QUANTIZATION OF LIE BIALGEBRAS REVISITED

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Abstract. We describe a new method of quantization of Lie bialgebras, based
on a construction of Hopf algebras out of a cocommutative coalgebra and a
braided comonoidal functor.

1. Introduction

The problem of functorial/universal quantization of Lie bialgebras was solved by
Etingof and Kazhdan in [2]. They first quantize the double of the Lie bialgebra
by defining a monoidal structure on the forgetful functor from the corresponding
Drinfeld category and then define their quantization of the Lie bialgebra as a Hopf
subalgebra of the Hopf algebra quantizing the double.

We present a different solution of the problem. It is based on the fact that a
cocommutative coalgebra and a braided comonoidal functor give rise, under certain
invertibility conditions, to a Hopf algebra (Theorem 1). This method avoids the
need for quantization of the double, and is technically simpler than the method of
Etingof and Kazhdan.

2. Braided monoidal categories

In this section we recall some basic definitions and facts concerning monoidal
categories, in order to fix the terminology.

If $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor between monoidal categories, a comonoidal structure
on $F$ (also called colax monoidal structure) is a natural transformation

$$c_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)$$

such that

$$F((X \otimes Y) \otimes Z) \to F(X \otimes Y) \otimes F(Z) \to (F(X) \otimes F(Y)) \otimes F(Z)$$

$$F(X \otimes (Y \otimes Z)) \to F(X) \otimes F(Y \otimes Z) \to F(X) \otimes (F(Y) \otimes F(Z))$$

commutes, together with a morphism

$$c : F(1_{\mathcal{C}_1}) \to 1_{\mathcal{C}_2}$$

compatible with the units. The functor $F$ then maps coalgebras to coalgebras: if
$M \in \mathcal{C}_1$ is a coalgebra then the coproduct on $F(M)$ is the composition

$$F(M) \to F(M \otimes M) \to F(M) \otimes F(M)$$

and the counit is the composition

$$F(M) \to F(1_{\mathcal{C}_1}) \to 1_{\mathcal{C}_2}.$$ If $c_{X,Y}$’s and $c$ are isomorphisms then $F$ is strongly monoidal.

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If $\mathcal{C}$ is a braided monoidal category then the functor
\[ \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \]
is a strong monoidal functor, with the monoidal structure
\[ (X_1 \otimes Y_1) \otimes (X_2 \otimes Y_2) \to (X_1 \otimes X_2) \otimes (Y_1 \otimes Y_2) \quad (\forall X_1, X_2, Y_1, Y_2 \in \mathcal{C}) \]
given by the parenthesized braid
\[ (X_1 Y_1) (X_2 Y_2) \]
In particular, if $M$ and $M'$ are coalgebras in $\mathcal{C}$ then $M \otimes M'$ is a coalgebra as well, with the coproduct
\[ (1) \]
In this way the category of coalgebras in $\mathcal{C}$ becomes a monoidal category. If $F : \mathcal{D} \to \mathcal{C}$ is a braided comonoidal functor then $F(M \otimes M') \to F(M) \otimes F(M')$ is a morphism of coalgebras; $F$ thus becomes a comonoidal functor from the category of coalgebras in $\mathcal{D}_1$ to the category of coalgebras in $\mathcal{C}_2$.

An algebra (i.e. a monoid) $H$ in the category of coalgebras in $\mathcal{C}$ is called a **bialgebra** in $\mathcal{C}$. It is a **Hopf algebra** if it comes with an invertible morphism $S \in \text{Hom}_{\mathcal{C}}(H, H)$ such that
\[ m_H \circ (S \otimes \text{id}_H) \circ \Delta_H = m_H \circ (\text{id}_H \otimes S) \circ \Delta_H = \eta_H \circ \epsilon_H \]
where $\epsilon_H : H \to 1_{\mathcal{C}}$ is the counit, $\eta_H : 1_{\mathcal{C}} \to H$ the unit, and $m_H$ and $\Delta_H$ the product and the coproduct of $H$.

### 3. A Construction of Hopf Algebras

In this section we shall construct a Hopf algebra out of a braided comonoidal functor and of a cocommutative coalgebra. The rest of the paper is an application of this construction.

