On $\omega_3$-chains in $\mathcal{P}(\omega_1) \text{ mod finite}$

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Abstract

It is consistent that there exists a sequence $\langle X_\alpha \mid \alpha < \omega_3 \rangle$ of subsets $X_\alpha \subseteq \omega_1$ such that $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ is uncountable for all $\beta < \alpha < \omega_3$. Such a sequence is added by a ccc forcing which is constructed along a simplified $(\omega_1, 2)$-morass. The idea of the proof is to use a finite support iteration of countable forcings which is not linear but three-dimensional. In the same way it is possible to construct along a simplified $(\omega_1, 2)$-morass a ccc forcing which adds $\omega_3$ many distinct functions $f_\alpha : \omega_1 \rightarrow \omega$ such that $\{ \xi < \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi) \}$ is finite for all $\alpha < \beta < \omega_3$.

1 Introduction

Cardinality is one of the central notions in set theory. Accordingly, many questions in set theory ask how big something can be. The most prominent example is of course the question of whether the continuum hypothesis is true: How large is $\mathcal{P}(\omega)$? The method of forcing allows to construct (models with) large structures of a certain kind quite freely. This becomes considerably more complicated once a second cardinal comes into play. Examples of such structures are $\mathcal{P}(\omega_1)$ modulo finite, $\omega_1^{\omega_1}$ modulo finite and the like. Possible questions are how large $\mathcal{P}(\omega_1)$ modulo finite or $\omega_1^{\omega_1}$ modulo finite can be, and how long chains in $\mathcal{P}(\omega_1)$ modulo finite or $\omega_1^{\omega_1}$ modulo finite can be.

P. Koszmider [16] proved that it is consistent that there exists a sequence $\langle X_\alpha \mid \alpha < \omega_2 \rangle$ of subsets $X_\alpha \subseteq \omega_1$ such that $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ is uncountable for all $\beta < \alpha < \omega_2$. He uses S. Todorcevic’s method of ordinal walks [25]. It is also known as the method of $\rho$-functions [15] and provides a powerful tool to construct ccc forcings in the presence of $\Box_{\omega_1}$. Other applications are a ccc forcing that adds an $\omega_2$-Suslin tree [25], a ccc forcing for $\omega_2 \not\rightarrow (\omega : 2)^2_\omega$ [25], a ccc forcing to add a Kurepa tree [25, 26] and a ccc forcing to add a thin-very tall superatomic Boolean algebra [25]. The last forcing was first found by Baumgartner and Shelah [2] independently from $\rho$-functions. That there can be a ccc forcing for $\omega_2 \not\rightarrow (\omega : 2)^2_\omega$ was first observed by Galvin [14]. That $\Box_{\omega_1}$ implies the existence of a ccc forcing which adds a Kurepa tree was first proved by Jensen [12, 11].

All these examples add a structure on $\omega_2$. The natural question arises if something similar can be done for forcings that add a structure on a higher cardinal.
Using morasses instead of \(\square_{\omega_1}\) this seems possible. As an example, we show that if there exists a simplified \((\omega_1, 2)\)-morass, then there exists a ccc forcing which adds a sequence \(\langle X_\alpha \mid \alpha < \omega_3 \rangle\) of subsets \(X_\alpha \subseteq \omega_1\) such that \(X_\beta - X_\alpha\) is finite and \(X_\alpha - X_\beta\) is uncountable for all \(\beta < \alpha < \omega_3\). In the same way it is possible to construct along a simplified \((\omega_1, 2)\)-morass a ccc forcing which adds \(\omega_3\) many distinct functions \(f_\alpha : \omega_1 \to \omega\) such that \(\{\xi < \omega_1 \mid f_\alpha (\xi) = f_\beta (\xi)\}\) is finite for all \(\alpha < \beta < \omega_3\).

The second example looks simpler, because the order of the elements doesn’t play a role in this case. And indeed, it is known that there can be families \(\{f_\alpha : \omega_1 \to \omega \mid \alpha \in \kappa\}\) of arbitrary prescribed size such that \(\{\xi < \omega_1 \mid f_\alpha (\xi) = f_\beta (\xi)\}\) is finite for all \(\alpha < \beta < \kappa\). This was proved by J. Zapletal [30] using proper forcing and Todorcevic’s method of side conditions [23, 24]. Moreover, an old result of Baumgartner’s [1] is that for any given \(\kappa\) there is consistently a family of size \(\kappa\) of cofinal subsets of \(\omega_1\) with pairwise finite intersections.

We will construct our forcings by induction along the morass. In previous papers [9, 10], we introduced three methods of constructing forcings along a simplified morasses: Finite support systems along gap-1 morasses, FS systems along gap-2 morasses and local FS systems along gap-1 morasses. Using this terminology, we will use a local FS system along a gap-2 morass to add an \(\omega_3\)-chain in \(\mathcal{P}(\omega_1)\) mod finite or to add our family of strongly almost disjoint functions. As applications of (local) FS systems along simplified morasses, we proved [9, 10]: (1) There exists consistently a ccc forcing of size \(\omega_1\) which adds an \(\omega_2\)-Suslin tree. (2) There exists consistently a ccc forcing of size \(\omega_1\) that adds a 0-dimensional Hausdorff topology \(\tau\) on \(\omega_3\) which has spread \(\omega_1\). (3) There exists consistently a ccc forcing which adds a chain \(\langle X_\alpha \mid \alpha < \omega_2 \rangle\) such that \(X_\alpha \subseteq \omega_1\), \(X_\beta - X_\alpha\) is finite and \(X_\alpha - X_\beta\) has size \(\omega_1\) for all \(\beta < \alpha < \omega_2\). The second statement implies that the existence of a 0-dimensional Hausdorff \(X\) space with spread \(\omega_1\) and size \(2^{\text{spread}(X)}\) is consistent.

The basic idea of all our constructions is simple: We try to generalize iterated forcing with finite support (FS). Classical iterated forcing with finite support as introduced by Solovay and Tenenbaum [20] works with continuous, commutative systems of complete embeddings which are indexed along a well-order. The following holds: If every forcing of the system satisfies a chain condition, then also the direct limit does. Assume for example that all forcings of the system are countable. Then its direct limit satisfies ccc. Assume, moreover, that we want to construct a forcing of size \(\omega_2\). Then taking the direct limit will not work, because in our case the limit forcing has size \(\leq \omega_1\). To overcome this difficulty, we do not consider a linear system which is indexed along a well-order but a higher-dimensional system indexed along a simplified morass.

As the examples in [9, 10] show, often consistency statements like above cannot be extended by simply raising the cardinal parameters. The reason why such a generalization could not work is that the higher-gap case yields a higher-dimensional construction. Therefore, the finite conditions of our forcing have to fit together appropriately in more directions and that might be impossible. Hence if and how a statement generalizes to higher-gaps depends heavily on the
concrete conditions. However, in the case of almost disjoint functions we know from Zapletal’s \cite{30} work that the existence of a family of \( \kappa \) many distinct, almost disjoint functions \( f : \omega_1 \to \omega \) is consistent for all \( \kappa \). So the question arises if and how this can be proved by ccc forcings over \( L \). In the case of chains in \( \mathcal{P}(\omega_1) \) mod finite, we can even hope to get chains of arbitrary prescribed size. If \( \mathbb{P} \) is the limit of a finite support iteration indexed along \( \alpha \), then we can understand a \( \mathbb{P} \)-generic extension as being obtained successively in \( \alpha \)-many steps. Moreover, there are names for the forcings used in the single steps. In the case of FS systems, it is unclear what a similar analysis looks like, but if we had it, it was completely justified to think of our constructions as higher-dimensional FS forcing iterations.

Morasses were introduced by R. Jensen in the early 1970’s to solve the cardinal transfer problem of model theory in \( L \) (see e.g. Devlin \cite{3}). For the proof of the gap-2 transfer theorem a gap-1 morass is used. For higher-gap transfer theorems Jensen has developed so-called higher-gap morasses \cite{13}. In his Ph.D. thesis, the author generalized these to gaps of arbitrary size \cite{8, 7, 6}. The theory of morasses is very far developed and very well examined. In particular it is known how to construct morasses in \( L \) \cite{3, 5, 8, 6} and how to force them \cite{21, 22}. Moreover, D. Velleman has defined so-called simplified morasses, along which morass constructions can be carried out very easily compared to classical morasses \cite{27, 29, 28}. Their existence is equivalent to the existence of usual morasses \cite{4, 18}. The fact that the theory of morasses is so far developed is an advantage of the morass approach compared to historic forcing or \( \rho \)-functions. It allows canonical generalizations to higher cardinals, as shown below.

Finally, we should stress that not everything can be done by ccc forcings. For example, Koszmider proved that if CH holds, then there is no ccc forcing that adds a sequence of \( \omega_2 \) many functions \( f : \omega_1 \to \omega_1 \) which is ordered by strict domination mod finite. However, he is able to produce a proper forcing which adds such a sequence \cite{17} using his method of side conditions in morasses which is an extension of Todorcevic’s method of side conditions in models. More on the method can be found in Morgan’s paper \cite{19}. In the context of our approach, this raises the question if it is possible to define something like a countable support iteration along a morass.

\section{Two forcings}

We want to add a chain \( \langle X_\alpha \mid \alpha < \omega_3 \rangle \) such that \( X_\alpha \subseteq \omega_1 \), \( X_\beta - X_\alpha \) is finite and \( X_\alpha - X_\beta \) has size \( \omega_1 \) for all \( \beta < \alpha < \omega_3 \). The natural forcing to do this would be

\[ P := \{ p : a_p \times b_p \to 2 \mid a_p \times b_p \subseteq \omega_3 \times \omega_1 \text{ finite} \} \]

where we set \( p \leq q \) if and only if \( q \subseteq p \) and

\[ \forall \alpha_1 < \alpha_2 \in a_q \forall \beta \in b_p - b_q \ p(\alpha_1, \beta) \leq p(\alpha_2, \beta). \]
Obviously, we will set
\[ X_\alpha = \{ \beta \in \omega_1 \mid p(\alpha, \beta) = 1 \text{ for some } p \in G \} \]
for a \( P \)-generic \( G \).

