A SHARP INEQUALITY FOR THE STRICHTARZ NORM

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Abstract. Let $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ be the solution of the linear Schrödinger equation
\[
\begin{cases}
iu_t + \Delta u = 0 \\
u(0, x) = f(x).
\end{cases}
\]
In the first part of this paper we obtain a sharp inequality for the Strichartz norm $\|u(t, x)\|_{L^2_t L^2_k(R \times \mathbb{R}^n)}$, where $k \in \mathbb{Z}$, $k \geq 2$ and $(n, k) \neq (1, 2)$, that admits only Gaussian maximizers. As corollaries we obtain sharp forms of the classical Strichartz inequalities in low dimensions (works of Foschi [4] and Hundertmark-Zharnitsky [6]) and also sharp forms of some Sobolev-Strichartz inequalities. In the second part of the paper we express Foschi’s [4] sharp inequalities for the Schrödinger and wave equations in the broader setting of sharp restriction/extension estimates for the paraboloid and the cone.

1. Introduction

Let $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ be the solution of the linear Schrödinger equation
\[
\begin{cases}
iu_t + \Delta u = 0 \\
u(0, x) = f(x).
\end{cases}
\] (1.1)
The homogeneous Strichartz estimates (see [3]) are inequalities of the type
\[
\|u(t, x)\|_{L^q_t L^r_x(R \times \mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)},
\] (1.2)
with
\[
\|u(t, x)\|_{L^q_t L^r_x(R \times \mathbb{R}^n)} = \left[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} |u(t, x)|^r \, dx \right)^{q/r} \, dt \right]^{1/q}.
\]
The pair of exponents $(q, r)$ is admissible if
\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2},
\]
with $2 \leq q, r \leq \infty$ and $(q, r, n) \neq (2, \infty, 2)$. The sharp forms of the Strichartz inequalities were first investigated in a paper by Kunze [7], who showed the existence of maximizers in the case $n = 1$, $(q, r) = (6, 6)$, by concentration-compactness techniques. Later, Foschi [4] and Hundertmark-Zharnitsky [6] independently obtained the sharp constants in the cases $n = 1$, $(q, r) = (6, 6)$; and $n = 2$, $(q, r) = (4, 4)$; showing that the only maximizers are Gaussians. They conjectured that in the case $q = r = 2 + 4/n$, $n \geq 3$, the extremals for the Strichartz inequalities should be given by Gaussians. Recently, Shao [9] showed that maximizers do exist for the

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In dimension $n$ of the classical Strichartz inequalities in low dimensions.

We observe that the solution of (1.1) can be given in terms of the Fourier transform
\[ f(t, x) = \mathcal{F}(\hat{f}(\eta)) = \int_{\mathbb{R}^n} \hat{f}(\eta) e^{i\eta \cdot x} d\eta. \]

The maximizers in Theorem 1 should be understood in the following way: if $\hat{f}(\eta)$ occurs in (1.3), then $f(\eta_1, \eta_2, ..., \eta_k)$ must be a Gaussian, and so is $\mathcal{F}(\hat{f}(\eta))$, where $f$ is a Gaussian.

Throughout this paper we will adopt the definition of the Fourier transform of the function $f : \mathbb{R}^n \to \mathbb{C}$ given by
\[ \hat{f}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\omega \cdot x} f(x) dx. \]

We observe that the solution of (1.1) can be given in terms of the Fourier transform
\[ u(t, x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \omega} e^{-it|\omega|^2} \hat{f}(\omega) d\omega. \]

The maximizers in Theorem 1 should be understood in the following way: if $\hat{f}$ is a measurable function such that the right hand side of (1.3) is finite, and equality occurs in (1.3), then $\hat{f}$ must be a Gaussian, and so is $f$. Here we shall always refer as Gaussians the functions of the form
\[ f(x) = e^{A|x|^2 + b \cdot x + C}, \]
where $A, C \in \mathbb{C}$, $b \in \mathbb{C}^n$ and $\Re(A) < 0$. The term $A$ is the covariance of the Gaussian $f$.

Some interesting inequalities arise from Theorem 1. First, we present the sharp forms of the classical Strichartz inequalities in low dimensions.

**Corollary 2.** In dimension $n = 1$ we have
\[ \|u(t, x)\|_{L^4_tL^8_x(\mathbb{R} \times \mathbb{R})} \leq 12^{-1/12} \|f\|_{L^2(\mathbb{R})}, \]
and
\[ \|u(t, x)\|_{L^4_tL^4_x(\mathbb{R} \times \mathbb{R})} \leq 2^{-1/4} \|f\|_{L^2(\mathbb{R})}. \]

In dimension $n = 2$ we have
\[ \|u(t, x)\|_{L^4_tL^4_x(\mathbb{R} \times \mathbb{R}^2)} \leq 2^{-1/2} \|f\|_{L^2(\mathbb{R}^2)}. \]

These inequalities are sharp and equality occurs if and only if $f$ is a Gaussian.
In dimension $n$ and the equivalence of decay inequalities for the space-time norm of the solutions of certain evolution equations and restriction estimates for the Fourier transform over curved surfaces. The classical reference on the subject is Strichartz original paper [8], but seminal ideas can already be observed in the work of Hörmander [5, 6]. They are a direct consequence of Theorem 1. The persistence of the Gaussian maximizers in a case where $q \neq r$.

By using the fact that
\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} g(x) g(y) x \cdot y \, dx \, dy \geq 0,
\]
for any real valued function $g$, with equality for example if $g$ is radial, one obtains some sharp Sobolev-Strichartz inequalities in low dimensions.

**Corollary 3.** In dimension $n = 1$ we have
\[
\|u(t,x)\|_{L_t^{10} L_x^6(\mathbb{R} \times \mathbb{R})} \leq (2\sqrt{3\pi})^{-1/10} \|f\|_{L_t^1 L_x^2(\mathbb{R})}^{1/5} \|f\|_{L_t^4 L_x^2(\mathbb{R})}^{4/5}, \tag{1.11}
\]
\[
\|u(t,x)\|_{L_t^{12} L_x^6(\mathbb{R} \times \mathbb{R})} \leq (6\pi)^{-1/12} \|f\|_{L_t^4 L_x^2(\mathbb{R})}^{1/6} \|f\|_{L_t^2 L_x^2(\mathbb{R})}^{5/6}, \tag{1.12}
\]
and
\[
\|u(t,x)\|_{L_t^{16} L_x^4(\mathbb{R} \times \mathbb{R})} \leq (8\pi)^{-1/16} \|f\|_{L_t^8 L_x^2(\mathbb{R})}^{1/8} \|f\|_{L_t^2 L_x^2(\mathbb{R})}^{7/8}. \tag{1.13}
\]

