CONTRACTIVE PROJECTIONS IN ORLICZ SEQUENCE SPACES

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Abstract. We characterize norm one complemented subspaces of Orlicz sequence spaces $\ell_M$ equipped with either Luxemburg or Orlicz norm, provided that the Orlicz function $M$ is sufficiently smooth and sufficiently different from the square function. This paper concentrates on the more difficult real case, the complex case follows from previously known results.

1. Introduction

One of the main topics in the study of Banach spaces has been, since the inception of the field, the study of projections and complemented subspaces. Naturally, one of the most important topics of the isometric Banach space theory is the study of contractive projections (i.e. projections of norm one) and 1-complemented subspaces (i.e. ranges of norm one projections). They were also investigated from the approximation theory point of view, as part of of the study of minimal projections, i.e. projections onto the given subspace with the smallest possible norm, for an overview of this line of research see [4, 12]. Contractive projections are also closely related to the metric projections or nearest point mappings, and are a natural extension of the notion of orthogonal projections from the Hilbert space setting to general Banach spaces. We refer the reader to the survey [16] for an outline of the development and applications of this theory. Here we just indicate some main facts putting the results of the present paper in context.

It is well known that in Lebesgue spaces $L_p$ and $\ell_p$, $1 \leq p < \infty$, a subspace $Y$ is 1-complemented if and only if $Y$ is isometrically isomorphic to an $L_p$–space of appropriate dimension (see [1, 3]). This is no longer the case for other spaces. Lindberg [8] demonstrated that there exist classes of Orlicz sequence spaces $\ell_M$ containing 1-complemented subspaces which are not even isomorphic to $\ell_M$. In fact, he showed that for all $1 < a \leq b < \infty$, there exists a reflexive Orlicz sequence space $\ell_M$ so that for all $p \in [a, b]$ there is a contractive projection from $\ell_M$ onto a subspace isomorphic to $\ell_p$. This implies in particular that Orlicz sequence spaces can have continuum isomorphic types of 1-complemented subspaces and thus any attempt for a geometric characterization of 1-complemented subspaces seemed hopeless.

On the other hand, 1-complemented subspaces of $\ell_p$ are also characterized as subspaces which are spanned by a family of mutually disjoint elements of $\ell_p$ (see [9, 2]). Moreover all known examples of 1-complemented subspaces in symmetric Banach spaces with
1-unconditional bases, and sufficiently different from Hilbert spaces, are spanned by a family of mutually disjoint vectors. (Note here that, since in Hilbert spaces every subspace is 1-complemented, it is both natural and necessary to include in this context some kind of an assumption about the space being different from Hilbert space.) In particular, the above described example of Lindberg of 1-complemented subspaces of Orlicz sequence spaces which were pathological in the isomorphic sense, are not pathological in the sense that they are spanned by mutually disjoint vectors and the norm one projection is the most natural averaging projection. It was shown in [13] that indeed every 1-complemented subspace $Y$ in any complex Banach space $X$ with a 1-unconditional basis (not necessarily symmetric) which does not contain a 1-complemented isometric copy of a 2-dimensional Hilbert space $\ell_2^2$, has to be spanned by a family of disjointly supported elements of $X$ and the norm one projection from $X$ onto $Y$ has to be the averaging projection. In particular, this holds in complex Orlicz sequence spaces $\ell_M$ equipped with either the Luxemburg or the Orlicz norm when $M$ is sufficiently different from the square function (cf. Remark 4.5).

In the real case this statement in its full generality is false (cf. [13]). For real spaces we only had the following much less satisfactory result describing special 1-complemented subspaces of finite codimension in Orlicz sequence spaces $\ell_M$.

**Theorem 1.1.** [14, Theorem 7] Let $M$ be an Orlicz function such that $M(t) > 0$ for all $t > 0$ and $M$ is not similar to $t^2$ (i.e. there do not exist constants $C, t_0 > 0$ so that $M(t) = Ct^2$ for all $t < t_0$). Let $\ell_M$ be the Orlicz space equipped with either the Luxemburg or the Orlicz norm and $F \subset \ell_M$ be a subspace of finite codimension. If $F$ contains at least one basis vector and $F$ is 1-complemented in $\ell_M$ then $F$ is spanned by a family of disjointly supported vectors.

In the present paper we prove a much stronger result – we eliminate the assumption that the subspace should be of finite codimension. Namely we show that when $M$ is a sufficiently smooth Orlicz function which satisfies condition $\Delta_2$ and is sufficiently different from the square function, then every 1-complemented subspace of the real Orlicz space $\ell_M$ is spanned by a family of mutually disjoint vectors and every norm one projection in $\ell_M$ is an averaging projection (see Theorem 1.3 and Corollary 1.4). This result is valid in Orlicz spaces equipped with either the Luxemburg or the Orlicz norm.

Our method of proof is different from that of [14], it relies on new results characterizing averaging projections through properties related to and generalizing disjointness preserving operators [17].

Recently, Jamison, Kamińska and Lewicki [3] obtained (using different techniques) a generalization of Theorem 1.1 in another direction – they characterized 1-complemented subspaces of finite codimension in sufficiently smooth Musielak-Orlicz sequence spaces, whose Orlicz function is sufficiently different from the square function.

We follow standard definitions and notations as may be found in [7, 9].
2. Preliminary definitions

Orlicz spaces are one of the most natural generalizations of classical spaces $L_p$. They were first considered by Orlicz in 1930s. Since then they were extensively studied by many authors, see, for example the monographs [7, 18, 3]. Below we recall the basic definitions and facts about Orlicz spaces that will be important for the present paper.

