Bipartization of Graphs

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Abstract
A dominating set of a graph $G$ is a set $D \subseteq V_G$ such that every vertex in $V_G - D$ is adjacent to at least one vertex in $D$, and the domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. In this paper we provide a new characterization of bipartite graphs whose domination number is equal to the cardinality of its smaller partite set. Our characterization is based upon a new graph operation.

Keywords Bipartite graph · Bipartization · Domination number

Mathematics Subject Classification 05C69 · 05C76 · 05C05

1 Introduction and Notation

For notation and graph theory terminology we in general follow [2]. Specifically, let $G = (V_G, E_G)$ be a graph with vertex set $V_G$ and edge set $E_G$. For a subset $X \subseteq V_G$, the subgraph induced by $X$ is denoted by $G[X]$. For simplicity of notation, if $X = \{x_1, \ldots, x_k\}$, we shall write $G[x_1, \ldots, x_k]$ instead of $G[\{x_1, \ldots, x_k\}]$. For a vertex $v$ of $G$, its neighborhood, denoted by $N_G(v)$, is the set of all vertices adjacent to $v$, and the cardinality of $N_G(v)$, denoted by $\deg_G(v)$, is called the degree of $v$. The closed neighborhood of $v$, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. In general, the neighborhood of $X \subseteq V_G$, denoted by $N_G(X)$, is defined to be $\bigcup_{v \in X} N_G(v)$, and the closed neighborhood of $X$, denoted by $N_G[X]$, is the set $N_G(X) \cup X$. A vertex of degree one is called a leaf, and the only neighbor of a leaf is called its support vertex.
Observation. A weak support is a vertex adjacent to exactly one leaf. Finally, the set of leaves and the set of supports of $G$ we denoted by $L_G$ and $S_G$, respectively.

A subset $D$ of $V_G$ is said to be a dominating set of a graph $G$ if each vertex belonging to the set $V_G - D$ has a neighbor in $D$. The cardinality of a minimum dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. A subset $C \subseteq V_G$ is a covering set of $G$ if each edge of $G$ has an end-vertex in $C$. The cardinality of a minimum covering set of $G$ is called the covering number of $G$ and denoted by $\beta(G)$.

It is obvious that if $G = ((A, B), E_G)$ is a connected bipartite graph, then $\gamma(G) \leq \min\{|A|, |B|\}$. In this paper the set of all connected bipartite graphs $G = ((A, B), E_G)$ in which $\gamma(G) = \min\{|A|, |B|\}$ is denoted by $\mathcal{B}$. Some properties of the graphs belonging to the set $\mathcal{B}$ were observed in the papers [1,3–6], where all graphs with the domination number equal to the covering number were characterized. In this paper, inspired by results and constructions of Hartnell and Rall [3], we introduce a new graph operation, called the bipartization of a graph with respect to a function, study basic properties of this operation, and provide a new characterization of the graphs belonging to the set $\mathcal{B}$ in terms of this new operation.

2 Bipartization of a Graph

Let $\mathcal{K}_H$ denote the set of all complete subgraphs of a graph $H$. If $v \in V_H$, then the set $\{K \in \mathcal{K}_H : v \in V_K\}$ is denoted by $\mathcal{K}_H(v)$. If $X \subseteq V_H$, then the set $\bigcup_{v \in X} \mathcal{K}_H(v)$ is denoted by $\mathcal{K}_H(X)$, and it is obvious that $\mathcal{K}_H(X) = \{K \in \mathcal{K}_H : V_K \cap X \neq \emptyset\}$. Let $f : \mathcal{K}_H \to \mathbb{N}$ be a function. If $K \in \mathcal{K}_H$, then by $\mathcal{F}_K$ we denote the set $\{(K, 1), \ldots, (K, f(K))\}$ if $f(K) \geq 1$, and we let $\mathcal{F}_K = \emptyset$ if $f(K) = 0$. By $\mathcal{K}^f_H$ we denote the set of all positively $f$-valued complete subgraphs of $H$, that is, $\mathcal{K}^f_H = \{K \in \mathcal{K}_H : f(K) \geq 1\}$.

