Construction of Lumps with nontrivial interaction

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Abstract
We develop a method based upon the Singular Manifold Method that yields an iterative and analytic procedure to construct solutions for a Bogoyavlenskii-Kadomtsev-Petviashvili equation. This method allows us to construct a rich collection of lump solutions with a nontrivial evolution behavior.

1 Introduction
In recent years, it has been proven in several papers [1], [2], [3] that the KPI equation contains a whole manifold of smooth rationally decaying “lump” configurations associated with higher-order pole meromorphic eigenfunctions. These configurations have an interesting dynamics and the lumps may scatter in a nontrivial way. Furthermore, algorithmic methods, based upon the Painlevé property, have been developed in order to construct lump-type solutions for different equations such as a 2 + 1 NLS (Nonlinear Schrödinger equation) [4], [5] equation and the KPI (Kadomtsev-Petviashvili equation) and GDLW (Generalized Dispersive Wave Equation) equations [6].

The present contribution is related to the construction of lump solutions for the 2 + 1 dimensional equation [7]
\[ (4u_x t + u_{xxx} + 8u_x u_y + 4u_x u_y)_x + \sigma u_{yy} = 0, \quad \sigma = \pm 1 \] (1)
which represents a modification of the Calogero-Bogoyavlenskii-Schiff (CBS) equation [8], [9], [10]:
\[ 4u_x t + u_{xxx} + 8u_x u_y + 4u_x u_y = 0. \]

Equation (1) has often been called the Bogoyavlenskii-Kadomtsev-Petviashvili (KP-B) equation [11].

As in the case of the KP equation, there are two versions of (1), depending upon the sign of \( \sigma \). Here we restrict ourselves to the minus sign. Therefore:
\[ (4u_x t + u_{xxx} + 8u_x u_y + 4u_x u_y)_x - u_{yyy} = 0 \] (2)
or
\[ 4u_x t + u_{xxx} + 8u_x u_y + 4u_x u_y = \omega_{yy}, \quad u_y = \omega_x. \] (3)

We refer to (3) as KP-BI in what follows.

In Section 2 we summarize the results that the singular method provides for KP-BI. These results are not essentially new because they were obtained by the author in [11] for the KP-BII version of the equation. Section 3 is devoted to the construction of rational solitons.
2 The Singular Manifold Method for KP-BI

It has been proven that (3) has the Painlevé property [7]. Therefore, the singular manifold method can be applied to it. In this section we adapt previous results obtained in [11] for KP-BII to KP-BI. This is why we are only present the main results with no detailed explanation since this has been shown in our earlier paper.

2.1 The singular Manifold Method

This method [12] requires the truncation of the Painlevé series for the fields \( u \) and \( \omega \) of (3) in the following form:

\[
\begin{align*}
  u^{[1]} &= u^{[0]} + \frac{\phi_x^{[0]}}{\phi^{[0]}}, \\
  \omega^{[1]} &= \omega^{[0]} + \frac{\phi_y^{[0]}}{\phi^{[0]}},
\end{align*}
\]

where \( \phi^{[0]}(x,y,t) \) is the singular manifold and \( u^{[i]}, \omega^{[i]} \) \( (i = 0,1) \) are solutions of (3). This means that (4) can be considered as an auto-Bäcklund transformation. The substitution of (4) in (3) yields a polynomial in negative powers of \( \phi^{[0]} \) that can be handled with MAPLE. The result (see [11]) is that we can express the seed solution \( u^{[0]}, \omega^{[0]} \) in terms of the singular manifold as follows:

\[
\begin{align*}
  u_x^{[0]} &= \frac{1}{4} \left( -v_x - \frac{v^2}{2} - z_y + \frac{z_x}{2} \right), \\
  u_y^{[0]} &= \omega_x^{[0]} = \frac{1}{4} \left( -r - 2v_y + 2z_x z_y \right),
\end{align*}
\]

where \( v, r \) and \( z \) are related to the singular manifold \( \phi \) through the following definitions

\[
\begin{align*}
  v &= \frac{\phi_x^{[0]}}{\phi^{[0]}}, \quad r = \frac{\phi_t^{[0]}}{\phi^{[0]}}, \quad z_x = \frac{\phi_y^{[0]}}{\phi^{[0]}},
\end{align*}
\]

Furthermore, the singular manifold \( \phi^{[0]} \) satisfies the equation

\[
s_y + r_x - z_{yy} - z_x z_{xy} - 2z_y z_{xx} = 0
\]

where \( s = v_x - \frac{v^2}{2} \) is the Schwartzian derivative.

