SYMPLECTIC STRUCTURES ON 3-LIE ALGEBRAS

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Abstract. The symplectic structures on 3-Lie algebras and metric symplectic 3-Lie algebras are studied. For arbitrary 3-Lie algebra \( L \), infinite many metric symplectic 3-Lie algebras are constructed. It is proved that a metric 3-Lie algebra \((A, B)\) is a metric symplectic 3-Lie algebra if and only if there exists an invertible derivation \( D \) such that \( D \in \text{Der}_B(A) \), and is also proved that every metric symplectic 3-Lie algebra \((\tilde{A}, \tilde{B}, \tilde{\omega})\) is a \( T^*_\theta \)-extension of a metric symplectic 3-Lie algebra \((A, B, \omega)\). Finally, we construct a metric symplectic double extension of a metric symplectic 3-Lie algebra by means of a special derivation.

1. Introduction

The notion of 3-Lie algebra was introduced in [1]. It is a vector space with a ternary linear skew-symmetric multiplication satisfying the generalized Jacobi identity (or Filippov identity). 3-Lie algebras, especially, metric 3-Lie algebras are applied in many fields in mathematics and mathematical physics. Motivated by some problems of quark dynamics, Nambu [2] introduced a 3-ary generalization of Hamiltonian dynamics by means of the 3-ary Poisson bracket

\[
[f_1, f_2, f_3] = \det \left( \frac{\partial f_i}{\partial x_j} \right)
\]

which satisfies the generalized Jacobi identity

\[
[[f_1, f_2, f_3], g_2, g_2] = [[f_1, g_2, g_3], f_2, f_3] + [f_1, [f_2, g_2, g_3], f_3] + [f_1, f_2, [f_3, g_2, g_3]].
\]

Following this line, Takhtajan [3] developed systematically the foundation of the theory of \( n \)-Poisson or Nambu-Poisson manifolds. Metric 3-Lie algebras are applied to the study of the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes; the Bagger-Lambert theory has a novel local gauge symmetry which is based on a metric 3-Lie algebra [4, 5]. The generalized Jacobi identity can be regarded as a generalized Plucker relation in the physics literature [6, 7, 8].

Authors in [9] studied the structure of metric \( n \)-Lie algebras. It is an \( n \)-Lie algebra with a non-degenerate \( ad \)-invariant symmetric bilinear form. The ordinary gauge theory requires a positive-definite metric to guarantee that the theory possesses positive-definite kinetic terms and to prevent violations of unitarity due to propagating ghost-like degrees of freedom. But very few metric \( n \)-Lie algebras admit positive-definite metrics (see [8, 10]); Ho, et al. in [5] confirmed that there are no non-strong semisimple \( n \)-Lie algebras [11] with positive-definite metrics for \( n = 5, 6, 7, 8 \). They also gave examples of 3-Lie algebras whose metrics are not

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positive-definite and observed that generators of zero norm are common in 3-Lie algebras. Papers [12, 13] studied the module-extension of 3-Lie algebras and $T^*_0$-extension of $n$-Lie algebras. So we can obtain more metric 3-Lie algebras by 3-Lie algebras and their modules.

We know that Lie groups which admit a bi-invariant pseudo-Riemannian metric and also a left-invariant symplectic form are nilpotent Lie groups and their geometry (and, consequently, that of their associated homogeneous spaces) is very rich. In particular, they carry two left-invariant affine structures: one defined by the symplectic form (which is well-known) and another which is compatible with a left-invariant pseudo-Riemannian metric. The paper [16] studied quadratic Lie algebras over a field $K$ of null characteristic which admit, at the same time, a symplectic structure. It is proved that if $K$ is algebraically closed every such Lie algebra may be constructed as the $T^*$-extension of a nilpotent algebra admitting an invertible derivation and also as the double extension of another quadratic symplectic Lie algebra by the one-dimensional Lie algebra. In this paper we study the metric 3-Lie algebra which, at same time, admits a symplectic structure. We call it a metric symplectic 3-Lie algebra.

Throughout this paper, $F$ denotes an algebraically closed field $F$ of characteristic zero. Any bracket that is not listed in the multiplication of a 3-Lie algebra is assumed to be zero. The symbol $\oplus$ will be frequently used. Unless other thing is stated, it will only denote the direct sum of vector spaces.

2. Fundamental notions

A 3-Lie algebra [1] is a vector space $L$ over a field $F$ on which a linear multiplication $[\ ,\ ,\ ] : L \wedge L \wedge L \rightarrow L$ satisfying generalized Jacobi identity (or Filippov identity)

$$[[x_1, x_2, x_3], y_2, y_3] = \sum_{i=1}^{3} [x_1, \cdots, [x_i, y_2, y_3], \cdots, x_3], \forall x_1, x_2, x_3, y_2, y_3 \in L.$$ 

A subspace $A$ of $L$ is called a subalgebra (an ideal) of $L$ if $[A, A, A] \subseteq A ([A, L, L] \subseteq A)$. If $[A, A, A] = 0 ([A, A, L] = 0)$, than $A$ is called an abelian subalgebra (an abelian ideal) of $L$.

In particular, the subalgebra generated by the vectors $[x_1, x_2, x_3]$ for all $x_1, x_2, x_3 \in L$ is called the derived algebra of $L$, which is denoted by $L^1$. If $L^1 = 0$, $L$ is called an abelian algebra.

A derivation of a 3-Lie algebra $L$ is a linear mapping $D : L \rightarrow L$ satisfying

$$D[x, y, z] = [Dx, y, z] + [x, Dy, z] + [x, y, Dz], \forall x, y, z \in L.$$ 

All the derivations of $L$ is a linear Lie algebra, is denoted by $Der(L)$.

A 3-Lie algebra $L$ is said to be simple if $L^1 \neq 0$ and it has no ideals distinct from 0 and itself.
An ideal $I$ of an 3-Lie algebra $L$ is called \textit{nilpotent} [14], if $I^s = 0$ for some $s \geq 0$, where $I^0 = I$ and $I^s$ is defined as

$$I^s = [I^{s-1}, I, L], \text{ for } s \geq 1.$$ 

In the case $I = L$, $L$ is called a nilpotent 3-Lie algebra. The abelian ideal

$$Z(L) = \{x \in L \mid [x, L, L] = 0\}$$

is called the \textit{center} of $L$.

