ON THE MODULARITY OF WILDLY RAMIFIED GALOIS REPRESENTATIONS

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Abstract. We show that an infinite family of odd complex 2-dimensional Galois representations ramified at 5 having nonsolvable projective image are modular, thereby verifying Artin’s conjecture for a new case of examples. Such a family contains the original example studied by Buhler. In the process, we prove that an infinite family of residually modular Galois representations are modular by studying Λ-adic Hecke algebras.

1. Introduction

There is considerable interest in continuous homomorphisms

\[ \rho : G_{\mathbb{Q}} \to GL_2(K_\ell) \]

where \( G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \) is the absolute Galois group of \( \mathbb{Q} \) and \( \ell \) is a fixed rational prime. For example, \( \rho = \rho_{E,\ell} \) may be the \( \ell \)-adic representation of an elliptic curve \( E \) over \( \mathbb{Q} \), or \( \rho = \rho_f \) may be the \( \ell \)-adic representation associated to a modular form. The continuity of such Galois representations implies the image lies in \( GL_2(O) \) for some ring of integers \( O \) with maximal ideal \( \lambda \) in a finite extension \( K \) of \( \mathbb{Q}_\ell \); then \( k = O/\lambda \) is a finite extension of \( \mathbb{F}_\ell \). We define the residual representation \( \overline{\rho} \) as the composition\n
\[ \overline{\rho} : G_{\mathbb{Q}} \to GL_2(O) \to GL_2(\mathbb{F}_\ell). \]

For example, \( \overline{\rho} = \overline{\rho}_{E,\ell} \) may be the mod \( \ell \) representation of an elliptic curve, or \( \overline{\rho} = \overline{\rho}_f \) may be the mod \( \ell \) reduction of the representation associated to a modular form. We also consider the projective representation \( \overline{\rho} \) as the composition\n
\[ \overline{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_\ell) \to PGL_2(\mathbb{Q}_\ell). \]

A common question is: Given a continuous \( \ell \)-adic Galois representation \( \rho \) such that \( \overline{\rho} \) is modular i.e. \( \overline{\rho} \cong \overline{\rho}_f \), when is \( \rho \) modular i.e. \( \rho \cong \rho_f \)? In this paper the main result is the

Theorem 1.0.1. For \( \ell \) an odd prime, let \( \rho : G_{\mathbb{Q}} \to GL_2(O) \) be a continuous \( \ell \)-adic representation such that

1. \( \rho \) is ordinary and ramified at finitely many primes;
2. \( \overline{\rho} \) is absolutely irreducible when restricted to \( Gal(\mathbb{Q}/(\sqrt{-1})^{(\ell-1)/2}) \), modular, and wildly ramified at \( \ell \).

Then \( \rho \) is \( \ell \)-adically modular i.e. \( \rho \cong \rho_f \) for an \( \ell \)-adic cusp form \( f \).

The proof the theorem follows the ideas outlined in [15] and [10], but we generalize to \( \Lambda \)-adic modular forms. One application of this result is the

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Theorem 1.0.2. For \( \ell \) an odd prime, let \( \rho : G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{O}) \) be a continuous Galois representation such that

1. \( \rho \) is ramified at finitely many primes;
2. \( \mathcal{E} \) is absolutely irreducible when restricted to \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-1}(-1)^{(\ell-1)/2}\ell)) \), modular, and wildly ramified at \( \ell \);
3. \( \rho(G_{\ell}) \) is finite and \( \bar{\rho}(G_{\ell}) \) is a cyclic group of \( \ell \)-power order.

Then \( 1 \circ \rho : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{C}) \) is modular for each embedding \( \iota : K \hookrightarrow \mathbb{C} \).

The proof of this result uses new ideas outlined in [4] on “glueing” together two \( \ell \)-adic modular forms. Our paper is different from others in the literature (most notably [7]) in that we consider deformations of wildly ramified \( \ell \)-adic representations, not tamely ramified ones. We use a trick that exploits the wild ramification of the representations at hand.

Our third result is the

Theorem 1.0.3. Let \( \rho : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{C}) \) be a continuous representation with non-solvable image, and denote \( L/\mathbb{Q} \) as the extension cut out by \( \bar{\rho} \). If this extension is the splitting field of \( x^5 + Bx + C \) such that \( 75C^2/\sqrt{256B^5+3125C^2} \) is the square of a 5-adic unit, then \( \rho \) is (classically) modular.

The proof relies on the modularity of \( \mathbb{Q} \)-curves, as shown in [18] using ideas from [10]. One example that is covered in this result is the original example in [2]; the polynomials

\[
x^5 + 10x^3 + 10x^2 + 35x + 18 \quad \text{and} \quad 5x^5 + 20x + 16
\]

generate the same splitting field. An infinite family of examples are

\[
x^5 + 5\left(\frac{9 - 5u^4}{5u^4}\right)x + 4\left(\frac{9 - 5u^4}{5u^4}\right), \quad u \in \mathbb{Q} \cap \mathbb{Z}_{5}^\times.
\]

(When \( u = 1 \) we find the previous example.) This result gives the first known proof of infinitely many examples of icosahedral Galois representations satisfying Artin’s Conjecture which are ramified at 5 since the conditions in the theorem force \( \rho \) to be wildly ramified there. Other papers in the literature consider representations which are unramified at 5: [27], [24], and [6] consider finitely many representations ramified at either 2 or 3 yet unramified at 5, while [5] considers infinitely many representations unramified at both 2 and 5. Only [2] considers a representation ramified at 5.

We organize this exposition as follows. In section 2 we review the Galois cohomology involved in computing the dimension of the universal deformation ring. We consider ordinary representations such that the restriction \( G_{\ell} \to \text{GL}_2(\mathbb{F}_{\ell}) \) may have equal eigenvalues, so we explain why the universal deformation ring exists. In section 3 we review the theory of ordinary \( \Lambda \)-adic modular forms and how this gives a family of modular Galois representations \( G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{Q}_{\ell}[[X]]) \), following closely [23] Chap. 7. We then show that one may prove certain weight 1 representations are modular if one first proves certain weight 2 representations are modular, and we employ standard “lifting” arguments as in [12], [15] and [10] to show the universal deformation ring is indeed modular in the weight 2 case. In section 4 we present applications of the theorems above using ideas from [28] relating icosahedral Galois extensions with elliptic curves.
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2. Universal Deformation Ring

Every ℓ-adic Galois representation ρ : G_Q → GL_2(O) has a mod ℓ reduction \( \bar{\rho} : G_Q \to GL_2(O) \). Conversely, for a local O-algebra O’ which is complete, Noetherian, and has residue field \( k' = O'/\mathfrak{m} \), we say a representation \( \rho' : G_Q \to GL_2(O') \) is a deformation of \( \bar{\rho} \) if \( \rho' \simeq \bar{\rho} \otimes k' \) and \( \det \rho' = \det \rho \). When \( \rho \) ramifies at finitely many primes and \( \bar{\rho} \) is absolutely irreducible, there exists a universal deformation \( \rho^{univ} : G_Q \to GL_2(R) \) i.e. for any deformation \( \rho' \) there exists a unique O-algebra surjection \( \phi : R \to O' \) such that \( \rho' \simeq \phi \circ \rho^{univ} \), for more properties, see [31]. Unfortunately, the universal deformation ring \( R \) is too big to work with but we may consider a filtration of finitely generated deformation rings \( R_\Sigma \) corresponding to finite sets \( \Sigma \). Our goal in this section is to give an explicit formula for the topological dimension of \( R_\Sigma \) as an O-algebra.

2.1. Galois Theory. We briefly review some preliminaries to fix notation.

Given a place \( \nu \) of \( \mathbb{Q} \), each embedding \( \iota_\nu : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_\nu} \) induces an embedding \( \iota'_\nu : \text{Gal}(\overline{\mathbb{Q}_\nu}/\mathbb{Q}_\nu) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) given by \( \sigma \mapsto \sigma \circ \iota_\nu \). We consider \( G_\nu = \text{Gal}(\overline{\mathbb{Q}_\nu}/\mathbb{Q}_\nu) \) a subgroup of the absolute Galois group \( G_Q = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) and call it the decomposition group at \( \nu \). We define the inertia group \( I_\nu \) as follows: When \( \nu = p \) is a finite prime, \( I_p \) is the kernel of the map \( \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \) induced by
reduction modulo \( p \), and when \( \nu = \infty \) is the infinite prime, \( I_\infty = G_\infty = \text{Gal}(\mathbb{C}/\mathbb{R}) \) is the group generated by complex conjugation. In either case, the inertia group is a normal subgroup of the decomposition group, and the quotient \( G_\nu/I_\nu \) is a procyclic group with topological generator \( \text{Frob}_p \). For more properties, see \([39]\).

Consider the restriction of \( \rho \) to the inertia and decomposition groups. We say that \( \rho \) is unramified at \( \nu \) if its restriction to the inertia group at \( \nu \) is trivial, and ramified otherwise. (We consider only \( \ell \)-adic representations which are ramified at finitely many places.) We say \( \rho \) is wildly ramified at a finite prime \( \nu = p \) if its restriction to the Sylow pro-\( p \)-subgroup \( I^p \subseteq I_p \) is nontrivial. Note \( \rho \) is ramified at the infinite prime \( \nu = \infty \) if and only if \( \det \rho(c) = -1 \) on complex conjugation; we say that such representations are odd. In this case,

\[
\rho(c) \simeq \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

When \( \nu = \ell \) we say that \( \rho \) is ordinary if the decomposition group may be conjugated over \( \mathcal{O} \) to be upper-triangular with the upper entry on the diagonal an unramified character i.e. there exist homomorphisms \( \chi_0, \chi_\ell : G_\ell \to \mathcal{O}^\times \) such that

\[
\rho|_{G_\ell} \simeq \begin{pmatrix} \chi_\ell & * \\ \chi_0 \end{pmatrix} \quad \text{with} \quad \chi_0|_{I_\ell} = 1.
\]

The cyclotomic character \( \varepsilon_\ell : G_\mathbb{Q} \to \mathbb{Z}_\ell^\times \) is a canonical \( \ell \)-adic representation defined by acting on the \( \ell \)-power roots of unity: \( \sigma(\zeta) = \zeta^{\varepsilon_\ell(\sigma)} \). Note that \( \varepsilon_\ell \) is unramified at \( \nu \neq \ell, \infty \); in fact for \( \ell \) odd, the Frobenius element \( \text{Frob}_p \mapsto p \) for finite primes \( p \neq \ell \) and complex conjugation \( c \mapsto -1 \) at the infinite prime, yet the inertia group \( I_\ell \) has nontrivial image because there are no nontrivial \( \ell \)-power roots of unity in \( \mathbb{F}_\ell \). In particular this gives a nontrivial action on \( \ell \)-power roots of unity \( \mu_{p^\infty} \) so for any \( k \)-vector space \( V \) acted upon by the absolute Galois group we define \( V(1) = \mu_{p^\infty} \otimes V \) as the Tate twist of \( V \).

2.2. Adjoint Representation. The residual representation \( \overline{\rho} \) induces an action of the absolute Galois group on the \( k \)-vector space of \( 2 \times 2 \) matrices given by \( \sigma \cdot m = \overline{\rho}(\sigma) m \overline{\rho}(\sigma)^{-1} \); we denote this \( k \)-vector space with such an action by \( \text{ad}\overline{\rho} \).

Explicitly, denote the matrices

\[
(1) \quad m_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad m_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix};
\]

then \( \text{ad}\overline{\rho} \) is the \( k \)-vector space spanned by the \( m_j \). We have a filtration

\[
\{0\} = \text{ad}^3\overline{\rho} \subset \text{ad}^2\overline{\rho} \subset \text{ad}^1\overline{\rho} \subset \text{ad}^0\overline{\rho} \subset \text{ad}^{-1}\overline{\rho} = \text{ad}\overline{\rho}
\]

where \( \text{ad}^i\overline{\rho} \) is the \( (3 - i) \)-dimensional subspace spanned by the \( m_j \) for \( j \geq i \).

**Proposition 2.2.1.** Let \( \overline{\rho} : G_\mathbb{Q} \to GL_2(k) \) be a continuous Galois representation, and let \( \epsilon \) be an infinitesimal i.e. \( \epsilon^2 = 0 \). The equivalence classes of infinitesimal deformations \( \overline{\rho}_\epsilon : G_\mathbb{Q} \to GL_2(k[\epsilon]) \) satisfying \( \overline{\rho}_\epsilon \equiv \overline{\rho} \mod \epsilon k[\epsilon] \) and \( \det \overline{\rho}_\epsilon = \det \overline{\rho} \) are in one-to-one correspondence with the cohomology classes in \( H^1(G_\mathbb{Q}, \text{ad}\overline{\rho}) \).

**Proof.** We follow \([11] \S4\). Given \( \sigma \in G_\mathbb{Q} \) we may express an infinitesimal deformation in the form \( \overline{\rho}_\epsilon(\sigma) = (I_2 + \epsilon \xi(\sigma)) \overline{\rho}(\sigma) \) for some \( \xi(\sigma) \in \text{Mat}_2(k) \), where \( \xi(\sigma) \) must have trace zero since \( \det \overline{\rho}_\epsilon(\sigma) = \det \overline{\rho}(\sigma) = (1 + \epsilon \text{tr} \xi(\sigma)) \det \overline{\rho}(\sigma) \). We claim that equivalence classes of homomorphisms are in one-to-one correspondence with \( \xi \in H^1(G_\mathbb{Q}, \text{ad}\overline{\rho}) \).
For \( \overline{\rho} \) to be a homomorphism we must have
\[
1_2 = \overline{\rho}(\sigma \cdot \tau) \cdot \overline{\rho}(\tau)^{-1} \cdot \overline{\rho}(\sigma)^{-1}
= (1_2 + \epsilon \xi_{\sigma \tau}) \overline{\rho}(\sigma \cdot \tau) \cdot \overline{\rho}(\tau)^{-1} (1_2 - \epsilon \xi_{\tau}) \cdot \overline{\rho}(\sigma)^{-1} (1_2 - \epsilon \xi_{\sigma})
= (1_2 + \epsilon \xi_{\sigma \tau}) \cdot (1_2 - \epsilon \bigg[ \overline{\rho}(\sigma) \cdot \xi_{\tau} \cdot \overline{\rho}(\sigma)^{-1} \bigg]) \cdot (1_2 - \epsilon \xi_{\sigma})
= 1_2 + \epsilon \big[ \xi_{\sigma \tau} - \sigma \cdot \xi_{\tau} - \xi_{\sigma} \big].
\]
Hence \( \overline{\rho} \) is a homomorphism if and only if \( \xi \) is a cocycle i.e. \( \xi_{\sigma \tau} = \xi_{\sigma} + \sigma \cdot \xi_{\tau} \).

The equivalence class of \( \overline{\rho} \) consists of conjugates \( A \overline{\rho} A^{-1} \) where \( A = 1_2 + \epsilon \alpha \) for \( \alpha \in \text{Mat}_2(k) \). But \( A \overline{\rho}(\sigma) A^{-1} = (1_2 + \epsilon [\alpha - \sigma \cdot \alpha]) \overline{\rho}(\sigma) \) so equivalence classes yield \( \xi_{\sigma} \) modulo coboundaries i.e. \( \alpha - \sigma \cdot \alpha \). Hence \( \xi \in H^1(G_Q, \text{ad}^0 \mathcal{P}) \) as claimed. \( \square \)

### 2.3. Selmer Groups.

We wish to classify infinitesimal deformations even further by considering what happens when we restrict to the decomposition and inertia groups. Recall that for each place \( \nu \) of \( \mathbb{Q} \), we have restriction maps \( \text{res}_\nu : H^1(G_Q, \text{ad}^0 \mathcal{P}) \to H^1(G_\nu, \text{ad}^0 \mathcal{P}) \) as well as the “inflation-restriction” exact sequence
\[
0 \to H^1(G_\nu/I_\nu, (\text{ad}^0 \mathcal{P})^{I_\nu}) \to H^1(G_\nu, \text{ad}^0 \mathcal{P}) \to H^1(I_\nu, \text{ad}^0 \mathcal{P}).
\]
For more properties, see [33].

