Heat Kernel Asymptotics of Operators with Non-Laplace Principal Part

Ivan G. Avramidi

Department of Mathematics, The University of Iowa
Iowa City, IA 52242, USA

and

Department of Mathematics
New Mexico Institute of Mining and Technology
Socoro, NM 87801, USA

and

Thomas Branson

Department of Mathematics, The University of Iowa
Iowa City, IA 52242, USA

October 8, 2000

We consider second-order elliptic partial differential operators acting on sections of vector bundles over a compact Riemannian manifold without boundary, working without the assumption of Laplace-like principal part \(-\nabla^2\). Our objective is to obtain information on the asymptotic expansions of the corresponding resolvent and the heat kernel. The heat kernel and the Green’s function are constructed explicitly in the leading order. The first two coefficients of the heat kernel asymptotic expansion are computed explicitly. A new semi-classical ansatz as well as the complete recursion system for the heat kernel of non-Laplace type operators is constructed. Some particular cases are studied in more detail.
1 Introduction

The resolvent and the heat kernel of elliptic partial differential operators are of great importance in mathematical physics, differential geometry and quantum theory \[17, 12, 8, 11, 19\]. Of special interest in the study of elliptic operators are the so-called heat kernel asymptotics. It is well known \[12\] that for a second-order, elliptic, self-adjoint partial differential operator \(F\) with a positive definite leading symbol, acting on sections of a vector bundle over a compact, boundariless manifold \(M\) of dimension \(m\), an asymptotic expansion of the following form is valid as \(t \downarrow 0\):

\[
\text{Tr}_{L^2} \exp(-tF) \sim (4\pi t)^{-m/2} \sum_{k \geq 0} \frac{(-t)^k}{k!} A_k.
\]

The coefficients \(A_k\) are called the heat invariants, or heat kernel coefficients. They are spectral invariants of the operator \(F\), and encode information about the asymptotic properties of the spectrum. Note that our normalization of the coefficients \(A_k\) differs by the factor \((-1)^k/k!\) from that used in \[12\]. This normalization has been used in previous works of one of the authors (see the book \[5\], the papers \[2, 4, 3\] and others). It has the advantage that for Laplace type operators with a potential (see definition below), \(F = -\Delta + q\), the numerical coefficient of the term \(\int_M d\text{vol}(x) \text{tr} q^k\) in \(A_k\) is equal to 1 for any \(k\).

An important subclass of the class of operators described above are the operators of Laplace type: those for which the leading symbol takes the form \(g^{\mu\nu} \xi_\mu \xi_\nu\), where \(g^{\mu\nu}\) is a non-degenerate, positive definite metric on the cotangent bundle of \(M\). For such operators, the leading symbol naturally defines a Riemannian metric on \(M\) (the inverse \(g_{\mu\nu}\) of the leading symbol). Alternatively, one may take the Riemannian metric as given, and produce the many natural operators of Laplace type which are so important in Physics.

The assumption of Laplace type affords a considerable simplification in the study of spectral asymptotics. Partly as a result of this, much is known about the resolvent, the heat kernel, and the zeta function in this category of operators \[12, 8, 4, 3\]. In particular, the heat kernel coefficients \(A_k\) for Laplace type operators are known explicitly up to \(k = 4\) \[2, 3\] (for a review, see \[1\]).

In this paper, we take a Riemannian metric as given, and study the most general class of second-order operators \(F\), acting on sections of a vector bundle \(\mathcal{V}\), with positive definite
leading symbol. That is, we drop the assumption of Laplace type, and assume only that
\[ \sigma_2(F; x, \xi) = a^{\mu\nu}(x) \xi_\mu \xi_\nu, \]
where \( a^{\mu\nu} \) is a symmetric two-tensor valued in \( \text{End}(\mathcal{V}) \). (We do not assume that \( a^{\mu\nu} = g^{\mu\nu} \text{id}_\mathcal{V} \), nor that \( a^{\mu\nu} \) is factored as \( g^{\mu\nu} E \) for \( E \) a section of \( \text{Aut}(\mathcal{V}) \).) We shall sometimes call these “NLT” (for “non-Laplace type”) operators. Of course, Laplace type is a special case, so to be more precise, NLT operators are operators that are not necessarily of Laplace type. NLT operators arise naturally in such areas of mathematical physics as quantum gauge field theory and quantum gravity [11, 5], differential geometry, classical continuum mechanics [18] and others.

The most elementary examples that one can use to illustrate the class of operators in question are the weighted form Laplacians. If \( d \) is the exterior derivative and \( \delta \) its formal adjoint, the form Laplacian is \( \Delta = \delta d + d\delta \); this differs from the Bochner Laplacian \( g^{\mu\nu} \nabla_\mu \nabla_\nu \) by an operator of order zero, the so-called Bochner-Weitzenböck operator. If \( a, b \) are real constants, an operator of the form \( a\delta d + b d\delta \) may be termed a weighted form Laplacian; such operators are elliptic if \( a \neq 0 \neq b \), and have positive definite leading symbol if \( a, b > 0 \).

The non-Laplace type operators on differential forms have been studied extensively by many authors under the general name “nonminimal operators”, or “exotic operators” (see [13, 4, 14, 15, 10, 11, 1 and references therein). In particular, the paper [13] contains a complete discussion of the coefficients \( A_k \) of the weighted form Laplacian for all \( k \) and of \( A_0 \) and \( A_1 \) of the operator with a potential function (Th. 1.2 and Th 1.3, pp. 2089-2090). In papers [14, 15, 16] a computer algorithm has been developed that employs the calculus of pseudodifferential operators and the coefficients \( A_0 \), \( A_1 \) and \( A_2 \) (more precisely, the local (non-integrated) coefficients \( a_0, a_1 \) and \( a_2 \)) have been computed.

The \( \zeta \)-function for non-Laplace type operators acting on symmetric 2-tensors have been studied in [10] (for the origin of such operators, see [11, 5]). By restricting to maximally symmetric even dimensional manifolds the authors were able to compute the eigenvalues of such operators and evaluate the \( \zeta \)-function at zero, \( \zeta(0) \), (which is, in fact, equivalent to the coefficient \( A_{m/2} \) [12]) for a certain special case of the operator in Euclidean spaces and \( m \)-spheres in dimensions \( m = 2, 4, 6, 8, 10 \).

In general, the study of spin-tensor quantum gauge fields in arbitrary gauge necessarily leads to non-Laplace type operators acting on sections of general tensor-spinor bundles.
It is precisely these operators that are of prime interest in the present paper. The main examples here are the symmetric 2-tensor bundle (gravitational field, spin 2) and the spin-vector bundle (gravitino field, spin 3/2) (for a discussion of such operators in gauge field theories, see [7]).

Let us formulate our main result from the very beginning.

**Theorem 1** Let \( F : C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V}) \) be a self-adjoint elliptic second-order partial differential operator with a positive definite leading symbol, acting on sections of a tensor-spinor bundle \( \mathcal{V} \) of fiber dimension \( d \) over a compact manifold \( M \) of dimension \( m \) without boundary. Let \( F = -a^{\mu\nu}\nabla_\mu \nabla_\nu + q \), where \( \nabla \) is a connection on the vector bundle \( \mathcal{V} \), \( a^{\mu\nu} \) is a parallel symmetric two-tensor valued in \( \text{End}(\mathcal{V}) \) and \( q \) is an endomorphism of the bundle \( \mathcal{V} \). The curvature of the connection \( \nabla \) is defined by \( [\nabla_\mu, \nabla_\nu] \phi = R^\alpha_{\beta\mu\nu} T^\beta_{\alpha} \phi \), where \( R^\alpha_{\beta\mu\nu} \) is the Riemann curvature tensor and \( T^\alpha_{\beta} \) is given by the representation of \( \mathfrak{so}(m) \) which induces the bundle \( \mathcal{V} \). Let \( \xi \in T^*M \) be a cotangent vector, and let \( \lambda_i(\xi) = \mu_i|\xi|^2 \), \( (i = 1, \cdots, s) \), \( \mu_i > 0 \), be the eigenvalues of the leading symbol, \( \sigma_2(F) = a^{\mu\nu}\xi_\mu\xi_\nu \), with the multiplicities \( d_i \). Then the corresponding (orthogonal) eigenspace projections have the form

\[
\Pi_i(\xi) = \sum_{n=0}^{p} \Pi_{\mu_1\cdots\mu_{2n}}^{(2n)} \frac{1}{|\xi|^{2n}} \xi_{\mu_1} \cdots \xi_{\mu_{2n}}
\]

\[
= \sum_{k=1}^{s} c_{ik}(\xi^{\mu_1\mu_2} \cdots a^{\mu_{2k-3}\mu_{2k-2}}) \frac{1}{|\xi|^{2k-2}} \xi_{\mu_1} \cdots \xi_{\mu_{2k-2}} ,
\]

where the numbers \( s \) and \( p \) depend on the structure of the bundle \( \mathcal{V} \) and the leading symbol. The matrix of coefficients \( C = (c_{ik}) \) is inverse to the (Vandermonde) matrix of powers \( \mathcal{M} = (\kappa_{kj}) := (\mu_j^{k-1}). \)

Furthermore, the \( L^2 \) trace of the heat kernel has the following asymptotics as \( t \to 0 \)

\[
\text{Tr}_{L^2} \exp(-tF) = (4\pi t)^{-m/2} \left[ A_0 - t A_1 + O(t^2) \right].
\]

The coefficients \( A_0 \) and \( A_1 \) are defined by

\[
A_0 = \sum_{i=1}^{s} d_i \mu_i^{-m/2} \text{vol}(M),
\]

\[
A_1 = \int_M d\text{vol}(x) \left\{ \text{tr}_\mathcal{V} (a_0 q) + \beta R \right\},
\]
where $R$ is the scalar curvature,

$$a_0 = \sum_{i=1}^{s} \mu_i^{-m/2} < \Pi_i >,$$

$$< \Pi_i > = \Pi_{i(0)} = \sum_{k=1}^{s} c_{ik} \frac{\Gamma(m/2)(2k-2)!}{\Gamma(m/2 + k - 1)2^{2k-2}(k-1)!} \times g(\mu_1\mu_2 \cdots g_{\mu_{2k-3}\mu_{2k-2}}^\alpha a^{(\mu_1\mu_2} \cdots a^{\mu_{2k-3}\mu_{2k-2})},$$

(1.1)

and $\beta$ is a constant defined by

$$\beta = -\frac{1}{6m} \sum_{i=1}^{s} \mu_i^{-m/2} \text{tr}_V (< \Pi_i > g_{\mu\nu}a^{\mu\nu})$$

$$+ \frac{1}{12(m-1)} \sum_{1 \leq i,k \leq s; i \neq k} [\kappa_{ik}(3m-2) + 4(m+2)\mu_i\sigma_{ik}] \text{tr}_V < \Pi_i \tilde{J}^\alpha \Pi_k \tilde{J}_\alpha >$$

$$- \frac{1}{2(m-1)} \sum_{1 \leq i,k \leq s; i \neq k} \kappa_{ik} \text{tr}_V \left\{ \left[ < \tilde{J}^\nu \Pi_i \tilde{J}^\alpha \Pi_k > + < \Pi_i \tilde{J}^\alpha \Pi_k \tilde{J}^\nu > \right] T_{[\nu\alpha]} \right\},$$

with

$$\kappa_{ik} = -\frac{\Gamma(m/2 - 1)}{2\Gamma(m/2 + 1)} \left\{ m\frac{\mu_i^{-m/2} + \mu_k^{-m/2}}{(\mu_i - \mu_k)} + 2\frac{\mu_i^{-m/2+1} - \mu_k^{-m/2+1}}{(\mu_i - \mu_k)^2} \right\},$$

$$\sigma_{ik} = \frac{\Gamma(m/2 - 1)}{8\Gamma(m/2 + 2)} \left\{ 24\frac{\mu_i^{-m/2+1} - \mu_k^{-m/2+1}}{(\mu_i - \mu_k)^3} + 8(m-4)\frac{\mu_i^{-m/2} + \mu_k^{-m/2}}{(\mu_i - \mu_k)^2} + 4(m-2)\frac{\mu_i^{-m/2-1}}{(\mu_i - \mu_k)^2} + m(m-2)\frac{\mu_i^{-m/2-1}}{(\mu_i - \mu_k)} \right\},$$

$$< \Pi_i \tilde{J}_\alpha \Pi_k \tilde{J}_\beta > = \sum_{1 \leq n,j \leq s} \frac{\Gamma(m/2)(2n + 2j - 2)!}{\Gamma(m/2 + n + j - 1)2^{2n+2j-2}(n + j - 1)!}$$

$$\times g(\mu_1\mu_2 \cdots g_{\mu_{2n-3}\mu_{2n-2}g_{\nu_1\nu_2} \cdots g_{\nu_{2j-3}\nu_{2j-2}}^\gamma a^\gamma a^{\nu_1\nu_2} \cdots a^{\nu_{2j-3}\nu_{2j-2}} a^\delta),$$

$$\times a^{(\mu_1\mu_2} \cdots a^{\mu_{2n-3}\mu_{2n-2}} a^{a\gamma} a^{\nu_1\nu_2} \cdots a^{\nu_{2j-3}\nu_{2j-2}} a^{a^\delta}) \cdot$$
We shall prove this theorem in Section 9. Note that for Laplace type operators, when $a^{\mu\nu} = g^{\mu\nu}I_V$, these formulas simplify considerably. We have then just one eigenvalue $\mu_1 = 1$ with multiplicity $d_1 = d = \dim(V)$ and the projection $\Pi_1 = I_V$. Thus for Laplace type operators, $a_0 = \Pi_1 = I_V$, and $\beta = -d/6$, and we recover the well known result

$$A_0 = \text{vol}(M),$$

$$A_1 = \int_M d\text{vol}(x) \left( \text{tr}_V q - \frac{d}{6} R \right).$$

In the next section we shall give a detailed description of our operator class. We would like to stress from the beginning that the theory of NLT operators, despite being a theory of second order operators, is closely related to the theory of higher-order Laplace-like operators (for a discussion, see [3]); that is, operators with principal part a power of the Laplacian. In this sense, even the weighted form Laplacian example mentioned above does not capture the full flavor of the theory, since it (almost uniquely in this category) is accessible entirely through second-order methods (see [13, 16]).

To contrast with the previous result mentioned above, it is worth stressing once again that:

i) there are many areas in both physics and mathematics where general second-order non-Laplace type operators arise naturally,

ii) we are primarily interested not just in the weighted form Laplacians but in the general NLT operators (differential forms being a very particular case),

iii) our approach to computation of heat kernel asymptotics is completely different from that of the previous authors [13, 16]. Our method enables us to compute explicitly not just the heat trace asymptotics but also the heat kernel and resolvent in the leading order that describe the local off-diagonal behavior of the resolvent and the heat kernel. To best of our knowledge, such formulas for non-Laplace operators are presented here for the first time.

Despite the importance of second-order operators with non-Laplace principal part in gauge field theory and quantum gravity (see, for example [4, 8]), their study is still quite new, and the available methodology is still underdeveloped in comparison with the Laplace type theory. In this paper, and in the more explicitly representation-theoretic
treatment \[3\], we hope to lay the groundwork for a systematic attack on the spectral asymptotics of this larger class of operators.

2 Non-Laplace Type Differential Operators

2.1 General vector bundle setup

Let \( M \) be a smooth compact manifold without boundary of dimension \( m \), equipped with a (positive definite) Riemannian metric \( g \). Let \( \mathcal{V} \) be a smooth vector bundle over \( M \), with \( \text{End}(\mathcal{V}) \cong \mathcal{V} \otimes \mathcal{V}^* \) the corresponding bundle of endomorphisms. Given any vector bundle \( \mathcal{V} \), we denote by \( C^\infty(M, \mathcal{V}) \), or just \( C^\infty(\mathcal{V}) \), its space of smooth sections. We assume that the vector bundle \( \mathcal{V} \) is equipped with a Hermitian metric \( H \). This naturally identifies the dual vector bundle \( \mathcal{V}^* \) with \( \mathcal{V} \), and defines a natural \( L^2 \) inner product, using the invariant Riemannian measure \( d\text{vol}(x) \) on the manifold \( M \). The completion of \( C^\infty(M, \mathcal{V}) \) in this norm defines the Hilbert space \( L^2(M, \mathcal{V}) \) of square integrable sections.

