Finite automata and relations of multiple zeta values

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July 28, 2018

Abstract

The theory of finite automata applies to the study on relations of multiple zeta values.

1 Introduction

1.1 Multiple zeta values and Zagier - Broadhurst’s formula

Multiple zeta values (MZVs, for short) are real numbers defined by

\[ \zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}} \]  

(1.1)

where \( k_1, k_2, \ldots, k_n \in \mathbb{Z}_{\geq 1} \), and \( k_1 \geq 2 \).

The following formula was conjectured by Zagier \[Z\], and proved by Broadhurst \[B_2, B_3L\]:

\[ \zeta(\{3, 1\}_n) = \zeta(3, 1, \ldots, 3, 1) = \frac{2\pi^4 n}{(4n + 2)!} \]  

(1.2)

The proof refined by Zagier is as follows (cf. \[AK\]): The equality

\[ \sum_{n=0}^{\infty} Li_{3,1,\ldots,3,1}(x) t^{4n} = F\left(\frac{t}{1+i}, -\frac{t}{1+i}; 1; x\right) F\left(\frac{t}{1-i}, -\frac{t}{1-i}; 1; x\right) \]

can be proved by showing that the both sides is annihilated by the differential operator

\[ \left((1-x)\frac{d}{dx}\right)^2 \left(x\frac{d}{dx}\right)^2 - t^4. \]

Here, for \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 1} \),

\[ Li_{k_1,\ldots,k_n}(z) = \sum_{m_1 > \cdots > m_n > 0} \frac{z^{m_1}}{m_1^{k_1} \cdots m_n^{k_n}} \]  

(1.3)

are multiple polylogarithms of one variable (MPLs, for short). Hence we have

\[ \sum_{n=0}^{\infty} \zeta(3,1,\ldots,3,1) t^{4n} = F\left(\frac{t}{1+i}, -\frac{t}{1+i}; 1; 1\right) F\left(\frac{t}{1-i}, -\frac{t}{1-i}; 1; 1\right). \]

Applying the formula

\[ F(a,-a;1;1) = \frac{1}{\Gamma(1-a)\Gamma(1+a)} = \frac{\sin \pi a}{\pi a}, \]

we have

\[ F\left(\frac{t}{1+i}, -\frac{t}{1+i}; 1; 1\right) F\left(\frac{t}{1-i}, -\frac{t}{1-i}; 1; 1\right) = \sum_{n=0}^{\infty} \frac{2\pi^4 n}{(4n+2)!} t^{4n}. \]
1.2 Waldschmidt’s idea

The original proof by \[ B^2, B^3 \] is based upon some combinatorics, while Waldschmidt gave more simple one by means of the idea “finite automata”.

Let \( \mathfrak{H} = \mathbb{Q}(x, y) \) be a \( \mathbb{Q} \)-algebra of polynomials of non-commutative variables \( x \) and \( y \), and a subalgebra \( \mathfrak{H}^0 = \mathbb{Q} \cdot 1 + x\mathfrak{H}y \). Set \( z_k = x^{k-1} y \) (\( k \geq 1 \)). We define a \( \mathbb{Q} \)-linear map \( Z : \mathfrak{H}^0 \rightarrow \mathbb{R} \) by

\[
Z(z_1 z_2 \cdots z_n) = \zeta(k_1, k_2, \ldots, k_n) \quad (k_1, k_2, \ldots, k_n \in \mathbb{Z}_{\geq 1} \text{ and } k_1 \geq 2).
\] (1.4)

which coresponds to the iterative integral representation;

\[
\zeta(k_1, k_2, \ldots, k_n) = \int_0^1 \frac{dt}{t} \cdots \frac{dt}{t} \frac{dt}{1-t} \cdots \frac{dt}{1-t},
\] (1.5)

The shuffle product \( \mathfrak{H} \) is a notion abstracting the product of iterated integrals.

**Definition 1.1.** \( \mathfrak{H} \times \mathfrak{H} \rightarrow \mathfrak{H} \) is a \( \mathbb{Q} \)-bilinear operation satisfying the following conditions:

(i) For \( w \in \mathfrak{H} \), \( w \mathfrak{H} 1 = 1 \mathfrak{H} w = w \).

(ii) Let \( u_1, u_2 \) be \( x \) or \( y \). Then, for any words \( w_1, w_2 \in \mathfrak{H} \),

\[
(u_1 w_1) \mathfrak{H} (u_2 w_2) = u_1 (w_1 \mathfrak{H} u_2 w_2) + u_2 (u_1 w_1 \mathfrak{H} w_2).
\] (1.6)

It is known that \( (\mathfrak{H}, \mathfrak{H}) \) is a commutative algebra (cf. [AK]), and \( (\mathfrak{H}^0, \mathfrak{H}) \) is a commutative subalgebra, furthermore, \( Z \) is an algebra homomorphism, that is,

\[
Z(u_1 \mathfrak{H} w_2) = Z(u_1) Z(w_2), \quad (w_1, w_2 \in \mathfrak{H}^0).
\] (1.7)

Let \( \mathfrak{H} = \mathbb{Q}(x, y) \) be a \( \mathbb{Q} \)-algebra of formal power series of \( x, y \) which can be viewed as the dual of \( \mathfrak{H} \). For \( w \in x\mathfrak{H} + y\mathfrak{H} \), set

\[
w^* = \sum_{n=0}^{\infty} w^n
\] (1.8)

which is called Kleene’s closure of \( w \). This is the inverse element of \( w \) in \( \mathfrak{H} \):

\[
w^*(1-w) = (1-w)w^* = 1.
\] (1.9)

Using the idea of finite automata, Waldschmidt [W] showed that

\[(xy)^* \mathfrak{H} (-xy)^* = (-4x^2y^2)^*.
\] (1.10)

We should note that the shuffle product preserves the weight of words. Transforming the both sides of (1.10) via \( Z \) to MZVs, one obtains

\[
(-4)^n \zeta(\{3,1\}_n) = \sum_{p=0}^{2n} (-1)^{2n-p} \zeta(\{2\}_p) \zeta(\{2\}_{2n-p}).
\]

The RHS above equals to \( (-1)^n \zeta(\{4\}_n) \) (see Corollary [GW] [4.10]), and

\[
\zeta(\{4\}_n) = \frac{2 \cdot 4^n \pi^{4n}}{(4n + 2)!},
\] (1.11)

so that (1.12) is established.

The original proof of (1.10) will be reviewed in Section 3 after the preliminaries on automata theory.
1.3 The purpose of the paper

In this paper, we aim at extending the idea of Waldschmidt and deriving various relations of MZVs. Our central idea is to associate an “adjacency matrix” to each finite automaton. Through adjacency matrices, one can compute the shuffle product in a combinatorial way.

