Black Hole Entropy and Exclusion Statistics

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Abstract

We compute the entropy of systems of quantum particles satisfying the fractional exclusion statistics in the space-time of 2+1 dimensional black hole by using the brick-wall method. We show that the entropy of each effective quantum field theory with a Planck scale ultraviolet cutoff obeys the area law, irrespective of the angular momentum of the black hole and the statistics interpolating between Bose-Einstein and Fermi-Dirac statistics.

Keywords: BTZ black hole, Entropy, Exclusion statistics

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I. INTRODUCTION

Since the discovery of area law for black hole entropy by Bekenstein [1] and Hawking [2], considerable efforts have been made in order to derive the thermodynamic properties in the contexts of both Euclidean path-integral formulation [3] and microcanonical functional integral formulation [4]. During the last decade, the statistical origin of the entropy of a black hole has been discussed in connection with the information approach [4,11] and with the entanglement entropy [6]. It is widely believed that the entropy of quantum field in the black hole background satisfies the area law. One of the authors recently showed in Ref. [7] that the entropy of a quantum field with a Planck scale ultraviolet (UV) cutoff is proportional to the area of the black hole event horizon by using the brick wall method and the microcanonical ensemble approach.

In the 80’s, the 2+1 dimensional anti-de Sitter gravity attracted considerable attention because of its Chern-Simons gauge formulation [8]. In the early 90’s, Barañados, Teitelboim, and Zarnelli (BTZ) reported 2+1 dimensional black hole solutions in anti-de Sitter spacetime [9]. Extensive studies on the thermodynamics of BTZ black holes have followed [10]. Recently, systematic counting of states for the BTZ black hole was proposed, and the area law was recovered in the framework of statistical mechanics with some assumptions [11]. Another intriguing feature in 2+1 dimensional physics is spin-statistics relation, that is, anyons [12] or exclusion statistics [13]. Therefore, a natural question in 2+1 dimensional black hole physics is its possible connection with statistics.

More specifically, we ask whether the area law is satisfied for the entropy of the quantum particles satisfying the fractional exclusion statistics in the background of BTZ black hole. An immediate obstacle is that we do not have a specific quantum field theory example of elementary excitations satisfying exclusion statistics. Instead we assume the existence of such free field, so-called *exclusions* in the background of the BTZ black holes, and simply impose a mass shell condition (Klein-Gordon equation). Since ordinary particle physics based on a local quantum field theory fails near the horizon of a black hole [14,15], the valid
range of our effective field theory is restricted beyond Planck scale UV cutoff. In this paper we find that the area law is satisfied for each quantum statistics irrespective of the statistical parameter connecting boson and fermion.

This paper is organized as follows. In Sec. II, we calculate the phase volume of one particle outside the BTZ black hole by using the brick-wall method, and then consider many particle systems with the energy and the particle number fixed. Here we examine the statistical behavior of the system which follows the exclusion statistics. Next we remove the constraint on the particle number and take an ensemble sum over all particle number states for the quantum statistics. We conclude in Sec. III with a few discussions.

II. BLACK HOLE ENTROPY OF EXCLUSONS

Let us begin this section with a brief recapitulation of BTZ black hole solutions. The line element of BTZ black hole as a vacuum solution is described by

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\Phi^2 dt^2 + \Phi^{-2} dr^2 + r^2(N^\phi dt + d\phi)^2,$$

where $\Phi^2(r) = -M + r^2/l^2 + J^2/4r^2$, and $N^\phi(r) = -J/2r^2[9]$. Here $M$ and $J$ denote mass and angular momentum per unit mass of the BTZ black hole respectively, and they satisfy $M > 0$, and $|J| \leq Ml$. From Eq. (1), the outer horizon of the black hole is present at

$$r_+ = l \left[ \frac{M}{2} \left\{ 1 + \left[ 1 - \left( \frac{J}{Ml} \right)^2 \right]^{1/2} \right\} \right].$$