2.2 Lax pair

Equations (5) can be linearized through the following definition of \( \psi^{[0]}(x,y,t) \), \( \chi^{[0]}(x,y,t) \) functions.

\[
\begin{align*}
  v &= \frac{\psi_x^{[0]}}{\psi^{[0]}} + \frac{\chi^{[0]}}{\chi^{[0]}}, \\
  z_x &= i \left( \frac{\psi_x^{[0]}}{\psi^{[0]}} \frac{\chi_x^{[0]}}{\chi^{[0]}} \right).
\end{align*}
\]
When one combines (5), (6) and (7), the following Lax pair arises:

\[
\psi^{[0]}_{xx} = -i\psi^{[0]}_y - 2u^{[0]}_x \psi^{[0]}_x
\]
\[
\psi^{[0]}_t = 2i\psi^{[0]}_{yy} - 4u^{[0]}_y \psi^{[0]}_x + (2u^{[0]}_{xy} + 2i\omega^{[0]}_y) \psi^{[0]}
\] (9)
together with its complex conjugate

\[
\chi^{[0]}_{xx} = i\chi^{[0]}_y - 2u^{[0]}_x \chi^{[0]}
\]
\[
\chi^{[0]}_t = -2i\chi^{[0]}_{yy} - 4u^{[0]}_y \chi^{[0]} + (2u^{[0]}_{xy} - 2i\omega^{[0]}_y) \chi^{[0]}.
\] (10)

In terms of \(\chi^{[0]}\) and \(\psi^{[0]}\), the derivatives of \(\phi^{[0]}\) are:

\[
v = \frac{\phi^{[0]}_x}{\phi^{[0]}_y} = \frac{\psi^{[0]}_x}{\chi^{[0]}} + \frac{\chi^{[0]}_x}{\chi^{[0]}} \Rightarrow \phi^{[0]}_y = \psi^{[0]} \chi^{[0]},
\]
\[
r = \frac{\phi^{[0]}_y}{\phi^{[0]}_x} = -4u^{[0]}_y + 2\psi^{[0]}_y \psi^{[0]}_x + 2\chi^{[0]}_y \psi^{[0]}_x - 2\psi^{[0]}_y \psi^{[0]}_x - 2\chi^{[0]}_y \chi^{[0]},
\] (11)
which allows us to write \(d\phi^{[0]}\) as:

\[
d\phi^{[0]} = \psi^{[0]} \chi^{[0]} dx + i \left( \chi^{[0]} \psi^{[0]} - \psi^{[0]} \chi^{[0]} \right) dy +
\]
\[
+ \left( -4u^{[0]}_y \psi^{[0]} \chi^{[0]} + 2\psi^{[0]}_y \chi^{[0]} + 2\psi^{[0]}_x \chi^{[0]} - 2\psi^{[0]}_y \chi^{[0]} - 2\psi^{[0]}_x \chi^{[0]} \right) dt.\] (12)

It is easy to check that the condition of the exact derivative in (12) is satisfied by the Lax pairs (9) and (10).