This paper is organized as follows: In Section 2, we give preliminaries on the basic concept of finite automata, in particular, shuffle automata and adjacency matrices. In Section 3, we review in details the proof of (1.10) due to W, and give the definition of the harmonic product of MZVs and show basic formulas which will be used later. In Section 4, first we give a direct application of the Waldschmidt formula (1.10). Next we derive several relations of MZVs by means of adjacency matrices. In Section 5 we introduce harmonic automata, and prove the basic formula presented in Section 3. In Appendix we consider the values of \( \zeta(\{2k\}_n) \) (1 \( \leq k \leq 7 \)).

In [K] and [S], the first and the second authors of this article deal with more shuffle automata and harmonic automata with generalization to multiple L values. These subjects will be treated in the next paper.

Acknowledgement

This research started with inspired by the talk of Professor Waldschmidt at the conference on “Zeta Functions, Topology, and Quantum Physics” held at Kinki University, March 2003. The authors would like to express deep gratitude to Professor Waldschmidt and the organizing committee of ZTQ.

The third author is partially supported by JPSP Grant-in-Aid No. 15540050.

2 Finite automata

2.1 Definition of finite automata

**Definition 2.1.** (1) A finite automaton over \( H_C = \mathbb{C} \langle x, y \rangle \) is a quintuple \( (Q, \Sigma, \delta, q_1, F) \) where

- (i) \( Q = \{q_1, q_2, \ldots, q_m\} \) is a set of states;
- (ii) \( \Sigma \) is a finite subset of \( \mathbb{C}x + \mathbb{C}y \) which is called the alphabets of the automaton;
- (iii) \( \delta : Q \times \Sigma \rightarrow Q \) is the transition function;
- (iv) \( q_1 \) is the initial state;
- (v) \( F \subset Q \) is the set of final states.

(2) If \( w = u_1 \cdots u_n \in \mathcal{H}_C \) \( (u_1, \ldots, u_n \in \Sigma) \) satisfies

\[
\delta(\cdots(\delta(q_1, u_1)\cdots u_n)) = q_i \in F,
\]

then we say that the word \( w \) is accepted by the automaton.

(3) If, in the automaton, the states transit like

\[
q_i = q_{i_0} \rightarrow q_{i_1} \rightarrow \cdots q_{i_{n-1}} \rightarrow q_{i_n} = q_j,
\]

then we call this trajectory a path of length \( n \) from \( q_i \) to \( q_j \). If the word \( w \) is accepted by a path of length \( n \), then we call \( w \) a word of length \( n \), and denote it by \( l(w) = n \).

(4) If, for \( w \in \mathcal{H}_C = \mathbb{C}(\langle x, y \rangle) \), there exists an automaton \( M = (Q, \Sigma, \delta, q_1, F) \) such that

\[
w = \sum (a \text{ word accepted by the automaton}) \times (\text{the number of paths for which the word is accepted}),
\]

then we say that the element \( w = w(M) \) is accepted by the automaton \( M \).

**Remark 2.1.** In the ordinary theory of automata (cf. [An], [HMU]), what is accepted by an automaton is a “language”, but in our theory, it is an element in \( \mathcal{H}_C \).
Example 2.1. Let $M = (Q, \Sigma, \delta, q_1, F)$ be
\[
Q = \{q_1, q_2, q_3\},
\]
\[
\Sigma = \{x, y, -y\},
\]
\[
\delta : \delta(q_1, x) = q_2, \ \delta(q_1, -y) = q_3, \ \delta(q_2, x) = q_3, \ \delta(q_3, y) = q_1,
\]
\[F = \{q_3\}.
\]
The transition diagram of the automaton is

where the initial state is surrounded by a circle and the final state by a double rectangular. For this automaton, $w(M) = (x^2 - y)(yx^2 - y^2)^\ast$.

Now we present addition of elements, multiplication of elements by scalors, concatenation of elements in terms of automata. Let $w_1 = u_1 u_2 \ldots u_m$, $w_2 = v_1 v_2 \ldots v_n \in \Sigma^\ast (u_1, \ldots, u_m, v_1, \ldots, v_n \in \Sigma)$. These are represented by the following automata:

\[
w_1 : \quad q_1 \quad q_2 \quad q_3 \quad \ldots \quad u_m \quad q_{m+1}
\]

(2.1)

\[
w_2 : \quad q_1 \quad q_2 \quad q_3 \quad \ldots \quad v_n \quad q_{m+1}
\]

(2.2)

Then the sum $w_1 + w_2$ and the concatenation $w_1 w_2$ are represented by

\[
w_1 + w_2 : \quad q_1 \quad q_2 \quad q_3 \quad \ldots \quad u_m \quad q_{m+1}
\]

(2.3)

\[
w_1 w_2 : \quad q_1 \quad q_2 \quad q_3 \quad \ldots \quad u_m \quad q_{m+1} \quad q_{m+2} \quad \ldots \quad v_n \quad q_{m+n+1}
\]

(2.4)

The scalar multiplication ($k \in K$) and Kleene’s closure of $w_1$ are represented by

\[
k w_1 : \quad q_1 \quad q_2 \quad q_3 \quad \ldots \quad u_m \quad q_{m+1}
\]

(2.5)

\[
w_1^\ast : \quad q_1 \quad q_2 \quad q_3 \quad \ldots \quad q_m
\]

(2.6)

In (2.6), “the initial state = the final state” is surrounded by a double circle. We should note that the automata representing the addition and the scalar multiplication is not unique. For example, if $u_1 = v_1 = x$, then $w_1 + w_2$ is also represented by

\[
w_1 + w_2 : \quad q_1 \quad q_2 \quad q_3 \quad \ldots \quad u_m \quad q_{m+1}
\]

In this automaton, the number of the states is less than that of (2.3). To reduce the number of the states in an automaton is important for easy computation of the words accepted.
2.2 Shuffle automata and adjacency matrices

Automata representing the shuffle product of words and of Kleene’s closure of words are considered in [HMU] and [W].

**Proposition 2.1.** The shuffle product \( w_1 \shuffle w_2 \) and \((w_1)^* \shuffle (w_2)^* \) are represented by the transition diagrams (2.7) and (2.8), respectively:

![Diagram](2.7)

![Diagram](2.8)

We call (2.7), (2.8) the shuffle automata. To compute a shuffle product is nothing but computing the words accepted by the shuffle automaton. For this end, an adjacency matrix is a useful tool.

**Definition 2.2.** Let \( M = (Q, \Sigma, \delta, q_1, F) \) be a finite automaton. Then we define a matrix \( A = A(M) \) by

\[
A = (a_{ij}) \text{ where } a_{ij} = \sum_{a \in \Sigma : \delta(q_i, a) = q_j} a.
\]  

(2.9)

We call \( A(M) \) the adjacency matrix of \( M \). This satisfies the following property.

**Theorem 2.2.** Let us denote the \((i, j)\)-entry of \( A(M)^n \) by \( a_{ij}^{(n)} \). Then \( a_{ij}^{(n)} \) is the sum of all the words accepted by paths of length \( n \) from \( q_i \) to \( q_j \).

