Large deviations in a population dynamics with catastrophes

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Abstract

The large deviation principle on phase space is proved for a class of Markov processes known as random population dynamics with catastrophes. In the paper we study the process which corresponds to the random population dynamics with linear growth and uniform catastrophes, where an eliminating portion of the population is chosen uniformly. The large deviation result provides an optimal trajectory of large fluctuation: it shows how the large fluctuations occur for this class of processes.

Key words: Population models, Catastrophes, Large Deviation Principle, Local Large Deviation Principle

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1 Introduction

Stochastic models with catastrophes are studied since 70’s and recieved a great attention of probability community, see \cite{1} for the probably first systematic review about such processes, and see \cite{2} for short historical overview and more references. These models are
used in analyzing a growth of a population subject to catastrophes due large-scale death or emigrations of a population. According [1] the population dynamics considered here we will call population dynamics with linear growth and uniform catastrophes, where the eliminating portion is chosen uniformly. Typically researchers are interested in extinction probability, the mean time to extinction, invariant measures, convergence to invariant measures for these processes.

In [3] for the population dynamic, $\xi(t)$ defined by (1, 2) in the following, with linear growth and uniform catastrophes we proved the local large deviation principle (LLDP): we established a rough logarithmic asymptotic for the probability of the scaling version $\xi_T(t), t \in [0, 1]$, defined by (3), to be in a small neighborhood of a continuous function. Here, based on the work [3] we established a large deviation principle (LDP) on the state space at the end of the interval of observation of the process: we find the logarithmic asymptotic for the probability $P(\xi_T(1) > x)$. Moreover, our proof also provides an optimal trajectory – how such deviation occurs taking in account the evolution of the process. As far as our understanding there are no other large deviations results for such processes.

Throughout the paper we assume that all random elements are defined on a probability space $(\Omega, \mathcal{F}, P)$.

We construct our continuous time process $\xi(t), t \in \mathbb{R}^+$, in two steps. First, consider the discrete time Markov chain $\eta(k), k \in \mathbb{Z}^+, \mathbb{Z}^+ = \{0\} \cup \mathbb{N}$, with state space $\mathbb{Z}^+$ and transition probabilities

$$P(\eta(k+1) = j | \eta(k) = i) = \begin{cases} \frac{\lambda}{\lambda+\mu}, & \text{if } j = i + 1, \\ \frac{\mu}{\lambda+\mu}, & \text{if } 0 \leq j < i, i \neq 0, \\ 1, & \text{if } j = 1, i = 0, \end{cases}$$

where $\lambda$ and $\mu$ are positive constants. Let $\eta(0) = 0$. Second, let $\nu(t), t \in \mathbb{R}^+$, be the Poisson process with parameter $E\nu(t) = \alpha t$, where $\alpha$ is positive parameter. Suppose that the process $\nu(\cdot)$ and the chain $\eta(\cdot)$ are independent. Finally,

$$\xi(t) := \eta(\nu(t)), \ t \in \mathbb{R}^+. \quad (2)$$

In order to establish the large deviation result we consider the scaled process

$$\xi_T(t) := \frac{\xi(Tt)}{T}, \ t \in [0, 1], \text{ as } T \to \infty. \quad (3)$$

where $T$ is an increasing parameter, $T \to \infty$. We are interested in LDP for the family of random variables $\xi_T(1)$.

In the proof of LDP we use the standard implication (see, for example [4, Lemma 4.1.23]):

$$\text{LLDP and ET } \Rightarrow \text{ LDP},$$

where LLDP is Local Large Deviation Principle, and ET stands for Exponential tightness.

In the next section we recall definitions and formulate main result. In Section 3 we prove the main result, Theorem 2.5. Auxiliary results are proved in Section 4.
2 Definitions and main results

Recall the definitions we need.

**Definition 2.1.** A family of random variables $\xi_T(1)$ satisfies LLDP in $\mathbb{R}$ with the rate function $I = I(x) : \mathbb{R} \to [0, \infty]$ and the normalizing function $\psi(T) : \lim_{T \to \infty} \psi(T) = \infty$, if the following equality holds for any $x \in \mathbb{R}$

$$\lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{\psi(T)} \ln P(\xi_T(1) \in U_\varepsilon(x)) = \lim_{\varepsilon \to 0} \liminf_{T \to \infty} \frac{1}{\psi(T)} \ln P(\xi_T(1) \in U_\varepsilon(x)) = -I(x),$$

where $U_\varepsilon(x) := \{y \in \mathbb{R} : |x - y| < \varepsilon\}$.

