Warp drive dynamic solutions considering different fluid sources

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Alcubierre proposed in 1994 that the well-known special relativistic limitation that particles cannot travel with velocities bigger than the light speed can be bypassed when such trips are considered globally within specific general relativistic frameworks. Although initial results indicated this scenario as being unphysical, since it would seem to require negative mass-energy density, recent theoretical analyses suggest that such an unphysical situation may not always be necessarily true. In this paper, we present solutions of the Einstein equations using the original Alcubierre warp drive metric endowed with various matter-energy sources, namely dust, perfect fluid, anisotropic fluid, and perfect fluid within a cosmological constant spacetime. A connection of some of these solutions featuring shock waves described by the Burgers equation is also shown.

Keywords: warp drive solutions, perfect fluid, anisotropic fluid, shock waves, electromagnetic field, cosmological constant

1. Introduction

Alcubierre advanced a model based on a specific general relativistic spacetime geometry in which massive particles can travel at superluminal speeds if they are located inside a specially designed spacetime distortion. The physics of this possible propulsion method, named as warp drive (WD) after science fiction literature, consists of a special spacetime metric forming a spacetime distortion, called warp bubble, such that it generates an expansion behind the distortion and a contraction in front of it. That makes it possible for a massive particle to be in a local sub-luminal speed inside the bubble, as required by special relativity, whereas outside the particle is propelled at superluminal speeds. Therefore, employing of this spacetime distortion a massive particle travels between two points in spacetime with apparent time less than the time a light particle with zero mass would require to travel between the same two points.

The WD metric, as originally proposed by Alcubierre, is based on a general metric that uses the 3+1 formalism. It consists of a boost in the direction of one of the spatial coordinates, described by the shift vector, a function of spacetime coordinates and the product of two functions, the velocity of the center of the warp bubble and the shape function that controls the shape of the bubble. Although
Alcubierre did not solve the Einstein equations with his proposed spacetime geometry, he gave an example of a shape function in the form of a hat function. One of the main caveats of the Alcubierre WD spacetime is the requirement of negative mass-energy density and the violation of the dominant and weak energy conditions.

Ford and Roman derived quantum inequalities for free massless quantized scalar fields, electromagnetic fields, and massive scalar fields in four-dimensional Minkowski spacetime and concluded that these constraints have the form of an uncertainty principle limitation on the magnitude and duration of negative mass-energy densities and that the exotic solutions of Einstein equations such as wormholes and WD would have significant limitations in their viability. Pfenning and Ford followed this last work and calculated the upper bound limits necessary for the WD viability, concluding then that the energy required to create a warp bubble is ten orders of magnitude greater than the total mass of the entire visible universe, also negative.

Krasnikov approached the hyperfast interstellar travel problem in general relativity by discussing the possibility of whether or not a mass particle can reach a remote point in spacetime and return sooner than a photon would. Krasnikov argued that it is not possible for a mass particle to win this race under reasonable assumptions for globally hyperbolic spacetimes. He discussed in detail the specific spacetime topologies, but with the constraint that tachyons would be a requirement for superluminal travel to occur. He conjectured a spacetime modification device that can be used to make superluminal travel possible without tachyons. Such spacetime was named as Krasnikov tube by Everet and Roman, and they generalized the metric proposed by Krasnikov by hypothesizing a tube along the path of the particle connecting Earth to a distant star.

The Krasnikov metric cannot shorten the time for a one-way trip from Earth to a distant star, but it can make the time for a round trip arbitrarily short for clocks on Earth. However, the Everet and Roman extension of the Krasnikov metric has the property that inside the tube the spacetime is flat and the lightcones are opened in such a way that they allow the superluminal travel in one direction. They also mentioned that even though the Krasnikov tube does not involve closed timelike curves it is possible to construct a time machine with a system of two non-overlapping tubes, and demonstrated that the Krasnikov tube also requires thin layers of negative energy density and large total negative energies.

Lobo and Crawford discussed the Krasnikov metric in detail, addressed the violation of the weak energy condition and the viability of superluminal travel. These authors concluded that with the imposition of the weak energy condition the Olum theorem prohibits superluminal travel and pointed out the necessity of further research on spacetimes with closed timelike curves and the need for a precise definition of superluminal travel. They argued that one can construct metrics that allow superluminal travel but are flat Minkowski spacetimes. The quantum inequalities, brought from quantum field theory, were also discussed. Van de Broeck showed
how a minor modification of the Alcubierre WD geometry can reduce the total energy required for the creation of a warp bubble. He presented a modification of the original WD metric where the total negative mass would be of the order of just a few solar masses.

