Which quantile is the most informative? Markov switching quantile model with unknown quantile level

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Abstract. In this study, we propose a Markov regime-switching quantile regression model, which considers the quantile as an unknown parameter and estimate it jointly with other regression coefficients. The parameters are estimated by the maximum likelihood estimation (MLE) method. Our proposed model aims to address the problem about which quantile would be the most informative one among all the candidates. A simulation study of this proposed model is conducted covering various scenarios. The results show that the MLE method is efficient as the estimated parameters are close to their true values. An empirical analysis is also provided, which focuses on the risk measurement in United States and United Kingdom stock markets. The degree of risk is measured by the most informative quantile regression coefficients in each regime. The result shows that the Markov regime-switching quantile regression model with unknown quantile can explain the behavior of the data better and more accurately than the Markov regime-switching quantile regression model when in terms of the minimum Akaiki information criterion (AIC) and Bayesian information criterion (BIC).

1. Introduction
The Markov switching quantile regression (MSQR) model is extended form the linear quantile regression of [1]. It allows one to examine the effect of independent variables on the predictor at specific quantile level under different regimes [2]. In other words, this approach allows the location and scale to change with regime shifts within a quantile. So, the model is likely to be more robust against outliers in the response measurements.

MSQR is a one of the most recent approach built on the vast literature on Markov switching models. However, specifying the quantile level, for fitting to the data, is difficult since there is a wide range of quantile levels, say 0.01-0.99. In the conventional quantile regression approach, the model will be useful if we want to focus on some or particular quantile levels. Nonetheless, if we estimate the quantile model at a given range of quantile, say 0.1, 0.2.....0.9, we will obtain nine different results, one for each quantile level. The challenge is that what is the best quantile or which quantile is the most informative to explain the behavior of the data. When the model contains many quantile levels, the best fit quantile plays an important role in the model building process to obtain a better interpretation and to improve the precision of model. To solve this problem, we need some criteria to determine the quantile level of the parameter to be estimated. Recently, this problem has received increasing attention in the literature. [3] and [4] suggested viewing the quantile level as an unknown parameter.
and estimate it with other parameters in the model. In other words, they estimated the quantile parameter from the data and allowed the data to tell their own story. Nevertheless, their purpose is confined to the study in one regime or linear quantile regression model. This study, thus, will contribute to the recent literature by extending their works into the Regime-switching quantile regression with unknown quantile model which is able to accommodate many stylized facts such as structural breaks and nonlinearities in various fields of research using economic and financial time series.

In the model estimation, the maximum likelihood estimator is employed where the likelihood function of MSQR is considered to be Asymmetric-Laplace distribution (ALD) [5], [6] confirmed that this distribution satisfies the restrictive conditions of quantile regression and the inference for Markov-switching quantile regression can be made through the standard Hamilton filter approach [7]. Generally speaking, the estimated parameters including quantile parameter are governed by first order Markov chain and thereby allowing the regression coefficients and quantile parameter to vary when the Markov state changes.

In this paper, our contribution is twofold. First, we develop a Markov regime-switching quantile regression with unknown quantile model. Second, the financial contagion is detected using Capital Asset Pricing model using our proposed model. Moreover, in this study, we also apply our proposed model in a simulation study to assess its accuracy and reliable in the estimation.

The rest of this paper is organized as follows. The Asymmetric Laplace Distribution is presented in Section 2. In Section 3, the parameter estimation is explained. Section 4 presents the model simulations under various scenarios and we use real data for empirical estimation of MSQR with unknown quantile model. Final remarks are provided in Section 5.

2. Methodology
In this section, we start with a brief of Asymmetric-Laplace distribution. We then introduce a Markov-switching quantile regression model with unknown quantile, an extension of traditional Markov-switching quantile regression model of [2], in which the quantile level is needed to specify.

