Modes of a stellar system I: ergodic systems

Jun Yan Lau\textsuperscript{1,2,⋆} & James Binney\textsuperscript{2}†
\textsuperscript{1} UCL Mullard Space Sciences Laboratory, Holmbury St Mary, Surrey RH5 6NT
\textsuperscript{2} Rudolf Peierls Centre for Theoretical Physics, Clarendon Laboratory, Parks Road, Oxford, OX1 3PU, UK

22 June 2021

ABSTRACT
The excursions of star clusters and galaxies around statistical equilibria are studied. For a stable ergodic model Antonov’s Hermitian operator on six-dimensional phase space has the normal modes as its eigenfunctions. The excitation energy of the system is just the sum of the (positive) energies associated with each normal mode. Formulae are given for the DFs of modes, which are of the type first described by van Kampen rather than Landau, and Landau ‘modes’ can be expressed as sums of van Kampen modes. Each van Kampen mode comprises the response of non-resonant stars to driving by the gravitational field of stars on a group of resonant tori, so its structure is sensitive to the degree of self gravity. The emergence of global distortions in N-body models when particles are started from an analytical equilibrium is explained in terms of the interplay of normal modes. The positivity of modal energies opens the way to modelling the thermal properties of clusters in close analogy with those of crystals.

Key words: Galaxy: kinematics and dynamics – galaxies: kinematics and dynamics – methods: analytical

1 INTRODUCTION
Galaxies and star clusters are in approximate states of equilibrium and have for decades been fitted to models in which the distribution function $f(x, v)$ of their constituent particles (stars, dark-matter particles) are steady-state solutions of the collisionless Boltzmann equation (CBE). The advent of massive simulations of galaxy formation (Laporte et al. 2019) and detailed data from the Gaia mission (Gaia Collaboration & Brown 2018) and large integral field units such as MUSE (e.g. Vitral & Mamon 2021) have stimulated interest in non-equilibrium features of galaxies, especially the Milky Way (Antoja et al. 2018).

For almost a century observations of galaxies and star clusters have been interpreted in terms of ‘mean-field’ models, that is to say models in which fluctuations have been averaged away. In the case of globular clusters the community has been aware since at least the pioneering work of Hénon (1961) that fluctuations drive secular evolution of the system towards higher central concentration and lower mass (core collapse and evaporation) but observations have nonetheless been fitted to mean-field models on the grounds that clusters evolve through a series of mean-field models.

Fluctuations in the surface brightnesses of early-type galaxies form the basis for a standard technique for estimating their distances (Tonry & Schneider 1988), but the fluctuations are computed by imposing shot noise on equilibrium models rather than using a dynamical theory of fluctuations.

Perhaps the most exciting single discovery made in the Gaia DR2 data is the phase spiral that Antoja et al. (2018) uncovered in the distribution of stars in the $(z, v_z)$ plane. The spiral is surely a symptom of a macroscopic oscillation of the disc that has a significant component in the $z$ direction. If we had a credible dynamical model of this oscillation, we would be able to extract from the Gaia data information about the structure of the disc and the agent [likely the Sagittarius dwarf galaxy (Binney & Schönrich 2018; Laporte et al. 2019; Bland-Hawthorn & Tepper-Garcia 2020)] that excited it. Unfortunately, the disc’s self-gravity certainly plays an important role in the oscillation, and there’s little prospect of adequately modelling the disc’s oscillation until we have a better understanding of the global oscillations of stellar systems. This is the first in a series of papers that lay the foundations for such understanding by setting up an adequate theory of the normal modes of stellar systems.

Normal modes (in quantum mechanics ‘stationary states’) owe their usefulness to three key properties: (i) they are complete in the sense that any initial condition can be expressed as a linear combination of normal modes; (ii) they have the trivial time dependence $e^{-i\omega t}$; (iii) they are mutually orthogonal, with the consequence that the energy of the whole system is simply the sum of the energies invested in each normal mode.

Modes of stellar systems have received significant at-
tension since the work of Toomre (1964), Lin & Shu (1964), and Kalnajs (1965). That work was motivated by the desire to understand spiral structure so focused on razor-thin, rotating stellar discs. Two decades later the focus switched to hot, spherical systems from a desire to understand how and when radial bias in the velocity dispersion caused systems to lose spherical symmetry (Palmer & Papaloizou 1987; Saha 1991; Weinberg 1991). The standard reference for this work is the two volumes of Fridman & Polyachenko (1984), and a glance at the contents pages make clear that interest focused exclusively on the search for unstable normal modes. We show below that these modes are qualitatively different from the modes required to investigate, as we do, the excursions that stable systems make around equilibrium.

Fluctuations may be externally or internally driven. The Antoja spiral in our Galaxy and shells around early-type systems (Malin & Carter 1980) are surely externally driven. The secular evolution of globular clusters is largely driven by fluctuations that are internally driven by Poisson noise [although fluctuations driven externally by tidal fields are also significant (Lee & Ostriker 1987)]. Even after more than a half century of work, there is no consensus as to whether observed spiral structure is sometimes internally driven (Sellwood & Masters 2022), although some ‘grand-design’ spiral structure (e.g., that of M51) is certainly externally driven. Whatever the driving mechanism, the natural way to model fluctuations is as solutions to the linearised Boltzmann equation (CBE) coupled to the already linear Poisson equation.

This time-translation invariant pair of linear equations may be expected to have a complete set of solutions with time dependence $e^{-i\omega t}$ (with potentially complex $\omega$). In this paper we derive these solutions for the important special case that the unperturbed system is ergodic – that is has a distribution function (DF) of the form $f_0(H)$, where

$$H(x, v) = \frac{1}{2}v^2 + \Phi(x)$$

is the Hamiltonian of a single particle moving in the gravitational potential $\Phi$. The second paper in the series generalises many results to the case of a DF of the form $f_0(H)$, where $J$ is the vector of the action integrals of stars moving in the unperturbed potential. In the third paper in the series we develop an apparatus for decomposing an arbitrary initial condition of a system with $f_0(J)$ into its constituent normal modes. The present paper relies heavily on an Hermitian operator that Antonov (1961) introduced. This operator does not generalise straightforwardly from ergodic systems to more general ones, so Paper II obtains a restricted range of results with a simpler but less powerful technique. The fourth paper in the series generalises Antonov’s operator to DFs of the form $f(J)$.

The plan of this paper is as follows. Section 2 introduces basic concepts and establishes notation. Section 3 introduces the Hermitian operator $K$ on phase space whose eigenfunctions are the required normal modes of the cluster. If the system is stable, all its modes are van Kampen modes; they have real frequencies drawn from a continuous spectrum. If the system is unstable the spectrum contains isolated pure imaginary frequencies. We show that the energies of modes are additive, and give a very simple expression for a mode’s energy in terms of its DF. This expression implies that the energy of van Kampen modes is positive and that of modes with imaginary frequencies vanishes. We show also that $K$ gives rise to a slightly different conserved quantity that provides a means to establish stability. In Section 4.1 we show that $K$ commutes with the angular-momentum operator $L_z$ before in Section 4.2 obtaining an expression for the DF of a van Kampen mode. This contains a free function and parameters that can be computed from the free function by matrix algebra. In Section 4.4 we investigate the way in which the structures of a van Kampen mode depends on the extent to which a system is self-gravitating. In Section 4.5 we re-express a van Kampen mode’s energy in terms of the free function and the potential that the mode generates, and in Section 4.6 we discuss the emergence of system-scale fluctuations in N-body simulations. In Section 5 we argue that van Kampen modes rest on conceptual foundations which are as solid as those that are generally accepted as secure in other branches of physics. In Section 6 we stress the importance of the concept of particle dressing in stellar dynamics as in other branches of physics, and discuss the role that van Kampen modes play in dressing. Section 6.2 discusses the relationship between van Kampen and Landau modes, while Section 6.3 considers the prospect for using van Kampen modes to extend conventional statistical mechanics to stellar systems, and for understanding the role of thermal fluctuations within them. Section 7 sums up.

2 MATHEMATICAL BACKGROUND

Here we introduce essential mathematical tools and establish our notation. We focus on stable ergodic clusters, that is systems with unperturbed DFs $f_0(H)$ where

$$H(x, v), \equiv \frac{1}{2}v^2 + \Phi(x)$$

is the Hamiltonian of a single particle in the gravitational potential $\Phi(x)$. A necessary and sufficient condition for such a system to be stable is that the derivative $f'_0 < 0$ at all energies (Antonov 1961).

2.1 Variable degree of self gravity

In the following it proves helpful to be able to consider self-gravity to be a variable $\xi$ that runs from zero (stars move in the fixed potential of a specified density distribution) to unity (stars experience only their gravitational attraction to the other stars). It is straightforward to set up a numerical experiment for any given value of $\xi$ by sampling an analytic density distribution in the usual way and taking the force on each star to be $\xi$ times the force returned by an N-body solver plus $(1 - \xi)$ times the force provided by the analytic density.

