Model of bosonization by flux attachment on hamiltonian lattices of arbitrary dimension

A. Bochniak, B. Ruba

Institute of Theoretical Physics, Jagiellonian University in Kraków, prof. Łojasiewicza 11, 30-348 Kraków, Poland
E-mail: arkadiusz.bochniak@doctoral.uj.edu.pl, blazej.ruba@doctoral.uj.edu.pl

ABSTRACT: We present and prove the correctness of a bosonization prescription for fermionic lattice models in arbitrary dimensions. Our bosonized model is subject to constraints, which are interpreted in the language of lattice $\mathbb{Z}_2$ gauge theory. Complete solutions of the constraints is found in the case of even-even spatial lattices. Further possible relations with other topologically non-trivial lattice models are discussed.
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1 Introduction

Many fermionic systems admit bosonizations, i.e. dual descriptions formulated using bosonic operators. Such correspondences are especially abundant in spacetime dimension two [1–3]. They are of paramount importance, as they allow to construct analytic solutions of certain models [4, 5], to gain nonperturbative insights into dynamics of strongly coupled systems [6] and more recently to understand certain phases of topologically nontrivial fermionic matter [7]. Furthermore, there exist systems for which bosonization helps to overcome problems in numerical studies, such as the sign problem in Monte Carlo simulations or difficulties in implementation of operators acting on Hilbert spaces which do not factorize into tensor products of on-site Hilbert spaces. This last problem may also have some significance for the field of quantum information [8, 9].

The most well-known bosonization prescriptions apply only to two-dimensional systems. Some proposals valid in higher dimensions have been put forward [7, 10–17]. Nevertheless, each of these constructions introduces some additional difficulties not present for two-dimensional systems, such as non-locality or presence of complicated constraints. One might
argue that this problem is an inherent feature of models involving fermionic degrees of freedom. Better understanding of these phenomena might help to construct bosonization maps truly useful in practical calculations.

This paper is concerned with the study of method prescription proposed in [11]. In this approach fermion fields are replaced by on-site $\Gamma$ matrices. The price to pay is that one has to introduce certain constraints on physical states. Correctness of this prescription, at least in the case of free fermions, has been conjectured based on matching of braiding relations in these two models. The first proof of this (in a sense to be clarified in the main text) has been presented in [18]. It turned out that the correspondence is purely kinematical, i.e. that it is valid for any choice of hamiltonian for fermions. In this work we present a new proof, whose structure is inspired by techniques presented in [19]. It provides an interpretation of pertinent constraints in terms of $\mathbb{Z}_2$ gauge theory: they are seen as a flatness condition on the gauge field and choice of a spin structure. Our work parallels [20], which motivated our studies, allowed to formulate initial hypotheses and tested our results using symbolic algebra software.

Encouraged by the emerging physical interpretation we study the content of the theory described by $\Gamma$ matrices in the case that constraints are not imposed. Then the $\mathbb{Z}_2$ gauge field has to be regarded as a new degree of freedom. It turns out that it is not possible to completely decouple it from fermions. Indeed, there exists an exact relation between the total charge modulo 2 and certain combination of holonomies. This resembles certain properties of gauge systems with Chern-Simons couplings [17, 19, 21–29], but for now we do not offer a definitive answer about the precise relation.

Despite the use of similar methods, bosonization map discussed here is (at least superficially) distinct from the one presented in [10, 19]. The first difference is in geometric setups in these two approaches. In our model local degrees of freedom act on Hilbert spaces associated to lattice sites, while in the aforementioned formulation degrees of freedom associated to edges have been proposed. Therefore, it is not surprising that constraints arising from the two bosonization prescriptions appear to be different. Secondly, construction described here involves a cubical lattice on a torus. This is done for simplicity and in order to preserve the possibility to write down translationally invariant hamiltonians. As shown in [18] it is possible to generalize this framework. Nevertheless, it seems to be essential that there is an even number of edges emanating from every lattice site. The reason for that is these edges are put in one-to-one correspondence with generators of a Clifford algebra associated to the lattice site. Then we use the fact that product of all generators anti-commutes (rather than commutes) with every generator, forcing the number of generators to be even. This should be contrasted with [19], in which general triangulations are considered.

Organization of this paper is as follows. In Section 2 we introduce the language of homology and cohomology, which is a technical language used afterwards. Reader not at all familiar with these concepts may want to consult introductory books in algebraic topology [30] first. A part of this material not strictly necessary to understand the rest of the text is moved to the Appendix A. Section 3 is concerned mainly with the description of the algebra of even fermionic operators in terms of a convenient set of generators and relations. This is then used to establish that the $\Gamma$ model introduced in Section 4 is equivalent to fermions.
We investigate dependence of the $\Gamma$ model on a certain deformation parameter and describe structure of constraints and their solutions. This reveals a connection between this system and Kitaev’s toric code [8]. In Section 5 we discuss the role of constraints and interpret them in terms of gauge theory. An extended discussion of $\mathbb{Z}_2$ gauge theory from topological perspective is contained in Appendix B. We summarize and discuss future directions in Section 6.

2 Lattice geometry - homological language

In order to keep track of numerous sign factors appearing in manipulations of fermionic operators and their bosonic counterparts we will use the language of (co)chains, which we introduce in this section. Informally, they may be thought of as formal sums of certain geometric objects: points, lines, squares, cubes and their higher-dimensional analogues. For the purpose of understanding sign factors, it is sufficient to count them modulo 2. This means that we will consider vector spaces over the finite field $\mathbb{Z}_2$, with algebraic operations defined by the formulas

\[0 + 0 = 1 + 1 = 0, \quad 0 + 1 = 1, \quad 0 \cdot 0 = 0 \cdot 1 = 0, \quad 1 \cdot 1 = 1.\]

Furthermore, for any set $S$ we will denote the number of its elements by $|S|$.

We will consider systems on a toric lattice of dimension $d \geq 1$ and size $L_1 \times \ldots \times L_d$. This means that vertices of our lattice are in one-to-one correspondence with the set $V$ of $d$-tuples of integers, with two $d$-tuples identified if they differ by a vector whose $i$-th entry is a multiple of $L_i$ for every $i \in D := \{1, \ldots, d\}$. To avoid certain pathologies, we assume that $L_i \geq 3$ for each $i$.

For $i \in D$ we define a transformation $e_i$ acting on the set $V$ in the following way

\[e_i \cdot (v_1, \ldots, v_d) = (v_1, \ldots, v_i + 1, \ldots, v_d).\] (2.1)

These transformations generate a finite group, which we denote by $G$. For any $I \subseteq D$ we denote by $G_I$ the subgroup of $G$ generated by $\{e_i\}_{i \in I}$. Furthermore, we put $e_I = \prod_{i \in I} e_i$.

Now let $p \in \{0, 1, \ldots, d\}$. Subset $q \subseteq V$ is said to be a $p$-cube if there exist $v \in V$ and $I \subseteq D$ such that $|I| = p$ and $q = \{e_J \cdot v\}_{J \subseteq I}$. Notice that if $q$ is a $p$-cube, then $v \in V$ and $I \subseteq D$ are uniquely determined by the set $q$. We write $\text{in}(q) := v$, $\text{fin}(q) := e_I \cdot v$ and $I(q) := I$. The vertices $\text{in}(q)$ and $\text{fin}(q)$ are called the initial vertex and the final vertex of a $p$-cube $q$, respectively. We denote the set of all $p$-cubes by $Q_p$. One may identify $Q_0$ with $V$. Accordingly, 1-cubes will be called edges, and 2-cubes will be called faces.

We let $Q = \bigcup_{p=0}^d Q_p$ be the set of all cubes. We note the famous topological relation, expressing the fact that tori have vanishing Euler characteristic:

\[\sum_{p=0}^d |Q_p|(-1)^p = 0.\] (2.2)

For our purposes, $p$-chain is an expression of the form $c = \sum_{q \in Q_p} n_q q$ with some coefficients $n_q \in \mathbb{Z}_2$. The set of all $p$-chains is denoted by $C_p$. It is a vector space over the field
Z₂, with p-cubes serving as a distinguished basis. Space of all chains is defined as the direct sum \( C = \bigoplus_{p=0}^{d} C_p \).

For every \( v, v' \in V \) there is at most one cube with the property that its initial vertex is \( v \) and the final one is \( v' \). If such cube exists, we denote it by \( q(v, v') \). Otherwise we put \( q(v, v') = 0 \). Furthermore, we put \( q_I(v) = q(v, e_I \cdot v) \) for \( I \subseteq D \).

Boundary of a p-cube \( q \) is defined as \( \partial q = \sum q' \), with the sum taken over all \((p-1)\)-cubes \( q' \) contained in \( q \). This operation extends to whole \( C \) by linearity: \( \partial \sum q_n q = \sum q_n \partial q \).

It satisfies \( \partial^2 = 0 \).

We will say that an element \( c \in C_p \) is a p-cycle (resp. p-boundary) if \( \partial c = 0 \) (resp. \( c = \partial c' \) for some \( c' \in C_{p-1} \)). We denote the set of p-cycles by \( Z_p \) and the set of p-boundaries by \( B_p \). Equation \( \partial^2 = 0 \) implies that \( B_p \subseteq Z_p \). The converse is not true. To describe this phenomenon, one defines the homology \( H_p \) in degree \( p \) as \( H_p = \frac{Z_p}{B_p} \), i.e. the space of equivalence classes of p-cycles, with two p-cycles identified if they differ by a p-boundary.

Later on we will need to know the form of a general p-cycle, at least for \( p \in \{0, 1, 2\} \). By the very definition of homology, one may proceed as follows: choose one representing cycle for every equivalence class in \( H_p \). Then any given p-chain may be represented uniquely as the sum of the preferred representative of its homology class and of some boundary. Thus we proceed to describe the structure of \( H_p \).

