On minimizers of the maximal distance functional for a planar convex closed smooth curve

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Abstract

Fix a compact $M \subset \mathbb{R}^2$ and $r > 0$. A minimizer of the maximal distance functional is a connected set $\Sigma$ of the minimal length, such that
\[
\max_{y \in M} \text{dist} (y, \Sigma) \leq r.
\]
The problem of finding maximal distance minimizers is connected to the Steiner tree problem.

In this paper we consider the case of a convex closed curve $M$, with the minimal radius of curvature greater than $r$ (it implies that $M$ is smooth). The first part is devoted to statements on structure of $\Sigma$: we show that the closure of an arbitrary connected component of $B_r(M) \cap \Sigma$ is a local Steiner tree which connects no more than five vertices.

In the second part we “derive in the picture”. Assume that the left and right neighborhoods of $y \in M$ are contained in $r$-neighborhoods of different points $x_1, x_2 \in \Sigma$. We write conditions on the behavior of $\Sigma$ in the neighborhoods of $x_1$ and $x_2$ under the assumption by moving $y$ along $M$.

1 Introduction

For a given compact set $M \subset \mathbb{R}^2$ consider the maximal distance functional
\[
F_M(\Sigma) := \max_{y \in M} \text{dist} (y, \Sigma),
\]
where $\Sigma$ is a compact planar set, and $\text{dist} (y, \Sigma)$ stands for the Euclidean distance between $y$ and $\Sigma$. Also $F_M(\emptyset) := \infty$.

Consider the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ such that $F_M(\Sigma) \leq r$ for a given $r > 0$. We are interested in the properties of sets of minimal length (one-dimensional Hausdorff measure) $H^1(\Sigma)$ over the mentioned class. Further we call such sets minimizers.

It is known that the set of minimizers is non-empty. It is also known that every minimizer $\Sigma$ of positive length satisfies $F_M(\Sigma) = r$. Also in this case the set of minimizers coincides with the set of solutions of the corresponding dual problem: to minimize $F_M$ among the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ with the prescribed bound on the length $H^1(\Sigma) \leq l$ (that is the reason of calling the desired set minimizers of maximal distance functional). General statements and details of the mentioned results can be found in \cite{4}.

Let $B_r(x)$ be the open ball of radius $r$ centered at a point $x$. Let $B_r(M)$ be the open $r$-neighborhood of $M$:
\[
B_r(M) := \bigcup_{x \in M} B_r(x).
\]

1.1 Properties of $\Sigma$ for general $M$

In this subsection $M$ is an arbitrary planar compact.

Note that $\Sigma$ is bounded (and hence compact), since $\Sigma \subset \overline{B_r(\text{conv} M)}$, where conv $M$ stands for the convex hull of $M$.

Definition 1.1. A point $x \in \Sigma$ is called energetic, if for all $\rho > 0$ the set $\Sigma \setminus B_\rho(x)$ does not cover $M$ i.e.
\[
\text{dist} (M, \Sigma \setminus B_\rho(x)) > r.
\]
Denote the set of energetic points by $G_\Sigma$.

Every minimizer $\Sigma$ can be split into three disjoint subsets:
\[
\Sigma = E_\Sigma \sqcup X_\Sigma \sqcup S_\Sigma,
\]
where \( X_{\Sigma} \subseteq G_{\Sigma} \) is the set of isolated energetic points (i.e., every \( x \in X_{\Sigma} \) is energetic and there is a \( \rho > 0 \) such that \( B_{\rho}(x) \cap G_{\Sigma} = \{x\} \)), \( E_{\Sigma} := G_{\Sigma} \setminus X_{\Sigma} \) is the set of non-isolated energetic points and \( S_{\Sigma} := \Sigma \setminus G_{\Sigma} \) is the set of non-energetic points also called the Steiner part of \( \Sigma \).

The following basic properties of minimizers has been proved in [4] (for planar \( M \)) and in [6] (for \( M \subset \mathbb{R}^n \)):

1. minimizers contain no cycles (homeomorphic images of circumference).
2. For every energetic \( x \in G_{\Sigma} \) there is a point \( y \in M \), such that \(|x - y| = r\) and \( B_r(y) \cap \Sigma = \emptyset \). Further we call \( y \) corresponding to \( x \) and denote by \( y(x) \). Note that a corresponding point may be not unique.
3. For every non-energetic \( x \in S_{\Sigma} \) there is an \( \varepsilon > 0 \), such that \( \Sigma \cap B_\varepsilon(x) \) is either a segment or a regular tripod, i.e. the union of three line segments with an endpoint in \( x \) and relative angles of \( 2\pi/3 \).