**Definition 2.2.** A family of random variables $\xi_T(1)$ is exponentially tight on $\mathbb{R}$ if, for any $c < \infty$, there exists a compact set $K_c \in \mathbb{R}$ such that

$$\limsup_{T \to \infty} \frac{1}{\psi(T)} \ln P(\xi_T(1) \not\in K_c) < -c.$$

We denote the closure and interior of the set $B$ by $[B]$ and $(B)$, respectively.

**Definition 2.3.** A family of random variables $\xi_T(1)$ satisfies LDP on $\mathbb{R}$ with a rate function $I = I(f) : \mathbb{R} \to [0, \infty]$ and the normalizing function $\psi(T) : \lim_{T \to \infty} \psi(T) = \infty$, if for any $c \geq 0$ the set $\{x \in \mathbb{R} : I(x) \leq c\}$ is a compact set and, for any set $B \in \mathcal{B}(\mathbb{R})$ the following inequalities hold:

$$\limsup_{T \to \infty} \frac{1}{\psi(T)} \ln P(\xi_T(1) \in B) \leq -I([B]),$$

$$\liminf_{T \to \infty} \frac{1}{\psi(T)} \ln P(\xi_T(1) \in B) \geq -I((B)),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra on $\mathbb{R}$, $I(B) = \inf_{x \in B} I(x)$, $I(\emptyset) = \infty$.

Further we will use the following notations: $\overline{B}$ is a complement of the set $B$; $I(B)$ is the indicator function of the set $B$; $[a]$ is the integer part of the number $a$.

We recall Low of Large Numbers (LLN), it was proved in [3].

**Theorem 2.4.** (LLN) For any $\varepsilon > 0$ the following equality holds true

$$P \left( \lim_{T \to \infty} \sup_{t \in [0,1]} \xi_T(t) > \varepsilon \right) = 0.$$

LDP is the main theorem in the paper.
Theorem 2.5. (LDP) The family of random variables $\xi_T(1)$ satisfies LDP with normalized function $\psi(T) = T$ and with rate function

$$I(x) = \begin{cases} 
\infty, & \text{if } x \in (-\infty, 0), \\
x \ln \left( \frac{\lambda + \mu}{x} \right), & \text{if } x \in [0, \alpha), \\
x \ln \left( \frac{x(\lambda + \mu)}{\alpha \lambda} \right) - x + \alpha, & \text{if } x \in [\alpha, \infty).
\end{cases}$$

The proof of Theorem 2.5 provides the “most probable” trajectories of large deviations $\xi_T(1) > x$. If $x < \alpha$ then there exists the moment $t_{x,\alpha} = 1 - \frac{x}{\alpha} \in (0, 1)$ such that the process $\xi_T(\cdot)$ stays near the zero up to the time $t_{x,\alpha}$ and after that $\xi_T(t), t \geq t_{x,\alpha}$ increases according the straight line which starts at point $(t_{x,\alpha}, 0)$ and grows up to the point $(1, x)$ with the slope $\alpha$, see the function $f_1$ on Figure 1. If $x \geq \alpha$ then the process grows together with the straight line starting from origin up to the point $(1, x)$, i.e. its slope is $x$, the function $f_2$ on Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{The “most probable” trajectories which provide large fluctuations. If $x < \alpha$ then the large deviation occurs according the function $f_1$. If $x \geq \alpha$, then the large deviation trajectory is in the neighborhood of the straight line $f_2$.}
\end{figure}

3 Proof of Theorem 2.5

Random process $\xi(t)$ we represent in the following way

$$\xi(t) = \nu_1(t) - \sum_{k=0}^{\nu_2(t)} \zeta_k(\xi(\tau_k-)),$$  \hspace{1cm} (4)
where \( \nu_1(t) \), \( \nu_2(t) \) are independent Poisson point processes with parameters
\[
\mathbf{E}\nu_1(t) = \frac{\alpha \lambda}{\lambda + \mu} t, \quad \mathbf{E}\nu_2(t) = \frac{\alpha \mu}{\lambda + \mu} t;
\]

where \( 0 = \tau_0 < \tau_1 < \cdots < \tau_k \cdots \) — jump moments of the process \( \nu_2(t) \); random variables \( \zeta_k(m) \), \( k \in \mathbb{Z}^+ \), \( m \in \mathbb{Z}^+ \) are mutually independent and do not depend on \( \nu_1(t) \) and \( \nu_2(t) \); \( \zeta_0(m) = 0 \), for all \( m \in \mathbb{Z}^+ \); for fixed \( k, m \in \mathbb{N} \) the distribution of \( \zeta_k(m) \) is given by
\[
P(\zeta_k(m) = r) = \frac{1}{m}, \quad 1 \leq r \leq m,
\]
and \( \zeta_k(0) = -1 \), for all \( k \in \mathbb{N} \). Using representation \([4]\), we have
\[
\xi_T(t) = \frac{\nu_1(Tt)}{T} - \frac{1}{T} \sum_{k=0}^{\nu_2(T)} \zeta_k(\xi(\tau_k)) := \xi_T^+(t) - \xi_T^-(t).
\]