5
Proof. We prove by induction. The case of $n = 1$ is trivial. Assume that the case of $n - 1$ holds. Let $Q = (q_1, q_2, \ldots, q_m)$ be the set of states of $M$. Since $A^n = A^{n-1} A$, we have

$$a_{ij}^{(n)} = \sum_{k=1}^{m} a_{ik}^{(n-1)} a_{kj}^{(1)}.$$ 

On the other hand, there exist $m$ strings of paths of length $n$ like

$$q_k \rightarrow \cdots \rightarrow q_k \rightarrow q_j \quad (k = 1, 2, \ldots, m).$$

Hence $a_{ij}^{(n)}$ equals to the sum of all the words accepted by paths of length $n$ from $q_i$ to $q_j$. 

Theorem 2.3. The element $w(M)$ accepted by $M$ is given by

$$w(M) = \sum_{j : q_j \in F} \left( \sum_{n=0}^{\infty} a_{ij}^{(n)} \right).$$  \hspace{1cm} (2.10)$$

(Here, for convenience, we set the word of length 0 to be $1 \in \hat{S}_C$ which is accepted only by the paths from $q_i$ to $q_i$. ) In particular, in the case that $F = \{q_1\}$, it is given by

$$w(M) = \sum_{n=0}^{\infty} a_{11}^{(n)}. \quad (2.11)$$

3 The Waldschmidt formula and harmonic product of MZVs

3.1 The original proof of the Waldschmidt formula

In [W], Waldschmidt proved the formula (1.10) in the following manner: From (2.8), the shuffle automaton of $(xy)^* w (-xy)^*$ is represented by the following transition diagram:

\begin{center}
\begin{tikzpicture}
\node (q1) at (0,0) {$q_1$};
\node (q2) at (0,-1) {$q_2$};
\node (q3) at (1,0) {$q_3$};
\node (q4) at (1,-1) {$q_4$};
\draw (q1) edge[->] node[above] {$x$} (q3);
\draw (q2) edge[->] node[above] {$x$} (q4);
\draw (q1) edge[->] node[left] {$-y$} (q2);
\draw (q3) edge[->] node[left] {$-y$} (q4);
\end{tikzpicture}
\end{center}

(3.1)

Denote this automaton by $M = (Q, \Sigma, \delta, q_1, F)$. Let $M_k = (Q, \Sigma, \delta, q_k, F)$ and $S_k$ the element accepted by $M_k$. Then we have the following linear recursive equations;

$$\begin{cases}
S_1 = 1 + xS_3 + xS_4, & S_2 = -yS_3 + yS_4, \\
S_3 = yS_1 + xS_2, & S_4 = -yS_1 + xS_2.
\end{cases} \quad (3.2)$$

Solving this, we have

$$S_1 = 1 - 4x^2 y^2 S_4. \quad (3.3)$$

Hence $S_1 = (-4x^2 y^2)^*$. 

For any finite automaton, one has linear recursive equations like (3.2). But they cannot be solved (or, it is too difficult to solve) in general. By introducing adjacency matrices, one can avoid such difficulty.
3.2 Proof of the Waldschmidt formula via an adjacency matrix

The adjacency matrix $A$ of the automaton (3.1) is
\[
A = \begin{bmatrix}
0 & 0 & x & x \\
0 & 0 & y & -y \\
y & x & 0 & 0 \\
-y & x & 0 & 0
\end{bmatrix}.
\] (3.4)

Let us compute $a_{11}^{(n)}$. We have
\[
A^2 = \begin{bmatrix}
0 & 2x^2 & 0 & 0 \\
-2y^2 & 0 & 0 & 0 \\
0 & 0 & * & * \\
0 & 0 & * & *
\end{bmatrix},
\]
where * stand for certain elements of length 2, and
\[
A^4 = \begin{bmatrix}
-4x^2y^2 & 0 & 0 & 0 \\
0 & -4y^2x^2 & 0 & 0 \\
0 & 0 & * & * \\
0 & 0 & * & *
\end{bmatrix},
\]
where * stand for certain elements of length 4. From this computation, we see that $a_{11}^{(n)} = 0$ unless $n = 4k$, and $a_{11}^{(4k)} = (-4x^2y^2)^k$. Thus we obtain the Waldschmidt formula (1.10).

3.3 Harmonic product

Let $k_1, k_2 \geq 2$. Then one can compute product of zeta values $\zeta(k_1), \zeta(k_2)$ like
\[
\zeta(k_1)\zeta(k_2) = \left\{ \sum_{m_1 > m_2 > 0} + \sum_{m_2 > m_1 > 0} + \sum_{m_1 = m_2 > 0} \right\} \frac{1}{m_1^{k_1} m_2^{k_2}}
= \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2).
\]

Generalizing this, one can introduce the harmonic product $*$ on $\mathcal{H}^1 = \mathbb{Q} \cdot 1 + \mathcal{H}y$.

**Definition 3.1.** $*: \mathcal{H}^1 \times \mathcal{H}^1 \rightarrow \mathcal{H}^1$ is a $\mathbb{Q}$-bilinear operation satisfying the following conditions:

(i) For any $w \in \mathcal{H}^1$, $w * 1 = 1 * w = w$.

(ii) Let $z_k = x^{k-1}y$ $(k = 1, 2, \ldots)$. For any words $w_1, w_2$ in $\mathcal{H}^1$,
\[
(z_1w_1) * (z_2w_2) = z_i(1 * (z_jw_2)) + z_j((z_iw_1) * w_2) + z_{i+j}(w_1 * w_2).
\] (3.5)

Hoffman [H] showed that $(\mathcal{H}^1, *)$ is a commutative algebra generated by $z_k = x^{k-1}y$ $(k = 1, 2, \ldots)$, and that the map $Z: \mathcal{H}^0 \rightarrow \mathbb{R}$ is an algebra homomorphism, that is,
\[
Z(w_1 * w_2) = Z(w_1)Z(w_2), \quad (w_1, w_2 \in \mathcal{H}^0).
\] (3.6)

By (1.7) and (3.6), we have the finite double shuffle relation:

**Proposition 3.1 (Zagier, cf. [AK]).**
\[
Z(w_1 * w_2 - w_1 w w_2) = 0, \quad (w_1, w_2 \in \mathcal{H}^0).
\] (3.7)

The harmonic product of the Kleene closure of $z_k$ is computed as follows:
Theorem 3.2. We have
\[(z_k)^* \ast (-z_k)^* = (-z_{2k})^*.\] (3.8)

In general, letting \(\omega\) be a primitive \(m\)-th root of unity,
\[(z_k)^* \ast (\omega z_k)^* \ast \cdots \ast (\omega^{m-1} z_k)^* = ((-1)^{m-1} z_{mk})^*.\] (3.9)

We will prove this theorem in Section 5 after introducing the notion of "harmonic automata". Transforming (3.8), (3.9) to MZVs via \(Z\) yields the following formula:

Corollary 3.3. We have
\[\sum_{p=0}^{2n} (-1)^p \zeta(\{k\}_p) \zeta(\{k\}_{2n-p}) = (-1)^n \zeta(\{2k\}_n).\] (3.10)

