Pythagorean Triples and A New Pythagorean Theorem

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Abstract

Given a right triangle and two inscribed squares, we show that the reciprocals of the hypotenuse and the sides of the squares satisfy an interesting Pythagorean equality. This gives new ways to obtain rational (integer) right triangles from a given one.

1. Harmonic and Symphonic Squares

Consider an arbitrary triangle with altitude $\alpha$ corresponding to base $\beta$ (see Figure 1a). Assuming that the base angles are acute, suppose that a square of side $\eta$ is inscribed as shown in Figure 1b.

![Figure 1: Triangle and inscribed square](image)

Then $\alpha, \beta, \eta$ form a harmonic sum, i.e. satisfy (1). The equivalent formula $\eta = \frac{\alpha \beta}{\alpha + \beta}$ is also convenient, and is found in some geometry books.

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\eta} \quad (1)$$
Equation (1) remains valid if a base angle is a right angle, or is obtuse, save that in the last case the triangle base must be extended, and the square is not strictly inscribed (Figures 2a, 2c).

Starting with the right angle case (Figure 2a) it is easy to see the inscribed square uniquely exists (bisect the right angle), and from similar triangles we have proportion

\[(\beta - \eta) : \beta = \eta : \alpha\]

which leads to (1). The horizontal dashed lines are parallel, hence the three triangles of Figure 2 have the same base and altitude; and the three squares are congruent and unique. Clearly (1) applies to all cases.

A scalene triangle will have three squares, one for each side of the triangle. Each one has a side length that is the harmonic sum of the corresponding triangle side and altitude. Of course when the triangle is equilateral, the squares are congruent, but do not coincide.

When a right triangle with legs \(a, b\) and hypotenuse \(c\) is given, there are just two squares (Figures 3a, 3b), the “harmonic” square of side \(h\), and the “symphonic” square of side \(s\).

Using (1) twice gives solutions (2), since the altitude to the hypotenuse
is \( \kappa = \frac{ab}{c} \) (using similar right triangles).

\[
h = \frac{ab}{a+b}, \quad s = \frac{abc}{ab+c^2}
\]

(2)

In all right triangles (Exercise) \( c > h > s \).

Now we present a striking identity we call the Symphonic Theorem.

**The Symphonic Theorem:**

With respect to \( c, h, \) and \( s \) as defined above, the triple \( [c^2, h^2, s^2] \) is harmonic, and the triple \( \left[ \frac{1}{c}, \frac{1}{h}, \frac{1}{s} \right] \) is Pythagorean.

\[
\frac{1}{c^2} + \frac{1}{h^2} = \frac{1}{s^2}
\]

(3)

Equation (3) holds for all right triangles \([a, b, c]\) and squares \( h, s \) inscribed as indicated. The altitudes of the triangle \((a, b, \kappa)\), have a similar relationship.

\[
\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{\kappa^2}
\]

(4)

Substitute (2) into (3) and clear fractions:

\[
[c(a+b)]^2 + [ab]^2 = [ab + c^2]^2.
\]

(5)

On both sides of the equal sign, make the replacement

\[
c^2 = a^2 + b^2
\]

(6)

to get an algebraic identity:

\[
[a^2 + b^2][a + b]^2 + [ab]^2 = [ab + a^2 + b^2]^2.
\]

Now suppose that \([a, b, c]\) are integers without common factors (a *Primitive Pythagorean Triple*, or PPT). From equation (6), \( a, b, c \) are relatively prime in pairs. Then equation (2) shows that \([c(a+b), ab, ab + c^2]\) is also a PPT. Any common factor divides \( ab + c^2, ab, \) hence also \((ab + c^2) - ab\), which implies a non-trivial common factor of \( a \) or \( b \) and \( c \), contrary to assumption.

Take for example \([a, b, c] = [3, 4, 5]\) and calculate \( h = \frac{12}{7}, s = \frac{60}{37} \). From (3) we have a rational right triangle \( \left[ \frac{1}{h}, \frac{1}{c}, \frac{1}{s} \right] = \left[ \frac{7}{12}, \frac{1}{3}, \frac{37}{60} \right] \).
Multiply this by 60 to arrive at the triple [35, 12, 37], in accord with (5).

On the other hand, we can enlarge $a$, $b$, $c$ so that $h$, $s$ also are integers. Multiply by $(11)(37) = 259$, to get $[a, b, c] = [777, 1036, 1295]$, $h = 444$, and $s = 420$. This is clearly the smallest example.

If (3) is regarded as simply an equation in three unknowns, then the smallest solution in integers is [15, 20, 12], derived by dividing the [3, 4, 5] right triangle by a factor of sixty. More generally, given PT $[a, b, c]$, we can obtain $[bc, ac, ab]$ as an integer solution to $1x^2 + 1y^2 = 1z^2$.

The transformation $S: [3, 4, 5] \rightarrow [35, 12, 37]$ by (5) is an instance of

$$S: [a, b, c] \rightarrow [c(a + b), ab, c^2 + ab].$$  \hspace{1cm} (7)

Looking at this form one day, we found the following analogous transformation.

$$S' : [a, b, c] \rightarrow [c|a - b|, ab, c^2 - ab]$$ \hspace{1cm} (8)

If in the definitions (3) the smaller of the two values $a$, $b$ is replaced by its negative, defining $h'$, $s'$, then the same development which led to (7) now leads to (8).

The results (7), (8) are here named the Symphonic Derivatives, Major and Minor. One of us (Price) found the Theorem, both of us worked out the results, and one of us (Bernhart) found the Minor derivative.