Definition 1 Let $H$ be a graph and let $f : \mathcal{K}_H \to \mathbb{N}$ be a function. The bipartization of $H$ with respect to $f$ is the bipartite graph $B_f(H) = ((A, B), E_{B_f(H)})$ in which $A = V_H$, $B = \bigcup_{K \in \mathcal{K}_H} \mathcal{F}_K$, and where a vertex $x \in A$ is adjacent to a vertex $(K, i) \in B$ if and only if $x$ is a vertex of the complete graph $K$ ($i = 1, \ldots, f(K)$).

Example 1 Figure 1 presents a graph $H$ (for which $\mathcal{K}_H = \{H[a], H[b], H[c], H[d], H[a, b], H[a, c], H[b, c], H[c, d], H[a, b, c]\}$) and its two bipartizations $B_f(H)$ and $B_g(H)$ with respect to functions $f, g : \mathcal{K}_H \to \mathbb{N}$, respectively, where $f(H[a]) = 1$, $f(H[b]) = 1$, $f(H[c]) = 2$, $f(H[d]) = 0$, $f(H[a, b]) = 3$, $f(H[a, c]) = 0$, $f(H[b, c]) = 2$, $f(H[c, d]) = 3$, $f(H[a, b, c]) = 1$, while $g(H[v]) = 0$ for every vertex $v \in V_H$, $g(H[u, v]) = 1$ for every edge $uv \in E_H$, and $g(H[a, b, c]) = 0$. Observe that $B_g(H)$ is the subdivision graph $\mathcal{S}(H)$ of $H$ (i.e., the graph obtained from $H$ by inserting a new vertex into each edge of $H$).
3 Properties of Bipartizations of Graphs

It is clear from the above definition of the bipartization of a graph with respect to a function that we have the following proposition.

**Proposition 1** The bipartization of a graph with respect to a function has the following properties:

1. If \( B_f(H) = ((A, B), E_{B_f(H)}) \) is the bipartization of a graph \( H \) with respect to a function \( f : K_H \rightarrow \mathbb{N} \), then:
   
   (a) \( N_{B_f(H)}(v) = \bigcup_{K \in K_H(v)} F_K \) if \( v \in A \).
   
   (b) \( N_{B_f(H)}(X) = \bigcup_{K \in K_H(X)} F_K \) if \( X \subseteq A \).
   
   (c) \( N_{B_f(H)}((K, i)) = V_K \) if \( (K, i) \in B \) \((i = 1, \ldots, f(K))\).
   
   (d) \( |V_{B_f(H)}| = |V_H| + \sum_{K \in K_H} f(K) \) and \( |E_{B_f(H)}| = \sum_{K \in K_H} f(K) |V_K| \).

2. If \( H \) is a connected graph and \( f : K_H \rightarrow \mathbb{N} \) is a function such that every edge of \( H \) belongs to a positively \( f \)-valued complete subgraph of \( H \), then the bipartization \( B_f(H) \) is a connected graph.

3. If \( H \) is a graph and \( f, g : K_H \rightarrow \mathbb{N} \) are functions such that \( f(K) \geq g(K) \) for every \( K \in K_H \), then the graph \( B_g(H) \) is an induced subgraph of \( B_f(H) \).

Our study of properties of bipartizations we begin by showing that every bipartite graph is the bipartization of some graph with respect to some function.

**Theorem 1** For every bipartite graph \( G = ((A, B), E_G) \) there exist a graph \( H \) and a function \( f : K_H \rightarrow \mathbb{N} \) such that \( G = B_f(H) \).

