SUPER REWRITING THEORY AND NONDEGENERACY OF ODD CATEGORIZED $sl_2$

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ABSTRACT. We develop the rewriting theory for monoidal supercategories and 2-supercategories. This extends the theory of higher-dimensional rewriting established for (linear) 2-categories to the super setting, providing a suite of tools for constructing bases and normal forms for 2-supercategories given by generators and relations. We then employ this newly developed theory to prove the non-degeneracy conjecture for the odd categorification of quantum $sl(2)$ from $[18, 7]$. As a corollary, this gives a classification of dg-structures on the odd 2-category conjectured in $[32]$.  

1. INTRODUCTION

Higher representation theory studies the higher categorical structure present when an associative algebra $A$ acts on an additive/abelian category $V$, with algebra generators acting by additive or exact functors and algebra relations lifting to explicit natural isomorphisms of functors. In its most refined form, this involves a categorification of an algebra $A$ itself, lifting $A$ to a monoidal category $\mathcal{A}$. The algebra $A$ is categorified in the sense that there is an isomorphism from the (additive or abelian) Grothendieck group $K(A)$ to $A$. The monoidal structure equips $K(A)$ with the structure of an algebra, where the $[X \otimes Y] = [X] \cdot [Y]$ and the class $[1]$ of the unit in the monoidal category becomes the unit element for algebra.

If the algebra $A$ is equipped with a system of mutually orthogonal idempotents, the most natural setting for categorification is to lift $A$ to an additive linear 2-category. Since any monoidal category can be regarded as a 2-category with one object, the 2-categorical setting is often the most natural. In particular, the diagrammatic calculus of 2-categorical string diagrams often appear in categorification, where the 2-categories $\mathcal{A}$ are defined diagrammatically via generating 2-morphisms modulo certain diagrammatic relations. Then the categorification isomorphisms $K(\mathcal{A}) \cong A$ often requires significant effort to demonstrate that the diagrammatic presentation does not collapse. In particular, finding a basis for the spaces of 2-morphisms in $\mathcal{A}$ becomes a fundamental problem. This can be viewed as the higher representation theoretic analog of studying PBW bases and related bases for enveloping algebras. In the same way that those more traditional bases are a basic tool in the study of these algebras, the analogous bases for the spaces of 2-morphisms are equally relevant in higher representation theory.

Higher-dimensional rewriting theory applies the tools of rewriting theory in higher categorical settings. It provides a set of tools for determining when a presentation of a 2-category will be coherent and allows for a determination of a normal form for a given 2-morphism within a given rewriting class, constructively providing bases from a specific presentation of a 2-category. The techniques of higher-dimensional rewriting have been effectively applied in a number of important examples in higher-representation theory $[4, 3, 14, 13]$ including cases where a determination of these bases have eluded experts for some time $[13]$.

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More recently, the field of higher representation theory has taken on the categorification of superalgebras \( A \). Superalgebras no longer lift to monoidal categories or 2-categories. Rather, they lift to so-called monoidal supercategories or 2-supercategories where the familiar interchange law is replaced by a super interchange law that depends on an additional \( \mathbb{Z}_2 \)-grading on 2-morphisms [6, 7].

Monoidal supercategories and 2-supercategories are becoming increasingly common place in modern representation theory with examples ranging from categorification (Heisenberg categories [8, 24], super 2-Kac-Moody algebras [17, 18, 7, 29, 27, 28], affine oriented Brauer-Clifford supercategory [25], Frobenius nilHecke [1]), descriptions of the representation category of Lie superalgebras of Type \( Q \) [5], Deligne categories for periplectic superalgebras [23], and super analogs of modular/fusion tensor categories [2, 31, 41].

Here we extend the theory of higher-dimensional rewriting to the super setting, allowing for these techniques to be applied to monoidal supercategories and 2-supercategories. This allows for a constructive approach to constructing bases in 2-supercategories and provides a suite of techniques for identifying Grothendieck groups needed for categorification. As an application, we prove the non-degeneracy conjecture for the odd categorification of quantum \( \mathfrak{sl}_2 \).

The odd categorification of quantum \( \mathfrak{sl}_2 \) arose as an attempt to provide a higher representation theoretic explanation for a phenomena discovered in link homology theory. Ozsvath, Rasmussen, and Szabo showed that Khovanov’s categorification of the Jones polynomial was not unique [38]. They defined what they called odd Khovanov homology that was similar in many ways to ordinary Khovanov homology (the theories agree when coefficients are reduced modulo two), but rather than being based on 2D TQFT, this theory was based on a strange type of 2D TQFT where signs appear when heights of handles are interchanged [39]. These theories are inequivalent in the sense that each can distinguish knots the other cannot [40]. Since Khovanov homology has a higher representation theoretic interpretation coming from the categorification of quantum \( \mathfrak{sl}_2 \) [33, 43], Ellis, Khovanov, and the third author initiated a program [17] to define odd analogs of quantum \( \mathfrak{sl}_2 \) and related structures. The result was the discovery of odd, noncommutative, analogs of many of the structures appearing in connection with \( \mathfrak{sl}_2 \) categorification including odd analogs of the Hopf algebra of symmetric functions [16, 17], cohomologies of Grassmannians [17] and Springer varieties [34]. Subsequent work has shown these odd categorifications extend to arc algebras and constructions of odd Khovanov homology for tangles [37, 36, 26].

These investigations into odd categorification turned out to be closely connected with parallel investigations into super Kac-Moody algebra categorifications [29, 27, 28], with the odd categorification of \( \mathfrak{sl}_2 \) lifting the rank one super Kac-Moody algebra. These odd categorifications are also closely connected with the theory of covering Kac-Moody algebras [22, 12, 10, 11]. Covering algebras \( U_q,\pi(\mathfrak{g}) \) generalize quantum enveloping algebras, depending on an additional parameter \( \pi \) with \( \pi^2 = 1 \). When \( \pi = 1 \), it reduces to the usual quantum enveloping algebra \( U_q(\mathfrak{g}) \), while the \( \pi = -1 \) specialization recovers the quantum group of a super Kac-Moody algebra. Covering algebras, and the novel introduction of the parameter \( \pi \), allow for the first construction of canonical bases for lie superalgebras [11, 12].

In the rank one case, the \( \pi = 1 \) specialization is \( U_q(\mathfrak{sl}_2) \), while for \( \pi = -1 \) it gives the quantum group \( U_q(\mathfrak{osp}(1|2)) \) associated with the super algebra \( \mathfrak{osp}(1|2) \). Following a categorification of the positive parts of these algebra in [22], Ellis and the third author categorified the full rank one covering algebra proving a conjecture from [12]. In doing so, a 2-supercategory \( \mathcal{U} := \mathcal{U}(\mathfrak{sl}_2) \) was defined [18] for the rank one covering algebra whose Grothendieck group recovers \( U_q,\pi(\mathfrak{sl}_2) \). This categorification was later greatly simplified in [7], where the 2-supercategory formalism was better developed, building off of the work [6]. This covering formalism and the connection with \( \mathfrak{osp}(1|2) \) also informs the realization of odd Khovanov homology in theoretical physics [35].

Despite being able to establish the categorification isomorphism for \( \mathcal{U}(\mathfrak{sl}_2) \), a basis for the space of 2-morphisms was not achieved in [18]. A spanning set was given in [18] and conjectured to form
a basis – the non-degeneracy conjecture for odd categorified $\mathfrak{sl}_2$. The need for a basis result was highlighted in [32] where dg-structures were defined on $\mathcal{U}$ extending differentials on the positive part from [19]. These differentials make the dg-Grothendieck group of its compact derived category isomorphic to the small quantum group $\check{u}(-1)^{\frac{1}{2}}(\mathfrak{sl}_2)$ that plays a role in quantum approaches to the Alexander polynomial. Such dg-structures were conjecturally classified on $\mathcal{U}$ assuming the non-degeneracy conjecture [32, Proposition 7.1]. As a corollary of the basis results achieved here, we prove this conjectured classification is complete.

This paper is organized as follows. In Section 2 we adapt the theory of rewriting in linear 2-categories to the context of super 2-categories. In Section 3 we give a convergent presentation of the 2-supercategory we call odd isotopies. This is analogous to the polygraph of isotopies from [20, 14], but adapted to the context of 2-super Kac Moody algebras. Section 4 presents the 2-supercategory associated to the odd nilHecke algebra; the resulting norma form is shown to recover the basis of the odd nilHecke algebra from [17]. Section 5 gives a presentation of the odd 2-category $\mathcal{U}(\mathfrak{sl}_2)$. Finally, in Section 6 we show that the (3,2)-superpolygraph presenting $\mathcal{U}(\mathfrak{sl}_2)$ is quasi-terminating and confluent modulo. The resulting quasi-normal form proves the non-degeneracy conjecture for $\mathcal{U}(\mathfrak{sl}_2)$.

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2. Super rewriting theory

2.1. 2-supercategories. Here we review Brundan and Ellis [6, 7] notion of a 2-supercategory.

2.1.1. Super vector spaces. Let $k$ be a field with characteristic not equal to 2. A superspace is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$. For a homogeneous element $v \in V$, write $|v|$ for the parity of $v$.

Let $\mathbf{SVect}$ denote the category of superspaces and all linear maps. Note that $\text{Hom}_{\mathbf{SVect}}(V, W)$ has the structure of a superspace since a linear map $f : V \to W$ between superspaces decomposes uniquely into an even and odd map. The usual tensor product of $k$-vector spaces is again a superspace with $(V \otimes W)_0 = V_0 \otimes W_0 \oplus V_1 \otimes W_1$ and $(V \otimes W)_1 = V_0 \otimes W_1 \oplus V_1 \otimes W_0$. Likewise, the tensor product $f \otimes g$ of two linear maps between superspaces is defined by

$$ (f \otimes g)(v \otimes w) := (-1)^{|g||v|} f(v) \otimes g(w). $$

Note that this tensor product does not define a tensor product on $\mathbf{SVect}$, as the usual interchange law between tensor product and composition has a sign in the presence of odd maps

$$ (f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|} (f \circ h) \otimes (g \circ k). $$

This failure of the interchange law depending on parity is the primary structure differentiating monoidal supercategories from their non-super analogs.

If we set $\mathbf{SVect}$ to be the subcategory consisting of only even maps, then the tensor product equips $\mathbf{SVect}$ with a monoidal structure. The map $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$ makes $\mathbf{SVect}$ into a symmetric monoidal category.

2.1.2. Supercategories. Supercategories, superfunctors, and supernatural transformations are defined [6] via the theory of enriched categories by enriching over the symmetric monoidal category $\mathbf{SVect}$. See [30] for a review of the enriched category theory. Unpacking this definition we have the following.
Definition 2.1 (Supercategories).
A supercategory \( \mathcal{C} \) is a category enriched in the monoidal category \( \textbf{SVect} \). This consists of the data of a set \( C_0 \) called the objects, or 0-cells, of \( \mathcal{C} \) and

- For each \( x, y \in C_0 \) a superspace of 1-cells \( \mathcal{C}(x, y) \),
- For each \( x \in C_0 \) an identity assigning superfunctor \( i_x : I \to \mathcal{C}(x, x) \) where \( I \) is the superspace \( k \) concentrated in degree zero.
- For each \( x, y, z \in C_0 \), composition is given by a even linear map
  \[ \star_0^{yz} : \mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \to \mathcal{C}(x, z) \]
such that composition is associative and unital with respect to identities.

Superfunctors are functors between supercategories that give even linear maps on hom spaces. For more details see \([6, \text{Definition 1.1}]\).

2.1.3. 2-supercategories.

Definition 2.2 (2-supercategories).
A 2-supercategory \( \mathcal{C} \) is a category enriched in the monoidal category of supercategories \( \textbf{SCat} \). Namely, a 2-supercategory \( \mathcal{C} \) is the data of a set \( C_0 \) called the objects of \( \mathcal{C} \) and

- For each \( x, y \in C_0 \) a supercategory \( \mathcal{C}(x, y) \),
- For each \( x \in C_0 \) an identity assigning superfunctor \( i_x : I \to \mathcal{C}(x, x) \) where \( I \) is the supercategory with
  - one object \( I \);
  - \( \text{Hom}(I, I) = \mathbb{1} \) where everything is even;
  - Composition is the linear map \( \circ : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1} \) sending \( c \otimes d \to cd \).
- For each \( x, y, z \in C_0 \), a composition superfunctor \( \star_0^{xyz} : \mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \to \mathcal{C}(x, z) \)
such that
  \[ \star_0^{yzw} \circ (\star_0^{yz} \otimes \text{id}_{\mathcal{C}(z, w)}) = \star_0^{yw} \circ (\text{id}_{\mathcal{C}(x, y)} \otimes \star_0^{yzw}) \] (associativity of composition)
  \[ \star_0^{xyz} \circ (i_x \otimes \text{id}_{\mathcal{C}(x, y)}) \circ is_l = \text{id}_{\mathcal{C}(x, y)} = \star_0^{xy} \circ (\text{id}_{\mathcal{C}(x, y)} \times i_b) \circ is_r \] where \( is_l \) and \( is_r \) are the canonical isomorphisms \( C(a, b) \to I \otimes C(a, b) \) and \( C(a, b) \to C(a, b) \otimes I \) (unitors).

The objects of the hom supercategories \( \mathcal{C}(x, y) \) taken over all \( x \) and \( y \) define the set \( C_1 \) of 1-cells of \( \mathcal{C} \) and the 1-cells in \( \mathcal{C}(x, y) \) form the set \( C_2 \) of 2-cells in \( \mathcal{C} \). We use \( \star_1 \) to denote the composition operation in the supercategory \( \mathcal{C}(x, y) \) and call this vertical composition of 2-cells.

For \( p \) an object of the supercategory \( \mathcal{C}(x, y) \) we define the 0-source of \( p \) as \( s_0(p) = x \) and 0-target of \( p \) as \( t_0(p) = y \). The source and target maps in \( \mathcal{C}(x, y) \) give 1-source and 1-target maps \( s_1, t_1 : C_1 \to C_0 \).

The fact that composition is given by a monoidal superfunctor implies that the usual interchange axiom of a 2-category must be replaced by the superinterchange law. That is, given 2-cells \( u : p \Rightarrow q : x \to y, u' : p' \Rightarrow q' : y \to z, v : q \Rightarrow r : x \to y, v' : q' \Rightarrow r' : y \to z \), then the superinterchange equation
\[ (u \star_0 u') \star_1 (v \star_0 v') = (-1)^{|u'||v|} (u \star_1 v) \star_0 (u' \star_1 v') \]
holds in a 2-supercategory \( \mathcal{C} \).

Definition 2.3. A 2-supercategory \( \mathcal{C} \) with one object is a monoidal supercategory. The tensor product operation is given by the \( \star_0 \)-composition and composition of morphisms by \( \star_1 \). The unit for the monoidal structure is given by the identity morphism of the unique object. For more details see \([6, \text{Definition 1.4}]\).

Definition 2.4. A hom-basis for a 2-supercategory \( \mathcal{C} \) is a family of sets \( (B_{p,q})_{p,q \in C_1} \) such that \( B_{p,q} \) is a linear basis of the \( k \)-superspace \( C_2(p, q) \).
The standard 2-categorical string diagrams can be adapted to the super setting. The primary difference is that the interchange law is replaced by the superinterchange. Since odd parity 2-morphisms now skew commute with each other, this means that for 2-supercategories one must be careful with the heights of 2-morphisms. In particular, the superinterchange axiom \((2.3)\) implies that given 2-cells \(u : p \Rightarrow q : x \to y\) and \(v : p' \Rightarrow q' : y \to z\) then

\[
(2.4) \quad (\text{Id}_p \ast_0 v) \ast_1 (u \ast_0 \text{Id}_{q'}) = (-1)^{|u||v|} (u \ast_0 v) = (-1)^{|u||v|} (u \ast_0 \text{Id}_{q'}) \ast_1 (\text{Id}_q \ast_0 v)
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (p) at (0,0) [circular node] {p};
\node (p') at (3,0) [circular node] {p'};
\node (q) at (0,1) [circular node] {q};
\node (q') at (3,1) [circular node] {q'};
\node (x) at (1.5,0) {x};
\node (y) at (1.5,1) {y};
\node (z) at (1.5,2) {z};
\arrow {p} {x} {\ast}
\arrow {q} {y}
\arrow {x} {z}
\arrow {p'} {y}
\arrow {q'} {z}
\end{tikzpicture}
\end{array}
= (-1)^{|u||v|}
\begin{array}{c}
\begin{tikzpicture}
\node (p) at (0,0) [circular node] {p};
\node (p') at (3,0) [circular node] {p'};
\node (q) at (0,1) [circular node] {q};
\node (q') at (3,1) [circular node] {q'};
\node (x) at (1.5,0) {x};
\node (y) at (1.5,1) {y};
\node (z) at (1.5,2) {z};
\arrow {p} {x} {\ast}
\arrow {q} {y}
\arrow {x} {z}
\arrow {p'} {y}
\arrow {q'} {z}
\end{tikzpicture}
\end{array}
= (-1)^{|u||v|}
\begin{array}{c}
\begin{tikzpicture}
\node (p) at (0,0) [circular node] {p};
\node (p') at (3,0) [circular node] {p'};
\node (q) at (0,1) [circular node] {q};
\node (q') at (3,1) [circular node] {q'};
\node (x) at (1.5,0) {x};
\node (y) at (1.5,1) {y};
\node (z) at (1.5,2) {z};
\arrow {p} {x} {\ast}
\arrow {q} {y}
\arrow {x} {z}
\arrow {p'} {y}
\arrow {q'} {z}
\end{tikzpicture}
\end{array}
\]

Remark 2.5. Throughout this paper, we read our compositions cells as is common in higher category theory, just as the first author does in \([14, 13]\). This composition is read backwards from the more prevalent way of reading composition used by Brundan and Ellis \([6, \text{Definition 2.1}]\). That is, \(f \ast_i g\) in this paper translates to \(g \ast_i f\) in \([6]\). So for example, we would have

\[
\lambda + 4 \uparrow \lambda + 2 \uparrow \lambda = \lambda + 2 \uparrow \lambda \uparrow \lambda + 4 \uparrow \lambda + 2
\]

2.1.4. 2-supercategories and free 2-supercategories. We now introduce the notion of superpolygraphs extending the notion of linear polygraphs developed in \([4]\). The theory of linear polygraphs is quite general, providing presentations of linear \((n, p)\)-categories; these are defined using a combination of globular \(n\)-category objects and \(p\)-fold iterative enrichment (see \([4, \text{Definition 2.2.1} \& 2.2.2]\)) so that a linear \((n + 1, p + 1)\)-category is a category enriched in \((n, p)\)-categories, with the base case of linear \((n, 0)\)-category corresponding to an internal \(n\)-category in \(\text{Vect}\). This means that a linear \((1, 1)\)-category is a linear category, a linear \((1, 0)\)-category is a category object in vector spaces, and a linear \((2, 2)\)-category is a linear 2-category. Within the higher dimensional rewriting framework, a linear \((n, p)\)-category is presented by a linear \((n + 1, p)\)-polygraph.

Here we will need to extend several instances of the general linear \((n, p)\)-category framework to the super setting. This is because a \((2, 2)\)-supercategory is just a 2-supercategory as defined in Definition 2.2 and these will be presented by \((3, 2)\)-supercategories. It is not hard to generalize Alleaume’s theory of linear \((n, p)\)-polygraphs to the super setting more generally, but as we do not have interesting examples of these structures in higher dimensions, we focus on unpacking the general inductive definitions in the cases of interest. To ease the exposition in this article, we make use of the definitions and notation of linear \((n, p)\)-polygraphs from \([4, \text{Section 3.2}]\). We start with \((2, 2)\)-supercategories which will be used to form the free 2-supercategory on a given set of generating cells.

Following \([4]\), we will denote by \(P_n^r\) the free strict \(n\)-category on a globular set

\[
P_n \underbrace{s_{n-1}}_{t_{n-1}} \ldots P_p \underbrace{s_{p-1}}_{t_{p-1}} P_{p-1} \underbrace{s_{p-2}}_{t_{p-2}} \ldots \underbrace{s_0}_{t_0} P_0
\]

Definition 2.6. A \((2, 2)\)-superpolygraph is a collection \(P = (P_0, P_1, P_2)\) of sets equipped with set maps \(s_k, t_k : P_{k+1} \to P_k^r\) for \(k < 2\), such that:

- \((P_0, P_1)\) with \(s_j, t_j\) for \(j < 1\) is a 1-polygraph as defined in \([4, \text{Section 3}]\);
• $P_2$ is a super globular extension of the free 1-category $P_1^*$ on $(P_0, P_1)$, that is a $\mathbb{Z}_2$-graded set equipped with source and target maps $s_1, t_1 : P_2 \rightarrow P_1^*$ satisfying globular relations $s_0 \circ s_1 = s_0 \circ t_1$ and $t_0 \circ s_1 = t_0 \circ t_1$.

We sometimes refer to $(2, 2)$-superpolygraphs as 2-superpolygraphs for convenience.

**Definition 2.7.** A pasting diagram on $(2, 2)$-superpolygraph $P = (P_0, P_1, P_2)$ is a formal composite of elements of $P_2^r := P_2 \cup \{1_x : x \Rightarrow x \mid x \in P_1^r\}$ of the form:

- $u$ for $\alpha \in P_2'$,
- $u \star_1 v$ for $u, v$ pasting diagrams on $P$ with $t_1(u) = s_1(v)$,
- $u \star_0 v$ for $u, v$ pasting diagrams with $t_0 s_1(u) = s_0 s_1(v)$.

