A HOMOLOGICAL CHARACTERIZATION OF GENERALIZED
MULTINOMIAL COEFFICIENTS RELATED TO THE ENTROPIC
CHAIN RULE

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Abstract. There is an asymptotic relationship between the multiplicative relations among multinomial coefficients and the (additive) recurrence property of Shannon entropy known as the chain rule. We show that both types of identities are manifestations of a unique algebraic construction: a 1-cocycle condition in information cohomology, an algebraic invariant of phesheaves of modules on information structures (categories of observables). Baudot and Bennequin introduced this cohomology and proved that Shannon entropy represents the only nontrivial cohomology class in degree 1 when the coefficients are a natural presheaf of probabilistic functionals. The author obtained later a 1-parameter family of deformations of that presheaf, in such a way that each Tsallis $\alpha$-entropy appears as the unique 1-cocycle associated to the parameter $\alpha$. In this article, we introduce a new presheaf of combinatorial functionals, which are measurable functions of finite arrays of integers; these arrays represent histograms associated to random experiments. In this case, the only cohomology class in degree 0 is generated by the exponential function and 1-cocycles are Fontené-Ward generalized multinomial coefficients. As a byproduct, we get a simple combinatorial analogue of the fundamental equation of information theory that characterizes the generalized binomial coefficients. The asymptotic relationship mentioned above is extended to a correspondence between certain generalized multinomial coefficients and any $\alpha$-entropy, that sheds new light on the meaning of the chain rule and its deformations.

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1. Motivations and main results

It is well known that the multinomial coefficients\footnote{For integers $n, k_1, \ldots, k_s$ such that $\sum_{i=1}^{s} k_i = n$, one has
\[
\binom{n}{k_1, \ldots, k_s} := \frac{n!}{k_1! \cdots k_s!}.
\] This expression counts words $w \in \{a_1, \ldots, a_s\}^n$ where the symbol $a_i$ appears $k_i$ times, for each $i$.} are asymptotically related to Shannon entropy: if $(p_1, \ldots, p_s)$ is a probability vector and $n \in \mathbb{N}$,
\[
\binom{n}{p_1n, \ldots, p_sn} := \frac{\Gamma(n+1)}{\Gamma(p_1n+1) \cdots \Gamma(p_sn+1)} = \exp(nS_1(p_1, \ldots, p_s) + o(n))
\]
where $S_1$, or more precisely $S_1^{(n)}$, denotes Shannon entropy in nats,
\[
S_1(p_1, \ldots, p_s) := -\sum_{i=1}^{s} p_i \ln p_i.
\]
The fact that the multiplicative relations between these coefficients translate asymptotically into the entropic chain rule is however never mentioned. For instance, from the combinatorial identity
\[
\binom{n}{p_1n, p_2n, p_3n} = \binom{n}{(p_1+p_2)n, p_3n} \binom{n}{p_1n, p_2n},
\]
one can deduce—taking the logarithm of both sides, normalizing by $n$ and then letting $n \to \infty$—that
\[
S_1(p_1, p_2, p_3) = S_1(p_1 + p_2, p_3) + (p_1 + p_2)S_1 \left( \frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \right).
\]
Both equalities are induced by a grouping of the arguments (“coarse graining”), which can be represented as a surjection from $\{1, 2, 3\}$ to $\{1, 2\}$ that maps 1 to 1, 2 to 1, and 3 to 2.

The additive relations exemplified by $S_1$, known in information theory as the chain rule, serve as a fundamental property to algebraically characterize the entropy. Let us denote by $\Delta^n$ the standard simplex $\{(x_0, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=0}^{n} x_i = 1\}$, and $[n]$ the set $\{1, \ldots, n\}$. Shannon\footnote{This expression counts words $w \in \{a_1, \ldots, a_s\}^n$ where the symbol $a_i$ appears $k_i$ times, for each $i$.} proved that $\{S_1^{(n+1)} : \Delta^n \to \mathbb{R}\}_{n \in \mathbb{N}}$ are the only continuous functions (up to a multiplicative constant) that vanish on the vertexes of each simplex $\Delta^n$, make $S_1^{(n)}(1/n, \ldots, 1/n)$ monotonic in $n$, and satisfy the chain rule induced by any surjection $[n] \to [m]$. It is natural to ask if the multinomial coefficients can be algebraically characterized in an analogous way.

The equality $S_1$, together with the symmetry of the entropy, imply that
\[
s_1(x) := S_1^{(2)}(x, 1-x) = -x \ln x - (1-x) \ln(1-x)
\]
is a solution of the so-called fundamental equation of information theory (FEITH):
\[
\forall x, y \in [0, 1) \text{ such that } x + y \leq 1,
\]
\[
u(x) + (1-x)u \left( \frac{y}{1-x} \right) = u(y) + (1-y)u \left( \frac{x}{1-y} \right).
\]
This functional equation was first introduced by Tverberg\footnote{This expression counts words $w \in \{a_1, \ldots, a_s\}^n$ where the symbol $a_i$ appears $k_i$ times, for each $i$.}, who proved that every integrable and symmetric solution of it is a multiple of $s_1(x)$. The regularity condition can be weakened to mere measurability. Tverberg’s result

\[
\]
gives an alternative algebraic characterization of Shannon entropy. Furthermore, the fundamental equation is also relevant in other areas of mathematics: it appears in Cathelineau’s computations of the degree-one homology of $SL_2$ over a field of characteristic zero with coefficients in the adjoint action [3], as well as in subsequent work by Elbaz-Vincent and Gangl [4,5] and Bloch and Esnault [2] connected to polylogarithms and motives. Kontsevich [11] used a version of the FEITH to introduce the entropy modulo $p$ and also gave a cohomological interpretation of this functional equation.

The algebraic characterizations of entropy already mentioned accept a 1-parameter family of deformations. For any $\alpha > 0$, define $S_\alpha : \Delta^{s-1} \to \mathbb{R}$ by the formula

$$S_\alpha(p_1, ..., p_s) := \frac{1}{1 - \alpha} \sum_{i=1}^{s} p_i^{\alpha} - 1,$$

in such a way that $S_\alpha \to S_1$ when $\alpha \to 1$. This function was first introduced by Havrda-Charvát [9] as structural $\alpha$-entropy and nowadays it is mostly known as Tsallis $\alpha$-entropy. It satisfies a deformed chain rule where the weights in front of each term are raised to the power $\alpha$, e.g.

$$S_\alpha(p_1, p_2, p_3) = S_\alpha(p_1 + p_2, p_3) + (p_1 + p_2)^\alpha S_\alpha\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right).$$

In [9], this property plays a fundamental role in an algebraic characterization of $S_\alpha$ (up to a multiplicative constant) analogue to Shannon’s characterization of $S_1$. Along the same line, Daróczy introduced a generalized FEITH,

$$\forall x, y \in [0, 1) \text{ such that } x + y \leq 1,$$

$$u(x) + (1 - x)^\alpha u\left(\frac{y}{1 - x}\right) = u(y) + (1 - y)^\alpha u\left(\frac{x}{1 - y}\right),$$

with boundary condition $u(0) = u(1)$, and proved that its only solutions are multiples of

$$s_\alpha(x) = \frac{1}{1 - \alpha}(x^\alpha + (1 - x)^\alpha - 1),$$

without any hypothesis on the regularity of $u$.

Up to this point, there is no general combinatorial counterpart to the entropies $S_\alpha$ and their chain rule; the latter could be judged as a purely formal rule, without further implications. However, we showed in a previous article [15] that the Gaussian $q$-multinomial coefficients are asymptotically related to the 2-entropy, giving a concrete combinatorial meaning to the deformed chain rule (for a precise statement, see the examples after Proposition 7). Similar results may hold for other combinatorial quantities and other values of $\alpha$.

At the algebraic level, there is more than an “analogy” between the multiplicative relations among multinomial coefficients and the entropic chain rule: we establish in this article that both are particular cases of a general construction called information cohomology. This theory was first introduced by Baudot and Bennequin in [1] and further developed by the author in [16]. These works prove that each entropy $S_\alpha$, for $\alpha > 0$, is the unique 1-cocycle in information cohomology with coefficients in certain module of probabilistic functionals $\mathcal{F}_\alpha$; the 1-cocycle condition corresponds in this case to the chain rule—exemplified by (7)—for certain restricted family of surjections encoded by an information structure (a categori
object defined from a given collection of random variables). This result does not require assumptions like the symmetry of $S_\alpha$ under permutations or its asymptotic behavior. The construction is summarized in Section 2.

