SOLUTION TO THE
GRAD-SHAFRANOV
BOUNDARY VALUE PROBLEM
FOR A THERMONUCLEAR PLASMA
CONTAINED IN A TOROIDAL VASE
WITH A CONDUCTING WALL
UNDER THE ASSUMPTION OF
CONSTANT SOURCES TO THE
EQUILIBRIUM IN FLUX SPACE BY
THE METHOD OF EXPANSION OF
THE POLOIDAL FLUX FUNCTION IN
SERIES OF MULTIPOLE SOLUTIONS

by

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Abstract

A method is proposed to solve the Grad-Shafranov partial differential equation for the poloidal flux function associated with the equilibrium of a plasma magnetically confined in an axisymmetric torus under the assumption that the sources to the equilibrium (the gradient of the plasma pressure and the gradient of half the squared toroidal field function in flux space) are constant, and subjected to the condition of constant value of the poloidal flux at the toroidal boundary. The solution to the equation is written as the sum of the particular solution and a combination of \( N \) multipole solutions of the lowest orders in the variables of the toroidal-polar coordinate system having the pole at the centre of the torus cross section. The term of the multipole solution of the zeroth order does not bring about any magnetic field and its inclusion in the combination fulfills the purpose of providing it with a constant term whose value can be adjusted so as to make equal to zero the value taken by the poloidal flux at the boundary; the term of the multipole solution of the first order is associated with a (constant) vertical field, such as required for the attainment of the balance of forces in a confinement system of toroidal geometry, and has the effect of introducing a shift (the Shafranov shift) to the line of null magnetic poloidal field inside the plasma with regard to the centre line of the torus; the terms containing the multipole solutions of higher orders tailor the contours of the flux surfaces, modifying the shape they would take from the sole presence of the particular solution. Imposition of the boundary condition on the assumed solution leads to a
system of linear equations for the constants multiplying the multipole solutions, in which the driving terms are derived from the expression for the particular solution at the boundary. The number of equations exceeds the number of unknowns in this system, and, in principle, it admits of no solution. Relaxation of the strict terms of the condition of nullity and application of the criterium that the magnitude of the poloidal flux at the boundary be the least possible decides which of the equations are to be abandoned in order that a solution can be extracted from the set of the remaining ones. To make the Shafranov shift appear as a parameter in the final expression of the flux function and to have it included in the solving process since the first stages of its development, an equation, obtained by setting equal to zero the gradient of the poloidal flux function on the equator line of the torus cross section, in which the radial coordinate is identified with that of the magnetic axis (and thus with the measure of the Shafranov shift), is then added to the set of equations. The information concerning the equilibrium sources is made present in the system so constituted only through a parameter measuring the ratio these sources keep between themselves, which one is formally treated as an unknown while the Shafranov shift itself is considered to be a given quantity. The coefficients of the $N$ multipole solutions in the proposed solution to the Grad-Shafranov boundary value problem follow from the solution to this truncated and enlarged system of algebraic equations for which the matrix of coefficients is a known function of the inverse aspect ratio and of the Shafranov shift. The Shafranov shift ultimately appears as the solution of an isolated algebraic equation whose coefficients are functions of the squared inverse aspect ratio and whose independent term is the ratio of the equilibrium sources. This equation can be seen, conversely, as an explicit definition of the ratio between the equilibrium sources in terms of the values ascribed to the inverse aspect ratio and to the Shafranov shift. The poloidal flux function that results from the application of the method is expressed in terms of elementary functions only and satisfies exactly the Grad-Shafranov equation while satisfying the boundary condition in an approximate way. For the purpose of illustration solutions are derived for three configurations as defined by the signals and strengths of the equilibrium sources, respectively paramagnetic, diamagnetic and magnetically neutral. It is concluded that the error carried by the expression found for the poloidal flux function in general decreases as the number $N$ of multipole solutions entering its composition is increased. Other findings are: 1) there is a non-null value of the Shafranov shift even for a paramagnetic configuration of vanishingly small plasma pressure; 2) the magnetic axis is always placed between the centre line of the cross section and the outer edge of the torus, irrespective of the magnetic character of the configuration; 3) although the outermost magnetic surface shows to become a separatrix for a sufficiently high value of the Shafranov shift, all indications are that, by increasing the number $N$ of multipole solutions contained in the representation of the poloidal flux function, it reassumes the shape of a surface that is topologically equivalent to that of a torus.
I. INTRODUCTION

The mathematical tool of commonest use to investigate the plasma equilibrium in the toroidal pinch (the tokamak) is the equation that came to be known as the Grad-Shafranov equation, a second order partial differential equation of the elliptic type for the flux associated with the poloidal magnetic field, which can be solved under proper boundary conditions when the plasma pressure function and the toroidal field function (the sources to the poloidal flux) are specified in flux space.

In vector notation, the Grad-Shafranov writes as [1]:

\[ R^2 \nabla \cdot \left( \frac{1}{R^2} \nabla \Psi \right) = -\mu_0 R^2 \frac{dp}{d\Psi} - I \frac{dI}{d\Psi} \] (1.1)

where \( \Psi \) is the poloidal field magnetic flux divided by \( 2\pi \); \( p = p(\Psi) \) is the plasma pressure; \( I = I(\Psi) \) is the toroidal field function, also called poloidal current function, defined as:

\[ I \equiv RB_\phi, \] (1.2)

\( B_\phi \) being the toroidal magnetic field; \( R \) is the distance from the axis of rotational symmetry of the physical system to the point where the poloidal flux function is \( \Psi \); \( \mu_0 \) is the absolute permeability of vacuum.

An exact analytic solution in closed form to the Grad-Shafranov equation under the double assumption of a constant pressure gradient and a constant squared toroidal field function gradient in flux space, which has the effect of suppressing any dependence of the terms in the equation coming from the poloidal field sources on the poloidal flux function, has been presented in Ref. [2]. This solution, however, does not fit the geometry of any boundary we may expect to find in laboratory devices. In Ref. [3] the form assumed for the source functions in flux space is such that the terms they contribute to the Grad-Shafranov equation are proportional to the flux function. A perturbative solution is obtained by developing an expansion of the
poloidal flux function in powers of the reciprocal of the aspect ratio (\(\epsilon\)) of the torus about the corresponding cylindrical configuration (that is, the one with vanishing \(\epsilon\)), which satisfies the requirement that the normal component of the magnetic field vanishes at the wall of the plasma container, supposed to be made of a perfectly conducting material. These are examples of solutions to a Grad-Shafranov differential equation which, because of the postulations regarding the form of dependence of the source functions on the flux function, is linear. We have no knowing of references in the literature on plasma equilibrium that report the use of Fourier’s method to solve the boundary value problem constituted by any of the linear versions that can be given to the Grad-Shafranov equation and a set of local conditions on the solution, possibly because the equation has been found to be separable in none of the coordinate systems of common knowledge in which one family of coordinate surfaces comprises a surface that fits the toroidal boundary of the physical system. Regarding the nonlinear problem, the usual procedure consists in resorting first to the expansion of the source functions in Taylor series in \(\Psi\) about the magnetic axis where the flux function is assigned an arbitrary value \(\Psi = \Psi_M\); the solution to the equation at a point is then obtained in the form of a power series of the distance from the considered point to the magnetic axis with coefficients that are functions of the poloidal angle. The domain of validity of such a solution is restricted to the neighborhood of the magnetic axis and this makes it unsuitable to supporting the imposition of boundary conditions.

In the present paper we consider the equilibrium of a plasma enclosed in a toroidal chamber as a boundary value problem. The equilibrium is supposed to be governed by the Grad-Shafranov equation; the plasma pressure gradient and the gradient of half the squared toroidal field function are assumed to obey a flat profile in flux space; the wall of the containing vase, with which the plasma is in contact, to be made of a
perfectly conducting material. We have already observed that for a Grad-Shafranov equation shaped by identical hypotheses concerning the sources of equilibrium the solution presented in Ref. [2] is not able to meet the boundary conditions that appertain to any of the physical situations currently found in practice, and, in particular, to the one we propound to consider here.

The method we shall develop in the following pages takes as “particular” a solution to the equation whose set of level curves on a plane $\phi = \text{constant}$, analogously to that representing the Solovev’s, does not include any that fits the contour of the torus cross section; to get a solution to the equation that also satisfies the boundary condition, we add to it a “complementary” solution, which we build as a linear combination of functions belonging to the infinite set of those that solve the sourceless Grad-Shafranov equation. These functions, usually referred to in the literature of Plasma Physics by the name of multipole solutions, are available to us thanks to the methods expounded in Ref. [4].

This procedure of building the “complete” solution is permissible, of course, because the problem is linear; also because the multipole equation and the homogeneous equation associated with the version of the Grad-Shafranov’s for the case, under present consideration, of constant sources to the equilibrium in flux space, are the same.

The equilibrium of a plasma contained in a vase with a metallic lining on the internal side of its wall leads to just one boundary condition, and this is that the poloidal flux function remains constant when its coordinate variables are made to describe the surface of the wall. By applying this sole requirement to the complete solution we expect to be able to determine the whole set of constants that multiply the functions of reference participating in the constitution of the complementary solution.
In fact it is impossible to have it satisfied in the strictest terms. As imposed on the complete solution, the boundary condition translates by a set of constraints on the constants, and, since the number of constraints exceeds that of the constants, there is no possible choice for the constants that would make them capable to satisfy simultaneously all the constraints in the set – the form proposed for the solution, which does indeed solve the equation, seems to suffer, after all, from a prohibitive inconsistency with the boundary condition. In an effort to save that form for the advantages that it however presents, we accede to satisfy, not all of the constraints, but the largest possible number of them – and thus to give only an approximate solution to the boundary value problem. The criterium to decide which constraints are to be kept and which are to be abandoned in fixing the values of the constants is that the abandoned be the ones within the whole set that, in remaining unsatisfied, give margin to the arising of only the least possible error at the boundary (that is, the smallest departure of the flux function from a constant value thereat).

A thorough understanding of the last statement passes through the recognition that the need of using only a finite number of multipole solutions in the construction of the flux function is an inherent feature of the method, and this not solely because of the lack of a general expression that would give a unified representation to the infinite set of multipole solutions, but, more significantly than that, because of the very *modus operandi* of the mathematical machinery of the method, which involves the resolution of algebraic equations of high degrees and the evaluation of functional determinants of high orders. We have found nonetheless that the error at the boundary (and by consequence at all internal points) is consistently diminished as the number of multipole solutions that are included in the complementary solution is increased, suggesting that in the limit of infinitely many ones we should have generated a convergent series in which each term would satisfy the differential equa-
tion separately and whose sum at the boundary would reproduce the exact value the flux function is to take there. A theorem regarding the existence and uniqueness of the solution to a Grad-Shafranov boundary value problem of the same kind as that which is considered in the present paper \cite{5} makes us then be sure that the solution we have found is the only possible one.

This paper is organized as follows. Section II is devoted to introducing a convenient normalization of the physical and geometrical quantities that enter the formulation of the problem, and to state it in precise mathematical terms; the coordinate system of election is the toroidal-polar one. In Section III the particular solution to the Grad-Shafranov equation with sources to the equilibrium that are constant in flux space is derived by the method of expansion of the flux function in series of Chebyshev polynomials of the cosine of the polar angle. Section IV is the core of the paper, the one in which the method of solution to the Grad-Shafranov boundary value problem for constant equilibrium sources by series of multipole solutions is developed. Section V states certain conditions on the values that can be assumed by the Shafranov shift (or, rather, by a related quantity) and by the poloidal flux function at the magnetic axis in order that the solutions provided by the method describe true equilibrium configurations (in the physical sense) and not merely field lines structurings which might disregard some obvious requirements for plasma confinement; it also discusses the complex question of the appearance of branching points on the plasma bounding surface in connection with the limits of validity of the method, and concludes by defining the quantities appropriate to evaluate the accurateness of the approximate solutions that can be obtained through the use of it. The method is then applied in Section VI to solve the Grad-Shafranov boundary value problem for a number of equilibrium situations by using a combination of the least possible number of multipole solutions, which is three. It is solved again in
Section VII under the same physical terms of the previous Section by considering a combination of four multipole solutions, for the uses of an odd number and of an even number of multipole solutions imply a difference in the structures of an auxiliary function (called equilibrium function) that participates in the solving process, and there is need to show that this difference is not a source of incongruities between solutions to the same problem as obtained with distinct compositions of the complementary solution to the equation. Section VIII is devoted to expounding some general conclusions regarding the mathematical structures of the flux function and of some auxiliary functions that are required for the derivation of the former, and to discussing the accuracy that can be gained with the increase of the number of multipole solutions that are made to enter the combination aimed at representing the flux function. In Section IX a few equilibrium configurations characterized by having representative values of the equilibrium sources are described by using the solutions for the flux function obtained with the help of the method. Finally, Section X recapitulates the main results and findings of the paper.

II. STATEMENT OF THE PROBLEM

The solutions of the Grad-Shafranov equation in which we shall be interested in this paper are those that give the flux surfaces in the plasma region as nested tori, that is to say, a surface inside the other, the innermost one degenerating into a line, the magnetic axis, where the poloidal field vanishes. We place the pole of the toroidal-polar coordinate system on a meridian plane at the centre of the torus cross section, which, designated by the letter $C$ in Fig. 2.1, is located at the distance $R_C$ from the axis of rotational symmetry of the physical system (in Fig. 1 in Ref. [4] the pole of the toroidal-polar coordinate system on a meridian plane is a point arbitrarily
chosen, which we have designated by the letter $A$ and located at a distance $R_A$ from the axis of rotational symmetry. In borrowing the mathematical expressions, as they become required by the needs of the present paper, from the collections appearing in Ref. [4], we shall, therefore, be implicitly taking $A$ as $C$ and identifying $R_A$ with $R_C$.) In Fig. 2.1 we have also represented the intersection of the magnetic axis with the meridian plane as a point that we have denoted by $M$ and placed at a distance $R_M$ from the axis of rotational symmetry. The distance that separates the magnetic axis from the centre line of the torus is $\Delta$. The coordinates of a generic point $P$ in this coordinate system are $r$, $\theta$ and $\phi$, as shown in Fig. 2.1 of the present paper and also in Fig. 1 in Ref. [4], and then its distance to the axis of rotational symmetry is given by:

$$R = R_C + r \cos \theta . \quad (2.1)$$

**FIG. 2.1** The toroidal-polar coordinate system $(r, \theta, \Phi)$ for which the pole $C$ is located at the intersection of the torus centre line with the meridian plane coinciding with the plane of the figure. The intersection of the magnetic axis with the meridian plane is the point denoted by $M$. The distance from the axis of rotational symmetry $Oz$ to $C$ is $R_C$ and that to $M$ is $R_M$. The Shafranov shift for the configuration represented in the figure is $\Delta = R_M - R_C$. 

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We also introduce here the notation $B_{\phi C}$ for the strength of the toroidal magnetic field at the torus centre line.

To write the plasma equilibrium equation in dimensionless form, we introduce the normalized radial coordinate, the normalized flux, the normalized plasma pressure and the normalized toroidal field function according to the definitions respectively:

\begin{align*}
x & \equiv \frac{r}{R_C}, \\
\psi & \equiv \frac{\Psi}{B_{\phi C} R_C^2}, \\
\hat{p} & \equiv \frac{p}{B_{\phi C}^2 / 2 \mu_0}, \quad \text{and} \\
\hat{I} & \equiv \frac{I}{R_C B_{\phi C}}.
\end{align*}

Taking $\mu \equiv \cos \theta$ as variable in place of the polar angle $\theta$, Eq. (1.1) then writes in the toroidal-polar coordinate system as:

\begin{equation}
x^2 (1 + x \mu) \frac{\partial^2 \psi}{\partial x^2} + x \frac{\partial \psi}{\partial x} + (1 + x \mu)(1 - \mu^2) \frac{\partial^2 \psi}{\partial \mu^2} - (x + \mu) \frac{\partial \psi}{\partial \mu} = x^2 (1 + x \mu) \left[ (1 + x \mu)^2 s_P + s_I \right],
\end{equation}

where

\begin{equation}
s_P \equiv - \frac{d}{d\psi} \left( \frac{\hat{p}}{2} \right)
\end{equation}

and

\begin{equation}
s_I \equiv - \frac{d}{d\psi} \left( \frac{\hat{I}^2}{2} \right)
\end{equation}

are the sources to the equilibrium, which ones we shall consider in this paper to be constant.

We shall assume that the plasma is in contact with the wall of the containing vessel, which is made of a perfectly conducting material\footnote{It may seem physically artificial to assume that the plasma is in contact with the wall of the enclosing shell. As an alternative situation of more realistic appeal, which still preserves the terms in which the problem is formulated, it can be imagined that the plasma is surrounded by vacuum.} Thus there is no magnetic
field perpendicular to the internal surface of the wall, the field lines run tangent to it, and the plasma-wall interface coincides with a plasma flux surface:

\[ \psi(\text{plasma boundary}) = \text{constant}. \quad (2.9) \]

Since the flux function is undetermined by a constant, we are free to choose the constant in Eq. (2.9) as we please, and, following the common practice, which will prove in the course of future developments to be indeed the most convenient for our purposes, we take it to be zero. We shall see in later Sections that with such a choice the solutions for the flux function can assume only one sign in the plasma region, either positive or negative, being this no more than a matter of working preference, but cannot change sign.

In which regards the interior of the conducting wall and the outer side of the containing vessel, the magnetic field is assumed to be zero. The supposition that a perfectly conducting wall is placed at the plasma boundary has to be followed physically by the recognition that a surface current is induced at the plasma-wall interface by the fields arising from the currents flowing in the bulk of the plasma, and that this surface current in its turn creates a magnetic field which cancels the former ones in the interior of the conducting wall and in the outside of the container.

With this we may state the problem as represented by Eq. (2.6) subjected to the boundary condition of Eq. (2.9); the region of physical interest is restricted to the domain of integration of the partial differential equation. In terms of the variables of the toroidal-polar coordinate system, this region is defined by

\[ 0 \leq x \leq \epsilon, \quad -\pi \leq \theta \leq \pi, \quad (2.10) \]

and that a system of coils applies an external magnetic field with a configuration such designed that it exactly cancels the field the currents circulating in the plasma bulk generate in the outside vacuum region, while \textit{ipso facto} constraining the plasma surface to conform to the shape assumed in the boundary condition stated in Eq. (2.9). See Reference [6].
where
\[ \epsilon \equiv \frac{a}{R_C} \quad (2.11) \]
is the inverse aspect ratio of a torus of minor radius \( a \) and major radius \( R_C \).

**III. DERIVATION OF THE PARTICULAR SOLUTION TO THE GRAD-SHAFRANOV EQUATION WITH CONSTANT EQUILIBRIUM SOURCES**

We shall derive the particular solution to Eq. (2.6) by the method of expansion of the solution in series of Chebyshev polynomials. As a preliminary step to accomplish this end we shall take advantage of the formula that gives the representation of an integral power of a variable \( \mu \) in terms of the Chebyshev polynomials of the first kind \( T_j(\mu) \) in that variable, to know [7]:

\[ \mu^k = \frac{1}{2^{k-1}} \left[ T_k(\mu) + \binom{k}{1} T_{k-2}(\mu) + \binom{k}{2} T_{k-4}(\mu) + \cdots \right] , \quad (3.1) \]

where the series in brackets terminates with \( \binom{k}{m} T_1(\mu) \) for \( k = 2m + 1 \) or with \( \frac{1}{2} \binom{k}{m} T_0(\mu) \) for \( k = 2m \), and reexpress the right hand side of Eq. (2.6) as:

\[ \text{RHS} = (s_P + s_I) T_0(\mu) x^2 + (3s_P + s_I) T_1(\mu) x^3 + 3s_P \left[ \frac{1}{2} T_2(\mu) + \frac{1}{2} T_0(\mu) \right] x^4 \\
+ s_P \left[ \frac{1}{4} T_3(\mu) + \frac{3}{4} T_1(\mu) \right] x^5 . \quad (3.2) \]

In which regards its dependence on the radial variable, we may assume an expression of the form:

\[ \psi_p(x, \mu) = \psi_2(\mu) x^2 + \psi_3(\mu) x^3 + \psi_4(\mu) x^4 \quad (3.3) \]

for the flux function, since the inclusion of powers of \( x \) lower than the second and higher than the fourth would give rise to terms on the left hand side of Eq. (2.6)
that would not be balanced by terms of equal powers of $x$ on the right hand side. It
should be noted that the process of derivation of the solution based on an expression
like the one of Eq. (3.3) will generate not only the particular solution, but also the
multipole solution of the second order (see Eq. (A.8) in Appendix A and Eq. (B.2)
in Appendix B in Ref. [4]), in which we have no present interest, and, to have it
suppressed from the analytical developments, we must choose the arbitrary constant
by which it would otherwise appear multiplied along these to be zero at the step we
first recognize its emergence in the calculations.

We start by inserting Eq. (3.3) in Eq. (2.6). By recalling the definitions of the
operators $T_l$ and $H_l$, given by Eqs. (3.19) and (3.10) respectively in Ref. [4], we can
write the left hand side of Eq. (2.6) as:

$$LHS = \sum_{k=2}^{4} x^k T_k \{ \psi_k(\mu) \} + \sum_{k=3}^{5} x^k H_{k-1} \{ \psi_{k-1}(\mu) \} .$$

We now expand the angular functions $\psi_2(\mu)$, $\psi_3(\mu)$ and $\psi_4(\mu)$ into series of
Chebyshev polynomials $T_n(\mu)$ as:

$$\psi_k(\mu) = \sum_{n=0}^{\infty} C_{k,n} T_n(\mu) , \quad (k = 2, 3, 4)$$

where the coefficients $C_{k,n}$ are constants. By having recourse to the property of the
operator $T_l$ stated in Eq. (3.20) and to those of the operator $H_l$ stated in Eqs. (3.14)
and (3.15) in Ref. [4], we may easily prove that

$$T_k \{ \psi_k(\mu) \} = \sum_{n=0}^{\infty} (k^2 - n^2) C_{k,n} T_n(\mu)$$

and that

$$H_k \{ \psi_k(\mu) \} = \frac{1}{2} (k+1)(k-2) C_{k,1} T_0(\mu) + \left[ k(k-1) C_{k,0} + \frac{1}{2} (k+2)(k-3) C_{k,2} \right] T_1(\mu)
- \frac{1}{2} \sum_{n=2}^{\infty} \left[ (n+k+1)(n-k+2) C_{k,n+1} + (n-k-1)(n+k-2) C_{k,n-1} \right] T_n(\mu) .$$
We next equate the terms of equal powers of \( x \) on the left and right hand sides of Eq. (2.6). From those in \( x^2 \) we obtain:

\[
\begin{align*}
T_2\{\psi_2(\mu)\} &= \sum_{n=0}^{\infty} (4 - n^2)C_{2,n}T_n(\mu) \\
&= (s_P + s_I)T_0(\mu),
\end{align*}
\]

(3.8)

from which we conclude that

\[
C_{2,1} = C_{2,3} = C_{2,4} = \cdots = 0
\]

(3.9)

and

\[
4C_{2,0} = s_P + s_I,
\]

(3.10)

while \( C_{2,2} \) remains indeterminate. It is seen, however, that this constant, which multiplies \( x^2 \) and the Chebyshev polynomial \( T_2(\mu) \) in the expression proposed for the particular solution, is precisely the one that, kept in the subsequent analytical developments, would appear at the end, still indetermined, multiplying the multipole solution of the second order. We thus take \( C_{2,2} \) to be zero, as we have anticipated in the sequel of Eq. (3.3).

From the terms in \( x^3 \) in Eq. (2.6) we get:

\[
T_3\{\psi_3(\mu)\} + H_2\{\psi_2(\mu)\} = (3s_P + s_I)T_1(\mu).
\]

(3.11)

Since all the coefficients \( C_{2,n} \) vanish with the exception of \( C_{2,0} \), Eq. (3.7), with \( k \) taken equal to 2, shows that the second term on the left hand side of Eq. (3.11) consists of a multiple of a single Chebyshev polynomial, \( T_1(\mu) \). Using Eq. (3.6) with \( k = 3 \) for the first term, Eq. (3.11) then becomes:

\[
\sum_{n=0}^{\infty} (9 - n^2)C_{3,n}T_n(\mu) + 2C_{2,0}T_1(\mu) = (3s_P + s_I)T_1(\mu),
\]

(3.12)

from which we obtain:

\[
8C_{3,1} + 2C_{2,0} = 3s_P + s_I,
\]

(3.13)

\[
C_{3,0} = C_{3,2} = C_{3,4} = C_{3,5} = \cdots = 0,
\]

(3.14)
and no relation involving $C_{3,3}$.

The next equation to be considered is the one coming from the equality of the terms in the fourth power of $x$ on both sides of Eq. (2.6). Using the results already known for the coefficients $C_{3,k}$ to reduce the expression for $H_{3}\{\psi_{3}(\mu)\}$, this equation writes as:

$$
\sum_{n=0}^{\infty} (16 - n^2) C_{4,n} T_{n}(\mu) + 2C_{3,1} T_{0}(\mu) - (3C_{3,3} - 3C_{3,1}) T_{2}(\mu) = \frac{3s_p}{2} T_{0}(\mu) + \frac{3s_p}{2} T_{2}(\mu),
$$

(3.15)

and implies the conditions:

$$
16C_{4,0} + 2C_{3,1} = \frac{3s_p}{2},
$$

(3.16)

$$
12C_{4,2} - 3C_{3,3} + 3C_{3,1} = \frac{3s_p}{2},
$$

(3.17)

and

$$
C_{4,1} = C_{4,3} = C_{4,5} = C_{4,6} = \cdots = 0.
$$

(3.18)

We finally arrive at the balance of the terms in $x^5$, the highest power of $x$ to appear in Eq. (2.6). Taking into account the results for the coefficients $C_{4,k}$ expressed in Eq. (3.18), the left hand side of the balancing equation simplifies to:

$$
LHS = H_{4}\{\psi_{\mu}(\mu)\}
$$

$$
= (12C_{4,0} + 3C_{4,2}) T_{1}(\mu) - (4C_{4,4} - 5C_{4,2}) T_{3}(\mu),
$$

(3.19)

while the right hand side is given by:

$$
RHS = \frac{3s_p}{4} T_{1}(\mu) + \frac{s_p}{4} T_{3}(\mu).
$$

(3.20)

We have therefore:

$$
12C_{4,0} + 3C_{4,2} = \frac{3s_p}{4},
$$

(3.21)

$$
4C_{4,4} - 5C_{4,2} = -\frac{s_p}{4}.
$$

(3.22)
and at this last step, in which the operator $T_l$ is not called to a participation, no coefficient $C_{i,j}$ remains indeterminate, and the cycle of equations for them is closed.

Having established which coefficients $C_{k,n}$ of the series in Eq. (3.5) must vanish, we can write the angular functions $\psi_k(\mu)$ ($k = 2, 3, 4$) as finite combinations of Chebyshev polynomials, to know:

$$\psi_2(\mu) = C_{2,0} T_0(\mu), \quad (3.23)$$
$$\psi_3(\mu) = C_{3,1} T_1(\mu) + C_{3,3} T_3(\mu), \quad \text{and} \quad (3.24)$$
$$\psi_4(\mu) = C_{4,0} T_0(\mu) + C_{4,2} T_2(\mu) + C_{4,4} T_4(\mu). \quad (3.25)$$

The six constants $C_{i,j}$ that appear in the above equations can be determined by solving the linear system constituted by Eqs. (3.10), (3.13), (3.16), (3.17), (3.21) and (3.22).

This having been accomplished, we return to Eq. (3.3). Recalling the expressions for the Chebyshev polynomials $T_1(\mu), T_2(\mu), T_3(\mu)$ and $T_4(\mu)$ as given by Eqs. (2.9), (2.21), (4.49) and (4.50) respectively in Ref. [4], we can finally state the particular solution to the Grad-Shafranov equation with constant sources as:

$$\psi_p(x, \mu) = \frac{1}{4}(s_P + s_I)x^2 + \left[\frac{1}{2}(s_P - s_I)\mu + \frac{1}{4}(-s_P + 3s_I)\mu^3\right]x^3 + \left[\frac{1}{4}(s_P - s_I)\mu^2 + \frac{1}{16}(-3s_P + 5s_I)\mu^4\right]x^4. \quad (3.26)$$

An expression alternative to this that takes the form of a linear combination of trigonometric terms can be written down immediately by replacing the Chebyshev polynomials in the angular dependent coefficient functions of the particular solution as they appear defined in Eqs. (3.23) – (3.25) by their trigonometric representations, to know, $T_n(\cos \theta) = \cos n\theta$ ($n = 0, 1, 2, 3, 4$).

Because of the terms in which the problem has been posed, the expression obtained for the flux function as Eq. (3.26) has the form of a linear combination of two
constants \((s_P\) and \(s_I\)) with no independent term. This means that the functional
dependence of the particular solution on the source terms, assumed to be constant
in flux space, is actually given shape singly by the ratio between them, with one of
the constants assuming the role of a mere overall multiplicative factor. In fact this
assertion applies not only to the particular but also to the complete solution, and,
trivial as it seems to be, it warrants the problem to be treated as one of a single
input. We thus divide the expressions we have for the particular solution by one of
the constants, which we choose to be \(s_P\), and pass to consider a newly normalized
flux function, which, in the case of the representation by a finite trigonometric series,
assumes the form:

\[
\bar{\psi}_p(x, \theta) = \frac{1}{4}(1+\lambda)x^2 + \frac{1}{128}(7-\lambda)x^4 + \frac{1}{16}(5+\lambda)x^3 \cos \theta + \frac{1}{32}(1+\lambda)x^4 \cos 2\theta \\
+ \frac{1}{16}(-1 + 3\lambda)x^3 \cos 3\theta + \frac{1}{128}(-3 + 5\lambda)x^4 \cos 4\theta ,
\]

(3.27)

where

\[
\bar{\psi}_p(x, \theta) \equiv \frac{\psi_p(x, \theta)}{s_P},
\]

(3.28)

and \(\lambda\) is the ratio between constants:

\[
\lambda \equiv \frac{s_I}{s_P},
\]

(3.29)

which we shall refer to in future discussions as the *equilibrium parameter*.

In future developments we shall find it convenient to express the normalized
particular solution as:

\[
\bar{\psi}_p(x, \theta) = S_0(x) + S_1(x) \cos \theta + S_2(x) \cos 2\theta + S_3(x) \cos 3\theta + S_4(x) \cos 4\theta ,
\]

(3.30)

and the \(x\)-dependent functions that appear as coefficients in the above representation
as:

\[
S_i(x) = P_i(x) + \lambda Q_i(x) \quad (i = 0, 1, 2, 3, 4).
\]

(3.31)
The explicit forms of the $P_i(x)$’s and $Q_i(x)$’s are collected in Appendix A.

We shall also face the need to have an expression for the particular solution in the cylindrical coordinate system $(R, \phi, z)$ represented in Fig. 1 in Ref. [4]. The conversion of that in terms of the coordinates $r, \theta$ and $\phi$ of the toroidal-polar system we have just derived is most easily accomplished by resorting to the representation it finds as a trigonometrical polynomial in the variable $\mu = \cos \theta$. In this way, by introducing the transformation relations of Eq. (5.21) in Ref. [4] in Eq. (3.26), we obtain:

$$\bar{\psi}_p(\rho, Z) = \frac{1}{16}(1 + \lambda)(\rho^2 - 1)^2 + \frac{1}{4}Z^2[(1 - \lambda)\rho^2 + 2\lambda],$$ (3.32)

where $\rho \equiv R/R_C$ and $Z \equiv z/R_C$.\footnote{Note that the expressions for both the flux function and its gradient – and thus for the magnetic field associated with $\bar{\psi}_p(x, \theta)$ – in the toroidal-polar coordinate system vanish at the point $x = 0$. In cylindrical coordinates the point of concurrent vanishing of these quantities translates as $R = R_C, z = 0$.}

IV. THE PRINCIPLES OF THE METHOD OF SOLUTION TO THE GRAD-SHAFRANOV BOUNDARY VALUE PROBLEM BY SERIES OF MULTIPOLAR SOLUTIONS

This method applies only to the linear version of the problem in which the sources to the equilibrium are constant.