We denote by \( TM \) and \( T^*M \) the tangent and cotangent bundles of \( M \). We assume given a connection \( \nabla^\mathcal{V} \) on the vector bundle \( \mathcal{V} \). The covariant derivative on \( \mathcal{V} \) is then a map

\[
\nabla^\mathcal{V}: C^\infty(\mathcal{V}) \to C^\infty(T^*M \otimes \mathcal{V})
\]

which we assume to be compatible with the Hermitian metric on the vector bundle \( \mathcal{V} \), in the sense that \( \nabla H = 0 \). Here the connection is given its unique natural extension to bundles in the tensor algebra over \( \mathcal{V} \) and \( \mathcal{V}^* \); in particular, to the bundle \( \mathcal{V}^* \otimes \mathcal{V}^* \) of which \( H \) is a section. In fact, using the Levi-Civita connection \( \nabla^{\text{LC}} \) of the metric \( g \) together with \( \nabla^\mathcal{V} \), we naturally obtain connections on all bundles in the tensor algebra over \( \mathcal{V}, \mathcal{V}^*, TM, T^*M \); the resulting connection will usually be denoted just by \( \nabla \). It will usually be clear which bundle’s connection is being referred to, from the nature of the section being acted upon. We denote the curvature of \( \nabla^\mathcal{V} \) (a section of \( T^*M \otimes T^*M \otimes \mathcal{V} \)) by \( \mathcal{R} \):

\[
[\nabla_\alpha, \nabla_\beta]\varphi = \mathcal{R}_{\alpha\beta}\varphi, \quad \varphi \in C^\infty(\mathcal{V}).
\]

The formal adjoint of the covariant derivative of \( (2.1) \) is defined using the Riemannian
metric and the Hermitian structure on $\mathcal{V}$:
\[
(\nabla^\mathcal{V})^* : C^\infty(T^*M \otimes \mathcal{V}) \rightarrow C^\infty(\mathcal{V}),
\]
\[
\varphi_\alpha \mapsto -\nabla^\alpha \varphi_\alpha.
\]

Let
\[
a \in C^\infty(TM \otimes TM) \otimes \text{End}(\mathcal{V}), \quad b \in C^\infty(TM \otimes \mathcal{V}), \quad q \in C^\infty(\text{End}(\mathcal{V})).
\]

These sections define certain natural bundle maps by contraction, which, for simplicity, we denote by the same letters:

\[
a : T^*M \otimes \mathcal{V} \rightarrow TM \otimes \mathcal{V}, \quad (2.3)
\]
\[
b : \mathcal{V} \rightarrow TM \otimes \mathcal{V},
\]
\[
q : \mathcal{V} \rightarrow \mathcal{V}.
\]

Using these maps we can write the general second order operator
\[
F = \nabla^\star(a \nabla) + b \nabla + q, \quad \text{where} \quad a^{\mu\nu} = a^{\nu\mu}.
\]

Any formally self-adjoint second order operator may thus be written
\[
\nabla^\star(a \nabla) + \frac{1}{2}(b \nabla + (b \nabla)^*) + q,
\]
where we may assume that each $a^{\mu\nu}$ is Hermitian. (Here, if necessary, we clear the notation and redefine $q$ and $a^{\mu\nu}$.) In abstract index notation, any formally self-adjoint second order operator may be written
\[
F = -\nabla_\mu (a^{\mu\nu} \nabla_\nu) + b^\mu \nabla_\mu - \nabla_\mu (b^*)^\mu + q
\]
\[
= -a^{\mu\nu} \nabla_\mu \nabla_\nu + [b^\mu - (b^*)^\mu - a^{\nu\mu;\nu}] \nabla_\mu + q - (b^*)^\mu ; \mu.
\]

Hereafter we denote by ";" the covariant derivative. One may now restrict to the case when the endomorphism $b^*$ is anti-Hermitian, since the Hermitian part of $b$ only contributes at order zero. With this, we have redefined $b$ and $q$. Thus, henceforth,
\[
a^{\mu\nu} = a^{\nu\mu}, \quad b^* = -b.
\]
The formal adjoint to the operator $F$ reads

$$F^* = \nabla^*(a^* \nabla) + b \nabla + (b \nabla)^* + q^*.$$  

Hence, in addition to the other conditions posited so far, for the operator $F$ to be formally self-adjoint the endomorphism $q$ should be Hermitian. To sum up, the general formally self-adjoint second order operator is described by (2.5) with

$$a^\mu_\nu = a^\nu_\mu, \quad (a^\mu_\nu)^* = a^\mu_\nu, \quad (b^\mu)^* = -b^\mu, \quad q^* = q.$$

Let us consider the effect of a change of the connection, $\nabla \to \tilde{\nabla} = \nabla + A$, with a one form $A$ valued in $\text{End}(V)$. We have

$$\tilde{F} = -\nabla_\mu a^{\mu_\nu} \nabla_\nu + (b^\mu - A_\nu a^{\nu_\mu}) \nabla_\mu + \nabla_\mu (b^\mu - a^{\mu_\nu} A_\nu) + b^\mu A_\mu + A_\mu b^\mu - A_\mu a^{\mu_\nu} A_\nu + q.$$  

In many cases (but not always!) it is possible to choose $A$ in such a way that $A_\mu a^{\mu_\nu} = b^\nu$. Then the first order part drops out. The point is whether the map $a$ (2.3) is invertible, i.e. whether there is a solution, $a^{-1} \in C^\infty(T^*M \otimes T^*M) \otimes \text{End}(V))$, to the equation

$$a^{\mu_\nu} a^{-1}_{\nu_\lambda} = \delta_\lambda^\mu I_V. \quad (2.6)$$

This can be put in another form. Let $e_i \in C^\infty(T^*M \otimes V)$ be the basis in the space of one forms valued in $V$ and $e_i^* \in C^\infty(T^*M \otimes V^*)$ be the adjoint basis in the space of one forms valued in $V^*$. Then the equation (2.3) has a unique solution if and only if the bilinear form $B_{ij} = <e_j^*, a e_i>$ is nondegenerate, i.e. $\text{det} B_{ij} \neq 0$. If this condition is satisfied then one can always redefine the connection in such a way, viz. $A_\mu = b^\nu a^{-1}_{\nu_\mu}$, that $A_\mu a^{\mu_\nu} = b^\nu$ and the first order terms are not present. In this paper we assume that this is the case, so that without loss of generality one can set the vector-endomorphism $b$ to zero, $b = 0$. Moreover, we will assume that the tensor-endomorphism $a$ is parallel, $\nabla a = 0$. Thus the operator under consideration has the form

$$F = -a^{\mu_\nu} \nabla_\mu \nabla_\nu + q, \quad (2.7)$$

where

$$a^{\mu_\nu} = a^{\nu_\mu}, \quad a^{* \mu_\nu} = a^{\mu_\nu}, \quad \nabla a = 0, \quad q^* = q. \quad (2.8)$$
2.2 Tensor-spinor bundles

We now restrict attention to operators acting on tensor-spinor bundles. These bundles may be characterized as those appearing as direct summands of iterated tensor products of the cotangent and spinor bundles. Alternatively, they may be described abstractly as bundles associated to representations of $O(m)$, $SO(m)$, $Spin(m)$, or $Pin(m)$, depending on how much structure we assume of our manifold. These are extremely interesting and important bundles, as they describe the fields in Euclidean quantum field theory. More general bundles appearing in field theory are actually tensor products of these with auxiliary bundles, usually carrying another (gauge) group structure. The connection on the tensor-spinor bundles is built in a canonical way from the Levi-Civita connection and its curvature is:

$$\mathcal{R}_{\mu\nu} = R^\alpha_{\beta\mu\nu} T^\beta_{\alpha},$$

where $R^\alpha_{\beta\mu\nu}$ is the Riemann curvature tensor, and $T^\alpha_{\beta}$ is determined by the representation of $so(m)$ which induces the bundle $\mathcal{V}$. $T^\beta_{\alpha}$ is a tensor-spinor constructed purely from Kronecker symbols, together with the fundamental tensor-spinor $\gamma^\mu$ if spin structure is involved.

We study in this paper a special class of second-order operators of the form (2.7) with the coefficient $a$ built in a universal, polynomial way, using tensor product and contraction from the metric $g$ and its inverse $g^*$, together with (if applicable) the volume form $E$ and/or the fundamental tensor-spinor $\gamma$. Such a tensor-endomorphism $a$ is obviously parallel. ($E$ is available given $SO(m)$ or $Spin(m)$ structure; $\gamma$ is available given $Spin(m)$ or $Pin(m)$ structure.) We do not set any conditions on the endomorphism $q$, except that it should be Hermitian. An important subclass of this class of operators is the class of natural operators, when, in addition, $q$ is also built from the geometric invariants only; i.e. from $g$, $g^*$, $E$, and $\gamma$, together with the Riemann curvature and its iterated covariant derivatives. By Weyl’s invariant theory and dimensional analysis (i.e., a check of the homogeneity of each term under uniform dilation of the metric), it is clear that while $a$ must be built polynomially from $g$ and $g^*$, together with $E$ and/or $\gamma$ if applicable, the endomorphism $q$ must be a sum of terms linear in the curvature. However, we do not need the additional assumption that $q$ is constructed from the curvature. In general, it could be any smooth endomorphism.
3 Leading Symbol of an NLT Operator

Let us describe now more exactly the class of operators \( (2.7) \) we are working with. We have assumed the operator \( F \) to be self-adjoint leading to the conditions \( (2.8) \). Since we are going to study the heat kernel asymptotics of the operator \( F \), we now require in addition that the leading symbol of the operator \( F \),

\[
\sigma_2(F)(\xi) =: A(\xi) = a^{\mu\nu} \xi_\mu \xi_\nu, \quad \text{with } \xi \in T^*M,
\]

be positive definite, i.e. we have

\[
\xi \neq 0 \Rightarrow A(\xi) \text{ Hermitian and positive definite on } V.
\]

In particular, \( F \) is elliptic. Positive definiteness implies that the roots of the characteristic polynomial

\[
\chi_a(\xi)(\lambda) := \det_V(A(\xi) - \lambda)
\]

are positive functions on \( M \).

**Remark 1** An important point is that the eigenvalues \( \lambda_1, \ldots, \lambda_s \) and their multiplicities \( d_1, \ldots, d_s \) are independent of the point \( x \in M \). Indeed, all the bundles under consideration are vector bundles associated to the principal bundle \( S \) of spin frames via a finite-dimensional representation \((\varphi,V)\) of Spin\((m)\): \( V = S \times_\varphi V \). Let \( V_1 \) be the defining representation of SO\((n)\). We get the section \( a \), and the endomorphisms \( a^{\mu\nu} \xi_\mu \xi_\nu \) by “promoting” vectors \( a \in V_1 \otimes V_1 \otimes V \otimes V^* \) and \( a^{\mu\nu} \xi_\mu \xi_\nu \in V \otimes V^* \) to sections of the associated bundles. In particular, the cited eigenvalues and multiplicities may be computed at the level of the representation \((\varphi,V)\).

A useful way in which to perturb our operator \( F \) is to let it run through a one-parameter family \( F(\varepsilon) \) for which

\[
a(\varepsilon) = a + \varepsilon b,
\]

where \( b \) is a section of \( TM \otimes TM \otimes \text{End}(V) \) which is built from \( g \) and \( g^* \), and if applicable, \( E \) and/or \( \gamma \). (This \( b \) is not to be confused with the \( b \) of section 2.1.) To preserve formal self-adjointness, we need to assume that \( b^{\mu\nu} \xi_\mu \xi_\nu \) is self-adjoint on \( V \) for each \( \xi \in C^\infty(T^*M) \).

\[
F = -(a^{\mu\nu} + \varepsilon b^{\mu\nu}) \nabla_\mu \nabla_\nu + q.
\]
For \( \varepsilon \) in some interval about 0, the symbol \( a(\varepsilon) \) remains positive definite. (This last statement does not even depend on the compactness of \( M \); by an argument analogous to Remark [4], this interval is independent of the point \( x \in M \).) In particular, we might take perturbations about an operator with a leading symbol which is factored; i.e., one for which

\[
a^{\mu\nu} = g^{\mu\nu}c,
\]

where \( c \) is a section of \( \text{End}(\mathcal{V}) \) which is built invariantly from \( g \) and \( g^* \), and if applicable, \( E \) and/or \( \gamma \). As a special case of this, we could take \( c = I_{\mathcal{V}} \); that is, perturb an operator of Laplace type in NLT directions. In fact, in case \( \mathcal{V} \) is associated to an irreducible representation of \( \text{Spin}(m) \), the endomorphism \( c \) must be a multiple of the identity \( I_{\mathcal{V}} \) by Schur’s Lemma.

Given such a perturbation, one might hope that relevant spectral quantities could be expanded in powers of (very small) \( \varepsilon \), or at least that one could work with the \( \varepsilon \)-variation of such quantities.

Now consider the symmetric 2\( n \)-tensor quantity

\[
\text{tr} \, \mathcal{V} a^{(\mu_1 \nu_1) \cdots (\mu_n \nu_n)}.
\]

(3.2)

As usual, the parentheses denote complete symmetrization over all included indices. An index-free way of writing this is

\[
\text{tr} \, \mathcal{V} \vee^n a,
\]

where \( \vee^n \) is the symmetrized tensor power defined by

\[
\vee^n a \equiv a \vee \cdots \vee a.
\]

**Remark 2** We claim that

\[
\text{tr} \, \mathcal{V} \vee^n a = a_{(n)} \vee^n g^*
\]

(3.3)

with some constants \( a_{(n)} \). (Since \( \nabla a \) and \( \nabla g \) vanish, so must \( \nabla a_{(n)} \).)

Indeed, this quantity is built polynomially from \( g \) and \( g^* \), and \( a \) priori possibly \( E \) and/or \( \gamma \). Since it is a tensor, \( \gamma \) is not involved. Since

\[
E^{\mu_1 \cdots \mu_m} E_{\nu_1 \cdots \nu_m} = \delta[\mu_1 | \nu_1 \cdots \delta^{\mu_m}]_{\nu_m},
\]
where the brackets denote antisymmetrization over enclosed indices, \((3.2)\) is affine linear in \(E\); that is, it has the form

\[
\varphi(\mu_1\nu_1...\mu_n\nu_n) + \psi(\mu_1\nu_1...\mu_n\nu_n)|\alpha_1...\alpha_m|E_{\alpha_1...\alpha_m}.
\]

The tensor \(\psi\) is constructed purely from the metric \(g^*\) and is symmetric in the first 2\(n\) indices and antisymmetric in the last \(m\) indices. It is clear that there are no such tensors, so \(\psi = 0\). Similarly, the only symmetric 2\(n\)-tensor constructed from the metric is \(\vee^ng^*\), thus leading to \((3.3)\).

Contracting \((3.3)\) with \(\xi_\mu\xi_\nu ... \xi_\mu...\xi_\nu\), we get

\[
\text{tr}_V A^n(\xi) = a(n)|\xi|^{2n}.
\]

Furthermore, if we denote by \(\text{tr}_g\) the total trace of a symmetric 2\(n\)-tensor \(P\),

\[
\text{tr}_g P \equiv g_{\mu_1\mu_2}...g_{\mu_{2n-1}\mu_{2n}}P^{\mu_1...\mu_{2n}} \tag{3.4}
\]

then since

\[
\text{tr}_g \vee^ng^* = \frac{\Gamma(m/2 + n)}{\Gamma(m/2)} \frac{2^{2n}n!}{(2n)!},
\]

we have from \((3.3)\)

\[
a(n) = \frac{\Gamma(m/2)}{\Gamma(m/2 + n)} \frac{(2n)!}{2^{2n}n!} \text{tr}_V \text{tr}_g \vee^n a.
\]

It is clear that for an operator of Laplace type, when \(a = g^* \otimes \text{id}_V\), each \(a(n)\) is just the fiber dimension \(d\) of \(V\).

