \\
1595901604890990 
& 2593314100341849 
& 78516534132899 
& 761228351492153 \\
1205613016109835 
& 558135232148637 
& 2390636602324674 
& 5761875110410924 \\
3747237576160921 
& 3454421039045640 
& 4890861636786055 
& 5174343100186643 \\
952855682874523 
& 65057562646023 
& 4577938138290674 
& 322266508880611 \\
7279852850902435 
& 721573074418405 
& 762324974645155 
& 261314695034027 \\
\end{pmatrix},
\]

with extrapolation operators:

\[
s_0 = \begin{pmatrix}
219956639421467 \\
221701623013847 \\
520011369575820 \\
2023501160943527 \\
229581680152738 \\
382516009227375 \\
3263428492192819 \\
\end{pmatrix},
\]

and

\[
s_1 = \begin{pmatrix}
6468677186621298 \\
651504303370127 \\
1938644757277753 \\
3888075087011408 \\
2156856047680369 \\
459778283079677 \\
1368973594544742 \\
3737648447778637 \\
\end{pmatrix},
\]

Applying the RK characterization of these operators yields the RK coefficient matrices.
\[ A = \begin{bmatrix}
2599318764153512 & 0 & 0 & 0 \\
149934461489343 & 451745023939154 & 0 & 0 \\
2473578070162024 & 9219979975029715 & 1764415878696593 & 0 \\
6750728399753803 & 3692730966460721 & 3542726035846558 & 105768632824691 \\
4131064334435269 & 2645011466329672 & 3542726035846558 & 1353985321415663 \\
4528510858186538 & 27999515137588854 & 961964729315903 & 105768632824691 \\
3859839102288749 & 481090145637995 & 961964729315903 & 1353985321415663 \\
25242635249797 & 481090145637995 & 961964729315903 & 1353985321415663 \\
\end{bmatrix}, \tag{70}\]

and

\[ b^T = \begin{bmatrix}
1562981495606409 \\
2969396643654262 \\
1145371769462519 \\
814633083590597 \\
532986375614513 \\
368162540461966 \\
528309594905542 \\
1846498430457139 \\
\end{bmatrix}, \tag{71}\]

with abscissa \( c = t \). Again, since this RK scheme is constructed from a diagonal-norm GSBP operator, it is L-stable, linearly-stable, algebraically-stable and energy-stable. It is also interesting to note that fourth-order is the highest order possible for diagonally-implicit RK schemes which are algebraically-stable [27]. Therefore, to construct a diagonally-implicit GSBP-RK scheme of order greater than four, it must not be algebraically stable, and therefore cannot be based on a diagonal-norm GSBP operator.

### 7. Conclusions

This article extends the theory of dual-consistent time-marching methods based on generalized summation-by-parts (GSBP) operators and simultaneous approximation terms by showing that they are Runge-Kutta (RK) schemes. The RK connection provides the analytical tools to facilitate the construction of more favourable GSBP time-marching methods in terms of accuracy, stability and efficiency. This is particularly true for dense-norm operators.

Cast as RK schemes, GSBP time-marching methods are shown by definition to satisfy the simplifying order conditions up to the minimum guaranteed order of superconvergence shown previously. The significance is that the simplifying order conditions are for general nonlinear IVPs rather than solely those linear with respect to the solution. Furthermore, the simplifying order conditions provide the criteria required to construct higher-order dense-norm GSBP time-marching methods. Casting the GSBP time-marching methods
as RK schemes also enables the use of the full RK order conditions. These conditions are not always easy to solve for, but can be exploited to construct more efficient schemes than with the simplifying conditions alone.

The RK characterization of time-marching methods based on GSBP operators is used to extend the types of nonlinear stability satisfied by schemes associated with diagonal norms to include AN-stability and algebraic-stability. In addition, the RK characterization and the algebraic criteria for nonlinear stability are used to construct the conditions under which time-marching methods based on dense-norm GSBP operators are also nonlinearly stable.

Finally, these results are used to construct a few examples of known and novel RK time-marching methods which are based on GSBP operators. This includes a known discontinuous-collocation RK method and some novel diagonally-implicit GSBP-RK schemes.

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