2.2 Angle-action variables

The role that Cartesian variables play for homogeneous systems is played for spheroidal systems by angle-action variables $(\theta, J)$. The actions $J_\theta$ are constants of motion while their conjugate variables, the angles $\theta$, increase linearly in time, so $\theta(t) = \theta(0) + \Omega t$. The particles’ Hamiltonian $H(x, v)$ is a function $H(J)$ of the actions only and the frequencies $\Omega$, that control the rates of increase of the angles.
are given by $\Omega = \partial H / \partial J$. Angle-action variables are canonical, so the volume element of phase space $d^3w = d^3x d^3v = d^3\theta d^3J$. Poisson brackets can be computed as

$$\{f, g\} = \sum_i \left( \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial J_i} - \frac{\partial f}{\partial J_i} \frac{\partial g}{\partial \theta_i} \right).$$

Functions on phase space can be expressed as Fourier series:

$$h(w) = \sum_n h_n(J) e^{i n \theta}; h_n(J) = \int \frac{d^3\theta}{(2\pi)^3} e^{-i n \theta} h(w).$$

Note that for real $h$, $h_{-n} = h_n^*$. 

2.3 Potential-density pairs

Unfortunately, while the potential $\Phi(x)$ is a function of only $x$, it becomes a function of both $\theta$ and $J$. So while angle-action variables make dynamics trivial, they seriously complicate the solution of Poisson’s equation. Following Kalnajs (1976) this difficulty is finessed by introducing a basis of biorthogonal potential-density pairs. That is, a set of pairs $(\rho^{(\alpha)}, \Phi^{(\alpha)})$ such that

$$4\pi G \rho^{(\alpha)} = \nabla^2 \Phi^{(\alpha)}$$

and

$$\int \alpha^3 \Phi^{(\alpha)} \rho^{(\alpha')} = -\mathcal{E} \delta_{\alpha\alpha'},$$

where $\mathcal{E}$ is an arbitrary constant with the dimensions of energy. Given a density distribution $\rho(x)$, we expand it in the basis

$$\rho(x) = \sum_{\alpha} A_\alpha \rho^{(\alpha)}(x) \quad \Leftrightarrow \quad \Phi(x) = \sum_{\alpha} A_\alpha \Phi^{(\alpha)}(x),$$

where

$$A_\alpha = -\frac{1}{\mathcal{E}} \int d^6w \Phi^{(\alpha)}(x) f(w).$$

If $\rho$ and $\Phi$ are time-dependent, the $A_\alpha$ become time-dependent. From equations (5) and (6) one can obtain an expression for $\Phi$ in terms of $\rho$. Comparison of this relation with Poisson’s integral, yields

$$\frac{G}{|x' - x|} = \frac{1}{\mathcal{E}} \sum_{\alpha} \Phi^{(\alpha)}(x) \Phi^{(\alpha*)}(x').$$

3 ANTONOV’S OPERATOR K

We now derive for stable ergodic clusters the Hermitian operator $K$ introduced by Antonov (1961). The true normal modes of the system are eigenfunctions of $K$ with non-negative eigenvalues that prove to be the squares of the modes’ (real) frequencies. Following Antonov (1961) we split the perturbed DF $f_1$ into parts that are even and odd in $v$:

$$f_1(x, v) = f_+(x, v) + f_-(x, v) \quad (9)$$

where

$$f_\pm(x, v) \equiv \frac{1}{2} [f_1(x, v) \pm f_1(x, -v)]. \quad (10)$$

In the absence of a perturbation, $f_-$ vanishes, so this part of the DF isolates the effect of the perturbation. On the other hand the perturbation changes the potential only through $f_+$.\footnote{In Dirac’s seminal textbook, he argues that the second-order Klein-Gordon equation cannot stand in for the Schrödinger equation, which is first-order in time. So he factorises the Klein-Gordon operator into two first-order operators, by splitting the wavefunction into four parts. Antonov proceeded in the opposite direction: by splitting the DF he derived two first-order operators and then combined them to obtain a second-order equation.}

$$\left< g|f \right> = - \int d^6w \frac{g^* f}{f_0'(H)}, \quad (11)$$

where $H(J)$ is the unperturbed Hamiltonian and the leading minus reflects the fact that $f_0' < 0$. Since $d^6w$ has dimensions of mass and $f/f_0'$ has dimensions of $v^2$, $\left< g|f \right>$ has the dimensions of $Mv^2$, i.e., energy. When we Fourier expand the DFs we find

$$\left< g|f \right> = - (2\pi)^3 \int \frac{d^3J}{f_0'} \sum_k g^*_k f_k. \quad (12)$$

Notice that

$$\left< f_1|f_1 \right> = \left< f_-|f_- \right> + \left< f_+|f_+ \right>. \quad (13)$$

$H(J)$ is even in $v$ and the Poisson bracket operator is odd in $v$, so our division (9) of $f$ splits the linearised CBE, $\partial^2 f_1 + [f_1, H] + [f_0, \Phi_1] = 0$, into two equations

$$\frac{\partial^2 f_+}{\partial t^2} = -[f_-, H]; \quad \frac{\partial^2 f_-}{\partial t^2} = -[f_+, H] + [\Phi_1, f_0]. \quad (14)$$

Now $[\Phi_1, f_0] = f_0'(H)[\Phi_1, H]$, so

$$\frac{\partial}{\partial t} [\Phi_1, f_0](w) = f_0'(H) \left[ \frac{\partial \Phi_1}{\partial t}, H \right]$$

$$= -f_0'(H) \left[ \int d^6w' \frac{\xi G}{|x' - x|} \frac{\partial f_+(w')}{\partial t}, H(w) \right].$$

We differentiate the second of equations (14) wrt $t$ and use the first equation to eliminate $\partial f_+$ from the rhs to obtain

$$\frac{\partial^2 f_-}{\partial t^2} = [[f_-, H], H]$$

$$+ f_0'(H) \left[ \int d^6w' \frac{\xi G}{|x' - x|} [f_-(w'), H(w')], H(w) \right].$$

This equation is of the form

$$\frac{\partial^2 f_-}{\partial t^2} = -K f_-, \quad (17)$$

where the operator

$$K \equiv -[[f_-, H], H]$$

$$- f_0'(H) \left[ \int d^6w' \frac{\xi G}{|x' - x|} [f_-(w'), H(w')], H(w) \right].$$

In terms of angle-action coordinates $(\theta, J)$,

$$[f(\theta, J), H(J)] = \Omega \cdot \frac{\partial f}{\partial \theta}. \quad (19)$$

so $K$ can be written

$$K = - \left( \Omega \cdot \frac{\partial}{\partial \theta} \right)^2 f_-$$

$$- f_0'(H) \Omega \cdot \frac{\partial}{\partial \theta} \int d^6w' \frac{\xi G}{|x' - x|} \Omega' \cdot \frac{\partial f_-(w')}{\partial \theta}. \quad (20)$$
Inserting equation (8) into (20) yields

\[
K = - \left( \Omega \cdot \frac{\partial}{\partial \theta} \right)^2 f_+ - f_0^\dagger (H) \xi \frac{\partial}{\partial \theta} \sum_\alpha \Phi^{(\alpha)}(x) \Phi^{(\alpha)*}(x') \Omega' \cdot \frac{\partial f_-(w)}{\partial \theta}. \tag{21}
\]

At this point it's convenient to define

\[
\alpha f_1(t) = - \frac{i}{\xi} \int d^6w \Phi^{(\alpha)*}(x) \Omega' \frac{\partial f_-(w)}{\partial \theta}, \tag{22}
\]

because it allows us to write equation (21) in the form

\[
K = - \left( \Omega \cdot \frac{\partial}{\partial \theta} \right)^2 f_- - i \xi f_0^\dagger (H) \Omega' \sum_\alpha \Phi^{(\alpha)}(x) j_\alpha \alpha f_-. \tag{23}
\]

\(j_\alpha\) is a functional of \(f_1\) rather than just \(f_-\) because it can also be computed from \(f_+\): eliminating \(f_-\) using the first of equations (14) and equation (19) we obtain

\[
j_\alpha f_1(t) = - \frac{i}{\xi} \int d^6w \Phi^{(\alpha)*}(x) \frac{\partial f_+(w)}{\partial t}. \tag{24}
\]

Since by equation (7) the coefficient \(A_\alpha\) of the potential/density expansion is a linear functional of \(f_1\), and \(A[f_-] = 0\), the derivative of \(f_\sigma\) in equation (24) can be replaced by a derivative of \(A_\alpha\) to yield

\[
j_\alpha f_1(t) = - \frac{i}{\xi} \frac{\partial A_\alpha}{\partial t}. \tag{25}
\]

### 3.1 Energy of a disturbance

When we use equation (8) to eliminate \(|x - x'|\) from equation (5.130) of Binney & Tremaine (2008), we find that the energy associated with a linearised disturbance is

\[
E[f_1] = \frac{1}{2} \left\{ \int d^6w \left( \frac{\partial}{\partial \theta} f_- - \frac{\xi}{\xi} \sum_\alpha \int d^6w \Phi^{(\alpha)}(x) f_{\sigma}(w) \right)^2 \right\}

= \frac{1}{2} \left\{ \langle f_- | f_+ \rangle + \langle f_+ | f_- \rangle - \xi \sum_\alpha |A_\alpha[f_\sigma]|^2 \right\}. \tag{26}
\]

In the first line of this equation, the first and second terms on the right quantify the potential and kinetic energies of the perturbation, respectively. Hence the inner product \(\langle f_- | f_1 \rangle\) gives twice a perturbation’s kinetic energy. In the last term, \(A_\alpha\) is independent of the degree of self-gravity \(\xi\), so the final term is proportional to \(\xi\) as it should be.