From Theorem 2.4 follows that Theorem 2.5 holds true for \( x = 0 \). Prove the theorem for \( x > 0 \). Let us estimate from above \( P(\xi_T(1) \geq x) \), \( x > 0 \). For any \( \delta > 0 \) and for any \( n \in \mathbb{N} \) we obtain
\[
P(\xi_T(1) \geq x) = P\left(\nu_1(T) + \sum_{k=0}^{\nu_2(T)} \zeta_k(\xi(\tau_k)) \geq T x\right)
\]
\[
\leq \sum_{k=1}^{n-1} P\left(\nu_1(T) + \sum_{k=0}^{\nu_2(T)} \zeta_k(\xi(\tau_k)) \geq T x, \sup_{t \in \left[\frac{k-1}{n}, \frac{k}{n}\right]} \xi_T(t) \leq \delta, \sup_{t \in \left(\frac{k}{n}, 1\right]} \xi_T(t) > \delta \right) \quad (5)
\]
\[
+ P\left(\nu_1(T) + \sum_{k=0}^{\nu_2(T)} \zeta_k(\xi(\tau_k)) \geq T x, \sup_{t \in \left[\frac{n-1}{n}, 1\right]} \xi_T(t) \leq \delta \right) := P_1 + P_2.
\]

Estimate from above \( P_1 \).
\[
P_1 \leq (n-1) \max_{1 \leq k \leq n-1} P\left(A, \inf_{t \in \left[\frac{k-1}{n}, \frac{k}{n}\right]} \xi_T(t) \leq \delta, \inf_{t \in \left(\frac{k}{n}, 1\right]} \xi_T(t) > \delta \right) := (n-1) \max_{1 \leq k \leq n-1} P_{1k}, \quad (6)
\]

where
\[
A := \left\{ \omega : \nu_1(T) + \sum_{k=0}^{\nu_2(T)} \zeta_k(\xi(\tau_k)) \geq T x \right\}.
\]

Estimate from above \( P_{1k} \). For any \( c > 0 \) we have
\[
P_{1k} \leq P\left(\nu_1(T) - \nu_1\left(T\frac{k-1}{n}\right) \geq T(x - \delta), B_c \right)
\]
\[
+ P\left(A, \inf_{t \in \left[\frac{k-1}{n}, \frac{k}{n}\right]} \xi_T(t) \leq \delta, \inf_{t \in \left(\frac{k}{n}, 1\right]} \xi_T(t) > \delta, \overline{B_c} \right) := P_{1k}^1 + P_{1k}^2, \quad (7)
\]

where
\[
B_c := \left\{ \omega : \nu_2(T) - \nu_2\left(T\frac{k-1}{n}\right) \leq cT \right\}.
\]
Since $\nu_1(\cdot)$ and $\nu_2(\cdot)$ are independent, then

$$P_{1k}^1 = P\left(\nu_1(T) - \nu_1\left(T\frac{k-1}{n}\right) \geq T(x - \delta)\right)P(B_c).$$  \hspace{1cm} (8)

Estimate from above $P_{1k}^2$. For any $a > 0$ we obtain

$$P_{1k}^2 \leq P\left(\inf_{t \in \left[\frac{k-1}{n}, \frac{k}{n}\right]} \xi_T(t) \leq \delta, \inf_{t \in \left(\frac{k}{n}, 1\right]} \xi_T(t) > \delta, B_c, \sum_{l=1}^{[cT]} \zeta_k(\xi(\tau_k(-))) > aT\right)$$

$$+ P\left(\inf_{t \in \left(\frac{k}{n}, 1\right]} \xi_T(t) > \delta, B_c, \sum_{l=1}^{[cT]} \zeta_k(\xi(\tau_k(-))) \leq aT\right)$$

$$\leq P\left(\nu_1(T) - \nu_1\left(T\frac{k-1}{n}\right) \geq T(x + a - \delta)\right)$$

$$+ P\left(\sup_{t \in \left(\frac{k}{n}, 1\right]} \xi_T(t) > \delta, B_c, \sum_{l=1}^{[cT]} \zeta_k(\xi(\tau_k(-))) \leq aT\right)$$

$$\leq P\left(\nu_1(T) - \nu_1\left(T\frac{k-1}{n}\right) \geq T(x + a - \delta)\right) + P\left(\bigcap_{l=1}^{[cT]} C_l, B_c, \sum_{l=1}^{[cT]} \zeta_k(\xi(\tau_k(-))) \leq aT\right),$$

