Three- and four-body systems with the Functional Renormalization Group

To cite this article: Benjamín Raziel and Jaramillo Ávila 2016 J. Phys.: Conf. Ser. 761 012016

View the article online for updates and enhancements.

Related content

- Stabilities of Four-Body Systems against Arbitrary Dissociations
  Mohamed Assad Abdul-Raouf

- Four-body effects on 9Be+208Pb scattering and fusion around the Coulomb barrier
  P Descouvemont, T Druet, L F Canto et al.

- Peculiar features of the interaction potential between hydrogen and antihydrogen at intermediate separations
  Lee Teck-Ghee, Wong Cheuk-Yin and Wang Lee-Shien
Three- and four-body systems with the Functional Renormalization Group

Benjamín Raziel Jaramillo Ávila
División de Ciencias e Ingenierías, Universidad de Guanajuato, León, México
E-mail: raziel@fisica.ugto.mx

Abstract. The Efimov effect arises in three-body systems near the unitary limit. Some of its features are universal, while others are not. This article uses a Functional-Renormalization-Group approach to discuss the Efimov effect and four-body systems. In this context, the Efimov effect appears as a consequence of the Renormalization-Group flow of couplings. On the four-body system, we find three tetramers below each Efimov trimer, and no evidence of four-body universality breaking. Two of these tetramers are in agreement with quantum-mechanical calculations and experimental results.

1. Introduction
Different physical systems are formed by nonrelativistic particles with interactions that have a large scattering length. There, the range of interparticle interactions can be neglected when two conditions are met: the range is much smaller than the scattering length, and interactions occur at sufficiently low energy. When this happens, an interaction is solely described by one parameter, its scattering length.

Under these conditions, there is independence of the detailed form of the interparticle interaction. This can produce universality in the sense that different systems might display the same low-energy behavior. This happens even though the particles (and interactions) that form these systems are very different. For example, particles can be atoms in some cases and nucleons in others. A few parameters (including the scattering length) might be enough to describe these systems. References [1, 2] review systems with these interactions.

The particle-particle interaction is the starting point to describe these systems. This work is restricted to systems of identical scalar particles, hence there is only one particle-particle scattering length. When the absolute value of the scattering length is large, the interaction is close to a zero-energy two-particle bound state. There are two possible cases. a) If the scattering length is large and positive, the two-body system has a weakly bound state with energy $-\frac{1}{m a^2}$ ($a$ is the scattering length and $m$ is the mass of each particle). b) On the other hand, if $a$ is very negative, the two-body system does not support a bound state, but the interaction is almost strong enough to do so. There, two-particle subsystems play an important role in the dynamics of larger systems even though there are no two-particle bound states. A limit sits between these cases: when the scattering length approaches either $+\infty$ or $-\infty$, the system supports a zero-energy bound state. This is known as the unitary (or resonant) limit, which can also be expressed as $1/a \to 0$. 
Two physical systems have the features described above, ultracold gases of atoms and low-energy nucleon systems. Experimentalists have an impressive amount of control over ultracold gases of atoms, where temperatures reach $\lesssim 10^{-4}$ Kelvin. In particular, it is possible to control the atom-atom interaction (and its scattering length) in carefully prepared systems. This is achieved by setting an external magnetic field close to a Feshbach resonance [3]. There, the scattering length can be set very close to the resonance, both with positive and negative scattering lengths. Because of this, ultracold gases of atoms provide a benchmark to study low-energy systems and some of their universal behavior.

Low-energy nucleon systems also have some of the features described in this introduction. When the kinetic energy of each nucleon is around or below a few tens of MeV, it is not necessary to use pion-mediated nucleon interactions. Instead, the dynamics can be described by a pionless Effective Field Theory [4, 5]. There, a contact vertex describes the two-nucleon interaction. The S-wave component of this interaction (which dominates at low-energy) has two channels: each channel controlled by a scattering length. The spin-singlet channel $^1S_0$ has scattering length $a_s = -23.7\text{ fm}$, and the triplet channel $^3S_1$ has $a_t = +5.4\text{ fm}$. These scattering lengths are larger than the range of interaction, which is roughly 1 fm. Low-energy Effective Field Theories are useful to describe both few- [4, 5] and many-nucleon systems [6]. Furthermore, corrections due to the finite range of the interactions can be added as a series in $(r_{\text{eff}}/a)$, where $r_{\text{eff}}$ is the effective range of the interaction [7, 8, 4].

In 1970, V. Efimov showed that some three-body systems have a large sequence of weakly bound states, and that the energies of these states are arranged in a geometric sequence [9, 10]. Furthermore, the dimensionless constant in the geometric sequence depends on the ratios of the masses of the particles. For particles of equal mass, this constant is an universal number, $E^{(3)}_{i+1}/E^{(3)}_i \approx 515.0$.