Similarly, to add \( \omega_3 \) many distinct functions \( f_\alpha : \omega_1 \to \omega \) such that \{ \xi \in \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi) \} \) is finite for all \( \alpha < \beta < \omega_3 \), the natural forcing would be
\[ P := \{ p : a_p \times b_p \to \omega \mid a_p \subseteq \omega_3, b_p \subseteq \omega_1, p \text{ finite} \} \]
where \( p \leq q \) if \( q \subseteq p \) and
\[ \forall \alpha, \beta \in a_q \ \forall \xi \in b_p - b_q \ p(\xi, \alpha) \neq p(\xi, \beta). \]
We set \( F = \bigcup \{ p \mid p \in G \} \) for some \( P \)-generic \( G \) and \( f_\alpha(\xi) = F(\alpha, \xi) \) for all \( \alpha < \omega_3, \xi < \omega_1 \).

The forcings do not have ccc. Therefore, we want to thin out \( P \) so that the remaining forcing satisfies ccc. More precisely, we want to thin it out so that for every \( \Delta \subseteq \omega_3 \)
\[ P_\Delta := \{ p \in P \mid a_p \subseteq \Delta \} \]
satisfies ccc. Moreover, we want that there remain enough conditions that the following proof still works: Let \( A \) be an uncountable set of conditions. Let w.l.o.g. \( \{ a_p \mid p \in A \} \) be a \( \Delta \)-system with root \( \Delta \). Consider \( \{ p \mid (\Delta \times \omega_3) \mid p \in A \} \). Then there are \( p \neq q \in A \) such that \( p \upharpoonright (\Delta \times \omega_3) \) and \( q \upharpoonright (\Delta \times \omega_3) \) are compatible. Hence, \( p \) and \( q \) are compatible, too.

To thin out \( P \), we use morasses.

### 3 Morasses

In this section, we summarize the theory of simplified gap-2 morasses to make our paper self-contained and to introduce the notations which we will use in the rest of the paper. Simplified morasses were introduced by D. Velleman in his papers \[27, 29\] where one can also find most of the proofs of the following results (see also \[9, 10\]).

A simplified \((\kappa, 1)\)-morass is a structure \( \mathfrak{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathfrak{F}_{\alpha \beta} \mid \alpha < \beta \leq \kappa \rangle \rangle \) satisfying the following conditions:

(P0) (a) \( \theta_0 = 1, \theta_\kappa = \kappa^+, \forall \alpha < \kappa \ 0 < \theta_\alpha < \kappa. \)

(P1) \( |\mathfrak{F}_{\alpha \beta}| < \kappa \) for all \( \alpha < \beta < \kappa. \)

(P2) If \( \alpha < \beta < \gamma, \) then \( \mathfrak{F}_{\alpha \gamma} = \{ f \circ g \mid f \in \mathfrak{F}_{\beta \gamma}, g \in \mathfrak{F}_{\alpha \beta} \} \).

(P3) If \( \alpha < \kappa, \) then \( \mathfrak{F}_{\alpha, \alpha+1} = \{ id \mid \theta_\alpha, f_\alpha \} \) where \( f_\alpha \) is such that \( f_\alpha \upharpoonright \delta = id \upharpoonright \delta \) and \( f_\alpha(\delta) \geq \theta_\alpha \) for some \( \delta < \theta_\alpha. \)

(P4) If \( \alpha < \kappa \) is a limit ordinal, \( \beta_1, \beta_2 < \alpha \) and \( f_1 \in \mathfrak{F}_{\beta_1, \alpha}, f_2 \in \mathfrak{F}_{\beta_2, \alpha} \), then there
are a $\beta_1, \beta_2 < \gamma < \alpha$, $g \in \mathcal{F}_{\gamma \alpha}$ and $h_1 \in \mathcal{F}_{\beta_1 \gamma}$, $h_2 \in \mathcal{F}_{\beta_2 \gamma}$ such that $f_1 = g \circ h_1$ and $f_2 = g \circ h_2$.

(P5) For all $\alpha > 0$, $\theta_\alpha = \bigcup \{ f[\theta_\beta] \mid \beta < \alpha, f \in \mathcal{F}_{\beta \alpha} \}$.

Lemma 3.1
Let $\alpha < \beta \leq \kappa$, $\tau_1, \tau_2 < \theta_\alpha$, $f_1, f_2 \in \mathcal{F}_{\alpha \beta}$ and $f_1(\tau_1) = f_2(\tau_2)$. Then $\tau_1 = \tau_2$ and $f_1 \upharpoonright \tau_1 = f_2 \upharpoonright \tau_2$.

A simplified morass defines a tree $\langle T, \prec \rangle$.
Let $T = \{ \langle \alpha, \nu \rangle \mid \alpha \leq \kappa, \nu < \theta_\alpha \}$.
For $t = \langle \alpha, \nu \rangle \in T$ set $\alpha(t) = \alpha$ and $\nu(t) = \nu$.
Let $\langle \alpha, \nu \rangle \prec \langle \beta, \tau \rangle$ iff $\alpha < \beta$ and $f(\nu) = \tau$ for some $f \in \mathcal{F}_{\alpha \beta}$.
If $s \prec t$, then $f \upharpoonright (\nu(s) + 1)$ does not depend on $f$ by lemma 3.1. So we may define $\pi_{st} := f \upharpoonright (\nu(s) + 1)$.

Lemma 3.2
The following hold:
(a) $\prec$ is a tree, $ht_T(t) = \alpha(t)$.
(b) If $t_0 \prec t_1 \prec t_2$, then $\pi_{t_1 t_2} = \pi_{t_1 t_2} \circ \pi_{t_0 t_1}$.
(c) Let $s \prec t$ and $\pi = \pi_{st}$. If $\pi(\nu') = \tau'$, $s' = \langle \alpha(s), \nu' \rangle$ and $t' = \langle \alpha(t), \tau' \rangle$, then $s' \prec t'$ and $\pi_{s' t'} = \pi \upharpoonright (\nu' + 1)$.
(d) Let $\gamma \leq \kappa$, $\gamma \in \text{Lim}$. Let $t \in T_\gamma$. Then $\nu(t) + 1 = \bigcup \{ \text{rng}(\pi_{st}) \mid s \prec t \}$.

A fake gap-1 morass is a structure $\langle \langle \varphi_\zeta \mid \zeta \leq \theta \rangle, \langle \mathfrak{G}_{\zeta \xi} \mid \zeta < \xi \leq \theta \rangle \rangle$ which satisfies the definition of simplified gap-1 morass, except that $\theta$ need not be a cardinal and there is no restriction on the cardinalities of $\varphi_\zeta$ and $\mathfrak{G}_{\zeta \xi}$. Let $\mathfrak{G}_{\zeta \xi \theta + 1} = \{ id, b \}$. Then the critical point of $b$ is denoted by $\sigma_\zeta$ and called the split point of $\mathfrak{G}_{\zeta \xi \theta + 1} = \{ id, b \}$.

Suppose that $\langle \langle \varphi_\zeta \mid \zeta \leq \theta \rangle, \langle \mathfrak{G}_{\zeta \xi} \mid \zeta < \xi \leq \theta \rangle \rangle$ and $\langle \langle \varphi_\zeta' \mid \zeta \leq \theta' \rangle, \langle \mathfrak{G}_{\zeta \xi} \mid \zeta < \xi \leq \theta' \rangle \rangle$ are fake gap-1 morasses. An embedding from the first one to the second will be a function $f$ with domain

$$(\theta + 1) \cup \{ \langle \zeta, \tau \rangle \mid \zeta \leq \theta, \tau < \varphi_\zeta \} \cup \{ \langle \zeta, \xi, b \rangle \mid \zeta < \xi \leq \theta, b \in \mathfrak{G}_{\zeta \xi} \}$$

satisfying certain requirements. We will write $f_\zeta(\tau)$ for $f(\langle \zeta, \tau \rangle)$ and $f_{\zeta \xi}(b)$ for $f(\langle \zeta, \xi, b \rangle)$.

The properties are the following ones:
(1) $f \upharpoonright (\theta + 1)$ is an order preserving function from $\theta + 1$ to $\theta' + 1$ such that $f(\theta) = \theta'$.
(2) For all $\zeta \leq \theta$, $f_\zeta$ is an order preserving function from $\varphi_\zeta$ to $\varphi'_f(\zeta)$.
(3) For all $\zeta < \xi \leq \theta$, $f_{\zeta \xi}$ maps $\mathfrak{G}_{\zeta \xi}$ to $\mathfrak{G}'_{f(\zeta), f(\xi)}$. 

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(4) If $\zeta < \theta$, then $f_\zeta(\sigma_\zeta) = \sigma'_{f(\zeta)}$.

(5) If $\zeta < \xi \leq \theta$, $b \in \mathcal{G}_{\zeta\xi}$ and $c \in \mathcal{G}_{\zeta\theta}$, then $f_{\zeta\theta}(c \circ b) = f_{\zeta\theta}(c) \circ f_{\zeta\xi}(b)$.

(6) If $\zeta < \xi \leq \theta$ and $b \in \mathcal{G}_{\zeta\xi}$, then $f_\zeta \circ b = f_{\zeta\xi}(b) \circ f_\zeta$.

Let $\mathcal{F}$ be a family of embeddings from $\langle \langle \varphi_\zeta \mid \zeta \leq \theta \rangle, \langle \mathcal{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta \rangle \rangle$ to $\langle \langle \varphi'_{\zeta} \mid \zeta \leq \theta' \rangle, \langle \mathcal{G}'_{\zeta\xi} \mid \zeta < \xi \leq \theta' \rangle \rangle$. Assume that $\theta < \theta'$, $\varphi'_{\zeta} = \varphi_\zeta$ for $\zeta \leq \theta$ and $\mathcal{G}'_{\zeta\xi} = \mathcal{G}_{\zeta\xi}$ for $\zeta < \xi \leq \theta$. Let $f \mid \theta = id$, $f_\zeta = id$ for all $\zeta < \theta$ and $f_{\zeta\xi} = id$ for all $\zeta < \xi \leq \theta$. Let $f_\theta \in \mathcal{G}_{\theta\theta}$. Define an embedding as follows: If $\zeta < \theta$ and $b \in \mathcal{G}_{\zeta\theta}$, then $f_{\zeta\theta}(b) = f_\theta \circ b$. We call such an embedding $f$ a left-branching embedding. There are many left-branching embeddings, one for every choice of $f_\theta$.