where $\tau_{k_1}, \ldots, \tau_{k_{[cT]}}$ are the first $[cT]$ jump times of the process $\nu_2(Tt)$ for $t \in \left[\frac{k}{n}, 1\right]$, and

$$C_l := \{\omega : \xi_T(\tau_k(\omega)) > \delta\}. \hspace{1cm} (10)$$

For $T$ sufficiently large and $x + a - \delta \geq 1$ we obtain

$$P\left(\nu_1(T) - \nu_1\left(T\frac{k-1}{n}\right) \geq T(x + a - \delta)\right) = e^{-\frac{\alpha \lambda (n-1)}{n(\lambda + \mu)} T} \sum_{r=\lceil T(x+a-\delta) \rceil + 1}^{\infty} \frac{(\alpha \lambda (n - k + 1)T)^r}{r!n^r(\lambda + \mu)^r}$$

$$\leq e^{-\frac{\alpha \lambda (n-k+1)}{n(\lambda + \mu)} T} \left(\frac{\alpha \lambda (n - k + 1)T}{T(x+a/2-\delta)n(\lambda + \mu)}\right)^{[Ta]/2} \sum_{r=\lceil T(x+a/2-\delta) \rceil + 1}^{\infty} \frac{(\alpha \lambda (n - k + 1)T)^r}{r!n^r(\lambda + \mu)^r}, \hspace{1cm} (11)$$

$$\leq e^{-\frac{\alpha \lambda (n-k+1)}{n(\lambda + \mu)} T} \left(\frac{2\alpha \lambda}{([x+a/2-\delta])(\lambda + \mu)}\right)^{[Ta]/2} \sum_{r=\lceil T(x-\delta) \rceil + 1}^{\infty} \frac{(\alpha \lambda (n - k + 1)T)^r}{r!n^r(\lambda + \mu)^r}.$$

Since there exists $a^*(x, \alpha, \delta, \lambda, \mu)$ such that for all $a \geq a^*$ the inequality

$$\left(\frac{2\alpha \lambda}{([x+a/2-\delta])(\lambda + \mu)}\right)^{[Ta]/2} \leq e^{-\frac{\alpha \mu}{(\lambda + \mu)} T} \leq P(B_c)$$

holds true, then from inequality (11) it follows that for $a > a^*$

$$P\left(\nu_1(T) - \nu_1\left(T\frac{k-1}{n}\right) \geq T(x + a - \delta)\right) \leq P_{1k}^1.$$  \hspace{1cm} (12)
From Lemma 4.4 it follows that for any \( a > 0 \) and for \( T \) sufficiently large
\[
P\left( \bigcap_{l=1}^{[cT]} C_l, \overline{B}_c, \sum_{l=1}^{[cT]} \xi_l(\xi(\tau_l-)) \leq aT \right) \leq \left( \frac{1}{[\delta T]} \right)^{[cT]} \exp(aT) \leq P_{1k}.
\] (13)

Choose \( a > a^* \). Using (7)–(11), (12), (13), we obtain for sufficiently large \( T \)
\[
P_{1k} \leq 3P_{1k} = 3P\left( \nu_1(T) - \nu_1\left( \frac{T^{k-1}}{n} \right) \geq T(x - \delta) \right) P(B_c).
\] (14)

From (6), (14) it follows that for all \( n \in \mathbb{N}, \delta > 0, c > 0 \)
\[
P_1 \leq 3(n-1) \max_{1 \leq k \leq n-1} P\left( \nu_1(T) - \nu_1\left( \frac{T^{k-1}}{n} \right) \geq T(x - \delta) \right) P(B_c).
\] (15)

Estimate from above \( P_2 \). The following inequalities holds true for \( n \geq \exp\left( \frac{\alpha}{x - \delta} \right) \)
\[
P_2 = P\left( \nu_1(T) + \sum_{k=0}^{\nu_2(T)} \xi_k(\xi(\tau_k-)) \geq Tx, \inf_{t \in \left[\frac{x-1}{n}, 1\right]} \xi_T(t) \leq \delta \right)
\leq P\left( \nu_1(T) - \nu_1\left( \frac{T^{n-1}}{n} \right) \geq T(x - \delta) \right) = e^{-\frac{\alpha T}{n(\lambda + \mu)}} \sum_{k=T(x - \delta) + 1}^{\infty} \frac{(\alpha \lambda T)^k}{k!(\lambda + \mu)^k} \leq e^{-\alpha T} \sum_{k=T(x - \delta) + 1}^{\infty} \frac{(\alpha \lambda T)^k}{k!(\lambda + \mu)^k}
= e^{-\frac{\alpha T}{(\lambda + \mu)n}} T \sum_{k=T(x - \delta) + 1}^{\infty} \frac{(\alpha \lambda T)^k}{k!(\lambda + \mu)^k} \leq P_{11}.
\]

