GEOMETRIC PROPERTIES OF UPPER LEVEL SETS OF LELONG NUMBERS ON PROJECTIVE SPACES

DAN COMAN AND TUYEN TRUNG TRUONG

Abstract. Let $T$ be a positive closed current of unit mass on the complex projective space $\mathbb{P}^n$. For certain values $\alpha < 1$, we prove geometric properties of the set of points in $\mathbb{P}^n$ where the Lelong number of $T$ exceeds $\alpha$. We also consider the case of positive closed currents of bidimension (1,1) on multiprojective spaces.

1. Introduction

Let $T$ be a positive closed current of bidimension $(p,p)$ on a complex manifold $M$. For $\alpha \geq 0$, we consider the upper level sets of the Lelong numbers $\nu(T,z)$ of $T$,

$$
E_\alpha(T) = E_\alpha(T, M) = \{ z \in M : \nu(T,z) \geq \alpha \},
$$

$$
E_\alpha^+(T) = E_\alpha^+(T, M) = \{ z \in M : \nu(T,z) > \alpha \}.
$$

By [11], if $\alpha > 0$ then $E_\alpha(T)$ is an analytic subvariety of $M$ of dimension at most $p$, hence $E_0^+(T)$ is an at most countable union of analytic subvarieties of $M$ of dimension $\leq p$.

If $M = \mathbb{P}^n$ we denote by $\|T\|$ the mass (or the degree) of $T$ computed with respect to the Fubini-Study form $\omega_n$ on $\mathbb{P}^n$,

$$
\|T\| = \int_{\mathbb{P}^n} T \wedge \omega_n^p.
$$

It is well known that $\nu(T,z) \leq \|T\|$ for every $z \in \mathbb{P}^n$ (see e.g. [2]). Assume without loss of generality that $\|T\| = 1$. When $p = 1$, it was shown in [1] Theorem 1.1] that $E_{2/3}^+(T, \mathbb{P}^n)$ is contained in a (complex) line, while $E_{1/2}^+(T, \mathbb{P}^n)$ is either contained in a line or else it is a finite set such that $|E_{1/2}^+(T, \mathbb{P}^n) \setminus L| = 1$ for some line $L$. In dimension two, it was proved in [1] Theorem 1.2] that $E_{2/5}^+(T, \mathbb{P}^2)$ is either contained in a conic or else it is a finite set such that $|E_{2/5}^+(T, \mathbb{P}^2) \setminus C| = 1$ for some conic $C$. When $p = n - 1$, i.e. when $T$ has bidegree $(1,1)$, it was shown in [2] Proposition 2.2] that $E_{n/(n+1)}^+(T, \mathbb{P}^n)$ is contained in a hyperplane of $\mathbb{P}^n$. Moreover, these values of $\alpha$ are sharp with respect to these geometric properties.

In this paper we generalize these results to the case of currents of arbitrary bidimension on $\mathbb{P}^n$. Namely, we prove the following theorems.

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Theorem 1.1. If $T$ is a positive closed current of bidimension $(p, p)$ on $\mathbb{P}^n$, $0 < p < n$, with $\|T\| = 1$, then the set $E^+_{(p+1)/(p+2)}(T, \mathbb{P}^n)$ is contained in a $p$-dimensional linear subspace of $\mathbb{P}^n$.

Decreasing the value of $\alpha$ to $p/(p + 1)$ we show that $E^+_{p/(p+1)}(T, \mathbb{P}^n)$ has all but at most one point contained in a $p$-dimensional linear subspace of $\mathbb{P}^n$. More precisely, the following holds:

Theorem 1.2. If $T$ is a positive closed current of bidimension $(p, p)$ on $\mathbb{P}^n$, $0 < p < n$, with $\|T\| = 1$, then the set $E^+_{p/(p+1)}(T, \mathbb{P}^n)$ is either contained in a $p$-dimensional linear subspace of $\mathbb{P}^n$ or else it is a finite set and $|E^+_{p/(p+1)}(T, \mathbb{P}^n) \setminus L| = p$ for some line $L$.

Theorems 1.1 and 1.2 are proved in Section 2. We also prove there another property of the set $E^+_{p/(p+1)}(T, \mathbb{P}^n)$ (see Proposition 2.5), and we give examples of currents showing that the values of $\alpha$ in Theorems 1.1 and 1.2 are sharp with respect to the corresponding geometric property.

Decreasing the value of $\alpha$ further to $(3p - 1)/(3p + 2)$, we obtain a result analogous to Theorem 1.2 in [1]:

Theorem 1.3. Let $T$ be a positive closed current of bidimension $(p, p)$ on $\mathbb{P}^n$ such that $1 < p < n$, $\|T\| = 1$, and the set $E^+_{(3p-1)/(3p+2)}(T, \mathbb{P}^n)$ is not contained in a $p$-dimensional linear subspace of $\mathbb{P}^n$. If $W = \text{Span}(E^+_{(3p-1)/(3p+2)}(T, \mathbb{P}^n))$ then $\dim W = p + 1$ and there exist plane conics $C_j \subset W$ and points $z_j \in W$, $1 \leq j \leq N_p$, where $N_p = \binom{p+2}{3}$, such that $z_j$ lies in the plane containing $C_j$ and

$$E^+_{(3p-1)/(3p+2)}(T, \mathbb{P}^n) \subset C_1 \cup \ldots \cup C_{N_p} \cup \{z_1, \ldots, z_{N_p}\}.$$

Theorem 1.3 is proved in Section 3. We note there that the corresponding statement does not hold for currents of bidimension (1,1). However, we give in Theorem 3.3 a geometric description of the set $E^+_{2/5}(T, \mathbb{P}^n)$ when $T$ is a positive closed current of bidimension $(1, 1)$ on $\mathbb{P}^n$ such that $\|T\| = 1$. The proof of Theorem 3.3 uses ideas from [3, 10, 12] related to self-intersection inequalities for positive closed currents. Using similar ideas, we further show in Theorem 3.4 that if $T$ is a positive closed current of bidimension $(p, p)$ on a compact Kähler manifold $(X, \omega)$, then the set $E_c(T, X)$, $c > 0$, is contained in an analytic set of dimension $\leq p$ whose volume and number of irreducible components are bounded above by a positive constant which depends only on $\|T\|$ and $c$. In this case, the volumes of analytic subvarieties of $X$ and the mass $\|T\|$ are computed with respect to the fixed Kähler form $\omega$ on $X$.

In Section 4 we study similar geometric properties for upper level sets of positive closed currents of bidimension $(1, 1)$ on multi-projective spaces.