**Theorem 1.2** (Teplitskaya, [7, 8]). Let \( \Sigma \) be a maximal distance minimizer for a compact set \( M \subseteq \mathbb{R}^2, r > 0 \). We say that the ray \( (ax) \) is a tangent ray of the set \( \Sigma \) at the point \( x \in \Sigma \) if there exists non stabilized sequence of points \( x_k \in \Sigma \) such that \( x_k \to x \) and \( \angle x_kxa \to 0 \). Then

- (i) \( \Sigma \) is a union of a finite number of injective images of the segment \([0, 1] \);
- (ii) the angle between each pair of tangent rays at every point of \( \Sigma \) is greater or equal to \( 2\pi/3 \);
- (iii) the number of tangent rays at every point of \( \Sigma \) is not greater than 3. If it is equal to 3, then there exists such a neighbourhood of \( x \) that the arcs in it coincide with line segments and the pairwise angles between them are equal to \( 2\pi/3 \).

1.2 The class of \( M \), considered in the paper

Fix a positive real \( r \) and a closed convex curve \( M \) with the minimal radius of curvature \( R > r \) (this implies \( C^{1,1} \)-smoothness of \( M \)). Introduce the notation: \( N := \text{conv}(M) \); let \( M_r \) be the inner part of the boundary of \( B_r(M) \), and finally put \( N_r = \text{conv}(M_r) \). Note that \( M_r \) also is a closed convex curve \( M \) with the minimal radius of curvature \( R - r \).

![Figure 1: Definition of \( N \), \( M_r \) and \( N_r \)](image)

Further \( \Sigma \) denotes an arbitrary minimizer for \( M \).

1.3 The problem for particular \( M \)

Finding the set of minimizers for almost every particular \( M \) is quite difficult. There are the following results.

**Theorem 1.3** (Cherkashin – Teplitskaya, 2018 [1]). Let \( r \) be a positive real, \( M \) be a convex closed curve with the radius of curvature at least \( 5r \) at every point, \( \Sigma \) be an arbitrary minimizer for \( M \). Then \( \Sigma \) is a union of an arc of \( M_r \) and two segments, that are tangent to \( M_r \) at the ends of the arc (so-called horseshoe, see Fig. 2). In the case when \( M \) is a circumference with radius \( R \), the claim is true for \( R > 4.98r \).

We prepare the paper with the following theorem.

**Theorem 1.4.** Let \( M = A_1A_2A_3A_4 \) be a rectangle, \( 0 < r < r_0(M) \). Then a maximal distance minimizer has the following topology, depicted in the left part of Fig. 3. The middle part of the picture contains enlarged fragment of the minimizer near \( A_1 \); the labeled angles are equal to \( \frac{2\pi}{3} \). The rightmost part contains much more enlarged fragment of minimizer near \( A_1 \).

A minimizer consists of 21 segments; an approximation of the length of a minimizer is \( \text{Per} ≈ 8.473981r \), where \( \text{Per} \) is the perimeter of the rectangle.
Structure of the paper. Section 2 contains an introduction to the Steiner problem. Section 3 is devoted to structural properties of $\Sigma$. In Section 4 we “derive in the picture”. Finally, Section 5 contains applications of our methods and open questions.

2 Steiner tree problem

Consider a finite set of points $C := \{A_1, \ldots, A_n\} \subset \mathbb{R}^2$. A Steiner tree is a connected set $S \subset \mathbb{R}^2$, which contains $C$ and has minimal possible length. It is known that such $S$ always exists (but is not necessarily unique) and that it is the union of a finite set of segments. Thus, $S$ can be represented as a plane graph, such that its set of vertices contains $C$, and all its edges are straight line segments. This graph is connected and does not contain cycles, i.e. is a tree, which explains the naming of $S$. It is known that the maximum degree of this graph is no greater than 3. Moreover, only vertices $A_i$ can have degree 1 or 2, all the other vertices have degree 3 and are called Steiner points. There are no more than $n - 2$ Steiner points. The angle between any two adjacent edges is at least $2\pi/3$. That means that for a Steiner point the angle between any two edges incident to it is exactly $2\pi/3$. $S$ is called a full Steiner tree, if the degree of each $A_i$ is 1, or, equivalently, if the number of Steiner points is $n - 2$. A (full) Steiner forest is a set, each connected component of which is a (full) Steiner tree. Proof of the listed properties of Steiner trees and additional information on them can be found in the book [3] and in the article [2].