Let $L$ be a 3-Lie algebra, $V$ be a vector space, $\rho : L \wedge L \rightarrow \text{End}(V)$ be a linear mapping. The pair $(V, \rho)$ is called a \textit{representation} [14] (or $V$ is an $L$-module) of $L$ in $V$ if $\rho$ satisfies $\forall a_1, a_2, a_3, b_1, b_2 \in L$,

$$[\rho(a_1, a_2), \rho(b_1, b_2)] = \rho([a_1, a_2, b_1], b_2) + \rho(b_1, [a_1, a_2, b_2]),$$

$$\rho([a_1, a_2, a_3], b_1) = \rho(a_2, a_3)\rho(a_1, b_1) - \rho(a_1, a_3)\rho(a_2, b_1) + \rho(a_1, a_2)\rho(a_3, b_1).$$

Then $(V, \rho)$ is a representation of the 3-Lie algebra $L$ if and only if the vector space $Q = L \oplus V$ is a 3-Lie algebra in the following multiplication

$$[a_1 + v_1, a_2 + v_2, a_3 + v_3] = [a_1, a_2, a_3]_L + \rho(a_1, a_2)(v_3) - \rho(a_1, a_3)(v_2) + \rho(a_2, a_3)(v_1).$$

Therefore, $A$ is a subalgebra and $V$ is an abelian ideal of the 3-Lie algebra $L \oplus V$, respectively.

If $(V, \rho)$ is a representation of the 3-Lie algebra $L$, $V^*$ is the dual space of $V$. Then $(V^*, \rho^*)$ is also a representation of $L$, which is called the dual representation of $(V, \rho)$, where

$$\rho^* : L \wedge L \rightarrow \text{End}(V^*), \quad \rho^*(a, b)f(c) = -f(\rho(a, b)c), \quad \forall a, b, c \in L, \ f \in V^*.$$  

For 3-Lie algebra $L$, the joint representation $(L, ad)$ is

$$ad : L \wedge L \rightarrow \text{End}(L), \quad ad(x, y)(z) = [x, y, z], \quad \forall x, y, z \in L.$$  

Then we obtain the dual representation $ad^* : L \wedge L \rightarrow \text{End}(L^*)$,

$$(ad^*(x, y)f)(z) = -f(ad(x, y)z) = -f([x, y, z]), \quad \forall x, y, z \in L, \ f \in L^*.$$  

Let $L$ be a 3-Lie algebra, $B : L \times L \rightarrow F$ be a non-degenerate symmetric bilinear form on $L$. If $B$ satisfies

$$B([x_1, x_2, x_3], x_4) + B(x_3, [x_1, x_2, x_4]) = 0, \forall x_1, x_2, x_3, x_4 \in L, \quad (2.1)$$

then $B$ is called a \textit{metric} on 3-Lie algebra $L$, and $(L, B)$ is called a \textit{metric 3-Lie algebra} [9].

Let $(L, B)$ be a metric 3-Lie algebra. Denotes

$$\text{Der}_B(L) = \{D \in \text{Der}(L) \mid B(Dx, y) + B(x, Dy) = 0, \ \forall x, y \in L\} = \text{Der}(L) \cap \text{so}(L, B). \quad (2.2)$$

Let $W$ be a subspace of a metric 3-Lie algebra $(L, B)$. The \textit{orthogonal complement} of $W$ is defined by

$$W^\perp = \{x \in L \mid B(w, x) = 0 \text{ for all } w \in W\}.$$
Then $W$ is an ideal if and only if $W^\perp$ is an ideal and $(W^\perp)^\perp = W$. Notice that $W$ is a minimal ideal if and only if $W^\perp$ is maximal. If $W \subseteq W^\perp$, then $W$ is called isotropic.

The subspace $W$ is called nondegenerate if $B|_{W \times W}$ is nondegenerate, this is equivalent to $W \cap W^\perp = 0$ or $L = W \oplus W^\perp$. If an ideal $I$ satisfies $I = I^\perp$, then $I$ is called a completely isotropic ideal.

If $L$ does not contain nontrivial nondegenerate ideals, then $L$ is called $B$-irreducible. For a metric 3-Lie algebra $(L,B)$, it is not difficult to see

$$L^1 = [L,L,L] = Z(L)^\perp.$$  

3. Symplectic 3-Lie algebras

**Definition 3.1** Let $L$ be a 3-Lie algebra over a field $F$, linear mapping $\omega : L \times L \rightarrow F$ be non-degenerate. If $\omega$ satisfies

$$\sum_{i=1}^{4} \omega([x_1, \ldots, \hat{x}_i, \ldots, x_4],(-1)^{i-1}x_i) = 0, \ \forall x_i \in L, i = 1, 2, 3, 4, \quad (3.1)$$

then $\omega$ is called a symplectic structure on $L$, and $(L,\omega)$ is called a symplectic 3-Lie algebra.

An ideal $I$ of a symplectic 3-Lie algebra $(L,\omega)$ is called an lagrangian ideal if and only if it coincides with its orthogonal with respect to the form $\omega$.

If there exists a metric $B$ and a symplectic structure $\omega$ on 3-Lie algebra $L$, respectively, then $(L,B,\omega)$ is called a metric symplectic 3-Lie algebra.

By the above definition, if $(L,\omega)$ is a symplectic 3-Lie algebra, then the dimension of $L$ is even.

**Theorem 3.1** Let $(L,B)$ be a metric 3-Lie algebra. Then there exists a symplectic structure on $L$ if and only if there exists a skew-symmetric invertible derivation $D \in \text{Der}_B(L)$.

