Non-commutative disintegrations:
existence and uniqueness in finite dimensions

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Abstract

Motivated by advances in categorical probability, we introduce non-commutative almost everywhere (a.e.) equivalence and disintegrations in the setting of $C^*$-algebras. We show that $C^*$-algebras (resp. $W^*$-algebras) and a.e. equivalence classes of 2-positive (resp. positive) unital maps form a category. We prove non-commutative disintegrations are a.e. unique whenever they exist. We provide an explicit characterization for when disintegrations exist in the setting of finite-dimensional $C^*$-algebras, and we give formulas for the associated disintegrations.

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1 Introduction and outline

Regular conditional probabilities, optimal hypotheses, disintegrations of one measure over another consistent with a measure-preserving map, conditional expectations, perfect error-correcting codes, and sufficient statistics are all examples of a single mathematical notion. We call this notion a disintegration. Although we only make the connection between our definition of disintegration

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and the first three examples listed, relationships to the other notions are described in [34], and further connections will be made in subsequent work. In this paper, our primary focus is to provide necessary and sufficient conditions for the existence and uniqueness of disintegrations in the setting of finite-dimensional $C^*$-algebras.

Developing this and related ideas is part of a larger program in extending Bayesian statistics to the non-commutative setting [34–36] in such a way so that it is compatible with a recently developed categorical framework for classical statistics [5,14]. These recent advances in classical categorical functional analysis and measure theory provide a suitable notion of disintegration [2,5,7,14,17,20,23,44,47,48], whose diagrammatic formulation can be transferred from a category of probability spaces to a category of states on $C^*$-algebras. This is achieved by utilizing a fully faithful (contravariant) functor from the former to the latter [15,31]. This categorical perspective offers a candidate for generalizing disintegrations to non-commutative probability theory without relying on the specific measure-theoretic details of classical probability theory. Since a disintegration is a special kind of Bayesian inverse [5,34,36], this article serves as a step towards a theory of non-commutative Bayesian inversion.

Briefly, the definition of a disintegration of a state $\omega$ over another state $\xi$ consistent with a unital $^*$-homomorphism $F$ preserving these states is a completely positive unital map $R$ in the reverse direction that is both state preserving and a left inverse of $F$ modulo the null space of $\xi$. If $^*$-homomorphisms are written as straight arrows $\rightarrow$ and completely positive unital maps are written as squiggly arrows $\rightsquigarrow$, this definition of a disintegration can be summarized diagrammatically as

\[
\begin{array}{c}
A 
\xrightarrow{F} & B \\
\omega & \downarrow & \xi \\
 & R & \\
 & \downarrow & \\
C & \xi & \\
\end{array}
\quad \text{such that} \quad \begin{array}{c}
A 
\xrightarrow{R} & B \\
\xi & \downarrow & \omega \\
 & \downarrow & \\
C & \omega & \\
\end{array}
\quad \text{and} \quad \begin{array}{c}
B 
\xrightarrow{F} & B \\
\xi & \downarrow & \omega \\
 & \downarrow & \\
A & \omega & \\
\end{array}
\quad (1.1)
\]

in the category of finite-dimensional $C^*$-algebras and completely positive unital maps. The rightmost diagram commutes almost everywhere (a.e.), in a sense that we make precise in this article. We introduce and develop non-commutative a.e. equivalence in order to properly address the uniqueness properties of disintegrations.

The interpretation of completely positive unital maps as quantum conditional probabilities is not new [24], but we take this perspective further and include the relationships between states and partially reversible dynamics analogous to what regular conditional probabilities accomplish in classical statistics. Our core result is Theorem 4.3, which specializes to the case where $A$ and $B$ are matrix algebras and $F$ sends $B \in B$ to $\text{diag}(B,\ldots,B)$. If we express our states $\omega$ and $\xi$ in terms of density matrices $\rho$ and $\sigma$, respectively, Theorem 4.3 says that a unique disintegration exists if and only if there exists a density matrix $\tau$ such that $\rho = \tau \otimes \sigma$. This is closely related to a well-known result on the existence of state-preserving conditional expectations [39], but our notion generalizes it due to our weakened assumption of a.e. equivalence.

Our subsequent results are generalizations of this theorem and culminate in Theorem 5.108, which assumes $A$ and $B$ are arbitrary finite-dimensional $C^*$-algebras and $F$ is an arbitrary unital $^*$-homomorphism. We provide explicit formulas for disintegrations and we analyze several examples, including one involving entanglement, which has its origins in the work of Einstein, Podolsky, and
Rosen [10]. In Example 5.99, we show how the standard classical theorem on the existence and uniqueness of disintegrations (Theorem 5.1) is a direct corollary of our theorem. We conclude by exploring consequences of our characterization theorem in the context of measurement in quantum information theory. Finally, Appendix A reviews stochastic maps (Markov kernels) and justifies our usage of the terminology ‘disintegration’ by showing that the diagrammatic notion agrees with a general measure-theoretic one.

2 Non-commutative a.e. equivalence

For classical probability spaces, a.e. equivalence specifies the degree of uniqueness of disintegrations, Bayesian inverses, and conditional distributions. The same is true in the quantum/non-commutative setting. In this section, we first recall some relevant definitions involving states on $C^*$-algebras and completely positive maps from Paulsen [38] and Sakai [43] to establish notation and terminology. Afterwards, we define a.e. equivalence for linear maps between $C^*$-algebras in Definition 2.9. We provide a more computationally useful definition for finite-dimensional $C^*$-algebras in Lemma 2.26. In the rest of this section, we analyze several properties of a.e. equivalence.

Definition 2.1. A $C^*$-algebra is an algebra $\mathcal{A}$ equipped with a unit $1_{\mathcal{A}}$, an involution $^* : \mathcal{A} \to \mathcal{A}$, and a norm $\| \cdot \| : \mathcal{A} \to \mathbb{R}$ such that it is a $^*$-algebra, it is closed with respect to the topology induced by its norm, and it satisfies the $C^*$-identity, which says $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. Given a $C^*$-algebra $\mathcal{A}$, a positive element of $\mathcal{A}$ is an element $a \in \mathcal{A}$ for which there exists an $x \in \mathcal{A}$ such that $a = x^*x$. The set of positive elements in $\mathcal{A}$ is denoted by $\mathcal{A}^+$. Given another $C^*$-algebra $\mathcal{B}$, a positive map $\varphi : \mathcal{B} \rightsquigarrow \mathcal{A}$ is a linear map such that $\varphi(\mathcal{B}^+) \subseteq \mathcal{A}^+$. A linear map $\varphi : \mathcal{B} \rightsquigarrow \mathcal{A}$ is unital if $\varphi(1_{\mathcal{B}}) = 1_{\mathcal{A}}$. A state on a $C^*$-algebra $\mathcal{A}$ is a positive linear unital functional $\omega : \mathcal{A} \rightsquigarrow \mathbb{C}$. A $^*$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a function $f : \mathcal{A} \to \mathcal{B}$ preserving the $C^*$-algebra structure, namely $f$ is linear, $f$ is multiplicative $f(aa') = f(a)f(a')$, $f$ is unital $f(1_{\mathcal{A}}) = 1_{\mathcal{B}}$, and $f(a^*) = f(a)^*$ for all $a, a' \in \mathcal{A}$. If $\omega : \mathcal{A} \rightsquigarrow \mathbb{C}$ and $\xi : \mathcal{B} \rightsquigarrow \mathbb{C}$ are states, then a linear map $\varphi : \mathcal{B} \rightsquigarrow \mathcal{A}$ is said to be state-preserving whenever $\varphi \circ \omega = \xi$, and the notation $(\mathcal{B}, \xi) \xrightarrow{\varphi} (\mathcal{A}, \omega)$ will be used to indicate this.

All $C^*$-algebras and $^*$-homomorphisms will be unital unless specified otherwise. Note that positive (and linear) maps on $C^*$-algebras are denoted with squiggly arrows $\rightsquigarrow$, while $^*$-homomorphisms are denoted with straight arrows $\to$.

Example 2.2. For each $n \in \mathbb{N}$, let $\mathcal{M}_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. The involution applied to $A \in \mathcal{M}_n(\mathbb{C})$ is given by the conjugate transpose and is written as $A^\dagger$ instead of $A^*$ to be consistent with the standard notation used in quantum theory. A matrix algebra is a $C^*$-algebra of the form $\mathcal{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$. If $\mathcal{A}$ is another $C^*$-algebra, then $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A} \cong \mathcal{M}_n(\mathcal{A})$, the algebra of $n \times n$ matrices with entries in $\mathcal{A}$, admits a $C^*$-algebra structure by matrix operations and a norm that can be obtained in many ways (cf. Chapter 1 in Paulsen [38]).
Our convention for the tensor product (also called the Kronecker product) of matrices will be

\[
\begin{bmatrix}
a_{11} & \cdots & a_{1m} \\
\vdots & & \vdots \\
a_{m1} & \cdots & a_{mm}
\end{bmatrix} \otimes \begin{bmatrix}
b_{11} & \cdots & b_{1n} \\
\vdots & & \vdots \\
b_{n1} & \cdots & b_{mn}
\end{bmatrix} = \begin{bmatrix}
a_{11} & \cdots & b_{1n} \\
\vdots & & \vdots \\
a_{m1} & \cdots & b_{mn}
\end{bmatrix}
\begin{bmatrix}
a_{11} & \cdots & a_{1m} \\
\vdots & & \vdots \\
a_{m1} & \cdots & a_{mm}
\end{bmatrix} \tag{2.3}
\]

which is induced by the isomorphism \( \mathbb{C}^m \otimes \mathbb{C}^n \to \mathbb{C}^{mn} \) determined by

\[
\vec{e}_1 \otimes \vec{e}_1 \mapsto \vec{e}_1, \quad \ldots \quad \vec{e}_1 \otimes \vec{e}_n \mapsto \vec{e}_n, \quad \vec{e}_2 \otimes \vec{e}_1 \mapsto \vec{e}_{n+1}, \quad \ldots \quad \vec{e}_m \otimes \vec{e}_n \mapsto \vec{e}_{mn}. \tag{2.4}
\]

Here, \( \vec{e}_i \) denotes the standard \( i \)-th unit vector in \( \mathbb{C}^n \) regardless of \( n \).

**Definition 2.5.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( C^* \)-algebras. Given \( n \in \mathbb{N} \), a linear map \( \varphi : \mathcal{B} \to \mathcal{A} \) is \( n \)-**positive** iff \( \text{id}_{\mathcal{M}_n(\mathbb{C})} \otimes \varphi : \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{B} \to \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A} \) is positive. The map \( \varphi \) is **completely positive** iff \( \varphi \) is \( n \)-positive for all \( n \in \mathbb{N} \). A completely positive (unital) map will be abbreviated as a CP (CPU) map.

The Choi–Kraus theorem gives a characterization of completely positive maps between matrix algebras. This will be used often, so we state it here to set notation [6, 22].

**Theorem 2.6.** Fix \( n, m \in \mathbb{N} \). A linear map \( R : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C}) \) is completely positive if and only if there exists a finite collection \( \{ R_i : \mathbb{C}^n \to \mathbb{C}^m \} \) of linear maps such that

\[
R = \sum_i \text{Ad}_{R_i}. \tag{2.7}
\]

Here, \( \text{Ad}_{R_i}(A) := R_i A R_i^\dagger \) for all \( A \in \mathcal{M}_n(\mathbb{C}) \). The map \( R \) is CPU if and only if, in addition, \( \sum_i R_i R_i^\dagger = 1_m \).

A collection \( \{ R_i \} \) satisfying (2.7) is called a **Kraus decomposition** for \( R \).

**Remark 2.8.** The standard assumption in quantum information theory is to work with completely positive trace-preserving maps instead of unital maps. The former class, typically called quantum operations/channels (cf. Section 8.2 in Nielsen and Chuang [28]) is used in the Schrödinger representation when transforming physical states, while the latter is used in the Heisenberg representation when transforming physical observables. We briefly explain the relationship between the two. A completely positive map \( R : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C}) \) is unital if and only if the dual map \( R^* : \mathcal{M}_m(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}) \) is trace-preserving. The dual map \( R^* \) is defined with respect to the **Hilbert–Schmidt** (a.k.a. **Frobenius**) inner product on a matrix algebra, which is given by \( \langle A, B \rangle := \text{tr}(A^\dagger B) \) for all square matrices (of the same dimension) \( A \) and \( B \). Therefore, \( R^* \) is the unique map satisfying \( \langle R^*(A), B \rangle = \langle A, R(B) \rangle \) for all \( A \in \mathcal{M}_m(\mathbb{C}) \) and all \( B \in \mathcal{M}_n(\mathbb{C}) \). If \( R = \sum_i \text{Ad}_{R_i} \) is a Kraus decomposition of \( R \), then \( R^* = \sum_i \text{Ad}_{R_i^\dagger} \) is a Kraus decomposition of
Let $\omega : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathbb{C}$ be a state, then its dual $\omega^* : \mathbb{C} \rightarrow \mathcal{M}_n(\mathbb{C})$ is determined by the image $\omega^*(1)$ of the unit 1 in $\mathbb{C}$, which is positive. Furthermore, since $\omega$ is unital, $\omega^*$ is trace-preserving. Hence, $\text{tr}(\omega^*(1)) = 1$. In other words, $\omega^*(1)$ is a trace 1 positive matrix. This is called the density matrix associated to $\omega$. Finally, $\omega = \text{tr}(\omega^*(1) \cdot)$ as states on $\mathcal{M}_n(\mathbb{C})$.

We now proceed to defining a.e. equivalence of linear maps on $C^*$-algebras.

**Definition 2.9.** Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras, let $F, F' : \mathcal{B} \rightarrow \mathcal{A}$ be two linear maps, and let $\omega : \mathcal{A} \rightarrow \mathbb{C}$ be a state on $\mathcal{A}$ (or more generally a positive linear functional). Let

$$\mathcal{N}_\omega := \{ a \in \mathcal{A} : \omega(a^*a) = 0 \}$$

(2.10)

denote the null space of $\omega$. Since $\mathcal{N}_\omega$ is a left ideal of $\mathcal{A}$ (see Construction 3.1 in [32] for details) denote the quotient vector space by $\mathcal{A}/\mathcal{N}_\omega$. The maps $F$ and $F'$ are said to be equal almost everywhere (a.e.) with respect to $\omega$ or equal $\omega$-a.e. iff the diagram (in the category of vector spaces and linear maps)

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{F} & \mathcal{A} \\
& \searrow & \swarrow \\
& \mathcal{A}/\mathcal{N}_\omega & \\
& \xleftarrow{F'} & \\
\end{array}$$

(2.11)

commutes, i.e. iff $F(b) - F'(b) \in \mathcal{N}_\omega$ for all $b \in \mathcal{B}$. The map $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{N}_\omega$ in (2.11) is the quotient map of $\mathcal{A}$ onto $\mathcal{A}/\mathcal{N}_\omega$. When $F$ and $F'$ are equal $\omega$-a.e., the notation $F = F'$ will be used.

The justification for the above terminology of a.e. equivalence is explained in the following illustrative example of finite probability spaces (cf. Appendix A for terminology).

**Example 2.12.** Let $\mathcal{A} := \mathbb{C}^X$ and $\mathcal{B} := \mathbb{C}^Y$ be the commutative $C^*$-algebras of complex-valued functions on the finite sets $X$ and $Y$, respectively, let $P : \mathbb{C}^X \rightarrow \mathbb{C}$ be a state on $X$, and let $F, G : \mathbb{C}^Y \rightarrow \mathbb{C}^X$ be two positive unital maps. Then there exists a unique probability measure $\mu$ on $X$ such that

$$\sum_{x \in X} \varphi(x) \mu_x = P(\varphi)$$

for all $\varphi \in \mathbb{C}^X$ (see Section 2.6 of [31] for details). Namely, $\mu_x := P(e_x)$, where $e_x$ is the function on $X$ defined by $X \ni x' \mapsto e_x(x') := \delta_{xx'}$. Similarly, there exist unique stochastic maps $f, g : X \rightarrow Y$ such that

$$\left(F(\psi)\right)(x) = \sum_{y \in Y} \psi(y) f_{yx} \quad \forall \psi \in \mathbb{C}^Y, \forall x \in X,$$

(2.13)

namely

$$X \ni x \mapsto \left(Y \ni y \mapsto f_{yx} := F(e_y)(x)\right)$$

(2.14)

and similarly for $G$ with $g$. One can show that the null space of $P$ is given by

$$\mathcal{N}_P := \{ \varphi \in \mathbb{C}^X : P(\varphi^*\varphi) = 0 \} = \{ \varphi \in \mathbb{C}^X : \varphi|_{X \setminus \mathcal{N}_P} = 0 \} = \text{span} \left( \bigcup_{x \in \mathcal{N}_P} \{ e_x \} \right),$$

(2.15)
where \( N_p \subseteq X \) is the measure-theoretic null space of \( p \) and \( \varphi|_{X \setminus N_p} \) denotes the restriction of \( \varphi \) to \( X \setminus N_p \). Hence, the quotient \( \mathbb{C}^X/N_p \) is isomorphic to functions on \( X \setminus N_p \) by the isomorphism

\[
\mathbb{C}^X/N_p \rightarrow \mathbb{C}^{X \setminus N_p},
\]

\[
[\varphi] \mapsto \varphi|_{X \setminus N_p}. \tag{2.16}
\]

As a result, the two positive unital maps \( F,G : \mathbb{C}^Y \to \mathbb{C}^X \) are equal P.a.e. if and only if the associated stochastic maps \( \tilde{f}, \tilde{g} : X \setminus N_p \to Y \) defined by the restrictions of \( f \) and \( g \) to \( X \setminus N_p \), respectively, are equal. This precisely means \( f \equiv g \).

We now proceed to establishing several important facts regarding non-commutative a.e. equivalence. First, if two maps are a.e. equivalent in terms of some state, then they pullback that state to the same state.

**Lemma 2.17.** Let \( A \) and \( B \) be \( C^* \)-algebras, let \( \xi : B \to \mathbb{C} \) be a state (or more generally a positive functional), and let \( \varphi, \psi : A \to B \) be linear maps. If \( \varphi = \psi \), then \( \xi \circ \varphi = \xi \circ \psi \).

**Proof.** Let \( a \in A \). Then

\[
|\xi(\varphi(a) - \psi(a))|^2 \leq \xi((\varphi(a) - \psi(a))^*(\varphi(a) - \psi(a))) = 0
\]

by the Cauchy–Schwarz inequality for positive functionals (cf. Proposition 5.2.1 in Fillmore [11]) and because \( \varphi(a) - \psi(a) \in N_{\xi} \). Hence, \( \xi(\varphi(a)) = \xi(\psi(a)) \). Since \( a \) was arbitrary, \( \xi \circ \varphi = \xi \circ \psi \). \( \blacksquare \)

The support of a state will also be useful in when formulating and proving our disintegration theorem.

**Lemma 2.19.** Let \( \omega : A \to \mathbb{C} \) be a state on a finite-dimensional \( C^* \)-algebra \( A \) (or more generally a \( W^* \)-algebra). Then there exists a unique projection \( P_\omega \in A \) (this means \( P_\omega^* = P_\omega \) and \( P_\omega^2 = P_\omega \)) such that \( N_\omega = A(1_A - P_\omega) \). Equivalently, \( P_\omega \) is characterized by

\[
\omega(a) = \omega(aP_\omega) = \omega(P_\omega a) = (\omega \circ \text{Ad}_{P_\omega})(a) \quad \forall a \in A. \tag{2.20}
\]

**Proof.** See Section 1.14 of Sakai [43]. \( \blacksquare \)

**Remark 2.21.** If \( A \) is not a finite-dimensional \( C^* \)-algebra in Lemma 2.19, then such a projection \( P_\omega \) for a state \( \omega \) need not exist. Indeed, if \( A = C(X) \), continuous complex-valued functions on a connected compact Hausdorff space \( X \), then there are no non-trivial projections and yet there are many states generating non-trivial null spaces. Such a projection does exist, however, if \( A \) is a \( W^* \)-algebra. Hence, several (but not all) of the results that follow involving such supports also hold for \( W^* \)-algebras.

**Definition 2.22.** Using the same notation from Lemma 2.19, \( P_\omega \) is called the **support of** \( \omega \). Its complement will be denoted by \( P_\omega^\perp := 1_A - P_\omega \).

**Example 2.23.** When \( A = M_n(\mathbb{C}) \) is a matrix algebra with a state \( \omega : M_n(\mathbb{C}) \to \mathbb{C} \), then \( \omega = \text{tr}(\rho \cdot) \) for some unique density matrix \( \rho \in M_n(\mathbb{C}) \) (cf. Remark 2.8). In this case, \( P_\omega^\perp \) is the projection onto the zero eigenspace of \( \rho \) and \( P_\omega \) satisfies \( P_\omega \rho = \rho = \rho P_\omega \).
Lemma 2.24. Let $P$ be a projection in a $C^*$-algebra (or $W^*$-algebra) $\mathcal{B}$ and let $B \in \mathcal{B}$. Then $BP = 0$ implies $B \in \mathcal{B}P^\perp$.

Proof. This follows from the fact that every $B \in \mathcal{B}$ can be uniquely expressed as a sum of four terms

$$B = (P + P^\perp)B(P + P^\perp) = PBP + PBP^\perp + P^\perp BP + P^\perp BP^\perp,$$

(2.25)

the non-zero ones of which are linearly independent. ■

The decomposition (2.25) will be used frequently in this work, particularly in conjunction with the support of a state. For example, we have the following alternative and computationally useful characterization of a.e. equivalence.

Lemma 2.26. Let $A$ and $B$ be finite-dimensional $C^*$-algebras, let $\xi : \mathcal{B} \to \mathbb{C}$ be a state, and let $\varphi, \psi : A \to \mathcal{B}$ be linear maps. Let $P_\xi \in \mathcal{B}$ denote the support of $\xi$.

i. Then $\varphi = \psi$ if and only if $\varphi(A)P_\xi = \psi(A)P_\xi$ for all $A \in A$.

ii. If $\varphi = \psi$, then $\text{Ad}_{P_\xi} \circ \varphi = \text{Ad}_{P_\xi} \circ \psi$.

Proof. For the first claim, if $\varphi = \psi$, then $\varphi(A) - \psi(A) = BP_\xi$ for some $B \in \mathcal{B}$ since $\mathcal{N}_\xi = \mathcal{B}P_\xi$ by Lemma 2.19. Multiplying by $P_\xi$ on the right gives item i. Conversely, if $\varphi(A)P_\xi = \psi(A)P_\xi$ holds, then $(\varphi(A) - \psi(A))P_\xi = 0$. Hence, $\varphi(A) - \psi(A) \in \mathcal{B}P_\xi^\perp$ by Lemma 2.24. Item ii follows from item i by multiplying $\varphi(A)P_\xi = \psi(A)P_\xi$ on the left by $P_\xi$. ■

Since CP maps between matrix algebras have particularly simple forms (cf. Theorem 2.6), it will also be useful to have a more quantitative version of Lemma 2.26. To state it, we first recall a general fact about the relationship between two Kraus decompositions of a CP map of matrix algebras.

Lemma 2.27. Let $\varphi : \mathcal{M}_m(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ be a CP map and suppose

$$\sum_{i=1}^p \text{Ad}_{V_i} = \varphi = \sum_{j=1}^q \text{Ad}_{W_j}$$

(2.28)

are two Kraus decompositions of $\varphi$ with $p \geq q$. Then there exists a $q \times p$ matrix $U$ that is a coisometry (meaning $UU^\dagger = 1_q$, i.e. the rows of $U$ are orthonormal) such that $V_i = \sum_{j=1}^q u_{ji}W_j$ for all $i \in \{1, 2, \ldots, p\}$. Here $u_{ji}$ denotes the $ji$-th entry of $U$.

---

1The usage of such a decomposition is certainly not new. More recently, they have made an appearance in the study of Pierce and corner algebras, also in the context of conditional expectations [40]. We thank Chris Heunen for informing us of this reference.