Suppose $M$ is a coalgebra in a braided monoidal category $\mathcal{D}$. Even if $M$ is cocommutative (i.e. if $\beta_{M,M} \circ \Delta_M = \Delta_M$, where $\Delta_M : M \to M \otimes M$ is the coproduct and $\beta_{M,M} : M \otimes M \to M \otimes M$ the braiding), the coalgebra $M \otimes M$ may be non-cocommutative. If $F : \mathcal{D} \to \mathcal{C}$ is a braided comonoidal functor to another braided monoidal category $\mathcal{C}$ then the coalgebra $F(M)$ is cocommutative, but $F(M \otimes M)$ may not be.

Somewhat surprisingly, under certain compatibility conditions on $M$ and $F$, the coalgebra $F(M \otimes M)$ is a Hopf algebra.

**Definition 1.** Let $\mathcal{D}$ and $\mathcal{C}$ be braided monoidal categories and $(M, \Delta_M, \epsilon_M)$ a cocommutative coalgebra in $\mathcal{D}$. A braided comonoidal functor
\[ F : \mathcal{D} \to \mathcal{C} \]

is $M$-adapted if it satisfies these invertibility conditions: the composition
\[ F(M) \xrightarrow{F(\varepsilon_M)} F(1_D) \rightarrow 1_C \]
is an isomorphism, and for every objects $X,Y \in \mathcal{D}$ the morphism
\[ \tau_{X,Y} : F((X \otimes M) \otimes Y) \rightarrow F(X \otimes M) \otimes F(M \otimes Y), \]
defined as the composition
\[ F((X \otimes M) \otimes Y) \xrightarrow{F((id_X \otimes \Delta_M) \otimes id_Y)} F((X \otimes (M \otimes M)) \otimes Y) \rightarrow F((X \otimes M) \otimes (M \otimes Y)) \rightarrow F(X \otimes M) \otimes F(M \otimes Y), \]
is an isomorphism.

Remark. The functor $\mathcal{D} \rightarrow \mathcal{C}$, $X \mapsto M \otimes X$, is comonoidal (since $\otimes : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ is strongly monoidal and $M$ is a coalgebra); explicitly, the comonoidal structure $M \otimes (X \otimes Y) \rightarrow (M \otimes X) \otimes (M \otimes Y)$ is
\[
\begin{array}{c}
(M \otimes X) \quad (M \otimes Y) \\
\downarrow \quad \downarrow \\
M \quad (X \otimes Y)
\end{array}
\]

A braided comonoidal functor $F$ is $M$-adapted iff the composition $\mathcal{D} \xrightarrow{M \otimes} \mathcal{D} \xrightarrow{F} \mathcal{C}$, which is a priori comonoidal, is strongly monoidal.

**Theorem 1.** Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be an $M$-adapted functor. Then $F(M \otimes M)$ is a Hopf algebra in $\mathcal{C}$, with the structure given as follows.

- The coalgebra structure on $F(M \otimes M)$ is inherited from the coalgebra structure on $M \otimes M$, with the coproduct $\Delta_M$ and counit $\varepsilon_M \otimes \varepsilon_M$.
- The product on $F(M \otimes M)$ is the composition
  \[ F(M \otimes M) \otimes F(M \otimes M) \xrightarrow{\tau_{M,M}} F((M \otimes M) \otimes M) \xrightarrow{F(id_M \otimes \varepsilon_M \otimes id_M)} F(M \otimes M). \]
- The unit is
  \[ 1_C \cong F(M) \xrightarrow{F(\varepsilon_M)} F(M \otimes M). \]
- The antipode is
  \[ F(M \otimes M) \xrightarrow{F(\beta^{-1}_{M,M})} F(M \otimes M) \]
  where $\beta_{M,M} : M \otimes M \rightarrow M \otimes M$ is the braiding in $\mathcal{D}$.

