VARIATION OF SINGULAR KÄHLER-EINSTEIN METRICS

by

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Introduction

Let $p : X \to Y$ be a proper, surjective holomorphic map with connected fibers between Kähler manifolds. It is a central question in complex geometry to relate the geometry of $X$ to the one of $Y$ and the fibers $X_y$ of $p$. An important instance of such a situation is when the (log)-canonical bundle of the generic fiber of $p$ has certain positivity properties. In this case, one can endow $X_y$ with a (singular) Kähler-Einstein metric and study the the geometry of $X$ induced by the
properties of the resulting family of metrics. This is the common theme for the three main results of this paper and their corollaries listed below, which are explained in more details further in the introduction.

Main results. — The setting of our results is the following. The map \( p : X \to Y \) is a proper, holomorphic fibration between two Kähler manifolds, and let \( B = \sum b_i B_i \) be a \( \mathbb{Q} \)-divisor on \( X \) with generically snc support and coefficients \( b_i \in (0, 1) \). Let \( Y^\circ \subset Y \) be the locus where \( p \) is smooth, \( B|_{Y^\circ} \) has snc support and set \( X^\circ := p^{-1}(Y^\circ) \).

**Theorem A.** — In the above set-up, assume that the \( \mathbb{Q} \)-line bundle \( K_X + B \) is \( p \)-big. Then, the relative Kähler-Einstein metrics \( (\omega_{KE,y})_{y \in Y^\circ} \) in the sense of Definition 1.1 induce a metric \( e^{-\phi_{KE}} \) on \( K_{X^\circ/Y^\circ} + B|_{X^\circ} \) which is positively curved and extends canonically across \( X \setminus X^\circ \) to a positively curved metric on \( K_{X/Y} + B \).

Theorem A is a special case of the more general Theorem 1.3. That result deals precisely with the setting that is needed to analyze the more general case when \( K_{X^y} + B_y \) has positive Kodaira dimension, in which case one has to deal with the relative version of the so-called canonical metrics introduced by Song and Tian [ST12] and generalized by Eyssidieux, Guedj and Zeriahi [EGZ16]. These metrics live on the base \( Z' \) of a birational model \( q' : X' \to Z' \) of the relative Iitaka fibration \( q : X \to Z \) over \( Y \), and can be related to the Narasimhan-Simha metrics on the direct image sheaf \( q'_*(m(K_{X/Z} + B')) \), cf. Section 1.1.2 for more details, and Section 1.1.4 for the proof.

**Corollary B.** — With the notation above, assume that for \( y \) generic, \( \kappa(K_{X^y} + B_y) > 0 \). Let \( f : X \to Z \) be the relative Iitaka fibration of \( K_{X/Y} + B \), and let \( f' : X' \to Z' \) a birational model of \( f \) such that \( X' \) and \( Z' \) are smooth.

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Z' \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Z \\
\downarrow & & \\
Y & \xleftarrow{p} & Y \\
\end{array}
\]

Let \( \omega_{can,y} \) be the canonical metric on \( Z'_y \) of the pair \( (X'_y, B'_y) \); it induces a current \( \omega_{can} \) over the smooth locus of \( Z' \to Y \).

Then, \( \omega_{can} \) is positive and extends canonically to a closed positive current on \( Z' \).

In the case where the fibers have zero log Kodaira dimension, we prove two types of theorem. The first one deals with the specific case where \( K_{X/Y} + B \) is numerically relatively trivial with flat direct image, cf. also Theorem 2.2.

**Theorem C.** — Let \( p : (X, B) \to Y \) be a map as above, and let \( \omega \) be a fixed Kähler metric on \( X \). Assume that the following conditions are satisfied.

\((0.0.1)\) For \( y \in Y^\circ \), the \( \mathbb{Q} \)-line bundle \( K_{X^y} + B_y \) is numerically trivial.

\((0.0.2)\) For some \( m \) large enough, the line bundle \( p_*(mK_{X^y/Y^\circ} + mB) \) is Hermitian flat with respect to the Narasimhan-Simha metric \( h \) on \( Y^\circ \), cf. (2.7).

Then we can construct a \((1, 1)\)-current \( \omega_{KE} \) such that the restriction \( \omega_y \) of \( \omega_{KE} \) to \( X_y \) is a smooth representative of \( \{\omega\}|_{X_y} \) and solves \( \text{Ric}(\omega_y) = [B]_y \). Moreover, we have

\((0.0.3)\) \( \omega_{KE} \) is a positive current on \( X^\circ \) and it extends canonically to a closed positive current \( \omega_{KE} \in \{\omega\} \) on \( X \).
(0.0.4) The fibration \( (X, B) \to Y \) is locally trivial over \( Y^0 \). Moreover, if \( p \) is smooth in codimension one and \( \text{codim}_X (B \setminus X^0) > 1 \), then \( p \) is locally trivial over the whole \( Y \).

The result above has many geometric applications, like for instance a Kahler version of a theorem of Ambro [Amb05], cf Corollary 2.3 and its proof given in page 45. We can also deduce the following positivity property of direct images of pluri-log canonical bundles, which to our knowledge has no algebraic proof yet, cf page 44 for a proof.

**Corollary D.** — Let \( p : (X, B) \to Y \) be a fibration between two compact Kahler manifolds and for \( y \in Y \) in the complement of an algebraic subset we have \( c_1 (K^y_X + B|_X^y) = 0 \). Assume moreover that the logarithmic Kodaira-Spencer map

\[
T_Y \to \mathcal{R}^1 p_* (T_{X/Y}^1 (B))
\]

is generically injective. Then the bundle \( p_* (m(K^y_X + B))^{**} \) is big.

Our second theorem in this set-up (i.e. \( \kappa(K^y_X + B_y) = 0 \)) is a regularity result for the relative Kahler-Einstein metric if no further assumptions on the basepoints of \( K^y_X + B_y \) or the flatness of the direct image of some power of \( K^y_{X/Y} + B \) are imposed. It is proved on page 60.

**Theorem E.** — In the above framework, let \( \omega \) be a fixed Kahler metric on \( X \) and assume that for \( y \) generic the Kodaira dimension of \( K^y_X + B_y \) equals zero. Let \( E \) be an effective \( \mathbb{Q} \)-divisor such that \( K^y_X + B_y \sim_{\mathbb{Q}} E_y \). Then there exists a current \( \omega^{\text{KE}}_{} \) of (1,1)-type whose restriction \( \omega_y := \omega^{\text{KE}}_{} - E_y \) is a representative of \( \{ \omega \} \mid X^y \) and solves the equation \( \text{Ric} \, \omega_y = - [E_y] + [B_y] \).

In addition, the local potentials of \( \omega^{\text{KE}}_{} \) are Lipschitz on \( X^0 \setminus \text{Supp}(B + E) \).

**Previously known results.** — In the paragraph that follows, we recall the definitions of relative (singular) Kahler-Einstein metrics and collect some earlier results. Most of them are concerned with the smooth setting.

**Relative smooth KE metrics.** — Let us consider a proper holomorphic map \( p : X \to Y \) between Kahler manifolds of relative dimension \( n \) such that the general fiber \( X_y \) of \( p \) admits a Kahler-Einstein metric \( \omega_y \), that is,

\[
\text{Ric} \, \omega_y = \lambda \omega_y
\]

for some fixed constant \( \lambda \in \mathbb{R} \). If \( \lambda = 0 \), one can choose to fix a (relative) Kahler form \( \omega \) on \( X \) and impose that \( \omega_y \in \{ \omega \} \mid X_y \) for \( y \) generic. This setting serves as a ground for a wide array of questions and problems, and drew a lot of attention over the years, cf. [Tsui10], [ST12], [Sch08, Sch12], [Tos10], [Piu12], [Ber13], [HT15], [Cho15], [BS17] to cite only a few. In this paper, we will mostly focus on the case \( \lambda \leq 0 \), which can be reduced to the two cases \( \lambda = -1 \) or \( \lambda = 0 \). We will denote by \( Y^0 \subset Y \) the Zariski open subset of \( Y \) consisting of regular values of \( p \), and we set \( X^0 := p^{-1}(Y^0) \).

One would like to construct a global, canonical object on \( X^0 \) induced by \( \omega_y \) on each fiber. One possibility is to consider the family of relative volume forms \( (\omega^n_y)_{y \in Y^0} \), which naturally induce a metric \( h^{\text{KE}}_{} = e^{-\phi^{\text{KE}}_{}(y)} \) on the relative canonical bundle \( K^{X^0/Y^0} = K^{X^0} \otimes K_Y^{1} \). At this point already, the situation is very different whether \( \lambda = -1 \) or \( \lambda = 0 \).

• Case \( \lambda = -1 \).

In this case, \( \omega_y \) can be recovered from \( \omega^n_y \) thanks to (0.6), so that \( \omega^{\text{KE}}_{} := dd^c \phi^{\text{KE}}_{} \) is a canonical smooth (1,1)-form on \( X^0 \) that restricts to \( \omega_y \) on \( X_y \) for each \( y \in Y^0 \).

• Case \( \lambda = 0 \).

In that case, the curvature form \( dd^c \phi^{\text{KE}}_{} \) above restricts to zero on each fiber \( X_y \), \( y \in Y^0 \), and \( dd^c \phi^{\text{KE}}_{} \) is up to a constant the pull-back of the Weil-Petersson metric on \( Y^0 \); therefore it is
not really related to the equation (0.6) itself. Instead, on each fiber $X_y$, one can write the Kähler-Einstein metric as $\omega_y = \omega + dd^c X_y \varphi_y$ for a unique function $\varphi_y \in C^\infty(X_y)$ such that $\int_{X_y} \varphi_y \omega_y^n = 0$. This in turns defines a unique smooth function $\varphi \in C^\infty(X^\circ)$, hence a smooth $(1,1)$-form $\omega_{KE} := \omega + dd^c X \varphi$ on $X^\circ$ that restricts to $\omega_y$ for any $y \in Y^\circ$.

So in any case, there is a way to canonically extend the forms $\omega_y$ to a global $(1,1)$-form $\omega_{KE}$ on $X^\circ$. Some of the most natural questions to study are: is $\omega_{KE}$ a positive form? What is its behavior near the singular fibers, or more roughly, can $\omega_{KE}$ be canonically extended to the whole $X$? What are the connections with the relative Bergman metrics?

Relative singular Kähler-Einstein metrics. — A singular Kähler-Einstein metric is a generic term to refer to a non-smooth, closed, positive, $(1,1)$-current $\omega$ that satisfies a Kähler-Einstein like equation in a weak sense. Among the most natural examples are: Kähler-Einstein with conic singularities, mentioned above, cf. also [Bre13, CGP13, JMR16], Kähler-Einstein metrics on singular varieties, cf. [EGZ09, BBE+11, BG14]. Down the line, these metrics are obtained by solving an equation of the form

$$\nabla \omega = \lambda \omega + T$$

on a compact Kähler manifold $X$, where $T$ is a closed $(1,1)$-current (e.g. the current of integration along a $\mathbb{R}$-divisor with coefficients in $(-\infty, 1]$). The cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ of $\omega$ is determined by the equation unless $\lambda = 0$, and it may be degenerate. That is, instead of being Kähler, $\alpha$ may be semipositive and big, or even merely big. The singularities of $\omega$ may then appear because of the singularities of $T$ or the non-Kählerness of $\{\omega\}$. The singularities of the first type are rather well known when $T$ is a current of integration along an effective divisor with snc support (one gets conic or cusp singularities, cf. e.g. [Kob84, TY87, Gue14]), but they are mostly mysterious in the second case with a few numbers of exceptions like when $X$ is a resolution of singularities of a variety $Y$ with orbifold singularities, or isolated conical singularities cf. [HS16], and $\alpha$ is the pull-back of a Kähler class on $Y$.

Now of course, one can study the relative analogue of (0.7) just as in the section above.

Earlier results. — If the generic fiber has ample canonical bundle, that is, if $K_{X_y}$ is ample for any $y \in Y^\circ$, then it follows from the Aubin-Yau theorem [Aub78] [Yau78] that one can endow each smooth fiber with a Kähler-Einstein metric with $\lambda = -1$. The induced $(1,1)$ form $\omega_{KE}$ mentioned above is smooth by an application of the implicit function theorem. Moreover, it has been showed to be semipositive by Schumacher [Sch08] and independently by Tsuji [Tsu11]. Schumacher’s proof consists in establishing that each of the eigenvalues $c_\alpha$ of $\omega_{KE}$ in the direction of the base satisfy a linear elliptic equation

$$(-\Delta + 1) c_\alpha = f_\alpha$$

with $f_\alpha \geq 0$, hence $c_\alpha \geq 0$ by the maximum principle. On the other hand, Tsuji proves that $\omega_{KE}$ is the limit of relative Bergman kernels whose variation is known to be semipositive cf. [BP08].

Tsuji also obtains for free that $\omega_{KE}$ extends canonically to a current $\omega_{KE} \in c_1(K_{X/Y})$ on $X$, using the analogous property for Bergman kernels established in the previously cited theorem.

Following Schumacher’s strategy, one obtains in [Pau12] a generalization of this result to the Kähler setting (including the extension property) only assuming that $K_{X_y} + \{\beta\}|_{X_y}$ is relatively ample for some smooth, semipositive, closed $(1,1)$-form $\beta$ on $X$. Of course the equation (0.6) has to be replaced by $\nabla \omega_y = -\omega_y + \beta|_{X_y}$. The extension property was then proved using a generalization of Ohsawa-Takegoshi theorem.

Based on this approach again, the second name author studied the conic analogue of these questions, cf. [Gue16]: let $B = \sum b_i B_i$ is a divisor with snc support on $X$ and coefficients in $(0,1)$ and assume that $K_{X_y} + B|_{X_y} |_{X_y}$ is ample for $y \in Y^\circ$, the the relative conical Kähler-Einstein metric solution of $\nabla \omega_y = -\omega_y + [B|_{X_y}]$ induces a singular $(1,1)$ current $\omega_{KE}$ on $X^\circ$ that is
positive, and extends canonical to a positive current $\omega_{\text{KE}} \in c_1(K_X/Y + B)$. The main difficulty is that one cannot directly make sense of (0.8) given the poor/unknown regularity of $\omega^\phi_{\text{KE}}$ in the base direction. However, because conic singularities are very well understood, one can eventually get enough regularity to make sense of the aforementioned equation essentially.

In the more general singular case, say when $K_X + B$ is big, the fiberwise Kähler-Einstein metrics pick up singularities that are yet to be understood, and the PDE based strategy explained above is bound to fail.

In the case where $\lambda = 0$, it is not known, at least to our knowledge, that $\omega^\phi_{\text{KE}}$ is semipositive on $X^\circ$, in full generality. Choi [Cho15] and Braun-Schumacher [BS17] manage to bound the negativity of $\omega_{\text{KE}}$ in terms of a lower bound of the Green function of $(X_y, \omega_y)$, so for instance in terms of an upper bound of $\text{diam}(X_y, \omega_y)$. The extension property is not known either. However, it is easy to see that if $c_1(K_X/Y)$ vanishes, then the family is locally trivial, and of course $\omega^\phi_{\text{KE}}$ is semipositive (it is even zero in the base directions). This observation relies for instance on the Kodaira-Spencer class of $p$ at the given point. However, if one wanted to derive the same conclusion for the conic Ricci-flat metrics, one would face the same severe regularity problem as in [Gue16] in order to set up the PDE and reach the expected conclusion.

Main steps of the proof. — We will describe next the outline of the proof of Theorems A, C and E above.

- As we explained above, the proof of Theorem A follows a strategy initiated by Tsuji, and amounts to realizing a Kähler-Einstein metric $\phi_{\text{KE}}$ for a pair $(X, B)$ with $K_X + B$ big as a weak limit of renormalized Bergman kernels on $\ell(K_X + B)$ when $\ell$ tends to $+\infty$. Then the conclusion would essentially follow from [BP08]. The process goes roughly as follows. Defined $\phi_1$ to be a psh weight on $K_X + B$ coming from a Zariski decomposition for instance. Then, if $\phi_\ell$ is a psh weight on $\ell(K_X + B)$, then it induces a $L^2$ metric on $(\ell + 1)(K_X + B) = K_X + \ell(K_X + B) + B$ using the canonical singular metric on $B$. In this way, one defines a sequence of psh weights $(\frac{1}{\ell}\phi_\ell)$ on $K_X + B$. The main result is to show that $\lim_{\ell \to +\infty} \frac{1}{\ell}\phi_\ell = \phi_{\text{KE}}$.

A first issue is that the iteration scheme is not correct as set up above. This is because $B$ is in general only a $\mathbb{Q}$-line bundle. So instead, one has to produce a sequence of iterations $(\phi_{\ell,m})_{\ell,m}$ parametrized by an integer $m$ and such that

\[(0.9) \quad \lim_{\ell \to +\infty} \frac{1}{\ell}\phi_{\ell,m} = \phi_m\]

where $\phi_m$ is a twisted Kähler-Einstein metric satisfying the equation (1.34) with $\Lambda = 0$, and it can be showed to converge to the genuine Kähler-Einstein metric $\phi_{\text{KE}}$ when $m \to +\infty$, cf. Proposition 1.16.

So everything comes down to proving (0.9). This is the content of Theorem 1.18 which takes up most of Section 1.5 and constitutes the heart of the proof of Theorem A. For each pair of integers $(\ell, m)$, one has to compare $\frac{1}{\ell}\phi_{\ell,m}$ and $\phi_{\text{KE}}$. It turns out that it is crucial to have a good control on the complex Hessian of $\phi_{\text{KE}}$ to do so, but this if way out of reach as of now. If instead, one compares $\frac{1}{\ell}\phi_{\ell,m}$ to a smooth approximate KE metric, then the estimates blow up. The key is to compare $\frac{1}{\ell}\phi_{\ell,m}$ to the partially regularized version $\phi_\ell$ of $\phi_{\text{KE}}$, solution of (1.32). Having only orbifold singularities, these singular metrics have a perfectly well understood complex Hessian, and yet they are singular enough to approximate the Bergman kernel type metric with a good precision.
• The first item of Theorem C is established by using two ingredients. The first one consists in showing that the conic Ricci-flat metric in $\{\omega_{X_y}\}$ on each fiber $X_y$ is the normalized limit of the unique solution of the family of equations of type (0.7)

\[(0.10) \quad \text{Ric} \rho_{\varepsilon} = -\rho_{\varepsilon} + \varepsilon \omega + [B] \]

on $X_y$ where $\rho_{\varepsilon} \in \varepsilon \{\omega_{X_y}\}$. We show that $\omega^0_{\text{KE}}|_{X_y}$ is obtained as limit of $\frac{1}{\varepsilon} \rho_{\varepsilon}$ as $\varepsilon \to 0$. On the other hand, the main result of [Gue16] shows that the family $\rho_{\varepsilon}$ has psh variation for each positive $\varepsilon > 0$ and the result follows (the flatness of the direct image is crucial in order to be able to use [Gue16]).

The arguments for the second item of Theorem C is more involved. We use a different type of approximation of the conic Ricci-flat metric, by regularizing the volume element. Let $\tau_\delta$ be the resulting family of metrics. The heart of the matter is to show that the horizontal lift with respect to $\tau_\delta$ of any local holomorphic vector field on the base has a holomorphic limit as $\delta \to 0$. This is a consequence of the estimates in [GP16] combined with the PDE satisfied by the geodesic curvature of $\tau_\delta$, cf. [Sch12]. Then we show that the geodesic curvature tends to a (positive) constant and as a consequence we finally infer that the horizontal lift of holomorphic vector fields with respect to $\omega^0_{\text{KE}}$ is holomorphic and tangent to $B$.

• The equation $\text{Ric} \omega = -[E] + [B]$ translates into a Monge-Ampère equation where the right-hand side has poles and zeros. Poles are relatively manageable in the sense that they induce conic metrics, that is we know relatively precisely the behavior of the complex Hessian of the solution. Zeros, however, are much more complicated to deal with for several reasons. First, it seems hard to produce a global degenerate model metric that should encode the behavior of the solution. Next, the regularized solutions of the Kähler-Einstein equation do not satisfy a Ricci lower bound, hence it seems difficult to estimate their Sobolev constant.

In Proposition 3.1, we establish a uniform (weak) Sobolev inequality where the measure in the right-hand side picks up zeros. Then, we get onto studying the regularity of families of such metrics. Despite having a rather poor understanding of the fiberwise metrics, we are still able to analyze the first order derivatives of the potentials in the transverse directions, leading to an $L^2$ estimate, yet with respect to a more degenerate volume form, cf Proposition 3.6. This is however enough to deduce the Lipschitz variation of the potentials away from $\text{Supp}(B + E)$.

Organization of the paper. —

• §1: We first recall some definitions and properties of singular Kähler-Einstein metrics in the absolute and relative case, when the log Kodaira dimension is positive. We explain how to deduce Corollary B from Theorem 1.3 and then prove that theorem by showing that the relative Kähler-Einstein metrics are limits of relative Bergman kernels.

• §2 We prove Theorem 2.2, and then derive successively Corollary 2.3 and Corollary D.

• §3: We obtain transverse regularity results for families of Monge-Ampère equations corresponding to adjoint linear systems having basepoints. This leads to Theorem E.

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1. Relative Kähler-Einstein metrics in the log general type case

1.1. Recap and complements on singular Kähler-Einstein metrics. —

1.1.1. The log general type case. — In this paragraph, $X$ will denote a connected, compact Kähler manifold of dimension $n$. We let $(L, h_L)$ be a $\mathbb{Q}$-line bundle endowed with a possibly singular hermitian metric $h_L = e^{-\phi_L}$ with positive curvature, that is, $\frac{i}{2\pi} \Theta_{h_L}(L) = dd^c \phi_L \geq 0$ in the sense of currents.

We now recall the definition of Kähler-Einstein metric for the pair $(X, L)$ when $K_X + L$ is big. This definition has been given by [BEGZ10, § 6] when $L = 0$, and can be readily adapted to our slightly more general context.

**Definition-Proposition 1.1.** — Let $X$ be a compact Kähler manifold, let $(L, h_L)$ be a $\mathbb{Q}$-line bundle endowed with a singular hermitian metric $h_L = e^{-\phi_L}$ with positive curvature, that is, $i \Theta_{h_L}(L) \geq 0$ in the sense of currents. We assume moreover that

1. (1.1.1) The $\mathbb{Q}$-line bundle $K_X + L$ is big.
2. (1.1.2) The algebra $R(X, L) = \bigoplus_{m \geq 0} H^0(X, [m(K_X + L)])$ is finitely generated.
3. (1.1.3) For every $p \in \mathbb{N}$ and every $s \in H^0(X, p(K_X + L))$, we have $\int_X |s|^2/p e^{-\phi_L} < +\infty$.

Then, there exists a unique closed, positive (1,1)-current $\omega_{KE}$ on $X$ which satisfies the following conditions.

1. (1.1.4) The current $\omega_{KE}$ belongs to the big cohomology class $c_1(K_X + L)$ and it has full mass, that is, $\int_X (\omega_{KE}^n) = \operatorname{vol}(K_X + L)$.
2. (1.1.5) The current $\omega_{KE}$ satisfies the following equation in the weak sense of currents

$$\operatorname{Ric}_{\omega_{KE}} = -\omega_{KE} + \frac{i}{2\pi} \Theta_{h_L}(L).$$

**Remark 1.2.** — Some remarks are in order.

(a) An important feature of this definition is that it is birationally invariant. More precisely, if $(X, L, e^{-\phi_L})$ satisfies conditions (1.1.1)-(1.1.3) and if $\pi : X' \to X$ is any birational proper morphism, then so does $(X', L', e^{-\phi_{L'}})$ where $L' := \pi^* L, \phi_{L'} := \pi^* \phi_L$. Furthermore, if $\omega_{KE}'$ is the Kähler-Einstein metric of $(X', L', e^{-\phi_{L'}})$, then $\omega_{KE}' = \pi^* \omega_{KE} + [K_{X'/X}]$.

(b) Conditions (1.1.2) and (1.1.3) are automatically satisfied if the multiplier ideal sheaf of $h_L$ is trivial, that is, if $\mathcal{I}(h_L) = \mathcal{O}_X$. This is clear for (1.1.3), and it follows from [BCHM10] for (1.1.2).

Indeed, write $K_X + L \sim_\mathbb{Q} A + E$ for $A$ an ample $\mathbb{Q}$-line bundle and $E$ an effective $\mathbb{Q}$-divisor. The question is birationally invariant, so by Demailly’s regularization theorem, one can suppose that we have an isometric isomorphism $L + \frac{1}{m} A \sim_\mathbb{Q} B + H$ where $B = \sum a_i [B_i]$ is an effective normal crossing $\mathbb{Q}$-divisor, $H$ is ample. Moreover, $\phi_L$ is more singular than $\phi_B$, where $\phi_B$ a canonical singular weight attached to $B$, cf proof of Lemma 1.14 for instance. From the solution of the openness conjecture, cf. [GZ15] and [Ber15], we have $\mathcal{I}(e^{-\phi_L} - \frac{1}{m} \delta_E) = \mathcal{O}_X$ for $m \gg 0$. This implies that $B + \frac{1}{m} E$ is klt. Now, observe that $(1 + \frac{1}{m})(K_X + L) \sim_\mathbb{Q} K_X + (B + \frac{1}{m} E) + H$ and use [BCHM10].

(c) If $L$ corresponds to an effective, klt $\mathbb{Q}$-divisor $B$ and $\phi_L$ is the canonical singular weight on $B$, then one recovers the standard log Kähler-Einstein metric whose existence follows essentially from [BEGZ10], cf. e.g. [Gue13, § 2.3].
(d) The condition (1.1.5) can be rewritten in terms of non-pluripolar Monge-Ampère equations as follows:

\[(dd^c \phi_{KE})^n = e^{\phi_{KE} - \phi_L}\]

where \(\phi_{KE}\) is a local weight for \(\omega_{KE}\) and where \(\langle .^n \rangle\) denotes the non-pluripolar Monge-Ampère operator, cf. [BEGZ10, Def.1.1 & Prop.1.6].

(e) We will see in the proof that \(\omega_{KE}\) has minimal singularities in the sense of [DPS01, Def.1.4].

(f) Assume that \(h_L\) is smooth on a non-empty Zariski open subset of \(X\). Then one can prove that \(\omega_{KE}\) is smooth on a Zariski open set by reducing the problem to the semiample and big case and use [EGZ09]. To our knowledge, there is still no purely analytical proof of the generic smoothness as explained in the few lines following [BEGZ10, Thm. C].

Proof of Definition-Proposition 1.1. — Set \(R(X, L) := \bigoplus_{m \geq 0} H^0(X, |m(K_X + L)|)\), and let us define \(X_{lc} := \text{Proj} R(X, L)\) to be the log canonical model of \(X\), cf. e.g. [BCHM10, Def. 3.6.7]. Taking a desingularization of the graph of the natural birational map \(f : X \rightarrow X_{lc}\), one get the following diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\nu} & X_{lc} \\
\mu \downarrow & & \downarrow \nu \\
X & \xrightarrow{f} & X_{lc}
\end{array}
\]

and the following formula

\[(1.6) \quad \mu^*(K_X + L) = \nu^*(K_{X_{lc}} + L_{lc}) + F\]

where \(L_{lc} := f_!L\) (recall that \(f\) does not contract any divisor) and \(F\) is an effective \(\nu\)-exceptional divisor. Clearly, \(K_{X_{lc}} + L_{lc}\) is ample. Thus, setting \(\tilde{L} := \mu^*L, A := \nu^*(K_{X_{lc}} + L_{lc})\) and \(E := F + K_{\tilde{X}/X}\), the decomposition

\[K_{\tilde{X}} + \tilde{L} = A + E\]

is a Zariski decomposition of \(K_{\tilde{X}} + \tilde{L}\), and we have \(\int_{\tilde{X}} |s|^2 e^{-\phi_{\tilde{L}}} < +\infty\) for any section \(s \in H^0(\tilde{X}, p(K_{\tilde{X}} + \tilde{L}))\) and where \(\phi_{\tilde{L}} := \mu^*\phi_L\). We want to solve the following equation

\[(dd^c \tilde{\phi})^n = e^{\tilde{\phi} + \phi_E - \phi_{\tilde{L}}}\]

for \(\tilde{\phi}\) a bounded psh weight on \(A\), where \(\phi_E\) is the canonical singular weight attached to \(E\). Thanks to the results in [Gue13, § 2.3], we are reduced to establishing the following property.

\[(1.8) \quad e^{\phi_E - \phi_{\tilde{L}}} \in L^{1+\varepsilon} \quad \text{for some } \varepsilon > 0.\]

Take \(p\) large enough so that \(p\tilde{L}, pE\) are integral and \(|pA|\) is basepoint free. Let \(\{\tau_1, \ldots, \tau_r\}\) be a basis of \(H^0(\tilde{X}, pA)\). Then \(\sum_{i=1}^r |\tau_i|^2\) is non-vanishing everywhere. Let \(s_{pE}\) be the canonical section of \(pE\). Thanks to (1.1.3), we have

\[\int_{\tilde{X}} \left(\sum_{i=1}^r |\tau_i|^2\right) \tilde{\phi}_p |s_{pE}|^2 e^{-\phi_{\tilde{L}}} < +\infty.\]

Together with the fact that \(\sum_{i=1}^r |\tau_i|^2\) is non vanishing everywhere, we get

\[\int_{\tilde{X}} e^{-\phi_{\tilde{L}} + \phi_E} < +\infty.\]

By applying [GZ15, Ber15], (1.8) holds.
Now, define $\phi := \tilde{\phi} + \phi_F$. From the Zariski decomposition (1.6), it follows that the psh weight $\phi$ on $\mu^*(K_X + L)$ has minimal singularities and it satisfies $\langle (dd^c\phi)^m \rangle = \langle (dd^c\tilde{\phi})^m \rangle = e^{\phi - \phi_L}$ as the operator $\langle . \rangle$ puts no mass on proper analytic sets. There exists a unique psh weight $\phi_{KE}$ on $K_X + L$ such that $\phi = \mu^*\phi_{KE}$. It has automatically minimal singularities and satisfies $\langle (dd^c\phi)_{KE}^m \rangle = e^{\phi_{KE} - \phi_L}$; this ends the proof. 