An embedding $f$ is right-branching if for some $\eta < \theta$,

1. $f \mid \eta = id$
2. $f(\eta + \zeta) = \theta + \zeta$ if $\eta + \zeta \leq \theta$
3. $f_\zeta = id$ for $\zeta < \eta$
4. $f_{\zeta\xi} = id$ for $\zeta < \xi < \eta$
5. $f_\eta \in \mathcal{G}_{\eta\theta}$
6. $f_{\zeta\xi}(\mathcal{G}_{\zeta\xi}) = \mathcal{G}'_{f(\zeta)f(\xi)}$ if $\eta \leq \zeta \leq \xi \leq \theta$.

An amalgamation is a family of embeddings that contains all possible left-branching embeddings, exactly one right-branching embedding and nothing else.

Let $\kappa \geq \omega$ be regular and $\langle \langle \varphi_\zeta \mid \zeta \leq \kappa^+ \rangle, \langle \mathcal{G}_{\zeta\xi} \mid \zeta < \xi \leq \kappa \rangle \rangle$ a simplified $(\kappa^+, 1)$-morass such that $\varphi_\zeta < \kappa$ for all $\zeta < \kappa$. Let $\langle \theta_\alpha \mid \alpha < \kappa \rangle$ be a sequence such that $0 < \theta_\alpha < \kappa$ and $\theta_\kappa = \kappa^+$. Let $\langle \mathcal{F}_{\alpha\beta} \mid \alpha < \beta \leq \kappa \rangle$ be such that $\mathcal{F}_{\alpha\beta}$ is a family of embeddings from $\langle \langle \varphi_\zeta \mid \zeta \leq \theta_\alpha \rangle, \langle \mathcal{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta_\alpha \rangle \rangle$ to $\langle \langle \varphi'_{\zeta} \mid \zeta \leq \theta_\beta \rangle, \langle \mathcal{G}'_{\zeta\xi} \mid \zeta < \xi \leq \theta_\beta \rangle \rangle$.

This is a simplified $(\kappa, 2)$-morass if it has the following properties:

1. $|\mathcal{F}_{\alpha\beta}| < \kappa$ for all $\alpha < \beta < \kappa$.
2. If $\alpha < \beta < \gamma$, then $\mathcal{F}_{\alpha\gamma} = \{ f \circ g \mid f \in \mathcal{F}_{\beta\gamma}, g \in \mathcal{F}_{\alpha\beta} \}$. Here $f \circ g$ is the composition of the embeddings $f$ and $g$, which are defined in the obvious way: $(f \circ g)\zeta = f_{g(\zeta)} \circ g_\zeta$ for $\zeta \leq \theta_\alpha$ and $(f \circ g)\xi = f_{g(\zeta)\xi} \circ g_{\xi}$ for $\zeta < \xi \leq \theta_\alpha$.
3. If $\alpha < \kappa$, then $\mathcal{F}_{\alpha, \alpha+1}$ is an amalgamation.
4. If $\alpha \leq \kappa$ is a limit ordinal, $\beta_1, \beta_2 < \alpha$ and $f_1 \in \mathcal{F}_{\beta_1\alpha}$, $f_2 \in \mathcal{F}_{\beta_2\alpha}$, then there are $\beta_1, \beta_2 < \gamma < \alpha$, $g \in \mathcal{F}_{\beta_1\gamma}$, $h \in \mathcal{F}_{\beta_2\gamma}$ such that $f_1 = g \circ h_1$ and $f_2 = g \circ h_2$.
5. For all $\alpha \leq \kappa$, $\alpha \in \text{Lim}$:
   a. $\theta_\alpha = \bigcup\{ f[\theta_\beta] \mid \beta < \alpha, f \in \mathcal{F}_{\beta\alpha} \}$.
   b. For all $\zeta \leq \theta_\alpha$, $\varphi_\zeta = \bigcup\{ f_\zeta[\varphi_\zeta] \mid \exists \beta < \alpha \ (f \in \mathcal{F}_{\beta\alpha} \text{ and } f(\zeta) = \zeta) \}$.
   c. For all $\zeta < \xi \leq \theta_\alpha$, $\mathcal{G}_{\zeta\xi} = \bigcup\{ f_{\zeta\xi}(\mathcal{G}_{\zeta\xi}) \mid \exists \beta < \alpha \ (f \in \mathcal{F}_{\beta\alpha} \text{ and } f(\zeta) = \zeta \text{ and } f(\xi) = \xi) \}$. 


Theorem 3.3
(a) If $V = L$, then there is a simplified $(\kappa, 2)$-morass for all regular $\kappa \geq \omega$.
(b) If $\kappa \geq \omega$ is regular, then there is a forcing $P$ which preserves cardinals and cofinalities such $P \models$ (there is a simplified $(\kappa, 2)$-morass).

Since $\langle \langle \varphi_\zeta \mid \zeta \leq \kappa^+ \rangle, \langle \Theta_{\zeta \xi} \mid \zeta < \xi \leq \kappa^+ \rangle \rangle$ is a simplified $(\kappa^+, 1)$-morass, there is a tree $\langle T, \prec \rangle$ with levels $T_\eta$ for $\eta \leq \kappa^+$ like in lemma 1.2. And there are maps $\pi_{st}$ for $s \prec t$. Moreover, if we set $\overline{\Phi}_{\alpha\beta} = \{ f \upharpoonright \theta_\alpha \mid f \in \overline{\Phi}_{\alpha\beta} \}$, then $\langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \Phi_{\alpha\beta} \mid \alpha < \beta \leq \kappa \rangle \rangle$ is a simplified $(\kappa, 1)$-morass. So there is also a tree $\langle T', \prec' \rangle$ with levels $T'_\eta$ for $\eta \leq \kappa$ like in lemma 3.2 on this morass. Improving lemma 3.1, the following holds:

Lemma 3.4
Suppose $\alpha < \beta \leq \kappa$, $f_1, f_2 \in \overline{\Phi}_{\alpha\beta}$, $\zeta_1, \zeta_2 < \theta_\alpha$ and $f_1(\zeta_1) = f_2(\zeta_2)$. Then $\zeta_1 = \zeta_2$, $f_1 \upharpoonright \zeta_1 = f_2 \upharpoonright \zeta_1$, $(f_1)_{\zeta} = (f_2)_{\zeta}$ for all $\zeta \leq \zeta_1$, and $(f_1)_{\zeta_2} = (f_2)_{\zeta_2}$ for all $\zeta < \eta \leq \zeta_1$.

Now, let $s = (\alpha, \nu) \in T'_\alpha$, $t = (\beta, \tau) \in T'_\beta$ and $s \prec' t$. Then there is some $f \in \overline{\Phi}_{\alpha\beta}$ such that $f(\nu) = \tau$. By lemma 3.4

$$f \upharpoonright ((\nu + 1) \cup \langle \langle \zeta, \tau \rangle \mid \zeta \leq \nu, \tau < \varphi_\zeta \rangle \cup \langle \langle \zeta, b \rangle \mid \zeta < \nu, b \in \Theta_{\zeta \xi} \rangle)$$

does not depend on $f$. So we may call it $\pi'_{st}$.

Lemma 3.5
(a) If $\zeta < \xi \leq \kappa^+$, then $id \upharpoonright \varphi_\zeta \in \Theta_{\zeta \xi}$.
(b) If $\alpha < \beta \leq \kappa$, then there is a $g \in \overline{\Phi}_{\alpha\beta}$ such that $g \upharpoonright \theta_\alpha = id \upharpoonright \theta_\alpha$.

In addition to the maps $f \in \overline{\Phi}_{\alpha\beta}$, we need maps $\tilde{f}$ that are associated to $f$. For a set of ordinals $X$, let $ssup(X)$ be the least $\alpha$ such that $X \subseteq \alpha$. And let $\tilde{f}(\zeta) = ssup(f(\zeta)) \leq f(\zeta)$.

Lemma 3.6
For every $\alpha < \beta \leq \kappa$, $f \in \overline{\Phi}_{\alpha\beta}$ and $\zeta \leq \theta_\alpha$, there are unique functions $\tilde{f}_\zeta : \varphi_\zeta \rightarrow \varphi_{f(\zeta)}$, $\tilde{f}_{\zeta \xi} : \Theta_{\zeta \xi} \rightarrow \Theta_{f(\zeta) f(\xi)}$ for all $\xi < \zeta$, and $f^#(\zeta) \in \Theta_{f(\zeta) f(\xi)}$ such that:

1. $\tilde{f}_\zeta = f^#(\zeta) \circ \tilde{f}_\zeta$
2. $\forall \xi < \zeta \forall b \in \Theta_{\zeta \xi}$ $f_{\zeta \xi}(b) = f^#(\zeta) \circ \tilde{f}_{\zeta \xi}(b)$.

Furthermore, these functions have the following properties:

3. If $\xi < \tilde{f}(\zeta)$ and $b \in \Theta_{\zeta \xi}$, then $\exists \eta < \zeta \exists c \in \Theta_{\zeta \eta} \exists d \in \Theta_{\xi \eta}$ $b = \tilde{f}_{\zeta \xi}(c) \circ d$.
4. $\forall \xi < \zeta \forall b \in \Theta_{\zeta \xi}$ $\tilde{f}_\zeta \circ b = \tilde{f}_{\zeta \xi}(b) \circ \tilde{f}_\zeta$.
5. If $\eta < \xi < \zeta$, $b \in \Theta_{\zeta \xi}$ and $c \in \Theta_{\zeta \eta}$, then $\tilde{f}_{\zeta \xi}(b \circ c) = \tilde{f}_{\xi}(b) \circ \tilde{f}_{\zeta \xi}(c)$.
6. If $\alpha < \beta < \gamma \leq \kappa$, $f \in \overline{\Phi}_{\alpha\beta}$, $g \in \overline{\Phi}_{\beta\gamma}$ and $\zeta \leq \theta_\alpha$, then $$(f \circ g)_{\zeta} = \tilde{f}_{\zeta}(\zeta) \circ \tilde{g}_{\zeta}$$

$(f \circ g)^#(\zeta) = \tilde{f}_{\zeta}(\zeta) g_{\zeta}((f^#(\zeta)) \circ f^#(\tilde{g}(\zeta)))$ and
From the previous lemma, we get of course also maps \((\pi_{st})_\zeta\) for \(s \prec t\) and \(\zeta \leq \nu(t)\).