**Proof.** To simplify notation, let us replace $\mathcal{D}$ and $\mathcal{C}$ with equivalent strict monoidal categories. The sequence of objects $X_n := M^{\otimes (n+1)}$ ($n = 0, 1, 2, \ldots$) is a simplicial object of $\mathcal{D}$, with degeneracies $id_M^{\otimes k} \otimes \Delta_M^{\otimes (n-k)}$ and faces $id_M^{\otimes k} \otimes \varepsilon_M \otimes id_M^{\otimes (n-k)}$ (it is the cobar construction of the coalgebra $M$). Since $M$ is cocommutative, $\Delta_M : M \rightarrow M \otimes M$ is a morphisms of coalgebras, and thus $X_n$ is a simplicial coalgebra in $\mathcal{D}$. As a result (using cocommutality of $F$), $Y_n := F(X_n)$ is a simplicial coalgebra in $\mathcal{C}$.

By repeatedly using invertibility of $\tau_{X,Y}$’s we know that the composition
\[ Y_n = F(M^{\otimes (n+1)}) \xrightarrow{F(id_M^{\otimes k} \otimes \Delta_M^{\otimes (n-1)} \otimes id_M^{\otimes n})} F(M^{\otimes 2n}) \rightarrow F(M \otimes M)^{\otimes n} \]
is an isomorphism. Since $M$ is cocommutative and $F$ is braided, both arrows are morphisms of coalgebras, hence we have an isomorphism of coalgebras in $\mathcal{C}$
\[ Y_n \cong F(M \otimes M)^{\otimes n}. \]
In terms of this isomorphism, the face maps of $Y_\bullet$ are given by the product on $F(M \otimes M)$ and the degeneracy maps are given by including the unit of $F(M \otimes M)$. This shows that the product is associative and that the unit is a unit of the product. The object $F(M \otimes M)$ is thus an algebra in the category of coalgebras in $C$, i.e. it is a bialgebra in $C$.

Finally, the fact that $F(\beta^{-1}_{M,M})$ is an antipode for the bialgebra $F(M \otimes M)$ follows easily from the definitions. □

Remark. If $D_{univ}$ is universal among braided monoidal categories with a chosen cocommutative coalgebra then one can show that for any Hopf algebra $H$ in any braided monoidal category $C$ there is an $M$-adapted functor $F : D_{univ} \to C$ (where $M \in D_{univ}$ is the coalgebra) such that $H = F(M \otimes M)$ as a Hopf algebra, and also that $F$ is unique up to an isomorphism.

4. Infinitesimally braided categories

In this section we recall Drinfeld’s construction of braided monoidal categories via associators. We also observe how cocommutative coalgebras and braided comonoidal functors arise in this construction, as we need to feed them to Theorem [1].

Let us fix a field $K$ with char $K = 0$. By a $K$-linear category we mean a category enriched over $K$-vector spaces, i.e. $\text{Hom}(X,Y)$ should be a vector space over $K$ and the composition map $\text{Hom}(X,Y) \times \text{Hom}(Y,Z) \to \text{Hom}(X,Z)$ should be bilinear.

An infinitesimally braided category (i-braided category for short) is a $K$-linear symmetric monoidal category $C$ together with a natural transformation $t_{X,Y} : X \otimes Y \to X \otimes Y$

such that $t_{X,Y \otimes Z} = t_{X,Y} \otimes \text{id}_Z + t_{X,Z} \otimes \text{id}_Y$

(with symmetry implicitly applied in the second term) and $t_{Y,X} \circ \sigma_{X,Y} = \sigma_{X,Y} \circ t_{X,Y}$ and $t_{X,1_e} = 0$.

The transformation $t_{X,Y}$ defines a deformation of the symmetric monoidal structure of $C$ to a braided monoidal structure: if $\epsilon$ is a formal parameter with $\epsilon^2 = 0$ and $C_{\epsilon}$ is the same as $C$ but with $\text{Hom}_{C_{\epsilon}}(X,Y) = \text{Hom}_{C}(X,Y)[\epsilon]$ then $\beta_{X,Y} = \sigma_{X,Y} \circ (\text{id}_X \otimes \epsilon + t_{X,Y}/2)$ is a braiding on $C_{\epsilon}$ (with the monoidal structure inherited from $C$).

Example 1. Let $\mathfrak{g}$ be a Lie algebra over $K$ and let $t \in (S^2 \mathfrak{g})^\circ$. The category of $U\mathfrak{g}$-modules is infinitesimally braided, with $t_{X,Y}$ given by the action of $t \in \mathfrak{g} \otimes \mathfrak{g} \subseteq U\mathfrak{g} \otimes U\mathfrak{g}$.

