Baryogenesis and helical magnetogenesis
from the electroweak transition of the minimal Standard Model

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We argue that the observed baryon asymmetry of the Universe can be explained within the minimal Standard Model, from the “sphaleron freezeout” reached when the electroweak sphaleron rate becomes equal to the Hubble rate of the Universe expansion. This freezeout drives the system out of equilibrium, and prevents the sphalerons from washing out the baryon asymmetry; we find that this mechanism can explain the observed magnitude of baryon asymmetry in the Universe. The test of the proposed scenario is possible through the study of magnetic helicity at intergalactic scales, as the baryon asymmetry appears tightly linked to the magnetic helicity at intergalactic scales.

I. INTRODUCTION

One of the central problems of cosmology is explanation of the Baryon Asymmetry of the Universe (BAU). It is commonly believed that solving this problem requires physics beyond the Standard Model (SM). In this paper, we propose a mechanism that allows to explain the observed BAU in terms of the minimal SM, through the interplay of the electroweak sphalerons and the Hubble expansion.

Since the problem involves many areas of physics, our introduction will be split into several parts. Let us begin from the discussion of the common setting – the cosmological Electroweak Phase Transition (EWPT), whereby the universe undergoes a transition from a symmetric phase to a broken phase with a nonzero vacuum expectation value (VEV) for the Higgs field.

Most of the studies of the EWPT, from early works till now, assumed it to be the first order transition, producing bubbles with large-scale deviations from equilibrium \cite{1}. In particular, most studies of gravitational wave emission were carried out in this setting. However, lattice calculations have shown that the SM can only undergo a first order transition for small Higgs masses, well below the 125 GeV mass observed at the LHC. The first order transition remains possible only in the models that go beyond the standard model (BSM); we will not discuss such models in this work.

Alternative scenario of the EWPT is the “hybrid” or “cold” scenario, in which that the broken symmetry phase occurs at the end of the inflation epoch. Here, the label “cold” refers to the fact that at the end of the reheating and equilibration of the Universe, the temperature becomes of the order of $T \approx 30 - 40$ GeV, well below the critical electroweak temperature $T_{EW} \approx 160$ GeV. Violent deviations from equilibrium occur in this scenario \cite{8, 9}. Detailed numerical studies of them \cite{9, 10} revealed “hot spots”, filled with strong gauge field, later identified \cite{10} with certain multi-quanta bags containing gauge quanta and top quarks. We will not consider this scenario as well.

What we will discuss here is the least violent scenario for EWPT, a smooth crossover transition of the Minimal Standard Model (MSM) which is consistent with the experimentally observed mass of the Higgs boson. The main cosmological parameters of the EWPT are by now well established. For completeness they are briefly summarized in Appendix A.
A. Sphalerons and the EWPT

Sphaleron transitions are topologically non-trivial fluctuations of the gauge field, in which Chern-Simons number \( N_{CS} \) is changed. They can be viewed as a slow “climb” up the effective potential \( V(N_{CS}) \), along the so-called sphaleron path, from an integer value \( N_{CS} = n \) to a half-integer value \( N_{CS} = n \pm \frac{1}{2} \) on top of the potential barrier separating vacuum sectors with different \( N_{CS} \). At the top of the potential barrier the gauge configuration is a static, purely magnetic, soliton of the equations of motion that is known as the “sphaleron” [12].

If perturbed, this configuration leads to a time-dependent solution known as the “sphaleron explosion”. Classically, gauge fields are rolling downhill, converting their potential energy into kinetic one. Quantum effects (chiral anomaly) tell us that changing the Chern-Simons number must lead to creation of quarks and leptons of specific types. Furthermore, quantum fermion loop including CKM matrix of mixing of all quark types leads to CP and T (and thus imaginary) part of the corresponding action, causing unequal probability to decay into positive and negative direction of the Chern-Simons number. Some configurations related to sphalerons and their explosions are collected in Appendix B.

At \( T > T_c \), in the symmetric (unbroken, with zero Higgs VEV) phase, the sphaleron rate is \( \Gamma/T^4 \sim \alpha_{EW}^5 \sim 10^{-7} \). In the broken phase, after the EWPT, it becomes exponentially suppressed, due to the sphaleron mass proportional to increasing Higgs VEV \( v(T) \). Some basic information about the electroweak sphalerons is given in the Appendix B. The overall sphaleron rate after EWPT is also available from lattice calculations.

The new material in the part of this work related to sphalerons is based on the discussion of the distribution in sphaleron size \( \rho \). At small sizes, the distribution is cut off because the sphaleron mass \( m \) increases as \( m \sim 1/\rho \) (by dimensional counting), and the solution is given by the gauge configuration discussed in refs [18] and [19]. At large sizes, the limiting factor is the magnetic screening mass which we will extract from lattice calculation [15]. The applications discussed below make use of both “large size” and “small size sphalerons”.

B. Generation of sounds and gravity waves

The “sphaleron explosion” is described by a time-dependent solution of the classical Yang-Mills equations. A number of such solutions have been obtained numerically. Analytic solutions for pure-gauge sphalerons have been obtained in [18] and [19]. We will use the latter one. Some details on how it was obtained and some basic formulae are summarized in the Appendix B.

As we will see, the word “explosion” is not really a metaphor here. Indeed, the time evolution of the stress tensor \( T^{\mu\nu}(t, \vec{x}) \) shows an expanding shell of energy density (and other components of the stress tensor). Although we have not developed its interaction with ambient matter in any detail, it is clear that a significant fraction of the energy in the shell will end up in spherical sound waves.

For \( T > T_{EW} \), in the symmetric phase, the sphaleron explosion is spherically symmetric. It does not sustain a quadrupole deformation and therefore cannot radiate gravitational waves directly. However, the indirect gravitational waves can still be generated at this stage, through the process sound+sound \( \rightarrow \) gravity wave, as pointed out in [13].

After EWPT, at \( T < T_{EW} \), the sphalerons and their explosions are no longer spherically symmetric, with a nonzero quadrupole moment, so that direct gravitational radiation becomes possible. We will calculate its matrix elements in section III.

C. Baryon Asymmetry of the Universe (BAU)

Any explanation of the baryon asymmetry in the universe (BAU) needs, as noted by Sakharov long ago [2], three famed conditions: 1) deviation from equilibrium; 2) baryon number violation; 3) CP violation.

It is well known that the SM provides the conditions 2 and 3 “in principle”. However, the current consensus is that the SM is unable to reproduce the key observed BAU parameter, the
baryon-to-photon ratio

\[
\frac{n_B}{n_\gamma} \sim 6 \cdot 10^{-10} \tag{1}
\]

As a result, the mainstream BAU studies focus mostly on “beyond the Standard Model” (BSM) scenarios, in which new sources of CP violation are introduced, e.g. axion fields, or extended Higgs or neutrino sectors with large CP violation. Leptogenesis scenarios are based on superheavy neutrino decays, occurring at very high scales, and satisfying both large CP and out-of-equilibrium requirements, with lepton asymmetry then transformed into baryon asymmetry at the electroweak scale. While one of these BSM scenarios may ultimately well turn out to be the explanation for BAU, at this time they still remain purely hypothetical and are not directly supported by experiment.