For \( I \subseteq D \) and \( v \in V \) we define an element \( l_I(v) \in C_{\mid I\mid} \) by

\[
l_I(v) = \sum_{g \in G_I} q_I(g \cdot v). \tag{2.3}
\]

It is easy to check that \( l_I(v) \) is a cycle: \( \partial l_I(v) = 0 \). Its homology class, which we denote by \( \ell_I \), does not depend on \( v \). It is well known that the set of all \( \ell_I \) with \( \mid I\mid = p \) forms a basis of \( H_p \). Among all homology classes, the so-called fundamental class \( \ell_D \) is distinguished by the fact that it admits a canonical representative:

\[
\text{vol} = \sum_{q \in Q_d} q. \tag{2.4}
\]

We define a non-degenerate symmetric bilinear form on \( C \) by \( (q, q') = \delta^q_{q'} \) for \( q, q' \in Q \). Using it we can define the coboundary operator \( \delta \) by the formula

\[
(c, \delta c') = (\delta c, c'), \tag{2.5}
\]

for \( c, c' \in C \). More explicitly, for \( q \in Q_p \) we have \( \delta q = \sum q' \), with the sum taken over all \( q' \in Q_{p+1} \) such that \( q \subseteq q' \). Alternatively, one can think of \( \delta \) as the boundary operator of the dual lattice. It follows from \( \delta^2 = 0 \) that we have \( \delta^2 = 0 \). Proceeding as for \( \partial \), we will call an element \( c \in C_p \) a p-cocycle (and write \( c \in Z^p \)) if \( \delta c = 0 \) and a p-coboundary (and write \( c \in B^p \)) if \( c = \delta c' \) for some \( c' \in C_{p-1} \). The set of all p-cocycles, with two cocycles identified if they differ by a coboundary, will be denoted by \( H^p \) and called the cohomology in degree \( p \).
It is easy to check directly from the definition that cycles are orthogonal to coboundaries and that boundaries are orthogonal to cocycles. This means that the bilinear form on $C$ induces a pairing between $H_p$ and $H^p$. It turns out that this pairing is nondegenerate. Therefore, the formula
\[
(\tilde{\ell}_I, \ell_J) = \delta^J_I
\] (2.6)
well-defines the cohomology classes $\tilde{\ell}_I$, which form a basis of cohomology.

It will be useful to have an explicit representative of the class $\tilde{\ell}_I$. To do this, put $J = \mathcal{D} \setminus I$, choose some $v \in V$ and define
\[
\tilde{\ell}_I(v) = \sum_{g \in G_J} q_I(g \cdot v).
\] (2.7)
One easily checks that $\tilde{\ell}_I(v)$ is a cocycle representing the class $\tilde{\ell}_I$.

3 Fermions - generators and relations

Here we consider a specific class of fermionic models, defined below. We emphasize those aspects of its mathematical structure which are used to prove validity of our bosonization prescription. In particular, we describe the algebra of even fermionic operators in terms of generators and relations. This result is similar to [19], with the statement and the proof adjusted to the fact that we work with finite, not simply-connected lattices. We do not consider any particular hamiltonian, so our considerations are independent of dynamics.

First, let us denote by $\mathcal{A}$ the complex $*$-algebra generated by elements $\phi^*(v)$ and $\phi(v)$ (called creation and annihilation operators located at the vertex $v$) with $v \in V$, subject to the canonical anticommutation relations
\[
\{\phi(v), \phi(v')\} = \{\phi^*(v), \phi^*(v')\} = 0,
\]
\[
\{\phi(v), \phi^*(v')\} = \delta^v_{v'}.
\] (3.1a, 3.1b)

By construction, every element of $\mathcal{A}$ may be written down as a linear combination of products of creation and annihilation operators. It is often useful to use a different set of generators of $\mathcal{A}$, e.g. the so-called Majorana operators:
\[
X(v) = \phi(v) + \phi^*(v), \quad Y(v) = i(\phi(v) - \phi^*(v)).
\] (3.2)

Defining relations (3.1) are equivalent to
\[
\{X(v), Y(v')\} = 0,
\]
\[
\{X(v), X(v')\} = \{Y(v), Y(v')\} = 2\delta^v_{v'}.
\] (3.3a, 3.3b)

This shows that $\mathcal{A}$ is a Clifford algebra on $2|V|$ generators, and hence $\mathcal{A}$ is isomorphic to the algebra $M_{2|V|}(\mathbb{C})$ of $2^{|V|} \times 2^{|V|}$ complex matrices. In particular, it has a unique (up to isomorphism) irreducible representation, which is faithful and has dimension $2^{|V|}$. Every finite-dimensional representation of $\mathcal{A}$ is a direct sum of finitely many copies of the irreducible representation.
The unique irreducible representation of \( \mathcal{A} \) is, of course, the Fock space, here denoted by \( \mathcal{F} \). It is a Hilbert space with a distinguished element \( |0\rangle \) (called the vacuum state), determined uniquely up to phase by the conditions \( \phi(v)|0\rangle = 0 \) and \( (0)|0\rangle = 1 \). Other states, labeled by 0-chains \( e \), are defined by acting with creation operators on the vacuum:

\[
|e\rangle = \prod_{v \in V} \phi^*(v)^{(e,v)}|0\rangle.
\] (3.4)

This vector actually depends on the ordering of vertices in the product, but different orderings give rise to states differing only by a factor \( \pm 1 \). To well-define elements \( |e\rangle \), fix any total order on \( V \) once and for all. The set of all vectors \( |e\rangle \) is an orthonormal basis of \( \mathcal{F} \).

Let us define the grading element of \( \mathcal{A} \):

\[
\gamma = \prod_{v \in V} (1 - 2\phi^*(v)\phi(v)).
\] (3.5)

It satisfies \( \gamma = \gamma^* = \gamma^{-1} \). For each \( \alpha \in \mathbb{Z}_2 \) we define

\[
\mathcal{F}_\alpha = \{ \psi \in \mathcal{F} | \gamma \psi = (-1)^\alpha \psi \},
\]

\[
\mathcal{A}_\alpha = \{ T \in \mathcal{A} | \forall \alpha' \in \mathbb{Z}_2 \ T \mathcal{F}_{\alpha'} \subseteq \mathcal{F}_{\alpha+\alpha'} \}.
\] (3.6a, 3.6b)

Equivalently, \( \mathcal{A}_\alpha = \{ T \in \mathcal{A} | \gamma T = (-1)^\alpha T \gamma \} \). Subspace \( \mathcal{A}_0 \) is a subalgebra of \( \mathcal{A} \). Its action on \( \mathcal{F} \) has two nontrivial invariant subspaces: \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \), which are both of dimension \( 2^{|V|-1} \).

It follows from the Artin-Weddeburn theory [31] that the algebra \( \mathcal{A}_0 \) is semisimple, with two simple factors \( \mathcal{A}_{0\alpha} = \text{End}_\mathbb{C}(\mathcal{F}_\alpha), \alpha \in \mathbb{Z}_2 \). This means that every finite-dimensional representation \( V \) of \( \mathcal{A}_0 \) is isomorphic to \( \bigoplus_{\alpha \in \mathbb{Z}_2} \mathcal{F}_\alpha^{V:F_\alpha} \), where multiplicity \( [V : \mathcal{F}_\alpha] \) is given by the formula

\[
[V : \mathcal{F}_\alpha] = \frac{1}{\dim_\mathbb{C}(\mathcal{F}_\alpha)} \text{tr}_V \left( \frac{1 + (-1)^\alpha \gamma} {2} \right).
\] (3.7)

The even subalgebra \( \mathcal{A}_0 \) is of our main interest here. First, we define elements \( \{ \gamma(v) \}_{v \in V} \) and \( \{ s(e) \}_{e \in Q_1} \) in \( \mathcal{A}_0 \) by

\[
\gamma(v) = 1 - 2\phi^*(v)\phi(v),
\] (3.8a)

\[
s(e) = X(\text{in}(e))X(\text{fin}(e)).
\] (3.8b)

We refer to \( \gamma(v) \) as fermionic parity operator at vertex \( v \), and to \( s(e) \) as hopping operator on the edge \( e \). It is not difficult to check that these operators generate the algebra \( \mathcal{A}_0 \).

Our next goal is to find a complete set of relations satisfied by our chosen generators\(^1\).

Direct calculation shows that the following relations hold in \( \mathcal{A}_0 \):

\[
\gamma(v) = \gamma(v)^* = \gamma(v)^{-1},
\] (3.9a)

\[
-s(e) = s(e)^* = s(e)^{-1},
\] (3.9b)

\[
\gamma(v)\gamma(v') = \gamma(v')\gamma(v),
\] (3.9c)

\[
\gamma(v)s(e) = (-1)^{(\partial e,v)}s(e)\gamma(v),
\] (3.9d)

\[
s(e)s(e') = (-1)^{(\partial e,\partial e')}s(e')s(e).
\] (3.9e)

\(^1\)More precisely, \( \mathcal{A}_0 \) is isomorphic to the free algebra on letters \( \gamma(v), s(e) \) divided by some two-sided ideal \( I \). We are looking for a set of generators of \( I \).
Before we write down the last relation, we need to introduce additional operators:

\[ \gamma(\epsilon) = \prod_{v \in V} \gamma(v)^{\epsilon,v}, \quad \text{for } \epsilon \in C_0, \]  
\[ s(\tau) = \prod_{e \in Q_1} s(e)^{\epsilon,e\tau}, \quad \text{for } \tau \in C_1. \]

(3.10a) \hspace{1cm} (3.10b)

As for the states |\epsilon\rangle, the sign of the operator \( s(\tau) \) depends on a choice of an ordering of the set of edges. We fix one such ordering once and for all.

The final relation in \( A_0 \) may be formulated as follows: there exists a unique function \( \Phi : Z_1 \to Z_2 \) such that for any \( \tau \in Z_1 \) we have

\[ s(\tau) = (-1)^{\Phi(\tau)}. \]  

(3.11)

Notice that not all of these relations are independent. Indeed, suppose that some algebra \( B \) contains elements \( \gamma(v) \) and \( s(e) \) satisfying (3.9) and such that relation (3.11) holds for \( \tau \) running through some set of generators \( \{\tau_i\}_{i=1}^s \) of \( Z_1 \). Then using relations (3.9) only, we can rewrite \( s\left(\sum_{i=1}^s c_i \tau_i\right) \) as the product

\[ \prod_{i=1}^s s(c_i \tau_i) = (-1)^{f(c_1,\ldots,c_s)+\sum_{i=1}^s \Phi(c_i \tau_i)}, \]  

(3.12)

for some uniquely determined function \( f : \prod_{i=1}^s Z_2 \to Z_2 \), possibly depending on the choice of ordering of \( Q_1 \). Performing the same calculation in \( A_0 \) we get that

\[ (-1)^{\Phi\left(\sum_{i=1}^s c_i \tau_i\right)} = (-1)^{f(c_1,\ldots,c_s)+\sum_{i=1}^s \Phi(c_i \tau_i)}. \]  

(3.13)

Hence (3.11) is satisfied in \( B \) for all \( \tau \).

