Abstract
We obtain an explicit characterization of linear maps, in particular, quantum channels, which are covariant with respect to an irreducible representation (\( U \)) of a finite group (\( G \)), whenever \( U \otimes U^c \) is simply reducible (with \( U^c \) being the contragradient representation). Using the theory of group representations, we obtain the spectral decomposition of any such linear map. The eigenvalues and orthogonal projections arising in this decomposition are expressed entirely in terms of representation characteristics of the group \( G \). This in turn yields necessary and sufficient conditions on the eigenvalues of any such linear map for it to be a quantum channel. We also obtain a wide class of quantum channels which are irreducibly covariant by construction. For two-dimensional irreducible representations of the symmetric group \( S(3) \), and the quaternion group \( Q \), we also characterize quantum channels which are both irreducibly covariant and entanglement breaking.

1 Introduction
Quantum channels are fundamental building blocks of any quantum communication- or information-processing system. A channel (classical or quantum) is inherently noisy, and its potential for communication is quantified by its capacity, i.e. the maximum rate at which information can be transmitted reliably through it. Unlike its classical counterpart, a quantum channel has a number of different capacities. These depend on various factors, e.g. the nature of information transmitted (classical, private classical or quantum), the nature of the input states (entangled or product), the absence or presence of any additional resource, e.g. prior shared entanglement between the sender and the receiver, the nature of the measurements, if any, done on the output states (collective or individual) etc. Evaluating the capacities of quantum channels is one of the most important and challenging problems in quantum information theory. The problem becomes more tractable, however, if the channel in question satisfies certain symmetries. Suitably exploiting these symmetries leads to a simplification of the problem and allows us to infer more about the properties of the channel.

To elucidate this, let us consider the simplest communication scenario, namely, the transmission of classical information through a memoryless quantum channel (say, \( \Phi \)). The latter is a channel...
for which each use is independent and identical to all prior uses. Hence there is no correlation in
the noise acting on the inputs to successive uses of the channel. The classical capacity, \( C(\Phi) \), is the
maximum number of bits that can be reliably transmitted per use of the channel. It is evaluated
in the limit \( n \to \infty \) (where \( n \) denotes the number of uses of \( \Phi \)), under the constraint that the
error incurred in the protocol vanishes in this limit. Thanks to the celebrated Holevo-Schumacher-
Westmoreland theorem \cite{14},\cite{27}, the classical capacity \( C(\Phi) \), of any memoryless quantum channel
\( \Phi \) is known to be given by the following \textit{regularized} expression:

\[
C(\Phi) = \lim_{n \to \infty} \frac{1}{n} \chi^*(\Phi^\otimes n),
\]  

where \( \chi^*(\Phi) \) is called the \textit{Holevo capacity} of the channel. The latter is equal to the classical capacity
of \( \Phi \) evaluated under the constraint that the input states are product states. It is given by the
following single-letter expression:

\[
\chi^*(\Phi) := \sup_{\{p_i, \rho_i\}} \left\{ S \left( \sum_i p_i \Phi(\rho_i) \right) - \sum_i p_i S(\Phi(\rho_i)) \right\},
\]

where the supremum\(^1\) is taken over ensembles of quantum states \( \rho_i \) occurring with probabilities \( p_i \),
and \( S(\rho) := -\text{Tr}(\rho \log \rho) \) denotes the von Neumann entropy of a state \( \rho \).

Unfortunately, due to the regularization present in it, the expression (1) of the (unconstrained)
classical capacity, \( C(\Phi) \), is in general intractable, and cannot be computed. If, however, the Holevo
capacity of the channel is additive, then the expression for the classical capacity reduces to a
single letter one: \( C(\Phi) = \chi^*(\Phi) \), and in this case, it is clear that using entangled inputs does
not provide any advantage in the transmission of classical information through \( \Phi \). By providing a
counterexample to the so-called \textit{additivity conjecture}, which had been the focus of active research
for more than a decade, Hastings \cite{12} proved that the Holevo capacity of a quantum channel is not
necessarily additive.

Determining whether the Holevo capacity of a given channel is additive or not is a rather non-
trivial problem. However, there are important families of channels satisfying certain symmetry
properties, for which this problem can be simplified. These are the so-called \textit{irreducibly covariant}
channels, defined below. If \( \Phi \) is such a channel, then its Holevo capacity, \( \chi^*(\Phi) \), is linearly related to
its minimum output entropy \( S_{\min}(\Phi) := \min_\rho S(\Phi(\rho)) \). Hence the problem of determining whether
\( \chi^*(\Phi) \) is additive reduces to the relatively simpler problem of determining whether \( S_{\min}(\Phi) \) is
additive. The additivity of the Holevo capacity has been successfully established for various families
of irreducibly covariant channels (see e.g. \cite{4, 5, 6, 8, 10} and references therein), by proving that
the corresponding minimum output entropy is additive.

Let us introduce the definition of irreducibly covariant channels. Let \( G \) be a finite (or compact)
group and for every \( g \in G \), let \( U(g) \) and \( V(g) \) be unitary representations in the Hilbert spaces \( H \)
and \( K \) respectively. Then a quantum channel \( \Phi \), with input Hilbert space \( \mathcal{H} \) and output Hilbert
space \( \mathcal{K} \), is said to be \textit{covariant} with respect to these representations if for any input state \( \rho \),

\[
\Phi \left( U(g)\rho U(g)^\dagger \right) = V(g)\Phi(\rho)V(g)^\dagger \quad \forall \ g \in G.
\]  

\(^1\)If the output Hilbert space (\( \mathcal{K} \), say) of the channel is finite dimensional, then the supremum in eq. (2) is a
maximum, and is attained for an ensemble of at most \( (\dim \mathcal{K})^2 \) states.
Moreover, if the representations $U$ and $V$ are irreducible, then the channel is said to be irreducibly covariant. For an irreducibly covariant channel $\Phi$, the Holevo capacity and minimum output entropy satisfy the following linear relation [13]: $\chi^*(\Phi) = \log d - S_{\min}(\Phi)$, where $d := \dim K$.

The symmetries underlying (irreducibly) covariant channels make them amenable to analysis in various information-theoretic problems, e.g.

establishing strong converse properties of classical capacities, channel discrimination, obtaining second order asymptotics for entanglement assisted- and private classical communication etc. Köenig and Wehner [19] proved that the classical capacity of any covariant quantum channel for which the minimum output entropy is additive, satisfies the so-called strong converse property. That is, any communication protocol with rate larger than the classical capacity fails with certainty (i.e. the probability of error in transmission converges to one) in the limit of asymptotically many uses of the channel. In [7], the strong converse property of the entanglement-assisted classical capacity of covariant channels was established. Recently, Jencova proved [18] that two irreducibly covariant channels can be discriminated using an optimal scheme employing a maximally entangled input state. Wilde et al [31] obtained second order expansions of relative entropy of entanglement bounds for private communication rates for covariant channels. These bounds also hold in the standard setting of private communication through a quantum channel, in which the sender and receiver have access to unlimited public classical communication. The bounds are useful for establishing converse bounds for quantum key distribution protocols conducted over these channels. Finally covariance with respect to some finite or compact group can be used to investigate asymmetry properties of pure quantum states [20]. The above discussion and examples illustrate the relevance of (irreducibly) covariant channels, and highlights the importance of understanding the structure and properties of such channels.

There have been some notable results related to the characterization of covariant channels, and a study of their properties in particular for some low dimensional compact groups. For example, Scutaru [28] proved a Stinespring type theorem, in the $C^*$-algebraic framework, for any completely positive linear map which is covariant with respect to a unitary representation of a locally compact group. In [23] covariance of quantum channels with respect to the $SU(2)$ group was investigated. In [23] the notion of EPOSIC channels was introduced, and as an application some new positive maps, which are not completely positive, were derived. It is worth mentioning here, that imposing covariance with respect to $SU(2)$ gives a way to understand more about entanglement in quantum spin systems [26] and also allows us to prove an extended version of the Lieb-Mattis-Schultz theorem by use of Matrix Product States [25]. Moreover, taking the group $SU(n)$, we can obtain direct proof of dimerization of quantum spin chains [22]. Mendl and Wolf [21] studied unital channels which are covariant with respect to the real orthogonal group, and determined the subset of these channels which are convex combinations of unitaries. In [4] complementarity and additivity of various covariant channels, such as the depolarizing and Weyl-covariant channels, were studied.

In this paper we obtain a detailed mathematical description of channels which are irreducibly covariant with respect to a finite group $G$ in the case in which: (i) the input and output Hilbert spaces of the channel are the same, and (ii) if $U$ is the particular unitary irreducible representation (irrep) considered, then $U \otimes U^c$ is simply reducible (or multiplicity free), where $U^c$ denotes the contragradient representation, i.e. for every $g \in G$, $U^c(g) = U(g^{-1})^T \equiv \overline{U}(g)$. Firstly, we obtain the spectral decomposition of the Choi-Jamiolkowski image of any linear map which is irreducibly covariant. The eigenvalues and orthogonal projections arising in this decomposition are expressed entirely in terms of representation characteristics of the group. This in turn yields necessary and sufficient conditions on the eigenvalues of the linear map, for which it is a quantum channel.
(i.e. a completely positive and trace-preserving map). We also obtain explicit expressions for the Kraus operators of such channels. See Theorem 40 and Theorem 41 of Section 6. Moreover we give geometrical interpretation of the set of the solutions for which given irreducibly covariant linear map (ICLM) is a irreducibly covariant quantum channel (ICQC). Namely we show, that all eigenvalues of the Choi-Jamiołkowski image of an ICQC necesarliy lie in the intersection of contracted simplex and certain subspace defined by the matrix obtained spectral analysis of the projectors appearing in the decomposition of an ICLM. See Proposition 43, and Proposition 47 of Section 7.

Using all above-mentioned characterization, for any finite, multiplicity free group, we obtain a wide class of quantum channels which are irreducibly covariant by construction. In addition, we provide explicit examples of ICQCs for certain multiplicity free groups, namely, the symmetric groups, $S(3)$ and $S(4)$, and the quaternion group $Q$. In each case we present both the matrix representation and the Kraus representation of the ICQC. Further, for the case of $S(3)$ and $Q$, using the Peres-Horodecki or positive partial transpose (PPT) criterion [16, 24], we also obtain the condition under which the ICQC is an entanglement breaking channel (EB) [29].

To give a flavour of our results, let us consider the example of the symmetric group $G = S(3)$, and the family of channels which are irreducibly covariant with respect to the two-dimensional irrep $U$ characterised by the partition $\lambda = (2, 1)$ using the so-called $\epsilon$-representation [1]. We prove (see Section 8.3) that the Kraus operators for this family of channels is given by:

$$
K_1(\lambda) = \sqrt{\frac{1}{2}(1 - l_{\text{sgn}})} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_2(\lambda) = \sqrt{\frac{1}{2}(1 - l_{\text{sgn}})} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
$$

$$
K_3(\text{sgn}) = \sqrt{\frac{1}{2}(1 + l_{\text{sgn}} - 2l_{\lambda})} \begin{pmatrix} -\sqrt{\frac{1}{2}} & 0 \\ 0 & \sqrt{\frac{1}{2}} \end{pmatrix}, \quad K_4(\text{id}) = \sqrt{\frac{1}{2}(1 + l_{\text{sgn}} + 2l_{\lambda})} \begin{pmatrix} \sqrt{\frac{1}{2}} & 0 \\ 0 & \sqrt{\frac{1}{2}} \end{pmatrix},
$$

where id, sgn, $\lambda$ denote inequivalent irreps of the symmetric group $S(3)$. In the eq. (4), $l_{\text{sgn}}$ and $l_{\lambda}$ are two real parameters, which are constrained to take values in the triangle shown in the left panel of Figure 1. Further, using the Peres-Horodecki or PPT criterion [16, 24], we show that irreducibly covariant channels for which the values of these parameters lie in the grey region shown in the right panel of Figure 1, are necessarily entanglement breaking (see Section 8.5).

## 2 Notations and Definitions

We first define the concepts of vectorization and matrix representation of a linear map. Let $\mathbb{M}(n, \mathbb{C})$ denote the space of $n \times n$ complex matrices and let $\{E_{ij}\}_{i,j=1}^n$, where $E_{ij} \equiv |i\rangle \langle j|$, denote a basis of $\mathbb{M}(n, \mathbb{C})$. Consider a matrix $A := (a_{ij})_{i,j=1}^n \equiv (a_{ij}) \in \mathbb{M}(n, \mathbb{C})$, with $a_{ij}$ denoting the $(i, j)^{th}$ matrix element of $A$. The matrix $A$ can be expressed as a vector in $\mathbb{C}^{n^2}$ using the standard matrix-vector isomorphism, referred to as vectorization.

**Definition 1 (Vectorization).** Consider a map $\text{vec} : \mathbb{M}(n, \mathbb{C}) \rightarrow \mathbb{C}^{n^2}$, such that for any $A = (a_{ij}) \in \mathbb{M}(n, \mathbb{C})$

$$
\text{vec}(A) = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn})^T \in \mathbb{C}^{n^2},
$$

where the superscript $^T$ denotes the transpose. In Dirac notation, $\text{vec}(E_{ij}) \equiv \text{vec}(|i\rangle \langle j|) = |i\rangle \otimes |j\rangle \in \mathbb{C}^{n^2}$. The map vec is an isomorphism between the linear spaces $\mathbb{M}(n, \mathbb{C})$ and $\mathbb{C}^{n^2}$, and we refer to it as vectorization.
Consider a linear map $\Phi \in \text{End} \left[ \mathcal{M}(n, \mathbb{C}) \right]$, i.e. $\Phi : \mathcal{M}(n, \mathbb{C}) \rightarrow \mathcal{M}(n, \mathbb{C})$. Its Choi-Jamiołkowski image $J(\Phi)$ is given by [17, 3]:

$$J(\Phi) := \sum_{i,j=1}^{n} E_{ij} \otimes \Phi(E_{ij}) \in \mathcal{M}(n^2, \mathbb{C}). \quad (6)$$

It is well-known that a linear map $\Phi \in \text{End} \left[ \mathcal{M}(n, \mathbb{C}) \right]$ is completely positive map if and only if its Choi-Jamiołkowski image $J(\Phi)$ is a positive semidefinite matrix, i.e. $J(\Phi) \succeq 0$.

The matrix resulting from the action of $\Phi$ on any basis element $E_{ij} \in \mathcal{M}(n, \mathbb{C})$ can be expressed as follows:

$$\Phi(E_{ij}) = \sum_{k,l=1}^{n} \phi_{kl,ij} E_{kl}. \quad (7)$$

The coefficients $\phi_{kl,ij}$ can be viewed as elements of an $n^2 \times n^2$ matrix, and hence we use the notation:

$$\text{mat}(\Phi) \equiv (\phi_{kl,ij}) \in \mathcal{M}(n^2, \mathbb{C}). \quad (8)$$

Further, using the isomorphism $\mathcal{M}(n^2, \mathbb{C}) \simeq \mathcal{M}(n, \mathbb{C}) \otimes \mathcal{M}(n, \mathbb{C})$, the matrix $\text{mat}(\Phi) = (\phi_{kl,ij}) \in$
\( \mathbb{M}(n^2, \mathbb{C}) \) may be written in the form:

\[
\text{mat}(\Phi) = \sum_{\nu=1}^{m} A^{\nu} \otimes B^{\nu}; \quad \phi_{kl,ij} = \left( \sum_{\nu=1}^{m} A^{\nu} \otimes B^{\nu} \right)_{kl,ij} = \sum_{\nu=1}^{m} a^{\nu}_{ki} b^{\nu}_{lj},
\]

for some \( A^{\nu} = (a^{\nu}_{ki}), B^{\nu} = (b^{\nu}_{lj}) \in \mathbb{M}(n, \mathbb{C}), \) and \( m \leq n^2 \). In the above, we use the notation \( \overline{X} = (\overline{x}_{ij}) \) to denote (element-wise) complex conjugation of a matrix \( X \in \mathbb{M}(n, \mathbb{C}) \), and \( (X \otimes Y)_{kl,ij} = x_{ki}y_{lj} \) for any \( X, Y \in \mathbb{M}(n, \mathbb{C}) \). From this we easily obtain:

\[
\Phi(E_{ij}) = \sum_{\nu=1}^{m} \sum_{k,l=1}^{n} a^{\nu}_{ik} E_{kl} b^{\nu}_{lj}, \quad (10)
\]

and

\[
\Phi(X) = \sum_{\nu=1}^{m} A^{\nu} X (B^{\nu})^\dagger, \quad (11)
\]

for any \( X = \sum_{ij} x_{ij} E_{ij} \in \mathbb{M}(n, \mathbb{C}) \).

The above observations are summarized in the following lemma:

**Lemma 2.** For any \( \Phi \in \text{End}[\mathbb{M}(n, \mathbb{C})] \), there exist sets of matrices \( A^{\nu} = (a^{\nu}_{ki}), B^{\nu} = (b^{\nu}_{lj}) \in \mathbb{M}(n, \mathbb{C}), \) with \( \nu = 1, \ldots, m \) (and \( m \leq n^2 \)), such that:

\[
\forall \ X \in \mathbb{M}(n, \mathbb{C}), \quad \Phi(X) = \sum_{\nu=1}^{m} A^{\nu} X (B^{\nu})^\dagger. \quad (12)
\]

From Lemma 2 and eq. (9) we get:

\[
(\Phi(X))_{ij} = \sum_{\nu=1}^{m} \sum_{k,l=1}^{n} a^{\nu}_{ik} x_{kl} b^{\nu}_{lj} = \sum_{\nu=1}^{m} \sum_{k,l=1}^{n} a^{\nu}_{ik} b^{\nu}_{lj} x_{kl} = \sum_{k,l=1}^{n} \sum_{\nu=1}^{m} (A^{\nu} \otimes B^{\nu})_{il,kj} x_{kl}. \quad (13)
\]

Further, using the vectorization map \( \text{vec} : \mathbb{M}(n, \mathbb{C}) \rightarrow \mathbb{C}^{n^2} \), and eq. (12), we obtain:

\[
\text{vec}[\Phi(X)] \equiv \text{vec} \left( \sum_{\nu=1}^{m} A^{\nu} X (B^{\nu})^\dagger \right) = \sum_{\nu=1}^{m} (A^{\nu} \otimes B^{\nu}) \text{vec}(X), \quad (14)
\]

where \( \sum_{\nu=1}^{m} (A^{\nu} \otimes B^{\nu}) \in \mathbb{M}(n^2, \mathbb{C}) \) and \( \text{vec}(X), \text{vec}[\Phi(X)] \in \mathbb{C}^{n^2} \).

This leads to the following natural definition of the **matrix representation** of a linear map \( \Phi \in \text{End}[\mathbb{M}(n, \mathbb{C})] \).