Let $C$ be an i-braided category and let $C_\hbar$ be as $C$, with $\text{Hom}_{C_\hbar}(X,Y) = \text{Hom}_C(X,Y)[\hbar]$. Following Drinfeld [1], we can make $C_\hbar$ to a braided monoidal category (extending the first order deformation $C_\epsilon$) in the following way. Let $\Phi \in K\langle x, y \rangle$

be an element which is group-like element w.r.t the coproduct $\Delta x = x \otimes 1 + 1 \otimes x$, $\Delta y = y \otimes 1 + 1 \otimes y$. 

Let us define a new braiding on $\mathcal{C}_h$ by

$$\beta_{X,Y} = \sigma_{X,Y} \circ e^{ht_{X,Y}/2}$$

and the new associativity constraint $\gamma_{X,Y,Z}$ by

$$(X \otimes Y) \otimes Z \xrightarrow{\Phi(h_{X,Y},h_{Y,Z})} (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z).$$

**Remark.** If $\mathcal{C}$ is enriched over coalgebras and $t_{X,Y}$ are primitive then the new braidings and associativity constraints are group-like. This is the reason for demanding $\Phi$ to be group-like and also for choosing $e^{ht_{X,Y}/2}$ among all power series $1 + ht_{X,Y}/2 + \ldots$ in $t_{X,Y}$.

The pentagon and hexagon relations for $\beta_{X,Y}$’s and $\gamma_{X,Y,Z}$ translate to the following properties of $\Phi$:

**Proposition 1 (Π).** The category $\mathcal{C}_h$ with the natural transformations $\beta$ and $\gamma$ is a braided monoidal category provided $\Phi$ is a Drinfeld associator, i.e. if it satisfies the relations

$$\Phi(y, x) = \Phi(x, y)^{-1},$$

$$e^{x/2} \Phi(y, x) e^{h/2} \Phi(z, y) e^{z/2} \Phi(x, z) = 1 \text{ where } z = -x - y,$$

$$\Phi^{i,j,k} \Phi^{j,k,l} \Phi^{k,l,i} = \Phi^{i,k,l} \Phi^{j,l,i} \Phi^{k,i,j} \Phi^{1,2,3,4} \Phi^{1,2,3,4} = \Phi^{1,2,3,4} \Phi^{1,2,3,4}.$$

The last relation takes place in the algebra generated by symbols $t^{i,j}$ $(1 \leq i, j \leq 4, i \neq j, t^{i,j} = t^{j,i})$ modulo the relations

$$[t^{i,j}, t^{i,k} + t^{j,k}] = 0 \text{ and } [t^{i,j}, t^{k,l}] = 0 \text{ if } \{i, j\} \cap \{k, l\} = \emptyset,$$

and $\Phi^{i,j,k} := \Phi(t^{i,k}, t^{j,k})$ with $t^{i,j} := \sum_{i \in A, j \in B} t^{i,j}$. See $\Pi$ for details, and also for a proof of existence of Drinfeld associators for every $K$.

We shall denote the category $\mathcal{D}_h$ with its new braided monoidal structure by $\mathcal{D}_h^\Phi$.

Let us now define infinitesimal versions of commutative coalgebras and of braided comonoidal functors.

**Definition 2.** Let $\mathcal{D}$ be an $i$-braided category. An $i$-cocommutative coalgebra in $\mathcal{D}$ is an object $M$ which is a cocommutative coalgebra in the symmetric monoidal category $\mathcal{D}$, and which satisfies $t_{M,M} \circ \Delta_M = 0$. If $\mathcal{C}$ is another $i$-braided category, an $i$-braided comonoidal functor $F : \mathcal{D} \rightarrow \mathcal{C}$ is a $K$-linear symmetric comonoidal functor $F : \mathcal{D} \rightarrow \mathcal{C}$ such that

$$F(X \otimes Y) \xrightarrow{\Phi} F(X) \otimes F(Y)$$

$$F(t^{X,Y}_h)$$

$$F(X \otimes Y) \xrightarrow{\Phi} F(X) \otimes F(Y)$$

commutes. $F$ is $M$-adapted if it is $M$-adapted as a braided comonoidal functor between the symmetric monoidal categories $\mathcal{D}$ and $\mathcal{C}$.