Function \( \Phi \) is rather complicated. In particular, it depends on the arbitrary choice of ordering of the set of edges. Fortunately, in verifying relations satisfied by \( s(\tau) \) constructed in bosonized theories it is sufficient to restrict attention to the relation (3.11) with \( \tau \) running through some convenient generating set of \( Z_1 \). We may take any representatives of some basis of \( H_1 \), say \( \{l_i(v)\}_{i=1}^d \), and any (not necessarily linearly independent) generating set of \( B_1 \), such as \( \{\partial q\}_{q \in Q_2} \).

Our goal is to show that there are no other relations, i.e. that (3.9) and (3.11) generate all relations in the algebra \( A_0 \). Our proof is inspired by techniques used in [19]. It will be necessary to understand how our distinguished elements of \( A_0 \) act on the Fock space. Firstly, we have the eigenvalue equation

\[ \gamma(\epsilon)|\epsilon\rangle = (-1)^{\epsilon,\epsilon'}|\epsilon\rangle. \]  

(3.14)

Secondly, there exists a (unique) function \( \chi : C_1 \times C_0 \to Z_2 \) such that

\[ s(\tau)|\epsilon\rangle = (-1)^{\chi(\tau,\epsilon)}|\epsilon + \partial \tau\rangle. \]  

(3.15)
As for $\Phi$, precise form of the function $\chi$ depends on the choice of ordering of vertices and edges. We shall not need it.

As a first step to prove the assertion, notice that using relations (3.9) only any monomial in the generators $s(\epsilon)$ and $\gamma(v)$ may be rewritten (perhaps up to a sign) as a product $\gamma(\epsilon)s(\tau)$ for some $\epsilon \in C_0$ and $\tau \in C_1$.

Now let $s$ be a section of $\partial : C_1 \to B_0$, i.e. a linear map $B_0 \to C_1$ such that $\partial s = 1_{B_0}$. Notice that such $s$ is guaranteed to exist, because $\partial$ is a linear map between vector spaces, with image $B_0$. However it is by no means unique.

For any $\tau \in C_1$ let $z(\tau) = \tau - s\partial \tau \in C_1$. Then we have $\tau = s\partial \tau + z(\tau)$ and $\partial z(\tau) = 0$, so $s(\tau)$ coincides with $s(s\partial \tau)$, possibly up to a sign. This means that, up to a sign, monomial $\gamma(\epsilon)s(\tau)$ depends on $\tau$ only through $\partial \tau$.

Using relations described so far, any relation in $A_0$ may be reduced to

$$\sum_{\epsilon \in C_0} \sum_{\epsilon' \in B_0} c_{\epsilon,\epsilon'} \gamma(\epsilon)s(\epsilon') = 0,$$

(3.16)

where $c_{\epsilon,\epsilon'}$ are complex coefficients.

Acting with the operator on the left hand side on the vector $|\epsilon''\rangle$ we obtain

$$\sum_{\epsilon \in C_0} \sum_{\epsilon' \in B_0} c_{\epsilon,\epsilon'} (-1)^{\epsilon(\epsilon' + \epsilon'')} (-1) \chi(s(\epsilon'),\epsilon'') |\epsilon'' + \epsilon'\rangle = 0.$$

(3.17)

Since the set $\{|\epsilon''\rangle\}_{\epsilon'' \in B_0}$ is linearly independent in $F$, each term of the summation over $\epsilon'$ vanishes separately. Therefore we have

$$\sum_{\epsilon \in C_0} c_{\epsilon,\epsilon'} (-1)^{\epsilon(\epsilon' + \epsilon'')} = 0,$$

(3.18)

Now let $\epsilon''' = \epsilon' + \epsilon''$, take any $\tau \in C_0$ and multiply this equation by $(-1)^{\epsilon(\epsilon'')).$ Summing over $\epsilon'''$ and using the Fourier inversion formula for the group $\mathbb{Z}_2$ we get

$$c_{\tau,\epsilon'} = 0.$$

(3.19)

Since $\tau$ and $\epsilon'$ were arbitrary, all coefficients $c$ vanish. We have shown that any relation in $A_0$ follows already from (3.9) and (3.11), which completes the proof.

4 Constrained $\Gamma$ model

We will now construct a bosonic model equivalent to the fermionic one discussed in the previous section. Relations (3.9) will be satisfied as operator equations, but (3.11) will be imposed only as a constraint on physical states. Due to the presence of $\Gamma$ matrices in its formulation, we will refer to it as the $\Gamma$ model [11].

4.1 Definition of the model

Our basic degrees of freedom are $2d$-dimensional Euclidean $\Gamma$ matrices $\{\Gamma_{i}(v)\}_{i=1}^{d}$ assigned to vertices $v \in V$. In other words, for fixed $v \in V$ operators $\Gamma_{i}(v)$ satisfy

$$\{\Gamma_i(v), \Gamma_j(v)\} = 2\delta_{ij} \quad \text{for } i, j \in \{\pm 1, \ldots, \pm d\},$$

(4.1)
but $[\Gamma_i(v), \Gamma_j(v')] = 0$ for $v \neq v'$. For each $v$ the set of all $\Gamma_i(v)$ acts on its unique irreducible representation $\mathcal{H}_i^\Gamma(v)$, which is of dimension $2^d$. Therefore, the Hilbert space of this system $\mathcal{H}^\Gamma = \bigotimes_{v \in V} \mathcal{H}_i^\Gamma(v)$ is of dimension $2^d|V|$.

Now we choose arbitrarily\footnote{2Dependence of the content of this model on the choice of $\eta$ is discussed in section 4.2.} some chain $\eta \in C_0$ and introduce operators

$$\Gamma_s(v) = i^d(-1)^{\langle \eta, v \rangle} \prod_{i=1}^d \Gamma_i(v)\Gamma_{-i}(v), \quad (4.2a)$$

$$\Gamma_s = \prod_{v \in V} \Gamma_s(v), \quad (4.2b)$$

and more generally $\Gamma_s(\epsilon) = \prod_{v \in V} \Gamma_s(v)\epsilon(v)$ for $\epsilon \in C_0$. Secondly, we let

$$S(e) = -i\Gamma_i(\text{fin}(e))\Gamma_{-i}(\text{fin}(e)), \quad \text{for } I(e) = \{i\}, \quad (4.3)$$

and put $S(\tau) = \prod_{e \in Q_1} S(e)^{(\tau,e)}$ for $\tau \in C_1$.

Simple calculation shows that mapping

$$\gamma(v) \mapsto \Gamma_s(v), \quad s(e) \mapsto S(e) \quad (4.4)$$

is compatible with (3.9). However, we do not have (3.11) as an operator equation,

$$S(\tau) \not\equiv (-1)^{\Phi(\tau)}, \quad \text{for } \tau \in Z_1. \quad (4.5)$$

Therefore we would like to impose (3.11) as a constraint on physical states:

$$S(\tau)|\text{phys}\rangle = (-1)^{\Phi(\tau)}|\text{phys}\rangle \quad \text{for } \tau \in Z_1. \quad (4.6)$$

This is not unthinkable, because operators $\{S(\tau)\}_{\tau \in Z_1}$ are all unitary, commute with each other and satisfy $S(\tau)^2 = 1$. We will discuss the set of solutions of these constraints later. On the invariant subspace $\mathcal{H}_0^\Gamma \subseteq \mathcal{H}^\Gamma$ defined by conditions (4.6) operators $\Gamma_s(v)$ and $S(e)$ satisfy all relations obeyed by their fermionic counterparts. This means that prescription (4.4) well-defines an action of the algebra $A_0$ on $\mathcal{H}_0^\Gamma$.

We will now describe the structure of the Hilbert space $\mathcal{H}_0^\Gamma$. Firstly, representation theory of $A_0$ implies that we have a decomposition

$$\mathcal{H}_0^\Gamma \cong \mathcal{F}_0^{\otimes m_0} \oplus \mathcal{F}_1^{\otimes m_1} \quad (4.7)$$

for some $m_0, m_1 \in \mathbb{N}$. We claim that we have $m_p = 1$ and $m_{p+1} = 0$, where

$$p = \sum_{v \in V} \langle \eta, v \rangle + \sum_{i \in D} \prod_{j \in D \setminus \{i\}} L_j \in \mathbb{Z}_2. \quad (4.8)$$

Indeed, to show that we have

$$\Gamma|\mathcal{H}_{0}^\Gamma = (-1)^p, \quad (4.9)$$

it is sufficient to compare two ways to compute the product $\prod_{i \in D} \prod_{g \in G_{D \setminus \{i\}}} S(l_i(g \cdot v))$: first using the eigenvalue equations (4.6) and second by manipulating $\Gamma$ matrices.

