Explicit approximations for nonlinear switching diffusion systems in finite and infinite horizons

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Abstract

Focusing on hybrid diffusion dynamics involving continuous dynamics as well as discrete events, this article investigates the explicit approximations for nonlinear switching diffusion systems modulated by a Markov chain. Different kinds of easily implementable explicit schemes have been proposed to approximate the dynamical behaviors of switching diffusion systems with local Lipschitz continuous drift and diffusion coefficients in both finite and infinite intervals. Without additional restriction conditions except those which guarantee the exact solutions possess their dynamical properties, the numerical solutions converge strongly to the exact solutions in finite horizon, moreover, realize the approximation of long-time dynamical properties including the moment boundedness, stability and ergodicity. Some simulations and examples are provided to support the theoretical results and demonstrate the validity of the approach.

Keywords: Explicit scheme; Switching diffusion systems; Local Lipschitz condition; Strong convergence; Stability; Invariant measure

1 Introduction

The switching diffusion systems (SDSs) modulated by Markov chains involving continuous dynamics and discrete events, have drawn more and more attention to many researches. Much of the study

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originated from applications arising from biological systems, financial engineering, manufacturing systems, wireless communications (see, e.g., [1, 2, 3, 4, 5] and the references therein). Compared with those of the subsystems the dynamics of SDSs are seemingly much different. For instance, considering a predator-prey ecosystem switching between two environments randomly, Takeuchi et al. in [4] revealed that both subsystems develop periodically but switching between them makes them neither permanent nor dissipative. Pinsky and his coauthors in [6, 7] provided several interesting examples to show that the switching system is recurrent (resp. transient) even if its subsystems are transient (resp. recurrent). Due to the coexistence of continuous dynamics and discrete events, the dynamics of SDSs are full of uncertainty and challenge.

Since solving SDSs is almost unavailable, numerical scheme or approximation techniques become viable alternatives. The explicit Euler-Maruyama (EM) scheme is popular for approximating diffusion systems and SDSs with global Lipschitz coefficients [2, 8, 5]. However, the coefficients of many important diffusion systems and SDSs are only locally Lipschitz and superlinear (see, e.g., [9, 10, 5, 3] and the references therein). Hutzenthaler et al. [10, Theorem 2.1] showed that the absolute moments of the EM approximation for a large class of diffusion systems with superlinear growth coefficients diverge to infinity at a finite time point $T \in (0, \infty)$. The implicit EM scheme is better than the explicit EM scheme in that its numerical solutions converge strongly to the exact solutions of diffusion and switching diffusion systems with the one-sided Lipschitz drift coefficient and global Lipschitz diffusion coefficient (see Higham et al. [11], Mao and Yuan [2, p.134-153]). Nevertheless, additional computational efforts are required for its implementation since the solution of an algebraic equation has to be found before each iteration. Due to the advantages of explicit schemes (e.g., simple structure and cheap computational cost), a few modified EM methods have been developed for diffusion systems with nonlinear coefficients including the tamed EM method [12, 9, 13, 14], the tamed Milstein method [15], the stopped EM method [16] and the truncated EM method [17]. These modified EM methods have shown their abilities to approximate the solutions of nonlinear diffusion systems. However, to the best of our knowledge these methods are not developed, even unavailable for a large class of nonlinear SDSs. For instance, Hamilton [18] remarked that the economy may either be in a fast growth or slow growth phase with the regime switching governed by the outcome of a Markov chain. Consider a two-dimensional nonlinear stochastic volatility model switching randomly between a fast growth phase

$$dX(t) = 2.5X(t)\left(1 - |X(t)|\right)dt + \left(\begin{array}{c}
-1 \\
\sqrt{2}
\end{array}\right)\frac{|X(t)|^{3/2}}{\sqrt{2}}dB(t),$$  \hspace{1cm} (1.1)
and a slow growth phase
\[ dX(t) = \left((1, 2)^T - X(t)\right) dt + \left(\begin{array}{cc} 0.2 & -0.5 \\ 1 & 0.4 \end{array}\right) |X(t)| dB(t), \]  
modulated by a Markov chain \( r(t) \). This SDS is one of popular volatility models used for pricing option in Finance (see, e.g., [18, 1, 19, 3] and the references therein), especially, for pricing VIX options. Based on its importance and no closed-form, one of our aims is to construct the explicit scheme available for this kind of SDSs and to study its convergence in the \( p \)th moment as well as the rate of convergence.

On the other hand, long-time behaviors of SDSs are also the major concerns in stochastic processes, systems theory, control and optimization (see, e.g., the monographs [2, 5] and the references therein). So far, the dynamical properties of SDSs are investigated deeply including stochastic stability (see, e.g., [2, 5, 20, 21]), invariant densities (see, e.g., [22, 23]), recurrence and transience (see, e.g., [5, 24]) and so on. Although the finite-time convergence is one of the fundamental concerns, how to approximate long-time behaviors of SDSs is significant and challenging. Recently, Higham et al. [25] by using the EM scheme, Mao and Yuan [2, pp.229-249] by using the backward EM (BEM) scheme showed respectively that the numerical solutions preserve the mean-square exponential stability of SDSs with globally Lipschitz continuous coefficients. Mao et al. [26] gave a counter-example that the EM numerical solutions don’t share the underlying almost surely exponential stability of SDSs with the nonlinear growth drift term, but the BEM numerical solutions do. Mao et al. [27], Yuan and Mao [28] and Bao et al. [29] made use of the EM method with a constant stepsize to approximate the underlying invariant measure of SDSs with linear growth coefficients while Yin and Zhu [5, p.159-179] did that by using the EM scheme with the decreasing step sizes. In the above mentioned works, the diffusion coefficients of SDSs are always required to be globally Lipschitz continuous. For the further development of numerical schemes for SDSs, we refer readers to [30], for example, and the references therein. Accommodating many applications although these systems are more realistically addressing the demands, the nontraditional setup makes the discrete approximations of the long-time behaviors for nonlinear SDSs more difficult. Thus, in order to close the gap, the other aim is to construct an appropriate explicit scheme for a large class of nonlinear SDSs such that the numerical solutions realize the underlying infinite-time dynamical properties, such as the \( p \)th moment (\( p > 0 \)) boundedness, stability as well as approximate the underlying invariant measure.

Motivated by the truncation idea in [17], together with the novel approximation technique, we have constructed a new explicit scheme and get the convergence with 1/2 order rate in finite time interval. Then we go further to improve the scheme according to the structure of the SDSs such
that it is easily implementable for approximating the underlying invariant measure (resp. stability). The schemes proposed in this paper are obviously different from those of [2, 13, 17, 31]. More precisely, the numerical solutions at the grid points are modified before each iteration according to the growth rate of the drift and diffusion coefficients such that the numerical solutions keep the underlying excellent properties of the exact solutions of SDSs. Our contributions are as follows.

- We construct an easily implementable scheme for the SDSs with only local Lipschitz drift and diffusion coefficients and establish finite-time moment convergence results. The rate of convergence is also estimated under slightly stronger conditions.

- Using novel techniques (i.e., combining the Lemma 4.1 with the Perron-Frobenius theorem (see [32, p.6]) to construct appropriate Lyapunov functions depending on the states and analyzing their asymptotic properties), we obtain the criterion on the existence and uniqueness of invariant measures of the SDSs as well as the $p$th moment ($p > 0$) and the almost surely exponential stability; see Theorem 4.6 and Corollary 4.5.

- We reconstruct the explicit scheme which numerical solutions admit a unique invariant measure converging to the underlying one in the Wasserstein distance. The restrictions on the coefficients by using the EM scheme (c.f. [28, 29, 33]) are relaxed. Thus this scheme is more suitable for the measure approximation of nonlinear SDSs.

- Without extra restrictions the numerical solutions of the appropriate explicit scheme stay in step of dynamical properties with the exact solutions.

The rest of the paper is arranged as follows. Section 2 begins with notations and preliminaries on the properties of the exact solutions. Section 3 constructs an explicit scheme, and yields the convergence in $p$th moment and the rate. Two examples are given to illustrate the availability of this scheme. Section 4 focuses on the analysis of invariant measures. The other explicit scheme is constructed preserving the stability in distribution, which numerical invariant measure approximates the underlying one in the Wasserstein distance. Several numerical experiments are presented to illustrate the results. Section 5 gives some concluding remarks to conclude the paper.

2 Notations and preliminaries

Throughout this paper, we use the following notations. Let $d$, $m$ and $n$ denote finite positive integers, $\| \cdot \|$ denote the Euclidean norm in $\mathbb{R}^n := \mathbb{R}^{n \times 1}$ and the trace norm in $\mathbb{R}^{n \times d}$. If $A$ is a vector or matrix, its transpose is denoted by $A^T$ and its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. For vectors or
matrixes $A$ and $B$ with compatible dimensions, $AB$ denotes the usual matrix multiplication. If $A$ is a symmetric matrix, denote by $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ its largest and smallest eigenvalue, respectively. For any $c = (c(1), \ldots, c(m))$ (or $= (c_1, \ldots, c_m)$) define $\hat{c} = \min_{1 \leq i \leq m} c(i)(c_i)$ and $\check{c} = \max_{1 \leq i \leq m} c(i)(c_i)$. For any $a, b \in \mathbb{R}$, $a \lor b := \max\{a, b\}$, and $a \land b := \min\{a, b\}$. If $\mathcal{D}$ is a set, its indicator function is denoted by $I_{\mathcal{D}}$, namely $I_{\mathcal{D}}(x) = 1$ if $x \in \mathcal{D}$ and 0 otherwise. Let $C_i$ and $C$ denote two generic positive real constants respectively, whose value may change in different appearances, where $C_i$ is dependent on $i$ and $C$ is independent of $i$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\mathbb{E}$ denotes the expectation corresponding to $\mathbb{P}$. Let $B(t)$ be a $d$-dimensional Brownian motion defined on this probability space. Suppose that $\{r(t)\}_{t \geq 0}$ is a right-continuous Markov chain with finite state space $\mathbb{S} = \{1, 2, \ldots, m\}$ and independent of the Brownian motion $B(\cdot)$. Suppose $\{\mathcal{F}_t\}_{t \geq 0}$ is a filtration defined on this probability space satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) such that $B(t)$ and $r(t)$ are $\mathcal{F}_t$ adapted. The generator of $\{r(t)\}_{t \geq 0}$ is denoted by $\Gamma = (\gamma_{ij})_{m \times m}$, so that for a sufficiently small $\delta > 0$,

$$
\mathbb{P}\{r(t + \delta) = j | r(t) = i\} = \begin{cases} 
\gamma_{ij}\delta + o(\delta), & \text{if } i \neq j, \\
1 + \gamma_{ii}\delta + o(\delta), & \text{if } i = j,
\end{cases}
$$

where $o(\delta)$ satisfies $\lim_{\delta \to 0} o(\delta)/\delta = 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from $i$ to $j$ if $i \neq j$ while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$. It is well known that almost every sample path of $r(t)$ is right-continuous step functions with a finite number of simple jumps in any finite interval of $\mathbb{R}_+ := [0, +\infty)$(c.f. [26]).

In this paper, we consider the $n$-dimensional SDS described by a hybrid stochastic differential equation (HSDE)

$$
dX(t) = f(X(t), r(t))dt + g(X(t), r(t))dB(t) \tag{2.1}
$$

with an initial value $(X(0), r(0)) = (x_0, \ell) \in \mathbb{R}^n \times \mathbb{S}$, where the drift and diffusion terms

$$
f : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n, \quad g : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n \times d},
$$

are locally Lipschitz continuous, that is, for any $N > 0$ there exists a positive constant $C_N$ such that, for any $x, y \in \mathbb{R}^n$ with $|x| \lor |y| \leq N$ and any $i \in \mathbb{S}$,

$$
|f(x, i) - f(y, i)| \lor |g(x, i) - g(y, i)| \leq C_N|x - y|.
$$

Let $C^2(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(x, i)$ on $\mathbb{R}^n \times \mathbb{S}$ which are continuously twice differentiable in $x$. For each $V(x, i) \in C^2(\mathbb{R}^n \times \mathbb{S}; \mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{S}$ to $\mathbb{R}$ by

$$
\mathcal{L}V(x, i) = V_x(x, i)f(x, i) + \frac{1}{2}\text{trace} \left( g^T(x, i)V_{xx}(x, i)g(x, i) \right) + \sum_{j=1}^m \gamma_{ij} V(x, j), \tag{2.2}
$$
where
\[ V_x(x,i) = \left( \frac{\partial V(x,i)}{\partial x_1}, \ldots, \frac{\partial V(x,i)}{\partial x_n} \right), \quad V_{xx}(x,i) = \left( \frac{\partial^2 V(x,i)}{\partial x_j \partial x_l} \right)_{n \times n}. \]

In order for the existence of exact regular solutions we impose the following assumption.

**Assumption 1** For some \( \bar{p} > 0 \) and each \( i \in S \), there exists a symmetric positive-definite matrix \( Q_i \in \mathbb{R}^{n \times n} \) and a constant \( \alpha_i \) such that
\[
\limsup_{|x| \to \infty} \frac{(1 + x^T Q_i x) \psi(x,i) - (2 - \bar{p}) x^T Q_i g(x,i)]^2}{(x^T Q_i x)^2} \leq \alpha_i, \quad \forall x \in \mathbb{R}^n, \tag{2.3}
\]
where \( \psi(x,i) := 2x^T Q_i f(x,i) + \text{trace}[g^T(x,i)Q_ig(x,i)] \).

Now we give the regularity of exact solutions as well as the estimation on their \( p \)th moment.

**Theorem 2.1** Under Assumption 1, SDS (2.1) with any initial value \((x_0, \ell) \in \mathbb{R}^n \times S\) has a unique regular solution \( (X(t),r(t)) \) satisfying
\[
\sup_{0 \leq t \leq T} \mathbb{E}|X(t)|^p \leq C \tag{2.4}
\]
for any \( p \in (0, \bar{p}] \), \( T > 0 \), where the positive constant \( \bar{p} \) is given by Assumption 1.

**Proof.** For any \( i \in S \) and \( 0 < \kappa_i < 1 \), Assumption 1 together with the continuity of \( f \) and \( g \) implies that there exists a positive constant \( C_i \) such that
\[
(1 + x^T Q_i x) \psi(x,i) - (2 - \bar{p}) |x^T Q_i g(x,i)]^2 \leq \left( \alpha_i + \frac{\kappa_i}{2} \right) (1 + x^T Q_i x)^2 + C_i \tag{2.5}
\]
for any \( x \in \mathbb{R}^n \). For any fixed \( 0 < p \leq \bar{p} \), define \( V(x,i) = (1 + x^T Q_i x)^{\frac{p}{2}} \). Direct calculation, together with (2.5), leads to
\[
\mathcal{L}V(x,i) = \frac{p}{2} (1 + x^T Q_i x)^{\frac{p}{2} - 2} \left\{ (1 + x^T Q_i x) \left[ 2x^T Q_i f(x,i) + \text{trace} (g^T(x,i)Q_ig(x,i)) \right] \right. \\
- \left. (2 - p) |x^T Q_i g(x,i)]^2 \right\} + \sum_{j=1}^{m} \gamma_{ij} (1 + x^T Q_j x)^{\frac{p}{2}} \\
\leq \frac{p}{2} \left( \alpha_i + \frac{\kappa_i}{2} \right) (1 + x^T Q_i x)^{\frac{p}{2}} + \sum_{j=1}^{m} \gamma_{ij} (1 + x^T Q_j x)^{\frac{p}{2}} + \frac{pC_i}{2} (1 + x^T Q_i x)^{\frac{p}{2} - 2}. \tag{2.6}
\]
For any \((x,i) \in \mathbb{R}^n \times S\) and any \( p > 0 \), using the Young inequality, we obtain
\[
\frac{pC_i}{2} (1 + x^T Q_i x)^{\frac{p}{2} - 2} \leq \frac{p\kappa_i}{4} (1 + x^T Q_i x)^{\frac{p}{2}} + C_i. \tag{2.7}
\]
Therefore it follows from (2.6) and (2.7) that
\[
\mathcal{L}V(x, i) \leq \left[ \frac{p}{2} (\alpha_i + \kappa_i) + \gamma_{ii} + \sum_{j \neq i}^{m} \gamma_{ij} (\hat{\lambda}_{\hat{\lambda}} - 1)^{\frac{q}{2}} \right] (1 + x^T Q_i x)^{\frac{p}{2}} + C_i, \tag{2.8}
\]
where \( \hat{\lambda} := \max_{i \in S} \{\lambda_{\max}(Q_i)\}, \hat{\lambda} := \min_{i \in S} \{\lambda_{\min}(Q_i)\} \). The next proof is standard (see e.g., Mao and Yuan [2, Theorem 3.19, p.95]) and hence is omitted. \( \Box \)

Define \( \tau_N = \inf \{t \geq 0 : |X(t)| \geq N \} \). By the virtue of Theorem 2.1, we have
\[
\mathbb{P}\{\tau_N \leq T\} \leq \frac{C}{N^p} \tag{2.9}
\]
for all \( p \in (0, \hat{p}] \), where \( C \) is a constant independent of \( N \).

## 3 Moment estimate and strong convergence

In this section, we aim to construct an easily implementable explicit scheme and show the strong convergence under Assumption 1. Given a stepsize \( \Delta > 0 \) and let \( t_k = k\Delta, r_k = r(t_k) \) for \( k \geq 0 \), and one-step transition probability matrix \( P(\Delta) = (P_{ij}(\Delta))_{m \times m} = \exp(\Delta \Gamma) \). The discrete Markov chain \( \{r_k, k = 0, 1, \ldots\} \) can be simulated by the technique in [2, p.112]. To define appropriate numerical solutions, we firstly choose several strictly increasing continuous functions \( \varphi_i : [1, \infty) \rightarrow \mathbb{R}_+ \) such that \( \varphi_i(u) \rightarrow \infty \) as \( u \rightarrow \infty \) and
\[
\sup_{|x| \leq u} \left( \frac{|f(x, i)|}{1 + |x|} \vee \frac{|g(x, i)|^2}{(1 + |x|)^2} \right) \leq \varphi_i(u), \quad \forall i \in S, \ u \geq 1. \tag{3.1}
\]
We note that \( \varphi_i \) is well defined since \( f(\cdot, i) \) and \( g(\cdot, i) \) are locally Lipschitz continuous. Denote by \( \varphi_i^{-1} \) the inverse function of \( \varphi_i \), obviously \( \varphi_i^{-1} : [\varphi_i(1), \infty) \rightarrow \mathbb{R}_+ \) is a strictly increasing continuous function. We also choose a strictly decreasing \( h : (0, 1] \rightarrow [\hat{\varphi}(|x_0| \vee 1), \infty) \) such that
\[
\lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/2} h(\Delta) \leq K, \quad \forall \Delta \in (0, 1], \tag{3.2}
\]
where \( K \) is a positive constant independent of the iteration order \( k \) and the stepsize \( \Delta \). For any given \( \Delta \in (0, 1] \), each \( i \in S \), define a truncation mapping \( \pi_{\Delta}^i : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by
\[
\pi_{\Delta}^i(x) = \left( \frac{|x| \land \varphi_i^{-1}(h(\Delta))}{|x|} \right) \frac{x}{|x|},
\]
where we use the convention \( \frac{x}{|x|} = 0 \) when \( x = 0 \in \mathbb{R}^n \). Obviously, for any \( (x, i) \in \mathbb{R}^n \times S \),
\[
|f(\pi_{\Delta}^i(x), i)| \leq h(\Delta) (1 + |\pi_{\Delta}^i(x)|), \quad |g(\pi_{\Delta}^i(x), i)|^2 \leq h(\Delta) (1 + |\pi_{\Delta}^i(x)|)^2. \tag{3.3}
\]
Remark 3.1 If there exists a state \( i \in S \) such that \( |f(x, i)| \leq C(1 + |x|) \), \( |g(x, i)|^2 \leq C(1 + |x|)^2 \), \( \forall x \in \mathbb{R}^n \), we choose \( \varphi_i(u) \equiv C \) for any \( u \geq 1 \), and let \( \varphi_i^{-1}(u) \equiv +\infty \) for any \( u \in [C, +\infty) \). Then \( \pi^t_{\Delta}(x) = x \), and (3.3) hold always.

