SURVEY ON THE D-MODULE $f^\bullet$

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WITH AN APPENDIX BY ANTON LEyKIN

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In this survey we discuss various aspects of the singularity invariants with differential origin derived from the $D$-module generated by $f^\bullet$. We should like to point the reader to some other works: [193] for $V$-filtration, Bernstein–Sato polynomials, multiplier ideals; [49] for all these and Milnor fibers; [216] and [161] for homogeneity and free divisors; [208] on details of arrangements, specifically their Milnor fibers, although less focused on $D$-modules.

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1. Introduction

Notation 1.1. In this article, $X$ will denote a complex manifold. Unless indicated otherwise, $X$ will be $\mathbb{C}^n$.

Throughout, let $R = \mathbb{C}[x_1,\ldots,x_n]$ be the ring of polynomials in $n$ variables over the complex numbers. We denote by $D = R(\partial_1,\ldots,\partial_n)$ the Weyl algebra.

Key words and phrases. Bernstein–Sato polynomial, b-function, hyperplane, arrangement, zeta function, logarithmic comparison theorem, multiplier ideal, Milnor fiber, algorithmic, free.

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In particular, $\partial_i$ denotes the partial differentiation operator with respect to $x_i$. If $X$ is a general manifold, $\mathcal{O}_X$ (the sheaf of regular functions) and $\mathcal{D}_X$ (the sheaf of $\mathbb{C}$-linear differential operators on $\mathcal{O}_X$) take the places of $R$ and $D$.

If $X = \mathbb{C}^n$ we use Roman letters to denote rings and modules; in the general case we use calligraphic letters to denote corresponding sheaves.

By the ideal $J_f$ we mean the $\mathcal{O}_X$-ideal generated by the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$: this ideal varies with the choice of coordinate system in which we calculate. In contrast, the Jacobian ideal $\text{Jac}(f) = J_f + (f)$ is independent.

The ring $D$ (resp. the sheaf $\mathcal{D}_X$) is coherent, and both left- and right-Noetherian; it has only trivial two-sided ideals [26, Thm. 1.2.5]. Introductions to the theory of $D$-modules as we use them here can be found in [120, 24, 26, 25].

The ring $D$ admits the order filtration induced by the weight $x_i \to 0$, $\partial_i \to 1$. The order filtration (and other good filtrations) leads to graded objects $\text{gr}^n M$, see [199]. The graded objects obtained from ideals are ideals in the polynomial ring $\mathbb{C}[x, \xi]$, homogeneous in the symbols of the differentiation operators; their radicals are closed under the Poisson bracket, and thus the corresponding varieties are involutive [116, 121]. For a $D$-module $M$ and a component $C$ of the support of $\text{gr}^n_{(0,1)}(M)$, attach to the pair $(M, C)$ the multiplicity $\mu(M, C)$ of $\text{gr}^n_{(0,1)}(M)$ along $C$. The characteristic cycle of $M$ is $\text{charC}(M) = \sum_C \mu(M, C) \cdot C$, an element of the Chow ring on $T^* \mathbb{C}^n$. The module is holonomic if it is zero or if its characteristic variety is of dimension $n$, the minimal possible value.

Throughout, $f$ will be a regular function on $X$, with divisor $\text{Var}(f)$. We distinguish several homogeneity conditions on $f$:

- $f$ is locally (strongly) Euler-homogeneous if for all $p \in \text{Var}(f)$ there is a vector field $\theta_p$ defined near $p$ with $\theta_p \bullet (f) = f$ (and $\theta_p$ vanishes at $p$).
- $f$ is locally (weakly) quasi-homogeneous if near all $p \in \text{Var}(f)$ there is a local coordinate system $\{x_i\}$ and a positive (resp. non-negative) weight vector $a = \{a_1, \ldots, a_n\}$ with respect to which $f = \sum \nolimits_{i=1}^n a_i x_i \partial_i (f)$.
- We reserve homogeneous and quasi-homogeneous for the case when $X = \mathbb{C}^n$ and $f$ is globally homogeneous or quasi-homogeneous.

To any non-constant $f \in R$, one can attach several invariants that measure the singularity structure of the hypersurface $f = 0$. In this article, we are primarily interested in those derived from the (parametric) annihilator $\text{ann}_{D[s]}(f^*)$ of $f^*$:

**Definition 1.2.** Let $s$ be a new variable, and denote by $R_f[s] \cdot f^*$ the free module generated by $f^*$ over the localized ring $R_f[s] = R[f^{-1}, s]$. Via the chain rule

$$
\partial_i \bullet \left( \frac{g}{f^k} f^* \right) = \partial_i \bullet \left( \frac{g}{f} f^* \right) f^* + \frac{sg}{f^k+1} \cdot \frac{\partial f}{\partial x_i} f^*
$$

for each $g(x, s) \in R[s]$, $R_f[s] \cdot f^*$ acquires the structure of a left $D[s]$-module. Denote by

$$\text{ann}_{D[s]}(f^*) = \{ P \in D[s] \mid P \bullet f^* = 0 \}$$

the parametric annihilator, and by

$$\mathcal{M}(s) = D[s]/\text{ann}_{D[s]}(f^*)$$

the cyclic $D[s]$-module generated by $1 \cdot f^* \in R_f[s] \cdot f^*$.
Bernstein’s functional equation [23] asserts the existence of a differential operator
$P(x, \partial, s)$ and a nonzero polynomial $b_{f,p}(s) \in \mathbb{C}[s]$ such that

\begin{equation}
(1.2) \quad P(x, \partial, s) \cdot f^{s+1} = b_{f,p}(s) \cdot f^s,
\end{equation}

i.e. the existence of the element $P \cdot f - b_{f,p}(s) \in \text{ann}_{\mathbb{C}[s]}(f^s)$. Bernstein’s result
implies that $D[s] \cdot f^s$ is $D$-coherent (while $R_f[s]f^s$ is not).

**Definition 1.3.** The monic generator of the ideal in $\mathbb{C}[s]$ generated by all $b_{f,p}(s)$
appearing in an equation (1.2) is the *Bernstein–Sato polynomial* $b_f(s)$. Denote $ho_f \subseteq \mathbb{C}$
the set of roots of $b_f(s)$.

Note that the operator $P$ in the functional equation is only determined up to
$\text{ann}_{\mathbb{C}[s]}(f^s)$. See [25] for an elementary proof of the existence of $b_f(s)$. Alternative
(and more general) proofs are given in [120]; see also [24, 151, 169].

The $\mathbb{C}[s]$-module $\mathcal{M}_f(s)/\mathcal{M}_f(s+1)$ is precisely annihilated by $b_f(s)$. It is an
interesting problem to determine for any $q(s) \in \mathbb{C}[s]$ the ideals $a_{f,q(s)} = \{ g \in R \mid q(s)g^s \in D[s] \cdot f^{s+1} \}$ from [230]. By [146], $a_{f,s+1} = R \cap (\text{ann}_{\mathbb{C}[s]}(f^s) + D[s] \cdot (f, J_f))$.

**Question 1.4.** Is $a_{f,s+1} = J_f + (f)$?

A positive answer would throw light on connections between $b_f(s)$ and cohomology
of Milnor fibers.

**Remark 1.5.** At the 1954 International Congress of Mathematics in Amsterdam,
I.M. Gel’fand asked the following question. Given a real analytic function $f: \mathbb{R}^n \to \mathbb{R}$, the assignment $(s \in \mathbb{C})$

\[ f(x)^s = \begin{cases} 
 f(x)^s & \text{if } f(x) > 0, \\
 0 & \text{if } f(x) \leq 0
\end{cases} \]

is continuous in $x$ and analytic in $s$ where the real part of $s$ is positive. Can one
analytically continue $f(x)^s$? Sato introduced $b_f(s)$ in order to answer Gel’fand’s
question; Bernstein [23] established their existence in general.

**Remark 1.6.** Let $m \in M$ be a nonzero section of a holonomic $D$-module. Generaliz-
ing the case $1 \in R$ there is a functional equation

\[ P(x, \partial, s) \cdot (mf^{s+1}) = b_{f,p;m}(s) \cdot mf^s \]

with $b_{f,p;m}(s) \in \mathbb{C}[s]$ nonzero. The monic generator of the ideal \{ $b_{f,p;m}(s)$ \} is the
$b$-function $b_{f,m}(s)$, [117].

2. **Parameters and Numbers**

For any complex number $\gamma$, the expression $f^\gamma$ represents, locally outside $\text{Var}(f)$,
a multi-valued analytic function. Via the chain rule as in (1.1), the cyclic $R_f$-module
$R_f \cdot f^\gamma$ becomes a left $D$-module, and we set

\[ \mathcal{M}_f(\gamma) = D \cdot f^\gamma \cong D/\text{ann}_D(f^\gamma). \]

There are natural $D[s]$-linear maps

\[ \text{ev}_f(\gamma): \mathcal{M}_f(s) \to \mathcal{M}_f(\gamma), \quad P(x, \partial, s) \cdot f^s \mapsto P(x, \partial, \gamma) \cdot f^\gamma, \]

and $D$-linear inclusions

\[ \text{inc}_f(s): \mathcal{M}_f(s+1) \to \mathcal{M}_f(s), \quad P(x, \partial, s) \cdot f^{s+1} \mapsto P(x, \partial, s) \cdot f \cdot f^s \]
with cokernel \( \mathcal{M}_f(s) = \mathcal{M}_f(s)/\mathcal{M}_f(s+1) \cong D[s]/(\text{ann}_D(s^*f + D[s]f)) \), and

\[
\text{inc}_f(\gamma): \mathcal{M}_f(\gamma + 1) \to \mathcal{M}_f(\gamma), \quad P(x, \partial) \cdot f^{\lambda+1} \mapsto P(x, \partial) \cdot f \cdot f^{\lambda}
\]

with cokernel \( \mathcal{M}_f(\gamma) = \mathcal{M}_f(\gamma)/\mathcal{M}_f(\gamma + 1) \cong D/(\text{ann}_D(s^*f) + D \cdot f) \).