In Appendix A we show that \(K\) is Hermitian and that

\[
\langle f_- | K | f_- \rangle = \langle f_\sigma | f_\sigma \rangle - \xi \sum_\alpha |A_\alpha[f_\sigma]|^2. \tag{27}
\]

When we use this equation to simplify equation (26), we discover the energy associated with an eigenfunction of \(K\) identically vanishes in the unstable case \(\omega^2 < 0\), and in the stable case is

\[
E[f_1] = \langle f_- | f_- \rangle = - (2\pi)^3 \int \frac{d^3k}{k_0} |f_-|_k^2, \tag{28}
\]

where \(f_-\) denotes the \(k\) component of \(f_-\). Remarkably, the degree of self-gravity \(\xi\) does not appear in equation (28). Changes to \(\xi\) do affect \(E\), however, by changing the \(f_-\).

The right side of equation (28) is inherently positive and vanishes only if \(f_-\) vanishes. Moreover, by equation (14) \(\omega f_+ = k \cdot \Omega f_-\), so \(f_+\) must also vanish if \(f_-\) does. Hence the energy of every mode of a stable ergodic system is positive.

Since \(K\) is Hermitian it has a complete set of orthogonal eigenfunctions. Expressing an arbitrary disturbance as a linear combination of eigenfunctions, 

\[
f_- = \sum_\alpha \xi_\alpha f_\alpha^{(\beta)}
\]

and inserting this expansion into equation (28), we conclude that the disturbance’s energy is the sum of the energies of its component eigenfunctions

\[
E[f_1] = \sum_\alpha |\xi_\alpha|^2 |f_\alpha^{(\beta)}|^2. \tag{29}
\]

In view of these results it is natural to identify the true modes of an ergodic system with the eigenfunctions of \(K\). It follows that the frequencies of true modes are either real or pure imaginary. No true mode of an ergodic system has negative energy; oscillatory modes have positive energy and growing/decaying modes (if any) have zero energy. Equation (26), from which we started, does not make evident the non-negativity of energies.

### 3.2 Antonov’s conserved quantity

The rather involved argument just given starts from equation (26) for the energy of a disturbance, which Binney & Tremaine (2008) derive by considering the work that must be done to initiate the disturbance. A much simpler argument based on the Hermitian nature of \(K\) yields the closely related conserved quantity,

\[
\tilde{E} = \frac{1}{2} \left\{ \langle f_- | f_- \rangle + \langle f_- | K | f_- \rangle \right\}. \tag{30}
\]

Indeed,

\[
\frac{d}{dt} \tilde{E} = \langle f_- | f_- \rangle + \langle f_- | f_- \rangle + \langle f_- | K | f_- \rangle + \langle f_- | K | f_- \rangle = 0. \tag{31}
\]

An eigenfunction of \(K\) with eigenvalue \(\omega^2\) has \(\tilde{E} = \omega^2 \langle f_- | f_- \rangle\), so in this case conservation of \(\tilde{E}\) implies conservation of \(E = \tilde{E}/\omega^2\). \(\tilde{E}\) does not have the dimensions of energy, however.

Since \(\langle f_- | f_- \rangle > 0\), instability, and thus systematic growth of \(f_-\), is excluded by conservation of \(\tilde{E}\) unless \(\langle f_- | K | f_- \rangle < 0\) for some function \(f_- (x, v)\). In fact positivity of \(\langle f_- | K | f_- \rangle\) for all functions in the natural space is a necessary and sufficient condition for stability (Laval et al. 1965; Kulsrud & Mark 1970).

### 4 DFS OF VAN KAMPEN MODES

If the system is stable, the Hamiltonian’s time-reversal invariance ensures that \(\omega\) is real because the existence of exponentially decaying solutions would imply the existence of growing solutions. van Kampen (1955) applied these arguments to an electrostatic plasma and deduced some properties of the normal modes of a plasma, which are known as van Kampen modes. The corresponding modes of a stellar system have received little attention, although Vandervoort (2003) derived some of their properties. We now examine the van Kampen modes of a stellar system in their role as eigenfunctions of the operator \(K\).
4.1 Operators that commute with \( K \)

Finding the eigenfunctions of \( K \) is facilitated by identifying operators that commute with \( K \) and seeking eigenfunctions of \( K \) that are also eigenfunctions of these operators. \( K \) is the operator associated with the time-translation invariance of the underlying equilibrium. The Hamiltonian \( H \) is invariant under increments in the angle variables \( \theta_i \) but in the presence of self gravity (\( \xi > 0 \)), \( K \) does not share the invariance with respect to increments in \( \theta_1 \) and \( \theta_2 \) conjugate to the radial action \( J_r \) and the modulus \( L \equiv |L| \) of the angular momentum vector \( \mathbf{L} \) because \( \Phi_\perp(\mathbf{x}) \) lacks this invariance. Fortunately, \( K \) is always invariant under increments of the angle variable \( \theta_3 \) conjugate to \( L_z \). To see this it is best to return to the definition of \( K \) in equation (18). Since \([H, L_z] = 0\), we have

\[
[K f_-, L_z] = -[[[f_-, L_z], H], H] - f_0(H) \times \left[ \int d^6w' \left[ \frac{\xi G}{|x' - x|} L_z \right] [f_-(w'), H(w')], H(w') \right].
\]

(32)

The operator \([., L_z] \) rotates the orbital plane on which \( x \) lies (by incrementing the ascending node \( \Omega \)). The operator \([., L_z + L'_z] \), where \( L'_z \) operates on \( x' \), rotates \( x \) and \( x' \) through the same angle, so

\[
[[x' - x], L_z + L'_z] = 0.
\]

(33)

Hence taking advantage of the fact that for any \( f, g, h \),

\[
\int d^6w [f, g]h = \int d^6w [f, g] h
\]

we have

\[
\int d^6w' \left[ \frac{\xi G}{|x' - x|} L_z \right] [f_-(w'), H(w')]
= - \int d^6w' \left[ \frac{\xi G}{|x' - x|} L'_z \right] [f_-(w'), H(w')]
= \int d^6w' \left[ \frac{\xi G}{|x' - x|} \right] [f_-(w'), H(w')], L'_z
= \int d^6w' \left[ \frac{\xi G}{|x' - x|} \right] [f_-(w'), L'_z], H(w')
\]

(34)

When this result is used in equation (32), we obtain

\[
[K f_-, L_z] = K[f_-, L_z],
\]

(35)

so \( K \) commutes with the operator \([., L_z] \). From these commutations it follows that the eigenfunctions of \( A \) provide representations of the group of translations around tori that is generated by \([., L_z] \). This compact Abelian group has only one-dimensional irreps, which can be reduced to multiplication by \( e^{i\alpha} \). Hence the eigenfunctions of \( A \) can be indexed by the integer \( n_3 \) associated with \( \theta_3 \).

4.2 Derivation of the DF

Bearing in mind that for an eigenfunction equation (25) yields \( j_\alpha(f_1) = -\omega A_\alpha \), from equation (23) we have for an eigenfunction

\[
-\omega^2 f_+ = \left( \Omega \cdot \frac{\partial}{\partial \theta} \right)^2 f_- - i\xi \omega f_0(H) \sum_\alpha A_\alpha \Omega \frac{\partial}{\partial \theta} \Phi_\alpha(x). \quad (36)
\]

Now we apply the derivative \( \Omega \cdot \frac{\partial}{\partial \theta} \) to both sides and use equations (14) and (19) to eliminate \( f_- \) in favour of \( f_+ \). Collecting terms with \( f_+ \) on the left we then have

\[
\left( \omega^2 + \left( \Omega \cdot \frac{\partial}{\partial \theta} \right)^2 \right) f_+ = \xi f_0(H) \sum_\alpha A_\alpha \left( \Omega \cdot \frac{\partial}{\partial \theta} \right)^2 \Phi_\alpha(x). \quad (37)
\]

Fourier decomposed in angle variables this becomes

\[
\{ \omega^2 - (n \cdot \Omega)^2 \} f_{n+} = -\xi f_0(H)(n \cdot \Omega)^2 \sum_\alpha A_\alpha \Phi_\alpha^{(n)}.
\]

(38)

This equation is analogous to the standard equation for the Laplace-transformed DF \( \bar{T} \equiv \int d^6w \, f_0^{(t)} e^{i\omega t} \) with \( 3(\omega) > 0 \),

\[
i(n \cdot \Omega - \omega) \bar{T}_n(J, \omega) = -f_0(H) n \Omega \sum_\alpha A_\alpha(\omega) \Phi_\alpha^{(n)} + f_0(J, 0) \quad (39)
\]

in that it relates the DF to the driving potential, but there are two significant differences:

1. The rhs of equation (38) doesn’t contain an initial condition analogous to \( f_0(J, 0) \) on the rhs of (39). It’s not there because we are seeking a normal mode rather than the solution to an initial-value problem.

2. Equation (38) starts with a factor \( \omega^2 - (n \cdot \Omega)^2 \) while equation (39) starts with \( i(n \cdot \Omega - \omega) \).