Natário proposed a new version of the WD theory with zero expansion and questioned the effects that occur in the WD in opposition to his newly proposed metric, namely, the zero expansion Natário WD metric. Natário discussed the nature of the WD spacetime symmetries with a series of propositions and corollaries such as the proof that the WD spacetime is flat whenever the tangent vector to the Cauchy surfaces is a Killing vector field for the Euclidean metric despite being time-dependent and as a particular case. In addition, it is also flat wherever the tangent vector is spatially constant. He also showed that nonflat WD spacetime violate both the weak energy condition (WEC) and the strong energy condition (SEC), as well as that the WD metric can be obtained from the Natário metric by a particular choice of coordinates. The spherical coordinates were the choice of charts for the zero expansion of the Natário WD spacetime.

Lobo and Visser pointed out that the WD theory is an example of reverse engineering of the solutions of the Einstein equations where one defines a specific spacetime metric and then one finds the matter distribution responsible for the respective geometry. These authors verified that the class of WD spacetimes necessarily violate the classical energy conditions even for low warp bubble velocity. Hence, this is the case of a geometric choice and not of superluminal properties. They proposed a more realistic WD by applying linearized gravity to the weak WD with nonrelativistic warp bubble velocities and argued that for the Alcubierre WD and its version proposed by Natário, the center of the bubble must be massless. They found that even for low velocities the negative energy stored in the warp fields must be a significant fraction of the particle’s mass at the center of the warp bubble.

White described how a warp field interferometer could be implemented at the Advanced Propulsion Physics Laboratory with the help of the original WD ideas. He also pointed out that the expansion behind the warp bubble and contraction in front of it is due to the nature of the WD functions, and that the distortion of spacetime may be interpreted as a kind of Doppler effect or stress and strain on spacetime. Lee and Cleaver argued that external radiation might affect the WD, and that the warp field interferometer proposed by White could not detect spacetime distortions. Mattingly et al. discussed curvature invariants in the Natário WD.

Bobrick and Martire proposed a general WD spacetime that encloses all WD definitions removing any alleged issues with the original Alcubierre WD. They presented a general subluminal model with spherical symmetry and positive energy solutions that satisfies the quantum energy inequalities such that it reduces two orders of magnitude the WD requirement for negative energy density. They claimed that any type of WD, including the original one, is a place of regular or exotic
material moving in inertial form with a certain speed.

Lentz\textsuperscript{18} claimed that the original warp bubble proposed by Alcubierre\textsuperscript{1} can be physically interpreted as hyper-fast gravitational solitons and presented the first solution for superluminal solitons in general relativity satisfying the WEC and the momentum conditions for conventional sources of stress, like energy and momentum, that do not require large amounts of negative energy. The basis for his work is to assume that the shift vector components obey a kind of wave equation giving rise to a positive energy geometry. Fell and Heisenberg\textsuperscript{19} also addressed the WD as gravitational solitons and shed light on the Eulerian energies and their relation to the WEC, raising the possibility for superluminal spacetimes with viable amounts of energy density.

Quarra\textsuperscript{20} established that within the scope of general relativity certain gravitational waveforms can result in geodesics that arrive at distant points earlier than light signals in flat spacetime, and presented an example waveform that can be used to manifest superluminal behavior.

Santiago et al.\textsuperscript{21} claimed that generic WD metrics violate the null energy conditions (NEC) and discussed how Eulerian observers are privileged, meaning that these observers may perceive positive energy densities causing the impression of viable WD. They also argue that for the WD to become a possibility it would require that all timelike observers observe positive energy densities. They stated that any WD spacetime will unavoidably violate the energy conditions. In a subsequent work\textsuperscript{22} they claimed that exotic spacetimes, such as the WD one will always violate energy conditions. They also provided other examples of such spacetimes relating them to wormholes, tractor beams and stress beams.

For a detailed discussion on the basics of WD theory, the reader is referred to Alcubierre and Lobo.\textsuperscript{23} For a nice exposition on WD theory one is referred to the lecture notes of the course given by Shoshany.\textsuperscript{24} It is particularly noteworthy the recent results concerning the possibility of WD features in negative energy density distribution of an experimental Casimir cavity,\textsuperscript{25} results that open the striking possibility of ingenious experimental avenues on how to create the WD phenomena in a laboratory environment.