### 4 The thinning-out – gap-1 step

In the following we thin out the forcing \(P\) to a forcing \(\mathbb{P}\) which satisfies ccc. For the thinning out we assume that a simplified \((\omega_1, 2)\)-morass \(\langle \theta_\alpha \mid \alpha \leq \omega_1, \langle \mathfrak{F}_{\alpha, \beta} \mid \alpha < \beta \leq \omega_1 \rangle \rangle\) is given. Let \(\langle \langle \varphi_\zeta \mid \zeta \leq \omega_2 \rangle, \langle \mathfrak{G}_{\zeta \xi} \mid \zeta < \xi \leq \omega_2 \rangle \rangle\) be the simplified \((\omega_2, 1)\)-morass such that \(\mathfrak{F}_{\alpha, \beta}\) is a family of embeddings from \(\langle \langle \varphi_\zeta \mid \zeta \leq \theta_\alpha \rangle, \langle \mathfrak{G}_{\zeta \xi} \mid \zeta < \xi \leq \theta_\alpha \rangle \rangle\) to \(\langle \langle \varphi_\zeta \mid \zeta \leq \theta_\beta \rangle, \langle \mathfrak{G}_{\zeta \xi} \mid \zeta < \xi \leq \theta_\beta \rangle \rangle\). Now, we proceed in two steps. In the first step, we thin out \(P\) along \(\langle \langle \varphi_\zeta \mid \zeta \leq \omega_2 \rangle, \langle \mathfrak{G}_{\zeta \xi} \mid \zeta < \xi \leq \omega_2 \rangle \rangle\) to a forcing \(P_{\omega_3}\). In the second step we thin out \(P_{\omega_3}\) along \(\langle \langle \theta_\alpha \mid \alpha \leq \omega_1 \rangle, \langle \mathfrak{F}_{\alpha, \beta} \mid \alpha < \beta \leq \omega_1 \rangle \rangle\) to a forcing \(\mathbb{P}\) which satisfies ccc.

The first step of the thinning-out procedure is as the construction of a ccc forcing which adds a chain \(\langle X_\alpha \mid \alpha < \omega_2 \rangle\) such that \(X_\alpha \subseteq \omega_1\), \(X_\beta - X_\alpha\) is finite and \(X_\alpha - X_\beta\) has size \(\omega_1\) for all \(\beta < \alpha < \omega_2\) in [9]. Following the terminology of [9], the first step applies the method of local FS systems along gap-1 morasses. The second step is then a repetition of that method, but this time along the gap-2 morass. The relationship between this “local FS system along a gap-2 morass” and local FS systems along gap-1 morasses is like the relationship between FS systems along gap-1 and gap-2 morasses. Therefore, it might be helpful, but not necessary to know the construction in [9] of a ccc forcing of size \(\omega_1\) that adds a 0-dimensional Hausdorff topology \(\tau\) on \(\omega_3\) which has spread \(\omega_1\).

Let us first consider the forcing which adds an \(\omega_3\) chain in \(\mathfrak{P}(\omega_1)\) mod finite.

In the recursive definition of \(\mathbb{P}\), we use the morass tree \(s \prec t\) and the mappings \(\pi_{st}\) to map conditions. Let more generally \(\pi : \tilde{\theta} \to \theta\) be any order-preserving map. Then \(\pi : \tilde{\theta} \to \theta\) induces maps \(\pi : \theta \times \omega_1 \to \theta 	imes \omega_1\) and \(\pi : (\theta \times \omega_1) \times \omega \to (\theta \times \omega_1) \times \omega\) in the obvious way:

\[
\pi : \tilde{\theta} \times \omega_1 \to \theta \times \omega_1, \quad (\gamma, \delta) \mapsto (\pi(\gamma), \delta)
\]

\[
\pi : (\tilde{\theta} \times \omega_1) \times \omega \to (\theta \times \omega_1) \times \omega, \quad (x, \epsilon) \mapsto (\pi(x), \epsilon).
\]

We define a system \(\langle \langle P_\eta \mid \eta \leq \omega_3 \rangle, \langle \sigma_{st} \mid s \prec t \rangle \rangle\) by induction on the levels of \(\langle \langle \varphi_\zeta \mid \zeta \leq \omega_1 \rangle, \langle \mathfrak{G}_{\zeta \xi} \mid \zeta < \xi \leq \omega_2 \rangle \rangle\) which we enumerate by \(\beta \leq \omega_2\).

**Base Case:** \(\beta = 0\)

Then we need only to define \(P_1\).

Let \(P_1 := \{ p \in P \mid a_p \times b_p \subseteq 1 \times 1 \}\).
Successor Case: $\beta = \alpha + 1$

We first define $P_{\varphi_{\beta}}$. Let it be the set of all $p \in P$ such that:

1. $a_p \times b_p \subseteq \varphi_{\beta} \times \beta$.
2. $f_{\alpha}^{-1}[p] \restriction (\varphi_{\alpha} \times \alpha), p \restriction (\varphi_{\alpha} \times \alpha) \in P_{\varphi_{\alpha}}$ where $f_{\alpha}$ is like in (P3) of the definition of a simplified gap-1 morass.
3. If $\alpha \in b_p$, then $p(\gamma, \alpha) \leq p(\delta, \alpha)$ for all $\gamma < \delta \in a_p$, i.e. $p \restriction (\varphi_{\beta} \times \{\alpha\})$ is monotone.

For all $\nu \leq \varphi_{\alpha}$, $P_{\nu}$ is already defined. For $\varphi_{\alpha} < \nu \leq \varphi_{\beta}$ set

$$P_{\nu} = \{ p \in P_{\varphi_{\beta}} \mid a_p \times b_p \subseteq \nu \times \beta \}.$$  

Set

$$\sigma_{st} : P_{\nu(s)+1} \rightarrow P_{\nu(t)+1}, p \mapsto \pi_{st}[p].$$

Limit Case: $\beta \in \text{Lim}$

For $t \in T_{\beta}$ set $P_{\nu(t)+1} = \bigcup \{ \sigma_{st}[P_{\nu(s)+1}] \mid s < t \}$ and $P_{\lambda} = \bigcup \{ P_{\eta} \mid \eta < \lambda \}$ for $\lambda \in \text{Lim}$ where $\sigma_{st} : P_{\nu(s)+1} \rightarrow P_{\nu(t)+1}, p \mapsto \pi_{st}[p]$.

Let us now consider the forcing which adds $\omega_3$ many distinct functions $f_{\alpha} : \omega_1 \rightarrow \omega$ such that $\{ \xi < \omega_1 \mid f_{\alpha}(\xi) = f_{\beta}(\xi) \}$ is finite for all $\alpha < \beta < \omega_3$. In this case we replace (3) in the successor case of the definition by:

3. If $\alpha \in b_p$, then $p(\gamma, \alpha) \neq p(\delta, \alpha)$ for all $\gamma < \delta \in a_p$, i.e. $p \restriction (\varphi_{\beta} \times \{\alpha\})$ is injective.

Lemma 4.1

For $p \in P$, $p \in P_{\omega_3}$ iff for all $\alpha < \omega_2$ and all $f \in \mathfrak{C}_{\alpha+1, \omega_2}$

$$f^{-1}[p] \restriction (\varphi_{\alpha+1} \times \{\alpha\}) \text{ is monotone.}$$

Proof: By induction on $\gamma \leq \omega_2$ we prove the following

Claim: $p \in P_{\omega_3}$ iff for all $\alpha < \gamma$ and all $f \in \mathfrak{C}_{\alpha+1, \gamma}$

$$f^{-1}[p] \restriction (\varphi_{\alpha+1} \times \{\alpha\}) \text{ is monotone.}$$

Base case: $\gamma = 0$

Then there is nothing to prove.

Successor case: $\gamma = \beta + 1$

Assume first that $p \in P_{\varphi_{\beta}}$. Then, by (2) in the successor step of the definition
of $P_{\omega_3}$, $f_{\alpha}^{-1}[p] \upharpoonright (\varphi_\alpha \times \alpha), p \upharpoonright (\varphi_\alpha \times \alpha) \in P_{\varphi_\alpha}$. Now assume $f \in G_{\alpha+1, \gamma}$ and $\alpha < \beta$. Then $f = f_\beta \circ f'$ or $f = f'$ for some $f' \in G_{\alpha+1, \beta}$ by (P2) and (P3). So by the induction hypothesis

$$f^{-1}[p] \upharpoonright (\varphi_\alpha \times \{\alpha\}) \text{ is monotone}$$

for all $f \in G_{\alpha+1, \gamma}$ and all $\alpha < \beta$. Moreover, if $\alpha = \beta$ then the identity is the only $f \in G_{\alpha+1, \gamma}$. In this case

$$f^{-1}[p] \upharpoonright (\varphi_\alpha \times \{\alpha\}) \text{ is monotone}$$

by (3) in the successor case of the definition of $P_{\omega_3}$.