Now let us consider the characteristic polynomial \((3.1)\) in more detail. Applying the above remarks to the coefficients of \(\chi_a(\xi)(\lambda)\), we find that \(\chi_a(\xi)\) depends on \(\xi\) only through \(|\xi|^2\). As a result, the dependence of the eigenvalues \(\lambda_i\) on \(\xi\) is only through \(|\xi|^2\). Since \(A(\xi)\) is 2-homogeneous in \(\xi\), the \(\lambda_i\) must be also:

\[
\lambda_i(\xi) = |\xi|^{2i} \mu_i,
\]

for some positive real numbers \(\mu_1, \ldots, \mu_s\) which are independent of the point \((x,\xi) \in T^*M\), and, in fact, independent of the specific Riemannian manifold \((M, g)\), depending only on the representation \((\varphi, V)\) and the vector \(a \in V_1 \otimes V_1 \otimes V \otimes V^*\). Computing the trace \(\text{tr}_V A(\xi)^n\) we get a sequence of equations,

\[
\sum_{i=1}^{s} d_i \mu_i^n = a(n), \quad n \in \mathbb{N}, \tag{3.5}
\]
relating the eigenvalues, multiplicities, and the quantities $a_{(n)}$. When $n = 0$, (3.3) is just $a_{(0)} = d$; this was immediate from the definition of $a_{(n)}$.

Let $\Pi_i(\xi)$ be the orthogonal projection onto the $\lambda_i$-eigenspace. The $\Pi_i$ satisfy the conditions

$$
\begin{align*}
\Pi_i^2 &= \Pi_i, \\
\Pi_i\Pi_k &= 0 \quad (i \neq k), \\
\sum_{i=1}^{s} \Pi_i &= I_V, \\
\text{tr}_V \Pi_i &= d_i.
\end{align*}
$$

In contrast to the eigenvalues, the projections depend on the direction $\xi/|\xi|$ of $\xi$, rather than on the magnitude $|\xi|$. In other words, they are 0-homogeneous in $\xi$. Furthermore, they are polynomial in $\xi/|\xi|$: $\Pi_i(\xi) = \sum_{n=0}^{2p} P_{i(n)}(\xi)$, for some $p$, where

$$
P_{i(n)}(\xi) = \frac{1}{|\xi|^n} \xi_{\mu_1} \cdots \xi_{\mu_n} \Pi_{i(n)}^{\mu_1 \cdots \mu_n}.
$$

Here the $\Pi_{i(n)}$ are some $\text{End}(V)$-valued symmetric $n$-tensors that do not depend on $\xi$.

There is, however, quite a bit of ambiguity in the definition (3.7) of the homogeneous polynomials $P_{i(n)}(\xi)$, since multiplication of an $n$-homogeneous polynomial $q(\xi)$ by $|\xi|^2$ produces an $(n + 2)$-homogeneous polynomial $\tilde{q}(\xi)$, without changing the associated 0-homogeneous function $q(\xi/|\xi|)$. We can remove the ambiguity by requiring that $P_{i(n)}$ have no $|\xi|^2$ factor. This is equivalent to requiring that $P_{i(n)}(\xi)$ be a harmonic polynomial in $\xi$, which in turn is equivalent to requiring that its restriction to the unit $\xi$-sphere is an $n^{th}$-order $(\text{End}(V)$-valued) spherical harmonic. Yet another equivalent formulation is to require that $\Pi_{i(n)}$ is trace free in all its indices. At any rate, with this requirement, we have uniquely defined quantities $\Pi_{i(n)}$.

Note that the explicit formula

$$
\Pi_i = \frac{(A - \lambda_1)(A - \lambda_{i-1})(A - \lambda_{i+1}) \cdots (A - \lambda_s)}{(\lambda_i - \lambda_1)(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \cdots (\lambda_i - \lambda_s)}
$$

exhibits each projection as a polynomial of degree $2(s - 1)$. By homogeneity,

$$
\Pi_i(\xi) = \frac{(A(\xi/|\xi|) - \mu_1)(A(\xi/|\xi|) - \mu_{i-1})(A(\xi/|\xi|) - \mu_{i+1}) \cdots (A(\xi/|\xi|) - \mu_s)}{(\mu_i - \mu_1)(\mu_i - \mu_{i-1})(\mu_i - \mu_{i+1}) \cdots (\mu_i - \mu_s)}.
$$
We can take this expression and expand as a product of homogeneous terms:

\[
\Pi_i(\xi) = \sum_{k=1}^{s} c_{ik} A^{k-1} \left( \frac{\xi}{|\xi|} \right) = \sum_{k=1}^{s} \sum_{\mu_1 \mu_2 \cdots \mu_2k-2} c_{ik} a^{\mu_1 \mu_2 \cdots \mu_2k-3 \mu_2k-2} \frac{\xi_{\mu_1} \cdots \xi_{\mu_2k-2}}{|\xi|^{2k-2}},
\]

(3.8)

where \( c_{ik} \) are numerical constants depending only on the \( \mu_j \). Since \( A(\xi) \) is 2-homogeneous it follows that all homogeneity orders are even, i.e. \( P_{i(2n+1)}(\xi) = \Pi_{i(2n+1)} = 0 \), and therefore,

\[
\Pi_i(\xi) = \sum_{n=0}^{p} P_{i(2n)}(\xi).
\]

(3.9)

In turn, by writing \( A(\xi/|\xi|) = \sum_{i=0}^{s} \mu_i \Pi_i(\xi) \) we compute powers of \( A \) in terms of projections:

\[
A^k(\xi/|\xi|) = \sum_{i=1}^{s} \mu_i^k \Pi_i(\xi).
\]

Substituting this into (3.8), we get

\[
\sum_{k=1}^{s} c_{ik} \mu_j^{k-1} = \delta_{ij}.
\]

In other words, the matrix of coefficients \( C := (c_{ik}) \) is inverse to the (Vandermonde) matrix of powers \( M = (\kappa_{kj}) := (\mu_j^{k-1}) \):

\[
(c_{ik}) := (\kappa_{kj})^{-1}.
\]

(3.10)

Though the number \( s \) depends on the particular leading symbol \( a \), there is an upper bound for \( s \) which depends only on the representation \( (\varphi, V) \) to which the bundle \( V \) is associated. This is described in detail in [6]. In particular, in case \( (\varphi, V) \) is irreducible, the algebra generated by restrictions to the unit \( \xi \)-sphere of equivariant leading symbols of all orders is commutative, and thus simultaneously diagonalizable. The resulting projections diagonalize equivariant leading symbols of any (not just second) order. The number of projections, i.e. the dimension of the algebra just described, may be described in terms of representation-theoretic parameters. For reducible \( (\varphi, V) \), similar considerations are valid, but the algebra is not commutative.

Now write the leading symbol in terms of projections

\[
A(\xi) = |\xi|^2 \sum_{i=1}^{s} \mu_i \Pi_i(\xi) = |\xi|^2 \sum_{i=1}^{s} \sum_{n=0}^{p} \mu_i P_{i(2n)}(\xi).
\]

(3.11)
Since $A(\xi)$ is 2-homogeneous, it is a sum of $\text{End}(\mathcal{V})$-valued spherical harmonics in $\xi$ of degrees 2 and 0. Thus the spherical harmonics on the right of all other orders vanish:

$$\sum_{i=1}^{s} \mu_i P_{i(2n)}(\xi) = \sum_{i=1}^{s} \mu_i \Pi_{i(2n)} = 0, \quad n \neq 0, 1.$$ 

Furthermore, it is clear that the decomposition of $A(\xi)$ into $n = 2$ and $n = 0$ contributions comes upon taking

$$A(\xi) = a^{\mu\nu} \xi_\mu \xi_\nu = |\xi|^2 b_0 + b_2^{\mu\nu} \xi_\mu \xi_\nu,$$

where $b_2^{\mu\nu} \xi_\mu \xi_\nu$ is a second-order spherical harmonic in $\xi$; i.e., $b_2^{\mu\nu}$ is trace free in its two indices. This gives

$$a^{\mu\nu} = g^{\mu\nu} b_0 + b_2^{\mu\nu},$$

where

$$b_0 = \frac{1}{m} g_{\mu\nu} a^{\mu\nu} = \sum_{i=1}^{s} \mu_i \Pi_{i(0)}, \quad b_2^{\mu\nu} = \sum_{i=1}^{s} \mu_i \Pi_{i(2)}.$$ 

4 Symmetric Two-Tensors

It is instructive at this point to consider two examples: the bundle $\mathcal{S}^2$ of symmetric two-tensors, and the subbundle $\mathcal{S}^2_0$ of trace-free symmetric two-tensors. In particular, in these examples, one begins to glimpse the differences and relations between the cases of reducible and irreducible $\mathcal{V}$. We may compute with tensors valued in either complex or real tensor bundles; in fact, most of the following discussion holds in either setting. But for the sake of definiteness, and with a view toward applications, let us assume that all our tensors are real.

First note that a basis of the 0-homogeneous symbols in our class is given by

$$(X_1 \varphi)_{\alpha\beta} = g_{\alpha\beta} \varphi^\mu_\mu,$$

$$(X_2 \varphi)_{\alpha\beta} = \frac{1}{|\xi|^2} \xi^\mu \xi_{(\alpha} \varphi_{\beta)}_{\mu},$$

$$(X_3 \varphi)_{\alpha\beta} = \frac{1}{|\xi|^2} \xi_\alpha \xi_\beta \varphi^\mu_\mu,$$

$$(X_4 \varphi)_{\alpha\beta} = \frac{1}{|\xi|^2} g_{\alpha\beta} \xi^\mu \xi^\nu \varphi_{\mu\nu}.$$
\[(X_5 \varphi)_{\alpha \beta} = \frac{1}{|\xi|^4} \xi_\alpha \xi_\beta \xi^\mu \xi^\nu \varphi_{\mu \nu}.\]

Note that in the inner product \((\varphi, \psi) = \varphi_{\mu \nu} \psi^{\mu \nu}\), we have

\[X_1^* = X_1, \quad X_2^* = X_2, \quad X_3^* = X_4, \quad X_5^* = X_5.\]

In addition, the traces of these endomorphisms are

\[\text{tr}\, S^2 I_{S^2} = \frac{m(m-1)}{2},\]
\[\text{tr}\, S^2 X_1 = m,\]
\[\text{tr}\, S^2 X_2 = m + 1,\]
\[\text{tr}\, S^2 X_3 = \text{tr}\, S^2 X_4 = \text{tr}\, S^2 X_5 = 1.\]

The multiplication table of the algebra generated by \(I_V\) and the \(X_i\) is shown in Table 1. The arrow indicates that the left factor is to be found in the leftmost column.

| \(\rightarrow\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4\) | \(X_5\) |
|-----------------|--------|--------|--------|--------|--------|
| \(X_1\)        | \(mX_1\) | \(X_4\) | \(X_1\) | \(mX_4\) | \(X_4\) |
| \(X_2\)        | \(X_3\) | \(\frac{1}{2}(X_2 + X_5)\) | \(X_3\) | \(X_5\) | \(X_5\) |
| \(X_3\)        | \(mX_3\) | \(X_5\) | \(X_3\) | \(mX_5\) | \(X_5\) |
| \(X_4\)        | \(X_1\) | \(X_4\) | \(X_1\) | \(X_4\) | \(X_4\) |
| \(X_5\)        | \(X_3\) | \(X_5\) | \(X_3\) | \(X_5\) | \(X_5\) |

Table 1: Multiplication table

Example, \(X_3 X_4 = m X_5\).

The leading symbol of any even-order operator has the form

\[A(\xi) = |\xi|^{2p} (\alpha_0 I_{S^2} + \alpha_1 X_1 + 2\alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5),\]

for some \(p\) and some numerical parameters \(\alpha_i\). If \(A\) is the leading symbol of a second-order operator, then \(\alpha_5\) vanishes. If in addition \(A\) is self-adjoint, then \(\alpha_3 = \alpha_4\). Thus for a second-order self-adjoint leading symbol \(A\),

\[A(\xi) = |\xi|^2 (\alpha_0 I_{S^2} + \alpha_1 X_1 + 2\alpha_2 X_2 + \alpha_3 (X_3 + X_4))\, \text{(4.1)}\]

If we further require that \(A\) be positive definite, we get additional inequality constraints on the \(\alpha_i\); these are discussed below.
4.1 Trace-free symmetric two-tensors

Now consider the bundle $S^2_0$ of trace-free symmetric two-tensors. The symbol

$$P := I_S - \frac{1}{m}X_1$$

is the self-adjoint projection onto $S^2_0$. Thus we may define a spanning set of symbols $Y_i$ on $S^2_0$ by

$$Y_i = PX_iP.$$ 

According to Table 1,

$$Y_1 = Y_3 = Y_4 = 0,$$

and

$$Y_2 = X_2 - \frac{1}{m}(X_3 + X_4) + \frac{1}{m^2}X_1,$$

$$Y_5 = X_5 - \frac{1}{m}(X_3 + X_4) + \frac{1}{m^2}X_1.$$ 

Clearly both $Y_2$ and $Y_5$ are self-adjoint. Together with the identity symbol $I_{S^2_0}$, the symbols $Y_2$ and $Y_5$ form a basis of the symbol algebra on $S^2_0$. For purposes of explicit computation, it is useful to note that

$$PX_1 = X_1P = PX_4 = X_3P = 0. \quad (4.2)$$

We compute that

$$Y_2Y_5 = Y_5Y_2 = Y_5^2 = \frac{m-1}{m}Y_5, \quad (4.3)$$

$$Y_2^2 = \frac{1}{2}Y_2 + \frac{m-2}{2m}Y_5. \quad (4.4)$$

This points up a very important and useful fact: that the symbol algebra over $S^2_0$ is commutative. In fact, as shown in [6], this is a general feature of the case of an irreducible bundle. In the case of a bundle, like $S^2$, that is reducible under its structure group, the symbol algebra may be noncommutative.

The commutativity of the symbol algebra over $S^2_0$ significantly simplifies the computation of the projections. First note that all symbols will be simultaneously diagonalizable, so the discussion on this level is independent of the particular symbol $A$. From (4.3) we immediately obtain one projection, namely

$$\Pi_3 = \frac{m}{m-1}Y_5. \quad (4.5)$$
Since $I_{S_0^2}$, $Y_2$, and $Y_5$ form a basis of the symbol algebra, (4.4) shows that $I_{S_0^2}$, $Y_2$, and $Y_2^2$ are also a basis. Note that

$$Y_2^3 = \frac{3m - 2}{2m}Y_2^2 - \frac{m - 1}{2m}Y_2,$$

and

$$Y_2^4 = \frac{7m^2 - 10m + 4}{4m^2}Y_2^2 - \frac{(3m - 2)(m - 1)}{4m^2}Y_2.$$

If $\Pi := aI_{S_0^2} + bY_2 + cY_2^2$, we thus get

$$\Pi^2 = a^2I_{S_0^2} + \left\{2ab - \frac{bc(m - 1)}{m} - \frac{c^2(3m - 2)(m - 1)}{4m^2}\right\}Y_2$$

$$+ \left\{2ac + b^2 + \frac{bc(3m - 2)}{m} + \frac{c^2(7m^2 - 10m + 4)}{4m^2}\right\}Y_2^2.$$

There are $8 = 2^3$ solutions $(a, b, c)$ of the projection equation $\Pi^2 = \Pi$. Two of these are

$$(a_2, b_2, c_2) := \left(0, \frac{4(m - 1)}{m - 2}, -\frac{4m}{m - 2}\right)$$

and

$$(a_3, b_3, c_3) := \left(0, -\frac{m^2}{(m - 1)(m - 2)}, \frac{2m^2}{(m - 1)(m - 2)}\right);$$

these in fact correspond to fundamental projections $\Pi_2$ and $\Pi_3$. ($\Pi_3$ is the same as the projection found by inspection in (4.5).)