**Definition 3** (Matrix representation of a linear map). Let \( \Phi \in \text{End}[\mathbb{M}(n, \mathbb{C})] \) such that \( \forall X \in \mathbb{M}(n, \mathbb{C}) \) \( \Phi(X) = \sum_{\nu=1}^{m} A^{\nu} X (B^{\nu})^\dagger \), for some matrices \( A^{\nu} = (a^{\nu}_{ki}), B^{\nu} = (b^{\nu}_{lj}) \in \mathbb{M}(n, \mathbb{C}), \) (for \( \alpha = 1, \ldots, m \)). Then the matrix \( \text{mat}(\Phi) \equiv \sum_{\nu=1}^{m} (A^{\nu} \otimes B^{\nu}) \in \mathbb{M}(n^2, \mathbb{C}) \) is a matrix representant of the map \( \Phi \). For this representation we have:

\[
[\text{mat}(\Phi) \text{vec}(X)]_{ij} = \Phi(X)_{ij}, \quad i,j = 1, \ldots, n. \quad (15)
\]
The vector space $\mathcal{M}(n, \mathbb{C})$ becomes a Hilbert space when equipped with the Hilbert-Schmidt scalar product:

$$\forall X, Y \in \mathcal{M}(n, \mathbb{C}) \quad (X, Y) \equiv \text{Tr}(X^\dagger Y)$$

(16)

with induced Hilbert-Schmidt norm:

$$||X||_2 \equiv \sqrt{\text{Tr}(X^\dagger X)}.$$  

(17)

A linear map $\Phi \in \text{End}[\mathcal{M}(n, \mathbb{C})]$ acts as an operator on this Hilbert space, and its adjoint $\Phi^*$ is defined through the relation:

$$\forall X, Y \in \mathcal{M}(n, \mathbb{C}) \quad (\Phi^*(X), Y) = (X, \Phi(Y)).$$

(18)

Hence, from eq. (11) and the above, it follows that:

$$\Phi^*(X) = \sum_{\nu=1}^m (A^\nu)^\dagger X B^\nu, \quad X \in \mathcal{M}(n, \mathbb{C}).$$

(19)

Further, the following proposition holds.

**Lemma 4.** Taking the adjoint of an element in $\text{End}[\mathcal{M}(n, \mathbb{C})]$ induces Hermitian conjugation of its matrix representant in the space $\mathcal{M}(n^2, \mathbb{C})$. Namely, for any $\Phi \in \text{End}[\mathcal{M}(n, \mathbb{C})]$ we have

$$\text{mat}(\Phi^*) = (\text{mat}(\Phi))^\dagger.$$  

(20)

**Proof.** From eq. (19) we have

$$\Phi^*(X) = \sum_{\nu=1}^m (A^\nu)^\dagger X B^\nu = \sum_{\nu=1}^m (A^\nu)^\dagger X ((B^\nu)^\dagger)^\dagger. $$

(21)

Then, using Definition 3 we get

$$\text{mat}(\Phi^*) = \sum_{\nu=1}^m (A^\nu)^\dagger \otimes ((B^\nu)^\dagger)^\dagger = \sum_{\nu=1}^m (A^\nu \otimes (B^\nu))^\dagger = (\text{mat}(\Phi))^\dagger.$$  

(22)

$\square$

### 3 Irreducibly covariant linear maps and quantum channels

Let $G$ be a finite group and let:

$$U : G \to \mathcal{M}(n, \mathbb{C}), \quad \text{i.e. } U(g) = (u_{ij}(g)) \in \mathcal{M}(n, \mathbb{C}) \quad \forall g \in G,$$

(23)

be a unitary irreducible representation (irrep, in short) of $G$. The contra-gradient representation $U^c : G \to \mathcal{M}(n, \mathbb{C})$ is given by

$$U^c(g) = U(g^{-1})^T \equiv \overline{U}(g) \quad \forall g \in G.$$  

(24)

The map $\text{Ad}_U^G : G \to \text{End}[\mathcal{M}(n, \mathbb{C})]$ is called the *adjoint representation* of the group $G$ with respect to the unitary irrep $U$, and is defined through its action on any $X \in \mathcal{M}(n, \mathbb{C})$ as follows:

$$\text{Ad}_U(g)(X) \equiv U(g)XU^\dagger(g) \quad \forall g \in G.$$  

(25)

Obviously, $\text{Ad}_U(g) \in \text{End}[\mathcal{M}(n, \mathbb{C})]$.
Definition 5 (Commutant of the adjoint representation). Let $\text{Int}_G(\text{Ad}_U)$ denote the set of intertwiners of $\text{Ad}_U$, i.e. the set of maps in $\text{End} [\mathbb{M}(n, \mathbb{C})]$ whose action commutes with that of $\text{Ad}_U$:

$$\text{Int}_G(\text{Ad}_U) = \{ \Psi \in \text{End} [\mathbb{M}(n, \mathbb{C})] : \Psi \circ \text{Ad}_U = \text{Ad}_U \circ \Psi \}. \quad (26)$$

Note that for any $\text{Ad}_U(g) \in \text{End}[\mathbb{M}(n, \mathbb{C})]$ (with $g \in G$) $\forall X = (x_{ij}) \in \mathbb{M}(n, \mathbb{C})$, we have:

$$\text{Ad}_U(g)(X)_{ij} = \left( U(g)XU(g)^\dagger \right)_{ij} = \sum_{kl} \left[ U(g) \otimes \overline{U}(g) \right]_{il,kj} x_{kl}, \quad (27)$$

so that:

$$\text{mat}(\text{Ad}_U(g)) = U(g) \otimes \overline{U}(g). \quad (28)$$

Thus the operator $\text{Ad}_U(g) \in \text{End}[\mathbb{M}(n, \mathbb{C})]$ may be represented as a matrix $U(g) \otimes \overline{U}(g) \in \mathbb{M}(n^2, \mathbb{C})$.

Lemma 6. The character of the adjoint representation $\text{Ad}_G^U : G \rightarrow \text{End} [\mathbb{M}(n, \mathbb{C})]$, denoted by $\chi^{\text{Ad}_G^U} : G \rightarrow \mathbb{C}$, is given by

$$\chi^{\text{Ad}_G^U}(g) := \text{Tr} (\text{mat}(\text{Ad}_U(g))) = \text{Tr} (U(g) \otimes \overline{U}(g)), \quad \forall g \in G. \quad (29)$$

In particular, we have

$$\chi^{\text{Ad}}(g) = |\chi^U(g)|^2, \quad \forall g \in G, \quad (30)$$

where $\chi^U : G \rightarrow \mathbb{C}$ is the character of the representation $U : G \rightarrow \mathbb{M}(n, \mathbb{C})$, i.e. $\chi^U(g) = \text{Tr} (U(g))$, $\forall g \in G$.

Definition 7 (Irreducibly covariant- linear maps (ICLM) and quantum channels (ICQC)). A linear map $\Phi \in \text{End} [\mathbb{M}(n, \mathbb{C})]$ is said to be irreducibly covariant with respect to the unitary irrep $U : G \rightarrow \mathbb{M}(n, \mathbb{C})$ of a finite group $G$, i.e. $\Phi \in \text{Int}_G(\text{Ad}_U)$, if

$$\forall g \in G, \quad \forall X \in \mathbb{M}(n, \mathbb{C}) \quad \text{Ad}_U(g)[\Phi(X)] = \Phi[\text{Ad}_U(g)(X)], \quad (31)$$

i.e. $\Phi \in \text{Int}_G(\text{Ad}_U)$. Further, if the linear map $\Phi$ is completely positive and trace-preserving, then it is referred to as an irreducibly covariant quantum channel. We denote an irreducibly covariant linear map by the acronym ICLM, and an irreducibly covariant quantum channel by the acronym ICQC.

Remark 8. Note that the set of ICLMs form an algebra with composition of linear maps as a product.

4 Tools and results from group representation theory

Using tools from group representation theory, it is easy to prove the following proposition:

Lemma 9. A linear map $\Phi \in \text{End} [\mathbb{M}(n, \mathbb{C})]$ is irreducibly covariant with respect to the irrep $U : G \rightarrow \mathbb{M}(n, \mathbb{C})$ of a finite group $G$ (i.e. $\Phi \in \text{Int}_G(\text{Ad}_U)$) if and only if

$$\text{mat}(\Phi) \in \text{Int}_G(U \otimes U^\dagger) = \text{Int}_G(U \otimes \overline{U}). \quad (32)$$
For sake of completeness, we include the proof of this lemma in Appendix A. The above lemma implies that instead of studying the structure of the commutant $\text{Int}_G(\text{Ad}_U)$ in the space $\text{End} [\mathbb{M}(n, \mathbb{C})]$, it suffices to study the structure of the commutant $\text{Int}_G (U \otimes U^c)$ in the matrix space $\mathbb{M}(n^2, \mathbb{C})$, which is simpler because we have to deal with matrices only.

It is known that the representation [1]:

$$U \otimes U^c : G \to \mathbb{M}(n^2, \mathbb{C})$$

(33)

is not irreducible and we have:

$$U \otimes U^c = \bigoplus_{\alpha} m_\alpha \varphi^\alpha,$$

(34)

where $\varphi^\alpha$ are unitary irreps of the group $G$: $\forall g \in G, \varphi^\alpha(g) = (\varphi^\alpha_{ij}(g)) \in \mathbb{M}(|\varphi^\alpha|, \mathbb{C})$, and $m_\alpha$ is the multiplicity of the irrep $\varphi^\alpha$ of dimension $|\varphi^\alpha| \equiv \text{dim} \varphi^\alpha$. The multiplicity $m_\alpha$ is given by the following expression [1]:

$$m_\alpha = \frac{1}{|G|} \sum_{g \in G} \chi^\alpha (g^{-1}) \chi^{\text{Ad}}(g),$$

(35)

where $\chi^\alpha$ denotes the character of the irrep $\varphi^\alpha(g)$, and $\chi^{\text{Ad}}(g) = |\chi^U(g)|^2$ is the character of the adjoint representation $\text{Ad}_U^G$ defined in Lemma 6. From this it follows that the commutant of the representation $U \otimes U^c$:

$$\text{Int}_G (U \otimes U^c) = \{ A \in \mathbb{M}(n^2, \mathbb{C}) : \forall g \in G \ A(U(g) \otimes U^c(g)) = (U(g) \otimes U^c(g)) A \}$$

(36)

is nontrivial i.e. it is not one-dimensional.

In fact, from the theory of group representations [1], one can deduce the following:

**Lemma 10.** Let $U : G \to \mathbb{M}(n, \mathbb{C})$ be a unitary irreducible representation of a given finite group $G$. Then we have

$$U \otimes U^c = \varphi^\text{id} \oplus_{\alpha \neq \text{id}} m_\alpha \varphi^\alpha,$$

(37)

i.e. the identity irrep, $\varphi^\text{id}$, is always included in the representation $U \otimes U^c$ with multiplicity one. Moreover,

$$\text{dim} [\text{Int}_G (U \otimes U^c)] = \frac{1}{|G|} \sum_{g \in G} |\chi^U(g)|^4,$$

(38)

where $\chi^U : G \to \mathbb{C}$ is the character of the representation $U : G \to \mathbb{M}(n, \mathbb{C})$, and $|G|$ is the cardinality of the group $G$.

We illustrate the above lemma by the following two examples for the symmetric group $S(n)$ for $n = 3, 4$. For the basics and notations of the representation theory of $S(n)$ see for example [2].

**Example 11.** For $G = S(3)$ and its two-dimensional, unitary irrep $U = \varphi^{(2,1)}$ characterised by the partition $\lambda = (2, 1)$ we have

$$U \otimes U^c = \varphi^\text{id} \oplus \varphi^\text{sgn} \oplus \varphi^{(2,1)}, \quad \text{dim} [\text{Int}_{S(3)} (U \otimes U^c)] = 3.$$  

(39)
**Example 12.** For $G = S(4)$ and its two-dimensional, unitary irrep $U = \varphi^{(2,2)}$ characterised by the partition $\lambda = (2, 2)$ we have

$$U \otimes U^c = \varphi^{\text{id}} \oplus \varphi^{\text{sgn}} \oplus \varphi^{(2,2)}, \quad \dim [\text{Int}_{S(4)}(U \otimes U^c)] = 3. \quad (40)$$

In these examples all the multiplicities are equal to one so that the decomposition of the representation $U \otimes U^c$ is simply reducible. This holds also for all other irreps of the group $G = S(4)$. However, in the case of the group $S(5)$ there are irreps $\varphi^\alpha$ for which the representation $U \otimes U^c$ is not simply reducible.

**Notation**

For notational simplicity, henceforth, elements of $\text{End} [M(n, \mathbb{C})]$, will be denoted by Greek letters, whereas their matrix representations in $M(n^2, \mathbb{C})$ will be denoted by the same Greek letters but with a tilde. For example,

$$\Gamma = \sum_{g \in G} a_g \text{Ad}_g U(g) \in \text{End} [M(n, \mathbb{C})], \quad a_g \in \mathbb{C}, \quad (41)$$

and

$$\tilde{\Gamma} \equiv \text{mat}(\Gamma) = \sum_{g \in G} a_g U(g) \otimes \overline{U}(g) \in M(n^2, \mathbb{C}). \quad (42)$$

The Schur orthogonality relations [11], given below, are useful in proving the results in the following sections:

- $$\sum_{g \in G} \varphi^\alpha_{ij} (g^{-1}) \varphi^\beta_{kl} (g) = \frac{|G|}{|\varphi^\alpha|} \delta^\alpha \beta \delta_{jk} \delta_{il}, \quad (43)$$

  where $|G|$ denotes cardinality of the group $G$.

- $$\frac{1}{|G|} \sum_{g \in G} \chi^\alpha (g^{-1}) = \begin{cases} 1 & \text{if } \alpha = \text{id}, \\ 0 & \text{if } \alpha \neq \text{id}, \end{cases} \quad (44)$$

  where id denotes the identity irrep of the group $G$.

- $$\sum_{\alpha \in \hat{G}} \chi^\alpha(h) \chi^\alpha(g^{-1}) = \frac{|G|}{|K(h)|} \delta_{K(h)K(g)}, \quad (45)$$

  where by $K(g)$ we denote the conjugacy class of the element $g \in G$,

  $$K(g) = \{ a \in G \mid \exists h \in G \text{ with } a = hgh^{-1} \}, \quad (46)$$

  and sum in the eq. (45) runs over the set of all inequivalent irreps $\hat{G}$. 

10
5 Intermediate Results

In this section we state some propositions and corollaries which are required to prove our main results. Their proofs are given in Appendix B.

**Proposition 13.** Suppose that an unitary irrep \( U : G \to M(n, \mathbb{C}) \), of a finite group \( G \) is such that \( U \otimes U^c \) is multiplicity-free, i.e.

\[
U \otimes U^c = \bigoplus_{\alpha \in \Theta} \varphi^\alpha,
\]

where \( \Theta \) is the index set of those irreps \((\varphi^\alpha)\) of the group \( G \) which appear (with multiplicity one) in the above decomposition of \( U \otimes U^c \). Then,

\[
\text{Int}_G(U \otimes U^c) = \text{span}_\mathbb{C} \left\{ \tilde{\Pi}^\alpha : \alpha \in \Theta \right\} \quad \text{and} \quad \dim[\text{Int}_G(U \otimes U^c)] = |\Theta|,
\]

where

\[
\tilde{\Pi}^\alpha = \left| \varphi^\alpha \right| \frac{1}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1}) \ U(g) \otimes \overline{U}(g) \in M(n^2, \mathbb{C}).
\]

The matrices \( \tilde{\Pi}^\alpha \) have the following properties:

\[
\tilde{\Pi}^\alpha \tilde{\Pi}^\beta = \delta_{\alpha\beta} \tilde{\Pi}^\alpha, \quad (\tilde{\Pi}^\alpha)^\dagger = \tilde{\Pi}^\alpha, \quad \sum_{\alpha \in \Theta} \tilde{\Pi}^\alpha = 1_{n^2},
\]

where \( 1_{n^2} \) is the identity operator on \( \mathbb{C}^{n^2} \). The set \( \left\{ \tilde{\Pi}^\alpha : \alpha \in \Theta \right\} \) is a complete set of orthogonal projectors and \( \text{Tr} \tilde{\Pi}^\alpha = |\varphi^\alpha| \).

**Corollary 14.** The identity irrep \( \text{id} \) always occurs in the decomposition eq. (47), so \( \text{id} \in \Theta \). Moreover

\[
\gamma \notin \Theta \Rightarrow \tilde{\Pi}^\gamma = 0,
\]

where \( \tilde{\Pi}^\gamma \) are the projectors defined through eq. (49).

This can be easily seen as follows: One can notice, that from eq. (49) we can deduce that:

\[
\text{Tr}(\tilde{\Pi}^\gamma) = m_\gamma |\varphi^\gamma|,
\]

where \( m_\gamma \) is the multiplicity given in eq. (35), therefore if \( \gamma \notin \Theta \), then \( m_\gamma = 0 \) and \( \text{Tr}(\tilde{\Pi}^\gamma) = 0 \). The latter implies, that \( \tilde{\Pi}^\gamma = 0 \) because \( \tilde{\Pi}^\gamma \) is a projector.

From Proposition 13 we see why an assumption of the multiplicity freeness is so useful. This is because in the general scenario, i.e. when multiplicities occurring in decomposition (34) satisfy \( m_\alpha \geq 1 \), the spanning set of \( \text{Int}_G(U \otimes U^c) \) is larger than the set given through expression eq. (48), since the multiplicity space is not trivial any more. From the Proposition 13 and Lemma 9 we get the following corollary.

**Corollary 15.** A linear map \( \Phi \in \text{End}[M(n, \mathbb{C})] \), which is irreducibly covariant with respect to a unitary irrep \( U : G \to M(n, \mathbb{C}) \) of a finite group \( G \), can be expressed in the form

\[
\Phi = l_{\text{id}} \tilde{\Pi}^\text{id} + \sum_{\alpha \in \Theta, \alpha \neq \text{id}} l_\alpha \tilde{\Pi}^\alpha : \quad l_\alpha \in \mathbb{C},
\]

\( 11 \)
where
\[ \Pi^\alpha = \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \chi^\alpha (g^{-1}) \text{Ad}_{U(g)} \in \text{End} [M(n, \mathbb{C})], \quad \alpha \in \Theta, \] (54)

and the operators \( \Pi^\alpha \) have the same properties as their matrix representants \( \tilde{\Pi}^\alpha \equiv \text{mat}(\Pi^\alpha) \), i.e.
\[ \Pi^\alpha \Pi^\beta = \delta_{\alpha\beta} \Pi^\alpha, \quad (\Pi^\alpha)^* = \Pi^\alpha, \quad \sum_{\alpha \in \Theta} \Pi^\alpha = \text{id}_{\text{End}[M(n, \mathbb{C})]}, \] (55)

where \( \text{id}_{\text{End}[M(n, \mathbb{C})]} \) denotes the identity map in \( \text{End} [M(n, \mathbb{C})] \).

