THE INFLUENCE OF ROTATION IN RADIATION-DRIVEN WIND FROM HOT STARS:
NEW SOLUTIONS AND DISK FORMATION IN Be STARS

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Received 1999 December 28; accepted 2004 June 18

ABSTRACT

The theory of radiation-driven wind including stellar rotation is reexamined. After a suitable change of variables, a new equation for the mass-loss rate is derived analytically. The solution of this equation remains within 1% confidence when compared with numerical solutions. In addition, a nonlinear equation for the position of the critical (singular) point is obtained. This equation shows the existence of an additional critical point besides the standard m-CAK critical point. For a stellar rotation velocity larger than ~0.7−0.8\(v_{\text{break}}\), there exists only one critical point, located away from the star’s surface. Numerical solutions crossing through this new critical point are attained. In these cases, the wind has a very low terminal velocity and therefore a higher density. Disk formation in Be stars is discussed in the framework of this new line-driven stellar wind solution.

Subject headings: hydrodynamics — methods: analytical — stars: early-type — stars: mass loss — stars: rotation — stars: winds, outflows

1. INTRODUCTION

The theory of radiation-driven winds has been very successful at describing the observed terminal velocities and mass-loss rates from hot stars ever since the pioneering work of Castor et al. (1975, hereafter CAK), who realized that the force due to line absorption in a rapidly expanding envelope can be calculated using the Sobolev approximation (Castor 1974). They developed a simple parameterization of the line force and were able to construct an analytical wind model.

Simultaneously and independently, Friend & Abbott (1986, hereafter FA86) and Pauldrach et al. (1986, hereafter PPK86) calculated the influence of the finite cone angle correction on the dynamics of the wind. They found a better agreement between the improved or modified theory (hereafter m-CAK) and the observations for both \(M\) (mass-loss rate) and \(v_{\infty}\) (terminal velocity) in a large domain in the Hertzsprung-Russell diagram. Furthermore, Kudritzki et al. (1989, hereafter KPPA89) developed analytical formulae for the localization of the critical point, mass-loss rate, and terminal velocity with an agreement within 5% for \(v_{\infty}\) and 10% for \(M\) when compared with numerical calculations.

The influence of the star’s rotation was investigated by Castor (1979) and Marlborough & Zamir (1984). Both studies concluded that adding a centrifugal force term in the CAK equation results in a lower terminal velocity wind, but the mass-loss rate is not substantially affected. Similar results were found by FA86 and PPK86.

Concerning Be stars, the m-CAK theory gives a good description for the polar wind, but it fails to account for a slowly accelerating flow like the one in the equator of Be stars. Therefore, several additional mechanisms have been proposed on the basis of the radiatively driven wind theory to explain the equatorial flow of these objects. The influence of magnetic fields and rotation was studied by Friend & MacGregor (1984; see review from Cassinelli 1998). A model with azimuthal symmetry, rotation, and viscous force was developed by de Araujo et al. (1994), and Koninx & Hearn (1992) incorporated sound waves in the wind dynamics of Be stars. A model driven mainly by thin lines has been proposed by de Araujo (1995); he obtains an outflow with a shallow expansion and a large mass flux when the line force parameters are considered as free and not in a self-consistent mode (Pauldrach 1987). A two-dimensional hydrodynamic model from Bjorkman & Cassinelli (1993) and Owocki et al. (1994) concluded that a meridional current may be responsible for the concentration of matter toward the equator, forming a wind-compressed disk (WCD). An assumption of the WCD model, that the line force is strictly central, was published by Owocki et al. (1996). They conclude that non-radial line force components, together with gravity-darkening effects, can inhibit the formation of a WCD structure.

Despite all the efforts, the most important problem that pervades all these models is the strong equatorial expansion that they exhibit. Not only is the terminal velocity too high (~1000 km s\(^{-1}\)), but there is also a sharp increase of the velocity field. The fitting of observed \(H\alpha\) line profiles requires terminal velocities of about 200 km s\(^{-1}\) or less (Poeckert & Marlborough 1978), while from Fe \(ii\) line profiles, Hanuschik (1994; see also Waters & Marlborough 1994) concluded that the disk expansion velocity must not be much larger than the Doppler width.

When rapid rotators (such as Be stars) are studied theoretically, many authors report the appearance of numerical problems when the rotational speed is about 0.8 times the breakup speed (see, e.g., FA86; Poe & Friend 1986; de Araujo & de Freitas Pacheco 1989; Boyd & Marlborough 1991). In view of the above-mentioned difficulties, our purpose is to perform a reanalysis including the rotational centrifugal force term in radiation-driven wind theory.

In § 2, we carry out an analytical treatment with the inclusion of the star’s rotational speed. In § 3, after a suitable coordinate transformation, exact formulae for the location of the critical (singular) point(s) and for the mass-loss rate are obtained. The roots of the critical point function define the number of singular (critical) points and their location. We show the existence of a new family of singular points in addition to the standard one (the m-CAK solution family). A simple, approximative treatment for the location of the singular point(s) and the value of the corresponding mass-loss rate (eigenvalue) is introduced in § 4. Furthermore, in this section, numerical and analytical
results for the standard m-CAK solution are compared, showing the high confidence of this analytical approximation. Section 5 is devoted to finding a numerical solution from the momentum equation of the wind, which starts at the stellar surface and reaches infinity after passing through a critical point that belongs to this new family of singular points. In addition, we compare the numerical and analytical results in order to show the accuracy of this analytic treatment. Finally, the applicability of this model to explain disk formation in Be stars is discussed in § 7. Conclusions and future lines of research are summarized in § 8.