The magnetic poloidal fields that are derived from the flux functions corresponding to the multipole solutions vanish all at a single point which is located by the

\footnote{The Solovev solution, in a form equivalent to that in which it is quoted in Ref. [2], is obtained by combining the particular solution, as given by Eq. (3.32), with the multipole solutions of the three lowest orders: $\bar{\psi}_{Sol}(\rho, Z) = \bar{\psi}_p + K_0\varphi^{(0)}(\rho, Z) + K_1\varphi^{(1)}(\rho, Z) + K_2\varphi^{(2)}(\rho, Z)$, and by taking the value of the constant multiplying the multipole solution of order $n = 2$ as $K_2 = (\lambda - 1)/4$. The values of the constants $K_0$ and $K_1$ remain free, so that they can be chosen as to fix the flux surface where the pressure is zero (the plasma border) and the position of the magnetic axis.}
coordinate $x = 0$, exception being taken to the multipole solution $\varphi^{(0)}(x, \theta)$, the constant one, which gives a null field everywhere in space, and to that of the first order, $\varphi^{(1)}(x, \theta)$, which gives a uniform vertical field. As we have seen in Section III, the poloidal field that is associated with the particular solution to the Grad-Shafranov equation for constant equilibrium sources also vanishes at that point where the multipole fields vanish. Thus the sum of the particular solution $\psi_p(x, \theta)$ and a combination of a number of multipole solutions $\varphi^{(n)}(x, \theta)$ of the orders $n = 0, 2, 3, \ldots$ is a solution to the Grad-Shafranov equation and describes a magnetic configuration that admits of a null poloidal magnetic field at the pole of the toroidal-polar coordinate system, which we have placed at the centre of the torus cross section. The occurrence of a line inside the torus lying on its equatorial plane and extending all the way around the axis of rotational symmetry where the poloidal field vanishes is an essential feature of the equilibrium solutions in which we have a dominant interest, these solutions being the ones that appear as a system of level surfaces encircling a magnetic axis in the usual pictorial representation of a magnetic configuration. It is well known, however, that the toroidal equilibrium of the physical arrangement we are considering is accomplished in general by effect of a shift of this “centre” of the family of nested flux surfaces (id est, the magnetic axis) from the geometrical centre line towards the outer side of the torus [8], usually referred to in the literature of Plasma Physics as the Shafranov shift. In a tokamak experiment this requirement for equilibrium is achieved by applying a vertical field to the plasma, this field being generated either by a set of external coils or by the image current induced by the currents circulating in the plasma bulk on the surface of a perfectly conducting shell placed around the plasma [9]. In the mathematical treatment of the equilibrium problem, taking after the experimental procedure, we shall include the multipole solution of order $n = 1$ in the combination of multipole solutions, such
that, by giving origin to a uniform vertical field, it will produce the desired effect of
shifting the locus of null poloidal field from the centre line of the torus at \( R = R_C \)
to some other circle with centre at the major axis, having the radius larger than \( R_C \), and also lying on the midplane of the torus. The representation we obtain in
this way for the flux function is:

\[
\psi^{(N)}(x, \theta) = \psi_p(x, \theta) + K_0 + \sum_{i=1}^{N-1} K_i \varphi^{(i)}(x, \theta),
\]

(4.1)

where the \( K_i \)'s \((i = 0, 1, 2, \ldots, N - 1)\) are constants, and, consistent with the nor-
malization adopted for the particular solution, we assume the complete solution to
be equally normalized by the constant \( s_P \):

\[
\overline{\psi}(x, \theta) \equiv \frac{\psi(x, \theta)}{s_P}.
\]

(4.2)

We shall refer to the representation of Eq. (4.1), which comprises the multipole
solutions of the \( N \) lowest orders in combination, as the approximation of order \( N \)
to the normalized flux function or the normalized partial flux function of order \( N \).

In general, by the reasons we have discussed in Section I, there is no choice
of the \( N \) constants \( K_0, K_1, \ldots, K_{N-1} \) capable to make the solution stated as Eq.
(4.1) satisfy the requirement that it vanishes at the boundary for any finite \( N \). We
shall show, however, that a criterium can be set up to determine the \( K_i \) constants
\((i = 0, 1, 2, \ldots, N - 1)\) such that the boundary condition \( \overline{\psi}^{(N)} = 0 \), at least for
some equilibria among which are included those of the greatest theoretical and
practical importance, can be approximately satisfied, and that the deviation of the
partial flux function from the null value at the boundary can be reduced to any
prescribed magnitude by increasing the number \( N \) of multipole solutions that enter
the combination of Eq. (4.1) to a sufficiently high value.

We write the multipole solutions in Eq. (4.1) in the form they are stated in Eq.
(A.1) in Ref. [4], here reproduced:

$$\varphi^{(i)}(x, \theta) = \sum_{j=0}^{2i} M_{ji}(x) \cos j\theta , \quad (4.3)$$

and then, by interchanging the order of summations over the indices $i$ and $j$, the double sum into which the last term on the right hand side is thereby converted can be brought to assume the following form:

$$\sum_{i=1}^{N-1} K_i \varphi^{(i)}(x, \theta) = \sum_{n=1}^{2N-2} \left[ \sum_{i=1}^{N-1} K_i M_{ni}(x) \right] \cos n\theta . \quad (4.4)$$

Writing further the particular solution as in Eq. (3.30) we are conducted to reexpress Eq. (4.1) as:

$$\psi^{(N)}(x, \theta) = G_0^{(N)}(x) + \sum_{n=1}^{2N-2} G_n^{(N)}(x) \cos n\theta . \quad (4.5)$$

In this representation of the normalized partial flux of order $N$ the angle independent term is given by:

$$G_0^{(N)}(x) = S_0(x) + K_0 + \sum_{i=1}^{N-1} M_{0i}(x) K_i , \quad (4.6)$$

where, as in Eq. (3.30), we have written $S_0(x)$ for $S_0(\lambda; x)$; the coefficients of the harmonics of the poloidal angle can be conveniently represented by a single formula:

$$G_n^{(N)}(x) = S_n(x) + \sum_{i=1}^{N-1} M_{ni}(x) K_i \quad (n = 1, 2, \ldots, 2N - 2) , \quad (4.7)$$

whose generality of scope in terms of the index $n$ presupposes that the following definitions apply to the two sets of functions $S_n(x) \equiv S_n(\lambda; x)$ and $M_{ni}(x)$ entering its composition:

$$S_n(x) = 0 \quad \text{for} \quad n \geq 5 \quad (4.8)$$

and

$$M_{ni}(x) = 0 \quad \text{for} \quad n > 2i \quad , \quad (4.9)$$
respectively, in compliance, the first with the expression for the particular solution in Eq. (3.30), and the second with that of the multipole solutions in Eq. (4.3).

Now, at the boundary, which is defined in terms of the normalized radial coordinate by \( x = \epsilon \), the normalized partial flux must vanish:

\[
\bar{\psi}^{(N)}(\epsilon, \theta) = 0 , \tag{4.10}
\]
a requirement that can be satisfied only if

\[
G_{0}^{(N)}(\epsilon) = G_{1}^{(N)}(\epsilon) = \cdots = G_{2N-2}^{(N)}(\epsilon) = 0 . \tag{4.11}
\]

Considering the expressions for \( G_{0}(x) \) and for \( G_{n}^{(N)}(x) \) \((n = 1, 2, \ldots, 2N - 2)\) as given by Eq. (4.6) and by Eq. (4.7) respectively, this last condition translates by:

\[
\begin{align*}
K_{0} + M_{01}(\epsilon)K_{1} + M_{02}(\epsilon)K_{2} + \cdots + M_{0,N-1}(\epsilon)K_{N-1} &= -S_{0}(\epsilon) \tag{4.12} \\
M_{11}(\epsilon)K_{1} + M_{12}(\epsilon)K_{2} + \cdots + M_{1,N-1}(\epsilon)K_{N-1} &= -S_{1}(\epsilon) \\
M_{21}(\epsilon)K_{1} + M_{22}(\epsilon)K_{2} + \cdots + M_{2,N-1}(\epsilon)K_{N-1} &= -S_{2}(\epsilon) \\
& \vdots \vdots \vdots \\
M_{N-1,1}(\epsilon)K_{1} + M_{N-1,2}(\epsilon)K_{2} + \cdots + M_{N-1,N-1}(\epsilon)K_{N-1} &= -S_{N-1}(\epsilon) \\
M_{N,1}(\epsilon)K_{1} + M_{N,2}(\epsilon)K_{2} + \cdots + M_{N,N-1}(\epsilon)K_{N-1} &= -S_{N}(\epsilon) \\
& \vdots \vdots \vdots \\
M_{2N-2,1}(\epsilon)K_{1} + M_{2N-2,2}(\epsilon)K_{2} + \cdots + M_{2N-2,N-1}(\epsilon)K_{N-1} &= -S_{2N-2}(\epsilon),
\end{align*}
\]

a system of linear algebraic equations for the constants \( K_{0}, K_{1}, K_{2}, \ldots, K_{N-1} \). The constant \( K_{0} \) may be considered separately from the set of the remaining ones since it appears solely in the equation generated by the condition imposed on \( G_{0}^{(N)}(\epsilon) \), and its value is accordingly to be determined from Eq. (4.12) in isolation once the constants \( K_{1}, K_{2}, \ldots, K_{N-1} \) have been found by solving the system of equations associated
with Eq. (4.13). This decoupling of the equation for $K_0$ from the equations for the other constants is of course an aspect of the arbitrariness of the value ascribable to the flux function at the boundary.

Since the $K$’s in Eq. (4.13) count as $N - 1$ and the equations as $2N - 2$, the system we have in hand comprehends $N - 1$ equations in excess to the number of unknowns, and we may expect it to be incompatible, that is to say, not admitting of a solution. Of course, this is just a manifestation of the inability of the coordinate system we have adopted in fulfilling the purpose of serving as supporting frame to the generation of a reference set of solutions to the Grad-Shafranov equation that be capable to accommodate themselves to the shape of the given boundary.

Now, from the general expressions for the coefficients of the harmonics that combine to make up the multipole solutions, given by Eqs. (A.2) and (A.4) in Ref. [4], it is apparent that, for a fixed first suffix $i$, the dominant coefficient is that for which the second suffix is $j = i$, and such that

$$M_{ii}(\epsilon) \sim \epsilon^i.$$  \hspace{1cm} (4.14)

Thus, considered two any equations in the set defined by Eq. (4.13), for the same departures of the values respective to the constants $K_1, K_2, \ldots, K_{N-1}$ from those by which they can be both simultaneously satisfied, the smallest between the errors that will result for the values of the terms on the right hand side of one and the other equations will be associated with the equation of the highest order (that is to say, the equation having the largest suffix $i$ for the coefficients $M_{ij}, j = 1, 2, \ldots, N - 1$). In this trend of the error to diminish with the increasing of the order of the equation we find the possibility of extracting a meaningful information concerning the boundary value problem for the flux function from the system represented by Eq. (4.13). Indeed, if we shall abandon the equations of the highest orders in this system, we
can arrive at a reduced system in which the number of equations will equal the
number of unknowns, which will be solvable, and whose solution for the coefficients
\( K_i \) when substituted in Eq. (4.1) will cause the annihilation at the boundary of
those trigonometric components of the flux function having the largest amplitudes.
In this way we shall have an exact solution to the partial differential equation that
will satisfy approximately the condition at the boundary; the error it will carry
along the frontier of the domain of integration will be given by the sum of the
trigonometrical components of the partial flux function whose amplitudes will not
have contributed to the generation of equations in the system for the constants
\( K_i \), the leading term of which will be dominated by a factor having the form of
the inverse aspect ratio raised to a power equal to the lowest of the orders of the
equations that will have been neglected in the system.

We thus disregard the equations of the orders \( i = N \) to \( i = 2N - 2 \), which
are below the dashed line in Eq. (4.13), and consider those of the orders \( i = 1 \)
to \( i = N - 1 \) as the ones that contain the profitable information concerning the
constants \( K_1, K_2, \ldots, K_{N-1} \). For the system of \( N - 1 \) equations and \( N - 1 \) unknowns
to which Eq. (4.13) is in this way reduced, the determinant is:

\[
D_A^{(N)}(\epsilon) = \begin{vmatrix}
M_{11}(\epsilon) & M_{12}(\epsilon) & \cdots & M_{1,N-1}(\epsilon) \\
M_{21}(\epsilon) & M_{22}(\epsilon) & \cdots & M_{2,N-1}(\epsilon) \\
\vdots & \vdots & \ddots & \vdots \\
M_{N-1,1}(\epsilon) & M_{N-1,2}(\epsilon) & \cdots & M_{N-1,N-1}(\epsilon)
\end{vmatrix},
\tag{4.15}
\]

and the solution, given by Cramer’s rule, is:

\[
K_j^{(N)} = \frac{\begin{vmatrix}
M_{11}(\epsilon) & \cdots & M_{1,j-1}(\epsilon) & -S_1(\lambda; \epsilon) & M_{1,j+1}(\epsilon) & \cdots & M_{1,N-1}(\epsilon) \\
M_{21}(\epsilon) & \cdots & M_{2,j-1}(\epsilon) & -S_2(\lambda; \epsilon) & M_{2,j+1}(\epsilon) & \cdots & M_{2,N-1}(\epsilon) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
M_{N-1,1}(\epsilon) & \cdots & M_{N-1,j-1}(\epsilon) & -S_{N-1}(\lambda; \epsilon) & M_{N-1,j+1}(\epsilon) & \cdots & M_{N-1,N-1}(\epsilon)
\end{vmatrix}}{D_A^{(N)}(\epsilon)},
\tag{4.16}
\]
(j = 1, 2, \ldots, N − 1), where the superscripts attached to symbols (here $D_A^{(N)}(\epsilon)$ and $K_j^{(N)}$) in general denote the number of multipole solutions that enter the evaluation of the quantities the symbols represent through the expression adopted for the partial flux function. The integer corresponding to the order of the column constituted by the driving terms to the equations being solved simultaneously (put within a frame) in the determinant appearing as the numerator of the expression for $K_j^{(N)}$ according to Cramer’s equals the suffix $j$ of this constant. Note that the constants $K_j^{(N)}$, as given by Eq. (4.16), depend linearly on $\lambda$ through the dependence the functions $S_n(\lambda; \epsilon)$ keep on this last quantity.

Besides being calculable by means of Eq. (4.12) it is possible to give $K_0$ an expression that does not presume the knowledge of the constants $K_1, K_2, \ldots, K_{N−1}$. Indeed, if we shall add Eq. (4.12) as first equation to the ensemble made up by the $N−1$ equations of the lowest orders in Eq. (4.13), we shall have a composite system for which the determinant will still be given by Eq. (4.15), since its first column will be constituted by unity for the top element and by zero for all the ones below it; the solution for $K_0$ will be written as:

$$K_0^{(N)} = \frac{-S_0(\lambda; \epsilon) M_{01}(\epsilon) \cdots M_{0,N−1}(\epsilon)}{D_A^{(N)}(\epsilon)} \frac{-S_1(\lambda; \epsilon) M_{11}(\epsilon) \cdots M_{1,N−1}(\epsilon)}{D_A^{(N)}(\epsilon)} \cdots \frac{-S_{N−1}(\lambda; \epsilon) M_{N−1,1}(\epsilon) \cdots M_{N−1,N−1}(\epsilon)}{D_A^{(N)}(\epsilon)}.$$  

(4.17)

When the expressions for $\bar{\psi}_p(x, \theta)$, for $K_0$ and for $K_i$ (i = 1, 2, \ldots, N − 1), as given by Eqs. (3.27), (4.17) and (4.16) respectively, are used in Eq. (4.11), the expression that will result for the flux function will display an explicit dependence on $\lambda$, which is the input parameter of the problem (besides the one having a purely geometrical meaning, $\epsilon$). From a physical standpoint, however, a parameter having
a greater interest than \( \lambda \) (and one that can always be associated with a configuration irrespectively of the shapes of the profiles that can be assumed for the sources to the equilibrium in flux space) is the Shafranov shift \( \Delta \). This appears as one of the equilibrium quantities that can be determined once the expression for the flux function is known. We shall show next, however, that it is also possible to conduct the solving process to the boundary value problem in a way that the Shafranov shift is made present since the very first stages of it, and such that the expression it generates for the flux function contains an explicit reference to \( \Delta \) (or rather to a normalized version of \( \Delta \)) instead of to \( \lambda \).

In the magnetic configurations in which we have interest the innermost of the flux surfaces degenerates into a line, the magnetic axis, which, because of the symmetry up-down of the physical system with regard to the equatorial plane of the torus, is always located on that plane. The field lines on any flux surface intercept the equatorial plane at right angles, and, correspondingly, the radial component of the magnetic poloidal field, as evaluated with the help of the expression for the flux function, must vanish at all points lying on that plane between the inner and the outer edge of the torus. This condition, which translates by

\[
\frac{\partial \psi}{\partial \theta}(x, \theta = 0 \text{ or } \pi) = 0 ,
\]

is automatically satisfied by the flux function by virtue of the properties of symmetry, consequent to the geometrical assumptions on the boundary, with which this function is endowed. The location of the magnetic axis hence follows solely from the proposition that the polar component of the magnetic poloidal field must vanish at the points along the circumference of a circle lying on the equatorial plane. This last condition writes as:

\[
\frac{\partial \psi}{\partial x}(x = \delta, \theta = 0) = 0 ,
\]
where
\[ \delta \equiv \frac{\Delta}{RC} \] (4.20)
is the Shafranov shift normalized to the torus major radius. Note that in Eq. (4.19) we have specified the angular coordinate as \( \theta = 0 \), anticipating a property of the equilibrium pinch to be evinced in later Sections, to know, that the shifting of the magnetic axis always proceeds from the centre line towards the outer edge of the torus.

From Eq. (4.5), the normalized partial flux linked with the annular region \( 1 \leq x \leq 1 + \epsilon \) distributes itself along the radial coordinate according to:
\[ \Phi^N(x, \theta = 0) = G^N_0(x) + \sum_{n=1}^{2N-2} G^N_n(x) . \] (4.21)

Introducing \( G^N_0(x) \) and \( G^N_n(x) \) as given by Eq. (4.6) and (4.7) respectively in Eq. (4.21), the resulting expression for \( \Phi^N(x, \theta = 0) \) can be put in the form:
\[ \Phi^N(x, \theta = 0) = \sum_{n=0}^{2N-2} S_n(x) + K_0 + \sum_{n=1}^{N-1} \left[ \sum_{j=1}^{N-1} M_{nj}(x)K_j \right] . \] (4.22)

Recalling Eq. (3.31) for \( S_n(x) \) we can write the first summation term on the right hand side of Eq. (4.22) as:
\[ \sum_{n=0}^{2N-2} S_n(x) = P(x) + \lambda Q(x) , \] (4.23)
where we have made use of the following definitions:
\[ P(x) = \sum_{n=0}^{2N-2} P_n(x) , \] (4.24)
\[ Q(x) = \sum_{n=0}^{2N-2} Q_n(x) , \] (4.25)
in which, consistent with the values specified for \( S_n(x) \) in Eq. (4.8), there are implied
the assumptions respectively:

\[
\begin{align*}
P_n(x) &= 0 \\
Q_n(x) &= 0
\end{align*}
\]  
for \( n \geq 5 \). \((4.26)\)

Note that, since the least number of multipole solutions to enter a combination aimed to approximate the flux function is \( N = 3 \), the sums defining \( P(x) \) and \( Q(x) \) in Eqs. \((4.24)\) and \((4.25)\), for any \( N \), include the totality of the functions \( P_n(x) \) and \( Q_n(x) \) that are non-null according to Eq. \((4.26)\), and there is no need to attach a superscript \( N \) to \( P(x) \) and \( Q(x) \) in order to have the order of the partial flux from which they proceed indicated. In Appendix A expressions for both these functions show that \( P(x) = Q(x) \).

With regard to the term of double summation that appears on the right hand side of Eq. \((4.22)\), by interchanging the order in which sums over the indices \( j \) and \( n \) are performed, it becomes:

\[
\sum_{n=0}^{2N-2} \left[ \sum_{j=1}^{N-1} M_{nj}(x)K_j \right] = \sum_{j=1}^{N-1} V_j^{(N)}(x)K_j ,
\] \((4.27)\)

where we have introduced the functions:

\[
V_j^{(N)}(x) = \sum_{n=0}^{2N-2} M_{nj}(x) \quad (j = 1, 2, \ldots, N - 1).
\] \((4.28)\)

Substituting the two summation terms on the right hand side of Eq. \((4.22)\) by the representations of Eq. \((4.23)\) and Eq. \((4.27)\) respectively, we reach:

\[
\overline{\psi}^{(N)}(x, \theta = 0) = P(x) + \lambda Q(x) + K_0 + \sum_{j=1}^{N-1} K_j V_j^{(N)}(x) .
\] \((4.29)\)

Application of the condition stated as Eq. \((4.19)\) to Eq. \((4.29)\) leads us to an algebraic relation connecting the normalized radial coordinate \( x = \delta \) of the magnetic
axis, the equilibrium parameter $\lambda$ and the $N - 1$ constants $K_1, K_2, \ldots, K_{N-1}$. Written in the form that most conveniently adapts itself to our future purposes, this relation is:

$$\sum_{j=1}^{N-1} V_j^{(N)}(\delta)K_j + Q'(\delta)\lambda = -P'(\delta), \quad (4.30)$$

where primes denote derivatives with respect to the argument. Note that, since the highest power of $x$ to appear in the multipole solution $\varphi^{(i)}(x, \theta)$ is $2i$ according to Eq. (2.32) in Ref. [4], the partial flux of order $N$ in Eq. (4.29) is a complete polynomial of the degree $2N - 2$ in $x$, and Eq. (4.30) is an algebraic relation of the $(2N - 3)$th degree in $\delta$ whose coefficients exhibit linear dependences on the parameter $\lambda$ and on the constants $K_i$ $(i = 1, 2, \ldots, N - 1)$. Concrete forms of the last mentioned equation for any particular $N$ show in general that its terms on both of its sides contain $1 + \delta$ as a common factor$^3$.

Equation (4.30) can be used to find a representation of $\lambda$ in terms of $\delta$, and then, by substituting it in the expression for the particular solution $\overline{\psi}_p(x, \theta)$ and in those for the constants $K_j$’s $(j = 0, 1, 2, \ldots, N - 1)$ in the forms they are originally generated by application of Eqs. (4.17) and (4.16), the flux function can be made to appear as dependent on $\delta$ instead of on $\lambda$. A more direct approach than this to accomplish the same end is however possible and in point of fact preferable.

We start by viewing the parameter $\lambda$ momentarily as an unknown of the problem, on the same footing as the constants $K_j$ $(j = 0, 1, 2, \ldots, N - 1)$, and take $\delta$ as a given quantity. Writing the driving terms in the equations placed above the dashed line in Eq. (4.13) in accordance with the pattern of representation of the functions $S_i(x)$ in Eq. (3.31), these same equations can be concisely restated as:

$$\sum_{j=1}^{N-1} M_{ij}(\epsilon)K_j + Q_i(\epsilon)\lambda = -P_i(\epsilon) \quad (i = 1, 2, \ldots, N - 1), \quad (4.31)$$

---

$^3$This is a consequence of the fact that the dependences of the functions $V^{(N)}(\delta), Q(\delta)$ and $P(\delta)$ on $\delta$ are mediated by the function of $\delta$ that defines the variable $\chi$ in Eq. (4.38) ahead.
which is a form appropriate to make them appear as a system of \( N - 1 \) equations for the \( N \) unknowns \( K_1, K_2, \ldots, K_{N-1} \) and \( \lambda \). To close the set, we add Eq. (4.30) to it as last equation. The determinant of the linear system of \( N \) unknowns and \( N \) equations in this way composed is:

\[
D_B^{(N)}(\epsilon, \delta) = \begin{vmatrix}
M_{11}(\epsilon) & M_{12}(\epsilon) & \cdots & M_{1,N-1}(\epsilon) & Q_1(\epsilon) \\
M_{21}(\epsilon) & M_{22}(\epsilon) & \cdots & M_{2,N-1}(\epsilon) & Q_2(\epsilon) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
M_{N-1,1}(\epsilon) & M_{N-1,2}(\epsilon) & \cdots & M_{N-1,N-1}(\epsilon) & Q_{N-1}(\epsilon) \\
V_1^{(N)'}(\delta) & V_2^{(N)'}(\delta) & \cdots & V_{N-1}^{(N)'}(\delta) & Q'(\delta)
\end{vmatrix}. \tag{4.32}
\]

We write the solution for \( \lambda \) in the form:

\[
F^{(N)}(\epsilon, \delta) = \lambda, \tag{4.33}
\]

where

\[
F^{(N)}(\epsilon, \delta) = \frac{\begin{vmatrix}
M_{11}(\epsilon) & M_{12}(\epsilon) & \cdots & M_{1,N-1}(\epsilon) & -P_1(\epsilon) \\
M_{12}(\epsilon) & M_{22}(\epsilon) & \cdots & M_{2,N-1}(\epsilon) & -P_2(\epsilon) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
M_{N-1,1}(\epsilon) & M_{N-1,2}(\epsilon) & \cdots & M_{N-1,N-1}(\epsilon) & -P_{N-1}(\epsilon) \\
V_1^{(N)'}(\delta) & V_2^{(N)'}(\delta) & \cdots & V_{N-1}^{(N)'}(\delta) & -P'(\delta)
\end{vmatrix}}{D_B^{(N)}(\epsilon, \delta)}. \tag{4.34}
\]

Equation (4.33) of course gives us the solution for \( \lambda \) in the same form that it would be provided by Eq. (4.30) if in this latter equation the constants \( K_j \) would first be expressed in the form that stems from their evaluation according to the formula in Eq. (4.16). Since the terms on the one and the other side of Eq. (4.30), as we have seen, include \( 1 + \delta \) as a common factor, which would be stricken out in a process of obtaining the solution for \( \lambda \), we conclude that \( F^{(N)}(\epsilon, \delta) \), as given by Eq. (4.34), is a rational function of \( \delta \) whose numerator and denominator are polynomials of the degree \( 2N - 4 \) in \( \delta \).
To express the particular solution in terms of the relative Shafranov shift instead of in terms of the equilibrium parameter, we substitute $\lambda$ by the function $F(N)(\epsilon, \delta)$ in Eq. (3.27). This substitution procedure can also be applied to achieve analogous end with respect to the constants $K_1, K_2, \ldots, K_{N-1}$ as evaluated according to the rule of Eq. (4.16), but a more direct approach than such is here made possible by appeal to that same linear system that provides us with the connection between $\lambda$ and $\delta$. Indeed, the solution for the constants $K_j$ ($j = 1, 2, \ldots, N - 1$), as extracted from the system constituted by Eq. (4.31) and Eq. (4.30), writes as:

$$K_j^{(N)} = \begin{vmatrix}
M_{11}(\epsilon) & \cdots & M_{1,j-1}(\epsilon) & -P_1(\epsilon) & M_{1,j+1}(\epsilon) & \cdots & M_{1,N-1}(\epsilon) & Q_1(\epsilon) \\
M_{21}(\epsilon) & \cdots & M_{2,j-1}(\epsilon) & -P_2(\epsilon) & M_{2,j+1}(\epsilon) & \cdots & M_{2,N-1}(\epsilon) & Q_2(\epsilon) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
M_{N-1,1}(\epsilon) & \cdots & M_{N-1,j-1}(\epsilon) & -P_{N-1}(\epsilon) & M_{N-1,j+1}(\epsilon) & \cdots & M_{N-1,N-1}(\epsilon) & Q_{N-1}(\epsilon)
\end{vmatrix}
$$

$$D_B^{(N)}(\epsilon, \delta) = \begin{vmatrix}
V_1^{(N)'}(\delta) & \cdots & V_{j-1}^{(N)'}(\delta) & -P'(\delta) & V_{j+1}^{(N)'}(\delta) & \cdots & V_{N-1}^{(N)'}(\delta) & Q'(\delta)
\end{vmatrix}
$$

\[(j = 1, 2, \ldots, N - 1), \quad (4.35)\]

and contains reference solely to $\delta$ in place of to $\lambda$. In what regards $K_0$, the kind of representation that is now desired can be most simply obtained by resorting to a procedure parallel to that adopted to derive its representation in terms of $\lambda$. We consider a system of $N + 1$ equations consisting of those expressed by Eqs. (4.31) and (4.30) plus the following one, taken as the first of the set:

$$K_0 + \sum_{j=1}^{N-1} M_{0j}(\epsilon)K_j + Q_0(\epsilon)\lambda = -P_0(\epsilon), \quad (4.36)$$

which is not another one but Eq. (4.12) with $S_0(\epsilon, \lambda)$ replaced by its definition according to Eq. (3.31), and written in a manner that brings $\lambda$ to appear as an unknown the same as the $N$ constants $K_0, K_1, \ldots, K_{N-1}$. Since Eq. (4.36) is the only equation within the whole set to contain $K_0$, the determinant of the system reduces to that given by Eq. (4.32). The solution for $K_0$ is then:
With the constants $K_0, K_1, \ldots, K_{N-1}$ and the equilibrium parameter in the particular solution expressed in terms of the relative Shafranov shift, all elements are given to make possible a representation of the flux function in which $\delta$ appears as the one physical parameter to characterize globally the equilibrium, assuming a role that belonged to $\lambda$ in the original formulation of the problem. An appreciable simplification in the mathematical development of the solution and in the form of the flux function can however be achieved if, in place of $\delta$, another quantity will be taken for reference, which is:

$$
\chi \equiv (1 + \delta)^2 - 1 .
$$

(4.38)

Because of the central role it shows to play in the theory of representation of equilibrium configurations by series of multipole solutions, we shall reserve the name of displacement variable to $\chi$. The relative Shafranov shift can be obtained from $\chi$ through the relation:

$$
\delta = \sqrt{1 + \chi} - 1 .
$$

(4.39)

Note that, being $\delta$ positive, as it indeed is, $\chi$ is also a positive quantity, and that, in the limit of small values, the two quantities obey a relation of proportionality between themselves: $\chi \simeq 2\delta$, vanishing one when the other vanishes.

A simple geometrical interpretation can be given to $\chi$ as follows. The defining

\[
K_0^{(N)} = \begin{bmatrix}
-P_0(\epsilon) & M_{01}(\epsilon) & M_{02}(\epsilon) & \cdots & M_{0,N-1}(\epsilon) & Q_0(\epsilon) \\
-P_1(\epsilon) & M_{11}(\epsilon) & M_{12}(\epsilon) & \cdots & M_{1,N-1}(\epsilon) & Q_1(\epsilon) \\
-P_2(\epsilon) & M_{21}(\epsilon) & M_{22}(\epsilon) & \cdots & M_{2,N-1}(\epsilon) & Q_2(\epsilon) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-P_{N-1}(\epsilon) & M_{N-1,1}(\epsilon) & M_{N-1,2}(\epsilon) & \cdots & M_{N-1,N-1}(\epsilon) & Q_{N-1}(\epsilon) \\
-P'(\delta) & V_1^{(N)'}(\delta) & V_2^{(N)'}(\delta) & \cdots & V_{N-1}^{(N)'}(\delta) & Q'(\delta) \\
\end{bmatrix} 
\]

\[
D_B^{(N)}(\epsilon, \delta) .
\]

(4.37)
relation for the displacement variable can be stated in the form:

$$\chi = \delta \left(1 + \frac{R_M}{R_C}\right).$$

(4.40)

Also, if we introduce a quantity $Z_{MC}$ with the dimension of length by means of the relation:

$$\chi \equiv \frac{Z_{MC}^2}{R_C^2},$$

(4.41)

from the above definition of $\chi$ we find that

$$Z_{MC}^2 = R_M^2 - R_C^2,$$

(4.42)

which shows that $Z_{MC}$ is the measure of half the segment determined on a vertical line passing through $C$ by an arc of circumference of radius $R_M$ drawn with centre at $O$ in Fig. 2.1.

To accomplish the transformation of $F^{(N)}(\epsilon, \delta)$, $K_j^{(N)} (j = 1, 2, \ldots, N - 1)$ and $K_0^{(N)}$ from functions of $\delta$ to functions of $\chi$ we have first to proceed to the transformations:

$$\begin{cases}
V_j^{(N)}(\delta) \rightarrow V_j^{(N)}(\chi) \quad (j = 1, 2, \ldots, N - 1) \\
P(\delta) \rightarrow P(\chi) \\
Q(\delta) \rightarrow Q(\chi)
\end{cases}$$

(4.43)

by using Eq. (4.39) for $\delta$ in the expressions for the functions listed on the left hand side of Eq. (4.43); next, taking the derivatives of the functions listed on the right hand side with respect to $\chi$, we obtain the new elements that are to substitute those forming the bottom rows of the determinants in Eqs. (4.32), (4.34) (4.35) and (4.37).

Note that in this scheme of substitution we have omitted the “inverse scale factor” of the transformation of derivatives with respect to $\delta$ to derivatives with respect to $\chi$, given by

$$\frac{d\chi}{d\delta} = 2(1 + \delta)$$
which is indeed unnecessary to be written out, for it would appear in both the numerator and the denominator of the expression for each of the quantities we are seeking to have transformed from one representation to the other, and would be cancelled. Note also that in Eq. (4.43) we have used, in conformity with the common practice, the same symbol for a function of $\delta$ and for its transformed version as a function of $\chi$, although the dependences they keep each on the respective variables are different. In Appendix A the reader will find the expressions for $P(\chi)$, $Q(\chi)$, $P'(\chi)$ and $Q'(\chi)$, and in Appendix C those for $V_{j}^{(N)'}(\delta)$ and $V_{j}^{(N)'}(\chi)$ ($j = 1, 2, \ldots, 9$).

One of the advantages offered by the displacement variable with respect to the relative Shafranov shift as working quantity is that the former is able to reduce the degrees of the polynomials entering the expression of $F^{(N)}(\epsilon, \delta)$ to the half. That is to say, if in the function defined by Eq. (4.34) we replace $\delta$ by $\chi$ according to Eq. (4.39), the resulting function, which we shall denote by $F^{(N)}(\epsilon, \chi)$, will be given by the quotient of two polynomials, each of the degree $N - 2$ in $\chi$. In substitution to Eq. (4.33), the solution for $\lambda$ will be written as:

$$F^{(N)}(\epsilon, \chi) = \lambda.$$  \hfill (4.45)

We shall call the function $F^{(N)}(\epsilon, \chi)$ the equilibrium function of order $N$ and Eq. (4.45) the equilibrium equation in the approximation of $N$ multipoles. Reverting now to the view of $\lambda$ as a parameter given and considering that $\epsilon$ is also a datum of the problem, then Eq. (4.45) is an algebraic equation of the $(N - 2)$-th degree for $\chi$. In a graphical interpretation of this equation, the equilibrium value (or possibly the equilibrium values) of the displacement variable $\chi$ for a fixed aspect ratio $\epsilon$ of the toroidal chamber appears as determined by the intersection of the curve $F^{(N)}(\epsilon, \chi)$ versus $\chi$ with the straight line drawn parallel to the axis of the
abscissas at the ordinate \( \lambda \). Once a value of the displacement variable \( \chi \) is known, the corresponding Shafranov shift can be evaluated by means of Eq. (4.39), and the location of a magnetic axis be set.