In Section 3 we introduce a new module of coefficients $\mathcal{G}$ made of “combinatorial” functionals, and show that 1-cocycles are in this case Fontené-Ward generalized multinomial coefficients: given any sequence $D = \{D_i\}_{i \geq 1}$ such that $D_1 = 1$, these coefficients are defined for any integers $k_1, \ldots, k_s \in \mathbb{N}$ by

$$\{ n \}_{D}^{k_1, \ldots, k_s} := \frac{[n]_D!}{[k_1]_D! \cdots [k_s]_D!},$$

where $[n]_D! := D_n D_{n-1} \cdots D_1$, $[0]_D! := 1$, and $n = \sum_{i=1}^s k_i$. Again, the 1-cocycle condition implies all the multiplicative relations akin to (3) for a given family of surjections encoded by the information structure.

The generalized binomial coefficients were first introduced by Fontené in 1915 [6], and later rediscovered by Ward [17], who developed a “calculus of sequences” analogue to the quantum calculus introduced by Jackson. The multinomial case was already treated by Gould [7]. To our knowledge, three particular cases appear in the literature under their own name:

(i) $D_n = n$ gives the usual multinomial coefficients.

(ii) $D_n = (q^n - 1)/(q - 1)$ gives the Gaussian $q$-multinomial coefficients, usually denoted $\left[\begin{array}{c} n \\hfill \vdots \hfill \vdots \hfill \vdots \hfill \end{array}\right]_q$. See [15].

(iii) When $D$ is the Fibonacci sequence and $s = 2$, the expressions (9) are called Fibonacci coefficients.

The functions $f_D(\nu_1, \nu_2) = \{\nu_1 + \nu_2\}_D$ are the only solutions of the functional equation

$$\forall (\nu_0, \nu_1, \nu_2) \in \mathbb{N}^3 \setminus \{(0, 0, 0)\}, \quad \frac{f(\nu_0 + \nu_1, \nu_2)}{f(\nu_0, \nu_2)} = \frac{f(\nu_1, \nu_0 + \nu_2)}{f(\nu_1, \nu_0)}.$$

This equation can be seen as a combinatorial version of the FEITH in view of the parallelism between Proposition 3 below and [10] Prop. 3.10.

We also prove that, for every $\alpha > 0$, there is a generalized multinomial coefficient asymptotically related to the corresponding $\alpha$-entropy. In fact, if $D'_n = \exp\{K(n^{\alpha-1} - 1)\}$, for some $K \in \mathbb{R}$, then

$$\left\{ \begin{array}{c} n \\ p_1 n, \ldots, p_s n \end{array} \right\}_{D'_n} = \exp \left\{ n^\alpha \frac{K}{\alpha} S_\alpha(p_1, \ldots, p_s) + O(n^\alpha) \right\}.$$

Since the Fontené-Ward multinomial coefficients satisfy the same multiplicative relations as the usual multinomial coefficients, their logarithms (properly normalized) are connected in the limit $n \to \infty$ to the deformed chain rule (1), as we already showed for the particular case $D_n = n$, which correspond to $\alpha = 1$ and Shannon

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2Already Fontené [6], in 1915, noted that $\{ n \}_{k, D} := \{ \begin{array}{c} n \\ k \end{array} \}_{D}$ verifies the additive recurrence formula

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\}_{D} - \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\}_{D} = \frac{D_n - D_{n-k}}{D_k},$$

with boundary conditions $\left\{ \begin{array}{c} n \\ 0 \end{array} \right\}_{D} = \left\{ \begin{array}{c} n \\ n \end{array} \right\}_{D} = 1$ for $n \geq 0$. Hence, for each sequence $D$, the corresponding $D$-binomial coefficients are associated to certain Pascal triangle defined in terms of $D$, see [7] p. 25].
entropy. This gives an asymptotic correspondence between some of these new combinatorial 1-cocycles and the old probabilistic 1-cocycles, which is the subject of the last section.

2. Information Structures and Their Cohomology

An information structure is a pair \((\mathbf{S}, \mathcal{E})\), made of a small category \(\mathbf{S}\) and a functor \(\mathcal{E} : \mathbf{S} \to \text{Meas}_{\text{surj}}\), whose codomain is the category of measurable sets and measurable surjections between them. We denote by \((E_X, \mathcal{B}_X)\) the image of an object \(X\) under \(\mathcal{E}\). The category \(\mathbf{S}\) is supposed

(i) to be a partially ordered set (poset): given any two objects \(A\) and \(B\) of \(\mathbf{S}\), there is at most one arrow from \(A\) to \(B\), and if \(A \to B\) and \(B \to A\), then \(A = B\) (strict equality);

(ii) to have a terminal object \(1\), and

(iii) to be “conditionally cartesian”: for any diagram \(X \leftarrow Z \to Y\) in \(\mathbf{S}\), the categorical product \(X \land Y\) exists.

In turn, the functor \(\mathcal{E}\) is conservative (it does not turn nonidentity arrows into isomorphisms) and satisfies:

(i) \(E_1 \cong \{\ast\}\),

(ii) for all \(X \in \text{Ob} \mathbf{S}\), the \(\sigma\)-algebra \(\mathcal{B}_X\) contains all the singletons \(\{x\} \subset E_X\), and

(iii) for every diagram \(X \xleftarrow{x} X \land Y \xrightarrow{\sigma} Y\) in \(\mathbf{S}\), the measurable map \(E_{X \land Y} \to E_X \times E_Y\), \(z \mapsto (x(z), y(z)) := (\mathcal{E} x(z), \mathcal{E} \sigma(z))\) is an injection.

Information structures are combinatorial objects that accept a probabilistic interpretation, under which the objects of \(\mathbf{S}\), denoted \(X, Y, Z, \ldots\), represent random variables, and the functor \(\mathcal{E}\) represents the possible outcomes of each variable. For any \(X \in \text{Ob} \mathbf{S}\) and \(A \in \mathcal{B}_X\), there is an event \(\{X \in A\}\). The arrows \(\pi : X \to Y\) in \(\mathbf{S}\) correspond to the notion of refinement, which is implemented by the measurable map \(\mathcal{E} \pi : \mathcal{E} X \to \mathcal{E} Y\): the event \(\{Y \in A\}\) can also be defined in terms of \(X\), as \(\{X \in \mathcal{E} \pi^{-1}(A)\}\). The product \(X \land Y\) represents the joint measurement of \(X\) and \(Y\), and the event \(\{X \land Y = z\}\) gives an interpretation to the probabilistic notation \(\{X = x(z), Y = y(z)\}\).

There is an appropriate notion of morphism between information structures. This and some properties of the corresponding category are treated in \([16]\).

The information structure is finite if each set \(E_X\) is finite; in this case, the algebra \(\mathcal{B}_X\) is necessarily the atomic \(\sigma\)-algebra and can be omitted from the notation. A treatment of the infinite case for gaussian random variables can be found in \([16]\).

Given an information structure \((\mathbf{S}, \mathcal{E})\), one can define a presheaf (i.e. a contravariant functor) of monoids that maps \(X \in \text{Ob} \mathbf{S}\) to the the set \(\mathcal{S}_X := \{Y \in \text{Ob} \mathbf{S} : X \to Y\}\) equipped with the product \((Y, Z) \mapsto YZ := Y \land Z\); an arrow \(X \to Y\) is mapped to the inclusion \(\mathcal{S}_Y \hookrightarrow \mathcal{S}_X\). The associated presheaf of induced algebras \(X \mapsto \mathbb{R}[\mathcal{S}_X]\) is denoted by \(A\).

