Projective non-Abelian Statistics of Dislocation Defects in a $\mathbb{Z}_N$ Rotor Model

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Non-Abelian statistics is a phenomenon of topologically protected non-Abelian Berry phases as we exchange quasiparticle excitations. In this paper, we construct a $\mathbb{Z}_N$ rotor model that realizes a self-dual $\mathbb{Z}_N$ Abelian gauge theory. We find that lattice dislocation defects in the model produce topologically protected degeneracy. Even though dislocations are not quasiparticle excitations, they resemble non-Abelian anyons with quantum dimension $\sqrt{N}$. Exchanging dislocations can produces topologically protected projective non-Abelian Berry phases. The dislocations, as projective non-Abelian anyons can be viewed as a generalization of the Majorana zero modes.

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Introduction— Searching for Majorana fermions (or more precisely, Majorana zero modes) in condensed matter systems have attracted increasing research interests recently. But what really is the Majorana zero mode? In fact, the so called “Majorana zero mode” is actually a phenomenon of topologically protected degeneracy in the presence of certain topological defects (such as vortices in 2D $p_x + i p_y$ superconductors). In the race for finding Majorana zero modes, much attention has been paid to the fermion systems. However, the boson/spin systems also have topologically protected degeneracies, which may also be ascribed to Majorana zero modes or their generalizations.

An 1D example of emergent Majorana zero modes in the spin system arises from the transverse field Ising chain, whose ground state degeneracy in the ferromagnetic phase can be viewed as the Majorana zero modes at both ends of the chain. A 2D example is found in the toric code model, where lattice dislocations are braided and fused as if they were Majorana zero modes which resemble non-Abelian anyons of quantum dimension $\sqrt{2}$. The toric code model can be generalized to a $\mathbb{Z}_N$ rotor model, whose low energy effective theory is a self-dual $\mathbb{Z}_N$ gauge theory. In this paper, we study the topologically protected degeneracy associated with the extrinsic topological defects, namely lattice dislocations in the $\mathbb{Z}_N$ rotor model, and found that these defects are of quantum dimension $\sqrt{N}$, which can be viewed as a generalization of the “Majorana zero mode”. Braiding topological defects with protected degeneracy will lead to topologically protected projective non-Abelian Berry phase, which may allow us to perform decoherence free quantum computations.

We like to remark that the dislocations in our $\mathbb{Z}_N$ rotor model are not non-Abelian anyons, since the non-Abelian anyons must be excitations of the Hamiltonian, while the dislocations are not the excitations in this sense. The dislocations do not really carry non-Abelian statistics since the non-Abelian Berry phase from exchanging dislocations is topologically protected only up to a total phase. We say the dislocations carry a projective non-Abelian statistics. Another example of projective non-Abelian statistics for dislocations in fractional quantum Hall states on lattice can be found in Ref. 31.

$\mathbb{Z}_N$ plaquette model— The $\mathbb{Z}_N$ plaquette model is a rotor model on a two-dimensional square lattice (see Fig. 1). On each site $i$, define a $\mathbb{Z}_N$ rotor with $N$ basis states $|m_i\rangle$, labeled by the angular momentum $m_i = 0, 1, \cdots, (N-1)$. For each rotor, introduce $U_i$ to measure the angular momentum by $U_i |m_i\rangle = e^{i\theta_N m_i}|m_i\rangle$ with $\theta_N \equiv 2\pi/N$, and $V_i$ to lower the angular momentum by one $|m_i\rangle \equiv |(m_i - 1)_{\text{mod } N}\rangle$. Both $U_i$ and $V_i$ are unitary operators $U_i U_i = |V_i V_i = 1$, satisfying $V_i U_i = e^{i\theta_N \delta_{i,\nu}} U_i V_i$.

The $\mathbb{Z}_N$ plaquette model is given by the Hamiltonian

$$H = - \sum_p O_p + \text{h.c.,}$$

where the operator $O_p$ describes a kind of ring coupling among the rotors on the corner sites of each plaquette $p$,

$$O_p = \prod_{i = 1}^{3} U_i V_i U_i^\dagger V_i^\dagger.$$

Here we adopt the graphical representation for the operators: $U_i = \mathcal{M}_i, V_i = \mathcal{P}_i, U_i^\dagger = \mathcal{M}_i, V_i^\dagger = \mathcal{P}_i$, by drawing directed string represent a product of $U_i$ and/or $V_i$ operators on the sites along the string. The operator on each site is specified by the string direction (see text).
rected strings going through the site. Because these operators only connect diagonal plaquettes, a string starting from the even plaquette will never enter the odd plaquette (and vice versa). So we can locally distinguish two different types of strings: e-string (m-string) if it lives in the even (odd) plaquettes (see Fig. 1). The assignment of even/odd to the plaquettes can be reversed under the translation of one lattice spacing, so the interchange of e- and m-strings could be realized by the curvature of the lattice as will be seen later.