The Efimov effect arises as a consequence of a large separation between two physical scales. Particularly, a separation between a (very small) two-body interaction range and a (very large) scattering length. Efimov states, or trimers, appear in scales between this range and scattering length. For simplicity, assume we are dealing with nonrelativistic systems of indistinguishable scalar particles with mass $m$. In natural units, the two-body interaction range, $R$, sets a high-energy scale, $E_R = \frac{R^2}{m}$, and the scattering length, $a$, sets a low-energy scale, $E_a = \frac{a^2}{m}$. If $R \ll |a|$, then $E_R \gg E_a$. All Efimov states have energy between these two scales, $E_R \gg |E^{(3)}_i| \gg E_a$.

Since the energies of Efimov states appear in a geometric sequence, the energy of the lowest state determines the energies of the rest. This introduces a dimensionful, three-body physical parameter, the Efimov parameter. Its value is determined by the detailed form of the microscopic interactions, and in this low-energy framework it is an independent parameter.

The Efimov effect breaks a scale-invariance symmetry. This is because, if Efimov states did not exist, there would not be any physical parameters to set a scale between $E_R$ and $E_a$. There, the system would be independent of the detailed form of the interactions, and the behavior of the three-body system would be universal. However, the Efimov effect breaks the continuous scaling symmetry into a discrete symmetry realized by the Efimov sequence. In that sense, the Efimov parameter breaks the universality of the three-body system.

Four-body systems are also influenced by the Efimov effect. Numerical quantum-mechanical calculations find two four-body states for each Efimov three-body state (or trimer) [11, 12, 13] and no four-body universality-breaking parameter. Experiments on ultracold gases of atoms also reveal the presence of two four-body states close to an Efimov trimer.

The Functional-Renormalization-Group approach sketched in this article finds three

\[ ^1 \text{We work in systems where even though } E_R \text{ is the high-energy scale, it is sufficiently low to use a nonrelativistic framework.} \]
tetramers for each Efimov trimer [15, 16], but one of them might be an artifact of the approach. The energies of the other two tetramers are in agreement with those in quantum-mechanical calculations [12, 13] and with experimental results [14]. Furthermore, we find no evidence of four-body universality-breaking parameters.

2. Low-energy systems and the Functional Renormalization Group

The Functional Renormalization Group (FRG) is a nonperturbative Quantum-Field-Theoretical tool to integrate low-momentum quantum fluctuations. It was developed by C. Wetterich [17, 18] based on ideas by Wilson. Reference [19] and Chapter 2 in [20] are reviews on the topic.

The FRG uses artificial mass terms, called regulators, to suppress low-momentum quantum fluctuations. The regulators depend on an unphysical momentum scale, $k$, and they suppress fluctuations with momentum below $k$.

The FRG starts with an (unphysical) running action at an ultra violet scale, $K_0$. Fluctuations below $K_0$ are suppressed from this action, but physical information from scales above $K_0$ is included. A flow equation describes the change in the running action as $k$ changes. We use this equation to evolve the running action, $\Gamma_k$, from $k = K_0$ to the physical limit, $k \to 0$. In this limit there are no suppressed fluctuations and the running action becomes the effective action. The latter contains the full dynamics, including all quantum fluctuations.

The flow equation,

$$ \partial_k \Gamma_k = -\frac{i}{2} \text{Str}\left( \partial_k R \cdot \left( \Gamma_k^{(2)} - R \right)^{-1} \right), $$

depends on the regulators, $R$, and on the second derivatives of the running action with respect to the fields, $\Gamma_k^{(2)}$. This is a second-order functional differential equation, and there are no general techniques to solve it. However, in some cases, if we use an ansatz for the running action, $\Gamma_k$, the flow equation reduces to a system of ordinary differential equations. The functions that solve this system of equations are $k$-dependent parameters, known as running functions; their $k$-dependence is given by the FRG flow. In the physical limit, these functions contain dynamical information about the system.

The use of an ansatz instead of solving the full FRG flow equation means that the results are not exact. However, the FRG can provide sensible physical results if the running action ansatz retains the main features of the (unknown) full effective action. Furthermore, FRG results are nonperturbative in the sense that they are obtained without using a series expansion around a small parameter.

3. Running action and three-body physics

To study two-, three-, and four-particle systems we use a running action ansatz that contains nonrelativistic kinetic terms and contact-interaction terms. The ansatz contains a one-particle field, $\psi(x)$; a two-particle (or dimer) field, $\phi(x)$; and a three-particle (or trimer) field, $\chi(x)$. All of these fields have nonrelativistic kinetic terms. Dimer kinetic terms approximately represent the energy dependence of two-particle subsystems, and trimer kinetic terms do a similar job for three-particle subsystems.