![ALD plot](image)

**Figure 1.** ALD plot.

2.1. Asymmetric Laplace distribution
As discussed in [8], the likelihood function based on ALD can be shown as follows

\[
L(y | \mu, \tau) = \frac{\tau(1-\tau)}{\sigma} \exp\left\{ \rho_\tau\left( \frac{y - \mu}{\sigma} \right) \right\},
\]

(2.1)

where \( \mu \) is location parameter, \( \sigma > 0 \) is considered as a scale parameter and skewness parameter or quantile level \( \tau \) where \( 0 < \tau < 1 \). \( \rho_\tau(u) = L(\tau - I(u < 0)) \) is called the check function. \( I(\cdot) \) is the usual indicator function. Figure 1 shows how the ALD changes when altering tau parameter.
example, when \( \tau = 0.1 \) almost all the mass of the ALD is situated in the right tail (black line), or conversely situated in the left tail (red line) when \( \tau = 0.9 \). In the case when \( \tau = 0.5 \), both tails of the ALD becomes symmetric (blue line). (\( \sigma = 1 \) is assumed for all cases).

2.2. Markov switching quantile regression using asymmetric Laplace distribution

The traditional quantile regression introduced by [1] was extended into the Markov Switching model of [7] and thereby forming a Markov switching quantile regression [2]. Then, we consider the model given by

\[
y_i = \beta^h_i(\tau) + \sum_{h=1}^{H} \beta^h_i(\tau)x^h + \epsilon_{s_i,t}(\tau), \quad t = 1, \ldots, T
\]  

(2.2)

where \( y_i \) is a dependent variable and \( x^h \) is a \( T \times H \) matrix of independent variables, \( \epsilon_i \) are independent and identically distributed (i.i.d) random errors with probability density function as shown in equation 2.1. The model (2.2) is a quantile regression with regime-switching coefficients. In other words, the coefficients \( \beta^h_i(\tau) \), which represents a vector of unknown parameters which determines a relationship between vector \( x_i \) and the \( r \)th conditional quantile function of \( y_i \), are regime dependent or the state variable(\( s_i \)) as we mentioned earlier, the quantile level \( \tau \) is unknown and treated as another parameter to be estimated and it is assumed to be a regime independent and restricted to have the value in [0,1].

In this study, the state variable(\( s_i \)) is an ergodic homogeneous Markov chain on a finite set \( s_i = \{1, \ldots, k\} \), with a transition matrix \( P \)

\[
P = \begin{bmatrix}
p_{11} & p_{12} & L & p_{1k} \\
p_{21} & p_{22} & L & p_{2k} \\
M & M & L & M \\
p_{lk} & p_{2l} & L & p_{lk}
\end{bmatrix}
\]  

(2.3)

where \( p_{ij} \) denotes the probability of transition from regime \( i \) to \( j \), \( i = 1, \ldots, k \); \( j = 1, \ldots, k \) and \( \sum p_{ij} = 1 \).

In the estimation, let parameter set \( \Theta = (\beta^h_i(\tau), \epsilon_{s_i,t}(\tau), \tau) \) can be estimated by the maximum likelihood, which is similar to the case of the conventional Markov quantile regression. And suppose we have two regime model (\( s_i = 1, 2 \)), the sample conditional likelihood function of the MSQR with unknown quantile model with 2 regimes can be defined as

\[
L(\Theta|y, x) = \prod_{t=1}^{T} \frac{\tau(1-\tau)}{\sigma_{s,t}} \exp\left(\frac{y_i - \beta^h_{s,t,0}(\tau) - \sum_{h=1}^{H} \beta^h_{s,t,h}(\tau)x^h}{\sigma_{s,t}}\right)(Pr(s_i|\Omega_{t-1};\Theta)),
\]  

(2.4)

where \( \Omega_{t-1} \) is all available information set of the model at time \( t-1 \), and \( (Pr(s_i|\Omega_{t-1};\Theta) \) is weighted probabilities computed recursively from the Hamilton’s filter algorithm [7]. Thus, filtered probabilities of each state are computed recursively by the following formula

\[
Pr(s_i|\Omega_{t-1};\Theta) = \left\{ \begin{array}{ll}
p_{i1}Pr(s_i = i | \Omega_{t-1};\Theta) + p_{22}Pr(s_i = j | \Omega_{t-1};\Theta),
\end{array} \right.
\]  