The kernel of the morphism \( \text{ev}_f(\gamma) \) contains the (two-sided) ideal \( D[s](s-\gamma) \); the containment can be proper, for example if \( \gamma = 0 \). If \( \{\gamma - 1, \gamma - 2, \ldots\} \) is disjoint from the root set \( \rho_f \) then \( \ker \text{ev}_f(\gamma) = D[s] \cdot (s-\gamma) \), [117]. If \( \gamma \notin \rho_f \) then \( \text{inc}_f(\gamma) \) is an isomorphism because of the functional equation; if \( \gamma = -1 \), or if \( bf(\gamma) = 0 \) while \( \rho_f \) does not meet \( \{\gamma - 1, \gamma - 2, \ldots\} \) then \( \text{inc}_f(\gamma) \) is not surjective [230].

**Question 2.1.** Does \( \text{inc}_f(\gamma) \) fail to be an isomorphism for all \( \gamma \in \rho_f \)?

In contrast, the induced maps \( \mathcal{M}_f(s)/(s-\gamma - 1) \to \mathcal{M}_f(s)/(s-\gamma) \) are isomorphisms exactly when \( \gamma \notin \rho_f \), [26, 6.3.15]. The morphism \( \text{inc}_f(s) \) is never surjective as \( s + 1 \) divides \( bf(s) \). One sets

\[
b_f(s) = \frac{bf(s)}{s+1}.
\]

By [217, 4.2], the following are equivalent for a section \( m \neq 0 \) of a holonomic module:

- the smallest integral root of \( b_{f,m}(s) \) is at least \( -\ell \);
- \( (D \cdot m) \otimes_R R[f^{-1}] \) is generated by \( m/f^\ell = m \otimes 1/f^\ell \);
- \( (D \cdot m) \otimes_R R[f^{-1}]/D \cdot (m \otimes 1) \) is generated by \( m/f^\ell \);
- \( D[s] \cdot mf^* \to (D \cdot m) \otimes_R R[f^{-1}], P(s) \cdot (mf^*) \mapsto P(-\ell) \cdot (mf^*) \) is an epimorphism with kernel \( D[s] \cdot (s+\ell)mf^* \).

**Definition 2.2.** We say that \( f \) satisfies condition

- \( (A_1) \) (resp. \( (A_s) \)) if \( \text{ann}_D(1/f) \) (resp. \( \text{ann}_D(f^*) \)) is generated by operators of order one;
- \( (B_1) \) if \( R_f \) is generated by \( 1/f \) over \( D \).

Condition \((A_1)\) implies \((B_1)\) in any case [214]. Local Euler-homogeneity, \((A_s)\) and \((B_1)\) combined imply \((A_1)\) [216], and for Koszul free divisors (see Definition 4.7 below) this implication can be reversed [214].

Condition \((A_1)\) does not imply \((A_s)\): \( f = xy(x+y)(x+yz) \) is free (see Definition 4.1), and locally Euler-homogeneous and satisfies \((A_1)\) and \((B_1)\) [60, 61, 59, 67, 214], but \( \text{ann}_D(s^*f^*) \) and \( \text{ann}_D(f^*) \) require a second order generator.

Condition \((A_1)\) implies local Euler-homogeneity if \( f \) has isolated singularities [213], or if it is Koszul-free or of the form \( z^n - g(x,y) \) for reduced \( g \) [214]. In [73] it is shown that for certain locally weakly quasi-homogeneous free divisors \( \text{Var}(f) \), \((A_1)\) holds for high powers of \( f \), and even for \( f \) itself by [161, Rem. 1.7.4].

For an isolated singularity, \( f \) has \((A_1)\) if and only if it has \((B_1)\) and is quasi-homogeneous [213]. For example, a reduced plane curve (has automatically \((B_2)\)) and has \((A_1)\) if and only if it is quasi-homogeneous. See [201] for further results.

Condition \((B_1)\) is equivalent to \( \text{inc}_f(-2), \text{inc}_f(-3), \ldots \) all being isomorphisms, and also to \(-1\) being the only integral root of \( b_f(s) \), [117]. Locally quasi-homogeneous free divisors satisfy condition \((B_1)\) at any point, [66].

3. **V-filtration and Bernstein–Sato polynomials**

3.1. **V-filtration.** The articles [191, 145, 47, 49] are recommended for material on \( V \)-filtrations.
3.1.1. Definition and basic properties. Let $Y$ be a smooth complex manifold (or variety), and let $X$ be a closed submanifold (or -variety) of $Y$ defined by the ideal sheaf $\mathcal{I}$. The $V$-filtration on $\mathcal{D}_Y$ along $X$ is, for $k \in \mathbb{Z}$, given by

$$V^k(\mathcal{D}_Y) = \{ P \in \mathcal{D}_Y | P \cdot \mathcal{I}^k \subseteq \mathcal{I}^{k+k'} \quad \forall k' \in \mathbb{Z} \}$$

with the understanding that $\mathcal{I}^k = \mathcal{O}_Y$ for $k' \leq 0$. The associated graded sheaf of rings $\text{gr}_V(\mathcal{D}_Y)$ is isomorphic to the sheaf of rings of differential operators on the normal bundle $T_X(Y)$, algebraic in the fiber of the bundle.

Suppose that $Y = \mathbb{C}^n \times \mathbb{C}$ with coordinate function $t$ on $\mathbb{C}$, and let $X$ be the hyperplane $t = 0$. Then $V^k(\mathcal{D}_Y)$ is spanned by $\{ x^a \partial^p t^q \partial^k | a - b \geq k \}$. Given a coherent holonomic $\mathcal{D}_Y$-module $M$ with regular singularities in the sense of [122], Kashiwara and Malgrange [147, 114] define an exhaustive decreasing rationally indexed filtration on $M$ that is compatible with the $V$-filtration on $D_Y$ and has the following properties:

1. each $V^\alpha(M)$ is coherent over $V^0(D_Y)$ and the set of $\alpha$ with nonzero $\text{gr}_V^\alpha(M) = V^\alpha(M)/V^{\alpha+1}(M)$ has no accumulation point;
2. for $\alpha \gg 0$, $V^\alpha(D_Y) V^\alpha(M) = V^{\alpha+1}(M)$;
3. $t \partial_t - \alpha$ acts nilpotently on $\text{gr}_V^\alpha(M)$.

The $V$-filtration is unique and can be defined in somewhat greater generality [47]. Of special interest is the following case considered in [147, 114].

Notation 3.1. Denote $R_{x,t}$ the polynomial ring $R[t], t$ a new indeterminate, and let $D_{x,t}$ be the corresponding Weyl algebra. Fix $f \in R$ and consider the regular $D_{x,t}$-module

$$\mathcal{B}_f = H^1_{-t}(R[t]),$$

the unique local cohomology module of $R[t]$ supported in $f-t$. Then $\mathcal{B}_f$ is naturally isomorphic as a $D_{x,t}$-module to the direct image (in the $D$-category) $i_+(R)$ of $R$ under the graph embedding

$$i : X \to X \times \mathbb{C}, \quad x \mapsto (x, f(x)).$$

Moreover, extending (1.1) via

$$t \cdot (g(x,s)f^{s-k}) = g(x,s+1)f^{s+1-k}; \quad \partial_t \cdot (g(x,s)f^{s-k}) = -sg(x,s-1)f^{s-1-k},$$

the module $R_f[s] \otimes f^s$ becomes a $D_{x,t}$-module extending the $D[s]$-action where $-\partial_t t$ acts as $s$.

The existence of the $V$-filtration on $\mathcal{B}_f = i_+(R)$ is equivalent to the existence of generalized $b$-functions $b_{f,\eta}(s)$ in the sense of [117], see [118, 147]. In fact, one can recover one from the other:

$$V^\alpha(\mathcal{B}_f) = \{ \eta \in \mathcal{B}_f | [b_{f,\eta}(-c) = 0] \Rightarrow [\alpha \leq c] \}$$

and the multiplicity of $b_{f,\eta}(s)$ at $\alpha$ is the degree of the minimal polynomial of $s - \alpha$ on $\text{gr}_V(D[s]f^s/D[s]f^{s+1})$, [182]. For more on this “microlocal approach” see [191].

3.2. The log-canonical threshold. By [125], see also [135, 235], the absolute value of the largest root of $b_f(s)$ is the log-canonical threshold $\text{lct}(f)$ given by the supremum of all numbers $s$ such that the local integrals

$$\int_{U \ni p} \frac{|dx|}{|f|^{2s}}$$
converge for all \( p \in X \) and all small open \( U \) around \( p \). Smaller \( \text{let} \) corresponds to worse singularities; the best one can hope for is \( \text{let}(f) = 1 \) as one sees by looking at a smooth point. The notion goes back to Arnol’d, who called it (essentially) the complex singular index \([10]\).

The point of \textit{multiplier ideals} is to force the finiteness of the integral by allowing moderating functions in the integral:

\[
\mathcal{I}(f, \lambda)_P = \{ g \in \mathcal{O}_X \mid \frac{g}{f^\lambda} \text{ is } L^2\text{-integrable near } p \in \text{Var}(f) \}
\]

for \( \lambda \in \mathbb{R} \). By \([90]\), there is a finite collection of \textit{jumping numbers} for \( f \) of rational numbers \( 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_\ell = 1 \) such that \( \mathcal{I}(f, \alpha) \) is constant on \([\alpha_i, \alpha_{i+1})\) but \( \mathcal{I}(f, \alpha_i) \neq \mathcal{I}(f, \alpha_{i+1}) \). The log-canonical threshold appears as \( \alpha_1 \). These ideas had appeared previously in \([137, 139]\).