2. HYDRODYNAMIC FORMULATION

The standard model for radiation-driven stellar winds considers a one-component isothermal fluid in a stationary regime with spherical symmetry, neglecting the effect of viscosity, heat conduction, and magnetic field (CAK; FA86; PPK86). The continuity equation reads

\[ 4\pi r^2 \rho v = M, \]

and the momentum equation is given by

\[ \frac{dv}{dr} = -\frac{1}{\rho} \frac{dp}{dr} - \frac{GM(1 - \Gamma)}{r^2} + \frac{v^2(r)}{r} + g_{\text{line}} \left( \rho, \frac{dv}{dr}, n_E \right), \]

where \( v \) is the fluid velocity; \( \frac{dv}{dr} \) is the velocity gradient; \( \rho \) is the mass density; \( M \) is the star’s mass-loss rate; \( p \) is the fluid pressure; \( v_0 = \frac{v_{\text{rot}}}{r} \), where \( v_{\text{rot}} \) is the star’s rotational speed at the equator; \( \Gamma \) is the radiative acceleration caused by Thomson scattering in terms of gravitational acceleration, and \( g_{\text{line}}(\rho, \frac{dv}{dr}, n_E) \) is the acceleration due to the lines. The standard parameterization of the line force (Abbott 1982; PPK86; FA86) is

\[ g_{\text{line}} = \frac{C}{r^2} C F \left( r, v, \frac{dv}{dr} \right) \left( \frac{v^2(r)}{r} \right)^\alpha \left[ \frac{n_E}{W(r)} \right]^\delta. \]

The coefficient \( C \) is given by

\[ C = \Gamma GMk \left( \frac{4\pi}{\sigma_{\text{E}} v_{\text{th}} M} \right)^\alpha, \]

where \( v_{\text{th}} \) is the thermal velocity of the protons, \( \sigma_E \) is the Thomson scattering absorption coefficient per density and \( n_E \) is the electron number density in units of \( 10^{-11} \text{ cm}^{-3} \).

\[ W(r) = \frac{1}{2} \left[ 1 - \sqrt{1 - \left( \frac{R_s}{r} \right)^2} \right] \]

is the dilution factor, \( C F \) is the correction factor (see Appendix B), and all the other quantities have their usual meaning (see, e.g., PPK86).

We introduce the following change of variables:

\[ u = \frac{R_s}{r}, \]

\[ w = \frac{v}{a}, \]

\[ w' = \frac{dv}{du}, \]

where \( R_s \) is the star’s radius and \( a \) is the isothermal sound speed; i.e., \( p = \alpha^2 \rho \).

The momentum equation (2) with the line force (3) becomes

\[ F(u, w, w') \equiv \left( 1 - \frac{1}{w^2} \right) w \frac{dw}{du} + A \frac{2}{u} + a^2 u \]

\[ - C' CF g(u) w^{-\delta} \left( \frac{dw}{du} \right)^\alpha = 0, \]

where

\[ A = \frac{GM(1 - \Gamma)}{a^2 R_s} = \frac{v_{\text{esc}}^2}{2a^2}, \]

\[ C' = C \left( \frac{MD}{2\pi a R_s^2} \right) \left( a^2 R_s \right)^{\alpha-1}, \]

\[ g(u) = \left( \frac{u^2}{1 - \sqrt{1 - u^2}} \right)^\delta, \]

\[ a_{\text{rot}} = \frac{v_{\text{rot}}}{a}, \]

where \( v_{\text{esc}} \) is the escape velocity and \( D \) is defined by

\[ D = \frac{(1 + Z_{\text{He}} Y_{\text{He}})}{(1 + Z_{\text{He}} Y_{\text{He}}) m_1^3}, \]

where \( Y_{\text{He}} \) is the helium abundance relative to hydrogen, \( Z_{\text{He}} \) is the amount of free electrons provided by helium, \( A_{\text{He}} \) is the atomic mass number of helium, and \( m_1 \) is the mass of the proton.

The standard method for solving this nonlinear differential equation (7) together with the constant \( C' \) (the eigenvalue of the problem) is to require the solution to pass through the singular (critical) point (see CAK for details), together with a constraint on the optical depth integral:

\[ \int_{R_s}^{\infty} \sigma_E \rho(r) dr = \frac{2}{3}. \]

Other authors prefer to use an equivalent lower boundary condition, e.g., setting the density at the stellar surface to a specific value,

\[ \rho(R_s) = \rho_s. \]

De Araujo & de Freitas Pacheco (1989) use \( \rho_s = 10^{-11} \text{ g cm}^{-3} \), while Friend & MacGregor (1984) use \( \rho_s = 1.5 \times 10^{-9} \text{ g cm}^{-3} \).

A critical point is located where the singularity condition is satisfied:

\[ \frac{\partial}{\partial w'} F(u, w, w') = 0. \]

At this specific point, regularity is imposed, namely,

\[ \frac{d}{du} F(u, w, w') = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial w} w' = 0. \]

3. LOCATION OF THE CRITICAL POINT(S)

In this section, after a coordinate transformation we obtain a set of analytical equations for the eigenvalue and the location of the singular point. For the latter, we show that a new family of
singular points exist. We also prove that the standard m-CAK solution vanishes for stars with high rotational velocity.

3.1. Coordinate Transformation

In order to solve equations (7), (12), and (13), we use the following coordinate transformation:

\[ Y = \frac{w}{w'}, \quad Z = \frac{w}{w'}, \]  

with these two new coordinates, the equations become

\[ \left(1 - \frac{1}{YZ}\right) Y + A + 2 \frac{u}{u} + a_{rot}^2 u - C' f_1(u, Z) g(u) Z^{-\delta/2} \gamma^{\alpha-\delta/2} = 0, \]  

\[ \left(1 - \frac{1}{YZ}\right) Y - C' f_2(u, Z) g(u) Z^{-\delta/2} \gamma^{\alpha-\delta/2} = 0, \]  

\[ \left(1 + \frac{1}{YZ}\right) Y - \frac{2Z}{u} + a_{rot}^2 Z - C' f_3(u, Z) g(u) Z^{-\delta/2} \gamma^{\alpha-\delta/2} = 0; \]  

derivation details and definitions of \( f_1(u, Z), f_2(u, Z), \) and \( f_3(u, Z) \) are summarized in Appendix A.

Solving for \( Y \) and \( C' \) from the set of equations (15a), (15b), and (15c), we obtain

\[ Y = \frac{1}{Z} \left[ \frac{f_2}{f_1 - f_2} \left( A + 2 \frac{u}{u} + a_{rot}^2 u \right) \right], \]

\[ C'(\hat{M}) = \frac{1}{g f_2} \left( 1 - \frac{1}{YZ} \right) Z^{\delta/2} \gamma^{1-\alpha+\delta/2}. \]

These equations are generalizations of the relations found by KPPA89 (see their eqs. [21] and [34] for \( Y \) and eqs. [20] and [44] for the eigenvalue) now including the rotational speed of the star.