**Proof.** Let $(L,B,\omega)$ be a symplectic 3-Lie algebra. Defines $D : L \rightarrow L$ by

$$B(Dx,y) = \omega(x,y), \ \forall x,y \in L. \quad (3.2)$$

Then $D$ is invertible, and from Eq.(3.1), for $\forall x_1,x_2,x_3,x_4 \in L$,

$$B([Dx_1,x_2,x_3],x_4) + B([x_1,Dx_2,x_3],x_4) + B([x_1,x_2,Dx_3],x_4) - B([x_1,x_2,x_3],x_4)$$

$$= -B([x_2,x_3,x_4],Dx_1) + B([x_1,x_3,x_4],Dx_2) - B([x_1,x_2,x_4],Dx_3) + B([x_1,x_2,x_3],Dx_4)$$

$$= \sum_{i=1}^{4} \omega([x_1, \ldots, \hat{x}_i, \ldots, x_4],(-1)^{i-1}x_i) = 0.$$  

Therefore, $D$ is a skew-symmetric invertible derivation of $(L,B)$, that is, $D \in \text{Der}_B(L)$.

Conversely, if $D \in \text{Der}_B(L)$ is invertible. Defines $\omega : L \times L \rightarrow F$ by Eq.(3.2). Then by the above discussion, $\omega$ is non-degenerate, and satisfies Eq.(3.1). The result follows. $\square$
Remark 1 One might thus think that every symplectic 3-Lie algebra \((A, \omega)\) admitting an invertible derivation which is skew-symmetric for \(\omega\) carries a metric structure; but this is not the case. Let \(A\) be a 4-dimensional 3-Lie algebra, the multiplication in a basis \(\{x_1, x_2, x_3, x_4\}\) be defined by

\[
[x_1, x_2, x_4] = x_3.
\]

Then the non-degenerate skew-symmetric bilinear form on \(A\) given by

\[
\omega(x_1, x_4) = \omega(x_2, x_3) = 1
\]

provides a symplectic structure on \(A\), and the linear endomorphism of \(A\) given by

\[
D(x_1) = 2x_1, D(x_2) = -x_2, D(x_3) = -x_3, D(x_4) = -2x_4
\]

is a skew-symmetric derivation of \((A, \omega)\). But for every symmetric bilinear form \(B : A \times A \to F\) satisfying Eq.(2.1), \(B\) satisfies

\[
B(x_3, x_3) = B(x_3, x_1) = B(x_3, x_2) = B(x_3, x_4) = 0.
\]

Therefore, \(B\) is degenerated. It follows that there does not exist metric structure on the 3-Lie algebra \(A\).

Under the assumptions of Theorem 3.1, the skew-symmetric derivation \(D \in \text{Der}_B(L)\) is also skew-symmetric with respect to the symplectic form \(\omega\) since for all \(x, y \in L\),

\[
\omega(Dx, y) = B(D^2x, y) = -B(Dx, Dy) = -\omega(x, Dy).
\]

Now for arbitrary 3-Lie algebra \(L\) and a positive integer \(n (n > 2)\), we construct a metric symplectic 3-Lie algebra. Let \(N\) be the set of all non-negative integers,

\[
F[t] = \{f(t) = \sum_{i=0}^{m} a_i t^i \mid a_i \in F, m \in N\}
\]

be the algebra of polynomials over \(F\). We consider the tensor product of vector spaces

\[
L_n = L \otimes (tF[t]/t^nF[t]),
\]

where \(tF[t]/t^nF[t]\) is the quotient space of \(tF[t]\) module \(t^nF[t]\). Then \(L_n\) is a nilpotent 3-Lie algebra in the following multiplication

\[
[x \otimes t^p, y \otimes t^q, z \otimes t^r] = [x, y, z]_L \otimes t^{p+q+r}, x, y, z \in L; p, q, r \in N \setminus \{0\}.
\]

Defines endomorphism \(D\) of \(L_n\) by

\[
D(x \otimes t^p) = p(x \otimes t^p), \forall x \in L, p = 1, \ldots, n - 1.
\]

Then \(D\) is an invertible derivation of the 3-Lie algebra \(L_n\).

Let \(\tilde{L}_n = L_n \oplus L_n^*\), where \(L_n^*\) is the dual space of \(L_n\). Then \((\tilde{L}_n, B)\) is a metric 3-Lie algebra with the multiplication

\[
[x + f, y + g, z + h] = [x, y, z]_{L_n} + ad^*(y, z)f - ad^*(x, z)g + ad^*(x, y)h,
\]
for $x, y, z \in L_n, f, g, h \in L_n^*$, and the bilinear form
\[ B(x + f, y + g) = f(y) + g(x). \] (3.6)

Defines linear mapping $\tilde{D} : \tilde{L}_n \to \tilde{L}_n$ by
\[ \tilde{D}(x + f) = Dx + D^*f, \forall x \in L_n, f \in L_n^* \] (3.7),
where $D^*f = -fD$. Then $\tilde{D}$ is an invertible, and by the direct computation, we have $\tilde{D} \in \text{Der}_B(\tilde{L}_n)$. Hence the metric 3-Lie algebra $(\tilde{L}_n, B)$ admits a symplectic structure $\omega$ as follows
\[ \omega(x + f, y + g) = B(\tilde{D}(x + f), y + g) = -f(Dy) + g(Dx). \] (3.8)

**Remark 2** By above discussion, from an arbitrary 3-Lie algebra, we can construct infinitely many metric symplectic 3-Lie algebras.

### 4. Symplectic Structures of $T^*_\theta$-extensions

In papers [12, 13], authors studied the extensions and module-extensions of 3-Lie algebras. In this section we need $T^*_\theta$-extension of 3-Lie algebras to describe the symplectic structures.

**Lemma 4.1** [12] Let $A$ be a 3-Lie algebra over a field $F$, $A^*$ be the dual space of $A$, $\theta : A \wedge A \wedge A \to A^*$ be a linear mapping satisfying
\[ \theta([x, u, v], y, z) + \theta([y, u, v], z, x) + \theta(x, y, [z, u, v]) = \theta([x, y, z], u, v). \] (4.1)

Then $T^*_\theta A = A \oplus A^*$ is a 3-Lie algebra in the following multiplication
\[ [x + f, y + g, z + h] = [x, y, z]_A + \theta(x, y, z) + ad^*(y, x) + ad^*(z, x)g + ad^*(x, y)h, \] (4.2)
where $x, y, z \in A, f, g, h \in A^*$. The 3-Lie algebra $T^*_\theta A$ is called the $T^*_\theta$-extension of the 3-Lie algebra $A$ by means of $\theta$.

