Solutions to the Einstein Constraint Equations with a Small TT-Tensor and Vanishing Yamabe Invariant

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Abstract. In this note, we prove an existence result for the Einstein conformal constraint equations for metrics with vanishing Yamabe invariant assuming that the mean curvature satisfies an explicit near-CMC condition and that the TT-tensor is small in $L^2$.

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1. Introduction

The conformal method and one of its generalization and the conformal thin sandwich (CTS) method (described, e.g., in [3] or [17]) are historically the main methods to solve the Einstein constraint equations, despite recent evidences that they fail at parameterizing correctly the full set of initial data (see, e.g., [5,9,16]).
Initial data for the Cauchy problem are generally given as a triple \((M, \hat{g}, \hat{K})\), where \(M\) is a \(n\)-dimensional manifold, \(\hat{g}\) is a metric on \(M\) and \(\hat{K}\) is a symmetric 2-tensor that correspond, respectively, to the metric induced by the spacetime (we are to find) metric on \(M\) and the second fundamental form of \(M\) as a hypersurface in the spacetime. The interested reader can consult, e.g., [20] for more information.

The strategy of the conformal method and of the CTS method is to decompose in a certain manner \((M, \hat{g}, \hat{K})\) as a given part and an unknown part that has to be adjusted in order to fulfill the constraint equations. To keep things simple, we will consider only the vacuum case and restrict to compact Cauchy surfaces \(M\). We fix

- a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\),
- a function \(\tau: M \to \mathbb{R}\),
- a symmetric traceless 2-tensor \(\sigma\) such that \(\text{div}_g \sigma = 0\) (such a tensor will be called a TT-tensor in what follows),
- a positive function \(\eta: M \to \mathbb{R}_+\),

and seek for

- a positive function \(\phi: M \to \mathbb{R}_+\),
- a vector field \(W\),

so that

\[
\hat{g} := \phi^{N-2} g \quad \text{and} \quad \hat{K} := \frac{\tau}{n} \hat{g} + \phi^{-2} \left( \sigma + \frac{1}{2\eta} \mathbb{L}_g W \right)
\]

(1.1)
satisfy the constraint equations

\[
\text{Scal}_{\hat{g}} + (\text{tr}_{\hat{g}} \hat{K})^2 - |\hat{K}|^2_{\hat{g}} = 0, \quad (1.2a)
\]

\[
\text{div}_{\hat{g}} \hat{K} - d(\text{tr}_{\hat{g}} \hat{K}) = 0. \quad (1.2b)
\]

Here, we have introduced the following notations:

\[
N := \frac{2n}{n-2} \quad \text{and} \quad \mathbb{L}_g W := \mathcal{L}_W g - \frac{\tau g}{n} \mathcal{L}_g W g,
\]

where \(\mathcal{L}\) denotes the Lie derivative. The operator \(\mathbb{L}_g\) is commonly known as the conformal Killing operator or as the Ahlfors operator. Note that \(\tau = \hat{g}^{ij} \hat{K}_{ij}\) so \(\tau\) corresponds to the mean curvature of the embedding of \(M\) into the spacetime.

The decomposition (1.1) relies on York’s splitting of symmetric 2-tensors [23]. TT-tensors were introduced first in a paper by R. Arnowitt, S. Deser and C. Misner in 1962 (see the reprint of this article in [2]). We refer the reader to [17] for more information about the history of the conformal method and of the conformal thin sandwich.

The system (1.2) is equivalent to the following:

\[
- \frac{4(n-1)}{n-2} \Delta_g \phi + \text{Scal}_g \phi = - \frac{n-1}{n} \tau^2 \phi^{N-1} + \left| \sigma + \frac{1}{2\eta} \mathbb{L}_g W \right|^2 \phi^{-N+1}, \quad (1.3a)
\]

\[
\Delta_{\mathbb{L}_g,\eta} W = \frac{n-1}{n} \phi^N \nabla \tau, \quad (1.3b)
\]
where we set
\[ \Delta_{L^g, \eta} := -\frac{1}{2} L^g_* \left( \frac{1}{2\eta} L^g \right). \]

Equation (1.3a) is commonly known as the Lichnerowicz equation, while Eq. (1.3b) bears no particular name. We will call it the vector equation. Hence, solving (1.3) is equivalent to solving (1.2).

The conformal method corresponds to the particular choice \( 2\eta \equiv 1 \) in the previous equations. As indicated in [3], allowing for more general \( \eta \) in (1.3) does not introduce new technical difficulties so theoretical studies have mostly concentrated on the conformal method.

Initial work was limited to the constant mean curvature (CMC) case (i.e., constant \( \tau \)) and to the near-CMC case (see [1] and references therein). But two constructions were introduced in 2009 by M. Holst, G. Nagy, G. Tsogtgerel and D. Maxwell (HNTM), see [10,11,15], and in 2011 by M. Dahl, E. Humbert and the author in [4] to solve (1.3). The interested reader can consult [6,18] for an overview and a comparison of both techniques.

We will focus on the HNTM method. It requires two things: that the Yamabe invariant \( Y_g \) of \( g \) is positive (see, e.g., [13] for the definition of the Yamabe invariant and the solution of the related Yamabe problem) and that \( \sigma \) is nonzero but small in a certain sense (see also [19]). This construction was interpreted as perturbative in a non-trivial sense in [6]. Despite the fact that the point of view introduced in [6] gives a result weaker than the original one in [15], it provides a quick way to test whether the HNTM method works in more general situations. This has been used in [7] for the Einstein-scalar field conformal method and in [8] for variants of the conformal method introduced by D. Maxwell in [14].

In this paper, we show that the HNTM construction extends to the case \( Y_g = 0 \) at the price of imposing an explicit condition on \( \tau \), see (3.11). This condition is satisfied in the near-CMC regime, i.e., if \( d\tau \) is sufficiently small in some well-chosen norm.

The main difficulty in this paper is that the conformal Laplacian
\[ -\frac{4(n-1)}{n-2} \Delta_g + \text{Scal} \]
has a 1-dimensional kernel so the behavior of \( \phi \) in the direction of this kernel is different than in the \( (L^2) \)-orthogonal direction.

The outline of the paper is as follows. Section 2 contains existence and uniqueness results for the Lichnerowicz equation and for the vector equation in a weak regularity context. Section 3 follows the construction in [6]. This gives an idea of what goes on and prepares for Sect. 4 where we prove existence of solutions to (1.3) when \( \sigma \) is small in the spirit of [7,15,19].

The main difference between Theorem 3.3 and Theorem 4.4 is that, in 3.3, we have no control on how \( \lambda_0 \) depends on \( \tilde{\sigma} \) so the theorem gives a weaker result existence, yet the proof is based on the implicit function theorem so is constructive. The proof of Theorem 4.4, however, is based on the Schauder fixed point theorem and is non-constructive by essence.
2. Preliminaries

The aim of this section is to reprove well-known existence results in a weak regularity context. Here and in what follows, we fix a value $p > n$.

We let $\phi_0$ denote the unique positive function such that $\|\phi_0\|_{L^2(M, \mathbb{R})} = 1$ and

$$- \frac{4(n-1)}{n-2} \Delta \phi_0 + \text{Scal} \phi_0 = 0.$$  \hspace{1cm} (2.1)

**Proposition 2.1.** Given $g \in W^{2,p/2}(M, S_2 M)$, $\tau \in L^p(M, S_2 M)$ and $A \in L^p(M, \mathbb{R})$ both nonzero, there exists a unique positive function $\phi \in W^{2,p/2}(M, \mathbb{R})$ solving the Lichnerowicz equation

$$- \frac{4(n-1)}{n-2} \Delta \phi + \text{Scal} \phi = - \frac{n-1}{n} \tau^2 \phi^{N-1} + A^2 \phi^{-N-1}.  \hspace{1cm} (2.2)$$

Further the mapping $A \mapsto \phi$ is continuous from $L^p(M, \mathbb{R})$ to $W^{2,p/2}(M, \mathbb{R})$.

It should be noted that if either $A \equiv 0$ or $\tau \equiv 0$, there cannot be any nonzero solution for a simple reason. Multiplying the Lichnerowicz equation by $\phi_0$ and integrating over $M$, the conformal Laplacian disappears by (formal) self-adjointness leaving the following equality

$$\frac{n-1}{n} \int_M \tau^2 \phi_0 \phi^{N-1} d\mu^g = \int_M A^2 \phi_0 \phi^{-N-1} d\mu^g.$$

If $A$ or $\tau$ vanishes, this identity leads to a contradiction if Eq. (2.1) admits a positive solution $\phi$. There is one exception to this fact, namely when both $\tau$ and $A$ vanish (compare with [12]). In this case, the solutions to (2.1) are the $\lambda \phi_0$, $\lambda \in \mathbb{R}_+$. We will not consider these cases anymore.

**Proof of Proposition 2.1.** We use a variational approach. Note that the functional

$$I(\phi) := \int_M \left( \frac{2(n-1)}{n-2} |d\phi|^2 + \frac{\text{Scal}}{2} \phi^2 + \frac{n-1}{Nn} \tau^2 \phi^N + \frac{A^2}{N \phi^N} \right) d\mu^g \hspace{1cm} (2.2)$$

is ill-defined on $W^{1,2}(M, \mathbb{R})$ since the term $\tau^2 \phi^N$ does not belong to $L^1(M, \mathbb{R})$. For any positive integer $k$, we set $\tau_k := \min\{|\tau|, k\} \in L^\infty$ so that $\tau_k \to |\tau|$ in $L^p$ and $\epsilon_k := 1/k$, and we introduce the family of functionals

$$I_k(\phi) := \int_M \left( \frac{2(n-1)}{n-2} |d\phi|^2 + \frac{\text{Scal} + \epsilon_k}{2} \phi^2 + \frac{n-1}{Nn} \tau_k^2 \phi^N + \frac{A^2}{N(\phi + \epsilon_k)^N} \right) d\mu^g.$$

$I_k$ is well defined, continuous and convex on the closed set

$$C_k := \{ \phi \in W^{1,2}(M, \mathbb{R}), \phi \geq \epsilon_k/2 \text{ a.e.} \}$$

(details can be found in [7]). We claim that there exists $\mu_k > 0$ so that $I_k(\phi) \geq \mu_k \|\phi\|^2_{W^{1,2}(M, \mathbb{R})}$ for all $\phi \in C_k$. Indeed, it suffices to prove that there exists $\mu_k > 0$ such that