Next we propose an explicit scheme to approximate the exact solution of SDS (2.1). To avoid the arbitrarily large excursions from Brownian paths, we truncate the grid point value according to the growth rate of the drift and diffusion functions. Define

\[
\begin{align*}
\dot{Y}_0 &= x_0, \quad r_0 = \ell, \\
Y_k &= \pi^t_{\Delta}(\tilde{Y}_k), \\
\tilde{Y}_{k+1} &= Y_k + f(Y_k, r_k)\Delta + g(Y_k, r_k)\Delta B_k,
\end{align*}
\]

(3.4)

for any integer \( k \geq 0 \), where \( \Delta B_k = B(t_{k+1}) - B(t_k) \). We call this iteration the truncated EM scheme. This method prevents the diffusion term from producing extra-ordinary large value. Additionally, the drift term and diffusion term have the linear property

\[
|f(Y_k, r_k)| \leq h(\Delta)(1 + |Y_k|), \quad |g(Y_k, r_k)|^2 \leq h(\Delta)(1 + |Y_k|)^2.
\]

(3.5)

To obtain the continuous-time approximations, define \( \tilde{Y}(t) \), \( Y(t) \) and \( \tilde{r}(t) \) by

\[
\tilde{Y}(t) := \tilde{Y}_k, \quad Y(t) := Y_k, \quad \tilde{r}(t) := r_k, \quad \forall t \in [t_k, t_{k+1}).
\]

For convenience, denote by \( G_{tk} \) the \( \sigma \)-algebra generated by \( \{F_{tk}, r_{k+1}\} \). Obviously, \( F_{tk} \subseteq G_{tk} \). The following lemma will play an important role in the proof of the moment boundedness of the numerical solutions.

Lemma 3.2 For any measurable functions \( \phi : \mathbb{R}^n \times S \times S \to \mathbb{R}, \tilde{\phi} : \mathbb{R}^n \times S \times S \to \mathbb{R}^{1 \times d} \) and \( \tilde{\phi} : \mathbb{R}^n \times S \times S \to \mathbb{R}^{d \times d} \), we have

\[
E[\phi(Y_k, r_k, r_{k+1})|F_{tk}] = \phi(Y_k, r_k, r_k) + \sum_{j \in S} \phi(Y_k, r_k, j)[\gamma_{rkj}\Delta + o(\Delta)],
\]

(3.6)

\[
E[\tilde{\phi}(Y_k, r_k, r_{k+1})\Delta B_k|F_{tk}] = 0,
\]

(3.7)

and

\[
E[\Delta B_k^T \tilde{\phi}(Y_k, r_k, r_{k+1})\Delta B_k|F_{tk}] = \text{trace}(\tilde{\phi}(Y_k, r_k, r_k))\Delta + \sum_{j \in S} \text{trace}(\tilde{\phi}(Y_k, r_k, j))[\gamma_{rkj}\Delta^2 + o(\Delta^2)].
\]

(3.8)
Proof. First of all, note that \( Y_k \) and \( r_k \) are \( \mathcal{F}_{t_k} \)-measurable. Then, by the Markov property, we derive that

\[
\mathbb{E}\left[ \phi(Y_k, r_k, r_{k+1}) \big| \mathcal{F}_{t_k} \right] = \mathbb{E}\left[ \phi(Y_k, r_k, r_{k+1}) I_{\{r_{k+1} = r_k\}} \big| \mathcal{F}_{t_k} \right] + \mathbb{E}\left[ \phi(Y_k, r_k, r_{k+1}) I_{\{r_{k+1} \neq r_k\}} \big| \mathcal{F}_{t_k} \right]
\]

\[
= \sum_{i \in \mathbb{S}} I\{r_k = i\} \phi(Y_k, i, i) \mathbb{P}\{r_{k+1} = i | r_k = i\} + \sum_{i,j \in \mathbb{S}, j \neq i} I\{r_k = i\} \phi(Y_k, i, j) \mathbb{P}\{r_{k+1} = j | r_k = i\}
\]

\[
= \sum_{i \in \mathbb{S}} I\{r_k = i\} \phi(Y_k, i, i) \left(1 + \gamma_{ii} + o(\Delta)\right) + \sum_{i \in \mathbb{S}} I\{r_k = i\} \sum_{j \in \mathbb{S} - \{i\}} \phi(Y_k, i, j) \left(\gamma_{ij} + o(\Delta)\right)
\]

\[
= \phi(Y_k, r_k, r_k) + \sum_{j \in \mathbb{S}} \phi(Y_k, r_k, j) [\gamma_{r_kj} + o(\Delta)].
\]

Thus, the required assertion (3.6) follows. Note also that \( Y_k, r_k \) and \( r_{k+1} \) are \( \mathcal{G}_{t_k} \)-measurable. Using the properties

\[
\mathbb{E}[\Delta B_k | \mathcal{G}_{t_k}] = \mathbb{E}(\Delta B_k) = 0, \quad \mathbb{E}[\Delta B_k \Delta B_k^T | \mathcal{G}_{t_k}] = \mathbb{I}_d \Delta,
\]

where \( \mathbb{I}_d \) denotes the \( d \times d \) identity matrix. Thus, we obtain that

\[
\mathbb{E}\left[ \phi(Y_k, r_k, r_{k+1}) \Delta B_k \big| \mathcal{F}_{t_k} \right] = \mathbb{E}\left[ \mathbb{E}(\phi(Y_k, r_k, r_{k+1}) \Delta B_k | \mathcal{G}_{t_k}) \big| \mathcal{F}_{t_k} \right]
\]

\[
= \mathbb{E}\left[ \phi(Y_k, r_k, r_{k+1}) \mathbb{E}(\Delta B_k | \mathcal{G}_{t_k}) \big| \mathcal{F}_{t_k} \right] = 0,
\]

which implies that (3.7) holds. Moreover,

\[
\mathbb{E}\left[ \Delta B_k^T \phi(Y_k, r_k, r_{k+1}) \Delta B_k \big| \mathcal{F}_{t_k} \right] = \mathbb{E}\left[ \mathbb{E}\left( \text{trace}(\phi(Y_k, r_k, r_{k+1}) \Delta B_k \Delta B_k^T) \big| \mathcal{G}_{t_k} \right) \big| \mathcal{F}_{t_k} \right]
\]

\[
= \text{trace}\left\{ \mathbb{E}\left[ \phi(Y_k, r_k, r_{k+1}) \mathbb{E}(\Delta B_k \Delta B_k^T | \mathcal{G}_{t_k}) \big| \mathcal{F}_{t_k} \right] \right\}
\]

\[
= \mathbb{E}\left[ \text{trace}(\phi(Y_k, r_k, r_{k+1})) \big| \mathcal{F}_{t_k} \right] \Delta.
\]

Making use of (3.6) we arrive at (3.8). The proof is complete. \( \square \)

In order to estimate the \( p \)th moment of the numerical solution \( Y(t) \), we prepare an elementary inequality.

Lemma 3.3 For any given \( 2k < p \leq 2(k+1) \) \( (k \) is a nonnegative integer) the following inequality

\[
(1 + u)^p \leq 1 + \frac{p}{2} u + \frac{p(p-2)}{8} u^2 + u^3 a_k(u)
\]

holds for any \( u > -1 \), where \( a_k(u) \) represents a \( k \)th-order polynomial of \( u \) which coefficients depend only on \( p \).
Proof. Applying the Taylor formula, for \( k = 0 \), namely, \( 0 < p \leq 2 \), we have
\[
(1 + u)^{\frac{p}{2}} \leq 1 + \frac{p}{2} u + \frac{p(p - 2)}{8} u^2 + \frac{p(p - 2)(p - 4)}{48} u^3, \quad \forall u > -1.
\] (3.11)

Then using the above inequality, for \( k = 1 \), namely, \( 2 < p \leq 4 \), we have
\[
(1 + u)^{\frac{p}{2}} = (1 + u)(1 + u)^{\frac{p}{2} - 1}
\leq (1 + u) \left( 1 + \frac{p - 2}{2} u + \frac{(p - 2)(p - 4)}{8} u^2 + \frac{(p - 2)(p - 4)(p - 6)}{48} u^3 \right)
= 1 + \frac{p}{2} u + \frac{p(p - 2)}{8} u^2 + \frac{p(p - 2)(p - 4)}{48} u^3 + \frac{(p - 2)(p - 4)(p - 6)}{48} u^4
\]
for any \( u > -1 \). Generally, for \( 2k < p \leq 2(k + 1) \) and any \( u > -1 \), we have
\[
(1 + u)^{\frac{p}{2}} = (1 + u)^k(1 + u)^{\frac{p}{2} - k}
\leq (1 + u)^k \left( 1 + \frac{p - 2k}{2} u + \frac{(p - 2k)(p - 2k - 2)}{8} u^2 
+ \frac{(p - 2k)(p - 2k - 2)(p - 2k - 4)}{48} u^3 \right)
= 1 + \frac{p}{2} u + \frac{p(p - 2)}{8} u^2 + u^3 a_k(u).
\]

Therefore the desired result follows. \( \Box \)

### 3.1 Moment estimate

Let us begin to establish the criterion on the \( p \)th moment boundedness of the numerical solutions of Scheme (3.4).

**Theorem 3.4** Under Assumption 1, Scheme (3.4) has the property that
\[
\sup_{0 < \Delta \leq 1} \left( \sup_{0 \leq k \Delta \leq T} \mathbb{E} |Y_k|^p \right) \leq C
\] (3.12)

for any \( p \in (0, \bar{p}] \) and any \( T > 0 \), where \( C \) is a constant independent of the iteration order \( k \) and the stepsize \( \Delta \).

**Proof.** For any integer \( k \geq 0 \), we have
\[
(1 + Y_{k+1}^T Q_{r_{k+1}} Y_{k+1})^{\frac{p}{2}} = \left[ 1 + (Y_k + f(Y_k, r_k) \Delta + g(Y_k, r_k) \Delta B_k)^T Q_{r_{k+1}} \right.
\times \left. (Y_k + f(Y_k, r_k) \Delta + g(Y_k, r_k) \Delta B_k) \right]^{\frac{p}{2}} = (1 + Y_k^T Q_{r_k} Y_k)^{\frac{p}{2}} (1 + \zeta_k)^{\frac{p}{2}},
\] (3.13)
where

\[
\zeta_k = (1 + Y_k^T Q_{rk} Y_k)^{-1} \left( Y_k^T Q_{rk+1} Y_k - Y_k^T Q_{rk} Y_k + 2 Y_k^T Q_{rk+1} f(Y_k, r_k) \Delta 
+ \Delta B_k^T g^T(Y_k, r_k) Q_{rk+1} g(Y_k, r_k) \Delta B_k + 2 Y_k^T Q_{rk+1} g(Y_k, r_k) \Delta B_k 
+ f^T(Y_k, r_k) Q_{rk+1} f(Y_k, r_k) \Delta^2 + 2 f^T(Y_k, r_k) Q_{rk+1} g(Y_k, r_k) \Delta B_k \Delta \right),
\]

and we can see that \( \zeta_k > -1 \). By the virtue of Lemma 3.3, without loss the generality we prove (3.12) only for \( 0 < p \leq 2 \). It follows from (3.11) and (3.13) that

\[
\mathbb{E} \left[ \left( 1 + Y_{k+1}^T Q_{rk+1} Y_{k+1} \right)^{\frac{p}{8}} \big| \mathcal{F}_{tk} \right] 
\leq (1 + Y_k^T Q_{rk} Y_k)^{\frac{p}{2}} \mathbb{E}(\zeta_k | \mathcal{F}_{tk}) + \frac{p(p-2)}{8} \mathbb{E}(\zeta_k^2 | \mathcal{F}_{tk}) + \frac{p(p-2)(p-4)}{48} \mathbb{E}(\zeta_k^3 | \mathcal{F}_{tk}). \tag{3.14}
\]

It follows from Lemma 3.2 and (3.5) that

\[
\mathbb{E}[Y_k^T Q_{rk} Y_k | \mathcal{F}_{tk}] = Y_k^T Q_{rk} Y_k + \sum_{j \in S} Y_k^T Q_j Y_k [\gamma_{rj} \Delta + o(\Delta)] \leq Y_k^T Q_{rk} Y_k + C |Y_k|^2 \Delta,
\]

and

\[
\mathbb{E}[2 Y_k^T Q_{rk+1} f(Y_k, r_k) \Delta + \Delta B_k^T g^T(Y_k, r_k) Q_{rk+1} g(Y_k, r_k) \Delta B_k | \mathcal{F}_{tk}] 
= \psi(Y_k, r_k) \Delta + \sum_{j \in S} \left( 2 Y_k^T Q_j f(Y_k, r_k) + \text{trace} \left( g^T(Y_k, r_k) Q_j g(Y_k, r_k) \right) \right) \left( \gamma_{rj} \Delta^2 + o(\Delta^2) \right) 
\leq \psi(Y_k, r_k) \Delta + C(1 + |Y_k|)^2 h(\Delta) \Delta^2 
\leq \psi(Y_k, r_k) \Delta + C(1 + |Y_k|)^2 \Delta^\frac{3}{2}.
\]

Thus both of the above inequalities implies

\[
\mathbb{E} \left[ \zeta_k | \mathcal{F}_{tk} \right] 
\leq (1 + Y_k^T Q_{rk} Y_k)^{-1} \left( \psi(Y_k, r_k) \Delta + C |Y_k|^2 \Delta + C |f(Y_k, r_k)|^2 \Delta^2 + C(1 + |Y_k|)^2 \Delta^\frac{3}{2} \right) 
\leq (1 + Y_k^T Q_{rk} Y_k)^{-1} \psi(Y_k, r_k) \Delta + C \Delta. \tag{3.15}
\]

Using Lemma 3.2 and (3.5) again yields

\[
\mathbb{E} \left[ \zeta_k^2 | \mathcal{F}_{tk} \right] 
\geq (1 + Y_k^T Q_{rk} Y_k)^{-2} \left\{ \mathbb{E} \left[ \mathbb{E} \left[ \left| Y_k^T Q_{rk+1} g(Y_k, r_k) \Delta B_k \right|^2 \big| \mathcal{G}_{tk} \right] \big| \mathcal{F}_{tk} \right] 
+ 4 \mathbb{E} \left[ \mathbb{E} \left[ \left( Y_k^T Q_{rk+1} g(Y_k, r_k) \Delta B_k \right)^T \left( Y_k^T Q_{rk+1} Y_k - Y_k^T Q_{rk} Y_k \right) 
+ \Delta B_k^T g^T(Y_k, r_k) Q_{rk+1} g(Y_k, r_k) \Delta B_k + f^T(Y_k, r_k) Q_{rk+1} f(Y_k, r_k) \Delta^2 
+ 2 Y_k^T Q_{rk+1} f(Y_k, r_k) \Delta + 2 f^T(Y_k, r_k) Q_{rk+1} g(Y_k, r_k) \Delta B_k \Delta \big| \mathcal{G}_{tk} \right] \big| \mathcal{F}_{tk} \right] \right\} 
\geq (1 + Y_k^T Q_{rk} Y_k)^{-2} \left[ 4 \Delta \mathbb{E} \left[ \left| Y_k^T Q_{rk+1} g(Y_k, r_k) \right|^2 \big| \mathcal{F}_{tk} \right] - C(1 + |Y_k|)^4 h^2(\Delta) \Delta^2 \right].
\]
\[ \geq 4(1 + Y_k^T Q_{r_k} Y_k)^2 |Y_k^T Q_{r_k} g(Y_k, r_k)|^2 \Delta - C\Delta. \] (3.16)

Using the properties
\[ \mathbb{E}(|\Delta B_k|^{2j}|\mathcal{F}_{t_k}) = C\Delta^j, \quad \mathbb{E}((A\Delta B_k)^{2j-1}|\mathcal{F}_{t_k}) = 0, \quad j = 1, 2, \ldots \] (3.17)
for any \( A \in \mathbb{R}^{1 \times d} \) as well as Lemma 3.2, we deduce that
\[
\mathbb{E}[\zeta_k^3|\mathcal{F}_{t_k}] = (1 + Y_k^T Q_{r_k} Y_k)^{-3} \mathbb{E}\left[ \left( Y_k^T Q_{r_k+1} Y_k - Y_k^T Q_{r_k} Y_k + 2Y_k^T Q_{r_k+1} f(Y_k, r_k) \Delta 
+ \Delta B_k g^T(Y_k, r_k) Q_{r_k+1} g(Y_k, r_k) \Delta B_k + 2Y_k^T Q_{r_k+1} g(Y_k, r_k) \Delta B_k 
+ f^T(Y_k, r_k) Q_{r_k+1} f(Y_k, r_k) \Delta^2 + 2 f^T(Y_k, r_k) Q_{r_k+1} g(Y_k, r_k) \Delta B_k \Delta \right)^3 |\mathcal{F}_{t_k}\right] 
\leq C(1 + |Y_k|^4) \Delta^3.
\]

By virtue of Lemma 3.2, for any integer \( l \geq 1 \),
\[
\mathbb{E}\left[ |Y_k^T Q_{r_k+1} Y_k - Y_k^T Q_{r_k} Y_k|^2 |\mathcal{F}_{t_k}\right] = \sum_{j \in S} |Y_k^T Q_{r_k} Y_k - Y_k^T Q_{r_k} Y_k|I_{\{r_k \neq j\}} (\gamma_{r_k, j} \Delta + o(\Delta)) 
\leq C(1 + |Y_k|^2) \Delta.
\]

Making use of the above inequality and (3.5) yields
\[ \mathbb{E}[\zeta_k^3|\mathcal{F}_{t_k}] \leq C(1 + Y_k^T Q_{r_k} Y_k)^{-3} \left\{ (1 + |Y_k|)^6 \left[ \Delta + h^3(\Delta) \Delta^3 (2 + h^3(\Delta) \Delta^3) 
+ h(\Delta) \Delta^2 (1 + h^2(\Delta) \Delta^2) (1 + 2h(\Delta) + h^2(\Delta) \Delta) \right]\right\} \leq C\Delta. \] (3.18)

Similarly, we can also prove that for any integer \( l > 3 \), \( \mathbb{E}[|\zeta_k|^l|\mathcal{F}_{t_k}] \leq C\Delta \). Combining (3.14)-(3.16) and (3.18), using (2.5) and (2.7), we obtain that
\[
\mathbb{E}\left[ (1 + \tilde{Y}_{k+1}^T Q_{r_{k+1}} \tilde{Y}_{k+1})^{\frac{p}{2}} |\mathcal{F}_{t_k}\right] 
\leq (1 + Y_k^T Q_{r_k} Y_k)^{\frac{p}{2}} \left\{ 1 + C\Delta + \frac{p\Delta}{2} \left[ \frac{(1 + Y_k^T Q_{r_k} Y_k)\psi(Y_k, r_k) + (p - 2)|Y_k^T Q_{r_k} g(Y_k, r_k)|^2}{(1 + Y_k^T Q_{r_k} Y_k)^2} \right]\right\} 
\leq (1 + Y_k^T Q_{r_k} Y_k)^{\frac{p}{2}} (1 + C\Delta) + C\Delta
\]
for any integer $k \geq 0$. The truncation property of Scheme (3.4)

$$Y_k^T Q_{r_k} Y_k = \left( \pi_{\Delta r_k}^* (\hat{Y}_k) \right)^T Q_{r_k} \pi_{\Delta r_k}^* (\hat{Y}_k) = \left( \frac{|\hat{Y}_k| \wedge \varphi_{r_k}^{-1}(h(\Delta))}{|\hat{Y}_k|} \right)^2 \hat{Y}_k^T Q_{r_k} \hat{Y}_k \leq \hat{Y}_k^T Q_{r_k} \hat{Y}_k,$$

implies that

$$\mathbb{E} \left[ (1 + Y_k^T Q_{r_k} Y_k)^{\frac{p}{2}} | \mathcal{F}_{t_k} \right] \leq (1 + Y_k^T Q_{r_k} Y_k)^{\frac{p}{2}} (1 + C\Delta) + C \Delta.$$

Repeating this procedure we obtain

$$\mathbb{E} \left( (1 + Y_k^T Q_{r_k} Y_k)^{\frac{p}{2}} | \mathcal{F}_0 \right) \leq (1 + C\Delta)^k \left( 1 + x_0^T Q_1 x_0 \right)^{\frac{p}{2}} + C \Delta \sum_{i=0}^{k-1} (1 + C\Delta)^i.$$

Taking expectations on both sides, for any integer $k$ satisfying $0 \leq k\Delta \leq T$, then we have

$$\mathbb{E} \left( (1 + Y_k^T Q_{r_k} Y_k)^{\frac{p}{2}} \right) \leq C (1 + C\Delta)^k \leq C \exp(CK\Delta) \leq C \exp(CT).$$

Therefore the desired result follows. $\square$

In order to establish the strong convergence theory of Scheme (3.4), we give the following lemma.

