Unitary mixing scalar–vector in $\xi$ gauge

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Abstract

We study the effect of unitary mixing of scalar and vector fields in general $\xi$ gauge. This effect takes place for non–conserved vector currents and $\xi$ gauge generates some additional problems with unphysical scalar field. We obtained solutions of Dyson-Schwinger equations and perfomed the renormalization of full propagators. The key feature of renormalization is the usage of Ward identity, which relates some different Green functions. We found that using Ward identity leads to disappearing of $\xi$-dependence in renormalized matrix element.
1 Introduction

The mixing of scalar and vector fields (S-V mixing) appears at the loop level if the non-diagonal loop connecting scalar and vector propagators exists. This effect takes place when vector current is not conserved.

Similar effect was noticed before [1, 2] when researching the Standard Model in $\xi$ gauge where appears the mixing between gauge boson field and unphysical field (so-called Higgs ghost) with propagator pole at the point $p^2 = \xi M^2$. However physical scalar fields also can participate in mixing. Thus, in [3] this effect in system $\pi - a_1$ was considered, and in [4] the S-V mixing between gauge bosons and Higgs particles in extended electroweak models was investigated. However in [4] the problem of renormalization which in this case is rather non-trivial, as well as the problem of gauge dependency were not investigated. Note that consideration of S-V mixing in $\xi$ gauge leads to interesting effect [5]: the full propagators will have another type of singularity then the bare ones. Namely, simple pole of bare propagator at the point $p^2 = \xi M^2$ turns into a double pole of full propagator. After that the question of whether the Standard Model is renormalizable in this gauge arises [3].

In the present work we consider the unitary mixing of physical scalar and physical vector in $\xi$ gauge. We focus on renormalization of matrix element and its dependence on $\xi$. We consider both boson and fermion loops that have got some special features. Particularly our consideration is applicable for electroweak models with extended Higgs sector.

$\xi$ gauge is defined by adding the following gauge fixing term to lagrangian

$$L_{gf} = -\frac{1}{2\xi} (\partial^\mu A_\mu)^2.$$ (1)

The mixing of three bare propagators appears when taking into account the loop contributions:

- scalar particle propagator
  $$\pi_{11} = \frac{1}{p^2 - \mu^2},$$ (2)

- vector field propagator in $\xi$ gauge
  $$\pi_{22}^{\mu\nu} = \frac{1}{p^2 - M^2} \left\{ -g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2 - \xi M^2} (1 - \xi) \right\},$$ (3)

- and ghost propagator
  $$\pi_{33} = \frac{1}{p^2 - \xi M^2} .$$ (4)

$^2$This field has different names but we shall call it below just ghost.
It is useful to divide vector propagator into transversal and longitudinal parts

\[ \pi_{22}^{\mu\nu} = T^{\mu\nu} \cdot \frac{1}{p^2 - M^2} + L^{\mu\nu} \cdot \frac{\xi}{\xi M^2 - p^2}, \quad T^{\mu\nu} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}, \quad L^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}. \] (5)

We should mention again that only longitudinal part of vector propagator, which is not connected with spin J=1, can mix with scalar fields. Vector current, with which the vector field interacts, should not be conserving to acquire this effect. In \( \xi \) gauge the longitudinal part of propagator has the unphysical pole at the point \( p^2 = \xi M^2 \). However, it was noticed before (see, e.g., [3]) that contribution of this pole in matrix element has an opposite sign in comparison with scalar meson contribution. Thus, for its cancellation at tree level it is enough to add scalar field with propagator of the kind (4). At loop level there arises a mixing of above mention propagators and, moreover, the full non-diagonal propagators, which do not exist at tree level, appear. So, first of all, we should study the problem of renormalization of coupled propagators and S-matrix (in)dependence on the gauge parameter.

2 The Dyson-Schwinger equations system

In case of mixing the propagators and loops acquire matrix structure and the Dyson-Schwinger equation takes the form \(^3\).

\[
\Pi_{11} = \pi_{11} - \Pi_{11} J_{11} \pi_{11} - \Pi_{12} J_{21} \pi_{11} - \Pi_{13} J_{31} \pi_{11}
\]

\[
\Pi_{12} = -\Pi_{11} J_{12} \pi_{12} - \Pi_{12} J_{22} \pi_{12} - \Pi_{13} J_{32} \pi_{12}
\]

\[
\Pi_{13} = -\Pi_{11} J_{13} \pi_{33} - \Pi_{12} J_{23} \pi_{33} - \Pi_{13} J_{33} \pi_{33}
\]

\[
\Pi_{21} = -\Pi_{21} J_{11} \pi_{11} - \Pi_{22} J_{12} \pi_{11} - \Pi_{23} J_{13} \pi_{11}
\]

\[
\Pi_{22}^{\mu\nu} = \pi_{22}^{\mu\nu} - \Pi_{21}^{\mu} J_{12}^{\nu} \pi_{22} - \Pi_{22}^{\nu} J_{12}^{\mu} \pi_{22} - \Pi_{23}^{\mu} J_{32}^{\nu} \pi_{22} - \Pi_{23}^{\nu} J_{32}^{\mu} \pi_{22}
\]