In general,
\[\sum_{\substack{p_1 + \cdots + p_m = mn, \\ p_1, \ldots, p_m \geq 0}} \omega^{0 \cdot p_1 + 1 \cdot p_2 + \cdots + (m-1) \cdot p_m} \zeta(\{k\}_{p_1}) \cdots \zeta(\{k\}_{p_m}) = (-1)^{mn-n} \zeta(\{mk\}_n).\] (3.11)

4 Variation of Zagier-Broadhurst’s formula

4.1 The shuffle automaton of \((-xy)^* \omega (xy)^* x^2(xy)^*)

As an application of Waldschmidt’s formula (1.10), we show
\[4n+2 \sum_{p=0}^{n-1} (-1)^{p+1} \frac{4^{2n-p+1} B_{4n-4p+2}}{(4p+2)!(4n-4p+2)!} \quad \text{(weight= 4n + 2)}\]
\[= -2 \sum_{p=0}^{n-1} (-4)^p \zeta(\{3,1\}_p, 3, 3, \{2\}_{2(n-p-1)}) - 3 \sum_{p=0}^{n-1} (-4)^p \zeta(\{3,1\}_p, 4, \{2\}_{2(n-p-1)}) \]
\[+ 2(-4)^n \sum_{p=0}^{n-1} \zeta(\{3,1\}_p, 5, 1, \{3,1\}_{n-p-1}).\] (4.1)

(ii)
\[\frac{(-1)^n 2 \cdot 4^{n+1}(n+1)n^{4n+4}}{(4n+6)!} \quad \text{(weight= 4n + 4)}\]
\[= 2 \sum_{p=0}^{n-1} (-4)^p \zeta(\{3,1\}_p, 3, 3, \{2\}_{2(n-p-1)}) + 3 \sum_{p=0}^{n} (-4)^p \zeta(\{3,1\}_p, 4, \{2\}_{2(n-p)}) \]
\[+ 2(-4)^n \sum_{p=0}^{n-1} \zeta(\{3,1\}_p, 3, 4, 1, \{3,1\}_{n-p-1}) - 2(-4)^n \sum_{p=0}^{n} \zeta(\{3,1\}_p, 3, 1, \{3,1\}_{n-p}).\] (4.2)

where \(B_k\)‘s are the Bernoulli numbers;
\[\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{te^t}{e^t-1},\] (4.3)

These relations are derived from the following shuffle automata:

Proposition 4.2. We have
By Waldschmidt’s formula \((4.10)\), \(S_{1,1} = (-4x^2y^2)^*\). Other terms are calculated as

\[
S_{1,1} = 2x^2(1 + yxS_{1,2} - yxS_{1,2}) = 2x^2,
\]

\[
S_{2,1,1} = x,
\]

\[
S_{2,1,1} = -y(-xy)^*.
\]

\[
S_{1,1,1,1} = x + 2xS_{1,3},
\]

\[
S_{1,1,1,1} = -y + yxS_{1,3} - yxS_{1,3} = -y,
\]

\[
S_{3,1,1,1} = x,
\]

\[
S_{1,1,1,1} = (-xy)^*.
\]

Hence we have \((4.4)\).

(ii) The transition diagram of the shuffle automaton \((-xy)^* \omega (xy)^*x^2(xy)^*\) is

\[
\[
\]

By similar consideration, we obtain \((4.5)\).

Proof of Theorem 4.1. Let \((w)^+ = w(w)^*\) for a word \(w\). From \((4.4)\) and \((4.5)\), we have

\[
(-xy)^* \omega (xy)^*x^2(xy)^+ = (-xy)^* \omega (xy)^*x^2(xy)^+ - (-xy)^* \omega (xy)^*x^2
\]

\[
= (-4x^2y^2)^*x^2(-4x^2y^2)^* - 4(-4x^2y^2)^*x^2y(-4x^2y^2)^* - 2(-4x^2y^2)^*x^2y(-4x^2y^2)^* + 2(-4x^2y^2)^*x^2y(-xy)^+ - (-4x^2y^2)^*x^2(-xy)^+.
\]
Note that, since the shuffle product preserves the length of words, the elements of the same length in both sides are equal. Picking up the terms of length $4n + 2$ from the RHS, we have

$$Z \text{(the sum of the terms of length } 4n + 2 \text{ in the RHS)}$$

$$= -2 \sum_{p=0}^{n-1} (-4)^p \zeta((3, 1)_p, 3, 3, (2)_{2(n-p-1)}) - 3 \sum_{p=0}^{n-1} (-4)^p \zeta((3, 1)_p, 4, (2)_{2(n-p)})$$

$$+ 2(-4)^n \sum_{p=0}^{n-1} \zeta((3, 1)_p, 5, 1, (1)_n, (3, 1)_{n-p-1}).$$

On the other hand, by Lemma 4.3.

$$Z \text{(the sum of the terms of length } 4n + 2 \text{ in the LHS)}$$

$$= \sum_{p=1}^{2n} (-1)^{p-1} \left( \zeta((2)_{p-1}) \sum_{q=1}^{2n-p+1} \zeta((2)_{q-1}, 4, (2)_{2n-p+1-q}) \right).$$

Lemma 4.3. We have

$$\sum_{p=1}^{2n-2p+1} (-1)^{p-1} \left( \zeta((2)_{p-1}) \sum_{q=1}^{2n-p+1} \zeta((2)_{q-1}, 4, (2)_{2n-p+1-q}) \right) = (-1)^n \sum_{p=1}^{n} \zeta((4)_{p-1}, 6, (4)_{n-p}) \quad (4.8)$$

Proof. By the definition of the harmonic product (1.6), we have

$$\sum_{q=1}^{2n-p+1} \zeta((2)_{q-1}, 4, (2)_{2n-p+1-q}) = \zeta(2) \zeta((2)_{2n-p+1}) - (2n - p + 2) \zeta((2)_{2n-p+2}).$$

Hence

the LHS of (4.8) = \sum_{p=1}^{2n} (-1)^{p-1} \left\{ \zeta(2) \zeta((2)_{p-1}) \zeta((2)_{2n-p+1}) - (2n - p + 2) \zeta((2)_{2n-p+1}) \zeta((2)_{2n-p+2}) \right\}.

