Boundedness of meta-conformal two-point functions in one and two spatial dimensions

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Abstract

Meta-conformal invariance is a novel class of dynamical symmetries, with dynamical exponent $z = 1$, and distinct from the standard ortho-conformal invariance. The meta-conformal Ward identities can be directly read off from the Lie algebra generators, but this procedure implicitly assumes that the co-variant correlators should depend holomorphically on time- and space coordinates. Furthermore, this assumption implies un-physical singularities in the co-variant correlators. A careful reformulation of the global meta-conformal Ward identities in a dualised space, combined with a regularity postulate, leads to bounded and regular expressions for the co-variant two-point functions, both in $d = 1$ and $d = 2$ spatial dimensions.
1 Introduction

Dynamical symmetries have been an extremely useful tool in the analysis of strongly interacting many-body systems. Probably the most striking example are the (ortho-)conformal transformations in (1 + 1)-dimensional time-space [11], for textbooks see [30, 34] or [54, 13, 38].

Expressed in orthogonal light-cone coordinates $z = t + i r$, $\bar{z} = t - i r$, these transformations are isomorphic to the direct product of holomorphic transformations $f(z)$ and anti-holomorphic transformation $\bar{f}(\bar{z})$, see table 1 whose Lie algebra is the direct sum of two Virasoro algebras. For ortho-conformal invariance, a convenient Lie algebra basis is spanned by the generators $\ell_1 = \frac{1}{2}(\partial_t + \partial_r)$, $\ell_0 = \frac{1}{2}(\partial_t - \partial_r)$ for the variable $z$ with the weight $\Delta = 1$, while $\ell_{N-1} = \frac{1}{2}(\partial_t + \partial_r)$ and $\ell_{N} = \frac{1}{2}(\partial_t - \partial_r)$ for the variable $\bar{z}$ with the weight $\Delta = 1$. These transformations are rotation-invariant. The most simple example of this is the conformally invariant $(1 + 1)D$ Laplace equation $\partial_r \partial_t \phi(z, \bar{z}) = 0$.

The one-particle generators $\ell_j$ can be lifted to $N$-particle generators $\ell_{j}^{[N]} = \sum_{p=1}^{N} \ell_j^{(p)}$, where $\ell_j^{(p)}$ gives the action of the generator $\ell_j$ on the $p$th scaling operator $\phi_p = \phi_p(z_p, \bar{z}_p)$. One of the most immediate applications of such symmetries is the derivation of co-variant $N$-point functions $C^{[N]} = C^{[N]}(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = \langle \phi_1(z_1, \bar{z}_1)\ldots\phi_N(z_N, \bar{z}_N) \rangle$ built from physical scaling operators $\phi_i(z_i, \bar{z}_i)$, $i = 1, \ldots, N$. The co-variance of the $N$-point function is then expressed through a closed set of linear first-order differential equations, the ortho-conformal Ward identities [11]. For $N = 2$ or $N = 3$, it is enough to consider merely the maximal finite-dimensional Lie sub-algebra, when these equations are of the form $\ell_{j}^{[N]} C^{[N]} = 0$ (with $j = -1, 0, 1$). Their solutions are holomorphic in the variables $z_p$ (or anti-holomorphic in the variables $\bar{z}_p$) [46]. For example, the ortho-conformal two-point function has the well-known form ($C_0$ is a normalisation constant)

$$C_{\text{ortho}}^{[2]}(z_1, \bar{z}_1, z_2, \bar{z}_2) = C_0 \delta_{\Delta_1, \Delta_2} \delta_{\Delta_1, \Delta_2} (z_1 - z_2)^{-2\Delta_1} (\bar{z}_1 - \bar{z}_2)^{-2\Delta_1}$$

In this work, we are interested in new types of (meta-)conformal invariance [35, 44, 57] where the implicit hypothesis of holomorphy in the physical coordinates $z_p, \bar{z}_p$ (or equivalently $t_p, r_p$, with $p = 1, \ldots, N$) is not necessarily satisfied. Meta-conformal invariance arises as a dynamical symmetry of the simple equation $S\phi(t, r) = \left(-\mu \partial_t + \partial_r \right)\phi(t, r) = 0$ of ballistic transport, which distinguishes a single preferred direction $[46]$, with coordinate $r_\parallel$, from the transverse direction(s), with coordinate $r_\perp$. This is sketched in figure 1. A preferred spatial direction

![Figure 1: Schematic illustration of ballistic transport in a channel, with the spatial coordinates $r_\parallel, r_\perp$. The directional bias is indicated.](image)


In table 1 several examples of time-space coordinate changes with rotation-invariance, hence co-variance. The correlator shown in table 1, for \( \bar{\text{correlator}} \) and in the 2 \( \text{corrrelator} \), is incompatible with rotation-invariance, hence \textit{a fortiori} incompatible with ortho-conformal invariance. A simple example from statistical physics of a system with a dynamical meta-conformal invariance is provided by the kinetic Glauber-Ising model with a bias and long-ranged initial conditions \([44,58]\). Ballistic transport occurs in innumerable closed quantum systems, see e.g. \([12,15,16,17,19,20,22,23,24,32,48,53]\) as well as in classical non-equilibrium dynamics, see e.g. \([59,21,18,28,31,56,60,4]\). In table 1 several examples of time-space coordinate transformations are listed, including ortho- and meta-conformal transformations. It can be seen that \((1+1)D\) ortho-conformal transformations and \(1D\) meta-conformal transformations have the same underlying abstract algebraic structure, namely isomorphic Lie algebras, see \([35]\). However, they refer to different physical choices which represent the time-space variables. As is further shown in table \( \| \) for \( d = 1 \) and \( d = 2 \) space dimensions, the Lie group of meta-conformal transformations is infinite-dimensional. The Lie algebra generators of meta-conformal invariance are read off from table \( \| \) as follows, in terms of the physical time-space coordinates \( t \) and \( r \) (respectively \( r_\|, r_\perp \)). In the 1D case, in terms of time- and space-coordinates \([35]\) (with \( n \in \mathbb{Z} \))

\[
\ell_n = -t^{n+1} \left( \partial_t - \frac{1}{\mu} \partial_r \right) - (n+1) \left( \delta - \frac{\gamma}{\mu} \right) t^n
\]

\[
\bar{\ell}_n = -t \left( t + \mu r \right) \partial_r - (n+1) \frac{\gamma}{\mu} \left( t + \mu r \right)^n
\]

(1.3)

(\text{notice the differences with respect to (1.1) and in the 2D case})

\[
A_n = -t^{n+1} \left( \partial_t - \frac{1}{\mu} \partial_\| \right) - (n+1) \left( \delta - \frac{2\gamma_\|}{\mu} \right) t^n
\]

\[
B_n^\pm = -\frac{1}{2 \mu} \left( t + \mu (r_\| \pm i r_\perp) \right)^{n+1} \left( \partial_\| \mp i \partial_\perp \right) - (n+1) \frac{\gamma_\| \mp i \gamma_\perp}{\mu} \left( t + \mu (r_\| \pm i r_\perp) \right)^n
\]

(1.4)
with the short-hands \( \partial_\parallel = \frac{\partial}{\partial x_\parallel} \) and \( \partial_\perp = \frac{\partial}{\partial x_\perp} \). The constants \( \delta \) and \( \gamma \) (respectively \( \gamma_\parallel, \gamma_\perp \)) are the space dimension and the space-time scaling operators on which these generators act and \( \mu^{-1} \) is a constant with the dimension of a velocity. Each of the infinite families of generators in \([3,3,3,4]\) produces a Virasoro algebra (with zero central charge). Therefore, the 1D meta-conformal algebra is isomorphic to a direct sum of two Virasoro algebras. In the 2D case, there is an isomorphism with the direct sum of three Virasoro algebras. Their maximal finite-dimensional Lie sub-algebras (isomorphic to a direct sum of two or three \( \mathfrak{s}(2,\mathbb{R}) \) algebras) fix the form of two-point correlators \( C(t, r) = \langle \varphi_1(t, r)\varphi_2(0, 0) \rangle \) built from quasi-primary scaling operators. Since the generators \([3,3,3,4]\) already contain the terms which describe how the scaling operators \( \varphi = \varphi(t, r) \) transform under their action, the global meta-conformal Ward identities can simply be written down. The requirement of meta-conformal co-variance leads to \([44]\)

\[
C^{[2]}_{\text{meta}}(t, r) = \langle \varphi(t, r)\varphi(0, 0) \rangle
\]

\[
= \left\{ \begin{array}{ll}
\delta_1, \delta_2 \delta_{\gamma_1, \gamma_2} C_0 t^{-2\delta_1} \left( 1 + \mu_\parallel^2 \right)^{-2\gamma_1/\mu} & ; \text{if } d = 1 \\
\delta_1, \delta_2 \delta_{\gamma_\parallel, \gamma_\perp, \gamma_{\parallel, \perp}} C_0 t^{-2\delta_1} \left( 1 + \mu_{\parallel, \perp}^2 \right)^{-2\gamma_1/\mu} & ; \text{if } d = 2
\end{array} \right.
\]

and where \( r = r \in \mathbb{R} \) for \( d = 1 \) and \( r = (r_\parallel, r_\perp) \in \mathbb{R}^2 \) for \( d = 2 \).