In this work, we present a summary of the results we have recently obtained when solving the Einstein equations with the assumption of simple fluid distributions embedded in the original Alcubierre WD spacetime geometry. New results such as the vacuum solutions connecting the WD to shock waves via the Burgers equation are presented.\textsuperscript{26} We used the perfect fluid energy-momentum tensor\textsuperscript{27} (EMT) which disclosed new exact solutions of the Einstein equations. We also demonstrated that starting with simple forms of mass and energy sources, vacuum solutions for the WD spacetime arise when we impose that the WEC is trivially satisfied by a choice of gauge on the shift vector and the spacetime coordinates.\textsuperscript{27} Further results of the Alcubierre WD metric endowed with a charged dust EMT that considers an electromagnetic tensor in curved spacetime with the cosmological constant are
Finally, adding the cosmological constant to the Einstein equations coupled with the Alcubierre WD spacetime having perfect fluid as a source has the effect of leading the energy density to possibly becoming positive depending on both the value and sign of the cosmological constant.

The paper is organized as follows. Section 2 reviews very briefly the Alcubierre WD metric. Section 3 depicts the perfect and anisotropic fluid cases and Section 4 discusses the relationship between the WD spacetime, vacuum solutions, and shock waves described by the Burgers equations. Section 5 presents a discussion on the role of the cosmological constant in the Einstein equation solutions for the WD spacetime. Section 6 presents the conclusions.

2. The Alcubierre Warp Drive Spacetime Metric

Alcubierre originally used the Arnowitt-Deser-Misner (ADM) approach of general relativity, a formalism where the spacetime is described by a foliation of space-like hypersurfaces of constant time coordinate. The general form of the WD metric in this formalism is described by the following equation,

\[ ds^2 = -d\tau^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\left(\alpha^2 - \beta_i \beta^i\right) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j, \]

where \( d\tau \) is the lapse of proper time, \( \alpha \) is the lapse function, \( \beta^i \) is the space-like shift vector, and \( \gamma_{ij} \) is the spatial metric for the hypersurfaces. The Greek indices range from 0 to 3, whereas the Latin ones indicate the space-like hypersurfaces and range from 1 to 3. The lapse function \( \alpha \) and the shift vector \( \beta^i \) are functions to be determined, \( \gamma_{ij} \) is a positive-definite metric on each of the space-like hypersurfaces, for all values of time, a feature that makes the spacetime globally hyperbolic. The lapse function \( d\tau \) is a measure of proper time between two adjacent hypersurfaces by observers moving along the normal direction to the hypersurfaces (Eulerian observers). The shift vector is a tangent vector to the hypersurfaces that relates the spatial coordinate systems on different hypersurfaces.

Alcubierre assumed the following ad hoc particular choices for the parameters of the general ADM metric,

\[ \alpha = 1, \]
\[ \beta^1 = -v_s(t) f \left[ r_s(t) \right], \]
\[ \beta^2 = \beta^3 = 0, \]
\[ \gamma_{ij} = \delta_{ij}. \]

Rewriting Eq. (1) with these definitions yields the Alcubierre WD spacetime metric below,

\[ ds^2 = -\left(1 - \beta^2\right) dt^2 - \beta dx dt + dx^2 + dy^2 + dz^2. \]
3. Einstein Equations Solutions for Simple Fluids

We to find solutions to the Einstein equations with the Alcubierre WD metric as the underlying geometry by coupling simple matter-energy distributions, namely the dust of particles, the perfect fluid and the anisotropic fluid with off-diagonal metric terms to see if those EMTs could lead to a viable WD bubble.

Starting with incoherent fluid, or dust, vacuum solutions of the Einstein equations for the WD metric were recovered, which led to the connection of the WD with shock waves via Burgers equation. Vacuum solutions mean that all the energy conditions are trivially satisfied, but the possibility of vacuum shock waves that could physically represent warp bubbles indicated that other known matter-energy distributions could result in more complex solutions of Einstein equations. Under this perception we proposed two other energy-momentum tensors, the perfect and “parametrized” fluids, where the latter is, in fact, an anisotropic fluid with heat flux.