The main theorem of this first section is the following

**Theorem 1.3.** — Let $p : X \to Y$ be a proper fibration between two smooth Kähler manifolds. Let $(L, h_L)$ be a holomorphic hermitian $\mathbb{Q}$-line bundle on $X$ such that $i\Theta_{h_L}(L) \geq 0$. Let $Y^0 \subset Y$ be a Zariski open subset such that where $p$ is smooth over $Y^0$, and set $X^0 := p^{-1}(Y^0)$. Assume that the additional conditions are satisfied.

(1.3.9) The $\mathbb{Q}$-line bundle $K_X + L$ is $p$-big.

(1.3.10) For every $y \in Y^0$ algebra $R(X_y, L) = \bigoplus_{m \geq 0} \mathcal{H}^0(X_y, |m(K_X + L)|)$ is finitely generated.

(1.3.11) Let $y \in Y^0$. For every $p \in \mathbb{N}$ and every $s \in \mathcal{H}^0(X_y, p(K_X + L))$, we have $\int_{X_y} |s|^2_{p^*L} < +\infty$.

Then, the KE metrics $(\omega_{KE, y})_{y \in Y^0}$ on the smooth fibers in the sense of Definition 1.1 induce a metric $e^{-\phi_{KE}}$ on $K_{X^0/Y^0} + L|_{X^0}$ such that:

(1.3.12) The metric $e^{-\phi_{KE}}$ is positively curved on $X^0$.

(1.3.13) The metric $e^{-\phi_{KE}}$ extends canonically across $X \setminus X^0$ to a positively curved metric on $K_{X/Y} + L$.

1.1.2. The case of intermediate Kodaira dimension. — For simplicity and naturality, we assume in this paragraph that $(L, h_L) = (B, e^{-\phi_B})$ for some $\mathbb{Q}$-effective klt divisor $B$ with its canonical singular metric $e^{\phi_B}$. We assume in this subsection that $0 < \kappa(K_X + B) < \dim X$.

Let $Z$ be the canonical model of $(X, B)$ and let $f : X \to Z$ be the Iitaka fibration with respect to the linear system $|m(K_X + B)|$ for $m$ large and divisible enough. Thanks to [BCHM10] in the projective case and [Fuj13] for the Kähler case, we know that $Z$ is normal. After some desingularisation of $f$, we get a fibration between two compact Kähler manifolds. For simplicity, we still denote it by $f : X \to Z$. In general, $f_*(m(K_X/Z + B))$ is not locally free on $Z$. We should take the reflexive hull $(f_*(m(K_X/Z + B)))^\vee$ to make it to be locally free. We now recall the definition of the Narashimhan-Simha metric on $(f_*(m(K_X/Z + B)))^\vee$.

**Definition 1.4.** — Let $Z_0 \subset Z$ be a locus such that $f$ is smooth over $Z_0$ and $B|_{X_{z_0}}$ is klt for every $z_0 \in Z_0$. Let $z_0 \in Z_0$ and put $s \in (f_*(m(K_X/Z + B)))_z = H^0(X_z, mK_{X_z} + mB)$. We define the Narashimhan-Simha metric

$$\|s\|_{h_m}^2 := \int_{X_z} |s|^2_{h_{B_m}}$$

where $h_B$ is the canonical singular hermitian metric with respect to the divisor $B$. Thanks to [BP08], $h_m$ can be canonically extended as a possible singular metric on

$$(Z, (f_*(m(K_X/Z + B)))^\vee)$$

We call it the $m$-th Narashimhan-Simha metric.

Let $h_m$ be the Narashimhan-Simha metric on $(f_*(m(K_X/Z + B)))^\vee$. We can easily check that the weight of $h_m$ is locally integrable over the locus where $f_*(m(K_X/Z + B))$ is locally free. Moreover, the pair $(Z, \frac{1}{m}(f_*(m(K_X/Z + B)))^\vee, h^\frac{1}{m}_{B_m})$ satisfies (1.1.1) and (1.1.2). However, it does not satisfy in general (1.1.3). We need the following standard trick.
By using Hironaka’s flattening theorem, cf. [Vie83, Lemma 7.3], we can find a morphism \( f' : X' \rightarrow Z' \) between two compact Kähler manifolds and satisfies the following commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Z' \\
\downarrow \pi & & \downarrow \nu \\
X & \xrightarrow{f} & Z
\end{array}
\]

such that the morphisms \( \pi \) and \( \mu \) are bimeromorphic, and moreover, each hypersurface \( W \subset X' \) such that \( \text{codim}_Y f'(W) \geq 2 \) is \( \pi \)-contractible, i.e., \( \text{codim}_X \pi(W) \geq 2 \).

We denote denote by \( \hat{B} \) the strict transform of \( B \) by \( \pi \), and write \( K_{X'} + \hat{B} = \pi^*(K_X + B) + \sum a_i E_i \). We set \( B' := \hat{B} + \sum_{a_i < 0} (-a_i) E_i \). Then \( (X', B') \) is klt. Let us choose \( m \) large enough so that \( \mathcal{F}_m := f'_* (m(K_{X'}/Z' + B')) \) is non-zero. Then, \( \mathcal{F}_m \) is a torsion free sheaf of rank one on \( Z' \) and its reflexive hull \( L := \mathcal{F}_m^* \) is a line bundle that we can equip with \( h_m = e^{-m\phi_m} \) the \( m \)-th Narashimhan-Simha metric. Define formally the \( \mathbb{Q} \)-line bundle

\[ L_m := \frac{1}{m} L = \frac{1}{m} f'_*(m(K_{X'}/Z' + B'))^{**} \]

It is endowed with the NS-type metric \( e^{-\phi_m} \).

We have the following proposition, which relies on [BP08].

**Proposition 1.5.** — In the above setting, let \( m \in \mathbb{N} \) sufficiently divisible. The following holds.

(1.5.14) \( \Theta_{\phi_m}(L_m) \geq 0 \) on \( Z' \).

(1.5.15) \( K_{Z'} + L_m \) is big and the algebra

\[ R(Z', L_m) := \bigoplus_{p \geq 0} H^0(Z', pm(K_{Z'} + L_m)) \]

is finitely generated.

(1.5.16) For every \( p \in \mathbb{N} \) and every \( s \in H^0(Z', pm(K_{Z'} + L_m)) \), we have

\[ \int_{Z'} |s|^{2m} e^{-\phi_m} < +\infty. \]

**Proof.** — The first item is a direct consequence of [BP08].

For the second term, let \( m \in \mathbb{N} \) sufficiently divisible such that for every \( p \in \mathbb{N} \), \( H^0(X', pm(K_{X'} + B')) \) is generated by \( \otimes^p H^0(X', m(K_{X'} + B')) \). By the construction of \( L_m \), there exist two effective divisors \( E_+ \) and \( E_- \) on \( X' \) such that

\[ m(K_{X'} + B') + E_- = m \cdot (f')^*(K_{Z'} + L_m) + E_+, \]

and for every \( \tau \in H^0(X', m(K_{X'} + B')) \), \( \tau \) vanishes over \( E_+ \). As a consequence, for every \( s \in H^0(X', pm(K_{X'} + B')) \), \( s \) vanishes over \( p[E_+] \). Note that \( (f')_* (E_-) \) is supported in the non locally free locus of \( f'_*(m(K_{X'}/Z' + B')) \). Then \( E_- \) is \( \pi \)-contractible. Together with the above argument, for every \( p \in \mathbb{N} \), we have the natural isomorphisms

\[ H^0(Z', pm(K_{Z'} + L_m)) = H^0(X', pm(K_{X'} + B') + pE_-) = H^0(X, pm(K_X + B)). \]

As a consequence, \( K_{Z'} + L_m \) is big and the algebra

\[ R(Z', L_m) := \bigoplus_{p \geq 0} H^0(Z', pm(K_{Z'} + L_m)) \]

is finitely generated.
For the third term, let \( s_{E^+} \) be the canonical section of \( E^+ \) and let \( h_{B'} \) (resp. \( h_{E_-} \)) be the canonical singular metric on \( B' \) (resp. \( E_- \)). Thanks to \((1.17)\), we have
\[
(f')^*(s) \otimes s_{E^+}^{p^p} \in H^0(X', p(mK_{X'} + mB' + E_-)).
\]
By the definition of the Narashimhan-Simha metric, we have
\[
\int_{\mathcal{Z}'(\pi)} |s|^{\frac{2}{m}} e^{-\phi_m} = \int_{X'} |(f')^*(s) \otimes s_{E^+}^{p^p}|^{\frac{2}{m}} h_{B'}, h_{E_-}.
\]
Note that \( E_- \) is \( \pi \)-contractible, \((f')^*(s) \otimes s_{E^+}^{p^p} \) vanishes along \( E_- \) of order at least \( pE_- \). Together with the fact that \( B' \) is klt, we have
\[
\int_{X'} |(f')^*(s) \otimes s_{E^+}^{p^p}|^{\frac{2}{m}} h_{B'}, h_{E_-} < +\infty.
\]
The proposition is proved.

\[\Box\]

**Remark 1.6.** — Using the above argument, we know that \( e^{-\phi_m} \) is in \( L_{\text{loc}}^1(\mathcal{Z}' \setminus f'(E_-)) \).

Together with Proposition 1.5, we get

**Corollary 1.7.** — With the above notation, \( Z' \) admits a natural Kähler-Einstein metric \( \omega_{\text{KE}} \) in the sense of Definition 1.1. This metric satisfies
\[
(1.18) \quad \text{Ric}_{\omega_{\text{KE}}} = -\omega_{\text{KE}} + \frac{i}{2\pi} \Theta_{\text{NS}}(L_m) \quad \text{on } Z'.
\]
where \( i\Theta_{\text{NS}}(L_m) \) is the Chern curvature of the Narasimhan-Simha metric \( e^{-\phi_m} \) on the \( \mathbb{Q} \)-line bundle \( L_m = \frac{1}{m} f'^*(m(K_{X'} + B'))^{**} \).

**1.1.3. Relation with the canonical metrics.** — Let \( X \) be a compact Kähler manifold and let \( B \) be a \( \mathbb{Q} \)-divisor with snc support such that \((X, B)\) is klt. We suppose that \( \kappa(K_X + B) \geq 1 \). Thanks to \([\text{BCHM}10, \text{Fuj}13]\), the canonical model \( Z \) of \((X, B)\) is normal. After blowing up the indeterminacy locus of the Iitaka fibration, we can suppose that the Iitaka fibration of \( K_X + B \) induces a morphism \( f : X \to Z \) and there is an ample \( \mathbb{Q} \)-line bundle \( A \) on \( Z \) such that
\[
K_X + B = f^*A + E_X
\]
is a Zariski decomposition for some effective \( \mathbb{Q} \)-divisor \( E_X \) with normal crossing support on \( X \). In that context, the analogue of Kähler-Einstein metrics for the pair \((X, B, E_X)\), sometimes called canonical metrics, are objects that are singular metrics \( \omega_{\text{can}} \) on \( Z \) satisfying a ”canonical” Monge-Ampère equation. They were first introduced by Song-Tian when \( B = E_X = 0 \) \([\text{ST}12]\) and later generalized by Eyssidieux-Guedj-Zeriahi \([\text{EGZ}16, \text{Definition 2.2, 2.7}]\).

Let us recall the definition of the canonical metric in this setting. One first picks a smooth hermitian \( h_A = e^{-\phi_A} \) on \( A \) with positive curvature \( \chi := df^*\phi_A \). Then, one introduces a measure \( \mu_{h_A, h_E} \) on \( X \) by setting
\[
\mu_{h_A, h_E} := \frac{(\sigma \wedge \ov{\sigma})^{\frac{1}{N}} e^{-\phi_B}}{|\sigma|^{2/N} f^*_{h_A, h_E}}
\]
where \( \sigma \) is a local trivialization of \( N(K_X + B) \) for \( N \) divisible enough, \( \phi_B \) is the canonical singular weight on \( B \) and \( h_E \) is the canonical singular metric on \( E \). Finally, one defines \( \omega_{\text{can}} := \chi + df^*\varphi_{\text{can}} \) as the unique positive current on \( Z \) with bounded potentials such that
\[
(1.19) \quad (\chi + df^*\varphi_{\text{can}})^{\dim Z} = e^{\varphi_{\text{can}}} f_*\mu_{h_A, h_E}.
\]
Note that the singularity of \( h_E \) gives rise the zero locus of \( \mu_{h_A, h_E} \). One can check that the measure \( f_*\mu_{h_A, h_E} \) has \( L^{1+\varepsilon} \) density with respect to a smooth volume form, cf. \([\text{EGZ}16, \text{Lemma} \])
2.1]. Moreover, the canonical metric $\omega_{\text{can}}$ is independent of the choice the hermitian metric $h_A$, cf. [EGZ16, Lemma 2.4].

By applying the construction in subsection 1.1.2 to $f : X \to Z$, we can find a morphism $f' : X' \to Z'$ between two compact Kähler manifolds and satisfies the following commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{f'} & Z' \\
\downarrow \pi & & \downarrow \mu \\
X & \xrightarrow{f} & Z
\end{array}
$$

such that the morphisms $\pi$ and $\mu$ are bimeromorphic, and moreover, each hypersurface $W \subset X'$ such that $\text{codim}_Y f'(W) \geq 2$ is $\pi$-contractible, i.e., $\text{codim}_X \pi(W) \geq 2$. Following the notations in Proposition 1.5, for $m$ large enough, we can equip $L_m := \frac{1}{m}f'_* (m(K_{X'} + B'))^*$ with the $m$-th Narashimhan-Simha type metric $e^{-\phi_m}$. Thanks to Corollary 1.7, we can find the "NS-type" Kähler-Einstein metric $\omega_{\text{KE}}$ on $Z'$ which satisfies

$$\text{Ric} \omega_{\text{KE}} = -\omega_{\text{KE}} + \frac{i}{2\pi} \Theta_{\text{NS}}(L_m) \quad \text{on } Z'.$$

We now establish a relation between the "NS-type" Kähler-Einstein metric $\omega_{\text{KE}}$ and the canonical metric $\omega_{\text{can}}$ as is follows.

**Proposition 1.8.** — With the notation above, let $\omega_{\text{KE}}$ be the Kähler-Einstein metric on $Z'$ solution of (1.18) and let $\omega_{\text{can}}$ be the canonical metric on $Z$ solution of (1.19). Then, one has $\mu_* \omega_{\text{KE}} = \omega_{\text{can}}$.

**Proof.** — Recall that by (1.17), we have

$$m(K_{X'} + B') + E_- = (f')^*(m(K_{Z'} + L_m)) + E_+, \quad (1.20)$$

and by construction, we have

$$m(K_{X'} + B' - E_{X'}) = (f' \circ \mu)^* mE_0, \quad (1.21)$$

for some $\mathbb{Q}$-effective divisor $E_{X'}$ such that $\pi_*(E_{X'}) = E_X$.

We first establish the relation between $\mu^* A$ and $K_{Z'} + L_m$. Remember that $f'_* \mathcal{O}_{X'}(kE_{X'}) \cong \mathcal{O}_{Z'}$ for any integer $k$ such that $kE_{X'}$ has integral coefficients. We deduce that $m(K_{Z'} + L_m) = \mu^* (mA) \otimes f'_* (mE_{X'} + E_-)$. In particular, we get that $f'_* (mE_{X'} + E_-)$ is a locally trivial sheaf of rank one, hence associated to a divisor $mE_{Z'}$; it is clearly effective and $\mu$-exceptional, as $\text{Supp}(f'_*) (mE_{X'}) \subset \mu^{-1}(Z_{\text{sing}})$ and $\text{codim}_{Z'} f'(E_-) \geq 2$. Then we have the Zariski decomposition

$$K_{Z'} + L_m \equiv \mathbb{Q} \mu^* A + E_{Z'}, \quad (1.22)$$

By construction, we have $f'^*(mE_{Z'}) + E_+ = mE_{X'} + E_-$.

Now, take $U \subset Z'$ a small coordinate open subset, and let $e_{\mu^* mA} \in H^0(U, \mu^* mA)$ and $e_{mE_{Z'}} \in H^0(U, mE_{Z'})$ be trivializations of $\mu^* mA$ and $mE_{Z'}$ respectively. Let $d\bar{z}^m$ be a trivialisation of $K_{Z'}$ over $U$. They induce a trivialization $e \in H^0(U, mL_m)$ of $mL_m$ such that

$$d\bar{z}^m \otimes e = e_{\mu^* mA} \otimes e_{mE_{Z'}}. \quad (1.23)$$

Set $e^{-\varphi_{\mu^* A}} := |e_{\mu^* mA}|^2_{\mu^* mA} \otimes e^{-\varphi_{E_{Z'}}} := |e_{mE_{Z'}}|^{2/m}_{e^{-\varphi_{E_{Z'}}}}$, and $e^{-\phi m} := |e|^{2/m}_{e^{-\phi m}}$. Let

$$\sigma := (f')^* e_{\mu^* mA} \in H^0(f'^{-1}(U), m(K_{X'} + B' - E_{X'})).$$
Thanks to (1.20) and (1.23), we have
\[ \tau := \sigma \otimes (f^* \mu_{E_n}) \otimes s_{E_+} \in H^0(f^{-1}(U), m(K' + B' + E_+)), \]
and
\[ e^{-\varphi_m}(z) = \int_{X'} |\sigma|^{2/m} e^{-\varphi_{E'}} - \frac{1}{m} \varphi_{E'} \cdot \int_{X} |\sigma|^{2/m} e^{-\varphi_{E'}}, \]  
(1.24)

The canonical measure \( \nu \) on \( Z' \) has density with respect to the Lebesgue measure \( d\lambda = |d\vec{z}|^2 \) given by the formula
\[ \frac{d\nu}{d\lambda}(z) = \int_{X'} (\sigma \wedge \bar{\sigma})^{1/m} e^{-\varphi_{E'}} + \varphi_{E'} \cdot \frac{1}{m} \int_{X} |\sigma|^{2/m} e^{-\varphi_{E'}}, \]
for \( z \in Z' \) generic. Together with (1.24), we get
\[ \frac{d\nu}{d\lambda} = e^{\varphi_m \cdot A - \varphi_m + \varphi_{E'}}, \]

Therefore, \( \omega := \mu^* \omega_{can} + [mE_{Z'}] \) satisfies
\[ \text{Ric} \omega = -\mu^* \omega_{can} + dd^c \varphi_m - [E_{Z'}] = -\omega + \frac{i}{2\pi} \Theta_{NS}(L_m) \]

As \( \omega \in c_1(K_{Z'} + L_m) \) has minimal singularities by the Zariski decomposition (1.22) and satisfies the same Monge-Ampère equation as \( \omega_{KE} \), one deduces that \( \omega = \omega_{KE} \), i.e.,
\[ \omega_{KE} = \mu^* \omega_{can} + [E_{Z'}]. \]

As \( E_{Z'} \) is \( \mu \)-exceptional, the proposition is proved. \( \square \)

**Remark 1.9.** — The proof of Proposition 1.8 above shows the more precise identity \( \omega_{KE} = \mu^* \omega_{can} + E_{Z'} \) for some explicit divisor \( E_{Z'} \) on \( Z' \).

### 1.1.4. Relative Kähler-Einstein, Bergman kernel and canonical metrics.

Let \( p : X \to Y \) be a proper fibration between two Kähler manifolds of relative dimension \( n \). Let now \( (L, h_L) \) be an hermitian holomorphic \( \mathbb{Q} \)-line bundle. We denote by \( \phi_L \) the possibly singular weight of \( h_L \).

Let \( Y^\circ \) denote a Zariski open subset of regular values of \( p \) such that for some \( m \in \mathbb{N} \) sufficiently large and divisible, the dimension \( h^0(X_y, m(K_{X_y} + L_y)) \) is constant with respect to \( y \in Y^\circ \). Let \( X^\circ := p^{-1}(Y^\circ) \) and \( L_y := L|_{X_y} \) for \( y \in Y \).

**Case a.**

If \( (L, h_L) \) is such that for any \( y \in Y^\circ \), one has

- (i) The curvature current of \( (L, h_L) \) has positive curvature, that is, \( dd^c \phi_L \geq 0 \).
- (ii) The \( \mathbb{Q} \)-line bundle \( K_{X_y} + L_y \) is big and the algebra \( R(X_y, L_y) = \bigoplus_{m \geq 0} H^0(X_y, m(K_{X_y} + L_y)) \) is finitely generated.
- (iii) For any integer \( p \) divisible enough, every section \( s \in H^0(X_y, p(K_{X_y} + L_y)) \) satisfies
\[ \int_{X_y} |s|^2/p e^{-\varphi_L} < +\infty. \]

Under the assumptions (i)-(iii) above, one knows that each smooth fiber \( X_y \) can be endowed with a singular Kähler-Einstein metric \( \omega_{KE, y} = dd^c \phi_{KE, y} \in c_1(K_{X_y} + L_y) \) in the sense of Definition 1.1. This means that \( \omega_{KE, y} \) has full Monge-Ampère mass, \( \int_{X_y} \omega_{KE, y}^n = \text{vol}(K_{X_y} + L_y) \), and that \( \text{Ric} \omega_{KE, y} = -\omega_{KE, y} + \Theta_{h_L}(L)|_{X_y} \). The reciprocal \( (\omega_{KE, y}^n)^{-1} \) of the singular volume form induced on \( X_y \) by \( \omega_{KE, y} \) is a singular hermitian metric on \( K_{X/Y}|_{X_y} \), so that \( e^{-\varphi_{KE}} := (\omega_{KE, y}^n)^{-1} e^{-\varphi_L} = e^{-\varphi_{KE, y}} \) is a singular hermitian metric on \( (K_{X/Y} + L)|_{X^\circ} \). The goal of the first section is to show that this metric has positive curvature and extends canonically across \( X \setminus X^\circ \).
The extension property will be proved using the positivity of the curvature along with an argument based on Ohsawa-Takegoshi extension theorem essentially contained in [Păun12, §3.3], cf. Proposition 1.11. As for the positivity of the curvature, it will rely on the analogue result for Bergman kernels due to Berdntsson and Păun [BP08, Thm. 0.1], cf next paragraph.

Case b.
If \((L, h_L)\) is a holomorphic line bundle (and not merely a \(\mathbb{Q}\)-bundle) such that

(i)’ The curvature current of \((L, h_L)\) has positive curvature, that is, \(dd^c \phi \geq 0\).

(ii)’ For some \(y \in Y^0\), there exists a non-zero section \(s \in H^0(X_y, K_{X_y} + L_y)\) satisfying

\[
\int_{X_y} |s|^2 e^{-\phi} < +\infty,
\]

then the relative Bergman kernel metric \(e^{-\phi_{\text{Ber}}} |_{X^0}\) on \((K_{X/Y} + L)|_{X^0}\) is not identically \(-\infty\). It has positive curvature and extends canonically across \(X \smallsetminus X^0\). Note that in restriction to \(y \in Y^0\), this metric is defined by \(\phi_{\text{Ber}, y} := \log(\sum u_i \wedge \bar{u}_i)\) where \((u_i)\) is an orthonormal basis of \(H^0(X_y, K_{X_y} + L_y)\) with respect to the \(L^2\) metric induced by \(e^{-\phi_{\text{Ber}, y}}\).

The strategy to prove the psh variation of the relative Kähler-Einstein metrics dates back to Tsuji [Tsu10], and it amounts to producing a sequence of metrics \(e^{-\phi_k}\) of Bergman type on \(K_{X/Y} + L_k = k(K_{X/Y} + L)\) for some suitable metric \(h_{L_k}\) on \(L_k\) with positive curvature such that \(\phi_k/k\) converges to \(\phi_{\text{KE}}\) in \(L^1_{\text{loc}}(X^0)\). The sequence \(\phi_k\) is defined by an iterative process using the decomposition \((k + 1)(K_{X/Y} + L) = K_{X/Y} + (k - 1)(K_{X/Y} + L) + L\) and endowing this bundle with the \(L^2\) metric induced by \(e^{-\phi_{k-1}-\phi_L}\). Unfortunately, this very compelling idea turns out to be very technical to put to practice. Among the main difficulties that arise: \(L\) is merely a \(\mathbb{Q}\)-line bundle and singular Kähler-Einstein metrics in this generality behave rather poorly. This forces us to set up a more complicated double iteration and to use a quite intricate approximation scheme for singular Kähler-Einstein metrics.

Case c.
Assume that \((L, h_L)\) satisfies

(iii)’ \((L, h_L) = (B, h_B)\) where \(B\) is an effective klt \(\mathbb{Q}\)-divisor with snc support and \(h_B\) is the canonical singular weight on \(O_X(B)\).

(iv)’ For any \(y \in Y^0\), the Kodaira dimension of \(K_{X_y} + B_y\) is positive.

Under the assumption (iii)’ by using [BCHM10, Fuj13], the canonical ring \(R(X_y, L_y) = \bigoplus_{m \geq 0} H^0(X_y, m(K_{X_y} + L_y))\) is finitely generated. Together with the fact that \(h^1(X_y, m(K_{X_y} + L))\) is constant with respect to \(y \in Y^0\) for some \(m\) large enough, one can construct the relative Iitaka fibration \(f: X \longrightarrow Z := \text{Proj}(p_* (m(K_{X/Y} + B)))\). Thanks to the subsection 1.1.3, we can find a desingularization \(f': X' \rightarrow Z'\) fitting the commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Z' \\
\downarrow \pi & & \downarrow \mu \\
X & \xrightarrow{f} & Z \\
\downarrow p & & \downarrow q \\
Y & \xleftarrow{\mu} & Y
\end{array}
\]

such that we have a \(f'\)-Zariski decomposition over \(Y_0\)

\[
K_{X'} + B' \equiv (f'\circ \mu)^* A + E \quad \text{on} \quad (f')^{-1}(Y_0),
\]

where \(A\) is \(q\)-ample.
Let \( q' := q \circ \mu : Z' \to Y \) be the projection to the base. Each fiber \( Z'_y \) for \( y \in Y^0 \) can be endowed with a canonical metric \( \omega_{\text{can}, y} \in c_1(\mu^*A_y) \) and a Kähler-Einstein metric \( \omega_{\text{KE}, y} \in c_1(K_{Z'_y} + L_{m,y}) \), where \( L_{m,y} := L_m|_{Z'_y} \) of the \( \mathbb{Q} \)-line bundle \( L_m := \sum f'_m(m(K_{X'/Z'} + B'))^{**} \) to \( Z'_y \) with the corresponding restriction of the Narasimhan-Simha metric on \( L_m \).

In particular, these fiberwise metrics induce singular hermitian metrics \( e^{-\phi_{\text{can}}} \) on \( \mu^*A_{|q^{-1}(Y^0)} \) and \( e^{-\phi_{\text{KE}}} \) on \( L_m|_{q'^{-1}(Y^0)} \) respectively. As seen in the subsection 1.1.3, there exists a \( \mu \)-exceptional effective divisor \( E_{Z'} \) on \( Z' \) such that \( \phi_{\text{KE}} = \phi_{\text{can}} + |E_{Z'}| \). By Theorem 1.3, \( e^{-\phi_{\text{KE}}} \) is a positively curved metric on \( (K_{Z'/Y} + L_m)|_{q'-1(Y^0)} \) that extends canonically to a positively curved metric on \( K_{Z'/Y} + L_m \) on the whole \( Z' \). As \( \phi_{\text{can}} \) comes from \( Z \) and \( E_{Z'} \) is \( \mu \)-exceptional, if follows that \( e^{-\phi_{\text{can}}} \) is a positively curved metric on \( \mu^*A_{|q^{-1}(Y^0)} \) that extends canonically to \( Z' \). This proves Corollary B.

**Remark 1.10.** — An alternative possible approach to Corollary B that would not use NS metrics is the following. Consider for the moment that \( Y \) is a point and assume that \( K_X + B \) is semipositive. Then, one can consider \( \omega_c \) the Kähler-Einstein metric of \( (X, B + \varepsilon A, \phi_B + \varepsilon \phi_A) \) where \( A \) is some ample line bundle on \( X \), and \( \phi_A \) is smooth strictly psh weight on \( A \). If one could prove the conic analogue of [Tos10], then one would obtain that \( \omega_c \) converges weakly when \( \varepsilon \to 0 \) to the pull-back of the canonical metric \( \omega_{\text{can}} \) on the base \( Z \) of the Itaka fibration of \( K_X + B \). Now, if \( Y \) is a manifold, then it follows from Theorem 1.3 that the metrics \( \omega_{Y, \varepsilon} \) glue to semipositive current on \( K_{X/Y} + B + \varepsilon A \) on \( X \). Passing to the limit when \( \varepsilon \to 0 \) would then give the expected result.

### 1.2. Proof the extension property and two reductions.

In this subsection, we first establish (1.3.13) in Theorem 1.3 assuming (1.3.12). Then, we will explain how to reduce the proof of (1.3.12) to a situation when the singularities of \( (L, h_L) \) are more explicit.

**The extension property.**

The proof of the extension property (1.3.13) follows very closely [Pāu12, §3.3], so we will mostly sketch the proof. Note that (1.3.13) follows at once from (1.3.12) and Proposition 1.11 below.

**Proposition 1.11.** — With the notation of Theorem 1.3 above, the local weights of \( \phi_{\text{KE}} \) are locally bounded above near \( X \times X^0 \).