2In the case where $A = \mathcal{M}_m(\mathbb{C})$ and $B = \mathcal{M}_n(\mathbb{C})$, this says that the two maps $\varphi(A)$ and $\psi(A)$ agree when restricted to the subspace $P_\xi \mathbb{C}^n \subseteq \mathbb{C}^n$.  

7
Proof. The reader is referred to Sections 6 and 7 of [33] for any unexplained details and terminology. First note that every such Kraus decomposition \( \varphi = \sum_{i=1}^{p} \text{Ad}_{V_i} \) can be expressed as a Stinespring representation \( \varphi = \pi \circ \text{Ad}_V \), where \( \pi \) and \( V \) are defined by

\[
M_m(\mathbb{C}) \ni A \mapsto 1_{p} \otimes A \in M_p(\mathbb{C}) \otimes M_m(\mathbb{C}) \tag{2.29}
\]

and

\[
\mathbb{C}^p \otimes \mathbb{C}^m \cong \mathbb{C}^{pm} \overset{V:=[V_1 \cdots V_p]}{\longrightarrow} \mathbb{C}^n, \tag{2.30}
\]

respectively (and similarly for \( \varphi = \sum_{j=1}^{q} \text{Ad}_{W_j} \) and \( W := [W_1 \cdots W_q] \)). By the universal property of Stinespring representations (see Theorem 6.29 and the end of Section 7 in [33]), there exists a coisometry\(^3\) \( \mathbb{C}^p \overset{U}{\to} \mathbb{C}^q \) such that the diagram

\[
\begin{array}{ccc}
\mathbb{C}^n & \xleftarrow{U} & \mathbb{C}^q \\
\downarrow{W} & & \downarrow{V} \\
\mathbb{C}^p \otimes \mathbb{C}^m & \xrightarrow{U \otimes I_m} & \mathbb{C}^p \otimes \mathbb{C}^m
\end{array}
\tag{2.31}
\]

commutes. Writing

\[
U \otimes 1_m = \begin{bmatrix}
u_{11}1_m & \cdots & u_{1p}1_m \\
\vdots & & \vdots \\
u_{q1}1_m & \cdots & u_{qp}1_m
\end{bmatrix},
\tag{2.32}
\]

we see that commutativity of (2.31) gives

\[
[V_1 \cdots V_p] = [W_1 \cdots W_q] (U \otimes 1_n), \tag{2.33}
\]

which is the result claimed. \(\blacksquare\)

**Lemma 2.34.** Fix a positive integer \( n \), let \( \xi : M_n(\mathbb{C}) \to \mathbb{C} \) be a state, and let \( \varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be a CPU map such that \( \varphi = \xi \circ \text{id}_{M_n(\mathbb{C})} \). Let \( P_{\xi} \in M_n(\mathbb{C}) \) denote the support of \( \xi \) and let \( \varphi \) have a Kraus decomposition \( \varphi = \sum_{i=1}^{p} \text{Ad}_{V_i} \). Then there exist complex numbers \( \{\alpha_i\}_{i \in \{1,\ldots,p\}} \) such that

\[
P_{\xi}V_i = \alpha_i P_{\xi} \quad \forall i \in \{1,\ldots,p\} \quad \text{and} \quad \sum_{i=1}^{p} |\alpha_i|^2 = 1. \tag{2.35}
\]

**Proof.** Since \( \varphi = \xi \circ \text{id}_{M_n(\mathbb{C})} \), Lemma 2.26 implies

\[
\text{Ad}_{P_{\xi}} \circ \varphi = \sum_{i=1}^{p} \text{Ad}_{P_{\xi}V_i} = \text{Ad}_{P_{\xi}}. \tag{2.36}
\]

Furthermore, Lemma 2.27 implies there exists a \( 1 \times p \) coisometry \( U \) such that \( P_{\xi}V_i = u_{1i}P_{\xi} \), where \( u_{1i} \) is the \( 1i \)-th entry of \( U \). Set \( \alpha_i := u_{1i} \). Since \( U \) is a coisometry, the first (and only) column of \( U \) is a unit vector, i.e. \( \sum_{i=1}^{p} |u_{1i}|^2 = 1 \), which proves the claim. \(\blacksquare\)

\(^3\)Technically, the theorem referenced claims there exists a partial isometry. However, this partial isometry can be extended to a coisometry by similar techniques to those employed in Example 7.27 and Theorem 7.30 in [33].
Remark 2.37. Using the notation of Lemma 2.34, \( P_\xi V_i = \alpha_i P_\xi \) says
\[
V_i = \alpha_i P_\xi + P_\xi V_i P_\xi + P_\xi V_i P_\xi^\perp. \tag{2.38}
\]
If we choose a basis in which the density matrix \( \xi^*(1) \) is diagonal with its non-zero eigenvalues all appearing on the top left, then (2.38) reads
\[
V_i = \begin{bmatrix} \alpha_i & 0 \\
V^\text{bl}_i & V^\text{br}_i
\end{bmatrix}, \tag{2.39}
\]
where \( r \) is the rank of \( \xi^*(1) \), and where \( V^\text{bl}_i \) is an \((n-r) \times r\) matrix while \( V^\text{br}_i \) is an \((n-r) \times (n-r)\) matrix.

It turns out that Lemma 2.34 holds even when the support \( P_\xi \) from the equations is removed (cf. Theorem 2.48 below). To prove this, it seems convenient to recall the notion of a pre-Hilbert \( C^\ast \)-algebra module due to Paschke [37].

Definition 2.40. Let \( \mathcal{A} \) be a (unital) \( C^\ast \)-algebra. A pre-Hilbert \( \mathcal{A} \)-module is a left\(^4 \) \( \mathcal{A} \)-module \( \mathcal{E} \) together with a linear-conjugate linear map\(^5 \) \( \langle \langle \cdot, \cdot \rangle \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A} \) satisfying the following properties

i. \( \langle \langle s, s \rangle \rangle \geq 0 \) for all \( s \in \mathcal{E} \),
ii. \( \langle \langle s, t \rangle \rangle = \langle \langle t, s \rangle \rangle^\ast \) for all \( s, t \in \mathcal{E} \),
iii. \( \langle \langle as, t \rangle \rangle = a \langle \langle s, t \rangle \rangle \) for all \( s, t \in \mathcal{E} \) and \( a \in \mathcal{A} \), and
iv. \( \langle \langle s, s \rangle \rangle = 0 \) if and only if \( s = 0 \) (this is called non-degeneracy of \( \langle \langle \cdot, \cdot \rangle \rangle \)).

\( \langle \langle \cdot, \cdot \rangle \rangle \) is called the \( \mathcal{A} \)-valued inner product on \( \mathcal{E} \).

Remark 2.41. It follows from this definition that \( \langle \langle s, at \rangle \rangle = \langle \langle s, t \rangle \rangle a^\ast \) for all \( s, t \in \mathcal{E} \) and \( a \in \mathcal{A} \).

Lemma 2.42. Let \( \mathcal{A} \) be a \( C^\ast \)-algebra and \( \mathcal{E} \) a pre-Hilbert module over \( \mathcal{A} \). Then
\[
\mathcal{E} \ni s \mapsto \| s \|_{\mathcal{E}} := \sqrt{\| \langle \langle s, s \rangle \rangle \|} \tag{2.43}
\]
defines a norm on \( \mathcal{E} \). Furthermore,
\[
\langle \langle t, s \rangle \rangle \langle \langle s, t \rangle \rangle \leq \| t \|^2_{\mathcal{E}} \langle \langle s, s \rangle \rangle \tag{2.44}
\]
for all \( s, t \in \mathcal{E} \).

Proof. See Proposition 2.3 in Paschke [37] or Section 3.14 in Fillmore [11].

\(^4\)Paschke defines a \textit{right} module structure instead of a left one. This does change some properties, but we have modified them appropriately. One such property is the Paschke–Cauchy–Schwarz inequality in (2.44).

\(^5\)This means \( \langle \langle s + t, u \rangle \rangle = \langle \langle s, u \rangle \rangle + \langle \langle t, u \rangle \rangle \) for all \( s, t, u \in \mathcal{E} \). The other properties usually associated with sesquilinearity (with conjugate linearity in the second coordinate) follow from the other conditions in the definition since the algebra \( \mathcal{A} \) is unital.
Example 2.45. Fix $n, m, p \in \mathbb{N}$. Let $\mathcal{M}_n(\mathbb{C})$ denote the vector space of $m \times n$ complex matrices. Let $\mathcal{A} := \mathcal{M}_m(\mathbb{C})$ and $\mathcal{E} := \mathcal{M}_n(\mathbb{C})^p$, the vector space direct sum of $p$ copies of $\mathcal{M}_n(\mathbb{C})$. Denote elements of $\mathcal{E}$ by $\bar{A} := (A_1, \ldots, A_p)$ so that $A_i \in \mathcal{M}_n(\mathbb{C})$ for all $i \in \{1, \ldots, p\}$. Define the left $\mathcal{A}$-module structure on $\mathcal{E}$ to be $B\bar{A} := (BA_1, \ldots, BA_p)$ for all $\bar{A} \in \mathcal{E}$ and $B \in \mathcal{A}$. Define the $\mathcal{A}$-valued inner product by

$$\mathcal{E} \times \mathcal{E} \ni (\bar{A}, \bar{B}) \mapsto \left\langle \bar{A}, \bar{B} \right\rangle := \sum_{i=1}^{p} A_i B_i^\dagger.$$  \hfill (2.46)

Straightforward matrix algebra shows $\mathcal{E}$ is indeed a pre-Hilbert $\mathcal{A}$-module with these structures. In fact, $\mathcal{E}$ is also a right $\mathcal{M}_n(\mathbb{C})$-module satisfying

$$\left\langle \bar{A}C, \bar{B} \right\rangle = \left\langle \bar{A}, \bar{B}C^\dagger \right\rangle \quad \forall \bar{A}, \bar{B} \in \mathcal{E} \text{ and } C \in \mathcal{M}_n(\mathbb{C}).$$  \hfill (2.47)

However, $\mathcal{E}$ is not a (right) pre-Hilbert module with respect to this action.

Theorem 2.48. Fix $n \in \mathbb{N}$, let $\xi : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathbb{C}$ be a state, and let $\varphi : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})$ be a CPU map such that $\varphi = \text{id}_{\mathcal{M}_n(\mathbb{C})}$. Let $\varphi$ have a Kraus decomposition $\varphi = \sum_{i=1}^{p} \text{Ad}_{V_i}$. Then there exist complex numbers $\{\alpha_i\}_{i \in \{1, \ldots, p\}}$ such that

$$V_i = \alpha_i 1_n \quad \forall i \in \{1, \ldots, p\} \quad \text{and} \quad \sum_{i=1}^{p} |\alpha_i|^2 = 1. \quad (2.49)$$

In particular, $\varphi = \text{id}_{\mathcal{M}_n(\mathbb{C})}$.

Proof. Let $P_\xi \in \mathcal{M}_n(\mathbb{C})$ denote the support of $\xi$. In order to proceed avoiding as many indices and sums as possible, we will first introduce a certain pre-Hilbert $C^*$-algebra module based on the number $p$ of Kraus operators assumed for $\varphi$. Set $\mathcal{E} := \mathcal{M}_n(\mathbb{C})^p$ and equip this with the pre-Hilbert $\mathcal{M}_n(\mathbb{C})$-module structure from Example 2.45. By Remark 2.37,

$$\bar{V} = P_\xi \bar{\alpha} + \bar{V}^{\text{bl}} + \bar{V}^{\text{br}}, \quad (2.50)$$

where

$$\bar{\alpha} := (\alpha_1 1_n, \ldots, \alpha_p 1_n), \quad \bar{V}^{\text{bl}} := P_\xi^{\perp} \bar{V} P_\xi, \quad \text{and} \quad \bar{V}^{\text{br}} := P_\xi^{\perp} \bar{V} P_\xi. \quad (2.51)$$

Note that $C\bar{\alpha} = \bar{\alpha} C$ for all $C \in \mathcal{M}_n(\mathbb{C})$. The identities

$$\left\langle \langle \bar{\alpha}, \bar{\alpha} \right\rangle \right\rangle = 1_n, \quad \left\langle \langle P_\xi \bar{\alpha}, \bar{V}^{\text{bl}} \right\rangle \right\rangle = 0, \quad \text{and} \quad \left\langle \langle \bar{V}^{\text{bl}}, \bar{V}^{\text{br}} \right\rangle \right\rangle = 0 \quad (2.52)$$

follow directly from the definitions. The fact that $\varphi$ is unital means $1_n = \sum_i V_i V_i^\dagger$. In terms of the $\mathcal{M}_n(\mathbb{C})$-valued inner product, this becomes

$$1_n = \left\langle \langle \bar{V}, \bar{V} \right\rangle \right\rangle \quad (2.50) \quad \left\langle \langle P_\xi \bar{\alpha} + \bar{V}^{\text{bl}} + \bar{V}^{\text{br}}, P_\xi \bar{\alpha} + \bar{V}^{\text{bl}} + \bar{V}^{\text{br}} \right\rangle \right\rangle$$

$$\left\langle \langle P_\xi \bar{\alpha}, P_\xi \bar{\alpha} \right\rangle \right\rangle + \left\langle \langle P_\xi \bar{\alpha}, \bar{V}^{\text{bl}} \right\rangle \right\rangle + \left\langle \langle \bar{V}^{\text{bl}}, P_\xi \bar{\alpha} \right\rangle \right\rangle + \left\langle \langle \bar{V}^{\text{bl}}, \bar{V}^{\text{bl}} \right\rangle \right\rangle + \left\langle \langle \bar{V}^{\text{br}}, \bar{V}^{\text{br}} \right\rangle \right\rangle. \quad (2.53)$$
Since
\[ P_\xi \bar{V}^{bl} = \bar{0} \quad \text{and} \quad P_\xi \bar{V}^{br} = \bar{0}, \quad (2.54) \]
it follows that
\[ \langle \bar{X}, \bar{V}^{bl} \rangle P_\xi = 0 \quad \text{and} \quad \langle \bar{X}, \bar{V}^{br} \rangle P_\xi = 0 \quad \forall \ \bar{X} \in \mathcal{E} \quad (2.55) \]
by the properties of the pre-Hilbert module structure (see Remark 2.41). Hence, multiplying (2.53) by \( P_\xi \) on the right and simplifying gives \( 0 = \langle \bar{V}^{bl}, P_\xi \bar{a} \rangle \) and similarly \( \langle P_\xi \bar{a}, \bar{V}^{bl} \rangle = 0 \). Furthermore, \( \langle \bar{V}^{bl}, P_\xi \bar{a} \rangle = 0 \) follows immediately from the definition of \( \bar{V}^{bl} \). Putting these two together gives
\[ \langle \bar{V}^{bl}, \bar{a} \rangle = 0. \quad (2.56) \]
Hence, the unitality of \( R \) condition (2.53) simplifies to
\[ P_\xi^\perp = \langle \bar{V}^{bl}, \bar{V}^{bl} \rangle + \langle \bar{V}^{br}, \bar{V}^{br} \rangle. \quad (2.57) \]
Now, write \( A \in \mathcal{M}_n(\mathbb{C}) \) as (cf. Equation (2.25))
\[ A = P_\xi A P_\xi + P_\xi A P_\xi^\perp + P_\xi^\perp A P_\xi + P_\xi^\perp A P_\xi^\perp \quad (2.58) \]
in terms of the support \( P_\xi \) of \( \xi \) and its orthogonal complement \( P_\xi^\perp \). Using this decomposition,
\[ \varphi(A)P_\xi = \sum_{i=1}^p V_i A V_i^\dagger P_\xi = \langle \bar{V} A, \bar{V} \rangle P_\xi \overset{\text{Rmk 2.41}}{=} \langle \bar{V} A, P_\xi \bar{V} \rangle \overset{\text{(2.50)}}{=} \langle \bar{V} A, P_\xi \bar{a} \rangle \]
\[ \overset{(2.46)}{=} \langle \bar{V}, \bar{a} \rangle A P_\xi \overset{(2.52) \& (2.56)}{=} (P_\xi + \langle \bar{V}^{br}, \bar{a} \rangle) A P_\xi \]
\[ \overset{(2.58)}{=} \left( P_\xi + \langle \bar{V}^{br}, \bar{a} \rangle \right) \left( P_\xi A P_\xi + P_\xi^\perp A P_\xi \right) \overset{(2.55)}{=} P_\xi A P_\xi + P_\xi^\perp A P_\xi \]
for all \( A \in \mathcal{M}_n(\mathbb{C}) \). But since \( \varphi = \text{id}_{\mathcal{M}_n(\mathbb{C})} \), this equals \( P_\xi A P_\xi + P_\xi^\perp A P_\xi \) by Lemma 2.26 item i.

Identifying terms, \( \langle \bar{V}^{br}, \bar{a} \rangle P_\xi^\perp A P_\xi = P_\xi^\perp A P_\xi \) for all \( A \in \mathcal{M}_n(\mathbb{C}) \), i.e. \( \langle \bar{V}^{br}, \bar{a} \rangle \) acts as the identity on \( n \times n \) matrices of the form \( P_\xi^\perp A P_\xi \). This combined with the fact that \( \langle \bar{V}^{br}, \bar{a} \rangle = P_\xi^\perp \langle \bar{V}^{br}, \bar{a} \rangle P_\xi^\perp \) implies
\[ \langle \bar{V}^{br}, \bar{a} \rangle = P_\xi^\perp. \quad (2.60) \]
This implies \( \langle \bar{a}, \bar{V}^{br} \rangle^* \langle \bar{a}, \bar{V}^{br} \rangle = P_\xi^\perp \). Hence, by the Paschke–Cauchy–Schwarz inequality (Lemma 2.42),
\[ P_\xi^\perp \leq \| \bar{a} \|_2^2 \langle \bar{V}^{br}, \bar{V}^{br} \rangle \overset{(2.52)}{=} \langle \bar{V}^{br}, \bar{V}^{br} \rangle. \quad (2.61) \]
On the other hand, (2.57) entails \( \langle \bar{V}^{br}, \bar{V}^{br} \rangle \leq P_\xi^\perp \) by condition i in Definition 2.40. These two inequalities force
\[ \langle \bar{V}^{br}, \bar{V}^{br} \rangle = P_\xi^\perp \quad \text{and} \quad \langle \bar{V}^{bl}, \bar{V}^{bl} \rangle = 0. \quad (2.62) \]
By non-degeneracy of the \( \mathcal{M}_n(\mathbb{C}) \)-valued inner product on \( \mathcal{E} \), this forces \( \bar{V}^{bl} = \bar{0} \), i.e.
\[ P_\xi V_i P_\xi = 0 \quad \forall \ i \in \{1, \ldots, p\}. \quad (2.63) \]
Finally, using these relations and the properties of the \( \mathcal{M}_n(\mathbb{C}) \)-valued inner product,
\[ \langle P_\xi^\perp \bar{a} - \bar{V}^{br}, P_\xi \bar{a} - \bar{V}^{br} \rangle \overset{(2.52)}{=} P_\xi^\perp - \langle \bar{a}, \bar{V}^{br} \rangle - \langle \bar{V}^{br}, \bar{a} \rangle + \langle \bar{V}^{br}, \bar{V}^{br} \rangle \overset{(2.60) \& (2.62)}{=} 0, \quad (2.64) \]
which, by non-degeneracy of the $\mathcal{M}_n(\mathbb{C})$-valued inner product, proves

$$
\vec{V}^{br} = P_{\xi}^\perp \vec{\alpha},
$$

i.e.

$$
P_{\xi}^\perp V_i P_{\xi}^\perp = \alpha_i P_{\xi}^\perp \quad \forall \ i \in \{1, \ldots, p\}.
$$

Putting this all together gives

$$
\vec{V} = \vec{\alpha}, \quad \text{i.e.} \quad V_i = \alpha_i \mathbb{1}_n \quad \forall \ i \in \{1, \ldots, p\}.
$$

The fact that $\varphi = \text{id}_{\mathcal{M}_n(\mathbb{C})}$ follows immediately from this result.

\[ \blacksquare \]

**Remark 2.68.** It should be stressed how surprising Theorem 2.48 is. Even if $\xi$ is a pure state, so that its support is a rank one projection, a.e. equivalence of a CPU map to the identity is strong enough to enforce equality of that CPU map to the identity, regardless of how large the dimension of the Hilbert space is. We feel this gives us a precise sense of how “probability zero” objects (the projection onto the orthogonal complement of the support of $\xi$ in this case) in quantum theory cannot be disregarded in the way that they can be in classical probability theory [19].

**Remark 2.69.** Theorem 2.48 might seem to suggest that if $\xi : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathbb{C}$ is a state and if $\varphi, \psi : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})$ are two CPU maps, then $\varphi = \psi$ implies $\varphi = \psi$. The following example shows this is false in general. Let $s \in \{1, \ldots, m - 1\}$ and let $\chi : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathbb{C}$ be the map that takes the trace of the bottom right part of a matrix, namely

$$
\chi(A) := \sum_{i=s+1}^m a_{ii},
$$

where $a_{ii}$ is the $i$-th entry of $A$. Note that this map is positive and therefore CP since the codomain of $\chi$ is $\mathbb{C}$ (cf. Theorem 3 in Stinespring [45]). Note, however, that $\chi$ is not unital. Similarly, the trace map $\text{tr} : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathbb{C}$ is CP (but not unital). Now, consider the following two maps

$$
\mathcal{M}_m(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})
$$

\begin{align*}
A \mapsto & \frac{1}{m} \text{tr}(A) \mathbb{1}_n \quad \text{and} \\
A \mapsto & \frac{1}{m} \text{tr}(A) P_{\xi} + \frac{1}{m-s} \chi(A) P_{\xi}^\perp.
\end{align*}

Note that $\varphi$ and $\psi$ are not equal. Nevertheless, $\varphi$ and $\psi$ are $\xi$-a.e. equivalent because $\varphi(A) P_{\xi} = \psi(A) P_{\xi}$ for all $A \in \mathcal{M}_m(\mathbb{C})$. A simple calculation shows $\psi$ and $\varphi$ are unital. Furthermore, they are both completely positive as their $p$-ampliations are

$$
\varphi_p(\cdot) = \frac{1}{m} \text{tr}_p(\cdot) \otimes \mathbb{1}_n \quad \text{and} \quad \psi_p(\cdot) = \frac{1}{m} \text{tr}_p(\cdot) \otimes P_{\xi} + \frac{1}{m-s} \chi_p(\cdot) \otimes P_{\xi}^\perp,
$$

respectively. Here, $\text{tr}_p$ and $\chi_p$ are the $p$-ampliations of $\text{tr}$ and $\chi$, which are positive.
Remark 2.73. The conclusion of Theorem 2.48 is false if $\varphi$ is assumed to only be CP but not unital. A simple counter-example is the CP map

$$\mathcal{M}_n(\mathbb{C}) \xrightarrow{\varphi} A + \text{tr}(A) P_{\xi}$$

Here, $\varphi$ is $\xi$-a.e. equivalent to $\text{id}_{\mathcal{M}_n(\mathbb{C})}$ but is not equal to it.

Remark 2.75. Using the same notation and assumptions as in Theorem 2.48, if $\varphi$ is $\xi$-a.e. equivalent to a $^*$-isomorphism, then it equals that $^*$-isomorphism. However, if $\xi : \mathcal{M}_{mp}(\mathbb{C}) \xrightarrow{\varphi} \mathbb{C}$ is a state and $\varphi : \mathcal{M}_m(\mathbb{C}) \xrightarrow{\varphi} \mathcal{M}_{mp}(\mathbb{C})$ is a CPU map that is $\xi$-a.e. equivalent to a $^*$-homomorphism, then it is not necessarily equal to that $^*$-homomorphism (unless $p = 1$). A simple counter-example is $\varphi(B) := \text{diag}(B, \text{tr}(B) 1_m, \ldots, \text{tr}(B) 1_m)$ and $\xi$ the state represented by the density matrix $\frac{1}{m} \text{diag}(1_m, 0, \ldots, 0)$. Then $\varphi$ is $\xi$-a.e. equivalent to the $^*$-homomorphism $B \mapsto \text{diag}(B, \ldots, B)$, but it is not equal to it (unless $m = 1$ or $p = 1$).