Generalizing Kollár’s approach, each \( \alpha_i \) is a root of \( b_f(s) \), \([90]\). In \([193, \text{Thm. 4.4}]\) a partial converse is shown for locally Euler-homogeneous divisors. Extending the idea of jumping numbers to the range \( \alpha > 1 \) one sees that \( \alpha \) is a jumping number if and only if \( \alpha + 1 \) is a jumping number, but the connection to the Bernstein–Sato polynomial is lost in general. For example, if \( f(x, y) = x^2 + y^3 \) then jumping numbers are \( \{5/6, 1\} + \mathbb{N} \) while \( b_f(s) = (s + 5/6)(s + 1)(s + 7/6) \).

### 3.3. Bernstein–Sato polynomial

The roots of \( b_f(s) \) relate to an astounding number of other invariants, see for example \([125]\) for a survey. However, besides the functional equation there is no known way to describe \( \rho_f \).

#### 3.3.1. Fundamental results

Let \( p \in \mathbb{C}^n \) be a closed point, cut out by the maximal ideal \( \mathfrak{m} \subseteq \mathcal{O} \). Extending \( \mathcal{O} \) to the localization \( \mathcal{O}_{\mathfrak{m}} \) (or even the ring of holomorphic functions at \( p \)) one arrives at potentially larger sets of polynomials \( b_{f, p}(s) \) that satisfy a functional equation \((1.2)\) with \( P(x, \partial, s) \) now in the correspondingly larger ring of differential operators. The \textit{local} (resp. \textit{local analytic}) Bernstein–Sato polynomial \( b_{f, p}(s) \) (resp. \( b_{f, p, \rho}(s) \)) is the generator of the resulting ideal generated by the \( b_{f, p}(s) \) in \( \mathbb{C}[s] \). We denote by \( \rho_{f, p} \) (resp. \( \hat{\rho}_{f, p} \)) the root set of \( b_{f, p, \rho}(s) \) (resp. \( b_{f, p, \rho}(s)/(s + 1) \)). From the definitions and \([143, 38, 36]\)

\[
(3.1) \quad b_{f, p, \rho}(s)|b_{f, p}(s)|b_{f}(s) = \text{lcm}_{p \in \text{Var}(f)} b_{f, p}(s) = \text{lcm}_{p \in \text{Var}(f)} b_{f, p, \rho}(s),
\]

and the function \( \mathbb{C}^n \ni p \mapsto \text{Var}(b_f(s)) \), counting with multiplicity, is upper semicontinuous in the sense that for \( p' \) sufficiently near \( p \) one has \( b_{f, p'}(s)|b_{f, p}(s) \). The underlying reason is the coherence of \( D \).

The Bernstein–Sato polynomial \( b_f(s) \) factors over \( \mathbb{Q} \) into linear factors, \( \rho_f \subseteq \mathbb{Q} \), and all roots are negative \([146, 117]\). The proof uses resolution of singularities over \( \mathbb{C} \) in order to reduce to simple normal crossing divisors, where rationality and negativity of the roots is evident. For this Kashiwara proves a comparison theorem \([117, \text{Thm. 5.1}]\) that establishes \( b_f(s) \) as a divisor of a shifted product of the least common multiple of the local Bernstein–Sato polynomials of the pullback of \( f \) under the resolution map. There is a refinement by Lichtin \([135]\) for plane curves. The roots of \( b_f(s) \), besides being negative, are always greater than \(-n \), \( n \) being the minimum number of variables required to express \( f \) locally analytically \([221, 191]\).

#### 3.3.2. Constructible sheaves from \( f^\times \)

Let \( V = V(n, d) \) be the vector space of all complex polynomials in \( x_1, \ldots, x_n \) of degree at most \( d \). Consider the function \( \beta: V \ni f \mapsto b_f(s) \). By \([143, 36]\), there is an algebraic stratification of \( V \) such that on each
stratum the function $\beta$ is constant. For varying $n, d$ these stratifications can be made to be compatible.

3.3.3. Special cases. If $p$ is a smooth point of $\text{Var}(f)$ then $f$ can be used as an analytic coordinate near $p$, hence $b_{f,p,n}(s) = s + 1$, and so $b_f(s) = s + 1$ for all smooth hypersurfaces. By Proposition 2.6 in [35], an extension of [37], the equation $b_f(s) = s + 1$ implies smoothness of $\text{Var}(f)$. Explicit formulæ for the Bernstein–Sato polynomial are rare; here are some classes of examples.

- $f = \prod x_i^{a_i}$: $P = \prod \partial_i^{a_i}$ up to a scalar, $b_f(s) = \prod \prod_{j=1}^{n}(s + j/a_j)$.
- $f$ (quasi-)homogeneous with isolated singularity at zero: $b_f(s) = \text{lcm}(s + \deg g, \deg f)$, where $g$ runs through a (quasi-)homogeneous standard basis of $J_f$ by work of Kashiwara, Sato, Miwa, Malgrange, Kochman [146, 235, 215, 124]. Note that the Jacobian ring of such a singularity is an Artinian Gorenstein ring, whose duality operator implies symmetry of $\rho_f$.
- $f = \det(x_i,j)^{a_i}$: $P = \det(\partial_i,j)^{a_i}$, $b_f(s) = (s + 1) \cdots (s + n)$. This is attributed to Cayley, but see the comments in [63].
- For some hyperplane arrangements, $b_f(s)$ is known, see [230, 56].
- There is a huge list of examples worked out in [235].

If $V$ is a complex vector space, $G$ a reductive group acting linearly on $V$ with open orbit $U$ such that $V \setminus U$ is a divisor $\text{Var}(f)$, Sato’s theory of prehomogeneous vectors spaces [198, 156, 197, 234] yields a factorization for $b_f(s)$. For reductive linear free divisors, [97, 203] discuss symmetry properties of Bernstein–Sato polynomials. In [162] this theme is taken up again, investigating specifically symmetry properties of $\rho_f$ when $D[s] \cdot f^s$ has a Spencer logarithmic resolution (see [66] for definitions).

This covers locally quasi-homogeneous free divisors, and more generally free divisors whose Jacobian is of linear type. The motivation is the fact that roots of $b_f(s)$ seem to come in strands, and whenever roots can be understood the strands appear to be linked to Hodge-theory.

There are several results on $\rho_f$ for other divisors of special shape. Trivially, if $f(x) = g(x_1, \ldots, x_k) \cdot h(x_{k+1}, \ldots, x_n)$ then $b_f(s) \mid b_g(s) \cdot b_h(s)$: the question of equality appears to be open. In contrast, $b_f(s)$ cannot be assembled from the Bernstein–Sato polynomials of the factors of $f$ in general, even if the factors are hyperplanes and one has some control on the intersection behavior, see Section 8 below. If $f(x) = g(x_1, \ldots, x_k) + h(x_{k+1}, \ldots, x_n)$ and at least one is locally Euler-homogeneous then there are Thom–Sebastiani type formulæ [191]. In particular, diagonal hypersurfaces are completely understood.

3.3.4. Relation to intersection homology module. Suppose $Y = \text{Var}(f_1, \ldots, f_k) \subseteq X$ is a complete intersection and denote by $\mathcal{H}^k_Y(\mathcal{O}_X)$ the unique (algebraic) local cohomology module of $\mathcal{O}_X$ along $Y$. Brylinski–Kashiwara [42, 43] defined $\mathcal{L}(X,Y) \subseteq \mathcal{H}^k_Y(\mathcal{O}_X)$, the intersection homology $\mathcal{D}_X$-module of $Y$, the smallest $\mathcal{D}_X$-module equal to $\mathcal{H}^k_Y(\mathcal{O}_X)$ in the generic point(s). See also [19]. The module $\mathcal{L}(X,Y)$ contains the fundamental class of $Y$ in $X$ [20].

Question 3.2. When is $\mathcal{L}(X,Y) = \mathcal{H}^k_Y(\mathcal{O}_X)$?

Equality is equivalent to $\mathcal{H}^k_Y(\mathcal{O}_X)$ being generated by the cosets of $\Delta/\prod_{i=1}^k f_i$ over $\mathcal{D}_X$ where $\Delta$ is the ideal generated by the $k$-minors of the Jacobian matrix of $f_1, \ldots, f_k$. A necessary condition is that $1/\prod_{i=1}^k f_i$ generates $\mathcal{H}^k_Y(\mathcal{O}_X)$, but this
is not sufficient: consider $xy(x + y)(x + yz)$, where $\rho_f = \{-1/2, 3/4, 1, 1, 5/4\}$. Indeed, by [217], equality can be characterized in terms of functional equations, as the following are equivalent at $p \in X$:

- (1) $\mathcal{L}(X, Y) = \mathcal{H}^k_{X}(\mathcal{O}_X)$ in the stalk;
- (2) $\rho_{f,p} \cap \mathbb{Z} = \emptyset$;
- (3) $1$ is not an eigenvalue of the monodromy operator on the reduced cohomology of the Milnor fibers near $p$.