### 2.3 Darboux transformations

Let \((\psi_1^{[0]} , \chi_1^{[0]}), (\psi_2^{[0]} , \chi_2^{[0]})\) be two pairs of eigenfunctions of the Lax pair (9)-(10) corresponding to the seed solution \(u^{[0]}\), \(\omega^{[0]}\)

\[
\psi^{[0]}_{j,xx} = -i\psi^{[0]}_{j,y} - 2u^{[0]}_x \psi^{[0]}_x
\]
\[
\psi^{[0]}_{j,t} = 2i\psi^{[0]}_{j,yy} - 4u^{[0]}_y \psi^{[0]}_x + (2u^{[0]}_{xy} + 2i\omega^{[0]}_y) \psi^{[0]}
\] (13)

\[
\chi^{[0]}_{j,xx} = i\chi^{[0]}_{j,y} - 2u^{[0]}_x \chi^{[0]}
\]
\[
\chi^{[0]}_{j,t} = -2i\chi^{[0]}_{j,yy} - 4u^{[0]}_y \chi^{[0]} + (2u^{[0]}_{xy} - 2i\omega^{[0]}_y) \chi^{[0]}
\] (14)
where \(j = 1, 2\). These Lax pairs can be considered as nonlinear equations between the fields and the eigenfunction (11). This means that the Painlevé expansion of the fields

\[
u^{[1]} = u^{[0]} + \frac{\phi^{[0]}_1 x}{\phi^{[0]}_1 y}
\]
\[
\omega^{[1]} = \omega^{[0]} + \frac{\phi^{[0]}_1 y}{\phi^{[0]}_1 x}
\] (15)
should be accompanied by an expansion of the eigenfunctions and the singular manifold itself. These expansions are

\[
\psi_2^{[1]} = \psi_2^{[0]} - \psi_1^{[0]} \frac{\Omega_{1,2}}{\phi_1^{[0]}}, \\
\chi_2^{[1]} = \chi_2^{[0]} - \chi_1^{[0]} \frac{\Omega_{2,1}}{\phi_2^{[0]}}, \\
\phi_2^{[1]} = \phi_2^{[0]} - \frac{\Omega_{1,2}\Omega_{2,1}}{\phi_1^{[0]}},
\]

(16)

Substitution of (16) in (13-14) yields

\[
d\Omega_{i,j} = \psi_1^{[0]} \chi_1^{[0]} dx + i \left( \chi_1^{[0]} \psi_{j,x}^{[0]} - \psi_1^{[0]} \chi_{j,x}^{[0]} \right) dy + \left(-4u_y^0 \psi_1^{[0]} \chi_j^{[0]} + 4\psi_{j,y}^{[0]} \chi_1^{[0]} + 2\psi_{j,y}^{[0]} \chi_{j,x}^{[0]} - 2\chi_{1,y}^{[0]} \psi_{j,y}^{[0]} - 2\chi_{1,x}^{[0]} \psi_{j,x}^{[0]} \right) dt.
\]

(17)

Direct comparison of (12) and (17) affords \( \phi_1^{[0]} = \Omega_{i,j} \). Therefore, knowledge of the two seed eigenfunctions \( (\psi_j^{[0]}, \chi_j^{[0]}), j = 1, 2 \), allows us to compute the matrix elements \( \Omega_{i,j} \), which yields the Darboux transformation (15-16).

### 2.4 Iteration: \( \tau \)-functions

According to the above results, \( \phi_2^{[1]} \) is a singular manifold for the iterated fields \( u^{[1]}, \omega^{[1]} \). Therefore, the Painlevé expansion for these iterated fields can be written as

\[
u^{[2]} = u^{[1]} + \phi_{2,x}^{[1]} \phi_2^{[1]}, \\
\omega^{[2]} = \omega^{[1]} + \phi_{2,y}^{[1]} \phi_2^{[1]},
\]

(18)

which combined with (15) is:

\[
u^{[2]} = u^{[0]} + \frac{(\tau_{1,2})_x}{\tau_{1,2}}, \\
\omega^{[2]} = \omega^{[0]} + \frac{(\tau_{1,2})_y}{\tau_{1,2}},
\]

(19)

where \( \tau_{1,2} = \phi_2^{[1]} \phi_1^{[0]} \), which according to (15) allows us to write it as

\[
\tau_{1,2} = \phi_2^{[0]} \phi_1^{[0]} - \Omega_{1,2} \Omega_{2,1} = \det(\Omega_{i,j}).
\]

(20)

