Graph Matching with Partially-Correct Seeds

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Abstract

The graph matching problem aims to find the latent vertex correspondence between two edge-correlated graphs and has many practical applications. In this work, we study a version of the seeded graph matching problem, which assumes that a set of seeds, i.e., pre-mapped vertex-pairs, is given in advance. Most previous work on seeded graph matching requires all seeds to be correct. In contrast, we study the setting where the seeds are partially correct. Specifically, consider two correlated graphs whose edges are sampled independently with probability $s$ from a parent Erdős-Rényi graph $G(n, p)$. Furthermore, a mapping between the vertices of the two graphs is provided as seeds, of which an unknown $\beta$ fraction is correct. This problem was first studied in [LS18] where an algorithm is proposed and shown to perfectly recover the correct vertex mapping with high probability if $\beta \geq \max \left\{ \frac{8}{3} p, \frac{16 \log n}{n ps^2} \right\}$. We improve their condition to $\beta \geq \max \left\{ \frac{30 \sqrt{\frac{\log n}{n(1-p)s^2}}}{n p(1-p)s^2}, \frac{45 \log n}{n p(1-p)s^2} \right\}$, which is more relaxed when the parent Erdős-Rényi graph is dense, i.e., when $p = \Omega\left(\sqrt{\frac{\log n}{ns^2}}\right)$. However, when $p = O\left(\sqrt{\frac{\log n}{ns^2}}\right)$, our improved condition still requires that $\beta$ must increase inversely proportional to $np$. In order to improve the matching performance for sparse graphs, we propose a new algorithm that uses “witnesses” in the 2-hop neighborhood, instead of only 1-hop neighborhood as in [LS18]. We show that when $np^2 \leq \frac{1}{135 \log n}$, our new algorithm can achieve perfect recovery with high probability if $\beta \geq \max \left\{ \frac{900 \sqrt{np^2(1-s) \log n}}{s}, \frac{600 \sqrt{\log n}}{ns^4}, \frac{1200 \log n}{n^2 p^2 s^4} \right\}$ and $nps^2 \geq 128 \log n$. Numerical experiments on both synthetic and real graphs corroborate our theoretical findings and show that our 2-hop algorithm significantly outperforms the 1-hop algorithm when the graphs are relatively sparse.

1 Introduction

Graph matching aims to find a bijective mapping between the vertex sets of two edge-correlated graphs so that their edge sets are maximally aligned. Graph matching is motivated by many practical applications, such as social network de-anonymization [NS09], computational biology [SXB08, KHGM16], computer vision [CFSV04, SS05], and natural language processing [HNM05]. For instance, from one anonymized version of the follow relationships graph on the Twitter microblogging service, researchers were able to re-identify the users by matching the anonymized graph to a correlated cross-domain auxiliary graph, i.e., the contact relationships graph on the Flickr photo-sharing service, where user identities are known [NS09].

Existing algorithms to find the correct matching can be divided into two categories, seedless and seeded matching algorithms. Seedless matching algorithms do not rely on any additional side information, but instead only use the topological information to match the two graphs. The

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quadratic assignment problem (QAP) \cite{PRW94, BCPP98} is perhaps the most natural idea to be applied here. Let $A$ and $B$ denote the adjacency matrices of two correlated graphs with $n$ vertices, respectively. The QAP can be formulated as the following optimization problem:

$$\max_{\Pi \in S_n} \langle A, \Pi B \Pi^T \rangle,$$

where $S_n$ is the set of permutation matrices in $\mathbb{R}^{n \times n}$, and $\langle \cdot, \cdot \rangle$ denotes the matrix inner-product.

However, QAP is NP-hard to solve or to approximate within an approximation ratio that even grows with $n$ \cite{MMS10}. To reduce the computational complexity, one may rewrite the objective function (1) as $\min_{\Pi \in S_n} \| A \Pi - B \Pi \|_F^2$ (where $\| \cdot \|_F$ is the Frobenius matrix norm), which is a quadratic function of $\Pi$, and relax the set $S_n$ to its convex hull (the set of doubly stochastic matrices), arriving at a quadratic programming (QP) problem. The experimental results in \cite{ABK15, VCL15, LFF16, DML17} demonstrate the effectiveness of the QP relaxation, but its theoretical understanding remains elusive. Other seedless matching algorithms have been proposed based on degree information \cite{DCKG18}, spectral method \cite{Ume88, FQM19, FMWX19a, FMWX19b} or random walk \cite{GMS05}. However, to the best of our knowledge, these algorithms either only succeed when the fraction of edges that differ between the two graphs is low, i.e., on the order of $O \left( \frac{1}{\log^2 n} \right)$ \cite{DMWX18} or require quasi-polynomial runtime ($n^{O(\log n)}$) \cite{BCL18}. The only exception is the neighborhood tree matching algorithm recently proposed in \cite{GM20}, which can output a partially-correct matching in polynomial-time when two graphs are sparse and differ a constant fraction of edges.

The other category of matching algorithms is seeded matching algorithms \cite{PG11, YG13, KL14, LFP13, FAP18, SGE17, MX19}. These algorithms require “seeds”, which are a set of pre-mapped vertex-pairs. Let $G_1$ and $G_2$ denote the two correlated graphs, respectively. For each pair of vertices $(u, v)$, where $u$ is in $G_1$ and $v$ is in $G_2$, a seed $(w, w')$ becomes a witness for $(u, v)$ if $w$ is a neighbor of $u$ in $G_1$ and $w'$ is a neighbor of $v$ in $G_2$. The basic idea of seeded matching algorithm is that a candidate pair of vertices should have more witnesses if they are a true pair than if they are a fake pair. Assuming that the seeds are correct, seeded matching algorithms can find the correct matching for the remaining vertices more efficiently than seedless matching algorithm. In social network de-anonymization, such initially matched seeds are often available, thanks to users who have explicitly linked their accounts across different social networks. For other applications, the seeds can be matched by prior knowledge or manual labeling.

However, most existing seeded matching algorithms require all seeds to be correct, which may be difficult to guarantee in practice. The authors of \cite{LS18} extend the idea of seeded matching algorithm to allow incorrect seeds. For example, the seeds may be provided by seedless matching algorithms, which will produce some incorrect seeds. The seeded matching algorithm in \cite{LS18} first counts the numbers of the witnesses for each candidate pair of vertices, and then uses Greedy Maximum Weight Matching (GMWM) to find the vertex correspondence between the two graphs such that the total number of witnesses is large. When the two graphs are correlated Erdős-Rényi graphs, whose edges are independently sub-sampled with probability $s$ from a parent Erdős-Rényi graph $G(n, p)$, and there are $\beta$ fraction of correct seeds, \cite{LS18} shows the proposed algorithm can correctly match all vertices with high probability if

$$\beta \geq \max \left\{ \frac{8}{3p}, \frac{16 \log n}{nps^2} \right\}. \tag{2}$$

However, the performance guarantee in (2) is somewhat conservative for the following two reasons. First, as $p$ increases, i.e., the parent graph gets dense, the number of witnesses increases.
Thus, we expect that the graphs should be easier to match, and hence it seems unnecessary to require $\beta \geq \frac{8}{3}p$ in (2). This observation thus raises the first question:

*Can we get a sufficient condition on $\beta$ tighter than condition (2) for the success of the seeded matching algorithm of [LS18] on dense graphs?*

Second, condition (2) suggests that $\beta$ must increase inversely proportional to $np$. Thus, if the graph is sparse, i.e., $p = \Theta \left(\frac{\log n}{n s^2}\right)$, $\beta$ would be required to be constant, which might be hard to obtain in practice. Indeed, our numerical experiments show that the algorithm in [LS18] fails when $\beta = o \left(\frac{\log n}{nps^2}\right)$. This is because their algorithm only uses witnesses that are in the 1-hop neighborhood. When $p$ is low, i.e., the parent graph is sparse, there are only a small number of 1-hop neighbors for each vertex, which cannot provide sufficiently many 1-hop witnesses to distinguish the true pairs from the fake pairs. This observation raises the second open question:

*Can we develop an efficient seeded matching algorithm that further relaxes the requirement on $\beta$ in relatively sparse graphs?*

In this paper, we provide new results that answer both questions affirmatively. For the first question, we provide a new analysis showing that the algorithm in [LS18] can correctly match all vertices with high probability if

$$\beta \geq \max \left\{ 30 \sqrt{\frac{\log n}{n(1-p)^2s^2}}, \frac{45\log n}{np(1-p)^2s^2} \right\}. \quad (3)$$

Note that the requirement for $\beta$ to grow with $p$ is removed, and the new condition (3) is more relaxed than condition (2) for dense graphs when $p = \Omega \left(\sqrt{\frac{\log n}{n s^2}}\right)$. The key observation is that the analysis of [LS18] only considers the correct seeds and ignores the incorrect seeds, when counting the number of witnesses for the true pairs. The reason that they ignore the incorrect seeds is because the events that the incorrect seeds become witnesses for a true pair depend on each other. Our analysis takes into account of the incorrect seeds for the true pairs and carefully deals with the dependency using concentration inequalities for dependent random variables [Jan04].

For the second question, we propose a new algorithm that can exactly match all vertices with high probability if

$$\beta \geq \max \left\{ 900\sqrt{n p^3(1-s)\log n}{s}, 600\sqrt{\frac{\log n}{n s^4}}, \frac{1200\log n}{n^2p^2s^4} \right\} \text{ and } nps^2 \geq 128\log n. \quad (4)$$

Note that our condition (4) only requires $\beta$ to increase inversely to $n^2p^2$ when $p = O \left(\sqrt{\frac{\log n}{n s^4}}\right)^{\frac{1}{2}}$, and is more relaxed than condition (3) for sparse graphs when $p = O \left(\sqrt{\frac{\log n}{n s^4(1-s)}}\right)^{\frac{1}{2}}$. A key idea of our algorithm is to match vertices by comparing the number of witnesses in the 2-hop neighborhood. In sparse graphs, most vertices have more 2-hop neighbors than 1-hop neighbors. Thus, compared to the algorithm in [LS18] that uses only 1-hop witnesses, our algorithm can leverage more witnesses to distinguish the true pairs from the fake pairs. The idea of using multi-hop neighborhoods to match vertices is used and analyzed previously when all seeds are correct [MX19]. In comparison, our analysis with incorrect seeds is significantly more challenging, as we need to take care of the dependency on the size of the 1-hop neighborhood and the dependency between incorrect seeds. Finally, using numerical experiments on both synthetic random graphs and real graphs, we show that our algorithm achieves better performance than the algorithm in [LS18] on sparse graphs, which agrees with our theoretical analysis.
The rest of the paper is organized as follows: In Section 2, we formally introduce the correlated Erdős-Rényi model and the problem statement. Section 3 describes the matching algorithms, presents our theoretical guarantees, and highlights the analysis challenges. Section 4 contains empirical evaluations of the various algorithms on both synthetic and real graphs. Section 5 concludes the paper with remarks on future directions. Additional numerical experiments and full proof are deferred to appendices.

2 Model

In this section, we formally introduce the model and the graph matching problem with partially-correct seeds.

We let $G(V, E)$ denote the parent graph with $n$ vertices, where $V$ is the set of vertices and $E$ is the set of edges. The parent graph $G(V, E)$ is generated from the Erdős-Rényi model $G(n, p)$, i.e., we start with an empty graph on $n$ vertices and connect any pair of two vertices independently with probability $p$. Then, we obtain a sub-sampled graph $G_1(V, E_1)$ by sampling each edge of $G$ into $E_1$ independently with probability $s$. Repeat the same sub-sampling process independently and relabel the vertices according to an unknown permutation $\pi : V \to V$ to construct another sub-sampled graph $G_2(\pi(V), E_2)$. Throughout the paper, we denote a vertex-pair by $(u, \pi(v))$, where $u \in V$ and $\pi(v) \in \pi(V)$. For each vertex-pair $(u, \pi(v))$, if $u = v$, then $(u, \pi(v))$ is a true pair; if $u \neq v$, then $(u, \pi(v))$ is a fake pair. The goal of graph matching is to recover this unknown permutation $\pi$ based on $G_1$ and $G_2$.

Prior literature proposes various algorithms to recover $\pi$ based on $G_1$ and $G_2$. The output of these graph matching algorithms can be interpreted as a set of partially correct seeds. Taking these partially correct seeds as input, we wish to efficiently correct all of the errors. However, it is difficult to perfectly model the correlation between the output of these algorithms and the graphs $G_1, G_2$. One way of getting around this issue is to treat these partially correct seeds as adversarially chosen and to design a matching algorithm that with high probability corrects all errors for all possible initial error patterns. However, the existing theoretical guarantees in this adversarial setting are rather pessimistic, requiring the fraction of incorrectly matched seeds is $o(1)$ (cf. [BCL+18, Lemma 3.21] and [DMWX18, Lemma 5]).

In this paper, we adopt a mathematically more tractable model introduced by [LS18], where the partially correct seeds are assumed to be generated independently from the graphs $G_1, G_2$. More specifically, we use $\hat{\pi} : V \to V$ to denote an initial mapping and generate $\hat{\pi}$ in the following way. For $\beta \in [0, 1]$, we assume that $\hat{\pi}$ is uniformly randomly chosen from all the permutations $\sigma : V \to V$ such that $\sigma(u) = \pi(u)$ for exactly $\beta n$ vertices. The benefit of this model is that $\hat{\pi}$ is independent of the graph $G$ and the sampling processes that generate $G_1$ and $G_2$, and it is convenient for us to obtain theoretical results. Then, for each seed $(u, \hat{\pi}(u))$, let $\pi(v)$ be the underlying vertex matched to $u$, i.e., $\hat{\pi}(u) = \pi(v)$. If $u = v$, then $(u, \hat{\pi}(u))$ is a correct seed; if $u \neq v$, then $(u, \hat{\pi}(u))$ is an incorrect seed. Thus, only $\beta$ fraction of the seeds are correct.

Given $G_1, G_2$ and $\hat{\pi}$, our goal is to find a mapping $\tilde{\pi} : V \to V$ such that $\lim_{n \to \infty} \mathbb{P}\{\tilde{\pi} = \pi\} = 1$.

3 Main Results

3.1 Algorithm Description

In this section, we present the general class of algorithms, shown in Algorithm 1, that we will use to recover $\pi$. As in [LS18], our algorithm also uses the notion of “witnesses”. However, unlike
With any constant \(L_{18}\), our algorithm leverages witnesses that are \(j\)-hop away. Given any graph \(G\) and two vertices \(u, v\) in \(G\), we denote the length of the shortest path from \(u\) to \(v\) in \(G\) by \(d^G(u, v)\). Then, for each vertex-pair \((u, \pi(v))\), the seed \((w, \hat{\pi}(w))\) becomes a \(j\)-hop witness for \((u, \pi(v))\) if \(d^{G_1}(u, w) = j\) and \(d^{G_2}(\pi(v), \hat{\pi}(w)) = j\). The total number of witnesses of every vertex-pair can be efficiently calculated as follows (see Algorithm 1).

We define the \(j\)-hop adjacency matrices \(A_j \in \{0, 1\}^{n \times n}\) of \(G_1\), which indicates whether a pair of vertices are \(j\)-hop neighbors in graph \(G_1\), i.e.,

\[
A_j(u, v) = \begin{cases} 
1 & \text{if } d^{G_1}(u, v) = j, \\
0 & \text{otherwise.}
\end{cases}
\]

Similarly, let \(B_j \in \{0, 1\}^{n \times n}\) denote the \(j\)-hop adjacency matrix of \(G_2\). Equivalently express the seed mapping \(\hat{\pi}\) by forming a permutation matrix \(\hat{\Pi} \in \{0, 1\}^{n \times n}\), where \(\hat{\Pi}(u, v) = 1\) if \(\hat{\pi}(u) = v\), and \(\hat{\Pi}(u, v) = 0\) otherwise. We can then count the number of \(j\)-hop witnesses for all vertex-pairs by computing \(W_j = A_j \hat{\Pi} B_j\), where the \((u, v)\)-th entry of \(W_j\) is equal to the number of \(j\)-hop witnesses for the vertex-pair \((u, \pi(v))\). This step has computational complexity of \(O(n^{2.373})\) [Gal14]. As we have mentioned, a true pair tends to have more witnesses than a fake pair, and thus we want to find the vertex correspondence between the two graphs that maximizes the total number of witnesses. In other words, given a weighted bipartite graph \(G_m\) with the vertex set being \(V \cup \pi(V)\), the edges connecting every possible vertex-pairs, and weight of an edge defined as \(w(u, \pi(v)) = W_j(u, \pi(v))\), we want to find the maximum weight matching in \(G_m\). To reduce computational complexity, we approximate the maximum weight matching by Greedy Maximum Weight Matching (GMWM) with computational complexity \(O(n^2 \log n)\). GMWM first chooses the vertex-pair with the largest weight from all candidate vertex-pairs in \(G_m\), removes all edges adjacent to the chosen vertex-pair, and then chooses the vertex-pair with the largest weight among the remaining candidate vertex-pairs, and so on. The total computational complexity of Algorithm 1 for any constant \(j\) is \(O(n^{2.375})\).

**Algorithm 1** Graph Matching based on Counting \(j\)-hop Witnesses.

1: **Input:** \(G_1, G_2, \hat{\pi}, j\)
2: Generate \(j\)-hop adjacency matrices \(A_j\) and \(B_j\) based on \(G_1\) and \(G_2\)
3: Generate \(\hat{\Pi}\) based on \(\hat{\pi}\)
4: \(W_j = A_j \hat{\Pi} B_j\)
5: \(\hat{\pi} = GMWM(W_j)\)
6: **Return** \(\hat{\pi}\)

**3.2 Results**

In this section, we present the performance guarantees for the 1-hop (i.e., \(j = 1\)) and 2-hop (i.e., \(j = 2\)) algorithms.

In [LS18], the authors show that if condition (2) holds, their 1-hop algorithm can exactly recover \(\pi\) with high probability. However, the analysis in [LS18] only takes into account the contribution of correct seeds when counting the 1-hop witnesses for true pairs. In this paper, we show a more relaxed bound of \(\beta\) by also considering the contribution of incorrect seeds.

**Theorem 1.** If condition (3) holds and \(n\) is sufficiently large, then Algorithm 1 with \(j = 1\) outputs a permutation \(\hat{\pi}\) such that \(\mathbb{P}\{\hat{\pi} = \pi\} \geq 1 - n^{-1}\).
Condition (3) is more relaxed than condition (2) for dense graphs when \( p = \Omega \left( \sqrt{\log n / ns^2} \right) \). However, when \( p = O \left( \sqrt{\log n / ns^2} \right) \), condition (3) still requires \( \beta \) to grow inversely proportional to \( np \). As we have discussed, this is because when the graph is sparse, there are not enough 1-hop witnesses among the true pairs. In order to reach improved performance for sparse graphs, we need to leverage witnesses in a larger neighborhood to match vertices. Algorithm 1 with \( j = 2 \) utilizes the 2-hop witnesses. We show that it recovers \( \pi \) with high probability under a more relaxed condition for sparse graphs.