The representation of the flux function that takes the equilibrium parameter as the reference quantity to characterize an equilibrium configuration depends linearly on \( \lambda \), whereas the representation that expresses itself in terms of the displacement variable exhibits a more complicated pattern of dependence on the quantity it takes for reference than the linear one, being as it is a rational function of \( \chi \). The latter representation, however, results in general simpler in aspect than the former, or at least more compact. In applying in later Sections the method of solution to the Grad-Shafranov boundary value problem by expansion of the poloidal flux function in series of multipole solutions to obtain the description of a particular equilibrium configuration we shall have occasion to illustrate the use of one and the other of these two possible representations of the flux function.

It is opportune to note here that, since the particular solution \( \bar{\psi}_p(x, \theta) \) and the multipole solutions \( \varphi^{(n)}(x, \theta) \) (\( n = 1, 2, \ldots \)) vanish all at the coordinate \( x = 0 \), Eq. (4.1) shows that \( K_0^{(N)} \) physically means the approximation of order \( N \) to the normalized magnetic poloidal flux at the centre of the torus cross section:

\[
K_0^{(N)} = \bar{\psi}_C^{(N)} \equiv \bar{\psi}^{(N)}(x = 0, \theta) .
\] (4.46)

We have noted, in the sequel of Eq. (3.32), that the two components of the gradient of the particular solution, and thus of the magnetic poloidal field generated by it, vanish at the coordinate \( x = 0 \). Also noticed, in Ref. [4], that the poloidal fields derived from the multipole solutions vanish all at the coordinate \( x = 0 \), exception being taken to the field proceeding from the solution of order \( n = 1 \). The multipole solution of this last mentioned order, which is expressed in cylindrical coordinates
by:

$$\varphi^{(1)}(R, z) = \frac{1}{4} \left( \frac{R^2}{R_C^2} - 1 \right),$$

(4.47)
gives origin to a poloidal magnetic field whose radial component is null and whose axial component is constant everywhere in space. We thus conclude, by Eq. (4.1), that the poloidal magnetic field associated with the approximation of order $N$ to the poloidal flux function at the centre of the torus cross section is given by (see Eq. (9.10) farther on):

$$\vec{B}^{(N)}_v = \frac{1}{2} s_P K_1^{(N)} B_{\phi_C} \vec{k},$$

(4.48)
an expression for which, to have it written in a way such as it has been here, we have to have remembered that $\bar{\psi}^{(N)}(x, \theta)$ embodies a double normalization, that by $B_{\phi_C} R_C^2$ (introduced by Eq. (2.3)) and that by $s_P$ (introduced by Eq. (4.2)). The field of Eq. (4.48) is obviously the vertical magnetic field that has to be applied to the toroidal pinch by external means to adjust the position of the plasma column inside the containing chamber.

It is also of interest to have an expression for the magnetic poloidal flux at the magnetic axis. From the expression for the flux function that identifies itself with the particular solution to the Grad-Shafranov equation, given in cylindrical coordinates by Eq. (3.32), and from the expressions that are identical with those for the multipole solutions, given in cylindrical coordinates in Appendix C in Ref. [4], we see that, for $z = 0, R = R_M$ and $R_A \equiv R_C$, they reduce all to a term proportional to some power of $(R_M^2 - R_C^2)/R_C^2$. By referring to Eqs. (4.41) and (4.42) we learn that this quantity equals $\chi$. We can thus write the expression for the normalized magnetic poloidal flux at the magnetic axis within the approximation of order $N$ to the poloidal flux function as:

$$\bar{\psi}^{(N)}_M = \frac{1}{16} \left[ 1 + F^{(N)}(\epsilon, \chi) \right] \chi^2 + K_0^{(N)} + \sum_{i=1}^{N-1} K_i^{(N)}(-1)^{i-1}(N.F.)_i \chi^i,$$

(4.49)
where we have replaced $\lambda$ by $F^{(N)}(\epsilon, \chi)$ according to Eq. (4.45), and $(N.F.)_i$ is the numerical factor multiplying the first term on the right hand side of the expression for the multipole solution $\varphi^{(i)}(\xi, \nu)$ in Appendix C in Ref. \[4\]: $(N.F.)_1 = 1/4$, $(N.F.)_2 = 1/4$, $(N.F.)_3 = 1/8$, $(N.F.)_4 = 1/20$, etc.

V. THE CONSTRAINTS ON THE VALUES OF THE DISPLACEMENT VARIABLE AND THE LIMITATIONS OF THE METHOD OF SERIES OF MULTIPOLE SOLUTIONS

Although the physical parameters that appear as input to the Grad-Shafranov equation are the pressure gradient and the gradient of half the squared toroidal field function, which, if uniform in flux space as we are considering them to be in this paper, can be expressed under proper normalizations as the constants $s_P$ and $s_I$, and although for the resolution by the method of series of multipole solutions the relevant physical parameter shows to be the ratio of $s_I$ to $s_P$, which we have called the equilibrium parameter and denoted by $\lambda$, the analysis and reasonings on the relations between the conceptual tools and operational devices of the method on one side and the physical properties of the equilibrium on the other are better conducted in terms of the parameter that we have denominated the displacement variable and have denoted by $\chi$. The value of $\chi$ for a given order $N$ of approximation to the flux function is fixed by the value of $\lambda$ through Eq. (4.45), which, however, given the mathematical complexity of the equilibrium function, is not a connection such as to admit of a prompt interpretation. Under this perspective, the relative Shafranov shift $\delta$, which relates to $\chi$ in a more transparent way than $\lambda$ does, becomes the most significant single physical parameter to spring out from the method.

In the remaining Sections of this paper we shall be pursuing the objective of
illustrating the method of series of multipole solutions by applying it to obtain approximate solutions to the Grad-Shafranov boundary value problem that satisfy certain preconditions, the effect of which is to place them in the class of those that partake of the greatest physical interest. Such preconditions, which can in a way be considered as complementary to the boundary conditions and can be stated in simple mathematical terms, prevent from the onset of the solving process the appearance of solutions corresponding to magnetic configurations that in some manner are unable to restrain the plasma from drifting to the walls of the containing vase and thus do not contrive to achieve its confinement. It should be remembered, however, that the equilibrium quasi-states we shall be deriving and describing still may or may not be dynamically stable to small perturbations, a concern that falls outside the scope of the present paper.

The first precondition the solutions we are interested in must satisfy is that they admit a magnetic axis that falls inside the chamber. Since the position of the null poloidal field with respect to the centre of the torus cross section is directly specified by the displacement variable, this precondition signifies that there is a maximal allowed value $\chi_{\text{max}}$ for $\chi$, at which the magnetic axis, on the equatorial plane, would be placed at the outer edge of the torus:

$$\chi < \chi_{\text{max}} \equiv (1 + \epsilon)^2 - 1.$$  \hspace{1cm} (5.1)

The second precondition stems from the recognition that the incorporation of the pressure gradient parameter into the quantity the problem is solved for by way of the relation stated in Eq. (4.2) has the effect of uniquely defining the sign that the solution must take in the domain enclosed by the boundary. Indeed, Eq. (2.7), which introduces the constant $s_P$ as the rate of decay of the pressure with the flux
coordinate $\psi$, transforms to
\[
\frac{d}{d\psi} \left( \frac{\hat{p}}{2} \right) = -s_P^2
\]  
(5.2)
when the coordinate variable is changed from $\psi$ to $\overline{\psi}$. Equation (5.2) above defines
the quantity that we have placed on its left hand side as negative, independently of
the sign of $s_P$. Thus, if we agree that the pressure is greater in the interior than at
the edge of the plasma, we must have $\overline{\psi}$ negative within the frontiers of the domain
of the solution, and, in particular, at the point associated with the location of the
magnetic axis. If the sign of $s_P$ is supposed to be positive, with this definition for
the sign of $\overline{\psi}$, both $\psi$ and $\Psi$ are negative quantities (except at the boundary, where
they are null), while a negative sign of $s_P$ would give them positive signs. The sign
of $s_P$ is ultimately irrelevant as it has no physical implication whatever. We shall see
in the Sections to come that the requirement of a negative sign for $\overline{\psi}$ in the plasma
region has the double consequence that the magnetic axis is always shifted from the
geometrical centre line of the torus outwardly, and that even for a vanishingly small
plasma pressure a nonvanishing shift is needed for equilibrium. In other words, this
means that, considered all possible physical conditions for equilibrium, there is a
minimal positive value $\chi_{\text{min}}$ that can be taken by the displacement variable,
\[
\chi \geq \chi_{\text{min}} > 0 ,
\]  
(5.3)
which is determined by the geometry of the confining field (that is, by the inverse
aspect ratio of the magnetic configuration).

Finally, the third precondition is that the surface found for $\psi = 0$ does not
exhibit singular points, to borrow the denomination by which they are referred to
in Ref. [10], at which the gradient of the flux function vanishes:
\[
\nabla \psi = 0 ,
\]  
(5.4)
and where a field line, as projected on the plane of the torus cross section, branches into two field lines, since this would lead to loss of the ability of the magnetic configuration to confine the plasma.

The meaning of this last precondition is subtler than those of the two first ones. For a sequence of equilibrium quasi-states generated by increasing gradually the displacement variable since small values while the inverse aspect ratio is kept constant we have found that branching points break out on the boundary for some critical shift of the magnetic axis relative to the centre of the torus cross section as the culmination of a process of distortion of the surface $\psi = 0$, which then becomes a separatrix of the magnetic configuration, and ceases to be topologically equivalent to a toroidal surface. There are reasons, however, to believe that this evolvement of the boundary surface into a separatrix we have observed might not be inherent to the solutions that comply with the first two preconditions, but merely a product of an unduly low number of multipole solutions that have been employed to approximate the flux function for such a high value of $\chi$, and, in this sense, an artificial effect of the method. Indeed, as the order of approximation is increased, we observe that the critical value of $\chi$ is also increased, the distortion of the flux surfaces in the vicinity of the boundary, which precedes the emergence of the branching points, becomes less and less pronounced, and when they finally break out the loop described by the segments of field lines comprised between them shows a tendency to become more and more shortened and tighter, suggesting that they would fuse into a single point that at last would disappear if the number $N$ of multipole solutions entering the composition of the flux function would become infinite. In brief, all indications are that the occurrence of points on the surface $\psi = 0$ whose coordinates verify the singularity condition stated in Eq. (5.4) can be suppressed by simply including an adequate number of multipole solutions in the approximation to the flux function.
Thus the true content of the third precondition is the setting of an upper limit to the allowed range of variation of the displacement variable for a given order $N$ of the partial flux as the value of $\chi$ that triggers off structural alterations in the geometry of the boundary as they appear portrayed by the method itself. We shall see in later Sections that the limits we encountered according to this criterium are much more stringent than the one fixed by Eq. (5.1) for all values of $N$ that have been considered.

A question that naturally arises concerns the properties of convergence of the series of multipoles, to know, if by increasing the number $N$ of multipole solutions the error associated with the partial flux always decreases and the approximate solution such flux represents then tends to a well defined limit (in which case the series is convergent), or if the error diminishes with the increase of $N$ up to an optimal value of the number of multipole solutions and starts to grow without limit thereafter by further increase of $N$ (which typifies an asymptotic series), or, if regardless of the value of the approximate solution that is obtained for the minimal value of $N$ (which is 3), any new term that is added to the series makes the error amplify (in which case the series is divergent).

The reply we may give to this question as supported by examples worked out in the present paper that apply the method to a variety of equilibrium situations with a succession of values of $N$ to approximate the flux function is that the third alternative (the divergence of the series) should be excluded and that of the two remaining ones the results consistently favour that of convergence.

The error carried by the partial flux of order $N$ is of course to be evaluated at the boundary, where the values that would be assumed by an exact solution are known. Two are the quantities we shall use to measure the accuracy of the solutions obtained for the flux function. The first, which we shall call simply the error,
given by the value of the flux itself at the normalized radial coordinate $x = \epsilon$, since were $\bar{\psi}(x, \theta; \chi, \epsilon)$ an exact solution to the boundary value problem, this would be zero. As we intend to compare the accurateness reached by solutions to different equilibria among themselves and also compare mutually those for the same equilibria as obtained with distinct orders of approximation to the flux function, we find it convenient to map the range of variation of $\bar{\psi}$, which is changeable according to the value of $\lambda$ (or $\chi$) and $N$, onto a fixed one, and introduce a new normalization to the flux function, namely:

$$\hat{\psi}(x, \theta; \chi, \epsilon) \equiv \frac{\psi(x, \theta; \chi, \epsilon)}{-\bar{\psi}_M(\chi, \epsilon)},$$

(5.5)

where $\bar{\psi}_M(\chi, \epsilon)$, we recall, is the (normalized) flux at the magnetic axis and must be a negative quantity according to the second precondition. In this way the interval of variation of $\hat{\psi}(x, \theta; \chi, \epsilon)$ is comprised between $-1$ (at the point where it takes its minimal value, the magnetic axis) and zero (at the boundary, where it takes its maximal value). The value of $\hat{\psi}(x, \theta; \chi, \epsilon)$ at $x = \epsilon$, which turns out to be zero only for an exact solution, is a function of the angle $\theta$ for the approximate ones and gives us the normalized or relative error, which we shall denote by $E$, defined over a common scale of reference:

$$E \equiv E(\theta; \chi, \epsilon) \equiv \hat{\psi}(x = \epsilon, \theta; \chi, \epsilon).$$

(5.6)

The second quantity we shall use to characterize the degree of precision of a solution to the Grad-Shafranov boundary value problem, which we shall refer to by the name of relative deviation, is the ratio between the departure of the contour of the surface $\hat{\psi} = 0$, as it is described by the actual approximate solution, from the one of circular shape, such as it would be obtained at the boundary if $\hat{\psi}$ were the exact solution, and the minor radius of the torus cross section. In terms of the
normalized radial coordinate this writes as:

\[ D \equiv \frac{x(\hat{\psi} = 0) - \epsilon}{\epsilon}. \]  

Note that the relative deviation, as much as the relative error, is a function of the poloidal angle \( \theta \).

To illustrate the use of the method of solution to the Grad-Shafranov boundary value problem by series of multipole solutions we have elected a tokamak of inverse aspect ratio \( \epsilon = 2/5 \), which, as it can be seen from Fig. 5.1, makes neither a thin nor a compact torus, but one of an intermediate thickness such as that which can be found in typical laboratory devices. For this tokamak we have chosen three equilibrium situations to study, characterized by having \( \lambda \) equal to 0, 1 and \(-1/5\) respectively. The first value of these for the equilibrium parameter corresponds to an equilibrium that has been widely studied \([1]\) and can serve therefore as a ground reference to estimate the merits and limitations of the method. The second value (\( \lambda = 1 \)), positive, provides us with the example of an equilibrium in which the plasma as a whole displays the behaviour of a paramagnetic body, while by the third (\( \lambda = -1/5 \)), negative, we are led to contemplate the case of an equilibrium that is reached with the plasma column assuming the characteristics of a diamagnetic body. All of them will be studied in the pages to follow with expressions for the partial flux that include combinations of a variable number, ranking from \( N = 3 \) to \( N = 10 \), of multipole solutions of the orders \( n = 0, 1, 2, \ldots, 9 \), which are the ones made available to us by Ref. \([4]\). We anticipate that, of the three equilibria considered, the largest relative error and the largest relative deviation are found for the one with \( \lambda = -1/5 \), and that, for a combination of \( N = 10 \) multipole solutions in the making up of the partial flux, at the angular positions where they attain their respective maximum
values, these two measures of departure from exactness are:

\[ E \approx 0.66 \times 10^{-2}, \quad D \approx 0.21 \times 10^{-2}. \quad (5.8) \]

Besides for these three we have determined the solutions for one or two more equilibria at each of the orders of approximation considered, as these were found to afford the possibility of obtaining expressions for the flux functions in which the terms carrying the dominant sources of error could be eliminated by a judicious choice of \( \lambda \), and appeared then as equilibria that could be described with exceptional accuracy within the approximation of a restricted number of multipole solutions.

All calculations, analytical and numerical, were performed with the help of a computer and the use of the program Maple (TM) [11].

![FIG. 5.1 The torus of inverse aspect ratio 2/5.](image)

VI. SOLUTION TO THE GRAD-SHAFRANOV BOUNDARY VALUE PROBLEM IN THE APPROXIMATION OF \( N = 3 \) MULTipoLE SOLUTIONS

In this and in the next Section we shall apply the method of series of multipole solutions to obtain solutions to the Grad-Shafranov boundary value problem in the
approximations of order $N = 3$ and $N = 4$ respectively, a task we shall carry out in some detail to take full advantage of the relatively simple mathematical terms in which it presents itself, and to illustrate techniques to obtain results of interest that are in general also employable when the order of the approximation is higher than those of the two lowest ones.

According to the pattern of development proposed by Eq. (4.1), the poloidal flux function in the order $N = 3$ of approximation consists of the sum of the particular solution and a combination of the multipole solutions of the three lowest orders, to know:

$$
\psi^{(3)}(x, \theta) = \psi_p(x, \theta) + K_0 \varphi^{(0)}(x, \theta) + K_1 \varphi^{(1)}(x, \theta) + K_2 \varphi^{(2)}(x, \theta).
$$

(6.1)

Note that this is the least number of multipole solutions to be included in a combination with the particular solution in order to obtain an approximate representation of the flux function that be minimally meaningful, since the term involving the zeroth order multipole solution, being a constant, gives zero magnetic field, and the term comprising the first order multipole solution merely generates a uniform vertical field, without modifying the shape of the field lines associated with the field brought about by the particular solution. Note also that the second is the lowest of the orders of the multipole solutions for which one of these shows to contain a power of $x$ equal to the highest power, and a harmonic of the poloidal angle of order equal to the highest order, of $x$ and of the harmonics of the poloidal angle, respectively, to be present in the particular solution. In fact, the particular solution and the multipole solution of the second order are bivariate polynomials with terms respectively of the same degrees both in $x$ and $\mu$.

We shall start by searching an expression for the poloidal flux function that takes $\lambda$ as the quantity defining the input datum to an equilibrium problem. In this case
the formulae we resort to as the appropriate ones for the evaluation of the constants $K^{(3)}_0$, $K^{(3)}_1$ and $K^{(3)}_2$ are those expressed by Eqs. (4.17) and (4.16). We have in this way:

$$K^{(3)}_0 = \frac{| -S_0(\lambda, \epsilon) \ M_{01}(\epsilon) \ M_{02}(\epsilon) |}{D_A^{(3)}(\epsilon)}, \quad (6.2)$$

$$K^{(3)}_1 = \frac{| -S_1(\lambda, \epsilon) \ M_{12}(\epsilon) |}{D_A^{(3)}(\epsilon)}, \quad (6.3)$$

$$K^{(3)}_2 = \frac{| M_{11}(\epsilon) \ -S_1(\lambda, \epsilon) |}{D_A^{(3)}(\epsilon)}, \quad (6.4)$$

the denominator $D_A^{(3)}(\epsilon)$ common to these three formulae, as we learn from Eq. (4.15), being given by:

$$D_A^{(3)}(\epsilon) = \frac{| \epsilon \ 0 \ -\epsilon^3/4 \ |}{\epsilon^2/8 - 1/8 \epsilon^4 - \epsilon^2}, \quad (6.5)$$

The expressions for the functions $S_0(\lambda, \epsilon)$, $S_1(\lambda, \epsilon)$ and $S_2(\lambda, \epsilon)$ and the expressions for the several $M_{ij}(\epsilon)$’s that enter Eqs. (6.2) – (6.5) as elements of the determinants on their right hand sides can be written down by referring to those that constitute the matter of Appendices A and B respectively. With this the determinant $D_A^{(3)}(\epsilon)$, for example, assumes the definite form:

$$D_A^{(3)}(\epsilon) = \begin{vmatrix} \epsilon & -\epsilon^3/4 \\ \epsilon^2/8 & -1/8 \epsilon^4 - \epsilon^2 \end{vmatrix}, \quad (6.6)$$

from which we extract:

$$D_A^{(3)}(\epsilon) = -\frac{\epsilon^3}{32}(\epsilon^2 + 16). \quad (6.7)$$
We merely quote the results that are obtained for the three constants:

\[
K_0^{(3)} = -\frac{\epsilon^2}{32(\epsilon^2 + 16)} \left[ \lambda + 1 \right] \left( 128 - 4\epsilon^2 \right) - 3\epsilon^4 ,
\]

\[ (6.8) \]

\[
K_1^{(3)} = -\frac{\epsilon^2}{\epsilon^2 + 16} (2\lambda + \epsilon^2 + 10) ,
\]

\[ (6.9) \]

\[
K_2^{(3)} = \frac{\epsilon^2}{4(\epsilon^2 + 16)} (\lambda - 3) .
\]

\[ (6.10) \]

Inserting the expression for \(\psi_p(x, \theta)\), as given by Eq. (3.27), those for \(K_0^{(3)}\), \(K_1^{(3)}\) and \(K_2^{(3)}\), as given by Eqs. (6.8), (6.9) and (6.10), and the formulae for the multipole solutions \(\varphi^{(0)}(x, \theta)\), \(\varphi^{(1)}(x, \theta)\) and \(\varphi^{(2)}(x, \theta)\), found in Appendix A in Ref. [4], in Eq. (6.1), we obtain the expression of the flux function normalized to \(s_P\) having \(\lambda\) as the characteristic equilibrium parameter in the approximation of three multipole solutions:

\[
\overline{\psi}^{(3)}(x, \theta) = \frac{\epsilon^2 - x^2}{\epsilon^2 + 16} \left\{ \frac{1}{32} \left[ (4\lambda - \epsilon^2 - 28)x^2 - (\lambda + 1)(128 - 4\epsilon^2) + 3\epsilon^4 \right] \\
- \frac{1}{2}(2\lambda + \epsilon^2 + 10)x \cos \theta - \frac{1}{8}(4\lambda + \epsilon^2 + 4)x^2 \cos 2\theta \right\} + \frac{1}{2} \frac{(6\lambda + \epsilon^2 - 2)}{\epsilon^2 + 16} x^3 \cos 3\theta + \frac{1}{32} \frac{(20\lambda + 3\epsilon^2 - 12)}{\epsilon^2 + 16} x^4 \cos 4\theta .
\]

\[ (6.11) \]

Notice the factor \(\epsilon^2 - x^2\) making the terms independent of the poloidal angle and those dependent on \(\cos \theta\) and \(\cos 2\theta\) respectively, which are grouped inside the keys, vanish at the boundary, where \(x = \epsilon\).

To find the position of the magnetic axis we have first to determine the equilibrium function. Choosing to represent it in terms of the relative Shafranov shift, we refer to Eq. (4.34), which, when the order of approximation to the flux function is the third, takes the form:

\[
F^{(3)}(\epsilon, \delta) = \begin{vmatrix}
M_{11}(\epsilon) & M_{12}(\epsilon) & -P_1(\epsilon) \\
M_{21}(\epsilon) & M_{22}(\epsilon) & -P_2(\epsilon) \\
V_1'(\delta) & V_2'(\delta) & -P'(\delta)
\end{vmatrix},
\]

\[ D_B^{(3)}(\epsilon, \delta) \]

\[ (6.12) \]
where the denominator, following Eq. (4.32), is:

\[ D_B^{(3)}(\epsilon, \delta) = \left| \begin{array}{ccc} M_{11}(\epsilon) & M_{12}(\epsilon) & Q_1(\epsilon) \\ M_{21}(\epsilon) & M_{22}(\epsilon) & Q_2(\epsilon) \\ V'_1(\delta) & V'_2(\delta) & Q'(\delta) \end{array} \right|. \tag{6.13} \]

Using for \( P_1(x = \epsilon) \), \( P_2(x = \epsilon) \), \( Q_1(x = \epsilon) \), \( Q_2(x = \epsilon) \), \( P'(\delta) \) and \( Q'(\delta) \) the expressions given in Appendix A, for the several \( M_{ij}(x = \epsilon)'s \) those given in Appendix B and for \( V'_1(\delta) \) and \( V'_2(\delta) \) those in Appendix C, we are able to evaluate the equilibrium function as:

\[ F^{(3)}(\epsilon, \delta) = -\frac{1}{2} \frac{12(4 + \epsilon^2)\delta(\delta + 2) - 10\epsilon^2 - \epsilon^4}{4\delta(\delta + 2) - \epsilon^2}. \tag{6.14} \]

By replacing \( \delta(\delta + 2) \) by \( \chi \) in conformity with the definition introduced in Eq. (4.38), we obtain the equilibrium function in terms of the displacement variable, which we can write in a convenient form as:

\[ F^{(3)}(\epsilon, \chi) = -(1 + \chi_P^{(3)}) \chi - \chi_Z^{(3)}, \tag{6.15} \]

where

\[ \chi_Z^{(3)} = \frac{5}{4} \epsilon^2 \left( \frac{1 + \frac{\epsilon^2}{10}}{1 + \frac{\epsilon^2}{4}} \right) \tag{6.16} \]

and

\[ \chi_P^{(3)} = \frac{\epsilon^2}{4}. \tag{6.17} \]

It is useful to consider also the representation of \( F^{(3)}(\epsilon, \chi) \) in partial fractions:

\[ F^{(3)}(\epsilon, \chi) = -\left(1 + \frac{\epsilon^2}{4}\right) + \frac{\epsilon^2 \left(1 + \frac{\epsilon^2}{16}\right)}{\chi - \frac{\epsilon^2}{4}}, \tag{6.18} \]

which shows that the equilibrium function of the third order is graphically described by an equilateral hyperbola having a vertical asymptote passing through the point
that locates the pole \( \chi_P = \epsilon^2 / 4 \) on the horizontal axis, and a horizontal asymptote that intercepts the vertical axis at the ordinate \(- (1 + \epsilon^2 / 4)\).

Before exploring the information contained in the equilibrium function, as a necessary preliminary step, we determine the expression of the poloidal flux at the magnetic axis. Resorting to Eq. (4.49), we write for the third order of approximation to the flux function:

\[
\psi_M^{(3)}(\epsilon, \chi) = \frac{1}{16} \left[ 1 + F^{(3)}(\epsilon, \chi) \right] \chi^2 + K_0^{(3)} \chi - \frac{1}{4} K_2^{(3)} \chi^2 ,
\]

from which, by using Eq. (6.18) for \( F^{(3)}(\epsilon, \chi) \) and Eqs. (6.8), (6.9) and (6.10) for \( K_0^{(3)} \), \( K_1^{(3)} \) and \( K_2^{(3)} \) respectively with \( \lambda \) replaced by \( F^{(3)}(\epsilon, \chi) \), we obtain:

\[
\psi_M^{(3)}(\epsilon, \chi) = -\frac{\epsilon^2}{32} \frac{8\chi(\chi - \epsilon^2) + 32\epsilon^2 + \epsilon^4}{4\chi - \epsilon^2} .
\]

For \( \epsilon = 2/5 \), the above expression becomes:

\[
\psi_M^{(3)}(\epsilon = \frac{2}{5}, \chi) = -\frac{402 - 100\chi + 625\chi^2}{2500(25\chi - 1)} .
\]

From the pictorial representation of the function on the right hand side of Eq. (6.21), how it appears in Fig. 6.1, it becomes evident that negative values of the flux at the magnetic axis are reached only in the domain of the displacement variable extending beyond the point of abscissa \( \chi_P^{(3)} \) that locates its single pole (and also that of the equilibrium function \( F^{(3)}(\epsilon, \chi) \)) on the \( \chi \)-axis. Thus, combining the condition, formulated in mathematical terms through Eq. (5.1), that the magnetic axis must fall inside the plasma containing chamber with the one that the flux must be negative at the magnetic axis, we may state that, in so far these two criteria are concerned, the range of variation of the displacement variable has to be restricted to:

\[
\chi_P < \chi < (1 + \epsilon)^2 - 1 ,
\]
in order that a solution to the flux function be considered physically acceptable. We have abstained from writing the superscript that designates the order of approximation employed in the evaluation of $\chi_P$, since, as we are going to see in the Sections to come, the lower limit of the interval expressed by Eq. (6.22) is fixed in general by the value of the abscissa of the one real pole of the equilibrium function that is, among those lying on the positive side of the $\chi$-axis, placed in the tightest closeness to the origin. For a tokamak of inverse aspect ratio equal to $2/5$, as the one we are taking as the recipient of applications of our theoretical results, and a description of the equilibrium based on a solution to the flux function of the third order of approximation, Eq. (6.22) translates to numerical terms as:

$$0.04 < \chi < 0.96 \ . \quad (6.23)$$

In general, corresponding to a vanishingly small pressure gradient, there is an equilibrium “ground state”, defined, in terms of the equilibrium parameter, by the limit $\lambda \to +\infty$. For such a state, which can be shown to be paramagnetic in charac-
ter, the magnitude of the displacement variable under any order \( N \) of approximation to the flux function can be estimated as:

\[
\chi_{\text{min}} \simeq \frac{\epsilon^2}{4},
\]

provided that \( \epsilon \) is not too close to unity. This means that it is not possible to have an equilibrium configuration for which the relative Shafranov shift assumes a value smaller than that expressed generically by:

\[
\delta_{\text{min}} \simeq \sqrt{1 + \frac{\epsilon^2}{4}} - 1.
\]

For \( \epsilon = 2/5 \) we find the minimal value of the displacement variable and the minimal relative Shafranov shift to be:

\[
\begin{align*}
\chi_{\text{min}} & \simeq 0.04 \\
\delta_{\text{min}} & \simeq 0.0198
\end{align*}
\]

We now resume the considerations on the equilibrium function. To fix ideas, we refer to the particular form it assumes for a numerically defined aspect ratio of the torus, which we again choose to be 2/5. From Eqs. (6.15), (6.16) and (6.17) we have:

\[
F^{(3)}(\epsilon = \frac{2}{5}, \chi) = -\frac{1}{25} \frac{650\chi - 127}{25\chi - 1}.
\]

A graph of the above function is displayed in Fig. 6.2. It is seen that, in the domain of \( \chi \) extending from \( \chi = \chi_P = 0.04 \) to plus infinity, \( F^{(3)}(\epsilon, \chi) \) decreases monotonically from plus infinity to its minimal value, which is worth

\[
F^{(3)}(\epsilon = \frac{2}{5}, \chi = \mathcal{+\infty}) = \frac{-650}{625} = -1.04
\]

No local extremum occurs between these two points to allow for the existence of two different values of \( \chi \) for a given value of \( F^{(3)}(\epsilon, \chi) \). Since according to Eq. (4.45) the
FIG. 6.2 The equilibrium function of the third order $F^{(3)}(\epsilon, \chi)$ as a function of the displacement variable for the inverse aspect ratio $\epsilon$ equal to $2/5$. Only the portion of the curve corresponding to the physically significant range of values of $\chi$ ($0 < \chi_p < \chi < \infty$) is represented.

equilibrium equation in the third order of approximation to the flux function writes as

$$
\lambda = F^{(3)}(\epsilon, \chi),
$$

we conclude that, in general, the allowed range of variation of the equilibrium parameter is:

$$
\lambda_\infty < \lambda < \infty
$$

where $\lambda_\infty$, as obtained by carrying out the limit $\chi \to \infty$ in Eq. (6.18), is:

$$
\lambda_\infty = -\left(1 + \frac{\epsilon^2}{4}\right).
$$

If the restriction on the maximal value of $\chi$, as it is imposed by Eq. (6.22), is taken into account, then the minimal allowed value of $\lambda$ shifts from $\lambda_\infty$ to this one:

$$
\lambda_{\text{min}} = -\frac{16 - 2\epsilon + 4\epsilon^2 + \epsilon^3}{2(8 + 3\epsilon)}.
$$
The numerical value of this quantity for \( \epsilon = 2/5 \) is:

\[
\lambda_{\text{min}}(\epsilon = 2/5) = -\frac{497}{575} \approx -0.864
\]  

(6.33)

about 17% smaller in absolute value than that predicted by Eq. (6.28).

We now establish a direct connection between the displacement variable and the equilibrium parameter through the equilibrium equation, Eq. (6.29). Equating the expression for the \( F^{(3)}(\epsilon, \chi) \), as given by Eq. (6.18), to \( \lambda \), and then solving the resulting equation for \( \chi \), we obtain:

\[
\chi = \frac{\epsilon^2 2\lambda + \epsilon^2 + 10}{2 \left( 4\lambda + \epsilon^2 + 4 \right)} .
\]  

(6.34)

Table 6.1 exhibits the values of the displacement variable, as furnished by this relation, for the three values of the equilibrium parameter associated with the configurations we are taking as of reference; also the corresponding values of the relative Shafranov shift, as evaluated according to Eq. (4.39).

| \( \lambda \) | \( \chi^{(3)} \) | \( \delta^{(3)} \) |
|----------------|----------------|----------------|
| 1/5            | 152/1275 \( \approx 0.1192 \) | \( \sqrt{72777} \)/255 \( \approx 0.05793 \) |
| 0              | 127/650 \( \approx 0.1954 \) | \( \sqrt{20202} \)/130 \( \approx 0.09334 \) |
| \(-\frac{1}{5}\) | 122/525 \( \approx 0.2324 \) | \( \sqrt{13587} \)/105 \( \approx 0.1101 \) |

Table 6.1 The values of the displacement variable (\( \chi \)) and of the relative Shafranov shift (\( \delta \)) according to the third order of approximation to the flux function for \( \epsilon = 2/5 \) and the three reference values of the equilibrium parameter (\( \lambda \)).