We have shown that $m_{p+1} = 0$. Equality $m_p = 1$ is a special case of a more general statement proven in the Section 5.2.
4.2 Dependence on $\eta$ and translational invariance

In this section we investigate dependence of the theory on the choice of $\eta$. First we note that it is possible to change $\eta$ by redefining $\Gamma_i(v)$ operators:

\begin{align}
\Gamma_i(v) &\mapsto (-1)^{\alpha_i,v}\Gamma_i(v), & (4.10a) \\
\Gamma_{-i}(v) &\mapsto (-1)^{\beta_i,v}\Gamma_{-i}(v), & (4.10b)
\end{align}

with some $\{\alpha_i,\beta_i\}_{i=1}^{d} \subseteq C_0$. By uniqueness of the irreducible representation of Clifford algebras, such transformation may be implemented by a unitary operation, which is unique up to scalars. Hence it does not change the physical content of the theory. In order to preserve the form (4.3) of hopping operators we have to restrict attention to transformations (4.10) satisfying

\begin{equation}
(\beta_i, v) = (\alpha_i, e^{-1}_iv).
\end{equation}

Then $\eta$ changes according to the formula

\begin{equation}
(\eta, v) \mapsto (\eta, v) + \sum_{i \in D}(\alpha_i, v + e^{-1}_iv).
\end{equation}

Consider the 1-chain

\begin{equation}
\theta = \sum_{i \in D} \sum_{v \in V}(\alpha_i, v)q_i(v).
\end{equation}

Observe that the relation between $\theta$ and $\{\alpha_i\}_{i=1}^{d}$ is invertible:

\begin{equation}
(\alpha_i, v) = (\theta, q_i(v)),
\end{equation}

and hence one-to-one. Transformation rule for $\eta$ may be rephrased as

\begin{equation}
\eta \mapsto \eta + \partial \theta.
\end{equation}

Since $\eta$ is a 0-chain, it is trivially closed: $\partial \eta = 0$. Field redefinition shifts it by a boundary term, so the homology class of $\eta$ is unchanged. Structure of the considered model depends on this homology class. However, we may choose whatever representative of $[\eta]$ without changing the physical content of the theory. If $[\eta] = 0$, it is natural to pick $\eta = 0$. Situation is more complicated if $[\eta] \neq 0$. If $|V|$ is odd, there exists a translationally invariant representative $\sum_{v \in V}v$. If $|V|$ is even, this cycle is a boundary and it is not possible to choose a translationally invariant representative of the homology class. Nevertheless, it is still possible to construct translationally invariant hamiltonians. The subtle point is that translation operators are not the naive ones, but rather involve also a transformation of the form (4.10) needed to cancel the homological shift of $\eta$.

4.3 Solving the constraints

In this section we outline some methods to solve constraints in the proposed model. Firstly, let $f$ be a face lying in the plane spanned by directions $1 \leq i < j \leq d$, and with vertices $A, B, C, D$ ordered counterclockwise, starting from the south-west corner (see Fig. 1).
Figure 1. Labels of vertices for a face $f$ lying in the plane spanned by directions $i, j$.

In such situation constraint (4.6) with $\tau = \partial f$ takes the form

$$P(f)|\text{phys} = |\text{phys}\rangle,$$  \hspace{1cm} (4.16a)

$$P(f) = -\Gamma_{i,j}(A)\Gamma_{j,-i}(B)\Gamma_{-i,j}(C)\Gamma_{-j,i}(D),$$  \hspace{1cm} (4.16b)

where $\Gamma_{k,l}(v) := \Gamma_k(v)\Gamma_l(v)$ for $v \in V$ and $k, l \in \{\pm 1, \ldots, \pm d\}$. We note the mnemonic rule that in the above, indices $\pm i, \pm j$ labeling gamma matrices are arranged in a cycle. The only remaining constraints correspond to generators of $H_1$. We write them as

$$L_j(v)|\text{phys} = |\text{phys}\rangle, \hspace{1cm} j = 1, \ldots, d,$$  \hspace{1cm} (4.17a)

$$L_j(v) := -i^{L_j} \prod_{k=0}^{L_j-1} \Gamma_{j,-j}(e^k_j \cdot v),$$  \hspace{1cm} (4.17b)

where $v \in V$ is one arbitrarily chosen reference vertex. We note the properties:

$$L_i(v) = L_i(v)^{-1} = L_i(v)^*,$$  \hspace{1cm} (4.18a)

$$[L_i(v), L_j(v)] = 0.$$  \hspace{1cm} (4.18b)

Geometrically, $L_i(v)$ is an operator supported on the cycle $l_{\{i\}}(v)$. On the subspace on which plaquette constraints (4.16a) are satisfied, all $L_i(v)$ coincide. Thus one may think of them as operators associated with the homology class $\ell_i$.

In the discussion of solutions of constraints written down above we shall confine ourselves to the case of all $L_i$ even. Furthermore, for simplicity of presentation we first take $d = 2$. Higher dimensional generalizations are mentioned afterwards.

We define operators

$$\Xi_1(v) = \prod_{k=0}^{L_1-1} \Gamma_{1,-1}^{(2)}(e^k_2 \cdot v),$$  \hspace{1cm} (4.19a)

$$\Xi_2(v) = \prod_{k=0}^{L_2-1} \Gamma_{-1,-1}^{(2)}(e^k_2 \cdot v).$$  \hspace{1cm} (4.19b)

Both $\Xi$ operators commute\footnote{This is false if $L_1, L_2$ are not even.} with all $P(f)$ and $\Gamma_*(v)$. Moreover, they satisfy

$$\Xi_i(v) = \Xi_i(v)^{-1} = \Xi_i(v)^*,$$  \hspace{1cm} (4.20a)

$$[\Xi_i(v), \Xi_j(v)] = 0,$$  \hspace{1cm} (4.20b)

$$\Xi_i(v)L_j(v) = (-1)^{\delta_i^j} L_j(v)\Xi_i(v).$$  \hspace{1cm} (4.20c)
Combining this with properties of $L_i$ operators we obtain that pairs \( \{L_1(v), \Xi_1(v)\} \) and \( \{L_2(v), \Xi_2(v)\} \) generate two independent copies of the Pauli algebra. It follows that solutions of plaquette constraints are organized in quadruplets, each of which contains precisely one solution of the loop constraint (4.18a). Given any state in such a quadruplet, the desired state satisfying (4.18a) may be easily obtained by acting with an appropriate element of the algebra generated by $L_i$ and $\Xi_i$.

Two remarks about the above construction are in order. Firstly, we engineered the $\Xi$ operators in such a way that they obey algebraic relations discussed in the previous paragraph. However, these requirements do not determine the $\Xi$ operators uniquely. For example, one can multiply $\Xi_i(v)$ by any $\Gamma_\star(\epsilon)$ or $S(\tau)$ with $\partial \tau = 0$ without spoiling the algebra. Secondly, this construction generalizes with only minor adjustments to higher dimension. Geometrically, its essence is that $\Xi_i$ is a product of $\Gamma$ matrices supported on a cycle of homology class $\ell_{D\setminus\{i\}}$. The reason for this is that this homology class intersects nontrivially\(^4\) with $\ell_{\{i\}}$, on which $L_i$ is supported. One still has to figure out which $\Gamma$ matrices to put on vertices of the chosen cycles. In order to preserve the particle number, it is necessary to always choose an even number of matrices per vertex. Preservation of plaquette constraints is achieved by noting that matrix $\Gamma_i(v)$ flips the values of all plaquette operators in the $i$-th direction from $v$, so by correlating the choices of gamma matrices on adjacent sites one may cancel all plaquette flips. This is not always possible to achieve if some $L_i$ are odd. In this case it is still possible to construct $\Xi_i$-like operators satisfying all desired properties except that they flip the occupation number at a single vertex.

Having dealt with the loop constraints, we proceed to the analysis of plaquettes. It will be convenient to divide the lattice into two complementary alternating sublattices. For example we may declare $v = (v_1, v_2) \in \frac{\mathbb{Z}^2}{L_1\mathbb{Z} \times L_2\mathbb{Z}}$ to be even if both of its coordinates are even or both are odd. Otherwise we say that $v$ is an odd vertex. Furthermore, we will say that face $f$ is even (resp. odd) if its south-west corner is even (resp. odd). We note that this prescription is ill-defined if not all $L_i$ are even.

In the first step of the analysis we take $\eta = 0$ and restrict attention to the sector with no fermions. Therefore, at every site we have the relation

\[
\prod_{i=1}^{2} \Gamma_i(v)\Gamma_{-i}(v) = -1. \tag{4.21}
\]

Using this equation we simplify the plaquette operators to the form

\[
P(f) = \Gamma_{1,2}(A)\Gamma_{1,2}(C)\Gamma_{1,-2}(B)\Gamma_{1,-2}(D). \tag{4.22}
\]

Now we introduce new local operators by the formulas

\[
\sigma_3(v) = \begin{cases} 
   i\Gamma_{1,2}(v) & \text{for } v \text{ even,} \\
   i\Gamma_{1,-2}(v) & \text{for } v \text{ odd,}
\end{cases} \tag{4.23a}
\]

\[
\sigma_1(v) = \begin{cases} 
   -i\Gamma_{1,-2}(v) & \text{for } v \text{ even,} \\
   i\Gamma_{1,2}(v) & \text{for } v \text{ odd.}
\end{cases} \tag{4.23b}
\]

\(^4\)See the Appendix A for a formal definition of intersection of homology classes.
We note for future reference that with the definition $\sigma_2(v) = -i\sigma_3(v)\sigma_1(v)$ we have

$$\sigma_2(v) = i\Gamma_{1,1}(v) \quad \text{for every } v \in V.$$ (4.24)

Using the relation (4.21) one verifies that at every lattice site these operators obey the Pauli algebra, which justifies the chosen notation.

In terms of the new variables, plaquette operators take the form

$$P(f) = \begin{cases} 
\prod_{v \in \{A,B,C,D\}} \sigma_3(v) & \text{for } f \text{ even,} \\
\prod_{v \in \{A,B,C,D\}} \sigma_1(v) & \text{for } f \text{ odd.}
\end{cases}$$ (4.25)

In this form plaquette constraints are readily recognized as equations defining ground states in the famous Kitaev’s toric code [8]. It is well-known that there exist four solutions, corresponding to two values of $L_1$ and $L_2$. This is also in accord with our general finding about the $\Gamma$ model. For completeness we provide a prescription to construct these states in the next paragraph.

We work in the standard eigenbasis of $\sigma_3(v)$ operators, so our basis states are labeled by elements $\omega \in C_0$ and satisfy

$$\sigma_3(v)|\omega\rangle = (-1)^{(\omega,v)}|\omega\rangle,$$

$$\sigma_1(v)|\omega\rangle = |\omega + v\rangle.$$ (4.26a, b)

In order to have $P(f) |\omega\rangle = |\omega\rangle$ for even $f \in Q_2$ we need to have

$$(\omega, A + B + C + D) = 0,$$ (4.27)

where $A, B, C, D$ are the four vertices of an even face. Every such chain $\omega$ will be called admissible. We remark that geometrically this condition means that $\omega$ may be identified with a 1-cocycle on the lattice whose vertices are the even faces and edges are vertices of our lattice (see Fig. 2).

