A Star Product for Complex Grassmann Manifolds

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Abstract

We explicitly construct a star product for the complex Grassmann manifolds using the method of phase space reduction. Functions on $\mathbb{C}^{(p+q)p \ast}$, the space of $(p+q) \times p$ matrices of rank p, invariant under the right action of $Gl(p, \mathbb{C})$ can be regarded as functions on the Grassmann manifold $G_{p,q}(\mathbb{C})$, but do not form a subalgebra whereas functions only invariant under the unitary subgroup $U(p) \subset Gl(p, \mathbb{C})$ do. The idea is to construct a projection from $U(p)$- onto $Gl(p, \mathbb{C})$-invariant functions, whose kernel is an ideal. This projection can be used to define a star-algebra on $G_{p,q}(\mathbb{C})$ onto which this projection acts as an algebra-epimorphism.

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1 Introduction

The idea of deformation quantization is to modify the multiplication on the algebra of smooth complex valued functions in a noncommutative way, such that in zeroth order the normal multiplication is preserved, the commutator equals in first order the Poisson bracket, the constant function with value one acts again as one in the new algebra and the pointwise complex conjugation remains an antilinear involution. This concept of a quantization, first introduced by Berezin [1], later as a program formulated by F. Bayen, M. Flato, c. Fronsdal, A. Lichnerowicz D. Sternheimer [2], is obviously founded on the existence of this kind of a deformed product. This nontrivial question was settled for symplectic manifolds by M. DeWilde and P.B.A. Lecomte [3], independently by B. Fedosov [4, 5] and by H. Omori, Y. Maeda and A. Yoshioka [6].

Explicit examples of star products for nontrivial phase spaces of physical interest are still rare. Although Grassmann manifolds $G_{p,q}(\mathbb{C})$ may be seen as the reduced phase space of a system of $(p+q) \cdot p$ harmonic oscillators where certain energy and angular momentum sums are fixed, it is not really a physical example. So the aim of this paper is principally to show that phase space reduction may be used to construct star products on more difficult symplectic manifolds. The methods used might be relevant as well to the possible deformation quantization of constrained systems.

C. Moreno [7] has already stated a recursion formula for a star product on nonexceptional Kähler symmetric spaces. M. Cahen, S. Gutt and J. Rawnsley [8] defined a star product on every compact Hermitian symmetric space and proved the convergence for the algebra of representative functions. Regarding $G_{p,q}(\mathbb{C})$ as a coadjoint orbit of $U(p+q)$ Karabegov [9] has given a method of deformation quantization in terms of representative functions which probably could be specialized explicitly in this case. The existence of star products on coadjoint orbits was treated as well by D. Arnal, J. Ludwig and M. Masmoudi [10]. For the special case of the projective space a closed formula for a star product of Wick type was given by M. Bordemann et al. [11]. It was constructed as well by phase space reduction and will be generalized in the present paper. Whereas there an equivalence transformation of the Wick product on $\mathbb{C}^{q+1}\setminus\{0\}$ onto a new star product was constructed which could be projected to $\mathbb{C}P^q$ trivially, in the case of Grassmann manifolds $G_{p,q}(\mathbb{C})$ we will directly head for the projection, because a similar equivalence transformation does not exist since the group $Gl(p, \mathbb{C})$ is noncommutative for $p > 1$.

The paper will be organized as follows: In the next section we will show how to construct a projection of functions on $\mathbb{C}^{(p+q):p}\ast$ onto functions on $G_{p,q}(\mathbb{C})$ whose kernel is an ideal with respect to the Wick star product on $C^\infty(\mathbb{C}^{(p+q):p}\ast)$. Then we will calculate this projection for the well known case of the projective space $\mathbb{C}P^q = G_{1,q}(\mathbb{C})$, thereby verifying the result of Bordemann et. al [11] in a very simple way. We use this as a guideline in the fourth section where the formula for the star product on Grassmann manifolds is proved. The arising coefficients are treated in the last section, where we will state some open questions as well. Throughout this paper we use Einstein’s summation convention.
2 The decomposition of the space of invariant functions

Our aim is to construct a star product on Grassmann manifolds $G_{p,q}(\mathbb{C})$ by making use of the Wick product on $\mathbb{C}^{(p+q)\times p}$, the space of complex $(p + q) \times p$ matrices of rank $p$, from which the Grassmann manifolds can be obtained by projection $\pi$. Unfortunately the homogeneous functions on $\mathbb{C}^{(p+q)\times p}$ – functions invariant under the $GL(p, \mathbb{C})$-action, which are the bijective image of functions on $G_{p,q}(\mathbb{C})$ by the pullback $\pi^*$ – do not form a subalgebra. Yet the set of invariant functions – we reserve this term for functions invariant under the unitary subgroup $U(p)$ of $GL(p, \mathbb{C})$ – is a subalgebra which we will denote by $(C^\infty_0(\mathbb{C}^{(p+q)\times p})[[\lambda]], \ast)$. So the Wick star product of two homogeneous functions will only be an invariant function and we are forced to search for a certain “projection” $P := C^\infty_0(\mathbb{C}^{(p+q)\times p})[[\lambda]] \rightarrow C^\infty_{hom}(\mathbb{C}^{(p+q)\times p})[[\lambda]]$, which satisfies $P(P(f \ast g) \ast h) = P(f \ast P(g \ast h))$ for any three homogeneous functions $f, g, h$. Then we can obviously equip $C^\infty_{hom}(\mathbb{C}^{(p+q)\times p})[[\lambda]]$ with an algebra structure by defining $f \ast g := P(f \ast g)$. For the unitary function we would like the equation $f = f \ast 1 = P(f \ast 1) = Pf$ to hold, thus $P|_{C^\infty_{hom}(\mathbb{C}^{(p+q)\times p})[[\lambda]]} = \text{id}|_{C^\infty_{hom}(\mathbb{C}^{(p+q)\times p})[[\lambda]]}$ and $P^2 = P$, so the name projection for $P$ is justified. We can split the algebra $C^\infty_0(\mathbb{C}^{(p+q)\times p})[[\lambda]] = PC^\infty_0(\mathbb{C}^{(p+q)\times p})[[\lambda]] \oplus \ker P$ and notice that if $P$ satisfies the above rules and is thus an algebra epimorphism of $(C^\infty_0(\mathbb{C}^{(p+q)\times p})[[\lambda]], \ast)$ onto $(C^\infty_{hom}(\mathbb{C}^{(p+q)\times p})[[\lambda]], \ast)$, obviously $\ker P$ must be a $\ast$-ideal $\mathcal{J}_\ast$. This condition is in fact an equivalent formulation, the projection $P$ is an algebra epimorphism if and only if $\ker P$ forms a $\ast$-ideal $\mathcal{J}_\ast$. The decomposition of the space of invariant functions shall now be constructed.