If further, $\theta$ satisfies
\[ \theta(x_1, x_2, x_3)(x_4) + \theta(x_1, x_2, x_4)(x_3) = 0, \] (4.3)
for all $x_1, x_2, x_3, x_4 \in A$, then the symmetric bilinear form $B$ on $T^*_\theta A$ given by
\[ B(x + f, y + g) = f(y) + g(x), \ x, y \in A, f, g \in A^*, \] (4.4)
defines a metric structure on $T^*_\theta A$.

**Theorem 4.2** Let $A$ be a 3-Lie algebra admitting an invertible derivation $D$, and $\theta : A \wedge A \wedge A \to A^*$ be a linear mapping satisfying Eqs.(4.1) and (4.3). If there exists a linear mapping $\Psi : A \wedge A \to F$ satisfying for $x, y, z, u \in A$,
\[ \Theta(x, y, z, u) = \Psi([x, y, z]) - \Psi([y, x, u]) + \Psi([z, x, y]) - \Psi([u, x, y]), \] (4.5)
where
\[
\Theta(x, y, z, u) = \theta(Dx, y, z)u - \theta(Dy, z, u)x + \theta(Dz, u, x)y - \theta(Du, x, y)z,
\]  
then the metric 3-Lie algebra \( T^*_\theta A \) admits a symplectic structure.

**Proof.** Let \( B \) be the metric on the 3-Lie algebra \( T^*_\theta A \) defined in Eq.(4.4). By Theorem 3.1, it suffices to prove that the existence of an invertible skew-symmetric derivation of the metric 3-Lie algebra \( (T^*_\theta A, B) \).

Defines a linear mappings \( H: A \to A^* \) and \( \bar{D}: T^*_\theta A \to T^*_\theta A \), respectively, by
\[
B(Hx, y) = \Psi(x, y), \ \forall x, y \in A,
\]
and
\[
\bar{D}(x + f) = Dx - Hx - fD, \ \forall x \in A, f \in A^*.
\]

It is straightforward to see that \( \bar{D} \) is invertible, since \( D \) is so. And
\[
B(\bar{D}(x + f), y + g) = B(Dx - Hx - fD, y + g) = g(Dx) - f(Dy) - F(x, y),
\]
\[
B(x + f, \bar{D}(y + g)) = B(x + f, Dy - Hy - gD) = -g(Dx) + f(Dy) - F(y, x).
\]

Therefore, \( \bar{D} \) is skew-symmetric with respect to the metric \( B \).

Further, since \( D \) is a derivation of \( A \), for \( x, y, z \in A \) and \( f, g, h \in A^* \) we get
\[
[Dx - Hx - fD, y + g, z + h] + [x + f, \bar{D}(y + g), z + h]
\]
\[
+ [x + f, y + g, \bar{D}(z + h)] - \bar{D}[x + f, y + g, z + h]
\]
\[
= [Dx - Hx - fD, y + g, z + h] + [x + f, Dy - Hy - gD, z + h]
\]
\[
+ [x + f, y + g, Dz - Hz - hD] - \bar{D}([x, y, z] + \theta(x, y, z)
\]
\[
+ ad^*(y, z)f + ad^*(z, x)g + ad^*(x, y)h)
\]
\[
= [Dx, y, z] + \theta(x, y, z) - ad^*(y, z)(Hx + fD) + ad^*(z, Dx)g
\]
\[
+ ad^*(Dx, y)h + [x, Dy, z] + \theta(x, Dy, z) + ad^*(Dy, z)f
\]
\[
- ad^*(z, x)(Hy + gD) + ad^*(x, Dy)h + [x, y, Dz] + \theta(x, y, Dz)
\]
\[
+ ad^*(y, Dz)f + ad^*(Dz, x)g - ad^*(x, y)(Hz + hD) - D[x, y, z]
\]
\[
+ H[x, y, z] + \theta(x, y, z)D - D^*ad^*(y, z)f - D^*ad^*(z, x)g - D^*ad^*(x, y)h
\]
\[
= \theta(Dx, y, z) + \theta(x, Dy, z) + \theta(x, y, Dz) + \theta(x, y, z)D - ad^*(y, z)Hx
\]
\[
- ad^*(z, x)Hy - ad^*(x, y)Hz + H[x, y, z].
\]

From Eqs.(4.5) and (4.6) and \( \Psi(x, y) = B(Hx, y) = Hx(y) \) for all \( x, y \in A \), for arbitrary \( u \in A \),
\[
\theta(Dx, y, z)u + \theta(x, Dy, z)u + \theta(x, y, Dz)u + \theta(x, y, z)Du
\]
\[ +B(Hx, [y, z, u]) + B(Hy, [z, x, u]) + B(Hz, [x, y, u]) + B(H[x, y, z], u) \]
\[ = \Theta(x, y, z, u) + \Psi(x, [y, z, u]) - \Psi(y, [x, z, u]) + \Psi(z, [x, y, u]) - \Psi(u, [x, y, z]) = 0. \]

Therefore, \( \bar{D} \) is an invertible derivation of \( T^*_\theta A \). The proof is completed. \( \square \)

**Lemma 4.3** Let \( A \) be a nilpotent 3-Lie algebra over \( F \), \( I \) be a nonzero ideal of \( A \). Then \( I \cap Z(A) \neq 0 \).