More generally, presheaves of sets on \(\mathbf{S}\) are functors \(\mathcal{H} : \mathbf{S}^{\text{op}} \to \text{Sets}\); a morphism \(\phi : \mathcal{H} \to \mathcal{K}\) between presheaves is a natural transformation: a collection of mappings

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3In this article, boldface is associated to categories and caligraphic letters to functors. Given a functor \(F : \mathbf{C} \to \mathbf{D}\) and an object \(X\) of \(\mathbf{C}\), we denote by \(F_X\) the image of \(X\) under \(F\) wherever is possible—instead of the traditional notation \(F(X)\)—to avoid excessive parentheses.
\{ \phi_X : \mathcal{H}_X \rightarrow \mathcal{K}_X \} \) such that, for every \( \pi : X \rightarrow Y \) in \( S \), the diagram
\[
\begin{array}{ccc}
\mathcal{H}_Y & \xrightarrow{\phi_Y} & \mathcal{K}_X \\
\downarrow{\mathcal{H}_\pi} & & \downarrow{\mathcal{K}_\pi} \\
\mathcal{H}_X & \xrightarrow{\phi_X} & \mathcal{K}_X
\end{array}
\] (13)
in \textbf{Sets} commutes. One obtains in this way a category \( \widehat{S} \), which is a basic example of a Grothendieck topos, see [12]. The product between two presheaves \( \mathcal{H} \) and \( \mathcal{K} \) is the presheaf that associates to each \( X \in \text{Ob} S \) the set \( \mathcal{H}_X \times \mathcal{K}_X \) and to each arrow \( \pi \) the map \( \mathcal{H}_\pi \times \mathcal{K}_\pi \).

A \textit{presheaf of \( A \)-modules} is a presheaf of sets \( \mathcal{M} \) together with a morphism \( \phi : \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M} \) such that, for every \( X \in \text{Ob} S \), the set \( \mathcal{A}_X \) is an abelian group and \( \phi_X : \mathcal{A}_X \times \mathcal{M}_X \rightarrow \mathcal{M}_X \) defines a structure of \( \mathcal{A}_X \)-module on \( \mathcal{M}_X \). A morphism \( \psi : \mathcal{M} \rightarrow \mathcal{N} \) between sheaves of \( \mathcal{A} \)-modules \( (\mathcal{M}, \phi^\mathcal{M}) \) and \( (\mathcal{N}, \phi^\mathcal{N}) \) is a morphism of presheaves \( \psi : \mathcal{M} \rightarrow \mathcal{N} \) such that, for every \( X \in \text{Ob} S \), the mapping \( \psi_X \) is linear, and the diagram of presheaves
\[
\begin{array}{ccc}
\mathcal{A} \times \mathcal{M} & \xrightarrow{\phi^\mathcal{M}} & \mathcal{M} \\
\downarrow{1 \times \psi} & & \downarrow{\psi} \\
\mathcal{A} \times \mathcal{N} & \xrightarrow{\phi^\mathcal{N}} & \mathcal{N}
\end{array}
\] (14)
commutes. The set of \( \mathcal{A} \)-module morphisms from \( \mathcal{M} \) to \( \mathcal{N} \) is denoted by \( \text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \).

Information cohomology is a geometrical invariant associated to presheaves of \( \mathcal{A} \)-modules. It can be explicitly introduced as follows.

Let \( R_S \) denote the constant sheaf, which associates to every object \( X \) the vector space \( R \) with trivial \( S \)-action and to every morphism the identity map. The presheaves \( \{ B_i \}_{i \in \mathbb{N}} \) introduced above form a \textit{resolution} of \( R_S \), which means that there is a diagram of presheaves
\[
\begin{array}{ccc}
0 & \leftarrow & R_S \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} \\
B_0 & \leftarrow & B_1 \\
\downarrow{\partial_1} & & \downarrow{\partial_1} \\
B_1 & \leftarrow & B_2 \\
\downarrow{\partial_2} & & \downarrow{\partial_2} \\
& & \vdots
\end{array}
\] (15)
such that \( \text{im} \partial_1 = \ker \partial_{i-1} \) and \( \text{im} \theta_1 = \ker \varepsilon \). These morphisms are defined on generators by the formulae \( \varepsilon([]) = 1 \), and
\[
\partial_n([X_1|\ldots|X_n]) = X_1[X_2|\ldots|X_n] + \sum_{k=1}^{n-1} (-1)^k[X_1|\ldots|X_kX_{k+1}|\ldots|X_n] + (-1)^n[X_1|\ldots|X_{n-1}].
\] (16)

Given any presheaf \( \mathcal{M} \), we get a differential complex
\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_{\mathcal{A}}(R_S, \mathcal{M}) \\
\rightarrow & C^0(\mathcal{M}) & \xrightarrow{\delta^0} C^1(\mathcal{M}) \\
& \rightarrow & C^2(\mathcal{M}) \\
& \rightarrow \ldots 
\end{array}
\] (17)
\footnote{After this paragraph, this mapping \( \phi \) is always implicit: instead of \( \phi(a, m) \), we write \( a.m \).}
where $C^n(M)$ denotes $\text{Hom}_A(B_n, M)$ and each morphism $\delta^i : C^i(M) \to C^{i+1}(M)$ is given by the formula $\delta^i(\phi) := \phi \circ \partial_{i+1} : B_{i+1} \to M$. In general, this complex is not exact, but $\delta^{i+1} \circ \delta^i = 0$ still holds for every $i \in \mathbb{N}$.

The information cohomology of $S$ with coefficients in $M$, denoted $H^\bullet(S, M)$, is the cohomology of the differential complex (17), this is,

$$H^0(S, M) := \ker \delta^0 \quad \text{and} \quad H^n(S, M) := \ker \delta^n / \text{im} \delta^{n-1} \text{ when } n \in \mathbb{N}^*. $$

The elements of $C^n(M)$ are called $n$-cochains: they are $n$-cocyles when they belong to $Z^n(M) := \ker \delta^n$, and $n$-coboundaries when they belong to $\delta^n C^{n-1}$. We omit the superindex of $\delta$ if it is clear from context. Every $n$-coboundary is an $n$-cocycle, but the converse is not true. An $n$-cochain $\phi$ is by definition a collection $\{\phi_X : B_n(X) \to M_X\}$ of $A_X$-equivariant mappings, see (14). Therefore, it is enough to determine the image $\phi_X([X_1] \cdots [X_n])$ of each generator $[X_1] \cdots [X_n]$ of $B_n(X)$; to simplify notation, we write $\phi_X([X_1] \cdots [X_n])$. The naturality with respect to $X$—this is, the commutativity of (13)—translates into the following condition: for every arrow $\pi : X \to Y$ in $S$,

$$\phi_X[X_1] \cdots [X_n] = M\pi(\phi_Y[X_1] \cdots [X_n])$$

whenever $\{X_1, ..., X_n\} \subset S_Y \hookrightarrow S_X$. Remark that any variable $Y$ that refines $X_1, X_2, ..., X_n$, also refines their product $X_1 \cdots X_n$; thus (19) is equivalent to

$$\phi_X[X_1] \cdots [X_n] = M\rho(\phi_{X_1 \cdots X_n}[X_1] \cdots [X_n])$$

where $\rho$ is the arrow $X \to X_1 \cdots X_n$. According to this equation, $\phi_X[X_1] \cdots [X_n]$ only depends on its “localization” at $X_1 \cdots X_n$; consequently, we refer to (19) or (20) as joint locality.

Remark 1. The category of $A$-modules is abelian and has enough injectives. Therefore, one can introduce cohomological $\partial$-functors in the sense of $S$ (see also (13)). The functors $\{\text{Ext}^i(\mathbb{R}S, -)\}_{i \geq 0}$ are the right derived functors of $\text{Hom}(\mathbb{R}S, -)$. Information cohomology with coefficients $M$ can be defined as $H^\bullet(S, M) := \text{Ext}^\bullet(\mathbb{R}, M)$. These cohomology groups are naturally isomorphic to those introduced above, because it can be proved that each $B_i$ is a projective object in $\text{Mod}(A)$. For details, see [10 Sec. 2.4].

We introduce now a concrete example related to probabilistic functionals studied in [1] and [16]. We assume from now on that $(S, E)$ is a finite information structure.

Let $P$ be the functor that associates to any $X \in \text{Ob} S$ the set of probabilities

$$P_X := \left\{ p : E_X \to [0, 1] : \sum_{x \in E_X} p(x) = 1 \right\},$$

and to each morphism $\pi : X \to Y$, the mapping $P\pi : P_X \to P_Y$ given by

$$P\pi(p)(y) := \sum_{x \in E^{-1}(y)} p(x),$$

called marginalization. When $\pi$ is clear from context, we write $Y \pi$ instead of $P\pi(p)$.

