A Novel Estimator for the Equation of State of the IGM by Lyα Forest Tomography

Hendrik Müller,1,2★ Christoph Behrens,1 David J.E. Marsh,1
1 Institut für Astrophysik, Universität Göttingen, Germany
2 Present Address: Max-Planck-Institut für Radioastronomie at Bonn, Germany

ABSTRACT

We present a novel procedure to estimate the Equation of State of the intergalactic medium in the quasi-linear regime of structure formation based on Lyα forest tomography and apply it to 31 high quality quasar spectra from the UVES_SQUAD survey at redshift \( z = 2.5 \). Our estimation is based on a full tomographic inversion of the line of sight. We invert the data with two different inversion algorithms, the iterative Gauss-Newton method and the regularized probability conservation approach, which depend on different priors and compare the inversion results in flux space and in density space. In this way our method combines fitting of absorption profiles in flux space with an analysis of the recovered density distributions featuring prior knowledge of the matter distribution. Our estimates are more precise than existing estimates, in particular on small redshift bins. In particular, we model the temperature-density relation with a power law and observe for the temperature at mean density \( T_0 = 1300 \pm 1300 \) K and for the slope of the power-law (polytropic index) \( \gamma = 1.44 \pm 0.09 \) for the power-law parameters describing the temperature-density relation. Moreover, we measure an photoionization rate \( \Gamma_{\text{HII}} = 1.12 \pm 0.17 \). An implementation of the inversion techniques used will be made publicly available.

Key words: (cosmology:) large scale structure of Universe– quasars: absorption lines – (methods): data analysis

1 INTRODUCTION

In the cosmological standard model small initial density perturbations evolved to the large scale structure of matter in the Universe that is visible today (Mukhanov 2005; Planck Collaboration et al. 2016, 2020). This large scale structure consists of dense objects such as galaxies and galaxy clusters, and the intergalactic space which is diffusely populated with baryonic matter, the so-called intergalactic medium (IGM). The physics of the IGM has been targeted by many studies in the past. Among many applications, the clustering properties of matter in the IGM can be used to study dark matter (Seljak et al. 2006; Hui et al. 2017; Rogers & Peiris 2020). Moreover, the thermal history of the IGM provides information on the reionization epoch, when the first galaxies and stars formed at high redshift, and other related cosmic heating processes. (McQuinn et al. 2009; Compostella et al. 2013; D’Aloisio et al. 2015; McQuinn & Upton Sanderbeck 2016; Liu et al. 2020)

A powerful probe of the IGM is the Lyα forest. The Lyα forest consists of densely packed, narrow absorption lines at higher frequencies (bluewards) of the Lyα emission line in the spectra of distant galaxies. It appears due to intervening hydrogen along the line of sight, and thus having differently redshifted emission lines (Bahcall & Salpeter 1965; Gunn & Peterson 1965; Bi & Davidsen 1997). At every point along the line of sight the IGM absorbs a fraction of the light with an optical depth which is proportional to the density of neutral hydrogen. Since every point along the line of sight corresponds to a specific Hubble redshift, the normalized flux in the Lyα forest is a tracer of the neutral hydrogen density fluctuations in the IGM. (Bi et al. 1992; Hui & Gnedin 1997)

The profiles of the absorption lines contain information regarding thermal properties of the IGM and the density profile of the underlying absorbers. There is, however, a particular difficulty in studying these absorption features: The width of the absorption lines is a composition of an intrinsic width of the underlying density profile, and thermal broadening of the line, which is difficult to separate (compare Hui & Gnedin 1997; Garzilli et al. 2020). The Equation of State (EOS) is a relation between the overdensity and the temperature in the IGM, hence describing the thermal part of the line broadening. Thus, it is crucial to determine the EOS of the IGM, above all the IGM temperature, when studying the IGM in order to constrain fundamental physics by cluster properties in the underlying neutral hydrogen density field. Furthermore, constraining the EOS of the IGM is of interest on its own to understand the thermal evolution of the IGM and related heating processes.

There are many attempts to estimate the temperature of the IGM: Among others this includes Schaye et al. (1999, 2000); Mc-
Donald & Miralda-Escudé (2001); Viel & Haehnelt (2006); Bolton et al. (2008); Viel et al. (2009); Becker et al. (2011); Calura et al. (2012); Rudie et al. (2012); Garzilli et al. (2012); Boersa et al. (2014); Bolton et al. (2014); His et al. (2018); Ronor et al. (2018); Boera et al. (2019); Waller et al. (2019); Teikey et al. (2019); Garzilli et al. (2020); Gaikwad et al. (2020b,a). In these studies the thermal Doppler broadening and the intrinsic width of the filaments in the IGM are distinguished by Voigt-profile fitting, curvature methods, or studies of the Lyα forest power spectrum. The exact values recovered by the different groups vary, but coincide at temperatures \( T \sim 5000 – 30000 \) K at redshift \( z \sim 2 – 3 \). However, note that there remain some notable discrepancies between the different methods. In this work we present a novel estimate by a full inversion technique of the Lyα forest spectra observed in UVES‐SQUAD (Murphy et al. 2019) survey.

Fitting tools are only a part of the tomographic use of the Lyα forest. In the past, powerful inversion schemes for the reconstruction of the neutral hydrogen density along a single line of sight at high (echelle) spectral resolution were proposed (Nusser & Haehnelt 1999; Pichon et al. 2001; Gallerani et al. 2011; Müller et al. 2020) and shown to yield accurate reconstruction results (Müller et al. 2020). Moreover, also the neural network method currently presented in Huang et al. (2020) could be used to carry out such reconstructions, but has not been utilized in this way yet. Rollinde et al. (2001) demonstrated that inversion schemes are capable of constraining the EOS of the IGM. The quality of the reconstruction with the scheme proposed in Pichon et al. (2001) depends strongly on the a-priori chosen EOS. Moreover, the schemes proposed in Gallerani et al. (2011) and Müller et al. (2020) depend on weaker prior assumptions, in this case being independent from any choice for the EOS. In this paper, we compare the inversion results of these approaches while varying the prior assumptions on the EOS for the inversion with the method provided by Pichon et al. (2001). Our estimates are based on the best the best match to observational data and the agreement between these different inversion algorithms (which depend on different priors). In this way we additionally perform model fitting in the density domain (rather than on the observed flux). This provides a powerful pipeline for analysis of Lyα forest data regarding the thermal history of the IGM.

The plan for the rest of the paper is as follows: We present the analytic model of the Lyα forest in Sec. 2, we present our inversion and estimation techniques in Sec. 3 and test it on synthetic data in Sec. 4. We apply our novel method to observational data from the UVES-SQUAD survey in Sec. 5. Our final marginal distributions for \( \gamma \) and \( T_0 \) are shown in Fig. 11. Additionally we simulate synthetic data with the same spectral properties as the observational ones and exactly mimic the estimation done on observational data in Sec. 6 to perform a consistency test and to verify our estimation procedure. Finally we discuss our results in Sec. 7 and finish with our conclusions.

During the rest of this paper we use the Planck Collaboration et al. (2016) cosmology. The chosen cosmology mainly affects the relation between the neutral hydrogen density and the optical depth. However, the error introduced by the cosmological model is negligible compared to the accumulation of statistical and synthetic errors during the inversion procedure.

