Systematic Differential Renormalization to All Orders

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Abstract

We present a systematic implementation of differential renormalization to all orders in perturbation theory. The method is applied to individual Feynman graphs written in coordinate space. After isolating every singularity which appears in a bare diagram, we define a subtraction procedure which consists in replacing the core of the singularity by its renormalized form given by a differential formula. The organization of subtractions in subgraphs relies on Bogoliubov’s formula, fulfilling the requirements of locality, unitarity and Lorentz invariance. Our method bypasses the use of an intermediate regularization and automatically delivers renormalized amplitudes which obey renormalization group equations.
I.- Introduction

It is well known that the amplitudes of the perturbative expansion of an interacting quantum field theory have, in general, an ill-defined ultraviolet behavior. From a mathematical point of view, the core of this problem lies on the nature of these amplitudes, which are distribution-valued objects, since the product of distributions is known to be, in general, ill-defined. As stated in ref.[1], renormalization consists, thus, in finding a prescription to define the product of distributions so that amplitudes verify some desired requirements, namely, Lorentz invariance, locality and unitarity. This problem was shown to have a solution long ago. Among the different ideas to prove the existence of a consistent renormalization program, let us single out the approach started by Bogoliubov and Parasiuk [1] and definitively settled by Hepp [2] and Zimmermann [3]. The BPHZ (Bogoliubov, Parasiuk, Hepp and Zimmermann) method is based on the concept of counterterms and defines a recursive subtraction scheme which can be applied on individual Feynman graphs. From these works, we learn that the renormalization program proceeds as follows. Locate first the divergences occurring in the bare amplitude of the studied graph, apply then a subtraction procedure, that is, a method to eliminate such divergences, and finally organize the subtraction according to the topology of the graph. In this paper, we define a differential renormalization subtraction procedure to be implemented in the program described above.

Let us recall the simple principles of differential renormalization (DR), as presented in [11]. The procedure is defined in coordinate space and yields right away renormalized amplitudes. Divergent expressions are written as derivatives of less singular functions. We have then to solve a differential equation, extracting as many derivatives as necessary to obtain a power-counting-finite expression. The differential equation is promoted to be a definition of the amplitude in the sense of distributions. Derivatives are, in this context, naturally understood to act on test functions. When solving those differential equations, integration constants appear, which play the role of renormalization group scales. Differential renormalization amplitudes fulfill renormalization group equations, which are used to extract the coefficients of the perturbative expansion of renormalization group $\beta$ and $\gamma$ functions. The main advantages of the method are the relative ease of the calculations and the fact that the space-time dimension remains unchanged. It has been successfully applied to massless $\phi^4$ to three loops in four dimensions [11], supersymmetric theories [12], the study of the three-gluon vertex [13], massive theories [14], low dimensional theories [15], QED to two loops [16] and theories with $\gamma_5$ [17]. It has also been related to the standard dimensional regularization procedure in reference [18].

Our purpose is to fill the main gap of differential renormalization, namely, the lack of a systematic set of rules. In spite of its successes and achievements, one can doubt whether differential renormalization principles are sufficient to render finite any Feynman graph. Moreover, without a systematic procedure, it is impossible to prove that differential renormalization amplitudes are consistent and unitary to all orders. By consistent, we understand following from a counterterm structure or fulfilling renormalization group equations, which is equivalent. At the present level of differential renormalization, we cannot ensure, for instance, that overlapping divergences are treated correctly. Although differential renormalization amplitudes

\footnote{Many people have contributed to set up the basis of renormalization theory. Among many others let us mention the works of Dyson [4], Salam [5], Stuckelberg and Green [6], Weinberg [7], Epstein and Glaser [8], Callan, Blaer and Young [9] and Polchinski [10].}
obey renormalization group equations at a given order (which are used to compute $\beta$ and $\gamma$ renormalization group-functions), this is not sufficient to guarantee a counterterm structure since we can think of amplitudes fulfilling renormalization group equations up to a certain order in perturbation theory and yet not being consistent [19]. An attempt to clarify the existence of a counterterm structure behind differential renormalization amplitudes was made in reference [20]. By introducing a cut off $\varepsilon$ in massless four-dimensional $\phi^4$ theory to three loops, one finds that divergences organize correctly in counterterms, yielding the differential renormalization renormalized amplitudes. However, a check, no matter how thorough, is never a proof. Therefore, for the sake of completeness, a systematization of differential renormalization is needed. Then, the consistency and unitarity of the renormalized amplitudes can be checked to all orders.

We can summarize the general idea of our approach as follows. The differential renormalization procedure is defined in coordinate space. We distinguish between divergences arising from two points collapsing and those coming from three or more points simultaneously closing up. We define for the first our particular subtraction procedure which consists of applying the basic idea of “pulling out” derivatives. We actually organize the subtraction as a true replacement of the singularity with the renormalized form once the derivatives are pulled in front. For the second, we work recursively by first observing that no singularity appears when all points but one are brought together. This indicates that the global singularity reduces to the problem of bringing the last point on top of the rest, which is just a two-point problem again. This simplification is essentially due to the fact that the subtraction to be performed is local. No regulator is needed to define these subtractions, sharing one of the characteristic features of the BPHZ scheme. The subtraction of subdivergences is organized following Bogoliubov’s recursion formula. We therefore come up with a systematic version of differential renormalization which guarantees the desired properties of consistency and unitarity of the method. The expert reader should be aware of the fact that we do not attempt to present an exhaustively rigorous proof. It is a virtue of DR to remain an extremely simple method to apply, regardless of all the technicalities we are borrowing to prove its workability to all orders.