Given any probability $p \in P_X$, an arrow $\pi : X \to Y$, and $y \in E_Y$ such that $Y \pi (y) \neq 0$, the conditional probability $p|_{Y=y} : E_X \to [0, 1]$ is given by

$$p|_{Y=y}(x) := \begin{cases} \frac{p(x)}{Y \pi (y)} & \text{if } x \in E^{-1}(y) \\ 0 & \text{otherwise} \end{cases}.$$
Let \( \mathcal{F} \) be the presheaf that associates to each \( X \in \text{Ob}\ S \) be the real vector space of measurable functions \( f : \mathcal{P}_X \to \mathbb{R} \) and to each arrow \( \pi : X \to Y \) in \( \text{S} \), the map given by precomposition with the corresponding marginalization: \( \mathcal{F}_\pi(f) = f \circ \mathcal{P}_\pi \).

For any \( \alpha > 0 \), we define an action of \( \mathcal{S}_X \) on \( \mathcal{F}_X \) as follows: for each \( Y \in \mathcal{S}_X \), \( f \in \mathcal{F}_X \) and \( p \in \mathcal{P}_X \),

\[
(\mathcal{Y}, f)(p) = \sum_{y \in E_Y \atop Y, p(y) \neq 0} (Y, p(y))^{\alpha} \phi(p|_{y})
\]

Extended linearly, this turns \( \mathcal{F}_X \) into an \( \mathcal{A}_X \)-module. Since the action is natural, we obtain an \( \mathcal{A} \)-module denoted \( \mathcal{F}_\alpha \). See Proposition 3.1 and 3.2 in [16].

We call \( H^\alpha(\text{S}, \mathcal{F}_\alpha) \) probabilistic information cohomology. Probabilistic 0-cochains \( \phi \in C^0(\mathcal{F}_\alpha) \) are given by a collection of functions \( \{\phi_X\} \in \mathcal{F}_\alpha(X) \) \( X \in \text{ObS} \) that by joint locality must be constant: \( \phi_X([P_X]) = \phi_1([1_P X]) = \phi_1([\delta_s]) \in \mathbb{R} \). It is not difficult to see that \( Z^0(\mathcal{F}_1) = C^0(\mathcal{F}_1) \), hence \( H^0(\text{S}, \mathcal{F}_1) \cong \mathbb{R} \), whereas \( H^0(\text{S}, \mathcal{F}_\alpha) = 0 \) for every \( \alpha \neq 1 \). In turn, any probabilistic 1-cochain \( \phi \in C^1(\mathcal{F}_\alpha) \) satisfies \( \phi_X[Z](p) = \phi_Z[Z](Z, p) \) by joint locality (19). In fact, the collection of measurable functions \( \{\phi[Z] : \mathcal{P}_Z \to \mathbb{R}\}_{Z \in \text{ObS}} \) defines the 1-cochain. Hence each \( \alpha \)-entropy determines a 1-cochain \( S_\alpha \in C^1(\mathcal{F}_\beta) \), for any \( \beta > 0 \), through the formulae

\[
\forall p \in \mathcal{P}_X, \quad S_1(X)(p) := -\sum_{x \in E_X} p(x) \ln p(x)
\]

and

\[
\forall p \in \mathcal{P}_X, \quad S_\alpha(X)(p) = \frac{1}{1 - \alpha} \left( \sum_{x \in E_X} p(x)^\alpha - 1 \right),
\]

when \( \alpha \in (0, \infty) \setminus \{1\} \).

Moreover, \( S_\alpha \) is a 1-cocycle of type \( \alpha \) i.e. an element of \( Z^1(\mathcal{F}_\alpha) \). The cocycle condition \( \delta S_\alpha = 0 \) means that, for every \( X \in \text{ObS} \) and every \( Y, Z \in \mathcal{S}_X \), the equation

\[
0 = (\mathcal{Y}, S_\alpha)[Z] - (S_\alpha)_X[Y, Z] + (S_\alpha)_X[Y]
\]

holds, and this corresponds exactly to the \( \alpha \)-chain rule, cf. \[7\]. Conversely, the equation \( 0 = \mathcal{Y}, \phi[Z] - \phi[Y, Z] + \phi[Y] \) (where marginalizations are implicit) has in general a unique solution, provided that the product \( Y, Z \) is nondegenerate, which means that \( E_{YZ} \) is “close” to \( E_Y \times E_Z \) in a sense made precise by \[10\] Def. 3.12 or Definition 2 in Section 4.

**Proposition 1** ( [10] Prop. 3.13, see also [1]). Let \((\text{S}, \mathcal{E})\) be a finite information structure and \( X, Y \) two different variables in \( \text{ObS} \) such that \( XY \in \text{ObS} \). Let \( \phi \) be a 1-cocycle of type \( \alpha \). If \( XY \) is nondegenerate, there exists \( \lambda \in \mathbb{R} \) such that

\[
\phi[X] = \lambda S_\alpha[X], \quad \phi[Y] = \lambda S_\alpha[Y], \quad \phi[XY] = \lambda S_\alpha[XY].
\]

The following result specifies the global number of free constants. It applies to any poset \( \text{S} \) with bounded height\[^5\] we say in this case that \((\text{S}, \mathcal{E})\) is bounded.

\[^5\]The set \( \mathcal{P}_X \) can be naturally identified with the standard simplex \( \Delta^{(E_X | -1)} \), which equipped with its Borel \( \sigma \)-algebra is a measurable space.

\[^6\]Information theorists would write \( H(Y, Z) = H(Y) + H(Z|Y) \) in the case of Shannon entropy.

\[^7\]The height of a poset is the length of the longest chain of morphisms \( a_1 \to a_2 \to ... \to a_n \), where no arrow equals an identity map.
Proposition 2 ([16], Thm. 3.14]). Let \((S,E)\) be a bounded, finite information structure. Denote by \(S^*\) the full subcategory of \(S\) generated by \(\text{Ob}S \setminus \{1\}\). Suppose that every minimal object can be factored as a nondegenerate product. Then,

\[
H^1(S,F_\lambda) = \prod_{\{C\} \in \pi_0(S^*)} \mathbb{R} \cdot S^C\tag{31}
\]

and, when \(\alpha \neq 1\),

\[
H^1(S,F_\alpha) = \left( \prod_{\{C\} \in \pi_0(S^*)} \mathbb{R} \cdot S^C \right) / \mathbb{R} \cdot S_\alpha\tag{32}
\]

In the formulae above, \(C\) represents a connected component of \(S^*\), and

\[
S_\alpha^C[X] = \begin{cases} S_\alpha[X] & \text{if } X \in \text{Ob } C \\ 0 & \text{if } X \notin \text{Ob } C \end{cases}
\]

3. Counting functions

Let \((S,E)\) be a finite information structure, and \(C : S \to \text{Sets}\) a functor that associates to each object \(X\) the set

\[
C_X = \left\{ \nu : E_X \to \mathbb{N} : \sum_{x \in E_X} \nu(x) > 0 \right\}, \tag{33}
\]

and to each arrow \(\pi : X \to Y\), associated to a surjection \(E\pi : E_X \to E_Y\), the map \(C\pi : C_X \to C_Y\) that verifies \(C\pi(\nu)(y) = \sum_{x \in E\pi^{-1}(y)} \nu(x)\). To simplify notation, we write \(Y\nu\) instead of \(C\pi(\nu)\), whenever \(\pi\) is clear from context. The elements of \(C_X\) are called counting functions. For \(\nu_X \in C_X\), we define its support as \(\{ x \in E_X : \nu_X(x) \neq 0 \}\), and its magnitude as the quantity \(\|\nu\| := \sum_{x \in X} \nu(x)\).

For any subset \(A\) of \(X\), there is a restriction

\[
\nu|_A(x) := \begin{cases} \nu(x) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \tag{34}
\]

When \(\|\nu|_A\| > 0\), we call \(\nu|_A\) the restricted counting given \(A \subset X\). Given an arrow \(\pi : X \to Y\), the notation \(\nu|_{Y=y}\) stands for \(\nu|_{\pi^{-1}(y)}\). Remark that \(\nu_0 = 0\) and \(\|\nu|_{Y=y}\| = Y_\pi(y)\).