**Proof.** — Let \( y \in Y^0 \) and let us pick any point \( x \in X_y \). We choose a Stein neighborhood \( \Omega \) of \( x \) in \( X \); we write \( \Omega_y := \Omega \cap X_y \), choose a potential \( \tau_y \) of \( \omega_{\text{KE}, y} \) such that the equation satisfied by \( \tau_y \) on \( \Omega_y \) is

\[
\langle (dd^c \tau_y)^n \rangle = e^{\tau_y - \bar{\varphi}_L} \left| \frac{dz}{dt} \right|^2
\]

where \( \varphi_L \) is a local weight for \( h_L \) on \( \Omega \), and the coordinates \( (z_1, \ldots, z_n, t_1, \ldots, t_m) \) are chosen so that \( p(z,t) = \bar{t} \). We set \( H_{m,y} := \{ f \in \mathcal{O}(\Omega_y) : \int_{\Omega_y} |f|^2 e^{-m\tau_y} \langle (dd^c \tau_y)^n \rangle \leq 1 \} \). Note that \( e^{-m\tau_y} \langle (dd^c \tau_y)^n \rangle = e^{-(m-1)\tau_y - u} \left| \frac{dz}{dt} \right|^2 \) for some psh function \( u \) on \( \Omega \). Then, thanks to Demailly’s approximation theorem, one has

\[
\tau(y)(x) = \lim_{m \to \infty} \sup_{f \in H_{m,y}} \frac{1}{m} \log |f(x)|
\]

But for \( f \in H_{m,y} \), Hölder’s inequality yields

\[
(1.25) \quad \int_{\Omega_y} |f|^{2/m} e^{-\tau_y} \langle (dd^c \tau_y)^n \rangle \leq \left( \text{vol}(K_{X_y} + L_y) \right)^{\frac{m}{m-1}}
\]
The right hand side is bounded above independently of $y$ and $m$; this can be seen for instance by finding a birational model $\pi : X' \to X$ where $\pi^*(K_X + L)$ has a relative Zariski decomposition $A+E$ so that the volume of $K_{X_y} + L_y$ is simply the intersection number $(A^y)$ which is independent of $y \in Y^\circ$. Furthermore, the $L^{2/m}$ version of Ohsawa-Takegoshi extension theorem \cite{BP12} yields a holomorphic function $F$ on $\Omega$ that extends $f$ and such that

$$
|F(x)|^{2/m} \leq C \int \Omega |F|^{2/m} |dz|^2 \leq C \int_{\Omega_y} |f|^{2/m} \left| \frac{dz}{dt} \right|^2 \leq C' \int_{\Omega_y} |f|^{2/m} e^{-\tau_y} \langle (dd^c \varphi_y)^m \rangle
$$

as $\varphi_L$ is bounded above on $\Omega$. Moreover, the integral on the right hand side is bounded above uniformly in $y$ and $m$ by (1.25). Therefore $\sup_{X_y} \tau_y \leq C$ for a constant $C$ that uniform as long as $y \in Y^\circ$ varies in compact subsets of $Y$.

**Remark 1.12.** — Another way to prove the extension property (1.3.13) would be to use the fact that the relative Kähler-Einstein metrics are limits of relative Bergman kernel metrics on $X^\circ$, cf Theorem 1.18, and use the extension property for Bergman kernels. One has to be a little bit careful as the previous theorem requires additional assumptions on $(X, L, \phi_L)$. However, one can check using Jensen inequality that along all the approximation procedures (i.e. the reduction steps below and Theorem 1.18), the weights are uniformly bounded above locally near $X \setminus X^\circ$, hence the same is true for the limiting Kähler-Einstein weight by Hartogs lemma.

**Two important reductions.**

We should keep in mind in the following that, thanks to Proposition 1.11, Theorem 1.3 is local on the base. So we can assume from now on that the base $Y$ is a small Stein open set. In order to reduce the make the computation to come manageable, we will reduce the general case to the situation where $i\Theta_{h_L}(L)$ is essentially the current of integration along a klt divisor and where $K_X + L$ admits a relative Zariski decomposition.

**Lemma 1.13.** — It is enough to prove Item (1.3.12) of Theorem 1.3 when $K_X + L$ admits a relative Zariski decomposition $K_X + L = A + E$ with $A$ relatively semipositive and $E$ effective and such that the natural map $p^* p_* O_X(mA) \to p^* p_* O_X(mA + mE)$ is a sheaf isomorphism over $Y^\circ$ for any $m$ divisible enough.

**Proof.** — By assumption, the $O_Y$-algebra $E := \bigoplus_{n \geq 0} p_* (m(K_X/Y + L))$ is finitely generated. By blowing up the base locus of $E$, we can find a birational map $\mu : \widetilde{X} \to X$ such that on the generic fiber $\widetilde{X}_y$ of $\mu \circ p$, we have Zariski decomposition of $K_{\widetilde{X}_y} + \mu^* L|_{\widetilde{X}_y}$. Therefore, there exists a Zariski dense open subset $Y_0 \subset Y$ such that $K_{\widetilde{X}} + \mu^* L$ admits a relative Zariski decomposition $K_{\widetilde{X}} + \mu^* L = A + E$ on $(\mu \circ p)^{-1}(Y_0)$ and for any $m$ divisible enough, the natural map

$$(\mu \circ p)^*(\mu \circ p)_* O_{\widetilde{X}}(mA) \to (\mu \circ p)^*(\mu \circ p)_* O_{\widetilde{X}}(mA + mE)$$

is an isomorphism.

Now, let $\omega_{KE,\widetilde{X}}$ (resp. $\omega_{KE,X}$) be the relative Kähler-Einstein metric with respect to $(\widetilde{X}, \mu^* L, \mu^* \phi_L)$ (resp. $(X, L, \phi_L)$) over $Y_0$. If we can prove that $\omega_{KE,\widetilde{X}} \geq 0$ on $(\mu \circ p)^{-1}(Y_0)$, thanks to

$$\mu^* \omega_{KE,\widetilde{X}} = \omega_{KE,X}$$

on $(\mu \circ p)^{-1}(Y_0)$, we know that $\omega_{KE,X} \geq 0$ on $p^{-1}(Y_0)$. Together with Proposition 1.11, the lemma is proved.

**Proposition 1.14.** — It is enough to prove Item (1.3.12) of Theorem 1.3 when $(L, h_L) = (B + \Lambda, h_B|_\Lambda)$ where $B$ is an $\mathbb{Q}$-effective divisor and the restriction on the generic fiber $B|_{X_y}$ is klt with snc support, $h_B$ is the canonical singular metric on $O_X(B)$ and $(\Lambda, h_\Lambda)$ is a semipositive hermitian $\mathbb{Q}$-line bundle.
Proof. — We proceed in two steps.

Step 1. Reduction to the case where \(dd^c \phi_L\) is a Kähler current

By Lemma 1.13 above, one can assume that \(K_X + L = A + E\) is a relative Zariski decomposition of \(K_X + L\) on \(X\). As \(Y\) is Stein and \(K_X + L\) is \(p\)-big, \(K_X + L\) is big on \(X\). There exists thus a weight \(\phi_0\) of analytic singular on \((K_X + L)\) such that \(dd^c \phi_0 > 0\) on \(X\). Let us fix some small \(\delta > 0\). We set \(L_\delta := L + \delta(K_X + L)\) so that the relative Zariski decomposition of \(K_X + L_\delta\) is \((1 + \delta)A + (1 + \delta)E\). Let \(\phi_L + \delta \phi_0\) be the weight on \(L_\delta\). Then \(dd^c (\phi_L + \delta \phi_0)\) is a Kähler current.

To finish the proof of Step 1, it remains to prove that

(i) The triplet \((X, L_\delta, e^{-\phi_L - \delta \phi_0})\) admits a relative Kähler-Einstein metric \(\omega_{KE, \delta}\) for \(\delta > 0\) small enough.

(ii) We have \(\omega_{KE, \delta} \to \omega_{KE}\) in the weak topology when \(\delta\) approaches zero.

To make notation more tractable, we will –from now on and in this first step only – work on a fixed fiber \(X_y\) and drop all indices \(y\).

For (i), thanks to \((1.1.3)\), \(e^{\phi_E - \phi_L} \in L^p\) for some \(p > 1\). Then \(e^{(1+\delta)(\phi_E - (\phi_L + \delta \phi_0))} \in L^r\) for some \(1 < r < p\) as long as \(\delta\) is small enough. (i) is proved.

(ii) requires more work. Let \(\omega_A\) a smooth semipositive form in \(c_1(A)\). The Kähler-Einstein metric \(\omega_{KE, \delta}\) can be written as \(\omega_\delta = (1 + \delta) \omega_A + dd^c \phi_\delta \in c_1((1 + \delta)A)\) a positive current with bounded potentials such that

\[
((1 + \delta) \omega_A + dd^c \phi_\delta)^n = e^{\phi_\delta + (1 + \delta)(\phi_E - \phi_L - \delta \phi_0)} dV
\]

where \(\phi_0\) is a weight function of \(\phi_0\) and \(dV\) is a smooth volume from. Let us write \(d\mu := e^{\phi_E - \phi_L} dV\) and \(d\mu_\delta := e^{(1+\delta)(\phi_E - \phi_L - \delta \phi_0)} dV\); by dominated convergence we have \(\int_K gd\mu_\delta \to \int_K gd\mu\) when \(\delta \to 0\), for any bounded function \(g\) on a given Borel subset of \(X\).

Let \(\varphi\) be the weight function of \(\omega_{KE}\). Now, if \(C > 0\) is given, one considers \(U := \{\varphi_\delta < \varphi - C\}\). By the maximum principle, one gets

\[
\int_U ((1 + \delta) \omega_A + dd^c \varphi)^n \leq \int_U ((1 + \delta) \omega_A + dd^c \varphi_\delta)^n.
\]

As \((1 + \delta) \omega_A + dd^c \varphi \geq \omega_A + dd^c \varphi\), one finds

\[
\int_U e^{\varphi} d\mu \leq e^{-C} \int_U e^{\varphi_\delta} d\mu_\delta
\]

so that \(U\) is empty for \(\delta\) small enough, that is,

\[
\limsup_{\delta \to 0} (\varphi_\delta - \varphi) > 0 \tag{1.26}
\]

One can use the same type of argument with \(V := \{\varphi < \varphi_\delta - C\}\), observing in addition that \(\int_V ((1 + \delta) \omega_A + dd^c \varphi)^n = \int_V e^{\varphi} d\mu + O(\delta)\) by Bedford-Taylor theory, \(\varphi\) being bounded. In the end, we get

\[
\limsup_{\delta \to 0} (\varphi - \varphi_\delta) > 0 \tag{1.27}
\]

In conclusion, \((1.26)-(1.27)\) show that \(\varphi_\delta\) converges weakly to \(\varphi\), hence \(\omega_\delta\) converges to \(\omega\) as current and therefore \(\omega_{KE, \delta}\) converges to \(\omega_{KE}\). This argument was done fiberwise, but it clear that the weak convergence on the fiber implies the weak convergence in any small neighborhood of the given fiber as well. This proves (ii) and completes Step 1.

Step 2. Reduction to the case where \(\phi_L\) has analytic singularities

By Step 1, one can assume that \(dd^c \phi_L\) is a Kähler current. By Demailly regularization theorem [Dem92], \(\phi_L\) is the weak, decreasing limit of strictly psh weights \(\phi_{L, \delta}\) on \(L\) with analytic
singularities, say with singularities along the analytic set $Z_\varepsilon$. Taking a log resolution $\pi_\varepsilon : X_\varepsilon \to X$ of $(X, Z_\varepsilon)$, one can assume that $\pi^* \phi_{B,\varepsilon} = \phi_{B_\varepsilon} + \phi_{A,\varepsilon}$ where $\phi_{B_\varepsilon}$ is the canonical singular psh weight on an effective normal crossing $\mathbb{Q}$-divisor $B_\varepsilon$, and $\phi_{A,\varepsilon}$ is a smooth psh weight on some $\mathbb{Q}$-line bundle $A_\varepsilon$.

After passing to another birational model $\pi_\varepsilon : X_\varepsilon \to X$, one can assume that over a generic fiber, we have a Zariski decomposition

$$K_{(X_\varepsilon)_y} + \pi_\varepsilon^* L_\varepsilon|_{(X_\varepsilon)_y} = M_\varepsilon + E_\varepsilon,$$

and $B_\varepsilon|_{(X_\varepsilon)_y} + E_\varepsilon$ is normal crossing. Let $\Gamma_\varepsilon := B_\varepsilon \wedge E$ be the common part of $B$ and $E$. We have the following Zariski decomposition

$$K_{(X_\varepsilon)_y} + (B_\varepsilon|_{(X_\varepsilon)_y} - \Gamma_\varepsilon|_{(X_\varepsilon)_y}) + A_\varepsilon|_{(X_\varepsilon)_y} = M_\varepsilon + (E_\varepsilon - \Gamma_\varepsilon|_{(X_\varepsilon)_y}).$$

Furthermore, thanks to (1.3.11) and the decreasing property of $(\phi_{L,\varepsilon})$, we know that the divisor $(B_\varepsilon|_{(X_\varepsilon)_y} - \Gamma_\varepsilon|_{(X_\varepsilon)_y})$ is klt on $(X_\varepsilon)_y$. Now, the Kähler-Einstein metric $\omega_\varepsilon$ of $(X_\varepsilon, \pi_\varepsilon^* L, \pi_\varepsilon^* (\phi_{L,\varepsilon}))$ is related to the Kähler-Einstein metric $\omega_\varepsilon$ of $(X, (B_\varepsilon - \Gamma_\varepsilon) + A_\varepsilon, \phi_{B_\varepsilon} - \phi_{\Gamma_\varepsilon} + \phi_{A_\varepsilon})$ by the relation

$$\omega_\varepsilon = \omega_\varepsilon + [\Gamma_\varepsilon].$$

In conclusion, if Theorem 1.3 holds for $(X_\varepsilon, (B_\varepsilon - \Gamma_\varepsilon) + A_\varepsilon)$, then it holds for $(X, L, \phi_{L,\varepsilon})$, hence for $(X, L, \phi_L)$. Indeed when $\varepsilon$ converges to 0, the Kähler-Einstein metric of $(X, L, \phi_{L,\varepsilon})$ converges to the Kähler-Einstein metric of $(X, L, \phi_L)$ as a direct consequence of the comparison principle. 

1.3. The approximation process. — Starting from now, we are in the setting of Theorem 1.3 with the reductions from Lemmas 1.14 and 1.13. Fix a $y \in Y^\circ$ and we work on the fixed fiber $X_y$. On this fiber, we have the singular Kähler-Einstein metric $\omega_{\phi_y}$ of the triple $(X_y, L|_{X_y}, \phi_L|_{X_y})$ as explained in the previous paragraph.

One will approximate $\phi_y$ by a different weight depending on parameters $\varepsilon, \delta, m$ that will be specified later, and one will relate these new weights to some Bergman-type weights. The singularities of $\phi_y$ are due to several factors: first, $\phi_y$ lives in a non-Kähler class $A_1|_{X_y}$, and solutions of Monge-Ampère equations in non-Kähler classes pick up singularities that are not well understood yet. Moreover, the volume form $\omega^n_{\phi_y}$ has zeros along Supp $E$ and poles along Supp $B$. The most naive idea would be to regularize $\phi_y$ by solving a Monge-Ampère equation where the cohomology class is a Kähler class of the form $A_1|_{X_y} + \delta \{\omega_y\}$ and where the right-hand side is a smooth volume form approaching the original one (typically replace $|f_E|^2$ by $|f_E|^2 + \varepsilon^2$). Unfortunately, this naive regularization seems unfit to get a precise comparison between the approximate Kähler-Einstein metric and Bergman kernel type ones.

However, there is a way to approach $A_1|_{X_y}$ by Kähler classes slightly differently as well as a way to "regularize" the volume form $\omega^n_{\phi_y}$. Counterintuitively, one will increase the singularities of the volume form to make the metric orbifold with small cone angles along a certain divisor. Let us detail the procedure in the following lines.

Note that $A$ is $p$-big, and as $Y$ is chosen to be Stein, $A$ is globally big. Therefore, up to passing to a birational model, one has globally

$$A = A_X + E_X$$

where $A_X$ is an ample $\mathbb{Q}$-bundle on $X$ and $E_X$ is an effective $\mathbb{Q}$-divisor on $X$ such that $supp(E_X + E)$ is normal crossing. We have thus

$$K_X + L = A + E = A_\delta + E_\delta$$

and

$$\omega_{\phi_y} = \omega_{\phi_y} + [\Gamma_\varepsilon].$$
where $A_\delta := (1 - \delta)A + \delta A_X$ and $E_\delta := E + \delta E_X$.

Now we work on the fixed fiber $X_y$. As $y \in Y^\circ$ is now fixed, one will replace $X_y$ by $X$ and drop the index $y$ in the relevant line bundles and weights that will be considered in the following. We pick $\omega_0 \in c_1(A_X)$ a Kähler form and we choose a smooth globally semipositive representative $\omega_A$ in $c_1(A)$. Finally we consider for $\delta > 0$, the fiberwise Kähler form

$$\omega_\delta := (1 - \delta)\omega_A + \delta \omega_0 \in c_1(A_\delta)$$

Also, we will denote

$$E := \sum_{i=1}^k a_i E_i, \quad E_\delta = \sum_{i=1}^k a_i^\delta E_i \quad \text{and} \quad B := \sum_{j=1}^r (1 - b_j) B_j$$

With these notations,

(1.28) \hspace{1cm} a_i^\delta = a_i + \delta c_i

for some rational constant $c_i \geq 0$. Let us choose sections $s_i$ (resp. $t_j$) cutting out $E_i$ (resp. $B_j$) as well as smooth hermitian metrics $h_{E_i}$ (resp. $h_{B_j}$) on $\mathcal{O}_X(E_i)$ (resp. $\mathcal{O}_X(B_j)$). This defines the metric $h_E = \prod h_{E_i}^{a_i}$ (resp. $h_{E_\delta} = \prod h_{E_i}^{a_i^\delta}$, $h_B := \prod h_{B_j}^{b_j}$) on $E$ (resp. on $E_\delta, B$). Let $\omega$ be Kähler form on $X$ such that

(1.29) \hspace{1cm} - \text{Ric} \omega^n + \Theta(B, h_B) + \Theta(\Lambda, h_{\Lambda}) = \omega_A + \Theta(E, h_E)

The Kähler-Einstein metric $\omega_\varphi = \omega_A + dd^c \varphi$ of $(X, B + \Lambda, \phi_B + \phi_\Lambda)$ satisfies the following Monge-Ampère equation on $X$:

(1.30) \hspace{1cm} (\omega_A + dd^c \varphi)^n = |s_E|^2 e^{\varphi} \prod_i \frac{\omega^n}{|s_i|^{2(1-1/n)}}

If we work with approximate conic metrics, we will face positivity issues in the next sections involving the variation of Bergman kernels; and if we work with conic metrics directly, we will lack regularity to chose geodesic coordinates - which is a crucial part of the forthcoming argument. To overcome this difficulty, one "replaces" the divisor by an orbifold one. More precisely, one chooses a large integer $N$ such that $\frac{1}{N} < b_j$ for all $j$, and one sets

$$B_N := \sum_{j=1}^r \left(1 - \frac{1}{N}\right) B_j$$

The behavior of $\varphi$ is mostly unknown near the support of $B$ or $E$: so instead of working with that metric directly, one regularizes it partially. More precisely, one will make the cohomology class Kähler, remove the zeros of the RHS along $E$, and transform the poles into "orbifold poles", the crucial point of this approximation is that we get an orbifold metric (whose behavior is well known) whose Monge-Ampère is at least as singular as the initial Monge-Ampère of the Kähler-Einstein metric. More precisely, for $\delta, \varepsilon > 0$, one solves:

$$\omega_\delta + dd^c \varphi_{\varepsilon, \delta}^\varepsilon = (|s_E|^2 + \varepsilon^2) \prod_i \frac{(|s_i|^2 + \varepsilon^2)^{b_i}}{|s_i|^{2(1-1/n)}} \cdot \omega^n$$

or equivalently

(1.31) \hspace{1cm} (\omega_\delta + dd^c \varphi_{\varepsilon, \delta})^n = \frac{|s_E|^2 + \varepsilon^2 (|s_{BN} - B|^2 + \varepsilon^2)}{|s_{BN}|^2} e^{\varphi_{\varepsilon, \delta}} \omega^n.

Furthermore, $|s_E|^2 + \varepsilon^2$ has to be interpreted as $\prod_i (|s_i|^2 + \varepsilon^2)^{a_i}$, and similarly for the terms involving $B_N$ and $B$. For any $\varepsilon, \delta > 0$, $\omega_{\varphi_{\varepsilon, \delta}} := \omega_\delta + dd^c \varphi_{\varepsilon, \delta}$ is an orbifold metric with cone..
angles $\frac{2\pi}{m}$ along $\text{Supp}(B)$. Therefore, $e^{-\varphi_{\varepsilon,\delta}}\omega_{\varphi_{\varepsilon,\delta}}^m$ converges to $e^{-\varphi}\omega^m$ when $\varepsilon$ tend to zero. Better, comparing (1.30) to (1.31), one observes

**1.4. Approximate Kähler-Einstein metrics and Ricci iteration.** —

**1.4.1. Setting.** — Because none of the divisors $E, B, \Lambda$ are integral in general, one cannot directly set up a Bergman kernel iteration that will converge to the Kähler-Einstein metric. Instead, one performs for each integer $m \geq 1$ an iteration that will converge to a twisted Kähler-Einstein metric depending on $m$, and that sequence of metrics will itself converge to the genuine Kähler-Einstein metric when $m \to +\infty$.

Let us fix a integer $p \geq 1$ such that $pE, pB$ and $pL$ are integral, and set $k = \frac{p-1}{p}$. One defines $\varphi_{0,\varepsilon,\delta} := 0$ and solve inductively the Monge-Ampère equations for $\varepsilon, \delta > 0$:

$$
(1.32) \quad (p\omega_\delta + dd^c\varphi_{m,\varepsilon,\delta})^n = e^{\varphi_{m,\varepsilon,\delta}-k\varphi_{m-1,\varepsilon,\delta}} \left( \frac{|s_E|^2 + \varepsilon^2}{|s_B|^2} \right) \omega^n
$$

where the notations are borrowed from the previous section.

One can check that the induction is well defined as at each step, the potential $\varphi_{m,\varepsilon,\delta}$ is bounded; more precisely, $\omega_\delta + dd^c\varphi_{m,\varepsilon,\delta}$ defines a Kähler current with orbifold singularities along the divisor $B_N$. In particular, its lift to local covers associated with $B_N$ is a smooth Kähler metric. We will also use the notation $\phi_{m,\varepsilon,\delta}$ for the induced weight on $pA_\delta$, that is, $dd^c\varphi_{m,\varepsilon,\delta} = p\omega_\delta + dd^c\varphi_{m,\varepsilon,\delta}$.

Thanks to [EGZ09], one can also define the induction above for $\delta = \varepsilon = 0$, and get bounded potentials $\varphi_{m,0,0}$ for every integer $m$. We write $\varphi_m := \varphi_{m,0,0}, \phi_m := \phi_{m,0,0}$ and $\omega_m := p\omega_A + dd^c\varphi_m$. The function $\varphi_m$ is $p\omega_A$-psh and satisfies:

$$
(1.33) \quad (p\omega_\delta + dd^c\varphi_m)^n = \left( \frac{|s_E|^2 e^{\varphi_m - \frac{p-1}{p}\varphi_{m-1}}}{|s_B|^2} \right) \omega^n
$$

Therefore $\omega_m$ satisfied the following twisted Kähler-Einstein-like equation:

$$
(1.34) \quad \text{Ric} \omega_m = -\omega_m + \frac{p-1}{p}\delta \omega_{m-1} - [E] + [B] + \frac{i}{2\pi} \Theta_{K_h}(\Lambda).
$$

**1.4.2. Convergence of the approximate KE metrics.** — Our earlier notation is consistent with the following result:

**Proposition 1.15.** — For each $m$, the function $\varphi_m$ is the uniform limit of $\varphi_{m,\varepsilon,\delta}$ when $\varepsilon, \delta$ converge to 0, that is:

$$
||\varphi_m - \varphi_{m,\varepsilon,\delta}||_{L^\infty(X)} \underset{\varepsilon, \delta \to 0}{\longrightarrow} 0
$$

**Proof.** — The proof is by induction on $m$. Assume the conclusion holds for the index $m-1$. Set $d\mu = \frac{|s_E|^2 e^{-k\varphi_{m-1}}}{|s_B|^2} \omega^n$ and $d\mu_{\varepsilon,\delta} = \frac{e^{-k\varphi_{m-1,\varepsilon,\delta}} \left( |s_E|^2 + \varepsilon^2 \right) |s_B|^2}{|s_B|^2} \omega^n$, so that $(p\omega_A + dd^c\varphi_m)^n = e^{\varphi_m} d\mu$ and $(p\omega_\delta + dd^c\varphi_{m,\varepsilon,\delta})^n = e^{\varphi_{m,\varepsilon,\delta}} d\mu_{\varepsilon,\delta}$. Remember that $\omega_\delta = (1-\delta)\omega_A + \delta \omega_0$ where $\omega_0$ is Kähler.

The key observation is that although $\varphi_m$ need not be $p\omega_A$-psh, $(1-\delta)\varphi_m$ certainly is as $p\omega_\delta + dd^c(1-\delta)\varphi_m = (1-\delta)\omega_m + \delta \omega_0$ where $\omega_m = p\omega_A + dd^c\varphi_m$. In particular,

$$(p\omega_\delta + dd^c(1-\delta)\varphi_m)^n = (1-\delta)^n \omega_m^n + \sum_{k=1}^n (\delta p)^k (1-\delta)^{n-k} \omega_0^k \land \omega_m^{n-k}$$

We proceed in two steps.

**Step 1.**
Let $C > 0$ be any positive constant and let us consider $U := \{ \varphi_{m,\epsilon,\delta} < \varphi_m - C \}$. By the comparison principle, one has
\[
\int_U (p\omega_\delta + dd^c(1-\delta)\varphi_m)^n \leq \int_U (p\omega_\delta + dd^c\varphi_{m,\epsilon,\delta})^n
\]
and therefore
\[(1-\delta)^n \int_U e^{\varphi_m} d\mu \leq \int_U e^{\varphi_{m,\epsilon,\delta}} d\mu_{\epsilon,\delta}\]
By the choice of $U$, this implies:
\[(1-\delta)^n \int_U e^{\varphi_m} d\mu \leq e^{-C} \int_U e^{\varphi_m} d\mu_{\epsilon,\delta}\]
By dominated convergence and the induction assumption for $m-1$, $\int_U e^{\varphi_m} d\mu_{\epsilon,\delta}$ converges to $\int_U e^{\varphi_m} d\mu$ when $\epsilon, \delta \to 0$. We conclude that for $\epsilon, \delta$ small enough, $\mu_{\epsilon,\delta}(U) = 0$. This implies that $\varphi_{m,\epsilon,\delta} \geq \varphi_m - C$ almost everywhere with respect to $MA(\varphi_{m,\epsilon,\delta})$. By the domination principle, this implies that the inequality holds everywhere on $X$. We conclude that
\[(1.35) \quad \liminf_{\epsilon,\delta \to 0} (\varphi_{m,\epsilon,\delta} - \varphi_m) \geq 0.\]

**Step 2.**
Now, let $V := \{ \varphi_m < \varphi_{m,\epsilon,\delta} - C \}$. By the comparison principle, one has
\[
\int_V (p\omega_\delta + dd^c\varphi_{m,\epsilon,\delta})^n \leq \int_V (p\omega_\delta + dd^c(1-\delta)\varphi_m)^n
\]
and therefore
\[
\int_V e^{\varphi_{m,\epsilon,\delta}} d\mu_{\epsilon,\delta} \leq (1-\delta)^n \int_V e^{\varphi_m} d\mu + \sum_{k=1}^n (\delta p)^k (1-\delta)^{n-k} \int_V \omega_0^k \wedge \omega_m^{n-k}
\]
hence
\[
e^{C} \int_V e^{\varphi_m} d\mu_{\epsilon,\delta} \leq \int_V e^{\varphi_m} d\mu + O(\delta)
\]
Once again, one can use dominated convergence to get
\[(1.36) \quad \liminf_{\epsilon,\delta \to 0} (\varphi_m - \varphi_{m,\epsilon,\delta}) \geq 0.\]
Combining (1.35) and (1.36) yields the expected result for the integer $m$.

We are now left to prove the result for $m = 1$. But for $m = 1$, the measure $d\mu_{\epsilon,\delta}$ (which happens to depend only on $\epsilon$) clearly converges to $d\mu$, and the same argument as above can be reproduced. This ends the proof.

**1.4.3. Convergence of the Ricci iteration. —**

**Proposition 1.16.** — When $m$ tends to $+\infty$, the current $\frac{1}{p}\omega_m$ converges weakly to the (unique) twisted Kähler-Einstein metric $\omega_\infty \in c_1(A)$ solution of
\[-\text{Ric}_{\omega_\infty} + [B] + \frac{i}{2\pi} \Theta_h(A) = \omega_\infty + [E]\]

**Remark 1.17.** — Recall that $\omega_\infty$ is related to the Kähler-Einstein metric $\omega_{\text{KE}}$ of $(X, L, \phi_L)$ by the relation $\omega_{\text{KE}} = \omega_\infty + [E]$.
Proof. — Recall that \( \omega_m = p\omega_A + dd^c\varphi_m \) is solution of the Monge-Ampère equation
\[
(p\omega_A + dd^c\varphi_m)^n = e^{\varphi_m - \frac{p-1}{p} \varphi_{m-1}} d\mu
\]
where \( d\mu = \frac{|s_E|^2}{|s_B|^2} \omega^n \). We aim to show that for each \( m \geq 2 \), one has
\[
\text{(1.37)} \quad \|\varphi_m - \varphi_{m-1}\|_{L^\infty(X)} \leq \frac{p-1}{p} \|\varphi_{m-1} - \varphi_{m-2}\|_{L^\infty(X)}
\]
Let \( C_m := \sup_X (\varphi_m - \varphi_{m-2}) \), and let \( U_m = \{ \varphi_m > \varphi_{m-1} + \frac{p-1}{p} C_m \} \). An application of the comparison principle yields:
\[
\text{(1.38)} \quad \int_{U_m} e^{\varphi_m - \frac{p-1}{p} \varphi_{m-1}} d\mu \leq \int_{U_m} e^{\varphi_{m-1} - \frac{p-1}{p} \varphi_{m-2}} d\mu.
\]
On \( U_m \), one has:
\[
\varphi_m - \frac{p-1}{p} \varphi_{m-1} > \frac{1}{p} \varphi_{m-1} + \frac{p-1}{p} C_m = (\varphi_{m-1} - \frac{p-1}{p} \varphi_{m-2}) + \frac{p-1}{p} [C_m - (\varphi_{m-1} - \varphi_{m-2})] \geq \varphi_{m-1} - \frac{p-1}{p} \varphi_{m-2}.
\]
Together with (1.38), we know that \( U_m \) has measure zero with respect to \( d\mu \), hence also with respect to \( (p\omega_A + dd^c\varphi_m)^n \). By the domination principle, cf. e.g. \([\text{BEGZ10}, \text{Cor. 2.5}]\), we see \( U_m \) is empty, hence \( \varphi_m - \varphi_{m-1} \leq \frac{p-1}{p} \sup_X (\varphi_{m-1} - \varphi_{m-2}) \). Using an analogous argument, one can show that \( \varphi_m - \varphi_{m-1} \geq \frac{p-1}{p} \inf_X (\varphi_{m-1} - \varphi_{m-2}) \), which proves (1.37). It follows by iteration that
\[
\|\varphi_m - \varphi_{m-1}\|_{L^\infty(X)} \leq (\frac{p-1}{p})^{m-1} \|\varphi_1\|_{L^\infty(X)}
\]
and therefore the sequence \( (\varphi_m)_{m \geq 1} \) converges uniformly to a \( p\omega_A \)-psh function \( \varphi_\infty \). As Bedford-Taylor product is continuous with respect to uniform convergence, \( \varphi_\infty \) satisfies:
\[
(p\omega_A + dd^c\varphi_\infty)^n = e^{\frac{1}{p} \varphi_\infty} d\mu
\]
which proves the proposition. \( \square \)

1.5. Convergence of the Bergman kernel iteration and proof of Theorem 1.3. — In this paragraph, we prove that the twisted Kähler-Einstein metric \( \omega_\infty \) is the weak limit of iterated Bergman kernels. Let \( m, \ell \geq 1 \) be two positive integers. We assume that \( (X, L, h_L) \) satisfies
• \( K_X + L = A + E \) is a Zariski decomposition of the big line bundle \( K_X + L \)
• The hermitian \( \mathbb{Q} \)-line bundle \( (L, h_L) \) decomposes as \( (B + \Lambda, h_B \otimes h_\Lambda) \) where \( B \) is an effective \( \mathbb{Q} \)-divisor with \( \text{snc} \) support, \( h_B \) is the canonical singular metric on \( \mathcal{O}_X(B) \) and \( h_\Lambda \) is a smooth hermitian metric on \( \Lambda \) with semipositive curvature.