The complex Grassmann manifold $G_{p,q}(\mathbb{C})$ is the space of equivalence classes of the complex $(p + q) \times q$ matrices of rank $p$ where the equivalence class is defined as follows:

$$z_1 \sim z_2 \iff \exists g \in Gl(p, \mathbb{C}) \quad z_2 = z_1 g \quad z_1, z_2 \in \mathbb{C}^{(p+q)\times p} \tag{1}$$

The Grassmann manifold $G_{1,q}$ is the projective space $\mathbb{CP}^q$. Grassmann manifolds can also be regarded as the reduced phase space associated to the right action $\Phi$ of the unitary group $U(p)$:

$$\Phi : \mathbb{C}^{(p+q)\times p} \times U(p) \longrightarrow \mathbb{C}^{(p+q)\times p}, \quad (z, U) \longmapsto \Phi_U z = zU$$

This action is symplectic with respect to the Kähler symplectic form

$$\omega = \frac{i}{2} \text{tr} dz \wedge dz^\dagger = \frac{i}{2} \partial \bar{\partial} \text{tr} z^\dagger z = \frac{i}{2} \partial \bar{\partial} \text{tr} x \quad x := z^\dagger z \tag{2}$$

and there is an $\text{Ad}^*$-equivariant momentum mapping

$$J : \mathbb{C}^{(p+q)\times p} \longrightarrow u(p)^*, \quad J = \frac{i}{2} z^\dagger z = \frac{i}{2} x \tag{3}$$

where $u(p)^*$ denotes the dual of the space of antihermitian matrices. Since for any positive definite hermitian $\mu \in H^+(p)$ there exists a representative of $z/\sim$ on $J^{-1}(\frac{i}{2}\mu)$, namely $z(z^\dagger z)^{-\frac{1}{2}}\mu \frac{1}{2}$, the Grassmann manifold $G_{p,q}$ is the quotient $J^{-1}(\frac{i}{2}\mu)/\mu^{-\frac{1}{2}}U\mu^\frac{1}{2}$, and this is the reduced phase space if and only if $\mu$ is a positive scalar multiple of the unit matrix, which is
always the case for \( p = 1 \). So we choose in the following \( \mu \in \mathbb{R}^+ \) and elucidate the situation by the usual phase space reduction diagramm [12]:

\[
\begin{array}{c}
\mathbb{C}^{(p+q)\cdot p} \ast \\
\downarrow \pi \\
G_{p,q}(\mathbb{C})
\end{array}
\]

Additionally to the general case there exist the direct projections \( \pi \) and \( \bar{\rho} \). Identifying \( \mathbb{C}^{(p+q)\cdot p} \ast \) with \( J^{-1}(\frac{i}{2} \mu) \times H^+ \) by the global diffeomorphism

\[
\begin{array}{c}
z \mapsto (\zeta, x) = (z(z^\dagger z)^{-\frac{1}{2}} \mu^{\frac{1}{2}}, z^\dagger z)
\end{array}
\] (4)

the mappings \( i \) and \( \bar{\rho} \) adopt the explicit form

\[
\begin{array}{c}
i(\zeta) = (\zeta, \mu) \\
\bar{\rho}(\zeta, x) = \bar{\rho}(z) = \zeta = z(z^\dagger z)^{-\frac{1}{2}} \mu^{\frac{1}{2}}
\end{array}
\] (5)

and an invariant function \( f \in C_0^\infty(\mathbb{C}^{(p+q)\cdot p} \ast) \) satisfies

\[
\begin{array}{c}
f(z) = f(\zeta, x) = f(zU) = f(\zeta U, U^\dagger x U)
\end{array}
\] (6)

Looking for an appropriate choice of the star ideal and the associated projection we consider the trivial case of the pointwise product as a guideline. There is obviously a 1:1 mapping between invariant functions on the orbit \( J^{-1}(\frac{i}{2} \mu) \) and functions on \( G_{p,q}(\mathbb{C}) \), i.e. \( \pi_* \) is invertible for invariant functions. All invariant functions on the orbit may be obtained by restriction of invariant functions on \( \mathbb{C}^{(p+q)\cdot p} \ast \) and thus there is a simple projection \( P \).

\[
P : C_0^\infty(\mathbb{C}^{(p+q)\cdot p} \ast) \longrightarrow C^\infty_{\text{hom}}(\mathbb{C}^{(p+q)\cdot p} \ast), \quad P := \pi_* \circ \pi_*^{-1} \circ i^* = \bar{\rho}^* \circ i^* 
\] (7)

We can easily decompose an invariant function into its homogeneous and its ideal part, thereby describing the ideal:

\[
[(1 - P)F](z) = F(\zeta, x) - F(\zeta, \mu) = \left< \int_0^1 dt \frac{\partial F}{\partial x}(\zeta, tx + (1 - t)\mu), x - \mu \right>
\]

So \( \ker P = (1 - P)C_0^\infty(\mathbb{C}^{(p+q)\cdot p} \ast) \) is generated by \( x - \mu \):

\[
\ker P = \mathcal{J} = \langle x - \mu \rangle = \{ F \mid F = (x - \mu, G), \quad G \in C^\infty_{\text{eq}}(\mathbb{C}^{(p+q)\cdot p} \ast, u(p)) \}
\]

Note that in our case of invariant functions we can restrict the domain of functions paired with \( x - \mu \) onto \( \text{Ad-equivariant} \) functions since for

\[
H : C_0^\infty(\mathbb{C}^{(p+q)\cdot p} \ast) \longrightarrow C^\infty_{\text{eq}}(\mathbb{C}^{(p+q)\cdot p} \ast), \quad H(F)(\zeta, x) := \int_0^1 dt \frac{\partial F}{\partial x}(\zeta, tx + (1 - t)\mu)
\] (8)

holds

\[
H(F \circ \Phi_U) = \text{Ad}_{U^{-1}} \circ H(F) \circ \Phi_U = H(F)
\] (9)
The decomposition
\[ F = P F + (1 - P) F = \bar{\rho}^* i^* F + \langle H(F), x - \mu \rangle \] (10)
is unique whereas the function \( H \) is not, unless \( p = 1 \), because we can add to \( H(F) \) any function orthogonal to \( x - \mu \). We have chosen \( H \) here to be the integral along the straight line between \((\zeta, x)\) and \((\zeta, \mu)\), but any other curve connecting these points will do as well. For the solution a convenient choice of \( H(F) \) will be important as well as the proof that the decomposition is unique, such that the result is independent of the choice of \( H(F) \).