\[
\Pi_{23}^{\mu} = -\Pi_{21}^{\mu} J_{13}^{\nu} \pi_{33} - \Pi_{22}^{\mu} J_{23}^{\nu} \pi_{33} - \Pi_{23}^{\mu} J_{33}^{\nu} \pi_{33}
\]

\[
\Pi_{31} = -\Pi_{31}^{\nu} J_{11}^{\mu} \pi_{11} - \Pi_{32}^{\nu} J_{21}^{\mu} \pi_{11} - \Pi_{33}^{\nu} J_{31}^{\mu} \pi_{11}
\]

\[
\Pi_{32}^{\nu} = -\Pi_{31}^{\nu} J_{12}^{\mu} \pi_{22} - \Pi_{32}^{\mu} J_{22}^{\nu} \pi_{22} - \Pi_{33}^{\nu} J_{32}^{\mu} \pi_{22}
\]

\[
\Pi_{33} = \pi_{33} - \Pi_{31} J_{13} \pi_{33} - \Pi_{32} J_{23} \pi_{33} - \Pi_{33} J_{33} \pi_{33}
\] (6)

Here \( \pi_{ij} \) are bare propagators, \( \Pi_{ij} \) are full propagators, \( J_{ij} \) are the one particle irreducible loop contributions. In this equations values with two indexes need to be divided into transversal and longitudinal parts.

\[
\Pi_{22}^{\mu\nu} = T^{\mu\nu} \Pi_{22}^T(p^2) + L^{\mu\nu} \Pi_{22}^L(p^2)
\]

\(^3\)As compared with consideration of unitary gauge [3] we redefine non-diagonal propagators scalar-vector \( i\Pi_{12} \rightarrow \Pi_{12}, \ i\Pi_{21} \rightarrow \Pi_{21} \) for more symmetry. Furthermore, we do not assume any symmetry for non-diagonal transitions in advance - the symmetry relations are the consequence of form of interaction and can vary. Note that such changing of equations is just the redefinition of loops.
\begin{align*}
\pi_{22}^{\mu\nu} &= T^{\mu\nu} \pi_{22}^T(p^2) + L^{\mu\nu} \pi_{22}^L(p^2), \\
J_{22}^{\mu\nu} &= T^{\mu\nu} J_{22}^T(p^2) + L^{\mu\nu} J_{22}^L(p^2),
\end{align*}

where

\[ T^{\mu\nu} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}, \quad L^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}. \]

In values with one index we shall come to scalar functions according to

\[ \Pi_{12}^\mu(p) = p^\mu \Pi_{12}(p^2), \quad \Pi_{21}^\mu(p) = p^\mu \Pi_{21}(p^2), \]
\[ J_{12}^\mu(p) = p^\mu J_{12}(p^2), \quad J_{21}^\mu(p) = p^\mu J_{21}(p^2), \]
\[ \Pi_{23}^\mu(p) = p^\mu \Pi_{23}(p^2), \quad \Pi_{32}^\mu(p) = p^\mu \Pi_{32}(p^2), \]
\[ J_{23}^\mu(p) = p^\mu J_{23}(p^2), \quad J_{32}^\mu(p) = p^\mu J_{32}(p^2). \]

One can see that equation for the transversal component is separates itself from the system and has the same form as in the absence of scalar-vector mixing.

\[ \Pi_{22}^T = \pi_{22}^T - \Pi_{22}^T J_{22}^T \pi_{22}^T, \]

the solution of which is

\[ \Pi_{22}^T = \frac{1}{p^2 - M^2 + J_{22}^T}. \]

As for longitudinal components we have the following system of equations in \( \xi \) gauge \footnote{Further we work with longitudinal components and drop index L except as otherwise noted.}

\[ \begin{align*}
\Pi_{11} &= \pi_{11} - \Pi_{11} J_{11} \pi_{11} - p^2 \Pi_{12} J_{21} \pi_{11} - \Pi_{13} J_{31} \pi_{11} \\
\Pi_{12} &= -\Pi_{11} J_{12} \pi_{22} - \Pi_{12} J_{22} \pi_{22} - \Pi_{13} J_{32} \pi_{22} \\
\Pi_{21} &= -\Pi_{21} J_{11} \pi_{11} - \Pi_{22} J_{21} \pi_{11} - \Pi_{23} J_{31} \pi_{11} \\
\Pi_{22} &= \pi_{22} - p^2 \Pi_{21} J_{12} \pi_{22} - \Pi_{22} J_{22} \pi_{22} - p^2 \Pi_{23} J_{32} \pi_{22} \\
\Pi_{13} &= -\Pi_{11} J_{13} \pi_{33} - p^2 \Pi_{12} J_{23} \pi_{33} - \Pi_{13} J_{33} \pi_{33} \\
\Pi_{31} &= -\Pi_{31} J_{11} \pi_{11} - p^2 \Pi_{32} J_{21} \pi_{11} - \Pi_{33} J_{31} \pi_{11} \\
\Pi_{23} &= -\Pi_{21} J_{13} \pi_{33} - \Pi_{22} J_{23} \pi_{33} - \Pi_{23} J_{33} \pi_{33} \\
\Pi_{32} &= -\Pi_{31} J_{12} \pi_{22} - \Pi_{32} J_{22} \pi_{22} - \Pi_{33} J_{32} \pi_{22} \\
\Pi_{33} &= \pi_{33} - \Pi_{31} J_{13} \pi_{33} - p^2 \Pi_{32} J_{23} \pi_{33} - \Pi_{33} J_{33} \pi_{33}. \end{align*} \]