We will make all our implementations (in particular the used inversion schemes) publicly available as part of the reglyman tool-

### 2 ANALYTIC MODEL

The observable in the Lyα forest is the normalized flux \( F \), i.e. the ratio between the observed flux and the flux that would be observed at full transmission. The optical depth \( \tau \) is defined as the negative logarithm of the normalized flux, i.e. it is:

\[
F = \exp (-\tau)
\]

The optical depth in the Lyα forest is given by (Bahcall & Salpeter 1965; Gunn & Peterson 1965; Hui & Gnedin 1997):

\[
\tau (z) = \frac{1}{k_B T (x, z)} \frac{\exp \left( \frac{v_{\text{HI}} (z) - v_{\text{pec}} (x, z)}{k_B T (x, z)} \right)}{\exp \left( \frac{v_{\text{pec}} (x, z) - v_{\text{pec}} (0, z)}{k_B T (x, z)} \right)} \cdot (1 - \frac{\sigma_0 c}{1 + z})
\]

where \( \sigma_0 \) is the Lyα cross section, \( c \) the speed of light, \( n_{\text{HI}} \) the number density of neutral hydrogen, \( z \) the Hubble redshift, \( v_{\text{HI}} \) the differential Hubble velocity and \( v_{\text{pec}} \) peculiar velocities (redshift space distortions). \( b_T \) is the thermal broadening parameter which satisfies (Hui & Gnedin 1997):

\[
b_T (x, z) = \sqrt{\frac{2 k_B T (x, z)}{m_p}}
\]

where \( k_B \) denotes the Boltzmann-constant, \( T \) the actual temperature and \( m_p \) the proton mass. We assume a power law EOS as it is widely assumed in Lyα forest studies (Hui & Gnedin 1997; Upton Sanderbeck et al. 2016):

\[
T (x, z) = T_0 (z) \Delta \Gamma^{-1}
\]

where \( T_0 \) denotes the temperature at mean density and \( \Delta = \rho_\text{b} / (\rho_\text{H}) \) is the baryonic density perturbation. \( T_0 \) defines the scale of the temperature, and thus by Eq. (3) the scale of thermal broadening. The neutral hydrogen density \( n_{\text{HI}} \) and the baryonic density perturbation \( \Delta \) are related by (Hui & Gnedin 1997; Nusser & Haehnelt 1999):

\[
n_{\text{HI}} (x, z) = \hat{n}_{\text{HI}} (z) \Delta^\alpha (x, z),
\]

where \( \alpha = 2.7 – 0.7 \gamma \) and \( \hat{n}_{\text{HI}} \) denotes the neutral hydrogen density at mean density. For Eq. (5) we assumed that the hydrogen is highly ionized and in ionizing equilibrium. See Appendix A for a detailed derivation of this property. At redshift \( z = 2.5 \) it is found to be (see Appendix A):

\[
\hat{n}_{\text{HI}} (z = 2.5) = \frac{1.87 \times 10^{-14} \, \text{m}^{-3} \, \text{s}^{-1}}{T_0 (z = 2.5) \Gamma (z = 2.5) ^{0.7}}
\]

where \( \Gamma \) denotes the photo-ionization rate and \( T_0 \) is expressed in units of Kelvin. Often the shifted photoionization rate \( \Gamma_{-12} = \Gamma / (10^{-12} \, \text{s}^{-1}) \) is used instead of \( \Gamma \) in studies of the IGM.

\[1\] Available at https://github.com/hruellergo/reglyman
\[2\] https://github.com/regpy/regpy
\[3\] https://nbodykit.readthedocs.io
3 METHOD

It is a common approach in the analysis of the temperature of the IGM to measure the thermal line-width in the Ly$\alpha$ forest and to deduce $T_0$ from that. However, the absorption feature is also broadened by the intrinsic width of the underlying density field $\Delta$. Garzilli et al. (2020) for example modeled the baryonic density perturbation $\Delta$ as sum of Gaussian peaks with varying center and variation $b_I$. According to Garzilli et al. (2020) the width of the absorption lines $b$ in the Ly$\alpha$ forest is:

$$b^2 = b_I^2 + b_T^2.$$  

(7)

Our new method is motivated by Eq. (7). It is a difficult task to estimate the thermal broadening from the total broadening of the line. However, for Ly$\alpha$ forest tomography, i.e. the problem of recovering the density field in the Ly$\alpha$ forest with high spectral resolution, this problem is tackled by powerful inversion algorithms. It is not possible to study the thermal broadening parameter without constraining the underlying matter density field. Thus, both problems (i.e. inversion of the Ly$\alpha$ forest and constraining the EOS of the IGM) should be solved simultaneously. Such an approach was used by Rollinde et al. (2001). They used the iterative Gauss-Newton method (IRGN) proposed by Pichon et al. (2001) to invert the Ly$\alpha$ forest. The IRGN algorithm tries to recover the observed Ly$\alpha$ flux by modelling the absorption explicitly by Eq. (2). The basic idea now is that the quality of the inversion with the IRGN method depends strongly on which set of parameters $b_{HI}, T_0$ and $\gamma$ has been assumed prior to the inversion procedure, i.e. whether the true temperature-density relation has been used to model the Ly$\alpha$ forest. Rollinde et al. (2001) chose the set of parameters which match the observations best based on a reduced $\chi^2$ criterion for the recovered flux (i.e. the flux calculated from the recovered density). This method of estimating the temperature-density relation by comparing the recovered flux (where the density has been recovered by the IRGN inversion algorithm) and the observed flux will be called IRGN-$\chi^2$ throughout this paper.

Meanwhile, there are inversion schemes available that depend on less strong priors (Gallerani et al. 2011; Müller et al. 2020), namely the regularized probability conservation (RPC) method. This method recovers the neutral hydrogen density in the Ly$\alpha$ forest by a statistical approach which is not affected by any choice for $\gamma$ and $T_0$. It has been demonstrated that the RPC inversion provides a suitable estimator for the true density (Müller et al. 2020). Thus, it is also possible to compare the recovered density with the IRGN method for different models of the EOS with a density that approximates the true density reasonably well. Choosing the set of parameters that model the recovered density profile best will be called the RPC-IRGN method throughout this paper. Compared to the IRGN-$\chi^2$ the comparison between observational data and recovered data is not performed in flux space, but in density space.

3.1 Iterative Gauss-Newton Method

We outline the approach of Rollinde et al. (2001) within this section. This method is based on an inversion algorithm for recovering the logarithm of the overdensity $M = \log(\Delta)$ where the bold symbols denote vectors, i.e. the density perturbation and its logarithm discretized on the observed redshift bins. Let us define the forward operator $g$ by Eq. (2), i.e. $g(M) = D$, where $D$ is the normalized flux computed from $\Delta$ by Eq. (2), Eq. (3) and Eq. (4). Pichon et al. (2001) demonstrated that given particular data $D$, the posterior of the vector of parameters $M$ is:

$$p(M|D) \propto \exp \left\{ -\frac{1}{2} [D - g(M)]^T C_d^{-1} [D - g(M)] + \frac{1}{2} [M - M_0]^T C_0^{-1} [M - M_0] \right\}. \tag{8}$$

where $C_0$ and $C_d$ denote the covariance matrix of the initial guess $M_0$ and noise. Eq. (8) is only valid if $M$ is Gaussian distributed. However, this is a reasonable assumption since the density perturbation $\Delta$ can be approximated by a lognormal distribution (Coles & Jones 1991; Choudhury et al. 2001; Choudhury & Ferrara 2005; Gallerani et al. 2006). We introduce a linearization $g(M) = g(M_0) + G(M - M_0)$. Here $G$ denotes the matrix of functional derivatives of $g$. Then the estimator $\langle M \rangle$ which maximizes the posterior of the linearized problem satisfies:

$$\langle M \rangle = M_0 + C_0 G^T (C_d + G C_0 G^T)^{-1} \cdot [D + G\langle(M) - M_0\rangle - g\langle(M)\rangle]. \tag{9}$$

The implicit Eq. (9) is solved with a fixed point iteration, i.e. we apply the right hand side of Eq. (9) iteratively until convergence is achieved, i.e. until the residuum becomes noise-like. For more details on the derivation and application of Eq. (9) we refer to Pichon et al. (2001) and Müller et al. (2020). According to our nomenclature in Müller et al. (2020) we call this method "iterative Gauss-Newton method" (IRGN).

The forward operator $g$ (and thus also the derivative $G$) appearing in Eq. (9) depends strongly on the choice of the constants $T_0$ and $\gamma$ in Eq. (4). Hence $\gamma$ and $T_0$ affect the recovered logarithmic density perturbation $\langle M \rangle$ and the recovered normalized flux $g\langle(M)\rangle$.