Before we divide equation (38) by \( \omega^2 - (n \cdot \Omega)^2 \), we must recognise that when it vanishes, which for a range of frequencies it will over 2d resonant surfaces in action space, \( f_{n+} \) is unconstrained. This being so, after we’ve divided by \( \omega^2 - (n \cdot \Omega)^2 \), we should add a function that’s non-zero only on resonant surfaces. Then we have

\[
f_{n+}(J) = \xi f_0(H) \frac{(n \cdot \Omega)^2}{\omega^2 - (n \cdot \Omega)^2} \sum_\alpha A_\alpha \Phi_\alpha^{(n)}(J)
+ g_n(J) \delta(\omega^2 - (n \cdot \Omega)^2),
\]

(40)

where \( g_n \) is an arbitrary function. Adding this term makes it possible for \( f_{n+} \) to take whatever value \( g_n \) specifies on the resonant surfaces. (The values taken by \( g_n \) off resonant surfaces are immaterial.) Multiplying equation (40) by \( \int d^6w e^{i\alpha \cdot \theta} \Phi_\alpha^{(n)}(x) \) and summing over \( n \), we get

\[
\mathcal{E} A_{\alpha'} = \int d^3J d^3\theta \sum_\alpha \xi f_0(H) \frac{(n \cdot \Omega)^2}{\omega^2 - (n \cdot \Omega)^2} \sum_\alpha A_\alpha \Phi_\alpha^{(n)}(J)
+ g_n(J) \delta(\omega^2 - (n \cdot \Omega)^2)
= (2\pi)^3 \int d^3J d^3\theta \sum_\alpha \left\{ \xi f_0(H) \frac{(n \cdot \Omega)^2}{\omega^2 - (n \cdot \Omega)^2} \sum_\alpha A_\alpha \Phi_\alpha^{(n)} \right\}
+ \Phi_\alpha^{(n)} g_n(J) \delta(\omega^2 - (n \cdot \Omega)^2),
\]

(41)

where the integral over \( J \) is principal in the sense that actions at which \( n \cdot \Omega = \pm \omega \) are to be excluded and the
large values of the integrand as such points are approached largely cancel during integration. Equation (41) has the form
\[
\sum_{\alpha} M_{\alpha',\alpha} A_{\alpha} = -B_{\alpha'},
\]  
where
\[
M_{\alpha',\alpha}(\omega) = -\frac{(2\pi)^3}{\varepsilon} P \int d^3 J \sum_{n} \omega^2 (n \cdot \Omega)^2 \Phi_\alpha^{(\alpha')*} \Phi_\alpha^{(\alpha)} - \left( \frac{n \cdot \Omega}{\omega} \right)^2 M_{\alpha',\alpha}(\omega).
\]

We shall call modes with real frequencies and non-zero \( g \) and \( \omega \) van Kampen modes. Our recognition that there can be non-trivial distribution of stars on resonant tori gives rise to non-vanishing \( B_{\alpha} \), and therefore gives rise to the dynamics of all stars, including non-resonant stars. The dispersion relation is the condition for \( |M(\omega)| \) to vanish.

4.3 Relation to Landau modes

In a conventional normal-mode analysis we derive an equation \( \dot{M} = 0 \) which is homogeneous in the disturbance's amplitude \( a \) with the consequence that non-trivial solutions exist only when the determinant of the matrix \( M \) vanishes. The dispersion relation is the condition for \( |M(\omega)| \) to vanish. Our recognition that there can be non-trivial distributions of stars on resonant tori gives rise to non-vanishing \( B_{\alpha} \), and thus gives rise to non-trivial solutions associated with the true modes of a stellar system. In general there will be complex frequencies \( \omega_0 \) at which \( |M(\omega_0)| = 0 \), but unless \( \omega_0 \) lies on an axis of the complex plane, it cannot be the frequency of a true mode because all true modes have real \( \omega^2 \).

Landau modes occur at the frequencies \( \omega_0 \) at which a matrix \( M(\omega) \) has vanishing determinant. The Landau matrix \( M \) is closely related to \( M_0 \), and if \( |M(\omega_0)| = 0 \) then \( |M(\pm \omega_0)| = 0 \) also. Thus every Landau mode is associated with the possibility of solving equation (42) with \( B_{\alpha} = 0 \). Yet when \( \omega_0 \) lies on neither axis of the complex plane, such a solution should not be included in the set of true modes for two reasons:

i) The solution cannot be an eigenfunction of \( K \) because \( \omega_0^2 \) is not real, so it falls outside the complete set formed by the true modes. (It follows that it can be written as a sum of the true modes.)

ii) The solution associated with one of \( \pm \omega_0 \) will grow exponentially, so every system would be unstable if these solutions were included in the complete set of true modes.

If a system is stable, the frequencies of its Landau modes all lie below the real axis. In Section 4.2 we used this fact to argue that given any real \( \omega \), equation (42) has a unique solution for \( A_{\alpha} \) given \( B_{\alpha} \). Time-reversal symmetry is responsible for \( |M| \) vanishing above the real axis whenever it vanishes below the real axis and the failure of the determinant of the Landau matrix \( M_0 \) to behave in the same way is a consequence of the the violation of time-reversal symmetry inherent in the initial-value problem that leads to \( M \).

4.4 Modes and dressing

To understand the physical reality that underlies this mathematics, consider that in the absence of self gravity (\( \xi = 0 \)), a non-trivial distribution of stars with respect to \( \theta \) on resonant surfaces will generate oscillations in the density at frequency \( \omega \) that will persist for ever. The \( \delta \)-function component of the DF (40) of a van Kampen mode represents this phenomenon, and the \( B_{\alpha} \) quantify the spatial form of this driving structure. When \( \xi > 0 \), these oscillations affect the dynamics of all stars, including non-resonant stars. The regular part of the mode’s DF (40) describes these sympathetic oscillations of non-resonant stars. The \( A_{\alpha} \) quantify the spatial structure of this “dressed” response to the driver \( g \).

The functions \( g_n \) are arbitrary, so we may consider the case in which \( g_n \) vanishes for all vectors but one, \( N \). Consider now the effect of multiplying equation (40) by \( \int d^6 w e^{i n \cdot \theta \Phi^{(\alpha')*}_\alpha (x)} \) as in the derivation of equation (41).
but now not summing over \( n \). Then we have
\[
C_{n'} = (2\pi)^2 \int d^3 J \Phi_n^{(\alpha)} \{ \xi f_0 \left( \frac{\langle n \cdot \Omega \rangle^2}{\omega^2 - (n \cdot \Omega)^2} \right) \sum \alpha \hat{A}_{\alpha} \Phi_n^{(\alpha)} + g_n(J) \delta(\omega^2 - (n \cdot \Omega)^2) \},
\]
(47)
where
\[
C_{\omega n} \approx - \int d^3 w \Phi_n^{(\alpha)}(w) e^{i n \cdot \theta} f_{n+},
\]
(48)
so
\[
A_{\alpha} = \frac{1}{\varepsilon} \sum_n C_{n\alpha}
\]
(49)
When \( n \neq N \), the right side of equation (47) has only the term proportional to \( \xi \), so would vanish with \( \xi \). That is, without self-gravity, \( C_{n\alpha} \neq 0 \) only for \( n = N \); in this case the van Kampen modes could be labelled by \( N \). In the presence of self-gravity, we have no reason to expect \( C_{n\alpha} \) to vanish for \( n \neq N \) because equation (49) includes a contribution to \( A_{\alpha} \) from \( C_{N\alpha} \). Self-gravity has this impact because it stops \( K \) commuting with the operators \([\cdot, J]\) and \([\cdot, L]\) as discussed above. From the fact that \( K \) does commute with the operator \([\cdot, L]\) it follows that even when \( \xi > 0 \), \( A_{n\alpha} = 0 \) unless \( n_3 = N_3 \).

When a Landau mode is weakly damped, \(|\mathcal{M}| = 0 \) at a frequency \( \omega_0 \) that lies just below the real axis, and in consequence \( |\mathcal{M}| \) is small on the real axis just above this zero. In view of equation (42), \(|\mathcal{M}|/|\mathcal{B}|\) will be large in these circumstances. That is, van Kampen modes with frequencies close to \( \Re(\omega_0) \) are “heavily dressed”. This is the true significance of Landau modes. Put differently, while van Kampen modes exist at any \( \omega \), they have a bigger footprint in action space at frequencies that lie close to those of Landau modes.

Whereas a gas ball has at most a finite number of normal modes at a countable number of frequencies, a cluster has an infinite number of normal modes at every frequency. This difference is a consequence of the likely completeness of normal modes in each system (Section 5) and the fact that much more information is required to specify the DF of a cluster than the state of a gas ball: the disturbance has adiabatically deformed the latter from its equilibrium, so the velocity distribution remains Maxwellian and one only has to specify a mean and dispersion at each location \( x \). In a cluster we need to specify the DF in six-dimensional phase space.

Whereas the Hermiticity of \( K \) ensures that van Kampen modes for different \( \omega \) are orthogonal, it falls to us to select from all modes for any given \( \omega \) a complete set of mutually orthogonal modes. That is, to select a set of functions \( \{ g_n \} \) that generate modes that are orthogonal in the sense \( \langle g_n g_m \rangle = \delta_{nm} \). Since \( K \) commutes with \([\cdot, L]\), we can require modes to be eigenfunctions of this Hermitian operator, and identification of a complete set of modes is reduced to finding an orthogonal set of vectors whose components are indexed by \( n_1 \) and \( n_2 \). Rather than solving this problem, in Paper III we show how an arbitrary state of a stellar system can be decomposed into its constituent modes. That is, to express a given perturbation at time zero, \( F(w, 0) \), in the form
\[
F(w, 0) = \int d\omega f(w, \omega),
\]
(50)
where \( f(w, \omega) \) is a van Kampen mode with frequency \( \omega \). This done, the state of the perturbation at any other time can be obtained as
\[
F(w, t) = \int d\omega f(w, \omega) e^{-i\omega t}.
\]
(51)
These integrals over \( \omega \), eliminate the principal-value and Dirac \( \delta \)-function symbols in equation (40) for \( f \).