In this work we aim at providing some estimate of the BAU within the minimal SM, as accurately as possible at this time. We suggest that, contrary to popular opinion, the SM may actually explain the observed baryon asymmetry of the Universe. Throughout, we will adhere to a conservative and minimal SM (MSM) scenario, in which the EWPT is smooth, with gradual build-up of the Higgs VEV \(v(T)\) at \(T < T_c\). The needed “out-of-equilibrium” conditions, as discussed below, will be associated with “freezeout” of large-size sphalerons, achieved when the sphaleron rate becomes comparable to the Universe expansion rate.

**D. Baryon number and CP violation in the Standard Model; the sphaleron explosions**

The baryon number violation in the MSM occurs in a standard way, through sphaleron transitions. Each of the sphaleron decay produces 9 quarks and 3 leptons as required by the axial anomaly.

The CP violation in the SM is induced by the phase of the CKM matrix. Its magnitude is known to be strongly scale dependent. Naively, at \(T_{EW}\) all particle momenta are of the order of \(p \sim 3T \sim 300\) GeV, higher than all quark masses. As shown by Jarlskog [20], the magnitude of the CP violation needs to be proportional to a product of two different factors.

The first is the “Jarlskog determinant” containing sine of CP violating phase and sin and cos of all the mixing phases, with numerical value \(J \approx 3 \cdot 10^{-5}\). The second factor is the combination of up and down quark mass differences, ensuring the vanishing of the CP asymmetry when any two masses in each category become equal, divided by the appropriate power of the mass scale of the process. When this scale is typical EW scale \(\sim 100\) GeV, the asymmetry is very small, \(A_{CP} \sim 10^{-19}\). So, naively, application of this source of CP violation to BAU seems to be doomed.

However, the second mass-dependent factor changes greatly with the momentum scale of the process. For instance, in the originally discovered K decay, the CP asymmetry is \(\sim 10^{-3}\) and is even larger in some observed B decays. As we show below, the asymmetry turns out to be maximal at a scale \(p \sim 1\) GeV, in a “sweet spot” between the masses of light and heavy quarks, reaching there \(A_{CP} \sim 10^{-6}\) or so. Therefore, our focus below will be on establishing whether the relevant sphaleron size can be close to this “optimal” scale.

**E. Intergalactic magnetic fields and helical magnetogenesis from EWPT**

The EWPT has also been suggested to be a source for large scale magnetic fields in the universe. The existence and properties of intergalactic magnetic fields are hotly debated by observational astronomers, cosmologists and experimentalists specialized in the detection of very high energy cosmic rays. Currently, there are only lower and upper limits on the magnitude of these fields, and, needless to say, the chirality (linkage) of these fields remains an open question. Even the expected magnitude of their correlation is subject to debates, with suggestions ranging from larger than the visible size of the universe (in case of pre-inflation chiral fluctuations) to sub-Galaxy size.

Our main point in this paper is that the sphaleron-induced BAU must also be related with the chiral imbalance of quarks and leptons produced in sphaleron transitions. This chiral imbalance is then transferred to linkage of magnetic field lines. Since the linkage is expected to be conserved in magnetohydrodynamic plasma,
it may be observable today, via the prevalence of a certain magnetic helicity in the intergalactic magnetic fields.

II. SPHALERONS IN THE CROSSOVER EW TRANSITION

A. The temperature dependence of the sphaleron rates

To assess the temperature of the sphaleron rate, we first start in the symmetric phase with zero Higgs VEV and \( T > T_{EW} \). The change in the baryon number is related to the sphaleron rate as

\[
\frac{1}{N_B} \frac{dN_B}{dt} = 39 \frac{\Gamma}{4T^3}.
\]

(2)

The sphaleron rate calculated from earlier lattice studies and also derived from Bodeker model is

\[
\Gamma = \kappa \left( \frac{gT}{m_D} \right)^2 \alpha^5 W T^4,
\]

(3)

with \( \kappa \approx 50 \) extracted from the lattice fit. The lattice work [10] yields an accurate evaluation for the rate

\[
\frac{\Gamma}{T^4} = (18 \pm 4)\alpha^5_{EW} \approx 1.5 \cdot 10^{-7}
\]

(4)

While (4) appears small, its convolution with time up to the electroweak transition time \( t_{EW} \), down to the corresponding temperature \( T_{EW} \), is large:

\[
\frac{1}{N_B} \frac{dN_B}{dt} t_{EW} = 3.2 \cdot 10^9.
\]

(5)

Therefore, the baryon production rate in the symmetric phase strongly exceeds the expansion rate of the Universe \( H \sim 1/t_{EW} \), by 9 orders of magnitude! Therefore, prior to EWPT, \( T \geq T_{EW} \), the sphaleron transitions are in thermal equilibrium. According to Sakharov, this excludes the formation of BAU. In fact, this even suggests a total washout of the baryon-lepton (BL) asymmetry. This particular conclusion will be circumvented below, by the proposed here “sphaleron freezeout” phenomenon.

Another important result of the lattice work [11] is the temperature dependence of the sphaleron rate in the broken phase

\[
\log \left( \frac{\Gamma(T < T_{EW})}{T^4} \right) = -(147.7 \pm 1.9) + (0.83 \pm 0.01) \left( \frac{T}{\text{GeV}} \right)
\]

(6)

It would be useful for our subsequent discussion to re-parametrize this rate, expressing it in terms of the sphaleron mass through the temperature-dependent Higgs VEV \( v(T) \), namely

\[
\frac{\Gamma}{T^4} \sim \exp \left( -\frac{\Delta M_v}{T} \right),
\]

(7)

with

\[
\Delta M_v(T) \approx v(T)^2/9 \text{GeV}.
\]

(8)

By comparing this rate to the Hubble value for the Universe expansion rate at the time \( t_{EW} \), the authors of [11] concluded that the sphaleron transitions become irrelevant when the temperature is below

\[
T_{\text{decoupling}} = 131.7 \pm 2.3 \text{GeV}.
\]

(9)

Therefore our subsequent discussion is limited to the times when the temperature is in the range

\[
T_{EWPT} \approx 160 \text{GeV} < T < T_{\text{decoupling}} \approx 130 \text{GeV}
\]

Note that by this time, the Higgs VEV \( A_5 \) reaches only a fraction of its value today, in the fully broken phase, i.e. \( v(T = 0) \approx 246 \text{GeV} \).

B. The sphaleron size distribution

The lattice results recalled above give us valuable information on the mean sphaleron...
rates, and thus masses. However for the purposes of this work, we need to know also the sphaleron size distribution. As we will detail below, baryogenesis driven by CP violation is biased toward sphalerons of sizes larger than average, while gravity wave signal and seeds of magnetic clouds are biased to smaller sizes.

Small sizes: Let us start with the small-size part of the distribution in size $\rho$. In this regime, we can ignore the Higgs VEV, even when it is non-vanishing, a significant simplification. By dimensional argument it is clear that $M_{sph}(\rho) \sim 1/\rho$. It is also clear that small-size sphalerons should be spherically symmetric.