The solution of the system is

\[ \begin{align*}
\Pi_{11} &= \frac{1}{D} \left[ (\pi_{22}^{-1} + J_{22})(\pi_{33}^{-1} + J_{33}) - sJ_{23} J_{32} \right] \\
\Pi_{12} &= -\frac{1}{D} \left[ J_{12}(\pi_{33}^{-1} + J_{33}) - J_{13} J_{32} \right].
\end{align*} \]
\[ \Pi_{21} = - \frac{1}{D} \left[ J_{21} (\pi_{33}^{-1} + J_{33}) - J_{23} J_{31} \right] \]
\[ \Pi_{22} = \frac{1}{D} \left[ (\pi_{11}^{-1} + J_{11})(\pi_{33}^{-1} + J_{33}) - J_{31} J_{13} \right] \]
\[ \Pi_{13} = - \frac{1}{D} \left[ J_{13} (\pi_{22}^{-1} + J_{22}) - s J_{12} J_{23} \right] \]
\[ \Pi_{31} = - \frac{1}{D} \left[ J_{31} (\pi_{22}^{-1} + J_{22}) - s J_{32} J_{21} \right] \]
\[ \Pi_{23} = - \frac{1}{D} \left[ J_{23} (\pi_{11}^{-1} + J_{11}) - J_{21} J_{13} \right] \]
\[ \Pi_{32} = - \frac{1}{D} \left[ J_{32} (\pi_{11}^{-1} + J_{11}) - J_{31} J_{12} \right] \]
\[ \Pi_{33} = \frac{1}{D} \left[ (\pi_{11}^{-1} + J_{11})(\pi_{22}^{-1} + J_{22}) - s J_{21} J_{12} \right] , \] (12)

where \( s = p^2 \) and
\[
D(s) = (\pi_{11}^{-1} + J_{11})(\pi_{22}^{-1} + J_{22})(\pi_{33}^{-1} + J_{33}) - (\pi_{11}^{-1} + J_{11}) s J_{32} J_{23} -
-(\pi_{22}^{-1} + J_{22}) J_{31} J_{13} - (\pi_{33}^{-1} + J_{33}) s J_{21} J_{12} + s J_{12} J_{23} J_{31} + s J_{32} J_{21} J_{13} . \] (13)

We should mention that transversal and longitudinal parts of vector propagator are not fully independent. The condition \( J_{22}^T(0) + J_{22}^L(0) = 0 \) is necessary for the matrix element not to have the pole \( 1/p^2 \).

3 Boson loops: \( \pi - a_1 \) system

Here we consider the same model that was studied in [3] in unitary gauge: \( \pi - a_1 \) system, which is dressed by the \( \pi\sigma \) intermediate state.

Feynman rules for the given model are
To cancel unphysical poles in matrix element at tree level it is necessary to fix the ghost coupling constant:

\[ g_{G\pi\sigma} = \frac{g_{a\pi\sigma}}{M} \]  

(14)

Let us write down the result of calculations of loop contribution. Note that after determining of Feynman rules the loops should be agreed upon Dyson-Schwinger equations. We prefer to calculate loops using Landau-Cutkosky rules.

\[
J_{11}(p^2) = -i \frac{g_{\sigma\pi\pi}^2}{(2\pi)^4} \int \frac{d^4l}{(l^2 - \mu^2)((l - p)^2 - m^2)} \frac{1}{(l^2 - \mu^2)((l - p)^2 - m^2)},
\]

\[ J_{12}^\mu(p) = -g_{a_1\pi\sigma} g_{\sigma\pi\pi} \int \frac{d^4l}{(2\pi)^4} \frac{(2l - p)^\mu}{(l^2 - \mu^2)((l - p)^2 - m^2)}, \]

\[ J_{13}(p^2) = -i \frac{g_{\pi\pi\sigma} g_{a_1\pi\sigma}}{M} \int \frac{d^4l}{(2\pi)^4} \frac{(p \cdot (2l - p))}{(l^2 - \mu^2)((l - p)^2 - m^2)}, \]

