Spectra of the energy operator of four-electron systems in the triplete state in the Hubbard model

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Abstract. We investigate spectral properties of a four-electron system in the Hubbard Model framework in the \( \nu \)-dimensional lattice \( \mathbb{Z}^\nu \). We prove that the essential spectrum of the system in a quintet state consists of a single segment and the four-electron bound state or four-electron anti-bound state is absent. We show that the essential spectrum of the system in a triplete states is the union of at most three segments. We also prove that four-electron bound states or a four-electron anti-bound state exists in triplete states. We prove that in the system exists three triplete states and their spectrum are different.

1. Introduction
In the early 1970s, three papers \[2, 3, 8\], where a simple model of a metal was proposed that has become a fundamental model in the theory of strongly correlated electron systems. Note that these papers appeared almost simultaneously and independently. In that model, a single nondegenerate electron band with a local Coulomb interaction was considered. The model Hamiltonian contains only two parameters: the matrix element \( t \) of electron hopping from a lattice site to a neighboring site and the parameter \( U \) of the one-site Coulomb repulsion of two-electrons. In the secondary quantization representation, the Hamiltonian can be written as follows
\[
H = t \sum_{m, \tau, \gamma} a_{m, \gamma}^+ a_{m+\tau, \gamma} + U \sum_m a_{m, \uparrow}^+ a_{m, \uparrow} a_{m, \downarrow}^+ a_{m, \downarrow}, \tag{1}
\]
where \( a_{m, \gamma}^+ \) and \( a_{m, \gamma} \) denote Fermi operators of creation and annihilation of an electron with spin \( \gamma \) on a site \( m \) and the summation over \( \tau \) means summation over the nearest neighbors on the lattice.

The model proposed in \[3, 2, 8\] was called the Hubbard model after John Hubbard, who made a fundamental contribution to studying the statistical mechanics of that system, although the local form of Coulomb interaction was first introduced for an impurity model in a metal by Anderson \[1\]. We also recall that the Hubbard model is a particular case of the Shubin-Wonsowsky polaron model \[15\], which had appeared 30 years before \[3, 2, 8\]. In the Shubin-Wonsowsky model, along with the on-site Coulomb interaction, the interaction of electrons on neighboring sites is also taken into account.

The Hubbard model is an approximation used in solid state physics to describe the transition between conducting and insulating states. It is the simplest model describing particle interaction on a lattice. Its Hamiltonian contains only two terms: the kinetic term corresponding to the
tunneling (hopping) of particles between lattice sites and a term corresponding to the on-site interaction. Particles can be fermions, as in Hubbard's original work, and also bosons. The simplicity and sufficiency of Hamiltonian (1) have made the Hubbard model very popular and effective for describing strongly correlated electron systems.

The Hubbard model well describes the behavior of particles in a periodic potential at sufficiently low temperatures such that all particles are in the lower Bloch band and long-range interactions can be neglected. If the interaction between particles at different sites is taken into account, then the model is often called the extended Hubbard model. It was proposed for describing electrons in solids, and it remains especially interesting since then for studying high-temperature superconductivity. Later, the extended Hubbard model also found applications in describing the behavior of ultracold atoms in optical lattices.

In considering electrons in solids, the Hubbard model can be considered a sophisticated version of the model of strongly bound electrons, involving only the electron hopping term in the Hamiltonian. In the case of strong interactions, these two models can give essentially different results. The Hubbard model exactly predicts the existence of so-called Mott insulators, where conductance is absent due to strong repulsion between particles.

The Hubbard model is based on the approximation of strongly coupled electrons. In the strong-coupling approximation, electrons initially occupy orbitals in atoms (lattice sites) and then hop over to other atoms, thus conducting the current. Mathematically, this is represented by the so-called hopping integral. This process can be considered the physical phenomenon underlying the occurrence of electron bands in crystal materials. But the interaction between electrons is not considered in more general band theories. In addition to the hopping integral, which explains the conductance of the material, the Hubbard model contains the so-called on-site repulsion, corresponding to the Coulomb repulsion between electrons. This leads to a competition between the hopping integral, which depends on the mutual position of lattice sites, and the on-site repulsion, which is independent of the atom positions. As a result, the Hubbard model explains the metal-insulator transition in oxides of some transition metals. When such a material is heated, the distance between nearest-neighbor sites increases, the hopping integral decreases, and on-site repulsion becomes dominant.

The Hubbard model is currently one of the most extensively studied multielectron models of metals [9, 10, 11, 17, 4]. But little is known about exact results for the spectrum and wave functions of the crystal described by the Hubbard model, and obtaining the corresponding statements is therefore of great interest.

The spectrum and wave functions of the system of two electrons in a crystal described by the Hubbard Hamiltonian were studied in [9]. It is known that two-electron systems can be in two states, triplet and singlet [9, 10, 11, 17, 4]. It was proved in [9] that the spectrum of the system Hamiltonian $H^t$ in the triplet state is purely continuous and coincides with a segment $[m, M]$, and the operator $H^s$ of the system in the singlet state, in addition to the continuous spectrum $[m, M]$, has a unique antibound state for some values of the quasimomentum. For the antibound state, correlated motion of the electrons is realized under which the contribution of binary states is large. Because the system is closed, the energy must remain constant and large. This prevents the electrons from being separated by long distances. Next, an essential point is that bound states (sometimes called scattering-type states) do not form below the continuous spectrum. This can be easily understood because the interaction is repulsive. We note that a converse situation is realized for $U < 0$: below the continuous spectrum, there is a bound state (antibound states are absent) because the electrons are then attracted to one another.

For the first band, the spectrum is independent of the parameter $U$ of the on-site Coulomb interaction of two electrons and corresponds to the energy of two noninteracting electrons, being exactly equal to the triplet band. The second band is determined by Coulomb interaction to a much greater degree: both the amplitudes and the energy of two electrons depend on $U$, and
the band itself disappears as \( U \to 0 \) and increases without bound as \( U \to \infty \). The second band largely corresponds to a one-particle state, namely, the motion of the doublet, i.e., two-electron bound states.

The spectrum and wave functions of the system of three electrons in a crystal described by the Hubbard Hamiltonian were studied in [16].

Here, we consider the energy operator of four-electron systems in the Hubbard model and describe the structure of the essential and discrete spectra of the system for quintet and triplet states.

The Hamiltonian of the chosen model has the form

\[
H = A \sum_{m,\gamma} a_{m,\gamma}^+ a_{m,\gamma} + B \sum_{m,\tau,\gamma} a_{m,\gamma}^+ a_{m+\tau,\gamma} + U \sum_{m} a_{m,\gamma}^+ a_{m,\uparrow} a_{m,\downarrow}^+ a_{m,\downarrow}.
\] (2)

Here, \( A \) is the electron energy at a lattice site, \( B \) is the transfer integral between neighboring sites (we assume that \( B > 0 \) for convenience), \( \tau = \pm \epsilon j, j = 1, 2, \ldots, \nu \), where \( \epsilon j \) are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, \( U \) is the parameter of the on-site Coulomb interaction of two electrons, \( \gamma \) is the spin index, \( \gamma = \uparrow \) or \( \gamma = \downarrow \), \( \uparrow \) and \( \downarrow \) denote the spin values \( \frac{1}{2} \) and \( -\frac{1}{2} \), and \( a_{m,\gamma}^+ \) and \( a_{m,\gamma} \) are the respective electron creation and annihilation operators at a site \( m \in \mathbb{Z}^\nu \).

The energy of the system depends on its total spin \( S \). In the case of a saturated ferromagnetic state (\( S = \frac{N_e}{2} \), where \( N_e \) is the number of electrons in the system), the solution of the problem is exact and trivial for any admissible number of electrons \( N_e \). In that case, the system is an ideal Fermi gas of electrons with the same direction of the spin projections.

Along with the Hamiltonian, the \( N_e \) electron system is characterized by the total spin \( S \),

\[ S = S_{\text{max}}, S_{\text{max}} - 1, \ldots, S_{\min}, S_{\text{max}} = \frac{N_e}{2}, S_{\min} = 0, \frac{1}{2}. \]

Hamiltonian (2) commutes with all components of the total spin operator \( S = (S^+, S^−, S^z) \), and the structure of eigenfunctions and eigenvalues of the system therefore depends on \( S \). The Hamiltonian \( H \) acts in the antisymmetric Fock space \( \mathcal{H}_a \).

2. Quintet state

Let \( \varphi_0 \) be the vacuum vector in the space \( \mathcal{H}_a \). The quintet state corresponds to the free motion of four electrons over the lattice with the basis functions

\[
q_{m,n,p,r}^2 = a_{m\uparrow}^+ a_{m\downarrow}^+ a_{p\uparrow}^+ a_{r\downarrow}^+ \varphi_0.
\]

The subspace \( \widetilde{\mathcal{H}}^q_2 \), corresponding to the quintet state is the set of all vectors of the form

\[
\psi = \sum_{m,n,p,r} \tilde{f}(m,n,p,r) q_{m,n,p,r}^2, \quad \tilde{f} \in l^2_{\text{as}},
\]

where \( l^2_{\text{as}} \) is the subspace of antisymmetric functions in the space \( l^2(\mathbb{Z}^\nu) \).

**Theorem 2.1.** The subspace \( \mathcal{H}^q_2 \) is invariant under the operator \( H \), and the restriction \( H^q_2 = H/\mathcal{H}^q_2 \) of \( H \) to the subspace \( \mathcal{H}^q_2 \) is a bounded self-adjoint operator. It generates a bounded self-adjoint operator \( \overline{H}^q \), acting in the space \( l^2_{\text{as}} \) as

\[
(\overline{H}^q f)(p; q; r; t) = 4A f(p; q; r; t) + B \sum_{\tau} (f(p+\tau, q, r, t) + f(p, q+\tau, r, t) + f(p, q, r+\tau, t) + f(p, q, r, t+\tau)).
\] (3)
The operator $H^2_q$ acts on a vector $\psi \in H^2_q$ as

$$H^2_q \psi = \sum_{m,n,p,r} (\overline{H}^q f)(m,n,p,r)q^2_{m,n,p,r}. \quad (4)$$

**Proof.** We act with the Hamiltonian $H$ on vectors $\psi \in H^2_q$ using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, $\{a_{m,\gamma}, a^+_{n,\beta}\} = \delta_{m,n}\delta_{\gamma,\beta}$. $\{a_{m,\gamma}, a_{n,\beta}\} = \{a^+_{m,\gamma}, a^+_{n,\beta}\} = \theta$, and also take into account that $a_{m,\gamma}\varphi_0 = \theta$, where $\theta$ is the zero element of $H^2_q$. This yields the statement of the theorem. \qed

**Lemma 2.2.** The spectra of the operators $\overline{H}^q$ and $H^2_q$ coincide.

The proof follows by using the Weyl criterion [13].

We call $\overline{H}^q$ as the four-electron quintet operator.

We let $F$ denote the Fourier transform:

$$F : l_2((Z^\nu)^4) \rightarrow L_2((T^\nu)^4) \equiv H^2_q,$$

where $T^\nu$ is the $\nu$-dimensional torus endowed with the normalized Lebesgue measure $d\lambda$, $\lambda(T^\nu) = 1$.