In dimension $n = 2$ we have
\[
\|u(t,x)\|_{L_t^{12} L_x^5(\mathbb{R} \times \mathbb{R}^2)} \leq (12\pi)^{-1/6} \|\nabla f\|_{L_t^3 L_x^2(\mathbb{R}^2)}^{1/3} \|f\|_{L_t^3 L_x^2(\mathbb{R}^2)}^{2/3}, \tag{1.14}
\]
and
\[
\|u(t,x)\|_{L_t^4 L_x^4(\mathbb{R} \times \mathbb{R}^2)} \leq (16\pi)^{-1/8} \|\nabla f\|_{L_t^4 L_x^2(\mathbb{R}^2)}^{1/4} \|f\|_{L_t^3 L_x^2(\mathbb{R}^2)}^{3/4}. \tag{1.15}
\]

In dimension $n = 4$ we have
\[
\|u(t,x)\|_{L_t^4 L_x^4(\mathbb{R} \times \mathbb{R}^4)} \leq (32\pi)^{-1/4} \|\nabla f\|_{L_t^4 L_x^2(\mathbb{R}^4)}^{1/2} \|f\|_{L_t^4 L_x^2(\mathbb{R}^4)}^{1/2}. \tag{1.16}
\]
These inequalities are sharp and equality occurs if and only if $f$ is a Gaussian.

Inequalities (1.11), (1.13) and (1.16) follow directly from Theorem 1 and (1.10). To obtain (1.12) and (1.14) one should put $f(x,y) = g(x)g(y)$ in (1.14) and (1.15), respectively, and exploit the product structure. In an analogous manner one obtains (1.15) by putting $f(x,y,z,k) = g(x,y)g(z,k)$ in (1.16).

1.1. **Sharp restriction/extension estimates.** It has been known for a long time the equivalence of decay inequalities for the space-time norm of the solutions of certain evolution equations and restriction estimates for the Fourier transform over curved surfaces. The classical reference on the subject is Strichartz original paper [9], but seminal ideas can already be observed in the work of Hörmander [5, Corollary 1.3].

The Schrödinger and wave equations are related to the restriction problem for the paraboloid and cone, respectively,
\[
S_{\text{parab}} := \{(\tau,\omega) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\omega|^2\}, \tag{1.17}
\]
and
\[
S_{\text{cone}} := \{(\tau,\omega) \in \mathbb{R} \times \mathbb{R}^n : \tau = |\omega|\}. \tag{1.18}
\]
We endow these surfaces $S \subset \mathbb{R}^{n+1}$ with canonical measures $d\sigma$ given by
\[
\int_{S_{\text{parab}}} g(\tau,\omega) \, d\sigma = \int_{\mathbb{R}^n} g(|\omega|^2,\omega) \, d\omega, \tag{1.19}
\]
and
\[ \int_{S_{\text{cone}}} g(\tau, \omega) \, d\sigma = \int_{\mathbb{R}^n} g(|\omega|, \omega) \frac{d\omega}{|\omega|}. \quad (1.20) \]

In this setting, the restriction estimates are a priori inequalities of the form
\[ \|\hat{h} S\|_{L^{p'}(S; d\sigma)} \leq C_{p, q, S} \|h\|_{L^p(\mathbb{R}^{n+1})}, \quad (1.21) \]

The scaling invariance tells us that the global estimate (1.21) can only hold for \( p' = nq/(n+2) \) in the case of the paraboloid and \( p' = (n-1)q/(n+1) \) in the case of the cone. On the other hand, Knapp’s example shows that we must have \( q > (2n+2)/n \) for the paraboloid and \( q > 2n/(n-1) \) for the cone. The restriction conjecture asserts that these are sufficient conditions in each case for (1.21) to hold, and so far it has been proved for the range \( q > (2n+6)/(n+1) \) in both cases, the paraboloid by Tao [11] and the cone by Wolff [12]. We refer the reader to [10] for a survey on the recent progress on the restriction conjecture.

A duality argument using Parseval’s identity shows that
\[ C_{p, q, S} = \sup_{\|h\|_{L^{p'}(\mathbb{R}^{n+1})} = 1} \|\hat{h} S\|_{L^{p'}(S; d\sigma)} \]

\[ = \sup_{\|h\|_{L^{p'}(\mathbb{R}^{n+1})} = 1} \sup_{\|g\|_{L^p(S; d\sigma)} = 1} \left| \int_S \hat{h}(\tau, \omega) g(\tau, \omega) \, d\sigma \right| \]

\[ = \sup_{\|g\|_{L^p(S; d\sigma)} = 1} \sup_{\|h\|_{L^{p'}(\mathbb{R}^{n+1})} = 1} \left| \int_{\mathbb{R}^{n+1}} h(t, x) \hat{g}(t, x) \, dt \, dx \right| \]

\[ = \sup_{\|g\|_{L^p(S; d\sigma)} = 1} \|\hat{g} d\sigma\|_{L^q(\mathbb{R}^{n+1})}. \quad (1.22) \]

Therefore (1.21) is equivalent to the extension estimate
\[ \|\hat{g} d\sigma\|_{L^q(\mathbb{R}^{n+1})} \leq C_{p, q, S} \|g\|_{L^p(S; d\sigma)}, \quad (1.23) \]

for all smooth functions \( g \) on \( S \), where \( \hat{g} d\sigma \) is the Fourier transform of the measure \( g d\sigma \):
\[ \hat{(g d\sigma)}(t, x) := \frac{1}{(2\pi)^{n+1}/2} \int_S g(\tau, \omega) e^{-i(t\tau + \omega x)} \, d\sigma. \]

In the case of the paraboloid, from (1.15) we see that the solution of the Schrödinger equation (1.1) satisfies
\[ u(t, -x) = (2\pi)^{1/2} \hat{g} d\sigma(t, x), \]

with \( g(|\omega|^2, \omega) = \hat{f}(\omega) \). Therefore, (1.23) is equivalent to the inequality
\[ \|u(t, x)\|_{L^q_t L^2_x(\mathbb{R} \times \mathbb{R}^n)} \leq (2\pi)^{1/2} C_{p, q, S} \|\hat{f}\|_{L^{p'}(\mathbb{R}^n)}. \quad (1.24) \]

From the equivalence of (1.21), (1.23) and (1.24), the sharp forms (1.7) and (1.9) discovered by Foschi [4] and Hundertmark-Zharntitsky [6] immediately translate into sharp restriction/extension estimates for the paraboloid.