Formally, the procedure to derive \((1.5)\) is completely analogous to the used above for the derivation of \((1.2)\) from ortho-conformal co-variance. The explicit forms \((1.5)\) make it apparent that \( C_{\text{meta}}(t, r) \) is not necessarily bounded for all \( t \) or \( r \). In figure 2 we illustrate this for the 1D case – a spurious singularity appears whenever \( \mu r = -t \).

In the limit \( \mu \to 0 \), the meta-conformal algebras contract into the \textit{galilean conformal algebras} \([3,3,3,4]\). Carrying out this limit on the correlator \((1.5)\), one obtains, as has been stated countless times in the literature, see e.g. \([5,6,7,8,51]\)

\[
C^{[2]}_{\text{CGA}}(t, r) = \delta_1, \delta_2 \delta_{\gamma_1, \gamma_2} \left\{ \begin{array}{ll}
t^{-2\delta_1} \exp \left( -\frac{2\gamma r}{t} \right) & ; \text{if } d = 1 \\
t^{-2\delta_1} \exp \left( -4\frac{\gamma r}{t} \right) & ; \text{if } d = 2
\end{array} \right.
\]

with the definition \( \gamma = (\gamma_\parallel, \gamma_\perp) \). While this correlator decays in one spatial direction (‘down-stream’, where \( \gamma_1 r > 0 \) or \( \gamma_1 \cdot r > 0 \), respectively and assuming \( t > 0 \)), it diverges in the opposite (‘upstream’) direction, as defined in figure 1. Again, such a behaviour does not appear to be physical. In view of the large interest devoted to conformal galilean field-theory, see \([3,3,3,3,4,51,25,26,27,50,38]\) and refs. therein, it appears important to be able to formulate well-defined correlators which remain bounded everywhere in time-space.

We mention in passing that the 1D form of \((1.6)\) can also be formally obtained from 2D ortho-conformal invariance: it is enough to consider complex conformal weights \( \Delta = \frac{1}{2} (\delta - i\gamma/\mu) \) and \( \overline{\Delta} = \frac{1}{2} (\delta + i\gamma/\mu) \). Then \((1.2)\) can be rewritten as

\[
C^{[2]}_{\text{ortho}}(t, r) = t^{-2\delta_1} \left[ 1 + \left( \frac{\mu r}{t} \right)^2 \right]^{-\delta} \exp \left[ -\frac{2\gamma}{\mu} \arctan \frac{\mu r}{t} \right] \mu \to 0 \to t^{-2\delta} e^{-2\gamma r/t} \quad (1.7)
\]

In what follows, we shall describe a mathematically well-defined procedure how to find meta-conformal correlators bounded everywhere. Since the implicit assumption of holomorphicity in
(a) Spurious singularities arise in eq. (1.5).

(b) Regularised bounded real-valued form eq. (2.15).

(c) For $t = -0.6$, a complex-valued singularity occurs at $r = -t = 0.6$ in eq. (1.5).

(d) For $t = -0.6$, bounded real-valued behaviour throughout in eq. (2.15).

Figure 2: Real part (orange) and imaginary part (blue) of the 1D meta-conformally co-variant two-point function $C(t, r)$, with $\delta_1 = 0.22$, $\gamma_1 = 0.33$ and $\mu_1 = \mu_2 = 1$. The dark straight lines in figure 2(a,b) indicate the section $t = -0.6$ along which the correlator is shown again in figure 2(c,d).

the coordinates gave the unbounded results (1.5,1.6), we shall explore how to derive non-holomorphic correlators. The first case where this question could be treated was the one of the conformal galilean algebra, where the regularised form reads in $d \geq 1$ dimensions \[ C^{[2]}_{\text{CGA,reg}}(t, r) = \delta_{\delta_1, \delta_2} \delta_{\gamma_1, \gamma_2} |t|^{-2\delta_1} \exp \left( -2\alpha \left| \frac{\gamma_1 \cdot r}{t} \right| \right) F_0(\gamma_1^2) \] instead of (1.6), where the function $F_0$ remains undetermined and $\alpha = \alpha(d)$ is a possibly dimension-dependent constant. The derivation is briefly reproduced in appendix A for $d = 1$, for the convenience of the reader. In particular, this example shows (i) the importance of the dualisation technique to be explained in section 2 and (ii) the relevance of semi-infinite representation theorems derived from the theory of Hardy spaces [3, 55], see appendix B. Our treatment of the meta-conformal case follows [43], to be generalised to the case $d = 2$ when necessary. This in turn is inspired by methods developed for the Schrödinger algebra [36, 39].

Throughout, we shall admit rotation-invariance in the transverse directions, if applicable. Therefore, in more than three spatial dimensions, the consideration of the two-point function can be reduced to the case of a single transverse direction, $r_\perp$. Therefore, it should be enough to discuss explicitly either (i) the case of one spatial dimension, referred from now one as the
1D case (without a transverse direction), or else (ii) the case of two spatial dimensions, called the 2D case (with a single transverse direction).

This work is organised as follows. Since the straightforwardish implementation of the global meta-conformal Ward identities leads to un-physical singularities in the time-space behaviour of such correlators, which arise since the meta-conformally co-variant correlators are no longer holomorphic functions of their arguments, we shall present a more careful approach in sections 2 and 3. These cover, respectively, $d = 1$ and $d = 2$ spatial dimensions. Technically, this proceeds via a dualisation and a regularity requirement in dual space. Our main result is the explicit form of a meta-conformally co-variant two-point function which remains bounded everywhere, as stated in eqs. (4.1,4.2) in section 4. Appendix A outlines the construction for the limit case of the conformal galilean algebra CGA(1), including the derivation of the required regularity property. Appendix B contains mathematical background on Hardy spaces in restricted geometries, for both $d = 1$ and $d = 2$.

2 Regularised meta-conformal correlator: the 1D case

Non-holomorphic correlators can only be found by going beyond the local differential operators derived from the meta-conformal Ward identities. We shall do so in a few simple steps [43], restricting in this section to the 1D case. First, we consider the ‘rapidity’ $\gamma$ as a new variable. Second, it is dualised [39, 41, 42] through a Fourier transformation, which gives the quasi-primary scaling operator

$$\hat{\varphi}(\zeta, t, r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} d\gamma e^{i\gamma \zeta} \varphi(\gamma t, \gamma r)$$

(2.1)

This leads to the following representation of the dualised meta-conf ormal algebra (using (1.3) from section 1 with $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \bar{\ell}_n$ [35, 44])

$$X_n = \frac{i(n+1)}{\mu} \left[ (t + \mu r)^n - t^n \right] \partial_\zeta - t^{n+1} \partial_\xi - \frac{1}{\mu} \left[ (t + \mu r)^{n+1} - t^{n+1} \right] \partial_r - (n+1) \delta t^n$$

$$Y_n = \frac{i(n+1)}{\mu} (t + \mu r)^n \partial_\zeta - \frac{1}{\mu} (t + \mu r)^{n+1} \partial_r$$

(2.2)

such that the 1D meta-conf ormal Lie algebra is given by

$$[X_n, X_m] = (n - m) X_{n+m} , \ [X_n, Y_m] = (n - m) Y_{n+m} , \ [Y_n, Y_m] = (n - m) Y_{n+m}$$

(2.3)

This form will be more convenient for us than the one used in [43], since the parameter $\mu$ does no longer appear in the Lie algebra commutators (2.3). Third, it was suggested [36, 39, 43] to look for a further generator $N$ in the Cartan sub-algebra $\mathfrak{h}$, viz. $\text{ad}_N \mathcal{X} = \alpha_N \mathcal{X}$ for any meta-conf ormal generator $\mathcal{X}$ and $\alpha_N \in \mathbb{C}$. It can be shown that

$$N = -\zeta \partial_\zeta - r \partial_r + \mu \partial_\mu + i\kappa(\mu) \partial_\zeta - \nu(\mu)$$

(2.4)

is the only possibility [43], where the functions $\kappa(\mu)$ and $\nu(\mu)$ remain undetermined. Since in this generator, the parameter $\mu$ is treated as a further variable, we see the usefulness of the chosen normalisation of the generators in (2.2). On the other hand, the generator of spatial
translations now reads $Y_{-1} = -\mu^{-1} \partial_{\tau}$, with immediate consequences for the form of the two-point correlator. In dual space, the two-point correlator is defined as

$$\hat{F} = \langle \hat{\phi}_1(\zeta_1, t_1, r_1, \mu_1) \hat{\phi}_2(\zeta_2, t_2, r_2, \mu_2) \rangle = \hat{F}(\zeta_1, \zeta_2, t_1, t_2, r_1, r_2, \mu_1, \mu_2)$$  \quad (2.5)$$

Lifting the generators from the representation (2.2) to two-body operators, the global meta-conformal Ward identities (derived from the maximal finite dimensional sub-algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$) become a set of linear partial differential equations of first order $\mathcal{X}_i^{[2]} \hat{F} = 0$ for the function $\hat{F}$. While the solution will certainly be holomorphic in its (dual) variables, the back-transformation according to (2.1) will lead to a correlator bounded everywhere but can introduce non-holomorphic behaviour in the $t_i, r_i$.