**Proof.** If \( A \) is abelian, the result is evident.

If \( A \) is non-abelian, and \( I \) is a nonzero ideal of \( A \). Then for every \( x, y \in A \), the left multiplication \( ad(x, y) : A \to A \) is nilpotent ([14]). Therefore, the inner derivation algebra \( ad(A) \) of the 3-Lie algebra \( A \) is constituted by nilpotent mappings. Since \( ad(x, y)(I) \subseteq I \), for all \( x, y \in A \), by Theorem 3.3 in [15], there exists non-zero element \( z \in I \) such that \( ad(x, y)(z) = 0, \forall x, y \in L. \) Therefore, \( z \in I \cap Z(A). \) \( \square \)

**Lemma 4.4** Let \( (A, B) \) be a non-abelian nilpotent metric 3-Lie algebra over \( F \). Then there exists a non-zero isotropic ideal of \( A \).

**Proof.** Denotes \( J = A^1 \cap Z(A) \). By Lemma 4.3 \( J \) is a non-zero ideal of \( A \). Thanks to Lemma 2.3 in paper [9], \( Z(A)^\perp = A^1 = [A, A, A] \). Then, \( J \subseteq J^\perp \), that is, \( J \) is a non-zero isotropic ideal of \( A \). \( \square \)

**Lemma 4.5** [12] Let \( (L, B) \) be a nilpotent metric 3-Lie algebra of dimension \( m \). If \( J \) is an isotropic ideal of \( L \), then \( L \) contains a maximally isotropic ideal \( I \) of dimension \( \left\lceil \frac{m}{2} \right\rceil \) containing \( J \). Moreover,

1) If \( m \) is even, then \( L \) is isometric to some \( T^*_\theta \)-extension of \( L/I \).
2) If \( m \) is odd, then the ideal \( I^\perp \) is an abelian ideal of \( L \), and \( L \) is isometric to a non-degenerate ideal of codimension one in some \( T^*_\theta \)-extension of \( L/I \).

**Theorem 4.6** Let \( (L, B) \) be a non-abelian nilpotent metric 3-Lie algebra over an algebraically closed field \( F \) which admits a skew-symmetric invertible derivation \( \bar{D} \). Then there exists a 3-Lie algebra \( A \), an invertible derivation \( D \) of \( A \) and \( \theta : A \wedge A \wedge A \to A^* \) satisfying Eq.(4.1) such that \( L = T^*_\theta A \). And There exists \( \Psi : A \wedge A \to F \) such that \( \Theta(x, y, z, u) \) defined by Eq.(4.6) satisfying Eq.(4.5).

**Proof.** By Lemma 4.3 and Lemma 4.4, \( I = L^1 \cap Z(L) \) is a non-zero isotropic characteristically ideal of the 3-Lie algebra \( L \). From Theorem 3.1, there exists a non-degenerate skew-symmetric bilinear form \( \omega \) on \( L \) such that the invertible derivation \( \bar{D} \) satisfies

\[ \omega(\bar{D}x, y) + \omega(x, \bar{D}y) = 0. \]

Therefore, the dimension of the 3-Lie algebra \( L \) is even.
Since the 3-Lie algebra $L$ is nilpotent, the inner derivation algebra $Ad(L)$ is a nilpotent Lie algebra. Then the Lie algebra $T = Ad(L) \oplus F \bar{D}$ is solvable. By Lemma 3.2 in [10] and Lemma 4.5, there exists a maximal isotropic ideal $J$ containing the isotropic ideal $I = L^1 \cap Z(L)$, and $\theta : (L/J) \cap (L/J) \cap (L/J) \rightarrow (L/J)$ satisfying Eq.(4.1) such that the metric 3-Lie algebra $(L, B)$ is isomorphic to the $T^*_\theta$-extension $T^*_\theta(L/J)$, and $J$ is stable by $\bar{D}$. Let $J'$ be a complement of $J$ in the vector space $L$, that is, $L = J' \oplus J$. Then for every $x \in J$, $y \in J'$, we have $\bar{D}(x) \in J$ and $\bar{D}(y) = y_1 + y_2$, where $y_1 \in J'$ and $y_2 \in J$. Denotes the 3-Lie algebra $L/J$ by $A$. Then $A^*$ is isomorphic to $J$ as subspaces and it is stable by $\bar{D}$.

Therefore, we can define linear mappings $D_{11} : A \rightarrow A$, $D_{21} : A \rightarrow A^*$, and $D_{22} : A^* \rightarrow A^*$ by

$$D(x + f) = D_{11}x + D_{21}x + D_{22}f, \forall x \in A, f \in A^*. \quad (4.7)$$

Clearly, $D_{11}$ and $D_{22}$ must be invertible since $\bar{D}$ is so. And for every $x, y \in A, f, g \in A^*$

$$0 = B(\bar{D}(x + f), y + g) + B(x + f, \bar{D}(y + g))$$
$$= B(D_{11}x + D_{21}x + D_{22}f, y + g) + B(x + f, D_{11}y + D_{21}y + D_{22}g)$$
$$= g(D_{11}x) + D_{21}x(y) + D_{22}f(y) + f(D_{11}y) + D_{21}y(x) + D_{22}g(x). \quad (4.8)$$

From the above equation, we obtain that in the case $x = 0, g = 0$,

$$D_{22}f(y) = -fD_{11}(y), \forall y \in A, f \in a^*,$$

and in the case $f = g = 0$,

$$B(D_{21}x, y) + B(D_{21}y, x) = 0, \forall x, y \in A.$$

Let $H = -D_{21} : A \rightarrow A^*$ and $D = D_{11} : A \rightarrow A$. Since $\bar{D}$ is a derivation of $L$, by Eq.(4.2)

$$0 = [\bar{D}x, y, z] + [x, \bar{D}y, z] + [x, y, \bar{D}z] - \bar{D}[x, y, z]$$

$$= [Dx - Hx, y, z] + [x, Dy - Hy, z] + [x, y, Dz - Hz] - \bar{D}([x, y, z] + \theta(x, y, z))$$

$$= [Dx, y, z] + \theta(Dx, y, z) - ad^*(y, z)Hx + [x, Dy, z] + \theta(x, Dy, z) - ad^*(x, z)Hy$$

$$+ [x, y, Dz] + \theta(x, y, Dz) - ad^*(x, y)Hz - D[x, y, z] + H[x, y, z] + \theta(x, y, z)D$$

$$= [Dx, y, z] + [x, Dy, z] + [x, y, Dz] - D[x, y, z]$$

$$+ \theta(Dx, y, z) + \theta(x, Dy, z) + \theta(x, y, Dz) + \theta(x, y, z)D$$

$$- ad^*(y, z)Hx - ad^*(z, x)Hy - ad^*(x, y)Hz + H[x, y, z], \forall x, y, z \in A.$$