**Theorem 4.** Let \( S \) be the paraboloid defined in (1.17) endowed with the measure \( d\sigma \) defined in (1.19). We have
\[ \|\hat{g} d\sigma\|_{L^q(\mathbb{R}^2)} \leq (2\pi)^{-1/2} 12^{-1/12} \|g\|_{L^2(S; d\sigma)}, \quad (1.25) \]

and
\[ \|\hat{g} d\sigma\|_{L^q(\mathbb{R}^3)} \leq (4\pi)^{-1/2} \|g\|_{L^2(S; d\sigma)}. \quad (1.26) \]
These inequalities are sharp. Equality occurs in (1.25) and (1.26) if and only if
\[ g(|\omega|^2, \omega) = e^{A|\omega|^2 + b \cdot \omega + C}, \]  
where \( A, C \in \mathbb{C} \), \( b \in \mathbb{C}^n \) and \( \Re(A) < 0 \).

For simplicity, we presented above the sharp extension inequality. One can deduce the dual sharp restriction inequality (1.21) for the paraboloid and find the maximizing functions \( h(t, x) \) by using the condition for equality in the duality argument (1.22) (Hölder’s inequality)
\[ h = C \overline{g d\sigma} \overline{y}^{1/4} g d\sigma, \]  
for a complex constant \( C \) and \( g \) given by (1.27).

In the same spirit, sharp restriction/extension inequalities for the cone are implicit in Foschi’s work [4] for the wave equation.

**Theorem 5.** Let \( S \) be the cone defined in (1.18) endowed with the measure \( d\sigma \) defined in (1.20). We have
\[ \|g d\sigma\|_{L^6(\mathbb{R}^3)} \leq (2\pi)^{1/3} \|g\|_{L^2(S; d\sigma)}, \]  
and
\[ \|g d\sigma\|_{L^4(\mathbb{R}^4)} \leq (2\pi)^{1/4} \|g\|_{L^2(S; d\sigma)}. \]  
These inequalities are sharp. Equality occurs in (1.29) and (1.30) if and only if
\[ g(|\omega|, \omega) = e^{A|\omega|^2 + b \cdot \omega + C}, \]  
where \( A, C \in \mathbb{C} \), \( b \in \mathbb{C}^n \) and \( |\Re(b)| < -\Re(A) \).

We will give a brief proof of Theorem 5 in section 4, indicating the basic changes that have to be made in Foschi’s argument. Again, the maximizers \( h(t, x) \) for the dual restriction inequalities (1.21) can be obtained from the duality condition (1.22) with \( g \) given by (1.27). It would be a very interesting line of research to investigate other sharp constants in the broader setting of restriction/extension estimates and to understand the role that the special functions (1.27) and (1.31) play in these inequalities.

We shall see in this paper that the natural generalization of the argument of Hundertmark-Zharnitsky [6] leads to the inequality in Theorem 1 which maintains the Gaussian maximizers, but is weaker than (1.24). Indeed, one can show that for
\[ q = 2k \quad \text{and} \quad p = \frac{2nk}{2nk - n - 2}, \]  
the following inequality holds
\[ \|\hat{f}\|_{L^p(\mathbb{R}^n)} \leq C \left( \int_{\mathbb{R}^{nk}} |\hat{F}(\eta)|^2 K(\eta)^{\frac{n(k-1)-2}{2}} d\eta \right)^{1/2k}. \]  
This is a consequence of the following three inequalities:

1. A basic inequality for real numbers:
\[ K(\eta) \frac{n(k-1)-2}{2} \geq C \sum_{1 \leq i < j \leq n} |\eta_i - \eta_j|^{n(k-1)-2}; \]
Lemma 6

The heart of the matter is the following representation lemma.

Fix the vectors \( \alpha \) (1.1). Then the Schrödinger equation with the constant invariant under any orthonormal transformation (rotation here for short) \( R \) in the space \( L^2(\mathbb{R}^n) \) to be in \( L^2(\mathbb{R}^n) \), we suppose that \( F(\eta) = f(\eta_1)f(\eta_2)...f(\eta_k) \) and \( K(\eta) = \frac{1}{k}\sum_{1 \leq i < j \leq k} |\eta_i - \eta_j|^2 \). Let us write

\[
F_1(\eta) = \hat{F}(\eta)K(\eta)\frac{n(k-1) - 2}{4}.
\]

In the space \( L^2(\mathbb{R}^{nk}) \), let \( E \) be the closed subspace consisting of the functions invariant under any orthonormal transformation (rotation here for short) \( R \) that fixes the vectors \( \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{R}^{nk} \) given by

\[
\alpha_i = (e_i, e_i, ..., e_i) \quad (k \text{ times}),
\]

where \( e_i = (0, 0, ..., 1, ..., 0) \) is the \( i \)-th canonical vector in \( \mathbb{R}^n \). Denote by \( P_E : L^2(\mathbb{R}^{nk}) \to L^2(\mathbb{R}^{nk}) \) the orthogonal projection operator onto the subspace \( E \). The heart of the matter is the following representation lemma.

Lemma 6 (Representation Lemma). Let \( u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C} \) be the solution of the Schrödinger equation (1.1). Then

\[
\int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} \, dx \, dt = C_{n,k} \langle P_E(F_1), F_1 \rangle_{L^2(\mathbb{R}^{nk})},
\]

with the constant \( C_{n,k} \) defined in (1.32).

(ii) The reversed Hardy-Littlewood-Sobolev inequality due to W. Beckner [1]:

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} |g(x)|^2 |x - y|^\lambda |h(y)| \, dx \, dy \geq C(n, \lambda) \|g\|_{L^{\frac{2n}{2n+\lambda}}(\mathbb{R}^n)} \|h\|_{L^{\frac{2n}{2n+\lambda}}(\mathbb{R}^n)},
\]

where \( \lambda > 0 \), the sharp constant given by

\[
C(n, \lambda) = \pi^{\lambda/2} \frac{\Gamma(n/2 + \lambda/2)}{\Gamma(n + \lambda/2)} \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{1+\lambda/n},
\]

and the only maximizers being \( g(x) = c \, h(x), \ c \in \mathbb{C} \) a constant, and

\[
h(x) = A(B^2 + |x - x_0|^2)^{-(2n+\lambda)/2},
\]

for some \( A \in \mathbb{C}, 0 \neq B \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^n \). For our purposes it suffices to use this inequality in the following format

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} |\hat{f}(\eta_k)|^2 |\hat{f}(\eta_j)|^2 |\eta_k - \eta_j|^{n(k-1) - 2} \, d\eta_k \, d\eta_j \geq C \|\hat{f}\|_{L^r(\mathbb{R}^n)}^4,
\]

where \( r = 4n/(n(k + 1) - 2) \); (iii) Hölder’s inequality:

\[
\|\hat{f}\|_{L^r(\mathbb{R}^n)}^2 \leq \|\hat{f}\|_{L^p(\mathbb{R}^n)}^4 \|\hat{f}\|_{L^2(\mathbb{R}^n)}^{2k-4}.
\]

Inequality (1.32) will be used later in section 3.