**Theorem 2.** Suppose that \( np^2 \leq \frac{1}{135} \). If condition (4) holds and \( n \) is sufficiently large, then Algorithm 1 with \( j = 2 \) outputs a permutation \( \tilde{\pi} \) such that \( \Pr \{ \tilde{\pi} = \pi \} \geq 1 - n^{-1} \).

Figure 1: Comparison of the conditions on \( \beta \) given in Theorem 1, Theorem 2, and [LS18], when \( s \) is a fixed constant, and \( p = o(1) \).

Note that, for small \( p \), condition (4) only needs \( \beta \) to grow inversely proportional to \( n^2p^2 \). To compare the conditions on \( \beta \) given in Theorem 1, Theorem 2 and [LS18], we plot them as a function of \( p \) when \( s \) is a fixed constant, and \( p = o(1) \). The blue dashed curve depicts the condition (2) in [LS18]. The two segments correspond to: when \( p = O \left( \sqrt{\log n / n} \right) \), \( \frac{\log n}{np} \) dominates the right-hand side of condition (2); When \( p = \Omega \left( \sqrt{\log n / n} \right) \), \( p \) dominates the right-hand side of condition (2). The red curve depicts the condition (3) in Theorem 1. The two segments correspond to: when \( p = O \left( \sqrt{\log n / n} \right) \), \( \frac{\log n}{np} \) dominates the right-hand side of condition (3); When \( p = \Omega \left( \sqrt{\log n / n} \right) \), \( \sqrt{np^3 \log n} \) dominates the right-hand side of condition (3). Clearly, while the two conditions are comparable for small \( p \), condition (3) is more relaxed than condition (2) for dense graphs when \( p = \Omega \left( \sqrt{\log n / n} \right) \).

The black curve depicts the condition (4) in Theorem 2. Condition (4) has three segments:

- When \( p = O \left( \left( \frac{\log n}{n^3} \right)^{\frac{1}{4}} \right) \), \( \frac{\log n}{n^{2p^2}} \) dominates the right-hand side of condition (4).

- When \( p = \Omega \left( \left( \frac{\log n}{n^3} \right)^{\frac{1}{4}} \right) \) and \( p = O \left( n^{-2/3} \right) \), \( \sqrt{\frac{\log n}{n}} \) dominates the right-hand side of condition (4).
• When \( p = \Omega \left( n^{-2/3} \right) \) and \( p = O \left( n^{-1/2} \right) \), there are two cases. The first case is when \( s \) is a fixed constant smaller than 1, in which case \( \sqrt{np^3 \log n} \) dominates the right-hand side of condition (4). The other case is when \( s = 1 \), in which case \( \sqrt{np^3 \log n} = 0 \) and \( \sqrt{\log n} \) dominates the right-hand side of condition (4). This bifurcation arises because if \( s < 1 \), then the two vertices of a true pair would have different 2-hop neighbors, which renders it harder to distinguish the true pairs based on the number of 2-hop witnesses when \( p = \Omega (n^{-\frac{1}{2}}) \).

Clearly, condition (4) is more relaxed than condition (2) and (3) for sparse graphs, provided that \( p = O \left( \left( \frac{\log n}{n^s} \right)^{\frac{1}{3}} \right) \) when \( s \) is a constant smaller than 1 and that \( p = O \left( n^{-1/2} \right) \) when \( s = 1 \).

### 3.3 Intuition and Analysis Challenges

In this section, we explain the intuition and analysis challenges for Theorem 1 and Theorem 2.

To understand the intuition behind Theorem 1 and why it provides a better result than [LS18], recall that the 1-hop algorithm will succeed (in recovering \( \pi \)) if the number of 1-hop witnesses for any true pair is larger than the number of 1-hop witnesses for any fake pair. For any correct seed, it is a 1-hop witness for a true pair with probability \( ps^2 \) and is a 1-hop witness for a fake pair with probability \( p^2 s^2 \). In contrast, for any incorrect seed, it is a 1-hop witness with probability \( p^2 s^2 \) for both true pairs and fake pairs. Since there are \( n\beta \) seeds that are correct, it follows that

\[
W_1(u, \pi(v)) \sim \begin{cases} \text{Binom} \left( n\beta, ps^2 \right) + \text{Binom} \left( n(1 - \beta), p^2 s^2 \right) & \text{if } u = v, \\ \text{Binom}(n, p^2 s^2) & \text{if } u \neq v. \end{cases}
\]

where \( \sim \) denotes “approximately distributed”. In (5), the second Binom distribution for \( u = v \) is not precise because the events that each incorrect seed becomes a witness for a true pair are dependent on each other. We address this issue using the concentration inequality for dependent random variables [Jan04]. Please refer to Appendix C for details.

Therefore, in order to make the 1-hop algorithm work, we need the difference between the expected values of the two cases in (5) to be greater than both standard deviations, i.e.,

\[
\begin{align*}
& n\beta ps^2 + n(1 - \beta)p^2 s^2 - np^2 s^2 \geq \sqrt{n\beta ps^2} + \sqrt{np^2 s^2} \\
\iff & n\beta p(1 - p)s^2 \geq \sqrt{n\beta ps^2} + \sqrt{np^2 s^2} \\
\iff & n\beta p(1 - p)s^2 \geq 2\sqrt{n\beta ps^2} \text{ and } n\beta p(1 - p)s^2 \geq 2\sqrt{np^2 s^2}.
\end{align*}
\]

This immediately leads to \( \beta \geq \max \left\{ \frac{4}{n\beta p(1 - p)s^2}, \frac{1}{2\sqrt{n(1 - p)^3 s^2}} \right\} \), which differ from the condition (3) in Theorem 1 only by the \( \log n \) factor. Adding the \( \log n \) factor ensures that the 1-hop algorithm will succeed with high probability. This argument suggests that condition (3) is also close to necessary for the 1-hop algorithm to succeed, which is confirmed by our simulation results in Appendix A.

We next explain condition (4) in Theorem 2. First, we need that the average degree of every vertex should be high enough so that there is no isolated vertex in \( G_1 \) or \( G_2 \). For the correlated Erdős-Rényi model, \( nps^2 - \log n \rightarrow +\infty \) ensures the intersection graph \( G_1 \land G_2 \) to be connected [ER59]. Then, analogous to the 1-hop algorithm, we derive the condition on \( \beta \) by comparing the expected values and standard deviations of the number of 2-hop witnesses for true pairs and for fake pairs. However, the dependency issue is more severe here when we bound the number of 2-hop witnesses. Specifically, in the analysis of (5) (see Appendix C), the event that a seed becomes a
1-hop witness for a true pair is at most dependent on that of two other incorrect seeds. For here, any two seeds could be dependent through the 1-hop neighborhood of the candidate vertex-pair (see Appendix D). Thus, directly using the concentration inequality in [Jan04] will lead to a poor bound. To address this new difficulty, we will condition on the 1-hop neighborhood first. After this conditioning, the remaining dependency becomes more manageable, which is handled by either ignoring a small fraction of seeds or by applying the concentration inequality in [Jan04] again. Please refer to Appendix D for details.

Towards this end, given any graph $G$, for any vertex $u$ in graph $G$, we use $N_j^G(u)$ to denote the set of $j$-hop neighbors of $u$ in $G$, i.e.,

$$N_j^G(u) = \{v \in G : d^G(u, v) = j\}.$$

For any two vertices $u, v$ in graph $G_1$, let $C_1(u, \pi(v))$ denote the set of 1-hop “common” neighbors of $u$ and $\pi(v)$ across $G_1$ and $G_2$, i.e.,

$$C_1(u, \pi(v)) = \{(w, \pi(w)) : w \in N_1^{G_1}(u), \pi(w) \in N_1^{G_2}(\pi(v))\}.$$

Conditioning on the 1-hop neighborhoods, we calculate the probability that a seed becomes a 2-hop witness by calculating the probability that the seed connects to the 1-hop neighbors. For any correct seed, it is a 2-hop witness for a true pair $(u, \pi(u))$ with probability about $|C_1(u, \pi(u))| ps^2$ and is a 2-hop witness for a fake pair $(u, \pi(v))$ with probability about $|N_1^{G_1}(u)| |N_1^{G_2}(\pi(v))| ps^2$. In contrast, for any incorrect seed, it is a 2-hop witness for a true pair $(u, \pi(u))$ with probability about $|N_1^{G_1}(u)| |N_1^{G_2}(\pi(v))| ps^2$ and is a 2-hop witness for a fake pair $(u, \pi(v))$ with probability about $|N_1^{G_1}(u)| |N_1^{G_2}(\pi(v))| ps^2$. Then, we have

$$W_2(u, \pi(v)) \sim \begin{cases} \text{Binom} \left(n\beta, |C_1(u, \pi(u))| ps^2\right) & \text{if } u = v, \\ + \text{Binom} \left(n(1 - \beta), |N_1^{G_1}(u)| |N_1^{G_2}(\pi(v))| ps^2\right) & \text{if } u \neq v. \end{cases} \quad (7)$$

The number of 1-hop neighbors of a certain vertex in $G_1$ or $G_2$ is approximately $nps \pm O\left(\sqrt{nps}\right)$, and the number of common 1-hop neighbors for a true pair is approximately $nps^2 \pm O\left(\sqrt{nps^2}\right)$. In order to make the 2-hop algorithm work, we need the difference between the conditional expected values of the two cases in (7) to be greater than both conditional standard deviations. We first ignore the fluctuation of the size of 1-hop neighborhood and use only the expected values of $|C_1(u, \pi(u))|, |N_1^{G_1}(u)|, |N_1^{G_2}(\pi(u))|$ and $|N_1^{G_2}(\pi(v))|$. Then, we need

$$n\beta \left[nps^2 ps^2\right] + n(1 - \beta) \left[(nps)(nps)p^2 s^2\right] - n(nps)(nps)p^2 s^2 \geq \sqrt{n\beta(nps^2) ps^2} + \sqrt{n(nps)(nps)p^2 s^2}. \quad (8)$$

Analogous to (6), (8) leads to the sufficient condition

$$\beta \geq \max \left\{ \frac{16}{n^2 p^2 s^4}, 4\sqrt{\frac{1}{ns^4}} \right\}. \quad (9)$$

Next, we consider the fluctuation of the size of 1-hop neighborhood. To guarantee the difference between the conditional expected values of the two cases in (7) to be greater than zero, it suffices
to ensure that
\[
\begin{align*}
\beta \left( nps^2 - \sqrt{nps^2} \right) ps^2 + n(1 - \beta)(nps - \sqrt{nps})^2 p^2 s^2 - n (nps + \sqrt{nps})^2 p^2 s^2 \\
= n\beta \left( nps^2 - \sqrt{nps^2} \right) ps^2 - n\beta(nps - \sqrt{nps})^2 p^2 s^2 - 4n^2 p^3 s^3 \sqrt{nps}
\end{align*}
\]
\begin{align*}
&\approx n^2 \beta^2 p^2 s^4 - n^3 \beta p^4 s^4 - 4n^2 p^3 s^3 \sqrt{nps} \\
&\approx n^2 \beta^2 p^2 s^4 - 4n^2 p^3 s^3 \sqrt{nps} \geq 0,
\end{align*}
\]
where the approximation in the step (a) is based on \( nps^2 \gg 1 \), and the step (b) is based on \( np^2 \ll 1 \).

The fluctuation of the number of 1-hop neighbors has a negligible impact on the standard deviation of the number of 2-hop witnesses. Thus, combining (8) and (10) yields an approximately sufficient condition for 2-hop algorithm to succeed,
\[
\beta \geq \max \left\{ 4 \sqrt{\frac{np^3}{s}}, 4 \sqrt{\frac{1}{ns^4}}, \frac{16}{n^2 p^2 s^4} \right\}.
\]

However, condition (11) does not give us the desirable result in condition (4) because we have considered a very strict criteria for GMWM to succeed, which requires the numbers of 2-hop witnesses of any true pair to be greater than the numbers of 2-hop witnesses of any fake pair. Indeed, the GMWM algorithm may succeed even when the above criteria does not hold. For example, consider the case when \( u \) and \( \pi(u) \) both have few 1-hop neighbors, while \( \pi(v) \) has many 1-hop neighbors (see Fig. 2). Then, the fake pair \((u, \pi(v))\) may have more 2-hop witnesses than the true pair \((u, \pi(u))\). Thus, we can not guarantee the 2-hop algorithm to succeed (and inequality (10) will not hold) if we use the above criteria. However, note that the 1-hop neighbors of \( v \) and \( \pi(v) \) should overlap significantly, and thus in this case \( v \) is also likely to have many 1-hop neighbors. As a result, the true pair \((v, \pi(v))\) is likely to have more 2-hop witnesses than the fake pair \((u, \pi(v))\). Then, GMWM will still select the true pair \((v, \pi(v))\) and eliminate the fake pair \((u, \pi(v))\). From the above example, we can see that, in order to make the 2-hop algorithm work, it suffices to consider the new criteria that each fake pair, \((u, \pi(u))\), has fewer 2-hop witnesses than either the true pair \((u, \pi(u))\) or the true pair \((v, \pi(v))\).

![Figure 2: The fake pair \((u, \pi(v))\) has more 2-hop witnesses than the true pair \((u, \pi(u))\), but it has fewer 2-hop witnesses than the true pair \((v, \pi(v))\). The 2-hop algorithm still works in this case.](image)

Next, we use the above new criteria to analyze when the 2-hop algorithm succeeds. Since \( N_{G^1}(u) \) and \( N_{G^2}(\pi(u)) \) are both generated by sampling with probability \( s \) the 1-hop neighbors of \( u \) in the parent graph \( G \), the difference between \( |N_{G^1}(u)| \) and \( |N_{G^2}(\pi(u))| \) is bounded by roughly...
\( \sqrt{2 |N^G_1(u)| s(1-s)} \leq 2 \sqrt{nps(1-s)} \) with high probability. Building upon this observation, for any two vertices \( u \neq v \) in graph \( G_1 \), we can derive that
\[
\left( |N^G_1(u)| - |N^G_1(v)| \right) \leq \left( |N^G_2(u)| - |N^G_2(v)| \right) \leq 4 \sqrt{nps(1-s)}.
\]

This inequality implies that either \( |N^G_1(u)| - |N^G_1(v)| \) or \( |N^G_2(u)| - |N^G_2(v)| \) is no larger than \( 2 \sqrt{nps(1-s)} \). Without loss of generality, we assume below that \( |N^G_2(u)| - |N^G_2(v)| \leq 2 \sqrt{nps(1-s)} \). Then, by comparing the conditional expected value of \( W_2(u, \pi(v)) \) and that of \( W_2(u, \pi(v)) \), we can relax the condition (10). Specifically, it suffices to ensure that
\[
n \beta \left( nps^2 - \sqrt{nps^2} \right) ps^2 + n(1 - \beta) \left| N^G_1(u) \right| \left| N^G_2(\pi(u)) \right| ps^2
\]
\[
- n \left| N^G_1(u) \right| \left| N^G_2(\pi(v)) \right| p^2s^2
\]
\[
\approx n^2 \beta p^2s^4 + n \left| N^G_1(u) \right| \left( \left| N^G_2(\pi(u)) \right| - \left| N^G_2(\pi(v)) \right| \right) p^2s^2
\]
\[
\geq n^2 \beta p^2s^4 - 4n^2p^3s^4 \sqrt{nps(1-s)} \geq 0,
\]
where the approximation in step (a) is based on \( n \beta \left| N^G_1(u) \right| \left| N^G_2(\pi(u)) \right| p^2s^2 \approx n^3 \beta p^4s^4 \ll n^2 \beta p^2s^4 \).

This immediately leads to the condition \( \beta \geq 4 \sqrt{\frac{nps(1-s)}{s}} \). Combining with (9) leads to condition (4) in Theorem 2 except for the \( \log n \) factor. Adding the \( \log n \) factor ensures the 2-hop algorithm to succeed with high probability. Note that, when \( 1 - s = o(1) \), condition (4) given by the new criteria is more relaxed than (11) given by the old criteria.

4 Numerical Experiments

In this section, we conduct numerical studies to compare the performance of the 1-hop and 2-hop algorithms. The experiment results show that the 2-hop algorithm outperforms the 1-hop algorithm in both synthetic and real graphs when the graphs are sparse.

Moreover, we verify our theoretical results, Theorem 1 and Theorem 2, are not only sufficient, but also close to necessary for the 1-hop and 2-hop algorithms to succeed, respectively. These experiment results are deferred to Appendix A due to space constraints.

4.1 Performance Comparison with Synthetic Data

For our experiments using synthetic data, we generate \( G_1 \), \( G_2 \) and \( \pi \) according to the correlated Erdős-Rényi model. We calculate the accuracy rate as the median of the proportion of vertices that are correctly matched, taken over 10 independent simulations. In Fig. 3, we plot the accuracy rates of the 1-hop and 2-hop algorithms for \( p = n^{-\frac{3}{4}} \) and \( s = 0.8 \). We vary the number of vertices from 2000 to 8000, and fix \( s = 0.8 \). From the results, we observe that the 2-hop algorithm significantly outperforms the 1-hop algorithm.
4.2 Performance Comparison with Real Data

We use the Autonomous Systems dataset from [LKF05] as real data graphs. The dataset consists of 9 graphs of Autonomous Systems (AS) peering information inferred from Oregon route-views between March 31, 2001, and May 26, 2001. Since some vertices and edges are changed over time, these nine graphs can be viewed as correlated versions of each other. The number of vertices of the 9 graphs ranges from 10,670 to 11,174 and the number of edges from 22,002 to 23,409. Thus, the average degree is about 3, and the graphs are sparse. To test the graph matching methods, we consider 10,000 vertices of the network that are present in all nine graphs. We apply the 1-hop algorithm and 2-hop algorithm to match each graph to that on March 31, with vertices randomly permuted. The performance comparison of the two algorithms is plotted in Fig. 4 for $\beta = 0.3, 0.6, 0.9$. We observe that our proposed 2-hop algorithm significantly outperforms the 1-hop algorithm. Note that the accuracy rate decays in time because over time the graphs become less correlated with the initial one on March 31.

5 Conclusion

In this work, we tackle the graph matching problem with partially-correct seeds. Under the correlated Erdős-Rényi model, we first present a sharper characterization of the condition for the 1-hop algorithm to perfectly recover the vertex matching, which is more relaxed than the prior art for dense graphs. Then, by exploiting 2-hop neighbourhoods, we proposed a more efficient 2-hop algorithm that exactly recovers the true vertex correspondence more effectively for sparse graphs. Experimental results validate the our theoretical analysis. Possible directions for future work includes finding more efficient algorithms to recover true vertex correspondence in dense graphs, analyzing the performance of algorithms using neighborhoods of a larger number of hops, and studying graph matching under other random graph models beyond Erdős-Rényi random graph.
Appendix A Numerical Experiments to Verify Our Theoretical Results

In this section, we conduct numerical studies to verify that our theoretical results, Theorem 1 and Theorem 2, are not only sufficient, but also close to necessary for the 1-hop and 2-hop algorithms to succeed, respectively.