By (by no means not accidental) analogy with usual gauge theory, we conclude that there exist $\frac{L_1L_2}{4} + 1$ admissible chains. Now consider the state

$$|\text{ref}\rangle = 2^{-\frac{L_1L_2}{4}} \sum_{\omega \text{ admissible}} |\omega\rangle.$$ (4.28)

Clearly we have $P(f)|\text{ref}\rangle = |\text{ref}\rangle$ for every $f \in Q_2$ and $\langle \text{ref}|\text{ref}\rangle = 1$.

State $|\text{ref}\rangle$ constructed in the previous paragraph satisfies all plaquette constraints, but does not satisfy loop constraints. In this paragraph we solve this difficulty. As a first step towards this goal, we express $L$ and $\Xi$ operators in terms of Pauli matrices. Without loss of generality we can take the reference vertex $v$ to be even. Then

$$L_1(v) = -\prod_{k=0}^{L_1-1} \sigma_2(e_1^k \cdot v), \quad \Xi_1(v) = (-1)^{\frac{L_2}{2}} \prod_{k=0}^{L_2-1} \sigma_3(e_2^k \cdot v),$$ (4.29a)

$$L_2(v) = -\prod_{k=0}^{L_2-1} \sigma_2(e_2^k \cdot v), \quad \Xi_2(v) = (-1)^{\frac{L_1}{2}} \prod_{k=0}^{L_1-1} \sigma_3(e_1^k \cdot v).$$ (4.29b)

\footnote{See the discussion above equation (5.2) for an analogous calculation spelled out in detail.}
Using the above and the definition of $|\text{ref}\rangle$ we obtain eigenvalue equations

$$L_1(v)\Xi_2(v)|\text{ref}\rangle = L_2(v)\Xi_1(v)|\text{ref}\rangle = -|\text{ref}\rangle.$$ \hspace{1cm} (4.30)

This eigensystem combined with the relations obeyed by $L$ and $\Xi$ operators implies that projection of $|\text{ref}\rangle$ onto the joint eigenspace of $L_1$ and $L_2$ to eigenvalue 1 has norm $\frac{1}{2}$.

To obtain a properly normalized state, we multiply this projection by 2:

$$|0\rangle = 2 \cdot \frac{1 + L_1(v)}{2} \frac{1 + L_2(v)}{2} |\text{ref}\rangle.$$ \hspace{1cm} (4.31)

Under our bosonization prescription, state $|0\rangle$ in the $\Gamma$ model corresponds to the fermionic vacuum. Images of all other states from $F_0$ in the $\Gamma$ model Hilbert space may be obtained by acting on the vacuum with creation operators of pairs of fermions. Bosonic counterparts of these operators may be easily written down explicitly using the bosonization prescription (4.4).

So far we have considered only the case of $\eta = 0$. If $\eta$ is nonzero, procedure presented above may be repeated almost verbatim. The only difference is that the state $|0\rangle$ obtained this way is not empty. Instead it satisfies

$$\Gamma_\ast(v)|0\rangle = (-1)^{\langle \eta, v \rangle} |0\rangle.$$ \hspace{1cm} (4.32)

This is slightly inconvenient, but there is no way around this difficulty. Indeed, if the homology class of $\eta$ is nonzero, Hilbert space $H_{\Gamma_0}$ does not contain any states with an even number of fermions at all.

We close this section with the remark that the assumption that all $L_i$ are even seems to be essential in the presented construction. We do not know similarly explicit formulas for states satisfying constraints if $L_i$ are not all even. Secondly, the fact that restricting attention to empty states reduces dimension of on-site Hilbert spaces to two is peculiar to $d = 2$. It is not clear how to proceed in higher dimensions. One should probably somehow use the (Bianchi) identity $\prod_{i=1}^{6} P(f_i) = 1$, satisfied whenever $f_1,\ldots,f_6$ are six faces of some
cube $q \in Q_3$. In any case, one can always choose some reference state and act on it with the projection operator \( \prod_{i=1}^{d} \frac{1 + L_i}{2} \left( \prod_{f \in Q_2} \frac{1 + P(f)}{2} \right) \). This construction produces a state described as a sum of contributions labeled by all possible lattice loops.

5 Unconstrained $\Gamma$ model

It was shown in Section 4 that the $\Gamma$ model with physical Hilbert space defined by the equation (4.6) is equivalent to a purely fermionic system with a fixed value of fermionic parity. In this section we will show that lifting the constraint (4.6) is equivalent to coupling the system to an external static and purely magnetic $Z_2$ gauge field, up to a subtle interplay between the value of the gauge field and value of the fermionic parity, different than that dictated by the standard Gauss’ law.

5.1 Gauging the fermionic parity

We will now explain what we mean by coupling fermions to a $Z_2$ gauge field. Such a procedure is naturally present in lattice gauge theories [32]. By a $Z_2$ gauge field we shall mean an element $A \in C_1$. Mapping $A \rightarrow A + \delta \theta$ with any $\theta \in C_0$ will be called a gauge transformation. Gauge fields related by a gauge transformation are said to be gauge equivalent. We define the holonomy along $\tau \in C_1$ by $U(\tau) = (-1)^{(A,\tau)}$. Under gauge transformations holonomies transform according to the rule

\[
U(\tau) \rightarrow U(\tau)(-1)^{(\partial \tau, \theta)}.
\]

In particular, $U(\tau)$ is gauge-invariant if and only if $\partial \tau = 0$. For any $\sigma \in C_2$ we have $U(\partial \sigma) = (-1)^{(B,\sigma)}$, where $B = \delta A$. We call $B$ the magnetic field. Notice that it is gauge-invariant. Moreover, it satisfies $\delta B = 0$ and $(\tilde{\ell}, B) = 0$ for any cohomology class $\tilde{\ell}$. Conversely, if $B$ is any 2-cochain satisfying these two conditions, then $B = \delta A$ for some $A$, unique up to addition of an element of $Z^1$. Notice that $Z^1$ is strictly bigger than $B^1$, so it is not true that $A$ is determined by $B$ uniquely up to gauge transformations. To make it unique, one has to specify the values of $U(\tau)$ for any set of $\tau \in Z_1$ whose homology classes form a basis of $H_1$.

We remark that there exist $2^{(d-1)|V|+1}$ gauge equivalence classes of gauge fields. Indeed, gauge equivalence classes are elements of the $Z_2$-vector space $C_1 / \delta C_0$. Thus it is sufficient to compute its dimension. We have $\dim Z_2(C_1) = d|V|$ and

\[
\dim(\delta C_0) = \dim \left( \frac{C_0}{Z^0} \right) = \dim(C_0) - \dim(Z^0) = |V| - 1.
\]

Minimal coupling of fermions to $A$ consists of replacing all occurrences of $S(\tau)$ in the hamiltonian by the decorated operators

\[
\mathfrak{s}_A(\tau) = U(\tau)\mathfrak{s}(\tau).
\]
Dressed operators span the same algebra as the undressed ones, but satisfy slightly different relations. Namely, (3.11) is modified to

\[ s_A(\tau) = (-1)^{\Phi(\tau)+(A,\tau)} \] if \( \partial \tau = 0 \). (5.4)

Gauge fields considered so far have been purely magnetic and time independent. Situation changes once the gauge field is promoted to a dynamical degree of freedom. This may be done by introducing operators \( V(e) = (-1)^{E(e)} \) and \( V(\tau) = \prod_{e \in Q_1} V(e)^{\tau,e} \) "conjugate" to \( U(e) \), in the sense that we have

\[
\begin{align*}
V(e)^* &= V(e)^{-1} = V(e), \quad (5.5a) \\
U(\tau)V(\tau') &= (-1)^{(\tau,\tau')}V(\tau')U(\tau). \quad (5.5b)
\end{align*}
\]

Additionally, we require that the Hilbert space associated with an edge \( e \) is an irreducible representation of the algebra generated by \( U(e) \) and \( V(e) \). It follows that gauge field operators may be represented (uniquely up to a unitary equivalence) by Pauli matrices associated to edges of the lattice, say \( U(e) = \sigma^3(e) \) and \( V(e) = \sigma^1(e) \).

The simplest natural choice for the gauge field hamiltonian is

\[
H = -g_1 \sum_{f \in Q_2} V(\partial f) - g_2 \sum_{e \in Q_1} V(e),
\]

where \( g_1, g_2 \) are real coupling constants. It is translationally and gauge invariant.

Physical Hilbert space is defined by restricting attention to gauge-invariant states. To substantiate this definition, we introduce the Gauss’ operators

\[ \mathcal{G}(\theta) = \gamma(\theta)V(\delta \theta). \] (5.7)

They satisfy

\[
\begin{align*}
\mathcal{G}(\theta)^* &= \mathcal{G}(\theta)^{-1} = \mathcal{G}(\theta), \quad (5.8a) \\
\mathcal{G}(\theta)U(\tau) &= (-1)^{(\delta \tau,\theta)}U(\tau)\mathcal{G}(\theta). \quad (5.8b)
\end{align*}
\]

Physical Hilbert space is defined by the equation

\[ \mathcal{G}(\theta)|\text{phys}\rangle = |\text{phys}\rangle. \] (5.9)

We claim that this Hilbert space is of dimension \( 2^{|V|+1} \). Indeed, in eigenbases of \( \{V(e)\}_{e \in E} \) the Gauss’ operators are diagonal, so counting is trivial.

One important consequence of gauging of fermionic parity is that states with an odd number of fermions are removed from the theory. Indeed, we have the relation

\[ |\text{phys}\rangle = \prod_{v \in V} \mathcal{G}(v)|\text{phys}\rangle = \prod_{v \in V} \gamma(v)|\text{phys}\rangle = \gamma|\text{phys}\rangle. \] (5.10)
5.2 Structure of the Hilbert space

Our next goal is to describe the structure of the Hilbert space $\mathcal{H}^\Gamma$ in the $\Gamma$ model without restricting attention to the subspace $\mathcal{H}_{0}^{\Gamma}$ defined by the constraint (4.6) [18]. First, let us observe that we have a decomposition

$$\mathcal{H}^\Gamma = \bigoplus_{[A] \in \mathcal{C}_1 \delta \mathcal{C}_0} \mathcal{H}_{[A]}^\Gamma, \quad (5.11)$$

where vectors $|\psi\rangle$ in $\mathcal{H}_{[A]}^\Gamma$ are defined by the modified constraint

$$U(\tau)|\psi\rangle = (-1)^{([A],\tau)}|\psi\rangle, \quad \text{for } \tau \in \mathbb{Z}_1. \quad (5.12)$$

Notice that the quantity $([A],\tau) \in \mathbb{Z}_2$ is well-defined because for $\partial \tau = 0$ scalar product $(A, \tau)$ is gauge invariant.