Consider now the multiplicative abelian group \(G_X\), whose elements are \(\mathbb{R}_+^*\)-valued measurable functions defined on \(C_X\). By \(\mathbb{R}_+^*\) we mean \(\{ x \in \mathbb{R} : x > 0 \}\). (The multiplicative notation is convenient, because multinomial coefficients appear directly as cocycles.) The group \(G_X\) becomes a real vector space if we define \((r.g)(\nu) := (g(\nu))^r\), for each \(g \in G_X\) and each \(r \in \mathbb{R}\), \(^8\) For each \(Y \in S_X\) and each \(g \in G_X\), set

\[
(Y.g)(\nu) := \prod_{y \in E_X, g(y) \neq 0} g(\nu|_{Y=y})\tag{35}
\]

Finally, define \((aY).g := a.(Y.g) = Y.(a.g)\). As a consequence of the following proposition, these formulae give an homomorphism \(\rho_X : A_X \to \text{End}(G_X)\), that turns \(G_X\) into an \(A_X\)-module.

\[^8\text{In principle this is a right action, but this is immaterial because }\mathbb{R}\text{ is commutative.}\]
Proposition 3. Given variables $Y, Z \in \mathcal{S}_X$ and $g \in \mathcal{G}_X$, 
\begin{equation}
ZY.g = Z.(Y.g).
\end{equation}

Proof. Set $W$ equal to $ZY := Z \wedge Y$. Since in $S$ we have the commutative diagram

we obtain the following commutative diagram of sets

where the upper triangle is explained by the functoriality of $\mathcal{E}$ (to simplify notation, we write $\pi$ instead of $\mathcal{E}\pi$) and the lower one by the universal property of products in $\text{Sets}$. The mapping $\iota$ is an injection by definition of an information structure.

Note that
\begin{equation}
Z.(Y.g)(\nu) = \prod_{z \in E_Z} \left( Y.g \right)(\nu_{|Z=z})
\end{equation}
\begin{equation}
= \prod_{z \in E_Z} \prod_{y \in E_Y} g((\nu_{|Z=z})(y))
\end{equation}

From the definition of conditioning, we deduce that $(\nu_{|Z=z})(y) = \nu_{A(y,z)} \cap \{Y=y\} = \nu_{A(y,z)}$, where we have set

$A(y, z) := \pi_{YX}^{-1}(y) \cap \pi_{ZX}^{-1}(z) = \pi_{WX}^{-1}(x) \cap \pi_{ZW}^{-1}(z)$.

If $(y, z) \not\in \iota$, $A(y, z)$ is empty, so $\nu_{A(y,z)} = 0$, as well as $\|\nu_{A(y,z)}\| = Y_*\nu_{Z=z}(y) = 0$. Therefore, the product in (35) can be restricted to pairs $(y, z) \in \iota$, and the condition $Y_*\nu_{Z=z}(y) = \|\nu_{A(y,z)}\| \neq 0$ translates into $W_*\nu_{\iota^{-1}(y,z)} = \|\nu_{A(y,z)}\| \neq 0$. Since there is a bijection $E_W \cong \iota$, upon relabeling we obtain the desired equality. \hfill $\square$

To any arrow $\pi : X \to Y$, we associate the map $\mathcal{G}\pi : \mathcal{G}_Y \to \mathcal{G}_X$ such that $\mathcal{G}\pi(g) = g \circ C(\pi)$. Then $\mathcal{G} : S \to \text{Sets}$ is a contravariant functor. In fact, it is a presheaf of $\mathcal{A}$-modules: it is not difficult to prove that the commutativity of (14) holds, cf. [16] Prop. 3.2.


4. Combinatorial information cohomology

In this section, we compute the information cohomology of $S$ with coefficients in $G$, which we call combinatorial information cohomology. See Section 2.

The elements of $C^n(G) := \text{Hom}_A(B_n, G)$ are called combinatorial $n$-cochains. The coboundary of $\psi \in C^n(G)$ is the $(n + 1)$-cochain $\delta \psi : B_{n+1} \to G$ defined on the generators of $B_{n+1}$ by

$$\delta \psi[X_1|...|X_{n+1}] =$$

$$(X_1, \psi[X_2|...|X_{n+1}]) \left( \prod_{k=1}^{n} (\psi[X_1|...|X_kX_{k+1}|...|X_n])^{(-1)^k} \right) \psi[X_1|...|X_n]^{(-1)^{n+1}},$$

because we are using multiplicative notation for $G$. A combinatorial $n$-cocycle is an element $\psi$ in $C^n(S, G)$ that verifies $\delta \psi = 1$; the submodule of all $n$-cocycles is denoted by $Z^n(G)$. The image under $\delta$ of $C^{n-1}$ forms another submodule of $C^n(G)$, denoted $\delta C^{n-1}(G)$; its elements are called combinatorial $n$-coboundaries.

4.1. Computation of $H^0$. The 0-cochains are given by a collection of functions \( \{\psi_X\}_{X \in \text{Ob}S} \) (the image of the generator \([\ )\] under $\psi$ over each $X$). Joint locality implies that, for every $X \in \text{Ob}S$, $\psi_X(\nu) = \psi_1(1, \nu_X) = \psi_1(\|\nu_X\|)$. Hence, 0-cochains are in one-to-one correspondence with measurable functions of the magnitude, $\Psi := \psi_1 : \mathbb{N}^* \to \mathbb{R}_+^*$.

A 0-cocycle $\psi$ must verify, for each $Y$ coarser than $X$, the equation $(\delta \psi)_X[Y] = (Y, \psi_X)(\psi_X)^{-1} = 1$, which is equivalent to

$$\Psi(\|\nu_X\|) = \prod_{\nu|_{Y, \nu(y)\neq 0}} \Psi(\|\nu|_{Y,y}\|).$$

Whenever $|Y| \geq 2$, this means in particular that

$$\Psi(x + y) = \Psi(x)\Psi(y)$$

for every $x, y \in \mathbb{N}^*$. Setting $a := \Psi(1) > 0$, one easily concludes by recurrence that $\Psi(n) = a^n = \exp(n \ln(a))$. The function $\Psi(x) = \exp(kx)$, for arbitrary $k \in \mathbb{R}$, is a general solution of (37), because $\|\nu_X\| = \sum_{\nu|_{Y, \nu(y)\neq 0}} \|\nu|_{Y,y}\|$. We have proved the following proposition.

**Proposition 4.** Let $\text{Exp} \in \text{Hom}_A(\ast, G)$ be the section defined by

$$\text{Exp}_X : C_X \to \mathbb{R}^*_+,$$

$$\nu \mapsto \exp(\|\nu\|).$$

Then $H^0(S, G) = \langle \text{Exp} \rangle_{\mathbb{R}}$.

4.2. Computation of $H^1$. For any 1-cochain $\psi$, we set $\psi[Z] := \psi_Z[Z] = \psi_X[Z]$, the last equality being valid for any $X$ such that $X \to Z$ by joint locality.

In order to compute the 1-cocycle, we prove first an auxiliary result.

**Lemma 1.** Let $\psi \in Z^1(G)$. For every $X \in \text{Ob}S$, if $\nu \in C_X$ verifies $\nu = \nu|_{X=x_0}$ for some $x_0 \in E_X$, then $\psi[X](\nu) = 1$.

In particular, $\psi[1] \equiv 1$. 

Proof. The cocycle condition implies in particular that \( \psi[Xx] = (X.\psi[X])\psi[X] \), this is
\begin{equation}
1 = \prod_{x \in E_X \atop \nu(x) \neq 0} \psi[X](\nu|_{x=x}) = \psi[X](\nu|_{x=x_0}).
\end{equation}
\[ \square \]

The following result will be essential for the characterization of all the 1-cocycles. It is the combinatorial analogue of [16, Prop. 3.10], where a variant of the fundamental equation of information theory [8] appears. Consequently, (40) and (42) can be seen as combinatorial generalizations of this functional equation.

**Proposition 5** (Combinatorial FEITH). Let \( f_1, f_2 : \mathbb{N} \setminus \{(0, 0)\} \to \mathbb{R}_+ \) be two unknown functions. The functions \( f_1, f_2 \) satisfy the conditions
\begin{enumerate}
  \item for every \( n \in \mathbb{N}^* \), \( f(n, 0) = f(0, n) = 1 \).
  \item for every \( \nu_0, \nu_1, \nu_2 \in \mathbb{N} \) such that \( \nu_0 + \nu_1 + \nu_2 \neq 0 \),
\end{enumerate}
\begin{equation}
(40) \quad f_1(\nu_0 + \nu_2, \nu_1)f_2(\nu_0, \nu_2) = f_2(\nu_0 + \nu_1, \nu_2)f_1(\nu_0, \nu_1).
\end{equation}
if, and only if, there exists a sequence of numbers \( D = \{D_i\}_{i \geq 1} \subset \mathbb{R}_+ \), such that \( D_1 = 1 \), and
\begin{equation}
(41) \quad f(\nu_1, \nu_2) = \frac{[\nu_1 + \nu_2]_D!}{[\nu_1]_D! [\nu_2]_D!}
\end{equation}
where \( [n]_D! = D_n D_{n-1} \cdots D_1 \) whenever \( n > 0 \), and \( [0]_D! = 1 \).