$$\mu_k \|\phi\|^2_{W^{1,2}(M, \mathbb{R})} \leq \int_M \left( \frac{2(n-1)}{n-2} |d\phi|^2 + \frac{\text{Scal} + \epsilon_k}{2} \phi^2 \right) d\mu^g.$$
for all $\phi \in W^{1,2}(M, \mathbb{R})$. Since $\text{Scal} \in L^{p/2}(M, \mathbb{R})$ with $p > n$, we have
\[
\left\| \left( \frac{\text{Scal} + \epsilon_k}{2} - \frac{2(n-1)}{n-2} \right) \phi^2 \right\|_{L^1(M, \mathbb{R})} \leq \frac{1}{2} \left[ \|\text{Scal}\|_{L^{p/2}(M, \mathbb{R})} + \left( \frac{2(n-1)}{n-2} - \epsilon_k \right) \text{Vol}(M, g)^{2/p} \right] \|\phi\|_{L^q(M, \mathbb{R})}^2,
\]
with $q = \frac{2p}{p-2} < N$. Set
\[
c := \frac{1}{2} \left[ \|\text{Scal}\|_{L^{p/2}(M, \mathbb{R})} + \frac{2(n-1)}{n-2} \text{Vol}(M, g)^{2/p} \right]
\]
so that
\[
\left\| \left( \frac{\text{Scal} + \epsilon_k}{2} - \frac{2(n-1)}{n-2} \right) \phi^2 \right\|_{L^1(M, \mathbb{R})} \leq c\|\phi\|_{L^q(M, \mathbb{R})}^2.
\]
By interpolation, for any $\epsilon > 0$, there is a constant $\Lambda_\epsilon > 0$ such that, for all $\phi \in W^{1,2}(M, \mathbb{R})$,
\[
\|\phi\|_{L^q(M, \mathbb{R})}^2 \leq \epsilon\|\phi\|_{W^{1,2}(M, \mathbb{R})}^2 + \Lambda_\epsilon\|\phi\|_{L^2(M, \mathbb{R})}^2.
\]
Choose $\epsilon = (n-1)/(c(n-2))$. We have
\[
\int_M \left( \frac{2(n-1)}{n-2} |d\phi|^2 + \frac{\text{Scal} + \epsilon_k}{2} \phi^2 \right) d\mu^g = \frac{2(n-1)}{n-2} \int_M (|d\phi|^2 + \phi^2) d\mu^g + \int_M \left( \frac{\text{Scal} + \epsilon_k}{2} - \frac{2(n-1)}{n-2} \right) \phi^2 d\mu^g \\
\geq \frac{2(n-1)}{n-2} \|\phi\|_{W^{1,2}(M, \mathbb{R})}^2 - \left\| \left( \frac{\text{Scal} + \epsilon_k}{2} - \frac{2(n-1)}{n-2} \right) \phi^2 \right\|_{L^1(M, \mathbb{R})} \\
\geq \frac{2(n-1)}{n-2} \|\phi\|_{W^{1,2}(M, \mathbb{R})}^2 - c\|\phi\|_{L^q(M, \mathbb{R})}^2 \\
\geq \frac{n-1}{n-2} \|\phi\|_{W^{1,2}(M, \mathbb{R})}^2 - c\Lambda_\epsilon\|\phi\|_{L^2(M, \mathbb{R})}^2.
\]
However, from the definition of the Yamabe invariant, we have
\[
\int_M \left( \frac{2(n-1)}{n-2} |d\phi|^2 + \frac{\text{Scal} + \epsilon_k}{2} \phi^2 \right) d\mu^g \geq 0
\]
for all functions $\phi \in W^{1,2}(M, \mathbb{R})$. As a consequence,
\[
\int_M \left( \frac{2(n-1)}{n-2} |d\phi|^2 + \frac{\text{Scal} + \epsilon_k}{2} \phi^2 \right) d\mu^g \geq \frac{\epsilon_k}{2}\|\phi\|_{L^2(M, \mathbb{R})}^2.
\]
Finally, combining both estimates, we obtain
\[
\left( 1 + \frac{2c\Lambda_\epsilon}{\epsilon_k} \right) \int_M \left( \frac{2(n-1)}{n-2} |d\phi|^2 + \frac{\text{Scal} + \epsilon_k}{2} \phi^2 \right) d\mu^g \\
\geq \frac{n-1}{n-2} \|\phi\|_{W^{1,2}(M, \mathbb{R})}^2 - c\Lambda_\epsilon\|\phi\|_{L^2(M, \mathbb{R})}^2 + \frac{2c\Lambda_\epsilon}{\epsilon_k} \frac{\epsilon_k}{2}\|\phi\|_{L^2(M, \mathbb{R})}^2 \\
\geq \frac{n-1}{n-2} \|\phi\|_{W^{1,2}(M, \mathbb{R})}^2.
\]
This proves that for all $\phi \in C_k$, we have $I_k(\phi) \geq \mu_k \|\phi\|_{W^{1,2}(M, \mathbb{R})}^2$ with
\[
\mu_k = \frac{n - 1}{n - 2} \left( 1 + \frac{2c\Lambda_k}{\epsilon_k} \right)^{-1}.
\]
In particular, any minimizing sequence $(\phi_i)$ for $I_k$ is bounded in $W^{1,2}(M, \mathbb{R})$ since the norm of $\phi_i$ eventually becomes less than $\mu_k^{-1} I_k(1)$. It is then a standard fact that there exists a minimizer $\phi_k$ for $I_k$ in $C_k$, and since $I_k$ is strictly convex, $\phi_k$ is unique.

At this point, we remark that for any $\phi \in C_k$, $I_k(|\phi|) \leq I_k(\phi)$, so $\phi_k \geq 0$. It should be noted that $C_k$ has empty interior in $W^{1,2}(M, \mathbb{R})$ so it makes no sense to speak of the (Gâteau) differential of $I_k$. However, if $f$ is a smooth (more generally, if $f \in W^{1,2}(M, \mathbb{R}) \cap L^\infty(M, \mathbb{R})$), we can define the directional derivative of $I_k$ in the direction $f$. This is sufficient to conclude that $\phi_k$ is a weak solution to
\[
-\frac{4(n - 1)}{n - 2} \Delta \phi + (\text{Scal} + \epsilon_k)\phi = -\frac{n - 1}{n} \tau^2_k \phi^{N - 1} + A^2(\phi + \epsilon_k)^{N - 1}. \tag{2.3}
\]
Note that the right-hand side of this equation belongs to $L^{p/2}(M, \mathbb{R})$ so, by elliptic regularity, we have $\phi_k \in W^{2,p/2}(M, \mathbb{R})$ and from Harnack’s inequality (see, e.g., [21]), $\phi_k > 0$.

We now let $k$ tend to infinity. We first prove that the functions $\phi_k$ are uniformly bounded from below by constructing suitable subsolutions. Let $u \in W^{2,p/2}(M, \mathbb{R})$ denote the solution to the following equation
\[
-\frac{4(n - 1)}{n - 2} \Delta u + \text{Scal} u + \frac{n - 1}{n} \tau^2 u = A^2.
\]
$u$ can be obtained by minimizing the functional
\[
J(u) := \int_M \left[ \frac{2(n - 1)}{n - 2} |du|^2 + \left( \frac{\text{Scal}}{2} + \frac{n - 1}{2n} \tau^2_0 \right) u^2 - A^2 u \right] d\mu^g,
\]
and, as before $J(|u|) \leq J(u)$ so $u \geq 0$ and $u > 0$ by Harnack’s inequality. (Note that we overcame the ill-definiteness of the $\tau$-term by changing the exponent.) Let $u_k$ denote the solution to
\[
-\frac{4(n - 1)}{n - 2} \Delta u_k + (\text{Scal} + \epsilon_k) u_k + \frac{n - 1}{n} \tau^2 u_k = A^2.
\]
We let the reader convince himself that $u_k \in W^{2,p/2}(M, \mathbb{R})$, $u_k > 0$ and $u_k \to u$ in $W^{2,p/2}(M, \mathbb{R})$ as $k$ tends to infinity. Since $W^{2,p/2}(M, \mathbb{R})$ embeds continuously in $L^\infty(M, \mathbb{R})$, there exist constants $c_-, c_+ > 0$ such that
\[
c_- \leq u_k \leq c_+
\]
for all $k$. We now look for $\lambda_+ > 0$ so that $\phi_{k,-} = \lambda_- u_k$ is a subsolution to Eq. (2.3). We want
\[
-\frac{4(n - 1)}{n - 2} \Delta \phi_{k,-} + (\text{Scal} + \epsilon_k) \phi_{k,-} \leq -\frac{n - 1}{n} \tau^2_k \phi_{k,-}^{N - 1} + A^2(\phi_{k,-} + \epsilon_k)^{-N - 1}. \tag{2.4}
\]
Equivalently,
\[
\lambda_- \left( -\frac{4(n-1)}{n-2} \Delta u_k + (\text{Scal} + \epsilon_k) u_k \right) + \frac{n-1}{n} \tau_k^2 \lambda_-^{N-1} u_k^{N-1} \leq A^2 (\lambda_- u_k + \epsilon_k)^{-N+1}
\]
which can be rewritten as follows:
\[
\frac{n-1}{n} \tau_k^2 \lambda_-^{N-1} u_k^{N-1} - \frac{n-1}{n} \tau^2 \lambda_- u_k \leq A^2 (\lambda_- u_k + \epsilon_k)^{-N+1} - \lambda_- A^2.
\]
Since \( \tau_k \leq \tau \), the left-hand side is non-positive if \( \lambda_-^{N-1} u_k^{N-1} \leq \lambda_- u_k \), i.e., \( \lambda_- \leq (c_+)^{-1} \). On the other hand, the right-hand side is nonnegative if
\[
\lambda_- (\lambda_- u_k + \epsilon_k)^{N+1} \leq 1
\]
Since \( \epsilon_k \leq 1 \), we see that the previous inequality holds when \( \lambda_- \leq (1 + c_+)^{-N-1} \). We have proven that if
\[
\lambda_- \leq \min\{(c_+)^{-1}, (1 + c_+)^{-N-1}\},
\]
\( \phi_{-,k} \) is a subsolution to Eq. (2.3). We now prove that \( \phi_k \geq \phi_{-,k} \). We compute the difference between (2.3) and (2.4), multiply it by \( (\phi_k - \phi_{-,k})_- = \min\{0, \phi_k - \phi_{-,k}\} \) and integrate over \( M \):
\[
\int_M \left( \frac{4(n-1)}{n-2} |d(\phi_k - \phi_{-,k})_-|^2_g + (\text{Scal} + \epsilon_k) |(\phi_k - \phi_{-,k})_-|^2_g \right) d\mu^g \\
\leq -\frac{n-1}{n} \int_M \tau_k^2 \left( \phi_k^{N-1} - \phi_{-,k}^{N-1} \right) (\phi_k - \phi_{-,k})_- d\mu^g \\
+ \int_M A^2 \left[ (\phi_k + \epsilon_k)^{-N-1} - (\phi_{-,k} + \epsilon_k)^{-N-1} \right] (\phi_k - \phi_{-,k})_- d\mu^g.
\]
The right-hand side is non-positive, while the left-hand side is nonnegative. This imposes that
\[
\mu_k \| (\phi_k - \phi_{-,k})_- \|_{W^{1,2}(M,\mathbb{R})} \\
\leq \int_M \left( \frac{4(n-1)}{n-2} |d(\phi_k - \phi_{-,k})_-|^2_g + (\text{Scal} + \epsilon_k) |(\phi_k - \phi_{-,k})_-|^2_g \right) d\mu^g = 0
\]
So \( (\phi_k - \phi_{-,k})_- \equiv 0 \), which means that \( \phi_k \geq \phi_{-,k} \geq \lambda_- c_- > 0 \). This ends the proof of the fact that the functions \( \phi_k \) are uniformly bounded from below. Let \( \phi_+ \in W^{2,p/2}(M,\mathbb{R}) \) denote the positive solution to
\[
-\frac{4(n-1)}{n-2} \Delta \phi_+ + \text{Scal} \phi_+ + \frac{n-1}{n} \tau_1^2 \phi_+^{N-1} = \frac{A^2}{(\lambda_- c_-)^{N+1}}.
\]
(We remind the reader that \( \tau_1 = \min\{\tau, 1\} \).) By similar arguments, we can prove that \( \phi_k \leq \phi_+ \) so the sequence of functions \( \phi_k \) is uniformly bounded from above and from below:
\[
\lambda_- c_- \leq \phi_k \leq \max \phi_+.
\]
We rewrite Eq. (2.3) as
\[
-\frac{4(n-1)}{n-2} \Delta \phi_k + \phi_k = (1 - \text{Scal} + \epsilon_k) \phi_k - \frac{n-1}{n} \tau_k^2 \phi_k^{N-1} + A^2 (\phi_k + \epsilon_k)^{-N+1},
\]
(2.5)
and notice that the right-hand side is bounded in $L^{p/2}(M, \mathbb{R})$, so, by elliptic regularity, $(\phi_k)_k$ is uniformly bounded in $W^{2,p/2}(M, \mathbb{R})$. Since $W^{2,p/2}(M, \mathbb{R})$ compactly embeds in $L^\infty(M, \mathbb{R})$, we can assume that $\phi_k$ converges strongly to some $\phi_\infty$ in $L^\infty(M, \mathbb{R})$ and, from (2.5), $\phi_\infty \in W^{2,p/2}(M, \mathbb{R})$ solves the Lichnerowicz equation (2.1).

Now, the functional $I$ introduced in (2.2) makes perfect sense on the (open) subset $\Omega_+ = \{ \phi \in W^{2,p/2}(M, \mathbb{R}), \phi > 0 \}$. It is differentiable and strictly convex. Furthermore, $\phi_\infty$ is a critical point for $I$ on $\Omega_+$. So it must be the unique minimum of $I$ on $\Omega_+$. Since a strictly convex functional can only have a single critical point and critical points of $I$ are exactly the solutions of (2.1), we conclude that $\phi_\infty$ is the unique solution to (2.1).