The function $\hat{F}$ is obtained as follows \[43\] [44]. First, co-variance under $X_{-1}$ and $Y_{-1}$ gives

$$\hat{F} = \hat{F}(\zeta_1, \zeta_2, t, \xi, \mu_1, \mu_2) ; \quad t = t_1 - t_2 \ , \ \xi = \mu_1 r_1 - \mu_2 r_2$$  \quad (2.6)$$

The action of the generators $Y_0$ and $Y_1$ on $\hat{F}$ is used best by introducing the new variables $\eta := \mu_1 \zeta_1 + \mu_2 \zeta_2$ and $\zeta := \mu_1 \zeta_1 - \mu_2 \zeta_2$. Then the corresponding Ward identities become

$$(2i \partial_{\eta} - (t + \xi) \partial_{\xi}) \hat{F} = 0 \ , \ \partial_{\xi} \hat{F} = 0$$  \quad (2.7)$$

Finally, the Ward identities coming from the generators $X_0$ and $X_1$ become

$$(-t \partial_t - \xi \partial_\xi - \delta_1 - \delta_2) \hat{F} = 0 \ , \ t (\delta_1 - \delta_2) \hat{F} = 0$$  \quad (2.8)$$

The second of these gives the constraint $\delta_1 = \delta_2$. The two remaining equations have the general solution

$$\hat{F} = (t_1 - t_2)^{-2\delta_1} \hat{\mathcal{F}} \left( \frac{1}{2} (\mu_1 \zeta_1 + \mu_2 \zeta_2) + i \ln \left( 1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2} \right) ; \mu_1, \mu_2 \right)$$  \quad (2.9)$$

with an undetermined function $\hat{\mathcal{F}}$. Spatial translation-invariance only holds in a more weak form, which could become useful for the description of physical situations where the propagation speed of each scaling operator can be different.$^1$

In \[43\] [44], we tried to use co-variance under the further generator $N$ in order to fix the function $\hat{\mathcal{F}}$, in close analogy with the conformal galilean algebra, see appendix A. However, therein a choice of basis in the meta-conformal Lie algebra was used where the parameter $\mu$ appears in the structure constants. In this way, $\hat{\mathcal{F}}$ is fixed and furthermore one can show that $\hat{F}$ with respect to the variable $\eta$ is in the Hardy space $H^2_{\eta}$, see appendix B for the mathematical details. If we want to consider $\mu$ as a further variable, as it is necessary because of the explicit form of $N$, objects such as “$\mu Y_{n+m}$” which arise in the commutators are not part of the meta-conformal Lie algebra. Therefore, it is necessary, to use the normalisation (2.2) which leads to the Lie algebra (2.3) which is independent of $\mu$. In order to illustrate the generic consequences, let $\nu = \nu(\mu)$ and $\sigma = -\mu \kappa(\mu)$ be constants (hence independent of $\mu_{1,2}$). The co-variance condition $N \hat{F} = 0$ gives

$$\hat{\mathcal{F}}(w; \mu_1, \mu_2) = (\mu_1 \mu_2)^\nu \hat{\mathcal{F}} \left( w + i \sigma \frac{\mu_1 + \mu_2}{2} \frac{\mu_1}{\mu_2} \right)$$  \quad (2.10)$$

$^1$See \[52\] for inhomogeneous ortho-conformal invariance without translation-invariance.
where the function $\hat{F}$ remains undetermined. In contrast to our earlier treatment, we can no longer show that $\hat{F}$ had to be in the Hardy space $H^+_2$. On the other hand, this mathematical property had turned out to be very useful for the derivation of bounded correlators, at least in the conformal galilean case. This motivates the following.

First, return to the result (2.9), rewritten as follows (with the constraint $\delta_1 = \delta_2$)

$$
\hat{F} = (t_1 - t_2)^{-2\delta_1} \tilde{F}(\zeta_+ + i\lambda), \quad \zeta_+ := \frac{\mu_1 \zeta_1 + \mu_2 \zeta_2}{2}, \quad \lambda := \ln \left(1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right) \quad (2.11)
$$

and we denote $\tilde{F}_\lambda(\zeta_+) := \tilde{F}(\zeta_+ + i\lambda)$. Then, we require (see appendix B for details):

**Postulate.** If $\lambda > 0$, then $\tilde{F}_\lambda \in H^+_2$ and if $\lambda < 0$, then $\tilde{F}_\lambda \in H^-_2$.

The Hardy spaces $H^+_2$ on the upper and lower complex half-planes $\mathbb{H}_{\pm}$ are defined in appendix B. There, it is also shown that, under mild conditions, that if $\lambda > 0$ and if there exist finite positive constants $\tilde{F}^{(0)} > 0, \varepsilon > 0$ such that $|\tilde{F}(\zeta_+ + i\lambda)| < \tilde{F}^{(0)} e^{-\varepsilon \lambda}$, then $\tilde{F}_\lambda$ is indeed in the Hardy space $H^+_2$. Physically, this amounts to a requirement that the dual correlator should decay algebraically, viz. $|\tilde{F}| \leq \tilde{F}^{(0)} \left|1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right|^{-\varepsilon}$, with respect to the scaling variable. The above postulate also appears natural since in the $\mu \to 0$ limit of conformal galilean invariance, it can be derived from the condition of co-variance under the extra generator $N$ [39] [41] [42].

The utility of our postulate is easily verified, following [43]. From Theorem 1 of appendix B, especially [B.3], we can write

$$
\tilde{F}_\lambda(\zeta_+) = \Theta(\lambda) \int_0^\infty d\gamma_+ e^{i(\zeta_+ + i\lambda)\gamma_+} \tilde{F}_+(\gamma_+) + \Theta(-\lambda) \int_0^\infty d\gamma_- e^{-i(\zeta_+ + i\lambda)\gamma_-} \tilde{F}_-(\gamma_-) \quad (2.12)
$$

where the Heaviside functions $\Theta(\pm \lambda)$ select the two cases. For $\lambda > 0$, we find

$$
\hat{F} = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\zeta_1 d\zeta_2 e^{-i\gamma_1 \zeta_1 - i\gamma_2 \zeta_2} \hat{F} = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} d\zeta_1 d\zeta_2 t^{-2\delta_1} \int_0^\infty d\gamma_+ e^{-i\gamma_1 \zeta_1 - i\gamma_2 \zeta_2} e^{i(\mu_1 \zeta_1 + \mu_2 \zeta_2 + 2i\lambda)\gamma_+ / 2} \tilde{F}_+(\gamma_+) \nonumber
$$

$$
= \frac{\sqrt{32\pi}}{\mu_1 \mu_2} t^{-2\delta_1} \int_0^\infty d\gamma_+ e^{-\gamma_+ \delta} \left(\gamma_+ - \frac{2\gamma_1}{\mu_1}\right) \delta \left(\gamma_+ - \frac{2\gamma_2}{\mu_2}\right) \tilde{F}_+(\gamma_+) \nonumber
$$

$$
= \frac{\sqrt{32\pi}}{\mu_1 \mu_2} t^{-2\delta_1} \delta_{\gamma_1 / \mu_1, \gamma_2 / \mu_2} \int_0^\infty d\gamma_+ e^{-\gamma_+ \delta} \left(\gamma_+ - \frac{2\gamma_1}{\mu_1}\right) \tilde{F}_+(\gamma_+) \nonumber
$$

$$
= \text{cste. } \delta_{\gamma_1 / \mu_1, \gamma_2 / \mu_2} (t_1 - t_2)^{-2\delta_1} \left(1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right)^{-2\gamma_1 / \mu_1} \Theta \left(\frac{\gamma_1}{\mu_1}\right) \quad (2.13)
$$
where the definitions (2.11) were used. Similarly, for $\lambda < 0$ we obtain

$$
F = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\zeta_1 d\zeta_2 e^{-i\gamma_1 \zeta_1 - i\gamma_2 \zeta_2} \hat{F}
$$

$$
= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} d\zeta_1 d\zeta_2 t^{-2\delta_1} \int_0^\infty d\gamma_- e^{-i\gamma_1 \zeta_1 - i\gamma_2 \zeta_2} e^{-i(\mu_1 \zeta_1 + \mu_2 \zeta_2 + 2i\lambda)\gamma_-/2} \hat{\varphi}_-(\gamma_-)
$$

$$
= \sqrt{\frac{32\pi}{\mu_1 \mu_2}} t^{-2\delta_1} \int_0^\infty d\gamma_- e^{\lambda \gamma_-} \delta \left( \gamma_- + \frac{2\gamma_1}{\mu_1} \right) \delta \left( \gamma_- + \frac{2\gamma_2}{\mu_2} \right) \hat{\varphi}_-(\gamma_-)
$$

$$
= \text{este.} \delta_{\gamma_1/\mu_1, \gamma_2/\mu_2} (t_1 - t_2)^{-2\delta_1} \left( 1 - \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2} \right)^{-2|\gamma_1/\mu_1|} \Theta \left( -\frac{\gamma_1}{\mu_1} \right)
$$