We set $\tilde{H}^q = \overline{F}H^2_qF^{-1}$. In the quasi-momentum representation, the operator $\overline{H}^q$ acts in the Hilbert space $L^2_{2\nu}((T^\nu)^4)$ as follows

$$\tilde{H}^q \tilde{f}(x,y,z,t) = h(x,y,z,t)\tilde{f}(x,y,z,t), \quad \tilde{f} \in L_2((T^\nu)^4),$$

where $L^\nu_{2\nu}$ is the subspace of antisymmetric functions in $L_2((T^\nu)^4)$ and

$$h(x,y,z,t) = 4A + 2B \sum_{i=1}^{\nu} (\cos x_i + \cos y_i + \cos z_i + \cos t_i). \quad (5)$$

It is obvious that the spectrum of $\tilde{H}^q$ is purely continuous and coincides with the value set of the function $h(x,y,z,t)$, i.e., with the set

$$Imh(x,y,z) = [m_\nu, M_\nu] = [4A - 8B\nu, 4A + 8B\nu],$$

where

$$m_\nu = \min_{x,y,z,t \in T^\nu} h(x,y,z,t), \quad M_\nu = \max_{x,y,z,t \in T^\nu} h(x,y,z,t).$$

Therefore, the quintet state spectrum is independent of the Coulomb interaction parameter $U$ and is the set of energies of four noninteracting electrons moving in the crystal. This result is totally natural because the quintet state cannot contain states with two electrons at a site. Hence, in the quintet state, the spectrum of four-electron systems can be evaluated exactly and is purely continuous. The spectral problem that we consider here is a particular case of the problem of finding the spectrum of a system of $N$ noninteracting electrons in a crystal lattice.

By Hunds rule, the minimum-energy state in an $N-$electron system is the state where all spins are directed upward, i.e., the state $\uparrow \uparrow \cdots \uparrow$. By the Pauli exclusion principle, this state cannot contain states with two electrons at a site. In this case, the spectrum of the system is independent of the Coulomb interaction parameter $U$ and is the band energy of $N$ noninteracting electrons moving in the crystal. The spectrum of the system is then purely continuous.
3. First triplet state

In the system there exist three triplet states. The triplet state corresponds to the basis functions

\[ 1t_{m,n,p,r}^1 = a_{m_1}^+ a_{n_1}^+ a_{p_1}^+ a_{r_1}^+ \varphi_0, \quad 2t_{m,n,p,r}^1 = a_{m_2}^+ a_{n_2}^+ a_{p_2}^+ a_{r_2}^+ \varphi_0, \quad 3t_{m,n,p,r}^1 = a_{m_3}^+ a_{n_3}^+ a_{p_3}^+ a_{r_3}^+ \varphi_0. \]

We see that there are three such states, and they have different origins.

The subspace \( \mathcal{H}_1^k, k = 1, 2, 3 \) corresponding to the triplet states in the set of all vectors of the forms \( \psi = \sum_{m,n,p,r} f(m,n,p,r)k_{m,n,p,r}, k = 1, 2, 3, f \in \mathcal{L}_2^{\text{as}}, \) where \( \mathcal{L}_2^{\text{as}} \) is the subspace of antisymmetric functions in the space \( \mathcal{L}_2((\mathbb{Z}^\nu)^4). \)

The restriction \( \mathcal{H}_1^k = H/\mathcal{H}_1^k, k = 1, 2, 3 \) of \( H \) to the subspace \( \mathcal{H}_1^k \) is a bounded self-adjoint operator.

**Theorem 3.1.** The subspace \( \mathcal{H}_1^1 \) is invariant under the operator \( H \), and the operator \( \mathcal{H}_1^1 \) is a bounded self-adjoint operator. It generates a bounded self-adjoint operator \( \mathcal{H}_1^1 \) acting in the space \( \mathcal{L}_2^{\text{as}} \) as

\[ (\mathcal{H}_1^1 \psi)(m,n,p,r) = 4A f(m,n,p,r) + 2B \sum_{\tau} |f(m+\tau,n,p,r)+f(m,n+\tau,p,r)+ \]

\[ + f(m,n,p+r,r)+f(m,n,p+r,r)|+U[\delta_{m,\tau} f(m,n,p,r)+\delta_{n,\tau} f(m,n,p,r)+\delta_{p,\tau} f(m,n,p,r)], \] \( \delta_{k,j} \) is the Kronecker symbol. The operator \( \mathcal{H}_1^1 \) acts on a vector \( \psi \in \mathcal{H}_1^1 \) as

\[ \mathcal{H}_1^1 \psi = \sum_{m,n,p,r} (\mathcal{H}_1^1 f)(m,n,p,r) t_{m,n,p,r}^1. \]

**Proof.** The proof of the theorem can be obtained from the explicit form of the action of \( H \) on vectors \( \Psi \in \mathcal{H}_1^1 \) using the standard anti-commutation relations between electron creation and annihilation operators.

We set \( \tilde{H}_1^1 = \mathcal{F} \mathcal{H}_1^1 \mathcal{F}^{-1} \). In the quasi-momentum representation, the operator \( \mathcal{H}_1^1 \) acts in the Hilbert space \( \mathcal{L}_2^{\text{as}}((\mathbb{Z}^\nu)^4) \) as

\[ (\mathcal{H}_1^1 f)(\lambda, \mu, \gamma, \theta) = h(\lambda, \mu, \gamma, \theta) f(\lambda, \mu, \gamma, \theta) + U \int_{\mathbb{Z}^\nu} f(s, \mu, \gamma, \lambda + \theta - s, \theta) ds + \]

\[ + \int_{\mathbb{Z}^\nu} f(s, \lambda, \mu + \theta - t) dt + \int_{\mathbb{Z}^\nu} f(s, \lambda, \mu + \theta - t) dr, \] \( f \in \mathcal{L}_2^{\text{as}}((\mathbb{Z}^\nu)^4), \)

where \( \mathcal{L}_2^{\text{as}} \) is the subspace of antisymmetric functions in \( L_2((\mathbb{Z}^\nu)^4) \), and \( h(\lambda, \mu, \gamma, \theta) \) has a form (5).

Using that the function \( f(\lambda, \mu, \gamma, \theta) \) is the antisymmetrical function, we verify that the operator \( \tilde{H}_1^1 \) can be represented in the form

\[ \tilde{H}_1^1 = \mathcal{H}_1 = I \bigotimes \mathcal{H}_2, \]

where \( I \) is the unit operator in the two-electron space,

\[ (\mathcal{H}_1 f)(\lambda, \theta) = \{2A + 2B \sum_{i=1}^{\nu} [\cos \lambda_i + \cos \theta_i] \} f(\lambda, \theta) + U \int_{\mathbb{Z}^\nu} f(s, \lambda + \theta - s) ds, \]

and \( (\mathcal{H}_2 f)(\mu, \gamma) = \{2A + 2B \sum_{i=1}^{\nu} [\cos \mu_i + \cos \gamma_i] \} f(\mu, \gamma). \)
We must therefore investigate the spectrum and bound states of the operators \( \tilde{H}_1 \) and \( \tilde{H}_2 \), respectively.

Let the total quasimomentum of the two-electron system \( \lambda + \theta = \Lambda_1 \) be fixed. We let \( L_2(\Gamma_\Lambda) \) denote the space of functions that are square integrable on the manifold \( \Gamma_\Lambda = \{(\lambda, \theta) : \lambda + \theta = \Lambda_1\} \).

Let \( \mathcal{H}', (.) \) be a separable Hilbert space, and \( < M, \mu > \) be a measurable space with \( \sigma \)–finite measure \( \mu \).

The mapping \( f : M \rightarrow \mathcal{H}' \) is called weakly measurable, if \( (f(m), \eta)_\mathcal{H}' \) is a measurable function on \( < M, \mu > \) for all \( \eta \in \mathcal{H}' \).

By \( \mathcal{H} \) we denote a linear space of all weakly measurable mappings \( f : M \rightarrow \mathcal{H}' \), for which \( \int_M ||f(m)||^2_{\mathcal{H}'}d\mu < \infty \) (equally nearly everywhere mappings are identify). The space \( \mathcal{H} \) is a Hilbert space with the norm \( ||f||_{\mathcal{H}'} \equiv \left( \int_M ||f(m)||^2_{\mathcal{H}'}d\mu \right)^{1/2} \). This space \( \mathcal{H} \) is called direct integral of spaces with identical layers and it is denoted by \( \mathcal{H} = \bigoplus \int_M \mathcal{H}'d\mu \).

Let \( \mathcal{L}(\mathcal{H}') \) be the algebra of all bounded linear mappings acting in \( \mathcal{H}' \). A mapping \( A : M \rightarrow \mathcal{L}(\mathcal{H}') \) is called measurable, if numerical function \( (\varphi, A(\cdot)\psi) \) is measurable in \( < M, \mu > \) for all \( \varphi, \psi \in \mathcal{H}' \). We denote by \( L^\infty(M, d\mu, \mathcal{L}(\mathcal{H}')) \) linear space of all measurable mappings from \( M \) into \( \mathcal{L}(\mathcal{H}') \), for which \( ||A||_\infty = \operatorname{esssup}||A(m)||_{\mathcal{L}(\mathcal{H}')} \) is decomposable into direct integral, if exists \( A(\cdot) \in L^\infty(M, d\mu, \mathcal{L}(\mathcal{H}')) \) such that

\[
(A\psi)(m) = A(m)\psi(m) \quad \text{a.e.}
\]

for all \( \psi \in \mathcal{H}' \). In this case \( A \) is called decomposable and will be written as \( A = \bigoplus \int_M A(m)d\mu(m) \).

The following result is a well-known theorem about the description of decomposable operators as direct integral.

**Theorem 3.2.** [14]. Let \( \mathcal{H} = \bigoplus \int_M \mathcal{H}'d\mu \), where \( < M, \mu > \) is separable measurable spaces with \( \sigma \)–finite measure and \( \mathcal{H}' \) is separable. Let \( A \) be an algebra of decomposable operators with layers, which multiple to the identity operator. In this case, \( A \in \mathcal{L}(\mathcal{H}') \) is decomposable if and only if \( A \) is commuted \( \text{whit} \) every operators from \( \mathcal{A} \).

Since \( \tilde{H}_1U_t = U_t\tilde{H}_1 \) for all \( s \in T^v \), where \( (U_s f)(t) = f(s-t), t, s \in T^v \), then, by Theorem 3.2 there exists the decomposition \( H_1 = \bigoplus \int_{T^v} \tilde{H}_1d\Lambda_1 \) of Hilbert spaces \( H_1 \) in the direct integral of spaces \( H_{1\Lambda_1}, \Lambda_1 \in T^v \), such that \( H_1 = \bigoplus \int_{T^v} \tilde{H}_1d\Lambda_1 \) and \( U_t = \bigoplus \int_{T^v} e^{iA_1 t} I_{\Lambda_1} d\Lambda_1 \), where \( \tilde{H}_1d\Lambda_1 \) are operators on \( H_{1\Lambda_1} \) and \( I_{\Lambda_1} \) are identity operator on \( H_{1\Lambda_1} \), and \( T^v \) are \( v \)–dimensional torus.

It is known [12, 14] that the operator \( \tilde{H}_1 \) and the space \( H_1 \equiv L_2((T^v)^2) \) can be decomposed into a direct integral of operators \( \tilde{H}_{1\Lambda_1} \) and spaces \( H_{2\Lambda_1} = L_2(\Gamma_{\Lambda_1}) \),

\[
\tilde{H}_1 = \bigoplus \int_{T^v} \tilde{H}_{1\Lambda_1}d\Lambda_1, \quad H_2 = \bigoplus \int_{T^v} H_{2\Lambda_1}d\Lambda_1,
\]
such that the spaces \( H_{2\Lambda_1} \) are then invariant under the operators \( \tilde{H}_{1\Lambda_1} \) and each operator \( \tilde{H}_{1\Lambda_1} \) acts in \( H_{2\Lambda_1} \) as \( (\tilde{H}_{1\Lambda_1}f_{\Lambda_1})(\lambda) = \{2A + 4B \sum_{i=1}^{\nu} \cos \frac{A_i}{2} \cos(\frac{A_i}{2} - x_i)\}f_{\Lambda_1}(\lambda) + U \int_{T^v} f_{\Lambda_1}(s)ds \), where \( f_{\Lambda_1}(\lambda) = f(\lambda, A_1 - \lambda) \).