\[ J_{22}^\mu(p) = -i \frac{g_{a_1\pi\sigma}^2}{(2\pi)^4} \frac{(2l - p)^\mu (2l - p)^\nu}{(l^2 - \mu^2)((l - p)^2 - m^2)}, \]

\[ J_{23}^\mu(p) = \frac{g_{a_1\pi\sigma}^2}{M} \int \frac{d^4l}{(2\pi)^4} \frac{(p \cdot (2l - p))(p \cdot (2l - p))}{(l^2 - \mu^2)((l - p)^2 - m^2)}, \]

\[ J_{33}(p^2) = -i \frac{g_{a_1\pi\sigma}^2}{M^2} \int \frac{d^4l}{(2\pi)^4} \frac{(p \cdot (2l - p))(p \cdot (2l - p))}{(l^2 - \mu^2)((l - p)^2 - m^2)}. \]  

(15)

Here \( m = m_\sigma, \mu = m_\pi \). Feynman rules lead to symmetry relations for non-diagonal loops.

\[ J_{21}^\mu = -J_{12}^\mu, \]

\[ J_{32}^\mu = -J_{23}^\mu, \]

\[ J_{31} = J_{13}. \]  

(16)

All loops are expressed in terms of one function \( H(p^2) \) with some subtractive polynomials which must be defined at renormalization.  

\[ H(p^2) = \frac{1}{\pi} \int \frac{ds}{s(s - p^2)} \left( \frac{\lambda(s, m^2, \mu^2)}{s^2} \right)^{1/2} \]  

(17)

\[ J_{11}(s) = g_1^2 [P_{11} + sH(s)] \]

\[ J_{12}(s) = -ig_1g_2 [P_{12} + H(s)] \]

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5Here and below we do indicate limits of integration in dispersion integrals: they are from threshold to infinity.
\[ J_{13}(s) = \frac{g_1 g_2}{M} [P_{13} + sH(s)] \]
\[ J_{22}(s) = g_2^2 [P_{22} + H(s)] \]
\[ J_{23}(s) = i \frac{g_2^2}{M} [P_{23} + H(s)] \]
\[ J_{33}(s) = \frac{g_2^2}{M^2} [P_{33} + sH(s)] \]

where \( P_{ij} \) are polynomials by \( s \) with real coefficients. We introduced notations: 
\[ g_1 = \frac{g_{\sigma\pi\pi}}{\sqrt{16\pi}}, \quad g_2 = (\mu^2 - m^2) g_{a_1\pi\pi}/\sqrt{16\pi}, \quad \lambda(a, b, c) = (a - b - c)^2 - 4bc. \]

The matrix element \( \pi \sigma \to \pi \sigma \) with full propagators has the form:
\[ \frac{1}{16\pi} M^I = -g_1^2 \Pi_{11} - 2i g_1 g_2 \Pi_{12} - 2i \frac{g_1}{M} g_2 \Pi_{13} + 2i \frac{g_2^2}{M} \Pi_{23} - \frac{g_2^2}{s} \Pi_{22} - \frac{g_2^2}{M^2} \Pi_{33}. \]

**Renormalization of pion pole**

We will use the renormalization scheme with subtraction on mass shell. It is clear that procedure is more complicated due to mixing of propagators. Requirements for renormalization of pion pole can be formulated in the most simple way:

- Function \( D(s) \) has a simple zero at the point \( s = \mu^2 \) at any values of coupling constants which are supposed to be independent.

- Full pion propagator \( \Pi_{11} \) has pole with unit residue like the bare \( \pi_{11} \). It means that the sum of all loop insertion to external pion line is equal to zero.

These requirements lead to conditions on loops at the point \( s = \mu^2 \) i.e. on subtractive polynomials.

\[ J_{11}(\mu^2) = J'_{11}(\mu^2) = 0, \]
\[ J_{12}(\mu^2) = 0, \]
\[ J_{13}(\mu^2) = 0. \]

**Renormalization of \( \xi \)**

As far as the mass of the vector particle is renormalized in transversal part of vector propagator we can consider \( M \) as renormalized mass. Thus, renormalization of unphysical pole at the point \( s = \xi M^2 \) is renormalization of gauge parameter \( \xi \).

Let us try to act by the analogy with pion pole and to formulate renormalization requirements in a following way:

\[ \text{(20)} \]

\(^6\)These requirements are minimal and mean that after the dressing propagators have the same type of singularity. Since ghost appears only in propagators but not as external lines we do not set any requirements to residues of propagators.
• Function $D(s)$ has a zero of second order at the point $s = \xi M^2$ at any values of coupling constants.

• Full propagators $\Pi_{22}$ and $\Pi_{33}$ have simple pole at this point.