Combining these two forms gives our final 1D two-point correlator

$$
F = \delta_{\delta_1, \delta_2} \delta_{\gamma_1/\mu_1, \gamma_2/\mu_2} \left( 1 + \left| \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2} \right| \right)^{-2|\gamma_1/\mu_1|} (2.14)
$$

up to normalisation. As illustrated in figure 2, this is real-valued and bounded in the entire time-space, although not a holomorphic function of the time-space coordinates.

Finally, it appears that our original motivation for allowing the $\mu_j$ to become free variables, is not very strong. We might have fixed the $\mu_j$ from the outset, had not included a factor $1/\mu$ into the generators $Y_n$ (such that the spatial translations are generated by $Y_{-1} = -\partial_r$ and continue immediately with our Postulate. Since a consideration of the meta-conformal three-point function shows that $\mu_1 = \mu_2 = \mu_3$ [37 chap. 5], we can then consider $\mu^{-1}$ as an universal velocity$^\Delta$.

### 3 Regularised meta-conformal correlator: the 2D case

The derivation of the 2D meta-conformal correlator starts essentially along the same lines as in the 1D case, but is based now on the generators $[1, 4]$. The dualisation is now carried out with respect to the chiral rapidities $\gamma = \gamma_\parallel - i\gamma_\perp$ and $\bar{\gamma} = \gamma_\parallel + i\gamma_\perp$ and we also use the light-cone coordinates $z = r_\parallel + i r_\perp$ and $\bar{z} = r_\parallel - i r_\perp$. The dualisation proceeds as follows

$$
\hat{\varphi}(\zeta, \bar{\zeta}, t, r) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\gamma d\bar{\gamma} e^{i(\zeta + \bar{\zeta})} \varphi_{\gamma, \bar{\gamma}}(t, r)
$$

Taking the translation generators $A_-1, B^\pm _{-1}$ into account, we consider the dual correlator

$$
\hat{F} = \hat{F} (\zeta_1, \zeta_2, \bar{\zeta}_1, \bar{\zeta}_2, t, \xi, \bar{\xi}, \mu_1, \mu_2)
$$

where we defined the variables

$$
t = t_1 - t_2 , \quad \xi = \mu_1 z_1 - \mu_2 z_2 \quad \bar{\xi} = \mu_1 \bar{z}_1 - \mu_2 \bar{z}_2
$$

$^\Delta$In the conformal galilean limit $\mu \rightarrow 0$, recover the bounded result $F \sim \exp (-2|\gamma_1 r|/t)$ [43].

In complete analogy with the 1D case, we further define the variables
\[
\eta = \mu_1 \zeta_1 + \mu_2 \zeta_2 , \quad \bar{\eta} = \mu_1 \bar{\zeta}_1 + \mu_2 \bar{\zeta}_2
\] (3.4)
such that the correlator \( \hat{F} = \hat{F}(\eta, \bar{\eta}, t, \xi, \bar{\xi}, \mu_1, \mu_2) \) obeys the equations
\[
(2i\partial_\eta - (t + \xi)\partial_\xi) \hat{F} = 0 , \quad (2i\partial_{\bar{\eta}} - (t + \bar{\xi})\partial_{\bar{\xi}}) \hat{F} = 0 , \quad (t\partial_t + \xi\partial_\xi + \bar{\xi}\partial_{\bar{\xi}} + 2\delta_1) \hat{F} = 0 \tag{3.5}
\]
along with the constraint \( \delta_1 = \delta_2 \). The most general solution of this system is promptly obtained. However, in order to obtain real-valued results in terms of the physical coordinates \( t, r_\parallel, r_\perp \), we expand the logarithms and go back to parallel and perpendicular dual coordinates, via \( \zeta_j = \frac{1}{2}(\zeta_{||,j} + i\zeta_{\perp,j}) \) and \( \bar{\zeta}_j = \frac{1}{2}(\zeta_{||,j} - i\zeta_{\perp,j}) \). Since the coordinates \( \xi, \bar{\xi} \) which occur are complex, we need the complex logarithm
\[
\ln(a + ib) = \frac{1}{2} \ln(a^2 + b^2) + i \arctan \frac{b}{a} \tag{3.6}
\]
and find (the first step is the solution of eqs. [3.5])
\[
\hat{F} = t^{-2\delta_1} \hat{\mathcal{F}} \left( \eta + i \ln(1 + \xi/t), \bar{\eta} + i \ln(1 + \bar{\xi}/t) \right)
= t^{-2\delta_1} \hat{\mathcal{F}} \left( \frac{\mu}{2} (\zeta_{||,1} + \zeta_{||,2}) + i\lambda_\parallel, \frac{\mu}{2} (\zeta_{\perp,1} + \zeta_{\perp,2}) + i\lambda_\perp \right)
= t^{-2\delta_1} \hat{\mathcal{F}} (u + i\lambda_\parallel, \bar{u} + i\lambda_\perp) \tag{3.7}
\]
with the abbreviations (\( \bar{u} \) is obtained from \( u \) by replacing \( \zeta_{||,j} \mapsto \zeta_{\perp,j} \))
\[
u := \frac{\mu}{2} (\zeta_{||,1} + \zeta_{||,2}) , \quad \lambda_\parallel := \frac{1}{2} \ln \left[ \left( 1 + \frac{\mu r_\parallel}{t} \right)^2 + \left( \frac{\mu r_\perp}{t} \right)^2 \right] \quad , \quad \lambda_\perp := \arctan \frac{\mu r_\perp/t}{1 + \mu r_\parallel/t} \tag{3.8}
\]
and we simplified the notation by letting \( \mu_1 = \mu_2 = \mu \) and assumed translation-invariance in time and space. As before, we expect that a Hardy space will permit to derive the boundedness, see appendix B for details. Define \( \hat{\mathcal{F}}_{\lambda_\parallel,\lambda_\perp} (u, \bar{u}) := \hat{\mathcal{F}} (u + i\lambda_\parallel, \bar{u} + i\lambda_\perp) \) and require:

**Postulate:** If the parameters \( \lambda_\parallel > 0 \text{ and } \lambda_\perp > 0 \), then \( \hat{\mathcal{F}}_{\lambda_\parallel,\lambda_\perp} \in H^+_2 \)

\[\lambda_\parallel > 0 \text{ and } \lambda_\perp < 0 \), then \( \hat{\mathcal{F}}_{\lambda_\parallel,\lambda_\perp} \in H^-_2 \)

\[\lambda_\parallel < 0 \text{ and } \lambda_\perp > 0 \), then \( \hat{\mathcal{F}}_{\lambda_\parallel,\lambda_\perp} \in H^+_2 \)

\[\lambda_\parallel < 0 \text{ and } \lambda_\perp < 0 \), then \( \hat{\mathcal{F}}_{\lambda_\parallel,\lambda_\perp} \in H^-_2 \)

The important point is that these postulates are made with respect to the parallel and perpendicular dual coordinates \( \zeta_\parallel = \zeta + \bar{\zeta} \) and \( \zeta_\perp = \frac{1}{2}(\zeta - \bar{\zeta}) \).