Dimer fields are introduced at the UV via a Hubbard-Stratonovich transformation [21]. At the UV, dimers are auxiliary (non dynamical) fields, but the FRG flow gives them dynamics. Trimer fields can also be introduced at the UV as non dynamical fields.
The explicit form of the running action ansatz is

\[
\Gamma_k = \int_x \left\{ \psi^\dagger \left( i \partial_0 + \frac{\nabla^2}{2m} \right) \psi + Z_d \phi^\dagger \left( i \partial_0 + \frac{\nabla^2}{4m} - \frac{u_d}{Z_d} \right) \phi + Z_t \chi^\dagger \left( i \partial_0 + \frac{\nabla^2}{6m} - \frac{u_t}{Z_t} \right) \chi \right.
\]

\[
- \frac{g}{2} (\phi^\dagger \psi \psi + \psi^\dagger \phi^\dagger \phi) - \lambda \phi^\dagger \phi^\dagger \phi \psi - h (\chi^\dagger \phi \psi + \phi^\dagger \psi^\dagger \phi \chi)
\]

\[
- \frac{w}{4} \phi^\dagger \psi^\dagger \phi \psi - \frac{v_d}{4} (\phi^\dagger \phi^\dagger \phi \psi + \phi^\dagger \psi^\dagger \psi^\dagger \phi \phi) - \frac{u_{dd}}{2} (\phi^\dagger \phi^\dagger \phi \psi + \psi^\dagger \psi^\dagger \phi \phi)
\]

\[
- \frac{v_t}{2} (\phi^\dagger \chi^\dagger \psi \phi \psi + \chi^\dagger \phi^\dagger \psi \phi \psi) - \frac{u_{dt}}{2} (\phi^\dagger \phi^\dagger \chi \psi + \chi^\dagger \psi^\dagger \phi \phi)
\]

- \frac{u_{tt}}{2} \chi^\dagger \psi^\dagger \chi \psi \}

This contains several \( k \)-dependent functions which are determined by solving the FRG flow equation. The objects \( Z_d, u_d, g \) are parameters associated with two-body systems. \( Z_d \) is the \( k \)-dependent dimer field renormalization, \( u_d \) is a \( k \)-dependent dimer field bilinear, and \( g \) is a constant coupling between particles and dimers. The quantities \( Z_t, u_t, \lambda, h \) are three-body parameters, and each of them is \( k \)-dependent. \( Z_t \) and \( u_t \) are a renormalization and a bilinear for the trimer (respectively), whereas \( \lambda \) and \( h \) are couplings. Finally, \( w, v_d, u_{dd}, v_t, u_{dt}, u_{tt} \) are \( k \)-dependent four-body coupling constants.

The flow of two-body parameters is analytically solvable, and it must satisfy one condition in the physical limit: two-particle scattering must produce the correct scattering length [1, 2, 15, 21]. Explicitly, two-particle scattering amplitude at zero energy must yield

\[
T_{\psi\psi} \rightarrow \frac{8\pi a}{m}.
\]

Using the running action ansatz, the same amplitude yields

\[
T_{\psi\psi} = \frac{-g^2}{u_d(k \rightarrow 0)}.
\]

The two expressions for the amplitude match when

\[
u_d(k \rightarrow 0) = -\frac{g^2 m}{8\pi a},
\]

which is a boundary condition on \( u_d \).

The flow of three-body running functions reflects the Efimov effect. In the unitary limit, these functions oscillate periodically in \( t = \log(k/K_0) \). The periodicity in \( t \) corresponds to geometric scaling in units of energy (or momentum). When the flow of couplings oscillates indefinitely in a cycle we say there is a limit cycle. This is in contrast with flows that approach a fixed point.

In the present approach, with the regulators in [15], the Efimov period in \( t \) is \( \frac{60\pi}{\sqrt{535}} \approx 6.79 \). These cycles are larger than the physical ones, which have period 6.25. In FRG approaches, the exact value of this period depends on an unphysical choice: the explicit form of the regulators, \( R \).

References [15, 21] and Chapter 3 of [22] study these three-body systems in an FRG approach.

Out of the unitary limit, the finite scattering length breaks the Efimov periodicity in the flow. The cycles stop when \( k \sim 1/|a| \) (in natural units). For fluctuations with momenta smaller than \( 1/|a| \), the three-body functions settle.
4. Four-body physics

The flow of four-body systems inherits the Efimov periodicity of three-body systems. In the unitary limit, the four-body couplings display an infinite sequence of peaks inside each Efimov cycle \[15, 22\]. This is known as super Efimov effect, and it arises from the structure of the flow equations. However, not all these peaks correspond to physical four-body resonances.