These requirements with usage of solutions (12) will give the following conditions

$$J_{22}(\xi M^2) = J_{33}(\xi M^2) = J_{12}(\xi M^2) = J_{13}(\xi M^2) = J_{23}(\xi M^2) = 0.$$  \hspace{1cm} (21)

It is easy to see that among full propagators, besides $\Pi_{22}$ and $\Pi_{33}$, only $\Pi_{23}$ can have the pole at the point $s = \xi M^2$. Therefore, it is enough to write down only these contributions to trace the unphysical pole in the matrix element.

$$\frac{1}{16\pi} \hat{M}^{T=0} = -\frac{g_2^2}{p^2} \Pi_{22}(p^2) - \frac{g_2^2}{M^2} \Pi_{33}(p^2) + 2i \frac{g_2^2}{M} \Pi_{23}(p^2).$$ \hspace{1cm} (22)

Using solutions of Dyson-Schwinger equations and requirements (21) we can find the necessary condition for absence of unphysical pole in matrix element: the function $Y(s)$

$$Y(s) = M^2 J_{33}(s) + s J_{22}(s) + 2i Ms J_{23}(s)$$ \hspace{1cm} (23)

have a second order zero at the point $\xi M^2$. This is condition on subtractive polynomials since as we can see from (18) loops function $H(p^2)$ is cancelled in (23).

Let us recall that the absence of pole $1/p^2$ in matrix element relates $J_{22}^T$ and $J_{22}^L$ is

$$J_{22}^T(0) + J_{22}^L(0) = 0.$$ \hspace{1cm} (24)

With accounting (24) polynomial in the loop $J_{22}^L$ must have the following form

$$P_{22} = E \left(1 - \frac{s}{\xi M^2}\right) - \frac{s}{\xi M^2} H(\xi M^2),$$ \hspace{1cm} (25)

where $E$ is some fixed constant which is defined in the transversal part of loop $J_{22}$. Now it is possible to write out renormalized loops satisfying the condition (23). Now it is possible to write out renormalized loops satisfying the condition (23).

$$J_{22} = \left[E \left(1 - \frac{s}{\xi M^2}\right) - \frac{s}{\xi M^2} H(\xi M^2) + H(s)\right],$$

$$J_{23} = i \frac{g_2^2}{M} \left[H(\xi M^2) - H(s)\right],$$

$$J_{33} = \frac{g_2^2}{M^2} \left[-\xi M^2 H(\xi M^2) - E \xi M^2 \left(1 - \frac{s}{\xi M^2}\right) + s H(s)\right].$$ \hspace{1cm} (26)
The rest of loops which do not take part in function $Y(s)$ (23):

\[
J_{11} = g_1^2 \left[ -sH(\mu^2) - \mu^2 H'(\mu^2)(s - \mu^2) + sH(s) \right],
\]

\[
J_{12} = -ig_1g_2 \left[ \frac{\xi M^2 H(\mu^2) - \mu^2 H(\mu^2)}{\mu^2 - \xi M^2} + s \frac{H(\xi M^2) - H(\mu^2)}{\mu^2 - \xi M^2} + H(s) \right],
\]

\[
J_{13} = g_1g_2M \left[ \frac{\mu^2 \xi M^2 H(\mu^2) - \mu^2 H(\mu^2)}{\mu^2 - \xi M^2} - \frac{\xi M^2 H(\xi M^2) - \mu^2 H(\mu^2)}{\mu^2 - \xi M^2} - sH(s) \right].
\] (27)

Now, after we defined subtractive polynomials in the loops, we can calculate matrix element. Substituting full propagators we obtain cumbersome expression which has evident dependence on gauge parameter $\xi$. So we can conclude that renormalization of unphysical pole by analogy with physical can not be done and must be realized in other way.

Ward identity.

The key feature in renormalization is the usage of Ward identity, which relates some different Green functions. It is obtained in [3] and has the form

\[
\langle 0 | T(\partial_\mu A^\mu(x) - \xi M \varphi(x))(\partial_\nu A^\nu(y) - \xi M \varphi(y)) | 0 \rangle = 0,
\] (28)

where $\varphi(x)$ is ghost field.

Let us recall that Ward identity (28) was obtained in [3] in the simpler case with the help of BRST transformation. For any case we will obtain it in a different way.

The above mentioned Feynman rules correspond to the following lagrangian $^7$

\[
\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}M^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial A)^2 + \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}\xi M^2 \varphi^2 + A_\mu J^\mu + \frac{1}{M} \partial_\mu \varphi J^\mu.
\] (29)

Here $J^\mu$ is vector current, we will not concretize it. We wrote here only terms with vector and ghost fields. Motion equations have the form

\[
(\partial_\alpha \partial^\alpha + M^2)A_\mu - (1 - \frac{1}{\xi})\partial_\mu (\partial A) = -J_\mu
\] (30)

\[
(\partial_\alpha \partial^\alpha + \xi M^2)\varphi = -\frac{1}{M} (\partial J).
\] (31)

Consequence of these equations is

\[
(\partial_\alpha \partial^\alpha + \xi M^2)((\partial A) - \xi M \varphi) = 0.
\] (32)