We will use these maps to define subgroups \( H^1_j(G_\nu, \text{ad}^0 \mathcal{P}) \subseteq H^1(G_\nu, \text{ad}^0 \mathcal{P}) \) that will encode information about infinitesimal deformations. When \( \nu \neq \ell \) each deformation \( \overline{\rho}_\nu \) should be (un)ramified when \( \mathcal{P} \) is (un)ramified so the restriction of a class from \( H^1(G_\nu, \text{ad}^0 \mathcal{P}) \) to \( H^1(I_\nu, \text{ad}^0 \mathcal{P}) \) should be trivial. Define
\[
H^1_j(G_\nu, \text{ad}^0 \mathcal{P}) = \ker \left[ H^1(G_\nu, \text{ad}^0 \mathcal{P}) \to H^1(I_\nu, \text{ad}^0 \mathcal{P}) \right] \quad \text{for } \nu \neq \ell.
\]

When \( \nu = \ell \) and \( \mathcal{P} \) is ordinary i.e. the restriction of \( \mathcal{P} \) to \( G_\ell \) is upper-triangular, each deformation \( \overline{\rho}_\ell \) should be ordinary as well. We choose
\[
H^1_j(G_\ell, \text{ad}^0 \mathcal{P}) \subseteq \ker \left[ H^1(G_\ell, \text{ad}^0 \mathcal{P}) \to H^1(G_\ell, \text{ad}^0 \mathcal{P}/\text{ad}^0 \mathcal{P}) \right]
\]
\[
= \text{im} \left[ H^1(G_\ell, \text{ad}^0 \mathcal{P}) \to H^1(G_\ell, \text{ad}^0 \mathcal{P}/\text{ad}^0 \mathcal{P}) \right].
\]

When restricted to the inertia group, the diagonal terms of \( \mathcal{P}_\ell \) should be the same as those of \( \mathcal{P} \) so the restriction of a class from \( H^1(G_\ell, \text{ad}^0 \mathcal{P}) \) to \( H^1(I_\ell, \text{ad}^0 \mathcal{P}/\text{ad}^0 \mathcal{P}) \) should be trivial. Define
\[
H^1_j(G_\ell, \text{ad}^0 \mathcal{P}) \quad (2)
\]
\[
= \text{im} \left[ \ker \left[ H^1(G_\ell, \text{ad}^0 \mathcal{P}) \to H^1(I_\ell, \text{ad}^0 \mathcal{P}/\text{ad}^0 \mathcal{P}) \right] \to H^1(G_\ell, \text{ad}^0 \mathcal{P}) \right] .
\]

We explain how this definition for \( \nu = \ell \) compares to the usual one. In general we have the exact sequence
\[
H^1_j(G_\ell, \text{ad}^0 \mathcal{P}) \longrightarrow H^1(G_\ell, \text{ad}^0 \mathcal{P}) \longrightarrow H^1(I_\ell, \text{ad}^0 \mathcal{P}/\text{ad}^0 \mathcal{P})
\]
but the map \( H^1(G_\ell, \text{ad}^0 \mathcal{P}) \to H^1(G_\ell, \text{ad}^0 \mathcal{P}) \) is not an injection. However when \( \det \rho = \varepsilon_\ell \) is the cyclotomic character, the group \( H^0(G_\ell, \text{ad}^0 \mathcal{P}/\text{ad}^0 \mathcal{P}) \) is trivial, so we recover the usual definition as considered in [12] and [14]:
\[
H^1_j(G_\ell, \text{ad}^0 \mathcal{P}) = \ker \left[ H^1(G_\ell, \text{ad}^0 \mathcal{P}) \to H^1(I_\ell, \text{ad}^0 \mathcal{P}/\text{ad}^0 \mathcal{P}) \right] .
\]
Fix a finite set $\Sigma$ of places that does not contain $\ell$. We define the Selmer group for $\text{ad}^0\mathfrak{p}$ with respect to $\Sigma$ as

$$H^1_\Sigma(\mathbb{Q}, \text{ad}^0\mathfrak{p}) = \ker\left[H^1(G_\mathbb{Q}, \text{ad}^0\mathfrak{p}) \to \bigoplus_{\nu \notin \Sigma} H^1(G_\nu, \text{ad}^0\mathfrak{p}) / H^1_j(G_\nu, \text{ad}^0\mathfrak{p})\right]$$

i.e. the collection of classes $\xi \in H^1(G_\mathbb{Q}, \text{ad}^0\mathfrak{p})$ such that $\text{res}_\nu(\xi) \in H^1_j(G_\nu, \text{ad}^0\mathfrak{p})$ for all places $\nu \notin \Sigma$.

Define $H^1_j(G_\nu, \text{ad}^0\mathfrak{p}(1))$ as the annihilator of $H^1_j(G_\nu, \text{ad}^0\mathfrak{p})$ under the pairing

$$H^1(G_\nu, \text{ad}^0\mathfrak{p}) \times H^1(G_\nu, \text{ad}^0\mathfrak{p}(1)) \to H^2(G_\nu, k(1)) \simeq k$$

and define the dual Selmer group $H^1_\Sigma(\mathbb{Q}, \text{ad}^0\mathfrak{p}(1))$ as classes $\xi \in H^1_\Sigma(G_\mathbb{Q}, \text{ad}^0\mathfrak{p}(1))$ such that $\text{res}_\nu(\xi) \in H^1_j(G_\nu, \text{ad}^0\mathfrak{p}(1))$ for all places $\nu \notin \Sigma$, yet $\text{res}_\nu(\xi) = 0$ otherwise.

**Proposition 2.3.1.** For $\ell$ an odd prime, let $\mathfrak{p} : G_\ell \to GL_2(k)$ be a continuous, ordinary mod $\ell$ representation that is wildly ramified. Then

$$\dim_k H^1_j(G_\ell, \text{ad}^0\mathfrak{p}) = 1 + \dim_k H^0_\ell(G_\ell, \text{ad}^0\mathfrak{p}).$$

**Proof.** As $\mathfrak{p}$ is ordinary, assume that a given $\sigma \in G_\ell$ maps to

$$\mathfrak{p}(\sigma) = \begin{pmatrix} a & b \\ d & b \end{pmatrix}$$

for some $a, d \in k^\times, b \in k$.

Recall that $\text{ad}^0\mathfrak{p}$ is spanned by $m_j$ as in (4) for $j \geq i$, and note the identities

$$\sigma \cdot m_0 = \frac{d}{a} m_0 + \frac{b}{a} m_1 + \frac{b^2}{ad} m_2, \quad \sigma \cdot m_1 = m_1 + \frac{2b}{d} m_2, \quad \sigma \cdot m_2 = \frac{a}{d} m_2.$$

In particular, $G_\ell$ acts trivially on $V = \text{ad}^1\mathfrak{p}/\text{ad}^2\mathfrak{p}$. To prove the proposition, note

$$H^1_j(G_\ell, \text{ad}^0\mathfrak{p}) \simeq \frac{K_1}{K_1 \cap K_2}$$

where $K_1 = \ker[H^1(G_\ell, \text{ad}^1\mathfrak{p}) \to H^1(I_\ell, V)]$ and $K_2 = \ker[H^1(G_\ell, \text{ad}^1\mathfrak{p}) \to H^1(G_\ell, \text{ad}^0\mathfrak{p})]$;

we will compute the dimension of $K_1$ and show $K_1 \cap K_2$ is trivial.

As $\mathfrak{p}$ is wildly ramified, there exists $\sigma_0 \in I_\ell^w$ (wild inertia) such that

$$\mathfrak{p}(\sigma_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for some $b_0 \in k^\times$.

When $j > i$ the elements of $\text{ad}^1\mathfrak{p}/\text{ad}^0\mathfrak{p}$ fixed by $\sigma_0$ form a 1-dimensional space that only depends on $j$:

$$H^0_\ell(G_\ell, \text{ad}^0\mathfrak{p}) \subset H^0_\ell(I_\ell, \text{ad}^1\mathfrak{p}/\text{ad}^0\mathfrak{p}) \subset H^0_\ell(I_\ell^w, \text{ad}^1\mathfrak{p}/\text{ad}^2\mathfrak{p}) = \text{ad}^{i-1}\mathfrak{p}/\text{ad}^0\mathfrak{p}.$$

We compute the dimension of $K_1$, the kernel of the composition $H^1(G_\ell, \text{ad}^1\mathfrak{p}) \to H^1(G_\ell, V) \to H^1(I_\ell, V)$. Wild inertia acts trivially on $\text{ad}^0\mathfrak{p}$ and hence nontrivially on $\text{ad}^2\mathfrak{p}(1)$ so $H^2(G_\ell, \text{ad}^2\mathfrak{p}) \simeq H^0(G_\ell, \text{ad}^2\mathfrak{p}(1)) \subset H^0_\ell(I_\ell^w, \text{ad}^2\mathfrak{p}(1))$ is trivial. Then the exact sequence $0 \to \text{ad}^2\mathfrak{p} \to \text{ad}^1\mathfrak{p} \to V \to 0$ implies the exact sequence

$$0 \to H^0_\ell(G_\ell, V) \to H^1(G_\ell, \text{ad}^0\mathfrak{p}) \to H^1(G_\ell, \text{ad}^1\mathfrak{p}) \to H^1(G_\ell, \text{ad}^1\mathfrak{p}) \to H^1(G_\ell, V) \to 0.$$

In particular, the map $H^1(G_\ell, \text{ad}^1\mathfrak{p}) \to H^1(G_\ell, V)$ is surjective. On the other hand, the map $H^1(G_\ell, V) \to H^1(I_\ell, V)$ has kernel $H^1(G_\ell/I_\ell, V^I_\ell) \simeq \text{Hom}(k, k)$ of
dimension 1 because $G_{\ell}/I_{\ell}$ is a procyclic group with trivial action on $V$. Hence
\[
\dim_k K_1 = \dim_k \ker [H^1(G_{\ell}, \text{ad}^1\mathcal{P}) \to H^1(G_{\ell}, V)] \\
+ \dim_k \ker [H^1(G_{\ell}, V) \to H^1(I_{\ell}, V)] \\
= [\dim_k H^1(G_{\ell}, \text{ad}^2\mathcal{P}) - \dim_k H^0(G_{\ell}, V)] + \dim_k H^1(G_{\ell}/I_{\ell}, V^{1i}) \\
= \dim_k H^1(G_{\ell}, \text{ad}^2\mathcal{P}) = 1 + \dim_k H^0(G_{\ell}, \text{ad}^2\mathcal{P}) \\
= 1 + \dim_k H^0(G_{\ell}, \text{ad}^0\mathcal{P}).
\]

We show $K_1 \cap K_2$ is trivial. As $H^0(G_{\ell}, \text{ad}^1\mathcal{P}) = H^0(G_{\ell}, \text{ad}^0\mathcal{P})$ we have the exact sequence $0 \to H^0(G_{\ell}, \text{ad}^0\mathcal{P}/\text{ad}^1\mathcal{P}) \to H^1(G_{\ell}, \text{ad}^1\mathcal{P}) \to H^1(G_{\ell}, \text{ad}^0\mathcal{P})$, so
\[
K_2 \simeq H^0(G_{\ell}, \text{ad}^0\mathcal{P}/\text{ad}^1\mathcal{P}) \subseteq H^0(I_{\ell}, \text{ad}^0\mathcal{P}/\text{ad}^1\mathcal{P}).
\]
Either $K_2$ is 1-dimensional and we have equality or $K_2$ is trivial. A brief chase of the exact diagram
\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & K_1 \cap K_2 & K_2 \\
\downarrow & \downarrow & \downarrow \\
0 & K_1 & H^1(G_{\ell}, \text{ad}^1\mathcal{P}) \\
\downarrow & \downarrow & \downarrow \\
H^1(G_{\ell}, \text{ad}^0\mathcal{P}) & H^1(G_{\ell}, \text{ad}^0\mathcal{P}) & H^1(G_{\ell}, \text{ad}^0\mathcal{P})
\end{array}
\]
shows $K_1 \cap K_2$ is trivial in either case. \hspace{1cm} \Box

2.4. Deformations of type $\Sigma$. Again, fix a finite set $\Sigma$ of places that does not contain $\ell$. Assume that $\rho : G_Q \to GL_2(O)$ is ordinary and ramified at finitely many primes, and its residual representation $\mathcal{P} : G_Q \to GL_2(k)$ is absolutely irreducible and wildly ramified at $\ell$. Denote by $O'$ a complete, local Noetherian $\mathcal{O}$-algebra with field of fractions $K'$ having residue field $k' = O'/\lambda'$. We say a representation $\rho' : G_Q \to GL_2(O')$ is a deformation of $\mathcal{P}$ of type $\Sigma$ if

1. $\mathcal{P} \simeq \mathcal{P} \otimes_k k'$ and $\det \rho' = \det \rho$;
2. $\rho'|_{I_\nu} \simeq \rho|_{I_\nu} \otimes_K K'$ for all $\nu \notin \Sigma$; and
3. $\rho'|_{G_{\ell}} \simeq \begin{pmatrix} \chi_{\ell} & * \\ \chi_0 & 1 \end{pmatrix}$ where $\chi_{\ell} = \det \rho \cdot \chi_0^{-1}$ and $\chi_0|_{I_{\ell}} = 1$.

These conditions reflect via Proposition 2.4[1] precisely the cohomology classes in the Selmer group $H^1_{2,\Sigma}(\mathbb{Q}, \text{ad}^0\mathcal{P})$. Indeed, an equivalence class of deformations $\rho'$ satisfying condition (1) corresponds to $\xi \in H^1(G_Q, \text{ad}^0\mathcal{P})$; condition (2) corresponds to $\text{res}_{\nu}(\xi) \in H^1(G_{\nu}, \text{ad}^0\mathcal{P})$; and condition (3) corresponds to $\text{res}_{\ell}(\xi) \in H^1(G_{\ell}, \text{ad}^0\mathcal{P})$.

The local conditions above are also reflected in the Artin conductor. Denote the conductor of $\rho$ by $N_{\rho}$. The condition at the infinite place $\nu = \infty$ is equivalent to saying a deformation $\rho'$ is odd precisely when $\rho$ is odd, so consider the finite places $\nu = p$. If $\rho$ is unramified at $p \notin \Sigma$ then so is $\rho'$; in particular $\rho'$ is ramified at finitely many primes. For $p \in \Sigma$ there are no conditions on the restriction of $\rho'$ at $I_p$ so at worst the $p$-part of the conductor is $p^2$. Hence the conductor of $\rho'$ is divisible
by $N_0$ yet divides $N_\Sigma = N_0 \prod \mathfrak{p}^2$ in terms of the product over $p \in \Sigma$ not dividing $N_0$. In particular, when $= N_0 = N(\mathfrak{p}) \ell$ in terms of the Serre conductor defined in [17], the local conditions above force a deformation $\rho'$ to be minimally ramified in the sense of [14, Definition 3.1].

The following proposition shows that the “deformation conditions” above are “representable” i.e. there exists a universal deformation ring $R_\Sigma$.

**Proposition 2.4.1.** Let $\rho : G_\mathbb{Q} \to GL_2(\mathcal{O})$ be a continuous $\ell$-adic representation such that $\rho$ is ordinary and ramified at finitely many primes, while $\mathfrak{p}$ is absolutely irreducible and wildly ramified at $\ell$. Fix a finite set $\Sigma$ of places that does not contain $\ell$. There exists a universal deformation $\rho_{\text{univ}}^\Sigma : G_\mathbb{Q} \to GL_2(R_\Sigma)$ of $\mathfrak{p}$ of type $\Sigma$ in the sense that if $\rho' : G_\mathbb{Q} \to GL_2(\mathcal{O}')$ is any deformation of $\mathfrak{p}$ of type $\Sigma$, then there exists a unique $\mathcal{O}$-algebra homomorphism $\phi : R_\Sigma \to \mathcal{O}'$ such that $\rho' \simeq \phi \circ \rho_{\text{univ}}^\Sigma$.

It may appear that such a result follows from the well-known properties of deformation rings associated to absolutely irreducible residual representations, but we do not impose the typical local condition that $\chi_\ell$ is ramified at $\ell$, as in [32, p. 457] and [32, §30]. Indeed, if $\chi_\ell$ and $\chi_0$ were trivial when restricted to the decomposition group at $\ell$ – and hence both unramified at $\ell$ – then the centralizer of the residual representation $\overline{\rho}|_{G_\ell}$ would be non-scalar, and hence a desired local “universal” deformation $G_\ell \to GL_2(R_\ell)$ constructed out of the local representation would be “versal” at best. We exploit the fact that the residual representation is wildly ramified at $\ell$ to produce a global universal deformation $G_\mathbb{Q} \to GL_2(R_\mathbb{Q})$.

**Proof.** We follow the exposition in [32]. Consider the functor

$$D_\Sigma : \left\{ \begin{array}{l} \text{artinian } \mathcal{O}\text{-algebras } A \\
\text{having residue field } \mathbb{k} \\
\text{with a local } \mathcal{O}\text{-homomorphism } A \to \mathcal{O} \\
\end{array} \right\} \to \left\{ \begin{array}{l} \text{equivalence classes of deformations } \rho_A \\
\text{of } \mathfrak{p} \text{ of type } \Sigma \\
\end{array} \right\}.$$  

We say this functor is representable if there exists a complete, local Noetherian $\mathcal{O}$-algebra $R_\Sigma$ depending only on the representation $\rho$ and the set $\Sigma$ such that

$$D_\Sigma (A) \simeq \text{Hom}_\mathcal{O} (R_\Sigma, A).$$

If this is the case, then the image $D_\Sigma (R_\Sigma)$ associated with the trivial homomorphism $R_\Sigma \to R_\Sigma$ corresponds to a deformation $\rho_{\text{univ}}^\Sigma$ of $\mathfrak{p}$ of type $\Sigma$. Moreover, the isomorphism above implies that any image $D_\Sigma (A)$ may be associated with a unique $\mathcal{O}$-algebra homomorphism $\phi_A : R_\Sigma \to A$, so that $\rho_A \simeq \phi_A \circ \rho_{\text{univ}}^\Sigma$. When $A = \mathcal{O}'/\Lambda^n$ we have the inverse limit

$$\mathcal{O}' = \projlim_{n \to \infty} \mathcal{O}'/\Lambda^n \implies \rho' \simeq \phi \circ \rho_{\text{univ}}$$

so that the proposition follows. Hence it suffices to show that $D_\Sigma$ is representable.

Now consider the functor $D_{\mathfrak{p}}$ sending artinian $\mathcal{O}$-algebras $A$ to deformations $\rho_A$ satisfying just the deformation condition (1) above. It is well-known that this functor is representable because $\mathfrak{p}$ is absolutely irreducible and ramified at finitely many primes; see [31, §1.2, Proposition 1] and [31, §1.3, (a.3)]. To show $D_\Sigma$ is representable, we use [32, §26, Proposition 1] and [32, §23, Corollary] to note that it suffices to show that the local conditions in (2) and (3) are “deformation conditions” in the sense of [32, §23] defined as follows: Fix a place $\nu$ of $\mathbb{Q}$ and a free rank two $\mathcal{O}$-module $V$ such that $\rho : G_\nu \to GL(V)$. Consider a category whose objects are pairs $(A, V_A)$ of artinian $\mathcal{O}$-algebras $A$ along with free rank two $A$-modules $V_A$ endowed with a continuous action by $G_\nu$. Note that we may consider an object to be
an assignment \( A \sim \rho_A \) sending an artinian \( \mathcal{O} \)-algebra \( A \) to a 2-dimensional representation \( \rho_A : G_\nu \to GL(V_A) \) defined by this action. A morphism \((A, V_A) \to (C, V_C)\) in this category consists of an \( \mathcal{O} \)-algebra homomorphism \( A \to C \) along with an \( A \)-module homomorphism \( V_A \to V_C \) inducing the isomorphism \( V_C \simeq V_A \otimes_A C \). We say that a category \( \mathcal{D}_\nu \) is a deformation condition for \( \mathcal{P} \) if the following four properties hold for all objects:

DC1: \((k, V \otimes_O k) \in \mathcal{D}_\nu \) where the action on \( V \otimes O k \) by \( G_\nu \) is given by \( \mathcal{P}|_{G_\nu} \).