In the rotating frame with a constant azimuthal velocity $\Omega_H$ which is the angular velocity of the horizon, the metric (1) becomes

$$ds^2 = g'_{tt} dt'^2 + 2g'_{\phi\phi} \left( \Omega_H + \frac{g_{t\phi}}{g_{\phi\phi}} \right) dt' d\phi' + g_{\phi\phi} d\phi'^2 + g_{rr} dr^2,$$

where $g'_{tt} = -\Phi^2 + r^2(N^\phi + \Omega_H)^2$. Suppose that there is a quantum mechanical particle of mass $m$ outside the black hole horizon. If it carries energy $E$ and momentum $(p_r, p_\phi)$, it satisfies the following dispersion relation:
\[
\frac{-g'_{tt}}{-D} \left( p_{\phi} + \frac{g_{t\phi} + \Omega_{tt} g_{\phi\phi}}{g_{tt}} E \right)^2 + \frac{p_r^2}{g_{rr}} = \frac{E^2 - m^2}{-g'_{tt}},
\]
\begin{equation}
(4)
\end{equation}

where \(-D = (g_{t\phi})^2 - g_{tt} g_{\phi\phi} = r^2 \Phi^2\).

Let us consider \(N\) quantum mechanical particles satisfying Eq. (4) in a box bounded by two concentric cylinders with radii \(r_+ + h\) and \(L\). The nature of the particles’ spin is assumed to appear in their mutual statistics, and these \(N\) particles satisfy exclusion statistics. In this system of \(N\) exclusions, energy levels \(E_i\) and occupation numbers \(n_i\) are specified by an index \(i\) so that the total energy \(E\) and the particle number \(N\) satisfy the conditions:

\[
E = \sum_i E_i n_i, \quad N = \sum_i n_i.
\]
\begin{equation}
(5)
\end{equation}

Then the number of accessible states of this system is given by the sum of the number of states of the system corresponding to the set of occupation numbers \(\{n_i\}\), i.e.,

\[
g_N(E) = \sum_{\{n_i\}} W(\{n_i\}).
\]
\begin{equation}
(6)
\end{equation}

For the fractional exclusion statistics, \(W(\{n_i\})\) is given by \[13\]

\[
W(\{n_i\}) = \prod_{i=1}^N \frac{[g_1(E_i) + (n_i - 1)(1 - \alpha)]!}{n_i![g_1(E_i) - \alpha n_i - (1 - \alpha)]!},
\]
\begin{equation}
(7)
\end{equation}

where the statistics parameter \(\alpha\) interpolates between boson (\(\alpha = 0\)) and fermion (\(\alpha = 1\)).

If the governing relativistic quantum mechanical equation is local for each exclusion, then the energy-momentum dispersion relation (4) implies that the wave function of each particle should satisfy the Klein-Gordon equation irrespective of its statistics:

\[
(\nabla_\mu \nabla^\mu - m^2) \Psi = 0.
\]
\begin{equation}
(8)
\end{equation}

In the context of semiclassical approximation, the WKB solution has the following form;

---

1Since the spacetime structure of the BTZ black hole is that of a \((2+1)\)-D anti de Sitter spacetime, it is not asymptotically flat. Therefore, though the identification of \(E\) as the particle energy is not exact, we will attach the name “energy” to this constant of motion of the timelike Killing vector.
\[ \Psi = e^{-iEt + ip\phi + iR(r)}, \]  

where \( R(r) \) is obtained from \( p_r = \partial R/\partial r \). In a box normalization with an appropriate boundary condition as in the brick wall method \[3\] with its location at \( r = r_+ + h \), a discrete momentum eigenvalue corresponds to one quantum state per unit volume. Then the sum over the quantum states can be rewritten as the integral over the phase space. The proper distance \( \epsilon \) from the horizon to the box is given by

\[ \epsilon = \int_{r_+}^{ r_+ + h} dr \sqrt{g_{rr}} = \frac{2\sqrt{h}}{\sqrt{\frac{2}{\kappa^2} \Phi^2|_{r=r_+}}}, \]  

and the area of the inner brick wall \( A \) is given by

\[ A = 2\pi r_+. \]  

Note that when the inner radius of the box starts at \( r = r_+ \), i.e., \( h \to 0 \), the inverse of proper distance \( \epsilon \) diverges and so do the many physical quantities (see below). Since this boundary condition is a natural choice in the black hole geometry, a UV cutoff is unavoidable in our local quantum field theory, and we shall relate \( \epsilon \) in Eq. (10) with the Planck scale \[14,15\].