By using (3.10) and (3.5), it is easy to see

the LHS of (4.8) = (-1)^n \sum_{p=1}^{n} \zeta((4)_{p-1}, 6, (4)_{n-p}) + (-1)^n \sum_{p=1}^{n+1} \zeta((4)_{p-1}, 2, (4)_{m+1-p})

$$+ \sum_{p=0}^{n} (-1)^{p+1}(2n + 1 - 2p) \zeta((2)_{p}) \zeta((2)_{2n+1-p}).$$

Furthermore, by induction, one can prove

$$(-1)^{n+1} \sum_{p=1}^{n+1} z_4^{p-1} z_2 z_4^{n+1-p} = \sum_{p=0}^{n} (-1)^{p+1}(2n + 1 - 2p) z_2^p * z_2^{2n+1-p}. \quad (4.9)$$

Thus the proof is completed.  □

By calculating harmonic product, we have

$$(-1)^n \sum_{p=1}^{n} \zeta((4)_{p-1}, 6, (4)_{n-p}) = (-1)^n \left( \zeta(6) \zeta((4)_{n-1}) - \sum_{p=1}^{n-1} \zeta((4)_{p-1}, 10, (4)_{n-p-1}) \right)$$

$$= (-1)^n \zeta(6) \zeta((4)_{n-1}) + (-1)^{n+1} \zeta(10) \zeta((4)_{n-2}) + (-1)^{n+2} \sum_{p=1}^{n-2} \zeta((4)_{p-1}, 10, (4)_{n-p-2})$$

$$+ \ldots + \sum_{p=0}^{n-1} (-1)^{p+1} \zeta((4)_p) \zeta(4n - 4p + 2).$$

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Here substituting the formulas

\[
\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}}{2(2k)!},
\]

(4.10)

\[
\zeta({\{4\}_n}) = \frac{2 \cdot 4^n \pi^{4n}}{(4n + 2)!},
\]

(4.11)

we obtain the LHS of (4.1).

We can prove (4.2) in a similar way. This completes the proof.

\[\square\]

4.2 The shuffle automaton of \((x^2y)^* \mathbb{w} (-x^2y)^*\)

Through consideration on the shuffle automaton of \((x^2y)^* \mathbb{w} (-x^2y)^*\), we show

Theorem 4.4. We have

\[
\sum_{\varepsilon, \varepsilon_i' = 0,1} \frac{12^n}{2^{e_1 + \cdots + e_n + \varepsilon_1 + \cdots + e_n}} \zeta(\{5 - \varepsilon_{i'-1} - \varepsilon_i, 1 + \varepsilon_i + \varepsilon_i'\}_{i=1}^n) = \frac{6(2\pi)^{6n}}{(6n + 3)!}
\]

(4.12)

where \(\varepsilon_0 = \varepsilon_n' = 0\).

First we show the following lemma:

Lemma 4.5. Assume that \(w \in \mathcal{H}\) be a word of length \(l\). Let \(A = (a_{ij})\) be the adjacency matrix of the finite automaton of \(w^* \mathbb{w} (-w)^*\) and \(A^n = (a_{ij}^{(n)})\). Then

\(n \in \mathbb{Z}_{\geq 0} \implies a_{11}^{(n)} = 0.\)

Proof. It is obvious that \(n \notin \mathbb{Z}_{\geq 0} \implies a_{11}^{(n)} = 0.\) Suppose \(n = (2k + 1)l\) \((k \in \mathbb{Z}_{\geq 0}).\) Then

\[
\sum_{i=0}^{2k+1} w^i \mathbb{w} (-w)^{2k+1-i} = \sum_{i=0}^{k} \{ w^i \mathbb{w} (-w)^{2k+1-i} + w^{2k+1-i} \mathbb{w} (-w)^i \}
\]

\[
= \sum_{i=0}^{k} \{(-1)^{2k+1-i} + (-1)^i\} w^i \mathbb{w} w^{2k+1-i}
\]

\[= 0.\]

\[\square\]

The shuffle automaton of \((x^2y)^* \mathbb{w} (-x^2y)^*\) is represented as

![Diagram](image-url)
The adjacency matrix is

\[ A = \begin{bmatrix} 0 & P_1 & 0 \\ 0 & 0 & P_2 \\ P_3 & 0 & 0 \end{bmatrix} \]  \hspace{1cm} (4.14)

where

\[ P_1 = \begin{bmatrix} x & x & 0 \\ 0 & y & x \\ -y & 0 & x \end{bmatrix}, \quad P_2 = \begin{bmatrix} x & x & 0 \\ 0 & x & x \\ -y & 0 & y \end{bmatrix}, \quad P_3 = \begin{bmatrix} y & x & 0 \\ 0 & 0 & x \\ -y & 0 & x \end{bmatrix}. \]

From Lemma 4.4 and Proposition 2.2, the words accepted by this automaton are \( a_{11}^{(2n)} \) \((n \in \mathbb{Z}_{\geq 0})\). Because of the form of \( A \ (4.14) \), it is the \((1, 1)\)-entry of \( \{P_1 P_2 P_3\}^{2n} \). Compute \( P := \{P_1 P_2 P_3\}^2 \):

\[ P = \left\{ \begin{bmatrix} x & x & 0 \\ 0 & y & x \\ -y & 0 & x \end{bmatrix}, \begin{bmatrix} x & x & 0 \\ 0 & x & x \\ -y & 0 & y \end{bmatrix}, \begin{bmatrix} y & x & 0 \\ 0 & 0 & x \\ -y & 0 & x \end{bmatrix} \right\}^2 = \begin{bmatrix} p & q & -q \\ r & s & t \\ -r & t & s \end{bmatrix} \] \hspace{1cm} (4.15)

where

\[ \begin{cases} p = -3(4x^4y^2 + 2x^3yxy), \\ q = -3(2x^3y + x^3yxy), \\ r = -3(2xy^3y^2 + xy^2xxy), \\ s = -3(2xy^2x^3 + xyxyx^3 + yx^3yx + xyxyx^2 + yx^2yx^2), \\ t = -3(2xy^2x^3 + xyxyx^3 - yx^3yx + xyxyx^2). \end{cases} \] \hspace{1cm} (4.16)

**Lemma 4.6.** There exist elements \( p_n, q_n, r_n, s_n, t_n \in \mathcal{S} \) such that

\[
   P^n = \begin{bmatrix} p_n & q_n & -q_n \\ r_n & s_n & t_n \\ -r_n & t_n & s_n \end{bmatrix}.
\]

**Proof.** From (4.15), this statement is true for \( n = 1 \). Now compute \( P^2 \):

\[
P^2 = \begin{bmatrix} p^2 + 2qr & pq + q(s - t) & -\{pq + q(s - t)\} \\ rp + (s - t)r & rq + s^2 + t^2 & -rq + st + ts \\ -\{rp + (s - t)r\} & -rq + st + ts & rq + s^2 + t^2 \end{bmatrix}.
\]

**Hence it is represented as**

\[
   \begin{bmatrix} p_2 & q_2 & -q_2 \\ r_2 & s_2 & t_2 \\ -r_2 & t_2 & s_2 \end{bmatrix}.
\]

For \( n \geq 3 \), it is proved by induction. \( \square \)