Proof. Setting \( \nu_0 = 0 \), we conclude first that \( f_1(\nu_2, \nu_1) = f_2(\nu_1, \nu_2) \). Define \( f(x, y) := f_1(x, y) \); it satisfies the equation
\begin{equation}
(42) \quad \frac{f(\nu_0 + \nu_1, \nu_2)}{f(\nu_0, \nu_2)} = \frac{f(\nu_1, \nu_0 + \nu_2)}{f(\nu_1, \nu_0)}.
\end{equation}
for any \( \nu_0, \nu_1, \nu_2 \in \mathbb{N} \) such that \( \nu_0 + \nu_1 + \nu_2 \neq 0 \). In particular, if \( \nu_0 = t > 0 \), and \( \nu_1 = \nu_2 = s > 0 \),
\begin{equation}
(43) \quad \frac{f(t, s)}{f(s, t)} = \frac{f(t + s, s)}{f(s, t + s)}.
\end{equation}
Thus, for any \( n > 1 \),
\begin{equation}
(44) \quad \frac{f(n, 1)}{f(1, n)} = \frac{f(n-1, 1)}{f(1, n-1)} = \cdots = \frac{f(1, 1)}{f(1, 1)} = 1.
\end{equation}
Let \( D_{n+1} \) be the common value of \( f(n, 1) \) and \( f(1, n) \). From Equation (42), setting \( \nu_0 = n, \nu_1 = 1, \) and \( \nu_2 = k \), we can obtain a recurrence formula for \( f(n + 1, k) \):
\begin{equation}
(45) \quad f(n + 1, k) = \frac{D_{n+k+1}}{D_{n+1}} f(n, k).
\end{equation}
By repeated application of this recurrence, we conclude that
\begin{equation}
(46) \quad f(n, k) = \frac{D_{n+k}}{D_n} \cdot \frac{D_{n+k-1}}{D_{n-1}} \cdots \frac{D_{k+1}}{D_1} f(0, k).
\end{equation}
Remark that \( D_1 = f(0, 1) = 1 \), and \( f(0, k) = 1 \) (Lemma [1]). Therefore, \( f \) can be rewritten as
\begin{equation}
(47) \quad f(\nu_1, \nu_2) = \frac{[\nu_1 + \nu_2]_D!}{[\nu_1]_D! [\nu_2]_D!}.
\end{equation}
This formula still make sense when $\nu_1 = 0$ or $\nu_2 = 0$. Conversely, for any sequence $D = \{D_i\}_{i \geq 1}$, with $D_1 = 1$, the assignment $f_1 = f_2 = f$ satisfies (52), and thus represents the most general solution.

**Example 1.** Let $(S, \mathcal{E})$ be an information structure defined as follows: $S$ is the poset represented by the graph

```
X_1 \xleftarrow{1} X_2
\text{and } E \text{ is the functor defined at the level of objects by } E(X_1) = \{x_{(1)}, x_{(0,2)}\}, \quad E(X_2) = \{x_{(2)}, x_{(0,1)}\}, \quad \text{and } E(X_1 X_2) = \{x_{(1)}, x_{(2)}, x_{(3)}\}; \text{ for each arrow } \pi : X \to Y, \text{ the map } \pi_* : E(X) \to E(Y) \text{ sends } x_I \to x_J \text{ iff } I \subset J.
```

For this structure, the cocycle condition give the equations

$$
\psi[X_1 X_2](\nu_0, \nu_1, \nu_2) = \psi[X_2](\nu_0 + \nu_1, \nu_2)\psi[X_1](\nu_0, \nu_1)\psi[X_1](\nu_2, 0),
$$

$$
\psi[X_2 X_1](\nu_0, \nu_1, \nu_2) = \psi[X_1](\nu_0 + \nu_2, \nu_1)\psi[X_2](\nu_0, \nu_2)\psi[X_2](\nu_1, 0).
$$

Since $X = X_1 X_2 = X_2 X_1$,

$$
\psi[X_2](\nu_0 + \nu_1, \nu_2)\psi[X_1](\nu_0, \nu_1) = \psi[X_1](\nu_0 + \nu_2, \nu_1)\psi[X_2](\nu_0, \nu_2)
$$

where we have taken into account that $\psi[X_1](\nu_2, 0) = \psi[X_2](\nu_1, 0) = 0$. This is exactly Equation (50), and the condition (i) in the statement is also met, therefore

$$
\psi[X_1](\nu_0, \nu_1) = \psi[X_2](\nu_0, \nu_1) = \begin{cases} \nu_0 + \nu_1 & \text{if } \nu_1 \neq 0, \\ \nu_0 & \text{if } \nu_1 = 0 \end{cases}
$$

for some sequence $D$. From (48), we conclude that

$$
\psi[X](\nu_0, \nu_1, \nu_2) = \begin{cases} \nu_0 + \nu_1 + \nu_2 & \text{if } \nu_1 \neq 0, \\ \nu_0 + \nu_1 & \text{if } \nu_1 = 0 \end{cases}
$$

\begin{equation}
=: \frac{[\nu_0 + \nu_1 + \nu_2]}{[\nu_0 + \nu_1][\nu_1][\nu_2]} D^1.
\end{equation}

**Definition 1.** Given any sequence $D = \{D_i\}_{i \geq 1}$ verifying $D_1 = 1$ (called admissible sequence), the corresponding Fontene-Ward multinomial coefficient is the 1-cochain given by

$$
\forall \nu \in \mathcal{C}(X), \quad W_D[X](\nu) = \frac{[\nu]}{\prod_{x \in E_X} [\nu(x)] D^1}.
$$

To characterize the 1-cocycles $\psi$ in the general case, we introduce a notion of nondegenerate product analogous to [10] Def. 3.12 [4]. Its definition is better understood reading the proof of Proposition [6]. The idea is to determine the function $\psi[XY]$, for given variables $X$ and $Y$, applying the same kind of reasoning used in the previous example. One obtains first the recursive formulae (57) and (58) for the functions $\psi[X]$ and $\psi[Y]$; these are based on a chosen total order of the sets $E_X$ and $E_Y$, and the steps of the recursion are coded by a path in $\mathbb{Z}^2$. Both recursive formulae are a simplification of the symmetric equation (56) for particular laws $\tilde{\nu}$ given by the Condition (i) in Definition 2 that make one of the factors trivial. These

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9 Both notions coincide when $Q$ in [10] is the functor $P$ introduced by [21].
recursive formulae involve a term where \( \psi[X] \) and \( \psi[Y] \) have only two nonzero arguments, and one recovers “locally” the combinatorial FEITH of Proposition 5 the fact that three different integers are involved in this equation is ensured by Condition \( \mathbf{3} \) in Definition 2.

**Definition 2.** Let \( X \) and \( Y \) be two objects of \( S \), such that \( |E_X| = k \) and \( |E_Y| = l \). Let \( \iota \) be the inclusion \( E_{XY} \hookrightarrow E_X \times E_Y \). We call the product \( XY \) **nondegenerate** if there exist enumerations \( \{x_1, ..., x_k\} \) of \( E_X \) and \( \{y_1, ..., y_l\} \) of \( E_Y \), and a North-East (NE) lattice path \( (\gamma_i)_{i=1}^m \) on \( \mathbb{Z}^2 \) going from \((1,1)\) to \((k,l)\) such that

(i) If \( \gamma_i = (a,b) \) and \( \gamma_{i+1} - \gamma_i = (1,0) \), we ask that for every counting function \( \nu \in C_X \) such that \( \text{supp} \nu \subset \{x_i : a \leq i \leq k\} \), there exists a counting function \( \tilde{\nu} \in C_{XY} \) whose support is contained in

\[
\iota^{-1}(\{(x_a, y_b+1)\} \cup \{(x_i, y_b) : a+1 \leq i \leq k\}) \cup \iota^{-1}(\{(x_a, y_b)\} \cup \{(x_i, y_b+1) : a+1 \leq i \leq k\})
\]

and such that \( \nu = X_\iota \tilde{\nu} \). Remark that, for such values of \( XY \), the value of the \( Y \)-component completely determine the \( X \)-component.