We define a local Steiner tree as a connected compact acyclic set $S$, which contains $C$, and such that for any $x \in S \setminus C$ there is a neighborhood $U \ni x$ such that $S \cap U$ coincides with the Steiner tree on the set of points $S \cap \partial U$. A local Steiner tree retains the following properties of a Steiner tree: it is the union of a finite set of segments; the angle between any two adjacent segments is at least $2\pi/3$. We are going to use the following fact: a connected closed subset of a local Steiner tree is itself a local Steiner tree.

For a given tree $T$ we denote the set of its vertices of degree 1 or 2 as $\partial T$.

**Definition 2.1.** Define a “wind rose” as a set of six rays starting at the origin point with angle $\pi/3$ between any two adjacent rays; each ray is given a weight (a real number), which satisfy the following property: the weight of a ray is the sum of weights of two rays adjacent to it. (It follows, in particular, that the sum of the weights of two opposite rays (the ones forming a line) is zero.)

By full Steiner pseudo-network let us call a connected set $S$ which contains $C$, if for any wind rose $\mathcal{R}$ such that

(i) $S$ consists of finite number of segments which are parallel to $\mathcal{R}$

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the following holds:

(ii) for any \( x \in S \setminus C \) and small enough \( \varepsilon > 0 \), sum of weights of rays of \( R \) which are parallel to rays of the form \([xy]\), \( y \in \partial B_r(x) \cap S \), is zero.

It is clear that full (local) Steiner tree is a full Steiner pseudo-network.

For a given pseudo-network \( T \) let us denote by \( \partial T \) set of vertices of degree 1.

Remark 2.2. Suppose that \( T \) is a full Steiner pseudo-network, and \( R \) is an arbitrary wind rose satisfying (i). Let us assign to each vertex \( x \in \partial T \) a weight of a ray of \( R \), which is parallel to a directed segment of \( T \) entering \( x \) (such segment is unique by definition of \( \partial T \)). Then sum of assigned numbers over all \( x \in \partial T \) is zero.

Lemma 2.3. Let \( T \) be a full Steiner pseudo-network, \( l \) be a line such that \( T \not\subset l \). Then

\[
\sharp(\partial T \cap l) \leq 2\sharp(\partial T \setminus l).
\]

Proof of Lemma 2.3. Let \( l^+, l^- \) be the two open half-planes bounded by \( l \). Note that it is sufficient to prove the inequality for a closure of an arbitrary component of \( \overline{T} \cap l^+ \) and \( \overline{T} \cap l^- \), denote such closure as \( S \).

Consider a wind rose with the origin in the same open half-plane as \( S \), such that the rays with positive weights are exactly the ones intersecting \( l \): such wind rose exists, because \( l \) intersects either 2 or 3 consecutive (in the counter-clockwise order) rays. In the former case we give these rays weights 1, 1, in the latter case — 1, 2, 1. The remaining rays will have weights 0, \(-1, -1, 0 \) or \(-1, -2, -1 \) (the weights are listed in counter-clockwise order in each case). We assign weights to all leaf vertices in the way described in Remark 2.2. Then the sum of weights over the leaf vertices lying on \( l \) is at least \( \sharp(S \cap l) \). Since, according to Remark 2.2 the sum over all leaf vertices should be zero, there are at least \( \frac{\sharp(S \cap l)}{2} \) leaf vertices not lying on \( l \).

Remark 2.4. Let \( T \) be a full Steiner pseudo-network fully lying on one side of line \( l \), such that equality in Lemma 2.3 is achieved. Then all leaf vertices in \( \partial T \setminus l \) have weight \(-2 \), therefore all segments of \( T \) incident to vertices from \( \partial T \setminus l \) are pairwise collinear.

3 Structural properties of minimizers

Recall that we work in the setting from Subsection 1.2.

Note that \( \Sigma \subset N \) (\( N \) is convex, so one can project the part of \( \Sigma \) belonging to \( \mathbb{R}^2 \setminus N \) on \( N \) and length of \( \Sigma \) will strictly decrease).

Consider the closure of an arbitrary connected component of \( \Sigma \setminus N_r \); denote it by \( S \). Points from \( S \cap M_r \) are called entering points. Connectedness of \( S \) implies that \( \overline{B_r}(S) \cap M \) is a closed arc; denote it by \( q(S) \).