3.2. The Critical-Point Function \( R(u, Z) \)

It is not possible to obtain the location of the critical point from this set of equations because we have only three equations and four unknowns \((Y, C', Z, \) and \( u)\). However, from equations (15a), (15b), and (15c), we obtain a function, \( R(u, Z) \), defined by

\[ R(u, Z) = -\frac{2}{Z} + \frac{2Z}{u^2} - a_{rot}^2 + f_{123}(u, Z) \left( A + 2 \frac{u}{u} + a_{rot}^2 u \right), \]

where \( f_{123}(u, Z) \) is defined by

\[ f_{123}(u, Z) = \frac{f_2(u, Z) - f_3(u, Z)}{f_2(u, Z) - f_1(u, Z)}. \]

The root(s) of this function \( R(u, Z) \) give(s) the location of the critical (singular) point(s) \( u_{\text{crit}} \). Notice that no approximation whatsoever has been used in the derivation of the above equations.

In order to find the range of the variable \( Z \) for a typical hot-star wind, we have performed a full numerical calculation; i.e., we solve without any approximation the momentum equation (7), together with the singularity condition, equation (12); the regularity condition, equation (13); and a lower boundary condition, equation (10) or equation (11). Our typical star is an O5 V star with the following stellar parameters: \( T_{\text{eff}} = 45,000 \) K, \( \log g = 4.0 \), \( R/R_\odot = 12 \), and \( v_{\text{rot}} = 0 \); while the line force parameters are \( k = 0.124, \alpha = 0.64 \), and \( \delta = 0.07 \). Figure 1 shows the behavior of \( Z \), which ranges between 0 and 2, for the whole wind. This figure also shows two \( \beta \)-field approximations (see below).

Now knowing the range of the variable \( Z \), we may plot \( R(u, Z) \) in terms of \( u \) and \( Z \). Figures 2 and 3 show two different visualization of the function \( R(u, Z) \) for this O5 V star with \( v_{\text{rot}}/v_{\text{bkup}} = 0.5 \). Figure 4 shows, in addition to \( R(u, Z) \), the “zero plane” defined by \( R \equiv 0 \). Thus, the root(s) of \( R(u, Z) \) is (are) the curve(s) defined by the intersection of this surface \( R(u, Z) \) with the zero plane. Figure 4 shows both surfaces; their intersections are two families of critical points. The standard m-CAK family of solutions (CAK: locus of singular points) is located in the zone defined approximately by \( u \in [-1, -0.3] \) and \( Z \in [0, 3] \).

When full numerical calculations are carried out, the lower boundary condition, equation (10) or equation (11), fixes one point in this family of critical points, and in this way a unique numerical solution for the m-CAK wind is achieved. As the rotational velocity of the star increases, this m-CAK locus no longer intersects the zero plane because the function \( R(u, Z) \) becomes negative in this region, as Figure 5 shows for the case \( v_{\text{rot}}/v_{\text{bkup}} = 0.9 \). Thus, a standard m-CAK solution does not exist for stars with high rotational speeds, e.g., Be stars. This is the reason why many authors (see, e.g., de Araujo et al. 1994; Boyd & Marlborough 1991; FA86) have reported numerical difficulties in finding the location of the singular point for high rotational velocities: it does not exist.

Furthermore, carefully inspecting Figure 5, we find a “new” family of solutions in the region defined approximately by \( u \in [-0.2, 0] \) and \( Z \in [0, 3] \). This family of solutions has been always present, even for lower rotational velocities (see the case of Fig. 4).

When both families of solutions are present, e.g., when speeds are not highly rotational, the family that satisfies the
lower boundary condition, equation (10), is the m-CAK locus. An investigation of the existence of a physical solution that crosses through a singular point that belongs to the new family, when both families are present, is the focus of a forthcoming article.

We are now ready to look for numerical solutions that start at the stellar surface and then cross through a singular point located at this new locus and reach infinity, for the special case when only the new family is present, e.g., for highly rotational speeds. We perform this task in § 6.

4. APPROXIMATIVE TREATMENT

In this section we develop an approximative treatment of the critical point function \( R(u, Z) \). After introducing a \( \beta \)-field approximation, the function \( R(u, Z) \) transforms to \( R_{\text{app}}(u) \). The

In the next section, we first introduce a simple approximation for the function \( R(u, Z) \). This approximation allows both the standard and the new singular point to be easily located.

**Fig. 4.**—Same as Fig. 3, but with the addition of \( R(u, Z) \), the zero plane. The intersection of both surfaces gives the family of solutions that simultaneously satisfy the momentum equation and both singularity and regularity conditions.

**Fig. 5.**—Same as Fig. 4, but with \( \nu_{\text{rot}}/\nu_{\text{bup}} = 0.9 \). This figure shows clearly that no solution exists for the standard m-CAK critical point for this rotational velocity, while a new family of solutions is found in the region where \( u \leq -0.2 \).
zeros (roots) of this approximative function are the location of the critical points. Knowing the location of the critical point(s), it is straightforward to calculate the eigenvalues, or equivalently, the mass-loss rate of the star. These approximative calculations are in much better agreement with full numerical calculations than previous analytical treatments and include the influence of star’s rotation.

4.1. The Approximative Critical-Point Function \( R_{app}(u) \)

From numerical calculations, PPK86 found that the velocity field, for stars with effective temperatures between 40,000 and 50,000 K, can be approximated to the so-called \( \beta \)-field approximation, namely,

\[
v = v_\infty (1 + u)^\beta,
\]

with \( \beta = 0.8 \). This relationship is broadly used for stellar wind diagnostics and it is justified a posteriori by the quality of the results achieved (Kudritzki & Puls 2000).

Applying this approximation for the \( Z \) variable, it becomes

\[
Z = \frac{(1 + u)}{\beta}.
\]

Figure 1 shows the \( Z \) versus \( u \) profile from numerical calculations and the behavior of equation (21) for two different values of the \( \beta \)-parameter (\( \beta = 0.8 \) and 1.0). This approximation holds, for the \( Z \) variable, up to \( \sim 2.5 R_\star \) (\( u \sim -0.4 \)) for \( \beta = 0.8 \) and up to \( \sim 1.1 R_\star \) (\( u \sim -0.8 \)) for \( \beta = 1.0 \).