**Definition 2.1.** We say that a function $M : \mathbb{R} \to [0, \infty)$ is an Orlicz function if $M$ is even, continuous, convex, $M(0) = 0$, $M(1) = 1$, $\lim_{u \to 0} M(u)/u = 0$ and $\lim_{u \to \infty} M(u)/u = \infty$.

Note that since the Orlicz function $M$ is convex, it has the right derivative $M'$. Let $q$ be the right inverse of $M'$. Then we call $M^*(v) = \int_{0}^{[v]} q(s)ds$ the complementary function of $M$. Function $M^*$ is also an Orlicz function.

**Definition 2.2.** We say that the Orlicz function $M$ satisfies the $\Delta_2$ condition near zero ($M \in \Delta_2$) if there exist constants $k > 0$ and $u_0 \geq 0$ such that for all $u$ with $|u| \leq u_0$

$$M(2u) \leq kM(u).$$

Note that $M \in \Delta_2$ does not imply that $M^* \in \Delta_2$.

The Orlicz function $M$ generates the modular defined for scalar sequences $x = (x_j)_{j \in \mathbb{N}}$ by:

$$\rho_M(x) = \sum_{j=1}^{\infty} M(x_j).$$

The Orlicz sequence space $\ell_M$ is the space of sequences $x$ such that there exists $\lambda > 0$ with $\rho_M(\lambda x) < \infty$. If $M \in \Delta_2$ then $\ell_M = \{x : \rho_M(\lambda x) < \infty \text{ for all } \lambda \in \mathbb{R}\}$. The Orlicz sequence space $\ell_M$ is usually equipped with one of the two following equivalent norms:

1. the **Luxemburg norm** defined by:
   $$\|x\|_M = \inf \{\lambda : \rho_M\left(\frac{x}{\lambda}\right) \leq 1\},$$
2. the **Orlicz norm** defined by:
   $$\|x\|_{O_M} = \sup\{\sum_{j=1}^{\infty} x_j y_j : \rho_M^*(y) \leq 1\}.$$

If $M \in \Delta_2$ then these norms are dual to each other in the following sense:

$$\left(\ell_M, \| \cdot \|_M\right)^* = \left(\ell_{M^*}, \| \cdot \|_{O_{M^*}}\right),$$

and

$$\left(\ell_M, \| \cdot \|_{O_M}\right)^* = \left(\ell_{M^*}, \| \cdot \|_{M^*}\right).$$

We say that two Orlicz functions $M_1$ and $M_2$ are **equivalent** if there exist $u_0 > 0$, $k, l > 0$ such that for all $u$ with $|u| \leq u_0$

$$M_2(ku) \leq M_1(u) \leq M(lu).$$
This condition is of importance since Orlicz spaces $\ell_{M_1}, \ell_{M_2}$ are isomorphic if and only if
the Orlicz functions $M_1, M_2$ are equivalent. We note that if an Orlicz function $M$ satisfies
the condition $\Delta_2$ near zero then every Orlicz function $M_1$ equivalent to $M$ also satisfies the
condition $\Delta_2$ near zero.

Krasnoselskii and Rutickii proved the following characterization of the $\Delta_2$-condition in
terms of the right derivative $M'$ of $M$.

**Proposition 2.3.** [7, Theorem 4.1] A necessary and sufficient condition that the Orlicz
function $M(u)$ satisfy the $\Delta_2$-condition near zero is that there exist constants $\alpha$ and $u_0 \geq 0$
such that, for $0 \leq u \leq u_0$

$$uM'(u) < \alpha,$$

(2.3)

where $M'$ denotes the right derivative of $M$.

Moreover, if (2.3) is satisfied then $M(2u) \leq 2^\alpha M(u)$ for $0 \leq u \leq u_0/2$.

In [15] we introduced another condition which on one hand is very similar to (2.3), but on
the other hand is in its nature of “smoothness type”, as we explain below.

**Definition 2.4.** Assume that the Orlicz function $M$ is twice differentiable and that $M$
satisfies the $\Delta_2$–condition near zero. We say that $M$ satisfies condition $\Delta_{2+}$ near zero if
there exist constants $\beta > 0$ and $u_0 \geq 0$ such that for all $u \leq u_0$

$$uM''(u) < \beta.$$

(2.4)

Condition $\Delta_{2+}$ is of “smoothness type” in the following sense:

(i) for every function $M$ which satisfies condition $\Delta_2$ there exists an equivalent Orlicz
function $M_1$ which does satisfy $\Delta_{2+}$; However, we do not know whether for every $\varepsilon > 0$
it is possible to choose $M_1$ so that it is $(1 + \varepsilon)$–equivalent with $M$,

(ii) for every Orlicz function $M$ which satisfies $\Delta_{2+}$ there exists an equivalent (even up to
an arbitrary $\varepsilon > 0$) Orlicz function $M_1$ which does not satisfy $\Delta_{2+}$.

We say that a Banach space $X$ is smooth if every element $x \in X$ has a unique norming
functional $x^* \in X^*$, i.e. the functional with the property that $\|x^*\|_{X^*} = \|x\|_X = x^*(x)$ is
determined uniquely for every $x \in X$.

If $M \in \Delta_2$ then an Orlicz space $\ell_M$ is smooth whenever $M$ is differentiable everywhere.