DC2: For any morphism \((A, V_A) \to (C, V_C)\), if \((A, V_A) \in \mathcal{D}_\nu \) then \((C, V_C) \in \mathcal{D}_\nu \).

DC3: For any morphism \((A, V_A) \to (C, V_C)\), if \((C, V_C) \in \mathcal{D}_\nu \) and \( A \to C \) then \((A, V_A) \in \mathcal{D}_\nu \).

DC4: Say that the following diagram of morphisms is cartesian:

\[
\begin{array}{ccc}
D & \xleftarrow{A} & B \\
\alpha & \downarrow & \beta \\
C & \xrightarrow{\beta} & \nu
\end{array}
\]

that is, \( D \simeq A \times C B \) is the fiber product. The object \((D, V_D) \in \mathcal{D}_\nu \) if and only if both \((A, V_A), (B, V_B) \in \mathcal{D}_\nu \).

For each place \( \nu \), consider the full subcategory \( \mathcal{D}_\nu \subseteq \mathcal{D}_\mathcal{P} \) of objects \((A, V_A)\) such that the action by inertia is given by \( \rho|_{I_\ell} \otimes_O A \). It is clear that a deformation \( \rho_A \) of \( \mathcal{P} \) satisfies conditions (1) and (2) if and only if \((A, V_A) \in \mathcal{D}_\nu \) for all \( \nu \not\in \Sigma \). We claim that a deformation \( \rho_A \) of \( \mathcal{P} \) satisfies conditions (1) and (3) if and only if \((A, V_A) \in \mathcal{D}_\ell \). To this end, choose an object \((A, V_A) \in \mathcal{D}_\ell \), and consider the \( A \)-linear submodule

\[
(V_A)_{I_\ell} = \left\{ v \in V_A \mid \sigma \cdot v = \det \rho(\sigma) \cdot v \text{ for all } \sigma \in I_\ell \right\}.
\]

This submodule is nontrivial because there exists an \( A \)-basis of \( V_A \) such that

\[
\rho_A|_{I_\ell} \simeq \rho|_{I_\ell} \otimes_O A \simeq \left(\begin{array}{cc}
\det \rho & * \\
0 & 1
\end{array}\right);
\]

yet the submodule is not all of \( V_A \) because \( \rho \) is wildly ramified at \( \ell \) i.e. * is not identically zero. We identify \((V_A)_{I_\ell} \simeq V_{I_\ell} \otimes O A \) as the free rank one \( I_\ell \)-module of \( I_\ell \)-coinvariants because inertia acts trivially on the quotient \((V_A)_{I_\ell} = V_A/(V_A)_{I_\ell} \simeq V_{I_\ell} \otimes O A \) of \( I_\ell \)-invariants. We mention in passing that \( \text{ad} \mathcal{P} \simeq \text{Hom}_k (V \otimes O k, V \otimes O k) \) and \( \text{ad}^2 \mathcal{P} \simeq \text{Hom}_k (V_{I_\ell} \otimes O k, V_{I_\ell} \otimes O k) \).

Since \( I_\ell \) is a normal subgroup of \( G_\ell \), the \( A \)-submodules \((V_A)_{I_\ell} \) and \((V_A)^{I_\ell} \) are stable under action by \( G_\ell \). To see why, choose \( \tau \in G_\ell \) and \( v \in (V_A)_{I_\ell} \); we must show that \( \sigma \cdot (\tau \cdot v) = \det \rho(\sigma) \cdot (\tau \cdot v) \) for all \( \sigma \in I_\ell \). Indeed, we have

\[
\sigma \cdot (\tau \cdot v) = \tau \cdot \sigma' \cdot v = \tau \cdot \det \rho(\sigma') \cdot v = \tau \cdot \det \rho(\sigma) \cdot v = \det \rho(\sigma) \cdot (\tau \cdot v);
\]

where \( \sigma' = \tau^{-1} \cdot \sigma \cdot \tau \in I_\ell \). This shows that \( \rho_A|_{G_\ell} \) acts on \((V_A)_{I_\ell} \) and \((V_A)^{I_\ell} \) by some characters \( \chi_\ell \) and \( \chi_0 \), respectively. It is clear that \( \chi_0 \cdot |_\ell = \det \rho_A = \det \rho \) and \( \chi_0 \) is unramified. Hence \( \rho_A \) is a deformation of \( \mathcal{P} \) of type \( \Sigma \) and is only if \((A, V_A) \) is in \( \mathcal{D}_\nu \) for all places either \( \nu \not\in \Sigma \) or \( \nu = \ell \).

It suffices show that \( \mathcal{D}_\nu \) is a deformation condition for \( \mathcal{P} \) for all places \( \nu \). It is clear that \((k, V \otimes O k) \) is in \( \mathcal{D}_\nu \) i.e. (DC1) is satisfied. Say that we have a morphism \((A, V_A) \to (C, V_C)\) where \((A, V_A) \) is in \( \mathcal{D}_\nu \). Since \( V_C \simeq V_A \otimes_A C \) as \( A \)-modules, the
action by inertia must be given by \((\rho|_{I_v} \otimes \mathcal{O}) \otimes_A C \simeq \rho|_{I_v} \otimes \mathcal{O} C\); hence \((C, V_C)\) is in \(D_v\) as well. This shows that (DC2) is satisfied. Say that we have a morphism \((A, V_A) \to (C, V_C)\) where \((C, V_C)\) is in \(D_v\) and we have the exact sequence of \(A\)-modules:

\[
0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0.
\]

We tensor this sequence over \(A\) with the free – and hence flat – \(A\)-module \(V_A\) to find the exact sequence

\[
0 \longrightarrow V_A \longrightarrow V_C \longrightarrow V_A \otimes_A C/A \longrightarrow 0.
\]

Fix \(\sigma \in I_v\) and \(v \in V_A\) and let \(w = (\sigma - \rho(\sigma)) \cdot v \in V_A\). By assumption on the object \((C, V_C)\) we have \(w \in \ker(V_A \to V_C)\). However \(V_A\) injects into \(V_C\) so \(w = 0\) i.e. the action of inertia on \(V_A\) is given by \(\rho|_{I_v} \otimes \mathcal{O} A\). This shows that (DC3) is satisfied.

To show (DC4) is satisfied, assume that we have \(\mathcal{O}\)-algebra homomorphisms \(\alpha : A \to C\) and \(\beta : B \to C\), and denote \(D = A \times_C B\) as the fiber product. Fix a free 2-dimensional \(D\)-module \(V_D\), and denote \(V_A\) and \(V_B\) as the tensor products of \(V_D\) with respect to the projections \(1 \times_C \beta\) and \(\alpha \times_C 1\), respectively; then \(V_D \simeq V_A \times_{V_C} V_B\). If \((D, V_D)\) is in \(D_v\), then the canonical morphisms \((D, V_D) \to (A, V_A)\) and \((D, V_D) \to (B, V_B)\) show that both \((A, V_A)\) and \((B, V_B)\) are in \(D_v\). Conversely, if the action of inertia on both \(V_A\) and \(V_B\) are specified, then it is also specified on \(V_C \simeq V_A \otimes_A C \simeq V_B \otimes_B C\) and hence it is specified on \(V_D \simeq V_A \times_{V_C} V_B\) as well. This shows that if \((A, V_A)\) and \((B, V_B)\) are in \(D_v\) then so is \((D, V_D)\). \(\square\)

2.5. Dimension Computations. We conclude this section by computing the dimension of the universal deformation ring. Note that this is the only proposition in this section which assumes the residual representation is odd.

**Proposition 2.5.1.** For \(\ell\) an odd prime, let \(\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O})\) be a continuous \(\ell\)-adic representation such that \(\rho\) is ordinary and ramified at finitely many primes, while \(\overline{\rho}\) is odd, absolutely irreducible, and wildly ramified at \(\ell\). If \(\ell = 3\) assume moreover that \(\overline{\rho}\) remains absolutely irreducible when restricted to \(Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt[3]{3}))\).

1. Fix a finite set \(\Sigma\) of places that does not contain \(\ell\). The universal deformation ring \(R_{\Sigma}\) of \(\overline{\rho}\) can be topologically generated as an \(\mathcal{O}\)-algebra by \(\dim_k H^1_\Sigma(\mathbb{Q}, \text{ad}^0 \overline{\rho})\) elements.

2. Fix a finite set \(\Sigma\) of finite places \(q \equiv 1 \mod \ell\) such that \(\overline{\rho}\) is unramified at \(q\) and \(\overline{\rho}(\text{Frob}_q)\) has distinct \(k\)-rational eigenvalues. Then

\[
\dim_k H^1_\Sigma(\mathbb{Q}, \text{ad}^0 \overline{\rho}) = \#\Sigma + \dim_k H^1_{\Sigma}(\mathbb{Q}, \text{ad}^0 \overline{\rho}(1)) = \dim_k H^1_{\Sigma}(\mathbb{Q}, \text{ad}^0 \overline{\rho}(1)).
\]

**Proof.** The Selmer group of \(\text{ad}^0 \overline{\rho}\) with respect to \(\Sigma\) may also be thought of as the kernel of the map from \(H^1(G_{\mathbb{Q}}, \text{ad}^0 \overline{\rho})\) to \(\bigoplus_\nu H^1(G_{\nu}, \text{ad}^0 \overline{\rho})/L_\nu\) where \(L_\nu = H^1_\nu(G_{\nu}, \text{ad}^0 \overline{\rho})\) when \(\nu \not\in \Sigma\) and \(L_q = H^1(G_q, \text{ad}^0 \overline{\rho})\) otherwise. In order to prove the first statement, it suffices to exhibit an isomorphism

\[
\text{Hom}_k(m_{\Sigma}/(\lambda, m_{\Sigma}/(\lambda, m_{\Sigma}/2), k) \simeq H^1_{\Sigma}(\mathbb{Q}, \text{ad}^0 \overline{\rho})
\]

with \(m_{\Sigma}\) the maximal ideal of \(R_{\Sigma}\). A \(k\)-linear homomorphism \(m_{\Sigma}/(\lambda, m_{\Sigma}/2) \to k\) defines an \(\mathcal{O}\)-algebra homomorphism \(R_{\Sigma} \to k[\xi]\) (and vice-versa) so we find a deformation \(G_{\mathbb{Q}} \to GL_2(R_{\Sigma}) \to GL_2(k[\xi])\) which gives a class in \(H^1(G_{\mathbb{Q}}, \text{ad}^0 \overline{\rho})\). It is clear from the construction that \(\text{res}_\nu(\xi) \in L_\nu\) for all places \(\nu\) so we have the desired map.
We now prove the second statement. By (12) we have the identity
\[
\frac{\# H^1_{\Sigma}(\mathbb{Q}, \text{ad}^0q)}{\# H^1_{\Sigma}(\mathbb{Q}, \text{ad}^0(1))} = \frac{\# H^0(G_\mathbb{Q}, \text{ad}^0q)}{\# H^0(G_\mathbb{Q}, \text{ad}^0(1))} \prod_\nu \frac{\# L_\nu}{\# H^0(G_\nu, \text{ad}^0q)}.
\]

First consider \(\nu \notin \Sigma\). When \(\nu = p \neq \ell\) is a finite prime we have \(\# L_\nu = \# H^1(G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}, \text{ad}^0q)(p)\). When \(\nu = \infty\) is the infinite prime \(\# L_\infty = 1\) since \(G_\infty = I_\infty\). Consider \(H^0(G_{\text{ad}}, \text{ad}^0q)\) i.e. the elements fixed by complex conjugation. By assumption \(q\) is odd so \(c \cdot m_j = (-1)^{j-1} m_j\) with notation as in (11); hence \(\# H^0(G_\infty, \text{ad}^0q) = \# k\).

Now consider \(q \in \Sigma\). By the Local Duality theorems,
\[
\frac{\# H^1(G_q, \text{ad}^0q)}{\# H^0(G_q, \text{ad}^0q)} = \# H^2(G_q, \text{ad}^0q) = \# H^0(G_q, \text{ad}^0(1)) = \prod_{\nu \in \Sigma} \frac{\# H^0(G_\nu, \text{ad}^0q)}{\# H^0(G_\nu, \text{ad}^0(1))}.
\]

As \(q \equiv 1 (\ell)\) the decomposition group \(G_q\) acts trivially on the \(\ell\)-power roots of unity: \(H^0(G_q, \text{ad}^0(1)) = H^0(G_q, \text{ad}^0q(1))\). Explicitly denote
\[
\overline{\rho}(\text{Frob}_q) = \left(\frac{\alpha_q}{\beta_q}\right) \Rightarrow \text{Frob}_q \cdot m_j = \left(\frac{\alpha_q}{\beta_q}\right)^{j-1} m_j, \quad j = 0, 1, 2.
\]

As \(\alpha_q \neq \beta_q\) we must have \(\# H^0(G_q, \text{ad}^0q) = \# k\).

Globally \(H^0(G_\mathbb{Q}, \text{ad}^0q)\) and \(H^0(G_\mathbb{Q}, \text{ad}^0(1))\) are trivial since \(q\) is absolutely irreducible which is not true when \(\ell = 3\) and \(\overline{\rho}\) is not absolutely irreducible over \(\mathbb{Q}(\sqrt{-3})\) – and Proposition 2.3.3 states \(\# H^1(G_\ell, \text{ad}^0q) = \# k \cdot \# H^0(G_\ell, \text{ad}^0q)\), so putting all of this together we have the desired formula \(\# H^1_{\Sigma}(\mathbb{Q}, \text{ad}^0q) = \prod_{q \in \Sigma} \# k \cdot \# H^1_{\Sigma}(\mathbb{Q}, \text{ad}^0q(1))\).

3. Modular Deformation Ring

In this section we introduce a different type of deformation ring using modular forms. Our goal is to show under certain hypotheses that this modular deformation ring is the same as the universal deformation ring introduced in the previous section. For an odd prime \(\ell\), we continue to fix an \(\ell\)-adic representation \(\rho : G_\mathbb{Q} \to GL_2(\mathcal{O})\) that is continuous, ordinary, and ramified at finitely many primes.

3.1. \(\ell\)-adic Modular Forms. Fix a positive integer \(\kappa\), a positive integer \(N = N_0 \ell\) in terms of an integer \(N_0\) to \(\ell\), and a Dirichlet character \(\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times\). A holomorphic function \(f : \mathcal{H} \to \mathbb{C}\) on the upper-half plane is called a classical modular form of weight \(\kappa\), level \(N\), and nebentype \(\chi\) if
\[
f \left( \frac{a \tau + b}{c \tau + d} \right) = \chi(d) (c \tau + d)^{\kappa} f(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \quad c \equiv 0 (N).
\]

Such modular forms have a \(q\)-expansion \(f(\tau) = \sum_n a_n q^n\) in terms of \(q = e^{2 \pi \sqrt{-1} \tau}\) and \(a_n \in \mathbb{Q} \to \mathbb{Q}_l\) for some embedding. We define the space of classical cusp forms \(S_\kappa(N, \chi)\) as those modular forms with coefficients in \(\mathcal{O}\) such that \(a_n = 0\) when \(n \leq 0\). (Equivalently, we assume \(\mathcal{O}\) is always large enough to contain the coefficients of the modular forms of a fixed level \(N\).) For more properties, see [3, Chap. 1].

Denote \(\Lambda = \mathcal{O}[[X]]\) as the power series ring in the variable \(X\). For each positive rational integer \(\kappa\), we have a specialization map \(\varphi_\kappa : \Lambda \to \mathcal{O}\) defined by \(1 + X \mapsto \)
$$(1 + \ell)^\kappa.$$ (We can actually define this map for \(\ell\)-adic integers \(\kappa \in \mathbb{Z}_\ell\), but we will not need this for what follows.) Denote the Teichmueller character \(\omega_\ell : \mathbb{Z}_\ell^* \to \mathbb{Z}_\ell^\times\) defined by \(\omega_\ell(d) = \lim_{n \to \infty} d^\ell^n\); the image is the \((\ell - 1)\)-th roots of unity since \((\mathbb{Z}_\ell/\ell^{n+1}\mathbb{Z}_\ell)^\times\) has order \(\ell^n(\ell - 1)\). Consider a collection of forms \(\sum a_n^{(\kappa)} q^n \in S_\kappa(N, \chi_\omega_\ell^{1-\kappa})\) for each \(\kappa\). We call such a collection a Hida family if for each \(n\) there exist power series \(a_n(X) \in \Lambda\) such that \(a_n^{(\kappa)} = a_n((1 + \ell)^\kappa - 1)\) for all but finitely many \(\kappa\). The collection \(S(N, \chi)\) of formal series \(F(X; \tau) = \sum a_n(X) q^n\) are called \(\Lambda\)-adic cusp forms of level \(N\) and nebentype \(\chi\) if \(F((1 + \ell)^\kappa - 1; \tau) \in S_\kappa(N, \chi_\omega_\ell^{1-\kappa})\) for all but finitely many \(\kappa\). For all \(\kappa\), the specializations \(\varphi_\kappa \circ F\) are called \(\ell\)-adic modular forms – even when they are not classical. For more properties see [23 Chap. 7], although we have altered the notation regarding the nebentype.