2. Proofs of Theorems 1.1 and 1.2

2.1. Proof of Theorem 1.1. Let us start by recalling some terminology. If $A \subset \mathbb{P}^n$ we denote by $\text{Span}(A)$ the smallest linear subspace of $\mathbb{P}^n$ containing $A$. We say that the points $x_1, \ldots, x_{k+1} \in \mathbb{P}^n$, $k \leq n$, are linearly independent if they span a $k$-dimensional
linear subspace of \( \mathbb{P}^n \). We say that \( k > n + 1 \) points of \( \mathbb{P}^n \) are in general position if any \( n + 1 \) of them are linearly independent. We will need the following lemma:

**Lemma 2.1.** Let \( T \) be a positive closed current of bidimension \((p, p)\) on \( \mathbb{P}^n \), \( 0 < p < n \), let \( \alpha > 0 \), and let \( V \) be a \((p + 1)\)-dimensional linear subspace of \( \mathbb{P}^n \). There exists a positive closed current \( S \) of bidegree \((1, 1)\) on \( V \equiv \mathbb{P}^{p+1} \) such that \( \|S\| = \|T\| \) and \( E^+_{\alpha}(T, \mathbb{P}^n) \cap V \subset E^+_{\alpha}(S, V) \).

**Proof.** Assume without loss of generality that \( \|T\| = 1 \). By [10], there exists a positive closed current \( T' \) of bidegree \((1, 1)\) on \( \mathbb{P}^n \) such that \( \|T'\| = 1 \) and \( \nu(T', z) = \nu(T, z) \) for every \( z \in \mathbb{P}^n \). Demailly’s regularization theorem [3, Proposition 3.7] yields a sequence of positive closed currents \( T'_m \) of bidegree \((1, 1)\) on \( \mathbb{P}^n \) with \( \|T'_m\| = 1 \), such that each \( T'_m \) is smooth outside an analytic subset contained in \( E^+_0(T') \) and \( \lim_{m \to \infty} \nu(T'_m, x) = \nu(T', x) \) at each \( x \in \mathbb{P}^n \). We write \( T'_m = \alpha_n + dd^c\alpha_m \) for some \( \alpha_n \)-plurisubharmonic function \( \alpha_m \) on \( \mathbb{P}^n \) (see e.g. [9]). Note that by [11], \( E^0_0(T') = E^0_0(T) \) is a countable union of analytic subsets of dimension at most \( p \), so \( V \cap E^+_0(T') \neq \emptyset \). Since \( \varphi_m \) is smooth at each point of \( V \cap E^+_0(T') \) the pull-back \( S_m \) of \( T'_m \) to \( V \), \( S_m = \varphi_n|_V + dd^c(\varphi_m|_V) \), is an absolute positive closed current of bidegree \((1, 1)\) on \( V \) with \( \|S_m\| = 1 \). By passing to a subsequence, we may assume that \( S_m \) converges weakly to a positive closed current \( S \) of bidegree \((1, 1)\) on \( V \). Then \( \|S\| = 1 \) and

\[ \nu(S, z) \geq \limsup_{m \to \infty} \nu(S_m, z) \geq \lim_{m \to \infty} \nu(T'_m, z) = \nu(T', z), \]

for all \( z \in V \). It follows that \( E^+_{\alpha}(T, \mathbb{P}^n) \cap V \subset E^+_{\alpha}(S, V) \). \( \square \)

**Proof of Theorem 2.2.** Assume for a contradiction that \( \dim \text{Span}(E^+_{(p+1)/(p+2)}(T, \mathbb{P}^n)) \geq p + 1 \), so there exist linearly independent points \( x_1, \ldots, x_{p+2} \in E^+_{(p+1)/(p+2)}(T, \mathbb{P}^n) \). Let \( V \equiv \mathbb{P}^{p+1} \) be the linear subspace spanned by these points. By Lemma 2.1 there exists a positive closed current \( S \) of bidegree \((1, 1)\) on \( V \) such that \( \|S\| = 1 \) and \( x_1, \ldots, x_{p+2} \in E^+_{(p+1)/(p+2)}(S, V) \). This is in contradiction to [2, Proposition 2.2], which shows that \( E^+_{(p+1)/(p+2)}(S, V) \) must be contained in a hyperplane of \( V \). \( \square \)

### 2.2. Proof of Theorem 1.2

We prove first Theorem 1.2 for currents of bidegree \((1, 1)\) on \( \mathbb{P}^n \). This is the contents of Theorem 2.3. Let \( \alpha_n = (n - 1)/n \). We begin with the following lemma:

**Lemma 2.2.** Let \( T \) be a positive closed current of bidegree \((1, 1)\) on \( \mathbb{P}^n \) such that \( \|T\| = 1 \) and \( E^+_{\alpha_n}(T, \mathbb{P}^n) \) contains a set \( A = \{x_1, \ldots, x_{n+1}\} \) of linearly independent points. Then:

(i) For every subset \( B \subset A \) with \( |B| = k + 1 \), \( k \geq 1 \), there exists a positive closed current \( R_B \) of bidegree \((1, 1)\) on \( \text{Span}(B) \equiv \mathbb{P}^k \) such that \( \|R_B\| = 1 \) and \( E^+_{\alpha_n}(T, \mathbb{P}^n) \cap \text{Span}(B) \subset E^+_{\alpha_n}(R_B, \text{Span}(B)) \).

(ii) \( E^+_{\alpha_n}(T, \mathbb{P}^n) \subset \bigcup_{1 \leq j < k \leq n+1} L_{jk} \), where \( L_{jk} \) is the line spanned by \( x_j \) and \( x_k \).

**Proof.** (i) It suffices to prove \( i \) for \( k = n - 1 \). Then we apply this inductively to obtain the result for arbitrary \( k \). Assume without loss of generality that \( B = \{x_1, \ldots, x_n\} \) and let \( H \) be the hyperplane spanned by \( B \). Siu’s decomposition theorem [11] implies that \( T = a[H] + R \), where \([H] \) denotes the current of integration along \( H \), \( 0 \leq a \leq 1 \), and \( R \)
is a positive closed current of bidegree \((1, 1)\) on \(\mathbb{P}^n\) with generic Lelong number 0 along \(H\). We have

\[
1 - a = \|R\| \geq \nu(R, x_{n+1}) = \nu(T, x_{n+1}) > \alpha_n, \quad \text{so } a < 1 - \alpha_n.
\]

The current \(R' = R/(1 - a)\) has mass \(\|R'\| = 1\), generic Lelong number 0 along \(H\), and if \(z \in E^+_{\alpha_n}(T, \mathbb{P}^n) \cap H\) then, since \(a < 1 - \alpha_n\),

\[
\nu(R', z) = \frac{\nu(T, z) - a}{1 - a} > \frac{\alpha_n - a}{1 - a} > \frac{2\alpha_n - 1}{\alpha_n} = \alpha_n - 1.
\]

By [3] Proposition 3.7 there exists a sequence of positive closed currents \(R'_m\) of bidegree \((1, 1)\) on \(\mathbb{P}^n\) with analytic singularities, such that \(\|R'_m\| = 1\), \(\nu(R'_m, x) \leq \nu(R', x)\) and \(\lim_{m \to \infty} \nu(R'_m, x) = \nu(R', x)\) for all \(x \in \mathbb{P}^n\). It follows that \(R'_m\) is smooth at each point of \(H\) outside an analytic subset of \(H\), so the pull-back \(R'_m|_H\) of \(R'_m\) to \(H\) is well-defined. Arguing as in the proof of Lemma 2.1 we obtain the current \(R_B\) that verifies the desired properties as a weak limit point of \(\{R'_m|_H\}\).