Such a composite inherits a $\mathbb{Z}_2$-grading determined by the parity of elements in $P_2$ as follows: $|1_u| = 0$, and $|u \star_k v| = |u| + |v|$ for $k = 0, 1$. We define a source $s_1(D)$ and target $t_1(D)$ of a composition $D$ iteratively by

- $s_1(u)$ and $t_1(u)$ are the normal 1-source and 1-target for $u \in P_2'$,
- $s_1(u \star_1 v) = s_1(u)$, $t_1(u \star_1 v) = t_1(v)$,
- $s_1(u \star_0 v) = s_1(u) \star_0 s_1(v)$, $t_1(u \star_0 v) = t_1(u) \star_0 t_1(v)$.

Then pasting diagrams on $P$ are such formal compositions quotiented by associativity of $\star_0$ and $\star_1$:

$$u \star_k (v \star_k w) = (u \star_k v) \star_k w \quad \text{for} \ k = 0, 1$$

We can now define the free $(2, 2)$-supercategory on a $(2, 2)$-superpolygraph by adapting the definition [21, Def. 2.4.3]. A $(2, 2)$-supercategory is the same thing as a 2-supercategory, so we will interchange freely between these two terminologies.

**Definition 2.8.** Let $P$ be a 2-supercategory. The free $(2, 2)$-supercategory over $P$, denoted by $P_2^s$, is defined as follows:

- the 0-cells of $P_2^s$ are the 0-cells of $P_0$,
- for all 0-cells $x$ and $y$ of $P$, $P_2^s(x, y)$ is the supercategory whose
  - 0-cells are the 1-cells $f \in P_1^s(x, y)$, where $P_1^s$ is the free 1-category generated by the 1-polygraph $(P_0, P_1)$,
  - set of 1-cells is the disjoint union of superspaces $P_2^s(p, q) := \text{Past}(p, q)$ where $\text{Past}(p, q)$ is the free superspace on the set of pasting diagrams with 1-source $p$ and 1-target $q$ for any $p, q \in P_1^s(x, y)$,

and quotiented by the congruence generated by the cellular extensions made of all the possible

$$(u \star_0 v) \star_1 (u' \star_0 v') = -1|v||u'|(u \star_1 u') \star_0 (v \star_1 v'), \quad 1_{s_1(u)} \star_1 u = u = u \star_1 1_{t_1(u)}$$

for all pasting diagrams $u, v, u', v'$ composable in this way. The 0-cells (resp. 1-cells) of the hom supercategories $P_2^s(x, y)$ will be the 1-cells (resp. 2-cells) of $P_2^s$. For any 0-cells $p, q$ and $r$ in $P_2^s(x, y)$, there is an even linear map $\star_1 : P_2^s(p, q) \otimes P_2^s(q, r) \rightarrow P_2^s(p, r)$ given by gluing two 2-cells $u : p \Rightarrow q$ and $v : q \Rightarrow r$ in $P_2^s$ along their common 1-cell $q$. For any 0-cells $x, y, z \in P_0$, there is a composition map $\star_0 : P_1^s(x, y) \otimes P_1^s(y, z) \rightarrow P_1^s(x, z)$ defined as the composition map on $P_1^s$. Let $p, q, r, s$ be any 0-cells in the supercategories $P_2^s(x, y)$ and $P_2^s(y, z)$ respectively. Then there is an even linear map $\star_0 : P_2^s(p, q) \otimes P_2^s(r, s) \rightarrow P_2^s(p \star_0 r, q \star_0 s)$ given by gluing two 2-cells $u : p \Rightarrow q$ and $v : r \Rightarrow s$ in $P_2^s$ along their common 0-cell $y$. The $\star_0$ maps above give the data of a composition superfunctor $\star_0 : P_2^s(x, y) \otimes P_2^s(y, z) \rightarrow P_2^s(x, z)$. For any 1-cells $u_1, \ldots, u_m$ in $P_2^s(x, y)$ and $v_1, \ldots, v_n$ in $P_2^s(y, z)$, these compositions satisfy

$$(u_1 \star_1 \cdots \star_1 u_m) \star_0 (v_1 \star_1 \cdots \star_1 v_n) = (u_1 \star_0 s(v_1)) \star_1 \cdots \star_1 (u_m \star_0 s(v_1)) \star_1 (t(u_m) \star_0 v_1) \star_1 \cdots \star_1 (t(u_m) \star_0 v_n)$$
Remark 2.9. If the $\mathbb{Z}_2$-grading of $P_2$ in a $(2,2)$-superpolygraph $P$ is all concentrated in even parity, then a $(2,2)$-superpolygraph is just a linear $(2,2)$-polygraph [4, Definition 3.2.3], and the free $(2,2)$-supercategory $P_s^s$ generated by $P$ will be a linear $(2,2)$-category $P_s^f$ defined as in [4, Definition 3.2.4].

Notation 2.10. Let $P$ be a 2-superpolygraph. Consider a subset $Q_2$ of the set $P_2$ of generating 2-cells. For a given 2-cell $u$ of $P_s^s$, denote by $|u|_{Q_2}$ the number of generating cells of $Q_2$ appearing in $u$. When $Q_2 = \{w\}$ is a singleton, $|u|_{Q_2}$ counts the number of occurrences of the generating 2-cell $w$ in $u$.

The notion of monomial in a free 2-supercategory is defined by disregarding the $\mathbb{Z}_2$-grading and utilizing the definition of monomial for free linear 2-categories from [4, Definition 4.1.4].

Definition 2.11. Let $P = (P_0, P_1, P_2)$ be a 2-superpolygraph and let $U(P)$ be the linear $(2,2)$-polygraph obtained by forgetting the parity of the elements $P_2$. Then a monomial of the free 2-supercategory $P_s^s$ is a monomial of the free linear $(2,2)$-category $U(P)_f^s$ equipped with a parity determined by $P_2$.

The set of monomials of $U(P)_2^f$ is the set of 2-cells of the free 2-category $U(P)_2^s$, so equipping each element in the set of 2-cells of $U(P)_2^s$ with the parity determined by $P_2$ gives the monomials of $P_s^s$.

Remark 2.12. For a 2-superpolygraph $P$, let $A$ be a set of 2-cells of $U(P)_2^s$ containing one element from each exchange equivalence class of pasting diagrams of $U(P)$, where 2-cells $u, v \in U(P)_2^s$ are in the same exchange equivalence class if $u = v$ via the exchange and identity relations. Then every 2-cell of $U(P)_2^s$ is equal to a unique element in $A$ by exchange and identity relations, so $A$ is the set of monomials of $U(P)_2^f$ and a linear combination of elements in $A$ is a monomial decomposition. We then obtain a set $B$ of 2-cells of $P_s^s$ by assigning to each element of $A$ the parity determined by $P_2$. Then $B$ is the set of monomials of $P_s^s$ and so a linear combination of elements in $B$ is a monomial decomposition.

It is known from [4, Definition 4.1.4] that every 2-cell of $U(P)_2^f$ has a unique monomial decomposition. This is true because there are no relations in $U_2^f$ other than the exchange and identity relations and no two elements of $A$ are related via these relations. We now prove a lemma that gives this result for 2-supercategories using similar principles.

Lemma 2.13. Every 2-cell in the free 2-supercategory $P_s^s$ generated by a 2-superpolygraph $P$ admits a unique monomial decomposition.

Proof. Let $A$ and $B$ be the sets of monomials of $U(P)_2^f$ and $P_s^s$ described in Remark 2.12. If $u = \pm v$ in $B$ by superinterchange, then there are corresponding elements $u'$ and $v'$ in $A$ that satisfy $u' = v'$ by exchange and identity relations. But we know that no two elements of $A$ are equal, so there are no two elements of $B$ that are scalar multiplies of each other by superinterchange and identity relations. Hence, $B$ is a linearly independent set of 2-cells because $P_s^s$ has no other relations other than the superinterchange and identity relations. Furthermore, every pasting diagram of $P$ is equal as a 2-cell by the superinterchange and identity relations to an element in $B$ up to a sign, so every 2-cell admits a decomposition as a linear combination of elements of $B$ by construction. Hence, every 2-cell of $P_s^s$ admits a unique decomposition into a linear combination of elements of $B$. □

Given a 2-cell $u$ of the free 2-supercategory $P_s^s$ expressed as a linear combination of monomials $u = \sum \lambda_i u_i$, we set

$$\text{Supp}(u) := \{u_i \mid u_i \text{ appears in the monomial decomposition of } u\}.$$
Definition 2.14. Let $C$ be a 2-(super)category. For a $k$-cell $f$ in $C$, with $k = 1, 2$, define the boundary of $f$ as the ordered pair of $(k - 1)$-cells defined by $\partial f := (s_{k-1}(f), t_{k-1}(f))$. A $k$-sphere of $C$ is a pair of $k$-cells $(f, g)$ such that $\partial f = \partial g$. That is, $s_{k-1}(f) = s_{k-1}(g)$ and $t_{k-1}(f) = t_{k-1}(g)$.

Let us recall some key definitions needed to prove termination using the derivation method from [20]: that of a context of a 2-category.

Definition 2.15. A context of a 2-category $C$ is a pair $(S, c)$ where $S$ is a 1-sphere of $C$ and $c$ is a 2-cell in the 2-category $C[S]$, defined as $C$ extended by a formal 2-cell tiling the sphere $S$ as in [20, Section 1.3] such that this 2-cell occurs exactly once in $c$. In other words, it is a 2-cell $c$ that contains one 'hole' with boundary the sphere $S$.

When $C$ is a 2-category freely generated by a 2-polygraph, a context of $C$ has the form $c = m_1 \ast_1 (m_2 \ast_0 S \ast_0 m_3) \ast_1 m_4$ where $m_i$ are monomials of $C$. For a 2-cell $u$ in $C_2$ such that $\partial u = S$, we denote by $c[u]$ the 2-cell $m_1 \ast_1 (m_2 \ast_0 u \ast_0 m_3) \ast_1 m_4$ in $C_2$.

Definition 2.16. Let $C$ be a 2-category. Then define the category of contexts $C(C)$ as the category with

- Objects: 2-cells in $C$
- Morphisms: $\text{Hom}(u, v)$ is the set of contexts $(\partial u, c)$ of $C$ such that $c[u] = v$.
- Composition: If $x = (\partial u, c) \in \text{Hom}(p, q)$ and $y = (\partial v, c') \in \text{Hom}(q, r)$, then $x \circ y := (\partial u, c' \circ c) \in \text{Hom}(p, r)$ where $(c' \circ c)[w] := c'[w]$. For any object $u$, there is an identity morphism $1_u := (S = \partial u, c = S \in C[S])$. For $w \in C_2$ with $\partial w = \partial u$, $\partial u[w] = w$, so $(c \circ \partial u)[w] = c[w]$.

In order to define rewriting steps of $(3, 2)$-polygraph we need to extend Definition 2.15 to the case of contexts of 2-supercategories.

Definition 2.17. A context of a 2-supercategory $C$ is a pair $(S, c)$ where $S := (p, q)$ is a 1-sphere of $C$ and $c$ is a 2-cell in the 2-supercategory $C[S]$, defined as the 2-supercategory $C$ extended with additional even 2-cells $\lambda_w$, for $\lambda \in k$, tiling the sphere $S$ such that one of these 2-cells appears exactly once in $c$.

In the case where $C$ is freely generated by a 2-supercategory, that is $C = P^{s}_2$, a context of $P^{s}_2$ has the form $c = \lambda m_1 \ast_1 (m_2 \ast_0 S \ast_0 m_3) \ast_1 m_4 + u$ for some scalar $\lambda$, monomials $m_i$ in $P^{s}_2$ and a 2-cell $u$ in $P^{s}_2$. For a 2-cell $v$ of $P^{s}_2$ with $\partial v = (p, q)$, denote by $c[v]$ the 2-cell $\lambda m_1 \ast_1 (m_2 \ast_0 v \ast_0 m_3) \ast_1 m_4 + u$ in $P^{s}_2$.

2.2. $(3, 2)$-supercategories. We now define $(3, 2)$-supercategories as a means of presenting $(2, 2)$-supercategories. This extends linear $(3, 2)$-polygraphs from [4, Definition 3.2.4].

Definition 2.18. A $(3, 2)$-supercategory is the data of $P = (P_0, P_1, P_2, P_3)$ where $(P_0, P_1, P_2)$ is a 2-supercategory and $P_3$ is a super globular extension of the free 2-supercategory $P^{s}_2$ on $(P_0, P_1, P_2)$, that is $P_3$ is a $\mathbb{Z}_2$-graded set equipped with even set maps $s_2, t_2 : P_3 \to P^{s}_2$ such that $s_1 \circ s_2 = s_1 \circ t_2$ and $t_1 \circ s_2 = t_1 \circ t_2$ where $s_1, t_1$ are the 1-source and 1-target set maps of $P^{s}_2$.

The evenness of the set maps $s_2$ and $t_2$ in the definition of a $(3, 2)$-supercategory implies they preserve the $\mathbb{Z}_2$ parity, so that the elements in $P_3$ with even parity have even sources and targets, while the elements in $P_3$ with odd parity have odd source and target.

2.3. $(3, 2)$-supercategory.

Definition 2.19. A $(1, 0)$-supercategory is a category object in $\mathbf{SVect}$. A $(2, 1)$-supercategory is a category enriched in $(1, 0)$-supercategories. A $(3, 2)$-supercategory is a category enriched in $(2, 1)$-supercategories.

We will unpack these definitions in the cases of interest below.
2.3.1. Free $(3,2)$-supercategory.

**Definition 2.20.** A pasting diagram on a $(3,2)$-superpolygraph $P = (P_0, P_1, P_2, P_3)$ is a formal composite of elements of the form

- $\alpha$ for $\alpha \in P_0^0 := P_3 \cup \{1_u : u \Rightarrow u \mid u \in P_2^s\}$
- $f \star_2 g$ for pasting diagrams $f, g$ with $t_2(f) = s_2(g)$
- $f \star_1 g$ for pasting diagrams $f, g$ with $t_1t_2(f) = s_1s_2(g)$
- $f \star_0 g$ for pasting diagrams $f, g$ with $t_0t_1t_2(f) = s_0s_1s_2(g)$

quotiented by associativity relations for $\star_0, \star_1$ and $\star_2$:

$$f \star_k (g \star_k h) = (f \star_k g) \star_k h \text{ for } 0 \leq k \leq 2.$$  

The source $s_2(f)$ and target $t_2(f)$ of a such a composition is defined by

- $s_2(f \star_2 g) = s_2(f)$, $t_2(f \star_2 g) = t_2(g)$
- $s_2(f \star_i g) = s_2(f) \ast_i s_2(g)$ for $i \in \{0, 1\}$
- $t_2(f \star_i g) = t_2(f) \ast_i t_2(g)$ for $i \in \{0, 1\}$

The parity of such a composition is defined by $|f \star_k g| = |f| + |g|$ for $k = 0, 1$ and $|f \star_2 g| = |f| = |s_2(f)|$.

**Definition 2.21.** Let $P = (P_0, P_1, P_2, P_3)$ be a $(3,2)$-superpolygraph. The free $(3,2)$-supercategory generated by $P$, denoted by $P_3^\ast$, is defined as follows: its 0-cells are the 0-cells of $P_0$. For any 0-cells $x$ and $y$ of $P$, we define the Hom $(1,0)$-supercategory $P_3^\ast(x, y)$ as follows:

- its 0-cells are the 1-cells $p \in P_1^1(x, y)$, where $P_1^1$ is the free 1-category generated by the 1-polygraph $(P_0, P_1)$,
- for any 0-cells $p$ and $q$ in $P_3^\ast(x, y)$, let us define the 2Hom $(1,0)$-supercategory $P_3^\ast(p, q)$ as follows:
  - its set of 0-cells $P_2(p, q)$ is given by the superspace $P_2^s(p, q)$ of 2-cells of the free $(2,2)$-supercategory $P_2^s$ with 1-source $p$ and 1-target $q$,
  - its set of 1-cells $P_3(p, q)$ is the superspace given by the free superspace on $(3,2)$-pasting diagrams with 1-source $p$ and 1-target $q$ quotiented by relations

$$\left( f \star_i g \right) \ast_j h \ast_k \left( g \star_j k \right) = (-1)^{|i| \ast |k|} \left( f \star_j h \right) \ast_i \left( g \star_j k \right), \quad 1_{s_1(f)} \ast f = f = f \ast 1_{t_1(f)}$$  

for any $0 \leq i < j < k \leq 2$ and for all pasting diagrams $f, g, k, h$ composable in this way.

The $\star_0$-composition for 1-cells and $\ast_1$-composition for 2-cells of $P_3^\ast$ are defined as in the free $(2,2)$-supercategory $P_2^s$. For any 0-cells $p, q$ in $P_3^\ast(x, y)$ and $r, s$ in $P_3^\ast(y, z)$, there is an even linear map $\star_0 : P_3(p, q) \otimes P_3(r, s) \to P_3(p \star_0 r, q \ast_0 s)$ given by gluing two 3-cells along their common 0-cell $y$. For any 0-cells $p, q, r$ in $P_3^\ast(x, y)$, there is an even linear map $\star_1 : P_3(p, q) \otimes P_3(q, r) \to P_3(p, r)$ given by gluing two 3-cells along their common 1-cell $y$. For any 0-cells $p, q, r$ in $P_3^\ast(x, y)$, there is an even linear map $\star_2 : P_3(p, q) \times P_3(p, q) \to P_3(p, q)$ given by gluing two 1-cells $f : i \Rightarrow r$ and $g : r \Rightarrow w$ of the 2Hom $(1,0)$-supercategory $P_3^\ast(p, q)$ along their common 0-cell $v \in P_3(p, q)$. For any 2-cells $f_1, \ldots, f_n, g_1, \ldots, g_n$ in $P_3^\ast(x, y)$, these compositions satisfy

$$\left( f_1 \ast_2 \cdots \ast_2 f_m \right) \ast_1 \left( g_1 \ast_2 \cdots \ast_1 g_n \right) = \left( f_1 \ast_1 s(g_1) \right) \ast_2 \cdots \ast_2 \left( f_m \ast_1 s(g_1) \right) \ast_2 \left( t(f_m) \ast_1 g_1 \right) \ast_2 \cdots \ast_2 \left( t(f_m) \ast_1 g_n \right).$$

**Remark 2.22.** When the $\mathbb{Z}_2$-grading on the sets $P_2$ and $P_3$ are concentrated in degree zero, then a $(3,2)$-superpolygraph and $(3,2)$-supercategory reduce to a linear $(3,2)$-polygraphs and linear $(3,2)$-categories from [4].
2.4. Presenting 2-supercategories by (3,2)-superpolygraphs.

Definition 2.23. Let $P$ be a (3,2)-superpolygraph, and let $P_3^s$ be the free (3,2)-supercategory on $P$. Define an equivalence relation $\equiv$ on $P_3^s$ by

$$u \equiv v \quad \text{if there is a 3-cell } f \in P_3^s \text{ such that } s_2(f) = u \text{ and } t_2(f) = v.$$ 

We say that a 2-supercategory $C$ is presented by the (3,2)-superpolygraph $P$ if $C$ is isomorphic to the quotient 2-supercategory $P_2^s/\equiv$.

Definition 2.24. A rewriting step of a (3,2)-superpolygraph $P$ is a 3-cell $c[\alpha] \in P_3^s$ of the form

$$c[\alpha]: c[s_2(\alpha)] \to c[t_2(\alpha)]$$

where $\alpha \in P_3$ is a generating 3-cell, and $c = \lambda m_1 \ast_1 (m_2 \ast_0 S \ast_0 m_3) \ast_1 m_4 + u$ is a context of $P_3^s$ such that the monomial $m_1 \ast_1 (m_2 \ast_0 s_2(\alpha) \ast_0 m_3) \ast_1 m_4$ does not appear in the monomial decomposition of $u$. A rewriting sequence is a sequence of rewriting steps. A 3-cell $f$ of $P_3^s$ is called positive if it is an identity 3-cell or a $\ast_2$-composition $f = f_1 \ast_2 \cdots \ast_2 f_n$ of rewriting steps of $P$. The length of a positive 3-cell $f$ in $P_3^s$, denoted by $t(f)$, is the number of rewriting steps of $P$ needed to write $f$ as a $\ast_2$-composition of these rewriting steps. As a consequence, the terminologies rewriting path of $P$ (resp. rewriting step of $P$) and positive 3-cell of $P_3^s$ (resp. positive 3-cell of $P_3$ of length 1) can both be used to represent the same notion.

2.5. Termination and confluence. A branching (resp. local branching) of a (3,2)-superpolygraph $P$ is a pair of rewriting sequences (resp. rewriting steps) of $P$ which have the same 2-cell as 2-source. Such a branching (resp. local branching) is confluent if it can be completed by rewriting sequences $f'$ and $g'$ of $P$ as follows:

A (3,2)-superpolygraph $P$ is said to be:

i) left-monomial if for any $\alpha$ in $P_3$, $s_2(\alpha)$ is a monomial of $P_3^s$.

ii) terminating if there is no infinite rewriting sequences in $P$.

iii) quasi-terminating if for each sequence $(u_n)_{n \in \mathbb{N}}$ of 2-cells such that there is a rewriting step from $u_n$ to $u_{n+1}$ for each $n$ in $\mathbb{N}$, the sequence $(u_n)_{n \in \mathbb{N}}$ contains an infinite number of occurrences of the same 2-cell.

iv) confluent (resp. locally confluent) if all the branchings (resp. local branchings) of $P$ are confluent.

v) convergent if it is both terminating and confluent.

From now on, we will only consider left-monomial (3,2)-superpolygraphs. Let us fix a (3,2)-superpolygraph $P$. A normal form of $P$ is a 2-cell $u$ that cannot be rewritten by any rewriting step of $P$. When $P$ is terminating, any 2-cell admits at least one normal form, and exactly one when it is also confluent. A quasi-normal form is a 2-cell $u$ such that for any rewriting step from $u$ to another 2-cell $v$, there exists a rewriting sequence from $v$ to $u$.