4.5 Energy of a mode

Equation (28) says that the energy of a mode is just the norm of the odd part of its DF, and equation (40) gives the even part of a mode’s DF from which the odd part follows trivially. So the natural next step is to substitute from the second of these equations into the first and express the mode’s energy in terms of its parameters \( g_n \) and its potential \( \Phi[J] \). This exercise proves long, and is made intricate by the singular denominator \( \omega^2 - (n \cdot \Omega)^2 \) in equation (40) for the DF.

In Appendix B we compute \( \langle f \, | \tilde{f} \rangle \) for modes with frequencies \( \omega \) and \( \tilde{\omega} \). The result is
\[
\langle f \, | \tilde{f} \rangle = -(2\pi)^3 \delta(\omega^2 - \tilde{\omega}^2) \int d^3 J \sum_{n_1, n_2} \left( \pi^2 \omega^2 f_0^2 \Phi_n[f] \Phi_n[\tilde{f}] + \frac{g_n^2}{f_0^2} \right) \delta((n \cdot \Omega)^2 - \tilde{\omega}^2).
\]
(52)
A pleasing feature of this expression is that it has no reference to potential/density basis functions. The factor \( \delta(\omega^2 - \tilde{\omega}^2) \) on the rhs reflects the orthogonality of modes of different frequencies that follows from the Hermiticity of \( K \). A remarkable feature of equations (52) is that its action-space integral is confined to the resonant tori, even though the modes very much involve non-resonant stars.

When we set \( f = \tilde{f} \), the inner product is divergent for finite \( g \) because finite \( g \) generates an \( f_0 \) that diverges as the resonant surface in action space is approached, and the divergences on opposite sides of the surface do not cancel because energy density is proportional to \( f^2 \). This result signals that we can have only an infinitesimal number of stars on any resonant surface. When use equation (52) to compute the energy of a physical disturbance (50), we find
\[
E[F] = \langle F \, | \tilde{F} \rangle = \left( \int d\omega f_\omega(\omega) \int d\omega' f_{\omega'}(\omega') \right)
= \int d\omega d\omega' \langle f(\omega) | f(\omega') \rangle
= -(2\pi)^3 \int \frac{d\omega}{2\omega} \int d^3 J \left( \pi^2 \omega^2 f_0^2 |\Phi_n[f]|^2 + \frac{|g_n|^2}{f_0^2} \right) \times \delta((n \cdot \Omega)^2 - \omega^2)
= -(2\pi)^3 \int \frac{d^3 J}{4} \left( \pi^2 |n \cdot \Omega|^2 f_0^2 |\Phi_n[f]|^2 + \frac{|g_n|^2}{(n \cdot \Omega)^2 f_0^2} \right).
\]
(53)
There is a striking similarity between the first term in the

4 In equation (52) \( n_3 \) has the same, fixed value for both \( f \) and \( \tilde{f} \).

5 The appearance of \( \omega^2 \) rather than \( \omega \) reflects time-reversal symmetry. From a practical perspective, it ensures that real time dependence (\( \cos \omega t \)) is possible.
expression (53) for $E$ and the expression $\rho_E = \frac{1}{2} \omega^2 \epsilon_0 A^2$ for the energy density contributed by an electromagnetic wave with vector-potential amplitude $A$. In the electromagnetic case one factor of $\omega$ arises from the quantisation condition $E = \hbar \omega$ and the other arises from the canonical momentum $\omega \mathbf{A}$ of the field $\mathbf{A}$. The second term in equation (53) has a different structure, however, and this term is arguably more important than the first because $\Phi[f]$ is driven by $g$.

A natural question to ask is why the coefficient of $[\Phi_n[f]]^2$ in equation (53) is positive, given that gravitational potential energy is inherently negative. The explanation must be that this term encapsulates all the energy, kinetic as well as potential, that’s tied up in the disturbance in non-resonant stars that is excited by the resonant stars. In the absence of self-gravity, the non-resonant stars are not disturbed, so this contribution to the energy vanishes with $\Phi[f]$.

The second term in the integrand of equation (53) is ultimately limited by Poisson noise and can be considered a given, while the first term depends on the system’s dynamics. The ratio of the two terms is proportional to $|<\mathbf{n}, \mathbf{\Omega}| f|^2|$, so the relative contributions to $E$ from the resonant driver $g$ and the non-resonant response $\Phi$ are sensitive to this factor. In principle $|\mathbf{n}, \mathbf{\Omega}|$ can be made as small as we please at given $\mathbf{J}$, but only by going to large $|\mathbf{n}|$, and at large $|\mathbf{n}|$ the projection of $g_n(\mathbf{J})$ into real space (measured by $B_{\alpha}$) will be small and thus the response (measured by $A_{\alpha}$ and $\Phi[f]$) will be small. Hence the first term in the integrand of equation (53) will be significant only for small $|\mathbf{n}|$. The fundamental dipole mode most obviously satisfies this criterion.

This consideration draws attention to short vectors $\mathbf{n}$ that make $|\mathbf{n}, \mathbf{\Omega}|$ small (if it is small on any torus, it will be small on many tori) because for these vectors the noise component $g_n$ generates a response at the least cost in energy, so the response is likely to be large. We saw above, moreover, that the response is enhanced when $|\mathbf{n}, \mathbf{\Omega}|$ is close to the real part of the frequency of a weakly damped Landau mode, because then $|M|$ is small. The fundamental dipole mode has been shown to be weakly damped in typical models (Weinberg 1994; Saha 1991; Hamilton et al. 2018).

### 4.6 Initialisation of N-body models

Suppose we set up a cluster by randomly sampling an analytic DF. When the selection is complete, the actual DF will differ from the analytic DF by virtue of Poisson noise, so the $g_n$ will be non-zero and the cluster’s van Kampen modes will be excited. The coefficients $B_{\alpha}$ that quantify the noisiness of the density distribution are unambiguously fixed by the Monte-Carlo selection, but the potential that are generated from them $\xi_n^{\alpha} = P^{(\alpha)} \mathcal{M}^{-1}(\xi, \omega) B_{\alpha}$ will vary with the degree of self-gravity $\xi$. Hence the DFs of the modes that sum to the sampled phase-space distribution will depend on $\xi$, but their sum must produce the DF sampled regardless of $\xi$. When $\xi = 1$, the modes are heavily dressed and yet produce the same small (Poisson) fluctuations in density as in the case $\xi = 0$ of vanishing self-gravity because the contributions of different modes cancel to a considerable extent. This cancellation is particularly pronounced in the case of low-order modes (small $|\mathbf{n}|$), and it occurs because when $\xi = 1$ the phases of modes are correlated, whereas when $\xi = 0$ they are probably uncorrelated.

Once we start moving the stars with $\xi$ set to unity, the phase differences between modes with different frequencies will tend towards uniform distribution in $(0, 2\pi)$ and cancellations between perturbations to the density will diminish. Consequently, the heavily dressed individual modes will become manifest and the system will become less spherical as Lau & Binney (2019) found empirically. By contrast, when stars are moved in the analytic potential, the initially uniformly distributed phases of the modes obtained with $\xi = 0$ remain uniformly distributed and no significant change in the density fluctuations will be observed.

The larger the value of $\xi$, the larger will be the values of $A_{\alpha}$ that are generated by the given $B_{\alpha}$, so the greater will be the departures from spherical symmetry once the phases of modes have decorrelated.

To obtain a self-consistent realisation of a self-gravitating system one needs to excite the modes for $\xi = 1$ with random phases, and it is not clear how this can be done without computing the system’s modes.

### 5 Completeness of modes

We have defined the true modes of a stellar system to be the eigenfunctions of Antonov’s Hermitian operator $K$. In quantum mechanics it is conventional to assume that the eigenfunctions of any Hermitian operator form a complete set although proof of completeness requires the operator to be bounded (e.g. Dieudonné 1969, §11.5), which some operators of physical interest are not. Similarly, much of condensed-matter physics relies on Bloch’s theorem that there is a complete set of stationary states for an electron in a crystal that have wavefunctions of the form $\psi(x) = e^{iKx}u(x)$ with $u(x+a) = u(x)$ for any lattice vector $a$. The standard derivation of Bloch’s theorem (e.g. Elliott & Dawber 1989, §14.4) starts from the observation that if $\psi(x)$ is a stationary state, then so is $\psi(x+a)$. Hence the stationary states of a given energy provide a representation of the Abelian translation group. Such groups only have one-dimensional irreducible representations, so the action of the group can be reduced to multiplication by $e^{iKx}$. That is, any functions providing a representation of the translation group can be reduced to functions satisfying $\psi(x+a) = e^{iKx}\psi(x)$, a relation that is clearly satisfied by $\psi(x) = e^{iKx}u(x)$.

Analogously, we might argue that the time-translation invariance of $\partial_t^2 + K$ implies that if $f(w,t)$ satisfies $(\partial_t^2 + K)f = 0$, then so does $f(w,t+\tau)$ and it follows that any set of solutions can be reduced to ones that satisfy $f(w,t+\tau) = e^{-i\omega\tau}f(w,t)$. Then setting $t$ to zero we infer that solutions of the form $f(w,\tau) = e^{-i\omega\tau}f(w)$ are complete.