Now suppose that

$$f^{-1}[p] \upharpoonright (\varphi_\alpha \times \{\alpha\}) \text{ is monotone}$$

for all $\alpha < \gamma$ and all $f \in G_{\alpha+1, \gamma}$. We have to prove that (2) and (3) in the successor step of the definition of $P_{\omega_3}$ hold. (3) obviously holds by the assumption because the identity is the only function in $G_{\gamma, \gamma} = G_{\beta+1, \gamma}$. For (2), it suffices by the induction hypothesis to show that

$$f^{-1}[f^{-1}[p]] \upharpoonright (\varphi_\alpha \times \{\alpha\}) \text{ is monotone}$$

and

$$f^{-1}[(id \upharpoonright \varphi_\beta)[p]] \upharpoonright (\varphi_\alpha \times \{\alpha\}) \text{ is monotone}$$

for all $f \in G_{\alpha+1, \beta}$. This, however, holds by (P2) and the assumption.

Limit case: $\gamma \in Lim$

Assume first that $p \in P_{\varphi_\gamma}$. Let $\alpha < \gamma$ and $f \in G_{\alpha+1, \gamma}$. We have to prove that

$$f^{-1}[p] \upharpoonright (\varphi_\alpha \times \{\alpha\}) \text{ is monotone.}$$

By the limit step of the definition of $P_{\omega_3}$, there are $\beta < \gamma$, $g \in G_{\beta, \gamma}$ and $\bar{p} \in P_{\varphi_\beta}$ such that $p = g[\bar{p}]$. By (P4) there are $\alpha + 1, \beta < \delta < \gamma$, $g' \in G_{\beta, \delta}$, $f' \in G_{\alpha+1, \delta}$ and $h \in G_{\beta, \gamma}$ such that $g = h \circ g'$ and $f = h \circ f'$. Let $p' := g'[\bar{p}]$. Then, by the induction hypothesis

$$(f')^{-1}[p'] \upharpoonright (\varphi_\alpha \times \{\alpha\}) \text{ is monotone.}$$

However, $(f')^{-1}[p'] = (f')^{-1}[h^{-1}[p]] = f^{-1}[p]$ and we are done.

Now assume that

$$f^{-1}[p] \upharpoonright (\varphi_\alpha \times \{\alpha\}) \text{ is monotone}$$

for all $\alpha < \gamma$ and all $f \in G_{\alpha+1, \gamma}$. We have to prove that $p \in P_{\varphi_\gamma}$, i.e. that there exist $\beta < \gamma$, $f \in G_{\beta, \gamma}$ and $\bar{p} \in P_{\varphi_\beta}$ such that $p = f[\bar{p}]$. However, since $p : a_p \times b_p \to 2$ is finite, there exist $\beta < \gamma$ and $g \in G_{\beta, \gamma}$ such that $p \in \operatorname{rng}(g)$. Hence by the induction hypothesis it suffices to prove that $\bar{p} := g^{-1}[\bar{p}] \in P_{\varphi_\beta}$, i.e. that

$$f^{-1}[\bar{p}] \upharpoonright (\varphi_\alpha \times \{\alpha\}) \text{ is monotone}$$
for all $\alpha < \beta$ and all $f \in \mathcal{G}_{\alpha+1,\beta}$. So let $f \in \mathcal{G}_{\alpha+1,\beta}$. Then

$$f^{-1}[p] \restriction (\varphi_{\alpha+1} \times \{ \alpha \}) = f^{-1}[g^{-1}[p]] \restriction (\varphi_{\alpha+1} \times \{ \alpha \}) =$$

which is monotone by our assumption. □

Respectively, for the forcing which adds $\omega_3$ many distinct functions $f_\alpha : \omega_1 \to \omega$ such that $\{ \xi < \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi) \}$ is finite for all $\alpha < \beta < \omega_3$ the following holds:

For $p \in P$, $p \in P_{\omega_3}$ iff for all $\alpha < \omega_2$ and all $f \in \mathcal{G}_{\alpha+1,\omega_2}$

$$f^{-1}[p] \restriction (\varphi_{\alpha+1} \times \{ \alpha \})$$

is injective.

Let $\Delta \subseteq \omega_3$ be finite and $P_\Delta = \{ p \in P(\omega_3) \mid a_p \subseteq \Delta \}$. We want to represent every $p \in P_\Delta$ as a function $p^* : [\alpha_0, \omega_2] \to P$ such $p^*(\alpha) \in P_{\varphi_\alpha}$ for all $\alpha_0 \leq \alpha < \omega_2$: Set

$\eta = \max(\Delta)$

$t = \langle \omega_2, \eta \rangle$

$s_0 = \min\{ s < t \mid \Delta \subseteq \text{rng}(\pi_{st}) \}$

$\alpha_0 = \alpha(s_0)$

$p^*(\alpha) = \pi_{st}^{-1}[p \restriction (\omega_3 \times \alpha)]$ for $\alpha_0 \leq \alpha < \omega_2$ where $s \in T_\alpha$, $s < t$

$\text{supp}(p) = \{ \alpha + 1 \mid \alpha_0 \leq \alpha < \omega_1, p^*(\alpha + 1) \neq p^*(\alpha), p^*(\alpha + 1) \neq f_\alpha[p^*(\alpha)] \} \cup \{ \alpha_0 \}$

where $f_\alpha$ is like in (P3) of the definition of a simplified gap-1 morass.

Note that, by lemma 3.2, $\text{supp}(p)$ is finite, since $p$ is finite.

**Lemma 4.2**

If $p, q \in P_\Delta$ and $p^*(\alpha), q^*(\alpha)$ are compatible in $P_{\varphi_\alpha}$ for $\alpha = \max(\text{supp}(p) \cap \text{supp}(q))$, then $p$ and $q$ are compatible in $P_\Delta$.

**Proof:** Suppose $p$ and $q$ are like in the lemma, but incompatible. Let $(\text{supp}(p) \cup \text{supp}(q)) = \alpha = \{ \gamma_n < \ldots < \gamma_1 \}$. We prove by induction on $1 \leq i \leq n$, that $p^*(\gamma_i)$ and $q^*(\gamma_i)$ are incompatible for all $1 \leq i \leq n$. Since $\gamma_n = \alpha$, this yields the desired contradiction.

Note first, that $p^*(\gamma_1)$ and $q^*(\gamma_1)$ are incompatible because otherwise $p = \pi_{st}[p^*(\gamma_1)]$ and $q = \pi_{st}[q^*(\gamma_1)]$ were incompatible (for $s \in T_{\gamma_1}$, $s < t$). If $\gamma_1 = \alpha$, we are done. So assume that $\gamma_1 \neq \alpha$. Then either $p^*(\gamma_1) = \pi_{ss}[p^*(\gamma_1 - 1)]$ or $q^*(\gamma_1) = \pi_{ss}[q^*(\gamma_1 - 1)]$ where $s < t, s \in T_{\gamma_1-1}$ and $s \in T_{\gamma_1}$. We assume in the following that $p^*(\gamma_1) = \pi_{ss}[p^*(\gamma_1 - 1)]$. Mutatis mutandis, the other case works the same.

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Claim: \( p^*(\gamma_1 - 1) \) and \( q^*(\gamma_1 - 1) \) are incompatible in \( \mathbb{P}_{\varphi_{\gamma_1-1}} \).

Assume not. Then there is \( \bar{r} \leq p^*(\gamma_1 - 1), q^*(\gamma_1 - 1) \) in \( \mathbb{P}_{\varphi_{\gamma_1-1}} \) such that \( a_{\bar{r}} = a_{p^*(\gamma_1 - 1)} \cup a_{q^*(\gamma_1 - 1)} \). Let \( r' := \pi_{a_{\bar{r}}}[\bar{r}] \). Then \( r' \leq \pi_{a_{\bar{r}}}[p^*(\gamma_1 - 1)] = p^*(\gamma_1) \) and \( r' \leq \pi_{a_{\bar{r}}}[q^*(\gamma_1 - 1)] = q^*(\gamma_1) \upharpoonright (\varphi_{\gamma_1} \times \gamma_1 - 1) \). In the following we will construct an \( r \leq p^*(\gamma_1), q^*(\gamma_1) \) which yields the contradiction we were looking for. By (2) in the definition of \( \mathbb{P}_{\varphi_{\gamma_1}} \), \( \bar{q}(\eta, \gamma_1) \leq \bar{q}(\delta, \gamma_1) \) for all \( \eta < \delta \in \tilde{a} \), where \( \tilde{a} := q^*(\gamma_1) \). Let \( \bar{\delta} = \max\{\delta \in a_{\tilde{a}} \mid \tilde{q}(\delta, \gamma_1) = 0\} \) if the set is not empty. Otherwise, set \( \bar{\delta} = 0 \). Set

\[
 r = r' \cup \{\langle \delta, \gamma_1 \rangle, 0 \mid \delta \leq \bar{\delta}, \delta \in a_r\} \cup \{\langle \delta, \gamma_1 \rangle, 1 \mid \bar{\delta} < \delta, \delta \in a_r\}.
\]

Then \( r \) is as wanted. This proves the claim.

It follows from the claim, that \( p^*(\gamma_2) \) and \( q^*(\gamma_2) \) are incompatible. Hence we can prove the lemma by repeating this argument inductively finitely many times. \( \square \)

Of course, lemma 4.2 also holds for the forcing which adds the disjoint functions. The proof is easy to adjust.

We could use this to prove ccc (like in lemma 5.2 of [9]), if every \( P_{\varphi_\alpha} \) was countable. This is however not the case. Therefore, we must thin out our forcing further.

5 The thinning-out – gap-2 step

Let \( {}^*P_\Delta := \{p^* \mid \text{supp}(p) \in P_\Delta\} \) and

\[
 Q := \bigcup\{ {}^*P_\Delta \mid \Delta \subseteq \omega_3 \text{ finite} \}.
\]

For \( p, q \in Q \), we set \( p \leq q \) iff \( \text{dom}(q) \subseteq \text{dom}(p) \) and \( p(\eta) \leq q(\eta) \) for all \( \eta \in \text{dom}(q) \).

We are going to thin out \( Q \) along the gap-2 morass.

To map \( p \in Q \) along \( s \prec t \) we use \( \pi_{st}^p \).