(ii) Let \(H_j\) denote the hyperplane spanned by \(A \setminus \{x_j\}\). We show first that

\[
E^+_{\alpha_n}(T, \mathbb{P}^n) \subset \bigcup_{j=1}^{n+1} H_j.
\]

Assume that there exists \(x_{n+2} \in E^+_{\alpha_n}(T, \mathbb{P}^n) \setminus \bigcup_{j=1}^{n+1} H_j\) and choose \(x_0 \in \mathbb{P}^n \setminus \{x_1, \ldots, x_{n+2}\}\). So that the points \(x_0, \ldots, x_{n+2}\) are in general position and \(\nu(T, x_0) = 0\). By [3] Proposition 3.7 there exists a positive closed current \(T'\) of bidegree \((1, 1)\) on \(\mathbb{P}^n\) with analytic singularities, such that \(\|T'\| = 1\), \(\nu(T', x_j) > \alpha_n, j = 1, \ldots, n+2\), and \(T'\) is smooth near \(x_0\). Let \(C\) be the unique rational normal curve passing through the points \(x_0, \ldots, x_{n+2}\) (see [7] p. 530). It follows by [4] and [6] that the measure \(T' \wedge [C]\) is well defined, where \([C]\) denotes the current of integration along \(C\). Since \(C\) has degree \(n\) and using [4] Corollary 5.10, we obtain

\[
n = \int_{\mathbb{P}^n} T' \wedge [C] \geq \sum_{j=1}^{n+2} T' \wedge [C]\{(x_j)\} \geq \sum_{j=1}^{n+2} \nu(T', x_j) \nu([C], x_j) > (n+2) \alpha_n,
\]

a contradiction. This proves (2).

Let now \(B_j = A \setminus \{x_j\}\). Applying (2) to the current \(R_{B_j}\) given by (i) we obtain

\[
E^+_{\alpha_n}(T, \mathbb{P}^n) \cap H_j \subset E^+_{\alpha_n-1}(R_{B_j}, H_j) \subset \bigcup_{k=1, k \neq j}^{n+1} \text{Span}(A \setminus \{x_j, x_k\})
\]

Together with (2) this implies that

\[
E^+_{\alpha_n}(T, \mathbb{P}^n) \subset \bigcup_{1 \leq j < k \leq n+1} \text{Span}(A \setminus \{x_j, x_k\})
\]

Repeating this argument inductively yields (ii). □
Theorem 2.3. If $T$ is a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^n$ with $\|T\| = 1$, then the set $E^+_{\alpha_n}(T,\mathbb{P}^n)$ is either contained a hyperplane or else it is a finite set and $|E^+_{\alpha_n}(T,\mathbb{P}^n) \setminus L| = n - 1$ for some line $L \subset \mathbb{P}^n$.

Proof. If $E^+_{\alpha_n}(T,\mathbb{P}^n)$ is not contained in a hyperplane then $\dim \text{Span}(E^+_{\alpha_n}(T,\mathbb{P}^n)) = n$ and there exists a set of linearly independent points $A = \{x_1, \ldots, x_{n+1}\} \subset E^+_{\alpha_n}(T,\mathbb{P}^n)$. Let $L_{jk}$ denote the line spanned by $x_j, x_k$.

If $E^+_{\alpha_n}(T,\mathbb{P}^n) = A$ then $|E^+_{\alpha_n}(T,\mathbb{P}^n) \setminus L_12| = n - 1$ and we are done. Suppose that there exists $x \in E^+_{\alpha_n}(T,\mathbb{P}^n) \setminus A$. By Lemma 2.2 we have, after relabeling points if necessary, that $x \in L_12$.

We show that $E^+_{\alpha_n}(T,\mathbb{P}^n) \subset A \cup L_{12}$. Assume for a contradiction that there exists $y \in E^+_{\alpha_n}(T,\mathbb{P}^n) \setminus (A \cup L_{12})$. Then $y \in L_{jk}$ for some $3 \leq j < k \leq n + 1$. Indeed, if $y \in L_{1k}$ or if $y \in L_{2k}$, $k \geq 3$, then let $B = \{1,2,k\}$ and $R_B$ be the current on $\text{Span}(B) \equiv \mathbb{P}^2$ provided by Lemma 2.2. Then $\{x, y, x_1, x_2, x_k\} \subset E^+_{1,2/3}(R_B,\mathbb{P}^2)$, and the set $\{x, y, x_1, x_2, x_k\}$ has at least two points outside each complex line. This is in contradiction to [1] Theorem 1.1. Hence after relabeling points if necessary we have that $y \in L_{34}$.

Consider now the set $B = \{x_1, x_2, x_3, x_4\}$ and the current $R = R_B$ on $\text{Span}(B) \equiv \mathbb{P}^3$ given by Lemma 2.2 so $\{x, y, x_1, x_2, x_3, x_4\} \subset E^+_{2/3}(R,\mathbb{P}^3)$. If $V_1 = \text{Span}(\{x_2, x_3, x_4\})$, $V_3 = \text{Span}(\{x_1, x_2, x_4\})$, we write, using [3], $R = a[V_1] + b[V_3] + R'$, where $R'$ has generic Lelong number 0 on $V_1 \cup V_3$, $\|R'\| = 1 - a - b$, and

$$\nu(R', x_1) > \frac{2}{3} - b, \quad \nu(R', x_3) > \frac{2}{3} - a, \quad \nu(R', x_j) > \frac{2}{3} - a - b, \quad j = 2, 4,$$

$$\nu(R', x) > \frac{2}{3} - b, \quad \nu(R', y) > \frac{2}{3} - a.$$  

Note that $a + b < 1$. By [3] Proposition 3.7 there exists a positive closed current with analytic singularities $S$ of bidegree $(1,1)$ on $\mathbb{P}^3$ with $\|S\| = 1 - a - b$ and such that the Lelong numbers of $S$ satisfy the same inequalities as those of $R'$ at the points $x, y, x_1, x_2, x_3, x_4$. Moreover, $S$ is smooth at each point where $R'$ has 0 Lelong number. Let $C_1$ be an irreducible conic in $V_1$ passing through $x_2, x_3, y$ and a point $w_1 \in V_1$ where $\nu(R', w_1) = 0$. Let $C_3$ be an irreducible conic in $V_3$ passing through $x_1, x_4, x$ and a point $w_3 \in V_3$ where $\nu(R', w_3) = 0$. Then the measures $S \wedge [C_j], j = 1, 3,$ are well defined and

$$4(1 - a - b) = \int_{\mathbb{P}^3} S \wedge ([C_1] + [C_3]) \geq \nu(S, x) + \nu(S, y) + \sum_{j=1}^{4} \nu(S, x_j) > 4 - 4a - 4b,$$

a contradiction.

