FROBENIUS GREEN FUNCTORS

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INTRODUCTION

These notes provide an informal introduction to a type of Mackey functor that arises naturally in algebraic topology in connection with Morava K-theory of classifying spaces of finite groups. The main aim is to identify key algebraic aspects of the Green functor structure obtained by applying a Morava K-theory to such classifying spaces. This grew out of joint work with Birgit Richter [1]. Of course the Morava K-theory and Lubin-Tate theory of such classifying spaces were important amongst subjects of [9,10], and our work is very much a footnote to those from a topological viewpoint, but we hope the algebraic structures encountered here are of wider interest. Since the potential audience is varied, we have tried to provide necessary background material.

For each prime $p$ and each natural number $n$ there is a multiplicative cohomology theory $K(n;p)^*(-) = K(n)^*(-)$ defined on spaces which on a point takes the value

$$K(n)^*(point) = K(n)^* = \mathbb{F}_p[v_n, v_n^{-1}],$$

where $v_n \in K(n)^{2-2p^n}$. If $p$ is odd, the values on spaces are graded commutative $K(n)^*$-algebras, while for $p = 2$ there is a mild deviation from graded commutativity, namely for a space $X$, and $u,v \in K(n)^*(X)$ of odd degree,

$$uv - vu = v_n Q(u)Q(v)$$

for a certain cohomology operation $Q$ on $K(n)^*(-)$; this is referred to as quasi-commutativity in [19]. In these notes we focus on aspects of the structure of $K(n)^*(BG)$ where this complication does not significantly affect our results.

In practise, it is useful to modify $K(n)^*(-)$ to obtain a 2-periodic theory $K_n^*(-)$ where

$$K_n^*(point) = K_n^* = \mathbb{F}[u, u^{-1}],$$

where $u \in K_n^{-2}$ and $\mathbb{F}$ is either $\mathbb{F}_{p^n}$ or $\overline{\mathbb{F}}_p$. For this theory, the grading is essentially over $\mathbb{Z}/2$ (or $\mathbb{Z}/2(p^n-1)$) so the ideas of [9] may prove useful. There is a Galois-theoretic relationship between $K_n^*(-)$ and $K(n)^*(-)$, corresponding to extending from $\mathbb{F}_p$ to $\mathbb{F}$, and adjoining a $(p^n-1)$-st root of $v_n$; therefore passage in either direction between these theories is well-understood. Note that $K_n^*$ is a graded field in the obvious sense, so computations are often simplified with the aid of a strict Künneth formula for products:

$$K_n^*(X \times Y) \cong K_n^*(X) \otimes_{K_n^*} K_n^*(Y).$$

Remarkably, despite $BG$ being a large space (for example it can usually be modelled by an infinite dimensional CW complex or manifold) there is a finiteness result due to Ravenel [16]: for
every finite group $G$, $K_n^*(BG)$ is finite dimensional over $K_n^*$. This is one of the key foundational results, the other being an observation that $K_n^*(BG)$ is self-dual as a $K_n^*(BG)$-module and therefore it is self-injective. Since $K_n^*(BG)$ is also a local $K_n^*$-algebra, this means that $K_n^*(BG)$ is a local Frobenius algebra (and the choice of Frobenius structure is in some sense functorial).

This circle of ideas was explored by Strickland [19]. An important observation (whose proof I learnt from Nick Kuhn) is that $\text{Tr}^G_1(1) \neq 0$, and it easily follows that $\text{Tr}^G_1(1)$ is a basis element for the socle of $K_n^*(BG)$.

Using properties of transfers for covering spaces, we can consider $K_n^*(B\text{--})$ as a Green functor on subgroups of a fixed finite group $G$. In fact, it extends to a globally defined Green functor on all finite groups. Here we exploit the fact that $K_n^*(BG) = K_n^*$ precisely when $p \nmid |G|$ to produce pushforward maps $(B\alpha)_*: K_n^*(BH) \to K_n^*(BK)$ whenever $\alpha: H \to K$ is a homomorphism for which $p \nmid |\ker \alpha|$.

In Appendix A we provide a brief review of material on local Frobenius algebras.

1. Recollections on Mackey and Green functors

We refer to Webb [21] for a convenient general overview, other useful references are Bouc [23].

Let $R$ be a commutative ring (in practise in our work it will be a field).

A Mackey functor $M$ on the subgroups of a finite group $G$ and taking values in the category of left $R$-modules $R_\text{mod}$, is an assignment

$$(H \leq G) \mapsto M(H) \in R_\text{mod}$$

together with morphisms

$$\text{res}_K^H: M(H) \to M(K), \quad \text{ind}_K^H: M(K) \to M(H), \quad c_g: M(H) \to M(gHg^{-1})$$

for $K \leq H \leq G$ and $g \in G$ which satisfy the following axioms.

(MF1) For $H \leq G$ and $h \in H$,

$$\text{res}_H^H = \text{ind}_H^H = c_h = \text{id}: M(H) \to M(H).$$

(MF2) For $L \leq K \leq H \leq G$,

$$\text{res}_L^K \text{res}_K^H = \text{res}_L^H, \quad \text{ind}_K^H \text{ind}_L^K = \text{ind}_L^H.$$

(MF3) For $g_1, g_2 \in G$ and $H \leq G$,

$$c_{g_1} c_{g_2} = c_{g_1 g_2}: M(H) \to M(g_1 g_2 H g_2^{-1} g_1^{-1}).$$

(MF4) For $K \leq H \leq G$ and $g \in G$,

$$\text{res}_{gKg^{-1}}^{gHg^{-1}} c_g = c_{gKg^{-1}} \text{res}_{K}^{H}, \quad \text{ind}_{gKg^{-1}}^{gHg^{-1}} c_g = c_{gKg^{-1}} \text{ind}_{K}^{H}.$$

(MF5) (Mackey double coset/decomposition formula) For $H \leq G$ and $K \leq H \geq L$,

$$\text{res}_L^H \text{ind}_K^H = \sum_{g: L \cap gKg^{-1} \subseteq L \cap K} \text{ind}_{L \cap gKg^{-1}}^L c_g \text{res}_{g^{-1} Lg \cap K}^K,$$

where the sum is over a complete set of representatives for the set of double cosets $L \setminus G / K$.

Such a Mackey functor $A$ is a Green functor if furthermore $A(H)$ is an $R$-algebra for $H \leq G$, and the following are satisfied.

(GF1) For $K \leq H \leq G$ and $g \in G$, $\text{res}_K^H$ and $c_g$ are $R$-algebra homomorphisms.
(GF2) (Frobenius axiom) For $K \leq H \leq G$, $x \in A(K)$ and $y \in A(H)$,
\[
\text{ind}_K^H(x \text{ res}_K^H(y)) = \text{ind}_K^H(x)y, \quad \text{ind}_K^H(\text{res}_K^H(y)x) = y \text{ ind}_K^H(x).
\]

**Remark 1.1.** Notice that when $K \leq H \leq G$, $A(K)$ becomes both a left and a right $A(H)$-module via $\text{res}_K^H(y)$, so that for $x \in A(K)$ and $y \in A(H)$,
\[
y \cdot x = \text{res}_K^H(y)x, \quad x \cdot y = x \text{ res}_K^H(y).
\]

Then the Frobenius axiom simply asserts that $\text{ind}_K^H : A(K) \to A(H)$ is both a left and a right $A(H)$-module homomorphism.

Now suppose that $\mathcal{X}, \mathcal{Y}$ are two collections of finite groups where $\mathcal{X}$ satisfies the following conditions. We say that $K$ is a *section* of a group $G$ if there is a subgroup $H \leq G$ and an epimorphism $H \to K$.