First of all let us compute the entropy of one particle system, and then extend it to that of \( N \) particles. The phase volume of a classical particle with fixed energy \( E \) in the frame \[3\] is the volume of a hypersurface satisfying \( H(p, x) = E \), that is, \( g_1(E) = \int d^2p d^2x \delta(E - H(p, x)) \), and is obtained by \( \partial \Gamma/\partial E \), where \( \Gamma(E) = \int d^2p d^2x \theta(E - H(p, x)) \). By use of the dispersion relation \[4\], \( \Gamma(E) \) is calculated to be

\[ \Gamma(E) = \pi \int_{\text{box}} d^2x \sqrt{-g_{rr}} \frac{D}{-g_{tt}'} \left( \frac{E^2}{-g_{tt}'} - m^2 \right). \]  

We integrate out the coordinate space integrals in Eq. (12), and express it in terms of the distance \( \epsilon \) \[10\] and the area \( A \) \[11\]. In the limit of \( h \to 0 \), these quantities are divergent;

\[ \Gamma(E) \sim \frac{4E^2}{\left( \frac{\partial}{\partial r} \Phi^2 \right)^2} \frac{2\pi r_+}{\epsilon} = \frac{E^2 A}{\kappa^2 \epsilon}, \]  

\[ g_1(E) \sim \frac{2E A}{\kappa^2 \epsilon}, \]  

\[ 5 \]
where $\kappa = \frac{\partial}{\partial r} \Phi^2 / 2 |_{r=r_+}$ is the surface gravity at the horizon \[10\]. Here let us recall that the model of our interest is an effective local quantum field theory, and it is valid only with a UV cutoff in the BTZ black hole background. Inserting Eq. \[14\] into the definition of the entropy of a particle, we get

$$S_1 \equiv \ln g_1(E) = \ln \frac{2E A}{\kappa^2 \epsilon}. \quad (15)$$

Note that the obtained entropy is not seemed to be proportional to the area $A$. Since the one particle system does not constitute a thermodynamical system and statistics of particles is not necessary, the area law needs not to be preserved in this case.

Now let us extend our calculation to many particle system. The number of accessible states $g_N(E)$ for the system of total energy $E$ and total number of particles $N$ is estimated by the maximal entropy principle when $N$ is sufficiently large. The value of $g_N(E)$ can be replaced by the maximal value of $W(\{\bar{n}_i\})$, where $\{\bar{n}_i\}$ is a set of occupation numbers that maximize $W(\{n_i\})$ subject to Eq. \[5\]:

$$\bar{n}_i = \frac{g_1(E_i)}{\omega(\epsilon E - \mu) + \alpha}, \quad (16)$$

where the temperature $1/\beta$ and chemical potential $\mu$ are determined as functions of the total energy $E$ and number of particles $N$ in Eq. \[3\]. The function $\omega(\zeta)$ satisfies the functional equation

$$\omega(\zeta)^\alpha [1 + \omega(\zeta)]^{1-\alpha} = \zeta \equiv e^{\beta (E - \mu)}, \quad (17)$$

or its equivalent one which is obtained by differentiation

$$\frac{\omega'(\zeta)}{\omega(\zeta)[\omega(\zeta) + 1]} = \frac{1}{\zeta [\alpha + \omega(\zeta)]}. \quad (18)$$

Note that $\omega(\zeta) = \zeta - 1$ for bosons ($\alpha = 0$) and $\omega(\zeta) = \zeta$ for fermions ($\alpha = 1$).