The classical sphaleron-path configurations in pure gauge theory were analytically found in \[18\]. The method used is “constrained minimization” of the energy, keeping their size $\rho$ and their Chern-Simons number $N_{CS}$ fixed. This gave the explicit shape of the sphaleron barrier. At the highest point of the barrier $N_{CS} = \frac{1}{2}$, the sphaleron mass is

$$M_{sph}(\rho) = \frac{3\pi^2}{g^2 \rho}$$

Later the same solutions were obtained in \[19\] by a different method, via an off-center conformal transformation of the Euclidean solution (the instanton) of the Yang-Mills equation. Some of the results are reviewed in Appendix B. It provides not only a static sphaleron configuration, but the whole sphaleron explosion process in relatively simple analytic form, to be used below.

Large sizes: Now we turn to the opposite limit of large-size sphalerons. Since the sphaleron itself is a magnetic configuration, at large $\rho$ one should consider magnetic screening effects. Unlike the simpler electric screening, the magnetic screening does not appear in perturbation theory \[21\]. It is purely nonperturbative, and likely due to magnetic monopoles.

The magnetic mass $M_m$ conjectured by Polyakov to scale as $M_m = O(g^2 T)$, was confirmed by lattice studies. While in the QCD plasma the coupling is large and the difference between the electric and magnetic masses is only a factor of two or so, in the electroweak plasma the coupling is small $\alpha_{EW} \sim 1/30$, and therefore the magnetic screening mass is smaller than the thermal momenta by about two order of magnitude

$$\frac{M_m}{3T} \sim \frac{\alpha_{EW}}{3} \sim 10^{-2}$$

The key consequence for the sphalerons is that their sizes would be about two orders of magnitude larger than the interparticle distances in the electroweak plasma. This conclusion, in turn, will have dramatic consequences for the magnitude of the CP violation.

The part of the gauge action related with the screening mass is

$$\Delta S_{\text{screening}} = \frac{M_m^2}{2} \int d^4 x (A_i^2)^2$$

For static sphalerons, the integral over the Matsubara time is trivial, giving $1/T$. Parametrically, we have $M_m \sim g^2 T, A \sim 1/g \rho$, so that

$$M_m^2 \int d^4 x (A_i^2)^2 \sim (g^2 T)^2 \left( \frac{1}{g \rho} \right) \frac{\rho^3}{T} \sim g^2 T \rho$$

At high temperature, the pure SU(2) lattice simulations in \[15\] give

$$M_m(T) \approx 0.457 g^2 T$$

Inserting (14) in (12) and using the pure gauge sphaleron configuration yield the screening factor for large size sphalerons

$$\frac{\Gamma}{T^4} \sim \exp \left( - (0.457)^2 \pi^2 g^2 T \rho \right)$$

The sphaleron size distribution can now be constructed using the mean mass \[8\], the small and large size limits \[10\] and \[15\]. More specifically, the proposed distribution interpolates between the small and large size distributions, which are forced to merge at $\rho = \rho_{\text{mid}} = 0.8 \text{ GeV}$ to give \[7\]
FIG. 1: The sphaleron probability distribution as a function of the sphaleron size $\rho(\text{GeV}^{-1})$. The curves correspond to $T = 159, 150, 140, 130 \text{ GeV}$, top to bottom. The horizontal line separates the tail which is out of the Hubble expansion rate.

\begin{align*}
P(\rho, T) &\sim \exp\left(-\frac{v(T)^2}{(9 \text{ GeV})T}\right) \\
&\times \exp\left(-\frac{3\pi^2}{g^2 T}\left(\frac{1}{\rho} - \frac{1}{\rho_{\text{mid}}}ight)\right) \\
&\times \exp\left(-\frac{(0.457)^2 \pi^2 g^2 T}{\rho - \rho_{\text{mid}}}\right) \\
&\text{(16)}
\end{align*}

In Fig. 1 we show the size distribution for four temperatures in the range $130 \text{ GeV} \leq T_{EW} \leq 159 \text{ GeV}$. The lowest temperature $T_L \approx 130 \text{ GeV}$ corresponds to a sphaleron rate that is below the Universe expansion rate (Hubble). The intercept of the curves with the horizontal line give the largest size sphalerons that are still cosmologically exploding, at the corresponding temperatures. For example, for $T = 140 \text{ GeV}$ (solid line) the largest size is $\rho_{\text{max}} \approx 10 \text{ GeV}^{-1}$. At $T = 130 \text{ GeV}$ they are about $1 \text{ GeV}^{-1}$.

### III. SPHALERON EXPLOSIONS: PRODUCTION OF SOUND AND GRAVITY WAVES

Most of the studies on the gravity wave generation by the EWPT focus on scenarios based on the first order transition or the “cold” transition, as those usually yield large stress tensor fluctuations. To our knowledge, the smooth cross over transition of the minimal SM has not been considered.

Since the sphaleron explosions give rise to significant deviations from a homogeneous stress tensor of the plasma

$$\Delta T^{\mu\nu} \sim G^{\mu\lambda}G^{\nu}_{\lambda} \sim \frac{1}{g^4 T^4} \quad (17)$$

one may expect radiation of the gravity waves. The stress tensor from the analytically known sphaleron field yields long expressions which are not suitable for reproduction here. Instead, we show in Fig. 2 the behavior of $T^{00}(t, r)$ (the energy density) and $T^{33}(t, r)$ (the pressure), which illustrates the time-development of the exploding sphaleron in a spherical shell.

FIG. 2: Components of the stress tensor (times $r^2$, namely $r^2 T^{00}(t, r)$ upper plot, $r^2 T^{33}(t, r)$ lower plot) as a function of $r$, the distance from the center, at times $t/\rho = 0.1, 1, 2$, left to right.
The key point here is to assess the scale dependence of both the sound and gravity waves triggered by the explosion, which can be expressed using the power-per-volume $dE/d^3x$. Dimensional reasoning shows that the average scale is shifted to smaller sphaleron sizes. The measure for small size sphalerons

$$d\rho \rho^5 P(\rho) = d\rho \exp\left(-\frac{3\pi^2}{9g^2T \rho} - 5\log(\rho)\right)$$

is peaked at

$$\rho_\ast = \frac{3\pi}{20\alpha_{EW} T} \approx \frac{1}{10 \text{GeV}}$$

which is about an order of magnitude smaller than at the peak of the distribution (Fig.1).

Also, for $T > T_c$, we do not expect direct gravitation emission from the sphaleron explosion. In this regime the Higgs VEV vanishes, nothing breaks the rotational symmetry of the gauge field leading to spherically symmetric sphaleron explosions. As a result, these explosions cannot directly generate gravitational waves no matter how violent they are. This is not the case for $T < T_c$ as we discuss below.

There is an indirect way to gravitational signal as discussed in [13]. Spherical sphaleron explosions do excite the underlying medium through hydrodynamical sound waves and vortices. Of course, the medium viscosity will eventually kill them, but since the damping rate scales as $\Gamma \sim \eta k^2$, at small $k$ (large wavelength) this time can be long. Random set of sound sources creates acoustic turbulence. Under certain conditions it may turn into the regime of inverse cascade and propagate many orders of magnitude, perhaps to the infrared cutoff, the horizon size of the Universe as suggested in [13]. It is a $2 \rightarrow 1$ generic process [13]

$$\text{sound} + \text{sound} \rightarrow \text{gravity wave}$$

which operates during the whole lifetime of the sound (which, we remind, is proportional to $1/k^2$ and can be long for small momenta $k$.)

Just after the transition, at $T < T_{EW}$, a nonzero Higgs VEV leads to different masses of various quarks, leptons and gauge bosons. This “mass separator” split expanding spherical shell of the explosion into separate subshells.