One decomposes the twisted pluricanonical bundle as follows:
\[
(\ell + 1)p(K_X + L) = K_X + \ell p(K_X + L) + (p - 1)(K_X + L) + L
\]
Given the decomposition above, one can define for any \( \ell \geq 1 \) a singular metric \( h_{\ell, m} \) on the line bundle \( \ell p(K_X + L) \) by induction on \( \ell \) in the following fashion:
\[
\text{(1.39)} \quad h_{\ell+1, m} := K_{\ell+1, m}^{-1} \quad K_{\ell+1, m} := K(X, (\ell + 1)p(K_X + L), h_{\ell, m} e^{- (p-1) \left( \frac{\varphi_m}{p} + \phi_E \right)}, e^{-\phi_L})
\]
is the Bergman kernel of \((\ell + 1)p(K_X + L)\) endowed with the metric above. Recall that \(\phi_m\) is the bounded psh weight on \(pA\) solving the Monge-Ampère equation \((1.33)\) with the convention that \(\phi_0 = \phi_A\) and \(\phi_E\) is a singular weight on \(\mathcal{O}_X(E)\) such that \(dd^c\phi_E = [E]\). The aim of this subsection is to prove the following result, from which Theorem 1.18 will follow easily.

**Theorem 1.18.** — Assuming that \((X, L, h_L)\) satisfies the assumptions above, the renormalized Bergman kernels \((\ell^{-n}K_{\ell,m})^{1/\ell}\) converge to \(e^{\phi_m + \phi_E}/\ell!\) when \(\ell \to +\infty\).

Before proving the proposition, we need some preparation. We first study the singularities of \(h_{\ell,m}\). Because \(K_X + L = A + E\) is a Zariski decomposition, one knows that if \(p\) is divisible enough, the multiplication by \(s_{pE}\) induces an isomorphism \(H^0(X, pA) \to H^0(X, p(K_X + L))\). Therefore all sections \(s \in H^0(X,\ell p(K_X + L))\) vanish along \(pE\) at order at least \(\ell\) and as \(A\) is semi-ample, there exists always one such section that vanishes exactly at order \(\ell\) along \(pE\). An easy induction shows that every section \(s \in H^0(X, (\ell + 1)p(K_X + L))\) is square integrable with respect to \(h_{\ell,m}. e^{-(\ell - 1)[s_{pE}]} = \cdot e^{-\phi_L}\).

We choose smooth hermitian metrics \(h_{E_i}\) on \(\mathcal{O}_X(E_i)\) along with global sections \(s_{E_i} \in H^0(X, \mathcal{O}_X(E_i))\) such that \(\text{div}(s_{E_i}) = E_i\). Then, we define

\[
|s_{E_i}|^2 := \prod_i |s_{E_i}|^{2(\ell p a_i^\delta)}
\]

where \(|s_{E_i}|^2\) denotes the squared norm of \(s_{E_i}\) with respect to \(h_i\). For each \(i\), let us denote by \(\phi_i\) the weight of \(h_{E_i}\). It induces a smooth weight \(\phi_{\ell i}^{\ell p} := \sum_i (\ell p a_i^\delta) \phi_i\) on \(\ell p E_\delta\), hence \(e^{\phi_{\ell i}^{\ell p}}\) is a metric on \(\ell p E_\delta\). It is also used the notation \(\phi_{E_i}\) to denote the singular weight on the \(Q\)-line bundle \(\mathcal{O}_X(E_\delta)\) defined by \(\phi_{E_i} := \sum_i a_i^\delta \phi_{E_i}\), where \(\phi_{E_i}\) is a singular weight on \(E_i\) such that \(dd^c\phi_{E_i} = [E_i]\) and \(\sum a_i \phi_{E_i} = \phi_E\).

Later in the proof, it will be important to have local equations of \(E\) at hand. More precisely, we will be working on a small open set \(U \subset X\) as \(pE\) has integral coefficients, one can choose a multivalued equation \(f_E\) of \(E\) on \(U\) such that \(f_E^p \in \mathcal{O}_X(U)\) is well defined. For \(E_\delta\) though, it is not possible to do a similar thing for a fixed power \(p\). However for any \(p \geq 0\), one can still define the quantity

\[
|f_{E_\delta}|^{2(\ell p a_i^\delta)} := \prod_i |f_{E_i}|^{2(\ell p a_i^\delta)}
\]

as well as its norm analogue

\[
|f_{E_\delta}|^{2(\ell p a_i^\delta)} := \prod_i |f_{E_i}|^{2(\ell p a_i^\delta)}
\]

where \(f_{E_\delta} \in \mathcal{O}_X(U)\) a holomorphic equation of \(E_i\).

In this section, one will choose \(\delta = \varepsilon\), and denote by

\[
\phi_{m,\varepsilon} := \phi_{m,\varepsilon,\varepsilon} \quad \text{and} \quad \omega_{m,\varepsilon} := \omega_{m,\varepsilon,\varepsilon}
\]

With these notations, \(|s_{E_i}|^{-2(\ell p) e^{-\phi_{m,\varepsilon} - \phi_{pE_i}}}\) defines a singular metric on \(\ell p(A_\varepsilon + E_\varepsilon) = \ell p(K_X + B + \Lambda)\).

As we explained above, \(K_{\ell,m}\) behaves like \(|f_E|^{-2\ell p}\) near \(E\) and is bounded above elsewhere. As \(E_\varepsilon \geq E\), the quantity

\[
C_{\ell,\varepsilon} := \inf_X \frac{K_{\ell,m}}{|s_{E_i}|^{2(\ell p) e^{-\phi_{m,\varepsilon} - \phi_{pE_i}}}}
\]

is strictly positive. The following key proposition gives some information about its asymptotics when \(\ell \to +\infty\):
Proposition 1.19. — For every $m$ fixed, there exists $\kappa_{\epsilon,t}>0$ such that:

$$C_{\ell,t} \geq \kappa_{\epsilon,t} \frac{\ell^n}{n!} C_{\ell-1,t}$$

and

$$\lim_{\epsilon \to 0} \lim_{t \to +\infty} \left( \prod_{k=1}^{t} \kappa_{\epsilon,k} \right)^{1/t} = 1.$$

Proof. — The existence of the constant $\kappa_{\epsilon,t}>0$ is guaranteed by the observation above. We are left to prove the statement about the limit. The proof is divided in four main steps.

Step 1. Choice of an appropriate local section $u$.

Let $x \in X \setminus \text{Supp}(B)$, and let $U$ be a small neighborhood of $x$. Let $e_U \in H^0(U, p(K_X + L))$ be a non-vanishing section. Let $f_{B_j}$ be a holomorphic function on $U$ whose divisor is $B_j \cap U$. There exist smooth functions $\psi_{E_i}, \psi_{B_j}$ on $U$ such that

$$|s_{E_i}|^2 = |f_{E_i}|^2 e^{-\psi_{E_i}} \quad \text{and} \quad |s_{B_j}|^2 = |f_{B_j}|^2 e^{-\psi_{B_j}} \quad \text{on } U.$$ 

As in the above section, $|s_{E_i}|$ (resp. $|s_{B_j}|$) denotes the norm of $s_{E_i}$ (resp. $s_{B_j}$) with respect to a $h_{E_i}$ (resp. $h_{B_j}$). Up to multiplying $f_{E_i}, f_{B_j}$ by a constant, we can suppose that

$$\psi_{E_i}(x) = \psi_{B_j}(x) = 0.$$ 

In the following, we set $\psi_E := \sum_i a_i \psi_{E_i}$ and $\psi_B := \sum_j (1 - b_j) \psi_{B_j}$.

Let $\psi_{m,\epsilon}$ be the weight function defined by

$$|e_U|^2 |\phi_{m,\epsilon} + p\psi_{E_i}| = |f_{E_i}|^{-2p} e^{-\psi_{m,\epsilon}} \quad \text{on } U,$$

Similarly, one can define a weight function $\psi_{m-1}$ such that $|e_U|^2 |\phi_{m-1} + p\psi_{E_i}| = |f_{E_i}|^{-2p} e^{-\psi_{m-1}}$. Let us choose some coordinates $(z_i)$ on $U$ inducing a trivializing section $dz$ of $K_X$ over $U$. Then $e_U \otimes (dz)^{\otimes -1} \in H^0(U, (p - 1)(K_X + L) + B + \Lambda)$, hence one defines a smooth weight function $\psi_{\Lambda}$ on $U$ such that

$$|e_U|^2 e^{-(p-1)(\phi_{m-1} + \psi_{\Lambda}) - \phi_B - \psi_{\Lambda}} = e^{-\frac{p-1}{p} \psi_{m-1} - \psi_{\Lambda} |f_{E_i}|^{-2p} |f_B|^{-2} |dz|^2} \quad \text{on } U.$$

The Monge-Ampère equation satisfied by $\psi_{m,\epsilon}$ on $U$ reads:

$$(dd^c \psi_{m,\epsilon})^n = \frac{(|s_{E_i}|^2 + \epsilon^2)(|s_{B_j}|^2 + \epsilon^2)}{|s_{B_j}|^2} e^{\psi_{m,\epsilon} - k \psi_{m-1,\epsilon} e^{\psi_{E_i} - \psi_B - \psi_{\Lambda} + G} |dz|^2}$$

for some pluriharmonic function $G$ on $U$. Up to changing the coordinates $(z_i)$, one can assume that $G(x) = 0$. Note that this operation will change the weight $\psi_{\Lambda}$.

Finally, set

$$h := \psi_{m,\epsilon}(x) + 2 \sum_{i=1}^{n} \partial_i \psi_{m,\epsilon}(x) \cdot z_i + 4 \sum_{1 \leq i < j \leq n} \partial_{ij} \psi_{m,\epsilon}(x) \cdot z_i z_j \quad \text{on } U,$$

and let us define:

$$u := e^{(\ell+1)h/2} f_{E_i}^{[(\ell+1)p]} e_U^{[(\ell+1)]} \in H^0(U, (\ell + 1)p(K_X + B + \Lambda)).$$

It is easy to see that

$$|u|^2(x) = |s_{E_i}|^{2[(\ell+1)p]} e^{(\ell+1)(\phi_{m,\epsilon} + p\psi_{E_i})}(x),$$

which is another way to say that the squared norm of $u$ at $x$ with respect to the singular metric $|s_{E_i}|^{-2[(\ell+1)p]} e^{-(\ell+1)(\phi_{m,\epsilon} + p\psi_{E_i})}$ is equal to 1.
Step 2. A pointwise estimate of $|u|^2$.

We define two metrics on the line bundle $(\ell + 1)p(K_X + B + \Lambda) - K_X$. First, let

$$h_1 := h_{\ell,m} e^{-(p-1)\left(\frac{\phi_m}{p} + \phi_E\right)} e^{-\phi_B - \phi}\,,$$

bet the Bergman kernel type metric defined in (1.39), and let

$$h_{\ell,\varepsilon} := |s_{E,\varepsilon}|^{-2(p)} e^{-\ell \phi_{m,\varepsilon} - \ell \psi_E, \varepsilon} e^{-(p-1)\left(\frac{\phi_m}{p} + \phi_E\right)} e^{-\phi_B - \phi}\,.$$ 

By (1.40), we have

$$(1.47) \quad h_1 \leq C_{\ell,\varepsilon}^{-1} h_{\ell,\varepsilon}\,.$$

We would like to estimate the volume form $|u|^2_{h_{\ell,\varepsilon}}$ on $U$. By (1.43) and (1.44), we have

$$|u|^2_{h_{\ell,\varepsilon}} = H \cdot e^{-(\ell+1)(\psi_m - \text{Re}(h))} e^{\psi_m - k\psi_{m-1} - \psi_B} e^{k(\psi_{m-1} - \psi_m - \psi_B)} |f_E|^2 |dz|^2$$

where $H := |s_{E,\varepsilon}|^{-2(p)} \prod_i |f_{E_i}|^2(\ell(1 + p\varepsilon) - \ell p_a - p_a)$. One can rewrite $H$ using (1.28) and (1.41) to get

$$H = e^{\psi_{(\ell,\varepsilon)}} \prod_i |f_{E_i}|^2(\ell(1 + p\varepsilon c_i) - \ell p c_i)$$

In particular, if follows from (1.42) that the following inequality holds

$$(1.48) \quad H(x) \leq 1\,.$$

Thanks to (1.45), there exists a constant $\tilde{C}_{\varepsilon}$ converging to 1 when $\varepsilon \to 0$ such that

$$(dd^c \psi_{m,\varepsilon})^n \geq \tilde{C}_{\varepsilon} |s_E|^2 \frac{2}{|s_B|^2} e^{\psi_m - k\psi_{m-1} - \psi_B} e^{G}|dz|^2.$$

Therefore

$$(1.49) \quad \frac{|s_E|^2}{|s_B|^2} e^{\psi_m - k\psi_{m-1} - \psi_B} e^{G}|dz|^2 \leq \tilde{C}_{\varepsilon} e^{-G} (dd^c \psi_{m,\varepsilon})^n.$$

Note that $|s_E|^2, |s_B|^2, e^{\psi_B - \psi_E} = |f_E|^2, |f_B|^2$ so that we get a pointwise estimate:

$$(1.50) \quad |u|^2_{h_{\ell,\varepsilon}} \leq \tilde{C}_{\varepsilon} \gamma_{\varepsilon} H e^{-G} e^{-(\ell+1)(\psi_m - \text{Re}(h))} (dd^c \psi_{m,\varepsilon})^n$$

on $U$, where $\gamma_{\varepsilon} := \sup_{x} e^{k(\psi_{m-1} - \psi_m - \psi_B)}$ and recall that by Proposition 1.15, $\gamma_{\varepsilon} \to 1$ when $\varepsilon$ tend to 0. We set $F := e^{-G} H$; this is a continuous function on $U$ that satisfies $F(x) \leq 1$. Note that $F = F_1$ actually depends on $\ell$ but the dependence is essentially harmless, in the sense that the family of functions $(F_\ell)_{\ell \geq 1}$ is equicontinuous on $U$.

Now, let us pick a number $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and let $U_{\ell-\alpha}$ be the ball centered at $x$ of radius $\ell^{-\alpha}$ with respect to $dd^c \psi_{m,\varepsilon}$.

**Claim 1.20.** — For every $\varepsilon > 0$ fixed, there exists a sequence $a_\ell$ converging to 0 (independent of $x \in X$), such that

$$(1.51) \quad \frac{(\ell + 1)^n}{n!} \int_{U_{\ell-\alpha}} F e^{-(\ell+1)(\psi_m - \text{Re}(h))} (dd^c \psi_{m,\varepsilon})^n \leq 1 + a_\ell \quad \text{for every } \ell \in \mathbb{N},$$

and

$$(1.52) \quad (\ell + 1)^{n+2(\alpha+1)} \int_{U_{\ell-\alpha} \setminus U_{\frac{1}{2} \ell-\alpha}} F e^{-(\ell+1)(\psi_m - \text{Re}(h))} (dd^c \psi_{m,\varepsilon})^n \leq a_\ell \quad \text{for every } \ell \in \mathbb{N}.$$
An important point of the claim is that the sequence \( \{ a_\ell \} \) is independent of \( x \) but depends strongly on \( \varepsilon \). We will see from the proof that \( a_\ell \) is of order \( O(\ell^{1-3\alpha}) \). For the sake of clarity, the claim will be established right after the end of the proof of Proposition 1.19, on page 27.

**Step 3. Construction of a global section.**

For every \( \varepsilon > 0 \) fixed, we will construct in this step a section \( v_\ell \in H^0(\calX, (\ell + 1)p(K_X + L)) \) such that

\[
\frac{(\ell + 1)^n}{n!} \cdot \int_X |v_\ell|_{h_x, \varepsilon}^2 \leq \tilde{C}_\varepsilon \gamma_\varepsilon (1 + b_\ell) \quad \text{and} \quad v_\ell(x) = u(x).
\]

for a sequence \( \{ b_\ell \}_{\ell = 1}^{+\infty} \) converging to 0 which is independent of \( x \in \calX \).

Let \( \rho \) be a smooth function on \( \calX \setminus \{ x \} \) which behaves like \( n \log d_{\omega_{m, \varepsilon}}(\cdot, x) \) near \( x \). For every \( \ell \), we can find a cut-off function \( \chi_\ell \) for \( U_{\ell-\alpha} \), namely \( \chi_\ell \equiv 1 \) on \( U_{\frac{1}{2} \ell-\alpha} \) and \( \chi_\ell \equiv 0 \) on \( \calX \setminus U_{\ell-\alpha} \) such that

\[
e^{-\rho} \partial \overline{\partial} \chi_\ell \omega_{m, \varepsilon}^2 \leq M_\varepsilon (2\ell + 2)^n \quad \text{on} \quad U_{\ell-\alpha} \setminus U_{\frac{1}{2} \ell-\alpha}
\]

for some constant \( M_\varepsilon \) independent of \( \ell \) and \( x \in \calX \). Moreover, by construction, for \( \ell \) large enough, we have

\[
\frac{i}{2\pi} \Theta_{h_x, \varepsilon} + dd^c \rho \geq \omega_{m, \varepsilon} . \quad \text{on} \quad \calX.
\]

Thanks to (1.50) and (1.51), we have

\[
\frac{(\ell + 1)^n}{n!} \cdot \int_X |\chi_\ell u|_{h_x, \varepsilon}^2 \leq \tilde{C}_\varepsilon \gamma_\varepsilon (1 + a_\ell).
\]

Thanks to (1.50) and the construction of \( \chi_\ell \), we have

\[
\int_X |\partial \overline{\partial} (\chi_\ell u)|_{h_x, \varepsilon}^2 \omega_{m, \varepsilon}^n e^{-\rho} \leq \tilde{C}_\varepsilon \gamma_\varepsilon \int_{U_{\ell-\alpha} \setminus U_{\frac{1}{2} \ell-\alpha}} e^{-\rho} |\partial \overline{\partial} \chi_\ell \omega_{m, \varepsilon}^2} \cdot F \cdot e^{-((\ell + 1)(\psi_{m, \varepsilon} - \Re(h))) (dd^c \psi_{m, \varepsilon})^n}.
\]

Together with (1.52) and (1.54), we get

\[
(\ell + 1)^n \int_X |\partial \overline{\partial} (\chi_\ell u)|_{h_x, \varepsilon}^2 \omega_{m, \varepsilon}^n e^{-\rho} \leq a_\ell \cdot M_\varepsilon.
\]

Applying Hörmander estimates, thanks to (1.55), for \( \ell \) large enough (independent of \( x \in \calX \)), one can find a \( u_\ell \) such that \( \partial u_\ell = \partial \chi_\ell u \) and

\[
(\ell + 1)^n \int_X |u_\ell|_{h_x, \varepsilon}^2 e^{-\rho} \leq a_\ell \cdot M_\varepsilon.
\]

Because of the non-integrability of \( e^{-\rho} \) at \( x \), one has \( u_\ell(x) = 0 \). As a consequence, the section \( v_\ell := \chi_\ell u - u_\ell \in H^0(\calX, (\ell + 1)p(K_X + L)) \) and satisfies (1.53).

**Step 4. Conclusion.**

Thanks to (1.47) and (1.53), we have

\[
\frac{(\ell + 1)^n}{n!} \cdot \int_X |v_\ell|_{h_1}^2 \leq C_{\ell, \varepsilon}^{-1} \cdot \tilde{C}_\varepsilon \gamma_\varepsilon (1 + b_\ell).
\]

Together with (1.46) and the definition of the Bergman kernel \( K_{\ell+1,m} \) (1.39), we obtain that the following inequality

\[
K_{\ell+1,m} \geq \frac{C_{\ell, \varepsilon}}{\tilde{C}_\varepsilon \gamma_\varepsilon \cdot (1 + b_\ell)} \frac{(\ell + 1)^n}{n!} \cdot |s_{E_x}|^{2((\ell + 1)p) e^{(\ell + 1)(\phi_{m, \varepsilon} + p\phi_{E_x})}}
\]

holds for every \( \ell \) large enough.
holds at \( x \). So far, \( x \) has been an arbitrary point of \( X \setminus \text{Supp}(B) \). As \( b_\ell \) is independent of \( x \in X \setminus \text{Supp}(B) \), by continuity, the identity above holds everywhere on \( X \), and therefore

\[
C_{\ell+1,\varepsilon} \geq \kappa_{\varepsilon,\ell} \cdot \frac{(\ell + 1)^n}{n!} \cdot C_{\varepsilon,\ell},
\]

where \( \kappa_{\varepsilon,\ell} = (\bar{C}_{\varepsilon} \gamma_{\varepsilon}(1 + b_\ell))^{-1} \). Although the sequence \( \{b_\ell\} \) depends on \( \varepsilon \), since it tends to 0, we have

\[
\lim_{\ell \to +\infty} \left( \prod_{k=1}^{\ell} \kappa_{\varepsilon,k} \right)^{1/\ell} = \bar{C}_{\varepsilon} \gamma_{\varepsilon}.
\]

Therefore

\[
\lim_{\varepsilon \to 0} \lim_{\ell \to +\infty} \left( \prod_{k=1}^{\ell} \kappa_{\varepsilon,k} \right)^{1/\ell} = 1.
\]

The proposition is proved.

It remains to prove Claim 1.20 stated in Step 3.

**Proof of Claim 1.20.** — Up to replacing \( \psi_{m,\varepsilon} \) by \( \psi_{m,\varepsilon} - \text{Re}(h) \) (which doesn’t change the metric \( d\bar{F} \psi_{m,\varepsilon} \)), one can assume that \( \psi_{m,\varepsilon} \) has no polyharmonic terms of order two or less in its expansion near \( x \), and \( F(x) = 1 \).

For fixed \( \varepsilon > 0 \), \( d\bar{F} \psi_{m,\varepsilon} \) is an orbifold Kähler metric on \( U \). In particular, one can pull it back to a finite Galois cover \( \pi_1 : V \to U \) of group \( G \), ramified along \( B_N \) to get a smooth \( G \)-invariant Kähler metric \( d\bar{F} \pi_1^* \psi_{m,\varepsilon} \) on \( V \). Set \( x_1 \) to be a preimage of \( x \) in \( V \) and let \( V_{\ell-a} \) the disc center at \( x_1 \) of radius \( \ell^{-a} \) with respect to \( \pi_1^* d\bar{F} \psi_{m,\varepsilon} \). For \( \ell \) large enough (independent of \( x \in X \setminus B \)), we have \( \pi_1(V_{\ell-a}) = U_{\ell-a} \). Therefore

\[
\int_{U_{\ell-a}} F e^{-(\ell+1)\psi_{m,\varepsilon}} (d\bar{F} \psi_{m,\varepsilon})^n \leq \int_{V_{\ell-a}} \pi_1^* F e^{-(\ell+1)\psi_{m,\varepsilon}} (\pi_1^* d\bar{F} \psi_{m,\varepsilon})^n.
\]

It is an equality if \( \pi_1 : V_{\ell-a} \to U_{\ell-a} \) is generically injective.

Let us introduce \((w_1, \ldots, w_n)\) a system of coordinates near \( x_1 \). Note that the choice of the coordinates \((z_i)\) downstairs guarantees that \( \pi_1^* \psi_{m,\varepsilon} \) has no order 2 polyharmonic terms in its series expansion near \( x_1 \) and its order three terms in its expansion are uniformly bounded on \( V \) independently of the choice of coordinates \((z_i)\) hence independently on \( x \in X \setminus B \).

Therefore

\[
\pi_1^* \psi_{m,\varepsilon} = \sum_{j,k} a_{j,k} w_j \bar{w}_k + O(|w|^3)
\]

where the order three terms \( O(|w|^3) \) has uniform bounded coefficients, and the matrix \( A = (a_{j,k}) \) is uniformly positive definite (and bounded) as well.

After the change of variable \( \tilde{w} := \sqrt{\ell + 1} \sqrt{A} w \), we have becomes

\[
\int_{V_{\ell-a}} e^{-[(\ell+1)\psi_{m,\varepsilon}]} (d\bar{F} \pi_1^* \psi_{m,\varepsilon})^n \leq \frac{n!}{(\ell + 1)^n} \int_D e^{-|\tilde{w}|^2 + O(\ell^{-1/2} |\tilde{w}|^3)} (1 + O(\ell^{-1})) \frac{|d\tilde{w}|^2}{\pi^n}
\]

where \( D \) is the image of \( V_{\ell-a} \) by the transformation. Recall that \( \alpha \in \left( \frac{1}{3}, \frac{1}{2} \right) \). Therefore, on \( D \), one has \( \ell^{-1/2} |\tilde{w}|^3 \sim \ell |w|^3 \leq \ell^{1-3\alpha} = o(1) \). In conclusion,

\[
\int_{V_{\ell-a}} F e^{-[(\ell+1)\psi_{m,\varepsilon}]} (d\bar{F} \psi_{m,\varepsilon})^n \leq \frac{n!}{(\ell + 1)^n} \pi^n \left( \int_{|\tilde{w}|^2 < \ell^{1-3\alpha}} e^{-|\tilde{w}|^2} + O(\ell^{1-3\alpha}) \right) = \frac{n!}{(\ell + 1)^n} (1 + O(\ell^{1-3\alpha}))
\]

where the correction factor \( O(\ell^{1-3\alpha}) \) is upper bounded by \( C \ell^{1-3\alpha} \) for some constant \( C \) independent of \( x \in X \) (but depends strongly on \( \varepsilon \)). (1.51) is proved.
As for \((1.52)\), note that \(\psi_{m,\varepsilon} - \text{Re}(h) = \sum_{j,k} a_{j,k} w_j \bar{w}_k + O(|w|^3)\). We have
\[
e^{-(\ell+1)(\psi_{m,\varepsilon} - \text{Re}(h))} \leq e^{-C\frac{\ell+2}{2n}} \quad \text{on } U_{\ell-a} \setminus U_{\frac{1}{2}\ell-a},
\]
for some uniform constant \(C > 0\). Therefore
\[
(\ell + 1)^{n+2(2n+2)\alpha} \int_{U_{\ell-a} \setminus U_{\frac{1}{2}\ell-a}} F e^{-(\ell+1)(\psi_{m,\varepsilon} - \text{Re}(h))}(dd^c \psi_{m,\varepsilon})^n \leq (\ell + 1)^{2n+2\alpha + n} e^{-C\frac{\ell+2}{2}n} \frac{1}{\ell^{2n}}.
\]
Together with the fact that \(\alpha < \frac{1}{2}\), \((1.52)\) is proved. \(\square\)

We will also need an integral upper estimate of \(K_{\ell,m}\); it follows almost easily from the definition of the Bergman kernel.

**Proposition 1.21.** — One has the following upper bound:
\[
\limsup_{\ell \to +\infty} \int_X (\ell! - n K_{\ell,m})^{1/\ell} e^{-(p-1)\left(\frac{m-1}{p} + \phi_E\right)} e^{-\phi_B - \phi_\Lambda} \leq \frac{(pA)^n}{n!}
\]

**Proof.** — First, observe that \((\ell! - n K_{\ell,m})^{1/\ell} e^{-(p-1)\left(\frac{m-1}{p} + \phi_E\right)} e^{-\phi_B - \phi_\Lambda}\) behaves like a volume form, so that the claim is licit. Let \((u_1, \ldots, u_{N_\ell})\) be an orthonormal basis of \(H^0(X, \ell p (K_X + L))\) with respect to the Bergman \(L^2\) metric
\[
K_{\ell-1,m}^{-1} e^{-(p-1)\left(\frac{m-1}{p} + \phi_E\right)} e^{-\phi_B - \phi_\Lambda}\]

One observed already that because of the Zariski decomposition for \(p(K_X + L) = pA + pE\), every (pluri)-section is \(L^2\) with respect to the Bergman metric. In particular, as \(A\) is semi-ample and big,
\[(1.56) \quad N_\ell = \dim H^0(X, \ell pA) = \frac{(pA)^n}{n!} \ell^n (1 + O(\ell^{-1}))\]
thanks to Riemann-Roch formula. In order to lighten notation, let us write (in this proof only) \(e^{-\phi} := e^{-(p-1)\left(\frac{m-1}{p} + \phi_E\right)} e^{-\phi_B - \phi_\Lambda}\). One has
\[
\int_X K_{\ell,m}^{-1} e^{-\phi} = N_\ell
\]
Therefore, applying Hölder’s inequality with \(p = \ell\) and \(q = \frac{\ell}{\ell - 1}\), one gets
\[
\int_X K_{\ell,m}^{-1} e^{-\phi} \leq \left( \int_X K_{\ell,m}^{-1} e^{-\phi} \right)^{\frac{1}{\ell}} \left( \int_X K_{\ell-1,m}^{-1} e^{-\phi} \right)^{\frac{\ell - 1}{\ell - \ell}} \leq N_\ell^{\frac{1}{\ell}} \left( \int_X K_{\ell-1,m}^{-1} e^{-\phi} \right)^{\frac{\ell - 1}{\ell - \ell}}
\]
By induction, one gets:
\[
\int_X K_{\ell,m}^{-1} e^{-\phi} \leq \left( \prod_{i=1}^\ell N_i \right)^{\frac{1}{\ell}}
\]
Now, thanks to \((1.56)\), the right-hand side of this inequality is equal to
\[
\frac{(pA)^n}{n!} \ell^\frac{1}{\ell} \left( 1 + O\left( \frac{\log \ell}{\ell} \right) \right)
\]
which concludes the proof of the proposition. \(\square\)

Now we can prove the main result of this subsection.
Proof of Theorem 1.18. — Thanks to Proposition 1.21 and Jensen inequality, \( \{(l!^{-n}K_{\ell,m})^{1/\ell}\}_{\ell=1}^{\infty} \) is a family of upper bounded psh weights. Therefore, to prove the Theorem, it is sufficient to prove that any convergent subsequence of \( \{(l!^{-n}K_{\ell,m})^{1/\ell}\}_{\ell=1}^{\infty} \) converges to \( \frac{e^{\phi_m + p\phi_E}}{n!} \).