The remaining solutions correspond to the third fundamental projection $\Pi_1 := I_{S_0^2} - \Pi_2 - \Pi_3$, and to $0$, $I_{S_0^2}$, $\Pi_2 + \Pi_3$, $\Pi_1 + \Pi_3$, and $\Pi_1 + \Pi_2$. The fundamental projections have particularly simple expressions in terms of $Y_2$ and $Y_5$:

$$\Pi_2 = \frac{4}{m - 2}(-mY_2^2 + (m - 1)Y_2) = 2(Y_2 - Y_5)$$

$$\Pi_3 = \frac{m^2}{(m - 1)(m - 2)}(2Y_2^2 - Y_2) = \frac{m}{m - 1}Y_5$$

$$\Pi_1 = I_{S_0^2} - 2Y_2 + \frac{m - 2}{m - 1}Y_5$$

The general self-adjoint second-order symbol $A_0$ on $S_0^2$ may be viewed as the compression $PAP$ of a second-order symbol $A$ on $S^2$, which, in view of (4.1), is

$$A_0(\xi) = |\xi|^2 \left(\alpha_0I_{S_0^2} + 2\alpha_2Y_2\right)$$
\[ |\xi|^2 (\mu_1 \Pi_1 + \mu_2 \Pi_2 + \mu_3 \Pi_3) \]
\[ = |\xi|^2 \left\{ \mu_1 I_{S^3} + 2(\mu_2 - \mu_1) Y_2 \right\} \]
\[ + \frac{1}{m - 1} \left[ (m - 2)\mu_1 - 2(m - 1)\mu_2 + m\mu_3 \right] Y_5. \]

From this we obtain the eigenvalues in terms of the parameters \( \alpha_0 \) and \( \alpha_2 \):

\[ \mu_1 = \alpha_0, \quad \mu_2 = \alpha_0 + \alpha_2, \quad \mu_3 = \alpha_0 + \frac{2(m - 1)}{m} \alpha_2. \]

Thus, self-adjoint positive definite second-order symbols are in one-to-one correspondence with choices of \( (\alpha_0, \alpha_2) \) for which

\[ \alpha_0, \alpha_0 + \alpha_2, m\alpha_0 + 2(m - 1)\alpha_2 > 0. \]

Note that for \( m \geq 2 \), the first and third conditions together imply the second one.

Our projections have the following interpretation. After choosing the distinguished direction \( \xi \), one can distinguish three subspaces of the trace-free symmetric two-tensors. First, there are tensors in the direction of the trace-free part \( (\xi \otimes \xi)_0 \) of \( \xi \otimes \xi \). This is clearly the range of \( \Pi_3 \), since \( Y_5 \phi \) is a scalar multiple of \( (\xi \otimes \xi)_0 \). Next, there is the subspace consisting of tensors \( \xi \vee \zeta \), where \( \zeta \perp \xi \). This is the range of \( \Pi_2 \), since for

\[ \zeta_\beta := \frac{1}{|\xi|} \xi^\lambda \phi_{\beta \lambda} - \frac{1}{|\xi|^2} \xi^\beta \xi^\lambda \xi^\mu \phi_{\lambda \mu}, \]

we have

\[ (Y_2 - Y_5) \phi = \xi \vee \zeta, \quad \zeta^\lambda \zeta_\lambda = 0. \]

The remaining subspace is the range of \( \Pi_1 \); it may be described as the space generated by tensors \( (\zeta \vee \theta)_0 \), where \( \zeta \perp \xi \perp \theta \). The traces of the projections are thus

\[ \text{tr}_\gamma \Pi_1 = \frac{(m + 1)(m - 2)}{2}, \quad \text{tr}_\gamma \Pi_2 = m - 1, \quad \text{tr}_\gamma \Pi_3 = 1. \]

### 4.2 Symmetric two-tensors (not necessarily trace-free)

To leave the irreducible setting, consider the bundle \( S^2 \) of symmetric two-tensors (unrestricted as to trace); this is equivariantly isomorphic to the direct sum of the trace-free
symmetric two-tensors and the scalars. In fact, the direct sum decomposition is implemented by the projections $P$ and $I_{S^2} - P = (1/m)X_1$. By noting that $X_1 = m(I_{S^2} - P)$ and using Table 1, we obtain in addition to (4.2)\[ PX_2(I_V - P) = \frac{1}{m} \left( X_3 - \frac{1}{m}X_1 \right), \]
\[ (I_V - P)X_2P = \frac{1}{m} \left( X_4 - \frac{1}{m}X_1 \right), \]
\[ PX_3(I_V - P) = X_3 - \frac{1}{m}X_1, \]
\[ (I_V - P)X_4P = X_4 - \frac{1}{m}X_1, \]
\[ (I_V - P)X_2(I_V - P) = \frac{1}{m}(I_V - P), \]
\[ (I_V - P)X_3(I_V - P) = (I_V - P)X_4(I_V - P) = (I_V - P). \]

Let $\Pi_1$, $\Pi_2$ and $\Pi_3$ be the fundamental projections defined by (4.6). Denoting
\[ T = \frac{1}{\sqrt{m-1}} \left( X_3 - \frac{1}{m}X_1 \right), \quad T^* = \frac{1}{\sqrt{m-1}} \left( X_4 - \frac{1}{m}X_1 \right), \]
and using the equations (4.9), we obtain the decomposition
\[ A(\xi) := |\xi|^2 \left[ \mu_1 \Pi_1 + \mu_2 \Pi_2 + \mu_3 \Pi_3 + \kappa (T + T^*) + q(I_{S^2} - P) \right], \]
where $\kappa$ and $q$ are real constants defined by
\[ \kappa = \sqrt{m-1} \left( \frac{2}{m} \alpha_2 + \alpha_3 \right), \]
\[ q = \alpha_0 + m \alpha_1 + \frac{2}{m} \alpha_2 + 2 \alpha_3. \]

Note the useful relations
\[ TP = T\Pi_1 = T\Pi_2 = T\Pi_3 = 0, \]
\[ PT^* = \Pi_1 T^* = \Pi_2 T^* = \Pi_3 T^* = 0. \]
\[ T^2 = (T^*)^2 = 0. \]
\[ T^*T = I_V - P, \quad TT^* = \Pi_3. \]

\[ \Pi_3 T = T, \quad T^* \Pi_3 = T^*. \] (4.10)

Observe also the obvious non-commutativity; the projections will now depend on the particular leading symbol we are studying. We see that for our new symbol on $S^2$, the first two projections $\Pi_1$, $\Pi_2$ computed above are untouched, and there are two more projections $Z_3$, $Z_4$ onto one-dimensional subspaces. $\Pi_1$ and $\Pi_2$ are independent of the particular symbol we are diagonalizing, while and $Z_3$ and $Z_4$ depend on it. $Z_3$ and $Z_4$ take the form

\[ Z = a\Pi_3 + b(T + T^*) + c(I_V - P) \]

By (4.10), we have

\[ Z^2 = (a^2 + b^2)\Pi_3 + b(a + c)(T + T^*) + (b^2 + c^2)(I_{S^2} - P). \]

The projection equation $Z^2 = Z$ gives

\[ (2a - 1)^2 + (2b)^2 = (2c - 1)^2 + (2b)^2 = 1, \quad b(a - c + 1) = 0. \]

Aside from the solutions

\[ (a, b, c) = (0, 0, 0), \quad (1, 0, 1), \] (4.11)

all solutions have the form

\[ a = \frac{1}{2}(1 + \cos \theta), \quad b = \frac{1}{2}\sin \theta, \quad c = \frac{1}{2}(1 - \cos \theta), \]

where $\theta$ is an arbitrary real parameter; conversely, all choices of $\theta$ give solutions. The solutions (4.11) give 0- and 2-dimensional projections, so can be discarded from the present point of view, where we seek complementary one-dimensional projections. If we define

\[ (a_3, b_3, c_3) : = \left( \frac{1}{2}(1 + \cos \theta), \frac{1}{2}\sin \theta, \frac{1}{2}(1 - \cos \theta) \right), \]

\[ (a_4, b_4, c_4) : = \left( \frac{1}{2}(1 - \cos \theta), -\frac{1}{2}\sin \theta, \frac{1}{2}(1 + \cos \theta) \right), \]

we get a set of complementary projections $Z_3$ and $Z_4$; that is, we have $Z_3Z_4 = Z_4Z_3 = 0$.

We still need to find a value of $\theta$ adapted to our given symbol $A$. Denote by $\nu_3$ and $\nu_4$ the eigenvalues of $A$ in the ranges of $Z_3$ and $Z_4$ respectively:

\[ \mu_3 \Pi_3 + \kappa (T + T^*) + q(I_V - P) = \nu_3 Z_3 + \nu_4 Z_4. \]
We find that
\[(\nu_3 + \nu_4) + (\nu_3 - \nu_4) \cos \theta = 2\mu_3,\]
\[(\nu_3 - \nu_4) \sin \theta = 2\kappa,\]
\[(\nu_3 + \nu_4) - (\nu_3 - \nu_4) \cos \theta = 2q.\]

From these equations we first determine the eigenvalues
\[\nu_3 = \rho + \omega, \quad \nu_4 = \rho - \omega,\]
where
\[\rho = \frac{1}{2}(\mu_3 + q) = \alpha_0 + \frac{m}{2}\alpha_1 + \alpha_2 + \alpha_3,\]
\[\omega^2 = \kappa^2 + \left(\frac{\mu_3 - q}{2}\right)^2\]
\[= \frac{1}{4m^2}[-m^2\alpha_1 + 2(m - 2)\alpha_2 - 2m\alpha_3]^2 + 4(m - 1)(2\alpha_2 + m\alpha_3)^2.\]

The positivity of the leading symbol is translated now into the condition \(\omega^2 < \rho^2\), or \(\kappa^2 < q\mu_3\). So, in addition to \(\alpha_0 > 0\) and \(\alpha_0 + \alpha_2 > 0\) we have
\[(m - 1)(2\alpha_2 + m\alpha_3)^2 < (m\alpha_0 + m^2\alpha_1 + 2\alpha_2 + 2m\alpha_3)[m\alpha_0 + 2(m - 1)\alpha_2].\]

The parameter \(\theta\) is now determined by
\[\cos \theta = \frac{\mu_3 - q}{2\omega} = \frac{1}{2m\omega}[-m^2\alpha_1 + 2(m - 2)\alpha_2 - 2m\alpha_3],\]
\[\sin \theta = \frac{\kappa}{\omega} = \frac{\sqrt{m - 1}}{m\omega}(2\alpha_2 + m\alpha_3).\]

Therefore, the projections \(Z_3\) and \(Z_4\) have the form
\[Z_{3,4} = \frac{1}{2} \left(1 \pm \frac{\mu_3 - q}{2\omega}\right) \Pi_3 \pm \frac{\kappa}{2\omega}(T + T^*) + \frac{1}{2} \left(1 \mp \frac{\mu_3 - q}{2\omega}\right)(I_Y - P),\]
and indeed depend on the symbol \(A\).
5 The resolvent and the heat kernel

Let $F$ be a self-adjoint second-order partial differential operator on a compact manifold with positive definite leading symbol. (In particular, $F$ is elliptic.) Then $F$ has discrete real eigenvalue spectrum which is bounded below by some (possibly negative) real number $c$. If $\lambda$ is a complex number with $\text{Re} \lambda < c$, then the \textit{resolvent} $(F - \lambda I)^{-1}$ is well defined: if $\{(\lambda_j, \varphi_j)\}$ is a spectral resolution, with the $\varphi_j$ forming a complete orthonormal set in $L^2(V)$, then

$$(F - \lambda I)^{-1}\varphi_j = (\lambda_j - \lambda)^{-1}\varphi_j.$$  

The resolvent is a bounded operator on $L^2(V)$, and in fact is a compact operator by the Rellich Lemma, since it carries $L^2(V)$ to the Sobolev section space $L^2_2(V)$ continuously.

The resolvent kernel is a section of the external tensor product of the vector bundles $V$ and $V^*$ over the product manifold $M \times M$, and satisfies the equation

$$(F - \lambda I)G(\lambda|x,y) = \delta(x,y)$$

where $\delta(x,y)$ is the Dirac distribution (which in turn is the kernel function of the identity operator). Here and below, all differential operators will act in the first ($x$, as opposed to $y$) argument of any kernel functions to which they are applied. The resolvent is well-defined as long as the null space of $F - \lambda I$ vanishes; i.e., as long as $\lambda$ is not one of the $\lambda_j$. As a consequence of self-adjointness, we have

$$G^\dagger(\lambda|x,y) = G(\lambda|y,x).$$

Similarly, for $t > 0$ the heat operator $U(t) = \exp(-tF) : L^2(M,V) \to L^2(M,V)$ is well defined. $U(t)$ is a \textit{smoothing} operator; that is, it carries $L^2$ sections to sections in $\cap_{k \in R} L^2_k = C^\infty$. The kernel function of this operator, called the \textit{heat kernel}, satisfies the equation

$$(\partial_t + F)U(t|x,y) = 0$$

with the initial condition

$$U(0^+|x,y) = \delta(x,y),$$

and the self-adjointness condition

$$U^\dagger(t|x,y) = U(t|y,x).$$
As is well known [12], the heat kernel and the resolvent kernel are related by the Laplace transform:

\[ G(\lambda) = \int_0^\infty dt \, e^{t \lambda} U(t), \]

\[ U(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\lambda \, e^{-t \lambda} G(\lambda). \]

It is also well known [12] that the heat kernel \( U(t|x,y) \) is a smooth function near the diagonal \( \{x = y\} \) of \( M \times M \), with the diagonal values integrating to the functional trace:

\[ \text{Tr}_{L^2} \exp(-tF) = \int_M d\text{vol}(x) \text{tr}_V U(t|x,x). \]

Moreover, there is an asymptotic expansion of the heat kernel as \( t \to 0^+ \),

\[ \text{Tr}_{L^2} \exp(-tF) \sim (4\pi t)^{-m/2} \sum_{k \geq 0} \frac{(-t)^k}{k!} A_k, \]

and a corresponding expansion of the resolvent as \( \lambda \to -\infty \):

\[ \text{Tr}_{L^2} \partial^n \lambda G(\lambda) \sim (4\pi)^{-m/2} \sum_{k \geq 0} \frac{(-1)^k}{k!} \Gamma[(k - m)/2 + n + 1](-\lambda)^{m/2-k-n-1} A_k, \]

for \( n \geq m/2 \). Here \( A_k \) are the global, or integrated heat coefficients, sometimes called the Minakshisundaram-Pleijel coefficients.

There is additional information in the local asymptotic expansion of the heat kernel diagonal,

\[ U^{\text{diag}}(t) := U(t|x,x) \sim (4\pi t)^{-m/2} \sum_{k \geq 0} \frac{(-t)^k}{k!} a_k. \]  

(5.1)

The local heat coefficients \( a_k \) integrate to the global ones:

\[ A_k = \int_M d\text{vol}(x) \text{tr}_V a_k. \]  

(5.2)

One can, in fact, get access to the local heat coefficients via a functional trace, by taking advantage of the principle that a function (or distribution) is determined by its integral against an arbitrary test section \( f \in C^\infty(\text{End}(V)) \):

\[ \text{Tr}_{L^2} f \exp(-tF) \sim (4\pi t)^{-m/2} \sum_{k \geq 0} \frac{(-t)^k}{k!} A_k(f, F). \]
We have
\[ A_k = A_k(1, F), \quad A_k(f, F) = \int_M d\text{vol}(x) f \text{ tr}_V a_k. \]

The local heat coefficients \( a_k \) have been calculated for Laplace type operators up to \( a_4 \) \[2\]. For non-Laplace type operators, some of them are known only in the very specific case of differential form bundles \[9\].

We shall calculate below the coefficients \( A_0 \) and \( A_1 \) for NLT operators in terms of the projections introduced in the previous sections. Our tactic will be to construct an approximation to the heat kernel \( U(t|x, y) \), that is a parametrix. The important information in the parametrix for small \( t \) is carried by its values near the diagonal, since the heat kernel vanishes to order \( \infty \) off the diagonal as \( t \to 0^+ \). Since the heat and resolvent kernels are related by the Laplace transform, this is equivalent to studying an approximation to the resolvent kernel \( G(\lambda|x, y) \) near the diagonal for large negative Re \( \lambda \).

Let us stress here that our purpose is not to provide a rigorous construction of the resolvent with estimates; for this we rely on the standard references \[12\]. Rather, given that the existence of resolvent and heat parametrices is known, our aim is to compute various aspects of it; in particular, information on leading order terms sufficient to determine some of the heat kernel coefficients \( A_k \).

