A LINEARLY EXPANDING UNIVERSE WITH VARIABLE $\Lambda$ AND $G$

Arbab I. Arbab

Department of Physics, Faculty of Science, University of Khartoum, Khartoum 11115, SUDAN

ABSTRACT

We have studied a cosmological model with a cosmological term of the form $\Lambda = 3\alpha \dot{R}^2 + \beta \ddot{R} + \gamma R^2$, where $\alpha$, $\beta$, $\gamma$ are constants. The scale factor ($R$) is found to vary linearly with time for both radiation and matter dominated epochs. The cosmological constant is found to decrease as $t^{-2}$ and the rate of particle creation is smaller than the Steady State value. The model gives $\Omega^\Lambda = \frac{1}{3}$ and $\Omega^m = \frac{2}{3}$ in the present era, $\Omega^\Lambda = \Omega^r = \frac{1}{3}$ in the radiation era. The present age of the universe ($t_p$) is found to be $t_p = H_p^{-1}$, where $H_p$ is the Hubble constant. The model is free from the main problems of the Standard Model. Since the scale factor $R \propto t$ during the entire evolution of the universe the ratio of the cosmological constant at the Planck and present time is $\frac{\Lambda_p}{\Lambda_p} = 10^{120}$. This decay law justifies why, today, the cosmological constant is exceedingly small.

KEY WORDS: Cosmology, Variable $\Lambda$, Inflation

The present interest in the flat cosmological constant models is motivated by the fact that a non-zero $\Lambda$ term helps to reconcile inflation with observation. This term could be responsible for the missing mass in the universe. It also helps to obtain, for a flat universe, a theoretical age in the observed range, even for a high value of the Hubble constant. It has been announced by NASA group that they have found the age of the universe to be about 12 billion years old [1]. In an attempt to reconcile this value with FRW universe, we introduce a cosmological term ($\Lambda$) of the form $\Lambda = 3\alpha \frac{\dot{R}^2}{R^2} + \beta \frac{\ddot{R}}{R} + \gamma R^2$. We have shown that the scale factor ($R$) varies as $R \propto t$ in both radiation and matter epochs. The model has then no horizon, age or flatness problem. Our analysis shows that $H_p = 81 \text{ km s}^{-1} \text{ Mpc}^{-1}$, a result that is allowed by current observations. It turns out that the $\beta$ does not affect our cosmology if the universe is singular.
2. THE MODEL

In a Robertson Walker metric, the Einstein’s field equations with variable cosmological and gravitational ‘constants’ and a perfect fluid yield [2]

\[ \frac{3}{R^2} \frac{\dot{R}^2}{R^2} + \frac{3k}{R^2} = 8\pi G \rho + \Lambda , \]  
\[ \frac{2}{R} \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G p + \Lambda , \]  
where \( \rho \) is the fluid energy density and \( p \) its pressure. The equation of the state is usually given by

\[ p = \omega \rho , \]  
where \( \omega \) is a constant. Elimination of \( \ddot{R} \) gives

\[ 3(p + \rho) \dot{R} = -\left(\frac{\dot{G}}{G} \rho + \dot{\rho} + \frac{\dot{\Lambda}}{8\pi G}\right)R. \]  

3.a PARTICLE CREATION

3.a.1 Matter-Dominated Universe

For a pressure-less \( (p = 0) \) universe, eq.(4) reads

\[ \frac{d(\rho R^3)}{dt} = -\frac{R^3}{8\pi G} \frac{d\Lambda}{dt} . \]  

In this paper we discuss flat cosmological models with a \( \Lambda \) term varying as [3]

\[ \Lambda = 3\alpha \frac{\dot{R}^2}{R^2} + \beta \frac{\dot{R}}{R} + 3\gamma \frac{R^2}{R^2} , \]  
where \( \alpha, \beta \) and \( \gamma \) are arbitrary constants. This model extends the Carvalho et al. model to include the possibility of variable gravitational constant [4]. Eqs.(2) and (6) yield

\[ (2 - \beta)\ddot{R}R + (1 - 3\alpha)\dot{R}^2 - \frac{3\gamma}{R^2} = 0 , \]  
which yields

\[ \dot{R}^2 = \frac{3\gamma}{(1 - 3\alpha)} + A R^{-2(1-3\alpha)/(2-\beta)} , \]  
where \( A = \text{constant} \). If \( A=0 \) (singular solution), one obtains,

\[ R = \sqrt{\frac{3\gamma}{(1 - 3\alpha)}} t , \alpha < \frac{1}{3} . \]  