For our experiments using synthetic data, we generate $G_1$, $G_2$ and $\hat{\pi}$ according to the correlated Erdős-Rényi model. We vary the number of vertices from 2000 to 8000, and fix $s = 0.8$. We calculate the accuracy rate as the median of the proportion of vertices that are correctly matched, taken over 10 independent simulations.

We first verify that the condition (3) in Theorem 1 is both sufficient and close to necessary for the 1-hop algorithm to exactly recover $\pi$. We simulate the performance of the 1-hop algorithm for $p = n^{-\frac{1}{4}}$ and $p = n^{-\frac{3}{4}}$. The results are presented in Fig. 5(a) and Fig. 6(a) as a function of $\beta$. Since Theorem 1 predicts that the 1-hop algorithm succeeds in exact recovery with high probability at $\beta \gtrsim \sqrt{\frac{\log n}{n}}$ when $p = \Omega\left(\sqrt{\frac{\log n}{n}}\right)$ and at $\beta \gtrsim \frac{\log n}{np}$ when $p = O\left(\sqrt{\frac{\log n}{n}}\right)$, we rescale the x-axis in Fig. 5(b) and Fig. 6(b) as $\beta/\sqrt{\frac{\log n}{n}}$ for $p = n^{-\frac{1}{4}}$ and $\beta/\left(\frac{\log n}{np}\right)$ for $p = n^{-\frac{3}{4}}$. As we can see in Fig. 5(b) and Fig. 6(b), the curves for different $n$ align well with each other, which suggests that condition (3) is both sufficient and close to necessary for the 1-hop algorithm to succeed.

Next, we verify that the condition (4) in Theorem 2 is both sufficient and close to necessary for the 2-hop algorithm to exactly recover $\pi$. We simulate the performance of the 2-hop algorithm for $p = n^{-\frac{1}{4}}$, $p = n^{-\frac{3}{4}}$, and $p = n^{-\frac{7}{5}}$. The results are presented in Fig. 7(a), Fig. 8(a) and Fig. 9(a). Since Theorem 2 predicts that the 2-hop algorithm succeeds in exact recovery with high probability when $\beta \gtrsim \max\left\{\sqrt{np\log n}, \sqrt{\frac{\log n}{n}}, \frac{\log n}{np}\right\}$, we rescale the x-axis in Fig. 7(b), Fig. 8(b)
and Fig. 9(b) as $\beta/\sqrt{np^3 \log n}$ for $p = n^{-\frac{1}{3}}$, $\beta/\sqrt{\log n}$ for $p = n^{-\frac{2}{3}}$ and $\beta/\left(\log n / np\right)$ for $p = n^{-\frac{4}{5}}$. As we can see in Fig. 7(b), Fig. 8(b) and Fig. 9(b), the curves for different $n$ align well with each other, which suggests that condition (4) is both sufficient and close to necessary for the 2-hop algorithm to succeed.

In addition, if $s = 1$ and $p = n^{-\frac{1}{3}}$, we show in Fig. 10 that the curves for different $n$ align well when we rescale the x-axis as $\beta/\sqrt{\log n}$, but they do not align well with each other when the x-axis is rescaled as $\beta/\sqrt{np^3 \log n}$. This result agrees with Theorem 2, demonstrating that the condition (11) derived from the old criteria is not tight.

**Appendix B  Preliminary Results for Proofs**

In this section, we present some preliminary results that are useful for the proofs of Theorem 1 and Theorem 2.

**Notation** For any positive integer $n$, let $[n] = \{1, 2, ..., n\}$.

**Theorem 3.** Chernoff Bound ([DP09]): Let $X = \sum_{i \in [n]} X_i$, where $X_i$, $i \in [n]$, are independent
random variables taking values in \( \{0, 1\} \). Then, for \( \delta \in (0,1) \),

\[
P\{X \leq (1-\delta)\mathbb{E}(X)\} \leq \exp \left( -\frac{\delta^2}{2} \mathbb{E}(X) \right),
\]
\[
P\{X \geq (1+\delta)\mathbb{E}(X)\} \leq \exp \left( -\frac{\delta^2}{3} \mathbb{E}(X) \right).
\]

**Theorem 4.** **Bernstein’s Inequality ([DP09]):** Let \( X = \sum_{i\in[n]} X_i \), where \( X_i, i \in [n], \) are independent random variables such that \( |X_i| \leq K \) almost surely. Then, for \( t > 0 \), we have

\[
P\{X \geq \mathbb{E}(X) + t\} \leq \exp \left( -\frac{t^2}{2(\sigma^2 + Kt/3)} \right),
\]

where \( \sigma^2 = \sum_{i\in[n]} \text{var}(X_i) \) is the variance of \( X \). It follows then for \( \gamma > 0 \), we have

\[
P\left\{ X \geq \mathbb{E}(X) + \sqrt{2\sigma^2\gamma} + \frac{2K\gamma}{3} \right\} \leq \exp(-\gamma).
\]

The obtained estimate holds for \( P\left\{ X \leq \mathbb{E}(X) - \sqrt{2\sigma^2\gamma} - \frac{2K\gamma}{3} \right\} \) too (by considering \(-X\)), i.e.,

\[
P\left\{ X \leq \mathbb{E}(X) - \sqrt{2\sigma^2\gamma} - \frac{2K\gamma}{3} \right\} \leq \exp(-\gamma).
\]
Figure 9: The 2-hop algorithm with varying \(n\) and \(p = n^{-\frac{4}{5}}\). Fix \(s = 0.8\).

Figure 10: The 2-hop algorithm with varying \(n\) and \(p = n^{-\frac{1}{2}}\). Fix \(s = 1\).

**Corollary 1.** Let \(X\) denote a random variable such that \(X \sim \text{Binom}(n_x, \alpha)\). If \(n_x \in [n_{\text{min}}, n_{\text{max}}]\), then for \(\gamma > 0\),

\[
P \left\{ X \leq n_{\text{min}}\alpha - \sqrt{2n_{\text{max}}\alpha\gamma} - \frac{2\gamma}{3} \right\} \leq \exp(-\gamma),
\]

and

\[
P \left\{ X \geq n_{\text{max}}\alpha + \sqrt{2n_{\text{max}}\alpha\gamma} + \frac{2\gamma}{3} \right\} \leq \exp(-\gamma).
\]

**Proof.** Since \(X \sim \text{Binom}(n_x, \alpha)\), applying Bernsteins inequality in Theorem 4 yields

\[
P \left\{ X \leq n_x\alpha - \sqrt{2n_x\alpha(1-\alpha)\gamma} - \frac{2\gamma}{3} \right\} \leq \exp(-\gamma). \tag{12}
\]

Since \(n_x \in [n_{\text{min}}, n_{\text{max}}]\), we have

\[
n_x\alpha - \sqrt{2n_x\alpha(1-\alpha)\gamma} - \frac{2\gamma}{3} \geq n_{\text{min}}\alpha - \sqrt{2n_{\text{max}}\alpha\gamma} - \frac{2\gamma}{3}. \tag{13}
\]

and

\[
P \left\{ X \leq n_{\text{min}}\alpha - \sqrt{2n_{\text{max}}\alpha\gamma} - \frac{2\gamma}{3} \right\} \leq \exp(-\gamma). \tag{14}
\]
Similarly,

\[ P \left\{ X \leq n_x \alpha + \sqrt{2n_x \alpha} + \frac{2\gamma}{3} \right\} \leq \exp (-\gamma). \tag{15} \]

Since \( n_x \in [n_{\min}, n_{\max}] \), we have

\[ n_x \alpha + \sqrt{2n_x \alpha(1 - \alpha)\gamma} + \frac{2\gamma}{3} \leq n_{\max} \alpha + \sqrt{2n_{\max} \alpha \gamma} + \frac{2\gamma}{3}. \tag{16} \]

and

\[ P \left\{ X \leq n_{\max} \alpha + \sqrt{2n_{\max} \alpha(1 - \alpha)\gamma} + \frac{2\gamma}{3} \right\} \leq \exp (-\gamma). \tag{17} \]

**Definition 1.** Given random variables \( \{X_i\}, i \in [n] \), the dependency graph for \( \{X_i\} \) is a graph \( \Gamma \) with vertex set \([n]\) such that if \( i \in [n] \) is not connected by an edge to any vertex in \( J \) for \( J \subset [n] \), then \( X_i \) is independent of \( \{X_j\} \) for \( j \in J \).

**Theorem 5.** ([Jan04]) Let \( X = \sum_{i \in [n]} X_i \), where \( X_i \), \( i \in [n] \) are random variables such that \( X_i - \mathbb{E}[X_i] \leq K \), \( i \in [n] \) for some \( K > 0 \). We let \( \Gamma \) denote a dependency graph for \( \{X_i\} \) and \( \Delta(\Gamma) \) denote the maximum degree of \( \Gamma \). Let \( \sigma^2 = \sum_{i \in [n]} \text{var}(X_i) \). Then, for \( t \geq 0 \), we have

\[ P \left\{ X \geq \mathbb{E}[X] + t \right\} \leq \exp \left( -\frac{8t^2}{25\Delta_1(\Gamma)(\sigma^2 + Kt/3)} \right), \]

where \( \Delta_1(\Gamma) = \Delta(\Gamma) + 1 \). It follows then for \( \gamma > 0 \), we have

\[ P \left\{ X \geq \mathbb{E}[X] + \sqrt{\frac{25\Delta_1(\Gamma)}{8}\sigma^2\gamma + \frac{25\Delta_1(\Gamma)K\gamma}{24}} \right\} \leq \exp (-\gamma). \]

If the assumption \( X_i - \mathbb{E}[X_i] \leq K \) is reversed to \( X_i - \mathbb{E}[X_i] \geq -K \), the obtained estimate holds for \( P \left\{ X \leq \mathbb{E}[X] - \sqrt{\frac{25\Delta_1(\Gamma)}{8}\sigma^2\gamma - \frac{25\Delta_1(\Gamma)K\gamma}{24}} \right\} \) too (by considering \(-X\)), i.e.,

\[ P \left\{ X \leq \mathbb{E}[X] - \sqrt{\frac{25\Delta_1(\Gamma)}{8}\sigma^2\gamma - \frac{25\Delta_1(\Gamma)K\gamma}{24}} \right\} \leq \exp (-\gamma). \]

**Theorem 6.** For \( r \geq 0 \), every real number \( x \in (0,1) \) and \( rx \leq 1 \), it holds that

\[ r \log (1 - x) \leq \log (1 - \frac{rx}{2}). \]

**Proof.** We set

\[ f(x) = r \log (1 - x) - \log (1 - \frac{rx}{2}), \]

and have

\[ f(0) = r \log 1 - \log 1 = 0. \]
Then, we can get the derivative of $f(x)$,

$$f'(x) = \frac{-r}{1-x} + \frac{r}{2-rx}$$

$$= \frac{r(rx - x) - 1}{(2-rx)(1-x)}$$

$$\leq 0.$$

It shows that $f(x)$ decreases and $f(0) = 0$. Thus, we conclude that the statement is true for $r > 0$, $x \in (0, 1)$ and $rx \leq 1$.

\[ \square \]

### Appendix C  Proof of Theorem 1

We prove Theorem 1 by proving that, with high probability, $W_1(u, \pi(u)) > W_1(v, \pi(w))$ for all vertices $u$, $v$ and $w$ in graph $G_1$. This follows from the following two lemmas. The proofs of the lemmas are deferred to Appendix C.1 and C.2, respectively.

**Lemma 1.** For any vertex $u$ in graph $G_1$ and sufficiently large $n$, the following holds

$$\mathbb{P}\left\{ W_1(u, \pi(u)) > x_{min} + y_{min} \right\} \geq 1 - n^{-\frac{7}{2}}.$$

where $x_{min} = (n\beta - 1)p s^2 - \sqrt{5n\beta p s^2 \log n} - \frac{5}{3}\log n$ and $y_{min} = (n(1-\beta)-2)p s^2 - 5\sqrt{np^2 s^2 \log n} - \frac{37}{3}\log n$.

**Lemma 2.** For any two vertices $v \neq w$ in graph $G_1$ and sufficiently large $n$, the following holds

$$\mathbb{P}\left\{ W_1(v, \pi(w)) < z_{max} \right\} \geq 1 - n^{-\frac{7}{2}}.$$

where $z_{max} = np^2 s^2 + \sqrt{7np^2 s^2 \log n} + \frac{7}{2}\log n + 2$.

**Proof of Theorem 1.** Based on Lemma 1 and the union bound, we have

$$\mathbb{P}\left\{ \bigcap_{u \in V} \{ W_1(u, \pi(u)) > x_{min} + y_{min} \} \right\} \geq 1 - n \cdot n^{-\frac{7}{2}} = 1 - n^{-\frac{9}{2}}.$$

Based on Lemma 2 and the union bound, we have

$$\mathbb{P}\left\{ \bigcap_{v, w \in V} \{ W_1(v, \pi(w)) < z_{max} \} \right\} \geq 1 - n^2 \cdot n^{-\frac{7}{2}} = 1 - n^{-\frac{9}{2}}.$$

It remains to verify $x_{min} + y_{min} - z_{max} \geq 0$ under the condition of Theorem 1. Note that

$$x_{min} + y_{min} - z_{max} = n\beta p(1-p)s^2 - \sqrt{5n\beta p s^2 \log n} - (5 + \sqrt{7})\sqrt{np^2 s^2 \log n} - 2p(1+p)s^2 - \frac{37}{3}\log n - 2.$$

First, by assumption that $\beta \geq \frac{45\log n}{np(1-p)s^2}$, we have

$$\frac{1}{3}n\beta p(1-p)s^2 \geq \sqrt{\frac{45\log n}{np(1-p)s^2}} \cdot \frac{1}{3}n\sqrt{\beta p(1-p)s^2} = \sqrt{5n\beta p s^2 \log n}.$$  \hspace{1cm} (18)
Second, by assumption that $\beta \geq 30\sqrt{\frac{\log n}{n(1-p)s^2}}$, we have

$$\frac{1}{3}n\beta p(1-p)s^2 \geq 30\sqrt{\frac{\log n}{n(1-p)s^2}} \cdot \frac{1}{3}np(1-p)s^2 \geq (5 + \sqrt{7})\sqrt{np^2s^2 \log n}.$$  

(19)

Third, by assumption that $\beta \geq \frac{45\log n}{np(1-p)s^2}$ and $n$ is sufficiently large, we have

$$\frac{1}{3}n\beta p(1-p)s^2 \geq \frac{45\log n}{np(1-p)s^2} \cdot \frac{1}{3}np(1-p)s^2 \geq \frac{37}{3} \log n + 2p(1+p)s^2.$$  

(20)

Combing (18)-(20), we have $x_{\min} + y_{\min} - z_{\max} \geq 0$. Taking a union bound, we have

$$\mathbb{P}\left\{ \min_{u \in V} W_1(u, \pi(u)) > \max_{v,w \in V} W_1(v, \pi(w)) \right\}$$

$$\geq 1 - \mathbb{P}\left\{ \bigcup_{u \in V} \{ W_1(u, \pi(u)) \leq x_{\min} + y_{\min} \} \right\} - \mathbb{P}\left\{ \bigcup_{v,w \in V} \{ W_1(v, \pi(w)) \geq z_{\max} \} \right\}$$

$$\geq 1 - n^{-\frac{4}{3}} - n^{-\frac{3}{4}} \geq 1 - n^{-1}.$$  

Thus, GMWM outputs $\tilde{\pi}$ with $\mathbb{P}\{ \tilde{\pi} = \pi \} \geq 1 - n^{-1}$ if we use the 1-hop algorithm to match graphs.

\[\square\]

\section*{C.1 Proof of Lemma 1}

Fix a true pair $(u, \pi(u))$, we next bound the number of 1-hop witnesses for $(u, \pi(u))$. For each seed $(u_i, \hat{\pi}(u_i))$, let $\pi(v_i)$ be the underlying vertex matched to $u_i$, i.e., $\hat{\pi}(u_i) = \pi(v_i)$. Then, $(u_i, \pi(v_i))$ is a correct seed if $u_i = v_i$ and is an incorrect seed if $u_i \neq v_i$. Among all seeds, some of them may be of the form that $u_i = u$ or $v_i = u$. Then, they can not become 1-hop witnesses for $u$ and $\pi(u)$. The number of such seeds is at most 2. In the following, we exclude such seeds.

We first count the contribution to $W_1(u, \pi(u))$ by correct seeds. For any correct seed $(u_i, \pi(u_i))$, let $X_i$ be a binary random variable such that $X_i = 1$ if $(u_i, \pi(u_i))$ is a 1-hop witness for $u$ and $\pi(u)$, and $X_i = 0$ otherwise. Then, we have $\mathbb{P}\{ X_i = 1 \} = ps^2$ because $X_i = 1$ if and only if the edge $(u, u_i)$ is in $G$, and is sampled into both $G_1$ and $G_2$. Note that the edges $(u, u_i)$’s are different for different $u_i$ in the parent graph $G$, and the sampling process is independent on each edge. Thus, $X_i$’s are mutually independent across $u_i$. Let $X$ denote the number of 1-hop witnesses contributed by the correct seeds. Since there are at least $n\beta - 1$ correct seeds that could be 1-hop witnesses for $u$ and $\pi(u)$, it follows that $X \geq \sum_{i=1}^{n\beta-1} X_i \sim \text{Binom}(n\beta - 1, ps^2)$. Recall that $x_{\min} = (n\beta - 1)ps^2 - \sqrt{5n\beta ps^2 \log n} - \frac{5}{3} \log n$. It follows that

$$\mathbb{P}\{ X \leq x_{\min} \} \leq \mathbb{P}\left\{ \sum_{i=1}^{n\beta-1} X_i \leq x_{\min} \right\}$$

$$\leq \mathbb{P}\left\{ \sum_{i=1}^{n\beta-1} X_i \leq (n\beta - 1)ps^2 - \sqrt{5(n\beta - 1)ps^2(1 - ps^2)} \log n - \frac{5}{3} \log n \right\}$$

$$\leq \exp\left( -\frac{5}{2} \log n \right) = n^{-\frac{5}{4}}.$$  

(21)
where that last inequality follows from Bernstein’s inequality given in Theorem 4 with $\gamma = \frac{5}{2} \log n$ and $K = 1$.