By making use of the Stirling’s formula, the entropy defined by $S_N = \ln W(\{\bar{n}_i\})$ becomes

$$S_N = \sum_{i=1}^{N} \left[ \bar{n}_i \ln \left( 1 + \frac{g_1(E_i) - \alpha \bar{n}_i - (1 - \alpha)}{\bar{n}_i} \right) \right] + \left[ g_1(E_i) - \alpha \bar{n}_i - (1 - \alpha) \right] \ln \left( 1 + \frac{\bar{n}_i}{g_1(E_i) - \alpha \bar{n}_i - (1 - \alpha)} \right). \quad (19)$$
Again a change from the summation to an integral is applied to Eq. (19) and some rearrangement of the terms in Eq. (19) with the aid of Eqs. (17) and (18) yields

\[ S_N = \frac{2A}{\kappa \beta^2 \epsilon} [3f_{a,3}(\mu \beta) - \mu \beta f_{a,2}(\mu \beta)], \quad (20) \]

where

\[ f_{a,n}(\mu \beta) = \frac{1}{(n-1)!} \int_0^\infty dx \frac{x^{n-1}}{\omega(e^{-x \mu \beta} + \alpha)}. \quad (21) \]

Note that, for \( n \geq 2 \), \( f_{a,n} \) satisfies the following recursion relation:

\[ \frac{\partial}{\partial(\mu \beta)} f_{a,n} = f_{a,n-1}. \quad (22) \]

Furthermore, \( f_{a,n}(\mu \beta) \) is monotonic decreasing function of \( \alpha \) because

\[ \frac{\partial}{\partial\alpha} [\alpha + \omega(\mu \beta)] = 1 + \frac{\omega(1 + \omega)}{\omega + \alpha} \ln[\omega(1 + \omega)] \geq 0 \text{ for } \omega > 0, \ 1 \geq \alpha \geq 0. \quad (23) \]

The energy and the number of particles of the system given by the constraint equations (5) are rewritten in a form of integrals:

\[ E = \frac{2A}{\kappa \beta^2 \epsilon} \frac{2}{\beta} f_{a,3}(\mu \beta), \quad N - n_0 = \frac{2A}{\kappa \beta^2 \epsilon} f_{a,2}(\mu \beta). \quad (24) \]

Here \( n_0 \) stands for the number of ground state particles, i.e., \( n_0 = 1/(\omega(e^{-\beta \mu}) + \alpha) \) from Eq. (16), and this term should be taken into account for the case of bosons exclusively. The entropy (19) can be expressed in terms of the energy \( E \) and the number of particles \( N \) in Eq. (24) as

\[ S_N = \beta \left[ \frac{3}{2} E - \mu(N - n_0) \right]. \quad (25) \]

By use of Eq. (24), we have

\[ \frac{\partial}{\partial E} S_N \bigg|_N = \frac{1}{T} = \beta, \quad \frac{\partial}{\partial N} S_N \bigg|_E = -\mu \beta. \quad (26) \]

Since any local quantum field theory of point particles in black hole background is believed to include divergences \[ \text{[14,15]}, \] we defined our theory as an effective quantum field...
theory with an external UV cutoff. The obtained results in Eqs. (19) and (24) also contain the cutoff $1/\epsilon$ explicitly so that we have to adjust the cutoff. We are interested in the question whether or not the dependence on the statistics parameter $\alpha$ appears in the expression of the Bekenstein-Hawking entropy of the black hole $(4\pi r_+)$. Therefore, an appropriate choice of the UV cutoff is to use the Planck scale which is the only relevant scale in our theory:

$$4\pi^2\epsilon = 3f_{0,3}(0)l_P.$$  \hspace{1cm} (27)