For \( f \in {}^{3}\alpha\beta \) and \( p \in Q \) with \( \text{dom}(q) \subseteq \theta_\alpha \), we may define \( \bar{f}[p] \) with \( \text{dom}(\bar{f}[p]) = \bar{f}[\text{dom}(p)] \) by setting

\[
 \bar{f}[p](\bar{f}(\eta)) = \bar{f}_\eta[p(\eta)] \text{ for all } \eta \in \text{dom}(p)
\]

where \( \bar{f}, \bar{f}_\eta \) are like above.

In the same way we may define \( \pi_{\alpha\eta}^p[p] \).

Set \( Q(\nu) := \{p \in Q \mid \text{dom}(p) \subseteq \nu\} \).

Lemma 5.1

Let \( \eta \leq \theta_\alpha, \bar{f}(\bar{\eta}) = \eta, \Delta \subseteq \varphi_{\bar{\eta}} \) and \( \Delta = \bar{f}_\eta[\Delta] \). Assume \( \bar{p} \in {}^*P_\Delta \). Set
Again, \( \bar{\eta} \) and \( \bar{\omega} \). We define a system \( \beta \) the gap-2 morass which we enumerate by 5.1 would show that \( \bar{\eta} \) problem. There we simply include by definition all if we could also prove, that \( \bar{\omega} \) and we are done.

By (4) in the definition of embedding, we have that

\[
\begin{align*}
\bar{p}(\alpha_i - 1) &= \pi_{s^{\alpha_i-1}}[\bar{p}(\alpha_i)].
\end{align*}
\]

Hence

\[
\begin{align*}
p(\beta_i + 1 - 1) &= \pi_{s_{\beta_i-1}^{\beta_i+1}}[\bar{p}(\beta_i)].
\end{align*}
\]

**Proof:** Since \( \bar{p} \in Q \),

\[
\bar{p}(\alpha_i - 1) = \pi_{s^{\alpha_i-1}}[\bar{p}(\alpha_i)].
\]

By (6) in the definition of embeddings where \( h_i \mid \nu(\bar{s}_\alpha_i) + 1 = \pi_{s^{\alpha_i-1}}[\bar{p}(\alpha_i)]. \)

However, \( f_{\alpha_i} = f^\#(\alpha_i) \circ \bar{f}_{\alpha_i} \) by lemma 3.6 (1). So

\[
\begin{align*}
f_{\alpha_i,\alpha_i+1}(h_i) \circ f_{\alpha_i}[\bar{p}(\alpha_i)] &= f_{\alpha_i,\alpha_i+1}(h_i) \circ f^\#(\alpha_i) \circ \bar{f}_{\alpha_i}[\bar{p}(\alpha_i)]
\end{align*}
\]

by (6) in the definition of embeddings where \( h_i \mid \nu(\bar{s}_\alpha_i) + 1 = \pi_{s^{\alpha_i-1}}[\bar{p}(\alpha_i)]. \)

and we are done. □

If we could also prove, that \( p(\beta_i) \notin \text{rng}(\sigma_{s_{\beta_i}^{\beta_i} s_{\beta_i}^{\beta_i}}) \) for all \( i < n \), then lemma 5.1 would show that \( \bar{p} \in Q(\theta_\beta) \). However, this is not true, and only our definition for the second thinning-out along the gap-2 morass will solve this problem. There we simply include by definition all \( \sigma_{st}[p] \) into \( Q_{s(t)+1} \).

For \( f \in \mathcal{F}_{\theta_\beta} \) and \( p \in Q(\theta_\beta) \) we define \( \bar{f}^{-1}[p] \) as a function with \( \text{dom}(\bar{f}^{-1}[p]) \subseteq \theta_\alpha \) and \( \text{rng}(\bar{f}^{-1}[p]) \subseteq P_\omega \) by setting

\[
\begin{align*}
\text{dom}(\bar{f}^{-1}[p]) &:= \bar{f}^{-1}[\text{dom}(p)]
\end{align*}
\]

\[
\begin{align*}
\bar{f}^{-1}[p](\eta) &:= \bar{f}_\eta^{-1}[p(\bar{f}(\eta)) \mid (\omega_3 \times \alpha)] \text{ for } \eta \in \text{dom}(\bar{f}^{-1}[p]).
\end{align*}
\]

Again, \( \bar{f}^{-1}[p] \) is not necessarily an element of \( Q(\theta_\alpha) \).

We define a system \( \langle \langle \mathcal{Q}_\alpha, \sigma'_{st} \mid \eta \leq \omega_2 \rangle, \langle \sigma'_{st} \mid s \prec t \rangle \rangle \) by induction on the levels of the gap-2 morass which we enumerate by \( \beta \leq \omega_1 \).
Base Case: $\beta = 0$

Then we need only to define $Q_1$.

Let $Q_1 := \{p \in Q \mid \text{dom}(p) \subseteq 1\}$.

Successor Case: $\beta = \alpha + 1$

We first define $Q_{\theta_\beta}$. Let it be the set of all $p \in Q$ such that:

1. $\text{dom}(p) \subseteq \theta_\beta$.
2. $f_{\alpha}^{-1}[p], (\text{id} \upharpoonright \theta_\alpha)^{-1}[p] \in Q_{\theta_\alpha}$ where $f_\alpha$ is the unique right-branching embedding in $\mathcal{F}_{\alpha \beta}$.
3. $\text{dom}(p(\eta)) \subseteq \varphi_\eta \times \beta$ and $p(\eta) \upharpoonright (\varphi_\eta \times \{\alpha\})$ is monotone for all $\eta \in \text{dom}(p)$.

For all $\nu \leq \theta_\alpha$, $Q_\nu$ is already defined. For $\theta_\alpha < \nu \leq \theta_\beta$ set

$$Q_\nu = \{p \in Q_{\varphi_\beta} \mid \text{dom}(p) \subseteq \nu\}.$$ 

Set

$$\sigma_{st}' : Q_{\nu(t)+1} \rightarrow Q_{\nu(t)+1}, p \mapsto \pi_{st}[p].$$

Limit Case: $\beta \in \text{Lim}$

For $t \in T_3$ set $Q_{\nu(t)+1} = \bigcup\{\sigma_{st}'[Q_{\nu(s)+1}] \mid s \prec t\}$ and $Q_\lambda = \bigcup\{Q_\eta \mid \eta < \lambda\}$ for $\lambda \in \text{Lim}$ where $\sigma_{st}' : Q_{\nu(s)+1} \rightarrow Q_{\nu(t)+1}, p \mapsto \pi_{st}[p]$.

If we want to add $\omega_3$ many distinct functions $f_\alpha : \omega_1 \rightarrow \omega$ such that $\{\xi < \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi)\}$ is finite for all $\alpha < \beta < \omega_3$, then we replace (4) in the successor case of the definition by:

4. $\text{dom}(p(\eta)) \subseteq \varphi_\eta \times \beta$ and $p(\eta) \upharpoonright (\varphi_\eta \times \{\alpha\})$ is injective for all $\eta \in \text{dom}(p)$.

Define

$$P := \{p \in P_{\omega_3} \mid p^* \upharpoonright \text{supp}(p) \in Q_{\omega_2}\} \quad \text{and} \quad P_\eta = P_\eta \cap P.$$

Lemma 5.2

For $p \in P$, $p \in P$ iff for all $\alpha < \omega_1$ and all $f \in \mathcal{F}_{\alpha+1, \omega_1}$

$$f_{\theta_\alpha+1, \omega_1}^{-1}[p] \upharpoonright (\varphi_{\theta_\alpha+1} \times \{\alpha\}) \text{ is monotone.}$$

Proof: The proof is as in lemma 4.1. By induction on $\gamma \leq \omega_1$ we prove the following

Claim: $p \in P_{\varphi_\gamma}$ iff for all $\alpha < \gamma$ and all $f \in \mathcal{F}_{\alpha+1, \gamma}$

$$f_{\theta_\alpha+1, \gamma}^{-1}[p] \upharpoonright (\varphi_{\theta_\alpha+1} \times \{\alpha\}) \text{ is monotone.}$$

Base case: $\gamma = 0$

Then there is nothing to prove.
**Successor case:** $\gamma = \beta + 1$

Assume first that $p \in P_{\theta \gamma}$. Then, by (2) in the successor step of the definition of $Q_{\omega 2}$, $f^{-1}_\alpha[p], (id \upharpoonright \theta \alpha)^{-1}[p] \in Q_{\theta \alpha}$ where $f_\alpha$ is the unique right-braching embedding in $F_{\alpha \beta}$. Now assume $f \in F_{\alpha \gamma}$ and $\alpha < \beta$. Then $f_{\alpha+1} = (f_\beta)_{\theta \beta} \circ f'_{\alpha+1}$ or $f_{\alpha+1} = f \circ f'_{\alpha+1}$ where $f_\beta$ is the unique right-braching embedding in $F_{\beta \gamma}$, $f' \in F_{\alpha+1, \gamma}$ and $f \in \mathcal{G}_{\theta \beta \theta}$ by (3) in the definition of a simplified gap-2 morass. So by the induction hypothesis

$$f^{-1}_{\theta \alpha+1}[p] \upharpoonright (\varphi_{\theta \alpha+1} \times \{\alpha\}) \text{ is monotone}$$

for all $f \in F_{\alpha+1, \gamma}$ and all $\alpha < \beta$. Moreover, if $\alpha = \beta$ then the identity is the only $f \in F_{\alpha+1 \gamma}$. In this case

$$f^{-1}_{\theta \alpha+1}[p] \upharpoonright (\varphi_{\theta \alpha+1} \times \{\alpha\}) \text{ is monotone}$$

by (3) in the successor case of the definition of $Q_{\omega 2}$.