We then count the contribution to $W_1(u, \pi(u))$ by incorrect seeds. For any incorrect seed $(u_i, \pi(v_i))$, let $Y_i$ be a binary random variable such that $Y_i = 1$ if $(u_i, \pi(v_i))$ is a 1-hop witness for $u$ and $\pi(u)$, and $Y_i = 0$ otherwise. Then, we have $\mathbb{P}\{Y_i = 1\} = p^2s^2$ because $Y_i = 1$ if and only if the two edges $(u, u_i)$ and $(u, v_i)$ are both in $G$, and are sampled in $G_1$ and $G_2$, respectively. Let $Y$ denote the number of 1-hop witnesses contributed by the incorrect seeds. Since there are $n(1 - \beta)$ incorrect seeds, the number of incorrect seeds that could be 1-hop witness for $(u, \pi(u))$ is no less than $n(1 - \beta) - 2$. Thus, $Y \geq \sum_{i=1}^{n(n(1-\beta) - 2)} Y_i$. Note that $Y_i$ are dependent. Specifically, the event that $(u_i, \pi(v_i))$ becomes a 1-hop witness for $(u, \pi(u))$ is dependent on $(u_j, \pi(u_i))$ and $(v_i, \pi(v_j))$ (See Fig. 11 for an example). Then, we cannot apply Bernstein’s Inequality in Theorem 4.

Fortunately, the event $(u_j, \pi(v_j))$ becomes a 1-hop witness for $(u, \pi(u))$ if $u_j = v_i$ or $v_j = u_i$. Thus, the event that $(u_i, \pi(v_i))$ becomes a 1-hop witness for $(u, \pi(u))$ depends on at most two other seeds. Then, we apply the concentration inequality for the sum of dependent random variables given in Theorem 5. Specifically, we construct a dependency graph $\Gamma$ for $\{Y_i\}$. The maximum degree of $\Gamma$, $\Delta(\Gamma)$, equals two. Thus, we apply Theorem 5 with $\Delta(\Gamma) = \Delta(\Gamma) + 1 = 3$, $K = 1$, $\sigma^2 = (n(1-\beta) - 2)p^2s^2(1-p^2s^2)$ and $\gamma = \frac{8}{3} \log n$. Recall that $\gamma_{min} = (n(1-\beta) - 2)p^2s^2 - 5\sqrt{np^2s^2 \log n} - \frac{25}{3} \log n$. We then get

$$
\mathbb{P}\{Y \leq y_{min}\} \\
\leq \mathbb{P}\left\{\sum_{i=1}^{n(n(1-\beta) - 2)} Y_i \leq y_{min}\right\} \\
\leq \mathbb{P}\left\{\sum_{i=1}^{n(n(1-\beta) - 2)} Y_i \leq (n(1-\beta) - 2)p^2s^2 - 5\sqrt{(n(1-\beta) - 2)p^2s^2(1-p^2s^2)} \log n - \frac{25}{3} \log n\right\} \\
\leq \exp\left(-\frac{8}{3} \log n\right) = n^{-\frac{8}{3}}.
$$

Figure 11: The event that $(u_i, \pi(v_i))$ becomes a 1-hop witness for $(u, \pi(u))$ is dependent on $(u_j, \pi(u_i))$ and $(v_i, \pi(v_j))$.

Finally, since $W_1(u, \pi(u)) = X + Y$ and $n$ is sufficiently large, (21) and (22) yield that

$$
\mathbb{P}\{W_1(u, \pi(u)) \leq x_{min} + y_{min}\} \leq \mathbb{P}\{X \leq x_{min}\} + \mathbb{P}\{Y \leq y_{min}\} \leq n^{-\frac{2}{3}} + n^{-\frac{8}{3}} < n^{-\frac{7}{3}}.
$$
C.2 Proof of Lemma 2

Fix a fake pair \((v, \pi(w))\), we next bound the number of 1-hop witnesses for \((v, \pi(w))\). For each seed \((u_i, \hat{\pi}(u_i))\), let \(\pi(v_i)\) be the underlying vertex matched to \(u_i\), i.e., \(\hat{\pi}(u_i) = \pi(v_i)\). Then, \((u_i, \pi(v_i))\) is a correct seed if \(u_i = v_i\) and is an incorrect seed if \(u_i \neq v_i\). The seed \((u_i, \pi(v_i))\) could be a 1-hop witness for \((v, \pi(w))\) only if \(u_i \neq v\) and \(\pi(v_i) \neq \pi(w)\). Besides, if \(u_i = w\) or \(\pi(v_i) = \pi(v)\), then the event \(u_i \in N_1^{G_1}(v)\) is dependent on the event \(\pi(v_i) \in N_1^{G_2}(\pi(w))\) (see Fig. 12 for details).

Fortunately, there is at most two such seed. Thus, we exclude this seed and consider the remaining seeds.

![Figure 12: If \(u_i = w\) and \(\pi(v_i) = \pi(v)\), then the event \(u_i \in N_1^{G_1}(v)\) would be dependent on the event \(\pi(v_i) \in N_1^{G_2}(\pi(w))\).](image)

Let \(Z_i\) be a binary random variable such that \(Z_i = 1\) if \((u_i, \pi(v_i))\) is a 1-hop witness for \(v\) and \(\pi(w)\), and \(X_i = 0\) otherwise. Then, we have \(\mathbb{P}\{Z_i = 1\} = p^2s^2\) because \(Z_i = 1\) if and only if the two edges \((v, u_i)\) and \((w, v_i)\) are both in \(G\), and are sampled in \(G_1\) and \(G_2\), respectively. Note that the edges \((v, u_i)\) and \((w, v_i)\) are different in the parent graph \(G\) for different seeds \((u_i, \pi(v_i))\); Otherwise \(u_i = w\) or \(v_i = v\), but we have excluded such seeds. Thus, \(Z_i\)'s are mutually independent because the sampling process is independent on each edge. Since there are at most \(n\) seeds which could be 1-hop witnesses for \((v, \pi(w))\), we have \(W_1(v, \pi(w)) = 2 + \sum_{i=1}^{n} Z_i\) and \(\sum_{i=1}^{n} Z_i \sim \text{Binom}(n, p^2s^2)\). Recall \(z_{\text{max}} = np^2s^2 + \sqrt{7np^2s^2 \log n} + \frac{7}{3}\log n + 2\). We then get

\[
\mathbb{P}\{W_1(v, \pi(w)) \geq z_{\text{max}}\} \leq \mathbb{P}\left\{\sum_{i=1}^{n} Z_i \geq z_{\text{max}} - 2\right\} \\
\leq \mathbb{P}\left\{\sum_{i=1}^{n} Z_i \geq np^2s^2 + \sqrt{7np^2s^2(1 - p^2s^2) \log n} + \frac{7}{3}\log n\right\} \\
\leq \exp\left(-\frac{7}{2}\log n\right) = n^{-\frac{7}{2}}.
\]

where the last inequality follows from Bernsteins inequality given in Theorem 4 with \(\gamma = \frac{7}{2}\log n\) and \(K = 1\).

Appendix D Proof of Theorem 2

In this section, we prove Theorem 2 for the correlated Erdős-Rényi model \(G(n, p; s)\) when \(np^2 \leq \frac{1}{135 \log n}\) and \(nps^2 \geq 128 \log n\). We set

\[
\epsilon = \sqrt{\frac{12 \log n}{(n - 1)ps^2}} \leq \frac{1}{3}.
\] (23)
Before proving Theorem 2, we need some lemmas to bound the number of 2-hop witnesses. In order to count the number of 2-hop witnesses, we first bound the number of 1-hop neighbors of any vertex in \( G_1 \) and \( G_2 \).

**Lemma 3.** Given any two vertices \( u \neq v \) in graph \( G_1 \), let \( R_{uv} \) denote the event such that the followings hold simultaneously:

\[
(1 - \epsilon)(n - 1)ps < |N_{G_1}^1(u), N_{G_2}^1(\pi(u)), N_{G_1}^1(v), N_{G_2}^1(\pi(v))| < (1 + \epsilon)(n - 1)ps,
\]

\[
(1 - \epsilon)(n - 1)ps^2 < |C_1(u, \pi(u)), C_1(v, \pi(v))| < (1 + \epsilon)(n - 1)ps^2,
\]

where \( \epsilon \) is given in (23), i.e., \( \epsilon = \sqrt{\frac{12 \log n}{(n - 1)ps^2}} \leq \frac{1}{3} \). Then

\[
\mathbb{P}\{R_{uv}\} \geq 1 - n^{-\frac{5}{2}}.
\]

**Remark 1.** The number of 1-hop neighbors in \( G_1 \) or \( G_2 \) is approximately \( nps \pm O\left(\sqrt{np}^s\right) \), and the number of common 1-hop neighbors is approximately between \( nps^2 \pm O\left(\sqrt{np}^s\right) \). Since \( |C_1(u, \pi(v))|, W_1(v, \pi(u)) \sim \text{Binom}(n, p^2s^2) \), they can be bounded by sub-exponential tail bounds. See Appendix D.2 for the proof.

We next bound the number of 2-hop witnesses for the true pair \( (u, \pi(u)) \), and the fake pair \( (u, \pi(v)) \), by conditioning on their 1-hop neighbors. We set \( \delta_1 = \frac{6ps}{\beta} \) and \( \delta_2 = \frac{9ps}{1 - \beta} \). The proofs of Lemma 4 and Lemma 5 are deferred to Appendix D.3 and D.4, respectively.

**Lemma 4.** Given any two vertices \( u \neq v \) in graph \( G_1 \), we use \( Q_{uv} \) to collect all information of 1-hop neighborhood of \( u \) and \( v \), i.e.,

\[
Q_{uv} = \left\{ N_{G_1}^1(u), N_{G_2}^1(\pi(u)), N_{G_1}^1(v), N_{G_2}^1(\pi(v)) \right\}.
\]

For sufficiently large \( n \),

\[
\mathbb{P}\left\{ W_2(u, \pi(u)) \leq l_{min} + m_{min}\mid Q_{uv} \right\} \cdot \mathbb{1}(R_{uv}) \leq n^{-\frac{5}{2}},
\]

where

\[
l_{min} = \frac{7}{24}(1 - \delta_1)n^2\beta p^2s^4 - \sqrt{\frac{35}{16}}n^2\beta p^2s^4 \log n - \frac{5}{2} \log n, \tag{24}
\]

and

\[
m_{min} = (1 - \delta_2)n(1 - \beta) \left(1 - (1 - ps)\left|N_{G_1}^1(u)\setminus\{u\}\right|\right) \left(1 - (1 - ps)\left|N_{G_2}^1(\pi(u))\setminus\{\pi(u)\}\right|\right) - \frac{15}{2} \sqrt{\frac{3}{2}}n^3p^4s^4 \log n - \frac{25}{2} \log n. \tag{25}
\]

**Remark 2.** Lemma 4 provides a lower bound on the number of 2-hop witnesses for the true pair, \( u \) and \( \pi(u) \), conditioned on \( Q_{uv} \). Note that \( l_{min} \) is contributed by the correct seeds in \( \pi \) and \( m_{min} \) is contributed by the incorrect seeds in \( \pi \). Conditional on the 1-hop neighbors, a correct seed becomes a 2-hop witness for the true pair \( (u, \pi(u)) \) with probability about \( |C_1(u, \pi(u))|ps^2 \approx np^2s^4 \). An incorrect seed becomes a 2-hop witness for the true pair \( (u, \pi(u)) \) with probability about
(1 - (1 - ps)|N_i^{G_1}(u)|)(1 - (1 - ps)|N_i^{G_2}(\pi(u))|). Both the expressions of \(l_{\min}\) and \(m_{\min}\) consist of three parts. Specifically, the first term of (24) and (25) is a lower bound of the expectation, the second term is due to the sub-Gaussian tail bound, and the third term is due to the sub-exponential tail bound. We introduce \(\delta_1\) and \(\delta_2\) to exclude seeds that are 1-hop neighbors of \(u\) or \(\pi(u)\), which simplify the expressions.

**Lemma 5.** For any two vertices \(u \neq v\) in graph \(G_1\) and sufficiently large \(n\),

\[
P \left\{ W_2(u, \pi(v)) \geq x_{\text{max}} + y_{\text{max}} + 2z_{\text{max}} + 109 \log n \bigg| Q_{uv} \right\} \cdot 1 (R_{uv}) \leq n^{-\frac{7}{4}}. \tag{26}
\]

where

\[
x_{\text{max}} = 2n\beta \left( 3ps^2 \log n + \frac{9}{4} n^2 p^4 s^4 \right),
\]

\[
y_{\text{max}} = n(1 - \beta) \left( 1 - (1 - ps)|N_i^{G_1}(u)| \right) \left( 1 - (1 - ps)|N_i^{G_2}(\pi(u))\backslash\{\pi(v)\}| \right) + \frac{15}{2} \sqrt{\frac{3}{2} n^2 p^4 s^4 \log n},
\]

and

\[
z_{\text{max}} = \frac{19}{2} n^2 p^3 s^3.
\]

**Remark 3.** Lemma 5 provides an upper bound on the number of 2-hop witnesses for the fake pair, \(u\) and \(\pi(v)\), conditioned on \(Q_{uv}\). Let \((u_i, \pi(v_i))\) denote any seed. Note that if \(u\) and \(v\) are connected in \(G_1\), conditioning on \(Q_{uv}\) changes the probability that those seeds with \(u_i \in N_1^{G_1}(v)\) become 2-hop witnesses for \(u\) and \(\pi(v)\). Thus, we have to divide the seeds into several types based on whether \(u_i \in N_1^{G_1}(v)\) or \(\pi(v_i) \in N_1^{G_2}(\pi(u))\), and consider their contribution to the number of 2-hop witnesses separately.

1) \(x_{\text{max}} + y_{\text{max}}\) is contributed by the seeds such that \(u_i \notin N_1^{G_1}(v) \cup \{v\}\) and \(\pi(v_i) \notin N_1^{G_2}(\pi(u)) \cup \{\pi(u)\}\) (see Fig. 13(a) for example). In addition, we divide the seeds in the first type into two cases: \(x_{\text{max}}\) is contributed by the correct seeds in \(\tilde{\pi}\), and \(y_{\text{max}}\) is contributed by the incorrect seeds in \(\tilde{\pi}\).

2) One multiple of \(z_{\text{max}}\) in (26) is contributed by the seeds such that \(u_i \in N_1^{G_1}(v)\) and \(\pi(v_i) \notin N_1^{G_2}(\pi(u)) \cup \{\pi(u)\}\) (see Fig. 13(b) for example). There are roughly \(nps\) seeds, \((u_i, \pi(v_i))\), such that \(u_i\) is 1-hop neighbor of \(v\) in graph \(G_1\). If \(u\) and \(v\) are connected, these \(u_i\) have been 2-hop neighbors of \(u\). The probability that \(\pi(v_i)\) becomes a 2-hop neighbor of \(\pi(v)\) is approximately \(np^2s^2\). Thus, the expected number of 2-hop witnesses contributed by this type of seeds, \((u_i, \pi(v_i))\), is approximately \(n^2p^3s^3\). The other multiple of \(z_{\text{max}}\) in (26) is for the opposite case: it is contributed by the seeds such that \(u_i \notin N_1^{G_1}(v) \cup \{v\}\) and \(\pi(v_i) \in N_1^{G_2}(\pi(u))\).

3) The term \(109 \log n\) in (26) is contributed by the seeds such that \(u_i \in N_1^{G_1}(v)\) and \(\pi(v_i) \in N_1^{G_2}(\pi(u))\) (see Fig. 13(c) for example) and the sub-exponential tail bounds. This type of seeds, \((u_i, \pi(v_i))\), are 1-hop witnesses for \(v\) and \(\pi(u)\). Since there are no more than \(3 \log n\) 1-hop witnesses for \(v\) and \(\pi(u)\) according to Lemma 3, we obtain the upper bound \(109 \log n\).
We now present the following lemma, which shows that the lower bound \( l_{\min} + m_{\min} \) is no smaller than the upper bound \( x_{\max} + y_{\max} + 2z_{\max} + 109 \log n \) if \( |N_1^{G_2}(\pi(v))| - |N_1^{G_2}(\pi(u))| \leq 2\sqrt{5nps(1-s)}\log n + \frac{109}{2} \log n \). See Appendix Section D.5 for the proof.

**Lemma 6.** Given any two vertices \( u \neq v \) in graph \( G_1 \), if \( R_{uv} \) occurs, \( |N_1^{G_2}(\pi(v))| - |N_1^{G_2}(\pi(u))| \leq \tau \) with \( \tau \leq 2\sqrt{5nps(1-s)}\log n + \frac{109}{2} \log n \), and condition (4) holds, then for sufficiently large \( n \),

\[
l_{\min} + m_{\min} \geq x_{\max} + y_{\max} + 2z_{\max} + 109 \log n.
\]

where the definitions of \( l_{\min}, m_{\min}, x_{\max}, y_{\max} \) and \( z_{\max} \) are given in Lemma 4 and Lemma 5.

Lemma 6 gives the condition when the number of 2-hop witnesses for \( u \) and \( \pi(u) \) is larger than that for \( u \) and \( \pi(v) \). However, \( |N_1^{G_2}(\pi(v))| - |N_1^{G_2}(\pi(u))| \leq \tau \) is not always satisfied. We present Lemma 7 to show that either \( |N_1^{G_1}(u)| - |N_1^{G_1}(v)| \) or \( |N_1^{G_2}(\pi(v))| - |N_1^{G_2}(\pi(u))| \) is no larger than \( \tau \) with high probability. See Appendix D.6 for the proof.

**Lemma 7.** Given any \( u, v \in G_1 \), let \( T_{uv} \) denote the event:

\[
T_{uv} = \left\{ |N_1^{G_1}(u)| - |N_1^{G_1}(v)| \leq \tau \right\} \cup \left\{ |N_1^{G_2}(\pi(v))| - |N_1^{G_2}(\pi(u))| \leq \tau \right\}. \tag{27}
\]
Then,
\[ \mathbb{P}(T_{uv}) \geq 1 - n^{-\frac{5}{2}}. \]

Clearly, if the sub-event \( \left\{ \left| N_{1}^{G_2}(\pi(v)) \right| - \left| N_{1}^{G_2}(\pi(u)) \right| \leq \tau \right\} \) in (27) occurs, we can apply Lemma 6 directly. If the other sub-event \( \left\{ \left| N_{1}^{G_1}(u) \right| - \left| N_{1}^{G_1}(v) \right| \leq \tau \right\} \) occurs, we can use an analogous result of Lemma 6 that compares the number of 2-hop witnesses for \( u \) and \( \pi(v) \) with that for \( v \) and \( \pi(v) \) instead of comparing to the number of 2-hop witnesses for \( u \) and \( \pi(u) \). This leads to the proof of Theorem 2, which shows that, with high probability, either \( W_2(u, \pi(u)) > W_2(u, \pi(v)) \) or \( W_2(v, \pi(v)) > W_2(u, \pi(v)) \) for any \( u \neq v \), and as a result, GMWM must succeed.