Eqs.(1), (6) and (9) yield

\[ H_p = \frac{1}{t_p} , \Lambda(t) = \frac{1}{t_p^2} , \rho = \frac{1}{4\pi G t_p^2} . \]
where $H = \frac{\ddot{R}}{R}$ is the Hubble constant. Hereafter the subscript ‘p’ denotes the present value of the quantity. The vacuum energy density ($\rho_v$) is given by

$$\rho_v(t) = \frac{\Lambda}{8\pi G} = \frac{1}{8\pi G \frac{1}{t^2}}.$$ (11)

The deceleration parameter ($q$) is defined as

$$q \equiv -\frac{\dddot{R}}{\dot{R}^2} = 0$$ (12)

The density parameter of the universe ($\Omega^m$) is given by

$$\Omega^m = \frac{\rho}{\rho_c} = \frac{2}{3},$$ (13)

where $\rho_c = \frac{3H^2}{8\pi G}$. The density parameter due to vacuum contribution is defined as $\Omega^\Lambda = \frac{\Lambda}{3H^2}$. Using eq.(10) this yields

$$\Omega^\Lambda = \frac{1}{3}.$$ (14)

It is interesting that the above cosmological parameters are independent of the value of $\alpha, \beta$ or $\gamma$. We shall define $\Omega_{\text{total}}$ as

$$\Omega_{\text{total}} = \Omega^m + \Omega^\Lambda.$$ (15)

Hence eqs.(13), (14) and (15) give $\Omega_{\text{total}} = 1$. The inflationary paradigm requires this solution. A very recent age of the universe is shown to be 12 billion years. This would imply that the Hubble constant $H_p = 81 \text{ km s}^{-1} \text{Mpc}^{-1}$. This value is consistent with the recent Hubble Space Telescope determination of $h = 0.80 \pm 0.17$ [5]. We now turn to calculate the rate of particle creation (annihilation) $n$, which is defined as [6]

$$n = \frac{1}{R_p^3} \frac{d(\rho R^3)}{dt}|_p.$$ (16)

Using eqs.(5), (9) and (10), eq.(16) yields

$$n_p = \rho_p H_p.$$ (17)

We remark that this rate is less than that of the Steady State model ($= 3\rho_p H_p$).

**A model with variable $G$:**

We now consider a model in which both $G$ and $\Lambda$ vary with time in such a way the usual energy conservation law holds. Equation (4) can be split to give [2]

$$\dot{\rho} + 3H \rho = 0,$$ (18)

and

$$\dot{\Lambda} + 8\pi \dot{G} \rho = 0.$$ (19)
Using eqs.(9) and (10), eqs.(18) and (19) yield

\[ \rho(t) = A't^{-3} , \]  

and

\[ G(t) = \frac{1}{4\pi GA'} t , \quad A' = \text{const.} \]  

This solution is obtained by Berman and Som [7] and can be obtained from Berman’s model [8] with \([m = 1, \alpha = 0]\). It has been shown that the development of the large-scale anisotropy is given by the ratio of the shear \(\sigma\) to the volume expansion \((\theta = 3\frac{\dot{R}}{R})\) which evolves as [9]

\[ \frac{\sigma}{\theta} \propto t^{-2} . \]  

The present observed isotropy of the Universe requires this anisotropy to be decreasing as the universe expands. Thus an increasing \(G\) guarantees an isotropized universe.