The EMT for an anisotropic and dissipative fluid may be written by the following equation,

\[ T_{\alpha\beta} = \mu u^\alpha u^\beta + p h_{\alpha\beta} + u^\alpha q^\beta + u^\beta q^\alpha + \pi^{\alpha\beta}, \]

(7)

where

\[ h_{\alpha\beta} = g_{\alpha\beta} + u_a u_b \]

(8)

projects tensors onto hypersurfaces orthogonal to \( u^\alpha \), \( \mu \) is the matter density, \( p \) is the fluid static pressure, \( q^\alpha \) is the heat flux vector and \( \pi^{\alpha\beta} \) is the viscous shear tensor. The world lines of the fluid elements are the integral curves of the four-velocity vector \( u^\alpha \). The heat flux vector and the viscous shear tensor are transverse to the world lines, that is,

\[ q_a u^a = 0, \quad \text{and} \quad \pi_{ab} u^b = 0. \]

(9)

For simplicity we depicted the EMT of the anisotropic fluid in the following matrix form,

\[ T_{\alpha\sigma} = \begin{pmatrix} \mu + \beta^2 p & -\beta D & 0 & 0 \\ -\beta D & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & C \end{pmatrix}, \]

(10)

where \( p, A, B \) and \( C \) are anisotropic static pressures, \( D \) is the momentum density parameter that represents the heat flux for this fluid, and \( \mu \) is the particles’ matter density. Notice that if we choose all pressures and momentum density as being
equal to the static pressure $p$ we recover the perfect fluid form shown below,

$$
T_{\alpha\sigma} = \begin{pmatrix}
\mu + \beta^2 p & -\beta p & 0 & 0 \\
-\beta p & p & 0 & 0 \\
0 & 0 & p & 0 \\
0 & 0 & 0 & p
\end{pmatrix},
$$

(11)

The EMT for a perfect fluid can be written in terms of the tensor notation as follows,

$$
T_{\alpha\beta} = (\mu + p) \ u_\alpha u_\beta + p \ g_{\alpha\beta},
$$

(12)

where $\mu$ is the matter density, $p$ is the fluid static pressure, $g_{\alpha\beta}$ is the metric tensor and $u_\alpha$ is the 4-velocity of an observer inside the fluid. Perfect fluids have no shear stress, rotation, heat conduction or viscosity. To recover the EMT for dust of particles, we have to choose the static fluid pressure to be zero, and the matrix form for the dust is given by the following equation,

$$
T_{\alpha\sigma} = \begin{pmatrix}
\mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

(13)

which can be written in tensor notation as follows,

$$
T_{\alpha\beta} = \mu \ u_\alpha u_\beta,
$$

(14)

where $\mu$ is the matter density represented by a scalar function of the spacetime coordinates, and $u^\alpha$ are the observers 4-velocity components.

Table 1 lists the solutions of the Einstein equations for the anisotropic fluid defined by Eq. (10), where the two solutions subsets 1b and 2b are considered unphysical because the resulting EMT leads to the anisotropic static pressures $A$ and $D$ equal to zero, whereas $B = C$. The energy density in the $T_{00}$ component of the EMT is equal to zero, resulting in the following equation relating the matter density $\mu$, the shift vector $\beta$ and the static pressure $p$,

$$
\mu = -\beta^2 p.
$$

(15)

We must point out that we made a notational change on the shift vector definition $\beta^1$ so that it has the negative sign of the original parameter defined by Alcubierre, yielding the shift vector $\beta$ used in here as being given as below,

$$
\beta = -\beta^1 = u_\alpha(t)f[r_\alpha(t)].
$$

(16)

Finally, one should also notice that solutions 1b and 2b brought about the following Burgers equation in dissipative form,

$$
\frac{\partial}{\partial x} \left[ \frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (\beta^2) \right] = -64\pi B.
$$

(17)
Table 1: WD solutions of the Einstein equations with anisotropic fluid as source.