Let \( \{(l!^{-n}K_{\ell,m})^{1/\ell}\}_{\ell=1}^{\infty} \) be a convergent subsequence of \( \{(l!^{-n}K_{\ell,m})^{1/\ell}\}_{\ell=1}^{\infty} \) and let \( \Gamma \) be the limit. Thanks to Proposition 1.19, one infers:

\[
(l!^{-n}K_{\ell,m})^{1/\ell} \geq \left( \prod_{k=1}^{\ell} K_{\varepsilon,k} \right)^{1/\ell} \frac{1}{n!} |s_E| \cdot e^{\phi_m + p\phi_E}.
\]

Letting \( \ell \) tend to \(+\infty\), and then letting \( \varepsilon \) tend to zero, Proposition 1.15 and Proposition 1.19 yield:

\[
\lim_{\ell \to +\infty} (l!^{-n}K_{\ell,m})^{1/\ell} \geq \frac{e^{\phi_m + p\phi_E}}{n!}.
\]

Therefore, we have

\[
\Gamma \geq \frac{e^{\phi_m + p\phi_E}}{n!}.
\]

Note that

\[
\int_X e^{-(p-1)\left( \frac{\phi_m - 1}{p} + \phi_E \right)} e^{-\psi - \phi_A} = \lim_{\ell \to +\infty} \int_X (l!^{-n}K_{\ell,m})^{1/\ell} e^{-(p-1)\left( \frac{\phi_m - 1}{p} + \phi_E \right)} e^{-\psi - \phi_A}.
\]

Combining this with Proposition 1.21, we get

\[
\int_X e^{-(p-1)\left( \frac{\phi_m - 1}{p} + \phi_E \right)} e^{-\psi - \phi_A} \leq \frac{(pA)^n}{n!} = \frac{1}{n!} \int_X e^{\phi_m + p\psi} e^{-(p-1)\left( \frac{\phi_m - 1}{p} + \phi_E \right)} e^{-\psi - \phi_A},
\]

where the last equality comes from (1.33). Together with (1.58), we get \( \Gamma = \frac{e^{\phi_m + p\phi_E}}{n!} \) and the theorem is proved.

Remark 1.22. — The convergence \( (l!^{-n}K_{\ell,m})^{1/\ell} \rightarrow \frac{e^{\phi_m + p\phi_E}}{n!} \) has been proved for a fixed fiber \( X_y \), but it readily implies convergence in \( L^1_{\text{loc}}(X^\circ) \). Indeed, as \( (A^n_y) \) is independent of \( y \in Y \), Proposition 1.21 coupled with Jensen inequality show that the weights of the metric \( (l!^{-n}K_{\ell,m})^{-1/\ell} \) are uniformly bounded above locally near \( X \times X_0 \), hence the pointwise convergence almost everywhere on \( X^\circ \) implies convergence in \( L^1_{\text{loc}}(X^\circ) \).

Now, we can finally give the proof of Theorem 1.3.

Proof of Theorem 1.3. — Thanks to the reduction steps, we can suppose that on the fibers over \( y \in Y \), we have a Zariski decomposition

\[
(K_{\ell/m} + B + \Lambda)|_{X_y} = A_y + E_y,
\]

where \( A \) is semi-ample and big, \( B_y + E_y \) has suc support and \( h_A \) is smooth with semipositive curvature.

We first prove by induction that for every \( m \in \mathbb{N} \), \( \phi_m + p\phi_E \) is a psh weight on \( X^0 \).

For \( m = 1 \): We get a sequence of metrics \( (h_{1,1})_{\ell \geq 1} \) on \( p\ell(K_X + L) \) defined by (1.39). Recall that \( \phi_0 \) is the weight of \( \omega_A \). Thanks to [BP08, Thm. 0.1], \( h_{1,1} \) has positive curvature on the total space \( X \). We suppose by induction that \( h_{1,1} \) has positive curvature. Then \( h_{1,1} \cdot e^{-(p-1)\left( \frac{\phi_0}{p} + \phi_E \right)} e^{-\psi} \) has also positive curvature. By applying [BP08, Thm. 0.1] again, \( h_{\ell+1,1} \) has positive curvature on the total space \( X \). As a consequence, \( h_{\ell,1} \) has positive curvature on the total space \( X \) for all \( \ell \). Together with Theorem 1.18, the limit \( \phi_1 + p\phi_E \) is a psh weight on \( X^0 \).
Then we apply the same process again to $m = 2$, and get a sequence of metrics $(h_{\ell,2})_{\ell \geq 1}$ on $p\ell(K_X + L)$ with positive curvature, and therefore the limit $\phi_2 + p\phi_E$ is psh. By induction on $m$, we know that $\phi_m + p\phi_E$ is psh for any $m \geq 1$.

We can now prove the theorem. Thanks to Proposition 1.16 and Remark 1.17, $\phi_m + p\phi_E$ converges to the relative Kähler-Einstein metric of $(X^o, L, \phi_L)$. As $\phi_m + p\phi_E$ has positive curvature, the relative Kähler-Einstein metric has also positive curvature.
2. Relative Ricci-flat conic metrics

2.1. Setting. — Let \( p : X \to Y \) a holomorphic proper map of relative dimension \( n \) between Kähler manifolds. We denote by \( Y^0 \subset Y \) the set of regular values of \( p \), and let \( X^0 := p^{-1}(Y^0) \) so that \( p_{|X^0} : X^0 \to Y^0 \) is a smooth fibration. For \( y \in Y^0 \), one writes \( X_y := p^{-1}(X_y) \) the fiber over \( y \). Let \( B \) be an effective \( \mathbb{Q} \)-divisor on \( X \) that has coefficients in \((0,1)\) and whose support has snc. Our assumption throughout the current section will be that for each \( y \in Y^0 \) we have

\[
\c_1(K_{X_y} + B_y) = 0 \in H^{1,1}(X_y, \mathbb{Q}).
\]

Thanks to the log abundance in the Kähler setting, cf. Corollary 2.18 on page 46, we know that \( K_{X_y} + B_y \) is \( \mathbb{Q} \)-effective. Combining this with Ohsawa-Takegoshi extension theorem in its Kähler version, cf. [Cao14], one can assume that there exists \( m \geq 1 \) such that \( m(K_{X_y} + B_y) \simeq O_{X_y} \) for all \( y \in Y^0 \).

In this context the main result we obtain here shows that the flatness of the direct image \( p_* (mK_{X/Y} + mB) \) implies the local isotriviality of the family \( p : (X, B) \to Y \). By this we mean that there exists a holomorphic vector field \( v \) on \( X^0 \) whose flow identifies the pairs \((X_y, B_y)\) and \((X_w, B_w)\) provided that \( y, w \in Y^0 \) are close enough. This is the content of Theorem 2.2 below. Prior to stating our theorems in a formal manner, we need to recall a few notions and facts.

Given a point \( y \in Y^0 \), there exists a coordinate ball \( U \subset Y^0 \) containing \( y \) and a nowhere vanishing holomorphic section

\[
\begin{equation}
\Omega \in H^0 \left( X_U, m(K_{X/Y} + B) |_{X_U} \right)
\end{equation}
\]

by our assumption (2.1), where \( X_U := p^{-1}(U) \).

If \( f_B \) is a local multivalued holomorphic function cutting out the \( \mathbb{Q} \)-divisor \( B \), then the form \( \frac{(\Omega_y \wedge \overline{\Omega_y})^{\frac{1}{m}}}{|f_B|^2} \) induces a volume element on the fibers of \( p \) over \( U \). We fix a Kähler class \( \{\omega\} \in H^{1,1}(X, \mathbb{R}) \). Up to renormalizing \( \omega \), one can assume that the constant function

\[
Y^0 \ni y \mapsto \int_{X_y} \omega^n
\]

is identically equal to 1. We also define \( V_y := \int_{X_y} \frac{(\Omega_y \wedge \overline{\Omega_y})^{\frac{1}{m}}}{|f_B|^2} \); this is a Hölder continuous function of \( y \in Y^0 \).

Let \( \rho_y \) be the unique positive current on \( X_y \) which is cohomologous to \( \omega_y \) and satisfies

\[
\rho_y^n = \frac{(\Omega_y \wedge \overline{\Omega_y})^{\frac{1}{m}}}{V_y |f_B|^2},
\]

cf. [Yau75]. One can write \( \rho_y = \omega|_{X_y} + dd^c \varphi_y \), where the function \( \varphi_y \) is uniquely determined by the normalization

\[
\int_{X_y} \varphi_y \frac{(\Omega_y \wedge \overline{\Omega_y})^{\frac{1}{m}}}{|f_B|^2} = 0.
\]

For each \( y \in U \subset Y^0 \), the current \( \rho_y \) is reasonably well understood: it has Hölder potentials, and it is quasi-isometric to a metric with conic singularities along \( B \), cf. [GP16].

We analyze next its regularity properties in the "base directions"; this will allow us to derive a few interesting geometric consequences.

The function \( \varphi \) defined on \( X^0 \) by \( \varphi(x) := \varphi_{p(x)}(x) \) is a locally bounded function on \( X^0 \) (by the family version of Kołodziej’s estimates cf. [DDG+14]) hence it induces a \((1,1)\) current

\[
\rho := \omega + dd^c \varphi
\]
on $X^0$. Let $\Delta \subset Y^0$ be a small, 1-dimensional disk. If $\Delta$ is generic enough, then the inverse image $X := p^{-1}(\Delta)$ is non-singular, and the restriction map $p : X \to \Delta$ is a submersion. We denote by $t$ a holomorphic coordinate on the disk $\Delta$. Following [Siu86] we recall next the expression of the horizontal lift of the local vector field $\frac{\partial}{\partial t}$. For the moment, this is a vector field $v_p$ with distribution coefficients on the total space $X$ given by the expression

$$v_p := \frac{\partial}{\partial t} - \sum_{\alpha} \rho^m_{\alpha} \rho_{\bar{\beta}} \frac{\partial}{\partial z_{\alpha}},$$

where the notations are as follows. We denote by $(z_1, \ldots, z_n, t)$ a coordinate system centered at some point of $X$, and $\rho_{\bar{\beta}}$ is the coefficient of $dt \wedge d\bar{z}_{\beta}$. We denote by $(\rho_{\alpha \bar{\beta}})$ the coefficients of the inverse of the matrix $(\rho_{\alpha \beta})$.

The reflexive hull of the direct image

$$F_m := p \ast (m(K_{X/Y} + B))^{**}$$

plays a key role in the study of the geometry of algebraic fiber spaces. It admits a positively curved singular metric whose construction we next recall, cf. [BP08, PT14] and the references therein.

Let $\sigma \in H^0(U, F_m|_U)$ be a local holomorphic section of the line bundle $F_m$ defined over a small coordinate set $U \subset Y^0$. The expression

$$\|\sigma\|^2 := V_y^{m-1} \int_{X_y} |\sigma|^2 \frac{\Omega_y^{2m-1}}{\Omega_y} e^{-\phi_B}$$

defines a metric $h$ on $F_m|_{Y^0}$. It is remarkable that this metric extends across the singularities of the map $p$, and it has semi-positive curvature current, see loc. cit. for more complete statements.

2.2. Main results. — This subsection aims to the proof of the following results.

**Theorem 2.1.** — Let $p : (X, B) \to Y$ be a proper holomorphic map between Kähler manifolds as in (2.1). We assume moreover that the curvature of $F_m$ with respect to the metric in (2.7) equals zero when restricted to $X^0$. Then the $(1,1)$-current $\rho$ defined on $X^0$ by (2.4) is semipositive and it extends canonically to a closed positive current on $X$ in the cohomology class $[\omega]$.

For example, if we assume that $Y$ is compact, then the curvature of $F_m$ will automatically be zero if $c_1(F_m) = 0$ thanks to the properties of the metric (2.7) discussed above, cf [CP17, Thm. 5.2].

What we mean by the word canonical in the Theorem 2.1 above is that the local potential $\varphi$ of $\rho$ are locally bounded above across $X \setminus X^0$.

We equally prove the next statement.

**Theorem 2.2.** — We assume that the hypothesis in Theorem 2.1 are satisfied. Then, $p$ is locally trivial over $Y^0$, that is, for every $y \in Y^0$, there exists a neighborhood $U \subset Y^0$ of $y$ such that $$(p^{-1}(U), B) \cong (X_y, B|_{X_y}) \times U.$$ Moreover, if $p$ is smooth in codimension one, then $p$ is locally trivial over the whole $Y$ provided that $\text{codim}_{X \setminus X^0}(B \setminus X^0) > 0$.

In particular, under the assumptions in the second part of Theorem 2.2 the map $p$ is automatically a locally isotrivial submersion.

As an application, we establish the following result; it partially generalizes to the Kähler case a theorem of F. Ambro [Amb05].
Corollary 2.3. — Let \( p : X \to Y \) be a fibration between two compact Kähler manifolds. Let \( B \) be a \( \mathbb{Q} \)-effective klt divisor on \( X \) with snc support.

(2.3.8) If \( -(K_X + B) \) is nef, then \( -K_Y \) is pseudo-effective.

(2.3.9) Moreover, if \( c_1(K_X + B) = 0 \) and \( c_1(Y) = 0 \), then \( p \) is locally trivial, that is, for every \( y \in Y \), there exists a neighborhood \( U \subset Y \) of \( y \) such that

\[
(p^{-1}(U), B) \cong (X_y, B|_{X_y}) \times U.
\]

In particular, if \( c_1(K_X + B) = 0 \), the Albanese map \( p : X \to Alb(X) \) is locally trivial.

2.3. Proof of Theorem 2.1. — We will proceed by approximation, mainly using the following lemma combined with the results in [Gue16].

The next statement will enable us to reduce the problem to the canonically polarized pairs.

Lemma 2.4. — Let \( X \) be a compact Kähler manifold and let \( B \) be an effective divisor such that \( (X, B) \) is klt. We assume that \( c_1(K_X + B) = 0 \). Let \( \omega \) be Kähler form on \( X \). For every \( \varepsilon > 0 \), let \( \rho_\varepsilon \in \varepsilon \{\omega\} \) be the unique twisted conic Kähler-Einstein metric such that

\[
\text{Ric}(\rho_\varepsilon) = -\rho_\varepsilon + \varepsilon \omega + [B].
\]

Let \( \rho \in \{\omega\} \) be the unique conic Kähler-Einstein metric such that \( \text{Ric}(\rho) = [B] \). Then

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \rho_\varepsilon = \rho
\]

where the convergence is smooth outside \( \text{Supp}(B) \).

Proof. — Let \( m \in \mathbb{N} \) such that \( m(K_X + B) \) is effective. Let \( \Omega \in H^0(X, m(K_X + B)) \) be a holomorphic section normalized such that

\[
\int_X \frac{(\Omega \wedge \overline{\Omega})^\frac{1}{m}}{|f_B|^2} = 1.
\]

There exists a unique function \( \varphi_\varepsilon \) on \( X \) such that

\[
\rho_\varepsilon = \varepsilon \omega + dd^c \varphi_\varepsilon \quad \text{(2.4.12)}
\]

\[
\rho_\varepsilon^n = \varepsilon^n e^{\varphi_\varepsilon} \frac{(\Omega \wedge \overline{\Omega})^\frac{1}{m}}{|f_B|^2} \quad \text{(2.4.13)}
\]

Now, let us set \( \psi_\varepsilon := \frac{1}{\varepsilon} \varphi_\varepsilon \). One has \( \frac{1}{\varepsilon} \rho_\varepsilon = \omega + dd^c \psi_\varepsilon \) and

\[
(\omega + dd^c \psi_\varepsilon)^n = e^{\varepsilon \psi_\varepsilon} \frac{(\Omega \wedge \overline{\Omega})^\frac{1}{m}}{|f_B|^2} \quad \text{(2.14)}
\]

As \( \frac{(\Omega \wedge \overline{\Omega})^\frac{1}{m}}{|f_B|^2} \) and \( (\frac{1}{\varepsilon} \rho_\varepsilon)^n \) are probability measures and \( \psi_\varepsilon \) is \( \omega \)-psh, Jensen inequality yields

\[
\int_X (\varepsilon \psi_\varepsilon) \frac{(\Omega \wedge \overline{\Omega})^\frac{1}{m}}{|f_B|^2} \leq 0,
\]

and therefore

\[
\int_X \psi_\varepsilon \frac{(\Omega \wedge \overline{\Omega})^\frac{1}{m}}{|f_B|^2} \leq 0. \quad \text{(2.15)}
\]

As the measure \( \frac{(\Omega \wedge \overline{\Omega})^\frac{1}{m}}{|f_B|^2} \) integrates every quasi-psh function, it follows from standard results in pluripotential theory that there exists a constant \( C \) such that

\[
\sup_X \psi_\varepsilon \leq C \quad \text{(2.16)}
\]

By (2.14)-(2.16) and Kołodziej’s estimate, one gets

\[
\text{osc}_X \psi_\varepsilon \leq C \quad \text{(2.17)}
\]
As \( \frac{(\Omega \wedge \Omega)_{\frac{1}{m}}}{|fB|^\frac{1}{2}} \) and \( (\frac{1}{\varepsilon} \rho_\varepsilon)^n \) are probability measures again, (2.14) shows that
\[
\inf_X \psi_\varepsilon \leq 0 \leq \sup_X \psi_\varepsilon.
\]
Combining this information with (2.17), we obtain the inequality
\[
(2.18) \quad ||\psi_\varepsilon||_{L^\infty(X)} \leq C.
\]
Moreover, Jensen inequality applied to the equation \( \frac{(\Omega \wedge \Omega)_{\frac{1}{m}}}{|fB|^\frac{1}{2}} = e^{-\varepsilon \psi_\varepsilon} \) yields
\[
(2.19) \quad \int_X \psi_\varepsilon (\omega + dd^c \psi_\varepsilon)^n \geq 0
\]
From (2.14) and (2.18), we get uniform estimates at any order for \( \psi_\varepsilon \) outside \( B \). If \( \psi \) is a subsequential limit of the family \( (\psi_\varepsilon)_{\varepsilon>0} \), it will satisfy
\[
(\omega + dd^c \psi)^n = \frac{(\Omega \wedge \Omega)_{\frac{1}{m}}}{|fB|^\frac{1}{2}}
\]
Combining this information with (2.15) and (2.19), we find
\[
\int_X \psi_\varepsilon (\omega + dd^c \psi_\varepsilon)^n = 0
\]
Therefore \( \psi \) is uniquely determined, and the whole family \( (\psi_\varepsilon)_{\varepsilon>0} \) converges to \( \psi \). The lemma is thus proved. \( \square \)

Now we can prove Theorem 2.1.

**Proof of Theorem 2.1.** — We fix a reference Kähler form \( \omega \) on \( X \), and let \( U \) be some small topological open set of \( Y^\circ \). By hypothesis, the curvature of the bundle \( F_m|U \) is identically zero. This is equivalent to the existence of a section
\[
(2.20) \quad s \in H^0 \left( X_U, mK_{X/Y} + mB|\right) \]
whose norm is a constant function on \( U \), cf. [CP17] or [HPS16]. In a word, the section \( s \) is constructed by using the Ohsawa-Takegoshi theorem with optimal constant (which is the same as saying that we construct \( s \) by parallel transport).

We therefore have \( \|s\|_{h} \left( y \right) = 1 \) for every \( y \in U \). Let
\[
\Omega_y := s|_X \in H^0(X_y, mK_{X_y} + mB_y)
\]
be the restriction of \( s \) to the fibers of \( p \).

Since \( c_1(K_{X_y} + B_y) + \varepsilon \omega|_{X_y} \) is a Kähler class for each \( \varepsilon > 0 \) and for each \( y \in Y^\circ \), there exists a unique \( \varphi_\varepsilon \) such that
\[
(\varepsilon \omega + dd^c \varphi_\varepsilon)^n = \varepsilon^n e^{\varepsilon \varphi_\varepsilon} \frac{(\Omega_y \wedge \Omega_y)_{\frac{1}{m}}}{|fB|^\frac{1}{2}} \quad \text{on } X_y.
\]
Since \( y \in U \) is a regular value, this is equivalent to
\[
\text{Ric } \rho_{\varepsilon,y} = -\rho_{\varepsilon,y} + \varepsilon \omega + [B_y] \quad \text{on } X_y,
\]
where \( \rho_{\varepsilon,y} = \varepsilon \omega + dd^c \varphi_\varepsilon|_{X_y} \).

Next, the section \( s \) is holomorphic hence the relative \( B \)-valued volume forms \( \Omega_y \wedge \Omega_y \) induce a metric with zero curvature on \( K_{X/Y} + B \) over \( p^{-1}(U) \). Because of that,
\[
\rho_\varepsilon := \varepsilon \omega + dd^c \varphi_\varepsilon
\]
coincides with the current studied in [Gue16] and the content of the main theorem in loc. cit. is that \( \rho_\varepsilon \) is positive on \( p^{-1}(U) \). Thanks to Lemma 2.4, \( \rho \) is the fiberwise weak limit on \( p^{-1}(U) \).
of the fiberwise twisted Kähler-Einstein metrics $\frac{1}{\varepsilon}\rho_\varepsilon$: moreover, the estimate (2.18) is uniform over $U$, so that $\rho$ is actually the global weak limit of the metrics $\frac{1}{\varepsilon}\rho_\varepsilon$ on $p^{-1}(U)$. In particular, $\rho > 0$ on $p^{-1}(U)$, hence on $X^0$.

As for the extension property, it is proved in [Gue16] that $\rho_\varepsilon$ extends canonically to the whole $X$ as a positive current in $\{\varepsilon \omega\}$. This means that given any small neighborhood $U$ of a point $x \in X \setminus X^0$, one has $\sup_{U \cap X^0} \psi_\varepsilon < +\infty$. In other words, $\psi_\varepsilon$ extends to an $\omega$-psh function on $X$. Now, let us fix $U$ as above. The family of $\omega$-psh functions $(\tilde{\psi}_\varepsilon)_{\varepsilon > 0}$ on $U$ defined by $\tilde{\psi}_\varepsilon := \psi_\varepsilon - \sup_U \psi_\varepsilon$ is relatively compact. In particular one can find a sequence $\varepsilon_k \to 0$ and an $\omega$-psh function $\tilde{\psi}$ on $U$ such that $\tilde{\psi}_{\varepsilon_k} \to \tilde{\psi}$ a.e. in $U$. Moreover, we know that $\psi_{\varepsilon_k} = \tilde{\psi}_{\varepsilon_k} + \sup_U \psi_{\varepsilon_k}$ converges to the $\omega$-psh function $\varphi$ a.e. in $U \cap X^0$. This implies that $\sup_U \psi_{\varepsilon_k}$ converges when $k \to +\infty$. By Hartogs lemma, this implies that $\sup_{U \cap X^0} \varphi < +\infty$, which had to be proved.

2.4. Proof of Theorem 2.2. — We will proceed in a few steps, roughly as follows.

- We start by approximating $\rho$ by smoothing the volume element. Let $\tau_\delta$ be the resulting $C^\infty$ form. Then we have $\lim_\delta \tau_\delta = \rho$ in weak sense.
- We analyze next the behavior of the geodesic curvature of $\tau_\delta$. The main tools are the Laplace equation satisfied by this quantity, cf. [Sch12], and the $C^2$-estimates for conic Monge-Ampère equations, cf. [GP16]. As a consequence, we first show that we can extract a limit of the horizontal lift $v_\delta$ (corresponding to $\tau_\delta$) which is holomorphic on the fibers of $p$. Afterwards we show that the geodesic curvature of $\tau_\delta$ converges (on $X \setminus \text{Supp}(B)$) to a constant as $\delta \to 0$.
- Finally, we infer that $v_\delta$ converges to $v_\rho$ uniformly on the complement of the divisor $B$.
- After completing the previous steps, we show that $v_\rho$ is in fact holomorphic on the total space $X$ by using a few arguments borrowed from [Ber09].

- Finally, we show that $v_\rho$ extends across the singular locus of $p$ provided that $X$ is compact and $p$ is smooth in codimension one.

2.4.1. Approximation. — This is a fairly standard and widely used procedure, so we will be very brief.

By hypothesis, we have $B = \sum a_j B_j$ where $a_j \in (0, 1)$ and $\cup B_j$ has simple normal crossings. We consider a smooth metric $e^{-\phi_j}$ on the bundle associated to $B_j$; it induces a smooth metric $e^{-\phi_B} := e^{-\sum a_j \phi_j}$ on the $Q$-line bundle associated to $B$. For any $\delta \geq 0$ we define the quantity $C_{\delta,y}$ as follows

$$e^{-C_{\delta,y}} = \int_{X_y} \frac{(\Omega_y \wedge \overline{\Omega_y})^{\frac{1}{m}}}{\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{e_j}}.$$  

Here $\Omega$ is a section of $\mathcal{F}_m|_\Delta$ whose norm is equal to one at each point, and $f_j$ is a local holomorphic function cutting out $B_j$. The expression $\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{-e_j}$ is then a globally defined smooth metric on the $Q$-line bundle associated to $B$. Finally, we let $s_j$ be the canonical section of $\mathcal{O}_X(B_j)$, and we will denote by $|s_j|^2$ the squared norm of $s_j$ with respect to $e^{-\phi_j}$.

Let us further define the smooth $(1,1)$-form

$$\tau_\delta = \omega + dd^c u_\delta$$

on $X^0$ such that $u_\delta|_{X_y}$ is solution of the following system of equations

$$\left\{ \begin{array}{l}
\int_{X_y} u_\delta \frac{(\Omega_y \wedge \overline{\Omega_y})^{\frac{1}{m}}}{\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{e_j}} = 0 \\
(\omega + dd^c u_\delta)^n = e^{C_{\delta,y}} \frac{(\Omega_y \wedge \overline{\Omega_y})^{\frac{1}{m}}}{\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{e_j}} \\
\end{array} \right.$$

(2.22)
By the family version of Ko/łodziej’s estimates [DDG+14], one can easily see that for any relatively compact subset \( U \subseteq Y^\circ \), there exists a constant \( C > 0 \) independent of \( \delta \in (0,1) \) such that

\[
\sup_{y \in U} ||u_\delta||_{L^\infty(X_y)} \leq C
\]

As a consequence, we get the following easy result, cf (2.4) for the definition of \( \rho \) and \( \varphi \).

**Lemma 2.5.** — When \( \delta \) approaches zero, \( \tau_\delta \) converges weakly to \( \rho \) on \( X^\circ \). More precisely, one has \( u_\delta \to \varphi \) in \( L^1_{\text{loc}}(X^\circ) \).

**Proof.** — The convergence \( u_\delta \to \varphi \) in \( L^1_{\text{loc}}(X^\circ) \) follows from Ko/łodziej’s stability theorem [Kol05, Thm. 4.1] (one even gets uniform convergence). The convergence on the total space then follows from Lebesgue’s dominated convergence theorem coupled with (2.23).

### 2.4.2. Uniformity properties of \((\tau_\delta)_{\delta>0}\). — In this subsection we will only consider the restriction of our initial family of manifolds above a disk in the complex plane

\[
p : \mathcal{X} \to \Delta
\]

where we recall that \( \Delta \subseteq Y^\circ \) is generic and \( \mathcal{X} = p^{-1}(\Delta) \).

The coordinate on \( \Delta \) will be denoted by \( t \). We recall that the geodesic curvature of the form \( \tau_\delta \) is the function defined by the equality

\[
\tau_\delta^{n+1} = c(\tau_\delta)\tau_\delta^n \land \sqrt{-1}dt \land dt
\]

If \( v_\delta \) is the horizontal lift of \( \frac{\partial}{\partial t} \) with respect to \( \tau_\delta \), then it is easy to verify that we have

\[
c(\tau_\delta) = \langle v_\delta, v_\delta \rangle_{\tau_\delta}.
\]

For each \( \delta > 0 \), the form \( \tau_\delta \) induces a metric say \( h_\delta \) on the relative canonical bundle \( K_{\mathcal{X}/\Delta} \) as follows. Let \( z_1, \ldots, z_n, z_{n+1} \) be a coordinate system defined on the set \( W \subseteq \mathcal{X} \). Recall that \( t \) is a coordinate on \( \Delta \). This data induces in particular a trivialization of \( K_{\mathcal{X}/\Delta} \), with respect to which the weight of \( h_\delta \) is given as follows

\[
e^{\Psi_\delta(z,t)}dz_1 \land \cdots \land dz_{n+1} = \tau_\delta^n \land \sqrt{-1}dt \land dt.
\]

The curvature of \( (K_{\mathcal{X}/\Delta}, h_\delta) \) is the Hessian of the weight

\[
\Theta_\delta(K_{\mathcal{X}/\Delta})|_W = dd^c\Psi_\delta.
\]

We have the following result, relating the various quantities defined above.