Continuity of $\phi_\infty$ with respect to $A$ follows from the implicit function theorem in a way that is similar to the one presented in the next section so we omit the proof of it. \hfill \Box

**Proposition 2.2.** Let $g \in W^{2,p/2}(M, S_2 M)$, $\eta \in W^{1,p}(M, \mathbb{R})$, $\eta > 0$ and $\xi \in L^q(M, TM)$ be given, for some $q \in (1, p/2)$. Assume that $g$ has no conformal Killing vector field, i.e., no non-trivial vector field $V$ such that $\mathbb{L}_g V = 0$. There exists a unique $W \in W^{2,q}(M, TM)$ solving

$$\Delta_{L_g, \eta} W = \xi. \quad (2.6)$$

Further, the mapping $\xi \mapsto W$ is continuous.

**Proof.** As in the proof of the previous proposition, we introduce the functional

$$J(W) := \frac{1}{2} \int_M \frac{|\mathbb{L}_g W|^2}{2\eta} |g d\mu^g - \int_M \langle W, \xi \rangle_g d\mu^g$$

for any $W \in W^{1,2}(M, TM)$. Since in any coordinate system

$$\nabla_i W_j = \partial_i W_j - \Gamma^k_{ij} W_k,$$

where $\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}) \in W^{1,p/2} \subset L^n$ denotes the Christoffel symbol of $g$, we have that, for any $W \in W^{1,2}(M, TM)$, $\Gamma^k_{ij} W_k \in L^2(M, \mathbb{R})$ as a sum of products of functions in $L^n$ and in $L^N(M, \mathbb{R})$. Hence, $\nabla W \in L^2(M, T^{\otimes 2} M)$ and, in particular, $\mathbb{L}_g W \in L^2(M, S_2 M)$.

We claim that the quadratic part of $J$ is coercive on $W^{1,2}(M, TM)$. Indeed, there exists a constant $\Lambda > 0$ so that $2\eta \leq \Lambda$. It follows from the Bochner formula for $\mathbb{L}_g$ that

$$\frac{1}{2} \int_M \frac{|\mathbb{L}_g W|^2}{2\eta} |g d\mu^g \geq \frac{1}{2\Lambda} \int_M |\mathbb{L}_g W|^2 |g d\mu^g \geq \frac{1}{\Lambda} \int_M \left[ |\nabla W|^2 g + \left( 1 - \frac{2}{n} \right) (\text{div } W)^2 - \text{Ric}(W, W) \right] d\mu^g \geq \frac{1}{\Lambda} \left[ \int_M |\nabla W|^2 |g d\mu^g - \| \text{Ric} \|_{L^{p/2}(M, S_2 M)} \| W \|_{L^q(M, TM)}^2 \right],$$

where $\text{Ric}$ is the Ricci curvature.
where \( q = \frac{2p}{p-2} < N \). Now, assume that for all \( k \in \mathbb{N}_+ \), there exists a nonzero \( W_k \in W^{1,2}(M, TM) \) such that

\[
\frac{1}{2} \int_M \frac{|\nabla_g W|^2}{2\eta} d\mu^g \leq \frac{1}{k} \|W_k\|^2_{L^q(M, TM)}.
\]

Without loss of generality, we can assume that \( \|W_k\|_{L^q(M, TM)} = 1 \). Note that, due to the Sobolev embedding, the norm

\[
\|W\|^2 := \int_M |\nabla W|^2 g d\mu^g + \|W\|^2_{L^q(M, TM)}
\]

is equivalent to the usual Sobolev norm on \( W^{1,2}(M, TM) \), so for some constant \( \delta > 0 \), we have

\[
\|W\|^2_{W^{1,2}(M, TM)} \leq \delta\|W\|^2
\]

for all \( W \in W^{1,2}(M, TM) \). Hence, from the previous estimates

\[
\frac{1}{\Lambda} \left[ \int_M |\nabla W_k|^2_g d\mu^g - \|\text{Ric}\|_{L^{p/2}(M,S_2 M)} \|W_k\|^2_{L^q(M, TM)} \right] \\
\leq \frac{\delta}{k} \left( \int_M |\nabla W_k|^2_g d\mu^g + \|W_k\|^2_{L^q(M, TM)} \right).
\]

Equivalently, using \( \|W_k\|_{L^q(M, TM)} = 1 \),

\[
\left( \frac{1}{\Lambda} - \frac{\delta}{k} \right) \int_M |\nabla W_k|^2_g d\mu^g \leq \frac{\|\text{Ric}\|_{L^{p/2}(M,S_2 M)}}{\Lambda} + \frac{\delta}{k}.
\]

It follows that \((W_k)\) is bounded in \( W^{1,2}(M, TM) \). Since \( W^{1,2}(M, TM) \) compactly embeds into \( L^q(M, TM) \), we can assume that \( W_k \) converges to \( W_\infty \) strongly in \( L^q(M, TM) \) and weakly in \( W^{1,2}(M, TM) \). We have \( \|W_\infty\|_{L^q(M, TM)} = 1 \) so \( W_\infty \) is nonzero but

\[
\int_M \frac{|\nabla_g W_\infty|^2}{2\eta} d\mu^g = \lim_{k \to \infty} \int_M \frac{\langle \nabla_g W_\infty, \nabla_g W_k \rangle_2}{2\eta} d\mu^g \\
\leq \left( \int_M \frac{|\nabla_g W_\infty|^2}{2\eta} d\mu^g \right)^{1/2} \left( \lim_{k \to \infty} \int_M \frac{|\nabla_g W_k|^2}{2\eta} d\mu^g \right)^{1/2} = 0.
\]

Namely, we obtained a contradiction.

It then follows from the Lax–Milgram theorem that the functional \( J \) admits a unique minimizer \( W \in W^{1,2}(M, TM) \) which is then a weak solution to (2.6). Elliptic regularity then implies that \( W \in W^{2,q}(M, TM) \). \( \Box \)

### 3. An Implicit Function Argument

In this section, we make the following regularity assumptions:

- \( g \in W^{2,p/2}(M, S_2 M) \),
- \( \tau \in W^{1,p/2}(M, \mathbb{R}) \),
- \( \sigma \in L^p(M, S_2 M) \),
- \( \eta \in W^{1,p}(M, \mathbb{R}) \), \( \eta > 0 \)
for some \( p > n \). The idea in [6] is to introduce a parameter \( \lambda > 0 \) in the system (1.3). Namely, we set \( \phi = \lambda \tilde{\phi} \) and \( W = \lambda^N \tilde{W} \) so the system becomes

\[
- \frac{4(n-1)}{n-2} \Delta_g \tilde{\phi} + \text{Scal} \tilde{\phi} = -\frac{n-1}{n} \lambda^{N-2} \tau^2 \tilde{\phi}^{N-1} + \left| \lambda^{-\frac{N+2}{2}} \sigma + \frac{\lambda^{\frac{N-2}{2}}}{2\eta} \| \nabla g \tilde{W} \|_g \right|^2 \tilde{\phi}^{N-1}, \tag{3.1a}
\]

\[
\Delta_{Lg,\eta} \tilde{W} = \frac{n-1}{n} \tilde{\phi}^N \nabla \tau. \tag{3.1b}
\]

(Note that the rescaling we present here is different from the one in [6].) Setting

\[
\sigma = \lambda^{\frac{N+2}{2}} \tilde{\sigma}, \tag{3.2}
\]

the Lichnerowicz equation (3.1a) reads

\[
- \frac{4(n-1)}{n-2} \Delta_g \tilde{\phi} + \text{Scal} \tilde{\phi} = -\frac{n-1}{n} \lambda^{N-2} \tau^2 \tilde{\phi}^{N-1} + \left| \tilde{\sigma} + \frac{\lambda^{\frac{N-2}{2}}}{2\eta} \| \nabla g \tilde{W} \|_g \right|^2 \tilde{\phi}^{N-1}. \tag{3.3a}
\]

Letting \( \lambda \) go to zero, we see that the system (3.1) is a perturbation of

\[
- \frac{4(n-1)}{n-2} \Delta_g \phi_0 + \text{Scal} \phi_0 = |\tilde{\sigma}|^2 \phi_0^{N-1}, \tag{3.3a}
\]

\[
\Delta_{Lg,\eta} \tilde{W}_0 = \frac{n-1}{n} \phi_0^N \nabla \tau. \tag{3.3b}
\]

So \( \tilde{W} \) has disappeared from Eq. (3.3a). Solving Eq. (3.3a) requires that the Yamabe invariant of \((M, g)\) be positive since the metric \( g = \phi_0^{N-2} g \) has scalar curvature \( \text{Scal}_0 = |\tilde{\sigma}|^2 \phi_0^{-2N} \), which is nonnegative and nonzero. This explains why the method was limited to \( Y_g > 0 \).

As we indicated before, in the case \( Y_g = 0 \), the conformal Laplacian

\[
u \mapsto - \frac{4(n-1)}{n-2} \Delta_g u + \text{Scal} u
\]

has a 1-dimensional kernel generated by a positive function \( \phi_0 \in W^{2,p/2}(M, \mathbb{R}) \) which we normalize so that

\[
\int_M \phi_0^2 d\mu^g = 1.
\]

Since the conformal Laplacian is Fredholm with index zero and formally self-adjoint, the equation

\[
- \frac{4(n-1)}{n-2} \Delta_g u + \text{Scal} u = f
\]

with \( f \in L^{p/2}(M, \mathbb{R}) \) is solvable iff

\[
\int_M f \phi_0 d\mu^g = 0.
\]
The solution $u \in W^{2,p/2}(M, \mathbb{R})$ is unique up to the addition of a constant multiple of $\phi_0$ so it is unique if we impose further that

$$\int_M u \phi_0 d\mu_g = 0.$$ 

If we change the scaling law (3.2) of $\sigma$ to

$$\sigma = \lambda N \tilde{\sigma}, \quad (3.4)$$

the system (1.3) reads now

$$\begin{cases}
-4(n-1) \frac{n-2}{n-2} \Delta_g \tilde{\phi} + \text{Scal} \tilde{\phi} = \lambda^{N-2} \left( -\frac{n-1}{n} \tau^2 \tilde{\phi}^{N-1} + \left| \tilde{\sigma} + \frac{1}{2\eta} L_g \tilde{W} \right|^2_g \tilde{\phi}^{-N-1} \right), \\
\Delta_{L_g, \eta} \tilde{W} = \frac{n-1}{n} \tilde{\phi}^N \nabla \tau.
\end{cases} \quad (3.7)$$

Hence, setting

$$\tilde{\phi} := c_\lambda \phi_0 + \lambda^{N-2} \psi_\lambda, \quad (3.5)$$

where $\psi_\lambda$ belongs to the space $\tilde{W}^{2,p/2}(M, \mathbb{R})$ of $W^{2,p}(M, \mathbb{R})$-functions orthogonal to $\phi_0$ for the $L^2$-product:

$$\tilde{W}^{2,p/2}(M, \mathbb{R}) := \left\{ u \in W^{2,p/2}(M, \mathbb{R}), \int_M u \phi_0 d\mu_g = 0 \right\} \quad (3.6)$$

(More generally, for any function space $F$ such that $F \hookrightarrow L^2(M, \mathbb{R})$, we will denote by $\tilde{F}$ the set of functions $u$ belonging to $F$ that are $L^2$-orthogonal to $\phi_0$.) we finally arrive at

$$\begin{cases}
-4(n-1) \frac{n-2}{n-2} \Delta_g \psi_\lambda + \text{Scal} \psi_\lambda = \left( -\frac{n-1}{n} \tau^2 \tilde{\phi}^{N-1} + \left| \tilde{\sigma} + \frac{1}{2\eta} L_g \tilde{W} \right|^2_g \tilde{\phi}^{-N-1} \right), \\
\Delta_{L_g, \eta} \tilde{W} = \frac{n-1}{n} \tilde{\phi}^N \nabla \tau.
\end{cases} \quad (3.7a)$$

$$n-1 \int_M \tau^2 \phi_0 \tilde{\phi}^{N-1} d\mu_g = \int_M \phi_0 \left| \tilde{\sigma} + \frac{1}{2\eta} L_g \tilde{W} \right|^2_g \tilde{\phi}^{-N-1} d\mu_g, \quad (3.7b)$$

$$\Delta_{L_g, \eta} \tilde{W} = \frac{n-1}{n} \tilde{\phi}^N \nabla \tau. \quad (3.7c)$$
3.1. The Limit $\lambda = 0$