The following corollary of Theorem 2.48 is similar to a fact used frequently in the area of reversible quantum operations (cf. the proof of Theorem 2.1 in Nayak and Sen [27]).

Corollary 2.76. Let $F : \mathcal{M}_m(\mathbb{C}) \xrightarrow{\varphi} \mathcal{M}_n(\mathbb{C})$ and $R : \mathcal{M}_n(\mathbb{C}) \xrightarrow{\varphi} \mathcal{M}_m(\mathbb{C})$ be CPU maps with Kraus decompositions

$$R = \sum_{i=1}^p \text{Ad}_{R_i} \quad \text{and} \quad F = \sum_{j=1}^q \text{Ad}_{F_j}.$$  

If $R \circ F = \xi \text{id}_{\mathcal{M}_m(\mathbb{C})}$ for some state $\xi : \mathcal{M}_n(\mathbb{C}) \xrightarrow{\varphi} \mathbb{C}$, then there exist complex numbers $\{\alpha_{ij}\}_{i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}}$ such that

$$R_i F_j = \alpha_{ij} 1_m \quad \text{and} \quad \sum_{i,j} |\alpha_{ij}|^2 = 1.$$  

In particular, $R \circ F = \text{id}_{\mathcal{M}_m(\mathbb{C})}$.

Proof. This follows immediately from Theorem 2.48.

3 Categories of $C^*$-algebras, states, and morphisms

We prove that non-commutative probability spaces, $C^*$-algebras equipped with states, and a.e. equivalence classes of CPU maps (in fact, 2-positive unital maps) form a category. In fact, finite-dimensional $C^*$-algebras and a.e. equivalence classes of positive unital maps form a category. The following Cauchy–Schwarz type inequality, due to Kadison [21], for positive unital and 2-positive unital maps is useful in proving many of these claims.

Lemma 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and let $\varphi : \mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ be a positive unital map.

i. If $a \in \mathcal{A}$ is self-adjoint, then $\varphi(a)^2 \leq \varphi(a^2)$.

ii. If $\varphi$ is 2-positive, then $\varphi(a)^* \varphi(a) \leq \varphi(a^* a)$ for all $a \in \mathcal{A}$.

Proof. See Theorem 1.3.1 and Corollary 1.3.2 in [46] and Proposition 3.3 in [38].
Proposition 3.2. Let $C, B,$ and $A$ be $C^*$-algebras, let $\omega : A \rightarrow C$ be a state on $A$, and let $G, G' : C \rightarrow B$ and $F, F' : B \rightarrow A$ be $2$-positive (or Schwarz-positive) unital maps. If $F = \omega$ and $G = G'$, where $\xi := \omega \circ F$, then $F \circ G = F' \circ G'$.

Proof. By assumption, the diagrams

$$
\begin{array}{ccc}
C & \xrightarrow{G} & B \\
\downarrow{G'} & & \downarrow{F} \\
B & \xrightarrow{F'} & A \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
B/N_\xi & \xrightarrow{F} & A/N_\omega \\
\downarrow{F'} & & \downarrow{G} \\
B & \xrightarrow{G'} & A \\
\end{array}
$$

(3.3)

both commute. For the composite, we have

$$
\begin{array}{ccc}
C & \xrightarrow{G} & B/N_\xi \\
\downarrow{G'} & & \downarrow{F} \\
B & \xrightarrow{F'} & A/N_\omega \\
\end{array}
$$

(3.4)

The left part of this diagram commutes by commutativity of the left diagram in (3.3). It would be convenient to have a function $B/N_\xi \rightarrow A/N_\omega$ to fill in the diagram. In this regard, let $\tilde{F}, \tilde{F}' : B/N_\xi \rightarrow A/N_\omega$ be the functions defined by

$$
B/N_\xi \ni [b]_\xi \mapsto \tilde{F}([b]_\xi) := [F(b)]_\omega \quad \text{and} \quad B/N_\xi \ni [b]_\xi \mapsto \tilde{F}'([b]_\xi) := [F'(b)]_\omega.
$$

(3.5)

To see that $\tilde{F}$ is well-defined, let $b \in N_\xi$, i.e. $\xi(b^*b) = 0$. Then

$$
\omega(F(b)^*F(b)) \leq \omega(F(b^*b)) = \xi(b^*b) = 0
$$

(3.6)

by Lemma 3.1 applied to $F$ and the fact that $F$ is state-preserving so that $\omega \circ F = \xi$. A similar conclusion can be made for $\tilde{F}'$. Since $F(b)^*F(b) \geq 0$ and $\omega$ is a positive functional, this shows $F(b) \in N_\omega$, which proves $\tilde{F}$ and $\tilde{F}'$ are well-defined. In fact, by commutativity of the right diagram in (3.3), $\tilde{F} = \tilde{F}'$. Hence, all the subdiagrams in the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{G} & B/N_\xi \\
\downarrow{G'} & & \downarrow{F} \\
B & \xrightarrow{F'} & A/N_\omega \\
\end{array}
$$

(3.7)

commute so that $F \circ G = F' \circ G'$.

Assuming finite-dimensionality, we can prove more. Although the previous proposition is enough for the sequel, the following theorem is an interesting result in its own right.
Theorem 3.8. Let $\mathcal{C}, \mathcal{B},$ and $\mathcal{A}$ be finite-dimensional $C^*$-algebras, let $\omega : \mathcal{A} \to \mathbb{C}$ be a state on $\mathcal{A}$, and let $G, G' : \mathcal{C} \to \mathcal{B}$ and $F, F' : \mathcal{B} \to \mathcal{A}$ be positive unital maps with $G = G'$ and $F = F'$, where $\xi := \omega \circ F = \omega \circ F'$. Then $F \circ G = F' \circ G'$.

We will break up this proof into several lemmas, some of which are of independent interest. For the first lemma (the proof of which is immediate), recall that if $\varphi : \mathcal{A} \to \mathcal{B}$ is a linear map between $C^*$-algebras, then $\varphi$ is self-adjoint iff $\varphi(a)^* = \varphi(a^*)$ for all $a \in \mathcal{A}$. Also, a vector subspace $V \subseteq \mathcal{B}$ is self-adjoint iff $v \in V$ implies $v^* \in V$.

Lemma 3.9. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a linear map. If $\varphi$ is self-adjoint, then the image is a self-adjoint subspace of $\mathcal{B}$.

Lemma 3.10. Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional $C^*$-algebras, let $\xi : \mathcal{B} \to \mathbb{C}$ be a state, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a self-adjoint linear map. If $\varphi$ is $\xi$-a.e. equivalent to $0$, then $\text{Im}(\varphi) \subseteq P_{\xi}^+ B P_{\xi}$.

Proof of Lemma 3.10. By assumption, $\text{Im}(\varphi) \subseteq B P_{\xi}$. If $\varphi(a) = P_{\xi} b P_{\xi}^* + P_{\xi} b P_{\xi}$ for some $b \in \mathcal{B}$, then $\varphi(a^*) = \varphi(a)^* \in \mathcal{N\xi}$ as well by Lemma 3.9. But $\varphi(a)^* = P_{\xi} b^* P_{\xi} + P_{\xi} b^* P_{\xi}$. Hence $P_{\xi} b^* P_{\xi} = 0$. By taking the adjoint of this, we get $P_{\xi} b P_{\xi} = 0$. Thus, $\varphi(a) = P_{\xi} b P_{\xi}$ for some $b \in \mathcal{B}$. ■

Lemma 3.11. $F : \mathcal{B} \to \mathcal{A}$ be a positive map between $C^*$-algebras and let $P$ be a projection in $\mathcal{A}$. If $b \in \mathcal{B}$, then $F(b)$ can be uniquely decomposed as

$$F(b) = F^{\text{tl}}(b) + F^{\text{tr}}(b) + F^{\text{hl}}(b) + F^{\text{br}}(b),$$

where

$$F^{\text{tl}}(b) := P F(b) P, \quad F^{\text{tr}}(b) := P F(b) P^*, \quad F^{\text{hl}}(b) := P^* F(b) P, \quad F^{\text{br}}(b) := P^* F(b) P^*.$$  \hspace{1cm} (3.12)

Furthermore, $F^{\text{tr}}(b)^* = F^{\text{hl}}(b^*)$ for all $b \in \mathcal{B}$ and the maps $F^{\text{tl}}, F^{\text{br}} : \mathcal{B} \to \mathcal{A}$ are positive.

Proof. The decomposition itself is just (2.25). From this and self-adjointness of $F$,

$$F^{\text{tr}}(b)^* = (P F(b) P^*)^* = P^* F(b)^* P = P^* F(b) P = F^{\text{hl}}(b^*).$$  \hspace{1cm} (3.14)

Furthermore, $F^{\text{tl}}$ and $F^{\text{br}}$ are positive maps since $F^{\text{tl}} = \text{Ad}_P \circ F$ and $F^{\text{br}} = \text{Ad}_{P^*} \circ F$ are composites of positive maps. ■

Lemma 3.15. Let $(\mathcal{B}, \xi)$ and $(\mathcal{A}, \omega)$ be finite-dimensional $C^*$-algebras equipped with states and let $F : \mathcal{B} \to \mathcal{A}$ be a positive unital and state-preserving. Then $F(P_{\xi}^+ BP_{\xi}) \subseteq P_{\omega} A P_{\omega}$. In particular $F(P_{\xi}^+ BP_{\xi}) \subseteq \mathcal{N\omega}.

Proof. Let $b \in P_{\xi}^+ BP_{\xi}$. First, assume $b$ is self-adjoint. Then

$$\omega(F(b)^* F(b)) = \omega(F(b)^2) \leq \omega(F(b^2)) = \xi(b^2) = \xi(b^* b) = 0,$$

where the inequality follows from part i of Lemma 3.1, the equality after it follows from the fact that $F$ is state-preserving, and the final equality follows from $b \in \mathcal{N\xi}$. This proves that $F(b) \in \mathcal{N\omega}$ for self-adjoint $b \in P_{\xi}^+ BP_{\xi}$. Hence, $F^{\text{tl}}(b) = 0$ and $F^{\text{hl}}(b) = 0$ for $b$ self-adjoint by Lemma 3.11.
Second, assume $b$ is skew-adjoint. Then $b = ib'$ for some self-adjoint $b' \in \mathcal{B}$ (namely, $b' := -ib$). Then

$$F_{\text{tl}}(b) = F_{\text{tl}}(ib') = iF_{\text{tl}}(b') = 0$$

and similarly

$$F_{\text{bl}}(b) = 0$$

for skew-adjoint $b$ by the previous fact since $F_{\text{tl}}$ and $F_{\text{bl}}$ are linear. Since every $b$ can be decomposed as the linear combination of a self-adjoint and skew-adjoint element, this proves $F_{\text{tl}}$ and $F_{\text{bl}}$ are both equal to the zero map. Finally, for any $b \in P_{\xi}^{\perp} B P_{\xi}^{\perp}$,

$$F_{\text{tr}}(b) = F_{\text{tr}}((b^*)^*) = F_{\text{bl}}(b^*) = 0$$

by Lemma 3.11 and the facts just proved.

Proof of Theorem 3.8. We are required to prove $F(G(c)) P_\omega = F'(G'(c)) P_\omega$ for all $c \in \mathcal{C}$. First, note that

$$F'(G'(c)) P_\omega = F(G'(c)) P_\omega \text{ since } F = F'$$

$$= F(G'(c) P_{\xi} + G'(c) P_{\xi}^\perp) P_\omega$$

Therefore,

$$F(G(c)) P_\omega - F'(G'(c)) P_\omega = F\left((G(c) - G'(c)) P_{\xi} + (G(c) - G'(c)) P_{\xi}^\perp\right) P_\omega \text{ by (3.19)}$$

$$= F\left((G(c) - G'(c)) P_{\xi}^\perp\right) P_\omega \text{ since } G = G'$$

$$= F\left(\text{Ad}_{P_{\xi}^\perp} (G(c) - G'(c))\right) P_\omega \text{ by Lemma 3.10}$$

$$= \text{Ad}_{P_{\omega}^\perp} \left(F\left(\text{Ad}_{P_{\xi}^\perp} (G(c) - G'(c))\right)\right) P_\omega \text{ by Lemma 3.15}$$

$$= 0.$$

This proves that composition of a.e.-equivalence classes of positive unital maps between finite-dimensional $C^*$-algebras is well-defined.

Remark 3.21. By Lemma 2.19 and Remark 2.21, Theorem 3.8 holds if $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are $W^*$-algebras.

Definition 3.22. A non-commutative probability space is a pair $(\mathcal{A}, \omega)$, with $\mathcal{A}$ a $C^*$-algebra and $\omega$ a state on $\mathcal{A}$. A state-preserving map $(\mathcal{B}, \xi) \rightsquigarrow (\mathcal{A}, \omega)$ is a map (linear, positive, CP, $\ast$-homomorphism, etc.) $\mathcal{B} \overset{\xi}{\sim}\mathcal{A}$ such that $\xi = \omega \circ F$.

Corollary 3.23. The following facts hold.

i. The collection of non-commutative probability spaces and state-preserving maps forms a category.

ii. The collection of non-commutative probability spaces and a.e. equivalence classes of 2-positive unital maps forms a category.

iii. The collection of finite-dimensional non-commutative probability spaces (or non-commutative probability spaces on $W^*$-algebras) and a.e. equivalence classes of $PU$ maps forms a category.
iv. The opposite of the category of finite probability spaces and probability-preserving stochastic maps embeds fully into the category of non-commutative probability spaces and state-preserving PU maps. It is an equivalence on the subcategory of finite-dimensional commutative C*-algebras.

v. Two probability-preserving stochastic maps are a.e. equivalent if and only if their associated PU maps are a.e. equivalent.

The functor in item iv is uniquely determined by sending a finite set $X$ to the C*-algebra $\mathbb{C}^X$ and sending a stochastic map $f : X \rightarrow Y$ to the PU map $\mathbb{C}^Y \rightarrow \mathbb{C}^X$ uniquely determined by sending the basis vector $e_y$ to the function $\sum_{x \in X} f_{yx} e_x \in \mathbb{C}^X$ (cf. Example 2.12).

Proof. This follows from Example 2.12, Proposition 3.2, Theorem 3.8, and Section 2 of [31].

Remark 3.24. Note that when $f : X \rightarrow Y$ is a function, the functor from Corollary 3.23 item iv produces the *-homomorphism $\mathbb{C}^Y \rightarrow \mathbb{C}^X$ sending $\varphi \in \mathbb{C}^Y$ to $\varphi \circ f$, the pullback of $\varphi$ along $f$.

Remark 3.25. The diagrammatic definition of a.e. equivalence discussed in Remark A.21 cannot be transferred to our categories of C*-algebras and positive maps. To see this, first note that the cartesian product of sets also goes to the tensor product of C*-algebras (up to a natural isomorphism), i.e. the (contravariant) functor sending stochastic maps to PU maps extends to a monoidal functor (the product of stochastic maps is defined by the product of the associated probability measures). In particular, the diagonal map $\Delta_X : X \rightarrow X \times X$ becomes the multiplication map $\mathbb{C}^X \otimes \mathbb{C}^X \rightarrow \mathbb{C}^X$. Therefore, a natural candidate for $A \otimes A \rightarrow A$ would be the linear map that takes the product of the elements. However, this map is not positive in general. This is closely related to the no-cloning/no-broadcasting theorem in quantum mechanics [3, 8, 9, 30, 34, 49]. As a result, it is not a morphism in any of our categories. Our definition of a.e. equivalence in Definition 2.9, though not explicitly categorical, gives a direct definition of a.e. equivalence in terms of null spaces and is suitable for our purposes of non-commutative probability. Nevertheless, it has recently been proven that this result does agree with the categorical definition of a.e. equivalence when instantiated in quantum Markov categories [34, Theorem 5.12].

4 Non-commutative disintegrations on matrix algebras

Here, we define optimal hypothesis, disintegration, and regular conditional probability in the non-commutative setting. In Theorem 4.3, we provide a necessary and sufficient condition for a disintegration to exist on matrix algebras. The state on the initial algebra must be separable with the induced state as a factor. This result holds for *-homomorphisms of a special kind. In this same theorem, it is shown that a disintegration is unique whenever one exists. In the proof, we construct an explicit formula for any disintegration on matrix algebras. Theorem 4.30 covers the more general case of arbitrary *-homomorphisms between matrix algebras. Briefly, the existence no longer requires the initial state to be separable. However, it is separable after a specific unitary operation that transforms the *-homomorphism to one of the kind discussed in Theorem 4.3.
Definition 4.1. Given a state-preserving *-homomorphism \( (B, \xi) \xrightarrow{E} (A, \omega) \) on \( C^* \)-algebras, a hypothesis for \( (B, \xi) \xrightarrow{E} (A, \omega) \) is a CPU map \( R : A \rightarrow B \) such that \( \xi \). A hypothesis for \( (B, \xi) \xrightarrow{E} (A, \omega) \) is optimal if \( \xi \circ R = \omega \). A CPU map \( R : A \rightarrow B \) is a disintegration of \( \omega \) over \( \xi \) if \( \xi \circ R = \omega \) holds. A CPU map \( R : A \rightarrow B \) is a disintegration of \( \omega \) over \( \xi \) consistent with \( F \) if \( R \) is a disintegration of \( \omega \) over \( \xi \) such that \( R \circ F = \text{id}_B \). More concisely, a disintegration refers to a disintegration of \( \omega \) over \( \xi \) consistent with \( F \).

The following example illustrates how Definition 4.1 extends the classical definition of a disintegration to the non-commutative setting.

Example 4.2. Let \( X \) and \( Y \) be finite sets (with the discrete \( \sigma \)-algebras) with probability measures \( p : \{\bullet\} \rightarrow X \) and \( q : \{\bullet\} \rightarrow Y \). Let \( f : X \rightarrow Y \) be a function and let \( r : Y \rightarrow X \) be a stochastic map. Let \( P : C^X \rightarrow C^Y \), \( Q : C^Y \rightarrow C^X \), \( F : C^Y \rightarrow C^X \), and \( R : C^X \rightarrow C^Y \) denote the corresponding PU maps (cf. Example 2.12). Functoriality as discussed after Corollary 3.23 immediately implies the following.

i. \( F \) is state-preserving if and only if \( f \) is measure-preserving.

ii. \( R \) is a disintegration of \( P \) over \( Q \) consistent with \( F \) in the sense of Definition 4.1 if and only if \( r \) is a disintegration of \( p \) over \( q \) consistent with \( f \) in the sense of Appendix A (see also Theorem 5.1 later for a simpler description in terms of finite sets).

Several natural questions arise when comparing our definition of disintegration to the one from finite probability spaces, measure-preserving maps, and stochastic maps. First of all, given a state-preserving *-homomorphism \( (B, \xi) \xrightarrow{E} (A, \omega) \), does there exist a disintegration \( R \) over \( \omega \) consistent with \( \xi \)? Second, if a disintegration exists, is it unique or at least unique up to a.e. equivalence? Third, is a disintegration of a *-isomorphism (a.e. equivalent to) the inverse? All of these are true in the commutative case. Before addressing the general case of finite-dimensional \( C^* \)-algebras, the present section focuses on the setting of matrix algebras.

Theorem 4.3. Fix \( n, p \in \mathbb{N} \). Let \( F \) be the *-homomorphism given by the block diagonal inclusion

\[
\mathcal{M}_n(\mathbb{C}) \ni B \xrightarrow{E} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \equiv 1_p \otimes B \in \mathcal{M}_{np}(\mathbb{C}) \cong \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})
\]  

(4.4)

and let \( (\mathcal{M}_n(\mathbb{C}), \text{tr}(\rho \cdot \cdot)) \equiv \omega \) \( \xrightarrow{E} (\mathcal{M}_{np}(\mathbb{C}), \xi \equiv \text{tr}(\sigma \cdot \cdot)) \) be state-preserving. Then the following facts hold.

i. A disintegration \( R \) of \( \omega \) over \( \xi \) consistent with \( F \) exists if and only if there exists a density matrix \( \tau \in \mathcal{M}_p(\mathbb{C}) \) such that \( \rho = \tau \otimes \sigma \).

---

\(^6\)This definition of hypothesis is a non-commutative generalization of the definition from [1]. In [1], the definition also requires equality rather than a.e. equality, so our notion is also a weakening in this sense.
ii. When such a $\tau$ exists, the disintegration is unique and is given by the formula

$$\mathcal{M}_{np}(\mathbb{C}) \ni A \equiv \begin{bmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pp} \end{bmatrix} \mapsto R(A) := \sum_{j,k=1}^{p} \tau_{kj}A_{jk} \equiv \text{tr}_{\mathcal{M}_p(\mathbb{C})}( (\tau \otimes 1_n)A ),$$

(4.5)

where $A_{jk}$ is the $jk$-th $n \times n$ block of $A$ using the isomorphisms $\mathcal{M}_{np}(\mathbb{C}) \cong \mathcal{M}_p(\mathcal{M}_n(\mathbb{C})) \cong \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ and $\text{tr}_{\mathcal{M}_p(\mathbb{C})} : \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C}) \hookrightarrow \mathcal{M}_n(\mathbb{C})$ is the partial trace, uniquely determined by sending $C \otimes B \in \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_n(\mathbb{C})$ to $\text{tr}(C)B$. Furthermore, $R \circ F = \text{id}_{\mathcal{M}_n(\mathbb{C})}$.

iii. When such a $\tau$ exists, a Kraus decomposition of $R$ is given by

$$R = \text{Ad}[1_n \ 0 \ \cdots \ 0]_{(\sqrt{\tau} \otimes 1_n)} + \cdots + \text{Ad}[0 \ \cdots \ 1_n]_{(\sqrt{\tau} \otimes 1_n)}.$$  

(4.6)

As a consequence of uniqueness, we prove all disintegrations on matrix algebras are strict left inverses of their associated $^*$-homomorphism. Note that uniqueness is meant in the literal sense, not in the a.e. sense. This is surprising due to Remark 2.69, which says two a.e. equivalent CPU maps on matrix algebras need not be equal. The additional conditions for a disintegration are strong enough to imply equality. The following proof also provides a construction of the density matrix $\tau$ from $R$.