### 3.a.2 Radiation-Dominated Universe

This is defined by the equation of the state \(p = \frac{1}{3}\rho\ (\omega = \frac{1}{3})\). Eqs.(1), (2) and (6) yield

\[ (1 - 2\alpha) \frac{\dot{R}^2}{R^2} + (1 - 2\frac{2}{3}\beta) \frac{\ddot{R}}{R} - 2\frac{\gamma}{R^2} = 0 , \]  

which gives

\[ \dot{R}^2 = \frac{2\gamma}{(1 - 2\alpha)} + BR^{-2(1-2\alpha)/(1-\frac{2}{3}\beta)} , \]  

where \(B = \text{const.}\) For \(B = 0\) (singular solution), one obtains

\[ R = \sqrt{\frac{2\gamma}{(1 - 2\alpha)}} t , \quad \alpha < \frac{1}{2} . \]  

Eqs.(1) and (6) give

\[ \Lambda = \frac{3}{2} t^2 , \quad \rho = \frac{3}{16\pi G t^2} . \]  

It has been pointed out that we need not have an inflationary phase (exponential law for \(R\)) because there is no horizon problem with the above solution [8]. We see from eqs.(10) and (26) that the ration of the cosmological constant at the Planck’s time \((t_{\text{Pl}} = 10^{-43} \text{ sec})\) and the present present time \((t_p = 10^{17} \text{ sec})\) is \(\Lambda_{\text{Pl}} / \Lambda_p = (\frac{10^{43}}{10^{17}})^2 = 10^{120}\). Thus \(\Lambda\) today is exceedingly small because the universe is too old! Eqs.(15) and (26) yield

\[ \Omega^\Lambda = \Omega^\Lambda = \frac{1}{2} . \]  

Again we see that the above cosmological parameters are independent of the value of \(\alpha, \beta\) or \(\gamma\). Equation (27) shows that the radiation and vacuum contribute equally to the total energy density.
A model with variable $G$:

Equation (4) now reads
\[ \dot{\rho} + 4H\rho = 0 , \] (28)

and
\[ \dot{\Lambda} + 8\pi \dot{G}\rho = 0 . \] (29)

Employing eqs.(1) and (6), eqs.(28) and (29) yield
\[ \rho = B't^{-4} , \quad G = \frac{3}{16\pi B'}t^2 , \quad B' = \text{const}. \] (30)

Recently, Abdel Rahman [2] and Berman [7,8,10] (with $[m = 1, \alpha = \frac{1}{3}]$) have found that $R \propto t$, $G \propto t^2$, $\rho \propto t^{-4}$ in the radiation era.

CONCLUSION

We have analyzed in this paper the effect of the assumed decay law for the cosmological constant. The only possible solution is the $R \propto t$ for the radiation and matter dominated epochs. This solution is obtained by Berman (for Brans-Dicke models with a time dependent cosmological term) for the matter and the radiation epochs and by Abdel Rahman for the radiation epoch. There is no horizon or age problem associated with this solution. Our model predicts that $H_p = 81 \text{ kms}^{-1}\text{Mpc}^{-1}$ if the age of the universe is 12 billion years. Thus by introducing the cosmological constant it is possible for a flat model to be in accordance with the universe age even with high values of $H_p$. We stress that the decay law $\Lambda \propto t^{-2}$ indeed plays an important role in cosmology as remarked by Berman. A natural extension of this work would be to investigate different scenarios by considering non-singular solutions, i.e., solutions with $A$ and $B$ negative.

ACKNOWLEDGMENT

I would like to thank the University of Khartoum for financial support.

REFERENCES

1- NASA GROUP of astronomers, 25 May 1999
2- Abdel-Rahman, A. -M. M., 1990. *Gen. Rel. Gravit.* **22**, 655
3- Al-Rawaf, A. S., 1998. *Mod. Phys. Lett.A* **13**, 429
4- Carvalho, J. C., Lima, J. A. S., and Waga, I., 1992. *Phys. Rev.* **D46**, 2404
5- Freedman, W. L., 1994. *Nature* **371**, 757
6- Matyjasek, J., 1995. *Phys. Rev.* **D51**, 4154
7- Berman, M. S., and Som, M. M., 1990. *Int. J. Theor. Phys.* **29**, 1411
8- Berman, M. S., 1991. *Gen. Rel. Gravit.* **23**, 465
9- Barrow, J. D., 1978. *Mon. Not. astr. Soc.* **184**, 677
10- Berman, M. S., 1990. *Int. J. Theor. Phys.* **29**, 1419