**Lemma 3.5** Under Assumption 1, define

$$\rho_\Delta =: \inf \{ t \geq 0 : |\hat{Y}(t)| \geq \varphi_{r(t)}^{-1}(h(\Delta)) \}, \quad (3.19)$$

then for any $T > 0$,

$$\mathbb{P} \{ \rho_\Delta \leq T \} \leq \frac{C}{\left( \hat{\varphi}^{-1}(h(\Delta)) \right)^p} \quad (3.20)$$

for all $p \in (0, \bar{p}]$, where $C$ is a constant independent of $k$ and $\Delta$.

**Proof.** Define $\beta =: \inf \left\{ k \geq 0 : |\hat{Y}_k| \geq \varphi_{r_k}^{-1}(h(\Delta)) \right\}$, then $\rho_\Delta = \Delta \beta$. Obviously, $\rho_\Delta$ and $\beta$ are $\mathcal{F}_t$, $\mathcal{F}_{t_k}$ stopping time, respectively. For $\omega \in \{ \beta \geq k + 1 \}$, we have $Y_k = \hat{Y}_k$ and

$$\hat{Y}_{(k+1)\wedge \beta} = \hat{Y}_{k+1}, \quad \hat{Y}_{(k+1)\wedge \beta}^T Q_{r_{(k+1)\wedge \beta}} \hat{Y}_{(k+1)\wedge \beta} = \hat{Y}_{k+1}^T Q_{r_{k+1}} \hat{Y}_{k+1}.$$

On the other hand, for $\omega \in \{ \beta < k + 1 \}$, we have $\beta \leq k$ and hence

$$\hat{Y}_{(k+1)\wedge \beta} = \hat{Y}_\beta = \hat{Y}_{k\wedge \beta}, \quad \hat{Y}_{(k+1)\wedge \beta}^T Q_{r_{(k+1)\wedge \beta}} \hat{Y}_{(k+1)\wedge \beta} = \hat{Y}_{k\wedge \beta}^T Q_{r_{k\wedge \beta}} \hat{Y}_{k\wedge \beta}.$$

Therefore, we derive from (3.4) that for any integer $k \geq 0$,

$$\hat{Y}_{(k+1)\wedge \beta} = \hat{Y}_{k\wedge \beta} + \left[ f(\hat{Y}_k, r_k) \Delta + g(\hat{Y}_k, r_k) \Delta B_k \right] I_{[0, \beta]}(k + 1).$$

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Then
\[
\left( 1 + \tilde{Y}_{(k+1)\wedge \beta}^T Q_{r(k+1)\wedge \beta} \tilde{Y}_{(k+1)\wedge \beta} \right)^{\frac{p}{2}} = \left( 1 + \tilde{Y}_{k\wedge \beta}^T Q_{r_k\wedge \beta} \tilde{Y}_{k\wedge \beta} \right)^{\frac{p}{2}} \left( 1 + \tilde{c}_k I_{[[0,\beta]]}(k+1) \right)^{\frac{p}{2}},
\] (3.21)
where
\[
\tilde{c}_k = \left( 1 + \tilde{Y}_k^T Q_{r_k} \tilde{Y}_k \right)^{-1} \left( \tilde{Y}_k^T Q_{r_k} \tilde{Y}_k - \tilde{Y}_k^T Q_{r_k} \tilde{Y}_k + 2 \tilde{Y}_k^T Q_{r_k+1} f(\tilde{Y}_k, r_k) \Delta + 2 \tilde{Y}_k^T Q_{r_k+1} g(\tilde{Y}_k, r_k) \Delta B_k \right.
\]
\[+ \left. f^T(\tilde{Y}_k, r_k) Q_{r_k+1} f(\tilde{Y}_k, r_k) \Delta^2 + 2 f^T(\tilde{Y}_k, r_k) Q_{r_k+1} g(\tilde{Y}_k, r_k) \Delta B_k \Delta \right).
\]

By the virtue of Lemma 3.3, without loss the generality we prove the required result only for \(0 < p \leq 2\). It follows from (3.11) and (3.21) that
\[
\mathbb{E} \left[ \left( 1 + \tilde{Y}_k^T Q_{r_k} \tilde{Y}_k \right)^{\frac{p}{2}} | F_{t_k\wedge \beta} \right]
\leq \left( 1 + \tilde{Y}_k^T Q_{r_k} \tilde{Y}_k \right)^{\frac{p}{2}} \left\{ 1 + \frac{p}{2} \mathbb{E} \left[ \tilde{c}_k I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right] + \frac{p(p - 2)}{8} \mathbb{E} \left[ \tilde{c}_k^2 I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right]
\]
\[+ \frac{p(p - 2)(p - 4)}{48} \mathbb{E} \left[ \tilde{c}_k^3 I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right].
\] (3.22)

By Lemma 3.2 one observes that
\[
\mathbb{E} \left[ (\tilde{Y}_k^T Q_{r_k} \tilde{Y}_k) I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right]
\leq \mathbb{E} \left[ (\tilde{Y}_k^T Q_{r_k} \tilde{Y}_k + \sum_{j \in S} \tilde{Y}_k^T Q_{j} \tilde{Y}_k \gamma_{r_k\wedge \beta} \Delta + o(\Delta)) I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right]
\leq \mathbb{E} \left[ (\tilde{Y}_k^T Q_{r_k} \tilde{Y}_k + C|\tilde{Y}_k|^2 \Delta) I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right],
\] (3.23)

Note that \(\Delta B_k I_{[[0,\beta]]}(k+1) = B(t_{(k+1)\wedge \beta}) - B(t_{k\wedge \beta})\). Since \(B(t)\) is a continuous local martingale, by the virtue of the Doob martingale stopping time theorem, we know that \(\mathbb{E} \left[ \Delta B_k I_{[[0,\beta]]}(k+1) | G_{t_k\wedge \beta} \right] = 0\) and \(\mathbb{E} \left[ \Delta B_k \Delta B_k^T I_{[[0,\beta]]}(k+1) | G_{t_k\wedge \beta} \right] = \Delta I_d \mathbb{E} \left[ I_{[[0,\beta]]}(k+1) | G_{t_k\wedge \beta} \right].\) Hence
\[
\mathbb{E} \left[ (\Delta B_k^T g(\tilde{Y}_k, r_k) Q_{r_k+1} g(\tilde{Y}_k, r_k) \Delta B_k) I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right]
\leq \mathbb{E} \left[ \text{trace} \left( g^T(\tilde{Y}_k, r_k, r_k, \beta) Q_{r_k+1} g(\tilde{Y}_k, r_k, r_k, \beta) \mathbb{E} \left[ \Delta B_k \Delta B_k^T I_{[[0,\beta]]}(k+1) | G_{t_k\wedge \beta} \right] \right) | F_{t_k\wedge \beta} \right]
\leq \mathbb{E} \left[ \text{trace} \left( g^T(\tilde{Y}_k, r_k) Q_{r_k+1} g(\tilde{Y}_k, r_k) I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right).\right.
\]

The above equality together with Lemma 3.2 implies
\[
\mathbb{E} \left[ (2 \tilde{Y}_k^T Q_{r_k+1} f(\tilde{Y}_k, r_k) \Delta + \Delta B_k^T g(\tilde{Y}_k, r_k) Q_{r_k+1} g(\tilde{Y}_k, r_k) \Delta B_k) I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right]
\leq \mathbb{E} \left[ \left( 2 \tilde{Y}_k^T Q_{r_k+1} f(\tilde{Y}_k, r_k) + \text{trace} \left( g^T(\tilde{Y}_k, r_k) Q_{r_k+1} g(\tilde{Y}_k, r_k) \right) I_{[[0,\beta]]}(k+1) | F_{t_k\wedge \beta} \right).\right.
\]

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Thus the above inequality and (3.23) imply
\[\begin{align*}
&\mathbb{E}\left[\tilde{\zeta}_k I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right] = \mathbb{E}\left[\mathbb{E}\left(\tilde{\zeta}_k I_{[0,\beta]}(k + 1) \mid \mathcal{G}_{t_{k+\beta}}\right) \mid \mathcal{F}_{t_{k+\beta}}\right] \\
&\leq \mathbb{E}\left[(1 + \tilde{Y}_k^T Q_{r_k} \tilde{Y}_k)^{-1}\left(\psi(\tilde{Y}_k, r_k) \Delta + C|\tilde{Y}_k|^2 \Delta + C|f(\tilde{Y}_k, r_k)|^2 \Delta^2
\right.ight. \\
&\quad + C(1 + |\tilde{Y}_k|^2 h(\Delta) \Delta^2) I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\left.\right]\right] \\
&\leq C \Delta \mathbb{E}\left[I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right]. \tag{3.24}
\end{align*}\]

Using the techniques in the proof of Theorem 3.4, we show that
\[\begin{align*}
&\mathbb{E}\left[\tilde{\zeta}_k^2 I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right] = \mathbb{E}\left[\mathbb{E}\left(\tilde{\zeta}_k^2 I_{[0,\beta]}(k + 1) \mid \mathcal{G}_{t_{k+\beta}}\right) \mid \mathcal{F}_{t_{k+\beta}}\right] \\
&\geq \mathbb{E}\left[(4(1 + \tilde{Y}_k^T Q_{r_k} \tilde{Y}_k)^{-2}|\tilde{Y}_k^T Q_{r_k} \tilde{g}(\tilde{Y}_k, r_k)|^2 \Delta - C \Delta) I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right], \tag{3.25}
\end{align*}\]

and
\[\begin{align*}
&\mathbb{E}\left[\tilde{\zeta}_k^3 I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right] = \mathbb{E}\left[\mathbb{E}\left(\tilde{\zeta}_k^3 I_{[0,\beta]}(k + 1) \mid \mathcal{G}_{t_{k+\beta}}\right) \mid \mathcal{F}_{t_{k+\beta}}\right] \\
&\quad \leq C \Delta \mathbb{E}\left[I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right]. \tag{3.26}
\end{align*}\]

Similarly, we can also show that for any integer \(l > 3\),
\[\begin{align*}
&\mathbb{E}\left[\tilde{\zeta}_k^l I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right] \leq C \Delta \mathbb{E}\left[I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right].
\end{align*}\]

Combining (3.22) and (3.24)-(3.26), using (2.5) and (2.7), for any integer \(k \geq 0\),
\[\begin{align*}
&\mathbb{E}\left[\left(1 + \tilde{Y}_{k+1,\beta} Q_{r_{k+1,\beta}} \tilde{Y}_{k+1,\beta}\right)^\frac{\beta}{2} \mid \mathcal{F}_{t_{k+\beta}}\right] \\
&\leq \left\{1 + \mathbb{E}\left[\left(p \Delta \left(1 + \tilde{Y}_k^T Q_{r_k} \tilde{Y}_k\right) \psi(\tilde{Y}_k, r_k) + (p - 2)|\tilde{Y}_k^T Q_{r_k} \tilde{g}(\tilde{Y}_k, r_k)|^2 \right.ight. \\
&\quad + C \Delta) I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\left.\right]\right\} \left(1 + \tilde{Y}_{k+\beta} Q_{r_{k,\beta}} \tilde{Y}_{k+\beta}\right)^\frac{\beta}{2} \\
&\leq \left(1 + \tilde{Y}_{k,\beta} Q_{r_{k,\beta}} \tilde{Y}_{k,\beta}\right)^\frac{\beta}{2} \left\{1 + \mathbb{E}\left[\left(p \Delta (\alpha_{r_k} + \frac{k_{r_k}}{2}) + C \Delta\right) I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right] \right\} \\
&\quad + \frac{p \Delta}{2} (1 + \tilde{Y}_{k,\beta} Q_{r_{k,\beta}} \tilde{Y}_{k,\beta})^{\frac{\beta}{2} - 2} \mathbb{E}\left[C_{r_k} I_{[0,\beta]}(k + 1) \mid \mathcal{F}_{t_{k+\beta}}\right] \\
&\leq (1 + \tilde{Y}_{k,\beta} Q_{r_{k,\beta}} \tilde{Y}_{k,\beta})^\frac{\beta}{2} (1 + C \Delta) + C \Delta.
\end{align*}\]

Repeating this procedure we obtain
\[\begin{align*}
&\mathbb{E}\left[(1 + \tilde{Y}_{k,\beta} Q_{r_{k,\beta}} \tilde{Y}_{k,\beta})^\frac{\beta}{2}\right] \leq (1 + C \Delta)^k (1 + x_0^T Q_{r_0} x_0)^\frac{\beta}{2} + C \Delta \sum_{i=0}^{k-1} (1 + C \Delta)^i \leq C (1 + C \Delta)^k
\end{align*}\]
for any integer \( k \) satisfying \( 0 \leq k \Delta \leq T \). Therefore the required assertion follows from that

\[
(\hat{\varphi}^{-1}(h(\Delta)))^p \mathbb{P}\{\rho_\Delta \leq T\} \leq \mathbb{E}[|\bar{Y}(T \wedge \rho_\Delta)|^p] = \mathbb{E}[|\bar{Y}_{[\frac{T}{\Delta}]\wedge \rho_\Delta}|^p] \leq C \exp(CT),
\]

where \( \lfloor \frac{T}{\Delta} \rfloor \) represents the integer part of \( T/\Delta \). The proof is complete. \( \square \)

### 3.2 Strong convergence

In this subsection, we give the convergence result of Scheme (3.4).

**Theorem 3.6** If Assumption 1 holds with \( \bar{p} > 0 \), then for any \( q \in (0, \bar{p}) \),

\[
\lim_{\Delta \to 0} \mathbb{E}|Y(T) - X(T)|^q = 0, \quad \forall \ T > 0.
\]

**Proof.** Let \( \tau_N \) and \( \rho_\Delta \) be defined as before. Define

\[
\theta_{N,\Delta} = \tau_N \wedge \rho_\Delta, \quad e_\Delta(T) = X(T) - Y(T).
\]

For any \( l > 0 \), using the Young inequality we obtain that

\[
\mathbb{E}|e_\Delta(T)|^q = \mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{N,\Delta} \geq T\}}\right) + \mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{N,\Delta} \leq T\}}\right)
\]

\[
\leq \mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{N,\Delta} \geq T\}}\right) + \frac{ql}{\bar{p}} \mathbb{E}\left(|e_\Delta(T)|^\bar{p}\right) + \frac{\bar{p} - q}{\bar{p}l^{q/(\bar{p} - q)}} \mathbb{P}\{\theta_{N,\Delta} \leq T\}.
\]

(3.28)

It follows from Theorem 2.1 and Theorem 3.4 that

\[
\mathbb{E}|e_\Delta(T)|^\bar{p} \leq 2^\bar{p} \mathbb{E}|X(T)|^\bar{p} + 2^\bar{p} \mathbb{E}|Y(T)|^\bar{p} \leq C.
\]

Now let \( \varepsilon > 0 \) be arbitrary. Choose \( l > 0 \) small sufficiently such that \( Cql/\bar{p} \leq \varepsilon/3 \), then we have

\[
\frac{ql}{\bar{p}} \mathbb{E}\left(|e_\Delta(T)|^\bar{p}\right) \leq \frac{\varepsilon}{3}.
\]

(3.29)

Then choose \( N > 1 \) large sufficiently such that \( \frac{C(\bar{p} - q)}{Np^q/(\bar{p} - q)} \leq \frac{\varepsilon}{6} \). Choose \( \Delta^* \in (0, 1] \) small sufficiently such that \( \hat{\varphi}^{-1}(h(\Delta^*)) \geq N \). Then for any \( \Delta \in (0, \Delta^*] \), it follows from Lemma 3.5 that \( \mathbb{P}\{\rho_\Delta \leq T\} \leq \frac{C}{Np} \). This together with (2.9) implies

\[
\frac{\bar{p} - q}{\bar{p}l^{q/(\bar{p} - q)}} \mathbb{P}\{\theta_{N,\Delta} \leq T\} \leq \frac{\bar{p} - q}{\bar{p}l^{q/(\bar{p} - q)}} \left( \mathbb{P}\{\tau_N \leq T\} + \mathbb{P}\{\rho_\Delta \leq T\} \right) \leq \frac{2C(\bar{p} - q)}{Np^q/(\bar{p} - q)} \leq \frac{\varepsilon}{3}.
\]

(3.30)

Combining (3.28), (3.29) and (3.30), we know that for the chosen \( N \) and all \( \Delta \in (0, \Delta^*] \),

\[
\mathbb{E}|e_\Delta(T)|^q \leq \mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{N,\Delta} \geq T\}}\right) + \frac{2\varepsilon}{3}.
\]
If we can show that
\[
\lim_{\Delta \to 0} \mathbb{E} \left( |e_{\Delta}(T)|^q I_{\{\theta_{N,\Delta} \geq T\}} \right) = 0,
\]
the required assertion follows. For this purpose we define the truncated functions
\[
f_N(x,i) = f \left( (|x| \wedge N) \frac{x}{|x|}, i \right), \quad \text{and} \quad g_N(x,i) = g \left( (|x| \wedge N) \frac{x}{|x|}, i \right)
\]
for any \((x,i) \in \mathbb{R}^n \times \mathbb{S}\). Consider the truncated SDS
\[
dz(t) = f_N(z(t), r(t)) dt + g_N(z(t), r(t)) dB(t)
\]
with the initial value \(z(0) = x_0\) and \(r(0) = \ell\). For the chosen \(N\), Assumption 1 implies that \(f_N(\cdot, \cdot)\) and \(g_N(\cdot, \cdot)\) are globally Lipschitz continuous with the Lipschitz constant \(C_N\). Therefore, SDS (3.32) has a unique regular solution \(z(t)\) satisfying
\[
X(t \wedge \tau_N) = z(t \wedge \tau_N) \quad \text{a.s.} \quad \forall \ t \geq 0.
\]
On the other hand, for each \(\Delta \in (0, \Delta^*]\), we apply the EM method to (3.32) and denote by \(u(t)\) the piecewise constant EM solution (see [2]) which has the property
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |z(t) - u(t)|^q \right) \leq C_N \Delta^{q/2}, \quad \forall \ T > 0.
\]
Due to \(\varphi_i^{-1}(h(\Delta)) \geq N\), we have \(Y(t \wedge \theta_{N,\Delta}) = u(t \wedge \theta_{N,\Delta})\) a.s. This together with (3.33) implies
\[
\mathbb{E} \left( |e_{\Delta}(T)|^q I_{\{\theta_{N,\Delta} \geq T\}} \right) = \mathbb{E} \left( |e_{\Delta}(T \wedge \theta_{N,\Delta})|^q I_{\{\theta_{N,\Delta} \geq T\}} \right) \leq \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \theta_{N,\Delta}} |z(t) - u(t)|^q \right).
\]
Thus the desired assertion (3.31) follows from (3.34). The proof is complete.

### 3.3 Convergence Rate

In this subsection our aim is to establish the rate of convergence. The rate is optimal similar to the standard results of the explicit EM scheme for SDSs with globally Lipschitz coefficients, see [2, p.115]. To estimate the rate, we need somewhat stronger conditions compared with the convergence alone, which are stated as follows.