A diagram for triangle $[a, b, c]$ may be embellished with lines showing how to find $h'$ and $s'$, but these constructions appear highly artificial, compared with the elegant beauty of Figure 3! Query: can a “natural” construction for (8) be devised?

There is one clue, a challenge problem posed by Sastry in The College Mathematics Journal [10]. The editors made a brief composite of five independent solutions, and appended a long list of other solvers. The question starts with the two triples $[3, 4, 5]$ and $[5, 12, 13]$, and seeks to generalize to $[a, b, c]$ and $[c, ab, ab+1]$. This is the special case of the minor derivative when $a - b = 1$! This last condition defines a family of primitive triples studied by Fermat (see Eckert [6]).

That family is closely related to a venerable problem of recreational mathematics, the square-triangle problem. The challenge is to find pairs of positive integers $x, y$ such that a square arrangement of marbles, $x$ on a side, equals a triangular arrangement, $y$ on a side.

The question amounts to finding integer solutions to $2x^2 = y(y+1)$, which can be converted to a Pellian and solved (see Barbeau [2]). The pairs $(x_i, y_i)$
form an infinite family, but we shall focus on

\[(x_i) = (1, 6, 35, 204, 1189, \ldots, a, b, 6b-a, \ldots),\]
easily extended by the embedded rule \(c = 6a-b\). Pick two consecutive values, such as 6, 35 or 35, 204. Take the sum and difference, but split the sum into two consecutive integers:

\[
35 - 6 = 29, \quad 35 + 6 = 20 + 21 \Rightarrow [20, 21, 29]
\]
\[
204 - 35 = 169, \quad 204 + 35 = 119 + 120 \Rightarrow [119, 120, 169].
\]

We summarize the effects of “Symphonic derivation” on a few smaller Pythagorean triangles.

- **S, S**: \([3, 4, 5] \rightarrow [35, 12, 37], [5, 12, 13]\)
- **S, S**: \([5, 12, 13] \rightarrow [221, 60, 229], [91, 60, 109]\)
- **S, S**: \([15, 8, 17] \rightarrow [391, 120, 409], [119, 120, 169]\)
- **S, S**: \([7, 24, 25] \rightarrow [775, 168, 793], [425, 168, 457]\)

2. **Raising the Standard**

It is very helpful to study symphonic derivation from the point of view of the standard generators of triples. From one point of view, a pair \((p, q)\) of parameters with certain restrictions is used to generate a triple \([a, b, c]\). From another viewpoint, a proper fraction \(\frac{q}{p}\) in lowest terms, is the generator – with a definite geometric meaning. We shall treat the parameter pair and the fraction as equivalent, interchangeable objects, with a few adjustments. A brief review will be given (details and proofs omitted). A fuller treatment is found in [4], [5].

Primitive triple \([a, b, c]\) has two generators. We assume without loss that \(a\) is odd. **Primary** and **secondary** generators \(t_1, t_2\) are obtained as follows.

\[
t_1 = \frac{b}{c+a} = \frac{c-a}{b} = \frac{q_1}{p_1}, \quad t_2 = \frac{a}{c+b} = \frac{c-b}{a} = \frac{q_2}{p_2}.
\]

Quotients \(t_1, t_2\) are geometrically the **half-angle tangents** of the right triangle. The generators for \([3, 4, 5]\) are:

\[
t_1 = \frac{4}{3+5} = \frac{1}{2}, \quad t_2 = \frac{3}{4+5} = \frac{1}{3}
\]
The key sequence \([q_2, q_1, p_1, p_2]\) is a Fibonacci-Rule sequence, \(q_2 + q_1 = p_1, q_1 + p_1 = p_2\) with the additional conditions that the first member \((q_2)\) is odd, and the first two members \((q_1, q_2)\) are relatively prime (and of course all four members are positive integers). Some possibilities are as follows.

\[
[1, n, \ldots] \ [3, 1, \ldots] \ [3, 2, \ldots] \ [3, 4, \ldots] \ [5, 1, \ldots] \ [5, 2, \ldots] = [5, 2, 7, 9]
\]

We can also make a key sequence from any positive proper fraction \(\frac{q}{p}\). If \(q+p\) is even, we use the “template” \([q, *, *, p]\), but if \(q + p\) is odd, we use the template \([*, q, p, *]\). In either case, the template can be completed uniquely.

For example, fractions \(\frac{2}{3}, \frac{1}{5}\) each give the same sequence, \([1, 2, 3, 5]\), and fractions \(\frac{1}{4}, \frac{3}{5}\) each give \([3, 1, 4, 5]\).

The two common parametric solutions are (9) and (10). The first is correct if \(\frac{q}{p} = \frac{q_1}{p_1}\) is primary, and the second is correct if \(\frac{q}{p} = \frac{q_2}{p_2}\) is secondary.

\[
a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2 \quad (9)
\]

\[
b = \frac{p^2 - q^2}{2}, \quad a = pq, \quad c = \frac{p^2 + q^2}{2} \quad (10)
\]

In our work we found the following mixed solution(s) quite convenient.

\[
a = p_2 q_2, \quad b = 2p_1 q_1, \quad c = p_1 p_2 - q_1 q_2 = p_1 q_2 + p_2 q_1 \quad (11)
\]

For example, using key sequence \([1, 1, 2, 3]\):

\[
a = 3 = (1 \cdot 3), \quad b = 4 = 2(1 \cdot 2), \quad c = 5 = (2 \cdot 3) - (1 \cdot 1) = (2 \cdot 1) + (3 \cdot 1).
\]

Write \(G : \frac{q}{p} \rightarrow [a, b, c]\) to indicate that \(\frac{q}{p}\) is a generator of \([a, b, c]\). Whether \(\frac{q}{p}\) is primary or secondary will be stated, if not evident from the context.