3.2 Regularized Probability Conservation Approach

In this section we outline the recent development in Ly$\alpha$ forest tomography presented in Müller et al. (2020) based on the approach by Gallerani et al. (2011) and sketch a novel way of using this approach for constraining the EOS of the IGM. If thermal broadening would be absent, Eq. (2) and Eq. (5) would describe a one-to-one correspondence between the baryonic density perturbation and the observed optical depth. In the presence of thermal broadening this one-to-one correspondence is not satisfied since $\Delta$ also appears in the standard deviation of the Gaussian. Nevertheless, Gallerani et al. (2011) proposed to also assume this one-to-one relation in this case. Let us denote the probability density function of the observed flux by $P_T$ (which can be measured from the sample of observed lines of sight) and the probability density function of $\Delta$ by $P_\Delta$ (which has to be assumed prior to the inversion procedure). Let $F_{\text{max}}$ be the maximal flux that can be distinguished from full emission. Then this flux can be identified by an overdensity $\Delta_b$ corresponding to the bright limit:

$$\int_{F_{\text{max}}}^{1} F_T dF = \int_0^{\Delta_b} P_\Delta d\Delta. \tag{10}$$

According to Gallerani et al. (2011) the flux $F_\gamma$ in every bin is now identified with a density perturbation $\Delta_\gamma$ in that bin by the equation:

$$\int_{F_\gamma}^{F_{\text{max}}} F_T dF = \int_0^{\Delta_\gamma} P_\Delta d\Delta. \tag{11}$$

Similar equations are derived for the minimal flux that can be distinguished from full absorption (Gallerani et al. 2011). The assumption that the overdensity is lognormal distributed is a good model for $P_\Delta$.
which was successfully used by Gallerani et al. (2011) and the consecutive work Kitaura et al. (2012). We demonstrated in Müller et al. (2020) that the reconstruction results are improved by reformulating Eq. (11) as an optimization problem and adding a penalty term, i.e. by solving the problem:

\[
\Delta_\alpha \in \text{argmin}_\Delta \left\{ \Psi(\Delta) = \frac{1}{2} \int_{F_*}^{F_{\text{max}}} P_F dF - \Phi(\Delta) \right\} + \frac{\alpha}{2} \left\| \frac{\partial \Delta}{\partial \sigma} \right\|_{L^2}^2 \right\} \tag{12}
\]

instead of Eq. (11). Here, \( \Phi \) denotes the operator \( \Phi : \Delta_\alpha \mapsto \int_\Delta P_\Delta d\Delta \) where \( \Delta_\alpha \) is the vector of density perturbations and \( \Phi(\Delta) \) is evaluated pointwise. Eq. (12) is solved by a gradient descent algorithm with small numerical computation time. This method is called the “regularized probability conservation approach” (RPC). For more details on the implementation and the mathematical details of the RPC method we refer to Müller et al. (2020).

In short, the first term of \( \Psi(\Delta) \) is the mathematical details of the RPC method we refer to Müller et al. (2020). In short, the first term of \( \Psi(\Delta) \) is the optimization of a new method for performing such estimates. We propose two alternative fitting procedures: We compare the IRGN inversion results to the observed data in flux space (we call this approach IRGN_\( x^2 \)) and we compare the IRGN and RPC inversion in density space (we call this method RPC_\( IRGN \)). Both are needed to find statistically significant estimates for the EOS of the IGM.

We invert the Ly\( \alpha \) forest data at a high spectral resolution with the IRGN method by assuming \( [y', T_0] \) in a uniform sample of values which cover the whole range of reasonable parameters, e.g. \( T_0 \in [5000, 30000] \) K and \( y \in [1.2, 2] \). The selection of the exact values tested in this sample plays only a minor role as long the whole range is sufficiently covered. We store the recovered density perturbations \( \Delta' \) for later usage. We invert the same spectra with the RPC method and recover the density profile \( \Delta \). We pick those values \( [y', T_0] \) for which the correspondence between \( \Delta' \) and \( \Delta \) is maximal. This approach is driven by the assertion that \( \Delta \) mimics reasonably well the true density profile which has been demonstrated in Müller et al. (2020). We examine the correspondence between \( \Delta' \) and \( \Delta \) based on the \( L^2 \)-distance of the logarithms of the density perturbations, i.e. we minimize:

\[
d_{\log} = \sqrt{\sum_j \left( \log(\Delta'_j) - \log(\Delta_j) \right)^2},
\]

where \( j \) runs over all bins in the spectra. Taking the logarithm is needed to compute a meaningful distance. The reconstruction methods perform badly at large overdensities and small underdensities due to line saturation. The standard \( L^2 \)-distance would be dominated by the reconstruction of large overdensities and the information regarding similarity would be lost. Therefore, we only use the pixels with moderate overdensities to perform the comparison between the reconstruction with the RPC method and with the IRGN method.

Concerning the estimation with the IRGN_\( x^2 \) method, we first compute the recovered flux \( D' = g(\log(D'))^4 \) for every set of parameters from our inversion results \( \Delta' \) with the IRGN method. Rollinde et al. (2001) proposed to examine the recovered flux based on the reduced \( x^2 \):

\[
x^2 = \frac{1}{N_{\text{pix}}} \sum_{j=1}^{N_{\text{pix}}} \left( D'_j - D_{i,j}^{\text{obs}} \right)^2,
\]

where \( N_{\text{pix}} \) denotes the number of pixels in each line of sight, \( \sigma_j \) the standard deviation of noise in pixel \( j \) and \( D_{i,j}^{\text{obs}} \) the observed, noisy data. It is well known that \( x^2 > 1 \) means that the structure in the flux data is badly resolved, while \( x^2 < 1 \) characterizes overfitting of noise. According to Rollinde et al. (2001) the exact values \( y, T_0 \) are identified with \( x^2 \approx 1 \). If we overestimate thermal broadening (e.g. by overestimating the temperature \( T_0 \)), this promotes smoothing in data space. Therefore, the small scale structures in the normalized flux are inaccurately estimated, such that \( x^2 \) becomes bigger than one. Similarly understanding thermal broadening leads to \( x^2 < 1 \).

As demonstrated by Pichon et al. (2001) and Rollinde et al. (2001) the IRGN_\( x^2 \) approach suffers from a degeneracy between \( T_0 \) and \( y \). \( x^2 = 1 \) defines a line in the 2D parameter space of \( T_0 \) and \( y \). As will be shown in Sec. 4 the RPC_\( IRGN \) approach is to first
order only sensitive to $\gamma$. Consequently, combining both estimates can remove the degeneracy between $\gamma$ and $T_0$. We will discuss this in more details in Sec. 4. Our whole estimation procedure is summarized in Fig. 1.

The inversion schemes (IRGN, RPC) try to decompose the thermal profile from the profile of the underlying density field. Hence, they are only applicable for echelle-resolution spectra. More precisely, this decomposition only carries strong information about the thermal history of the IGM for high resolution reconstructions (when thermal broadening and instrumental broadening are of similar order). We examined the resilience of these algorithms against observational noise in Müller et al. (2020). Although both algorithms show robustness against noise, for the purpose of estimating $T_0$ and $\gamma$ only high quality data (high signal to noise ratio) should be used. For larger noise contributions the algorithms typically start fitting small scale noise instead of properly fitting the thermal profile.

We test the impact of noise on the estimation in Sec. 4.5.

3.4 Fitting Procedure

We discuss now the IRGN, $\chi^2$ step in the procedure presented above (find the parameters for which $\chi^2 \approx 1$). When applying our estimation to real observational spectra additional sources of uncertainties occur which could bias the estimated parameters. In fact, a high $\chi^2$ could be the consequence of poorly fitting only a part of the spectrum or underestimating the error (in particular for large overdensities). Moreover, the iterative inversion procedure could have stopped too early or the density distribution in the observed sample shows a weaker (stronger) correlation than was assumed as a prior. All these uncertainties shift the computed reduced $\chi^2$ towards larger (smaller) values. We fight these errors by masking out the wave-lengths at which only bad fits to the observed data are available, Moreover, instead of just taking the set of parameters for which $\chi^2$ drops to one, we fit the curve $\chi^2(T_0)$ for every $\gamma$.

When the temperature goes to zero the reduced $\chi^2$ typically converges to a constant lower limit $\chi^2 \approx 0.85$. This is expected. The Gaussian kernel in (2) turns into a $\delta$ distribution for very small temperatures. The forward operator turns into a one-to-one correspondence between the optical depth and the neutral hydrogen density as described by the fluctuating Gunn-Peterson approximation (Eq. (13)):

$$\tau(z) \propto n_{HI}(z).$$

Hence, if the prior in Eq. (8) would be absent, we would be able to exactly (over-)fit the noisy data, i.e. to achieve $\chi^2 \approx 0$. But as the second term in Eq. (8) is present, i.e. we assume a specific auto-correlation as prior, for very small temperatures the IRGN method computes a best estimator for the optical depth which satisfies the prior covariance $\mathbf{C}_0$. Thus, we compute a smooth curve fit to the noisy observed data. Therefore, the reduced $\chi^2$ converges for small temperatures to a lower limit which is unequal to zero. The exact value of this lower limit depends on the prior covariance, $\mathbf{C}_0$, the noise distribution, $\mathbf{C}_d$ and the explicit choice of the forward model.