We shall employ the standard scaling device for the resolvent \( G(\lambda|x, y) \) and heat kernel \( U(t|x, y) \) when \( x \to y, \lambda \to -\infty \) and \( t \to 0 \). This means that one introduces a small expansion parameter \( \varepsilon \) reflecting the fact that the points \( x \) and \( y \) are close to each other, the parameter \( t \) is small, and the parameter Re \( \lambda \) is negative and large. This can be done by fixing a point \( x' \), choosing normal coordinates at \( x' \) (with \( g_{\mu\nu}(x') = \delta_{\mu\nu} \)), scaling according to
\[ x \to x_{\varepsilon} = x' + \varepsilon(x - x'), \quad y \to y_{\varepsilon} = x' + \varepsilon(y - x'), \quad t \to t_{\varepsilon} = \varepsilon^2 t, \quad \lambda \to \lambda_{\varepsilon} = \varepsilon^{-2}\lambda, \]
and expanding in an asymptotic series in \( \varepsilon \). If one uses a local Fourier transform, then the corresponding momenta \( \xi \in T^*M \) are large and scale according to
\[ \xi \to \xi_{\varepsilon} = \varepsilon^{-1}\xi. \]

This construction is standard \[12\]. This procedure can be done also in a completely covariant way \[2, 3\].
In the case of Laplace type operators, the most convenient form of the off-diagonal asymptotics as \( t \to 0 \), among many equivalent forms, is [2, 3]

\[
U(t|x, y) \sim (4\pi t)^{-m/2} \exp \left(-\frac{\sigma}{2t} \right) \Delta^{1/2} \sum_{k \geq 0} \frac{(-t)^k}{k!} b_k(x, y),
\]

(5.3)

where \( \sigma = \sigma(x,y) = r^2(x,y)/2 \) is half the geodesic distance between \( x \) and \( y \), and \( \Delta = \Delta(x,y) = |g|^{-1/2}(x) |g|^{-1/2}(y) \det (-\partial_\mu^\sigma \partial_\nu^\sigma \sigma(x,y)) \) is the corresponding Van Vleck-Morette determinant. The functions \( b_k(x,y) \) are called the off-diagonal heat coefficients. These coefficients satisfy certain differential recursion relations. Expanding each coefficient in a covariant Taylor series near the diagonal, one gets a recursively solvable system of algebraic equations on the Taylor coefficients [2]. The diagonal values give the local heat kernel coefficients \( a_k(x) = b_k(x,x) \). However, in the general case of non-Laplace type operators, it is very difficult to follow this approach, since the Ansatz for the off-diagonal heat kernel asymptotics (5.3) does not apply; the correct Ansatz would be much more complicated. For this reason, we employ the approach of pseudo-differential operators (or, roughly speaking, local Fourier transforms). An alternative approach, which generalizes the Ansatz (5.3), is developed in Section 10.

6 Gaussian integrals

The Gaussian average of a function \( f(\xi) \) on \( \mathbb{R}^m \) is

\[
\langle f \rangle \equiv \int_{\mathbb{R}^m} \frac{d\xi}{\pi^{m/2}} e^{-|\xi|^2} f(\xi).
\]

The Gaussian average of an exponential function gives the generating function

\[
I_0(x) = \int_{\mathbb{R}^m} \frac{d\xi}{\pi^{m/2}} \exp \left(-|\xi|^2 + i\xi \cdot x \right) = \langle \exp(i\xi \cdot x) \rangle = \exp \left(-\frac{|x|^2}{4} \right).
\]

Expansion of \( I_0(x) \) in a power series in \( x \) generates the Gaussian averages of polynomials

\[
\langle 1 \rangle = 1, \quad \langle \xi_\mu \rangle = 0, \quad \langle \xi_\mu \xi_\nu \rangle = \frac{1}{2} g_{\mu\nu},
\]

\[
\langle \xi_{\mu_1} \cdots \xi_{\mu_{2n+1}} \rangle = 0,
\]

\[
\langle \xi_{\mu_1} \cdots \xi_{\mu_{2n}} \rangle = \frac{(2n)!}{2^n n!} g_{\mu_1 \mu_2} \cdots g_{\mu_{2n-1} \mu_{2n}}.
\]
Furthermore, using the relation
\[
\frac{1}{|\xi|^{2p}} = \frac{1}{\Gamma(p)} \int_0^\infty ds \ s^{p-1} e^{-s|\xi|^2}, \quad \text{Re} \ p > 0 \quad (6.1)
\]
and analytic continuation, one can obtain the more general formulas
\[
\left\langle \frac{\xi_{\mu_1} \cdots \xi_{\mu_2n}}{|\xi|^{2p}} \right\rangle = 0, \quad (6.2)
\]
\[
\left\langle \frac{\xi_{\mu_1} \cdots \xi_{\mu_2n}}{|\xi|^{2n}} \right\rangle = \frac{\Gamma(m/2 + n - p) \ (2n)!}{\Gamma(m/2) \ 2^{2n} n!} g(\mu_1 \cdots \mu_{2n}) \quad (6.3)
\]
for any \( p \) with \( \text{Re} \ p < n + m/2 \). This means, in particular, that
\[
\left\langle \frac{\xi_{\mu_1} \cdots \xi_{\mu_2n}}{|\xi|^{2n}} \right\rangle = \frac{\Gamma(m/2 + n - p) \ (2n)!}{\Gamma(m/2 + n) \ 2^{2n} n!} g(\mu_1 \cdots \mu_{2n-1} \mu_{2n}).
\]

Note that if a function \( f \) depends only on \( \xi/|\xi| \), that is, if it is homogeneous of order 0, then the average introduced above is a constant multiple of the average over the unit \((m-1)\)-sphere \( S^{m-1} \).

We will also need to compute Fourier integrals of the form
\[
I_{\mu_1 \cdots \mu_{2n}}(x) = \int_{R^m} \frac{d\xi}{\pi^m/2} \exp(i \xi \cdot x - |\xi|^2) \frac{\xi_{\mu_1} \cdots \xi_{\mu_{2n}}}{|\xi|^{2n}}
\]
\[
= \left\langle \frac{\xi_{\mu_1} \cdots \xi_{\mu_{2n}}}{|\xi|^{2n}} \exp(i \xi \cdot x) \right\rangle \quad (6.4)
\]

The trace of the symmetric \( 2n \)-form \( I_{2n} \) over any two indices is \( 2(n-1) \)-form \( I_{2n-2} \):
\[
g^{\mu_{2n-1} \mu_{2n}} I_{\mu_1 \cdots \mu_{2n-2} \mu_{2n-1} \mu_{2n}} = I_{\mu_1 \cdots \mu_{2n-2}}.
\]

Therefore, the total trace of the symmetric form \( I_{2n} \) is \( I_0 \):
\[
\text{tr}_g I_{2n}(x) = I_0(x) = \exp \left( -\frac{|x|^2}{4} \right)
\]
By rescaling \( \xi \to \sqrt{t} \xi \) one can show that this integral satisfies the equation
\[
t^{m/2} \partial_t^{m/2} \left[ t^{-m/2} I_{\mu_1 \cdots \mu_{2n}} \left( \frac{x}{\sqrt{t}} \right) \right] = \partial_{\mu_1} \cdots \partial_{\mu_{2n}} \exp \left( -\frac{|x|^2}{4t} \right).
\]
Taking into account the obvious asymptotic condition $\lim_{t \to \infty} [t^{-m/2}I_n(x/\sqrt{t})] = 0$, by multiple integration we get

$$I_{\mu_1, \ldots, \mu_{2n}}(x) = \frac{(-1)^n}{(n-1)!} \int_1^\infty ds \, (s-1)^{n-1} s^{-m/2} \partial_{\mu_1} \cdots \partial_{\mu_{2n}} \exp \left(-\frac{|x|^2}{4s}\right).$$

By changing the variable, $s = 1/u$, we can rewrite this in the form

$$I_{\mu_1, \ldots, \mu_{2n}}(x) = \frac{(-1)^n}{(n-1)!} \int_0^1 du \, u^{n/2-(n-1)} (1-u)^{n-1} \partial_{\mu_1} \cdots \partial_{\mu_{2n}} \exp \left(-\frac{|x|^2}{4} u\right). \quad (6.5)$$

If $m$ is even this can be computed in elementary functions.

In the case $n < m/2$, one can interchange the order of the integration and differentiation. Then by using the formula

$$\int_0^1 du \, u^{a-1}(1-u)^{b-a-1} e^{zu} = \frac{\Gamma(a) \Gamma(b-a)}{\Gamma(b)} \, 1_F(1; a; b; z),$$

(with $\Re b > \Re a > 0$) where

$$1_F(1; a; b; z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k) \Gamma(b)}{\Gamma(a) \Gamma(b+k) k!} z^k$$

is the confluent hypergeometric function, we obtain

$$I_{\mu_1, \ldots, \mu_{2n}}(x) = (-1)^n \partial_{\mu_1} \cdots \partial_{\mu_{2n}} \tilde{\Phi}_n \left(-\frac{|x|^2}{4}\right),$$

where

$$\tilde{\Phi}_n(z) = \frac{\Gamma(m/2 - n)}{\Gamma(m/2)} \, 1_F \left(\begin{array}{c} m/2 - n \\ \frac{m}{2} \end{array}; \frac{m}{2} + n + 1; z\right) = \sum_{k=0}^{\infty} \frac{\Gamma(m/2 - n + k)}{\Gamma(m/2 + k) k!} z^k$$

(with $n < m/2$).

If $n \geq m/2$, then this formula cannot be applied directly. However, it is still valid if one does analytic continuation in the dimension $m$. The physical (integer) value of the dimension should be put after computing the derivatives. Alternatively, one can do the differentiation in (6.5) explicitly before the integration. This is equivalent to subtracting the first $n$ terms of the Taylor series of the exponential. This subtraction is clearly harmless, since the $2n$-th derivatives of these terms vanish. However, this makes the integral finite, and justifies the interchange of the differentiation and integration. The result applies for any $n$. In this way one obtains

$$I_{\mu_1, \ldots, \mu_{2n}}(x) = (-1)^n \partial_{\mu_1} \cdots \partial_{\mu_{2n}} \Phi_n \left(-\frac{|x|^2}{4}\right), \quad (6.6)$$
where $\Phi_n(z)$ is obtained from the function $\tilde{\Phi}_n(z)$ by substracting the first $n$ terms of the power series:

$$
\Phi_n(z) = \frac{\Gamma(m/2 - n)}{\Gamma(m/2)} {}_1F_1 \left( \frac{m}{2} - n; \frac{m}{2}; z \right) - \sum_{k=0}^{n-1} \frac{\Gamma(m/2 - n + k)}{\Gamma(m/2 + k) k!} z^k
$$

$$
= \sum_{k=n}^{\infty} \frac{\Gamma(m/2 - n + k)}{\Gamma(m/2 + k) k!} z^k
$$

(6.7)

7 Leading order off-diagonal heat kernel asymptotics

In general, the leading order resolvent and the heat kernel are determined by the leading symbol of the operator $F$:

$$
G_0(\lambda|x, y) = \int \frac{d\xi}{(2\pi)^m} e^{i\xi(x-y)} [A(\xi) - \lambda I_y]^{-1},
$$

$$
U_0(t|x, y) = \int \frac{d\xi}{(2\pi)^m} e^{i\xi(x-y)} \exp[-tA(\xi)],
$$

where $\xi \in T^*M$, $\xi \cdot (x - y) = \xi \mu(x^\mu - y^\mu)$, and $d\xi$ is Lebesgue measure on $R^m$. Here and everywhere below all integrals over $\xi$ below will be over the whole $R^m$.

Writing the leading symbol in terms of the projections from (3.11), it is not difficult to obtain

$$
G_0(\lambda|x, y) = \sum_{i=1}^{s} \int \frac{d\xi}{(2\pi)^m} e^{i\xi(x-y)} \frac{\Pi_i(\xi)}{\mu_i |\xi|^2 - \lambda},
$$

$$
U_0(t|x, y) = \sum_{i=1}^{s} \int \frac{d\xi}{(2\pi)^m} e^{i\xi(x-y) - t\mu_i |\xi|^2} \Pi_i(\xi).
$$

The problem is now to compute these integrals. If we are only interested in traces, then by (3.6), we can easily compute

$$
\text{tr}_y U_0(t|x, y) = \sum_{i=1}^{s} d_i (4\pi t \mu_i)^{-m/2} \exp \left( -\frac{|x - y|^2}{4t \mu_i} \right).
$$

That is, the trace of the leading order term in the heat kernel asymptotics is a weighted linear combination of the scalar leading order heat kernel term with scaled times, $t \rightarrow \mu_i t$. Similarly, we can relate the trace of the resolvent at leading order to resolvents of Laplace type operators:

$$
\text{tr}_y G_0(\lambda|x, y) = \sum_{i=1}^{s} d_i (2\pi \mu_i)^{-m/2} \left( -\frac{\lambda \mu_i}{|x - y|^2} \right)^{(m-2)/4} K_{(m-2)/2} \left( \sqrt{\frac{-\lambda}{\mu_i}} |x - y| \right),
$$

$$
\left( \sqrt{\frac{-\lambda}{\mu_i}} |x - y| \right),
$$
where $K_p(z)$ is the modified Bessel function.

Applying the methods described in the previous subsection we can compute the heat kernel and resolvent before taking the trace:

$$U_0(t|x, y) = \sum_{i=1}^{s} (4\pi t \mu_i)^{-m/2} \left\langle \exp \left[ i \xi \cdot \frac{(x - y)}{\sqrt{t \mu_i}} \right] \Pi_i(\xi) \right\rangle.$$  

Using the decomposition of the projections into spherical harmonics (3.9), we obtain

$$U_0(t|x, y) = \sum_{1 \leq i, k \leq s} (4\pi t \mu_i)^{-m/2} c_{ik} a^{(\mu_1 \mu_2 \cdots \mu_{2k-3} \mu_{2k-2})} (-t \mu_i)^{k-1}$$

$$\times \partial_{\mu_1} \cdots \partial_{\mu_{2k-2}} \Phi_{k-1} \left( -\frac{|x - y|^2}{4t \mu_i} \right),$$

$$= \sum_{1 \leq i, k \leq s} (4\pi t \mu_i)^{-m/2} c_{ik} (t \mu_i F_0)^{k-1} \Phi_{k-1} \left( -\frac{|x - y|^2}{4t \mu_i} \right), \quad \text{(7.1)}$$

$$G_0(\lambda|x, y) = \sum_{1 \leq i, k \leq s} (4\pi \mu_i)^{-m/2} c_{ik} (\mu_i F_0)^{k-1}$$

$$\times \int_0^\infty dt \, t^{\lambda t - m/2 + k-1} \Phi_{k-1} \left( -\frac{|x - y|^2}{4t \mu_i} \right),$$

where $F_0 = \sigma^{\mu\nu} \partial_\mu \partial_\nu$ and the functions $\Phi_n(z)$ are defined in previous section; they are given by (6.7). We would like to stress that these formulas can be presented locally in a “covariantized” form. This is effectively achieved by replacing $|x - y|^2$ by $2\sigma(x, y)$ and $(x - y)^\mu$ by $\nabla^\mu \sigma(x, y)$ and adding a factor $\Delta^{1/2}(x, y)$ (for details, see [2]); the objects $\sigma$ and $\Delta$ being defined after equation (5.3).

8 The heat kernel coefficient $a_0$

The leading order term of the diagonal heat kernel asymptotics can now be written as

$$U_0^{\text{diag}}(t) := U_0(t|x, x) = (4\pi t)^{-m/2} \sum_{i=1}^{s} \mu_i^{-m/2} \langle \Pi_i \rangle.$$  

This gives the coefficient $a_0$:

$$a_0 = \sum_{i=1}^{s} \mu_i^{-m/2} \langle \Pi_i \rangle. \quad \text{(8.1)}$$
The Gaussian average is computed by using (6.3):

\[
<\Pi_i> = \Pi_i(0) = \sum_{k=1}^{s} c_{ik} \frac{\Gamma(m/2)(2k-2)!}{\Gamma(m/2 + k - 1)2^{2k-2}(k-1)!} \text{tr}_g \nabla^{k-1} a, \quad (8.2)
\]

where \(\nabla^n\) is the symmetrized tensor power of symmetric form, \(\text{tr}_g\) is the total trace of a symmetric form defined in (3.4) and the constants \(c_{ik}\) are defined by (3.10).

From this, we have the formula

\[
\text{tr}_V a_0 = \sum_{i=1}^{s} \frac{d_i}{\mu_i^{m/2}}
\]

for the trace of the heat coefficient.

Note that if our bundle \(V\) is associated to an irreducible representation of Spin\((m)\), the averages of the projections, being Spin\((m)\)-invariant endomorphisms, are proportional to the identity, by Schur’s Lemma. The exact proportionality constant may be determined by taking the trace. We have:

\[
<\Pi_i> = \frac{d_i}{d} I_V, \quad a_0 = \frac{1}{d} \sum_{i=1}^{s} \frac{d_i}{\mu_i^{m/2}} I_V.
\]

These formulas point up a new feature of non-Laplace type operators; one which complicates life somewhat. Whereas the dimension dependence of the heat coefficients of Laplace type operators is isolated in the overall factor of \((4\pi)^{-m/2}\), the dimension dependence for NLT operators is more complicated.

