Decomposition theorem on matchable distributive lattices*

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Abstract

A distributive lattice structure $M(G)$ has been established on the set of perfect matchings of a plane bipartite graph $G$. We call a lattice \textit{matchable distributive lattice} (simply MDL) if it is isomorphic to such a distributive lattice. It is natural to ask which lattices are MDLs. We show that if a plane bipartite graph $G$ is elementary, then $M(G)$ is irreducible. Based on this result, a decomposition theorem on MDLs is obtained: a finite distributive lattice $L$ is an MDL if and only if each factor in any cartesian product decomposition of $L$ is an MDL. Two types of MDLs are presented: $J(m \times n)$ and $J(T)$, where $m \times n$ denotes the cartesian product between $m$-element chain and $n$-element chain, and $T$ is a poset implied by any orientation of a tree.

\textbf{Key words:} Perfect matching, Plane bipartite graph, $Z$-transformation graph, Distributive lattice, Decomposition theorem.

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1 Introduction

Perfect matching of graphs is significant for theoretical chemistry and theoretical physics. This graph-theoretical concept coincides with that of the Kekulé structure of organic molecules. The Kekulé structure count can be used to predict the stability of benzenoid hydrocarbons. The carbon-skeleton of a benzenoid hydrocarbon is a hexagonal system, i.e. 2-connected plane graph every interior face of which is a regular hexagon of side length

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unit. Since 1980’s there have been developed a combinatorial object, the $Z$-transformation graph (or resonance graph) \cite{23, 24} on the set perfect matchings of a hexagonal system, late extended to a general plane bipartite graph \cite{27, 33, 34, 35}; see a recent survey \cite{29}. Randić \cite{15, 16} showed that the leading eigenvalue of the resonance graphs has a quite satisfactory correlation with the resonance energy of benzenoid hydrocarbons.

A domino tiling of a polygon in the plane corresponds to a perfect matching of a related graph. In theoretical physics, a domino is seen as a dimer, a diatomic molecule (as the molecule of hydrogen), and each tiling is seen as a possible state of a solid or a fluid. In 2003 Fournier \cite{5} reintroduced $Z$-transformation graph under name “perfect matching graph” in investigating domino tiling spaces of Saldanha et al. \cite{19}. E. Rémiña \cite{17, 18} established the distributive lattice structure on the set of domino tilings of a polygon by using Thurston’s height function. In general, a distributive lattice on the set of perfect matchings of a plane bipartite graph was presented in terms of $Z$-transformation digraph and the unit decomposition of alternating cycle systems with respect to a perfect matching \cite{10}.

Let $G$ be a finite and simple graph with vertex-set $V(G)$ and edge-set $E(G)$. A perfect matching or $1$-factor of $G$ is a set of independent edges which saturate all vertices of $G$. Let $\mathcal{M}(G)$ denote the set of $1$-factors of $G$. A plane bipartite graph $G$ is elementary \cite{11} if $G$ is connected and every edge is contained in some $1$-factor; further weakly elementary \cite{34, 20} if every alternating cycle with respect to some $1$-factor together with its interior form an elementary subgraph.

For a plane bipartite graph $G$, the $Z$-transformation graph $Z(G)$ is defined as a graph on $\mathcal{M}(G)$: $M, M' \in \mathcal{M}(G)$ are joined by an edge if and only if they differ only in one cycle that is the boundary of an inner face of $G$.

To give an acyclic orientation of $Z(G)$ \cite{33}, a proper 2-coloring (white-black) of bipartite graph $G$ is specified. For $M \in \mathcal{M}(G)$, a cycle $C$ is said to be $M$-alternating if the edges of $C$ appear alternately in and off $M$; further proper (improper) \cite{32} if every edge of $C$ belonging to $M$ goes from white (black) end-vertex to black (white) end-vertex along the clockwise orientation of $C$. Now $Z$-transformation digraph $\vec{Z}(G)$ is the orientation of $Z(G)$: an edge $M_1M_2$ of $Z(G)$ is oriented from $M_1$ to $M_2$ if the symmetric difference $M_1 \oplus M_2$ form a proper $M_1$- and improper $M_2$-alternating cycle (the boundary of an inner face).

Since $\vec{Z}(G)$ has no directed cycles \cite{33}, it naturally implies a partial ordering on $\mathcal{M}(G)$. This poset is denoted by $\mathcal{M}(G)$. Then its Hasse diagram is isomorphic to $\vec{Z}(G)$. Lam and Zhang \cite{10} showed that $\mathcal{M}(G)$ is a finite distributive lattice (FDL) if $G$ is weakly elementary. Further the first author of the present paper showed \cite{28} that $\mathcal{M}(G)$ is direct
sum of at least two distributive lattices if $G$ is non-weakly elementary. By applying such a lattice structure, Zhang et al. showed [30] that every connected resonance graph of plane bipartite graphs is a median graph, and extended Klavžar et al.’s result [9] in the case of cata-condensed benzenoid systems.

In different ways, Propp [14] established a distributive lattice structure on the set of $c$-orientations of a plane bipartite graph $G$; Pretzel [13] provided a new proof to Propp’s result. Similar structures were also given on the set of reachable configurations of an edge firing game [12], $\alpha$-orientations of a planar graph [4], and flows of a planar graph [7].

In this paper we propose a problem: which distributive lattices are isomorphic to distributive lattice $M(G)$ on the set of 1-factors of a plane bipartite graphs $G$? A lattice is called matchable distributive lattice (simply MDL) if it is isomorphic to such a distributive lattice $M(G)$. Non-matchable distributive lattices exist. We show that if a plane bipartite graph $G$ is elementary, then $M(G)$ is irreducible. Based on this result, a decomposition theorem on MDL is obtained (Theorem 3.8): a finite distributive lattice $L$ is an MDL if and only if each factor in any cartesian product decomposition of $L$ is an MDL. Finally, we present two types of irreducible MDLs by applying the fundamental theorem for finite distributive lattices (FTFDL): $J(m \times n)$ and $J(T)$, where $m \times n$ denotes the cartesian product between $m$-element chain and $n$-element chain, and $T$ is a poset implied by any orientation of a tree. Meantime, we also show that for any order ideal $W$ of $m \times n$, $J(W)$ is an MDL.