From this lemma and \( P^n = P^{n-1} P \), we have the following recursive equations for \( p_n, q_n \):

\[
   \begin{cases}
   p_n = -6p_{n-1}(2x^4y^2 + x^3yxy) - 6q_{n-1}y(2x^3y^2 + x^2yxy), \\
   q_n = -3p_{n-1}(2x^4y^2 + x^3y^2) - 3q_{n-1}y(2x^3yx + x^2yx^2).
   \end{cases} \hspace{1cm} (4.17)
\]
Set \( w_n = (x^4y^2)^n \) and \( w'_n = (x^4y^2)^n x^4y(x_0 = w'_0 = 1) \). We define the operations \( \sigma_i, \tau_j \) by

\[
\begin{align*}
\sigma_i(w_n^{(\epsilon_i)}) &= w_{i-1}x^3yxyw_{n-i}^{(\epsilon_i)} \quad (1 \leq i \leq n-1), \quad \sigma_n(w_n') = w_{n-1}x^3yxx \\
\tau_j(w_n^{(\epsilon_j)}) &= w_{j-1}x^3yxyx^3y^2w_{n-j}^{(\epsilon_j)} \quad (1 \leq j \leq n-2) \quad \tau_{n-1}(w_n') = w_{n-2}x^4yxyx.
\end{align*}
\] (4.18)

where the underlines designate the position of exchanging the order of \( x \) and \( y \). Solving the recursive equations (4.17) in terms of these operations, we have the following proposition:

**Proposition 4.7.**

\[
\begin{align*}
p_n &= \sum_{\epsilon_1, \ldots, \epsilon_n = 0, 1, \epsilon_1', \ldots, \epsilon_{n-1}'} \frac{(-12)^n}{2^{\epsilon_1 + \cdots + \epsilon_n + \epsilon_1' + \cdots + \epsilon_{n-1}'}} \sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n} \tau_1^{\epsilon_1'} \cdots \tau_{n-1}^{\epsilon_{n-1}'}(w_n), \\
q_n &= \sum_{\epsilon_1, \ldots, \epsilon_n = 0, 1, \epsilon_1', \ldots, \epsilon_{n-1}'} \frac{(-12)^n}{2^{\epsilon_1 + \cdots + \epsilon_n + \epsilon_1' + \cdots + \epsilon_{n-1}'}} \sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n} \tau_1^{\epsilon_1'} \cdots \tau_{n-1}^{\epsilon_{n-1}'}(w_n').
\end{align*}
\] (4.19) (4.20)

**Proof.** We prove by induction. For \( n = 1 \), we have

\[
p_1 = -12 x^4 y^2 - 6 x^3 yxy = -12\left(x^4 y^2 + \frac{1}{2} x^4 y^2\right),
\]
\[
q_1 = -6 x^4 yx - 3 x^3 yx^2 = -12\left(\frac{1}{2} x^4 yx + \frac{1}{4} x^4 yx\right).
\]

So the statement is true. Next assume (4.19) to be true for \( p_n, q_n \). As for \( p_{n+1} \), from (4.19), it follows that

\[
p_{n+1} = -12 p_n x^4 y^2 - 6 p_n x^3 yxy - 12 q_n x y^3 y^2 - 6 q_n x y^2 yxy = \sum_{\epsilon_1, \ldots, \epsilon_n = 0, 1, \epsilon_1', \ldots, \epsilon_{n-1}'} \frac{(-12)^{n+1}}{2^{\epsilon_1 + \cdots + \epsilon_n + \epsilon_1' + \cdots + \epsilon_{n-1}'}} \sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n} \tau_1^{\epsilon_1'} \cdots \tau_{n-1}^{\epsilon_{n-1}'}(w_n)^n + 1
\]

\[
+ \sum_{\epsilon_1, \ldots, \epsilon_n = 0, 1, \epsilon_1', \ldots, \epsilon_{n-1}'} \frac{(-12)^{n+1}}{2^{\epsilon_1 + \cdots + \epsilon_n + \epsilon_1' + \cdots + \epsilon_{n-1}'}} \sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n} \tau_1^{\epsilon_1'} \cdots \tau_{n-1}^{\epsilon_{n-1}'} \sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n} \tau_1^{\epsilon_1'} \cdots \tau_{n-1}^{\epsilon_{n-1}'}(w_n)^n + 1
\]

\[
+ \sum_{\epsilon_1, \ldots, \epsilon_n = 0, 1, \epsilon_1', \ldots, \epsilon_{n-1}'} \frac{(-12)^{n+1}}{2^{\epsilon_1 + \cdots + \epsilon_n + \epsilon_1' + \cdots + \epsilon_{n-1}'}} \sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n} \tau_1^{\epsilon_1'} \cdots \tau_{n-1}^{\epsilon_{n-1}'} \sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n} \tau_1^{\epsilon_1'} \cdots \tau_{n-1}^{\epsilon_{n-1}'} \sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n} \tau_1^{\epsilon_1'} \cdots \tau_{n-1}^{\epsilon_{n-1}'}(w_n)^n + 1
\]

\[
= \sum_{\epsilon_1, \ldots, \epsilon_n = 0, 1, \epsilon_1', \ldots, \epsilon_{n+1}'} \frac{(-12)^{n+1}}{2^{\epsilon_1 + \cdots + \epsilon_n + \epsilon_1' + \cdots + \epsilon_{n+1}'}} \sigma_1^{\epsilon_1} \cdots \sigma_n^{\epsilon_n} \tau_1^{\epsilon_1'} \cdots \tau_{n+1}^{\epsilon_{n+1}'}(w_{n+1}).
\]

Hence (4.19) holds for \( p_{n+1} \). The case for \( q_{n+1} \) is proved in a similar way.

**Proof of Theorem 4.4.** Translating the RHS of (4.19) via \( Z \), we obtain the LHS (4.12). By the finite double shuffle relation Proposition 3.7 and the formula (3.10) (Corollary 3.5), we have

\[
Z(\text{the sum of the terms of } (x^2y)^* w (-x^2y)^* \text{ of length } 6n) = \zeta(\{6\}_n).
\]
Furthermore, we have the formula (see Appendix)

\[ \zeta(\{6\}) = \frac{6(2\pi)^6}{(6n+3)!}. \]  

Thus the proof is completed. \[\Box\]

4.3 The shuffle automaton of \((xy)^* w (\omega xy)^* w (\omega^2 xy)^*\)

Let \(\omega\) be a primitive cubic root of unity. Through consideration on the shuffle automaton of \((xy)^* w (\omega xy)^* w (\omega^2 xy)^*\),

Theorem 4.8. We have

\[
\sum_{\varepsilon_0 = 0, 1, \ldots, \varepsilon_n = 0, 1} 36^n \frac{\zeta(\{4 - \varepsilon_1 - \varepsilon_{n-1}, 1 + \varepsilon_1, 1 + \varepsilon_{n}\})}{(6n+3)!} = \frac{6(2\pi)^6}{(6n+3)!}.
\]

where \(\varepsilon_0 = \varepsilon_n = 0\).

The transition diagram and the adjacency matrix of this shuffle automaton are as follows:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & x & x & x & 0 \\
0 & 0 & 0 & 0 & \omega y & y & 0 & x \\
0 & 0 & 0 & 0 & \omega^2 y & 0 & y & x \\
y & x & x & 0 & 0 & 0 & 0 & 0 \\
\omega y & x & 0 & x & 0 & 0 & 0 & 0 \\
\omega^2 y & 0 & x & x & 0 & 0 & 0 & 0 \\
0 & \omega^2 y & \omega y & y & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Let \(A^n = (a_{ij}^{(n)})\). The following lemma is an analogy of Lemma 4.5.