With this cutoff the energy $E$ and the number of particles $N$ in Eq. (24) become

$$E = \frac{4}{3\beta_H} \frac{f_{\alpha,3}(\mu \beta_H) 2\pi r_+}{f_{0,3}(0) l_P}, \quad N - n_0 = \frac{2f_{\alpha,2}(\mu \beta_H) 2\pi r_+}{3f_{0,3}(0) l_P},$$ \hspace{1cm} (28)

where $\beta_H = 2\pi/\kappa$ is on-shell temperature. We attempt an expansion of the entropy (19) for small chemical potential with $E$ fixed:

$$S_N = \frac{4\pi r_+ f_{\alpha,3}(0)}{l_P f_{0,3}(0)} \left[ 1 - \frac{1}{2} \left( \frac{f_{\alpha,1}(0)f_{\alpha,3}(0)}{f_{\alpha,2}(0)} \right) - 2 \left( \frac{f_{\alpha,2}(0)}{3f_{0,3}(0)} \right)^2 (\mu \beta_H)^2 + \cdots \right].$$ \hspace{1cm} (29)

The area law is satisfied only when the chemical potential $\mu$ vanishes.

Now we remove the conservation of total particle number $N$ of the system and allow the transition between states of different particle numbers in order to describe the many body theory of exclusons. Therefore, when we count the accessible states in the microcanonical ensemble for a quantum field, we have to take into account all possible particle number states with the total energy being fixed:

$$E = \sum_i n_i E_i.$$ \hspace{1cm} (30)

The number of states $g(E)$ can be obtained by summing $g(N)$ over all possible $N$:

$$g(E) = \sum_{N=0}^{\infty} g_N(E).$$ \hspace{1cm} (31)

This summation may be approximated appropriately by the peak value of $\bar{N}$ which maximizes $S = \ln g_N(E)$ by the following equation
\[ dS_N|_E = \mu \beta d(N - n_0) = 0. \] (32)

The unique solution to this equation is \( \mu = 0 \) and \( \bar{N} = n_0 + \frac{2f_{\alpha,2}(0)2\pi r_+}{3f_{0,3}(0)l_P} \) from Eq. (28). Now, the total energy is replaced by the expression in Eq. (24) with zero on-shell chemical potential \( \mu_H = 0 \), and then the entropy \( S = \ln g(E) \) of the system with the constraint is given by

\[ S \approx S_{\bar{N}} = \frac{6A}{(\kappa\beta)^2\epsilon}f_{\alpha,3}(0) = \frac{3}{2}\beta E|_{\mu=0}. \] (33)

Under the on-shell temperature \( \beta_H \) and the UV cutoff, the maximum entropy reproduces the area law:

\[ S = \frac{3}{2}\beta_H E|_{\mu=0} = \frac{f_{\alpha,3}(0)}{f_{0,3}(0)} \frac{4\pi r_+}{l_P}. \] (34)

A remark on the \( \alpha \)-dependence should be placed: the coefficient \( f_{\alpha,3}(0)/f_{0,3}(0) \) is a monotonic decreasing function of \( \alpha \), and reproduces successfully the values at both ends, i.e., 1 for bosons (\( \alpha = 0 \)) and 3/4 for fermions (\( \alpha = 1 \) [13]).

**III. CONCLUSION**

In this paper, we have computed the entropy of quantum particles obeying the fractional exclusion statistics in the background of the BTZ black holes with or without angular momentum. The only assumption we have made was the existence of exclusons whose dispersion relation is consistent with the locality of the corresponding field. It has been shown for quantum statistical systems that the area law is satisfied irrespective of the species of particles, and the spin dependence comes through an over-all proportionality constant.

Since our calculation was based on the brick-wall method with external UV cutoff, a genuine defect of this method also appeared in our formula: the entropy contained an *ad hoc* cutoff introduced in order to control the divergence from infinite phase volume around the event horizon. We regularized it by setting this UV cutoff to be a specific value of Planck length scale which must be the only natural cutoff scale in our model.
A final comment is in order: Since the fractional exclusion statistics holds in any space-time dimension [13], the area law may also be derived for the system of exclusions in both 1+1 and 3+1 dimensions.

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