(ii) For each \( \gamma_i = (a,b) \), the set

\[
\iota^{-1}\{(x_i, y_j) : a \leq i \leq a+1 \text{ and } b \leq j \leq b+1\}
\]

contains at least three different elements.

**Proposition 6.** Let \( (S, E) \) be a finite information structure and \( X, Y \) two different variables in \( \text{Ob}\, S \) such that \( XY \in \text{Ob}\, S \). Let \( \psi \) be a combinatorial 1-cocycle i.e. an element of \( Z^1(S, G) \). If \( XY \) is nondegenerate, there exists an admissible sequence \( D \), such that

\[
\psi[X] = W_D[X], \quad \psi[Y] = W_D[Y], \quad \psi[XY] = W_D[XY].
\]

**Proof.** Since \( \psi \) is a 1-cocycle, it satisfies the two equations derived from (36)

\[
(54) \quad Y.\psi[X]\psi[Y] = \psi[XY], \quad (55) \quad X.\psi[Y]\psi[X] = \psi[XY],
\]

and therefore the symmetric equation

\[
(56) \quad (X.\psi[Y])\psi[X] = (Y.\psi[X])\psi[Y].
\]

For any counting function \( \nu \), we write

\[
\begin{pmatrix}
  s & t & u & \ldots \\
  p & q & r & \ldots 
\end{pmatrix}
\]

if \( \nu(s) = p, \nu(t) = q, \nu(u) = r \), etc. and the images of the unwritten parts are zero.

---

10 A North-East (NE) lattice path on \( \mathbb{Z}^2 \) is a sequence of points \( (\gamma_i)_{i=1}^m \subset \mathbb{Z}^2 \) such that \( \gamma_{i+1} - \gamma_i \in \{(1,0), (0,1)\} \) for every \( i \in \{1, ..., m-1\} \).
Fix an order \((x_1, \ldots, x_k)\) and \((y_1, \ldots, y_l)\) that satisfies the definition of nondegenerate product, and let \(\{\gamma_i\}_{i=0}^m\) be the corresponding path. If \(\gamma_i = (a, b)\) and \(\gamma_{i+1} - \gamma_i = (1, 0)\), we are going to show that the following recursive formula holds:

\[
(57) \quad \psi[X] \left( \begin{array}{cccc}
  x_a & \cdots & x_k \\
  \mu_a & \cdots & \mu_k 
\end{array} \right) = \psi[X] \left( \begin{array}{cccc}
  x_{a+1} & \cdots & x_k \\
  \mu_{a+1} & \cdots & \mu_k 
\end{array} \right) \psi[X] \left( \begin{array}{cccc}
  x_a & x_{a+1} \\
  \mu_a & \parallel \mu || - \mu_a 
\end{array} \right).
\]

Analogously, if \(\gamma_i = (a, b)\) and \(\gamma_{i+1} - \gamma_i = (0, 1)\),

\[
(58) \quad \psi[Y] \left( \begin{array}{cccc}
  y_b & \cdots & y_l \\
  \nu_b & \cdots & \nu_l 
\end{array} \right) = \psi[Y] \left( \begin{array}{cccc}
  y_{b+1} & \cdots & y_l \\
  \nu_{b+1} & \cdots & \nu_l 
\end{array} \right) \psi[Y] \left( \begin{array}{cccc}
  y_b & y_{b+1} \\
  \nu_b & \parallel \nu || - \nu_b 
\end{array} \right).
\]

Suppose that \(\gamma_i = (a, b)\) and \(\gamma_{i+1} - \gamma_i = (1, 0)\). Let

\[
\mu = \left( \begin{array}{cccc}
  x_a & \cdots & x_k \\
  \mu_a & \cdots & \mu_k 
\end{array} \right)
\]

be a counting function in \(\mathcal{C}_X\). By Definition \([\text{20}]\) above, we know that \(\mu\) has a preimage under marginalization \(\tilde{\mu}\), whose support is such that \((X, \psi[Y])(\tilde{\mu}) = 1\), cf. Lemma \([\text{4}]\). Equation \((56)\) then reads

\[
(59) \quad \psi[X] \left( \begin{array}{cccc}
  x_{a+1} & \cdots & x_k \\
  \mu_{a+1} & \cdots & \mu_k 
\end{array} \right) \psi[Y] \circ \tau \left( \parallel \mu || - \mu_a \right) = \psi[X] \left( \begin{array}{cccc}
  x_a & \cdots & x_k \\
  \mu_a & \cdots & \mu_k 
\end{array} \right),
\]

where \(\tau\) is the identity or the transposition of the nontrivial arguments of \(\mu\). In any case, setting \(\mu_{a+1} = \parallel \mu || - \mu_a\) and \(\mu_{a+2} = \ldots = \mu_k = 0\), we conclude that

\[
(60) \quad \psi[X] \left( \begin{array}{cccc}
  x_a & x_{a+1} \\
  n_1 & n_2 
\end{array} \right) = \psi[Y] \circ \tau \left( \begin{array}{cccc}
  y_b & y_{b+1} \\
  n_1 & n_2 
\end{array} \right),
\]

which combined with \((59)\) implies \((57)\). The identity \((58)\) can be obtained analogously.

To determine

\[
f_a(n_1, n_2) := \psi[X] \left( \begin{array}{cccc}
  x_a & x_{a+1} \\
  n_1 & n_2 
\end{array} \right) \quad \text{and} \quad g_b(n_1, n_2) := \psi[Y] \left( \begin{array}{cccc}
  y_b & y_{b+1} \\
  n_1 & n_2 
\end{array} \right),
\]

for \((n_1, n_2) \in \mathbb{N}^2 \setminus \{(0, 0)\}\), consider the three different elements \(w_1, w_2, w_3\) in \(E_{XY} \subset E_X \times E_Y\) given by the property \([\text{21}]\) of a nondegenerate product. The symmetric equation \((56)\) evaluated on \(\nu_1 \delta_{w_1} + \nu_2 \delta_{w_2} + \nu_3 \delta_{w_3} \in \mathcal{C}_{XY}\) gives the equation that appears in Proposition \([\text{3}]\) which implies that \(f_a(n_1, n_2) = g_b(n_1, n_2) = \left\{ \left\{ \left\{ n_1, n_2 \right\} \right\}_{D} \right\}_{D'}\) for certain admissible sequence \(D\) (the eventual permutations of the arguments in the unknowns become irrelevant, because the solution is symmetric).

When considering \(\gamma_{i+1} = \gamma_i\) one finds the functions \(f_a\) and \(g_{b+1}\), or the functions \(f_{a+1}\) and \(g_b\), since \(\gamma_{i+1} - \gamma_i\) is either \((1, 0)\) or \((0, 1)\). This ensures that the admissible sequence \(D\) that appears for each \(\gamma_i\) is always the same, as proved in Lemma \([\text{2}]\). The recurrence relations \((57)\) and \((58)\) then imply the desired result. \(\square\)

**Lemma 2.** Let \(D, D'\) be two admissible sequences. If for all \(n_1, n_2 \in \mathbb{N}^2\)

\[
(61) \quad \left\{ \begin{array}{l}
  n_1 + n_2 \\
  n_1, n_2
\end{array} \right\}_{D} \quad \text{and} \quad \left\{ \begin{array}{l}
  n_1 + n_2 \\
  n_1, n_2
\end{array} \right\}_{D'}
\]

then \(D = D'\).
Proof. Just remark that
\begin{equation}
\left\{ \begin{array}{l}
1 \\
n
\end{array} \right\}_D = \left\{ \begin{array}{l}
n \\
n, n-1
\end{array} \right\}_D = [n]_D!
\end{equation}
so we have $[n]_D! = [n]_D!$ for all $n \in \mathbb{N}$.
\hfill \Box

As in the continuous case, the number of admissible sequences that appear in the computation of the 1-cocycles $\mathcal{C}(\mathcal{S}, \mathcal{G})$ depends on the number of connected components of $\mathcal{S}^*$, that is $\mathcal{S}$ deprived of its final element. In addition, a choice of 0-cochain $\psi$ induces globally a Fontené-Ward coefficient $\delta \psi$ for a unique admissible sequence $D_g$. Therefore, $\mathcal{C}^1(\mathcal{G})$ and $\mathcal{C}^{01}(\mathcal{G})$ are both infinite dimensional. If $\mathcal{S}^*$ is connected, the quotient is trivial; otherwise it is infinite: $|\pi_0(\mathcal{S}^*)| - 1$ admissible sequences remain arbitrary. This is the combinatorial version of Proposition 2.