Thus, from (5), (15) it follows that for \( n \geq \exp\left( \frac{\alpha}{x - \delta} \right) \)
\[
P(\xi_T(1) \geq x) \leq 4(n-1) \max_{1 \leq k \leq n-1} P\left( \nu_1(T) - \nu_1\left( \frac{T^{k-1}}{n} \right) \geq T(x - \delta) \right) P(B_c).
\] (16)

Using Lemma 4.2, Lemma 4.3, Definition 2.3 and inequality (16) we obtain for \( n \geq \exp\left( \frac{\alpha}{x - \delta} \right), \delta > 0, c > 0 \)
\[
\lim_{T \to \infty} \frac{1}{T} \ln P(\xi_T(1) \geq x) \leq \max_{1 \leq k \leq n-1} \sup_{y \geq x} \left( -y - \delta \ln \left( \frac{y(\delta)(\lambda + \mu)n}{\alpha \lambda(n - k + 1)} \right) + (y - \delta) \right.
- \frac{\alpha \lambda(n - k + 1)}{\lambda + \mu n} - \frac{\alpha \mu(n - k)}{\lambda \mu n} + \frac{\alpha \mu(n - k)c}{\lambda \mu n} - c \ln c \right)
\]

When \( \delta \to 0, c \to 0 \) we obtain
\[
\lim_{T \to \infty} \frac{1}{T} \ln P(\xi_T(1) \geq x) \leq \max_{1 \leq k \leq n-1} \sup_{y \geq x} \left( -y \ln \left( \frac{y}{\alpha \lambda(n - k + 1)} \right) + y - \frac{\alpha \lambda(n - k + 1)}{\lambda + \mu n} - \frac{\alpha \mu(n - k)}{\lambda + \mu n} \right).
\]
And when \( n \to \infty \) we obtain

\[
\limsup_{T \to \infty} \frac{1}{T} \ln P(\xi_T(1) \geq x) \leq \sup_{z \in (0, 1)} \sup_{y \geq x} \left( -y \ln \left( \frac{y(\lambda + \mu)}{\alpha \lambda z} \right) + y - \frac{\alpha \lambda z}{(\lambda + \mu)} - \frac{\alpha \mu z}{(\lambda + \mu)} \right).
\]

Finding the maximum of the function

\[
f(y, z) = -y \ln \left( \frac{y(\lambda + \mu)}{\alpha \lambda z} \right) + y - \alpha z
\]

in the domain \( y \geq x, \ z \in [0, 1] \) we obtain

\[
\limsup_{T \to \infty} \frac{1}{T} \ln P(\xi_T(1) \geq x) \leq -I(x). \tag{17}
\]

Estimate from below \( P(\xi_T(1) > x) \). Since the processes \( \nu_1(\cdot) \) and \( \nu_2(\cdot) \) are independent, each with independent increments, then for all \( \varepsilon > 0, \ z \in (0, 1) \)

\[
P(\xi_T(1) > x) = P \left( \nu_1(T) + \sum_{k=0}^{\nu_2(T)} \zeta_k(\xi(\tau_k-)) \geq Tx \right)
\]

\[
\geq P \left( \xi_T(1-z) < \varepsilon, \nu_1(T) - \nu_1(T(1-z)) > Tx, \nu_2(T) - \nu_2(T(1-z)) = 0 \right)
\]

\[
= P(\xi_T(1-z) < \varepsilon) P(\nu_1(T) - \nu_1(T(1-z)) > Tx) P(\nu_2(T) - \nu_2(T(1-z)) = 0).
\]

From Theorem 2.4 and Lemma 4.2 it follows that for any \( z \in (0, 1) \)

\[
\liminf_{T \to \infty} \frac{1}{T} \ln P(\xi_T(1) > x) \geq \sup_{y \geq x} \left( -y \ln \left( \frac{y(\lambda + \mu)}{\alpha \lambda z} \right) + y - \frac{\alpha \lambda z}{(\lambda + \mu)} - \frac{\alpha \mu z}{(\lambda + \mu)} \right).
\]

Since it holds for any \( z \in (0, 1) \), then

\[
\liminf_{T \to \infty} \frac{1}{T} \ln P(\xi_T(1) > x) \geq \sup_{z \in (0, 1)} \sup_{y \geq x} \left( -y \ln \left( \frac{y(\lambda + \mu)}{\alpha \lambda z} \right) + y - \alpha z \right) = -I(x). \tag{18}
\]