2 Preliminaries

Terms on poset and distributive lattice used in this paper can be found in [1, 6, 21]. If $P$ and $Q$ are posets, then the direct (cartesian) product of $P$ and $Q$ is the poset $P \times Q$ on the set $\{(x, y) : x \in P$ and $y \in Q\}$ such that $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq x'$ in $P$ and $y \leq y'$ in $Q$.

Let $L$ be an FDL with the greatest element $\hat{1}$ and the least element $\hat{0}$. If $L$ can be expressed as the direct product of a series of FDLs $L_j (j \in J)$, i.e. $L = \prod_{j \in J} L_j$, then we say that $L$ has a (direct product) decomposition $\prod_{j \in J} L_j$. A lattice with exactly one element is viewed as a trivial lattice. An FDL is irreducible if it cannot be expressed as direct product of at least two non-trivial FDLs. A decomposition $L = \prod_{j \in J} L_j$ is called irreducible if each $L_j (j \in J)$ is non-trivial and irreducible.

For a decomposition $L = \prod_{i=1}^n L_i$, let $\hat{1}_i$ and $\hat{0}_i$ denote the greatest element and the least element of $L_i$, respectively. Then $\hat{0} = (\hat{0}_1, \hat{0}_2, \cdots, \hat{0}_n)$ and $\hat{1} = (\hat{1}_1, \hat{1}_2, \cdots, \hat{1}_n)$. If each $L_i$
is non-trivial and \( n \geq 2 \), \( e_i = (\hat{0}_1, \cdots, \hat{0}_{i-1}, \hat{1}_i, \hat{0}_{i+1}, \cdots, \hat{0}_n) \) is called a central element of \( L \). For \( x, y \in L \), \( x \) is called a complement of \( y \) if \( x \lor y = \hat{1} \) and \( x \land y = \hat{0} \). The complement of \( x \), when it exists, is unique. For example, two central elements \((2,1)\) and \((1,3)\) of \( L = 2 \times 3 \) are complementary each other (see Fig. 1). For a positive integer \( n \), \( \{1, 2, \ldots, n\} \) with its usual order forms an \( n \)-element chain, denoted by \( n \).

**Lemma 2.1.** \([1]\) Any central element of an FDL has a unique complement.

**Lemma 2.2.** \([1]\) Any FDL has a unique irreducible decomposition, i.e. if both \( \prod_{i=1}^{n} L_i \) and \( \prod_{j=1}^{m} L'_j \) are irreducible decompositions of \( L \), then \( m = n \) and there exists a permutation \( \pi \) of \([n]\) such that \( L_i = L'_{\pi(i)}(i = 1, 2, \cdots, n) \).

For an FDL \( L \), its rank function \([21]\) satisfies

\[
\rho(x) + \rho(y) = \rho(x \land y) + \rho(x \lor y),
\]

for any \( x, y \in L \). For a pair of complementary elements \( x \) and \( y \) of \( L \), we have

\[
\rho(x) + \rho(y) = \rho(\hat{0}) + \rho(\hat{1}) = \rho(\hat{1}) = \rho(L).
\]

**Lemma 2.3.** Let \( L \) be an FDL of rank \( k \) and \( y \) the complement of \( x \in L \). If \( \rho(x) = r \geq 1 \) and \( \rho(y) = k - r \geq 1 \), then \( L \) has a sublattice \((r + 1) \times (k - r + 1)\) containing \( x \) and \( y \).

**Proof.** \( L \) has at least two saturated chains between \( \hat{0} \) with \( x \) and \( y \), respectively:

\[
P_1 : \hat{0} = x_0 \ll x_1 \ll x_2 \ll \cdots \ll x_r = x,
\]

and

\[
P_2 : \hat{0} = y_0 \ll y_1 \ll y_2 \ll \cdots \ll y_{k-r} = y.
\]

Then \( x_i \land y_j = \hat{0} \), for any \( 0 \leq i \leq r, 0 \leq j \leq k - r \), since \( x_i \land y_j \leq x \land y = \hat{0} \). Hence \( P_1 \) and \( P_2 \) have no common elements except for \( \hat{0} \).
Let $L' = \{a_{ij} : a_{ij} = x_i \lor y_j, 0 \leq i \leq r, 0 \leq j \leq k - r\}$. Then $L'$ satisfies the following three properties:

1. $a_{ij} = a_{i'j'}$ if and only if $i = i'$ and $j = j'$. If $a_{i'j'} \leq a_{ij}$, i.e. $x_i \lor y_{j'} \leq x_i \lor y_j$, then $(x_i \lor y_{j'}) \land y_{j'} \leq (x_i \lor y_j) \land y_{j'}$. By distributive laws and $x_i \land y_j = 0$, we have that $y_{j'} \leq y_j \land y_{j'}$ and $j' \leq j$. Similarly we have $i' \leq i$. So the property holds.

2. $L'$ forms a sublattice of $L$ and $L' = \langle x_1, \ldots, x_r, y_1, \ldots, y_{k-r} ; \land, \lor \rangle$. It suffices to show that $L'$ is closed under meet and join operations $\land$ and $\lor$ of $L$.

   (i) $a_{ij} \lor a_{i'j'} = (x_i \lor y_j) \lor (x_{i'} \lor y_{j'}) = (x_i \lor x_{i'}) \lor (y_j \lor y_{j'}) = x_{i''} \lor y_{j''} = a_{i''j''} \in L'$, where $i'' = \max\{i, i'\}, j'' = \max\{j, j'\}$;

   (ii) $a_{ij} \land a_{i'j'} = a_{ij} \land (x_i \lor y_{j'}) = (a_{ij} \land x_i) \lor (a_{ij} \land y_{j'})$

   $= ((x_i \lor y_j) \land x_i) \lor ((x_i \lor y_j) \land y_{j'})$

   $= ((x_i \land x_{i'}) \lor (y_j \land x_i)) \lor ((x_i \land y_{j'}) \lor (y_j \land y_{j'}))$

   $= (x_{i''} \lor 0) \lor (0 \lor y_j \land y_{j'})$

   $= x_{i''} \lor y_{j''} = a_{i''j''} \in L'$,

where $i'' = \min\{i, i'\}, j'' = \min\{j, j'\}$.