For temperatures slightly bigger than the exact temperature, the reduced $\chi^2$ drops nearly linearly. The exact value for the temperature lies in between these two regimes: the regime of linear decay at slightly higher temperatures and the regime of a constant lower limit at slightly smaller temperatures (see the illustration in Fig. 2). To find a proper estimate for the temperature we fit the borderline, the asymptotic lower limit (green line in Fig. 2) and the linear regime (yellow line in Fig. 2) separately. We take the intersection between these two lines as estimator for the temperature.

This procedure has the special advantage that it is unaffected by shifting the reduced $\chi^2$ for one of the reasons stated above (e.g. in the situation that a small fraction of the line of sight is badly estimated). This is also demonstrated in Fig. 2 by dashed lines. Due to the constant shift there is no intersection with the $\chi^2 = 1$ line anymore. But as the fitted borderline and also the linear fit function are shifted in the same manner, the estimate with our fitting procedure does not change.

We study the precision of this fitting procedure on synthetic data in Sec. 4.3.

4 NUMERICAL TESTS ON SYNTHETIC DATA

4.1 Synthetic Data

We create synthetic data with the semi-analytic approach described by Choudhury et al. (2001) and Gallerani et al. (2006) which is based on the lognormal model for the density (Coles & Jones 1991). The lognormal model is well motivated (Coles & Jones 1991; Bi et al. 1992; Bi & Davidsen 1997) and has been frequently used to describe the distribution of overdensities. In fact the model was used for modelling the cosmic density field (e.g. Choudhury et al. 2001; Viel et al. 2002; Gallerani et al. 2006; McDonald et al. 2006; Font-Ribera et al. 2012; Hand et al. 2018; Karacayli et al. 2020) and for tomographic reconstruction (Pichon et al. 2001; Kitaura & Enßlin 2008; Caucci et al. 2008; Kitaura et al. 2012; Aita et al. 2015; Müller et al. 2020) of the density field with the Ly$\alpha$ forest. In particular, the $nbodkit$ toolbox (Hand et al. 2018) which implements the lognormal model is currently used in a large number of publications as a reference model of the cosmic density field. The lognormal model, however, is inadequate for describing highly non-linear overdensities, or overdensities at very small scale (Choudhury & Ferrara 2005), although the latter might hold for dark matter primarily and lesser for the ordinary matter distribution (Gallerani et al. 2006).

As both regimes are not accessible to Ly$\alpha$ forest inversion anyhow (due to line saturation and line broadening) the lognormal model is a simple and sufficient analytic description of the overdensity distribution.

An implementation of our simulation is available within the reglyman toolbox which makes strong use of the $nbodkit$ toolbox too. In a nutshell, we create Gaussian white noise, multiply this with the matter power spectrum and take the inverse Fourier transform to compute the linear overdensity field. We project the linear density perturbation to the non-linear density perturbation by taking the exponential and normalizing, such that the non-linear density perturbation is lognormal distributed (as the linear density perturbation was constructed by a Gaussian distribution). We compute the peculiar velocities by the Zel`dovich approximation (Zel`dovich 1970; White 2014) from the linear density field. Lastly, we compute the Ly$\alpha$ forest from the density perturbation according to Eq. (2).

Our simulation is very fast and allows for the computation of small scales (binning in the forest) and large scales (box width) in parallel due to its semi-analytic nature. Moreover, the speed of the simulation enables us to test varying physical parameters (such as Hubble constant, matter power spectrum or Jeans length) easily.

Among the chosen cosmology we additionally have to specify a Jeans-length for this procedure that describes biasing between dark matter and ordinary matter. The baryonic matter power spectrum $P_B$ is related to the dark matter power spectrum $P_{DM}$ by the relation:

$$P_B(k, z) = \frac{P_{DM}(k, z)}{(1 + \delta_B^2)^2},$$

(18)
where \(s_b\) denotes the Jeans length (compare the prescriptions in Choudhury et al. 2001; Gallerani et al. 2006). We adopt the value that was used by Choudhury et al. (2001) and which is compatible to the findings in Zaroubi et al. (2006). The Jeans length affects the prior density distribution of the overdensity field to the findings in Zaroubi et al. (2006). The Jeans length affects the prior density distribution of the overdensity field.

4.2 Estimation Methods

For our synthetic data we assume a temperature of \(T_0 = 10^4\) K, \(\gamma = 1.4\) and \(\delta = 3.5 \cdot 10^{-5}\) m\(^3\). We compute 100 independent lines of sight of 10 h\(^{-1}\) Mpc comoving length at redshift \(z = 2.5\). Moreover we use a high signal to noise level \(SNR = 50\) and a small noise contribution \(\sigma_0 = 0.01\). For this subsection we ignore peculiar velocities in the computation of the lines of sights. We will quantify in Sec. 4.4 how much this affects our estimation.

We first recover \(\hat{n}_{HI}\) from our set of synthetic data. We simulate the neutral hydrogen density fluctuation in a second box with the IRGN algorithm, but it turns out to play only a minor role for the results with the IRGN algorithm. The expected error in \(s_b\) will be added to the error budget during the analysis of experimental data.

We compute independent lines of sight at redshift \(z = 2.5\) and spectral resolution \(R = \frac{\lambda_a}{\Delta \lambda} = 100000\). To mimic instrumental artifacts we add noise according to the noise model:

\[
\sigma_F^2 = \frac{F^2}{SNR^2} + \sigma_0^2\n\]

with signal-to-noise ratio \(SNR\) and a small noise contribution \(\sigma_0\) which dominates at very small fluxes \(F\). Finally we rebinned our spectra to a spectral resolution \(R = 50000\), which is compatible to high resolution spectra taken with the UVES (Dekker et al. 2000) or the HIRES (Vogt et al. 1994) spectrometers.

**Figure 1.** Flowchart of estimation method.
Figure 2. Schematic illustration of the fitting procedure used to find the temperature $T_0$ at which $\chi^2 = 1$ (red horizontal line). $\chi^2(T_0)$ (blue line) typically converges to a lower limit for small temperatures (green line). For slightly greater temperatures $\log(\chi^2)$ could be estimated well by a linear fit (orange line). The intersection between the linear fit and the $\chi^2 = 1$ borderline roughly matches the exact value (black line). The dashed curves represent a case where $\chi^2$ is overestimated. Our method, however, still recovers the correct value of $T_0$.


grid of values $\gamma \in [1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8]$ and $T_0 \in [2500, 5000, 7500, 10000, 12500, 15000, 17500, 20000]$ K.

We compute the joint reduced $\chi^2$ and the joint absolute distance for all 100 lines of sight in our sample of synthetic lines of sight. Our results are shown in Fig. 3 and Fig. 4. We use a Gaussian interpolation on our grid of parameters in Fig. 3 and Fig. 4 (and will use this interpolation also on Fig. 5 later). This interpolation is purely for plotting purposes and will not be needed for the analysis of observed data.

Similar to the results proposed in Rollinde et al. (2001) $\chi^2 \approx 1$ defines a line in the two dimensional parameter space (green line in Fig. 3). In the most likely region $1.3 \leq \gamma \leq 1.6$ this line is well approximated by a linear relation between $\gamma$ and $T_0$. The slope of this line is characteristic for the chosen redshift $z = 2.5$. Future works should extend this study to other redshifts. If a global relation between $\gamma$ and $T_0$ could be established, the degrees of freedom in estimating the thermal history of the IGM would be reduced by one parameter.