It is well known (see e.g. [3]) that any Orlicz function $M$ can be “smoothed out”, that is
for any $M$ there exists an equivalent Orlicz function $M_1$ such that $M_1$ is twice differentiable
everywhere, $M_1''$ is continuous on $\mathbb{R}$ and $M_1''(u) > 0$ for all $u > 0$. We recall here that a class
of functions whose second derivative exists and is continuous on $\mathbb{R}$ is denoted by $C^2$. Thus
$M_1$ above belongs to $C^2$. Moreover, given any $\varepsilon > 0$ it is possible to choose $M_1$ so that $\ell_M$
and $\ell_{M_1}$ are $(1 + \varepsilon)$–isomorphic to each other [3].
Maleev and Troyanski [10] considered a stronger notion of smoothness in Orlicz spaces which guarantees the differentiability of the norm. We recall the relevant definitions and results.

**Definition 2.5.** [1] (cf. [9, p. 143]) To every Orlicz function $M$ we associate the following Matuszewska-Orlicz index:

$$
\alpha_0 M = \sup \{ p : \sup \{ \frac{M(\lambda t)}{t^p M(\lambda)} : \lambda, t \in (0, 1] \} < \infty \}.
$$

**Definition 2.6.** [10] We say that an Orlicz function $M$ belongs to the class $AC^k$ at zero if

(i) $\alpha_0 M > k$,

(ii) $M^{(k)}$ is absolutely continuous in every finite interval,

(iii) $t^{k+1} |M^{(k+1)}(t)| \leq c M(ct)$ a.e. in $[0, \infty)$ for some $c > 0$.

**Definition 2.7.** Let $X, Y$ be Banach spaces. The function $\varphi : X \to Y$ is said to be $k$–times differentiable at $\varphi \in X$ if for every $j$, $1 \leq j \leq k$, there exists a continuous symmetric $j$–linear form $T_j^f : X \times \cdots \times X = X^{(j)} \to Y$ so that:

$$
\varphi(f + \alpha g) = \varphi(f) + \sum_{j=1}^k \alpha^j T_j^f(y, \ldots, y) + \sigma_f(||\alpha||^k)
$$

uniformly on $g$ from the unit sphere $S(X)$ of $X$.

For an open set $V \subset X$, $\varphi \in F^k(V, Y)$ means $\varphi$ is $k$–times differentiable at every point of $V$. If (2.5) is fulfilled uniformly on $f$ over a set $W \subset V$ we shall say that $\varphi$ is $k$–times uniformly differentiable over $W$ and shall write $\varphi \in UF^k(W, Y)$. We say that $X$ is $UF^k$–smooth if the norm in $X$ belongs to $UF^k(S(X), \mathbb{R})$.

Maleev and Troyanski proved the following results about the uniform smoothness of Orlicz sequence spaces $\ell_M$:

**Theorem 2.8.** [10, Theorem 6] Let $M$ be an Orlicz function satisfying condition $\Delta_2$ at zero and such that $M \in AC^k$ at zero. Then $\ell_M$ equipped with the Luxemburg norm is $UF^k$-smooth.

**Theorem 2.9.** [10, Corollary 10] Let $M$ be an Orlicz function satisfying condition $\Delta_2$ at zero. Then for every $k \in \mathbb{N}$ such that $k < \alpha_0 M$ there exists an Orlicz function $\widetilde{M}$ equivalent to $M$ at zero so that $\ell_{\widetilde{M}}$ (with the Luxemburg norm) is $UF^k$-smooth. (In particular $\ell_{\widetilde{M}}$ is isomorphic to $\ell_M$.)

We do not know whether in Theorem 2.9 it is possible for any $\varepsilon > 0$ to select $\widetilde{M}$ so that $M$ and $\widetilde{M}$ are $(1 + \varepsilon)$–equivalent.

Next we recall that a Banach lattice $X$ is called strictly monotone if $\|x + y\| > \|x\|$ for all $x, y \geq 0, y \neq 0$, in $X$.

An Orlicz space $\ell_M$ with either the Luxemburg or the Orlicz norm is strictly monotone whenever $M$ is strictly increasing on $[0, \infty)$.
3. Tools

In this section we gather our main tools – facts about contractive projections and about disjointness in Orlicz spaces.

We will say that a projection $P$ on a purely atomic Banach lattice $X$ is an *averaging projection* if there exist mutually disjoint elements $\{u_j\}_{j \in J}$ in $X$ and functionals $\{u_j^*\}_{j \in J}$ in $X^*$ so that $u_j^*(u_k) = 0$ if $j \neq k$, $u_j^*(u_j) = 1$ for all $j \in J$ and for each $f \in X$

$$Pf = \sum_{j \in J} u_j^*(f)u_j.$$ 

First we recall two abstract conditions that we introduced in [17] in our study of averaging projections in purely atomic Banach lattices.

**Definition 3.1.** [17] Let $X$ be a Banach lattice and $P : X \to X$ be a linear operator on $X$. We say that the operator $P$ is

1. **semi band preserving** if and only if for all $f, g \in X$,

$$\text{supp}(Pf) \cap \text{supp}(g) = \emptyset \implies \text{supp}(Pf) \cap \text{supp}(Pg) = \emptyset.$$ (3.1)

2. **semi containment preserving** if and only if for all $f, g \in X$,

$$\text{supp} g \subset \text{supp} Pf \implies \text{supp} Pg \subset \text{supp} Pf.$$ (3.2)

In the above statement all set relations are considered modulo sets of measure zero.