**Proof of Theorem 4.3.**

i. ($\Rightarrow$) Suppose a disintegration $R : \mathcal{M}_{np}(\mathbb{C}) \hookrightarrow \mathcal{M}_n(\mathbb{C})$ exists. Let

$$R = \sum_{i=1}^{n^2p} \text{Ad}[V_{i1} \ \cdots \ V_{ip}]$$

(4.7)

be a Kraus decomposition of $R$ with $V_{ij} \in \mathcal{M}_n(\mathbb{C})$ for all $i \in \{1, \ldots, n^2p\}$ and $j \in \{1, \ldots, p\}$ ($n^2p$ is the minimal number of Kraus operators needed in this case). For the moment, let $R_i := [V_{i1} \ \cdots \ V_{ip}]$. Also note that $F$ has a Kraus decomposition $F = \sum_{j=1}^{p} \text{Ad}F_j$, where the (adjoint of the) Kraus operators are given by

$$F_j^\dagger := [0 \ \cdots \ 1_n \ \cdots \ 0]$$

(4.8)

with $1_n$ in the $j$-th $n \times n$ block. By Corollary 2.76, there exist numbers $\{\alpha_{ij} \in \mathbb{C}\}_{i \in \{1, \ldots, n^2p\}, j \in \{1, \ldots, p\}}$ such that

$$V_{ij} = R_iF_j = \alpha_{ij}1_n \quad \forall \ i, j \quad \text{and} \quad \sum_{i,j} |\alpha_{ij}|^2 = 1.$$  

(4.9)

due to the form of our matrices in (4.7) and (4.8). We now impose the condition $\xi \circ R = \omega$, which is equivalent to $R^*(\sigma) = \rho$, where

$$R^* = \sum_{i=1}^{n^2p} \text{Ad}_{R_i^\dagger}$$

(4.10)
is the dual or $R$ with respect to the Hilbert–Schmidt inner product (cf. Remark 2.8). Therefore,

$$
\rho = R^*(\sigma) \overset{(4.10)}{=} \sum_{i=1}^{n^2} A_d R^i_1(\sigma) = \sum_{i=1}^{n^2} \left[ \begin{array}{c}
V_{i1}^\dagger \\
\vdots \\
V_{ip}^\dagger
\end{array} \right] \sigma \left[ \begin{array}{c}
V_{i1} \\
\vdots \\
V_{ip}
\end{array} \right]
$$

$$
(4.10)
$$

$$
\overset{(4.9)}{=}
\sum_{i=1}^{n^2} \left[ \begin{array}{c}
\bar{\alpha}_{i1} 1_n \\
\vdots \\
\bar{\alpha}_{ip} 1_n
\end{array} \right] \sigma \left[ \begin{array}{c}
\alpha_{i1} 1_n \\
\vdots \\
\alpha_{ip} 1_n
\end{array} \right] = \sum_{i=1}^{n^2} \left[ \begin{array}{c}
|\alpha_{i1}|^2 \sigma \\
\vdots \\
|\alpha_{ip}|^2 \sigma
\end{array} \right]
$$

$$
(4.11)
$$

$$
\overset{(2.3)}{=}
\left( \sum_{i=1}^{n^2} \left[ \begin{array}{c}
|\alpha_{i1}|^2 \\
\vdots \\
|\alpha_{ip}|^2
\end{array} \right] \right) \otimes \sigma
$$

$$
\overset{=:\tau}{=}
$$

showing that $\rho$ is separable and has a tensor product factorization with $\sigma$ as a factor. Now, $\tau \in M_p(\mathbb{C})$ is a positive matrix because it is a positive sum of positive operators, namely

$$
\tau = \sum_{i=1}^{n^2} \left[ \begin{array}{c}
\bar{\alpha}_{i1} \\
\vdots \\
\bar{\alpha}_{ip}
\end{array} \right] \left[ \begin{array}{c}
\alpha_{i1} \\
\vdots \\
\alpha_{ip}
\end{array} \right].
$$

$$
(4.12)
$$

Furthermore, (4.9) implies

$$
\text{tr}(\tau) = \sum_{j=1}^{p} \sum_{i=1}^{n^2} |\alpha_{ij}|^2 = 1,
$$

$$
(4.13)
$$

which shows that $\tau$ is a density matrix.

($\Leftarrow$) Conversely, suppose there exists a density matrix $\tau$ such that $\rho = \tau \otimes \sigma$. Define $R : M_{np}(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ as in (4.5). The map $R$ is linear by construction and unital since

$$
R(1_{np}) = \sum_{j,k=1}^{p} \tau_{kj} \delta_{jk} 1_p = \sum_{j=1}^{p} \tau_{jj} 1_p = 1_p
$$

$$
(4.14)
$$

because $\text{tr}(\tau) = 1$. A similar calculation shows

$$
R(F(B)) = \sum_{j,k=1}^{p} \tau_{kj} \delta_{jk} B = B
$$

$$
(4.15)
$$

for all $B \in M_n(\mathbb{C})$. Hence, $R$ is actually a left inverse of $F$. In order for $R$ to preserve the states, it must be that $\omega \circ R = \xi$, i.e. $\text{tr}(\rho A) = \text{tr}(\sigma R(A))$ for all $A \in M_{np}(\mathbb{C})$. This follows
from
\[
\text{tr}(\rho A) = \text{tr}((\tau \otimes \sigma) A) = \text{tr}\left( \begin{bmatrix} \tau_{11} \sigma & \cdots & \tau_{1p} \sigma \\ \vdots & \ddots & \vdots \\ \tau_{p1} \sigma & \cdots & \tau_{pp} \sigma \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pp} \end{bmatrix} \right)
\]
\begin{equation}
= \text{tr}\left( \begin{bmatrix} \sum_{j=1}^{p} \tau_{j1} \sigma A_{j1} & \cdots & \sum_{j=1}^{p} \tau_{j1} \sigma A_{jp} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{p} \tau_{pj} \sigma A_{j1} & \cdots & \sum_{j=1}^{p} \tau_{pj} \sigma A_{jp} \end{bmatrix} \right) = \sum_{j,k=1}^{p} \tau_{kj} \text{tr}(\sigma A_{jk}) = \text{tr}(\sigma(R(A)).
\end{equation}

The final step is to prove \( R \) is CP. This follows from the fact that the partial trace satisfies a partially cyclic property, namely
\[
\text{tr}_{\mathcal{M}_p(\mathbb{C})}((\tau \otimes 1_n) A) = \text{tr}_{\mathcal{M}_p(\mathbb{C})}(A(\tau \otimes 1_n)).
\]
(4.17)

Thus,
\[
\text{tr}_{\mathcal{M}_p(\mathbb{C})}((\tau \otimes 1_n) A) = \text{tr}_{\mathcal{M}_p(\mathbb{C})}((\sqrt{\tau} \otimes 1_n) A(\sqrt{\tau} \otimes 1_n)) = (\text{tr}_{\mathcal{M}_p(\mathbb{C})} \circ \text{Ad}_{\sqrt{\tau}\otimes 1_n}) (A).
\]
(4.18)

shows that \( R \) is the composite of two CP maps, and is therefore CP. The claim also follows from showing that the Choi matrix associated to \( R \) is \( \Phi(R) = \tau^T \otimes \Phi(\text{id}_{\mathcal{M}_n(\mathbb{C})}) \), which is positive (the proof is omitted).

ii. Suppose \( R' \) is another disintegration of \( \omega \) over \( \xi \) consistent with \( F \). Let \( \{\alpha_{ij}\} \) and \( \{\alpha'_{ij}\} \) be coefficients obtained from Choi’s theorem as in (4.9). Construct the density matrices \( \tau \) and \( \tau' \) as in the proof of part i of this theorem. Then \( \tau' \otimes \sigma = \rho = \tau \otimes \sigma \), i.e. \( (\tau - \tau') \otimes \sigma = 0 \).

Since \( \sigma \) is non-zero, this means \( \tau - \tau' = 0 \), i.e. \( \tau' = \tau \). Hence, each of the entries of \( \tau \) and \( \tau' \) are equal, i.e. \( \tau_{jk} = \tau'_{jk} \) for all \( j, k \), or in terms of the \( \alpha \)'s and \( \alpha' \)'s,
\[
\sum_{i=1}^{n^2p} \alpha_{ij} \alpha_{ik} = \sum_{i=1}^{n^2p} \alpha'_{ij} \alpha'_{ik} \quad \forall j, k \in \{1, \ldots, p\}.
\]
(4.19)

Now, let \( A \) be a matrix in \( \mathcal{M}_{np}(\mathbb{C}) \equiv \mathcal{M}_p(\mathcal{M}_n(\mathbb{C})) \) as in (4.5) so that each \( A_{jk} \in \mathcal{M}_n(\mathbb{C}) \). Then, after some algebra
\[
R(A) = \sum_{i=1}^{n^2p} \begin{bmatrix} \alpha_{i1} 1_n & \cdots & \alpha_{ip} 1_n \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pp} \end{bmatrix} \begin{bmatrix} \alpha_{i1} 1_n \\ \vdots \\ \alpha_{ip} 1_n \end{bmatrix}
\begin{equation}
= \sum_{j,k=1}^{p} \left( \sum_{i=1}^{n^2p} \alpha_{ij} \alpha_{ik} \right) A_{jk} \overset{(4.19)}{=} \sum_{j,k=1}^{p} \left( \sum_{i=1}^{n^2p} \alpha'_{ij} \alpha'_{ik} \right) A_{jk} = R'(A),
\end{equation}
\end{equation}

which shows that \( R = R' \). Hence, disintegrations are unique when they exist. The fact that \( R \circ F = \text{id}_{\mathcal{M}_n(\mathbb{C})} \) follows from uniqueness of disintegrations and Corollary 2.76.
iii. The formula for the Kraus decomposition follows from the results just proven and a Kraus decomposition for the partial trace.

**Example 4.21.** Let
\[
\rho := \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad (4.22)
\]
be the density matrix on \( \mathbb{C}^4 \) corresponding to the projection operator onto the one-dimensional subspace of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) spanned by the vector\(^7\)
\[
\vec{u}_{\text{EPR}} := \frac{1}{\sqrt{2}} (\vec{e}_1 \otimes \vec{e}_2 - \vec{e}_2 \otimes \vec{e}_1). \quad (4.23)
\]
Let \( F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C}) \) be the map defined by
\[
\mathcal{M}_2(\mathbb{C}) \ni B \mapsto F(B) := \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad (4.24)
\]
which corresponds to the assignment \( \mathcal{M}_2(\mathbb{C}) \ni B \mapsto \mathbb{1}_2 \otimes B \in \mathcal{M}_2(\mathbb{C}) \otimes \mathcal{M}_2(\mathbb{C}) \) under the isomorphism from (2.4). Let \( \sigma \) be the density matrix on \( \mathbb{C}^2 \) given by
\[
\sigma := \frac{1}{2} \begin{bmatrix} p_1 + p_3 & 0 \\ 0 & p_2 + p_4 \end{bmatrix}, \quad (4.25)
\]
be density matrices with associated states given by \( \omega := \text{tr}(\rho \cdot) \) and \( \xi := \text{tr}(\sigma \cdot) \) be the corresponding states. Then, \( N_\xi = \{0\} \) and \( (\mathcal{M}_2(\mathbb{C}), \text{tr}(\rho \cdot) \equiv \omega) \xrightarrow{F} (\mathcal{M}_4(\mathbb{C}), \xi \equiv \text{tr}(\sigma \cdot)) \) is state-preserving, but there does not exist a disintegration of \( \omega \) over \( \xi \) consistent with \( F \).

**Example 4.25.** Fix \( p_1, p_2, p_3, p_4 \geq 0 \) with \( p_1 + p_2 + p_3 + p_4 = 1, p_1 + p_3 > 0, \) and \( p_2 + p_4 > 0 \). Let
\[
\rho = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \quad \text{and} \quad \sigma = \begin{bmatrix} p_1 + p_3 & 0 \\ 0 & p_2 + p_4 \end{bmatrix} \quad (4.26)
\]
be density matrices with associated states given by \( \omega := \text{tr}(\rho \cdot) \) and \( \xi := \text{tr}(\sigma \cdot) \), respectively. Let \( F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C}) \) be the diagonal inclusion from (4.24). Then \( \xi = \omega \circ F \) and \( N_\xi = \{0\} \). Furthermore, a CPU disintegration \( R : \mathcal{M}_4(\mathbb{C}) \twoheadrightarrow \mathcal{M}_2(\mathbb{C}) \) of \( \omega \) over \( \xi \) consistent with \( F \) exists if and only if \( p_1 p_4 = p_2 p_3 \). When this holds, the map
\[
R = \text{Ad}_{\sqrt{p_1 + p_2}} \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{bmatrix} + \text{Ad}_{\sqrt{p_3 + p_4}} \begin{bmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix}, \quad (4.27)
\]
is the unique disintegration of \( \omega \) over \( \xi \) consistent with \( F \). Furthermore, the density matrix \( \tau \in \mathcal{M}_2(\mathbb{C}) \) given by
\[
\tau = \begin{bmatrix} \frac{p_1}{p_1 + p_3} & 0 \\ 0 & \frac{p_4}{p_2 + p_4} \end{bmatrix} \quad (4.28)
\]
satisfies \( \tau \otimes \sigma = \rho \).

---

\(^7\)This is the spin EPR state discussed in Section 1.3.6 in Nielsen and Chuang [28].
Remark 4.29. Theorem 4.3 reproduces a well-known result in quantum information theory in the special case when the density matrices $\rho$ and $\sigma$ are invertible (see Example 9.6 in Petz’s text for example [39]). The surprising result we have shown is the fact that this still holds regardless of the sizes of the null-spaces associated to the density matrices and, moreover, the disintegration is uniquely determined.

The following result is a generalization of Theorem 4.3 on the existence of disintegrations to allow for $^\ast$-homomorphisms $F$ that are not necessary of the block diagonal form.

Theorem 4.30. Fix $n, p \in \mathbb{N}$. Let $(\mathcal{M}_n(\mathbb{C}), \text{tr}(\rho \cdot)) \equiv (\mathcal{M}_n(\mathbb{C}), \xi \equiv \text{tr}(\sigma \cdot))$ be a state-preserving $^\ast$-homomorphism. A disintegration of $\omega$ over $\xi$ consistent with $F$ exists if and only if there exists a unitary $U \in \mathcal{M}_{np}(\mathbb{C})$ and a density matrix $\tau \in \mathcal{M}_p(\mathbb{C})$ such that $F = \text{Ad}_U \circ i$ and $U^\dagger \rho U = \tau \otimes \sigma$. Here $i : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_{np}(\mathbb{C})$ is the block diagonal inclusion (4.4). Furthermore, if a disintegration exists, it is unique.

Proof. For any unital $^\ast$-homomorphism $F : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_{np}(\mathbb{C})$, there exists a unitary $U \in \mathcal{M}_{np}(\mathbb{C})$ such that $F = \text{Ad}_U \circ i$ (cf. Section 1.1.2 of Fillmore [11]). Hence, the diagram

\[
\begin{array}{c}
\mathcal{M}_{np}(\mathbb{C}) \xrightarrow{i} \mathcal{M}_n(\mathbb{C}) \\
\downarrow{\omega \circ \text{Ad}_U = \text{tr}(U^\dagger \rho U \cdot)} \downarrow{\xi = \text{tr}(\sigma \cdot)} \\
\mathbb{C}
\end{array}
\]

(4.31)

commutes. By Theorem 4.3, a disintegration $R : \mathcal{M}_{np}(\mathbb{C}) \twoheadrightarrow \mathcal{M}_n(\mathbb{C})$ of $\omega \circ \text{Ad}_U$ over $\xi$ consistent with $i$ exists if and only if there exists a density matrix $\tau$ such that $U^\dagger \rho U = \tau \otimes \sigma$. Explicitly, this means $\xi \circ R = \omega \circ \text{Ad}_U$ and $R \circ i = \text{id}_{\mathcal{M}_n(\mathbb{C})}$. Setting $R_U := R \circ \text{Ad}_{U^\dagger} : \mathcal{M}_{np}(\mathbb{C}) \twoheadrightarrow \mathcal{M}_n(\mathbb{C})$ and applying $\text{Ad}_{U^\dagger}$ to the right of $\xi \circ R = \omega \circ \text{Ad}_U$ gives $\xi \circ R_U = \omega$. Similarly, $R \circ i = \text{id}_{\mathcal{M}_n(\mathbb{C})}$ holds if and only if $R \circ \text{Ad}_{U^\dagger} \circ \text{Ad}_U \circ i = \text{id}_{\mathcal{M}_n(\mathbb{C})}$ holds, i.e. $R_U \circ F = \text{id}_{\mathcal{M}_n(\mathbb{C})}$. The map $R_U$ is CPU if and only if $R$ is CPU. Thus, $R_U$ defines a disintegration of $\omega$ over $\xi$ consistent with $F$ if and only if $U^\dagger \rho U = \tau \otimes \sigma$. Finally, the uniqueness of $R_U$ follows from the uniqueness of $R$ by part ii of Theorem 4.3. 

Note that an immediate consequence of this theorem is when $F$ is a $^\ast$-isomorphism, then $F^{-1}$ is the unique disintegration. Thus, a disintegration can be viewed as a generalization of time reversal.

Remark 4.32. Theorem 4.30, says there exists a tensor factorization $\tau \otimes \sigma = U^\dagger \rho U$ if and only if a disintegration exists. It is not necessary for $\rho$ to be separable in this case (compare this to Theorem 4.3, where $\rho = \tau \otimes \sigma$ was separable). This is because $U^\dagger \rho U$ is separable does not imply $\rho$ is separable in general—the unitary evolution of a separable state can cause that state to become entangled due to interactions between subsystems.

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8The techniques we have used to prove our results do not use Takesaki’s theorem nor the modular group (see [39, Theorem 9.2]). Instead, we worked directly with Kraus operators, a familiar tool in the quantum information theory community. A deeper analysis relating conditional expectations to disintegrations will be presented in forthcoming work.
Remark 4.33. Theorem 4.30 bears a striking resemblance to Theorem 2.1 in the work of Nayak and Sen [27]. However, there are three main differences. First, they work with completely positive trace-preserving (not necessarily unital) maps \( F : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C}) \), where \( m \leq n \), while we focus on the class of unital \(*\)-homomorphisms. Second, they assume \( R : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_m(\mathbb{C}) \) is a strict left inverse of \( F \) while we initially assume \( R \) is a left inverse up to a.e. equivalence. We showed this condition is actually equivalent for matrix algebras in Corollary 2.76 but we will see that a.e. equivalence is necessary for arbitrary finite-dimensional \( C^* \)-algebras. Third, and most importantly, Nayak and Sen do not require \( R \) and \( F \) to preserve any specified states while we do. This forces an additional constraint that our map \( R \) must satisfy making it even less obvious whether such a CP map \( R \) exists. Therefore, it seems that neither of our results subsume each other but are complementary and cover different situations.

If a deterministic process (a \(*\)-homomorphism) evolves a pure state into a mixed state between matrix algebras, is it possible for there to exist a disintegration that evolves the mixed state back into the pure state? The following corollary is a “no-go theorem” for such disintegrations.

**Corollary 4.34.** Given a state-preserving \(*\)-homomorphism \((\mathcal{M}_n(\mathbb{C}), \text{tr}(\sigma \cdot)) \xrightarrow{F} (\mathcal{M}_{np}(\mathbb{C}), \text{tr}(\rho \cdot)),\) with \( \rho \) pure, if a disintegration exists, then \( \sigma \) must necessarily be pure as well.

**Proof.** By Theorem 4.30, there exist a unitary \( U \in \mathcal{M}_{np}(\mathbb{C}) \) and a density matrix \( \tau \in \mathcal{M}_p(\mathbb{C}) \) such that 
\[
F(A) = U \text{diag}(A, \ldots, A) U^\dagger \quad \text{and} \quad \rho = U^\dagger (\tau \otimes \sigma) U.
\]
Since \( \rho \) is pure, it is a rank 1 projection operator. Its rank also equals \( \text{rank}(\rho) = \text{rank}(\tau) \text{rank}(\sigma) \), which equals 1 if and only if both \( \text{rank}(\tau) \) and \( \text{rank}(\sigma) \) are equal to 1. Hence \( \tau \) and \( \sigma \) are pure. \( \square \)

**Remark 4.35.** One might object to the conclusion of Corollary 4.34 and ask a more elementary question without referring to disintegrations. Namely, does there exist a mixed state \( \xi : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathbb{C} \) and a CPU map \( \varphi : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C}) \) such that \( \xi \circ \varphi \) is a pure state? The reason to ask such a question is that if its answer is no, then one does not even need a disintegration for it to be impossible to evolve a mixed state into a pure state. The following example addresses this. Let \( \omega : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathbb{C} \) be any pure state and let \( \xi : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathbb{C} \) be any mixed state. Set \( \varphi(A) := \omega(A) \mathbb{1}_n \), which is a CPU map satisfying \( \omega = \xi \circ \varphi \). Note that this situation is described by the diagram

\[
\begin{array}{ccc}
\mathcal{M}_m(\mathbb{C}) & \xrightarrow{\varphi} & \mathcal{M}_n(\mathbb{C}) \\
\omega & \simeq & 1 \\
\mathbb{C} & \xrightarrow{!} & \mathbb{C} \\
\xi & \simeq & \xi
\end{array}
\]

i.e. \( \varphi \) factors through \( \mathbb{C} \). In this diagram, \( ! \) is the unique unital map from \( \mathbb{C} \) into any (unital) \( C^* \)-algebra.

## 5 Disintegrations on finite-dimensional \( C^* \)-algebras

In the present section, we will extend Theorems 4.3 and 4.30 to the case of \(*\)-homomorphisms between arbitrary finite-dimensional \( C^* \)-algebras, which are all isomorphic to finite direct sums
of matrix algebras. We begin by analyzing CP maps between such direct sums in Lemma 5.15, their adjoints with respect to a generalized Hilbert–Schmidt inner product in Lemma 5.21, and the general form of states on direct sums in Lemma 5.27. After these preliminary results are established, we study the structure of Kraus decompositions of hypotheses in Lemma 5.40 and Lemma 5.56. Proposition 5.67 provides a generalization of the “tracing out” operation for direct sums, i.e. the induced state via pull-back from a *-homomorphism and a state on the target. After all this preparation, our main result, Theorem 5.76, is provided. Theorem 5.108 generalizes this disintegration theorem to arbitrary (unital) *-homomorphisms. But first, we recall the classical disintegration theorem.

**Theorem 5.1.** Let \((X, p) \xrightarrow{f} (Y, q)\) be a probability-preserving function. Then the following facts hold.

i. The assignment

\[
X \times Y \ni (x, y) \mapsto r_{xy} := \begin{cases} 
p_x \delta_{yf(x)}/q_y & \text{if } q_y > 0 \\
1/|X| & \text{otherwise} \end{cases}
\]  

(5.2)

defines a disintegration \(r : Y \rightsquigarrow X\) of \(p\) over \(q\) consistent with \(f\).

ii. The stochastic map \(r\) is the unique one up to a set of measure zero with respect to \(q\) satisfying \(f \circ q = \text{id}_Y\) and \(r \circ q = p\), i.e. \(r = r'\) for any other disintegration \(r' : Y \rightsquigarrow X\).

iii. Suppose \(f'\) is another measure-preserving function satisfying \(f = f'\). Let \(r\) be a disintegration of \(f\) and let \(r'\) be a disintegration of \(f'\). Then \(r = r'\).

We will omit the details of this proof, which are neither difficult nor new. However, some of the lemmas used in proving it provide insight into the proof of our main theorem on non-commutative disintegrations in Theorem 5.76. These lemmas motivate the formula (5.2) and also assist in proving a.e. uniqueness. They show that a measure-preserving function \((X, p) \xrightarrow{f} (Y, q)\) is surjective onto a set of full measure, and they illustrate that a hypothesis \(r : Y \rightsquigarrow X\) forces \(r_y\) to be supported on \(f^{-1}(\{y\})\) for almost all \(y \in Y\). This allows us to think of a disintegration more visually as follows. First, a probability space can be viewed as a finite number of water droplets, each of which has some volume (probability); the total volume is normalized to one. One can visualize a morphism \((X, p) \xrightarrow{f} (Y, q)\) as combining some of the water droplets, summing their volumes in the process. A hypothesis is a choice of physically splitting the water droplets back to the original set, but possibly with different volumes. A perfect splitting of the water droplets in which the volumes are reproduced exactly is an optimal hypothesis. From a topologist’s point of view, a hypothesis is a stochastic section of \(f\), which assigns a probability measure on the fiber (as opposed to a specific element) over each point that has non-zero \(q\) measure.

---

9We learned this point of view from Gromov [18].
Lemma 5.3. Let $r : Y \leadsto X$ be a hypothesis for $(X, p) \xrightarrow{f} (Y, q)$. Then the probability measure $r_y$ is supported on $f^{-1} \{ y \}$ for all $y \in Y \setminus N_q$.

Lemma 5.4. Let $(X, p) \xrightarrow{f} (Y, q)$ be a morphism in FinProb. Then, for each $y \in Y \setminus N_q$, there exists an $x \in X$ such that $f(x) = y$, i.e. $f$ is surjective onto a set of full $q$-measure.