Analogously we would like to define a star ideal \( J_\natural \) generated by \( x - \mu \). First of all we have of course to extend our considerations to power series in a formal parameter \( \lambda \) with coefficients in the respective function spaces. The ideal
\[ J_\natural := \langle x - \mu \rangle_\natural = \{ F | = \langle G; x - \mu \rangle = G_j i^* (x - \mu)^j, \ G \in C^\infty_c(\mathbb{C}^{(p+q) \ast} \ast u(p))[[\lambda]] \} \]
may be defined independently of a global basis for \( \mathbb{C}^{(p+q) \ast} \) and is indeed a twosided ideal because the Wick star product is strongly \( U(p) \)-invariant
\[ f \ast J - J \ast f = \frac{i\lambda}{2} \{ f, J \} \quad \forall f \in C^\infty(\mathbb{C}^{(p+q) \ast}) , \] (11)
while \( U(p) \) is compact – note that \( G \) is only equivariant. Now we can formulate the important

**Lemma 2.1** There exists a unique decomposition of invariant functions
\[ C^\infty_0(\mathbb{C}^{(p+q) \ast})[[\lambda]] = C^\infty_{hom}(\mathbb{C}^{(p+q) \ast})[[\lambda]] \oplus J_\natural \]
into a homogeneous and a \( \natural \)-ideal part, where the ideal is spanned by \( x - \mu \).

As an immediate consequence of the proposition and the remarks in the introduction we obtain the following

**Corollary 2.1** There exists a star product on \( G_{p,q}(\mathbb{C}) \) and an algebra epimorphism
\[ \pi^{-1} \circ P : (C^\infty(\mathbb{C}^{(p+q) \ast})[[\lambda]], \natural) \rightarrow (C^\infty(G_{p,q}(\mathbb{C}))[[\lambda]], \ast) . \]

**Proof:** We start with the point multiplicative decomposition and start deforming the usual product in the following way:
\[ F = \bar{\rho}^* i^* F + \langle H(F), x - \mu \rangle = \bar{\rho}^* i^* F + \langle H(F) \natural, x - \mu \rangle + \lambda \Delta(F) \] (12)
\[ \Delta(F) := \frac{1}{\lambda} \left( \langle (H(F) \natural, x - \mu) - \langle H(F), x - \mu \rangle \right) \] (13)
\( \Delta(F) \) is a power series in \( \lambda \) since the coefficient of \( \lambda^{-1} \) vanishes and we can proceed decomposing \( \Delta(F) \). Using induction, keeping in mind the linearity of \( \bar{\rho}^*, i^*, H \) and \( \Delta \) yields:
\[ F = \bar{\rho}^* i^* \left( \frac{1}{1 + \lambda \Delta} F \right) + \left\langle H \left( \frac{1}{1 + \lambda \Delta} F \right), x - \mu \right\rangle \] (14)
So there is a decomposition, but since \( H \) is not uniquely defined, \( \Delta \) is not either, so uniqueness might be lost during the deformation process. To ensure uniqueness assume a function \( F \) in the \( \tilde{\ast} \)-ideal \( J_{\ast} \) is simultaneously homogeneous, i.e.

\[
\tilde{\rho}^s \varphi = F = \langle G, x - \mu \rangle , \quad G \in C_{eq}^\infty (\mathbb{C}^{(p+q)p \ast}, u(p)) \quad [\lambda] \quad \varphi \in C_0^\infty \left( J^{-1} \left( \frac{i}{2} \mu \right) \right) [\lambda]
\]

It has to be shown that \( F \) vanishes identically, thus both sides of the above equation. We can make use of the fact, that the series for the Wick product with \( x \) breaks off after the first term:

\[
\langle G, x \rangle = \langle G, x \rangle + \lambda \tilde{M}_1 \langle G, x \rangle , \quad \tilde{M}_1 \langle G, x \rangle := z^A \partial_{\tilde{A}_j} G_{i} \quad G \in C_{eq}^\infty (\mathbb{C}^{(p+q)p \ast}, u(p)) \quad (15)
\]

Let \( \varphi_\nu \in C_0^\infty (J^{-1} \left( \frac{i}{2} \mu \right)) \) and \( G_\nu \in C_{eq}^\infty (\mathbb{C}^{(p+q)p \ast}, u(p)) \) be the coefficients of \( \varphi \) and \( G \) respectively:

\[
\tilde{\rho}^s \varphi = \sum_{\nu=0}^\infty \lambda^\nu \tilde{\rho}^s \varphi_\nu = \sum_{\nu=0}^\infty \lambda^\nu \left( \langle G_\nu, x - \mu \rangle + \tilde{M}_1 \langle G_{\nu-1}, x \rangle \right) \quad G_\nu \equiv 0
\]

Considering all orders separately we obtain after restriction to \( x = \mu \)

\[
\varphi_\nu = i^s \tilde{M}_1 \langle G_{\nu-1}, x - \mu \rangle \quad \varphi_0 = 0 . \quad (16)
\]

The strategy is the following: Assume \( \varphi_\nu = 0 \) for \( \nu < n \). Consequently it holds

\[
\tilde{M}_1 \langle G_{\nu-1}, x - \mu \rangle = -\langle G_\nu, x - \mu \rangle \quad \text{for} \quad \nu < n . \quad (17)
\]

Express \( \varphi_n = i^s \tilde{M}_1 \langle G_{\nu-1}, x \rangle \) by derivatives of \( \langle G_{\nu-1}, x - \mu \rangle \) and use the \( n - 1 \) equations (17) successively in order to write \( \varphi_n \) as derivatives of \( \langle G_\nu, x - \mu \rangle \) which vanishes identically. Then \( \varphi_0 = 0 \) and by induction \( \varphi_\nu = 0 \) for all \( \nu \) what completes the proof.