Out of the unitary limit we find three four-body peaks inside each Efimov cycle. They are labeled as (0), (1), and (2), where (0) is the deepest one, and (2) is extremely shallow. At zero energy these states appear when the two-particle scattering length is

\[
\frac{a_4}{a_3}^{(0)} = 0.438, \quad \frac{a_4}{a_3}^{(1)} = 0.877, \quad \frac{a_4}{a_3}^{(2)} = 0.997,
\]

where \(a_3\) is the scattering length that produces the closest Efimov three-body state above the four-body resonances. The scattering lengths for the deepest of these peaks, (0) and (1), are in 5% agreement with quantum-mechanical calculations \[12, 13\], and they are also consistent with experimental results \[14\]. The shallow resonance, (2), quickly merges with the particle-trimer threshold and might be an artifact of using an ansatz. Figure 1 shows a plot for the energies and scattering lengths that produce the four-body resonances.

**Figure 1.** This plot shows the features of the four-body system that appear in the present approach. It displays four-body resonances and particle-trimer thresholds of several Efimov cycles. The blue straight lines below the shaded region represent the dimer-dimer threshold. The black solid curves are the particle-trimer thresholds. The red dot-dashed curve is the lowest four-body resonance below its corresponding Efimov trimer, (0). The green short-dashed curve is the four-body resonance (1). The magenta long-dashed curve is the shallow tetramer, (2), which quickly merges with the particle-trimer threshold. The horizontal axis is the inverse of the two-particle scattering length, \(1/a\), but it is normalized by (the absolute value of) the inverse of a two-particle scattering length that produces a trimer, \(1/|a_3|\). The vertical axis is proportional to \(\text{sign}(E) \sqrt{|E/m|}\), which (in natural units) has units of inverse length. This axis is also normalized by \(1/|a_3|\). On the horizontal axis, the ratio between four-body resonances should be 29.8 in our truncation (the physical value is \(22.7 \simeq \sqrt{515.0}\)). Instead, this number was artificially replaced by 5 to show several Efimov cycles in one plot.
5. Conclusions

The FRG can describe few-body systems with reasonable accuracy. However, to improve these results, it might be necessary to use a more general energy dependence in the running action. It is also necessary to study how much do FRG results depend on unphysical choices in the FRG, in particular, the choice of regulators.

The FRG can be a powerful and computationally-cheap tool to study many-body systems, and few-body FRG studies can provide foundations for many-body studies.

References

[1] Braaten E and Hammer H W 2006 Phys. Rept. 428 259
[2] Platter L 2009 Few Body Syst. 46 139
[3] Chin C, Grimm R, Julienne P and Tiesinga E 2010 Rev. Mod. Phys. 82 1225
[4] Bedaque P F and van Kolck U 2002 Annu. Rev. Nucl. Part. Sci. 52 339
[5] Epelbaum E, Hammer H W and Meiβner U G 2009 Rev. Mod. Phys. 81 1773
[6] Drews M and Weise W 2015 Phys. Rev. C 91 035802
[7] Birse M C and McGovern J A and Richardson K G 1999 Phys. Lett. B 464 169
[8] Kaplan D B, Savage M J and Wise M B 1998 Nucl. Phys. B 534 329
[9] Efimov V 1970 Phys. Lett. B 33 563
[10] Efimov V 1971 Sov. J. Nucl. Phys. 12 589
[11] Hammer H W and Platter L 2007 Eur. Phys. J. A 32 113
[12] von Stecher J, D'Incao J P and Greene C H 2009 Nature Phys. 5 417
[13] Deltuva A 2010 Phys. Rev. A 82 040701
[14] Ferlaino F, Knoop S, Berninger M, Harm W, D'Incao J P, Nægerl H C and Grimm R 2009 Phys. Rev. Lett. 102 130401
[15] Jaramillo Avila B and Birse M C 2013 Phys. Rev. A 88 043613
[16] Jaramillo Avila B and Birse M C 2015 Phys. Rev. A 92 023601
[17] Wetterich C 1991 Nucl. Phys. B 352 529
[18] Wetterich C 1993 Phys. Lett. B 301 90
[19] Berges J, Tetradis N and Wetterich C 2002 Phys. Rept. 363 223
[20] Schwenk A and Polonyi J (editors) 2012 Renormalization Group and Effective Field Theory Approaches to Many-Body Systems (Berlin Springer-Verlag)
[21] Schmidt R and Moroz S 2010 Phys. Rev. A 81 052709
[22] Jaramillo Avila B 2015 Functional Renormalisation Group and Nuclear Matter (University of Manchester PhD thesis)