The following lemma is proved in \( \square \) (the proof of these statements does not use the additional requirement \( R > 5r \), which is inherited from the main theorem of the paper \( \square \)).

Lemma 3.1. Let \( S \) be the closure of a connected component of \( \Sigma \setminus N_r \). Then

(i) \( S \) is a local Steiner tree connecting the set of entering points of \( S \) and energetic points of \( S \);

(ii) \( S \) contains one or two energetic points.

(iii) Suppose that \( S \) contains 2 energetic points \( x_1 \) and \( x_2 \). Then

(i) there are unique points \( y(x_1) \) and \( y(x_2) \);

(ii) if \( x_i \) has degree 1 (i.e. \( x_i \) is the end of a line segment \([z_i x_i] \subset \Sigma \)), then \( z_i, x_i \) and \( y(x_i) \) are collinear;

(iii) if \( x_i \) has degree 2 (i.e. \( x_i \) is the end of a line segments \([z_i^1 x_i], [x_i z_i^2] \subset \Sigma \)), then ray \([y(x_i) x_i]\) contains the bisector of \( z_i^1 x_i z_i^2 \).

In this section we prove the following statement.

Proposition 3.2. Let \( S \) be the closure of a connected component of \( \Sigma \setminus N_r \). Then

(i) the convex hull of \( S \) is a line segment, a triangle or a quadrangle; the vertices of convex hull are only energetic or entering points of \( S \), the latter no more than 2;

(ii) \( S \) has at most 3 entering points.
Proof of (i). Since $S$ is a local Steiner tree for its entering and energetic points, every other point $x$ is a convex combination of points from a neighborhood of $x$. So it is enough to show that all entering points except at most two lie in the interior of $\text{conv}(S)$.

Suppose the contrary and consider maximal (by inclusion) arc $A \subset M_r$ ending by entering points of $S$ (further we call them extreme), that $A \subset B_r(q(S))$. Consider an arbitrary entering point $x$ lying in the interior of the arc and set the tangent line to $M_r$ at $x$. Since $N_r$ is convex, connected component $S$ contains points in the both sides of the tangent line, say $t_1$ and $t_2$. Then $x$ is a convex combination of $t_1$ and $t_2$; which is a contradiction.

Proof of (ii). Denote extreme (defined analogously to the previous proof) entering points by $Y_1$ and $Y_2$. For every other entering point $Y \in S$ denote by $R = R(Y)$ a continuation of a segment of $S$ which contains $Y$ beyond the point $Y$. Let us show that $R$ intersects with a line $Y_1Y_2$. Note that $Y_1$, $Y$, $Y_2$ are contained in arc $M_r \cap B_r(q(S)) = Q(S)$.

Suppose the contrary, that is $R$ neither meets again with $M_r$ at $U \in Q(S)$ or is tangent to the $M_r$. Let us show that on the arc $YU \subset Q(S)$ there will be an entering point $Y'$ that belongs to the closure of another connected component $S'$. If $R$ is tangent to $M_r$ in $Y$, then $Y$ lies in a closure of another component, therefore we can put $Y' = Y$. Otherwise $Y$ lies in a closure of a component of $\Sigma \cap \text{Int}(N_r)$, denote this component as $T$. Since $\Sigma \cap \text{Int}(N_r)$ lies in the Steiner part of $\Sigma$, a closure of any component of $\Sigma \cap \text{Int}(N_r)$ is a full local Steiner tree. Then $\partial T \setminus \{Y\}$ contains a point in each of closed half-planes divided by $YU$; since $\partial T \subset M_r$, there is a vertex $Y'' \in \partial T$ on the arc $YU$, since $\Sigma$ is acyclic, $Y'$ is not contained in $S$.

But then $q(S') \subset Q(S)$ or $S \cap S' \neq \emptyset$. It follows from the first option that $S'$ contains no energetic points; the second option is impossible by definition.

For each entering point $Y$ denote by $I(Y)$ an intersection of $R(Y)$ with $Y_1Y_2$. By $St(S)$ we denote the union of $S$ and all segments $[YI(Y)]$.

Let us consider two cases.

(a) Let $S$ have one energetic point $x$.