**Proof** We say that vertices \( x \) and \( y \) of \( G \) are similar if \( N_G(x) = N_G(y) \). It is obvious that this similarity is an equivalence relation on \( B \) (as well as on \( A \) and \( A \cup B \)). Let \( B_1, \ldots, B_l \) be the equivalence classes of this relation on \( B \), say \( B_i = \{b_i^1, b_i^2, \ldots, b_i^{k_i}\} \) for \( i = 1, \ldots, l \). It follows from properties of the equivalence classes that \( |B_1| + \cdots + |B_l| = |B|, N_G(b_i^1) = N_G(x) \) for every \( x \in B_i \), and \( N_G(b_i^1) \neq N_G(b_j^1) \) if \( i, j \in \{1, \ldots, l\} \) and \( i \neq j \).
Fig. 2 Graph $G$ is the bipartization of the two non-isomorphic graphs $H$ and $F$

Now, let $H = (V_H, E_H)$ be a graph in which $V_H = A$ and two vertices $x$ and $y$ are adjacent in $H$ if and only if they are at distance two apart from each other in $G$. Let $\mathcal{K}_H$ be the set of all complete subgraphs of $H$, and let $f : \mathcal{K}_H \to \mathbb{N}$ be a function such that $f(K) = |\{b \in B : N_G(b) = V_K\}|$ for $K \in \mathcal{K}_H$. Next, let $K_i$ be the induced subgraph $H[N_G(b_i')]$ of $H$. It follows from the definition of $H$ that $K_i$ is a complete subgraph of $H$. In addition, from the definition of $f$ and from properties of the classes $B_1, \ldots, B_l$, it follows that $f(K_i) = |B_i| > 0$ ($i = 1, \ldots, l$), and $f(K) = 0$ if $K \in \mathcal{K}_H - \{K_1, \ldots, K_l\}$. Consequently, $\mathcal{K}_H^f = \{K_1, \ldots, K_l\}$.

Finally, consider the bipartite graph $B_{f}(H) = ((X, Y), E_{B_{f}(H)})$ in which $X = V_H = A$, $Y = \bigcup_{K \in \mathcal{K}_H} F_K = \bigcup_{K \in \mathcal{K}_H^f} F_K = \bigcup_{j=1}^{l} \{(K_i, 1), \ldots, (K_i, k_i)\}$, and where $N_{B_{f}(H)}((K_i, j)) = V_{K_i} = N_G(b_i^j)$ for every $(K_i, j) \in Y$. Now, one can observe that the function $\varphi : A \cup B \to X \cup Y$, where $\varphi(x) = x$ if $x \in A$, and $\varphi(b_i^j) = (K_i, j)$ if $b_i^j \in B$, is an isomorphism between graphs $G$ and $B_{f}(H)$.

We have proved that a bipartite graph $G = ((A, B), E_G)$ is the bipartization $B_{f}(H)$ of a graph $H = (V_H, E_H)$ (in which $V_H = A$ and $E_H = \{xy : x, y \in A \text{ and } d_G(x, y) = 2\}$) with respect to a function $f : \mathcal{K}_H \to \mathbb{N}$, where $f(K) = |\{b \in B : N_G(b) = V_K\}|$ for $K \in \mathcal{K}_H$. The same graph $G$ is also the bipartization $B_{g}(F)$ of a graph $F = (V_F, E_F)$ (in which $V_F = B$ and $E_F = \{xy : x, y \in B \text{ and } d_G(x, y) = 2\}$) with respect to a function $g : \mathcal{K}_F \to \mathbb{N}$, where $g(K) = |\{a \in A : N_G(a) = V_K\}|$ for $K \in \mathcal{K}_F$. Consequently, every bipartite graph may be the bipartization of two non-isomorphic graphs.

**Example 2** Figure 2 depicts the bipartite graph $G$ which is the bipartization of the non-isomorphic graphs $H$ and $F$ with respect to functions $\overline{f} : \mathcal{K}_H \to \mathbb{N}$ and $\overline{g} : \mathcal{K}_F \to \mathbb{N}$, respectively, which non-zero values are displayed in the figure.