**Lemma 2.6.** — Let \( \Delta''_\delta \) be the Laplace operator corresponding to the metric \( \tau_\delta|_{\mathcal{X}_t} \). Then we have the equality

\[
-\Delta''_\delta(c(\tau_\delta)) = \langle \bar{\partial}v_\delta \rangle_{\tau_\delta}^2 - \Theta_\delta(K_{\mathcal{X}/\Delta})(v_\delta, v_\delta).
\]

We will not prove Lemma 2.6 in detail because this type of results appear in many articles, cf. [Sch12] or [P˘ au12]. The main steps are as follows: we have \( \Psi_\delta = \log \det(g_{\alpha\beta}) \) where we denote \( g_{\alpha\beta} := \tau_{\delta,\alpha\beta} \) and a few simple computations show that the Hessian of \( \Psi_\delta \) evaluated in the \( v_\delta \)-direction equals
\[ \partial \bar{\partial} \log \det(g_{\alpha \beta})(v_\delta, \overline{v}_\delta) = g^{\alpha \overline{\gamma}} g_{\overline{\gamma} \alpha \overline{\beta}} - g^{\alpha \overline{\gamma}} g_{\alpha \overline{\gamma} \beta} g_{\overline{\beta} \alpha \overline{\gamma}} + g^{\alpha \overline{\gamma}} g_{\alpha \overline{\gamma} \beta} g_{\overline{\beta} \alpha \overline{\gamma}} g_{\overline{\beta} \alpha \overline{\gamma}}. \] (2.30)

In the rhs term in (2.30) we recognize the beginning of \( \Delta^\beta_\delta (c(\tau_\delta)) \) (cf. the 1st term), and in the end this gives (2.29). Again, we refer to [CLP16], pages 18-19 for a detailed account of these considerations.

**Remark 2.7.** — The equation (2.29) can be seen as the analogue of the usual \( C^2 \) estimates in “normal directions”. By this we mean the following: the \( C^2 \) estimates are derived by evaluating the Laplace of the (log of the) sum of eigenvalues of the solution metric with respect to the reference metric. Vaguely speaking, in (2.29) we compute the Laplace of the normal eigenvalue.

The following result is an important step towards the proof of Theorem 2.2.

**Proposition 2.8.** — Let \( t \in \Delta \) be fixed. For any sequence \( \delta_j \to 0 \), there exists a holomorphic vector field \( w \) on \( X_t \setminus \text{Supp}(B) \) such that, up to extracting a subsequence, the sequence \( (v_{\delta_j}|_{X_t})_{j \geq 0} \) converges locally smoothly outside \( \text{Supp}(B) \) to the vector field \( w \).

**Remark 2.9.** — At this point, it is not obvious that \( w \) is independent of the sequence \( \delta_j \) and that it should coincide with the lift \( \tilde{v} \) of \( \frac{\partial}{\partial \tau} \) with respect to \( \rho|_{X_t \setminus \text{Supp}(B)} \).

Before giving the proof of Proposition 2.8 we collect here a few results concerning the family of forms \( (\tau_\delta)_{\delta > 0} \) taken from [GP16] and [Gue16].

(a) It follows from [GP16, Sect. 5.2] that \( \tau_\delta|_{X_y} \) has “uniform regularized conic singularities” in the sense that if on a small coordinate open set \( \Omega \subset X' \), the divisor \( B \) is given by \( B = \sum_j a_j B_j \) where \( B_j \) is defined by \( \{ z_j = 0 \} \), then there is a constant \( C \) independent of \( \delta \) such that for any \( y \in U \), we have

\[ C^{-1} \left( \sum_{k=1}^{r} \frac{i dz_k \wedge d \overline{z}_k}{|z_k|^2 + \delta^2} \right) \leq \tau_\delta|_{X_y \cap \Omega} \leq C \left( \sum_{k=1}^{r} \frac{i dz_k \wedge d \overline{z}_k}{|z_k|^2 + \delta^2} \right) + \sum_{k=r+1}^{n} \frac{i dz_k \wedge d \overline{z}_k}{|z_k|^2 + \delta^2} \] (2.31)

(b) The estimates [Gue16, (3.13), Prop. 4.1&4.2] go through for \( u_\delta \), that is, for any integer \( k \geq 0 \), there exists \( C_k \) > 0 independent of \( \delta \in (0, 1) \) such that

\[ \sup_{t \in \Delta} ||\partial_t u_\delta||_{C^k(\Omega \cap X_t)} \leq C_k \] (2.32)

and there exists a constant \( C \) > 0 such that the following global estimate holds:

\[ \sup_{t \in \Delta} \int_{X_t} |v_\delta|^2 \tau_\delta^n \leq C \] (2.33)

One also gets

\[ \lim_{\delta \to 0} \sup_{t \in \Delta} \int_{X_t \cap (\cup \{ |s_j|^2 < \delta \})} |v_\delta|^2 \tau_\delta^n = 0 \] (2.34)
Again, we will not reproduce here the arguments for (2.32)-(2.34), but let us comment e.g. (2.33) for the comfort of the reader/referee. The main observation is that in local coordinates this amounts to obtaining a bound of \( |\nabla^\delta(\partial_t u_\delta)|^2 \) with respect to the volume element \( \tau_\delta^n \) on \( X_t \). Here \( |\cdot|^2 \) is measured with respect to the reference metric \( \omega \), and \( \nabla^\delta \) is the gradient corresponding to \( \tau_\delta \). By (2.31) this is smaller than \( |\nabla^\delta(\partial_t u_\delta)|^2 \) up to a uniform constant. This new quantity is controlled by taking the derivative of the Monge-Ampère equation verified by \( \tau_\delta \) in normal directions and integration by parts. Of course, the real proof is much more involved and we refer to loc. cit. for the details.

We see immediately that (2.33)-(2.34) imply the next statement.

**Lemma 2.10.** — *One has the following*

\[
\limsup_{\delta \to 0} \sup_{t \in \Delta} \int_{X_t} \left( \sum_j \frac{\delta^2}{|s_j|^2 + \delta^2} \right) |v_\delta|^2 \tau_\delta^n = 0.
\]

The proof of Lemma 2.10 is very elementary and we skip it. We present next the arguments for Proposition 2.8.

**Proof.** — Recall that in local coordinates,

\[
v_\delta = \frac{\partial}{\partial t} - \sum_{\alpha,\beta} \tau_\delta^{\alpha\beta} \tau_\delta^{\alpha t} \frac{\partial}{\partial z_\alpha}.
\]

By (2.32), the family \((v_\delta|_{X_t})_{\delta > 0}\) is relatively compact in the \( C^\infty_{\text{loc}}(X_t \setminus \text{Supp}(B)) \) topology. Let \( \delta_j \) a sequence converging to zero such that \((v_{\delta_j}|_{X_t})_{j \geq 0}\) converges locally smoothly outside \( \text{Supp}(B) \) to a vector field \( w \).

Now, the geodesic curvature \( c(\tau_\delta) \) of \( \tau_\delta \) satisfies the following equation

\[
-\Delta_{\tau_\delta} c(\tau_\delta) = \|\hat{\partial}v_\delta\|^2 - \Theta_\delta(K_{\tau_\delta})(v_\delta, \bar{v}_\delta)
\]

by Lemma 2.6. In our setting (cf. (2.21) and the definition of \( \tau_\delta \)) the curvature term in (2.31) becomes

\[
\frac{\partial^2 C_\delta(t)}{\partial t \partial \tau} - \sum_j a_j \delta^2 \sqrt{-1} \langle \partial s_j, \partial s_j \rangle (v_\delta, \bar{v}_\delta) \frac{|v_\delta|^2}{|s_j|^2 + \delta^2} + \sum_j a_j \delta^2 \Theta_j(v_\delta, \bar{v}_\delta) \frac{|v_\delta|^2}{|s_j|^2 + \delta^2}
\]

where \( \Theta_j \) above is the curvature of the hermitian line bundle \((\mathcal{O}_X(B_j), e^{-\phi_j})\).

Integrating (2.31) against \( \tau_\delta^n \) yields

\[
\limsup_{\delta \to 0} \sup_{t \in \Delta} \left( \int_{X_t} \|\hat{\partial}v_\delta\|^2 \tau_\delta^n + \sum_j a_j \int_{X_t} \delta^2 \sqrt{-1} \langle \partial s_j, \partial s_j \rangle (v_\delta, \bar{v}_\delta) \tau_\delta^n \right) = \limsup_{\delta \to 0} \frac{\partial^2 C_\delta(t)}{\partial t \partial \tau}.
\]

Indeed, thanks to Lemma 2.10 the third term in (2.32) vanishes as \( \delta \to 0 \).

We show next that we have

\[
\limsup_{\delta \to 0} \frac{\partial^2 C_\delta(t)}{\partial t \partial \tau} = 0
\]

and this will end the proof of Proposition 2.8. Recall that the expression of the function in (2.34) is

\[
C_\delta(t) = -\log \int_{X_t} \frac{(\Omega_{\underline{\underline{\Omega}}}^{\underline{\underline{\Omega}}} g) \frac{1}{\bar{\Omega}_j}}{\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})}.
\]
and given that the norm of $\Omega$ is equal to one at each point of $\Delta$, we have

$$C_\delta(t) = -\log \left(1 - \int_{X_t} \frac{\prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{a_j}}{\prod_j |f_j|^{2a_j} \prod_j (|f_j|^2 + \delta^2 e^{\phi_j})^{a_j}} (\Omega_y \wedge \Omega_y)^{\frac{1}{2}} \right).$$

With the same notations as in (2.31), the restriction of the function under the sum sign in (2.36) on a coordinate set $W_\alpha$ reads as

$$F_{\alpha,\delta}(z,t) := \frac{\prod_j (|z_j|^2 + \delta^2 e^{\phi_j})^{a_j}}{\prod_j |z_j|^{2a_j} \prod_j (|z_j|^2 + \delta^2 e^{\phi_j})^{a_j}}$$

and then the integral in (2.36) becomes

$$\sum_\alpha \int_{W_\alpha \cap X_t} \theta_\alpha F_{\alpha,\delta}(z,t) e^{f_\alpha} \omega^n$$

where $\theta_\alpha$ is a partition of unit and the $f_\alpha$ are given smooth functions. If $v$ is the horizontal lift of $\frac{\partial}{\partial t}$ with respect to the reference metric $\omega$, then we have the usual formula

$$\frac{\partial}{\partial t} \sum_\alpha \int_{X_t} \theta_\alpha F_{\alpha,\delta}(z,t) e^{f_\alpha} \omega^n = \sum_\alpha \int_{X_t} v(\theta_\alpha F_{\alpha,\delta}(z,t) e^{f_\alpha}) \omega^n.$$

The formula (2.35) shows that $\frac{\partial F_{\alpha,\delta}}{\partial t}$ converges to zero as $\delta \to 0$ because only the weights $\phi_j$ depend on $t$ and the coefficients $a_j$ are strictly smaller than 1. Indeed, we have

$$\frac{\partial F_{\alpha,\delta}}{\partial t} = \sum_j \frac{\delta^2 e^{\phi_j} \partial_t \phi_j}{(|z_j|^2 + \delta^2 e^{\phi_j})^{1+a_j} \prod_{i \neq j} (|z_i|^2 + \delta^2 e^{\phi_i})^{a_i}}$$

and our claim follows since $\int_{\mathbb{C},0} \frac{\delta^2}{(|z|^2 + \delta^2)^{1+a}} d\lambda(z) \to 0$ as $\delta \to 0$ for any $a < 1$.

As for terms involving $\frac{\partial F_{\alpha,\delta}}{\partial z_i}$ we infer the same conclusion (i.e. they tend to zero) by using integration by parts as we explain next. The corresponding terms in (2.39) have the following shape

$$\int_{X_t} \frac{\partial F_{\alpha,\delta}}{\partial z_i}(z) \tau_\alpha(z) d\lambda(z)$$

where $\tau_\alpha$ is a smooth function with compact support in $W_\alpha \cap X_t$. The integral (2.41) is equal to

$$-\int_{X_t} \frac{\partial \tau_\alpha}{\partial z_i}(z) F_{\alpha,\delta}(z) d\lambda(z)$$

and this tends to zero by dominated convergence.

The same type of arguments apply for the second order derivatives of $C_\delta(t)$; the claim (2.34) follows.

As $v_\delta \to w$ in the $C_0^\infty(X_t \setminus \text{Supp}(B))$ topology when $j \to +\infty$, it follows from the identity (2.33) above that $w|_{X_t \setminus \text{Supp}(B)}$ is holomorphic.

The next proposition is equally very important in the analysis of the uniformity properties of $(v_\delta)_{\delta > 0}$.
Proposition 2.11. — Let $t \in \Delta$ be fixed. Then the identity
\begin{equation}
\lim_{\delta \to 0} \left( c(\tau_\delta) - \int_{X_t} c(\tau_\delta) \tau_\delta^n \right) = 0
\end{equation}
holds on $X_t \setminus \text{Supp}(B)$.

Proof. — Let $G_\delta : X_t \times X_t \to \mathbb{R}$ be the Green function of $(X_t, \tau_\delta)$. Let $x \in X_t \setminus \text{Supp}(B)$; by definition, one has
\begin{equation}
c(\tau_\delta)(x) - \int_{X_t} c(\tau_\delta) \tau_\delta^n = \int_{X_t} -\Delta_{\tau_\delta} c(\tau_\delta) \cdot G_\delta(x, \cdot) \tau_\delta^n
\end{equation}
Clearly, $\text{Vol}(X_t, \tau_\delta) = \int_{X_t} \tau_\delta^n = \int_{X_t} \omega^n = 1$ is independent of $\delta$. Moreover, by (2.31), there exists a constant $C_1 > 0$ independent of $\delta$ such that $\text{diam}(X_t, \tau_\delta) \leq C_2$. Therefore, it follows from [Siu87, A.2] that
\begin{equation}
G(x, y) \geq -C_2
\end{equation}
for some $C_2 > 0$ independent of $\delta$. Now recall that $G_\delta(x, y) = \int_0^{\infty} G_\delta(x, y, s)ds$ where $G_\delta(x, y, s)$ satisfies
\begin{equation}
G_\delta(x, y, s) \leq \begin{cases} 
C_3 s^{-n} e^{-d_{\tau_\delta}(x,y)/5s} & \text{if } 0 < s < 1 \\
C_4 s^{-n} & \text{for any } 0 < s < +\infty
\end{cases}
\end{equation}
where $d_{\tau_\delta}$ is the geodesic distance induced by $\tau_\delta$ on $X_t$. This follows respectively by [Dav88, Thm. 16] and [Siu87, p.139] – recall that the Ricci curvature of $\tau_\delta$ is uniformly bounded below thanks to (2.31). Integrating the above inequalities, one gets
\begin{equation}
G(x, y) \leq C_3 d_{\tau_\delta}(x,y)^{2-2n}
\end{equation}
for some uniform $C_3 > 0$. Let $I_\delta(x) := c(\tau_\delta)(x) - \int_{X_t} c(\tau_\delta) \tau_\delta^n$, and let $C_4 > 0$ be large enough so that $\pm \Theta_\delta \leq C_4 \omega$. One has successively:
\begin{align*}
|I_\delta(x)| &= \left| \int_{X_t} -\Delta_{\tau_\delta} c(\tau_\delta) \cdot (G_\delta(x, \cdot) + C_2) \tau_\delta^n \right| \\
&\leq \int_{X_t} \left( \|\bar{\partial} v_\delta\|^2 + C_4 \left( \sum_j \frac{\delta^2}{|s_j|^2 + \delta^2} |v_\delta|_\omega^2 \right) \cdot (G_\delta(x, \cdot) + C_2) \tau_\delta^n \right. \\
&\quad + \left. \int_{X_t} \left( \sum_j a_j \delta^2 \sqrt{1 - \frac{\langle \bar{\partial}s_j, \bar{\partial}s_j \rangle}{(|s_j|^2 + \delta^2)^2}} \right) \cdot (G_\delta(x, \cdot) + C_2) \tau_\delta^n \right) \\
&\leq C_5 \int_{X_t} \left( \|\bar{\partial} v_\delta\|^2 + \left( \sum_j \frac{\delta^2}{|s_j|^2 + \delta^2} |v_\delta|_\omega^2 \right) \cdot d_{\tau_\delta}(x, \cdot)^{2-2n} \tau_\delta^n \right. \\
&\quad + \left. \int_{X_t} \left( \sum_j a_j \delta^2 \sqrt{1 - \frac{\langle \bar{\partial}s_j, \bar{\partial}s_j \rangle}{(|s_j|^2 + \delta^2)^2}} \right) \cdot d_{\tau_\delta}(x, \cdot)^{2-2n} \tau_\delta^n \right)
\end{align*}
We claim that the right hand side converges to 0 when $\delta \to 0$, uniformly on $x$ belonging to a fixed compact subset of $X_t \setminus \text{Supp}(B)$. To see this, it is enough to check that out of any sequence $\delta_j \to 0$, one has $\lim_{j \to +\infty} I_{\delta_j}(x) = 0$ uniformly on $x$, up to extracting a subsequence. Thanks to Lemma 2.8, one can assume that $v_\delta$ converges locally smoothly to a holomorphic vector field $w$ on $X_t \setminus \text{Supp}(B)$. Let us pick $\varepsilon > 0$.

By the estimates and observations above, one can find a small neighborhood $U_x \Subset X_t \setminus \text{Supp}(B)$ and a constant $C = C(x) > 0$ such that:
\[ (i) \ |v_\delta|_2^2 \leq C; \ |\bar{\partial}v_\delta|_2^2 \leq \varepsilon, \text{ and } |s_j|^2 \geq C^{-1} \text{ hold on } U_x \text{ for any index } j; \]
\[ (ii) \int_{U_x} d_{\tau_6}(z, \cdot)^{2-2n} \leq C; \]
\[ (iii) \ d_{\tau_6}(z, w)^{2-2n} \leq C \text{ for any } w \notin U_x. \]

The rest of the proof is easy: we split the integral into two pieces on \( U_x \) and its complement.

- On the complement of \( U_x \) we use the item (iii) so that we can replace the function \( d_{\tau_6}(x, \cdot)^{2-2n} \) in the inequalities above by a constant independent of \( \delta \). The proof of Proposition 2.8 shows that the integral of the remaining terms tends to 0 and \( \delta \to 0 \).

- On the set \( U_x \) we are ‘far’ from the support of \( B \). Combined with the items (i) and (ii) above, this finishes the proof of Proposition 2.11.

In fact, Proposition 2.11 shows that the limit (2.43) is uniform on compact sets contained in the complement of the divisor \( B \). We intend to couple this with the elliptic equation satisfied by \( c(\tau_6) \) in order to obtain bounds for the derivatives of this function in the fiber directions. To this end, we need the following statement.

**Proposition 2.12.** — There exists a constant \( C > 0 \) independent of \( \delta > 0 \) such that

\[
\left| \int_{X_t} c(\tau_6) r_{\delta}^0 \right| \leq C
\]

*Proof.* — This statement can be seen as a by-product of the considerations in the article [Gue16, (5.3) & Prop. 5.4]. Therefore we will content ourselves to highlight the main steps.

To start with, we recall that the normalization of \( u_\delta \) is as follows

\[
(2.47) \quad \int_{X_t} u_\delta \frac{(\Omega_y \wedge \overline{\Omega_y})^m}{\prod_j (|f_j|^2 + \delta^2 e_\theta)^{v_j}} = 0
\]

and this can be re-written as

\[
(2.48) \quad \int_{X_t} u_\delta e^{F_\delta} \omega_\delta^n = 0
\]

where \( \omega_\delta \) is a metric with conic singularities on \( X \), whose multiplicities along the components of \( B \) are \( 1 > b_j \geq \max(a_j, 1/2) \) (notations as in (2.31)). Note that \( F_\delta \) in (2.48) has an explicit expression, being the log of the ratio \( \frac{\omega_\delta^n}{\omega_\delta^0} \).

Let \( V_\delta \) be the horizontal lift of \( \frac{\partial}{\partial t} \) with respect to \( \omega_\delta \). By applying the \( \frac{\partial^2}{\partial t \partial t} \) operator in (2.48) we obtain

\[
\int_{X_t} V_\delta (\nabla_\delta(u_\delta)) e^{F_\delta} \omega_\delta^n = \int_{X_t} V_\delta(u_\delta) \nabla_\delta(f_\delta) e^{F_\delta} \omega_\delta^n - \int_{X_t} \nabla_\delta(u_\delta) V_\delta(F_\delta) e^{F_\delta} \omega_\delta^n \]

\[
(2.49) \quad - \int_{X_t} u_\delta V_\delta(\nabla_\delta(f_\delta)) e^{F_\delta} \omega_\delta^n - \int_{X_t} u_\delta |V_\delta(F_\delta)|^2 e^{F_\delta} \omega_\delta^n
\]

Now the point is that, up to terms for which we have a uniform estimate already, the function \( V_\delta(\nabla_\delta(u_\delta)) \) is “the same” as \( c(\tau_6) \). Hence the absolute value of the lhs of (2.49) is equivalent to

\[
\left| \int_{X_t} c(\tau_6) r_{\delta}^0 \right|.
\]

The terms on the rhs of (2.49) are uniformly bounded, as it is proved in the reference indicated at the beginning of the proof.

We can now prove that the vector field \( v_\rho \) is holomorphic when restricted to the fibers of \( p \).

\[ \]
Corollary 2.13. — Let $t \in \Delta$ be fixed. The family $(v_\delta|_{X_t})_{\delta>0}$ converges locally smoothly outside $\text{Supp}(B)$ to the lift $v$ of $\frac{\partial}{\partial t}$ with respect to $\rho|_{X^0 \setminus \text{Supp}(B)}$. In particular, $v|_{X^0 \setminus \text{Supp}(B)}$ is holomorphic.

Proof. — Combining Propositions 2.11 and 2.12, one sees that $c(\tau_\delta)$ is locally uniformly bounded on $X_t \setminus \text{Supp}(B)$. Given the elliptic equation satisfied by $c(\tau_\delta)$, it implies local bound at any order (in the fiber directions).

Let $W \subset X$ be a coordinate open subset of $X$ such that $W \cap \text{Supp}(B) = \emptyset$. In local coordinates, this implies that
\begin{equation}
\frac{\partial^2 u_\delta}{\partial t \partial \overline{t}}
\end{equation}
is bounded on $W$ by a constant independent of $\delta$. Since we already dispose of this type of bounds for any other mixed second order derivatives of $u_\delta$, we infer that we have
\begin{equation}
|\Delta'' u_\delta| \leq C_W
\end{equation}
where $\Delta''$ is the Laplace operator corresponding to the flat metric on $W$ and $C_W$ is a constant independent of $\delta$.

This implies that the global function $u_\delta$ admits $C^{1,\alpha}$ bounds locally on $X \setminus \text{Supp}(B)$ for any $\alpha < 1$. By Arzela-Ascoli theorem and Lemma 2.5, it implies that $u_\delta$ converges to $\varphi$ in $C^{1,\alpha}_{\text{loc}}(X \setminus \text{Supp}(B))$. In particular, $\varphi$ is differentiable in the $t$ variable outside $\text{Supp}(B)$, and on this locus, $\partial_t \varphi_t = \lim \partial_t u_\delta$ in the $C^{\alpha}_{\text{loc}}$ topology. Now, (2.32) shows that the convergence actually takes places in $C^\infty_{\text{loc}}(X_t \setminus \text{Supp}(B))$. In particular, outside $\text{Supp}(B)$, $v_\rho|_{X_t}$ is the smooth limit of $v_\delta|_{X_t}$ when $\delta \to 0$. Corollary 2.13 is now a consequence of Proposition 2.8.

Corollary 2.14. — Let $t \in \Delta$ be fixed. Then $dc(\tau_\delta)|_{X_t}$ converges locally uniformly to 0 on the compact subsets of $X_t \setminus \text{Supp}(B)$.

Proof. — Let $K \subset X_t \setminus \text{Supp}(B)$. By the proof of Corollary 2.13 and given (2.29), $c(\tau_\delta)|_K$ is bounded in $L^\infty$ norm hence in any $C^k_{\text{loc}}$ norm on $K$. This implies that family $dc(\tau_\delta)|_K$ is relatively compact in the smooth topology, and the claim follows from Proposition 2.11.

Lemma 2.15. — The vector field $v$ on $X \setminus \text{Supp}(B)$ is holomorphic and extends across $\text{Supp}(B)$.

Proof. — This first assertion follows from a simple computation in [Ber09, Lem. 2.5]. In our setting, this yields on $X_t \setminus \text{Supp}(B_t)$:
\begin{equation}
\partial_t v_\delta - \tau_\delta = \partial c(\tau_\delta) - i\tau_\delta (\partial v_\delta, \overline{v_\delta})
\end{equation}
As on $X_t \setminus \text{Supp}(B_t)$, $\tau_\delta$ and $v_\delta$ converge locally smoothly to $\rho$ and $v$ respectively, one deduces from Corollary 2.14 above that $v$ is holomorphic (hence smooth, too) in the $t$ variable as well, outside $\text{Supp}(B)$.

For the second assertion, let us first observe that $\tau_\delta^n \wedge idt \wedge d\overline{t}$ dominates a smooth volume form $dV$ on $X$. Therefore, it follows from (2.33) that
\begin{equation}
\int_{p^{-1}(U) \setminus \text{Supp}(B)} |v_\delta|^2_{\omega_p} dV \leq C
\end{equation}
An application of Fatou lemma gives:
\begin{equation}
\int_{p^{-1}(U) \setminus \text{Supp}(B)} |v|^2_{\omega_p} dV < +\infty
\end{equation}
By Hartog’s theorem, it follows that $v$ extends to a holomorphic vector field across $\text{Supp}(B)$.
**Lemma 2.16.** — The vector field $v$ preserves $\rho$, hence its flow preserves $B$.

**Proof.** — On $\mathcal{X} \setminus \text{Supp}(B)$, we obtain the equality
\[(2.53) \quad \mathcal{L}_v \rho = 0\]
as a consequence of (2.52).

We show next that (2.53) extends in the sense of currents on $\mathcal{X}$. Indeed, if so then we claim that the flow of $v$ produces the biholomorphic maps $F_t = X_0 \to X_1$ such that $F_0$ is the identity and such that $F_t^* \omega_t = \omega_0$. It is for this equality that we need (2.53) to hold on $\mathcal{X}$ in the sense of currents: it gives
\[(2.54) \quad \frac{d}{dt} F_t^* \omega_t = 0\]
in weak sense on $\mathcal{X}$, but this is enough to conclude that $F_t^* \omega_t = \omega_0$.

If one pulls back the Kähler-Einstein equation satisfied by $\omega_t$ by $F_t$, one gets
\[\text{Ric} F_t^* \omega_t = -F_t^* \omega_t + F_t^* [B_t]\]
where $[B_t] = \sum_k a_k [B_{t,k}]$ if $B_{t,k}$ are the irreducible components of $\text{Supp}(B)$. Because $F_t^* \omega_t = \omega_0$, we obtain
\[F_t^* [B_t] = [B_0]\]
In particular, the local flow of $v$ preserves $\text{Supp}(B)$.

Let us now prove that $v \cdot \rho$ is zero on $\mathcal{X}$. First, let us observe that $\rho$ being a positive current, its coefficients are locally defined complex measures. We claim that these measures put no mass on $\text{Supp}(B)$.

Indeed, by e.g. [Dem12, Proposition 1.14] the "mixed terms" of $\rho$ are dominated by the trace of $\rho$ (the sum of the diagonal coefficients). Therefore everything boils down to showing that if $\omega$ is a smooth Kähler form on $\mathcal{X}$, then the positive measure $\rho \wedge \omega^n$ does not charge $\text{Supp}(B)$. But it is easy to produce a family of cut-off function $\chi_d$ such that $\chi_d$ tends to the characteristic function of $\text{Supp}(B)$ and such that $||\nabla \omega \chi_d||_{L^2(\omega^n+1)}$ and $||\Delta \omega \chi_d||_{L^1(\omega^n+1)}$ tends to 0. We refer to e.g. [CGP13, §9] for this classic construction. Finally, let us introduce $\eta$ a smooth positive function with compact support on $\mathcal{X}$. One can assume that on $\text{Supp}(\eta)$, $\rho = dd^c \psi$ admits a local (bounded) potential. Performing an integration by parts, one obtains:
\[
\int_X \eta \chi_d \rho \wedge \omega^n = \int_X \eta \chi_d dd^c \psi \wedge \omega^n \\
= \int_X \eta \psi dd^c \chi_d \wedge \omega^n + \int_X \chi_d dd^c \eta \wedge \omega^n + \int_X \psi dd \chi_d \wedge \omega^n \\
\leq ||\psi||_\infty \left(||\eta||_\infty ||\Delta \omega \chi_d||_{L^1} + ||\Delta \eta||_\infty \right) \int_{\text{Supp}(\chi_d)} \omega^{n+1} + ||\nabla \eta||_{L^2} ||\nabla \chi_d||_{L^2} 
\]
which tends to 0.

In conclusion, the coefficients of $\rho$ and hence those of $v \cdot \rho$ are complex measures which do not charge $B$. As $v \cdot \rho = 0$ outside $\text{Supp}(B)$, this identity extends across $\text{Supp}(B)$, which is what we wanted to prove. 

If we sum up the results obtained so far, we can find near any $y \in Y^c$ a sufficiently small polydisk $U \subset Y^c$ with coordinates $(t_1, \ldots, t_m)$ centered around $y$ as well as holomorphic vector fields $v_1, \ldots, v_m$ on $p^{-1}(U)$ lifting $\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_m}$ which are tangent to $\text{Supp}(B)$. Up to shrinking $U$, one can assume that the flow of the vector fields $v_\underline{a} := \sum a_i v_i$ for $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{D}^m$ exists at least up to time one. Here $\mathbb{D}$ is the unit disk in $\mathbb{C}$. Then one has a holomorphic map $f : X_y \times \mathbb{D}^m \to p^{-1}(U)$ which sends $(x, \underline{a})$ to $\phi^a_0(x)$ where $(\phi^a_0)_t$ is the flow of $v_\underline{a}$. It is easy to see that $f$ is an isomorphism onto its image, cf e.g. [MK06].
To conclude the proof of Theorem 2.2, we need to show that $v_p$ extends across the singular locus of $p$ provided that $X$ is compact and $p$ is smooth in codimension one. The argument goes as follows.

End of the proof of Theorem 2.2. — Let $n$ be the relative dimension of $p$ and let $m := \dim Y$. Let $Y^0 \subset Y$ be the smooth locus of $p$, and let $X^0 := p^{-1}(Y^0)$. Let $\Omega \in H^0(X, m(K_X/Y + B))$. Let $\rho = \omega + dt^\Psi$ be the positive current constructed in Theorem 2.1, and let $y \in Y \setminus Y^0$.