3.2. Hecke Algebras. Given \(f \in S_\kappa(N, \chi_\omega_\ell^{1-\kappa})\) define the endomorphisms

\[
(f|T_p)(\tau) = \frac{1}{p} \sum_{i=0}^{p-1} f \left( \frac{\tau + i}{p} \right) + \chi(p) \langle p \rangle^{\kappa-1} f(p \tau) \quad \text{for } p \nmid N \quad \text{and}
\]

\[
(f|U_p)(\tau) = \frac{1}{p} \sum_{i=0}^{p-1} f \left( \frac{\tau + i}{p} \right) \quad \text{for } p | N,
\]

in terms of the character \(\langle \cdot \rangle : \mathbb{Z}_\ell^\times \to 1 + \ell \mathbb{Z}_\ell\) mapping \(d \mapsto d/\omega_\ell(d)\); and define the \(\ell\)-adic Hecke algebra \(h_\kappa(N, \chi_\omega_\ell^{1-\kappa})\) as the \(\mathbb{O}\)-algebra generated by these operators. Following [23 pg. 209], we extend this action to \(\Lambda\)-adic modular forms \(F(X; \tau) = \sum a_n(X) q^n\) by considering the \(q\)-series expansions:

\[
(F|T_p)(X; \tau) = \sum_{p^n | n} a_n(X) q^{n/p} + \chi(p) \sigma_\ell(p) \sum_n a_n(X) q^{pn} \quad \text{for } p \nmid N \quad \text{and}
\]

\[
(F|U_p)(X; \tau) = \sum_{p^n | n} a_n(X) q^{n/p} \quad \text{for } p | N,
\]

in terms of the character \(\sigma_\ell : \mathbb{Z}_\ell^\times \to \Lambda^\times\) mapping \(d \mapsto ((1 + X)/(1 + \ell))^{s(d)}\) where \(s(d) = \log(\langle d \rangle/\log(1 + \ell)) \in \mathbb{Z}_\ell\); and define the \(\Lambda\)-adic Hecke algebra \(h(N, \chi)\) as the \(\Lambda\)-algebra generated by these operators. One checks that the composition \(\varphi_\kappa \circ \sigma_\ell\)

\(p \mapsto \langle p \rangle^{\kappa-1}\) for \(p \nmid N\), so \(\ker \varphi_\kappa = (\sigma_\ell(1 + \ell) - (1 + \ell)^{\kappa-1}) \Lambda\) is a prime and for any operator \(T\) and \(\Lambda\)-adic cusp form \(F\) we have \((\varphi_\kappa \circ F)|T = \varphi_\kappa \circ (F|T)\).

The full \(\Lambda\)-adic Hecke algebra is too big for our purposes, so following [23 pg. 202], we consider a subalgebra. Set \(e = \lim_{n \to \infty} U_{\ell^n} \in h(N, \chi)\) as an idempotent and define the projection \(h^0(N, \chi) = e \cdot h(N, \chi)\) as the ordinary part of the Hecke algebra. Say \(F\) is an eigenform with \(F|U_\ell = a_\ell(X) F\); either \(a_\ell(X) \in \Lambda^\times\) is a unit so that \(F|e = F\) or \(a_\ell(X)\) is in the maximal ideal so \(F|e = 0\). We define ordinary cusp forms as follows: The pairing \(S(N, \chi) \times h(N, \chi) \to \Lambda\) defined by mapping a modular form \(F\) and an endomorphism \(T\) to the first term in the \(q\)-expansion of \(F|T\) induces an \(\Lambda\)-linear functional \(h(N, \chi) \to \Lambda\) defined by \(\pi_F = \langle F, \cdot \rangle\). In particular if \(F\) is a normalized eigenform for \(h(N, \chi)\) i.e. \(a_1(X) = 1\) then \(\pi_F(T_p) = a_p(X)\) for all \(p \nmid N\). (\(\chi\) must be odd i.e. \(\chi(-1) = -1\) for normalized eigenforms to exist.) We now define the ordinary \(\Lambda\)-adic cusp forms \(S^0(N, \chi)\) as those cusp forms which
make the following diagram commute for all but finitely many $\kappa$:

$$
\begin{array}{ccc}
S^0(N, \chi) & \xrightarrow{\sim} & \text{Hom}_\Lambda(h^0(N, \chi), \Lambda) \\
\downarrow & & \downarrow \\
S(N, \chi) & \xrightarrow{\sim} & \text{Hom}_\Lambda(h(N, \chi), \Lambda) \\
\downarrow_{\varphi_\kappa} & & \downarrow_{\varphi_\kappa} \\
S_k(N, \chi \omega_{\ell}^{1-\kappa}) & \xrightarrow{\sim} & \text{Hom}_\mathcal{O}(h_k(N, \chi \omega_{\ell}^{1-\kappa}), \mathcal{O})
\end{array}
$$

3.3. Modular Galois Representations. We wish to associate Galois representations to modular forms. To this end, for $\chi$ an odd Dirichlet character modulo $N$ we identify the composition

$$
G_Q \longrightarrow \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathcal{O}^\times
$$

as the Galois representation associated to $\chi$, and identify the compositions $\omega_\ell \circ \varepsilon_\ell$ and $\sigma_\ell \circ \varepsilon_\ell$ in terms of the cyclotomic character as the Galois representations associated to $\omega_\ell$ and $\sigma_\ell$, respectively. We continue to denote these Galois representations by $\chi$, $\omega_\ell$, and $\sigma_\ell$. Note that $\omega_\ell$ and $\sigma_\ell$ are unramified at all places $\nu \neq \ell$, $\infty$; and we have the relations $\omega_\ell \equiv \varepsilon_\ell (\ell)$ and $\varphi_\kappa \circ \sigma_\ell = (\varepsilon_\ell / \omega_\ell)^{\kappa-1}$.

Associated to each ordinary normalized classical eigenform $f(\tau) = \sum_n a_n q^n$ of weight $\kappa$, level $N$ and nebentype $\chi \omega_{\ell}^{1-\kappa}$ there is a continuous $\ell$-adic Galois representation $\rho_f : G_Q \to GL_2(\overline{\mathbb{Q}}_\ell)$ with the properties

1. $\rho_f$ is unramified outside of the primes that divide $N$;
2. $tr \rho_f (\text{Frob}_p) = a_p$ for $p \nmid N$; and
3. $det \rho_f = \chi \cdot \sigma_\ell$; and
4. $\rho_f$ is ordinary.

(Recall that $\ell$ divides $N$ by assumption.) This is due to Eichler and Shimura [21] Chap. 7] when $\kappa = 2$; Deligne and Serre [12] when $\kappa = 1$; and Deligne [1] for $\kappa > 2$. For proofs, see [28]. Similarly, associated to each ordinary normalized $\Lambda$-adic eigenform $F(X; \tau) = \sum_n a_n(X) q^n$ of level $N$ and nebentype $\chi$ there is a continuous $\Lambda$-adic Galois representation $\rho_F : G_Q \to GL_2(\overline{\mathbb{Q}}_\ell[[X]])$ with the properties

1. $\rho_F$ is unramified outside of the primes that divide $N$;
2. $tr \rho_F (\text{Frob}_p) = a_p(X)$ for $p \nmid N$;
3. $det \rho_F = \chi \cdot \sigma_\ell$; and
4. $\rho_F$ is ordinary.

This is due to Hida [21] Theorem 2.1]; see also [24] §7.5, Theorem 3]. As $F$ specializes to $\ell$-adic cusp forms $f = \varphi_\kappa \circ F$ the Galois representation $\rho_F$ specializes to $\ell$-adic representations $\rho_f = \varphi_\kappa \circ \rho_F$.

Fix a finite set $\Sigma$ of places that does not contain $\ell$. Let us consider an $\ell$-adic representation $\rho : G_Q \to GL_2(\mathcal{O})$ that is ordinary and ramified at finitely many primes where $\overline{\mathfrak{p}}$ is odd, absolutely irreducible and wildly ramified at $\ell$. Fix $\kappa$ and $\chi$ by the relation $\chi \cdot (\varepsilon_\ell / \omega_\ell)^{\kappa-1} = det \rho$. We say that an ordinary normalized $\Lambda$-adic eigenform $F$ is a $\chi$-cusp form for $\overline{\mathfrak{p}}$ of type $\Sigma$ if $\varphi_\kappa \circ \rho_F$ is a deformation of $\overline{\mathfrak{p}}$ of type $\Sigma$. The level $N_\ell$ of such a $\chi$-cusp form is divisible by $N_0 = N(\overline{\mathfrak{p}}) \ell$ yet divides $N_\Sigma = N_0 \prod_p p^2$ in terms of the product over $p \in \Sigma$ not dividing $N_0$. The assumption that $\overline{\mathfrak{p}}$ is wildly ramified at $\ell$ forces $\ell$ to divide the level $N_0$; this manifest in the concept of companion forms. For more properties, see [20] or [9].
Define a map \( \pi : h^0(N_{\Sigma}, \chi) \to \prod_F \text{type } \Sigma \Lambda \) by

\[
\pi : \quad T_p \mapsto (\ldots, \pi_F(T_p), \ldots) = (\ldots, \text{tr } \rho_F(\text{Frob}_p), \ldots) \quad \text{for } p \nmid N_{\Sigma};
\]

where each component corresponds to a cusp form for \( \overline{\rho} \) of type \( \Sigma \). We define the modular deformation ring \( T_{\Sigma} \) to be the \( \Lambda \)-algebra generated by the images of \( T_p \) for \( p \nmid N_{\Sigma} \). Note there is a continuous representation

\[
\rho_{\Sigma}^{\text{mod}} : G_{\mathbb{Q}} \to GL_2(\mathbb{T}_{\Sigma})
\]

where \( \rho_{\Sigma}^{\text{mod}} \simeq \prod_F \text{type } \Sigma \rho_F \) is a deformation of \( \overline{\rho} \) of type \( \Sigma \).

### 3.4. Modularity Criteria

In order to apply the results of the last section, we list a few properties of the modular deformation ring \( T_{\Sigma} \).

**Proposition 3.4.1.** For \( \ell \) an odd prime, let \( \rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O}) \) be a continuous \( \ell \)-adic representation such that \( \rho \) is ordinary and ramified at finitely many primes, while \( \overline{\rho} \) is modular, absolutely irreducible, and wildly ramified at \( \ell \). Fix a finite set \( \Sigma \) of places that does not contain \( \ell \).

1. There exist \( \Lambda \)-adic cusp forms for \( \overline{\rho} \) of type \( \Sigma \).
2. \( T_{\Sigma} \simeq h^0(N_{\Sigma}, \chi)_{m_{\Sigma}} \) is the localization at some maximal ideal \( m_{\Sigma} \).
3. \( T_{\Sigma} \) is a finitely generated, torsion-free, local \( \Lambda \)-algebra.

**Proof.** We prove (1). With \( \kappa, \chi \), and \( N_0 \) as above, fix an integer \( \kappa(\overline{\rho}) \) satisfying \( 2 \leq \kappa(\overline{\rho}) \leq \ell \) and \( \kappa(\overline{\rho}) \equiv \kappa \mod (\ell - 1) \). By the “level lowering” arguments in \[34\] and \[36\] we can find \( f \in S_{\kappa(\overline{\rho})}(N_0, \chi \omega^{1-\kappa(\overline{\rho})}) \) such that \( \overline{\rho} \simeq \overline{\rho}_f \). Choose a \( \Lambda \)-adic cusp form \( F \in S^0(N_0, \chi) \) such that \( f = \varphi_{\kappa(\overline{\rho})} \circ F \). Then \( F \) is a cusp form of type \( \emptyset \), and hence of type \( \Sigma \).

We show how (3) follows from (2). The localization \( h^0(N_{\Sigma}, \chi)_{m_{\Sigma}} \) is finitely generated since \( h^0(N_{\Sigma}, \chi) \) is finite-dimensional. It is well-known that \( h^0(N_{\Sigma}, \chi) \) is torsion-free; see \[22\] Theorem 3.1 or \[23\] §7.3, Theorem 1. Hence the statements follow for \( T_{\Sigma} \simeq h^0(N_{\Sigma}, \chi)_{m_{\Sigma}} \).

We now prove (2). First we express the \( \Lambda \)-adic Hecke algebra as a “complete intersection.” Denote \( L \) as the field of fractions of \( \Lambda \). Fix a cusp form \( F(X; \tau) = \sum_n a_n(X) q^n \in S^0(N_{\Sigma}, \chi) \), but say that it is a newform of level \( N_F \). It is well-known that there is a decomposition \( S^0(N_{\Sigma}, \chi) \otimes \Lambda L = \bigoplus_p S_F \) over such newforms, where \( S_F \) is the \( L \)-linear span of \( \{ F(X; d\tau) \mid d \text{ divides } N_{\Sigma}/N_F \} \) of dimension

\[
d_F = \# \left\{ d \mid d \text{ divides } \frac{N_{\Sigma}}{N_F} \right\} = \prod_{p|N_{\Sigma}/N_F} (m_p + 1) \quad \text{with} \quad \frac{N_{\Sigma}}{N_F} = \prod_{p|N_{\Sigma}/N_F} p^{m_p}.
\]

Let \( A_F = L[t : p \text{ divides } N_{\Sigma}/N_F] \) be a polynomial ring over \( L \), and consider the map \( A_F \to \text{End}_L S_F \) defined by \( t_p \mapsto T_p \). For each divisor \( d_0 \) of \( N_{\Sigma}/N_F \) prime to \( p \), one checks that the characteristic polynomial \( T_p \) on the \( (m_p + 1) \)-dimensional space \( \{ F(X; d_0 p^m \tau) \mid m \leq m_p \} \) is \( P_p(t) = t^{m_p+1} - a_p(X) t^{m_p} + (\chi \cdot \sigma(p))(p) t^{m_p-1} \), setting \( \chi(p) = 0 \) if \( p \) divides \( N_F \). Hence the kernel contains the ideal \( I_F = \langle P_p(t_p) : p \text{ divides } N_{\Sigma}/N_F \rangle \). But \( \dim_L A_F/I_F = d_F = \dim_L \text{End}_L S_F \) so

\[
\pi : h^0(N_{\Sigma}, \chi) \otimes \Lambda L \xrightarrow{\sim} \prod_F \text{End}_L S_F \xrightarrow{\sim} \prod_F A_F/I_F
\]

where \( \pi = \prod_F \pi_F \) has components \( \pi_F \) (corresponding to each newform) mapping \( T_p \mapsto t_p \mod I_F \) if \( p \) divides \( N_{\Sigma}/N_F \) and \( T_p \mapsto a_p(X) \) otherwise.
Next we show $T_{\Sigma} \otimes_{\Lambda} \mathcal{L}$ is a localization of the $\Lambda$-adic Hecke algebra. For each maximal ideal $m$ in $h^0(N_{\Sigma}, \chi)$ there is a natural isomorphism of localizations
\[
(5) \quad h^0(N, \chi)_m \otimes_{\Lambda} \mathcal{L} \simeq \prod_p h^0(N, \chi)_p \prod_{F_{\text{type } \Sigma}} (A_F/I_F)_{J_F/I_F}
\]
where the product is over newforms $F$ and primes $J_F/I_F \subseteq A_F/I_F$ corresponding to primes $p = \pi_F^{-1}(J_F/I_F) \subseteq h^0(N_{\Sigma}, \chi) \otimes_{\Lambda} \mathcal{L}$ such that $p \cap h^0(N_{\Sigma}, \chi) \subseteq m$. Both $\rho$ and $\rho_F$ are unramified at $p$ for $p \nmid N_{\Sigma}$, so Tate–Shafarevich density implies $F$ is a cusp form for $\mathcal{P}$ of type $\Sigma$ if and only if the composition
\[
\pi_F : T_p \mapsto \mathcal{P}(\text{Frob}_p) \quad \text{for all } p \nmid N_{\Sigma}. \quad \text{(Recall that } \kappa \text{ is the weight associated to } \rho.\text{)}
\]
Choose $m_{\Sigma}$ as the maximal ideal of $h^0(N_{\Sigma}, \chi)$ containing $\pi_F$ for such cusp forms. Then the product in $(5)$ is actually over newforms $F$ of type $\Sigma$, where $J_F = (t_p : p \text{ divides } N_{\Sigma}/N_F) \supseteq I_F$. But then $A_F/J_F \simeq \mathcal{L}$ implies $(A_F/I_F)_{J_F/I_F} \simeq \mathcal{L}$ so we have the isomorphism
\[
\rho : G_Q \xrightarrow{\rho_{\text{mod}}} GL_2(T_{\Sigma}) \xrightarrow{\phi_{\kappa}} GL_2(\mathcal{L})
\]
which shows $\pi_F(T_p) = a_p(X) = \text{tr} (\rho_{\text{mod}})^{t_p} (\text{Frob}_p) = \text{tr} (\rho_F^\text{mod})^{t_p} (\text{Frob}_p)$ is an element of $T_{\Sigma} \cap \prod_{F_{\text{type } \Sigma}} (A_F/I_F)_{J_F/I_F}$. Hence $h^0(N, \chi)_{m_{\Sigma}} \simeq T_{\Sigma}$.

Finally we show $h^0(N, \chi)_{m_{\Sigma}} \to T_{\Sigma}$. For $p \nmid N_{\Sigma}$, each $T_p \in h^0(N, \chi)_{m_{\Sigma}}$ maps to $\pi(T_p) \in T_{\Sigma} \cap \prod_{F_{\text{type } \Sigma}} (A_F/I_F)_{J_F/I_F}$, so it suffices to show the same for $p$ dividing $N_{\Sigma}$. Fix a cusp form $F(X_0, X_1) = \sum a_n(X) q^n$ of type $\Sigma$. If $p$ divides $N_{\Sigma}/N_F$ then $T_p \mapsto 0$ so consider $p$ that divides $N_F$ but not $N_{\Sigma}/N_F$. The projection $T_{\Sigma} \to \Lambda$ gives the composition
\[
\rho_F : G_Q \xrightarrow{\rho_{\text{mod}}} GL_2(T_{\Sigma}) \xrightarrow{\phi_{\kappa}} GL_2(\mathcal{L})
\]
which shows $\pi_F(T_p) = a_p(X) = \text{tr} (\rho_F)^{t_p} (\text{Frob}_p) = \text{tr} (\rho_F^\text{mod})^{t_p} (\text{Frob}_p)$ is an element of $T_{\Sigma} \cap \prod_{F_{\text{type } \Sigma}} (A_F/I_F)_{J_F/I_F}$. Hence $h^0(N, \chi)_{m_{\Sigma}} \simeq T_{\Sigma}$. \qed

The modular deformation ring $T_{\Sigma}$ is a complete, Noetherian, local $\Lambda$-algebra so there is a unique $\Lambda$-algebra surjection $\phi_{\Sigma} : R_{\Sigma} \to T_{\Sigma}$ of the universal deformation ring such that $\rho_{\text{mod}} \simeq \phi_{\Sigma} \circ \rho_{\text{univ}}$. The following result states that this map is an isomorphism if and only if it is an isomorphism upon specializing the weight.