Remark 16. The expression given in eq. (53) of Corollary 15 is the spectral decomposition of \( \Phi \): the coefficients \( l_\alpha \) are its eigenvalues, with \( \Pi^\alpha \) being the corresponding projectors.

In the next step we will need the following statement describing the structure of the projectors \( \tilde{\Pi}^\alpha \) which span the commutant \( \text{Int}_G (U \otimes U^c) \).

Proposition 17 (Spectral decomposition of the projectors \( \tilde{\Pi}^\alpha \)). Let \( \tilde{\Pi}^\alpha \) be a projector as in Proposition 13. It has the following spectral decomposition:
\[ \tilde{\Pi}^\alpha = \sum_{i=1}^{\left| \varphi^\alpha \right|} \tilde{\Pi}^\alpha_i, \quad \tilde{\Pi}^\alpha_i \tilde{\Pi}^\beta_j = \delta_{\alpha\beta} \delta_{ij} \tilde{\Pi}^\alpha_i, \quad \left( \tilde{\Pi}^\alpha_i \right)^\dagger = \tilde{\Pi}^\alpha_i, \quad \text{Tr} \left( \tilde{\Pi}^\alpha_i \right) = 1, \] (56)

where
\[ \tilde{\Pi}^\alpha_i = \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \varphi^\alpha_{ii} (g^{-1}) U(g) \otimes \bar{U}(g) \in M(n^2, \mathbb{C}). \] (57)

Remark 18. In particular as a remark of Proposition 17 we can say that for \( p,q,s,t \in \{1, \ldots, n\} \) we have:
\[ \tilde{\Pi}^\alpha_{pq, st} = \left( \frac{1}{|U|} \delta_{pq} \delta_{st} \right) \Rightarrow \tilde{\Pi}^\alpha_{pq, pq} = \left( \frac{1}{|U|} \delta_{pq} \right), \] (58)

where \( |U| \equiv \dim U \).

Hence, the spectral decomposition of \( \tilde{\Pi}^\alpha \) is explicitly given by characteristics of the representation \( U \otimes U^c \) of the group \( G \) (which are known for any given irrep \( U \)).

Corollary 19. The linear maps \( \Pi^\alpha_i \in \text{End} [M(n, \mathbb{C})] \) such that \( \text{mat}(\Pi^\alpha_i) = \tilde{\Pi}^\alpha_i \), are given by:
\[ \Pi^\alpha_i = \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \varphi^\alpha_{ii} (g^{-1}) \text{Ad}_{U(g)} \cdot \] (59)

and satisfy relations analogous to eq. (56) of their matrix representants \( \tilde{\Pi}^\alpha_i \):
\[ \Pi^\alpha = \sum_{i=1}^{\left| \varphi^\alpha \right|} \Pi^\alpha_i, \quad \Pi^\alpha_i \Pi^\beta_j = \delta_{\alpha\beta} \delta_{ij} \Pi^\alpha_i, \quad (\Pi^\alpha_i)^* = \Pi^\alpha_i. \] (60)

The following proposition gives the spectral decomposition of the rank one projectors \( \Pi^\alpha_i \in \text{End} [M(n, \mathbb{C})] \).
**Proposition 20.** Let \( V^\alpha_i \in \mathbb{M}(n, \mathbb{C}) \) denote the normalised in the Hilbert-Schmidt norm eigenvectors of the projector \( \Pi^\alpha_i \in \text{End}[\mathbb{M}(n, \mathbb{C})] \), corresponding to the eigenvalue 1, i.e.

\[
\Pi^\alpha_i V^\beta_j = \delta^{\alpha\beta} \delta_{ij} V^\alpha_i.
\] (61)

Then \( V^\alpha_i \) has the following form: there exists a pair \((s, t)\) with \( s, t \in \{1, \ldots, n\} \) such that:

\[
V^\alpha_i \equiv V^\alpha_i(s, t) = \frac{1}{\sqrt{\left| \Pi^\alpha_i \right|}} \sum_{g \in G} \varphi_{ii}^\alpha(g^{-1}) U_{C}(g) U_{R}(g^{-1}) \neq 0,
\] (62)

where \( U_{C}(g) \) and \( U_{R}(g) \) respectively denote the \( s^{th} \) column and the \( t^{th} \) row of the matrix \( U(g) \in \mathbb{M}(n, \mathbb{C}) \). If \((s, t)\) and \((p, q)\) (with \( s, t, p, q \in \{1, \ldots, n\} \)) are pairs for which \( V^\alpha_i(s, t) \neq 0 \) and \( V^\alpha_i(p, q) \neq 0 \), then the following orthonormality relation holds:

\[
(V^\alpha_i(s, t), V^\beta_j(p, q)) = \delta^{\alpha\beta} \delta_{ij} e^{i\zeta},
\] (63)

where \( i^2 = -1 \) and \( \zeta \) is some phase factor.

**Corollary 21.** In eq. (63) of Proposition 20, the phase factor \( \zeta \) depends on the indices \( s, t, p, q \in \{1, \ldots, n\} \) and if \((s, t) = (p, q)\) then \( \zeta = 0 \), so that

\[
V^\alpha_i(p, q) = e^{i\zeta} V^\alpha_i(s, t).
\] (64)

Note that from eq. (62) of Proposition 20 we can deduce that

\[
V^{id}(s, s) = \frac{1}{\sqrt{|U|}} 1_n.
\] (65)

For any \( \beta \in \Theta \) and \( i \in \{1, 2, \ldots, |\varphi^\beta|\} \), let us define the set

\[
S_{\beta,i} := \left\{ (s, t) \in \{1, \ldots, n\} \times \{1, \ldots, n\} : \left( \tilde{\Pi}^\beta_{st} \right)_{st} \neq 0 \right\}.
\] (66)

Then for any \( \beta \in \Theta \) and \( i \in \{1, 2, \ldots, |\varphi^\beta|\} \) the vector \( V^\beta_i \) is uniquely parametrized (up to a phase) by a given pair in the set \( S_{\beta,i} \). The phase turns out to be irrelevant in our characterization of irreducibly covariant linear maps or quantum channels (see Remark 32). The set \( S_{\beta,i} \) parametrizes the non-zero vectors \( V^\beta_i(s, t) \) because we have (see eq. (157) in Theorem 58):

\[
\left| V^\beta_i(s, t) \right|^2 = \left\| \tilde{\Pi}^\beta_{st} \right\|_{st},
\] (67)

so if \( \left( \tilde{\Pi}^\beta_{st} \right)_{st} \neq 0 \) then the corresponding vector \( V^\beta_i(s, t) \) is well defined.

**Corollary 22.** The set of \( n^2 \) matrices

\[
\left\{ V^\beta_i \equiv V^\beta_i(s, t) : (s, t) \in S_{\beta,i}, \beta \in \Theta, i \in \{1, 2, \ldots, |\varphi^\beta|\} \right\},
\] (68)

constitute an orthonormal basis of the linear space \( \mathbb{M}(n, \mathbb{C}) \).
Remark 23. In order to construct the basis given in eq. (68) of Corollary 22 one has to construct the projectors $\tilde{\Pi}^\beta_i$ defined in eq. (57) of Proposition 17, and then choose the indices $(s,t)$ such that $(\tilde{\Pi}^\beta_i)_{st,st} \neq 0$. In particular $\forall \beta \in \Theta$ and $\forall i = 1, \ldots, |\varphi^\beta|$ by fixing a pair $(s,t)$ from the above set we also fix the basis.

From Proposition 20, Definition 1 and Definition 3 we obtain the following corollary:

Corollary 24. The vector $\text{vec}(V^\alpha_i) \in \mathbb{C}^{n^2}$ is an eigenvector of $\text{mat}(\Pi^\alpha_i) \equiv \tilde{\Pi}^\alpha_i \in M(n^2, \mathbb{C})$ with eigenvalue 1, i.e.

$$\tilde{\Pi}^\alpha_i \text{vec}(V^\alpha_i) = \text{vec}(V^\alpha_i).$$

The following proposition gives a necessary and sufficient condition for an irreducibly covariant linear map (ICLM) to be trace-preserving.

Proposition 25. An ICLM $\Phi = l_{id}\Pi_{id} + \sum_{\alpha \in \Theta, \alpha \neq id} l_{\alpha} \Pi^\alpha \in \text{Int}_G(\text{Ad}_U)$ is trace preserving if and only if $l_{id} = 1$, so that it is of the form:

$$\Phi = \Pi_{id} + \sum_{\alpha \in \Theta, \alpha \neq id} l_{\alpha} \Pi^\alpha,$$

where the coefficient $l_{\alpha}$ for $\alpha \in \Theta$, with $\alpha \neq id$, can be arbitrary.

To establish a necessary and sufficient condition for an irreducibly covariant linear (ICLM) map $\Phi \in \text{End} [M(n, \mathbb{C})]$ to be completely positive, it is convenient to consider the Choi-Jamiołkowski image $J(\Phi)$ of the map defined through eq. (6), since the complete positivity of $\Phi$ is equivalent to positive semi-definiteness of $J(\Phi)$. Restricting ourselves an ICLM, $\Phi$, which is trace-preserving, by the linearity of the Choi-Jamiołkowski transformation we get

$$J(\Phi) = J\left(\Pi_{id}\right) + \sum_{\alpha \in \Theta, \alpha \neq id} l_{\alpha} J(\Pi^\alpha),$$

an explicit form for which is given in the following proposition.

Proposition 26. The Choi-Jamiołkowski image of a trace-preserving ICLM $\Phi \in \text{End} [M(n, \mathbb{C})]$ (as given by Proposition 25) is given by

$$J(\Phi) = \frac{1}{|U|} \mathbb{I}_n \otimes \mathbb{I}_n + \frac{1}{|G|} \sum_{ij} E_{ij} \otimes \sum_{g \in G} \left( \sum_{\alpha \in \Theta, \alpha \neq id} l_{\alpha} |\varphi^\alpha| \chi^\alpha(g^{-1}) \right) U_C(i)(g) (U_C(j)(g))^\dagger,$$

where $U_C(i)(g) = (u_{ki}(g))_k$ denotes the $i^{th}$ column of the matrix $U(g)$.

Remark 27. The trace of $J(\Phi)$ from eq. (72) of Proposition 26 depends only on the dimension of the irrep $U$, and is independent of the ICLM $\Phi$, i.e.

$$\text{Tr} (J(\Phi)) = |U|.$$

The above relation can be easily obtained by a direct calculation of the trace, using the orthogonality relation, eq. (44) for irreducible characters.

The following proposition deals with the eigenvalue problem of the Choi-Jamiołkowski images of the projectors $\Pi^\alpha$ arising in the decomposition in eq. (70) of the trace-preserving ICLM $\Phi$. It provides the first step towards finding a necessary and sufficient condition for the positive semi-definiteness of $J(\Phi)$. Let us start from the following definition

Remark 27.
Remark 32. From the structure of the right hand side of eq. (77) of Proposition 20, it follows that the eigenvalues \( \mu_i(\alpha, \beta) \) do not depend on the particular choice of the pair \((s, t) \in S_{\beta, i}\) used to parametrize \(V_i^\beta \equiv V_i^\beta(s, t)\).

From Definition 28, Lemma 29 and Proposition 30 we obtain the following corollary:

**Corollary 33.** The operators \( J(\Pi^\alpha) \in \mathbb{M}(n^2, \mathbb{C}) \), \( \alpha \in \Theta \) are normal.

Using eq. (62) of Proposition 20, the eigenvalues \( \mu_i(\alpha, \beta) \) (given in Proposition 30) can also be expressed equivalently as follows:

**Remark 34.** For any \((s, t) \in S_{\beta, i}\), defined through eq. (66),

\[
\mu_i(\alpha, \beta) = \frac{1}{(\tilde{\Pi}_i^\beta)_{s,t}} \frac{||\varphi^\alpha||}{|G|^2} \sum_{g, h \in G} \chi^\alpha (g^{-1}) \varphi_{ii}^\beta (h^{-1}) u_{st} (g^{-1}) u_{st} (hgh^{-1}), \quad \alpha, \beta \in \Theta. \tag{79}
\]

From Remark 32 it follows that the right hand side of eq. (79), does not depend on the particular choice of the pair \((s, t) \in S_{\beta, i}\).
By Proposition 30, the Choi-Jamiołkowski image $J(\Phi)$ given by eq. (71) is a linear combination of mutually commuting matrices whose eigenvalues are known (i.e. given by eq. (77)). Hence, we can explicitly write down the conditions for positive-semidefiniteness of $J(\Phi)$ as follows:

**Corollary 35.** The Choi-Jamiołkowski image $J(\Phi)$ given by eq. (71) is positive semi-definite if and only if its eigenvalues, which we denote by $\epsilon_i^\beta$, are non-negative, i.e. for any $\beta \in \Theta$, and $i = 1, \ldots, |\phi^\beta|$

$$
\epsilon_i^\beta \equiv \sum_{\alpha \in \Theta} l_\alpha \mu_i(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \left( \sum_{\alpha \in \Theta} l_\alpha |\varphi^\alpha| \chi^\alpha (g^{-1}) \right) \left| \text{Tr} \left( V_i^\beta U^\dagger (g) \right) \right|^2 \geq 0,
$$

(80)

where $V_i^\beta \in \mathbb{M}(n, \mathbb{C})$ is the normalized eigenvector (see Proposition 20) of the projector $\Pi_i^\beta$.

**Remark 36.** From eq. (73) of Remark 27 and eq. (80) of Corollary 35 it follows, that

$$
\sum_{\beta \in \Theta} \sum_{i = 1, \ldots, |\beta|} \epsilon_i^\beta = |U|.
$$

(81)

Using eq. (62) of Proposition 20, we get the following equivalent expression for the eigenvalues $\epsilon_i^\beta$.

**Lemma 37.** For some $k, l \in \{1, \ldots, n\}$ for which $\left( \bar{\Pi}_i^\beta \right)_{st,st} \neq 0$, we have

$$
\epsilon_i^\beta = \frac{1}{|G|^2} \sum_{g \in G} \sum_{\alpha \in \Theta} l_\alpha |\varphi^\alpha| \chi^\alpha (g^{-1}) \left| \frac{|\varphi^\beta|}{\sqrt{\left( \bar{\Pi}_i^\beta \right)_{st,st}}} \sum_{h \in G} \varphi_i^\beta (h^{-1}) u_{ts} (h^{-1} g^{-1} h) \right|^2 \geq 0,
$$

(82)

or more explicitly

$$
\epsilon_i^\beta = \sum_{\alpha \in \Theta} l_\alpha \mu_i(\alpha, \beta) = \frac{1}{\left( \bar{\Pi}_i^\beta \right)_{st,st}} \sum_{g, h \in G} \left[ \sum_{\alpha \in \Theta} l_\alpha |\varphi^\alpha| \chi^\alpha (g^{-1}) \right] \varphi_i^\beta (h^{-1}) u_{ts} (g^{-1}) u_{st} (hgh^{-1}) \geq 0,
$$

(83)

for $\beta \in \Theta$, $i = 1, \ldots, |\varphi^\beta|$. 

**Remark 38.** From Proposition 20 and Corollary 21 it follows that the right hand side of the above equations do not depend on the particular choice of the pair $s, t \in S_{\beta,i}$.

The properties of the eigenvalues $\mu_i(\alpha, \beta)$ depend on the properties of the particular irrep (of the group $G$) which is considered, as discussed below.

**Lemma 39 (Hermiticity of the Choi-Jamiołkowski image).** The Choi-Jamiołkowski image $J(\Phi)$ of an ICLM $\Phi \in \text{End} (\mathbb{M}(n, \mathbb{C}))$ is Hermitian if and only if

$$
\forall X \in \mathbb{M}(n, \mathbb{C}), \quad \Phi(X)^\dagger = \Phi(X^\dagger).
$$

(84)
Note that for any $X \in \mathbb{M}(n, \mathbb{C})$, we have
\[
\Pi^\alpha(X)^\dagger = \frac{1}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1}) U(g) X^\dagger U(g^{-1})
\tag{85}
\]
and
\[
\Pi^\alpha \left( X^\dagger \right) = \frac{1}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1}) U(g) X^\dagger U(g^{-1}).
\tag{86}
\]

From the above, we can infer that if the irreducible characters of the group $G$ are real then the Choi-Jamiołkowski image $J(\Pi^\alpha)$ of any projector $\Pi^\alpha$, appearing in Corollary 15, is Hermitian and hence the corresponding eigenvalues $\mu_i(\alpha, \beta)$ are real. For example, all characters for the symmetric group $S(n)$ and quaternion group $Q$ are given by real numbers, so in this case eigenvalues $\mu_i(\alpha, \beta)$ are always real.

6 Main Results

The results stated in the previous section (and proved in Appendix B) are summarized in the following theorem (Theorem 40). The latter gives an explicit description of an irreducibly covariant quantum channel (ICQC), corresponding to an irrep $U$ of a finite group for which $U \otimes U^c$ is simply reducible. In addition, in Theorem 41 we obtain explicit expressions for the Kraus operators of any ICQC. These theorems are based on the following assumption:

Assumption 1. Suppose that a unitary irrep $U : G \to \mathbb{M}(n, \mathbb{C})$ (of a finite group $G$) is such that $U \otimes U^c$ is simply reducible, i.e.,
\[
U \otimes U^c = \bigoplus_{\alpha \in \Theta} \varphi^\alpha,
\tag{87}
\]
where $\Theta$ is the index set of those irreps of $G$ which appear in the above decomposition.