The above arguments for the completeness of Bloch waves and eigenfunctions of $A$ are open to the objection that the theorem regarding the decomposition of representations into irreducible representations requires the group to be compact, which translation groups are not.\footnote{The reduction theorem applies only to unitary representations, which associates each group member $g$ with a unitary operator $T_g$ on a vector space. If a group is compact, Maschke’s operator $S^2 = \sum_g T_g^* T_g$ can be used to establish that any representation is isomorphic to a unitary representation. In the non-compact case the sum over $g$ is ill-defined.} Hence, the
completeness of van Kampen modes cannot be rigorously established by the group-theoretic argument, although similar arguments are widely accepted in physics. Case (1959) established the completeness of the van Kampen modes of an electrostatic plasma by direct demonstration that any DF \( f(w) \) can be written as a sum of van Kampen modes. The corresponding exercise for stellar systems will be presented in Paper III of this series.

6 DISCUSSION

6.1 Particle dressing

The concept of particle dressing has been central to high-energy physics for over half a century, but has been slow to catch on in stellar dynamics. In galactic dynamics it can be traced at least as far back as Julian & Toomre (1966), who showed that a gas cloud in a galactic disc would attract an entourage of passing stars \( \sim \)ten times more massive than itself. Toomre & Kalnajs (1991) showed that individual disc stars also enhance their masses tenfold by attracting an (ever-changing) entourage of other stars. Sellwood & Carlberg (2014) showed that large-amplitude spiral structure emerges from Poisson noise through successive spiral instabilities, but this important process was only firmly connected to particle dressing by Fouvrè et al. (2015). Hamilton (2021) made the connection between the BL equation and particle dressing beautifully clear via Rostoker’s principle: that it is permissible to compute the effects of discreteness as from the interaction of uncorrelated but dressed particles (Rostoker 1964). Here we have interpreted a van Kampen mode as the result of dressing not one star but an ensemble of stars on a group of resonant tori.

Given that the CBE is the first equation in the BBGKY hierarchy of equations with the two-particle correlation function set to zero, the importance of dressing for the structure of van Kampen modes may seem paradoxical. The CBE is a mean-field approximation akin to the Weiss theory of magnetism, and embraces correlations that are induced by perturbing fields. Hence it embraces the dressing of resonant tori involved in van Kampen modes. Moreover, when a simulation is started, its DF is inevitably perturbed from the underlying analytic DF and thus its van Kampen modes are excited.

6.2 van Kampen vs Landau modes

Normal modes are perhaps the most important single tool in theoretical physics – quantum field theory has even taught us to see particles as excitations of normal modes of the vacuum. Modal analyses have played a significant role in stellar dynamics since the seminal work of Kalnajs (1965), Toomre (Toomre 1964, 1981) and later the prescient work of Weinberg (Weinberg 1993, 1994, 1998, 2001), but in all these studies the modes considered were those of Landau. These ‘modes’ lack key properties of true modes: (i) completeness in the sense that any initial condition can be expressed as a linear combination of modes, and (ii) additivity of energies. These two properties are essential for the use of modes in physics and engineering outside stellar dynamics. The contents pages of the two volumes of Fridman & Polyachenko (1984) explain the focus on Landau modes: the community wanted to establish which equilibrium models are stable, rather than to investigate, as we do, the excursions that stable systems make around equilibrium.

Doubt is sometimes cast on the physical standing of van Kampen modes because their DFs contain a \( \delta \)-function. Actually this feature is a natural consequence of their forming a continuum. Testable predictions of van Kampen modes will always emerge after integration over \( \omega \), just as in the familiar quantum-mechanical treatment of radiative transitions sensible results emerge only after integration over the frequency of the electromagnetic field, and the \( \delta \)-functions will disappear in the process. Landau modes are superpositions of van Kampen modes, and they decay as their constituent van Kampen modes drift apart in phase (Case 1959). If you run a decayed Landau mode back in time, the phases move back into alignment for a finite time before drifting apart, so the disturbance grows for only a limited time.

Hamilton & Heinemann (2020) take a fresh approach to relaxation in stellar systems that involves Landau modes in an essential way. Binney & Lacey (1988) showed that the diffusion tensor of the action-space Fokker-Planck equation follows immediately from the temporal power spectrum of the gravitational potential. Hamilton & Heinemann (2020) argue that both dressed two-particle interactions and normal modes of the entire system contribute to the power spectrum. They assume that the modes in question are Landau modes, which they imagine to reach an equilibrium level of excitation through their native damping being offset by constant excitation by Poisson noise. This picture involves a transfer of energy from Landau modes to the underlying heat bath, and then back to the Landau modes via Poisson noise. The mechanism by which Poisson noise draws energy from the heat bath is unclear.

In a simpler picture each van Kampen mode has a fixed amplitude and energy and a phase that advances at its own steady rate. Modes with frequencies that lie close to the real parts of a weakly damped Landau mode have large amplitudes because they are heavily dressed. From time to time their phases yield constructive interference and the Landau mode appears to be highly excited. The excitation decays as shifts in relative phase spoil the constructive interference. At a later time the phases again align favourably, and the process repeats.

6.3 Thermodynamics of star clusters

When a stellar system is born, its van Kampen modes are assigned particular amplitudes and phases. At birth the phases may be highly correlated, but they will decorrelate on a dynamical timescale. This decorrelation may manifest itself through the emergence of system-scale fluctuations in the density, wandering of the point of highest central density, etc (Lau & Binney 2019; Heggie et al. 2020). On a longer timescale non-linear terms in the CBE will mediate exchanges of energy between modes. We know that the amplitudes of modes are invariant at linear order, and we expect them to evolve at quadratic order in the perturbations.

\(^7\) When deriving Fermi’s golden rule, one analogously integrates over the energies of final states.
The two-body timescale is precisely the timescale associated with terms of quadratic order (Chavanis 2012), so van Kampen modes are expected to exchange energy on the two-body timescale.

There is no reason to believe that when a cluster is first realised the amplitudes of its van Kampen modes conform to the Gibbs distribution. We expect exchanges of energy between modes to drive the distribution towards the Gibbs distribution and equipartition of energy between modes, which is to say that the actual DF is \( F = \int d\omega f(\omega) \) with \( f(\omega) \) of van-Kampen form (40) and (cf. eqn 53)

\[
E[f] = \langle f_\omega | f_\omega \rangle = \text{constant.} \tag{54}
\]

Formally, modes exchange energy and equipartition can be approached on the same (two-body) timescale on which core collapse and evaporation change the mean-field model, but since core collapse occurs after \( \sim 300 \) central two-body times (Binney & Tremaine 2008, §7.5.3), it is plausible that a good approximation to the Gibbs distribution will be achieved before core collapse occurs. Hence a seductive programme of work is to assume equipartition of energy and random phases between the modes of a particular mean-field model and to compare the observables predicted thus with observational data and N-body simulations.

In this regard it is instructive to compare the applicability of thermodynamics to star clusters and to classical systems that also have access to states of very high entropy, for example a mixture of two parts hydrogen and one part oxygen, or a diamond, both of which have higher entropy states (water vapour and graphite) that can only be reached by climbing over a significant energy barrier. On account of this barrier, a hydrogen/oxygen mixture and a diamond will extensively explore the configurations accessible with thermal energy regardless of the existence of states of much higher entropy. A stellar system is not denied access to states of high energy by an energy barrier but by a glacial rate of energy transport.

Nevertheless, computing the thermodynamics of a cluster is feasible because the positivity of model energy ensures that the cluster’s constant-energy surfaces have finite volume \( \Omega \), and it would be very interesting to examine its predictions.

In such a theory the DF would itself be a random variable in addition to the coordinates of stars, which are the random variables whose probability distribution the DF specifies. Testable predictions would emerge from the theory as double expectations: first \( \langle \mathcal{O} \rangle_f = \int d^3\mathbf{w} f(\mathbf{w}) \mathcal{O}(\mathbf{w}) \) and then an average of these averages weighted by the probability of each \( f \). Hence a prerequisite of the theory is the ability to assign probabilities to DFs in a rational way. In particular, the probability assigned to a group of DFs must remain unchanged as a cluster evolves under the CBE. In standard statistical mechanics the analogous requirement, that a probability density on phase space be invariant under Hamiltonian evolution, is satisfied by making a priori probability proportional to the measure of phase-space volume \( d^3q \) \( d^3p \) that canonical coordinates \( (q,p) \) deliver. In a companion paper (Lau & Binney 2021) we extend this idea to the space of distribution functions by defining canonical coordinates for this space. It turns out that the energy of a van Kampen mode then takes the form of a sum of Hamiltonians of simple-harmonic oscillators.

### 6.4 Prior work

To our knowledge the van Kampen modes of a stellar system have previously been considered only by Vandervoort (2003), who followed van Kampen (1955) in deriving the modes directly from the CBE, Antonov (1961) obtained the second-order, Hermitian differential operator \( K \) by splitting the DF into parts even and odd in \( v \), but he was focused on proving his stability principle and didn’t show that the eigenfunctions of \( K \) are the van Kampen modes. He didn’t take advantage of angle-action variables or compute excitation energies. Polyachenko et al. (2021) discussed the relation of Landau and van Kampen modes in the context of the periodic cube, though principally in the unphysical case that the cube’s mass exceeds the Jeans mass so the system is unstable. Their work makes very clear that Landau ‘modes’ lack essential properties of modes, and also illustrates what a treacherous arena the complex plane is: physically ill-motivated changes in the contour of integration over velocity give rise to solutions with radically different properties. In particular they show that Landau’s choice of contour breaks the time-reversal symmetry of the underlying problem.

Polyachenko et al. (2021) introduced the nomenclature ‘true mode’ for a member of the complete set of modes and like us restricted the term ‘van Kampen mode’ to true modes with real frequencies. The principal point made by Polyachenko et al. (2021) is that the unstable Jeans mode at \( \omega = iy \) is accompanied by a decaying mode at \( \omega = -iy \). This is a trivial consequence of time-reversibility but Polyachenko et al. (2021) show that some effort is required to explain why Landau’s analysis misses this mode.