- If $G \in \mathcal{X}$ and $K$ is a section of $G$, then $K \in \mathcal{X}$.
- Let $G', G'' \in \mathcal{X}$. If
  \[
  1 \to G' \to G \to G'' \to 1,
  \]
  is a short exact sequence, then $G \in \mathcal{X}$.

A *globally defined Mackey functor* with respect to $\mathcal{X}, \mathcal{Y}$ on finite groups and taking values in $R$-modules, is an assignment of an $R$-module $M(G)$ to each finite group $G$, for each homomorphism $\alpha : G \to H$ with $\ker \alpha \in \mathcal{X}$ a homomorphism $\alpha^* : M(H) \to M(G)$, and for each homomorphism $\beta : K \to L$ with $\ker \alpha \in \mathcal{Y}$ a homomorphism $\beta_* : M(K) \to M(L)$, satisfying the following conditions.

(GD1) When these are defined, $(\alpha_1 \alpha_2)^* = \alpha_2^* \alpha_1^*$ and $(\beta_1 \beta_2)_* = (\beta_1)_*(\beta_2)_*$.

(GD2) If $\gamma : G \to G$ is an inner automorphism, then $\gamma^* = \text{id} = \gamma_*$. 

(GD3) Given a pullback diagram of finite groups
\[
\begin{array}{ccc}
J & \downarrow \gamma & G \\
\beta & \downarrow \alpha & \\
K & \downarrow \delta & H
\end{array}
\]
the following hold:
- if $\ker \alpha = \ker \beta \in \mathcal{Y}$, then $\delta^* \alpha_* = \beta_* \gamma^*$;
- if $\ker \alpha = \ker \beta \in \mathcal{X}$, then $\alpha^* \delta_* = \gamma_* \beta^*$.

(GD4) For a commutative diagram of epimorphisms of finite groups
\[
\begin{array}{ccc}
G & \xrightarrow{\beta} & K \\
\downarrow{\alpha} & & \downarrow{\gamma} \\
H & \xrightarrow{\delta} & G/\ker \alpha \ker \beta
\end{array}
\]
with $\ker \alpha \in \mathcal{Y}$ and $\ker \beta \in \mathcal{X}$, we have $\alpha_* \beta^* = \delta^* \gamma_*$. 

(GD5) (Mackey formula) For subgroups $H \leq G \geq K$ with inclusion maps $t^G_K : H \rightarrow G$, etc, and $c_g$ induced by $g^{-1}(-)g : K \cap gHg^{-1} \rightarrow g^{-1}Kg \cap H$,
\[
(t^G_K)^* (t^G_H)_* = \sum_{g : K \cap G/H} (t^K_{K \cap gHg^{-1}})_* c_g (t^H_{g^{-1}Kg \cap H})^*.
\]

Such a globally defined Mackey functor is a globally defined Green functor if

(GD6) Whenever $\alpha : G \rightarrow H$ is a homomorphism for which $\alpha^*$ is defined, then $\alpha^* : A(H) \rightarrow A(G)$ is an $A$-algebra homomorphism.

(GD7) (Frobenius axiom) Suppose that $\beta : K \rightarrow L$ is a homomorphism for which $\beta^*$ and $\beta_*$ are both defined. Note that $\beta^*$ induces left and right $A(H)$-module structures on $A(G)$ (these coincide when $A(G)$ is commutative). Then $\beta_* : A(G) \rightarrow A(H)$ is a homomorphism of left and right $A(H)$-modules.

For our purposes we will take $\mathcal{X}$ to consist of all finite groups, and $\mathcal{Y}$ to consist of either trivial groups or all groups of order not divisible by some prime $p > 0$.

2. Local Artinian Green functors and globally defined extensions

Suppose that $A$ is a Green functor on the subgroups of $G$ taking values in the category of local Artinian $k$-algebras, where $k$ is a field with char$k = p \geq 0$. For $H \leq G$, we will write $m(H) \circ A(H)$ for the unique maximal left ideal of $A(H)$; this agrees with the Jacobson radical, $m(H) = \text{rad} A(H)$, which is a two-sided ideal. If $K \leq H \leq G$, then $\text{res}^H_K$ is a local algebra homomorphism, i.e., $\text{res}^H_K m(H) \subseteq m(K)$. We will call such a Green functor local Artinian.

More generally, we can suppose that $A$ is a globally defined Green functor taking values in the category of local Artinian $k$-algebras. In the notation of [21 section 8], we will take $\mathcal{X}, \mathcal{Y}$ to consist of suitable collections of finite groups. Then the restriction/inflation homomorphism $\alpha^* : A(H) \rightarrow A(G)$ associated to a homomorphism $\alpha : G \rightarrow H$ with ker$\alpha \in \mathcal{X}$ is a local algebra homomorphism, and the induction homomorphism $\beta_* : A(K) \rightarrow A(L)$ associated to a homomorphism $\beta : K \rightarrow L$ with ker$\beta \in \mathcal{Y}$ is a left and right $A(L)$-module homomorphism with respect to the $A(L)$-module structure induced on $A(K)$ by $\beta^*$. When $\beta$ is the inclusion of a subgroup we will also write $\beta^* = \text{res}^L_K$ and $\beta_* = \text{ind}^L_K$.

From now on we assume $A$ is a globally defined local Artinian Green functor, where we take $\mathcal{X}$ to consist of all finite groups and $\mathcal{Y}$ to consist of the trivial groups. One of our goals is to extend $A$ to a Green functor where $\mathcal{Y}$ is enlarged to include all groups of order not divisible by $p$.

We make some further assumptions.

Assumption (A). For any trivial group 1, $A(1) = k$.

Notice that for any finite group $G$ and any trivial group 1, the diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & G \\
\end{array}
\]

induces

\[
\begin{array}{ccc}
k = A(1) & \longrightarrow & A(G) \\
\unit & & \aug \end{array}
\]

such that $A(1) = k$. 

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where \( \text{aug} = \text{res}_1^G \). Therefore \( \ker \text{aug} = \mathfrak{m}(G) \) and

\[
(2.1) \quad A(G)/\mathfrak{m}(G) = k.
\]

**Assumption (B).** For any finite group \( G \), \( 0 \neq \text{ind}_1^G(1) \in A(G) \).

When \( H \leq K \), we can use the restriction \( \text{res}_K^H \) to induce left and right \( A(K) \)-module structures on \( A(H) \) and by the Frobenius axiom, \( \text{ind}_K^K : A(K) \longrightarrow A(K) \) is an \( A(K) \)-module homomorphism. If \( z \in \ker \text{res}_K^H \), then for any \( x \in A(H) \),

\[
(2.2) \quad z \text{ind}_K^H(x) = 0 = \text{ind}_K^H(x)z.
\]

More generally these remarks apply to the restriction \( \alpha^* \) associated to an arbitrary group homomorphism \( \alpha \), and to the associated \( \alpha^* \) if it is defined.

We can now give a basic result. Recall that \( p \geq 0 \) is the characteristic of \( k \).

**Lemma 2.1.** Suppose that \( H \leq K \) and \( p \nmid |K : H| \).

(a) \( \text{ind}_K^H(1) \in A(K)^\times \).

(b) \( \text{res}_K^K : A(K) \longrightarrow A(H) \) is a split monomorphism. In fact \( \text{ind}_K^H \text{res}_K^H \) is right or left multiplication by a unit in \( A(K) \).

**Proof.** (a) By the Mackey double coset formula for the subgroups \( 1 \leq K \geq H \),

\[
\text{aug}(\text{ind}_K^H(1)) = \text{res}_1^K \text{ind}_K^H(1) = \sum_{k: 1 \backslash K/H} \text{ind}_1^k c_k \text{res}_1^H(1)
\]

\[
= \sum_{k: 1 \backslash K/H} 1 = |K : H|.
\]

Thus

\[
(2.3) \quad \text{ind}_K^H(1) - |K : H| \in \mathfrak{m}(K) = \text{rad} A(K),
\]

so if \( p \nmid |K : H| \), \( \text{ind}_K^H(1) \in A(K)^\times \).