The mixed form solution (11) can be made more symmetric with the aid of the following definitions.

\[
r_1 = q_1 q_2, \quad r_2 = q_1 p_2, \quad r_3 = q_2 p_1, \quad r_4 = p_1 p_2.
\]

Circle Theorem:

Triangle \([a, b, c]\) has an in-circle with radius \(r_1\) and three ex-circles with radii \(r_2, r_3, r_4\). Moreover, (i) \(r_1 + r_2 + r_3 = r_4\), \(r_1 \cdot r_4 = r_2 \cdot r_3\). (ii) \(a = r_1 + r_2 = r_4 - r_3\), \(b = r_1 + r_3 = r_4 - r_2\), \(c = r_2 + r_3 = r_4 - r_1\).
It follows from \((ii)\) that four circles with radii \(r_i\) form a tangent system with their centers at the corners of a rectangle of dimensions \(a \times b\). Proof of these claims is elementary. A more detailed correlation between tangent circles and ex/in-circles and full proofs can be found in [4]. See also Akhtar [1].

The key sequence \([3,1,4,5]\) was an example above. From it we can find the radii by multiplying out the product \((3 + 5)(1 + 4) \rightarrow 3, 12, 5, 20\). Then the radii give the sides of the triangle by \((ii)\):
\[
a = 3 + 12 = 20 - 5, \quad b = 3 + 5 = 20 - 12, \quad c = 12 + 5 = 20 - 3.
\]
The radii are useful here, primarily by means of the following. We assume \(a < b\).

Symphonic Corollary:

\((i)\) The primary and secondary generators \(T, T'\) for the symphonic major derivative are

\[
T = \frac{q_1q_2}{p_1p_2} = \frac{ab}{(c+a)(c+b)} = \frac{r_1}{r_4},
\]
\[
T' = \frac{(r_2+r_3)}{(r_4+r_1)} = \frac{c}{(a+b)}.
\]

\((ii)\) In the case of the minor derivative we have generators \(T, T'\) given by

\[
T = \frac{q_2p_1}{p_2q_1} = \frac{ab}{(c-a)(c+b)} = \frac{r_2}{r_3},
\]
\[
T' = \frac{(r_3-r_2)}{(r_3+r_2)} = \frac{(b-a)}{c}.
\]

We have put the most convenient expressions last on each line.

Remark: If \(a > b\), then in part \((ii)\) we must exchange \(a, b\) and exchange \(r_2, r_3\). Fairly routine computation using equations already given is all that is needed for proof.

Take as an example
\[
S : [3,4,5] \rightarrow [35,12,37], \quad S' : [3,4,5] \rightarrow [5,12,13].
\]

Secondary and primary generators for \([3,4,5]\) are \(1/3: 1/2\). Their product, \(1/6\), and quotient, \(2/3\), are the primary generators for the major and minor derivatives! Alternately, the radial quotients \(\frac{r_4}{r_2}, \frac{r_2}{r_3}\) give the same results.
We also check that $\frac{c}{a+b} = \frac{5}{7}$ and $\frac{b-a}{c} = \frac{1}{5}$ are the secondary generators. Note that $\frac{1}{6}, \frac{5}{7}$ combine in the key sequence $[5, 1, 6, 7]$, and similarly $\frac{2}{3}, \frac{1}{5}$ combine in $[1, 2, 3, 5]$.

3. A Symphonic Family History

The set of PPT's has a family structure. It was independently discovered, seven years apart, by Barning [3] and Hall [7], so we refer to it as the Barning-Hall Tree. It has often been rediscovered. The smallest triple $[3, 4, 5]$ is the root and only occupant of level zero. Level $n$ is converted to level $n+1$ by replacing each triple $[a, b, c]$ by its three immediate successors (its direct descendants, or “children”). Hence level $n$ has $3^n$ members. Linking each triple with its three immediate successors creates a ternary, ordered, plane tree (compare Figure 4).

Given the key sequence $[q_2, q_1, p_1, p_2]$ the three successors, called left, middle, right are obtained by completing $[p_2, q_1, \cdots], [p_2, p_1, \cdots], [q_2, p_1, \cdots]$. Thus our diagram agrees with Hall, and with Eckert [6] after it is rotated 90 degrees.

Level one is $[15, 8, 17], [21, 20, 29], [5, 12, 13]$. Level two is

\[
\begin{align*}
a & : 35 & 65 & 33 & | & 77 & 119 & 39 & | & 45 & 55 & 7  
b & : 12 & 72 & 56 & | & 36 & 120 & 80 & | & 28 & 48 & 24  
c & : 37 & 97 & 65 & | & 85 & 169 & 89 & | & 53 & 73 & 25 
\end{align*}
\]

The (ternary) tree is simpler to display, and to analyze, when each triple is “abbreviated” to its primary generator: Let $\frac{q}{p}$ be any proper fraction. Consider the quotient $\frac{a}{p-2q}$. This new quotient can be turned into a simpler positive proper fraction by (if necessary) changing sign, and taking the reciprocal, unless $\frac{q}{p} = \frac{1}{2}$ or $\frac{q}{p} = \frac{1}{3}$.