We conclude that $E^+_{\alpha_n}(T,\mathbb{P}^n) \subset A \cup L_{12}$, hence $|E^+_{\alpha_n}(T,\mathbb{P}^n) \setminus L_{12}| = n - 1$. If $B = \{x_1, x_2, x_3\}$ and $R_B$ is the current on $\text{Span}(B) \equiv \mathbb{P}^2$ given by Lemma 2.2 then $E^+_{\alpha_n}(T,\mathbb{P}^n) \cap L_{12} \subset E^+_{1/2}(R_B,\mathbb{P}^2)$. By [1] Theorem 1.1, the set $E^+_{1/2}(R_B,\mathbb{P}^2)$ is finite since it is not contained in a complex line. It follows that $E^+_{\alpha_n}(T,\mathbb{P}^n)$ is a finite set. □

Theorem 1.2 for arbitrary $p$ follows at once from Theorem 2.3 and the next proposition.
Proposition 2.4. Let $T$ be a positive closed current of bidimension $(p, p)$ on $\mathbb{P}^n$, $0 < p < n - 1$, with $\|T\| = 1$ and such that $E^+_{p/(p+1)}(T, \mathbb{P}^n)$ is not contained in a $p$-dimensional linear subspace of $\mathbb{P}^n$. If $W = \text{Span}(E^+_{p/(p+1)}(T, \mathbb{P}^n))$ then $\dim W = p + 1$ and there exists a positive closed current $R$ of bidegree $(1, 1)$ on $W \equiv \mathbb{P}^{p+1}$ such that $\|R\| = 1$ and $E^+_{p/(p+1)}(T, \mathbb{P}^n) \subset E^+_{p/(p+1)}(R, W)$.

Proof. By hypothesis $\dim W \geq p+1$. Assume for a contradiction that there exist linearly independent points $x_1, \ldots, x_{p+2} \in E^+_{p/(p+1)}(T, \mathbb{P}^n)$. Let $U = \text{Span}(\{x_1, \ldots, x_{p+2}\})$ and pick $y \in U$ so that the points $x_1, \ldots, x_{p+2}, y$ are in general position in $U \equiv \mathbb{P}^{p+1}$. We will construct a positive closed current $S$ of bidegree $(1, 1)$ on $U$ such that $\|S\| = 1$ and $\{x_1, \ldots, x_{p+2}, y\} \subset E^+_{p/(p+1)}(S, U)$. By Lemma 2.2 (ii), $y$ must lie in a line spanned by some $x_j, x_k$, $1 \leq j < k \leq p+2$. This contradicts the fact that the points $x_1, \ldots, x_{p+2}, y$ are in general position in $U$. The construction of $S$ is as follows. Choose a sequence of points $y_m \in W \setminus U$ such that $y_m \to y$. Then the points $x_1, \ldots, x_{p+2}, y_m$ are linearly independent. Let $F_m$ be an automorphism of $\mathbb{P}^n$ such that $F_m(x_j) = x_j$, $1 \leq j \leq p+2$, $F_m(x_{p+3}) = y_m$ and set $T_m = (F_m)_{*}T$. These are positive closed currents of bidimension $(p, p)$ on $\mathbb{P}^n$ with $\|T_m\| = 1$ and $\nu(T_m, x_j) = \nu(T, x_j)$, $1 \leq j \leq p+2$, $\nu(T_m, y_m) = \nu(T, x_{p+3})$. By passing to a subsequence we may assume that $T_m$ converge weakly to a current $T'$. Then $\|T'\| = 1$ and by [4],

\[
\nu(T', x_j) \geq \limsup_{m \to \infty} \nu(T_m, x_j) = \nu(T, x_j) > \frac{p}{p+1}, \quad 1 \leq j \leq p+2,
\]

\[
\nu(T', y) \geq \limsup_{m \to \infty} \nu(T_m, y_m) = \nu(T, x_{p+3}) > \frac{p}{p+1}.
\]

Now Lemma 2.1 applied to $T'$ and $U$ with $\alpha = p/(p+1)$ yields the desired current $S$.

Hence we have shown that $\dim W = p + 1$. Lemma 2.1 yields a positive closed current $R$ of bidegree $(1, 1)$ on $W \equiv \mathbb{P}^{p+1}$ such that $\|R\| = 1$ and $E^+_{p/(p+1)}(T, \mathbb{P}^n) = E^+_{p/(p+1)}(R, \mathbb{P}^n) \cap W \subset E^+_{p/(p+1)}(R, W)$ and the proposition is proved. \hfill $\Box$

2.3. Remarks. We start with some examples showing that Theorems 1.1 and 1.2 are sharp. Let $0 < p < n$ and $A = \{x_1, \ldots, x_{p+2}\}$ be a set of linearly independent points of $\mathbb{P}^n$. We set $V_j = \text{Span}(A \setminus \{x_j\})$ and denote by $L_{jk}$ the line spanned by $x_j, x_k$.

Let $T_1 = \frac{1}{p+2} \sum_{j=1}^{p+2} [V_j]$. Then $\|T_1\| = 1$ and $E_{p/(p+2)}(T_1, \mathbb{P}^n) = A$ is not contained in a $p$-dimensional linear subspace of $\mathbb{P}^n$, so the value $\alpha = (p+1)/(p+2)$ in Theorem 1.1 is sharp.

If $p = 1$ the value $\alpha = 1/2$ in Theorem 1.2 was shown to be sharp in [1]. Assume that $2 \leq p \leq n - 1$, choose points $x \in L_{12} \setminus A$, $y \in L_{34} \setminus A$, and let $V_x$, resp. $V_y$, denote the $p$-dimensional linear subspace of $\mathbb{P}^n$ spanned by $(A \cup \{x\}) \setminus \{x_1, x_2\}$, resp. by $(A \cup \{y\}) \setminus \{x_3, x_4\}$. Note that

\[\{x, y\} \subset V_x \cap V_y \cap V_j \quad \text{for} \quad j \geq 5, \quad \{x_1, x_2\} \subset V_y \setminus V_x, \quad \{x_3, x_4\} \subset V_x \setminus V_y,\]

\[x \in (V_3 \cap V_4) \setminus (V_1 \cup V_2), \quad y \in (V_1 \cap V_2) \setminus (V_3 \cup V_4).\]
It follows that the bidimension \((p, p)\) current

\[
T_2 = \frac{1}{2(p+1)} \left( \sum_{j=1}^{4} [V_j] + [V_2] + [V_9] \right) + \frac{1}{p+1} \sum_{j=5}^{p+2} [V_j]
\]

has mass \(\|T_2\| = 1\) and \(\nu(T_2, x) = \nu(T_2, y) = \nu(T_2, x_j) = p/(p+1)\), \(1 \leq j \leq p+2\). Thus \(E_{p/(p+1)}(T_2, \mathbb{P}^n) \supset A \cup \{x, y\}\), so \(|E_{p/(p+1)}(T_2, \mathbb{P}^n) \setminus L| \geq p+1\) for every line \(L \subset \mathbb{P}^n\).