For the purpose of drawing flux maps the expression for the flux function of Eq. (6.11) is perfectly adequate, being enough to substitute \( \epsilon \) and \( \lambda \) by the numerical
values relative to the configurations being considered. The position of the magnetic axis is always a datum of relevance and must be supplied independently by evaluating the relative Shafranov shift with the help of the equilibrium equation. In this paper, however, for the reasons we have expounded in Section V, we shall favour in general the use of the flux function normalized to the absolute value of the magnetic flux at the magnetic axis, and this requires that the expression of Eq. (6.11) be divided by minus that of Eq. (6.20). We should then have a statement for the doubly normalized flux function having reference to two physical parameters, $\lambda$ and $\chi$, which, however, are not independent between themselves but interrelated through the equilibrium equation. It would be clearly preferable to this situation to dispose of an expression for the normalized flux function depending on only one characteristic parameter, as it is the equilibrium problem itself, either $\lambda$ or $\chi$. In the case of the approximation of the third order to the flux function, which we are presently considering, the connection between these two quantities can be translated into an algebraic equation that is linear both in $\lambda$ and $\chi$, from which explicit definitions of either parameter in terms of the other can be extracted, but in approximations of higher orders the equilibrium function is a quotient of polynomials in the displacement variable of degrees higher than the first, and this makes it impossible in general to have $\chi$ stated explicitly as a function of $\lambda$. The inverse alternative (to express $\lambda$ in terms of $\chi$) is however always possible, being as it is the statement of the equilibrium equation itself. This is one of the reasons why the convenient parameter to which to refer in defining an equilibrium problem be the displacement variable rather than the equilibrium parameter, although it is the second of these two quantities the one to participate ostensibly in the mathematical formulation of it.

We thus substitute $\lambda$ by $F^{(3)}(\epsilon, \chi)$, as given by Eq. (6.18), in Eq. (6.11), divide the result by minus the expression of the poloidal flux at the magnetic axis, as given
by Eq. (6.20), and obtain the normalized flux function as:

\[
\hat{\psi}^{(3)}(x, \theta; \chi, \epsilon) = \frac{\epsilon^2 - x^2}{d^{(3)}} \left[ (3\epsilon^2 - 8\chi)x^2 + 8\chi\epsilon^2 - 32\epsilon^2 - \epsilon^4 - 32\chi x \cos \theta - 4\epsilon^2 x^2 \cos 2\theta \right] \\
+ \frac{1}{d^{(3)}} \left[ 32(\epsilon^2 - \chi)x^3 \cos 3\theta + (7\epsilon^2 - 8\chi)x^4 \cos 4\theta \right],
\]

(6.35)

where

\[
d^{(3)} \equiv d^{(3)}(\epsilon, \chi) = \epsilon^2(8\chi^2 - 8\chi\epsilon^2 + 32\epsilon^2 + \epsilon^4),
\]

(6.36)

and we have used the notation introduced in Eq. (5.5) for the left hand side.

We now replace \(\epsilon\) by \(2/5\) and \(\chi\) by the three numerical values appearing in Table 6.1 in Eq. (6.35) to obtain the expressions for the flux function corresponding to the three values we are considering for the equilibrium parameter. For the purposes of the present paper it is sufficient to have reproduced here just the expression we get relative to the equilibrium with \(\lambda = 0\):

\[
\hat{\psi}^{(3)}(x, \theta; \lambda = 0, \epsilon = 2/5) = -\frac{16250}{274673} \left[ \left( -x + \frac{2}{5} \right) \left( x + \frac{2}{5} \right) \left( \frac{2486}{25} + 22x^2 \\
+ 127x \cos \theta + 13x^2 \cos 2\theta \right) + 23x^3 \cos 3\theta + 9x^4 \cos 4\theta \right].
\]

(6.37)

Plots of the level curves portraying the flux function in the approximation of the third order in the domain comprised between the surface identified by \(\hat{\psi}^{(3)}(x, \theta) = -1\) (which reduces to the magnetic axis) and that identified by \(\hat{\psi}^{(3)}(x, \theta) = 0\) for \(\epsilon = 2/5\) and \(\lambda = 1, 0\) and \(-1/5\) are given in Figs. 6.3(a), (b) and (c) respectively. Superimposed on each flux map, a circle allows a visual comparison between the contour of the surface \(\hat{\psi}^{(3)}(x, \theta) = 0\) and that of the boundary of the cross section of a torus of inverse aspect ratio equal to 2/5 and centre line placed at \(x = 0\). The most salient discrepancy between both contours is seen to belong to the map for \(\lambda = -1/5\), and can be attributed to the high value assumed by the displacement.
\[ \lambda = 1 \]  
\[ \lambda = 0 \]  
\[ \lambda = -1/5 \]

**FIG. 6.3** Flux surfaces in the approximation of order \( N = 3 \) to the flux function for the equilibria in a torus of inverse aspect ratio 2/5 and equilibrium parameter \( \lambda \) equal to (a) 1; (b) 0; (c) \(-1/5\). In terms of the normalized flux function, the spacing between two neighboring flux surfaces in flux space is \( \Delta \hat{\psi}^{(3)} (x, \theta) = 1/20 \). The outermost of the flux surfaces, as seen in the figure, corresponds to the flux \( \hat{\psi}^{(3)} (x, \theta) = 0 \) and the innermost one, which defines the magnetic axis, corresponds to the flux \( \hat{\psi}^{(3)} (x, \theta) = -1 \). The circle drawn in thick line represents the contour of the cross section of a torus of inverse aspect ratio equal to 2/5 and centre line located at \( x = 0 \).
variable in this case. The error carried by the flux function in the approximation of
the third order, as by that of any order of approximation in general, is not, however,
a monotonic function of the value assumed by $\chi$.

The function that expresses the normalized error associated with the flux func-
tion $\hat{\psi}^{(3)}(x, \theta; \chi, \epsilon)$, according to the definition introduced by Eq. (5.6), is:

$$E^{(3)}(\theta; \chi, \epsilon) = -\frac{\epsilon}{8{\chi}^2 - 8{\chi}\epsilon^2 + 32{\epsilon}^2 + {\epsilon}^4} \left[32(\chi - {\epsilon}^2) \cos 3\theta + \epsilon(8\chi - 7{\epsilon}^2) \cos 4\theta\right].$$

(6.38)

A graph of $E^{(3)}(\theta; \chi, \epsilon)$ for $\epsilon = 2/5$ and the value of $\chi$ equal to that corresponding
to $\lambda = 0$ is displayed in Fig. 6.4. Note that the maximum error in absolute value
occurs at the angle $\theta = 0$. Also for the equilibrium configurations labelled by $\lambda = 1$
and $\lambda = -1/5$ (with $\epsilon = 2/5$) the errors of maximum magnitudes occur at the outer
dege of the torus. Table 6.2 displays the values reached by the function $E^{(3)}(\theta; \chi, \epsilon = 2/5)$ at the angular position $\theta = 0$ for the three values we are taking as of reference
for the equilibrium parameter. As it should be expected from the inspection of
the flux maps, the largest error is that which accompanies the configuration with
$\lambda = -1/5$. Table 6.2 also shows that, when $\lambda$ decreases from unity to zero, the
maximum error in absolute value decreases, although the relative Shafranov shift be
then increased.

Of course there is nothing of necessity in that the error attains its maximum
absolute value at $\theta = 0$ as it happens to attain for the three cases just examined.
It is possible, for example, to make $E^{(3)}(\theta; \chi, \epsilon)$ vanish at this same angular position
by choosing the displacement variable to be:

$$\chi = \frac{\epsilon^2(32 + 7\epsilon)}{32 + 8\epsilon}$$

(6.39a)

$$= \frac{87}{550} \simeq 0.1582,$$

the numerical values applying to the value $\epsilon = 2/5$ for the inverse aspect ratio. For
such an equilibrium the values of $\delta$ and $\lambda$ are:

$$\delta = \frac{7\sqrt{286}}{110} - 1 \simeq 0.07619 , \quad (6.39b)$$

$$\lambda = \frac{532}{1625} \simeq 0.3274 . \quad (6.39c)$$

FIG. 6.4 The poloidal angle ($\theta$) dependence of the function $\mathcal{E}^{(3)}(\theta; \chi, \epsilon)$ describing the normalized error relative to the $N = 3$ order of approximation to the flux function along the circular contour $x = \epsilon$ of the torus cross section for the equilibrium characterized by having $\epsilon = 2/5$ and $\lambda = 0$.

$$\mathcal{E}(\theta; \epsilon = 2/5, \lambda = 0)$$

$\theta$ (degrees)

| $\lambda$   | 1     | 0     | $-1/5$ |
|-------------|-------|-------|--------|
| $\mathcal{E}^{(3)}(\theta = 0; \chi, \epsilon)$ | 0.1074 | -0.0107 | -0.1979 |

Table 6.2 The values of the function describing the normalized errors associated with the third order of approximation to the flux function ($\mathcal{E}^{(3)}$) at the outer border of the torus ($\theta = 0$) for $\epsilon = 2/5$ and the three reference values of the equilibrium parameter ($\lambda$).

A graph depicting the behaviour of the error as a function of the poloidal angle in this case is given in Fig. 6.5. The maximum absolute value is reached at $\theta = 180^\circ$ and it is worth only 0.9050% of the flux at the magnetic axis. The flux function writes as:
The normalized error $E^{(3)}(\theta; \epsilon = 2/5, \chi = 87/550)$ as a function of the poloidal angle.

\[
\hat{\psi}^{(3)}(x, \theta; \chi = 87/550, \epsilon = 2/5) = -\frac{275}{7203} \left( \frac{4}{25} - x^2 \right) \left( \frac{472}{3} + 25x^2 + \frac{1450}{9}x \cos \theta \right) \\
+ \frac{550}{27} x^2 \cos 2\theta + \frac{6875}{194481} \left( 2x^3 \cos 3\theta - 5x^4 \cos 4\theta \right).
\]

A map of the flux surfaces that is implied by this expression is given in Fig. 6.6; on account of the smallness of the error it is barely possible to distinguish visually the contour of the surface $\hat{\psi}^{(3)} = 0$ from the one of circular shape this surface would assume if the boundary condition were exactly satisfied.

Other situations of interest are those in which one of the two terms of the normalized error, by appropriate choice of \( \chi \), is made to vanish. Thus, if we take $\chi = \epsilon^2$, we obtain from Eq. (6.38):

\[
E^{(3)}(\theta; \chi = \epsilon^2) = -\frac{\epsilon^2}{32 + \epsilon^2} \cos 4\theta,
\]

the term of the third harmonic in the poloidal angle of the flux function having been suppressed at the boundary.
FIG. 6.6  Flux surfaces as described by the flux function in the approximation of order \( N = 3 \) for the equilibrium characterized by having \( \epsilon = 2/5 \) and \( \chi = 87/550 \).

Similarly, by choosing \( \chi = \frac{7\epsilon^2}{8} \), the term of the fourth harmonic of the poloidal angle in the expression for the flux function at \( x = \epsilon \) is made to vanish, and Eq. (6.38) for the error becomes:

\[
\mathcal{E}^{(3)}(\theta; \epsilon, \chi = \frac{7\epsilon^2}{8}) = \frac{32\epsilon}{256 + \epsilon^2} \cos 3\theta .
\]  

Table 6.3 exhibits the numerical values of the main parameters characterizing these two equilibria whose flux functions contain just one harmonic of the poloidal angle at the boundary. As it could be expected, the maximum error drops sensibly when the term of the third harmonic is absent from the function \( \mathcal{E}^{(3)}(\theta; \chi, \epsilon) \) in comparison with that which results when the harmonic made to be absent is the fourth.

It may be of some interest to determine the local extrema of the function \( \mathcal{E}^{(3)}(\theta; \chi, \epsilon) \), their magnitudes, and the angles where they occur. Because of the symmetry of the flux function with respect to the equatorial plane of the torus, two of such positions are defined by the angles \( \theta = 0 \) and \( \theta = \pi \). The remaining ones can be obtained by deriving Eq. (6.38) with respect to \( \theta \) and equating the result to zero;
Number of the harmonic of the poloidal angle present in the function \( \mathcal{E}^{(3)}(\theta; \chi, \epsilon) \) & \( \lambda \) & \( \chi \) & \( \delta \) & \( |\mathcal{E}^{(3)}(\theta, \chi, \epsilon)|_{\text{max}} \) \\
3 & \( \frac{72}{125} \approx 0.5760 \) & 0.1400 & 0.06771 & 0.04997 \\
4 & \( \frac{23}{75} \approx 0.3067 \) & 0.1600 & 0.07703 & 0.004975 \\

**Table 6.3** Data concerning the equilibrium configurations and the maximum values of the relative errors associated with the approximation of the order \( N = 3 \) to the flux function when the function \( \mathcal{E}^{(3)}(\theta; \chi, \epsilon) \) contains either one of the third and the fourth harmonics of the poloidal angle.

After dividing by \( \sin \theta \) and expressing the angular functions of the arcs multiple of \( \theta \) in terms of \( \cos \theta \), we are led to the following algebraic equation for \( \cos \theta \equiv \mu \):

\[
2\epsilon\alpha_0\mu^3 + 24\alpha_1\mu^2 - \epsilon\alpha_0\mu - 6\alpha_1 = 0 ,
\]

where

\[
\alpha_0 \equiv 8\chi - 7\epsilon^2
\]

and

\[
\alpha_1 \equiv \chi - \epsilon^2
\]

As an example of results obtained from the use of this equation, Table 6.4 exhibits the data regarding the local extrema of the function \( \mathcal{E}^{(3)}(\theta; \chi, \epsilon) \) when the value attributed to the equilibrium parameter is \( \lambda = 0 \).

Another measure of the degree of accuracy reached by an approximate solution, and one which bears a closer correspondence with its visual representation as provided by the flux map than does the relative error, is the relative deviation, already
introduced in Section V by Eq. \((5.7)\).

| \(\theta\) (degrees) | 0   | 57.02 | 116.11 | 180  |
|---------------------|-----|-------|--------|------|
| \(E^{(3)}(\theta; \chi, \epsilon)\) | -0.1007 | 0.09513 | -0.08189 | 0.07346 |

**Table 6.4** Measures of the poloidal angle and values assumed by \(E^{(3)}(\theta; \chi, \epsilon)\) for \(\epsilon = 2/5\) and \(\lambda = 0\) at the angular positions where this function reaches a local extremum.

Figure 6.7 depicts the dependence of the normalized radial coordinate \(x\) on the poloidal angle \(\theta\) at the flux surface specified by \(\tilde{\psi}^{(3)}(x, \theta; \chi, \epsilon) = 0\) for the equilibrium having \(\epsilon = 2/5\), \(\lambda = 0\); it is just the plot of \(x\), taken as a function implicitly defined by the flux function of Eq. \((6.37)\) under the constraint that the value of this last-mentioned function be kept at its boundary value, **versus** the angle \(\theta\), but while in drawing the flux contour the variables of the couple \((x, \theta)\) are interpreted as polar coordinates, here the cartesian ones are employed. Of course, if \(\tilde{\psi}(x, \theta)\) in general were the exact solution to the boundary value problem, a graph like this would reduce to a straight line parallel to the horizontal axis passing through the ordinate \(x = \epsilon\); with the fluctuations, as those seen in Fig. 6.7, it shows the extent to which the contour of the boundary that is implied by the solution actually found departs from the shape of the one it intends to be an approximation.

The same as for the relative error, it is of interest to determine the local maxima and minima of the relative deviation. Again by reasons of symmetry the outer and the inner edges of the torus are two positions at which \(x\) and \(D\) go through extrema of values. To find which are these values we solve the equations:

\[
8x^4 \pm 32x^3 + 16(2 - \chi)x^2 \mp 32x(\chi x - [8(4 - \chi) + \epsilon^2]x) = 0,
\]

which follow from Eq. \((6.35)\) with \(\tilde{\psi}^{(3)}(x, \theta; \chi, \epsilon)\) taken to be zero, the one with the upper signs by putting \(\theta = 0\) and the one with the lower signs by putting \(\theta = \pi\). The
solutions with physical meaning for the one and the other equations are respectively:

\[ x = \mp 1 \pm \sqrt{1 + \chi \pm \sqrt{\chi^2 - \chi \epsilon^2 + 4 \epsilon^2 + \frac{\epsilon^4}{8}}}. \] (6.47)

Not always, however, the dominant extremum of \( x \) is located at \( \theta = 0 \) or \( \theta = \pi \).

To determine the local extrema of \( x \) inside the interval comprised between these two angular positions we note that, upon infinitesimal variations \( dx \) and \( d\theta \) of \( x \) and \( \theta \), a flux function of general order of approximation \( \widehat{\psi}^{(N)}(x, \theta; \chi, \epsilon) \) undergoes an infinitesimal change \( d\widehat{\psi}^{(N)} \) given by:

\[ d\widehat{\psi}^{(N)} = \left. \frac{\partial \widehat{\psi}^{(N)}}{\partial x} \right|_\theta \, dx + \left. \frac{\partial \widehat{\psi}^{(N)}}{\partial \theta} \right|_x \, d\theta. \] (6.48)

Now, the flux surface we are referring to is defined by:

\[ \widehat{\psi}^{(N)}(x, \theta; \chi, \epsilon) = \text{constant} = 0, \] (6.49)

so that the differential \( d\widehat{\psi}^{(N)} \) on the left hand side of Eq. (6.48) is zero. If \( x \) is to pass through an extreme value as \( \theta \) is increased by \( d\theta \), then \( dx \) in Eq. (6.48) also
vanishes. We are thus left with:

\[
\frac{\partial \hat{\psi}^{(N)}}{\partial \theta} \bigg|_{x=0} = 0 .
\]  

(6.50)

Equations (6.49) and (6.50) form a system of two equations for \( x \) and \( \theta \) (or trigonometric functions of \( \theta \)) which specify both the angular positions where \( x \) reaches local extrema and the values of these on the fixed surface \( \hat{\psi}^{(N)} = 0 \).

We now consider specifically the case \( N = 3 \) using Eq. (6.35) for the normalized flux function. Expressing the trigonometric functions whose arguments are multiples of the poloidal angle in terms of \( \cos \theta \equiv \mu \), taken as the free variable, the equation that translates the condition that \( \hat{\psi}^{(N)} (x, \theta; \chi, \epsilon) \) be the outermost magnetic surface can be put in the form:

\[
8\alpha_0 x^4 \mu^4 + 128\alpha_1 x^3 \mu^3 - 8(2\alpha_2 x^2 - \epsilon^4)x^2 \mu^2 - 32(\alpha_2 x^2 - \chi \epsilon^2)x \mu + 8\epsilon^2 (2\chi - 4 - \epsilon^2)x^2 - (8\chi - 32 - \epsilon^2)\epsilon^4 = 0,
\]  

(6.51)

where

\[
\alpha_2 \equiv 4\chi - 3\epsilon^3 = \alpha_0 - 4\alpha_1
\]  

(6.52)

and \( \alpha_0 \) and \( \alpha_1 \) are as defined in Eqs. (6.44) and (6.45).

Under similar manipulation the equation that expresses the requirement that \( x \) be an extremum, after being divided by \( \sin \theta \), can be conducted to assume the shape:

\[
2\alpha_0 x^3 \mu^3 + 24\alpha_1 x^2 \mu^2 - (2\alpha_2 x^2 - \epsilon^4)x \mu - 2\alpha_2 x^2 + 2\chi \epsilon^2 = 0.
\]  

(6.53)

It is possible to obtain the solutions for \( x \) and \( \mu \) of the system constituted by Eqs. (6.51) and (6.53) by solving just one equation. We first introduce a transformation of the unknown \( \mu \) according to:

\[
\mu = \frac{y}{x} ,
\]  

(6.54)
obtaining in place of Eqs. (6.51) and (6.53) two equations for \( y \) and \( x \) in which \( x \) appears only to the second power. Isolating \( x^2 \) from the transformed version of Eq. (6.53) we have:

\[
x^2 = \frac{2\alpha_0 y^3 + 24\alpha_1 y^2 + \epsilon^4 y + 2\chi\epsilon^2}{2\alpha_2(y + 1)} .
\]  

(6.55)

Upon substitution of \( x^2 \) as given by this expression in the transformed version of Eq. (6.51), we are led to the following equation for the quantity \( y \):

\[
c_0 y^5 + c_1 y^4 + c_2 y^3 + c_3 y^2 + c_4 y + c_5 = 0 ,
\]  

(6.56)

where

\[
\begin{align*}
c_0 &= 8\alpha_0\alpha_2 , \\
c_1 &= 8\alpha_2(32\chi - 29\epsilon^2) , \\
c_2 &= 128(8 - \epsilon^2)\chi^2 - 16\epsilon^2(96 - 11\epsilon^2)\chi + 8\epsilon^4(68 - 7\epsilon^2) , \\
c_3 &= -8\epsilon^2[32\chi^2 - 2(24 + 23\epsilon^2)\chi + 48\epsilon^2 + 15\epsilon^4] , \\
c_4 &= \epsilon^4(4\chi - 16 - \epsilon^2)\alpha_0 , \\
c_5 &= \epsilon^4[16\chi^2 - 4(24 + 5\epsilon^2)\chi + 3\epsilon^2(32 + \epsilon^2)] .
\end{align*}
\]  

(6.57)

Once Eq. (6.56) has been solved, Eq. (6.55) can be used to determine the values of \( x \) associated with those found for \( y \), and then, from Eq. (6.54), the values of \( \mu \) (and \( \theta \)) corresponding to the pairs \((y, x)\). Of course not all of the five solutions obtained for \( y \) will give rise to physically meaningful solutions for the pair \((x, \mu)\).

Numerical programs like the ones built in Maple 7 are able to handle the system constituted by Eqs. (6.51) and (6.53) keeping its original form of two coupled equations for \( x \) and \( \mu \), but as a rule they require that the intervals where the roots are located be specified. For this a graphical solution to the system can be helpful. For example, for \( \epsilon = 2/5 \) and \( \lambda = 0 \), the two equations corresponding to Eq. (6.51) and Eq. (6.53) can be written respectively as:

\[
x^4\mu^4 + \frac{23}{18}x^3\mu^3 + \left(\frac{13}{225} - \frac{49}{36}x^2\right)x^2\mu^2 + \left(\frac{127}{450} - \frac{49}{18}x^2\right)x\mu + \frac{1243}{5625} - \frac{49}{36}x^2 = 0
\]  

(6.58)

and

\[
x^3\mu^3 + \frac{23}{24}x^2\mu^2 + \left(\frac{13}{450} - \frac{49}{72}x^2\right)x\mu + \frac{127}{1800} - \frac{49}{72}x^2 = 0 .
\]  

(6.59)
Figure 6.8 displays the two curves for $x$, as they are each implicitly defined by one and the other of these two equations as a function of $\mu$, on the same graph. By inspection we are able to find that one of the roots of $x$ of interest belongs to the interval $(0.42, 0.44)$ and the other to the interval $(0.38, 0.40)$. These informations are sufficient for the program to solve the system.

As an example of the results obtained by use of these equations, Table 6.5 exhibits the angular positions and values of the extrema of the relative deviation associated with the third order of approximation to the flux function for the equilibrium characterized by having $\epsilon = 2/5$ and $\lambda = 0$. Table 6.6 is a collection of the data regarding the maximum absolute values reached by the relative deviations pertaining to this same order of approximation to the flux function for the equilibria labelled by $\epsilon = 2/5$ and the several values of $\lambda$ we have been considering in this Section.

For those equilibria for which $|D|_{\text{max}} \ll 1$ it is possible to find an explicit representation of the coordinate $x$ in terms of the poloidal angle $\theta$ that approximates...
the description of the surface defined by \( \hat{\psi}(x, \theta; \chi, \epsilon) = 0 \) and thus an approximate expression for the deviation as a function of \( \theta \). Indeed, if the quantity \( x - \epsilon \) remains small at all points along the contour \( \hat{\psi}(x, \theta; \chi, \epsilon) = \) constant, then it suffices to retain the terms up to the linear one in the Taylor expansion of \( \hat{\psi}(x, \theta; \chi, \epsilon) \) about the toroidal surface on which \( x = \epsilon \); that is to say, we may approximate:

\[
\hat{\psi}(x, \theta; \chi, \epsilon) \simeq \hat{\psi}(x = \epsilon, \theta; \chi, \epsilon) + (x - \epsilon) \frac{\partial \hat{\psi}(x, \theta; \chi, \epsilon)}{\partial x} \bigg|_{x=\epsilon}.
\]

(6.60)

| \( \theta \) (degrees) | 0  | 59.2466 | 119.6028 | 180 |
|------------------------|----|---------|----------|-----|
| \( D^{(3)} \)         | 0.03702 | -0.03580 | 0.06573 | -0.05942 |

Table 6.5 Angular positions (\( \theta \)) and values of the extrema of the relative deviation \( D \) associated with the flux function \( \psi^{(3)}(x, \theta; \chi, \epsilon) \) for the equilibrium \( \epsilon = 2/5, \lambda = 0 \).

| \( \lambda \) | \( \theta \) (degrees) | \( |D^{(3)}|_{\text{max}} \) |
|--------------|-----------------------|---------------------|
| 1            | 180                   | 0.08522             |
| 0            | 119.6028              | 0.06573             |
| \(-\frac{1}{5}\) | 121.8675             | 0.2078              |
| \(\frac{23}{75}\) | 180                   | 0.004167            |
| \(\frac{72}{125}\) | 180                   | 0.04280             |
| \(\frac{532}{1625}\) | 180                   | 0.007595            |

Table 6.6 Angular positions (\( \theta \)) and magnitudes of the maxima of the extrema reached by the relative deviations (\( |D|_{\text{max}} \)) associated with the flux functions \( \psi^{(3)}(x, \theta; \chi, \epsilon) \) in the angular interval \([0^\circ, 180^\circ]\) for the equilibria having \( \epsilon = 2/5 \) and the values of \( \lambda \) displayed in the left column.
Now, if the flux function on the left hand side is made to coincide with the function specifying the surface \( \hat{\psi}(x, \theta; \chi, \epsilon) = 0 \), then an approximate value of the coordinate \( x \) along this surface is obtained from the above expression as:

\[
x \simeq \epsilon - \frac{\hat{\psi}(x, \theta; \chi, \epsilon)}{\partial \hat{\psi}(x, \theta; \chi, \epsilon)/\partial x} \bigg|_{x=\epsilon} \quad (6.61)
\]

and the relative deviation is uniformly approximated by:

\[
D \equiv D(\theta; \chi, \epsilon) \simeq -\frac{1}{x \partial \ln \hat{\psi}(x, \theta; \chi, \epsilon)/\partial x} \bigg|_{x=\epsilon} \quad . \quad (6.62)
\]

This expression holds valid whatever be the order of approximation to be used for the flux function. If that is the third one, taking for \( \hat{\psi}(x, \theta; \chi, \epsilon) \) the expression given by Eq. (6.35), the function \( D(\theta; \chi, \epsilon) \) can be evaluated to be:

\[
D^{(3)}(\theta; \chi, \epsilon) \simeq \epsilon \frac{32(\chi - \epsilon^2) \cos 3\theta + \epsilon(8\chi - 7\epsilon^2) \cos 4\theta}{4 \epsilon(16 - \epsilon^2) + 16\chi \cos \theta + 2\epsilon^3 \cos 2\theta - 24(\chi - \epsilon^2) \cos 3\theta - 8\epsilon(8\chi - 7\epsilon^2) \cos 4\theta} . \quad (6.63)
\]

The above expression can be used to determine the extrema of the deviation in an approximate way without the need of solving a pair of coupled equations, as it is required by the exact calculation. The content of Table 6.7 is an example of the results obtained by such an application of the explicit representation of the deviation given by Eq. (6.63), in which the equilibrium parameters were taken to be \( \epsilon = 2/5 \) and \( \lambda = 0 \), and should be compared with that of Table 6.5, which shows the results of the exact calculation.

To conclude this Section we determine the condition of breakdown of the boundary condition for the flux function as expressed according to its third order of approximation, that is to say, the critical value of \( \chi \) at which the surface \( \psi^{(3)}(x, \theta; \epsilon, \chi) = 0 \)
\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
\(\theta\) (degrees) & 0 & 59.3234 & 119.3787 & 180 \\
\hline
\(\mathcal{D}^{(3)}\) & 0.03823 & -0.03481 & 0.06613 & -0.05955 \\
\hline
\end{tabular}
\end{center}

Table 6.7 Angular positions \((\theta)\) and values of the extrema of the relative deviation \(\mathcal{D}\) in the interval \([0^\circ, 180^\circ]\) for the equilibrium \(\epsilon = 2/5, \lambda = 0\) as evaluated with the help of the approximate formula for \(\mathcal{D}\) given by Eq. (6.63).

For the kind of magnetic configuration we are considering, a magnetic surface can cease to be topologically equivalent to that of a torus, and becomes a separatrix within the set of all magnetic surfaces implied by the flux function. For this purpose we find it convenient to have the flux function represented in terms of the normalized cylindrical coordinate variables \(\rho\) and \(Z\) rather than in terms of the coordinate variables of the toroidal-polar system we have been using throughout. By using the transformation relations stated in Eq. (5.21) in Ref. [4], Eq. (6.35) assumes the form:

\[
\tilde{\psi}^{(3)}(\rho, Z; \chi, \epsilon) = \frac{1}{\epsilon^2 [8 \chi (\chi - \epsilon^2) + 32 \epsilon^2 + \epsilon^4] \left\{ 8 \epsilon^2 \rho^4 + 16 [(4 \chi - 3 \epsilon^2) Z^2 - (\chi + 1) \epsilon^2] \rho^2 
- 8 [2 \chi (4 + \epsilon^2) - 10 \epsilon^2 - \epsilon^4] Z^2 + \epsilon^2 [8 (2 + \epsilon^2) \chi + 8 - 32 \epsilon^2 - \epsilon^4] \right\}}, \tag{6.64}
\]

For the kind of magnetic configuration we are considering, a magnetic surface can be identified as a separatrix if it includes points (called singular points) where the poloidal field (and thus the gradient of the flux function) vanishes [10]. The critical value \(\chi_c\) of the displacement variable at which the plasma outermost surface becomes a separatrix of the magnetic configuration and the coordinates of the singular points are therefore determined by:

\[
\frac{\partial \tilde{\psi}^{(3)}(\rho, Z; \chi, \epsilon)}{\partial \rho} = 0, \tag{6.65}
\]
\[
\frac{\partial \tilde{\psi}^{(3)}(\rho, Z; \chi, \epsilon)}{\partial Z} = 0 \tag{6.66}
\]
and

\[ \hat{\psi}^{(3)}(\rho, Z; \chi, \epsilon) = 0. \]  \hspace{1cm} (6.67)

Using Eq. (6.64) for \( \hat{\psi}^{(3)}(\rho, Z; \chi, \epsilon) \), Eqs. (6.65) and (6.66) become respectively:

\[ (4\chi - 3\epsilon^2)Z^2 + \epsilon^2 \rho^2 - \epsilon^2(1 + \chi) = 0 \] \hspace{1cm} (6.68)

and

\[ 2(4\chi - 3\epsilon^2)\rho^2 - 2(4 + \epsilon^2)\chi + 10\epsilon^2 + \epsilon^4 = 0, \] \hspace{1cm} (6.69)

while the equation corresponding to the condition stated in Eq. (6.67) is obtained by putting the expression between keys on the right hand side of Eq. (6.64) equal to zero.