Let $x \in X$ be a generic point of $p^{-1}(y)$. Take a small neighborhood $U$ of $x$, and set $D := p(U)$. If $p$ is smooth on codim 1, $p$ is smooth on $U$. We can thus fix a coordinate system $(t_1, z_1, \ldots, z_n)$ of $U$, such that $t_1$ represents the horizontal directions and $\frac{\partial}{\partial t_1}$ is in the fiber direction. The notation $\underline{t}$ means that $\underline{t} = (t_1, \ldots, t_m)$. There is a slight abuse of notation: the coordinate of the base is also $\underline{t}$. But as $p$ is smooth on $U$, we just mean that $p_\ast \left( \frac{\partial}{\partial t_1} \right) = \frac{\partial}{\partial t_1}$, where the former is on $X$ and the later is on $Y$. Finally, we set $p_\ast(id \underline{t} \wedge d \underline{t}) := \wedge_{k=1}^m id \underline{t}_k \wedge d \underline{t}_k$.

Let $v_k$ be the holomorphic vector field on $X^0 \cap p^{-1}(D)$ constructed in the proof of Theorem 2.2, attached to $\frac{\partial}{\partial t_k}$, where $1 \leq k \leq m$.

\[(2.55) \quad \rho^n \wedge p_\ast(id \underline{t} \wedge d \underline{t}) = \left( \frac{\Omega \wedge (\Omega)^\perp_{f_B}}{|f_B|^2} \right)^n \wedge p_\ast(id \underline{t} \wedge d \underline{t}) \quad \text{on } U.\]

We know that $\iota_{v_k} \rho$ is proportional to $dt_k$, from which it follows that

\[(2.56) \quad \iota_{v_1, \bar{v}_1} \cdots \iota_{v_m, \bar{v}_m} (\rho^n \wedge p_\ast(id \underline{t} \wedge d \underline{t})) = \rho^n\]

Combining (2.55) and (2.56), one gets

\[\iota_{v_1, \bar{v}_1} \cdots \iota_{v_m, \bar{v}_m} \left[ \frac{\Omega \wedge (\Omega)^\perp_{f_B}}{|f_B|^2} \right]^n \wedge p_\ast(id \underline{t} \wedge d \underline{t}) = \rho^n\]

One can find a Kähler form $\omega_X$ on $X$ such that $\frac{\Omega \wedge (\Omega)^\perp_{f_B}}{|f_B|^2} \wedge p_\ast(id \underline{t} \wedge d \underline{t}) \geq \omega_X^{n+m}$. Given that $\omega_X^n \wedge [\iota_{v_1, \bar{v}_1} \cdots \iota_{v_m, \bar{v}_m} (\omega_X^{n+m})] = (\prod_k |v_k|^2_{\omega_X}) \cdot \omega_X^{n+m}$ (maybe up to some constant), we eventually get that

\[\int_{U \cap X^0} \left( \prod_{k=1}^m |v_k|^2_{\omega_X} \right) \cdot \omega_X^{n+m} \leq \int_{U \cap X^0} \rho^n \wedge \omega_X^n\]

and the right hand side is finite, dominated by $\int_X (\rho^n \wedge \omega_X^n) \leq \{\omega\}^n \cdot \{\omega_X\}^m$ by [BEGZ10, Prop. 1.6 & 1.20], given that $\rho$ is a closed, positive current on $X$ in the cohomology class $\{\omega\}$. As $|v_k|^2_{\omega_X}$ is uniformly bounded below by a positive constant on $p^{-1}(D) \cap X^0$, one deduce that $v_k \in L^2(p^{-1}(D) \cap X^0, \omega_X)$. By Riemann extension theorem the holomorphic vector fields $v_k$ extend to holomorphic vector fields on $p^{-1}(D)$ whose flow provide the expected trivialization. Indeed, the $v_k$ are tangent to $B$ on $X^0$, hence they are tangent to $B$ everywhere by the assumptions in 2.2.

As application of Theorem 2.2 we can prove Corollary D.

Proof of Corollary D. — Our proof follows the same line of arguments as in [Kol87].

We proceed by contradiction: assume that $\mathcal{F}_m$ is not big. In any case, this bundle can be endowed with a metric (used several times in the current subsection) with semi-positive curvature form denoted by $\theta$, and smooth on a Zariski open subset $V \subset Y$ as $B$ is generically transverse to the fibers. Then we claim that we have

\[\theta|_V \wedge \dim(Y) = 0\]
at each point of $V$. Indeed, if (2.57) is not true, then there exists a point $y_0 \in V$ such that all the eigenvalues of $\theta_{y_0}$ are strictly positive. By the singular version of holomorphic Morse inequalities (cf. [Bou02, Cor. 3.3]) this implies that $\mathcal{F}_m$, and we have assumed that this is not the case.

It follows that the kernel of $\theta$ is non-trivial at each point of $V$. Since $\theta|_V$ is smooth and closed, locally near each point of $V$ its kernel defines a foliation whose leaves are analytic sets, cf [Kol87] and the references therein. We choose a smooth holomorphic disk $\Delta$ contained in such a leaf; the restriction of $p$ to $p^{-1}(\Delta) := X_\Delta$ is a submersion, and the curvature of the direct image of the relative pluricanonical bundle is identically zero. By Theorem 2.2 we infer that the vector $v_{\rho}$ is holomorphic. On the other hand, $\bar{\partial} v_{\rho}$ is a representative of the image of the tangent vector $\frac{\partial}{\partial t} \in T\Delta$ by the map (0.5). Since by hypothesis this map is injective, we obtain a contradiction.

We finish the current section with the proof of Corollary 2.3.

Proof of Corollary 2.3. — The statement (2.3.8) is a direct consequence of [Gue16] applied to the right hand side term of the equality

$$-p^*(K_Y) = K_{X/Y} + (-K_X - B) + B.$$  

By hypothesis the class $-c_1(K_X + B)$ is in the closure of the Kähler cone of $X$ and one can use loc. cit.

Given Theorem 2.2, it would be enough to prove that $p$ is smooth in codimension one. We use the following elegant argument due to Q. Zhang, cf. [Zha05]. Assume that there exists some codimension one subvariety $D \subset X$ such that $p_*(D)$ is of codimension at least two. Let $\tau : Y' \to Y$ be the composition of the blow-up of the closed analytic set $p_*(D)$ with a resolution of singularities of the resulting complex space. There exists an effective divisor $E_{Y'}$ whose support is contained in the $\tau$-exceptional locus such that we have

$$K_{Y'} \sim E_{Y'}.$$  

Let $p' : X' \to Y'$ be a resolution of indeterminacies of $X \dashrightarrow Y'$. As $c_1(K_X + B) = 0$, we have

$$(p')^*(-K_{Y'}) + E_{X'} \equiv_Q K_{X'/Y'} + B',$n

where $E_{X'}$ is supported in exceptional locus of $\pi : X' \to X$. By [Gue16], $K_{X'/Y'} + B'$ is pseudo-effective. Therefore the direct image $\pi_*(p')^*(-K_{Y'}) + E_{X'} = \pi_*(-E_{Y'})$ is pseudo-effective as well. However by construction we have $\pi_*(E_{Y'}) \geq |D|$, and we obtain a contradiction.

We prove next that the map $p$ is reduced in codimension one. Let $E \subset Y$ be a divisor. Its $p$-inverse image can be written as

$$p^{-1}(E) = \sum_i a_i [D_i]$$  

where $D_i \subset X$ are irreducible divisors. It is well know that (cf. [CP17, Thm. 2.4] or also [Tak16])

$$K_{X/Y} + B \geq \sum_i (a_i - 1)_+ \cdot [D_i],$$  

where $(a_i - 1)_+ := \max\{a_i - 1, 0\}$.

Therefore we must have $a_i = 1$ for every $i$, since by assumption $K_{X/Y} + B \equiv_Q 0$. Corollary 2.3 is proved.  

\end{document}
2.5. Log abundance in the Kähler setting. — In this section, we briefly explain how to prove the log abundance for klt Kähler pairs \((X, B)\) such that \(B\) has snc support. This is based on the following lemma, which is a consequence of [Bud09] and [Wan16, Cor 1.4] (cf. also [CKP12, Lem.1.1] and [CP11] and the references therein). For the reader’s convenience, we recall briefly the proof here. \(^{(1)}\)

Lemma 2.17. — Let \(X\) be a compact Kähler manifold and \(\Delta = \sum a_iB_i\) be an effective klt \(\mathbb{Q}\)-divisor with simple normal crossing support. Assume that \(\Delta \sim L_1\) for some \(L_1 \in \text{Pic}(X)\). For each integer \(k \geq 0\), define \(L_k := kL_1 - [k\Delta]\). Then for each \(k, i\) and \(q\), the set

\[
V^q_i(f, L_k) = \{\lambda \in \text{Pic}^0(X) : h^q(X, K_X + L_k + \lambda) \geq i\}
\]

is a finite union of torsion translates of complex subtori of \(\text{Pic}^0(X)\).

Proof. — Let \(N\) be the minimal number such that \(N \cdot a_i \in \mathbb{N}\) for every \(i\). Let \(\sigma : \tilde{X} \to X\) be the \(N\)-cyclic cover of \(L_1\) along the canonical section \(NL_1\). One can check that \(\tilde{X}\) has analytic quotient singularities, hence rational singularities by [Bur74, Prop. 4.1]. This implies in turn that for any resolution \(\pi : \tilde{X} \to X\), one has \(\pi_*\mathcal{O}_{\tilde{X}}(K_{\tilde{X}}) = \mathcal{O}_X(K_X)\) thanks to e.g. [FZ03, Prop. 1.17 (a)]. Therefore, if we define \(f := \sigma \circ \pi : \tilde{X} \to X\), we have

\[
H^q(\tilde{X}, K_{\tilde{X}} + f^*\lambda) \simeq \bigoplus_{k=0}^{N-1} H^q(X, K_X + L_k + \lambda)
\]

for any line bundle \(\lambda\) on \(X\).

Let \(g : \text{Pic}^0(X) \to \text{Pic}^0(\tilde{X})\) be the natural morphism induced by \(f\) and set

\[
V^q_i(f) := \{\rho \in \text{Pic}^0(X), h^q(\tilde{X}, K_{\tilde{X}} + f^*\rho) \geq i\}
\]

and

\[
V^q_i := \{\rho \in \text{Pic}^0(\tilde{X}), h^q(\tilde{X}, K_{\tilde{X}} + \rho) \geq i\}.
\]

Then we have

\[
V^q_i(f) = g^{-1}(V^q_i).
\]

Thanks to [Wan16], \(V^q_i\) is a finite union of torsion translates of complex subtori of \(\text{Pic}^0(\tilde{X})\). Together with (2.60), this shows that \(V^q_i(f)\) has the same structure. Thanks to (2.59), we have

\[
V^q_i(f) \cap \bigcup_{k=0}^{N-1} V^q_i(f, L_k) = \bigcup_{k=0}^{N-1} V^q_i(f, L_k),
\]

where \(V^q_i(f, L_k) := \{\rho \in \text{Pic}^0(X), h^q(X, K_X + L_k + \rho) \geq i\}\). As \(V^q_i(f)\) is the finite union of torsion translates of complex subtori, by using (2.61), we can prove that ([CKP12, Lemma 1.1]) \(V^q_i(f, L_k)\) has the same structure. \(\square\)

Corollary 2.18. — Let \(X\) be a compact Kähler manifold and \(\Delta = \sum a_iB_i\) be an effective klt \(\mathbb{Q}\)-divisor with simple normal crossing support. If \(c_1(K_X + \Delta) = 0 \in H^{1,1}(X, \mathbb{Q})\), then \(K_X + \Delta\) is \(\mathbb{Q}\)-effective.

Proof. — Note that \(\text{Pic}^0(X)\) is a torus, then \(K_X + \Delta \sim L\) for some line bundle \(L \in \text{Pic}^0(X)\). Let \(\pi : X' \to X\) be a log resolution of \((X, \Delta)\). We can thus find a klt divisor \(\Delta'\) on \(X'\) with normal crossing support such that

\[
K_{X'} + \Delta' \equiv_{\mathbb{Q}} \pi^*(K_X + \Delta) + E
\]

\(^{(1)}\)We would like to thank Botong Wang for telling us the following nice application of his result.
for some $\mathbb{Q}$-effective divisor $E$ supported in the exceptional locus of $\pi$. As $c_1(K_X + \Delta) = 0$, we have $\kappa(K_{X'} + \Delta' + \rho) \geq 0$ for some $\rho \in \text{Pic}^0(X')$. By applying Lemma 2.17 to $K_{X'} + \Delta'$, we obtain that $\kappa(K_{X'} + \Delta') \geq 0$. Then $\kappa(K_X + \Delta) \geq 0$. The Corollary is proved. \qed
3. Transverse regularity of singular Monge-Ampère equations

In this section our main goal is to prove Theorem E. This will be achieved as a consequence of a few intermediate results which we state in a general setting.

The main source of difficulties in the proof of E arise from the fact that the set of base points of pluricanonical sections may be non-zero. The determinant of the metric adapted to this geometric setting vanishes along the said base points so in particular the Ricci curvature of this metric is not bounded from below. Unfortunately under these circumstances we were not able to obtain a complete analogue of the Sobolev and Poincaré inequalities (which are needed for the study of the regularity properties of Monge-Ampère equations). We will therefore start this section with a weak version of these results.

3.1. Weak Sobolev and Poincaré inequalities. — In this section we will derive a version of the usual Poincaré and Sobolev type inequalities which are needed in our context. As it is well known, they are playing a crucial role in the regularity questions for the Monge-Ampère equations. The set-up is as follows: let \((X, \omega)\) be a compact Kähler manifold of dimension \(n\), and let

\[
E := \sum_{\alpha \in I} e_\alpha E_\alpha \quad B := \sum_{\beta \in J} b_\beta B_\beta
\]

be two effective divisors on \(X\) without common components, such that \(e_\alpha \in \mathbb{Q}_+\), \(b_\beta \in [0, 1]\) and such that the support of \(E + B\) is snc. We assume that the manifold \(X\) is covered by a fixed family of coordinate sets \((\Omega_j)\) such that

\[
\Omega_j \cap \text{Supp}(E + B) = (z_j^1 \ldots z_j^d = 0)
\]

where \((z_j)\) are coordinates on \(\Omega_j\).

Let \(\sigma_i, s_i\) be the canonical section of the Hermitian bundle \((\mathcal{O}(E_i), h_i)\) and \((\mathcal{O}(B_j), g_i)\) respectively, where \(h_i\) and \(g_i\) are non-singular reference metrics. For each positive \(\varepsilon \geq 0\) and each multi-index \(q\) we introduce the following volume element

\[
d\mu_{q}^{(\varepsilon)} := \prod_{\alpha \in I} (\varepsilon^2 + |\sigma_\alpha|^2)^{q_\alpha} \prod_{\beta \in J} (\varepsilon^2 + |s_\beta|^2)^{b_\beta} dV_\omega
\]

where \(dV_\omega\) is the volume element corresponding to the reference metric \(\omega\). Also, for each positive real number \(p \leq 2\) we define the multi-index \(q_p\) whose components are

\[
\left(1 - \frac{p}{2}\right) q_\alpha.
\]

Then we have the following statements.

**Proposition 3.1.** — There exists a constant \(C > 0\) independent of \(\varepsilon\) (but depending on everything else) such that for every smooth function \(f\) on \(X\) we have

\[
\left(\int_X |f|^{2np} d\mu_q^{(\varepsilon)}\right)^{\frac{2n-p}{2np}} \leq C \left(\int_X |\nabla_\varepsilon f|^p d\mu_{q_p}^{(\varepsilon)} + \int_X |f|^p d\mu_{q_p}^{(\varepsilon)}\right)^{\frac{1}{p}}
\]

where \(1 \leq p < 2\) is any real number, and the gradient \(\nabla_\varepsilon\) corresponds to the \(\varepsilon\)-regularization of a fixed metric with conic singularities along the divisor \(\sum_{\beta \in J} b_\beta B_\beta\).

As we can see, there is an important difference between the Proposition 3.1 and the standard weighted Sobolev inequalities: the volume element in the left hand side of (3.5) is not the same as the one in the right hand side term.

In a similar vein, we have the next version of the Poincaré inequality.
Proposition 3.2. — There exists a constant $C > 0$ as above such that for any smooth function $f$ on $X$ we have

$$\int_X |f - VM_\mu(f)|^p \, d\mu^{(e)} \leq C \int_X |\nabla_\varepsilon f|^p \, d\mu^{(e)},$$

where $p \geq 1$ is a real number, and where we use the notation

$$VM_\mu(f) := \int_X f \, d\mu^{(e)}.$$ 

We first prove the statement 3.1; the arguments which will follow have been “borrowed” from the book [HKM06, Chap. 15].

Proof of Proposition 3.1. — We first assume that $B = 0$ because the arguments for the general case are practically identical.

A first remark is that it is enough to consider the local version of the statement, as follows. Let $\Omega$ be one of the domains covering $(X, E)$ as mentioned in (3.2); we denote by $(z_1, \ldots, z_n)$ the corresponding coordinate system. We will assume that we have

$$\Omega = \prod_j (|z_j| < 1)$$

and that the function $f$ has compact support in $\Omega$.

In terms of this local setting, the quantity to be evaluated becomes

$$\int_\Omega |f|^{2np} \prod_{\alpha=1}^d (\varepsilon^2 + |z_\alpha|^2)^{q_\alpha} \, d\lambda$$

(since $b_\ell = 0$). Let $B := (|t| < 1) \subset \mathbb{C}$ be the unit disk in the complex plane. We consider the function

$$F_\varepsilon(t) = \left(\frac{\varepsilon^2 + |t|^2}{1 + \varepsilon^2}\right)^{q/2}$$

where $q > 0$ is a real number. It turns out that $F_\varepsilon$ is a diffeomorphism and the square of the absolute value of its Jacobian $dF_\varepsilon \wedge dF_\varepsilon$ verifies the inequality

$$C^{-1}(\varepsilon^2 + |t|^2)^q \leq \frac{dF_\varepsilon \wedge dF_\varepsilon}{dt \wedge dt} \leq C(\varepsilon^2 + |t|^2)^q$$

where $C$ is a constant independent of $\varepsilon$ (it can be explicitly computed). Let $G_\varepsilon$ be the inverse of $F_\varepsilon$. The implicit function theorem shows that we have

$$|dG_\varepsilon(t)| \leq \frac{C}{(\varepsilon^2 + |t|^2)^{q/2}}.$$ 

By the change of variables formula we have

$$\int_\Omega |f(z)|^{2np} \prod_{\alpha=1}^d (\varepsilon^2 + |z_\alpha|^2)^{q_\alpha} \, d\lambda \leq C \int_\Omega |\tilde{f}(w)|^{2np} \, d\lambda(w)$$

where by definition we set

$$\tilde{f}(w) := f(G_\varepsilon(w_1), \ldots, G_\varepsilon(w_d), w_{d+1}, \ldots, w_n);$$

it is a function defined on the “same” poly-disk $\Omega$, and it has compact support.

Therefore, by the usual version of the Sobolev inequality we obtain
We proceed as in the previous proof: we have

\[(3.15) \quad \left( \int_{\Omega} |\tilde{f}(w)|^{\frac{2np}{2n-p}} d\lambda(w) \right)^{\frac{2n-p}{2n}} \leq C \int_{\Omega} |\nabla \tilde{f}(w)|^{p} d\lambda(w). \]

We use the relation (3.14), together with the change of coordinates \(w_{\alpha} = F_{\varepsilon}(z_{\alpha})\) for \(\alpha = 1, \ldots d \) and we infer that we have

\[(3.16) \quad \int_{\Omega} |\nabla \tilde{f}(w)|^{p} d\lambda(w) \leq C \int_{\Omega} |\nabla f(z)|^{p} \prod_{\alpha=1}^{d} (\varepsilon^{2} + |z_{\alpha}|^{2})^{q_{\alpha}(1-\frac{p}{2n})} d\lambda. \]

In conclusion we have

\[(3.17) \quad \left( \int_{\Omega} |f(z)|^{\frac{2np}{2n-p}} \prod_{\alpha=1}^{d} (\varepsilon^{2} + |z_{\alpha}|^{2})^{q_{\alpha}} d\lambda \right)^{\frac{2n-p}{2n}} \leq C \int_{\Omega} |\nabla f(z)|^{p} \prod_{\alpha=1}^{d} (\varepsilon^{2} + |z_{\alpha}|^{2})^{q_{\alpha}(1-\frac{p}{2n})} d\lambda. \]

that is to say, we have established the local version of the inequality (3.1). The general case follows by a partition of unit argument which we skip. \(\square\)

The same scheme of proof applies to Proposition 3.2: we will first show that the local version of this statement holds by using a change of coordinates and the classical version of Poincaré inequality, and then we show that the global version (3.6) is true by a well-chosen covering of \(X\).

**Proof.** — The inequality (3.6) is easily seen to follow provided that we are able to establish the following relation

\[(3.18) \quad \int_{X \times X} |f(x) - f(y)|^{p} d\mu_{q}^{(x)}(x) d\mu_{q}^{(y)}(y) \leq C \int_{X} |\nabla f|^{p} \mu_{q}^{(x)} \]

for any \(1 \leq p \leq 2\). This is very elementary and we will not provide any additional explanation.

Assume that we have a covering of \(X\)

\[(3.19) \quad X = \bigcup_{i} U_{i} \]

where each \(U_{i}\) is a coordinate open set. In order to obtain a bound as in (3.18), it would be enough to analyze the quantities

\[(3.20) \quad \int_{U_{i} \times U_{j}} |f(x) - f(y)|^{p} d\mu_{q}^{(x)}(x) d\mu_{q}^{(y)}(y) \]

for each couple of indexes \(i, j\) which is what we do next.

To start with, let \(\Omega\) be one of the coordinate sets \(U_{i}\); we will show next that the following local version of (3.6) holds true

\[(3.21) \quad \int_{\Omega \times \Omega} |f(x) - f(y)|^{p} d\mu_{q}^{(x)}(x) d\mu_{q}^{(y)}(y) \leq C \int_{\Omega} |\nabla f|^{p} \mu_{q}^{(x)}. \]

We proceed as in the previous proof: we have

\[(3.22) \quad \int_{\Omega \times \Omega} |f(x) - f(y)|^{p} d\mu_{q}^{(x)}(x) d\mu_{q}^{(y)}(y) \leq C \int_{\Omega \times \Omega} |\tilde{f}(z) - \tilde{f}(w)|^{p} d\lambda(z, w) \]

by a change of coordinates as indicated in (3.10). Now we have

\[(3.23) \quad \tilde{f}(z) - \tilde{f}(w) = \int_{0}^{1} \frac{d}{dt} \tilde{f}((1-t)z + tw)) dt \]
and it follows that we have
\begin{equation}
\int_{\Omega \times \Omega} \left| \tilde{f}(z) - \tilde{f}(w) \right|^p d\lambda(z, w) \leq C \int_0^1 dt \int_{\Omega \times \Omega} \left| \nabla \tilde{f}((1 - t)z + tw) \right|^p d\lambda(z, w),
\end{equation}
where the constant $C > 0$ in (3.24) depends on the diameter of $\Omega$ measured with respect to the Euclidean metric.

Then we invoke the usual trick: we split the integral above in two—the first part is as follows
\begin{equation}
\int_0^{1/2} dt \int_{\Omega \times \Omega} \left| \nabla \tilde{f}((1 - t)z + tw) \right|^p d\lambda(z, w) \leq C \int_{\Omega} \left| \nabla \tilde{f}(z) \right|^p d\lambda(z)
\end{equation}
where up to a numerical constant, $C$ in (3.25) only depends on the volume of $\Omega$. We have a similar estimate for the integral corresponding to the interval $[1/2, 1]$, so all in all we infer
\begin{equation}
\int_{\Omega \times \Omega} \left| \tilde{f}(z) - \tilde{f}(w) \right|^p d\lambda(z, w) \leq C \int_{\Omega} \left| \nabla \tilde{f}(z) \right|^p d\lambda(z).
\end{equation}
Changing the coordinates back, together with the considerations in the proof of weak Sobolev inequality show that (3.21) is proved.

The general case follows by choosing a covering $(U_j)$ of $X$, such that the following properties are satisfied.

1. If $U_p \cap U_q \neq \emptyset$ and if at least one of them intersects the support of the divisor $E$, then the union $U_p \cup U_q$ is contained in a coordinate set endowed with coordinates adapted to $(X, E)$ (as in the beginning of this section).
2. If $U_p \cap U_q \neq \emptyset$ and if neither of $U_i$ or $U_j$ intersects Supp$(E)$, then the union $U_p \cup U_q$ is contained in a coordinate ball which is disjoint of Supp$(E)$.
3. The $d\mu^{(\varepsilon)}$-volume of the coordinate sets containing $U_i \cup U_j$ in (1) and (2) is bounded from above and below by constants which are independent of $\varepsilon$.

It is clear that such cover exists, and we fix one denoted by $\Lambda$ for the rest of the proof. Note that this cover is independent of $\varepsilon$. Next, given any couple $U_i, U_j$ of sets belonging to $\Lambda$, we consider a collection
\begin{equation}
\Xi_{ij} = (\Omega_1, \Omega_2, \ldots, \Omega_N)
\end{equation}
of elements of $\Lambda$ such that the following properties are verified.

(a) We have $\Omega_1 := U_i$ and $\Omega_N := U_j$, and all of the intermediate $\Omega$’s are elements of $\Lambda$.
(b) For any $r = 1, \ldots, N - 1$ we have $\Omega_r \cap \Omega_{r+1} \neq \emptyset$.

Again, there are many choices for such $\Xi_{ij}$, but we just pick one of them for each pair of indexes $(i, j)$.

We are now ready to analyze the quantities (3.20): for each couple $(i, j)$ we consider the collection $\Xi_{ij}$. Given
\begin{equation}
(x_1, \ldots, x_N) \in \Omega_1 \times \cdots \times \Omega_N
\end{equation}
we have
\begin{equation}
|f(x_1) - f(x_N)|^\alpha \leq C \sum_q |f(x_q) - f(x_{q+1})|^\alpha
\end{equation}
for some numerical constant $C > 0$.

We consider now the following expression
\begin{equation}
\int_{\Omega_1 \times \cdots \times \Omega_N} |f(x_1) - f(x_N)|^p d\mu^{(\varepsilon)}(x_1) \ldots d\mu^{(\varepsilon)}(x_N);
\end{equation}
on one hand, up to a constant this is simply (3.20). One the other hand (3.30) is bounded from above by
\( C \sum_q \int_{\Omega_1 \times \cdots \times \Omega_N} |f(x_q) - f(x_{q_1})|^p d\mu_q^\varepsilon(x_1) \cdots d\mu_q^\varepsilon(x_N) \)

The last observation is that each term of the sum (3.31) is of type (3.21)—here we are using the properties (1)-(3) and (a), (b) above—for which we have already shown the desired Poincaré inequality. This ends the proof of the case \( B = 0 \).

We will not detail the proof of the general statement, because the arguments are identical to the ones already given. The only change is that we will work with geodesics with respect to the model conic metric
\( \sqrt{-1} \sum_{\alpha \in J} dz_\alpha \wedge d\bar{z}_\alpha (e^2 + |z_\alpha|^2)b_\alpha + \sqrt{-1} \sum_{\alpha \notin J} dz_\alpha \wedge d\bar{z}_\alpha \)

instead of straight lines \((1 - t)x + ty\). The same proof works because the Ricci curvature of the metric (3.32) is bounded from below by some constant independent of \( \varepsilon \). For a complete treatment of this point we refer to \[SC02\], pages 177-179.

3.2. Lie derivative of fiberwise Monge-Ampère Equations. — In this subsection we consider the restriction of our initial family \( p \) to a generic disk contained in the base, together with a family of Monge-Ampère equations of its fibers. Let \( \mathbb{D} \subset Y \) be a one-dimensional germ of submanifold contained in a coordinate set of \( Y \), and let \( X := p^{-1}(\mathbb{D}) \) (notations as in Theorem E).

The resulting map \( p : X \to \mathbb{D} \) will be a proper submersion, provided that \( \mathbb{D} \) is generic. We recall that the total space \((X, \omega)\) of \( p \) is a Kähler manifold. We denote by \( t \) a coordinate on the unit disk \( \mathbb{D} \), and let
\( v = \frac{\partial}{\partial t} + v^\alpha \frac{\partial}{\partial z_\alpha} \)

be the local expression of a smooth vector field which projects into \( \frac{\partial}{\partial t} \).

Another piece of data is the following fiberwise Monge-Ampère equation
\( (\omega + d\bar{d}^c \varphi)^n = e^{\lambda \varphi + f} \omega^n \)
on each \( X_t \). Here \( \lambda \geq 0 \) is a positive real number, and \( f \) is a smooth function on \( X \). We can write this globally as follows
\( (\omega + d\bar{d}^c \varphi)^n \wedge \sqrt{-1} dt \wedge d\bar{t} = e^{\lambda \varphi + f} \omega^n \wedge \sqrt{-1} dt \wedge d\bar{t} \)
on \( X \), where the meaning of \( d\bar{d}^c \) and of \( \varphi \) is not the same as in (3.34), but...

We take the Lie derivative \( \mathcal{L}_v \) of (3.35) with respect to the vector field \( v \), and then restrict to a fiber \( X_t \). The Lie derivative of the left-hand side term of (3.35) equals
\( n \mathcal{L}_v (\omega + d\bar{d}^c \varphi) \wedge (\omega + d\bar{d}^c \varphi)^{n-1} \wedge \sqrt{-1} dt \wedge d\bar{t} \)
because we have \( \mathcal{L}_v (\sqrt{-1} dt \wedge d\bar{t}) = 0 \), given the expression (3.33).