Thus, replacing equation (21) in equation (18), \( R(u, Z) \) transforms to \( R_{app}(u) \), defined as

\[
R_{app}(u) \equiv -\frac{2\beta}{(1 + u)} + \frac{2(1 + u)}{u^2\beta} - \frac{a^2}{\beta} \frac{(1 + u)}{\beta} + f_{123}(u)\left( A + \frac{2}{u} + \frac{a^2}{\beta} u \right).
\]

The function \( f_{123}(u, Z) \) transforms to \( f_{123}(u) \), and its behavior is shown in Figure 6 for different values of the free parameter \( \beta \). Once the root(s) of \( R_{app}(u) \) in the interest domain (the interval \( -1 \leq u \leq 0 \) or \( R_\star \leq r \leq \infty \)) is (are) obtained, the values of \( Y \) and \( C' = C'(M) \) are obtained from equations (16) and (17), respectively.

4.2. An Example: CAK with Rotation

It could be thought that this simple case is only of academic interest, but as we see below, these results contribute to our understanding of the dynamic of rapid rotators. For the CAK model (with \( \delta = 0 \)), the \( f \)'s functions are \( f_1 = 1, f_2 = f_3 = \alpha \), and \( f_{123}(u) = 0 \); then the \( R_{app}(u) \) \( \equiv R_{CAK}(u) \) function reads

\[
R_{CAK}(u) \equiv \frac{2}{(1 + u)} + \frac{2(1 + u)}{u^2\beta} - \frac{a^2}{\beta} \frac{(1 + u)}{\beta}.
\]

This is a fourth-order polynomial in \( u \). Contrary to expectations, the nonrotational case, \( R_{CAK}(u) \), depends neither on the stellar parameters \( T_{\text{eff}}, M_\star \), and \( R_\star \) nor on the line force parameters \( k, \alpha \), and \( \delta \); it depends exclusively on \( \beta \). Furthermore, for the rotational case it depends on the stellar parameters and the free parameter \( \beta \) but not on the line force parameters.

CAK and KPPA89 have shown that the behavior of the velocity profile is given approximately by

\[
v(u) = v_\infty \sqrt{1 + u}.
\]
one critical point exists in the domain of interest. Bjorkman (1995) arrived at the same conclusion in his topological study of the nonrotating CAK model.

As the rotational speed increases, Figure 7 shows that the location of the CAK singular point is shifted downstream, the same conclusion reached by Castor (1979) and Marlborough & Zamir (1984).

The zero \( u_{\text{crit}} \) of equation (23) in the interval \(-1 \leq u \leq 0\) is given by the analytical formula

\[
u_{\text{crit}} = \frac{1}{2} (t_7 - t_6 - 1),
\]

where the \( t \)-coefficients are

\[
\begin{align*}
t_0 &= -2 + \frac{r_{\text{rot}}^2}{2} + 2\beta^2, \\
t_1 &= 432a_{\text{rot}}^2 - 216a_{\text{rot}}^4 + 216a_{\text{rot}}^2t_0 + 2t_0^3, \\
t_2 &= \left( t_1 + \sqrt{-4t_0^6 + t_1^2} \right)^{1/3}, \\
t_3 &= \frac{21/3/2}{3a_{\text{rot}}^2t_0}, \\
t_4 &= \frac{t_2}{3a_{\text{rot}}^2t_0^{2/3}}, \\
t_5 &= -8 + \frac{32}{a_{\text{rot}}^2} + \frac{8t_0}{a_{\text{rot}}^2}, \\
t_6 &= \frac{2t_0}{3a_{\text{rot}}^2}, \\
t_7 &= \sqrt{1 - t_6 + t_3 + t_4}, \\
t_8 &= \sqrt{2 - 2t_6 - t_3 - t_4 + \frac{t_5}{4t_7}}.
\end{align*}
\]

For the simple case of \( a_{\text{rot}} = 0 \), a straightforward and exact solution for equation (18) is

\[
u_c = -Z_c,
\]

and \( Z_c \) from full numerical calculations.\(^1\) For the nonrotational case, the best value is \( \beta = 1/2 \), confirming the results of previous analyses. The KPPA89 “cooking recipe” for \( M_{\text{CAK}} \) (see KPPA89 eq. [27]) gives \( M_{\text{CAK}} = 3.074 \times 10^{-6} \ M_\odot \text{yr}^{-1} \), the same as the nonrotational case with \( \beta = 1/2 \). While the value of the mass-loss rate is almost independent of the value of \( \beta \), the location of the critical point is very sensitive to the value of \( \beta \), but in the presence of rotation, this sensitivity disappears. The reason is that here the rotational term (the third term in eq. [23]) is the dominant one, and therefore the dependence on \( \beta \) becomes minimal. Furthermore, the larger the rotational speed, the larger is the \( \beta \)-value that best fits the numerical calculations.

The terminal velocity of the wind, \( v_{\infty} \), can now be computed from direct integration of equation (16). This follows because \( \dot{I} \) is the unique solution of equation (7) or equation (15a).

4.3. The \( m \)-CAK Standard Solution

Now we analyze the influence of the finite cone-angle effect. The main difference between this approach and the one of KPPA89 is that they introduced the \( \beta \)-field approximation before calculating the derivatives of the singularity and the regularity conditions. Therefore, they could not derive the equations (15a), (15b), and (15c). In fact, the latter two equations become the same in the KPPA89 approximation.