| Case | Conditions | Results |
|------|------------|---------|
| 1) $\partial \beta / \partial z = 0$ | 1a) $\partial \beta / \partial x = 0$ | $\mu = \beta^2(2D - A - p) + A/3$  
$\beta = \beta(t, y)$  
$B = -C = A/3$  
$\left( \frac{\partial \beta}{\partial y} \right)^2 = 32\pi C$  
$\partial^2 \beta / \partial y^2 = 16\pi \beta(A - D)$ |
| 1) $\partial \beta / \partial z = 0$ | 1b) $\partial \beta / \partial y = 0$ | $\mu = -\beta^2 p$  
$\beta = \beta(t, x)$  
$B = C$  
$A = D = 0$  
$\partial \left[ \partial \beta / \partial t + \frac{1}{2} \partial / \partial x (\beta^2) \right] = -64\pi B$  
$\to$ solution dismissed as unphysical |
| 2) $\partial \beta / \partial y = 0$ | 2a) $\partial \beta / \partial x = 0$ | $\mu = \beta^2(2D - A - p) + A/3$  
$\beta = \beta(t, z)$  
$B = -C = A/3$  
$\left( \frac{\partial \beta}{\partial z} \right)^2 = 32\pi C$  
$\partial^2 \beta / \partial z^2 = 16\pi \beta(A - D)$ |
| 2) $\partial \beta / \partial y = 0$ | 2b) $\partial \beta / \partial z = 0$ | $\mu = -\beta^2 p$  
$\beta = \beta(t, x)$  
$B = C$  
$A = D = 0$  
$\partial \left[ \partial \beta / \partial t + \frac{1}{2} \partial / \partial x (\beta^2) \right] = -64\pi B$  
$\to$ solution dismissed as unphysical |
Perfect fluid solutions have equations of state given by the expression below:

\[ p = p(\mu) = (\gamma - 1)\mu, \quad (18) \]

where \( \gamma = \) is a constant which for ordinary fluids can be approximated by \( 1 \leq \gamma \leq 2 \). The incoherent matter, or dust, corresponds to \( \gamma = 1 \), and radiation corresponds to \( \gamma = \frac{4}{3} \). For solutions 1b and 2b in Table 1 it is clear that \( \gamma \) is a function of the shift vector given by the following equation,

\[ \gamma = 1 - \frac{1}{\beta^2}. \quad (19) \]

Notice that since the shift vector is a function of the warp velocity and it is also the regulating function, we are facing a problem with a discontinuity when \( \beta = 0 \) since outside the warp bubble the regulating function approaches to zero, but inside the warp bubble \( f(r_s) = 1 \). So Eq. (19) takes the following form,

\[ \gamma = 1 - \frac{1}{v_s^2}, \quad (20) \]

where \( v_s = v_s(t) \) is the warp bubble velocity. Solutions 1a and 2a shown in Table 1 are very similar, except that for solution 1a the shift vector is a function of spacetime coordinates \( (t, y) \), whereas solution 2a is a function of \( (t, z) \) coordinates. However, in both of these cases, the shift vector may be a complex-valued function depending on the values of the static pressure \( C \). For these solutions the equations of state relating the matter density \( \mu \) and the fluid pressures are identical and given by the following expression,

\[ \mu = \beta^2(2D - A - p) + \frac{A}{3}. \quad (21) \]

The perfect fluid solutions listed in Table 2 are a special case of the anisotropic fluid if we chose all pressures to be equal to \( p \). Table 3 presents the special dust case (\( p = 0 \)), which leads to vacuum.

4. Warp Drive and Shock Waves

In previous studies\(^{26,27}\) it was found an intrinsic relationship between the WD concept and shock waves via the Burgers equation arising from vacuum solutions of Einstein equations for the Alcubierre WD metric. Such a result may mean that the warp bubble could be interpreted as shock waves in vacuum. In other words, the Burgers equation can be seen as a vacuum solution of Einstein equations connecting the Alcubierre WD metric to shock waves.

In the case of the anisotropic fluid, we considered as unphysical the solution which led to a Burgers equation\(^{17}\) because the parameter values resulted in an EMT with only two static pressures (diagonal terms) and no energy density. Nevertheless, for the perfect fluid and dust EMTs we found the Burgers equation
Table 2: WD solutions of the Einstein equations with perfect fluid as source.