If $P$ is a terminating (3,2)-superpolygraph, Newman’s lemma states that its confluence is equivalent to its local confluence. Following [4, Section 4], branchings of a (3,2)-superpolygraph may be divided into four families: aspherical branchings, additive branchings, Peiffer branchings and overlapping branchings. A critical branching of $P$ is an overlapping local branching that is minimal for the order $\sqsubseteq$ on monomials of $P_2^s$ defined by $f \sqsubseteq g$ if there exists a context $c$ of the free 2-category $U(P)^s_2$ generated by $P$ such that $g = c[f]$. 

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Following [4], we prove that a terminating (3, 2)-superpolygraph is locally confluent if and only if its critical branchings are confluent. Indeed, the proofs of [4, Lemma 4.2.12 & Theorem 4.2.13] would remain the same: first proving that additive branchings are confluent and then prove that confluence of critical branchings implies confluence of all the overlapping branchings using implicit rewriting modulo superinterchange instead of the usual interchange. Moreover, with the definition of monomials from Definition 2.11, we obtain that if \( P \) is a convergent (3, 2)-polygraph presenting a (2, 2)-supercategory \( C \), then the set of monomials in normal form with respect to \( P \) gives a hom-basis of \( C \) in the sense of Definition 2.4. Indeed, the same linear algebra argument as in the proof of [4, Prop. 4.2.15] would apply in this context since monomials of \( P^s_2 \) are defined in such a way that a 2-cell of \( P^s_2 \) admits a unique monomial decomposition.

**2.5.1. Termination by Derivation.** Recall from [20] a method to prove termination of a 3-polygraph using derivations of a 2-category. The first author extended this method to the setting of linear 2-categories in [14, 13], giving a method to prove termination of a (3, 2)-linear polygraph using derivations of a 2-category. Inspired by this extension to the linear setting, we describe a method to prove termination of a (3, 2)-superpolygraph using derivations of a 2-category in this subsection.

The linear extension of proving termination by derivation from [14, 13] utilizes monomials of a linear (2, 2)-category, which up to parity, are the same as monomials of supercategories. This suggests the following definition.

**Definition 2.25.** Let \( P = (P_0, P_1, P_2, P_3) \) be a (3, 2)-superpolygraph. Then define \( U(P) \) as the linear (3, 2)-polygraph with

1. \( U(P)_i = P_i \) except that we forget the parity of elements
2. The same source and target maps as in \( P \) (forgetting parity of elements sends map \( s_2, t_2 : P_3 \to P^s_2 \) to maps \( s_2, t_2 : U(P)_3 \to U(P)^s_2 \)).

**Definition 2.26.** Let \( C \) be a 2-category. A \( C \)-module is a functor \( M : C(C) \to \text{Ab} \) where \( C(C) \) is the category of context from Definition 2.16 and \( \text{Ab} \) is the category of abelian groups.

Let \( \text{Ord} \) denote the category of partially ordered sets and monotone maps. This is a monoidal category under the cartesian product. As in [20], thinking of \( \text{Ord} \) as a 2-category with one object, we build examples of \( C \)-modules as follows:

**Definition 2.27.** Let \( C \) be a 2-category, \( G \) be an internal abelian group in \( \text{Ord} \), and \( X : C \to \text{Ord} \) and \( Y : C^{\text{op}} \to \text{Ord} \) be 2-functors. Then we can define \( C \)-module \( M := M_{X,Y,G} \) as follows:

- Every 2-cell \( u : p \Rightarrow q \) in \( C \) is sent to the abelian group of morphisms \( M(u) = \text{Hom}_{\text{Ord}}(X(p) \times Y(q), G) \).
- If \( p, q \) are 1-cells of \( C \) and \( c = p' \star_0 S \star_0 q' \) is a context from \( u : p \Rightarrow q \) to \( v' \star_0 u \star_0 q' \), then \( M(c) \) sends a morphism \( a : X(p) \times Y(q) \to G \) in \( \text{Ord} \) to the morphism \( X(p') \times X(q') \times Y(p') \times Y(q') \times G \) in \( \text{Ord} \) sending \( (x', x, x'', y', y, y'') \to a(x, y) \).
- If \( u : p' \Rightarrow p, w : q \Rightarrow q' \), are 2-cells and \( c = u \star_1 x \star_1 w \) is a context from a 2-cell \( v : p \Rightarrow q \) to \( u \star_1 v \star_1 w \), then \( M(c) \) sends morphism \( a : X(p) \times Y(q) \to G \) in \( \text{Ord} \) to the morphism to \( a \circ (X \times Y) \), which is the map \( X(p') \times Y(q') \to G \) sending \( (x, y) \to a(X(q)(x), Y(h)(y)) \).

When \( C = U(P)^s_2 \) is freely generated by a 2-polygraph \( U(P)_{\leq 2} \), then such a \( C \)-module is uniquely determined by \( X(p) \) and \( Y(p) \) for \( p \in P_1 \) and the morphisms \( X(u) : X(p) \to X(q) \) and \( Y(u) : Y(q) \to Y(p) \) for every generating 2-cell \( u : p \Rightarrow q \) in \( U(P)_2 \).

We also recall the notion of a derivation of a 2-category:

**Definition 2.28.** A derivation of a 2-category \( C \) into a \( C \)-module \( M \) is a map sending every 2-cell \( u \) in \( C \) to an element \( d(u) \in M(f) \) such that

\[
d(u \star_i v) = u \star_i d(v) + d(u) \star_i v
\]
where \( u \star_i d(v) = M(u \star_i x)(d(v)) \) and \( d(u) \star_i v = M(x \star_i v)(d(u)) \).

Then following [14], we get the following result:

**Theorem 2.29.** Let \( P \) be a \((3, 2)\)-superpolygraph and \( U(P) \) be the linear \((3, 2)\)-polygraph defined in Definition 2.25. Then if there exist

1. Two 2-functors \( X : U(P)_2^* \to \text{Ord} \) and \( Y : (U(P)_2^*)^{\text{op}} \to \text{Ord} \) such that for every 1-cell \( p \) in \( P_1 \), the sets \( X(p) \) and \( Y(p) \) are non-empty and for every generating 3-cell \( \alpha \) in \( P_3 \), the inequalities \( X(s_2(\alpha)) \geq X(h) \) and \( Y(s_2(\alpha)) \geq Y(h) \) hold for every \( h \in \text{Supp}(t_2(\alpha)) \).
2. An abelian group \( G \) in \( \text{Ord} \) whose addition is strictly monotone in both arguments and such that every decreasing sequence of non-negative elements of \( G \) is stationary.
3. A derivation of \( U(P)_2^* \) into the \( U(P)_2^* \)-module \( M_{X,Y,G} \) such that for every 2-cell of \( u \in U(P)_2^* \), we have \( d(u) \geq 0 \), and for every generating 3-cell \( \alpha \) in \( P_3 \), \( d(s_2(\alpha)) > d(h) \) for every \( h \in \text{Supp}(t_2(\alpha)) \).

Then the \((3, 2)\)-superpolygraph \( P \) terminates.

**Proof.** Like in the linear setting, the proof works in a similar manner to the proof of termination by derivation given in [20, Theorem 4.2.1]. \( \square \)

**Remark 2.30.** Usually we take internal abelian group \( G = \mathbb{Z} \) and consider derivations with values into a \( C \)-module of the form \( M_{X,Y,G} \). We often consider \( C \)-module where \( X \) or \( Y \) are the trivial 2-functor and write \( M_{X,*,G} \) or \( M_{*,Y,G} \).

2.5.2. **Termination by context stable maps.** Derivations were introduced in order to define termination orders by requiring some inequalities on sources and targets of generating 3-cells; the properties of derivations make this order stable by context of 2-categories. Instead of a derivation, we can equivalently use maps \( d : C_2 \to \mathbb{N} \) that are stable under context, that is \( d(a) \geq d(b) \) implies \( d(c[a]) \geq d(c[b]) \) for any context \( c \) of \( C \).

2.5.3. **Derivation by steps.** The process of proving termination can be achieved in steps, proving termination for subsets of generating 3-cells at a time.

**Lemma 2.31.** Let \( P = (P_0, P_1, P_2, P_3) \) be a superpolygraph with \( P_3 = A \sqcup B \) and let \( d : C_2 \to \mathbb{N} \) be a context stable map satisfying the inequalities

\[
d(s_2(f)) > d(t_2(f)) \quad \text{for } f \in A, \quad d(s_2(g)) \geq d(t_2(g)) \quad \text{for } g \in B.
\]

Then \( P \) terminates if \( P' = (P_0, P_1, P_2, B) \) terminates.

**Proof.** Suppose \( P' \) terminates and

\[
v_1 \xrightarrow{f_1} v_2 \xrightarrow{f_2} v_3 \to \ldots
\]

is an infinite rewriting sequence in \( P \). Define \( d(u) := \max\{d(u') \mid u' \in \text{Supp}(u)\} \). Then since \( P' \) terminates, there are an infinite number of \( f_i \) that are in \( A \). Then consider the non-increasing infinite sequence \( (d(v_n))_{n \in \mathbb{N}} \) of natural numbers. The inequality is strict for \( f_n \in A \) and \( \text{Supp}(u) \) is a finite set, so \( d(v_n) \) must decrease after a finite number of rewriting steps from \( A \). Hence, there is an infinite subsequence \( (d(v_{n_k}))_{k \in \mathbb{N}} \) of natural numbers that is strictly decreasing giving a contradiction. \( \square \)

Lemma 2.31 allows us to prove termination, progressively eliminating 3-cells. When one of these steps is constructed from a context stable map arising from a derivation, we will need the conditions

\[
X(s_2(f)) \geq X(t_2(f)), \quad Y(s_2(f)) \geq Y(t_2(f)) \quad \text{for all } f \in P_3
\]

to hold at each step for the 2-functors used in defining the derivations.
One can view the process of proving derivations in steps as defining a termination lexicographic order. If we denote the context stable map used at step \( j \) by \( d_j \), then a \( k \) step procedure amounts to considering one large context stable map \( d = (d_1, d_2, \ldots, d_k) \) satisfying
\[
(d_1(s_2(\alpha)), d_2(s_2(\alpha)), \ldots, d_k(s_2(\alpha))) \succ_{\text{lex}} (d_1(t_2(\alpha)), d_2(t_2(\alpha)), \ldots, d_k(t_2(\alpha)))
\]
for any generating 3-cell \( \alpha \) of the (3,2)-superpolygraph \( P \), where \( \succ_{\text{lex}} \) denotes the lexicographic order on \( \mathbb{N}^k \). Each of these components being stable by context, we thus obtain that if there is an infinite rewriting sequence
\[
u_1 \rightarrow \nu_2 \rightarrow \ldots
\]
with respect to \( P \), this yields an infinite strictly decreasing sequence
\[
(d_1(\nu_1), d_2(\nu_1), \ldots, d_k(\nu_1)) \succ_{\text{lex}} (d_1(\nu_2), d_2(\nu_2), \ldots, d_k(\nu_2)) \succ_{\text{lex}} \ldots
\]
for the lexicographic order on \( \mathbb{N}^k \), which is impossible since this order is well-founded.

2.6. (3,2)-superpolygraphs modulo. In this section we introduce the notion of rewriting modulo in 2-supercategories extending the work of the second author [14, 13]. This is tool for breaking termination and confluence arguments into incremental steps. We utilize this to first prove that ‘odd isotopies’ have a convergent presentation. We then study presentations of the odd 2-category \( \Omega \) modulo these odd isotopies.

A (3,2)-superpolygraph modulo is a data \((R, E, S)\) made of two (3,2)-superpolygraphs \( R \) and \( E \) such that \( R \leq_1 = E \leq_1 \) and \( E_2 \subseteq R_2 \), and a cellular extension \( S \) of the free 2-supercategory generated by \( R \leq_2 \) satisfying \( \tilde{R} \subseteq S \subseteq E_{R \leq_2} \), where the cellular extension \( E_{R \leq_2} \) is made of elements of triples of the form \((e, f, e')\) for 3-cells \( e, e' \) in \( E_2^s \) and a rewriting step \( f \) of \( R \) such that \( t_2(e) = s_2(f) \) and \( t_2(f) = s_2(e') \) as follows:

\[
\begin{array}{c}
\text{u} \\
\xrightarrow{f} \\
\text{v}
\end{array}
\]

The rewriting sequences with respect to \( E_{R \leq_2} \) thus correspond to application of rewriting sequences of \( R \) by allowing sources and targets of 3-cells to be transformed by a zig-zag sequence of rewriting steps of \( E \). We refer to [15] for a detailed definition of higher-dimensional polygraphs modulo. Given a (3,2)-superpolygraph modulo \((R, E, S)\), the data of \( R \leq_2 \) and \( S \) gives a (3,2)-superpolygraph, that we denote by \( S \) in the sequel.

2.6.1. Branchings and confluence modulo. A branching modulo \( E \) of a (3,2)-superpolygraph \((R, E, S)\) is a triple \((f, e, g)\) where \( f \) and \( g \) are rewriting sequences of \( S \), with \( f \) non-identity, and \( e \) is a 3-cell in \( E_3^s \) such that \( s_2(f) = s_2(e) \) and \( s_2(g) = t_2(e) \). Such a branching modulo is confluent modulo \( E \) if there exist rewriting sequences \( f' \) and \( g' \) of \( S \), and a 3-cell \( e' \) in \( E_3^s \) as in the following diagram:

\[
\begin{array}{c}
u \xrightarrow{f} \nu' \\
\downarrow \\
\text{v}
\end{array} \quad \xrightarrow{f'} \quad \begin{array}{c}
u' \xrightarrow{g} \nu'' \\
\downarrow \\
\text{v}'
\end{array}
\]

We then say that the triple \((f', e', g')\) is a confluence modulo \( E \) of the branching \((f, e, g)\) modulo \( E \). The (3,2)-superpolygraph \( S \) is confluent modulo \( E \) if all its branchings modulo \( E \) are confluent modulo \( E \). A branching \((f, e, g)\) modulo \( E \) is local if \( f \) is a rewriting step of \( S \), \( g \) is a positive
3-cell of $S^3$ and $e$ is a 3-cell of $E^3_2$ such that $\ell(g) + \ell(e) = 1$. Following \cite[Section 2.2.6]{14}, local branchings are classified in the following families: local aspherical, local Peiffer, local additive, local Peiffer modulo, local additive modulo and overlappings modulo which are all the remaining local branchings modulo. A critical branching modulo $E$ is an overlapping branching modulo which is minimal for the order $\subseteq$ defined by $(f, e, g) \subseteq (c[f], c[e], c[g])$ for any context $c$ of the 2-supercategory $R^2_2$.

2.6.2. (Quasi)-Normal forms modulo. Let us consider a $(3, 2)$-superpolygraph modulo $(R, E, S)$ such that $S$ is confluent modulo $E$. If $S$ is terminating (resp. quasi-terminating), each 2-cell $u$ of $R^2_2$ admits at least one normal form (resp. quasi-normal form) with respect to $S$, and all these normal forms (resp. quasi-normal forms) are congruent modulo $E$ by confluence of $S$ modulo $E$. We fix such a normal form (resp. quasi-normal form), that we denote by $\tilde{u}$. By convergence of $E$, any 2-cell $u$ of $R^2_2$ admits a unique normal form with respect to $E$, that we denote by $\hat{u}$. Note that when $S$ is confluent modulo $E$, the element $\hat{u}$ does not depend on the chosen normal form $\tilde{u}$ for $u$ with respect to $S$, since two normal forms of $u$ being equivalent with respect to $E$, they have the same $E$-normal form. A normal form for $(R, E, S)$ (resp. quasi-normal form for $(R, E, S)$) of a 2-cell $u$ in $R^2_2$ is a 2-cell $v$ such that $v$ appears in the monomial decomposition of $\tilde{u}$, where $w$ is a monomial in the support of $\tilde{u}$. Such a set is obtained by reducing a 2-cell $u$ in $R^2_2$ into its chosen normal form (resp. quasi-normal form) with respect to $S$, then taking all the monomials appearing in the $E$-normal form of each element in $\text{Supp}(\hat{u})$.

2.6.3. Decreasingness modulo. The property of decreasingness modulo has been introduced in \cite{14} following Van Oostrom’s abstract decreasingness property for a rewriting system to give confluence criteria with respect to a well-founded labelling on the rewriting steps of a linear $(3, 2)$-polygraph modulo. When this polygraph is quasi-terminating, one may consider the quasi-normal form labelling, given by measuring the distance between a 2-cell and a fixed quasi-normal form. It is proven in \cite{14} that if a linear $(3, 2)$-polygraph is decreasing with respect to this labelling, which can be proved by proving the confluence of its critical branchings, it is confluent modulo. Note that this extends to the case of $(3, 2)$-superpolygraphs since it is an abstract property. Another proof of the critical branching lemma modulo in the quasi-terminating setting may be found in \cite{9}, based on induction on the distance to the quasi-normal form.

2.7. Linear bases from confluence modulo. Given a $(3, 2)$-superpolygraph $P$, we define a splitting of $P$ as a pair $(E, R)$ of $(3, 2)$-superpolygraphs such that:

i) $E$ is a sub-superpolygraph of $P$ such that $E_{\leq 1} = P_{\leq 1}$ and $E_2 \subseteq P_2$,

ii) $R$ is a $(3, 2)$-superpolygraph such that $R_{\leq 2} = P_{\leq 2}$ and $P_3 = R_3 \coprod E_3$.

Such a splitting is called convergent if we require that $E$ is convergent. The data of a splitting of a $(3, 2)$-superpolygraph $P$ gives two distinct $(3, 2)$-superpolygraphs $E$ and $R$ from which we can construct $(3, 2)$-superpolygraphs modulo. Then, since the definition of monomials imply that every 2-cell $u$ of $P_2^s$ admits a unique monomial decomposition, we prove in the same fashion as in the non-super setting the following statement:

**Theorem 2.32.** Let $P$ be a $(3, 2)$-superpolygraph presenting a $(2, 2)$-supercategory $C$, $(E, R)$ a convergent splitting of $P$ and $(R, E, S)$ a $(3, 2)$-superpolygraph modulo such that

i) $S$ is terminating (resp. quasi-terminating),

ii) $S$ is confluent modulo $E$,

then the set of all normal forms (resp. of all quasi-normal forms) for $(R, E, S)$ is a hom-basis of $C$.

**Remark 2.33.** Note that we require $E$ to be convergent to ensure that any quasi-normal form with respect to the polygraph modulo $S$ admits a unique normal form with respect to $E$. However,
even if we will still require $E$ to be terminating, the whole confluence assumption can be weakened. In particular, when $E$ is convergent with a set of 2-cells that does not contain all the generating 2-cells of $P$, the generating 2-cells of $P_2 - E_2$ could create new indexed critical branchings, and thus obstructions to confluence. But confluence outside of these indexed critical branchings might be enough provided that these obstructions can be removed using the 3-cells of $S$, so that any (quasi-)normal form with respect to $S$ still admit a unique normal form with respect to $E$. This is the case for the $(3, 2)$-superpolygraph $\text{Osl}(2)$ in which the $(3, 2)$-superpolygraph will be confluent outside of crossing indexations as in (5.15), but the polygraph modulo $E R$ admits 3-cells allowing the removal of self-intersections, as explained in Section 5.3.

3. A CONVERGENT PRESENTATION OF THE SUPER ISOTOPY CATEGORY

3.1. Definition of supercategory of super isotopies. Let $I$ be a possibly infinite index set equipped with a parity function

$$ I \rightarrow \mathbb{Z}/2, \quad i \mapsto |i| $$

We say that $i \in I$ is odd if $|i| = 1$ and even if $|i| = 0$.

Let $(-d_{ij})_{i,j \in I}$ be a generalized Cartan matrix with $d_{ii} = -2$, $d_{ij} \geq 0$ for $i \neq j$, and $d_{ij} = 0$ if and only if $d_{ji} = 0$. Under the additional assumption that $d_{ij}$ is even whenever $i$ is odd, Brundan and Ellis define a super 2-Kac-Moody algebra as a certain 2-super category $\text{g}$ determined by the generalized Cartan matrix $(-d_{ij})_{i,j \in I}$. In particular, associated to this Cartan matrix pick one can choose a complex vector space $\mathfrak{h}$ and linearly independent subsets $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$, $\{h_i \mid i \in I\} \subset \mathfrak{h}$, such that the natural pairing $\mathfrak{h}^* \times \mathfrak{h} \rightarrow \mathbb{Z}$ is given by $\langle h_i, \alpha_j \rangle = -d_{ij}$ for all $i, j \in I$. We denote the weight lattice of $\mathfrak{g}$ by $X = \{\lambda \in \mathfrak{h}^* \mid \langle h_i, \lambda \rangle \in \mathbb{Z} \text{ for all } i \in I\}$ and the root lattice by $Y = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$. We sometimes write $\lambda_i := \langle h_i, \lambda \rangle$.

In what follows we consider a certain sub super 2-category of the super 2-Kac-Moody category $\text{U(2)}$ defined by Brundan and Ellis [7, Definition 1.5]. This can be thought of as a super analog of the 2-category of pearls from [20].