In the limit $\lambda = 0$, we have $\tilde{\phi} = c_0 \phi_0$ so the system (3.7) becomes

$$-\frac{4(n-1)}{n-2} \Delta_g \psi_0 + \text{Scal} \psi_0 = -\frac{n-1}{n} \tau^2 \tilde{\phi}^{-N-1} + \left| \tilde{\sigma} + \frac{\|g \tilde{W}\|}{2 \eta} \right|^2_g \tilde{\phi}^{-N-1}, \quad (3.8a)$$

$$\frac{n-1}{n} c_0^2 \int_M \tau^2 \phi_0^N d\mu^g = \int_M \phi_0^{-N} \left| \tilde{\sigma} + \frac{\|g \tilde{W}\|}{2 \eta} \right|^2_g d\mu^g, \quad (3.8b)$$

$$\Delta_{L_g, \eta} \tilde{W}_0 = \frac{n-1}{n} c_0^N \phi_0^N \nabla \tau. \quad (3.8c)$$

We solve this system from bottom to top. Namely, from Proposition 2.2, there exists a unique solution $W \in W^{2,p/2}(M,TM)$ to

$$\Delta_{L_g, \eta} W = \frac{n-1}{n} c_0^N \phi_0^N \nabla \tau. \quad (3.9)$$

so we have $\tilde{W}_0 = c_0^N W$. Inserting it into Equation (3.8b), we find

$$\frac{n-1}{n} c_0^2 \int_M \tau^2 \phi_0^N d\mu^g = \int_M \left( |\tilde{\sigma}|^2 + \frac{1}{\eta} c_0^N \langle \tilde{\sigma}, L_g W \rangle + c_0^2 \frac{\|g \tilde{W}\|^2}{4 \eta^2} \right) \phi_0^{-N} d\mu^g. \quad (3.10)$$

This is a second-order equation in $c_0^N$ which we have to assume has a positive solution. From Descartes’ rule of signs (see, e.g., [22]), Eq. (3.10) has a unique positive solution provided that

$$\frac{n-1}{n} \int_M \tau^2 \phi_0^N d\mu^g > \int_M \frac{\|g \tilde{W}\|^2}{4 \eta^2} \phi_0^{-N} d\mu^g. \quad (3.11)$$

Note, however, that there might exist situations in which Equation (3.10) has two positive solutions. We plan to investigate this question later.

Having fulfilled the last two equations, we can finally solve Eq. (3.8a) for $\psi_0 \in \tilde{W}^{2,p/2}(M, \mathbb{R})$. We have thus proven

**Proposition 3.1.** Under the assumption (3.11), there exists a unique solution $(c_0, \psi_0, \tilde{W}_0) \in \mathbb{R}_+ \times \tilde{W}^{2,p/2}(M, \mathbb{R}) \times \tilde{W}^{2,p/2}(M,TM)$ to the system (3.8) where $\tilde{W}^{2,p/2}(M, \mathbb{R})$ is defined in (3.6).

3.2. Extending to $\lambda > 0$

Let $\Omega \subset \mathbb{R} \times \mathbb{R} \times \tilde{W}^{2,p/2}$ be the following open subset:

$$\Omega := \{(\lambda, c, \psi) \in \mathbb{R} \times \mathbb{R} \times \tilde{W}^{2,p/2}, \text{ s.t. } c \phi_0 + \lambda^{N-2} \psi > 0\}$$

We define the operator

$$\Phi : \Omega \times \tilde{W}^{2,p/2}(M,TM) \to \tilde{L}^2 \times \mathbb{R} \times L^p(M,TM)$$
as follows:

\[
\Phi_\lambda(c, \psi, \tilde{W}) := \left( \begin{array}{c}
\Pi \left( -\frac{4(n-1)}{n-2} \Delta_g \psi + \text{Scal} \psi + \frac{n-1}{n} \tau^2 \hat{\phi}^N - 1 - \left| \tilde{\sigma} + \frac{L_g \hat{W}}{2\eta} \right|^2 \hat{\phi}^N - 1 \right) \\
\frac{n-1}{n} \int_M \tau^2 \phi_0 \hat{\phi}^N - 1 d\mu^g - \int_M \phi_0 \tilde{\sigma} + \frac{L_g \hat{W}}{2\eta} \right|^2 \hat{\phi}^N - 1 d\mu^g \\
\Delta_{L_g, \eta} \tilde{W} - \frac{n-1}{n} \hat{\phi}^N \nabla_\tau 
\end{array} \right),
\]

(3.12)

where we used \( \hat{\phi} = c\phi_0 + \lambda N^{-2} \psi \) as a shorthand (see (3.5)) and where \( \Pi \) denotes the \( L^2 \)-orthogonal projection onto \( \hat{L}^2(M, \mathbb{R}) \):

\[
\Pi(f) = f - \left( \int_M f \phi_0 d\mu^g \right) \phi_0.
\]

Solving the system (3.7) is then equivalent to finding solutions to

\[
\Phi_\lambda(c, \psi, \tilde{W}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

It is routine to check that \( \Phi \) is well defined and \( C^1 \). To apply the implicit function theorem, we only need to check that the differential of \( \Phi_\lambda \) with \( \lambda = 0 \) kept fixed is invertible at the point \((c_0, \psi_0, \tilde{W}_0)\). Since \( \hat{\phi} = c\phi_0 \) when \( \lambda = 0 \), \( \Phi_0 \) reads

\[
\Phi_0(c, \psi, \tilde{W}) = \left( \begin{array}{c}
\Pi \left( -\frac{4(n-1)}{n-2} \Delta_g \psi + \text{Scal} \psi + \frac{n-1}{n} cN^{-1} \tau^2 \phi_0^{-1} - 1 - \left| \tilde{\sigma} + \frac{L_g \hat{W}}{2\eta} \right|^2 c^{-N} \phi_0^{-1} - 1 \right) \\
\frac{n-1}{n} cN^{-1} \int_M \tau^2 \phi_0^{-1} d\mu^g - cN^{-1} \int_M \tilde{\sigma} + \frac{L_g \hat{W}}{2\eta} \right|^2 c^{-N} d\mu^g \\
\Delta_{L_g, \eta} \tilde{W} - \frac{n-1}{n} cN \phi_0^{-1} \nabla_\tau 
\end{array} \right),
\]

Its differential at \((c_0, \psi_0, \tilde{W}_0)\) can be computed:

\[
D\Phi_0(c_0, \psi_0, W) \begin{pmatrix} \psi' \\ c' \\ W' \end{pmatrix} = \begin{pmatrix}
-\frac{4(n-1)}{n-2} \Delta_g + \text{Scal} & \Pi(F) & \Pi(\ell(\cdot)) \\
0 & 0 & -N \frac{n-1}{n} cN^{-1} \phi_0^{-1} \nabla_\tau \\
0 & \int_M \phi_0 F d\mu^g & \int_M \phi_0 \ell(\cdot) d\mu^g
\end{pmatrix} \begin{pmatrix} \psi' \\ c' \\ W' \end{pmatrix}
\]

(3.13)

where we used the following notations:

\[
F := \frac{n-1}{n} (N-1)cN^{-2} \tau^2 \phi_0^{-1} + (N+1) \left| \tilde{\sigma} + \frac{L_g \hat{W}}{2\eta} \right|^2 \phi_0^{-N-1} c^{-N-2},
\]

\[
\ell(W') := -2c^{-N} \phi_0^{-N} \left( \tilde{\sigma} + \frac{L_g \hat{W}}{2\eta}, \frac{L_g \hat{W}'}{2\eta} \right)_g.
\]

The matrix of the differential in (3.13) is not upper triangular as is the case in [6]. However, the conformal Laplacian appearing in the upper left corner of the matrix is an isomorphism from \( \hat{W}^{2, p/2}(M, \mathbb{R}) \) to \( \hat{L}^{p/2}(M, \mathbb{R}) \) so it suffices
to check that the lower $2 \times 2$ block is invertible. We show that for any $d \in \mathbb{R}$ and any $V \in L^p(M, TM)$ there exists a unique solution to the system
\begin{align}
\begin{cases}
d = c' \int_M \phi_0 F d\mu^g + \int_M \phi_0 \ell(W') d\mu^g, \\
V = -c' N \frac{n-1}{n} c_0^{-1} \phi_0^N \nabla \tau + \Delta_{L_g, \eta} W'.
\end{cases}
\end{align}
(3.14)

The second equation can be solved explicitly for $W'$:
\[ W' = N c_0^{-1} \phi_0^N + \frac{1}{\Delta_{L_g, \eta} N^{-1} V}, \]
where $(\Delta_{L_g, \eta})^{-1} : L^{p/2}(M, TM) \to W^{2, p/2}(M, TM)$ denotes the inverse of $\Delta_{L_g, \eta}$. Injecting into the first equation, we get
\[ c' \int_M \phi_0 F d\mu^g - 2 c_0^{-1} \int_M \phi_0^{-1} \left( \bar{\sigma} + \frac{L_g \bar{W}}{2\eta} \right) \phi_0^{-N} \frac{1}{2\eta} L_g \left( N c_0^{-1} \phi_0^N + (\Delta_{L_g, \eta})^{-1} V \right) g \neq \int_M \left( \bar{\sigma}, \frac{L_g \bar{W}}{2\eta} \right)_g \phi_0^{-N} d\mu^g. \]

This equation can be solved for $c'$ if and only if
\[ \left( \frac{n-1}{n} \int_M \tau^2 \phi_0^N d\mu^g - \int_M \left| \frac{L_g W}{2\eta} \right|_g^2 \phi_0^{-N} d\mu^g \right) c_0^N \neq \int_M \left( \bar{\sigma}, \frac{L_g \bar{W}}{2\eta} \right)_g \phi_0^{-N} d\mu^g, \]
where we used Eq. (3.10). This condition is to be expected, it means that $c_0^N$ is not a double root of Eq. (3.10) seen as a second-order equation in $c_0^N$. We have thus proven the following proposition:

**Proposition 3.2.** Under the assumption (3.11), there exist $\lambda_0 > 0$ and a continuous curve of solutions $(c_\lambda, \psi_\lambda, \bar{W}_\lambda)$ to the system (3.7).

Using now the rescaling presented at the beginning of the section, we obtain the first theorem of this paper:

**Theorem 3.3.** Given $(M, g, \tau, \eta)$ and $\bar{\sigma} \in L^p(M, S_2 M)$, $\bar{\sigma} \neq 0$, where the regularities are indicated at the beginning of the section, and assuming further that
\[ \frac{n-1}{n} \int_M \tau^2 \phi_0^N d\mu^g > \int_M \left| \frac{L_g W}{4\eta^2} \phi_0^{-N} d\mu^g, \right. \]
there exists a $\lambda_0 > 0$ such that the system (1.3) with $\sigma \equiv \lambda \bar{\sigma}$ has at least a solution $(\phi, W) \in W^{2, p/2}(M, \mathbb{R}) \times W^{2, p/2}(M, TM)$ for all $\lambda \in (0, \lambda_0)$.

### 4. A Small TT-Tensor Argument

In this section, we will require stronger regularity for $\tau$ than in the previous section. Namely, we assume that, for some $p > n$,

- $g \in W^{2, p/2}(M, S_2 M)$,
- $\eta \in W^{1, p}(M, \mathbb{R})$, $\eta > 0$,
- $\sigma \in L^p(M, S_2 M)$,
• $\tau \in W^{1,t}(M, \mathbb{R})$, where $t > t_0$ with

$$t_0 = \frac{2n(n - 1)}{3n - 2}. \quad (4.1)$$

The reason why we have to impose stronger regularity for $\tau$ will become apparent in the course of the proof. We also take advantage of the fact that the CTS method is conformally covariant (see [17]) to enforce the condition $\text{Scal} \equiv 0$. In particular, $\phi_0 \equiv 1$. We will also assume that $(M, g)$ has volume one:

$$\text{Vol}(M) = 1.$$  

So, if $p \leq q$,

$$\|f\|_{L^p(M, \mathbb{R})} \leq \|f\|_{L^q(M, \mathbb{R})}$$

for any measurable function. This can be achieved by rescaling the metric by some (constant) factor.