Using this operation to define “parent” the proper fractions are formed into two ternary trees, one for primary generators and one for secondary generators. Only the first is needed or displayed here.

Any location on the tree can be reached by a path from the top, consisting of $n$ steps to reach the $n^{th}$ level. One step down-left (A), straight-down (B), or down-right (C), is given by

\[
A: \frac{q}{p} \rightarrow \frac{q}{p+2q}, \quad B: \frac{q}{p} \rightarrow \frac{p}{2p+q}, \quad C: \frac{q}{p} \rightarrow \frac{p}{2p-q}. \quad (12)
\]

Let us take the four triples from levels zero and one, and extract the major derivative. The primary generator is noted as a fraction.

$\frac{1}{6} [35, 12, 37]; \quad \frac{3}{20} [391, 120, 409]; \quad \frac{6}{35} [1189, 420, 1261]; \quad \frac{2}{15} [221, 60, 229]$
Likewise, the minor derivatives are as follows.

\[
\frac{2}{3} [5, 12, 13]; \quad \frac{5}{12} [119, 120, 169]; \quad \frac{14}{15} [29, 420, 421]; \quad \frac{3}{10} [91, 60, 109]
\]

The figure below shows the location of all eight of these on the tree. “Rho” (ρ) marks the root. The major/minor derivatives are shown as small dark circles/rectangles (each circle is labeled with a generator).

We can locate a fraction (\(\frac{6}{35}\) say) on the tree by regression.

\[
\frac{6}{35} \rightarrow \frac{6}{23} \rightarrow \frac{6}{11} \rightarrow \frac{1}{6} \rightarrow \frac{1}{4} \rightarrow \frac{1}{2}.
\]

Analyzing and reversing the steps gives us the code AACAA for traveling “down” from the root, to arrive at the triple. The path codes for the major derivatives are AA, CBAA, AACAA, CAAA. Those for the minor derivatives are C, BB, C^{13}, CCA.
By employing (12), these codes can be used as operators from the right side, e.g.

\[
\left(\frac{1}{4}\right) \text{CAA} = \left(\frac{4}{7}\right) \text{AA} = \left(\frac{4}{15}\right) \text{A} = \left(\frac{4}{23}\right) : [513, 184, 545].
\]

Evidently most of the locations on the tree do not belong to major or minor derivatives.

The ternary tree is beautiful because it contains every primitive triple, each in its own unique position. It is possible to obtain trees using derivatives as well. One can define a binary tree in which the two immediate successors of a triple are the major derivative and the minor derivative. The chief reason we will not pursue this course is that there would be not one, but many such trees. Each triple that is not itself a derivative (major or minor) is the root of a different tree.

It seems natural to pose certain questions:

(a) Can one characterize directly which triples are major or minor derivatives?

(b) Can a triple be a derivative in two different ways?

(c) For certain infinite sequences of triples, can we give the tree location of the major and minor derivatives?

All in all, it seems best to start with the third question. We now add the symbols \(S, S'\) to the one-step symbols \(A, B, C\), supplementing (12) with (13).

\[
S : \frac{q}{p} \rightarrow \frac{q(p-q)}{p(p+q)}, \quad S' : \frac{q}{p} \rightarrow \frac{q(p+q)}{p(p-q)}
\]  

(13)

In case of the latter expression, it is necessary to regard an improper fraction, say \(\left(\frac{3}{2}\right)\), as an alternate generator \(\frac{p}{q}\) equivalent to \(\frac{2}{p}\). (Thus \(p = 3, q = 2\) either way).

This notation makes it convenient to investigate special cases, \textit{viz.}

\[
\left(\frac{1}{2}\right) A^{n-1} S = \left(\frac{1}{2^n}\right) S = \frac{2^{n-1}}{2^n(2n+1)}.
\]

We have here a simple way of describing the symphonic major derivative of an arbitrary member of the family \([4n^2 - 1, 4n, 4n^2 + 1]\) attributed to Plato, and occupying the extreme left branch of the ternary tree. Likewise

\[
\left(\frac{1}{2}\right) C^{n-1} S = \left(\frac{n}{n+1}\right) S = \frac{n}{(n+1)(2n+1)}
\]
describes the symphonic major derivative of the family on the extreme right branch of the ternary tree, attributed to Pythagoras. The Platonic family is characterized by $A^{n-1}$ or $\frac{1}{2n}$, and the Pythagorean family is characterized by $C^{n-1}$, or $\frac{a}{n+1}$. It is inevitable that we include also the “Fermat” family, characterized by $B^{n-1}$ or $\frac{P_n}{P_{n+1}}$ (primary generator) or $\frac{Q_n}{Q_{n+1}}$ (secondary generator). Here the Pell sequences

\begin{align*}
(P_n)n &= 1, 2, \cdots = (1, 2, 5, 12, 29, \cdots, a, b, 2b + a, \cdots) \\
(Q_n)n &= 1, 2, \cdots = (1, 3, 7, 17, 41, \cdots, a, b, 2b + a, \cdots)
\end{align*}

are familiar recursive sequences, satisfying the indicated “embedded” recursion, which is not dissimilar to the Fibonacci recursion. We call attention to the key sequence formed from the generators:

$$[Q_n, P_n, P_{n+1}, Q_{n+1}].$$

Here the Fibonacci rule applies, and may be used to define both sequences. The ratios $\frac{Q_n}{P_n}$ converge to $\sqrt{2}$; this helps to explain the ubiquity and popularity of these numbers. Because the Pell sequences are the subject of a large literature, we do not need to regret giving them such a brief glance.