**Definition 3.1.** Define the 2-supercategory of $\mathfrak{g}$-valued isotopies $\text{Iso}_\mathfrak{g}(\mathfrak{g})$ to have

i) objects consisting of weights $\lambda \in X$ of the Kac-Moody algebra $\mathfrak{g}$;

ii) the 1-morphisms are generated by $\mathbb{1}_\lambda : \lambda \rightarrow \lambda$, $\mathcal{E}_i \mathbb{1}_\lambda : \lambda \rightarrow \lambda + \alpha_i$, $\mathcal{F}_i \mathbb{1}_\lambda : \lambda \rightarrow \lambda - \alpha_i$

iii) 2-morphisms generated by

\[
\begin{align*}
\lambda^i : & \mathcal{E}_i \mathbb{1}_\lambda \rightarrow \mathcal{E}_1 \mathbb{1}_\lambda \\
\downarrow & \downarrow \\
\lambda_i : & \mathcal{F}_i \mathbb{1}_\lambda \rightarrow \mathcal{F}_1 \mathbb{1}_\lambda
\end{align*}
\]

(3.1) (parity $|i|$)

\[
\begin{align*}
\lambda \mathcal{F}_i \mathcal{E}_i \mathbb{1}_\lambda & \rightarrow \mathbb{1}_\lambda \\
\lambda \mathcal{E}_i \mathcal{F}_i \mathbb{1}_\lambda & \rightarrow \mathbb{1}_\lambda
\end{align*}
\]

(parity $|i, \lambda|$)

where

\[
|i, \lambda| := |i| (\langle h_i, \lambda \rangle + 1) = |i| (\lambda_i + 1).
\]

These 2-morphisms are required to satisfy the following axioms.
a) Super zig-zag identities

\[
\begin{align*}
\bigcup_i \lambda &= \uparrow_i \lambda, & \bigcup_i \lambda &= \downarrow_i \lambda, & \bigcup_i \lambda &= (-1)^{|i|} \lambda_{\uparrow_i}, & \bigcup_i \lambda &= \downarrow_i \lambda,
\end{align*}
\]

b) For \( i \in I \) of parity 1, define the odd bubble by

\[
\bigotimes_i \lambda := \begin{cases} 
(-1)^{\frac{|\lambda|}{2}} & \text{if } \lambda \geq 0 \\
\lambda & \text{if } \lambda \leq 0
\end{cases}
\]

Then the odd ‘cyclicity’ relations

\[
\begin{align*}
\bigcup_i \lambda &= \uparrow_i \lambda, & \bigcup_i \lambda &= \downarrow_i \lambda, & \bigcup_i \lambda &= (-1)^{|i|} \lambda_{\uparrow_i}, & \bigcup_i \lambda &= \downarrow_i \lambda.
\end{align*}
\]

### 3.2. The super \((3,2)\)-polygraph \(S_{\text{Iso}}\)

In this section we define a \((3,2)\)-super polygraph presenting the super 2-category \(S_{\text{Iso}}(g)\) of \(g\)-valued isotopies. Let \(S_{\text{Iso}}(g)\) be the super \((3,2)\)-polygraph defined by:

i) the elements of \(S_{\text{Iso}}(g)_0\) are the weights \(\lambda \in X\) of the Kac-Moody algebra;

ii) the elements of \(S_{\text{Iso}}(g)_1\) are given by

\[
1_{\lambda}^{*} \mathcal{E}_{e_{i_{1}}} \cdots \mathcal{E}_{e_{i_{m}}} 1_{\lambda}
\]

for any signed sequence of vertices with \(e_{i_{\ell}} \in \{\pm\}\) and \(i_{\ell} \in I\) for all \(1 \leq \ell \leq m\). Here we identify \(\mathcal{E}_{+} := \mathcal{E}\) and \(\mathcal{E}_{-} := \mathcal{F}\). Such a 1-cell has for 0-source \(\lambda\) and 0-target \(\lambda' = \lambda + \sum_{\ell} e_{i_{\ell}} \alpha_{i_{\ell}}\), and

\[
1_{\lambda}^{*} \mathcal{E}_{e_{i_{1}}} \cdots \mathcal{E}_{e_{i_{m}}} 1_{\lambda} = 1_{\lambda'} \mathcal{E}_{e'_{i_{1}j_{1}}} \cdots \mathcal{E}_{e'_{i_{m}j_{m}}} 1_{\lambda'}
\]

iii) the elements of \(S_{\text{Iso}}(g)_2\) are the following generating 2-cells: for any \(i \in I\) and \(\lambda' \in X\),

\[
\begin{align*}
\bigcup_i \lambda &= \uparrow_i \lambda, & \bigcup_i \lambda &= \downarrow_i \lambda, & \bigcup_i \lambda &= (-1)^{|i|} \lambda_{\uparrow_i}, & \bigcup_i \lambda &= \downarrow_i \lambda.
\end{align*}
\]

with respective parity \(|i|, |i\|, |i, \lambda|, |i, \lambda|, 0, 0\).

iv) \(S_{\text{Iso}}(g)_3\) consists of the following 3-cells:

\[
\begin{align*}
\bigcup_i \lambda &= \uparrow_i \lambda, & \bigcup_i \lambda &= \downarrow_i \lambda, & \bigcup_i \lambda &= (-1)^{|i|} \lambda_{\uparrow_i}, & \bigcup_i \lambda &= \downarrow_i \lambda.
\end{align*}
\]
For the definition of an odd bubble in weight spaces $\lambda_i = \langle h_i, \lambda \rangle = 0$ with $|i| = 1$, we also add 3-cells

\[ \bigcirc^0_{i} \Rightarrow i^0 \]

and for any $i \in I$ of parity 1 and any endomorphism 2-cell $k$ of the identity $1_\lambda$ in normal form with respect to the set of 3-cells above:

\[ \bigcirc^{m+1}_{i} \Rightarrow \bigcirc^{m+1}_{i} \]

where the odd bubble $\bigotimes^{\lambda}_{i}$ is the 2-cell defined as in (3.2).

**Lemma 3.2.** One can define 3-cells

\[ \bigcirc^0_{i} \Rightarrow i^0 \]

from the generating 3-cells of $S\text{Iso}(g)$
Remark 3.3. Note that in every 3-cell except for \( \alpha_{m,k} \) and \( \beta_{m,k} \), every strand of the source and target are labeled by the same \( i \in I \). In \( \alpha_{m,k} \) and \( \beta_{m,k} \), the strands of \( k \) can be labeled with any \( j \in I \). However, we cannot rewrite \( k \) using any rewriting step since it is in normal form by definition. Knowing this, we can write \( \text{SIso}_3 = \bigsqcup_{i \in I} \text{SIso}_3^i \), where \( \text{SIso}_3^i \) is the set of 3-cells where all of the strands of the source and target are labeled by \( i \) along with the 3-cells \( \alpha_{m,k} \) and \( \beta_{m,k} \) where the strands of the odd bubble and the bubble surrounding \( k \) are labeled with \( i \). Then there can be no critical branchings between 3-cells in \( \text{SIso}_3^i \) and \( \text{SIso}_3^j \) unless \( i = j \).

We can prove that the (3,2)-superpolygraph \( (\text{SIso}_0, \text{SIso}_1, \text{SIso}_2, \text{SIso}_3) \) is convergent for any \( i \in I \) of parity \( |i| = 0 \) by using an argument similar to the proof that the polygraph of pearls from [20, Section 5.5] is convergent. Thus, if we prove that the (3,2)-superpolygraph \( (\text{SIso}_0, \text{SIso}_1, \text{SIso}_2, \text{SIso}_3) \) is convergent for an arbitrary \( i \in I \) of parity \( |i| = 1 \), then we will have proved that the entire (3,2)-superpolygraph \( \text{SIso}(g) \) is convergent.

3.2.1. Termination. We now prove the termination of the (3,2)-superpolygraph \( \text{SIso}(g) \) using the derivation method from Section 2.5.1.

Lemma 3.4. Let \( U(\text{SIso}(g)) \) be the linear (3,2)-polygraph given by \( U(\text{SIso}(g))_i = \text{SIso}(g)_i \) forgetting the parity of elements in \( \text{SIso}(g) \) as in Definition 2.25. Then the map \( d: U(\text{SIso}(g))_2 \to \mathbb{N} \) given by

\[
d(u) = ||u|| \{ \quad \quad \quad \quad \quad - 2 \text{ times the number of odd bubbles}
\]

is stable under contexts as described in 2.5.2.

Proof. For \( f \in \{ u_{\lambda,0}, d_{\lambda,0}, u'_{\lambda,0}, d'_{\lambda,0} \} \), we have \( d(s_2(f)) = 2 > 0 = d(t_2(f)) \). Furthermore, for any context \( c \) of \( U(\text{SIso}(g))_2 \) such that \( c[f] \) is defined, we have that

\[
||c[s_2(f)]|| \{ \quad \quad \quad \quad \quad + 2
\]

and \( c[t_2(f)] \) must have at least as many odd bubbles as \( c[s_2(f)] \). Thus, \( d(c[s_2(f)]) \geq d(c[t_2(f)]) + 2 > d(c[t_2(f)]) \) for \( f \in \{ u_{\lambda,0}, d_{\lambda,0}, u'_{\lambda,0}, d'_{\lambda,0} \} \).

For \( f \in \{ i_1, i_2 \} \), and any context \( c \) for which \( c[f] \) is defined, we have that \( d(c[s_2(f)]) \geq d(c[t_2(f)]) \) because \( c[s_2(f)] \) and \( c[t_2(f)] \) have the same number of caps and cups and \( c[s_2(f)] \) cannot have more odd bubbles than \( c[t_2(f)] \) by the definition of the odd bubble in 3.2. For \( f \in \{ i_1, i_2 \} \), the context \( c[t_2(f)] \) has two terms. We have \( d(c[s_2(f)]) \geq d(c[h]) \) for all \( h \in \text{Supp}(t_2(f)) \) using a similar argument for the first term of the target and observing that in the second term the target has exactly two more caps and cups and at least one more odd bubble than the source.

The remaining 3-cells are endomorphism 2-cells of the identity \( 1_\lambda \) and it is straightforward to verify the desired inequality. \( \square \)

Proposition 3.5. The (3,2)-superpolygraph \( \text{SIso}(g) \) terminates.

Proof. We prove the termination of \( \text{SIso}(g) \) in steps as described in Section 2.5.3.

Step 1. Using the context stable map from Lemma 3.4, we have that \( d(c[s_2(f)]) > d(c[t_2(f)]) \) for \( f \in \{ u_{\lambda,0}, d_{\lambda,0}, u'_{\lambda,0}, d'_{\lambda,0} \} \) and \( d(c[s_2(f)]) \geq d(c[t_2(f)]) \) for the remaining 3-cells. Hence, the map \( d \) allows us to reduce termination of \( \text{SIso}(g) \) to termination of

\[
\text{SIso}(g)' := (\text{SIso}(g)_0, \text{SIso}(g)_1, \text{SIso}(g)_2, \text{SIso}(g)_3 - \{ u_{\lambda,0}, d_{\lambda,0}, u'_{\lambda,0}, d'_{\lambda,0} \}).
\]
Step 2. Define 2-functors $X: U(S\text{Iso}(\mathfrak{g}'))_2 \to \text{Ord}$ and $Y: (U(S\text{Iso}(\mathfrak{g}'))_2^* \to \text{Ord}$ whose nonempty values are given on generators by

$$X \left( \begin{array}{c} \lambda \\ i \end{array} \right) = Y \left( \begin{array}{c} \lambda \\ i \end{array} \right) = \mathbb{N}, \quad X \left( \begin{array}{c} \lambda \\ i \\ \lambda \end{array} \right) = X \left( \begin{array}{c} \lambda \\ \lambda \\ i \end{array} \right) = (0,0), \quad Y \left( \begin{array}{c} \lambda \\ i \\ \lambda \end{array} \right) = Y \left( \begin{array}{c} \lambda \\ \lambda \\ i \end{array} \right) = (0,0),$$

$$X \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n) = Y \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n) = X \left( \begin{array}{c} \lambda \\ i \\ \lambda \end{array} \right) (n) = Y \left( \begin{array}{c} \lambda \\ i \\ \lambda \end{array} \right) (n) = n + 1$$

Then a derivation $d: U(S\text{Iso}(\mathfrak{g}'))_2 \to M_{X,Y,\mathbb{Z}}$ is defined from

$$d \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n,m) = 0 \quad d \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n,m) = 0$$

$$d \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n,m) = d \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n,m) = m \quad d \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n,m) = d \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n,m) = 0.$$

It is immediate from this definition that

$$d \left( \begin{array}{c} \lambda \\ i \end{array} \right) \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n,m) = m + a \quad d \left( \begin{array}{c} \lambda \\ i \end{array} \right) \left( \begin{array}{c} \lambda \\ i \end{array} \right) (n,m) = m + a \quad d \left( \begin{array}{c} \lambda \\ i \end{array} \right) \left( \begin{array}{c} \lambda \\ i \end{array} \right) = 0 \quad d \left( \begin{array}{c} \lambda \\ i \end{array} \right) \left( \begin{array}{c} \lambda \\ i \end{array} \right) = 0$$

Then for every generating 3-cell $x \in S\text{Iso}(\mathfrak{g}')_3$, the inequalities $X(s_2(x)) \geq X(h), Y(s_2(x)) \geq Y(h)$, and $d(s_2(x)) \geq d(h)$ hold for every $h \in \text{Supp}(t_2(x))$. Furthermore, for $f \in \{i_3^2, i_3^1\}$ we have the inequalities $X(s_2(x)) \geq X(h), Y(s_2(x)) \geq Y(h)$ and a strict inequality $d(s_2(x)) > d(h)$ for every $h \in \text{Supp}(t_2(x))$. This reduces the termination of $S\text{Iso}(\mathfrak{g}')$ to the termination of the $(3,2)$-superpolygraph $R$ with $R_{\leq 2} := S\text{Iso}(\mathfrak{g}')_{\leq 2}$ and $R_3 := S\text{Iso}(\mathfrak{g}')_3 - \{i_3^2, i_3^1\}$.

Step 3. To prove termination of $R$, consider the derivation $d$ into the trivial $U(R)_2^*$-module $M_{*,*,\mathbb{Z}}$ counting the number caps and cups, that is

$$d(u) = ||u|| \{ \begin{array}{c} 1 \\ \lambda \\ i \end{array} \}$$

for any 2-cell $u$ of $R_2^*$. For every generating 3-cell in $\alpha \in R_3$, we have the inequality $d(s_2(\alpha)) \geq d(h)$ for every $h \in \text{Supp}(t_2(\alpha))$.

$$d(s_2(i_3^1)) = 1 = d(t_2(i_3^1)) \quad d(s_2(i_3^2)) = 1 = d(t_2(i_3^2))$$

$$d(s_2(\alpha_{m,k})) = d(k) + 4 > d(k) + 2 = d(t_2(\alpha_{m,k})) \quad d(s_2(\beta_{m,k})) = d(k) + 4 > d(k) + 2 = d(t_2(\beta_{m,k}))$$

$$d(s_2(I_0)) = 2 = d(t_2(I_0))$$

Furthermore, for $\alpha \notin \{i_3^1, i_3^2, I_0\}$ we have strict inequalities $d(s_2(\alpha)) > d(h)$ for every $h \in \text{Supp}(t_2(\alpha))$. This reduces the termination of $R$ to the termination of the $(3,2)$-superpolygraph $R'$ with $R'_{\leq 2} := R_{\leq 2}$ and $R_3' := \{i_3^1, i_3^2, I_0\}$.

Step 4. Now consider 2-functors $X: U(R')_2^* \to \text{Ord}, Y: (U(R'))_2^* \to \text{Ord}$

$$X \left( \begin{array}{c} \lambda \\ i \end{array} \right) = Y \left( \begin{array}{c} \lambda \\ i \end{array} \right) = \mathbb{N}, \quad X \left( \begin{array}{c} \lambda \\ i \\ \lambda \end{array} \right) = X \left( \begin{array}{c} \lambda \\ i \\ \lambda \end{array} \right) = (0,0), \quad Y \left( \begin{array}{c} \lambda \\ i \\ \lambda \end{array} \right) = Y \left( \begin{array}{c} \lambda \\ i \\ \lambda \end{array} \right) = (0,0)$$
and derivation $d : U(R')_2^* \to M_{X,Y,Z}$ given by

$$d \left( \begin{array}{c} i \\ \lambda \\ i \end{array} \right) (n, m) = 0, \quad d \left( \begin{array}{c} i \\ \lambda \end{array} \right) (n, m) = 0,$$

$$d \left( \begin{array}{c} i \\ \lambda \\ i \end{array} \right) (n, m) = m, \quad d \left( \begin{array}{c} i \\ \lambda \\ i \end{array} \right) (n, m) = m, \quad d \left( \begin{array}{c} i \\ \lambda \\ i \end{array} \right) (n, m) = m.$$

Then we have the desired inequalities $X(s_2(\alpha)) \geq X(h), Y(s_2(\alpha)) \geq Y(h)$, and $d(s_2(\alpha)) \geq d(h)$ for every $h \in \text{Supp}(t_2(\alpha))$ for every generating 3-cell $\alpha$ of $R'$, with strict inequalities for $\alpha \in \{i_1^\lambda, i_2^\lambda\}$.

So termination of $R'$ reduces to termination of $R'' := (R'_0, R'_1, R'_2, \{I_0\})$.

**Step 5.** Consider the derivation $d$ into the trivial module $M_{*,*,*}$ defined by

$$d(u) = |u| \quad \text{.}$$

Then we have that $d(s_2(I_0)) = 1 > 0 = d(t_2(I_0))$, so $R''$ terminates. Therefore $S_{\text{Iso}}(g)$ terminates.

3.2.2. Convergence of $S_{\text{Iso}}(g)$.

**Proposition 3.6.** The $(3,2)$-superpolygraph $S_{\text{Iso}}(g)$ defined in Section 3.2 is convergent.

**Proof.** Since $S_{\text{Iso}}(g)$ is terminating, following [4, Theorem 4.2.13] its confluence is equivalent to the confluence of its critical branchings, that are all proved confluent in Appendix A. □

4. A CONVERGENT PRESENTATION OF THE ODD NILHECKE ALGEBRA

4.1. Definition of odd nilHecke 2-supercategory. Here we recall the odd nilHecke algebra and its associated 2-supercategory. This algebra appeared independently in [17, 27] and is closely related to the spin Hecke algebra associated to the affine Hecke-Clifford superalgebra appearing in earlier work of Wang [42].

**Definition 4.1.** Define the odd nilHecke 2-supercategory to have

i) one object $*$,

ii) 1-morphisms $n \in \mathbb{N}$,

iii) 2-morphisms generated by $\begin{array}{c} \cdot \end{array} : 1 \to 1$ and $\begin{array}{c} \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \end{array} : 2 \to 2$ both of parity $\bar{1}$

modulo the relations

$$= 0, \quad = \quad =, \quad + =, \quad + =.$$

4.2. The super $(3,2)$-polygraph ONH.

4.2.1. **Definition.** In this section we define a $(3,2)$-superpolygraph presenting the odd Nilhecke 2-supercategory. Let $\text{ONH}$ be the $(3,2)$-superpolygraph defined by

(1) one object denoted by $\lambda$,

(2) one generating 1-cell denoted 1, with $n$ denoting the $*_0$-composition of 1 with itself $n$ times.

Since there are only one 0-cell and one generating 1-cell, we omit them in the string diagrams below.
(3) generating 2-cells $\bullet$ and $\bigtriangleup$ both of parity $\bar{1}$,

(4) generating 3-cells

$\bigtriangleup \overset{dc}{\Rightarrow} 0$, $\bigtriangleup \overset{yb}{\Rightarrow} -$, $\bigtriangleup \overset{on_1}{\Rightarrow} -$, $\bigtriangleup \overset{on_2}{\Rightarrow} -$.

4.2.2. Termination. We closely follow [13, Section 2.3.3] to prove the termination of the $(3,2)$-superpolygraph $\text{ONH}$ in two steps.

**Proposition 4.2.** The $(3,2)$-superpolygraph $\text{ONH}$ terminates.

**Proof.** *Step 1.* Define a 2-functor $X : U(\text{ONH})^2 \to \text{Ord}$ by setting $X(i) = \mathbb{N}$, so that $X(i \circ_i i) = \mathbb{N} \times \mathbb{N}$, and on generating 2-cells of $\text{ONH}$ by:

$$X\left(\bigtriangleup\right)(n) = n \quad X\left(\bullet\right)(n) = n \quad X\left(\bigtriangleup\bigtriangleup\right)(n,m) = (m,n+1)$$

for all $n,m \in \mathbb{N}$. Define a derivation $d : U(\text{ONH})^2 \to M_{X,*}\mathbb{Z}$ on the generating 2-cells of $\text{ONH}$ by

$$d\left(\bigtriangleup\right)(n) = 0, \quad d\left(\bigtriangleup\bigtriangleup\right)(n,m) = m, \quad d\left(\bullet\right)(n) = 0$$

for any $n,m \in \mathbb{N}$. Then by the same calculation in [13, Section 2.3.3] for the even nilHecke algebra, we obtain the inequalities $X(s_2(f)) \geq X(t_2(f))$ and $d(s_2(f)) \geq d(t_2(f))$ for all 3-cells $f$ and $d(s_2(\alpha)) > d(t_2(\alpha))$ for $\alpha \in \{yb,dc\}$. Thus termination of $\text{ONH}$ is reduced to termination of $\text{ONH'} := (\text{ONH}_0,\text{ONH}_1,\text{ONH}_2,\{on_1, on_2\})$.

**Step 2.** Define a 2-functor $X : U(\text{ONH})^2 \to \text{Ord}$ on the generating 2-cells of $\text{ONH}$ by:

$$X\left(\bigtriangleup\right)(n) = n \quad X\left(\bullet\right)(n) = n \quad X\left(\bigtriangleup\bigtriangleup\right)(n,m) = (m+2,n+1)$$

for all $n,m \in \mathbb{N}$, and a derivation $d : U(\text{ONH'})^2 \to M_{X,*}\mathbb{Z}$ given by

$$d\left(\bigtriangleup\right)(n) = 0, \quad d\left(\bigtriangleup\bigtriangleup\right)(n,m) = n, \quad d\left(\bullet\right)(n) = n$$

for any $n,m \in \mathbb{N}$. Then by [13, Section 2.3.3], we obtain the desired inequalities $X(s_2(\alpha)) \geq X(t_2(\alpha))$ and $d(s_2(\alpha)) > d(t_2(\alpha))$ for $\alpha \in \{on_1, on_2\}$, so that Theorem 2.29 implies that $\text{ONH'}$ is terminating, and thus $\text{ONH}$ is terminating.