**Proposition 2.** Let $\mathcal{D}$ and $\mathcal{C}$ be $i$-braided categories. Let $\Phi$ be a Drinfeld associator. If $M \in \mathcal{D}$ is an $i$-cocommutative coalgebra then it is, with the same coproduct and counit, a cocommutative coalgebra in $\mathcal{D}_h^\Phi$. If $F : \mathcal{D} \rightarrow \mathcal{C}$ is an $i$-braided comonoidal functor then it is, with the same comonoidal structure, a braided comonoidal functor $\mathcal{D}_h^\Phi \rightarrow \mathcal{C}_h^\Phi$. If $F : \mathcal{D} \rightarrow \mathcal{C}$ is $M$-adapted then it remains $M$-adapted as a functor $\mathcal{D}_h^\Phi \rightarrow \mathcal{C}_h^\Phi$. 

Proof. If \( F : \mathcal{D} \to \mathcal{C} \) is i-braided comonoidal then the braided comonoidality of \( F : \mathcal{D}_h^\mathcal{K} \to \mathcal{C}_h^\mathcal{K} \) is an immediate consequence of the definitions.

Let \( \mathcal{I} \) be the symmetric monoidal category with a unique object \( I \) and with \( \text{Hom}(I, I) = K \). Let us make it i-braided via \( t_{I, I} = 0 \). An i-cocommutative coalgebra \( M \in \mathcal{D} \) is equivalent to an i-braided functor \( G : I \to \mathcal{D} \), with \( M = G(I) \). The functor \( G \) is thus braided comonoidal as a functor \( 1_h = 1_h^\mathcal{K} \to \mathcal{D}_h^\mathcal{K} \), which means that \( G(I) = M \) is a cocommutative coalgebra in \( \mathcal{D}_h^\mathcal{K} \).

**Example 2.** Let \( \mathfrak{d} \) be a Lie algebra and \( t \in (S^2 \mathfrak{d})^0 \). Let us suppose that \( \dim \mathfrak{d} < \infty \) and that \( t \) is non-degenerate, and thus defines a symmetric pairing \( (,): \mathfrak{d} \times \mathfrak{d} \to K \).

Let \( \mathfrak{g} \subset \mathfrak{d} \) be a Lie subalgebra which is Lagrangian w.r.t. the pairing, i.e. \( \mathfrak{g}^\perp = \mathfrak{g} \).

Then the functor
\[
F : U\mathfrak{d}\text{-}\text{Mod} \to \mathcal{V}ect, \quad F(V) = V/(\mathfrak{g} \cdot V),
\]
with the obvious comonoidal structure
\[
(V \otimes W)/(\mathfrak{g} \cdot (V \otimes W)) \to (V/(\mathfrak{g} \cdot V)) \otimes (W/(\mathfrak{g} \cdot W))
\]
is i-braided, as the projection of \( t \) to \( S^2(\mathfrak{d}/\mathfrak{g}) \) vanishes.

If \( \mathfrak{g}^\ast \subset \mathfrak{d} \) is another Lagrangian Lie subalgebra, such that \( \mathfrak{g} \cap \mathfrak{g}^\ast = 0 \), i.e. if \( (\mathfrak{g}, \mathfrak{g}^\ast \subset \mathfrak{d}) \) is a Manin triple, then
\[
M = U\mathfrak{d}/(U\mathfrak{d})\mathfrak{g}^\ast,
\]
with the coalgebra structure inherited from \( U\mathfrak{d} \), is i-cocommutative. The reason is again that the image of \( t \) in \( S^2(\mathfrak{d}/\mathfrak{g}^\ast) \) vanishes. The functor \( F \) is \( M \)-adapted.

The Hopf algebra \( F(M \otimes M) \) in \( \mathcal{V}ect_h \) (given by Proposition \( \mathcal{I} \) and Theorem \( \mathcal{I} \)) is a quantization of the Lie bialgebra \( \mathfrak{g} \). We discuss this quantization in detail in Section 5.