Now choose arbitrarily one representative $A$ of each gauge equivalence class. In other words, choose some embedding

$$\mathcal{g}_{fix} : \mathcal{C}_1 \delta \mathcal{C}_0 \hookrightarrow \mathcal{C}_1. \quad (5.13)$$

We may take $\mathcal{g}_{fix}$ to be a linear map.

Having chosen $\mathcal{g}_{fix}$ we define an action of the algebra $\mathcal{A}_0$ on $\mathcal{H}^\Gamma$:

$$\gamma(\epsilon)|\psi\rangle := \Gamma(\epsilon)|\psi\rangle, \quad (5.14a)$$

$$\mathcal{g}_{[A]}(\tau)|\psi\rangle := S(\tau)|\psi\rangle, \quad \text{for } \psi \in \mathcal{H}_{[A]}^\Gamma. \quad (5.14b)$$

As in (4.7), we have an isomorphism of $\mathcal{A}_0$-modules

$$\mathcal{H}_{[A]}^\Gamma \cong \mathcal{F}^{\oplus m_0[A]}_0 \oplus \mathcal{F}^{\oplus m_1[A]}_1 \quad (5.15)$$

for some natural numbers $m_0[A]$ and $m_1[A]$. Furthermore, generalizing the calculation that led to the formula (4.9) we obtain

$$\Gamma|\mathcal{H}_{[A]}^\Gamma = (-1)^{p[A]}, \quad (5.16)$$

where

$$p([A]) = \sum_{e \in Q_1} ([A], e) + \sum_{v \in V} (\eta, v) + \sum_{i \in D} \prod_{j \in D \setminus \{i\}} L_j \in \mathbb{Z}_2. \quad (5.17)$$

It follows that $m_{p[A]+1}[A] = 0$. Our next step is to show that $m_{p[A]}[A] = 1$ for every $[A] \in \mathcal{C}_1 \delta \mathcal{C}_0$, so that we have an isomorphism

$$\mathcal{H}_{[A]}^\Gamma \cong \mathcal{F}_{p[A]} \quad (5.18)$$

The first step is to calculate the dimension of $\mathcal{H}^\Gamma$. We have

$$2^{d[V]} = \dim(\mathcal{H}^\Gamma) = \sum_{[A]} \dim\left(\mathcal{H}_{[A]}^\Gamma\right) = 2^{[V]-1} \sum_{[A]} m[A]. \quad (5.19)$$
It follows that we have a sum rule for multiplicities:

\[ \sum_{[A]} m[A] = 2^{(d-1)|V|+1}. \]  

(5.20)

The right hand side of this equation coincides with the number of distinct gauge fields. The proof will be completed by showing that \( m[A] \) does not depend on \( [A] \). This can be achieved by demonstrating that for every two \( [A], [A'] \in \frac{C_1 \delta C_0}{Z_0} \) there exists a (non-unique) unitary operator on \( \mathcal{H}^{\Gamma} \) which carries \( \mathcal{H}^{\Gamma}_{[A]} \) to \( \mathcal{H}^{\Gamma}_{[A']} \). We do it in the forthcoming paragraph.

Let \( \{\mu_i, \nu_i\}_{i=1}^d \) be a collection of 0-chains. We define

\[ O_{\mu\nu} = \prod_{i \in D} \prod_{v \in V} \Gamma_i(v)^{(\mu_i,v)} \Gamma_{-i}(v)^{(\nu_i,v)}. \]  

(5.21)

Put \( \rho = \sum_{j=1}^d (\mu_j + \nu_j) \in C_0 \). The following braiding relations hold

\[ O_{\mu\nu} \Gamma_i(v) = (-1)^{(\mu_i,v)+(\rho,v)} \Gamma_i(v) O_{\mu\nu}, \]  

\[ O_{\mu\nu} \Gamma_{-i}(v) = (-1)^{(\nu_i,v)+(\rho,v)} \Gamma_{-i}(v) O_{\mu\nu}. \]  

(5.22a, 5.22b)

It follows that for an edge \( e \in Q_1 \) with \( \text{fin}(e) = e_i \cdot \text{in}(e) \) we have

\[ O_{\mu\nu} S(e) = (-1)^{(\mu_i,\text{fin}(e))+(\nu_i,\text{fin}(e))+(\rho,\partial e)} S(e) O_{\mu\nu}. \]  

(5.23)

Introduce the 1-chain

\[ \Upsilon = \sum_{i \in D} \sum_{v \in V} [(\mu_i,v) + (\nu_i, e_i \cdot v)] q_i(v). \]  

(5.24)

Relation (5.23) generalizes to

\[ O_{\mu\nu} S(\tau) = (-1)^{(\rho,\partial \tau)+(\Upsilon,\tau)} S(\tau) O_{\mu\nu}. \]  

(5.25)

In particular for \( \tau \in Z_1 \) we have \( (\rho, \partial \tau) = 0 \). Therefore \( O_{\mu\nu}(\mathcal{H}^{\Gamma}_{[A]}) \subseteq \mathcal{H}^{\Gamma}_{[A+\Upsilon]} \). It is possible to choose \( \mu, \nu \) so that we have \( \Upsilon = A' - A \), which completes the proof.

5.3 Comparison with gauge theory

We have shown in the previous section that the Hilbert space of the \( \Gamma \) model decomposes into subspaces labeled by gauge equivalence classes \( [A] \in \frac{C_0 \delta C_0}{Z_0} \), as in the fermionic theory with gauged fermionic parity. However, there is an important difference between the two models. In the latter case Gauss’ law asserts, among other things, that only states with an even number of fermions should be regarded as physical. In the \( \Gamma \) model it is also true that value of \( (-1)^N \) acts as a c-number in every subspace \( \mathcal{H}^{\Gamma}_{[A]} \), but its value depends on \( [A] \) in a way described by the formula (5.17). This is reminiscent of the Chern-Simons theory\(^6\), in which Gauss’ law relates the total charge to the total flux of the magnetic field.

Unfortunately, we have not been able to formulate a precise correspondence. In any case,

\(^6\)Say, with gauge group \( U(1) \) and spacetime dimension 3.
one may speculate that the unconstrained $\Gamma$ model is equivalent to a certain deformation of the gauge theory defined in (5.1), e.g. corresponding to some $\theta$ or Chern-Simons-like term. We discuss some possibilities to construct such deformations in the Appendix B.

We will now interpret calculations presented in Section 4.3 in terms of the picture outlined above. For simplicity we restrict attention to $\eta = 0$. The empty sector, i.e. the subspace of $\mathcal{H}^\Gamma$ defined by equations

$$\Gamma_i(v)|\text{empty}\rangle = |\text{empty}\rangle, \quad (5.26)$$

is the physical Hilbert space of a $\mathbb{Z}_2$ gauge theory. There are no spurious states and no Gauss’ law to be imposed on states, because the correspondence between possible values of holonomies and basis vectors is one-to-one (in a theory with residual gauge symmetry the number of distinct states would exceed the number of possible values of observables). This $\mathbb{Z}_2$ gauge field is not present in fermionic theory, so in order to obtain a purely fermionic model it is necessary to remove the gauge field by imposing constraints. Plaquette constraints implement a flatness condition for the connection, while loop contraints impose correct boundary conditions for fermions (or in more geometric terms, correct spin structure). Indeed, change of spin structure is equivalent to coupling fermions to an external flat $\mathbb{Z}_2$ gauge field.

In the unconstrained $\Gamma$ model we have two distinguished subspaces - the constrained $\Gamma$ model and the empty sector, corresponding to fermionic and flux excitations, respectively. However, the full Hilbert space does not factorize as a tensor product of a flux Hilbert space and fermionic Hilbert space, as shown by the formula (5.17), which relates the total number of fluxed with the total number of fermions. Such relation is not surprising if we observe that the elementary degree of freedom, the $\Gamma$ field, may be thought of as a composite of a fermion and some magnetic fluxes. We write schematically

$$\Gamma = \text{Fermion} \times \text{Fluxes}. \quad (5.27)$$

More precisely, braiding relations of our basic observables are such that the operator $\Gamma_i(v)$ flips the value of $\Gamma_i(v)$ (and hence creates or annihilates a fermion) as well as all plaquettes in the $i$-th direction from the vertex $v$ (see Fig. 3). The bosonic nature of the $\Gamma$ field (in the sense that fields separated in space commute) follows from existence of nontrivial Aharonov-Bohm phases: fermion circulating around a flux picks up a minus sign.

![Figure 3](image-url)  

Figure 3. Operator $\Gamma_1(v)$ flips values of plaquettes to the east of the vertex $v$, represented here by shaded squares.

6 Summary and outlook

We gave a new proof of correctness of the bosonization prescription proposed in [11] and reinterpreted it in terms of lattice $\mathbb{Z}_2$ gauge theory. Secondly, we presented a complete
solution of constraints present in the model in the case of lattices of dimension two and even lengths. It would be interesting to generalize these results, especially to higher dimensions. Another interesting problem is to relate our construction to that presented in [19].

Our analysis shows that the Γ model without constraints needed to make it equivalent to fermions resembles traditional gauge-Fermi systems in some respects, but there are some aspects more similar to the Chern-Simons theory. A natural next step would be to try to find some theory of fermions coupled to a $\mathbb{Z}_2$ gauge field which is exactly equivalent to the unconstrained Γ model.

Finally, it could be enlightening to obtain a path integral formulation of the Γ model in its constrained and/or unconstrained version. We leave this for the future research.

A Further remarks about cohomology

In this appendix we discuss further properties of homology on cohomology with $\mathbb{Z}_2$ coefficients on lattices which we work with. This language is natural for discussion of topological features of fermionic models.