3. $L'$ is isomorphic to $(r + 1) \times (k - r + 1)$. Let $\phi : L' \to (r + 1) \times (k - r + 1)$ be a bijection as $\phi(a_{ij}) = (i, j)$ for any $a_{ij} \in L'$. Then by Property 2(i) and (ii), we have that

   $\phi(a_{ij} \lor a_{i'j'}) = \phi(a_{i''j''}) = (i'', j'') = (i, j) \lor (i', j') = \phi(a_{ij}) \lor \phi(a_{i'j'})$,

and

   $\phi(a_{ij} \land a_{i'j'}) = \phi(a_{i''j''}) = (i'', j'') = (i, j) \land (i', j') = \phi(a_{ij}) \land \phi(a_{i'j'})$.

Hence $L' \cong (r + 1) \times (k - r + 1)$.

3 Some fundamental results on MDL

Let $G$ be a plane bipartite graph with a specific proper black-white coloring to vertices. An edge $e$ of a cycle (or an inner face) $C$ is proper if $e$ goes from the white end-vertex to the black endvertex along the clockwise direction of $C$. Let $\mathcal{F}(G)$ denote the set of all inner faces of $G$. Recall that $\mathcal{M}(G)$ denotes the set of all 1-factors of $G$.
Definition 1. A binary relation $\preceq$ on $\mathcal{M}(G)$ is defined as: $M_1 \preceq M_2$, $M_1, M_2 \in \mathcal{M}(G)$, if and only if $\vec{Z}(G)$ has a directed path from $M_2$ to $M_1$.

It is known that $\mathcal{M}(G) = (\mathcal{M}(G), \preceq)$ is a poset and a lattice structure on $\mathcal{M}(G)$ is revealed in the following two theorems.

Theorem 3.1. \cite{10} Let $G$ be a plane (weakly) elementary bipartite graph. Then $\mathcal{M}(G)$ is a finite distributive lattice, and its Hasse diagram is isomorphic to $\vec{Z}(G)$.

Theorem 3.2. \cite{28} Let $G$ be a plane bipartite graph with 1-factor. Then $\mathcal{M}(G)$ is direct sum of distributive lattices and the Hasse diagram is isomorphic to $\vec{Z}(G)$.

Definition 2. An FDL $L$ is called an matchable distributive lattice (MDL) if there exist a plane bipartite graph $G$ such that $L \cong \mathcal{M}(G)$.

Let $M^1$ and $M^0$ denote 1-factors of $G$ such that $G$ has neither improper $M^1$- nor proper $M^0$-alternating cycles, called source and root 1-factors of $G$ respectively. If $\mathcal{M}(G)$ is an FDL, then $M^1$ and $M^0$ are the greatest element and the least element, respectively.

Lemma 3.3. Let $G$ be a plane elementary bipartite graph with more than two vertices. Then the boundary of $G$ is proper $M^1$- and improper $M^0$-alternating cycle.

Proof. It is known that $G$ is 2-connected and the boundary is a cycle. For every proper edge $e = uv$ on the boundary of $G$, it suffices to show that $e \in M^1$. Otherwise, an edge $e'$ different from $e$ and incident to $u$ belongs to $M^1$. Since $G$ is elementary, it has a 1-factor $M$ such that $e \in M$. Then $M \oplus M^1$ has a cycle containing $e$ and $e'$, which is both improper $M^1$- and proper $M$-alternating cycle, a contradiction. Hence the boundary of $G$ is proper $M^1$-alternating cycle. Similarly, we can show that the boundary of $G$ is improper $M^0$-alternating cycle. \hfill $\square$

Let $G$ be a plane elementary bipartite graph with $M' \preceq M$ in $\mathcal{M}(G)$. For any $f \in \mathcal{F}(G)$, let $\Delta_\mathcal{C}(f)$ denote the number of proper $M$-alternating cycles in $\mathcal{C} := \mathcal{C}(M, M') = M \oplus M'$ with $f$ in their interiors minus the number of improper $M$-alternating cycles in $\mathcal{C}$ with $f$ in their interiors. Then $M' \preceq M$ implies that $\Delta_\mathcal{C}(f) \geq 0$ by Lemma 3.1 in \cite{28}. For any directed path $\vec{P} = M_0 (= M)M_1...M_t (= M')$ from $M$ to $M'$ of $\vec{Z}(G)$, let $s_i := M_{i-1} \oplus M_i$, $i = 1, ..., t-1$. Let $\delta_\mathcal{P}(f)$ denote the times of $f$ appearing in the face sequence corresponding to $s_1, ..., s_t$. Lemma 3.5 in Ref. \cite{28} implies the following result.

Lemma 3.4. Let $G$ be a plane elementary bipartite graph with $M' \preceq M$ in $\mathcal{M}(G)$. If $\vec{P}$ is a directed path from $M$ to $M'$ of $\vec{Z}(G)$ and $\mathcal{C} = M \oplus M'$, then $\delta_\mathcal{P}(f) = \Delta_\mathcal{C}(f)$ for each $f \in \mathcal{F}(G)$.
From Lemmas 2.3 and 3.3 we can derive the following critical result.

**Lemma 3.5.** For a plane elementary bipartite graph $G$ with more than two vertices, each element of $\mathcal{M}(G)$ has no complement except the greatest element $M^1$ and the least element $M^0$.

![Figure 2. Sublattice $(r + 1) \times (k - r + 1)$](image)

**Proof.** Suppose to the contrary that $\mathcal{M}(G)$ has a pair of mutually complementary elements $M$ and $M'$ except $M^1$ and $M^0$. Let $\rho(\mathcal{M}(G)) = k$ and $\rho(M) = r$. Then $\rho(M') = k - r$, and $k - 1 \geq r \geq 1$. By Lemma 2.3, $\mathcal{M}(G)$ has a sublattice $(r + 1) \times (k - r + 1)$ as shown in Fig. 2 containing the following two maximal chains:

$$M_0(= M^0) \prec M_1 \prec \cdots \prec M_r(= M) \prec \cdots \prec M_{k-1} \prec M_k(= M^1),$$

$$M'_0(= M^0) \prec M'_1 \prec \cdots \prec M'_{k-r}(= M') \prec \cdots \prec M'_{k-1} \prec M'_k(= M^1).$$

Put $M_{ij} := M_i \lor M'_j$, $i = 0, 1, \ldots, r$, $j = 0, 1, \ldots, k - r$, $s_i := M_i \oplus M_{i-1}(1 \leq i \leq r)$ and $s'_j := M'_j \oplus M'_{j-1}(1 \leq j \leq k - r)$. Since each maximal chain of $(r + 1) \times (k - r + 1)$ is a saturated chain of $\mathcal{M}(G)$, the $s_i$ and $s'_j$ are the boundaries of inner faces of $G$.