**Assumption 2** For some \(\tilde{p} > 2\), there exist positive constants \(L_i\) and \(l\) such that
\[
2(x - y)^T (f(x, i) - f(y, i)) + (\tilde{p} - 1)|g(x, i) - g(y, i)|^2 \leq L_i |x - y|^2,
\]
\[
|f(x, i) - f(y, i)| \leq L_i (1 + |x|^l + |y|^l) |x - y|, \quad \forall x, y \in \mathbb{R}^d, \quad i \in \mathbb{S}.
\]
Remark 3.7 One observes that if Assumption 2 holds, then
\[ |g(x, i) - g(y, i)|^2 \leq C_i(1 + |x|^l + |y|^l)|x - y|^2. \] (3.37)

One also knows that
\[ |f(x, i)| \leq |f(x, i) - f(0, i)| + |f(0, i)| \leq L_i(1 + |x|^l)|x| + |f(0, i)| \leq C_i(1 + |x|^{l+1}), \]
and by Young’s inequality,
\[ |g(x, i)| \leq C_i(|x|^2 + |x|^{l+2})^{1/2} + |g(0, i)| \leq C_i(1 + |x|^{l/2 + 1}). \] (3.39)

Remark 3.8 Under Assumption 2, we may define \( \varphi_i(u) = C_i(1 + u^l) \) for any \( u > 0 \) in (3.1), then \( \varphi_i^{-1}(u) = (u/C_i - 1)^{1/l} \) for all \( u > C_i \). In order to obtain the rate, we specify \( h(\Delta) = K\Delta^{-\tau} \) for any \( \Delta \in (0, \Delta^*], \) where \( \tau \in (0, 1/2] \) will be specified in Lemma 3.12. Thus, \( \pi_{\Delta}(x) = (|x| \wedge (K\Delta^{-\tau}/C_i - 1)^{1/l})x/|x| \) for any \( x \in \mathbb{R}^n, i \in \mathbb{S}. \)

Making use of Scheme (3.4), we define an auxiliary approximation process by
\[ \tilde{Y}(t) = Y_k + f(Y_k, r_k)(t - t_k) + g(Y_k, r_k)(B(t) - B(t_k)), \quad \forall t \in [t_k, t_{k+1}). \] (3.40)

Note that \( \tilde{Y}(t_k) = Y(t_k) = Y_k, \) that is \( \tilde{Y}(t) \) and \( Y(t) \) coincide with the discrete solution at the grid points.

Lemma 3.9 If Assumptions 1 and 2 hold, for any \( p \in (0, p/(l + 1)] \), the process defined by (3.40) has the property
\[ \sup_{0 \leq t \leq T} \mathbb{E}(|\tilde{Y}(t) - Y(t)|^p) \leq C \Delta^\frac{p}{2}, \quad \forall T > 0, \] (3.41)
where \( C \) is a positive constant independent of \( \Delta. \)

Proof. For any \( t \in [0, T], \) there is a nonnegative integer \( k \) such that \( t \in [t_k, t_{k+1}). \) Then,
\[
\mathbb{E}(|\tilde{Y}(t) - Y(t)|^p) = \mathbb{E}(|\tilde{Y}(t) - Y(t_k)|^p) \leq 2^p \mathbb{E}(|f(Y_k, r_k)|^p) \Delta^p + 2^p \mathbb{E}(|g(Y_k, r_k)|^p|B(t) - B(t_k)|^p)
\leq C \left( \mathbb{E}|f(Y_k, r_k)|^p \Delta^p + \mathbb{E}|g(Y_k, r_k)|^p \Delta^\frac{p}{2} \right).
\]

Due to (3.38), (3.39) and Theorem 3.4,
\[
\mathbb{E}(|\tilde{Y}(t) - Y(t)|^p) \leq C \mathbb{E}(1 + |Y_k|^{l+1})^p \Delta^p + C \mathbb{E}(1 + |Y_k|^{l/2+1})^p \Delta^\frac{p}{2}
\leq C \Delta^\frac{p}{2} + C \left( \mathbb{E}|Y_k|^p \right)^{(l+1)p} \Delta^p + C \left( \mathbb{E}|Y_k|^p \right)^{(l+2)p} \Delta^\frac{p}{2} \leq C \Delta^\frac{p}{2}.
\]

The required assertion follows. \( \square \)

Using the techniques in the proof of Theorem 3.4 and Lemma 3.5 yields the following lemmas.
Lemma 3.10 \textit{Under Assumption 1, the process defined by (3.40) has the property that}
\[
\sup_{0<\Delta \leq 1} \sup_{0 \leq t \leq T} \mathbb{E}|\hat{Y}(t)|^p \leq C, \quad \forall T > 0, \tag{3.42}
\]
for all \(p \in (0, \bar{p}]\), where \(C\) is a positive constant independent of \(\Delta\).

Lemma 3.11 \textit{Let Assumption 1 hold. Define}
\[
\bar{\rho}_\Delta := \inf\{t \geq 0 : |\hat{Y}(t)| \geq \varphi_{\hat{r}(t)}^{-1}(h(\Delta))\}. \tag{3.43}
\]
Then for any \(T > 0\),
\[
\mathbb{P}\{\bar{\rho}_\Delta \leq T\} \leq \frac{C}{(\varphi_{\hat{r}}^{-1}(h(\Delta)))^\bar{p}} \tag{3.44}
\]
for all \(p \in (0, \bar{p}]\), where \(C\) is a positive constant independent of \(\Delta\).

Lemma 3.12 \textit{If Assumptions 1 and 2 hold with} \(4(l+1) \leq \bar{p}\), \textit{the process defined by (3.40) with} \(\tau \in [l/(\bar{p}-2), 1/2]\) \textit{has the property that}
\[
\mathbb{E}|\hat{Y}(T) - X(T)|^2 \leq C\Delta, \quad \forall T > 0.
\]
\textbf{Proof.} Define \(\hat{\theta}_\Delta = \tau_{\varphi^{-1}(h(\Delta))} \land \rho_\Delta \land \bar{\rho}_\Delta\), \(\Omega_1 := \{\omega : \hat{\theta}_\Delta > T\}\), \(\bar{c}(t) = \hat{Y}(t) - X(t)\), for any \(t \in [0, T]\), where \(\tau_N\), \(\rho_\Delta\) and \(\bar{\rho}_\Delta\) are defined by (2.9), (3.19) and (3.43), respectively. Using the Young inequality, we have
\[
\mathbb{E}|\bar{c}(T)|^2 = \mathbb{E}\left(|\bar{c}(T)|^2 I_{\Omega_1}\right) + \mathbb{E}\left(|\bar{c}(T)|^2 I_{\bar{\Omega}_1}\right)
\leq \mathbb{E}\left(|\bar{c}(T)|^2 I_{\Omega_1}\right) + \frac{2\Delta}{\bar{p}} \mathbb{E}\left(|\bar{c}(T)|^p\right) + \frac{\bar{p} - 2}{\bar{p}\Delta^{2/(\bar{p}-2)}} \mathbb{P}(\Omega_1^c). \tag{3.45}
\]
It follows from the results of Theorem 2.1 and Lemma 3.10 that
\[
\frac{2\Delta}{\bar{p}} \mathbb{E}\left(|\bar{c}(T)|^p\right) \leq C\Delta. \tag{3.46}
\]
It follows from (2.9), (3.20) and (3.44) that
\[
\frac{\bar{p} - 2}{\bar{p}\Delta^{2/(\bar{p}-2)}} \mathbb{P}(\Omega_1^c) \leq \frac{\bar{p} - q}{\bar{p}\Delta^{2/(\bar{p}-2)}} \left(\mathbb{P}\{\tau_{\varphi^{-1}(h(\Delta))} \leq T\} + \mathbb{P}\{\rho_\Delta \leq T\} + \mathbb{P}\{\bar{\rho}_\Delta \leq T\}\right)
\leq \frac{3(\bar{p} - 2)}{\bar{p}\Delta^{2/(\bar{p}-2)}} \frac{C}{\varphi_{\hat{r}}^{-1}(h(\Delta)))^\bar{p}} = \frac{3(\bar{p} - 2)}{\bar{p}\Delta^{2/(\bar{p}-2)}} \frac{C}{(K\Delta^{-\tau}/C - 1)^{\bar{p}/l}} \leq C\Delta. \tag{3.47}
\]
On the other hand, note that for any \(t \in [0, T \land \hat{\theta}_\Delta]\),
\[
\bar{c}(t) = \int_0^t \left(f(X(s), r(s)) - f(Y(s), \bar{r}(s))\right)ds + \int_0^t \left(g(X(s), r(s)) - g(Y(s), \bar{r}(s))\right)dB(s).
\]
Using the generalised Itô formula (see, e.g., [2, Lemma 1.9, p.49]) yields that

\[ |\bar{e}(T \wedge \tilde{\vartheta}_\Delta)|^2 \]
\[ = \int_0^{T \wedge \tilde{\vartheta}_\Delta} 2\bar{e}^T(s)(f(X(s), r(s)) - f(Y(s), \bar{r}(s))) + |g(X(s), r(s)) - g(Y(s), \bar{r}(s))|^2 ds + M(T \wedge \tilde{\vartheta}_\Delta), \]

where \( M(t) = 2 \int_0^t \bar{e}^T(s)(g(X(s), r(s)) - g(Y(s), \bar{r}(s))) dB(s) \) is a local martingale (see, e.g., [2]). Choose a small constant \( \iota > 0 \) such that \( 1 + \iota \leq \bar{p} - 1 \), then the application of Young’s inequality implies

\[
\mathbb{E}(|\bar{e}(T \wedge \tilde{\vartheta}_\Delta)|^2) \\
\leq \mathbb{E} \int_0^{T \wedge \tilde{\vartheta}_\Delta} \left[ 2\bar{e}^T(s)(f(X(s), r(s)) - f(\bar{Y}(s), r(s))) + 2\bar{e}^T(s)(f(\bar{Y}(s), r(s)) - f(Y(s), \bar{r}(s))) \\
+ (1 + \iota)|g(X(s), r(s)) - g(\bar{Y}(s), r(s))|^2 + \left( 1 + \frac{1}{\iota} \right)|g(\bar{Y}(s), r(s)) - g(Y(s), \bar{r}(s))|^2 \right] ds \\
\leq \mathbb{E} \int_0^{T \wedge \tilde{\vartheta}_\Delta} \left[ 2\bar{e}^T(s)(f(X(s), r(s)) - f(\bar{Y}(s), r(s))) + (1 + \iota)|g(X(s), r(s)) - g(\bar{Y}(s), r(s))|^2 \\
+ |\bar{e}(s)|^2 + |f(\bar{Y}(s), r(s)) - f(Y(s), \bar{r}(s))|^2 + \left( 1 + \frac{1}{\iota} \right)|g(\bar{Y}(s), r(s)) - g(Y(s), \bar{r}(s))|^2 \right] ds. \ (3.48)\]

It follows from Assumption 2 and the elementary inequality that

\[
\mathbb{E}(|\bar{e}(T \wedge \tilde{\vartheta}_\Delta)|^2) \\
\leq C\mathbb{E} \int_0^{T \wedge \tilde{\vartheta}_\Delta} \left( |\bar{e}(s)|^2 + |f(\bar{Y}(s), r(s)) - f(Y(s), \bar{r}(s))|^2 + |g(\bar{Y}(s), r(s)) - g(Y(s), \bar{r}(s))|^2 \right) ds \\
\leq C\mathbb{E} \left[ \int_0^T |\bar{e}(s \wedge \tilde{\vartheta}_\Delta)|^2 ds + \int_0^T |f(\bar{Y}(s), r(s)) - f(Y(s), \bar{r}(s))|^2 + |g(\bar{Y}(s), r(s)) - g(Y(s), \bar{r}(s))|^2 ds \right] \\
\leq C\mathbb{E} \left[ \int_0^T |\bar{e}(s \wedge \tilde{\vartheta}_\Delta)|^2 ds + \int_0^T |f(\bar{Y}(s), r(s)) - f(Y(s), r(s))|^2 + |g(\bar{Y}(s), r(s)) - g(Y(s), r(s))|^2 ds \\
+ \int_0^T f(Y(s), r(s)) - f(Y(s), \bar{r}(s))|^2 + |g(Y(s), r(s)) - g(Y(s), \bar{r}(s))|^2 ds \right] \\
=: C\mathbb{E} \int_0^T |\bar{e}(s \wedge \tilde{\vartheta}_\Delta)|^2 ds + J_1 + J_2. \ (3.49)\]

Due to (3.36) and (3.37), by Jensen’s inequality and Lemma 3.9, one observes

\[
J_1 \leq C\mathbb{E} \int_0^T \left( |f(\bar{Y}(s), r(s)) - f(Y(s), r(s))|^2 + |g(\bar{Y}(s), r(s)) - g(Y(s), r(s))|^2 \right) ds \\
\leq C\mathbb{E} \int_0^T \left( (1 + |\bar{Y}(s)|^2 + |Y(s)|^2)|\bar{Y}(s) - Y(s)|^2 + (1 + |\bar{Y}(s)|^2 + |Y(s)|^2)|\bar{Y}(s) - Y(s)|^2 \right) ds \\
\leq C\mathbb{E} \int_0^T \left( (1 + |\bar{Y}(s)|^2 + |Y(s)|^2)|\bar{Y}(s) - Y(s)|^2 \right) ds
\]
\[
\leq C \int_0^T \left( \mathbb{E}(1 + |\tilde{Y}(s)|^{2l} + |Y(s)|^{2l})^2 \right)^{\frac{1}{2}} \left( \mathbb{E}|\tilde{Y}(s) - Y(s)|^4 \right)^{\frac{1}{2}} ds
\leq C\Delta. \tag{3.50}
\]

Let \( j = \lfloor T/\Delta \rfloor \), by (3.38), (3.39), using Young’s inequality yields
\[
J_2 \leq C \sum_{k=0}^j \int_{t_{k+1}}^{t_{k+1}} \mathbb{E}\left( |f(Y_k, r(s)) - f(Y_k, r_k)|^2 + |g(Y_k, r(s)) - g(Y_k, r_k)|^2 \right) ds
= C \sum_{k=0}^j \int_{t_{k+1}}^{t_{k+1}} \left[ \mathbb{E}\left( |f(Y_k, r(s)) - f(Y_k, r_k)|^2 + |g(Y_k, r(s)) - f(Y_k, r_k)|^2 \right) I_{\{r(s) \neq r_k\}} \right] ds
\leq C \sum_{k=0}^j \int_{t_{k+1}}^{t_{k+1}} \left[ (1 + |Y_k|^{l/2+1} + |Y_k|^{l+1})^2 \mathbb{E}(I_{\{r(s) \neq r_k\}} | F_{t_k}) \right] ds
\leq C \sum_{k=0}^j \int_{t_{k+1}}^{t_{k+1}} \left[ (1 + |Y_k|^{2l+2}) \mathbb{P}\{r(s) \neq r_k \} \right] ds.
\]

It follows from Theorem 3.4 and the property of Markov chain \( r(\cdot) \) that
\[
J_2 \leq C\Delta \sum_{k=0}^j \int_{t_{k+1}}^{t_{k+1}} \mathbb{E}(1 + |Y_k|^{2l+2}) ds \leq C\Delta. \tag{3.51}
\]

Substituting (3.50) and (3.51) into (3.49), applying the Gronwall inequality, we yield that
\[
\mathbb{E}(|\tilde{\bar{e}}(T)|^2 I_{\Omega_1}) \leq \mathbb{E}(|\tilde{\bar{e}}(T \wedge \tilde{\theta}_\Delta)|^2) \leq C \int_0^T \mathbb{E}(|\tilde{\bar{e}}(s \wedge \tilde{\theta}_\Delta)|^2) ds + C\Delta \leq C\Delta. \tag{3.52}
\]

Inserting (3.46), (3.47) and (3.52) into (3.45) yields the desired assertion. \( \square \)

By the virtues of Lemmas 3.9 and 3.12, we yield the optimal rate of strong convergence.

**Theorem 3.13** If Assumptions 1 and 2 hold with \( 4(l + 1) \leq \bar{p} \), the numerical solution of Scheme (3.4) with \( \tau \in [l/(\bar{p} - 2), 1/2] \) has the property that
\[
\mathbb{E}|Y(T) - x(T)|^2 \leq C\Delta, \quad \forall \ T > 0.
\]

**Remark 3.14** If we can find a uniform function \( \varphi(u) \) such that (3.1) holds for each \( i \in \mathbb{S} \), then we can find the uniform truncation function \( \pi_\Delta(x) \). The second equation in (3.4) degenerates to \( Y_k = \pi_\Delta(\tilde{Y}_k) \) independent of \( r_k \). Note that the results of Theorem 3.6 and Theorem 3.13 still hold for this special case. One observes that sometimes it is hard to find a uniformly continuous function \( \varphi(u) \geq \max_{i \in \mathbb{S}} \varphi_i(u) \) for \( u \in [1, \infty) \). However, for the given \( \Delta \) and \( h(\Delta) \), it is easy to find \( \varphi^{-1}(h(\Delta)) \), the minimum of all \( \varphi_i^{-1}(h(\Delta)) \). Let \( \pi_\Delta(x) = \left( |x| \wedge \varphi^{-1}(h(\Delta)) \right) x/|x| \). Clearly, the property (3.5) still holds for Scheme (3.4) with \( \pi_\Delta(x) \equiv \pi_\Delta(x) \), which implies Theorem 3.6 and Theorem 3.13 still hold. On the other hand, in view of computation, the cost of using the uniform scheme is the same as that of using the non-uniform one because the value of \( r_k \) is fixed before each iteration. More precisely, we illustrate it in Example 3.15.
3.4 Numerical examples

In order to illustrate the efficiency of Scheme (3.4) we recall the introductory example and present some simulations.

**Example 3.15** Consider the stochastic volatility model with random switching between (1.1) and (1.2) modulated by a Markov chain $r(t)$ with generator

$$
\Gamma = \begin{pmatrix}
-4 & 4 \\
0.2 & -0.2
\end{pmatrix},
$$

where initial value $x_0 = (1, 1)^T$, $\ell = 2$ and $B(t)$ is a two-dimensional Brownian motion. Obviously, its coefficients

$$
f(x, 1) = 2.5x(1 - |x|), \quad f(x, 2) = (1, 2)^T - x,
$$

$$
g(x, 1) = \begin{pmatrix}
-1 & \sqrt{2} \\
\sqrt{2} & 1
\end{pmatrix} |x|^{3/2}, \quad g(x, 2) = \begin{pmatrix}
0.2 & -0.5 \\
1 & 0.4
\end{pmatrix} |x|,
$$

are locally Lipschitz continuous for any $x \in \mathbb{R}^2$ and $|g(x, 1)|^2 = 6|x|^3$, $|g(x, 2)|^2 = 1.45|x|^2$.

In order to represent the simulations by Scheme (3.4), we divide it into five steps.

**Step 1.** Examine the hypothesis. Since that

$$
\limsup_{|x| \to \infty} \frac{(1 + |x|^2)^2}{|x|^{4}} \left[ 2x^T f(x, 1) + \text{trace}(g^T(x, 1)g(x, 1)) \right] - (2 - 5/3)|x^T g(x, 1)|^2 
\leq 5,
$$

and

$$
\limsup_{|x| \to \infty} \frac{(1 + |x|^2)^2}{|x|^{4}} \left[ 2x^T f(x, 2) + \text{trace}(g^T(x, 2)g(x, 2)) \right] - (2 - 5/3)|x^T g(x, 2)|^2 
\leq -0.64,
$$

Assumption 1 holds for any $0 < \bar{p} \leq 5/3$. By Theorem 2.1, the unique regular solution $X(t)$ exists.

**Step 2.** Choose $\varphi_i(\cdot)$ and $h(\cdot)$. For any $u \geq 1$, compute

$$
\sup_{|x| \leq u} \left( \frac{|f(x, 1)|}{1 + |x|} \lor \frac{|g(x, 1)|^2}{(1 + |x|)^2} \right) \leq 6u, \quad \sup_{|x| \leq u} \left( \frac{|f(x, 2)|}{1 + |x|} \lor \frac{|g(x, 2)|^2}{(1 + |x|)^2} \right) \leq \sqrt{10}.
$$

Then choose $\varphi_1(u) = 6u$, $\varphi_2(u) = \sqrt{10}$. Obviously, $\varphi_1^{-1}(u) = u/6$, $\varphi_2^{-1}(u) \equiv +\infty$. We set $h(\Delta) = 18\Delta^{-1/2}$ and then it satisfies (3.2) for any $\Delta \in (0, 1]$. We can therefore conclude by Theorem 3.6 that the numerical solution $Y(t)$ of Scheme (3.4) satisfies $\lim_{\Delta \to 0} \mathbb{E}|X(T) - Y(T)|^q = 0$ for any $T \in [0, +\infty)$, $0 < q < 5/3$.