Therefore, we have

$$[Dx, y, z] + [x, Dy, z] + [x, y, Dz] - D[x, y, z] = 0, \forall x, y, z \in A, \quad (4.9)$$

$$\theta(Dx, y, z) + \theta(x, Dy, z) + \theta(x, y, Dz) + \theta(x, y, z)D$$
Therefore, \( D \) is an invertible derivation of \( A \). Denotes
\[
\Theta(x, y, z, u) = \theta(Dx, y, z)u - \theta(Dy, z, u)x + \theta(Dz, u, x)y - \theta(Du, x, y)z, \quad \forall x, y, z, u \in A.
\]
Defines bilinear mapping \( \Psi : A \times A \to F \) by
\[
\Psi(x, y) = -B(Hx, y) = -Hx(y), \quad \forall x, y \in A.
\]
Then \( \Psi \) is skew-symmetric and satisfies \( \forall x, y, z, \omega \in A, \)
\[
\Theta(x, y, z, \omega) + (\Psi(x, [y, z, \omega]) - \Psi(y, [x, z, \omega]) + \Psi(z, [x, y, \omega]) - \Psi(\omega, [x, y, z])) = 0.
\]
The result follows. \( \square \)

The following result gives a characterization of 3-Lie algebras admitting an invertible derivation. Note that the result is valid for an arbitrary base field of characteristic zero (not necessarily algebraically closed).

**Theorem 4.7** Let \( A \) be a 3-Lie algebra over a field \( F \) with a characteristic zero. Then there exists an invertible derivation \( D \) of \( A \) if and only if \( A \) is isomorphic to the quotient 3-Lie algebra \( L/J \) of a metric symplectic 3-Lie algebra \((L, B, \omega)\) by a lagrangian and completely isotropic ideal \( J \).

**Proof.** If \( A \) admits an invertible derivation. From Theorem 4.2, let \( \theta = 0, \Psi = 0, H = 0 \) then the 3-Lie algebra \( L = A \oplus A^* \) obtained by \( T^*_0 \)-extension of \( A \) is a metric symplectic 3-Lie algebra.

We define
\[
\bar{D} : L \to L, \quad \bar{D}(x + f) = Dx - fD, \quad \forall x \in A, f \in A^*,
\]
and
\[
\omega(x + f, y + g) = B(\bar{D}(x + f), y + g) = g(Dx) - f(Dy), \quad x, y \in A, f, g \in A^*.
\]
Then \( J = A^* \) is a lagrangian ideal of the symplectic 3-Lie algebra \((L, \omega)\), and is a completely isotropic ideal of the metric 3-Lie algebra \((L, B)\), and \( A \) is isomorphic to the quotient 3-Lie algebra \( L/J \).

Conversely, suppose that the 3-Lie algebra \( A \) is isomorphic to \( L/J \), where \((L, B, \omega)\) is a metric symplectic 3-Lie algebra and \( J \) is a lagrangian completely isotropic ideal of \( L \). By Theorem 3.4 in [12], \( L \) is isometrically isomorphic to \( T^*_0(L/J) = T^*_0A \) since \( J \) is completely isotropic. From Theorem 3.1, there exists a skew-symmetric invertible derivation \( \bar{D} \) of the metric 3-Lie algebra \((L, B)\). From Eq.(3.2), \( \bar{D}(J) = J \). Then by the same argument used in the proof of Theorem 4.6, the projection \( \bar{D}|_A : A \to A \) provides a non-singular derivation of \( A \). \( \square \)
At last of the paper, we give the characterization of metric symplectic double extensions of 3-Lie algebras.

**Lemma 4.8** \([13]\) Let \((A, B)\) be a metric 3-Lie algebra, \(b\) be another 3-Lie algebra and \(\pi = ad^* : b \times b \to \text{End}(b^*)\) be the coadjoint representation of \(b\). Suppose that \((A, \psi)\) is a representation of \(b\), where \(\psi : b \wedge b \to \text{End}(A)\) satisfies \(\psi(b, b) \subseteq \text{Der}_B(A)\). Let \(\tilde{A} = b^* \oplus A \oplus b\), and \(\phi : A \otimes A \otimes b \to b^*\) defined by for any \(x_1, x_2 \in A, y, z \in b\)

\[
\phi(x_1, x_2)(y)(z) = -\phi(x_1, x_1, y)(z) = B(\psi(y, z)x_1, x_2).
\]

If \(\psi\) satisfies \(\psi(b^1, b)(A) = \psi(b, b)(A^1) = 0\). Then \((\tilde{A}, T)\) is a metric 3-Lie algebra in the following multiplication, \(\forall y_1, y_2, y_3 \in b, \forall x_1, x_2, x_3 \in A, \forall f_1, f_2, f_3 \in b^*\),

\[
[y_1 + x_1 + f_1, y_2 + x_2 + f_2, y_3 + x_3 + f_3] = [y_1, y_2, y_3]_b + [x_1, x_2, x_3]_A + \psi(y_2, y_3)x_1 - \psi(y_1, y_3)x_2 + \psi(y_1, y_2)x_3 + \pi(y_2, y_3)f_1 - \pi(y_1, y_3)f_2 + \pi(y_1, y_2)f_3 + \phi(x_1, x_2, y_3) - \phi(x_1, x_3, y_2) + \phi(x_2, x_3, y_1).
\]

(4.11)

\[
T(y_1 + x_1 + f_1, y_2 + x_2 + f_2) = B(x_1, x_2) + f_1(y_2) + f_2(y_1).
\]

(4.12)

\[
\Box
\]

In Lemma 4.8, if \(b = F_e_1 + F_e_2\) is a two-dimensional 3-Lie algebra, then

\[
\psi : b \wedge b \to A, \psi(e_1, e_2) = \delta \in \text{Der}_B(A).
\]