**Claim 1.** $M_{i,j} = M_{i-1,j} \oplus s_i = M_{i,j-1} \oplus s'_j$, and $s_i$ and $s'_j$ are disjoint, for $i = 1, 2, \ldots, r$, and $j = 1, 2, \ldots, k - r$.

**Proof.** We prove that $M_{i,j} = M_{i-1,j} \oplus s_i = M_{i,j-1} \oplus s'_j$ such that $s_i$ and $s'_j$ are proper $M_{i,j}$-alternating by induction on $(i,j) \geq (1,1)$. For $i = j = 1$, $s_1$ and $s'_1$ are improper $M^0$-alternating facial cycles, and are thus disjoint. Hence we have that $M_{1,1} = M_1 \oplus s'_1 = M'_1 \oplus s_1$.
by Lemma 2.3 and the required holds. Let $i \geq 2$ or $j \geq 2$. For the induction step, suppose that the assertion holds for smaller $i$ or $j$. By induction hypothesis, $M_{i,j-1} = M_{i-1,j-1} \oplus s_i$ and $M_{i-1,j} = M_{i-1,j-1} \oplus s'_j$, and $s_i$ and $s'_j$ are distinct and improper $M_{i-1,j-1}$-alternating facial cycles, and are disjoint. Hence $s_i$ and $s'_j$ are improper $M_{i,j-1}$-alternating and improper $M_{i-1,j}$-alternating, respectively, and $M_{i,j-1} \oplus s'_j$ and $M_{i-1,j} \oplus s_i$ cover $M_{i,j-1}$ and $M_{i-1,j}$, respectively. Obviously, $M_{i,j-1} \oplus s'_j = M_{i-1,j-1} \oplus s_j \oplus s'_j = M_{i-1,j-1} \oplus s'_j \oplus s_i = M_{i-1,j} \oplus s_i$. Hence $M_{i,j} = M_{i-1,j} \oplus s_i = M_{i,j-1} \oplus s'_j$ since $M_{i,j} = M_{i-1,j} \vee M_{i,j-1}$ by Lemma 2.3. The assertion holds for any $(i, j)$.

Claim 2. Let $f_i$ and $h_j$ denote the inner faces of $G$ bounded by $s_i$ and $s'_j$, respectively. Then $F(G) = \{f_1, f_2, \ldots, f_r, h_1, h_2, \ldots, h_{k-r}\}$.

Proof. Let $F_1 := \{f_1, f_2, \ldots, f_r\}$ and $F_2 := \{h_1, h_2, \ldots, h_{k-r}\}$. So we want to prove that $F(G) = F_1 \cup F_2$.

Let $C := M^1 \oplus M^0$. Then each cycle in $C$ is proper $M^1$ and improper $M^0$-alternating cycle, one being the boundary of $G$ by Lemma 3.3. Hence $\Delta_C(f) \geq 1$ for any $f \in F$.

Let $P := M_{r,k-r}(= M^1)M_{r,k-r-1} \cdots M_{r,0}M_{r-1,0} \cdots M_{0,0}(= M^0)$ be a directed path of $Z(G)$, corresponding to a maximal chain of $M(G)$. For any $f \in F$, by Lemma 3.4 we have that $\delta_P(f) = \Delta_C(f) \geq 1$. Hence $F(G) = \{f_1, f_2, \ldots, f_r, h_1, h_2, \ldots, h_{k-r}\}$.

Since $G$ is 2-connected, inner dual graph $G^\#$ of $G$ is connected. Let $f^*$ be a vertex of $G^\#$ corresponding to $f \in F$. Then there must exist a vertex $f^*_i$ in $\{f^*_1, \ldots, f^*_r\}$ being adjacent to a vertex $h^*_j$ in $V(G^*) \setminus \{f^*_1, \ldots, f^*_r\} = \{h^*_1, \ldots, h^*_{k-r}\}$. That means that $f_i$ and $h_j$ are adjacent, contradicting Claim 1.

From the above arguments, we have the following main results of this paper.

**Theorem 3.6.** For a plane elementary bipartite graph $G$, $M(G)$ is irreducible.

**Proof.** If $G = K_2$, it is trivial. Otherwise, $M(G)$ is a non-trivial FDL. By Lemma 3.5 every element of $M(G)$ has no complement except for $M^1$ and $M^0$. By Lemma 2.1 $M(G)$ has no central elements. Hence, $M(G)$ is irreducible.

**Elementary components** of a plane bipartite graph $G$ with 1-factor mean components other than $K_2$ of the subgraph obtained from $G$ by the removal of all forbidden edges (those edges not contained in any 1-factors).

**Corollary 3.7.** Let $G$ be a weakly elementary plane bipartite graph with elementary components $G_1, G_2, \ldots, G_k$. Then $M(G) = M(G_1) \times M(G_2) \times \cdots \times M(G_k)$ is an irreducible decomposition.
Theorem 3.8. \textit{(Decomposition Theorem)} Let $L$ be an FDL with a decomposition $L = \prod_{i=1}^{n} L_i$. Then $L$ is an MDL if and only if each $L_i (1 \leq i \leq n)$ is an MDL.

Proof. If each factor $L_i$ is an MDL, $1 \leq i \leq n$, then there exists a weakly elementary plane bipartite graph $G_i$ such that $M(G_i) \cong L_i$. We construct a weakly elementary plane bipartite graph $G$ by connecting $G_i$ to $G_{i+1}$ with a new edge in their exteriors for each $1 \leq i \leq n-1$. Then such new edges are forbidden edges of $G$. It follows that $M(G) \cong M(G_1) \times M(G_2) \times \cdots \times M(G_n) \cong L_1 \times L_2 \times \cdots \times L_n = L$. Hence $L$ is an MDL.

