Improved efficiency for explicit covering codes
matching the sphere-covering bound

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Abstract

A covering code is a subset of vectors over a finite field with the property that any vector in the space is close to some codeword in Hamming distance. Blinovsky [Bli90] showed that most linear codes have covering radius attaining the sphere-covering bound. Taking the direct sum of all \(2^\Omega(n^2)\) linear codes gives an explicit code with optimal covering density, which is, to our knowledge, the most efficient construction. In this paper, we improve the randomness efficiency of this construction by proving optimal covering property of a Wozencraft-type ensemble. This allows us to take the direct sum of only \(2^{O(n \log n)}\) many codes to achieve the same covering goodness. The proof is an application of second moment method coupled with an iterative random shift trick along the lines of Blinovsky.

1 Introduction

A \((n, R, d, r)_q\) covering code \(C\) is a subset of \(\mathbb{F}_q^n\) of size \(|C| = q^{nR}\) with minimum distance \(d\) and covering radius \(r\). In this paper, we are concerned with linear covering codes over the binary field \(\mathbb{F}_2\). Our results easily generalize to larger fields. As a counterpart of packing solid balls in Hamming space, which is the heart of error-correction theory, covering codes play a central role in rate distortion theory and source coding. On the other hand, as a combinatorial object, a covering code can be used as a net to approximate any point in the space and hence is also natural to study.

Unlike its dual problem packing whose optimal asymptotic density is widely open in coding theory, the best tradeoff between rate and covering radius is understood for large \(n\). An \(r\)-packing/error-correcting code is one such that Hamming balls of radii \(r\) around codewords are disjoint, or equivalently, any two

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codewords are at least $2r + 1$ apart. The goal is to maximize rate and packing radius simultaneously. As is well-known, the Gilbert–Varshamov (GV) [Gil52, Var57] bound says that a packing code of minimum distance $d = \delta n$ can achieve rate at least $1 - H(\delta)$ for large $n$, where $H(\cdot)$ denotes the binary entropy function. Surprisingly, such a naive bound is still more or less the best we know by now. The best upper bound on packing rate is given by linear programming (LP) bound [MRRW77] which does not match GV bound in general. On the other hand, the sphere-covering bound tells us that a code with covering radius $r = \gamma n$ cannot attain rate lower than $1 - H(\gamma)$. As a simple application of probabilistic method, it is not hard to show that a random code achieves sphere-covering bound with high probability (whp). Moreover, Blinovsky [Bli90] showed that most random linear codes have the same performance as well. The ultimate goal is to find fully explicit covering codes meeting the sphere-covering bound. Our interest is to take structured codes lying on the GV rate and consider their covering goodness. The packing radius of a $(n, R, d, r)$ code $C$ is defined as $r_{\text{pack}}(C) = \lfloor (d - 1)/2 \rfloor$. Obviously, by definition, $r_{\text{pack}} < r_{\text{cov}}$. As mentioned above, for a code of rate $R$, by GV bound, it is possible to make $r_{\text{pack}}$ as large as $\frac{1}{2}H^{-1}(1 - R)$; on the other hand, sphere-covering bound constrained us that there is no way to make $r_{\text{cov}}$ lower than $H^{-1}(1 - R)$. We wish to show that good ensembles $\{C_n\}_n$ have covering radius matching their packing radius up to a factor of $1/2$ asymptotically, i.e.,

$$\limsup_{n \to \infty} \frac{r_{\text{pack}}(C_n)}{n} = \liminf_{n \to \infty} \frac{r_{\text{cov}}(C_n)}{2n}.$$ 

Wozencraft codes $C_\alpha$ for some $\alpha \in \mathbb{F}_2^n$ are an ensemble of linear codes of rate $1/2$ defined as follows:

$$C_\alpha : \mathbb{F}_2^n \to \mathbb{F}_2^n \times \mathbb{F}_2^n,$$

$$x \mapsto (x, \alpha x).$$

It is known that most Wozencraft codes for $\alpha \in \mathbb{F}_2^n$ meet the Gilbert–Varshamov bound [Mas63] and achieve the Shannon channel (e.g., Binary Erasure Channel (BEC)) capacity. Moreover, they can be constructed in $2^{O(n)}$ time. This makes them possible to play a role as inner codes together with other algebraic codes (e.g., Reed–Solomon codes [RS60]) as outer codes in various ensembles of concatenated codes. This procedure gives rise to asymptotically good explicit codes. For instance, Forney codes [For65] are polynomial-time constructible and Justesen codes [Jus72] are locally polynomially computable.