| Case | Condition | Results |
|------|-----------|---------|
| 1)  | $\frac{\partial \beta}{\partial z} = 0$ | $p = 3\mu$  
$\beta = \beta(y, t)$  
$\frac{\partial \beta}{\partial y} = \pm \sqrt{-32\pi \mu}$ |
|  | 1a) $\frac{\partial \beta}{\partial x} = 0$ | $\beta = \beta(y, t)$  
$\frac{\partial \beta}{\partial y} = \pm \sqrt{-32\pi \mu}$ |
|  | 1b) $\frac{\partial \beta}{\partial y} = 0$ | $p = 3\mu = 0$  
$\beta = \beta(x, t)$  
$\frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(\beta^2) = h(t)$ |
| 2)  | $\frac{\partial \beta}{\partial y} = 0$ | $p = 3\mu$  
$\beta = \beta(z, t)$  
$\frac{\partial \beta}{\partial z} = \pm \sqrt{-32\pi \mu}$  
$\frac{\partial \beta}{\partial z} = \pm \sqrt{\pm 96\pi \mu}$ |
|  | 2a) $\frac{\partial \beta}{\partial x} = 0$ | $\beta = \beta(x, t)$  
$\frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(\beta^2) = h(t)$ |
|  | 2b) $\frac{\partial \beta}{\partial z} = 0$ | $p = 3\mu = 0$  
$\beta = \beta(x, t)$  
$\frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(\beta^2) = h(t)$ |

Table 3: WD solutions of the Einstein equations with dust source (leads to vacuum).

| Case | Consequence | Results |
|------|-------------|---------|
| 1)  | $\frac{\partial \beta}{\partial z} = 0$  
$\frac{\partial \beta}{\partial y} = 0$ | $\mu = 0$  
$\beta = \beta(t, x)$  
$\frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(\beta^2) = h(t)$ |
| 2)  | $\frac{\partial \beta}{\partial y} = 0$  
$\frac{\partial \beta}{\partial z} = 0$ | $\mu = 0$  
$\beta = \beta(t, x)$  
$\frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(\beta^2) = h(t)$ |

in its viscous form describing a dissipative system with the right-hand side being a
function of time coordinate only, according to the expression below,
\[ \frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (\beta^2) = h(t). \]  
(22)

Here \( h = h(t) \) is an arbitrary function to be determined by the boundary conditions. If \( h(t) = 0 \) Eq. (22) takes its homogeneous form describing a conservative system known as inviscid Burgers equation.

The Burgers or Bateman-Burgers equation is a well-known partial differential equation that models several physical systems such as gas dynamics, traffic flows, and even stock market, since, for the latter application, it is connected to the symmetries of the Black-Scholes equation. Its most notorious system description is, however, the phenomena arising from conservation laws and formation of shock waves, that is, discontinuities that appears after a finite time and then propagates in a regularly. The physics behind the field solutions of the Burgers equation can be seen as a current density.

The WD is depicted by the shift vector \( \beta = v_s(t)f(r_s) \), where \( v_s \) is the bubble velocity, \( f(r_s) \) is the regulating function of the warp bubble shape and the inviscid Burgers in Eq. (22) can be interpreted and representing a conservation law for this current density. Analyzing each term of Eq. (22), the first one on the l.h.s., \( \frac{\partial \beta}{\partial t} \) can be interpreted as a force per unit mass, i.e., the time derivative of momentum. The second term on the l.h.s., \( \frac{1}{2} \frac{\partial}{\partial x} (\beta^2) \), can be interpreted as the divergence of the total energy, which is entirely kinetic. Physically the result can be understood by considering the WD metric as conservation of both energy and momentum in the direction of the wave propagation. Calculating the divergence of the parametrized, or anisotropic, perfect fluid energy-momentum tensor, and demanding that it should be zero, one arrives at the following system.

\[ \begin{align*}
- \frac{\partial \beta}{\partial x} (D + \mu) - \frac{\partial \mu}{\partial t} - \beta \left( \frac{\partial D}{\partial x} + \frac{\partial \mu}{\partial x} + \frac{\partial \beta}{\partial t} (2p + A - 3D) \right) \\
+ \beta^2 \left[ \frac{\partial D}{\partial t} - \frac{\partial p}{\partial t} + 3 \frac{\beta}{\partial x} (D - p) \right] + \beta^3 \left( \frac{\partial D}{\partial x} - \frac{\partial \mu}{\partial x} \right) = 0, \\
\frac{\partial A}{\partial x} + \frac{\partial \beta}{\partial t} (D - A) + \beta \left[ 3 \frac{\beta}{\partial x} (D - A) + \frac{\partial D}{\partial x} - \frac{\partial A}{\partial t} \right] \\
+ \beta^2 \left( \frac{\partial D}{\partial x} - \frac{\partial A}{\partial x} \right) = 0,
\end{align*} \]

(23)

\[ \begin{align*}
\frac{\partial B}{\partial y} + \beta \frac{\partial \beta}{\partial y} (D - A) = 0, \\
\frac{\partial C}{\partial z} + \beta \frac{\partial \beta}{\partial z} (D - A) = 0.
\end{align*} \]