Moreover, we now prove the following result:

**Proposition 4.3.** The $(3,2)$-superpolygraph $\text{ONH}$ is convergent.

**Proof.** Since $\text{ONH}$ is terminating by Proposition 4.2, following [4, Theorem 4.2.13], its confluence is equivalent to the confluence of its critical branchings, whose classification follows from [13], and are all proved confluent in Appendix B.

4.2.3. Bases of $\text{ONH}$.

**Definition 4.4.** Define the normal form basis for the 2-supercategory $\text{ONH}$ to be the basis obtained from the convergent $(3,2)$-superpolygraph $\text{ONH}$. This basis is obtained by choosing a fixed representative for each equivalence class of normal forms modulo superinterchange.
In practice, to rewrite a 2-cell in \( \text{ONH}_2 \), one checks if there is a representative in its equivalence class modulo superinterchange that is reducible by a 3-cell. If there is more than one representative where a 3-cell can be applied, the convergence of the superpolygraph ensures that it does not matter which representative is chosen to apply a 3-cell. Then a 2-cell is in its normal form if and only if for any representative modulo superinterchange, this representative is irreducible using the set of 3-cells in the \((3,2)\)-superpolygraph \( \text{ONH} \).

In the case of the odd nilHecke algebra, we can further specify the resulting normal form basis by making a preferred choice of representative of the superinterchange class for the order of dots; for example, choosing that dots will decrease in height going from left to right. With our fixed choice or ordering of dots, we can represent these dot sequences as \( x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \) with \( \alpha_1 \) dots appearing on the first strand, \( \alpha_2 \) dots below these on the second, and so on.

The 3-cells in \( \text{ONH} \) ensure that all dots appearing in a given normal form 2-cell appear below any crossings. Then for each reduced expression of \( w = s_{i_1} \ldots s_{i_k} \) of a permutation in the symmetric group \( S_n \), there is a corresponding crossing diagram \( \partial_w = \partial_{i_1} \ldots \partial_{i_k} \) in the odd nilHecke algebra, where \( \partial_i \) is the crossing of the \( i \)th and \( (i+1) \)st lines. The crossings appearing at the top of a normal form diagram will have reduced expressions \( \partial_w \) where no equivalence class under superinterchange admits a reduction \( \partial_i \partial_{i+1} \partial_i \Rightarrow \partial_{i+1} \partial_i \partial_{i+1} \). The superinterchange equivalence class may still be undetermined if the reduced expression contains a subsequence of the form \( \partial_i \partial_j = -\partial_j \partial_i \) with \( |i - j| > 1 \). We can then uniquely specify a representative by choosing the ordering \( \partial_i \partial_j \) where \( i \leq j \). An example is given below with the reduced expression \( s_2 s_1 s_3 s_2 \), rather than \( s_2 s_1 s_3 s_2 \), illustrating this choice of ordering.

\[
\partial_2 \partial_1 \partial_3 \partial_2 x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4} := \alpha_1 \alpha_2 \alpha_3 \alpha_4
\]

In [17, Proposition 2.11], bases for the odd nilHecke algebra are defined by making a choice of a reduced expression for each element \( w \in S_n \) and considering elements \( \{ \partial_w x^\alpha \} \) or \( \{ x^\alpha \partial_w \} \) where \( \partial_w \).

**Proposition 4.5.** The superpolygraph \( \text{ONH} \) presents the odd nilHecke 2-supercategory. The resulting normal form basis recovers the basis \( \{ \partial_w x^\alpha \} \) from [17, Proposition 2.11] where the choice of reduced expressions cannot be simplified further by any application of the identity \( \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \) for any representative of the superinterchange equivalence class of \( \partial_w x^\alpha \).

### 5. Rewriting modulo in the odd 2-category

#### 5.1. Definition of the odd 2-category

Ellis and Brundan give a description of the odd 2-category \( \mathcal{U}(\mathfrak{sl}_2) \) involving a minimal number of relations by requiring the invertibility of certain maps lifting the \( \mathfrak{sl}_2 \)-relations. They show that the invertibility of these maps imply the relations given below. In the definition that follows we do not attempt to provide a minimal set of relations. In section 5.2 we will explain how to reduce the number of generating 2-morphisms and defining relations in a way that will be helpful for presenting this super 2-category by a \((3,2)\)-superpolygraph.

**Definition 5.1.** The odd 2-supercategory \( \mathcal{U} = \mathcal{U}(\mathfrak{sl}_2) \) is the 2-supercategory consisting of

- objects \( \lambda \) for \( \lambda \in \mathbb{Z} \),
- for a signed sequence \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \), with \( \varepsilon_1, \ldots, \varepsilon_m \in \{+, -\} \), define \( \varepsilon \) as \( \varepsilon_1 \varepsilon_2 \ldots \varepsilon_m \).
where $\mathcal{E}_+ := \mathcal{E}$ and $\mathcal{E}_- := \mathcal{F}$. A 1-morphism from $\lambda$ to $\lambda'$ is a formal finite direct sum of strings
\[\mathcal{E}_\pm 1_\lambda = 1_\lambda \mathcal{E}_\pm\]
for any signed sequence $\varepsilon$ such that $\lambda' = \lambda + 2 \sum_{j=1}^m \varepsilon_j 1$.

- 2-morphisms are generated by the $\mathbb{Z} \times \mathbb{Z}_2$-graded generating 2-morphisms
\[
\lambda + 2 \uparrow \lambda : \mathcal{E} 1_\lambda \to \mathcal{E} 1_\lambda
\]
\[
\lambda - 2 \downarrow \lambda : \mathcal{F} 1_\lambda \to \mathcal{F} 1_\lambda
\]
\[
\lambda \leftarrow \lambda : \mathcal{E} 1_\lambda \to \mathcal{E} 1_\lambda
\]
\[
\lambda \rightarrow \lambda : \mathcal{F} 1_\lambda \to \mathcal{F} 1_\lambda(-2)
\]
\[
\lambda \uparrow \lambda : \mathcal{F} 1_\lambda \to 1_\lambda(1 + \lambda)
\]
\[
\lambda \downarrow \lambda : 1_\lambda \to \mathcal{E} 1_\lambda(1 - \lambda)
\]
\[
\lambda \leftarrow \lambda : \mathcal{F} 1_\lambda \to 1_\lambda(1 + \lambda)
\]
\[
\lambda \rightarrow \lambda : \mathcal{E} 1_\lambda \to 1_\lambda(1 - \lambda)
\]
\[
\lambda \uparrow \lambda : \mathcal{F} 1_\lambda \to 1_\lambda(1 + \lambda)
\]
\[
\lambda \downarrow \lambda : 1_\lambda \to \mathcal{E} 1_\lambda(1 - \lambda)
\]
where we have indicated a $Q$-grading and parity as an ordered tuple $(x, \bar{y})$.

The identity 2-morphism of the 1-morphism $\mathcal{E} 1_\lambda$ is represented by an upward oriented line (likewise, the identity 2-morphism of $\mathcal{F} 1_n$ is represented by a downward oriented line).

Horizontal and vertical composites of the above diagrams are interpreted using the conventions for supercategories explained in Section 2.1.3. The rightmost region in our diagrams is usually colored by $\lambda$. The fact that we are defining a 2-supercategory means that diagrams with odd parity skew commute. The 2-morphisms satisfy the following relations (see [7] for more details).

1. **(Odd nilHecke)** The odd nilHecke relations from Definition 4.1 are satisfied for upward oriented strands and any $\lambda \in \mathbb{Z}$.

2. **(Odd isotopies)** The odd isotopy relations from Definition 3.1 for a Cartan data with a single odd $i \in I$.

3. **(Bubble relations)** Dotted bubbles of negative degree are zero, so that for all $m \geq 0$

\[
m \lambda = 0 \quad \text{if } m < \lambda - 1,
\]
\[
m \lambda = 0 \quad \text{if } m < -\lambda - 1.
\]

Dotted bubbles of degree 0 are equal to the identity 2-morphism:

\[
\lambda - 1 \quad \text{if } \lambda \geq 1,
\]
\[
\lambda - 1 \quad \text{if } \lambda \leq -1.
\]

The degree two bubble is given a special notation as in (3.2) and squares to zero by the superinterchange law.

We call a clockwise (resp. counterclockwise) bubble fake if $m + \lambda - 1 < 0$ and (resp. if $m - \lambda - 1 < 0$). These correspond to positive degree bubbles that are labeled by a negative
number of dots. These are to be interpreted as formal symbols recursively defined by the odd infinite Grassmannian relations

\[ 2n^+ \otimes \lambda := - \sum_{l=1}^{n} 2(n-\ell)^+ \otimes 2l^+ \quad \text{for } 0 \leq 2n < -\lambda, \]

\[ \lambda \otimes 2n^+ := - \sum_{l=1}^{n} 2l^+ \otimes 2(n-\ell)^+ \quad \text{for } 0 \leq 2n < \lambda, \]

\[ 2n^+ \otimes \lambda := \otimes \quad \text{for } 0 \leq 2n < -\lambda, \]

\[ \lambda \otimes 2n^+ := \otimes \quad \text{for } 0 \leq 2n + 1 < \lambda, \]

(4) (Centrality of odd bubbles) Odd bubbles are central

\[ \lambda \otimes = \lambda \otimes \]

(5) (Odd crossing cyclicity) The cyclic relations for crossings\(^1\) are given by

\[ \lambda := \quad = \quad \]

Sideways crossings satisfy the following identities:

\[ \lambda := \quad = \quad \lambda \]

(6) (Odd sl(2) relations)

\[ \lambda + \lambda = \sum_{f_1+f_2+f_3 = \lambda-1} (-1)^{f_2} \lambda_1 \]

Remark 5.2. Let Sym denote the algebra of symmetric functions over \( \mathbb{k} \). This algebra is generated by elementary symmetric functions \( e_r \) for \( r \geq 0 \) and by the complete symmetric functions \( h_s \) with

\(^1\)Equation 5.6 differs by a sign from \([7, \text{Equation (1.28)}]\), but is consistent with the original formulation of the odd 2-category from \([18]\).
s \geq 0. By convention \( e_0 = h_0 = 1 \). These generators are related by the equations

\[ \sum_{r+s=n} (-1)^s e_r h_s = 0 \quad \text{for all } n \geq 0. \]

Let \( \text{Sym}[d] \) be the supercommutative superalgebra obtained by placing \( \text{Sym} \) in even degree and adjoining an odd generator \( d \) with \( d^2 = 0 \). Then consider the unique surjective homomorphism

\[ (5.9) \quad \beta_\lambda : \text{Sym} \to \text{End}_\Omega(1_\lambda) \]

such that

\[ (5.10) \quad e_n \mapsto \lambda^{n+1} \circ \lambda \quad \text{if } n > -\frac{h}{2}, \quad h_n \mapsto (-1)^n \lambda^{\lambda+1-2n+2} \quad \text{if } n > \frac{h}{2}; \]

\[ (5.11) \quad de_n \mapsto \lambda^{n+1} \circ \lambda \quad \text{if } n > -\frac{h}{2}, \quad dh_n \mapsto (-1)^n \lambda^{\lambda+1-2n+2} \quad \text{if } n > \frac{h}{2}. \]

The relations in \( \Omega \) imply that this is a homomorphism and that the relations (5.4) defining the fake bubbles hold for all values of \( \lambda \) and for all \( n \geq 0 \), see [7, Proposition 5.1].

5.2. The super \((3,2)\)-polygraph \( Osl(2) \).

**Definition 5.3.** Let \( Osl(2) \) be the linear \((3,2)\)-polygraph defined by:

i) the elements of \( Osl(2)_0 \) are the weights \( \lambda \in \mathbb{Z} \) of \( \mathfrak{sl}_2 \),

ii) the elements of \( Osl(2)_1 \) are given by

\[ 1_\lambda e_{\varepsilon_1} \ldots e_{\varepsilon_m} 1_\lambda \]

for any sequence of signs \( (\varepsilon_1, \ldots, \varepsilon_m) \) and \( \lambda, \lambda' \) in \( \mathbb{Z} \). Such a 1-cell has for 0-source \( \lambda \) and 0-target \( \lambda' \), and

\[ 1_\lambda e_{\varepsilon_1} \ldots e_{\varepsilon_m} 1_\lambda \star_0 1_{\lambda''} e_{\varepsilon_1'} \ldots e_{\varepsilon_l} 1_{\lambda'} = 1_{\lambda''} e_{\varepsilon_1'} \ldots e_{\varepsilon_m} 1_\lambda \]

iii) the elements of \( Osl(2)_2 \) are the following generating 2-cells: for \( \lambda \in \mathbb{Z} \):

- \( \lambda \)
- \( \lambda \)
- \( \lambda \)
- \( \lambda \)
- \( \lambda \)
- \( \lambda \)
- \( \lambda \)

with respective parity 1, 1, 1, 1, \( \lambda + 1 \), \( \lambda + 1 \), 0, 0.

iv) \( Osl(2)_3 \) consists of the following 3-cells:

1) The odd nilHecke 3-cells, given by

\[ d e^\lambda \Rightarrow 0, \quad y b^\lambda \Rightarrow \]

with the rightmost region of the diagram being labeled \( \lambda \). When no confusion is likely to arise we often drop the \( \lambda \) subscript from this notation.

2) The super isotopy 3-cells of \( SISO_3 \),

3) The cyclicity 3-cell for the definition of the downward crossing:

\[ P_\lambda \Rightarrow - \lambda \]

\[ P_\lambda' \Rightarrow - \lambda \]
together with their respective images \( Q_\lambda \) and \( Q'_\lambda \) through the Chevalley involution \( \omega \) defined in [7, Proposition 3.5] giving the same cyclicity condition for the upward crossing in terms of the downward crossing.

4) The 3-cells for the degree conditions on bubbles: for every \( \lambda \in \mathbb{Z} \) and \( n \in \mathbb{N} \)

\[
\begin{align*}
\text{(5.12)} & \quad n \circ \lambda \xrightarrow{b_{\lambda}^1} \begin{cases} 1_{1\lambda} & \text{if } n = \lambda - 1 \\ 0 & \text{if } n < \lambda - 1 \end{cases} \\
\text{(5.13)} & \quad \lambda \circ n \xrightarrow{c_{\lambda}^1} \begin{cases} 1_{1\lambda} & \text{if } n = -\lambda - 1 \\ 0 & \text{if } n < -\lambda - 1 \end{cases}
\end{align*}
\]

5) The Infinite-Grassmannian 3-cells: for any \( \lambda \in \mathbb{Z} \) and \( n \geq 1 \) such that \( 2n + \lambda - 1 \geq 0 \)

\[
\text{(5.14)} & \quad 2n+* \quad \lambda \xrightarrow{i_{2n,\lambda}^*} - \sum_{l=1}^{n} 2(n-l)+* \quad 2l+* \quad \lambda \circ n
\]

6) Bubble Slide 3-cells

\[
\quad \lambda \xrightarrow{s_{\lambda,n}^+} \sum_{r \geq 0} (2r + 1) \quad 2r \quad n-2r+* \quad \lambda
\]

\[
\quad \lambda \circ n+* \xrightarrow{s_{\lambda,n}^-} - 3 \quad 2 \quad \lambda \circ n-2r+* \quad \lambda \quad 4 \quad \sum_{r \geq 2} (-1)^r \quad 2r \quad \lambda \circ n-2r+*
\]

and their reflections across the horizontal axis \( r_{\lambda,n}^+ \) and \( r_{\lambda,n}^- \), which allow a bubble to go through a downwards strand. The reflections correspond to the images of these relations via the Chevalley involution \( \omega \) defined in [7, Proposition 3.5]. By (3.2) and the definition of fake bubbles (5.4), we simplify notation and write \( s_{\lambda,1} = s_{\lambda,1}^+ = s_{\lambda,1}^- \) and \( r_{\lambda,1} = r_{\lambda,1}^+ = r_{\lambda,1}^- \). These are added to the presentation to reach confluence modulo.

7) The invertibility 3-cells:

\[
\quad \lambda \xrightarrow{F_{\lambda}} - (1)^{\lambda+1} \quad i \quad \lambda \xrightarrow{E_{\lambda}} - (1)^{\lambda+1} \quad i
\]

\[
\quad \lambda \xrightarrow{C_{\lambda}} \sum_{n=0}^{\lambda} (-1)^n \quad n \quad \lambda \xrightarrow{A_{\lambda}} \sum_{n=0}^{-\lambda} (-1)^n \quad n
\]

8) Remaining 3-cells:

\[
\quad \lambda \xrightarrow{B_{\lambda}} \sum_{n=0}^{-\lambda} (-1)^n \quad n \quad \lambda \xrightarrow{D_{\lambda}} \sum_{n=0}^{\lambda} (-1)^n \quad n
\]
Proposition 5.6. The proof goes in steps as explained in Section 2.5.3:

Step 1. Eliminate the zigzag 3-cells using the first step of the proof of termination of SISO.

Step 2. Eliminate $yb$ and $dc$ using the first step of the proof of termination of ONH, extending values of $X$ and $d$ by

$$X \left( \begin{array}{c} \cdot \hline \lambda \end{array} \right) = X \left( \begin{array}{c} \cdot \hline \lambda \end{array} \right) = (0, 0), \quad d \left( \begin{array}{c} \cdot \hline \lambda \end{array} \right) = d \left( \begin{array}{c} \cdot \hline \lambda \end{array} \right) = 0 \quad d \left( \begin{array}{c} \cdot \hline \lambda \end{array} \right) = (n, m) = d \left( \begin{array}{c} \cdot \hline \lambda \end{array} \right) = 0$$

so that the inequalities $d(s_2(\alpha)) \geq d(t_2(\alpha))$ hold for any $\alpha \in SISO - \{u_{\lambda,0}, u_{\lambda,0}', d_{\lambda,0}, d_{\lambda,0}'\}$.

Step 3. Finish the proof by eliminating the 3-cells in the same order as in the proof of termination of SISO.

Since $E_2 = SISO_2 \cup \{\begin{array}{c} \cdot \hline \lambda \end{array}\}$, additional indexed critical branchings appear in $E$ between $i^1_{\lambda}$ and $i^4_{\lambda}$ of the form

$$(5.15)$$

that are not confluent. However, we still have the following.

Lemma 5.7. Any 2-cell $u$ that does not contain a strand that self-intersects admits a unique decomposition into monomials in normal form with respect to $E$.

Proof. Let $u$ be a 2-cell that does not contain a self-intersecting strand, that is up to application of $yb$ that does not contain any element of the form $E_{3}(\lambda)$. Since $E$ is terminating and left-monomial, $u$ admits at least a linear decomposition into monomials in normal form with respect to $E$. If two such decompositions exist, then the two reductions leading to these results give a branching, that is

---

2The 3-cell $\Gamma_{\lambda}$ corrects a minor typo from [7, Equation (7.20)].
either a Peiffer branching or come from a critical branching in a context. However, since \( u \) does not contain a self-intersection, this critical branching is not given by a crossing indexation as in (5.15), and thus from confluence of critical branchings of \( \text{Siso} \) and \( \{yb, dc\} \), there exists a confluence of that branching, so that these two decompositions are equal. \( \square \)

Lemma 5.7 is enough to get the hom-basis of \( \mathcal{U} \) since the 3-cells \( A_\lambda, B_\lambda, C_\lambda \) and \( D_\lambda \) in \( ER^s \) can be used to remove all self-intersections, so that any quasi-normal form with respect to \( ER \) will admit a unique normal form with respect to \( E \).

5.4. Quasi termination of \( ER \). In this subsection, we will prove that the \((3,2)\)-superpolygraph \( R \) is terminating without bubble slide and cyclicity 3-cells, and quasi-terminating with these 3-cells. We also give a procedure showing that \( ER \) is quasi-terminating with rewriting cycles being induced by bubble slide cycles as in [4], isotopy cycles created by dots moving on cups and caps, and cyclicity for crossings.

5.4.1. Termination without bubble slide and cyclicity 3-cells.

**Lemma 5.8.** The \((3,2)\)-superpolygraph \( R' := R - \{s^+_\lambda, s^-_\lambda, r^+_\lambda, r^-_\lambda, P_\lambda, P'_\lambda, Q_\lambda, Q'_\lambda\} \) terminates.

**Notation 5.9.** For a 3-cell \( \alpha \), define \( d(t_2(\alpha)) := \max \{d(h) \mid h \in \text{Supp}(t_2(\alpha)) \} \) and similarly \( X(t_2(\alpha)) := \max \{X(h) \mid h \in \text{Supp}(t_2(\alpha)) \} \).

**Proof.** We prove termination in steps.

**Step 1.** First, consider the derivation \( d \) into the trivial \( U(R'_2)^s \)-module \( M_{*,*,\mathbb{Z}} \) given by

\[
d(u) = ||u||
\]

for any 2-cell \( u \) of \( R'_2^2 \). Then \( d(s_2(\alpha)) > d(t_2(\alpha)) \) for \( \alpha \in \{A_\lambda, B_\lambda, C_\lambda, D_\lambda, E_\lambda, F_\lambda\} \) and \( d(s_2(\alpha)) \geq d(t_2(\alpha)) \) for all other \( \alpha \) in \( R'_2^2 \). Thus, termination of \( R' \) is reduced to termination of \( R'' := (R'_0, R'_1, R'_2, R'_3 - \{A_\lambda, B_\lambda, C_\lambda, D_\lambda, E_\lambda, F_\lambda\}) = (R'_0, R'_1, R'_2, R'_3 - \{on_1, on_2, \Gamma, b^{n,0}_\lambda, b^{1}_\lambda, c^{n,0}_\lambda, c^{1}_\lambda, ig_{2n}\}) \).