If $1/\prod_{i=1}^{k} f_i$ generates $R[1/ \prod f_i]$ and $1/\prod_{i=1}^{k} f_i \in \mathcal{L}(X, Y)$ then $\tilde{b}_f(-1) \neq 0$, [217]. It seems unknown whether (irrespective of $1/\prod_{i=1}^{k} f_i$ generating $R[1/ \prod f_i]$) the condition $\tilde{b}_f(-1) \neq 0$ is equivalent to $1/\prod_{i=1}^{k} f_i$ being in $\mathcal{L}(X, Y)$. See also [149] for a topological viewpoint (by the Riemann–Hilbert correspondence of Kashiwara and Mebkhout [119, 150], $\mathcal{L}(X, Y)$ corresponds to the intersection cohomology complex of $Y$ on $X$ [42] and $\mathcal{H}^k_{X}(\mathcal{O}_X)$ to $\mathcal{C}_Y[n - k]$, [102, 117, 152]; equality then says: the link is a rational homology sphere). In [21], Barlet characterizes property (3) above in terms of currents for complexified real $f$. Equivalence of (1) and (3) for isolated singularities can be derived from [155, 39]; the general case can be shown using [189, 4.5.8] and the formalism of weights. For the case $k = 1$, (1) requires irreducibility; in the general case, there is a criterion in terms of $b$-functions [217, 1.6, 1.10].

4. LCT and Logarithmic Ideal

4.1. Logarithmic forms. Let $X = \mathbb{C}^n$ be the analytic manifold, $f$ a holomorphic function on $X$, and $Y = \text{Var}(f)$ a divisor in $X$ with $j: U = X \setminus Y \hookrightarrow X$ the embedding. Let $\Omega_X^\ast (\ast Y)$ denote the complex of differential forms on $X$ that are (at worst) meromorphic along $Y$. By [102], $\Omega_X^\ast (\ast Y) \to \mathbb{R}^j_* \mathbb{C}_U$ is a quasi-isomorphism.

A form $\omega$ is logarithmic along $Y$ if $f \omega$ and $f d \omega$ are holomorphic; these $\omega$ form the logarithmic de Rham complex $\Omega_X^\ast (\log Y)$ on $X$ along $Y$. The complex $\Omega_X^\ast (\log Y)$ was first used with great effect on normal crossing divisors by Deligne [83] in order to establish mixed Hodge structures, and later by Esnault and Viehweg in order to prove vanishing theorems [91]. A major reason for the success of normal crossings is that in that case $\Omega_X^i (\log Y)$ is a locally free module over $\mathcal{O}_X$. The logarithmic de Rham complex was introduced in [187] for general divisors.

4.2. Free divisors.

**Definition 4.1.** A divisor $\text{Var}(f)$ is free if (locally) $\Omega_X^1 (\log f)$ is a free $\mathcal{O}_X$-module.

For a non-smooth locally Euler-homogeneous divisor, freeness is equivalent to the Jacobian ring $\mathcal{O}_X/J_f$ being a Cohen–Macaulay $\mathcal{O}_X$-module of codimension 2; in general, freeness is equivalent to the Tjurina algebra $R/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ being of projective dimension 2 or less over $R$. See [187, 2] for relations to determinantal equations. Free divisors have rather big singular locus, and are in some ways at the opposite end from isolated singularities in the singularity zoo. If $\Omega_X^1 (\log f)$ is (locally) free, then $\Omega_X^i (\log f) \cong \bigwedge^i \Omega_X^1 (\log f)$ and also (locally) free, [187]. A weakening is

**Definition 4.2.** A divisor $\text{Var}(f)$ is tame if, for all $i \in \mathbb{N}$, (locally) $\Omega_X^i (\log f)$ has projective dimension at most $i$ as a $\mathcal{O}_X$-module.
ON THE D-MODULE $f^*$

Plane curves are trivially free; surfaces in 3-space are trivially tame. Normal crossing divisors are easily shown to be free. Discriminants of (semi)versal deformations of an isolated complete intersection singularity (and some others) are free, [2, 3, 141, 188, 77, 44]. Unitary reflection arrangements are free [212].

**Definition 4.3.** The logarithmic derivations $\text{Der}_X(-\log f)$ along $Y = \text{Var}(f)$ are the $\mathbb{C}$-linear derivations $\theta \in \text{Der}(\mathcal{O}_X; \mathbb{C})$ that satisfy $\theta \cdot f \in (f)$.

A derivation $\theta$ is logarithmic along $Y$ if and only if it is so along each component of the reduced divisor to $Y$ [187]. The modules $\text{Der}_X(-\log f)$ and $\Omega^i_X(\log f)$ are reflexive and mutually dual over $R$. Moreover, $\Omega^i_X(\log f)$ and $\Omega^{n-i}_X(\log f)$ are dual.

**4.3. LCT.**

**Definition 4.4.** If

$$\Omega^i_X(\log Y) \to \Omega^i_X(*Y)$$

is a quasi-isomorphism, we say that **LCT holds for $Y$**.

We recommend [161].

**Remark 4.5.** (1) This “Logarithmic Comparison Theorem”, a property of a divisor, is very hard to check explicitly. No general algorithms are known, even in $\mathbb{C}^3$ (but see [74] for $n = 2$).

(2) LCT fails for rather simple divisors such as $f = x_1x_2 + x_3x_4$.

(3) If $Y$ is a reduced normal crossing divisor, Deligne proved (4.1) to be a filtered (by pole filtration) quasi-isomorphism [82]; this provided a crucial step in the development of the theory of mixed Hodge structures [83].

(4) Limiting the order of poles in forms needed to capture all cohomology of $U$ started with the seminal article [99] and continues, see for example [81, 87, 113].

(5) The free case was studied for example in [72]. But even in this case, LCT is not understood.

(6) If $f$ is quasi-homogeneous with an isolated singularity at the origin, then LCT for $f$ is equivalent to a topological condition (the link of $f$ at the origin being a rational homology sphere), as well as an arithmetic one on the Milnor algebra of $f$, [104]. In [202], using the Gauß–Manin connection, this is extended to a list of conditions on an isolated hypersurface singularity, each one of which forces the implication $[D \text{ has LCT}] \Rightarrow [D \text{ is quasi-homogeneous}]$.

(7) For a version regarding more general connections, see [58].

A plane curve satisfies LCT if and only it is locally quasi-homogeneous, [61]. By [72], free locally quasi-homogeneous divisors satisfy LCT in any dimension. By [95], in dimension three, free divisors with LCT must be locally Euler-homogeneous. Conjecturally, LCT implies local Euler-homogeneity [61]. The converse is false, see for example [69]. The classical example of rotating lines with varying cross-ratio $f = xy(x + y)(x + yz)$ is free, satisfies LCT and is locally Euler-homogeneous, but only weakly quasi-homogeneous, [61]. In [73], the effect of the Spencer property on LCT is discussed in the presence of homogeneity conditions. For locally quasi-homogeneous divisors (or if the non-free locus is zero-dimensional), LCT implies $(B_1)$, [66, 216]. In particular, LCT implies $(B_1)$ for divisors with isolated singularities. In [96] quasi-homogeneity of isolated singularities is characterized in terms of a map of local cohomology modules of logarithmic differentials.
A free divisor is linear free if the (free) module \( \text{Der}_X (\log f) \) has a basis of linear vector fields. In [93], linear free divisors in dimension at most 4 are classified, and for these divisors LCT holds at least on global sections. In the process, it is shown that LCT is implied if the Lie algebra of linear logarithmic vector fields is reductive. The example of \( n \times n \) invertible upper triangular matrices acting on symmetric matrices [93, Ex. 5.1] shows that LCT may hold without the reductivity assumption. Linear free divisors appear naturally, for example in quiver representations and in the theory of prehomogeneous vector spaces and castling transformations [45, 196, 94]. Linear freeness is related to unfoldings and Frobenius structures [79].

Denote by \( \text{Der}_{X,0} (\log f) \) the derivations \( \theta \) with \( \theta \cdot f = 0 \). In the presence of a global Euler-homogeneity \( E \) on \( Y \) there is a splitting \( \text{Der}_X (\log f) \cong R \cdot E \oplus \text{Der}_{X,0} (\log f) \). Reading derivations as operators of order one, \( \text{Der}_{X,0} (\log f) \subseteq \text{ann}_D (f^*) \). We write \( S \) for \( \text{gr}_{(0,1)} (D) \); if \( y_t \) is the symbol of \( \partial_i \) then we have \( S = R[y] \).

**Definition 4.6.** The inclusion \( \text{Der}_{X,0} (\log f) \hookrightarrow \text{ann}_D (f^*) \), via the order filtration, defines a subideal of \( \text{gr}_{(0,1)} (\text{ann}_D (f^*)) \subseteq \text{gr}_{(0,1)} (D) = S \) called the logarithmic ideal \( L_f \) of \( \text{Var}(f) \).

Note that the symbols of \( \text{Der}_X (\log f) \) are in the ideal \( R \cdot y \) of height \( n \).

**Definition 4.7.** If \( \text{Der}_X (\log f) \) has a generating set (as an \( R \)-module) whose symbols form a regular sequence on \( S \), then \( Y \) is called Koszul free.

As \( \text{Der}_X (\log f) \) has rank \( n \), a Koszul free divisor is indeed free. Divisors in the plane [187] and locally quasi-homogeneous free divisors [59, 57] are Koszul free. In the case of normal crossings, this has been used to make resolutions for \( D[s] \cdot f^* \) and \( D[s]/D[s] (\text{ann}_D (f^*) , f) \), [101]. A way to distill invariants from resolutions of \( D[s] \cdot f^* \) is given in [9]. The logarithmic module \( \tilde{M}^{\log f} = D/D \cdot \text{Der}_X (\log f) \) has in the Spencer case (see [66, 62]) a natural free resolution of Koszul type.

For Koszul-free divisors, the ideal \( D \cdot \text{Der}_X (\log f) \) is holonomic [60]. By [93, Thm. 7.4], in the presence of freeness, the Koszul property is equivalent to the local finiteness of Saito’s logarithmic stratification. This yields an algorithmic way to certify (some) free divisors as not locally quasi-homogeneous, since free locally quasi-homogeneous divisors are Koszul free. Based on similar ideas, one may devise a test for strong local Euler-homogeneity [93, Lem. 7.5]. See [60] and [216, §2] for relations of Koszul freeness to perversity of the logarithmic de Rham complex.