**Proposition 3.4.2.** For $\ell$ an odd prime, let $\rho : G_Q \to GL_2(\mathcal{O})$ be a continuous $\ell$-adic representation such that $\rho$ is ordinary and ramified at finitely many primes, while $\mathcal{P}$ is absolutely irreducible, modular, and wildly ramified at $\ell$. For each positive integer $\kappa$, denote $T_{\Sigma}^{(\kappa)}$ and $R_{\Sigma}^{(\kappa)}$ as the images of $T_{\Sigma}$ and $R_{\Sigma}$ under the specialization map $\phi_{\kappa}$, respectively. Then the following are equivalent:

1. $\phi_{\Sigma} : R_{\Sigma} \to T_{\Sigma}$ is an isomorphism.
2. $\phi_{\Sigma}^{(\kappa)} : R_{\Sigma}^{(\kappa)} \to T_{\Sigma}^{(\kappa)}$ is an isomorphism for all positive integers $\kappa$.
3. $\phi_{\Sigma}^{(\kappa)} : R_{\Sigma}^{(\kappa)} \to T_{\Sigma}^{(\kappa)}$ is an isomorphism for some positive integer $\kappa$.

If any of the above hold for every finite set $\Sigma$ not containing $\ell$, then $\rho$ is $\ell$-adically modular i.e. $\rho \simeq \rho_f$ for some $\ell$-adic cusp form $f$.

**Proof.** We prove the last statement of the proposition assuming $\phi_{\Sigma} : R_{\Sigma} \to T_{\Sigma}$ is an isomorphism for every finite set $\Sigma$. Choose $\Sigma$ large enough so that $\rho$ is a deformation of $\mathcal{P}$ of type $\Sigma$, define $\kappa$ and $\chi$ by the relation $\det \rho = \chi \cdot (\omega / \omega_t)^{\kappa - 1}$. 

There is a surjection $\phi : R_\Sigma \to \Lambda$, so there is a map $\pi_\Sigma : T_\Sigma \to \Lambda$ such that $\phi = \pi_\Sigma \circ \phi_\Sigma$. However, the maps $T_\Sigma \to \Lambda$ all correspond to the canonical pairing $S(N_\emptyset, \chi) \times h(N_\emptyset, \chi) \to \Lambda$ so $\pi_\Sigma \simeq \pi_F$ for some cusp form $F$ of type $\Sigma$. Then $f = \varphi_\kappa \circ F$ is the desired $\ell$-adic cusp form because

$$\rho \simeq (\varphi_\kappa \circ \phi) \circ \rho_\Sigma^{\text{univ}} \simeq \varphi_\kappa \circ \pi_F \circ \rho_\Sigma^{\text{mod}} \simeq \varphi_\kappa \circ \rho_F \simeq \rho_f.$$

Now we prove the equivalence of the statements; we are motivated by the proof of [5, Proposition 3.4]. For each positive integer $\kappa$ we have the commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \ker \varphi_\kappa \cdot R_\Sigma & \longrightarrow & R_\Sigma & \longrightarrow & R_\Sigma^{(\kappa)} & \longrightarrow & 0 \\
& & \downarrow \phi_\Sigma & & \downarrow \phi_\Sigma & & \downarrow \phi_\Sigma^{(\kappa)} & & \\
0 & \longrightarrow & \ker \varphi_\kappa \cdot T_\Sigma & \longrightarrow & T_\Sigma & \longrightarrow & T_\Sigma^{(\kappa)} & \longrightarrow & 0
\end{array}
$$

where $\ker \varphi_\kappa = (\sigma_\ell(1 + \ell) - (1 + \ell)^{\kappa-1}) \Lambda$ is a prime ideal. Clearly from the diagram (1) implies (2), and (2) obviously implies (3). Assume (3) holds i.e. $\phi_\Sigma^{(\kappa)}$ is an isomorphism for some positive integer $\kappa$. Then $R_\Sigma / \ker \varphi_\kappa \cdot R_\Sigma \simeq T_\Sigma / \ker \varphi_\kappa \cdot T_\Sigma$ as integral domains, but by Proposition 3.4.1 both $R_\Sigma$ and $T_\Sigma$ are torsion-free so $R_\Sigma \simeq T_\Sigma$. \hfill \square

3.5. Specialization to Weight 2. We now show that under suitable hypotheses the modular deformation ring is the same as the universal deformation ring. This is the main result of the paper.

**Theorem 3.5.1.** For $\ell$ an odd prime, let $\rho : G_\mathbb{Q} \to GL_2(\mathcal{O})$ be a continuous $\ell$-adic representation such that

1. $\rho$ is ordinary and ramified at finitely many primes;
2. $\overline{\rho}$ is absolutely irreducible when restricted to Gal\(\overline{\mathbb{Q}}/\mathbb{Q}\((\sqrt{-1})^{(\ell-1)/2} \ell)\), modular, and wildly ramified at $\ell$.

Then $\rho$ is $\ell$-adically modular i.e. $\rho \simeq \rho_f$ for an $\ell$-adic cusp form $f$.

**Proof.** By Proposition 3.4.2 it suffices to show $\phi_\Sigma^{(2)} : R_\Sigma^{(2)} \to T_\Sigma^{(2)}$ (i.e. the weight $\kappa = 2$ case) is an isomorphism for all finite sets $\Sigma$ not containing $\ell$. We identify $T_\Sigma^{(2)}$ as the modular deformation ring associated to ordinary cusp forms $f(\tau) = \sum_{n} a_n q^n$ of weight 2, level $N_\Sigma$, and nebentype $\chi \cdot \omega_\ell^{-1}$ such that $\overline{\rho} \simeq \overline{\rho}_f$ and identify $R_\Sigma^{(2)}$ is its universal deformation ring. Recall that $N_\Sigma = N_0 \prod_p p^2$ in terms of the product over $p \in \Sigma$ not dividing $N_0 = N(\overline{\rho}) \ell$. The following commutative diagram is exact for the unique surjection $\phi$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \ker \phi & \longrightarrow & R_\Sigma^{(2)} & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\
& & \downarrow \phi_\Sigma^{(2)} & & \downarrow \phi_\Sigma^{(2)} & & \\
0 & \longrightarrow & \ker \pi_f & \longrightarrow & T_\Sigma^{(2)} & \longrightarrow & \mathcal{O} & \longrightarrow & 0
\end{array}
$$

We fix some notation. Choose a cusp form $f(\tau)$ of level $N_0$. We will denote the “tangent space” of $R_\Sigma^{(2)}$ as $\Phi_\Sigma = (\ker \phi) / (\ker \phi)^2$, as well as the ideals $p_\Sigma = \ker \pi_f$ and $I_\Sigma = \text{Ann}_{T_\Sigma^{(2)}} \ker \pi_f$ in $T_\Sigma^{(2)}$. We assume for the moment that there exists a family of $T_\Sigma^{(2)}$-modules $L_\Sigma$ satisfying the following properties:

HM1: $L_\Sigma$ is free over $\mathcal{O}$ where $\text{rank}_\mathcal{O} L_\Sigma = 2 \cdot \text{rank}_\mathcal{O} T_\Sigma^{(2)}$ and $\text{rank}_\mathcal{O} L_\Sigma[p_\Sigma] = 2$. 
HM2: There is a pairing $\langle \cdot, \cdot \rangle_{\Sigma} : L_\Sigma \times L_\Sigma \to \mathcal{O}$ such that $L_\Sigma \simeq \text{Hom}_\mathcal{O}(L_\Sigma, \mathcal{O})$.

HM3: For $\Sigma' \subseteq \Sigma$, there exist surjective $\mathbb{T}^{(2)}_{\Sigma'}$-homomorphisms $\beta_{\Sigma'} : L_\Sigma \to L_{\Sigma'}$.

The proof of this theorem will be divided into three steps: first we show how the minimal case follows from the existence of these Hecke modules, second we show how the general case follows from the minimal case, and third we construct these Hecke modules using modular curves. The arguments below follow the discussion in [15] and [10], where we use an axiomatic approach following [16].

The Minimal Case. We show $\phi^{(2)}_0 : R^{(2)}_0 \to \mathbb{T}^{(2)}_0$ is an isomorphism. Denote $R^{(2)}_0 = R^{(2)}_0 / \lambda R^{(2)}_0$ and $T^{(2)}_0 = T^{(2)}_0 / \lambda T^{(2)}_0$ as $k$-vector spaces. It is well-known that $R^{(2)}_0 \to T^{(2)}_0$ is an isomorphism of complete intersections if and only if $\mathbb{T}^{(2)}_0 \to \mathbb{T}^{(2)}_0$ is an isomorphism of complete intersections; see [11] Lemma 5.29. We show the criteria of [15] Theorem 2.1 are satisfied.

We construct a sequence of surjective maps. Denote $r = \dim_k H^1_{\emptyset} (\mathbb{Q}, \text{ad}^0 \rho(1))$. For each positive integer $n$, let $Q$ be a set of $r$ primes $q$ satisfying

1. $q \equiv 1 \mod \ell^n$;
2. $\rho$ is unramified at $q$ and $\rho(\text{Frob}_q)$ has distinct $k$-rational eigenvalues; and
3. $H^3_{\emptyset} (\mathbb{Q}, \text{ad}^0 \rho)$ may be embedded in $\bigoplus_{q \in Q} H^1 (G_q, \text{ad}^0 \rho)$.

Such a collection $Q$ exists since we assume $\rho$ remains absolutely irreducible over $\mathbb{Q}(\sqrt{-1})^{(1/4)}$; see [11] Theorem 2.49 and its proof. By Proposition 2.5.1 – note $\rho$ is odd because it is residually modular – we have $\dim_k H^1_{\emptyset} (\mathbb{Q}, \text{ad}^0 \rho(1)) = r + \dim_k H^1_{\emptyset} (\mathbb{Q}, \text{ad}^0 \rho(1))$.

$$H^1_{\emptyset} (\mathbb{Q}, \text{ad}^0 \rho(1)) = \ker \left( H^1_{\emptyset} (\mathbb{Q}, \text{ad}^0 \rho) \to \bigoplus_{q \in Q} H^1 (G_q, \text{ad}^0 \rho) \right)$$

is trivial so $\dim_k H^1_{\emptyset} (\mathbb{Q}, \text{ad}^0 \rho) = r$. Using Proposition 2.5.1, again we see $R^{(2)}_0$ may be generated by $r$ elements as an $\mathcal{O}$-algebra. The universal property implies that there is a surjection $R^{(2)}_0 \to R^{(2)}_0$, so $T^{(2)}_0$ has dimension at most $r$ as a $k$-vector space. Denote $\Delta_Q$ as the maximal quotient of $\text{Gal} \left( \prod_{q \in Q} \mathbb{Q}(\zeta_q)/\mathbb{Q} \right) \simeq \prod_{q \in Q} (\mathbb{Z}/q \mathbb{Z})^\times$ of $\ell$-power order. Given $r$ variables $X_i$ we have a surjection of the power series ring $\mathcal{O}[[X_1, \ldots, X_r]] \to \mathbb{T}^{(2)}_0$. Introduce a second power series ring $\mathcal{O}[[S_1, \ldots, S_r]]$ with prime ideal $\mathfrak{a} = (S_1, \ldots, S_r)$, and a map onto $\mathcal{O}[\Delta_Q]$ that sends $1 + S_i$ to one of its $r$ generators; the kernel of this map is contained in $\mathfrak{a}^n$.

The following diagram commutes:

$$\begin{array}{ccc}
\mathcal{O}[[S_1, \ldots, S_r]] & \longrightarrow & \mathcal{O}[\Delta_Q] \\
\downarrow & & \downarrow \\
\mathcal{O}[[X_1, \ldots, X_r]] & \longrightarrow & \mathbb{T}^{(2)}_0 \xrightarrow{\phi^{(2)}_0} \mathbb{T}^{(2)}_0
\end{array}$$

where the horizontal maps are surjections, and the vertical maps are chosen so that the image of $\mathfrak{a}$ in $R^{(2)}_0$ is trivial.

Now impose the following additional condition on the Hecke modules:

HM5: $L_Q$ is free over $\mathcal{O}[\Delta_Q]$ and $L_Q/\mathfrak{a} L_Q \simeq L_0$. 
(The numbering of this property will become clear later.) Denote $L'_Q = L_Q / \alpha^n L_Q$ so that $L'_Q / \alpha L'_Q \simeq L_0$. Property (HM5) implies $\operatorname{Ann}_O[[S_1, \ldots, S_r]] L_Q \subseteq \alpha^n$, so we actually have $\operatorname{Ann}_O[[S_1, \ldots, S_r]] L'_Q = \alpha^n$. We have the surjections $L_Q \to L'_Q$ and $O[\Delta Q] \to O[[S_1, \ldots, S_r]] / \alpha^n$, so $L'_Q$ is free over $O[[S_1, \ldots, S_r]] / \alpha^n$. Then by [15, Theorem 2.1] where we use property (HM1) and tensor with $k$, both $\overline{R}_\emptyset^{(2)}$ and $\overline{T}_\emptyset^{(2)}$ are complete intersections while $L_0 \otimes_O k$ is free over $\overline{R}_\emptyset^{(2)}$. However $\ker [\overline{R}_\emptyset^{(2)} \to \overline{T}_\emptyset^{(2)}] \subseteq \operatorname{Ann}_{\overline{R}_\emptyset^{(2)}} L_0 \otimes_O k = \{0\}$ so $\overline{R}_\emptyset^{(2)} \simeq \overline{T}_\emptyset^{(2)}$.

**Reduction to the Minimal Case.** We now show $\phi_\Sigma^{(2)} : R_\Sigma^{(2)} \to T_\Sigma^{(2)}$ is an isomorphism by verifying the criteria of [15, Theorem 2.4]. Denote $\operatorname{ad}^0 \rho_{f,n} = \operatorname{ad}^0 \rho_f \otimes_O \lambda^{-n} O/O$ so that as a generalization to (3),

$$\# \Phi_\Sigma = \lim_{n \to \infty} \# \operatorname{Hom}_O (\Phi_\Sigma, \lambda^{-n} O/O) = \lim_{n \to \infty} \# H^1_{\Sigma} (\mathbb{Q}, \operatorname{ad}^0 \rho_{f,n}).$$

Hence the change in size from $\Phi_\emptyset$ to $\Phi_\Sigma$ satisfies

$$\frac{\# \Phi_\Sigma}{\# \Phi_\emptyset} \leq \lim_{n \to \infty} \prod_{p \in \Sigma} \frac{\# H^1_{\Sigma} (G_p, \operatorname{ad}^0 \rho_{f,n})}{\# H^0_{\Sigma} (G_p, \operatorname{ad}^0 \rho_{f,n})} = \lim_{n \to \infty} \prod_{p \in \Sigma} \frac{\# H^0_{\Sigma} (G_p, \operatorname{ad}^0 \rho_{f,n}(1))}{\# H^1_{\Sigma} (G_p, \operatorname{ad}^0 \rho_{f,n}(1))},$$

because $\# H^1_{\Sigma} (G_p, \operatorname{ad}^0 \rho_{f,n}) = \# H^0_{\Sigma} (G_p, \operatorname{ad}^0 \rho_{f,n})$ when $p \neq \ell$, and we only consider finite sets $\Sigma$ not containing $\ell$ since $\mathfrak{p}$ is ramified at $\ell$.

We recall the value of $c_p = \lim_{n \to \infty} \# H^0_{\Sigma} (G_p, \operatorname{ad}^0 \rho_{f,n}(1))$. Denote $K \subseteq \overline{\mathbb{Q}}_p$ as the field of fractions of $O$, and fix a normalized valuation $v_\lambda : K^\times \to \mathbb{Z}$. Note that for $c_p \neq 0$ we have $v_\lambda (c_p) = v_\lambda (c_p')$ if and only if $c_p O = c_p' O$, so it suffices to compute $v_\lambda (c_p)$; we do so via determinants. If $\overline{\mathfrak{p}}$ is unramified at $p$ we have

$$\det [1_3 - \operatorname{ad}^0 \rho_f (\text{Frob}_p) \cdot X] = (1 - X) \left(1 + X \right)^2 - \frac{a_p^2}{(\chi \cdot \omega_\ell^{-1})(\text{Frob}_p)} \frac{X}{p},$$

so for any prime $p \notin \Sigma$ such that $p \neq \ell$ define $t_p \in T_\Sigma^{(2)}$ as the element

$$t_p = \begin{cases} 
(1 - p) \left((1 + p)^2 - T_p^2 / (\chi \cdot \omega_\ell^{-1})(\text{Frob}_p)\right) & \text{if dim}_k \overline{\mathfrak{p}}^{(p)} = 2; \\
1 - p^2 & \text{if dim}_k \overline{\mathfrak{p}}^{(p)} = 1, \det \overline{\mathfrak{p}}^{(p)} = 1; \\
1 - p & \text{if dim}_k \overline{\mathfrak{p}}^{(p)} = 1, \det \overline{\mathfrak{p}}^{(p)} \neq 1; \\
1 + p & \text{if dim}_k \overline{\mathfrak{p}}^{(p)} = 0, p \in P; \\
1 & \text{if dim}_k \overline{\mathfrak{p}}^{(p)} = 0, p \notin P;
\end{cases}$$

where we denote $P$ as the collection of primes $p \equiv -1 \pmod{\ell}$ such that $\overline{\mathfrak{p}}|_{G_p}$ is irreducible yet $\overline{\mathfrak{p}}|_{G_P}$ is reducible. It is straightforward to verify

$$v_\lambda (c_p) = v_\lambda \left(\det [1_3 - \operatorname{ad}^0 \rho_f (1) \cdot \text{Frob}_p] \right) = v_\lambda (t_p \mod \mathfrak{p}_\Sigma).$$

Now impose the following additional condition on the Hecke modules:

**HM4:** $\left(\hat{\beta}_\Sigma, \cdot \hat{\beta}_\Sigma \right) : L_{\Sigma} = \prod_{p \in \Sigma - \Sigma'} t_p \cdot L_{\Sigma'}$, where $\hat{\beta}_\Sigma : L_{\Sigma'} \to L_{\Sigma}$ is the adjoint.