2. Proof of Theorem 1 - the sharp inequality

The proof of Theorem 1 given here follows closely the outline of Hundertmark and Zharnitsky [9]. As we are interested in an a priori estimate, in this section we suppose that \( f \in C_0^\infty(\mathbb{R}^n) \). Throughout the proof of Theorem 1 we reserve the variables \( \eta \) and \( \xi \) to be in \( \mathbb{R}^{nk} \) and write \( \eta = (\eta_1, \eta_2, ..., \eta_k) \) with each \( \eta_k \in \mathbb{R}^n \). We have also defined \( F(\eta) = f(\eta_1)f(\eta_2)...f(\eta_k) \) and \( K(\eta) = \frac{1}{k}\sum_{1 \leq i < j \leq k} |\eta_i - \eta_j|^2 \). Let us write

\[
F_1(\eta) = \hat{F}(\eta)K(\eta)\frac{n(k-1) - 2}{4}.
\]
Proof. Using the representation (1.5) for the solution $u(t, x)$ we obtain
\[
|u(t, x)|^{2k} = \frac{1}{(2\pi)^{nk}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} e^{ix \cdot (\sum \eta_i - \sum \xi_i)} e^{-i(t - |\xi|^2)} \hat{F}(\eta) \hat{F}(\xi) \, d\eta \, d\xi,
\]
where $\eta = (\eta_1, \eta_2, ..., \eta_k)$ and $\xi = (\xi_1, \xi_2, ..., \xi_k)$, with each $\eta_i$ and $\xi_i$ in $\mathbb{R}^n$. Integrating with respect to $x$ and $t$ and using that, as distributions, the n-dimensional delta function $\delta_n(w) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix \cdot w} \, dx$, one arrives at
\[
\int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} \, dx \, dt
\]
\[
= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} \delta_n \left( \sum_{i=1}^k \eta_i - \sum_{i=1}^k \xi_i \right) \delta\left(|\eta|^2 - |\xi|^2\right) \hat{F}(\eta) \hat{F}(\xi) \, d\eta \, d\xi
\]
\[
= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} \left( \prod_{i=1}^n \delta\left((\eta - \xi) \cdot \alpha_i\right) \right) \delta\left(|\eta|^2 - |\xi|^2\right) \hat{F}(\eta) \hat{F}(\xi) \, d\eta \, d\xi.
\]

We will rewrite the last equation in the following strategic way
\[
\int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} \, dx \, dt
\]
\[
= \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk} \times \mathbb{R}^{nk}} \left( \prod_{i=1}^n \delta\left((\eta - \xi) \cdot \alpha_i\right) \right) \delta\left(|\eta|^2 - |\xi|^2\right) \hat{F}(\eta) \hat{F}(\xi) \, d\eta \, d\xi.
\]

The insight now is to recognize the last expression as a quadratic form associated to a self-adjoint operator. Indeed, for $G \in C_0^\infty(\mathbb{R}^{nk})$ define the operator
\[
AG(\xi) = \frac{1}{(2\pi)^{n(k-1)-1}} \int_{\mathbb{R}^{nk}} \left( \prod_{i=1}^n \delta\left((\eta - \xi) \cdot \alpha_i\right) \right) \delta\left(|\eta|^2 - |\xi|^2\right) \hat{F}(\eta) \hat{F}(\xi) \, d\eta.
\]

In this context we have
\[
\int_{\mathbb{R} \times \mathbb{R}^n} |u(t, x)|^{2k} \, dx \, dt = \langle AF_1, F_1 \rangle_{L^2(\mathbb{R}^{nk})}.
\]

Our objective is to show that the operator $A$ is a multiple of the projection operator $P_E$. We start by showing that $A$ is a bounded operator in $L^2(\mathbb{R}^{nk})$, via the following lemma.

Lemma 7. (i) For all $\xi \in \mathbb{R}^{nk}$ the measure
\[
m_\xi(d\eta) = \frac{k^{n/2} \Gamma \left( \frac{n(k-1)}{2} \right)}{\pi^{n(k-1)/2}} \left( \prod_{i=1}^n \delta\left((\eta - \xi) \cdot \alpha_i\right) \right) \delta\left(|\eta|^2 - |\xi|^2\right) \frac{d\eta}{(K(\eta)K(\xi))^{n(k-1)-2}}
\]

is a probability measure on $\mathbb{R}^{nk}$.

(ii) For all Borel measurable sets $B \subset \mathbb{R}^{nk}$, we have
\[
\int_{\mathbb{R}^{nk}} m_\xi(B) \, d\xi = |B|,
\]
where $|B|$ denotes the Lebesgue measure of $B$. 
Proof. Throughout this proof let us write
\[ C = \frac{k^{n/2} \Gamma \left( \frac{n(k-1)}{2} \right)}{\pi^{n(k-1)/2}}. \]

Observe that in the support of the delta functions we have \( \sum \eta_k = \sum \xi_i \) and \( |\eta|^2 = |\xi|^2 \). This implies that \( K(\eta) = K(\xi) \), since
\[
K(\eta) = \frac{1}{k} \sum_{1 \leq i < j \leq k} |\eta_i - \eta_j|^2
= |\eta|^2 - \left( \eta_1 + \eta_2 + \ldots + \eta_k \right)^2
= |\eta|^2 - \left( \sum_{j=1}^n (\eta \cdot \alpha_j)\right)^2.
\]

Therefore we have
\[
m_\xi(\mathbb{R}^n) = \frac{C}{K(\xi)} \int_{\mathbb{R}^n} \left( \sum_{j=1}^n \delta((\eta - \xi) \cdot \alpha_j) \right) \delta(|\eta|^2 - |\xi|^2) \, d\eta.
\]