(b) Consider \( \text{ind}_K^H \text{res}_K^H : A(K) \longrightarrow A(K) \). For \( x \in A(K) \), the Frobenius axiom gives

\[
x \text{ind}_K^H(1) = \text{ind}_K^H \text{res}_K^H(x) = \text{ind}_K^H(1)x.
\]

By (a) \( \text{ind}_K^H(1) \) is invertible in \( A(K) \) so we can use \( \text{ind}_K^H(1)^{-1} \) to show that \( \text{ind}_K^H \text{res}_K^H \) splits as a left or right \( A(K) \)-module homomorphism. \( \square \)

We also have the following.

**Proposition 2.2.** Let \( G \) be a finite group.

(a) \( 0 \neq \text{ind}_1^G(1) \in \soc A(G) \).

(b) If \( p \nmid |G| \), then \( \text{ind}_1^G(1) \in A(G)^\times \).

**Proof.** (a) Suppose that

\[
z \in \text{rad} A(G) = \mathfrak{m}(G) = \ker \text{res}_1^G.
\]

By (2.2), \( z \text{ind}_1^G(1) = 0 = \text{ind}_1^G(1)z \), so \( \text{ind}_1^G(1) \in \soc A(G) \).

(b) This follows from Lemma 2.1(a). \( \square \)

We also record
Proposition 2.3. If $P$ is a $p$-Sylow subgroup of $G$, then
\[
\text{res}_P^G(\text{ind}_I^G(1)) = |G : P| \text{ind}_I^P(1).
\]

Hence Assumption (B) holds for all finite groups if and only if it holds for all finite $p$-groups.

Proof. The Mackey formula (MF5) gives
\[
\text{res}_P^G(\text{ind}_I^G(1)) = \sum_{g : P \triangleleft G/I} \text{ind}_I^P(1) \circ \text{res}_I^1(z)
\]
\[
= \sum_{g : P \triangleleft G/I} \text{ind}_I^P(1)
\]
\[
= |G : P| \text{ind}_I^P(1) \neq 0.
\]

The conclusion about Assumption (B) is obvious. □

We can also deduce some useful results on induction when the prime $p$ divides the index.

Proposition 2.4. Suppose that $H \leq G$ and $p \mid |G : H|$. Then for $z \in A(H)$,
\[
\text{ind}_I^H(z) \in m(G).
\]

Proof. By the Mackey formula,
\[
\text{res}_I^G(\text{ind}_I^H(z)) = \sum_{g : \Gamma \triangleleft G/H} \text{ind}_I^\Gamma(1) \circ \text{res}_I^\Gamma(z)
\]
\[
= \sum_{g : \Gamma \triangleleft G/H} \text{res}_I^\Gamma(z)
\]
\[
= |G : H| \text{res}_I^\Gamma(z) = 0.
\]

So $\text{ind}_I^H(z) \in m(G)$. □

Here is another important consequence of Lemma 2.1(b) and Assumption (A). For a converse, see Theorem 5.5 which depends on more assumptions on the Green functor $A$.

Proposition 2.5. Suppose that $G$ is a group for which $p \nmid |G|$. Then $A(G) = k$.

Our next result is crucial in identifying when a Green functor can be extended to a globally defined Green functor.

Lemma 2.6. Suppose that $\text{char} k = p > 0$.
(a) Suppose that $P$ is a $p$-group, and that $N$ is a group for which $p \nmid |N|$ and $P$ acts on $N$ by automorphisms, so the semi-direct product $PN = P \rtimes N$ is defined. Then the inclusion $P \to PN$ induces an isomorphism
\[
\text{res}_P^{PN} : A(PN) \xrightarrow{\cong} A(P).
\]

(b) Suppose that $K \triangleleft G$ where $p \nmid |K|$. Then the quotient epimorphism $\pi : G \to G/K$ induces an isomorphism
\[
\pi^* : A(G/K) \xrightarrow{\cong} A(G).
\]
Proof. (a) Since
\[ |PN : P| = |N/P \cap N| = |N| \]
and \( PN/N \cong P \), Lemma 2.1 gives a commutative diagram
\[
\begin{array}{ccc}
A(PN/N) & \xrightarrow{\cong} & A(P) \\
\res^P_{PN} & & \res^P_{PN} \\
\ind^P_{PN} & & \ind^P_{PN}
\end{array}
\]
where \( \ind^P_{PN} \res^P_{PN} \) is the identity. It easily follows that \( \res^P_{PN} \) is both monic and epic, hence it is an isomorphism.

(b) Let \( Q \leq G/K \) be a \( p \)-Sylow subgroup. The pullback square
\[
\begin{array}{ccc}
\tilde{Q} & \xrightarrow{\bot} & Q \\
\downarrow & & \downarrow \\
G & \xrightarrow{\bot} & G/K
\end{array}
\]
defines a subgroup \( \tilde{Q} \leq G \) containing \( K \) and of order \( |Q||K| \). Let \( P \leq \tilde{Q} \leq G \) be a \( p \)-Sylow subgroup of \( \tilde{Q} \) (and so of \( G \)). Then \( \tilde{Q} \) is a \( p \)-nilpotent group with
\[ \tilde{Q} = PK = P \times K, \]
and the composition
\[ P \xrightarrow{\text{inc}} PK = \tilde{Q} \xrightarrow{\text{quo}} Q \]
is an isomorphism. By (a), the inclusion induces an isomorphism
\[ A(\tilde{Q}) \xrightarrow{\text{inc}^* = \res^\tilde{Q}_P} A(P), \]
hence \( \text{quo}^*: A(Q) \to A(\tilde{Q}) = A(PK) \) is also an isomorphism. Now we have a commutative diagram
\[
\begin{array}{ccc}
A(G/K) & \xrightarrow{\pi^*} & A(G) \\
\ind^{G/K}_{\tilde{Q}} & & \res^{G/K}_Q \\
\res^{G/K}_{PK} & & \ind^{G/K}_{PK}
\end{array}
\]
from which it follows that \( \pi^* \) is monic and epic, hence it is an isomorphism. \( \square \)

This result allows us to define \( \alpha_* = (\alpha^*)^{-1} \) for any such homomorphism \( \alpha: G \to G/K \), and more generally any homomorphism \( \beta: G \to H \) with \( p \nmid |\ker \beta| \). Thus we have the following extension result.

**Theorem 2.7.** There is a unique extension of \( A \) to a globally defined local Artinian Green functor for the pair \( \mathcal{X}, \mathcal{Y}' \), where \( \mathcal{Y}' \) consists of all finite groups of order not divisible by \( p \).
We now introduce a third condition which is suggested by this result. When $A(G)$ is Morava $K$-theory of $BG$ at the prime $p$, this condition is automatic since the groups for which $A(G) = k$ are precisely those of order not divisible by $p$. More generally, Theorem 5.5 provides conditions under which this holds. It is not clear if this always holds for local Artinian Green functors satisfying Assumptions (A) and (B).

**Assumption (C).** Suppose that $K \trianglelefteq G$ and $A(K) = k$, and $\pi: G \to G/K$ is the quotient homomorphism. Then $\pi$ induces an isomorphism $\pi^*: A(G/K) \to A(G)$.

Now we give a result on the effect of automorphisms of $G$ on the socle of $A(G)$; when $A(G)$ is a Frobenius algebra (see Section 4 for details), $\dim_k \text{soc} A(G)$ is 1-dimensional, and the full strength of this applies.