**Example 3.** More generally, let \( \mathfrak{p} \subset \mathfrak{d} \) be a coisotropic Lie subalgebra, i.e. such that \( \mathfrak{p}^\perp \subset \mathfrak{p} \). Notice that \( \mathfrak{p}^\perp \) is an ideal in \( \mathfrak{p} \). Let \( \mathfrak{h} = \mathfrak{p}/\mathfrak{p}^\perp \). The functor
\[
F : U\mathfrak{d}\text{-}\text{Mod} \to U\mathfrak{h}\text{-}\text{Mod}, \quad F(V) = V/(\mathfrak{p}^\perp \cdot V)
\]
is i-braided comonoidal.

Let \( \mathfrak{p} \subset \mathfrak{d} \) be another coisotropic Lie subalgebra such that \( \mathfrak{d} = \mathfrak{p} \oplus \mathfrak{p}^\perp \) as a vector space (for example, \( \mathfrak{d} \) can be semisimple with the Cartan-Killing form and \( \mathfrak{p} \) and \( \mathfrak{p}^\perp \) a pair of opposite parabolic subalgebras; \( \mathfrak{p}^\perp \subset \mathfrak{p} \) is then the nilpotent radical of \( \mathfrak{p} \)).

If we set
\[
M = U\mathfrak{d}/(U\mathfrak{d})\mathfrak{p},
\]
then \( F \) is \( M \)-adapted.

Proposition \( \mathcal{I} \) and Theorem \( \mathcal{I} \) now make \( F(M \otimes M) \) to a Hopf algebra in the braided monoidal category \( (U\mathfrak{h}\text{-}\text{Mod})^h_\mathcal{K} \). The object \( F(M \otimes M) \) can be naturally identified with \( U(\mathfrak{p}^\perp) \), and we get a deformation of the standard Hopf algebra structure on \( U(\mathfrak{p}^\perp) \).

5. Quantization of Lie bialgebras

Let us recall that if \( \mathfrak{g} \) is a Lie algebra and if \( m_h \) and \( \Delta_h \) are formal deformation of the product and of the coproduct on \( U\mathfrak{g} \), making \( U\mathfrak{g} \) (with the original unit and counit) to a bialgebra in \( \mathcal{V}ect_h \), then
\[
\delta(x) := (\Delta_h - \Delta_h^{op})(x)/h \mod h, \quad x \in \mathfrak{g} \subset U\mathfrak{g},
\]
is a Lie cobracket and that it makes \( \mathfrak{g} \) to a Lie bialgebra. The quantization problem of Lie bialgebras is the problem of constructing \( m_h \) and \( \Delta_h \) out of \( [,] \) and \( \delta \) in a functorial/universal way.
To solve the problem, let us reformulate Example 2 so that it works for infinite-dimensional Lie bialgebras and is functorial with respect to Lie bialgebra morphisms. Let $g$ be a Lie bialgebra over a field $K$ of characteristic 0, with cobracket $\delta : g \to g \otimes g$. Let $D$ be the category whose objects are vector spaces with an action of the Lie algebra $g$

$$\rho : g \otimes V \to V$$

and with a right coaction of the Lie coalgebra $g$

$$\tilde{\rho} : V \to V \otimes g$$

such that

$$\tilde{\rho} \circ \rho = (\rho \otimes \text{id}) \circ (\text{id} \otimes \tilde{\rho}) + (\rho \otimes \text{id}) \circ \sigma_{23} \circ (\delta \otimes \text{id}) + (\text{id} \otimes [\cdot , \cdot]) \circ \sigma_{12} \circ (\text{id} \otimes \tilde{\rho}),$$

or equivalently, such that the resulting map

$$(\rho + \tilde{\rho}^*) : (g \otimes g^*) \otimes V \to V$$

is an action of the double $\mathfrak{d} = g \otimes g^*$. If $\dim g < \infty$ then $D$ is simply the category of $U\mathfrak{d}$-modules; in general it is its (full) subcategory.