\(^1\) Here full numerical calculation solves eqs. (7), (12), and (13) and the lower boundary condition eq. (10).

| \( v_{\text{rot}}/v_{\text{knap}} \) | \( r_{\text{crit}}/R_c \) | \( \beta = \frac{1}{2} \) | \( \beta = 1 \) | \( \beta = 2 \) | \( \beta = \frac{1}{2} \) | \( \beta = 1 \) | \( \beta = 2 \) | \( \beta = \frac{1}{2} \) | \( \beta = 1 \) | \( \beta = 2 \) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| (1)            | (2)            | (3)            | (4)            | (5)            | (6)            | (7)            | (8)            | (9)            | (10)           | (11)           |
| 0.0            | 1.500          | 2.000          | 3.000          | 1.562          | 3.074          | 3.086          | 3.093          | 3.083          |                 |                 |
| 0.3            | 6.125          | 6.211          | 6.537          | 6.186          | 3.127          | 3.128          | 3.131          | 3.128          |                 |                 |
| 0.5            | 10.175         | 10.220         | 10.398         | 10.100         | 3.163          | 3.165          | 3.167          | 3.164          |                 |                 |
| 0.7            | 14.233         | 14.264         | 14.385         | 14.507         | 3.201          | 3.202          | 3.205          | 3.203          |                 |                 |
| 0.9            | 18.295         | 18.318         | 18.409         | 18.555         | 3.240          | 3.241          | 3.243          | 3.243          |                 |                 |

\[ \frac{Z_m}{w} \] as a function of \( u = R_*/r \) in the interval \([-1.0, -0.9] \). The plus sign shows the location of the critical point from full numerical calculations. Two \( \beta \)-field approximations for \( Z = 1 + u/\beta \) are overlapped: \( \beta = 0.8 \) (dashed line) and \( \beta = 1.0 \) (dotted line). Clearly, \( \beta = 1.0 \) intersects \( Z(u) \) almost at the critical point.
4.3.1. OB Stars

In order to obtain the critical point and the mass-loss rate, it is necessary to know the value of $\beta$. As discussed above, PPK86 found that a value of $\beta = 0.8$ is appropriate for OB stars. Figure 8 is the same as Figure 1, but here in the region near the photosphere, the plus sign indicates the location of the critical point at the $Z(u)$ profile. The $\beta$-field approximation with $\beta = 1$ intersects almost exactly this location, showing that for this type of stars $\beta = 1$ is a fairly good approximation for calculating the critical point location and the mass-loss rate.

Table 2 shows a comparison between analytical and full numerical calculations for the O5 V test star (see stellar and line force parameters in § 3.2). The best agreement for both analytical quantities, $r_{\text{crit}}$ and $\dot{M}$, with the numerical calculation occurs for $\beta = 1$. Furthermore, these predicted analytical values agree much better with the numerical calculations than the values given by the cooking recipe from KPPA89. This recipe gives for the nonrotational case $r_{\text{crit}} = 1.038R_*$ and $\dot{M} = 1.976 \times 10^{-6} M_\odot \text{ yr}^{-1}$ (with $\beta = 1$). When the analytic approximation of KPPA89 is compared with numerical calculations, it is found that the values of $\dot{M}$ are always smaller than the numerical values. For the nonrotational case, our approximation almost exactly matches the values of the numerical calculations, but for rotational cases it still shows values slightly less than the numerical ones.

4.3.2. Central Stars of Planetary Nebulae

Radiation-driven stellar winds are also present in central stars of planetary nebulae (CSPNs), i.e., for objects in which the effect of photospheric extension is important. Pauldrach et al. (1988) have computed radiation-driven wind models along evolutionary tracks for these objects. They found that the predicted terminal wind velocities are in agreement with the observational data, while the mass-loss rates are in qualitative agreement.

In their non-LTE analysis of CSPNs, Kudritzki et al. (1997) used a $\beta$-field approximation for the velocity, finding that $\beta = 1.5$ is the best value for these stars (see Table 1 from Kudritzki et al. 1997). Table 3 shows a comparison between analytical and numerical calculations for one of the models of Pauldrach et al. (1988). The stellar parameters for this model are $T_{\text{eff}} = 80,000$ K, log $g = 5.24$, $M_*/M_\odot = 0.565$, and $R/R_\odot = 0.3$; line force parameters are $k = 0.053$, $\alpha = 0.709$, and $\delta = 0.052$. This analysis confirms that a value of $\beta = 1.5$ is a fairly good one for nonrotational CSPNs, but as the rotational velocity increases, a slightly larger value must be assumed.

In this section, we have developed an analytical treatment for the location of the critical point (standard m-CAK) and mass-loss rate that improves upon previous investigations by including rotation and has about 1% confidence when compared with full numerical calculations.

5. ANALYTICAL TREATMENT OF THE NEW WIND SOLUTION

In previous sections, the existence of a new family of solutions that are different from the standard m-CAK has been established from the properties of the $R(u, Z)$ function. In this section we study the approximate function $R_{\text{app}}(u)$ to get information about the location of the singular point and the mass-loss rate.

5.1. Critical Points

5.1.1. The Standard m-CAK Critical Point

Figure 9 shows $R_{\text{app}}(u)$, with $\beta = 1$, for an O5 V star for different values of the star’s rotational speed for both models, CAK and m-CAK. From this figure it is clear that the inclusion of the finite disk correction factor in the wind momentum equation (7) “twists” the $R_{\text{CAK}}(u)$ function to $R_{\text{app}}(u)$. It is evident from this result that while $R_{\text{CAK}}(u)$ displays only one root in the interest domain, the number of roots of $R_{\text{app}}(u)$ depends on the rotational velocity.

For nonrotational cases (Fig. 9a) the CAK critical point shifts to a location close to the stellar surface; this is the standard m-CAK critical point. As the rotational speed increases, the location of the m-CAK singular point is shifted downstream (FA86; PPK86), but unlike in the CAK case, this occurs only up to a certain rotational speed.