**Theorem 40.** Under Assumption 1 a linear map $\Phi \in \text{End} \left[ \mathbb{M}(n, \mathbb{C}) \right]$, is an ICQC with respect to the irrep $U$ if and only if it has a decomposition of the following form:
\[
\Phi = l_{id} \Pi^{id} + \sum_{\alpha \in \Theta, \alpha \neq id} l_{\alpha} \Pi^{\alpha} \quad \text{with} \quad l_{id} = 1, \ l_{\alpha} \in \mathbb{C}; \quad \Pi^{id}, \Pi^{\alpha} \in \text{End} \left[ \mathbb{M}(n, \mathbb{C}) \right],
\tag{88}
\]
where $\Pi^{id}, \Pi^{\alpha}$ are the projectors are defined through eq. (54) and eq. (55); the coefficients $l_{\alpha}$ are eigenvalues of $\Phi$ and satisfy the following inequalities:
\[
\sum_{g \in G} \left( \sum_{\alpha \in \Theta} l_{\alpha} |\varphi^\alpha|^2 \chi^\alpha(g^{-1}) \right) \left| \text{Tr} \left( V^\beta_i U^i(g) \right) \right|^2 \geq 0, \quad \forall \beta \in \Theta, \quad i \in \{1, \ldots, |\varphi^\beta|\}.
\tag{89}
\]
In the above, $V^\beta_i \in \mathbb{M}(n, \mathbb{C})$ denote the normalized eigenvectors of rank-one projectors $\Pi^\beta_i \in \text{End} \left[ \mathbb{M}(n, \mathbb{C}) \right]$ such that $\Pi^\beta = \sum_i \Pi^\beta_i$, and are explicitly given in eq. (62).
The eigenvalue equation for the Choi-Jamiołkowski image $J(\Phi)$, defined through eq. (72), reads:

$$J(\Phi)|\upsilon_\beta^i\rangle = \epsilon_\beta^i|\upsilon_\beta^i\rangle,$$

with the eigenvalues $\epsilon_\beta^i$ being given by Corollary 35. From Theorem 40 and eq. (62) of Proposition 20 it follows that for some $(s, t)$ in the set $S_{\beta,i}$ defined through eq. (66):

$$\epsilon_\beta^i = \frac{1}{(\Pi_\beta^i)_{st,st}} |\varphi_\beta^i|^2 \sum_{g,h \in G} \left[ \sum_{\alpha \in \Theta} l_\alpha |\varphi_\alpha| |\chi_\alpha (g^{-1})| \right] \varphi_\beta^i (h^{-1}) u_{ts} (g^{-1}) u_{st} (gh^{-1}) \geq 0,$$

for $\beta \in \Theta$, $i = 1, \ldots, |\Theta|$. From Remark 32 it follows, that the above expression for $\epsilon_\beta^i$ does not depend on the particular choice of the pair $(s, t) \in S_{\beta,i}$.

Using the statement of Lemma 59 of Appendix C, and Proposition 30, we are able to give the Kraus decomposition of any ICQC which satisfies Assumption 1. This is stated in the following theorem:

**Theorem 41.** The Kraus operators of any ICQC $\Phi \in \text{End} \left[ \mathbb{M}(n, \mathbb{C}) \right]$, which satisfy Assumption 1, have the following form:

$$K_i(\beta) = \sqrt{\epsilon_\beta^i} \left( \upsilon_\beta^i \right)^T, \quad \beta \in \Theta, \quad i = 1, \ldots, |\Theta|,$$

where $\epsilon_\beta^i$ are eigenvalues of the Choi-Jamiołkowski image $J(\Phi)$ given by eq. (80) and $V_i(\beta) = \left( \text{vec}^{-1} \left[ |\upsilon_\beta^i| \right] \right)^T \in \mathbb{M}(n, \mathbb{C})$ with $|\upsilon_\beta^i|$ defined through eq. (74).

**Remark 42.** The matrices $K_i(\beta)$ given through eq. (92) depend on the indices $(s, t)$ in $V_i(\beta) \equiv V_i(\beta) (s, t)$ (see Proposition 20) but from eq. (63) it follows that the Kraus representation of ICQC $\Phi$:

$$\Phi(X) = \sum_{\beta \in \Theta} \sum_{i=1}^{2^n} K_i(\beta)X K_i^\dagger(\beta), \quad X \in \mathbb{M} (n, \mathbb{C})$$

(93)

do not depend on the choice of the indices $(s, t)$ in $V_i(\beta) \equiv V_i(\beta) (s, t)$.

7 Some Geometrical Properties of ICLMs and ICQCs

The system of linear equations (see eq. (80) in Corollary 35):

$$\epsilon_\beta^i = \sum_{\alpha \in \Theta} l_\alpha \mu_i(\alpha, \beta), \quad \beta \in \Theta, \quad i = 1, \ldots, |\varphi^\beta|,$$

(94)

without the assumption $l_{id} = 1$, describes a linear dependence between the vectors $L = (l_\alpha) \in \mathbb{C}^{|\Theta|}$ of pairwise distinct eigenvalues$^2$ of the ICLM $\Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha$, and the vectors $E = (\epsilon_\beta^i) \in \mathbb{C}^{n^2}$

$^2$For each $\alpha \in \Theta$ there is an eigenvalue $l_\alpha$. These have multiplicity greater than one whenever $\Pi^\alpha$ is not a rank-one projector. However, these multiplicities are not taken into account in defining the vector $L$. 

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describing all the eigenvalues (including multiplicities) of the Choi-Jamiołkowski image $J(\Phi)$. This system of linear equations may also be written in the matrix form:

$$E = ML, \quad M = (m_{\beta_i,\alpha}) \equiv (\mu_i(\alpha, \beta)) \in \mathbb{M}(n^2 \times |\Theta|, \mathbb{C}). \quad (95)$$

In the above, we choose a particular ordering of the irreps, such that the first column and row of the matrix $M$ is always labelled by the parameter $(\text{id})$ of the identity irrep. The matrix $M = (m_{\beta_i,\alpha}) \equiv (\mu_i(\alpha, \beta))$ has the following important property:

**Proposition 43.** The matrix $M = (m_{\beta_i,\alpha}) \equiv (\mu_i(\alpha, \beta))$ has maximal possible rank, equal to $|\Theta|$, which means that the columns $M_{C(\alpha)} \in \mathbb{C}^{n^2}$ of the matrix $M$ are linearly independent. This implies that the matrix $M$ is invertible from the left, and denoting the left inverse as $M^{\text{inv}}$, we have:

$$M^{\text{inv}} M = 1_{|\Theta|}, \quad M^{\text{inv}} = G^{-1}M^\dagger \in \mathbb{M}(|\Theta| \times n^2, \mathbb{C}), \quad (96)$$

where $G = ((M_{C(\alpha)}, M_{C(\alpha')}) \in \mathbb{M}(|\Theta|, \mathbb{C})$ is the Gram matrix of the columns $M_{C(\alpha)} \in \mathbb{C}^{n^2}$ of the matrix $M$.

**Proof.** The Choi-Jamiołkowski map $J$ when restricted to the linear subspace of the ICLM (see Remark 8) is also an isomorphism. Any ICLM $\Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha \in \text{End}[\mathbb{M}(n, \mathbb{C})]$ is, by construction, normal; it means that $\Phi = 0$ if and only if all its eigenvalues $\{l_\alpha : \alpha \in \Theta\}$ are zero. If the rank of the matrix $M$ was smaller than $|\Theta|$, then it would mean that the Choi-Jamiołkowski image $J(\Phi)$ of a nonzero ICLM $\Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha$ (with nonzero eigenvalues $L = (l_\alpha)$), would have all eigenvalues $E = (\epsilon_i^\beta)$ equal to zero. On the other hand, the matrix $J(\Phi)$, which is a linear combination of commuting normal matrices $J(\Pi^\alpha)$, is also normal (see Corollary 33), so it would mean that $J(\Phi)$ is zero, which is impossible because $J$ defines an isomorphism and therefore cannot take the value zero on a nonzero argument. The maximal rank of the matrix $M = (m_{\beta_i,\alpha}) \equiv (\mu_i(\alpha, \beta))$ implies that it is invertible. In fact, we have

$$M^\dagger M = G \in \mathbb{M}(|\Theta|, \mathbb{C}) : G = (M_{C(\alpha)}, M_{C(\alpha')}), \quad \alpha, \alpha' \in \Theta \quad (97)$$

i.e. the matrix $G$ is the Gram matrix of the columns $M_{C(\alpha)} \in \mathbb{C}^{n^2}$ of the matrix $M$. From the linear independence of these columns it follows that the Gram matrix $G$ is invertible, so we get

$$G^{-1}M^\dagger M = 1_{|\Theta|}, \quad (98)$$

where $1_{|\Theta|}$ denotes the $|\Theta| \times |\Theta|$ identity matrix. \hfill $\Box$

**Remark 44.** The first column of the matrix $M$ given in eq. (95), $M_{C(\text{id})} = (\mu_i(\text{id}, \beta))$, labelled by $\alpha = \text{id}$, is of the form

$$(M_{C(\text{id})})^T = \frac{1}{|U|}(1, 1, \ldots, 1) \in \mathbb{C}^{n^2}, \quad (99)$$

and the first row $M_{R(\text{id})}$, indexed by $\beta = \text{id}$ is given by:

$$M_{R(\text{id})} = (m_{\text{id},\alpha}) = (\mu(\alpha, \text{id})) = \left(\frac{|\varphi^\alpha|}{|U|}\right). \quad (100)$$
Corollary 45. The correspondence between pairwise distinct eigenvalues \( L = (l_\alpha) \in \mathbb{C}^{\lvert \Theta \rvert} \) of the ICLM \( \Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha \) and eigenvalues \( E = (\epsilon_i^\beta) \in \mathbb{C}^{n^2} \) of its Choi-Jamiołkowski image \( J(\Phi) \) is one-to-one. The linear space \( \mathbb{C}^{\lvert \Theta \rvert} \) of all possible pairwise distinct eigenvalues \( L = (l_\alpha) \) of an ICLM is transformed isomorphically to a subspace \( \mathcal{E} \) of dimension \( \lvert \Theta \rvert \) in \( \mathbb{C}^{n^2} \), spanned by vectors \( E = ML \) in the linear space \( \mathbb{C}^{n^2} \). It means that we have \( \mathcal{E} = M(\mathbb{C}^{\lvert \Theta \rvert}) \), where \( M(\mathbb{C}^{\lvert \Theta \rvert}) \) is the subspace generated by the action of the matrix \( M \) on the vectors in \( \mathbb{C}^{\lvert \Theta \rvert} \). Moreover, the condition \( E = (\epsilon_i^\beta) \in \mathcal{E} \) implies that

\[
M^{\text{inv}} E = L = (l_\alpha) \in \mathbb{C}^{\lvert \Theta \rvert}.
\]

Now the conditions of trace preservation and complete positivity, which an ICLM must satisfy in order to be an ICQC, imply the following conditions on the eigenvalues \( E = (\epsilon_i^\beta) \):

\[
\epsilon_i^\beta \geq 0, \quad \beta \in \Theta, \quad i = 1, \ldots, |\varphi^\beta|,
\]

\[
\sum_{\beta \in \Theta} \sum_{i=1}^{\lvert \varphi^\beta \rvert} \epsilon_i^\beta = |U|.
\] (102)

Let us first recall the definition of a simplex.

Definition 46. The set,

\[
\left\{ (x_1, x_2, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_i \geq 0, \sum_{i=1}^{d+1} x_i = 1 \right\}
\] (103)

is called a standard \( d \)-dimensional simplex.

Comparing eq. (102) with Definition 46 of the simplex we see that the conditions on the eigenvalues \( \epsilon_i^\beta \), with \( \beta \in \Theta, \ i = 1, \ldots, |\varphi^\beta| \), coincide with the conditions of a simplex scaled by a factor \( |U| \). Let us denote such a scaled simplex by \( \Sigma(U) \), so we have

\[
\Sigma(U) = \left\{ (x_1, x_2, \ldots, x_{n^2}) \in \mathbb{R}^{n^2} : x_i \geq 0, \sum_{i=1}^{n^2} x_i = |U| \right\} \subset \mathbb{C}^{n^2}.
\] (104)

From eq. (104) and the definition of subspace \( \mathcal{E} \) given in Corollary 45 we get the following:

Proposition 47. The eigenvalues \( E = (\epsilon_i^\beta) \) of the Choi-Jamiołkowski image \( J(\Phi) \) of an ICQC \( \Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha \) lie in the intersection of the scaled simplex \( \Sigma(U) \subset \mathbb{C}^{n^2} \) and the subspace \( \mathcal{E} \subset \mathbb{C}^{n^2} \), i.e.,

\[
E = (\epsilon_i^\beta) \in \Sigma(U) \cap \mathcal{E}.
\] (105)

From this statement and Corollary 45 we get the following characteristic of the ICQC \( \Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha \).

Corollary 48. An ICLM \( \Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha \) is an ICQC if and only if its vector of eigenvalues \( L = (l_\alpha) \in \mathbb{C}^{\lvert \Theta \rvert} \) is an inverse image, in the mapping \( M : \mathbb{C}^{\lvert \Theta \rvert} \to \mathcal{E} = M(\mathbb{C}^{\lvert \Theta \rvert}) \), of some vector \( E = (\epsilon_i^\beta) \in \Sigma(U) \cap \mathcal{E} \), i.e.

\[
L = (l_\alpha) = M^{\text{inv}}(E),
\] (106)

where \( M^{\text{inv}} \) is the left inverse of the linear mapping \( M : \mathbb{C}^{\lvert \Theta \rvert} \to \mathcal{E} = M(\mathbb{C}^{\lvert \Theta \rvert}) \) described in Proposition 43.

Remark 49. In particular one can check by direct computation, that the structure of the matrix \( M = (m_{\beta_i,\alpha}) \equiv (\mu_i(\alpha, \beta)) \) is such that, \( \sum_{\beta \in \Theta} \sum_{i=1}^{\lvert \varphi^\beta \rvert} \epsilon_i^\beta = |U| \) if and only if \( l_{id} = 1 \), i.e. the ICLM \( \Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha \) is trace preserving.
8 Examples of ICQC

In this section we give an explicit choice of the parameters $l_\alpha$ (occurring in eq. (88)), for which the inequality eq. (89), which gives the condition for complete positivity of an ICLM, is automatically satisfied. For this choice, the ICLM given by eq. (88) is therefore completely positive, and hence defines an ICQC if in addition $l_{id} = 1$.

In addition, we provide explicit examples of ICQCs for some fixed groups for which tensor product $U \otimes U^c$ is simply reducible for an irrep $U$. We focus on the quaternion group $Q$ and the symmetric groups $S(3)$ and $S(4)$. In each case we present both the matrix representation and the Kraus representation of the ICQC. Further, for the case of $S(3)$ and $Q$, using the Peres-Horodecki (or PPT) criterion [16, 24], we find the condition under which the ICQC is an entanglement breaking channel.

8.1 A wide class of ICQC

Here we give a wide class of ICQCs by providing explicit expressions for the eigenvalues $l_\alpha$ in the spectral decomposition of an ICLM $\Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha$ for which $\Phi$ is completely positive and trace-preserving. These eigenvalues form a class of solutions of eq. (89) and they are given in the following theorem:

**Theorem 50.** Let $K(g) \subset G$ be the conjugacy class of $g \in G$, defined through eq. (46), and let $f: G \to \mathbb{C}$ be a function on the group $G$ such that:

1. $\sum_{g \in G} f(g) = |G|$. \hfill (107)

2. For all conjugacy classes $K(g)$, $\sum_{h \in K(g)} f(h) \geq 0$. \hfill (108)

Then a family of ICQC (i.e. a family of quantum channels satisfying Theorem 40) are those for which the coefficients $l_\alpha$ in eq. (88) are given by:

$$l_\alpha = \frac{1}{|G| |\varphi^\alpha|} \sum_{g \in G} \chi^\alpha(g) f(g), \quad \alpha \in \Theta.$$ \hfill (109)

**Remark 51.** The first condition on the function $f: G \to \mathbb{C}$ given in eqs. (107) and (108) guarantees that the ICLM $\Phi$ is trace-preserving, whereas the second condition in eqs. (107) and (108) imply that $\Phi$ is also completely positive.

In order to prove Theorem 50 we first make the following trivial extension of the expression eq. (70), which is implied by Proposition 13 and Corollary 15.

**Corollary 52.** An ICLM $\Phi \in \text{End} [\mathbb{M}(n, \mathbb{C})]$ can be expressed as follows:

$$\Phi = \sum_{\gamma \in \widehat{G}} l_\gamma \Pi^\gamma = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha + \sum_{\gamma \notin \Theta} l_\gamma \Pi^\gamma : \quad l_\alpha, l_\gamma \in \mathbb{C},$$ \hfill (110)

where $\widehat{G}$ is the set of all irreps of the group $G$, and where the $l_\gamma$, with $\gamma \notin \Theta$, are arbitrary.
Finally, the inequalities given in eq. (89) of Theorem 40 may be equivalently formulated as follows:

\[ J(\Phi) = J \left( \sum_{\gamma \in \tilde{G}} l_\gamma \Pi^\gamma \right) = \frac{1}{|G|} \sum_{ij} E_{ij} \otimes \sum_{g \in G} \left( \sum_{\gamma \in \tilde{G}} l_\gamma |\varphi^\gamma| \chi^\gamma (g^{-1}) \right) U_{C(i)}(g)U_{R(j)}(g^{-1}), \]  

(111)

with \( l_{id} = 1 \). Obviously \( J(\Pi^\gamma) = 0 \) if \( \gamma \notin \Theta \), and one can check by a direct calculation that the corresponding eigenvalues vanish, i.e.

\[ \mu_i(\gamma, \beta) = \frac{|\varphi^\gamma|}{|G|} \sum_{g \in G} \chi^\gamma (g^{-1}) \left| \text{Tr} \left( V_i^\beta U^\dagger(g) \right) \right|^2 = 0, \quad \beta \in \Theta, \ i = 1, \ldots, |\varphi^\beta|. \]  

(122)

Finally, the inequalities given in eq. (89) of Theorem 40 may be equivalently formulated as follows:

\[ \sum_{g \in G} \left( \sum_{\gamma \in \tilde{G}} l_\gamma |\varphi^\gamma| \chi^\gamma (g^{-1}) \right) \left| \text{Tr} \left( V_i^\beta U^\dagger(g) \right) \right|^2 = \sum_{g \in G} x(g) \left| \text{Tr} \left( V_i^\beta U^\dagger(g) \right) \right|^2 \geq 0, \]  

(113)

where \( \beta \in \Theta, \ i = 1, \ldots, |\varphi^\beta| \), and \( x(g) \equiv \sum_{\gamma \in \tilde{G}} l_\gamma |\varphi^\gamma| \chi^\gamma (g^{-1}) \). This form of the inequalities is more convenient to deal with. In particular, the form of the coefficients \( x(g) \) allows us to apply the orthogonality relations for irreps. Now we focus our attention on these coefficients. The idea of the construction of the solution of the inequalities from eq. (113) is to find \( \{ l_\gamma : \gamma \in \tilde{G} \} \) such that the coefficients \( x(g) \) are non-negative for any \( g \in G \), which obviously implies that the inequalities from eq. (113) are satisfied. In order to study the properties of the coefficients \( x(g) \) let us introduce the following matrices:

**Definition 53.** Let \( T = (t_{g\gamma}) \equiv (\chi^\gamma (g^{-1})) \in \mathbb{M}(|G| \times |\tilde{G}|, \mathbb{C}) \), where \( g \in G \) and \( \gamma \in \tilde{G} \) be a rectangular matrix whose rows are indexed by the group elements \( g \in G \) and columns are indexed by irreps \( \gamma \in \tilde{G} \). We assume that the group elements are ordered in a way such that elements belonging to the same conjugacy class are grouped together. Let \( D \in \mathbb{M}(|\tilde{G}|, \mathbb{C}) \) be a square matrix defined as follows:

\[ D = (d_{\gamma \delta}) \equiv \text{diag}(|\varphi^{\gamma_1}|, |\varphi^{\gamma_2}|, \ldots, |\varphi^{\gamma_n}|) : \ \gamma_i \in \tilde{G}, \]  

(114)

and let us define the following set of vectors:

\[ \tilde{L} \equiv (l_\gamma) \in \mathbb{C}^{|	ilde{G}|}, \quad F \equiv (f(g)) \in \mathbb{C}^{|G|}. \]  

(115)

Using Schur’s orthogonality relations for irreducible characters given in eq. (43) and eq. (45), one can show that the matrix \( T \) satisfies the relations given in the following lemma. They can be verified by direct computations. It is well-known that for finite groups the number of inequivalent irreps is equal to the number of conjugacy classes [11]. This allows us to label the conjugacy classes \( K(g), \ g \in G \) with the indices \( \gamma \in \tilde{G} \) used for the irreps. Hence, we interchangeably denote the conjugacy classes as \( K(g) \) or \( K_\gamma \).
Lemma 54. Suppose that we are given with the matrix \( T = (t_{g\gamma}) = (\chi^\gamma (g^{-1})) \in \mathbb{M}(|G| \times |\hat{G}|, \mathbb{C}) \), where the group elements \( g \in G \) are ordered in such a way that the conjugated elements of the group are all grouped together. Then,

\[
\frac{1}{|G|} T^\dagger T = 1_{|\hat{G}|} \in \mathbb{M}(|\hat{G}|, \mathbb{C}), \quad \frac{1}{|G|} TT^\dagger = \bigoplus_{\gamma \in \hat{G}} 1_{|K_\gamma|} \in \mathbb{M}(|G|, \mathbb{C}),
\]

(116)

where the matrices \( 1_{|K_\gamma|} \in \mathbb{M}(|K_\gamma|, \mathbb{C}) \) and all their entries are equal to 1 and \( K_\gamma : \gamma \in \hat{G} \) are classes of conjugated elements. The matrix \( \frac{1}{|G|} TT^\dagger \) is a block-diagonal matrix with diagonal blocks equal to the matrices \( 1_{|K_\gamma|} \) and all remaining blocks are equal to zero.