### 3 Lumps

The iteration method described above can be started from the most trivial initial solution \( u^{[0]} = \omega^{[0]} = 0 \). In this case, the lax pair, is:

\[
\psi_{j,x}^{[0]} = -i\psi_{j,y}^{[0]}, \\
\psi_{j,y}^{[0]} = 2i\psi_{j,y}^{[0]}, \\
\chi_j^{[0]} = i\chi_j^{[0]}, \\
\chi_{j,x}^{[0]} = -2i\chi_{j,yy}^{[0]},
\]

(21)
It is trivial to prove that equations (21) have the following solutions

\[
\psi_1^{[0]} = P_m(x, y, t; k) \exp \{ Q_0(x, y, t; k) \}, \\
\chi_1^{[0]} = P_n(x, y, t; k) \exp \{- Q_0(x, y, t; k) \}, \\
\psi_2^{[0]} = \left( \chi_1^{[0]} \right)^*, \\
\chi_2^{[0]} = \left( \psi_1^{[0]} \right)^* ,
\]

(22)

where \( m, n \) are arbitrary integers and \( k \) an arbitrary complex constant.

\[
Q_0(x, y, t; k) = k x + i k^2 y + 2i k^4 t \Rightarrow (Q_0(x, y, t; k))^* = k^* x - i(k^*)^2 y - 2i(k^*)^4 t
\]

(23)

and \( P_j(x, y, t; k) \) is defined as:

\[
P_j(x, y, t; k) \exp \{ Q_0(x, y, t; k) \} = \frac{\partial^j \left( P_{j-1}(x, y, t; k) \exp \{ Q_0(x, y, t; k) \} \right)}{\partial k^j}, \quad P_0 = 1.
\]

(24)

These solutions are characterized by two integers, \( n \) and \( m \) that provide a rich collection of different solutions corresponding to the same wave number \( k \). Thus in our opinion, all of them should be considered as one-soliton solutions despite the different behaviors shown by the solutions corresponding to the different combinations of \( n \) and \( m \). We now present some of these cases.
3.1 Lump (0,0): \( n = 0, \quad m = 0 \)

The eigenfunctions (22) are:

\[
\begin{align*}
\psi_{1}^{[0]} &= \exp [Q_{0}(x, y, t, k)], \\
\chi_{1}^{[0]} &= \exp [-Q_{0}(x, y, t, k)], \\
\psi_{2}^{[0]} &= \exp [-Q_{0}^{*}(x, y, t, k)], \\
\chi_{2}^{[0]} &= \exp [Q_{0}^{*}(x, y, t, k)].
\end{align*}
\]

The matrix elements (17) can be integrated as:

\[
\begin{align*}
\phi_{1}^{[0]} &= \Omega_{1,1} = x + 2iky - 8ik^{3}t, \\
\phi_{2}^{[0]} &= \Omega_{2,2} = x - 2ik^{*}y + 8i(k^{*})^{3}t, \\
\Omega_{1,2} &= -\frac{1}{k + k^{*}} \exp [-Q_{0}(x, y, t, k)] \exp [-Q_{0}^{*}(x, y, t, k)], \\
\Omega_{2,1} &= \frac{1}{k + k^{*}} \exp [Q_{0}(x, y, t, k)] \exp [Q_{0}^{*}(x, y, t, k)].
\end{align*}
\]

Therefore, the \( \tau \)-function (23) is the positive defined expression

\[
\tau_{1,2} = X_{1}^{2} + Y_{1}^{2} + \frac{1}{4a_{0}^{2}},
\]

where

\[
\begin{align*}
&k = a_{0} + ib_{0}, \\
&X_{1} = x - 2b_{0}y + 8b_{0} (3a_{0}^{2} - b_{0}^{2}) t, \\
&Y_{1} = 2a_{0} (y + 4(3a_{0}^{2} - b_{0}^{2})t).
\end{align*}
\]

The profile of this solution is shown in Figure 1. It represents a lump (static in the variables \( X_{1}, Y_{1} \)) of height \( 8a_{0}^{2} \).