Conversely, suppose that $L$ is an MDL. Then there exists a plane weakly elementary bipartite graph $G$ such that $M(G) \cong L$. Let $G_1, \cdots, G_m$ be the non-trivial elementary components of $G$ ($m \geq 1$). By Corollary 3.7 $L \cong \prod_{j=1}^{m} M(G_j)$ is an irreducible decomposition. If $L = \prod_{i=1}^{n} L_i$ is irreducible, then by Lemma 2.2 $m = n$ and there exists a permutation $\pi$ of $[n]$ such that $L_i = M(G_{\pi(i)}) i = 1, 2, \cdots, n$. So each $L_i (1 \leq i \leq n)$ is an MDL. If $\prod_{i=1}^{n} L_i$ is not irreducible, then each factor $L_i$ is a direct product of some $M(G_j)$’s. So each factor $L_i$ is still an MDL. \hfill $\Box$

4 MDL $J(m \times n)$

From now on we will present two typical irreducible MDLs by the \textit{fundamental theorem for finite distributive lattice} (FTFDL).

Let $P$ be a finite poset. An order ideal (semi-ideal or down-set) $I$ of $P$ is a subset of $P$ if for every $x \in I$, $y \preceq x$ implies $y \in I$. The set $J(P)$ of order ideals of $P$, ordered by the set-inclusion, forms a poset $J(P)$. It is well known that $J(P)$ is indeed a distributive lattice. The FTFDL states that the converse is true.

Theorem 4.1 (FTFDL). \textit{([27])} Let $L$ be an FDL. Then there is a unique (up to isomorphism) finite poset $P$ for which $L \cong J(P)$.

In fact the above $P$ can be viewed as a subposet of $L$ consisting of all join-irreducible elements of $L$: an element $x$ of $L$ is said to be \textit{join-irreducible} if one cannot write $x = y \vee z$ where $y \prec x$ and $z \prec x$.

In this section we show that $J(W)$ are MDLs for any order ideal $W$ of $m \times n$. Let us introduce a type of hexagonal systems called truncated parallelogram \cite{2,3}: A \textit{truncated parallelogram}, simply denoted by $H := L(r_1, r_2, \cdots, r_m)$, consists of $m$ condensed linear chains (rows) of the length $r_1, \cdots, r_m$, $r_1 \geq r_2 \geq \cdots \geq r_m > 0$ and the first hexagons (conventionally drawn to the left) from all chains also form a linear chain, the first column;
In particular, \( L(m; n) = L(n, n, ..., n) \) is a parallelogram, and \( T_m := L(m, m-1, \cdots , 1) \) is a prolate triangle. For example, see Fig. 3. For convenience, all hexagonal systems considered in this section are drawn such that an edge-direction is vertical and the valleys are colored white.

Let \( L \) and \( B \) be the left and bottom perimeters of \( H \), respectively, which have a black vertex in common. The root 1-factor \( M^0 \) of \( H \) has all vertical edges in \( L \), and a series of parallel edges of \( B \) from left-low to right-up, and a series of parallel edges of \( H - L - B \) from left-up to right-lower. We can see that the boundary of \( H \) is an improper \( M^0 \)-alternating cycle. Hence \( H \) is elementary [34].

Since \( H \) has a forcing edge \( e \) (an edge contained in a unique 1-factor), each \( M^0 \)-alternating cycle must pass through \( e \); see [25] for details. For each 1-factor \( M \) of \( H \) other than \( M^0 \), \( C_M := M \oplus M^0 \) is an \( M^0 \)-alternating cycle of \( H \). Thus we have a bijection [26] between the 1-factors other than \( M^0 \) of \( H \) and the \( M^0 \)-alternating cycles of \( H \). Hence the subhexagonal system of \( H \) formed by \( C_M \) together with its interior is also a truncated parallelogram. Conversely, the perimeter of any sub-truncated parallelogram of \( H \) with edge \( e \) is an \( M^0 \)-alternating cycle. Hence each 1-factor \( M \) of \( H \) corresponds exactly to a sub-truncated parallelogram of \( H \) with edge \( e \), denoted by \( H_M \). However, \( H_{M^0} \) corresponds to the empty graph (without vertex), the degenerated sub-truncated parallelogram of \( H \).

Let \( P_M := (L \cup B) \oplus C_M \). Then \( P_M \) is an \( M \)-alternating path with both end-edges in \( M \) (see Fig. 4(a)). Note that \( C_{M^0} = \emptyset \) and \( P_{M^0} = L \cup B \). From \( M = M^0 \oplus C_M \), we have the following structure of \( M \).

**Proposition 4.2.** For each \( M \in \mathcal{M}(H) \), the edges in \( M \setminus E(P_M) \) have the same edge-direction from left-up to right-low. \( \square \)
Lemma 4.3. Let $M \in \mathcal{M}(H)$ and $h$ an $M$-alternating hexagon of $H$. Then $h$ intersects at three consecutive edges of $P_M$; Moreover, $h$ is proper if and only if $h \subseteq H_M$.

Proof. If $h$ is disjoint with $P_M$, then $h$ is not $M$-alternating by Proposition 4.2. Otherwise, $1 \leq |E(h \cap P_M)| \leq 3$. Since $h$ is $M$-alternating, $h$ intersects at three consecutive edges of $P_M$. So $h$ is proper $M$-alternating if and only if $e_2, e_4, e_6 \in M$. This holds if and only if $e_2, e_6 \in M \cap E(P_M)$. Thus $h$ and $P_M$ have exactly three common edges $e_1, e_2, e_6$ and $h \in H_M$. Similarly, $h$ is improper $M$-alternating if and only if $h$ and $P_M$ have exactly three common edges $e_3, e_4, e_5$ and $h \notin H_M$ (see Fig. 4(b) and(c)).

Lemma 4.4. Let $M, M' \in \mathcal{M}(H)$. Then $M' \preceq M$ in $\mathcal{M}(H)$ if and only if $H_{M'}$ is a sub-truncated parallelogram of $H_M$, namely $H_{M'} \subseteq H_M$.