- If $x$ has degree 1, then $St(S)$ is a full Steiner pseudo-network, then by application of Lemma 2.3 to $St(S)$ and $Y_1Y_2$, we have that $St(S)$ intersects with $Y_1Y_2$ at most two times, thus $S$ has at most two entering points.
- If $x$ has degree 2 then $St(S)$, being cut in the point $x$, falls apart into two full networks $St_1$ and $St_2$. Let one of them have at least two entering points (say, $St_1$). By Lemma 2.3 or $St_1$ and $Y_1Y_2$, it can be only a tripod; denote by $V_1$ the branching point of the tripod and by $U_1^1$ and $U_1^2$ the points of intersection with $Y_1Y_2$. Let $St_2$ also be a tripod with branching point $V_2$ and with points $U_2^1$ and $U_2^2$ on the line $Y_1Y_2$; without loss of generality we can assume that points $U_1^2$ and $U_2^1$ are lying between points $U_1^1$ and $U_2^2$. Then the sum of angles of pentagon $U_1^1V_1xV_2U_2^2$ is at least $10\pi/3$, because $\angle U_1^1V_1x = \angle xV_2U_2^2 = 2\pi/3$ (these angles are external for the pentagon and corresponding inside angles are equal to $4\pi/3$). That is a contradiction.

Summing up, no subtree can contain three entering points, and subtreess can not both contain two entering points simultaneously, which finishes this case.

(b) Let $S$ have two energetic points $x_1$ and $x_2$. Consider a polygon $P$ (see the right-hand side of Figure 4), that bounded by $S$, by segments $[x_1y(x_1)]$, $[x_2y(x_2)]$ and by tangents to $M$ in points $y(x_1)$ and $y(x_2)$ (by Lemma 2.3 points $y(x_1)$ and $y(x_2)$ are unique). Note that $P$ is convex and its angles at vertices from $S$ are at most $2\pi/3$. Since $B_r(y(x_1)) \cap \Sigma = B_r(y(x_2)) \cap \Sigma = \emptyset$, angles at $y(x_1)$ and $y(x_2)$ are at most $\pi/2$; by Lemma 5.1(iii) the line $y(x_1)x_1$ contains a side of $P$, thus angle at $y(x_1)$ is less than $\pi/2$. If $P$ has at least 3 vertices from $S$, then the sum of external angles of $P$ is strictly greater than $3\pi/3 + 2\pi/2 = 2\pi$, what is impossible. Therefore, $P$ contains no more than two vertices from $S$.

Let $x_1$ have degree 1. Then, if it is connected with $x_{3-i}$ by a segment of $\Sigma$, segment $[x_1x_2]$ can be removed from $St(S)$ and by Lemma 2.3 remaining full Steiner network has no more than two points of intersection with $Y_1Y_2$, so $S$ has no more than two entering points. In the other case $x_1$ is connected by a segment of $\Sigma$ with a branching point $V_i$. Let $v_i = V_i$, if $x_1$ has degree 1, and $v_i = x_1$, if $x_1$ has degree 2 (see the left-hand side of Figure 4). Then $v_1$ and $v_2$ are vertices of $P$. Since $P$ contains no more than 2 vertices from $S$, it turns out that either $v_1 = v_2$, or $[v_1v_2] \subset \Sigma$. 

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Figure 4: Illustration to the proof of Lemma 3.1(ii)
If $v_1 = v_2$ then after deleting line segments $[v_1x_1]$ and $[v_2x_2]$ from $S$, application of Lemma 2.3 to line $Y_1Y_2$ gives that $S$ has at most two entering points.

If $[v_1v_2] \subset \Sigma$, then we consider two cases.

- The case where both points $x_1$ and $x_2$ have degree 1; in this case $S$ is a full Steiner pseudo-network. The application of Lemma 2.3 to the line $Y_1Y_2$ gives that $S$ contains at most 4 entering points, moreover if $S$ has 4 entering points then equality in lemma is achieved and by Remark 2.4 rays $[x_1y(x_1))$ and $[x_2y(x_2))$ have similar direction. Then the pass between $x_1$ and $x_2$ in $P$ has 3 branching points but $P$ has at most 2 vertices from $S$; which is a contradiction.

- The case where at least one of points $x_1$ and $x_2$ has degree 2. Removing line segment $[v_1v_2]$ splits $S \varepsilon(S)$ into two subnetworks $S_{t_1}$ and $S_{t_2}$. Suppose that one of them has at least two entering points (say, $S_{t_1}$). Note that $S_{t_1} \setminus [x_1v_1]$ is a full pseudo-network, so by Lemma 2.3 it is a tripod; denote by $V$ the branching point of the tripod, and by $U_1, U_2$ the entering points in such a way that $M_r$ contains points $Y_1, U_1, U_2, Y_2$ in the mentioned order (some points may coincide). Then vector $U_1Y$ is directed away from line $Y_1Y_2$, hence vector $\overrightarrow{v_1v_2}$ is also directed away from line $Y_1Y_2$.