It is known that the continuous spectrum of \( \tilde{H}_{1\Lambda_1} \) is independent of \( U \) and coincides with the segment \([m_{\Lambda_1}, M_{\Lambda_1}]\), where \( m_{\Lambda_1} = \inf_{x \in T^v} h_{\Lambda_1}(x), M_{\Lambda_1} = \sup_{x \in T^v} h_{\Lambda_1}(x) \), here \( h_{\Lambda_1}(x) = 2A + 4B \sum_{i=1}^{\nu} \cos \frac{A_i}{2} \cos(\frac{A_i}{2} - x_i) \).

We set \( \Delta'_{\Lambda_1}(z) = 1 + U \int_{T^v} \frac{ds_1 ds_2 ds_3}{2A + 4B \sum_{i=1}^{\nu} \cos \frac{A_i}{2} \cos(\frac{A_i}{2} - x_i) - z} \).
Lemma 3.3. The number \( z = z_0 \) not belonging to the continuous spectrum of the operator \( \tilde{H}_1 \) is an eigenvalue of that operator if and only if it is a zero of the function \( \Delta_{\lambda_1}(z) \), i.e., \( \Delta_{\lambda_1}(z_0) = 0 \).

Proof. If \( z_0 \) is an eigenvalue of \( \tilde{H}_1 \), then

\[
\{2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_i}{2} \cos (\frac{\Lambda_i}{2} - x_i)\} \tilde{f}_{\lambda_1}(x) + U \int_{T^\nu} \tilde{f}_{\lambda_1}(s)ds_1ds_2...ds_\nu = z_0 \tilde{f}_{\lambda_1}(x),
\]

whence \( \Delta_{\lambda_1}(z_0) = 0 \).

Now let \( \Delta_{\lambda_1}(z_0) = 0 \). Then the homogeneous equation

\[
\tilde{f}_{\lambda_1}(x) + U \int_{T^\nu} \frac{\tilde{f}_{\lambda_1}(s)ds_1ds_2...ds_\nu}{2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_i}{2} \cos (\frac{\Lambda_i}{2} - s_i) - z} = 0
\]

has a nontrivial solution. It then follows that the number \( z = z_0 \) is an eigenvalue of \( \tilde{H}_1 \). \( \square \)

It is clear that the continuous spectrum of \( \tilde{H}_1 \) coincides with the segment

\[
[m_{\lambda_1}^1, M_{\lambda_1}^1] = [2A - 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_i}{2}, 2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_i}{2}].
\]

We consider the one-dimensional case.

Theorem 3.4. At \( \nu = 1 \) and \( U > 0 \) and for all values of the parameter of the Hamiltonian, the operator \( \tilde{H}_1 \) has a unique eigenvalue \( z = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}} \) that is above the continuous spectrum of \( \tilde{H}_1 \), i.e., \( z > M_{\lambda_1}^1 \).

Proof. In the one-dimensional case, the function \( \Delta_{\lambda_1}(z) \) increases monotonically outside the continuous spectrum domain of the singlet-state operator \( \tilde{H}_1 \). For \( z < m_{\lambda_1}^1 \), the function \( \Delta_{\lambda_1}(z) \) increases from 1 to \( +\infty \), and one has

\[
\Delta_{\lambda_1}(z) \to 1 \quad \text{as} \quad z \to -\infty
\]

\[
\Delta_{\lambda_1}(z) \to +\infty \quad \text{as} \quad z \to m_{\lambda_1}^1 - 0.
\]

Therefore, \( \Delta_{\lambda_1}(z) \) cannot vanish below the value \( m_{\lambda_1}^1 \). For \( z > M_{\lambda_1}^1 \), \( \Delta_{\lambda_1}(z) \) increases from \( -\infty \) to 1,

\[
\Delta_{\lambda_1}(z) \to -\infty \quad \text{as} \quad z \to M_{\lambda_1}^1 + 0,
\]

\[
\Delta_{\lambda_1}(z) \to +1 \quad \text{as} \quad z \to +\infty.
\]

Therefore, above the value \( M_{\lambda_1}^1 \), \( \Delta_{\lambda_1}(z) \) vanishes at a single point \( z = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}} \). \( \square \)

We now consider the two-dimensional case.

Theorem 3.5. At \( \nu = 2 \) and \( U > 0 \) and for all values of the parameter of the Hamiltonian, the operator \( \tilde{H}_1 \) has a unique eigenvalue \( \bar{z} \) that is above the continuous spectrum of \( \tilde{H}_1 \), i.e., \( z > M_{\lambda_1}^2 \).
Proof. In the two-dimensional case, the function $\Delta_{\Lambda_1}^2(z)$ increases monotonically outside the continuous spectrum domain of $\tilde{H}_1$. For $z < m_{\Lambda_1}^2$, the function $\Delta_{\Lambda_1}^2(z)$ increases from $1$ to $+\infty$,

$$\begin{align*}
\Delta_{\Lambda_1}^2(z) &\to 1 \quad \text{as} \quad z \to -\infty, \\
\Delta_{\Lambda_1}^2(z) &\to +\infty \quad \text{as} \quad z \to m_{\Lambda_1}^2 - 0.
\end{align*}$$

Therefore, $\Delta_{\Lambda_1}^2(z)$ cannot vanish below the value $m_{\Lambda_1}^2$. For $z > M_{\Lambda_1}^2$, $\Delta_{\Lambda_1}^2(z)$ increases from $-\infty$ to $1$,

$$\begin{align*}
\Delta_{\Lambda_1}^2(z) &\to -\infty \quad \text{as} \quad z \to M_{\Lambda_1}^2, \\
\Delta_{\Lambda_1}^2(z) &\to 1 \quad \text{as} \quad z \to +\infty.
\end{align*}$$

Therefore, above $M_{\Lambda_1}^2$, the function $\Delta_{\Lambda_1}^2(z)$ has a single zero at the point $\tilde{z} > M_{\Lambda_1}^2$. 

In the three-dimensional case, the operator $\tilde{H}_1$ for some values of the parameters in the Hamiltonian has a unique eigenvalue $\tilde{\delta}$, such that $\tilde{\delta} > M_{\Lambda_1}^3$.

We consider the Watson integral \[18\]

$$W = \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \int_0^\pi \frac{3dxdydz}{3 - \cos x - \cos y - \cos z} = 1,516.$$ 

Because the measure $\nu$ is normalized,

$$J = \int_{-\pi}^\pi \int_{-\pi}^\pi \int_{-\pi}^\pi \frac{dxdydz}{3 - \cos x - \cos y - \cos z} = \int_{-\pi}^\pi \int_{-\pi}^\pi \int_{-\pi}^\pi \frac{dxdydz}{3 + \cos x + \cos y + \cos z} = W.$$

**Theorem 3.6.** At $\nu = 3$ and $U > 0$ and the total quasimomentum $\Lambda_1$ of the system have the form

$$\Lambda_1 = (\Lambda_1^1; \Lambda_1^2; \Lambda_1^3) = (\Lambda_0^1; \Lambda_0^2; \Lambda_0^3).$$

Then the operator $\tilde{H}_1$ has a unique eigenvalue $\tilde{\delta}$ if

$$U > \frac{12B\cos\frac{\Lambda_0^0}{2}}{W}.$$ 

Otherwise, the operator $\tilde{H}_1$ has no eigenvalue.

**Proof.** In the three-dimensional case,

$$\begin{align*}
\Delta_{\Lambda_1}^3(z) &\to 1 \quad \text{as} \quad z \to -\infty, \\
\Delta_{\Lambda_1}^3(z) &\to Q > 1 \quad \text{as} \quad z \to m_{\Lambda_1}^3.
\end{align*}$$

Hence, the function $\Delta_{\Lambda_1}^3(z)$ cannot vanish below the continuous spectrum. Above the continuous spectrum,

$$\begin{align*}
\Delta_{\Lambda_1}^3(z) &\to 1 - \frac{UW}{12B\cos(\Lambda_0^0/2)} \quad \text{as} \quad z \to M_{\Lambda_1}^3 + 0, \\
\Delta_{\Lambda_1}^3(z) &\to 1 \quad \text{as} \quad z \to +\infty.
\end{align*}$$

Inspecting the equation $\Delta_{\Lambda_1}^3(z) = 0$ above the continuous spectrum domain of the operator $\tilde{H}_1$, we obtain the statement of the theorem. 

We note that the converse situation is realized for $U < 0$.

We must therefore investigate the spectrum of the operator $\tilde{H}_2$.

$$(H_2f)(\mu, \gamma) = \{2A + 2B\sum_{i=1}^\nu |\cos\mu_i + \cos\gamma_i|\}f(\mu, \gamma).$$

Now let $\Lambda_2 = \mu + \gamma$. Then $(H_{2\Lambda_2}f_{\Lambda_2})(\mu) = \{2A + 4B\sum_{i=1}^\nu |\cos\Lambda_0^i \cos(\Lambda_0^0/2 - \mu)|\}f_{\Lambda_2}(\mu).$
It is obvious that the spectrum of $\tilde{H}_{2\Lambda_2}$ is purely continuous and coincides with the value set of the function $h_{\Lambda_2}(\mu) = 2A + 4B \sum_{i=1}^{\nu} [\cos \frac{\Lambda_i}{2} \cos (\frac{\Lambda_i}{2} - \mu')]$, i.e., with the set $[2A - 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_i}{2}, 2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_i}{2}]$.

Now, using the obtained results and representation (9), we describe the structure of the essential spectrum and the discrete spectrum of the operator $1\tilde{H}^1_1$.

The spectrum of the operator $A \otimes I + I \otimes B$, where $A$ and $B$ are densely defined bounded linear operators, was studied in [5],[6],[7]. Explicit formulas were given there that express the essential spectrum $\sigma_{ess}(A \otimes I + I \otimes B)$ and the discrete spectrum $\sigma_{disc}(A \otimes I + I \otimes B)$ of $A \otimes I + I \otimes B$ in terms of the spectrum $\sigma(A)$ and the discrete spectrum $\sigma_{disc}(A)$ of $A$ and in terms of the spectrum $\sigma(B)$ and the discrete spectrum $\sigma_{disc}(B)$ of $B$:

$$\sigma_{disc}(A \otimes I + I \otimes B) = \{\sigma(A) \setminus \sigma_{ess}(A) + \sigma(B) \setminus \sigma_{ess}(B)\} \cup \{\sigma_{ess}(A) + \sigma_{ess}(B)\},$$

$$\sigma_{ess}(A \otimes I + I \otimes B) = (\sigma_{ess}(A) + \sigma(B)) \cup (\sigma(A) + \sigma_{ess}(B)).$$

It is clear that $\sigma(A \otimes I + I \otimes B) = \{\lambda + \mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}$.

**Theorem 3.7.** At $\nu = 1$ and $U > 0$ the essential spectrum of the system first triplet-state operator $1\tilde{H}^1_1$ is exactly the union of two segments: $\sigma_{ess}(1\tilde{H}^1_1) = [4A - 4B \cos \frac{\Lambda_1}{2} - 4B \cos \frac{\Lambda_2}{2}, 4A + 4B \cos \frac{\Lambda_1}{2} + 4B \cos \frac{\Lambda_2}{2}] \cup [4A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}, 4A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}} + 4B \cos \frac{\Lambda_2}{2}]$. The discrete spectrum of $1\tilde{H}^1_1$ is empty.

**Proof.** It follows from representation (9) that

$$\sigma(1\tilde{H}^1_1) = \{\lambda + \mu : \lambda \in \sigma(\tilde{H}_1), \mu \in \sigma(\tilde{H}_2)\},$$

and in the one-dimensional case, the continuous spectrum of $\tilde{H}_1$ is

$$\sigma_{cont}(\tilde{H}_1) = [2A - 4B \cos \frac{\Lambda_1}{2}, 2A + 4B \cos \frac{\Lambda_1}{2}],$$

and the discrete spectrum of $\tilde{H}_1$ consists of a single point $z = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}$.