$^7$We did not write here isotopic indexes because they were trivial for our model.
It means that appeared combination of fields \( (\partial A) - \xi M \varphi \) is non-interacting field. So the two point Green function of this combination should not change under interactions. For the case of bare propagators we have following expression

\[
\langle 0 | T \{((\partial A(x)) - \xi M \varphi(x))((\partial A(y)) - \xi M \varphi(y))\} | 0 \rangle = 0. \tag{33}
\]

To obtain it we must accurately differentiate T-product of vector fields. In this procedure there appear additional terms proportional to simultaneous commutators of interacting fields. However, it is well known that simultaneous commutative relations of interacting fields are coincided with the same for free fields, see e.g. citeBS. After some calculations we have

\[
\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \langle T\{A_\mu(x)A_\nu(y)\}\rangle_0 = \langle T\{\partial A(x)\partial A(y)\}\rangle_0 - i\xi \delta^4(x - y). \tag{34}
\]

As a result, we come to Ward identity in terms of full propagators

\[
s\Pi_{22}^L(s) - 2is\xi M \Pi_{23}(s) + \xi^2 M^2 \Pi_{33}(s) + \xi = 0. \tag{35}
\]

When substituting explicit form of full propagators \( \text{(12)} \) into the Ward identity \( \text{(35)} \) we shall get some relations for the loop contributions. Note that we consider the coupling constants \( g_1, g_2 \) as independent and \( \text{(35)} \) gives few conditions for loops.

\[
M^2 J_{33} + s J_{22} + 2isM J_{23} = 0,
\]

\[
J_{22} J_{33} + s(J_{23})^2 = 0,
\]

\[
2isMJ_{12} J_{13} - M^2 J_{13}^2 + s^2 J_{12}^2 = 0,
\]

\[
-J_{22} J_{13}^2 + sj_{33} J_{12}^2 + 2sj_{12} J_{13} J_{23} = 0. \tag{36}
\]

Having resolved this equations we get the simple relations between loops

\[
J_{33} = \frac{s}{M^2} J_{22},
\]

\[
J_{23} = \frac{i}{M} J_{22},
\]

\[
J_{13} = \frac{i}{M} J_{12}. \tag{37}
\]

We notice that the same function \( Y(s) \) \( \text{(23)} \) appears in \( \text{(36)} \) which guarantees the absence of unphysical pole in the matrix element.

Now we calculate the full propagators using relation \( \text{(36)} \) following from Ward identity. One can see at once that in the function \( D(s) \) dependence on gauge parameter is factorized. \footnote{Note that there exists a disagreement on exact form of this relation. In particular in \( \text{[8]} \) it was written without last term.}

\footnote{It is useful to express all loops via \( J_{22} \) because it has one more restriction from condition \( \text{(24)} \).}

\footnote{We note that usually under dressing the pole lying above threshold shifts to complex plane. But in our case under Ward identities the pole stays at real axis.}
\[ D(s) = -\frac{(s - \xi M^2)^2}{\xi M^2} \hat{D}(s), \]  

(38)

where appears function

\[ \hat{D}(s) = (\pi^{-1}_{11} + J_{11})(M^2 + J_{22}) + s(J_{12})^2, \]  

(39)

playing the same role in unitary gauge [3].

The full propagators acquire very simple form

\[
\begin{align*}
\Pi_{11} &= \frac{M^2 + J_{22}}{\hat{D}} \\
\Pi_{12} &= \frac{\xi M^2 J_{12}}{(s - \xi M^2)\hat{D}} \\
\Pi_{13} &= -i\frac{M \xi s J_{12}}{(s - \xi M^2)\hat{D}} \\
\Pi_{23} &= i\frac{\xi M\left[(\pi^{-1}_{11} + J_{11})J_{22} + s(J_{12}^2)\right]}{(s - \xi M^2)^2\hat{D}} \\
\Pi_{22} &= -\xi\frac{\left[(\pi^{-1}_{11} + J_{11})(M^2(s - \xi M^2) + sJ_{22}) + s^2(J_{12})^2\right]}{(s - \xi M^2)^2\hat{D}} \\
\Pi_{33} &= M^2\frac{\left[(\pi^{-1}_{11} + J_{11})((s - \xi M^2 - \xi J_{22}) - \xi s(J_{12})^2\right]}{(s - \xi M^2)^2\hat{D}}.
\end{align*}
\]  

(40)

When we substitute the full propagators into the matrix element \( \pi\sigma \rightarrow \pi\sigma \) (19) we shall find

\[ \frac{1}{16\pi} \mathcal{M}^{t=0} = -g_1^2\frac{(M^2 + J_{22})}{\hat{D}} + 2ig_1g_2\frac{J_{12}}{\hat{D}} - g_2^2\frac{(\pi^{-1}_{11} + J_{11})}{\hat{D}}. \]  

(41)

We see that dependence on gauge parameter \( \xi \) has disappeared and this expression coincides with matrix element in unitary gauge if there are no conditions on loops \( J_{11}, J_{12}, J_{22} \) at the point \( s = \xi M^2 \).