Let \( \{\hat{e}_j\}, 1 \leq j \leq nk \), be the canonical vectors in \( \mathbb{R}^{nk} \). Change the variable \( \eta \) in the integration [24] by a rotation \( R \) that sends \( \alpha_i \) to \( \sqrt{k} \hat{e}_i \) for \( 1 \leq i \leq n \). We obtain
\[
m_\xi(\mathbb{R}^n) = \frac{C}{K(\xi)} \int_{\mathbb{R}^n} \delta_n \left( \sqrt{k} \eta_1 - \sum \xi_i \right) \delta(|\eta|^2 - |\xi|^2) \, d\eta
\]
\[
= \frac{C}{k^{n/2} K(\xi)} \int_{\mathbb{R}^{n(k-1)}} \delta \left( \sum_{i=2}^k |\eta_i|^2 - K(\xi) \right) \, d\eta_2 \ldots d\eta_k
\]
\[
= \frac{C \left| S^{n(k-1)-1} \right|}{k^{n/2} K(\xi)} \int_0^\infty \delta(r^2 - K(\xi)) \, r^{n(k-1)-1} \, dr
\]
\[
= \frac{C \left| S^{n(k-1)-1} \right|}{2k^{n/2} K(\xi)} \int_0^\infty \delta(t - K(\xi)) \, t^{n(k-1)-2} \, dt
\]
\[
= \frac{C \left| S^{n(k-1)-1} \right|}{2k^{n/2}} = 1,
\]
and this proves (i). To prove (ii), just observe the symmetry of the measure \( m \) with respect to the variables \( \eta \) and \( \xi \),
\[
\int_{\mathbb{R}^n} m_\xi(B) \, d\xi = \int_{\mathbb{R}^n} \int_B C \left( \prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i) \right) \delta(|\eta|^2 - |\xi|^2) \, d\eta \, d\xi
\]
\[
= \int_B \int_{\mathbb{R}^n} C \left( \prod_{i=1}^n \delta((\eta - \xi) \cdot \alpha_i) \right) \delta(|\eta|^2 - |\xi|^2) \, d\xi \, d\eta
\]
\[
= \int_B m_\eta(\mathbb{R}^n) \, d\eta = \int_B \, d\eta = |B|.
\]
\( \square \)
We now return to the proof of the Representation Lemma \[6\]. Note the the operator \( A \) can be written as

\[ AG(\xi) = C_{n,k} \int_{\mathbb{R}^{nk}} G(\eta) m_\xi(d\eta). \]

The boundedness of the operator \( A \) in \( L^2(\mathbb{R}^{nk}) \) follows from an application of Lemma \[7\] and Jensen’s inequality

\[ \|AG\|_{L^2(\mathbb{R}^{nk})}^2 = C^2_{n,k} \int_{\mathbb{R}^{nk}} \left| \int_{\mathbb{R}^{nk}} G(\eta)m_\xi(d\eta) \right|^2 d\xi \leq C^2_{n,k} \int_{\mathbb{R}^{nk}} \int_{\mathbb{R}^{nk}} |G(\eta)|^2 m_\xi(d\eta) d\xi \]

\[ = C^2_{n,k} \int_{\mathbb{R}^{nk}} |G(\eta)|^2 \int_{\mathbb{R}^{nk}} m_\xi(d\eta) d\xi = C^2_{n,k} \int_{\mathbb{R}^{nk}} |G(\eta)|^2 d\eta = C^2_{n,k} \|G\|_{L^2(\mathbb{R}^{nk})}^2. \]

We thus arrive at

\[ \|AG\|_{L^2(\mathbb{R}^{nk})} \leq C_{n,k}\|G\|_{L^2(\mathbb{R}^{nk})}, \]

proving that the operator \( A \) extends to a bounded operator from \( L^2(\mathbb{R}^{nk}) \) to \( L^2(\mathbb{R}^{nk}) \). It remains to prove that \( A \) is a multiple of the projection operator \( P_E \). Let \( R \) be a rotation on \( \mathbb{R}^{nk} \) fixing the vectors \( \alpha_1, ..., \alpha_n \). It is clear from \[2.2\] and \[2.3\] that

\[ AG(Re) = AG(\xi), \]

therefore \( A \) maps \( L^2(\mathbb{R}^{nk}) \) into the subspace \( E \). From the fact that the operator \( A \) is self-adjoint we can show that \( A(E^\perp) = 0 \). It remains to prove that \( A \) acts like a multiple of the identity on \( E \). For this, consider a function \( H \in C^\infty(\mathbb{R} \times \mathbb{R} \times \times \mathbb{R} \times \mathbb{R}^+) \) and write

\[ G(\eta) = H(\eta \cdot \alpha_1, \eta \cdot \alpha_2, ..., \eta \cdot \alpha_n, |\eta|^2). \]  

Certainly \( G \) is a function in \( E \), and from definition \[2.2\] we find that, for a \( G \) of the form \[2.5\],

\[ AG(\xi) = C_{n,k}G(\xi). \]

Since the functions of the form \[2.5\] are dense in \( E \), we conclude that \( A = C_{n,k}I \) on \( E \). We have proved that \( A = C_{n,k}P_E \) and this concludes the lemma. \( \square \)

The proof of the inequality proposed in Theorem \[1\] is then a trivial consequence of the Representation Lemma \[6\]. In fact,

\[ \int_{\mathbb{R} \times \mathbb{R}^n} |u(t,x)|^{2k} dt \] \[ dx \]

\[ = C_{n,k} \|P_E(F_1), F_1\|_{L^2(\mathbb{R}^{nk})} \leq C_{n,k} \|F_1\|_{L^2(\mathbb{R}^{nk})}^2 \]

\[ = C_{n,k} \int_{\mathbb{R}^{nk}} |\hat{F}(\eta)|^2 K(\eta) \frac{n(k-1)-2}{2} d\eta. \]  

It remains to investigate when equality in \[2.6\] can be attained. A necessary and sufficient condition is that the function \( F_1(x) \) belongs to the subspace \( E \).

3. PROOF OF THEOREM \[1\] - GAUSSIAN MAXIMIZERS

We investigate here under which conditions the function

\[ F_1(\eta) = \hat{F}(\eta)K(\eta) \frac{n(k-1)-2}{4} \]

belongs to the subspace \( E \). Let us say that a measurable function \( G : \mathbb{R}^{nk} \to \mathbb{C} \) satisfies the property \((*)\) if \( G \) is invariant under all the rotations \( R \) that fix the vectors \( \alpha_1, \alpha_2, ..., \alpha_n \). In this setting, \( G \in E \) if and only if \( G \in L^2(\mathbb{R}^{nk}) \) and satisfies \((*)\).
From \([2.3]\) we see that \(K(x)\) satisfies \((\star)\). Therefore, we must have \(\hat{F}(\eta) = \hat{f}(\eta_1) f(\eta_2) \cdots f(\eta_k)\) satisfying \((\star)\), and we shall prove that under these symmetries \(\hat{f}\) must be a Gaussian. The proof will be divided in five steps.

**Step 1.** Let \(g : \mathbb{R}^n \to \mathbb{C}\) be a measurable function such that \(G(\eta) = g(\eta_1)g(\eta_2) \cdots g(\eta_k)\) satisfies
\[
\int_{\mathbb{R}^n} |G(\eta)|^2 K(\eta)^{\frac{n(k-1)-2}{2}} d\eta < \infty. \tag{3.1}
\]
Then \(g \in L^p(\mathbb{R}^n)\) for \(p = \frac{2nk}{2nk - n - 2}\).