9 The heat kernel coefficient \(A_1\)

As we have seen, even the computation of the leading order heat kernel requires significant effort in the case of non-Laplace principal part. This indicates that the calculation of higher-order coefficients will be a challenging task. In this paper we will compute the coefficient \(A_1\). Since \(A_1\) is the integral of \(\text{tr}_V a_1\) by \((\ref{9.1})\), it suffices to compute the local coefficient \(a_1\) modulo trace-free endomorphisms. By elementary invariant theory, \(a_1\) has the form

\[
a_1 = \sum_{k\geq 0} Q_{(k)} \mu_1 \cdots \mu_k q P_{(k)}^{\mu_1 \cdots \mu_k} + H_1 R + H_2^{\mu \nu} R_{\mu \nu} + H_3^{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} \quad (9.1)
\]

where \(Q_{(k)}\), \(P_{(k)}\), and \(H_i\) are End \((V)\)-valued tensors, and \(R\), \(R_{\mu \nu}\), and \(R_{\mu \nu \alpha \beta}\) are the scalar, Ricci, and Riemann curvatures.
For the trace we have a similar formula,

$$\text{tr}_V a_1 = \text{tr}_V(h_0q) + h_1 R + h_2^{\mu\nu} R_{\mu\nu} + h_3^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}$$

where $h_i \equiv \text{tr}_V H_i$ ($i = 1, 2, 3$) are some tensors. By invariant theory, we may conclude that these tensors have the following form:

$$h_0 = \sum_{k \geq 0} P_{(k)}^{\mu_1 \cdots \mu_k} Q_{(k) \mu_1 \cdots \mu_k},$$

$$h_1 = c_1, \quad h_2^{\mu\nu} = c_2 g^{\mu\nu}, \quad h_3^{\mu\nu\alpha\beta} = c_3 g^{\mu\alpha} g^{\nu\beta} + c_4 g^{\mu\beta} g^{\nu\alpha} + c_5 g^{\mu\nu} g^{\alpha\beta},$$

where $c_i$ are some constants.

Note that for irreducible representations the endomorphisms $Q_0, P_0, h_0$ and $H_1$ are proportional to the identity endomorphism; in particular, $h_0 = c_0 I_V, H_1 = (c_1/d) I_V$.

Taking this into account, we obtain

$$\text{tr}_V a_1 = \text{tr}_V(h_0q) + \beta R,$$

where

$$\beta = c_1 + c_2 + c_3 - c_4.$$

Thus it suffices to compute the constant $\beta$ and the endomorphism $h_0$.

To compute $h_0$, we can employ a by now standard variational principle. If the operator $F(\varepsilon)$ depends (in a suitably estimable way) on a parameter $\varepsilon$, then

$$\frac{\partial}{\partial \varepsilon} \text{Tr}_{L^2} \exp(-tF) = -t \text{Tr}_{L^2}(\partial_\varepsilon F) \exp(-tF).$$

If our variation comes from scaling $q$,

$$q \rightarrow \varepsilon q,$$

then $(\partial_\varepsilon F) = q$. Expanding both sides in powers of $t$ and comparing coefficients of like powers, we obtain

$$\partial_\varepsilon A_1 = \int_M d\text{vol}(x) \text{tr}_V(a_0q),$$

so that

$$A_1 = \int_M d\text{vol}(x) \left[ \text{tr}_V(a_0q) + \beta R \right],$$

and, therefore, $h_0 = a_0$, where $a_0$ is given by (8.1) and (8.2).
The computation of the constant $\beta$ is considerably more difficult. It is clear the $\beta$ is universal to our class of operators; thus we may compute it for any particular operator and manifold, or class of such. Accordingly, note that $\beta$ is equal to the coefficient $a_1$ when $q = 0$ and $R = 1$:

$$
\beta = \left. \text{tr} V a_1 \right|_{q=0, R=1}.
$$

The following considerations will be completely local. Fix a point $x'$, and compute in normal coordinates centered at $x'$, with $g_{\mu\nu}(0) = \delta_{\mu\nu}$. Furthermore, impose by gauge transformation the Fock-Schwinger gauge for the connection 1-form $A$. We then have

$$
[g_{\mu\nu}(x) - \delta_{\mu\nu}]x^\nu = 0, \quad A_{\mu}(x)x^\mu = 0.
$$

Further, we can expand all quantities in Taylor series about $x'$ and restrict our attention to terms linear in the curvature. This gives

$$
g_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{1}{3} R_{\mu\nu\alpha\beta} x^\alpha x^\beta + \ldots,
$$

$$
\det g_{\mu\nu}(x) = 1 - \frac{1}{3} R_{\alpha\beta} x^\alpha x^\beta + \ldots,
$$

$$
A_{\mu}(x) = -\frac{1}{2} R_{\mu\alpha} x^\alpha + \ldots.
$$

Here and below, the dots denote higher-order terms in the curvature. $R_{\mu\nu}$ is, of course, the curvature of the bundle $\mathcal{V}$, given by

$$
R_{\mu\nu} = R^\alpha_{\beta\mu\nu} T^\beta_{\alpha}.
$$

The Taylor expansion of the leading symbol section $a^{\mu\nu}$ is determined by the equation

$$
\nabla_\mu a^{\alpha\beta} = 0.
$$

From this, we get

$$
a^{\mu\nu}(x) = a^{\mu\nu} + \frac{1}{3} a^{(\mu} R^{\nu)}_{\alpha\beta} x^\alpha x^\beta + \ldots.
$$

Here and below, we denote $a^{\mu\nu}(0)$ simply by $a^{\mu\nu}$. By the above, the potential term $q$ will only enter the calculation through $q(0)$, which we denote simply by $q$:

$$
q(x) = q + \ldots.
$$

Since the constant $\beta$ is universal, we are free to compute in the case of a constant curvature metric, i.e.

$$
R_{\mu\nu\alpha\beta} = \frac{R}{m(m-1)} (g_{\mu\alpha} g_{\beta\nu} - g_{\mu\beta} g_{\alpha\nu}),
$$

(9.2)
\[ R_{\mu\nu} = \frac{R}{m} g_{\mu\nu}, \quad R = \text{const}. \]

Now let us take the total symbol of our operator in normal coordinates, \( \sigma(F|x, \xi) \), and expand in a Taylor series:

\[
\sigma(F|x, \xi) = \sigma_L(F|0, \xi) + \sigma(F_1|x, \xi) + \ldots,
\]

where

\[
\sigma_L(F|0, \xi) = a^{\mu\nu} \xi_\mu \xi_\nu \equiv A(\xi),
\]

\[
\sigma(F_1|x, \xi) = -X^{\mu\nu}_{\alpha\beta} x^\alpha x^\beta \xi_\mu \xi_\nu + iY^\mu_{\alpha} x^\alpha \xi_\nu + q,
\]

\[
X^{\mu\nu}_{\alpha\beta} = -\frac{1}{3} a^{\lambda(\mu} R^{\nu(\alpha|\lambda|\beta)},
\]

\[
Y^\mu_{\alpha} = \frac{2}{3} a^{\mu\lambda} R_{\alpha\lambda} - \frac{1}{2} [T^\sigma_{\rho\mu} a^{\nu\rho}] + R^\rho_{\sigma\alpha\nu},
\]

and \([A, B]_+ = AB + BA\) denotes the anticommutator.

There are many equivalent ways of constructing the heat kernel asymptotics on the diagonal locally. Using the Volterra series

\[
\exp(-tF) = \exp(-tF_0) - t \int_0^1 d\tau \exp[-t(1-\tau)F_0] F_1 \exp[-t\tau F_0] + \ldots
\]

and the formula for the heat kernel on the diagonal

\[
U^{\text{diag}}(t) = \int \frac{d\xi}{(2\pi)^m} e^{-\xi^2} \exp(-tF) e^{i\xi \cdot x} \bigg|_{x=0},
\]

we get

\[
U^{\text{diag}}(t) = \int \frac{d\xi}{(2\pi)^m} \left\{ e^{-tA(\xi)} - t \int_0^1 d\tau e^{-t(1-\tau)A(\xi)} \hat{F}_1 e^{-t\tau A(\xi)} + \ldots \right\},
\]

where

\[
\hat{F}_1 = \sigma(F_1|i\partial_\xi, \xi) = X^{\mu\nu}_{\alpha\beta} \partial_\xi^\alpha \partial_\xi^\beta \xi_\mu \xi_\nu - Y^\mu_{\alpha} \partial_\xi^\alpha \xi_\nu + q.
\]

Finally, by scaling the integration variable \( \xi \to t^{-1/2} \xi \), we obtain the standard asymptotic expansion of the heat kernel on the diagonal (5.1), with the coefficients

\[
a_0 = \int \frac{d\xi}{\pi^{m/2}} e^{-A(\xi)},
\]

\[
a_1 = \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau e^{-(1-\tau)A(\xi)} \hat{F}_1 e^{-\tau A(\xi)}.
\]
In particular, we recover the formula for $a_0$ derived in the previous section.

To integrate by parts in this formula, we need to know how to differentiate the exponential $e^A$. This can be done via the Duhamel formula

$$
\partial_\xi^\alpha e^{-\tau A(\xi)} = -2 \int_0^\tau ds \, e^{-s A(\xi)} J^\alpha (\xi) e^{-s A(\xi)},
$$

where

$$
J^\alpha (\xi) = a^{\alpha\beta} \xi_\beta.
$$

The contraction of this with $\xi$ leads to a much simpler formula:

$$
\xi_\alpha \partial_\xi^\alpha e^{-\tau A(\xi)} = -2 \tau A(\xi) e^{-\tau A(\xi)}.
$$

The exponential itself is computed by using the projections $\Pi_i$:

$$
e^{-\tau A(\xi)} = \sum_{i=1}^s e^{-\tau \mu_i |\xi|^2} \Pi_i (\xi).
$$

Note that when computing $\xi$-derivatives one can, using integration by parts, act in either direction. This trick can be used to avoid having to compute the second derivative of the exponential $e^A$. First, we rewrite $\hat{F}_1$ in the form

$$
\hat{F}_1 = \partial_\xi^\alpha X_{\alpha\beta} \partial_\xi^\beta + \partial_\xi^\alpha L_\alpha + L_\alpha \partial_\xi^\alpha + Q,
$$

where

$$
X_{\alpha\beta}(\xi) = X_{\mu\nu} a_{\alpha\beta} \xi_\mu \xi_\nu = -\frac{1}{3} a^{\gamma\rho} R_{(\alpha|\beta)} \xi_\mu \xi_\nu,
$$

$$
L_\alpha (\xi) = L^{\mu\alpha} \xi_\mu,
$$

$$
L^{\mu\alpha} = -\frac{1}{4} a^{\mu\lambda} R_{\lambda\alpha} + \frac{1}{12} a^{\lambda\nu} R^{\mu}_{\lambda\nu} + \frac{1}{4} R^{\mu}_{\sigma\alpha \nu} [T^\alpha_{\rho} a^{\mu\nu}]_+,
$$

$$
Q = q - \frac{1}{6} a^{\mu\nu} R_{\mu\nu}.
$$

Now, integrating by parts over $\xi$ (and changing $\tau$ to $1 - \tau$ in some places) we get

$$
a_1 = \int \frac{d\xi}{\pi^{m/2}} \left\{ -4 \int_0^1 d\tau \int_0^{1-\tau} ds_1 \int_0^\tau ds_2 e^{-(1-\tau-s_1) A} \times J^\alpha e^{-s_1 A} X_{\alpha\beta} e^{-(\tau-s_2) A} J^\beta e^{-s_2 A} + 2 \int_0^1 d\tau \int_0^\tau ds \left[ e^{-(\tau-s) A} J^\alpha e^{-s A} L_\alpha e^{-(1-\tau) A} - e^{-(1-\tau) A} L_\alpha e^{-(\tau-s) A} J^\alpha e^{-s A} \right] + \int_0^1 d\tau e^{-(1-\tau) A} Q e^{-\tau A} \right\}.
$$

(9.3)
By separating the curvature factors in (9.3), one can compute each of the tensors $H_i$ entering (9.1). Note that for a Laplace type operator (the case $a_{\mu\nu} = g_{\mu\nu}$) we have $A = |\xi|^2$, $J^\mu = \xi^\mu$, $X_{\alpha\beta} = -(1/3) R^\mu_{\alpha\nu} \xi^\mu \xi^\nu$, $L_\alpha = -(1/6) R^\mu_{\alpha\mu}$, $Q = q - (1/6) R I$. Therefore the first two terms (with $X$ and $L$) vanish in the Laplace type case, and we get the well known result

$$a_1 = q - \frac{1}{6} R I V.$$  

Note that the tensors $H_i$ depend only on the leading symbol, i.e. on $a_{\mu\nu}$. In principle, it is possible to compute them explicitly by using the representation of the $e^A$ in terms of projections and Gaussian averages.

We shall not do this explicitly, but rather compute only the trace of $a_1$. The number of projections involved in this calculation is less by one. Computing in the case of constant curvature (9.2), and attaching a tilde to $X$, $L$ and $Q$ in this case, we have

$$\tilde{X}_{\alpha\beta}(\xi) = -\frac{R}{3m(m-1)} \left[ A g_{\alpha\beta} - J_{(\alpha} \xi_{\beta)} \right],$$

$$\tilde{L}_\alpha(\xi) = -\frac{R}{12m(m-1)} \left\{ \bar{a} \xi_\alpha - (3m - 2) J_\alpha + 6 [T|_{\nu\alpha}], J^\nu + \right\},$$

$$\tilde{Q} = q - \frac{R}{6m} \bar{a},$$

(9.4)

where

$$\bar{a} = g_{\mu\nu} a^{\mu\nu}.$$  

Now taking the trace and changing $s$ to $\tau - s$ in the second integral, we obtain

$$\text{tr} V a_1 = \text{tr} V \int \frac{d\xi}{\pi^{m/2}} \left\{ -4 \int_0^1 d\tau \int_0^{1-\tau} ds_1 \int_0^{1-\tau} ds_2 e^{-(\tau-s_2)A} \right.$$

$$\times J^\beta e^{-(1-\tau-s_1+s_2)A} J^\alpha e^{-s_1 A} \tilde{X}_{\alpha\beta}$$

$$+ 2 \int_0^1 d\tau \int_0^{1-\tau} ds \left[ e^{-(1-s)A} J^\alpha e^{-sA} - e^{-sA} J^\alpha e^{-(1-s)A} \right] \tilde{L}_\alpha$$

$$+ e^{-A} \tilde{Q} \right\}. \tag{9.5}$$

In the second multiple, the $s$-integration may be accomplished explicitly, to give

$$\text{tr} V a_1 = \text{tr} V \int \frac{d\xi}{\pi^{m/2}} \left\{ -4 \int_0^1 d\tau \int_0^{1-\tau} ds_1 \int_0^{1-\tau} ds_2 e^{-(\tau-s_2)A} \right.$$

$$\times J^\beta e^{-(1-\tau-s_1+s_2)A} J^\alpha e^{-s_1 A} \tilde{X}_{\alpha\beta}$$
Now let us consider the different contributions separately.

**Q contribution.**

The Q contribution has the form

\[
\text{tr}_V \int \frac{d\xi}{\pi^{m/2}} e^{-A\tilde{Q}} = \text{tr}_V \sum_{i=1}^{s} \int \frac{d\xi}{\pi^{m/2}} e^{-\mu_i|\xi|^2}\Pi_i\tilde{Q}
\]

By changing the integration variable \(\xi\) to \((\mu_i)^{-1/2}\xi\), we obtain a Gaussian average:

\[
\text{tr}_V \sum_{i=1}^{s} \mu_i^{-m/2} < \Pi_i > \left( q - \frac{1}{6m} \bar{a}R \right) = \text{tr}_V a_0 \left( q - \frac{1}{6m} \bar{a}R \right).
\]

**L contribution.**

This contribution has the form

\[
2 \text{tr}_V \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau \tau \left[ e^{-\tau A} J^\alpha e^{-(1-\tau)A} - e^{-(1-\tau)A} J^\alpha e^{-\tau A} \right] \tilde{L}_\alpha.
\]  

Using (9.4), we get

\[
-\frac{R}{6m(m-1)} \text{tr}_V \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau \tau \left[ e^{-\tau A} J^\alpha e^{-(1-\tau)A} - e^{-(1-\tau)A} J^\alpha e^{-\tau A} \right]
\]

\[
\times (\bar{a}\xi_\alpha - (3m-2)J_\alpha + 6[T[\nu_\alpha], J^\nu]_+) .
\]  

(9.8)

Noting that \(J^\alpha \xi_\alpha = A\), we see that the term in \(L_\alpha\) proportional to \(\xi_\alpha\) does not contribute.