The continuous spectrum of the operator $\tilde{H}_2$ is the segment $[2A - 4B \cos \frac{\Lambda_1}{2}, 2A + 4B \cos \frac{\Lambda_1}{2}]$, and the discrete spectrum of $\tilde{H}_2$ is empty. Therefore, the essential spectrum of the system first triplet-state operator $1\tilde{H}^1_1$ is the union of two segments, and the first triplet-state operator $1\tilde{H}^1_1$ has no eigenvalue.

The following theorem is proved totally similarly to Theorem 3.6.

**Theorem 3.8.** At $\nu = 2$ and $U > 0$ the essential spectrum of the system first triplet-state operator $1\tilde{H}^1_1$ is exactly the union of two segments: $\sigma_{ess}(1\tilde{H}^1_1) = [4A - 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2} - 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2}, 4A + 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2} + 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2}] \cup [2A + \tilde{z} - 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2}, 2A + \tilde{z} + 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2}]$. The discrete spectrum of $1\tilde{H}^1_1$ is empty.

Let $\nu = 3$ and $\Lambda_1 = (\Lambda_1^0, \Lambda_1^1, \Lambda_1^2)$. 

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Theorem 3.9. The following statements hold:

1. Let

\[ U > \frac{12B\cos\frac{L_0}{2}}{W}. \]

Then the essential spectrum of the system first triplet-state operator \( \tilde{H}_1 \) is the union of two segments \( \sigma_{\text{ess}}(\tilde{H}_1) = [4A - 12B\cos\frac{L_0}{2} - 4B\sum_{i=1}^{3}\cos\frac{L_i}{2}, 4A + 12B\cos\frac{L_0}{2} + 4B\sum_{i=1}^{3}\cos\frac{L_i}{2}] \cup [\tilde{z} + 2A - 4B\sum_{i=1}^{3}\cos\frac{L_i}{2}, \tilde{z} + 2A + 4B\sum_{i=1}^{3}\cos\frac{L_i}{2}]. \) The discrete spectrum of \( \tilde{H}_1 \) is empty. Here and hereafter, \( \tilde{z} \) is an eigenvalue of \( \tilde{H}_1. \)

2. Let

\[ U < \frac{12B\cos\frac{L_0}{2}}{W}. \]

Then the essential spectrum of the system first triplet-state operator \( \tilde{H}_1 \) is the segment \( \sigma_{\text{ess}}(\tilde{H}_1) = [4A - 12B\cos\frac{L_0}{2} - 4B\sum_{i=1}^{3}\cos\frac{L_i}{2}, 4A + 12B\cos\frac{L_0}{2} + 4B\sum_{i=1}^{3}\cos\frac{L_i}{2}], \) and the discrete spectrum is empty.

Proof. The proof uses representation (9) and the results for the spectra of \( \tilde{H}_1 \) and \( \tilde{H}_2 \) in the case \( \nu = 3. \)

4. Second triplet state

Theorem 4.1. The subspace \( ^2\mathcal{H}_1 \) is invariant under the operator \( H \), and the operator \( ^2H_1 \) is a bounded self-adjoint operator. It generates a bounded self-adjoint operator \( ^2\overline{\mathcal{H}}_1 \) acting in the space \( L_{2}^{\alpha}(\mathcal{V}^4) \) as

\[
( ^2\overline{\mathcal{H}}_1 f)(m,n,p,r) = 4Af(m,n,p,r) + 2B\sum_{r}(f(m+\tau,n,p,r) + f(m,n+\tau,p,r) + f(m,n,p+r,r) + f(m,n,p+r,\tau) + U[\delta_m,pf(m,n,p,r) + \delta_n,pf(m,n,p,r) + \delta_p,r f(m,n,p,r)],
\]

where \( \delta_{k,i} \) is the Kronecker symbol. The operator \( ^2H_1 \) acts on a vector \( \psi \in ^2\mathcal{H}_1 \) as

\[
^2H_1\psi = \sum_{m,n,q,r} ( ^2\overline{\mathcal{H}}_1 f)(m,n,p,r) ^2_1(m,n,p,r).
\]

Proof. The proof of the theorem follows from the explicit form of the action of the Hamiltonian \( H \) on vectors \( \psi \in ^2\mathcal{H}_1 \) using standard anticommutation relations for electron creation and annihilation operators.

We set \( ^2\tilde{H}_1 = \mathcal{F} ^2\overline{\mathcal{H}}_1 \mathcal{F}^{-1} \). In the quasimomentum representation, the operator \( ^2\overline{\mathcal{H}}_1 \) acts in the Hilbert space \( L_{2}^{\alpha}((\mathcal{V}^4))) \) as

\[
( ^2\tilde{H}_1 f)(\lambda,\mu,\gamma,\theta) = h(\lambda,\mu,\gamma,\theta)f(\lambda,\mu,\gamma,\theta) + U[\int_{\mathcal{V}^4} f(s,\mu,\lambda + \gamma - s,\theta)ds +
\]

\[
+ \int_{\mathcal{V}^4} f(\lambda,t,\mu + \gamma - t,\theta)dt + \int_{\mathcal{V}^4} f(\lambda,\mu,\gamma + \theta - r,dr] \quad f \in L_{2}^{\alpha}((\mathcal{V}^4)),
\]

where \( L_{2}^{\alpha} \) is the subspace of antisymmetric functions in \( L_{2}((\mathcal{V}^4)), \) and \( h(\lambda,\mu,\gamma,\theta) \) has form (5).

Taking into account that the function \( f(\lambda,\mu,\gamma,\theta) \) is antisymmetric, we can rewrite formula (12) as

\[
( ^2\tilde{H}_1 f)(\lambda,\mu,\gamma,\theta) = h(\lambda,\mu,\gamma,\theta)f(\lambda,\mu,\gamma,\theta)
\]

\[
+ \int_{\mathcal{V}^4} f(\lambda,t,\mu + \gamma - t,\theta)dt + \int_{\mathcal{V}^4} f(\lambda,\mu,\gamma + \theta - r,dr] \quad f \in L_{2}^{\alpha}((\mathcal{V}^4)).
\]
The proof of the statements in the lemma 4.2 is similar to the proof of lemma 3.2.

Proof. We verify that the operator $\tilde{H}_1$ can be represented in the form

$$\tilde{H}_1 = \tilde{H}_3 \bigotimes I \bigotimes I + I \bigotimes \tilde{H}_4 \bigotimes I + I \bigotimes I \bigotimes \tilde{H}_5,$$

where $(\tilde{H}_3 f)(\lambda, \gamma) = \{-2A - 2B \sum_{i=1}^{\nu} [\cos \lambda_i + \cos \gamma_i]\} f(\lambda, \gamma)$

and $(\tilde{H}_4 f)(\mu, \gamma) = U \int_{T^\nu} f(s, \mu + \gamma - s) ds$,

and $(\tilde{H}_5 f)(\gamma, \theta) = \{-2A - 2B \sum_{i=1}^{\nu} [\cos \mu_i + \cos \theta_i]\} f(\mu, \theta)$

where $f(\lambda, \gamma) = \{-2A - 2B \sum_{i=1}^{\nu} [\cos \lambda_i + \cos \gamma_i]\} f(\lambda, \gamma) - U \int_{T^\nu} f(s, \lambda + \gamma - s) ds$.

We must therefore investigate the spectrum and bound states of the operators $\tilde{H}_3$, $\tilde{H}_4$, and $\tilde{H}_5$.

Let the total quasimomentum of the two-electron system $\lambda + \gamma = \Lambda_3$ be fixed.

We let $L_2(\Gamma_{\Lambda_3})$ denote the space of functions that are square integrable on the manifold $\Gamma_{\Lambda_3} = \{ (\lambda, \gamma) : \lambda + \gamma = \Lambda_3 \}$.

That the operator $\tilde{H}_3$ and the space $\mathcal{H}_2 \equiv L_2((T^\nu)^2)$ can be decomposed into a direct integral of operators $\tilde{H}_{2\Lambda_3}$ and spaces $\mathcal{H}_{2\Lambda_3} = L_2(\Gamma_{\Lambda_3})$,

$$\tilde{H}_3 = \bigoplus \int_{T^\nu} \tilde{H}_{2\Lambda_3} d\Lambda_3, \quad \mathcal{H}_2 = \bigoplus \int_{T^\nu} \mathcal{H}_{2\Lambda_3} d\Lambda_3,$$

such that the spaces $\mathcal{H}_{2\Lambda_3}$ are then invariant under the operators $\tilde{H}_{2\Lambda_3}$ and each operator $\tilde{H}_{2\Lambda_3}$ acts in $\mathcal{H}_{2\Lambda_3}$ as $(\tilde{H}_{2\Lambda_3} f_{\Lambda_3})(\lambda) = \{-2A - 4B \sum_{i=1}^{\nu} \cos \Lambda_3^i \cos (\Lambda_3^i - \lambda_i)\} f_{\Lambda_3}(\lambda) - U \int_{T^\nu} f_{\Lambda_3}(s) ds$, where $f_{\Lambda_3}(\lambda) = f(\lambda, \Lambda_3 - \lambda)$.

It is known that the continuous spectrum of $\tilde{H}_{2\Lambda_3}$ is independent of $U$ and coincides with the segment $[m_{\Lambda_3}, M_{\Lambda_3}]$, where $m_{\Lambda_3} = \inf_{x \in T^\nu} h_{\Lambda_3}(x)$, $M_{\Lambda_3} = \sup_{x \in T^\nu} h_{\Lambda_3}(x)$, here $h_{\Lambda_3}(x) = -2A - 4B \sum_{i=1}^{\nu} \cos \Lambda_3^i \cos \Lambda_3^i - x_i$.

We set $\Delta_{\Lambda_3}^\nu(z) = 1 - U \int_{T^\nu} \prod_{i=1}^{\nu} ds_i \prod_{i=1}^{\nu} \cos \Lambda_3^i \cos (\Lambda_3^i - s_i) - z$.

Lemma 4.2. The number $z = z_0$ not belonging to the continuous spectrum of the operator $\tilde{H}_3$ is an eigenvalue of that operator if and only if it is a zero of the function $\Delta_{\Lambda_3}^\nu(z)$, i.e., $\Delta_{\Lambda_3}^\nu(z_0) = 0$.

Proof. The proof of the statements in the lemma 4.2 is similar to the proof of lemma 3.2. □

It is clear that the continuous spectrum of $\tilde{H}_3$ coincides with the segment

$$[m_{\Lambda_3}^\nu, M_{\Lambda_3}^\nu] = [-2A - 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_3^i}{2}, -2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_3^i}{2}].$$

We consider the one-dimensional case.

Theorem 4.3. At $\nu = 1$ and $U > 0$ and for all values of the parameter of the Hamiltonian, the operator $\tilde{H}_3$ has a unique eigenvalue $z = -2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_3}{2}}$ that is below the continuous spectrum of $\tilde{H}_3$, i.e., $z < m_{\Lambda_3}^1$. 