In Fig. 4 we show the absolute distance between the inversion with the RPC and the IRGN method. This distance only depends very weakly on $T_0$. This is reasonable. We substituted the strong dependency of $r$ on $T_0$ in $n_{HI}$, such that $T_0$ only appears as free parameter in the standard deviation of the Gaussian kernel in Eq. (2). Hence, $T_0$ only plays a minor role for the overall scale of the estimated density field fluctuations amplitudes. On the other side $\gamma$ also affects the standard deviation of the Gaussian kernel (see Eq. (4)), but it appears outside of the kernel function too (see Eq. (5)). All in all, our special choice for the set of parameters allows us to find estimates for the slope parameter $\gamma$ independent of $T_0$ (at least in the significant signal region of $T_0 \in [10000, 15000]$ K). Lastly it should be mentioned that the minimum in the distance visible in Fig. 4 is not a sharp minimum. Hence, when applying to real observational data we expect a scatter in the estimation of $\gamma$ from different lines of sight.

The IRGN$_\chi^2$ and the RPC_IRGN method are complementary. Fig. 5 demonstrates that our method is working on synthetic data as the true values for $T_0$ and $\gamma$ (black diamond) lies well inside the estimated regions. For the RPC_IRGN method we fitted the minimum for every line of sight and took the mean of these estimates. Note that for this analysis of synthetic data we optimized the stopping rules and regularization parameters for the inversion procedure with the RPC method and IRGN procedure based on the known synthetic solution.

4.3 Fitting Procedure

We now verify the fitting procedure outlined in Sec. 3.4 on our set of synthetic data. For this we fit every line of sight in our sample of 100 synthetic lines of sight individually. We assume the correct $\gamma = 1.4$ at redshift $z = 2.5$ (i.e. the parameter that was used for the creation of synthetic data) and vary the temperature at mean density $T_0$. For every line of sight we perform the fitting procedure sketched
in Fig. 2. We show in Fig. 6 a histogram of all our estimates from our synthetic lines of sight. The distribution is well described by a Gaussian centered around \(T_0 = 10000\) K. The yellow curve is a Gaussian fit to the histogram: We find a mean \(\mu = 10220\) K and a standard deviation \(\sigma = 2190\) K.

### 4.4 Peculiar Velocities

As described by Eq. (2) the redshift space coordinate of an absorption feature has to independent components, the Hubble redshift \(v_H\) and peculiar velocities \(v_{\text{pec}}\). It is difficult to estimate the contribution of peculiar velocities from the flux along single lines of sight. Thus, for the inversions we have to ignore peculiar velocities. We quantify in this section how much this assumption is affecting our estimation. For this we take a reduced sample of 25 lines of sights out of our simulation, calculate the peculiar velocities from the density field in the box and model the Ly\(\alpha\) forest with respect to these peculiar velocities. Then we perform the inversions with the RPC and with the IRGN method while still assuming for the inversion algorithms that no redshift space distortions are present.

Peculiar velocities affect the spectrum in two different ways:
- They shift the absorption features in redshift space and they narrow/broaden the absorption lines. The latter effect is often ignored when fitting the absorption profile for the purpose of identifying the thermal history of the IGM. However, peculiar velocities are crucial for doing an inversion of the density field (Nusser & Haehnelt 1999). The lack of information regarding peculiar velocities inserts an additional bias in the estimation of the EOS from tomographic inversion.

We perform an analysis assuming no peculiar velocities on synthetic data that in fact include peculiar velocities. In this case the RPC algorithm recovers equation of state \(\gamma = 1.34\), compared to the true value \(\gamma = 1.4\). We thus include the resulting bias, \(\delta_\gamma = 0.06\), as an additional source of error (added in quadrature) in our analysis of real data with this method.

On the other hand, peculiar velocities play only a minor role for the IRGN_\(\chi^2\) method. We show in Fig. 5 with black lines the edges of the \(\chi^2 = 1\) (e.g. \(\chi^2 \in [0.98, 1.02]\)) region when finding estimates from synthetic data which had been created with redshift space distortions. To obtain these estimates we cut of all the pixels in which the recovered flux is clearly distorted from the observed normalized flux. This mainly occurs at the boundaries of the studied wavelength interval. These fluxes are clearly not well recovered as the convolution in Eq. (2) could not be computed from values outside of the density interval and the integration is not complete at the boundaries. When peculiar velocities are ignored in the creation of synthetic data, then this would only affect a very small number of pixels due to the relatively small thermal broadening parameter. However, when peculiar velocities are considered other parts of the spectrum could be shifted towards the boundary. Hence, we do not consider the flux in pixels which are less than roughly 100 km/s away from the boundary of the wavelength interval. The line profiles in Fig. 5 coincide very well, such that the bias introduced by not including peculiar velocities in the IRGN_\(\chi^2\) method is very small and will be ignored in the following. In particular, even in this case that peculiar velocities are ignored the profile shown by the black lines suggests a linear relation between \(\gamma\) and \(T_0\) in the most likely region of values for \(\gamma\).

There is a straightforward explanation for this finding that the IRGN_\(\chi^2\) estimation is unaffected by peculiar velocities. We do an error in the inversion when not including peculiar velocities in the inversion, i.e. when not respecting the broadening and narrowing of absorption lines by an additional velocity distribution. However, we make the same error when forwardly modeling the optical depth in the Ly\(\alpha\) forest from the inverted density field and both effects cancel out when taken together.

### 4.5 Noisy Spectra

In the former subsections we examined our estimation procedure at high signal to noise ratios. In this Section we discuss the estimation from noisy spectra. We show in Fig. 7 the fitting results with the IRGN_\(\chi^2\) method for smaller SNR. Based on Fig.7 we only suggest the IRGN_\(\chi^2\) method for high quality data SNR>10 as we now explain.
The IRGN method performs a minimization of the two terms in the exponential of Eq. (8). The first term (the so called data-fidelity term) measures the proximity of the observed data and the recovered data (i.e. computes the reduced $\chi^2$). The second term (the so called penalty term) measures the proximity of the recovered solution to a prior guess. The scale of the penalty term is set by $C_0$, the prior auto correlation. In an ideal reconstruction the auto-correlation function of the recovered density matches the prior correlation, and the reduced $\chi^2$ is exactly one. If thermal broadening is underestimated, the correlation of the density would be overestimated. In this sense, the penalty term evaluates over- or underestimation of thermal broadening. However, if there is a significant noise contribution, then the penalty term introduces effective smoothing to regularize the recovered density against noise rather than evaluating the over- or underestimation of thermal broadening.

In fact, as shown in Fig. 7, smaller SNR push the reduced $\chi^2$ of the exact set of parameters to smaller values. Consequently the $\chi^2 \approx 1$ lines would bias the estimation of $\gamma$ and $T_0$ towards larger values. Furthermore, the error of the estimation increases with decreasing SNR which is indicated by the larger width of the green overplotted lines ($\chi^2 \in [0.98, 1.02]$). However, for all SNR the behavior of $\chi^2$ as a function of $\gamma$ and $T_0$ is similar. Thus, it might be possible to also include lower quality data in the analysis and fight the introduced bias in a post-processing step. This idea is left for future refinements of our method.

5 APPLICATION TO OBSERVATIONAL DATA

We apply our estimation procedure to observational data in this section. Our final results are shown in Fig. 10 and Fig. 11.

5.1 Mean Neutral Hydrogen Density

We apply the mean optical depth model from Becker et al. (2013):

$$\tau_{\text{eff}} \approx 0.751 \cdot \left( \frac{1 + z}{4.5} \right)^{2.9} - 0.132. \quad (20)$$

This function was proven to adequately fit the observed effective optical depths. It is also compatible with alternative models such as the fits presented in Kirkman et al. (2005) and Faucher-Giguère et al. (2008). In particular for redshift $z = 2.5$ we get: $\tau_{\text{eff}}(z = 2.5) \approx 0.23$. We vary the parameter $\theta H_0$ create each time 100 lines of sight of 50 $h^{-1}$ Mpc length with our simulation outlined in Sec. 4.1 (with peculiar velocities) and compute the effective optical depth. We get $\theta H_0 = (22 \pm 3) \cdot 10^{-12}$ cm$^{-3}$. The error originates from the uncertainty in $\tau_{\text{eff}}$ from Becker et al. (2013), the variance of estimated $\theta H_0$ between different random seeds in our simulation box, and the variation in the selection of suitable Jeans scale drawn from the result for $z = 2.48$ in Zaroubi et al. (2006).