Using these matrices we may write the set of equations \( x(g) = \sum_{\gamma \in \hat{G}} l_\gamma |\varphi^\gamma| \chi^\gamma (g^{-1}) : g \in G \) in the matrix form:

\[
X = (x(g)) = TD\hat{L} \iff x(g) = \sum_{\gamma, \delta \in \hat{G}} t_{g\gamma} d_{\gamma\delta} l_\delta = \sum_{\gamma \in \hat{G}} l_\gamma |\varphi^\gamma| \chi^\gamma (g^{-1}) : g \in G.
\]

(117)

Now we are in the position to prove Theorem 50.

Proof of Theorem 50. We see that

\[
\hat{L} = \frac{1}{|G|} D^{-1} T^\dagger F \iff l_\gamma = \frac{1}{|G|} |\varphi^\gamma| \sum_{g \in G} \chi^\gamma (g)f(g) : \gamma \in \hat{G},
\]

(118)

then, using Definition 53 and the second statement of Lemma 54, we find that the coefficients

\[
x(g) \equiv \sum_{\gamma \in \hat{G}} l_\gamma |\varphi^\gamma| \chi^\gamma (g^{-1}) : g \in G
\]

(119)

occurring in eq. (113) become

\[
(x(g)) = TD\hat{L} = \frac{1}{|G|} TT^\dagger F \Rightarrow \forall g \in G \ x(g) = \frac{1}{|K(g)|} \sum_{h \in K(g)} f(h).
\]

(120)

From the above we see that for any \( g \in G \) the coefficient \( x(g) \) is non-negative by assumption on the function \( f : G \to \mathbb{C} \) and therefore the inequalities given in eq. (113) and eq. (89) are satisfied because on the LHS of these inequalities we have the sum of non-negative numbers. Summarizing, for a given function \( f : G \to \mathbb{C} \), which satisfies the assumptions of Theorem 50 we get a set of \( |\hat{G}| \) numbers

\[
l_\gamma = \frac{1}{|G|} |\varphi^\gamma| \sum_{g \in G} \chi^\gamma (g)f(g) : \gamma \in \hat{G}
\]

(121)

for which coefficients \( x(g) \) from eq. (119) are automatically non-negative. Since the sum in the decomposition of an ICQC \( \Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha \) runs over those \( \alpha \) which occur in the decomposition eq. (47) of \( U \otimes U^c \), we can restrict ourselves to the subset \( \{ l_\alpha : \alpha \in \Theta \} \).

Now we illustrate Theorem 50 by two examples:
Example 55. If \( f : G \to \mathbb{C} \) is such that \( \forall g \in G \ f(g) = 1 \), which obviously satisfies the conditions of Theorem 50, then using eq. (120) we obtain:

\[
\begin{align*}
l_{\text{id}} &= 1, \\
l_{\alpha} &= 0, \quad \alpha \neq \text{id}
\end{align*}
\]

and \( \Phi = \Pi^{\text{id}} \).

Example 56. It is clear that any function \( f : G \to \mathbb{C} \) such that \( \forall g \in G \ f(g) \geq 0 \) and \( \frac{1}{|G|} \sum_{g \in G} f(g) = 1 \) which satisfies the conditions of Theorem 50 defines a probability distribution. From this it follows that for any probability distribution on a finite group \( G \), one can define an ICQC.

8.2 Quaternion group \( Q \)

The quaternion group \( Q = \{ \pm Q_e, \pm Q_1, \pm Q_2, \pm Q_3 \} \) is a non-abelian group of order eight satisfying

\[
Q = \left\langle -Q_e, Q_1, Q_2, Q_3 \mid (-Q_e)^2 = Q_e, Q_1^2 = Q_2^2 = Q_3^2 = Q_1Q_2Q_3 = -Q_e \right\rangle.
\]

(123)

It possesses five inequivalent irreducible representations which we label by \( \text{id}, t_1, t_2, t_3, t_4 \), respectively. However, only one of them, labelled by \( t_4 \), has dimension greater than one and its dimension is equal to two. It is known that the quaternion group can be represented as a subgroup of \( GL(2, \mathbb{C}) \).

The matrix representation \( R : Q \to GL(2, \mathbb{C}) \) is given by

\[
\begin{align*}
Q_e &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & Q_1 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & Q_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & Q_3 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\end{align*}
\]

(124)

where \( i^2 = -1 \). In table 1 we present values of the characters for all irreducible representations of the group \( Q \).

| \( Q \)    | \( Q_e \) | \(-Q_e\) | \( Q_1\) | \( Q_2\) | \( Q_3\) | \(-Q_1\) | \(-Q_2\) | \(-Q_3\) |
|-----------|-----------|---------|---------|---------|---------|---------|---------|---------|
| \( \chi^{\text{id}} \) | 1 \quad 1 | 1 \quad 1 | 1 \quad 1 | 1 \quad 1 | 1 \quad 1 |
| \( \chi^{t_1} \) | 1 \quad 1 | -1 \quad 1 | -1 \quad 1 | -1 \quad 1 | -1 \quad 1 |
| \( \chi^{t_2} \) | 1 \quad 1 | 1 \quad -1 | -1 \quad 1 | 1 \quad -1 | -1 \quad 1 |
| \( \chi^{t_3} \) | 1 \quad 1 | -1 \quad 1 | -1 \quad 1 | -1 \quad 1 | 1 \quad -1 |
| \( \chi^{t_4} \) | 2 \quad -2 | 0 \quad 0 | 0 \quad 0 | 0 \quad 0 | 0 \quad 0 |

Table 1: Table of characters for the quaternion group \( Q \).

Now we can construct an ICQC for the quaternion group \( Q \) with respect to the two-dimensional irrep \( U = t_4 \). In this case the decomposition given by eq. (47) takes a form:

\[
U \otimes U^c = U^{\text{id}} \oplus U^{t_1} \oplus U^{t_2} \oplus U^{t_3}, \quad \dim [\text{Int}_Q(U \otimes U^c)] = 4,
\]

(125)

so \( \Theta = \{ \text{id}, t_1, t_2, t_3 \} \). The matrix representation \( \widetilde{\Phi}^{t_4} \) of the ICLM \( \Phi^{t_4} \) (see Corollary 15) is given by the following expression:

\[
\widetilde{\Phi}^{t_4} = \lambda t_{\text{id}} \Pi^{\text{id}} + t_1 \Pi^{t_1} + t_2 \Pi^{t_2} + t_3 \Pi^{t_3} = \frac{1}{2} \begin{pmatrix} l_{t_{\text{id}}} + l_{t_2} & 0 & 0 & 1 - l_{t_{\text{id}}} \\ 0 & l_{t_1} + l_{t_3} & l_{t_2} - l_{t_1} & 0 \\ 0 & l_{t_1} - l_{t_2} & l_{t_1} + l_{t_3} & 0 \\ 1 - l_{t_{\text{id}}} & 0 & 0 & l_{t_{\text{id}}} + l_{t_{\text{id}}} \end{pmatrix},
\]

(126)
where \( l_{\text{id}}, l_t, l_{t2}, l_{t3} \in \mathbb{R} \) (see explanation below Lemma 39). From Proposition 25, we know also that such a map \( \Phi^{t4} \) is trace-preserving if and only if \( l_{\text{id}} = 1 \), so its matrix representation in eq. (126) reduces to:

\[
\widetilde{\Phi}^{t4} = \widetilde{\Pi}^{\text{id}} + l_t \widetilde{\Pi}^{t1} + l_{t2} \widetilde{\Pi}^{t2} + l_{t3} \widetilde{\Pi}^{t3} = \frac{1}{2} \begin{pmatrix}
1 + l_{t2} & 0 & 0 & 1 - l_{t2} \\
0 & l_{t1} + l_{t3} & l_{t3} - l_{t1} & 0 \\
0 & l_{t3} - l_{t1} & l_{t1} + l_{t3} & 0 \\
1 - l_{t2} & 0 & 0 & 1 + l_{t2}
\end{pmatrix}.
\] (127)

A direct calculation gives the following explicit formula for the Choi-Jamiołkowski image of \( \Phi^{t4} \) given by eq. (137):

\[
J (\Phi^{t4}) = \sum_{i,j=1}^{2} E_{ij} \otimes \Phi^{t4} (E_{ij}) = \frac{1}{2} \begin{pmatrix}
1 + l_{t2} & 0 & 0 & l_{t1} + l_{t3} \\
0 & 1 - l_{t2} & l_{t3} - l_{t1} & 0 \\
l_{t3} - l_{t1} & 1 - l_{t2} & 0 & 0 \\
l_{t1} + l_{t3} & 0 & 0 & 1 + l_{t2}
\end{pmatrix}.
\] (128)

Further, to ensure that the trace-preserving ICLM \( \Phi^{t4} \) is completely positive (so that it is an ICQC) we require that \( J (\Phi^{t4}) \geq 0 \). This yields:

\[
\epsilon^{t1} = \frac{1}{2} (1 + l_{t1} - l_{t2} - l_{t3}) \geq 0,
\]

\[
\epsilon^{t2} = \frac{1}{2} (1 - l_{t1} + l_{t2} - l_{t3}) \geq 0,
\]

\[
\epsilon^{t3} = \frac{1}{2} (1 - l_{t1} - l_{t2} + l_{t3}) \geq 0,
\]

\[
\epsilon^{\text{id}} = \frac{1}{2} (1 + l_{t1} + l_{t2} + l_{t3}) \geq 0.
\] (129)

The expressions for the eigenvalues of \( J(\Phi^{t4}) \) from eq. (129) may be written in the matrix form using geometrical approach presented in Section 7. Namely constructing matrix \( M \) and the vectors \( E \) and \( L \) as in eq. (95) we get:

\[
\begin{pmatrix}
\epsilon^{\text{id}} \\
\epsilon^{t1} \\
\epsilon^{t2} \\
\epsilon^{t3}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix} \begin{pmatrix}
l_{\text{id}} \\
l_{t1} \\
l_{t2} \\
l_{t3}
\end{pmatrix}.
\] (130)

It easy to check that the matrix \( M \) is invertible (exactly it is orthogonal i.e. we have \( M^T = M \)). From Corollary 48 it follows that:

\[
\begin{pmatrix}
l_{\text{id}} \\
l_{t1} \\
l_{t2} \\
l_{t3}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1
\end{pmatrix} \begin{pmatrix}
\epsilon^{\text{id}} \\
\epsilon^{t1} \\
\epsilon^{t2} \\
\epsilon^{t3}
\end{pmatrix}.
\] (131)

In particular, because of the explanations given below Lemma 39 we have \( \forall \alpha \in \Theta \ l_{\alpha} \in \mathbb{R} \), and thanks to eq. (81) in Corollary 35 we are allowed to write:

\[
l_{\text{id}} = \frac{1}{2} (\epsilon^{\text{id}} + \epsilon^{t1} + \epsilon^{t2} + \epsilon^{t3}) = 1.
\] (132)
Equations 131 and 132 describe the all possible values of eigenvalues $L = (l_\alpha)$ of ICLM $\Phi = \sum_{\alpha \in \Theta} l_\alpha \Pi^\alpha$ for which $\Phi$ is trace-preserving. We see here that the case of quaternion group $Q$ is very particular, because we have equality $|U|^2 = 4 = |\Theta|$, and therefore the eigenvalues $L = (l_\alpha)$ are generated by all points $E = (e^\beta)$ of the scaled simplex $\Sigma(|U|)$ (see eq. (104) and Proposition 47).

Further from conditions $e^\beta \geq 0$ and eq. (132) the set of inequalities in eq. (129) reduces to:

$$|l_\alpha| \leq \frac{1}{2} \sum_{\beta \in \Theta} |e^\beta| = \frac{1}{2} \sum_{\beta \in \Theta} e^\beta = 1, \quad \forall \alpha \in \Theta \setminus \{\text{id}\}, \quad (133)$$

so all $l_\alpha$ where $\alpha \in \Theta \setminus \{\text{id}\}$ are included in a three-dimensional cube. The allowed values of the parameters $l_1, l_2, l_3$ satisfying the above constraints (for which $\Phi^t$ is an ICQC) are graphically presented on the left panel of fig. 3.

In the next step we construct Kraus operators for the ICQC $\Phi^t$. From the general considerations given in Proposition 30, we are able to compute orthogonalized eigensystem of the Choi-Jamiołkowski image $J(\Phi^t)$. Then, using Lemma 59 and its particular form given by Theorem 41, we can construct Kraus operators for the channel $\Phi^t$:

$$K(t_1) = \sqrt{\frac{e^t_1}{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K(t_2) = \sqrt{\frac{e^t_2}{2}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$K(t_3) = \sqrt{\frac{e^t_3}{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K(\text{id}) = \sqrt{\frac{e^{\text{id}}}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (134)$$

Moreover we have $\sum_{\beta \in \Theta} K(\beta)K^\dagger(\beta) = 1$ and $\sum_{\beta \in \Theta} K^\dagger(\beta)K(\beta) = 1$. Hence, the channel $\Phi^t$ is unital and trace-preserving.

### 8.3 Symmetric group $S(3)$

In the case of $G = S(3)$ we have three inequivalent irreducible representations (see [11], [2], [1]), one-dimensional identity representation (denoted by id), one-dimensional sign representation (denoted by sgn) and two-dimensional nontrivial representation (denoted by $\lambda$). Here we construct a trace-preserving ICLM and in particular ICQC for the irrep $U$ characterised by the $\lambda$ by using so called $\epsilon-$representation [1]. The generators of this representation are given by:

$$\varphi^\lambda(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \varphi^\lambda(23) = \begin{pmatrix} 0 & w^2 \\ w & 0 \end{pmatrix}, \quad (135)$$

where $w = \exp\left(\frac{2\pi i}{3}\right)$, and by (12), (23) we denote transpositions between respective elements. Decomposition given by eq. (47) in this case takes a form:

$$U \otimes U^c = U^{\text{id}} \oplus U^{\text{sgn}} \oplus U, \quad \text{and} \quad \dim \left[\text{Int}_{S(3)}(U \otimes U^c)\right] = 3. \quad (136)$$

We see that all possible irreps occur in the decomposition of $U \otimes U^c$, so $\Theta = \{\text{id}, \text{sgn}, \lambda\}$. The matrix representation $\tilde{\Phi}_\epsilon$ of the trace-preserving ICLM $\Phi_\epsilon$ (see Proposition 25) is given by the following expression:

$$\tilde{\Phi}_\epsilon = \tilde{\Pi}^{\text{id}} + l_{\text{sgn}} \tilde{\Pi}^{\text{sgn}} + l_\lambda \tilde{\Pi}^{\lambda} = \frac{1}{2} \begin{pmatrix} 1 + l_{\text{sgn}} & 0 & 0 & 1 - l_{\text{sgn}} \\ 0 & 2l_\lambda & 0 & 0 \\ 0 & 0 & 2l_\lambda & 0 \\ 1 - l_{\text{sgn}} & 0 & 0 & 1 + l_{\text{sgn}} \end{pmatrix}. \quad (137)$$

26
where \( l_{\text{sgn}}, l_{\lambda} \in \mathbb{R} \) (see explanation below Lemma 39). The corresponding Choi-Jamiołkowski image is given by:

\[
J(\Phi_{\epsilon}) = \sum_{i,j=1}^{2} E_{ij} \otimes \Phi_{\epsilon}(E_{ij}) = \begin{pmatrix}
\frac{1}{2}(1 + l_{\text{sgn}}) & 0 & 0 & l_{\lambda} \\
0 & \frac{1}{2}(1 - l_{\text{sgn}}) & 0 & 0 \\
0 & 0 & \frac{1}{2}(1 - l_{\text{sgn}}) & 0 \\
l_{\lambda} & 0 & 0 & \frac{1}{2}(1 + l_{\text{sgn}})
\end{pmatrix}.
\] (138)

Similarly as in Section 8.2 we can use results from Section 7 and construct set of linear constraints

\[E = ML:
\begin{pmatrix}
\epsilon_{\text{id}} \\
\epsilon_{\text{sgn}} \\
\epsilon_{1} \\
\epsilon_{2}
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
1 & 1 & 2 \\
1 & 1 & -2 \\
1 & -1 & 0 \\
1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
l_{\text{sgn}} \\
l_{\lambda}
\end{pmatrix}.
\] (139)

This gives us the spectrum of \( J(\Phi_{\epsilon}) \):

\[
\{ \epsilon_{1} = \frac{1}{2}(1 - l_{\text{sgn}}), \epsilon_{2} = \frac{1}{2}(1 - l_{\text{sgn}}), \epsilon_{\text{id}} = \frac{1}{2}(1 + l_{\text{sgn}}) + l_{\lambda}, \epsilon_{\text{sgn}} = \frac{1}{2}(1 + l_{\text{sgn}}) - l_{\lambda} \}.
\] (140)

From Corollary 48 it follows that

\[
\begin{pmatrix}
1 \\
l_{\text{sgn}} \\
l_{\lambda}
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\epsilon_{\text{id}} \\
\epsilon_{\text{sgn}} \\
\epsilon_{1} \\
\epsilon_{2}
\end{pmatrix}.
\] (141)

Conditions \( \epsilon_{i} \geq 0 \) and \( \frac{1}{2}(\epsilon_{\text{id}} + \epsilon_{\text{sgn}} + \epsilon_{1} + \epsilon_{2}) = 1 \) yields to the following statement:

\[
J(\Phi_{\epsilon}) \geq 0 \iff 1 \geq l_{\text{sgn}} \geq -1, \quad \frac{1}{2}(1 + l_{\text{sgn}}) \geq |l_{\lambda}|.
\] (142)

Graphical representation of the constraints given by the above inequalities is presented on the left panel of Figure 1 in the Introduction. The form of the matrix representation \( \Phi_{\epsilon} \) given by eq. (137) guarantees that the linear map \( \Phi_{\epsilon} \) is covariant with respect to the irrep \( U \) characterised by the irrep \( \lambda \) and trace-preserving. Moreover, under the constraints on parameters \( l_{\text{sgn}}, l_{\lambda} \) given by the inequalities (142), \( \Phi_{\epsilon} \) is completely positive map, hence an ICQC.