It is obvious from Theorem 1 that every tree is a bipartization. We are now interested in providing a simple characterization of graphs $H$ and functions $f : \mathcal{K}_H \to \mathbb{N}$ for which the bipartization $B_{f}(H)$ is a tree. We begin with the following notation: An alternating sequence of vertices and complete graphs $(v_0, F_1, v_1, \ldots, v_{k-1}, F_k, v_k)$ is said to be a positively $f$-valued complete $v_0 - v_k$ path if $v_{i-1}v_i$ is an edge in the complete graph $F_i$ for $i = 1, \ldots, k$. We now have the following two useful lemmas.

\[\text{Springer}\]
Lemma 1 Let $H$ be a connected graph, and let $f : \mathcal{K}_H \to \mathbb{N}$ be a function. If there are two vertices $u$ and $v$ and two distinct internally vertex-disjoint positively $f$-valued complete $u-v$ paths in $H$, then the bipartization $B_f(H)$ contains a cycle.

Proof If $(v_0 = u, F_1, v_1, \ldots, v_{m-1}, F_m, v_m = v)$ and $(v_0' = u, F_1', v_1', \ldots, v_{n-1}', F_n', v_n' = v)$ are distinct internally vertex-disjoint positively $f$-valued complete $u-v$ paths in $H$, then $(v_0, (F_1, 1), v_1, \ldots, v_{m-1}, (F_m, 1), v_m)$ and $(v_0', (F_1', 1), v_1', \ldots, v_{n-1}', (F_n', 1), v_n')$ are distinct $u-v$ paths in $B_f(H)$, and so they generate at least one cycle in $B_f(H)$. □

Let us recall first that a maximal connected subgraph without a cutvertex is called a block. A graph $H$ is said to be a block graph if each block of $H$ is a complete graph. The next lemma is probably known, therefore we omit its easy inductive proof.

Lemma 2 If $S$ is the set of all blocks of a graph $H$, then $\sum_{B \in S} (|V_B| - 1) = |V_H| - 1$.

Now we are ready for a characterization of graphs which bipartizations (with respect to some functions) are trees.

Theorem 2 Let $H$ be a connected graph, and let $f : \mathcal{K}_H \to \mathbb{N}$ be a function such that every edge of $H$ belongs to some positively $f$-valued complete subgraph of $H$. Then the bipartization $B_f(H)$ is a tree if and only if the following conditions hold:

(1) $f(K) \leq 1$ for every non-trivial complete subgraph $K$ of $H$.
(2) $H$ is a block graph.
(3) For a non-trivial complete subgraph $K$ of $H$ is $f(K) = 1$ if and only if $K$ is a block of $H$.

Proof Assume that $B_f(H)$ is a tree. The statement (1) is obvious, for if there were a non-trivial complete subgraph $K$ of $H$ for which $f(K) \geq 2$, then for any two vertices $u$ and $v$ belonging to $K$, the sequence $(u, (K, 1), v, (K, 2), u)$ would be a cycle in $B_f(H)$.

Suppose now that $H$ is not a block graph. Then there exists a block in $H$, say $B$, which is not a complete graph. Thus in $B$ there exists a cycle such that not all its chords belong to $B$. Let $C = (v_0, v_1, \ldots, v_l, v_0)$ be a shortest such cycle in $B$. Then $l \geq 3$ and we distinguish two cases. If $C$ is chordless, then, by Lemma 1, $B_f(H)$ contains a cycle. Thus assume that $C$ has a chord. We may assume that $v_0$ is an end-vertex of a chord of $C$, and then let $k$ be the smallest integer such that $v_0v_k$ is a chord of $C$. Now the choice of $C$ implies that the vertices $v_0, v_1, \ldots, v_k$ are mutually adjacent, and therefore, $k = 2$. Similarly, $v_0, v_k, \ldots, v_l$ are mutually adjacent, and so we must have $l = 3$. Consequently, $C = (v_0, v_1, v_2, v_3, v_0)$ and $v_0v_2$ is the only chord of $C$. Now it is obvious that there are at least two $v_0-v_2$ positively $f$-valued complete paths in $H$. From this and from Lemma 1 it follows that the bipartition $B_f(H)$ contains a cycle. This contradiction completes the proof of the statement (2).