It is clear that all averaging projections are both semi band preserving and semi containment preserving. In [17] we proved that in fact in “nice” purely atomic Banach spaces either of semi band or semi containment preservation characterizes averaging projections among contractive projections. More precisely, we have:

**Theorem 3.2.** [17] Let $X$ be a purely atomic strictly monotone Banach lattice and let $P : X \to X$ be a norm one projection which is semi band preserving or semi containment preserving. Then $P$ is an averaging projection.

This theorem will be very useful for our considerations since in [15] we obtained conditions which partially describe disjointness and containment of supports of elements in Orlicz spaces. These conditions will enable us to verify that contractive projections in Orlicz sequence spaces are semi band preserving or semi containment preserving.

We note here that all theorems in [15] were formulated and proved for Orlicz function spaces $L_M$, where $M$ is an Orlicz function satisfying conditions $\Delta_2$ and $\Delta_{2+}$ near infinity. However to adapt to the case of Orlicz sequence spaces $\ell_M$, where $M$ is an Orlicz function satisfying conditions $\Delta_2$ and $\Delta_{2+}$ near zero, the proofs require only very minor changes, if any. Thus in the following when we refer to the statements from [15] we will formulate them using $\ell_M$ instead of $L_M$, which is more appropriate for the present paper.
We stress that theorems in [15] are proven for Orlicz spaces equipped with the Luxemburg norm, and the analogs of most of the results from [15] are false in Orlicz spaces equipped with the Orlicz norm.

**Proposition 3.3.** [15, Proposition 3.1] Assume that $M$ is an Orlicz function which satisfies condition $\Delta_{2+}$ and such that $M''(0) = 0$ and $M''(t) > 0$ for all $t > 0$. Let $f, g \in \ell_M$ and $N(\alpha) = \|f + \alpha g\|_M$. Then

(a) If $f, g$ have disjoint supports, $\mu(\text{supp } g) < \infty$ and $g$ is bounded then $N'(0) = 0$ and $N''(\alpha) \to 0$ as $\alpha \to 0$ along a subset of $[0, 1]$ of full measure.

(b) If $N'(0) = 0$ and $N''(\alpha) \to 0$ as $\alpha \to 0$ along a subset of $[0, 1]$ of full measure then $f, g$ have disjoint supports.

**Proposition 3.4.** [15, Proposition 4.1] Assume that $M$ is an Orlicz function which satisfies condition $\Delta_{2+}$ near zero and such that $M''$ is a continuous function on $(0, \infty)$ with $\lim_{t \to 0} M''(t) = \infty$. Let $f, g \in \ell_M$ with $f, g \neq 0$ and $N(\alpha) = \|f + \alpha g\|_M$. Then

(a) If $\mu(\text{supp } g \setminus \text{supp } f) > 0$ then $N''(\alpha) \to \infty$, as $\alpha \to 0$ along a subset of $[0, 1]$ of full measure.

(b) If $g$ is simple and $\mu(\text{supp } g \setminus \text{supp } f) = 0$ then there exists a subset $E$ of $[0, 1]$ of full measure and $C > 0$ such that for all $\alpha \in E$

$$N''(\alpha) \leq C.$$

**Remark 3.5.** A careful reader may have noticed that Proposition 3.4 above appears slightly stronger than [15, Proposition 4.1]. However the differences between these two statements are minimal and result from a slight simplification of the proof of [15, Proposition 4.1] in the case of sequence Orlicz spaces. Also the formulation of Proposition 3.4 clarifies a slight ambiguity of the statement of [15, Proposition 4.1]. We leave the details, which are easy but require cumbersome notation, to the interested reader.

Finally we recall a result from [13] which describes the form of two dimensional 1-complemented subspaces of Orlicz sequence spaces, when the two spanning elements have disjoint supports. (We say that a subspace is 1-complemented if it is the range of a projection $P$ with $\|P\| = 1$.) This result will allow us to give a very detailed description of 1-complemented subspaces of any dimension of Orlicz sequence spaces.

**Theorem 3.6.** [13, Theorem 6.1] Let $M$ be an Orlicz function satisfying condition $\Delta_2$ and $\ell_M$ be a (real or complex) Orlicz sequence space equipped with either the Luxemburg or the Orlicz norm and let $x, y \in \ell_M$, be disjoint norm one elements such that $\text{span}\{x, y\}$ is 1-complemented in $\ell_M$. Then one of three possibilities holds:

1. $\text{card}(\text{supp } x) < \infty$ and $|x_i| = |x_j|$ for all $i, j \in \text{supp } x$; or
2. there exists $p, 1 \leq p \leq \infty$, such that $M(t) = C t^p$ for all $t \leq \|x\|_\infty$; or
(3) there exists \( p, 1 \leq p \leq \infty \), and constants \( C_1, C_2, \gamma \geq 0 \) such that \( C_2 t^p \leq M(t) \leq C_1 t^p \) for all \( t \leq \|x\|_{\infty} \) and such that, for all \( j \in \text{supp} \, x \),
\[
|x_j| = \gamma^{k(j)} \cdot \|x\|_{\infty}
\]
for some \( k(j) \in \mathbb{Z} \).

In particular, it follows from Theorems 3.6 that in “most” Orlicz spaces the only 1-complemented disjointly supported subspaces of any dimension are those spanned by a block basis with constant coefficients of some permutation of the original basis.

4. Main results

We start from a lemma which will allow us to apply Propositions 3.3 and 3.4 to study whether contractive projections in Orlicz sequence spaces are semi band preserving or semi containment preserving.