Finally using Theorem 41 we construct the Kraus operators of the ICQC \( \Phi_{\epsilon} \), they are of the following form:

\[
K_{1}(\lambda) = \sqrt{\epsilon_{1}^{\lambda}X_{1}^{\dagger}(\lambda)} = \sqrt{\epsilon_{1}^{\lambda}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad K_{2}(\lambda) = \sqrt{\epsilon_{2}^{\lambda}X_{2}^{\dagger}(\lambda)} = \sqrt{\epsilon_{2}^{\lambda}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

\[
K(\text{sgn}) = \sqrt{\epsilon_{\text{sgn}}^{\text{sgn}}}X_{1}^{\dagger}(\text{sgn}) = \sqrt{\epsilon_{\text{sgn}}^{\text{sgn}}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K(\text{id}) = \sqrt{\epsilon_{\text{id}}^{\text{id}}}X_{1}^{\dagger}(\text{id}) = \sqrt{\epsilon_{\text{id}}^{\text{id}}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (143)

The channel is easily checked to be unital.
8.4 Symmetric group $S(4)$

In this case we have three nontivial irrepes labelled by partitions $\lambda_1 = (3, 1)$, $\lambda_2 = (2, 2)$, $\lambda_3 = (2, 1, 1)$ and two one-dimensional denoted by id and sgn. Everything in this section is computed in the Young-Yamanouchi representation [2]. Here we construct a trace-preserving ICLM and in particular ICQC for the irrep $U$ characterised by the partition $\lambda_1$. The generators in the Young-Yamanouchi representation for the partition $\lambda_1$ have the following form:

$$
\varphi^{\lambda_1}(12) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix},
\varphi^{\lambda_1}(23) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & \sqrt{3}/2 \\
0 & \sqrt{3}/2 & -1/2
\end{pmatrix},
\varphi^{\lambda_1}(34) = \begin{pmatrix}
1 & \sqrt{3}/3 & 2/3 & 0 \\
\sqrt{3}/3 & 1/3 & 0 & 1
\end{pmatrix}.
$$

Decomposition given by eq. (47) in this case reads as:

$$
U \otimes U^c = U^{id} \oplus U^{\lambda_1} \oplus U^{\lambda_2} \oplus U^{\lambda_3}, \quad \text{and} \quad \dim \left[ \text{Int}_{S(4)} (U \otimes U^c) \right] = 4.
$$

We see that sgn irrep does not occur in the decomposition of $U \otimes U^c$. In this case we have $\Theta = \{\text{id}, \lambda_1, \lambda_2, \lambda_3\}$ and $\Phi^{\lambda_1} = \Pi^{id} + \lambda_1 \Pi^{\lambda_1} + \lambda_2 \Pi^{\lambda_2} + \lambda_3 \Pi^{\lambda_3}$ which is the trace-preserving ICLM (see Proposition 25). The matrix representation of the trace-preserving ICLM $\tilde{\Phi}^{\lambda_1}$ in the Young-Yamanouchi representation is given by:

$$
\tilde{\Phi}^{\lambda_1} = \begin{pmatrix}
a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_2 \\
0 & a_3 & 0 & a_4 & a_5 & 0 & 0 & 0 & -a_5 \\
0 & 0 & a_3 & 0 & 0 & -a_5 & a_4 & -a_5 & 0 \\
0 & a_4 & 0 & a_3 & a_5 & 0 & 0 & 0 & -a_5 \\
a_2 & a_5 & 0 & a_6 & 0 & 0 & 0 & a_7 & 0 \\
0 & 0 & -a_5 & 0 & 0 & a_8 & -a_5 & a_9 & 0 \\
0 & 0 & a_4 & 0 & 0 & -a_5 & a_3 & -a_5 & 0 \\
0 & 0 & -a_5 & 0 & 0 & a_9 & -a_5 & a_8 & 0 \\
a_2 & -a_5 & 0 & -a_5 & a_7 & 0 & 0 & 0 & a_6
\end{pmatrix},
$$

where

$$
a_1 = \frac{1}{3}(1 + 2l_{\lambda_1}), \quad x_2 = \frac{1}{3}(1 - l_{\lambda_1}),
$$

$$
a_3 = \frac{1}{6}(l_{\lambda_1} + 2l_{\lambda_2} + 3l_{\lambda_3}), \quad a_4 = \frac{1}{6}(l_{\lambda_1} + 2l_{\lambda_2} - 3l_{\lambda_3}),
$$

$$
a_5 = \frac{1}{3\sqrt{2}}(l_{\lambda_2} - l_{\lambda_1}), \quad a_6 = \frac{1}{6}(2 + 2l_{\lambda_1} + l_{\lambda_2}),
$$

$$
a_7 = \frac{1}{6}(2 - l_{\lambda_1} - l_{\lambda_2}), \quad a_8 = \frac{1}{6}(2l_{\lambda_1} + l_{\lambda_2} + 2l_{\lambda_3}),
$$

$$
a_9 = \frac{1}{6}(2l_{\lambda_1} + l_{\lambda_2} - 3l_{\lambda_3}),
$$

$$
$$
and \(l_{\lambda_1}, l_{\lambda_2}, l_{\lambda_3} \in \mathbb{R}\), since all characters for the group \(S(4)\) are real (see explanation below Lemma 39). The corresponding Choi-Jamiolkowski image (given in Proposition 26) can be written as:

\[
J \left( \Phi_{\chi}^{\lambda_1} \right) = \begin{pmatrix}
    a_1 & 0 & 0 & a_3 & 0 & 0 & a_3 \\
    0 & a_2 & 0 & a_6 & -a_9 & 0 & 0 \\
    0 & 0 & a_2 & 0 & 0 & a_9 & a_9 \\
    0 & a_6 & 0 & a_2 & a_9 & 0 & 0 \\
    a_3 & -a_9 & 0 & a_9 & a_4 & 0 & 0 \\
    0 & 0 & a_9 & 0 & 0 & a_7 & a_9 \\
    0 & 0 & a_6 & 0 & 0 & a_2 & a_9 \\
    a_3 & a_9 & 0 & a_9 & a_5 & 0 & 0 \\
\end{pmatrix},
\]

whose eigenvalues are given by (using Corollary 35):

\[
\begin{align*}
    \epsilon_1^{\lambda_1} &= \epsilon_2^{\lambda_1} = \epsilon_3^{\lambda_1} = \frac{1}{6} (2 + 3l_{\lambda_1} - 2l_{\lambda_2} - 3l_{\lambda_3}), \\
    \epsilon_1^{\lambda_2} &= \epsilon_2^{\lambda_2} = \frac{1}{6} (2 - 3l_{\lambda_1} + 4l_{\lambda_2} - 3l_{\lambda_3}), \\
    \epsilon_1^{\lambda_3} &= \epsilon_2^{\lambda_3} = \epsilon_3^{\lambda_3} = \frac{1}{6} (2 - 3l_{\lambda_1} - 2l_{\lambda_2} + 3l_{\lambda_3}), \\
    \epsilon_{id} &= \frac{1}{3} (1 + 3l_{\lambda_1} + 2l_{\lambda_2} + 3l_{\lambda_3}).
\end{align*}
\]

Note that \(J \left( \Phi_{\chi}^{\lambda_1} \right) \geq 0\) if and only if the parameters \(l_{\lambda_1}, l_{\lambda_2}, l_{\lambda_3}\) belong to the shaded region shown in Figure 2. The corresponding Kraus operators (see Lemma 59 and Theorem 41) are given as follows:

\[
K_1(\lambda_1) = \sqrt{\epsilon_1^{\lambda_1}} \begin{pmatrix}
    -\frac{2}{3} & \frac{1}{3\sqrt{2}} & 0 \\
    \frac{1}{3\sqrt{2}} & 0 & 0 \\
    0 & 0 & 2\frac{3}{\sqrt{2}}
\end{pmatrix}, \quad
K_2(\lambda_1) = \sqrt{\epsilon_2^{\lambda_1}} \begin{pmatrix}
    0 & 0 & \frac{1}{\sqrt{6}} \\
    \frac{1}{\sqrt{6}} & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix},
\]

\[
K_3(\lambda_1) = \sqrt{\epsilon_3^{\lambda_1}} \begin{pmatrix}
    -\frac{\sqrt{2}}{3} & -\frac{1}{\sqrt{3}} & 0 \\
    -\frac{1}{\sqrt{3}} & 0 & 0 \\
    0 & 0 & -\frac{1}{\sqrt{3}}
\end{pmatrix}, \quad
K_1(\lambda_2) = \sqrt{\epsilon_1^{\lambda_2}} \begin{pmatrix}
    0 & 0 & \frac{1}{\sqrt{6}} \\
    \frac{1}{\sqrt{6}} & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix},
\]

\[
K_2(\lambda_2) = \sqrt{\epsilon_2^{\lambda_2}} \begin{pmatrix}
    0 & 0 & -\frac{1}{\sqrt{6}} \\
    \frac{1}{\sqrt{6}} & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}, \quad
K_1(\lambda_3) = \sqrt{\epsilon_1^{\lambda_3}} \begin{pmatrix}
    0 & 0 & 0 \\
    0 & 0 & -\frac{1}{\sqrt{2}} \\
    0 & 0 & 0
\end{pmatrix},
\]

\[
K_2(\lambda_3) = \sqrt{\epsilon_2^{\lambda_3}} \begin{pmatrix}
    0 & 0 & 0 \\
    \frac{1}{\sqrt{2}} & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}, \quad
K_3(\lambda_3) = \sqrt{\epsilon_3^{\lambda_3}} \begin{pmatrix}
    0 & -\frac{1}{\sqrt{2}} & 0 \\
    \frac{1}{\sqrt{2}} & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix},
\]

\[
K(id) = \sqrt{\epsilon_{id}} \begin{pmatrix}
    \sqrt{3} & 0 & 0 \\
    0 & \frac{1}{\sqrt{3}} & 0 \\
    0 & 0 & \frac{1}{\sqrt{3}}
\end{pmatrix}.
\]

One can check, that the resulting channel is unital and trace-preserving.
8.5 Families of entanglement breaking ICQCs

In this section we present two families of entanglement breaking ICQCs based on $S(3)$ and the quaternion group $Q$. Let

$$S(\mathcal{H}) := \{\rho \in \mathcal{B}(\mathcal{H}) \mid \rho \geq 0, \text{Tr} \rho = 1\},$$

(151)

denote the set of all states (density matrices) acting on the Hilbert space $\mathcal{H} \simeq \mathbb{C}^d$. Here $\mathcal{B}(\mathcal{H})$ denotes the algebra of all linear operators acting on $\mathcal{H}$. Suppose now that we are dealing with two finite dimensional Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$.

A bipartite state $\rho_{AB} \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$ belongs to the set of separable states $\mathcal{SEP}$ if it can be written as $\rho_{AB} = \sum_i p_i \rho_i^A \otimes \sigma_i^B$, where $\rho_i^A, \sigma_i^B$ are states on $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, and $p_i$ are some positive numbers satisfying $\sum_i p_i = 1$. Otherwise the state $\rho_{AB}$ is entangled. A given quantum channel $\Phi$ (not necessarily an ICQC) is entanglement breaking (EB) [29] if and only if its Choi-Jamiołkowski image given by

$$(1 \otimes \Phi) (|\psi^+\rangle\langle\psi^+|) \equiv J(\Phi)$$

(152)

is separable for $|\psi^+\rangle = \frac{1}{\sqrt{d}} \sum_i |ii\rangle$. In general we do not have a unique criterion for checking separability, but it is known that in the case of quantum states on $S(\mathbb{C}^2 \otimes \mathbb{C}^2)$ and $S(\mathbb{C}^2 \otimes \mathbb{C}^3)$ necessary and sufficient conditions for separability are given in terms of partial transposition $1_A \otimes T_B$, where $T_B$ denotes standard transposition on $\mathcal{H}_B$. Namely, we have that $\sigma_{AB}$ is separable if and only if it has a positive partial transpose (PPT), i.e. $(1_A \otimes T_B)\sigma_{AB} \succeq 0$ [16, 24]. Since the maximal
possible dimension of irreps of both $S(3)$ and $Q$ is two, we can directly apply the above mentioned criterion to deduce when quantum channels, which are irreducibly covariant with respect to them, are also EB.

For the group $S(3)$ in the $\epsilon$–representation and partition $\lambda = (2,1)$ we have

$$J(\Phi_\epsilon) \in SEP \iff (1 \otimes T) J(\Phi_\epsilon) \geq 0 \iff -1 \leq l_{\text{sgn}} \leq 1, \frac{1}{2}(1 - l_{\text{sgn}}) \geq |l_\lambda|. \quad (153)$$

Comparing the above conditions for PPT with the conditions for CPTP of the map $\Phi_\epsilon$ given in eq. (142) we see that $\Phi_\epsilon$ is EB $\iff$

$$\begin{cases} 
-1 \leq l_{\text{sgn}} \leq 0, & |l_\lambda| \leq \frac{1}{2}(1 + l_{\text{sgn}}), \\
0 < l_{\text{sgn}} \leq 1, & |l_\lambda| \leq \frac{1}{2}(1 - l_{\text{sgn}}).
\end{cases} \quad (154)$$

The solution of the above inequalities is graphically represented on the right panel of Figure 1.

For the quaternion group $Q$ and irrep $t_4$ given in Section 8.2 the separability criterion of the Choi-Jamiołkowski image reads

$$J(\Phi^{t_4}) \in SEP \iff (1 \otimes T) J(\Phi^{t_4}) \geq 0 \iff \begin{cases} 
\frac{1}{2}(1 - l_{t_1} - l_{t_2} - l_{t_3}) \geq 0, \\
\frac{1}{2}(1 + l_{t_1} + l_{t_2} - l_{t_3}) \geq 0, \\
\frac{1}{2}(1 + l_{t_1} - l_{t_2} + l_{t_3}) \geq 0, \\
\frac{1}{2}(1 - l_{t_1} + l_{t_2} + l_{t_3}) \geq 0.
\end{cases} \quad (155)$$

Comparing the set of solutions of the above inequalities with the range of parameters $l_{t_1}, l_{t_2}, l_{t_3}$ for which the map $\Phi^{t_4}$ is CPTP, we get allowed triples $(l_{t_1}, l_{t_2}, l_{t_3})$ when $\Phi^{t_4}$ is an ICQC and EB channel. We present a graphical representation of allowed triples of parameters in Figure 3.

9 Additional results: spectral properties of the rank-one projectors $\Pi_i^\alpha$

In this section, we state properties of the eigenvectors $V_i^\alpha(s,t)$ of the rank-one projectors $\Pi_i^\alpha$, defined in Proposition 20. To formulate the main result of this section, which is stated in Theorem 58, we need the following lemma (which is proved in Appendix C).

**Lemma 57.** Suppose that $\gamma \notin \Theta$, where $U \otimes U^c = \bigoplus_{\alpha \in \Theta} \varphi^\alpha$, then

$$\forall s, t = 1, \ldots, n \quad V_i^\gamma(s,t) = \frac{|\varphi^\gamma|}{|G|} \sum_{g \in G} \varphi_i^\gamma g^{-1} U_C(s)(g)U_R(t)(g^{-1}) = 0. \quad (156)$$

**Theorem 58.** The eigenvectors $V_i^\alpha \equiv V_i^\alpha(s,t) \in \mathbb{M}(n, \mathbb{C})$ of the rank-one projectors $\Pi_i^\alpha \in \text{End} \left[ \mathbb{M}(n, \mathbb{C}) \right]$, defined in Proposition 20, satisfy the following properties:

1. $||V_i^\alpha(s,t)||^2 = (\Pi_i^\alpha)_{st,st}$, and hence $\sum_{s,t=1}^n ||V_i^\alpha(s,t)||^2 = 1. \quad (157)$
2. The following summation rules hold:

\[ \forall s, t = 1, \ldots, n \quad \sum_{\alpha \in \Theta} \sum_{i=1}^{\varphi^\alpha} V_i^\alpha(s, t) = E_{st}, \tag{158} \]

where \( \{E_{st}\}_{s,t=1}^n \) is a natural basis of \( M(n, \mathbb{C}) \), and

\[ \forall \alpha \in \Theta, \ i = 1, \ldots, |\varphi^\alpha| \quad \sum_{s=1}^{s=n} V_i^\alpha(s, s) = \delta^{\alpha, \text{id}} 1_n, \tag{159} \]

where \( 1_n \) denotes the identity matrix in \( M(n, \mathbb{C}) \).