The $Z_N$ plaquette model Eq. (1) is exact solvable, as evidenced from the commutation relation $[O_p, O_{p'}] = 0$, as $O_p O_{p'} = e^{iN q_p} e^{iN q_{p'}} = O_{p'} O_p$ for adjacent $p$ and $p'$, where the overlay of strings indicates the ordering of the operators, such as $V_i U_i = \mathcal{X}$ and $U_i V_i = \mathcal{X}$, with the algebra $\mathcal{X} = e^{i\theta_N} \mathcal{X}$.

Every $O_p$ operator has $N$ distinct eigenvalues $e^{i\theta_N q_p}$ labeled by $q_p = 0, 1, \cdots, (N - 1)$, as inferred from the fact that $O_p^N = 1$. The energy will be minimized if all $O_p$'s take the eigenvalue 1 ($q_p = 0$). Therefore the ground states are the common eigenstates that satisfy $O_p|\text{grnd}\rangle = |\text{grnd}\rangle$ for all $p$'s, and is free of any $Z_N$ charges.

Intrinsic anyon excitations— The excited states can be obtained by applying open string operators to the ground state, which create opposite $Z_N$ charge excitations in pairs at both ends of the string. These excitations and can be detected by the close string operator (like $O_p$) surrounding them in the counterclockwise direction. Take $S$ in Fig. 1 for example, $O_p S|\text{grnd}\rangle = 0 \cdots |\text{grnd}\rangle = e^{i\theta_N} \cdots |\text{grnd}\rangle$, showing that a charge $q_p = +1$ is created at the end of the open string by the action of $S$. One can show that the opposite charge $q_p = -1$ is created at the other end.

Because $Z_N$ charge excitations are the ends of open strings, their statistics are inherited from the algebra of the string operators. According to $\mathcal{X} = e^{i\theta_N} \mathcal{X}$, braiding a $q_e$ e-charge with a $q_m$ m-charge would acquire a phase $\exp(\theta_N / 2 q_e q_m)$. In this sense, these excitations are Abelian anyons. However we must stress that these anyons are intrinsic, as they are collective motions of rotors, described by the excited state within the rotor Hilbert space. This is to be distinguished from the extrinsic anyons introduced later as lattice dislocations, which does not belong to the rotor Hilbert space. Note that both the phase $\exp(\theta_N / 2 q_e q_m)$ and the excitation energy are invariant under the exchange of $e$ and $m$. This manifests the self-duality of the $Z_N$ plaquette model, and can be realized by lattice translation.

Ground state degeneracy— The degeneracy of the ground states of the $Z_N$ plaquette model depends on the topology of the lattice. Let us consider the torus topology by setting the model on a $L_x \times L_y$ sized lattice with periodic boundary condition in both directions. The total number of states is $N^{N_{\text{site}}}$, with $N_{\text{site}} = L_x L_y$ being the number of sites. To count the ground states, we note that they are constrained by $\forall p : O_p = 1$. Consider a particular $O_p$ operator and the subspaces labeled by its different eigenvalues. Those subspaces all have the same dimension, because any open string operator that ends in the plaquette $p$ can be used to perform a unitary transform that rotates these subspaces into each other. So each time imposing $O_p = 1$ on a particular plaquette will reduce the available Hilbert space dimension by a factor of $N$. However the $O_p$ operators are not independent. Because e-charges (m-charges) are created in opposite pairs, summing over the lattice, e-charges and m-charges must be neutralized respectively, i.e. $\prod_{p \in \text{even}} O_p = \prod_{p \in \text{odd}} O_p = 1$. This is true on an even\times even lattice (i.e. both $L_x$, $L_y$ are even), which reduces the number of independent $O_p$ constraints to $(N_{\text{plaq}} - 2)$, with $N_{\text{plaq}} = L_x L_y$ being the number of plaquettes. So after restricting the full Hilbert space to the ground state subspace, the remaining dimension is $N^{N_{\text{site}} - N_{\text{plaq}} + 1} = N^2$, meaning the ground state degeneracy of the $Z_N$ plaquette model is $N^2$ on the even\times even lattice.

However for the even\times odd or odd\times odd lattices (i.e. $L_x$ or $L_y$ is odd), e-string and m-string can be continued into each other by going along the odd direction, thus e-charge and m-charge are made identical. So they are no longer required to be neutralized separately, but only neutralized as a whole. Therefore we only have one relation $\prod_{p} O_p = 1$, which reduces the number of independent $O_p$ constraints to $(N_{\text{plaq}} - 1)$, and the resulting ground state degeneracy will be $N^{N_{\text{site}} - N_{\text{plaq}} + 1} = N$.