Therefore, \(\phi : A \otimes A \otimes b \to b^*\) defined by for any \(x_1, x_2 \in A, e_1, e_2 \in b\)

\[
\phi(x_1, x_2)(e_2) = -\phi(x_1, x_1, e_2) = B(\psi(e_1, e_2)x_1, x_2) = B(\delta x_1, x_2).
\]

(4.13)

\[
\phi(x_1, x_2)(e_1) = -B(\delta x_1, x_2), \ \phi(x_1, x_2)(e_1) = \phi(x_1, x_2, e_1)(e_2) = B(\delta x_1, x_2) = 0.
\]

(4.13')

Then we say that \((\tilde{A} = F_e_1 + F_e_2 \oplus A \oplus F_{e_1^*} + F_{e_2^*}, T)\) is the double extension of \(A\) by means of the derivation \(\psi(e_1, e_2) = \delta\), and the multiplication is for \(\forall x, y, z, x, y, z \in A, \alpha, \alpha', \beta, \beta', \gamma_1, \gamma_1', \gamma_2, \gamma_2' \in F, e_1^*, e_2^* \in b^*\) (where \(e_j^*(e_j) = \delta_{ij}, 1 \leq 1, j \leq 2\)),

\[
[\alpha e_1 + x + \alpha' e_1^*, \beta e_2 + y + \beta' e_2^*, \gamma_1 e_1 + \gamma_2 e_2 + z + \gamma_1' e_1^* + \gamma_2' e_2^*] = [x, y, z] + \delta(-\beta \gamma_1 x - \alpha \gamma_2 y + \alpha \beta z) + \phi(x, y, \gamma_1 e_1 + \gamma_2 e_2) - \phi(x, z, \beta e_2) + \phi(y, z, \alpha e_1),
\]

(4.14)

and the metric is

\[
T(\alpha e_1 + x + \alpha' e_1^*, \beta e_2 + y + \beta' e_2^*) = B(x, y) + \alpha \beta' + \beta \alpha'.
\]

(4.15)

By the above notations we have the following result.
Theorem 4.9 Let \((A, B)\) be a metric 3-Lie algebra, \(D\) be an invertible derivation of \(A\) and \(D \in \text{Der}_B(A)\), and \(\delta \in \text{Der}_B(A)\) satisfy
\[
\delta D - D\delta = 2\delta. \tag{4.16}
\]
Let \((\hat{A} = b \oplus A \oplus b^* , T)\) be the double extension of \(A\) by means of the derivation \(\delta\), where \(b = Fe_1 + Fe_2\) be the 2-dimensional 3-Lie algebra. Then the linear endomorphism \(\tilde{D}\) of \(\hat{A}\) defined by
\[
\tilde{D}|_A = D, \quad \tilde{D}e_i = -e_i, \quad \tilde{D}e_i^* = e_i^*, \quad i = 1, 2 \tag{4.17}
\]
is an invertible derivation of the 3-Lie algebra \((\hat{A}, T)\), and \(\tilde{D} \in \text{Der}_T(\hat{A})\).

Proof Let \(\psi : b \wedge b \to A\), \(\psi(e_1, e_2) = \delta \in \text{Der}_B(A)\). By the above discussion, \((\hat{A} = b \oplus A \oplus b^* , T)\) is the double extension of \(A\) by means of the derivation \(\delta\).

By Eq.(4.17), the linear mapping \(\tilde{D} : \hat{A} \to \hat{A}\) is invertible. From Lemma 4.8 and Eq.(4.14), \(\forall x, y, z \in A, \alpha, \alpha', \beta, \beta', \gamma_1, \gamma_1', \gamma_2, \gamma_2' \in F,\)
\[
\tilde{D}[\alpha e_1 + x + \alpha' e_1^*, \beta e_2 + y + \beta' e_2^*, \gamma_1 e_1 + \gamma_2 e_2 + z + \gamma_1' e_1^* + \gamma_2' e_2^*] = D[x, y, z] + D\delta(-\beta\gamma_1 x - \alpha\gamma_2 y + \alpha'\beta z) + \phi(x, y, \gamma_1 e_1 + \gamma_2 e_2) - \phi(x, z, \beta e_2) + \phi(y, z, \alpha e_1).
\]
Thanks to Eqs.(4.16) and (4.17),
\[
[D(x, y, z) + [x, Dy, z] + [x, y, Dz] + \delta D(-\beta\gamma_1 x - \alpha\gamma_2 y + \alpha'\beta z) - 2\delta(-\beta\gamma_1 x - \alpha\gamma_2 y + \alpha'\beta z) + \phi(Dx, y, \gamma_1 e_1 + \gamma_2 e_2) - \phi(Dx, z, \beta e_2) + \phi(y, z, -\alpha e_1) + \phi(x, Dy, \gamma_1 e_1 + \gamma_2 e_2) - \phi(x, z, -\beta e_2) + \phi(Dy, z, \alpha e_1) + \phi(x, y, -\gamma_1 e_1 - \gamma_2 e_2) - \phi(x, Dz, \beta e_2) + \phi(y, Dz, \alpha e_1)]
\]
\[
= D[x, y, z] + D\delta(-\beta\gamma_1 x - \alpha\gamma_2 y + \alpha'\beta z) + \phi(Dx, y, \gamma_1 e_1 + \gamma_2 e_2) - \phi(Dx, z, \beta e_2) + \phi(y, z, \alpha e_1) + \phi(x, Dy, \gamma_1 e_1 + \gamma_2 e_2) + \phi(x, z, \beta e_2) + \phi(Dy, z, \alpha e_1)
\]
\(-\phi(x, y, \gamma_1 e_1 + \gamma_2 e_2) - \phi(x, Dz, \beta e_2) + \phi(y, Dz, \alpha e_1)\).