The derived sequence $\frac{1}{2}P_{2n} = Q_nP_n$ was introduced above, just after the minor derivative, to explain the Fermat family in connection with Sastry’s challenge problem! Working out the relationships (for instance Hatch [8]) we leave to the reader. The book by Barbeau [2] is very helpful, and thorough.

We are now ready to continue, starting with the following.

$$\begin{align*}
(\frac{1}{2})^n B^{n-1} S &= \frac{P_n}{P_{n+1}} S = \frac{P_{2n}}{P_{2n+2}} \\
(\frac{1}{2})^n B^{n-1} S' &= \frac{P_n}{P_{n+1}} S' = \frac{P_{2n+1} - 1}{P_{2n+1} + 1}
\end{align*}$$

The final task is to specify the ABC path (eliminate $S, S'$ from the path code). It is helpful to start by calculating a few cases and generalizing. The results are presented in the six tables of Figures 6 and 7 below. Symbol $t$ is the primary generator, and $(t)S, (t)S'$ are the primary generators for the major/minor derivatives. Each has an associated path code.

One can verify from these tables that no duplication other than AA occurs, forecasting the answer to question (b). It is also curious to observe that the Platonic family AA... is the trickiest, requiring a separation into even and odd cases.
A look at the final column of these tables shows that B occurs only for the Platonic family and at most twice there!

When X, Y are path codes, write \(X \cong Y\) to mean \((\frac{1}{2})X = (\frac{1}{2})Y\). If X, Y are pure codes, made up of A, B, C only, then they are identical, letter for letter. But cases like \(AS' \cong BB\) are of interest here.

**Derivative Location Theorem:**

*The locations of the major and minor derivatives are given by the formula in the last row of the tables in Figures 6 and 7.*

We begin with the Pythagoras major derivative, claiming that

\[C^{n-1}S \cong C^{n-1}A^{n+1}.\]

This is simple, for \((\frac{1}{2})C^{n-1} = \frac{n}{n+1},\) and then \((\frac{n}{n+1})S = (\frac{n}{n+1}) (\frac{1}{2n+1})\), or alternately

\[(\frac{n}{n+1}) A^{n+1} = \frac{n}{(n+1) + 2n(n+1)} = \frac{n}{(2n+1)(n+1)}.\]

The minor derivative \(C^{n-1}S' \cong C^n A^{n-1}\) is similar. Note: direct descendents!

For the Fermat major derivative, recall Pell sequences \((P_n), (Q_n)\) and related sequence

\[(\frac{1}{2})P_{2n} = (P_nQ_n) = (1, 6, 35, 204, \ldots, a, b, c, \ldots),\]

where \(c = 6b - a\) is the recursion rule. First we have \((\frac{1}{2})B^{n-1} = \frac{P_n}{P_{n+1}}.\) Here \(T = \frac{P_n}{P_{n+1}}\) is the primary generator, and \(T' = \frac{P_{n+1} - P_n}{P_{n+1} + P_n} = \frac{Q_n}{Q_{n+1}}\) is the secondary generator. Then the major derivative is

\[\left(\frac{P_n}{P_{n+1}}\right)S = TT' = \left(\frac{P_n}{P_{n+1}}\right) \left(\frac{Q_n}{Q_{n+1}}\right) = \left(\frac{P_nQ_n}{P_{n+1}Q_{n+1}}\right).\]

Now \((\frac{1}{2}) AA = (\frac{1}{6}) = \frac{P_{n+1}Q_n}{P_{n+2}Q_{n+2}},\) and \((\frac{a}{b}) CAA = \frac{b}{6b-a} = \frac{b}{c}\), and so we get \(B^{n-1}S \cong AA(CAA)^{n-1},\) since \(\left(\frac{P_n}{P_{n+1}}\right) CAA = \left(\frac{P_{n+1}}{P_{n+2}}\right).\)

The Fermat minor derivative requires \(B^{n-1}S' \cong C^{k-1}.\) Firstly,

\[\left(\frac{P_n}{P_{n+1}}\right)S' = T' = \frac{P_{n+1}Q_{n+1}}{P_{n+1}Q_n} \quad n \text{ even}\]

\[\left(\frac{P_n}{P_{n+1}}\right)S' = T' = \frac{P_{n+1}Q_n}{P_nQ_{n+1}} \quad n \text{ odd}.\]
In both cases the final fraction has the form \( \frac{k}{k+1} = \left( \frac{1}{2} \right) C^{k-1} \) where
\[
k = \frac{1}{2}(P_{2n+1} - 1) = \min \{ P_n Q_{n+1}, \ P_{n+1} Q_n \}
\]
\[
k + 1 = \frac{1}{2}(P_{2n+1} + 1) = \max \{ P_n Q_{n+1}, \ P_{n+1} Q_n \}
\]