**Proposition 2.8.** Suppose that $\text{char } p > 0$. Let $G$ be a finite group and let $\alpha: G \to H$ be an isomorphism. Then

$$\alpha^* \text{ind}^H_1(1) = \text{ind}^G_1(1).$$

In particular, if $H = G$ so $\alpha$ is an automorphism and $\dim_k \text{soc} A(G) = 1$, then $\alpha^*: A(G) \to A(G)$ restricts to the identity function on $\text{soc} A(G)$.

**Proof.** Taking $\mathcal{X}$ and $\mathcal{Y}$ to consist of all finite groups and all trivial groups respectively, we can apply (GD3) to the diagram

$$\begin{array}{ccc}
1 & \to & G \\
\downarrow & & \downarrow \cong \\
1 & \to & H \\
\alpha & & \\
\end{array}$$

to obtain

$$\alpha^* \text{ind}^H_1(1) = \text{ind}^G_1(1).$$

If $\alpha$ is an automorphism, it induces a $k$-algebra automorphism $\alpha^*: A(G) \to A(G)$ which restricts to an automorphism $\text{soc} A(G) \to \text{soc} A(G)$. If $\text{soc} A(G) = 1$ is 1-dimensional, then by Proposition 2.2(a) it is spanned by $\text{ind}^G_1(1)$ which is fixed by $\alpha^*$. \qed

An automorphism of order $p^e$ must act as the identity on the 1-dimensional vector space $\text{soc} A(G)$, but an automorphism of order not divisible by $p$ might be expected to act non-trivially.

3. The stable elements formula

It is well-understood that analogues of the classic stable elements formula of Cartan & Eilenberg [5] often applies to compute Mackey functors. The very accessible introduction of Webb [21, section 3] provides the necessary background material which we will use. We have all the necessary ingredients for such a result when $A$ is a local Artinian Green functor which satisfies both of Assumptions (A) and (B).

We assume that $\text{char } k = p > 0$ and denote by $\mathcal{P}$ the collection of all non-trivial finite $p$-groups.

First note that for any finite group and a $p$-Sylow subgroup $P \leq G$, the induction homomorphism $\text{ind}^G_P: A(P) \to A(G)$ is surjective by Lemma 2.1(b).

**Proposition 3.1.** The Mackey functor $A$ on the subgroups of $G$ is $\mathcal{P}$-projective.
Proof. The coproduct of the maps \( \text{ind}_Q^G \),
\[
(\text{ind}_Q^G)_Q : \bigoplus_{Q \subseteq G, Q \in \mathcal{P}} A(Q) \rightarrow A(G),
\]
is surjective since the factor \( \text{ind}_P^G \) corresponding to a \( p \)-Sylow subgroup is. Now by Dress’ theorem [8], see [21, theorem 3.4], \( A \) is a \( \mathcal{P} \)-projective Mackey functor. \( \square \)

Given this result, the theory of resolutions using Amitsur complexes described in [21, section 3] can be applied. In particular, we can take the two diagrams \( D^*, D_* \) consisting of all morphisms of the form

\[
\begin{array}{ccc}
A(Q_1) & \xrightarrow{c_g \circ \text{res}^Q_1} & A(gQ_1g^{-1} \cap Q_2) \\
\downarrow & & \downarrow \\
A(Q_1) & \xrightarrow{\text{ind}_Q^G \circ c_g^{-1}} & A(gQ_1g^{-1} \cap Q_2)
\end{array}
\]

where \( Q_1, Q_2 \leq G \) with \( Q_1, Q_2 \in \mathcal{P} \) and \( g \in G \).

Then from proposition (3.6) and corollary (3.7) of [21], we obtain

**Proposition 3.2** (Stable elements formulae). For a finite group \( G \), \( A(G) \) can be computed from each of the formulae

\[
\lim_{D^*} A = A(G) = \text{colim}_{D_*} A.
\]

Since \( p \)-Sylow subgroups are cofinal in each case, we can take \( Q_1, Q_2 \) to be \( p \)-Sylow subgroups, then as they are all conjugate, we can consider the equivalent diagrams of morphisms under or over one particular \( p \)-Sylow subgroup, say \( P \leq G \). This recovers the formulae

\[
\begin{align*}
A(G) &= \bigcap_{g \in G} \ker \left( \text{res}_P^g \circ c_g^{-1} : A(P) \rightarrow A(gPg^{-1} \cap P) \right), \\
A(G) &= A(P)/ \bigcup_{g \in G} \text{im} \left( \text{ind}_P^g \circ c_g^{-1} : A(gPg^{-1} \cap P) \rightarrow A(P) \right).
\end{align*}
\]

When \( N \trianglelefteq G \), for each \( g \in G \), \( c_g \) restricts to an automorphism of \( A(N) \), and if \( g \in N \) this is the identity. Hence \( G/N \) acts on \( A(N) \) and we can form invariants \( A(N)^{GN} \) and coinvariants \( A(N)_{GN} \). Now we have
Proposition 3.3. Suppose that $G$ has the unique normal $p$-Sylow subgroup $P \triangleleft G$. Then

$$A(G) = A(P)^{G/P} = A(P)_{G/P}.$$  

Furthermore,

$$\text{res}_G^P(\text{ind}_G^P(1)) = |G : P| \text{ind}_P^G(1) \neq 0.$$  

Of course this can also be proved more directly by using Proposition 3.1 with the coproduct over all the $p$-Sylow subgroups of $G$, or even with just one of them.

4. Local Frobenius Green functors

We now assume that our local Artinian Green functor $A$ satisfies assumptions (A) and (B), and that char $k = p > 0$. We also require

Assumption (QF). For each finite group $G$, $A(G)$ is Frobenius, i.e., it is a finite dimensional $k$-algebra and there is an isomorphism of left $A(G)$-modules

$$A(G) \cong A(G)^* = \text{Hom}_k(A(G), k).$$

A choice of such an isomorphism determines a Frobenius form $\lambda \in A(G)^*$ which is the element corresponding to 1. We then refer to the pair $(A(G), \lambda)$ as a Frobenius algebra (structure) on $A(G)$. Such a Frobenius form is characterized by the requirement that $\ker \lambda$ contains no non-trivial left (or equivalently right) ideals. This Frobenius condition also implies the Gorenstein condition since $\dim_k \text{soc} A(G) = 1$; this follows from the self-duality of $A(G)$ and the resulting isomorphisms

$$\text{Hom}_{A(G)}(k, A(G)) \cong \text{Hom}_{A(G)}(A(G), k) \cong \text{Hom}_k(k, k) = k.$$  

Our assumptions (B) and (C) together imply that a linear form $\lambda \in A(G)^*$ is a Frobenius form if and only if $\lambda(\text{ind}_G^P(1)) \neq 0$ (since every non-zero left ideal intersects the socle $\text{soc} A(G)$ non-trivially). We do not require that the choice of Frobenius form should be contravariantly functorial with respect to $G$. However, the element $\text{ind}_G^P(1)$ is covariantly functorial since if $G \leq H$ then

$$\text{ind}_G^H(1) = \text{ind}_G^H \text{ind}_G^P(1).$$  

In general, if $\alpha : G \rightarrow H$ is a homomorphism, the restriction $\alpha^* : A(H) \rightarrow A(G)$ need not send $\text{soc} A(H)$ into $\text{soc} A(G)$, nor need it be non-zero on it. However, if $\alpha_* : A(G) \rightarrow A(H)$ is defined then it restricts to an isomorphism

$$\alpha_* : \text{soc} A(G) \cong \text{soc} A(H),$$  

since $\alpha_*(\text{ind}_G^P(1)) = \text{ind}_H^H(1) \neq 0$.