Let $B$ be a block of $H$. We have already proved that $B$ is a complete graph. Let $B'$ be a proper non-trivial complete subgraph of $B$. To prove (3), it suffices to observe that $f(B') = 0$. On the contrary, suppose that $f(B') \neq 0$. We now choose two distinct
vertices \( v \) and \( u \) belonging to \( B' \), and a vertex \( w \) belonging to \( B \) but not to \( B' \). This clearly forces that there are at least two \( v - u \) positively \( f \)-valued complete paths in \( H \). Consequently, by Lemma 1, \( B_f(H) \) contains a cycle, and this contradiction completes the proof of the statement (3).

Assume now that the conditions (1)–(3) are satisfied for \( H \) and \( f \). Since end-vertices of \( B_f(H) \), corresponding to positively \( f \)-valued one-vertex complete subgraphs of \( H \), are not important to our study of tree-like structure of \( B_f(H) \), we can assume without loss of generality that \( f(H[v]) = 0 \) for every vertex \( v \in V_H \). Consequently, \( H \) is a block graph and \( f(K) = 1 \) for every block \( K \) of \( H \), while \( f(K') = 0 \) for every other complete subgraph \( K' \) of \( H \). It remains to prove that \( B_f(H) \) is a tree. Since \( B_f(H) \) is a connected graph, it suffices to show that \(|E_{B_f(H)}| = |V_{B_f(H)}| - 1 \). Let \( S \) be the set of all blocks of \( H \). Then \( \mathcal{K}_H^f = S, |V_{B_f(H)}| = |V_H| + \sum_{K \in \mathcal{K}_H^f} f(K) = |V_H| + |S| \), and \(|E_{B_f(H)}| = \sum_{K \in \mathcal{K}_H^f} f(K)|V_K| = \sum_{K \in S} |V_K| = \sum_{K \in S} (|V_K| - 1) + |S| \).

Now, since \( \sum_{K \in S} (|V_K| - 1) = |V_H| - 1 \) (by Lemma 2), we finally have \(|E_{B_f(H)}| = (|V_H| - 1) + |S| = (|V_H| + |S|) - 1 = |V_{B_f(H)}| - 1 \).

**Corollary 1** For every connected graph \( H \), there exists a function \( f : \mathcal{K}_H \rightarrow \mathbb{N} \) such that the bipartization \( B_f(H) \) is a tree.

**Proof** Let \( F \) be a spanning block graph of \( H \) and let \( f : \mathcal{K}_F \rightarrow \{0, 1\} \) be a function such that \( f(K) = 1 \) if and only if \( K \) is a block of \( F \). Clearly, \( f \) satisfies the conditions (1)–(3) of Theorem 2, and so the bipartization \( B_f(H) \) is a tree. \( \square \)

**Example 3** Figure 2 shows the tree \( G \) which is the bipartization of two block graphs \( H \) and \( F \) with respect to functions \( \overline{f} \) and \( \overline{g} \), respectively, which non-zero values are listed in the same figure.

### 4 Graphs Belonging to the Family \( \mathcal{B} \)

In this section, we provide an alternative characterization of all bipartite graphs whose domination number is equal to the cardinality of its smaller partite set, that is, we prove that a connected graph \( G \) belongs to the class \( \mathcal{B} \) if and only if \( G \) is some bipartization of a graph. For that purpose, we need the following lemma.

**Lemma 3** [4] Let \( G = ((A, B), E_G) \) be a connected bipartite graph with \( 1 \leq |A| \leq |B| \). Then the following statements are equivalent:

1. \( \gamma(G) = |A| \).
2. \( \gamma(G) = \beta(G) = |A| \).
3. \( G \) has the following two properties:
   
   (a) Each support vertex of \( G \) belonging to \( B \) is a weak support and each of its non-leaf neighbors is a support.
   