Obviously we have to consider derivatives at \( J^{-1} \left( \frac{i}{2} \mu \right) \) transversally to this submanifold. Thus in what follows we set \( z = \zeta y \) and we investigate derivatives with respect to \( y \) at the point \( y = 1 \). In a tedious, but simple calculation one verifies the following formulas.

\[
\frac{\partial^r}{\partial y_{i_1} \cdots \partial y_{i_r}} \bigg |_{y=1} \left( \langle G_\nu, x - \mu \rangle (\zeta y) \right) = \mu \sum_{s=1}^r \frac{\partial^{r-1} G_\nu}{\partial z_{A_{i_s}} z_{A_{i_1}} \cdots z_{A_{i_{s-1}}} z_{A_{i_{s+1}}} \cdots z_{A_{i_r}}} (\zeta) \zeta_{A_{i_1}} \cdots \zeta_{A_{i_r}} . \quad (18)
\]

\[
\frac{\partial^r}{\partial y_{i_1} \cdots \partial y_{i_r}} \bigg |_{y=1} \left( \tilde{M}_1 \langle G_\nu, x - \mu \rangle (\zeta y) \right) = \delta^{r+1}_{i_{r+1}} \frac{\partial^{r+1} G_\nu}{\partial z_{A_{i_1}} \cdots z_{A_{i_{r+1}}}} (\zeta) \zeta_{A_{i_1}} \cdots \zeta_{A_{i_{r+1}}} + \sum_{s=1}^r \delta^{i_{r+1}} \frac{\partial^r G_\nu}{\partial z_{A_{i_1}} \cdots z_{A_{i_s}} z_{A_{i_{s+1}}} \cdots z_{A_{i_r}}} (\zeta) \zeta_{A_{i_1}} \cdots \zeta_{A_{i_r}} . \quad (19)
\]

Before proceeding with the proof let us summarize some notation for conjugacy classes of the symmetric group. By the \( r \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_r) \) we denote the conjugacy class in \( S_r \) having \( \alpha_i \) i-cycles in the cycle representation, by \( Con(S_r) \) the set of all conjugacy classes
Obviously it holds $\sum i\alpha_i = r$ and we define $|\alpha| = \alpha_1 + \ldots + \alpha_r$ as usual. The number of elements in the conjugacy class $\alpha$ is

$$h_\alpha = \frac{r!}{1^{\alpha_1}\alpha_1!\ldots r^{\alpha_r}\alpha_r!}.$$  \hfill (20)

Since $i - 1$ transpositions are necessary to compose an $i$-cycle from 1-cycles (resp. to decompose it into 1-cycles) the lowest number of transpositions, denoted by $\#$, that is needed to compose a certain permutation $\sigma \in \alpha$ from the identity (or vice versa to decompose it into the identity) is only dependent on the conjugacy class $\alpha$ and it holds

$$\#\alpha = \#[\sigma] = \sum_{i=1}^{r} (i - 1)\alpha_i = r - |\alpha|$$ \hfill (21)

We abbreviate by $D := \left( \frac{\partial}{\partial y_{ij}} \right)_{i,j=1\ldots p}$ and set $D_k = \text{tr}D^k := \frac{\partial^r}{\partial y_{i_1} \partial y_{i_2} \ldots \partial y_{i_k}}$. Now assume $\varphi_\nu \equiv 0$ for $\nu < n$ thus also equations (17). Then we find successively

$$\varphi_n = -\frac{1}{(-\mu)^k} \sum_{r=k+1}^{2k} \frac{1}{r!} \sum_{\alpha \in \text{Con}(S_r) \atop \alpha_1 = 0 \#\alpha = k} h_\alpha D_2^{\alpha_2} \ldots D_r^{\alpha_r} \langle G_{(n-k)}, x - \mu \rangle.$$ \hfill (22)

This formula can be proved by induction using the above formulas. Using it for $k = n$ shows that $\varphi_n$ vanishes identically and the proof is completed. \hfill $\square$

A final lemma will be useful in the next section:

**Lemma 2.2** Let $F = PF + \langle G; x - \mu \rangle$ be the unique decomposition of the invariant function $F$ and let $f$ be homogeneous $P f = f$. Then $f F = f PF + \langle f G; x - \mu \rangle$ is the decomposition of the product $f F$.

**Proof:** For a homogeneous function $f$ holds $z^A_i \frac{\partial}{\partial z^A_j} f = 0$ $i, j = 1 \ldots p$. Hence we have $\langle f G; x - \mu \rangle = f \langle G; x - \mu \rangle$. \hfill $\square$

### 3 The solution for the projective space $\mathbb{C}P^q$

The first step towards the construction of a star product on the quotient manifold for the projective space as well as for the general Grassmann manifolds consists in ”homogenizing” the expression for the Wick product of two homogeneous functions. The summands in the series are written as products of homogeneous differential operators – differential operators that assign two homogeneous functions another homogeneous function and that are thus well defined on the quotient manifold – and powers of a certain nonhomogeneous function. According to the lemma in the last paragraph it suffices to calculate the projection for this certain function and its powers. In case of the projective space $\mathbb{C}P^q$ the homogeneization is simple and was also used in [11] to find an explicit expression for the star product on $\mathbb{C}P^q$.

$$F \star G = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} x^{-r} \left( \frac{\partial^r F}{\partial z^A_1 \ldots \partial z^A_r} \cdot \frac{\partial^r G}{\partial z^A_1 \ldots z^A_r} \right)$$

$$= \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} x^{-r} M_r(F,G)$$ \hfill (23)
Obviously \( x^r \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x} \) is a homogeneous differential operator, assigning two homogeneous functions the tensor product of homogeneous functions, so its power (followed by multiplication) \( M_r \) are as well homogeneous. Now a star product for homogenous functions \( F, G \in C^\infty_{\text{hom}}(\mathbb{C}^{q+1}\setminus\{0\}) \) is easily constructed

\[
F \ast G = P(F \hspace{1pt} \tilde{\ast} \hspace{1pt} G) = \sum_{r=0}^{\infty} P(x^{-r} M_r(F,G)) = \sum_{r=0}^{\infty} P(x^{-r}) M_r(F,G) \quad ,
\]

where the last equation sign holds because of lemma 2.2. Following the program described in section 4 we obtain

\[
(H(x^{-r}), x - \mu) = x^{-r} - \tilde{\rho}^s i^s x^{-r} = x^{-r} - \mu^{-r} = -(x - \mu) \sum_{\nu=1}^{r} x^{-\nu\nu^{-r-1}}
\]

In this case \( H(F) \) is a uniquely determined invariant function.

\[
\Delta(x^{-r}) = z^A \frac{\partial}{\partial z^A} H(x^{-r}) = x^A \frac{\partial}{\partial x} H(x^{-r}) = \sum_{\nu=1}^{r} \nu x^{-\nu\nu^{-r-1}}
\]

Applying the general recursion formula

\[
P(F) = \tilde{\rho}^s i^s F - \lambda P(\Delta F)
\]

to the cases \( F = x^{-r} \) and \( F = \mu^{-1} x^{r+1} \) as well as to \( F = x^{-1} \) yields a recursion formula and its beginning:

\[
P \left( x^{-r} \right) = \frac{1}{\mu + \lambda r} P \left( x^{-(r-1)} \right) \quad \quad P \left( x^{-1} \right) = \frac{1}{\mu + \lambda}
\]

This proves the simple

**Lemma 3.1**

\[
P \left( x^{-r} \right) = \prod_{s=1}^{r} \frac{1}{\mu + \lambda s} = \lambda^{-r} \frac{1}{[c]_r},
\]

where \( [y]_r := y(y+1)\ldots(y+r-1) \), \( [y]_0 := 1 \) and \( c := \mu \lambda^{-1} + 1 \). \([\Box]\)

In the same way as described here one could calculate \( P(x^r) = \prod_{s=0}^{r-1} (\mu - \lambda s) \).