A nonzero Weinberg angle (the nonzero $g'$ coupling of the Higgs to the QED Abelian $U(1)$ field) also produces an elliptic deformation of the sphaleron explosion. It is created by the following part of the action

$$\Delta S_a = \frac{m_Z^2 - m_W^2}{2} \int d^4x \sqrt{g} g_{\mu\nu} Z^\mu Z^\nu$$

where the metric is explicitly shown. Writing it as a flat metric plus perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and expanding in $h_{\mu\nu}$ is the standard way to derive the corresponding stress tensor, which is

$$\Delta T_{\mu\nu} = \frac{m_Z^2 - m_W^2}{2} \left(-Z^\mu Z^\nu + \frac{\eta_{\mu\nu}}{2} Z^2\right)$$

Here, the pre-factor is proportional to $v^2(T)$, nonzero only after EWPT, at $T < T_{EW}$.

The power produced by the gravity wave is proportional to the squared matrix element $|M|^2$ of the Fourier resolved stress tensor by the gravity wave with momentum $\vec{k}$

$$\mathcal{M}(h, k) = \int d^4x \Delta T_{\mu\nu}^{\text{wave}}(x) h_{\mu\nu} \frac{e^{i\vec{k} \cdot x}}{r}$$

We recall that the polarization tensor for the gravity wave $h_{\mu\nu}$ is traceless, and transverse, i.e. nonzero only in the 2-d plane normal to $\vec{k}$. For example, for $\vec{k}$ in the 1-direction, the pertinent contributions in (22) are $T^{22} - T^{33}$ or $T^{23}$ for the respective polarizations.

The main part of the stress tensor gives vanishing matrix element, as it should, but the asymmetric part of the stress tensor produces gravitational radiation. In Fig.3 we show the dependence of the gravity wave matrix element as a function of $kr$. As expected, it is maximal at $kr \sim 1$. We have already evaluated the most important sphaleron size in (19). As a result, the expected gravitational wave momentum should be $k \sim 1/\rho_\ast \approx 10 \text{GeV}$. 
FIG. 3: The dimensionless matrix element $M_j/(M_2^2 - M_0^2)^2$ in (22) versus $k\rho$, for a gravity wave propagating in the 1-direction with transverse polarization giving $T^{22} - T^{33}$.

IV. CP VIOLATION AND THE SPHALERON EXPLOSIONS

A. Standard Model CP violation, from the phase of the CKM matrix

In this section we discuss whether the “minimal” CP violation in the SM, following from the experimentally well studied complex contribution of the CKM matrix, can generate the required level of asymmetry. Although this question was addressed many times, in literature and even textbooks, it is worth briefly reviewing it again here.

The CP-violating effects appear at one-loop level, with the contribution from all generations of up and down quarks. Multiple diagrams interfere in nontrivial ways, producing complete cancellations when any two masses in each set become equal. In principle, the calculation of effective CP-violating Lagrangian is straightforward, but due to technical difficulties it is not yet converged on a single consistent result.

Because of our focus on explosion of large-size sphalerons, the appropriate strategy appears to be an evaluation of the fermionic determinant in the “smooth” background of a W-field with small momenta. The determinant of the Dirac operator in such field $\log(\det(D))$ generates the effective action, induced by one-loop fermion process, which is similar to the well known Heisenberg-Euler effective action in QED, with the CP-violating part extracted from its imaginary part.

Studies along these lines have been carried, but the results are still (to our knowledge) inconclusive. One group [23] found no CP violation to the leading order $O(W^4)$, but reported a nonzero contribution to order $O(ZW^3DW)$ from a dimension-6 P-odd and C-even operator of the type

$$e^{\mu\nu\lambda\sigma} (Z_{\mu}W^{\nu+}_{\lambda}W^{-}_{\alpha} (W^{\nu}_{\alpha}W^{-}_{\sigma} + W^{+}_{\alpha}W^{-}_{\sigma}) + c.c.)$$

(23)

containing one neutral current vertex and the Z-boson field. Other operators, which are C-odd and P-even, were claimed to contribute in [3, 4].

It is not a trivial task to find an example of the field which would give a non-vanishing expectation value for this operator. In particular, it should be T-odd, and thus involving time evolution or electric field strength. We have checked using the analytic solution for the sphaleron explosion [18, 19] described in Appendix B, does give a non-vanishing expectation value for this operator. The formulae are unfortunately too long and the plots made were not found particularly instructive as well, so those are not given here.

B. Scale dependence of the CP violation

The main physical issue is not so much the operator expectation values, but rather the scale dependence of the Wilsonian OPE coefficients multiplying them. These coefficients are usually rather complicated functions of the quark masses, see e.g. the explicit form of the coefficient of the operator (23) in the Appendix of [23]. Instead of calculating the OPE coefficients for specific operators, we suggest a somewhat more general and universal approach, that will help us understand their scale dependence.

Consider a typical CP violating contribution to the effective action in some smooth gauge background $W_\mu(x)$, $Z_\mu(x)$ as vertices in a one-loop fermionic contribution. Each fermion line is characterized by a Dirac operator $\slashed{D}$ in the gauge background. Using left-right spinor notations it has the form
\[ \det \left( \begin{array}{cc} i\slashed{D} & M \\ M^\dagger & i\slashed{\partial} \end{array} \right) = \det(i\slashed{\partial}) \det \left( i\slashed{D} + \frac{1}{i\slashed{\partial}} M^\dagger \right) \]

where \( M \) is a mass matrix in flavor space and the slash here and below means the convolution with the Dirac matrices, e.g. \( \slashed{D} = D_\mu \gamma^\mu \). Let us use a representation in which this operator is diagonalized

\[ i\slashed{D} \psi_\lambda(x) = \lambda \psi_\lambda(x) \]

Its two sub-operators, \( \slashed{\partial} \) and \( \slashed{W} \) are not in general diagonal in this basis, but for our qualitative argument we will only include their diagonal parts

\[ \langle \lambda \left| i\slashed{\partial} \right| \lambda' \rangle \approx \rho \delta_{\lambda\lambda'}, \quad \langle \lambda \left| \slashed{W} \right| \lambda' \rangle \approx \xi \lambda \delta_{\lambda\lambda'} \]

where \( \rho, \xi \) are in general some functions of \( \lambda \). In this approximation the corresponding (Euclidean) propagator describing a quark of flavor \( f \) propagating in the background can be represented as the usual sum over modes

\[ S(x, y) \approx \sum_\lambda \frac{\psi^*_\lambda(y) \psi_\lambda(x)}{\lambda + M \rho^{-1} M^+} \]

where the right-handed operator \( i\slashed{\partial} \) is approximated by its diagonal matrix element in the \( \lambda \)-basis. Throughout, we will trade the geometric mean appearing in all expressions \( \sqrt{\rho \lambda} \to \lambda \) for simplicity.