Proof. We first show that $M$ covers $M'$ if and only if $H_M$ can be obtained from $H_{M'}$ by adding a hexagon. If $M$ covers $M'$, then there is a proper $M$-alternating hexagon $h$ such that $M' = M \oplus h$ by Theorem 3.1, and $C_M = M' \oplus M^0 = (M' \oplus M) \oplus (M \oplus M^0) = h \oplus C_M$. By Lemma 4.3 we have that $h \in H_M$ has exactly three edges of $P_M$. If $h = C_M$, the result is trivial. Otherwise, $C_{M'}$ is an improper $M^0$-alternating cycle, and the sub-truncated parallelogram $H_{M'}$ of $H$ bounded by $C_{M'}$ can be obtained by removing $h$ from $H_M$.

Conversely, assume that $H_{M'}$ can be obtained from $H_M$ by removing a hexagon $h$ of $H$. Since both $H_M$ and $H_{M'}$ are sub-truncated parallelograms of $H$, $h \in H_M$ must have exactly three edges of $P_M$. By Lemma 4.3 $h$ is proper $M$-alternating. Then $M' = M^0 \oplus C_{M'} = M^0 \oplus (C_M \oplus h) = M \oplus h$. Hence $M$ covers $M'$ in $\mathcal{M}(H)$.

We now show the lemma. If $M' \preceq M$ in $\mathcal{M}(H)$, we can show that that $H_{M'} \subseteq H_M$ by choosing a saturated chain between $M$ and $M'$ and applying repeatedly the above fact proved. If $H_{M'} \subseteq H_M$, there are a series of sub-truncated parallelograms of $H$: $H_1(= H_{M'})$, $H_2$, ..., $H_\ell(= H_M)$, such that each $H_i$ is obtained from $H_{i+1}$ by adding a hexagon.
Each $H_i$ corresponds to a 1-factor $M_i$ of $H$, $i = 1, 2, ..., t$. By the above fact we have $M_{i+1}$ covers $M_i$, $i = 1, 2, ..., t - 1$. Hence $M' \preceq M$.

Now, we define a poset on $\mathcal{F}(H)$, the set of hexagons of $H$. $h \in H$ is labeled with $h_{ij}$ if $h$ lies in the $i$-th row and $j$-th column, $1 \leq j \leq r_i, 1 \leq i \leq m$. For two hexagons $h_{ij}$ and $h_{kl}$, $h_{ij} \preceq h_{kl}$ if and only if $i \leq k$ and $j \leq l$. Then $\mathbf{F}(H) := (\mathcal{F}(H), \preceq)$ is a poset. If $H$ is a parallelogram, then $\mathbf{F}(H)$ is $m \times n$. In general, $\mathbf{F}(H)$ is an order ideal of $m \times n$. For example, see Fig. 5.

![Figure 5](image)

(a) $\mathbf{F}(L(r_1, r_2, \cdots, r_m))$, (b) $\mathbf{F}(L(m; n))$, and (c) $\mathbf{F}(T_m)$.

By Lemma 4.4, we can see that $M \in \mathbf{M}(H)$ is join-irreducible if and only if $H$ has a unique proper $M$-alternating hexagon lying in $H_M$, which is a sub-parallelogram of $H$. Let $\mathcal{I}(\mathbf{M}(H))$ denote the subposet of $\mathbf{M}(G)$ consisting of all join-irreducible elements.

**Lemma 4.5.** $\mathcal{I}(\mathbf{M}(H)) \cong \mathbf{F}(H)$.

**Proof.** A bijection $\psi : \mathcal{I}(\mathbf{M}(H)) \to \mathbf{F}(H)$ is defined as follows. For each $M \in \mathcal{I}(\mathbf{M}(H))$, let $\psi(M)$ denote the unique proper $M$-alternating hexagon of $H_M$, i.e. the right-up-most hexagon of $H_M$. Moreover, both $\psi$ is an isomorphism: for any $M, M' \in \mathcal{I}(\mathbf{M}(H))$, by Lemma 4.4 we have that $M' \preceq M$ in $\mathbf{M}(H) \Leftrightarrow H_{M'} \subseteq H_M \Leftrightarrow \psi(M') \preceq \psi(M)$ in $\mathbf{F}(H)$. □

By Theorem 4.1 and Lemma 4.5 we have a main theorem as follows.

**Theorem 4.6.** $\mathbf{M}(H) \cong \mathbf{J}(\mathbf{F}(H))$. □

When $H$ takes all over the truncated parallelograms for fixed $m$ and $n$, $\mathbf{F}(H)$ goes all order ideals of $m \times n$. From the above theorem we have an immediate consequence as follows.

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Corollary 4.7. Let \( W \) be any order ideal of \( m \times n \). Then \( J(W) \) is an irreducible MDL. \( \square \)

We can obtain a series of MDLs by applying the above theorem to the special truncated parallelograms, such as parallelogram, prolate triangle, etc. Let \( R_m := F(T_m) \). Here we give two special ones:

Corollary 4.8. \( J(m \times n) \) and \( J(R_m) \) are irreducible MDLs. Moreover

(1) \( J(m \times n) \cong M(L(m; n)) \), and

(2) \( J(R_m) \cong M(T_m) \). \( \square \)

Note that \([2, 3]\) the number of 1-factors of parallelogram \( L(m; n) \) and prolate triangle \( T_m \) are \( \binom{m+n}{m} \) and \( \frac{1}{m+2} \binom{2m+2}{m+1} \) (Catalan number), respectively.

5 MDL \( J(T) \)

In this section we will show that \( J(T) \) is an irreducible MDL with outerplane bipartite graphs for a poset \( T \) implied by any orientation of a tree. A connected plane graph \( G \) is outerplane if all vertices lie on the boundary of the outer face of \( G \). Let \( G \) be the set of all 2-connected outerplane bipartite graphs. Catacondensed hexagonal systems are typical members of \( G \) \([8]\).