3. Moreover,

\[ \text{Tr} V_i^\alpha(s, t) = \delta_{\alpha, \text{id}} \delta_{st}. \tag{160} \]

## 10 Conclusions and Open Questions

In this paper we present a detailed characterization of linear maps which are covariant with respect to an irreducible representation \( U \) of a finite group \( G \), in the case in which \( U \otimes U^c \) is simply
reducible (or multiplicity-free); here $U^c$ denotes the contragradient representation. We refer to such a linear map as an irreducibly covariant linear map (ICLM). We derive necessary and sufficient conditions under which an ICLM is trace-preserving and completely positive, and is thus an irreducibly covariant quantum channel (ICQC). These conditions are obtained by requiring that the Choi-Jamiołkowski image of the ICLM is positive semidefinite. We present explicit analytical expressions for the eigenvalues and the eigenvectors of the Choi-Jamiołkowski image, and give the conditions for complete positivity as a set of inequalities, which can be solved directly for any given finite group $G$. The resulting characterization of the ICQC is given entirely in terms of representation characteristic of the group $G$, and is valid for any finite group and its irreps $U$, as long as $U \otimes U^c$ is simply reducible. Moreover, we construct a wide class of non-trivial ICQCs for which the conditions for complete positivity are automatically satisfied. In addition, we obtain a geometrical interpretation of the spectrum of the Choi-Jamiołkowski image of an ICQC, by showing that it always lies in the intersection of a certain scaled simplex and a linear subspace, which are defined by the spectral properties of the projectors appearing in the spectral decomposition of an ICQC.

As a direct application of our result, we give the full description of ICQCs generated by irreps $(U)$ of certain finite groups, for which $U \otimes U^c$ is simply reducible. These are the quaternion group $Q$, and the symmetric groups $S(3)$ and $S(4)$. In each case we present both the matrix representation and the Kraus representation of the ICQC. Moreover, for covariance with respect to two-dimensional irreps of the groups $S(3)$ and $Q$, we obtain conditions under which the ICQCs are also entanglement breaking channels.

We expect to obtain an analogous characterization of quantum channels which are irreducibly covariant with respect to a compact group. However, such an analysis needs to be done carefully, since dealing with compact groups is technically more challenging than finite groups. This is the topic worthy of a separate project. In our opinion, however, it might be more interesting to first extend our analysis to the cases in which (i) $U \otimes U^c$ is not simple reducible, and (ii) the linear map is positive but not completely positive. In particular, the latter might lead to new classes of (covariant) entanglement witnesses.

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A Proofs of results from Section 4

Proof of Lemma 9. Suppose that \( \text{mat}(\Phi) \in \text{Int}_G(U \otimes \overline{U}) \), then \( \forall g \in G \) and \( \forall X \in \mathbb{M}(n, \mathbb{C}) \) by Lemma 2 we have the following chain of equivalences:

\[
\left( \sum_{\nu=1}^{m} A^\nu \otimes B^\nu \right) U(g) \otimes \overline{U}(g) = U(g) \otimes \overline{U}(g) \left( \sum_{\nu=1}^{m} A^\nu \otimes B^\nu \right) \iff \\
\left( \sum_{\nu=1}^{m} A^\nu \otimes B^\nu \right) U(g) \otimes \overline{U}(g) . \text{vec}(X) = U(g) \otimes \overline{U}(g) \left( \sum_{\nu=1}^{m} A^\nu \otimes B^\nu \right) . \text{vec}(X) \iff \\
\left( \sum_{\nu=1}^{m} A^\nu \otimes B^\nu \right) \text{vec} \left( U(g) X U^\dagger(g) \right) = U(g) \otimes \overline{U}(g) \text{vec} \left( \sum_{\nu=1}^{m} A^\nu X (B^\nu)^\dagger \right) \iff \\
\text{vec} \left( \sum_{\nu=1}^{m} A^\nu \text{Ad}_{U(g)}(X) (B^\nu)^\dagger \right) = \text{vec} \left( U(g) \sum_{\nu=1}^{m} (A^\nu X (B^\nu)^\dagger) U^\dagger(g) \right) \iff \\
\text{vec} \left( \Phi [\text{Ad}_{U(g)}(X)] \right) = \text{vec} \left( \text{Ad}_{U(g)} \left[ \sum_{\nu=1}^{m} A^\nu X (B^\nu)^\dagger \right] \right). 
\]

B Proofs of results from Section 5

Proof of Proposition 13. From the decomposition given by eq. (47) and Schur’s Lemma it is clear, that dimension of the commutant space of the representation \( U \otimes U^c \) is equal to number of irreps \( \varphi^\alpha \) appearing in this decomposition. From the orthogonality of the projectors \( \Pi^\alpha \) it follows that they are linearly independent, so they form a basis of the space \( \text{Int}_G(U \otimes U^c) \). The properties of the operators \( \Pi^\alpha \) are derived in [1], where in particular is shown that, for a given \( \varphi^\alpha \), the operator \( \Pi^\alpha \) (eq. (49)) is an orthogonal projector onto a subspace of \( \mathbb{C}^{n^2} \) containing all irreps \( \varphi^\alpha \) included in \( U \otimes U^c \). Because, by assumption, in our case each irrep \( \varphi^\alpha \) appears at most once in \( U \otimes U^c \), then in our case projectors \( \Pi^\alpha \) project onto the subspace of the irrep \( \varphi^\alpha \) in the representation space \( \mathbb{C}^{n^2} \). The orthogonality of the projectors \( \Pi^\alpha \) follows now from the direct sum decomposition in eq. (47). It can be also checked by direct calculation, using the orthogonality relations for irreducible characters eq. (43).

Now we prove that \( \Pi^\alpha \in \text{Int}_G(U \otimes U^c) \) for \( \alpha \in \Theta \). In fact we have for any \( h \in G \):

\[
U(h) \otimes \overline{U}(h) \Pi^\alpha U(h^{-1}) \otimes \overline{U}(h^{-1}) = \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1}) U(h g h^{-1}) \otimes \overline{U}(h g h^{-1}) \\
\frac{|\varphi^\alpha|}{|G|} \sum_{k \in G} \chi^\alpha(h^{-1} k^{-1} h) U(k) \otimes \overline{U}(k) = \frac{|\varphi^\alpha|}{|G|} \sum_{k \in G} \chi^\alpha(k^{-1}) U(k) \otimes \overline{U}(k) = \Pi^\alpha, 
\]

because characters are class functions.

Proof of Proposition 17. The hermiticity of matrices \( \Pi^\alpha \) follows directly from the unitarity of the
representation \( U : G \to \mathbb{M}(n, \mathbb{C}) \). We have also:

\[
\Pi_i^\alpha \Pi_j^\beta = \frac{|\varphi^\alpha|}{|G|^2} \sum_{g,h \in G} \varphi_{ii}^\alpha (g^{-1}) \varphi_{jj}^\alpha (h^{-1}) U(gh) \otimes U(gh)
\]

\[
= \frac{|\varphi^\alpha|}{|G|^2} \sum_{g,k \in G} \sum_l \delta_{il}\delta_{ij}\varphi_{jl}^\beta (k^{-1}) U(k) \otimes \overline{U}(k)
\]

Now using the Schur orthogonality relations for irreps, given by eq. (43), we get:

\[
\Pi_i^\alpha \Pi_j^\beta = \frac{|\varphi^\alpha|}{|G|^2} \sum_{k \in G} \varphi_{ii}^\alpha (g^{-1}) \varphi_{jj}^\alpha (k^{-1}g) U(k) \otimes \overline{U}(k)
\]

\[
= \frac{|\varphi^\alpha|}{|G|^2} \sum_{k \in G} \delta_{ij} \varphi_{jl}^\beta (k^{-1}) U(k) \otimes \overline{U}(k) = \delta_{ij} \Pi_i^\alpha.
\]

Similarly using the orthogonality relation eq. (43) for irreps, and the decomposition in eq. (47) one can prove that \( \text{Tr}(\Pi_i^\alpha) = 1 \).

**Proof of Corollary 19.** The first property in eq. (60) is obvious from the explicit form of the projectors \( \Pi_i^\alpha \) given in eq. (59).

Now we prove directly the second property from eq. (60). Namely \( \forall X \in \mathbb{M}(n, \mathbb{C}) \) we have:

\[
\Pi_i^\alpha \Pi_j^\beta (X) = \frac{|\varphi^\alpha|}{|G|^2} \sum_{g,h \in G} \varphi_{ii}^\alpha (g^{-1}) \varphi_{jj}^\beta (h^{-1}) U(gh) X U (h^{-1}g^{-1})
\]

Setting \( s = gh \) we get the following:

\[
\Pi_i^\alpha \Pi_j^\beta (X) = \frac{|\varphi^\alpha|}{|G|^2} \sum_{g,s \in G} \varphi_{ii}^\alpha (g^{-1}) \varphi_{jj}^\beta (s^{-1}g) U(s) X U (s^{-1})
\]

\[
= \frac{|\varphi^\alpha|}{|G|^2} \sum_{g,s \in G} \sum_l \varphi_{ii}^\alpha (g^{-1}) \varphi_{jl}^\beta (s^{-1}) \varphi_{ij}^\beta (g) U(s) X U (s^{-1}).
\]

Using the orthogonality relation eq. (43) for irreps, we get for any \( X \in \mathbb{M}(n, \mathbb{C}) \):

\[
\Pi_i^\alpha \Pi_j^\beta (X) = \frac{|\varphi^\alpha|}{|G|} \sum_{s \in G} \sum_l \delta^{\alpha \beta} \delta_{ij} \varphi_{jl}^\beta (s^{-1}) U(s) X U (s^{-1})
\]

\[
= \frac{|\varphi^\alpha|}{|G|} \sum_{s \in G} \delta^{\alpha \beta} \delta_{ij} \varphi_{jl}^\alpha (s^{-1}) U(s) X U (s^{-1}) = \delta^{\alpha \beta} \delta_{ij} \Pi_i^\alpha (X).
\]

To prove the third property from eq. (60), which is in fact, a conjugation of operators in the space
Comparing the right hand side of eq. (170) with the corresponding matrix element of \( \Pi^\alpha_r \) we use general considerations from Section 2. We have \( \forall X, Y \in \mathcal{M}(n, \mathbb{C}) \):

\[
(X, \Pi^\alpha_r(Y)) = \text{Tr} \left( X^* \left| \frac{\varphi^\alpha}{G} \sum_{g \in G} \varphi^\alpha_{ii} (g^{-1}) \ U(g)YU(g^{-1}) \right| \right)
\]

\[
= \text{Tr} \left( \left| \frac{\varphi^\alpha}{G} \sum_{g \in G} \varphi^\alpha_{ii} (g^{-1}) \ U(g^{-1}) \ X^\dagger U(g) \right| \right)
\]

\[
= \text{Tr} \left( \left| \frac{\varphi^\alpha}{G} \sum_{g \in G} \varphi^\alpha_{ii} (g^{-1}) \ U(g^{-1}) \ XU(g) \right| \right)^\dagger
\]

\[
= \text{Tr} \left( \left| \frac{\varphi^\alpha}{G} \sum_{h \in G} \varphi^\alpha_{ii} (h^{-1}) \ U(h)XU(h^{-1}) \right| \right)^\dagger
= (\Pi^\alpha_r(X), Y).
\]

\[ \square \]

Proof of Proposition 20. First we prove that the matrices \( V^\alpha_r(s, t) \) satisfy the eigenvalue equation for the projector \( \Pi^\alpha_r \). For any \( s, t = 1, \ldots, n \) we can write

\[
\Pi^\alpha_r(V^\alpha_r(s, t)) = \frac{|\varphi^\alpha|^2}{|G|^2} \sum_{g,h \in G} \varphi^\alpha_{rr} (h^{-1}) \varphi^\alpha_{rr} (g^{-1}) \ U(h)U_{c(s)}(g)U_{R(t)} (g^{-1}) \ U (h^{-1})
\]

\[
= \frac{|\varphi^\alpha|^2}{|G|^2} \sum_{g,h \in G} \varphi^\alpha_{rr} (h^{-1}) \varphi^\alpha_{rr} (g^{-1}) \ U_{c(s)}(hg)U_{R(t)} (g^{-1}h^{-1})
\]

\[
= \frac{|\varphi^\alpha|^2}{|G|^2} \sum_{w,h \in G} \varphi^\alpha_{rr} (h^{-1}) \varphi^\alpha_{rr} (hw^{-1}) \ U_{c(s)}(w)U_{R(t)} (w^{-1})
\]

\[
= \frac{|\varphi^\alpha|^2}{|G|} \sum_{w \in G} \varphi^\alpha_{rr} (w^{-1}) \ U_{c(s)}(w)U_{R(t)} (w^{-1}) = V^\alpha_r(s, t),
\]

where in the last step we have used the orthogonality relation eq. (43) for irreducible representations.

Now it remains to prove that not all matrices \( V^\alpha_r(s, t) \) are equal to zero. This can be done by calculating the norm of the matrices \( V^\alpha_r(s, t) \).

\[
||V^\alpha_r(s, t)||^2 = \left| \frac{\varphi^\alpha}{G} \sum_{g \in G} \varphi^\alpha_{rr} (g^{-1}) U_{ss}(g) \overline{U}_{tt}(g) \right|
\]

Comparing the right hand side of eq. (170) with the corresponding matrix element of \( \Pi^\alpha_r \),

\[
(\Pi^\alpha_r)_{st, st} = \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \varphi^\alpha_{rr} (g^{-1}) \left[ U(g) \otimes \overline{U}(g) \right]_{st, st} = \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \varphi^\alpha_{rr} (g^{-1}) U(g)_{ss} \overline{U}(g)_{tt},
\]

(171)
we see that the norms \( ||V^\alpha_i(s,t)||_2^2 \) are equal to the diagonal terms of the matrix \( \tilde{\Pi}_i^\alpha \in \mathbb{M}(n^2, \mathbb{C}) \). As \( \tilde{\Pi}_i^\alpha \) is a projector of rank one, its diagonal terms are non-negative and at least one of them is positive. Otherwise the projector \( \tilde{\Pi}_i^\alpha \) would be zero, which is impossible since \( \alpha \in \Theta \).

In order to prove the orthonormality relation given in eq. (63) it is enough to use the explicit form of the eigenvectors given in eq. (62) for those pairs of indices for which eigenvectors are non-zero. Thanks to this we have

\[
\left( V_i^\alpha(s,t), V_j^\beta(p,q) \right) = \text{Tr} \left[ \left( V_i^\alpha(s,t) \right)^\dagger V_j^\beta(p,q) \right] =
\]

\[
= \frac{1}{\sqrt{\left( \tilde{\Pi}_i^\alpha \right)_{st, st}}} \frac{1}{\sqrt{\left( \tilde{\Pi}_j^\beta \right)_{pq, pq}}} \frac{|\varphi_i^\alpha| \varphi_i^\beta}{|G|^2} \sum_{g,h} \varphi_{ii}^\alpha(g) \varphi_{jj}^\beta(h^{-1}) u_{qt}(gh^{-1}) u_{sp}(g^{-1}h) . \tag{172}
\]

Using the substitution \( gh^{-1} = w^{-1} \), the orthogonality relation eq. (43) for irreps, and the expression for the matrix element of the projector \( \tilde{\Pi}_i^\alpha \) from eq. (171), the right hand side of eq. (172) can be expressed as follows:

\[
1 \frac{1}{\sqrt{\left( \tilde{\Pi}_i^\alpha \right)_{st, st}}} \frac{1}{\sqrt{\left( \tilde{\Pi}_j^\beta \right)_{pq, pq}}} \frac{|\varphi_i^\alpha| \varphi_i^\beta}{|G|^2} \sum_{g,h} \varphi_{ii}^\alpha(g) \varphi_{jj}^\beta(g^{-1}w^{-1}) u_{qt}(w^{-1}) u_{sp}(w) =
\]

\[
= \frac{1}{\sqrt{\left( \tilde{\Pi}_i^\alpha \right)_{st, st}}} \frac{1}{\sqrt{\left( \tilde{\Pi}_j^\beta \right)_{pq, pq}}} \frac{|\varphi_i^\alpha| \varphi_i^\beta}{|G|^2} \sum_w \sum_r \left( \sum_g \varphi_{jr}^\beta(g^{-1}) \varphi_{ii}^\alpha(g) \right) \varphi_{rj}^\beta(w^{-1}) \bar{u}_{tq}(w) u_{sp}(w) =
\]

\[
= \frac{1}{\sqrt{\left( \tilde{\Pi}_i^\alpha \right)_{st, st}}} \frac{1}{\sqrt{\left( \tilde{\Pi}_j^\beta \right)_{pq, pq}}} \frac{|\varphi_i^\alpha|}{|G|} \sum_w \varphi_{ii}^\alpha(w^{-1}) u_{sp}(w) \bar{u}_{tq}(w) = \sqrt{\left( \tilde{\Pi}_i^\alpha \right)_{st, pq}} \frac{\left( \tilde{\Pi}_i^\alpha \right)_{st, pq}}{\left( \tilde{\Pi}_j^\beta \right)_{pq, pq}} \frac{\left( \tilde{\Pi}_j^\beta \right)_{pq, pq}}{\left( \tilde{\Pi}_i^\alpha \right)_{st, pq}} . \tag{173}
\]

From the [15] we know that the result of the calculations in eq. (173) has modulus equal to one, so it can be expressed as \( e^{i \zeta} \), where the parameter \( \zeta \equiv \zeta(s,t,p,q) \) for some \( s,t,p,q \in \{1, \ldots, n\} \). This proves the orthonormality relation eq. (63).

**Proof of Corollary 21.** From the argumentation presented in the above proof it is clear that when \( (s,t) = (p,q) \), then \( \zeta = 0 \). Moreover, for a fixed \( \alpha \in \Theta \) and \( i = 1, \ldots, |\varphi^\alpha| \), the vectors \( V_i^\alpha(s,t) \) and \( V_i^\alpha(p,q) \) can differ only by a phase \( e^{i \zeta} \), i.e. they lie in the same ray.

**Proof of Proposition 25.** We have for any \( X \in \mathbb{M}(n, \mathbb{C}) \):

\[
\text{Tr}[\Phi(X)] = \text{Tr} \left( \sum_{g \in G} \sum_{\alpha \in \Theta} \frac{|\varphi^\alpha|}{|G|} l_\alpha \chi^\alpha(g^{-1}) U(g) X U^\dagger(g) \right) =
\]

\[
= \sum_{\alpha \in \Theta} l_\alpha \left( \sum_{g \in G} \frac{|\varphi^\alpha|}{|G|} \chi^\alpha(g^{-1}) \right) \text{Tr}(X) = l_{id} \text{Tr}(X), \tag{175}
\]

where we use the orthogonality relation for irreducible characters given by eq. (44).