Castro-Jiménez and Ucha established conditions for \( Y = \text{Var}(f) \) to have LCT in terms of \( D \)-modules [67, 66, 68] for certain free \( f \). For example, LCT is equivalent to \((A_1)\) for Spencer free divisors. Calderón-Moreno and Narváez-Macarro [62] proved that free divisors have LCT if and only if the natural morphism \( \mathcal{D}_X \otimes_{\mathcal{O}_Y (\mathcal{D}_X)} \mathcal{O}_X (Y) \to \mathcal{O}_X (sY) \) is a quasi-isomorphism, \( \mathcal{O}_X (Y) \) being the meromorphic functions with simple pole along \( f \). For Koszul free \( Y \), one has at least \( \mathcal{D}_X \otimes_{\mathcal{V}_0 (\mathcal{D}_X)} \mathcal{O}_X (Y) \cong \mathcal{D}_X \otimes_{\mathcal{V}_0 (\mathcal{D}_X)} \mathcal{O}_X (Y) \). A similar condition ensures that the logarithmic de Rham complex is perverse [60, 62]. The two results are related by duality between logarithmic connections on \( \mathcal{D}_X \) and the \( \mathcal{V} \)-filtration [66, 62, 75].

It is unknown how LCT is related to \((A_1)\) in general, but for quasi-homogeneous polynomials with isolated singularities the two conditions are equivalent, [216].

### 4.4. Logarithmic linearity.

**Definition 4.8.** We say that \( f \in R \) satisfies \((L_s)\) if the characteristic ideal of \( \text{ann}_D (f^*) \) is generated by symbols of derivations.
Condition \( (L_s) \) holds for isolated singularities [235], locally quasi-homogeneous free divisors [59], and locally strongly Euler-homogeneous holonomic tame divisors [231]. Also, \( (L_s) \) plus \( (B_1) \) yields \( (A_1) \) for locally Euler-homogeneous \( f \) by [117], see [216].

The logarithmic ideal supplies an interesting link between \( \Omega_X^* (\log f) \) and \( \text{ann}_D(f^*) \) via approximation complexes: if \( f \) is holonomic, strongly locally Euler-homogeneous and also tame then the complex \((\Omega_X^* (\log f)[y], y dx)\) is a resolution of the logarithmic ideal \( L_f \), and \( S/L_f \) is a Cohen–Macaulay domain of dimension \( n + 1 \); if \( f \) is in fact free, \( S/L_f \) is a complete intersection [161, 231].

**Question 4.9.** For locally Euler-homogeneous divisors, is \( \text{ann}_D(f^*) \) related to the cohomology of \((\Omega_X^* (\log f)[y], y dx)\)?

5. **Characteristic variety**

We continue to assume that \( X = \mathbb{C}^n \). For \( f \in R \) let \( U_f \) be the open set defined by \( df \neq 0 = f \). Because of the functional equation, \( \mathcal{M}_f(s) \) is coherent over \( D \) [23, 117], and the restriction of \( \text{char} V(D[s] \cdot f^*) \) to \( U_f \) is

\[
\left\{ (\xi, s \frac{df(\xi)}{f(\xi)}) \mid \xi \in U_f, s \in \mathbb{C} \right\} \text{Zariski},
\]

an \((n + 1)\)-dimensional involutive subvariety of \( T^* U_f \), [120]. Ginsburg [92] gives a formula for the characteristic cycle of \( D(s) \cdot m f^* \) in terms of an intersection process for holonomic sections \( m \).

In favorable cases, more can be said. By [59], if the divisor is reduced, free and locally quasi-homogeneous then \( \text{ann}_{D[s]}(f^*) \) is generated by derivations, both \( \mathcal{M}_f(s) \) and \( \mathcal{N}_f(s) \) have Koszul–Spencer type resolutions, and in particular the characteristic varieties are complete intersections. In the more general case where \( f \) is holonomic, locally strongly Euler-homogeneous and tame, \( \text{ann}_D(f^*) \) is still generated by order one operators and the ideal of symbols of \( \text{ann}_D(f^*) \) (and hence the characteristic ideal of \( \mathcal{M}_f(s) \) as well) is a Cohen–Macaulay prime ideal, [231]. Under these hypotheses, the characteristic ideal of \( \mathcal{N}_f(s) \) is Cohen–Macaulay but usually not prime.

5.1. **Stratifications.** By [115], the resolution theorem of Hironaka can be used to show that there is a stratification of \( \mathbb{C}^n \) such that for each holonomic \( D \)-module \( M \), \( \text{char} C(M) = \bigsqcup_{\sigma \in \Sigma} \mu(M, \sigma) T^*_\sigma \) where \( T^*_\sigma \) is the closure of the conormal bundle of the smooth stratum \( \sigma \) in \( \mathbb{C}^n \) and \( \mu(M, \sigma) \in \mathbb{N} \).

For \( D[s] \cdot f^*/D[s] \cdot f^{*+1} \) Kashiwara proved that if one considers a Whitney stratification \( S \) for \( f \) (for example the “canonical” stratification in [78]) then the characteristic variety of the \( D \)-module \( \mathcal{N}_f(s) \) is in the union of the conormal varieties of the strata \( \sigma \in S \), [235].

If one slices a pair \((X,D)\) of a smooth space and a divisor with a hyperplane, various invariants of the divisor will behave well provided that the hyperplane is not “special”. A prime example are Bertini and Lefschetz theorems. For \( D \)-modules, Kashiwara defined the notion of non-characteristic restriction: the smooth hypersurface \( H \) is non-characteristic for the \( D \)-module \( M \) if it meets each component of the characteristic variety of \( M \) transversally (see [177] for an exposition). The condition assures that the inverse image functor attached to the embedding \( H \hookrightarrow X \) has no higher derived functors for \( M \). In [86] these ideas are used to show that the
5.2. Deformations. Varchenko proved, via establishing constancy of Hodge numbers, that in a $\mu$-constant family of isolated singularities, the spectrum is constant [223]. In [86] it is shown that the formation of the spectrum along the divisor $Y \subseteq X$ commutes with the intersection with a hyperplane transversal to any stratum of a Whitney regular stratification of $D$, and a weak generalization of Varchenko’s constancy results for certain deformations of non-isolated singularities is derived.

In contrast, the Bernstein–Sato polynomial may not be constant along $\mu$-constant deformations. Suppose $f(x) + \lambda g(x)$ is a 1-parameter family of plane curves with isolated singularities at the origin. If the Milnor number $\dim\mathcal{C}(R/J_{f+\lambda g})$ is constant in the family, the singularity germs in the family are topologically equivalent [218]; for discussion see [88, §2]. However, in such a family $b_f(s)$ can vary, as it is a differential invariant. Indeed, $f + \lambda g = x^4 + y^5 + \lambda xy^4$ has constant Milnor number 20, but the general curve (not quasi-homogeneous in any coordinate system, as $\rho_{f+\lambda g}$ is not symmetric about $-1$, see Subsection 3.3 above) has $-\rho_{f+\lambda g} = \{1\} \cup \frac{1}{20}\{9, 11, 13, 14, 17, 18, 19, 21, 22, 23, 26, 27\}$ while the special curve has $-\rho_f = -\rho_{f+\lambda g} \cup \{-31/20\} \setminus \{-11/20\}$. See [64] for details and similar examples based on Newton polytope considerations, and [205] for deformations of plane diagonal curves.

6. Milnor fiber and monodromy

6.1. Milnor fibers. Let $B(p, \varepsilon)$ denote the $\varepsilon$-ball around $p \in \text{Var}(f) \subseteq \mathbb{C}^n$. Milnor [155] proved that the diffeomorphism type of the open real manifold

$$M_{p,t_0,\varepsilon} = B(p, \varepsilon) \cap \text{Var}(f - t_0)$$

is independent of $\varepsilon, t_0 \in \mathbb{R}$ as long as $0 < |t_0| \ll \varepsilon \ll 1$. For $0 < \tau \ll \varepsilon \ll 1$ denote by $M_p$ the fiber of the bundle $B(p, \varepsilon) \cap \{q \in \mathbb{C}^n \mid 0 < |f(q)| < \tau\} \to f(q)$.

The direct image functor for $D$-modules to the projection $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}, (x, t) \mapsto t$ turns the $D_x$-module $\mathscr{B}_f$ into the Gauß–Manin system $\mathcal{H}_f$. The $D$-module restriction of $H^k(\mathcal{H}_f)$ to $t = t_0$ is the $k$-th cohomology of the Milnor fibers along $\text{Var}(f)$ for $0 < |t_0| < \tau$.

Fix a $k$-cycle $\sigma \in H_p(\text{Var}(f - t_0))$ and choose $\eta \in H^k(\mathcal{H}_f)$. Deforming $\sigma$ to a $k$-cycle over $t$ using the Milnor fibration, one can evaluate $\int_{\sigma_t} \eta$. The Gauß–Manin system has Fuchsian singularities and these periods are in the Nilsson class [148]. For example, the classical Gauß hypergeometric function saw the light of day the first time as solution to a system of differential equations attached to the variation of the Hodge structure on an elliptic curve (expressed as integrals of the first and second kind) [41]. In [177] this point of view is taken to be the starting point. The techniques explained there form the foundation for many connections between $f^s$ and singularity invariants attached to $\text{Var}(f)$.

In [46], a bijection (for $0 < \alpha \leq 1$) is established between a subset of the jumping numbers of $f$ at $p \in \text{Var}(f)$ and the support of the Hodge spectrum [207]

$$\text{Sp}(f) = \sum_{\alpha \in \mathbb{Q}} n_\alpha(f)t^\alpha,$$

with $n_\alpha(f)$ determined by the size of the $\alpha$-piece of Hodge component of the cohomology of the Milnor fiber of $f$ at $p$. See also [190, 221], and [206] for a survey on
Hodge invariants. We refer to [49, 194] for many more aspects of this part of the story.