Theorem 2 in appendix B, especially (B.12), then states that
\[
\hat{\mathcal{F}}_{\lambda_\parallel,\lambda_\perp} (u, \bar{u}) = \frac{\Theta(\lambda_\parallel)\Theta(\lambda_\perp)}{2\pi} \int_0^\infty d\tau \int_0^\infty d\bar{\tau} \ e^{i(u + i\lambda_\parallel)\tau + i(\bar{u} + i\lambda_\perp)\bar{\tau}} \hat{\mathcal{F}}_{++}(\tau, \bar{\tau})
+ \frac{\Theta(\lambda_\parallel)\Theta(-\lambda_\perp)}{2\pi} \int_0^\infty d\tau \int_0^\infty d\bar{\tau} \ e^{i(u + i\lambda_\parallel)\tau - i(\bar{u} + i\lambda_\perp)\bar{\tau}} \hat{\mathcal{F}}_{+-}(\tau, \bar{\tau})
+ \frac{\Theta(-\lambda_\parallel)\Theta(\lambda_\perp)}{2\pi} \int_0^\infty d\tau \int_0^\infty d\bar{\tau} \ e^{-i(u + i\lambda_\parallel)\tau + i(\bar{u} + i\lambda_\perp)\bar{\tau}} \hat{\mathcal{F}}_{-+}(\tau, \bar{\tau})
+ \frac{\Theta(-\lambda_\parallel)\Theta(-\lambda_\perp)}{2\pi} \int_0^\infty d\tau \int_0^\infty d\bar{\tau} \ e^{-i(u + i\lambda_\parallel)\tau - i(\bar{u} + i\lambda_\perp)\bar{\tau}} \hat{\mathcal{F}}_{--}(\tau, \bar{\tau}) \tag{3.9}
\]
For example, we can write the two-point function in the case $\lambda_\parallel > 0$ and $\lambda_\perp > 0$, with the short-hand $\mathcal{D} \zeta := d\zeta_{\parallel,1} d\zeta_{\perp,1} d\zeta_{\parallel,2} d\zeta_{\perp,2}$ and the abbreviations from (3.8)

\[
F = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \mathcal{D} \zeta \ e^{-i\gamma_{\parallel,1} \zeta_{\parallel,1} - i\gamma_{\perp,1} \zeta_{\perp,1} - i\gamma_{\parallel,2} \zeta_{\parallel,2} - i\gamma_{\perp,2} \zeta_{\perp,2}} \hat{F}
\]

\[
= \frac{t^{-2\delta_1}}{(2\pi)^3} \int_{\mathbb{R}^4} \mathcal{D} \zeta \ e^{-i\gamma_{\parallel,1} \zeta_{\parallel,1} - i\gamma_{\perp,1} \zeta_{\perp,1} - i\gamma_{\parallel,2} \zeta_{\parallel,2} - i\gamma_{\perp,2} \zeta_{\perp,2}} \times
\]

\[
\times \int_0^\infty d\tau \int_0^\infty d\bar{\tau} \ e^{i(\mu(\zeta_{\parallel,1} + \zeta_{\perp,2}))\tau/2 - \lambda_\perp \tau} e^{i(\mu(\zeta_{\parallel,1} + \zeta_{\perp,2}))\bar{\tau}/2 - \lambda_\perp \bar{\tau}} \hat{F}_{++}(\tau, \bar{\tau})
\]

\[
= \frac{t^{-2\delta_1}}{(2\pi)^3} \int_0^\infty d\tau \int_0^\infty d\bar{\tau} \ e^{-\lambda_\parallel \tau - \lambda_\perp \bar{\tau}} \times
\]

\[
\times \int_{\mathbb{R}^4} \mathcal{D} \zeta \ e^{i(\mu(\tau - 2) + \zeta_{\parallel,1})\kappa_{\parallel,1}} e^{i(\mu(\tau - 2) - \zeta_{\perp,2})\kappa_{\perp,2}} e^{i(\mu(\tau - 2) - \zeta_{\parallel,1})\kappa_{\parallel,1}} e^{i(\mu(\tau - 2) - \zeta_{\perp,2})\kappa_{\perp,2}}
\]

\[
= \text{cste.} \ t^{-2\delta_1} \delta_{\gamma_{\parallel,1}, \mu, \gamma_{\perp,2}/\mu} \delta_{\gamma_{\perp,1}, \mu, \gamma_{\perp,2}/\mu} \exp \left[ -2\lambda_\parallel \frac{\gamma_{\parallel,1}}{\mu} - 2\lambda_\perp \frac{\gamma_{\perp,1}}{\mu} \right] \Theta \left( \frac{\gamma_{\parallel,1}}{\mu} \right) \Theta \left( \frac{\gamma_{\perp,1}}{\mu} \right)
\]

Herein, variables were changed according to $\zeta_{\parallel,1} = \zeta_{\parallel,1} + \zeta_{\parallel,2}$ and $\zeta_{\perp,2} = \zeta_{\perp,2} - \zeta_{\parallel,2}$ and similarly for the $\zeta_{\perp,1}$. The other cases are treated in the same manner, in complete analogy with the 1D situation. For example, if $\lambda_\parallel < 0$ and $\lambda_\perp < 0$, we find

\[
F = \frac{t^{-2\delta_1}}{(2\pi)^3} \int_{\mathbb{R}^4} \mathcal{D} \zeta \ e^{-i\gamma_{\parallel,1} \zeta_{\parallel,1} - i\gamma_{\perp,1} \zeta_{\perp,1} - i\gamma_{\parallel,2} \zeta_{\parallel,2} - i\gamma_{\perp,2} \zeta_{\perp,2}} \times
\]

\[
\times \int_0^\infty d\tau \int_0^\infty d\bar{\tau} \ e^{-i(\mu(\zeta_{\parallel,1} + \zeta_{\perp,2}))\tau/2 + \lambda_\perp \tau} e^{-i(\mu(\zeta_{\parallel,1} + \zeta_{\perp,2}))\bar{\tau}/2 + \lambda_\perp \bar{\tau}} \hat{F}_{- -}(\tau, \bar{\tau})
\]

\[
= \text{cste.} \ t^{-2\delta_1} \delta_{\gamma_{\parallel,1}, \mu, \gamma_{\perp,2}/\mu} \delta_{\gamma_{\parallel,1}, \mu, \gamma_{\perp,2}/\mu} \exp \left[ -2\lambda_\parallel \frac{\gamma_{\perp,1}}{\mu} - 2\lambda_\perp \frac{\gamma_{\parallel,1}}{\mu} \right] \Theta \left( -\frac{\gamma_{\parallel,1}}{\mu} \right) \Theta \left( -\frac{\gamma_{\perp,1}}{\mu} \right)
\]

We also observe the implied positivity conditions on the $\gamma_{\parallel,1}$ and $\gamma_{\perp,1}$.

In order to understand the meaning of these expressions, we return to the physical interpretation of the conditions $\lambda_{\parallel,1} > 0$, or $\lambda_{\parallel,1} < 0$. From (3.8), we see first that $\lambda_\perp > 0$ if and only if $\mu r_\perp / t [1 + \mu r_\parallel / t]^{-1} > 0$. On the other hand, concerning the condition $\lambda_\parallel > 0$, the most restrictive case occurs for $r_\perp = 0$. Then $\lambda_\parallel > 0$ is equivalent to $r_\parallel / t > 0$. Summarising, we conclude that

\[
F = \delta_{\delta_1, \delta_2} \delta_{\gamma_{1, \parallel,1}, \gamma_{1, \parallel,2}} \delta_{\gamma_{1, \perp,1}, \gamma_{1, \perp,2}} t^{-2\delta_1} \left( 1 + \left( \frac{\mu r_\parallel}{t} \right)^2 \right) \left( \frac{\mu r_\perp}{t} \right)^{2 - |\gamma_{\parallel,1}/\mu|} \exp \left[ -\frac{2\gamma_{\perp,1}}{\mu} \right. \left( \frac{\mu r_\perp}{t} \right) \left. + \frac{\mu r_\parallel}{t} \right] \right] \left( \frac{\mu r_\parallel}{t} \right) \left( \frac{\mu r_\perp}{t} \right)
\]

up to normalisation, is the final form for the 2D meta-conformally co-variant two-point correlator which is bounded in the entire time-space with points $(t, r_\parallel, r_\perp) \in \mathbb{R}^3$.

In figure [3] we show that the shape of this correlator (3.12) smoothly interpolates between the preferred and the transverse direction, when the transverse rapidity $\gamma_{\perp,1} = 0$. This is
Figure 3: Shape of the 2D meta-conformal two-point correlator \( C(r \cos \phi, r \sin \phi) = C_{\text{meta}, \, 2D}(1, 0, r \cos \phi, 0, r \sin \phi, 0) \) of eq. (4.2), with rapidities \( \gamma_{\parallel,1} = \frac{1}{4} \) and \( \gamma_{\perp,1} = 0 \) and with the scales set to \( \mu_1 = \mu_2 = 1 \), for the angles \( \phi = [0^\circ, 30^\circ, 60^\circ, 90^\circ] \). The inset shows the algebraic decay \( C(r) \sim r^{-1/2} \) for large distances.

qualitatively very similar to the 1D meta-conformal correlator, which has been described before [44, 45]. For \( \phi \neq 90^\circ \), the cusp at the origin indicates a non-analyticity.