3.2 Lump (1,0): \( n = 1, \quad m = 0 \)

The eigenfunctions (22) are:

\[
\begin{align*}
\psi_{1}^{[0]} &= P_{1}(x, y, t; k) \exp [Q_{0}(x, y, t, k)], \\
\chi_{1}^{[0]} &= \exp [-Q_{0}(x, y, t, k)], \\
\psi_{2}^{[0]} &= \exp [-Q_{0}^{*}(x, y, t, k)], \\
\chi_{2}^{[0]} &= \{P_{1}(x, y, t; k)\}^{*} \exp [Q_{0}^{*}(x, y, t, k)],
\end{align*}
\]

where according to (24) we have:

\[
P_{1}(x, y, t; k) = x + 2iky - 8ik^{3}t.
\]
In this case, the matrix elements (17) are:

\[
\phi_{1}^{[0]} = \frac{x^2}{2} + iy + 2ixyk - 2y^2k^2 - 12itk^2 - 8ixtk^3 + 16ytk^4 - 32tk^6 =
\]

\[
= \frac{X_1^2 - Y_1^2}{2} + i\frac{2a_0X_1Y_1 + Y_1 - 16a_0^2t}{2a_0},
\]

\[
\phi_{2}^{[0]} = \left(\phi_{1}^{[0]}\right)^*,
\]

\[
\Omega_{1,2} = \frac{1}{2a_0} \exp \left[-Q_0(x, y, t; k)\right] \exp \left[-Q_0^*(x, y, t; k)\right],
\]

\[
\Omega_{2,1} = \frac{1 - 2a_0X_1 + 2a_0^2(X_1^2 + Y_1^2)}{4a_0^4} \exp \left[Q_0(x, y, t; k)\right] \exp \left[Q_0^*(x, y, t; k)\right].
\]

Therefore, the \(\tau\)-function (20) is the positive defined expression

\[
\tau_{1,2} = \left(\frac{X_1^2 - Y_1^2}{2}\right)^2 + \left(\frac{2a_0X_1Y_1 + Y_1 - 16a_0^2t}{2a_0}\right)^2 + \left(\frac{X_1 - \frac{1}{2a_0}}{2a_0}\right)^2 + \left(\frac{Y_1}{2a_0}\right)^2 + \frac{1}{16a_0^4}.
\]

The profile of this solution is shown in Figure 2. If we wish to show its behavior when \(t \to \pm\infty\), we need to look along the lines

\[
X_1 = \hat{X}_1 + c_1 t^{1/2}
\]

\[
Y_1 = \hat{Y}_1 + c_2 t^{1/2}
\]

such that (32), when \(t \to \pm\infty\), is different from 0

- For \(t < 0\), the possibilities are \(c_1 = \pm 2a_0\sqrt{-2}, \ c_2 = -c_1\) which yields two lumps approaching with opposite velocities along the lines

\[
X_1 = \hat{X}_1 \pm 2a_0(-2t)^{1/2}
\]

\[
Y_1 = \hat{Y}_1 \pm 2a_0(-2t)^{1/2},
\]

and the limit of \(\tau_{1,2}\) along these lines is

\[
\tau_{1,2} = \left(\hat{X}_1 + \frac{1}{4a_0}\right)^2 + \left(\hat{Y}_1 - \frac{1}{4a_0}\right)^2 + \frac{1}{4a_0^4}.
\]

- For \(t > 0\), the possibilities are \(c_1 = \pm 2a_0\sqrt{2}, \ c_2 = c_1\) which yields two lumps with opposite velocities along the lines

\[
X_1 = \hat{X}_1 \pm 2a_0(2t)^{1/2}
\]

\[
Y_1 = \hat{Y}_1 \pm 2a_0(2t)^{1/2},
\]

and the limit of \(\tau_{1,2}\) along these lines is

\[
\tau_{1,2} = \left(\hat{X}_1 + \frac{1}{4a_0}\right)^2 + \left(\hat{Y}_1 + \frac{1}{4a_0}\right)^2 + \frac{1}{4a_0^4}.
\]
3.3 Lump (1,1): \( n = 1, \quad m = 1 \)