Hence the value \(\alpha = p/(p+1)\) in Theorem 1.2 is sharp.

One can construct a positive closed current \(T_3\) of bidimension \((p, p)\) on \(\mathbb{P}^n\) with \(\|T_3\| = 1\), for which \(E_{p/(p+1)}(T_3, \mathbb{P}^n)\) is a countable union of linear subspaces of dimension at most \(p-1\) contained in a \(p\)-dimensional linear subspace of \(\mathbb{P}^n\). Indeed, let \(V, V_j, j \geq 1\), be distinct \(p\)-dimensional linear subspaces of \(\mathbb{P}^n\) such that \(V \cap V_j \neq \emptyset\) for all \(j\), and set

\[
T_3 = \frac{p}{p+1} [V] + \frac{1}{p+1} \sum_{j=1}^{\infty} 2^{-j} [V_j].
\]

Finally, given any \(k \geq 2\), one can construct a positive closed current \(T_4\) of bidimension \((p, p)\) on \(\mathbb{P}^n\) such that \(\|T_4\| = 1\), \(|E_{p/(p+1)}(T_4, \mathbb{P}^n) \setminus L| = p\) and \(|E_{p/(p+1)}(T_4, \mathbb{P}^n) \cap L| = k\), for some line \(L\). Indeed, pick distinct points \(y_j \in L_{12} \setminus \{x_1, x_2\}, 1 \leq j \leq k-2\), and let \(W_j = \text{Span} (\{y_j, x_3, \ldots, x_{p+2}\})\). If \(0 < \varepsilon < \frac{1}{k-1}\) let

\[
T_4 = \frac{p - \varepsilon}{p(p+1)} \sum_{j=3}^{p+2} [V_j] + \frac{1}{k(p+1)} \left( [V_1] + [V_2] + \sum_{j=1}^{k-2} [W_j] \right)
\]

Then \(\|T_4\| = 1\) and

\[
\nu(T_4, x_1) = \nu(T_4, x_2) = \nu(T_4, y_j) = \frac{p - \varepsilon}{p+1} + \frac{1 + \varepsilon}{k(p+1)} = \frac{p}{p+1} + \frac{1 - (k-1)\varepsilon}{k(p+1)} > \frac{p}{p+1},
\]

\[

\nu(T_4, x_j) = \frac{(p - 1)(p - \varepsilon)}{p(p+1)} + \frac{1 + \varepsilon}{p+1} = \frac{p}{p+1} + \frac{\varepsilon}{p(p+1)} > \frac{p}{p+1}, \quad j \geq 3.
\]

Hence \(T_4\) satisfies the desired properties with \(L = L_{12}\).

We conclude this section by showing the following property of the set \(E_{p/(p+1)}(T, \mathbb{P}^n)\):

**Proposition 2.5.** Let \(T\) be a positive closed current of bidimension \((p, p)\) on \(\mathbb{P}^n\), \(0 < p < n\), with \(\|T\| = 1\), such that the set \(E_{p/(p+1)}(T, \mathbb{P}^n)\) contains the linearly independent points \(x_1, \ldots, x_{p+1}\). If \(V = \text{Span} (\{x_1, \ldots, x_{p+1}\})\) and \(c\) is the generic Lelong number of \(T\) along \(V\) then \(c > 0\).

**Proof.** Assume that \(c = 0\). Applying [10] and [3, Proposition 3.7] as in the proof of Lemma 1.1, we obtain a positive closed current \(S\) of bidegree \((1, 1)\) on \(\mathbb{P}^n\), with analytic singularities, such that \(\|S\| = 1\), \(\nu(S, x_j) > p/(p+1)\) for \(1 \leq j \leq p+1\), and \(S\) is smooth at each point of \(V \equiv \mathbb{P}^p\) outside an analytic subset of \(V\). Then the pull-back \(R\) of \(S\) to \(V\) is well defined, it has unit mass and Lelong number \(> p/(p+1)\) at the linearly independent points \(x_j\). This contradicts Theorem 1.1 (or [2, Proposition 2.2]). \(\blacksquare\)
3. Proof of Theorem 1.3

We prove first Theorem [I.3] for currents of bidegree \((1,1)\) on \(\mathbb{P}^n\), \(n \geq 3\). This is done in the following lemma. Let \(\beta_n = (3n - 4)/(3n - 1)\).

**Lemma 3.1.** Let \(T\) be a positive closed current of bidegree \((1,1)\) on \(\mathbb{P}^n\), \(n \geq 3\), such that \(\|T\| = 1\) and \(E_{\beta_n}^+(T, \mathbb{P}^n)\) contains a set \(A = \{x_1, \ldots, x_{n+1}\}\) of linearly independent points. Then:

(i) For every subset \(B \subset A\) with \(|B| = k + 1, k \geq 2\), there exists a positive closed current \(R_B\) of bidegree \((1,1)\) on \(\operatorname{Span}(B) \equiv \mathbb{P}^k\) such that \(\|R_B\| = 1\) and \(E_{\beta_n}^+(T, \mathbb{P}^n) \cap \operatorname{Span}(B) \subset E_{\beta_n}^+(R_B, \operatorname{Span}(B))\).

(ii) \(E_{\beta_n}^+(T, \mathbb{P}^n) \subset \bigcup_{1 \leq j < k < l \leq n+1} P_{jkl}\), where \(P_{jkl} = \operatorname{Span}\{x_j, x_k, x_l\}\).

(iii) There exist conics \(C_{jkl} \subset P_{jkl}\) and points \(z_{jkl} \in P_{jkl}\) such that \(E_{\beta_n}^+(T, \mathbb{P}^n) \subset \bigcup_{1 \leq j < k < l \leq n+1} (C_{jkl} \cup \{z_{jkl}\})\).

**Proof.** Assertions (i) and (ii) are shown exactly as in the proof of Lemma 2.2, using in (I) the fact that \((2\beta_n - 1)/\beta_n = \beta_{n-1}\), and in (III) the fact that \(n > (n+2)\beta_n\) implies \(n \leq 2\), which contradicts the assumption that \(n \geq 3\).

(iii) Let \(B = \{x_j, x_k, x_l\}\), where \(1 \leq j < k < l \leq n+1\). By (i) there exists a positive closed current \(R\) of bidegree \((1,1)\) on \(P_{jkl}\) such that \(\|R\| = 1\) and \(E_{\beta_n}^+(T, \mathbb{P}^n) \cap P_{jkl} \subset E_{2/5}^+(R, P_{jkl})\). Theorem 1.2 in [I] shows that there exist a conic \(C_{jkl} \subset P_{jkl}\) and a point \(z_{jkl} \in P_{jkl}\) such that \(E_{\beta_n}^+(T, \mathbb{P}^n) \subset C_{jkl} \cup \{z_{jkl}\}\). Hence (iii) follows from (ii).