We prove an analog of the result in [7,19], namely the existence of a solution to the system (1.3) when $\sigma$ is small in $L^2(M, \mathbb{R})$ under the assumption that Inequality (3.11) is fulfilled, i.e., that

$$\frac{n - 1}{n} \int_M \tau^2 d\mu^g > \int_M \frac{|\nabla_g W|^2}{4\eta^2} g d\mu^g. \quad (4.2)$$

To keep expressions short, we adopt at some points a probabilistic notation and denote

$$\mathbb{E}[f] := \int_M f d\mu^g$$

and

$$\mathbb{E}_\tau[f] := \frac{1}{\mathbb{E}[\tau^2]} \mathbb{E}[\tau^2 f] = \frac{1}{\int_M \tau^2 d\mu^g} \int_M \tau^2 f d\mu^g$$

for any function $f$ for which this makes sense (e.g., $f \in L^2(\tau^2)$). The strategy is similar to the previous ones in [7,10,11,15,19]. The previous section suggests that we have to decompose $\phi$ as $\phi = c + \hat{\phi}$ where $c$ is a constant and $\hat{\phi}$ has zero average. Yet, for technical reasons, it appears more useful not to decompose $\phi$ itself but $\phi^N$ and to do it in a way that involves $\tau^2$. For any function space $X$, we set

$$\tilde{X} := \{ f \in X, \mathbb{E}_\tau[f] = 0 \}.$$  

Let $p_0$ be defined as follows:

$$\frac{1}{p_0} = \frac{2}{p} - \frac{1}{t}. \quad (4.3)$$

Given $c_{\text{max}} > 0$ and $r > 0$ to be chosen later, we let $C_0 \subset L^{p_0}(M, \mathbb{R})$ be the following subset

$$C_0 := \{ u \in L^{p_0}(M, \mathbb{R}), \ u \geq 0, \mathbb{E}_\tau[u] \leq c_{\text{max}} \text{ and } \|u - \mathbb{E}_\tau[u]\|_{L^{\frac{N+1}{N}}(M, \mathbb{R})} \leq r \}. \quad (4.4)$$

The reason why we work in the Lebesgue $L^{\frac{N}{N}+1}$-norm will become apparent later. We construct a mapping $\Psi : C_0 \to L^{p_0}(M, \mathbb{R})$ as follows:
Given $u = c + \psi \in C_0$ ($c \in \mathbb{R}$ and $\psi \in \tilde{L}^{p_0}(M, \mathbb{R})$), we let $W \in W^{2, p/2}(M, TM)$ be the unique solution to the following equation

$$\Delta_{L_g, \eta} W = \frac{n-1}{n} u \nabla \tau,$$

see Proposition 2.2. Note that $p_0$ is chosen so that the right-hand side belongs to $L^{p/2}(M, TM)$. So, according to the notation introduced in (3.9), we have

$$W = cW + W_\psi,$$

where $W_\psi$ denotes the solution to (4.5) with $u$ replaced by $\psi$.

We next solve the Lichnerowicz equation for $\phi > 0$ with the $W$ we found in the first step. The solution $\phi \in W^{2, p/2}(M, \mathbb{R})$ is known to exist from Proposition 2.1.

Finally, we set $\Psi(u) = \phi^N = c' + \psi'$ where $c' \in \mathbb{R}$ and $\psi' \in \tilde{L}^{p_0}(M, \mathbb{R})$. $\Psi$ is the composition of three continuous mappings and, hence, continuous, and its fixed points correspond to solutions of the system (1.3). Our first aim is to show that if $\|\sigma\|_{L^2(M, S^2_M)}$ is small enough, we can adjust $c_{\text{max}}$ and $r$ so that $\Psi(C_0) \subset C_0$.

To estimate $\phi$, we multiply the Lichnerowicz equation (1.3a) by $\phi^N + 1$ and integrate over $M$:

$$- \frac{4(n-1)}{n-2} \int_M \phi^{N+1} \Delta \phi d\mu^g + \frac{n-1}{n} \int_M \tau^2 \phi^{2N} d\mu^g = \int_M \sigma + \frac{\|L_g \psi\|_{L^2}}{2\eta} \left|\frac{d}{g}\right|^2 d\mu^g.$$  

Integrating by parts the first term, we have

$$- \frac{4(n-1)}{n-2} \int_M \phi^{N+1} \Delta \phi d\mu^g = \frac{4(n-1)}{n-2} \int_M \langle d\phi^{N+1}, d\phi \rangle_g d\mu^g$$

$$= \frac{4(n-1)}{n-2} \frac{(N+1)}{2} \int_M \langle \phi^{N/2} d\phi, \phi^{N/2} d\phi \rangle_g d\mu^g$$

$$= \frac{4(n-1)}{n-2} \frac{(N+1)}{2} \int_M \left|d\left(\phi^{N/2} + 1\right)\right|^2_g d\mu^g$$

$$= \frac{3n-2}{n-1} \int_M \left|d\left(\phi^{N/2} + 1\right)\right|^2_g d\mu^g.$$  

So the estimate for $\phi$ reads

$$\frac{3n-2}{n-1} \int_M \left|d\left(\phi^{N/2} + 1\right)\right|^2_g d\mu^g + \frac{n-1}{n} \int_M \tau^2 \phi^{2N} d\mu^g = \int_M \sigma + \frac{\|L_g \psi\|_{L^2}}{2\eta} \left|\frac{d}{g}\right|^2 d\mu^g.$$  

(4.6)

We prove the following variant of the Sobolev inequality:

**Lemma 4.1.** There exists a constant $s = s(M, g, \tau) > 0$ such that for any function $f \in W^{1, 2}(M, \mathbb{R})$, we have

$$\|f - \mathbb{E}_r[f]\|_{L^\infty(M, \mathbb{R})}^2 \leq s \|df\|_{L^2(M, \mathbb{R})}^2.$$  

(4.7)
Proof. We argue by contradiction and assume that there exists no constant $s > 0$ such that Inequality (4.7) holds for all $f$. There exists a sequence $(f_k)_{k \in \mathbb{N}}$ such that

$$\|f_k - E_{\tau}[f_k]\|_{L^N(M,\mathbb{R})}^2 \geq k \|df_k\|_{L^2(M,\mathbb{R})}^2$$

for all $k$. From the Sobolev embedding theorem, there exists $s_0 > 0$ such that

$$\|f_k - E_{\tau}[f_k]\|_{L^N(M,\mathbb{R})}^2 \leq s_0 \left( \|df_k\|_{L^2(M,\mathbb{R})}^2 + \|f_k - E_{\tau}[f_k]\|_{L^2(M,\mathbb{R})}^2 \right).$$

As a consequence, we have that

$$\|f_k - E_{\tau}[f_k]\|_{L^2(M,\mathbb{R})}^2 \geq \frac{k - s_0}{s_0} \|df_k\|_{L^2(M,\mathbb{R})}^2$$

By replacing $f_k$ by $f_k - E_{\tau}[f_k]$, we can assume that $E_{\tau}[f_k] = 0$ for all $k$, and, by rescaling $f_k$, we impose that $\|f_k\|_{L^2(M,\mathbb{R})} = 1$ for all $k$. We finally also assume that there exists $f_\infty \in W^{1,2}(M,\mathbb{R})$ such that $(f_k)$ converges to $f_\infty$ weakly in $W^{1,2}(M,\mathbb{R})$ and strongly in $L^2(M,\mathbb{R})$. This comes from the fact that $W^{1,2}(M,\mathbb{R})$ is a Hilbert space (hence reflexive) and that the injection $W^{1,2}(M,\mathbb{R}) \hookrightarrow L^2(M,\mathbb{R})$ is compact. In particular, we have that, for any $u \in W^{1,2}(M,\mathbb{R})$,

$$\left| \int_M \langle du, df_k \rangle d\mu^g \right| \leq \|u\|_{W^{1,2}(M,\mathbb{R})} \|df_k\|_{L^2(M,\mathbb{R})} \leq \frac{s_0}{k - s_0} \|u\|_{W^{1,2}(M,\mathbb{R})} \rightarrow_{k \rightarrow \infty} 0.$$

Thus,

$$\int_M \langle du, df_\infty \rangle d\mu^g = 0$$

for all $u \in W^{1,2}(M,\mathbb{R})$. Namely, $f_\infty$ is a weak solution to $\Delta f_\infty = 0$ so $f_\infty$ is a constant. Since $\|f_k\|_{L^2(M,\mathbb{R})} = 1$ for all $k$ and $f_k \rightarrow f_\infty$ strongly in $L^2(M,\mathbb{R})$, we have $\|f_\infty\|_{L^2(M,\mathbb{R})} = 1$. So $f_\infty \equiv \pm 1$ is a nonzero constant function. Yet,

$$f_\infty = E_{\tau}[f_\infty] = \lim_{k \rightarrow \infty} E_{\tau}[f_k] = 0,$$

a contradiction. This proves the lemma. \hfill \Box

Applying eagerly the previous lemma to estimate (4.6), we get:

\begin{equation}
\frac{3n - 2}{n - 1} \|\phi^{\frac{n}{2} + 1} - E_{\tau}[\phi^{\frac{n}{2} + 1}]\|_{L^N(M,\mathbb{R})}^2 + \frac{n - 1}{n} \int_M \tau^2 \phi^{2N} d\mu^g \\
\leq \int_M \left[ \sigma + \frac{LgW}{2\eta} \right] g d\mu^g.
\end{equation}

But, as we indicated at the beginning of the section, we want to decompose $\phi^N$ not $\phi^{\frac{n}{2} + 1}$! So we need a second lemma:
Lemma 4.2. For any $\beta > 1$, any $\alpha \in (1, 2)$ and any positive function $f$, we have

$$\|f^\alpha - E_\tau[f^\alpha]\|_{L^\beta(M, \mathbb{R})} \leq \alpha E_\tau[f^\alpha]^{\frac{\alpha - 1}{\alpha}} \left[ 1 + \frac{\|\tau\|_{L^{2\gamma}(M, \mathbb{R})}^{\frac{2}{\alpha}}}{\|\tau\|_{L^2(M, \mathbb{R})}^{\frac{2}{\alpha}}} \|f - E_\tau[f]\|_{L^{\alpha\beta}(M, \mathbb{R})}^{\frac{\alpha - 1}{\alpha}} \right]$$

where $\gamma$ satisfies

$$\frac{1}{\beta} + \frac{1}{\gamma} = 1.$$

Proof. Before getting into the proof of the lemma, we state and prove the following inequality:

$$\forall x \in \mathbb{R}_+, \ |x^\alpha - 1| \leq \alpha |x - 1| + |x - 1|^\alpha. \quad (4.10)$$

First assume that $x \in (0, 1)$. Then, since $\alpha > 1$, we have,

$$|x^\alpha - 1| = 1 - x^\alpha = \alpha \int_x^1 y^{\alpha - 1} dy \leq \alpha \int_x^1 dy = \alpha (1 - x) \leq \alpha |x - 1| + |x - 1|^\alpha.$$

Next, for $x > 1$, the function

$$h : y \mapsto \frac{y^{\alpha - 1} - 1}{(y - 1)^{\alpha - 1}}$$

has derivative

$$h'(y) = \frac{\alpha - 1}{(y - 1)^\alpha} (1 - y^{\alpha - 2})$$

so, since $\alpha \in (1, 2)$, $h$ is increasing on the interval $(1, \infty)$ and tends to 1 at infinity. Thus, $h(y) \leq 1$ for all $y \in (1, \infty)$. This inequality can be rewritten

$$\alpha y^{\alpha - 1} \leq \alpha + \alpha (y - 1)^{\alpha - 1}.$$

Integrating from $y = 1$ to $y = x$, we obtain Inequality (4.10) for all $x > 1$. This concludes the proof of Inequality (4.10).