Lemma 4.9. Let \(w \in \mathcal{F}\) be a word of length \(l\). Let \(A = (a_{ij})\) be the adjacency matrix of the shuffle automaton of \(w^* w (\omega w)^* w (\omega^2 w)^*\) and \(A^n = (a_{ij}^{(n)})\). Then

\(n \not\in 3\mathbb{Z} \implies a_{11}^{(n)} = 0.\)
Proof. Let us show $a_{11}^{(l(3m+1))} = 0$:

$$
d_{11}^{(l(3m+1))} = \sum_{i_1+i_2+i_3=3m+1} w^{i_1} \omega w^{i_2} \omega^2 w^{i_3} = \sum_{i_1+i_2+i_3=3m+1} \omega^{i_2+2i_3} w^{i_1} w w^{i_3} = \sum_{j_1+j_2+j_3=3m+1, j_1 > j_2 > j_3} \alpha(\omega) w^{j_1} w w^{j_2} w w^{j_3} + \sum_{2j_1+j_2=3m+1, j_1 \neq j_2} \beta(\omega) w^{j_1} w w^{j_2},$$

where

$$\begin{align*}
\alpha(\omega) &= \omega^{2j_1+j_2} + \omega^{2j_1+j_3} + \omega^{2j_2+j_1} + \omega^{2j_2+j_3} + \omega^{2j_3+j_1} + \omega^{2j_3+j_2}, \\
\beta(\omega) &= \omega^{j_1+2j_2} + \omega^{j_1+2j_3} + \omega^{j_2+2j_1}.
\end{align*}$$

One can easily show $\alpha(\omega) = \beta(\omega) = 0$ by noting that $\omega^{j_1+j_2+j_3-1} = 1$ for $j_1 + j_2 + j_3 = 3m + 1$ and $\omega^{2j_1+j_2-1} = 1$ for $2j_1 + j_2 = 3m + 1$. Thus $a_{11}^{(l(3m+1))} = 0$.

One can prove $a_{11}^{(l(3m+2))} = 0$ in a similar way. \( \square \)

From this lemma and the form of $A \{124\}$ we see that $a_{11}^{(n)} = 0$ ($n \notin 6\mathbb{Z}$) and $a_{11}^{(6n)}$ is the $(1,1)$-entry of $\hat{P}^n$,

$$\hat{P} = \begin{pmatrix}
    x & x & x & 0 \\
    \omega y & y & 0 & x \\
    0 & \omega^2 y & \omega y & x \\
    0 & 0 & 0 & \omega y
\end{pmatrix}^3$$

$$= \begin{pmatrix}
    p & -\omega q & -\omega^2 q & -q \\
    -\omega^2 r & s + t & s + t' & s + t'' \\
    -\omega r & s + t'' & s + t & s + t'' \\
    -r & s + t'' & s + t & s + t''
\end{pmatrix}.$$  \hspace{1cm} (4.25)

where

$$\begin{align*}
p &= 12(3x^3y^3 + x^2yxy^2) & t &= 4yxy^2x + 4yx^2y^2x + 4xyxyx \\
q &= 12x^3y^2x + 4x^2yxyx & t' &= 4\omega xyxy^2 + 4\omega^2 y^2 xy^2 x \\
r &= 12\omega x^3 y^2 x + 4\omega xy^2 xy^2 & t'' &= 4\omega^2 y^2 xy^2 y + 4\omega^2 y^2 x^2 y^2 x \\
s &= 12(y^3x^2 y x + xy^3 y x) + 4(\omega y^2 y x y x + \omega y^2 y x y x)
\end{align*}$$

(4.26)

By induction, one can show

Lemma 4.10. There exist $p_n, q_n, r_n, s_n, t_n, t'_n, t''_n \in \mathbb{N}$ such that

$$\hat{P}^n = \begin{pmatrix}
p_n & -\omega q_n & -\omega^2 q_n & -q_n \\
-\omega^2 r_n & s_n + t_n & s_n + t''_n & s_n + t'_n \\
-\omega r_n & s_n + t''_n & s_n + t_n & s_n + t''_n \\
-r_n & s_n + t''_n & s_n + t'_n & s_n + t''_n
\end{pmatrix}.$$  \hspace{1cm} (4.27)

The elements $p_n, q_n$ satisfy the recursive relations

$$\begin{align*}
p_n &= 36 \left\{ p_{n-1} \left( x^3 y^3 + \frac{1}{3} x^2 y^2 y^2 + q_{n-1} \left( y^2 x^2 y^2 + \frac{1}{3} y x y x y x \right) \right) \right\}, \\
q_n &= 36 \left\{ \omega q_{n-1} q_{n-1} \left( \frac{1}{2} x^3 y^2 x + \frac{1}{3} x^2 y^2 y x y x \right) + q_{n-1} \left( \frac{1}{2} y x^2 y^2 x + \frac{1}{3} x y y x y x \right) \right\}.
\end{align*}$$

(4.27)

Set $w_n = (x^3 y^3)^n$ and $w'_n = (x^3 y^3)^{n-1} - x^3 y^2 x$ ($w_0 = w'_0 = 1$). Let us define the operations $\sigma_i, \tau_j$ by

$$\begin{align*}
\sigma_i(w'_n) &= w_{i-1} x^i y^i y^i y^i w_{n-i} \quad (1 \leq i \leq n - 1), & \sigma_n(w'_n) &= w_{n-1} x^2 y x y \\
\tau_j(w'_n) &= w_{j-1} x^j y^j y^j y^j w_{n-1-j} \quad (1 \leq j \leq n - 2), & \tau_{n-1}(w'_n) &= w_{n-2} x^3 y^2 y x y x y x y x y x y x y x y x.
\end{align*}$$

(4.28)
Similarly as in the previous subsection, one can show

**Proposition 4.11.** We have

\[ p_n = \sum_{\varepsilon_1, \ldots, \varepsilon_n = 0,1} \frac{(36)^n}{3^{1+\cdots+\varepsilon_n+\varepsilon'_1+\cdots+\varepsilon'_{n-1}}} \varepsilon_1 \cdots \varepsilon_n \tau_1' \cdots \tau_{n-1}' (w_n), \tag{4.29} \]

\[ q_n = \sum_{\varepsilon_1, \ldots, \varepsilon_n = 0,1} \frac{(36)^n}{3^{1+\cdots+\varepsilon_n+\varepsilon'_1+\cdots+\varepsilon'_{n-1}}} \varepsilon_1 \cdots \varepsilon_n \tau_1' \cdots \tau_{n-1}' (w'_n). \tag{4.30} \]

**Proof of Theorem 4.8.** Transforming (4.29) via \( Z \) to MZVs yields the LHS of (4.22). From Proposition 3.1 and Theorem 3.2, it follows that

\[
\begin{align*}
Z ((xy)^* \omega (\omega xy)^*) &= Z ((xy)^* (\omega xy)^* (\omega^2 xy)^*) \\
&= Z ((x^5 y)^*) \\
&= \text{the RHS of (4.22)}.
\end{align*}
\]

\[ \Box \]