(24)

(25)

(26)

The perfect fluid zero divergence is recovered by letting
\[ p = A = B = C = D, \]
(27)
hence, Eqs. (23) to (26) become,
\[ -\frac{\partial \mu}{\partial t} - \beta \left( \frac{\partial p}{\partial x} + \frac{\partial \mu}{\partial x} \right) = 0, \]  
(28)
\[ \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0, \]  
(29)
which means that the static pressure \( p \) of the perfect fluid does not depend on the spatial coordinates, and Eq. (28) reduces to the expression below,
\[ \frac{\partial \mu}{\partial t} + \beta \frac{\partial \mu}{\partial x} = 0, \]  
(30)
which is a continuity equation, \( \mu \) is the fluid density, and \( \beta \) is the flow velocity vector field. Notice that for constant density the fluid has an incompressible flow. Therefore, all partial derivatives of \( \beta \) in terms of the spatial coordinates vanish, and the flow velocity vector field has zero divergence, this being a classical fluid dynamics scenario. The local bubble volume expansion rate is zero, and the WD metric becomes the Minkowski metric in this scenario.

If we want to find these results for the dust of particles as a particular case, we can see that the partial time derivative of the matter-density is zero and we have
\[ \mu \frac{\partial \beta}{\partial x} = 0, \]  
(31)
which is immediately satisfied since the matter density is zero for the dust EMT and the Einstein equations solution for the WD is the vacuum one. This is also true for the sets of solutions 1b and 2b found for the perfect fluid, as shown in Table 2. This result for the vacuum solutions of the Einstein equations with the WD metric as a background suggests that the necessary energy to create the associate shock wave is purely geometrical. It is interesting evidence that the WD metric can be understood as a spacetime motion equivalent to a shock wave moving in a fluid, where spacetime itself plays the role of the fluid.

Considering all these results one can speculate about vacuum energy, quantum fluctuations and dark matter as fuel for the feasibility of the warp bubble, but one has to overcome possible difficulties caused by the event horizon of the bubble that would require particles with imaginary mass like tachyons.23

5. The Cosmological Constant

Bearing in mind the results above led to the next step of adding the cosmological constant in the Einstein equation to seek solutions for Alcubierre WD spacetime geometry.29 The motivation came from Eq. (19) in the original Alcubierre work1 which was obtained by applying the WEC considering Eulerian observers, contracting the EMT and writing the final expression \( G_{\mu \nu} = \kappa T_{\mu \nu} \) in the Einstein equation, whose result reads as follows,
\[ T^{\mu \nu} u_\mu u_\nu = \alpha^2 T^{00} = G^{00} = -\frac{v^2 \rho^2}{4r_s^2} \left( \frac{df}{dr_s} \right)^2. \]  
(32)
Here $u^\mu$ and $u_\mu$ are respectively the contravariant and covariant components of the four-vector velocity of the Eulerian observers given by

$$u^\mu = \frac{1}{\alpha} (1, \beta^1, \beta^2, \beta^3), \quad u_\mu = -(\alpha, 0, 0, 0),$$

where $\alpha$ is the lapse time between the constant hypersurfaces of the ADM-formalism and $\beta^i$ are the three-components of the shift vector $\beta$. Using the original parameters from the $3 + 1$-formalism, its general metric and the chosen chart of coordinates we can write Eq. (32) as below,

$$T_{\alpha\beta} u^\alpha u^\beta = -\frac{1}{4} \left[ \left( \frac{\partial \beta}{\partial y} \right)^2 + \left( \frac{\partial \beta}{\partial z} \right)^2 \right].$$

The above expression is clearly a non-positive term everywhere, violating the WEC in a nontrivial way. We then showed that if we begin with simple energy and momentum sources and impose Eq. (34) to be as identically zero we arrive at vacuum solutions for the WD spacetime. However, if we include the cosmological constant in the Einstein equations the expression is changed, yielding,

$$T_{\alpha\beta} u^\alpha u^\beta = \Lambda -\frac{1}{4} \left[ \left( \frac{\partial \beta}{\partial y} \right)^2 + \left( \frac{\partial \beta}{\partial z} \right)^2 \right].$$

For positive and large enough values of the cosmological constant, it may be possible to construct a WD that does not violate the WEC, because the original negative mass-energy density necessary to create the warp bubble could become positive.