To be precise, the adjoint is with respect to the pairings: $\langle x, \cdot \hat{\beta}_\Sigma x' \rangle_{\Sigma} = \langle \hat{\beta}_\Sigma x, x' \rangle_{\Sigma'}$ for all $x \in L_{\Sigma}$ and $x' \in L_{\Sigma'}$. We’ve seen that property (HM5) implies $R_\emptyset^{(2)}$ is a complete intersection and $L_{\emptyset}$ is a free $R_\emptyset^{(2)}$-module, while property (HM1) implies $\mathfrak{p}_\Sigma$ and $\mathfrak{p}_\emptyset$ are in the support of $L_{\Sigma}$ and $L_{\emptyset}$, respectively. By property (HM2) we
have the isomorphism $L_\Sigma/L_\Sigma[I_\Sigma] \simeq \text{Hom}_\mathcal{O}(L_\Sigma[p_\Sigma], \mathcal{O})$, so for any basis $\{x_1, x_2\}$ of $L_\Sigma[p_\Sigma]$ we have

$$\Omega_\Sigma = \frac{L_\Sigma}{L_\Sigma[p_\Sigma] + L_\Sigma[I_\Sigma]} \simeq \text{Hom}_\mathcal{O}(L_\Sigma[p_\Sigma], \mathcal{O}) \simeq \frac{\mathcal{O}}{\det \langle x_i, x_j \rangle \mathcal{O}}.$$

(Compare with the discussion in [11 4.4].) By properties (HM3) and (HM4), the adjoint $\hat{\beta}_\Sigma$ has torsion-free cokernel, so we can choose the basis $\{x'_1, x'_2\}$ of $L_\Sigma[p_\Sigma]$ such that $\{x_1, x_2\} = \hat{\beta}_\Sigma\{x'_1, x'_2\}$ is a basis for $L_\Sigma[p_\Sigma]$. This gives

$$\langle x_i, x_j \rangle \mathcal{O} = \left( \langle \beta_\Sigma \circ \hat{\beta}_\Sigma \rangle x'_i, x'_j \right) \mathcal{O} = \prod_{p \in \Sigma - \Sigma'} c_p \cdot \langle x'_i, x'_j \rangle \mathcal{O},$$

which implies, for $\Sigma' = \emptyset$, the inequality

$$\#\Omega_\Sigma = \prod_{p \in \Sigma} \left[ \det \langle x'_i, x'_j \rangle \mathcal{O} : \det \langle x_i, x_j \rangle \mathcal{O} \right] = \prod_{p \in \Sigma} c_p^2 \geq \left(\frac{\#\Phi_\Sigma}{\#\Phi_\emptyset}\right)^2.$$

By [15 Theorem 2.4] we find $\#\Omega_\emptyset = (\#\Phi_\emptyset)^2$ so that we have the inequality $\#\Omega_\Sigma \geq (\#\Phi_\Sigma)^2$. Then the same theorem implies $R^{(2)}_\Sigma$ is a complete intersection and $L_\Sigma$ is a free $R^{(2)}_\Sigma$-module. Hence $\ker \left[ R^{(2)}_\Sigma \to T^{(2)}_\Sigma \right] \subseteq \text{Ann}_{R^{(2)}_\Sigma} L_\Sigma = \{0\}$ so $R^{(2)}_\Sigma \simeq T^{(2)}_\Sigma$.

**Construction of Hecke Modules.** To complete the proof, we construct a family of $T^{(2)}_\Sigma$-modules $L_\Sigma$, for finite sets $\Sigma$ not including $\ell$, satisfying the following properties:

HM1: $L_\Sigma$ is free over $\mathcal{O}$ where $\text{rank}_\mathcal{O} L_\Sigma = 2 \cdot \text{rank}_\mathcal{O} T^{(2)}_\Sigma$ and $\text{rank}_\mathcal{O} L_\Sigma[p_\Sigma] = 2$.

HM2: There is a pairing $\langle \cdot, \cdot \rangle : L_\Sigma \times L_\Sigma \to \mathcal{O}$ such that $L_\Sigma \simeq \text{Hom}_\mathcal{O}(L_\Sigma, \mathcal{O})$.

HM3: For $\Sigma' \subseteq \Sigma$, there exist surjective $T^{(2)}_\Sigma$-homomorphisms $\beta_\Sigma : L_\Sigma \to L_{\Sigma'}$.

HM4: $\left( \beta_\Sigma \circ \hat{\beta}_\Sigma \right) L_{\Sigma'} = \prod_{p \in \Sigma - \Sigma'} t_p \cdot L_{\Sigma'}$, where $\hat{\beta}_\Sigma : L_\Sigma \to L_\Sigma$ is the adjoint.

HM5: $L_Q$ is free over $\mathcal{O}[\Delta_Q]$ and $L_Q/a L_Q \simeq L_\emptyset$.

We will construct the modules using modular curves.

First we recall a few facts about the modular curve $X(N)$ of level $N = \prod_{p} p^{v_p}$ in order to motivate a more general definition. A function $f : \mathcal{H} \to \mathbb{C}$ is a cusp form of weight 2 and level $N$ if and only if $f(\tau) d\tau$ is a holomorphic differential on the compact Riemann surface $X(N)$. The upper-half plane $\mathcal{H} \simeq SL_2(\mathbb{R})/SO_2(\mathbb{R})$, so using the Strong Approximation Theorem [14 Proposition 3.3.1], we have the double-coset decomposition

$$\Gamma(N) \backslash \mathcal{H} \simeq \Gamma(N) \backslash SL_2(\mathbb{R})/SO_2(\mathbb{R}) \simeq GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q})/U_N \cdot SO_2(\mathbb{R}) \mathbb{R}^\times$$

where $U_N = \prod_{p} \text{ker}[GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{Z}/p^{v_p}\mathbb{Z})]$. We may identify $X(N)$ as the compactification of this decomposition by adjoining the cusps. We may construct a cuspidal automorphic form $\pi : GL_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ of level $U_N$ from a cusp form $f$ of weight 2 and level $N$ by defining

$$\pi(g) = \frac{ad - bc}{(c\tau + d)^2} f \left( \frac{a\tau + b}{c\tau + d} \right)$$

where $\tau = \sqrt{-1}$, $g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(\mathbb{R})^+$, and $g = \gamma g_\infty u$ for $\gamma \in GL_2(\mathbb{Q})$ and $u \in U_N \cdot SO_2(\mathbb{R}) \mathbb{R}^\times$. In general, given an open, compact subgroup $U \subseteq GL_2(\mathbb{A}_\mathbb{Q})^\vee$, we define $X_U$ as the compactification of $GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q})/U \cdot SO_2(\mathbb{R}) \mathbb{R}^\times$. Write $N_\Sigma = \ell \prod_{p \neq \ell} p^{v_p}$, and consider in
particular the subgroup

$$U_\Sigma = \ker \left[ GL_2(\mathbb{Z}_\ell) \to GL_2(\mathbb{F}_\ell) \right] \times \prod_{p \in P} \ker \left[ GL_2(\mathbb{Z}_p) \to GL_2(\mathbb{Z}/p^{r/2}\mathbb{Z}) \right]$$

$$\times \prod_{p \in \Sigma - P} U_p(e_p + \dim_k \mathfrak{p}^\ell_x) \times \prod_{p \in \Sigma \cup \{\ell\}} U_p(e_p)$$

where we denote

$$U_p(e) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) \middle| c \equiv 0, \quad d^{p^n} \equiv 1 \pmod{p} \right\}.$$ 

We identify cusps forms of weight 2 and level $N_\Sigma$ having coefficients in $\mathcal{O}$ with those holomorphic differentials in $H^1(X_{U_\Sigma}, \mathcal{O})$.

Second we construct an $\mathcal{O}$-lattice $V_\Sigma$ with a pairing $(\cdot, \cdot)_\Sigma$ following [10]. Define the following subgroup of the finite quotient $GL_2(\mathbb{A}_\mathbb{Q}^\infty)/U_\Sigma$:

$$G_\Sigma = GL_2(\mathbb{F}_\ell) \times \prod_{p \in P} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}/p^{r/2}\mathbb{Z}) \middle| c \equiv 0 \pmod{p} \right\}$$

and consider absolutely irreducible representations $G_\Sigma \to GL(V_\Sigma)$ for some $\mathcal{O}$-module $V_\Sigma$; we will choose a specific representation associated to certain multiplicative characters $\chi_p$ of finite order prime to $\ell$ for either $p \in P$ or $p = \ell$. When $p \in P$ the restriction $\mathfrak{p}^\ell_p|G_p$ is induced from a character $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^x) \to \mathbb{F}^\times_\ell$ defined over the unramified quadratic extension $\mathbb{Q}_p^x$ of $\mathbb{Q}_p$; see [14]. Choose a homomorphism $\chi_p$ such that the restriction of this character to the inertia group at $p$ is the composition

$$I_p \longrightarrow W(\mathbb{F}_p^x) \cdot \chi_p \longrightarrow \mathcal{O}^\times \mod \lambda \longrightarrow \mathbb{F}_\ell^\times.$$ 

This character induces an irreducible $\phi(p^{r/2})$-dimensional representation $\Theta(\chi_p)$ of $GL_2(\mathbb{Z}/p^{r/2}\mathbb{Z})$; see [10] §3.2 for details on the construction. When $p = \ell$, the representation $\mathfrak{p}$ is ordinary so choose a homomorphism $\chi_\ell$ such that we have the composition

$$\det \mathfrak{p} : \quad I_\ell \overset{\omega_\ell}{\longrightarrow} \mathbb{F}_\ell^\times \overset{\chi_\ell}{\longrightarrow} \mathcal{O}^\times \mod \lambda \longrightarrow \mathbb{F}_\ell^\times.$$ 

If $\chi_\ell = \chi_0$ is the map such that $\chi_0 \mod \lambda$ is trivial we have a 1-dimensional representation of $GL_2(\mathbb{F}_\ell)$ given by the composition $\chi_0 \circ \det$; otherwise we have an irreducible $(\ell + 1)$-dimensional representation $I(\chi_\ell, \chi_0)$. We choose a $G_\Sigma$-invariant $\mathcal{O}$-lattice $V_\Sigma$ such that the irreducible representation $G_\Sigma \to GL(V_\Sigma \otimes_\mathcal{O} K)$ is the tensor product over $p \in P$ and $p = \ell$ of the irreducible representations described above. Enlarging $\mathcal{O}$ if necessary, we can choose $V_\Sigma$ such that there exists a non-degenerate pairing $(\cdot, \cdot)_\Sigma$ for which $V_\Sigma$ self-dual; see [10] Lemma 3.3.1.

We now define $L_\Sigma$ following the discussion in [10]. We recall that by Proposition 3.4.1 where we specialize to weight $\kappa = 2$, we have $T_\Sigma^{(2)} \simeq T_m^{(2)}$ as the localization of $T^{(\kappa)} = \varphi_\kappa(h^0(N_\Sigma, \chi))$ at the maximal ideal $m = \varphi_\kappa(m_\Sigma)$ containing $p_\Sigma = \ker \pi_f$. We define $L_\Sigma$ as a localization at this maximal ideal:

$$L_\Sigma = \text{Hom}_{\mathcal{O}[G_\Sigma]}(V_\Sigma, H^1(X_{U_\Sigma}, \mathcal{O}))_m.$$ 

Property (HM1) is essentially [10] Lemma 5.3.1. Properties (HM2), (HM3), and (HM4) are verified in [10] Proposition 5.5.1. Property (HM5) is verified in [10] Proposition 5.6.1. This completes the proof of Theorem 3.5.1. $\square$
3.6. Specialization to Weight 1. We have shown that under suitable hypotheses \(\rho\) is \(\ell\)-adically modular, but this cusp form may not be classical. Our goal in this section is to show when such a form is indeed classical. We will work with projective \(\ell\)-adic representations

\[
\bar{\rho} : G_Q \longrightarrow GL_2(\mathcal{O}) \longrightarrow PGL_2(\mathcal{O}).
\]

**Theorem 3.6.1.** For \(\ell\) an odd prime, let \(\rho : G_Q \rightarrow GL_2(\mathcal{O})\) be a continuous Galois representation such that

1. \(\rho\) is ramified at finitely many primes;
2. \(\overline{\rho}\) is absolutely irreducible when restricted to \(\text{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt[\ell]{-1})\right)\), modular, and wildly ramified at \(\ell\);
3. \(\rho(G_{\ell})\) is finite and \(\rho(G_{\ell})\) is cyclic of \(\ell\)-power order.

Then \(i \circ \rho : G_Q \rightarrow GL_2(\mathbb{C})\) is modular for each embedding \(i : K \hookrightarrow \mathbb{C}\).

The conditions above force \(\ell = 3, 5\): Since \(\overline{\rho}\) is wildly ramified its image is divisible by \(\ell\), and since it is absolutely irreducible its image must contain \(SL_2(F_{\ell})\). Such groups with a complex representation have \(\ell \leq 5\).

**Proof.** Let \(H \subseteq I_{\ell}\) be the kernel of \(\overline{\rho}|_{I_{\ell}}\) so that the restriction \(\rho|_H\) is scalar for some character \(\chi : H \rightarrow \mathbb{C}^*\) of finite order. We will choose special extensions of \(\chi\) to \(G_Q\). Say that \(\rho|_G\) has order \(\ell^n\) i.e. the image of the quotient \(G_{\ell}/H\) has order \(\ell^n\), and let \(\sigma_0 \in G_{\ell}\) be a generator. The characteristic polynomial of \(\rho(\sigma_0)\) is \((x - \alpha)(x - \beta)\) where \(\alpha/\beta\) is a primitive \(\ell^n\)th root of unity, and \(\rho(\sigma_0)\) is similar to its rational canonical form

\[
\begin{pmatrix}
0 & \alpha \\
-1 & \alpha + \beta
\end{pmatrix} = A_0 \begin{pmatrix}
\beta & 1 \\
\alpha & 1
\end{pmatrix} A_0^{-1} = A_\beta \begin{pmatrix}
\alpha & 1 \\
1 & \beta
\end{pmatrix} A_\beta^{-1} \text{ where } A_x = \begin{pmatrix}
x & -1 \\
1 & 0
\end{pmatrix}.
\]

(Since \(\overline{\rho}\) is wildly ramified the residual image cannot be diagonal.) In particular, \(\rho(G_{\ell})\) may be conjugated to be upper-triangular. Choose characters \(\chi_1, \chi_2 : G_Q \rightarrow \mathbb{C}^*\) such that

1. \(\chi_1|_H = \chi_2|_H = \chi\);
2. \(\chi_1(\sigma_0^m) = \alpha^m\) and \(\chi_2(\sigma_0^m) = \beta^m\) where \(m = [\rho(G_{\ell}) : \rho(I_{\ell})]\); and
3. \(\chi_1\) and \(\chi_2\) have finite order and are unramified outside of \(\ell\).

Now define the characters \(\chi_{\ell} = \chi_1^{-1}\chi_2\) and \(\chi_0 = \det \rho/(\chi_1 \chi_2)\) as well as the representations \(\rho_i = \chi_{\ell}^{-1} \otimes \rho\) for \(i = 1, 2\). Then \(\chi_{\ell}\) is wildly ramified at \(\ell\), \(\chi_0\) is unramified at \(\ell\), and hence \(\rho_i\) are all \(\ell\)-adically modular.

Each \(\rho_i\) is ordinary and ramified at finitely many primes; while \(\overline{\rho}\) is modular, absolutely irreducible, and wildly ramified at \(\ell\). Theorem 3.5.1 implies that \(\rho_1\) and \(\rho_2\) are all \(\ell\)-adically modular.