Now suppose that

$$f^{-1}_{\theta \alpha+1}[p] \upharpoonright (\varphi_{\theta \alpha+1} \times \{\alpha\}) \text{ is monotone}$$

for all $\alpha < \gamma$ and all $f \in F_{\alpha+1 \gamma}$. We have to prove that (2) and (3) in the successor step of the definition of $P_{\omega 3}$ hold. (3) obviously holds by the assumption because the identity is the only function in $F_{\gamma \gamma} = F_{\beta+1 \gamma}$. For (2) it suffices, by the argument of lemma 5.1, to prove that

$$(f_\beta)_{\theta \beta}^{-1}[p], g^{-1}[p] \in P_{\varphi \beta}$$

for all $g \in \mathcal{G}_{\theta \beta \theta}$ and $f_\beta$, the unique right-braching embedding in $F_{\beta \gamma}$. Hence

$$f^{-1}_{\theta \alpha+1}[(f_\beta)_{\theta \beta}^{-1}[p]] \upharpoonright (\varphi_{\theta \alpha+1} \times \{\alpha\}) \text{ is monotone}$$

and

$$f^{-1}_{\theta \alpha+1}[g^{-1}[p]] \upharpoonright (\varphi_{\theta \alpha+1} \times \{\alpha\}) \text{ is monotone}$$

for all $f \in F_{\alpha+1, \beta}$ and all $g \in \mathcal{G}_{\theta \beta \theta}$. This, however, holds by (2) and (3) of the definition of a simplified gap-2 morass and by the assumption.

**Limit case:** $\gamma \in \text{Lim}$

Assume first that $p \in P_{\varphi \gamma}$. Let $\alpha < \gamma$ and $f \in F_{\alpha+1 \gamma}$. We have to prove that

$$f^{-1}_{\theta \alpha+1}[p] \upharpoonright (\varphi_{\theta \alpha+1} \times \{\alpha\}) \text{ is monotone}$$

By the limit step of the definition of $Q_{\omega 2}$, there are $\beta < \gamma$, $g \in F_{\beta \gamma}$, and $\bar{p} \in P_{\varphi \beta}$ such that $p^* \upharpoonright \text{supp}(p) = \bar{g}[\bar{p}^* \upharpoonright \text{supp}(\bar{p})]$. By the argument of lemma 5.1, we may assume that $p = g_{\theta \beta}[\bar{p}]$. By (4) in the definition of a simplified gap-2 morass

$$f^{-1}_{\theta \alpha+1}[p] \upharpoonright (\varphi_{\theta \alpha+1} \times \{\alpha\}) \text{ is monotone}$$
morass, there are \( \alpha + 1, \beta < \delta < \gamma, g' \in \mathcal{F}_{\delta\gamma}, f' \in \mathcal{F}_{\alpha+1\delta} \) and \( h \in \mathcal{F}_{\delta\gamma} \) such that \( g = h \circ g' \) and \( f = h \circ f' \). Let \( p' := g'[\bar{p}] \). Then, by the induction hypothesis

\[
(f'_{\theta_{\alpha+1}})^{-1}[p'] \upharpoonright (\varphi_{\alpha+1} \times \{\alpha\}) \text{ is monotone.}
\]

However, \((f'_{\theta_{\alpha+1}})^{-1}[p'] = (f'_{\theta_{\alpha+1}})^{-1}[h_{\theta_{\delta}}[p]] = f_{\bar{\theta}_{\alpha+1}}^{-1}[p] \) and we are done.

Now assume that

\[
f_{\theta_{\alpha+1}}^{-1}[p] \upharpoonright (\varphi_{\theta_{\alpha+1}} \times \{\alpha\}) \text{ is monotone}
\]

for all \( \alpha < \gamma \) and all \( f \in \mathcal{F}_{\alpha+1, \gamma} \). We have to prove that \( p \in P_{\varphi_{\theta_{\beta}}} \), i.e. that there exist \( \beta < \gamma, f \in \mathcal{F}_{\beta\gamma} \) and \( p \in P_{\varphi_{\beta}} \) such that \( p^* \upharpoonright \text{supp}(p) = f[p^* \upharpoonright \text{supp}(\bar{p})] \).

However, since \( p : a_p \times b_p \to 2 \) is finite, there exist \( \beta < \gamma \) and \( g \in \mathcal{F}_{\beta\gamma} \) such that \( p \in \text{rng}(g_{\theta_{\beta}}) \). Hence by the induction hypothesis and the argument of lemma 5.1, it suffices to prove that \( \bar{p} := g_{\theta_{\beta}}^{-1}[p] \in P_{\varphi_{\beta}} \), i.e. that

\[
f_{\theta_{\alpha+1}}^{-1}[\bar{p}] \upharpoonright (\varphi_{\theta_{\alpha+1}} \times \{\alpha\}) \text{ is monotone}
\]

for all \( \alpha < \beta \) and all \( f \in \mathcal{F}_{\alpha+1, \beta} \). So let \( f \in \mathcal{F}_{\alpha+1, \beta} \). Then

\[
f_{\theta_{\alpha+1}}^{-1}[\bar{p}] \upharpoonright (\varphi_{\theta_{\alpha+1}} \times \{\alpha\}) = f_{\theta_{\alpha+1}}^{-1}[g_{\theta_{\beta}}^{-1}[p]] \upharpoonright (\varphi_{\alpha+1} \times \{\alpha\}) = (g \circ f)^{-1}_{\theta_{\alpha+1}}[p] \upharpoonright (\varphi_{\alpha+1} \times \{\alpha\})
\]

which is monotone by our assumption. \( \square \)

Respectively, for the forcing which adds \( \omega_3 \) many distinct functions \( f_{\alpha} : \omega_1 \to \omega \)

such that \( \{ \xi < \omega_1 \mid f_{\alpha}(\xi) = f_{\beta}(\xi) \} \) is finite for all \( \alpha < \beta < \omega_3 \) the following holds:

For \( p \in P, p \in P_{\omega_3} \) iff for all \( \alpha < \omega_2 \) and all \( f \in \mathcal{F}_{\alpha+1, \omega_2} \)

\[
f^{-1}[p] \upharpoonright (\varphi_{\alpha+1} \times \{\alpha\}) \text{ is injective.}
\]

Let \( \Delta \subseteq \omega_2 \) and \( \Gamma \subseteq \omega_3 \) both be finite. Let

\[
Q_{\Delta, \Gamma} := \{ p^* \upharpoonright \text{supp}(p) \in Q_{\omega_2} \mid \text{supp}(p) = \Delta, p \in P_{\Gamma}, a_p = \Gamma \}.
\]

We want to represent every \( p \in Q_{\Delta, \Gamma} \) as a function \( p^* : [\alpha_0, \omega_1] \to Q_{\omega_2} \) such \( p^*(\alpha) \in Q_{\theta_{\alpha}} \) for all \( \alpha_0 \leq \alpha < \omega_2 \).

Set

\[
\eta = \max(\Delta)
\]

\[
t = (\omega_1, \eta)
\]

\[
\tilde{\Gamma} = \pi_{s_0}^{-1}[\Gamma] \text{ where } t' = (\omega_2, \max(\Gamma)) \text{ and } s' < t', s' \in T_\eta.
\]

\[
s_0 = \min\{ s < t' \mid \Delta \subseteq \text{rng}(\pi_{s'}), \tilde{\Gamma} \subseteq \text{rng}((\pi_{s'}\eta)) \} \text{ where } \pi_{s'}(\eta) = \eta.
\]

\[
\alpha_0 = \alpha(s_0)
\]

\[
p^*(\alpha) = \pi_{s'}^{-1}[p] \text{ for } \alpha_0 \leq \alpha < \omega_1 \text{ where } s \in T_{\alpha}, s < t'
\]
supp(p) =
{α + 1 | α₀ ≤ α < ω₁, p⁺(α + 1) ≠ p⁺(α), p⁺(α + 1) ≠ f⁻₁[α]} ∪ {α₀}
where f⁺ is the unique right-branching embedding in ⃗F⁺α₁⁺₁.

Lemma 5.3
If p, q ∈ Q∆₁ and p⁺(α), q⁺(α) are compatible in Qθ₁, for α = max(supp(p) ∩ supp(q)), then p and q are compatible in Q∆₁.

Proof: We proceed like in the proof of lemma 4.2. Suppose p and q are like in the lemma, but incompatible. Let (supp(p) ∪ supp(q)) − α = {γ₁ < ⋯ < γ₁}. We prove by induction on 1 ≤ i ≤ n, that p⁺(γᵢ) and q⁺(γᵢ) are incompatible for all 1 ≤ i ≤ n. Since γ₁ = α, this yields the desired contradiction.

Note first, that p⁺(γ₁) and q⁺(γ₁) are incompatible because otherwise p = π∗[p⁺(γ₁)] and q = π∗[q⁺(γ₁)] were incompatible (for s ∈ T₁, s ≺ t). If γ₁ = α, we are done. So assume that γ₁ ≠ α. Then either p⁺(γ₁) = π∗[p⁺(γ₁ − 1)] or q⁺(γ₁) = π∗[q⁺(γ₁ − 1)] where s ≺′ s ≺ t, s ∈ T₁ and s ∈ T₁. We assume in the following that p⁺(γ₁) = π∗[p⁺(γ₁ − 1)]. Mutatis mutandis, the other case works the same.

Claim: p⁺(γ₁ − 1) and q⁺(γ₁ − 1) are incompatible in Qθ₁−₁
Assume not. Let η = max(Δ), t = (ω₁, η), s ∈ T₁−₁ and s ∈ T₁ such that s ≺ s ≺′ t. Let π⁺[η] = π⁺[η'] = η and p = p⁺(γ₁ − 1)(η), q = q⁺(γ₁ − 1)(η). Then there exists r ≤ p, q. For a contradiction it suffices to find an r ≤ p', q' where p' = p⁺(γ₁)(η') and q' = q⁺(γ₁)(η'). Let r := (fγ₁−₁[s]) where fγ₁−₁ is the unique right-branching embedding in ⃗Fγ₁−₁γ₁. Then by assumption p⁺ = π⁺[r]. Let r' = π⁺[r]. Then r' ≤ p' and r' ≤ π⁺[q'] = q' ∈ (ω₁ × γ₁ − 1). Hence we may define a condition r := p' ∪ q'. By (3) in the successor case of the definition of Q, r is as wanted. This proves the claim.

It follows from the claim, that p⁺(γ₂) and q⁺(γ₂) are incompatible. Hence we can prove the lemma by repeating this argument inductively finitely many times. □

Lemma 5.4
P satisfies ccc.