The next proposition is proved exactly like Proposition 2.4 by using Lemma 3.1 (ii).

**Proposition 3.2.** Let \(T\) be a positive closed current of bideimension \((p,p)\) on \(\mathbb{P}^n\) such that \(1 < p < n - 1\), \(\|T\| = 1\), and the set \(E_{(3p-1)/(3p+2)}^+(T, \mathbb{P}^n)\) is not contained in a \(p\)-dimensional linear subspace of \(\mathbb{P}^n\). If \(W = \operatorname{Span}(E_{(3p-1)/(3p+2)}^+(T, \mathbb{P}^n))\) then \(\dim W = p + 1\) and there exists a positive closed current \(R\) of bidegree \((1,1)\) on \(W \equiv \mathbb{P}^{p+1}\) such that \(\|R\| = 1\) and \(E_{(3p-1)/(3p+2)}^+(T, \mathbb{P}^n) \subset E_{(3p-1)/(3p+2)}^+(R, W)\).

Theorem [I.3] for arbitrary \(p \geq 2\) now follows as at once from Proposition 3.2 and from Lemma 3.1 (iii).

We now turn our attention to the case of currents of bideimension \((1,1)\). If \(L_1, L_2\) are non-concurrent lines in \(\mathbb{P}^n\) and \(T = ([L_1] + [L_2])/2\) then \(\dim \operatorname{Span}(E_{2/5}^+(T)) = 3\), and Theorem 1.3 does not hold for \(p = 1\). However, we have the following geometric property of the set \(E_{2/5}^+(T)\) in this setting:

**Theorem 3.3.** Let \(T\) be a positive closed current of bideimension \((1,1)\) on \(\mathbb{P}^n\) with \(\|T\| = 1\). If \(|E_{2/5}^+(T)| > 37\) then there exists a curve \(C \subset \mathbb{P}^n\) of degree at most 2 such that \(|E_{2/5}^+(T) \setminus C| \leq 1\).

**Proof.** We consider two mutually exclusive cases.

Case 1: The set \(E_{\alpha}^+(T)\) is infinite for some \(\gamma > 1/3\). Then, by [I], \(E_{\gamma}(T)\) contains an irreducible curve \(X\) and \(T = T' + \gamma[X]\), where \(T'\) is a positive closed current and
deg \( X \leq 1/\gamma < 3 \). If \( \deg X = 2 \) then, by [3, Proposition 0], \( X \) is an irreducible plane conic. Moreover, \( \| T' \| = 1 - 2\gamma < 1/3 \), so \( E_{2/3}^+(T) \subset X \). If \( \deg X = 1 \) then \( \| T' \| = 1 - \gamma < 2/3 \). It follows by [11, Theorem 1.1] that \( |E_{1/3}(T') \setminus L| \leq 1 \) for some line \( L \). Since \( E_{2/3}^+(T) \subset X \cup E_{1/3}(T') \) we conclude that \( |E_{2/3}^+(T) \setminus C| \leq 1 \), where \( C = X \cup L \).

Case 2: The set \( E_{2/3}^+(T) \) is finite for all \( \gamma > 1/3 \). By [10], there is a positive closed current \( S \) of bidegree \( (1, 1) \) on \( \mathbb{P}^n \) such that \( \| S \| = \| T' \| = 1 \), and \( S \) has the same Lelong number as \( T \) at every point. Fix \( \gamma \in (1/3, 2/3) \). By Demailly's regularization theorem applied to \( S \) (Main Theorem 1.1 in [3], where we can take \( u = 0 \) since we work on \( \mathbb{P}^n \)), for any \( \epsilon > 0 \) there is a positive closed current \( S_{\epsilon, \gamma} \) of bidegree \( (1, 1) \) on \( \mathbb{P}^n \) with the following properties:

(i) \( S_{\epsilon, \gamma} \) is smooth on \( \mathbb{P}^n \setminus E_{\gamma}(T) \), hence \( S_{\epsilon, \gamma} \) is smooth outside a finite set.

(ii) \( \| S_{\epsilon, \gamma} \| = 1 + \epsilon \) and \( \nu(S_{\epsilon, \gamma}, x) = \max\{\nu(T, x) - \gamma, 0\} \) at each \( x \in \mathbb{P}^n \).

Let \( A = E_{2/3}^+(T) \). Then \( \nu(S_{\epsilon, \gamma}, x) > 2/5 - \gamma \) for \( x \in A \). Since \( S_{\epsilon, \gamma} \) is smooth outside a finite set the measure \( S_{\epsilon, \gamma} \wedge T \) is well defined [4]. We estimate \( |A| \) as follows:

\[
1 + \epsilon = \int_{\mathbb{P}^n} S_{\epsilon, \gamma} \wedge T \geq \sum_{x \in A} \nu(S_{\epsilon, \gamma}, x)\nu(T, x) > \frac{2}{5} \left( \frac{2}{5} - \gamma \right) |A|.
\]

Choosing \( \epsilon > 0 \) very small and \( \gamma > 1/3 \) very close to \( 1/3 \) we find that \( |A| \leq 37 \). \( \square \)

The argument in Case 2 of the proof of Theorem 3.3 can be used to prove a more general result.

**Theorem 3.4.** Let \((X, \omega)\) be a compact Kähler manifold of dimension \( n \), and \( T \) be a positive closed current of bidimension \((p, p)\) on \( X \). For any \( c > 0 \), the set \( E_c(T) \) is contained in an analytic set of dimension \( \leq p \) whose volume and number of irreducible components are bounded above by a constant \( K(\|T\|, c) \) depending only on \( \|T\| \) and \( c \).

**Proof.** Recall that \( \|T\| = \int_X T \wedge \omega^p \) and if \( Z \) is an analytic subvariety of \( X \) then \( \text{vol} Z = \sum_V \int_V \omega^{\dim V} \), where the sum is over all irreducible components \( V \) of \( Z \). By Lelong’s theorem, there is a positive number \( \mu_0 \) such that any subvariety of \( X \) has volume at least \( \mu_0 \). Therefore the number of irreducible components of \( Z \) is \( \leq (\text{vol} Z)/\mu_0 \).

The proof is by induction on \( p \). If \( p = 0 \) then \( T \) is a measure and \( E_c(T) \) is a finite set whose cardinality is \( \leq \|T\|/c \).