Lemma 4.1. Suppose that $\lambda$ is a Frobenius form for $A(H)$. Then $\alpha^* \lambda = \lambda \circ \alpha_*$ is a Frobenius form for $A(G)$.

Proof. This works very generally. Suppose that $A$ and $B$ are local Frobenius algebras over a field $k$, and that $f : A \rightarrow B$ is a local algebra homomorphism, making $B$ into a left $A$-module by $a \cdot b = f(a)b$. Suppose that $f_* : B \rightarrow A$ is an $A$-module homomorphism for which $f_* \text{soc} B = \text{soc} A$. Then for any Frobenius form $\lambda$ on $A$, $f^* \lambda = \lambda \circ f_*$ is a Frobenius form on $B$. This is easy to see since

$$f^* \lambda \text{soc} B = \lambda \circ f_* \text{soc} B = \lambda \text{soc} A \neq 0,$$
and as every non-trivial ideal intersects soc $B$ non-trivially, the Frobenius condition holds for the linear form $f^* \lambda$.

In the situation of the proof, we may define an inner product on $A$ by $(x \mid y)_A = \lambda(xy)$; similarly, define an inner product on $B$ by $(x' \mid y')_B = f^* \lambda(x'y')$. It follows that for $a \in A$ and $b \in B$,

$$(f(a) \mid b)_B = \lambda \circ f^*(f(a)b) = \lambda(a f_*(b)) = (a \mid f_*(b))_A.$$ 

This is a version of Frobenius reciprocity.

5. Green functors on abelian groups and $p$-divisible groups

For general notions of groups schemes see [22]; for $p$-divisible groups see [7]. A general reference on formal schemes and formal groups is Strickland [18].

Let $k$ be a field of characteristic $\text{char } k = p > 0$. Recall that a finite dimensional commutative $k$-Hopf algebra $H$ represents a group scheme $\text{Spec}(H)$ is a group valued functor on the category of commutative $k$-algebras $\mathcal{C}A_k$ defined by

$$\text{Spec}(H)(A) = \mathcal{C}A_k(H, A).$$

Denoting the coproduct by $\psi: H \rightarrow H \otimes H$ and the product on $A$ by $\varphi: A \otimes A \rightarrow A$, the group structure is defined by

$$f \ast g = \varphi(f \otimes g) \psi$$

where $f, g \in \mathcal{C}A_k(H, A)$. The unit is given by the counit $\varepsilon \in \mathcal{C}A_k(H, k)$ and the inverse is given by an algebra automorphism $\chi \in \mathcal{C}A_k(H, H)$.

We remark that by the Larson-Sweedler theorem, such a Hopf algebra is a Frobenius algebra.

Recall that $H$ is connected if it has no non-trivial idempotents. In particular, this is true if $H$ is local. Of course this means that $\text{Spec}(H)$ is connected.

A commutative $k$-algebra valued Green functor $A$ is a K"unneth functor if it takes products of groups to pushouts of commutative $k$-algebras, i.e., it satisfies the strict K"unneth formula

$$A(G \times H) = A(G) \otimes A(H)$$

for every pair of finite groups $G, H$.

We will impose another condition.

Assumption (KF). The Green functor $A$ is a K"unneth functor.

When $H = G$, the diagonal homomorphism $\Delta: G \rightarrow G \times G$ induces the product on $A(G)$. If $G$ is an abelian group, the multiplication $G \times G \rightarrow G$ is a group homomorphism and so it induces an algebra homomorphism

$$A(G) \rightarrow A(G \times G) = A(G) \otimes A(G)$$

which is coassociative and counital, with antipode induced by the inverse map $G \rightarrow G$. Furthermore this coproduct is cocommutative. This shows that $A(G)$ is naturally a cocommutative Hopf algebra.

Remark 5.1. Recall the theory of integrals for finite dimensional Hopf algebras, as described in [13] definition 2.1.1] for example. For a finite dimensional local Hopf algebra $H$, taking a generator $z \in \text{soc } H$ we have for any $x \in H$,

$$xz = \text{aug}(x)z = zx,$$
hence $z$ is a left and right integral for $H$. Therefore

$$
soc H = \int_H^l = \int_H^r,
$$

so $H$ is unimodular and $soc H = \int_H$. Of course, if $H \neq k$ then $\text{aug}$ is not a Frobenius form.

Now recall that a sequence of finite abelian group schemes $G_r$ ($r \geq 1$) where $G_r$ has order $p^{rh}$ for some natural number $h \geq 1$ over the field $k$ forms a $p$-divisible group or Barsotti-Tate group of height $h$ if there are exact sequences of group schemes fitting into commutative diagrams

$$
G_r \rightarrow G_{r+s} \xleftarrow{i_{r,r+s}} G_{r+1} \xrightarrow{q_{r+s}} G_{r+s+1} \rightarrow G_r \quad \text{for all } r, s \geq 1,
$$

where $G_r \xrightarrow{i_{r,s}} G_{r+s}$ is a kernel for multiplication by $p^r$ on $G_{r+s}$. These are required to be compatible in the sense that there are commutative diagrams of the following forms.

If $G_r = \text{Spec}(H_r)$ for some cocommutative Hopf algebra $H_r$, then $\text{dim}_k H_r = p^{rh}$ and there are morphisms of Hopf algebras

$$
(5.1) \quad H_r \xleftarrow{i_{r,s}} H_{r+s} \xrightarrow{q_{r+s}} H_s
$$

inducing the diagram of group schemes

$$
(5.2) \quad G_r \xrightarrow{i_{r,s}} G_{r+s} \xrightarrow{q_{r,s}} G_s.
$$
We are interested in the case where each $G_r$ is connected, and this follows from the requirement that every $H_r$ is a local $k$-algebra. Then we have the following Borel algebra decomposition of each $H_r$.

**Proposition 5.2.** For each $r \geq 1$, there is an isomorphism of $k$-algebras of the form

$$H_r \cong k[x_1, x_2, \ldots, x_\ell]/(x_1^{q_1}, x_2^{q_2}, \ldots, x_\ell^{q_\ell}),$$

where $q_i = p^{d_i}$ for some $d_i \geq 1$ and $d_1 + d_2 + \cdots + d_\ell = rh$.

Notice that

$$\text{soc } H_r = k\{x_1^{q_1-1}x_2^{q_2-1} \cdots x_\ell^{q_\ell-1}\}.$$

The restriction epimorphisms $H_r \to H_{r-1}$ have a limit

$$H = \lim_r H_r = k[x_1, x_2, \ldots, x_\ell]$$

which has a topological coproduct $\psi: H \to H \hat{\otimes}_k H$ defining a cocommutative formal group over $k$ of dimension $\ell$. Each $H_r$ can be recovered from $H$ by forming the quotient with respect to the ideal generated by the image of the multiplication by $p^r$, expressible as a composition

$$H \xrightarrow{\psi(p)} H \hat{\otimes}_k \cdots \hat{\otimes}_k H \xrightarrow{\psi(p)} H.$$

We now introduce another assumption.

**Assumption (D).** The Hopf algebras $A(C_{p^s})$ with restriction and inflation homomorphisms $\text{res}_{C_{p^s}}$ and $\text{res}_{C_{p^{s+s}}}$ (induced by the canonical quotient homomorphism $C_{p^{s+s}} \to C_{p^s}$) give rise to a $p$-divisible group which satisfies $G_r = \text{Spec } A(C_{p^r})$.

We can immediately deduce a non-triviality result; a version of this for Morava $K$-theory appeared in John Hunton’s PhD thesis.

**Proposition 5.3.** Suppose that $\pi: G \to C_{p^s}$ be an epimorphism for $s \geq 1$. Then the induced homomorphism $\pi^*: A(C_{p^s}) \to A(G)$ is a monomorphism.