A very different behaviour is observed whenever \( \gamma_{\perp,1} \neq 0 \), as is illustrated in figure 4. For all values of \( \phi \), there is a cusp at the origin, but the decay for larger values of \( r \) is considerably changed with respect to figure 3 whenever \( \phi \neq 0^\circ \). Indeed, for a non-vanishing transverse rapidity \( \gamma_{\perp,1} \neq 0 \), the decay with \( r \) becomes more fast when the angle \( \phi \) increases, a contrario to what is seen in figure 3 where we have \( \gamma_{\perp,1} = 0 \). Instead of an algebraic decay, we rather find from figure 3 an exponentially decaying correlator and only when for very large distances the ratio \( r_{\perp}/(1 + r_{\parallel}) \to \tan \phi \), there is a cross-over to an algebraic decay\(^3\) \( C(r) \sim r^{-2\gamma_{\parallel,1}/\mu_1} \).

Several further comments are in order:

1. Only if the Hardy property is postulated on the parallel and perpendicular dual coordinates \( \zeta_\parallel \) and \( \zeta_\perp \), does one obtain a completely bounded expression. Analogous postulates on different coordinates will lead to more weak results (typically for only one of the parameters \( \lambda_\parallel \) or \( \lambda_\perp \)).
2. the 1D case is contained in (3.12) as the special case \( r_\perp = 0 \).
3. in the limit \( \mu \to 0 \), one recovers the expected result of the 2D conformal galilean algebra

\[
C_{\text{CGA}}^{[2]}(t, r_\parallel, r_\perp) = \lim_{\mu \to 0} C_{\text{meta}}^{[2]}(t, r_\parallel, r_\perp) = \delta_{t_1,t_2} \delta_{\gamma_1,\gamma_2} \delta_{\gamma_{\parallel,1},\gamma_{\parallel,2}} t^{-2\gamma_{\parallel}} \exp \left( -2 \frac{\gamma_{\parallel} r_{\parallel}}{t} - 2 \frac{\gamma_{\perp} r_{\perp}}{t} \right) \tag{3.13}
\]

\(^3\)In the extreme case \( \gamma_{\parallel,1} = 0 \), the (connected !) correlator \( C(r) \) tends towards a constant when \( r \to \infty \).
which could be recast into a vectorial form, as in (1.6), but now the correlator is bounded everywhere.

4. the special case \( r_\parallel = 0 \) reproduces the (non-unitary) ortho-conformal correlator (1.7), with the identifications \( \gamma = \gamma_{\perp,1} \) and \( \delta = |\gamma_{\parallel,1}/\mu| = \delta_1 \).

4 Conclusions

Having raised the question how to formulate sufficient criteria such that the meta-conformally co-variant two-point functions remain bounded in the entire time-space, we have shown that a refined form of the global Ward identities is needed. These are formulated in a dual space, where the dual variables are naturally confined to a tube on a half-space for \( d = 1 \) or on one of the quadrants of the dual plane for \( d = 2 \). Then the regularity condition, namely that these dual two-point functions belong to a certain Hardy space, is a sufficient condition for the construction of bounded two-point functions. This leads to

\[
C_{\text{meta}, 1D}(t_1, t_2, r_1, r_2) = \delta_{\delta_1, \delta_2} \delta_{\gamma_{\parallel,1}/\mu_1, \gamma_{\perp,2}/\mu_2} (t_1 - t_2)^{-2\delta_1} \left( 1 + \left| \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2} \right| \right)^{-2|\gamma_{\parallel,1}/\mu|} (4.1)
\]
(up to normalisation) in \( d = 1 \) spatial dimensions, see also figure 2 and

\[
C_{\text{meta, } 2D}^{(2)}(t_1, t_2, r_{\parallel,1}, r_{\parallel,2}, r_{\perp,1}, r_{\perp,2}) = \delta_{\delta_1, \delta_2} \delta_{\gamma_{\parallel,1}/\mu_1, \gamma_{\parallel,2}/\mu_2} \delta_{\gamma_{\perp,1}/\mu_1, \gamma_{\perp,2}/\mu_2} (t_1 - t_2)^{-2\delta_1} \times
\]

\[
\times \left[ \left( 1 + \frac{|\mu_1 r_{\parallel,1} - \mu_2 r_{\parallel,2}|}{t_1 - t_2} \right)^2 + \frac{\left( \mu_1 r_{\perp,1} - \mu_2 r_{\perp,2} \right)^2}{(t_1 - t_2)} \right]^{-|\gamma_{\parallel,1}/\mu_1|} \times
\]

\[
\times \exp \left[ -\frac{2\gamma_{\parallel,1}}{\mu_1} \arctan \left( \frac{(\mu_1 r_{\perp,1} - \mu_2 r_{\perp,2})/(t_1 - t_2)}{1 + (\mu_1 r_{\perp,1} - \mu_2 r_{\perp,2})/(t_1 - t_2)} \right) \right]
\]

(4.2)

in \( d = 2 \) spatial dimensions. We also see that the parallel and perpendicular rapidities \( \gamma_{\parallel,1} \) and \( \gamma_{\perp,1} \), respectively, enter into the \( 2D \) two-point correlator in very different ways. The qualitative shapes of the correlator have been shown in figures 3 and 4 which illustrate clearly the respective importance of the longitudinal and transverse rapidities, \( \gamma_{\parallel} \) and \( \gamma_{\perp} \), together with the richness of possible scaling forms, in the real physical coordinates. In general, one would have expected that rotation-invariance in the transverse space coordinates \( r_{\perp} \) should reduce the case of \( d \geq 3 \) dimensions to (4.2). Surprisingly, this only seems to work if the transverse rapidity \( \gamma_{\perp,1} = 0 \), when the two-point correlator takes indeed a form which is rotation-invariant in the \( d - 1 \) transverse directions. In that case, one may replace \( r_{\perp,j} \mapsto r_{\perp,j} \).

These explicit results clearly show that boundedness on the entire set of time-space points \((t, r_{\parallel}, r_{\perp}) \in \mathbb{R}^3\) is not compatible with a holomorphic dependence of the two-point function on the time- and space-coordinates. According to the Wiener-Khintchine theorem [29], the Fourier transform of a two-point correlator which obeys spatial translation-invariance must be positive. We have checked that eqs. (4.1,4.2) obey this necessary condition for a physical correlator.

The known regularised two-point correlators of the conformal galilean algebra [41, 43], see eq. (1.8), are reproduced in the \( \mu \rightarrow 0 \) limit.

Remarkably, questions of causality and boundedness appear to find different answers for the non-semi-simple Schrödinger and conformal galilean algebras than for the semi-simple conformal and meta-conformal algebras, see table 1. In the first case, it turned out to be sufficient to enlarge the Cartan sub-algebra \( h \) of the Lie algebra by a further generator \( N \). This either immediately yielded the physically required causality condition of the response functions whose form is described by the Schrödinger algebras [36, 39], or made it possible to demonstrate that the dualised correlator is in the Hardy space \( H^2_\pm \) (if \( d = 1 \)) [41, 42, 43], see appendix A. Although the extra generator \( N \) ceases to be useful for the meta-conformal algebra, the Hardy-space criterion does remain useful, even as a postulate. Of course, for ortho-conformal representations the implicit analyticity in the time-space coordinates permits to sidestep the question. It would be interesting to extend these results to the logarithmic representations [40] of conformal galilean and meta-conformal algebras.