6.2. Monodromy. The vector spaces $H^k(M_{p,t_0, \varepsilon}, \mathbb{C})$ form a smooth vector bundle over a punctured disk $\mathbb{C}^*$. The linear transformation $\mu_{f,p,k}$ on $H^k(M_{p,t_0, \varepsilon}, \mathbb{C})$ induced by $p \mapsto p \cdot \exp(2\pi i \lambda)$ is the $k$-th monodromy of $f$ at $p$. Let $\chi_{f,p,k}(t)$ denote the characteristic polynomial of $\mu_{f,p,k}$, set

$$e_{f,p,k} = \{ \gamma \in \mathbb{C} \mid \gamma \text{ is an eigenvalue of } \mu_{f,p,k} \}$$

and put $e_{f,p} = \bigcup e_{f,p,k}$.

For most (in a quantifiable sense) divisors $f$ with given Newton diagram, a combinatorial recipe can be given that determines the alternating product $\prod (\chi_{f,p,k}(t))^{(-1)^k}$ similarly to A’Campo’s formula in terms of an embedded resolution [1].

6.3. Degrees, eigenvalues, and Bernstein–Sato polynomial. By [147, 114], the exponential function maps the root set of the local analytic Bernstein–Sato polynomial of $f$ at $p$ onto $e_{f,p}$. The set $\exp(-2\pi i \hat{\rho}_{f,p})$ is the set of eigenvalues of the monodromy on the Grothendieck–Deligne vanishing cycle sheaf $\phi_f(C_{X,p})$. This was shown in [191] by algebraic microlocalization.

If $f$ is an isolated singularity, the Milnor fiber $M_f$ is a bouquet of spheres, and $H^{n-1}(M_f, \mathbb{C})$ can be identified with the Jacobian ring $R/J_f$. Moreover, if $f$ is quasi-homogeneous, then under this identification $R/J_f$ is a $\mathbb{Q}[s]$-module, $s$ acting via the Euler operator, and $\hat{\rho}_f$ is in bijection with the degree set of the nonzero quasi-homogeneous elements in $R/J_f$. For non-isolated singularities, most of this breaks down, since $R/J_f$ is not Artinian in that case. However, for homogeneous $f$, consider the Jacobian module

$$H^0(R/J_f) = \{ g + J_f \mid \exists k \in \mathbb{N}, \forall i, x_i^k g \in J_f \}$$

and the canonical $(n-1)$-form

$$\eta = \sum_i x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$ 

Every class in $H^{n-1}(M_f; \mathbb{C})$ is of the form $g\eta$ for suitable $g \in R$, and there is a filtration on $H^{n-1}(M_f, \mathbb{C})$ induced by integration of $\partial f_i$ along $\partial_1, \ldots, \partial_n$, with the following property: if $g \in R$ is the smallest degree homogeneous polynomial such that $g\eta$ represents a chosen element of $H^{n-1}(M_f, \mathbb{C})$ then $b_f(-\deg(g\eta))/\deg(f) = 0$, [230]. Moreover, let $\mathbf{7} \neq 0$ by a homogeneous element in the Jacobian module and suppose that its degree $\deg(g\eta) = \deg(g) + \sum_i \deg(x_i)$ is between $d$ and $2d$. Then, by [231], $g\eta$ represents a nonzero cohomology class in $H^{n-1}(M_f, \mathbb{C})$ as in the isolated case.

6.4. Zeta functions. The zeta function $Z_f(s)$ attached to a divisor $f \in R$ is the rational function

$$Z_f(s) = \sum_{I \subseteq S} \chi(E^*_I) \prod_{i \in I} \frac{1}{N_i s + \nu_i}$$

where $\pi: (Y, \bigcup E_i) \to (\mathbb{C}^n, \Var(f))$ is an embedded resolution of singularities, and $N_i$ (resp. $\nu_i - 1$) are the multiplicities of $E_i$ in $\pi^*(f)$ (resp. in the Jacobian of $\pi$). By results of Denef and Loeser [84], $Z_f(s)$ is independent of the resolution.

**Conjecture 6.1** (Topological Monodromy Conjecture).

(SMC) Any pole of $Z_f(s)$ is a root of the Bernstein–Sato polynomial $b_f(s)$.
(MC) Any pole of $Z_f(s)$ yields under exponentiation an eigenvalue of the monodromy operator at some $p \in \text{Var}(f)$.

The strong version (SMC) implies (MC) by [146, 114]. Each version allows a generalization to ideals.

(SMC), formulated by Igusa [110] and Denef–Loeser [84] holds for

- reduced curves by [138] with a discussion on the nature of the poles by Veys [226, 225, 227];
- certain Newton-nondegnerate divisors by [140];
- some hyperplane arrangements (see Section 8);
- monomial ideals in any dimension by [106].

Additionally, Conjecture (MC) holds for

- bivariate ideals by Van Proeyen and Veys [220];
- all hyperplane arrangements by [54, 56];
- some partial cases: [11, 127] some surfaces; [13] quasi-ordinary power series; [136, 140] in certain Newton non-degenerate cases; [109, 123] for invariants of prehomogeneous vector spaces; [126] for nondegenerate surfaces.

Strong evidence for (MC) for $n = 3$ is procured in [228]. The articles [180, 164] explore what (MC) could mean on a normal surface as ambient space and gives some results and counterexamples to naive generalizations. See also [85] and the introductions of [31, 32] for more details in survey format.

A root of $b_f(s)$, a monodromy eigenvalue, and a pole of $Z_f(s)$ may have multiplicity; can the monodromy conjecture be strengthened to include multiplicities?

This version of (SMC) was proved for reduced bivariate $f$ in [138]; in [153, 154] it is proved for certain nonreduced bivariate $f$, and for some trivariate ones.

A different variation, due to Veys, of the conjecture is the following. Vary the definition of $Z_f(s)$ to $Z_{f,g}(s) = \sum_{I \subseteq \mathcal{S}} \chi(E_I') \prod_{i \in I} \frac{1}{\alpha_{i,s} \nu_i'}$ where $\nu_i'$ is the multiplicity of $E_i$ in the pullback along $\pi$ of some differential form $g$. (The standard case is when $g$ is the volume form). Two questions arise: (1) varying over a suitable set $G$ of forms $g$, can one generate all roots of $b_f(s)$ as poles of the resulting zeta functions? And if so, can one (2) do this such that the pole sets of all zeta functions so constructed are always inside $\rho_f$, so that

$$\rho_f = \{ \alpha \mid \exists g \in G, \lim_{s \to \alpha} Z_{f,g}(s) = \infty \}?$$

Némethi and Veys [163, 164] prove a weak version: if $n = 2$ then monodromy eigenvalues are exponentials of poles of zeta functions from differential forms.

The following is discussed in [30]. For some ideals with $n = 2$, (1) is false for the topological zeta function (even for divisors: consider $xy^5 + x^3y^2 + x^4y$). For monomial ideals with two generators in $n = 2$, (1) is correct; with more than two generators it can fail. Even in the former case, (2) can be false.

7. Multi-variate versions

If $f = (f_1, \ldots, f_r)$ defines a map $f : \mathbb{C}^n \to \mathbb{C}^r$, several $b$-functions can be defined:

(1) The univariate Bernstein–Sato polynomial $b_f(s)$ attached to the ideal $(f) \subseteq R$ from [51].

(2) The multi-variate Bernstein–Sato polynomials $b_{f,i}(s)$ of all $b(s) \in \mathbb{C}[s_1, \ldots, s_r]$ such that there is an equation $P(x, \partial, s) \bullet f_i f^* = b(s)f^*$ in multi-index notation.
(3) The multi-variate Bernstein–Sato ideal \( B_{f,\mu}(s) \) for \( \mu \in \mathbb{N}^r \) of all \( b(s) \in \mathbb{C}[s_1, \ldots, s_r] \) such that there is an equation \( P(x, \partial_s) \cdot f^{s+\mu} = b(s)f^s \) in multi-index notation. The most interesting case is \( \mu = 1 = (1, \ldots, 1) \).

(4) The multi-variate Bernstein–Sato ideal \( B_{f,\Sigma}(s) \) of all \( b(s) \in \mathbb{C}[s_1, \ldots, s_r] \) that multiply \( f^s \) into \( \sum D[s]f^s \) in multi-index notation.

The Bernstein–Sato polynomial in (1) above has been studied in the case of a monomial ideal in [52] and more generally from the point of view of the Newton polygon in [53]. While the roots for monomial ideals do not depend just on the Newton polygon, their residue classes modulo \( \mathbb{Z} \) do.

Non-triviality of the quantities in (2)-(4) have been established in [184, 185, 183], but see also [17]. The ideals \( B_{f,\mu}(s) \) and \( B_{f,\Sigma}(s) \) do not have to be principal, [219, 18]. In [184, 103] it is shown that \( B_{f,\mu}(s) \) contains a polynomial that factors into linear forms with non-negative rational coefficients and positive constant term. Bahloul and Oaku [18] show that these ideals are local in the sense of (3.1).

The following would generalize Kashiwara’s result in the univariate case as well as the results of Sabbah and Gyoja above.

**Conjecture 7.1** ([48]). The Bernstein–Sato ideal \( B_{f,\mu}(s) \) is is generated by products of linear forms \( \sum \alpha_i s_i + a \) with \( \alpha_i \), a non-negative rational and \( a > 0 \).