**Proof.** Lifting a generator of $C_{p^s}$ to $G$, we obtain a commutative diagram of the form

$$C_{p^{s+s}} \xrightarrow{\text{quo}} G \xrightarrow{\pi} C_{p^s}$$

and on applying $A(-)$ this gives

$$A(C_{p^{s+s}}) \xrightarrow{\text{quo}^*} A(G) \xrightarrow{\pi^*} A(C_{p^s})$$

where we know from (5.1) that $\text{quo}^* = \text{res}_{C_{p^s}}$ is monic and so $\pi^*$ is also monic. □

Of course, if $G$ is a non-trivial $p$-group, such epimorphisms always exist.

**Corollary 5.4.** If $G$ is a non-trivial $p$-group, then $A(G)$ is non-trivial, $A(G) \neq k$. In particular, $0 \neq \text{ind}_1^G(1) \notin k$. 13
Proof. For the statement about \( \text{ind}^G_1(1) \), recall that \( 0 \neq \text{ind}^G_1(1) \in \text{soc} A(G) \) and since \( A(G) \neq \mathbb{k} \) we must have \( \text{soc} A(G) \neq \mathbb{k} \). \( \square \)

**Theorem 5.5.** Let \( G \) be a finite group. Then \( A(G) = \mathbb{k} \) if and only if \( p \mid |G| \).

**Proof.** We must show that if \( G \) is a non-trivial finite group whose order is divisible by \( p \), then \( A(G) \neq \mathbb{k} \). We know this holds for \( p \)-groups, so suppose that \( G \) is not a \( p \)-group. Let \( P \subseteq G \) be a \( p \)-Sylow subgroup. Then the Mackey formula gives

\[
\text{res}^G_P \text{ind}^G_1(1) = \sum_{g \in P \setminus G/1} \text{ind}^P_1 c_g \text{res}^1(1) = \sum_{g \in P \setminus G/1} \text{ind}^P_1(1) = |G : P| \text{ind}^P_1(1).
\]

By Corollary 5.4, \( |G : P| \text{ind}^P_1(1) \notin \mathbb{k} \), hence \( \text{ind}^G_1(1) \notin \mathbb{k} \). This shows that \( A(G) \neq \mathbb{k} \). \( \square \)

If we restrict attention to finite abelian \( p \)-groups, then the following result holds, see [9, proposition 2.4] for the Morava \( K \)-theory version.

**Theorem 5.6.** Suppose that \( G, H, K \) are finite abelian \( p \)-groups.

(a) If \( \varphi \colon G \to H \) is an epimorphism, then \( \varphi^* \colon A(H) \to A(G) \) is monic.

(b) If \( \theta \colon K \to G \) is a monomorphism, then \( \theta^* \colon A(G) \to A(K) \) is epic.

In each case, the converse also holds.

**Proof.** As a starting point, we recall that Assumption (D) implies this for the canonical epi-morphisms \( C_{p^{r+s}} \to C_{p^r} \) and monomorphisms \( C_{p^r} \to C_{p^{r+s}} \).

(a) Since \( H \) is isomorphic to a product of cyclic groups,

\[
H \cong C_{p^{r_1}} \times \cdots \times C_{p^{r_k}}
\]

we can choose lifts of the generators to elements of \( G \) and then define a homomorphism

\[
C_{p^{r_1+s_1}} \times \cdots \times C_{p^{r_k+s_k}} \to G
\]

and a factorisation of the canonical quotient

\[
\begin{array}{ccc}
C_{p^{r_1+s_1}} \times \cdots \times C_{p^{r_k+s_k}} & \longrightarrow & G \\
\downarrow \cong & & \downarrow \varphi \\
C_{p^{r_1}} \times \cdots \times C_{p^{r_k}} & \cong & H \\
\end{array}
\]

to which we can apply \( A \). In the resulting diagram

\[
\begin{array}{ccc}
A(C_{p^{r_1+s_1}}) \otimes \cdots \otimes A(C_{p^{r_k+s_k}}) & \longrightarrow & A(G) \\
\downarrow \cong & & \downarrow \varphi^* \\
A(C_{p^{r_1}}) \otimes \cdots \otimes A(C_{p^{r_k}}) & \cong & A(H)
\end{array}
\]
we see that $\varphi^*$ must be monic.

(b) Taking the Pontrjagin dual of $\theta$ we obtain an exact sequence

$$0 \leftarrow \text{Hom}(K, C_{p^\infty}) \leftarrow \theta^* \text{Hom}(G, C_{p^\infty}),$$

where $C_{p^\infty} = \colim_r C_{p^r} \subseteq S^1$, which is an injective $\mathbb{Z}$-module. Since $K$ is a product of cyclic groups,

$$K \cong C_{p^1} \times \cdots \times C_{p^\ell},$$

and each projection to a factor $C_{p^i}$ gives a homomorphism

$$\lambda_i : K \to C_{p^1} \times \cdots \times C_{p^\ell} \to C_{p^i} \to C_{p^\infty} \to S^1$$

we obtain algebra generators for

$$A(C_{p^1} \times \cdots \times C_{p^\ell}) \cong A(C_{p^1}) \otimes \cdots \otimes A(C_{p^\ell})$$

by applying $\lambda_i^*$ to the generators of $H$ given in (5.3). Since each $\lambda_i$ factors through $G$, this shows that the algebra generators of $A(K)$ are all in the image of $\theta^*$, therefore $\theta^*$ is epic.

The converse statements are easily verified. \qed

The next result follows easily using standard facts about commutative groups schemes.

**Corollary 5.7.** Suppose that

$$1 \to G' \xrightarrow{f} G \xrightarrow{g} G'' \to 1$$

is a short exact sequence of finite abelian $p$-groups. Then the induced homomorphisms of Hopf algebras

$$A(G'') \xrightarrow{g^*} A(G) \xrightarrow{f^*} A(G')$$

induce a short exact sequence of commutative groups schemes

$$1 \to \text{Spec}(A(G')) \to \text{Spec}(A(G)) \to \text{Spec}(A(G'')) \to 1.$$

6. **Examples: Honda formal groups and $p$-divisible groups**

For each $n \geq 1$, there is a $p$-divisible group of height $n$, where

$$H_r = \mathbb{F}[x_r]/(x_r^{q^r}),$$

where $q$ is a power of $p$ or $q = \mathbb{F}_p$, $\mathbb{F}_q$ or $\mathbb{F}_{p\infty} = \mathbb{F}_p$. Here the coproduct on $H_1$ has the form

$$\psi(x_1) = x_1 \otimes 1 + 1 \otimes x_1 - \sum_{1 \leq i \leq p-1} \frac{1}{p} \binom{p}{i} x_1^{ip^{n-1}} \otimes x_1^{(p-i)p^{n-1}}.$$

Furthermore, the natural homomorphisms

$$H_r \leftarrow H_{r+s} \leftarrow H_s$$

are given by

$$H_{r+s} \to H_r: \quad x_{r+s} \mapsto x_r$$

and

$$H_s \to H_{r+s} \quad x_s \mapsto x_{r+s}^{q^r}.$$

We also have

$$\text{soc } H_r = \mathbb{F}\{x_r^{q^r-1}\}.$$
Passing to the limit we obtain

\[ H = \lim_{r} H_r = \mathbb{F}[x], \]

and the formal group is known as the Honda formal group.