---

4There is no obvious rotation-invariant generalisation of the last exponential factor in (4.2), beyond a single transverse dimension.
Appendix A. Bounded correlators for the conformal galilean algebra

To make this work more self-contained, we briefly recall the corresponding argument for the limit case \( \mu \to 0 \) of the conformal galilean algebra CGA(1) \([41, 42, 43]\). In this case, the Hardy-space property of the dualised correlator can be derived and need not be postulated. In order to carry out the required contraction, we start from the 1D meta-conformal generators (2.2) and rescale the generators \( \mu Y_n \to Y_n \). This changes the commutators (2.3) into

\[
[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m] = (n-m)Y_{n+m}, \quad [Y_n, Y_m] = (n-m)\mu Y_{n+m}
\] (A.1)

Taking now the limit \( \mu \to 0 \) in (2.2) gives the contracted generators

\[
X_n = i(n+1)t^n r \partial_\zeta - t^{n+1} \partial_t - (n+1)t^n r \partial_r - (n+1)\delta t^n
\]
\[
Y_n = i(n+1)t^n \partial_\zeta - t^{n+1} \partial_r
\] (A.2)

of the non-semi-simple conformal galilean algebra CGA(1), with the commutators

\[
[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m] = (n-m)Y_{n+m}, \quad [Y_n, Y_m] = 0
\] (A.3)

We proceed as in section 2 of the main text. Add an extra generator \( N \) to the Cartan subalgebra, which must read

\[
N = -\zeta \partial_\zeta - r \partial_r - \nu
\] (A.4)

where \( \nu \) is a constant. As in section 2 of the main text, the dualised two-point correlator must take the form \( \widehat{F} = \delta_{\delta_1, \delta_2} |t|^{-2\delta_1} \widehat{F}(\zeta_+ + iv/t) \). Recall the notation \( \widehat{F}(\zeta_+) := \widehat{F}(\zeta_+ + i\lambda) \) and \( \zeta_+ = \frac{1}{2} (\zeta_1 + \zeta_2) \). In the case at hand co-variance under \( N \) leads to the specific form \([41, 42]\)

\[
\widehat{F}(u) = u^{-2\nu}, \quad 2\nu := \nu_1 + \nu_2
\] (A.5)

up to an unspecified normalisation constant. We can now make contact with the Hardy spaces \( H^+_2 \) introduced in appendix B.

**Proposition** \([41, 42]\). Let \( \nu > \frac{1}{4} \). If \( \lambda > 0 \), then \( \widehat{F}_\lambda \in H^+_2 \) and if \( \lambda < 0 \), then \( \widehat{F}_\lambda \in H^-_2 \).

**Proof:** \( \widehat{F}_\lambda \) is clearly analytic in the complex half-planes \( \mathbb{H}_\pm \), respectively and we must check the bound (B.1). Observe that \( |\widehat{F}_\lambda(u+iv)| = |(u+i(v+\lambda))^{-2\nu}| = (u^2 + (v+\lambda)^2)^{-\nu} \). Since \( \nu > \frac{1}{4} \) the integral in (B.1) converges and we have explicitly, for \( \lambda > 0 \)

\[
M^2 = \sup_{v>0} \int \mathbb{R} du \left| \widehat{F}_\lambda(u+iv) \right|^2 = \frac{\sqrt{\pi} \Gamma(2\nu - \frac{1}{2})}{\Gamma(2\nu)} \sup_{v>0} (v+\lambda)^{1-4\nu} < \infty
\]

as required (\( \Gamma \) denotes Euler’s Gamma function \([41]\)). For \( \lambda < 0 \) the argument is analogous. \( \square \)

The final correlator is found from the Hardy space representation theorem (B.3). For \( \lambda > 0 \), this gives (with the constraint \( \delta_1 = \delta_2 = \frac{1}{2} \))

\[
F = \frac{|t|^{-2\delta_1}}{\pi \sqrt{2\pi}} \int_{\mathbb{R}^2} d\zeta_+ d\zeta_- e^{-i(\gamma_1 + \gamma_2)\zeta_+} e^{-i(\gamma_1 - \gamma_2)\zeta_-} \int_{\mathbb{R}} d\gamma_+ \Theta(\gamma_+) \widehat{F}_+(\gamma_+) e^{-\gamma_+ \lambda} e^{i\gamma_+ \zeta_+}
\]

\[
= \frac{|t|^{-2\delta_1}}{\pi \sqrt{2\pi}} \int_{\mathbb{R}} d\gamma_+ \Theta(\gamma_+) \widehat{F}_+(\gamma_+) e^{-\gamma_+ \lambda} \int_{\mathbb{R}} d\zeta_- e^{-i(\gamma_1 - \gamma_2)\zeta_-} \int_{\mathbb{R}} d\zeta_+ e^{i(\gamma_+ - \gamma_1 - \gamma_2)\zeta_+}
\]

\[
= \delta(\gamma_1 - \gamma_2) \Theta(\gamma_1) F_{0,+}(\gamma_1) e^{-2\gamma_1 \lambda} |t|^{-2\delta_1},
\] (A.6)
Herein, $F_{0,+}$ contains the unspecified dependence on the positive constant $\gamma_1$. Similar, for $\lambda < 0$, we find $F = \delta(\gamma_1 - \gamma_2)\Theta(-\gamma_1)F_{0,-}(\gamma_1)e^{2\gamma_1|t|^{-2\delta_1}}$.

Before writing down a single combined form, we generalise to $G\Lambda(d)$ in $d > 1$ dimensions. Using a vector notation, the dualised generators are (with $n \in \mathbb{Z}$, and $1 \leq j, k \leq d$)

\begin{align}
X_n &= +i(n+1)nt^{n-1}\mathbf{r} \cdot \partial_\zeta - t^{n+1}\partial_t - (n+1)t^n \mathbf{r} \cdot \partial_\mathbf{r} - (n+1)\delta^n \\
Y_n &= +i(n+1)t^n\partial_\zeta - t^{n+1}\partial_t \\
R^{(jk)}_n &= -t^n(r_j\partial_{r_k} - r_k\partial_{r_j}) - t^n(\zeta_j \partial_{\zeta_k} - \zeta_k \partial_{\zeta_j}) \\
N &= -\zeta \cdot \partial_\zeta - \mathbf{r} \cdot \partial_\mathbf{r} - \nu
\end{align}

(A.7)

Taking also rotation-invariance (notice the specific form of the dualised spatial rotation generator $R^{(jk)}_0$) into account, we recover eq. (1.8) of the main text.

**Appendix B. Background on Hardy spaces**

In the main text (and appendix A), we need precise statements on the Fourier transform on semi-infinite spaces. These can be conveniently formulated in terms of Hardy spaces, where we restrict to the special case $H_2$. Our brief summary is based on [3, 55].

We begin with the case of functions of a single complex variable $z$, defined in the upper half-plane $\mathbb{H}^+ := \{z \in \mathbb{C} | z = x + iy \text{ and } y > 0\}$.

**Definition 1:** A function $f : \mathbb{H}^+ \to \mathbb{C}$ belongs to the Hardy space $H_2^+$ if it is holomorphic on $\mathbb{H}^+$ and if

\[ M^2 := \sup_{y > 0} \int_{-\infty}^{\infty} dx |f(x + iy)|^2 < \infty \]  

(B.1)

The main results of interest to us can be summarised as follows.

**Theorem 1:** [3] Let $f : \mathbb{H}^+ \to \mathbb{C}$ be a holomorphic function. Then the following statements are equivalent:

1. $f \in H_2^+$

2. there exists a function $f : \mathbb{R} \to \mathbb{C}$, which is square-integrable $f \in L^2(\mathbb{R})$, such that $\lim_{y \to 0^+} f(x + iy) = f(x)$ and

\[ f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\xi \frac{f(\xi)}{\xi - z} , \quad 0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\xi \frac{f(\xi)}{\xi - z^*} \]  

(B.2)

where $z^* = x - iy$ denotes the complex conjugate of $z$. For notational simplicity, one often writes $f(x) = \lim_{y \to 0^+} f(x + iy)$, with $x \in \mathbb{R}$.

3. there exists a function $\hat{f} : \mathbb{R}_+ \to \mathbb{C}$, $\hat{f} \in L^2(\mathbb{R}_+)$, such that for all $y > 0$

\[ f(z) = f(x + iy) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} d\zeta e^{i(x+iy)\zeta} \hat{f}(\zeta) \]  

(B.3)
The property (B.3) is of major interest to us in the main text.

Clearly, any function \( f : \mathbb{H}_+ \to \mathbb{C} \) which also admits a representation (B.3) with \( y > 0 \) and \( \hat{f} \in L^2(\mathbb{R}_+) \) is in the Hardy space \( H^+_2 \), since

\[
\int_{\mathbb{R}} dx \ |f(x+iy)|^2 = \frac{1}{2\pi} \int_0^\infty d\zeta \int_0^\infty d\zeta' \hat{f}(\zeta) \hat{f}^*(\zeta') \int_{\mathbb{R}} dx \ e^{i(x+iy)\zeta - i(x-iy)\zeta'}
\]

\[
= \frac{1}{2\pi} \int_0^\infty d\zeta \int_0^\infty d\zeta' \hat{f}(\zeta) \hat{f}^*(\zeta') e^{-2i\zeta(\zeta' + \zeta)} \int_{\mathbb{R}} dx \ e^{ix(\zeta' - \zeta)}
\]

\[
= \int_0^\infty d\zeta \ |\hat{f}(\zeta)|^2 e^{-4\delta^2} \leq \frac{1}{\pi} \int_0^\infty d\zeta \ |\hat{f}(\zeta)|^2 =: M^2 < \infty \quad (B.4)
\]

The more difficult part is to show that any \( f \in H^+_2 \) indeed admits such a representation. For the proof, (B.2) is needed \[3\]. Elements of a Hardy space enjoy certain limit behaviours, e.g. if \( f \in H^+_2 \), it can easily be shown that \[3\]

\[
\lim_{y \to \infty} f(x + iy) = 0 \quad \text{uniformly for all } x \in \mathbb{R} \quad (B.5a)
\]

\[
\lim_{x \to \pm \infty} f(x + iy) = 0 \quad \text{uniformly with respect to } y \geq y_0 > 0 \quad (B.5b)
\]

Indeed, eq. (B.5a) follows from the bound (which in turn can be obtained from (B.3) \[3\] \[49\])

\[
|f(x + iy)| \leq f_\infty y^{-1/2} \quad (B.5c)
\]

which holds for all \( x \in \mathbb{R} \) and where the constant \( f_\infty > 0 \) depends on the function \( f \). A simple sufficient criterion establishes whether a given function \( f \) is in the Hardy space \( H^+_2 \):

**Lemma:** If the complex function \( f(z) = f(x + iy) \) is holomorphic for all \( y \geq 0 \), obeys the bound \( |f(z)| < f_0 e^{-\delta y} \), with constants \( f_0 > 0 \) and \( \delta > 0 \), and if \( \int_{-\infty}^{\infty} dx \ |f(x)|^2 < \infty \), then \( f \in H^+_2 \).