   (b) If \( x \) and \( y \) are vertices belonging to \( A - (L_G \cup S_G) \) and \( d_G(x, y) = 2 \), then there are at least two vertices \( \overline{x} \) and \( \overline{y} \) in \( B \) such that \( N_G(\overline{x}) = N_G(\overline{y}) = \{x, y\} \).
We are ready to establish our main theorem that provides an alternative characterization of the graphs belonging to $B$ in terms of the bipartization of a graph.

**Theorem 3** Let $G = ((A, B), E_G)$ be a connected bipartite graph with $1 \leq |A| \leq |B|$. Then $\gamma(G) = |A|$ if and only if $G$ is the bipartization $B_f(H)$ of a connected graph $H$ with respect to a non-zero function $f : \mathcal{K}_H \to \mathbb{N}$ and $f$ has the following two properties:

1. If $uv \in E_H$ and $f(H[u, v]) = 0$, then $f(H') > 0$ for some complete subgraph $H'$ of $H$ containing the edge $uv$.
2. If $uv \in E_H$ and $f(H[u]) = f(H[v]) = 0$, then $f(H[u, v]) \geq 2$.

**Proof** Assume first that $\gamma(G) = |A|$. Then $G$ has the properties (3a) and (3b) of Lemma 3. Let $H = (V_H, E_H)$ be a graph in which $V_H = A$ and $E_H = \{xy : x, y \in A$ and $d_G(x, y) = 2\}$, and let $f : \mathcal{K}_H \to \mathbb{N}$ be a function such that $f(K) = |\{x \in B : N_G(x) = V_K\}|$ for each $K \in \mathcal{K}_H$. Then $G$ is the bipartization $B_f(H)$ of $H$ with respect to $f$ as we have shown in the proof of Theorem 1. It is obvious that if $H = K_1$, then $\mathcal{K}_H = \{H\}$ and it must be $f(H) \geq 1$ (as otherwise $G = B_f(H)$ would be a graph of order one). Thus assume that $H$ is non-trivial. Now it remains to prove that $f$ has the properties (1) and (2).

Let $uv$ be an edge of $H$ such that $f(H[u, v]) = 0$. Suppose on the contrary that $f(H') = 0$ for every complete subgraph $H'$ containing the edge $uv$. Then the vertices $u$ and $v$ do not share a neighbor in $B_f(H) = G$, so $d_G(u, v) > 2$ and $uv$ is not an edge in $H$, a contradiction. This proves the property (1).

Now let $uv$ be an edge of $H$ such that $f(H[u]) = f(H[v]) = 0$. From these assumptions it follows that $d_G(u, v) = 2$ and neither $u$ nor $v$ is a support vertex in $G = B_f(H)$. Now we shall prove that none of the vertices $u$ and $v$ is a leaf in $G$. First, because $u, v \in A$ and they have a common neighbor, it follows from the first part of the property (3a) of Lemma 3 that at least one of the vertices $u$ and $v$ is not a leaf in $G$. Suppose now that exactly one of the vertices $u$ and $v$ is a leaf in $G$, say $u$ is a leaf. Then it follows from the second part of the property (3a) of Lemma 3 that $v$ is a support vertex in $G = B_f(H)$ and, therefore, $f(H[v]) > 0$, a contradiction. Consequently, both $u$ and $v$ are elements of $A - N_G[L_G]$. Thus, since $d_G(u, v) = 2$, the property (3b) of Lemma 3 implies that there are at least two vertices $\bar{u}, \bar{v} \in B$ such that $N_G(\bar{u}) = N_G(\bar{v}) = \{u, v\}$. Therefore $f(H[u, v]) = |\{x \in B : N_G(x) = \{u, v\}\}| \geq |\{\bar{u}, \bar{v}\}| = 2$ and this proves the property (2).