Lemma 4.1. Suppose that \( \varphi, \psi : \mathbb{R} \to [0, \infty) \) are convex functions, differentiable everywhere and such that \( \varphi(0) = \psi(0), \varphi(\alpha) \leq \psi(\alpha) \) for all \( \alpha \in \mathbb{R} \).

(i) Then \( \psi'(0) = \varphi'(0) \).

(ii) If \( \varphi''(0), \psi''(0) \) exist and \( \psi''(0) = 0 \), then \( \varphi''(0) = 0 \).

(iii) Suppose that \( \varphi' \) and \( \psi' \) are absolutely continuous on \([0, 1]\). Then, if \( \varphi''(\alpha) \to \infty \) as \( \alpha \to 0 \) along a subset of \([0, 1]\) of full measure, then for every \( C > 0 \)
\[
\mu(\{\alpha \in [0, 1] : \varphi''(\alpha) \text{ exists and } \varphi''(\alpha) \leq C\}) < 1.
\]

Proof. To prove (i) observe that, since \( \varphi(0) = \psi(0) \), we have for all \( \alpha \in \mathbb{R} \)
\[
\varphi(\alpha) - \varphi(0) \leq \psi(\alpha) - \psi(0).
\]
Thus for \( \alpha > 0 \)
\[
\frac{\varphi(\alpha) - \varphi(0)}{\alpha} \leq \frac{\psi(\alpha) - \psi(0)}{\alpha}, \quad (4.1)
\]
and for \( \alpha < 0 \)
\[
\frac{\varphi(\alpha) - \varphi(0)}{\alpha} \geq \frac{\psi(\alpha) - \psi(0)}{\alpha}, \quad (4.2)
\]
Since \( \varphi'(0) \) and \( \psi'(0) \) exist we have, by (1.1),
\[
\varphi'(0) = \lim_{\alpha \to 0^+} \frac{\varphi(\alpha) - \varphi(0)}{\alpha} \leq \lim_{\alpha \to 0^+} \frac{\psi(\alpha) - \psi(0)}{\alpha} = \psi'(0),
\]
and, by (1.2),
\[
\varphi'(0) = \lim_{\alpha \to 0^-} \frac{\varphi(\alpha) - \varphi(0)}{\alpha} \geq \lim_{\alpha \to 0^-} \frac{\psi(\alpha) - \psi(0)}{\alpha} = \psi'(0),
\]
Thus \( \varphi'(0) = \psi'(0) \) and (i) is proved.

To prove (ii), we consider the set \( A = \{\alpha > 0 : \varphi'(\alpha) = \psi'(\alpha)\} \).
If \( \inf \{ \alpha \in A \} = 0 \), then there exists a sequence \( \{ \alpha_n \}_{n=1}^\infty \subset A \) so that \( \lim_{n \to \infty} \alpha_n = 0 \). Since \( \varphi''(0) \) and \( \psi''(0) \) exist, and by \((i)\), we obtain:

\[
\varphi''(0) = \lim_{n \to \infty} \frac{\varphi'(\alpha_n) - \varphi'(0)}{\alpha_n} = \lim_{n \to \infty} \frac{\psi'(\alpha_n) - \psi'(0)}{\alpha_n} = \psi''(0) = 0,
\]

So \((ii)\) is proved.

If \( \inf \{ \alpha \in A \} > 0 \) (this includes the case that \( A = \emptyset \) and then we say \( \inf \{ \alpha \in A \} = \infty > 0 \)), then there exists \( \varepsilon, 0 < \varepsilon < \inf \{ \alpha \in A \} \) so that \( \varphi' (\alpha) \neq \psi'(\alpha) \) for all \( \alpha \in (0, \varepsilon) \).

Let \( h = \psi - \varphi \). Then \( h(\alpha) \geq 0 \) for all \( \alpha \in \mathbb{R} \), \( h(0) = 0 \) and \( h'(\alpha) \neq 0 \) for all \( \alpha \in (0, \varepsilon) \).

Since \( h' \) satisfies the Darboux property, we get either:

\[
(4.3) \quad h'(\alpha) > 0 \quad \text{for all} \quad \alpha \in (0, \varepsilon),
\]
or

\[
(4.4) \quad h'(\alpha) < 0 \quad \text{for all} \quad \alpha \in (0, \varepsilon).
\]

But \( h(0) = 0 \) and \( h(\varepsilon) \geq 0 \), so by the Mean Value Theorem there exists \( \alpha_0 \in (0, \varepsilon) \) so that

\[
h'(\alpha_0) = \frac{h(\varepsilon)}{\varepsilon} \geq 0.
\]

Thus \((4.3)\) has to hold. This implies that, since \( \psi''(0) \) exists, \( \psi''(0) \geq 0 \). This means:

\[
0 = \psi''(0) \geq \varphi''(0).
\]

Since \( \varphi \) is convex, we also get

\[
\varphi''(0) \geq 0.
\]

Thus \( \varphi''(0) = 0 \) and \((ii)\) is proved.

To prove \((iii)\) we denote by \( E_1 = \{ \alpha \in [0, 1] : \varphi''(\alpha) \text{ exists} \} \).