This is the special case of Sastry discussed earlier where triple \([a, b, c]\) with \(b - a = 1\) has minor derivative \([c, ab, ab+1]\). We also have \(k = x + y\) where
\[
(x, y) = (1, 1), (6, 8), (35, 49), (204, 288), \ldots
\]
is any solution of the square-triangle problem: \(x^2 = y(y + 1)\). The Platonic derivatives remain, and they require a distinction between odd and even. But the proof is relatively easy. To show \(A_{n-1}S ∼ C A^{k-1}B A^n\) when \(n = 2k\) is even, and \(A^{n-1}S ∼ BA^{k-1}B A^n\) when \(n = 2k + 1\) is odd, calculate
\[
\left( \frac{1}{2} \right) A^{n-1}S = \left( \frac{1}{2n} \right) \left( \frac{2n}{2n+1} \right) = \left( \frac{2n-1}{2n(2n+1)} \right),
\]
\[
\left( \frac{1}{2} \right) C A^{k-1} = \left( \frac{2}{3} \right) A^{k-1} = \left( \frac{2}{3+4(k-1)} \right) = \left( \frac{2}{4k-1} \right) = \left( \frac{2}{2n-1} \right); \ n = 2k,
\]
\[
\left( \frac{1}{2} \right) B A^{k-1} = \left( \frac{2}{5} \right) A^{k-1} = \left( \frac{2}{5+4(k-1)} \right) = \left( \frac{2}{4k+1} \right) = \left( \frac{2}{2n-1} \right); \ n = 2k + 1,
\]
\[
\left( \frac{2}{2n-1} \right) B A^n = \left( \frac{2n-1}{4n-2+2} \right) A^n = \left( \frac{2n-1}{4n} \right) A^n = \left( \frac{2n-1}{2n(2n+1)} \right).
\]

The case of \(A^{n-1}S' ∼ BA^{k-1}B A^{n-2}\), \(n = 2k\); \(A^{n-1}S' ∼ CA^{k}B A^n\), \(n = 2k + 1\) is quite similar, and also routine. Q.E.D.

Taking a symphonic derivative at least doubles the length of the path code, which supports the empirical conclusion that these derivatives are quite sparse. The Fermat minor derivatives are especially far down the tree!

4. Musical Aptitude

Just how improbable is it for an arbitrary triple to be symphonic? This is another approach to question (a) above. The following development shows that often it is easy to conclude that a triple is not symphonic.

For any PPT \([a, b, c]\) the product \(abc\) is divisible by 60. This fact is long known (Sierpinski [11]) but apparently not always well known (Monaghan [9]). Recall that we assume that \(a\) is odd. Write \(x|y\) to say that integer \(x\) divides integer \(y\). Then
\[
4|b, \ 3|ab, \ 5|abc. \quad (14)
\]
By the mixed solution \( (1) \), we can write
\[
b = 2q_1p_1, \quad ab = 2q_1p_1q_2 p_2, \quad abc = 2q_1p_1 q_2 p_2 (q_1 p_2 + p_1 q_2).
\]

Let \( Q = (\ldots, x, y, z, u, v, \ldots) \) be a Fibonacci rule sequence. Say it is \( k \)-primitive if not every term is divisible by \( k \). Then if \( Q \) is 2-primitive, \( 2 \mid xyz \) and every third item is even. If \( Q \) is 3-primitive, \( 3 \mid xyzu \) and every fourth item is a multiple of three, and if \( Q \) is 5-primitive, either \( 5 \mid xyzuv \) and every fifth item is a multiple of five, or else no item is a multiple of five. The last possibility must equal \( (\ldots, 1, -2, -1, 2, 1, \ldots) \mod(5) \).

Since key sequence \( (q_2, q_1, p_1, p_2) \) has the Fibonacci property, and \( q_2 \) is odd, it is easy to show that \( b \) is a multiple of four. Also \( ab \) is divisible by three (and \( c \) therefore cannot be). Finally \( 5 \mid abc \), – the trickiest case.

The key sequence may be extended indefinitely at either end, yielding a longer sequence with the Fibonacci property. If \( 5 \mid ab \), we are done. Else (i) the sequence is a part of a longer sequence \( (5x, q_2, q_1, p_1, p_2, 5y) \) and is equivalent \( \mod(5) \) to \( (0, x, x, 2x, 3x, 0) \), or (ii) extends indefinitely without including a multiple of five.

Suppose case (i). Given that \( c = (q_1 p_2 + p_1 q_2) \), this is \( \mod(5) \) congruent to \( (x)(2x) + (x)(3x) = x^2(2 + 3) = 0 \). Now suppose (ii). The key sequence is a four term subsequence of \( (\ldots, 1, -2, -1, 2, 1, \ldots) \). Wherever the starting point, in \( \mod(5) \) we get \( c = (2)(-2) + (1)(-1) = 0 \). Q.E.D.

Based on the location of the factors, we can list six classes of PPT’s:

| 3|a | 3|b |
|---|---|---|---|
| 5|c | T1: [3x, 4x, 5x] | T2: [x, 12y, 5z] |
| 5|a | T3: [15x, 4x, x] | T4: [5x, 12y, z] |
| 5|b | T5: [3x, 20y, z] | T6: [x, 60y, z] |

The simplest member of each (infinite!) class is:

| 1/2 | 3/4 | 1/4 | 2/3 | 2/5 | 3/6 |
|---|---|---|---|---|---|
| [3,4,5] | [7,24,25] | [15,8,17] | [5,12,13] | [21,20,29] | [11,60,61] |

A symphonic derivative can only be in class T4 or T6. More specifically, using \( S, S' : [a, b, c] \rightarrow [A, B, C] = [|a \pm b|c, \ ab, \ c^2 \pm ab] \) we see that factors of \( c \) are transferred to \( A \), and factors of \( a, b \) are transferred to \( B \).
Symphonic Factor Theorem:

*The symphonic product* \([A, B, C]\) *is in class T4 if the original* \([a, b, c]\) *is in T1 or T2, but otherwise is in class T6. Hence after two steps, a symphonic product is in T6.*

Are all class T6 PPT’s symphonic derivatives? We explore this question making use of a famous PPT studied by Fermat. According to Sierpinski [11], Fermat wrote a letter to Mersenne in the year 1643, in which he asserted that the PPT

\[
\begin{bmatrix}
456 & 54860 & 27761 \\
106 & 16522 & 93520 \\
468 & 72986 & 10289
\end{bmatrix}
\]

is the *smallest* PPT in which the hypotenuse and the sum of the legs are both squares. Just for starters, we compute the primary generator and use it to locate the tree position of Fermat’s triple.