The eigenfunctions (22) are:

\[
\begin{align*}
\psi_1^{[0]} &= P_1(x, y, t; k) \exp [Q_0(x, y, t; k)], \\
\chi_1^{[0]} &= P_1(x, y, t; k) \exp [-Q_0(x, y, t; k)], \\
\psi_2^{[0]} &= \{P_1(x, y, t; k)\}^* \exp [-Q_0^*(x, y, t; k)], \\
\chi_2^{[0]} &= \{P_1(x, y, t; k)\}^* \exp [Q_0^*(x, y, t; k)].
\end{align*}
\] (38)

This yields the following matrix elements according to (17):

\[
\begin{align*}
\phi_1^{[0]} &= X_1 \left( \frac{X_1^2 - 3Y_1^2}{3} \right) + i \left( \frac{X_1^2Y_1 - Y_1^3}{3} + 8a_0 t \right), \\
\phi_2^{[0]} &= (\phi_1^{[0]})^*, \\
\Omega_{1,2} &= -\frac{1 + 2a_0X_1 + 2a_0^2 (X_1^2 + Y_1^2)}{4a_0^3} \exp [-Q_0(x, y, t; k)] \exp [-Q_0^*(x, y, t; k)], \\
\Omega_{2,1} &= \frac{1 - 2a_0X_1 + 2a_0^2 (X_1^2 + Y_1^2)}{4a_0^3} \exp [Q_0(x, y, t; k)] \exp [Q_0^*(x, y, t; k)].
\end{align*}
\] (39)
Therefore, the $\tau$-funcion (20) is the positive defined expression

$$
\tau_{1,2} = \left( \frac{X_1 (X_1^2 - 3Y_1^2)}{3} \right)^2 + \left( X_1^2 Y_1 - \frac{Y_1^3}{3} + 8a_0 t \right)^2 + \left( \frac{X_1^2 + Y_1^2}{2a_0} \right)^2 + \left( \frac{Y_1}{2a_0^2} \right)^2 + \frac{1}{16a_0^2}. \quad (40)
$$

The profile of this solution is shown in Figure 3. The asymptotic behavior of this solution can be obtained by considering the transformation

$$
X_1 = \hat{X}_1 + c_1 t^\frac{1}{2},
Y_1 = \hat{Y}_1 + c_2 t^\frac{1}{2}. \quad (41)
$$

There are three possible solutions for $c_i$. For all of them $\tau_{1,2}$ is

$$
\tau_{1,2} \rightarrow X_1^2 + Y_1^2 + \frac{1}{4a_0^2}. \quad (42)
$$

- $c_1 = 0, c_2 = 2(3a_0)^{\frac{1}{2}}$. This corresponds to a lump moving along the line

$$
X_1 = \hat{X}_1, \quad Y_1 = \hat{Y}_1 + 2(3a_0)^{\frac{1}{2}}. \quad (43)
$$

- $c_1 = -\sqrt{3}(3a_0)^{\frac{1}{2}}, c_2 = (-3a_0)^{\frac{1}{2}}$. This corresponds to a lump moving along the line

$$
X_1 = \hat{X}_1 - \sqrt{3}(-3a_0)^{\frac{1}{2}}, \quad Y_1 = \hat{Y}_1 + (-3a_0)^{\frac{1}{2}}. \quad (44)
$$
\( c_1 = \sqrt{3}(-3a_0)^{\frac{1}{3}}, c_2 = 2(3a_0)^{\frac{1}{3}} \). This corresponds to a lump moving along the line
\[
X_1 = \hat{X}_1 + \sqrt{3}(-3a_0t)^{\frac{1}{3}} \\
Y_1 = \hat{Y}_1 + (-3a_0t)^{\frac{1}{3}}.
\]

(45)

4 Conclusions

The Singular Manifold Method allows us to derive an iterative method to construct lump solutions characterized by two integers whose different combinations yield a rich possibilities of nontrivial self-interactions between the components of the solution.

Acknowledgements

This research has been supported in part by the DGICYT under project FIS2009-07880.

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