**Corollary 3.1** There exists a star product on \( G_{1,q}(\mathbb{C}) = \mathbb{C} P^q \) defined by (\( f, g \in C^\infty(\mathbb{C} P^q) \))

\[
\pi^s(f \ast g) := P(\pi^s f \hspace{1pt} \tilde{\ast} \hspace{1pt} \pi^s g) = \sum_{r=0}^{\infty} \frac{1}{r! [c]_r} M_r(\pi^s f, \pi^s g) \quad ,
\]

where the homogeneous operators \( M_r \) are defined by

\[
M_r(F,G) := x^r \frac{\partial^r F}{\partial z^{A_1} \ldots \partial z^{A_r}} \cdot \frac{\partial^r G}{\partial z^{A_1} \ldots \partial z^{A_r}} = \mu^r \tilde{\rho}^s i^s \left( \frac{\partial^r F}{\partial z^{A_1} \ldots \partial z^{A_r}} \cdot \frac{\partial^r G}{\partial z^{A_1} \ldots \partial z^{A_r}} \right) \quad .
\]

At this point it seems to be only a matter of curiosity that for this star product not only the limit \( \lambda \to 0 \), but also the opposite one, \( \lambda \to \infty \) is well defined as a formal power series. And at a glance the space of functions for which this product converges is nonempty. This strange possibility of a "strong quantum limit" will be preserved in the Grassmann case.
4 The projection for Grassmann manifolds

We have to find a transfered version of the homogeneization procedure (23), since \( x \) is no longer a scalar function. First note that according to the usual properties differentiation lowers the degree of homogeneity by one, so for a homogeneous function \( F \in C^\infty_{\text{hom}}(\mathbb{C}^{(p+q)\cdot p \cdot *}) \) which is more precisely a homogeneous of degree 0 holds:

\[
F(zg) = F(z) \quad \forall g \in \text{Gl}(p, \mathbb{C}) \quad \implies \quad g^i_j \frac{\partial F}{\partial z^A_j}(zg) = \frac{\partial F}{\partial z^A_i}(z) \quad i = 1 \ldots p \quad A = 1 \ldots p + q
\]

Thus the expression \( z^R_i \frac{\partial F}{\partial z^A_i} \) is again homogeneous and this suggests to rewrite the expression for the Wick product for the case of homogeneous functions \( F, G \in C^\infty_{\text{hom}}(\mathbb{C}^{(p+q)\cdot p \cdot *}) \) in the following way:

\[
F \ast G = m \circ \exp \left( \lambda S^D_{C \cdot C} z^C_i \frac{\partial}{\partial z^j_{\scalebox{0.7}{A}}} \otimes \frac{\partial}{\partial z^{j \cdot D}} \right) (F \otimes G)
\]

\[
= m \circ \exp \left( \lambda S^D_{C \cdot C} z^C_i \frac{\partial}{\partial z^j_{\scalebox{0.7}{A}}} \right) (F \otimes G) = \sum_{r=0}^\infty \frac{\lambda^r}{r!} \left( S^{\otimes r}, M_r(F, G) \right)
\]

where

\[
S = zx^{-2}z^\dagger, \quad M_r(F, G)^{C_1 \ldots C_r}_{D_1 \ldots D_r} = z^{C_1}_{i_1} \ldots z^{C_r}_{i_r} \frac{\partial^r F}{\partial z^{A_1}_{i_1} \ldots \partial z^{A_r}_{i_r}} \frac{\partial^r G}{\partial z^{A_1}_{j_1} \ldots \partial z^{A_r}_{j_r}} z^{j_1}_{D_1} \ldots z^{j_r}_{D_r},
\]

\[
m(F \otimes G) = F \cdot G,
\]

and \((\cdot, \cdot)\) is the extension of the usual hermitian product \((A, B) = \text{tr} A^\dagger B\) to tensor products. The matrix \( S \) is obviously hermitian and in contrast to the momentum mapping \( J \), which is only equivariant with respect to the \( U(p) \)-action, even \( U(p) \)-invariant. The homogeneous differential operators take values in the \( r \)-fold tensor product of the defining representation of \( U(p + q) \).

Formally the construction of a star product for homogeneous functions on \( \mathbb{C}^{(p+q)\cdot p \cdot *} \) is completely analogous to the \( \mathbb{C}P^n \)-case of the last section

\[
F \ast G := P(F \ast G) = \sum_{r=0}^\infty \frac{\lambda^r}{r!} P \left( S^{\otimes r}, M_r(F, G) \right) = \sum_{r=0}^\infty \frac{\lambda^r}{r!} \left( P(S^{\otimes r}), M_r(F, G) \right), \quad (29)
\]

where again the last equation sign is justified by lemma (5). Yet the evaluation of the projection \( P(S^{\otimes r}) \) shows a new aspect for which we introduce the notion of the action of the symmetric group on the tensor product space \((\mathbb{C}^s)^{\otimes r}\):

**Definition 4.1** For \( \sigma \in S_r \) let \( \rho(\sigma) : (\mathbb{C}^s)^{\otimes r} \rightarrow (\mathbb{C}^s)^{\otimes r} \) be the representation defined by linear extension of the prescription \( \rho(\sigma)(e_{A_1} \otimes \ldots \otimes e_{A_s}) = e_{A_{\sigma(1)}} \otimes \ldots \otimes e_{A_{\sigma(r)}} \) \( 1 \leq A_i \leq s \), where \( \{e_A\}_{A=1 \ldots s} \) is a basis of \( \mathbb{C}^s \). This representation can linearly be extended to the group algebra. As well let for \( S \in \text{Gl}(s, \mathbb{C}) \) be \( D(S) : (\mathbb{C}^s)^{\otimes r} \rightarrow (\mathbb{C}^s)^{\otimes r} \) the representation defined by \( D(S) := S^{\otimes r} \), which in a basis reads \( D(S)(e_{A_1} \otimes \ldots \otimes e_{A_s}) := e_{B_1} \otimes \ldots \otimes e_{B_s} S_{A_1}^{B_1} \ldots S_{A_s}^{B_s} \).
Obviously the two representations commute, that means $\rho(\sigma) \circ D(S) = D(S) \circ \rho(\sigma)$ for all $\sigma \in S_r$ and $S \in \text{Gl}(s, \mathbb{C})$. Linear extension of $D$ to a representation of the endomorphism algebra $\text{End}(\mathbb{C}^s)$ leads to the subalgebra of all mappings that commute with permutations. We will denote this subalgebra as $\mathbb{C}[D(\text{Gl}(s, \mathbb{C}))]$.