The category $D$ is i-braided, with

$$t_{V,W} : V \otimes W \to V \otimes W$$

given in terms of $\rho$ and $\tilde{\rho}$ as

$$t_{V,W} = r_{V,W} + \sigma_{V,W} \circ r_{W,V} \circ \sigma_{V,W},$$

where

$$r_{V,W} : V \otimes W \to V \otimes W$$

is the composition

$$V \otimes W \xrightarrow{\tilde{\rho} \otimes \text{id}_W} V \otimes g \otimes W \xrightarrow{\text{id}_V \otimes \rho_W} V \otimes W.$$

Let us now define an i-cocommutative coalgebra $M$ in $D$. Let

$$M = U\mathfrak{g}$$

with the $g$-action

$$\rho_M(x \otimes y) = xy \quad (x \in g, y \in U\mathfrak{g})$$

and with the coaction $\tilde{\rho}_M$ uniquely determined by

$$\tilde{\rho}_M(1) = 0;$$

in particular,

$$\tilde{\rho}_M(x) = \delta(x) \text{ for } x \in g.$$ The usual coproduct $\Delta : M \to M \otimes M$ and counit $\epsilon : M \to K$ of $U\mathfrak{g}$ make $M$ to an i-cocommutative coalgebra in $D$.

**Remark.** The coalgebra $M$ plays an important role in the quantization of Etingof and Kazhdan [2], where it is denoted $M_-$. Despite this similarity, the relation between these two quantization methods is unclear to me. For technical reasons Etingof and Kazhdan had to replace $D$ with a somewhat complicated category of equicontinuous $U\mathfrak{d}$-modules.

Let $\text{Vect}$ denote the symmetric monoidal category of vector spaces over $K$ (we can see it as i-braided with $t_{X,Y} = 0$ for all $X, Y \in \text{Vect}$). Let $F : D \to \text{Vect}$ be given by

$$F(V) = V/(g \cdot V).$$

It is an i-braided comonoidal functor, which is $M$-adapted, with the obvious comonoidal structure

$$(V \otimes W)/(g \cdot (V \otimes W)) \to V/(g \cdot V) \otimes W/(g \cdot W).$$
We have a linear bijection
\[ F(M \otimes M) \cong U\mathfrak{g} \]
given by
\[ [x \otimes y] \mapsto S_0(x)y \quad (x, y \in U\mathfrak{g}) \]
where \( S_0 \) is the usual antipode on \( U\mathfrak{g} \) and \([x \otimes y]\) denotes the class of \( x \otimes y \in M \otimes M \) in \( F(M \otimes M) \). The inverse of this bijection is given by \( x \mapsto [1 \otimes x], x \in U\mathfrak{g} \).

Let is now choose a Drinfeld associator \( \Phi \) over \( K \) and consider the braided monoidal category \( \mathcal{D}^h_{\text{br}} \). By Theorem 1 \( F(M \otimes M) \) becomes a Hopf algebra in \( \mathcal{V}ect_h \).

**Theorem 2.** The Hopf algebra structure on \( F(M \otimes M) \cong U\mathfrak{g} \) in \( \mathcal{V}ect_h \) is a deformation of the cocommutative Hopf algebra \( U\mathfrak{g} \), and its classical limit is the Lie bialgebra structure on \( \mathfrak{g} \). It is functorial with respect to Lie bialgebra morphisms.

**Proof.** Let us identify \( F((M \otimes M) \otimes M) \) with \( U\mathfrak{g} \otimes U\mathfrak{g} \) via the linear map
\[ x \otimes y \mapsto [S_0(x) \otimes 1 \otimes y] \quad (x \otimes y \in U\mathfrak{g} \otimes U\mathfrak{g}) \]
The isomorphism in \( \mathcal{V}ect_h \)
\[ \tau_{M,M} : F((M \otimes M) \otimes M) \rightarrow F(M \otimes M) \otimes F(M \otimes M) \]
then becomes an isomorphism (of vector spaces)
\[ U\mathfrak{g} \otimes U\mathfrak{g} \rightarrow U\mathfrak{g} \otimes U\mathfrak{g}, \]
which is, by the definition of \( \tau_{M,M} \) and by (2), of the form
\[ x \otimes y \mapsto x \otimes y + O(h). \]
On the other hand, the map
\[ F((M \otimes M) \otimes M) \xrightarrow{F(\text{id} \otimes \otimes \text{id})} F(M \otimes M) \]
becomes under these identifications
\[ x \otimes y \mapsto xy, \quad U\mathfrak{g} \otimes U\mathfrak{g} \rightarrow U\mathfrak{g}. \]
The product on \( F(M \otimes M) \cong U\mathfrak{g} \) is thus \( x \otimes y \mapsto xy + O(h) \).