The generic fourth-order diagram in the weak interactions, contain at least four CKM matrices, and takes in the coordinate representation the form

\[ \int \prod_i^4 d^4x_i \, \text{Tr} \left( \slashed{W}(x_1) \hat{V} \hat{S}_u(x_1, x_2) \right. \\
\times \slashed{W}(x_2) \hat{V}^\dagger \hat{S}_d(x_2, x_3) \slashed{W}(x_3) \hat{V} \hat{S}_u(x_3, x_4) \\
\left. \times \slashed{W}(x_4) \hat{V}^\dagger \hat{S}_d(x_4, x_1) \right) \]

Here \( \hat{V} \) is the CKM matrix, the propagators labels \( u, d \) indicating that they are \( 3 \times 3 \) matrices in flavor subspace. The trace is over both spin and flavor indices. If one considers the next order diagrams, with \( Z, \phi \) field vertices, the expressions are generalized straightforwardly.

The spin-Lorentz structure of the resulting effective action is very complicated. However, to understand the scale dependence we will make a second strong simplifying assumption. Specifically, one can use the orthogonality condition of the different \( \lambda \)-modes and perform the integration over coordinates, to obtain a simple expression, with a single sum over eigenvalues

\[ \sum_\lambda F(\lambda) \]

with

\[ F(\lambda) = \lambda^4 \text{Tr} \left( \hat{V} S_d \hat{V}^\dagger S_d \hat{V}^\dagger S_d \right) \]

This is the box diagram in the "\( \lambda \)-representation", which generalizes the momentum representation valid only for constant fields. Unlike momenta, the spectrum of the
Dirac eigenvalues $\lambda$ may have various spectral densities $d(\lambda)$. In particular, there is a zero mode, corresponding to the zero mode in the original 4-dimensional symmetric case, describing the fermion production. However, whatever the spectral density $d(\lambda)$ may be for a particular field, one can perform the multiplication of the flavor matrices and extract some universal function of $\lambda$, that describes the dependence of the CP violation contribution on the given scale. Using the standard form of the CKM matrix $\hat{V}$, in terms of the known three angles and the CP-violating phase $\delta$, and also the six known quark masses, one can perform the multiplication of these 8 flavor matrices and identify the lowest order CP-violating term of the result. Performing the multiplication in the combination above one finds a complicated expression which does not have $O(\delta)$ term, so there is no lowest order CP violation. However, all higher order diagrams have such contributions. We have generated a number of those. The simplest turned out to be the sixth-order diagram with four $W$ vertices and two $Z$. Substituting the CKM matrix and propagators one can do the explicit summation over the quark types, with the following result

$$\text{Im} F(\lambda) = \lambda^6 \text{Im} \text{Tr} \left( \hat{V} S_d \hat{V}^\dagger S_u \hat{V} S_d Z S_d \hat{V}^\dagger S_u Z S_u \right)$$

$$= 2 J_0 \left[ (m_b^2 - m_d^2)(m_b^2 - m_s^2)(m_d^2 - m_s^2)(m_c^2 - m_t^2)(m_c^2 - m_u^2)(m_t^2 - m_u^2) \right] \Pi_{f=1,6}(\lambda^2 + m_f^2)^2$$

We recall that $\hat{V}$ is the CKM matrix, $S_u, S_d$ are propagators of the up-type and down-type quarks, and $J$ is the Jarlskog combination of the CKM cos and sin of all angles times the sin of the CP violating phase.

In Fig.4 we show (29) as a function of the eigenvalue scale $\lambda$. It is clear that the magnitude of the CP violation depends on the absolute scale very strongly. When the momentum scale is at the electroweak value $\sim 100$ GeV (the r.h.s. of the plot), it is $\sim 10^{-19}$. But in the "sweet spot", between the masses of the light and heavy quark $0.2 - 2$ GeV, the asymmetry is suppressed only by about $10^{-6}$.

The momentum scale of the spaleron gauge field is $\lambda \approx 1/\rho$. From Fig.4 one may naively conclude that the largest spalerons do indeed correspond to this “sweet spot”. However, this is not exactly the case since our preceding evaluation of the CP violation referred to the physical quark masses calculated in the fully broken vacuum, with the VEV $v = v(0)$. In the cross over region of temperatures $150 - 131$ GeV, the VEV $v(T)$ varies from zero to only a fraction of its vacuum value. The CP scale should be compared to quark masses in this regime, after scaling them by $v(T)/v(0)$. Since the $\lambda$-axis in Fig.4 is logarithmic, this rescaling amounts to sliding the curve in the horizontal direction to the left with

$$\lambda_{\text{eff}} = \frac{v(T)}{v(0)} \rho$$

The way it works is explained in the Table where we have recorded few particular cases. It follows that the effective scale $\lambda_{\text{eff}}$ characterizing the mean spaleron size falls inside the “sweet spot” of CP violation, for temperatures in the cross-over region $155 - 130$ GeV, where the CP asymmetry is at its maximal value of about $10^{-6}$.

V. BARYOGENESIS

A. Which sphaleron transitions are out of equilibrium?

Before we discuss freezeout of the sphaleron transitions, it is instructive to recall an analogous case of freezeout of the “little Bang” in heavy ion collisions. A good example is the
production of antinucleons $\bar{N}$. In the 1990’s the cascade codes predicted small yield of $\bar{N}$, based on the fact that on average many baryons surround an anti-nucleon. Since the annihilation cross section $\sigma_{NN}$ is large, the antinucleon lifetime $\tau \sim 1/(n_{N}\sigma_{NN}(v))$ must be quite short. However, the data showed otherwise, with a number of produced anti-nucleons much larger than predicted by the numerical codes. The explanation was given in \[22\]. The annihilation creates multi-pion final states with $\pi\pi\pi\pi\pi\pi$ \[\sigma\] surrounding an anti-nucleon. Since the annihilation creates multi-pion final states with $\pi\pi\pi\pi\pi\pi$, it is large, the anti-nucleon lifetime $\tau \sim 1/(n_{N}\sigma_{NN}(v))$ must be quite short. However, the data showed otherwise, with a number of produced anti-nucleons much larger than predicted by the numerical codes. The explanation was given in \[22\]. The annihilation creates multi-pion final states with $\pi\pi\pi\pi\pi\pi$, and the inverse reaction, like $\pi\pi\pi\pi\pi\pi \rightarrow N\bar{N}$ was ignored because of certain prejudice, that the multi-particle collision has negligible rate. Explicit calculations showed otherwise, in agreement with detailed balance in thermal equilibrium.

This equilibrium is only violated after the so called \textit{chemical freezeout}, when the rate $\Gamma_{\text{inelastic}}$ of the inelastic reactions changing $N_{\pi}$ and $N_{N}$ gets smaller than the expansion rate of the fireball $H = \partial_{\mu}u^{\mu}$ (the Hubble of the Little Bang).

(In this particular problem, there are elastic scatterings still going on, so a different equilibrium ensemble is established. Since after chemical freezeout the particle numbers no longer change, the thermal state of the expanding fireball is described via time-dependent chemical potentials, for all particle types like $\mu_{\pi}(t)$ and $\mu_{N}(t)$. The annihilation channel contains the fugacity factor $\exp(-2\mu_{N}/T)$, while the inverse reaction channel contains the fugacity $\exp(-N_{\pi}\mu_{\pi}/T)$. Since $\mu = 0$ due to $CP$ violation, same as $T$-symmetry violation, even the probability to decay into two sides of the barrier is not in fact the same!