Using the notation introduced in lemma 2.1 concerning the conjugacy classes of the symmetric group we can formulate the essential

**Lemma 4.1** For the projection $P(S^{\otimes r})$ the following recursion relation holds:

$$
\lambda^r \rho \left( \sum_{\alpha \in \text{Con}(S_r)} c^{[\alpha]} k_\alpha \right) P(S^{\otimes r}) = T^{\otimes r} ,
$$

where $T := \mu \rho^* i^* S = \mu^{-1} \zeta^\dagger \zeta$, $c := \lambda^{-1} \mu + p$ and $k_\alpha := \sum_{\sigma \in \alpha} \sigma$ is defined to be the sum over all permutations of the conjugation class $\alpha$ in the group algebra $\mathbb{C}[S_r]$.

**Proof:** The formula will be proved by induction. For any constant hermitian $p + q$ matrix $\Phi$ holds:

$$
\langle H((S, \Phi)), x - \mu \rangle = -\mu^{-1}(zx^{-1}(x - \mu)x^{-1}z^\dagger, \Phi) = -\mu^{-1}(x^{-1}z^\dagger \Phi zx^{-1}, x - \mu)
$$

and we can choose

$$
H((S, \Phi)) = -\mu^{-1}z^\dagger \Phi zx^{-1} .
$$

The calculation of $\Delta(S, \Phi)$ is simplified by the fact that $z^A \frac{\partial}{\partial z^A}$ is the fundamental vector field for the holomorphic $\text{Gl}(p, \mathbb{C})$-action, i.e. $((z, z^\dagger), g) \mapsto (z g, z^\dagger)$ under which the term $zx^{-1}$ is invariant.

$$
\Delta((S, \Phi)) = z^A \frac{\partial}{\partial z^A} H^i_j(S, \Phi) = \mu^{-1} \rho(S, \Phi) \implies \Delta(S) = \mu^{-1} p S
$$

Using the recursion formula (25) one finds $\lambda c P(S) = T$, which proves the start of the induction. Now assume the formula is proved for $i < r$. A possible choice for $H$ is

$$
H(S^{\otimes r}) = S^{\otimes r-1} \otimes H(S) + H(S^{\otimes r-1}) \otimes \mu^{-1} T
$$

and it follows

$$
\Delta(S^{\otimes r}) = S^{\otimes r-1} \otimes \Delta(S) + (z^A \frac{\partial}{\partial z^A} S^{\otimes r-1}) \otimes H^i_j(S) + \Delta(S^{\otimes r-1}) \otimes \mu^{-1} T
$$

$$
= \mu^{-1} p S^{\otimes r} + \sum_{s=1}^{r-1} \rho(\tau_{sr}) S^{\otimes r} + \mu^{-1} \Delta(S^{\otimes r-1}) \otimes T ,
$$

where $\tau_{sr}$ is the transposition of $s$ and $r$. Comparing the recursion formula (25) analogously to the proof of lemma (3.1) for $P(S^{\otimes r})$ and $\mu^{-1} P(S^{\otimes r-1}) \otimes T$ taking into consideration that $P$ and $\rho$ commute, yields:

$$
(1 + \lambda \mu^{-1} P + \lambda \mu^{-1} \sum_{s=1}^{r-1} \rho(\tau_{sr})) P(S^{\otimes r}) = \mu^{-1} P(S^{\otimes r-1}) \otimes T
$$
Using the induction hypothesis and the relation (21) leads to
\[ \lambda^r (c + \sum_{s=1}^{r-1} \rho(\tau_{sr})) \left( \sum_{\sigma \in S_{r-1} \subseteq S_r} c^{r-1-\#[\sigma]} \rho(\sigma) \right) \lambda^r \sum_{\sigma \in S_r} c^{r-\#[\sigma]} \rho(\sigma) P(S^{\otimes r}) = T^{\otimes r}, \]
which proves the lemma. □

Remark: In the same way one can show that for the positive powers holds
\[ P \left( z z^\dagger^{\otimes r} \right) = \lambda \sum_{\alpha \in Con(S_r)} \left( -\lambda \mu^{-1} \right) c_{\alpha} k_{\alpha} \left( \zeta \zeta^\dagger^{\otimes r} \right). \]

Since \( \{k_{\alpha}\}_{\alpha \in S_r} \) is a basis of the center of the group algebra \( \mathbb{C}[S_r] \), \( \sum_{\alpha \in Con(S_r)} c_{\alpha} k_{\alpha} \) is an element of the center which will be invertible for generic \( c \). For the inverse element we can write
\[ \left( \sum_{\alpha} c_{\alpha} k_{\alpha} \right)^{-1} =: \sum_{\alpha} s_{\alpha}(c) k_{\alpha}, \]
where \( s_{\alpha} \) are rational functions which are to be determined. First we formulate a trivial corollary of lemma 4.1.

Corollary 4.1 There exists a star product on \( G_{p,q}(\mathbb{C}) \) defined by \((f,g) \in C^\infty(G_{p,q}(\mathbb{C}))[\lambda] \)
\[ \pi^*(f * g) = P(\pi^* f \ast \pi^* g) = \sum_{r=0}^{\infty} \mu^r \rho^r \left\langle \left( \sum_{\alpha \in Con(S_r)} s_{\alpha}(c) k_{\alpha} \right) \frac{\partial^r \pi^* f}{\partial z_{A_1} \cdots \partial z_{A_r}}, \frac{\partial^r \pi^* g}{\partial z_{A_1} \dagger \cdots \partial z_{A_r} \dagger} \right\rangle, \]
where the rational functions \( s_{\alpha} \) are defined by (33) and \( c = \lambda^{-1} \mu + p \). Here we regard the derivatives \( \frac{\partial^r F}{\partial z_{A_1} \dagger \cdots \partial z_{A_r} \dagger} \) as \((i_1, \ldots, i_r)\)-component of a tensor in \((\mathbb{C}^p)^{\otimes r}\) and \( \langle \cdot, \cdot \rangle \) denotes the hermitian inner product of \((\mathbb{C}^p)^{\otimes r}\). □