We begin with the definition of a linear functional on $C$, called the integral:

$$\int : C \ni c \mapsto \int c = (\text{vol}, c) \in \mathbb{Z}_2.$$  \hspace{1cm} (A.1)

By construction, $\int$ is nonzero only in degree $d$. Furthermore we have

$$\int \delta c = 0.$$ \hspace{1cm} (A.2)

It follows that integration of cohomology classes makes sense. One may also introduce integration over arbitrary cycles, or even chains with a nontrivial boundary. Then Stokes’ formula holds by the very definition of $\delta$.

Then we introduce the cup product on $C$. It is defined on basis vectors by

$$q \smile q' = \delta_{\text{fin}(q')}^{\text{in}(q)} \cdot q(\text{in}(q), \text{fin}(q')) \quad \text{for } q, q' \in Q.$$ \hspace{1cm} (A.3)

Cup product defined as above is associative and satisfies the Leibniz rule, that is

$$\delta(c \smile c') = \delta c \smile c' + c \smile \delta c'.$$ \hspace{1cm} (A.4)

Furthermore, $1 = \sum_{v \in V} v$ is the unit element:

$$c \smile 1 = 1 \smile c = c.$$ \hspace{1cm} (A.5)

Indeed, associativity is simple. As for the Leibniz rule, it is sufficient to prove it for a product of two cubes. Furthermore the identity is nontrivial if the two cubes do not share any common direction. We may then renumber the coordinate axes so that the cubes are of the form $q = \{e_{k_1} \cdots e_{k_p}v\}_{k_1,\ldots,k_p=0}$, $q' = \{e_{k_{p+1}}^{p+1} \cdots e_{p+q}^{p+q}v'\}_{k_{p+1},\ldots,k_{p+q}=0}$ for some $v, v' \in V$. Then we have

$$q \smile q' = \delta^{v'}_{e_{1} \cdots e_{p+q}} q(v, e_{1} \cdots e_{p+q} v).$$ \hspace{1cm} (A.6)
Now evaluate explicitly
\[
\delta q \cir q' + q \cir \delta q' = \sum_{r=p+q+1}^d \left( \delta^v_{e_1\ldots e_{p+q}} q(v, e_1\ldots e_{p+q} e_r v) + \delta^v_{e_1\ldots e_{p+q}} q(e_r^{-1} v, e_1\ldots e_{p+q} v) \right)
\]
\[
+ \delta^v_{e_1\ldots e_{p+q}} q(v, e_1\ldots e_{p+q} e_r v) + \delta^v_{e_1\ldots e_{p+q}} q(v, e_1\ldots e_{p+q} e_r v)
\]
\[
= \delta^v_{e_1\ldots e_{p+q}} \sum_{r=p+q+1}^d (q(e_r^{-1} v, e_1\ldots e_{p+q} v) + q(v, e_1\ldots e_{p+q} e_r v))
\]
\[
= \delta^v_{e_1\ldots e_{p+q}} \delta (q(v, e_1\ldots e_{p+q} v)) = \delta (q \cir q').
\] (A.7)

It follows from the Leibniz rule that \(\cir\) descends to a product on cohomology. Moreover, cup product of cohomology classes is commutative. The cohomology ring is the exterior algebra over \(\mathbb{Z}_2\) in generators \(\{\tilde{\ell}_i\}_{i=1}^d\), with unit element \(\tilde{\ell}_\emptyset\). To show that, first choose \(I, J \subseteq D\). If \(I \cap J \neq \emptyset\), then
\[
\tilde{\ell}_I(v) \cir \tilde{\ell}_J(v) = 0.
\] (A.8)

Otherwise we have
\[
\tilde{\ell}_I(v) \cir \tilde{\ell}_J(v) = \tilde{\ell}_{I\cup J}(v).
\] (A.9)

Hence the claimed results follow.

Bilinear form \(C \times C \to \mathbb{Z}_2\), called the integral pairing, is defined by
\[
(c, c') \mapsto \int c \cir c'.
\] (A.10)

Calculating its values for a pair of cubes we get:
\[
\int q_I(v) \cir q_J(v') = \delta^v_{e_1\ldots e_{p+q}} \delta^{D\setminus I}_{D\setminus J}.
\] (A.11)

From this we see explicitly that the integral pairing is non-degenerate, but not symmetric. It satisfies the integration by parts formula
\[
\int \delta c \cir c' = \int c \cir \delta c'.
\] (A.12)

It follows that integration descends to a bilinear form on the cohomology ring. Furthermore, integration pairing on the cohomology ring is non-degenerate and symmetric. In particular we have Poincaré duality on the (co)homology level:
\[
H_p \cong \text{Hom}_{\mathbb{Z}_2}(H^p, \mathbb{Z}_2) \cong H^{d-p}.
\] (A.13)

Indeed, notice that
\[
\int \tilde{\ell}_I \cir \tilde{\ell}_J = \delta^{D\setminus J}_{D\setminus I}.
\] (A.14)

Another object derived from the cup product is the (right) cap product \(\cap_R\): \(C \times C \to C\), which is given by the formula
\[
(c \cap_R c', c'') = (c, c' \cir c'').
\] (A.15)
Associativity of the cup product is equivalent to the relation
\[(c \smile_R c') \smile_R c'' = c \smile_R (c' \smile_R c'')\],
(A.16)
which combined with \(c \smile_R 1 = c\) may be interpreted as the statement that the cap product defines on \(C\) a structure of a right module over \(C\) equipped with the cup product.

We evaluate explicitly the cap product of two cubes:
\[q \smile_R q' = \delta_{\text{fin}(q')} q(\text{fin}(q'), \text{fin}(q))\].
(A.17)

Leibniz rule satisfied by the cup product is equivalent to the identity
\[\partial(c \smile_R c') = \partial c \smile_R c' + c \smile_R \delta c'\],
(A.18)
which implies that homology is a right module over the cohomology ring. Explicit computation shows that we have
\[\ell_I \smile_R \ell_J = \begin{cases} \ell_{I \cap J} & \text{if } J \subseteq I, \\ 0 & \text{otherwise}. \end{cases}\]
(A.19)

We can define also the left cap product by the formula
\[(c, c' \smile_L c'') = (c \smile c', c'')\],
(A.20)
which defines on \(C\) the structure of a left \((C, \smile)\)-module. We have an identity
\[\partial(c \smile_L c') = \delta c \smile_L c' + c \smile_L \partial c'\],
(A.21)
so there is an induced left \(H^*\)-module structure on homology. Since the cohomology ring is commutative, there is a canonical identification between left and right \(H^*\)-module structures. With this identification, the two \(H^*\)-module structures on \(H_*\) coincide. Situation is more complicated on the (co)chain level, because the cup product is not commutative. However we do have the identity
\[(c \smile_L c') \smile_R c'' = c \smile_L (c' \smile_R c'')\],
(A.22)
which may be read as a statement that left and right multiplication operators commute, or in other words that \(C\) is a \((C, \smile)\)-bimodule. We finish this paragraph with a formula for the left cap product of two cubes:
\[q \smile_L q' = \delta_{\text{fin}(q')} q(\text{fin}(q'), \text{fin}(q))\].
(A.23)
We note that there is a pleasant symmetry between this formula and (A.17).

Our next goal is to generalize the Poincaré duality to the chain level. In other words, we would like to find an operator on \(C\) which behaves like the Hodge star. Actually it does not seem to be possible to find one such operator, due to the non-commutativity of the cup product (which should be contrasted with graded commutativity of the wedge product...
of differential forms). Instead, one can construct two distinct operators obeying a certain compatibility condition. Indeed, let us define

\[ \text{PD}_L(c) = c \hookrightarrow_L \text{vol}, \]  
\[ \text{PD}_R(c) = \text{vol} \hookrightarrow_R c. \]  
(A.24a, b)

Equivalently, we may define the operators \( \text{PD}_L, \text{PD}_R \) by the formula

\[ \int c \hookrightarrow c' = (\text{PD}_R(c), c') = (c, \text{PD}_L(c')), \]  
(A.25)

which makes manifest the fact that \( \text{PD}_L \) and \( \text{PD}_R \) are adjoint with respect to the symmetric pairing on \( C \). Since the integral pairing is non-degenerate, we see also that \( \text{PD}_L \) and \( \text{PD}_R \) are bijective. On the other hand (A.24), combined with \( \partial \text{vol} = 0 \) and the properties of the cap product, implies that

\[ \partial \circ \text{PD}_L = \text{PD}_L \circ \delta, \]  
\[ \partial \circ \text{PD}_R = \text{PD}_R \circ \delta, \]  
(A.26a, b)

\[ \text{PD}_L(c \hookrightarrow c') = c \hookrightarrow_L \text{PD}_L(c'), \]  
(A.26c)

\[ \text{PD}_R(c \hookrightarrow c') = \text{PD}_R c \hookrightarrow_R c'. \]  
(A.26d)

The first two equations imply that \( \text{PD}_L, \text{PD}_R \) descend to operators \( H^\bullet \to H_\bullet \). Simple check shows that both give what we want on the cohomology level. The second two relate various \((C, \hookrightarrow)\)-module structures on \( C \). Our next step is to calculate Poincaré duals of cubes. The results are

\[ \text{PD}_L(q') = q (e_D^{-1}\text{fin}(q'), \text{in}(q')) , \]  
(A.27a)

\[ \text{PD}_R(q') = q (\text{fin}(q'), e_D\text{in}(q')) , \]  
(A.27b)

or equivalently

\[ \text{PD}_L(q_I(v)) = q_D^{-1}(e_D^{-1}e_I v), \]  
(A.28a)

\[ \text{PD}_R(q_I(v)) = q_D(e_I v). \]  
(A.28b)

With those formulas in hand, we calculate compositions:

\[ (\text{PD}_L)^2(q_I(v)) = q_I(e_D v), \]  
(A.29a)

\[ (\text{PD}_R)^2(q_I(v)) = q_I(e_D^{-1} v), \]  
(A.29b)

\[ \text{PD}_L \text{PD}_R(q_I(v)) = \text{PD}_R \text{PD}_L(q_I(v)) = q_I(v). \]  
(A.29c)

In other words, \( \text{PD}_L = (\text{PD}_R)^{-1} \). However, squares of these operators do not coincide with the identity, but rather by a translation, either by \( e_D \) or \( e_D^{-1} \). We note these transformations do coincide with the identity operator on the (co)homology level. Using the fact that the two Poincaré dualities are inverses of each other, we obtain new properties of the Poincaré
dualities from (A.26):

\[
\begin{align*}
\delta \circ \text{PD}_L &= \text{PD}_L \circ \partial, \\
\delta \circ \text{PD}_R &= \text{PD}_R \circ \partial, \\
\text{PD}_L(c \rightsquigarrow_R c') &= \text{PD}_L(c) \rightsquigarrow c', \\
\text{PD}_R(c \rightsquigarrow_L c') &= c \rightsquigarrow \text{PD}_R(c').
\end{align*}
\]

(A.30)

In particular there is an induced map from cohomology to homology, which is clearly the inverse of the map from cohomology to homology introduced earlier.