The form \( \omega + d\bar{d}^c \varphi \) is closed, hence by Cartan formula we have
\( \mathcal{L}_v (\omega + d\bar{d}^c \varphi) = d(i_v \cdot (\omega + d\bar{d}^c \varphi)) \)
where \( i_v \cdot \omega \) is the contraction of \( \omega \) with respect to the vector field \( v \). We evaluate next the quantity
\( d(i_v \cdot d\bar{d}^c \varphi) \wedge \sqrt{-1} dt \wedge d\bar{t} \)
by a point-wise computation, as follows. With respect to the local coordinates as in (3.33), we write
\begin{equation}
    dd^c \varphi = \varphi_{\beta \gamma} \sqrt{-1}dt \wedge d\overline{\varphi} + \varphi_{\beta \gamma} \sqrt{-1}dz^\beta \wedge d\overline{\varphi} \wedge \sqrt{-1}dz \wedge d\overline{\varphi};
\end{equation}
in the expression above we are using the Einstein convention. Then we have
\begin{equation}
    d \left( iv \cdot dd^c \varphi \right) \equiv \left( \varphi_{\beta \gamma} + \varphi_{\gamma \beta} v^\gamma + \varphi_{\gamma \beta} v^\gamma \right) dz^\beta \wedge dz \wedge d\overline{\varphi}
\end{equation}
where \( \equiv \) means that we are only consider the terms of (1,1)-type which do not contain \( dt \) or its conjugate.

On the other hand, the coefficients of the Hessian of the function
\begin{equation}
    v(\varphi) = \varphi_t + \varphi_\gamma v^\gamma
\end{equation}
in the fibers direction are equal to
\begin{equation}
    (\varphi_\gamma v^\gamma + \varphi_\gamma v^\gamma d\overline{\varphi}) dz^\beta \wedge dz \wedge d\overline{\varphi} = \partial (\bar{\partial}v \cdot \varphi).
\end{equation}
Here \( \bar{\partial}v \) is a (0,1)-form with values in \( T_{X_t} \) and then \( \bar{\partial}v \cdot \varphi \) is a form of (0,1) type on \( X_t \).

On the other hand, if we denote by \( \Delta \varphi = \text{Tr}_\varphi \sqrt{-1}d\bar{\partial} \) the Laplace operator corresponding to the metric \( \omega_{\varphi} := \omega + dd^c \varphi \) on the fibers of \( p \), then we can rewrite the equation (3.37) as follows
\begin{equation}
    \left( \Delta \varphi v(\varphi) - \text{Tr}_\varphi \partial (\bar{\partial}v \cdot \varphi) + \Psi_{\varphi,v} \right) \omega_{\varphi}^\alpha \wedge \sqrt{-1}dt \wedge d\overline{\varphi}.
\end{equation}
In the expression (3.44) we denote by \( \text{Tr}_\varphi \) the trace with respect to \( \omega_{\varphi} \) on \( X_t \), and we denote by \( \Psi_{\varphi,v} \) the function on \( X \) such that the equality
\begin{equation}
    \Psi_{\varphi,v} \omega_{\varphi}^\alpha \wedge \sqrt{-1}dt \wedge d\overline{\varphi} = L_v(\omega) \wedge \omega_{\varphi}^{\alpha-1} \wedge \sqrt{-1}dt \wedge d\overline{\varphi}
\end{equation}
holds on \( X \).

As for the right hand side of (3.35), the expression of the Lie derivative reads as follows
\begin{equation}
    (\lambda v(\varphi) + v(f) + \Psi_v) \omega_{\varphi}^\alpha \wedge \sqrt{-1}dt \wedge d\overline{\varphi}
\end{equation}
where -as before- the function \( \Psi_v \) is defined by the equality
\begin{equation}
    \Psi_v \omega_{\varphi}^\alpha \wedge \sqrt{-1}dt \wedge d\overline{\varphi} = L_v(\omega) \wedge \omega_{\varphi}^{\alpha-1} \wedge \sqrt{-1}dt \wedge d\overline{\varphi}.
\end{equation}
In conclusion, for each \( t \in \mathbb{D} \) we obtain the equality
\begin{equation}
    \Delta \varphi v(\varphi) - \text{Tr}_\varphi \partial (\bar{\partial}v \cdot \varphi) + \Psi_{\varphi,v} = \lambda v(\varphi) + v(f) + \Psi_v
\end{equation}
which is the identity we intended to obtain in this subsection. \( \square \)
3.3. Regularity in transverse directions. — In this section we will apply the results above in order to analyze the transversal regularity of the solution of the equation

\[(\omega + dd^c \varphi_t)^n = e^{\lambda \varphi_t + f} \prod_{i \in I} |\sigma_i|^{2e_i} \prod_{j \in J} |s_j|^{2b_j} \omega^n\]

on $X_t$. Here $\lambda \geq 0$ is a positive real, and the parameters $e_i, b_j$ are chosen as above. In case we have $\lambda = 0$, the normalization we choose for the solution is

\[\int_{X_t} \varphi_t \omega^n = 0.\]

The function $f$ in (3.49) is supposed to be smooth on the total space $X$.

We consider the family of approximations of (3.49)

\[(\omega + dd^c \varphi_\varepsilon)^n = e^{\lambda \varphi_\varepsilon + f} \prod_{i \in I} (\varepsilon^2 + |\sigma_i|^2)^{e_i} \prod_{j \in J} (\varepsilon^2 + |s_j|^2)^{b_j} \omega^n\]

on $X_t$. By general results in MA theory, the function $\varphi_\varepsilon$ obtained by glueing the fiberwise solutions of (3.51) is smooth. We will analyze in the next subsections the uniformity with respect to $\varepsilon$ of several norms of $\varphi_\varepsilon$.

We recall the following important result whose origins can be found in [Yau78].

**Theorem 3.3.** — For any strictly smaller disk $D' \subset D$ there exists a constant $C > 0$ such that we have

\[\|\varphi_\varepsilon\|_{C^1(X_t)} \leq C\]

for all $t \in D'$, where the $C^1$ norm above is with respect to a fixed metric which is quasi-isometric to (3.32).

If $b_j = 0$, this is a consequence of [Yau78], cf. also the version established in [Pău08], stating that

\[\omega + dd^c \varphi_\varepsilon \leq C\omega|_{X_t}.\]

The conic case is much more involved and we refer to Theorem 3.7 and the few lines following that statement, on page 58. Note that inequality (3.53) is still true provided that we replace the RHS with $C\omega_{B,\varepsilon}|_{X_t}$, where $\omega_{B,\varepsilon}$ is the regularization of a conic metric corresponding to $(X, B)$ which is quasi-isometric with (3.32).

During the rest of the current subsection we assume that $\lambda = 0$, which is anyway what we need for the proof of Theorem E. We will explain along the way how to adapt our method to the case $\lambda > 0$.

3.3.1. Mean value of the t-derivative. — Let $v$ be a smooth vector field on $X$ of (1,0)-type, which has the following properties.

(i) It is a lifting of $\partial / \partial t$, i.e. we have

\[dp(v) = \frac{\partial}{\partial t}\]

(with the usual abuse of notation...).

(ii) We write $v$ locally as in (3.33); then on $\Omega_j$ we have

\[|v^\alpha(z_j)| \leq C|z_j^\alpha|\]

(we use the notations/conventions as in (3.2)) for all $\alpha = 1, \ldots, d$. This means that $v$ is a smooth section of the logarithmic tangent space of $(X, E_{red} + B_{red})$. 
Such a vector field $v$ is easy to construct, by a partition of unit of local lifts of $\frac{\partial}{\partial t}$. We consider the coordinate sets $\Omega_j$ and the $z_j$ adapted to the pair $(\mathcal{X}, B + E)$. Then the particular form of the transition implies (ii).

In this context we have the following statement.

**Lemma 3.4.** — There exists a constant $C > 0$ independent of $\varepsilon$ such that we have

$$\int_{\mathcal{X}_t} v(\varphi_{\varepsilon}) \omega^n_{\varphi_{\varepsilon}} \leq C$$

for any $t \in \mathbb{D}'$.

**Proof.** — We consider a covering of $\mathcal{X}$ by coordinate sets $(U_i, (z_i, t))$, where the last coordinate $t$ is given by the map $p$. The normalization condition (3.50) can be written as

$$\sum_i \int_{\|z_i\| < 1} \theta_i(z_i, t) \varphi_{\varepsilon}(z_i, t) \prod_{\alpha \in I} \left( e^{\varepsilon^2 + |z_i|^2 e^{\phi_{\varepsilon}(z_i, t)}} \right)^{e_\alpha} \prod_{\beta \in J} \left( e^{\varepsilon^2 + |z_i|^2 e^{\psi_{\varepsilon}(z_i, t)}} \right)^{b_\beta} \omega^n_{F_i(t, \varepsilon)} \, d\lambda(z_i)$$

where $\theta_i$ is a partition of unit, $I \cap J = \emptyset$ and $\omega^n_{F_i(t, \varepsilon)} \, d\lambda(z_i)$ is the volume element $\omega^n$ restricted to $\mathcal{X}_t$. We take the $t$-derivative of (3.55) and we have

$$\sum_i \int_{\|z_i\| < 1} \theta_i(z_i, t) \frac{\partial \varphi_{\varepsilon}(z_i, t)}{\partial t} \prod_{\alpha \in I} \left( e^{\varepsilon^2 + |z_i|^2 e^{\phi_{\varepsilon}(z_i, t)}} \right)^{e_\alpha} \prod_{\beta \in J} \left( e^{\varepsilon^2 + |z_i|^2 e^{\psi_{\varepsilon}(z_i, t)}} \right)^{b_\beta} \omega^n_{F_i(t, \varepsilon)} \, d\lambda(z_i) = O(1)$$

where $O(1)$ above is uniform with respect to $t, \varepsilon$ by the $C^0$ estimates for $\varphi_{\varepsilon}$. Now by the construction of the vector $v$ above the LHS of (3.56) is precisely (3.54), so the lemma follows. \hfill \Box

**3.3.2. $L^2$-bound of the $t$-derivative.** — We rewrite the relation corresponding to (3.48) in our setting; during the next computations, we denote by

$$\tau := v(\varphi_{\varepsilon})$$

and then we have

$$\Delta_{\varphi_{\varepsilon}} \tau - \operatorname{Tr} \varphi_{\varepsilon} \partial (\partial v \cdot \varphi_{\varepsilon}) + \Psi_{\varphi_{\varepsilon}, v} = \lambda \tau + v(f) + \sum_j e_j v\left( \log(\varepsilon^2 + |\sigma_j|^2) \right) - \sum_i b_i v\left( \log(\varepsilon^2 + |s_i|^2) \right) + \Psi_v.$$  

The equality (3.58) will be used in order to establish the following statement.

**Proposition 3.5.** — There exists a constant $C > 0$ such that we have

$$\int_{\mathcal{X}_t} |\nabla_{\varepsilon} \tau|^2 \omega^n_{\varphi_{\varepsilon}} \leq C \left( 1 + \int_{\mathcal{X}_t} |\tau| \omega^n_{\varphi_{\varepsilon}} \right)$$

for any $\varepsilon > 0$. The operator $\nabla_{\varepsilon}$ is the gradient corresponding to the metric $\omega^n_{\varphi_{\varepsilon}}$.

**Proof.** — In order to establish (3.59) we multiply the relation (3.58) with $\tau$ and then we integrate the result on $\mathcal{X}_t$ against the measure $\omega^n_{\varphi_{\varepsilon}}$. A few observations are in order.

- We have

$$\sup_{\mathcal{X}_t} \left( |v(f)| + \sum_j e_j v\left( \log(\varepsilon^2 + |\sigma_j|^2) \right) + \sum_i b_i v\left( \log(\varepsilon^2 + |s_i|^2) \right) + |\Psi_v| \right) \leq C$$

uniformly with respect to $\varepsilon$, by the property (ii) of the vector field $v$ and the definition (3.47) of the function $\Psi_v$. 

• Since the constant $\lambda$ is positive, the $L^2$ norm of $\sqrt{\lambda} \tau$ will be on the left-hand side part of (3.59), hence the presence of a strictly positive $\lambda$ would reinforce the inequality we want to obtain.

The terms
\[ (3.60) \quad \text{Tr} \, \varphi \partial (\bar{\partial} v \cdot \varphi), \quad \Psi \varphi \varepsilon, v \]
are kind of troublesome, because we do not have a $L^\infty$ bound for them. Nevertheless, we recall that we only intend to establish an inequality between $L^p$ norms, and we will use integration by parts to deal with (3.60).

For the first term in (3.60) we argue as follows: integration by parts gives
\[ (3.61) \quad \int_{X_t} \bar{\partial} \partial (\bar{\partial} v \cdot \varphi) \wedge \omega_{\varphi \varepsilon}^{n-1} = - \int_{X_t} \partial \bar{\partial} \bar{\partial} v \cdot \varphi \wedge \omega_{\varphi \varepsilon}^{n-1} \]
and then we use Cauchy-Schwarz: the $L^2$ norm of $\bar{\partial} \partial$ is what we are after, but on the right hand side term we have it squared. The $L^2$ norm of $\bar{\partial} v \cdot \varphi$ is completely under control, because it only involves the fiber-direction derivatives of $\varphi$.

The second term is tamed in a similar manner. By definition of $\Psi \varphi \varepsilon, v$ we have
\[ (3.62) \quad \int_{X_t} \bar{\partial} \Psi \varphi \varepsilon, v \wedge \omega_{\varphi \varepsilon}^n = \int_{X_t} \bar{\partial} \partial v \wedge \omega_{\varphi \varepsilon}^{n-1} \]
and by Cartan formula this is equal to
\[ (3.63) \quad \int_{X_t} \bar{\partial} d (i_v \cdot \omega) \wedge \omega_{\varphi \varepsilon}^{n-1} = \int_{X_t} \bar{\partial} \partial (i_v \cdot \omega) \wedge \omega_{\varphi \varepsilon}^{n-1}. \]
By Stokes formula the term (3.63) is equal to
\[ (3.64) \quad \int_{X_t} \partial \bar{\partial} (i_v \cdot \omega) \wedge \omega_{\varphi \varepsilon}^{n-1} \]
and now things are getting much better, in the sense that the $(0,1)$–form $i_v \cdot \omega$ is clearly smooth, so its $L^2$ norm with respect to $\omega_{\varphi \varepsilon}$ is dominated by $C \int_{X_t} \omega \wedge \omega_{\varphi \varepsilon}^{n-1} \leq C'$ and we use the Cauchy-Schwarz inequality.

All in all, we infer the existence of two constants $C_1$ and $C_2$ such that we have
\[ (3.65) \quad \int_{X_t} |\nabla_\varepsilon \tau|^2 \omega_{\varphi \varepsilon}^n \leq C_1 \int_{X_t} |\tau| \omega_{\varphi \varepsilon}^n + C_2 \left( \int_{X_t} |\nabla_\varepsilon \tau|^2 \omega_{\varphi \varepsilon}^n \right)^{1/2} \]
for any positive $\varepsilon > 0$. The inequality (3.59) follows.

We infer the following statement.

**Theorem 3.6.** — There exists a positive integer $N \in \mathbb{Z}_+$ and a positive constant $C$ such that we have
\[ (3.66) \quad \int_{X_t} |\tau|^2 d\mu_{N \varepsilon}^{(\varepsilon)} \leq C \]
for every positive $\varepsilon$.

**Proof.** — The arguments which will follow are absolutely standard, by combining the Sobolev and Poincaré inequalities with (3.59). Prior to this, we recall that we have
\[ (3.67) \quad \omega \leq C \omega_{B, \varepsilon} \]
on each $X_t$ for some constant $C$ which is uniform with respect to $\varepsilon$ and with respect to $t \in \mathbb{D}'$. On the RHS of (3.67) we have $\omega_{B,\varepsilon}$ which stands for any metric quasi-isometric with (3.32). In particular, for any function $f$ we have

\[(3.68) \quad |\nabla f| \leq C|\nabla \varepsilon f|\]

where the symbols $|\cdot|, \nabla$ and $|\cdot|\varepsilon, \nabla \varepsilon$ correspond to the metric $\omega_{B,\varepsilon}$ and $\omega_{\varepsilon}$ respectively.

Now, Poincaré inequality 3.2 applied for $\alpha = 1$ combined with Lemma 3.4 gives

\[(3.69) \quad \int_{X_t} |\tau| \, d\mu^{(\varepsilon)}_\varepsilon \leq C(1 + \int_{X_t} |\nabla \tau| \, d\mu^{(\varepsilon)}_{\varepsilon/2})\]

On the other hand we have

\[
\int_{X_t} |\nabla \tau| \, d\mu^{(\varepsilon)}_{\varepsilon/2} \leq C \int_{X_t} |\nabla \varepsilon \tau| \, d\mu^{(\varepsilon)}_{\varepsilon/2}
\leq C \left( \int_{X_t} |\nabla \varepsilon \tau|^2 \, d\mu^{(\varepsilon)}_{\varepsilon/2} \right)^{1/2}
\leq C + C \left( \int_{X_t} |\tau| \, d\mu^{(\varepsilon)}_{\varepsilon} \right)^{1/2}
\]

where we have used Proposition 3.5 for the last inequality. When combined with (3.69), this implies

\[(3.70) \quad \int_{X_t} |\tau| \, d\mu^{(\varepsilon)}_{\varepsilon} \leq C\]

for any $\varepsilon > 0$.

We define next the sequence of rational numbers

\[(3.71) \quad p_1 = 1, \quad p_{k+1} := \frac{2n p_k}{2n - p_k}\]

as well as the sequence

\[(3.72) \quad q_1 = e, \quad q_{k+1} := \frac{2}{2 - p_k} q_k.\]

One can actually find a closed formula for $p_k = \frac{2n}{2n-k+1}$ holding for $1 \leq k \leq 2n$. It also follows that $p_k < 2$ as long as $1 \leq k \leq n$ which is thus the range of integers for which $q_{k+1}$ is defined; one can also check the formula $q_{k+1} = \frac{(2n)!(n-k)!}{n!(2n-k)!} \cdot q$. In particular $q_{n+1} = \frac{(2n)!}{n!^2} \cdot q$. This is the factor $N$ in the statement of the proposition.

We observe that for $k = 1, \ldots, n$ the components of $q_k$ are positive rational numbers, greater than the respective components of $q$.

The Sobolev inequality 3.1 gives

\[(3.73) \quad \left( \int_{X_t} |\tau|^{p_{k+1}} \, d\mu^{(\varepsilon)}_{q_{k+1}} \right)^{\frac{1}{p_{k+1}}} \leq C \left( \int_{X_t} |\nabla \tau|^2 \, d\mu^{(\varepsilon)}_{\varepsilon} + \int_{X_t} |\tau|^2 \, d\mu^{(\varepsilon)}_{\varepsilon} \right)^{\frac{1}{p_k}}\]

We iterate (3.64) for $k = 1, \ldots, n$ and the Proposition 3.6 is proved by observing that the following holds.

- We have $\int_{X_t} |\nabla \tau|^2 \omega^n_{\varepsilon, \varepsilon} \leq C$, by Proposition 3.5, combined with (3.70) and the fact that the quotient of the two measures

\[(3.74) \quad \omega^n_{\varepsilon, \varepsilon}, \, d\mu^{(\varepsilon)}_{\varepsilon}\]

is uniformly bounded both sides.
3. For each \( k = 1, \ldots, n \) we have
\[
(3.75) \quad \int_{X_t} |\nabla^2 \tau|^p_{q_k} d\mu^{(e)}_{q_k} \leq C \left( \int_{X_t} |\nabla^2 \tau|^2 d\mu^{(e)}_{q_k} \right)^{p_k/2} \leq C
\]
where the first inequality is simply Cauchy-Schwarz, and the second one is due to the fact that we have
\[
(3.76) \quad d\mu^{(e)}_{q_k} \leq C \omega^n_{\varphi_k}
\]
because \( \frac{q_k}{p_k} \geq \frac{q}{2} \). This last inequality follows by induction given that \( \frac{q_k+1}{p_k+1} = \frac{2n-p_k}{2n-np_k} q_k/ p_k \). \( \square \)

3.4. A gradient estimate in the conic case. —

**Theorem 3.7.** — Let \((X, \omega)\) be a compact Kähler manifold, and let \( \omega_\varphi := \omega + dd^c \varphi \) be a Kähler metric satisfying
\[
\omega_\varphi^n = e^{\lambda \varphi + F} \omega^n
\]
for some \( F \in \mathcal{C}^\infty(X) \) and \( \lambda \in \mathbb{R} \). We assume that there exists \( C > 0 \) and a smooth function \( \Psi, \Phi \) such that:

1. \( \sup X |\varphi| \leq C \)
2. \( \sup X |\Psi| \leq C \) and for any \( \delta > 0 \), there exists \( C_\delta \) such that
   a. \( dd^c \Psi \geq \delta^{-1} dd^c \varphi \land dd^c \Phi - C_\delta \omega \)
   b. \( \Delta_\omega \Psi \geq \delta^{-1} |\nabla F|_\omega - C_\delta \)
3. \( i\Theta_\omega(T_X) \geq -(C \omega + dd^c \Phi) \otimes \text{Id} \)
4. \( |\omega_\varphi - \omega_\varphi'| \leq C \omega \)

Then there exists a constant \( A > 0 \) depending only on \( C \) and \( n \) such that \( |\nabla \varphi_\omega| \leq C \).

As a corollary of this result, the gradient estimate (3.52) in Theorem 3.3 holds.

**Proof of Theorem 3.3.** — Let us rewrite the equation (3.51) as
\[
(\omega_\varphi + dd^c u_\varphi)^n = e^{\lambda u_\varphi + f_\varphi} \prod_{i \in I} (\varepsilon^2 + |s_i|^2)^{e_i} \omega^n
\]
where the reference metric \( \omega_\varphi \in \{\omega\} \) is an approximate conic metric along the divisor \( B \), and \( u_\varphi \) differs from \( \varphi_\varphi \) by a function whose \( L^\infty \) norm as well as gradient and complex Hessian are uniformly bounded with respect to \( \omega_\varphi \). Therefore it is sufficient to establish (3.52) for \( u_\varphi \). We check successively that conditions (i) - (iv) are satisfied.

The bound (i) follows from Kolodziej’s estimate. It is straightforward when \( \lambda = 0 \), and when \( \lambda > 0 \), it requires an additional step easily achieved with Jensen inequality. Next, we choose \( \Psi_\varepsilon := C(\sum_{i} |s_i|^2 + \varepsilon^2)^\rho + \sum_{j} (|s_j|^2 + \varepsilon^2)^\rho) \) for large enough and \( \rho > 0 \) small enough. Condition (ii).a can be checked independently for each summand of \( \Psi_\varepsilon \) of \( \Psi_\varepsilon \) in which case if follows from the fact that \( \Psi_\varepsilon \) is uniformly quasi-psh (hence \( C\omega_\varepsilon\)-psh). Condition (ii).b is an easy computation combined with [GP16, Sect. 5.2]. Condition (iii).a is showed in [GP16, Sect. 4], while (iii).b is the content of [GP16, Prop. 1]. To be more precise, op. cit. assumes an upper and lower bound on \( f_\varepsilon + \sum e_i \log(|s_i|^2 + \varepsilon^2) \) in order to get a two-sided inequality for \( \omega_\varepsilon \); however one only needs an upper bound for the previous quantity if one only wishes to prove the one-sided inequality (iv). \( \square \)

**Proof of Theorem 3.7.** — Let \( \beta := |\nabla \varphi|^2 \) (computed with respect to \( \omega \)) and \( \alpha := \log \beta - \gamma \circ \varphi \) where \( \gamma \) is a function to specify later. Without loss of generality, one can assume inf \( \varphi = 0 \), and we set sup \( \varphi =: C_0 \). We use the local notation \((g_\beta)\) for \( \omega \). We work at a point \( y \in X \) where \( \alpha + 2\Psi \) attains its maximum, and we choose a system of geodesic coordinates for \( \omega \) such that
\[ g_{ij}(y) = \delta_{ij}, \quad dg_{ij}(y) = 0, \] and \( \varphi_{ij} \) is diagonal. We set \( u_{ij} = g_{ij} + \varphi_{ij} \) the components of the metric \( \omega_{\varphi} \). As \( \alpha_p = \frac{2p}{\beta} - \varphi \circ \varphi_p \) and \( \alpha_p(y) = -2\Psi_p(y) \), one has

\[ \beta_p(y) = (\gamma' \circ \varphi(y))\varphi_p(y) - 2\Psi_p(y) \] (3.77)

Moreover, some computations show that

\[ \alpha_{pp} = \frac{1}{\beta} \left( R_{jkpq} \varphi_{jq} \varphi_k + 2\text{Re} \sum_j u_{pqj} \varphi_j + \sum_j |\varphi_{jp}|^2 + \varphi_{pp} \right) - \frac{|\beta_p|^2}{\beta^2} - 2\lambda - \gamma''|\varphi_p|^2 - \gamma'\varphi_{pp} \]

Therefore at \( y \), one gets from (3.77) the following inequality:

\[ \alpha_{pp} \geq \frac{1}{\beta} \left( R_{jkpq} \varphi_{jq} \varphi_k + 2\text{Re} \sum_j u_{pqj} \varphi_j + \sum_j |\varphi_{jp}|^2 + \varphi_{pp} \right) - 2\lambda - \gamma''|\varphi_p|^2 - \gamma'\varphi_{pp} - |\gamma' \varphi_p - 2\Psi_p|^2 \]

so at \( y \), the RHS is non-positive.

**Step 1. The curvature term**

By the assumption (iii), we have for all \( a, b \): 

\[ R_{jkpq} a_k b_q \hat{a}_k \hat{b}_q \geq -(C|a_j|^2 + \Psi_j a_j \hat{a}_k |b|^2 \] and by symmetry of the curvature tensor, we get

\[ R_{jkpq} a_k b_q \hat{a}_k \hat{b}_q \geq -(C|b_j|^2 + \Psi_{pqj} b_j \hat{b}_j |a|^2 \]

We apply that to \( a = \nabla \varphi \) and \( b \) the vector with only non-zero component the \( p \)-th one, equal to \( \sqrt{u_{pp}} \), we get:

\[ u_{pp} R_{jkpp} \varphi_{jq} \varphi_k \geq -(C u_{pp} + u_{pp} \Psi_{pp}) |\nabla \varphi|^2 \]

As a consequence,

\[ \frac{1}{\beta} \sum_p u_{pp} R_{jkpp} \varphi_{jq} \varphi_k \geq -C \sum_p u_{pp} - \sum_p u_{pp} \Psi_{pp} \]

Therefore, Equation (3.78) becomes, at \( y \in X \):

\[ \Delta' (\alpha + \Psi) \geq (\gamma - C) \text{tr}_{\omega_{\varphi}} \omega + \frac{1}{\beta} \sum_p u_{pp} \left( 2\text{Re} \sum_j u_{pqj} \varphi_j + \sum_j |\varphi_{jp}|^2 \right) - \gamma'' |\nabla \omega|^2 \omega_{\varphi} - n \gamma' - \sum_p u_{pp} |\gamma' \varphi_p - 2\Psi_p|^2 - C \]

(3.80)

**Step 2. The gradient term**

The next term to analyze is

\[ \frac{1}{\beta} \sum_p u_{pp} \left( 2\text{Re} \sum_j u_{pqj} \varphi_j \right) = \frac{2}{\beta} \text{Re} \sum_j F_j \varphi_j \]

by [Blo09, 1.13], and this term is dominated (in norm) by \( 2|\nabla F|\beta^{-1/2} \) and at the point \( y \), \( \beta \) can always be assumed to be larger than 1 so that our term is bigger that \(-2|\nabla F| \). In particular, one gets at \( y \):

\[ \Delta' (\alpha + \Psi) \geq (\gamma - C) \text{tr}_{\omega_{\varphi}} \omega + \sum_p u_{pp} \left( \frac{1}{\beta} \sum_j |\varphi_{jp}|^2 - |\gamma' \varphi_p - 2\Psi_p|^2 \right) - \gamma'' |\nabla \omega|^2 \omega_{\varphi} - n \gamma' - 2|\nabla F| - C \]

(3.82)
Step 3. Using the second derivatives

Recall that 
\[ \beta_p = \sum_j \phi_{jp} \phi_j + \phi_p (u_{pp} - 1) \]
At \( y \), 
\[ \frac{\partial}{\partial y} \beta_p - \gamma' \phi_p = -2\Psi_p \]
so that at this point, one has 
\[ \sum_j \phi_{jp} \phi_j = (\gamma' + 1 - u_{pp}) \phi_p - 2\beta \Psi_p \]
hence 
\[ |\sum_j \phi_{jp} \phi_j| = \beta |(\gamma' \phi_p - 2\Psi_p) + \beta^{-1}(1 - u_{pp}) \phi_p| \]
By Schwarz inequality, 
\[ \left| \sum_j \phi_{jp} \phi_j \right|^2 \leq \beta \sum_j |\phi_{jp}|^2 \]
and therefore
\[ \sum_p u_{pp} \left( \frac{1}{\beta} \sum_j |\phi_{jp}|^2 - |\gamma' \phi_p - 2\Psi_p|^2 \right) \geq -C(\text{tr}_{\omega_\varphi} \omega + |\nabla \Psi|_{\omega_\varphi}^2) \]
Combining this last inequality with (3.82), we get at \( y \):
\[ 0 \geq \Delta'(\alpha + 2\Psi) \geq (\gamma' - C)\text{tr}_{\omega_\varphi} \omega - \gamma''|\nabla \omega_\varphi|^2 - n\gamma' + \left( \Delta' \Psi - C|\nabla \Psi|_{\omega_\varphi}^2 - 2|\nabla F|_{\omega} \right) - C \]
As \( \Psi \) is quasi-psh and \( \omega_\varphi \leq C \omega \), we have \( \Delta' \Psi \geq C^{-1} \Delta \Psi - C \text{tr}_{\omega_\varphi} \omega \) so by (ii).b, \( \Delta' \Psi \geq 4|\nabla F|_{\omega} - C(1 + \text{tr}_{\omega_\varphi} \omega) \). Using (ii).a, one ends up with the following inequality at \( y \):
\[ (\gamma' - C)\text{tr}_{\omega_\varphi} \omega - \gamma''|\nabla \omega_\varphi|^2 - n\gamma' \leq C \]
Choosing \( \gamma(t) = (C + 1)t - ||\varphi||_\infty^2 t^2 \) enables to conclude just as in [Blo09].

Proof of Theorem E. — It is a combination of our preceding considerations. The equation which gives \( \omega_{\text{KE}} \) fiberwise is of the same type as (3.49) (with \( \lambda = 0 \)). We conclude by Theorem 3.3 and Theorem 3.6.
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