Let \(f(\tau) = \sum_n a_n q^n\) denote the \(\ell\)-adic form associated with \(\rho_1\) and \(g(\tau) = \sum_n b_n q^n\) denote the \(\ell\)-adic form associated with \(\rho_2\). We have

1. \(f\) and \(g\) are ordinary cusp forms of weight \(\kappa = 1\);
2. \(f\) and \(g\) have nebentype det \(\rho_1 = \chi_0 \cdot \chi_{\ell}\) and det \(\rho_2 = \chi_0 \cdot \chi_{\ell}^{-1}\);
3. \(\rho_1 = \chi_{\ell} \otimes \rho_2\) i.e. \(a_p = \chi_{\ell}(\text{Frob}_p) \cdot b_p\) for almost all \(p \neq \ell\); and
4. \(a_{\ell} = b_{\ell} = \chi_0(\text{Frob}_\ell)\).
Note that the projective representations \( \tilde{\rho} \simeq \tilde{\rho}_f \simeq \tilde{\rho}_g \), so they all have the same prime-to-\( \ell \) part of the Artin conductor \( N_\emptyset \). It will follow from [41, Theorem 11.1] that such forms are classical modular forms once we verify that \( N_\emptyset \geq 5 \). The kernel of \( \tilde{\rho} : G_\Q \to PGL_2(\C) \) fixes an extension \( L/\Q \) such that \( \text{Gal}(L/\Q) \) contains \( PSL_2(\F_\ell) \); the only possibilities for the Galois group are \( A_4 \simeq PSL_2(\F_3) \), \( S_4 \simeq PGL_2(\F_3) \), and \( A_5 \simeq PSL_2(\F_5) \). But consulting the tables at [26] and [25] we see that there are no such number fields with prime-to-\( \ell \) discriminant \( N_\emptyset \leq 4 \). □

4. Icosahedral Galois Representations

In this section we specialize to \( \ell = 5 \). For any continuous complex Galois representation \( \rho \), there is a finite extension \( L/\Q \) which makes the following diagram commute:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \text{Gal}(\Qbar/L) & \longrightarrow & \text{Gal}(\Qbar/\Q) & \longrightarrow & \text{Gal}(L/\Q) & \longrightarrow & 1 \\
\downarrow & & \downarrow \rho & & \downarrow \tilde{\rho} & & \downarrow & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & GL_2(\mathbb{C}) & \longrightarrow & PGL_2(\mathbb{C}) & \longrightarrow & 1 \\
\end{array}
\]

The various possibilities for \( \tilde{\rho}(G_\Q) \) were classified in [23]: it is either cyclic, dihedral, tetrahedral \( (A_4 \simeq PSL_2(\F_3)) \), octahedral \( (S_4 \simeq PGL_2(\F_3)) \), or icosahedral \( (A_5 \simeq PSL_2(\F_5)) \). We will focus on the latter, so that \( L/\Q \) is an \( A_5 \)-extension. Our goal in this section is to find a family of Galois representations \( \rho \) which satisfy the hypotheses of Theorem 3.6.1 by focusing on the extensions \( L/\Q \). We introduce elliptic curves to give a family residually modular representations.

4.1. Quintics and Elliptic Curves. We begin by showing the explicit relationship between a certain class of quintics and elliptic curves. Most of the exposition that follows in this section is motivated by both [29] and [30].

The group \( A_5 \) may be realized as the group of rotations of the icosahedron. This Platonic Solid has 12 vertices which may be inscribed on the unit sphere, so that after projecting to the extended complex plane we have a natural group action on the complex numbers

\[
(\zeta_5 + \zeta_5^4) \zeta_5^\nu, \quad (\zeta_5^2 + \zeta_5^3) \zeta_5^\nu \quad \text{for } \nu \in \F_5; \quad \infty; \quad \text{and} \quad 0;
\]

generated by the three fractional linear transformations

\[
S \cdot z = \zeta_5 \cdot z, \quad T \cdot z = \frac{\varepsilon \cdot z + 1}{z - \varepsilon}, \quad \text{and} \quad U \cdot z = \frac{1}{z};
\]

where \( \zeta_5 \) is a primitive fifth root of unity and \( \varepsilon = \zeta_5 + \zeta_5^4 \) is a fundamental unit. Consider the rational functions

\[
\lambda(z) = \frac{[z^2 + 1]^2 [z^2 - 2 \varepsilon z - 1]^2 [z^2 + 2 \varepsilon^{-1} z - 1]^2}{-z (z^{10} + 11 z^5 - 1)}; \quad \mu(z) = \frac{-125 z^5}{z^{10} + 11 z^5 - 1}.
\]

The first has nontrivial action by \( S \) so we may associate a polynomial of degree 5, while the second has trivial action by \( S \) so we may associate a polynomial of degree 6. The following is the fundamental result in [29], with the last statement being motivated by the degree 6 resolvent first considered in [30].

Proposition 4.1.1. Fix a quintic \( q(x) = x^5 + A x^2 + B x + C \) over \( \Q \) such that \( A^4 - 5B^3 + 25ABC \) is nonzero, and denote \( L \) as its splitting field.
(1) The rational function
\[ j(z) = (\lambda + 3)^3 (\lambda^2 + 11 \lambda + 64) = \frac{(\mu^2 + 10 \mu + 5)^3}{\mu} \]
is invariant under \(A_5 \cong \langle S, T, U \rangle\). For each \(m, n \in \mathbb{Q}\), the resolvents
\[ x_n = \frac{m}{\lambda(\zeta_5^2 z) + 3} + \frac{n}{|\lambda(\zeta_5^2 z) + 3|[\lambda(\zeta_5^2 z)^2 + 10 \lambda(\zeta_5^2 z) + 45]} \]
are roots of the quintic \(x^5 + A_{m,n,j} x^2 + B_{m,n,j} x + C_{m,n,j}\) where
\[
A_{m,n,j} = -\frac{20}{j} \left[ (2m^3 + 3m^2 n) + 432 \frac{6mn^2 + n^3}{1728 - j} \right], \\
B_{m,n,j} = -\frac{5}{j} \left[ m^4 - 864 \frac{3m^2 n^2 + 2mn^3}{1728 - j} + 559872 \frac{n^4}{(1728 - j)^2} \right], \\
C_{m,n,j} = -\frac{1}{j} \left[ m^5 - 1440 \frac{m^3 n^2}{1728 - j} + 62208 \frac{15mn^4 + 4n^5}{(1728 - j)^2} \right].
\]

(2) There exist \(m, n, j \in \mathbb{Q}(\sqrt{5} \text{Disc}(q))\) such that \(A = A_{m,n,j}, B = B_{m,n,j}\) and \(C = C_{m,n,j}\).

(3) There exists a curve \(E\) over \(\mathbb{Q}(\sqrt{5} \text{Disc}(q))\) such that \(L(\sqrt{5} \text{Disc}(q))\) is the field generated by sum \(x_P + x_{2P}\) of \(x\)-coordinates of the 5-torsion of \(E\).

\textbf{Proof.}\ The properties stated in the first part are easily verified using a symbolic calculator. For the second, define \(\delta, \gamma_4, \gamma_6\) and \(\text{Disc}(q)\) by
\[
5^4 \cdot \delta = A^4 - 5B^3 + 25ABC \\
12^2 5^5 \cdot \gamma_4 = 128 A^4 B^2 - 192 A^3 C - 600 A B^3 C + 1000 A^2 B^2 C^2 \\
\quad - 144 B^5 + 3125 C^4 \\
12^3 5^{10} \cdot \gamma_6 = 1728 A^{10} + 10400 A^6 B^3 + 405000 A^2 B^6 - 180000 A^7 B C \\
\quad - 1170000 A^3 B^4 C + 1725000 A^4 B^2 C^2 - 1800000 A^5 C^3 \\
\quad + 2812500 A B^3 C^3 - 4687500 A^2 B^2 C^4 \\
\quad - 2025000 B^5 C^2 - 9765625 C^6 \\
\text{Disc}(q) = -27 A^4 B^2 + 108 A^5 C - 1600 A B^3 C + 2250 A^2 B^2 C^2 \\
\quad + 256 B^5 + 3125 C^4
\]
By eliminating \(m\) and \(n\) in the system above, we find that \(j\) is a root of the equation
\[
5^5 j^2 - 1728 (\gamma_4^3 - \gamma_6^2 + \delta^5) j + 1728^2 \gamma_4^3 = 0.
\]
The statement follows since \(\mathfrak{R}\) has discriminant \(5 \cdot \text{Disc}(q)\) and \(m\) and \(n\) may be expressed in terms of \(j\). For the third statement, fix \(j\) as a root of \(\mathfrak{R}\) so that \(L(\sqrt{5} \text{Disc}(q))\) is the splitting field of \(q'(-\mu) = (\mu^2 + 10 \mu + 5)^3 - j \mu\), and let \(E\) be denote the elliptic curve \(y^2 = x^3 + 3j/(1728 - j)x + 2j/(1728 - j)\) with invariant \(j\). Given a 5-torsion point \(P\) on \(E\),
\[
x_P + x_{2P} = -2 \frac{\mu^2 + 10 \mu + 5}{\mu^2 + 4 \mu - 1} \quad \text{for some root of } q'(-\mu)
\]
so that \( L(\sqrt{5\text{Disc}(q)}) \) contains the field generated by the sum of the \( x \)-coordinates of the 5-torsion, and conversely, given a root of \( q(\mu) \),

\[
\mu = \frac{31104 x^3}{(x - 2)^5 j - 1728 x^3 (x^2 - 10 x + 34)} \quad \text{where } x = x_P + x_{2P}
\]

for some 5-torsion \( P \) on \( E \). Hence \( L(\sqrt{5\text{Disc}(q)}) \) is generated as claimed. \( \square \)

4.2. \( \mathbb{Q} \)-Curves. We specialize to a family of quintics with particularly nice properties. The motivation for the following result comes from the classical work of Bring and Jerrard that any quintic can be brought to the form \( x^5 + Bx + C \). We say an elliptic curve \( E \) is associated to \( q(x) \) whenever statement (3) of Proposition 4.1.1 holds. We remind the reader that a \( \mathbb{Q} \)-curve is an elliptic curve without complex multiplication which is isogenous to each of its Galois conjugates, as first considered in [35].

**Proposition 4.2.1.** For \( t \in \mathbb{Q}^\times \), define the quintic and the curve

\[
q_t(x) = x^5 + 5 \left( \frac{9 - 5 t^2}{5 t^2} \right) x + 4 \left( \frac{9 - 5 t^2}{5 t^2} \right); \quad E_t : y^2 = x^3 + 2 x^2 + 3 + \frac{\sqrt{5}}{2} t.
\]

We have the following.

1. \( q_t(x) \) has Galois group contained in \( A_5 \). If \( t \) is square of a rational number then \( q_t(x) \) has Galois group \( A_5 \).
2. If \( t \) is the square of a 5-adic unit, then the decomposition, inertia, and wild inertia groups at 5 are cyclic of order 5. All higher ramification groups are trivial.
3. \( E_t \) is a 2-isogenous \( \mathbb{Q} \)-curve that is associated to \( q_t(x) \). Moreover, the isogeny is defined over \( \mathbb{Q}(\sqrt{-2}, \sqrt{5}) \).
4. Given a quintic \( q(x) = x^5 + Bx + C \) over \( \mathbb{Q} \) with Galois group \( A_5 \), the curve \( E_t \) is associated to \( q(x) \) when \( t = 75 C^2 / \text{Disc}(q) \).

Before we show the proof, we show an application of this result. Let \( L/\mathbb{Q} \) be an \( A_5 \)-extension which is unramified outside of \( \{2, 5, \infty\} \). By [25], there are only five such \( A_5 \)-extensions of \( \mathbb{Q} \), so it is a computational exercise to show that the splitting fields actually come from principal quintics:

| Original Quintic | Principal Quintic | Parameter \( t \) |
|------------------|-------------------|-------------------|
| \( x^5 + 20x - 16 \) | \( 4x^5 - 25x + 50 \) | \( 3/5, 15/11 \) |
| \( x^5 + 10x^3 - 10x^2 + 35x - 18 \) | \( 5x^5 + 20x + 16 \) | \( 1 \) |
| \( x^5 - 10x^3 + 110x - 116 \) | \( 5x^5 - 20x + 16 \) | \( 3 \) |
| \( x^5 + 10x^3 - 40x^2 + 60x - 32 \) | \( 5x^5 - 5x + 4 \) | \( 3/2 \) |
| \( x^5 - 10x^3 - 20x^2 + 10x + 216 \) | \( 5x^5 + 5x + 8 \) | \( 4/3 \) |

The associated \( \mathbb{Q} \)-curves can now be found via Proposition 4.2.1. In particular, the quintic \( x^5 + 10x^3 - 10x^2 + 35x - 18 \) first studied in [2] has associated \( \mathbb{Q} \)-curve

\[
y^2 = x^3 + \left( 5 - \sqrt{5} \right) x^2 + \sqrt{5}x.
\]

For more information, see [12].

**Proof.** The discriminant \( \text{Disc}(q_t) = 2^83^2 t^{-10} \left( 9 - 5 t^2 \right)^4 \) is a square, so its Galois group is contained in \( A_5 \). By [37], the only solvable trinomials in the form \( x^5 +
\[ Bx + C \] with square determinant satisfy
\[ B = 20 \frac{(v^2 + v - 1)(v^2 - v - 1)}{(v^2 + 1)^2} w^4, \quad C = 16 \frac{(v^2 + v - 1)(2v^2 + 3v - 2)}{(v^2 + 1)^2} w^5; \]
for some rational \( v \) and \( w \). If \( q(t) \) is solvable by radicals we must have
\[ w = \frac{v^2 - v - 1}{2v^2 + 3v - 2} \quad \text{and} \quad t = \frac{3(v^2 + 1)(2v^2 + 3v - 2)^2}{5(2v^3 + 2v^2 - v + 1)(v^3 + v^2 + 2v - 2)}. \]
Say \( t = u^2 \) for some \( u \in \mathbb{Q}^\times \); then the Galois group is properly contained in \( A_5 \) if and only if we have a rational point on the hyperelliptic curve
\[ y^2 = 15(x^2 + 1)(2x^3 + 2x^2 - x + 1)(x^3 + x^2 + 2x - 2). \]
This curve has no rational points as verified using \textsc{Magma} so the quintic has nonsolvable Galois group.

Now we consider the ramification groups at 5. Fix an embedding \( \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_5 \), and say \( t = u^2 \) for some \( u \in \mathbb{Z}_5^\times \). Then
\[ y = \frac{4u}{5 \sqrt{(1 + 5(u^2 - 2))}} \in \mathbb{Q}_5 \implies q(t) = \left( \frac{x}{5y/4} \right) = \left( \frac{x^5 - x - y}{(5y/4)^4} \right); \]
so that Galois group over \( \mathbb{Q}_5 \) is that of the Artin-Schreier quintic \( x^5 - x - y \). As \( y \) has 5-adic valuation -1, the results in [39, page 72, exercise 5] show that the decomposition, inertia, and wild inertia groups at 5 are cyclic of order 5 while the higher ramification groups are trivial.

Set \( r = (3 + \sqrt{5}t)/(2\sqrt{5}t) \) so that \( E_t \) is in the form \( y^2 = x^3 + 2x^2 + rx \). This curve is singular only if \( t = 0, \pm 3/\sqrt{5} \) which never happens when \( t \in \mathbb{Q}^\times \) so \( E_t \) is indeed an elliptic curve. Denote \( \sigma \in \text{Gal} (\mathbb{Q}(\sqrt{-2}, \sqrt{5})/\mathbb{Q}(\sqrt{-2})) \) as the nontrivial automorphism, and consider the isogeny \( \phi : E_t \to E_t^\sigma \) defined by
\[ \phi(x, y) = \left( \frac{1 - y^2}{x^2}, \frac{1 - y(r - x^2)}{x^2} \right) \implies \phi \circ \phi^\sigma = [-2]. \]
It follows that \( E_t \) is 2-isogenous to its Galois conjugate, so it is a \( \mathbb{Q} \)-curve. The fact that \( E_t \) is associated to \( q(t) \) comes from checking the system of equations in Proposition 4.1.1. It is easy to check that \( j(E_t) \) is a solution to the quadratic equation in \( [\overline{\mathbb{Q}}(\sqrt{5})] \) when \( t = 75C^2/\sqrt{\text{Disc}(q)} \), so \( E_t \) is indeed an elliptic curve over \( \mathbb{Q}(\sqrt{5}) \) which is associated to \( q(x) \). \hfill \Box

4.3. Geometric Galois Representations. The extension \( L/\mathbb{Q} \) is associated to the projective representation \( \bar{\rho} \) so we construct, under suitable hypotheses, a canonical complex Galois representation as a lift. We exploit the fact that mod 5 representations of \( \mathbb{Q} \)-curves defined over \( \mathbb{Q}(\sqrt{5}) \) fit into the composition
\[ \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{5})) \xrightarrow{\bar{C}_{E_t}} \mathbb{Z} \cdot \text{SL}_2(\mathbb{F}_5) \xrightarrow{\pi} \text{GL}_2(\mathbb{C}) \]
where \( \pi \) is some finite group representation and \( Z(\mathbb{F}_5) \) is the center of \( \text{GL}_2(\mathbb{F}_5) \). Recall that \( \varepsilon = \zeta_5 + \zeta_5^4 \) is a fundamental unit in \( \mathbb{Q}(\sqrt{5}) \).

**Proposition 4.3.1.** Let \( E = E_t \) be a \( \mathbb{Q} \)-curve as in Proposition 4.2.2.

1. There exists a faithful representation \( \pi : Z(\mathbb{F}_5) \cdot \text{SL}_2(\mathbb{F}_5) \to \text{GL}_2(\mathbb{C}) \) and a prime ideal \( \lambda \subseteq \mathbb{Z}[\varepsilon, \sqrt{-1}] \) above 5 such that \( \pi \equiv 1 \mod \lambda \).
(2) There exists a 1-dimensional representation \( \omega : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{5})) \to \mathbb{C}^\times \) such that the twists \( \rho^{(1)}_E = \omega \otimes (\pi \circ \overline{\rho}_{E,5}) \) and \( \rho^{(2)}_E = \omega \otimes \rho_{E,5} \) are continuous Galois representations, ramified at finitely many primes, which are restrictions of 2-dimensional representations of \( G_Q \).

(3) As representations of \( G_Q \), the determinant \( \det \rho^{(\kappa)}_E = \chi \cdot (\varepsilon_{5}/\omega_{5})^{\kappa - 1} \) for \( \kappa = 1, 2 \); where \( \chi \) is the nontrivial quadratic character which factors through \( \text{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \), and \( \omega_{5} \) is a character of order 4 which factors through \( \text{Gal}(\mathbb{Q}(\zeta_{5})/\mathbb{Q}) \). In particular, \( \rho^{(1)}_E \) is odd.

(4) If \( q_1(x) \) has Galois group \( A_5 \), then the residual representation \( \overline{\rho}_E \equiv \rho^{(1)}_E \equiv \rho^{(2)}_E \mod \lambda, \) as a representation \( G_Q \), has image \( Z(\mathbb{F}_{5}) \cdot SL_2(\mathbb{F}_{5}) \). In particular, the restriction of \( \overline{\rho}_E \) to \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{5})) \) is absolutely irreducible.