In general, for other orders of approximation to the flux function, the algebraic equations that translate the conditions analogous to those given by Eq. (6.65), Eq. (6.66) and Eq. (6.67) have to be solved numerically in the form they are originally stated of three simultaneous equations for the unknowns \( \chi \), \( \rho \) and \( Z \). For the present case of the third order of approximation, however, it is possible to reduce the problem to the solution of a single equation. Indeed, from Eq. (6.69), in which no term containing reference to the variable \( Z \) appears, we can isolate \( \rho^2 \) as:

\[ \rho^2 = \frac{(8 + 2\epsilon^2)\chi - 10\epsilon^2 - \epsilon^4}{2(4\chi - 3\epsilon^2)}. \] \hspace{1cm} (6.70)

Substituting this expression for \( \rho^2 \) in Eq. (6.68) and then solving it for \( Z^2 \) we obtain:

\[ Z^2 = \frac{\epsilon^2(8\chi^2 - 8\epsilon^2\chi + 4\epsilon^2 + \epsilon^4)}{2(4\chi - 3\epsilon^2)}. \] \hspace{1cm} (6.71)

Finally, the expression in terms of \( \chi \) solely that we have been able to determine for \( \rho^2 \) is inserted in the equation that expresses the vanishing of \( \hat{\psi}^{(3)}(\rho, Z; \chi, \epsilon) \). The terms depending on \( Z \) in this equation are cancelled out and we are led to the
following equation for $\chi$:

$$\chi^3 - 3 \left(2 + \frac{5}{8} \epsilon^2\right) \chi^2 + \epsilon^2 (10 + \epsilon^2) \chi + \frac{\epsilon^2}{64} (32 - 272 \epsilon^2 - 7 \epsilon^4) = 0 . \quad (6.72)$$

An approximate solution to the above equation for the root with physical significance in the form of a series of powers of the inverse aspect ratio can be obtained as:

$$\chi = \sqrt{\frac{3}{6}} \epsilon + \frac{121}{144} \epsilon^2 - \frac{103\sqrt{3}}{20736} \epsilon^3 - \frac{49}{15552} \epsilon^4 - \frac{30715\sqrt{3}}{4775744} \epsilon^5 + \ldots \quad (6.73)$$

For $\epsilon = 2/5$ Eq. (6.72) becomes:

$$\chi^3 - 63 \frac{10}{625} \chi^2 + \frac{1016}{625} \chi - \frac{457}{15625} = 0 . \quad (6.74)$$

The root of interest of Eq. (6.74) to four decimal places, obtained either by solving it numerically or by using the expansion about $\epsilon = 0$ given by Eq. (6.73), is:

$$\chi \equiv \chi_c^{(3)} \simeq 0.2493 . \quad (6.75)$$

Substituting this value for $\chi$ and $2/5$ for $\epsilon$ in Eqs. (6.70) and (6.71) we determine the normalized cylindrical coordinates of the singular point as:

$$\rho_c \simeq 0.6584$$

$$Z_c \simeq 0.5024 . \quad (6.76)$$

Comparing the upper limit of $\chi$ as dictated by the requirement that the magnetic axis be internal to the plasma containing vase with the critical value $\chi_c^{(3)}$ just determined we conclude that the allowed domain of variation of the displacement variable previously stated in Eq. (6.23) has to be replaced by this:

$$0.04000 < \chi < 0.2493 . \quad (6.77)$$

The corresponding interval of variation of the relative Shafranov shift is:

$$0.01980 < \delta < 0.1177 , \quad (6.78)$$
and that of the equilibrium parameter $\lambda$ is:

$$-0.2678 < \lambda < \infty . \quad (6.79)$$

The limits found for $\chi$, $\delta$ and $\lambda$ are, of course, those that stem from the approximation of order $N = 3$ to the solution to the Grad-Shafranov boundary value problem. Although the figures might change according to the order of approximation, we shall see in the next Sections that it remains true that the allowed range of variation of $\chi$ is defined to be:

$$\chi_P^{(N)} < \chi < \chi_c^{(N)} , \quad (6.80)$$

where $\chi_P^{(N)}$ is the real pole of the equilibrium function $F^{(N)}(\epsilon, \chi)$ of the least positive value, and $\chi_c^{(N)}$ is the critical value of $\chi$ under the approximation of order $N$ to the flux function. Figure 6.9 represents graphically the dependence of the critical value of the displacement variable on the inverse aspect ratio; also shown are the curve for the maximal value of $\chi$ according to the criterium that the magnetic axis must be internal to the toroidal chamber, and the curve for the minimal value of $\chi$ according to the criterium that the poloidal flux must be a negatively defined quantity at the magnetic axis, $N$ assumed to be 3.

Figure 6.10 provides us with a view of the structural changes that the bounding surface of a magnetic configuration of intermediate inverse aspect ratio, as portrayed by a low order approximation to the flux function, experiences when the value of the displacement variable increases from subcritical to critical and then to supercritical.

The two closed lines that appear on the two sides of Fig. 6.10(a), one as the reflection of the other about the vertical centre line, represent the (cartesian) plot of the contour $Z = Z(\rho)$ of the plasma cross section as it is implied by the equation $\hat{\psi}^{(3)}(\rho, Z) = 0$, for $\epsilon = 2/5$ and $\chi$ a little smaller than the critical value $\chi_c^{(3)}$.

The lines that are seen on the top and at the bottom of Fig. 6.10(a), which arise as branches of
Distorted as it is, it remains nonetheless topologically equivalent to a circle and concepts like error and deviation are still applicable.

\[ \chi_{\min}^{(3)} \]

\[ \chi_{\max}^{(3)} \]

\[ \epsilon \]

**FIG. 6.9** The curve in thick line represents the dependence of the critical value of the displacement variable \( \chi \) on the inverse aspect ratio \( \epsilon \) according to the approximation of order \( N = 3 \) to the flux function, as given implicitly by Eq. (6.72). The upper curve in thin line represents the variation of the maximal allowed value of \( \chi \), as defined in Eq. (5.1), and the lower curve in thin line, the variation of the minimal value of \( \chi \), as given by Eq. (6.24), when the inverse aspect ratio is increased from \( \epsilon = 0 \) to \( \epsilon = 1 \). The point marked on the thick line defines the critical condition for the configuration having \( \epsilon = 2/5 \).

Figure 6.10(b) shows the configuration assumed by the field lines when \( \chi \) takes on the precise critical value \( \chi_c \). The singular points (which in Ref. [1] are graphically

the level curve for \( \hat{\psi}^{(3)}(\rho, Z) = 0 \) in addition to the lines representing the plasma contour, cannot be lent the meaning of “fields lines” even if we disregard the effect of the metallic case in shielding the outside of the container from any magnetic fields internally generated, for they do not close upon themselves and extend to infinity, the magnitudes of the axial components of the magnetic fields along their paths growing without limit as they depart from the plasma neighborhood. Lacking physical sense as they do, it is interesting however to keep them represented on the flux maps to watch their advance towards the plasma vicinity as the value of \( \chi \) is increased bit to bit from \( \chi < \chi_c \) to \( \chi_c \), then the emergence of singular points as \( \chi \) reaches the critical value \( \chi_c \) and they meet the field lines running along the plasma boundary (Fig. 6.10(b)), and finally the rupture of the “magnetic wall” around the plasma as \( \chi \) is further increased beyond \( \chi_c \) and they connect to separate branches of the field lines, which, once closed and encircling the plasma, appear now as broken into disjoined pieces (Fig. 6.10(c)).
(a) $\chi = 0.24$

FIG. 6.10 Level curves of $\psi^{(3)}(\rho, Z; \chi, \epsilon = 2/5) = 0$ for (a) $\chi = 0.24$; (b) $\chi = 0.2493$ (critical value of the displacement variable); (c) $\chi = 0.25$. 
referred to as “X-points”) appear as those, of coordinates \((\rho_c, \pm Z_c)\), where the field lines coming from remote regions outside the plasma region meet the field lines that run along the plasma boundary, now converted into branches of the separatrix of the configuration. At a singular point the field vanishes, providing the plasma with an aperture where to escape from the trap contrived by the magnetic architecture of the toroidal pinch; confinement is thereby lost and we can no longer speak properly of equilibrium.

Finally Fig. 6.10(c) shows into which becomes the level curve corresponding to 
\[ \hat{\psi}^{(3)}(\rho, Z) = 0 \] for \(\chi\) slightly greater than \(\chi_c\). The branch that kept surrounding the magnetic axis with progressive distortion in shape along the process of gradual increase of \(\chi\) since values close but smaller than \(\chi_c\) is now broken into two disconnected ones and any kinship the boundary has once had with a toroid geometry is destroyed. With the breakdown of the boundary condition the flux function ceases to be the mathematical correspondent to the state of a physical system and measures telling of accuracy like the relative error and the relative deviation lose all meaning.

VII. SOLUTION TO THE GRAD-SHAFRANOV BOUNDARY VALUE PROBLEM ACCORDING TO THE FOURTH ORDER OF APPROXIMATION TO THE FLUX FUNCTION

In this Section and in the remaining ones in this paper we shall adhere to the representation of the flux function that employs the displacement variable \((\chi)\) as the quantity to characterize globally an equilibrium configuration rather than to those that employ instead the relative Shafranov shift \((\delta)\) or the equilibrium parameter \((\lambda)\).

With the inclusion of a term proportional to the multipole solution of order \(n = 3\)
in the expression for the partial flux, the equilibrium function passes to be defined as:

\[
F^4(\epsilon, \chi) = -\frac{\begin{vmatrix}
M_{11}(\epsilon) & M_{12}(\epsilon) & M_{13}(\epsilon) & -P_1(\epsilon) \\
M_{21}(\epsilon) & M_{22}(\epsilon) & M_{23}(\epsilon) & -P_2(\epsilon) \\
M_{31}(\epsilon) & M_{32}(\epsilon) & M_{33}(\epsilon) & -P_3(\epsilon) \\
V'_1(\chi) & V'_2(\chi) & V'_3(\chi) & -P'(\chi)
\end{vmatrix}}{D^4(\epsilon, \chi)}
\] (7.1)

with

\[
D^4(\epsilon, \chi) = \frac{\begin{vmatrix}
M_{11}(\epsilon) & M_{12}(\epsilon) & M_{13}(\epsilon) & Q_1(\epsilon) \\
M_{21}(\epsilon) & M_{22}(\epsilon) & M_{23}(\epsilon) & Q_2(\epsilon) \\
M_{31}(\epsilon) & M_{32}(\epsilon) & M_{33}(\epsilon) & Q_3(\epsilon) \\
V'_1(\chi) & V'_2(\chi) & V'_3(\chi) & Q'(\chi)
\end{vmatrix}}{144(1-\epsilon^2)^2}. 
\] (7.2)

Using the formulae for \(P_i(x)\) \((i = 1, 2, 3)\) and \(P'(\chi)\), the formulae for \(Q_i(x)\) \((i = 1, 2, 3)\) and \(Q'(\chi)\) given in Appendix A for the elements in the last column of one and the other of the two above determinants respectively, the formulae for \(M_{ij}(x)\) \((i, j = 1, 2, 3)\) given in Appendix B for the elements \(M_{ij}(\epsilon)\) in the first three rows and three columns of both determinants, and those given in Appendix C for the elements \(V'_i(\chi)\) \((i = 1, 2, 3)\) that belong to the first three columns in the last rows idem, we are able to arrive at the following expression for the equilibrium function of the fourth order:

\[
F^4(\epsilon, \chi) = \frac{1}{3} \left(1 - \frac{\epsilon^2}{2}\right) \left(\frac{16}{9} + \frac{5}{12} \epsilon^2\right) \chi^2 - \frac{1}{2} \left(1 - \frac{\epsilon^2}{2}\right) \frac{1 - \epsilon^2}{2} \chi^2 - \frac{1}{2} \left(1 + \frac{5}{12} \epsilon^2\right) \chi^2 + \frac{4}{9} \epsilon^2 + \frac{7}{48} \epsilon^4
\] (7.3)

This rational function of \(\chi\) has two zeros, which are:

\[
\chi^{(4)}_{Z1, Z2} = \frac{8/3}{1 - \frac{\epsilon^2}{2}} \left(\pm \sqrt{1 + \frac{45}{32} \epsilon^2 - \frac{627}{4096} \epsilon^4 + \frac{27}{4096} \epsilon^6 - \frac{195}{16384} \epsilon^8 - 1 - \frac{15}{64} \epsilon^2 - \frac{9}{128} \epsilon^4}\right), 
\] (7.4)

the first one corresponding to the positive sign in front of the square root on the right hand side and the second to the negative sign. The zero \(\chi^{(4)}_{Z1}\) vanishes for
\[ \epsilon = 0, \text{ is positive for } 0 < \epsilon \leq 1 \text{ and for this range of values of } \epsilon \text{ falls in the interval of variation of } \chi \text{ for which values this same variable is capable to satisfy the demands of physical provenience that are put on it. The zero } \chi^{(4)}_{Z_2}, \text{ in contradistinction, is large and negative, and placed on the } \chi \text{-axis in a position far from those that can be possibly associated with an equilibrium configuration.} \]

The equilibrium function \( F^{(4)}(\epsilon, \chi) \) has also two poles, which are:

\[ \chi_{P_1, P_2}^{(4)} = \frac{8}{9} \left( 1 \mp \sqrt{1 - \frac{3}{32} \epsilon^2 - \frac{531}{4096} \epsilon^4} \right) + \frac{5}{24} \epsilon^2. \]  

(7.5)

The pole \( \chi_{P_1}^{(4)} \), the one for which the square root is to be taken with the minus sign, is always positive, vanishing as \( \epsilon \) goes to zero, and the interval of variation it defines on the \( \chi \)-axis as \( \epsilon \) ranges from zero to unity covers the interval in which the displacement variable assumes the values belonging to the set of those that can be put in correspondence with a situation of physical significance. The pole \( \chi_{P_2}^{(4)} \), also positive for all values of \( \epsilon \), lies on the \( \chi \)-axis beyond the point that defines the maximal value the displacement variable can assume if the limits set on its scale of variation by considerations of physical order are to be respected.

Neither the zero \( \chi_{Z_2}^{(4)} \) nor the pole \( \chi_{P_2}^{(4)} \) of the equilibrium function \( F^{(4)}(\epsilon, \chi) \) have counterparts in the structure of the equilibrium function \( F^{(3)}(\epsilon, \chi) \), but while the zero does not introduce a significant change in the aspect of the graph of \( F^{(4)}(\epsilon, \chi) \) with regard to that of \( F^{(3)}(\epsilon, \chi) \), the pole causes a discrepant behaviour between the curves of the two functions, which, however, as we have seen, is observed only in a domain of the displacement variable that lies beyond that of physical interest.

The zero \( \chi_{Z_1}^{(4)} \) and the pole \( \chi_{P_1}^{(4)} \) of \( F^{(4)}(\epsilon, \chi) \) are essentially the same as the zero \( \chi_{Z}^{(3)} \) and the pole \( \chi_{P}^{(3)} \) of \( F^{(3)}(\epsilon, \chi) \), only shifted in position on the \( \chi \)-axis by effect of the nonzero value of the inverse aspect ratio. Indeed, using Eqs. (7.4) and (6.16) for the corresponding zeros of the two equilibrium functions, and Eqs. (7.5) and (6.17)
for the corresponding poles, it is possible to show that:

$$\chi Z_1^{(4)} = \chi Z_1^{(3)} \left( 1 - \frac{61}{320} \epsilon^2 + O(\epsilon^4) \right)$$ (7.6)

and

$$\chi P_1^{(4)} = \chi P_1^{(3)} \left( 1 + \frac{15}{64} \epsilon^2 + O(\epsilon^4) \right).$$ (7.7)

In the interval of variation of $\chi$ of physical interest, $F^{(4)}(\epsilon, \chi)$ does not pass through a maximum nor a minimum, but exhibits an inflexion point whose position is given as the solution to an equation of the third degree in $\chi$ with $\epsilon$-dependent coefficients. For $\epsilon \ll 1$ it is possible to find a representation in series of fractional powers of the inverse aspect ratio for this solution, the first few terms of which are:

$$\chi_I^{(4)} = \frac{4}{3} \epsilon^{2/3} - \epsilon^{4/3} + \epsilon^2 - \frac{65}{144} \epsilon^{8/3} + \frac{5}{24} \epsilon^{10/3} + O(\epsilon^4).$$ (7.8)

At the inflexion point the value assumed by the equilibrium function, under the same assumption that the inverse aspect ratio is small, can be expanded as:

$$F^{(4)}(\chi_I) = -1 - \epsilon^{2/3} + \frac{3}{4} \epsilon^{4/3} + \frac{3}{4} \epsilon^2 + \frac{13}{96} \epsilon^{8/3} - \frac{1}{256} \epsilon^{10/3} + O(\epsilon^4).$$ (7.9)

Comparison of this expression with that of Eq. $6.18$ shows that, for $\epsilon$ small, the value of $F^{(4)}(\chi_I)$ departs little from that at which $F^{(3)}(\chi)$ reaches its asymptotic limit as $\chi \to \infty$.

Figures 7.1(a), (b) and (c) show the graphical representations of the equilibrium function $F^{(4)}(\epsilon, \chi)$ for $\epsilon = 1/50$, 2/5 and 4/5 respectively. For small values of the inverse aspect ratio an approximate representation of the equilibrium function in the form of a partial fraction expansion proves to be helpful to clarify the connection between this and the equilibrium function of the third order. Keeping only terms of order $\epsilon^2$ in the numerator and in the denominator we have:

$$F^{(4)}(\chi, \epsilon) \approx \frac{1}{3} \left( 1 - \frac{\epsilon^2}{2} \right) + \frac{\epsilon^2}{\chi^2} + \frac{64}{27} \frac{1 - \frac{5}{16} \epsilon^2}{\chi - \frac{16}{9} - \frac{\epsilon^2}{6}}.$$ (7.10)
FIG. 7.1 Plots of the equilibrium function of the fourth order $F^{(4)}(\epsilon, \chi)$ for the inverse aspect ratio $\epsilon$ equal to (a) $1/50$; (b) $2/5$; (c) $4/5$. 
The second term on the right hand side of Eq. (7.10), which becomes the dominant one in the region of physical interest, is essentially the same as the second term on the right hand side of Eq. (6.18). We note that, for negligible values of \( \epsilon \) and \( \chi \), the first term on the right hand side of Eq. (7.10) adds to the third term to produce \( F^{(4)}(\epsilon, \chi) \sim -1 \), which is the value taken by the first term of \( F^{(3)}(\epsilon, \chi) \) in the representation of Eq. (6.18) under equal assumptions. Thus it appears that the first term in the equilibrium function in the passing from the third to the fourth order of approximation unfolds into two terms whose contributions to \( F^{(4)}(\epsilon, \chi) \) in the region of physical interest combine to reproduce approximately the value that would be given by the unsplit term of the original equilibrium function. This explains why the equilibrium equations as stated for the same values of the equilibrium parameter on the right hand side and the two equilibrium functions on the left hand side respectively do not lead to widely discrepant values for the displacement variable, although \( F^{(3)}(\epsilon, \chi) \) and \( F^{(4)}(\epsilon, \chi) \) are structurally distinct functions.

According to Eq. (4.1), the expression of the partial flux for \( N = 4 \) comprises the particular solution and the multipole solutions of the four lowest orders in combination. For the evaluation of the constant \( K_0^{(4)}(\epsilon, \chi) \) that multiplies the multipole solution \( \varphi^{(0)}(x, \theta) = 1 \) we refer to Eq. (4.37), and for the evaluation of the constants \( K_1^{(4)}(\epsilon, \chi) \), \( K_2^{(4)}(\epsilon, \chi) \) and \( K_3^{(4)}(\epsilon, \chi) \) that multiply the multipole solutions \( \varphi^{(1)}(x, \theta) \), \( \varphi^{(2)}(x, \theta) \) and \( \varphi^{(3)}(x, \theta) \), to Eq. (4.35), applying the substitution scheme stated as Eq. (4.43) to write the elements in the last rows of the determinants that appear as numerators and denominators in these two equations as functions of \( \chi \). By having recourse to the collections of expressions that constitute the contents of Appendices A, B and C we may then give definite forms to all the elements of the determinants involved in the resulting formulae for the four constants \( K_i^{(4)}(\epsilon, \chi) \) \((i = 0, 1, 2, 3)\). The equilibrium parameter can be eliminated from the final expression for the flux.
function in favour of $\chi$ if we substitute $\lambda$ by $F^{(4)}(\epsilon, \chi)$, as given by Eq. (7.3), in the expression of the particular solution $\psi_p(x, \theta)$ (Eq. (3.27)).

The flux function at the magnetic axis, $\psi^{(4)}_M(\epsilon, \chi)$, can be determined either directly, by using the general formula of Eq. (4.49) appropriately specialized to the case $N = 4$, or by evaluating the expression to which is reduced that of the flux function, $\psi^{(4)}(x, \theta; \epsilon, \chi)$, once it have been obtained, at the magnetic axis, where the coordinate variables assume the values $\theta = 0$, $x = (1 + \chi)^{1/2} - 1$. In a graphical representation, $\psi^{(4)}_M(\epsilon, \chi)$ would show to participate of the same general aspect as that of $\psi^{(3)}_M(\epsilon, \chi)$, how it can be apprehended, for example, from the graph in Fig. 6.1, which applies to $\epsilon = 2/5$. In particular, it would make apparent that the flux at the magnetic axis is negative for $\chi > \chi^{(4)}_P$ and positive for $\chi < \chi^{(4)}_P$.

The flux function normalized to minus the magnetic flux at the magnetic axis is obtained as:

$$
\psi^{(4)}(x, \theta; \epsilon, \chi) = \frac{x^2 - \epsilon^2}{d^{(4)}(\epsilon, \chi)} \left[Y_0^{(4)}(x; \epsilon, \chi) + xY_1^{(4)}(x; \epsilon, \chi) \cos \theta + x^2Y_2^{(4)}(x; \epsilon, \chi) \cos 2\theta + x^3Y_3^{(4)}(\epsilon, \chi) \cos 3\theta + \frac{1}{d^{(4)}} \left[x^4W_4^{(4)}(x; \epsilon, \chi) \cos 4\theta + x^5W_5^{(4)}(\epsilon, \chi) \cos 5\theta + x^6W_6^{(4)}(\epsilon, \chi) \cos 6\theta \right] \right], \quad (7.11)
$$

where

$$
d^{(4)} \equiv d^{(4)}(\epsilon, \chi) = \chi^4 - 4\epsilon^2 \chi^3 - \left(8 - \frac{55}{16}\epsilon^2\right)\epsilon^2 \chi^2 - \frac{7}{8}\epsilon^6 \chi + 16\epsilon^4 - \frac{\epsilon^6}{4} + \frac{21}{128}\epsilon^8, \quad (7.12)
$$

$$
Y_0^{(4)}(x; \epsilon, \chi) = \frac{1}{8}(\chi - \epsilon^2)x^4 - \left[\frac{15}{8}\chi^2 - \left(4 + \frac{17}{16}\epsilon^2\right)\chi + \frac{3}{2}\epsilon^2 + \frac{71}{128}\epsilon^4 \right]x^2 - \left(12 - \frac{21}{8}\epsilon^2\right)\chi^2 - \frac{7}{8}\epsilon^4 + 16 \epsilon^2 - \frac{\epsilon^4}{4} + \frac{21}{128}\epsilon^6, \quad (7.13)
$$

$$
Y_1^{(4)}(x; \epsilon, \chi) = -\frac{1}{2}(\chi - \epsilon^2)x^2 - 12\chi^2 + \left(16 + \frac{13}{4}\epsilon^2\right)\chi, \quad (7.14)
$$

$$
Y_2^{(4)}(x; \epsilon, \chi) = -\frac{5}{16}(\chi - \epsilon^2)x^2 - \frac{3}{2}\chi^2 + \left(4 - \frac{5}{16}\epsilon^2\right)\chi - 2\epsilon^2 + \frac{23}{32}\epsilon^4, \quad (7.15)
$$

$$
Y_3^{(4)}(\epsilon, \chi) = \frac{15}{4}(\chi - \epsilon^2), \quad (7.16)
$$
\[ W_4^{(4)}(x; \epsilon, \chi) = \frac{7}{8}(\chi - \epsilon^2)x^2 + \frac{3}{8} \chi^2 + \left(16 - \frac{15}{16} \epsilon^2\right) \chi - \frac{33}{2} \epsilon^2 + \frac{107}{128} \epsilon^4, \quad (7.17) \]
\[ W_5^{(4)}(\epsilon) = \frac{35}{4}\left(\chi - \epsilon^2\right), \quad (7.18) \]
\[ W_6^{(4)}(\epsilon; \chi) = \frac{21}{16}(\chi - \epsilon^2). \quad (7.19) \]

Once we are in possession of the expressions for the equilibrium function and for the flux function, the fourth order solution to the equilibrium problem for any given value of the equilibrium parameter proceeds in complete analogy with the third order solution as it was developed in Section VI. Table 7.1 summarizes the results for the equilibrium configurations having the inverse aspect ratio equal to \( \epsilon = 2/5 \) and equilibrium parameters equal to \( \lambda = 1, 0 \) and \(-1/5\) respectively.

| \( \lambda \) | \( (\chi)^{(4)} \) | \( (\delta)^{(4)} \) | \( (\mathcal{E}_{\max}^{(4)} \times 10^2) \) | \( (\mathcal{D}_{\max}^{(4)} \times 10^2) \) |
|----------------|-----------------|----------------|-------------------------|-------------------------|
| 1              | 0.1214          | 0.05896        | -5.500                  | 2.935 (180)             |
| 0              | 0.1905          | 0.09112        | 3.770                   | 2.075 (132.92)          |
| -1/5           | 0.2190          | 0.1041         | 8.120                   | 5.947 (133.83)          |

**Table 7.1**  Values assumed by the displacement variable \( (\chi) \) and by the relative Shafranov shift \( (\delta) \), extreme values of the relative error \( (\mathcal{E}_{\max}^{(4)}) \), relative deviations of the normalized radial variable from \( x = \epsilon \) at the surface \( \psi^{(4)} = 0 \) whose absolute values are maximal \( (\mathcal{D}_{\max}^{(4)}) \) and the poloidal angles where they are observed for the configurations characterized by having the equilibrium parameter \( \lambda \) equal to 1, 0 and \(-1/5\) respectively and aspect ratio fixed as \( \epsilon = 2/5 \), when the fourth order of approximation to the flux function is employed. The errors \( \mathcal{E}_{\max}^{(4)} \) occur at the poloidal angle \( \theta = 0^\circ \) for all the three values of \( \lambda \) listed.

The comparison between the data exhibited in Table 7.1 with those appearing in Tables 6.2 and 6.7 for the same values of \( \lambda \), and in particular for \( \lambda = -1/5 \), shows that the relative errors and the relative deviations are much decreased when
the order of approximation to the flux function is increased from \( N = 3 \) to \( N = 4 \). Level curves of the flux function as obtained with the latter order of approximation for \( \epsilon = 2/5 \) and the three reference values of \( \lambda \) are given in Figs. 7.2(a), (b) and (c) respectively.

The expression for the flux function \( \tilde{\psi}^{(4)}(x, \theta; \epsilon, \chi) \), as given by Eq. (7.11), shows that the error at the boundary associated with the fourth harmonic can be eliminated if the coefficient \( W_4^{(4)}(x; \epsilon, \chi) \) is made to vanish for \( x = \epsilon \). From Eq. (7.17) the condition by which this can be made to happen translates as:

\[
\chi^2 + \left( \frac{128}{3} - \frac{\epsilon^2}{6} \right) \chi - 44\epsilon^2 - \frac{5}{48}\epsilon^4 = 0 ,
\]

and thus \( \chi \) must assume the value:

\[
\chi = \frac{64}{3} \left( \sqrt{1 + \frac{91}{1024}\epsilon^2 + \frac{\epsilon^4}{4096}} - 1 \right) + \frac{\epsilon^2}{12}.
\]

Similarly, the errors at the boundary coming from the terms of the fifth and sixth harmonics can be simultaneously suppressed by choosing for the displacement variable the value:

\[
\chi = \epsilon^2.
\]

Results obtained for the equilibrium configurations derived from these choices for \( \chi \) are summarized in Table 7.2.

To determine the equilibrium configuration for which branching points appear on the boundary surface, as in the study undertaken with the third order of approximation to the flux function, it is preferable to use cylindrical rather than toroidal-polar coordinates. The system to be solved is that which translates the condition of simultaneous vanishing of \( \tilde{\psi}^{(4)}(\rho, Z; \chi, \epsilon) \) and of the two components of its gradient. In this case of a partial flux of the order \( N = 4 \) it is still possible to combine the three equations by which it is constituted to establish a single relation connecting the
FIG. 7.2 Flux surfaces according to the approximation of order $N = 4$ to the flux function for the equilibria in a torus of inverse aspect ratio 2/5 and equilibrium parameter $\lambda$ equal to (a) 1; (b) 0; (c) $-1/5$. 

(a) $\lambda = 1$

(b) $\lambda = 0$

(c) $\lambda = -1/5$
critical value of the displacement variable and the value of the inverse aspect ratio of the configuration, the steps to be followed to reach this end we pass to indicate briefly.

| Numbers of the harmonics of the poloidal angle that are present in the function $\mathcal{E}^{(4)}(\theta, \chi)$ | $\lambda$ | $\lambda^{(4)}$ | $\delta^{(4)}$ | $\mathcal{E}_{\max}^{(4)} \times 10^2$ | $\mathcal{D}_{\max}^{(4)} \times 10^2$ |
|---|---|---|---|---|---|
| 5, 6 | 0.2524 | 0.1645 | 0.07913 | 0.1152 | 0.08621 |
| 4 | 0.3067 | 0.1600 | 0.07703 | 0.4975 | 0.4167 |

Table 7.2 Data concerning the equilibrium configurations of inverse aspect ratio $\epsilon = 2/5$ as portrayed by the fourth order of approximation to the flux function when, by appropriate choice of the values of the displacement variable, one or more of the harmonics of the poloidal angle are made to disappear from the function $\mathcal{E}^{(4)}(\theta; \epsilon, \chi)$ that describes the error at the boundary $x = \epsilon$. The quantities displayed are the same as those of Table 7.1. The maximum relative errors and the maximum relative deviations occur at the poloidal angles $\theta = 0^\circ$ and $\theta = 180^\circ$ respectively.

We first substitute the coordinate variables in the three equations of the original system by the variables $\varrho = \rho^2$ and $\zeta = Z^2$. Observing that the equation proceeding from the condition $\partial \psi^{(4)}(\rho, Z)/\partial Z = 0$ is linear in $\zeta$, we may solve it to obtain a representation of $\zeta$ in terms of $\varrho$ and $\chi$. By replacing this expression for $\zeta$ in the two other equations, we are led to a system for the quantities $\varrho$ and $\chi$ of the fourth degree in each of them. The appropriate combination of these two equations to obtain a single equation for $\chi$ can be achieved by a method due to Sylvester and known under the name of dyalitic elimination [12], which in this case concerns specifically the variable $\varrho$. This task is facilitated by the capability that has the
program Maple 7 \[11\] to construct, in obedience to the order of execution of the command \texttt{SylvesterMatrix} contained in its package \texttt{LinearAlgebra}, the proper Sylvester matrix, the determinant of which furnishes the equation for $\chi$ that is satisfied by the common root belonging to the two equations of the reduced system for any given value of the inverse aspect ratio. After factoring out two terms that would give rise to only spurious roots (of which the distinct ones are in number of three), we are led to an equation of the 18th degree in $\chi$, the coefficients of its terms being polynomials in $\epsilon^2$ whose degrees vary from the third to the 20th. For the reader reference, the equation for $\epsilon = 2/5$ can be found in Appendix E. Note that, of the fourteen roots that this equation admits as real, only one is to meet the requirements that make $\chi$ be critical according to our definitions.

A graph of the critical value of the displacement variable as a function of the inverse aspect ratio, obtained by applying the command \texttt{Implicitplot} of Maple 7 \[11\] to the equation whose derivation we have just outlined, is given in Fig. 7.3. For values of $\epsilon$ less than $1/4$, certainly because of the large number of roots, many of them having small and close values, Maple is not able to distinguish the path followed by a particular root as the value of $\epsilon$ is varied from the paths followed by the others, and all it appears on the plot is a blot.

The high degree of the equation that gives the critical value of the displacement variable for an arbitrary value of the inverse aspect ratio within the framework of the approximation of order $N = 4$ to the flux function warns us that equations of parallel purpose cannot in practice be derived for partial flux functions of order superior to the fourth. The recourse that then remains to us is to fix the numerical value of $\epsilon$ at the start and solve the system of three equations that defines the value of $\chi, \rho$ and $Z$ at the critical condition numerically. Results for $\epsilon = 2/5$, obtained by using any of the methods that prove to be suitable for the order $N = 4$, are
displayed in Table 7.3. The level curve for the critical boundary surface is presented in Fig. 7.4.

FIG. 7.3 Dependence of the critical value of the displacement variable with the inverse aspect ratio within the framework of the approximation of order \( N = 4 \) to the flux function. The point marked on the curve defines the critical condition for the configuration having \( \epsilon = \frac{2}{5} \).

| \( \chi^{(4)}_c \) | \( \delta^{(4)}_c \) | \( \lambda^{(4)}_c \) | \( \rho^{(4)}_c \) | \( Z^{(4)}_c \) |
|-----------------|-----------------|----------------|----------------|----------------|
| 0.2543          | 0.1200          | -0.3840        | 0.6241         | ±0.3645        |

Table 7.3 Critical values of the displacement variable (\( \chi_c \)), relative Shafranov shift (\( \delta_c \)) and equilibrium parameter (\( \lambda_c \)), and values of the radial and axial cylindrical coordinates (\( \rho_c \) and \( Z_c \)) of the branching points on the flux surface \( \psi^{(4)} = 0 \) for \( \epsilon = 2/5 \), as obtained from the fourth order of approximation to the flux function.
FIG. 7.4 The level curve associated with the surface $\tilde{\psi}^{(4)}(\rho, Z; \epsilon = 2/5, \chi) = 0$ at the critical value of the displacement variable, $\chi = \chi_c^{(4)}$.

VIII. THE STRUCTURES OF THE EQUILIBRIUM FUNCTION
AND OF THE PARTIAL FLUX FUNCTION, THE DILATION OF
THE LIMITS OF VALIDITY AND THE EVOLUTION OF THE
MEASURES OF ACCURACY OF THE APPROXIMATE
SOLUTIONS TO THE GRAD-SHAFRANOV BOUNDARY VALUE
PROBLEM WITH THE INCREASE OF THE ORDER OF
APPROXIMATION TO THE PARTIAL FLUX FUNCTION

By applying the procedure expounded in Section IV we have determined the expressions for the equilibrium functions $F^{(N)}(\epsilon, \chi)$ of all orders ranging from $N = 3$ to $N = 10$. The expression for $F^{(3)}(\epsilon, \chi)$ is given by Eq. (6.15). For the equilibrium functions of orders 5, 7, 9 and in general odd and higher than the third, it was found that they can all be written under the following common guise:

$$F^{(N)}(\epsilon, \chi) = A^{(N)}(\epsilon^2) \frac{[\chi - \chi_1^{(N)}(\epsilon^2)][\chi - \chi_2^{(N)}(\epsilon^2)][\chi - \chi_3^{(N)}(\epsilon^2)]f^{(N)}_{N-5}(\epsilon^2, \chi)}{[\chi - \chi_P^{(N)}(\epsilon^2)]g^{(N)}_{N-3}(\epsilon^2, \chi)},$$

(8.1)
where \( f_{N-5}^{(N)}(\epsilon^2, \chi) \) and \( g_{N-3}^{(N)}(\epsilon^2, \chi) \) are polynomials of the form:

\[
\begin{align*}
  f_{N-5}^{(N)}(\epsilon^2, \chi) &= \chi^{N-5} + a_1(\epsilon^2)\chi^{N-6} + \cdots + a_{N-6}(\epsilon^2)\chi + a_{N-5}(\epsilon^2), \\
  g_{N-3}^{(N)}(\epsilon^2, \chi) &= \chi^{N-3} + b_1(\epsilon^2)\chi^{N-4} + \cdots + b_{N-4}(\epsilon^2)\chi + b_{N-3}(\epsilon^2),
\end{align*}
\]

(8.2) 

(8.3)

with coefficients \( a_1(\epsilon^2), \ldots, a_{N-5}(\epsilon^2) \) and \( b_1(\epsilon^2), \ldots, b_{N-3}(\epsilon^2) \) that depend on the square of the inverse aspect ratio, and whose roots, for \( \epsilon \) positive and less than unity, are all complex. The function \( A^{(N)}(\epsilon^2) \), which we shall call the amplitude function of order \( N \), equally depends on the square of the inverse aspect ratio.