Finally if we group the terms in the matrix element (19) in the following way

\[ \frac{1}{16\pi} \mathcal{M}^{t=0} = -g_1^2\Pi_{11} - 2ig_1g_2(\Pi_{12} - \frac{i}{M}\Pi_{13}) - g_2^2(\frac{1}{s}\Pi_{22} + \frac{1}{M^2}\Pi_{33} - \frac{2i}{M}\Pi_{23}), \]  

(42)

we find that not only sum but each of this three addends do not depend on gauge parameter \( \xi \).

4 Fermion loops: W,Z – Higgs mixing in extended electroweak models

Unitary mixing between gauge bosons and Higgs particles is possible only in the extended electroweak models since pseudoscalar or charged Higgs particles are required for that. In
Standard Model where exists only one scalar Higgs this effect is absent. We do not define concretely the model but just fix form of vertex.

Mixing $W^\pm$ - scalar Higgs.

Interaction vertexes have the form

$$J_{\mu
u} = \int \frac{d^4l}{(2\pi)^4} Sp \left\{ \gamma^\mu (1 + \gamma^5) \frac{1}{l - p - m_2} \gamma^\nu (1 + \gamma^5) \frac{1}{l - m_1} \right\}.$$ (43)

Loops take the form

$$J_{11}(p^2) = -ig_1^2 \int \frac{d^4l}{(2\pi)^4} Sp \left\{ I \frac{1}{l - p - m_2} I \frac{1}{l - m_1} \right\}$$

$$J_{12}^\mu(p) = -ig_1g_2 \int \frac{d^4l}{(2\pi)^4} Sp \left\{ I \frac{1}{l - p - m_2} \gamma^\mu (1 + \gamma^5) \frac{1}{l - m_1} \right\}$$

$$J_{13}(p^2) = \frac{g_1g_2}{M} \int \frac{d^4l}{(2\pi)^4} Sp \left\{ I \frac{1}{l - p - m_2} \hat{p} (1 + \gamma^5) \frac{1}{l - m_1} \right\}$$

$$J_{22}^\mu(p) = -ig_2^2 \int \frac{d^4l}{(2\pi)^4} Sp \left\{ \gamma^\mu (1 + \gamma^5) \frac{1}{l - p - m_2} \gamma^\nu (1 + \gamma^5) \frac{1}{l - m_1} \right\}$$

$$J_{23}^\mu(p) = \frac{g_2^2}{M} \int \frac{d^4l}{(2\pi)^4} Sp \left\{ \hat{p} (1 + \gamma^5) \frac{1}{l - p - m_2} \hat{p} (1 + \gamma^5) \frac{1}{l - m_1} \right\}.$$ (43)

Symmetry properties become rather different as compared with boson loops.

$$J_{21}^\mu(p) = J_{12}^\mu(p)$$
$$J_{31}(p) = -J_{13}(p)$$
$$J_{32}^\mu(p) = -J_{23}^\mu(p)$$ (44)
In the fermion case all longitudinal loops are expressed in terms of the two functions $H_1(p^2)$, $H_2(p^2)$ with some subtractive polynomials.

\[
H_1(p^2) = \frac{1}{\pi} \int \frac{(m_1 + m_2)^2 - s}{s(s - p^2)} \left( \lambda(s, m^2, \mu^2) \right)^{1/2} ds
\]
\[
H_2(p^2) = \frac{p^2}{\pi} \int \frac{(m_1 - m_2)^2 - s(m_1^2 + m_2^2)}{s^2(s - p^2)} \left( \lambda(s, m^2, \mu^2) \right)^{1/2} ds
\]

\[
J_{11} = f_1^2 [P_{11} + sH_1(s)] , \quad J_{12} = f_1f_2 [P_{12} + H_1(s)]
\]
\[
J_{13} = i\frac{f_1f_2}{M} [P_{13} + sH_1(s)] , \quad J_{22} = \hat{f}_2^2 [P_{22} + H_2(s)]
\]
\[
J_{23} = i\frac{\hat{f}_2^2}{M} [P_{23} + H_2(s)] , \quad J_{33} = \hat{f}_2^2 [P_{33} + sH_2(s)]
\]

(45)

where $P_{ij}$ are polynomials by $s$ with real coefficients. Notations are $f_1 = g_1/\sqrt{8\pi}$, $f_2 = (m_1^2 - m_2^2)g_2/\sqrt{8\pi}$, $\hat{f}_2 = g_2/\sqrt{4\pi}$.

If to look at relation (37) following from Ward identity it is easy to see that Ward identity puts constraints only on subtractive polynomials, as for loop integrals $H_1$, $H_2$ they identically satisfy (37). If these relations are satisfied we obtain simple $\xi$ dependence on propagators and we only need to trace the $\xi$ dependence in the matrix element.