**Step 2.** Consider the 2-functor \( X : U(R'')^s_2 \rightarrow \text{Ord} \) and derivation \( d : U(R'')^s_2 \rightarrow \mathbb{Z} \) defined by extending the second derivation used for ONH as follows:

\[
X(\uparrow^\lambda)(n) = X(\downarrow^\lambda)(n) = n, \quad X(\uparrow)(n) = n, \quad X(\bigcirc)(n, m) = (m + 2, n + 1),
\]

\[
X(\uparrow^\lambda)(n) = n + 1, \quad X(\bigcirc)(n, m) = (m, n), \quad X(\bigcup^\lambda) = X(\bigcup) = (0, 0)
\]

\[
d(\text{\textcdot})(n) = 0, \quad d(\uparrow^\lambda)(n) = n, \quad d(\bigcirc)(n, m) = n, \quad d(\downarrow^\lambda)(n) = n, \quad d(\bigcup^\lambda)(n, m) = n + m
\]

\[
d(\bigcup) = d(\bigcup^\lambda) = 0, \quad d(\bigcup)(n, m) = d(\bigcup^\lambda)(n, m) = 0.
\]

Then we have \( X(s_2(\alpha)) \geq X(t_2(\alpha)) \) and \( d(s_2(\alpha)) \geq d(t_2(\alpha)) \) for all 3-cells \( \alpha \in R''_3 \). Furthermore, \( d(s_2(\alpha)) > d(t_2(\alpha)) \) for \( \alpha \in \{on_1, on_2, \Gamma\} \). Thus, termination of \( R'' \) is reduced to termination of the \((3,2)\)-superpolygraph

\[
\hat{R} := (R'_0, R'_1, R'_2, \{b^{n,0}_\lambda, b^{1}_\lambda, c^{n,0}_\lambda, c^{1}_\lambda, ig_{2n}\})
\]

**Step 3.** To prove termination of \( \hat{R} \), we use a context stable map as in Section 2.5.2. For any \( u \in U(\hat{R})_2^2 \), define a map \( \hat{d}' : U(\hat{R})_2^2 \rightarrow \mathbb{N} \) by

\[
\hat{d}'(u) := \text{number of bubbles in } u + \sum_{\pi \text{ clockwise bubble in } u} |\text{deg}(\pi)|
\]

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where the sum is over all clockwise bubbles appearing in $u$ and $|\text{deg}(\pi)|$ denotes the absolute value of the $\mathbb{Z}$-grading defined in Definition 5.1.

For $\alpha \in \check{R}_3$, we have $d'(s_2(\alpha)) > d'(t_2(\alpha))$. Let $c$ be any context of $U(\check{R})^*_2$ such that $c[\alpha]$ is defined. Then we have $d'(c[s_2(\alpha)]) = d'(s_2(\alpha)) + d'(c[1_1]) > d'(t_2(\alpha)) + d'(c[\text{Id}_3]) = d'(c[t_2(\alpha)])$ since both $s_2(\alpha)$ and $t_2(\alpha)$ are endomorphism 2-cells on the identity 1-cell $1_\lambda$. Therefore, $\check{R}$ terminates, implying $R'$ also terminates. □

5.4.2. Indexed cycles. The super $(3, 2)$-polygraph $R' = R - \{ s^\pm_3, r_\lambda^\pm, P_\lambda, P'_\lambda, Q_\lambda, Q'_\lambda \}$ terminates by Lemma 5.8. However, $E R'$, and thus $E R$ do not. Closing off crossing diagrams with caps and cups can create cycles where a dot slides around a closed strand and arrives back where it started as in the configurations:

\[ \lambda \quad \lambda \]

for $k > 0$ even and $l \geq 1$ odd, where the label $n$ stands for a $\star_1$-composition of $n$ crossings. By successive application of $on_1$ and $on_2$, these give a rewriting cycle in $E R$. However, for $k$ being even and $l \neq 1$ they do not have to be taken into account since the whole diagram will become 0 when taking the normal form with respect to $E$. The case $l = 1$ gives a rewriting cycle as follows:

\[ (i_2^*, s_{\lambda,1})_{\lambda} \quad (i_2^*, s_{\lambda,1})_{\lambda} \quad (i_2^*, s_{\lambda,1})_{\lambda} \quad (i_2^*, s_{\lambda,1})_{\lambda} \]

Further sliding the dot term produces

\[ (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \]

further sliding the dot term produces

\[ (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \]

further sliding the dot term produces

\[ (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \]

further sliding the dot term produces

\[ (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \quad (-1)^{\lambda+1} \]
The term with the odd bubble cancels with the corresponding term in (5.17). Continuing with the dot term we have

\[(5.18) \quad (-1)^{\lambda+1} \]

The double bubble term combines with the corresponding term in (5.17) with coefficient \((1+(-1)^\lambda)\), so for \(\lambda\) odd these cancel. But since negative degree bubbles vanish, this diagram is only nonzero if \(\lambda = 0\) in which the two bubbles are both multiples of the odd bubbles that squares to zero. Sliding the remaining dot term completes the cycle.

\[(5.19) \quad \]

This may seem like a special coincidence that the cycle completed, however the diagram that we started with vanishes unless \(\lambda = 0, -1\) using 3-cells \(C_{\lambda}\) and \(B_{\lambda}\), so that an element of the form (5.17) will never appear in a quasi-normal form with respect to \(E\). In general, if there are more dots inside the figure (5.17) the cycle can be shown to complete more generally. In fact, simplifying a diagram of this form with additional dots leads directly to the odd infinite Grassmannian equation. The cycles built in this way are called \textit{indexed cycles}, and are rewriting cycles proper to the context of rewriting modulo.

5.4.3. \textit{Quasi reduced monomials.} Alleaume showed in [4] that linear 2-categories with bubble slide relations cannot be presented by terminating polygraphs, but rather by quasi-terminating polygraphs. For the same reason, 2-supercategories with bubble slide relations can’t be presented with terminating superpolygraphs, but rather quasi-terminating superpolygraphs. Furthermore, rewriting modulo isotopies with the existence of cyclicity 3-cells for crossings imply the existence of cycles of the form:

\[(5.20) \quad \]

The image of these cycles through the Chevalley involution \(\omega\) give rise to similar cycles for the downward crossings. If we consider sideways crossings as defined in (5.7) in terms of upward crossings, we can derive their definition using downward crossing using \(P_{\lambda}\), \(P'_{\lambda}\), and come back to the upward version using \(Q_{\lambda}\), \(Q'_{\lambda}\). As a consequence, the cyclicity 3-cells provide cycles from any kind of crossing to itself. A monomial in \(\mathcal{P}\) is \textit{quasi-reduced} if it is not \(E\)-equivalent to 0 and, up to
indexed cycles, it can be rewritten only using rewriting cycles generated by (5.20) and (5.21) and cycles that slide a bubble through a cap or cup.

\[ \begin{array}{cccc}
\lambda & \Rightarrow & s_{\lambda-2,1} & \Rightarrow \\
\lambda & \Rightarrow & r_{\lambda,1} & \Rightarrow \\
\lambda & \Rightarrow & s_{\lambda} & \Rightarrow
\end{array} \]

Remark 5.10. No quasi-reduced monomial in \( P^n_2 \) can be rewritten as a linear combination of other non-equivalent quasi-reduced monomials.

5.4.4. Weight functions and quasi-normal forms.

Definition 5.11. Let \( C \) be a 2-supercategory, then a weight function on \( C \) is a function \( \tau: C_2 \rightarrow \mathbb{N} \) such that

1. \( \tau(u \star_1 v) = \tau(u) + \tau(v) \)
2. \( \tau(u) = \max\{\tau(u_i) \mid u_i \in \text{Supp}(u)\} \)

When \( C \) presented by (3, 2)-superpolygraph \( P \), such a weight function is uniquely determined by its values on generating 2-cells \( u \). This allows us to define a quasi-ordering \( \succeq \) on \( P^n_2 \) by \( u \succeq v \) if \( \tau(u) \geq \tau(v) \).

We define a weight function on \( \text{Osl}(2)_2^s \) by:

\[ \tau(\bigcup_{\lambda}) = \tau(\bigcap_{\lambda}) = \tau(\bigcap_{\lambda}) = 0, \tau(\bigcap_{\lambda}) = \tau(\bigcup_{\lambda}) = 3 \]

Then for all 3-cells \( \alpha \in E_3 \setminus\{dc\} \), we have \( \tau(s_2(\alpha)) = \tau(h) \) for all \( h \in \text{Supp}(t_2(\alpha)) \), so that all isotopy 3-cells but \( dc \) preserve the weight function. In the procedure below, we only use \( dc \) from left to right, and stop the procedure whenever a 2-cell \( u \) is 0. Then starting with a monomial \( u \) of \( \text{Osl}(2)_2^s \) that does not contain any negative degree bubble, and that is not \( E \)-equivalent to 0:

- While \( u \) is not 0 and can be rewritten with respect to \( E_R \) into a 2-cell \( u' \) such that \( \tau(u) > \tau(u') \), then assign \( u \) to \( u' \).
- While \( u \) is not 0 and can be rewritten with respect to \( E_R \) into a 2-cell \( u' \) without any of the rewriting sequences in the definition of quasi-reduced monomial, namely \( \Gamma_\lambda, on_1, on_2 \) outside of indexed cycles, infinite Grassmannians, reduction of bubbles of degree 0, bubble slide with a through strand, assign \( u \) to \( u' \).

This procedure terminates since \( \succeq \) is well-founded, \( R = \{s_{\lambda,1}, r_{\lambda,1}', P_\lambda, P_\lambda', Q_\lambda, Q_\lambda'\} \) is terminating by Lemma 5.8 and a bubble can only go through a finite number of through strands. It produces a linear combination of quasi-reduced monomials in \( \text{Osl}(2)_2^s \), on which one can only apply cycles generated by (5.20) and (5.21) and bubble slide through a cap or cup. Thus, \( E_R \) is quasi-terminating. Moreover, we will fix a choice of preferred quasi-normal form with respect to these cycles by the following:

- slide the bubble outside of caps and cups, and slide them to the rightmost region of the diagram,
- keep sideways crossings using their definition in terms of upward crossings (5.7), use the cyclicity 3-cell \( P_\lambda' \) provided the number of leftward caps and cups is decreasing, and replace every downward crossing with its value in terms of upward crossings rightward caps and cups as in (5.6) using \( Q_\lambda' \).
5.5. **Confluence modulo.** In this section, we will prove that the \((3, 2)\)-superpolygraph modulo \(E_R\) is confluent modulo \(E\) by showing decreasing confluence of its critical branchings with respect to the quasi-normal form labelling for the quasi-normal forms fixed in Section 5.4.4. We first start by enumerating many 3-cells that can be derived from the generating 3-cells of \(\text{Osl}(2)\), and that will be helpful for the proof of confluence of critical branchings and for the determination of the basis elements.

5.5.1. **Additional 3-cells.** From the definition of the \((3, 2)\)-superpolygraph \(\text{Osl}(2)\) we can derive the following 3-cells in \(E_s\) or \(E_R\). We will often simplify summations involving bubbles by removing the terms involving negative degree bubbles by applying \(b_0^\lambda\) or \(c_0^\lambda\) to each term in a summation containing a negative bubble. To make these types of 3-cells transparent in our notation we introduce a shorthand \(b'_\lambda\) or \(c'_\lambda\) to denote such application of \(b_0^\lambda\) or \(c_0^\lambda\). For example,

\[
\sum_{n=0}^{\lambda} n \lambda \quad \Rightarrow \quad \lambda \quad + \quad \lambda
\]

demonstrates how we will utilize this notation.

For \(\lambda > 0\), define \(A'_\lambda\) to be the 3-cell:

\[
\lambda \quad \Rightarrow \quad \begin{cases} 
0 & \text{if } n < \lambda \\
(-1)^{\lfloor \frac{n+1}{2}\rfloor} \lambda & \text{if } n = \lambda
\end{cases}
\]

We can use this to describe another 3-cell \(A''_\lambda\) for \(\lambda > 0\), defined by:

\[
\lambda \quad \Rightarrow \quad \begin{cases} 
0 & \text{if } n < \lambda \\
\lambda & \text{if } n = \lambda
\end{cases}
\]

For \(\lambda > 0\), let \(B'_\lambda\) be the 3-cell:

\[
\lambda \quad \Rightarrow \quad \begin{cases} 
0 & \text{if } n < \lambda \\
\lambda & \text{if } n = \lambda
\end{cases}
\]

For \(\lambda < 0\), let \(C'_\lambda\) be the 3-cell:

\[
\lambda \quad \Rightarrow \quad \begin{cases} 
0 & \text{if } n < -\lambda \\
\lambda & \text{if } n = -\lambda
\end{cases}
\]

For \(\lambda < 0\), let \(D'_\lambda\) be the 3-cell:

\[
\lambda \quad \Rightarrow \quad \begin{cases} 
0 & \text{if } n < -\lambda \\
\lambda & \text{if } n = -\lambda
\end{cases}
\]

As an illustration of how to derive these 3-cells, let us actually describe the process for creating \(A'_\lambda\). Given that \(\lambda > 0\), we slide the dots in the source of \(A'_\lambda\) through all crossings possible.

\[
\lambda \quad \Rightarrow \quad \begin{cases} 
0 & \text{if } n < \lambda \\
(-1)^{\lfloor \frac{n+1}{2}\rfloor} \lambda & \text{if } n = \lambda
\end{cases}
\]

The first term rewrites to 0 by the 3-cell
For $n < \lambda$, the second term rewrites to 0 by $b'_{\lambda}$

$$
\sum_{r+s = n-1} (-1)^{\left\lceil \frac{n}{2} \right\rceil + s+1} \bigcirc \lambda \Rightarrow 0
$$

For $n = \lambda$, only the $s = \lambda - 1$ term remains non-zero after applying $b'_{\lambda}$ and we can apply $b_{1\lambda}$ to this term to obtain

$$
(-1)^{\left\lceil \frac{\lambda}{2} \right\rceil + \lambda} \bigcirc \lambda \Rightarrow (-1)^{\left\lceil \frac{\lambda+1}{2} \right\rceil} \bigcirc \lambda
$$

Hence, for $\lambda > 0$, we obtain a 3-cell $A'_{\lambda}$ given by

$$
\bigcirc \lambda \Rightarrow \left\{ \begin{array}{ll}
0 & \text{if } n < \lambda \\
(-1)^{\left\lceil \frac{\lambda+1}{2} \right\rceil} \bigcirc \lambda & \text{if } n = \lambda
\end{array} \right.
$$

Using the bubble slide 3-cells of 5.14, we define a 3-cell $s'_{\lambda,n}$ that appears in some of the more complicated computations.

\begin{equation}
\sum_{r \geq 0} (2r + 1) \bigcirc_{n+2r} \Rightarrow \lambda \Rightarrow \bigcirc n+\ast
\end{equation}

We have a 3-cell in $E^s$ given by

\begin{equation}
\text{which allows, up to isotopy and using sideways crossings as defined in (5.7), to give an orientation for the Yang-Baxter relation for upward-upward-downward strands, corresponding to [7, Equation (3.8)]:}
\end{equation}

$$
\lambda \Rightarrow \lambda
$$

We actually can derive such 3-cells either in $E^s$ using $yb$ or in $E^{R^s}$ using $\Gamma_{\lambda}$ to fix an orientation for all the possible configurations of Yang-Baxter 3-cells.

Using the 3-cell $D'_\lambda$ to convert a downward crossing into an upward crossing with rightward caps and cups, and the odd isotopy 3-cells along with the 3-cells from superpolygraph $\text{ONH}$, one can derive the following 3-cells of $E^{R^s}$:

\begin{align*}
dc_{\lambda} \Rightarrow 0, \\
yb_{\lambda} \Rightarrow - \\
\text{on}_{1,\lambda} \Rightarrow - \\
\text{on}_{2,\lambda} \Rightarrow -
\end{align*}
5.5.2. Critical branchings modulo of $\text{Osl}(2)$. We prove that $\mathcal{E} \mathcal{R}$ is confluent modulo $E$ by showing that its critical branchings modulo are confluent and decreasing with respect to the quasi-normal form labelling for the fixed quasi-normal forms. All its critical branchings are proved confluent in Appendix C, and every rewriting step in these decrease the labelling to the quasi-normal form by 1. The classification of critical branchings modulo follows from [13]. Note that from the convergent presentation of the odd nilHecke 2-supercategory given in Section 4.2, all the critical branchings modulo involving two odd nilHecke 3-cells are confluent. There is no critical branching implying the degree condition 3-cells and infinite Grassmannians since these only reduce bubbles of positive degree by assumption, and branchings between degree condition 3-cells and bubble slide 3-cells are trivially confluent since the degree remains negative. There are critical branchings between infinite Grassmannians and bubble slide 3-cells, that are proved confluent in Appendix C.2. Moreover, the critical branchings implied by $P_\lambda$ or $P'_\lambda$ with another 3-cell given by modifying an upward crossing are trivially confluent, since there is a way to deform again the new crossing into the upward one, so that one gets back to the original 2-cell and can apply the other 3-cell of the branching to reach a confluence.

The remaining critical branchings are split into two families:

- branchings coming from the odd nilHecke 3-cells, that is, those involving a 3-cell of $\text{Osl}(2)$ and $o n_1$ or $o n_2$, and branchings that are given by applying two 3-cells on terms that are equal modulo application of $yb$. These branchings are proved confluent in Appendix C.1.
- branchings between the 3-cells $A_\lambda, B_\lambda, C_\lambda, D_\lambda, E_\lambda, F_\lambda$ and $\Gamma_\lambda$. These ones are proved confluent modulo $E$ in Appendix C.2.

6. A BASIS THEOREM FOR ODD CATEGORIZED $\mathfrak{sl}(2)$

Split the $(3,2)$-superpolygraph $\text{Osl}(2)$ into $E$ and $R$ as defined in section 5.3. We have proved the following statement:

**Theorem 6.1.** The $(3,2)$-superpolygraph $\mathcal{E} \mathcal{R}$ is quasi-terminating and confluent modulo $E$.

The quasi-normal forms resulting from the $(3,2)$-superpolygraph modulo $\mathcal{E} \mathcal{R}$ can be described in a diagrammatic fashion. The space 2-morphisms from $\mathcal{E}_2 \mathcal{I}_\lambda = \mathcal{E}_{\varepsilon_1} \ldots \mathcal{E}_{\varepsilon_k} \mathcal{I}_\lambda$ to $\mathcal{E}_2 \mathcal{I}_\lambda = \mathcal{E}_{\varepsilon_1} \ldots \mathcal{E}_{\varepsilon_m} \mathcal{I}_\lambda$, when nonzero, consists of planar diagrams with $k$ points at the bottom equipped with upward/downward oriented collar neighborhoods for each $+$ sign $\varepsilon_1, \ldots, \varepsilon_k$, and $m$ points at the top with collar neighborhoods determined by signs $\varepsilon_1, \ldots, \varepsilon_m$. These endpoints are connected by smoothly immersed directed strands whose endpoints connect the $(k + m)$ vertices compatibly with the orientation on the collar neighborhoods. Further,

- we require that there are no triple intersections and no tangencies;
- no strand intersects itself, and intersects any other strand at most once;
- dots on a given strand appear only in a small interval near the negatively oriented endpoint of a strand connecting the vertices;
- all closed diagrams have been reduced to a product of non-nested dotted bubbles with a counterclockwise orientation. Dots on bubbles are pushed to the rightmost edge of each bubble.
- If any three strands are such that each strand intersects the other two to create a triangle, then the triangle must be in the normal form with respect to the $(3,2)$-superpolygraph $\text{SISO}$ given by one of the following:

- ![Diagram](attachment:triangle.png)
We can further reduce the ambiguity of our chosen basis by making a preferred choice of each super interchange class of diagram. For example, choosing dots and crossings to decrease in height from right to left, with dots appearing above crossings when related by super interchange.

An example of the normal form of a 2-morphism from $E_\lambda E_\lambda E_\lambda E_\lambda 1\lambda$ to $E_\lambda E_\lambda E_\lambda E_\lambda E_\lambda 1\lambda$ is given in the first diagram below, while the second would not be in normal form as it does not have the correct Yang-Baxter representative.

Hence, we have proven the non-degeneracy conjecture for the odd 2-category $\mathcal{U}$ from [7, Section 8].

**Theorem 6.2** (Nondegeneracy Conjecture). Fixing a choice of representative for each super interchange class of elements from the quasi normal form of the $(3,2)$-superpolygraph $\text{Osl}(2)$ gives a basis for each Hom space $\text{Hom}_\mathcal{U}(E_{\alpha_1} E_{\alpha_2} E_{\beta_1}, E_{\alpha_3} E_{\alpha_4} E_{\beta_2})$. In particular, $\text{Hom}_\mathcal{U}(E_{\alpha_1} E_{\alpha_2}, E_{\alpha_3} E_{\alpha_4})$ is a free right $\text{Sym}[d]$-module with $\text{Sym}[d]$ the bubble algebra defined in Remark 5.2.

**Corollary 6.3.** The conjectural classification of dg-structures on the super 2-category $\mathcal{U}(\mathfrak{sl}_2)$ from [32, Proposition 7.1] is a complete classification.

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A.1. **Regular critical branchings.** Here we verify the critical branchings for the $(3,2)$-superpolygraph $SIso(g)$. For every 3-cell other than $\alpha_{m,k}$ and $\beta_{m,k}$, every strand in both the source and target is labelled with $i$, so for branchings that don’t use $\alpha_{m,k}$ and $\beta_{m,k}$, we often write $(-1)^{\lambda_i}$ instead of $(-1)^{\lambda_i}$.