**Proof of Theorem 2.** Given any vertices \( u,v \) in graph \( G_1 \) and \( u \neq v \), we let \( W_{uv} \) denote
\[ W_{uv} = \left\{ W_2(u, \pi(u)) > W_2(u, \pi(v)) \right\} \cup \left\{ W_2(v, \pi(v)) > W_2(u, \pi(v)) \right\}. \]
We will prove \( W_{uv} \) happens with high probability. We condition on \( Q_{uv} = \left\{ N_{1}^{G_1}(u), N_{1}^{G_2}(\pi(u)), N_{1}^{G_1}(v), N_{1}^{G_2}(\pi(v)) \right\} \) such that the event \( R_{uv} \) in Lemma 3 is true. Then, we consider two cases: \( \left| N_{1}^{G_2}(\pi(u)) \right| - \left| N_{1}^{G_2}(\pi(v)) \right| \leq \tau \) and \( \left| N_{1}^{G_1}(u) \right| - \left| N_{1}^{G_1}(v) \right| \leq \tau \).

**Case 1:** \( \left| N_{1}^{G_2}(\pi(v)) \right| - \left| N_{1}^{G_2}(\pi(u)) \right| \leq \tau. \)

Let \( w_{\text{min}} \) denote \( l_{\min} + m_{\min} \) and \( w_{\text{max}} \) denote \( x_{\text{max}} + y_{\text{max}} + 2z_{\text{max}} + 109 \log n \). According to Lemma 4 and Lemma 5, \( W_2(u, \pi(u)) > w_{\text{min}} \) with high probability, and \( W_2(u, \pi(v)) < w_{\text{max}} \) with high probability. Since \( w_{\text{min}} \geq w_{\text{max}} \) according Lemma 6, we get that \( W_2(u, \pi(u)) > W_2(u, \pi(v)) \) with high probability. More precisely, if \( R_{uv} \) occurs,
\[ \mathbb{P} \left\{ W_2(u, \pi(u)) \leq W_2(u, \pi(v)) \mid Q_{uv} \right\} \]
\[ \leq \mathbb{P} \left\{ \{ W_2(u, \pi(u)) \leq w_{\text{min}} \} \cup \{ W_2(u, \pi(v)) \geq w_{\text{max}} \} \mid Q_{uv} \right\} \]
\[ \leq \mathbb{P} \left\{ W_2(u, \pi(u)) \leq w_{\text{min}} \mid Q_{uv} \right\} + \mathbb{P} \left\{ W_2(u, \pi(v)) \geq w_{\text{max}} \mid Q_{uv} \right\} \]
\[ \leq 2 \cdot n^{-\frac{5}{2}}. \]

where the inequality (a) is based on \( w_{\text{min}} \geq w_{\text{max}} \) and the De Morgan’s laws, which states \( R \cup S = \overline{R} \cap \overline{S} \). The inequality (b) is based on the union bound. The inequality (c) is based on Lemma 4 and Lemma 5.

Since \( \{ W_2(u, \pi(u)) > W_2(u, \pi(v)) \} \subset W_{uv} \), it follows that,
\[ \mathbb{P} \left\{ W_{uv} \mid Q_{uv} \right\} \leq \mathbb{P} \left\{ W_2(u, \pi(u)) \leq W_2(u, \pi(v)) \mid Q_{uv} \right\} \leq 2 \cdot n^{-\frac{5}{2}}. \]

**Case 2:** \( \left| N_{1}^{G_1}(u) \right| - \left| N_{1}^{G_1}(v) \right| \leq \tau. \)

We can lower bound the number of 2-hop witnesses for \( v \) and \( \pi(v) \) analogous to Lemma 4, and prove that the lower bound is no smaller than the upper bound of the number of 2-hop witnesses for \( u \) and \( \pi(v) \) in this case. Then,
\[ \mathbb{P} \left\{ W_{uv} \mid Q_{uv} \right\} \leq \mathbb{P} \left\{ W_2(v, \pi(v)) \leq W_2(u, \pi(v)) \mid Q_{uv} \right\} \leq 2 \cdot n^{-\frac{5}{2}}. \]
Since \( T_{uv} = \left\{ \left| N_{1}^{G_{1}}(u) \right| - \left| N_{1}^{G_{1}}(v) \right| \leq \tau \right\} \cup \left\{ \left| N_{1}^{G_{2}}(\pi(v)) \right| - \left| N_{1}^{G_{2}}(\pi(u)) \right| \leq \tau \right\} \), applying the union bound yields that

\[
\mathbb{P}\left\{ W_{uv} \right\} \leq \mathbb{P}\left\{ Q_{uv} \right\} \cdot \mathbb{1}(R_{uv} \cap T_{uv}) + \mathbb{P}\left\{ \overline{W_{uv}} \right\} \cdot \mathbb{1}(R_{uv} \cap T_{uv})
\]

\[
\leq \mathbb{P}\left\{ Q_{uv} \right\} \cdot \mathbb{1}(R_{uv} \cap T_{uv}) + \mathbb{E}_{\overline{Q_{uv}}} \left[ \mathbb{1}(R_{uv} \cap T_{uv}) \right] + \mathbb{E}_{\overline{Q_{uv}}} \left[ \mathbb{1}(R_{uv} \cap T_{uv}) \right]
\]

\[
\leq 6 \cdot n^{-\frac{3}{2}}.
\]

Then, applying Lemma 3 and Lemma 7 yields that

\[
\mathbb{P}\left\{ \overline{W_{uv}} \right\} = \mathbb{E}_{\overline{Q_{uv}}} \left[ \mathbb{P}\left\{ Q_{uv} \right\} \cdot \mathbb{1}(R_{uv} \cap T_{uv}) + \mathbb{P}\left\{ \overline{W_{uv}} \right\} \cdot \mathbb{1}(R_{uv} \cap T_{uv}) \right]
\]

\[
\leq \mathbb{E}_{\overline{Q_{uv}}} \left[ \mathbb{P}\left\{ \overline{W_{uv}} \right\} \cdot \mathbb{1}(R_{uv} \cap T_{uv}) \right] + \mathbb{E}_{\overline{Q_{uv}}} \left[ \mathbb{1}(R_{uv} \cap T_{uv}) \right] + \mathbb{E}_{\overline{Q_{uv}}} \left[ \mathbb{1}(R_{uv} \cap T_{uv}) \right]
\]

\[
\leq 6 \cdot n^{-\frac{3}{2}}.
\]

Finally, applying the union bound over all pairs \((u, v)\) with \(u \neq v\), we get that

\[
\mathbb{P}\left\{ \bigcap_{u,v \in V, u \neq v} W_{uv} \right\} = 1 - \mathbb{P}\left\{ \bigcup_{u,v \in V, u \neq v} \overline{W_{uv}} \right\} \geq 1 - n^{2}\mathbb{P}\left\{ \overline{W_{uv}} \right\} \geq 1 - 6 \cdot n^{-\frac{3}{2}} \geq 1 - n^{-1}.
\]

Assuming \( \bigcap_{u,v \in V, u \neq v} W_{uv} \) is true, we next show that the output of GMWM, \( \overline{\pi} \), must be equal to \( \pi \).

We prove this by contradiction. Suppose in contrary that \( \overline{\pi} \neq \pi \). Assume the first fake pair is chosen by GMWM in the \( k \)-th iteration, which implies that GMWM selects true pairs in the first \( k - 1 \) iterations. We let \( (u^k, \pi(u^k)) \) denote the fake pair chosen at the \( k \)-th iteration. Because \( \bigcap_{u,v \in V, u \neq v} W_{uv} \) is true, we have \( W_{2}(u^k, \pi(u^k)) > W_{2}(u^k, \pi(v^k)) \) or \( W_{2}(u^k, \pi(v^k)) > W_{2}(u^k, \pi(v^k)) \). We consider two cases. The first case is that \( (u^k, \pi(u^k)) \) or \( (v^k, \pi(v^k)) \) has been selected in the first \( k - 1 \) iterations, in which case the fake pair \( (u^k, \pi(v^k)) \) would have been eliminated before the \( k \)-th iteration. The second case is that \( (u^k, \pi(u^k)) \) and \( (v^k, \pi(v^k)) \) have not been selected in the first \( k - 1 \) iterations. Then, GMWM would select one of them instead of \( (u^k, \pi(v^k)) \) in the \( k \)-th iteration. Thus, both cases contradict to the assumption that GMWM picks a fake pair in the \( k \)-th iteration.

Hence, GMWM outputs \( n \) true pairs. Then, we have \( \mathbb{P}\left\{ \overline{\pi} = \pi \right\} \geq \mathbb{P}\left\{ \bigcap_{u,v \in V, u \neq v} W_{uv} \right\} \geq 1 - n^{-1}.

\[\blacksquare\]

### D.1 A Supporting Lemma

In this section, we present a supporting lemma that is useful for the proofs of Lemma 3 and Lemma 7.
Lemma 8. Let $X$ denote a random variable such that $X \sim \text{Binom}(n - 1, \alpha)$. If $\alpha \in [ps^2, 1]$, then
\[
\mathbb{P}\{X \leq (1 - \epsilon)(n - 1)\alpha\} \leq n^{-6}, \quad \mathbb{P}\{X \geq (1 + \epsilon)(n - 1)\alpha\} \leq n^{-4},
\]
where $\epsilon$ is given in (23), i.e., $\epsilon = \sqrt{\frac{12 \log n}{(n - 1)ps^2}} \leq \frac{1}{3}$.

Proof. Since $X \sim \text{Binom}(n - 1, \alpha)$ and $\epsilon = \sqrt{\frac{12 \log n}{(n - 1)ps^2}} < \frac{1}{3}$, applying Chernoff bound in Theorem 3 yields
\[
\mathbb{P}\{X \leq (1 - \epsilon)(n - 1)\alpha\} \leq \exp\left(-\frac{\epsilon^2(n - 1)\alpha}{2}\right) = \exp\left(-\frac{6\alpha \log n}{ps^2}\right) \leq n^{-6},
\]
and
\[
\mathbb{P}\{X \leq (1 + \epsilon)(n - 1)\alpha\} \leq \exp\left(-\frac{\epsilon^2(n - 1)\alpha}{3}\right) = \exp\left(-\frac{4\alpha \log n}{ps^2}\right) \leq n^{-4}.
\]

\[\square\]

D.2 Proof of Lemma 3

In the sub-sampled graph $G_1$, for any vertex $u_i \in V \setminus \{u\}$, $u_i$ and $u$ are connected with probability $ps$. Then, we have $|N_{G_1}(u)| \sim \text{Binom}(n - 1, ps)$. Since $ps \geq ps^2$, we apply Lemma 8 in Appendix D.1 and get
\[
\mathbb{P}\left\{\left|N_{G_1}(u)\right| \leq (1 - \epsilon)(n - 1)ps\right\} \leq n^{-6}, \quad \mathbb{P}\left\{\left|N_{G_1}(u)\right| \geq (1 + \epsilon)(n - 1)ps\right\} \leq n^{-4}.
\]

The same lower and upper bound hold for $|N_{G_1}(v)|$, $|N_{G_2}(u)|$, $|N_{G_2}(v)|$ by similar proof.

In the sub-sampled graph $G_1$, for any vertex $u_i \in V \setminus \{u\}$, we have $\mathbb{P}\{(u_i, \pi(u_i)) \in C_1(u, \pi(u))\} = ps^2$ because $(u_i, \pi(u_i)) \in C_1(u, \pi(u))$ if and only if the edge $(u, u_i)$ is in $G$, and is sampled into both $G_1$ and $G_2$. Note that the edges $(u, u_i)$'s are different for different $u_i$ in the parent graph $G$, and the sampling process is independent on each edge. Thus, $(u_i, \pi(u_i)) \in C_1(u, \pi(u))$ are mutually independent across $u_i$. Then, we have $|C_1(u, \pi(u))| \sim \text{Binom}(n - 1, ps^2)$. Applying Lemma 8 in Appendix D.1 implies that
\[
\mathbb{P}\{|C_1(u, \pi(u))| \leq (1 - \epsilon)(n - 1)ps^2\} \leq n^{-6}, \quad \mathbb{P}\{|C_1(u, \pi(u))| \geq (1 + \epsilon)(n - 1)ps^2\} \leq n^{-4}.
\]

The same lower and upper bound hold for $|C_1(v, \pi(v))|$ by similar proof.

According to Lemma 2, we have
\[
\mathbb{P}\left\{W_1(v, \pi(u)) \leq np^2s^2 + \sqrt{7np^2s^2 \log n + \frac{7}{3} \log n + 2}\right\} \geq 1 - n^{-\frac{2}{3}}.
\]

Since $np^2 \leq \frac{1}{135 \log n}$ and $n$ is sufficiently large, we have
\[
np^2s^2 + \sqrt{7np^2s^2 \log n + \frac{7}{3} \log n + 2}
\]
\[ \leq \frac{1}{135 \log n} + 2 + \sqrt{\frac{1}{19} + \frac{7}{3} \log n} \leq 3 \log n. \]

Hence,

\[ \mathbb{P}\{W_1(v, \pi(u)) \geq 3 \log n\} \leq n^{-\frac{7}{2}}. \]

In the sub-sampled graph \( G_1 \), for any vertex \( u_i \in V \setminus \{u, v\} \), we have \( \mathbb{P}\{(u_i, \pi(u_i)) \in C_1(u, \pi(v))\} = p^2 s^2 \) because \( u_i \in N_i^{G_1}(u) \) and \( \pi(u_i) \in N_i^{G_2}(\pi(v)) \) if and only if the two edges \((u, u_i) \) and \((v, u_i) \) are both in \( G \), and are sampled into \( G_1 \) and \( G_2 \), respectively. Note that the two edges \((u, u_i) \) and \((v, u_i) \) are different for different \( u_i \) in the parent graph \( G \), and the sampling process is independent on each edge. We then get \((u_i, \pi(u_i)) \in C_1(u, \pi(v))\) are mutually independent across \( u_i \). Then, we have \(|C_1(u, \pi(v))| \sim \text{Binom}(n - 2, p^2 s^2)\). Applying Bernstein’s inequality in Theorem 4 with \( \gamma = \frac{7}{2} \log n \) and \( K = 1 \) yields

\[ \mathbb{P}\{|C_1(u, \pi(v))| \geq (n - 2)p^2 s^2 + \sqrt{7(n - 2)p^2 s^2(1 - p^2 s^2)} \log n + \frac{7}{3} \log n\} \leq \exp\left(-\frac{7}{2} \log n\right) = n^{-\frac{7}{2}}. \]

Since \( np^2 \leq \frac{1}{135 \log n} \) and \( n \) is sufficiently large, we have

\[ (n - 2)p^2 s^2 + \sqrt{7(n - 2)p^2 s^2(1 - p^2 s^2)} \log n + \frac{7}{3} \log n \]

\[ \leq \frac{1}{135 \log n} + \sqrt{\frac{1}{19} + \frac{7}{3} \log n} \]

\[ \leq 3 \log n. \]

Hence,

\[ \mathbb{P}\{|C_1(u, \pi(v))| \geq 3 \log n\} \leq n^{-\frac{7}{2}}. \]

Taking the union bound over all these inequalities yields that

\[ \mathbb{P}\{R_{uv}\} \geq 1 - 4(n^{-6} + n^{-4}) - 2(n^{-6} + n^{-4}) - 2n^{-\frac{7}{2}} \geq 1 - n^{-\frac{10}{3}}. \]

### D.3 Proof of Lemma 4

Fixing any two vertices \( u \neq v \) in \( G_1 \), we condition on \( Q_{uv} \) and assume \( R_{uv} \) is true. For each seed \((u_i, \pi(v_i))\), \( \pi(v_i) \) be the underlying vertex matched to \( u_i \), i.e., \( \pi(u_i) = \pi(v_i) \). Then, \((u_i, \pi(v_i))\) is a correct seed if \( u_i = v_i \) and is an incorrect seed if \( u_i \neq v_i \). Among all seeds, some of them may be of the form that \( u_i \in N_i^{G_1}(u) \cup \{u\} \) or \( \pi(v_i) \in N_i^{G_2}(\pi(u)) \cup \{\pi(u)\} \). They could not become 2-hop witnesses for \((u, \pi(v))\). The number of such seeds is bounded by \( |N_i^{G_1}(u)| + |N_i^{G_2}(\pi(u))| \leq 2 + 2(n - 1)ps + 2 \leq 3nps \), where the first inequality holds because \( R_{uv} \) is true, and the second inequality holds because \( \epsilon \leq \frac{1}{3} \) and \( nps \geq 6 \). Then, for each remaining seed \((u_i, \pi(v_i))\), we will estimate the probability that it becomes a 2-hop witness for \( u \) and \( \pi(u) \) by calculating the probability that \( u_i \) connects to the 1-hop neighbours of \( u \) and \( \pi(v_i) \) connects to the 1-hop neighbours of \( \pi(u) \). However, if \( u \) and \( v \) are connected in \( G_1 \), then any 1-hop neighbour of \( v \) would become the 2-hop neighbour of
u. Hence, conditioning on $Q_{uv}$ changes the probability that those seeds, $(u_i, \pi(v_i))$, become 2-hop witnesses for $u$ and $\pi(u)$ if $u_i$ is a 1-hop neighbor of $v$. Similarly if $\pi(u)$ and $\pi(v)$ are connected in $G_2$ and $\pi(v_i)$ is a 1-hop neighbor of $\pi(v)$. Therefore, we will also exclude all those seeds such that $u_i \in N_1^{G_1}(v) \cup \{v\}$ or $\pi(v_i) \in N_1^{G_2}(\pi(v)) \cup \{\pi(v)\}$ so that we can avoid this difficulty. The number of such seeds is bounded by $\left| N_1^{G_1}(v) \right| + \left| N_1^{G_2}(\pi(v)) \right| + 2 \leq 2(1 + \epsilon)(n - 1)ps + 2 \leq 3nps$, where the first inequality holds because $R_{uv}$ is true, and the second inequality holds because $\epsilon \leq \frac{1}{3}$ and $nps \geq 6$. Fortunately, since the total number of seeds we exclude is bounded by $6nps$, which is far less than $n$, we can still get a pretty tight lower bound.