### 7 CONCLUSIONS

As the completeness and precision of astronomical data grow, the oscillations of stellar systems around equilibrium configurations will increase in observational significance. The natural way to produce theoretical predictions of these phenomena is to adapt the techniques of statistical mechanics to stellar dynamics. This paper takes a step in this direction for ergodic stellar systems by focusing attention on the van Kampen modes of stellar systems, which have hitherto been eclipsed by Landau modes.

We showed for the first time that van Kampen modes of an ergodic system are the eigenfunctions with positive eigenvalues of Antonov’s second-order Hermitian operator on phase space. In consequence, the true modes of an ergodic stellar system are either purely sinusoidal or exponentially growing/decaying; there are no over-stable modes or modes comprising decaying oscillations. The frequencies of oscillating modes form a continuum, and the DFs of these modes contain \( \delta \)-functions which disappear when testable predictions are extracted by integrating over frequencies. Any exponentially growing/decaying modes are isolated in frequency space and their DFs do not contain \( \delta \)-functions. The energy of an oscillating mode is just the norm of the odd part of its DF. From this it follows that these modes have positive energy. The energy of a growing/decaying mode is identically zero.

We interpreted van Kampen modes as dressed sets of...
resonant tori. How heavily they are dressed increases with the extent to which the system is self-gravitating, and with proximity in frequency space to a zero of the response matrix $\mathcal{M}(\omega)$ – the zeroes of this matrix come in pairs with each pair that is not on the imaginary axis associated with a Landau ‘mode’. Landau modes are not members of the complete set of true modes and hence are linear combinations of true modes.

A star cluster has many more true modes than the equivalent gas ball because much more information is required to specify a DF than to specify the density and pressure in a ball of gas. For this reason one hesitates to enumerate a cluster’s van Kampen modes, except possibly in the much simplified case of vanishing self-gravity. Paper III shows that we can avoid this enumeration by showing how to decompose any initial state $F(w)$ of the system into a linear combination $F = \int d\omega f(\omega)$ of van Kampen DFs $f(w, \omega)$. This decomposition automatically identifies the particular mode at frequency $\omega$ that is required to synthesise the given DF.

When a model cluster is realised, its van Kampen modes acquire non-zero amplitudes by virtue of Poisson noise. The phases of modes evolve on a dynamical timescale while their amplitudes probably evolve on the two-body timescale. Consequently, there is an early phase of relaxation in the evolution of a simulated cluster in which system-scale distortions emerge. Consideration of the way modes depend of the degree of self-gravity explains why system-scale distortions are less prominent in simulations that are less self-gravitating.

The positivity of the energies of van Kampen modes opens the door to the application of standard statistical physics to stellar systems; while in the long term systems will drift through core collapse and evaporation to states of ever higher entropy, in the medium term disturbed systems may relax to distributions of energy among van Kampen modes that maximise entropy.

ACKNOWLEDGEMENTS

We thank an anonymous referee for an exceptionally careful reading and numerous constructive suggestions, and we thank J. Magorrian and B. Kocsis for helpful comments on early drafts. Our discussions of the completeness of van Kampen modes benefited from discussions with U. Tillmann. Jun Yan Lau gratefully acknowledges support from University College London’s Overseas and Graduate Research Scholarships. James Binney is supported by the UK Science and Technology Facilities Council under grant number ST/N000091/1 and by the Leverhulme Trust through an Emeritus Fellowship.

DATA AVAILABILITY

No new data was generated or analysed in support of this research.

REFERENCES

Antoja T. et al., 2018, Nat, 561, 360
Antonov V. A., 1961, SovA, 4, 859
Binney J., Lacey C., 1988, MNRAS, 230, 597
Binney J., Schönrich R., 2018, MNRAS, 481, 1501
Binney J., Tremaine S., 2008, Galactic Dynamics: Second Edition. Princeton University Press
Bland-Hawthorn J., Tepper-Garcia T., 2020, arXiv e-prints, arXiv:2009.02434
Case K. M., 1959, Annals of Physics, 7, 349
Chavanis P.-H., 2012, Physica A, 391, 3680
Dieudonné J., 1969, Foundations of modern analysis. Academic Press
Elliot J. P., Dawber P. G., 1989, Symmetry in physics. Macmillan
Fouvy J. B., Pichon C., Magorrian J., Chavanis P. H., 2015, A&A, 584, A129
Fridman A. M., Polyachenko V. L., 1984, Physics of gravitating systems, vols I and II. Springer
Gaia Collaboration, Brown A. G. A. et al., 2018, A&A, 616, A1
Hamilton C., 2021, MNRAS, 501, 3371
Hamilton C., Fouvy J.-B., Binney J., Pichon C., 2018, MNRAS, 481, 2041
Hamilton C., Heinemann T., 2020, arXiv e-prints, arXiv:2011.14812
Hoggie D. C., Breen P. G., Varri A. L., 2020, MNRAS, 492, 6019
Hénon M., 1961, Annales d’Astrophysique, 4, 369
Julian W. H., Toomre A., 1966, ApJ, 146, 810
Kahaja J. A., 1965, PhD thesis, Harvard University.
Kahaja J. A., 1976, ApJ, 205, 745
Kulsrud R. M., Mark J. W. K., 1970, ApJ, 160, 471
Laporte C. F. P., Minchev I., Johnston K. V., Gómez F. A., 2019, MNRAS, 485, 3134
Lau J. Y., Binney J., 2019, MNRAS, 490, 478
Lau J. Y., Binney J., 2021, arXiv e-prints, arXiv:xx
Laval G., Mercier C., Pellat R., 1965, Nucl.Fus., 5, 156
Lee H. M., Ostriker J. P., 1987, ApJ, 322, 123
Lin C. C., Shu F. H., 1964, ApJ, 140, 646
Malin D. F., Carter D., 1980, Nat, 285, 643
Palmer P. L., Papaloizou J., 1987, MNRAS, 224, 1043
Polyachenko E. V., Shukhman I. G., Borodina O. I., 2021, MNRAS, 503, 660
Ramos J. J., White R. L., 2018, Physics of Plasmas, 25, 034501
Rostoker N., 1964, Physics of Fluids, 7, 479
Saha P., 1991, MNRAS, 248, 494
Sellwood J. A., Carlberg R. G., 2014, ApJ, 785, 1377
Tonry J., Schneider D. P., 1988, AJ, 96, 807
Toomre A., 1964, ApJ, 139, 1217
Toomre A., 1981, in Structure and Evolution of Normal Galaxies, Fall S. M., Lynden-Bell D., eds., pp. 111–136
Toomre A., Kahaja J. A., 1991, in Dynamics of Disc Galaxies, Sundelius B., ed., p. 341
van Kampen N. G., 1955, Physica, 21, 949
Vandervoort P. O., 2003, MNRAS, 339, 537
Vitral E., Mamon G. A., 2021, A&A, 646, A63
Weinberg M. D., 1991, ApJ, 368, 66
Weinberg M. D., 1993, ApJ, 410, 543
Weinberg M. D., 1994, ApJ, 421, 481
Weinberg M. D., 1998, MNRAS, 297, 101
Weinberg M. D., 2001, MNRAS, 328, 311
APPENDIX A: MATRIX ELEMENTS OF ANTONOV’S OPERATOR

Here we compute a general matrix element $\langle \tilde{f}_-|K|f_-\rangle$ of Antonov’s operator, with $f$ and $\tilde{f}$ any two DFs. Equation (23) yields

$$\langle \tilde{f}_-|K|f_-\rangle = \int \frac{d^6w}{f_0} \langle \tilde{f}_- | \Omega \cdot \frac{\partial}{\partial \theta} | f_- \rangle^2 f_-$$

$$+ i \frac{\xi}{2} \int d^6w f_0 \frac{\partial}{\partial \theta} \sum_{\alpha} \Phi^{(a)}(x) j_\alpha[f_1]. \quad (A1)$$

Since we can write $d^6w = d^3J d^3\theta$, we can shift the derivatives wrt $\theta$ around by partial integration and obtain

$$\langle \tilde{f}_-|K|f_-\rangle = - \int \frac{d^6w}{f_0} \left( \Omega \cdot \frac{\partial f_-}{\partial \theta} \right) \left( \Omega \frac{\partial f_-}{\partial \theta} \right)$$

$$- i \int d^6w \sum_{\alpha} \Phi^{(a)}(x) \Omega \frac{\partial \tilde{f}_-}{\partial \theta} j_\alpha[f_1]$$

$$- i \frac{\xi}{2} \sum_{\alpha} j_\alpha \tilde{f}_1 j_\alpha[f_1]. \quad (A2)$$

The symmetry of the rhs wrt $f$, $\tilde{f}$ implies that $K$ is Hermitian. Since $K$ is Hermitian, all its eigenvalues $\omega^2$ are real and $K$’s eigenfunctions are either sinusoidal or show pure exponential growth/decay.