For \( n = 2 \), partial results by Cassou-Noguès and Libgober exist [65]. In [48] it is further conjectured that the Malgrange–Kashiwara result, exponentiating \( \rho_{f,p} \) gives \( e_{f,p} \), generalizes: monodromy in this case is defined in [224], and Sabbah’s specialization functor \( \psi_f \) from [186] takes on the rôle of the nearby cycle functor, and conjecturally exponentiating the variety of \( B_{f,\mu}(s) \) yields the uniform support (near \( p \)) of Sabbah’s functor. The latter conjecture would imply Conjecture 7.1.

Similarly to the one-variable case, if \( V(n, d, m) \) is the vector space of (ordered) \( m \)-tuples of polynomials in \( x_1, \ldots, x_n \) of degree at most \( d \), there is an algebraic stratification of \( V(n, d, m) \) such that on each stratum the function \( V \ni f = (f_1, \ldots, f_m) \mapsto b_f(s) \) is constant. Corresponding results for the Bernstein–Sato ideal \( B_{f,1}(s) \) hold by [38].

## 8. Hyperplane Arrangements

A **hyperplane arrangement** is a divisor of the form

\[
\mathcal{A} = \prod_{i \in I} \alpha_i
\]

where each \( \alpha_i \) is a polynomial of degree one. We denote \( H_i = \text{Var}(\alpha_i) \). Essentially all information we are interested in is of local nature, so we assume that each \( \alpha_i \) is a form so that \( \mathcal{A} \) is central. If there is a coordinate change in \( \mathbb{C}^n \) such that \( \mathcal{A} \) becomes the product of polynomials in disjoint sets of variables, the arrangement is decomposable, otherwise it is indecomposable.

A **flat** is any (set-theoretic) intersection \( \bigcap_{i \in J} H_i \) where \( J \subseteq I \). The **intersection lattice** \( L(\mathcal{A}) \) is the partially ordered set consisting of the collection of all flats, with order given by inclusion.

### 8.1. Numbers and parameters

Hyperplane arrangements satisfy \((B_1)\) everywhere [230]. Arrangements satisfy \((A_1)\) everywhere if they decompose into a union of a generic and a hyperbolic arrangement [214], and if they are tame [231]. Terao
conjectured that all hyperplane arrangements satisfy $(A_1)$; some of them fail $(A_s)$, [231].

Apart from recasting various of the previously encountered problems in the world of arrangements, a popular study is the following: choose a discrete invariant $I$ of a divisor. Does the function $\mathcal{A} \mapsto I(\mathcal{A})$ factor through the map $\mathcal{A} \mapsto L(\mathcal{A})$? Randell showed that if two arrangements are connected by a one-parameter family of arrangements which have the same intersection lattice, the complements are diffeomorphic [178] and the isomorphism type of the Milnor fibration is constant [179]. Rybnikov [181, 12] showed on the other hand that there are arrangements (even in the projective plane) with equal lattice but different complement. In particular, not all isotopic arrangements can be linked by a smooth deformation.

8.2. LCT and logarithmic ideal. The most prominent positive result is by Brieskorn: the Orlik–Solomon algebra $OS(\mathcal{A}) \subseteq \Omega^\bullet(\log \mathcal{A})$ generated by the forms $d\alpha_i/\alpha_i$ is quasi-isomorphic to $\Omega^\bullet(\ast \mathcal{A})$, hence to the singular cohomology algebra of $U_\mathcal{A}$, [40]. The relation with combinatorics was given in [175, 176]. For a survey on the Orlik–Solomon algebra, see [237]. The best known open problem in this area is

Conjecture 8.1 ([211]). $OS(\mathcal{A}) \to \Omega^\bullet(\log \mathcal{A})$ is a quasi-isomorphism.

While the general case remains open, Wiens and Yuzvinsky [232] proved it for tame arrangements, and also if $n \leq 4$. The techniques are based on [72].

8.3. Milnor fibers. There is a survey article by Suciu on complements, Milnor fibers, and cohomology jump loci [208], and [49] contains further information on the topic. It is not known whether $L(\mathcal{A})$ determines the Betti numbers (even less the Hodge numbers) of the Milnor fiber of an arrangement. The first Betti number of the Milnor fiber $M_\mathcal{A}$ at the origin is stable under intersection with a generic hyperplane (if $n > 2$). But it is unknown whether the first Betti number of an arrangement in 3-space is a function of the lattice alone. By [89], this is so for collections of up to 14 lines with up to 5-fold intersections in the projective plane. See also [134] for the origins of the approach. By [50], a lower combinatorial bound for the $\lambda$-eigenspace of $H^1(M_\mathcal{A})$ is given under favorable conditions on $L$. If $L$ satisfies stronger conditions, the bound is shown to be exact. In any case, [50] gives an algebraic, although perhaps non-combinatorial, formula for the Hodge pieces in terms of multiplier ideals.

By [174], the Betti numbers of $M_\mathcal{A}$ are combinatorial if $\mathcal{A}$ is generic. See also [76].

8.4. Multiplier ideals. Mustaţă gave a formula for the multiplier ideals of arrangements, and used it to show that the log-canonical threshold is a function of $L(\mathcal{A})$. The formula is somewhat hard to use for showing that each jumping number is a lattice invariant; this problem was solved in [55]. Explicit formulas in low dimensional cases follow from the spectrum formulas given there and in [236]. Teitler [210] improved Mustaţă’s formula for multiplier ideals to not necessarily reduced hyperplane arrangements [158].

8.5. Bernstein–Sato polynomials. By [230], $\rho_\mathcal{A} \cap \mathbb{Z} = \{-1\}$; by [192], $\rho_\mathcal{A} \subseteq (-2, 0)$. There are few classes of arrangements with explicit formulae for their Bernstein–Sato polynomial:

• Boolean (a normal crossing arrangement, locally given by $x_1 \cdots x_k$);
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- hyperbolic (essentially an arrangement in two variables);
- generic (central, and all intersections of $n$ hyperplanes equal the origin).

The first case is trivial, the second is easy, the last is [230] with assistance from [193]. Some interesting computations are in [56], and [48] has a partial confirmation of the multi-variable Kashiwara–Malgrange theorem. The Bernstein–Sato polynomial is not determined by the intersection lattice, [231].

8.6. Zeta functions. Budur, Mustaţă and Teitler [54] show: (MC) holds for arrangements, and in order to prove (SMC), it suffices to show the following conjecture.

**Conjecture 8.2.** For all indecomposable central arrangements with $d$ planes in $n$-space, $b_{-n/d} = 0$.

The idea is to use the resolution of singularities obtained by blowing up the dense edges from [200]. The corresponding computation of the zeta function is inspired from [107, 108]. The number $-n/d$ does not have to be the log-canonical threshold. By [54], Conjecture 8.2 holds in a number of cases, including reduced arrangements in dimension 3. By [231] it holds for tame arrangements.

Examples of Veys (in 4 variables) show that (SMC) may hold even if Conjecture 8.2 were false in general, since $-n/d$ is not always a pole of the zeta function [56]. However, in these examples, $-n/d$ is in fact a root of $b(f(s))$.

For arrangements, each monodromy eigenvalue can be captured by zeta functions in the sense of Némethi and Veys, see Subsection 6.4, but not necessarily all of $\rho_{-d}$ (Veys and Walther, unpublished).

9. Positive characteristic

Let here $\mathbb{F}$ denote a field of characteristic $p > 0$. The theory of $D$-modules is rather different in positive characteristic compared to their behavior over the complex numbers. There are several reasons for this:

1. On the downside, the ring $D_p$ of $\mathbb{F}$-linear differential operators on $R_p = \mathbb{F}[x_1, \ldots, x_n]$ is no longer finitely generated: as an $\mathbb{F}$-algebra it is generated by the elements $\partial(\alpha), \alpha \in \mathbb{N}^n$, which act via $\partial(\alpha) \cdot (x^\beta) = (\beta, \alpha)x^{\beta-\alpha}$.

2. As a trade-off, one has access to the Frobenius morphism $x_i \mapsto x_i^p$, as well as the Frobenius functor $F(M) = R^p \otimes_R M$ where $R^p$ is the $R-R$-bimodule on which $R$ acts via the identity on the left, and via the Frobenius on the right. Lyubeznik [142] created the category of $F$-finite $F$-modules and proved striking finiteness results. The category includes many interesting $D_p$-modules, and all $F$-modules are $D_p$-modules.

3. Holonomicity is more complicated, see [29].

A most surprising consequence of Lyubeznik’s ideas is that in positive characteristic the property $(B_1)$ is meaningless: it holds for every $f \in R_p$, [6]. The proof uses in significant ways the difference between $D_p$ and the Weyl algebra. In particular, the theory of Bernstein–Sato polynomials is rather different in positive characteristic. In [159] a sequence of Bernstein–Sato polynomials is attached to a polynomial $f$ assuming that the Frobenius morphism is finite on $R$ (e.g., if $\mathbb{F}$ is finite or algebraically closed); these polynomials are then linked to test ideals, the finite characteristic counterparts to multiplier ideals. In [28] variants of our modules $\mathcal{M}_f(\gamma)$ are introduced and [168] shows that simplicity of these modules detects the
F-thresholds from [160]. These are cousins of the jumping numbers of multiplier ideals and related to the Bernstein–Sato polynomial via base-p-expansions; see also [233]. The Kashiwara–Brylinski intersection homology module was shown to exist in positive characteristic by Blickle in his thesis, [27].

10. Appendix: Computability (by A. Leykin)

Computations around $f^s$ can be carried out by hand in special cases. Generally, the computations are enormous and computers are required (although not often sufficient). One of the earliest such approaches are in [34, 4], but at least implicitly Buchberger’s algorithm in a Weyl algebra was discussed as early as [70]. An algorithmic approach to the isolated singularities case [144] preceded the general algorithms based on Gröbner bases in a non-commutative setting outlined below.