1.1 Prior work

Previously, explicit codes with low covering radius were constructed by Pach and Spencer [PS88]. When the covering radius is fixed, the asymptotic dependence

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1A linear code is said to be polynomial-time constructible if its generator matrix can be computed in poly(n) time. Such a property is also referred to as explicitness in the literature.

2We say that a linear code is locally polynomially computable if each entry of the generator matrix can be computed in poly(log n) time. People also call such codes fully explicit. It is a more stringent notion than polynomial-time constructability.
on field size of covering radius was investigated in [DGMP09]. Covering codes under different error models and with respect to (wrt) different metrics were also studied in the literature [KS14, CEK02]. Covering codes were also used in covert communication and steganography [ZWZ07, BF08].

Beyond the scope of coding theory and information theory, covering codes also found their applications in cryptography [GJL14], complexity theory [Liu18], etc. Although we focus on combinatorial aspects of coverings, computational issues [Slo86] arising in coverings also received significant attention. Furthermore, a perhaps more familiar object to the readers, the Euclidean covering number, exhibits mysterious behaviours and is poorly understood, despite being researchers’ long obsession.

1.2 Notation and preliminaries

For a prime power \( q \), we use \( \mathbb{F}_q \) to denote the finite field of order \( q \). \( \mathbb{F}_q^\times \) denotes the multiplicative group associated to \( \mathbb{F}_q \), i.e., \( \mathbb{F}_q^\times := \mathbb{F}_q \setminus \{0\} \) consists of all nonzero elements.

Since \( \mathbb{F}_{2^n} \cong \mathbb{F}_2^n \), multiplication by \( \alpha \in \mathbb{F}_{2^n} \) can be thought as action of a matrix \( M_\alpha \in \mathbb{F}_2^n \times \mathbb{F}_2^n \). Hence the map \( x \mapsto (x, \alpha x) \) can be written in a matrix form \( v_x \mapsto v_x G \), where

\[
x = (x(1), \ldots, x(n)) \mapsto v_x = \sum_{i=1}^{n} x(i) 2^{i-1}
\]

is the natural additive group isomorphism from \( \mathbb{F}_{2^n} \) to \( \mathbb{F}_2^n \). Denote \( \langle G \rangle := \{(x, \alpha x) \mid x \in \mathbb{F}_{2^n}\} \) which is nothing but the row span of \( G \), and let \( A \) be a Hamming ball \( B(0, r) \subset \mathbb{F}_{2^n} \) of radius \( r \) centered at \( 0 \) such that \( |A| \geq n^3 \cdot 2^n \). For notational brevity, let \( \mathbb{F} := \mathbb{F}_{2^n} \).

Suppose \( X_1, \ldots, X_n \) are independent random variables taking values in \( \{0, 1\} \). Let \( X := \sum X_i \) and \( \mu := \mathbb{E}[X] \). Then for any \( 0 \leq \delta \leq 1 \),

\[
\Pr(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}, \quad \Pr(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}.
\]

For a nonnegative random variable \( X \), it is a corollary of Chebyshev inequality that

\[
\Pr(X = 0) \leq \frac{\text{Var}(X)}{\mathbb{E}[X]^2}.
\]

For a code \( C \subset \mathbb{F}_2^n \), the rate of \( C \) is \( R(C) := \frac{\log |C|}{n} \) and the minimum distance of \( C \) is \( d(C) := \min_{x \neq y \in C} d(x, y) \), where \( d(\cdot, \cdot) \) denotes the Hamming metric. The covering radius \( r_{cov}(C) \) of \( C \) is the smallest number \( r \) such that \( C + B(0, r) = \mathbb{F}_2^n \), where \( B(x, r) \) denotes a Hamming ball of radius \( r \) centered around \( x \).

\[
r_{cov}(C) := \min \{ r : \forall y \in \mathbb{F}_2^n, \exists x \in C, d(y, x) \leq r \}.
\]

Alternatively, from a dual view,

\[
r_{cov}(C) := \max_{y \in \mathbb{F}_2^n} d(y, C), \quad (1)
\]
where the distance of point $y$ to a set $S$ is defined as $d(y, S) := \min_{x \in S} d(y, x)$. The optimal asymptotic tradeoff between rate and covering radius is given by the following definitions.

$$r_{cov}(R) := \liminf_{n \to \infty} \min_{C \subset \mathbb{F}_2^n : |C| \leq 2^n} r(C),$$

$$R(r) := \liminf_{n \to \infty} \min_{C \subset \mathbb{F}_2^n : r(C) \leq r} R(C).$$

We use log and ln to denote logarithms to the base two and $e$, respectively.