We first count the contribution to $W_2(u, \pi(u))$ by the correct seeds. We use $n_R$ to denote the number of correct seeds remained after we exclude the above kind of seeds. Since there are a constant fraction $\beta$ of correct seeds in $\pi$, $n_R$ is no less than $n\beta - 6nps$. Let $n_R^+$ denote max $\{\lceil n(\beta - 6ps) \rceil, 0\}$. Then, there are at least $n_R^+$ correct seeds which could be a 2-hop witness for $(u, \pi(u))$. Next, we lower bound the number of 2-hop witnesses contributed by the correct seeds. Let $L_i$ be a binary random variable such that $L_i = 1$ if $(u_i, \pi(u_i))$ is a 2-hop witness for $(u, \pi(u))$ and $L_i = 0$ otherwise. Let $C_i$ be a binary random variable such that $C_i = 1$ if $u_i$ and $\pi(u_i)$ connect to the “common” 1-hop neighbours of $u$ and $\pi(u)$, respectively, and $C_i = 0$ otherwise. Since we have excluded $(u_i, \pi(u_i))$ such that $u_i \in N_1^{G_1}(v)$ or $\pi(u_i) \in N_1^{G_2}(\pi(v))$, the remaining $u_i$ and $\pi(u_i)$ can not connect to $v$ and $\pi(v)$, respectively. Thus, $C_i = 1$ can be expressed as

$$\{C_i = 1\} = \bigcup_{\{w, \pi(w)\} \in C_1(u, \pi(u)) \setminus \{(u, \pi(u))\}} \left( \left\{ u_i \in N_1^{G_1}(w) \right\} \cap \left\{ \pi(u_i) \in N_1^{G_2}(\pi(w)) \right\} \right)$$

Then, we can bound $P \left\{ C_i = 1 \left| Q_{uv} \right. \right\}$ as follows:

$$P \left\{ C_i = 1 \left| Q_{uv} \right. \right\}$$

$$\overset{(a)}{=} 1 - P \left\{ \bigcap_{\{w, \pi(w)\} \in C_1(u, \pi(u)) \setminus \{(u, \pi(u))\}} \left\{ (u_i, \pi(u_i)) \notin C_1(w, \pi(w)) \right\} \left| Q_{uv} \right. \right\}$$

$$\overset{(b)}{=} 1 - \prod_{\{w, \pi(w)\} \in C_1(u, \pi(u)) \setminus \{(u, \pi(u))\}} P \left\{ (u_i, \pi(u_i)) \notin C_1(w, \pi(w)) \left| Q_{uw} \right. \right\}$$

$$= 1 - \prod_{\{w, \pi(w)\} \in C_1(u, \pi(u)) \setminus \{(u, \pi(u))\}} \left( 1 - P \left\{ (u_i, \pi(u_i)) \in C_1(w, \pi(w)) \left| Q_{uw} \right. \right\} \right)$$

$$\overset{(c)}{=} 1 - \left( 1 - ps \right)^{|C_1(u, \pi(u)) \setminus \{(u, \pi(u))\}|}$$

$$\overset{(d)}{\geq} 1 - \left( 1 - \frac{1}{2} \left( |C_1(u, \pi(u))| - 1 \right) ps^2 \right)$$

$$= \frac{1}{2} \left( |C_1(u, \pi(u))| - 1 \right) ps^2$$
We use the De Morgan’s laws, which states \( \overline{R \cup S} = \overline{R} \cap \overline{S} \), in the equality (a). Because the edges \((u_i, w_i)\)'s are different in the parent graph \( G \) across different \( w \), and the sampling process is independent on each edge, which implies that \( \{(u_i, \pi(u_i)) \notin C_1(w, \pi(w))\} \) are independent across different \( w \) conditional on \( Q_{uw} \). Hence, the equality (b) holds. Because \( \{(u_i, \pi(u_i)) \in C_1(w, \pi(w))\} \) if and only if the edge \((u_i, w)\) is in \( G \), and is sampled into both \( G_1 \) and \( G_2 \), the equality (c) holds. Because \( |C_1(u, \pi(u))| \geq 0 \), \( ps^2 \in (0, 1) \) and \( |C_1(u, \pi(u))|ps^2 \leq (1 + \varepsilon)(n - 1)p^2s^4 \leq \frac{2}{3}np^2s^4 < 1 \), the inequality (d) holds based on Theorem 6. The inequality (e) holds because \( |C_1(u, \pi(u))| - 1 \geq (1 - \varepsilon)(n - 1)ps^2 - 1 \geq \frac{2}{3}(n - 1)ps^2 - 1 \geq \frac{7}{12}np^2s^2 \).

If \( C_i = 1 \), then \((u_i, \pi(v_i))\) is a 2-hop witness for \((u, \pi(u))\). Thus, \( \{C_i = 1\} \subset \{L_i = 1\} \), i.e.,

\[
\Pr \left\{ L_i = 1 \mid Q_{uw} \right\} \geq \Pr \left\{ C_i = 1 \mid Q_{uw} \right\} \geq \frac{7}{24}np^2s^4.
\]

(28)

Note that we have known the common neighbors \((w, \pi(w)) \in C_1(u, \pi(u))\) conditional on \( Q_{uw} \), and the edges \((u_i, w_i)\)'s are different for different \( u_i \) in the parent graph \( G \), then \( C_i \)'s are mutual independent of each other conditional on \( Q_{uw} \). Then, let \( L \) denote the number of 2-hop witnesses contributed by correct seeds. Since \( n^+_{R} \in [1 - \delta_1]n\beta, n\beta] \), we have \( L = \sum_{i=1}^{n^+_{R}} L_i \geq \sum_{i=1}^{n^+_{R}} C_i \geq C' \), where \( C' \) is a random variable such that \( C' \sim \text{Binom}(n^+_{R}, \frac{7}{24}np^2s^4) \). Recall that \( l_{\min} = \frac{7}{24}(1 - \delta_1)n^2\beta p^2s^4 - \sqrt{\frac{35}{16}n^2\beta p^2s^4 \log n} - \frac{5}{2} \log n \). We then get

\[
\Pr \left\{ L \leq l_{\min} \mid Q_{uw} \right\} \leq \Pr \left\{ C' \leq l_{\min} \mid Q_{uw} \right\} \leq \exp \left( -\frac{15}{4} \log n \right) = n^{-\frac{15}{4}},
\]

(29)

where the last inequality follows from Corollary 1 with \( \gamma = \frac{15}{4} \log n \).

We then count the contribution to \( W_2(u, \pi(u)) \) by the incorrect seeds. For any incorrect seed \((u_i, \pi(v_i))\), if \( \pi(u_i) \in N_{1}^{G_2}(\pi(u)) \) and \( v_i \in N_{1}^{G_1}(u) \), then the event that \( u_i \) connects to \( v_i \) would be dependent on the event that \( \pi(v_i) \) connects to \( \pi(u_i) \) (See Fig. 14 for example). Thus, we also exclude such seeds to avoid the difficulty of calculating probability. We use \( n_{W} \) to denote the number of incorrect seeds remained after we exclude the seeds shown above. Then, \( n_{W} \) is no less than \( n(1 - \beta - 6np_\delta - |N_{1}^{G_1}(u)| - |N_{1}^{G_2}(\pi(u))| \geq n(1 - \beta) - 9np_\delta \), where the inequality holds because \( R_{uw} \) is true. Let \( n_{W}^+ \) denote \( \max \{\lfloor n(1 - \beta - 9np_\delta)\rfloor, 0\} \), then there are at least \( n_{W}^+ \) incorrect seeds \((u_i, \pi(v_i))\) which could be a 2-hop witness for \((u, \pi(u))\).

\[
\end{equation}

Figure 14: If \( \pi(u_i) \in N_{1}^{G_i}(\pi(u)) \) and \( v_i \in N_{1}^{G_i}(u) \), then the event that \( u_i \) connects to \( v_i \) would be dependent on the event that \( \pi(v_i) \) connects to \( \pi(u_i) \).

Let \( M_i \) be a binary random variable such that \( M_i = 1 \) if \((u_i, \pi(v_i))\) is a 2-hop witness for \((u, \pi(u))\) and \( M_i = 0 \) otherwise. Then, we let \( \lambda = \Pr \left\{ M_i = 1 \mid Q_{uw} \right\} \) and have

\[
\lambda \triangleq \Pr \left\{ u_i \in N_{2}^{G_i}(u) \mid Q_{uw} \right\} \cdot \Pr \left\{ \pi(v_i) \in N_{2}^{G_i}(\pi(u)) \mid Q_{uw} \right\}
\]

29
\[ (b) \left(1 - \mathbb{P}\left\{ u_i \sim N_1^{G_1}(u) \mid Q_{uv}\right\}\right) \left(1 - \mathbb{P}\left\{ \pi(v_i) \sim N_1^{G_2}(\pi(u)) \mid Q_{uv}\right\}\right) \]
\[ (c) \left(1 - (1 - ps)^{|N_1^{G_1}(u)\setminus\{v\}|}\right) \left(1 - (1 - ps)^{|N_1^{G_2}(\pi(u))\setminus\{\pi(v)\}|}\right), \]

where for any graph \(G(V, E), A \subset V\) and \(u \in V\setminus A\), let \(u \sim A\) denote the event that \(u\) does not connect to any node in \(A\), i.e.,

\[ u \sim A = \bigcap_{v \in A} \{ u \notin N_1^{G}(v) \}. \]

The equality \((a)\) holds because \(u_i \in N_2^{G_1}(u)\) is independent of \(\pi(v_i) \in N_2^{G_2}(\pi(u))\); Otherwise, \(\pi(u_i) \in N_1^{G_2}(\pi(u))\) and \(v_i \in N_1^{G_1}(u)\) since \(u_i \neq v_i\), but we have neglected such seeds. The equality \((b)\) shows the probability that \(u_i\) and \(v_i\) connect to at least one 1-hop neighbour of \(u\) and \(\pi(u)\), respectively. Because we have excluded \((u_i, \pi(v_i))\) such that \(u_i \in N_1^{G_1}(v) \cup \{v\}\), it is impossible that \(u_i\) connects \(v\). Thus, when we calculate \(\mathbb{P}\left\{ u_i \sim N_1^{G_1}(u) \mid Q_{uv}\right\}\), we have to exclude \(v\) from \(N_1^{G_1}(u)\). And \(\mathbb{P}\left\{ \pi(v_i) \sim N_1^{G_2}(\pi(u)) \mid Q_{uv}\right\}\) is similar. Then, equality \((c)\) holds.

Figure 15: The event that \((u_i, \pi(v_i))\) becomes a 2-hop witness for \((u, \pi(u))\) conditional on \(Q_{uv}\) is dependent on \((u_j, \pi(u_i))\) and \((v_i, \pi(v_j))\).

Note that \(M_i\) are dependent. Specifically, the event that \((u_i, \pi(v_i))\) becomes a 2-hop witness for \((u, \pi(u))\) conditional on \(Q_{uv}\) is dependent on \((u_j, \pi(u_i))\) and \((v_i, \pi(v_j))\) (See Fig. 15 for an example). Then, we cannot apply Bernsteins Inequality in Theorem 4. Fortunately, the event that \((u_j, \pi(v_j))\) becomes a 2-hop witnesses for \((u, \pi(u))\) conditional on \(Q_{uv}\) is dependent on \((u_i, \pi(v_i))\) only if \(u_j = v_i\) or \(v_j = u_i\). Thus, the event that \((u_i, \pi(v_i))\) becomes a 2-hop witness for \((u, \pi(u))\) conditional on \(Q_{uv}\) depends on at most two other seeds. Thus, we apply the concentration inequality for the sum of dependent random variables given in Theorem 5. Specifically, we construct a dependency graph \(\Gamma\) for \(\{M_i\}\). The maximum degree of \(\Gamma\), \(\Delta(\Gamma)\), equals to two. Thus, we apply Theorem 5 with \(\Delta_1(\Gamma) = \Delta(\Gamma) + 1 = 3\), \(K = 1\), \(\sigma^2 = n_W^+ \lambda (1 - \lambda)\) and \(\gamma = 4 \log n\),

\[ \mathbb{P}\left\{ \sum_{i=1}^{n_W^+} M_i \leq n_W^+ \lambda - 5\sqrt{\frac{3}{2} n_W^+ \lambda (1 - \lambda) \log n - \frac{25}{2} \log n} \mid Q_{uv}\right\} \leq \exp (-4 \log n) = n^{-4}. \]
Since $n_W^+ \in [n(1 - \beta - 9ps), n]$, we have
\[
n_W^+ \lambda(1 - \lambda) \leq n \left( (1 - (1-ps)|N_1^{G_1}(u)|) \right) \left( (1 - (1-ps)|N_1^{G_2}(\pi(u))|) \right)
\]
\[
\leq n \left| N_1^{G_1}(u) \right| ps \left| N_1^{G_2}(\pi(u)) \right| ps
\]
\[
\leq \frac{9}{4} n^3 p^4 s^4.
\]
The inequality (a) is based on Bernoulli’s Inequality which states that $(1+x)^r \geq 1 + rx$ for every integer $r \geq 0$ and every real number $x \geq -2$. The equality (b) holds because $\left| N_1^{G_1}(u) \right|, \left| N_1^{G_2}(\pi(u)) \right| \leq (1+\epsilon)(n-1)ps \leq \frac{3}{2} np$. 

Let $M$ denote the number of 2-hop witnesses contributed by the incorrect seeds. We have
\[
M \geq \sum_1^{n_W^+} M_i. \text{ Recall that } m_{min} = (1 - \delta_2)n(1-\beta)\lambda - \frac{15}{2} \sqrt{\frac{3}{2} n^3 p^4 s^4 \log n - \frac{25}{2} \log n}. \text{ We then get}
\]
\[
\mathbb{P}\left\{ M \leq m_{min} \left| Q_{uv} \right. \right\} \leq \mathbb{P}\left\{ \sum_1^{n_W^+} M_i \leq n_W^+ \lambda - 5 \sqrt{\frac{3}{2} n_W^+ \lambda(1 - \lambda) \log n - \frac{25}{2} \log n} \left| Q_{uv} \right. \right\} \leq n^{-4}. \tag{30}
\]

Due to $W_2(u, \pi(u)) = L + M$ and sufficiently large $n$, taking an union bound over (29) and (30) yields that
\[
\mathbb{P}\left\{ W_2(u, \pi(u)) \leq l_{min} + m_{min} \left| Q_{uv} \right. \right\} \cdot \mathbb{1}\left( R_{uv} \right) \leq n^{-\frac{15}{4}} + n^{-4} < n^{-\frac{7}{4}}.
\]

**Remark 4.** In (28), even though we neglect the case that $u_i$ and $v_i$ connect to different 1-hop neighbours of $u$ and $\pi(u)$, the lower bound is still tight. This is because
\[
\mathbb{P}\left\{ \{L_i = 1\} \setminus \{C_i = 1\} \left| Q_{uv} \right. \right\} \approx \mathbb{P}\left\{ u_i \in N_1^{G_1}(u) \left| Q_{uv} \right. \right\} \mathbb{P}\left\{ \pi(v_i) \in N_1^{G_2}(\pi(u)) \left| Q_{uv} \right. \right\}
\]
\[
\approx \left| N_1^{G_1}(u) \right| \left| N_1^{G_2}(\pi(u)) \right| p^2 s^2 \leq \frac{9}{4} n^3 p^4 s^4.
\]
We can observe $\mathbb{P}\left\{ C_i = 1 \left| Q_{uv} \right. \right\} \gg \mathbb{P}\left\{ \{L_i = 1\} \setminus \{C_i = 1\} \left| Q_{uv} \right. \right\}$ because $np^2 \leq \frac{1}{100 \log n}$, and $n$ is sufficiently large. Thus, we can say $\frac{7}{24} np^2 s^4$ is a tight lower bound of $\mathbb{P}\left\{ L_i = 1 \left| Q_{uv} \right. \right\}$.

**D.4 Proof of Lemma 5**

Fixing any two vertices $u \neq v$ in $G_1$, we condition on $Q_{uv}$ and assume $R_{uv}$ is true. For each seed $(u_i, \pi(u))$, let $\pi(v_i)$ be the underlying vertex matched to $u_i$, i.e., $\pi(u_i) = \pi(v_i)$. Then, $(u_i, \pi(v_i))$ is a correct seed if $u_i = v_i$ and is an incorrect seed if $u_i \neq v_i$. For any seed $(u_i, \pi(v_i))$, it could be a 2-hop witness for $(u, \pi(v))$ if $u_i \notin N_1^{G_1}(u) \cup \{u\}$ and $\pi(v_i) \notin N_1^{G_2}(\pi(v)) \cup \{\pi(v)\}$. Then, there are five types of the remaining seeds we need consider (see Fig. 13 for example):

**Type 1:** $u_i \notin N_1^{G_1}(v) \cup \{v\}$ and $\pi(v_i) \notin N_1^{G_2}(\pi(u)) \cup \{\pi(u)\}$.

We first consider the correct seeds of this type. For any correct seed $(u_i, \pi(u_i))$ in Type 1 which could be a 2-hop witness for $u$ and $\pi(v)$, let $X_i$ be a binary random variable such that $X_i = 1$
if \((u_i, \pi(u_i))\) is a 2-hop witness for \((u, \pi(v))\) and \(X_i = 0\) otherwise. Let \(D_i\) be a binary random variable such that \(D_i = 1\) if \((u_i, \pi(u_i)) \in C_1(u, \pi(v))\). Thus, \(D_i = 1\) can be expressed as

\[
\{D_i = 1\} = \bigcup_{(w, \pi(w)) \in C_1(u, \pi(v))} \left\{ u_i \in N_1^{G_1}(w) \right\} \cap \left\{ \pi(u_i) \in N_1^{G_2}(\pi(w)) \right\}
\]

Then, we have

\[
P \left\{ D_i = 1 \mid Q_{uv} \right\} = 1 - P \left\{ (u_i, \pi(u_i)) \notin C_1(w, \pi(w)) \mid Q_{uv} \right\}
\]

\[
\overset{(b)}{=} 1 - \prod_{(w, \pi(w)) \in C_1(u, \pi(v))} P \left\{ (u_i, \pi(u_i)) \notin C_1(w, \pi(w)) \right\}
\]

\[
\overset{(c)}{=} 1 - (1 - |C_1(u, \pi(v))| ps^2)
\]

\[
\overset{(d)}{=} |C_1(u, \pi(v))| ps^2
\]

\[
\overset{(e)}{\leq} 3ps^2 \log n.
\]

The equality \((a)\) is based on the De Morgan’s laws, which states \(\overline{R \cup S} = \overline{R} \cap \overline{S}\). Because \(\{(u_i, \pi(u_i)) \in C_1(w, \pi(w))\}\) only depends on the edge \((u_i, w)\), and the edges \((u_i, w)\) are different in the parent graph \(G\) across different \(w\), \(\{(u_i, \pi(u_i)) \in C_1(w, \pi(w))\}\) are independent across different \(w\) conditional on \(Q_{uv}\). Hence, the equality \((b)\) holds. Because \((u_i, \pi(u_i)) \in C_1(w, \pi(w))\) if and only if the edge \((u_i, w)\) is in \(G\), and is sampled into both \(G_1\) and \(G_2\), the equality \((c)\) holds. The inequality \((d)\) is based on Bernoulli’s Inequality which states that \((1 + x)^r \geq 1 + rx\) for every integer \(r \geq 0\) and every real number \(x \geq -2\). The inequality \((e)\) holds based on \(|C_1(u, \pi(v))| < 3 \log n\).