A comparison with corollary 3.1 yields as a condition for \( s_{\alpha} \):
\[ \sum_{\sigma \in S_r} s_{|\sigma|}(c) = \sum_{\alpha \in Con(S_r)} h_{\alpha} s_{\alpha}(c) = \frac{1}{[c]_r}. \]

5 The irreducible differential operators and their coefficients

This section is devoted to the determination of the rational functions \( s_{\alpha} \) defined by (33) that arise in the star product on Grassmann manifolds (34). According to Wedderburn’s theorem the group algebra \( \mathbb{C}[G] \) of a finite group \( G \) decomposes directly into simple endomorphism rings \( \text{End}(\mathbb{C}^{n_a}), \ a = 1 \ldots f \) of dimension \( n_a^2 \) such that \( \sum n_a^2 \) equals the order \( |G| \) of the group. Each simple ring \( \text{End}(\mathbb{C}^{n_a}) \) may be seen as an isotypical submodule of the regular representation of the group \( G \) on its group algebra \( \mathbb{C}[G] \) obtained by left multiplication. These isotypical submodules \( \text{End}(\mathbb{C}^{n_a}) \) are associated to a certain irreducible representation
\[ \rho^a, \text{ which is therein contained with multiplicity } n_a. \text{ To sum up, all irreducible representations } \rho^a \text{ of a finite group } G \text{ are contained in the group algebra and their multiplicity equals their dimension } n_a. \text{ The number } f \text{ of inequivalent irreducible representations equals the number of conjugacy classes, which is in case of the symmetric group } S_r \text{ the number of partitions of } r. \text{ So the center of the group ring } \mathbb{C}[G] \text{ has two natural bases, the conjugation classes } k_\alpha \text{ and the units } e_\alpha \text{ of the simple rings } End(C^{n_a}) \text{ into which the group algebra decomposes. Between both there is a simple base transformation in terms of characters [13], which for the case of the symmetric group } S_r \text{ reads}
\]

\[ e_a = \frac{n_a}{r!} \sum_{\alpha=1}^{f} \chi^a_\alpha k_\alpha \quad \quad k_\alpha = h_\alpha \sum_{a=1}^{f} \frac{\chi^a_\alpha}{n_a} e_a \tag{36} \]

Here \( \chi^a_\alpha \) is the character of the irreducible representation \( \rho^a \) evaluated at an element of the conjugacy class \( \alpha \). Note that for the symmetric group a group element and its inverse are in the same class, such that the character is real. Of course the basis \( \{e_a\}_{a=1...f} \) is more appropriate for considering the inverse (33):

\[ \left( \sum_a e^{(\alpha)} k_\alpha \right)^{-1} = \sum_a \frac{n_a}{\sum_a h_\alpha \chi^a_\alpha e^{(\alpha)}} e_a =: \sum_a \frac{1}{t_a(c)} e_a \quad \quad t_a(c) := \frac{1}{n_a} \sum_\alpha h_\alpha \chi^a_\alpha e^{(\alpha)} \tag{37} \]

The relation between the rational functions \( s_\alpha \) introduced in (33) and the polynomials \( t_a \) is

\[ s_\alpha(c) = \sum_a \frac{n_a \chi^a_\alpha}{r!} \frac{1}{t_a(c)} \tag{38} \]

As noted above any isotypical submodule \( End(C^{n_a}) \) of the group ring or equivalently any irreducible representation \( \rho^a \) (up to isomorphism) or any unit \( e_a \) is characterised by a partition, which is usually represented by a frame. We quote the dimension formula [13]:

**Proposition 5.1** The dimension \( n_{[m]} \) of an irreducible representation \( \rho^{[m]} \) of \( S_r \) characterised by a frame \( [m] = [m_1, \ldots, m_k] \) with \( k \) rows of lengths \( m_1 \geq m_2 \geq \ldots \geq m_k \geq 0 \), \( [m] := \sum_{i=1}^{k} m_i = r \), and thus the multiplicity in the group algebra is given by

\[ n_{[m]} = r! \frac{\prod_{i<j} (l_i - l_j)}{l_1! \ldots l_k!} \quad \quad l_i = m_i + k - i \tag{39} \]

Note that the dimension is of course invariant if one adds certain rows of length zero, so the right hand side is in fact unchanged under \( [m] \mapsto [m'] = [m, 0] \).

Now we consider the decomposition of the \( r \)-fold product of the vector space \( \mathbb{C}^s \) into symmetry classes. \( (\mathbb{C}^s)^{\otimes r} \) carries the representation \( \rho \) defined above of the symmetric group \( S_r \) and is thus decomposable into irreducible components characterised by a frame. The irreducible components may be contained with multiplicity \( d_a \) (possibly 0).

\[ (\mathbb{C}^s)^{\otimes r} = d_1 \mathbb{C}^{n_1} \oplus \ldots \oplus d_f \mathbb{C}^{n_f} \text{ w.r.t. } S_r \tag{40} \]

Since the action \( D \) of \( Gl(s, \mathbb{C}) \) does commute with the action of the symmetric group Schur’s lemma states, that an element of \( Gl(s, \mathbb{C}) \) can only act as a multiple of unity between the
irreducible summands of \((10)\) which has to be zero between \(\mathbb{C}^{n_i}\) and \(\mathbb{C}^{n_j}\) if \(i \neq j\). This leads to another decomposition
\[
(\mathbb{C}^s)^{\circ r} = n_1 \mathbb{C}^{d_1} \oplus \ldots \oplus n_f \mathbb{C}^{d_f} \quad \text{w.r.t. } \text{Gl}(s, \mathbb{C}) ,
\] (41)
where here \(\mathbb{C}^{d_a}\) are irreducible representation spaces of an irreducible representation \(D^a\) of \(\text{Gl}(s, \mathbb{C})\), which arise with multiplicity \(n_a\). As a trivial consequence the isotypical components are identical for both actions:
\[
(\mathbb{C}^s)^{\circ r} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{n_1} \oplus \ldots \oplus \mathbb{C}^{d_f} \otimes \mathbb{C}^{n_f}
\] (42)

The units \(e_a\) of the simple components of the group ring \(\mathbb{C}[S_r]\) act as projectors into the isotypical components here as well. A unit \(e_a\) associated to a frame whose number of rows exceeds \(s\), acts as an annihilator, \(\rho(e_a) = 0\), since the degree of antisymmetrisation is too large, so \(n_a\) must be 0 in this case. So the representations \(D^a, a = 1 \ldots r\) of \(\text{Gl}(s, \mathbb{C})\) are as well characterised by frames. We quote again the dimension formula [13]:

**Proposition 5.2** For a frame with more than \(s\) rows of nonzero length the dimension of the associated representation \(D^{[m]}\) of \(\text{Gl}(s, \mathbb{C})\) is zero. So let \([m]\) be a frame with \(s\) rows of length \(m_1 \geq \ldots \geq m_s \geq 0\), where the lowest rows may be of length zero. Then the dimension \(d_{[m]}\) of the associated representation \(D^a\) of \(\text{Gl}(s, \mathbb{C})\) is
\[
d_{[m]} = \frac{\prod_{i<j}(l_i - l_j)}{(s-1)!((s-2)! \ldots 1)!} \quad l_i = m_i + s - i .
\]

(43)

Finally we recapitulate the considerations leading to Frobenius’ formula relating the characters of the respective representations \(\rho^a\) and \(D^a\) of \(S_r\) and \(\text{Gl}(s, \mathbb{C})\). Consider the composite actions of \(A \in \text{Gl}(s, \mathbb{C})\) and \(\sigma \in S_r\) on \((\mathbb{C}^s)^{\circ r}\) and assume \(\sigma\) belongs to the conjugacy class \(\alpha\). Using the isotypical decomposition (42) the mapping decomposes as
\[
D(A) \circ \rho(\sigma) = \rho(\sigma) \circ D(A) = D^1(A) \otimes \rho^1(\sigma) \oplus \ldots \oplus D^f(A) \otimes \rho^f(\sigma)
\]
and for its trace one finds on the one hand
\[
\text{tr}\left( D(A) \circ \rho(\sigma) \right) = \sum_{a=1}^f \text{tr}D^a(A) \cdot \text{tr}\rho^a(\sigma) = \sum_{a=1}^f \varphi(A)\chi^a_{\alpha} ,
\]
where \(\varphi^a\) is the character of the representation \(D^a\), and on the other hand according to definition 4.1
\[
\text{tr}\left( D(A) \circ \rho(\sigma) \right) = a_{i_1}^{i_{\sigma(1)}} \ldots A_{i_{\sigma(r)}}^{i_r} = a_1^{a_{1}} \ldots a_r^{\alpha_r}
\]
where \(a_k := \text{tr}A^k\) is the trace of the powers of \(A\). Finally using the orthogonality relation, which may be read from equations (36) yields Frobenius’

**Proposition 5.3** The relation between the characters of the irreducible representations \(D^{[m]}\) of \(\text{Gl}(s, \mathbb{C})\) and \(\rho^{[m]}\) of \(S_r\) associated to the same frame \([m] = [m_1, \ldots, m_s]\), \(|m| = r\) is given by
\[
\varphi^{[m]}(A) = \frac{1}{r!} \sum_{\alpha \in \text{Con}(S_r)} h_{\alpha} a_1^{\alpha_1} \ldots a_r^{\alpha_r} \chi^{[m]}_{\alpha},
\]
(44)

where \(a_i := \text{tr}A^i\) and \(h_{\alpha}\) is the number of element in the conjugacy class \(\alpha\). \(\Box\)
Summing up all informations the proof of the following corollary is trivial.

**Corollary 5.1** The polynomial \( t_{[m]} \) introduced by (47) associated to the frame \([m]\) is given by

\[
t_{[m]}(c) = [c]m_{1}[c-1]m_{2} \cdots [c-p+1]m_{p}
\]

**Proof:** It suffices to know the values of the polynomial for all natural values \( s \geq p \). Then

\[
t_{[m]}(s) = \frac{r!}{m_{[m]}} \varphi^{[m]}(1_s) = [s]m_{1}[s-1]m_{2} \cdots [1]m_{s}
\]

But the frames under consideration in our case have at most \( p \) rows, since otherwise \( e_{[m]} \) acts as annihilator on tensors in \((\mathbb{C}^p)^{\otimes r}\). Thus in our case \( m_{p+1} = \ldots m_s = 0 \) and the formula is proved.

This formula has a simple graphical interpretation: Consider an arbitrary frame:

For any box lying on the main diagonal \( t_{[m]} \) contains a factor \( c \), for any box on the first upper resp. lower diagonal it contains a linear factor \( c+1 \) resp \( c-1 \) and so forth. The relation (33) is now recognized as a direct consequence of equation (38) and the orthogonality relation.

The star product on the Grassmann manifolds \( G_{p,q}(\mathbb{C}) \) of corollary (44) then takes the form:

**Proposition 5.4** There exists a star product on \( G_{p,q}(\mathbb{C}) \) such that for \( f, g \in C^\infty(G_{p,q}(\mathbb{C}))[[\lambda]] \):

\[
\pi^*(f \ast g) = P(\pi^* f \ast \pi^* g) = \sum_{r=0}^{\infty} \frac{\mu^r}{r!} \sum_{\substack{|m| = r \\\text{and} \\\lim_{m \to 0} m_{p+1} = 0}} 1 t_{[m]}(c) \rho^r \left( \frac{\partial^r \pi^* f}{\partial z_{A_1} \cdots \partial z_{A_r}} \rho(e_{[m]}) \frac{\partial^r \pi^* g}{\partial z_{A_1} \cdots \partial z_{A_r}} \right)
\]

(46)

Here \( \rho(e_{[m]}) \) denotes the projector of \((\mathbb{C}^p)^{\otimes r}\) onto the symmetry class characterised by the frame \([m]\) and \( t_{[m]}(c) = [c]m_{1}[c-1]m_{2} \cdots [c-p+1]m_{p} \). In the sum over the frames only such frames yield a nonzero contribution whose number of rows is not greater than \( p \).}

Obviously this theorem contains Corollary (44) as a special case for \( p = 1 \). The product remains well defined in the limit \( \lambda \to \infty \), which is here even more remarkable, since frames containing one factor \( c + p - 1 \) do arise, but those containing the factor \( c - p = \mu \lambda^{-1} \) do not. Karabegov already stated in Theorem 3 (3) that for representative functions \( f, g \) the dependence of \( f \ast g \) on \( \lambda^{-1} \) is rational with no poles for \( \lambda \to 0 \). For arbitrary smooth functions formula (46) states that the dependence on \( \lambda \) of the product is given by an infinite sum of rational functions, obviously in accordance with Karabegov’s result. By a description of the star product via representative functions as it is done for the projective space in [14]...
one could verify Karabegov’s proposition directly for Grassmann manifolds. The expected isomorphy of the star algebras \((C^\infty(G_{p,q}(\mathbb{C})), \ast)\) and \((C^\infty(G_{q,p}(\mathbb{C})), \ast)\) can probably as well be established most easily in terms of representative functions. This would extend the duality of \(G_{p,q}(\mathbb{C})\) and \(G_{q,p}(\mathbb{C})\) to their star algebras.

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