(2.5)

where

\[
Pr(s_i = i | \Omega_{t-1};\Theta) = \frac{f(y_i|s_i = i|\Omega_{t-2};\Theta)Pr(s_i = i|\Omega_{t-2};\Theta)}{\sum_{k=1}^{2} f(y_i|s_i = k|\Omega_{t-2};\Theta)Pr(s_i = k|\Omega_{t-2};\Theta)}, \quad i = 1, 2.
\]  

(2.6)

Finally, we can then take a logarithm to likelihood function (2.4), and then the estimated parameters can be obtained from maximizing this log-likelihood function.
3. Simulation study

A simulation study of the proposed model is described in the section. This simulation study is conducted to evaluate the performance and accuracy of our proposed model. The details are given below. The simulated data are generated from the Markov-switching quantile regression model where the error term is assumed to have asymmetric Laplace distribution (ALD) \( \varepsilon(\tau) \sim ALD(0, \sigma^2, \tau) \). Here, we consider two regimes \( s = 1, 2 \). To this end, we consider the following equation:

\[
y_i = \beta_i^0(\tau) + \beta_i^1(\tau)x_i + \varepsilon_{s_i,i}(\tau), \quad s_i \in \{1, 2\},
\]  

where \( \beta_i^0(\tau) = 1 \), \( \beta_i^1(\tau) = 2 \), \( \beta_{s=2}^0(\tau) = 2 \) and \( \beta_{s=2}^1(\tau) = 3 \). We simulated the independent variables \( x_i \) from \( N(0, 1) \). The Monte Carlo experiment with 100, 200 and 500 data points generated from the model. In addition, the simulated filter probabilities for two-regime model is generated from \( U[0,1] \) where the transition probabilities \( p_{11} = p_{22} = 0.95 \). For the error term, we remark that the asymmetric Laplace distribution quantile parameters \( \tau \) are set to be invariant as Markov regime switches. We consider two scenarios as follows: Case 1: \( N = 100, 200, 500 \). Case 2: \( \tau = 0.2, 0.5, 0.7 \)

Table 1. Simulation results.

| Parameters | True Value | N=100     | N=200     | N=500     |
|------------|------------|-----------|-----------|-----------|
| \( \beta_{s=1}^0(\tau) \) | 1          | 0.9362 (0.0307) | 1.2861 (0.0342) | 1.1753 (0.0310) |
| \( \beta_{s=1}^1(\tau) \) | 2          | 2.9536 (0.0302) | 2.1162 (0.0234) | 1.9501 (0.0179) |
| \( \beta_{s=2}^0(\tau) \) | 2          | 1.3492 (0.0318) | 1.6966 (0.0385) | 2.3617 (0.0628) |
| \( \beta_{s=2}^1(\tau) \) | 3          | 3.0897 (0.0605) | 3.1493 (0.0300) | 2.9379 (0.0837) |
| \( p_{11} \) | 0.95       | 0.8144 (0.0944) | 0.8138 (0.0989) | 0.9417 (0.0225) |
| \( p_{22} \) | 0.95       | 0.8244 (0.1047) | 0.8079 (0.1187) | 0.9475 (0.0210) |
| \( \tau \) | 0.2        | 0.2191 (0.0614) | 0.2533 (0.0249) | 0.1998 (0.0146) |