Since \( \varphi \) and \( \psi \) are convex, \( \mu(E_1) = 1 \). Without loss of generality we can also assume that

\[
(4.5) \quad \varphi''(\alpha) \to \infty \quad \text{as} \quad \alpha \to 0 \quad \text{and} \quad \alpha \in E_1.
\]

Suppose, for contradiction, that there exists \( C > 0 \) so that the set

\[
E_2 = \{ \alpha \in [0, 1] : \psi''(\alpha) \text{ exists and} \ \psi''(\alpha) \leq C \}
\]

has full measure. Let \( E = E_1 \cap E_2 \). By \((4.3)\) there exists \( \varepsilon > 0 \) so that:

\[
(4.6) \quad \varphi''(\alpha) > C \quad \text{for every} \quad \alpha \in E \cap (0, \varepsilon).
\]

Now consider the set \( A = \{ \alpha > 0 : \varphi'(\alpha) = \psi'(\alpha) \} \) similarly as we did in the proof of \((ii)\). If \( \inf \{ \alpha \in A \} = 0 \), then there exist \( \alpha_1, \alpha_2 \in (0, \varepsilon) \) so that \( \alpha_1 \neq \alpha_2 \) and

\[
(4.7) \quad \varphi'(\alpha_1) = \psi'(\alpha_1), \quad \varphi'(\alpha_2) = \psi'(\alpha_2).
\]
But, since \( \varphi' \) is absolutely continuous on \([0, 1]\), and by (4.6), we also have:
\[
\varphi'(\alpha_1) - \varphi'(\alpha_2) = \int_{\alpha_1}^{\alpha_2} \varphi''(\alpha)d\alpha = \int_{[\alpha_1, \alpha_2] \cap E} \varphi''(\alpha)d\alpha > C(\alpha_1 - \alpha_2).
\]

On the other hand, by the absolute continuity of \( \psi' \) on \([0, 1]\) and the definition of \( E_2 \) we have:
\[
\psi'(\alpha_1) - \psi'(\alpha_2) = \int_{\alpha_1}^{\alpha_2} \psi''(\alpha)d\alpha = \int_{[\alpha_1, \alpha_2] \cap E} \psi''(\alpha)d\alpha \leq C(\alpha_1 - \alpha_2).
\]

This is a contradiction since (4.7) implies that \( \varphi'(\alpha_1) - \varphi'(\alpha_2) = \psi'(\alpha_1) - \psi'(\alpha_2). \)

Now let us consider the case that \( \inf \{\alpha \in A\} \neq 0 \), i.e. \( \inf \{\alpha \in A\} > 0 \) (this, as in \( \text{(ii)} \), includes the possibility that \( A = \emptyset \) in which case we say that \( \inf \{\alpha \in A\} = \infty \)). We showed in the proof of \( \text{(ii)} \) (cf. (4.3)) that in this case there exists \( \varepsilon_1 \), \( 0 < \varepsilon_1 < \inf \{\alpha \in A\} \), so that
\[
(4.8) \quad \psi'(\alpha) > \varphi'(\alpha) \quad \text{for all } \alpha \in (0, \varepsilon_1).
\]

By \( \text{(i)} \), \( \varphi'(0) = \psi'(0) \). Let \( \alpha_0 \in (0, \varepsilon) \cap (0, \varepsilon_1) \). Then, similarly as in the previous case, since \( \varphi' \) is absolutely continuous on \([0, 1]\), by (4.6), we obtain:
\[
\varphi'(\alpha_0) - \varphi'(0) = \int_0^{\alpha_0} \varphi''(\alpha)d\alpha = \int_{[0, \alpha_0] \cap E} \varphi''(\alpha)d\alpha > C\alpha_0.
\]

On the other hand, again by the absolute continuity of \( \psi' \) and the definition of \( E_2 \):
\[
\psi'(\alpha_0) - \psi'(0) = \int_0^{\alpha_0} \psi''(\alpha)d\alpha = \int_{[0, \alpha_0] \cap E_2} \psi''(\alpha)d\alpha \leq C\alpha_0.
\]

Thus
\[
\psi'(\alpha_0) < \varphi'(\alpha_0),
\]

which contradicts (4.8) and ends the proof of \( \text{(iii)} \).

We are now ready for our main results.

**Theorem 4.2.** Let \( M \) be an Orlicz function which satisfies condition \( \Delta_{2+} \) near zero and let \( \ell_M \) be the real Orlicz sequence space equipped with the Luxemburg norm. Suppose that \( P : \ell_M \to \ell_M \) is a contractive projection. Then the following hold:

1. If \( M \in AC^2 \), \( M''(0) = 0 \) and \( M''(t) > 0 \) for all \( t > 0 \), then \( P \) is semi band preserving;
2. If \( M \in AC^1 \) near zero, \( M'' \) is continuous on \((0, \infty)\) and \( \lim_{t \to 0} M''(t) = \infty \), then \( P \) is semi containment preserving.

**Proof.** Since bounded functions with finite supports are linearly dense in \( \ell_M \), to show that \( P \) is semi band preserving or semi containment preserving, respectively, it is enough to verify that (3.11) or (3.2), resp., are satisfied with the additional assumption that \( g \) is a bounded function and \( \mu(\text{supp } g) < \infty \).

For any functions \( f, g \in \ell_M \) we define
\[
\psi(\alpha) = \| Pf + \alpha g \|_M
\]
\[ \varphi(\alpha) = \| Pf + \alpha Pg \|_M \]

for all \( \alpha \in \mathbb{R} \). Then \( \varphi \) and \( \psi \) are convex functions and \( \psi(0) = \| Pf \| = \varphi(0) \). Moreover, by Theorem 2.8 in both cases (a) and (b), \( \varphi \) and \( \psi \) are differentiable everywhere. Since \( P \) is a contractive projection, we also get \( \varphi(\alpha) \leq \psi(\alpha) \) for all \( \alpha \in \mathbb{R} \).