**Step 3.** Matlab code. Next we specify the Matlab code for simulating $Y(T)$:
Step 4. Approximating the error $\mathbb{E}|X(T) - Y(T)|^q$. Due to no closed-form of the solution, using Scheme (3.4), we regard the better approximation with $\Delta = 2^{-19}$ as the exact solution $X(t)$ and compare it with the numerical solution $Y(t)$ with $\Delta = 2^{-8}, 2^{-9}, \ldots, 2^{-17}$. To compute the approximation error, we run $M$ independent trajectories where $X^{(j)}(t)$ and $Y^{(j)}(t)$ represent the $j$th trajectories of the exact solution $X(t)$ and the numerical solution $Y(t)$ respectively. Thus

$$
\mathbb{E}|X(T) - Y(T)|^q = \frac{1}{M} \sum_{j=1}^{M} |X^{(j)}(T) - Y^{(j)}(T)|^q.
$$

Step 5. The log-log error plot with $M = 1000$. The simulation procedure is carried out by steps 3 and 4. The blue solid line depicts log-log error while the red dashed is a reference line of slope $1/2$ in Figure 1. Figure 1 depicts the approximation error $\mathbb{E}|X(10) - Y(10)|$ of the exact solution and the numerical solution of Scheme (3.4) as the function of stepsize $\Delta \in \{2^{-8}, 2^{-9}, \ldots, 2^{-17}\}$. One observes that the schemes proposed in [17, 13, 12] are quite sensitive to the high nonlinearity of the diffusion coefficient, which don’t work for the above equation. However, the performance of Scheme (3.4) is very nice for this case.

On the other hand, as the claim in Remark 3.14, we can also use the uniform truncation mapping $\pi_{\Delta}(x) = (|x| \wedge 3\Delta^{-1/2})x/|x|$. Now let us compare the simulation time of Scheme (3.4) with the
Figure 1: The approximation error $\mathbb{E}|X(10) - Y(10)|$ of the exact solution and the numerical solution by Scheme (3.4) as the function of stepsize $\Delta \in \{2^{-8}, 2^{-9}, \ldots, 2^{-17}\}$.

Figure 2: The approximation error $\mathbb{E}|X(1) - Y(1)|$ of the exact solution, and the numerical solutions by the truncated EM scheme dependent on states and by the uniformly truncated EM scheme, respectively, as functions of the runtime with $\Delta \in \{2^{-13}, 2^{-14}, \ldots, 2^{-17}\}$.

truncation mapping dependent on the states and with the uniform one $\pi^x_{\Delta}(x) \equiv \pi_{\Delta}(x)$. Figure 2 depicts the approximation error $\mathbb{E}|X(1) - Y(1)|$ of the exact solution, and the numerical solutions by the truncated scheme and by the uniformly truncated scheme, respectively, as functions of the runtime with $\Delta \in \{2^{-13}, 2^{-14}, \ldots, 2^{-17}\}$ and $M = 1000$. When $\Delta = 2^{-17}$, the runtime of the truncated scheme achieving the accuracy 0.005479 on the computer with Intel Core 2 duo CPU 2.20GHz, is about 3884 seconds while the runtime of the uniformly truncated scheme achieving the accuracy 0.005479 is about 3885 seconds on the same computer (see the enlargement in Figure 2). Thus, the computational cost of the non-uniform scheme is the same as that of the uniform one.
Let us discuss another example to compare our scheme with the implicit EM scheme.

**Example 3.16** Consider the scalar hybrid cubic SDE (i.e. the stochastic Ginzburg-Laudau equation (4.52) in [8, p.125])

\[
dX(t) = (a(r(t))X(t) + b(r(t))X^3(t))dt + \sigma(r(t))X(t)dB(t), \quad t \geq 0,
\]

where \(r(\cdot)\) is the Markov chain taking values in \(\mathbb{S} = \{1, 2\}\) with generator matrix

\[
\Gamma = \begin{pmatrix}
-\gamma_{12} & \gamma_{12} \\
\gamma_{21} & -\gamma_{21}
\end{pmatrix},
\]

and \(b(i) \leq 0\) for any \(i \in \mathbb{S}\). Obviously, for any \(i \in \mathbb{S}\), \(f(x,i) = a(i)x + b(i)x^3\); \(g(x,i) = \sigma(i)x\) are locally Lipschitz continuous. Assumption 1 holds for any \(\bar{\rho}\) with \(\gamma\) (3.4) converges to the exact solution in the root mean square with error estimate \(\triangle\) holds with any \(\tilde{\rho}\).

In the same way as in [8], we get its closed-form

\[
X(t) = \frac{x_0 \exp \left\{ \int_0^t \left[ a(r(s)) - \frac{1}{2} \sigma^2(r(s)) \right] ds + \int_0^t \sigma(r(s))dB(s) \right\}}{\sqrt{1 - 2x_0^2 \int_0^t b(r(s)) \exp \left\{ \int_s^t \left[ 2a(r(u)) - \sigma^2(r(u)) \right] du + \int_0^t 2\sigma(r(u))dB(u) \right\} ds}},
\]

Clearly, for any \(i \in \mathbb{S}\), \(\sup_{|x| \leq u} \left( \frac{|f(x,i)|}{1 + |x|} \vee \frac{|g(x,i)|^2}{(1 + |x|)^2} \right) \leq c(u^2 + 1)\) for any \(u \geq 1\), where \(c = (|\tilde{b}| \vee |\tilde{\sigma}| \vee |\tilde{a}|, |\tilde{b}|\) = \(\max_{i \in \mathbb{S}} |b(i)|\). Thus, we define \(\varphi(u) = c(u^2 + 1), \forall u \geq 1\), and \(h(\Delta) = \varphi(|x_0|)\Delta^{-0.2}, \forall \Delta \in (0, 1]\). Compute \(\varphi^{-1}(u) = (u/c - 1)^{1/2}\) for any \(u > c\). Note that Assumption 3 holds with any \(\tilde{\rho} > 2\) and \(l = 2\). By the virtue of Theorem 3.13 the numerical solution of Scheme (3.4) converges to the exact solution in the root mean square with error estimate \(\Delta^{1/2}\).

We compare the simulations by the implicit EM scheme and by Scheme (3.4) for SDS (3.53) with \(\gamma_{12} = 1, \gamma_{21} = 4; a(1) = 1, b(1) = -1, \sigma(1) = 2; a(2) = 0.5, b(2) = -1, \sigma(2) = 1; x_0 = 20, \ell = 1\). We specify the truncated EM scheme

\[
\begin{cases}
Y_0 = x_0, \quad r_0 = \ell, \\
\tilde{Y}_{k+1} = Y_k + [a(r_k)Y_k + b(r_k)(Y_k)^3]\Delta + \sigma(r_k)Y_k\Delta B_k, \\
Y_{k+1} = \left( |\tilde{Y}_{k+1}| \wedge \sqrt{(x_0^2 + 1)\Delta^{-0.2} - 1} \right) \tilde{Y}_{k+1} / |\tilde{Y}_{k+1}|, 
\end{cases}
\]

and the implicit EM scheme

\[
\begin{cases}
\tilde{Y}_0 = x_0, \quad r_0 = \ell, \\
\tilde{Y}_{k+1} = \tilde{Y}_k + [a(r_k)\tilde{Y}_{k+1} + b(r_k)(\tilde{Y}_{k+1})^3]\Delta + \sigma(r_k)\tilde{Y}_k\Delta B_k, 
\end{cases}
\]
for any \(k = 0, 1, \ldots, \bar{N} - 1\), where \(T = 2, \bar{N} = \lceil T/\Delta \rceil \geq 7\). Thanks to Cardano’s method, roots of one-dimensional polynomials of degree three are known explicitly. Thus the implicit scheme (3.56) becomes

\[
\begin{cases}
\bar{Y}_0 = x_0, & r_0 = \ell, \\
D_k = \bar{Y}_k(1 + \sigma(r_k)\Delta B_k)/(-2b(r_k)\Delta), \\
H_k = \sqrt{D_k^2 - (1/\Delta - a(r_k))^3/(3b(r_k))^3}, \\
\bar{Y}_{k+1} = (H_k + D_k)^{1/3} - (H_k - D_k)^{1/3}.
\end{cases}
\] (3.57)

Figure 3 depicts the root mean square approximation error with 1000 sample points by different schemes as the functions of runtime, which reveals that the runtime of the implicit EM scheme (3.56) achieving the accuracy \(2/1000\) is 1.43 time than that of the truncated EM scheme (3.55) with the same stepsize.

Figure 3: The black trajectory depicts the root mean square approximation error \((\mathbb{E}|X(2) - \bar{Y}_{\bar{N}}|^2)^{1/2}\) of the exact solution of SDS (3.54) and the numerical solution by the implicit EM scheme (3.57) while the red trajectory depicts the root mean square approximation error \((\mathbb{E}|X(2) - Y_N|^2)^{1/2}\) of the exact solution of SDS (3.54) and the numerical solution by the truncated EM scheme (3.55) as the functions of the runtime with \(\Delta \in \{2^{-10}, 2^{-11}, \ldots, 2^{-19}\}\).

4 Approximation of invariant measure

In this section, we are concerned with the existence and uniqueness of invariant measures of the exact and numerical solutions for SDS (2.1). We reconstruct an explicit scheme and show the existence and uniqueness of the numerical invariant measure converging to the underlying exact one in the Wasserstein metric. Our results cover a large kind of nonlinear SDSs with only locally
Lipschitz continuous coefficients. For the convenience of invariant measure study we introduce some notations as well as an assumption. We write \((X^{n,t}, r^t_i)\) in lieu of \((X(t), r(t))\) to highlight the initial data \((X(0), r(0)) = (x_0, \ell)\). For each \(N > 0\), let \(B_N(0) = \{x \in \mathbb{R}^n : |x| \leq N\}\). Let \(\mathcal{B}(\mathbb{R}^n)\) denote the family of all Borel sets in \(\mathbb{R}^n\) and \(\mathcal{P}(\mathbb{R}^n \times S)\) denote the family of all probability measures on \(\mathbb{R}^n \times S\). For any \(p \in (0, 1]\), define a metric on \(\mathbb{R}^n \times S\) as follows
\[
d_p((x, i), (y, j)) := |x - y|^p + I_{\{i \neq j\}}, \quad (x, i), (y, j) \in \mathbb{R}^n \times S,
\]
and the corresponding Wasserstein distance between \(\mu, \bar{\mu} \in \mathcal{P}(\mathbb{R}^n \times S)\) by
\[
W_p(\mu, \bar{\mu}) := \inf_{\pi \in \mathcal{C}(\mu, \bar{\mu})} \int_{\mathbb{R}^n \times S} \int_{\mathbb{R}^n \times S} d_p(x, y)\pi(dx, dy),
\]
where \(\mathcal{C}(\mu, \bar{\mu})\) denotes the collection of all probability measures on \((\mathbb{R}^n \times S) \times (\mathbb{R}^n \times S)\) with marginal measures \(\mu\) and \(\bar{\mu}\). The Wasserstein distance \(W_p\) admits a duality representation (see, e.g, [34]), i.e.
\[
W_p(\mu, \bar{\mu}) = \sup_{\psi:\text{Lip}(\bar{\psi}) \leq 1} \left\{ \int_{\mathbb{R}^n \times S} \bar{\psi}(x, i)d\mu - \int_{\mathbb{R}^n \times S} \bar{\psi}(y, j)d\bar{\mu} \right\},
\]
where \(\bar{\psi}\) is a continuous function on \(\mathbb{R}^n \times S\) and
\[
\text{Lip}(\bar{\psi}) := \sup \left\{ \frac{\bar{\psi}(x, i) - \bar{\psi}(y, j)}{d_p((x, i), (y, j))} : \forall (x, i) \neq (y, j) \in \mathbb{R}^n \times S \right\}.
\]
From this section we always assume \(\gamma(t)\) is irreducible, namely, the following linear equation
\[
\pi \Gamma = 0, \quad \sum_{i=1}^m \pi_i = 1,
\]
has a unique solution \(\pi = (\pi_1, \ldots, \pi_m) \in \mathbb{R}^1 \times m\) satisfying \(\pi_i > 0\) for each \(i \in S\). This solution is termed a stationary distribution. For any vector \(u = (u_1, \ldots, u_m)^T\), any constant \(p > 0\), define
\[
\text{diag}(u) := \text{diag}(u_1, \ldots, u_m), \quad \Gamma_{p,u} := \Gamma + \frac{p}{2}\text{diag}(u), \quad \eta_{p,u} := -\max_{\nu \in \text{spec}(\Gamma_{p,u})} \text{Re}(\nu),
\]
where \(\text{spec}(\Gamma_{p,u})\) and \(\text{Re}(\nu)\) denote the spectrum of \(\Gamma_{p,u}\) (i.e. the multiset of its eigenvalues) and the real part of \(\nu\), respectively. In order to examine the properties of the exact solution of SDS (2.1), we prepare the following lemma.

**Lemma 4.1** Assume that \(p_u := \sum_{i \in S} \pi_i u_i < 0\). Then for any \(p \in (0, p_u^*)\), \(\eta_{p,u} > 0\), and there is a vector \(\xi_{p,u} := (\xi_1^{p,u}, \ldots, \xi_m^{p,u})^T \gg 0\) such that
\[
\sum_{j=1}^m \gamma_{ij} \xi_{p,u}^{p,u} + \frac{pu_i}{2} \xi_{p,u}^{p,u} = -\eta_{p,u} \xi_{p,u}^{p,u} < 0, \quad i = 1, \ldots, m,
\]
where
\[
\left\{ \begin{array}{ll}
p_u^* = +\infty, & \text{if } \hat{u} \leq 0, \\
p_u^* \in (0, \min_{i \in S, u_i > 0} \{-2\gamma_{ii}/u_i\}), & \text{if } \hat{u} > 0.
\end{array} \right.
\]
Proof. Fix any \( p \in (0, p_u^*) \). Recalling (4.3) and utilizing [35, Proposition 4.2] yields \( \eta_{p,u} > 0 \). Let \( \Gamma_{p,u}(t) = \exp(t \Gamma_{p,u}) \). Then, the spectral radius of \( \text{Ria}(\Gamma_{p,u}(t)) \) (i.e. \( \text{Ria}(\Gamma_{p,u}(t)) = \sup_{\nu \in \text{spec}(\Gamma_{p,u}(t))} |\nu| \)) equals to \( \exp(-\eta_{p,u}t) \). Since all coefficients of \( \Gamma_{p,u}(t) \) are positive, by the Perron-Frobenius theorem (see, e.g., [32, p.6]), \( \exp(-\eta_{p,u}t) \) is a simple eigenvalue of \( \Gamma_{p,u}(t) \) and there exists an eigenvector \( \xi_{p,u} = (\xi_{1,p,u}^p, \ldots, \xi_{m,p,u}^p)^T \gg 0 \) corresponding to \( \exp(-\eta_{p,u}t) \), where \( 0 \in \mathbb{R}^m \) is a zero vector, and \( \xi_{p,u} \gg 0 \) means each component \( \xi_{i,p,u}^p > 0 \) for any \( 1 \leq i \leq m \). Note that eigenvector of \( \Gamma_{p,u}(t) \) corresponding to \( \exp(-\eta_{p,u}t) \) is also an eigenvector of \( \Gamma_{p,u} \) corresponding to \( -\eta_{p,u} \). Thus, we obtain that \( \Gamma_{p,u} \xi_{p,u} = -\eta_{p,u} \xi_{p,u} \ll 0 \). Therefore the desired result follows. \( \square \)

4.1 Boundedness

Since it is related closely to the tightness as well as the ergodicity we go further to investigate the moment boundedness of solutions in infinite time interval. Firstly, we establish the criterion on the moment boundedness of the exact solutions of SDS (2.1). Then we show that the numerical solutions of Scheme (3.4) keep this property very well.

**Theorem 4.2** Suppose that Assumption 1 with matrix \( Q_i \equiv Q \) and \( \pi \alpha < 0 \) hold, for any \( p \in (0, p_u^*) \cap (0, \bar{p}] \), the solution \( X(t) \) of SDS (2.1) satisfies

\[
\sup_{t \geq 0} \mathbb{E}[|X(t)|^p] \leq C, \tag{4.5}
\]

where \( C \) is independent of \( t \) and \( i \), each \( \alpha_i \) is given by Assumption 1, and \( \alpha := (\alpha_1, \ldots, \alpha_m)^T \).

**Proof.** For the given \( \alpha \) and any \( p > 0 \), \( \text{diag}(\alpha) \), \( \eta_{p,\alpha} \), \( p_u^* \) are defined as (4.3) and (4.4). For any \( p \in (0, p_u^*) \cap (0, \bar{p}] \), Lemma 4.1 implies that \( \eta_{p,\alpha} > 0 \) and there exists \( \xi_{p,\alpha} \gg 0 \) such that

\[
\sum_{j=1}^{m} \gamma_{ij} \xi_{j,\alpha} + \frac{p \alpha_i}{2} \xi_{i,\alpha} = -\eta_{p,\alpha} \xi_{i,\alpha} < 0, \quad i = 1, \ldots, m \tag{4.6}
\]

holds. Choose \( 0 < \kappa < (\eta_{p,\alpha}/p) \land 1 \). Using the generalised Itô formula (see, e.g., [2, Lemma 1.9, p.49]) together with (4.6), (2.5)-(2.7), yields

\[
\mathbb{E}\left[ e^{\frac{\eta_{p,\alpha}}{2} t} \left( 1 + X^T(t)QX(t) \right)^{\frac{p}{2}} \xi_{p,\alpha} \right] \\
= \mathbb{E}\left[ (1 + x_0^TQx_0)^{\frac{p}{2}} \xi_{p,\alpha} \right] + \mathbb{E} \int_0^t e^{\frac{\eta_{p,\alpha}}{2} s} \left[ \eta_{p,\alpha} \left( 1 + X^T(s)QX(s) \right)^{\frac{p}{2}} \xi_{p,\alpha} \right. \\
+ \mathcal{L}\left( (1 + X^T(s)QX(s))^{\frac{p}{2}} \xi_{p,\alpha} \right) \bigg] ds \\
\leq (1 + x_0^TQx_0)^{\frac{p}{2}} \xi_{p,\alpha} + \mathbb{E} \int_0^t \frac{p \kappa r_s - \eta_{p,\alpha} e^{\frac{-\eta_{p,\alpha}}{2} s} (1 + X^T(s)QX(s))^{\frac{p}{2}} \xi_{p,\alpha}}{2} ds + \mathbb{E} \int_0^t C_r e^{\frac{-\eta_{p,\alpha}}{2} s} ds
\]
\[ \leq (1 + x_0^T Q x_0) \frac{p}{2} \xi_{r_i}^{p,\alpha} + C \int_0^t e^{-\frac{\eta_{r_i}}{2} s} ds, \]

where

\[ \mathcal{L} \left( (1 + x^T Q x) \frac{p}{2} \xi_{r_i}^{p,\alpha} \right) = \frac{p}{2} (1 + x^T Q x) \frac{p}{2} \xi_{r_i}^{p,\alpha} + \sum_{j=1}^m \gamma_{ij} (1 + x^T Q x) \frac{p}{2} \xi_{r_j}^{p,\alpha} \leq \frac{p}{2} ( \alpha_i + \kappa_i ) \xi_{r_i}^{p,\alpha} + \sum_{j=1}^m \gamma_{ij} \xi_{r_j}^{p,\alpha} \]

One observes that \( \mathbb{E} \left[ (1 + X^T(t)QX(t)) \frac{p}{2} \xi_{r_i}^{p,\alpha} \right] \leq Ce^{-\frac{\eta_{r_i}}{2} t} + C. \) Therefore the desired assertion follows from that

\[ \mathbb{E} |X(t)|^p \leq (\lambda_{\min}(Q))^{-\frac{p}{2}} \left( \xi_{r_i}^{p,\alpha} \right)^{-1} \mathbb{E} \left[ (1 + X^T(t)QX(t)) \frac{p}{2} \xi_{r_i}^{p,\alpha} \right] \leq C. \]

The proof is complete. \( \square \)

In order to show that the numerical solutions of Scheme (3.4) preserve this asymptotic boundedness perfectly, we further require the function \( h(\triangle) \) satisfying

\[ \triangle^{1/2-\theta} h(\triangle) \leq K, \quad \forall \, \triangle \in (0, 1] \]

for some \( 0 < \theta < \frac{1}{2} \). Then the asymptotic boundedness of the numerical solution is obtained as follows.