(In equilibrium, the principle of the detailed balance requires that the inverse reaction with $t \rightarrow -t$, has the same rate. It means that, contrary to prejudice it may still take place, where a large number of gauge quanta plus the 12 fermions required by the anomaly relation, can collide together, putting the field back on top of the sphaleron hill. As Sakharov argued, the presence of CP and thus T-violation

### TABLE I: Effective scale $\lambda_{\text{eff}}$ characterizing the mean sphaleron size in the cross-over region of the EWPT.

| $T$ (GeV) | $v(T)/v(0)$ | 1/$\rho_{\text{max}}$ (GeV) | $(v(T)/v(0))/\rho_{\text{max}}$ (GeV) | $(v(T)/v(0))/\rho_{\text{max}}$ (GeV) |
|-----------|-------------|-----------------------------|---------------------------------|---------------------------------|
| 159       | 0           | 0.03                        | 0                               | 0                               |
| 155       | 0.29        | 0.04                        | 0.29                            | 0.011                           |
| 150       | 0.43        | 0.05                        | 0.43                            | 0.021                           |
| 140       | 0.58        | 0.1                         | 0.58                            | 0.058                           |
| 130       | 0.67        | 1                           | 0.67                            | 0.67                            |
in the process matrix element does not matter. Thermal occupation factors depend only on masses/energies, which are CP invariant.

However, since the Universe is expanding at a Hubble rate $H \sim 1/t_{EW}$, some of these transitions involve particle changing rates smaller than the Hubble rate. They are out of equilibrium! Earlier, we have shown that as a function of the sphaleron size $\rho$, the sphaleron decays get frozen when

$$\exp\left(-B_{\text{sph}}\rho v(T)^2\right) < 10^{-9} \quad (32)$$

We now argue that the inverse process is frozen differently, so that the equilibrium condition and its detailed balance become violated. More specifically, for the large-$\rho$ tail and in the small-$v$ regime near $T_c$, the inverse reaction of multi-quanta collisions gets frozen first. The argument is based on the observation that the corrections to the sphaleron mass $\Delta E_{\text{sph}} = C_{\text{sph}}\rho v^2$ is smaller than the modification of the thermal Boltzmann factor of the inverse reaction. The latter can be written as corrections to ultra-relativistic energies of W bosons due to their mass $E_p \approx p + M_W^2/2p$, so the energy in their thermal exponent changes by

$$\Delta E_W = \sum_{i=1}^{N_W} \Delta E_i \approx \frac{N_W}{2p} \left( \frac{M_W(0)}{v(0)/v(T)} \right)^2 \quad (33)$$

after rescaling the W-mass. Since $1/p \sim \rho$, this correction is of order $\sim \rho v^2$, but the coefficient $N_W$ is parametrically larger with $N_W \sim 100 = O(1/\alpha_{EW})$.

B. Contribution to BAU from out-of-equilibrium sphalerons

Our next step is to calculate the BAU produced by the large-size sphalerons which are out of equilibrium. As detailed above, this requires moving to the freezeout point, thereby sacrificing 9 orders of magnitude in the rate with $T_{\text{freezout}} \sim 3 \cdot 10^{-9}$. This is the regime where sphalerons decay but are not regenerated. Each electroweak sphaleron changes the baryon number by 3 units, i.e. 9 quarks each carrying $\frac{1}{3}$ baryon charge. The baryon number density normalized to the entropy density of matter, follows by integrating the rate over the freezeout time $\Delta t_{FO}$

$$\left(\frac{n_B}{s}\right) = 3A_{CP} \left( \frac{\Gamma_{\text{freezout}}}{T_{EW}s_{EW}} \right) \times \left( T_{EW}t_{EW} \right) \left( \frac{\Delta t_{FO}}{t_{EW}} \right) \quad (34)$$

Here $A_{CP}$ is the CP asymmetry, the relative difference between baryon number production and annihilation in a single sphaleron transition. The contribution is the out-of-equilibrium sphaleron rate normalized to the total entropy. The last contribution is the fraction of the effective time of the freezeout normalized to the total time. In the cross over region we have $\Delta T \approx 10 \text{ GeV}$. Using Friedmann evolution $|\Delta t/t| = 2\Delta T/T$, we obtain $\Delta t_{FO}/t_{EW} = \frac{1}{8}$.

Since the entropy in the adiabatic expansion of the Universe is conserved, it is the same at the BBN time which is mostly black body photons. Standard Bose gas relation between the entropy density and the photon density is $n_{\gamma} = 0.1388s_{\gamma}$. Substituting all these estimates in (34) gives the baryon-to-photon ratio

$$\left(\frac{n_B}{n_{\gamma}}\right) = 1.3 \cdot 10^{-5} A_{CP} \quad (35)$$

The phenomenological value for this ratio, from the BBN fits, is

$$\left(\frac{n_B}{n_{\gamma}}\right)_{BBN} = 6 \cdot 10^{-10} \quad (36)$$

We conclude that the amount of CP violation necessary to produce the observed BAU is

$$A_{CP} \approx 3.7 \cdot 10^{-6} \quad (37)$$

Within the accuracy of the estimates (say an order of magnitude) this magnitude of CP violation can be obtained, provided that the sphaleron scale is in the “sweet spot” defined in Fig. A. So, contrary to many negative conclusions from previous papers (including our own) we now see that out-of-equilibrium large
Multiple other sources of CP violation have been proposed in the literature, but instead of considering some specific models we found it instructive to focus on a scenario in which CP violation is instead maximal. We would not speculate here what particular mechanism – CP violating $\theta$ field, lepton number asymmetry from the neutrino sector, or others – can lead to that.

VI. HELICAL MAGNETOGENESIS

The symmetry breaking by the Higgs VEV at $T < T_c$ leads to mass separation of the original non-Abelian field $A_{\mu}^3$ into a massive $Z_{\mu}$ and a massless $a_{\mu}$, related by a rotation involving the Weinberg angle. The expanding outer shell of the sphaleron explosion contains massless photons and near-massless quarks and leptons $u,d,e,\nu$.

The anomaly relation implies that the non-Abelian Chern-Simons number during the explosion defines the chiralities of the light fermions, which can be transferred to the so-called “magnetic helicity” (Chern-Simons three-form):

$$\int d^3x \vec{A}\vec{B} \sim B^2 \xi^4.$$  \hfill (38)

The configurations with nonzero \ref{38} correspond to chiral knots of magnetic flux, and are called helical.

The chiral anomaly allows the transfer of fermion chirality to the chirality of the gauge fields, and thus to magnetic helicity. Indeed, the time derivative of magnetic helicity yields the Chern-Pontryagin number $\int d^3x \vec{E}\vec{B}$ that is related by the chiral anomaly to the time derivative of fermion chirality. Because the transfer of chirality from fermions to magnetic helicity is energetically favorable, it induces a “chiral magnetic instability” resulting in an inverse cascade.

Microscopically, this instability can be attributed to the chiral magnetic current along the lines of magnetic field generated by the chiral imbalance of fermions. This current backreacts on magnetic field by increasing the magnetic helicity; at the same time, it reduces the chiral asymmetry stored in fermions. It has been found that this inverse cascade is self-similar, with exponents corresponding to a diffusive growth of size $L$ with time $t$, $L^2 \sim t$.

We conclude that the primordial sphaleron explosions may seed the helical clouds of primordial magnetic fields. Since the sphaleron rate is small, $\Gamma/T^4 < 10^{-7}$, these seeds are produced independently from each other, as spherical shells expanding luminally.