5.2 Data Selection

We apply our estimation method to a subsample of the UVES_SQUAD sample (Murphy et al. 2019). This is a survey of 467 quasar spectra in the redshift range $z = 0 \sim 5$ observed with the high resolution UVES instrument (Dekker et al. 2000). The spectra are fully reduced and continuum fitted by Murphy et al. (2019). The spectra are reproducing from raw data. The reduction pipeline only made use of publicly available software including UVES_POPLER (Murphy 2016). For more details on the reduction process we refer to Murphy et al. (2019). The complete survey ranges from quasar spectra with continuum to noise ratio (CNR) of CNR $> 4$ to CNR $= 342$ at 2.5 km/s pixels at wavelength $\lambda = 5500 \AA$.

In this publication we investigate $z = 2.5$. Moreover, we only use spectra with a sufficiently small noise contribution. In fact we only used spectra with CNR $> 30$ at wavelength $\lambda = 4500 \AA$. Our sample of QSO spectra is summarized in Tab. 1. For redshift 2.5 we only use the wavelength range $\lambda \in [4235, 4275] \AA$, such that uncertainties in redshift are negligible. We divide these spectra in four sets of 10.4 $\AA$ in length. According to the prescription in Rollinde et al. (2001) we only use the sets for inversion in which the minimal normalized flux drops below 0.2 for the inversion with the IRGN method, i.e. we only use the sets for inversion if they contain a significant absorption feature. For the RPC method we use all spectra. Lastly we evaluate all the reconstructions and reject every set for which the inversion failed, i.e. we were not able to reasonably fit the observed flux.

5.3 RPC_IRGN

As explained in Sec. 4.2 the distance of the logarithms of the recovered density with the IRGN method and the RPC method is nearly independent of $T_0$. To save computation time it is therefore reasonable to assume a reasonable value for the temperature $T_0$ and only vary $\gamma$ to perform the RPC_IRGN step. We perform the inversion assuming $T_0 = 10000 K$ and vary $\gamma$ on a uniform grid $\gamma \in [1.25, 1.3, 1.35, \ldots 1.95, 2.0]$. We performed the inversion for Murphy et al. (2019). The redshift, CNR and the dispersion are taken from Murphy et al. (2019). The CNR is reported at wavelength of 4500Å.

The Dispersion is given in km/s.

| QSO         | Redshift | CNR | Dispersion [km/s] |
|-------------|----------|-----|------------------|
| J000448-415728 | 2.76     | 113 | 2               |
| J005758-264314 | 3.655    | 44  | 2.5             |
| J010311+131617 | 3.478    | 22  | 2.5             |
| J014214+002324 | 3.37     | 39  | 2               |
| J015327-431137 | 2.74     | 119 | 2.5             |
| J033108-252443 | 2.685    | 64  | 2.5             |
| J033244-445557 | 2.6      | 34  | 2               |
| J034943-381030 | 3.205    | 62  | 1.3             |
| J040718-441013 | 3        | 57  | 1.3             |
| J042243-384452 | 3.11     | 77  | 2.5             |
| J045214-164016 | 2.6      | 40  | 2.5             |
| J045523-421617 | 2.66     | 90  | 1.3             |
| J064326-504112 | 3.09     | 33  | 2.25            |
| J091127+055054 | 2.798    | 36  | 2.5             |
| J091613+070224 | 2.786    | 72  | 2.5             |
| J094253-110426 | 3.054    | 115 | 2.5             |
| J101155+294141 | 2.64     | 36  | 2.5             |
| J103909-231326 | 3.13     | 39  | 2.5             |
| J111350-153333 | 3.37     | 55  | 2.5             |
| J114254+265457 | 2.625    | 99  | 2.5             |
| J133209-052335 | 3.7      | 46  | 2.5             |
| J133258-135559 | 3.19     | 35  | 2.5             |
| J151352+085555 | 2.901    | 40  | 2.5             |
| J162116-004250 | 3.703    | 52  | 2.5             |
| J193957-100241 | 3.787    | 38  | 2.5             |
| J214159-441325 | 3.17     | 42  | 2.5             |
| J223408+000001 | 3.025    | 38  | 2.5             |
| J224708-601545 | 3.005    | 66  | 2               |
| J233446-090812 | 3.317    | 36  | 2.5             |
| J235034-432559 | 2.885    | 131 | 1.5             |
| J235129-142756 | 2.94     | 45  | 2.5             |

Table 1. Sample of high quality QSO spectra from the UVES_SQUAD survey (Murphy et al. 2019). The redshift, CNR and the dispersion are taken from Murphy et al. (2019). The CNR is reported at wavelength of 4500Å. The Dispersion is given in km/s.
We apply our RPC_IRGN analysis with sets of parameters applied to the quasar J091613-070224. The two asymptotic regions of a constant $\chi^2$ at smaller temperatures than the estimated $T_0$ and of a linear decay at larger temperatures are clearly visible. We fit both with linear functions (dashed and dotted black lines in the middle panels). The intersection point (solid, vertical black lines) fits well the temperatures at which $\chi^2 \approx 1$ is expected. Moreover, as expected from our analysis on synthetic data in Sec. 4, the estimated temperature decreases with increasing $\gamma$. This becomes clearly visible when plotting the $\chi^2$($T_0$) curves in one figure, see the upper left panel (red framed) in Fig. 9. The lines from $\gamma = 1.3$ to $\gamma = 1.6$ are ordered from lowest to uppermost. We find the corresponding estimate for the temperature for fixed $\gamma$ by averaging the estimates from all studied lines of sight.

Our seven estimates for the temperature $T_0$ are plotted in Fig. 10. The errorbars were computed by the standard deviation of the temperature estimates for each individual line of sight. Based on our proposition in Sec. 4.2 we fit the $\chi^2 = 1$ line with a linear fit $T_0 = m \cdot \gamma + b$. This linear fit is also plotted in Fig. 10. However, the line seems to be over-correlated indicating that the errorbars for the temperatures may be overestimated. The magenta shaded area indicates the $1-\sigma$ predictive distribution of our linear fit parameters. In particular we find: $m = -1.634$ and $b = 3.668$, where $m$ and $b$ are
reported in units of $10^4$ K. The estimated covariance of the linear fit parameters is printed in Tab. 2.

### 5.5 Joint Estimation

The estimates with the RPC_IRGN method (green shaded in Fig. 10) and the IRGN$_x^2$ method (magenta shaded in Fig. 10) can be combined to find joint estimates for $\gamma$ and $T_0$. The joint probability density function $p(\gamma, T_0)$ is plotted in Fig. 11. We also sketch the marginal distributions for $T_0$ and $\gamma$ in Fig. 11. These two plots present our final estimation results. The marginal distribution $p(T)$ differs slightly from a Gaussian as it is not fully symmetric around the maximum of the distribution. We estimate $T_0$ by the maximum of the distribution and estimate the error towards larger and towards smaller temperatures by the temperature at which the distribution drops below smaller temperatures by the temperature at which the distribution flank of the marginal distribution. We find for the marginalized the maximum of the distribution. We estimate $T_0$ and $\gamma$ by using the maximum of the distribution.

As pointed out in Sec. 3 we now use this estimate for the temperature at mean density $T_0$ and our estimate for the mean neutral hydrogen density to constrain the photoionization rate $\Gamma$ by Eq. (6). We get $\Gamma_{-12} = 1.12 \pm 0.17$.

### 6 MOCK ESTIMATION

We demonstrated in Sec. 4 that our estimation procedure works on high signal-to-noise synthetic spectra. However, for the analysis in Sec. 4 we assumed no systematic uncertainties. In this Section we create mock data with exactly the same parameters $(\gamma, T_0, \Gamma_{-12}, \chi, \Sigma)$ as they were estimated in Sec. 5. We exactly mimic the steps that we did in Sec. 5 to perform a consistency check of our estimation and to verify our analysis, in particular the slope of the linear fit, on mock data similar to the observational data set.