Assume now that $H$ is a connected graph, and $f : \mathcal{K}_H \to \mathbb{N}$ is a non-zero function having the properties (1) and (2). We shall prove that in the bipartization $B_f(H) = ((A, B), E_{B_f(H)})$, where $A = V_H$ and $B = \bigcup_{K \in \mathcal{K}_H} \mathcal{F}_K$, is $|A| \leq |B|$ and $\gamma(B_f(H)) = |A|$. This is obvious if $H$ is a graph of order 1. Thus assume that $H$ is a graph of order at least 2. From the property (1) it follows that $B_f(H)$ is a connected graph. We first prove the inequality $|A| \leq |B|$. To prove this, it suffices to show that $B_f(H)$ has an $A$-saturating matching. We begin by dividing $A = V_H$ into two subsets $V^1_H = \{v \in V_H : f(H[v]) \geq 1\}$ and $V^0_H = \{v \in V_H : f(H[v]) = 0\}$. It is obvious that the edge-set $M^1 = \{v(H[v]) \geq 1 : v \in V^1_H\}$ is a $V^1_H$-saturating matching in $B_f(H)$. Next, we order the set $V^0_H$ in an arbitrary way, say $V^0_H = \{v_1, \ldots, v_n\}$. Now,
depending on this order, we consecutively choose edges \(e_1, \ldots, e_n\) in such a way that 
\(M^1 \cup \{e_1, \ldots, e_i\}\) is a \((V^1_H \cup \{v_1, \ldots, v_i\})\)-saturating matching in \(B_f(H)\).

Assume that we have already chosen a \((V^1_H \cup \{v_1, \ldots, v_{i-1}\})\)-saturating matching 
\(M^1 \cup \{e_1, \ldots, e_{i-1}\}\) in \(B_f(H)\), and consider the next vertex \(v_i \in V^0_H, f(N_H(v_i)) \neq \emptyset\), say 
\(v_j \in N_H(v_i) \cap V^0_H\), then \(f(H[v_j]) = 0\) and therefore \(f(H[v_j]) \geq 2\) (by the property (2)) and the edge \(e_i = v_i(H[v_i, v_j], 1)\) if \(j < i\), \(e_i = v_i(H[v_i, v_j], 2)\) if \(j < i\) together with \(M^1 \cup \{e_1, \ldots, e_{i-1}\}\) form a \((V^1_H \cup \{v_1, \ldots, v_i\})\)-saturating matching in \(B_f(H)\). Thus assume that \(N_H(v_i) \subseteq V^0_H\). Let \(v\) be a neighbor of \(v_i\) in 
\(H\). If \(f(H[v_i, v]) \geq 1\), then the edge \(e_i = v_i(H[v_i, v], 1)\) has the desired property. Finally, if 
f \((H[v_i, v]) = 0\), then \((H') > 0\) for some complete subgraph \(H'\) of \(H\) containing 
the edge \(v_i v\) (by the property (1)) and in this case the edge \(e_i = v_i(H', 1)\) has the desired property (as \(N_H(v_i) \subseteq V^1_H\)). Repeating this procedure as many times 
as needed, an \(A\)-saturating matching in \(B_f(H)\) can be obtained.