**Theorem 4.3** Under the conditions of Theorem 4.2, for any \( p \in (0, p^*_\alpha) \cap (0, \bar{p}] \), there is a constant \( \triangle^* \in (0, 1] \) such that the scheme (3.4) has the property that

\[ \sup_{0 < \triangle \leq \triangle^*} \sup_{k \geq 0} \mathbb{E} |Y_k|^p \leq C, \]

where \( C \) is independent of \( k \) and \( \triangle \).

**Proof.** For any \( p \in (0, p^*_\alpha) \cap (0, \bar{p}] \), by the virtue of (3.4), we have

\[ (1 + Y_{k+1}^T Q Y_{k+1}) \frac{p}{2} = (1 + Y_k^T Q Y_k) \frac{p}{2} (1 + \zeta_k) \frac{p}{2} \]

for any integer \( k \geq 0 \), where

\[ \zeta_k = (1 + Y_k^T Q Y_k)^{-1} \left( 2Y_k^T Q f(Y_k, r_k) \Delta + \Delta B_k^T g(Y_k, r_k) Q g(Y_k, r_k) \Delta B_k + 2Y_k^T Q g(Y_k, r_k) \Delta B_k \right) f^T(Y_k, r_k) Q f(Y_k, r_k) \Delta^2 + 2f^T(Y_k, r_k) Q g(Y_k, r_k) \Delta B_k \Delta. \]
One observes $\zeta_k > -1$. By Lemma 3.3, without loss the generality, we prove (4.8) only for $0 < p \leq 2$. It follows from (3.11) and (4.9) that

$$\begin{align*}
\mathbb{E}\left[\left(1 + \tilde{Y}_k^T Q Y_{k+1}\right)^{\frac{p}{2}} \xi_{r_k+1} | \mathcal{F}_{t_k}\right] \\
\leq (1 + Y_k^T Q Y_k)^{\frac{p}{2}} \mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{F}_{t_k}\right] + \frac{p}{2} \mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{F}_{t_k}\right] + \frac{p(p - 2)}{8} \mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{F}_{t_k}\right] \\
+ \frac{p(p - 2)(p - 4)}{48} \mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{F}_{t_k}\right].
\end{align*}$$

(4.10)

where the vector $\xi_{r_k} = (\xi_1^{p,\alpha}, \ldots, \xi_m^{p,\alpha})^T \gg 0$ is given in the proof of Theorem 4.2. It follows from Lemma 3.2 that $\mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{F}_{t_k}\right] = \xi_{r_k} + \sum_{j \in \mathbb{S}} \xi_{r_k} (\gamma_{r_k,j} \triangle + o(\triangle))$. Then, making use of the techniques in the proof of Theorem 3.4 as well as (4.7) yields

$$\begin{align*}
\mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{F}_{t_k}\right] &= \mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{G}_{t_k}\right] | \mathcal{F}_{t_k}\right] \\
\leq \frac{\psi(Y_k, r_k) \cdot \triangle}{1 + Y_k^T Q Y_k} + C \Delta^{1+2\theta} \left[\sum_{j \in \mathbb{S}} \xi_{r_k} (\gamma_{r_k,j} \triangle + o(\triangle))\right] \\
&\leq \frac{\psi(Y_k, r_k) \cdot \triangle}{1 + Y_k^T Q Y_k} + C \Delta^{1+2\theta}.
\end{align*}$$

(4.11)

One further observes that

$$\begin{align*}
\mathbb{E}\left[\xi_{r_k+1}^2 | \mathcal{F}_{t_k}\right] &= \mathbb{E}\left[\xi_{r_k+1}^2 | \mathcal{G}_{t_k}\right] | \mathcal{F}_{t_k}\right] \\
&\leq 4(1 + Y_k^T Q Y_k)^{-2} \left[(Y_k^T Q g(Y_k, r_k))^2 \Delta - 2|Q|^2(1 + |Y_k|)^4 h^2(\Delta) \Delta^2\right] \mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{F}_{t_k}\right] \\
&\geq \frac{4|Y_k|^2 Q g(Y_k, r_k)|^2 \Delta}{(1 + Y_k^T Q Y_k)^2} - C \Delta^{1+2\theta}.
\end{align*}$$

(4.12)

From (3.5), (3.17) and (4.7), one obtains

$$\begin{align*}
\mathbb{E}\left[\xi_{r_k+1}^3 | \mathcal{F}_{t_k}\right] &= \mathbb{E}\left[\xi_{r_k+1}^3 | \mathcal{G}_{t_k}\right] | \mathcal{F}_{t_k}\right] \\
&\leq C(1 + Y_k^T Q Y_k)^{-3} \left[|Y_k|^3 |f(Y_k, r_k)|^3 \Delta^3 + |g(Y_k, r_k)|^6 \Delta^3 + |f(Y_k, r_k)|^6 \Delta^6 \\
&+ |g(Y_k, r_k)|^2 \left(|Y_k|^2 + |f(Y_k, r_k)|^2 \Delta^2\right) \Delta^2 \left(|Y_k||f(Y_k, r_k)| + |g(Y_k, r_k)|^2 + |f(Y_k, r_k)|^2 \Delta\right)\right] \mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{F}_{t_k}\right] \\
&\leq C(1 + Y_k^T Q Y_k)^{-3} \left(1 + |Y_k|^6 \right) \left[h^3(\Delta) \Delta^3 + h^6(\Delta) \Delta^6 \\
&+ h^2(\Delta) \Delta^2 \left(1 + h^2(\Delta) \Delta^2\right) \left(1 + h(\Delta)\Delta\right)\right] \mathbb{E}\left[\xi_{r_k+1}^\alpha | \mathcal{F}_{t_k}\right] \leq C \Delta^{1+2\theta}.
\end{align*}$$

(4.13)

Similarly, we can also prove that for any integer $l > 3$, $\mathbb{E}\left[|\xi_{r_k+1}^l | \mathcal{F}_{t_k}\right] \leq C \Delta^{1+2\theta}$. For any integer $k \geq 0$, substituting (4.11)-(4.13) into (4.10), we derive from (4.6), (2.5) and (2.7) that

$$\mathbb{E}\left[\left(1 + \tilde{Y}_k^T Q Y_{k+1}\right)^{\frac{p}{2}} \xi_{r_k+1} | \mathcal{F}_{t_k}\right]$$
\[
\leq (1 + Y_k^T Q Y_k)^{\frac{p}{2}} \left\{ \xi_{\rho_k}^{p,\alpha} + \sum_{j \in S} \xi_{\gamma_{r_k,j}}^{p,\alpha} \Delta + o(\Delta) \right\} + \frac{p\Delta (1 + Y_k^T Q Y_k)\psi(Y_k, r_k) + (p - 2)|Y_k^T Q g(Y_k, r_k)|^2}{(1 + Y_k^T Q Y_k)^2} \xi_{\rho_k}^{p,\alpha} \n\]
\[
\leq (1 + Y_k^T Q Y_k)^{\frac{p}{2}} \left[ \xi_{\rho_k}^{p,\alpha} - \frac{\eta_{p,\alpha}}{2} \xi_{\rho_k}^{p,\alpha} \Delta + o(\Delta) \right] + C\Delta.
\]

Choose \( \Delta^* \in (0, 1] \) sufficiently small such that \( \Delta^* < 4/\eta_{p,\alpha}, o(\Delta^*) \leq \eta_{p,\alpha} \hat{\xi}_{p,\alpha} \Delta^*/4. \) Taking expectations on both sides, for any \( \Delta \in (0, \Delta^*], \) yields
\[
\mathbb{E} \left[ (1 + Y_{k+1}^T Q Y_{k+1})^{\frac{p}{2}} \xi_{\rho_k}^{p,\alpha} \right] \leq \mathbb{E} \left[ (1 + Y_{k+1}^T Q Y_{k+1})^{\frac{p}{2}} \xi_{\rho_k}^{p,\alpha} \right] \leq \left( 1 - \frac{\eta_{p,\alpha}}{4} \right) \mathbb{E} \left[ (1 + Y_k^T Q Y_k)^{\frac{p}{2}} \xi_{\rho_k}^{p,\alpha} \right] + C\Delta.
\]
for any integer \( k \geq 0. \) Repeating this procedure arrives at
\[
\mathbb{E} \left[ (1 + Y_k^T Q Y_k)^{\frac{p}{2}} \xi_{\rho_k}^{p,\alpha} \right] \leq \mathbb{E} \left[ (1 + Y_k^T Q Y_k)^{\frac{p}{2}} \xi_{\rho_k}^{p,\alpha} \right] \leq \frac{4C}{\eta_{p,\alpha}} \left[ 1 - \left( 1 - \frac{\eta_{p,\alpha}}{4} \right)^k \right] \leq C.
\]
Therefore, \( \sup_{k \geq 0} \mathbb{E}|Y_k|^p \leq C. \) The desired assertion follows. \( \square \)

### 4.2 Invariant measure

In this subsection, we discuss the criterion on the existence and uniqueness of the invariant measure for SDS (2.1). It follows from SDS (2.1) that
\[
d(X_t^{x_0,\ell} - X_t^{\bar{x}_0,\ell}) = \left( f(X_t^{x_0,\ell}, r_t^{\ell}) - f(X_t^{\bar{x}_0,\ell}, r_t^{\ell}) \right) dt + \left( g(X_t^{x_0,\ell}, r_t^{\ell}) - g(X_t^{\bar{x}_0,\ell}, r_t^{\ell}) \right) dB(t)
\]
\[
= F(X_t^{x_0,\ell}, X_t^{\bar{x}_0,\ell}, r_t^{\ell}) dt + G(X_t^{x_0,\ell}, X_t^{\bar{x}_0,\ell}, r_t^{\ell}) dB(t) \tag{4.14}
\]
for any two initial values \((x_0, \ell), (\bar{x}_0, \ell)\) \( \in K \times S, \) where \( K \) is a compact set in \( \mathbb{R}^n. \) For convenience we impose the following hypothesis.

**Assumption 3** There exists a positive constant \( \rho > 0 \) such that
\[
|f(x) - f(y)|^2 \leq \beta_i |x - y|^4, \quad \forall x, y \in \mathbb{R}^n, \quad i \in S,
\]
where \( f(x) = 2(x - y)^T F(x, y, i) + |G(x, y, i)|^2, \) and each \( \beta_i \) is a constant.

In order for the unique invariant measure we provide the asymptotic attractivity of SDS (2.1).

**Lemma 4.4** Suppose that Assumption 1 with \( Q_i \equiv Q \) and Assumption 3 hold, \( \pi \alpha < 0 \) and \( \pi \beta < 0. \) Then for any \( p \in (0, p^*_\alpha \wedge p^*_\beta \wedge \tilde{p}) \cap (0, \rho], \) any compact \( K \subset \mathbb{R}^n, \) any compact \( S \subset \mathbb{R} \) respectively satisfy
\[
\lim_{t \to \infty} \mathbb{E}|X_t^{x_0,\ell} - X_t^{\bar{x}_0,\ell}|^p = 0, \tag{4.15}
\]
where each \( \beta_i \) is given by Assumption 3 and \( \beta := (\beta_1, \ldots, \beta_m)^T. \)
Proof. For the given \( \beta \) and any \( p > 0, \) \( \text{diag}(\beta), \Gamma_{p,\beta}, \eta_{p,\beta}, P_{\beta}^p \) are defined as (4.3) and (4.4). Fix \( p \in (0, p_0^\ast) \cap (0, \rho) \). Lemma 4.1 implies that \( \eta_{p,\beta} > 0 \) and there exists \( \xi_{p,\beta} > 0 \) such that
\[
\sum_{j=1}^{m} \gamma_{ij} \xi_{p,\beta}^j + \frac{p\beta_i}{2} \xi_{p,\beta}^i = -\eta_{p,\beta} \xi_{p,\beta} < 0, \quad i = 1, \ldots, m
\] (4.16)
holds. By generalised Itô’s formula (see e.g., [2, Theorem 1.45, p.48]), we obtain
\[
e^{\eta_{p,\beta} t} |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^p \xi_{p,\beta} \sum_{r(t)}
= |x_0 - \bar{x}_0|^p \xi_{p,\beta} + \int_0^t e^{\eta_{p,\beta} s} \left\{ \eta_{p,\beta} |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^p \xi_{p,\beta} + \mathcal{L} \left( |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^p \xi_{p,\beta} \right) \right\} ds + M(t),
\] (4.17)
where
\[
\mathcal{L} \left( |x - y|^p \xi_{p,\beta} \right) = |x - y|^p \left[ \frac{p \xi_{p,\beta}}{2} |x - y|^2 \psi(x, y, i) - (2 - p) |(x - y)^T G(x, y, i)|^2 + \sum_{j=1}^{m} \gamma_{ij} \xi_{p,\beta}^j \right],
\]
and
\[
M(t) = \int_0^t e^{\eta_{p,\beta} s} \sum_{r(s)} \left( |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^p \xi_{p,\beta} - \xi_{p,\beta} \xi_{r(s)} \right) \mu(ds, dl)
\]
where \( \mu(ds, dl) = \nu(ds, dl) - \mu(dl)ds \) is a martingale measure, the definition of \( h \) can be found in [2, p.48], and \( M(t) \) is a real-valued local continuous martingale (see [2]) with \( M(0) = 0 \). Thus, taking expectations on both sides, using (4.16) and Assumption 3 implies
\[
\mathbb{E} |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^p \leq C \exp \left( -\eta_{p,\beta} t \right)
\] (4.18)
for any two initial values \((x_0, \ell), (\bar{x}_0, \ell) \in \mathbb{K} \times \mathbb{S}\). Define a stopping time \( \bar{\tau} = \inf \{ t \geq 0 : r_t = r_\ell \} \). Due to the irreducibility of \( r(\cdot) \), there exists a constant \( \bar{\lambda} > 0 \) such that
\[
\mathbb{P}(\bar{\tau} > t) \leq e^{-\bar{\lambda} t}, \quad \forall t > 0.
\] (4.19)
For any \( p \in (0, p_\alpha^* \wedge p_\rho^* \wedge \bar{p}) \cap (0, \rho) \), choose \( q > 1 \) such that \( pq \in (0, p_\alpha^*) \cap (0, \bar{p}) \). Using Hölder’s inequality, (4.18) and (4.19) yields
\[
\mathbb{E} |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^p \leq \mathbb{E} \left( |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^p I_{\{\bar{\tau} > t/2\}} \right) + \mathbb{E} \left( |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^p I_{\{\bar{\tau} \leq t/2\}} \right)
\leq \left( \mathbb{E} |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^{pq} \right)^{\frac{1}{q}} \left[ \mathbb{P}(\bar{\tau} > t/2) \right]^{1-\frac{1}{q}} + \mathbb{E} \left( |X_{t}^{x_0, \ell} - X_{t}^{x_0, \ell}|^p |\mathcal{F}_\tau \right)
\]
Thus the desired assertion (4.15) follows from (4.20). The proof is complete.

As well as we know the stability of the trivial solution is one of the major concerns in many applications. As a corollary, the result on the stability follows from the proof of Lemma 4.4 directly. For clarity we impose the following assumption.

**Assumption 4** If

\[ f(0, i) = 0, \quad g(0, i) = 0, \quad \forall i \in S \]  

holds, and for some \( \rho > 0 \), there exists a symmetric positive-definite matrix \( Q \in \mathbb{R}^{n \times n} \) such that

\[ (x^T Q x) \psi(x, i) - (2 - \rho) |x^T Q g(x, i)|^2 \leq \beta_i (x^T Q x)^2, \quad \forall x \in \mathbb{R}^n, \quad i \in S \]  

holds, where \( \psi(x, i) := 2x^T Q f(x, i) + \text{trace}(g^T(x, i) Q g(x, i)) \), and \( \beta_i \) is a constant.

Due to the above assumption, one observes that \( X(t) \equiv 0 \) is the trivial solution of SDS (2.1) with initial values \( X(0) = 0 \), any \( r(0) \in \mathbb{S} \).

**Corollary 4.5** Suppose that Assumption 4 holds and \( \pi \beta < 0 \). Then for any \( p \in (0, p^*_\beta) \cap (0, \rho] \), the solution \( X(t) \) of SDS (2.1) has the property that

\[ \limsup_{t \to \infty} \frac{\log(\mathbb{E}|X(t)|^p)}{t} \leq -\frac{\eta_{p, \beta}}{p}, \quad \text{a.s.} \]  

where each \( \beta_i \) is given by Assumption 4, \( \beta = (\beta_1, \ldots, \beta_m)^T \) and \( \eta_{p, \beta} \) is defined by (4.3).

**Proof.** In the same way as the proof of Lemma 4.4, for any initial data \( (x_0, \ell) \in \mathbb{R}^n \times \mathbb{S} \), the solution \( X_{t}^{x_0, \ell} \) satisfies inequality (4.18) with \( X_{t}^{x_0, \ell} \equiv 0 \). Then (4.23) follows directly. On the other hand, due to Assumption 4 and (4.17) with \( X_{t}^{x_0, \ell} \equiv 0 \), by the nonnegative semimartingale convergence theorem (see, e.g., [2, p.18, Theorem 1.10]), we obtain \( \limsup_{t \to \infty} \hat{\xi}^p_{t} e^{\eta_{p, \beta t}} |X_{t}^{x_0, \ell}|^p < \infty \text{ a.s.} \). Therefore the other required assertion follows. The proof is complete. \( \square \)
Next we give the existence and uniqueness of the invariant measure for the solution \((X^{x_0,\ell}_t, r^\ell_t)\) of SDS (2.1). Let \(P_t(x_0, \ell; dx \times \{i\})\) be the transition probability kernel of the pair \((X^{x_0,\ell}_t, r^\ell_t)\), a time homogeneous Markov process (see, e.g., [2, Theorem 3.27, pp.104-105]). Recall that \(\mu \in \mathcal{P}(\mathbb{R}^n \times S)\) is called an invariant measure of \((X^{x_0,\ell}_t, r^\ell_t)\) if

\[
\mu(\mathcal{Y} \times \{\ell\}) = \sum_{i=1}^m \int_{\mathbb{R}^n} P_t(x, i; \mathcal{Y} \times \{\ell\}) \mu(dx \times \{i\}), \quad \forall t \geq 0, \; \mathcal{Y} \in \mathcal{B}(\mathbb{R}^n), \; \ell \in S
\]

holds.

**Theorem 4.6** Under the conditions of Lemma 4.4, the solution \((X^{x_0,\ell}_t, r^\ell_t)\) admits a unique invariant measure \(\mu \in \mathcal{P}(\mathbb{R}^n \times S)\).