To summarize, the ground state degeneracy of the $Z_N$ plaquette model on a torus follows from the general formula

$$GSD = N^{N_{\text{site}} - N_{\text{plaq}}},$$

where $N$ denotes the number of species of the intrinsic excitations that are supported by the lattice topology. On the even\times even lattice, we have totally $N = N^2$ distinct excitations by combination of e- and m-charges. When it comes to the even\times odd or odd\times odd lattice, e- and m-charges are no longer distinct, and the number of excitation species is reduced to $N = N$. The topological order in the ground state is now evidenced from the protected ground state degeneracy on torus,\cite{10,11} and from the dependence of the ground state degeneracy on the parity of the lattice periodicity.

Dislocations— One can change the lattice periodicity by first generating a pair of edge dislocations with opposite unit length Burger’s vectors, and moving them in the direction perpendicular to their Burger’s vectors all the way around the lattice, then annihilating them as they meet again at the periodic boundary. During this process, the ground state degeneracy must have changed. This motivates us to introduce dislocations as shown in
Fig. 2(a) to probe the topological order by looking at the degeneracy associated to them. With dislocations, one can no longer globally color the plaquettes consistently. *Branch cuts* must be left behind between each pairs of dislocations. Going around a dislocation exchange the \( e \)- and \( m \)-charges, as \( e \)- and \( m \)-strings are transmuted into each other across the branch cut. The self-duality is made explicit by dislocations.

In the presence of dislocations, the \( \mathbb{Z}_N \) plaquette model is still defined by the Hamiltonian in Eq. (1), with the same ring operator \( O_p \) in Eq. (2) for quadrangular plaquettes (including those on the branch cuts). Only around the pentagonal plaquettes (at the dislocations), the ring operator \( O_p \) should be redefined as

\[
O_p = -e^{i \frac{\pi}{N}} \mathbf{5} \equiv -e^{i \frac{\pi}{N}} U_1 V_2 U_3 V_4 U_5 V_6. \tag{4}
\]

The phase factor \(-e^{i \pi/N/2}\) is to guarantee that \( O_p^N \equiv 1 \) holds for the pentagonal plaquette as well. The pentagonal ring operator \( O_p \) commutes with all the other ring operators, so the exact solvability of the model is preserved. The ground states are again common eigenstates of \( \forall \rho : O_\rho |\text{grnd}\rangle = |\text{grnd}\rangle \). The dislocations are topological defects that do not belong to the model Hilbert space. To distinguish from those *intrinsic* \( \mathbb{Z}_N \) charges, we will call the dislocations as the *extrinsic* defects.

With the branch cuts, \( e \)-charge and \( m \)-charge are indistinguishable, so the species of intrinsic excitations count to \( \mathcal{N} = N \). According to Eq. (3), the ground state degeneracy will be given by \( N^{\mathcal{N}_{\text{cuts}} + \mathcal{N}_{\text{plaq}} + 1} \) in general. To count the number of sites and plaquettes, we first establish a correspondence between them by mapping each plaquette to its bottom-left corner site, as indicated by the arrows in Fig. 2(b). Between a pair of dislocations, only one of them will hold a site that has no plaquette correspondence (see Fig. 2(b)), so the introduction of every pair of dislocations will give rise to one extra site (with respect to the number of plaquettes). Therefore if there are \( n \) dislocations on the lattice, there will be \( N_{\text{site}} - N_{\text{plaq}} = n/2 \) more sites than plaquettes, and the ground state degeneracy of the \( \mathbb{Z}_N \) plaquette model will be \( \text{GSD} = N^{n/2+1} \).

This ground state degeneracy is topologically protected indeed. To better understand the topology, we start from the even\( \times \)even periodic lattice without dislocations, i.e. a torus with no branch cut. In this case, the \( e \)-strings and \( m \)-strings are distinct, and can never be deformed into each other, as if they were living on two different layers of the torus. So the topological space is the disjoint union of two separate torus. Introducing a pair of dislocations, the two layers will be connected: strings on one layer can be carried on into the other layer through the branch cut. So the topological space becomes a doubled torus under the diffeomorphism[32] as shown in Fig. 3.

All the operators that act within the ground state subspace are closed-string (cycle) operators, as they commute with the Hamiltonian. Note that the contractable cycles act trivially (as \( O_p = 1 \)). Only non-contractable cycles can be used to label the different ground states and to perform unitary transforms among them. On the double torus topology as in Fig. 4(a) one can specify 4 non-contractable cycles: \( C_{ex}, C_{ey}, C_{mx}, C_{my} \), as the canonical homology basis. Their operator forms are given explicitly according to their graphical representations depicted in Fig. 2(a). We now study the representation of these cycle operators in the ground state subspace. First we find the following commutation relations \( [C_{ex}, C_{ey}] = [C_{mx}, C_{my}] = [C_{ex}, C_{mx}] = [C_{ey}, C_{my}] = 0 \), and two independent algebras \( C_{ey} C_{mx} = e^{i \theta N} C_{mx} C_{ey}, C_{my} C_{ex} = e^{i \theta N} C_{ex} C_{my} \). Each algebra requires an \( N \)-dimensional representation space, so the 4 cycle operators...
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