### 1.3 Our results

We show that a variant of Wozencraft codes with a slightly larger amount of randomness is covering with high probability.

**Theorem 1.1.** There exists a constant $c > 0$, $t = c \log n$, such that a linear code $C$ defined as the row span of the matrix $G_0 \in \mathbb{F}_2^{(n+t) \times (2^n)}$ is covering with high probability, i.e.,

$$\Pr(C + A = \mathbb{F}_2^{2n}) = 1 - o(1) \quad \text{as } n \to \infty,$$

where the probability is taken over the random construction of $G_0$ defined as follows.

$$G_0 = \begin{bmatrix} G \\ M \end{bmatrix} = \begin{bmatrix} I \\ M_\alpha \end{bmatrix},$$

where $I \in \mathbb{F}_2^{n \times n}$ is the identity matrix, $M_\alpha$ is the matrix corresponding to multiplication by a uniformly random element $\alpha$ in $\mathbb{F}_2^n$, and $M \in \mathbb{F}_2^{t \times (2^n)}$ is a uniformly random matrix with each entry independent and identically distributed (iid) according to Bern(1/2).

Note that our code has rate $\frac{n+t}{2^n} \to \frac{1}{2}$. Also recall that $A \subset \mathbb{F}_2^{2n}$ is taken to be a ball of volume $n^3 \cdot 2^{n^3} \log n = 2^{n^3} n^{3 \log n}$. In other words, $A$ has relative radius $H^{-1}(n^3 \cdot 2^{n^3} n^{3 \log n} \cdot (1 \pm o(1)))^{n^3 \to \infty} H^{-1}(1/2)$. Our main theorem alternatively reads that the code defined above of rate 1/2 has relative covering radius $\gamma = H^{-1}(1/2)$ almost surely, which is the best one can hope for by the sphere-covering bound.

The proof of our main theorem is inspired by Blinovsky’s [Bli90] proof of the covering goodness of random linear codes, but is conducted in a somewhat cleaner way. Blinovsky’s proof technique can also be adapted to $L^2$ codes over $\mathbb{R}^n$. In [ELZ05], it was shown that random Construction–A lattices are covering $\mathbb{R}^n$ with high probability. Gábor [FT09] utilized Blinovsky’s idea to obtain (lower order) improvement over the previous bound due to Rogers [Rog57] on the covering number of an Euclidean convex body. Gábor’s bound still remains the state-of-the-art.
2 Proof of Theorem

First, we prove that a regular Wozencraft code without extra random rows appended to its generator matrix almost surely covers most points of $\mathbb{F}_{2^n}^2$.

Lemma 2.1. For $G$ and $A$ as defined above, we have

$$\Pr(|\mathbb{F}_{2^n}^2 \setminus ((G) + A)| \geq (1/n)2^{2n}) \leq \frac{1}{n^2}.$$ 

Proof. Denote $W := (G)$. Instead of working with $W$, we will look at $W' := (G) + b$ for a randomly chosen $b \in \mathbb{F}_{2^n}^2$ of the form $b = (0, b^{(2)})$, where $b^{(2)} \in \mathbb{F}_{2^n}$. It is easy to see that $|W + A| = |W' + A|$, and this shift provides us with some additional randomness that will help in the analysis. For any $u \in \mathbb{F}_{2^n}^2$, let us define $A_u := A + \{u\}$.