Summing up, no subtree can contain three entering points, and subtrees can not both contain two entering points simultaneously, which finishes the proof.

4 Derivation in the picture

Consider a point $y \in M$ such that $B_r(y) \cap \Sigma = \emptyset$. Suppose there exists an energetic point $x \in \partial B_r(y) \setminus M_r$. Our goal is to determine how the length of $\Sigma$ in the vicinity of point $x$ changes with the infinitesimal movement of $y$ along $M$ (and the corresponding movement of $x$). We are going to consider all possible options for the local structure of $\Sigma$ in the vicinity of $x$. Since radius of curvature of $M$ is greater than $r$, each $x$ corresponds to no more than two distinct $y$. Additionally, the degree of $x$ is either 1 or 2. Therefore, there are 4 cases to consider.

In all cases below we are going to move point $y$ along $M$ a distance $\varepsilon$ in such direction that the length of the arc covered by point $x$ increases (it changes a minimizer in a neighborhood of $x$). The substitution of negative $\varepsilon$ corresponds to moving $y$ along $M$ in the opposite direction.

**Case 1.** The degree of point $x$ is 1 (so $x$ is the end of some segment $[zx] \subset \Sigma$) and $y(x)$ is unique (look at the left half of Fig. 5). Points $z, x,$ and $y(x)$ lie on one line by Lemma 3.1. Let $|zx| = l$, let $\alpha$ be the angle between $(yz(x))$ and $M$. We obtain the point $y(x_\varepsilon)$ by moving $y(x)$ a sufficiently small distance $\varepsilon$ along $M$; let $x_\varepsilon := [zy(x_\varepsilon)] \cap \partial B_r(y(x_\varepsilon)))$. $M$ is smooth, so the distance between point $y(x_\varepsilon)$ and the tangent to $M$ at point $y(x)$ is $o(\varepsilon)$. By cosine rule for triangle $zy(x)y(x_\varepsilon)$,

$$|zy(x_\varepsilon)| = \sqrt{|zy(x)|^2 + 2|zy(x)||z\varepsilon| + o(\varepsilon^2) = |zy(x)| + |z\varepsilon| + o(\varepsilon^2).$$

Therefore the derivative of the length of $\Sigma$ in the vicinity of $x$ with respect to the movement of $y(x)$ along $M$ in this case is $\cos \alpha$.

![Figure 5: The first and second cases](image-url)
Therefore, by the cosine rule for triangle $y$.

Denote the angle between $\angle \beta$ between the last line and $\angle \delta$.

Combining (1) and (2), we conclude that the derivative is

$$l_{new} = \sqrt{2 + (\varepsilon \cos \alpha + o(\varepsilon))^2 + 2((\varepsilon \cos \alpha + o(\varepsilon) \varepsilon \sin \delta + o(\varepsilon)) \cos \beta = l + \varepsilon \cos \alpha \cos \beta + o(\varepsilon).$$

Combining (1) and (2), we conclude that the derivative is

$$2 \cos \alpha \cos \beta.$$

In cases 3 and 4 $x$ corresponds to two points: $y_1(x)$ and $y_2(x)$. Let $y_1(x) = y_1(x_e)$, and let the point $y_2(x_e)$ be obtained from $y_2(x)$ by moving it along $M$ a (possibly negative) distance $\varepsilon$. This uniquely determines the point $x_e := \partial B_r(y_1(x)) \cap \partial B_r(y_2(x_e)) \cap N$. Let us find this point explicitly (look at the left half of Fig. 6).

The triangle $xy_1(x)y_2(x)$ is isosceles with two sides of length $\varepsilon$; let $\angle xy_1(x)y_2(x) = \angle xy_2(x)y_1(x) =: \alpha$, $\angle xy_1(x)y_2(x) = \angle xy_2(x)y_1(x) =: \alpha_e$.