In order to have a better picture of the approximation $Z = (1 + u)/\beta$ we have to recall the function $R(u, Z)$ (see, e.g.,

| Table 2 |
|---|---|---|---|
| $r_{\text{crit}}/R_*$ | $M \times 10^{-6} M_\odot \text{ yr}^{-1}$ |
| $\beta = 0.8$ | $\beta = 1$ | $\beta = 2$ | $\beta = 0.8$ | $\beta = 1$ | $\beta = 2$ |
| 0.0 | 1.026 | 1.034 | 1.084 | 1.033 | 2.089 | 2.131 | 2.364 | 2.129 |
| 0.3 | 1.027 | 1.035 | 1.088 | 1.036 | 2.235 | 2.280 | 2.520 | 2.280 |
| 0.5 | 1.029 | 1.038 | 1.098 | 1.040 | 2.566 | 2.618 | 2.903 | 2.623 |
| 0.7 | 1.036 | 1.048 | 1.129 | 1.051 | 3.372 | 3.437 | 3.782 | 3.452 |

| Table 3 |
|---|---|---|---|
| $r_{\text{crit}}/R_*$ | $M \times 10^{-9} M_\odot \text{ yr}^{-1}$ |
| $\beta = 1$ | $\beta = 1.5$ | $\beta = 2$ | $\beta = 1$ | $\beta = 1.5$ | $\beta = 2$ |
| 0.0 | 1.042 | 1.069 | 1.102 | 1.074 | 4.638 | 4.881 | 5.140 | 4.916 |
| 0.3 | 1.044 | 1.073 | 1.108 | 1.080 | 4.882 | 5.141 | 5.418 | 5.189 |
| 0.5 | 1.050 | 1.082 | 1.123 | 1.092 | 5.424 | 5.716 | 6.024 | 5.791 |
| 0.7 | 1.064 | 1.107 | 1.163 | 1.126 | 6.678 | 7.021 | 7.372 | 7.137 |
This is a surface in the phase space defined by the independent coordinates $u$ and $Z$. The approximation $Z = (1 + u)/\beta$ is a vertical plane in this phase space that cuts the $R$ surface defined by equation (18). The projection of this intersection curve over the vertical plane $(u, R)$ is the $R_{app}$ function shown in Figure 9.

Figure 10 is a contour plot of the $R$ surface at the zero plane, the $(u, Z)$ plane. Both solid lines correspond to the two families (loci) of singular points, discussed in § 5.1, while the dashed line is the $\beta$-field approximation ($\beta = 1$). This approximation cuts the standard m-CAK family at two points. The standard m-CAK solution passes through the first intersection point (nearest to the stellar surface, $u = -1$) and not through the second intersection point as a result of imposing the lower boundary condition equation (10). The second intersection point is only a consequence of the $\beta$-field approximation, and a solution (if one exists) that passes only through this point would probably have no physical meaning for radiation-driven winds.

5.1.2. The New m-CAK Critical Point

In addition to the standard m-CAK critical point, Figures 9 and 10 show a second critical point (root of $R_{app}$), the last intersection point. As the rotational speed increases, for this O5 V star, the standard m-CAK critical point now disappears from the integration domain when $v_{rot}/v_{b_{\text{up}}}$ $\geq$ 0.9 (Fig. 9d; see also Fig. 5). In this case, the CAK critical point is shifted far downstream in the wind. At the critical point, $u_{\text{crit}}$, the function $f_{123}(u_{\text{crit}}) \to 0$, so $R_{app}(u)$ is almost the same for the CAK and m-CAK models (see eqs. [22] and [23]). Thus, for the fast-rotational case, equation (25) gives a very good approximation for the location of the critical point and therefore the mass-loss rate, as well. We conclude that the CAK model with rotation is
applicable for high rotational velocities to calculate the mass-loss rate and the location of the singular point of the m-CAK model.

Because we are now interested in stars with high rotational velocities, from now on our test star is a typical B1 V star with the stellar parameters $T_e = 25,000$ K, $\log g = 4.03$, and $R/R_\odot = 5.3$ (Slettebak et al. 1980) and line force parameters $k = 0.3$, $\alpha = 0.5$, and $\delta = 0.07$. For this star, Figure 11 shows $R_{app}(u)$ for two different values of $\beta$. In both cases, as the rotational speed increases $R_{app}(u)$ goes to a scenario where there is only one critical point, the new one. For $\beta = 2.5$, a lower value of the rotational speed is required to get to this situation, i.e., $v_{rot}/v_{bkup} \geq 0.6$, while for $\beta = 1$, the required value is $v_{rot}/v_{bkup} \geq 0.8$. This last value is lower than that required for the O5 V star (see Fig. 9).

In order to solve the nonlinear wind momentum differential equation (7) for a fast-rotational case, we do not use the “standard” numerical method, according to which, once the singular point location, its velocity, and its velocity gradient are approximately obtained, direct integration up- and downstream is performed (e.g., using the Runge-Kutta or Burlish-Stoer method). After this process is done, the lower boundary condition is calculated and the whole procedure is repeated until convergence is achieved.

Instead, here we use a finite-difference method, modified for handling singular points (Nobili & Turolla 1988). This method uses a trial solution as its initial guess. This initial solution then relaxes, using the Newton method, toward the numerical solution (see Nobili & Turolla 1988 for details). The main advantage of this method is the fact that it is not necessary to give a priori the location of the critical point, as the standard method requires, so no guess value for $\beta$ is needed to solve the momentum equation.

6. NUMERICAL RESULTS

It has been established in the previous sections that in a fast-rotational case, the standard m-CAK critical point vanishes and there exists a second critical point. In this section we obtain full numerical solutions that, starting at the star’s surface, reach infinity passing through this new critical point.

6.1. The Numerical Method

In order to solve the nonlinear wind momentum differential equation (7) for a fast-rotational case, we do not use the “standard” numerical method, according to which, once the singular point location, its velocity, and its velocity gradient are approximately obtained, direct integration up- and downstream is performed (e.g., using the Runge-Kutta or Burlish-Stoer method). After this process is done, the lower boundary condition is calculated and the whole procedure is repeated until convergence is achieved.

Instead, here we use a finite-difference method, modified for handling singular points (Nobili & Turolla 1988). This method uses a trial solution as its initial guess. This initial solution then relaxes, using the Newton method, toward the numerical solution (see Nobili & Turolla 1988 for details). The main advantage of this method is the fact that it is not necessary to give a priori the location of the critical point, as the standard method requires, so no guess value for $\beta$ is needed to solve the momentum equation.
6.2. The Fast Solution

First we calculate the nonrotational case, \( v_{\text{rot}} = 0 \), for our test star using \( \tau = \frac{1}{2} \) (eq. [10]) as the lower boundary condition. In Figure 12 the velocity profile for this case (solid line) is shown. The numerical results are as follows: critical point location at \( r_{\text{crit}} = 1.014 R_\star \), mass-loss rate \( \dot{M} = 3.178 \times 10^{-9} M_\odot \text{ yr}^{-1} \), and terminal velocity \( v_\infty = 2025 \text{ km s}^{-1} \).