We close this appendix with a definition. Intersection \(x \cdot y\) of two homology classes \(x, y\) is defined as the integral pairing of the corresponding Poincaré dual cohomology classes. Thus we have

\[
\ell_I \cdot \ell_J = \delta_{IJ}^{D}\delta_{IJ}.
\]

(A.31)

B Topology of \(\mathbb{Z}_2\) gauge theory

This appendix is devoted to a discussion of topological properties of \(\mathbb{Z}_2\) gauge theory. In contrast to the rest of the paper, we use language typical for continuum physics. In this context gauging of discrete symmetries is typically thought of as, informally speaking, twisting boundary conditions by action of the pertinent symmetry. It is purely topological and in particular it does not introduce new local degrees of freedom. As soon as we explain all that, we proceed to comparison with lattice gauge theory.

Let \(X\) be a manifold and \(G\) a Lie group. Gauge field on \(X\) with gauge group \(G\) is the same as a principal \(G\)-bundle \(P \to X\) together with a choice of a connection, i.e. a \(\dim(X)\)-dimensional distribution on \(P\) which is transverse to fibers and \(G\)-equivariant. If this distribution is integrable, we say that the connection is flat. This situation is characterized by the statement that the holonomy \(\text{hol}(\gamma) \in G\) along a loop \(\gamma\) depends only on the homotopy class of \(\gamma\).

Suppose that \(X\) is connected and \(x_0 \in X\) is a reference point. Isomorphism classes of principal bundles with a flat connection are in one-to-one correspondence with elements of the coset space \(\text{Hom}(\pi_1(X, x_0), G) / G\), where \(G\) acts on representations of the fundamental group by conjugation. This correspondence is given by mapping a gauge field to the conjugacy class of the homomorphism \(\pi_1(X, x_0) \ni [\gamma] \mapsto \text{hol}(\gamma) \in G\). If \(G\) is abelian, the conjugation action is trivial and we have

\[
\text{Hom}(\pi_1(X, x_0), G) / G \cong \text{Hom}(\pi_1(X, x_0), G) \cong \text{Hom}(H_1(X, \mathbb{Z}), G) \cong H^1(X, G),
\]

(B.1)

where the Hurewicz theorem and the universal coefficient theorem were used to obtain the second and the third isomorphism, respectively. Thus the gauge field is fully characterized by a cohomology class \(A \in H^1(X, G)\) such that, for any loop \(\gamma\), holonomy \(\text{hol}(\gamma)\) coincides with the canonical pairing of \(A\) with the homology class in \(H_1(X, \mathbb{Z})\) represented by \(\gamma\).

Now consider the case \(G = \mathbb{Z}_2\). Principal \(\mathbb{Z}_2\)-bundle over \(X\) is the same as a two-sheeted regular covering space \(P \to X\). Over sufficiently small open subsets of \(X\) space \(P\)
is isomorphic to a product of the base and a set with two elements, so there exists a unique connection. This connection is flat, so it is represented by a cohomology class in \( H^1(X, \mathbb{Z}_2) \).

We will now try to answer the question what sort of topological terms may arise in a \( \mathbb{Z}_2 \) gauge theory. For this we need to briefly review the formalism of classifying spaces. Let \( G \) be a discrete group\(^7\). One can show [34] that there exists a topological space \( BG \) together with a principal \( G \)-bundle \( EG \to BG \) with the following universal property: given a paracompact space \( X \) and a principal \( G \)-bundle \( P \to X \) there exists a continuous map \( \phi : X \to BG \), unique up to homotopy, such that \( P \) is isomorphic to the pullback \( \phi^*EG \).

Now fix a principal \( G \)-bundle \( P \to X \). We let \( A \) be an abelian group and \( \alpha \) a cohomology class in \( H^p(BG, A) \) for some \( p \in \mathbb{N} \). Then the pullback \( \phi^*\alpha \) is a cohomology class in \( H^p(X, A) \) associated in a canonical way with \( P \). Such cohomology classes are called characteristic classes. The group \( A \) relevant for the construction of topological terms is \( \mathbb{R}/\mathbb{Z} \) [35, 36], since one wants to obtain a cohomology class which paired with some cycle gives a real number \( S \) defined modulo integers, so that \( e^{2\pi iS} \) is a well-defined weight for path integrals.

Since we are interested in the case \( G = \mathbb{Z}_2 \), we need to consider cohomology groups \( H^p(B\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) \). Fortunately, all answers are well-known. Space \( E\mathbb{Z}_2 \) may be taken as an infinite-dimensional sphere \( S^\infty \) (defined as the direct limit of finite dimensional spheres) with \( B\mathbb{Z}_2 \) - the infinite dimensional projective space \( \mathbb{R}P^\infty \), i.e. the quotient of \( S^\infty \) by the antipodal map. Cohomology groups of \( \mathbb{R}P^\infty \) take the form

\[
H^p(\mathbb{R}P^\infty, A) \cong \begin{cases} 
A & \text{for } p = 0, \\
A/2A & \text{for } p > 0 \text{ even}, \\
\{a \in A | 2a = 0\} & \text{for } p \text{ odd},
\end{cases}
\]

which may be easily obtained using a cellular decomposition of \( \mathbb{R}P^\infty \). Specifying \( A = \mathbb{R}/\mathbb{Z} \) we obtain vanishing cohomology groups for \( p \neq 0 \) even and equal to \( \mathbb{Z}_2 \) for \( p > 0 \) odd. Considering the long exact sequence of cohomology groups associated to the short exact sequence of groups

\[
0 \to \mathbb{Z}_2 \to \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \to 0
\]

we obtain that for every \( p > 0 \) map \( H^p(\mathbb{R}P^\infty, \mathbb{Z}_2) \to H^p(\mathbb{R}P^\infty, \mathbb{R}/\mathbb{Z}) \) induced from the embedding of \( \mathbb{Z}_2 \to \mathbb{R}/\mathbb{Z} \) is an epimorphism\(^8\). Therefore, there will be no loss of generality in working with cohomology with coefficients in \( \mathbb{Z}_2 \). This is convenient, because \( \mathbb{Z}_2 \) is not only an abelian group but also a ring, so \( \mathbb{Z}_2 \)-valued cohomology forms a ring with the cup product operation \( \smile \). It turns out (see e.g. [30] - Section 3.2, or [37] - Section 4) that there is a ring isomorphism

\[
H^\bullet(\mathbb{R}P^\infty, \mathbb{Z}_2) \cong \mathbb{Z}_2[\bar{x}],
\]

where \( \bar{x} \) is a generator of cohomological degree 1. This generator is precisely the cohomology class which characterizes the universal bundle \( S^\infty \to \mathbb{R}P^\infty \) in the sense described below

\(^7\)Discussion that follows admits a suitable generalization to groups carrying a nontrivial topology, but we shall not need it.

\(^8\)This is also easy to prove directly by working with cellular cochains.
All other $\mathbb{Z}_2$-valued (and hence also all $\mathbb{R}/\mathbb{Z}$-valued) cohomology classes on $\mathbb{R}P^\infty$ may be constructed by taking iterated cup products of this class with itself. This means that the most general Feynman weight that may be obtained by pulling back classes on the classifying space and integrating is of the form

$$e^{2\pi i S} = (-1)^{(A,c_1)+(A\sim A,c_2)+(A\sim A\sim A,c_3)+...},$$

where each $c_p$ is a $p$-cycle in $X$ with coefficients in $\mathbb{Z}_2$.

Going back to lattice systems, it is striking that lattice $\mathbb{Z}_2$ gauge theory as considered in the main text does not resemble purely topological models defined above. In fact discretization introduces a large number of holes in the approximated space. Due to the presence of spurious holes, nonzero curvature is possible. In the main text we have considered a lattice, gave a suitable definition of cohomology and defined gauge field as a $\mathbb{Z}_2$-valued 1-cochain modulo coboundaries. This is the same as a cohomology class in degree one provided that we truncate the space $X$ to its 1-skeleton $X\leq 1$ - topological space obtained by gluing together vertices and edges of the lattice ignoring all higher dimensional cubes. If we do not impose the constraint $\delta A = 0$ (here $\delta$ is the coboundary as defined in the Section 2), it is not possible to extend the corresponding principal $\mathbb{Z}_2$-bundle to one defined over the 2-skeleton of $X$. This means that in Feynman weights of the form (B.5) we have to take $c_p = 0$ for $p > 1$. As for $c_1$, there exists one canonical choice: $c_1 = \sum_{c \in Q_1} e$.

Based on the heuristics above, we propose a deformation of the usual lattice $\mathbb{Z}_2$ gauge theory [32, 38] described by the partition function

$$Z = \sum_{A \in C_1} (-1)^{\sum_{c \in Q_1} (A,c)} \exp \left( -g \sum_{f \in Q_2} (-1)^{(A,\partial f)} \right),$$

where we refer to 1- and 2-cubes in a $d+1$-dimensional Euclidean spacetime lattice. It would be interesting to clarify relation between this model and the $\Gamma$ model (if any relation exists), but for now we do not have a useful hamiltonian formulation of the former or path integral description of the latter.

Instead of relying on high-brow concepts such as classifying spaces one may also try to directly mimic Lagrangians known from continuum theory. If $d = 2n$ it is possible to form Chern-Simons-like weights

$$e^{2\pi i S_{CS}} = (-1)^{\int A \sim (\delta A)^n},$$

which are easily checked to be gauge-invariant. For $n = 1$ there is also a known hamiltonian approach [39, 40]. Since these theories exist only for odd spacetime dimensions, it seems unlikely that they are directly relevant for the $\Gamma$ model.

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