## 5 Harmonic automata

**Theorem 5.1.** The automaton accepting the harmonic product \( w_1 \ast w_2 \) where \( w_1 = z_{p_1} \cdots z_{p_m} \) and \( w_2 = z_{q_1} z_{q_2} \cdots z_{q_n} \) is represented by the following transition diagram (5.1). We call this the harmonic automaton of \( w_1 \ast w_2 \):

![Diagram](5.1)

**Proof.** In the definition of harmonic product

\[
\begin{align*}
z_{p_1} z_{p_2} \cdots z_{p_m} \ast z_{q_1} z_{q_2} \cdots z_{q_n} &= z_{p_1} (z_{p_2} \cdots z_{p_m} \ast z_{q_1} z_{q_2} \cdots z_{q_n}) + z_{q_1} (z_{p_1} z_{p_2} \cdots z_{p_m} \ast z_{q_2} \cdots z_{q_n}) \\
&+ z_{p_1+q_1} (z_{p_2} \cdots z_{p_m} \ast z_{q_2} \cdots z_{q_n}),
\end{align*}
\]

each term designates

(1) the first term: the transition from the state \( q_1 \) to the state \( q_2 \) inputting \( z_{p_1} \),

(2) the second term: the transition from the state \( q_1 \) to the state \( q_{(m+1)+1} \) inputting \( z_{q_1} \),

(3) the third term: the transition from the state \( q_1 \) to the state \( q_{(m+1)+2} \) inputing \( z_{p_1+q_1} \).

Thus we have (5.1). \[ \Box \]
Corollary 5.2. The harmonic automaton of $w_1^* \ast w_2^*$ is represented by the following transition diagram:

Proof of Theorem 5.2. We prove the case of $m = 3$: Let $\omega$ be a primitive cubic root of unity. The harmonic automaton representing $z_k^* \ast (\omega z_k)^*$ is as follows:

Hence we have $z_k^* \ast (\omega z_k)^* = ((1 + \omega)z_k + \omega z_{2k})^*$. The harmonic product of $(z_k^* \ast (\omega z_k)^*) \ast \omega^2 z_k$ is represented by

Therefore the element accepted by this automaton is $(z_{3k})^*$.

A On the value of $\zeta(\{2k\}_n)$

Let $\omega = \omega_{2k}$ be a primitive $2n$-th root of unity. From

$$\frac{\sin \pi x}{\pi x} = \prod_{r=1}^{\infty} \left(1 - \frac{x^2}{r^2}\right),$$

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we have
\[
\frac{\sin \pi x \sin \pi \omega x \cdots \sin \pi \omega^{k-1} x}{\pi^k \omega^k (k-1)/2 x^k} = \prod_{r=1}^{\infty} \left( 1 - \frac{x^{2k}}{r^{2k}} \right). \tag{A.1}
\]

The Taylor expansion of the RHS above is
\[
\sum_{n=0}^{\infty} \zeta \{(2k)_n\} (-x^{2k})^n,
\]
so it is easy to see
\[
\zeta \{(2)_n\} = \frac{\pi^{2n}}{(2n+1)!}, \quad \zeta \{(4)_n\} = \frac{2^{2n+1} \pi^{4n}}{(4n+2)!}.
\tag{A.3}
\]

Comparing the Taylor expansion of the LHS
\[
\frac{1}{(2i)^k \pi^k \omega^k (k-1)/2 x^k} \sum_{\varepsilon_0 \cdots \varepsilon_{k-1} = \pm 1} \varepsilon_0 \cdots \varepsilon_{k-1} e^{\pi i x (\varepsilon_0 + \varepsilon_1 \omega + \cdots + \varepsilon_{k-1} \omega^{k-1})}
\]
with \(\zeta \{(2k)_n\}\) we obtain the following proposition:

**Proposition A.1.** We have
\[
\zeta \{(2k)_n\} = \frac{(-1)^{(k+1)n} \pi^{2kn}}{2 \omega^k (k-1)/2 (2kn+k)!} \sum_{\varepsilon_0 \cdots \varepsilon_{k-1} = \pm 1} \varepsilon_0 \cdots \varepsilon_{k-1} (\varepsilon_0 + \varepsilon_1 \omega + \cdots + \varepsilon_{k-1} \omega^{k-1})^{2kn+k}. \tag{A.4}
\]

From this proposition, as the explicit formula for \(\zeta \{(2k)_n\}\) \((k = 3, 4, 5, 6)\), we have
\[
\zeta \{(6)_n\} = \frac{6(2\pi)^6}{(6n+3)!}, \tag{A.5}
\]
\[
\zeta \{(8)_n\} = \frac{2^{6n+2} \pi^{8n} \{3 + 2 \sqrt{2} \}^{2n+1} + (3 - 2 \sqrt{2})^{2n+1}}{(8n+4)!}, \tag{A.6}
\]
\[
\zeta \{(10)_n\} = \frac{5 \cdot 2^{8n} \pi^{10n} \{2^{2n+1} + (11 + 5 \sqrt{5})^{2n+1} + (11 - 5 \sqrt{5})^{2n+1}\}}{(10n+5)!}, \tag{A.7}
\]
\[
\zeta \{(12)_n\} = \frac{3 \cdot 2^{12n+2} \pi^{12n} \{2^{6n+3} + (26 + 15 \sqrt{3})^{2n+1} + (26 - 15 \sqrt{3})^{2n+1}\}}{(12n+6)!}. \tag{A.8}
\]

These formulas reflect the fact that \(\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\omega_8)\), \(\mathbb{Q}(\sqrt{5}) \subset \mathbb{Q}(\omega_{10})\), \(\mathbb{Q}(\sqrt{3}) \subset \mathbb{Q}(\omega_{12})\). To represent \(\zeta \{(14)_n\}\), we need the imaginary quadratic field \(\mathbb{Q}(\sqrt{-7})\) and the cubic field \(\mathbb{Q}(2 \cos \frac{2\pi}{14})\) which are subfields of \(\mathbb{Q}(\omega_{14})\): Let \(\alpha_1, \beta_1\) be the roots of the quadratic equation
\[
\lambda^2 - 13\lambda + 128 = 0,
\]
and \(\alpha_2, \beta_2, \gamma_2\) be the roots of the cubic equation
\[
\lambda^3 - 57\lambda^2 + 103\lambda - 1 = 0.
\]

Then
\[
\zeta \{(14)_n\} = \frac{7 \cdot 2^{14n+1} \pi^{14n}}{(14n+7)!} \left\{1 + \alpha_1^{2n+1} + \beta_1^{2n+1} + \alpha_2^{2n+1} + \beta_2^{2n+1} + \gamma_2^{2n+1}
+ \alpha_2^{-2n-1} + \beta_2^{-2n-1} + \gamma_2^{-2n-1}\right\}. \tag{A.9}
\]
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