Lemma 5.5. Let $r : Y \leadsto X$ be a disintegration of $(X, p) \xrightarrow{f} (Y, q)$. Then
\[ r_{xf(x)}q_{f(x)} = p_x \quad \forall x \in X. \quad (5.6) \]

Notation 5.7. Throughout the rest of this section, let
\[ A := \mathcal{M}_{m_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{m_s}(\mathbb{C}) \quad \text{and} \quad B := \mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_t}(\mathbb{C}) \quad (5.8) \]
denote direct sums of matrix algebras. An element $\tilde{A} \in A$ will be denoted as a column vector
\[ \tilde{A} \equiv \begin{pmatrix} A_1 \\ \vdots \\ A_s \end{pmatrix} \quad (5.9) \]
and similarly for elements of $B$. The vector notation is often used for emphasis. An arbitrary linear map $\varphi : A \leadsto B$ will be written in matrix form as
\[ \varphi \equiv \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1s} \\ \vdots & \ddots & \vdots \\ \varphi_{t1} & \cdots & \varphi_{ts} \end{pmatrix}, \quad (5.10) \]
where $\varphi_{ij} : \mathcal{M}_{m_i}(\mathbb{C}) \leadsto \mathcal{M}_{n_j}(\mathbb{C})$ is a linear map for all $i, j$. The notation indicates the action of $\varphi$ on $\tilde{A}$ as
\[ \varphi(\tilde{A}) = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1s} \\ \vdots & \ddots & \vdots \\ \varphi_{t1} & \cdots & \varphi_{ts} \end{pmatrix} \begin{pmatrix} A_1 \\ \vdots \\ A_s \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^s \varphi_{1i}(A_i) \\ \vdots \\ \sum_{i=1}^s \varphi_{ti}(A_i) \end{pmatrix}. \quad (5.11) \]
Let $(B, \xi) \xrightarrow{F} (A, \omega)$ be a state-preserving *-homomorphism defined by\(^{10}\)
\[ B \ni \tilde{B} \mapsto F \left( \begin{pmatrix} B_1 \\ \vdots \\ B_t \end{pmatrix} \right) := \begin{pmatrix} \underbrace{\text{\#\#\# \#\# \#\#\# \#\#} \cdots \underbrace{\text{\#\#\# \#\# \#\#\# \#\#}}}_{c_{11} \text{ times}} & \underbrace{\text{\#\#\# \#\# \#\#\# \#\#}} \cdots \underbrace{\text{\#\#\# \#\# \#\#\# \#\#}}}_{c_{1t} \text{ times}} \\
\vdots & \vdots \\
\underbrace{\text{\#\#\# \#\# \#\#\# \#\#}} \cdots \underbrace{\text{\#\#\# \#\# \#\#\# \#\#}}}_{c_{t1} \text{ times}} & \underbrace{\text{\#\#\# \#\# \#\#\# \#\#}} \cdots \underbrace{\text{\#\#\# \#\# \#\#\# \#\#}}}_{c_{tt} \text{ times}} \end{pmatrix}. \quad (5.12) \]

\(^{10}\)We will work with more general *-homomorphisms later, but we will see that all (unital) *-homomorphisms are unitarily equivalent to ones of this form. Hence, we do not lose much generality by focusing on these.
where the non-negative integer \( c_{ij} \) is called the multiplicity of \( F \) of the factor \( M_{n_j} \) inside \( M_{m_i} \) (cf. Section 1.1.2 and 1.1.3 in Fillmore [11]). In particular, the dimensions are related by the formula

\[
m_i = \sum_{j=1}^{t} c_{ij} n_j \quad \forall i \in \{1, \ldots, s\}. \tag{5.13}
\]

Since \( F \) is linear, it also has a matrix representation

\[
F \equiv \begin{pmatrix}
F_{11} & \cdots & F_{1t} \\
\vdots & \ddots & \vdots \\
F_{s1} & \cdots & F_{st}
\end{pmatrix} \tag{5.14}
\]

with \( F_{ij} : M_{n_j} \to M_{m_i} \) a (not necessarily unital) \(^*\)-homomorphism for all \( i, j \).

**Lemma 5.15.** Let \( A, B, \) and \( \varphi \) be as in (5.8) and (5.10). Then \( \varphi \) is CP if and only if \( \varphi_{ji} \) is CP for all \( i \in \{1, \ldots, s\} \) and \( j \in \{1, \ldots, t\} \). Furthermore, \( \varphi \) is unital if and only if

\[
1_{n_j} = \sum_{i=1}^{s} \varphi_{ji}(1_{m_i}) \quad \forall j \in \{1, \ldots, t\}. \tag{5.16}
\]

A Kraus decomposition of \( \varphi_{ji} \) in this case will be expressed as

\[
\varphi_{ji} = \sum_{l_{ji}=1}^{m_{in_j}} \text{Ad} V_{ji;l_{ji}}
\]

where the \( V_{ji;l_{ji}} : \mathbb{C}^{m_i} \to \mathbb{C}^{n_j} \) are linear maps. This allows the unitality condition (5.16) to be expressed as

\[
1_{n_j} = \sum_{i=1}^{s} \sum_{l_{ji}=1}^{m_{in_j}} V_{ji;l_{ji}} V_{ji;l_{ji}}^\dagger \quad \forall j \in \{1, \ldots, t\}. \tag{5.18}
\]

The following facts are easy to check and are analogous to what happens in the usual matrix algebra case. We include them here for completeness.

**Lemma 5.19.** Let \( \mathcal{A} := \mathcal{M}_{m_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{m_s}(\mathbb{C}) \). Then the assignment

\[
\mathcal{A} \times \mathcal{A} \ni (\tilde{A}, \tilde{A}') \mapsto \langle \tilde{A}, \tilde{A}' \rangle := \sum_{i=1}^{s} \text{tr}(A_i^\dagger A_i')
\]

defines an inner product on \( \mathcal{A} \). This is called the Hilbert–Schmidt (a.k.a. Frobenius) inner product on \( \mathcal{A} \).

**Lemma 5.21.** Let \( \mathcal{A}, \mathcal{B}, \) and \( \varphi \) be as in (5.8) and (5.10). Then there exists a unique linear map \( \varphi^* : \mathcal{B} \to \mathcal{A} \) satisfying

\[
\langle \tilde{B}, \varphi(\tilde{A}) \rangle = \langle \varphi^*(\tilde{B}), \tilde{A} \rangle \quad \forall \tilde{A} \in \mathcal{A}, \tilde{B} \in \mathcal{B}. \tag{5.22}
\]
The linear map $\varphi^*$ is called the adjoint of $\varphi$. Furthermore,

$$\varphi^* = \left( \varphi_{11}^* \cdots \varphi_{1t}^* \\ \vdots \quad \vdots \\ \varphi_{s1}^* \cdots \varphi_{st}^* \right), \quad (5.23)$$

where $\varphi_{ji}^*: \mathcal{M}_{nj}(\mathbb{C}) \rightarrow \mathcal{M}_{ni}(\mathbb{C})$ is the usual (Hilbert–Schmidt) adjoint of $\varphi_{ji}$. In particular, if $\varphi$ is a CP map where $\varphi_{ji}$ has Kraus decomposition as in (5.17), then

$$\varphi_{ji}^* = \sum_{l_{ji}=1}^{m_{nj}} \text{Ad}_{V_{ji,l_{ji}}^*}. \quad (5.24)$$

Finally, $\varphi$ is CPU if and only if $\varphi^*$ is CP and trace-preserving in the sense that

$$\langle \bar{B}, 1_B \rangle = \langle \varphi^*(\bar{B}), 1_A \rangle \quad \forall \bar{B} \in \mathcal{B}, \quad (5.25)$$

i.e.

$$\sum_{j=1}^{t} \text{tr}(B_j) = \sum_{i=1}^{s} \sum_{j=1}^{t} \text{tr}(\varphi_{ji}^*(B_j)) \quad (5.26)$$

in terms of the components of $\varphi^*$ and $\bar{B}$.

**Lemma 5.27.** Let $\xi: \mathcal{B} \rightarrow \mathbb{C}$ be a state with $\mathcal{B}$ as in (5.8). Then there exists unique non-negative real numbers $q_1, \ldots, q_t$ and (not necessarily unique) density matrices $\sigma_1 \in \mathcal{M}_{n_1}(\mathbb{C}), \ldots, \sigma_t \in \mathcal{M}_{n_t}(\mathbb{C})$ such that

$$\sum_{j=1}^{t} q_j = 1 \quad \text{and} \quad \xi(\bar{B}) = \sum_{j=1}^{t} q_j \text{tr}(\sigma_j B_j) \quad \text{for all } \bar{B} \in \mathcal{B}. \quad (5.28)$$

Furthermore, for every $j$ such that $q_j > 0$, the density matrix $\sigma_j$ is the unique one satisfying these conditions.

**Proof.** Since $\xi$ is a state, it is CPU. The adjoint $\xi^*: \mathbb{C} \rightarrow \mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_t}(\mathbb{C})$ of $\xi$ is CP and trace-preserving by Lemma 5.21. Let $\zeta := \xi^*(1)$, let $\zeta_j \in \mathcal{M}_{nj}(\mathbb{C})$ denote the $j$-th component of $\zeta$, and set $q_j := \text{tr}(\zeta_j)$. By the trace-preserving condition of $\xi^*$, the first equation in (5.28) holds. By the positivity of $\xi^*$, each $\zeta_j$ is a non-negative matrix. If $q_j > 0$ set

$$\sigma_j := \frac{\zeta_j}{q_j}. \quad (5.29)$$

Otherwise, if $q_j = 0$, then $\zeta_j$ is the zero matrix. In this case, let $\sigma_j$ be any density matrix. The conclusions of this lemma follow from these assignments. \qed

**Notation 5.30.** Due to Lemma 5.27, a state $\xi$ as above might occasionally be denoted by $\xi \equiv \sum_{j=1}^{t} q_j \text{tr}(\sigma_j \cdot )$, where $\xi(\bar{B})$ is understood to be given as in (5.28). Furthermore, the subset $N_q := \{ j \in \{1, \ldots, t\} : q_j = 0 \}$ will occasionally be used.

$^{11}$ $\varphi_{ji}^*$ will be the notation used for the dual of $\varphi_{ji}$ as opposed to the more precise $(\varphi_{ji})^*$. It is the dual of the $ij$-th entry of $\varphi^*$, which itself could be denoted by $(\varphi^*)_ij$. Hence, $(\varphi^*)_ij = \varphi_{ji}^* = (\varphi_{ji})^*$. 

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Lemma 5.31. Using the same notation from Lemma 5.27, the support $P_\xi$ of $\xi$ is given by the vector of matrices whose $j$-th component is given by

$$(P_\xi)_j = \begin{cases} P_{\xi_j} & \text{if } q_j > 0 \\ 0 & \text{if } q_j = 0 \end{cases}, \quad (5.32)$$

where $P_{\xi_j}$ is the support of $\sigma_j$ on $\mathcal{M}_{n_j}(\mathbb{C})$.

Notation 5.33. Let $\mathcal{B}$ be as in (5.8). For each $j, k \in \{1, \ldots, t\}$, let

$$\iota_j : \mathcal{M}_{n_j}(\mathbb{C}) \hookrightarrow \mathcal{B} \quad \text{and} \quad \pi_k : \mathcal{B} \twoheadrightarrow \mathcal{M}_{n_k}(\mathbb{C}) \quad (5.34)$$

be the inclusion of the $j$-th factor and projection of the $k$-th factor, respectively.

Lemma 5.35. Given $A, B, F, \xi, P_\xi, (P_\xi)_k, \iota_j$, and $\pi_k$ as in (5.8), (5.12), (5.34), and Lemma 5.31, a CPU map $R : A \rightsquigarrow B$ satisfies $R \circ F = \xi \text{id}_B$ if and only if

$$R(\xi)_k \circ \pi_k \circ R \circ F \circ \iota_j = \begin{cases} R(\xi)_k & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}, \quad (5.36)$$

for all $j, k \in \{1, \ldots, t\}$. Here, $R(\xi)_k$ is the right-multiplication map defined by $R(\xi)_k(B_k) := B_k(\xi)_k$ for all $B_k \in \mathcal{M}_{n_k}(\mathbb{C})$.

Proof. The condition $R \circ F = \text{id}_B$ holds if and only if (cf. Lemma 2.26) $R(\xi)_k \circ R \circ F = R(\xi)_k$, which holds if and only if $R(\xi)_k \circ R \circ F \circ \iota_j = R(\xi)_k \circ \iota_j$ for all $j \in \{1, \ldots, t\}$. Finally, this is equivalent to

$$\pi_k \circ R(\xi)_k \circ R \circ F \circ \iota_j = \pi_k \circ R(\xi)_k \circ \iota_j \quad \forall j, k \in \{1, \ldots, t\}, \quad (5.37)$$

which is equivalent to the claim (5.36) because

$$\pi_k \circ \iota_j = \begin{cases} \text{id}_{\mathcal{M}_{n_k}(\mathbb{C})} & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}. \quad (5.38)$$

It may be helpful to visualize the map $\pi_k \circ R \circ F \circ \iota_j \equiv (R \circ F)_{kj}$ as the following composite of CP (not necessarily unital) maps

$$\mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_t}(\mathbb{C}) \xrightarrow{\iota_j} \mathcal{M}_{n_j}(\mathbb{C}),$$

$$\mathcal{M}_{m_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{m_s}(\mathbb{C}) \xrightarrow{R} \mathcal{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_t}(\mathbb{C}) \xrightarrow{\pi_k} \mathcal{M}_{n_k}(\mathbb{C}) \quad (5.39)$$

The following is an analogue of Lemma 2.34 to direct sums of matrix algebras.
Lemma 5.40. Using the same notation as in Lemma 5.35 and assuming $R \circ F = \text{id}_B$, write

$$
R = \begin{pmatrix}
R_{11} & \cdots & R_{1s} \\
\vdots & \ddots & \vdots \\
R_{t1} & \cdots & R_{ts}
\end{pmatrix}
$$

and

$$
F = \begin{pmatrix}
F_{11} & \cdots & F_{1t} \\
\vdots & \ddots & \vdots \\
F_{s1} & \cdots & F_{st}
\end{pmatrix}
$$

where the completely positive (not necessarily unital) maps $R_{ki} : \mathcal{M}_{m_i}(\mathbb{C}) \rightarrow \mathcal{M}_{n_k}(\mathbb{C})$ and $F_{ij} : \mathcal{M}_{n_j}(\mathbb{C}) \rightarrow \mathcal{M}_{m_i}(\mathbb{C})$ have Kraus decompositions,\(^{12}\)

$$
R_{ki} = \sum_{\beta_{ki}=1}^{n_km_i} \text{Ad}_{R_{ki},\beta_{ki}} \quad \text{and} \quad F_{ij} = \sum_{\gamma_{ij}=1}^{m_in_j} \text{Ad}_{F_{ij},\gamma_{ij}}
$$

with $R_{ki;\beta_{ki}} : \mathbb{C}^{m_i} \rightarrow \mathbb{C}^{n_k}$ and $F_{ij;\gamma_{ij}} : \mathbb{C}^{n_j} \rightarrow \mathbb{C}^{m_i}$ linear maps.

i. For each $k \in \{1, \ldots, t\} \setminus N_q$, there exist a collection of complex numbers $\{\alpha_{ki;\beta_{ki},\gamma_{ik}}\}$; indexed by $\gamma_{ik} \in \{1, \ldots, c_{ik}\}$, $\beta_{ki} \in \{1, \ldots, n_km_i\}$, $i \in \{1, \ldots, s\}$, such that

$$
P_{\xi_k} R_{ki;\beta_{ki}} F_{ik;\gamma_{ik}} = \alpha_{ki;\beta_{ki},\gamma_{ik}} P_{\xi_k}
$$

for all $\beta_{ki} \in \{1, \ldots, n_km_i\}$, $\gamma_{ik} \in \{1, \ldots, n_km_i\}$, $i \in \{1, \ldots, s\}$ and

$$
\sum_{i=1}^{s} \sum_{\beta_{ki}=1}^{n_km_i} \sum_{\gamma_{ik}=1}^{m_in_k} |\alpha_{ki;\beta_{ki},\gamma_{ik}}|^2 = 1.
$$

ii. For every pair $j \in \{1, \ldots, t\}$ and $k \in \{1, \ldots, t\} \setminus N_q$ with $j \neq k$,

$$
P_{\xi_k} R_{ki;\beta_{ki}} F_{ij;\gamma_{ij}} = 0
$$

for all $\beta_{ki} \in \{1, \ldots, n_km_i\}$, $\gamma_{ij} \in \{1, \ldots, n_jm_i\}$, $i \in \{1, \ldots, s\}$.

Proof. Computing $\pi_k \circ R \circ F \circ \iota_j$ for $j, k \in \{1, \ldots, t\}$ gives

$$
\pi_k \circ R \circ F \circ \iota_j = \sum_{i=1}^{s} R_{ki} \circ F_{ij} = \sum_{i=1}^{s} \sum_{\beta_{ki}=1}^{n_km_i} \sum_{\gamma_{ij}=1}^{m_in_j} \text{Ad}_{R_{ki;\beta_{ki}} F_{ij;\gamma_{ij}}},
$$

Suppose $q_k > 0$. Then, (5.36) becomes

$$
\mathcal{R}_{P_{\xi_k}} \circ \pi_k \circ R \circ F \circ \iota_j = \begin{cases}
\mathcal{R}_{P_{\xi_k}} & \text{if } k = j \\
0 & \text{if } k \neq j
\end{cases}
$$

i. In the case $k = j$, (5.47) entails

$$
\text{Ad}_{P_{\xi_k}} \circ \pi_k \circ R \circ F \circ \iota_k = \text{Ad}_{P_{\xi_k}}
$$

upon multiplying by $P_{\xi_k}$ on the left. Combining this with (5.46) and Lemma 2.27 (by following a similar proof to that of Lemma 2.34), there exist complex numbers $\alpha_{ki;\beta_{ki},\gamma_{ik}}$ satisfying (5.43) and (5.44).

\(^{12}\)We will see in the text surrounding (5.54) that there exists a Kraus decomposition of $F_{ij}$ such that the index $\gamma_{ij}$ runs from 1 to $c_{ij}$ instead of $m_in_j = \sum_{k=1}^{t} c_{ik}n_kn_j$. This just means that $F_{ij;\gamma_{ij}}$ is zero when $\gamma_{ij}$ exceeds $c_{ij}$.
ii. In the case \( k \neq j \), (5.47) becomes
\[
\text{Ad}_{P_{\xi k}} \circ \pi_k \circ R \circ F \circ \iota_j = 0
\]  
(5.49)
upon multiplying by \( P_{\xi k} \) on the left. This implies
\[
P_{\xi k} R_{ki;\beta_{ki}} F_{ij;\gamma_{ij}} = 0
\]  
(5.50)
by (5.46) and Lemma 2.27.

At this point, it is helpful to make the conclusions of Lemma 5.40 even more explicit by further explicating \( P_{\xi k} R_{ki;\beta_{ki}} F_{ij;\gamma_{ij}} \). The Kraus operator \( R_{ki;\beta_{ki}} : \mathbb{C}^{m_i} \to \mathbb{C}^{n_k} \) can be partitioned into block sums of matrices based on the multiplicity of \( F \) in the following way
\[
R_{ki;\beta_{ki}} = \begin{bmatrix}
V_{ki;\beta_{ki};11} & \cdots & V_{ki;\beta_{ki};1c_{i1}} \\
V_{ki;\beta_{ki};21} & \cdots & V_{ki;\beta_{ki};2c_{i1}} \\
\vdots & \ddots & \vdots \\
V_{ki;\beta_{ki};t1} & \cdots & V_{ki;\beta_{ki};tc_{i1}}
\end{bmatrix}
\]  
(5.51)
due to (5.13). Based on this partitioning, the unitality condition on \( R \) reads
\[
1_{n_k} = \sum_{i=1}^{s} \sum_{\beta_{ki}=1}^{m_i} \sum_{j=1}^{t} \sum_{\gamma_{ij}=1}^{c_{ij}} V_{ki;\beta_{ki};j\gamma_{ij}} V_{ki;\beta_{ki};j\gamma_{ij}}^\dagger \quad \forall k \in \{1, \ldots, t\}. 
\]  
(5.52)
due to (5.18). Furthermore, the definition of \( F \) from (5.12) says
\[
F_{ij}(B_j) = \text{diag}(0, \ldots, 0, \ldots, B_j, \ldots, 0, \ldots, 0) \quad \forall B_j \in \mathcal{M}_{n_j}(\mathbb{C}).
\]  
(5.53)
This implies that the (adjoint of the) Kraus operators \( F_{ij;\gamma_{ij}} : \mathbb{C}^{n_j} \to \mathbb{C}^{m_i} \) of \( F_{ij} : \mathcal{M}_{n_j}(\mathbb{C}) \to \mathcal{M}_{m_i}(\mathbb{C}) \) have the following partitioned form
\[
F_{ij;\gamma_{ij}}^\dagger = \begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 1_{n_j} \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix}
\]  
(5.54)
where the identity matrix \( 1_{n_j} \) is in the \( \gamma_{ij} \)-th \( n_j \times n_j \) subblock inside the \( n_j \times (c_{ij}n_j) \) block indicated (and all other entries are 0). In particular, the index \( \gamma_{ij} \) runs from 1 to \( c_{ij} \) (as opposed to \( m_in_j \)). Therefore, the product \( R_{ki;\beta_{ki}} F_{ij;\gamma_{ij}} \) is
\[
R_{ki;\beta_{ki}} F_{ij;\gamma_{ij}} = V_{ki;\beta_{ki};j\gamma_{ij}},
\]  
(5.55)
which is an \( n_k \times n_j \) matrix. The following result is a generalization of Equation (4.9) to the direct sum case.