A. The “inverse cascade” of magnetic fields

The requirement for the inverse cascade effect is chiral unbalance which is at the origin of the CME. Locally the trapped and co-moving light fermions produced by the sphaleron explosion are chiral. The time during which chirality is conserved is given by the appropriate fermion masses. For magnetic fields it is the electron mass, which at the sphaleron freezeout time is

$$m_e(T_{FO}) = m_e \frac{v(T_{FO})}{v(0)} \sim 20 \text{KeV}$$  \hfill (39)

The size growth of the chiral (linked) magnetic cloud is diffusive. For a magnetically driven plasma with a large electric conductivity $\sigma$, a typical magnetic field $\vec{B}$ diffuses as

$$\frac{d\vec{B}}{dt} = D \nabla^2 \vec{B}$$  \hfill (40)

with the diffusion constant $D = 1/(4\pi\sigma) \sim 1/T$. It follows that the magnetic field size grows as

$$R^2(t) = D \Delta t \sim \frac{\Delta t}{T}$$  \hfill (41)

where the inverse cascade time $\Delta t$ is limited by the electron mass

$$\Delta t \sim 1/m_e(T_{FO})$$  \hfill (42)

As a result, the size of the chiral magnetic cloud is
\[ R(\Delta t) \sim \left( \frac{1}{m_c(1_{FO})T} \right)^{\frac{1}{2}} \sim 5 \text{fm} \]  

We note that this is two orders of magnitude larger than the UV scale of the problem \( 1/T \sim 0.02 \text{ fm} \), and far from the IR cutoff of the problem, the horizon at \( \sim 0.3 \text{ mm} \).

B. CP violation results in the helical asymmetry of magnetic clouds

One of the chief observation in section IV B is that the magnitude of CKM induced CP violation is strongly scale dependent. It increases with the sphaleron size to a maximum as large as \( \max P_{CP} \sim 10^{-6} \). Therefore, the sphaleron seeded magnetic clouds would start with such an initial asymmetry. Their subsequent evolution goes beyond the scope of this work. However, we expect that during the evolution the left- and right-linked clouds to annihilate. Since helicity in magneto-hydrodynamics is conserved, we expect the asymmetry to grow with time.

After the CME is switched off, ordinary magneto-hydrodynamical evolution continues to expand the cloud size and to decrease its field strength. This evolution is stopped only when the matter is no longer a plasma, that is at the recombination era.

VII. SUMMARY

The main purpose of this paper is to revive discussion of the cosmological EWPT, in connection to generation of the baryon asymmetry and helical magnetic clouds. In contrast to many other works, we have restricted our analysis to within the minimal SM, using the established by lattice simulations fact that the transition is a smooth cross-over. The Higgs VEV in it is gradually growing, instead of abruptly as in the first order scenarios.

We have focused on the primordial dynamics of the sphaleron explosions. By now, their overall rate is more or less understood, both in the symmetric and slightly broken phases, from lattice simulations. We have used this knowledge to study the sphaleron size distribution, by constraining the small and large \( \rho \)-tail distribution to known results.

The small-size end of the sphaleron size distribution, at \( \rho \sim 1/(10 \text{ GeV}) \) was found to dominate the production of sound waves, as well as direct gravitational radiation. These sound waves may or may not be involved in the inverse acoustic cascade, advocated in [13]. However if they do, long wave-length sounds would reach the horizon at the time and then be converted to gravity waves, in a frequency range accessible by eLISA.

The large-size end of the sphaleron size distribution, at \( \rho \sim 1/(1 \text{ GeV}) \), is the most intriguing part of our study. In a specific time range during the cross-over region of the EWPT, we showed that all three Sakharov conditions are satisfied in principle, and actually can generate the needed BAU ratio. Admittedly, our treatment of the sphaleron explosions, freeze-out and especially the CP violation contribution are crude.

Small sphaleron explosion rate can be compensated by the long production time during which the asymmetry is generated. As a result, the BAU accumulates over a long period of time. Most fortunately, the momentum scale of the sphaleron fields \( \sim 1/\rho \sim 1 \text{ GeV} \) happens to be in a such a relation to the quark masses so as to generate the maximal CP asymmetry, \( \sim 10^{-6} \).

Although we do not claim to have found a definitive solution to the BAU problem, we have shown that it may well be possible to find this solution purely within the framework of the minimal SM. Clearly, more studies in this direction are needed.

Finally, we have shown that like the BAU, CP asymmetry should be the origin of the helical magnetic fields. The baryon number produced per sphaleron (9 quarks, 3 baryons) is directly connected with the chirality of the electrons, which is transferred to linkage of the magnetic clouds. This linkage is one of very few potential observables telling us something about the cosmological EWPT. More specifically, if observed, the magnitude and the chirality of the intergalactic magnetic fields would directly confirm or reject the electroweak sphaleron mechanism.
Appendix A: Basics of Electroweak phase transition

The transition temperature for the EWPT follows from lattice studies \[11\]

\[ T_{EW} = (159 \pm 1) \text{ GeV} \quad (A1) \]

The temperature of Universe today is \( T_{\text{now}} = 2.73K \). The ensuing redshift \( z_{\text{EW}} \) is

\[ z_{\text{EW}} = \frac{T_{EW}}{T_{\text{now}}} \approx 6 \times 10^{14} \quad (A2) \]

During the radiation dominated era, the relation of time to temperature is given by Friedmann

\[ t = \left( \frac{90}{32\pi^3 N_{\text{DOF}}(t)} \right)^\frac{1}{2} \frac{M_P}{T^2} \quad (A3) \]

Inserting the Planck Mass \( M_P \), the transition temperature and the effective number of degrees of freedom \( N_{\text{DOF}} \), we find the time after Big Bang to be

\[ t_{\text{EW}} \sim 0.3 \cdot 10^{-11} \text{s} \quad (A4) \]

or \( ct_{\text{EW}} \approx 0.1 \text{ mm} \).

As we already mentioned, lattice studies of the in SM have excluded a first order transition. Therefore we focus on the cross-over, in which the Higgs VEV grows gradually by some smooth function \( v(T) \) for \( T < T_{\text{EW}} \). Following \[11\], a relatively sharp cross-over is observed at \( T_{\text{EW}} = (159 \pm 1) \text{ GeV} \). The squared Higgs VEV below this temperature grows approximately linearly

\[ \frac{v^2(140 \text{ GeV} < T < T_{\text{EW}})}{T^2} \approx 9 \left( 1 - \frac{T}{T_{\text{EW}}} \right) \quad (A5) \]

This scaling is consistent with the naive Landau-Ginzburg treatment of the Higgs potential. The coefficient is also in agreement with the two-loop perturbative calculations.

In the symmetric phase \( T > T_{\text{EW}} \), the normalized sphaleron rate remains constant, which according to \[11\] is

\[ \frac{\Gamma}{T^3} \approx 1.5 \cdot 10^{-7} \quad (A6) \]

consistent with expected magnitude of \( 18e_5^{\text{EW}} \) from perturbative calculations.