The classification of critical branchings is analogous to that of the 3-polygraph of pearls [20, Section 5.5], with one extra regular critical branching involving the 3-cells $I_0$ and $\alpha_{0,1} - 2$, and two extra indexed critical branchings involving the 3-cells $\alpha_{m,k}$ and $\beta_{m,k}$, coming from the definition of the odd bubble.

\[ (-1)^{(\lambda+1)^2} \]

since $(-1)^{(\lambda+1)^2} = (-1)^{\lambda^2+1} = (-1)^{\lambda+1}$. Diagrams with reverse orientations give the same critical branchings as in the even case, since the use of superinterchange do not create any sign.

\[ (-1)^{\lambda+1} \]

where the * symbol before the rewriting step $d'_{\lambda,0}$ means that we used a superinterchange relation, between the odd bubble and the leftward cup before applying the rewriting step, creating the sign.
Moreover, the critical branching involving $I_0$ and $\alpha_{0,1_{-2}}$ is proved confluent as follows:

\[
\begin{array}{c}
\lambda = 0 \\
\otimes \\
\alpha_{0,1_{-2}} \\
I_0 \\
\end{array}
\]

**Shortened notation for critical branchings.** In order to avoid drawing all the critical branchings entirely, we introduced a shortened diagrammatic representation for these, encoding the minimal amount of data that we need in order to reconstruct the actual branching. We only draw the diagrammatic source and the diagrammatic normal form (or chosen quasi-normal form) of the critical branchings, and indicate between brackets the two rewriting sequences that lead from the source to the common target. If one has to apply super-interchange relations at the source of the critical branching, we will indicate this by adding the element SInt at the beginning of one of the rewriting paths. If one has to apply super-interchange relation in the middle of a rewriting path, we will indicate this by writing a symbol $\ast$ before applying the rewriting step with the correct sign brought by super interchange. Later, when rewriting modulo isotopy, we will indicate using a 3-cell $e$ of the super-isotopy polygraph $E$ before applying a rewriting step $f$ of $R$ by $e \cdot f$.

For example, the last critical branching above is depicted in our shorthand as follows:

\[
\begin{array}{c}
\{ \text{SInt}, (-1)^{\lambda+1} d'_{\lambda,0} \} \\
\{ i^4_{\lambda}, (-1)^{h+1} d'_{\lambda,0} + 2(-1)^{\lambda+1} d'_{\lambda,0} \} \\
\end{array}
\]

We assume that if the two different reductions on a given diagram are applied at different heights, the upper branch of the critical branchings will represent the rewriting sequence corresponding to the application of the uppermost first rewriting step. From now on, unless reconstructing the final result of a given critical branching is difficult for one branch of reductions, we will represent the critical branchings and critical branchings modulo using this notation.

**A.2. Indexed critical branchings.** The classification of indexed critical branchings follows from the indexed critical branchings for the 3-polygraphs of pears in [20], for all possible orientation of strands. Let us draw the ones that differ from the even case, labelled by some odd $i \in I$.

\[
\begin{array}{c}
\{ i^3_{\lambda}, i^1_{\lambda}, u'_{\lambda,0}, \ast (-1)^{h+1} u_{\lambda,0} \} \\
\{ \text{SInt}, (-1)^{\lambda+1} i^3_{\lambda} - u'_{\lambda,1} + 2(-1)^{h+1} u'_{\lambda,0} - 2u_{\lambda,0} - (-1)^{\lambda+2} u_{\lambda,0} \} \\
\end{array}
\]

\[
\begin{array}{c}
\{ i^3_{\lambda}, (-1)^{\lambda} d'_{\lambda,1} + 2(-1)^{h+1} d'_{\lambda,0}, (-1)^{\lambda} (2d_{\lambda,0} - d_{\lambda,0}) + 2(-1)^{\lambda+1} d_{\lambda,0} \} \\
\{ \text{SInt}, (-1)^{\lambda+1} i^3_{\lambda}, (-1)^{\lambda+1} i^2_{\lambda}, (-1)^{\lambda+1} d'_{\lambda,0} + (-1)^{\lambda+1} d_{\lambda,0} \} \\
\end{array}
\]

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B.1. Helpful 3-cells of ONH. We introduce some additional 3-cells in ONH that will be helpful in analyzing the critical branchings. To simplify the description of these critical branchings we make use of the following 3-cells obtained by iterative application of on and on2, see also [7, Lemma 3.1].

\[ (B.1) \quad \bigotimes_n \stackrel{on_1^n}{\Rightarrow} \bigotimes_n (-1)^n + \sum_{a+b=n-1} (-1)^b a b \]

\[ \bigotimes_n \stackrel{on_2^n}{\Rightarrow} \bigotimes_n (-1)^n + \sum_{a+b=n-1} (-1)^a a b \]  

Using these 3-cells we also introduce 3-cells

\[ (B.3) \quad \Theta_{x,y} := \begin{pmatrix} \quad \bigotimes_y \stackrel{on_2^x}{\Rightarrow} (-1)^x \bigotimes_x + \sum_{a+b=x-1} (-1)^b a b \quad \end{pmatrix} \]

\[ \{ on_2^x + Sint \} \bigotimes_x \Rightarrow (-1)^{x+y} \bigotimes_x + \sum_{a+b=x-1} (-1)^a a b \bigotimes_x + \sum_{a+b=x-1} (-1)^b a y \bigotimes_y \]

\[ \Rightarrow \begin{pmatrix} \quad \bigotimes_x \Rightarrow (-1)^{x+y} \bigotimes_x + \sum_{a+b=x-1} (-1)^a a b \bigotimes_x + \sum_{a+b=x-1} (-1)^b a y \bigotimes_y \quad \end{pmatrix} \]

\[ = (-1)^{x+y} \bigotimes_x + \sum_{a+b=x-1} (-1)^a a b \bigotimes_x + \sum_{a+b=x-1} (-1)^b a y \bigotimes_y \]

\[ = (-1)^{x+y} \bigotimes_x + \sum_{a=\min(x,y)} (-1)^{x+a+ay+b} \bigotimes_{x+y-a} \]

\[ \Phi_1 := \begin{pmatrix} \quad \bigotimes_n \stackrel{on_1^n}{\Rightarrow} (-1)^n \bigotimes_n + \sum_{a+b=n-1} (-1)^b a b \quad \end{pmatrix} \]

\[ \{ dc,= \} \Rightarrow \sum_{a+b=n-1} (-1)^b a b \]

\[ \Phi_2 := \begin{pmatrix} \quad \bigotimes_n \stackrel{on_2^n}{\Rightarrow} (-1)^n \bigotimes_n + \sum_{a+b=n-1} (-1)^a a b \quad \end{pmatrix} \]

\[ \{ dc,= \} \Rightarrow \sum_{a+b=n-1} (-1)^a a b \]

and more generally we have

\[ (B.6) \quad \Phi_{x,y} := \begin{pmatrix} \quad \bigotimes_x \stackrel{\Theta_{x,y}}{\Rightarrow} (-1)^{x+y} \bigotimes_x + \sum_{a=\min(x,y)} (-1)^{x+a+ay+\delta_{\max(x,y),x}} \bigotimes_{x+y-a} \quad \end{pmatrix} \]

\[ \Theta_{x,y} \bigotimes_x \Rightarrow \sum_{a=\min(x,y)} (-1)^{x+a+ay+\delta_{\max(x,y),x}} \bigotimes_{x+y-a} \]
It will also be convenient to define a 3-cell $\Upsilon$ given by

(B.7)

\[
\Upsilon := \left( \begin{array}{c}
\begin{array}{cccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \right) + \sum_{a+b = -n-1} (-1)^{a+b} \left( \begin{array}{c}
\begin{array}{cccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \right)
\]

where in the first summation is zero unless $b > c$ and the second is zero unless $b < c$. We will show that the target of this 3-cell is zero. Swapping the $b, c$ variables in the second summation the right hand side can be written

(B.8)

\[
\sum_{a+b+c = n-2} (-1)^{b+1} \left( \begin{array}{c}
\begin{array}{cccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \right) \Phi_{b,c} \sum_{a+b+c, j=c} \sum_{j=c} (-1)^{j+c} \left( \begin{array}{c}
\begin{array}{cccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \right)
\]

where

\[
\sum_{a+b+c = n-2} (-1)^{j+c} - (-1)^{j+b} = \begin{cases} 
2, & \text{if } b \text{ is even, } c \text{ is odd, and } j \text{ is odd,} \\
-2, & \text{if } b \text{ is odd, } c \text{ is even, and } j \text{ is odd,} \\
0 & \text{otherwise.}
\end{cases}
\]

Breaking the $b$ and $c$ summations in (B.9) into a sum over even and odd terms, the only nonvanishing terms are

\[
\sum_{\ell_1=0}^{\lfloor n/2 \rfloor} \sum_{\ell_2=0}^{\lfloor 2\ell_1-1 \rfloor} \sum_{j=2\ell_1+1} (-1)^j \left( \begin{array}{c}
\begin{array}{cccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \right)
\]

\[
+ \sum_{\ell_1=0}^{\lfloor n/2 \rfloor} \sum_{\ell_2=0}^{\ell_1} \sum_{j=2\ell_2} (-1)^j \left( \begin{array}{c}
\begin{array}{cccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\end{array} \right)
\]

Now observe that since $j$ is assumed to be odd, we can remove the $\ell_2 = \ell_1$ term in the second summation since $2\ell_2 \leq j \leq 2\ell_1$ would imply $j$ was even. Similarly, since $j$ is odd the $j$ summation index in the second term can start at $2\ell_2 + 1$ and end at $2\ell_1 - 1$ so that the above terms cancel out and the target of the 3-cell from (B.8) is zero.
B.2. Regular critical branchings of ONH. In this section we study the critical branchings of the (3,2)-superpolygraph ONH. We begin with the relatively straightforward regular branchings using the shorthand notation introduced in Appendix A.1.

\[ \{on_1, -on_2\} \xrightarrow{\text{SInt, } -on_2, on_1, \text{SInt}} \]  
\[ \{dc\} \xrightarrow{\text{SInt, } -yb, -on_2, \text{SInt}} \]

B.3. Indexed critical branchings of ONH. We now verify the indexed critical branchings of ONH, whose classification is the same as in [13], by spelling out in greater detail the required steps as they are somewhat subtle and differ notably from the corresponding calculations in the even setting. The first indexed critical branching is obtained by reducing the diagram

\[ \sum_{a+b+c=n-2} (-1)^{n-1-b} \]

in two possible ways; the top branch is obtained by first applying the Yang-Baxter 3-cell to the bottom half, then sliding the \( n \) dots to the bottom, while the bottom branch is obtained by first
doing super interchange law, applying Yang-Baxter to the top half of the diagram, and sliding the $n$ dots to the bottom. In detail, the top branch is given by the following.

\[
\begin{align*}
- \sum_{a+b=n-1} (-1)^b - \sum_{a+b+c=n-2} (-1)^b \equiv (-1)^n
\end{align*}
\]

The bottom branch is given by

\[
\begin{align*}
- \sum_{a+b+c=n-2} (-1)^{n-1-b}
\end{align*}
\]

where the first summand reduces by $\Phi_2$ into the right-hand side of (B.10) recovering the normal form obtained in the top part of the branching, and the second summand reduces to 0 using the 3-cell in (B.8).
The final indexed critical branching for ONH is obtained from two branches obtained from the left-hand side below

\[
\begin{align*}
\sum_{a+b=n-1} (-1)^b a &- \sum_{a+b+c=n-2} (-1)^b a,b,c \\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\} 
\end{align*}
\]

obtained by applying the Yang-Baxter 3-cell to the top of the diagram, then sliding the \( n \) dots to the bottom using 3-cells \( on_2 \). The bottom branch is obtained by applying super interchange, applying the Yang-Baxter 3-cell to the bottom of the diagram, then sliding the \( n \) dots to the bottom of the diagram using \( on_2 \). In more detail, the top branching is

\[
\begin{align*}
\sum_{a+b=n-1} (-1)^b a &- \sum_{a+b+c=n-2} (-1)^b a,b,c \\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\\} 
\end{align*}
\]

Note that the last summand is a normal form, and the first three terms reduce respectively using the rewriting paths \{yb, SInt, yb, yb\}, \{yb\} and \{SInt, −yb, −dc\} so that this branch of the branching gives

\[
(B.11) 
\begin{align*}
\sum_{a+b=n-1} (-1)^b a &- \sum_{a+b+c=n-2} (-1)^b a,b,c 
\end{align*}
\]

The bottom branch is given by applying the 3-cells below.

\[
(B.12) 
\begin{align*}
\sum_{a+b=n-1} (-1)^b a &- \sum_{a+b+c=n-2} (-1)^b a,b,c 
\end{align*}
\]

Now we relate the terms in (B.12) to those appearing in the top branch (B.11).

- The first summand of (B.12) reduces using the rewriting path \{SInt, −yb, SInt, yb, yb\} to the first summand of (B.11).
• The second summand of (B.12) reduces using the rewriting path \( \{SInt, \sum_{a+b=n-1} \Phi_{1,b} \} \) into

\[
\sum_{a+c+d=n-2} (-1)^{d+1}
\]

(B.13)

• The third summand of (B.12) reduces as follows:

\[
\sum_{a+b=n-1} (-1)^{b+n+1} \xrightarrow{\text{on}^5} \sum_{a+b=n-1} \text{SInt} = \sum_{a+c+d=n-2} (-1)^c
\]

so that the first sum gives the second sum of (B.11), and the second sum gives the third sum of (B.11). As a consequence, using the first and third summand on the bottom branch, we recover all the elements from the top branch.

• The fourth summand of (B.12) reduces using the 3-cell \( \Theta_{b,c} \) as follows:

\[
\begin{aligned}
\Theta_{b,c} \xrightarrow{\text{on}^5} & \sum_{a+b+c=n-2} (-1)^b + \sum_{a+b+c=n-2} \max(b,c)-1 \\
& \sum_{a+b+c=n-2} (-1)^{c+b+j+c+\delta_{\max(b,c),b}}
\end{aligned}
\]

(B.14)

where, after applying the 3-cell \( yb \), the first term on the right-hand side cancels with (B.13), and the second summation above reduces to zero as in the computation of (B.8). The second sum

\[
\sum_{a+b+c=n-2} \sum_{j=\min(b,c)}^{\max(b,c)-1} (-1)^{c+b+j+c+\delta_{\max(b,c),b}}
\]

reduces to 0 as follows:

\[
\sum_{a+b+c=n-2} \sum_{j=\min(b,c)}^{\max(b,c)-1} (-1)^{c+b+j+c+\delta_{\max(b,c),b}} =
\]
Swapping the $b, c$ variables in the first summation, this is equal to
\[
\sum_{a+b+c} \sum_{j} (-1)^{b+j+c} [(-1)^{j} b - (-1)^{j} c] = \sum_{a+b+c} \sum_{j=1}^{n-2} (-1)^{b+c+j} [(-1)^{j} b - (-1)^{j} c]
\]
where
\[
[(-1)^{j} c - (-1)^{j} b] = \begin{cases} 
2, & \text{if } b \text{ is even, } c \text{ is odd, and } j \text{ is odd}, \\
-2, & \text{if } b \text{ is odd, } c \text{ is even, and } j \text{ is odd}, \\
0 & \text{otherwise}.
\end{cases}
\]
In particular, $b + c$ must be odd, so we write this summation as
\[
(B.17) \quad \sum_{b=1}^{n-2} \sum_{c=1}^{b-1} \sum_{j} (-1)^{j} [(-1)^{j} c - (-1)^{j} b]
\]
Breaking the $b$ and $c$ summations in (B.17) into a sum over even and odd terms, the only non vanishing terms are
\[
\sum_{\ell_1=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{\ell_2=0}^{\lfloor \frac{b-1}{2} \rfloor} \sum_{j=2\ell_2+1}^{2\ell_1-1} (-1)^{j} \left[ (-1)^{j(2\ell_2+1)} - (-1)^{j(2\ell_1)} \right]
\]
\[
+ \sum_{\ell_1=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{\ell_2=0}^{\lfloor \frac{b-1}{2} \rfloor} \sum_{j=2\ell_2}^{2\ell_1} (-1)^{j} \left[ (-1)^{j(2\ell_2)} - (-1)^{j(2\ell_1)} \right]
\]
Now observe that since $j$ is assumed to be odd, we can remove the $\ell_2 = \ell_1$ term in the second summation since $2\ell_1 \leq j \leq 2\ell_1$ would imply $j$ was even. Similarly, since $j$ is odd the $j$ summation index in the second term can start at $2\ell_2 + 1$ and end at $2\ell_1 - 1$ so that the above terms cancel out and the target of the 3-cell from (B.15) is zero.

**APPENDIX C. CRITICAL BRANCHINGS MODULO FOR THE FULL 2-CATEGORY**

In this Section, we prove that the critical branchings modulo for the $(3, 2)$-superpolygraph $\text{Osl}(2)$ are confluent modulo $E$.

**C.1. CRITICAL BRANCHINGS FROM 3-CELLS OF ONH.** We prove that the critical branchings implying a 3-cell of $\text{Osl}(2)$ with $on_1$, $on_2$ and two 3-cells of $\text{Osl}(2)$ on two terms that are equal up to $yb$ are confluent modulo $E$.

**Critical branchings** $(A_\lambda, on_{1,\lambda-2})$ For any $\lambda$ in $\mathbb{Z}$, the critical branchings $(A_\lambda, on_{1,\lambda-2})$ are confluent modulo super isotopies as follows:
The last 3-cell used in a bottom sequence is a 3-cell of $E^n$, needed to close the confluence diagram modulo on the right. Note that, when applying the 3-cell $(i^1_\lambda)^{-}\cdot A_\lambda$, we obtain the following 2-cell:

$$\sum_{n=0}^{\lambda} (-1)^n \sum_{n=0}^{\lambda} (-1)^n$$

which reduces using the 3-cell $\sum_{n=0}^{\lambda} (-1)^n \alpha_{-n-1,1_\lambda}$ on the second summand into

$$\sum_{n=0}^{\lambda} \sum_{n=0}^{\lambda} (-1)^n \sum_{n=0}^{\lambda} (-1)^n$$

We can then use the isotopy 3-cell $\sum_{n=0}^{\lambda} i^1_\lambda$ to move dots on the right of the cup of the first summand, and obtain the following term:

$$\sum_{n=0}^{\lambda} \sum_{n=0}^{\lambda} (-1)^n \sum_{n=0}^{\lambda} (-1)^n$$

If $\lambda$ is even, this quantity is

$$\sum_{n=0}^{\lambda} \sum_{n=0}^{\lambda} (-1)^n \sum_{n=0}^{\lambda} (-1)^n$$

If $\lambda$ is odd, this quantity is

$$\sum_{n=0}^{\lambda} \sum_{n=0}^{\lambda} (-1)^n \sum_{n=0}^{\lambda} (-1)^n$$

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which proves that the result is the same as in the top branch, so that this branching is confluent modulo \( E \).

**Critical branchings** \((B_\lambda, (i_1^2) \cdot \text{on}_{1,\lambda-2})\)

\[
\begin{align*}
\lambda & \\
\xrightarrow{B_\lambda, \ast} & \sum_{n=0}^\lambda 2\alpha, \sum_{n=0}^\lambda 2s_{\lambda,1}, \sum_{n=0}^\lambda \alpha_{n-1} & \\
\xrightarrow{\text{Slat}, i_1^2 \cdot (\text{on}_{1,\lambda-2} + 2s_{\lambda,1}), (i_1^2 \ast 2(i_1^2)) \cdot \text{on}_{2,\lambda-2}, \ast B_\lambda} & \sum_{n=0}^\lambda (-1)^n -n-1 \cdot \lambda_{n+1} - (1 + (-1)^\lambda) \cdot \lambda
\end{align*}
\]

Note that after using the odd bubble slides in the top sequence, we use similar arguments as above to prove that the target of the rewriting step is equal to the expected result.

**Critical branchings** \((C_\lambda, (i_1^2) \cdot \text{on}_{2,\lambda-2})\)

\[
\begin{align*}
\lambda & \\
\xrightarrow{C_\lambda} & \sum_{n=0}^\lambda (-1)^n \lambda -n-1 \\
\xrightarrow{\lambda} & \sum_{n=0}^\lambda (-1)^n \lambda -n-1
\end{align*}
\]

where \( X \) is the rewriting sequence given by

\[
\{\text{Slat}, - (i_1^2) \cdot \text{on}_{2,\lambda-2}, -\gamma, \ast (-1)\lambda C_\lambda + \ast 2(-1)^\lambda C_\lambda, \ast 2(-1)^\lambda \sum_{n=0}^\lambda (-1)^n r_{\lambda,1}, \ast 2 \sum_{n=0}^\lambda \beta_{n-1,1_{\lambda+2}}, \ast \sum_{n=0}^\lambda i_1^2 \ast \},
\]

where the 3-cell \( \gamma \) is defined as follows:

\[
\begin{align*}
\xrightarrow{\text{on}_{1,\lambda-2} + 2s_{\lambda,1}} & \lambda (-1)^\gamma \xrightarrow{(i_1^2) \cdot \lambda} \lambda + (-1)^{\lambda+1} \xrightarrow{-2 \cdot \lambda} \lambda
\end{align*}
\]

and the proof that the final result of the bottom sequence is the same as the one obtained in the top branch is made similarly, using bubble slide through a downward strand to make the odd bubble go back into the regular bubble before applying the 3-cell \( 2 \sum_{n=0}^\lambda \beta_{n-1,1_{\lambda+2}} \).