The term proportional to \(J_\alpha\) gives

\[
\frac{R}{6m(m-1)} \sum_{1 \leq i,k \leq s} \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau \tau \left[ e^{-\mu_i\tau + \mu_k(1-\tau)}|\xi|^2 \right. \\

\left. \quad -e^{-\mu_k\tau + \mu_i(1-\tau)}|\xi|^2 \right] \text{tr}_V \Pi_i J^\alpha \Pi_k J_\alpha.
\]
By changing the integration variable $\xi$ to $\left[\mu_i \tau + \mu_k (1 - \tau)\right]^{-1/2} \xi$, we may express this in terms of a Gaussian average:

$$R \frac{3m-2}{6m(m-1)} \sum_{1 \leq i, k \leq s} \kappa_{ik} \text{tr} \mathcal{V} \langle \Pi_i J^\alpha \Pi_k J_\alpha \rangle,$$

where

$$\kappa_{ik} = \int_0^1 d\tau \left\{\left[\mu_i \tau + \mu_k (1 - \tau)\right]^{-(m+2)/2} - \left[\mu_k \tau + \mu_i (1 - \tau)\right]^{-(m+2)/2}\right\}.$$

Clearly $\kappa_{ik} = -\kappa_{ki}$; hence, for $i = k$

$$\kappa_{ii} = 0.$$

For $i \neq k$ we compute

$$\kappa_{ik} = -\frac{\Gamma(m/2-1)}{\Gamma(m/2+1)(\mu_i - \mu_k)} \left\{ \frac{m}{2} \left( \mu_i^{-m/2} + \mu_k^{-m/2} \right) + \frac{\mu_i^{-m/2+1} - \mu_k^{-m/2+1}}{\mu_i - \mu_k} \right\}.$$

Similarly, the contribution of the term with $T$ is computed to be

$$-\frac{R}{m(m-1)} \text{tr} \mathcal{V} \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau \left[ e^{-\tau A} J^\alpha e^{-(1-\tau)A} - e^{-(1-\tau)A} J^\alpha e^{-\tau A} \right] [T_{[\nu \alpha]}, J^\nu]_+$$

$$= -\frac{R}{m(m-1)} \text{tr} \mathcal{V} \sum_{1 \leq i, k \leq s} \kappa_{ik} \langle \Pi_i J^\alpha \Pi_k J_\alpha \rangle [T_{[\nu \alpha]}, J^\nu]_+$$

$$= -\frac{R}{m(m-1)} \text{tr} \mathcal{V} \sum_{1 \leq i, k \leq s} \kappa_{ik} \left( \langle J^\nu \Pi_i J^\alpha \Pi_k \rangle + \langle \Pi_i J^\alpha \Pi_k J^\nu \rangle \right) T_{[\nu \alpha]}.$$

Thus the total contribution of $L$ is

$$R \frac{1}{6m(m-1)} \text{tr} \mathcal{V} \sum_{1 \leq i, k \leq s} \kappa_{ik} \left\{ (3m-2) \langle \Pi_i J^\alpha \Pi_k J_\alpha \rangle - 6 (\langle J^\nu \Pi_i J^\alpha \Pi_k \rangle + \langle \Pi_i J^\alpha \Pi_k J^\nu \rangle) T_{[\nu \alpha]} \right\}.$$

$X$ contribution.
This term has the form
\[ R \frac{4}{3m(m-1)} \text{tr} V \int \frac{d\xi}{\pi^{m/2}} \left\{ \int_0^1 d\tau \int_0^{1-\tau} ds_1 \int_0^\tau ds_2 \ e^{-(\tau-s_2)A} \right. \]
\[ \times J^\beta e^{-(1-\tau-s_1+s_2)A} J^\alpha e^{-s_1A} \left[ A g_{\alpha\beta} - J_{(\alpha\xi\beta)} \right]. \]

We consider first the term proportional to \( A \). Because \( A \) commutes with the exponent, we have
\[ R \frac{4}{3m(m-1)} \text{tr} V \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau \int_0^{1-\tau} ds_1 \int_0^\tau ds_2 \ A e^{-(\tau-s_2+s_1)A} \]
\[ \times J^\alpha e^{-(1-\tau-s_1+s_2)A} J^\alpha. \quad (9.10) \]

Introducing the projections again, we get
\[ R \frac{4}{3m(m-1)} \text{tr} V \sum_{1 \leq i, k \leq s} \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau \int_0^{1-\tau} ds_1 \int_0^\tau ds_2 \]
\[ \times e^{-[(\tau-s_2+s_1)\mu_i + (1-\tau-s_1+s_2)\mu_k]}/2} J^\alpha J^\alpha J^\alpha. \]
Scaling, \( \xi \rightarrow [(\tau-s_2+s_1)\mu_i + (1-\tau-s_1+s_2)\mu_k]^{-1/2} \xi \), we obtain
\[ R \frac{4}{3m(m-1)} \text{tr} V \sum_{1 \leq i, k \leq s} \rho_{ik} \mu_i \langle \xi^{2} J^\alpha J^\alpha J^\alpha \rangle, \]
where
\[ \rho_{ik} = \int_0^1 d\tau \int_0^{1-\tau} ds_1 \int_0^\tau ds_2 \left[ (\tau-s_2+s_1)\mu_i + (1-\tau-s_1+s_2)\mu_k \right]^{-(m+4)/2}. \]
By changing \( \tau \) to \( 1-\tau \) and switching the roles of \( s_1 \) and \( s_2 \), we see that
\[ \rho_{ik} = \rho_{ki}. \]
For \( i = k \), we easily obtain
\[ \rho_{ii} = \frac{1}{6} \mu_i^{-m/2-2}. \]
For \( i \neq k \) we compute that
\[ \rho_{ik} = \frac{\Gamma(m/2-1)}{\Gamma(m/2+2) (\mu_i - \mu_k)^2} \left\{ (m-4) \left( \mu_i^{-m/2} + \mu_k^{-m/2} \right) \right. \]
\[ + 2 \frac{\mu_i^{-m/2+1} - \mu_k^{-m/2+1}}{\mu_i - \mu_k} \right\}. \quad (9.11) \]
Furthermore, the term proportional to $\xi$ in the $X$ contribution is

$$-R\frac{4}{3m(m-1)} \text{tr } V \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau \int_0^{1-\tau} ds_1 \int_0^\tau ds_2 e^{-\left(\tau-s_2\right)A} J^\beta e^{-\left(1-\tau-s_1+s_2\right)A} J^\alpha e^{-s_1 A} J_{(\alpha\beta)}.$$

Remembering that $J^\alpha \xi_\alpha = A$ and changing the variables of integration, we have

$$-R\frac{4}{3m(m-1)} \text{tr } V \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau \frac{1}{2} \tau^2 A e^{-\tau A} J^\alpha e^{-\left(1-\tau\right)A} J_\alpha.$$

Introducing the projections $\Pi_i$, this becomes

$$-R\frac{4}{3m(m-1)} \text{tr } V \sum_{1 \leq i,k \leq s} \int \frac{d\xi}{\pi^{m/2}} \int_0^1 d\tau \frac{1}{2} \tau^2 \mu_i e^{-\left[\tau \mu_i + (1-\tau) \mu_k\right] |\xi|^2} |\xi|^2 \Pi_i J^\alpha \Pi_k J_\alpha.$$

Finally, scaling $\xi \to [\tau \mu_i + (1-\tau) \mu_k]^{-1/2} \xi$, we get

$$-R\frac{4}{3m(m-1)} \text{tr } V \sum_{1 \leq i,k \leq s} \gamma_{ik} \mu_i \langle |\xi|^2 \Pi_i J^\alpha \Pi_k J_\alpha \rangle,$$

where

$$\gamma_{ik} = \int_0^1 d\tau \frac{1}{2} \tau^2 [\tau \mu_i + (1-\tau) \mu_k]^{-m/2-2}.$$

For $i = k$, we have

$$\gamma_{ii} = \frac{1}{6} \mu_i^{-m/2-2}.$$

For $i \neq k$, we compute that

$$\gamma_{ik} = \frac{\Gamma(m/2-1)}{8\Gamma(m/2+2)(\mu_i - \mu_k)} \left\{ -m(m-2) \mu_i^{-m/2-1} - 4(m-2) \frac{\mu_i^{-m/2}}{\mu_i - \mu_k} - 8 \frac{\mu_i^{-m/2+1} - \mu_k^{-m/2+1}}{(\mu_i - \mu_k)^2} \right\}.$$

Thus, the total $X$ contribution is

$$R\frac{4}{3m(m-1)} \text{tr } V \sum_{1 \leq i,k \leq s} \mu_i \sigma_{ik} \langle |\xi|^2 \Pi_i J^\alpha \Pi_k J_\alpha \rangle,$$

where

$$\sigma_{ik} = \rho_{ik} - \gamma_{ik}.$$
Clearly
\[ \sigma_{ii} = 0. \]

For \( i \neq k \),
\[
\sigma_{ik} = \frac{\Gamma(m/2 - 1)}{8 \Gamma(m/2 + 2)} \left\{ 24 \frac{\mu_i^{m/2 + 1} - \mu_k^{m/2 + 1}}{(\mu_i - \mu_k)^3} + 8(m - 4)\frac{\mu_i^{m/2} + \mu_k^{m/2}}{(\mu_i - \mu_k)^2} + 4(m - 2)\frac{\mu_i^{m/2} - \mu_k^{m/2}}{(\mu_i - \mu_k)^2} + m(m - 2)\frac{\mu_i^{m/2 - 1}}{(\mu_i - \mu_k)} \right\}.
\]

(9.12)

**Result.**

Summing up the various contributions, we get the main result:
\[
\text{tr}_\mathcal{V} \alpha_1 = \sum_{i=1}^s \mu_i^{-m/2} \text{tr}_\mathcal{V} < \Pi_i > q + \beta R,
\]

where
\[
\beta = -\frac{1}{6m} \sum_{i=1}^s \mu_i^{-m/2} \text{tr}_\mathcal{V} < \Pi_i > \bar{a}
\]

\[
+ \frac{1}{6m(m - 1)} \sum_{1 \leq i, k \leq s; i \neq k} \kappa_{ik} \left\{ (3m - 2) \text{tr}_\mathcal{V} < \Pi_i J^\alpha \Pi_k J^\beta > \right\}
\]

\[
- 6 \text{tr}_\mathcal{V} \left[ (J^{\nu} \Pi_i J^\alpha \Pi_k) + (\Pi_i J^\alpha \Pi_k J^{\nu}) \right] T^{[\nu\alpha]} \right\}
\]

\[
+ \frac{4}{3m(m - 1)} \sum_{1 \leq i, k \leq s; i \neq k} \mu_i \sigma_{ik} \text{tr}_\mathcal{V} \left\{ |\xi|^2 \Pi_i J^\alpha \Pi_k J^\beta > \right\},
\]

with the constants \( \kappa_{ik} \) and \( \sigma_{ik} \) given by (9.9) and (9.12).

Next we need to compute the Gaussian averages. This can be easily done by using the formulas (8.2). First note that the radial part can be always separated by
\[
\langle |\xi|^{2p} f(\xi/|\xi|) \rangle = \frac{\Gamma(m/2 + p)}{\Gamma(m/2)} \langle f(\xi/|\xi|) \rangle.
\]

This gives
\[
\langle \Pi_i J^\alpha \Pi_k J^\beta > = \frac{m}{2} \langle \Pi_i J^\alpha \Pi_k J^\beta >,
\]
\[
< J^\alpha \Pi_i J^\beta \Pi_i > = \frac{m}{2} < J^\alpha \Pi_i \bar{J}^\beta \Pi_i >,
\]
\[
< \xi^2 \Pi_i J^\alpha \Pi_k J^\beta > = \frac{m(m+2)}{4} \left< \Pi_i \bar{J}^\alpha \Pi_k \bar{J}^\beta \right>,
\]
where
\[
\bar{J}^\alpha = \frac{J^\alpha}{|\xi|} = a^{\alpha \beta} \xi^\beta.
\]
This allows to simplify the result somewhat:
\[
\beta = -\frac{1}{6m} \sum_{i=1}^{s} \mu_i^{-m/2} \text{tr}_V < \Pi_i > \bar{a}
\]
\[
+ \frac{1}{12(m-1)} \sum_{1 \leq i, k \leq s; i \neq k} [\kappa_{ik}(3m-2) + 4(m+2)\mu_i \sigma_{ik}] \text{tr}_V \left< \Pi_i \bar{J}^\alpha \Pi_k \bar{J}_\alpha \right>
\]
\[
- \frac{1}{2(m-1)} \sum_{1 \leq i, k \leq s; i \neq k} \kappa_{ik} \text{tr}_V \left[ \left< \bar{J}^{\nu} \Pi_i \bar{J}^\alpha \Pi_k \right> + \left< \Pi_i \bar{J}^\alpha \Pi_k \bar{J}^{\nu} \right> \right] T_{[\nu \alpha]}.
\]
Note that for irreducible representations the endomorphisms \( < \Pi_i >, \bar{a} = a^{\mu \mu}, \) and \( < \Pi_i J^\alpha \Pi_k J_\alpha > \) are proportional to the identity \( I_V. \)

Finally, for the sake of completeness, we list the averages explicitly. By using the formulas (6.3) we get
\[
< \Pi_i \bar{J}_\alpha \Pi_k \bar{J}_\beta > = \sum_{1 \leq n, j \leq s} \frac{\Gamma(m/2)(2n+2j-2)!}{\Gamma(\frac{m}{2} + n + j - 1)2^{2n+2j-2}(n+j-1)!} \times g(\mu_1 \mu_2 \cdots g_{\nu_{2n-3}} \nu_{2n-2} \nu_{1} \cdots g_{\nu_{2j-3}} \nu_{2j-2} g_{\nu \delta})
\]
\[
\times a^{\mu_1 \mu_2 \cdots a^{\nu_{2n-3}} \nu_{2n-2} a^{\gamma \alpha} a^{\nu_{1} \nu_2 \cdots a^{\nu_{2j-3}} \nu_{2j-2} a^\delta}_\beta}.
\]
Denoting by \( \vee \) the symmetric product of symmetric forms and by \( \text{tr}_g \) the total trace of a symmetric form, we can rewrite this in the compact form
\[
< \Pi_i \bar{J}_\alpha \Pi_k \bar{J}_\beta > = \sum_{1 \leq n, j \leq s} \frac{\Gamma(m/2)(2n+2j-2)!}{\Gamma(\frac{m}{2} + n + j - 1)2^{2n+2j-2}(n+j-1)!} \times \text{tr}_g \left[ (\vee^{n-1} a) \vee \bar{a}_\alpha (\vee^{j-1} a) \vee \bar{a}_\beta \right],
\]
where \( \bar{a}_\alpha \) is a vector defined by \( \bar{a}_\alpha = a^{\nu \alpha} \partial_\mu. \)

This completes the general calculation of heat kernel coefficient \( \text{tr}_V a_1. \) More explicit formulas may be obtained in particular cases by using explicit formulas for the projections.
10 The covariant semi-classical approximation

In this section we show how the standard semi-classical method can be adapted to compute the small-$t$ heat kernel asymptotics of a non-Laplace type operator. In treating the semi-classical approximation, we shall follow [19].

The object of study is the fundamental solution of the heat equation

$$ \left( \frac{1}{\varepsilon} \partial_t + F \right) U(t|x,x') = 0 $$

for a non-Laplace type operator

$$ F = -a^{\mu\nu} \nabla_{\mu} \nabla_{\nu} + q. $$

with a singular initial condition

$$ U(t|x,x') \bigg|_{t \to 0} = \delta(x,x'). \quad (10.1) $$

Here $\varepsilon$ is a small formal parameter.