Assume now that $a, b \in \mathbb{R}_+$. We set $x = a/b$ in (4.10) and multiply it by $b^\alpha$. We obtain the homogeneous form of (4.10):

$$\forall a, b \in \mathbb{R}_+, \ |a^\alpha - b^\alpha| \leq \alpha \alpha^{-1} |a - b| + |a - b|^\alpha. \quad (4.11)$$

We now prove Inequality (4.9). We first compare $E_\tau[f^\alpha]$ with $E_\tau[f]^\alpha$. It follows from Jensen’s inequality that

$$E_\tau[f]^\alpha \leq E_\tau[f^\alpha].$$

For the opposite direction, we use Minkowski’s inequality:

$$E_\tau[f^\alpha]^{1/\alpha} = E_\tau[|f - E_\tau[f] + E_\tau[f]|_g^\alpha]^{1/\alpha} \leq E_\tau[|f - E_\tau[f]|_g^\alpha]^{1/\alpha} + E_\tau[E_\tau[f]^\alpha]^{1/\alpha}$$
We then use Hölder’s inequality:

\[ \mathbb{E}_r[|f - \mathbb{E}_r[f]|^\alpha]^{1/\alpha} + \mathbb{E}_r[f] \leq \mathbb{E}_r[|f - \mathbb{E}_r[f]|^\alpha]^{1/\alpha} + \mathbb{E}_r[f]. \]

We apply Inequality (4.11) to the left-hand side of (4.9):

\[ \mathbb{E}_r[f] \leq \mathbb{E}_r[f^\alpha] \leq \left( \mathbb{E}_r[f] + \frac{\mathbb{E}[\tau^{2\gamma}]^{1/(\alpha\gamma)}}{\mathbb{E}[\tau^2]^{1/\alpha}} \mathbb{E}[|f - \mathbb{E}_r[f]|_{g}^{\alpha\beta}]^{1/(\alpha\beta)} \right)^\alpha. \]

As a consequence, we have proven

\[ \mathbb{E}_r[f^\alpha] \leq \mathbb{E}_r[f^\alpha] \leq \left( \mathbb{E}_r[f] + \frac{\mathbb{E}[\tau^{2\gamma}]^{1/(\alpha\gamma)}}{\mathbb{E}[\tau^2]^{1/\alpha}} \mathbb{E}[|f - \mathbb{E}_r[f]|_{g}^{\alpha\beta}]^{1/(\alpha\beta)} \right)^\alpha. \]

In particular,

\[ \left| \mathbb{E}_r[f^\alpha]^{1/\alpha} - \mathbb{E}_r[f] \right|_g \leq \frac{\mathbb{E}[\tau^{2\gamma}]^{1/(\alpha\gamma)}}{\mathbb{E}[\tau^2]^{1/\alpha}} \mathbb{E}[|f - \mathbb{E}_r[f]|_{L^{\alpha\beta}(M,\mathbb{R})}]^\alpha. \]

We apply Inequality (4.11) to the left-hand side of (4.9):

\[ \|f^\alpha - \mathbb{E}_r[f^\alpha]\|_{L^{\beta}(M,\mathbb{R})} \leq \|\alpha \mathbb{E}_r[f]^{\alpha-1} |f - \mathbb{E}_r[f]|_g + |f - \mathbb{E}_r[f]|_g^\alpha\|_{L^{\beta}(M,\mathbb{R})} \]

\[ \leq \alpha \mathbb{E}_r[f]^{\alpha-1} \|f - \mathbb{E}_r[f]\|_{L^{\beta}(M,\mathbb{R})} + \|f - \mathbb{E}_r[f]|_g^\alpha\|_{L^{\beta}(M,\mathbb{R})} \]

\[ \leq \alpha \mathbb{E}_r[f]^{\alpha-1} \|f - \mathbb{E}_r[f]\|_{L^{\beta}(M,\mathbb{R})} + \|f - \mathbb{E}_r[f]|_g^\alpha\|_{L^{\beta}(M,\mathbb{R})}. \]

Finally, combining with (4.12), we get

\[ \|f^\alpha - \mathbb{E}_r[f^\alpha]\|_{L^{\beta}(M,\mathbb{R})} \leq \|f^\alpha - \mathbb{E}_r[f^\alpha]\|_{L^{\beta}(M,\mathbb{R})} + \|\mathbb{E}_r[f^\alpha] - \mathbb{E}_r[f]|_g^\alpha\|_{L^{\beta}(M,\mathbb{R})} \]

\[ \leq \alpha \mathbb{E}_r[f]^{\alpha-1} \|f - \mathbb{E}_r[f]\|_{L^{\beta}(M,\mathbb{R})} + \|f - \mathbb{E}_r[f]|_g^\alpha\|_{L^{\beta}(M,\mathbb{R})} \]

\[ + \alpha \mathbb{E}_r[f]^{\alpha-1} \frac{\mathbb{E}[\tau^{2\gamma}]^{1/(\alpha\gamma)}}{\mathbb{E}[\tau^2]^{1/\alpha}} \mathbb{E}[|f - \mathbb{E}_r[f]|_{L^{\alpha\beta}(M,\mathbb{R})}^\alpha] \]

\[ + \frac{\mathbb{E}[\tau^{2\gamma}]^{1/\gamma}}{\mathbb{E}[\tau^2]} \|f - \mathbb{E}_r[f]|_{L^{\alpha\beta}(M,\mathbb{R})}^\alpha \]

\[ \leq \alpha \mathbb{E}_r[f]^{\alpha-1} \left( 1 + \frac{\mathbb{E}[\tau^{2\gamma}]^{1/(\alpha\gamma)}}{\mathbb{E}[\tau^2]^{1/\alpha}} \right) \|f - \mathbb{E}_r[f]|_{L^{\beta}(M,\mathbb{R})}^\alpha \]

\[ + \left( 1 + \frac{\mathbb{E}[\tau^{2\gamma}]^{1/\gamma}}{\mathbb{E}[\tau^2]} \right) \|f - \mathbb{E}_r[f]|_{L^{\alpha\beta}(M,\mathbb{R})} \].

This ends the proof of the lemma. \(\square\)
In view of estimate (4.8), we choose \( f = \phi^{\frac{N}{2}+1} \), \( \alpha = \frac{N}{N/2+1} = \frac{n}{n-1} \) and \( \beta = \frac{N}{2} + 1 \) so \( \alpha \beta = N \) and \( \gamma = 2^{\frac{n-1}{n}} \). We remind the reader that, according to our notation, \( \mathbb{E}_r[\phi^N] = c' \) and \( \psi' = \phi^{N} - c' \). We obtain:

\[
\|\psi'\|_{L^\alpha(M,\mathbb{R})} \leq \alpha(c')^{-\frac{n-1}{2}} \left[ 1 + \frac{\|\tau\|_{L^{2\alpha}(M,\mathbb{R})}^{2/\alpha}}{\|\tau\|_{L^2(M,\mathbb{R})}^{2/\alpha}} \right] \left[ \|\phi^{\frac{N}{2}+1} - \mathbb{E}_r[\phi^{\frac{N}{2}+1}]\|_{L^N(M,\mathbb{R})}^{\alpha} \right]
\]

\[ + \left[ 1 + \frac{\|\tau\|_{L^{2\gamma}(M,\mathbb{R})}^{2/\alpha}}{\|\tau\|_{L^2(M,\mathbb{R})}^{2/\alpha}} \right] \left[ \|\phi^{\frac{N}{2}+1} - \mathbb{E}_r[\phi^{\frac{N}{2}+1}]\|_{L^N(M,\mathbb{R})}^{\alpha} \right]. \tag{4.13}
\]

Note that \( 2\gamma = 4^{\frac{n-1}{n}} \leq n \), since, multiplying by \( n\), this inequality is nothing but \( (n-2)^2 \geq 0 \). As we assumed \( \tau \in W^{1,1}(M,\mathbb{R}) \subset L^n(M,\mathbb{R}) \), all norms of \( \tau \) appearing in estimate (4.13) are finite.

Returning to estimate (4.8), remark that

\[
\frac{n-1}{n} \int_M \tau^2 \phi^{2N} \, d\mu = \frac{n-1}{n} \int_M \tau^2 ((c')^2 + 2c'\psi' + (\psi')^2) \, d\mu
\]

\[
= \frac{n-1}{n} \int_M \tau^2 ((c')^2 + (\psi')^2) \, d\mu
\]

due to our choice of decomposition. As a consequence, estimate (4.8) implies

\[
\left\{ \begin{array}{l}
\frac{3n-2}{n-1} \\|\phi^{\frac{N}{2}+1} - \mathbb{E}_r[\phi^{\frac{N}{2}+1}]\|_{L^N(M,\mathbb{R})}^2 \leq \int_M \left| \sigma + \frac{\mathbb{L}_gW}{2\eta} \right|_g^2 \, d\mu, \\
\frac{n-1}{n} \int_M \tau^2 \, d\mu ((c')^2) \leq \int_M \left| \sigma + \frac{\mathbb{L}_gW}{2\eta} \right|_g^2 \, d\mu.
\end{array} \right. \tag{4.14}
\]

Hence, the first line of estimate (4.14) together with (4.13) implies

\[
\|\psi'\|_{L^{\frac{N}{2}+1}} \leq c_1 (c')^{1/n} \left( \int_M \left| \sigma + \frac{\mathbb{L}_gW}{2\eta} \right|_g^2 \, d\mu \right)^{1/2} + c_2 \left( \int_M \left| \sigma + \frac{\mathbb{L}_gW}{2\eta} \right|_g^2 \, d\mu \right)^{\frac{n}{n-1}}. \tag{4.15}
\]

for some constants \( c_1, c_2 \) depending only on \( (M,g,\tau) \) and \( p \). The right-hand side of Estimates (4.14) can be bounded from above as follows:

\[
\int_M \left| \sigma + \frac{\mathbb{L}_gW}{2\eta} \right|_g^2 \, d\mu = \int_M \left| \sigma + c \frac{\mathbb{L}_gW}{2\eta} + \frac{\mathbb{L}_gW}{2\eta} \right|_g^2 \, d\mu
\]

\[
= \int_M \left| \sigma \right|^2 + 2c \left\langle \sigma, \frac{\mathbb{L}_gW}{2\eta} \right\rangle + c^2 \left| \frac{\mathbb{L}_gW}{2\eta} \right|_g^2
\]

\[
+ \left\langle \frac{\mathbb{L}_gW}{2\eta}, 2\sigma + 2c \frac{\mathbb{L}_gW}{2\eta} + \frac{\mathbb{L}_gW}{2\eta} \right\rangle_\eta \, d\mu
\]

\[
\leq x^2 + 2cxA_1 + c^2A_1^2
\]

\[
+ \left\| \frac{\mathbb{L}_gW}{2\eta} \right\|_{L^2(M,\mathbb{R})} \left( 2x + 2cA_1 + \left\| \frac{\mathbb{L}_gW}{2\eta} \right\|_{L^2(M,\mathbb{R})} \right). \tag{4.16}
\]
with
\[ x := \left( \int_M |\sigma|^2 d\mu^g \right)^{1/2}, \quad A_1 := \left\| \frac{\mathbb{L}_g W}{2\eta} \right\|_{L^2(M,\mathbb{R})}. \]