...
**Proof.** In the one-dimensional case, the function $\Delta^1_{A_3}(z)$ decreases monotonically outside the continuous spectrum domain of the operator $\hat{H}_3$. For $z < m^1_{A_3}$, the function $\Delta^1_{A_3}(z)$ decreases from 1 to $-\infty$,

$$\Delta^1_{A_3}(z) \to 1 \quad \text{as} \quad z \to -\infty,$$

$$\Delta^1_{A_3}(z) \to -\infty \quad \text{as} \quad z \to m^1_{A_3} - 0.$$

Therefore, below the value $m^1_{A_3}$, $\Delta^1_{A_3}(z)$ vanishes at a single point $z = -2A - \sqrt{U^2 + 16B^2\cos^2\frac{A_3}{2}}$. For $z > M^1_{A_3}$, $\Delta^1_{A_3}(z)$ decreases from $+\infty$ to 1,

$$\Delta^1_{A_3}(z) \to +\infty \quad \text{as} \quad z \to M^1_{A_3} + 0,$$

$$\Delta^1_{A_3}(z) \to +1 \quad \text{as} \quad z \to +\infty.$$ 

Therefore, above the value $M^1_{A_3}$, $\Delta^1_{A_3}(z)$ cannot vanish.  

We now consider the two-dimensional case.

**Theorem 4.4.** At $\nu = 2$ and $U > 0$ and for all values of the parameter of the Hamiltonian, the operator $\hat{H}_3$ has a unique eigenvalue $\tilde{z}$ that is below the continuous spectrum of $\hat{H}_3$, i.e., $z < m^2_{A_3}$.

**Proof.** In the two-dimensional case, the function $\Delta^2_{A_3}(z)$ decreases monotonically outside the continuous spectrum domain of $\hat{H}_3$. For $z < m^2_{A_3}$, the function $\Delta^2_{A_3}(z)$ decreases from 1 to $-\infty$,

$$\Delta^2_{A_3}(z) \to 1 \quad \text{as} \quad z \to -\infty,$$

$$\Delta^2_{A_3} \to -\infty \quad \text{as} \quad z \to m^2_{A_3} - 0.$$ 

Therefore, below $m^2_{A_3}$, the function $\Delta^2_{A_3}(z)$ has a single zero at the point $\tilde{z} < m^2_{A_3}$. For $z > M^2_{A_3}$, $\Delta^2_{A_3}(z)$ decreases from $+\infty$ to 1,

$$\Delta^2_{A_3}(z) \to +\infty \quad \text{as} \quad z \to M^2_{A_3} = 0,$$

$$\Delta^2_{A_3}(z) \to +1 \quad \text{as} \quad z \to +\infty.$$ 

Therefore, above $M^2_{A_3}$, the function $\Delta^2_{A_3}(z)$ cannot vanish.

In the three-dimensional case, the operator $\hat{H}_3$ for some values of the parameters in the Hamiltonian has a unique eigenvalue $\tilde{z}$, such that $\tilde{z} < m^3_{A_3}$.

**Theorem 4.5.** At $\nu = 3$ and $U > 0$ and the total quasimomentum $A_3$ of the system have the form

$$A_3 = (A_3^1; A_3^2; A_3^3) = (A_3^0; A_3^0; A_3^0).$$

Then the operator $\hat{H}_3$ has a unique eigenvalue $\tilde{z}$ if

$$U > \frac{12B \cos \frac{A_3^0}{2}}{W}.$$ 

Otherwise, the operator $\hat{H}_3$ has no eigenvalue.

**Proof.** In the three-dimensional case,

$$\Delta^3_{A_3}(z) \to 1 \quad \text{as} \quad z \to -\infty,$$

$$\Delta^3_{A_3} \to 1 - \frac{UW}{12B\cos(A_3/2)} \quad \text{as} \quad z \to m^3_{A_3} - 0.$$ 

Hence, the function $\Delta^3_{A_3}(z)$ has a single zero below the continuous spectrum, if $U > \frac{12B \cos \frac{A_3^0}{2}}{W}$. Above the continuous spectrum,

$$\Delta^3_{A_3}(z) \to +\infty \quad \text{as} \quad z \to M^3_{A_3} + 0,$$

$$\Delta^3_{A_3}(z) \to +1 + \frac{UW}{12B\cos(A_3/2)} \quad \text{as} \quad z \to +\infty.$$ 

Inspecting the equation $\Delta^3_{A_3}(z) = 0$ above the continuous spectrum domain of the operator $\hat{H}_3$, we obtain the statement of the theorem.

\[\square\]
We note that the converse situation is realized for $U<0$.

The spectrum of operator $(H_5f)(\mu, \gamma) = U \int_{T^\nu} f(s, \mu + \gamma - s) ds$ is purely discrete and consists of a single point $z = U$, for arbitrary value $\nu$.

Now we investigated the spectrum of the operator

\[(H_5f)(\gamma, \theta) = \{-2A - 2B \sum_{i=1}^{\nu} \cos \mu_i + \cos \theta_i\} f(\mu, \theta) + U \int_{T^\nu} f(s, \gamma + \theta - s) ds.
\]

We let $A = \mu + \theta$, and $A_5 = \gamma + \theta$.

It is known that the continuous spectrum of $\tilde{H}_5$ is independent of $U$ and coincides with the segment $[m_{A_4}, M_{A_4}^\nu]$, where $m_{A_4}^\nu = -2A - 4B \sum_{i=1}^{\nu} \cos \frac{A_i}{2}$, $M_{A_4}^\nu = -2A + 4B \sum_{i=1}^{\nu} \cos \frac{A_i}{2}$.

We let $L_2(\Gamma_{A_4})$ denote the space of functions that are square integrable on the manifold $\Gamma_{A_4} = \{(\mu, \theta) : \mu + \theta = A_4\}$.

That the operator $\tilde{H}_5$ and the space $H_2 \equiv L_2((T^\nu)^2)$ can be decomposed into a direct integral of operators $\tilde{H}_{5A_4}$ and spaces $H_{2A_4} = L_2(\Gamma_{A_4})$.

\[\tilde{H}_5 = \bigoplus \int_{T^\nu} \tilde{H}_{5A_4} d\Lambda_4, \quad H_2 = \bigoplus \int_{T^\nu} H_{2A_4} d\Lambda_4,\]

such that the spaces $H_{2A_4}$ are then invariant under the operators $\tilde{H}_{5A_4}$ and each operator $\tilde{H}_{5A_4}$ acts in $H_{2A_4}$ as $(\tilde{H}_{5A_4} f_{\Lambda_4})(\lambda) = \{-2A - 4B \sum_{i=1}^{\nu} \cos \frac{A_i}{2} \cos (\frac{A_i}{2} - \lambda_i)\} f_{\Lambda_4}(\lambda) + U \int_{T^\nu} f_{\Lambda_4}(s) ds$, where $f_{\Lambda_4}(\lambda) = f(\lambda, A_4 - \lambda)$.

We set $\Delta_{A_4}(z) = 1 + U \int_{T^\nu} \frac{ds_1 ds_2 \ldots ds_{\nu}}{-2A - 4B \sum_{i=1}^{\nu} \cos \frac{A_i}{2} \cos (\frac{A_i}{2} - s_i)}$.

**Lemma 4.6.** The number $z = z_0$ not belonging to the continuous spectrum of the operator $\tilde{H}_4$ is an eigenvalue of that operator if and only if it is a zero of the function $\Delta_{A_4}(z)$, i.e., $\Delta_{A_4}(z_0) = 0$.

**Proof.** The proof of the statements in the lemma 4.6 is similar to the proof of Lemma 3.2. \qed

We consider the one-dimensional case.

**Theorem 4.7.** At $\nu = 1$ and $U > 0$ and for all values of the parameter of the Hamiltonian, the operator $\tilde{H}_5$ has a unique eigenvalue $z = -2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{A}{2}}$ that is above the continuous spectrum of $\tilde{H}_5$, i.e., $z > M_{A_4}^1$.

**Proof.** In the one-dimensional case, the function $\Delta_{A_4}(z)$ increases monotonically outside the continuous spectrum domain of the operator $\tilde{H}_5$. For $z < M_{A_4}^1$, the function $\Delta_{A_4}(z)$ increases from 1 to $+\infty$,

\[\Delta_{A_4}(z) \rightarrow 1 \quad \text{as} \quad z \rightarrow -\infty,\]
\[\Delta_{A_4}(z) \rightarrow +\infty \quad \text{as} \quad z \rightarrow M_{A_4}^1 - 0.\]

Therefore, below the value $M_{A_4}^1$, the function $\Delta_{A_4}(z)$ cannot vanish. For $z > M_{A_4}^1$, the function $\Delta_{A_4}(z)$ increases from $-\infty$ to 1,

\[\Delta_{A_4}(z) \rightarrow -\infty \quad \text{as} \quad z \rightarrow M_{A_4}^1 + 0,\]
\[\Delta_{A_4}(z) \rightarrow +1 \quad \text{as} \quad z \rightarrow +\infty.\]

Therefore, above the value $M_{A_4}^1$, the function $\Delta_{A_4}(z)$ has a single zero at the point $z = -2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{A}{2}}$. \qed

We now consider the two-dimensional case.

**Theorem 4.8.** At $\nu = 2$ and $U > 0$ and for all values of the parameter of the Hamiltonian, the operator $\tilde{H}_5$ has a unique eigenvalue $z'$ that is above the continuous spectrum of $\tilde{H}_5$, i.e., $z' > M_{A_4}^2$. 

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Proof. In the two-dimensional case, the function $\Delta^2_{\Lambda_5}(z)$ increases monotonically outside the continuous spectrum domain of $\tilde{H}_5$. For $z < m^2_{\Lambda_4}$, the function $\Delta^2_{\Lambda_5}(z)$ increases from 1 to $+\infty$, 
\[ \Delta^2_{\Lambda_5}(z) \to 1 \quad \text{as} \quad z \to -\infty \]
\[ \Delta^2_{\Lambda_5} \to +\infty \quad \text{as} \quad z \to m^2_{\Lambda_4} - 0. \]
Therefore, below $m^2_{\Lambda_4}$, the function $\Delta^2_{\Lambda_5}(z)$ cannot vanishes. For $z > M^2_{\Lambda_4}$, the function $\Delta^2_{\Lambda_5}(z)$ increases from $-\infty$ to 1,
\[ \Delta^2_{\Lambda_5}(z) \to -\infty \quad \text{as} \quad z \to M^2_{\Lambda_4} + 0, \]
\[ \Delta^2_{\Lambda_5}(z) \to 1 \quad \text{as} \quad z \to +\infty. \]
Therefore, above $M^2_{\Lambda_4}$, the function $\Delta^2_{\Lambda_5}(z)$ has a single zero at the point $\tilde{z} > M^2_{\Lambda_4}$.

In the three-dimensional case, the operator $\tilde{H}_5$ for some values of the parameters in the Hamiltonian has a unique eigenvalue $\tilde{z}$, such that $\tilde{z} > M^3_{\Lambda_4}$.

**Theorem 4.9.** At $\nu = 3$ and $U > 0$ and the total quasimomentum $\Lambda_5$ of the system have the form
\[ \Lambda_5 = (\Lambda_5^1; \Lambda_5^2; \Lambda_5^3) = (\Lambda_3^0; \Lambda_3^0; \Lambda_3^0). \]
Then the operator $\tilde{H}_5$ has a unique eigenvalue $z''$ if
\[ U > \frac{12B \cos \Lambda_3^0}{W}. \]
Otherwise, the operator $\tilde{H}_5$ has no eigenvalue.

Proof. In the three-dimensional case,
\[ \Delta^3_{\Lambda_5}(z) \to 1 \quad \text{as} \quad z \to -\infty \]
\[ \Delta^3_{\Lambda_5}(z) \to 1 + \frac{UW}{12B \cos(\Lambda_5^0/2)} \quad \text{as} \quad z \to m^3_{\Lambda_4} - 0. \]
Hence, the function $\Delta^3_{\Lambda_5}(z)$ cannot vanishes below the continuous spectrum. Above the continuous spectrum,
\[ \Delta^3_{\Lambda_5}(z) \to 1 - \frac{UW}{12B \cos(\Lambda_5^0/2)} \quad \text{as} \quad z \to M^3_{\Lambda_4} + 0, \]
\[ \Delta^3_{\Lambda_5}(z) \to 1 \quad \text{as} \quad z \to +\infty. \]
Inspecting the equation $\Delta^3_{\Lambda_5}(z) = 0$ above the continuous spectrum domain of the operator $\tilde{H}_5$, we obtain the statement of the theorem.