Let $\sigma^\mu(x,x')$ denote the tangent vector to the geodesic connecting the points $x$ and $x'$ at the point $x'$, the norm of which is equal to the length of that geodesic. In this section we describe a systematic method for constructing the local formal asymptotic solution of the heat equation as $\varepsilon \to 0$. The initial condition suggests the following Ansatz:

$$ U(t|x,x') = J(t|x,x') \exp \left( -\frac{1}{\varepsilon} S(t|x,x') \right) \Omega(t|x,x'), $$

where $S$ is a scalar function, $J$ is another scalar function defined by

$$ J(t|x,x') = \frac{1}{\sqrt{\det g_{\mu\nu}(x)}} \det \left[ -\frac{1}{2\pi \varepsilon} \nabla_\mu \nabla_{\nu'} S(t|x,x') \right] \frac{1}{\sqrt{\det g_{\mu\nu}(x')}}, $$

and $\Omega(t)$ has the expansion

$$ \Omega(t|x,x') \sim \sum_{k \geq 0} \varepsilon^k \phi_k(t|x,x') $$

in powers of $\varepsilon$. The leading asymptotics as $\varepsilon \to 0$ require

$$ \left( \dot{S} + a^{\mu\nu} S_{,\mu} S_{,\nu} \right) \phi_0 = 0 $$
where $\dot{S} = \partial_t S$ and $S_{\mu} \equiv \nabla_\mu S$. From this it follows that the function $S$ is determined by the Hamilton-Jacobi equation

$$\det v [\partial_t S + a^{\mu\nu} S_{\mu} S_{\nu}] = 0.$$  

Recalling the eigenvalues of the leading symbol, we have

$$\prod_{i=1}^{s} [\partial_t S + \mu_i g^{\mu\nu} S_{\mu} S_{\nu}]^{d_i} = 0.$$  

This has $s$ different solutions, one for each eigenvalue, determined by

$$\frac{1}{\mu_i} \partial_t S_i + g^{\mu\nu} S_{i\mu} S_{i\nu} = 0.$$  

These solutions differ by scaling $t \to t_i \equiv \mu_i t$, i.e. $S(t|x,x') = S_0(\mu_i t|x,x')$, where $S_0$ is determined by the equation

$$\partial_t S_0 + g^{\mu\nu} S_{0\mu} S_{0\nu} = 0.$$  

This is the Hamilton-Jacobi equation for a particle moving in a curved manifold. There is a Hamiltonian system that corresponds to each Hamilton-Jacobi equation (10.2). These Hamiltonian systems describe the geodesics parametrized by $t_i = t \mu_i$. The solution of this equation is given by the action along the geodesics connecting the points $x$ and $x'$ and parameterized so that $x(0) = x'$, $x(t) = x$. Thus

$$S_0(t|x,x') = \frac{\sigma(x,x')}{2t}, \quad S_i(t|x,x') = \frac{\sigma(x,x')}{2t \mu_i},$$

where $\sigma(x,x')$ is half the square of the length of the geodesic. This means that

$$J_i(t|x,x') = (4\pi t \mu_i \varepsilon)^{-m} \Delta(x,x'),$$

where $\Delta$ is the Van Vleck-Morette determinant. As an immediate consequence of the Liouville Theorem, this satisfies the equation

$$\frac{1}{\mu_i} \partial_t J_i + 2 \nabla_\mu (S_i^{\mu} J_i) = 0,$$

or

$$(D + \sigma^{\mu}_{\mu}) J_i^{1/2} = 0$$

where

$$D = t \partial_t + \sigma^{\mu} \nabla_\mu.$$
Here and below we denote the derivatives of $\sigma$ simply by adding indices to it, i.e. $\sigma_{\mu} \equiv \nabla_{\mu} \sigma$, $\sigma_{\mu \nu} \equiv \nabla_{\nu} \nabla_{\mu} \sigma$, etc. Recall that

$$D \Delta^{1/2} = \frac{1}{2}(m - \sigma^\mu_{\mu})\Delta^{1/2}. \quad (10.3)$$

Thus the semiclassical approximation as $\varepsilon \to 0$ polarizes along different eigenvalues. That is, all quantities become dependent on the eigenvalue, and the total solution is the superposition of all particular solutions for all eigenvalues. Thus, our final Ansatz is:

$$U(t|x, x') = \sum_{i=1}^{s} (4\pi t \mu_i \varepsilon)^{-m/2} \Delta^{1/2} \exp \left( -\frac{\sigma}{2t \mu_i \varepsilon} \right) \Omega_i(t|x, x'),$$

$$\Omega_i(t|x, x') \sim \sum_{k \geq 0} t^k \varepsilon^k \phi(i)_k(t|x, x').$$

For the function $\Omega_i(t)$, we get a transport equation

$$\left( \frac{1}{\varepsilon^2 t^2} N_i + \frac{1}{\varepsilon t} L_i + M \right) \Omega_i(t) = 0,$$

where

$$N_i = \frac{1}{2 \mu_i} \left( \sigma - \frac{1}{2 \mu_i} a^{\mu \nu} \sigma_{\mu} \sigma_{\nu} \right),$$

$$L_i = t \partial_t - \frac{m}{2} + \frac{1}{\mu_i} \Delta^{-1/2} a^{\mu \nu} \sigma_{\mu} \nabla_{\nu} \Delta^{1/2} + \frac{1}{2 \mu_i} a^{\mu \nu} \sigma_{\mu \nu},$$

$$M = \Delta^{-1/2} F \Delta^{1/2}.$$

The initial condition for $\Omega_i(t)$ is determined by the diagonal value of the heat kernel

$$\Omega_i(0|x, x) = < \Pi_i >,$$

where $< \Pi_i >$ is defined by (8.2). While for irreducible representations this is proportional to the identity matrix, for a general reducible bundle it is not. By using the expansion of $\Omega_i(t)$, we get recursion relations

$$N_i \phi(i)_0 = 0,$$

$$N_i \phi(i)_1 = -L_i \phi(i)_0,$$

$$N_i \phi(i)_k = -(L_i + k - 1) \phi(i)_{k-1} - M \phi(i)_{k-2} \quad k \geq 2. \quad (10.5)$$
Note that $N_i$ is just an endomorphism (i.e., has order zero as a differential operator).

Now we have a quadratic form $a^\mu\nu\sigma_\mu\sigma_\nu$ which can be expanded in terms of the same projections as before. The role of the covector argument $\xi$ of each projection is now played by $\nabla_\sigma$, and we denote

$$P_i \equiv \Pi_i(\nabla_\sigma) = \sum_{n=0}^{p} \Pi_{\mu_1\ldots\mu_n}^{(2n)} \frac{\sigma_{\mu_1} \cdots \sigma_{\mu_{2n}}}{(2\sigma)^n}$$

$$= \sum_{k=1}^{s} \sum_{i=0}^{s} c_{ik} a^{(\mu_1\mu_2 \ldots \mu_{2k-3}\mu_{2k-2})} \frac{\sigma_{\mu_1} \cdots \sigma_{\mu_{2k-2}}}{(2\sigma)^{k-1}},$$

(10.6)

$$a^{\mu\nu}\sigma_\mu\sigma_\nu = 2\sigma \sum_{i=0}^{s} \mu_i P_i.$$

Recall that $\nabla_\lambda a^{\mu\nu} = 0$; this implies that the tensors $\Pi_{i(\alpha)}$ (not $P_i$) are covariantly constant:

$$\nabla \Pi_{i(\alpha)} = 0.$$
Now, by multiplying this recursion by $P_n$, $n \neq i$, and using \((10.7)\), we obtain
\[
\frac{\sigma}{2\mu_i^2}(\mu_i - \mu_n)P_n\chi(i)_0 = 0,
\]
\[
\frac{\sigma}{2\mu_i^2}(\mu_i - \mu_n)P_n\chi(i)_1 = -P_nL_i\psi(i)_0,
\]
\[
\frac{\sigma}{2\mu_i^2}(\mu_i - \mu_n)P_n\chi(i)_k = -P_nL_i\psi(i)_{k-1} - P_n(L_i + k - 1)\chi(i)_{k-1}
- P_nM(\psi(i)_{k-2} + \chi(i)_{k-2}^\prime) .
\]
\[(10.9)\]

This recursion determines $\chi(i)_k$ algebraically in terms of $\psi(i)_{k-1}$, $\psi(i)_{k-2}$, $\chi(i)_{k-1}$ and $\chi(i)_{k-2}$. In particular, we find that
\[
\chi(i)_0 = 0,
\]
\[
\chi(i)_1 = -2\sigma\sum_{1 \leq n \neq i \leq s} \frac{\mu_i^2}{\mu_i - \mu_n}P_nL_i\psi(i)_0.
\]

The recursion does not determine the $\psi(i)_k$ however. These are determined by another differential recursion that is obtained by multiplying \((10.8)\) by $P_i$,
\[
P_iL_i\psi(i)_0 = 0,
\]
\[
(P_iL_i + k)\psi(i)_k = -P_iL_i\chi(i)_k - P_iM(\psi(i)_{k-1} + \chi(i)_{k-1}) , \quad k \geq 1.
\]

Now let us compute the operator $P_iL_iP_i$ entering this recursion. We have
\[
P_iL_iP_i = t\partial_t + \frac{1}{\mu_i}P_i\Delta^{-1/2}a^{\mu\nu}\sigma_\mu P_i\nabla_\nu\Delta^{1/2}
\]
\[+ \frac{1}{2\mu_i}(P_i a^{\mu\nu}\sigma_\mu P_i;_\nu + P_i a^{\mu\nu}\sigma_\mu P_i) - \frac{1}{2}mP_i \quad (10.10)\]

where $P_i;_\nu = \nabla_\nu P_i$. Further, noting that
\[
P_i a^{\mu\nu}\sigma_\mu\sigma_\nu P_i = 2\mu_i\sigma P_i ,
\]
we get
\[
P_i a^{\mu\nu}\sigma_\nu P_i = \mu_i\sigma^\mu P_i .
\]
Taking into account the equation for the Van Vleck-Morette determinant (10.3), we obtain

\[ P_iL_iP_i = P_iD + K_i, \]

where

\[ D = t\partial_t + \sigma^\mu \nabla_\mu \]

is a first-order differential operator, and

\[ K_i = \frac{1}{2\mu_i} P_i [2a^{\mu\nu} \sigma_\mu P_{i\nu} + (a^{\mu\nu} - \mu_i g^{\mu\nu}) \sigma_{\mu\nu}] P_i \]

is some endomorphism. Note that \( \sigma^\mu = tdx^\mu/dt \). Therefore, the operator \( D \) is expressed in terms of the operator of total differentiation along the geodesics,

\[ D = t \frac{d}{dt}. \]

The operator \( D \) commutes with any function that depends only on “angular” coordinates \( \sigma^\mu/\sqrt{\sigma} \); in particular, it commutes with the projections:

\[ DP_i = P_i D. \]

Thus the recursion for the \( \psi \)'s takes the form

\[
(D + K_i) \psi(i)_0 = 0,
\]

\[
(D + k + K_i) \psi(i)_k = -P_iL_i\chi(i)_{k-1} - P_iM(\psi(i)_{k-1} + \chi(i)_{k-1}), \quad k \geq 1.
\]

But these are exactly the transport equations along geodesics. They may be integrated with the appropriate initial conditions determined by (10.4). In particular, the first coefficient \( \psi(i)_0 \) is

\[ \psi(i)_0 = \exp \left( - \int_0^t \frac{d\tau}{\tau} K_i \right) < \Pi_i >, \]

where the integration is along the geodesic connecting the points \( x' \) and \( x \), parametrized so that \( x = x(\tau) \) that \( x(0) = x', x(t) = x \). In the standard case, i.e. for a Laplace type operator, we have \( K_i = 0 \). Therefore, the first coefficient is just \( \varphi_0 = I \).

Thus, the recursion relations determine the asymptotic solution completely. The algorithm is the following: at each step determine first the \( \chi(i)_k \) by (10.9), and then \( \psi(i)_k \) by (10.11).
11 Concluding remarks

Let us summarize the results of this paper. We have studied in detail a general class of non-Laplace type operators, i.e. elliptic second-order partial differential operators acting on sections of a tensor-spinor vector bundle over a compact manifold without boundary. The only essential assumptions that have been made are: i) the positivity of the leading symbol, $a^{\mu\nu} \xi_\mu \xi_\nu > 0$ (in the sense of endomorphisms) for $\xi \neq 0$, and ii) the covariant constancy of the tensor $a^{\mu\nu}$, i.e. $\nabla a = 0$. We constructed the leading order resolvent and the heat kernel and computed the first two coefficients of the heat kernel asymptotic expansion explicitly. In the last section we developed an alternative approach for the off-diagonal heat kernel asymptotics for by making use of the general theory of the semi-classical approximation. This enabled us to construct a new ansatz for the heat kernel, as well as to find a complete set of recursion relations for the coefficients of the off-diagonal asymptotic expansion. This generalizes a well known ansatz for the heat kernel of Laplace type operators (e.g. see [2]). In contrast to the Laplace type case, the off-diagonal heat kernel for the non-Laplace type case exhibits some essentially new features, notably polarization along the different eigenvalues of the leading symbol. As an explicit example of a non-Laplace type operator we considered the most general second-order operator acting on the bundle of symmetric two-tensors. We computed the eigenvalues of the leading symbol, the multiplicities and the corresponding projections.

References

[1] Alexandrov, S., Vassilevich, D.: Heat kernel for nonminimal operators on a Kaehler manifold, J. Math. Phys. 37, 5715–5718 (1996)

[2] Avramidi, I.G.: A covariant technique for the calculation of the one-loop effective action, Nucl. Phys. B 355, 712–754 (1991)

[3] Avramidi, I.G.: Green functions of higher-order differential operators, J. Math. Phys. 39, 2889–2909 (1998)

[4] Avramidi, I.G.: Covariant techniques for computation of the heat kernel, Rev. Math. Phys. 11, 947–980 (1999)
[5] Avramidi, I.G.: *Heat Kernel and Quantum Gravity*. Lecture Notes in Physics, Series Monographs, LNP:m64. Berlin, New York: Springer-Verlag, 2000

[6] Avramidi, I.G., Branson, T.: A discrete leading symbol and spectral asymptotics for natural differential operators, Irvin Segal memorial volume, (2001), to appear

[7] Avramidi, I.G., Esposito, G.: Gauge theories on manifolds with boundary, Comm. Math. Phys. **200**, 495–543 (1999)

[8] Berline, N., Getzler, E., Vergne, M.: *Heat Kernels and Dirac Operators*. Berlin, New York: Springer-Verlag, 1992.

[9] Branson, T.P, Gilkey, P.B., Pierzhalski, A.: Heat equation asymptotics of elliptic differential operators with non-scalar leading symbol, Math. Nachr. **166**, 207–215 (1994)

[10] Cho, H.T., and Kantowski, R.: ζ-functions for nonminimal operators, Phys. Rev. D **52**, 4588–4599 (1995)

[11] De Witt, B.S.: *The Spacetime Approach to Quantum Field Theory*. In: De Witt, B.S., Stora, R. (eds.) *Relativity, Groups and Topology II*, pp. 383–738. Amsterdam: North Holland, 1984

[12] Gilkey, P.B.: *Invariance Theory, the Heat Equation and the Atiyah–Singer Index Theorem*. Boca Raton: CRC Press, 1995

[13] Gilkey, P.B., Branson, T.P., Fulling, S.A.: Heat equation asymptotics of “nonminimal” operators on differential forms, J. Math. Phys., **32**, 2089–2091 (1991)

[14] Gusynin, V.P.: Asymptotics of the heat kernel for nonminimal differential operators, Ukrainian Math. Zh., **43**, 1541–1551 (1991)

[15] Gusynyn, V.P., Heat kernel technique for nonminimal operators. In: Fulling, S.A. (ed.) *Heat Kernel Techniques and Quantum Gravity*. Proceedings, Winnipeg, Manitoba, 1994. Discourses in Mathematics and Its Applications, vol. 4, pp. 65–86. College Station, Texas: Texas A& M University, 1995
[16] Gusynin, V.P., Kornyak, V.V.: Complete computation of the De Witt-Seeley-Gilkey coefficient $E_4$ for nonminimal operator on curved manifolds, Fundamental and Applied Mathematics, 5, 649–674 (1999)

[17] Hadamard, J.: Lectures on Cauchy’s Problem. In: Linear Partial Differential Equations. New Haven: Yale University Press, 1923

[18] Kupradze, V.D.: Potential Methods in the Theory of Elasticity. Jerusalem: Israel program of Scientific Translation, 1965

[19] Maslov, V.P., Fedoriuk, M.V.: Semi-classical Approximation in Quantum Mechanics. Boston: D. Reidel Pub. Co, 1981