**Critical branchings** \((D_\lambda, \text{on}_2^3)\)

\[
\begin{align*}
\lambda & \\
\xrightarrow{D_\lambda} & \sum_{n=0}^\lambda (-1)^n n+1 \cdot \lambda -n-1 \\
\xrightarrow{\lambda} & \sum_{n=0}^\lambda (-1)^n n+1 \cdot \lambda -n-1
\end{align*}
\]

where \( X \) is the rewriting path defined by

\[
\{\text{Slat}, (-1)^{\lambda+1} \cdot \text{on}_{2,\lambda-2}, (-1)^\lambda \delta, i_1^3 \cdot \ast D_\lambda, (i_1^3) \cdot \ast -2(-1)^\lambda \sum_{n=0}^h r_{\lambda,1}, -2(-1)^\lambda \sum_{n=0}^\lambda \beta_{n-1,1_{\lambda+2}} \}
\]
and the 3-cell \( \delta \) is defined as follows:

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{image1.png}
\end{array}
\end{align*}
\]

and we then prove that we obtain the same result as in the top branch by using \( i_1^3 \) on the first summand, creating an extra term that cancel the third summand above, and reducing the first summand with \( D_\lambda \). After applying these 3-cells, it remains

\[
\sum_{n=0}^\lambda (-1)^n \sum_{r,s,t \geq 0} (-1)^{r+s+t} \lambda_{-n-1}^{r+s} + (1 + (-1)^\lambda) \lambda_{-1}^{r+s+t} + (-1)^{\lambda+1} \lambda_{-1}^{s+t+1} + \sum_{n=0}^\lambda (-1)^{n+r} \lambda_{-n-2}^{n+r} + \sum_{n=0}^\lambda (-1)^{n+r} \lambda_{-n-2}^{n+r}
\]

and we prove after using \( \sum_{n=0}^\lambda (i_1^3)^- \) to place all dots on the left that the final result is equal to the top result using similar arguments.

**Critical branchings** (\( \Gamma_\lambda, \text{on}_{1,\lambda} \)) This critical branching has source

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{image2.png}
\end{array}
\end{align*}
\]

One can use superinterchange and move the dots to the bottom of the diagram: this process gives minus a diagram on which we can apply \( \Gamma \) with a dot at the bottom of the leftmost strand, and two extra terms on which we can apply the 3-cell \( E_\lambda \). By applying these 3-cells, we obtain up to isotopy the following terms

\[
(C.1)
\]

For the other branch, we first apply the 3-cell \( \Gamma \) and then move the dot to the bottom, creating two extra terms on which we can apply the 3-cell \( F_{\lambda+2} \), giving

\[
(C.2)
\]
Indeed, consider for instance the case $\lambda < 0$. In (C.2), the third term reduces to 0 using degree of bubble 3-cells, and 6th and 7th terms are 0 since sums are increasing. In (C.1), the third term also reduces to 0. Moreover, changing variables to $s' = s + 1$ and $t' = t - 1$ gives a sum that is similar to the second element of (C.2), up to extra terms given by $-\lambda$ the term for $s' = -1$ and $+\lambda$ the term for $t' = 0$, which cancel the 6th and 7th terms of (C.1). We proceed similarly for $\lambda > 0$, where the second element of (C.1) reduces to 0 using bubble 3-cells. Note that there is another critical branching implying $\Gamma$ and $on_{1,\lambda}$, given by putting a dot on top of the other upward oriented strand, however this one would be proved confluent in a similar manner.

**Critical branchings** ($F_{\lambda, on_{2,\lambda-2}}$ and $E_{\lambda, on_{1,\lambda-2}}$). Let us denote by $on_{\lambda-2}$ the following composition of 3-cells of $E R^2$:

We then prove the critical branching ($F_{\lambda, on_{2,\lambda-2}}$) confluent modulo $E$ as follows:

where the 3-cell $c'_\lambda$ is the 3-cell defined in 5.5.1. Similarly, the critical branching ($E_{\lambda, on_{1,\lambda-2}}$) is proved confluent modulo $E$ as follows:

where the 3-cell $b'_\lambda$ is the 3-cell defined in 5.5.1.

**Critical branchings** ($u_{\lambda,0} \ast_2 u_{\lambda,0} \cdot F_{\lambda+2, on_{2,\lambda}}$). Starting from here, whenever we write $A \equiv B$ in the source of a branching between 3-cells $f$ and $e \cdot g$ we take $A$ to be the source and $B$ to be the result after applying $e$ to $A$. 

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where

\[ X := \left\{ \eta_{\lambda}, \lambda, -dc + \sum_{n=0}^{\lambda+2} \eta_{\lambda,0} \right\} \]

\[ Y := \left\{ (u_{\lambda,0}' \ast_2 u_{\lambda,0})^\ast \cdot F_{\lambda+2}, (u_{\lambda,0}' \ast_2 u_{\lambda,0})^\ast \cdot s_{\lambda,\lambda+2}', c_{\lambda+2}' \right\}. \]

and the 3-cell \( \eta_{\lambda} := (u_{\lambda,0}') \ast C_{\lambda+2} \) in \( \mathcal{E} \mathcal{R}^s \) is defined as the first rewriting step in the following \( \ast_2 \)-composition of rewriting steps of \( \mathcal{E} \mathcal{R} \).

\[ (C.3) \]

Introduce the shorthand

\[ g(n) := \sum_{n=0}^{\lambda+2} (1)^n \]

\[ h(n) := \sum_{n=0}^{\lambda+2} (1)^n \]

Then the result of the top branch (and the critical branching) is \( \sum_{n=0}^{\lambda+2} (1)^n g(n) \).

In the bottom branch, the result after applying the steps up to and including \( (u_{\lambda,0}' \ast_2 u_{\lambda,0}) \) is

\[ h(0) - \sum_{n=0}^{\lambda+1} \sum_{r\geq 0} (1)^r g(n + r + 1) \]

We can write this as follows:

\[ h(0) - \sum_{n=0}^{\lambda+1} \sum_{r\geq 0} (1)^r g(n + r + 1) = h(0) - \left( \sum_{z\geq 1} 2zg(2z) + \sum_{z\geq 0} g(2z + 1) \right) \]

\[ = h(0) - \sum_{z\geq 0} (2z + 1)g(2z) + \sum_{t\geq 0} (-1)^tg(t) \]

Then \( h(0) - \sum_{z\geq 0} (2z + 1)g(2z) \Rightarrow 0 \), so we have that

\[ h(0) - \sum_{n=0}^{\lambda+1} \sum_{r\geq 0} (1)^r g(n + r + 1) \Rightarrow \sum_{t\geq 0} (-1)^tg(t) \Rightarrow \sum_{t=0}^{\lambda+2} (-1)^t g(t) = \sum_{n=0}^{\lambda+2} (-1)^t g(n) \]
Critical branching \((F_{\lambda,s}(u_{\lambda+2,0} \ast 2 u_{\lambda+2,0} \ast 2 yb \ast 2 u_{\lambda+2,0}) \ast C_{\lambda+4})\) Consider the critical branching
\[
\begin{array}{ccc}
\lambda & \equiv & \lambda \\
\xrightarrow{X} & \xrightarrow{Y} & - \lambda + 3 \sum_{r=0}^{\lambda+4} \sum_{s=0}^{\lambda+4} (-1)^{r+s} \\
\end{array}
\]
with
\[
X := \left\{ (u'_{\lambda+2,0} \ast 2 u_{\lambda+2,0} \ast 2 yb \ast 2 u_{\lambda+2,0}) \ast C_{\lambda+4}, *u_{\lambda+2,0} \sum_{n=0}^{\lambda+4} (-1)^{n} a_{n}^{n}, \sum_{n=0}^{\lambda+4} de^{\lambda} \right\}
\]
\[
Y := \left\{ \text{SInt}, -F_{\lambda+4}, *u'_{\lambda+2,0} \ast 2 u_{\lambda+2,0} \sum_{n=0}^{\lambda+4} (-1)^{n+r} \sum_{r=0}^{\lambda+4} \sum_{r+1}^{\lambda+4} a_{n}^{n} \right\}
\]
where the 3-cell \(\eta_{\lambda}\) is defined in \((C.3)\) and the 3-cell \(\gamma\) in the bottom branch will be defined in \((C.7)\) below.

Let us denote by \(f(a,b)\) the monomial
\[
f(a,b) := \bigcirc_{\lambda+2-(a+b)+*}^{\lambda} \uparrow_{a} \downarrow_{b}
\]
Then using \(s'_{\lambda+2,\lambda+2-a}\), we get:
\[
(C.6) \sum_{a=0}^{\lambda+2} \sum_{b=0}^{\lambda+2} (-1)^{a} (2b + 1) f(2b, a) \\ s'_{\lambda+2,\lambda+2-a} \Rightarrow \sum_{n=0}^{\lambda+3} \sum_{r=0}^{\lambda+3} (-1)^{n} \bigcirc_{n-1}^{n} \xrightarrow{\lambda}
\]
and thus in particular we get that there is rewriting sequence \(\gamma := \{\eta_{\lambda}, u_{\lambda+2,0} \cdot s'_{\lambda+2,\lambda+2-a}\}\) of \(E_{R}S\) obtained from \(\eta_{\lambda}\) and \((C.6)\) as follows:
\[
(C.7) \bigcirc_{\lambda}^{\lambda} - \sum_{a=0}^{\lambda+2} \sum_{b=0}^{\lambda+2} (-1)^{a} (2b + 1) f(2b, a) \\ \gamma \Rightarrow 0
\]
Note that before applying the 3-cell \(\gamma\) in the bottom branch, we have obtained the polynomial
\[
(C.8) \left(\begin{array}{c}
-1\lambda+1 \\
\end{array}\right) \bigcirc_{\lambda}^{\lambda} - \sum_{n=0}^{\lambda+3} \sum_{r=0}^{\lambda+3} (-1)^{n} \bigcirc_{n-r+*}^{n-r} \xrightarrow{\lambda} + \left(\begin{array}{c}
-1\lambda+1 \\
\end{array}\right) \sum_{n=0}^{\lambda+3} \sum_{r=0}^{\lambda+3} (-1)^{n+r} \sum_{s=0}^{\lambda+3} \bigcirc_{n+r-1}^{n+r} \xrightarrow{\lambda}
\]
We now show that the first summand of \((C.8)\) cancels the third using the 3-cell \(\gamma\) from \((C.7)\). Using the 3-cells \(c_{\lambda}\) to remove the terms containing bubbles of negative degree, the last term reduces to
\[
(-1)^{\lambda} \sum_{n=0}^{\lambda+3} \sum_{r=0}^{\lambda+3} \sum_{s=0}^{r} (-1)^{n+r} f(n + r - s, r)
\]
\[
= (-1)^{\lambda} \sum_{n=0}^{\lambda+3} \sum_{r=0}^{\lambda+3} \sum_{a=0}^{\lambda+2} (-1)^{a+n} \bigcirc_{n-r}^{n-r} f(\lambda + 2 - r' - a, a)
\]
\[
= (-1)^{\lambda} \sum_{a=0}^{\lambda+2} \sum_{r=0}^{\lambda+2} \sum_{n=0}^{\lambda+2} (-1)^{a+n} f(\lambda + 2 - r', a)
\]
\[
= (-1)^{\lambda+2} \sum_{a=0}^{\lambda+2} (-1)^a f(\lambda + 2 - r'/a) \left( \sum_{n=0}^{\lambda+2-r'/a} (-1)^{(\lambda+2-r'/a)n} \right)
\]

where we set \(r' = \lambda + 2 - n - r\) and \(s = a\) in the second equality and exchanged the summation order in the third. Now let \(b' = \lambda + 2 - a - r'\)

\[(C.9)\]

\[
= (-1)^{\lambda} \sum_{a=0}^{\lambda+2} \sum_{b'=0}^{\lambda+2-a} (-1)^a f(b', a) \left( \sum_{n=0}^{\lambda+2-a} (-1)^{b'n} \right)
\]

When \(b'\) is odd the \(n\) summation gives zero, but when \(b'\) is even it gives a coefficient \(b + 1\). Keeping only the nonzero terms gives

\[
= (-1)^{\lambda} \sum_{a=0}^{\lambda+2-a} (-1)^a (2b + 1) f(2b, a)
\]

so that

\[
(-1)^{\lambda} \sum_{n=0}^{\lambda+2} \sum_{r \geq 0} \sum_{s=0}^{r} (-1)^{r+s} r^s f(n + r - s, s) = (-1)^{\lambda} \sum_{a=0}^{\lambda+2} \sum_{b \geq 0} (-1)^a (2b + 1) f(2b, a).
\]

Therefore, after applying the 3-cell \(\gamma\) from (C.7) to (C.8), only the second term remains

\[
- \sum_{n=0}^{\lambda+3} \sum_{r \geq 0} (-1)^n \sum_{r \geq 0} \sum_{s \geq 0} (-1)^{r+s} r^s \left( \begin{array}{c} \lambda+3-r+s \lambda+3-r-s \\ n-r-s \end{array} \right) \left( \begin{array}{c} \lambda \lambda \end{array} \right) = \sum_{r \geq 0} \sum_{s \geq 0} (-1)^{r+s} r^s \left( \begin{array}{c} \lambda+3-r+s \lambda+3-r-s \\ n-r-s \end{array} \right) \left( \begin{array}{c} \lambda \lambda \end{array} \right)
\]

agreeing with the result in (C.5) of the top branch, establishing that this critical branching is confluent modulo \(E\).

**Critical branching** \((E_\lambda, (u_{\lambda} *_2 y_{b} *_2 (u_{\lambda,0})_s *_2 u_{\lambda+2,0})_s \cdot \Gamma_\lambda)\) Recalling the definition of sideways crossings from 5.7, we describe a critical branching between \(E_\lambda\) and \((u_{\lambda} *_2 y_{b} *_2 (u_{\lambda,0})_s *_2 u_{\lambda+2,0})_s \cdot \Gamma_\lambda\)

\[
\equiv (-1)^{\lambda+1} \sum_{x,y,r \geq 0} (-1)^{x+r} \sum_{x-y-r=3} \Gamma_\lambda
\]

with

\[
X := \left\{ E_\lambda, s u_{\lambda,0} \cdot \sum_{n=0}^{\lambda-1} \sum_{r \geq 0} s on_2^n, dc \right\}
\]

\[
Y := \left\{ \text{SInt}, (u_{\lambda} *_2 y_{b} *_2 (u_{\lambda,0})_s *_2 u_{\lambda+2,0})_s \cdot \Gamma_\lambda, \Omega \right\}
\]

Where \(\Omega\) is the rewriting sequence described below. The 3-cell \(\Omega\) has source
The second term rewrites to 0 by either $A'_{\lambda+2}$ for $\lambda > -2$ or $c_{\lambda+2}^0$ for $\lambda \leq -2$. The first term rewrites by $E_{\lambda+2}$ to

$$(1)^\lambda$$

plus an extra sum which is reduced to

$$-\lambda-3 \sum_{n=0}^{\frac{-n-k}{2}} (-1)^{n+k}$$

via the rewriting sequence $\{ u_{\lambda+2,0} \cdot on_1, (u'_{\lambda,0})^{-} \cdot F_{\lambda+2} + C'_{\lambda+2} \}$.

Hence, first term rewrites to:

(C.10) $$(1)^\lambda$$

For the third term, we use super isotopy and $on_2$ to move the $s$ dots through the sideways crossing and then move them below the $t$ dots to obtain

$$\sum_{x,y,r} (-1)^{x+r} \sum_{a,b,r,t} (-1)^{r+a+bt-t}$$

And one can check that the second term of this cancels with the second term of C.10 once we apply bubble slides.

C.2. Critical branchings from odd $sl(2)$-relations. We prove that the critical branching between two 3-cells of the set $\{A_{\lambda}, B_{\lambda}, C_{\lambda}, D_{\lambda}, E_{\lambda}, F_{\lambda}, \Gamma_{\lambda}\}$ are confluent modulo $E$.

A and C

i) For $\lambda < 0$:

$$\sum_{x,y,r} (-1)^{x+r}$$

ii) For $\lambda = 0$:

$$\sum_{a,b,r,t} (-1)^{r+a+bt-t}$$

iii) For $\lambda > 0$, the calculation is similar to the case $\lambda < 0$, except $A_{\lambda}$ takes it to 0 instead of $C_{\lambda}$.

A and F

i) For $\lambda < 0$:
ii) For $\lambda = 0$: 

$$\{A_0, b_1^0, D_0, c_1^0\} \rightarrow \{A_0\} \rightarrow 0$$

iii) For $\lambda > 0$, 

$$\{F_\lambda, b_1^\lambda, c_1^\lambda, b_1^\lambda, c_1^\lambda\} \rightarrow 0$$

**B and D**

i) For $\lambda < 0$, 

$$\{B_\lambda, c_1^\lambda, c_1^\lambda + b_1^\lambda\} \rightarrow 0$$

ii) For $\lambda = 0$: 

$$\{B_0, b_1^0\} \rightarrow \{D_0, c_1^0, I_0\}$$

iii) For $\lambda > 0$, we get a similar calculation as for $\lambda < 0$ except $B_\lambda$ takes it to 0 instead of $D_\lambda$.

**E and F**

i) For $\lambda < 0$: 

$$\{E_\lambda, \sum_{n=0}^{\lambda-1} \sum_{r \geq 0} D_r^\lambda\} \rightarrow \{E_0\} \rightarrow (-1)^\lambda$$

ii) For $\lambda = 0$: 

$$\{E_0\} \rightarrow \{E_0\}$$
iii) For $\lambda > 0$:

\[ \{F_\lambda, \sum_{n=0}^{\infty} \sum_{r \geq 0} B'_\lambda\} \xrightarrow{\lambda} (-1)^{\lambda} \xrightarrow{E_\lambda} \]

The other family of critical branchings with $F_\lambda$ and $E_\lambda$ would be proved to be confluent modulo $E$ in a similar manner.

**B and F**

i) For $\lambda > 0$:

\[ \{F_\lambda, b'_\lambda, c'_\lambda, c^1_\lambda, b^1_\lambda\} \xrightarrow{B_\lambda} 0 \]

ii) For $\lambda = 0$:

\[ \{B_0, b^1_0, c^1_0, c^0_0\} \xrightarrow{F_0} \]

iii) For $\lambda < 0$:

\[ \{B_\lambda, \sum_{n=0}^{\lambda} C'_\lambda, b^1_\lambda\} \xrightarrow{E_\lambda} (-1)^{\lambda} \xrightarrow{(-1)^{\lambda}} \]

**E and D**

i) For $\lambda \geq 0$:

\[ \{D_\lambda, \sum_{n=0}^{\lambda} A'_\lambda, c^1_\lambda\} \xrightarrow{E_\lambda} (-1)^{\lambda} \xrightarrow{(-1)^{\lambda}} \]

ii) For $\lambda < 0$:

\[ \{E_\lambda, c'_\lambda, b'_\lambda, b^1_\lambda, c^1_\lambda\} \xrightarrow{D_\lambda} 0 \]

**C and E**
i) For $\lambda > 0$:

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram1.png}}
\end{array}
\end{align*}
\]

ii) For $\lambda = 0$:

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram2.png}}
\end{array}
\end{align*}
\]

iii) For $\lambda < 0$:

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram3.png}}
\end{array}
\end{align*}
\]

Critical branching $(\Gamma_\lambda, C_\lambda)$

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram4.png}}
\end{array}
\end{align*}
\]

Note that there is also a critical branching between $\Gamma_\lambda$ and $D_\lambda$ given by attaching to the source of $\Gamma_\lambda$ a rightward cup on bottom on the rightmost two strands. This one is proven confluent in a similar manner.

Critical branching $(\Gamma_\lambda, F_\lambda)$

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram5.png}}
\end{array}
\end{align*}
\]

where $B'_\lambda$ is the 3-cell that reduces the term

\[
\sum_{n=0}^{\lambda-1} \sum_{r=0}^{\lambda-1} (-1)^{n+r} \sum_{a+b=n-1} (-1)^a b_{\lambda}
\]
into 0, as defined in 5.5.1. The 3-cell $\Omega$ is defined as the following composition of $E_R$-rewriting steps: when applying $\Gamma_\lambda$ we obtain the polynomial

$$\lambda - \sum_{r,s,t \geq 0} (-1)^{\lambda+r+s+1} \lambda^r t^s t^t - \sum_{r,s,t \geq 0} (-1)^{\lambda+r+s+t} \lambda^r t^s t^t$$

The third term reduces to 0 using the 3-cell $D'_\lambda$ into 0 since $s < \lambda$, the first term reduces using \{(s(u'_{\lambda,0})*2yb*2(u'_{\lambda,0}^-))\cdot sF_{\lambda+2}\} into

$$(-1)^\lambda$$

plus an extra term that one might check is cancelled by the term obtained from the second summand when using super isotopies and making the $r$ dots move to the bottom of the crossing, so that it only remains the terms where the dots break the crossing, giving the summand

$$\sum_{a,b,s,t \geq 0} (-1)^{s+\lambda+b+(\lambda+a+b)(a+t)}$$

and one checks that this reduces using bubble slide 3-cells $s'_{\lambda,n}$ into the second term of the final result. Note that there also is a critical branching between $\Gamma_\lambda$ and $E_\lambda$ given by attaching to the source of $\Gamma_\lambda$ a rightward crossing on bottom on the rightmost two strands. This one would be proved confluent in a similar manner.

**Critical branching** ($ig_{2n}, s^+_{2n,\lambda}$)

$$2n+\lambda \xrightarrow{\{s^+_{\lambda,2n,\lambda} \mid \sum_{r=0}^{n-1} \sum_{r=0}^{n-1} ig_{2n-2r,\lambda}\}} (2n+1) \xrightarrow{\{ig_{2n,\lambda+2,s^+_{\lambda,2n-2r,\lambda}} \mid \sum_{r=0}^{n-1} \sum_{r=0}^{n-1} ig_{2n-2r,\lambda}\}} 2n-2r-2\ell+\lambda$$

where the last $ig$ 3-cell in the bottom branch is only applied to terms without a counter-clockwise bubble of positive degree. Note that there is a similar branching between $ig_{2n}$ and $r_{2n,\lambda}$ given by changing the upward strand to the right of the bubble in the source of the last branching to a downward strand. This would be proved confluent in a similar manner.

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