The generator turns out to be \(246792/2150905\). Now we back up through the tree 41 steps! (see Table 1)

Reversing and collating the sequence of letters gives the path code locator for Fermat’s enormous triple. It has an interesting structure that may perhaps be typical.

\[
\begin{array}{cccccc}
B & C & C & C & B & A
\end{array}
\]

\[
\begin{array}{cccccc}
A & A & A & A & A & A
\end{array}
\]

\[
\begin{array}{cccccc}
C & A & A & B
\end{array}
\]

\[
\begin{array}{cccccccc}
C & C & C & C & C & C & C & C
\end{array}
\]

\[
\begin{array}{ccccccc}
B & C & C & B & A
\end{array}
\]

\[
\begin{array}{ccccccc}
A & A & A & A & A & A
\end{array}
\]

\[
\begin{array}{cccccccc}
C & A & A & B
\end{array}
\]

\[
\begin{array}{cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc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The 41 Steps

| A  | 246792 / 1657321 (Fermat) | C  | 3755 / 5778 |
|----|--------------------------|----|------------|
| A  | 246792 / 1163737         | C  | 732 / 3755 |
| A  | 246792 / 670153          | B  | 291 / 1732 |
| B  | 176569 / 246792          | A  | 291 / 1150 |
| C  | 106346 / 176569          | A  | 291 / 568  |
| C  | 36123 / 106346           | C  | 14 / 291   |
| B  | 34100 / 36123            | A  | 14 / 263   |
| C  | 32077 / 34100            | A  | 14 / 235   |
| C  | 30054 / 32077            | A  | 14 / 207   |
| C  | 28031 / 30054            | A  | 14 / 179   |
| C  | 26008 / 28031            | A  | 14 / 151   |
| C  | 23985 / 26008            | A  | 14 / 123   |
| C  | 21962 / 23985            | A  | 14 / 95    |
| C  | 19939 / 21962            | A  | 14 / 67    |
| C  | 17916 / 19939            | A  | 14 / 39    |
| C  | 15893 / 17916            | B  | 11 / 14    |
| C  | 13870 / 15893            | C  | 8 / 11     |
| C  | 11847 / 13870            | C  | 5 / 8      |
| C  | 9824 / 11847             | C  | 2 / 5      |
| C  | 7801 / 9824              | B  | 1 / 2      (Root) |
| C  | 5778 / 7801              | .... | q/p       |

Table 1: Path code to Fermat’s triangle

This can not be the square \((a - b)^2\) thus Fermat’s triple is not “major”.
Alternately we find that \(c = 31 \cdot 239 \cdot 257\), which contains isolated primes of form \(4k + 3\), contrary to known results. With the necessary adjustments, a similar argument establishes that the Fermat triple is not a minor derivative as well.

Finally, we conclude with some results on query (b).

Anti-derivative Theorem

*If* \([a, b, c] \rightarrow [A, B, C]\) *by either* \(S\) *or* \(S'\) *then* \([A, B, C]\) *determines* \([a, b, c]\).*
Let \( \frac{Q}{P} \) be the primary generator of the derivative \([A, B, C]\). Suppose that 

\[
[A, B, C] = [P^2 - Q^2, 2PQ, P^2 + Q^2] = [c(a + b), \ ab, \ c^2 + ab].
\]

Then: \( P - Q = c, \ P + Q = a + b, \ 2PQ = ab. \)

The quadratic \( x^2 - (P + Q)x + 2PQ \) has roots \( a, b \). Now suppose

\[
[A, B, C] = [P^2 - Q^2, 2PQ, P^2 + Q^2] = [c(b - a), \ ab, \ c^2 - ab].
\]

Then: \( P + Q = c, \ P - Q = b - a, \ 2PQ = ab. \)

(We have assumed \( a < b \) which could have a side affect – the exchange of radii \( r_2, r_3 \).) From quadratic \( x^2 + (P - Q)x - 2PQ \) we find \( (b, -a) \).

This result allows us to compute “anti-derivatives” for any primitive triple! The triple \( T = [15, 8, 17] \) is not a major or a minor derivative. Undaunted, we produce real and complex surds as the anti-derivatives.

\[
S^{-1}(T) = [a, b, 3], \quad a, b = \frac{1}{2} \left( 5 \pm \sqrt{-7} \right), \\
S'^{-1}(T) = [a, b, 5], \quad -a, b = \frac{1}{2} \left( 3 \pm \sqrt{41} \right).
\]

These are “right triangles” that transform by \( S, S' \) to \([15, 8, 17]\).

**Major/Minor Theorem:**

*There is no triple \([A, B, C]\) which is both major and minor.*

Proceeding in a similar manner, suppose \([a, b, c]\) is a major anti-derivative, and suppose \([a', b', c']\) is a minor anti-derivative for \([A, B, C]\). Then

\[
P + Q = c' = a + b, \quad P - Q = c = b' - a', \quad 2PQ = ab = a'b'.
\]

The last pair of equations imply that the two anti-derivatives are triangles with the same area. The first two equations can be added and subtracted to show that \( \{r_1, r_4\} = \{r'_2, r'_3\} \) which with the equations \( r_1r_4 = r_2r_3, \ r'_1r'_4 = r'_2r'_3 \) implies \( \{r'_1, r'_4\} = \{r_2, r_3\} \). But this is impossible, since \( r_2, r_3 < r_4 \) and \( r'_2, r'_3 < r'_4 \).
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### Major Derivatives of Three Families (AAA.. BBB.. CCC..)