Proof: Let A ⊆ P be a set of size ω₁. By the Δ-lemma, we may assume that \{bₚ | p ∈ A\} forms a Δ-system with root D. Since for every α ∈ D there are only countably many possibilities for

\[ f_{θₐ+1}⁻¹[p] \upharpoonright (φ_{θₐ+1} × \{α\}) \]

we may moreover assume that for all α ∈ D, all f ∈ ⃗Fₐ+₁ω₁ and all p, q ∈ A

\[ f_{θₐ+1}⁻¹[p] \upharpoonright (φ_{θₐ+1} × \{α\}) = f_{θₐ+1}⁻¹[q] \upharpoonright (φ_{θₐ+1} × \{α\}) \]

By the Δ-system lemma, we may assume that \{aₚ | p ∈ A\} ⊆ ω₁ forms a Δ-system with root Δ₁. Consider A' := \{p | (Δ₁ × ω₁) | p ∈ A\}. By the
\[ \Delta \text{-system lemma we may also assume that } \{ supp(p) \mid p \in A' \} \subseteq \omega_2 \text{ forms a } \Delta \text{-system with root } \Delta_2. \text{ Let } B := \{ p^* \downarrow \Delta_2 \mid p \in A' \} \subseteq \mathbb{Q}. \text{ Again we may assume that } \{ supp(q) \mid q \in B \} \subseteq \omega_1 \text{ is a } \Delta \text{-system with root } \Delta_3. \text{ Let } \alpha = \max(\Delta_3). \text{ Since } \mathbb{Q}_{\alpha_\ast} \text{ is countable, there are } q_1 \neq q_2 \in B \text{ such that } q_1^* = q_2^*(\alpha). \text{ Hence } q_1 \neq q_2 \in B \text{ are compatible by lemma 5.3. Assume that } q_1 = p_1^* \uparrow \Delta_2 \text{ and } q_2 = p_2^* \uparrow \Delta_2 \text{ with } p_1, p_2 \in A'. \text{ Then } p_1 \neq p_2 \text{ are compatible by lemma 4.2. Moreover, there exist } r_1, r_2 \in A \text{ such that } p_1 = r_1 \uparrow (\Delta_1 \times \omega_1) \text{ and } p_2 = r_2 \uparrow (\Delta_1 \times \omega_1). \text{ Hence we can define } r \leq r_1, r_2 \text{ as follows: } a_r = a_{r_1} \cup a_{r_2}, b_r = b_{r_1} \cup b_{r_2}, r \uparrow (a_r \times b_r) = r_1, r \uparrow (a_{r_2} \times b_{r_2}) = r_2. \text{ We still need to define } r \text{ on } (a_r \times b_r) - ((a_{r_1} \times b_{r_1}) \cup (a_{r_2} \times b_{r_2})). \text{ Let } (\alpha, \beta) \in (a_r \times b_r) - ((a_{r_1} \times b_{r_1}) \cup (a_{r_2} \times b_{r_2})). \text{ Then } \beta \in D. \text{ Hence either } \beta \notin b_{p_1} \text{ or } \beta \notin b_{p_2}. \text{ Assume first that } \beta \in b_{p_1} - b_{p_2}. \text{ Then let } \gamma \text{ be minimal such that } p_1(\gamma, \beta) = 1, \text{ if such a } \gamma \text{ exists. Otherwise set } \gamma = 0. \text{ Then let } r(\alpha, \beta) = 1 \text{ if } \alpha \geq \gamma, \text{ and } r(\alpha, \beta) = 0 \text{ if } \alpha < \gamma. \text{ Now, let } \beta \in b_{p_2} - b_{p_1}. \text{ Then let } \gamma \text{ be minimal such that } p_2(\gamma, \beta) = 1, \text{ if such a } \gamma \text{ exists. Otherwise set } \gamma = 0. \text{ Then let } r(\alpha, \beta) = 1 \text{ if } \alpha \geq \gamma, \text{ and } r(\alpha, \beta) = 0 \text{ if } \alpha < \gamma. \text{ Obviously } r \leq r_1, r_2. \text{ It remains to prove that } r \in \mathbb{P}. \text{ For this we use lemma 5.2. That is, we have to show for all } \alpha < \omega_1 \text{ and all } f \in \mathfrak{S}_{\alpha + 1, \omega_1} \frac{f_{\theta_{\alpha+1}}^{-1}}{\downarrow} (\varphi_{\theta_{\alpha+1}} \times \{ \alpha \}) \text{ is monotone.}

\] However, if } \alpha \in D, \text{ then this holds by our first thinning-out of } A. \text{ If } \alpha \notin D, \text{ then it holds because of the way in which we defined } r \text{ on } (a_r \times b_r) - ((a_{r_1} \times b_{r_1}) \cup (a_{r_2} \times b_{r_2})). \Box

\] It is easy to change the proof so that it shows the ccc for the forcing which adds } \omega_3 \text{ many distinct functions } f_\alpha : \omega_1 \rightarrow \omega \text{ such that } \{ \xi < \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi) \} \text{ is finite for all } \alpha < \beta < \omega_3. \text{ In the definition of } r \leq r_1, r_2 \text{ we only have to take care that } r(\alpha, \xi) \neq r(\beta, \xi) \text{ for all } (\alpha, \xi) \in (a_r \times b_r) - ((a_{r_1} \times b_{r_1}) \cup (a_{r_2} \times b_{r_2})) \text{ and } (\beta, \xi) \in a_r \times b_r.

\] Lemma 5.5

(a) Let } p \in \mathbb{P} \text{ and } \max(a_p) < \alpha. \text{ Then there exists a } q \leq p \text{ such that } \alpha \in a_q.

(b) Let } p \in \mathbb{P}, \alpha < \gamma \in a_p \text{ and } \beta \notin b_p. \text{ Then there exists } q \leq p \text{ such that } q(\alpha, \beta) = 0 \text{ and } q(\gamma, \beta) = 1.\n
\textbf{Proof:} (a) Let } a_q = a_p \cup \{ \alpha \}, b_q = b_p \text{ and } q \uparrow (a_p \times b_p) = p. \text{ We have to define } q(\alpha, \beta) \text{ for all } \beta \in b_p. \text{ For those simply set } q(\alpha, \beta) = 1. \text{ Then obviously } q \leq p \text{ and } q \in \mathbb{P} \text{ by lemma 5.2.}

(b) Let } a_q = a_p, b_q = b_p \cup \{ \beta \} \text{ and } q \uparrow (a_p \times b_p) = p. \text{ We have to define } q(\delta, \beta) \text{ for all } \delta \in a_q. \text{ For those set } q(\delta, \beta) = 1 \text{ if } \delta \geq \gamma \text{ and } q(\delta, \beta) = 0 \text{ if } \delta < \gamma. \text{ Then obviously } q \leq p \text{ and } q \in \mathbb{P} \text{ by lemma 5.2.} \Box

\] If we want to construct a ccc forcing which adds } \omega_3 \text{ many distinct functions } f_\alpha : \omega_1 \rightarrow \omega \text{ such that } \{ \xi < \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi) \} \text{ is finite for all } \alpha < \beta < \omega_3, \text{ the following is enough:

For all } p \in \mathbb{P} \text{ and all } (\alpha, \xi) \in \omega_3 \times \omega_1 \text{ there exists a } q \leq p \text{ such that } (\alpha, \xi) \in
dom(q).

To see this, we define q by setting $a_q = a_p \cup \{\alpha\}$, $b_q = b_p \cup \{\xi\}$ and $q \restriction (a_p \times b_p) = p$. We still need to define q on $(a_q \times b_q) - (a_p \times b_p)$. Do this in such a way that $q(\alpha, \xi) \neq q(\beta, \zeta)$ for all $\langle \alpha, \xi \rangle \in (a_q \times b_q) - (a_p \times b_p)$, $\langle \beta, \zeta \rangle \in (a_q \times b_q)$, $\langle \alpha, \xi \rangle \neq \langle \beta, \zeta \rangle$. Then, by lemma 5.2, $q \in P$. Moreover, it is clear that $q \leq p$ and $\langle \xi, \alpha \rangle \in \text{dom}(q)$.

**Theorem 5.6**

(a) If there exists a simplified $(\omega_1, 2)$-morass, then there is a ccc-forcing $P$ which adds a sequence $\langle X_\alpha \mid \alpha < \omega_3 \rangle$ of subsets $X_\alpha \subseteq \omega_1$ such that $X_\beta - X_\alpha$ is finite and $X_\alpha - X_\beta$ is uncountable for all $\beta < \alpha < \omega_3$.

(b) If there exists a simplified $(\omega_1, 2)$-morass, then there is a ccc-forcing $P$ which adds $\omega_3$ many distinct functions $f_\alpha : \omega_1 \to \omega$ such that $\{\xi < \omega_1 \mid f_\alpha(\xi) = f_\beta(\xi)\}$ is finite for all $\alpha < \beta < \omega_3$.

**Proof:** (a) Of course, $P$ is the forcing which we defined above. Let $G$ be $P$-generic, $A = \bigcup \{a_p \mid p \in G\}$ and $X_\alpha = \{\beta \in \omega_1 \mid p(\alpha, \beta) = 1 \text{ for some } p \in G\}$ for $\alpha \in A$. By lemma 5.4, cardinals are preserved. By lemma 5.5 (a), $\text{card}(A) = \omega_3$. By lemma 5.5 (b), $X_\alpha - X_\beta$ is uncountable for all $\beta < \alpha \in A$. By the definition of $\leq$, $X_\beta - X_\alpha$ is finite for all $\beta < \alpha \in A$.

(b) $P$ is now the forcing which we defined parallel with the forcing for (a). Let $G$ be $P$-generic. Set $F = \bigcup \{p \mid p \in G\}$ and $f_\alpha(\xi) = F(\alpha, \xi)$ for all $\alpha < \omega_3$, $\xi < \omega_1$. By lemma 5.4, cardinals are preserved. By lemma 5.5, $f_\alpha : \omega_1 \to \omega$ is defined for all $\alpha \in \omega_3$. By the definition of $\leq$, the $f_\alpha$ are as wanted. □

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