This example is closely connected with the \( n \)-th Morava \( K \)-theory \( K^*_n(\mathbb{Q}) \). More precisely, the 2-periodic version has

\[ K^*_n = \mathbb{F}_{p^n}[u, u^{-1}] \]

which is a graded field with \( u \in K^{-2}_n \), and

\[ H_r = K^0_n(BC_{p^r}). \]

Since \( C_{p^r} \) is an abelian group, \( BC_{p^r} \) is a commutative \( H \)-space and so \( K^0_n(BC_{p^r}) \) is a cocommutative graded Hopf algebra over \( \mathbb{F}_{p^n} \). For a suitable choice of compatible generators \( x_r \), this agrees with the above algebraic example. This example is also special because the dimension of the \( p \)-divisible group is 1; in general the dimension satisfies \( 1 \leq \dim G \leq h \). Because of the way Morava \( K \)-theory and other topological examples arise, they always give 1-dimensional \( p \)-divisible groups.

**Appendix A. Some recollections on Frobenius algebras**

We will use the phrase Frobenius algebra to indicate that an algebra \( A \) has at least one Frobenius structure, \( (A, \Phi) \), where \( \Phi: A \xrightarrow{\cong} A^* \) is a left \( A \)-module isomorphism. It might be better to use Nakayama’s terminology Frobeniusean of [14], but as remarked by Lam [12, comments on page 453], this has fallen out of fashion. For our purposes it is useful to allow flexibility over the choice of Frobenius structure on such an algebra. Our usage differs from that of Koch [11] who requires the Frobenius structure as well as the underlying algebra.

Throughout we assume that \( k \) is a field and set

\[ \dim = \dim_k, \quad \otimes = \otimes_k, \quad \text{Hom} = \text{Hom}_k. \]

We assume that \( A \) is a finite dimensional local \( k \)-algebra: here local means that \( A \) has a unique maximal left, or equivalently right, ideal which agrees with the Jacobson radical \( \text{rad} A \), and the quotient \( A/\text{rad} A \) is the unique simple \( A \)-module, and we will also assume that \( A \) is augmented over \( k \), therefore \( A/\text{rad} A \cong k \) is the unique simple \( A \)-module. We will use some basic facts about the radical, in particular it is nilpotent, say \( (\text{rad} A)^e = 0 \) and \( (\text{rad} A)^{e-1} \neq 0 \). The socle of \( A \) is the right annihilator of \( \text{rad} A \),

\[ \text{soc} A = \{ z \in A : (\text{rad} A)z = 0 \} \supseteq (\text{rad} A)^{e-1}, \]

which is known to be the sum of all the simple left \( A \)-submodules of \( A \). In fact, by choosing a minimal set of simple submodules \( V_i \) with \( \text{soc} A = V_1 + \cdots + V_{\ell} \), we find that

\[ \text{soc} A = V_1 \oplus \cdots \oplus V_{\ell}. \]

Of course, for all \( i \) there is an isomorphism of \( A \)-modules \( V_i \cong k \).

Let \( A^* = \text{Hom}_k(A, k) \) be the \( k \)-linear dual of \( A \). Then \( A^* \) is a left \( A \)-module with scalar multiplication \( \cdot \) given by

\[ a \cdot f(x) = f(xa) \quad (a, x \in A, \ f \in A^*), \]

and it is also a right \( A \)-module with scalar multiplication

\[ (f \cdot a)(x) = f(ax) \quad (a, x \in A, \ f \in A^*). \]
In either case \( A^* \) is an injective \( A \)-module.

We recall various aspects of a Frobenius algebra structure on \( A \) and their properties; details can be found in [11,12]. In particular, we will follow recent tradition in using cobordism diagrams to express relationships between the various structure morphisms associated with a Frobenius algebra structure.

For \( \Phi \in \text{Hom}_A(A, A^*) \cong \text{Hom}_k(k, A^*) \cong A^* \), the pair \( (A, \Phi) \) is a Frobenius algebra over \( k \) if \( \Phi \) is an isomorphism. Since \( A \cong A^* \), \( A \) is self-injective. It is immediate that if \( (A, \Theta) \) is also a Frobenius algebra then there is a unit \( u \in A^\times \) such that \( \Theta(-) = \Phi(-u^{-1}) \), i.e., for all \( a \in A \),

\[
\Theta(a) = \Phi(au^{-1}).
\]

This shows that

**Proposition A.1.** Let \( (A, \Phi) \) be a Frobenius algebra. Then the set of all Frobenius algebras \( (A, \Theta) \) is 1-1 correspondence with the set of units \( A^\times \).

Given a Frobenius structure \( \Phi \) on \( A \), the linear form \( \varepsilon = \Phi(1) \in A^* \) has the following property:

- \( \ker \varepsilon \) contains no non-trivial left ideals.

Notice that \( \varepsilon \) can never be a \( k \)-algebra homomorphism if \( \text{dim} \, A > 1 \).

We call \( \varepsilon \) the counit of the Frobenius algebra \( (A, \Phi) \) and sometimes indicate it diagrammatically by \( \subseteq \).

There is an associated \( k \)-bilinear form \( \langle -| - \rangle : A \otimes A \to k \) given by

\[
\langle x|y \rangle = \varepsilon(xy)
\]

for \( x, y \in A \). This is called the Frobenius pairing and is denoted \( \subseteq \). It satisfies the Frobenius associativity condition:

- for \( x, y, z \in A \), \( \langle x|yz \rangle = \langle xy|z \rangle \).

The linear mapping

\[
\lambda : A \to A^*; \quad a \mapsto \langle -|a \rangle
\]

is non-degenerate since \( Aa \subseteq \ker \varepsilon \). By finite dimensionality, the linear mapping

\[
\rho : A \to A^*; \quad a \mapsto \langle a|-\rangle
\]

is also non-degenerate, and these two linear mappings give \( k \)-linear isomorphisms \( A \cong A^* \).

The first of these is actually a left \( A \)-module isomorphism \( \lambda : A \cong A^* \), while the second is a right \( A \)-module isomorphism. Of course we can recover \( \varepsilon \) from \( \langle -| - \rangle \) by using the functional identities

\[
\varepsilon(-) = \langle -|1 \rangle = \langle 1|-\rangle.
\]

Denoting the unit \( k \to A \) by \( \subseteq \), these amount to the identities of the following cobordism diagram.

\[
\subseteq = \subseteq = \subseteq
\]

The three structures \( \Phi, \varphi, \langle -| - \rangle \) with the above properties give equivalent information and any one determines the others, see [11, section 2.2].

**Proposition A.2.** If \( (A, \Phi) \) is a local Frobenius algebra, then \( \text{dim} \, \text{soc} \, A = 1 \) and so there is an isomorphism of \( A \)-modules \( \text{soc} \, A \cong k \). In particular, \( \text{soc} \, A \) is a simple \( A \)-module.
Proof. This amounts to verifying the degree zero Gorenstein condition. Here we have a unique maximal ideal \( m \triangleleft A \) with \( A/m \cong k \) which gives \( k \) a unique \( A \)-module structure. Since \( A \cong A^* \) is self-injective,

\[
\text{Hom}_A(k, \text{soc} A) = \text{Hom}_A(k, A) \cong \text{Hom}_A(k, A^*) = \text{Hom}_A(k, \text{Hom}_k(A, k)) \\
\cong \text{Hom}_k(A \otimes_A k, k) \\
\cong \text{Hom}_k(k, k) \cong k.
\]

Therefore \( \dim \text{soc} A = 1. \) \( \square \)

Let \( B \) be a \( k \)-algebra. Recall that there is a sequence of isomorphisms

\[
(A.1) \quad B^* = \text{Hom}_k(B, k) \cong \text{Hom}_k(B \otimes B, k) \cong \text{Hom}_B(B, B^*),
\]

where we use left \( B \)-module structures except for in the tensor product \( B \otimes B \) where we use the right module structure on the first factor and the left module structure on the second factor. Under this composition, \( \theta \in A^* \) corresponds to \( \Theta \in \text{Hom}_A(A, A^*) \) characterized by \( \Theta(1) = \theta. \)

Lemma A.3. Suppose that \( B \) is a finite dimensional \( k \)-algebra for which \( \dim \text{soc} B = 1. \) Suppose that \( \theta \in B^* \) corresponds to \( \Theta \in \text{Hom}_B(B, B^*) \) under the composition of the isomorphisms of \( (A.1) \). Then \( (B, \Theta) \) is a Frobenius algebra if and only if \( \theta \) is non-trivial on \( \text{soc} B \).