**Proof:** Since \( f(z) \) is holomorphic on the closure \( \overline{\mathbb{H}_+} \) (which includes the real axis), one has the Cauchy formula

\[
f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dw \ \frac{f(w)}{w - z} = \frac{1}{2\pi i} \int_{-R}^{R} dw \ \frac{f(w)}{w - z} + \frac{1}{2\pi i} \int_{\mathcal{C}_{\text{sup}}} dw \ \frac{f(w)}{w - z} =: F_1(z) + F_2(z)
\]

where the integration contour \( \mathcal{C} \) consists of the segment \([-R, R]\) on the real axis and the superior semi-circle \( \mathcal{C}_{\text{sup}} \). One may write \( w = u + iv = Re^{i\theta} \in \mathcal{C}_{\text{sup}} \). It follows that on the superior semi-circle \( |f(w)| < f_0 e^{-\delta v} = f_0 e^{-\delta R \sin \theta} \). Now, for \( R \) large enough, one has \( |w - z| = |w(1 - z/w)| \geq R^2/2 \) for \( z \in \mathbb{H}_+ \) fixed and \( w \in \mathcal{C}_{\text{sup}} \). One estimates the contribution \( F_2(z) \) of the superior semi-circle, as follows

\[
|F_2(z)| \leq \frac{1}{2\pi} \int_{\mathcal{C}_{\text{sup}}} |dw| \frac{|f(w)|}{|w(1 - z/w)|} \leq \frac{1}{2\pi} \int_0^\pi d\theta \ \frac{f_0 e^{-\delta R \sin \theta}}{R} \leq \frac{2f_0}{\pi} \int_0^{\pi/2} d\theta \ \exp \left(-\frac{2\delta}{R}\right) \leq \frac{f_0}{\delta} \frac{1}{R} \to 0 \quad \text{for } R \to \infty
\]

Hence, for \( R \to \infty \), one has the integral representation \( f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} dw \ f(w)(w - z)^{-1} \). Since \( f \in L^2(\mathbb{R}) \), the assertion follows from eq. (B.2) of theorem 1. \( \square \)
One may also define a Hardy space $H_2^+$ for functions holomorphic on the lower complex half-plane $\mathbb{H}_-$, by adapting the above definition. All results transpose in an evident way.

Further conceptual preparations are necessary for the generalisation of these results to higher dimensions, although we shall merely treat the 2D case, which is enough for our purposes (and generalisations to $n > 2$ will be obvious). We denote $z = (z_1, z_2) \in \mathbb{C}^2$ and write the scalar product $z \cdot w = z_1 w_1 + z_2 w_2$ for $z, w \in \mathbb{C}^2$. Following [55], $H_2$-spaces can be defined as follows.

**Definition 2:** If $B \subset \mathbb{R}^2$ is an open set, the tube $T_B$ with base $B$ is

$$
T_B := \{ z = x + iy \in \mathbb{C}^2 \mid y \in B, x \in \mathbb{R}^2 \}
$$

(A.6)

A function $f : T_B \to \mathbb{C}$ which is holomorphic on $T_B$ is in the Hardy space $H_2(T_B)$ if

$$
M^2 := \sup_{y \in B} \int_{\mathbb{R}^2} dx |f(x + iy)|^2 < \infty
$$

(A.7)

However, it turns out that this definition is too general. More interesting results are obtained if one uses cônes as a base of the tubes.

**Definition 3:** (i) An open cône $\Gamma \subset \mathbb{R}^n$ satisfies the properties $0 \not\in \Gamma$ and if $x, y \in \Gamma$ and $\alpha, \beta > 0$, then $\alpha x + \beta y \in \Gamma$. A closed cône is the closure $\overline{\Gamma}$ of an open cône $\Gamma$.

(ii) If $\Gamma$ is a cône, and if the set

$$
\Gamma^* := \{ x \in \mathbb{R}^n \mid x \cdot t \geq 0 \text{ with } t \in \Gamma \}
$$

(A.8)

has a non-vanishing interior, then $\Gamma^*$ is the dual cône with respect to $\Gamma$. The cône $\Gamma$ is called self-dual, if $\Gamma^* = \overline{\Gamma}$.

For illustration, note that in one dimension ($n = 1$) the only cône is $\Gamma = \{ x \in \mathbb{R} \mid x > 0 \} = \mathbb{R}_+$. It is self-dual, since $\Gamma^* = \overline{\Gamma} = \mathbb{R}_{0,+}$. In two dimensions ($n = 2$), consider the cône $\Gamma^{++} := \{ x \in \mathbb{R}^2 \mid x = (x_1, x_2) \text{ with } x_1 > 0, x_2 > 0 \}$ which is the first quadrant in the 2D plane. Since

$$
\Gamma^{++} := \{ x \in \mathbb{R}^2 \mid x \cdot t \geq 0, \text{ for all } t \in \Gamma^{++} \} = \mathbb{R}_{0,+} \oplus \mathbb{R}_{0,+} = \overline{\Gamma^{++}}
$$

(A.9)

the cône $\Gamma^{++}$ is self-dual.

Hardy spaces defined on the tubes $T_{\Gamma^{++}}$ of the first quadrant provide the required structure.

**Definition 4:** [55] If $\Gamma^{++}$ denotes the first quadrant of the plane $\mathbb{R}^2$, a function $f : T_{\Gamma^{++}} \to \mathbb{C}$ holomorphic on $T_{\Gamma^{++}}$ is in the Hardy space $H_2^{++}$ if

$$
M^2 := \sup_{y \in \Gamma^{++}} \int_{\mathbb{R}^2} dx |f(x + iy)|^2 < \infty
$$

(A.10)

**Theorem 2:** [55] Let the function $f : T_{\Gamma^{++}} \to \mathbb{C}$ be holomorphic. Then the following statements are equivalent:

1. $f \in H_2^{++}$

2. there exists a function $f : \mathbb{R}^2 \to \mathbb{C}$, which is square-integrable $f \in L^2(\mathbb{R}^2)$, such that

$$
\lim_{y \to 0^+} f(x + iy) = f(x) \text{ and }
$$

$$
f(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dw \frac{f(w)}{w - z}, \quad 0 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dw \frac{f(w)}{w - z^*}
$$

(B.11)
where \((w - z)^{-1} := (w_1 - z_1)^{-1}(w_2 - z_2)^{-1}\) and \(z^* = x - iy\) denotes the complex conjugate of \(z\). For notational simplicity, one often writes \(f(x) = \lim_{y \to 0^+} f(x + iy)\), with \(x \in \mathbb{R}^2\).

3. there exists a function \(\hat{f} : \mathbb{R}_+ \oplus \mathbb{R}_+ \to \mathbb{C}\), with \(\hat{f} \in L^2(\mathbb{R}_+ \oplus \mathbb{R}_+)\) such that for all \(z_i \in \mathbb{H}_+\)
\[
f(z) = \frac{1}{2\pi} \int_{\Gamma^+} dt \ e^{iz \cdot t} \hat{f}(t) = \frac{1}{2\pi} \int_0^\infty dt_1 \int_0^\infty dt_2 \ e^{i(z_1 t_1 + z_2 t_2)} \hat{f}(t)
\]
(B.12)

The property (B.12) is of major interest to us in the main text. Summarising, the restriction to the first quadrant \(\Gamma^++\) allows to carry over the known results from the 1D case, separately for each component.

Hardy spaces \(H_2^{++}\), \(H_2^{+-}\), \(H_2^{--}\) on the other quadrants can be defined in complete analogy.

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