Let us prove the LLDP for the family of the random variables \( \xi_T(1) \). Applying (17), we obtain

\[
\limsup_{\varepsilon \to 0} \frac{1}{T} \ln P(\xi_T(1) \in (x - \varepsilon, x + \varepsilon)) \leq \limsup_{\varepsilon \to 0} \frac{1}{T} \ln P(\xi_T(1) \geq x - \varepsilon)
\]

\[
\leq - \lim_{\varepsilon \to 0} I(x - \varepsilon) = -I(x).
\]
Using (17), (18) and the fact that the function \( I(x) \) is continuously differentiable function for \( x > 0 \), we obtain

\[
\lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln P(\xi_T(1) \in (x - \varepsilon, x + \varepsilon)) = \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln \left( P(\xi_T(1) > x - \varepsilon) - P(\xi_T(1) \geq x + \varepsilon) \right) = \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln \left( 1 - \frac{P(\xi_T(1) \geq x + \varepsilon)}{P(\xi_T(1) > x - \varepsilon)} \right)
\]

\[
\geq \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln P(\xi_T(1) > x - \varepsilon) + \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln \left( 1 - e^{-T(I(x+\varepsilon)-I(x-\varepsilon)+o(1))} \right)
\]

\[
= -I(x) + \lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \ln \left( 1 - e^{-T(2\varepsilon I'(\tilde{x})+o(1))} \right) = -I(x),
\]

where \( \tilde{x} \in (x - \varepsilon, x + \varepsilon) \) is the point where \( I'(\tilde{x}) = (I(x+\varepsilon) - I(x-\varepsilon))/2\varepsilon \). Thus LLDP is proved.

Exponential tightness follows from (17) and the fact that for any \( c \geq 0 \) the set \( I(x) \leq c \) is compact. □

### 4 Auxiliary results

Here we will prove several auxiliary lemmas.

**Lemma 4.1.** For any \( a \in \mathbb{R} \) the following inequality holds true

\[
P\left( \sum_{l=1}^{[cT]} \zeta_{k_l}([\delta T]) \leq aT \right) \leq \left( \frac{1}{[\delta T]} \right)^{[cT]} \exp(aT).
\]

**Proof.** Since \( \zeta_{k_l}([\delta T]) \), \( 1 \leq l \leq [cT] \) are i.i.d., using Chebyshev inequality we obtain

\[
P\left( \sum_{l=1}^{[cT]} \zeta_{k_l}([\delta T]) \leq aT \right) = P\left( \exp \left( - \sum_{l=1}^{[cT]} \zeta_{k_l}([\delta T]) \right) \geq \exp(-aT) \right)
\]

\[
\leq \frac{E \exp \left( - \sum_{l=1}^{[cT]} \zeta_{k_l}([\delta T]) \right)}{\exp(-aT)} \leq \frac{E \left( E \exp(-\zeta_{k_1}([\delta T])) \right)^{[cT]}}{\exp(-aT)}
\]

\[
= \frac{\left( \frac{1}{[\delta T]} \sum_{r=1}^{[\delta T]} \exp(-r) \right)^{[cT]}}{\exp(-aT)} \leq \left( \frac{1}{[\delta T]} \right)^{[cT]} \exp(aT).
\]

□
Lemma 4.2. The family of random variables \( \frac{1}{T} (\nu_1(T) - \nu_1(T\Delta)) \), \( \Delta \in [0, 1) \) satisfies LDP with normalized function \( \psi(T) = T \) and rate function

\[
I_1(x) = \begin{cases} 
\infty, & \text{if } x \in (-\infty, 0), \\
x \ln \left( \frac{x(\lambda+\mu)}{\alpha(1-\Delta)} \right) - x + \frac{\alpha\lambda(1-\Delta)}{\lambda+\mu}, & \text{if } x \in [0, \infty). 
\end{cases}
\]

Proof. Random variable \( \frac{1}{T} (\nu_1(T) - \nu_1(T\Delta)) \) can be represented as a sum of independent random variables which have the same distribution as \( \nu_1(1-\Delta) \). Thus from [4, Theorem 2.2.3] it is enough to show that the Legendre transform of exponential moment of random variable \( \nu_1(1-\Delta) \) has the following form

\[
\Lambda(x) = \sup_{y \in \mathbb{R}} (xy - \ln \mathbb{E} e^{y\nu_1(1-\Delta)}) = x \ln \left( \frac{x(\lambda+\mu)}{\alpha(1-\Delta)} \right) - x + \frac{\alpha\lambda(1-\Delta)}{\lambda+\mu}, \quad x \geq 0.
\]