Consider any $a = (a^{(1)}, a^{(2)}) \in \mathbb{F}^2$. By definition of $W'$, we have that $a \in W'$ means there is some $x \in \mathbb{F}$ such that $(a^{(1)}, a^{(2)}) = (x, \alpha x) + (0, b^{(2)})$. Therefore, we have

$$\Pr(a \in W') = \sum_\tau \Pr(\alpha = \tau) \Pr(b^{(2)} = a^{(2)} - \tau a^{(1)})$$

and so by linearity of expectation,

$$\mathbf{E}[|W' \cap A_u|] = |A_u| \cdot \frac{1}{2^n}.$$ 

We also have:

$$\mathbf{E}[|W' \cap A_u|^2] = \sum_{a_1, a_2 \in A_u} \Pr(a_1 \in W' \land a_2 \in W').$$ 

For distinct $a_1, a_2$, we observe that the event $\{a_1 \in W' \land a_2 \in W'\}$ is equivalent to that there are distinct $x_1, x_2 \in \mathbb{F}$ such that

1. $(a^{(1)}_1, a^{(2)}_1) = (x_1, \alpha x_1) + (0, b^{(2)})$;
2. $(a^{(1)}_2, a^{(2)}_2) = (x_2, \alpha x_2) + (0, b^{(2)})$.

Clearly, this gives us that $x_1 = a^{(1)}_1$ and $x_2 = a^{(1)}_2$. Note that in order for both condition 1 and 2 to hold, $b$ has to simultaneously satisfy the following equations

$$b^{(2)} = a^{(2)}_1 - \alpha a^{(1)}_1, \quad b^{(2)} = a^{(2)}_2 - \alpha a^{(1)}_2.$$

For pairs $(a^{(1)}_1, a^{(2)}_1)$ and $(a^{(1)}_2, a^{(2)}_2)$, if exactly one of them is equal, then there is no feasible $b$. Otherwise such $b$ exists if $a^{(2)}_1 - \alpha a^{(1)}_1 = a^{(2)}_2 - \alpha a^{(1)}_2$. 

5
The number of distinct \(a_1, a_2 \in A\) such that \(a^{(1)}_1 = a^{(1)}_2\) is

\[
|S| := \sum_{r \leq j \leq i} \sum_{i \leq j} \binom{n}{i} \binom{n}{j - i} (\binom{n}{j - i} - 1)
\]

\[
= \sum_{i \leq j} \sum_{r \leq j} n_i \binom{n}{j - i} \cdot o \left( \binom{n}{n/2} \right)
\]

\[
\leq \sum_{i \leq j} \sum_{r \leq j} n_i \binom{n}{j - i} \cdot o(2^n)
\]

\[
= o(2^n) \cdot \sum_{j \leq r} \binom{2n}{j}
\]

\[
= o(2^n \cdot |A_u|),
\]

where Eqn. 2 follows since the maximal value of \(j - i\) is at most \(2n \cdot H^{-1}(1/2) \cdot (1 + o(1))\) which itself is less than \(2n \cdot (1/4) = n/2\).

Therefore, continuing the second moment, we have:

\[
\mathbb{E}[|W' \cap A_u|^2] = \sum_{a_1, a_2 \in A_u} \Pr(a_1 \in W' \land a_2 \in W')
\]

\[
= \sum_a \Pr(a \in W') + \sum_{a_1 \neq a_2} \sum_\tau \Pr(\alpha = \tau) \Pr(a_1 \in W' \land a_2 \in W'| \alpha = \tau)
\]

\[
= \mathbb{E}[|W' \cap A_u|] + \sum_\tau \Pr(\alpha = \tau) \frac{1}{2^{2n}}
\]

\[
\leq |A_u| \cdot \frac{1}{2^n} + (|A_u|(|A_u| - 1) - |S|) \cdot \frac{1}{2^{2n}}
\]

\[
t = \frac{|A_u|^2}{2^{2n}} + \frac{|A_u|}{2^n} - \frac{|A_u| + |S|}{2^{2n}},
\]

where inequality 3 follows by dropping the last condition \(a^{(2)}_1 - \alpha a^{(1)}_1 = a^{(2)}_2 - \alpha a^{(1)}_2\).

Therefore, we have

\[
\text{Var}(|W' \cap A_u|) = \mathbb{E}[|W' \cap A_u|^2] - \mathbb{E}^2[|W' \cap A_u|]
\]

\[
= \frac{|A_u|}{2^n} - \frac{|A_u| + |S|}{2^{2n}}
\]

\[
= \frac{|A_u|}{2^n} - \frac{|A_u| + o(2^n |A_u|)}{2^{2n}}
\]

\[
= \frac{|A_u|}{2^n} - o \left( \frac{|A_u|}{2^n} \right)
\]

\[
= \frac{|A|}{2^n} (1 - o(1)).
\]
Since the variance is small, a corollary of Chebyshev inequality gives us that:

$$\Pr(W' \cap A_u = \emptyset) \leq \frac{\text{Var}(|W' \cap A_u|)}{\mathbb{E}^2[|W' \cap A_u|]} \lesssim \frac{2^n}{|A|}$$

modulo lower order term.