Matrix element $f_1(q_1)\overline{f}_2(q_2) \rightarrow f_1(k_1)\overline{f}_2(k_2)$ have form

\[
\mathcal{M}^{f=0} = -g_1^2 \Pi_{11} \overline{\nu}(q_2)u(q_1) \cdot \overline{\nu}(k_1)v(k_2) - \\
-g_1g_2 \left( \Pi_{12} - i\frac{1}{M}\Pi_{13} \right) \overline{\nu}(q_2)u(q_1) \cdot \overline{\nu}(k_1)\not{p}(1 + \gamma^5)v(k_2) - \\
-g_1g_2 \left( \Pi_{21} + \frac{i}{M}\Pi_{31} \right) \overline{\nu}(q_2)\not{p}(1 + \gamma^5)u(q_1) \cdot \overline{\nu}(k_1)v(k_2) - \\
-g_2^2 \left( \frac{i}{s}\Pi_{22} + \frac{1}{M^2}\Pi_{33} - i\frac{1}{M}\Pi_{23} + i\frac{1}{M}\Pi_{32} \right) \overline{\nu}(q_2)\not{p}(1 + \gamma^5)u(q_1) \cdot \overline{\nu}(k_1)\not{p}(1 + \gamma^5)v(k_2).
\]

(46)

It is possible to simplify this expression using motion equation for spinors but it is clear that different spinor matrix elements in (47) are accompanied by $\xi$ independent factors (see (42)). Thus, the dependence on gauge parameter in matrix element disappears.

**Mixing $W^\pm(Z^0) -$ pseudoscalar Higgs**

Vertexes of interaction Higgs with fermions have form

\[
\begin{array}{c}
\text{---} \quad m_1 \\
\text{---} \quad m_2 \\
\text{---} \quad \text{---}
\end{array}
\quad \begin{array}{c}
\nu \quad \text{---} \\
\gamma^5 \quad \text{---} \\
\text{---} \quad \nu
\end{array}
\quad \begin{array}{c}
\nu \quad \text{---} \\
\gamma^5 \quad \text{---} \\
\text{---} \quad \nu
\end{array}
\quad \begin{array}{c}
\text{---} \quad m_1 \\
\text{---} \quad m_2 \\
\text{---} \quad \text{---}
\end{array}
\]

The matrix element $f_1(q_1)f_2(q_2) \rightarrow f_1(k_1)f_2(k_2)$ slightly changes

\[
\mathcal{M}^{f=0} = -g_1^2 \Pi_{11} \overline{\nu}(q_2)\gamma^5u(q_1) \cdot \overline{\nu}(k_1)\gamma^5v(k_2) - \\
\]
Some loops have changed in comparison with scalar Higgs.

\[ -g_1 g_2 \left( \Pi_{12} - \frac{i}{M} \Pi_{13} \right) \overline{\psi}(q_2) \gamma^5 u(q_1) \cdot \overline{\psi}(k_1) \hat{\rho}(1 + \gamma^5) v(k_2) - \]

\[ -g_1 g_2 \left( \Pi_{21} + \frac{i}{M} \Pi_{31} \right) \overline{\psi}(q_2) \hat{\rho}(1 + \gamma^5) u(q_1) \cdot \overline{\psi}(k_1) \gamma^5 v(k_2) - \]

\[ -g_2^2 \left( \frac{1}{s} \Pi_{22} + \frac{1}{M^2} \Pi_{33} - \frac{i}{M} \Pi_{23} + \frac{i}{M} \Pi_{32} \right) \overline{\psi}(q_2) \hat{\rho}(1 + \gamma^5) u(q_1) \cdot \overline{\psi}(k_1) \hat{\rho}(1 + \gamma^5) v(k_2). \]  

(47)

5 Summery

We investigated the effect of unitary mixing scalar-vector in general $\xi$ gauge and found that under usage of Ward identity the renormalized matrix element does not depend on gauge
parameter. The interesting feature noted in [5] in simpler case consist in changing of singularity type after dressing. Simple pole $1/(p^2 - \xi M^2)$ in bare propagators after dressing turns into double pole. Such possibility always exists in mixing of two bare propagators with same masses but it is realized only at definite relations between loops which follow from the Ward identity.

In [3] the boson loop contributions were calculated and it was found that position of double unphysical pole is diverged under the usage of Ward identity. It is resulted in the opinion that Standard Model [11] is not renormalizable in $\xi$ gauge.

We see from above that Ward identity leads to simple relations between loops (37) and if these relations are fullfiled the position of double pole is ultraviolet stable. So we can suppose that in calculations of [5] the obtained loops do not satisfy the Ward identity although this identity is used in general form.

After our investigation it seems that the usage of $\xi$ gauge for extended Higgs model is not convinient. But this gauge(and its particulary cases) is widely spread in investigations of electroweak models. Particulary it is possible to control the correctness of calculations varying $\xi$ and tracing the variation in the matrix element.

The physical consequences of unitary mixing ”scalar—vector” in extended electroweak models deserves further investigation.

We are intended to V.V.Lyubushkin for verification of some formulae.

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\[11\]In [5] it was investigated mixing ”longitudinal part W — ghost” in the Standard Model, that is partial case of our consideration.