**Lemma 5.56.** Under the same assumptions as in Lemma 5.40, for every \( k \in \{1, \ldots, t\} \setminus N_q \), there exist a collection of complex numbers \( \{\alpha_{ki;\beta_{ki};\gamma_{ik}}\} \), indexed by \( \gamma_{ik} \in \{1, \ldots, c_{ik}\} \), \( \beta_{ki} \in \{1, \ldots, n_km_i\} \), and \( i \in \{1, \ldots, s\} \), such that
\[
R_{ki;\beta_{ki}} = \begin{bmatrix}
\frac{1}{n_k \times n_1} & \cdots & 0 \\
0 & \cdots & 0 \\
\frac{1}{n_k \times n_1} & \cdots & 0 \\
\frac{1}{n_k \times n_t} & \cdots & 0
\end{bmatrix}
\]  
(5.57)
for all $\beta_{ki} \in \{1, \ldots, n_km_i\}, i \in \{1, \ldots, s\}$ and

$$\sum_{i=1}^{s} \sum_{\beta_{ki}=1}^{n_km_i} \sum_{\gamma_{ik}=1}^{c_{ik}} |\alpha_{k;i,\beta_{ki},\gamma_{ik}}|^2 = 1. \quad (5.58)$$

**Proof.** In analogy to the proof of Theorem 2.48, for every $k \in \{1, \ldots, t\} \setminus N_q$, let $\mathcal{E}_k$ be the pre-Hilbert $\mathcal{M}_{n_k}(\mathbb{C})$-module consisting of vectors of $n_k \times n_k$ matrices whose vector components are labelled by the triple of indices $\gamma_{ik}, \beta_{ki}, i$. Let $\vec{V}_k$ be the vector whose vector components are the $n_k \times n_k$ matrices $V_{ki;\beta_{ki};k\gamma_{ik}}$. The first case of Lemma 5.40 implies there exists a vector $\vec{a}_k \in \mathcal{E}_k$ whose vector components are constant multiples of the identity matrix satisfying

$$\vec{V}_k = P_{\xi_k} \vec{a}_k + \vec{V}_{\text{br}}^k, \quad (5.59)$$

where

$$\vec{V}_{\text{bl}}^k := P_{\xi_k}^\perp \vec{V}_k P_{\xi_k}, \quad \vec{V}_{\text{br}}^k := P_{\xi_k}^\perp \vec{V}_k P_{\xi_k}^\perp, \quad \text{and} \quad \langle \langle \vec{a}_k, \vec{a}_k \rangle \rangle = 1_{n_k}. \quad (5.60)$$

Similarly, for every pair $(k, j) \in \{1, \ldots, t\} \times \{1, \ldots, t\}$ such that $j \neq k$ and such that $q_k > 0$, let $\mathcal{E}_{kj}$ be the pre-Hilbert $\mathcal{M}_{n_k}(\mathbb{C})$-module consisting of vectors of $n_k \times n_j$ matrices whose vector components are labelled by the triple of indices $\gamma_{ij}, \beta_{ki}, i$. Let $\vec{V}_{kj}$ be the vector of the $n_k \times n_j$ matrices whose components are given by $V_{ki;\beta_{ki};j\gamma_{ij}}$. The second case of Lemma 5.40 implies $\vec{V}_{kj} = \vec{V}_{\text{bl}}^{kj} + \vec{V}_{\text{br}}^{kj}$, where

$$\vec{V}_{\text{bl}}^{kj} := P_{\xi_k}^\perp \vec{V}_{kj} P_{\xi_j}, \quad \text{and} \quad \vec{V}_{\text{br}}^{kj} := P_{\xi_k}^\perp \vec{V}_{kj} P_{\xi_k}^\perp. \quad (5.61)$$

The equalities

$$\langle \langle P_{\xi_k}^\perp \vec{a}_k, \vec{V}_k \rangle \rangle = \langle \langle \vec{V}_{\text{bl}}^k, \vec{V}_{\text{bl}}^k \rangle \rangle = \langle \langle \vec{V}_{\text{br}}^k, \vec{V}_{\text{br}}^k \rangle \rangle = 0 \quad (5.62)$$

all follow immediately from the definitions. Unitality of $R$ takes on the form

$$1_{n_k} = \langle \langle \vec{V}_k, \vec{V}_k \rangle \rangle + \sum_{j=1,j \neq k}^{t} \langle \langle \vec{V}_{kj}, \vec{V}_{kj} \rangle \rangle \quad (5.63)$$

by (5.52). By expanding out (5.63) and multiplying on the right by $P_{\xi_k}$, completely similar arguments to those in the proof of Theorem 2.48, specifically the discussion surrounding Equations (2.53) through (2.56), prove $\langle \langle \vec{V}_{\text{br}}^k, \vec{a}_k \rangle \rangle = 0$. Hence, the unitality condition (5.63) simplifies to

$$P_{\xi_k}^\perp = \langle \langle \vec{V}_{\text{bl}}^k, \vec{V}_{\text{bl}}^k \rangle \rangle + \langle \langle \vec{V}_{\text{br}}^k, \vec{V}_{\text{br}}^k \rangle \rangle + \sum_{j=1,j \neq k}^{t} \left( \langle \langle \vec{V}_{kj}^\perp \vec{V}_{kj}^\perp \rangle \rangle + \langle \langle \vec{V}_{kj}^\perp \vec{V}_{kj}^\perp \rangle \rangle \right) \quad (5.64)$$

analogously to (2.53). Now, computing $\pi_k \circ R \circ F \circ t_j$ in terms of the pre-Hilbert module inner product gives

$$(\pi_k \circ R \circ F \circ t_j)(A_j) = \langle \langle \vec{V}_{kj} A_j, \vec{V}_{kj} \rangle \rangle \quad \forall \ A_j \in \mathcal{M}_{n_j}(\mathbb{C}) \quad (5.65)$$

for all $k, j$ (when $j = k$, remove one of the indices from $\vec{V}_{kk}$) by (5.55). When $j = k$, multiplying this equation on the right by $P_{\xi_k}$ (which equals $(P_k)_{kk}$ since $q_k > 0$) and combining this with Lemma 5.35 gives $\langle \langle \vec{V}_{\text{br}}^k, \vec{a}_k \rangle \rangle = P_{\xi_k}^\perp$ by following an argument exactly analogous to (2.59) and the text surrounding this equation. Similarly, combining this result with the Paschke–Cauchy–Schwarz
inequality gives $P_{\xi_k}^\perp \leq \langle \tilde{V}_k^{\text{br}}, \tilde{V}_k^{\text{br}} \rangle$. On the other hand, (5.64) says $P_{\xi_k}^\perp \geq \langle \tilde{V}_k^{\text{br}}, \tilde{V}_k^{\text{br}} \rangle$. Therefore, following analogous lines of thought to those from (2.62) to (2.65) gives
\[
\tilde{V}_k^{\text{br}} = P_{\xi_k}^\perp \tilde{a}_k, \quad \tilde{V}_k^{\text{bl}} = \tilde{0}, \quad \tilde{V}_k^{\text{bl}} = \tilde{0}, \quad \text{and} \quad \tilde{V}_k^{\text{br}} = \tilde{0}.
\] (5.66)
Therefore, $\tilde{V}_k = \tilde{a}_k$ and $\tilde{V}_k = \tilde{0}$. Expanding out the vector entries coming from the definitions of $E_k$ and $E_{k,j}$ completes the proof. ■

Given a state-preserving *-homomorphism $(B, \xi) \xrightarrow{F} (A, \omega)$, it may be useful to know how the density matrices associated to $\xi$ and $\omega$ are related. The following fact describes this relationship. It is a generalization of the “tracing out degrees of freedom” method in quantum theory.

**Proposition 5.67.** Let $A, B, F, \omega$, and $\xi$ be as in Notation 5.7, and let
\[
\omega \equiv \sum_{i=1}^s p_i \text{tr}(\rho_i \cdot) \quad \text{and} \quad \xi \equiv \sum_{j=1}^t q_j \text{tr}(\sigma_j \cdot)
\] (5.68)
be decompositions of the states $\omega$ and $\xi$ as described in Lemma 5.27. Then the following facts hold.

i. For each $i \in \{1, \ldots, s\}$, there exists a $j \in \{1, \ldots, t\}$ such that $c_{ij} > 0$.

ii. If there exists a $j \in \{1, \ldots, s\}$ such that $c_{ij} = 0$ for all $i \in \{1, \ldots, s\}$, then $q_j = 0$.

iii. Finally,
\[
q_j \sigma_j = \sum_{i=1}^s \sum_{\gamma_{ij}=1}^{c_{ij}} p_i \rho_{i;j;\gamma_{ij}} \quad \forall \ j \in \{1, \ldots, t\},
\] (5.69)
where $\rho_{i;j;\gamma_{ij}}$ is the $n_j \times n_j$ matrix obtained from $\rho_i$ in the following way. Since $m_i = \sum_{k=1}^t c_{ik} n_k$, each $m_i \times m_i$ matrix $\rho_i$ has a block matrix decomposition
\[
\rho_i = \begin{bmatrix}
\rho_{i;11} & \cdots & \rho_{i;1t} \\
\vdots & \ddots & \vdots \\
\rho_{i;t1} & \cdots & \rho_{i;tt}
\end{bmatrix},
\] (5.70)
where $\rho_{i;jk}$ is a $(c_{ij} n_j) \times (c_{ik} n_k)$ matrix. This matrix further breaks up into subblocks
\[
\rho_{i;jk} = \begin{bmatrix}
\rho_{i;jk;11} & \cdots & \rho_{i;jk;1c_{ik}} \\
\vdots & \ddots & \vdots \\
\rho_{i;jk;c_{ij}1} & \cdots & \rho_{i;jk;c_{ij}c_{ik}}
\end{bmatrix},
\] (5.71)
where $\rho_{i;jk;\gamma_{ij}\gamma_{jk}}$ is an $n_j \times n_k$ matrix.

**Remark 5.72.** The contrapositive of part ii of Proposition 5.67 will be used occasionally in certain technical points later. It states that if $q_j > 0$, there exists at least one $i \in \{1, \ldots, s\}$ such that $c_{ij} > 0$. In other words, $F$ is injective almost everywhere. This should be compared to Lemma 5.4. Furthermore, using partial traces, Equation (5.69) becomes
\[
q_j \sigma_j = \sum_{i=1}^s p_i \text{tr}_{M_{c_{ij} \langle c \rangle}}(\rho_{i;jj}),
\] (5.73)
where $\rho_i$ is decomposed as in (5.70).
Proof of Proposition 5.67.

i. Since $m_i > 0$ and $m_i = \sum_{j=1}^{t} c_{ij} n_j$, there must exist a non-zero $c_{ij}$ for some $j \in \{1, \ldots, t\}$.

ii. Suppose there exists a $j \in \{1, \ldots, s\}$ such that $c_{ij} = 0$ for all $i \in \{1, \ldots, s\}$. Then $F_{ij}(\mathbb{1}_{n_j}) = 0$ for all $i \in \{1, \ldots, s\}$. Since $\omega \circ F = \xi$, this shows $\xi(\mathbb{1}_{n_j}) = 0$. But $\xi(\mathbb{1}_{n_j}) = q_j \text{tr}(\sigma_j) = q_j$ so that $q_j = 0$.

iii. This follows from taking the adjoint of the equation $\omega \circ F = \xi$, which gives

$$F^* \circ \omega^* = \xi^* \implies F^*(\omega^*(1)) = \xi^*(1).$$

(5.74)

Expanding out these expressions and extracting the $j$-th term gives

$$q_j \sigma_j = \sum_{i=1}^{s} p_i F_{ij}^*(\rho_i) \quad \forall j \in \{1, \ldots, t\}.$$  

(5.75)

Applying (5.42) and (5.54) gives the desired result.

A consequence of Lemma 5.56 is the following fact regarding the existence and uniqueness of disintegrations on finite-dimensional $C^*$-algebras. It is a generalization of Theorem 4.3 to direct sums of matrix algebras and is the main theorem of the present work.

**Theorem 5.76.** Let $A, B, F, \omega, \text{and } \xi$ be as in Notation 5.7 and Proposition 5.67.

i. A disintegration $R$ of $\omega$ over $\xi$ consistent with $F$ exists if and only if for each $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$ there exist non-negative matrices $\tau_{ji} \in \mathcal{M}_{c_{ij}}(\mathbb{C})$ such that

$$\text{tr} \left( \sum_{i=1}^{s} \tau_{ji} \right) = 1 \quad \forall j \in \{1, \ldots, t\} \setminus N_q$$

(5.77)

and

$$p_i \rho_i = \text{diag}(q_1 \tau_{i1} \otimes \sigma_1, \ldots, q_t \tau_{it} \otimes \sigma_t) \quad \forall i \in \{1, \ldots, s\}. \quad (5.78)$$

ii. Furthermore, if $R'$ is another disintegration of $\omega$ over $\xi$ consistent with $F$, then $R' = R$ and $\xi$ and

$$R'_{ji} = R_{ji} \quad \forall i \in \{1, \ldots, s\} \quad \forall j \in \{1, \ldots, t\} \setminus N_q.$$  

(5.79)

iii. Finally, if such a disintegration $R$ exists, a formula for the $ji$-th component of the disintegration is given by

$$R_{ji}(A_i) = \text{tr}_{\mathcal{M}_{c_{ij}}(\mathbb{C})}((\tau_{ji} \otimes \mathbb{1}_{n_j})A_{i;jj})$$

(5.80)

for all $j \in \{1, \ldots, t\} \setminus N_q$ and for all $i \in \{1, \ldots, s\}$. Here, $A_{i;jj}$ is uniquely defined by the decomposition as a $t \times t$ matrix

$$A_i \equiv \begin{bmatrix} A_{i;11} & \cdots & A_{i;1t} \\ \vdots & \ddots & \vdots \\ A_{i;t1} & \cdots & A_{i;tt} \end{bmatrix}$$  

(5.81)

where the $kl$-th subblock, $A_{i;kl}$, is a $(c_{ik}n_k) \times (c_{il}n_l)$ matrix.

$\text{N}_q$ was introduced in Notation 5.30.
Proof. Proving the first item will provide proofs of the subsequent claims.

(⇒) Suppose a disintegration $R$ exists. The condition $\xi \circ R = \omega$ is equivalent to $R^* (\xi^*(1)) = \omega^*(1)$ by Lemma 5.21. Hence, using the notation from (5.88), this equation gives

$$R^* (\xi^*(1)) \equiv \begin{pmatrix} R_{11}^* & \cdots & R_{t1}^* \\ \vdots & \ddots & \vdots \\ R_{1s}^* & \cdots & R_{ts}^* \end{pmatrix} \begin{pmatrix} q_1 \sigma_1 \\ \vdots \\ q_t \sigma_t \end{pmatrix} = \begin{pmatrix} p_1 \rho_1 \\ \vdots \\ p_s \rho_s \end{pmatrix} \equiv \omega^*(1), \quad (5.82)$$

which is equivalent to

$$p_i \rho_i = \sum_{j=1}^{t} q_j R_{ji}^*(\sigma_j) \quad \forall \ i \in \{1, \ldots, s\}. \quad (5.83)$$

To compute $R_{ji}^*(\sigma_j)$, we can follow an analogous computation to that from (4.11). First, when $q_j > 0$, we obtain

$$R_{ji}^*(\sigma_j) = \sum_{\beta_{ji}=1}^{n_j m_i} \text{Ad}_{R_{ji}^* (\sigma_j)} \begin{pmatrix} 0 \\ \alpha_{j;i,\beta_{ji};1} 1_n_j \\ \vdots \\ \alpha_{j;i,\beta_{ji};c_j} 1_n_j \end{pmatrix} \begin{pmatrix} \sigma_j [0 \; \alpha_{j;i,\beta_{ji};1} 1_n_j \; \cdots \; \alpha_{j;i,\beta_{ji};c_j} 1_n_j \; 0] \end{pmatrix}, \quad (5.84)$$

where the top 0 block in the left matrix is a $\left( \sum_{k=1}^{j-1} c_{ik} n_k \right) \times n_j$ matrix and the bottom 0 block in the left matrix is a $\left( \sum_{k=j+1}^{t} c_{ik} n_k \right) \times n_j$ matrix. Keeping track of these sizes, we obtain

$$R_{ji}^*(\sigma_j) = \sum_{\beta_{ji}=1}^{n_j m_i} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & |\alpha_{j;i,\beta_{ji};1}|^2 \sigma_j & \cdots & \alpha_{j;i,\beta_{ji};1}^* \sigma_j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{j;i,\beta_{ji};c_j} \sigma_j & \cdots & |\alpha_{j;i,\beta_{ji};c_j}|^2 \sigma_j \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (5.85)$$

where the top-left 0 matrix is a $\left( \sum_{k=1}^{j-1} c_{ik} n_k \right) \times \left( \sum_{k=1}^{j-1} c_{ik} n_k \right)$ matrix and the bottom-right 0 matrix is a $\left( \sum_{k=j+1}^{t} c_{ik} n_k \right) \times \left( \sum_{k=j+1}^{t} c_{ik} n_k \right)$ matrix. Define the $c_{ij} \times c_{ij}$ matrix $\tau_{ji}$ to be

$$\tau_{ji} := \sum_{\beta_{ji}=1}^{n_j m_i} \begin{pmatrix} |\alpha_{j;i,\beta_{ji};1}|^2 & \cdots & \alpha_{j;i,\beta_{ji};1}^* \sigma_j \\ \vdots & \ddots & \vdots \\ \alpha_{j;i,\beta_{ji};c_j} \sigma_j & \cdots & |\alpha_{j;i,\beta_{ji};c_j}|^2 \sigma_j \end{pmatrix}, \quad (5.86)$$

so that the $\gamma_{ij} \eta_{ij}$-th entry of $\tau_{ji}$ is given by

$$\tau_{ji;\gamma_{ij}\eta_{ij}} = \sum_{\beta_{ji}=1}^{n_j m_i} \alpha_{j;i,\beta_{ji};\gamma_{ij}} \alpha_{j;i,\beta_{ji};\eta_{ij}}, \quad (5.87)$$
Notice that $\tau_{ji}$ is defined only when $c_{ij} > 0$ and when $q_j > 0$. Furthermore, when it is defined, $\tau_{ji}$ is a non-negative matrix and

$$\sum_{i=1}^s \text{tr}(\tau_{ji}) = \sum_{i=1}^s \sum_{\beta_{ji}=1} \sum_{\gamma_{ij}=1} c_{ij} |\alpha_{j;i,\beta_{ji},\gamma_{ij}}|^2 \overset{(5.58)}{=} 1,$$  

which shows $\sum_{i=1}^s \tau_{ji}$ is a density matrix (again, when $q_j > 0$). The sum in (5.88) is guaranteed to have at least one term due to Remark 5.72. Second, when $q_j = 0$, then $q_j R_{ji}^r(\sigma_j) = 0$ so that this term does not contribute to the sum in (5.83). Therefore, in this case, $\tau_{ji}$ can be chosen to be an arbitrary non-negative matrix provided that $c_{ij} > 0$. If $c_{ij} = 0$, then $\tau_{ji}$ does not exist and any expression involving such a $\tau_{ji}$ should be excluded. Then,

$$p_i \rho_i = \sum_{j=1}^t q_j R_{ji}^r(\sigma_j) \overset{(5.85)}{=} \begin{bmatrix} q_1 \tau_{1i} \otimes \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & q_t \tau_{ti} \otimes \sigma_t \end{bmatrix} \quad \forall \, i \in \{1, \ldots, s\}.$$  

Note that this sum after the first equality is not empty by part i of Proposition 5.67.

(⇐) For the converse, suppose the non-negative matrices $\tau_{ji} \in M_{c_{ij}}(\mathbb{C})$ satisfying (5.77) and (5.78) exist. Denote the $\gamma_{ij}\eta_{ij}$-th entry of $\tau_{ji}$ by $\tau_{ji;\gamma_{ij}\eta_{ij}}$. For each pair of indices $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\},$ define $R'_{ji} : M_{m_i}(\mathbb{C}) \rightarrow M_{n_j}(\mathbb{C})$ in the following way. Write an element $A_i \in M_{m_i}(\mathbb{C})$ as in (5.81). Then, write $A_{ikl}$ as a $c_{ik} \times c_{ul}$ matrix consisting of $n_k \times n_l$ matrices indexed as in $A_{ikl;\gamma_{ik}\eta_{kl}}$. Set

$$R'_{ji}(A_i) := \left\{ \begin{array}{ll} \sum_{\gamma_{ij}\eta_{ij}=1}^{c_{ij}} \tau_{ji;\gamma_{ij}\eta_{ij}} A_{ij j;\gamma_{ij}\eta_{ij}} & \text{if } q_j > 0 \\
\frac{1}{s m_i} \text{tr}(A_i) I_{n_j} & \text{if } q_j = 0 \end{array} \right.$$  

(5.90)

A direct calculation shows that this formula equals (5.80) when $j \in \{1, \ldots, t\} \setminus N_q$. Set $R' : \mathcal{A} \rightarrow \mathcal{B}$ to be the $t \times s$ matrix of linear maps whose $ji$-th entry is $R'_{ji}$ from (5.90). Then $R'$ is a disintegration of $\omega$ over $\xi$ consistent with $F$. The proof of this is similar to the proof of Theorem 4.3 though one must keep track of indices more carefully. Unitality of $R'$ follows from

$$\sum_{i=1}^s R'_{ji}(1_{m_i}) = \sum_{i=1}^s \sum_{\gamma_{ij}\eta_{ij}=1}^{c_{ij}} \tau_{ji;\gamma_{ij}\eta_{ij}} 1_{m_i \times j;\gamma_{ij}\eta_{ij}} = \sum_{i=1}^s \sum_{\gamma_{ij}=1}^{c_{ij}} \tau_{ji;\gamma_{ij}} 1_{n_j} = \sum_{i=1}^s \text{tr}(\tau_{ji}) 1_{n_j} = 1_{n_j}$$  

whenever $q_j > 0$ because $\sum_{i=1}^s \text{tr}(\tau_{ji}) = 1$. When $q_j = 0$, one obtains

$$\sum_{i=1}^s R'_{ji}(1_{m_i}) = \sum_{i=1}^s \frac{1}{s m_i} \text{tr}(1_{m_i}) 1_{n_j} = 1_{n_j}.$$  

(5.92)

---

\(^{14}\)Note that the swapping of the $\gamma_{ij}$ and $\eta_{ij}$ indices in Equation (5.90) is not a typo. In addition, note that if $q_j > 0$ and $c_{ij} = 0$, then the sum in the top case is empty and gives, by definition of an empty sum, 0.
We will now show $\pi_j \circ R' \circ F = \pi_j$ for all $j$ satisfying $q_j > 0$. First, note that

$$(R' \circ F)(\bar{B}) = \begin{pmatrix} R'_{t1} & \cdots & R'_{ts} \\ \vdots & \ddots & \vdots \\ R'_{t1} & \cdots & R'_{ts} \end{pmatrix} \begin{pmatrix} \text{diag}(B_1, \ldots, B_1, \ldots, B_t, \ldots, B_t) \\ \vdots \\ \text{diag}(B_1, \ldots, B_1, \ldots, B_t, \ldots, B_t) \end{pmatrix}. \quad (5.93)$$

Focusing on the $j$-th term when $q_j > 0$, one obtains

$$\sum_{i=1}^{s} R'_{ji} \left( \text{diag}(B_1, \ldots, B_1, \ldots, B_t, \ldots, B_j) \right) = \sum_{i=1}^{s} \sum_{\gamma_{ij}=1}^{c_{ij}} \tau_{ji;\gamma_{ij}} B_j = \sum_{i=1}^{s} \text{tr}(\tau_{ji}) B_j = B_j. \quad (5.94)$$

Although the equality $\pi_j \circ R' \circ F = \pi_j$ fails when $q_j = 0$, the equation $R' \circ F = \Pi^\xi$ still holds. Furthermore, $R'$ is state-preserving because

$$\omega(\bar{A}) = \sum_{i=1}^{s} \text{tr}(p_i \rho_i A_i) \xrightarrow{(5.78)} \sum_{i=1}^{s} \sum_{j=1}^{t} q_j \text{tr}(\tau_{ji} \otimes \sigma_j) A_{ij;jj}$$

$$= \sum_{i=1}^{s} \sum_{j=1}^{t} q_j \text{tr} \left( \sum_{\gamma_{ij}=1}^{c_{ij}} \tau_{ji;\gamma_{ij}} \sigma_j A_{ij;jj} \right) \quad (5.95)$$

$$\xrightarrow{(5.90)} \sum_{i=1}^{s} \sum_{j=1}^{t} q_j \text{tr}(\sigma_j R'_{ji}(A_i)) = (\xi \circ R')(\bar{A})$$

for all $\bar{A} \in \mathcal{A}$. To show $R'$ is CP, it suffices to show each $R'_{ji}$ is CP by Lemma 5.15. This follows from the equality between the formulas (5.80) and (5.90) when $j \in \{1, \ldots, t\} \setminus N_q$. The case when $j \in N_q$ gives a CP map as well since the trace in (5.90) is a CP map.

Finally, we prove the uniqueness condition (5.79) for disintegrations. The condition $R' = \xi R$ is equivalent to

$$\sum_{i=1}^{s} R'_{ji}(A_i)(P_\xi)_j = \sum_{i=1}^{s} R_{ji}(A_i)(P_\xi)_j \quad \forall \bar{A} \in \mathcal{A}, \; j \in \{1, \ldots, t\}. \quad (5.96)$$

When $q_j = 0$, this equality holds trivially because $(P_\xi)_j = 0$. When $q_j \neq 0$, Lemma 5.56 guarantees the existence of complex numbers $\{\alpha_{ji;i,j;i,j}\}$ and Kraus operators $\{R_{ji;i,j}\}$ for $R_{ji}$ satisfying the conditions in the statement of that theorem. Therefore, by carefully working out the matrix operations, one obtains

$$R_{ji}(A_i) = \sum_{\beta_{ji}=1}^{n_{ji}} \sum_{\gamma_{ij}=1}^{q_{ij}} \sum_{\eta_{ij}=1}^{s_{ij}} \alpha_{ji;i,j;i,j} R_{ji;i,j;jj} A_{ij;jj} \quad (5.97)$$

$$\xrightarrow{(5.87)} \sum_{\gamma_{ij}=1}^{c_{ij}} \sum_{\eta_{ij}=1}^{c_{ij}} \tau_{ji;\gamma_{ij} \gamma_{ij}} R_{ji;i,j;jj} \xrightarrow{(5.90)} R'_{ji}(A_i) \quad \forall A_i \in \mathcal{M}_{m_i}(\mathbb{C}).$$

This concludes the proof of the theorem.
Remark 5.98. By applying the trace to both sides of (5.78), one obtains \( p_i = \sum_{j=1}^t q_j \text{tr}(\tau_{ji}) \) for all \( i \in \{1, \ldots, s\} \).