Let $X_u$ denote the indicator random variable for the event $\{W' \cap A_u = \emptyset\}$, i.e., that $u$ is not covered by $W' + A$. Denoting $X := \sum_u X_u$, bound 4 give us that $\mathbb{E}[X] \lesssim 2^{2n} \cdot \frac{2^n}{|A|}$, and so by Markov’s inequality, we have that

$$\Pr(X \geq n^2(2^n/|A|)2^{2n}) \leq \frac{1}{n^2},$$

which gives us the desired claim by the choice of $|A|$.

In the second phase, we argue that the union of $t$ random translations of the almost covering Wozencraft code obtained in the previous phase will cover the whole space with high probability.

Let us call the uncovered points at the current stage $U = U_0 := \mathbb{F}_2^{2n} \setminus (W + A)$, and the let the covered points $C = C_0 := W + A$. For a positive integer $i$, and a random vector $u_i \in \mathbb{F}_2^{2n}$ independently chosen for each $i$, let $C_i := \text{span}\{C_{i-1}, u_i\} = C_{i-1} \cup (C_{i-1} + u_i)$, and $U_i := \mathbb{F}_2^{2n} \setminus C_i$ denote the set of covered and uncovered points at stage $i$, respectively. Note that at any stage $i$, $C_i$ and $U_i$ form a partition of the whole space, i.e., $C_i \sqcup U_i = \mathbb{F}_2^{2n}$. The following lemma completes the proof of Theorem 1.1.

**Lemma 2.2.** There is some constant $c > 0$ large enough, such that for $t \geq c \log n$, we have

$$\Pr(U_t \neq \emptyset) \leq \frac{1}{n^2}.$$  

**Proof.** First, we observe that

$$\mathbb{E}[|U_{i+1}| || U_i] = \frac{|U_i|^2}{2^{2n}}.$$  

Indeed, for any $u \in U_i$, denote $X_u$ as the indicator random variable of the event $\{u \notin U_{i+1}\}$. We have

$$\Pr(X_u = 1 || U_i) = \Pr(\exists v \in C_i, v + u_i = u || U_i) = \frac{|C_i|}{2^{2n}}$$

and linearity of expectation gives us the desired identity:

$$\mathbb{E}[|U_{i+1}| || U_i] = |U_i| \Pr(u \in U_{i+1} || U_i) = |U_i|(1 - |C_i|/2^{2n}) = |U_i|^2/2^{2n}.$$
For $i \geq 0$, denote $Y_i$ to be the indicator random variable for the event $\{|U_{i+1}| \leq 2 \cdot (|U_i|^2/2^{2n})\}$. We have by Markov’s inequality, $\Pr(Y_i = 1) \geq \frac{1}{2}$. Further, all the $\{Y_i\}_{i \geq 0}$ are independent random variables by the choice of $\{u_i\}_{i \geq 0}$. Let $Y := \sum_{i=1}^t Y_i$. First note that $\mu := E[Y] \geq t/2 = \frac{t}{2} \log n$. By Chernoff bound, we have that:

$$\Pr(Y \leq 2 \log n) = \Pr \left( Y \leq \left( 1 - \frac{\mu - 2 \log n}{\mu} \right) \mu \right)
= \Pr \left( Y \leq \left( 1 - \frac{c - 4}{c} \right) \mu \right)
\leq 2^{\log c - \frac{1}{2} \left( \frac{c-4}{c} \right)^2 \frac{1}{2} \log n}
\leq \frac{1}{n^2}, \quad (5)$$

where in the last inequality (Eqn. 5) we set $c \geq 4(1 + \sqrt{(\ln 2 + 2) \ln 2 + \ln 2})$.