The fitting results with the RPC_IRGN method and with the IRGN$_x^2$ algorithm are shown in Fig. 12 in a similar way as in Fig. 10. The error in the estimation of $\gamma$ is dominated by the systematic uncertainty due to the propagation of the error in the estimation of $n_H$. We also added the systematic uncertainty in the estimation of $\mu$ (the parameter of the prior lognormal model) to mimic the same error as in the estimation from observational data. All in all we slightly overestimate the true $\gamma = 1.4$. We estimate $\gamma = 1.47 \pm 0.09$. However, the exact value lies well inside the error interval. The linear relation that we expected from synthetic data and that we observed in the observational data set is again visible. By comparing Fig. 12 and Fig. 10 it becomes obvious that the estimates of the temperature for single values of $\gamma$ is more precise for our mock data sets (the error bars are significantly smaller). The dashed line in Fig. 12 shows the linear fit from Fig. 10. The dashed line lies inside the predictive distribution (blue shaded) from the linear fit from our mock data. This shows the consistency of our linear fit. The linear fit to observational data and the linear fit to our mock data coincide. In particular we estimate $T_0 = 12700^{+1400}_{-1300}$ K from our mock data which is well inside the error bars.

### 7 DISCUSSION

#### 7.1 Comparison With Recent Estimates

We compare our estimates with recent estimates for the temperature at mean density in Fig. 13, with recent estimates of the slope parameter $\gamma$ in Fig. 14 and with estimates of the photoionization rate in Fig. 15.

We predict a slightly larger temperature than in the current study by Walther et al. (2019), but our estimate for the temperature coincides very well with the measurements by Garzilli et al. (2012), Boera et al. (2014), Hiss et al. (2018) and Gaikwad et al. (2020a). Our results are more precise than existing estimates and comparably as the estimates by Becker et al. (2011), Boera et al. (2014) and Gaikwad et al. (2020a). However, in compatible high precision studies such as Becker et al. (2011), Boera et al. (2014), Hiss et al. (2018), Walther et al. (2019) and Gaikwad et al. (2020a) the precision in the temperature estimate comes at the cost of a large uncertainty in redshift. This is caused by accounting for redshift bins of width $\Delta z = 0.1 - 0.2$ to create an artificial sample of absorption lines. However, recent estimates suggest that $T_0$ might vary over the width of this bin (e.g. see the bin-to-bin variations in the estimates represented in Fig. 13). Our estimate is free of such uncertainties, because our redshift bin is 10 times narrower (this is discussed further below).

Moreover, our estimate for the slope parameter $\gamma$ of the temperature-density relation matches well the observations by Hiss et al. (2018) and Telikova et al. (2019) and is compatible with the lower limit by Garzilli et al. (2012). However, Walther et al. (2019) found larger values for the slope and Gaikwad et al. (2020a) found slightly smaller slopes. Again our estimate is competitive without suffering from averaging in the redshift domain.

Lastly, we compare our estimate for the photoionization rate at redshift $z = 2.5$ with the observations by Tytler et al. (2004), Bolton et al. (2005), Faucher-Giguère et al. (2008), Becker et al. (2013) and Telikova et al. (2019) in Fig. 15. Our result coincides best with the observation by Becker et al. (2013) and Bolton et al. (2005). However, there is some scatter between the different measurements visible. The error in $\Gamma_{-12}$ in our analysis is dominated by the uncertainty in the estimation of the mean density. Thus, we cannot improve the significance of the estimate as we did for the temperature at mean density. In particular, our estimate favors a large photoionization rate of $\Gamma_{12} \approx 1$.

#### 7.2 Discussion of increased Precision

We increase the precision of the estimation of $T_0$ by our method. Note that we only used a 40 Å portion of each spectrum (such that there is only a negligible redshift error introduced) and gained estimates with an error of only approximately $\approx 1000$ K, which is competitive to current measurements drawn from much greater redshift bins (and thus a greater sample of absorption lines). We
Figure 9. Exemplary application of our estimation procedure to the quasar J091613-070224 for wavelength $\lambda \in [4235, 4245]$ Å. The red framed panels show the results with the fitting results with the IRGN-$\chi^2$ method (upper left panel) and with the RPC_IRGN method (lower right panel). All other panels show $\chi^2$ as computed with the IRGN-$\chi^2$ method as function of $T_0$ for seven different values of $\gamma$. Upper left panel (red framed): $\chi^2$ as function of $T_0$ for different values of $\gamma$. For clarity we show only four out of seven curves. Middle panels: curves for one specific $\gamma$ (diamond) with fitted borderline (dashed line), fitted linear increase (dotted line) and fitted temperature $T_0$ (solid black line). Lower right panel: The logarithmic distance calculated with the RPC_IRGN method for different values of $\gamma$. For this LOS we favor $\gamma = 1.55$.

Figure 10. Joint estimation with the RPC_IRGN method (green shaded, 1–$\sigma$ confidence interval) and the estimates with the IRGN-$\chi^2$ method (black diamonds). The seven data points are fitted with a linear fit (black solid line) with corresponding predictive 1–$\sigma$ error (magenta shaded). We explain this improved precision in the estimation of $T_0$ using our method by the following three assertions:

- In comparison to simple Voigt-profile fitting we use a complete fitting approach which uses the explicit formulation of the forward model. On the one hand, this fitting increases the coverage of fitted wavelength (we do not fit only absorption lines, but every feature in the spectrum) and enables us to correctly take into account the profile of the underlying neutral hydrogen density. On the other hand, this procedure is much more computationally expensive because finding suitable inversion parameters (regularization parameter, stopping criterion) can be difficult.

- Among fitting of the absorption profiles, for the reconstruction with the RPC method and with the IRGN algorithm we also assume some prior distribution for matter. In comparison to purely fitting tools we therefore include much more prior knowledge in order to infer the temperature-density distribution. Additionally the approach of treating the set $\{n_{\text{HI}}, \gamma, T_0\}$ instead of the set of parameters $\{T, \gamma, T_0\}$ allows us to disentangle the degeneracy between $\gamma$ and $T_0$, i.e. we were able to estimate $\gamma$ solely without $T_0$ with the RPC_IRGN algorithm.
Figure 12. Same as in Fig. 10, but for an estimation from synthetic mock data. The dotted black line indicates the linear fit from Fig. 10, the solid black line shows the linear fit to the plotted data points on mock data. Both linear fits coincide indicating consistency of the estimation procedure.

Figure 13. Comparison of recent estimates for $T_0$. We show our estimate with a red diamond and plot the $3 - \sigma$ errorbounds of our estimate. Other data points (with $1 - \sigma$ errorbounds) are taken from Becker et al. (2011) (magenta circles, a fiducial $\gamma \sim 1.5$ assumed), Garzilli et al. (2012) (blue boxes), Boera et al. (2014) (cyan points, fiducial $\gamma \sim 1.5$ assumed), Hiss et al. (2018) (green stars), Telikova et al. (2019) (grey diamonds), Walther et al. (2019) (black boxes, strong prior results) and Gaikwad et al. (2020a) (orange boxes).

- The statistical analysis reduces the error. There is a reasonable scatter for the temperature estimates for different parameters $\gamma$, but the assumption of a linear relation within the IRGN$_\chi^2$ method allows us to reduce the error for the predictive distribution by cross correlating with estimates for other temperatures.

7.3 Inverted Equation of State

Some studies reported that the match between observational data and simulations could be improved by assuming $\gamma \approx 1$ or even an inverted EOS ($\gamma < 1$), e.g. Bolton et al. (2008); Furlanetto & Oh (2009). While this is within the errorbars of recent observations at redshift $z \approx 5$ (Boera et al. 2019; Gaikwad et al. 2020b), measurements at smaller redshifts tend to favor a positive $\gamma$, see Fig. 14. This fits the model of an asymptotic temperature-density relation (McQuinn & Upton Sanderbeck 2016). In fact, our findings do not support an inverted temperature density relation at redshift $z = 2.5$. The RPC_IRGN method shows a distinct minimum at values $\gamma > 1$ for the quasar spectra, compare for example the lower right panel in Fig. 9. In particular, for $\gamma = 0.9$ we measure logarithmic distances between the inversion results with the RPC method and with the IRGN method that are a couple of times larger than at the minimal value. The reason for that is that we underestimate the amplitude of variations in the neutral hydrogen density (while letting the mean density unaffected) when assuming small values for $\gamma$ in the inversion procedure. An example of this effect on the inversion procedure is shown for QSO J101155+294141 in Fig. 16. Only for $\gamma \approx 1.5$ (red line) does the recovered density field span over the whole range of recovered overdensities with the RPC method.
Inversions match on all scales, i.e. the amplitude of the variations in density line) and IRGN with varying

Figure 16. Inversion results for the QSO J101155+294141 with RPC (black line) and IRGN with varying $\gamma$. Only for $\gamma = 1.5$ the RPC and the IRGN inversions match on all scales, i.e. the amplitude of the variations in density space match while being overestimated for $\gamma = 2$ and underestimated for $\gamma = 1$.