Now to prove (a) assume that \( \mu(\text{supp } g) < \infty \) and \( \text{supp}(g) \cap \text{supp}(Pf) = \emptyset \). Since \( M \in AC^2 \), by Theorem 2.8, \( \varphi''(0) \) and \( \psi''(0) \) exist. By Proposition 3.3(a) we get \( \psi'(0) = 0 \) and \( \psi''(0) = 0 \). Hence by Lemma 4.1(i) and (ii), \( \varphi'(0) = 0 \) and \( \varphi''(0) = 0 \). Thus, by Proposition 3.3(b), we get that \( Pf \) and \( Pg \) have disjoint supports, which proves that \( P \) is semi band preserving.

To prove (b) assume, for contradiction, that there exist \( f, g \in \ell_M \) so that \( \mu(\text{supp } g) < \infty \), \( \text{supp}(g) \subseteq \text{supp}(Pf) \) and \( \text{supp}(Pg) \not\subseteq \text{supp}(Pf) \).

Note that since \( M \in AC^1 \), by Theorem 2.8, functions \( \varphi \) and \( \psi \) are differentiable everywhere, \( \varphi', \psi' \) are absolutely continuous and \( \varphi'', \psi'' \) exist almost everywhere. Further, by Proposition 3.3(b), there exists a subset \( E \) of \([0, 1]\) of full measure and \( C_0 > 0 \) such that for all \( \alpha \in E \)

\[ \psi''(\alpha) \leq C_0. \]

On the other hand, by Proposition 3.4(b), \( \psi''(\alpha) \to \infty \), as \( \alpha \to 0 \) along a subset of \([0, 1]\) of full measure. Hence, by Lemma 4.1(iii) for every \( C > 0 \)

\[ \mu(\{ \alpha \in [0, 1] : \psi''(\alpha) \text{ exists and } \psi''(\alpha) \leq C \}) < 1. \]

This contradicts (1.9) and ends the proof of part (b).

As a consequence we obtain the characterization of contractive projections in Orlicz sequence spaces.

**Theorem 4.3.** Suppose that \( M \) is an Orlicz function such that \( M \) satisfies condition \( \Delta_{2+} \) near zero and one of the following two conditions:

(i) \( M \in AC^2 \), \( M''(0) = 0 \) and \( M''(t) > 0 \) for all \( t > 0 \).

(ii) \( M \in AC^1 \) near zero, \( M'' \) is continuous on \((0, \infty)\) and \( \lim_{t \to 0} M''(t) = \infty \).

Let \( \ell_M \) be the real Orlicz sequence space equipped with the Luxembourg norm and let \( P \) be a contractive projection on \( \ell_M \). Then \( P \) is an averaging projection, i.e. there exist mutually disjoint elements \( \{ u_j \}_{j \in J} \) in \( \ell_M \) and functionals \( \{ u_j^* \}_{j \in J} \) in \((\ell_M)^*\) so that \( u_j^*(u_k) = 0 \) if \( j \neq k \), \( u_j^*(u_j) = 1 \) for all \( j \in J \) and for each \( f \in \ell_M \)

\[ Pf = \sum_{j \in J} u_j^*(f)u_j. \]

Moreover, one of the three possibilities holds:

1. \( \text{card}(\text{supp } u_j) < \infty \) for each \( j \in J \), and \( |(u_j)_k| = |(u_j)_l| \) for each \( k, l \in \text{supp}(u_j) \), \( j \in J \).

   (Here \( u_j = \sum_{k \in \text{supp } u_j} (u_j)_k e_k \); or

   2. \( \text{supp } u_j = \emptyset \) for all \( j \in J \).

   (Here \( u_j = 0 \); or

   3. \( \text{supp } u_j \neq \emptyset \) for all \( j \in J \).
(2) there exist \( p, 1 < p < \infty \), and \( C \in \mathbb{R} \), so that \( M(t) = C t^p \) for all \( t \leq \sup_{j \in J} \| u_j \|_\infty (\leq \infty) \); or

(3) there exist \( p, 1 < p < \infty \), and constants \( C_1, C_2, \gamma > 0 \), so that \( C_2 t^p \leq M(t) \leq C_1 t^p \) for all \( t \leq \sup_{j \in J} \| u_j \|_\infty (\leq \infty) \), \( \| u_j \|_\infty < \infty \) for all \( j \in J \), and

\[
| (u_j)_k | \in \{ \gamma^m : \| u_j \|_\infty : m \in \mathbb{Z} \}
\]

for all \( j \in J \) and \( k \in \text{supp}(u_j) \).

Proof. Note first that either condition (i) or (ii) implies that \( \ell_M \) is smooth and that \( M' \) is a strictly increasing function on \((0, \infty)\). Thus \( M \) is also strictly increasing on \((0, \infty)\) and \( \ell_M \) is strictly monotone. Hence the fact that \( P \) is an averaging projection follows immediately from Corollary 3.2 and Proposition 4.2.