| Parameters | True Value | N=100     | N=200     | N=500     |
|------------|------------|-----------|-----------|-----------|
| \( \beta_{s=1}^0(\tau) \) | 1          | 1.3188 (0.0274) | 0.9897 (0.1629) | 1.0815 (0.0711) |
| \( \beta_{s=1}^1(\tau) \) | 2          | 1.9614 (0.0251) | 2.0233 (0.1480) | 1.8415 (0.0595) |
| \( \beta_{s=2}^0(\tau) \) | 2          | 1.7953 (0.0500) | 2.4747 (0.0042) | 2.0225 (0.0288) |
| \( \beta_{s=2}^1(\tau) \) | 3          | 3.1232 (0.0433) | 3.0671 (0.0356) | 3.0227 (0.0275) |
| \( p_{11} \) | 0.95       | 0.9774 (0.0246) | 0.9887 (0.0112) | 0.9658 (0.0159) |
| \( p_{22} \) | 0.95       | 0.9495 (0.0572) | 0.9854 (0.0152) | 0.9396 (0.0259) |
| \( \tau \) | 0.5        | 0.4915 (0.0363) | 0.4921 (0.0324) | 0.4880 (0.0175) |

| Parameters | True Value | N=100     | N=200     | N=500     |
|------------|------------|-----------|-----------|-----------|
| \( \beta_{s=1}^0(\tau) \) | 1          | 0.2908 (0.8145) | 1.0805 (0.0349) | 0.7785 (0.0309) |
| \( \beta_{s=1}^1(\tau) \) | 2          | 0.7187 (0.8318) | 1.9437 (0.0206) | 1.9084 (0.0178) |
| \( \beta_{s=2}^0(\tau) \) | 2          | 1.7608 (0.0318) | 0.8655 (0.2015) | 1.4906 (0.0691) |
| \( \beta_{s=2}^1(\tau) \) | 3          | 2.8727 (0.0363) | 3.4077 (0.0325) | 2.3203 (0.0441) |
| \( p_{11} \) | 0.95       | 0.9995 (0.0458) | 0.9289 (0.0391) | 0.9623 (0.0214) |
| \( p_{22} \) | 0.95       | 0.8245 (0.1245) | 0.9015 (0.0515) | 0.9041 (0.0452) |
| \( \tau \) | 0.7        | 0.6857 (0.0331) | 0.6322 (0.0259) | 0.6678 (0.0173) |

*Note* (): denotes standard error
In this simulation study, we try to evaluate the performance of our proposed model and compare the results with the true values of each case. Table 1 shows the results of simulation results for assessing the maximum likelihood estimation of our proposed model. It can be seen that the estimated parameters are quite close to their true values with satisfactory standard derivations, meaning that we have an accurate and reliable estimation in our proposed model. In addition, it can be seen that the estimated parameters tend to be close to their true values when the sample size increases. According this simulation study, we can conclude that our proposed model is workable and should be useful in real data analysis.

4. Application to capital asset pricing model

4.1. Descriptive statistics

The data sets include monthly stock index from S&P500 (proxy of US index), London Stock Exchange (proxy of UK index), and Morgan Stanley Capital International (MSCI) (proxy of market index) during the period from November, 2007 to February, 2018. The data information is collected from Bloomberg. We use the stock indices (secondary data) to calculate the natural log returns. We choose the monthly yield of 10-year government bond of each country as a representative for risk free.

The descriptive statistics are shown in Table 2. We observe that UK has standard deviation higher than those of stock markets and US. The normality Jarque-Bera test suggest that all series are not normally distributed. The Augmented Dickey-Fuller (ADF) Unit root test is also employed and the result shows that all returns are stationary. Therefore, these variables can be used to estimate a Markov switching quantile regression CAPM model in the next step.

4.2. Estimation result

We fit the models as described in Section 2 and compare to the MSQR as a median regression ($\tau=0.5$). To compare their performance, we consider the Akaike and Bayesian information criteria. Table 3, shows the results of US and UK stocks fitted by MSQR CAPM model when $\tau=0.5$ and unknown quantile. The beta risks are introduced in both two regimes. $\beta_{s=1}(\tau)$ (bull) and $\beta_{s=2}(\tau)$ (bear). The results in table 3 show that the model with unknown quantile, because of lower AIC and BIC, can explain the behaviour of the data better and more accurately than the one with given $\tau=0.5$. The beta risks of US and UK stocks are pairwise lower than unity in both regimes, indicating that the stock moves less than the market does in the same direction so the stock return is theoretically less risky than the market [9]. We observe that the beta risk of US is 0.288 in bull market while it is 0.493 in bear market. Meanwhile, the beta risks of UK stock are 0.276 and 0.237 respectively, in bull and bear regimes. In brief, the beta risks of US stock are higher than US stock in both bull and bear regimes.