Let us now compute the coproduct \( \Delta_h \) on \( F(M \otimes M) \cong U\mathfrak{g} \) to first order in \( h \).

Let us recall that \( M \otimes M \) is a coalgebra in \( \mathcal{D}^h_{\text{br}} \), with the coproduct given by (1) (with \( M' = M \)). For \( x \in \mathfrak{g} \) we get
\[ \Delta_{M \otimes M}(1 \otimes x) = 1 \otimes x \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes x - \frac{h}{2} 1 \otimes \delta(x) \otimes 1 + O(h^2). \]
The coproduct \( \Delta_h \) on \( F(M \otimes M) \cong U\mathfrak{g} \) is thus
\[ \Delta_h(x) = x \otimes 1 + 1 \otimes x + \frac{h}{2} \delta(x) + O(h^2) \]
where the sign change comes from \((\text{id} \otimes S_0)(\delta(x)) = -\delta(x))\), hence
\[ (\Delta_h - \Delta^0_h)(x) = h \delta(x) + O(h^2), \]
as we wanted to show.

Functoriality of the deformed Hopf algebra structure on \( U\mathfrak{g} \) in Lie bialgebra morphisms follows from functoriality of \( \mathcal{D}, M, F, \) and of the isomorphism (2).

Let us stress that the Hopf algebra structure on \( F(M \otimes M) \cong U\mathfrak{g} \) depends on the associator \( \Phi \). Curiously, the antipode is independent of \( \Phi \).
Remark (Quantization of infinitesimally braided Lie bialgebras). If we apply Theorem 1 and Proposition 2 to Example 3, we make
\[ F(M \otimes M) \cong U(p^+) \]
to a Hopf algebra in the braided monoidal category \( Uh\text{-Mod}^\Phi \), deforming the standard Hopf algebra structure on \( U(p^+) \). This is a special case of quantization of Lie bialgebras in (Abelian) \( i \)-braided categories, which is a minor generalization of what we did in above.

By a Lie bialgebra in an \( i \)-braided category \( \mathcal{C} \) we mean an object \( g \in \mathcal{C} \), together with a Lie bracket \( \mu : g \otimes g \to g \) and a Lie cobracket \( \delta : g \to g \otimes g \) such that
\[
\delta \circ \mu = (\mu \otimes \text{id}) \circ (\text{id} \otimes \delta) \circ (\text{id} \otimes \sigma_{12}) + t_{g,g} \circ (\text{id} \otimes \sigma_{12})/2
\]
(besides the \( t_{g,g} \)-term, it is the standard definition of a Lie bialgebra). As an example, \( p^+ \) is a Lie bialgebra in the \( i \)-braided category \( Uh\text{-Mod} \), with the cobracket given by the Lie bracket on \( \bar{p} \).

Let \( \mathcal{D} \) be the category whose objects are objects \( V \) of \( \mathcal{C} \) together with a left action \( \rho : g \otimes V \to V \) and a right coaction \( \tilde{\rho} : V \to V \otimes g \), such that
\[
\tilde{\rho} \circ \rho = (\rho \otimes \text{id}) \circ (\text{id} \otimes \tilde{\rho}) + (\rho \otimes \text{id}) \circ \sigma_{23} \circ (\delta \otimes \text{id}) + (\text{id} \otimes \mu) \circ (\text{id} \otimes \tilde{\rho}) + \sigma_{g,V} \circ t_{g,V}.
\]
Category \( \mathcal{D} \) is \( i \)-braided, with
\[
t^\mathcal{D}_{V,W} = r_{V,W} + \sigma_{V,W} \circ r_{W,V} \circ \sigma_{V,W} + t^\mathcal{E}_{V,W}.
\]

Supposing that \( \mathcal{C} \) is an Abelian category, so that we can make sense of \( U\mathfrak{g} \) and of \( \mathfrak{g} \)-coinvariants, we define \( M = U\mathfrak{g} \in \mathcal{D} \) and \( F : \mathcal{D} \to \mathcal{C} \) as above, and \( F(M \otimes M) \) becomes a Hopf algebra in \( \mathcal{C}_h^\Phi \) deforming \( U\mathfrak{g} \).

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