The moreover part follows directly from [13, Theorem 6.1] (see Theorem 3.6). Indeed, since the elements \( \{ u_j \}_{j \in J} \) are mutually disjoint, for any \( j_1, j_2 \in J \) and any \( f \in \ell_M \) we have

\[
\| u_{j_1}^* (f) u_{j_1} + u_{j_2}^* (f) u_{j_2} \| \leq \| \sum_{j \in J} u_j^* (f) u_j \| = \| P f \| \leq \| f \|.
\]

Thus the projection \( Q : \ell_M \rightarrow \text{span}\{ u_{j_1}, u_{j_2} \} \) defined by \( Q f = u_{j_1}^* (f) u_{j_1} + u_{j_2}^* (f) u_{j_2} \), has \( \| Q \| = 1 \). Thus, by Theorem 3.6, conditions (1)-(3) in the statement of Theorem 4.3 are satisfied.

By duality we also obtain the description of contractive projections in real Orlicz sequence spaces equipped with the Orlicz norm.

**Corollary 4.4.** Suppose that \( M \) is an Orlicz function such that \( M \) satisfies condition \( \Delta_2 \) near zero and \( M^* \) satisfies condition \( \Delta_{2+} \) near zero and one of the following two conditions:

(i*) \( M^* \in AC^2 \) near zero, \( M'' \) is continuous on \((0, \infty)\), \( M''(t) > 0 \) for all \( t > 0 \) and \( \lim_{t \to 0} M''(t) = \infty \).

(ii*) \( M^* \in AC^1 \) near zero, \( M \in C^2 \), \( M''(t) > 0 \) for all \( t > 0 \) and \( M''(0) = 0 \)

Let \( \ell_M \) be the real Orlicz sequence space equipped with the Orlicz norm and let \( P \) be a contractive projection on \( \ell_M \). Then \( P \) has the form described in Theorem 4.3.

Proof. This follows from Theorem 4.3 by duality. Indeed, since \( M \in \Delta_2 \), by (2.2) we have \((\ell_M, \| \cdot \|_M^* )^* = (\ell_{M^*}, \| \cdot \|_{M^*}) \) and the dual projection \( P^* \) is contractive in \( \ell_{M^*} \) equipped with the Luxemburg norm. Further, either of the conditions (i*) or (ii*) implies that \( M^* \) is smooth, so the only thing that needs to be verified is that condition (i*) implies that \( M^* \) satisfies condition (i) and condition (ii*) implies that \( M^* \) satisfies condition (ii) from Theorem 4.3.

For that, note that by the definition of the complementary function \( M^* \) and since in either case (i*) or (ii*), \( M''(t) > 0 \) for \( t > 0 \), we have for all \( t > 0 \)

\[
(M^*)''(t) = \frac{1}{M''((M^*)'(t))}.
\]
Since $M''$ and $(M^*)'$ are both continuous on $(0, \infty)$ in either case $(i^*)$ or $(ii^*)$, we conclude that also $(M^*)''$ is continuous on $(0, \infty)$ and $(M^*)''(t) > 0$ for all $t > 0$. Moreover, since $\lim_{t \to 0} (M^*)'(t) = (M^*)'(0) = 0$, we have in case $(i^*)$:

$$\lim_{t \to 0} (M^*)'''(t) = \lim_{t \to 0} \frac{1}{M''((M^*)'(t))} = \lim_{s \to 0} \frac{1}{M''(s)} = 0.$$  

It is not difficult to check that this implies that $(M^*)'''(0) = 0$. Therefore condition $(i)$ is implied by $(i^*)$.

Similarly, in case $(ii^*)$ we have:

$$\lim_{t \to 0} (M^*)'''(t) = \lim_{t \to 0} \frac{1}{M''((M^*)'(t))} = \lim_{s \to 0} \frac{1}{M''(s)} = \infty.$$  

So condition $(ii)$ is implied by $(ii^*)$.

Hence, by Theorem 4.3, in either case $(i^*)$ or $(ii^*)$, $P^*$, and thus also $P$, have form (4.10) and the conditions (1) – (3) from Theorem 4.3 hold. \qed

Remark 4.5. We do not know whether the assumption about smoothness of $M$ is necessary for Theorem 4.3 and Corollary 4.4 to hold. We suspect that, similarly as in the complex case, smoothness of $M$ should not be necessary.

However it is clear that some assumption about a behavior of $M''$ near zero is necessary. Indeed in [13, Example 3] we showed that if $a \in (\sqrt{2/3}, 1)$ and

$$M_a(t) = \begin{cases} 
  t^2 & \text{if } 0 \leq t \leq a, \\
  (1 + a)t - a & \text{if } a \leq t \leq 1,
\end{cases}$$

then the real or complex 4-dimensional Orlicz space $\ell^4_{M_a}$ equipped with either the Luxemburg or the Orlicz norm contains a 2-dimensional 1-complemented isometric copy of $\ell^2_2$ which cannot be spanned by a family of disjoint vectors from $\ell^4_{M_a}$. It is not difficult to adjust this example so that if $a$ is any positive number then the real or complex Orlicz space $\ell_{M_a}$ (of infinite dimension) contains a 2-dimensional 1-complemented isometric copy of $\ell^2_2$ which cannot be spanned by a family of disjoint vectors from $\ell_{M_a}$.

It would be interesting to characterize what condition on $M$ is equivalent to the fact that $\ell_M$ (complex or real) does not contain a 2-dimensional 1-complemented isometric copy of $\ell^2_2$ (which cannot be spanned by a family of disjoint vectors from $\ell_M$). Either of the conditions $(i)$, $(ii)$, $(i^*)$ or $(ii^*)$ is clearly sufficient, but they all involve smoothness. We suspect that the right condition is that for all $a > 0$ the function $M(t)/t^2$ is not constant on the interval $(0, a)$.

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