It is left to observe that given $\sum_{i=1}^t Y_i \geq 2 \log n$, we have that $U_t = \emptyset$. Indeed, since

$$|U_i| \leq \left( \frac{2}{2^n} \right)^{1+2+2^2+\cdots+2^\log n - 1} |U_0|^{2^\log n}
= \left( \frac{2}{2^n} \right)^{2^{\log n} - 1} |U_0|^{2^\log n}
= \left( \frac{2|U_0|}{2^n} \right)^{2^{\log n}} \frac{2^{2n}}{2}
\leq \frac{1}{2} \cdot 2^{-n^2 \log \frac{2}{2} + 2n}
< 1.$$

Finally, to finish the proof, let

$$E := \{C + A = F_{2^n}^2\},
E_1 := \{|\langle G \rangle + A| > (1 - 1/n) \cdot 2^{2n}\}.$$

Overall we have that

$$\Pr(E) \geq \Pr(E_1) \Pr(E|E_1)
\geq (1 - 1/n^2) \cdot (1 - 1/n^2)
= 1 - o(1).$$

A note on other rates The above construction started off with an ensemble of codes $C$ of rate $\frac{1}{2}$ (Wozencraft). However, the construction can be generalized
for other rates in a standard way. The only thing that was used about the Wozencraft ensemble in the proof of Theorem 1.1 was that $C$ was supported on $2^n$ codes, and had the following property. Fix a message $m$, and choose a random code $C \in \mathcal{C}$, sends the message $m$ to $(m, x)$ where $x$ is a uniformly random element of $F_2^n$. One can check that the proof works verbatim when $C$ is supported on $2^k$ codes, and a random $C \in \mathcal{C}$ sends a message $m$ to $(m, x)$ where $x$ is uniform in $F_2^k$. Therefore, one can restrict the Wozencraft ensemble to a set of coordinates (or puncture it) to achieve different rates. For an $k \times n$ matrix $M$, for $S \subset [k]$ and $T \subset [n]$, we use $M[S, T]$ to denote the submatrix where the rows are indexed by $S$ and columns are indexed by $T$. For every generator matrix $G = [I|\alpha \cdot M]$ from the Wozencraft ensemble, denote $G_k = [I_n[[k],[k]] \mid M_\alpha[[k],[n]]]$. Given any message $m' \in F_2^k$, one can check that a randomly chosen $G_k$ takes $m'$ to $(m', x')$ where $x'$ is a uniform point in the row span of $M_\alpha[[k],[n]]$. To see this, note that since $\alpha m$ is uniform in $F_2^n$, take $m = (m'(1), \ldots, m'(k), 0, \ldots, 0) \in F_2^n$. The image of each such $m$ under $M_\alpha$ is exactly $m' \cdot M_\alpha[[k],[n]]$ which is equal to $x'$ and hence each $x'$ is equally likely to be output. Therefore, by a similar proof as above, one can check that the code generated by a randomly chosen matrix $G_0$ given by:

$$G_k = [I_n[[k],[k]] \mid M_\alpha[[k],[n]]]$$

is almost surely a good covering code of rate $\frac{n+\log n}{n+R} = \frac{nR+c \log n}{n+nR} \to \frac{R}{1+R}$ if we denote $k = nR$. Similar arguments show that one can truncate $G$ as

$$C^k = [I_n \mid M_\alpha[[n],[k]]]$$

to get a code of rate $\frac{n+c \log n}{n+R} \to \frac{1}{1+R}$.

2.1 Covering radius of concatenated codes

Here we show that covering codes behave well under concatenation.

The direct sum $C_1 \oplus C_2$ of a $(n_1, R_1, d_1, r_1)$ code $C_1$ and a $(n_2, R_2, d_2, r_2)$ code $C_2$ is defined as

$$C_1 \oplus C_2 = \{(x_1, x_2) \mid x_1 \in C_1, x_2 \in C_2\}.$$  

Clearly, $C_1 \oplus C_2$ has blocklength $n_1 + n_2$, rate $R_1 + R_2$ and minimum distance $\min\{d_1, d_2\}$. It is not hard to see that the covering radius of a direct sum code is the sum of its components, i.e., $r_{\text{cov}}(C_1 \oplus C_2) = r_{\text{cov}}(C_1) + r_{\text{cov}}(C_2)$. Indeed,
the claim follows from the very definition of covering radius (Eqn. 1),

$$r_{\text{cov}}(C_1 \oplus C_2) = \min_{(y_1, y_2) \in F_2^n} \max_{x_1, x_2 \in C_1 \oplus C_2} d((y_1, y_2), (x_1, x_2))$$

$$= \min_{y_1, y_2 \in F_2^n} \max_{x_1 \in C_1, x_2 \in C_2} d(y_1, x_1) + d(y_2, x_2)$$