From what we saw above, controlling the $L^2$-norm of the right-hand side in estimate (4.14) (which is the best thing we can do if we insist on imposing restrictions on the $L^2$-norm of $\sigma$ only) gives no more than an $L^{\frac{N}{2}+1}$-control on $\psi'$. This is why the restriction for the set $C_0$ only concerned this norm. Moreover, since $W_\psi$ solves
\[ \Delta_{L_g,\eta} W_\psi = \frac{n-1}{n} \psi \nabla \tau, \]
the best we can say from Proposition 2.2 is that $W_\psi$ is controlled in the $W^{2,q}$-norm for $q = \frac{2n(n-1)}{n^2 + 2n - 4} + O(p-n)$. From the Sobolev embedding theorem, it follows that
\[ \frac{\mathbb{L}_g W_\psi}{2\eta} \in L^{q'}(M, S_2 M) \]
with $q' = \frac{2n(n-1)}{n^2 - 2} + O(p-n)$. If $p$ is too close to $n$, we have $q' < 2$. As a consequence, we need to reinforce our assumption on $\nabla \tau$. To control the $L^2$-norm of $\frac{\mathbb{L}_g W_\psi}{2\eta}$, we need to impose that $\tau \in W^{1,t_0}(M, \mathbb{R})$ where $t_0$ is defined in (4.1). Indeed, from Hölder’s inequality, we then have that
\[ \|\psi \nabla \tau\|_{L^v(M,TM)} \leq \|\psi\|_{L^{\frac{N}{2}+1}(M,\mathbb{R})} \|\nabla \tau\|_{L^{t_0}(M,TM)} \]
with $v = \frac{n}{n+2}$. So, from Proposition 2.2, there exists a constant $\Lambda > 0$ so that
\[ \|W_\psi\|_{W^{2,v}(M,TM)} \leq \Lambda \|\psi\|_{L^{\frac{N}{2}+1}(M,\mathbb{R})} \|\nabla \tau\|_{L^{t_0}(M,TM)} \]
and, using the Sobolev embedding theorem, we get
\[ \left\| \frac{\mathbb{L}_g W_\psi}{2\eta} \right\|_{L^2(M,\mathbb{R})} \leq \Lambda' \|\psi\|_{L^{\frac{N}{2}+1}(M,\mathbb{R})} \|\nabla \tau\|_{L^{t_0}(M,TM)} \]
for some constant $\Lambda' = \Lambda'(M, g, \tau, \eta)$. For reasons that will become apparent later, we need to impose $\tau \in W^{1,t}(M, \mathbb{R})$ with $t > t_0$. We can now return to estimate (4.16). From what we just saw, we have
\[ \int_M \left| \sigma + \frac{\mathbb{L}_g W}{2\eta} \right|^2 g \, d\mu^g \leq x^2 + 2cx A_1 + c^2 A_1^2 \]
\[ + \Lambda' \|\psi\|_{L^{\frac{N}{2}+1}} \|\nabla \tau\|_{L^{t_0}} \left( 2x + c_\max A_1 + \Lambda \|\psi\|_{L^{\frac{N}{2}+1}} \|\nabla \tau\|_{L^{t_0}} \right) \]
\[ \leq x^2 + 2c_\max x A_1 + c_\max^2 A_1^2 + \Lambda'' x (2x + 2c A_1 + \Lambda'' r), \]
where we set
\[ \Lambda'' := \Lambda' \|\nabla \tau\|_{L^{t_0}}. \]
Defining
\[ A_0^2 := \frac{n-1}{n} \int_M \tau^2 d\mu^g, \]
Estimates (4.14) and (4.15) imply
\[ \begin{align*}
A_0^2 (c')^2 &\leq f(x, c_{\text{max}}, r) \\
\|\psi'\|_{L^{\frac{n}{2}+1}(M, \mathbb{R})} &\leq c_1 c_{\text{max}}^{1/n} f(x, c_{\text{max}}, r)^{1/2} + c_2 f(x, c_{\text{max}}, r)^{\frac{n}{2(n-1)}},
\end{align*} \tag{4.18} \]
where
\[ f(x, c_{\text{max}}, r) := x^2 + 2c_{\text{max}}x A_1 + c_{\text{max}}^2 A_1^2 + \Lambda'' r (2x + 2c A_1 + \Lambda'' r) \tag{4.19} \]
The set \( C_0 \) introduced in (4.4) will be stable provided that we choose \( c_{\text{max}} \) and \( r \) such that
\[ \begin{align*}
A_0^2 c_{\text{max}}^2 &\geq f(x, c_{\text{max}}, r) \\
r &\geq c_1 c_{\text{max}}^{1/n} f(x, c_{\text{max}}, r)^{1/2} + c_2 f(x, c_{\text{max}}, r)^{\frac{n}{2(n-1)}},
\end{align*} \tag{4.20} \]
for some positive constants \( a, b \) to be chosen later. This allows to keep track of the order of magnitude of both components of \( \phi^N \) as \( x \) tends to zero. The system (4.20) can be rewritten
\[ \begin{align*}
A_0^2 a^2 &\geq (1 + a A_1)^2 + O(x^\frac{1}{n}) \\
b &\geq c_1 a^\frac{1}{n} (1 + a A_1) + O(x^\frac{1}{n}),
\end{align*} \tag{4.21} \]
where the big O terms depend on \( a \) and \( b \). The idea is now to replace inequalities by equalities and use the implicit function theorem. When \( x = 0 \), the system
\[ \begin{align*}
A_0^2 a^2 &= (1 + a A_1)^2 \\
b &= c_1 a^\frac{1}{n} (1 + a A_1),
\end{align*} \tag{4.22} \]
admits a solution, namely \( a_0 = 1/(A_0 - A_1) \) and \( b_0 = c_1 a_0^\frac{1}{n} (1 + a_0 A_1) \) and the linearization of the system (4.22) has no non-trivial solution. Hence, for small \( x \) the system (4.21) admits a solution in a vicinity of \( (a_0, b_0) \). It should be noted that this is where the assumption (4.2) plays a role in the proof. Indeed, it can be rewritten as \( A_0^2 > A_1^2 \) so it ensures that \( a_0 \) is positive.

We pause at this point and summarize what we have proven so far:

**Proposition 4.3.** Assume that \( g \in W^{2, \frac{n}{2}}(M, S_2 M) \), \( \sigma \in L^2(M, \mathbb{R}) \) and \( \tau \in W^{1, t_0}(M, \mathbb{R}) \), where \( t_0 = \frac{2n(n-1)}{3n-2} \). Then, provided that (3.11) holds and
\[ x := \left( \int_M |\sigma|^2 d\mu^g \right)^{1/2} \]
is small enough, there exist constants \( c_{\text{max}} > 0 \) and \( r > 0 \) such that the set \( C_0 \) defined in (4.4) is stable for the mapping \( \Psi \).
We are not yet in a position to apply Schauder’s fixed point theorem since \( C_0 \) is not bounded. So, in what follows, we use a bootstrap argument to find nested closed subsets \( C_k \) (i.e., such that \( C_{k+1} \subset C_k \)) so that \( \Psi(C_k) \subset C_{k+1} \) eventually getting a bounded closed set. This point is inspired by [7, Proposition 4.6].

We construct sequences \( (q_i), (k_i), (r_i), (R_i) \) as follows. We choose \( q_0 = N^2 + 1 \). There exists a constant \( R_0 > 0 \) such that
\[
C_0 \subset \{ u \in L^{q_0}(M, \mathbb{R}), \| u \|_{L^{q_0}(M, \mathbb{R})} \leq R_0 \}.
\]
Assume now that for some \( i \geq 0 \), \( q_i \) and \( R_i \) are known (we just defined \( q_0 \) and \( R_0 \)). Then, from Young’s inequality, we have that for all \( u \in C_i \),
\[
\| u \nabla \tau \|_{L^{c_i}(M, \mathbb{R})} \lesssim R_i \quad \text{where} \quad c_i = \frac{1}{q_i} = \frac{1}{q_i} + \frac{1}{t}.
\]
Here, the notation \( A \lesssim B \) means that there exists a constant \( C > 0 \) that may vary from line to line but independent of \( u \) such that \( A \leq CB \).

By Proposition 2.2, we have
\[
\| W \|_{W^{2,c_i}} \lesssim R_i
\]
for all \( W \) solving (4.5). From the Sobolev embedding theorem, we get that
\[
\| \sigma + \frac{L_g W}{2\eta} \|_{L^{r_i}(M, S_2 M)} \lesssim \| \phi^{2k_i} \|_{L^{r_i-2}(M, \mathbb{R})}.
\]
We now multiply the Lichnerowicz equation by \( \phi^{N+1+2k_i} \) for some \( k_i \) to be chosen later and integrate over \( M \). We get
\[
\frac{4(n-1)}{n-2} \int_M \langle d\phi^{N+1+2k_i}, d\phi \rangle_g d\mu^g \leq \int_M \| \sigma + \frac{L_g W}{2\eta} \|_{L^{r_i}(M, S_2 M)}^2 \| \phi^{2k_i} \|_{L^{r_i-2}(M, \mathbb{R})}^2,
\]
or, equivalently,
\[
\frac{4(n-1)}{n-2} \frac{N + 1 + 2k_i}{N + \frac{1}{2} + k_i} \int_M \| \phi^{\frac{N}{2} + 1 + k_i} \|_{L^{r_i-2}(M, \mathbb{R})}^2 \| \phi^{2k_i} \|_{L^{r_i-2}(M, \mathbb{R})}^2 \leq \int_M \| \sigma + \frac{L_g W}{2\eta} \|_{L^{r_i}(M, S_2 M)}^2 \| \phi^{2k_i} \|_{L^{r_i-2}(M, \mathbb{R})}^2.
\]
Using Hölder’s inequality, we have that
\[
\int_M \| \sigma + \frac{L_g W}{2\eta} \|_{L^{r_i}(M, S_2 M)}^2 \| \phi^{2k_i} \|_{L^{r_i-2}(M, \mathbb{R})}^2 \leq \int_M \| \sigma + \frac{L_g W}{2\eta} \|_{L^{r_i}(M, S_2 M)}^2 \| \phi^{N} \|_{L^{r_i-2}(M, \mathbb{R})}^2 \frac{2k_i}{N r_i - 2}.
\]
We now choose \( k_i \) such that
\[
\frac{2k_i}{N r_i - 2} = q_i,
\]
We apply the Sobolev embedding theorem: For some constant $s_i$, we have that
\[
\left( \int_M \left| \phi \right|^{\frac{N}{2} + 1 + k_i} d\mu^g \right)^{2/N} \leq s_i \int_M \left| \sigma + \frac{L_g W}{2 \eta} \right|^2 \phi^{2k_i} d\mu^g + \int_M \phi^{N + 2 + 2k_i} d\mu^g,
\]
\[
\left\| \phi^N \right\|_{L^{\frac{N}{2} + 1 + k_i}(M, \mathbb{R})} \leq s_i \left\| \sigma + \frac{L_g W}{2 \eta} \right\|_{L^1(M, S_2 M)}^{2} \left\| \phi^N \right\|_{L^{n_i}(M, \mathbb{R})}^{\frac{2k_i}{N}} + \left\| \phi^N \right\|_{L^{n_i}(M, \mathbb{R})}^{1 + \frac{2}{N} + \frac{2k_i}{N}}.
\]

A straightforward calculation shows that $1 + \frac{2}{N} + \frac{2k_i}{N} < q_i$, so, since $g$ has volume one,
\[
\left\| \phi^N \right\|_{L^{n_i + 1}(M, \mathbb{R})} \leq s_i \left\| \sigma + \frac{L_g W}{2 \eta} \right\|_{L^1(M, S_2 M)}^{2} \left\| \phi^N \right\|_{L^{n_i}(M, \mathbb{R})}^{\frac{2k_i}{N}} + \left\| \phi^N \right\|_{L^{n_i}(M, \mathbb{R})}^{1 + \frac{2}{N} + \frac{2k_i}{N}}.
\]

Setting
\[
q_i + 1 = \left( N - 1 - \frac{N}{t} \right) q_i - \frac{2}{n - 2},
\]
we obtain that
\[
\left\| \phi^N \right\|_{L^{n_i + 1}(M, \mathbb{R})} \leq R_i
\]
for some well-chosen $R_i$ as we have bounded all the terms of the right-hand side.

Setting $C_{i+1} := \{ u \in C_i, \left\| u \right\|_{L^{n_i+1}(M, \mathbb{R})} \leq R_{i+1} \}$, we have that $C_{i+1} \subseteq C_i$ and $\Psi(C_i) \subseteq C_{i+1}$.

We now study in more details the sequence $(q_i)$. It is defined by the recurrence relation (4.24). Let $\bar{q}$ denote the solution to
\[
\bar{q} = \left( N - 1 - \frac{N}{t} \right) \bar{q} - \frac{2}{n - 2},
\]
namely
\[
\bar{q} = \frac{2}{n - 2 (N - 2) t - N}.
\]

We have
\[
q_i = \left( N - 1 - \frac{N}{t} \right)^i (q_0 - \bar{q}) + \bar{q}.
\]
If \( t \geq t_0 \), we have
\[
N - 1 - \frac{N}{t} \geq \frac{n}{n - 1} > 1.
\]
Yet if \( t = t_0 \), where \( t_0 \) is defined in (4.1), we have \( q = q_0 \) so the sequence \( (q_i) \) is constant. This is where we have to assume that \( t > t_0 \) to ensure \( q_i \to \infty \). There is an \( i_0 \) such that \( q_{i_0 + 1} > p_0 \). Then, \( C_{i_0 + 1} \) is bounded and closed in \( L^{p_0}(M, \mathbb{R}) \).

Even more is true. Assume that \( i_0 \) has been chosen so that \( q_{i_0 + 1} > \max\{p_0, N\} \). Set \( C := \Psi(C_i) \). We claim that \( C \) is precompact in \( L^{p_0}(M, \mathbb{R}) \).