10.1. Gröbner bases. The monomials $x^n\partial^\beta$ with $\alpha, \beta \in \mathbb{N}^n$ form a $\mathbb{C}$-basis of $D$; expressing $p \in D$ as linear combination of monomials leads to its normal form. The monomial orders on the commutative monoid $[x, \partial]$ for which for all $i \in [n]$ the leading monomial of $\partial_i x_i = x_i \partial_i + 1$ is $x_i \partial_i$, can be used to run Buchberger’s algorithm in $D$. Modifications are needed in improvements that exploit commutativity, but the naïve Buchberger’s algorithm works without any changes. See [112] for more general settings in polynomial rings of solvable type. Surprisingly, the worst case complexity of Gröbner bases computations in Weyl algebras is not worse than in the commutative polynomial case: it is doubly exponential in the number of indeterminates [14, 100].

10.2. Characteristic variety. A weight vector $(u, v) \in \mathbb{Z}^n \times \mathbb{Z}^n$ with $u + v \geq 0$ induces a filtration of $D$,

$$F_i = \mathbb{C} \cdot \{x^n\partial^\beta \mid u \cdot \alpha + v \cdot \beta \leq i\}, \quad i \in \mathbb{Z}.$$  

The $(u, v)$-Gröbner deformation of a left ideal $I \subseteq D$ is

$$\text{in}_{(u,v)}(I) = \mathbb{C} \cdot \{\text{in}_{(u,v)}(P) \mid P \in I\} \subseteq \text{gr}_{(u,v)} D,$$

the ideal of initial forms of elements of $I$ with respect to the given weight in the associated graded algebra. It is possible to compute Gröbner deformations in the homogenized Weyl algebra

$$D^h = D(h)/(\partial_i x_i - x_i \partial_i - h^2, x_i h - hx_i, \partial_i h - h \partial_i, |1 \leq i \leq n)$$

see [71, 172]. Gröbner deformations are the main topic of [195].

10.3. Annihilator. Recall the construction appearing in the beginning of §6.1: $D_{x,t}$ acts on $D[s]f^s$; in particular, the operator $-\partial_t$ acts as multiplication by $s$. It is this approach that lead Oaku to an algorithm for $\text{ann}_{D[s]}(f^s)$, $\text{ann}_D(f^s)$ and $b_f(s)$, [170]. We outline the ideas.

Malgrange observed that

$$\text{ann}_{D[s]}(f^s) = \text{ann}_{D_{x,t}}(f^s) \cap D[s],$$

(10.2) with

$$\text{ann}_{D_{x,t}}(f^s) = \langle t - f, \partial_1 + \frac{\partial f}{\partial x_1} \partial_1, \ldots, \partial_n + \frac{\partial f}{\partial x_1} \partial_1 \rangle \subseteq D_{x,t}. $$

The former can be found from the latter by eliminating $t$ and $\partial_t$ from the ideal

$$\text{(10.3)} \quad (s + t \partial_t) + \text{ann}_{D_{x,t}}(f^s) \subseteq D_{x,t}(s);$$

of course $s = -\partial_t t$ does not commute with $t, \partial_t$ here.
Oaku’s method for \( \text{ann}_{D[s]}(f^s) \) accomplished the elimination by augmenting two commuting indeterminates:

\[
\text{ann}_{D[s]}(f^s) = I_f' \cap D[s],
\]

(10.4)

\[
I_f' = \langle t - uf, \partial_1 + u \frac{\partial f}{\partial x_1} \partial_1, \ldots, \partial_n + u \frac{\partial f}{\partial x_n} \partial_1, uv - 1 \rangle \subseteq D_{x,t}[u, v].
\]

Now outright eliminate \( u, v \). Note that \( I_f' \) is quasi-homogeneous if the weights are \( t, u \sim -1 \) and \( \partial_1, v \sim 1 \), all other variables having weight zero. The homogeneity of the input and the relation \( [\partial_t, t] = 1 \) assures the termination of the computation. The operators of weight 0 in the output (with \(-\partial_t t\) replaced by \(s\)) generate \( I_f' \cap D[s] \).

A modification given in [33] and used, e.g., in [219], reduces the number of algebra generators by one. Consider the subalgebra \( D(s, \partial_t) \subset D_{x,t} \); the relation \( [s, \partial_t] = \partial_t \) shows that it is of solvable type. According to [33],

\[
\text{ann}_{D[s]}(f^s) = I_f'' \cap D[s],
\]

(10.5)

\[
I_f'' = \langle s + f \partial_t, \partial_1 + \frac{\partial f}{\partial x_1} \partial_1, \ldots, \partial_n + \frac{\partial f}{\partial x_n} \partial_1 \rangle \subset D(s, \partial_t).
\]

Note that \( I_f'' = \text{ann}_{D_{x,t}}(f^s) \cap D(s, \partial_t) \). The elimination step is done as in [170]; the decrease of variables usually improves performance. An algorithm to decide \( (A_1) \) for arrangements is given in [5].

10.4. Algorithms for the Bernstein–Sato polynomial. As the minimal polynomial of \( s \) on \( \mathcal{M}(f) \), \( b_f(s) \) can be obtained by means of linear algebra as a syzygy for the normal forms of powers of \( s \) modulo \( \text{ann}_{D[s]}(f^s) + D[s] \cdot f \) with respect to any fixed monomial order on \( D[s] \). Most methods follow this path, starting with [170]. Variations appear in [229, 171, 173]; see also [195].

A slightly different approach is to compute \( b_f(s) \) without recourse to \( \text{ann}_{D[s]}(f^s) \), via a Gröbner deformation of the ideal \( I_f = \text{ann}_{D_{x,t}}(f^s) \) in (10.2) with respect to the weight \((-w, w)\) with \( w = (0^n, 1) \in \mathbb{N}^{n+1} \): \( \langle b_f(s) \rangle = \text{in}_{(-w, w)}(I_f) \cap \mathbb{Q}[-\partial_t t]. \) Here again, computing the minimal polynomial using linear algebra tends to provide some savings in practice.

In [128] the authors give a method to check specific numbers for being in \( \rho_f \). A method for \( b_f(s) \) in the prehomogeneous vector space setup is in [157].

10.5. Stratification from \( b_f(s) \). The Gröbner deformation \( \text{in}_{(-w, w)}(I_f) \) in §10.4 can be refined as follows, see [22, Thm. 2.2]. Let \( b(x, s) \) be nonzero in the polynomial ring \( \mathbb{C}[x, s] \). Then \( b(x, s) \in (\text{in}_{(-w, w)} I_f) \cap \mathbb{C}[x, s] \) if and only if there exists \( P \in D[s] \) satisfying the functional equation \( b(x, s) f^s = Pf f^s \). From this one can design an algorithm not only for computing the local Bernstein–Sato polynomial \( b_{f,p}(s) \) for \( p \in \text{Var}(f) \), but also the stratification of \( \mathbb{C}^n \) according to local Bernstein–Sato polynomials; see [165, 22] for various approaches. Moreover, one can compute the stratifications from Subsection 3.3.2, see [131].

For the ideal case, [8] gives a method to compute an intersection of a left ideal of an associative algebra over a field with a subalgebra, generated by a single element. An application is a method for the computation of the Bernstein-Sato polynomial of an ideal. Another such was given by Bahloul in [15], and a version on general varieties in [16].

10.6. Multiplier ideals. Consider polynomials \( f_1, \ldots, f_r \in \mathbb{C}[x] \), let \( f \) stand for \( (f_1, \ldots, f_r) \), \( s \) for \( s_1, \ldots, s_r \), and \( f^s \) for \( \prod_{i=1}^r f_i^{s_i} \). In this subsection, let \( D_{x,t} = \mathbb{C}[x, t, \partial_x, \partial_t] \) be the \((n + r)\)-th Weyl algebra.
Consider $D_{x,t}(s) \cdot f^s \subseteq R_{x,t}[f^{-1}, s]\{f^s\}$ and put
\[
t_f(x, s_1, \ldots, s_j, \ldots, s_r) f^s = h(x, s_1, \ldots, s_j + 1, \ldots, s_r) f^s,
\]
\[
\partial t_f(x, s_1, \ldots, s_j, \ldots, s_r) f^s = -s_j h(x, s_1, \ldots, s_j - 1, \ldots, s_r) f_j^{-1} f^s,
\]
for $h \in \mathbb{C}[x][f^{-1}, s]$, generalizing the univariate constructions.

The generalized Bernstein–Sato polynomial $b_{f,g}(\sigma)$ of $f$ at $g \in \mathbb{C}[x]$ is the monic univariate polynomial of the lowest degree for which there exist $P_k \in D_{x,t}$ such that
\[
(10.6) \quad b(\sigma) g f^s = \sum_{k=1}^r P_k g f_k f^s, \quad \sigma = - \left( \sum_{i=1}^r \partial_i t_i \right).
\]

An algorithm for $b_{f,g}(\sigma)$ is an essential ingredient for the algorithms in [204, 22] that compute the jumping numbers and corresponding multiplier ideals for $I = \langle f_1, \ldots, f_r \rangle$. That $b_{f,g}(\sigma)$ is related to multiplier ideals was worked out in [51].

There are algorithms for special cases: monomial ideals [105], hyperplane arrangements [158], and determinantal ideals [111]. A Macaulay2 package MultiplierIdeals by Teitler collects all available (in Macaulay2 implementations. See also [47].

### 10.7. Software

Algorithms for computing Bernstein–Sato polynomials have been implemented in kan/sm1 [209], Risa/Asir [167], dmod.lib library [130] for Singular [80], and the $D$-modules package [133] for Macaulay2 [98]. The best source of information of these is documentation in the current versions of the corresponding software. A relatively recent comparison of the performance for several families of examples is given in [129].

The following are articles by developers discussing their implementations: [166, 165, 171, 7, 130, 132, 22].

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