From the above, it is clear that the ICLM \( \Phi \in \text{End} [\mathbb{M}(n, \mathbb{C})] \) is trace-preserving if and only if \( l_{id} = 1 \).
Proof of Proposition 26. We have:

\[
J(\Pi^\alpha) = \sum_{ij} E_{ij} \otimes \Pi^\alpha(E_{ij}) = \sum_{ij} E_{ij} \otimes \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1}) U_{C(i)}(g) U_{R(j)}^\dagger(g),
\]

where

\[
U_{R(j)}^\dagger(g) = U_{R(i)}(g^{-1}) = [U_{C(i)}(g)]^\dagger.
\]

In particular for \(\Pi^{id}\) we have:

\[
J(\Pi^{id}) = \sum_{ij} E_{ij} \otimes \frac{1}{|G|} \sum_{g \in G} U_{C(i)}(g) U_{R(j)}(g^{-1}) = \sum_{ij} E_{ij} \otimes \frac{1}{|U|} \delta_{ij} \mathbf{1}_n = \frac{1}{|U|} \mathbf{1}_n \otimes \mathbf{1}_n, \tag{178}
\]

where we have used the orthogonality relation eq. (43) for irreps.

Proof of Proposition 30. We prove the Proposition checking directly that the eigenvalue equation holds. We calculate:

\[
J(\Pi^\alpha) |v^\beta_i \rangle = \sum_j |j \rangle \otimes \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1}) \text{Tr} \left( V_i^\beta U^\dagger(g) \right) U(g) |j \rangle. \tag{179}
\]

The operator

\[
X^\alpha \left( V_i^\beta \right) \equiv \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1}) \text{Tr} \left( V_i^\beta U^\dagger(g) \right) U(g) \in \mathbb{M}(n, \mathbb{C}), \tag{180}
\]

which appears on RHS has the following important property, which one derives by direct calculation

\[
\Pi^\alpha_k \left[ X^\alpha \left( V_i^\beta \right) \right] = \delta^{\gamma \beta} \delta_{ki} X^\alpha \left( V_i^\beta \right) : \alpha, \beta, \gamma \in \Theta, \quad k = 1, \ldots, |\varphi^\alpha|, \quad i = 1, \ldots, |\varphi^\beta|, \tag{181}
\]

where \(\Pi^\alpha_k, \alpha \in \Theta, \quad k = 1, \ldots, |\varphi^\alpha|\) are rank one projectors described in Proposition 17, so for any \(\alpha \in \Theta\), the matrices \(X^\alpha \left( V_i^\beta \right)\) are eigenvectors of the rank one projectors \(\Pi^\beta_k\) and therefore these matrices must be proportional to the matrices \(V_i^\beta \in \mathbb{M}(n, \mathbb{C})\) which are eigenvectors of \(\Pi^\beta_k\) i.e. we have:

\[
X^\alpha \left( V_i^\beta \right) = \mu_i(\alpha, \beta) V_i^\beta. \tag{182}
\]

Now it remains to calculate the proportionality coefficients \(\mu_i(\alpha, \beta)\), which can be done using the normalization of the eigenvectors \(V_i^\beta\) of projectors \(\Pi^\beta_k\) written in their matrix representation:

\[
\text{Tr} \left( V_i^\beta \left( V_i^\beta \right)^\dagger \right) = 1. \tag{183}
\]

From this it follows:

\[
\mu_i(\alpha, \beta) = \text{Tr} \left( X^\alpha \left( V_i^\beta \right) \left( V_i^\beta \right)^\dagger \right) = \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1}) \text{Tr} \left( V_i^\beta U^\dagger(g) \right) \text{Tr} \left( U(g) \left( V_i^\beta \right)^\dagger \right) = \frac{|\varphi^\alpha|}{|G|} \sum_{g \in G} \chi^\alpha(g^{-1}) \left| \text{Tr} \left( V_i^\beta U^\dagger(g) \right) \right|^2. \tag{184}
\]

Finally using eq. (62) and Proposition 20 we get the last result. \(\square\)
Proof of Lemma 37. In order to prove eq. (82) it is enough to compute directly quantity \( \text{Tr} \left( V_i^\gamma U^\dagger (g) \right) \) using explicit form of the eigenvectors given by eq. (62) in Proposition 20.

To prove eq. (83) as a starting point we use eq. (82) for some fixed irrep \( \varphi^\alpha \). Expanding the square of the modulus, we get

\[
|\varphi^\alpha| |\varphi^\beta| \frac{1}{|G|^3} \left( \tilde{\Pi}^\beta \right)_{st,st} \sum_{g,h,w} \chi^\alpha (g^{-1}) \varphi^\beta_i (h^{-1}) \varphi^\beta_j (w) u_{ts} (h^{-1} g^{-1} h) u_{st} (w^{-1} g w). \tag{185}
\]

Now substituting \( r = h^{-1} g^{-1} h \) in the above, we get

\[
|\varphi^\alpha| |\varphi^\beta| \frac{1}{|G|^3} \left( \tilde{\Pi}^\beta \right)_{st,st} \sum_{h,r} \chi^\alpha (r) \varphi^\beta_i (h^{-1}) \varphi^\beta_j (w) u_{ts} (r) u_{st} (w^{-1} h r^{-1} h^{-1} w), \tag{186}
\]

since the character \( \chi^\alpha \) is a class function. Next, substituting \( f = w^{-1} h \) we get

\[
|\varphi^\alpha| |\varphi^\beta| \frac{1}{|G|^3} \left( \tilde{\Pi}^\beta \right)_{st,st} \sum_{f,h,r} \chi^\alpha (r) \varphi^\beta_i (h^{-1}) \left( \sum_a \varphi^\beta_{i a} (h) \varphi^\beta_{a j} (f^{-1}) \right) u_{ts} (r) u_{st} (f r^{-1} f^{-1}). \tag{187}
\]

Finally, using the orthogonality relation for irreps given in eq. (43) we obtain

\[
|\varphi^\alpha| |\varphi^\beta| \frac{1}{|G|^3} \left( \tilde{\Pi}^\beta \right)_{st,st} \sum_{f,h} \chi^\alpha (r) \varphi^\beta_i (r^{-1}) u_{ts} (r) u_{st} (f r^{-1} f^{-1}). \tag{188}
\]

Changing indeces in the sum \( r \to g^{-1} \) and \( f \to h \) we obtain the claim. \( \square \)

Proof of Lemma 39. If \( \Phi(X)^\dagger = \Phi(X^\dagger) \), then

\[
J(\Phi)^\dagger = \sum_{i,j} E_{ji} \otimes \Phi(E_{ij})^\dagger = \sum_{i,j} E_{ji} \otimes \Phi(E_{ji}) = J(\Phi). \tag{189}
\]

If \( J(\Phi)^\dagger = J(\Phi) \), then

\[
\sum_{i,j} E_{ji} \otimes \Phi(E_{ij})^\dagger = \sum_{i,j} E_{ji} \otimes \Phi(E_{ji}) = \sum_{i,j} E_{ji} \otimes \Phi(E_{ij})^\dagger. \tag{190}
\]

C Proofs of results from Section 9

Proof of Lemma 57. Suppose that the matrix \( V_i^\gamma (s,t) \) is nonzero, then by Proposition 30 we know that it satisfies the eigenvalue equation

\[
\Pi_i^\gamma (V_i^\gamma (s,t)) = V_i^\gamma (s,t), \tag{191}
\]

and this means that the projector \( \Pi_i^\gamma \) is nonzero. This implies that the projector \( \Pi^\gamma = \sum_{i=1}^{\|\omega^\gamma\|} \Pi_i^\gamma \) is also nonzero. The latter operator projects onto the subspace of irreps \( \varphi^\gamma \) in \( U \otimes U^c \), which does not exist in this representation because \( \gamma \notin \Theta \). So we get a contradiction assuming that the matrix \( V_i^\gamma (s,t) \) is nonzero when \( \gamma \notin \Theta \). \( \square \)
Proof of Theorem 58. The relations in eq. (157) and in eq. (160) can be verified directly using Schur’s orthogonality relations. The proof of the first summation rule from eq. (158) is based on Lemma 57. It is clear that for any \( k, l = 1, \ldots, n \) we have

\[
\sum_{\alpha \in \Theta} \sum_{i=1}^{G} V_i^{\alpha}(s, t) = \sum_{\alpha \in G} \sum_{i=1}^{G} V_i^{\alpha}(s, t) = \sum_{\alpha \in G} \sum_{i=1}^{G} \frac{|\varphi_i^\alpha|}{|G|} \sum_{g \in G} \varphi_i^\alpha (g^{-1}) U_{C(s)}(g) U_{R(t)}(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \sum_{\alpha \in G} \chi^\alpha(g) \chi^\alpha(g^{-1}) U_{C(s)}(g) U_{R(t)}(g^{-1}),
\]

(192)

where \( e \) denotes the identity element of the given group \( G \). Now using the orthogonality relation for irreps given by eq. (45) we get

\[
\sum_{\alpha \in \Theta} \sum_{i=1}^{G} V_i^{\alpha}(s, t) = U_{C(s)}(e) U_{R(t)}(e) = E_{st}.
\]

(193)

To prove the second summation rule in eq. (159) it is enough to observe that

\[
\forall g \in G \quad U_{C(s)}(g) U_{R(t)}(g^{-1}) = 1_n \in M(n, \mathbb{C})
\]

(194)

and that

\[
\frac{|\varphi_i^\alpha|}{|G|} \sum_{g \in G} \varphi_i^\alpha(g^{-1}) = \frac{|\varphi_i^\alpha|}{|G|} \sum_{g \in G} \varphi_i^{id}(g) \varphi_i^{\alpha}(g^{-1}) = \delta_{\alpha, id},
\]

(195)

which follows directly from the orthogonality relation for irreps given by eq. (44).

\[\square\]

D Connection between Kraus representation and Choi-Jamiołkowski image of a quantum channel

Lemma 59. Let us assume, that we are given with a quantum channel \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) with its Choi-Jamiołkowski image \( J(\Phi) \) with the normalised eigensystem \( \{\lambda_m, |x_m\rangle\} \). Then its Kraus operators are given by the following expression:

\[
K_m = \sqrt{\lambda_m} X_m^T,
\]

(196)

where \( X_m = \text{vec}^{-1}(|x_m\rangle) \) is the matrix obtained by the inverse vectorization of the eigenvector \( |x_m\rangle \).

Proof. At the first step let us calculate an action of the channel \( \Phi \) on some state \( \rho \):

\[
\Phi(\rho) = \sum_{kl} (\Phi(\rho))_{kl} E_{kl} = \sum_{kl} \text{Tr} \left[ E_{kl}^{\dagger} \Phi(\rho) \right] E_{kl} = \sum_{kl} \text{Tr} \left[ E_{lk} \Phi(\rho) \right] E_{kl} = \sum_{kl} \text{Tr} \left[ E_{lk} \left( \sum_{ij} \text{Tr}(E_{ji} \rho) E_{ij} \right) \right] E_{kl} = \sum_{ijkl} \text{Tr} \left[ E_{ij} \rho \right] \text{Tr} \left[ E_{lk} \Phi(E_{ij}) \right] E_{kl}.
\]

(197)

Notice that \( \langle k| \Phi(E_{ij}) |l \rangle = J(\Phi)_{ik, jl} \), indeed we have

\[
J(\Phi)_{ik, jl} = \langle ik| J(\Phi) |jl \rangle = \langle ik| \sum_{rs} E_{rs} \otimes \Phi(E_{rs}) |jl \rangle = \langle k| \Phi(E_{ij}) |l \rangle.
\]

(198)
We know that $J(\Phi) \geq 0$. Thanks to this we can write the spectral decomposition of $J(\Phi)$ as $J(\Phi) = \sum_m \lambda_m |x_m\rangle\langle x_m|$, where $\lambda_m$ are the eigenvalues of $J(\Phi)$ and $\{ |x_m\rangle\}$ is the set of its normalised eigenvectors. Now using the above mentioned decomposition let us compute the matrix element $(J(\Phi))_{ik, jl}$ in the operator basis $\{ E_{ij} \}$:

$$(J(\Phi))_{ik, jl} = \sum_m \lambda_m |i\rangle\langle k| |x_m\rangle\langle x_m| |j\rangle\langle l|.$$  \hfill (199)

Finally putting eq. (199) into eq. (197) we have:

$$\Phi(\rho) = \sum_{ijkl} \sum_m \lambda_m |i\rangle\langle k| |x_m\rangle\langle x_m| |j\rangle\langle l| = \sum_{ijkl} \sum_m \lambda_m |i\rangle\langle k| |x_m\rangle\langle x_m| |j\rangle\langle l|$$

$$= \sum_{ijkl} \sum_m \lambda_m |i\rangle\langle k| |x_m\rangle\langle x_m| |j\rangle\langle l| E_{ki} \rho E_{lj} = \sum_m \left( \sum_{ki} \sqrt{\lambda_m} |i\rangle\langle k| E_{ki} \right) \rho \left( \sum_{jl} \sqrt{\lambda_m} |j\rangle\langle l| E_{jl} \right)$$

$$= \sum_m K_m \rho K_m^\dagger, \hfill (200)$$

where

$$K_m := \sqrt{\lambda_m} \sum_{ki} |i\rangle\langle k| E_{ki}. \hfill (201)$$

It is easy to see that

$$\sum_{ki} |i\rangle\langle k| E_{ki} = \left[ \text{vec}^{-1}(|x_m\rangle) \right]^T = X_m^T, \hfill (202)$$

where vec$^{-1}(\cdot)$ is the inverse vectorisation, and hence eq. (196) holds.

We establish below that $\{ K_m \}$ is indeed a set of Kraus operators for the ICQC $\Phi$ by showing that $\sum_m K_m^\dagger K_m = 1$. Indeed we have

$$\sum_m K_m^\dagger K_m = \sum_m \lambda_m \sum_{ijkl} |x_m\rangle\langle x_m| |i\rangle\langle k| E_{ki} \rho E_{lj} = \sum_{ij} \sum_m \langle jk| x_m\rangle\langle x_m| i\rangle E_{ij} =$$

$$= \sum_{ijkl} \langle jk| E_{st} \otimes \Phi(E_{st}) |i\rangle \langle l| = \sum_{ijkl} \langle jk| \Phi(E_{ji}) |i\rangle \langle l| = \sum_{ij} \text{Tr}[\Phi(E_{ji})] E_{ij}. \hfill (203)$$

Since $\Phi$ is a CPTP map, we have $\text{Tr}[\Phi(E_{ji})] = \delta_{ji}$, since $\text{Tr}(E_{ji}) = \delta_{ji}$. This completes the proof. \qed
References

[1] M.A. Naimark and A.I. Stern. Theory of group representations. *Springer-Verlag New York*, 1982.

[2] J.-Q. Chen, J. Ping, and F. Wang. Group representation theory for physicists. *World Scientific*, 2002.

[3] M.D. Choi. Positive semidefinite biquadratic forms. *Linear Algebra and Its Applications*, 12(2):95–100, 1975.

[4] N. Datta, M. Fukuda, and A.S. Holevo. Complementarity and additivity for covariant channels. *Quantum Information Processing*, 5(2):179–207, 2006.

[5] N. Datta, A.S. Holevo, and Y.M. Suhov. On a sufficient condition for additivity in quantum information theory. *Problems in Information Transmission*, 41(2):76–90, 2005.

[6] N. Datta, A.S. Holevo, and Y.M. Suhov. Additivity for transpose depolarizing channels. *International Journal of Quantum Information*, 4(1):85–98, 2006.

[7] N. Datta, M. Tomamichel, and M.M. Wilde. On the second-order asymptotics for entanglement-assisted communication. *Quantum Information Processing*, 15(6):2569–2591, 2016.

[8] M. Fannes, B. Haegeman, M. Mosonyi, and D. Vanpeteghem. Additivity of minimal entropy output for a class of covariant channels. *arXiv:quant-ph/0410195*, October 2004.

[9] M. Fannes, B. Nachtergaele, and R. F. Werner. Quantum spin chains with quantum group symmetry. *Communications in Mathematical Physics*, 174:477–507, January 1996.

[10] M. Fukuda and G. Gour. Additive bounds of minimum output entropies for unital channels and an exact qubit formula. *arXiv:1502.06411v1*.

[11] W. Fulton and J. Harris. Representation theory: A first course. *Springer*, 2004.

[12] M.B. Hastings. Superadditivity of communication capacity using entangled inputs. *Nature Physics*, 5(4):255–257, 2009.

[13] A. S. Holevo. Remarks on the classical capacity of quantum channel. *arXiv:quant-ph/0212025v1*.

[14] A. S. Holevo. The capacity of the quantum channel with general signal states. *Information Theory, IEEE Transactions on*, 44(1):269–273, Jan 1998.

[15] R. A. Horn and Ch. R. Johnson. Matrix analysis. *Cambridge University Press*, February 1990.

[16] M. Horodecki, P. Horodecki, and R. Horodecki. Separability of mixed states: necessary and sufficient conditions. *Physics Letters A*, 223(1):1–8, 1996.

[17] A. Jamiolkowski. Linear transformations which preserve trace and positive semi-definiteness of operator. *Reports on Mathematical Physics*, 3(4):275–278, 1972.

42
[18] A. Jenčová and M. Plávala. Conditions for optimal input states for discrimination of quantum channels. arXiv:1603.01437.

[19] R. Köenig and S. Wehner. A strong converse for classical channel coding using entangled inputs. Physical Review Letters, 103(7):070504, 2009.

[20] I. Marvian and R. W. Spekkens. Asymmetry properties of pure quantum states. Physical Review A, 90(1):014102, July 2014.

[21] Ch. B. Mendl and M. M. Wolf. Unital quantum channels - convex structure and revivals of birkhoff’s theorem. Communication in Mathematical Physics, 289(3):1057–1096, 2009.

[22] B. Nachtergaele and D. Ueltschi. A direct proof of dimerization in a family of SU(n)-invariant quantum spin chains. ArXiv e-prints, January 2017.

[23] M. A. Nuwairan. SU(2)-Irreducibly covariant and EPOSIC channels. arxiv:quant-ph/1306.5321, June 2013.

[24] A. Peres. Separability criterion for density matrices. Physics Review Letters, 77(8):1413–1415, 1996.

[25] M. Sanz, M. M. Wolf, D. Pérez-García, and J. I. Cirac. Matrix product states: Symmetries and two-body Hamiltonians. Physical Review A, 79(4):042308, April 2009.

[26] J. Schliemann. Entanglement in SU(2)-invariant quantum spin systems. Physical Review A, 68(1):012309, July 2003.

[27] B. Schumacher and M. D. Westmoreland. Sending classical information via noisy quantum channels. Phys. Rev. A, 56(1):131–138, Jul 1997.

[28] H. Scutaru. Some remarks on covariant completely positive linear maps on C*-algebras. Reports on Mathematical Physics, 16(1):79–87, 1979.

[29] P. W. Shor, M. Horodecki, and M.B. Ruskai. Entanglement breaking channels. Reviews in Mathematical Physics, 15(6):629641, 2003.

[30] K. G. H. Vollbrecht and R. F. Werner. Entanglement measures under symmetry. Physical Review A, 64(6):062307, December 2001.

[31] M.M. Wilde, M. Tomamichel, and M. Berta. Converse bounds for private communication over quantum channels. arXiv:1602.08898.