(black line). At larger $\gamma$ (blue line) the range of possible values, i.e. the amplitude of density fluctuations, is overestimated, at smaller $\gamma$ (orange line) the range of possible values is underestimated. Overall our measurements do not support an inverted EOS.

7.4 Outlook

Our estimation method suffers from the major drawback that peculiar velocities are not included in the inversion algorithms along single lines of sight. This issue is common to most fitting algorithms used to recover the thermal history of the IGM. The unknown peculiar velocities introduce the largest error on $\gamma$ and thus induce the largest error in the estimation of $T_0$. However, if the peculiar velocities would be known a priori, they could be inserted in the IRGN inversion straightforward.

Although Pichon et al. (2001) studied a method to estimate the velocity field along a single line of sight in parallel, we were not able to reproduce these results. Gas streams towards large overdensities are possibly not detectable in the density along the line of sight if the overdensity is located transverse to the line of sight. In such cases the tangential projection of the stream leads to a redshift distortion which cannot be identified with an overdensity in the line of sight. However, these overdensities would be visible in neighboring lines of sight. Thus, the velocity field could be approximated by three dimensional Lyα forest tomography from a field of quasar spectra with small separation length, such as the COSMOS field (Lee et al. 2018; Ate et al. 2020). There are many recent approaches in recovering a three dimensional maps at low spectral resolution from the Lyα forest, among others Pichon et al. (2001); Kitaura et al. (2012); Lee et al. (2014); Ciesielski et al. (2014); Stark et al. (2015); Lee et al. (2016); Lee & White (2016); Lee et al. (2018); Krolewski et al. (2018); Horowitz et al. (2019); Japelj et al. (2019); Porqueres et al. (2019, 2020) which would carry information regarding redshift space distortions.

Lee et al. (2014) discussed observational requirements for such tomographic reconstructions. In a nutshell, at mean separation length of $1\,h^{-1}\text{Mpc}$ only moderate resolution ($R \approx 1000$) spectra are required. However, to fit the thermal profile of the absorption features in the Lyα forest by tomographic inversion we require high resolution spectra ($R > 50000$). We observe that the velocity field is typically only varying on larger bins. We therefore propose to observe a field of close quasars with small transverse separation length at small resolution, estimate the peculiar velocity field from these observations and then subsequently observe one quasar at the center of the field with very high resolution. Then we propose to use the recovered redshift space distortions as approximation to the real distortions and perform the inversions with respect to these distortions. We expect that with this strategy one would be able to infer the temperature with some 100 K precision using our estimation procedure.

8 CONCLUSION

We presented a novel estimation procedure to study the EOS of the IGM by Lyα forest tomography and applied this approach to a sample of 31 QSO spectra from the UVES_SQUAD sample. In particular, we observed $T_0 = 13000^{+13000}_{-12000}$ K, $\gamma = 1.44 \pm 0.09$ and $\Gamma_{-12} = 1.12 \pm 0.17$ at redshift $z = 2.5$. The corresponding (marginal) probability distributions are shown in Fig. 10 and Fig. 11. Our estimates are more precise than existing estimates while drawn from significantly smaller redshift bins.

In contrast to classical absorption line fitting approaches we utilize a full inversion of the Lyα forest at high spectral resolution ($\approx 2.5\,\text{km/s-bins}$) to constrain the EOS. In particular, we use the IRGN inversion algorithm (which strongly depends on the chosen model of the IGM) and the RPC method (which depends on less strong priors) to invert the spectra. The quality of the inversion results with the IRGN method depend strongly on whether the correct EOS has been used. We therefore compare in data space the recovered flux with the observed flux (IRGN$_\gamma^2$) and compare in density space the density recovered with the IRGN method and with the RPC method for different choices for the EOS (RPC_IRGN). We take the set of parameters for the EOS for which the observed flux and the recovered density are described best.

It turns out that the RPC_IRGN method is (to first order) only sensitive to $\gamma$, while the IRGN$_\gamma^2$ predicts a linear relation between the possible estimates $\gamma$ and $T_0$. Both methods solely are not capable of predicting the EOS due to a degeneracy between $\gamma$ and $T_0$. However, both methods together are complementary and provide unique estimates. The RPC_IRGN method breaks the degeneracy between $T_0$ and $\gamma$ in the IRGN$_\gamma^2$ method. In principle both methods also could be combined with the results by classical fitting approaches or curvature methods to constrain the EOS.

Our estimates, in particular for the temperature, are more precise than most existing estimates. While sometimes this accuracy is achieved by extending the redshift range, our estimate does not introduce a significant errors in redshift. We explain the achieved precision mainly by the fact that we combine several approaches to model the Lyα forest with varying assumptions. Hence, we combine prior knowledge regarding the model of the Lyα forest, the power spectrum, and the probability density distribution of matter overdensities.

The full power of our method would be achieved if the peculiar velocities along the studied lines of sight could be known at least approximately. This might be possible from recent surveys of QSO spectra with small mean transverse separation length. Our method is expected to provide temperature measurements with a precision of several 100 K for these data sets.
Nevertheless, suitable observations on which to apply our analysis are rare. They need to have a large spectral resolution and a high signal to noise ratio.

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**DATA AVAILABILITY**

The software to reproduce the analysis in this paper will be made publicly available under the url https://github.com/hmullergoe/reglyman and upon request. The studied spectra are available in the UVES_SQUAD survey (Murphy et al. 2019).

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Estimator for IGM EOS
APPENDIX A: NEUTRAL HYDROGEN DENSITY

We describe the neutral hydrogen density in the quasi-linear regime of structure formation by the neutral hydrogen density fraction:

\[ f_{\text{HI}} = \frac{n_{\text{HI}}}{n_b + n_p}, \]  

(A1)

with the baryonic number density \( n_b \) and proton number density \( n_p \). Here, the helium mass density is ignored. Moreover, it is in local equilibrium:

\[ \alpha n_e n_b = \Gamma n_{\text{HI}}, \]  

(A2)

with recombination rate \( \alpha \). All together it follows:

\[ n_{\text{HI}} = f_{\text{HI}} n_b = \frac{\alpha n_e n_b}{\alpha n_e n_b + \Gamma} \approx \frac{\alpha n_e n_b}{\Gamma} = \frac{\mu_e \alpha n_b^2}{\Gamma}, \]  

(A3)

where \( \mu_e = 2(2 - Y)/(4 - 3Y) \) with helium mass fraction \( Y \approx 0.24 \) accounts for helium in the IGM. The recombination rate is modeled as a power law (Black 1981; Rauch et al. 1997) which had been subsequently been used for simulations of the Ly\( \alpha \) forest (e.g. Choudhury et al. 2001; Gallerani et al. 2006):

\[ \alpha = \alpha_0 T_0^{-0.7}, \]  

(A4)

where the temperature is represented in units of Kelvin and \( \alpha_0 = 4.2 \times 10^{-1.2} \text{ cm}^3\text{s}^{-1} \). With the power-law EOS Eq. (4) it follows:

\[ \alpha = \alpha_0 T_0^{-0.7} \Delta^{0.7-0.7Y}. \]  

(A5)

Thus:

\[ n_{\text{HI}} = \frac{\mu_e \alpha_0 n_b^2}{\Gamma T_0^{0.7} \Delta^{2.7-0.7Y}}, \]  

(A6)

where \( n_b \) is the mean baryonic density defined by \( n_0(z) = \Omega_b \rho_c (1 + z)^3 \). Here \( \Omega_b \) is the baryonic density parameter, \( \rho_c \) the critical density, \( \mu \) the mean molecular weight and \( m_p \) the proton mass. Eq. (A6) shows Eq. (5). We get:

\[ \mu_e \alpha_0 n_b^2 \approx 1.871 \times 10^{-14} \text{ m}^{-3} \text{s}^{-1}, \]  

(A7)

at redshift \( z = 2.5 \) which proves Eq. (6).

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