Since

\[
\mathbb{E} e^{y\nu_1(1-\Delta)} = \exp \left\{ -\alpha\mu(1-\Delta)\frac{T}{\lambda+\mu} + \frac{\alpha\mu(1-\Delta)c}{\lambda+\mu}T - Tc \ln c \right\},
\]

then the differential calculus finishes the proof. \( \square \)

Lemma 4.3. For any \( c \in [0, 1) \), \( \Delta \in [0, 1] \) the following inequality holds true

\[
P(\nu_2(T) - \nu_2(\Delta T) \leq cT) \leq \exp \left\{ -\frac{\alpha\mu(1-\Delta)}{\lambda+\mu}T + \frac{\alpha\mu(1-\Delta)c}{\lambda+\mu}T - Tc \ln c \right\}. \tag{19}
\]

Proof. For any \( r > 0 \) by Chebyshev inequality we have

\[
P(\nu_2(T) - \nu_2(\Delta T) \leq cT) = P \left( \exp \left\{ -r(\nu_2(T) - \nu_2(\Delta T)) \right\} \right) \leq \frac{\mathbb{E} \exp \left\{ -r(\nu_2(T) - \nu_2(\Delta T)) \right\}}{\exp \left\{ -rcT \right\}} = \exp \left\{ -r \frac{\alpha\mu(1-\Delta)}{\lambda+\mu}T + \frac{\alpha\mu(1-\Delta)c}{\lambda+\mu}T + rcT \right\}.
\]

Choosing \( r = -\ln c \) we obtain inequality (19). \( \square \)

Lemma 4.4. The following inequality holds true

\[
P \left( \bigcap_{l=1}^{\lfloor cT \rfloor} C_l, \bigcup_{l=1}^{\lfloor cT \rfloor} \sum_{l=1}^{\lfloor cT \rfloor} \zeta_{k_i}(\xi(\tau_i -)) \leq aT \right) \leq \left( \frac{1}{\delta T} \right)^{\lfloor cT \rfloor} \exp(aT),
\]

where \( C_l, 1 \leq l \leq \lfloor cT \rfloor \) are defined by [10] on the previous section.

Proof. Define random variables

\[
\tilde{\zeta}_{k_i}(m_i) := \begin{cases} 
\zeta_{k_i}(m_i), & \text{if } \zeta_{k_i}(m_i) \leq [\delta T], \\
\gamma_i, & \text{if } \zeta_{k_i}(m_i) > [\delta T],
\end{cases}
\]

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where random variables $\gamma_l$, $1 \leq l \leq [cT]$ are mutually independent and do not depend on $\zeta_k(t_l), \xi(t_{k_l})$, $m_l \in \mathbb{N}$, $1 \leq l \leq [cT]$, $\nu_1(\cdot), \nu_2(\cdot)$ and $P(\gamma_l = r) = \frac{r}{[\delta T]}$, $1 \leq r \leq [\delta T]$. 

Since $\tilde{\zeta}_{k_l}(\xi(t_{k_l})) \leq \zeta_k(\xi(t_{k_l}))$, then 

$$P\left(\bigcap_{l=1}^{[cT]} C_l, \bar{B}_c, \sum_{l=1}^{[cT]} \zeta_{k_l}(\xi(t_{k_l})) \leq aT\right) \leq P\left(\bigcap_{l=1}^{[cT]} C_l, \bar{B}_c, \sum_{l=1}^{[cT]} \tilde{\zeta}_{k_l}(\xi(t_{k_l})) \leq aT\right). \quad (20)$$

Denote 

$$V := \left\{ v_1 \in \mathbb{Z}^+, \ldots, v_{[cT]} \in \mathbb{Z}^+ : v_1 + \cdots + v_{[cT]} \leq aT, \max_{1 \leq l \leq [cT]} v_l \leq [\delta T] \right\}. $$

We have 

$$P\left(\bigcap_{l=1}^{[cT]} C_l, \bar{B}_c, \sum_{l=1}^{[cT]} \tilde{\zeta}_{k_l}(\xi(t_{k_l})) \leq aT\right) = \sum_{v_1, \ldots, v_{[cT]} \in V} P\left(\bigcap_{l=1}^{[cT]} C_l, \tilde{\zeta}_{k_l}(\xi(t_{k_l})) = v_1, \ldots, \tilde{\zeta}_{k_{[cT]}}(\xi(t_{k_{[cT]}})) = v_{[cT]}, \bar{B}_c\right)$$

$$= \sum_{v_1, \ldots, v_{[cT]} \in V} P\left(\bigcap_{l=1}^{[cT]} C_l, \bigcap_{l=1}^{[cT]} D_l, \bar{B}_c\right) = \sum_{v_1, \ldots, v_{[cT]} \in V} P\left(\bigcap_{l=1}^{[cT]} (C_l \cap D_l) \bigg| \bar{B}_c\right) P(\bar{B}_c),$$

where 

$$D_l := \left\{ \omega : \tilde{\zeta}_{k_l}(\xi(t_{k_l})) = v_l \right\}, \quad 1 \leq l \leq [cT].$$