6.3. The Slow Solution

We have solved numerically the fast-rotational case for \( v_{\text{rot}}/v_{\text{bkup}} = 0.8 \). This new solution is denser than the fast solution (m-CAK), and we call it hereafter the "slow" solution. The critical point is at \( r_{\text{crit}} = 23.63 R_\star \), confirming the approximative calculations of the preceding sections, that the critical point is located far away in the wind. The terminal velocity of this new solution is \( v_\infty = 443 \text{ km s}^{-1} \). This value is a factor of 4 lower than the standard m-CAK solution (the fast solution; Fig. 12, dashed line). The mass-loss rate is \( \dot{M} = 6.854 \times 10^{-9} M_\odot \text{ yr}^{-1} \), approximately twice the value of the nonrotational case.

6.3.1. The Approximation \( Z = (1 + u)/\beta \)

Figure 13 shows \( Z \) versus \( u \) from full numerical calculations (solid line). In addition, the location of the critical point is shown with a plus sign in this figure. The approximation \( Z = (1 + u)/\beta \) (Fig. 13, straight line) is also plotted for two different values of the parameter \( \beta \). The line with \( \beta = 2 \) intersects \( Z(u) \) close to the location of the critical point, showing that for fast-rotational cases a value of \( \beta > 1 \) better matches the location of the critical point and the mass-loss rate (eigenvalue).

6.3.2. Functions \( R(u, Z) \) and \( R_{\text{app}}(u) \)

In order to validate the \( R_{\text{app}}(u) \) function, we have plotted in Figure 14 the function \( R(u, Z) \) from the results of the full numerical calculations (solid line) and compared it with the \( R_{\text{app}}(u) \) functions for two different values of the \( \beta \)-parameter (dashed lines). The behavior of \( R(u, Z) \) and \( R_{\text{app}}(u) \) in the region far away in the wind (\( u \gtrsim -0.05 \)) is similar for both values of \( \beta \), confirming that for this new solution, as well, the last root of the approximate function \( R_{\text{app}}(u) \) gives an accurate value for the location of the critical point, \( r_{\text{crit}} \), and through equations (16) and (17) the eigenvalue, \( C'(M) \), for the fast-rotational case.

6.4. \( v_\infty \) and \( \dot{M} \) from Slow Solutions

As de Araujo (1994) and Stee & de Araujo (1994 and references therein) argued, it is possible to obtain equatorial solutions with lower terminal velocities when the wind is driven mainly by thin lines, e.g., using a lower value of the line force parameter \( \alpha \). Thus, we have investigated the value of \( v_{\text{rot}} \) that makes the standard m-CAK critical point disappear, leaving only the slow solution.

Table 4 summarizes the value of this minimum rotational velocity from the \( R_{\text{app}}(u) \) function for the existence of the slow solution. The values of \( \alpha \) and \( k \) are taken from de Araujo (1994), and \( \delta = 0 \) was used. The values of \( v_\infty \) and \( \dot{M} \) come from full numerical solutions. This result suggests that in order to obtain a better modeling of this slow solution a self-consistent calculation (Pauldrach 1987) of the line force parameters is necessary. Table 4 shows that the thinner the lines driving the wind (lower \( \alpha \)), the lower is the required rotational velocity for slow solution and therefore the lower are the terminal velocity and mass-loss rate. This table also shows a weak dependence of the \( \beta \)-parameter on the minimal rotational velocity.

7. DISK FORMATION IN Be STARS

For our model it is natural to think that a fast-rotating star can have both wind solutions, e.g., Be stars. The polar direction is equivalent to the nonrotational case, while the equatorial flow corresponds to the fast-rotational case. In Figure 15 the density profiles for both directions, i.e., polar (dashed line) and equatorial (solid line), are plotted against the inverse radial coordinate \( u \).

The ratio of the equatorial to the polar densities is about a factor of 100 very close to the star’s surface but on average a factor of 10 for the entire wind. This density quotient value is still a factor of 10 smaller than the observational values. The disk apspar angle occurs when the m-CAK critical point

| Table 4 | Value of the Minimum Rotational Rate for the Case in Which Only One Solution Exists |
|---------|----------------------------------------------------------------------------------|
| \( \alpha \) | \( k \) | \( v_{\text{rot}}/v_{\text{bkup}} \) | \( v_\infty \) (km s\(^{-1}\)) | \( \dot{M} \) (10\(^{-9}\) \( M_\odot \text{ yr}^{-1} \)) |
| 0.5       | 0.3 | 0.88 | 0.87 | 0.86 | 454 | 6.86 |
| 0.4       | 0.7 | 0.82 | 0.81 | 0.79 | 375 | 3.72 |
| 0.3       | 1.5 | 0.75 | 0.72 | 0.69 | 307 | 1.15 |
| 0.2       | 3.2 | 0.64 | 0.56 | 0.56 | 245 | 0.13 |
| 0.1       | 6.0 | 0.44 | 0.38 | 0.34 | 187 | 0.0001 |
disappears from the \( R(u, Z) \) function, so this angle depends strongly on the assumed value of the line force parameter \( \alpha \). A detailed “two-dimensional” (latitude dependent) velocity and density profile and line formation are the subject of a forthcoming article.

8. CONCLUSIONS

After a suitable change of coordinates, we show that in addition to the standard m-CAK critical point there exists a second one when the star’s rotational speed is taken into account for a hot star with radiation-driven wind. Using a simple \( \beta \)-field approximation, we developed an analytical description for the location of the critical point and the value of the mass-loss rate of the star. Our solution remains within 1% confidence when compared with numerical calculations. This point is very important when trial solutions for numerical codes are selected.