When we set $\tilde{f} = f$, we obtain

$$\langle f_-|K|f_-\rangle = \int \frac{d^6w}{f_0} \left| \Omega \cdot \frac{\partial f_-}{\partial \theta} \right|^2 - \frac{\xi}{2} \sum_{\alpha} |j_\alpha[f_1]|^2. \quad (A3)$$

It is interesting to express the rhs of equation (A3) in terms of $f_+$ using equations (14), (19) and (25). The result is

$$\langle f_-|K|f_-\rangle = \int \frac{d^6w}{f_0} \left| \frac{\partial f_-}{\partial \theta} \right|^2 - \frac{\xi}{2} \sum_{\alpha} |\partial A_\alpha[f_+]|^2. \quad (A4)$$

In the case that $f_-$ is an eigenfunction of $K$ with eigenvalue $\omega^2$, we can replace time derivatives by $-i\omega$ (with $\omega$ potentially pure imaginary), and equation (A4) becomes

$$\langle f_-|K|f_-\rangle = \omega^2 \langle f_-|f_-\rangle$$

$$= |\omega|^2 \left\{ \int \frac{d^6w}{f_0} \left| f_- \right|^2 - \frac{\xi}{2} \sum_{\alpha} |A_\alpha[f_+]|^2 \right\}$$

$$= |\omega|^2 \left\{ \langle f_+|f_+\rangle - \frac{\xi}{2} \sum_{\alpha} |A_\alpha[f_+]|^2 \right\}. \quad (A5)$$

Hence $\langle f_-|\tilde{f}_-\rangle = 0$ unless there is some vector $n$ at which both $f_{n-}$ and $\tilde{f}_{n-}$ are non-zero. This fact confirms that in the absence of self-gravity, when $f_{n-}$ vanishes if $g_n$ vanishes, the sought-after basis modes for a given frequency are indexed by $n$ in the sense that the Fourier expansions of their DFs contain only $\pm n$. When $\xi > 0$, $f_{n-}$ is expected to be non-zero when $g_{n'} = 0$ providing $g_n \neq 0$ for a vector $n$ such that $n_3 = n'_3$.

Equation (40) gives $f_+$ for a van Kampen mode, and we have seen that in the case of a mode $\langle n|\Omega|f_{n+} = \omega f_{n+}$, so

$$\langle f_-|\tilde{f}_-\rangle = -(2\pi)^3 \int \frac{d^3J}{f_0} \sum_{n_1,n_2} \frac{\omega \bar{\omega}}{(n \cdot \Omega)^2} f_{n+} \tilde{f}_{n+}$$

$$= -(2\pi)^3 \omega \bar{\omega} \int \frac{d^3J}{f_0} \sum_{n_1,n_2} \frac{1}{(n \cdot \Omega)^2}$$

$$\times \left( \frac{(n \cdot \Omega)^2}{\omega^2 - (n \cdot \Omega)^2} \sum_{\alpha} A_{\alpha}^* \Phi^{(a)}_n(J) + g_{n}(J) (\bar{\omega}^2 - (n \cdot \Omega)^2) \right)$$

$$\times \left( \frac{(n \cdot \Omega)^2}{\bar{\omega}^2 - (n \cdot \Omega)^2} \sum_{\alpha} \bar{A}_{\alpha} \Phi^{(a)}_n(J) + \bar{g}_{n}(J) (\omega^2 - (n \cdot \Omega)^2) \right). \quad (B2)$$

where quantities associated with the mode $\tilde{f}$ are marked by tildes. When we multiply out the big brackets, we get a term with two denominators of the form $(n \cdot \Omega)^2 - \omega^2$. The integral over these is to be interpreted as a principal value, that is by excluding points at which the denominator vanishes. We use the identity (e.g. Ramos & White 2018)

$$P \frac{1}{x-x_1} P \frac{1}{x-x_2} = P \frac{1}{x-x_1} \left( P \frac{1}{x-x_1} - P \frac{1}{x-x_2} \right)$$

$$+ \pi^2 \delta(x-x_1) \delta(x_1-x_2) \quad (B3)$$

to rewrite this term as

$$\int d^3J \ldots = P \int \frac{d^3J}{f_0} \sum_{n_1,n_2} (n \cdot \Omega f_0)^2 \sum_{\alpha} A_{\alpha}^* \bar{A}_{\alpha} \Phi^{(a)}_n \Phi^{(b)}_n$$

$$\times \left\{ \frac{1}{\omega^2 - \bar{\omega}^2} \frac{1}{(n \cdot \Omega)^2 - \omega^2} - \frac{1}{(n \cdot \Omega)^2 - \bar{\omega}^2} \right\}$$

$$+ \pi^2 \delta((n \cdot \Omega)^2 - \omega^2) \delta(\omega^2 - \bar{\omega}^2) \right\}. \quad (B4)$$

The cross terms in the product of equation (B2) can be written

$$\int d^3J \ldots = \int \frac{d^3J}{f_0} \sum_{n_1,n_2} A_{\alpha}^* \Phi^{(a)}_n g_n \delta((n \cdot \Omega)^2 - \omega^2)$$

$$\times \left( \sum_{\beta} A_{\beta} \Phi^{(b)}_n \right) g_n \delta((n \cdot \Omega)^2 - \bar{\omega}^2)$$

$$- \sum_{\beta} A_{\beta} \Phi^{(a)}_n g_{n} \delta((n \cdot \Omega)^2 - \omega^2) \right\). \quad (B5)$$

while the term involving $g_n \bar{g}_n$ can be written

$$\int d^3J \ldots = \int \frac{d^3J}{f_0} \sum_{n_1,n_2} \tilde{g}_{n} \tilde{g}_{n} \delta((n \cdot \Omega)^2 - \omega^2) \delta(\omega^2 - \bar{\omega}^2). \quad (B6)$$

APPENDIX B: INNER PRODUCT OF VAN KAMPEN MODES

Here we compute the inner product of two van Kampen modes of the same frequency. We have

$$\langle f_-|\tilde{f}_-\rangle = - \int \frac{d^6w}{f_0} f_- \tilde{f}_-$$

$$= -(2\pi)^3 \int \frac{d^3J}{f_0} \sum_{n} f_{n-} \tilde{f}_{n-}. \quad (B1)$$
Adding these fragments together and reinstating the prefactor in equation (B2) we have

\[
\langle f_- | \tilde{f}_- \rangle = -(2\pi^3)\omega \bar{\omega} \mathcal{P} \int \frac{d^3 \mathbf{J}}{f_0^0} \sum_{n_1, n_2} \left\{ \int \frac{d^3 \mathbf{J}}{f_0} \sum_{\alpha} A_\alpha^* \Phi_\alpha^{(\alpha)_*} + g_\alpha^* \delta((\mathbf{n} \cdot \mathbf{\Omega})^2 - \omega^2) \right\} \sum_{\beta} \bar{A}_\beta \Phi_\beta^{(\beta)}
\]

\[
\times \left( \frac{(\mathbf{n} \cdot \mathbf{\Omega})^2 f_0^0}{(\mathbf{n} \cdot \mathbf{\Omega})^2 - \omega^2} \sum_{\alpha} A_\alpha^* \Phi_\alpha^{(\alpha)_*} + g_\alpha^* \delta((\mathbf{n} \cdot \mathbf{\Omega})^2 - \bar{\omega}^2) \right) \sum_{\beta} \bar{A}_\beta \Phi_\beta^{(\beta)}
\]

\[
+ \left( \pi^2 (\mathbf{n} \cdot \mathbf{\Omega})^2 \sum_{\alpha \beta} A_\alpha^* \bar{A}_\beta \Phi_\alpha^{(\alpha)_*} \Phi_\beta^{(\beta)} + \frac{g_\alpha^* g_n^*}{(\mathbf{n} \cdot \mathbf{\Omega})^2} \right)
\]

\[
\times \delta((\mathbf{n} \cdot \mathbf{\Omega})^2 - \omega^2) \delta(\omega^2 - \bar{\omega}^2) \right) \}
\]

\[
= -(2\pi^3)\frac{\omega \bar{\omega}}{\omega^2 - \bar{\omega}^2} \int d^3 \mathbf{J} \sum_{n_1, n_2} \left( f_{n+}^* + \sum_{\beta} \bar{A}_\beta \Phi_\beta^{(\beta)} \right)
\]

\[
- \bar{f}_{n+} \sum_{\alpha} A_\alpha^* \Phi_\alpha^{(\alpha)_*}
\]

\[
- (2\pi^3)\frac{\omega \bar{\omega}}{(\mathbf{n} \cdot \mathbf{\Omega})^2 f_0^0} \int d^3 \mathbf{J} \sum_{n_1, n_2} \left( \pi^2 (\mathbf{n} \cdot \mathbf{\Omega})^2 f_0^0 \sum_{\alpha \beta} A_\alpha^* \bar{A}_\beta \Phi_\alpha^{(\alpha)_*} \Phi_\beta^{(\beta)} 
\]

\[
+ \frac{g_n^* g_n^*}{(\mathbf{n} \cdot \mathbf{\Omega})^2 f_0^0} \right) \delta((\mathbf{n} \cdot \mathbf{\Omega})^2 - \bar{\omega}^2) \delta(\omega^2 - \bar{\omega}^2). \quad \text{(B7)}
\]

Now

\[
A_\alpha = \int d^6 \mathbf{w} \Phi^{(\alpha)_*} f_+
\]

\[
= \int d^3 \mathbf{J} d^3 \theta \sum_n \Phi_n^{(\alpha)_*} e^{-in \theta} f_+ = \sum_n \Phi_n^{(\alpha)_*} f_{n+}. \quad \text{(B8)}
\]

so the first integral in our final expression for \( \langle f_- | f_- \rangle \) vanishes because its integrand is \( \sum_{\alpha} (A_\alpha A_\alpha^* - A_\alpha^* \bar{A}_\alpha) \). With some further simplifications we can write the inner product in the form given by equation (52).