Moreover, the filtered probabilities of the bull regime are plotted in figures 2 and 3, for US and UK stocks, respectively. We can observe that our MSQR with unknown quantile has an ability to capture stocks, respectively.
the US financial crisis during 2008-2012 as the filtered probabilities are low in this period, while the MSQR cannot capture this crisis. For the case of UK stock, similar results are obtained.

| Table 3. The Markov switching quantile regression CAPM model. |
|---------------------------------|
| The Markov switching quantile regression CAPM model when $\tau = 0.5$ |
| Parameters | US | UK |
| $\beta^0_{s=1}(\tau)$ | 1.3428 (10.51) | 0.2122 (12.78) |
| $\beta^0_{s=2}(\tau)$ | 0.0088 (0.0056) | 0.0052 (0.0072) |
| $\beta^2_{s=2}(\tau)$ | 0.4730 (0.0276)*** | 0.2755 (0.016)*** |
| $\sigma_{s=1}(\tau)$ | 0.5587 (0.0024)*** | 0.8454 (0.9824) |
| $\sigma_{s=2}(\tau)$ | 0.1158 (0.0025)*** | 0.2585 (0.0145)*** |
| $p_{11}$ | 0.99 (0.083)*** | 0.99 (0.017)*** |
| $p_{22}$ | 0.75 (19.04) | 0.82 (4.827) |
| AIC | -293.97 | -462.97 |
| BIC | -268.52 | -437.52 |

The Markov switching quantile regression CAPM model with unknown quantile

| Parameters | US | UK |
|---------------------------------|
| $\alpha_{s=1}(\tau)$ | 0.015 (0.012) | 0.042 (12.143) |
| $\beta_{s=1}(\tau)$ | 0.288 (0.163)* | 0.276 (117.35) |
| $\alpha_{s=2}(\tau)$ | 0.009 (0.003)** | 0.014 (0.001)*** |
| $\beta_{s=2}(\tau)$ | 0.493 (0.037)*** | 0.237 (0.007)*** |
| $\sigma_{s=1}(\tau)$ | 0.4165 (0.0001)*** | 0.32154 (0.5547) |
| $\sigma_{s=2}(\tau)$ | 0.1001 (0.0010)*** | 0.2145 (0.1244)*** |
| $p_{11}$ | 0.99 (0.035)*** | 0.91 (1.621) |
| $p_{22}$ | 0.96 (0.036)*** | 0.99 (0.035)*** |
| $\tau$ | 0.53 (0.037)*** | 0.66 (0.035)*** |
| AIC | -307.88 | -472.07 |
| BIC | -282.42 | -446.61 |

Note ***,**,* are significant at 1 %, 5%, and 10 %level, respectively. () is standard error.

5. Conclusions

There are some drawbacks about the quantile level in MSQR model. The challenge is that what is the best quantile or which quantile is the most informative to explain the behavior of the data. When the model contains many quantile levels, the best fit quantile plays an important role in the model building process to obtain a better interpretation and to improve the precision of model. In this study, we propose a MSQR model, which considers the quantile as an unknown parameter and estimate it jointly with other regression coefficients. Both simulation and empirical analysis are provided. The simulation study result confirms the accuracy of our proposed model. In the application study, we focus on the risk measurement in US and UK stock markets. Our model can explain the behavior of the data better and more accurately than MSQR given $\tau = 0.5$, according to the minimum AIC and
BIC. In addition, MSQR with unknown quantile also has an ability to detect the US financial crisis during 2008-2012 which corresponding to the empirical work of [10].

![Figure 2](image2.png)

**Figure 2.** The filtered probabilities of regime 1: US stock.

![Figure 3](image3.png)

**Figure 3.** The filtered probabilities of regime 1: UK stock.

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