If the seeded magnetic field would be simply produced at the electroweak scale \( T_{\text{EW}} \), and then just grow with the Universe with the redshift factor \( z_{\text{EW}} \), its resulting spatial scale today would be

\[ \xi \approx \frac{z_{\text{EW}}}{T_{\text{EW}}} = 4 \times 10^{14} \times 10^{-17} \text{ m} \approx 4 \text{ mm} \quad (A7) \]

The primary phase of the inverse magnetic cascade can only reach from the micro scale of \( 1/T_{\text{EW}} \sim 0.01 \text{ fm} \) to the horizon at that time, \( ct_{\text{EW}} \), about 13 orders of magnitude away. If that would be the end of the inverse cascade, the correlation length of the magnetic chirality would be

\[ \xi \approx \frac{z_{\text{EW}}}{ct_{\text{EW}}} \sim 4 \times 10^{14} \times 3 \times 10^{-4} \text{ m} \approx 10^{11} \text{ m} \quad (A8) \]

This distance may appear large on a human scale, but in units used for intergalactic distances it is tiny \( 1/3 \times 10^{-11} \text{ Mpc} \). This scale is also the same as the predicted maximal wavelength of the gravity waves emitted at electroweak transition today, in the hypothetical inverse acoustic cascade.

Appendix B: Pure gauge sphalerons and their explosion

Both static and time-dependent exploding solutions for the pure-gauge sphaleron have been originally discussed by Carter, Ostrovsky and Shuryak (COS) \[18\]. Its simpler derivation, to be used below, has been discussed by Shuryak and Zahed \[19\]. The construction relies on an off-center conformal transformation of the \( O(4) \) symmetric Euclidean instanton solution, which is analytically continued to Minkowski space-time. The focus of the work in \[19\] was primarily the detailed description of the fermion production.

The original \( O(4) \)-symmetric solution is given by the following ansatz
\[ gA^\mu = \eta_{\mu\nu} \partial_\nu F(y) \]

\[ F(y) = 2 \int_0^{\xi(y)} d\xi' f(\xi') \quad \text{(B1)} \]

with \( \xi = \log(y^2/\rho^2) \) and \( \eta_{\mu\nu} \) the 't Hooft symbol. Upon substitution of the gauge fields in the gauge Lagrangian one finds the effective action for \( f(\xi) \)

\[ S_{\text{eff}} = \int d\xi \left[ \frac{f^2}{2} + 2f^2(1 - f)^2 \right] \quad \text{(B2)} \]

corresponding to the motion of a particle in a double-well potential. In the Euclidean formulation, as written, the effective potential is inverted

\[ V_E = -2f^2(1 - f)^2 \quad \text{(B3)} \]

and the corresponding solution is the well known BPST instanton, a path connecting the two maxima of \( V_E \), at \( f = 0, 1 \). Any other solution of the equation of motion following from \( S_{\text{eff}} \) obviously generalizes to a solution of the Yang-Mills equations for \( A^\mu_\tau(x) \) as well. The sphaleron itself is the static solution at the top of the potential between the minima with \( f = -1/2 \).

The next step is to perform an off-center conformal transformation

\[ (x + a)_\mu = \frac{2\rho^2}{(y + a)^2} (y + a)_\mu \quad \text{(B4)} \]

\[ f(\xi) = \frac{1}{2} \left[ 1 - \sqrt{1 + 2\epsilon} \, \text{dn} \left( \sqrt{1 + 2\epsilon}(\xi - K), \frac{1}{\sqrt{m}} \right) \right] \quad \text{(B7)} \]

where \( \text{dn}(z, k) \) is one of the elliptic Jacobi functions, \( 2\epsilon = E/E_s, 2m = 1 + 1/\sqrt{2\epsilon} \), and \( E = V(f_m) \) is the conserved energy of the mechanical system normalized to that of the sphaleron energy \( E_s = V(f = 1/2) = 1/8 \). Since the start with \( a_\mu = (0, 0, 0, \rho) \). It changes the original spherically symmetric solution to a solution of the Yang-Mills equation depending on the new coordinates \( x_\mu \), with separate dependences on time \( x_4 \) and the 3-dimensional radius \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \).

The last step is the analytic continuation to Minkowski time \( t \), via \( x_4 \to it \). The original parameter \( \xi \) in terms of these Minkowskian coordinates, which we still call \( x_\mu \), has the form

\[ \xi = \frac{1}{2} \log \left( \frac{y^2}{\rho^2} \right) = \frac{1}{2} \log \left( \frac{(t + i\rho)^2 - r^2}{(t - i\rho)^2 - r^2} \right) \quad \text{(B5)} \]

which is pure imaginary. To avoid carrying the extra \( i \), we use the real substitution

\[ \xi_E \to -i\xi_M = \arctan \left( \frac{2\rho t}{t^2 - r^2 - \rho^2} \right) \quad \text{(B6)} \]

and in what follows we will drop the suffix \( E \). Switching from imaginary to real \( \xi \), corresponds to switching from the Euclidean to Minkowski spacetime solution. It changes the sign of the acceleration, or the sign of the effective potential \( V_M = -V_E \), to that of the normal double-well problem.

The needed solution of the equation of motion has been given in \( [19] \) \( [25] \).
The small displacement $\kappa$ ensures that “rolling downhill” from the maximum takes a finite time and that the half-period $K$ – given by an elliptic integral – in the expression is not divergent. In the plots below we will use $\kappa = 0.01$, but the results dependent on its value very weakly.

The solution above describes a particle tumbling periodically between two turning points, and so the expression above defines a periodic function for all $\xi$. However, as it is clear from (B6), for our particular application the only relevant domain is $\xi \in [-\pi/2, \pi/2]$. The solution $f(\xi)$ in it is shown in Fig. 5. Using the first 3 nonzero terms of its Taylor expansion

\[
f \approx 0.49292875 - 0.0070691232 \xi^2 - 0.0011773 \xi^4 - 0.0000781531899 \xi^6
\]

we find a parametrization with an accuracy of $10^{-5}$, obviously invisible in the plot and more than enough for our considerations.

![FIG. 5: The function $f(\xi)$ in the needed range of its argument $\xi \in [-\pi/2, \pi/2]$](https://example.com/figure5.png)

The components of the gauge potentials have the form [19]

\[
gA^a_4 = -f(\xi) \frac{8t \rho x_a}{[(t - i\rho)^2 - r^2][(t + i\rho)^2 - r^2]}
gA^a_i = 4 \rho f(\xi) \frac{\delta_{ai}(t^2 - r^2 + \rho^2) + 2 \rho \epsilon_{aij} x_j + 2 x_i x_a}{[(t - i\rho)^2 - r^2][(t + i\rho)^2 - r^2]}
\]

which are manifestly real. From those potentials we have generated rather lengthy expressions for the electric and magnetic fields, and eventually for the CP-violating operators using Mathematica.

Let us only mention that for the sphaleron solution itself at $t = 0$, the static solution is purely magnetic with $gA^a_4 = 0$. The magnetic field squared is spherically symmetric and simple

\[
\vec{B}^2 = \frac{96 \rho^4}{(\rho^2 + r^2)^4}
\]

We note that the specific expressions for pure-gauge sphaleron explosions were compared with numerical real-time simulations [1] where they occur inside the “hot spots” with very good agreement [5]. In the “cold scenario” numerically studied the sphaleron size was not determined by the Higgs VEV in the broken phase, but by the size of the hot spots with the unbroken phase. Unfortunately, a large size tail of the sphaleron distribution on which we focused in this work cannot be studied in similar simulations, as their probability is prohibitively low to reach it statistically.

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