Let $D_0 := \Omega, C_0 := \Omega$. We will show that for $1 \leq l \leq [cT]$ the following inequality holds 

$$P\left(D_l, C_l \bigg| \bigcap_{d=0}^{l-1} (C_d \cap D_d), \bar{B}_c\right) \leq \frac{1}{[\delta T]} \quad (21)$$

Note that by definition, the family of random variables $\tilde{\zeta}_{k_l}(m_l), m_l \in \mathbb{N}$ do not depend on $\tilde{\zeta}_{k_1}(m_1), m_1 \in \mathbb{N}, \ldots, \tilde{\zeta}_{k_{l-1}}(m_{l-1}), m_{l-1} \in \mathbb{N}$ and $\xi(t_{k_l}), \ldots, \xi(t_{k_{l-1}}), \nu_2(\cdot)$. Thus,

$$P\left(D_l, C_l \bigg| \bigcap_{d=0}^{l-1} (C_d \cap D_d), \bar{B}_c\right)$$

$$= P\left(D_l \bigg| \bigcap_{d=0}^{l-1} C_d \cap D_d, \bar{B}_c\right) P\left(C_l \bigg| \bigcap_{d=0}^{l-1} (C_d \cap D_d), \bar{B}_c\right) \leq P\left(D_l \bigg| \bigcap_{d=0}^{l-1} C_d \cap D_d, \bar{B}_c\right)$$

$$= \sum_{r=[\delta T]}^{\infty} P\left(\tilde{\zeta}_{k_l}(r) = v_l \bigg| \xi(t_{k_l}) = r, \bigcap_{d=0}^{l-1} C_d \cap D_d, \bar{B}_c\right) P\left(\xi(t_{k_l}) = r \bigg| \bigcap_{d=0}^{l-1} C_d \cap D_d, \bar{B}_c\right)$$

$$= \sum_{r=[\delta T]}^{\infty} \frac{1}{[\delta T]} P\left(\xi(t_{k_l}) = r \bigg| \bigcap_{d=0}^{l-1} C_d \cap D_d, \bar{B}_c\right) = \frac{1}{[\delta T]},$$

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where we used the fact that, if \( r \geq \lfloor \delta T \rfloor \), then \( P(\tilde{\zeta}_k(r) = v_l) = \frac{1}{\lfloor \delta T \rfloor} \). Using (21) we obtain

\[
P\left( \bigcap_{t=1}^{[cT]} C_t, \bigcap_{t=1}^{[cT]} D_t, B_c \right) = \prod_{t=1}^{[cT]} P\left( D_t, C_t \bigg| \bigcap_{d=0}^{l-1} (C_d \cap D_d), B_c \right) P(B_c) \leq \left( \frac{1}{\lfloor \delta T \rfloor} \right)^{[cT]}.
\]

Thus,

\[
P\left( \bigcap_{t=1}^{[cT]} C_t, B_c, \sum_{t=1}^{[cT]} \tilde{\zeta}_k(\xi(\tau_{k_l}^-)) \leq aT \right) \leq \sum_{v_1, \ldots, v_{[cT]} \in V} \left( \frac{1}{\lfloor \delta T \rfloor} \right)^{[cT]} = P\left( \sum_{t=1}^{[cT]} \zeta_{k_l}(\lfloor \delta T \rfloor) \leq aT \right).
\]

Therefore, from (20) and Lemma 4.1 it follows that

\[
P\left( \bigcap_{t=1}^{[cT]} C_t, B_c, \sum_{t=1}^{[cT]} \tilde{\zeta}_k(\xi(\tau_{k_l}^-)) \leq aT \right) \leq \left( \frac{1}{\lfloor \delta T \rfloor} \right)^{[cT]} \exp(aT).
\]

\[
\square
\]

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**References**

[1] Brockwell, P. J., Gani, J., Resnick, S. I. (1982). Birth, immigration and catastrophe processes. Advances in Applied Probability, 14(4), 709-731.

[2] Ben-Ari, I., Roitershtein, A., Schinazi, R. B. (2017). A random walk with catastrophes. arXiv preprint arXiv:1709.04780.

[3] Logachev, A., Logacheva, O., Yambartsev, A. (2018) The local large deviation principle for random walk with catastrophes. arXiv preprint arXiv:1806.07459.

[4] Dembo, A., Zeitouni, 0. (1998). Large deviations techniques and applications. Applications of Mathematics (New York), 38.