Indeed, performing the analysis following (4.23) but without using the Sobolev embedding theorem, we have that, for any \( u = \phi^N \in \Psi(C_i) \),
\[
\|\phi^{q_{i+1}}\|_{W^{1,2}(M, \mathbb{R})} \leq R'
\]
for some constant \( R' > 0 \). Let us denote \( q = q_{i+1} \) for simplicity.

Let \((u_k)_k, u_k = \phi^N_k \in \Psi(C_i) \), be any given sequence. Since \( p_0 < q \), we have that \( \lambda := N\frac{p_0}{q} \leq N \) so the embedding \( W^{1,2}(M, \mathbb{R}) \hookrightarrow L^{\lambda}(M, \mathbb{R}) \) is compact. As a consequence, there exists a subsequence \((u_{\omega(k)})_k \) of \((u_k)_k \) such that \( \phi^{q_{\omega(k)}}_\omega \to \phi^q_\infty \) in \( L^{\lambda}(M, \mathbb{R}) \). We have \( u_\infty = \phi^q_\infty \in L^{p_0}(M, \mathbb{R}) \) so all we need to do is to check that
\[
uu_{\omega(k)} \to u_\infty \text{ in } L^{p_0}(M, \mathbb{R}).
\]
The idea is similar to the one for (4.9), yet simpler. Let \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) denote the function
\[
h(x) := \frac{|x^N - 1|_{p_0}^{q_0}}{|x^q - 1|^{\lambda}}.
\]
Since we chose \( q > N \), we have \( \lambda = \frac{N}{q}p_0 < p_0 \) and
\[
h(x) \sim \frac{N^{p_0}}{q^{\lambda}} |x - 1|^{p_0 - \lambda}
\]
ear \( x = 1 \). Since \( h(x) \) tends to 1 when \( x \) goes to 0 or to \( \infty \), we conclude that \( h \) is bounded on \( \mathbb{R}_+ \). Let \( A > 0 \) be an upper bound for \( h \): \( h(x) \leq A \) for all \( x \in \mathbb{R}_+ \). We have that
\[
|x^N - 1|_{q}^{p_0} \leq A |x^q - 1|^{\lambda}_q.
\]
Setting \( x = \phi_i/\phi_\infty \) and multiplying by \( \phi_\infty^{Np_0} = \phi_\infty^{q_0} \), we have that
\[
|\phi^N_i - \phi^q_\infty|_q^{p_0} \leq A |\phi_i^q - \phi_\infty^q|^{\lambda}_q.
\]
Integrating over \( M \), we obtain
\[
\|u_i - u_\infty\|_{L^{p_0}(M, \mathbb{R})} \leq A^{1/p_0} \|\phi_i^q - \phi^q_\infty\|^{\lambda/p_0}_{L^{\lambda}(M, \mathbb{R})}
\]
which shows that \( u_i \to u_\infty \) in \( L^{p_0}(M, \mathbb{R}) \). We have proven that \( C \) is (sequentially) precompact in \( L^{p_0}(M, \mathbb{R}) \).
Set $\overline{C} := \text{conv}(C)$ be the closed convex hull of $C$. Then, $\overline{C}$ is compact, convex and $\Psi(\overline{C}) \subset \Psi(C_{i_0}) \subset \overline{C}$. We can now apply the Schauder fixed point theorem to $\Psi$ and $\overline{C}$ and get the following theorem:

**Theorem 4.4.** Assume that $g \in W^{2,p/2}(M, S_2 M)$, $\sigma \in L^p(M, S_2 M)$, $\tau \in W^{1,1}(M, \mathbb{R})$ for some $t > t_0$, where $t_0$ is defined in (4.1). Then, provided that Condition (3.11) is satisfied:

$$\frac{n-1}{n} \int_M \tau^2 \phi_0^N d\mu_g > \frac{1}{n} \int_M \frac{\|L_g W\|^2}{4\eta^2} \phi_0^{-N} d\mu_g,$$

and if $\int_M |\sigma|^2 d\mu_g$ is small enough (as given in Proposition 4.3), there exists at least one solution to the system (1.3).

As pointed out by the referee, it is interesting to see how this result can be compared to those obtained in [16] and [5] and in particular to make explicit Condition (3.11) in that settings. The strategy of these papers is to study a model problem where the system (1.3) is replaced by a pair of coupled differential equations. This is achieved by choosing for $(M, g)$ a rectangular torus, i.e., the manifold is a Riemannian product $S^1(r_1) \times \cdots \times S^1(r_n)$ with $S^1(r)$ denoting the circle of radius $r$, and $\tau$ and $\sigma$ parallel in the $n-1$ first coordinates so that they only depend on the last coordinate. Despite the introduction of conformal Killing vector fields in the model, the vector equation still admits a unique solution if one restricts to $W = w(x_n) \partial_n$ and similarly $\phi = \phi(x_n)$, where $x_n$ denotes the $2\pi$-periodic coordinate on $S^1(r)$. As a consequence, all the analysis performed in this paper can be applied mutatis mutandis to this particular case.

Upon rescaling $g$, we can assume that $r_n = 1$ and we can choose $\phi_0 \equiv 1$ as $g$ is a flat metric$^1$. Some straightforward calculations then show that Eq. (3.9) can be rewritten as

$$\left(\frac{1}{\eta} w' \right)' = \tau' \quad (4.25)$$

where a prime denotes derivation with respect to $x_n$ and

$$\frac{\|L_g W\|^2}{4\eta^2} = 4 \frac{n-1}{n} (w')^2.$$

Thus, Condition (3.11) reads

$$\int_M \tau^2 d\mu_g > \int_M \frac{(w')^2}{\eta^2} d\mu_g,$$

or, equivalently,

$$\int_0^{2\pi} \tau^2 dx_n > \int_0^{2\pi} \left( \frac{w'}{\eta} \right)^2 dx_n. \quad (4.26)$$

$^1$Note that the fact that $\phi_0$ has $L^2$-norm equal to 1, as was imposed at the beginning of Sect. 2 is not used to obtain Condition (3.11).
Equation (4.25) imposes that

\[
\frac{1}{\eta} w' = \tau + c
\]

for some constant \( c \) which can be computed from the fact that \( w \) is a continuous function:

\[
0 = w(2\pi) - w(0) = \int_0^{2\pi} \eta(\tau + c) dx_n,
\]

so

\[
c = -\frac{\int_0^{2\pi} \eta \tau dx_n}{\int_0^{2\pi} \eta dx_n}.
\]

In [16], the author chooses \( \tau = t + \lambda \) where \( t \) is a constant and \( \lambda \) is defined by

\[
\lambda(x_n) = \begin{cases} 
1 & \text{if } x \in (0, \pi), \\
-1 & \text{if } x \in (\pi, 2\pi).
\end{cases}
\]

As a consequence, \( c = -t + \gamma \) where \( \gamma \) is defined by

\[
\gamma = -\frac{\int_0^{2\pi} 2\eta \lambda dx_n}{\int_0^{2\pi} 2\eta dx_n}
\]

(\( \gamma \) corresponds to the quantity denoted by \( \gamma_N \) in [16].) Since \( \lambda \) has null mean value, we have

\[
\int_0^{2\pi} \tau^2 dx_n = \int_0^{2\pi} (t + \lambda)^2 dx_n = 2\pi(t^2 + 1)
\]

and, similarly,

\[
\int_0^{2\pi} \left( \frac{w'}{\eta} \right)^2 dx_n = 2\pi(\gamma^2 + 1).
\]

Condition (4.26) then reads \( 2\pi(t^2 + 1) > 2\pi(\gamma^2 + 1) \) which is equivalent to \( |t| > |\gamma| \) so we recover the assumption of [16, Theorem 5].

In [5], the authors consider the usual conformal method, i.e., \( \eta \equiv 1/2 \), with \( \tau = 1 + a \cos(x_n) \) for some constant \( a \). Here, we get

\[
\int_0^{2\pi} \tau^2 dx_n = (a^2 + 2)\pi \quad \text{and} \quad \int_0^{2\pi} \left( \frac{w'}{\eta} \right)^2 dx_n = a^2 \pi,
\]

so Condition (4.26) is always fulfilled. This explains why the blue curve on [5, Figure 1] never crosses the axis \( \eta = 0 \) (in their notation \( \eta \) is proportional to the norm of \( \sigma \)). [5, Figure 2] shows that if \( a \) is large enough, our method constructs the solution with small volume but ignore the second one which has large volume.
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References

[1] Allen, P.T., Clausen, A., Isenberg, J.: Near-constant mean curvature solutions of the Einstein constraint equations with non-negative Yamabe metrics, Class. Quantum Grav. 25(7), 075009, 15 (2008)

[2] Arnowitt, Richard L., Deser, Stanley, Misner, Charles W.: The Dynamics of general relativity. Gen. Rel. Grav. 40, 1997–2027 (2008)

[3] Bartnik, R., Isenberg, J.: The Constraint Equations, The Einstein Equations and the Large Scale Behavior of Gravitational Fields, pp. 1–38. Birkhäuser, Basel (2004)

[4] Dahl, M., Gicquaud, R., Humbert, E.: A limit equation associated to the solvability of the vacuum Einstein constraint equations by using the conformal method. Duke Math. J. 161(14), 2669–2697 (2012)

[5] Dilts, J., Holst, M., Kozareva, T., Maxwell, D.: Numerical Bifurcation Analysis of the Conformal Method. arXiv:1710.03201

[6] Gicquaud, R., Ngô, Q.A.: A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature and small TT-tensor. Class. Quantum Grav. 31(19), 195014 (20pp) (2014)

[7] Gicquaud, R., Nguyen, C.: Solutions to the Einstein-scalar field constraint equations with a small TT-tensor. Calc. Var. Partial Differ. Equ. 55(2), Art. 29, 23 (2016)

[8] Holst, M., Maxwell, D., Mazzeo, R.: Conformal Fields and the Structure of the Space of Solutions of the Einstein Constraint Equations. arXiv:1711.01042

[9] Holst, M., Meier, C.: Non-uniqueness of Solutions to the Conformal Formulation. arXiv:1210.2156

[10] Holst, M., Nagy, G., Tsogtgerel, G.: Far-from-constant mean curvature solutions of Einstein’s constraint equations with positive Yamabe metrics. Phys. Rev. Lett. 100(16), 161101, 4 (2008)

[11] Holst, M., Nagy, G., Tsogtgerel, G.: Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions. Commun. Math. Phys. 288(2), 547–613 (2009)

[12] Isenberg, J.: Constant mean curvature solutions of the Einstein constraint equations on closed manifolds. Class. Quantum Grav. 12(9), 2249–2274 (1995)

[13] Lee, J.M., Parker, T.H.: The Yamabe problem. Bull. Am. Math. Soc. (N.S.) 17(1), 37–91 (1987)
[14] Maxwell, D.: Initial Data in General Relativity Described by Expansion, Conformal Deformation and Drift. arXiv:1407.1467

[15] Maxwell, D.: A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature. Math. Res. Lett. 16(4), 627–645 (2009)

[16] Maxwell, D.: A model problem for conformal parameterizations of the Einstein constraint equations. Commun. Math. Phys. 302(3), 697–736 (2011)

[17] Maxwell, D.: The conformal method and the conformal thin-sandwich method are the same. Class. Quantum Gravity 31(14), 145006, 34 (2014)

[18] Nguyen, T.C.: Nonexistence and Nonuniqueness Results for Solutions to the Vacuum Einstein Conformal Constraint Equations. Comm. Anal. Geom. 26(5), 1169–1194 (2018)

[19] Nguyen, T.C.: Applications of fixed point theorems to the vacuum Einstein constraint equations with non-constant mean curvature. Ann. Henri Poincaré 17(8), 2237–2263 (2016)

[20] Ringström, H.: The Cauchy problem in general relativity, ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich (2009)

[21] Trudinger, N.S.: Linear elliptic operators with measurable coefficients. Ann. Scuola Norm. Sup. Pisa 3(27), 265–308 (1973)

[22] Yap, C.K.: Fundamental Problems of Algorithmic Algebra. Oxford University Press, New York (2000)

[23] York Jr., J.W.: Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial-value problem of general relativity. J. Math. Phys. 14, 456–464 (1973)

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