Assume that the theorem is true for \( p = p_0 \). We need to prove it for \( p = p_0 + 1 \). Let us define \( A_{c/2, p_0+1}(T) \) to be the union of all irreducible components of dimension \( p_0 + 1 \) of the analytic set \( E_{c/2}(T) \). Set

\[
T' = T - \sum_{V \subset A_{c/2, p_0+1}(T)} \lambda_V(T)[V],
\]

where the sum is over all irreducible components \( V \) of \( A_{c/2, p_0+1}(T) \) and \( \lambda_V(T) \) is the generic Lelong number of \( T \) along \( V \). By [11] \( T' \) is a positive closed current of bidimension \((p_0 + 1, p_0 + 1)\) and \( \|T'\| \leq \|T\| \). Moreover the set \( E_{c/2}(T') \) has dimension at most \( p_0 \), since
$E_{c/2}(T') \subset E_{c/2}(T)$ and $T'$ does not charge any irreducible component $V$ of $A_{c/2,p_0+1}(T)$.

Since $\lambda_\nu(T) \geq c/2$, we have that

$$\text{vol} A_{c/2,p_0+1}(T) = \sum_{V \subset A_{c/2,p_0+1}(T)} [V] \leq 2\|T\|/c.$$  

By [12, Theorem 3.1] there is a positive closed current $R$ of bidegree $(1, 1)$ on $X$ which has the same Lelong number as $T'$ at every point and such that $\|R\| \leq C_1\|T'\|$, where $C_1 > 0$ is a constant depending only on $X$ and $\omega$. By Demailly's regularization theorem applied to $R$ (Main Theorem 1.1 in [3]), there is a positive closed current $R'$ of bidegree $(1, 1)$ on $X$ such that: $\|R'\| \leq C_2\|R\|$, where $C_2$ is a constant depending only on $X$ and $\omega$, $R'$ is smooth on $X \setminus E_{c/2}(T')$, and $\nu(R', x) = \max\{\nu(T', x) - c/2, 0\}$ for every $x \in X$. Since dim $E_{c/2}(T') \leq p_0$, $T_1 = R' \cap T'$ is a well defined positive closed current of bidimension $(p_0, p_0)$ by [3]. Moreover, $\|T_1\| \leq C_3\|T'\|\|R'\| \leq C\|T\|^2$, where $C = C_1C_2C_3$ and $C_3$ is a constant depending only on $X$ and $\omega$. By Demailly's comparison theorem for Lelong numbers [4] we have for $x \in E_c(T) \setminus A_{c/2,p_0+1}(T)$,

$$\nu(T_1, x) \geq \nu(R', x)\nu(T', x) \geq c^2/2.$$

Therefore, if $W$ is the union of all the irreducible components of $E_c(T)$ that are not contained in $A_{c/2,p_0+1}(T)$, then $W \subset E_{c/2}(T_1)$. The induction assumption implies that $E_{c^2/2}(T_1)$ is contained in an analytic subset of dimension $\leq p_0$ whose volume is $\leq K(C\|T\|^2, c^2/2)$. Thus the proof for the case $p = p_0 + 1$ is complete. 

We note that it is not true that the number of irreducible components of the set $E_c(T)$ itself is bounded by a constant depending only on $\|T\|$ and $c$, as the following simple example shows. Let $L_j$, $0 \leq j \leq k + 1$, be lines in $\mathbb{P}^n$ so that $L_0 \cap L_j = \{z_j\}$, $j \geq 1$, and no three of them pass through the same point. Let

$$T = \left(\frac{1}{2} - \frac{1}{2k}\right) [L_0] + \frac{1}{2k} \sum_{j=1}^{k+1} [L_j].$$

Then $\|T\| = 1$ and $E_{1/2}(T, \mathbb{P}^n) = \{z_1, \ldots, z_{k+1}\}$, provided that $k \geq 3$.  

4. **Positive closed currents on $\mathbb{P}^m \times \mathbb{P}^n$**

We prove here certain geometric properties of the upper level sets of Lelong numbers of positive closed currents of bidimension $(1, 1)$ on a multiprojective space

$$X = \mathbb{P}^m \times \mathbb{P}^n = \mathbb{P}^m_z \times \mathbb{P}^n_w.$$  

Let $\pi_z : X \to \mathbb{P}^m_z$, $\pi_w : X \to \mathbb{P}^n_w$, denote the canonical projections and

$$z = [z_0 : \ldots : z_m], \ w = [w_0 : \ldots : w_n],$$

denote the homogeneous coordinates on $\mathbb{P}^m$, respectively on $\mathbb{P}^n$. Set

$$\omega_z = \pi^*_z\omega_m, \ \omega_w = \pi^*_w\omega_n,$$
where $\omega_m$ and $\omega_n$ are the Fubini-Study forms on $\mathbb{P}^m$, respectively $\mathbb{P}^n$. The Dolbeault cohomology group $H^{m+n-1, m+n-1}(X, \mathbb{R})$ is generated by the forms $\omega_m \wedge \omega_n$ and $\omega_m \wedge \omega_n$. Let

$$\theta_{a,b} = a \omega_z \wedge \omega_w + b \omega_z \wedge \omega_w, \quad a, b \geq 0,$$

and let $T_{a,b}$ denote the space of positive closed currents of bidimension $(1, 1)$ on $X$ which lie in the cohomology class of $\theta_{a,b}$.

**Proposition 4.1.** If $T \in T_{a,b}$ then $E_{(a+b)/2}^+(T, X) \subset \pi_z^{-1}(x)$ for some $x \in \mathbb{P}^m$ or $E_{(a+b)/2}^+(T, X) \subset \pi_w^{-1}(y)$ for some $y \in \mathbb{P}^n$.

**Proof.** We may assume that $a + b = 1$ and that for any $x \in \mathbb{P}^m$ we have $E_{1/2}^+(T, X) \not\subset \pi_z^{-1}(x)$. Then there exist points $p_1, p_2 \in E_{1/2}^+(T, X)$ such that $\pi_z(p_1) \neq \pi_z(p_2)$.

We claim that $\pi_w(p_1) = \pi_w(p_2)$. Indeed, suppose $\pi_w(p_1) \neq \pi_w(p_2)$. Composing with an automorphism of $X$ we may assume that

$$p_1 = ([1 : 0 : \ldots : 0], [1 : 0 : \ldots : 0]), \quad p_2 = ([0 : \ldots : 0 : 1], [0 : \ldots : 0 : 1]).$$

Consider the function on $\mathbb{C}^{m+1} \times \mathbb{C}^{n+1}$,

$$u(z_0, \ldots, z_m, w_0, \ldots, w_n) = \max \left\{ \left( \sum_{j=1}^{m} |z_j|^2 \right) \left( \sum_{k=0}^{n-1} |w_k|^2 \right), \left( \sum_{j=0}^{m-1} |z_j|^2 \right) \left( \sum_{k=1}^{n} |w_k|^2 \right) \right\}.$$

The current $\frac{1}{\pi} dd^c \log u$ determines a positive closed current $S$ of bidegree $(1, 1)$ on $X$ in the cohomology class of $\omega_z + \omega_w$ (see e.g. [8, 2]). Note that $S$ has bounded local plurisubharmonic potentials on $X \setminus \{p_1, p_2\}$ and $\nu(S, p_1) = \nu(S, p_2) = 1$. Then

$$a + b = \int_X T \wedge S \geq T \wedge S(\{p_1\}) + T \wedge S(\{p_2\}) \geq \nu(T, p_1) + \nu(T, p_2) > 1,$$

a contradiction.