**Proof.** For arbitrary \(t > 0\), define a probability measure

\[
h_t(\mathcal{Y}) = \frac{1}{t} \int_0^t P_s(x, i; \mathcal{Y}) ds, \quad \mathcal{Y} \in \mathcal{B}(\mathbb{R}^n \times S).
\]

Then, for any \(\varepsilon > 0\), by Theorem 4.2 and Chebyshev’s inequality, there exists a positive integer \(N > 0\) sufficiently large such that

\[
h_t(B_N(0) \times S) = \frac{1}{t} \int_0^t P_s(x_0, \ell; B_N(0) \times S) ds \geq 1 - \frac{\sup_{t \geq 0} \mathbb{E}|X^{x_0,\ell}_t|^p}{N^p} \geq 1 - \varepsilon
\]

for any \(p \in (0, p_\alpha^* \cap (0, \bar{\rho}])\). Hence \(\{h_t\}_{t \geq 0}\) is tight since \(B_N(0)\) is a compact subset of \(\mathbb{R}^n\). Moreover, one observes that \((X^{x_0,\ell}_t, r^\ell_t)\) enjoys the Feller property (see, e.g., [5, Theorem 2.18, p.48]). Thus, the solution \((X^{x_0,\ell}_t, r^\ell_t)\) of SDS (2.1) has an invariant measure (see, e.g., [36, Theorem 4.14, p.128]). We have established the existence of the invariant measure, and next we go a further step to show its uniqueness. For any \(p \in (0, p_\alpha^* \land p_\beta^* \land \bar{\rho}) \cap (0, \rho]\), (4.15) and the irreducibility of \(r(\cdot)\) result in

\[
W_p(\delta_{(x_0,\ell)} P_t, \delta_{(x_0,\ell)} P_t) \leq \mathbb{E}|X^{x_0,\ell}_t - X^{x_0,\ell}_0|^p + \mathbb{P}(r^\ell_t \neq \bar{r}^\ell_t) \to 0, \quad t \to \infty, \quad (4.25)
\]

where \(\delta_{(x,i)}\) stands for the Dirac’s measure at the point \((x, i)\). Assume both \(\mu\) and \(\bar{\mu}\) are invariant measures, then we have

\[
W_p(\bar{\mu}, \mu) = W_p(\bar{\mu} P_t, \mu P_t) = \sup_{\tilde{\psi}: \text{Lip}(\tilde{\psi}) \leq 1} \left\{ \int_{\mathbb{R}^n \times S} \tilde{\psi}(x, i) d(\bar{\mu} P_t) - \int_{\mathbb{R}^n \times S} \tilde{\psi}(y, j) d(\mu P_t) \right\}
\]

\[
\leq \int_{\mathbb{R}^n \times S} \bar{\mu}(dx \times \{i\}) \mu(dy \times \{j\}) W_p(\delta_{(x,i)} P_t, \delta_{(y,j)} P_t).
\]

One then observes that, due to (4.25), \(W_p(\bar{\mu}, \mu) \to 0\) as \(t \to \infty\). The desired assertion follows. \(\square\)
4.3 The invariant measure of numerical solution

In order to approximate the invariant measure $\mu$ of SDS (2.1) we need to construct the appropriate scheme such that the numerical solutions are attractive in $\rho$th moment and admit a unique numerical invariant measure. However, the proposed mapping $\pi_\Delta^i(x)$ is not suitable for the avoidance of extra wide divergence in distance between two different numerical solutions. Thus we construct a new truncation mapping $\bar{\pi}_\Delta^i(x)$ according to the local Lipschitz continuity of the drift and diffusion coefficients. Then making use of the appropriate truncation mapping we give an explicit scheme. Finally we show that, it produces a unique numerical invariant measure $\bar{\mu}$ which tends to the invariant measure $\mu$ of SDS (2.1) as $\Delta \to 0$ in the Wasserstein metric.

For each $i \in S$ choose a strictly increasing continuous functions $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi_i(u) \to \infty$ as $u \to \infty$ and
\[
\sup_{|x| \leq u, x \neq y} \left( \left| f(x, i) - f(y, i) \right| \vee \left| g(x, i) - g(y, i) \right| \right) \leq \varphi_i(u), \quad \forall u \geq 1. \tag{4.26}
\]
Due to the local Lipschitz continuity of $f$ and $g$ the function $\varphi_i$ can be well defined as well as its inverse function $\varphi_i^{-1} : [\varphi_i(1), \infty) \to \mathbb{R}_+$. We choose a strictly decreasing $\bar{h} : (0, 1] \to [\hat{\varphi}(\bar{h}(0), \infty)$ such that
\[
\bar{h}(1) \geq \max_{i \in S} \{|f(0, i)| \vee |g(0, i)|^2\}, \quad \lim_{\Delta \to 0} \bar{h}(\Delta) = \infty, \quad \text{and} \quad \Delta^{1/2-\bar{\theta}} \bar{h}(\Delta) \leq K, \quad \forall \Delta \in (0, 1] \tag{4.27}
\]
for some $0 < \bar{\theta} < 1/2$ holds, where $K$ is a positive constant independent of $k$ and $\Delta$. For a given stepsizes $\Delta \in (0, 1]$, let us define the truncated mapping $\bar{\pi}_\Delta^i : \mathbb{R}^n \to \mathbb{R}^n$ by
\[
\bar{\pi}_\Delta^i(x) = \left( |x| \wedge \varphi_i^{-1}(\bar{h}(\Delta)) \right) \frac{x}{|x|}, \tag{4.28}
\]
where we let $\frac{x}{|x|} = 0$ when $x = 0 \in \mathbb{R}^n$. Obviously, for any $i \in S$ and $x, y \in \mathbb{R}^n$,
\[
|f(\bar{\pi}_\Delta^i(x), i) - f(\bar{\pi}_\Delta^i(y), i)| \leq \bar{h}(\Delta)|\bar{\pi}_\Delta^i(x) - \bar{\pi}_\Delta^i(y)|, \\
|g(\bar{\pi}_\Delta^i(x), i) - g(\bar{\pi}_\Delta^i(y), i)|^2 \leq \bar{h}(\Delta)|\bar{\pi}_\Delta^i(x) - \bar{\pi}_\Delta^i(y)|^2, \tag{4.29}
\]
and
\[
|f(\bar{\pi}_\Delta^i(x), i)| \leq \bar{h}(\Delta)(1 + |\bar{\pi}_\Delta^i(x)|), \quad |g(\bar{\pi}_\Delta^i(x), i)|^2 \leq \bar{h}(\Delta)(1 + |\bar{\pi}_\Delta^i(x)|)^2. \tag{4.30}
\]

Remark 4.7 If there exists the state $i \in S$ such that
\[
|f(x, i) - f(y, i)| \leq C|x - y|, \quad |g(x, i) - g(y, i)|^2 \leq C|x - y|^2, \quad \forall x, y \in \mathbb{R}^n, \tag{4.31}
\]
we choose $\varphi_i(u) \equiv C$ for any $u \geq 1$, and let $\varphi_i^{-1}(u) \equiv +\infty$ for any $u \in [C, +\infty)$. Then $\bar{\pi}_\Delta^i(x) = x$, (4.29) and (4.30) hold always.
For any given stepsize \( \Delta \in (0, 1] \), define a new truncated EM method scheme by

\[
\begin{cases}
  \tilde{Z}_0 = x_0, & r_0 = \ell, \\
  Z_k = \pi^\ell_k(\tilde{Z}_k), \\
  \tilde{Z}_{k+1} = Z_k + f(Z_k, r_k)\Delta + g(Z_k, r_k)\Delta B_k,
\end{cases}
\]  

(4.32)

for any integer \( k \geq 0 \). To obtain the continuous-time approximation, define \( Z(t) := Z_k \) for any \( t \in [t_k, t_{k+1}) \). We write \((Z_k^{x_0,\ell}, r_k^{\ell})\) in lieu of \((Z_k, r_k)\) to highlight the initial data \((Z_0, r_0) = (x_0, \ell)\). Then we have

\[
|f(Z_k^{x_0,\ell}, r_k^{\ell}) - f(Z_k^{x_0,\ell}, r_k^{\ell})| \leq \tilde{h}(\Delta)|Z_k^{x_0,\ell} - Z_k^{x_0,\ell}|, \\
|g(Z_k^{x_0,\ell}, r_k^{\ell}) - g(Z_k^{x_0,\ell}, r_k^{\ell})|^2 \leq \tilde{h}(\Delta)|Z_k^{x_0,\ell} - Z_k^{x_0,\ell}|^2,
\]

(4.33)

and linear property

\[
|f(Z_k^{x_0,\ell}, r_k^{\ell})| \leq \tilde{h}(\Delta)(1 + |Z_k^{x_0,\ell}|), \\
|g(Z_k^{x_0,\ell}, r_k^{\ell})|^2 \leq \tilde{h}(\Delta)(1 + |Z_k^{x_0,\ell}|)^2.
\]

(4.34)

Thus, the above properties support that the conclusion of Theorem 3.6 holds for Scheme (4.32). In order for the uniqueness of the numerical invariant measure we prepare the attractive property of the numerical solutions.

**Lemma 4.8** Under the conditions of Lemma 4.4, there is a constant \( \Delta^{**} \in (0, \Delta^*] \) such that for any \( p \in (0, p_0^* \wedge p_\beta^* \wedge p) \cap (0, \rho) \), Scheme (4.32) has the property that

\[
\lim_{k \to \infty} \mathbb{E}|Z_k^{x_0,\ell} - Z_k^{x_0,\ell}|^p = 0
\]

(4.35)

for any \( \Delta \in (0, \Delta^{**}] \), \((x_0, \ell), (\bar{x}_0, \bar{\ell}) \in \mathbb{K} \times \mathbb{S} \), where \( \Delta^* \) is given in Theorem 4.3.

**Proof.** Recall the definitions of \( F \) and \( G \) in (4.14), we know that

\[
\tilde{Z}_{k+1}^{x_0,\ell} - \tilde{Z}_{k+1}^{x_0,\ell} = Z_k^{x_0,\ell} - Z_k^{x_0,\ell} + F(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_k^{\ell})\Delta + G(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_k^{\ell})\Delta B_k.
\]

(4.36)

Without loss the generality, we prove (4.35) only for \( 0 < p \leq 2 \). For any \( p \in (0, p_0^* \cap (0, \rho \wedge 2) \), any \( \varepsilon > 0 \), we derive from (3.11) and (4.36) that

\[
\mathbb{E}\left[ (\varepsilon + |Z_k^{x_0,\ell} - Z_k^{x_0,\ell}|^2)^\frac{p}{2} \xi_{rk+1}^{p,\beta} | \mathcal{F}_k \right] \\
\leq \left(\varepsilon + |Z_k^{x_0,\ell} - Z_k^{x_0,\ell}|^2\right)^\frac{p}{2} \left\{ \mathbb{E}[\xi_{rk+1}^{p,\beta} | \mathcal{F}_k] + \frac{p(p-2)}{8} \mathbb{E}[\xi_{rk+1}^{p,\beta} | \mathcal{F}_k] + \frac{p(p-2)(p-4)}{48} \mathbb{E}[\xi_{rk+1}^{p,\beta} | \mathcal{F}_k] \right\},
\]

(4.37)
where \( \xi^{p,\beta} \) satisfying (4.16) is given in the proof of Lemma 4.4, and

\[
\bar{\varsigma}_k = \left( \varepsilon + \left| Z_{k_0}^{x_0,\ell} - Z_{k_0}^{x_0,\ell} \right|^2 \right)^{-1} \left[ 2 \left( Z_{k_0}^{x_0,\ell} - Z_k^{x_0,\ell} \right)^T F(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_\ell^k) \triangle + \left| G(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_\ell^k) \right|^2 \right] + 2 \left( Z_{k_0}^{x_0,\ell} - Z_k^{x_0,\ell} \right)^T G(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_\ell^k) \triangle B_k + \left| F(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_\ell^k) \right|^2 \triangle^2 + 2 F^T(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_\ell^k) G(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_\ell^k) \triangle B_k \triangle .
\]

Using the techniques in the proof of Theorem 4.3, by (4.33), we deduce that

\[
\mathbb{E}\left[ \bar{\varsigma}_k \xi^{p,\beta}_{\varsigma_{r_k+1}} | \mathcal{F}_{t_k} \right] = \mathbb{E}\left[ \bar{\varsigma}_k \xi^{p,\beta}_{\varsigma_{r_k+1}} | \mathcal{G}_{t_k} \right] \mathbb{E}\left[ \mathcal{F}_{t_k} \right] \leq \frac{\psi(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_\ell^k) \xi^{p,\beta}_{\varsigma_{r_k+1}} \triangle}{\varepsilon + \left| Z_{k_0}^{x_0,\ell} - Z_k^{x_0,\ell} \right|^2} + C \triangle^{1+2\bar{\beta}}, 
\]

(4.38)

\[
\mathbb{E}\left[ \bar{\varsigma}_k^2 \xi^{p,\beta}_{\varsigma_{r_k+1}} | \mathcal{F}_{t_k} \right] \geq \frac{4 \left( Z_{k_0}^{x_0,\ell} - Z_k^{x_0,\ell} \right)^T T G(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_\ell^k) \xi^{p,\beta}_{\varsigma_{r_k+1}}}{\left( \varepsilon + \left| Z_{k_0}^{x_0,\ell} - Z_k^{x_0,\ell} \right|^2 \right)} - C \triangle^{1+2\bar{\beta}}, 
\]

(4.39)

and

\[
\mathbb{E}\left[ \bar{\varsigma}_k^3 \xi^{p,\beta}_{\varsigma_{r_k+1}} | \mathcal{F}_{t_k} \right] = \mathbb{E}\left[ \mathcal{G}_{t_k} | \mathcal{F}_{t_k} \right] \leq C \triangle^{1+2\bar{\beta}}. 
\]

(4.40)

Similarly, we can also prove that for any integer \( l > 3 \), \( \mathbb{E}\left[ \bar{\varsigma}_k^l | \xi^{p,\beta}_{\varsigma_{r_k+1}} \right] \leq C \triangle^{1+2\bar{\beta}} \). For any integer \( k \geq 0 \), substituting (4.38)-(4.40) into (4.37), we deduce from (16) that

\[
\mathbb{E}\left[ \left( \varepsilon + \left| \tilde{Z}_{k_{\epsilon_0}}^{x_0,\ell} - \tilde{Z}_{k_0}^{x_0,\ell} \right|^2 \right)^{\frac{p}{2}} \xi^{p,\beta}_{\varsigma_{r_k+1}} | \mathcal{F}_{t_k} \right] \leq \left( \varepsilon + \left| Z_{k_0}^{x_0,\ell} - Z_k^{x_0,\ell} \right|^2 \right)^{\frac{p}{2}} \left\{ \xi^{p,\beta}_{\varsigma_{r_k+1}} - \frac{p \beta_k}{2} \xi^{p,\beta}_{\varsigma_{r_k+1}} \Delta - \eta_{p,\beta} \xi^{p,\beta}_{\varsigma_{r_k+1}} \Delta + o(\Delta) 
\right. 
\]

\[ 
\left. + \frac{\varepsilon \Delta^2}{2} \psi(Z_k^{x_0,\ell}, Z_k^{x_0,\ell}, r_\ell^k) \xi^{p,\beta}_{\varsigma_{r_k+1}} \right\} \]
Define $\tilde{\tau} = \inf\{k \geq 0 : r_k^\ell = r_0^\ell\}$. For any $p \in (0, p_\alpha^* \wedge p_\beta^* \wedge \bar{p}) \cap (0, \rho]$, choose $q > 1$ such that $pq \in (0, p_\alpha^*) \cap (0, \bar{p}]$. Thus, using Hölder’s inequality, (4.19) and (4.41) yields

$$
\mathbb{E}|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p \\
= \mathbb{E}\left(|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p I_{\{\tilde{\tau} > [k/2]+1\}}\right) + \mathbb{E}\left(|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p I_{\{\tilde{\tau} \leq [k/2]+1\}}\right) \\
\leq \left(\mathbb{E}|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p\right)^{\frac{1}{q}} \left(\mathbb{P}(\tilde{\tau} > [k/2] + 1)\right)^{1 - \frac{1}{q}} + \mathbb{E}\left[I_{\{\tilde{\tau} \leq [k/2]+1\}}\mathbb{E}\left(|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p | F_{\tilde{\tau}}\right)\right] \\
\leq \left(\mathbb{E}|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p\right)^{\frac{1}{q}} \left(\mathbb{P}(\tilde{\tau} > k/2)\right)^{1 - \frac{1}{q}} + \mathbb{E}\left[I_{\{\tilde{\tau} \leq [k/2]+1\}}\mathbb{E}\left(|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p \right)\right] \\
\leq e^{-\frac{q-1}{2q}h\Delta_k} \left(\mathbb{E}|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p\right)^{\frac{1}{q}} + C\mathbb{E}\left[I_{\{\tilde{\tau} \leq [k/2]+1\}} e^{-(\eta_{p,\gamma} - \varrho)(k-\tilde{\tau})} \mathbb{E}\left(|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p \right)\right] \\
\leq e^{-\frac{q-1}{2q}h\Delta_k} \left(\mathbb{E}|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p\right)^{\frac{1}{q}} + Ce^{-\frac{(\eta_{p,\gamma} - \varrho)(k-2)}{2}} \Delta_k \mathbb{E}\left(|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p \right) \rightarrow 0 \text{ as } k \rightarrow \infty.
$$

Due to (4.34), one observes that the conclusion of Theorem 4.3 holds for the new scheme (4.32).

Then $\sup_{k \geq 0} \mathbb{E}|Z_k^{x_0,\ell}|^p \leq C$, $\sup_{k \geq 0} \mathbb{E}|Z_k^{x_0,\ell}|^p \leq C$. One further observes that

$$
\mathbb{E}|Z_\tilde{\tau}^{x_0,\ell}|^p \leq \sum_{l=0}^{[k/2]+1} \mathbb{E}\left(|Z_l^{x_0,\ell}|^p I_{\{\tilde{\tau} \leq ([k/2]+1)l\}}(\omega)\right) \leq \sum_{l=0}^{[k/2]+1} \mathbb{E}|Z_l^{x_0,\ell}|^p \leq C(2 + k/2),
$$

which implies

$$
Ce^{-\frac{(\eta_{p,\gamma} - \varrho)(k-2)}{2}} \Delta_k \mathbb{E}\left(|Z_k^{x_0,\ell} - Z_k^{\bar{x}_0,\ell}|^p \right) \rightarrow 0 \text{ as } k \rightarrow \infty.
$$

Inserting this into (4.42) yields the desired assertion (4.35). \qed

Furthermore we can also use Scheme (4.32) to approximate the stability as follows.

**Corollary 4.9** Under the conditions of Corollary 4.5, for any $p \in (0, p_\alpha^* \cap (0, \rho]$, any $\varrho \in (0, \eta_{p,\gamma})$, there is a constant $\Delta^{**} \in (0, 1]$ such that $\forall \Delta \in (0, \Delta^{**})$, Scheme (4.32) satisfies

$$
\limsup_{k \rightarrow \infty} \frac{\log \left(\mathbb{E}|Z_k^{x_0,\ell}|^p \right)}{k\Delta} \leq -(\eta_{p,\gamma} - \varrho) < 0, \quad (4.43)
$$

and

$$
\limsup_{k \rightarrow \infty} \frac{\log \left(\mathbb{E}|Z_k^{x_0,\ell}|^p \right)}{k\Delta} \leq -\frac{\eta_{p,\gamma} - \varrho}{p} \quad \text{a.s.} \quad (4.44)
$$

**Proof.** Clearly, (4.43) follows from (4.41) directly. Moreover, an application of the Borel-Cantelli lemma (see, e.g., [2]) results in (4.44), please refer to [37, p.600]. \qed

In order for the ergodicity, we show the Markov property of $\{\left(Z_k^0, r_k^\ell\right)\}_{k \geq 0}$.