We define the following coordinate system: the middle point of the segment $y_1(x)y_2(x)$ is the origin $O$; the $x$ axis is collinear to the ray $[y_1(x)y_2(x)]$; the $y$ axis is collinear to the ray $[Ox]$. Then

$$O = (0, 0), \quad x = (0, r \sin \alpha), \quad y_1(x) = (-r \cos \alpha, 0), \quad y_2(x) = (r \cos \alpha, 0).$$

Denote the angle between $y_1(x)y_2(x)$ and $M$ as $\delta$. Then

$$y_2(x_e) = (r \cos \alpha + \varepsilon \cos \delta + o(\varepsilon), \varepsilon \sin \delta + o(\varepsilon)).$$

Therefore, by the cosine rule for triangle $y_1(x)y_2(x)y_2(x_e)$,

$$|y_1(x)y_2(x_e)| = \sqrt{(2r \cos \alpha + \varepsilon \cos \delta + o(\varepsilon))^2 + (\varepsilon \sin \delta + o(\varepsilon))^2} = 2r \cos \alpha + \varepsilon \cos \delta + o(\varepsilon).$$

Let $O_\varepsilon$ be the middle point of segment $[y_1(x_e)y_2(x_e)]$. Then

$$O_\varepsilon = \left(\frac{\varepsilon \cos \delta}{2} + o(\varepsilon), \frac{\varepsilon \sin \delta}{2} + o(\varepsilon)\right).$$

By the definition of the cosine function,

$$\alpha_\varepsilon = \arccos \left(\frac{|y_1(x)O_\varepsilon|}{r}\right) = \arccos \left(\cos \alpha + \varepsilon \cos \frac{\delta}{2r} + o(\varepsilon)\right) = \alpha - \frac{\cos \delta}{2r \sin \alpha} \varepsilon + o(\varepsilon).$$

Let $\Delta$ be the directed angle $\angle y_2(x)y_1(x)y_2(x_e)$ (so $\Delta < 0$ when $\varepsilon$ is negative). By the sine rule for the triangle $y_2(x)y_1(x)y_2(x_e)$,

$$\frac{\varepsilon}{\sin \Delta} = \frac{|y_1(x)y_2(x)|}{\sin(\delta - \Delta + o(\varepsilon))} \geq |y_1(x)y_2(x)|, \quad \text{so} \quad \Delta = O(\varepsilon).$$

Therefore,

$$\Delta = \sin \Delta + o(\varepsilon) = \frac{\varepsilon \sin(\delta + O(\varepsilon))}{|y_1(x)y_2(x)|} = \frac{\varepsilon \sin \delta}{2r \cos \alpha} + o(\varepsilon).$$
Writing out the sum of angles in the isosceles triangle $xy_1(x)ε$, we get

$$\angle xy_1(x)ε = \alpha - \alpha_ε - \Delta = \left(\frac{\cos \delta}{2r \sin \alpha} - \frac{\sin \delta}{2r \cos \alpha}\right) \varepsilon + o(\varepsilon) = \frac{\cos(\alpha + \delta)}{r \sin(2\alpha)} \varepsilon + o(\varepsilon).$$

It follows that

$$|xx_ε| = 2r \sin \frac{\angle xy_1(x)ε}{2} = \frac{\cos(\alpha + \delta)}{\sin(2\alpha)} \varepsilon + o(\varepsilon),$$

and the angle between the segment $xx_ε$ and the $x$ axis (look at the right half of Fig. 6) is

$$\pi - \alpha - \frac{\pi - \angle xy_1(x)ε}{2} = \frac{\pi}{2} - \alpha + \frac{\cos(\alpha + \delta)}{2r \sin(2\alpha)} \varepsilon + o(\varepsilon) = \frac{\pi}{2} - \alpha + o(1).$$

Case 3. The degree of point $x$ is 1 (so $x$ is the end of some segment $[zx] \subset \Sigma$) and there are two distinct points $y_1(x)$ and $y_2(x)$.

Let $\beta$ be the angle between $[zx]$ and the $x$ axis (look at the right half of Fig. 6). Then

$$\angle zxx_ε = \frac{3\pi}{2} - \alpha - \beta + o(1).$$

By the cosine rule for the triangle $zxx_ε$,

$$|xx_ε| = \sqrt{|zx|^2 - 2|xx_ε||zx| \cos \angle zxx_ε + |xx_ε|^2} = |zx| - |xx_ε| \cos \angle zxx_ε + o(\varepsilon) = |zx| + \frac{\cos(\alpha + \delta) \sin(\alpha + \beta)}{\sin(2\alpha)} \varepsilon + o(\varepsilon).$$

So the derivative is equal to

$$\frac{\cos(\alpha + \delta) \sin(\alpha + \beta)}{\sin(2\alpha)}.$$

Case 4. The degree of point $x$ is 2 (so it is the end of some segments $[z_1x], [xz_2] \subset \Sigma$) and there are two distinct points $y_1(x)$ and $y_2(x)$. Similar to the previous case, the derivative is equal to

$$\frac{\cos(\alpha + \delta)}{\sin(2\alpha)} (\sin(\alpha + \beta) + \sin(\alpha + \gamma)),$$

where $\beta$ and $\gamma$ are the angles between the $x$ axis and the segments $[z_1x]$ and $[z_2x]$, respectively.