Hence $\pi_w(p_1) = \pi_w(p_2) = y$. We show that $E_{1/2}^+(T, X) \subset \pi_w^{-1}(y)$. If not, there exists $p \in E_{1/2}^+(T, X)$ with $\pi_w(p) \neq y$. Since $\pi_z(p_1) \neq \pi_z(p_2)$ we may assume that $\pi_z(p) \neq \pi_z(p_1)$. Then we obtain a contradiction as above, working with the points $p, p_1$ instead of the points $p_1, p_2$.

Our next result is in analogy to that of Theorem 1.1. By vertical line in $X$ we mean a line $L \subset \pi_z^{-1}(x) \equiv \mathbb{P}^n$ for some $x \in \mathbb{P}^m$, while by horizontal line in $X$ we mean a line $L \subset \pi_w^{-1}(y) \equiv \mathbb{P}^m$ for some $y \in \mathbb{P}^n$.

**Proposition 4.2.** Let $T \in T_{a,b}$ and $\alpha = \max \{ \frac{2a+b}{3}, \frac{a+2b}{3} \}$. Then $E_{\alpha}^+(T, X)$ is contained in a vertical line in $X$ or in a horizontal line in $X$.

**Proof.** Since $\alpha \geq (a + b)/2$ it follows by Proposition 1.1 that $E_{\alpha}^+(T, X) \subset \pi_z^{-1}(x)$ for some $x \in \mathbb{P}^m$ or $E_{\alpha}^+(T, X) \subset \pi_w^{-1}(y)$ for some $y \in \mathbb{P}^n$. Without loss of generality we may assume that $E_{\alpha}^+(T, X) \subset \pi_z^{-1}(x)$, where $x = [1 : 0 : \ldots : 0] \in \mathbb{P}^m$. We will show that $E_{\alpha}^+(T, X)$ is contained in a line $\pi_z^{-1}(x) \equiv \mathbb{P}^n$.

Suppose for a contradiction that there exist non-collinear points $y^1, y^2, y^3 \in \mathbb{P}^n$ such that $p_j := (x, y^j) \in E_{\alpha}^+(T, X)$. We can find homogeneous quadratic polynomials
$P_1, \ldots, P_n$ on $\mathbb{C}^{n+1}$ such that the set $\{P_1 = 0\} \cap \ldots \cap \{P_n = 0\} \subset \mathbb{P}^n$ is finite and it contains the points $y^1, y^2, y^3$. Moreover, the $2\omega_m$-plurisubharmonic function on $\mathbb{P}^n$ determined by $\frac{1}{2}\log \left( \sum_{k=1}^{n} |P_k|^2 \right)$ has Lelong number 1 at each $y^j$. Consider the function on $\mathbb{C}^{m+1} \times \mathbb{C}^{n+1}$,
\[
 u(z, w) = \max \left\{ \left( \sum_{j=0}^{m} |z_j|^2 \right) \left( \sum_{k=1}^{n} |P_k(w)|^2 \right), \left( \sum_{j=1}^{m} |z_j|^2 \right) \left( \sum_{k=0}^{n} |w_k|^2 \right)^2 \right\}.
\]

The current $\frac{1}{2}dd^c \log u$ determines a positive closed current $S$ of bidegree $(1, 1)$ on $X$ in the cohomology class of $\omega_x + 2\omega_w$ (see e.g. [8, 2]). Note that $S$ has bounded local plurisubharmonic potentials on the complement of a finite subset of $X$ and $\nu(S, p_j) = 1$. Then
\[
 2a + b = \int_X T \wedge S \geq \sum_{j=1}^{3} T \wedge S(\{p_j\}) \geq \sum_{j=1}^{3} \nu(T, p_j) > 3\alpha,
\]
a contradiction. This completes the proof. \hfill \square

We end this section with some examples showing that Propositions 4.1 and 4.2 are sharp. Consider distinct points $x_1, x_2 \in \mathbb{P}^m$, $y_1, y_2 \in \mathbb{P}^n$, and let $p_{jk} = (x_j, y_k) \in X$. For $j = 1, 2$, denote by $V_j \subset \pi_x^{-1}(x_j)$ the vertical line determined by the points $p_{j1}, p_{j2}$, and by $H_j \subset \pi_w^{-1}(y_j)$ the horizontal line determined by the points $p_{1j}, p_{2j}$. Let $a, b > 0$ and $T_1, T_2 \in T_{a,b}$ be the currents
\[
 T_1 = \frac{a}{2}([V_1] + [V_2]) + b[H_1], \quad T_2 = a[V_1] + \frac{b}{2}([H_1] + [H_2]).
\]

Then
\[
 \{p_{11}, p_{21}\} \subset E^+_{(a+b)/2}(T_1, X) \subset H_1 \subset \pi_w^{-1}(y_1),
\]
\[
 \{p_{11}, p_{12}\} \subset E^+_{(a+b)/2}(T_2, X) \subset V_1 \subset \pi_x^{-1}(x_1).
\]

Hence the set $E^+_{(a+b)/2}(T, X)$ can be contained in a vertical fiber or in a horizontal fiber, regardless of how $a$ compares to $b$. Assume next that $a \geq b$ and let
\[
 T_3 = \frac{a + b}{2} [V_1] + \frac{a - b}{2} [V_2] + b[H_1], \quad \text{so} \quad V_1 \cup \{p_{21}\} \subset E_{(a+b)/2}(T_3, X).
\]

Hence $E_{(a+b)/2}(T_3, X) \not\subset \pi_x^{-1}(x)$ for any $x \in \mathbb{P}^m$, $E_{(a+b)/2}(T_3, X) \not\subset \pi_w^{-1}(y)$ for any $y \in \mathbb{P}^n$, and Proposition 4.1 is sharp.

The next example shows the sharpness of Proposition 4.2. Assume $a \geq b$, let $x \in \mathbb{P}^m$, let $y_1, y_2, y_3 \in \mathbb{P}^n$ be non-collinear points, and set $p_j = (x, y_j)$. Denote by $V_{jk} \subset \pi_x^{-1}(x)$ the vertical line determined by the points $p_j, p_k$, and by $H_j \in \pi_w^{-1}(y_j)$ a horizontal line containing $p_j$. Then
\[
 T_4 = \frac{a}{3} ([V_{12}] + [V_{23}] + [V_{13}]) + \frac{b}{3} ([H_1] + [H_2] + [H_3]) \in T_{a,b}
\]
has $\nu(T_4, p_j) = (2a + b)/3 = \alpha$, where $\alpha$ is as in Proposition 4.2. Thus $E_{\alpha}(T_4, X)$ is not contained in a vertical line or in a horizontal line in $X$. 

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