From Eqs. (4.13) and (4.16),

\[
(\phi(Dx, y, \gamma_1 e_1 + \gamma_2 e_2) + \phi(x, Dy, \gamma_1 e_1 + \gamma_2 e_2) - \phi(x, y, \gamma_1 e_1 + \gamma_2 e_2))(e_1)
= B(-\gamma_2 \delta Dx, y) + B(-\gamma_2 \delta x, Dy) + B(\gamma_2 \delta x, y)
= B(-\gamma_2 \delta Dx, y) + B(\gamma_2 D \delta x, y) + B(\gamma_2 \delta x, y)
= -\gamma_2 B((\delta D - D \delta - 2 \delta)x, y) - B(\gamma_2 \delta x, y)
\]

\[
= B(-\gamma_2 \delta x, y) = \phi(x, y, \gamma_1 e_1 + \gamma_2 e_2))(e_1),
\]

\[
(\phi(Dx, y, \gamma_1 e_1 + \gamma_2 e_2) + \phi(x, Dy, \gamma_1 e_1 + \gamma_2 e_2) - \phi(x, y, \gamma_1 e_1 + \gamma_2 e_2))(e_2)
= B(\gamma_1 \delta Dx, y) + B(\gamma_1 \delta x, Dy) - B(\gamma_1 \delta x, y)
= \gamma_1 B((D \delta - D \delta - 2 \delta)x, y) + B(\gamma_1 \delta x, y)
= B(\gamma_1 \delta x, y) = \phi(x, y, \gamma_1 e_1 + \gamma_2 e_2))(e_2).
\]

Then we have

\[
\phi(Dx, y, \gamma_1 e_1 + \gamma_2 e_2) + \phi(x, Dy, \gamma_1 e_1 + \gamma_2 e_2) - \phi(x, y, \gamma_1 e_1 + \gamma_2 e_2) = \phi(y, z, \gamma_1 e_1 + \gamma_2 e_2).
\]

Similarly,

\[
-\phi(Dx, z, \beta e_2) + \phi(x, z, \beta e_2) - \phi(x, Dz, \beta e_2) = -\phi(x, z, \beta e_2),
\]

\[
-\phi(y, z, \alpha e_1) + \phi(Dy, z, \alpha e_1) + \phi(y, Dz, \alpha e_1) = \phi(y, z, \alpha e_1).
\]

Therefore, \(\tilde{D}\) satisfies

\[
\tilde{D}[\alpha e_1 + x + \alpha' e_1^* + \beta e_2 + y + \beta' e_2^*, \gamma_1 e_1 + \gamma_2 e_2 + z + \gamma_1 e_1^* + \gamma_2 e_2^*]
= \tilde{D}[\alpha e_1 + x + \alpha' e_1^*, \beta e_2 + y + \beta' e_2^*, \gamma_1 e_1 + \gamma_2 e_2 + z + \gamma_1 e_1^* + \gamma_2 e_2^*]
+ [\alpha e_1 + x + \alpha' e_1^*, \tilde{D}(\beta e_2 + y + \beta' e_2^*), \gamma_1 e_1 + \gamma_2 e_2 + z + \gamma_1 e_1^* + \gamma_2 e_2^*]
+ [\alpha e_1 + x + \alpha' e_1^*, \beta e_2 + y + \beta' e_2^*, \tilde{D}(\gamma_1 e_1 + \gamma_2 e_2 + z + \gamma_1 e_1^* + \gamma_2 e_2^*)].
\]

Again by Eqs. (4.15) and (4.17),

\[
T(\tilde{D}(\alpha e_1 + \beta e_2 + \epsilon x + \alpha' e_1^* + \beta' e_2^*), \lambda e_1 + \mu e_2 + \nu y + \lambda' e_1^* + \nu' e_2^*)
+ T(\alpha e_1 + \beta e_2 + \epsilon x + \alpha' e_1^* + \beta' e_2^*, \tilde{D}(\lambda e_1 + \mu e_2 + \nu y + \lambda' e_1^* + \nu' e_2^*))
= T(-\alpha e_1 - \beta e_2 + \epsilon Dx + \alpha' e_1^* + \beta' e_2^*, \lambda e_1 + \mu e_2 + \nu y + \lambda' e_1^* + \nu' e_2^*)
+ T(\alpha e_1 + \beta e_2 + \epsilon x + \alpha' e_1^* + \beta' e_2^*, -\lambda e_1 - \mu e_2 + \nu Dy + \lambda' e_1^* + \nu' e_2^*)
\]

\[
= B(\epsilon Dx, \nu y) + B(\epsilon x, \nu Dy) - \alpha' \lambda' - \beta' \mu + \alpha' \lambda + \beta' \mu + \alpha \lambda + \beta \mu - \alpha' \lambda - \beta' \mu = 0.
\]

Summarizing above discussion, we obtain that \(\tilde{D}\) is an invertible derivation of the metric 3-Lie algebra \((\tilde{\mathcal{A}}, T)\) and \(\tilde{D} \in Der_T(\tilde{\mathcal{A}})\). \(\square\)
If \((A, B)\) be a metric 3-Lie algebra and \(D \in \text{Der}_B(A)\) is invertible. From Eq.(3.2), \((A, B, \omega)\) is a metric symplectic 3-Lie algebra, where \(\omega(x, y) = B(Dx, y), \forall x, y \in A\). Then we obtain the following result.

**Corollary** Let \((A, B)\) be a metric 3-Lie algebra, \(D\) be an invertible derivation of \(A, D \in \text{Der}_B(A)\) and \(\delta \in \text{Der}_B(A)\) satisfy Eq.(4.16). Then the 3-Lie algebra \((\tilde{A}, T, \tilde{\omega})\) is a metric symplectic 3-Lie algebra, which is called the metric symplectic double extension of \((A, B, \omega)\), where \((\tilde{A}, T)\) is the double extension of \((A, B)\) by means of \(\delta\), and \(\tilde{\omega}\) is defined by

\[
\tilde{\omega}(x, y) = \omega(x, y), \quad \tilde{\omega}(e_1, e_2^*) = \tilde{\omega}(e_2, e_1^*) = -1, \quad \forall x, y \in A.
\]  

(4.18)

**Proof** The result follows from Theorem 4.9 and Theorem 3.1, directly. □

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