Now, using the obtained results and representation (9), we describe the structure of the essential spectrum and the discrete spectrum of the operator $2\tilde{H}_1^1$.

**Theorem 4.10.** At $\nu = 1$ and $U > 0$ the essential spectrum of the system second triplet-state operator $2\tilde{H}_1^1$ is exactly the union of three segments: $\sigma_{\text{ess}}(2\tilde{H}_1^1) = [U - 4A - 4B \cos \frac{\Lambda_3}{2} - 4B \cos \frac{\Lambda_2}{2}, U - 4A + 4B \cos \frac{\Lambda_2}{2} + 4B \cos \frac{\Lambda_1}{2}] \cup [U - 4A - 4B \cos \frac{\Lambda_3}{2} - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}, U - 4A + 4B \cos \frac{\Lambda_1}{2} - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_3}{2}}] \cup [U - 4A - 4B \cos^2 \frac{\Lambda_2}{2} + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_1}{2}}, U - 4A + 4B \cos^2 \frac{\Lambda_1}{2} + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_3}{2}}]$. The discrete spectrum of the operator $2\tilde{H}_1^1$ is consists no more than one point.

Proof. It follows from representation (14) that
\[ \sigma(2\tilde{H}_1^1) = \{ \lambda + \mu + \theta : \lambda \in \sigma(\tilde{H}_3), \mu \in \sigma(\tilde{H}_4), \theta \in \sigma(\tilde{H}_5) \}. \]
The following statements hold:

1. Let \( \nu = 2 \) and \( U > 0 \) the essential spectrum of the system second triplet-state operator \( \tilde{H}_1 \) is the union of three segments: \( \sigma_{\text{ess}}(\tilde{H}_1) = [U - 4A - 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2} - 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}, U - 4A + 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2} + 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}] \cup [U - 2A + \bar{z} - 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}, U - 2A + \bar{z} + 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}] \cup [U - 2A + z' - 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}, U - 2A + z' + 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}] \). The discrete spectrum of the operator \( \tilde{H}_1 \) consists no more than one point.

2. Let \( \nu = 3 \) and \( \lambda_3 = (\lambda_0^0, \lambda_0^0, \lambda_0^0) \), and \( \lambda_4 = (\lambda_0^0, \lambda_0^0, \lambda_0^0) \).

Theorem 4.11. Let \( \nu = 2 \) and \( U > 0 \) the essential spectrum of the system second triplet-state operator \( \tilde{H}_1 \) is the union of three segments: \( \sigma_{\text{ess}}(\tilde{H}_1) = [U - 4A - 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2} - 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}, U - 4A + 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2} + 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}] \cup [U - 2A + \bar{z} - 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}, U - 2A + \bar{z} + 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}] \cup [U - 2A + z' - 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}, U - 2A + z' + 4B \sum_{i=1}^{2} \cos \frac{\lambda_i}{2}] \). The discrete spectrum of the operator \( \tilde{H}_1 \) consists no more than one point.

Theorem 4.12. The following statements hold:

1. Let \( U > \frac{12B \cos \frac{\lambda_0^0}{2}}{W}, \cos \frac{\lambda_0^0}{2} \geq \cos \frac{\lambda_0^0}{2} \). Then the essential spectrum of the system second triplet-state operator \( \tilde{H}_1 \) is the union of three segments \( \sigma_{\text{ess}}(\tilde{H}_1) = [U - 4A - 12B \cos \frac{\lambda_0^0}{2} - 12B \cos \frac{\lambda_0^0}{2}, U - 4A + 12B \cos \frac{\lambda_0^0}{2} + 12B \cos \frac{\lambda_0^0}{2}] \cup [\tilde{z} + U - 2A - 12B \cos \frac{\lambda_0^0}{2}, \tilde{z} + U - 2A + 12B \cos \frac{\lambda_0^0}{2}] \cup [z'' + U - 2A - 12B \cos \frac{\lambda_0^0}{2}, z'' + U - 2A + 12B \cos \frac{\lambda_0^0}{2}] \). The discrete spectrum of the operator \( \tilde{H}_1 \) consists no more than one point. Here and hereafter, \( \tilde{z} \) is an eigenvalue of the operator \( \tilde{H}_3 \), and \( z'' \) is an eigenvalue of the operator \( \tilde{H}_5 \).

2. Let \( \frac{12B \cos \frac{\lambda_0^0}{2}}{W} < U \leq \frac{12B \cos \frac{\lambda_0^0}{2}}{W} \). Then the essential spectrum of the system second triplet-state operator \( \tilde{H}_1 \) is the union of two segment \( \sigma_{\text{ess}}(\tilde{H}_1) = [U - 4A - 12B \cos \frac{\lambda_0^0}{2} - 12B \cos \frac{\lambda_0^0}{2}, U - 4A + 12B \cos \frac{\lambda_0^0}{2} + 12B \cos \frac{\lambda_0^0}{2}] \cup [\tilde{z} + U - 2A - 12B \cos \frac{\lambda_0^0}{2}, \tilde{z} + U - 2A + 12B \cos \frac{\lambda_0^0}{2}] \), and the discrete spectrum of the system second triplet-state operator \( \tilde{H}_1 \) is empty.

3. Let \( \frac{12B \cos \frac{\lambda_0^0}{2}}{W} < U \leq \frac{12B \cos \frac{\lambda_0^0}{2}}{W} \). Then the essential spectrum of the system second triplet-state operator \( \tilde{H}_1 \) is the union of two segment \( \sigma_{\text{ess}}(\tilde{H}_1) = [U - 4A - 12B \cos \frac{\lambda_0^0}{2} - 12B \cos \frac{\lambda_0^0}{2}, U - 4A + 12B \cos \frac{\lambda_0^0}{2} + 12B \cos \frac{\lambda_0^0}{2}] \cup [z'' +
U - 2A - 12B \cos \frac{N_0}{2}, z'' + U - 2A + 12B \cos \frac{N_0}{2}, and the discrete spectrum of the system second triplet-state operator $2 \tilde{H}_1^I$ is empty.

4. Let

$$U \leq \frac{12B \cos \frac{N_0}{2}}{W}, \cos \frac{N_0}{2} \leq \cos \frac{N_0}{2}.$$  

Then the essential spectrum of the system second triplet-state operator $2 \tilde{H}_1^I$ consists of single segment $\sigma_{ess}(2 \tilde{H}_1^I) = [U - 4A - 12B \cos \frac{N_0}{2} - 12B \cos \frac{N_0}{2}, U - 4A + 12B \cos \frac{N_0}{2} + 12B \cos \frac{N_0}{2}]$, and the discrete spectrum of the system second triplet-state operator $2 \tilde{H}_1^I$ is empty.

Proof. The proof uses representation (14) and the results for the spectra of $\tilde{H}_3, \tilde{H}_4$, and $\tilde{H}_5$ in the case $\nu = 3$.

We have considered the case $\cos \frac{N_0}{2} > 0$, and $\cos \frac{N_0}{2} > 0$, The case $\cos \frac{N_0}{2} < 0$, and $\cos \frac{N_0}{2} < 0$, and other case is investigated similarly. Hence, in second triplet state, the four-electron system with $U > 0$ or $U < 0$ respectively has at most one antibound state or at most one bound state. The essential spectrum of the system is the union of at most three segments.

5. Threed triplet state

Theorem 5.1. The subspace $3\mathcal{H}_1^I$ is invariant under the operator $H$, and the operator $3\mathcal{H}_1^I$ is a bounded self-adjoint operator. It generates a bounded self-adjoint operator $3\tilde{\mathcal{H}}_1^I$ acting in the space $l_2^{as}$ as

$$(3\tilde{\mathcal{H}}_1^I)(m,n,p,r) = 4Af(m,n,p,r) + 2B \sum_{\tau} [f(m+\tau,n,p,r) + f(m,n+\tau,p,r) + f(m,n,p,\tau,r) + f(m,n,p,r+\tau)] + U[\delta_{m,n}f(m,n,p,r) + \delta_{n,p}f(m,n,p,r) + \delta_{n,r}f(m,n,p,r)],$$

where $\delta_{k,l}$ is the Kronecker symbol. The operator $3\mathcal{H}_1^I$ acts on a vector $\psi \in 3\mathcal{H}_1^I$ as

$$3\mathcal{H}_1^I \psi = \sum_{m,n,q,r} (3\tilde{\mathcal{H}}_1^I)(m,n,p,r)\tilde{t}_1^{m,n,p,r}.$$  

Proof. The proof of the theorem follows from the explicit form of the action of the Hamiltonian $H$ on vectors $\psi \in 3\mathcal{H}_1^I$ using standard anticommutation relations for electron creation and annihilation operators.

We set $3\tilde{H}_1^I = \mathcal{F}3\tilde{\mathcal{H}}_1^I\mathcal{F}^{-1}$. In the quasimomentum representation, the operator $3\tilde{\mathcal{H}}_1^I$ acts in the Hilbert space $L_2^{as}((T^\nu)^4)$ as

$$(3\tilde{\mathcal{H}}_1^I)(\lambda, \mu, \gamma, \theta) = h(\lambda, \mu, \gamma, \theta)f(\lambda, \mu, \gamma, \theta) + U\int_{T^\nu} f(s, \lambda + \mu - s, \gamma, \theta)ds +$$

$$+ \int_{T^\nu} f(\lambda, t, \mu + \gamma - t, \theta)dt + \int_{T^\nu} f(\lambda, t, \mu, \gamma + \theta - t)dt, \quad f \in L_2^{as}((T^\nu)^4),$$

where $L_2^{as}$ is the subspace of antisymmetric functions in $L_2((T^\nu)^4)$, and $h(\lambda, \mu, \gamma, \theta)$ has form (5).

Taking into account that the function $f(\lambda, \mu, \gamma, \theta)$ is antisymmetric, we can rewrite formula (17) as

$$(3\tilde{\mathcal{H}}_1^I)(\lambda, \mu, \gamma, \theta) = \{2A + 2B \sum_{i=1}^{\nu} [\cos \lambda_i + \cos \mu_i] \} f(\lambda, \mu, \gamma, \theta) +$$
\[ + U \int_{T^2} f(s, \lambda + \mu - s, \gamma, \theta) ds + U \int_{T^2} f(\lambda, s, \mu + \gamma - s, \theta) ds + \]
\[ + \{2A + 2B \sum_{i=1}^{\nu} [\cos \gamma_i + \cos \theta_i] \} f(\lambda, \mu, \gamma, \theta) - U \int_{T^2} f(\lambda, \mu, s, \mu + \theta - s) ds. \]  

We verify that the operator \( 3\tilde{H}_1 \) can be represented in the form

\[ 3\tilde{H}_1 = \tilde{H}_6 \otimes I \otimes I + I \otimes \tilde{H}_7 \otimes I + I \otimes I \otimes \tilde{H}_8, \]

where \((\tilde{H}_6f)(\lambda, \mu) = \{2A + 2B \sum_{i=1}^{\nu} [\cos \lambda_i + \cos \mu_i] \} f(\lambda, \mu) + U \int_{T^2} f(s, \lambda + \mu - s) ds, \)

\[ (\tilde{H}_7f)(\mu, \gamma) = U \int_{T^2} f(s, \mu + \gamma - s) ds, \]

\[ (\tilde{H}_8f)(\gamma, \theta) = \{2A + 2B \sum_{i=1}^{\nu} [\cos \gamma_i + \cos \theta_i] \} f(\gamma, \theta) - U \int_{T^2} f(s, \mu + \theta - s) ds. \]

We must therefore investigate the spectrum and bound states of the operators \( \tilde{H}_6 \), and \( \tilde{H}_7 \), and \( \tilde{H}_8 \).