The real zero \( \chi_{Z}^{(N)}(\epsilon^2) \) and the real pole \( \chi_{P}^{(N)}(\epsilon^2) \), which are also functions of the square of the inverse aspect ratio, are positive definite for \( 0 \leq \epsilon \leq 1 \) and fall in the range of \( \chi \) of physical interest, meaning this that they are smaller than the critical value of the displacement variable as evaluated in accordance with the order \( N \) of approximation to the flux function. The real zero \( \chi_{1}^{(N)}(\epsilon^2) \) is large and negative, and, as \( N \) is increased to values larger than \( N = 5 \), it moves towards the origin, remaining however always greater than unity in absolute value. The real zero \( \chi_{2}^{(N)}(\epsilon^2) \) is large and positive, and also approaches the origin with the increase of \( N \), but keeping itself beyond the point of abscissa \( \chi = 1 \). Between the two real and positive zeros, \( \chi_{Z}^{(N)}(\epsilon^2) \) and \( \chi_{2}^{(N)}(\epsilon^2) \), the equilibrium function becomes negative, and exhibits a point of minimum located by an abscissa on the \( \chi \)-axis whose measure is larger than the critical value of the displacement variable. This last fact precludes the possibility of the arising of two magnetic axes for the same (negative) value of the equilibrium parameter.\(^5\)

The equilibrium functions of even orders (for \( N = 4, 6, 8, 10, \ldots \)) were found to

\(^5\)The equilibrium function \( F^{(3)}(\epsilon, \chi) \) has only one zero, that is \( \chi_{Z}^{(3)}(\epsilon^2) \), and, therefore, no local minimum occurs for \( \chi_{P}^{(3)}(\epsilon^2) < \chi < \infty \). This makes its behaviour for \( \chi > \chi_{Z}^{(3)} \) be distinct from the behaviour for \( \chi > \chi_{Z}^{(N)} \) common to the equilibrium functions of odd orders higher than the third, and also explains why the amplitude function \( A^{(3)}(\epsilon^2) \) is the only one to be negative among all the amplitude functions of odd orders.
obey in general a representation of the form:

\[ F^{(N)}(\epsilon, \chi) = A^{(N)}(\epsilon^2) \frac{[\chi - \chi_1^{(N)}(\epsilon^2)][\chi - \chi_{\pi}^{(N)}(\epsilon^2)][h_{N-4}^{(N)}(\epsilon^2, \chi)]}{[\chi - \chi_P(\epsilon^2)][\chi - \chi_{\pi}^{(N)}(\epsilon^2)][j_{N-4}^{(N)}(\epsilon^2, \chi)]}, \quad (8.4) \]

where \( h_{N-4}^{(N)}(\epsilon^2, \chi) \) and \( j_{N-4}^{(N)}(\epsilon^2, \chi) \) are complete polynomials in \( \chi \) of degrees \( N - 4 \) with coefficients that are functions of \( \epsilon^2 \) save those for the highest power of \( \chi \), which, as in \( f_{N-3}^{(N)}(\epsilon^2, \chi) \) and \( g_{N-3}^{(N)}(\epsilon^2, \chi) \) in Eqs. (8.2) and (8.3), are equal to unity. The real zero \( \chi_1^{(N)}(\epsilon^2) \), for a given even value of \( N \), locates itself on the \( \chi \)-axis between the two real zeros \( \chi_1^{(N-1)}(\epsilon^2) \) and \( \chi_1^{(N+1)}(\epsilon^2) \) of the equilibrium functions of orders \( N - 1 \) and \( N + 1 \) respectively, as these are defined for odd orders by Eq. (8.1), and therefore, like the two last-mentioned zeros, it approaches the origin from the left as \( N \) is increased to values above \( N = 4 \). The pole \( \chi_\pi^{(N)}(\epsilon^2) \) of the equilibrium functions of even orders, which is absent in the equilibrium functions of odd orders, is real and positive, and approaches the origin as \( N \) is increased from \( N = 4 \) to \( N = 10 \), yet remaining larger than unity and therefore outside the range of \( \chi \) of physical interest.

In the representation of Eq. (8.4) for the equilibrium functions of even orders, the polynomials \( h_{N-4}^{(N)}(\epsilon^2, \chi) \) and \( j_{N-4}^{(N)}(\epsilon^2, \chi) \) have only complex roots.

The amplitude function \( A^{(N)}(\epsilon^2) \) is the limit, for a given \( \epsilon \), of the equilibrium function \( F^{(N)}(\epsilon, \chi) \) as \( \chi \to \infty \). Except for \( N = 3 \), it is positive definite and less than unity, and, for a fixed \( \epsilon \), decreases monotonically with increasing \( N \), with all appearances approaching a non-null limiting value as \( N \) tends to infinity; for \( \epsilon = 2/5 \) and \( N = 10 \), it is worth 0.1978.

The pole \( \chi_P \) separates the values of the displacement variable that are associated with the configurations for which \( \bar{\psi} \) is positive at the magnetic axis (those that have \( \chi < \chi_P \)) from the values of the displacement variable that are associated with the configurations for which \( \bar{\psi} \) is negative at the magnetic axis (those that have \( \chi > \chi_P \)). The zero \( \chi_Z \) gives the value of the displacement variable that makes the
equilibrium configuration be magnetically neutral, that is to say, with null poloidal current density; equilibria for which \( \chi < \chi_Z \) are paramagnetic and those for which \( \chi > \chi_Z \) are diamagnetic. As \( N \) is increased from \( N = 3 \) to \( N = 10 \), the pole and the zero of the equilibrium function \( F^{(N)}(\epsilon, x) \) that belong to the interval of \( \chi \) of physical interest show a clear tendency to convergence as it can be seen in Table 8.1, where their values are given for \( \epsilon = 2/5 \).

In a plot the equilibrium functions of even orders show to follow the same pattern as that of \( F^{(4)}(\epsilon, \chi) \), while those of odd orders follow that of \( F^{(5)}(\epsilon, \chi) \). In Fig. 8.1, graphs of \( F^{(9)}(\epsilon = 2/5, \chi) \) and \( F^{(10)}(\epsilon = 2/5, \chi) \) for \( \chi > 0 \) are shown superimposed one on the other. It can be seen that the point where these two curves seem to depart from a common track to pursue opposite directions is close to the point of inflexion of \( F^{(10)}(\epsilon = 2/5, \chi) \).

| \( N \) | \( \chi_P^{(N)}(\epsilon = 2/5) \) | \( \chi_Z^{(N)}(\epsilon = 2/5) \) |
|---|---|---|
| 3  | 0.04000000 | 0.19538462 |
| 4  | 0.041512632 | 0.19053374 |
| 5  | 0.041571493 | 0.19101171 |
| 6  | 0.041578215 | 0.19094767 |
| 7  | 0.041578812 | 0.19095680 |
| 8  | 0.041578877 | 0.19095539 |
| 9  | 0.041578884 | 0.19095561 |
| 10 | 0.041578885 | 0.19095557 |

**Table 8.1** The pole and the zero of the equilibrium function \( F^{(N)}(\epsilon, \chi) \) within the range \( 0 < \chi < \chi_c^{(N)}(\epsilon) \) for \( \epsilon = 2/5 \) and \( N \) ranging from 3 to 10.

Also shown on the curves in Fig. 8.1 are the points \( (\chi_c^{(9)}, \lambda_c^{(9)}) \) and \( (\chi_c^{(10)}, \lambda_c^{(10)}) \) with abscissas and ordinates equal to the critical values of the displacement variable and of the equilibrium parameter respectively, at which branching points break on the contours of \( \tilde{\psi}^{(9)}(\rho, Z; \epsilon = 2/5, \chi) = 0 \) and \( \tilde{\psi}^{(10)}(\rho, Z; \epsilon = 2/5, \chi) = 0 \); owing to their nearness on the common display, they appear to be fused into a single one.
To be noticed is that these points of critical coordinates, which pinpoint the limits to which the equilibrium variable can be pushed without causing the boundary condition to collapse, are far moved back from the region where the curves start to diverge one from the other. This means that the useful parts of the equilibrium functions do not extend to the domain of $\chi$ where that of even order and that of odd order exhibit disparate behaviours. In general, in the “useful” domain $\chi < \chi_c^{(N)}(\epsilon)$, the locations and the profiles of the curves representing the equilibrium functions built from numbers of multipole solutions moderately high and high, even and odd, are sufficiently close and sufficiently similar to appear coincident on a common display, as do the curves for $F^{(9)}(\epsilon = 2/5, \chi)$ and $F^{(10)}(\epsilon = 2/5, \chi)$ on a strip next to the vertical axis in Fig. 8.1.

![Graph showing equilibrium functions](image)

**FIG. 8.1** The equilibrium functions $F^{(9)}(\epsilon = 2/5, \chi)$ and $F^{(10)}(\epsilon = 2/5, \chi)$. In order of increasing values of $\chi$ are seen on the curves: the points defining the critical equilibria at which $X$-points occur on the flux surfaces $\hat{\psi}^{(9)}(\rho, Z; \epsilon = 2/5, \chi) = 0$ and $\hat{\psi}^{(10)}(\rho, Z; \epsilon = 2/5, \chi) = 0$ (which appear as a single point on the scale of the figure); the point of inflexion of $F^{(10)}(\epsilon = 2/5, \chi)$; the point of minimum of $F^{(9)}(\epsilon = 2/5, \chi)\).
Table 8.2 summarizes the data concerning the equilibrium configurations whose outermost magnetic surfaces exhibit singular points as determined by use of the flux functions in the approximations of order $N = 3$ to $N = 10$; the parameter $\epsilon$ is taken to be equal to $2/5$. No pattern of convergence can be recognized here: as the number $N$ increases the values of the critical displacement and of the critical Shafranov shift are monotonically increased, while those of the critical equilibrium parameter become more and more negative.

| $N$ | $\chi_c^{(N)}$ | $\delta_c^{(N)}$ | $\lambda_c^{(N)}$ | $\rho_c^{(N)}$ | $Z_c^{(N)}$ |
|-----|----------------|-----------------|-------------------|--------------|-----------|
| 3   | 0.2493         | 0.1177          | -0.2678           | 0.6584       | 0.5024    |
| 4   | 0.2543         | 0.1200          | -0.3840           | 0.6241       | 0.3645    |
| 5   | 0.2817         | 0.1321          | -0.4717           | 0.6099       | 0.2869    |
| 6   | 0.2982         | 0.1394          | -0.5323           | 0.6042       | 0.2344    |
| 7   | 0.3116         | 0.1453          | -0.5717           | 0.6024       | 0.1959    |
| 8   | 0.3201         | 0.1490          | -0.5966           | 0.6022       | 0.1663    |
| 9   | 0.3258         | 0.1514          | -0.6122           | 0.6025       | 0.1428    |
| 10  | 0.3294         | 0.1530          | -0.6219           | 0.6029       | 0.1238    |

Table 8.2 Critical values of the displacement variable ($\chi_c$), relative Shafranov shift ($\delta_c$) and equilibrium parameter ($\lambda_c$) at which branching points, with normalized cylindrical coordinates ($\rho_c, Z_c$), appear on the surface $\hat{\psi}^{(N)}(\rho, Z; \epsilon = 2/5, \chi) = 0$, as evaluated by use of the approximations to the flux function of orders ranging from $N = 3$ to $N = 10$.

The critical values of the displacement variable, as obtained from the $N = 3$ order of approximation to the flux function, grow with the inverse aspect ratio faster than the critical values obtained from the next higher order of approximation ($N = 4$) and eventually surpass the latter ones, but this trend of relative growths between curves generated from representations to the flux function of two successive orders of approximation is observed only when that proceeding from the lowest of the two orders has it as $N = 3$. For higher orders of approximation to the flux function, for
any given value of $\epsilon$, the critical value of $\chi$ obtained by using the order $N$ seems to be consistently larger than the critical value obtained by use of the order $N - 1$. Table 8.3 shows the evolution of the critical values of the displacement variable with the order of approximation, from $N = 3$ to $N = 5$, for a range of values of the inverse aspect ratio.

| $\epsilon$ | 3     | 4     | 5     |
|-----------|-------|-------|-------|
| 0.4       | 0.24927 | 0.25432 | 0.28171 |
| 0.5       | 0.35309 | 0.34250 | 0.36697 |
| 0.6       | 0.47333 | 0.44658 | 0.46806 |
| 0.7       | 0.60985 | 0.56771 | 0.58773 |
| 0.8       | 0.76252 | 0.70672 | 0.72732 |
| 0.9       | 0.93114 | 0.86429 | 0.88717 |

**Table 8.3** Values of the critical displacement variable for the configurations with inverse aspect ratios listed in the first column on the left, as evaluated with the help of the approximations to the flux function of the orders $N = 3, 4$ and 5. For the value of the inverse aspect ratio in the second row, the critical values of the displacement variable, according to the orders of approximation $N = 3$ and $N = 4$, are equal.

As a matter of interest, Fig. 8.2 provides us with a view of the transformations undergone by the flux map, as drawn with the help of a relatively high order representation to the flux function, for a configuration having medium-sized inverse aspect ratio, when the displacement variable is increased from a value slightly less than the critical to the critical one, and then from this to a supercritical value.
(a) $\chi = 0.32$

(b) $\chi \approx 0.3294$
\( \chi = 0.35 \).

**FIG. 8.2** Evolution of the flux map for the plasma equilibrium in a torus of inverse aspect ratio \( \epsilon = 2/5 \) with the increase of the value taken by the displacement variable, according to the approximation of order \( N = 10 \) to the flux function. In (a) \( \chi \) is less than and in (c) greater than the critical value \( \chi_c \). The flux map for the equilibrium at \( \chi = \chi_c \) is shown in (b); the point marked on the horizontal axis represents the trace of the magnetic axis on the torus cross section, and the two points that are marked on symmetrical positions above and below the horizontal axis represent the branching points as they appear on the contour of the intersection of the surface \( \psi = 0 \) with a meridional plane.

We can now give form to a statement of convergence regarding the sequence of values that the method generates for the displacement variable. If the equilibrium equation (Eq. (4.45)) is solved for a fixed \( \lambda \) and a growing order \( N \) of approximation to \( F(N)(\epsilon, \chi) \), up to the highest for which such function is presently available, \( \epsilon \) being kept constant along the process, then the solutions found for \( \chi \) show a clear tendency to approach a definite limit, provided they remain always smaller than the critical value \( \chi_c^{(N)} \).

Before presenting the data in support to this claim we look at the question of convergence brought about by the partial flux function \( \hat{\psi}^{(N)}(x, \theta; \epsilon, \chi) \), which is
built, as the equilibrium function $F^{(N)}(\epsilon, \chi)$, from a number $N$ of multipole solutions which, at least in principle, can be increased indefinitely.

Evaluation of the partial fluxes from order $N = 3$ to $N = 10$ shows that they can in general be written in the form of a quotient,

$$
\psi^{(N)}(x, \theta; \epsilon, \chi) = \frac{N^{(N)}(x, \theta; \epsilon, \chi)}{d^{(N)}(\epsilon, \chi)},
$$

(8.5)

the numerator of which is given by the sum of two terms:

$$
N^{(N)}(x, \theta; \epsilon, \chi) = H^{(N)}(x, \theta; \epsilon, \chi) + R^{(N)}(x, \theta; \epsilon, \chi),
$$

(8.6)

whose forms are respectively:

$$
H^{(N)}(x, \theta; \epsilon, \chi) = (\epsilon^2 - x^2) \sum_{k=0}^{N} Y^{(N)}_k(x; \epsilon, \chi)x^k \cos k\theta
$$

(8.7)

and

$$
R^{(N)}(x, \theta; \epsilon, \chi) = \sum_{k=N+1}^{2N} W^{(N)}_k(x; \epsilon, \chi)x^k \cos k\theta,
$$

(8.8)

such that only the first one is apt to satisfy the vanishing condition at $x = \epsilon$ in any case. The functions that appear under the symbols of summation in Eqs. (8.7) and (8.8) are defined to be:

$$
Y^{(N)}_k(x; \epsilon, \chi) = \sum_{j=0}^{l} p^{(N)}_{k,2j}(\epsilon, \chi)x^{2j}
$$

(8.9)

and

$$
W^{(N)}_k(x; \epsilon, \chi) = \sum_{j=0}^{l+1} q^{(N)}_{k,2j}(\epsilon, \chi)x^{2j},
$$

(8.10)

with

$$
l = \begin{cases} 
N - \frac{k}{2} - 1 & \text{if } k \text{ even} \\
N - \frac{k + 1}{2} - 1 & \text{if } k \text{ odd},
\end{cases}
$$

(8.11)

$p^{(N)}_{k,2j}(\epsilon, \chi)$ and $q^{(N)}_{k,2j}(\epsilon, \chi)$ being polynomials in $\chi$ with coefficients that are polynomials in $\epsilon^2$. The denominator in Eq. (8.5) is of the form:

$$
d^{(N)}(\epsilon, \chi) = [\chi - \chi^{(N)}_{P}(\epsilon^2)]t^{(N)}(\epsilon, \chi),
$$

(8.12)
\( t^{(N)}(\epsilon, \chi) \) being a polynomial in \( \chi \) whose coefficients are polynomials in \( \epsilon^2 \).

The expression for the poloidal flux function normalized by minus the poloidal flux at the magnetic axis writes in general as:

\[
\hat{\psi}^{(N)}(x, \theta; \epsilon, \chi) = -\frac{\mathcal{N}^{(N)}(x, \theta; \epsilon, \chi)}{\mathcal{N}^{(N)}(x = \delta(\chi), \theta = 0; \epsilon, \chi)} .
\] (8.13)

All the results obtained from the application of the method to a number of equilibrium configurations, as characterized by the parameter \( \lambda \), are consistent with the supposition that, if the values taken by the displacement variable in the expression for the partial flux \( \hat{\psi}^{(N)}(x, \theta; \epsilon, \chi) \) are the same as those provided by the solution of the equilibrium equation in which the equilibrium function \( F^{(N)}(\epsilon, \chi) \) is also taken to be of order \( N \), then the remainder \( R^{(N)}(x, \theta; \epsilon, \chi) \) in Eq. (8.6) tends to zero as \( N \) tends to infinity. In other words, the sequence of partial fluxes constructed by increasing the number of multipole solutions that are made to enter their compositions, one by one since \( N = 3 \) until they include the totality of those that are presently available (\( N = 10 \)), seems to obey a pattern of convergence in which the exact solution to the Grad-Shafranov boundary value problem appears as the limit for \( N \) tending to infinity.

The data resulting from the solution to the equilibrium equation and those relative to the departure of the boundary flux surface \( (\hat{\psi} = 0) \) from the circular shape as the order of approximation to the flux function is varied from \( N = 3 \) to \( N = 10 \) are given for the equilibrium configurations having \( \lambda = 1, 0 \) and \( -1/5 \) with \( \epsilon \) fixed at \( 2/5 \) in Tables 8.4(a), 8.4(b) and 8.4(c) respectively. Note that the values of the displacement variable (and thus of the relative Shafranov shift) in the sequence that is generated when \( N \) is increased by unity in succession starting with \( N = 3 \) are alternately lower and higher than that of the limit to which the sequence tends.
### Tables 8.4 (a), (b), (c)

The values taken by the displacement variable ($\chi$), the relative Shafranov shift ($\delta$), the relative error of maximum absolute value ($\varepsilon_{\text{max}}$), the relative deviation of maximum absolute value ($D_{\text{max}}$) and by the poloidal angle where this deviation occurs ($\theta_D$) as the order of approximation to the flux function is increased from $N = 3$ to $N = 10$ for the equilibria with $\epsilon = 2/5$ and (a) $\lambda = 1$, (b) $\lambda = 0$ and (c) $\lambda = -1/5$ respectively. In all cases the poloidal angle where $\varepsilon_{\text{max}}$ occurs is $\theta = 0^\circ$.
As the order of approximation is increased while the value of the equilibrium parameter is kept fixed as \( \lambda = \lambda \), we observe that the absolute values of the relative error and of the relative deviation decrease monotonically and that the point with critical coordinates \((\chi_c^{(N)}, \lambda_c^{(N)})\) is removed away from the point on the curve for \(F^{(N)}(\epsilon, \chi)\) having ordinate equal to \(\lambda\). For \(N \geq 4\), the absolute value of the relative error is always greater than that of the relative deviation. Of the three reference equilibria, the largest values of one and the other measures of inaccuracy of the approximate solutions are found for \(\lambda = -1/5\). Note that, of the two configurations having \(\lambda = 1\) and \(\lambda = 0\) respectively, it is the former that has the smallest relative Shafranov shift, but it is to the latter that belong the smallest relative error and the smallest relative deviation whatever be the value of \(N\).

By judicious choice of the value of \(\chi\) it is possible to eliminate one of the harmonics of the poloidal angle in the expression of the “remainder” \(R^{(N)}(x, \theta; \epsilon, \chi)\) in Eq. (8.8) for \(x = \epsilon\), in this way determining a configuration that can be described with exceptionally high accuracy. Of course the harmonic to be suppressed in order to obtain the least possible maximum relative error varies according to the order \(N\) of approximation to the flux function. For \(N = 10\) and \(\epsilon = 2/5\), the smallest relative error and the smallest relative deviation we have been able to observe are those for the configuration in which the eleventh harmonic is absent at the plasma boundary; they are given together with the main parameters characterizing such equilibrium in Table 8.5.

To conclude this Section, we note that, for any order of approximation to the flux function, the relative departure of the equilibrium parameter from the value corresponding to that of the displacement variable as determined by the equilibrium equation, for a small variation of the displacement variable, is in general much larger than the relative variation of the displacement variable itself.
Table 8.5  Data relative to the equilibrium configuration for which the eleventh harmonic of the poloidal angle is missing in the expression of order \( N = 10 \) for the partial flux function, for \( x = \epsilon = 2/5 \). The poloidal angles for which the relative error and the relative deviation of maximum absolute values are observed are \( \theta_\epsilon \) and \( \theta_D \) respectively.

If the value of the displacement variable is changed from \( \chi \) by \( \Delta \chi \), and the corresponding change in the value of the equilibrium parameter \( \lambda \) is \( \Delta \lambda \), we have, from Eq. \((4.45)\), the following relation between the fractional variations of both quantities:

\[
\frac{\Delta \lambda}{\lambda} = S^{(N)}(\epsilon, \chi) \frac{\Delta \chi}{\chi},
\]

where \( S^{(N)}(\epsilon, \chi) \), defined as:

\[
S^{(N)}(\epsilon, \chi) = \frac{\chi}{F^{(N)}(\epsilon, \chi)} \frac{\partial F^{(N)}(\epsilon, \chi)}{\partial \chi},
\]

is a function with two distinct poles in the interval of \( \chi \) of physical interest, one at the zero \( \chi_Z^{(N)} \) and the other at the pole \( \chi_P^{(N)} \) of \( F^{(N)}(\epsilon, \chi) \).

Figure 8.3 is a plot of \( S^{(N)}(\epsilon, \chi) \) for \( \epsilon = 2/5 \) and \( N = 10 \), and Table 8.6 brings the numerical values taken by this function for the three reference equilibria. Note that, according to the graph of \( S^{(N)}(\epsilon, \chi) \), as far as \( \lambda \) remains small, any uncertainty in the value of \( \chi \) will lead to enormous errors in the value of \( \lambda \). This means that, for equilibria having \( \lambda \sim 0 \), it is virtually impossible to extract any reliable information concerning the value of \( \lambda \) from the position taken up by the magnetic axis.
FIG. 8.3 Dependence of the sensitivity function $S^{(N)}(\epsilon, \chi)$, as defined by Eq. (8.15), on the displacement variable $\chi$ for $\epsilon = 2/5$ and $N = 10$. The point with negative ordinate shown on the curve corresponds to the equilibrium $\lambda = 1$ and the point with positive ordinate to the equilibrium $\lambda = -1/5$.

| $\lambda$   | 1   | 0   | $-1/5$ |
|------------|-----|-----|--------|
| $S^{(10)}(\epsilon = 2/5, \chi)$ | $-3.18$ | $\pm\infty$ | 6.39 |

Table 8.6 Values of the sensitivity function $S^{(N)}(\epsilon, \chi)$ according to the order $N = 10$ of approximation to the flux function for the three reference equilibria.

IX. DESCRIPTION OF THE EQUILIBRIUM CONFIGURATIONS OF REFERENCE BY USE OF THE FLUX FUNCTION IN THE APPROXIMATION OF ORDER $N = 10$

In this Section we shall use the approximate solution to the Grad-Shafranov boundary value problem that comprises a superposition of $N = 10$ multipole solutions to derive expressions for the fluid pressure and for the magnetic fields and current densities, poloidal and toroidal, in the equilibrium configurations we have
been referring to throughout for illustrative purposes, namely, those for which the
inverse aspect ratio is $\epsilon = 2/5$ and the equilibrium parameter $\lambda$ is equal to 1, 0 and
$-1/5$ respectively; the knowledge of the spatial distributions of those quantities of
local definition will then be used to determine the numerical values of some parameters of global definition as poloidal beta and beta. Not every quantity or parameter,
local and global, that has a share in the composition of a complete picture of the
equilibrium or that is currently employed in stability analysis has been evaluated
as the objective here merely amounts to illustrating the recourses that are made
available by the analytical representation of the flux function.

The flux functions for $\lambda = 1$, $\lambda = 0$ and $\lambda = -1/5$, with the value of $\epsilon$ fixed
at 2/5, are given in the approximations of order $N = 10$ in Appendix D. They are
presented there in the forms they take in the cylindrical coordinate system, since
they look simpler in this than in the toroidal-polar system. The transformation from
the former to the latter system can be achieved with the help of the formulae also
given in Appendix D. The use of ordinary fractions for the coefficients of the powers
of the axial variable $Z$ was dictated by the intent of making to look less cumbersome
to the eye the representation of numbers that, in decimal notation, should require
a decimal point and ten places to have fully expressed the degree of accuracy with
which they were generated.

(1) \textit{Flux maps}

They are given in Fig. 9.1.

(2) \textit{The plasma pressure}

The plasma pressure profile in flux space can be obtained by integrating Eq.
(2.7) under the assumption that $s_P$ is constant, and since the plasma pressure must
vanish at the plasma boundary, where $\psi = 0$, the integration constant must be
FIG. 9.1 Flux surfaces as given by the $N = 10$ order of approximation to the flux function for the equilibrium configurations characterized by having $\epsilon = 2/5$ and (a) $\lambda = 1$; (b) $\lambda = 0$; (c) $\lambda = -1/5$. 
chosen to be null, with the result for \( \hat{\rho} \) that it is a quantity proportional to \( \psi \). From the normalizations for the flux function introduced by Eq. (4.2) and Eq. (5.5), we find that the connection between \( \psi \) and \( \hat{\psi} \) is given by:

\[
\psi(\rho, Z) = s_P(-\bar{\psi}_M)\hat{\psi}(\rho, Z),
\]

(9.1)

and we obtain a relation for the plasma pressure normalized by the magnetic pressure at the magnetic axis, \( \hat{\rho} \), as:

\[
\frac{\hat{\rho}}{s_P^2} = 2(-\bar{\psi}_M)(-\hat{\psi}(\rho, Z)).
\]

(9.2)

The ratio of the normalized plasma pressure \( \hat{\rho} \) at any point inside the cavity located by the coordinates \( \rho \) and \( Z \) to the normalized plasma pressure at the torus centre line, \( \hat{\rho}_C \), is:

\[
\frac{\hat{\rho}}{\hat{\rho}_C} = \frac{\hat{\psi}(\rho, Z)}{\hat{\psi}_C},
\]

(9.3)

where

\[
\hat{\psi}_C \equiv \hat{\psi}(\rho = 1, Z = 0)
\]

(9.4)

is the normalized flux function at the torus centre line.

Figure 9.2 shows the curves that represent the radial dependences of the quantity defined by the right hand side of Eq. (9.3) on the equatorial plane of the torus for the three reference configurations of equilibrium respectively, as obtained by using, for \( \hat{\psi}_C \) the values given in Table D.1 in Appendix D, and for \( \hat{\psi}(\rho, Z) \) the expressions for the flux function to which those given in that same Appendix are reduced by putting \( Z \) equal to zero.

(3) The toroidal magnetic field

The toroidal magnetic field can be obtained by integrating Eq. (2.8) in the variable \( \psi \), considering that \( s_I \) is constant by assumption and recalling that the
FIG. 9.2 The ratios of the plasma pressures along the intersection of the torus midplane with a meridian plane to the values they take at the torus centre line, for the three reference equilibria. In order of increasing peak values are seen the graphs for $\lambda = 1$, $\lambda = 0$ and $\lambda = -1/5$.

The normalized toroidal field function $\hat{I}(\psi)$ is defined by Eq. (2.5) together with Eq. (1.2). Replacing the flux $\psi$ by its normalized version according to Eq. (9.1), the integration constant can be determined from the condition that at the centre of the torus cross section, located by $R = R_C$, $Z = 0$, where $\hat{\psi} = \hat{\psi}_C$, the toroidal field $B_\phi$ equals $B_{\phi C}$. We are in this way conducted to the following expression for the toroidal field:

$$\frac{B_\phi}{B_{\phi C}} = \frac{1}{\rho} \sqrt{1 \mp L^2 (\hat{\psi}(\rho, Z) - \hat{\psi}_C)},$$  \hspace{1cm} (9.5)

where we have introduced the parameter:

$$L = s_p \sqrt{2|\lambda|(-\bar{\psi}_M)},$$  \hspace{1cm} (9.6)

and where the sign inside the symbol of square root depends on the sign of $\lambda$ in accordance with a scheme of discrimination stated in the discussion that follows. For $\lambda = 0$, the parameter $L$ vanishes; the toroidal field decays from the inner to the outer edge of the chamber in conformity with the $1/R$ law, undisturbed by the
plasma interposed between them, the toroidal field function is constant, and the poloidal current density is zero (see Eq. (9.18) farther on) also as if there were no material medium filling the cavity. For positive values of $\lambda$, the sign to be taken in Eq. (9.5) is the negative one; the expression under the symbol of square root then reaches its minimal value at the plasma boundary, where $\hat{\psi} = 0$, and its maximum value at the magnetic axis, where $\hat{\psi} = -1$. This means an enhancement of the toroidal field intensity at all points in the plasma bulk with respect to that which would exist within a magnetically inert configuration generated by a toroidal field function of equal strength at the plasma bounding surface, and the plasma as a whole thus behaves as a paramagnetic body: the poloidal current that is now induced flows in such a sense that the magnetic field it originates adds constructively to the applied toroidal field. Note that, as the parameter $L$ is increased, the toroidal field at the plasma boundary decreases and vanishes for $L = L_{\text{max},1}$, where

$$L_{\text{max},1} \equiv \frac{1}{\sqrt{|\hat{\psi}_C|}}. \quad (9.7)$$

For example, for $\lambda = 1$, using for $\hat{\psi}_C$ the numerical value given in Table D.1 in Appendix D, we obtain that the maximal allowed value for $L$ is $L_{\text{max},1} = 1.0115$, to which, by Eq. (9.6) and the numerical value of $\hat{\psi}_M$ given in that same Table, there corresponds a maximal allowed value for the pressure gradient parameter equal to $s_{P_{\text{max},1}} = 2.5195$. For negative values of $\lambda$, the sign to be taken in Eq. (9.5) is the positive one, and the equilibrium configuration reproduces the characteristics of a diamagnetic body, one that tends to cancel the applied magnetic field in its interior. In this case the expression under the sign of square root reaches its minimal value at the magnetic axis as $\rho$ runs between its two edge values, and remains positive at this and all locations inside the plasma as far as the value of the parameter $L$ is
smaller than $L_{\text{max},2}$, defined to be:

$$L_{\text{max},2} \equiv \frac{1}{\sqrt{1 - |\psi_C|}}. \tag{9.8}$$

For example, for $\lambda = -1/5$, using the data for the fluxes displayed in Table D.1 in Appendix D, we obtain that $L_{\text{max},2} = 3.5582$, and thus the value of the pressure parameter cannot exceed $s_{P_{\text{max},2}} = 30.4134$.

Figures 9.3(a) and 9.3(b) illustrate the behaviour of the profiles of the toroidal field intensities along the intersection of a meridian plane with the equatorial plane of the torus for a paramagnetic ($\lambda > 0$) and for a diamagnetic ($\lambda < 0$) configurations respectively, together with those for the magnetically inert configurations ($\lambda = 0$) having the same respective field strengths at the plasma-wall interface.