Note that if \(D_i = 1\), then \((u_i, \pi(u_i))\) is a 2-hop witnesses for \((u, \pi(v))\). Thus, \(\{D_i = 1\} \subset \{X_i = 1\}\), and \(\{X_i = 1\} \setminus \{D_i = 1\}\) can be can be expressed as

\[
\{X_i = 1\} \setminus \{D_i = 1\} = \bigcup_{A \subset N_1^{G_1}(u), A \neq \emptyset} \left( \bigcup_{B \subset N_1^{G_2}(\pi(v)) \setminus \pi(A), B \neq \emptyset} \left\{ u_i \sim_{N_1^{G_1}(u)} A, \pi(u_i) \sim_{N_1^{G_2}(\pi(v))} B \right\} \right)
\]

\[
\overset{(a)}{=} \bigcup_{A \subset N_1^{G_1}(u), A \neq \emptyset} \left( \left\{ u_i \sim_{N_1^{G_1}(u)} A \right\} \cap \bigcup_{B \subset N_1^{G_2}(\pi(v)) \setminus \pi(A), B \neq \emptyset} \left\{ \pi(u_i) \sim_{N_1^{G_2}(\pi(v))} B \right\} \right)
\]

where in any graph \(G(V, E)\), if \(U \subset V, A \subset V, u \in V \setminus A\), let \(u \sim A\) denote the event that \(u\) connects to all vertices in \(A\) but does not connect any vertex in \(U \setminus A\), i.e.,

\[
u \sim A = \left( \bigcap_{v \in A} \left\{ u \in N_1^G(v) \right\} \right) \cap \left( \bigcap_{u \in U \setminus A} \left\{ u \notin N_1^G(w) \right\} \right).
\]
The equality \((a)\) is based on the distribution law of set, which states that \((R \cap S) \cup (R \cap T) = R \cap (S \cup T)\).

Since \(A\) determines the 1-hop neighbors of \(u\) that \(u_i\) connects to, \(\{u_i \sim N_1^{G_1}(u) A\}\) are disjoint from each other across \(A\) conditional on \(Q_{uv}\). And \(\{\pi(u_i) \sim N_1^{G_2}(\pi(v)) B\}\) are disjoint from each other across \(B\) conditional on \(Q_{uv}\) for the same reason. In addition, when \(A\) is fixed, \(u_i \sim N_1^{G_1}(u) A\) is independent of \(\pi(u_i) \sim N_1^{G_2}(\pi(v)) B\) because \(B\) does not have the same vertex with \(\pi(A)\). Thus, \(\mathbb{P}\left\{\{X_i = 1\} \setminus \{D_i = 1\} \mid Q_{uv}\right\}\) could be bounded by

\[
\mathbb{P}\left\{\{X_i = 1\} \setminus \{D_i = 1\} \mid Q_{uv}\right\} = \sum_{A \subseteq N_1^{G_1}(u), A \neq \emptyset} \left( \mathbb{P}\left\{u_i \sim N_1^{G_1}(u) A \mid Q_{uv}\right\} \cdot \sum_{B \subseteq N_1^{G_2}(\pi(v)) \setminus \pi(A), B \neq \emptyset} \mathbb{P}\left\{\pi(u_i) \sim N_1^{G_2}(\pi(v)) B \mid Q_{uv}\right\} \right)
\]

\[
\leq \sum_{A \subseteq N_1^{G_1}(u), A \neq \emptyset} \left( \mathbb{P}\left\{u_i \sim N_1^{G_1}(u) A \mid Q_{uv}\right\} \cdot \sum_{B \subseteq N_1^{G_2}(\pi(v)) \setminus \pi(A), B \neq \emptyset} \mathbb{P}\left\{\pi(u_i) \sim N_1^{G_2}(\pi(v)) B \mid Q_{uv}\right\} \right)
\]

\[
= \mathbb{P}\left\{u_i \in N_2^{G_1}(u) \mid Q_{uv}\right\} \cdot \mathbb{P}\left\{\pi(u_i) \in N_2^{G_2}(\pi(v)) \mid Q_{uv}\right\}.
\]

where we relax the sum of probability that \(\pi(u_i)\) connects to some vertices belonging to \(N_1^{G_2}(\pi(v))\) in the inequality \((a)\) and take it out in the equality \((b)\). Then, we have

\[
\mathbb{P}\left\{u_i \in N_2^{G_1}(u) \mid Q_{uv}\right\} = 1 - \mathbb{P}\left\{\bigcap_{w \in N_1^{G_1}(u)} \{u_i \in N_1^{G_1}(u) \mid Q_{uv}\} \right\}
\]

\[
\stackrel{(a)}{=} 1 - (1 - ps) \left| N_1^{G_1}(u) \setminus \{v\} \right|
\]

\[
\leq 1 - \left(1 - \left| N_1^{G_1}(u) \right| ps \right)
\]

\[
\leq \left| N_1^{G_1}(u) \right| ps \tag{31}
\]

\[
\leq \frac{3}{2} n p^2 s^2.
\]

Because we have excluded \((u_i, \pi(v_i))\) such that \(u_i \in N_1^{G_1}(v) \cup \{v\}\), it is impossible that \(u_i\) connects \(v\). Thus, when we calculate the probability that \(u_i\) connects to the 1-hop neighbors of \(u\), we have to exclude \(v\) from \(N_1^{G_1}(u)\). The inequality \((b)\) is based on Bernoulli’s Inequality which states that \((1 + x)^r \geq 1 + rx\) for every integer \(r \geq 0\) and every real number \(x \geq -2\). The inequality
(c) is based on \(|N_1^{G_1}(u)|, |N_1^{G_2}(\pi(v))| < (1 + \epsilon)(n - 1)ps < \frac{3}{2}nps\). And

\[
P \left\{ \tau(u_i) \in N_2^{G_2}(\pi(v)) \bigg| Q_{uv} \right\} \leq \frac{3}{2}np^2s^2
\]

follows from the similar proof. Then, we have

\[
P \left\{ \{X_i = 1\} \setminus \{D_i = 1\} \bigg| Q_{uv} \right\} \leq \frac{9}{4}n^2p^4s^4.
\]

Thus,

\[
P \left\{ X_i = 1 \bigg| Q_{uv} \right\} \leq 3ps^2 \log n + \frac{9}{4}n^2p^4s^4 = \mu_1.
\]

Let \(X\) denote the number of 2-hop witnesses contributed by correct seeds in Type 1. Since there are a constant fraction \(\beta\) of correct seeds, the number of correct seeds which could be 2-hop witnesses is no larger than \(n\beta\). Thus, we have \(X \leq \sum_{i=1}^{n\beta} X_i\).

For incorrect seed \((u_i, \pi(v_i))\) in Type 1, if \(\pi(u_i) \in N_1^{G_2}(\pi(v))\) or \(v_i \in N_1^{G_1}(u)\), then the event \(u_i\) connects to \(v_i\) would be dependent on the event that \(\pi(v_i)\) connects to \(\pi(u_i)\) (see Fig. 16 for example). The number of such \((u_i, \pi(v_i))\) is no larger than \(3nps\) (using \(|N_1^{G_1}(u)| + |N_1^{G_2}(\pi(v))| < 2(1 + \epsilon)(n - 1)ps < 3nps\). Let \(\Psi_i\) be a binary random variable such that \(\Psi_i = 1\) if such \((u_i, \pi(v_i))\) is a 2-hop witness for \((u, \pi(v))\) and \(\Psi_i = 0\) otherwise.

![Figure 16: If \(\pi(u_i) \in N_1^{G_2}(\pi(u))\) and \(v_i \in N_1^{G_1}(v)\), then the event that \(u_i\) connects to \(v_i\) would be dependent on the event that \(\pi(v_i)\) connects to \(\pi(u_i)\).](image)

We use \(\mu_2\) to denote \(P \left\{ \Psi_i = 1 \bigg| Q_{uv} \right\} \) and have

\[
\begin{align*}
\mu_2 & \overset{(a)}{=} P \left\{ \Psi_i = 1 \bigg| Q_{uv} \cap \left\{ u_i \in N_1^{G_1}(v_i) \right\} \right\} P \left\{ u_i \in N_1^{G_1}(v) \bigg| Q_{uv} \right\} \\
& \quad + P \left\{ \Psi_i = 1 \bigg| Q_{uv} \cap \left\{ u_i \notin N_1^{G_1}(v_i) \right\} \right\} P \left\{ u_i \notin N_1^{G_1}(v) \bigg| Q_{uv} \right\} \\
& \overset{(b)}{=} \left( 1 - (1 - s)(1 - ps)|N_1^{G_2}(\pi(v))\setminus\{\pi(u)\}|^{-1} \right) ps \\
& \quad + \left( 1 - (1 - ps)|N_1^{G_1}(v)|^{-1} \right) \left( 1 - (1 - ps)|N_1^{G_2}(\pi(v))\setminus\{\pi(u)\}| \right) (1 - ps) \\
& \leq ps + \left( 1 - (1 - ps)|N_1^{G_1}(u)| \right) \left( 1 - (1 - ps)|N_1^{G_2}(\pi(v))| \right) \\
& \overset{(c)}{\leq} ps + \left( 1 - \left( 1 - |N_1^{G_1}(u)\big| ps \right) \right) \left( 1 - \left( 1 - |N_1^{G_2}(\pi(v))\big| ps \right) \right)
\end{align*}
\]
\[ (d) \quad \leq ps + \frac{9}{4} n^2 p^4 s^4. \]

We condition on two cases: \( u_i \in N_1^G(v_i) \) and \( u_i \notin N_1^G(v_i) \) in equality (a). If \( u_i \in N_1^G(v_i) \), then \( u_i \) is a 2-hop neighbor of \( u \), we only need to calculate the probability that \( \pi(v_i) \) becomes a 2-hop neighbor of \( \pi(v) \). If \( u_i \notin N_1^G(v_i) \), the event \( u_i \) connects to the other 1-hop neighbor of \( u \) would not be dependent on the event that \( \pi(v_i) \) connects to 1-hop neighbor of \( \pi(v) \). In addition, because we have excluded \((u_i, \pi(v_i))\) such that \( u_i \in N_1^G(v) \cup \{v\} \), it is impossible that \( u_i \) connects \( v \). Thus, when we calculate the probability that \( u_i \) connects to the 1-hop neighbors of \( u \), we have to exclude \( v \) from \( N_1^G(u) \). And the probability that \( \pi(v_i) \) connects to the 1-hop neighbors of \( \pi(v) \) is similar. Thus, equality (b) holds. The inequality (c) is based on Bernoulli’s Inequality which states that \( (1 + x)^r \geq 1 + r x \) for every integer \( r \geq 0 \) and every real number \( x \geq -2 \). The inequality (d) is based on \( |N_1^G(u)|, |N_2^G(\pi(v))| < (1 + \epsilon)(n - 1)ps < \frac{3}{2} nps \).

Then, we use \( \Psi \) to denote the number of 2-hop witnesses for \((u, \pi(v))\) contributed by the seeds \((u_i, \pi(v_i))\) such that \( \pi(u_i) \in N_1^G(\pi(v)) \) and \( v_i \in N_1^G(u) \). We have \( \Psi \leq \sum_{i=1}^{3nps} \Psi_i \).

For incorrect seed \((u_i, \pi(v_i))\) such that \( \pi(u_i) \notin N_1^G(\pi(v)) \) and \( v_i \notin N_1^G(u) \), \( u_i \in N_2^G(\pi(v)) \) would be independent of \( v_i \in N_2^G(\pi(v)) \). Let \( Y_i \) be a binary random variable such that \( Y_i = 1 \) if such seed \((u_i, \pi(v_i))\) is the 2-hop witnesses for \((u, \pi(v))\). We use \( \mu_3 \) to denote \( P \left\{ Z_i = 1 \bigg| Q_{uw} \right\} \) and have

\[
\mu_3 = P \left\{ u_i \in N_2^G(u), v_i \in N_2^G(\pi(v)) \bigg| Q_{uw} \right\} \\
\overset{(a)}{=} P \left\{ u_i \in N_2^G(u) \bigg| Q_{uw} \right\} \cdot P \left\{ v_i \in N_2^G(\pi(v)) \bigg| Q_{uw} \right\} \\
\overset{(b)}{=} \left( 1 - (1 - ps) \right)^{|N_1^G(u) \setminus \{v\}|} \left( 1 - (1 - ps) \right)^{|N_1^G(\pi(v)) \setminus \{\pi(u)\}|} \\
\overset{(c)}{\leq} \frac{9}{4} n^2 p^4 s^4. 
\]

The equality (a) holds because the event that \( u_i \in N_2^G(u) \) is independent of the event that \( v_i \in N_2^G(\pi(v)) \). Otherwise, \( \pi(u_i) \in N_1^G(\pi(v)) \) and \( v_i \in N_1^G(u) \) since \( u_i \neq v_i \), but we have discussed such seeds. The steps (b) and (c) follow from the similar proof in (31).

Let \( Y \) denote the number of 2-hop witnesses contributed by the incorrect seeds such that . We have \( Y \leq \sum_{i=1}^{n(1-\beta)} Y_i \).

**Type 2:** \( u_i \in N_1^G(v) \) and \( \pi(v_i) \notin N_1^G(\pi(u)) \cup \{\pi(u)\} \).

The number of the seeds in Type 2 is no larger than \( \frac{3}{2} nps \) (using \( |N_1^G(v)| \leq (1 + \epsilon)nps < \frac{3}{2} nps \)). Let \( Z_i^v \) denote a binary random variable such that \( Z_i^v = 1 \) if the seed, \((u_i, \pi(v_i))\), of this type is a 2-hop witness for \((u, \pi(v))\). If \( u \) and \( v \) are connected in \( G_1 \), then as \( u_i \in N_1^G(v) \), it follows that \( u_i \in N_2^G(u) \). Thus,

\[
P \left\{ Z_i^v = 1 \bigg| Q_{uw} \right\} = P \left\{ u_i \in N_2^G(u), \pi(v_i) \in N_2^G(\pi(v)) \bigg| Q_{uw} \right\} \\
\leq P \left\{ \pi(v_i) \in N_2^G(\pi(v)) \bigg| Q_{uw} \right\} \\
\leq \frac{3}{2} np^2 s^2. 
\]
The last inequality follows from the similar proof in (31).

We use $Z^u$ to denote the number of 2-hop witnesses contributed by this type of seeds. Then, we have $Z^u \leq \sum_{i=1}^{\lfloor \frac{3}{2}np^s \rfloor} Z_i^u$.

**Type 3:** $u_i \notin N_1^{G_1}(v) \cup \{v\}$ and $\pi(v_i) \in N_1^{G_2}(\pi(u))$.

Let $Z_i^u$ denote a binary random variable such that $Z_i^u = 1$ if the seed, $(u_i, \pi(v_i))$, in Type 3 is a 2-hop witness for $(u, \pi(v))$. Let $Z^u$ denote the number of 2-hop witnesses contributed by this type of seeds. Following the similar proof as in Type 2 cases, we can get $Z^u \leq \sum_{i=1}^{\lfloor \frac{3}{2}np^s \rfloor} Z_i^u$ and $\Pr \left\{ Z_i^u = 1 \mid Q_{uv} \right\} \leq \frac{3}{2}np^2s^2$.

**Type 4:** $u_i \in N_1^{G_1}(v) \cup \{v\}$ and $\pi(v_i) \in N_1^{G_2}(\pi(u)) \cup \{\pi(u)\}$.

There are at most $3 \log n$ seeds in Type 4 conditional on $Q_{uv}$ (using $W_1(v, \pi(u)) \leq 3 \log n$). Here, $(u_i, \pi(v_i))$ would be a 1-hop witness for $(v, \pi(u))$. If $v \in N_1^{G_1}(u)$ and $\pi(u) \in N_1^{G_2}(\pi(v))$, then $(u_i, \pi(v_i))$ would be a 2-hop witness for $(u, \pi(v))$.

**Type 5:** $u_i = v$ or $\pi(v_i) = \pi(u)$.

There are at most 2 such seeds.

After calculating the probability that each type of seeds become 2-hop witnesses for $(u, \pi(v))$, we upper bound the contribution to $W_2(u, \pi(v))$ by the correct and incorrect seeds, respectively. First, we consider the correct seeds. Let $W_R$ denote the number of 2-hop witnesses contributed by the correct seeds. We then have $W_R \leq X + Z^u + 3 \log n$. Note that, for any correct seed $(u_i, \pi(u_i))$, the event that $(u_i, \pi(u_i))$ becomes a 2-hop witness for $(u, \pi(v))$ is independent of any other seeds $(u_j, \pi(v_j))$. Otherwise, $u_i \in N_1^{G_1}(u)$, $u_j \in N_1^{G_1}(u)$ or $\pi(v_j) \in N_1^{G_2}(\pi(v))$, but we have excluded such seeds. We then can get

$$\Pr \left\{ W_R \geq x_{max} + 9n^2p^3s^3 + 18 \log n \mid Q_{uv} \right\} \leq \Pr \left\{ X + Z^u + Z^v \geq x_{max} + 9n^2p^3s^3 + 15 \log n \mid Q_{uv} \right\} \leq \Pr \left\{ X \geq x_{max} + 5 \log n \mid Q_{uv} \right\}$$

$$(a) \leq \Pr \left\{ Z^u \geq \frac{9}{2}n^2p^3s^3 + 5 \log n \mid Q_{uv} \right\} + 2 \cdot \Pr \left\{ Z^v \geq x_{max} + 9n^2p^3s^3 + 15 \log n \mid Q_{uv} \right\}$$

$$(b) \leq \Pr \left\{ X \geq n\beta \mu_1 \sqrt{\frac{15}{2}n\beta \mu_1 \log n} + 5 \log n \mid Q_{uv} \right\}$$

$$+ 2 \cdot \Pr \left\{ Z^u \geq \left[ \frac{3}{2}np^s \right] \frac{3}{2}np^2s^2 + \sqrt{\left[ \frac{3}{2}np^s \right] \frac{45}{4}np^2s^2 \left( 1 - \frac{3}{2}np^2s^2 \right) \log n} + 5 \log n \mid Q_{uv} \right\}$$

$$(c) \leq 3 \exp \left( -\frac{15}{4} \log n \right) = 3 \cdot n^{-\frac{15}{4}}.$$  

The inequality $(a)$ follows from the union bound. The inequality $(b)$ follows from the AM-GM inequality, which states that $2\sqrt{xy} \leq x + y$ for two non-negative numbers $x$ and $y$. The inequality $(c)$ follows from Bernsteins Inequality in Theorem 4 with $\gamma = \frac{15}{4} \log n$. 