Let the total quasimomentum of the two-electron system \( \lambda + \mu = \Lambda_6 \) be fixed.

By \( L_2(\Gamma_{\Lambda_6}) \) we denote the space of functions that are square integrable on the manifold \( \Gamma_{\Lambda_6} = \{(\lambda, \mu) : \lambda + \mu = \Lambda_6 \} \).

That the operator \( \tilde{H}_6 \) and the space \( \mathcal{H}_2 \equiv L_2((T')^2) \) can be decomposed into a direct integral of operators \( \tilde{H}_{6\Lambda_6} \) and spaces \( \mathcal{H}_{2\Lambda_6} = L_2(\Gamma_{\Lambda_6}), \)

\[ \tilde{H}_6 = \bigoplus \int_{T^2} \tilde{H}_{6\Lambda_6} d\Lambda_6, \quad \mathcal{H}_2 = \bigoplus \int_{T^2} \mathcal{H}_{2\Lambda_6} d\Lambda_6, \]

such that the spaces \( \mathcal{H}_{2\Lambda_6} \) are then invariant under the operators \( \tilde{H}_{6\Lambda_6} \) and each operator \( \tilde{H}_{6\Lambda_6} \) acts in \( \mathcal{H}_{2\Lambda_6} \) as \((\tilde{H}_{6\Lambda_6}f_{\Lambda_6})(\lambda) = \{2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_6^2}{2} \cos \left( \frac{\Lambda_6^2}{2} - \lambda_i \right) \} f_{\Lambda_6}(\lambda) + U \int_{T^2} f_{\Lambda_6}(s) ds, \)

where \( f_{\Lambda_6}(\lambda) = f(\lambda, \Lambda_6 - \lambda). \)

It is known that the continuous spectrum of \( \tilde{H}_{6\Lambda_6} \) is independent of \( U \) and coincides with the segment \([m_{\Lambda_6}, M_{\Lambda_6}], \) where \( m_{\Lambda_6} = \inf_{x \in T'} h_{\Lambda_6}(x), M_{\Lambda_6} = \sup_{x \in T'} h_{\Lambda_6}(x), \) here \( h_{\Lambda_6}(x) = 2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_6^2}{2} \cos \left( \frac{\Lambda_6^2}{2} - x_i \right). \)

We set \( \Delta_{\Lambda_6}^\nu(z) = z + U \int_{T^2} \frac{ds_1 ds_2 ds_n}{2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_6^2}{2} \cos \left( \frac{\Lambda_6^2}{2} - s_i \right) - z} \).

**Lemma 5.2.** The number \( z = z_0 \) not belonging to the continuous spectrum of the operator \( \tilde{H}_6 \) is an eigenvalue of that operator if and only if it is a zero of the function \( \Delta_{\Lambda_6}^\nu(z) \), i.e., \( \Delta_{\Lambda_6}^\nu(z_0) = 0. \)

**Proof.** The proof of the statements in the lemma 5.2 is similar to the proof of lemma 3.2. \( \square \)

It is clear that the continuous spectrum of \( \tilde{H}_6 \) coincides with the segment

\[ [m_{\Lambda_6}^\nu, M_{\Lambda_6}^\nu] = [2A - 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_6^2}{2}, 2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_6^2}{2}]. \]

We consider the one-dimensional case.

**Theorem 5.3.** At \( \nu = 1 \) and \( U > 0 \) and for all values of the parameter of the Hamiltonian, the operator \( \tilde{H}_6 \) has a unique eigenvalue \( z = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_6^2}{2}} \) that is above the continuous spectrum of \( \tilde{H}_6, \) i.e., \( z > M_{\Lambda_6}^1. \)
Proof. In the one-dimensional case, the function $\Delta_{\Lambda_6}^1(z)$ increases monotonically outside the continuous spectrum domain of the operator $\hat{H}_6$. For $z < m_{\Lambda_6}^1$, the function $\Delta_{\Lambda_6}^1(z)$ increases from 1 to $+\infty$,

$$
\Delta_{\Lambda_6}^1(z) \to 1 \quad \text{as} \quad z \to -\infty,
$$

$$
\Delta_{\Lambda_6}^1(z) \to +\infty \quad \text{as} \quad z \to m_{\Lambda_6}^1 - 0.
$$

Therefore, below the value $m_{\Lambda_6}^1$, $\Delta_{\Lambda_6}^1(z)$ cannot vanish. For $z > M_{\Lambda_6}^1$, the function $\Delta_{\Lambda_6}^1(z)$ increases from $-\infty$ to 1,

$$
\Delta_{\Lambda_6}^1(z) \to -\infty \quad \text{as} \quad z \to m_{\Lambda_6}^1 + 0,
$$

$$
\Delta_{\Lambda_6}^1(z) \to +1 \quad \text{as} \quad z \to +\infty.
$$

Therefore, above the value $M_{\Lambda_6}^1$, the function $\Delta_{\Lambda_6}^1(z)$ has a single zero at the point $z = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_6}{2}}$.

We now consider the two-dimensional case. at a single point

**Theorem 5.4.** At $\nu = 2$ and $U > 0$ and for all values of the parameter of the Hamiltonian, the operator $\hat{H}_6$ has a unique eigenvalue $\tilde{z}$ that is above the continuous spectrum of $\hat{H}_6$, i.e., $z > M_{\Lambda_6}^2$.

**Proof.** In the two-dimensional case, the function $\Delta_{\Lambda_6}^2(z)$ increases monotonically outside the continuous spectrum domain of $\hat{H}_6$. For $z < m_{\Lambda_6}^2$, the function $\Delta_{\Lambda_6}^2(z)$ increases from 1 to $+\infty$,

$$
\Delta_{\Lambda_6}^2(z) \to 1 \quad \text{as} \quad z \to +\infty,
$$

$$
\Delta_{\Lambda_6}^2(z) \to -\infty \quad \text{as} \quad z \to m_{\Lambda_6}^2 - 0.
$$

Therefore, below $m_{\Lambda_6}^2$, the function $\Delta_{\Lambda_6}^2(z)$ cannot vanish. For $z > M_{\Lambda_6}^2$, the function $\Delta_{\Lambda_6}^2(z)$ increases from $-\infty$ to 1,

$$
\Delta_{\Lambda_6}^2(z) \to -\infty \quad \text{as} \quad z \to M_{\Lambda_6}^2 + 0,
$$

$$
\Delta_{\Lambda_6}^2(z) \to +1 \quad \text{as} \quad z \to +\infty.
$$

Therefore, above $M_{\Lambda_6}^2$, the function $\Delta_{\Lambda_6}^2(z)$ has a single zero at the point $\tilde{z} > M_{\Lambda_6}^2$.

In the three-dimensional case, the operator $\hat{H}_6$ for some values of the parameters in the Hamiltonian has a unique eigenvalue $\tilde{z}$, such that $\tilde{z} > M_{\Lambda_6}^3$.

**Theorem 5.5.** At $\nu = 3$ and $U > 0$ and the total quasimomentum $\Lambda_6$ of the system have the form

$$
\Lambda_6 = (\Lambda_6^1; \Lambda_6^2; \Lambda_6^3) = (\Lambda_6^0; \Lambda_6^0; \Lambda_6^0).
$$

Then the operator $\hat{H}_6$ has a unique eigenvalue $\tilde{z}$ if

$$
U > \frac{12B \cos \frac{\Lambda_6^0}{2}}{W}.
$$

Otherwise, the operator $\hat{H}_6$ has no eigenvalue.

**Proof.** In the three-dimensional case,

$$
\Delta_{\Lambda_6}^3(z) \to 1 \quad \text{as} \quad z \to -\infty,
$$

$$
\Delta_{\Lambda_6}^3(z) \to 1 - \frac{UW}{12B \cos(\Lambda_6^0/2)} \quad \text{as} \quad z \to m_{\Lambda_6}^3 - 0.
$$

Hence, the function $\Delta_{\Lambda_6}^3(z)$ cannot vanish below the continuous spectrum. Above the continuous spectrum,

$$
\Delta_{\Lambda_6}^3(z) \to 1 - \frac{UW}{12B \cos(\Lambda_6^0/2)} \quad \text{as} \quad z \to M_{\Lambda_6}^3 + 0,
$$

$$
\Delta_{\Lambda_6}^3(z) \to +\infty \quad \text{as} \quad z \to +\infty.
$$

Inspecting the equation $\Delta_{\Lambda_6}^3(z) = 0$ above the continuous spectrum domain of the operator $\hat{H}_6$, we obtain the statement of the theorem.
We note that the converse situation is realized for $U < 0$.

The spectrum of operator $(H_f)(\mu, \gamma) = U \int_{\gamma} f(s, \mu + s) ds$ is purely discrete and consists of the two-dimensional case.

such that the spaces $H_{2\Lambda_5}$ are then invariant under the operators $H_{8\Lambda_5}$ and each operator $H_{8\Lambda_5}$ acts in $H_{2\Lambda_5}$ as (8H_{8\Lambda_5}f_{\Lambda_5})(\lambda) = \{2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_5}{2} \cos (\frac{\Lambda_5}{2} - \lambda_i)\} f_{\Lambda_5}(\lambda) - U \int_{\nu} f_{\Lambda_5}(s) ds,$ where $f_{\Lambda_5}(\lambda) = \int_{\nu} ds \frac{\cos \frac{\Lambda_5}{2} \cos (\frac{\Lambda_5}{2} - s_i) - 1}{2A + 4B \sum_{i=1}^{\nu} \cos \frac{\Lambda_5}{2} \cos \frac{\Lambda_5}{2} - s_i}.$

Lemma 5.6. The number $z = z_0$ not belonging to the continuous spectrum of the operator $H_8$ is an eigenvalue of that operator if and only if it is a zero of the function $\Delta_{\Lambda_5}^{\nu}(z)$, i.e., $\Delta_{\Lambda_5}^{\nu}(z_0) = 0$.

Proof. The proof of the statements in the lemma 5.6 is similar to the proof of lemma 3.2. \hfill \Box

We consider the one-dimensional case.

Theorem 5.7. At $\nu = 1$ and $U > 0$ and for all values of the parameter of the Hamiltonian, the operator $H_8$ has a unique eigenvalue $z = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_5}{2}}$ that is below the continuous spectrum of $H_8$, i.e., $z < m_{\Lambda_5}^{1}$.

Proof. In the one-dimensional case, the function $\Delta_{\Lambda_5}^{1}(z)$ decreases monotonically outside the continuous spectrum domain of the operator $H_5$. For $z < m_{\Lambda_5}^{1}$, the function $\Delta_{\Lambda_5}^{1}(z)$ decreases from 1 to $-\infty$,

$\Delta_{\Lambda_5}^{1}(z) \rightarrow 1$ as $z \rightarrow -\infty$

$\Delta_{\Lambda_5}^{1}(z) \rightarrow -\infty$ as $z \rightarrow m_{\Lambda_5}^{1} - 0$.

Therefore, below the value $m_{\Lambda_5}^{1}$, the function $\Delta_{\Lambda_5}^{1}(z)$ has a single zero at the point $z = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_5}{2}}$. For $z > M_{\Lambda_5}^{1}$, the function $\Delta_{\Lambda_5}^{1}(z)$ decreases from $+\infty$ to 1,

$\Delta_{\Lambda_5}^{1}(z) \rightarrow +\infty$ as $z \rightarrow M_{\Lambda_5}^{1} + 0$.

$\Delta_{\Lambda_5}^{1}(z) \rightarrow +1$ as $z \rightarrow +\infty$.

Therefore, above the value $M_{\Lambda_5}^{1}$, the function $\Delta_{\Lambda_5}^{1}(z)$ cannot vanish. \hfill \Box

We now consider the two-dimensional case.