#### Pythagoras CCC...

| $t$  | $(t)S = T$ | t-code  | T-code |
|------|------------|---------|--------|
| 1    | 1/2        | $\times$ | $AA$   |
| 2    | 2/3        | 2/15    | $C$    | $C$ $AAA$ |
| 3    | 3/4        | 3/20    | $CC$   | $CC$ $AAAA$ |
| 4    | 4/5        | 4/15    | $CCC$  | $CCC$ $AAAAA$ |
| 5    | 5/6        | 5/66    | $CCCC$ | $CCCC$ $AAAAAA$ |
| ...  | ...        | ...     | ...    | ...     |
| $n$  | $n(n+1)$   | $n(n+1)(2n+1)$ | $C^n$ | $C^n$ $S = C^n$ $A^{n^2}$ |

#### Plato AAA...

| $t$  | $(t)S = T$ | t-code  | T-code |
|------|------------|---------|--------|
| 1    | 1/2        | $\times$ | $AA$   |
| 2    | 1/3        | 3/20    | $A$    | $CB$ $AA$ |
| 3    | 1/6        | 5/42    | $AA$   | $BB$ $AAA$ |
| 4    | 1/8        | 7/72    | $AAA$  | $CAB$ $AAAA$ |
| 5    | 1/10       | 9/110   | $AAA$  | $BAB$ $AAAAA$ |
| ...  | ...        | ...     | ...    | ...     |
| $n=2k+1>1$ | $1/(2n)$ | $(2n-1)/2n(2n+1)$ | $A^{n-1}$ | $A^{n-1}S = BA^{n-1}BA^n$ |
| $n=2k$ | $1/(2n)$ | $(2n-1)/2n(2n+1)$ | $A^{n-1}$ | $A^{n-1}S = CA^{n-1}BA^n$ |

#### Fermat ("Pell") BBB...

| $t$  | $(t)S = T$ | t-code  | T-code |
|------|------------|---------|--------|
| 1    | 1/2        | $\times$ | $AA$   |
| 2    | 2/5        | 6/35    | $B$    | $AA$ $CAA$ |
| 3    | 5/12       | 35/204  | $BB$   | $AAC$ $AA$ |
| 4    | 12/29      | 204/1189| $BBB$  | $AA$ $CAA$ $AAC$ |
| 5    | 29/70      | 1189/6970| $BBBB$ | $AA$ $CAA$ $AAC$ $AA$ |
| ...  | ...        | ...     | ...    | ...     |
| $n$  | $P_nP_{n+1}$ | $P_{2n}/P_{2n+2}$ | $B^n$ | $B^n$ $S = AA(CAA)^{n-1}$ |

Figure 6: Three Tables: Major Derivative
Minor Derivatives of Three Families (AAA.. BBB.. CCC..)

**Pythagoras CCC...**

| t   | (tS' = T) | t-code | T-code |
|-----|-----------|--------|--------|
| 1   | 1/2       | 2/3    | C      |
| 2   | 2/3       | 3/10   | C      |
| 3   | 3/4       | 4/21   | CC     |
| 4   | 4/5       | 5/36   | CCC    |
| 5   | 5/6       | 6/55   | CCC    |
| ... | ...       | ...    | ...    |
| n   | n(n+1)    | (n+1)n(2n+1) | C^{-1} |
|     |           |        | C^{-1} S' = C^n A^{n-1} |

**Plato AAA...**

| t   | (tS' = T) | t-code | T-code |
|-----|-----------|--------|--------|
| 1   | 1/2       | 2/3    | C      |
| 2   | 1/4       | 5/12   | A      |
| 3   | 1/6       | 7/30   | AA     |
| 4   | 1/8       | 9/56   | AAA    |
| 5   | 1/10      | 11/90  | AAAA   |
| ... | ...       | ...    | ...    |
| n=2k+1 | 1/(2n) | (2n+1)/2n(2n-1) | A^{n-1} |
| n=2k  |         |        | A^{n-1} S' = C^A^B^A^{n-2} |
|       |         |        | A^{n-1} S' = B^A^{-1}B^{n+2} |

**Fermat ("Pell") BBB...**

| t   | (tS' = T) | t-code | T-code |
|-----|-----------|--------|--------|
| 1   | 1/2       | 2/3    | C      |
| 2   | 2/5       | 14/15  | B      |
| 3   | 5/12      | 104/65 | BB     |
| 4   | 12/29     | 492/293| BBB    |
| 5   | 29/70     | 2970/2871 | BBBB | C^{202} |
| ... | ...       | ...    | ...    |
| n   | P_nP_{n+1} | k/(k+1): | B^{n-1}: |
|     |           | k=½(P_{2n+1} - 1) | k=½(P_{2n+1} - 1) |
|     |           |        | B^{n-1} S' = C^{k-1}, |
|     |           |        | B^{n-1} S' = C^{k-1}, |
|     |           |        | B^{n-1} S' = C^{k-1}, |
|     |           |        | B^{n-1} S' = C^{k-1}, |

Figure 7: Three Tables: Minor Derivative