4) The poloidal field

The total magnetic field in the plasma can be written as:

$$\vec{B} = \vec{B}_\phi + \vec{B}_p, \tag{9.9}$$

where $\vec{B}_p$ is the poloidal field and $\vec{B}_\phi = B_\phi \vec{e}_\phi$ is the toroidal field. From the fact that the divergence of the unit vector pointing in the azimuthal direction $\vec{e}_\phi$ is null, and from the constancy of the toroidal field strength $B_\phi$ with the azimuthal angle, we can establish that the divergence of the toroidal field is zero. We thus arrive at that which can be called the fundamental property of the fields in axisymmetric configurations: the divergences of the toroidal and poloidal fields vanish separately. This means that Eqs. (1.4) and (1.5) in Ref. [4] for the radial and the axial components of a multipole field still hold for the poloidal field generated by a current-carrying plasma subjected to a toroidal field. Using vector notation we may thus express the poloidal field of the equilibrium toroidal pinch in generality as:
(a) $\lambda = 1$

\[
\frac{B_\phi}{B_{\phi c}} \bigg|_{z=0}
\]

(b) $\lambda = -1/5$

\[
\frac{B_\phi}{B_{\phi c}} \bigg|_{z=0}
\]

**FIG. 9.3** Profiles of the normalized toroidal field intensities along the intersection of the midplane with the torus cross section for $L = 4/5$, $L$ being the parameter defined by Eq. (9.6), and (a) $\lambda = 1$; (b) $\lambda = -1/5$. The thin lines represent the $1/R^\lambda$ decays of the toroidal field intensities in case of constant toroidal field functions, for which $\lambda = 0$.

\[
\vec{B}_p = \frac{1}{R} \vec{e}_\phi \times \nabla \Psi .
\]  

(9.10)

In terms of normalized quantities, the polar component of the poloidal field in the toroidal-polar coordinate system with the pole coinciding with the centre of the torus cross section is given by:

\[
(\vec{B}_p)_\theta = s_p B_{\phi c}(-\overline{\psi}_M) b_\theta(x, \theta)
\]  

(9.11)
with

\[ b_\theta(x, \theta) = \frac{1}{1 + x \cos \theta} \frac{\partial \hat{\psi}}{\partial x} \].

(9.12)

In units of the applied vertical field (which is identical with the polar component of the poloidal field at the centre line of the torus), the polar component of the poloidal field at any point in the plasma interior is expressed by:

\[ \frac{(\vec{B}_p)_\theta}{B_v} = \frac{b_\theta(x, \theta)}{b_\theta(x = 0, \theta = 0)} \].

(9.13)

Figure 9.4 displays the dependences of the right-hand side of Eq. (9.13) on the poloidal angle at the plasma boundary for the three reference equilibria. Note that, although the Shafranov shift is a positive quantity for all of them, for the paramagnetic configuration \((\lambda = 1)\) the poloidal field has the intensity increased as the inner edge of the torus is approached from the outer edge by following the path described by a field line close to the plasma boundary, while for the magnetically inert \((\lambda = 0)\) and the diamagnetic \((\lambda = -1/5)\) configurations the tendency it manifests to observation along the same path is the opposite one.

Since the flux function is constructed from the combination of a number of multipole solutions that is necessarily finite, the poloidal magnetic field it generates according to the formulae in Eqs. (9.11) and (9.12) will not be rigorously tangent to the boundary, but will admit a small radial component at the points lying on it. The ratio of this radial component to the tangential component of the poloidal magnetic field at the plasma toroidal surface can be calculated as:

\[ \left. \frac{(\vec{B}_p)_r}{(\vec{B}_p)_\theta} \right|_{\text{Plasma surface}} = -\frac{1}{x} \left. \frac{\partial \hat{\psi}(x, \theta)}{\partial x} \right|_{x=\epsilon} \].

(9.14)
FIG. 9.4 Magnitudes of the normalized poloidal field at the plasma boundary as functions of the poloidal angle for the three reference equilibria. The upper curve corresponds to the paramagnetic configuration ($\lambda = 1$), the lowest one to the diamagnetic configuration ($\lambda = -1/5$), and the curve between both to the magnetically inert configuration ($\lambda = 0$).

Figure 9.5 is the plot of the ratio on the left hand side of Eq. (9.14) as obtained by using the expression of the $N = 10$ order of approximation to the normalized flux function $\hat{\psi}(x, \theta)$, with $\epsilon = 2/5$ and $\lambda$ taken to be zero, on the right hand side. Since the poloidal field is a quantity physically more relevant than the magnetic flux is, Eq. (9.14) embodies a measure of error which is perhaps more suggestive of the merit of the partial flux function of order $N$ in approximating the solution for a given equilibrium than are the other measures we have introduced previously. Table 9.1 presents the errors attached to the description of the three reference equilibria obtained from the approximation of order $N = 10$ to the flux function according to this view.

Because of the up-down symmetry of the configuration, the longitudinal component $B_\rho$ of the poloidal field vanishes on the equatorial plane of the torus. Figure 9.6 shows the variations of the normalized axial components of the poloidal fields on the equatorial plane with the normalized radial coordinate $\rho$ from the inner to
the outer edge of the torus for the three reference equilibria. It is worth noting the remarkably linear pattern of decaying followed by the magnitude of the poloidal field as the magnetic axis is neared from any of the two extreme positions of the equator line on a meridian section of the torus in the equilibrium having \( \lambda = 1 \).

\[
\left. \frac{(B_p)_r}{(B_p)_\theta} \right|_{x=\epsilon}
\]

**FIG. 9.5** The ratio of the normal to the tangential component of the poloidal field at the plasma boundary \((x = \epsilon)\) for the equilibrium defined by \( \epsilon = 2/5 \) and \( \lambda = 0 \), according to the approximation of order \( N = 10 \) to the flux function.

| \( \lambda \) | Error \( \times 10^2 \) | \( \theta \) (degrees) |
|------------|-----------------|----------------|
| 1          | -2.42           | 7.88           |
| 0          | 0.998           | 8.06           |
| -1/5       | 2.34            | 8.13           |

**Table 9.1** Values of the errors introduced by the approximation of order \( N = 10 \) to the flux function for the equilibria defined by \( \lambda = 1, 0 \) and \(-1/5\) respectively, as measured by the maxima of the ratios of the normal to the polar components of the poloidal field at the boundary \( x = \epsilon = 2/5 \), and the angular positions where they occur.
FIG. 9.6 Variations of the axial components of the normalized poloidal fields with the normalized longitudinal coordinate $\rho$ along the intersection of the torus midplane with a meridian plane, for the three reference equilibria. The normalization quantity $B_{z_c}$ is the value taken by the axial component of the poloidal field $B_z$ at the torus centre line. In order of increasing values of the abscissas of the magnetic axes are seen the curves for the configurations having $\lambda = 1$, $\lambda = 0$ and $\lambda = -1/5$ respectively.

(5) *The poloidal current density*

To be applicable to the toroidal pinch, Eq. (1.2) in Ref. [4] has to be replaced by Ampère’s law in its general form for static fields, which is:

$$\nabla \times \vec{B} = \mu_0 \vec{J},$$

(9.15)

where $\mu_0$ is the permeability of vacuum and $\vec{J}$ is the total current density:

$$\vec{J} = \vec{J}_\phi + \vec{J}_p,$$

(9.16)

$\vec{J}_\phi = J_\phi \vec{e}_\phi$ being the toroidal current density and $\vec{J}_p$ the poloidal current density.

By writing down the longitudinal and the axial components of Eq. (9.15) in the cylindrical coordinate system ($R, \phi, z$) (see Fig. 1 in Ref. [4]), we see immediately that the poloidal current density vector can be expressed as:

$$\mu_0 \vec{J}_p = -\nabla \phi \times \nabla I,$$

(9.17)
where we have used the fact that $\nabla \phi = \vec{e}_\phi / R$. Since $I \equiv R B_\phi$ is a function of the flux function only, we may rewrite Eq. (9.17) as:

$$
\mu_0 \vec{J}_p = - \frac{dI}{d\Psi} \nabla \phi \times \nabla \Psi = - \frac{dI}{d\Psi} \vec{B}_p ,
$$

(9.18)

the second equality in Eq. (9.18) following from the expression for $\vec{B}_p$ stated in Eq. (9.10). The poloidal current density and the poloidal field are therefore parallel, the lines of force of one vector field coinciding with those of the other field, and of both with the level curves in the flux map. This is of course a statement that applies to any and all axisymmetric equilibria (except to the magnetically inert ones, which have constant toroidal field function and carry no poloidal current). The profiles of the longitudinal and axial components of the poloidal current density and the profiles of the longitudinal and axial components of the poloidal field along any direction in space, however, are not respectively parallel in general, since the ratio between corresponding components depends on the shape of the function $I(\Psi)$. Along the intersection of a meridian plane with the equatorial plane of the torus the longitudinal component of the poloidal current density is zero and the axial component is expressed in terms of normalized quantities as:

$$
\mu_0 J_{pz} = s_2 B_{pC} \frac{\lambda(-\psi_M)}{\sqrt{1 + L^2(\psi - \psi_C)}} \frac{1}{\rho} \frac{\partial \hat{\psi}}{\partial \rho} \bigg|_{Z=0} ,
$$

(9.19)

where the minus and the plus signs inside the square root symbol, as in Eq. (9.5), apply to the paramagnetic and the diamagnetic configurations respectively. The same as the $z$-component of the poloidal field, $J_{pz}$ vanishes at the magnetic axis. The graphs in Fig. 9.7 show the variations of the ratios $J_{pz}/J_{pzc}$, where $J_{pzc}$ is the axial component of the poloidal current density at the torus centre line, along the equator line for the two reference equilibria having non-null values of $\lambda$, assuming that the parameter $L$ takes the value $4/5$ for both.
FIG. 9.7  Dependences of the axial components of the poloidal current densities normalized to the values they respectively take at the torus centre line, $J_{pz}/J_{pzC}$, on the normalized longitudinal coordinate $\rho$ along the intersection of the torus midplane with a meridian plane, for the reference equilibria having $\lambda = 1$ (curve showing the smallest value of the abscissa for which $J_{pz}$ is null) and $\lambda = -1/5$ respectively, and a common value of the parameter $L$ equal to $4/5$.

Writing the toroidal field as:

$$\vec{B}_\phi = I(\Psi) \nabla \phi \quad (9.20)$$

and then taking the curl, by subsequent use of Eq. (9.18) for the poloidal current density, we can establish that:

$$\nabla \times \vec{B}_\phi = \mu_0 \vec{J}_p \quad (9.21)$$

The poloidal current density thus acts as the source to the toroidal field, as we indeed should expect intuitively from the rotational invariance property of the toroidal pinch in equilibrium. Equation (9.21) also implies that Ampère’s law for the toroidal pinch breaks into two separate laws of similar structures in which the toroidal and the poloidal symmetries belong to either one of the two respective vector fields they put in relation and which alternate between them the fields to which one
and other of both symmetries are ascribed. The partial version of Ampère’s law that makes a pair with that expressed by Eq. (9.21) states that the toroidal current density is the sole source to the poloidal field, and reads:

$$\nabla \times \vec{B}_p = \mu_0 \vec{J}_\phi . \quad (9.22)$$

(6) The toroidal current density

The expression for the toroidal current density that is best suited to the purpose of numerical evaluation is the one that follows directly from the pressure balance condition:

$$\nabla p = \vec{J} \times \vec{B} . \quad (9.23)$$

Considering that the pressure is a function of $\Psi$ and that the poloidal current density is parallel to the poloidal field, Eq. (9.23) can be written as:

$$\frac{dp}{d\Psi} \nabla \Psi = \vec{J}_\phi \times \vec{B}_p + \vec{J}_p \times \vec{B}_\phi . \quad (9.24)$$

By taking the cross product of the above relation with $\nabla \phi$ on the left, and then developing the two terms that constitute the right hand side of the ensuing equation in accordance with the rule of the vector triple product, we obtain:

$$\frac{dp}{d\Psi} \vec{B}_p = -\frac{1}{R} J_\phi \vec{B}_p + \frac{1}{R} B_\phi \vec{J}_p , \quad (9.25)$$

where in the left hand side we have made use of Eq. (9.10) for the poloidal field. We next substitute $\vec{J}_p$ on the right hand side by the expression for it given by Eq. (9.18). Cancelling the factor $\vec{B}_p$ that then appears in all terms of the resulting equation, and isolating the term of the toroidal current density, we have:

$$\mu_0 J_\phi = -\mu_0 R \frac{dp}{d\Psi} - \frac{I}{R} \frac{dI}{d\Psi} . \quad (9.26)$$

Note that, if the sources to the equilibrium are constant, as we are considering them to be, $J_\phi$ is independent of the axial coordinate.
In terms of normalized quantities the above expression can be made to become:

\[
\frac{J_\phi}{J_{\phi C}} = \frac{1}{1 + \lambda} \left( \rho + \frac{\lambda}{\rho} \right) \quad (9.27)
\]

where

\[
J_{\phi C} = (1 + \lambda) \frac{s_P B_{\phi C}}{\mu_0 R_C} \quad (9.28)
\]

is the toroidal current density at the torus centre line.

For \( \lambda = 0 \) the toroidal current density grows linearly from the inner to the outer side of the torus. For \( \lambda > 0 \) it is positive at all points in the plasma and passes through a minimum at the coordinate:

\[
\rho = \rho_m \equiv \sqrt{\lambda} \quad (9.29)
\]

where it assumes the value:

\[
(J_\phi)_{\text{min}} = \frac{2\sqrt{\lambda}}{1 + \lambda} J_{\phi C} \quad (9.30)
\]

For \( \lambda = 1 \) as the value of the equilibrium parameter we have been taking to consider a concrete example of a paramagnetic configuration, the minimum value of the toroidal current density is reached at the torus centre line. For \( \lambda < 0 \) no local minimum of \( J_\phi \) can occur; there can be current reversion, however, at the point of normalized longitudinal coordinate given by:

\[
\rho = \rho_R \equiv \sqrt{-\lambda} \quad (9.31)
\]

provided, of course, that this point falls inside the chamber for the equilibrium considered. Figure 9.8 shows the profiles of the toroidal current densities along the equatorial line on the torus cross section for the three reference equilibria.
FIG. 9.8 Profiles of the magnitudes of the normalized toroidal current densities, \( J_\phi / J_{\phi C} \), on a meridian plane of the torus, for the three reference equilibria. The normalization quantity \( J_{\phi C} \) is the toroidal current density on the cylindrical surface having the axis coincident with the axis of rotational symmetry of the torus and radius equal to \( R_C \). In order of increasing values of the ordinates at the abscissa associated with the location of the inner edge of the torus (\( \rho = 0.6 \)) are seen the curves relative to the equilibria having \( \lambda = -1/5 \), \( \lambda = 0 \) and \( \lambda = 1 \) respectively.

Before we pass to the consideration of examples of calculation of macroscopic parameters associated with a given state of equilibrium, let us observe that, alternative to the formula stated in Eq. (9.26), a relation for the toroidal current density which relies on operations on the flux function only can be established by exploiting the information contained in Ampère’s law for the poloidal field. Indeed, by taking the dot product of Eq. (9.22) with \( \hat{e}_\phi \), we have:

\[
\mu_0 J_\phi = R \nabla \phi \cdot \nabla \times \vec{B}_p
\]

\[
= R \nabla \cdot (\vec{B}_p \times \nabla \phi)
\]

(9.32)

where, to write the second equality, we have recalled a well known identity for the divergence of the cross product of two vectors. With \( \vec{B}_p \) written as in Eq. (9.10), application of the rule of the triple vector product to the expression between
parentheses in Eq. (9.32) conducts us at once to the result:

\[ \mu_0 J_\phi = R \nabla \left( \frac{1}{R^2} \nabla \psi \right). \] (9.33)

The equation that comes up from the comparison of Eq. (9.33) with Eq. (9.26) is the Grad-Shafranov’s.

(7) Poloidal beta

From a physical standpoint poloidal beta is probably the most useful of the macroscopic parameters that are costumarily associated with an equilibrium configuration, since there is an univocal correspondence between its numerical value and that of the equilibrium parameter \( \lambda \), while embodying a physical connotation of more prompt appeal than the latter. Loosely speaking, it is a measure of the ratio of the plasma thermal energy to the magnetic energy content of the poloidal field. A convenient mathematical definition is the one that adopts for itself the form of a term that arises naturally in the derivation of the average integral version of Grad-Shafranov’s equation. Following Ref. [13] we then define the poloidal beta as:

\[ \beta_p \equiv \frac{1}{V_S} \int_{V_S} p dV \frac{\langle B_p^2 \rangle / 2\mu_0}{}, \] (9.34)

where the integral of the pressure is taken over the plasma volume, \( V_S \) is the plasma volume, which for a torus of major radius \( R_C \) and inverse aspect ratio \( \epsilon \) is given by:

\[ V_S = 2\pi^2 R_C^3 \epsilon^2, \] (9.35)

and \( \langle B_p^2 \rangle \) means the average of the squared poloidal field magnitude over the plasma contour in the following sense:

\[ \langle B_p^2 \rangle \equiv \frac{\oint B_p dl}{\oint \frac{dl}{B_p}}, \] (9.36)
the integrations being performed along the poloidal contour of the plasma surface (which in our case coincides with the circular trace of the inner wall on the cross section of the containing chamber).

In terms of the normalized quantities we have been using throughout the expression for poloidal beta introduced by Eq. (9.34) translates as:

\[ \beta_p = \frac{4h}{\epsilon^2(-\psi_M)\langle b_\theta^2(\epsilon) \rangle}, \]  

(9.37)

where

\[ h = -\frac{1}{2\pi} \int_0^\epsilon dx \int_0^{2\pi} d\theta \, \hat{\psi}(x, \theta) \, x \, (1 + x \cos \theta) \]  

(9.38)

and

\[ \langle b_\theta^2(\epsilon) \rangle = \frac{\int_0^{2\pi} b_\theta(x = \epsilon, \theta) d\theta}{\int_0^{2\pi} d\theta} \]  

(9.39)

\[ b_\theta(x, \theta) \] being given by Eq. (9.12). The integrals appearing in the definitions of \( h \) and \( \langle b_\theta^2(\epsilon) \rangle \) are easily computed with the help of Maple V. Table 9.2 displays the results obtained for poloidal beta for our three reference equilibria. It should be noted that, based on the balancing the terms in the expression of the integral average of Grad-Shafranov’s equation derived in Ref. [13] must keep among themselves, the figures for \( \beta_p \) that we find in Table 9.2 for the three kinds of magnetically distinct configurations are all of the orders of magnitude we expect them to be, to know, \( \beta_p < 1 \) for \( \lambda > 0 \), \( \beta_p \sim 1 \) for \( \lambda = 0 \) and \( \beta_p > 1 \) for \( \lambda < 0 \).

| \( \lambda \) | \( \beta_p \) |
|---|---|
| 1 | 0.487 |
| 0 | 1.035 |
| -1/5 | 1.358 |

**Table 9.2** Values of poloidal beta (\( \beta_p \)) corresponding to the three configurations of reference as defined by the values of the equilibrium parameter \( \lambda \).
Beta

The global parameter known as beta is probably the stick of commonest use in assessing the merit of a thermonuclear plasma confining device as a prospective reactor, and in ruling the discussions on engineering and economic aspects of fusion research. In general terms it is a measure of the proportion of the plasma thermal energy to the total magnetic energy spent in confining the plasma. Since it is not put forward by any mathematical statement meaning a balance between physical quantities, as the poloidal beta is, its definition remains somewhat arbitrary. In this paper we shall adopt the same as that of Refs. [1] and [6], which is:

\[ \beta \equiv \frac{1}{V_S} \int_{V_S} \frac{p dV}{B_{\phi c}^2/2\mu_0}, \tag{9.40} \]

where \( B_{\phi c} \) is the toroidal magnetic field at the centre of the plasma cross section. In terms of normalized quantities Eq. (9.40) writes as:

\[ \beta = \frac{4s^2_p(-\bar{\psi}_M)h}{\epsilon^2}, \tag{9.41} \]

where \( h \) is defined as in Eq. (9.38). Note that the value of beta, distinctly from that of poloidal beta, is not uniquely defined by the value of the equilibrium parameter \( \lambda \), and thus cannot be linked in an unequivocal way to any particular equilibrium configuration and to the geometrical characteristics typefying the level curves in a given flux map. In other words, beta is not a state function and the value it assumes depends on the way the plasma is driven from its early formative stages to the final equilibrium state. Besides the implicit references it contains to \( \lambda \) through the dependences that \( -\bar{\psi}_M \) and \( h \) keep on that parameter, a formula for beta in general will call for the presence of another equilibrium quantity (in addition to that of the normalization constants, \( R_C \) and \( B_{\phi c} \)), which can be chosen among a multitude of possibilities. In the formula stated as Eq. (9.41) this is \( s_p \), but of course equivalent
expressions could be written as well in which, in place of $s_P$, we should have $s_I$ or the pressure at the centre line $\hat{p}_C$ or the toroidal current, etc. The convenient one to use in each case is that which incorporates as second parameter the quantity that is kept constant during the beta-raising process. Table 9.3 shows the values of beta for the three equilibrium configurations of reference as evaluated by means of Eq. (9.41) assuming that they have $s_P$ equal to unity.

| $\lambda$ | $\beta \times 10^2$ |
|-----------|---------------------|
| 1         | 8.08                |
| 0         | 4.12                |
| $-1/5$    | 3.33                |

Table 9.3 Values of beta in unities of $s_P^2$ for the reference equilibrium configurations.

X. SUMMARY

The problem considered in this paper is an instance of that of the plasma equilibrium in toroidal confinement systems as described by the Grad-Shafranov equation, the constraint under which it is to be solved being that the outermost of the flux surfaces in the continuum that structures the magnetic configuration represented by its solution must conform to the toroidal shape of the surface of the plasma bulk, with which that surface identifies itself. In terms of normalized quantities, as they are defined in Eqs. (2.2) – (2.5), the Grad-Shafranov equation writes in the toroidal-polar coordinate system as in Eq. (2.6), and the boundary condition on its solution as in Eq. (4.10). The sources to the equilibrium – the pressure gradient and the gradient of the squared toroidal field function – are formally defined by Eqs. (2.7) and (2.8). Under the assumption that these two quantities are constant in flux
space, by introducing a flux function normalized by the pressure gradient parameter as in Eq. (4.2), the problem can be made to admit of a single datum of input, the equilibrium parameter, defined in Eq. (3.29).

For the application of the method that is the theme of the present paper, the solution to the Grad-Shafranov equation is written as the sum of the particular solution and a linear combination of $N$ multipole solutions of the orders $n = 0, 1, 2, \ldots, N-1$ in the polar coordinate system having the pole located at the intersection of the centre line of the torus with a meridian plane, the azimuthal coordinate being ignorable (Eq. (4.1)). The particular solution is derived in Section III by the method of expansion of the flux function in series of Chebyshev polynomials, and is given under normalization by the pressure gradient constant in Eq. (3.27) (in toroidal-polar coordinates) and in Eq. (3.32) (in cylindrical coordinates). Imposition that the flux be null at the boundary generates a system of $2N - 1$ linear equations for the $N$ constants $K_0^{(N)}, K_1^{(N)}, \ldots, K_{N-1}^{(N)}$ multiplying the multipole solutions that enter the combination designated to represent the flux function. This is an overdetermined system, and, to extract a meaningful solution from it, the strict number of $N$ equations is retained, the $N-1$ abandoned ones being chosen as those that, in remaining unsatisfied, the error effect they will have on the value taken by the assumed flux function at the frontier of the domain of integration of the partial differential equation will be the least. The solution of the truncated system of $N$ equations for the $N$ constants $K_j^{(N)}$ is given by Eqs. (4.17) and (4.16) in conjunction with Eq. (4.15) in terms of the equilibrium parameter $\lambda$. It may be found advantageous to replace $\lambda$ by the relative Shafranov shift $\delta$, this being defined by Eq. (4.19), as the input datum to the problem. In this case the constants $K_j^{(N)}$ pass to be given by Eqs. (4.37) and (4.35) in conjunction with Eq. (4.32). The connection between $\lambda$ and $\delta$ is given by Eq. (4.33) together with Eq. (4.34).
A significant simplification to the mathematical expressions involved in the development of the solution can be gained by further substituting $\delta$ by the displacement variable $\chi$ as the input datum to the problem, this latter quantity being related to the former through Eq. (4.39). The value of $\chi$ can be evaluated from the value of $\lambda$ (or vice versa) by means of the equilibrium equation (Eq. (4.45)), which equates the equilibrium function $F^{(N)}(\epsilon, \chi)$ to $\lambda$. The equilibrium function itself is calculated from the formula given in Eq. (4.34) in conjunction with that of Eq. (4.32), replacing quantities defined in terms of $\delta$ by quantities defined in terms of $\chi$ according to the rules of transformation expounded in the paragraph containing Eq. (4.43). Similarly, the constants $K_j^{(N)}$ are still given by the formulae that apply when the input parameter is $\delta$, provided that the quantities appearing in them be subjected to this same scheme of substitution. Use of Eq. (4.45) in Eq. (3.27) permits us also to express the particular solution in terms of $\chi$.

The allowed interval of variation of $\chi$ is that specified by Eq. (6.22). The lower bound of it finds a reasonably good estimate in Eq. (6.24), provided that the torus is not too thick. There seems to be no physical constraint to set an upper bound to $\chi$, so that the extreme value it can take is that by which the magnetic axis is placed at the outer edge of the torus. In practice, however, as a feature of the method, branching points erupt on the boundary surface as $\chi$ reaches a critical value $\chi_c^{(N)}$, and this fixes the maximal value that can be attributed to $\chi$ in describing an equilibrium configuration within the framework of the order $N$ of approximation to the flux function. It is found that the value of $\chi_c^{(N)}$, which, together with those of the coordinates of the branching points, is obtained by simultaneously solving Eq. (6.65), Eq. (6.66) and Eq. (6.67) (with the superscript 3 replaced by $N$), increases with the increase of $N$, apparently without tending to a limiting value. A last normalization to the poloidal flux function (by the absolute value of the flux
at the magnetic axis) is introduced by Eq. (5.5). Section V also introduces the quantities that will be used in the subsequent ones to evaluate the accuracy of a solution obtained through the use of the method, to know, the relative error (defined by Eq. (5.6)) and the relative deviation (defined by Eq. (5.7)).

Three configurations, for which the common inverse aspect ratio is 2/5 and the equilibrium parameters are 1, 0 and −1/5 respectively, are elected to serve the purpose of testing the use of the method of series of multipole solutions. Solutions to the boundary value problem for these reference configurations are derived in Section VI by using the \(N = 3\) order of approximation, and again in Section VII by using the \(N = 4\) order of approximation to the flux function. Also obtained are the measures of accuracy of the solutions and the allowed intervals of variation of the displacement variables in each of these two orders of approximation.

In Section VIII the findings of the two previous Sections are generalized to characterize the structures of the equilibrium function and of the poloidal flux function that stem from a representation of the solution to the boundary value problem comprising any finite number \(N\) of multipole solutions. The general forms of \(F^{(N)}(\epsilon, \chi)\) for \(N\) odd and \(N\) even are given by Eqs. (8.1) and (8.4) (with Eqs. (8.2) and (8.3)), and that of \(\psi^{(N)}(x, \theta; \epsilon, \chi)\) by Eq. (8.5) (with Eqs. (8.6) – (8.12)). A systematic application of the method of solution by series of multipole solutions to the equilibria of reference with the order of approximation to the flux function ranging from \(N = 3\) to \(N = 10\) allows us to infer that the series is convergent.

Finally in Section IX the \(N = 10\) order of approximation to the poloidal flux function is employed to derive expressions for a number of quantities of interest in characterizing an equilibrium configuration (the plasma pressure, the toroidal and the poloidal fields, the toroidal and the poloidal current densities, beta and poloidal beta). These expressions are then applied to obtain a graphical and a numerical
description of the equilibria chosen to illustrate the use of the method as it was presented here.

APPENDIX A: EXPRESSIONS OF THE FUNCTIONS $P_i(x)$ AND $Q_i(x)$ ($i = 0, 1, 2, 3, 4$) CONSTITUENTS OF THE RADIAL DEPENDENT COEFFICIENTS $S_i(x)$ IN THE REPRESENTATION OF THE NORMALIZED PARTICULAR SOLUTION $\bar{\psi}_p(x, \theta)$ AS A COMBINATION OF HARMONICS OF THE POLOIDAL ANGLE $\theta$. EXPRESSIONS OF THE FUNCTIONS $P'(\delta)$ AND $Q'(\delta)$ AND OF THE FUNCTIONS $P'(\chi)$ AND $Q'(\chi)$ REQUIRED FOR THE EVALUATION OF THE EQUILIBRIUM FUNCTIONS, $F^{(N)}(\delta)$ AND $F^{(N)}(\chi)$, AND OF THE CONSTANTS MULTIPLYING THE MULTIPOLe SOLUTIONS, $K_j^{(N)}(\delta)$ AND $K_j^{(N)}(\chi)$, ($j = 0, 1, 2, \ldots, N - 1$), THAT ENTER THE REPRESENTATIONS OF THE PARTIAL FLUX OF ORDER $N$ IN TERMS OF THE RELATIVE SHAFRANOV SHIFT $\delta$ AND IN TERMS OF THE DISPLACEMENT VARIABLE $\chi$, RESPECTIVELY.

For the normalized particular solution $\bar{\psi}_p(x, \theta)$ written as in Eq. [3.30], and its $x$-dependent average value $S_0(x)$ and coefficient functions $S_i(x)$ of the harmonics of the poloidal angle, $\cos i\theta$ ($i = 1, 2, 3, 4$), expressed under the form of a two-term combination linear in $\lambda$, as in Eq. [3.31], the radial functions that constitute the terms in each of the combinations are respectively given by:

$$
\begin{align*}
P_0(x) &= \frac{1}{4}x^2 + \frac{7}{128}x^4, \quad (A.1) \\
Q_0(x) &= \frac{1}{4}x^2 - \frac{1}{128}x^4, \quad (A.2)
\end{align*}
$$
\[
\begin{align*}
P_1(x) &= \frac{5}{16}x^3, \quad (A.3) \\
Q_1(x) &= \frac{1}{16}x^3, \\
A.4
\end{align*}
\]

\[
\begin{align*}
P_2(x) &= \frac{1}{32}x^4, \quad (A.5) \\
Q_2(x) &= \frac{1}{32}x^4, \\
A.6
\end{align*}
\]

\[
\begin{align*}
P_3(x) &= -\frac{1}{16}x^3, \quad (A.7) \\
Q_3(x) &= \frac{3}{16}x^3, \\
A.8
\end{align*}
\]

\[
\begin{align*}
P_4(x) &= -\frac{3}{128}x^4, \quad (A.9) \\
Q_4(x) &= \frac{5}{128}x^4. \\
A.10
\end{align*}
\]

The functions \(P(x)\) and \(Q(x)\), according to the definitions of Eq. (4.24) and Eq. (4.25), assume then the following specific forms:

\[
\begin{align*}
P(x) &= P_0(x) + P_1(x) + P_2(x) + P_3(x) + P_4(x) \\
&= \frac{1}{4}x^2 + \frac{1}{4}x^3 + \frac{1}{16}x^4 \\
&= \frac{1}{16}x^2(x + 2)^2, \\
A.11
\end{align*}
\]

\[
\begin{align*}
Q(x) &= Q_0(x) + Q_1(x) + Q_2(x) + Q_3(x) + Q_4(x) \\
&= \frac{1}{16}x^2(x + 2)^2 = P(x). \\
A.12
\end{align*}
\]

For the evaluation of the equilibrium function \(F^{(N)}(\epsilon, \delta)\) and of the constants \(K_j^{(N)}(\epsilon, \delta)\) \((j = 1, 2, \ldots, N - 1)\) and \(K_0(\epsilon, \delta)\) that enter the making up of the partial flux of order \(N\), the formulae stated in Eqs. (4.32), (4.34), (4.35) and (4.37) require
the knowledge of the derivatives of the two above functions at the coordinate $x = \delta$. We have:

\[ P'(\delta) = Q'(\delta) = \frac{1}{4} \delta(\delta + 1)(\delta + 2) , \]  

(A.13)

where primes denote derivatives with respect to the argument.