5. ASYMPTOTIC RELATION WITH PROBABILISTIC INFORMATION COHOMOLOGY

Proposition 7. Let $\psi$ be a combinatorial $n$-cocycle. Suppose that, for every $X_1, \ldots, X_n \in \text{Ob} \mathcal{S}$ such that $X_1 \cdots X_n \in \text{Ob} \mathcal{S}$, there exists a measurable function
\[ \phi[X_1][X_n] : \mathcal{P}_{X_1} \cdots X_n \rightarrow \mathbb{R} \]
with the following property: for every sequence of counting functions \( \{ \nu_n \}_{n \geq 1} \subset \mathcal{C}_{X_1 \cdots X_n} \) such that
(i) $\| \nu_n \| \rightarrow \infty$, and
(ii) for every $z \in E_{X_1} \cdots X_n$, $\nu_n(z)/\| \nu_n \| \rightarrow p(z)$ as $n \rightarrow \infty$
the asymptotic formula
\[ \psi[X_1][X_n](\nu_n) = \exp(\| \nu_n \|^\alpha \phi[X_1][X_n](p) + o(\| \nu_n \|^\alpha)) \]
holds. Then $\phi$ is a $n$-cocycle of type $\alpha$, i.e. $\phi \in \mathcal{Z}^n(\mathcal{S}, \mathcal{F}_n(\mathcal{P}))$.

Proof. To simplify notation, we assume that $n = 1$; the proof is still valid in the general case. We must show that, for every $p \in \mathcal{P}_{XY}$,
\[ \phi[XY](p) = (X \phi[Y])(p) + \phi[X](X \ast p). \]
Let $\{ \nu_n \}_{n \geq 1}$ be a sequence of counting functions such that $\| \nu_n \| \rightarrow \infty$ and, for every $z \in E_{XY}$, $\nu_n(z)/\| \nu_n \| \rightarrow p(z)$. A sequence like this always exists: just consider a rational approximation of the values of $p$ with common denominator.

Since $\psi$ is a 1-cocycle, $\psi[XY] = (X \psi[Y]) \psi[X]$. Evaluate it at $\nu_n$, take the logarithm and divide by $\| \nu_n \|^{\alpha}$ in order to obtain
\begin{equation}
\frac{\ln \psi[XY](\nu_n)}{\| \nu_n \|^\alpha} = \sum_{X, \nu_n(z) \neq 0} \frac{\ln \psi[Y](\nu_n | x = z)}{\| \nu_n \|^{\alpha}} + \frac{\ln \psi[X](\nu_n)}{\| \nu_n \|^\alpha}.
\end{equation}
Recall that, for any counting function $\nu$, $\| \nu | x = z \| = X_i \nu(x)$. Hence,
\begin{equation}
\frac{\ln \psi[Y](\nu_n | x = z)}{\| \nu_n \|^\alpha} = \frac{\ln \psi[Y](\nu_n | x = z) (X_i \nu_n(x))^\alpha}{\| \nu_n | x = z \|^\alpha \| \nu_n \|^\alpha}.
\end{equation}
Plug this in (63) and take the limit as $n$ goes to infinity to conclude. \hfill \Box

We discuss now some important examples:
(i) The exponential $\text{Exp}^k : \nu \rightarrow \exp(k \| \nu \|)$ is the a combinatorial 0-cocycle, and it corresponds to the constant $k$ seen as a probabilistic 0-cocycle.
(ii) As we explained in Section 1

\[
\binom{n}{p_1n, \ldots, p_sn} = \exp(nS_1(p_1, \ldots, p_s) + o(n))
\]

The multinomial coefficient is a combinatorial 1-cocycle and \( S_1 \) defines an element of \( Z^1(F_1) \).

(iii) Whereas the previous examples are not surprising, Proposition 7 hints at new objects that are connected to the generalized \( \alpha \)-entropies and have gone unnoticed until now. For example, the \( q \)-multinomial coefficients are connected asymptotically to the 2-entropy (quadratic entropy),

\[
\left[ \begin{array}{c} n \\ p_1n, \ldots, p_sn \end{array} \right]_q = \exp(n^2 \frac{\ln q}{2} S_2(p_1, \ldots, p_s) + o(n^2)),
\]

see [15] Prop. 2]. These coefficients have a combinatorial interpretation: when \( q \) is a prime power and \( k_1, \ldots, k_s \) are integers such that \( \sum_{i=1}^s k_i = n \), the coefficient \( \left[ \begin{array}{c} n \\ k_1, \ldots, k_s \end{array} \right]_q \) counts the number of flags of vector spaces \( V_1 \subset V_2 \subset \ldots \subset V_n = F_q^n \) such that \( \dim V_i = \sum_{j=1}^i k_j \) (here \( F_q \) denotes the finite field of order \( q \)). In particular, the \( q \)-binomial coefficient \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q \) counts vector subspaces of dimension \( k \) in \( F_q^n \).

In [15], we push this parallel between \( S_1 \) and \( S_2 \) much further: we introduce a probabilistic model that generates vector spaces and study its concentration properties, in order to obtain a generalization of the Asymptotic Equipartition Property that involves the quadratic entropy.

It is quite natural to ask if, for any \( \alpha > 0 \), there exists a sequence \( D^\alpha = \{ D_i^\alpha \}_{i \geq 1} \) asymptotically related to the entropy \( S_\alpha \) through Proposition 7. The answer turns out to be yes.

**Proposition 8.** Consider any \( \alpha \in \mathbb{R}_+ \setminus \{1\} \). If \( D_n^\alpha = \exp\{K(n^{\alpha - 1})\} \), for any \( K \in \mathbb{R} \), then

\[
\left\{ \begin{array}{c} n \\ p_1n, \ldots, p_sn \end{array} \right\}_{D_n^\alpha} = \exp\left\{ n^\alpha \frac{K}{\alpha} S_\alpha(p_1, \ldots, p_s) + o(n^\alpha) \right\}.
\]

**Proof.** Remark that \( [n]_D! := \exp\{K(\sum_{i=1}^n i^{\alpha - 1} - n)\} \).

Suppose first that \( \alpha > 1 \). In this case, \( x \mapsto x^{\alpha - 1} \) is strictly increasing and

\[
\int_0^n x^{\alpha - 1} \, dx = \frac{n^\alpha}{\alpha} = \sum_{i=1}^n i^{\alpha - 1} < \int_1^{n+1} x^{\alpha - 1} \, dx = \frac{(n+1)^\alpha}{\alpha} - \frac{1}{\alpha}.
\]

Hence, if \( K > 0 \),

\[
\exp\left\{ K \left( \frac{n^\alpha}{\alpha} - n \right) \right\} < [n]_D! < \exp\left\{ K \left( \frac{(n+1)^\alpha}{\alpha} - \frac{1}{\alpha} - n \right) \right\}.
\]

This directly implies that

\[
\frac{[n]_D!}{[n_1]_D! \cdots [n_s]_D!} < \exp\left\{ K \left( \frac{n^\alpha}{\alpha} - n - \sum_{i=1}^s \frac{(n_i)^\alpha}{\alpha} - n_i \right) \right\}
\]

as well as

\[
\frac{[n]_D!}{[n_1]_D! \cdots [n_s]_D!} > \exp\left\{ K \left( \frac{n^\alpha}{\alpha} - n - \sum_{i=1}^s \frac{(n_i + 1)^\alpha}{\alpha} - n_i \right) \right\}
\]
from which the conclusion follows.

If $K < 0$, the inequalities (68), (69) and (69) must be reversed, but the result is the same. Similarly, when $0 < \alpha < 1$ the argument remains valid making the necessary modifications: all inequalities are reversed, since $x \mapsto x^{\alpha-1}$ is strictly decreasing. \hfill \Box

It is not known if these or similar coefficients related to $S_\alpha$, for $\alpha \in \mathbb{R}_+ \setminus \{1, 2\}$, have a combinatorial or statistical interpretation.

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