Proof. Suppose that \( (B, \Theta) \) is a Frobenius algebra and note that \( \text{soc} B \cong k. \) Then \( \theta \in B^* \) is the corresponding Frobenius form. Since \( \text{soc} B \) is a non-trivial left ideal in \( B, \) \( \text{soc} B \not\subset \ker \theta, \) therefore \( \theta \) must be non-trivial on \( \text{soc} B. \)

Now suppose that \( \theta \in B^* \) is non-trivial on \( \text{soc} B. \) If \( I \) is a non-trivial left ideal in \( B, \) then for some \( r \geq 1, (\text{rad } B)^r I = 0 \) and \( (\text{rad } B)^{r-1} I \neq 0. \) But then \( 0 \neq (\text{rad } B)^{r-1} I \subseteq \text{soc } B, \) and so \( \text{soc } B = (\text{rad } B)^{r-1} I \subseteq I \) since \( \dim \text{soc } B = 1. \) Hence \( \theta \) is not zero on \( I. \) This shows that \( \theta \) must be non-trivial on every non-trivial left ideal and it follows that \( (B, \Theta) \) is a Frobenius algebra. \( \square \)

Lemma A.4. Suppose that \( (A, \Theta) \) is a Frobenius algebra with Frobenius form \( \theta \in A^*. \) Let \( u_0 \in \text{soc } A \) and \( \theta(u_0) = 1, \) and let \( u_0, u_1, \ldots, u_{d-1} \) be a \( k \)-basis for \( A. \) Then given any sequence \( t_0, t_1, \ldots, t_{d-1} \in k \) with \( t_0 \neq 0, \) there is a Frobenius algebra \( (A, \Theta') \) whose Frobenius form \( \Theta' \) satisfies \( \Theta'(u_i) = t_i \) for every \( i. \)

Proof. By Lemma A.3 we have \( \theta(u_0) \neq 0, \) so for simplicity we will assume that \( \theta(u_0) = 1. \) Consider the dual basis with respect to the associated bilinear form \( \langle -| - \rangle, \) say \( v_0, v_1, \ldots, v_{d-1} \) where

\[
\langle u_i | v_j \rangle = \delta_{i,j}.
\]

In fact, if \( z \in \text{rad } A \) then \( \langle z| u_0 \rangle = \theta(zu_0) = 0; \) hence we can assume that \( v_0 = 1. \) Furthermore we then have \( v_i \in \text{rad } A \) for \( i \geq 1. \) Now define

\[
\theta' = (t_0 + \sum_{1 \leq i \leq d-1} t_i v_i) \cdot \theta \in A^*.
\]
Then for each $r$, 
\[
\theta'(u_t) = \theta(t_0 u_r + \sum_{1 \leq i \leq d-1} t_i u_r v_i) \\
= \langle u_r | t_0 \rangle + \sum_{1 \leq i \leq d-1} t_i v_i \\
= t_r.
\]
Then $\theta' \in A^*$ corresponds to $\Theta' \in \text{Hom}_A(A^*, A^*)$ where $(A, \Theta')$ is a Frobenius algebra. \qed

We mention a general algebraic result which appears in a special case in the topological context of Morava $K$-theory. Let $A$ and $B$ be two local Frobenius algebras over the field $k$. Then as $k$-vector spaces, 
\[
soc A \cong k \cong soc B.
\]
Suppose that $\theta: A \rightarrow B$ is a local algebra homomorphism. Then $B$ becomes a left $A$-module and the above isomorphisms can be taken to be isomorphisms of $A$-modules.

**Lemma A.5.** Given any isomorphism of $A$-modules $\alpha: soc B \rightarrow soc A$, there are extensions to $A$-module homomorphisms $\alpha': B \rightarrow A$.

\[
\begin{array}{c}
soc B \\
inc \\
B
\end{array} \xymatrix{ & soc A \ar[ld]_{\alpha} \ar[dd] \ar[rd]^{\alpha'} & \\
& B \ar[ld]_{inc} & \\
& A}
\]

**Proof.** Since $A$ is self-injective, the composition $\text{inc} \circ \alpha: soc B \rightarrow A$ has such extensions to $A$-module homomorphisms $B \rightarrow A$. \qed

We also note the following.

**Lemma A.6.** Suppose that the algebra homomorphism $\theta: A \rightarrow B$ is non-trivial on $soc A$. Then there is an element $b \in B$ for which

\[
b\theta soc A = soc B.
\]

**Proof.** Clearly $\theta soc A \cong k$, and $\{0\} \neq B\theta soc A \cap B$. As $(B\theta soc A) \cap soc B \neq \{0\}$, we must have $soc B \subseteq B\theta soc A$. Choosing any $b \in B$ for which $b\theta soc A \neq \{0\}$ we easily see that $b\theta soc A = soc B$ since $\dim soc B = 1$. \qed

**Proposition A.7.** Let $\alpha: B \rightarrow A$ be an $A$-module homomorphism for which $\alpha soc B = soc A$, and let $\lambda$ be a Frobenius form on $A$. Then $\lambda \circ \alpha$ is a Frobenius form on $B$.

**Proof.** Since

\[
\lambda \circ \alpha soc B = \lambda soc A \neq 0,
\]
this is immediate. \qed

**Proposition A.8.** Suppose that $\lambda_A$ and $\lambda_B$ are Frobenius forms for $A$ and $B$ respectively. Then there is an $A$-module homomorphism $\alpha: B \rightarrow A$ for which $\alpha soc B = soc A$. 

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Proof. The Frobenius forms give rise to an isomorphism of $A$-modules
\[ A^* \xrightarrow{\cong} A; \quad \lambda_A \mapsto 1, \]
and an isomorphism of $B$-modules (and hence of $A$-modules)
\[ B \xrightarrow{\cong} B^*; \quad 1 \mapsto \lambda_B. \]
Composing with $\theta$ we obtain a homomorphism of $A$-modules $\alpha$,
\[ B \xrightarrow{\cong} B^* \xrightarrow{\theta^*} A^* \xrightarrow{\cong} A, \]
where $\theta^*(f) = f \circ \theta$. Let $\alpha': B \longrightarrow A^*$ be the intermediate composition, given by
\[ \alpha'(b)(a) = \lambda_B(\theta(a)b) \]
for $b \in B$ and $a \in A$. In particular, if $z \in \text{soc} B$ and $w \in \text{rad} A$, then by definition of $w\alpha'(z)$ we have
\[ (w\alpha'(z))(a) = \lambda_B(\theta(aw)z) = \lambda_B(\theta(a)\theta(w)z) = 0, \]
since $\theta$ is local and therefore $\theta(w)z = 0$. It follows that $w\alpha(z) = 0$ for all $w \in \text{rad} A$ and so $\alpha \text{soc} B \subseteq \text{soc} A$. If $z \neq 0$,
\[ \alpha'(z)(1) = \lambda_B(\theta(1)z) = \lambda_B(z) \neq 0 \]
since $\dim \text{soc} B = 1$ and $\lambda_B$ is non-trivial on $\text{soc} B$. Hence we have $\alpha \text{soc} B = \text{soc} A$. □

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