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Next, we upper bound the contribution to $W_2(u, \pi(v))$ by the incorrect seeds. Let $W_W$ denote the number of 2-hop witnesses contributed by the correct seeds. We then have $W_W \leq Y + \Psi + Z^u + Z^v + 3 \log n + 2 \leq Y + \Psi + Z^u + Z^v + \frac{7}{2} \log n$. Note that, the event that the incorrect seed $(u_i, \pi(v_i))$ becomes a 2-hop witness for $(u, \pi(v))$ is dependent on other incorrect seeds. Specifically, $(u_i, \pi(v_i))$ is dependent on $(u_j, \pi(u_j))$ and $(v_i, \pi(v_j))$ (see Fig. 17 for example). Then, we cannot apply Bernsteins Inequality in Theorem 4. Fortunately, the event $(u_j, \pi(v_j))$ becomes a 2-hop witness for $(u, \pi(v))$ conditional on $Q_{uv}$ is dependent on $(u_i, \pi(v_i))$ only if $u_j = v_i$ or $v_j = u_i$. Thus, the event that $(u_i, \pi(v_i))$ becomes a 2-hop witness for $(u, \pi(u))$ conditional on $Q_{uv}$ depends on at most two other seeds. Thus, we apply the concentration inequality for the sum of dependent random variables given in Theorem 5. Specifically, we construct a dependency graph $\Gamma$ for $\{Y_i, \Psi_i, Z^u_i, Z^v_i\}$. The maximum degree of $\Gamma$, $\Delta(\Gamma)$, equals to two. Thus, we apply Theorem 5 with we have $\Delta_1(\Gamma) = 1 + 2 = 3$ and $\gamma = 4 \log n$.

\[
\mathbb{P} \left\{ W_W \geq \gamma \max + 10n^2 p^3 s^3 + 114 \log n \mid Q_{uv} \right\} \\
\leq \mathbb{P} \left\{ Y + \Psi + Z^u + Z^v \geq \gamma \max + 10n^2 p^3 s^3 + \frac{175}{2} \log n \mid Q_{uv} \right\} \\
\leq \mathbb{P} \left\{ Y \geq \gamma \max + \frac{25}{2} \log n \mid Q_{uv} \right\} \\
+ \mathbb{P} \left\{ \Psi \geq n^2 p^3 s^3 \left( \frac{6}{nps} + \frac{27}{2n^2 p^2 s^2} \right) + 25 \log n \mid Q_{uv} \right\} \\
+ 2 \cdot \mathbb{P} \left\{ Z^u \geq \frac{9}{2} n^2 p^3 s^3 + 25 \log n \mid Q_{uv} \right\} \\
\leq \mathbb{P} \left\{ Y \geq n(1 - \beta) \mu_3 + 5 \sqrt{\frac{3}{2} n(1 - \beta) \mu_3 (1 - \mu_3) \log n + \frac{25}{2} \log n} \,, Q_{uv} \right\} \\
+ \mathbb{P} \left\{ \Psi \geq \frac{3}{2} [n] \mu_2 + 5 \sqrt{\frac{3}{2} [n] \mu_2 (1 - \mu_2) \log n + \frac{25}{2} \log n} \,, Q_{uv} \right\} \\
+ 2 \cdot \mathbb{P} \left\{ Z^u \geq \frac{3}{2} [n] \frac{3}{2} n^2 p^2 s^2 + \frac{15}{2} \sqrt{\frac{3}{2} [n] n^2 p^2 s^2 \left( 1 - \frac{3}{2} n^2 p^2 s^2 \right) \log n + \frac{25}{2} \log n} \,, Q_{uv} \right\} \\
\leq 4 \exp \left( -4 \log n \right) = 4 \cdot n^{-4}.
\]

The inequality $a)$ follows from the union bound and $\frac{6}{nps} + \frac{27}{2} n^2 p^2 s^2 \leq 1$. The inequality $b)$ follows from the AM-GM inequality, which states that $2 \sqrt{xy} \leq x + y$ for two non-negative numbers $x$ and $y$.

Since $n$ is sufficiently large and $W_2(u, \pi(v)) = W_R + W_W$, taking an union bound on (32) and (33) yields

\[
\mathbb{P} \left\{ W_2(u, \pi(v)) \geq x_{\max} + y_{\max} + 2z_{\max} + 109 \log n \mid Q_{uv} \right\} \cdot \mathbb{1}(R_{uv}) \\
= \mathbb{P} \left\{ W_R + W_W \geq x_{\max} + y_{\max} + 2z_{\max} + 109 \log n \mid Q_{uv} \right\} \cdot \mathbb{1}(R_{uv}) \\
\leq 4 \cdot n^{-4} + 3 \cdot n^{-\frac{10}{7}} < n^{-\frac{7}{2}}.
\]
D.5 Proof of Lemma 6

\[
\begin{align*}
    l_{\min} + m_{\min} - x_{\max} - y_{\max} - 2z_{\max} - 109 \log n & \leq \frac{7}{24} n^2 \beta p^2 s^4 - \frac{7}{4} n^2 p^3 s^5 - \frac{35}{16} n^2 \beta p^2 s^4 \log n - 15 \sqrt{\frac{3}{2} n^2 p^4 s^4 \log n} \\
    & \quad - 2n \beta \left( 3ps^2 \log n + \frac{9}{4} n^2 p^4 s^4 \right) - 19n^2 p^3 s^3 - 139 \log n \\
    & \quad - 9n ps \left( 1 - (1 - ps) \left| N_1^{G_1}(u) \backslash \{v\} \right| \right) \left( 1 - (1 - ps) \left| N_1^{G_2}(\pi(u)) \backslash \{\pi(v)\} \right| \right) \\
    & \quad \quad + n(1 - \beta) \left( 1 - (1 - ps) \left| N_1^{G_1}(u) \backslash \{v\} \right| \right) \left( 1 - ps \right) \left| N_1^{G_2}(\pi(u)) \backslash \{\pi(v)\} \right| \\
    & \quad \quad \cdot \left( 1 - ps \right) \left| N_1^{G_2}(\pi(v)) \backslash \{\pi(v)\} \right| - \left| N_1^{G_2}(\pi(u)) \backslash \{\pi(v)\} \right| - 1. \tag{34}
\end{align*}
\]

In order to bound \( l_{\min} + m_{\min} - x_{\max} - y_{\max} - 2z_{\max} - 109 \log n \), we bound the last two terms in (34) firstly,

\[
- 9n ps \left( 1 - (1 - ps) \left| N_1^{G_1}(u) \backslash \{v\} \right| \right) \left( 1 - (1 - ps) \left| N_1^{G_2}(\pi(u)) \backslash \{\pi(v)\} \right| \right) \geq -9n \left| N_1^{G_1}(u) \right| \left| N_1^{G_2}(\pi(u)) \right| p^3 s^3
\]

\[
- \frac{n(1 - \beta) \left( 1 - (1 - ps) \left| N_1^{G_1}(u) \backslash \{v\} \right| \right) \left( 1 - ps \right) \left| N_1^{G_2}(\pi(u)) \backslash \{\pi(v)\} \right|}{p^3 s^3} \geq -16n^3 p^5 s^5
\]

The inequality (a) is based on the Bernoulli’s Inequality which states that \((1+x)^r \geq 1+rx\) for every integer \(r \geq 0\) and every real number \(x \geq -2\). The inequality (b) is based on \( \left| N_1^{G_1}(u) \right| \left| N_1^{G_2}(\pi(u)) \right| \leq (1+\epsilon)(n-1)ps \leq \frac{4}{3}nps \).

\[
\begin{align*}
    & \quad \quad \cdot \left( 1 - ps \right) \left| N_1^{G_2}(\pi(v)) \backslash \{\pi(v)\} \right| - \left| N_1^{G_2}(\pi(u)) \backslash \{\pi(v)\} \right| - 1
\end{align*}
\]

Figure 17: The event that \( (u_i, \pi(v_i)) \) is a 2-hop witness for \( (u, \pi(v)) \) is dependent on \( (u_j, \pi(u_i)) \) and \( (v_i, \pi(v_j)) \).
\[ (a) \quad \geq \min \left\{ -n(1 - \beta)(1 - ps)\left| N_i^G(z)\right|, \left| N_i^G(z)\right| - \left| N_i^G(z)\right| - 1 > 0 \right. \]

\[ (b) \quad \geq -\tau psn(1 - \beta) \left( 1 - (1 - ps)\left| N_i^G(z)\right| \right) \]

\[ (c) \quad \geq -\tau n\left| N_i^G(z)\right| p^2s^2 \]

\[ (d) \quad \geq -3n^2p^3s^3\sqrt{5nps(1 - s) \log n} - 5n^2p^3s^3 \log n. \]

For the inequality \((a)\), if \(\left| N_i^G(z)\right| < \left| N_i^G(z)\right|\), then \(1 - ps\left| N_i^G(z)\right| - \left| N_i^G(z)\right| - 1 > 0\). Otherwise, we apply Bernoulli’s Inequality. The inequality \((b)\) is based on \(\left| N_i^G(z)\right| - \left| N_i^G(z)\right| \leq \tau\). The inequality \((c)\) is based on \(\left| N_i^G(z)\right| < (1 + \epsilon)(n - 1)ps < \frac{3}{2}nps\). Then we can continue to bound \(l_{min} + m_{min} - x_{max} - y_{max} - 2z_{max} - 109 \log n\),

\[ l_{min} + m_{min} - x_{max} - y_{max} - 2z_{max} - 109 \log n \]

\[ \geq \frac{7}{24}n^2p^2s^4 - \frac{7}{4}n^2p^3s^5 - \sqrt{\frac{15}{8}n^2p^4s^4 \log n} - 15\sqrt{\frac{3}{2}n^3p^4s^4 \log n} \]

\[ - 2n\beta \left( 3ps^2 \log n + \frac{9}{4}n^2p^4s^4 \right) - 19n^2p^3s^3 - 139 \log n \]

\[ - 3n^2p^3s^3 \sqrt{5nps(1 - s) \log n} - 5n^2p^3s^3 \log n - 16n^3p^5s^5. \]

In view of \((35)\), we can guarantee \(l_{min} + m_{min} - x_{max} - y_{max} - 2z_{max} - 109 \log n \geq 0\) if inequalities \((36)-(41)\) hold. We next verify \((36)-(41)\) hold.

First, by assumption that \(\beta \geq 600\sqrt{\frac{\log n}{ns^4}}, ps \leq \frac{1}{135 \log n}\), and \(n\) is sufficiently large, we have

\[ \frac{1}{60}n^2p^2s^4 \geq \frac{1}{60}n^2p^2s^4 \cdot 600\sqrt{\frac{\log n}{ns^4}} \geq 10n^2p^2s^4 \cdot p \log n \]

\[ \geq n^2p^3s^3(5 \log n + 19 + \frac{7}{4}s^2 + 16np^2s^2), \]

Second, by assumption that \(\beta \geq \frac{1200\log n}{n^2ps^4}\), we have

\[ \frac{1}{24}n^2p^2s^4 \geq \frac{1}{24}n^2p^2s^4 \cdot \sqrt{\frac{1200 \log n}{n^2ps^4}} > \sqrt{\frac{35}{16}n^2p^2s^4 \log n}. \]

Third, by assumption that \(\beta \geq 600\sqrt{\frac{\log n}{ns^4}}\), we have

\[ \frac{1}{30}n^2p^2s^4 \geq \frac{1}{30}n^2p^2s^4 \cdot 600\sqrt{\frac{\log n}{ns^4}} > 15\sqrt{\frac{3}{2}n^3p^4s^4 \log n}. \]

Fourth, by assumption that \(\beta \geq \frac{1200\log n}{n^2ps^4}\), we have

\[ \frac{1}{8}n^2p^2s^4 \geq \frac{1}{8}n^2p^2s^4 \cdot \frac{1200 \log n}{n^2ps^4} > 139 \log n. \]
Fifth, by the assumption that \( np^2 \geq 128 \log n \), \( np^2 \leq \frac{1}{135 \log n} \), and \( n \) is sufficiently large, we have
\[
\frac{1}{15} n^2 \beta p^2 s^4 \geq \frac{1}{20} n \beta ps^2 \cdot 128 \log n + \frac{1}{60} n^2 \beta p^2 s^4 \cdot 135 np^2 \log n \\
\geq 2n \beta \left( 3ps^2 \log n + \frac{9}{4} n^2 p^4 s^4 \right),
\]
(40)

Sixth, by the assumption that \( \beta \geq 900 \sqrt{\frac{np^3(1 - s) \log n}{s}} \), we have
\[
\frac{1}{120} n^2 \beta p^2 s^4 \geq \frac{1}{120} n^2 p^2 s^4 \cdot 900 \sqrt{\frac{np^3(1 - s) \log n}{s}} \geq 3n^2 p^3 s^3 \sqrt{5np(1 - s) \log n}.
\]
(41)

Thus,
\[
l_{\min} + m_{\min} \geq x_{\max} + y_{\max} + 2z_{\max} + 109 \log n.
\]

D.6 Proof of Lemma 7

In the parent graph \( G \), for any vertex \( u_i \in V \setminus \{u\} \), \( u_i \) and \( u \) are connected with probability \( p \). Then, \(|N_1^G(u)| \sim \text{Binom}(n - 1, p)\). Since \( p \geq ps^2 \), applying Lemma 8 in Appendix D.1 implies,
\[
\mathbb{P}\left\{ |N_1^G(u)| \geq (1 + \epsilon)(n - 1)p \right\} \leq n^{-4}.
\]
(42)

Then, we use \( R_u^G \) to denote the event \( \{ |N_1^G(u)| < (1 + \epsilon)(n - 1)p \} \).

For any two vertices \( u \neq v \) in the parent graph \( G \), \( u \) and \( \pi(u) \) are the same vertex, and \( v \) and \( \pi(v) \) are the same vertex. Hence, \( N_1^G(u) = N_1^G(\pi(u)) \) and \( N_1^G(v) = N_1^G(\pi(v)) \). Then, we use \( E_{uv} \) to denote:
\[
E_{uv} = \{ N_1^G(u), N_1^G(v) \}.
\]
Conditioning on \( E_{uv} \), such that \( R_u^G \) and \( R_v^G \) are true, we have to consider two cases: \( |N_1^G(u)| \leq |N_1^G(v)| \) and \( |N_1^G(u)| > |N_1^G(v)| \).

**Case 1:** \( |N_1^G(u)| \leq |N_1^G(v)| \).

We let \( H_1, H_2, ..., H_{|N_1^G(u)|} \) denote the binary random variables such that \( H_i = 1 \) if the \( i \)-th 1-hop neighbour of \( u \) in \( G \) is still connected to \( u \) in \( G_1 \) and \( J_1, J_2, ..., J_{|N_1^G(v)|} \) denote the binary random variables such that \( J_i = 1 \) if the \( i \)-th 1-hop neighbour of \( v \) in \( G \) is still connected to \( v \) in \( G_1 \). For any \( i \in \{1, 2, ..., |N_1^G(u)|\} \), we have
\[
\mathbb{P}\left\{ H_i - J_i = k \mid E_{uv} \right\} = \begin{cases} s(1 - s) & \text{if } k = -1 \\ s^2 + (1 - s)^2 & \text{if } k = 0 \\ s(1 - s) & \text{if } k = 1 \end{cases}.
\]

Since \( H_i \)'s and \( J_i \)'s are generated by independent sub-sampling process, \( (H_i - J_i) \)'s are mutually independent conditional on \( E_{uv} \). Thus, we can apply Bernstein’s inequality given in Theorem 4. For \( i = 1, 2, ..., |N_1^G(u)| \),
\[
\mathbb{E}\left[ H_i - J_i \mid E_{uv} \right] = 0, |H_i - J_i| \leq 1 = K \text{ and } \sigma^2 = \sum_{i=1}^{|N_1^G(u)|} \text{var} \left( H_i - J_i \mid E_{uv} \right) = 2 |N_1^G(u)| s(1 - s) \text{.}
\]

We set \( \gamma = \frac{5}{1 + \epsilon} \log n \) and get
\[
\mathbb{P}\left\{ \sum_{i=1}^{|N_1^G(u)|} (H_i - J_i) \geq \sqrt{\frac{20}{1 + \epsilon} |N_1^G(u)| s(1 - s) \log n + \frac{10}{3(1 + \epsilon)} \log n \mid E_{uv}} \right\}
\]
40
where the first inequality is based on that \( R_G^G \) is true and \( \epsilon \geq 0 \). Then,

\[
\mathbb{P} \left\{ \sum_{i=1}^{N_G^G(u)} (H_i - J_i) \geq \tau \mid E_{uv} \right\} \leq \mathbb{P} \left\{ \sum_{i=1}^{N_G^G(u)} (H_i - J_i) \geq \tau \right\} \leq n^{-\frac{5}{1+\epsilon}}.
\]

Because \( |N_1^G(u)| - |N_1^G(v)| = \sum_{i=1}^{N_G^G(u)} H_i - \sum_{i=1}^{N_G^G(v)} J_i \leq \sum_{i=1}^{N_G^G(u)} (H_i - J_i) \) (using \( |N_1^G(u)| \leq |N_1^G(v)| \)), we have

\[
\mathbb{P} \left\{ |N_1^G(u)| - |N_1^G(v)| \geq \tau \mid E_{uv} \right\} \leq \mathbb{P} \left\{ \sum_{i=1}^{N_G^G(u)} (H_i - J_i) \geq \tau \mid E_{uv} \right\} \leq n^{-\frac{5}{1+\epsilon}}.
\]

Since \( \overline{T_{uv}} \subset \left\{ |N_1^G(u)| - |N_1^G(v)| \geq \tau \right\} \), then

\[
\mathbb{P} \left\{ \overline{T_{uv}} \mid E_{uv} \right\} \leq \mathbb{P} \left\{ |N_1^G(u)| - |N_1^G(v)| \geq \tau \mid E_{uv} \right\} \leq n^{-\frac{5}{1+\epsilon}}.
\]

**Case 2:** \( |N_1^G(u)| > |N_1^G(v)| \).

Following the similar proof, we can get

\[
\mathbb{P} \left\{ \overline{T_{uv}} \mid E_{uv} \right\} \leq \mathbb{P} \left\{ |N_1^G(\pi(v))| - |N_1^G(\pi(u))| \geq \tau \mid E_{uv} \right\} \leq n^{-\frac{5}{1+\epsilon}}.
\]

We combine the two cases and get

\[
\mathbb{P} \left\{ \overline{T_{uv}} \mid E_{uv} \right\} \cdot 1 \left( R_G^G \cap R_G^G \right) \leq n^{-\frac{5}{1+\epsilon}}.
\]  

(43)

Finally, since \( n \) is sufficiently large, applying (42), (43) and the union bound yields

\[
\mathbb{P} \left\{ \overline{T_{uv}} \right\} = \mathbb{E}_{E_{uv}} \mathbb{P} \left\{ \overline{T_{uv}} \mid E_{uv} \right\} \leq \mathbb{E}_{E_{uv}} \left[ \mathbb{P} \left\{ \overline{T_{uv}} \mid E_{uv} \right\} \cdot 1 \left( R_G^G \cap R_G^G \right) + \mathbb{P} \left\{ \overline{T_{uv}} \mid E_{uv} \right\} \cdot 1 \left( R_G^G \cap R_G^G \right) \right] \leq \mathbb{E}_{E_{uv}} \left[ \mathbb{P} \left\{ \overline{T_{uv}} \mid E_{uv} \right\} \cdot 1 \left( R_G^G \cap R_G^G \right) \right] + \mathbb{E}_{E_{uv}} \left[ 1 \left( R_G^G \cap R_G^G \right) \right] \leq n^{-\frac{5}{1+\epsilon}} + 2 \cdot n^{-4} \leq n^{-\frac{7}{2}}.
\]
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