The solutions for the WD metric which include the cosmological constant in the Einstein equations are presented in Table 4. Cases 1b and 2b are just the vacuum solutions connecting the WD spacetime to shock waves via the Burgers equations, and they also require a vanishing cosmological constant, since both the static fluid pressure ($p$) and the matter density ($\mu$) also vanish. The other two solutions subsets 1a and 2a are also similar to the equally labeled ones in Table 2, but now they include the cosmological constant and the solutions are no longer required to be complex if the following conditions are satisfied,

$$\Lambda - 8\pi \mu \geq 0,$$

$$\Lambda - 8\pi p \geq 0.$$  

Taking all these results into account imply that creating a warp bubble for superluminal travel of massive particles seems to require more complex forms of matter than the dust for stable solutions, and that the requirement of negative mass-energy may not be as strict as originally thought for superluminal spacetime travel with warp speeds. The shift vector in the direction of the warp bubble movement creates a coupling in Einstein equations that requires the off-diagonal source terms in the energy-momentum tensor that represent momentum densities.

Previous authors investigated the WD theory from the geometrical point of view and came up with other types of WD mechanics, but it now seems clear...
that it is also necessary to consider both the energy and momentum tensors to propose a superluminal propelling system in the WD theory.

Table 4: WD solutions of the Einstein equations with Λ and perfect fluid.

| Case | Condition | Results |
|------|------------|---------|
| 1) $\frac{\partial \beta}{\partial z} = 0$ | 1a) $\frac{\partial \beta}{\partial x} = 0$ | $\Lambda = 6\pi \left(\mu - \frac{\mu}{4}\right)$
$\beta = \beta(y,t)$
$\frac{\partial \beta}{\partial y} = \pm \sqrt{4(\Lambda - 8\pi\mu)}$
$\frac{\partial \beta}{\partial y} = \pm \sqrt{\frac{4}{3}(\Lambda - 8\pi p)}$
$\beta \frac{\partial \mu}{\partial x} + \frac{\partial \mu}{\partial t} = 0$ (null divergence) |
| | 1b) $\frac{\partial \beta}{\partial y} = 0$ | $\Lambda = 8\pi\mu = 8\pi p = 0$
$\beta = \beta(x,t)$
$\frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(\beta^2) = h(t)$
Null divergence is trivially satisfied
This is the solution found in Ref.26 |
| 2) $\frac{\partial \beta}{\partial y} = 0$ | 2a) $\frac{\partial \beta}{\partial x} = 0$ | $\Lambda = 6\pi \left(\mu - \frac{\mu}{4}\right)$
$\beta = \beta(y,t)$
$\frac{\partial \beta}{\partial z} = \pm \sqrt{4(\Lambda - 8\pi\mu)}$
$\frac{\partial \beta}{\partial z} = \pm \sqrt{\frac{4}{3}(\Lambda - 8\pi p)}$
$\beta \frac{\partial \mu}{\partial x} + \frac{\partial \mu}{\partial t} = 0$ (null divergence) |
| | 2b) $\frac{\partial \beta}{\partial z} = 0$ | $\Lambda = 8\pi\mu = 8\pi p = 0$
$\beta = \beta(x,t)$
$\frac{\partial \beta}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(\beta^2) = h(t)$
Null divergence is trivially satisfied
This is the solution found in Ref.26 |
6. Conclusions

In this work, we described new exact solutions of the Einstein equations endowed with the Alcubierre warp drive (WD) spacetime geometry having simple energy-momentum tensor (EMT) distributions of matter and energy. The incoherent matter, or dust, the simplest fluid source EMT used with the WD metric, led to vacuum solutions and the weak energy condition being satisfied with null energy density. In addition, we showed that the dynamics of the vacuum WD spacetime are governed by the Burgers equation, which links the WD bubble to shock waves. Both the perfect fluid EMT and an anisotropic fluid recover the Burgers equation as vacuum solutions, but some of these new sets of solutions were considered unphysical, and others showed that the shift vector can be a complex-valued function. The Einstein equations with the cosmological constant were used with the WD metric having the perfect fluid as the source, whose results showed that if the cosmological constant is big enough it would be possible to attain positive energy density for the WD spacetime. This result is in line with recent research activity in WD theory by other authors, who provide examples of different WD spacetimes geometries, which suggests that it may be possible to create warp bubbles with positive energy densities.

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