An immediate corollary of Theorem 5.76 is the standard existence and uniqueness theorem of regular conditional probabilities from classical finite probability. We work this out in full detail as an example.

Example 5.99. Using the notation from Theorem 5.76, suppose \( m_i = 1 \) and \( n_j = 1 \) for all \( i, j \). Then, \( \rho_i = 1 = \sigma_j \) for all \( i, j \). Furthermore, since each \( m_i = 1 \), the multiplicity is drastically restricted since \( m_i = \sum_{j=1}^t c_{ij} n_j \). By this equality, for each \( i \), there exists a unique \( j \) such that \( c_{ij} = 1 \) and all other \( c_{ik} = 0 \). In other words, there exists a unique function \( f : \{1, \ldots, s\} \to \{1, \ldots, t\} \) such that

\[
c_{ij} = \delta_{f(i)j} \equiv \begin{cases} 1 & \text{if } f(i) = j \\ 0 & \text{otherwise} \end{cases} \quad (5.100)
\]

This implies

\[
F = \begin{bmatrix} \delta_{f(1)1} & \cdots & \delta_{f(1)t} \\ \vdots & \ddots & \vdots \\ \delta_{f(s)1} & \cdots & \delta_{f(s)t} \end{bmatrix} \quad \text{and} \quad F^* = \begin{bmatrix} \delta_{f(1)1} & \cdots & \delta_{f(s)1} \\ \vdots & \ddots & \vdots \\ \delta_{f(1)t} & \cdots & \delta_{f(s)t} \end{bmatrix}.
\]

Hence,

\[
q_j \sum_{i=1}^s p_i F^*_{ij}(1) = \sum_{i=1}^s p_i \delta_{f(i)j} = \sum_{i \in f^{-1}(j)} p_i, \quad (5.102)
\]

which reproduces the probability-preserving condition \( (X, p) \xrightarrow{f} (Y, q) \), assuming \( X = \{1, \ldots, s\} \) and \( Y = \{1, \ldots, t\} \). In what follows, we will construct, without any additional assumptions, non-negative matrices \( \tau_{ji} \in \mathcal{M}_{c_{ij}}(\mathbb{C}) \) satisfying (5.77) and (5.78) as well as a disintegration

\[
R = \begin{bmatrix} r_{11} & \cdots & r_{1s} \\ \vdots & \ddots & \vdots \\ r_{t1} & \cdots & r_{ts} \end{bmatrix}. \quad (5.103)
\]

This will prove that a disintegration automatically exists in this case. First note that if \( j \neq f(i) \), the set \( \mathcal{M}_{c_{ij}}(\mathbb{C}) \) is just a singleton so that we only have a chance of constructing \( \tau_{ji} \) when \( j = f(i) \). In this case, \( c_{if(i)} = 1 \) and such a matrix will be a \( 1 \times 1 \) matrix, i.e. a non-negative number. We set

\[
\tau_{ji} := \begin{cases} p_i/q_j & \text{if } q_j > 0 \text{ and } j = f(i) \\ \# & \text{if } q_j = 0 \text{ and } j = f(i) \\ \text{DNE} & \text{if } c_{ij} = 0 \end{cases} \quad (5.104)
\]

where \( \# \) can be chosen to be any non-negative number. Note that if there exists a \( j \in \{1, \ldots, t\} \) for which \( c_{ij} = 0 \) for all \( i \in \{1, \ldots, s\} \), then \( q_j = 0 \) by part ii of Proposition 5.67. For such \( j \), \( \tau_{ij} \) cannot be defined for any \( i \in \{1, \ldots, s\} \). Nevertheless,

\[
\text{tr} \left( \sum_{i=1}^s \tau_{ji} \right) = \sum_{i=1}^s \tau_{ji} = \sum_{i \in f^{-1}(j)} \frac{p_i}{q_j} \quad (5.102) 
\quad \forall \ j \in \{1, \ldots, s\} \setminus N_q \quad (5.105)
\]
proves (5.77). Secondly, because there exists a unique \( j \) for each \( i \) such that \( c_{ij} = 1 \),
\[
\text{diag}(q_1\tau_{1i} \otimes \sigma_1, \ldots, q_t\tau_{ti} \otimes \sigma_t) = q_{f(i)}\tau_{f(i)i} = q_{f(i)} \begin{cases} p_i/q_{f(i)} & \text{if } q_{f(i)} > 0 \\ \# & \text{if } q_{f(i)} = 0 \end{cases}
\]
\[
= \begin{cases} p_i & \text{if } q_{f(i)} > 0 \\ 0 & \text{if } q_{f(i)} = 0 \end{cases} \quad \forall \ i \in \{1, \ldots, s\}. \tag{5.106}
\]
Note that if \( q_{f(i)} = 0 \), then \( p_i = 0 \) by (5.102). Hence, this proves (5.78). Although this already proves a disintegration exists via Theorem 5.76, it is fruitful to construct it based on the proof of Theorem 5.76 and compare it to the classical disintegration from Theorem 5.1. Using the construction of a disintegration from (5.90), we get
\[
\begin{align*}
\tau_{ji} & \quad \text{if } q_j > 0 \text{ and } c_{ij} = 1 \\
0 & \quad \text{if } q_j > 0 \text{ and } c_{ij} = 0 \\
1/s & \quad \text{if } q_j = 0
\end{align*}
\]
\[
\begin{align*}
p_i/q_j & \quad \text{if } q_j > 0 \text{ and } j = f(i) \\
0 & \quad \text{if } q_j > 0 \text{ and } j \neq f(i) \\
1/s & \quad \text{if } q_j = 0
\end{align*}
\]
\[
= \begin{cases} p_i\delta_{f(i)j}/q_j & \text{if } q_i > 0 \\ 1/s & \text{if } q_j = 0 \end{cases}
\tag{5.107}
\]
This reproduces formula (5.2) for an ordinary disintegration.

Finally, we end this section with a generalization of Theorems 5.76 and 4.30 by allowing for arbitrary (unital) \( * \)-homomorphisms \( F : \mathcal{B} \to \mathcal{A} \).

**Theorem 5.108.** Let \( \mathcal{A}, \mathcal{B}, F, \omega, \) and \( \xi \) be as in Notation 5.7 and Proposition 5.67 except that \( F \) is now an arbitrary (unital) \( * \)-homomorphism, but not necessarily of the form (5.12). Then, a disintegration \( R \) exists if and only if there exist unitary matrices \( U_i \in \mathcal{M}_{m_i}(\mathbb{C}) \) and non-negative matrices \( \tau_{ji} \in \mathcal{M}_{c_{ij}}(\mathbb{C}) \) such that \( \text{Ad}_{U_i} \circ F \) is of the form (5.12),
\[
\text{tr} \left( \sum_{i=1}^{s} \tau_{ji} \right) = 1 \quad \forall \ j \in \{1, \ldots, t\} \setminus N_q
\tag{5.109}
\]
and
\[
p_iU_i^\dagger p_iU_i = \text{diag}(q_1\tau_{1i} \otimes \sigma_1, \ldots, q_t\tau_{ti} \otimes \sigma_t) \quad \forall \ i \in \{1, \ldots, s\}. \tag{5.110}
\]
Furthermore, any two such disintegrations are unique \( \xi \)-a.e.

**Proof.** This follows from an argument analogous to the proof of Theorem 4.30. \( \square \)
6 Example: measurement in quantum mechanics

It is instructive to work out the following example due to its connection with measurement in quantum mechanics (it may be helpful at this point to review Example 2.12 for notation). We will also avoid using the results of Theorem 5.76 and will instead provide a self-contained analysis since this is simple enough in this special case. Fix \( m \in \mathbb{N} \), let \( A \in \mathcal{M}_m(\mathbb{C}) \) be a self-adjoint matrix with spectrum \( \sigma(A) \subseteq \mathbb{R} \), and let \( \omega = \text{tr}(\rho \cdot) : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathbb{C} \) be a state. The matrix \( A \) induces the \(^*\)-homomorphism uniquely determined by

\[
\mathbb{C}^{\sigma(A)} \xrightarrow{F} \mathcal{M}_m(\mathbb{C})
\]

\[
e_\lambda \mapsto P_\lambda,
\]

(6.1)

where \( P_\lambda \) is the orthogonal projection onto the \( \lambda \) eigenspace associated to \( A \). This pulls back the state \( \omega \) to a probability measure \( q \) on \( \sigma(A) \) whose evaluation on \( \lambda \in \sigma(A) \) will be denoted by \( q_\lambda \). The pullback state will be denoted by \( \langle q, \cdot \rangle \), where \( \langle \cdot, \cdot \rangle \) is the natural inner product on \( \mathbb{C}^{\sigma(A)} \) induced by the basis \( \{e_\lambda\}_{\lambda \in \sigma(A)} \). Physically, the number \( q_\lambda \) is interpreted as the probability that the state \( \omega \) takes the value \( \lambda \) when the observable \( A \) is measured. If a disintegration \( R : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathbb{C}^{\sigma(A)} \) exists, it is uniquely determined by the collection of PU maps \( R_\lambda \) defined by

\[
\mathcal{M}_m(\mathbb{C}) \xrightarrow{R} \mathbb{C}^{\sigma(A)} \xrightarrow{\text{ev}_\lambda} \mathbb{C}
\]

\[
A \xrightarrow{R_\lambda} \langle e_\lambda, R(A) \rangle
\]

(6.2)

and indexed by \( \lambda \in \sigma(A) \). Because these are states on \( \mathcal{M}_m(\mathbb{C}) \), they uniquely determine a density matrix \( \rho_\lambda \in \mathcal{M}_m(\mathbb{C}) \), i.e.

\[
R_\lambda = \text{tr}(\rho_\lambda \cdot), \quad \rho_\lambda \geq 0, \quad \text{tr}(\rho_\lambda) = 1, \quad \text{and} \quad R = \sum_{\lambda \in \sigma(A)} \text{ev}_\lambda^* \circ R_\lambda.
\]

(6.3)

Because \( R \) must be state-preserving to be a disintegration, this entails

\[
\text{tr}(\rho \cdot) = \langle q, R(\cdot) \rangle \overset{(6.3)}{=} \sum_{\lambda \in \sigma(A)} q_\lambda \text{tr}(\rho_\lambda \cdot) \implies \rho = \sum_{\lambda \in \sigma(A)} q_\lambda \rho_\lambda.
\]

(6.4)

The other condition for \( R \) to be a disintegration is \( R \circ F = \text{id}_{\mathbb{C}^{\sigma(A)}} \), which says

\[
\sum_{\lambda \in \sigma(A)} b_\lambda e_\lambda - R \left( \sum_{\lambda \in \sigma(A)} b_\lambda e_\lambda \right) \in \mathcal{N}_{\langle q, \cdot \rangle} \quad \forall \quad \sum_{\lambda \in \sigma(A)} b_\lambda e_\lambda \in \mathbb{C}^{\sigma(A)}.
\]

(6.5)

Expanding this out, relabeling indices, and using part i of Lemma 2.26 gives

\[
\sum_{\lambda \in \sigma(A) \setminus N_q} b_\lambda e_\lambda = \sum_{\lambda \in \sigma(A) \setminus N_q} \left( \sum_{\mu \in \sigma(A)} b_\mu \text{tr}(\rho_\mu P_\mu) \right) e_\lambda.
\]

(6.6)

Linear independence of the \( e_\lambda \) then gives the constraints

\[
b_\lambda = \sum_{\mu \in \sigma(A)} b_\mu \text{tr}(\rho_\mu P_\mu) \quad \forall \lambda \in \sigma(A) \setminus N_q.
\]

(6.7)
Since the \(b\)'s can be chosen arbitrarily and independently (indeed, set \(b_\mu := \delta_{\mu \nu}\) for various \(\nu\)), we conclude
\[
\text{tr}(\rho_\lambda P_\mu) = \delta_{\mu \nu} \quad \forall \mu, \lambda \in \sigma(A) \setminus N_q.
\]
(6.8)

Since \(\rho_\lambda\) is a positive matrix, \(\text{tr}(\rho_\lambda P_\mu) = \text{tr}(P_\mu \rho_\lambda P_\mu) = 0\) if and only if \(P_\mu \rho_\lambda P_\mu = 0\) whenever \(\mu \neq \lambda\).

In what follows, we will prove \(\rho_\lambda = P_\lambda \rho_\lambda P_\lambda\). To see this, first let \(\bar{u} \in \text{Im}(P_\mu)\) and \(\bar{v} \in \text{Im}(P_\nu)\), where \(\mu, \nu \in \sigma(A) \setminus \{\lambda\}\). Then \(P_\mu, P_\nu \leq P_\lambda^\perp\) and
\[
0 \leq \langle \bar{u} + \bar{v}, P_\lambda^\perp \rho_\lambda P_\lambda^\perp (\bar{u} + \bar{v}) \rangle
= \langle \bar{u}, P_\lambda^\perp \rho_\lambda P_\lambda^\perp \bar{u} \rangle + \langle \bar{u}, P_\lambda^\perp \rho_\lambda P_\lambda^\perp \bar{v} \rangle + \langle \bar{v}, P_\lambda^\perp \rho_\lambda P_\lambda^\perp \bar{u} \rangle + \langle \bar{v}, P_\lambda^\perp \rho_\lambda P_\lambda^\perp \bar{v} \rangle
= \langle \bar{u}, P_\mu \rho_\lambda P_\mu \bar{u} \rangle + \langle \bar{u}, P_\mu \rho_\lambda P_\nu \bar{v} \rangle + \langle \bar{v}, P_\nu \rho_\lambda P_\nu \bar{u} \rangle + \langle \bar{v}, P_\nu \rho_\lambda P_\nu \bar{v} \rangle
= 2\Re \langle \bar{u}, P_\mu \rho_\lambda P_\nu \bar{v} \rangle,
\]
(6.9)
where we have freely used the facts \(P_\lambda^\perp \bar{u} = P_\mu \bar{u} = \bar{u}\) and \(P_\lambda^\perp \bar{v} = P_\nu \bar{v} = \bar{v}\) together with the self-adjointness and orthogonality of these projections. Since \(\bar{u}\) and \(\bar{v}\) can be arbitrary, positivity of \(P_\lambda^\perp \rho_\lambda P_\lambda^\perp\) guarantees that \(P_\mu \rho_\lambda P_\nu = 0\) for all \(\mu, \nu \in \sigma(A) \setminus \{\lambda\}\). So far, we have shown \(\rho_\lambda = P_\lambda \rho_\lambda P_\lambda + P_\lambda^\perp \rho_\lambda P_\lambda + P_\lambda \rho_\lambda P_\lambda^\perp\). What is left to show is that \(P_\mu \rho_\lambda P_\lambda = 0\) for all \(\mu \in \sigma(A) \setminus \{\lambda\}\) (which would imply \(P_\lambda \rho_\lambda P_\mu = 0\) by taking the adjoint). Now, let \(\bar{u} \in \text{Im}(P_\mu)\) and \(\bar{v} \in \text{Im}(P_\lambda)\), where \(\mu \in \sigma(A) \setminus \{\lambda\}\). Positivity of \(\rho_\lambda\) gives
\[
0 \leq \langle \bar{u} + \bar{v}, \rho_\lambda (\bar{u} + \bar{v}) \rangle = 2\Re \langle \bar{u}, P_\mu \rho_\lambda P_\nu \bar{v} \rangle + \langle \bar{v}, \rho_\lambda \bar{v} \rangle
\]
(6.10)
by a similar calculation and using the previous result. Since \(\bar{u}\) can be chosen freely, it can be chosen so that the left term becomes arbitrarily negative unless \(P_\mu \rho_\lambda P_\lambda = 0\). This concludes the argument that \(\rho_\lambda = P_\lambda \rho_\lambda P_\lambda\). Thus, \(\rho_\lambda\) and \(\rho_{\lambda'}\) have mutually orthogonal supports for \(\lambda \neq \lambda'\) provided that \(\lambda, \lambda' \in \sigma(A) \setminus N_q\). Hence, we have no restrictions on \(\rho_\lambda\) when \(\lambda \in N_q\), we still obtain
\[
\rho \stackrel{(6.4)}{=} \sum_{\lambda \in \sigma(A)} q_\lambda \rho_\lambda = \sum_{\lambda \in \sigma(A) \setminus N_q} q_\lambda \rho_\lambda = \sum_{\lambda \in \sigma(A) \setminus N_q} q_\lambda P_\lambda \rho_\lambda P_\lambda,
\]
(6.11)
which agrees with the result (5.78) with respect to a spectral basis, or more accurately (5.110), in this special case since \(s = 1\) so that there is only one \(i\) index and \(\sigma_j = 1\) for all \(j\) because \(\sigma_j\) is a \(1 \times 1\) matrix. Thus, the \(\tau_{ji}\) matrices reduce to the \(\rho_\lambda\) matrices. To make a more explicit connection to quantum information theory, we recall the definition of a Lüders projection, which is a model for the ensemble of the induced states of a system after a measurement has taken place [25].

**Definition 6.12.** Let \(\rho \in \mathcal{M}_m(\mathbb{C})\) be a density matrix and let \(A \in \mathcal{M}_m(\mathbb{C})\) be self-adjoint with spectrum \(\sigma(A)\). The **Lüders projection of \(\rho\) with respect to the measurement of \(A\)** is the density matrix
\[
\rho' := \sum_{\lambda \in \sigma(A)} P_\lambda \rho P_\lambda.
\]
(6.13)

In summary, we have obtained the following theorem based on our above analysis.

**Theorem 6.14.** Let \(A \in \mathcal{M}_m(\mathbb{C})\) be a self-adjoint matrix with spectrum \(\sigma(A)\), let \(F : \mathbb{C}^{\sigma(A)} \to \mathcal{M}_m(\mathbb{C})\) be as in (6.1), and let \(\omega = \text{tr}(\rho \cdot \cdot) : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathbb{C}\) be a state with \(\langle q, \cdot \rangle := \omega \circ F\) the induced state on \(\mathbb{C}^{\sigma(A)}\). Then \(F\) has a disintegration of \(\omega\) over \(\langle q, \cdot \rangle\) consistent with \(F\) if and only if \(\rho\) equals its Lüders projection with respect to the measurement of \(A\).
**Proof.** We will use the same notation as earlier in this section.

(⇒) Assume a disintegration exists. By (6.11), \( P_\lambda \rho P_\lambda = q_\lambda \rho \) for all \( \lambda \in \sigma(A) \setminus N_q \). Hence,

\[
\rho_\lambda = \frac{P_\lambda \rho P_\lambda}{q_\lambda} = \frac{P_\lambda \rho P_\lambda}{\text{tr}(\rho P_\lambda)} \quad \forall \ \lambda \in \sigma(A) \setminus N_q
\]  

(6.15)

because \( \rho_\lambda \) is a density matrix. Furthermore, since a disintegration exists,

\[
\rho \overset{(6.4)}{\supseteq} \sum_{\lambda \in \sigma(A) \setminus N_q} q_\lambda \rho_\lambda \overset{(6.15)}{\supseteq} \sum_{\lambda \in \sigma(A) \setminus N_q} q_\lambda \frac{P_\lambda \rho P_\lambda}{q_\lambda} = \sum_{\lambda \in \sigma(A) \setminus N_q} P_\lambda \rho P_\lambda.
\]  

(6.16)

(⇐) Suppose \( \rho \) equals its Lüders projection, i.e. suppose

\[
\rho = \sum_{\lambda \in \sigma(A)} P_\lambda \rho P_\lambda.
\]  

(6.17)

For each \( \lambda \in \sigma(A) \), set \( R_\lambda : M_m(\mathbb{C}) \rightarrow \mathbb{C} \) to be the linear map defined by

\[
M_m(\mathbb{C}) \ni B \mapsto R_\lambda(B) := \begin{cases} 
\text{tr} \left( \frac{P_\lambda \rho P_\lambda}{q_\lambda} B \right) & \text{if } q_\lambda > 0 \\
\frac{1}{m} \text{tr}(B) & \text{if } q_\lambda = 0 
\end{cases}
\]  

(6.18)

Then the linear map \( R : M_m(\mathbb{C}) \rightarrow \mathbb{C}^{\sigma(A)} \) defined by \( R := \sum_{\lambda \in \sigma(A)} \text{ev}_\lambda^* \circ R_\lambda \) is a disintegration of \( F \).

To see this, first notice that \( R \) is positive, which implies it is CP since \( \mathbb{C}^{\sigma(A)} \) is commutative (cf. Theorem 3 in Stinespring [45]). Second, \( R \) is unital because

\[
R(1_m) = \sum_{\lambda \in \sigma(A) \setminus N_q} R_\lambda(1_m) e_\lambda + \sum_{\lambda \in N_q} R_\lambda(1_m) e_\lambda \overset{(6.18)}{=} \sum_{\lambda \in \sigma(A) \setminus N_q} e_\lambda + \sum_{\lambda \in N_q} e_\lambda = \sum_{\lambda \in \sigma(A)} e_\lambda
\]  

(6.19)

since \( \text{tr}(P_\lambda \rho P_\lambda) = q_\lambda \) for all \( \lambda \in \sigma(A) \). To show \( R \) satisfies \( R \circ F \overset{\langle q, \cdot \rangle}{\longrightarrow} \text{id}_{\mathbb{C}^{\sigma(A)}} \), we will show

\[
R(F(e_\mu)) - e_\mu \in \mathcal{N}_{\langle q, \cdot \rangle} \quad \text{for all } \mu \in \sigma(A).
\]

Setting \( d_\mu := \text{tr}(P_\mu) \), the degeneracy/multiplicity of \( \mu \in \sigma(A) \), we obtain

\[
R(F(e_\mu)) \overset{(6.1)}{=} R(P_\mu) = \sum_{\lambda \in \sigma(A) \setminus N_q} R_\lambda(P_\mu) e_\lambda + \sum_{\lambda \in N_q} R_\lambda(P_\mu) e_\lambda \quad \overset{(6.18)}{=} \sum_{\lambda \in \sigma(A) \setminus N_q} \delta_{\mu \lambda} e_\lambda + \sum_{\lambda \in N_q} \frac{d_\mu}{m} e_\lambda = \begin{cases} 
\sum_{\lambda \in \sigma(A) \setminus N_q} \delta_{\mu \lambda} e_\lambda + \sum_{\lambda \in N_q} \frac{d_\mu}{m} e_\lambda & \text{if } q_\mu > 0 \\
\sum_{\lambda \in N_q} \frac{d_\mu}{m} e_\lambda & \text{if } q_\mu = 0
\end{cases}
\]  

(6.20)

Hence, \( R(F(e_\mu)) - e_\mu \in \mathbb{C}^{N_q} \), which is the null space of \( \langle q, \cdot \rangle \). Thus, \( R \) is a disintegration. \( \blacksquare \)

### A Equivalence definitions of disintegration

In this appendix, we review the definition of a disintegration from measure theory (cf. Definition 452E in Fremlin [13]). Tables 1 and 2 provide two, a-priori different, definitions of a disintegration with varying input data and consistency conditions. This appendix serves to explain how
these definitions are related to each other. More precisely, Theorems A.12 and A.23 state that

the definitions in the respective tables are equivalent. Theorem A.34 says that this diagrammatic
definition of a disintegration is equivalent to the definition of a regular conditional probability (cf. Definition 2.1 in Panagaden [29]).

In all that follows, \((X, \Sigma, \mu)\) and \((Y, \Omega, \nu)\) are measure spaces with no additional assumptions
other than \(\mu\) and \(\nu\) are non-negative measures. Furthermore, \(f : X \to Y\) is taken to be measure-

preserving so that the pushforward \(f_*\mu\) of \(\mu\) along \(f\) is \(\nu\), i.e. \(\nu(F) = \mu(f^{-1}(F))\) for all \(F \in \Omega\).