An edge set \( T \) of a connected graph \( G \) is called a minimal edge-cut if \( G - T \) is not connected but \( G - T' \) remains connected for any proper subset \( T' \) of \( T \). For a plane graph \( G \), let \( e^* \) and \( f^* \) denote the edge and vertex of dual graph \( G^* \) corresponding to edge \( e \) and face \( f \) of \( G \), respectively; For \( T \subseteq E(G) \), put \( T^* := \{ e^* : e \in T \} \). Some edges in a plane graph \( G \) form a minimal edge-cut in \( G \) if and only if the corresponding dual edges form a cycle in \( G^* \) \([22]\). A minimal edge-cut \( T \) of a plane bipartite graph \( G \) is called elementary edge cut (e-cut for short) \([34]\) if all edges of \( T \) are incident with white vertices of one component of \( G - T \), called the white bank of \( T \), and the other component is the black bank of \( T \).

Lemma 5.1. \([31]\) Let \( T \) be a minimal edge-cut of \( G \in G \). Then \( T \) is an e-cut of \( G \) if and only if for any 1-factor \( M \) of \( G \), \( | M \cap T | = 1 \).

We now give an orientation \( \vec{G}^* \) of the dual \( G^* \): an edge \( e^* \) is oriented as an arc from \( f^*_1 \) to \( f^*_2 \) if one goes along \( e^* \) from \( f^*_1 \) to \( f^*_2 \) the white end-vertex of \( e \) lies right side. For example, see Figs. 6 and 7. We can see that a minimal edge-cut \( T \) is an e-cut of \( G \) if and only if \( T^* \) forms a directed cycle of \( \vec{G}^* \).
For $G \in \mathcal{G}$, we now give a poset on $\mathcal{F}(G)$. Let $\vec{G}^\#$ be the orientation of inner dual graph $G^\#$, obtained from directed dual graph $\vec{G}^*$ by deleting the vertex $f_0^*$ corresponding to the outer face of $G$. For $f_1, f_2 \in \mathcal{F}(G)$, we define “$f_1 \preceq f_2$” if $\vec{G}^\#$ has a directed path from $f_2^*$ to $f_1^*$. Since $\vec{G}^\#$ contains no directed cycles, $\mathcal{F}(G) := (\mathcal{F}(G), \preceq)$ is a poset.

For a plane elementary bipartite graph $G$ with $M, M' \in \mathcal{M}(G)$, if $M' \preceq M$, then there exists a saturated chain $M_0 (= M)M_1 \cdots M_k (= M')$ in $\mathcal{M}(G)$ between $M$ and $M'$. Then $M_{i-1}$ covers $M_i$, and $f_i := M_{i-1} \oplus M_i$ is a proper $M_{i-1}$-alternating face, $i = 1, 2, \ldots, k$. Then we say that $M_i$ is obtained from $M_{i-1}$ by a $Z$-transformation on the (proper $M_{i-1}$-alternating) face $f_i$, or simply by transforming $f_i$. Further, we also say that $M'$ is obtained from $M$ by a $Z$-transformation sequence on inner faces $f_1, f_2, \ldots, f_k$, and $f_1, f_2, \ldots, f_k$ is a face sequence of $G$ by a $Z$-transformations sequence from $M$ to $M'$. The $Z$-transformation of $G$ is simple if every inner face of $G$ is transformed at most once during any $Z$-transformation sequence of $G$.

**Lemma 5.2.** Let $G \in \mathcal{G}$. If $f' \preceq f$ in $\mathcal{F}(G)$, then $f'$ always appears after $f$ in any $Z$-transformation sequence of $G$ from $M^1$ to $M^0$. Hence $Z$-transformation of $G$ is simple.

**Proof.** Without loss of generality, suppose that $f$ covers $f'$ in $\mathcal{F}(G)$. That is, $(f, f')$ is an arc of directed inner dual $\vec{G}^\#$. Then $(f, f')$ can be extended to a maximal directed path of $\vec{G}^\#$, which can be further extended to a directed cycle of $\vec{G}^*$, denoted by $\vec{C} := f_0^*e_0^*f_1^*e_1^*\cdots f_t^*e_t^*f_0^*$; see Fig. 6. Then $T = \{e_0, e_1, \ldots, e_t\}$ is an e-cut of $G$, each edge $e_j$ is a common edge of $f_j$ and $f_{j+1}$ (script module $t + 1$), and each $e_j$ is a proper edge of $f_{j+1}$ and improper edge of $f_j$, $0 \leq j \leq t - 1$. For any $M \in \mathcal{M}(G)$, by Lemma 5.1, $| M \cap T | = 1$.

Let $P := M_1M_2\cdots M_s$ be a directed path in $\vec{Z}(G)$ from $M^1$ to $M^0$. Then $\delta_P(f) = 1$ for all $f \in \mathcal{F}(G)$ by Lemmas 3.3 and 3.4. Suppose that $M_{i+1}$ is obtained from $M_i$ by a $Z$-transformation on $f_j$. It is sufficient to show that $f_j, f_{j+1}, \ldots, f_t$ do not appear in $Z$-transformations from $M^1$ to $M_i$. We proceed by induction on $j$. If $j = 1$, then $e_0 \in M_1, M_2, \ldots, M_i$. Hence proper edge $e_{k-1}$ of each $f_k$, $k \geq 2$, does not belong to $M_1, M_2, \ldots, M_i$. Hence, the required holds. By induction hypothesis we have that $f_{j+1}, \ldots, f_t$ do not appear in $Z$-transformations from $M^1$ to $M_{i+1}$ through $M_i$. Suppose that $M_{i' + 1}$ is obtained from $M_{i'}$ by a $Z$-transformation on $f_{j+1}$. Then $i + 1 \leq i'$, and proper edge $e_j$ of $f_{j+1}$ belong to all $M_{i+1}, \ldots, M_{i'}$. That implies that proper edge $e_k$ of $f_{k+1}$ does not belong to $M_{i+1}, \ldots, M_{i'}$ for all $k > j$. Hence $f_{j+1}, \ldots, f_t$ do not appear in $Z$-transformations from $M^1$ to $M_{i'}$; that is, $f_{j+2}, \ldots, f_t$ do dot appear in $Z$-transformations from $M^1$ to $M_{i+1}$ through $M_{i'}$, as expected.

For $G \in \mathcal{G}$, we now define a mapping from $\mathcal{M}(G)$ to $J(\mathcal{F}(G))$. For any $M \in \mathcal{M}(G)$, let
Figure 6. An outerplane bipartite graph with e-cuts (the set of edges intersecting a dashed line).