To complete the proof, it remains to show that \(\gamma(B_f(H)) = |A|\). In a standard way, 
suppose to the contrary that \(\gamma(B_f(H)) < |A|\). Let \(D\) be a minimum dominating set 
of \(B_f(H)\) with \(|D \cap A|\) as large as possible. Since \(\gamma(B_f(H)) = |D|\), the inequality 
\(\gamma(B_f(H)) < |A|\) implies that \(|A - D| > |D \cap B| \geq 1\). In addition, since \(|D \cap A|\) is as 
large as possible, the set \(V^1_H = \{v \in V_H : f(H[v]) \geq 1\}\) is a subset of \(D \cap A\), while 
\(A - D\) is a subset of \(V^0_H = \{v \in V_H : f(H[v]) = 0\}\). Now, because \(|A - D| > |D \cap B|\) and 
each vertex of \(A - D\) has a neighbor in \(D \cap B\), the pigeonhole principle implies that there 
are two vertices \(x, y\) in \(A - D\) which are adjacent to the same vertex in 
\(D \cap B\). Hence, \(x\) and \(y\) are adjacent in \(H\) (by the definition of \(B_f(H)\)). Now, since 
f \((H[x]) = f(H[y]) = 0\), the property (2) implies that \(f(H[x, y]) \geq 2\). Next, since 
\(N_{B_f(H)}((H[x, y], 1)) = N_{B_f(H)}((H[x, y], 2)) = \{x, y\}\) and \(x, y \cap D = \emptyset\), the 
vertices \((H[x, y], 1)\) and \((H[x, y], 2)\) belong to \(D \cap B\). Consequently, it is easy to 
observe that the set \(D' = (D - (H[x, y], 1), (H[x, y], 2))\) \(\cup \{x, y\}\) is a dominating 
set of \(B_f(H)\), which is impossible as \(|D'| = |D|\) and \(|D' \cap A| > |D \cap A|\). This 
completes the proof.

\(\square\)

**Example 4** The graph \(H\) and the function \(f : \mathbb{K}_H \to \mathbb{N}\) given in Example 1 have the 
properties (1) and (2) of Theorem 3 and therefore the bipartization \(B_f(H)\) belongs to 
the family \(\mathcal{B}\), that is, \(\gamma(B_f(H)) = |A|\), where \(A\) is the smaller of two partite sets of 
\(B_f(H)\) shown in Fig. 1.

The graph \(F\) and the function \(\overline{f}\) given in Fig. 2 do not satisfy the condition (2) 
of Theorem 3. However, the bipartization \(G = B_{\overline{f}}(F)\) is a graph belonging to the 
family \(\mathcal{B}\) since \(G\) is also the bipartization \(B_{\overline{f}}(H)\), with \(H\) and \(\overline{f}\) given in Fig. 2 
and possessing properties (1) and (2) of Theorem 3.

It is obvious that the complete bipartite graph \(K_{m,n}\) is the bipartization of the 
complete graph \(K_m\) (resp. \(K_n\)) with respect to the function \(f : \mathbb{K}_{K_m} \to \{0, n\}\), where 
\(f(K) = 0\) if and only if \(K \in \mathbb{K}_{K_m} - \{K_m\}\), (resp. \(g : \mathbb{K}_{K_n} \to \{0, m\}\), where \(g(K) = 0\) 
if and only if \(K \in \mathbb{K}_{K_n} - \{K_n\}\)). It is also evident that if \(\min\{m, n\} \geq 3\), then \(K_{m,n}\) 
does not belong to the family \(\mathcal{B}\) (as \(\gamma(K_{m,n}) = 2 < \min\{m, n\}\)), and neither \(K_n\) and 
\(f\) nor \(K_n\) and \(g\) possess the property (2) of Theorem 3.
Finally, as an immediate consequence of Theorems 2 and 3 we have the following simple characterization of trees in which the domination number is equal to the size of a smaller of its partite sets. All such trees are bipartizations of block graphs.

**Corollary 2** Let $T = ((A, B), E_T)$ be a tree in which $1 \leq |A| \leq |B|$. Then $\gamma(T) = |A|$ if and only if $T$ is the bipartization $B_f(H)$ of a block graph $H$ with respect to a non-zero function $f : \mathcal{K}_H \to \mathbb{N}$ and $f$ has the following two properties:

1. $f(K) = 1$ if $K$ is a block of $H$, and $f(K') = 0$ if $K'$ is a non-trivial complete subgraph of $H$ which is not a block of $H$.
2. $\max\{f(H[u]), f(H[v])\} \geq 1$ for every edge $uv$ of $H$ (or, equivalently, the set $\{v \in V_H : f(H[v]) \geq 1\}$ is a covering set of $H$).

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