**Definition A.1.** Let \((X, \Sigma)\) and \((Y, \Omega)\) be measurable spaces. A **transition kernel** \(r\) from \((Y, \Omega)\) to \((X, \Sigma)\), written \(r : Y \rightrightarrows X\), is a function \(r : Y \times \Sigma \to [0, \infty]\) such that

i. \(r(y, \cdot) : \Sigma \to [0, \infty]\) is a measure for all \(y \in Y\) and

ii. \(r(\cdot, E) : Y \to [0, \infty]\) is measurable for all \(E \in \Sigma\).

The notation \(r_y(E) := r(y, E)\) will be implemented. A transition kernel as above is called a **stochastic map** (also **Markov kernel**) when \(r_y\) is a probability measure for all \(y \in Y\).

Transition kernels are generalizations of measurable functions in that they assign to each point in the source/domain a measure on the target/codomain (cf. Example A.5). If a function is to be thought of as a deterministic process, a transition kernel whose associated measures are probability measures can be interpreted as a non-deterministic (i.e. stochastic) process, where one only knows the probabilities associated with the possible outcomes of that process.

**Example A.2.** Let \((X, \Sigma)\) be a measurable space and let \(\{\bullet\}\) denote a one element set with the unique \(\sigma\)-algebra. There is a bijection between the set of measures on \((X, \Sigma)\) and the set of transition kernels \(\{\bullet\} \rightrightarrows X\) from \(\{\bullet\}\) to \(X\). This allows measures to be viewed as morphisms.

**Definition A.3.** Let \((X, \Sigma), (Y, \Omega),\) and \((Z, \Xi)\) be measurable spaces. Let \(\mu : X \rightrightarrows Y\) and \(\nu : Y \rightrightarrows Z\) be two transition kernels. The **composite** of \(\mu\) followed by \(\nu\), written as \(\nu \circ \mu : X \rightrightarrows Z\), is defined by

\[
X \times \Xi \ni (x, E) \mapsto (\nu \circ \mu)_x(E) := \int_Y \nu_y(E) \, d\mu_x(y). \tag{A.4}
\]

This equation is known as the **Chapman–Kolmogorov** equation.

The fact that the composite of transition kernels defines a transition kernel follows from the monotone convergence theorem [42, Theorem 1.26]. Rather than proving this here, we will recall techniques from analysis that can be used to prove this when we prove Theorem A.12 below.

**Example A.5.** Let \((X, \Sigma)\) and \((Y, \Omega)\) be measurable spaces and let \(\mu : \{\bullet\} \rightrightarrows X\) be a measure on \(X\) and let \(f : X \to Y\) be a measurable function. Then \(f\) can be viewed as the transition kernel \(f : X \rightrightarrows Y\) given by

\[
X \times \Omega \ni (x, E) \mapsto f_x(E) := \chi_E(f(x)) := \begin{cases} 
1 & \text{if } f(x) \in E \\
0 & \text{otherwise}
\end{cases}, \tag{A.6}
\]
Furthermore, \( f \circ \mu \) is the pushforward \( f_\ast \mu \) of the measure \( \mu \) along the map \( f \) because
\[
\begin{align*}
\Omega \ni E \mapsto (f \circ \mu)(E) &= \int_X f_x(E) \, d\mu(x) = \int_X (\chi_E \circ f) \, d\mu = \int_Y \chi_{f^{-1}(E)} \, d(f_\ast \mu) = \mu(f^{-1}(E)).
\end{align*}
\] (A.7)

A special case of this occurs for the diagonal map \( \Delta_Y : Y \to Y \times Y \). This pushes forward a probability measure \( q : \{\bullet\} \rightsquigarrow Y \) to the diagonal subset
\[
\Delta_Y(Y) := \{(y, y) \in Y \times Y : y \in Y\}
\] (A.8)
of \( Y \times Y \) so that \((\Delta_Y \circ q)(A \times B) = q(A \cap B)\) for all \( A, B \in \Omega \). This map is used to instantiate a categorical formulation of a.e. equivalence (cf. Remark A.21).

**Example A.9.** Let \( X, Y, \) and \( Z \) be finite sets equipped with the discrete \( \sigma \)-algebra and suppose that all transition kernels are stochastic maps. Then Definitions A.1 and A.3 reproduce the notion of *stochastic matrices* including their compositions. Indeed, since the \( \sigma \)-algebra on \( Y \) is discrete,
\[
f_x(E) = \sum_{y \in E} f_x(\{y\})
\] (A.10)
so that the probability measure \( f_x \) is determined by its values on points of \( Y \). We therefore write \( f_{yx} := f_x(\{y\}) \) to denote the \( yx \) entry of \( f \) in matrix form. Second, the composite \( X \overset{f}{\to} Y \overset{g}{\to} Z \) sends \( x \in X \) to the probability measure on \( Z \) determined by
\[
Z \ni z \overset{(g \circ f)(\bullet)}{\to} (g \circ f)_{zz} := \sum_{y \in Y} g_{zy} f_{yx}.
\] (A.11)

Following Example A.5, when \( Y \) is a finite set equipped with the discrete \( \sigma \)-algebra, the pushforward of \( q : \{\bullet\} \rightsquigarrow Y \) along \( \Delta_Y : Y \to Y \times Y \) simplifies to \((\Delta_Y \circ q)_{(y, y')} = \delta_{yy'} q_y\).

With these definitions in place, we can compare several definitions of disintegrations. Table 1 below describes three equivalent definitions of a disintegration of one measure over another together with a list of references that use said definition.

| Data | Functional | Measure-theoretic | Diagrammatic |
|------|------------|------------------|-------------|
|      | transition kernel | transition kernel | transition kernel |
|      | \( r : Y \rightsquigarrow X \) | \( r : Y \rightsquigarrow X \) | \( r : Y \rightsquigarrow X \) |
| Conditions | \( \int_X h \, d\mu = \int_Y (\int_X h \, dr_y) \, d\nu(y) \) \( \forall \) measurable \( h : X \to [0, \infty] \) | \( \mu(E) = \int_Y r_y(E) \, d\nu(y) \) \( \forall E \in \Sigma \) | \( \bullet \) | \( \mu \) | \( \nu \) | i.e. \( r \circ \nu = \mu \) |
| References | [16, 41] | [4, 13, 26, 48] | [7] |

Table 1: Three definitions of a disintegration of \((X, \Sigma, \mu)\) over \((Y, \Omega, \nu)\).

**Theorem A.12.** Given a transition kernel \( Y \overset{r}{\to} X \) from a measure space \((Y, \Omega, \nu)\) to a measure space \((X, \Sigma, \mu)\), the three conditions in Table 1 are equivalent.
Proof. The equivalence between the measure-theoretic definition and the diagrammatic definition is immediate from the definition of the composition of transition kernels. Therefore, it suffices to prove the equivalence between the measure-theoretic and functional definitions. By setting \( h := \chi_E \) with \( E \in \Sigma \), the measure-theoretic condition follows from the functional definition. The only slightly non-trivial part of the proof of this equivalence is showing that the measure-theoretic definition implies the functional one. First, a straightforward computation, using \( r \circ \nu = \mu \), shows

\[
\int_X s \, d\mu = \int_Y \left( \int_X s \, dr_y \right) d\nu(y) \tag{A.13}
\]

for all simple functions \( s : X \to [0, \infty) \). The general case for arbitrary measurable \( h : X \to [0, \infty] \) follows from the monotone convergence theorem, though one needs to be careful about how to choose a monotone sequence of simple functions \( (s_n) \) converging pointwise to \( h \). Such a sequence can be obtained as in the proof of Theorem 2.10 in Folland [12] (cf. Lemma 4.10 in [31]). From such a choice, it follows that

\[
N \ni n \mapsto \left( Y \ni y \mapsto \int_X s_n \, dr_y \right) \tag{A.14}
\]

is a monotone increasing sequence of measurable functions on \( Y \) (see Equation (4.94) in the proof of part iii of Proposition 4.79 of [31] for details). Using all of these facts gives

\[
\int_X h \, d\mu = \lim_{n \to \infty} \int_X s_n \, d\mu \quad \text{by definition of } \int \text{ w.r.t. } \mu \\
= \lim_{n \to \infty} \int_Y \left( \int_X s_n \, dr_y \right) d\nu(y) \quad \text{by (A.13) for simple } s_n \\
= \int_Y \lim_{n \to \infty} \left( \int_X s_n \, dr_y \right) d\nu(y) \quad \text{by the monotone convergence theorem} \\
= \int_Y \left( \int_X h \, dr_y \right) d\nu(y) \quad \text{by definition of } \int \text{ w.r.t. } r_y
\]

for arbitrary measurable \( h : X \to [0, \infty] \).

When one is equipped with the additional datum of a measure-preserving map \( f : X \to Y \), there is another coherence condition that can be enforced on disintegrations. This assumption is to demand that a disintegration \( r : Y \rightharpoonup X \) be consistent with the map \( f \). From our diagrammatic perspective, this means \( r \) is a (stochastic) section of \( f \) a.e. This is described in Table 2.

**Definition A.16.** Let \((X, \Sigma, \mu)\) and \((Y, \Omega, \nu)\) be two measure spaces. Two transition kernels \( f, g : X \rightharpoonup Y \) are said to be \( \mu \)-a.e. equivalent, written as \( f \equiv g \), iff for each \( F \in \Omega \), there exists a measurable set \( N_F \in \Sigma \) such that

\[
f_x(F) = g_x(F) \quad \forall \ x \in X \setminus N_F \quad \text{and} \quad \mu(N_F) = 0. \tag{A.17}
\]

**Example A.18.** The definition of a.e. equivalence takes a particularly simple form for finite sets. Let \( X \) and \( Y \) be finite sets and let \( \mu \) be a measure on \( X \). Let

\[
N_\mu := \{ x \in X : \mu_x = 0 \}
\]

(A.19)
denote the **null-set** of \((X, \mu)\). Two transition kernels \(f, g : X \leadsto Y\) are **\(\mu\)-a.e. equivalent** iff

\[
\{ x \in X : f_x \neq g_x \} \subseteq N_{\mu}.
\]  
(A.20)

The notation \(f = \mu g\) is used whenever \(f\) and \(g\) are \(\mu\)-a.e. equivalent. Here, \(f_x \neq g_x\) means \(f_x\) and \(g_x\) are different measures on \(Y\), i.e. there exists a \(y \in Y\) such that \(f_{yx} \neq g_{yx}\).

**Remark A.21.** The definition of a.e. equivalence in Definition A.16 is a bit subtle in the general measure-theoretic case. Another reasonable option would be to say \(f\) and \(g\) are \(\mu\)-a.e. equivalent iff there exists an \(N \in \Sigma\) such that \(f_x = g_x\) (equality of measures) for all \(x \in X \setminus N\) and \(\mu(N) = 0\). However, this definition is too strong for the conditions in Table 2 to be equivalent for arbitrary measure spaces. The definition we have chosen agrees with the diagrammatic definition of Cho and Jacobs [5, Section 5], which says that the diagram

\[
\begin{array}{ccccccc}
X & \xrightarrow{\Delta_X} & X \times X & \xrightarrow{id_X \times f} & X \times Y \\
\mu & \downarrow & \downarrow & \downarrow & \\
\{ \bullet \} & \xrightarrow{\Delta_X} & X \times X & \xrightarrow{id_X \times g} & X \\
\mu & \downarrow & \downarrow & \downarrow & \\
X & \xrightarrow{\Delta_X} & X \times X & \xrightarrow{id_X \times f} & X \times Y
\end{array}
\]  
(A.22)

commutes (the product of stochastic maps can be defined using joint probability measures as is done in Section 2 of [5]).

| Data | Measure-theoretic | Diagrammatic |
|------|------------------|--------------|
| Transition kernel | \(r : Y \leadsto X\) | \(r : Y \leadsto X\) |
| Conditions besides \(r\) is a disintegration of \(\mu\) over \(\nu\) | for each \(F \in \Omega\) \(\exists \nu\)-null set \(N_F \in \Omega\) s.t. \(r_y(f^{-1}(F)) = 1\) \(\forall y \in (Y \setminus N_F) \cap F\) | i.e. \(f \circ r = \text{id}_Y\) |

Table 2: Two definitions of a disintegration of \((X, \Sigma, \mu)\) over \((Y, \Omega, \nu)\) consistent with a measure-preserving measurable map \(f : X \to Y\).

More explicitly, the condition \(f \circ r = \text{id}_Y\) says that for each \(F \in \Omega\), there exists a \(\nu\)-null set \(M_F \in \Omega\) such that \((f \circ r)_y(F) = \chi_F(y)\) for all \(y \in Y \setminus M_F\). Expanding out \((f \circ r)_y(F)\) using Example A.5 and the definition of transition kernels, this is equivalent to \(r_y(f^{-1}(F)) = \chi_F(y)\) for all \(y \in Y \setminus M_F\). Therefore, it is immediate that the diagrammatic definition implies the measure-theoretic one.

**Theorem A.23.** Let \((Y, \Omega, \nu)\) and \((X, \Sigma, \mu)\) be measure spaces. Given a measure-preserving measurable map \(X \xrightarrow{f} Y\), together with a disintegration \(Y \xrightarrow{\sim} X\) of \(\mu\) over \(\nu\), the conditions in Table 2 are equivalent.
Proof. By the comment preceding the statement of this theorem, the equivalence will follow from proving the measure-theoretic definition implies the diagrammatic one, i.e. for each $F \in \Omega$, there exists a $\nu$-null set $M_F \in \Omega$ such that $r_y(f^{-1}(F)) = \chi_F(y)$ for all $y \in Y \setminus M_F$ (cf. [13, Proposition 452G]). In more detail, by assumption, there exist $\nu$-null sets $N_F, N_{Y\setminus F}, N_Y \in \Omega$ such that

$$r_y(X) = r_y(f^{-1}(Y)) = 1 \quad \forall y \in (Y \setminus N_Y) \cap Y \equiv Y \setminus N_Y, \quad (A.24)$$

$$r_y(f^{-1}(F)) = 1 \quad \forall y \in (Y \setminus N_F) \cap F, \quad (A.25)$$

and

$$r_y(f^{-1}(Y \setminus F)) = 1 \quad \forall y \in (Y \setminus N_{Y\setminus F}) \cap (Y \setminus F) \equiv Y \setminus (N_{Y\setminus F} \cup F). \quad (A.26)$$

Therefore,

$$1 - r_y(f^{-1}(F)) = r_y(X) - r_y(f^{-1}(F)) = r_y(f^{-1}(Y \setminus F)) = 1$$

$$\forall y \in (Y \setminus N_Y) \cap (Y \setminus (N_{Y\setminus F} \cup F)), \quad (A.27)$$

i.e.

$$r_y(f^{-1}(F)) = 0 \quad \forall y \in (Y \setminus N_Y) \cap (Y \setminus (N_{Y\setminus F} \cup F)) \equiv Y \setminus (N_Y \cup N_{Y\setminus F} \cup F). \quad (A.28)$$

Set

$$M_F := N_F \cup N_{Y\setminus F} \cup N_Y, \quad (A.29)$$

which, being the finite union of $\nu$-null sets, is $\nu$-null. If $y \in (Y \setminus M_F) \cap F$, then, in particular, $y \in (Y \setminus N_F) \cap F$ so that $r_y(f^{-1}(F)) = 1$. If $y \in (Y \setminus M_F) \cap (Y \setminus F) \equiv Y \setminus (N_Y \cup N_{Y\setminus F} \cup N_Y \cup F)$, then, in particular, $y \in Y \setminus (N_Y \cup N_{Y\setminus F} \cup F)$ so that $r_y(f^{-1}(F)) = 0$. Putting these together,

$$r_y(f^{-1}(F)) = \chi_F(y) \quad \forall y \in Y \setminus M_F. \quad (A.30)$$

Therefore, the measure-theoretic definition implies the diagrammatic one. ■

Remark A.31. The equality (A.24) says $r_y$ is a probability measure for all $y \in Y \setminus N_Y$.

A consistent disintegration is also related to the notion of a regular conditional probability.

Definition A.32. Let $(X, \Sigma, \mu)$ and $(Y, \Omega, \nu)$ be measure spaces and let $f : X \rightarrow Y$ be a measure-preserving map. A regular conditional probability is a transition kernel $r : Y \rightsquigarrow X$ for which there exists a $\nu$-null set $N \in \Omega$ such that $r_y$ is a probability measure for all $y \in Y \setminus N$ and

$$\mu(E \cap f^{-1}(F)) = \int_F r_y(E) \, d\nu(y) \quad \forall E \in \Sigma \text{ and } \forall F \in \Omega. \quad (A.33)$$

Theorem A.34. Let $(X, \Sigma, \mu)$ and $(Y, \Omega, \nu)$ be measure spaces and let $f : X \rightarrow Y$ be a measure-preserving map. $r : Y \rightsquigarrow X$ is a regular conditional probability if and only if it is a disintegration of $\mu$ over $\nu$ consistent with $f$. 

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Proof.

\((\Rightarrow)\) Suppose \(r\) is a regular conditional probability. Then

\[
\mu(E) = \mu(E \cap X) = \mu(E \cap f^{-1}(Y)) = \int_Y r_y(E) \, d\nu(y) = (r \circ \nu)(E) \tag{A.35}
\]

for all \(E \in \Sigma\). Now, fix \(F \in \Omega\) and let \(N \in \Omega\) be a \(\nu\)-null set such that \(r_y\) is a probability measure for all \(y \in Y \setminus N\). Then

\[
\begin{align*}
\mu(f^{-1}(F) \cap f^{-1}(F)) & \quad \overset{\text{(A.33)} \text{ with } E = f^{-1}(F)}{=} \\
\mu(f^{-1}(F)) & \quad \int_F r_y(f^{-1}(F)) \, d\nu(y) \\
\text{since } \nu = f \circ \mu & \quad \int_F d\nu
\end{align*}
\tag{A.36}
\]

Since \(r_y\) is a probability measure \(\nu\)-a.e., \(r_y(f^{-1}(F)) \leq 1\) for all \(y \in (Y \setminus N) \cap F\) so that the quantity in (A.36) is finite. This allows us to meaningfully take the difference of these terms. Therefore, (A.36) implies \(\int_F (1 - r_y(f^{-1}(F))) \, d\nu(y) = 0\). Furthermore, since the integrand is non-negative, there exists a \(\nu\)-null set \(M_F \in \Omega\) such that

\[
r_y(f^{-1}(F)) = 1 \quad \forall \ y \in (Y \setminus (N \cup M_F)) \cap F. \tag{A.37}
\]

Hence, \(f \circ r = \text{id}_Y\) so that \(r\) is a consistent disintegration.

\((\Leftarrow)\) Conversely, suppose \(r\) is a consistent disintegration. By Remark A.31, \(r_y\) is a probability measure \(\nu\)-a.e. Hence,

\[
\begin{align*}
\mu(E \cap f^{-1}(F)) & \quad \int_F r_y(E) \, d\nu(y) \\
\int_Y r_y(E \cap f^{-1}(F)) \, d\nu(y) & \quad \overset{\text{set theory}}{=} \\
\int_F r_y(E \cap f^{-1}(Y \setminus F)) \, d\nu(y) & \quad \overset{\text{(A.28) for } Y \setminus F}{=} \\
\int_F r_y(E \cap f^{-1}(F)) \, d\nu(y) & \quad + \int_F r_y(E \cap f^{-1}(Y \setminus F)) \, d\nu(y)
\end{align*}
\tag{A.38}
\]

for arbitrary \(E \in \Sigma\) and \(F \in \Omega\). This proves \(r\) is a regular conditional probability. \(\blacksquare\)

From this perspective, the results in this paper can be viewed as an approach to non-commutative regular conditional probabilities.
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References

[1] John C. Baez and Tobias Fritz, *A Bayesian characterization of relative entropy*, Theory and Applications of Categories 29 (2014), No. 16, 422–457, available at 1402.3067. ↑18

[2] John C. Baez, Tobias Fritz, and Tom Leinster, *A characterization of entropy in terms of information loss*, Entropy 13 (2011), no. 11, 1945–1957, available at 1106.1791. ↑2, 49

[3] Howard Barnum, Carlton M. Caves, Christopher A. Fuchs, Richard Jozsa, and Benjamin Schumacher, *Non-commuting mixed states cannot be broadcast*, Phys. Rev. Lett. 76 (1996), 2818–2821. ↑17

[4] Srishti Dhar Chatterji, *Disintegration of measures and lifting*, Vector and operator valued measures and applications (Proc. Sympos., Alta, Utah, 1972), 1973, pp. 69–83. ↑44

[5] Kenta Cho and Bart Jacobs, *Disintegration and Bayesian inversion via string diagrams*, Mathematical Structures in Computer Science (2019), 1–34, available at 1709.00322. ↑2, 46

[6] Man Duen Choi, *Completely positive linear maps on complex matrices*, Linear Algebra and Applications 10 (1975), 285–290. ↑4

[7] Florence Clerc, Vincent Danos, Fredrik Dahlqvist, and Ilias Garnier, *Pointless learning*, Foundations of software science and computation structures, 2017, pp. 355–369. ↑2, 44, 46

[8] Bob Coecke and Robert W. Spekkens, *Picturing classical and quantum Bayesian inference*, Synthese 186 (2012), no. 3, 651–696, available at 1102.2368. ↑17

[9] Dennis Dieks, *Communication by EPR devices*, Physics Letters A 92 (1982), no. 6, 271–272. ↑17

[10] Albert Einstein, Boris Podolsky, and Nathan Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Physical Review 47 (1935), 777–780. ↑3

[11] Peter A. Fillmore, *A user’s guide to operator algebras*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, Inc., New York, 1996. ↑6, 9, 23, 27

[12] Gerald B. Folland, *Real analysis*, Second, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication. ↑45

[13] D. H. Fremlin, *Measure theory. Vol. 4: Topological measure spaces. part i, ii, corrected second printing of the 2003 original*, Torres Fremlin, Colchester, 2006. Updated version (as of 23.3.10) available at https://www1.essex.ac.uk/maths/people/fremlin/cont45.htm. ↑42, 44, 46, 47

[14] Tobias Fritz, *A synthetic approach to markov kernels, conditional independence and theorems on sufficient statistics*, 2019. arXiv preprint: 1908.07021 [math.ST]. ↑2

[15] Robert Furber and Bart Jacobs, *From Kleisli categories to commutative C*-algebras: probabilistic Gelfand duality*, Logical Methods in Computer Science 11 (2015), no. 2, 1:5, 28, available at 1303.1115. ↑2
[38] Vern Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002. ↑3, 13

[39] Dénes Petz, *Quantum information theory and quantum statistics*, Theoretical and Mathematical Physics, Springer-Verlag, Berlin, 2008. ↑2, 23

[40] Robert Pluta, *Ranges of bimodule projections and conditional expectations*, Cambridge Scholars Publishing, 2013. ↑7

[41] Vladimir A. Rohlin, *On the fundamental ideas of measure theory*, American Mathematical Society Translations 1952 (1952), no. 71, 55. ↑44

[42] Walter Rudin, *Real and complex analysis*, Third ed., McGraw-Hill Book Co., New York, 1987. ↑43

[43] Shôichirô Sakai, *$C^*$-algebras and $W^*$-algebras*, Springer-Verlag, New York-Heidelberg, 1971. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60. ↑3, 6

[44] Zbigniew Semadeni, *Monads and their Eilenberg–Moore algebras in functional analysis*, Queen’s University, Kingston, Ont., 1973. Queen’s Papers in Pure and Applied Mathematics, No. 33. ↑2

[45] W. Forrest Stinespring, *Positive functions on $C^*$-algebras*, Proceedings of the American Mathematical Society 6 (1955), no. 2, 211–216. ↑12, 42

[46] Erling Størmer, *Positive linear maps of operator algebras*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013. ↑13

[47] Tadeusz Świątcz, *Monadic functors and convexity*, Bulletin de l’Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques 22 (1974), 39–42. ↑2

[48] Michael Wendt, *The category of disintegration*, Cahiers de Topologie et Géométrie Différentielle Catégoriques 35 (1994), no. 4, 291–308. ↑2, 44

[49] William K. Wootters and Wojciech H. Zurek, *A single quantum cannot be cloned*, Nature 299 (1982), no. 5886, 802–803. ↑17

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