Theorem 5.8. At $\nu = 2$ and $U > 0$ and for all values of the parameter of the Hamiltonian, the operator $H_8$ has a unique eigenvalue $z'$ that is below the continuous spectrum of $H_8$, i.e., $z' < m_{\Lambda_5}^{2}$.
Proof. In the two-dimensional case, the function $\Delta^2_{\Lambda_5}(z)$ decreases monotonically outside the continuous spectrum domain of $\tilde{H}_8$. For $z < m^2_{\Lambda_5}$, the function $\Delta^2_{\Lambda_5}(z)$ decreases from 1 to $-\infty$,

$$\Delta^2_{\Lambda_5}(z) \to 1 \quad \text{as} \quad z \to -\infty$$

$$\Delta^2_{\Lambda_5} \to -\infty \quad \text{as} \quad z \to m^2_{\Lambda_5} - 0.$$

Therefore, below $m^2_{\Lambda_5}$, the function $\Delta^2_{\Lambda_5}(z)$ has a single zero at the point $z' < m^2_{\Lambda_5}$. For $z > M^2_{\Lambda_5}$, the function $\Delta^2_{\Lambda_5}(z)$ decreases from $+\infty$ to 1,

$$\Delta^2_{\Lambda_5}(z) \to +\infty \quad \text{as} \quad z \to M^2_{\Lambda_5} + 0,$$

$$\Delta^2_{\Lambda_5}(z) \to 1 \quad \text{as} \quad z \to +\infty.$$  

Therefore, above $M^2_{\Lambda_5}$, the function $\Delta^2_{\Lambda_5}(z)$ cannot vanish. 

In the three-dimensional case, the operator $\tilde{H}_8$ for some values of the parameters in the Hamiltonian has a unique eigenvalue $z''$, such that $z'' < m^3_{\Lambda_5}$.

**Theorem 5.9.** At $\nu = 3$ and $U > 0$ and the total quasimomentum $\Lambda_5$ of the system have the form

$$\Lambda_5 = (\Lambda^1_5; \Lambda^2_5; \Lambda^3_5) = (\Lambda^0_5; \Lambda^0_3; \Lambda^0_3).$$

Then the operator $\tilde{H}_8$ has a unique eigenvalue $z''$ if

$$U > \frac{12B \cos \frac{\Lambda^0_5}{2}}{W}.$$  

Otherwise, the operator $\tilde{H}_8$ has no eigenvalue.

Proof. In the three-dimensional case,

$$\Delta^3_{\Lambda_5}(z) \to 1 \quad \text{as} \quad z \to -\infty$$

$$\Delta^3_{\Lambda_5}(z) \to 1 - \frac{UW}{12B \cos(\Lambda^3_5/2)} \quad \text{as} \quad z \to m^3_{\Lambda_5} - 0.$$

Hence, the function $\Delta^3_{\Lambda_5}(z)$ has a single zero at the point $z''$ below the continuous spectrum. Above the continuous spectrum,

$$\Delta^3_{\Lambda_5}(z) \to 1 + \frac{UW}{12B \cos(\Lambda^3_5/2)} \quad \text{as} \quad z \to M^3_{\Lambda_5} + 0,$$

$$\Delta^3_{\Lambda_5}(z) \to 1 \quad \text{as} \quad z \to +\infty.$$  

Inspecting the equation $\Delta^3_{\Lambda_5}(z) = 0$ above the continuous spectrum domain of the operator $\tilde{H}_8$, we obtain the statement of the theorem.  

Now, using the obtained results and representation (19), we describe the structure of the essential spectrum and the discrete spectrum of the operator $\tilde{H}_8$.

**Theorem 5.10.** At $\nu = 1$ and $U > 0$ the essential spectrum of the system third triplet-state operator $\tilde{H}_8^1$ is exactly the union of three segments: $\sigma_{\text{ess}}(\tilde{H}_8^1) = [U + 4A - 4B \cos \frac{\Lambda^1_5}{2} - 4B \cos \frac{\Lambda^1_5}{2}, U + 4A + 4B \cos \frac{\Lambda^1_5}{2} + 4B \cos \frac{\Lambda^1_5}{2}, U + 4A + 4B \cos \frac{\Lambda^1_5}{2} - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda^1_5}{2}}] \cup [U + 4A + 4B \cos \frac{\Lambda^1_5}{2} + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda^1_5}{2}}, U + 4A + 4B \cos^2 \frac{\Lambda^1_5}{2} + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda^1_5}{2}}]$. The discrete spectrum of the operator $\tilde{H}_8^1$ is consists no more than one point.

Proof. It follows from representation (19) that

$$\sigma(\tilde{H}_8^1) = \{\lambda + \mu + \theta : \lambda \in \sigma(\tilde{H}_6), \mu \in \sigma(\tilde{H}_7), \theta \in \sigma(\tilde{H}_8)\},$$
and in the one-dimensional case, the continuous spectrum of $\tilde{H}_6$ is

$$\sigma_{\text{cont}}(\tilde{H}_6) = [2A - 4B \cos \frac{\Lambda_6}{2}, 2A + 4B \cos \frac{\Lambda_6}{2}],$$

and the discrete spectrum of $\tilde{H}_6$ consists of a single point $z_1 = 2A + \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_6}{2}}$.

The spectra of the operator $\tilde{H}_7$ is a purely discrete and consists is a single point $z = U$, for arbitrary value $\nu$.

The continuous spectrum of the operator $\tilde{H}_8$ consists of the segment $[2A - 4B \cos \frac{\Lambda_8}{2}, 2A + 4B \cos \frac{\Lambda_8}{2}]$, and the discrete spectrum of $\tilde{H}_8$ consists of a single point $z_2 = 2A - \sqrt{U^2 + 16B^2 \cos^2 \frac{\Lambda_8}{2}}$. Therefore, the essential spectrum of the system third triplet-state operator $^3\tilde{H}_1$ is the union of three segments, and the third triplet-state operator $^3\tilde{H}_1$ has a single eigenvalue $z = z_1 + z_2 + U$. If these eigenvalue lie the outside the essential spectrum of operator $^3\tilde{H}_1$, then these eigenvalue is belonging the discrete spectrum of the operator $^3\tilde{H}_1$, otherwise, the discrete spectrum of operator $^3\tilde{H}_1$, is empty.

The following theorem is proved totally similarly to Theorem 5.10.

**Theorem 5.11.** At $\nu = 2$ and $U > 0$ the essential spectrum of the system third triplet-state operator $^3\tilde{H}_1$ is exactly the union of three segments: $\sigma_{\text{ess}}(^3\tilde{H}_1) = [U + 4A - 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2} - 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2}, U + 4A + 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2} + 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2}]$ or $[U + 2A + z - 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2}, U + 2A + z + 4B \sum_{i=1}^{2} \cos \frac{\Lambda_i}{2}]$. The discrete spectrum of the operator $^3\tilde{H}_1$ is consists no more than one point.

Let $\nu = 3$ and $\Lambda_5 = (\Lambda_5^0, \Lambda_5^1, \Lambda_5^2)$, and $\Lambda_6 = (\Lambda_6^0, \Lambda_6^1, \Lambda_6^2)$.

**Theorem 5.12.** The following statements hold:

1. Let

$$U > \frac{12B \cos \frac{\Lambda_6^0}{2}}{W}, \cos \frac{\Lambda_6^0}{2} \geq \cos \frac{\Lambda_6^0}{2}.$$

Then the essential spectrum of the system third triplet-state operator $^3\tilde{H}_1$ is the union of three segments $\sigma_{\text{ess}}(^3\tilde{H}_1) = [U + 4A - 12B \cos \frac{\Lambda_6^0}{2} - 12B \cos \frac{\Lambda_6^0}{2}, U + 4A + 12B \cos \frac{\Lambda_6^0}{2} + 12B \cos \frac{\Lambda_6^0}{2}] \cup [\tilde{z} + U + 2A + 12B \cos \frac{\Lambda_6^0}{2}, z'' + U + 2A + 12B \cos \frac{\Lambda_6^0}{2}].$ The discrete spectrum of the operator $^3\tilde{H}_1$ is consists no more than one point. Here and hereafter, $\tilde{z}$ is an eigenvalue of the operator $\tilde{H}_6$, and $z''$ is an eigenvalue of the operator $\tilde{H}_8$.

2. Let

$$\frac{12B \cos \frac{\Lambda_6^0}{2}}{W} < U \leq \frac{12B \cos \frac{\Lambda_6^0}{2}}{W}.$$

Then the essential spectrum of the system third triplet-state operator $^3\tilde{H}_1$ is the union of two segment $\sigma_{\text{ess}}(^3\tilde{H}_1) = [U + 4A - 12B \cos \frac{\Lambda_6^0}{2} - 12B \cos \frac{\Lambda_6^0}{2}, U + 4A + 12B \cos \frac{\Lambda_6^0}{2} + 12B \cos \frac{\Lambda_6^0}{2}] \cup [\tilde{z} + U + 2A + 12B \cos \frac{\Lambda_6^0}{2}, z'' + U + 2A + 12B \cos \frac{\Lambda_6^0}{2}],$ and the discrete spectrum of the system third triplet-state operator $^3\tilde{H}_1$ is empty.

3. Let

$$\frac{12B \cos \frac{\Lambda_6^0}{2}}{W} < U \leq \frac{12B \cos \frac{\Lambda_6^0}{2}}{W}.$$

Then the essential spectrum of the system third triplet-state operator $^3\tilde{H}_1$ is the union of two segment $\sigma_{\text{ess}}(^3\tilde{H}_1) = [U + 4A - 12B \cos \frac{\Lambda_6^0}{2} - 12B \cos \frac{\Lambda_6^0}{2}, U + 4A + 12B \cos \frac{\Lambda_6^0}{2} + 12B \cos \frac{\Lambda_6^0}{2}] \cup [z'' +
\[ U + 2A - 12B \cos \frac{\Lambda_0}{2}, z'' + U + 2A + 12B \cos \frac{\Lambda_0}{2}, \] and the discrete spectrum of the system second triplet-state operator \[ \tilde{\mathcal{H}}_1^2 \] is empty.

4. Let \[ U \leq \frac{12B \cos \frac{\Lambda_0}{2}}{W}, \cos \frac{\Lambda_0}{2} \leq \cos \frac{\Lambda_0}{2}. \]

Then the essential spectrum of the system third triplet-state operator \[ \tilde{\mathcal{H}}_1^3 \] is consists of single segment \[ \sigma_{\text{ess}}(\tilde{\mathcal{H}}_1^3) = [U + 4A - 12B \cos \frac{\Lambda_0}{2} - 12B \cos \frac{\Lambda_0}{2}, U + 4A + 12B \cos \frac{\Lambda_0}{2} + 12B \cos \frac{\Lambda_0}{2}], \]
and the discrete spectrum of the system second triplet-state operator \[ \tilde{\mathcal{H}}_1^2 \] is empty.

Proof. The proof uses representation (19) and the results for the spectra of \[ \tilde{\mathcal{H}}_6, \tilde{\mathcal{H}}_7, \] and \[ \tilde{\mathcal{H}}_8 \] in the case \( \nu = 3 \).

We have considered the case \( \cos \frac{\Lambda_0}{2} > 0 \), and \( \cos \frac{\Lambda_0}{2} > 0 \), The case \( \cos \frac{\Lambda_0}{2} < 0 \), and \( \cos \frac{\Lambda_0}{2} < 0 \), and other case is investigated similarly. Hence, in third triplet state, the four-electron system with \( U > 0 \) or \( U < 0 \) respectively has at most one antibound state or at most one bound state. The essential spectrum of the system is the union of at most three segments.

We see the spectra of these three triplet states are the different.

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