This was proved in \((1.32)\). From now on we fix \(p = \frac{2nk}{2nk - n - 2}\).

**Step 2.** Let \(g \in L^p(\mathbb{R}^n)\) be such that \(G(\eta)\) satisfies the property \((\star)\). Then \(g\) is a product of one-dimensional functions.

We shall write here each \(\eta_i \in \mathbb{R}^n\) as \(\eta_i = (\eta_{i1}, \eta_{i2}, \ldots, \eta_{in})\). If \(g \in L^p(\mathbb{R}^n)\) is nonzero, there exists a cube \(J = \prod_{i=1}^{k} [a_i, b_i] \subset \mathbb{R}^n\) such that
\[
\int_J g(y) \, dy = A \neq 0.
\]
Consider the orthonormal transformation \(R\) in \(\mathbb{R}^{nk}\) that simply switches the coordinates \(\eta_{11}\) and \(\eta_{21}\) on \(\eta = (\eta_1, \ldots, \eta_k)\). Naturally, this transformation fixes the vectors \(\alpha_i\) and thus the relation \(G(Rx) = G(x)\) implies
\[
g(\eta_{11}, \eta_{12}, \ldots, \eta_{1n})g(\eta_{21}, \eta_{22}, \ldots, \eta_{2n})g(\eta_{31}, \eta_{32}, \ldots, \eta_{3n}) \cdots g(\eta_{kn})
g(\eta_{21}, \eta_{22}, \ldots, \eta_{2n})g(\eta_{31}, \eta_{32}, \ldots, \eta_{3n}) \cdots g(\eta_{kn}). \tag{3.2}
\]
Integrating both sides of \((3.2)\) with respect to \(d\eta_2 d\eta_3 \cdots d\eta_k\) on \(J \times J \times \cdots \times J\) we find that
\[
A^{k-1} g(\eta_{11}, \eta_{12}, \ldots, \eta_{1n})
= A^{k-2} \int_{a_1}^{b_1} g(\eta_{21}, \eta_{12}, \ldots, \eta_{1n}) \, d\eta_{21} \int_J g(\eta_{11}, \eta_{22}, \ldots, \eta_{2n}) \, d\eta_{12}, \tag{3.3}
\]
where \(J' = \prod_{i=2}^{n} [a_i, b_i] \) and \(d\eta'_2 = d\eta_{22} d\eta_{23} \cdots d\eta_{2n}\). Expression \((3.3)\) plainly says that
\[
g(\eta_{11}, \eta_{12}, \ldots, \eta_{1n}) = w_1(\eta_{11}) h_1(\eta_{12}, \ldots, \eta_{1n}). \tag{3.4}
\]
By repeating this argument we arrive at
\[
g(\eta_{11}, \eta_{12}, \ldots, \eta_{1n}) = w_j(\eta_{1j}) h_j(\eta_{11}, \ldots, \eta_{(j-1)}, \eta_{1(j+1)}, \ldots, \eta_{1n}), \tag{3.5}
\]
for \(j = 2, \ldots, n\). Expressions \((3.4)\) and \((3.5)\) are sufficient to conclude that
\[
g(\eta_{11}, \eta_{12}, \ldots, \eta_{1n}) = g_1(\eta_{11}) g_2(\eta_{12}) \cdots g_n(\eta_{1n}).
\]

**Step 3.** Suppose that all \(g_i\)’s are smooth and non-vanishing. Then all \(g_i\)’s are Gaussians with the same covariance. Therefore \(g\) is itself a Gaussian.
Let $R_{12}$ be a rotation on $\mathbb{R}^{2n}$ fixing the vectors $\beta_i = \frac{1}{\sqrt{2}}(e_i, e_i)$, $i = 1, 2, \ldots, n$. Observe that the rotation on $\mathbb{R}^{nk}$ given by

$$R = \begin{bmatrix} R_{12} & I & 0 \\ I & I & \ddots \\ 0 & \ddots & I \end{bmatrix}$$

(3.6)

fixes the vectors $\alpha_i = (e_i, e_i, \ldots, e_i) \in \mathbb{R}^{nk}$. Among all the possible rotations $R$ given by this form, we will choose a simple rotation $R_{12}$ to work with. Let us denote the tensor product $a \otimes b$ of two vectors $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$ as the $n \times n$ matrix $[a_i b_j]$, corresponding to the linear transformation $x \mapsto (x, b)$. Consider the orthonormal basis of $\mathbb{R}^{2n}$ formed by the vectors $\beta_i = \frac{1}{\sqrt{2}}(e_i, e_i)$ and $\gamma_i = \frac{1}{\sqrt{2}}(e_i, -e_i)$, with $i = 1, 2, \ldots, n$, and let $R_{12}(\theta)$ be given by

$$R_{12}(\theta) = \sum_{i=1}^{n} \beta_i \otimes \beta_i + \sum_{i=3}^{n} \gamma_i \otimes \gamma_i + \cos(\theta)\gamma_1 \otimes \gamma_1 - \sin(\theta)\gamma_1 \otimes \gamma_2 + \sin(\theta)\gamma_2 \otimes \gamma_1 + \cos(\theta)\gamma_2 \otimes \gamma_2.$$

Let $R(\theta)$ be the rotation on $\mathbb{R}^{nk}$ given by the matrix (3.6) with the corresponding $R_{12}(\theta)$. From the fact that $G(R(\theta)\eta) = G(\eta)$ and $R(0) = I$ we obtain

$$0 = -2\left. \frac{\partial G(R(\theta)\eta)}{\partial \theta} \right|_{\theta=0}$$

$$= \sum_{i=1}^{n} (\eta_1 - \eta_2)\partial_{\eta_1} - (\eta_1 - \eta_2)\partial_{\eta_2} - (\eta_1 - \eta_2)\partial_{\eta_1} + (\eta_1 - \eta_2)\partial_{\eta_2} \right] G(\eta).$$

By introducing the logarithmic derivatives $h_i' = g_i' / g_i$ the last expression becomes

$$(\eta_1 - \eta_2)h_1'(\eta_1) - (\eta_1 - \eta_2)h_1'(\eta_2) - (\eta_1 - \eta_2)h_2'(\eta_1) + (\eta_1 - \eta_2)h_2'(\eta_2) = 0.$$ 

Differentiating with respect to the variable $\eta_1$ we obtain

$$(\eta_1 - \eta_2)h_1''(\eta_1) - h_1'(\eta_1) + h_2'(\eta_2) = 0.$$ 

Finally, differentiating with respect to $\eta_2$ yields

$$h_1''(\eta_1) = h_2''(\eta_2),$$

and since the variables $\eta_1$ and $\eta_2$ are independent we conclude that both logarithmic second derivatives are constant. The argument above can be reproduced for $\gamma_1$ and $\gamma_j$ yielding $h_j'' = C$ for all $j = 1, 2, \ldots, n$. This proves that all $g_i$’s are Gaussians with the same covariance, and thus $g$ will itself be a Gaussian.