A similar result is true for all \( \mathbb{Q} \)-curves; see [15] for proofs using Galois cohomology. Note that \( \rho^{(1)}_E : G_Q \to GL_2(\mathbb{C}) \) is a complex representation, whereas \( \rho^{(2)}_E : G_Q \to GL_2(\mathbb{Q}_5) \) is a 5-adic representation.

Proof. Let \( \varpi = (\omega_{5}(2) \cdot \varepsilon^{-1} - 1) \in \mathbb{Q}(\sqrt{5}, \sqrt{-1}) \) in terms of the fundamental unit \( \varepsilon \) and the Teichmüller character \( \omega_{5} : \mathbb{Z}_5^\times \to \mathbb{C}^\times \), and \( \lambda = \varpi \cdot Z[\varepsilon, \sqrt{-1}] \) be a prime above 5. Define the map \( \pi : Z(\mathbb{F}_{5}) \cdot SL_2(\mathbb{F}_{5}) \to GL_2(\mathbb{C}) \) by

\[
\pi : \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \to T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

\[
U(a, d) = \begin{pmatrix} \omega_{5}(a) \\ \omega_{5}(d) \end{pmatrix}.
\]

We take for granted that \( Z(\mathbb{F}_{5}) \cdot SL_2(\mathbb{F}_{5}) \) is generated by the three matrices on the left, so it suffices to check \( \pi \) on these generators. We follow the exposition in [15] page 405. It is a rather straightforward exercise to show that the orders of each generator match correctly i.e. \( S^3 = T^4 = U^4 = 1_2 \), so it suffices to consider the commutation relations among the image of these generators:

(1) \( T \cdot U(a, d) \cdot T^{-1} = U(d, a) \),

(2) \( U(a, d) \cdot S \cdot U(a, d)^{-1} = S^n \) where \( n \equiv a \cdot d^{-1} \) (5),

(3) \( T \cdot S^d \cdot T^{-1} = U(a, d) \cdot S^{-d} \cdot T^{-1} \cdot S^{-a} \) where \( a \cdot d \equiv 1 \) (5).

(Condition 2 is not satisfied if \( n \equiv \pm 2 \) (5); this explains why we must restrict to \( Z(\mathbb{F}_{5}) \cdot SL_2(\mathbb{F}_{5}) \) where \( n \equiv \pm 1 \).) It is also straightforward to verify these relations. The the congruence \( \pi \equiv 1 \) (\( \lambda \)) follows because \( 2 - \varepsilon = \varepsilon^2 \varpi \varpi^c, 2 - \omega_{5}(2) = \varepsilon \varpi (\varepsilon \varpi^c - 1), \) and \( \sqrt{5} = \varepsilon \varpi \varpi^c \) are elements of \( \lambda \). (Recall \( c \) is complex conjugation.)

We now prove statement (2) of the proposition. By Proposition 4.2.1 the isogeny for \( E \) is defined over \( \mathbb{Q}(\sqrt{-2}, \sqrt{5}) \) so \( a_{\sigma(p)} = (-2/N p) \cdot a_p \) in terms of the trace of Frobenius of \( \rho_{E,5} \), the Legendre symbol \( (-2/\ast) \), the norm map \( N \), and the nontrivial automorphism \( \sigma \in \text{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) \). Hence the twisted representations \( \rho^{(\kappa)}_E \) will be Galois invariant and defined over \( \mathbb{Q} \) if we exhibit a character \( \omega : Z[\varepsilon] \to \mathbb{C} \) such that \( \omega^{-1} = (-2/\ast) \circ N \). The field \( \mathbb{Q}(\sqrt{5}) \) has narrow class number 1 so we extend
the Dirichlet characters of conductors $4 \cdot Z[\varepsilon]$, $8 \cdot Z[\varepsilon]$, and $\sqrt{5} \cdot Z[\varepsilon]$ as follows:

$$
\omega_4 : \{ -1 \mapsto -1 \}, \quad \omega_8 : \{ 1 \mapsto -1 \}, \quad \omega_5 : \varepsilon \mapsto \zeta_4.
$$

It is straightforward to verify on the generators that

$$
\omega = \omega_4^3 \omega_8 \omega_5 \implies \omega^{-1} = \left( \frac{2}{*} \right) \circ N \quad \text{and} \quad \omega^2 = \left[ \left( \frac{1}{*} \right) \cdot \omega_5^{-1} \right] \circ N.
$$

Since $\omega$ is trivial on the totally positive units $\varepsilon^2 n$, we identify $\omega$ with a Galois representation through the composition

$$
\text{Gal}(\overline{Q}/Q(\sqrt{5})) \longrightarrow (Z[\varepsilon]/8\sqrt{5} \cdot Z[\varepsilon])^2 \longrightarrow C^\times
$$

via Class Field Theory. It is clear that $\rho_E^{(s)}$ is continuous and ramified at finitely many primes because the same is true for both $\rho_{E,5}$ and $\omega$.

We have $\omega^2 = \chi \cdot \omega_5^{-1}$ with $\chi$ the nontrivial quadratic character ramified at 2. It is well-known that $\det \rho_{E,5} \equiv \varepsilon_5$ so that

$$
\det \rho_E^{(1)} = \omega^2 \cdot \det \pi \circ \overline{\rho}_{E,5} = \chi \cdot (\omega_5/\omega_5), \quad \det \rho_E^{(2)} = \omega^2 \cdot \det \rho_{E,5} = \chi \cdot (\varepsilon_5/\omega_5);
$$

hence statement (3) follows.

To show statement (4) note that $Z(F_5) \cdot SL_2(F_5)$ is the subgroup of $GL_2(F_5)$ of index 2 consisting of those matrices with square determinants. Since $\det \overline{\rho}_E \equiv \chi (5)$ is a quadratic character, the image of $\overline{\rho}_E$ is contained in $Z(F_5) \cdot SL_2(F_5)$. If $q_\lambda(x)$ has Galois group $A_5$, the restriction $\overline{\rho}_E|_{Q(\sqrt{5})}$ is the twist of an absolutely irreducible mod 5 representation, which has image $Z(F_5) \cdot SL_2(F_5)$. This gives equality. □

### 4.4. Modularity Results.

We now apply the results from the previous section. Note that the following result imposes no local conditions at 3.

**Proposition 4.4.1.** Let $E = E_4$ be a $Q$-curve as in Proposition 4.2.1. Then $E$ is modular. In particular, both $\rho_E^{(2)}$ and $\overline{\rho}_E$ are modular.

**Proof.** We use the arguments in [10] and [18] on the modularity of representations which are potentially Barsotti-Tate, so we consider the local properties of $E$ at $\ell = 3$.

For a prime $\ell$, denote $\rho_\ell : G_Q \rightarrow GL_2(\overline{Q})$ as the $\ell$-adic representation associated to the Galois invariant twist $\omega \otimes \rho_{E,\ell}$ as in Proposition 2.3.4. When $\ell = 5$ we identify $\rho_\ell = \rho_E^{(2)}$, but it suffices to show $\rho_\ell$ is modular for $\ell = 3$. Recall by [18, Proposition 2.10] that there exists an abelian variety $A$ defined over $Q$ such that there exists an embedding $Z[\sqrt{-2}] \hookrightarrow \text{End}(A) \otimes Q$ and $\rho_3 \simeq \rho_{A,\lambda}$ can be realized as the $\lambda$-adic representation of the Tate module

$$
\text{proj}_{n \rightarrow \infty} \text{End}(A) \otimes Q \lambda \quad \text{where} \quad \lambda = (1 + \sqrt{-2}) \cdot Z[\sqrt{-2}].
$$

The elliptic curve $E$ has a Weierstrass equation of the form $y^2 = x^3 + 2x^2 + 3x$ where $r = (3 + \sqrt{5})/(2\sqrt{5})$ for some $t \in Q^\times$. Fix an embedding $Q \hookrightarrow Q_3$ as well as a normalized valuation $\nu : Q_3 \rightarrow Z$. If $\nu(t) \leq 1$ then both $r, 1 - r \in Z_3[\sqrt{5}]$ so that $E$ has good reduction at the primes in $Q(\sqrt{-2}, \sqrt{5})$ above 3. [18, Theorem 5.2] shows $E$ is modular. (Modularity essentially follows from [13, Theorem 5.3] because $\rho_\ell$ is crystalline.) Assume then that $\nu(t) \geq 2$, and write $t = 3^{n+2} u$ for
some \( u \in \mathbb{Z}_3^\times \) so that the twist of \( E \) by \( \sqrt{-3}^u \) has good reduction at the primes above 3. If \( n \equiv 3 \pmod{4} \) then \( E \) itself has good reduction at the primes above 3, so \( E \) is again modular. We assume that \( n \not\equiv 3 \pmod{4} \), and follow the arguments in \cite{K} §3 to show \( E \) is modular. Note that reduction of the twisted elliptic curve modulo the primes above 3 is \( y^2 = x^3 + D \) where \( D = (-1)^n/((\sqrt{5}u) \in \mathbb{F}_9^\times \), so the variety \( A \) has potential supersingular reduction.

Denote \( F = \mathbb{Q}_3(\sqrt{-3}) \) as the ramified quadratic extension of \( \mathbb{Q}_3 \), and \( \psi : G_F = \text{Gal}(\mathbb{Q}_3/\mathbb{Q}_3(\sqrt{-3})) \to \{\pm 1\} \) as the ramified quadratic character; the twisted curve \( A^\psi \) defined over \( F \) has good supersingular reduction and so the twisted representation \( \rho_3|_{G_F} \otimes \psi \) is unramified. (Compare with \cite{K} Lemma 3.4.) Clearly the restriction of \( \tilde{\rho}_3 \) to \( \text{Gal}(\mathbb{Q}_3(\sqrt{-3})/\mathbb{Q}_3(\sqrt{-3})) \) is absolutely irreducible, and so by \cite{K} Lemma 3.2 the residual representation \( \tilde{\rho}_3 \) is modular. By \cite{K} Lemma 3.5 the centralizer of the image \( \tilde{\rho}_3(G_3) \) consists of scalars. Following the notation in \cite{La} Appendix B], it suffices to show that the representation

\[
WD(\rho_3) : \quad W_Q \to GL_2(\mathbb{Q}_3)
\]

of the Weil group of \( \mathbb{Q} \) is “strongly acceptable” for \( \tilde{\rho}_3 \). (Compare with \cite{K} Lemma 3.6.) Denote the restriction \( \tau = WD(\rho_3)|_{I_F} \). Because twisted representation \( \rho_3|_{G_F} \otimes \psi \) is unramified and \( \psi \) is a quadratic character we have the identity

\[
WD(\rho_3|_{G_F} \otimes \psi) \simeq WD(\rho_3|_{G_F}) \otimes_{\mathbb{Q}_3} WD(\psi) \quad \implies \quad \tau|_{I_F} \simeq WD(\psi)|_{I_F}.
\]

That is, \( \tau|_{I_F} \) is a nontrivial quadratic character for \( I_F \). As in Proposition 4.3.1, the determinant \( \det \rho_\ell = \chi \cdot \varepsilon_\ell/\omega_5 \) so by \cite{La} §B.2 we have

\[
WD(\det \rho_3) \simeq \left( \chi/\omega_5 \right)|_{W_F} \otimes_{\mathbb{Q}_3(\sqrt{-3})} WD(\varepsilon_3).
\]

This character is unramified and \( \psi \) is a quadratic character so \( \det \tau = 1 \). We conclude that \( \tau = \phi^2 \otimes \phi \) in terms of the Teichmuller lift of the fundamental character \( \phi : G_Q \to \mathbb{F}_5^\times \) of level 2. If \( F' \) is an extension of \( Q_3 \) such that \( \tau|_{I_F'} \) is trivial, then \( F' \) contains the splitting field of \( x^4 - 3 \) and so \( F \subset F' \), so that \( \rho_3|_{I_F'} \simeq \rho_{A^{\psi}, \lambda}|_{I_F'} \) is Barsotti-Tate. It follows from the criteria of \cite{La} §1.2] that \( \rho_3 \) is a deformation of \( \tilde{\rho}_3 \) of type \( \tau \), and so it follows from \cite{La} Corollary 2.3.2] that \( \tau \) is “acceptable” for \( \tilde{\rho}_3 \). Since \( A \) is potentially supersingular at 3, we have \( \tilde{\rho}_3|_{I_3} \otimes \mathbb{F}_3 \simeq \phi \otimes \phi \) and so using the criteria in \cite{La} §1.2] we conclude that \( \tau \) is indeed “strongly acceptable” for \( \tilde{\rho}_3 \).

\[\text{Theorem 4.4.2.} \quad \text{Let } \tilde{\rho} : G_Q \to PGL_2(\mathbb{C}) \text{ be a projective icosahedral Galois representation, and assume that the field } L \text{ fixed by its kernel is the splitting field of a quintic } x^5 + B x + C \text{ with } 75 C^2/\sqrt{256 B^3} + 3125 C^4 \text{ the square of a 5-adic unit. Then for any continuous lift } \rho : G_Q \to GL_2(\mathbb{C}) \text{ of } \tilde{\rho}, \]

\begin{enumerate}
\item \( \rho \) is ramified at finitely many primes;
\item \( \rho \) is absolutely irreducible when restricted to \( \text{Gal}(\mathbb{Q}/\mathbb{Q}(\sqrt{5})) \), modular, and wildly ramified at 5;
\item \( \rho(G_5) \) is finite and \( \bar{\rho}(G_5) \) is cyclic of order 5.
\end{enumerate}

In particular, \( \rho \) is (classically) modular.

By Propositions 4.2.1 and 4.3.1 there are infinitely many projective representations that satisfy the hypotheses of the Theorem; for example, the family of quintics

\[
x^5 + 5 \left( \frac{9 - 5 u^4}{5 u^2} \right) + 4 \left( \frac{9 - 5 u^4}{5 u^2} \right), \quad u \in \mathbb{Q} \cap \mathbb{Z}_5^\times;
\]
would suffice to yield such representations.

Proof. The modularity of \( \rho \) would follow from Theorem 3.6.1 if we show the three statements hold true. Choose a 2-isogenous \( \mathbb{Q} \)-curve \( E \) associated to \( x^5 + Bx + C \) via Proposition 4.2.1 and consider the representation \( \rho_E^{(1)} : G_\mathbb{Q} \to GL_2(\mathbb{C}) \) as in Proposition 4.3.1. We will show that \( \rho \) is a twist of \( \rho_E^{(1)} \).

Denote \( L^{(1)} \) as the field fixed by the kernel of \( \rho_E^{(1)} \). First we show \( L^{(1)} = L \). It is well-known that the field fixed by the kernel of \( \overline{\rho}_{E,5} \) is contained in the field generated by the coordinates of the 5-torsion of \( E \), so the field fixed by the kernel of the composition

\[
\text{Gal} \left( \overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{5}) \right) \to \overline{\rho}_{E,5} \to \mathbb{Z}(\mathbb{F}_5) \cdot SL_2(\mathbb{F}_5) \to PSL_2(\mathbb{F}_5)
\]

must be generated by those coordinates of the 5-torsion which are fixed by action of the scalars \( \mathbb{Z}(\mathbb{F}_5) \) i.e. the field generated by sum \( xP + x2P \) of \( x \)-coordinates of the 5-torsion of \( E \). It follows from Proposition 4.4.1 that this field is \( L(\sqrt{5}) \). On the other hand, it follows from Proposition 13.1 that \( \overline{\rho}_{E,5} \) is surjective while \( \pi \) is injective so \( L(\sqrt{5}) \) is also the field fixed by the kernel of the composition

\[
\text{Gal} \left( \overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{5}) \right) \to \overline{\rho}_{E,5} \to GL_2(\mathbb{C}) \to PGL_2(\mathbb{C});
\]

hence \( L^{(1)}(\sqrt{5}) = L(\sqrt{5}) \). Note \( \sqrt{5} \notin L \) because \( L/\mathbb{Q} \) is a nonsolvable extension and has no subfields of degree 2, so \( L^{(1)} \supseteq L \). Similarly, \( L^{(1)} \subseteq L \). Now consider the following diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \text{Gal} \left( \overline{\mathbb{Q}}/L \right) & \longrightarrow & \text{Gal} \left( \overline{\mathbb{Q}}/\mathbb{Q} \right) & \longrightarrow & \text{Gal} \left( L/\mathbb{Q} \right) & \longrightarrow & 1 \\
& & \downarrow \chi_E & & \downarrow \rho, \rho_E^{(1)} & & \downarrow \overline{\rho}, \overline{\rho}_E^{(1)} & & \\
1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & GL_2(\mathbb{C}) & \longrightarrow & PGL_2(\mathbb{C}) & \longrightarrow & 1
\end{array}
\]

There are only two such projective representations so upon choosing a Galois conjugate of \( \pi \) as in Proposition 13.1, we have \( \overline{\rho} \simeq \rho_E^{(1)} \); without loss of generality we assume equality. To show \( \rho \) is a twist of \( \rho_E^{(1)} \) we consider the character \( \chi_E \) in the left most column. For automorphisms \( \sigma \) and \( \tau \), define the symbol \( \chi_E(\sigma) \) by the relation \( \chi_E(\sigma) \cdot 1_2 = \rho(\sigma) \cdot \rho_E^{(1)}(\sigma)^{-1} \). Then \( \chi_E \) is actually a multiplicative character since

\[
\chi_E(\sigma \tau) \cdot 1_2 = \rho(\sigma \tau) \rho_E^{(1)}(\sigma \tau)^{-1} = \rho(\sigma) \rho_E^{(1)}(\tau)^{-1} \rho_E^{(1)}(\sigma)^{-1} = \chi_E(\sigma) \chi_E(\tau) \cdot 1_2.
\]

Hence, \( \rho \simeq \chi_E \otimes \rho_E^{(1)} \). This character is of finite order because \( \rho \) is continuous i.e. has finite image. The statements that \( \overline{\mathbb{Q}} \) is wildly ramified at 5 and \( \overline{\rho}(G_5) \) is cyclic of order 5 now follow from Proposition 4.4.1 while the statement that \( \overline{\rho} \) is modular follows from Proposition 4.4.3.

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