We have studied new solutions for the case of extremely high rotational speed (applied to a Be star). There exists only one critical point, and numerical wind solutions were obtained. This new solution gives very slow terminal velocities (in the range \( \approx 190 \) to \( \approx 450 \) km s\(^{-1}\), depending on the assumed values of the line force parameters), and the density of the wind is about 10 times greater than the standard m-CAK wind solution. These results indicate that a new wind solution of the standard m-CAK model may be important in explaining the formation of the disk in Be stars. However, to improve this model for Be stars it is necessary that the following effects be taken into account: the dependence of the radius and temperature as a function of rotational speed and latitude, a self-consistent treatment of the line force parameters, a detailed study of the influence of the boundary conditions, a reanalysis of angular momentum conservation (including viscous forces), and the (ignored here) influence of magnetic fields.

Currently, a detailed study of the topology of the singular points is underway. There are probably more solutions that involve more than one singular point as in the case of magnetic fields.

I would like to express my gratitude to N. Zamorano for his continuous support and critical comments on the manuscript. This work has been partially supported by Universidad de Valparaíso, internal project DIUV 14/00 and 15/03.

APPENDIX A

COORDINATE TRANSFORMATION

Here the basic steps toward equations (15a), (15b), and (15c) are outlined. The reader should keep in mind the original derivation by CAK.

The partial derivatives of \( F(u, w, w') \) (eq. [15a]) with respect to \( u, w \) and \( w' \) are

\[
\frac{\partial F}{\partial u} = -\frac{2}{w'} + a_{\text{rot}}^2 - C' \left( \frac{\partial CF}{\partial u} g + CF \frac{\partial g}{\partial u} \right) w^{-\delta} (ww')^\alpha, \tag{A1}
\]

\[
\frac{\partial F}{\partial w} = \left( 1 + \frac{1}{w^2} \right) w' - C' \left( \frac{\partial CF}{\partial w} + \frac{CF}{w} - \delta \frac{CF}{w} \right) gw^{-\delta} (ww')^\alpha, \tag{A2}
\]

\[
\frac{\partial F}{\partial w'} = \left( 1 - \frac{1}{w^2} \right) w - C' \left( \frac{\partial CF}{\partial w'} + \frac{CF}{w'} \right) gw^{-\delta} (ww')^\alpha. \tag{A3}
\]

After using the new coordinates \( Y = ww' \) and \( Z = w/w' \), some derivative relation of the correction factor (see Appendix B), and defining

\[
\frac{dg(u)}{du} = g(u)h(u), \tag{A4}
\]

where

\[
h(u) = \delta \left[ \frac{2}{u} - \frac{u}{\sqrt{1-u^2}} \left( 1 - \sqrt{1-u^2} \right) \right], \tag{A5}
\]
the singularity condition \([w'(\partial F/\partial w') = 0]\) reads
\[
\left(1 - \frac{1}{YZ}\right) Y - C^f f_2(u, Z) g Z^{-\delta/2} Y^{\alpha-\delta/2} = 0, \tag{A6}
\]
where \(f_1(u, Z) \equiv CF(u, Z),\)
\[
f_2(u, Z) = \alpha f_1(u, Z) - u Z e(u, Z), \tag{A7}
\]
and \(e(u, Z)\) is a function defined in Appendix B.

The regularity condition \([Z(dF/du) = 0]\) transforms to
\[
\left(1 + \frac{1}{YZ}\right) Y - C^f f_3(u, Z) g(u) Z^{-\delta/2} Y^{\alpha-\delta/2} = + \frac{2Z}{u^2} - a^2 \omega Z, \tag{A8}
\]
with
\[
f_3(u, Z) = (3u + Z) Z e(u, Z) + f_1(u, Z) [h(u) Z + \alpha - \delta]. \tag{A9}
\]

**APPENDIX B**

**THE CORRECTION FACTOR**

The correction factor is defined by
\[
CF = \frac{2}{1 - \mu^2} \int_{r}^{1} \left[ \frac{(1 - \mu^2)v/r + \mu^2 v'}{v'} \right] \mu d\mu, \tag{B1}
\]
where \(\mu^2 = -1 - (r_0/r)^2\) and \(v' = dv/dr\). Integrating equation (B1) and changing the variables from \(r, v\) to \(u, w\) \((u = -R_e/r,\ w = v/\alpha,\) where \(\alpha\) is the thermal velocity), the finite disk correction factor transforms to
\[
CF(u, w/w') = \frac{1}{1 - \alpha} \frac{1}{w^2 + (1/w)(w/w')} \left[ 1 - \left(1 - u^2 - u \frac{w}{w'} \right)^{(1+\alpha)} \right], \tag{B2}
\]
where \(w' = dw/dr\). Because of the fact that \(CF\) depends on \(u\) and only the quotient \(Z \equiv w/w'\), by defining \(\lambda\) as
\[
\lambda \equiv u(u + Z), \tag{B3}
\]
we can rewrite \(CF(u, Z = w/w')\) as
\[
CF(\lambda) = \frac{1}{1 - \alpha} \frac{1}{Z} \left[ 1 - (1 - \lambda)^{(1+\alpha)} \right]. \tag{B4}
\]

The partial derivatives of \(CF\) with respect to \(u, w, w'\) are related to \(\partial CF/\partial \lambda\) via the chain rule, namely,
\[
\frac{\partial CF}{\partial u} = \frac{\partial CF}{\partial \lambda} \frac{\partial \lambda}{\partial u}, \tag{B5}
\]
\[
\frac{\partial CF}{\partial w} = \frac{\partial CF}{\partial \lambda} \frac{\partial \lambda}{\partial w}, \tag{B6}
\]
\[
\frac{\partial CF}{\partial w'} = \frac{\partial CF}{\partial \lambda} \frac{\partial \lambda}{\partial w'}. \tag{B7}
\]

Defining \(e(\lambda) \equiv e(u, \lambda) = \partial CF/\partial \lambda,\)
\[
e(\lambda) = \frac{(1 - \lambda)\alpha}{\alpha} - CF(\lambda). \tag{B8}
\]

Then (B5), (B6), and (B7) are related to (B8) by
\[
e(\lambda) = \frac{1}{2u + Z} \frac{\partial CF}{\partial u} = \frac{w'}{u} \frac{\partial CF}{\partial w} = \frac{w'}{uZ} \frac{\partial CF}{\partial w'}. \tag{B9}
\]

Approximating \(Z\) by a \(\beta\)-field \([Z = (1 + u)/\beta]\), we obtain \(CF\) and \(e\) as functions only of \(u\).
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