The two last steps (reduction to the smooth non-vanishing case) plainly follows the argument of Hundertmark and Zharnitsky [3]. This idea originally appeared in a paper by Carlen [2]. We denote by $P_\epsilon$ the convolution with the Gaussian kernel on $\mathbb{R}^{nk}$

$$\varphi_\epsilon(\eta) = \frac{1}{(2\pi \epsilon)^{nk/2}} e^{-|\eta|^2 / 2\epsilon},$$

and by $Q_\epsilon$ the convolution with the Gaussian kernel on $\mathbb{R}^n$

$$\phi_\epsilon(y) = \frac{1}{(2\pi \epsilon)^{n/2}} e^{-|y|^2 / 2\epsilon}.$$
Step 4. Let \( g \in L^p(\mathbb{R}^n) \) be such that \( G(\eta) \) satisfies the property (\( \ast \)). Assume \( Q_\epsilon(g) \) never vanishes as \( \epsilon \to 0 \). Then \( g \) is a Gaussian.

Observe that \( P_\epsilon(G) \) inherits the rotational symmetries of \( G \), and since

\[
P_\epsilon(G)(\eta) = Q_\epsilon(g)(\eta_1)Q_\epsilon(g)(\eta_2)\ldots Q_\epsilon(g)(\eta_k),
\]

and \( Q_\epsilon(g) \) is smooth and non-vanishing, we conclude by Step 3 that it must be a Gaussian. As \( g \in L^p(\mathbb{R}^n) \), we have \( g = \lim_{\epsilon \to 0} Q_\epsilon(g) \) and this implies that \( g \), being a limit of Gaussians, is also a Gaussian.

Step 5. Let \( g \in L^p(\mathbb{R}^n) \) be such that \( G(\eta) \) satisfies the property (\( \ast \)). Then \( Q_\epsilon(g) \) never vanishes as \( \epsilon \to 0 \).

Indeed, take absolute values in (3.7) and apply the convolution operator \( P_\lambda \) again

\[
P_\lambda[P_\epsilon(G)](\eta) = Q_\lambda|Q_\epsilon(g)|(\eta_1)Q_\lambda|Q_\epsilon(g)|(\eta_2)\ldots Q_\lambda|Q_\epsilon(g)|(\eta_k).
\]

Again, \( P_\lambda[P_\epsilon(G)] \) inherits all the rotational symmetries of \( P_\epsilon(G) \), in particular those of \( G \). Since \( Q_\lambda(g) \to g \) in \( L^p(\mathbb{R}^n) \), as \( \epsilon \to 0 \), we conclude that \( Q_\lambda(g) \) is not the zero function for small \( \epsilon \). Since convolution with a Gaussian improves positivity, \( Q_\lambda(g) \) is a strictly positive smooth function. By Step 4 we conclude that \( |Q_\epsilon(g)| \) is a Gaussian, and thus never vanishes for small \( \epsilon \).

By putting \( g = \hat{f} \) in Steps 1-5 we are led to the conclusion that \( \hat{f} \) must be a Gaussian, and then so is \( f \).

4. Proof of Theorem 5: sharp cone estimates

This final section is devoted to a brief proof of Theorem 5 in which we follow the basic ideas of Foschi [4, sections 5 and 6]. Let us prove first the case \( n = 3 \), \( q = 4 \), which corresponds to (1.30). From now on we shall write

\[
g(|\omega|, \omega) = f(\omega),
\]

and assume that \( f \) is a smooth, compactly supported function. Observe that

\[
|\hat{g}d\sigma|_{L^2(\mathbb{R}^4)}^2 = \|\hat{g}d\sigma\|_{L^2(\mathbb{R}^4)}^2 = \|g d\sigma * g d\sigma\|_{L^2(\mathbb{R}^4)},
\]

where, in the case of the cone, we identify

\[
g d\sigma(\tau, \omega) = f(\omega) \frac{\delta(\tau - |\omega|)}{|\omega|}.
\]

Therefore we can write

\[
g d\sigma * g d\sigma(\tau, \omega) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{f(\eta)f(\xi)}{|\eta||\xi|} \delta_3(\omega - \eta - \xi) \delta(\tau - |\eta| - |\xi|) d\eta d\xi, \tag{4.1}
\]

and we observe that \( g d\sigma * g d\sigma \) is supported in the closure of the region

\[
C_{++} = \{ (\tau, \omega) \in \mathbb{R} \times \mathbb{R}^3 : \tau > |\omega| \}.
\]

For each choice of \( (\tau, \omega) \in C_{++} \), we denote by \( \langle \cdot, \cdot \rangle_{(\tau, \omega)} \) the \( L^2 \)-inner product associated with the measure

\[
\mu_{(\tau, \omega)} := \delta_3(\omega - \eta - \xi) \delta(\tau - |\eta| - |\xi|) d\eta d\xi, \tag{4.2}
\]
and by $\|\cdot\|_{(\tau, \omega)}$ the corresponding norm. From (4.1) and Cauchy-Schwarz inequality we have
\[
\begin{align*}
g d\sigma \ast g d\sigma (\tau, \omega) &= \left( \frac{f(\eta) f(\xi)}{|\eta|^{1/2} |\xi|^{1/2}} \right) (\tau, \omega) \cdot \frac{1}{|\eta|^{1/2} |\xi|^{1/2}} (\tau, \omega) \\
&\leq \left\| \frac{f(\eta) f(\xi)}{|\eta|^{1/2} |\xi|^{1/2}} \right\|_{(\tau, \omega)} \left\| \frac{1}{|\eta|^{1/2} |\xi|^{1/2}} \right\|_{(\tau, \omega)}.
\end{align*}
\] (4.3)

In [4] Lemma 5.2 it is proved that for each $(\tau, \omega) \in C_{++}$
\[
\left\| \frac{1}{|\eta|^{1/2} |\xi|^{1/2}} \right\|_{(\tau, \omega)} = (2\pi)^{1/2}.
\] (4.4)