Transitions between the cases. Note that the second case can transform into the first case, and the other way around; similarly, the third case can turn into the fourth and vice versa. The value of the derivative does not change in such transitions because

$$2 \cos \beta \cos \alpha = \cos \alpha$$ when $\beta = \pi/3;$
Proposition 4.1. Let \( x \in \Sigma \) be an energetic point, \( y(x) \in M \) be an arbitrary corresponding point. Then the derivative of length of \( \Sigma \) in a neighborhood of \( x \) in the moving \( y \) along \( M \) is nonnegative.

Proof. Suppose the contrary. Then one may shift \( y \) along \( M \) and the length of \( \Sigma \) will strictly decrease. Note that \( \Sigma \) is still connected and \( M \) is still covered by \( \Sigma \); which is a contradiction. \( \square \)

Proposition 4.2. Let \( y \in M \) be a point such that \( B_r(y) \cap \Sigma = \emptyset \) and \( \partial B_r(y) \) contains energetic points \( x_1 \) and \( x_2 \). Define \( Y = \partial B_r(y) \cap M_r \). Then

(i) points \( x_1 \) and \( x_2 \) lie on the opposite sides of the line \( (yY) \);
(ii) derivative of length of \( \Sigma \) in neighborhoods of \( x_1 \) and \( x_2 \) in the moving \( y \) along \( M \) are equal.

Proof. Suppose the contrary to item (i); without loss of generality, \( \angle Y x_1 > \angle Y x_2 \). Then

\[
B_r(B_r(x_1) \cap \Sigma) \cap M \subset B_r(x_2),
\]

where \( \rho > 0 \) is small enough. Thus \( x_1 \) is not energetic, which is a contradiction.

Now suppose the contrary to item (ii). Without loss of generality, the derivative of the length of \( \Sigma \) in a neighborhood of \( x_1 \) is bigger than the derivative in a neighborhood of \( x_2 \). Then after a shifting of \( y \) along \( M \) from \( x_2 \) to \( x_1 \) the length of \( \Sigma \) strictly decreases. Note that \( \Sigma \) is still connected and \( M \) is still covered by \( \Sigma \); this gives a contradiction. \( \square \)

5 Applications and open problems

- Sometimes it is possible to “derive in the picture” in the case of a partially smooth \( M \). For this purpose one has to clarify the behavior of a considered competitor in a neighborhood of \( B_r(y) \), with \( y \) lying in the smooth part of \( M \).
  
  For instance we use an analog of Statement 4.2 during the pruning of cases in the proof of Theorem 1.4.

- Miranda, Paolini and Stepanov [5] conjectured that all the minimizers for a circumference of radius \( R > r \) are horseshoes. Theorem 1.3 proves this conjecture with assumption \( R > 4.98r \); for \( 4.98r \geq R > r \) the conjecture remains open.

- At the same time, the statement of Theorem 1.3 for general \( M \) needs an assumption on the minimal radius of curvature as we show below.

Define a stadium as the boundary of the \( R \)-neighborhood of a segment. By the definition, stadium has the minimal radius of curvature \( R \). If \( R < 1.75r \) and a stadium is long enough, then there is a connected set \( \Sigma' \) that has smaller length than an arbitrary horseshoe and covers \( M \).

![Figure 7: Horseshoe is not a minimizer for long enough stadium with \( R < 1.75r \).](image)

Define \( \Sigma_0 \) as a locally Steiner tree depicted in Fig. 7. Let \( \Sigma' \) consist of copies of \( \Sigma_0 \), glued at points \( A \) and \( B \) along the length of the stadium. In the case \( R < 1.75r \) the length of \( \Sigma_0 \) is strictly smaller than \( 2|AB| \). Thus for long enough stadium \( \Sigma' \) has length \( cL + O(1) \), where \( L \) is the length of the stadium and \( c < 2 \) is a constant depend on \( \Sigma_0 \) and \( R \). Obviously, any horseshoe has length \( 2L + O(1) \).

This example leads to the following problems.

**Problem 5.1.** Find the minimal \( c \) such that Theorem 1.3 holds with the replacement of \( 5r \) with \( cr \).

**Problem 5.2.** Find the set of minimizers for a given stadium.
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