**Lemma 4.10** $\{\left(Z_k, r_k\right)\}_{k \geq 0}$ is a time homogeneous Markov chain.
Proof. One observes

\[ Z_{k+1} = \pi_{\triangle}^{r_{k+1}}(x + f(x, i)\Delta + g(x, i)\Delta B_k), \quad \text{and} \quad Z_1 = \pi_{\triangle}^x(x + f(x, i)\Delta + g(x, i)\Delta B_0). \]

Since \( \Delta B_k \) and \( \Delta B_0 \) are identical in probability law, comparing the two formulas above, we know that \((Z_{k+1}, r_{k+1})\) and \((Z_1, r_1)\) are identical in probability law under \( Z_k = x, r_k = i \) and \( x_0 = x, r_0 = i \) respectively. Thus

\[ \mathbb{P}\left( (Z_{k+1}, r_{k+1}) \in \mathbb{D} \times \{j\} \mid (Z_k, r_k) = (x, i) \right) = \mathbb{P}\left( (Z_1, r_1) \in \mathbb{D} \times \{j\} \mid (x_0, r_0) = (x, i) \right) \]

for any \( \mathbb{D} \in \mathcal{B} (\mathbb{R}^n) \), \( j \in \mathcal{S} \), which is the desired homogenous property. For any \( k \geq 0, \Delta \in (0, 1], x \in \mathbb{R}^n, i \in \mathcal{S} \), define

\[ \lambda^i_{k+1} := i + r_{k+1} - r_k, \quad \theta^{x,i}_{k+1,j} := \pi_{\triangle}^j(x + f(x, i)\Delta + g(x, i)\Delta B_k). \]

By (4.32) we know that \( r_{k+1} = \lambda^i_{k+1} \) and \( Z_{k+1} = \theta^{Z_k, r_k}_{k+1, r_{k+1}} \). Note that \( \lambda^i_{k+1} \) and \( \theta^{x,i}_{k+1,j} \) are bounded measurable random functions independent of \( \mathcal{F}_k \). Hence, for any \( \mathbb{D} \times \{j\} \in \mathcal{B} (\mathbb{R}^n) \times \mathcal{S} \), using [2, Lemma 3.2, p.104] with \( \bar{h}(x, i, \omega) = I_{\mathbb{D} \times \{j\}}(\theta^{x,i}_{k+1, \lambda_{k+1}^i}, \lambda_{k+1}^i) \) yields

\[ \mathbb{P}\left( (Z_{k+1}, r_{k+1}) \in \mathbb{D} \times \{j\} \mid \mathcal{F}_k \right) = \mathbb{E}\left( I_{\mathbb{D} \times \{j\}}(Z_{k+1}, r_{k+1}) \mid \mathcal{F}_k \right) \]

\[ = \mathbb{E}\left( I_{\mathbb{D} \times \{j\}}(\theta^{Z_k, r_k}_{k+1, r_{k+1}}, \lambda^i_{k+1}) \mid \mathcal{F}_k \right) \]

\[ = \mathbb{E}\left( I_{\mathbb{D} \times \{j\}}(\theta^{x,i}_{k+1, \lambda_{k+1}^i}) \mid x = Z_k, i = r_k \right) \]

\[ = \mathbb{P}\left( (Z_{k+1}, r_{k+1}) \in \mathbb{D} \times \{j\} \mid (Z_k, r_k) \right), \]

which is the desired Markov property. \( \Box \)

Let \( P_{k\triangle}^\mathcal{S}(x_0, \ell; dx \times \{i\}) \) be the transition probability kernel of the pair \( (Z^{x_0, \ell}_k, r^\ell_k) \), a time homogeneous Markov chain. If \( \mu^\triangle \in \mathcal{P}(\mathbb{R}^n \times \mathcal{S}) \) satisfies

\[ \mu^\triangle(\mathcal{Y} \times \{\ell\}) = \sum_{i=1}^m \int_{\mathbb{R}^n} P_{k\triangle}^\Delta(x, i; \mathcal{Y} \times \{\ell\}) \mu^\Delta(dx \times \{i\}), \quad \mathcal{Y} \in \mathcal{B}(\mathbb{R}^n), \ell \in \mathcal{S} \]

for any \( k \geq 0 \), then \( \mu^\triangle \) is called an invariant measure of \( (Z^{x_0, \ell}_k, r^\ell_k) \). Moreover, such an invariant measure \( \mu^\triangle \in \mathcal{P}(\mathbb{R}^n \times \mathcal{S}) \) is also called a numerical invariant measure. Next we give the existence and uniqueness of the numerical invariant measure for SDS (2.1) using Scheme (4.32).

**Theorem 4.11** Under the conditions of Theorem 4.6, for any \( \triangle \in (0, \Delta^{**}] \), the numerical solutions of Scheme (4.32) admit a unique invariant measure \( \mu^\triangle \in \mathcal{P}(\mathbb{R}^n \times \mathcal{S}). \)
Moreover, by Theorem 3.6, there is a \( \Delta \). The proof is complete.

Proof. For arbitrary integer \( l > 0 \), define a measure sequence \( \{h_l\} \) that

\[
h_l(\Upsilon \times \{i\}) := \frac{1}{l + 1} \sum_{k=0}^{l} \mathbb{P}\left((Z_k^{x_0,\ell}, r_k^{\ell}) \in \Upsilon \times \{i\}\right)
\]

for any \( \Upsilon \in \mathcal{B}(\mathbb{R}^n) \) and \( i \in \mathbb{S} \). For any \( p \in (0, p^{*}_\alpha) \cap (0, \bar{p}) \) and \( \Delta \in (0, \Delta^{**}) \), by Theorem 4.3 and Chebyshev’s inequality, we derive that \( \{h_l\} \) is tight, then one can extract a subsequence which converges weakly to an invariant measure. Thus, the numerical solution \( (Z_k^{x_0,\ell}, r_k^{\ell}) \) has an invariant measure \( \mu^\Delta \). For any \( \Delta \in (0, \Delta^{**}) \), we have established the existence of the numerical invariant measures, and now we further show its uniqueness. For any \( p \in (0, p^*_\alpha \wedge p^*_\beta \wedge \bar{p}) \cap (0, \rho] \), it follows from (4.19) and (4.35) that

\[
W_p(\delta(\mu, \mu^\Delta)) \leq W_p(\delta(\Delta_{k,\Delta}, \delta(\mu^\Delta))) + W_p(\delta(\Delta_{k,\Delta}, \delta(\mu^\Delta))) + W_p(\delta(\mu^\Delta, \mu^\Delta)).
\]

Assume both \( \mu^\Delta \) and \( \bar{\mu}^\Delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S}) \) are invariant measures, then we have

\[
W_p(\mu^\Delta, \bar{\mu}^\Delta) \leq W_p(\delta(x,i, \mu^\Delta), \mu^\Delta) + W_p(\delta(x,i, \mu^\Delta), \delta(y,j, \mu^\Delta)) + W_p(\delta(y,j, \mu^\Delta, \bar{\mu}^\Delta)).
\]

Due to (4.45), \( W_p(\mu^\Delta, \bar{\mu}^\Delta) \to 0 \) holds as \( k \to \infty \). The desired assertion follows.

The following theorem reveals that numerical invariant measure \( \mu^\Delta \) converges in the Wassertein distance to the underlying one \( \mu \).

**Theorem 4.12** Under the conditions of Theorem 4.6, \( \lim_{\Delta \to 0} W_p(\mu, \mu^\Delta) = 0 \).

**Proof.** By the virtues of Theorems 4.6 and 4.11, we have

\[
W_p(\delta(x,i, \mu^\Delta), \mu) \leq \int_{\mathbb{R}^n \times \mathbb{S}} \mu(dy \times \{j\}) W_p(\delta(x,i, \mu^\Delta), \delta(y,j, \mu^\Delta)),
\]

and

\[
W_p(\delta(x,i, \mu^\Delta), \mu^\Delta) \leq \int_{\mathbb{R}^n \times \mathbb{S}} \mu^\Delta(dy \times \{j\}) W_p(\delta(x,i, \mu^\Delta), \delta(y,j, \mu^\Delta))
\]

for any \( \Delta \in (0, \Delta^{**}) \). Hence

\[
\lim_{k \to \infty} W_p(\delta(x,i, \mu^\Delta), \mu) = \lim_{k \to \infty} W_p(\delta(x,i, \mu^\Delta), \mu^\Delta) = 0.
\]

Thus for any \( \epsilon > 0 \), there exists a positive integer \( \bar{k} > 0 \) sufficiently large such that

\[
W_p(\delta(x,i, \mu^\Delta), \mu) + W_p(\delta(x,i, \mu^\Delta), \mu^\Delta) \leq \frac{\epsilon}{2}, \quad \forall \Delta \in (0, \Delta^{**}).
\]

Moreover, by Theorem 3.6, there is a \( \Delta_1 \in (0, \Delta^{**}) \) such that \( W_p(\delta(x,i, \mu^\Delta), \delta(x,i, \mu^\Delta)) < \epsilon/2 \) for any \( \Delta \in (0, \Delta_1] \). Therefore, for any \( \Delta \in (0, \Delta_1] \),

\[
W_p(\mu, \mu^\Delta) \leq W_p(\delta(x,i, \mu^\Delta), \mu) + W_p(\delta(x,i, \mu^\Delta), \mu^\Delta) + W_p(\delta(x,i, \mu^\Delta, \delta(x,i, \mu^\Delta)) < \epsilon.
\]

The proof is complete. \( \Box \)
Remark 4.13 For the given stepsize $\Delta$ and $\bar{h}(\Delta)$, it is easy to find $\hat{\phi}^{-1}(\bar{h}(\Delta))$, the minimum of all $\varphi_i^{-1}(\bar{h}(\Delta))$. Define the uniform truncation mapping $\bar{\pi}(x) = \left(\frac{|x| \wedge \hat{\phi}^{-1}(\bar{h}(\Delta))}{|x|}\right)x/|x|$. Clearly, the properties (4.33) and (4.34) still hold for Scheme (4.32) with each $\bar{\pi}_i(x) = \bar{\pi}(x)$. This implies that all results on numerical solutions in this section still hold for this uniformly truncated scheme.

4.4 Numerical examples

Before closing this section we carry out some simulations to illustrate the efficiency of the scheme (4.32) in the approximation of invariant measures.

Recall Example 3.15. Assumption 1 holds with $\bar{p} = 5/3$ and $\alpha_1 = 5$, $\alpha_2 = -0.64$. Due to more computations Assumption 3 holds with $\rho = 0.8$ and $\beta_1 = 5$, $\beta_2 = -0.898$. Solving the linear equation (4.2) results in $\pi = (\pi_1, \pi_2) = (1/21, 20/21)$. Clearly, $\pi_\alpha = \pi_1 \alpha_1 + \pi_2 \alpha_2 = -7.8/21 < 0$, $\pi_\beta = \pi_1 \beta_1 + \pi_2 \beta_2 = -12.96/21 < 0$.

It follows from Theorem 4.6 that exact solutions of stochastic volatility model with random switching between (1.1) and (1.2) admit a unique invariant measure $\mu \in \mathcal{P}(\mathbb{R}^n \times S)$. By the virtues of Theorems 4.11 and 4.12 Scheme (4.32) produces a unique numerical invariant measure $\mu^\Delta \in \mathcal{P}(\mathbb{R}^n \times S)$ converging to $\mu$ in the Wasserstein metric. To the best of our knowledge almost all numerical methods in the literatures such as [28, 29, 33] cannot treat this case.

Secondly we carry out some numerical experiments. Basing on the model structure, we begin to specify the explicit truncated EM scheme (4.32).

Step 1. Choose $\hat{\varphi}_i(\cdot)$ and $\bar{h}(\cdot)$. Compute

$$
\sup_{|x| \vee |y| \leq u, x \neq y} \left( \frac{|f(x, 1) - f(y, 1)|}{|x - y|} \vee \frac{|g(x, 1) - g(y, 1)|^2}{|x - y|^2} \right) \leq 18u, \quad \forall u \geq 1,
$$

and

$$
\sup_{|x| \vee |y| \leq u, x \neq y} \left( \frac{|f(x, 2) - f(y, 2)|}{|x - y|} \vee \frac{|g(x, 2) - g(y, 2)|^2}{|x - y|^2} \right) \leq 1.45, \quad \forall u \geq 1.
$$

Then choose $\hat{\varphi}_1(u) = 18u$, $\hat{\varphi}_2(u) = 1.45$, $\forall u \geq 1$, which implies

$$
\hat{\varphi}_1^{-1}(u) = u/18, \quad \hat{\varphi}_2^{-1}(u) = +\infty, \quad \forall u \geq 18.
$$

Let $\bar{h}(\Delta) = 54\Delta^{-0.4}$, $\forall \Delta \in (0, 1]$. Thus (4.27) holds.

Step 2. MATLAB code. Next we specify the MATLAB code for calculating the truncated EM approximation $Z(t)$:
%MATLAB code for calculating the truncated EM approximation \( Z(t) \)

```matlab
clear all;
T=100; dt=2^(-9); Gam=[-4 4;0.2 -0.2]; c=expm(Gam*dt); h=54*dt^(-2/5);
Z=zeros(2,T/dt+1,'double'); r=zeros(1,T/dt+1,'double'); Z(:,1)=[1;1];
r(1)=2; dB=sqrt(dt)*randn(2,T/dt); v=h/18; %Obviously, \( v > \| Z(:,1) \| \);
for n=1:T/dt
    if r(n)==1
        Z(:,n+1)=Z(:,n)+2.5*Z(:,n)*(1-norm(Z(:,n)))*dt+...
            [-1 sqrt(2);sqrt(2) 1]*norm(Z(:,n))^(3/2)*dB(:,n);
    else
        Z(:,n+1)=Z(:,n)+([1;2]-Z(:,n))*dt+...
            [0.2 -0.5 1 0.4]*norm(Z(:,n))*dB(:,n);
    end
    if rand<=c(r(n),1)
        r(n+1)=1;
        if norm(Z(:,n+1))>v
            Z(:,n+1)=v*Z(:,n+1)/norm(Z(:,n+1));
        end
    else
        r(n+1)=2;
    end
end
```

Figure 4: (a) Computer simulation of a single path of Markov chain \( r(t) \). (b) A sample path of exact solution \( X(t) \) in 3D settings. (c) A sample path of numerical solution \( Z(t) \) in 3D settings. The red trajectory represents the exact solution (i.e. the numerical solution of Scheme (4.32) with \( \Delta = 2^{-18} \)) while the blue trajectory represents the numerical solution of Scheme (4.32) with \( \Delta = 2^{-9} \).
Figure 5: (a) The empirical density of $\mu^{\triangle}$ in 2D and 3D settings. (b) The empirical density of $\mu$ in 2D and 3D settings.

**Step 3.** MATLAB simulations. Without the closed-form, the more precise numerical solution $X(t) = (X_1(t), X_2(t))^T$ with $\triangle = 2^{-18}$ is a good substitute of the exact solution. We compare it with the other numerical solution $Z(t) = (Z_1(t), Z_2(t))^T$ with stepsize $\triangle = 2^{-9}$. Figure 4 depicts the paths of the Markov chain (Figure 4(a)), $X(t)$ (Figure 4(b)) and $Z(t)$ (Figure 4(c)) for $t \in [0, 100]$. With 51201 iterations, Figure 5(a) depicts the empirical density of $\mu^{\triangle}$ in 2D and 3D settings while Figure 5(b) depicts the empirical density of $\mu$ in 2D and 3D settings. It is evident to see that these two density pictures are very similar. To support the theoretical results deeply, we further plot the empirical cumulative distribution function (ECDF) of $Z(t)$ with the blue dashed line and the ECDF of $X(t)$ with the red solid line. To measure the similarity quantitatively, we use the Kolmogorov-Smirnov test [38] to test the alternative hypothesis that the exact and numerical invariant measures are from different distributions against the null hypothesis that they are from the same distribution for each component. With 2% significance level, the Kolmogorov-Smirnov test indicates that we cannot reject the null hypothesis. So the numerical invariant measure approximates the underlying exact invariant measure very well.

To further illustrate the validity, we reanalyze the existence of the invariant measure and its approximation of SDS (3.53).
Figure 6: (a) The ECDFs of $X_1(t), Z_1(t)$. (b) The ECDF of $X_2(t), Z_2(t)$. The red solid line represents the exact solution of the switching system while the blue dashed line represents the numerical solution of the switching system.

Example 4.14 Let $r(t)$ be a Markov chain with the state space $S = \{1, 2\}$ and the generator

$$
\Gamma = \begin{pmatrix}
-\gamma & \gamma \\
3 & -3
\end{pmatrix}
$$

for some $\gamma > 0$. Its unique stationary distribution $\pi = (\pi_1, \pi_2) \in \mathbb{R}^{1 \times 2}$ is given by $\pi_1 = 3/(3 + \gamma)$, $\pi_2 = \gamma/(3 + \gamma)$. Consider the scalar hybrid cubic SDS (3.53) with the initial value $(x_0, \ell) = (0.5, 2)$, and coefficients

$$a(1) = 1, \; b(1) = -1, \; \sigma(1) = 2; \; a(2) = 2, \; b(2) = 0, \; \sigma(2) = -1.$$

Thus, Assumption 1 holds with any negative constant $\alpha_1$ and $\alpha_2 = 3 + \bar{p}$ for any $\bar{p} > 0$. Moreover, one observes Assumption 3 holds with $\beta_1 = 4\rho - 2$, $\beta_2 = 3 + \rho$ for any $\rho > 0$. Let $\rho = 0.1$, then we know $\pi\alpha = \pi_1\alpha_1 + \pi_2\alpha_2 < 0$, $\pi\beta = \pi_1\beta_1 + \pi_2\beta_2 < 0$ holds with $\gamma \in (0, 1.548]$. It follows from Theorem 4.6 that the exact solutions of SDS (3.53) admit a unique invariant measure $\mu \in \mathcal{P}(\mathbb{R} \times S)$. Moreover, by Theorems 4.11 and 4.12, the unique numerical invariant measure $\mu^\triangle \in \mathcal{P}(\mathbb{R} \times S)$ of the truncated EM scheme exists and converges to $\mu$ in the Wasserstein metric.

Next we begin to construct the explicit scheme to approximate the underlying invariant measure of SDS (3.53). For any $i \in S$, compute

$$
\sup_{|x| \vee |y| \leq u, x \neq y} \left( \frac{|f(x, i) - f(y, i)|}{|x - y|} \vee \frac{|g(x, i) - g(y, i)|^2}{|x - y|^2} \right) \leq (3|b_i|u^2 + |a_i|) \vee \sigma_i^2, \quad \forall \; u \geq 1.
$$
Thus, for any $u \geq 1$, we choose $\bar{\varphi}_1(u) = 3u^2 + 1$, $\bar{\varphi}_2(u) = 2$, which implies $\bar{\varphi}_1^{-1}(u) = ((u - 1)/3)^{1/2}$, $\forall u \geq 4$, $\bar{\varphi}_2^{-1}(u) = +\infty$, $\forall u \geq 2$. Let $\bar{h}(\Delta) = 6\Delta^{-0.4}$. Clearly, (4.26) and (4.27) hold for any $\Delta \in (0, 1]$. Then, we give the truncated EM scheme

\[
\begin{cases}
Z_0 = x_0, & r_0 = \ell, \\
\tilde{Z}_{k+1} = Z_k + [a(r_k)Z_k + b(r_k)(Z_k)^3]\Delta + \sigma(r_k)Z_k\Delta B_k, \\
Z_{k+1} = \left(\left|\tilde{Z}_{k+1}\right| \wedge \bar{\varphi}^{-1}_{r_{k+1}}(\bar{h}(\Delta))\right)\frac{\tilde{Z}_{k+1}}{|\tilde{Z}_{k+1}|}
\end{cases}
\]

for any $k = 0, 1, \ldots, \bar{N} - 1$. Let $\gamma = 1.5$, $T = 100$ and stepsize $\Delta = 10^{-4}$. We implement Scheme (4.47) in the numerical experiments. We simulate 100 paths by using MATLAB. On the computer running at Intel Core i3-4170 CPU 3.70 GHz, the runtime of the truncated EM scheme (4.47) is about 22.137421 seconds while the the runtime of the backward EM scheme (3.56) is about 38.229964 seconds on the same computer. Thus we know that the speed of the truncated EM scheme (4.47) is 1.727 times faster than that of the backward EM scheme. Figure 7 depicts 10 paths of the numerical solution of Scheme (4.47). Figure 8(a) depicts the path of the Markov chain, Figure 8(b) further compares the path of the exact solution $X(t)$ with that of the numerical solution $Z(t)$ while Figure 8(c) compares the ECDF of the exact solution with that of the numerical solution. The similarity between the paths as well as the distributions is significant. Thus the numerical invariant measure approximates the underlying one very well.

Figure 7: 10 trajectories of the numerical solution of Scheme (4.47) with $x_0 = 0.5$, $\ell = 2$ and stepsize $\Delta = 10^{-4}$. 

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Figure 8: (a) Computer simulation of a sample path of Markov chain \( r(t) \). (b) Sample paths of the exact solution (the red solid line) and numerical solution (the blue dashed line). (c) ECDFs for the exact solution (the red solid line) and numerical solution (the blue dashed line).

5 Concluding Remarks

This paper investigates the approximation methods for the SDSs without globally Lipschitz continuous coefficients. A novelty is to construct two explicit schemes approximating the dynamical properties of SDSs. In finite horizon, by one scheme, we show the boundedness of the numerical solutions, obtain the convergence in the \( p \)th moment and estimate the rate of convergence. On the other hand, in infinite horizon we use the other scheme to approximate the underlying invariant measure of exact solutions in the Wasserstein distance. Moreover, this scheme is also suitable to realize the stability of SDSs. Our exploiting schemes perform the dynamical behaviors of exact solutions very well but don’t require more restrictions except the structure conditions which guarantee the exact solutions posses some propety such as stability, moment boundedness and ergodicity. Some simulation examples are provided to support the theoretical results and demonstrate the validity of approaches.
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