$\sigma(M)$ denote the set of faces in the face sequence by a Z-transformation sequence from $M$ to $M^0$. By Lemma 3.4 we have

$$\sigma(M) = \{ f \in \mathcal{F}(G) \mid f \text{ is contained in the interior of some cycle in } M \oplus M^0 \}.$$ 

In particular, $\sigma(M^0) = \emptyset$, and $\sigma(M^1) = \mathcal{F}(G)$ since $M^1 \oplus M^0$ is just the boundary of $G$ by Lemma 3.3.

**Lemma 5.3.** $\sigma : \mathcal{M}(G) \to J(\mathcal{F}(G))$ is an injective mapping.

**Proof.** For $M \in \mathcal{M}(G)$, let $f \in \sigma(M)$. If $f' \prec f$ in $\mathcal{F}(G)$, then by Lemma 5.2, $f'$ always appears after $f$ in any Z-transformation sequence of $G$ from $M^1$ to $M^0$ passing through $M$. So $f' \in \sigma(M)$, and $\sigma(M)$ is an order ideal of $\mathcal{F}(G)$. That is, $\sigma$ is a mapping from $\mathcal{M}(G)$ to $J(\mathcal{F}(G))$. Further it is clear that $\sigma$ is injective. \hfill $\square$

Further, we will show that $\sigma$ is an isomorphism between $\mathcal{M}(G)$ and $J(\mathcal{F}(G))$ in the following theorem.

**Theorem 5.4.** For each $G \in \mathcal{G}$, $\mathcal{M}(G) \cong J(\mathcal{F}(G))$.

**Proof.** We first show that, for each $M \in \mathcal{M}(G)$, if $M_1, M_2, \cdots, M_k$ are the 1-factors covered by $M$, then the order ideals $\sigma(M_1), \sigma(M_2), \cdots, \sigma(M_k)$ in $\mathcal{M}(G)$ are exactly the ones covered by $\sigma(M)$. By the fact that the order ideals in a finite poset $\mathcal{P}$ covered in $J(\mathcal{P})$ by an order ideal $Y$ are exactly the sets $Y \setminus \{ x \}$ for all maximal elements $x$ of $Y$, it is sufficient to prove that the faces which can be properly transformed in 1-factor $M$ are exactly the maximal elements of $\sigma(M)$.

Let $f_i$ denote properly $M$-alternating facial cycle such that $f_i = M \oplus M_i, i = 1, 2, \ldots, k$. For convenience, we also use $f_i$ to denote the corresponding inner face. Hence $\sigma(M_i) =$
σ(M) \ {f_i}. By Lemma 5.2, all faces which greater than \( f_i \) must be transformed during any transforming sequence from \( M^1 \) to \( M \). So each \( f_i \) is an maximal element in \( \sigma(M) \), and \( \sigma(M_i) = \sigma(M) \setminus \{f_i\} \) is covered by \( \sigma(M) \).

If \( f \) is a maximal element of \( \sigma(M) \), then by Lemma 5.2, all elements which greater than \( f \) in \( F(G) \) have been transformed from \( M^1 \) to \( M \). Let \( e \) be any proper edge of \( f \). Let \( f' \) be the face of \( G \) that has a common edge \( e \) with \( f \). If \( f' \) is the outer face of \( G \), by Lemma 5.3 \( e \) remains unchanged in any \( Z \)-transformation from \( M^1 \) to \( M \). Otherwise, \( f' \in F(G) \setminus \sigma(M) \). Then \( e \in M \) since \( f' \) has been transformed but \( f \) not from \( M^1 \) to \( M \) and \( e \) is an improper edge of \( f' \). Hence, all proper edges of \( f \) belong to \( M \); that is, \( f \) is proper \( M \)-alternating, as expected.

Further, \( \sigma \) is surjective since \( F(G) = \sigma(M^1) \) is the maximum element of \( J(F(G)) \). Therefore \( \sigma \) is an isomorphism between \( M(G) \) and \( J(F(G)) \).

Given an undirected tree \( T = (V,E) \), \( \vec{T} \) is any orientation of \( T \). Of course, directed tree \( \vec{T} \) has no (directed) cycles. Similar to \( F(G) \), we could consider \( \vec{T} \) as the Hasse diagram of a poset \( T \). As a consequence, we obtain another irreducible MDL described in the following result.

**Theorem 5.5.** Let \( T \) be a poset derived from any orientation \( \vec{T} \) of a tree \( T \). Then \( J(T) \) is an irreducible MDL.

**Proof.** By Theorem 5.4, it is sufficient to show that there is a \( G \in \mathcal{G} \) such that \( \vec{G}^\# \cong \vec{T} \). If \( |V(T)| \leq 2 \), it is obvious. So let \( |V(T)| \geq 3 \). Let \( \Delta \) denote the maximum degree of \( T \). We now construct such a graph \( G \) as follows. For any vertex \( v \) of \( T \), we gave an inner face \( f_v \) bounded by a cycle of length \( 2\Delta \). If a vertex \( u \) of \( T \) is adjacent to \( v \), then we place the corresponding inner face \( f_u \) outside \( f_v \) by overlapping their edges \( e' \in f_u \) and \( e'' \in f_v \) to a new edge \( e \in G \), satisfying the orientation rule of \( F(G) \): \((u,v)\) is an arc from \( u \) to \( v \) if and only if \( e \) goes from the black end-vertex to the white one along the clockwise orientation of \( f_u \). Since \( f_v \) has \( 2\Delta \) edges and \( v \) has at most \( \Delta \) going-out (going-in) arcs in the directed \( \vec{T} \), for all other neighbors of \( v \) we can proceed similarly. By repeating the above process, one can construct an outerplane bipartite graph \( G \in \mathcal{G} \) such that \( \vec{G}^\# \cong \vec{T} \). For example, see Fig. 7.

In fact, in the above construction the face degree of \( f_v \) may be smaller than \( 2\Delta \). The least value of face degree of \( f_v \) may reach \( 2 \max\{\Delta^-_v, \Delta^+_v, 2\} \), where \( \Delta^-_v, \Delta^+_v \) are in- or out-degree of \( v \) in \( \vec{T} \). During the process, we may need to exchange the order of in- and
out-edges such that the edges surrounded by $f_v$ are alternately in- and out-edges as many as possible; See Fig. 7(b).

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