If instead of the relative Shafranov shift it is the displacement variable the quantity favoured for playing the role of working parameter along the analytical development of the solution, the aforementioned formulae can still be used, provided that the functions $V_j'(\delta)$ ($j = 1, 2, \ldots, N - 1$), $P'(\delta)$ and $Q'(\delta)$ that constitute the elements in the bottom rows of the determinants in those formulae be first substituted by the functions $V_j'(\chi)$, $P'(\chi)$ and $Q'(\chi)$, as explained in the following of Eq. (4.43). By having recourse to Eq. (4.39) to achieve the conversion of the representation in terms of $\delta$ to a representation in terms of $\chi$, we obtain for the functions $P$ and $Q$ proceeding from the particular solution:

\[ P(\chi) \equiv P(x = \delta(\chi)) = \frac{1}{16} \chi^2 , \]  

(A.14)

\[ Q(\chi) \equiv Q(x = \delta(\chi)) = \frac{1}{16} \chi^2 , \]  

(A.15)

and thus

\[ P'(\chi) = Q'(\chi) = \frac{1}{8} \chi . \]  

(A.16)
APPENDIX B: THE FUNCTIONS $M_{ij}(x)$ REQUIRED FOR THE EVALUATION OF THE EQUILIBRIUM FUNCTION AND OF THE CONSTANTS THAT MULTIPLY THE MULTIPOLE SOLUTIONS IN THE EXPRESSIONS OF THE PARTIAL FLUXES OF ORDER $N = 3$ TO ORDER $N = 10$

In this Appendix we shall denote the matrix element $M_{ij}(x)$ of the main text by $M(i, j)$, the indices there appearing as arguments here, and the argument $x$ there being suppressed here.

\begin{align*}
M(0, 0) & = 1. \\
M(0, 1) & = \frac{x^2}{8}. \\
M(1, 1) & = \frac{x}{2}. \\
M(2, 1) & = \frac{x^2}{8}. \\
M(3, 1) & = M(4, 1) = M(5, 1) = M(6, 1) = M(7, 1) \\
& = M(8, 1) = M(9, 1) = M(10, 1) = M(11, 1) \\
& = M(12, 1) = M(13, 1) = M(14, 1) = M(15, 1) \\
& = M(16, 1) = M(17, 1) = M(18, 1) = 0. \\
M(0, 2) & = \frac{x^4}{32}. \\
M(1, 2) & = -\frac{x^3}{4}. \\
M(2, 2) & = -\frac{1}{8}x^4 - x^2. \\
M(3, 2) & = -\frac{3}{4}x^3. \\
M(4, 2) & = -\frac{5}{32}x^4. \\
M(5, 2) & = M(6, 2) = \cdots = M(18, 2) = 0. \\
M(0, 3) & = \frac{x^6}{128}. \\
\end{align*}

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\[ M(1, 3) = -\frac{x^5}{32}. \quad \text{(B.13)} \]
\[ M(2, 3) = -\frac{5}{256} x^6 + \frac{1}{4} x^4. \quad \text{(B.14)} \]
\[ M(3, 3) = \frac{15}{64} x^5 + x^3. \quad \text{(B.15)} \]
\[ M(4, 3) = \frac{7}{128} x^6 + \frac{5}{4} x^4. \quad \text{(B.16)} \]
\[ M(5, 3) = 35 x^5. \quad \text{(B.17)} \]
\[ M(6, 3) = \frac{21}{256} x^6. \quad \text{(B.18)} \]
\[ M(7, 3) = M(8, 3) = \cdots = M(18, 3) = 0. \quad \text{(B.19)} \]
\[ M(0, 4) = \frac{x^8}{512}. \quad \text{(B.20)} \]
\[ M(1, 4) = \frac{x^7}{160}. \quad \text{(B.21)} \]
\[ M(2, 4) = -\frac{7}{1600} x^8 + \frac{1}{40} x^6. \quad \text{(B.22)} \]
\[ M(3, 4) = \frac{21}{800} x^7 - \frac{1}{5} x^5. \quad \text{(B.23)} \]
\[ M(4, 4) = \frac{21}{3200} x^8 - \frac{7}{25} x^6 - \frac{4}{5} x^4. \quad \text{(B.24)} \]
\[ M(5, 4) = \frac{21}{160} x^7 - \frac{7}{5} x^5. \quad \text{(B.25)} \]
\[ M(6, 4) = \frac{189}{200} x^8 - \frac{33}{1600} x^6. \quad \text{(B.26)} \]
\[ M(7, 4) = \frac{231 x^7}{800}. \quad \text{(B.27)} \]
\[ M(8, 4) = \frac{429 x^8}{12800}. \quad \text{(B.28)} \]
\[ M(9, 4) = M(10, 4) = \cdots = M(18, 4) = 0. \quad \text{(B.29)} \]
\[ M(0, 5) = \frac{x^{10}}{2048}. \quad \text{(B.30)} \]
\[ M(1, 5) = \frac{5 x^9}{3584}. \quad \text{(B.31)} \]
\[ M(2, 5) = -\frac{15}{14336} x^{10} + \frac{1}{224} x^8. \quad \text{(B.32)} \]
\[ M(3, 5) = \frac{9}{1792} x^9 - \frac{1}{56} x^7. \quad \text{(B.33)} \]
\[ M(4, 5) = \frac{33}{25088} x^{10} - \frac{3}{112} x^8 + \frac{1}{7} x^6. \quad \text{(B.34)} \]
\begin{align*}
M(5,5) & = -\frac{165}{2544}x^9 + \frac{15}{56}x^7 + \frac{4}{7}x^5 . \quad \text{(B.35)} \\
M(6,5) & = -\frac{297}{200704}x^{10} + \frac{9}{1568}x^8 + \frac{9}{7}x^6 . \quad \text{(B.36)} \\
M(7,5) & = \frac{33}{7168}x^9 + \frac{28}{7}x^7 . \quad \text{(B.37)} \\
M(8,5) & = \frac{429}{100352}x^{10} + \frac{429}{784}x^8 . \quad \text{(B.38)} \\
M(9,5) & = \frac{6435x^9}{50176} . \quad \text{(B.39)} \\
M(10,5) & = \frac{2431x^{10}}{200704} . \quad \text{(B.40)} \\
M(11,5) & = M(12,5) = \cdots = M(18,5) = 0 . \quad \text{(B.41)} \\
M(0,6) & = \frac{x^{12}}{8192} . \quad \text{(B.42)} \\
M(1,6) & = -\frac{x^{11}}{3072} . \quad \text{(B.43)} \\
M(2,6) & = -\frac{11}{43008}x^{12} + \frac{5}{5376}x^{10} . \quad \text{(B.44)} \\
M(3,6) & = \frac{50176}{5176}x^{11} - \frac{1}{336}x^9 . \quad \text{(B.45)} \\
M(4,6) & = \frac{2408448}{715}x^{12} - \frac{11}{2352}x^{10} + \frac{1}{84}x^8 . \quad \text{(B.46)} \\
M(5,6) & = -\frac{301056}{715}x^{11} + \frac{55}{21}x^9 - \frac{2}{49}x^7 . \quad \text{(B.47)} \\
M(6,6) & = -\frac{1806336}{715}x^{12} + \frac{429}{25088}x^{10} - \frac{11}{49}x^8 - \frac{8}{21}x^6 . \quad \text{(B.48)} \\
M(7,6) & = \frac{129024}{2431}x^{11} - \frac{143}{715}x^9 - \frac{22}{588}x^7 . \quad \text{(B.49)} \\
M(8,6) & = \frac{3612672}{2431}x^{12} - \frac{7056}{715}x^{10} - \frac{715}{588}x^8 . \quad \text{(B.50)} \\
M(9,6) & = -\frac{100352}{4199}x^{11} - \frac{4704}{225792}x^9 . \quad \text{(B.51)} \\
M(10,6) & = -\frac{1806336}{46189}x^{12} - \frac{60775}{225792}x^{10} . \quad \text{(B.52)} \\
M(11,6) & = -\frac{903168}{1032192} . \quad \text{(B.53)} \\
M(12,6) & = -\frac{4199x^{12}}{1032192} . \quad \text{(B.54)} \\
M(13,6) & = M(14,6) = M(15,6) = M(16,6) = M(17,6) \\
& = M(18,6) = 0 . \quad \text{(B.55)}
\end{align*}
\[
M(0, 7) = \frac{x^{14}}{32768}.
\] (B.56)

\[
M(1, 7) = -\frac{7x^{13}}{90112}.
\] (B.57)

\[
M(2, 7) = -\frac{91}{1441792} x^{14} + \frac{7}{33792} x^{12}.
\] (B.58)

\[
M(3, 7) = \frac{360448}{455} x^{13} - \frac{5}{8448} x^{11}.
\] (B.59)

\[
M(4, 7) = \frac{6488064}{1625} x^{14} - \frac{65}{67584} x^{12} + \frac{1}{528} x^{10}.
\] (B.60)

\[
M(5, 7) = -\frac{3244032}{1105} x^{13} + \frac{65}{16896} x^{11} - \frac{1}{132} x^9.
\] (B.61)

\[
M(6, 7) = -\frac{12976128}{1547} x^{14} + \frac{22528}{455} x^{12} - \frac{13}{704} x^{10} + \frac{2}{33} x^8.
\] (B.62)

\[
M(7, 7) = \frac{1622016}{4199} x^{13} - \frac{25344}{1105} x^{11} + \frac{91}{528} x^9 + \frac{8}{33} x^7.
\] (B.63)

\[
M(8, 7) = \frac{35684352}{29393} x^{14} - \frac{221}{4199} x^{12} + \frac{1584}{325} x^{10} + \frac{26}{33} x^8.
\] (B.64)

\[
M(9, 7) = -\frac{1982464}{29393} x^{13} + \frac{848}{230072} x^{11}.
\] (B.65)

\[
M(10, 7) = -\frac{142737408}{29393} x^{14} + \frac{104975}{2230272} x^{12} + \frac{5525}{6336} x^{10}.
\] (B.66)

\[
M(11, 7) = \frac{3244032}{52003} x^{13} + \frac{50688}{29393} x^{11}.
\] (B.67)

\[
M(12, 7) = \frac{71368704}{67039} x^{14} + \frac{247808}{247808} x^{12}.
\] (B.68)

\[
M(13, 7) = \frac{35684352}{142737408} x^{13}.
\] (B.69)

\[
M(14, 7) = \frac{185725}{142737408} x^{14}.
\] (B.70)

\[
M(15, 7) = M(16, 7) = M(17, 7) = M(18, 7) = 0.
\] (B.71)

\[
M(0, 8) = \frac{x^{16}}{131072}.
\] (B.72)

\[
M(1, 8) = -\frac{5x^{15}}{53248}.
\] (B.73)

\[
M(2, 8) = -\frac{319488}{35} x^{16} + \frac{7}{146432} x^{14}.
\] (B.74)

\[
M(3, 8) = \frac{585728}{119} x^{15} - \frac{4912}{54912} x^{13}.
\] (B.75)

\[
M(4, 8) = \frac{7028736}{595} x^{16} - \frac{35}{164736} x^{14} + \frac{5}{13728} x^{12}.
\] (B.76)

\[
M(5, 8) = -\frac{5271552}{858} x^{15} + \frac{125}{164736} x^{13} - \frac{1}{858} x^{11}.
\] (B.77)
\begin{align*}
M(6,8) &= -\frac{2261}{115974144}x^{16} + \frac{85}{146432}x^{14} - \frac{5}{1716}x^{12} + \frac{2}{429}x^{10}. \quad \text{(B.78)} \\
M(7,8) &= -\frac{119}{57987072}x^{15} + \frac{35}{41184}x^{13} - \frac{16}{2574}x^{11} - \frac{64}{429}x^{9}. \quad \text{(B.79)} \\
M(8,8) &= -\frac{26512}{463896576}x^{16} + \frac{5148}{1615}x^{14} - \frac{1287}{51}x^{12} - \frac{429}{8}x^{10}. \quad \text{(B.80)} \\
M(9,8) &= -\frac{1507663872}{6443008}x^{15} + \frac{286}{151008}x^{13} - \frac{143}{186}x^{11}. \quad \text{(B.81)} \\
M(10,8) &= -\frac{312}{52003}x^{16} + \frac{14496768}{16754208}x^{14} - \frac{8075}{1812096}x^{12} - \frac{1190}{1287}x^{10}. \quad \text{(B.82)} \\
M(11,8) &= -\frac{11305}{68530176}x^{15} + \frac{2261}{164736}x^{13} - \frac{261}{2574}x^{11}. \quad \text{(B.83)} \\
M(12,8) &= -\frac{3015327744}{185725}x^{16} + \frac{2617472}{1812096}x^{14} - \frac{151008}{188457984}x^{12}. \quad \text{(B.84)} \\
M(13,8) &= -\frac{52003}{167518208}x^{15} - \frac{2617472}{9100525}x^{13}. \quad \text{(B.85)} \\
M(14,8) &= -\frac{83759104}{557175}x^{15} - \frac{188457984}{1077205}x^{13}. \quad \text{(B.86)} \\
M(15,8) &= -\frac{1077205}{2680291328}x^{16}. \quad \text{(B.87)} \\
M(16,8) &= -\frac{1077205}{2680291328}x^{16}. \quad \text{(B.88)} \\
M(17,8) &= M(18,8) = 0. \quad \text{(B.89)} \\
M(0,9) &= \frac{x^{18}}{524288}. \quad \text{(B.90)} \\
M(1,9) &= -\frac{3x^{17}}{655360}. \quad \text{(B.91)} \\
M(2,9) &= -\frac{x^{18}}{13107200} + \frac{3}{266240}x^{16}. \quad \text{(B.92)} \\
M(3,9) &= -\frac{x^{17}}{10649600} - \frac{21}{732160}x^{15}. \quad \text{(B.93)} \\
M(4,9) &= -\frac{x^{18}}{234291200} - \frac{357}{732160}x^{16} + \frac{7}{91520}x^{14}. \quad \text{(B.94)} \\
M(5,9) &= -\frac{x^{17}}{257720320} + \frac{119}{732160}x^{15} - \frac{1}{4576}x^{13}. \quad \text{(B.95)} \\
M(6,9) &= -\frac{x^{18}}{10308812800} + \frac{51}{161075200}x^{16} - \frac{91520}{91520}x^{14} + \frac{1}{1430}x^{12}. \quad \text{(B.96)} \\
M(7,9) &= -\frac{x^{17}}{110789} + \frac{2261}{4026880}x^{15} + \frac{119}{57200}x^{13} - \frac{2}{715}x^{11}. \quad \text{(B.97)} \\
M(8,9) &= -\frac{x^{18}}{67007283200} + \frac{16}{715}x^{10}. \quad \text{(B.98)}
\end{align*}
\[ M(9, 9) = -\frac{468027}{6700728320} x^{17} + \frac{6783}{4026880} x^{15} - \frac{8721}{629200} x^{13} + \frac{306}{3575} x^{11} \]
\[ + \frac{64}{715} x^9 . \] (B.99)

\[ M(10, 9) = -\frac{37145}{5360582656} x^{18} + \frac{52003}{83759104} x^{16} - \frac{2261}{201344} x^{14} \]
\[ + \frac{2261}{15730} x^{12} + \frac{7429}{715} x^{10} . \] (B.100)

\[ M(11, 9) = \frac{60915712 x^{17}}{66861} - \frac{52003}{9518080} x^{15} + \frac{15827}{114400} x^{13} + \frac{2584}{3575} x^{11} . \] (B.101)

\[ M(12, 9) = \frac{6700728320 x^{18}}{31654} - \frac{66861}{41879552} x^{16} + \frac{1092063}{13087360} x^{14} \]
\[ + \frac{39325}{39325} x^{12} . \] (B.102)

\[ M(13, 9) = -\frac{66861}{257720320} x^{17} + \frac{52003}{1610752} x^{15} + \frac{364021}{629200} x^{13} . \] (B.103)

\[ M(14, 9) = -\frac{107211653120 x^{18}}{5816907} + \frac{3276189}{41879552} x^{16} + \frac{364021}{1308736} x^{14} . \] (B.104)

\[ M(15, 9) = \frac{5360582656 x^{17}}{3535767} + \frac{5234944}{1938969} x^{15} . \] (B.105)

\[ M(16, 9) = \frac{5360582656 x^{18}}{60108039 x^{17}} + \frac{104698880}{104698880} x^{16} . \] (B.106)

\[ M(17, 9) = \frac{26802913280 x^{17}}{1178589 x^{18}} . \] (B.107)

\[ M(18, 9) = \frac{9746513920}{1178589 x^{18}} . \] (B.108)
APPENDIX C: THE FUNCTIONS $V_j^j(\delta)$ AND $V_j^j(\chi)$ THAT APPEAR AS ELEMENTS OF THE DETERMINANTS IN THE NUMERATORS AND DENOMINATORS OF THE EXPRESSIONS FOR THE EQUILIBRIUM FUNCTIONS $F^{(N)}(\epsilon, \delta)$ AND $F^{(N)}(\epsilon, \chi)$ RESPECTIVELY, AND IN THE NUMERATOR AND DENOMINATOR OF THE EXPRESSION FOR THE CONSTANTS $K_i^{(N)}$ THAT MULTIPLY THE MULTIPOLe SOLUTIONS $\varphi^{(i)}(x, \theta)$ IN THE REPRESENTATIONS OF THE FLUX FUNCTIONS OF THE ORDERS $N = 3$ TO $N = 10$.

The functions $V_j^{(N)}(x)$ are formally defined by Eq. (4.28), which, here reproduced, writes as:

$$V_j^{(N)}(x) = \sum_{n=0}^{2N-2} M_{nj}(x) \ (j = 1, 2, \ldots, N-1). \quad (4.28)$$

Considering that, according to Eq. (4.9) the matrix elements $M_{nj}(x)$ in general vanish for $n$ greater than $2j$, the definition of $V_j^{(N)}(x)$ can be restated as:

$$V_j^{(N)}(x) = \sum_{n=0}^{2j} M_{nj}(x) \ (j = 1, 2, \ldots, N-1). \quad (C.1)$$

The replacement of $2N - 2$ by $2j$ as the upper limit of the summation is permissible because, even for the least value of $N$, which is 3, all non-null matrix elements $M_{nj}(x)$ that are summoned to contribute to the sum according to the original definition of $V_j^{(N)}(x)$ are also taken into account by this newest one. Thus, except for the specification of the range of variation of $j$ and of the number $N - 1$ of functions that are implied by a given order $N$ of approximation, the definition of $V_j^{(N)}(x)$ is actually independent of $N$, and, the same as for the functions $P(x)$ and $Q(x)$ of Appendix A, we can omit the superscript $N$ without incurring the risk that any
ambiguity be introduced into the meaning of the symbol that denotes the quantity defined by Eq. (C.1). Another consequence of this lack of dependence of $V_j^{(N)}(x)$ on $N$ is that, in passing from the order $N$ to the order $N + 1$ of approximation to the flux function, there is need to carry out the calculation as new only of the function $V_j^{(N+1)}(x)$, all the functions of lower subscripts, $V_1^{(N+1)}(x), V_2^{(N+1)}(x), \ldots, V_{N-1}^{(N+1)}(x)$, remaining the same as those previously calculated for the order $N$, to know, $V_1^{(N)}(x), V_2^{(N)}(x), \ldots, V_{N-1}^{(N)}(x)$. The functions listed in the present Appendix are sufficient in number to compute the flux functions $\psi^{(N)}(x, \theta)$ up to the order $N = 10$.

It may be helpful to observe that, adopting the pattern of notation used in Appendix B to denote the matrix elements, the evaluation of the sums to obtain the functions $V_j(x)$ according to the rule of Eq. (C.1) can be most conveniently accomplished by use of MAPLE’s command add.

(a) The functions $V_j(\delta)$ and their derivatives $V_j'(\delta)$ for $1 \leq j \leq 9$

The functions $V_j(\delta)$ can be obtained from the functions $V_j(\chi)$ given in (b) ahead by the replacement $\chi \to \delta(\delta + 2)$.

\[
\begin{align*}
V'_1(\delta) &= \frac{1}{2}(\delta + 1)
V'_2(\delta) &= -\delta(\delta + 1)(\delta + 2)
V'_3(\delta) &= \frac{3}{4}\delta^2(\delta + 1)(\delta + 2)^2
V'_4(\delta) &= -\frac{2}{5}\delta^3(\delta + 1)(\delta + 2)^3
V'_5(\delta) &= \frac{5}{28}\delta^4(\delta + 1)(\delta + 2)^4
V'_6(\delta) &= -\frac{1}{14}\delta^5(\delta + 1)(\delta + 2)^5
V'_7(\delta) &= \frac{7}{264}\delta^6(\delta + 1)(\delta + 2)^6
V'_8(\delta) &= -\frac{4}{429}\delta^7(\delta + 1)(\delta + 2)^7
\end{align*}
\]
\[ V_j'(\delta) = \frac{9}{2860} \delta^8 (\delta + 1)(\delta + 2)^8. \]  

(b) The functions \( V_j(\chi) \) and their derivatives \( V_j'(\chi) \) for \( 1 \leq j \leq 9 \)

\[
V_1(\chi) = \frac{1}{4} \chi, \quad \text{(C.11)} \\
V_2(\chi) = -\frac{1}{4} \chi^2, \quad \text{(C.12)} \\
V_3(\chi) = \frac{1}{8} \chi^3, \quad \text{(C.13)} \\
V_4(\chi) = -\frac{1}{20} \chi^4, \quad \text{(C.14)} \\
V_5(\chi) = \frac{1}{56} \chi^5, \quad \text{(C.15)} \\
V_6(\chi) = -\frac{1}{168} \chi^6, \quad \text{(C.16)} \\
V_7(\chi) = \frac{1}{528} \chi^7, \quad \text{(C.17)} \\
V_8(\chi) = -\frac{1}{1716} \chi^8, \quad \text{(C.18)} \\
V_9(\chi) = \frac{1}{5720} \chi^9, \quad \text{(C.19)}
\]

\[
V_1'(\chi) = \frac{1}{4}, \quad \text{(C.20)} \\
V_2'(\chi) = -\frac{1}{2} \chi, \quad \text{(C.21)} \\
V_3'(\chi) = \frac{3}{8} \chi^2, \quad \text{(C.22)} \\
V_4'(\chi) = -\frac{1}{5} \chi^3, \quad \text{(C.23)} \\
V_5'(\chi) = \frac{5}{56} \chi^4, \quad \text{(C.24)} \\
V_6'(\chi) = -\frac{1}{28} \chi^5, \quad \text{(C.25)} \\
V_7'(\chi) = \frac{7}{528} \chi^6, \quad \text{(C.26)} \\
V_8'(\chi) = -\frac{2}{429} \chi^7, \quad \text{(C.27)} \\
V_9'(\chi) = \frac{9}{5720} \chi^8. \quad \text{(C.28)}
\]
APPENDIX D: VALUES OF THE NORMALIZED MAGNETIC POLOIDAL FLUX AT THE MAGNETIC AXIS AND VALUES OF THE NORMALIZED MAGNETIC POLOIDAL FLUX AT THE CENTRE LINE OF THE TORUS FOR THE EQUILIBRIUM CONFIGURATIONS WITH INVERSE ASPECT RATIO $\epsilon = 2/5$ AND VALUES OF THE EQUILIBRIUM PARAMETER $\lambda$ EQUAL TO 1.0 AND $-1/5$ RESPECTIVELY. EXPRESSIONS FOR THE NORMALIZED PARTIAL FLUX FUNCTIONS OF THE ORDER $N = 10$ IN NORMALIZED CYLINDRICAL COORDINATES $(\rho, Z, \phi)$ FOR THE THREE REFERENCE EQUILIBRIA.

(a) The dimensionless normalized magnetic poloidal flux at the magnetic axis is defined as:

$$\psi_M \equiv \frac{\psi(x = \delta, \theta = 0)}{s_p}, \quad (D.1)$$

where $\psi(x, \theta)$ is related to the dimensional magnetic poloidal flux $\Psi(x, \theta)$ through Eq. (2.3). The dimensionless normalized magnetic poloidal flux at the centre line of the torus is defined as:

$$\hat{\psi}_C \equiv \frac{\psi(x = 0, \text{independent of } \theta)}{(-\psi_M)} \quad \text{or} \quad \frac{\psi(\rho = 1, Z = 0)}{(-\psi_M)}, \quad (D.2)$$

where $\psi(x, \theta)$ is related to the dimensionless magnetic poloidal flux $\psi(x, \theta)$ through Eq. (4.2) and $\psi(\rho, Z)$ is its representation in terms of the normalized cylindrical coordinates $(\rho, Z)$. These definitions hold for any order $N$ of approximation to the poloidal flux function. Table D.1 to follow displays the numerical values for the two normalized fluxes $\psi_M$ and $\hat{\psi}_C$ for the three reference equilibria as evaluated by use of the approximation of order $N = 10$ to the poloidal flux function.
| \( \lambda \) | \( \psi_M \) | \( \psi_C \) |
|---|---|---|
| 1  | -0.08058778256 | -0.9773960062 |
| 0  | -0.04180788277 | -0.9422073888 |
| -1/5 | -0.03421957455 | -0.9210167079 |

Table D.1 Values of the normalized dimensionless flux at the magnetic axis \( \psi_M \) and values of the normalized dimensionless flux at the centre line of the torus \( \psi_C \) for the three reference equilibria according to the approximation of order \( N = 10 \) to the flux function.

(b) The normalized cylindrical coordinates \( \rho \) and \( Z \) are defined in the following of Eq. (3.32). In these coordinates the expression for the normalized flux in the approximation of order \( N = 10 \) writes in general as:

\[
\hat{\psi}^{(10)}(\rho, Z, \phi) = C_0 + C_2 \rho^2 + C_4 \rho^4 + \cdots + C_{18} \rho^{18}.
\]  

(D.3)

The coefficients \( C_l \) \((l = 0, 2, \ldots, 18)\) are functions of \( Z^2 \) and depend on the values of the displacement variable \( \chi \) and of the squared inverse aspect ratio \( \epsilon^2 \). The formulae for the \( C_l \)'s given here apply all to \( \epsilon = 2/5 \).

\( b.1 \) \( \lambda = 1 \)

\[
C_0 = \frac{31202}{5029} Z^2 + \frac{6669}{4402}, \tag{D.4}
\]

\[
C_2 = \frac{-245677}{260681} Z^{16} - \frac{62049}{7630} Z^{14} - \frac{299573}{9654} Z^{12} - \frac{444275}{6471} Z^{10} - \frac{400731}{4133} Z^8
\]

\[
- \frac{180008}{83601} Z^6 - \frac{429281}{12408} Z^4 - \frac{173077}{25848} Z^2 - \frac{137}{14006}, \tag{D.5}
\]

\[
C_4 = \frac{2326383}{82282} Z^{14} + \frac{130801}{707} Z^{12} + \frac{469742}{1269} Z^{10} + \frac{487374}{631} Z^8 + \frac{612877}{903} Z^6
\]

\[
+ \frac{1042381}{1264} Z^4 + \frac{173077}{25848}, \tag{D.6}
\]
\begin{align*}
C_6 &= -\frac{2818793}{13147} Z^{12} - \frac{2125655}{2089} Z^{10} - \frac{1067543}{556} Z^8 - \frac{702869}{390} Z^6 - \frac{941713}{1110} Z^4 \\
&\quad - \frac{9563}{3003} Z^2 - \frac{13147}{1366}, \quad (D.7) \\
C_8 &= \frac{3180978}{5395} Z^{10} + \frac{8282194}{1277} Z^8 + \frac{835538}{4341} Z^6 + \frac{886471}{877} Z^4 + \frac{539364}{2543} Z^2 \\
&\quad + \frac{72219}{10207}, \quad (D.8) \\
C_{10} &= -\frac{847057}{1277} Z^8 - \frac{1312826}{983} Z^6 - \frac{4409255}{5249} Z^4 - \frac{263069}{1557} Z^2 - \frac{53719}{10131}, \quad (D.9) \\
C_{12} &= \frac{531185}{1716} Z^6 + \frac{912167}{2732} Z^4 + \frac{92822}{1105} Z^2 + \frac{50147}{17808}, \quad (D.10) \\
C_{14} &= -\frac{81367}{1472} Z^4 - \frac{92557}{3881} Z^2 - \frac{46411}{46410}, \quad (D.11) \\
C_{16} &= \frac{50119}{16925} Z^2 + \frac{3987}{18724}, \quad (D.12) \\
C_{18} &= -\frac{2795}{135916}. \quad (D.13) \\
(b.2) \lambda = 0 \\
C_0 &= -\frac{Z^2}{250156325} + \frac{32177}{39153}, \quad (D.14) \\
C_2 &= \frac{19849}{44309} Z^{16} + \frac{37471}{9662} Z^{14} + \frac{1084410}{72947} Z^{12} + \frac{62779}{1896} Z^{10} + \frac{93829}{1985} Z^8 + \frac{216053}{4795} Z^6 \\
&\quad + \frac{138829}{4735} Z^4 + \frac{185449}{10569} Z^2 + \frac{27950}{27950}. \quad (D.15)
\end{align*}
\[ \begin{align*}
C_4 &= -\frac{54885}{4084} Z^{14} - \frac{519755}{368180} Z^{12} - \frac{222473}{2179} Z^{10} - \frac{470470}{907} Z^8 - \frac{4223723}{27339} Z^6, \\
C_6 &= \frac{190169}{1866} Z^{12} + \frac{1037966}{549651} Z^{10} + \frac{2037394}{20861} Z^8 + \frac{345061}{397} Z^6 + \frac{656803}{1588} Z^4 \\
&\quad + \frac{279419}{6506} Z^2 + \frac{997}{22526}, \\
C_8 &= -\frac{2915185}{3204} Z^{10} - \frac{1607533}{1498} Z^8 - \frac{620371}{1142} Z^6 - \frac{105986}{1025} Z^4 \\
&\quad - \frac{279419}{79295} Z^2 - \frac{22526}{22526}, \\
C_{10} &= \frac{826381}{2621} Z^8 + \frac{647091}{1016} Z^6 + \frac{459161}{1141} Z^4 + \frac{120679}{1481} Z^2 + \frac{36560}{14143}, \\
C_{12} &= -\frac{334441}{2273} Z^6 - \frac{350773}{2203} Z^4 - \frac{590873}{14683} Z^2 - \frac{34775}{25606}, \\
C_{14} &= \frac{142407}{5420} Z^4 + \frac{148159}{13027} Z^2 + \frac{50073}{104521}, \\
C_{16} &= -\frac{99455}{70658} Z^2 - \frac{13315}{131122}, \\
C_{18} &= \frac{6100}{624061}.
\end{align*} \]

(b.3) \( \lambda = -1/5 \)

\[ \begin{align*}
C_0 &= -\frac{25012}{8559} Z^2 + \frac{57683}{117405}.
\end{align*} \]
\[
C_2 = \frac{46241}{42012} Z^{16} + \frac{107618}{11309} Z^{14} + \frac{30075}{826} Z^{12} + \frac{303631}{3754} Z^{10} + \frac{355945}{3096} Z^8 \\
+ \frac{337021}{3100} Z^6 + \frac{450663}{6497} Z^4 + \frac{172946}{4883} Z^2 - \frac{12927}{42241}, \quad (D.25)
\]

\[
C_4 = -\frac{523959}{15868} Z^{14} - \frac{206317}{225946} Z^{12} - \frac{613388}{5509} Z^{10} - \frac{856237}{1505} Z^8 - \frac{228559}{13992} Z^6 \\
- \frac{953}{72781} Z^4 - \frac{156591}{541} Z^2 - \frac{1021}{13992}, \quad (D.26)
\]

\[
C_6 = \frac{278195}{1111} Z^{12} + \frac{2111123}{268054} Z^{10} + \frac{3706011}{62489} Z^8 + \frac{1895975}{7207} Z^6 + \frac{544236}{1315} Z^4 \\
+ \frac{953}{1315} Z^2 + \frac{1773}{7207}, \quad (D.27)
\]

\[
C_8 = -\frac{647974}{941} Z^{10} - \frac{1308289}{586} Z^8 - \frac{1235337}{470} Z^6 - \frac{578559}{436} Z^4 - \frac{462500}{1839} Z^2 \\
- \frac{108003}{12716}, \quad (D.28)
\]

\[
C_{10} = \frac{232403}{300} Z^8 + \frac{854853}{547} Z^6 + \frac{2027464}{2057} Z^4 + \frac{371618}{1867} Z^2 + \frac{79592}{12659}, \quad (D.29)
\]

\[
C_{12} = -\frac{1031043}{2852} Z^6 - \frac{1100213}{2816} Z^4 - \frac{254394}{2581} Z^2 - \frac{67984}{20493}, \quad (D.30)
\]

\[
C_{14} = \frac{198059}{3068} Z^4 + \frac{113359}{4062} Z^2 + \frac{43360}{36953}, \quad (D.31)
\]

\[
C_{16} = -\frac{245199}{70900} Z^2 - \frac{17064}{68483}, \quad (D.32)
\]

\[
C_{18} = \frac{605}{25191}. \quad (D.33)
\]

Transformation from the normalized cylindrical coordinates \((\rho, Z, \phi)\) to the normalized toroidal-polar coordinates \((x, \theta, \phi)\) can be accomplished by means of the
relations:
\[
\begin{align*}
\rho &= 1 + x \cos \theta , \\
Z &= x \sin \theta , \\
\phi &= \phi .
\end{align*}
\]

**APPENDIX E: EQUATION FOR THE CRITICAL VALUE OF THE DISPLACEMENT VARIABLE FOR AXISYMMETRIC EQUILIBRIA OF INVERSE ASPECT RATIO 2/5 UNDER THE APPROXIMATION OF ORDER \( N = 4 \) TO THE FLUX FUNCTION**

The equation is:

\[
\chi^{18} - 119.6972128\chi^{17} + 64.65882500\chi^{16} + 6630.186141\chi^{15} + 1633.052456\chi^{14} \\
- 20702.02206\chi^{13} + 5737.175591\chi^{12} + 15141.43088\chi^{11} - 7192.561804\chi^{10} \\
- 3778.002560\chi^9 + 2659.627640\chi^8 + 166.2328382\chi^7 - 378.2158119\chi^6 \\
+ 48.19578700\chi^5 + 17.57526666\chi^4 - 5.141304959\chi^3 + 0.3449459767\chi^2 \\
+ 0.01733079721\chi - 0.002048736713 = 0 .
\]

\[(E.1)\]

The root with physical significance is:

\[
\chi \equiv \chi_c^{(4)} = 0.2543144486 .
\]

\[(E.2)\]
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DEDICATION

This paper is dedicated to the memory of Professor Gumercindo Lima who, as professor of the author in high school, taught him the principles of linear algebra that are applied in the theory here presented.

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