Therefore, combining (4.3) and (4.4) we obtain
\[
\|\widehat{g d\sigma}\|_{L^4(\mathbb{R}^4)} = \left\| \int C_{++} \frac{f(\eta) f(\xi)}{|\eta|^{1/2} |\xi|^{1/2}} d\tau d\omega \right\|_{L^2(\mathbb{R}^4)} \leq 2\pi \int C_{++} \left\| \frac{f(\eta) f(\xi)}{|\eta|^{1/2} |\xi|^{1/2}} \right\|^2 d\tau d\omega
\]
\[
= 2\pi \int_{\mathbb{R}^3} \left( \frac{|f(\eta)|}{|\eta|} \right)^2 \left( \frac{|f(\xi)|}{|\xi|} \right)^2 d\eta d\xi = 2\pi \|g\|_{L^2(S; d\sigma)}^4,
\]
and this proves (1.30). From the Cauchy-Schwarz condition, we know that equality in (1.31) can only be attained if there is a function $F : C_{++} \rightarrow \mathbb{C}$ such that
\[
f(\eta) f(\xi) = F(|\eta| + |\xi|, \eta + \xi),
\] (4.6)
for almost all $(\eta, \xi)$ (with respect to the measure (4.2)) in the support of the measure (4.2), and almost all $(\tau, \omega) \in C_{++}$, with respect to the Lebesgue measure in $\mathbb{R} \times \mathbb{R}^3$.

This means that $f(\eta) f(\xi) = F(|\eta| + |\xi|, \eta + \xi)$, for almost all $\eta, \xi \in \mathbb{R}^3$. The locally integrable functions $f$ satisfying property (1.6) were characterized by Foschi in [4] Proposition 7.2] and they turn out to be
\[
g(|\omega|, \omega) = f(\omega) = e^{A|\omega| + b \omega + C},
\]
where $A, C \in \mathbb{C}, b \in \mathbb{C}^3$ and $|\Re(b)| < -\Re(A)$ (this last condition to ensure that $g \in L^2(S; d\sigma)$).

The proof for the case $n = 2, q = 6$, which corresponds to (1.29), follows exactly the same outline. Here we will have
\[
\|\widehat{g d\sigma}\|_{L^6(\mathbb{R}^2)}^3 = \|\widehat{g d\sigma}\|^3_{L^2(\mathbb{R}^3)} = \|gd\sigma \ast g d\sigma \ast g d\sigma\|_{L^2(\mathbb{R}^3)},
\]
where $gd\sigma \ast g d\sigma \ast g d\sigma (\tau, \omega) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{f(\eta) f(\xi) f(\zeta)}{|\eta| |\xi| |\zeta|} d\eta d\xi d\zeta$. For each $(\tau, \omega)$ on the region $C_+ = \{(\tau, \omega) \in \mathbb{R} \times \mathbb{R}^2 : \tau > |\omega|\}$, consider the measure
\[
\nu(\tau, \omega) = \delta_2(\omega - \eta - \xi - \zeta) \delta(\tau - |\eta| - |\xi| - |\zeta|) d\eta d\xi d\zeta.
\]
As in (4.3), by Cauchy-Schwarz inequality
\[
gd\sigma \ast gd\sigma \ast gd\sigma (\tau, \omega) \leq \left\| \frac{f(\eta) f(\xi) f(\zeta)}{|\eta|^{1/2} |\xi|^{1/2} |\zeta|^{1/2}} \right\|_{(\tau, \omega)} \left\| \frac{1}{|\eta|^{1/2} |\xi|^{1/2} |\zeta|^{1/2}} \right\|_{(\tau, \omega)}.
\]
In [4, Lemma 6.1] it is proved that for each \((\tau, \omega) \in C_+\)
\[
\frac{1}{|\eta|^{1/2} |\xi|^{1/2} |\zeta|^{1/2}} = 2\pi.
\] (4.7)

Therefore
\[
\|g d\sigma\|_{L^6(S^2)}^6 \leq \|g d\sigma * g d\sigma * g d\sigma\|_{L^2(S^2)}^2
\leq 4\pi^2 \int_{C_+} \left\| \frac{f(\eta) f(\xi) f(\zeta)}{|\eta|^{1/2} |\xi|^{1/2} |\zeta|^{1/2}} \right\|^2 d\tau d\omega
= 4\pi^2 \int_{\mathbb{R}^3} \frac{|f(\eta)|^2 |f(\xi)|^2 |f(\zeta)|^2}{|\eta| |\xi| |\zeta|} d\eta d\xi d\zeta = 4\pi^2 \|g\|_{L^2(S^2)}^6,
\] (4.8)

which proves (1.29). As in the previous case, equality happens in (4.8) if and only if there is a function \(F : C_+ \to \mathbb{C}\) such that
\[
f(\eta)f(\xi)f(\zeta) = F(|\eta| + |\xi| + |\zeta|, \eta + \xi + \zeta),
\] (4.9)
for almost all \(\eta, \xi, \zeta \in \mathbb{R}^2\). The locally integrable functions \(f\) satisfying (4.9) were also characterized in [4, Proposition 7.19], and they are
\[
g(|\omega|, \omega) = f(\omega) = e^{A|\omega| + b \cdot \omega + C},
\]
where \(A, C \in \mathbb{C}, b \in \mathbb{C}^2\) and \(|\Re(b)| < -\Re(A)\). This concludes the proof.

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References

[1] W. Beckner, Personal communication.
[2] E. Carlen, Superadditivity of Fisher’s information and logarithmic Sobolev inequalities, J. Funct. Anal. 101 (1991), no. 1, 194–211.
[3] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes, American Mathematical Society, New York, 2003.
[4] D. Foschi, Maximizers for the Strichartz inequality, J. Eur. Math. Soc. (JEMS) 9 (2007), no. 4, 739–774.
[5] L. Hörmander, Oscillatory integrals and multipliers on \(FL^p\), Ark. Mat. 11, 1–11. (1973).
[6] D. Hundertmark and V. Zharnitsky, On sharp Strichartz inequalities in low dimensions, Int. Math. Res. Not. 2006, Art. ID 34080, 18 pp.
[7] M. Kunze, On the existence of a maximizer for the Strichartz inequality, Comm. Math. Phys. 243 (2003), no. 1, 137–162.
[8] R. Strichartz, Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations, Duke Math. J. 44 (1977), 705–774.
[9] S. Shao, Maximizers for the Strichartz inequalities and the Sobolev-Strichartz inequalities for the Schrödinger equation, preprint arXiv:0809.0153.
[10] T. Tao, Some recent progress on the restriction conjecture, Fourier analysis and convexity, 217–243, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2004.
[11] T. Tao, A sharp bilinear restriction estimate on paraboloids, Geom. Funct. Anal. 13 (2003), no. 6, 1359–1384.
[12] T. Wolff, A sharp bilinear cone restriction estimate, Annals of Math. 153 (2001), 661–698.

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