$$= \min_{y_1 \in F_2^n} \max_{x_1 \in C_1} d(y_1, x_1) + \min_{y_2 \in F_2^n} \max_{x_2 \in C_2} d(y_2, x_2)$$

$$= r_{\text{cov}}(C_1) + r_{\text{cov}}(C_2).$$

If $C_1$ and $C_2$ are linear codes generated by matrices $G_1 \in F_{n_1}^{r_1 \times n_1}$ and $G_2 \in F_{n_2}^{r_2 \times n_2}$, then the direct sum has a block diagonal generator matrix $G$ of the following form:

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}.$$

Since we known that our construction is covering almost surely, the direct sum operation allows us to construct explicit covering codes by concatenating all Wozencraft-type codes. Let $n' := n + t$. We just put all matrices $\{G_i\}_{i=1}^N$ of the form $G_0$ defined in Theorem 1 along the diagonal and get a matrix $G$ of size $n' \cdot N$ by $(2n) \cdot N$. These matrices generate the Wozencraft-type ensemble $\{C_i\}_i$ and there are $N := 2^n \cdot 2^{2^n} = 2^{O(n \log n)}$ many such matrices in total. This operation results in a code $C$ with generator matrix $G$ of blocklength $(2n) \cdot N$ and rate $\frac{n' \cdot N}{2n \cdot N} = \frac{n - t \cdot \log n}{2n}$. The relative covering radius of the direct sum is at most

$$r_{\text{cov}} \left( \bigoplus_{i=1}^N C_i \right) \leq (1 - 1/n^2)^2 \cdot H^{-1} \left( \frac{n + 3 \log n}{2n} \cdot (1 + o(1)) \right) + (1 - (1 - 1/n^2)^2) \cdot 1 \rightarrow H^{-1}(1/2),$$

as $n$ approaches infinity.

Previously the best randomness efficiency of the construction of explicit covering codes is achieved by concatenating all linear codes. This attains sphere-covering bound since most linear codes do by [Bli90]. If a random linear code is operating at a rate, say, 1/2 for comparison with our result, then its generator matrix is supported on $2^{n \times (2n)} = 2^{O(n^2)}$ many matrices in $F_2^{n \times 2n}$. In our construction, only those Wozencraft-type codes are concatenated and there are $2^{O(n \log n)}$ many of them. Although we are still far from a fully explicit construction, this improves the best known bound.

3 Open problems

Although Blinovsky’s technique for proving covering property is ingenious and powerful, the second phase of his proof corresponds to padding uniformly random rows to the generator matrix of codes we are interested in. This is fine
only for uniformly random linear codes but could destroy the explicitness of our codes with further structures. We believe that the random shift technique is just an artifact for the analysis. Given that the majority of random linear codes is covering anyway, it may be instructive to rediscover Blinovsky’s result in a one-shot manner. Such a proof can be potentially generalized to study various other ensembles whose covering properties we are unable to show, e.g., codes generated by circulant matrices, low-density parity-check (LDPC) codes, etc.

A quasicyclic code $C \leq F_2^n$ is a linear code of rate $1/2$ spanned by the rows of a matrix of the form $G = [I|M]$, where $I \in F_2^{n \times n}$ and $M \in F_2^{n \times n}$ is a circulant matrix

$$M = \begin{bmatrix}
-r_1 \\
-r_2 \\
\vdots \\
-r_n
\end{bmatrix}.$$

For any $i \in [n-1]$, the $(i+1)$-th row is a one-bit right-shift of the $i$-th row, i.e., $r_{i+1} = \sigma(r_i)$ where $\sigma \in S_n$ is a permutation

$$\sigma = \begin{pmatrix}
1 & 2 & 3 & \cdots & n \\
1 & 2 & 3 & \cdots & n-1
\end{pmatrix}.$$

If we sample a row $r$ uniformly at random from $F_2^n$ and construct a corresponding code $C$, then $C$ is known [GZ08] to attain GV bound with high probability. Actually, it beats GV bound by some lower order factor and is the best asymptotic existence result in the constant relative minimum distance regime. However, we are unable to show its covering property. One challenge among others is that there is only a small amount of randomness in the construction. The whole matrix $G$ is completely determined once any row or column of $M$ is sampled. Indeed, a one-shot analysis is in demand.

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