The organization of our paper goes as follows. In section II, we first recall the basic ideas of the renormalization proof adapting them to our coordinate space approach. We then sketch two examples of how subtractions will be defined and proceed to fully present the general procedure. Section III is devoted to non-trivial illustrations of the method, going from a three-loop non-planar diagram to a six-loop one. We end up with a series of comments on the differences between our method and the standard BPHZ, and its applicability to theories with supplementary symmetries. Two appendices are devoted to technical questions.

II.- Systematic Differential Renormalization

1.- Organization of the Renormalization Procedure

A correct renormalization procedure must provide a method to define sensible Green functions while preserving locality, unitarity and Lorentz invariance. In momentum space, Green functions are ill-defined when loop integrals blow up at high momenta. Therefore, renormalization amounts to eliminating those infinities appearing in loop integrals. In coordinate space, divergences arise when vertices come close. We have then to smear out such divergent behavior so that amplitudes end up being (tempered) distributions.
As we have already said, this program can be achieved by a three-step process. Let us briefly comment on each step, with a preview of how differential renormalization will adapt to them.

Given a Feynman graph, we have first to know whether it needs renormalization. Relying on Weinberg’s theorem [7], power counting techniques provide the tools for this job. Weinberg’s theorem states that a graph which is power counting finite and whose subgraphs are also power counting finite is finite. Therefore, to locate potential divergences in coordinate space, we must investigate the superficial degree of divergence of every set of vertices closing up. Once a divergence is detected we have to obtain its precise form. For instance, the method of BPHZ would instruct us to expand the loop integrands in Taylor series of the external momenta. Instead, we propose a simple coordinate space method to find an equivalent divergence which depends only on two points but has the same relevant singular behavior.

The following step is to define a subtraction procedure to cure the divergences from the bare amplitude. For instance, BPHZ proceeds by eliminating the divergent terms in the Taylor expansion used in the first step. (If, alternatively, we were using the minimal subtraction scheme in dimensional regularization we would analytically continue the loop integrals in terms of the space-time dimension, expand them in Laurent series around the pole and subtract the pole). In any case, such subtractions must be local for primitively divergent graphs in order to preserve locality. We recognize a local subtraction in momentum space by being polynomial in the external momenta. In coordinate space a local subtraction has support only on the coinciding vertices. We actively make use of this last observation and construct a recursive subtraction of the divergences isolated in step one. Each subtraction is done through a differential formula in such a way that the renormalized form of the initial singularity is delivered.

Different subtraction procedures define different renormalization procedures. However, they must all organize the subtraction of the subdivergences of a given Feynman graph according to its topology (see for instance references [21], [22], [23], [24] and [25]), following Bogoliubov’s recursion formula [1][2], in order to eliminate all such subdivergences and still preserve locality. This is proved by induction in the number of loops showing that, if we can consistently eliminate all subdivergences up to a given order, the remaining overall divergence coming out at the next order is local. A particular solution of Bogoliubov’s formula was given by Zimmermann [3], who defined the well known forests of renormalization. This formulation can easily be shown to come from a counterterm structure [1][25], provided the subtraction procedure is local for primitively divergent graphs. Recall that, when divergences organize in a counterterm structure, they can be absorbed in the parameters of the theory (fields and couplings constants). From counterterms we define bare fields and couplings. Divergences are then cancelled when bare fields and couplings are written in terms of physical quantities. In this picture, the renormalization scale appears to be a new parameter needed to separate divergent from finite parts, so bare amplitudes are independent of it. This statement of independence leads to renormalization group equations for renormalized amplitudes. RGE show that changes in the renormalization scale are also absorbed into redefinitions of the couplings and fields. Since the form of the lagrangian remains the same, Lorentz invariance is preserved. On the other hand, if counterterms are hermitian, the lagrangian maintains, at least formally, its hermiticity and, thus, the $S$-matrix remains unitary.

Let us summarize this brief review. A subtraction procedure which is local for primitively divergent graphs and which is implemented into Bogoliubov’s formula, ensures the renormalization of any Feynman diagram and the existence of a counterterm structure, which, if hermitian, ensures the unitarity of renormalized amplitudes. Such procedure is therefore a correct renormalization procedure.
2.- Basic Examples of the Differential Renormalization Subtraction Procedure

Let us start by recalling the differential renormalization (DR) techniques in two illustrative examples of four-dimensional Euclidean massless $\lambda \phi^4$. From them, we devise a subtraction operation that yields the same DR amplitudes and preserves Euclidean invariance and locality.

The 1PI one-loop four point amplitude of Euclidean massless $\lambda \phi^4$, see Fig.1, is

$$\Gamma^{bare}(x, y, z, w) = \frac{\lambda^2}{2} \delta(x - z) \delta(y - w) \left(\Delta(x - y)\right)^2 + (2 - \text{perm.}), \quad (2.1)$$

where

$$\Delta(x) = \frac{1}{4\pi^2 x^2}, \quad (2.2)$$

is the propagator. We can set $y = 0$, due to translational invariance. Even if the propagator (2.2) is a well-defined distribution, its square is not. This problem manifests itself in the fact that the factor $1/x^4$ upon Fourier transformation produces a logarithmic divergence. To treat this divergence following DR one has to solve the differential equation

$$\frac{1}{x^4} = \Box A(x). \quad (2.3)$$

The solution of Eq. (2.3) is

$$\frac{1}{x^4} = -\frac{1}{4} \ln \frac{x^2 M^2}{x^2}. \quad (2.4)$$

The degree of divergence of the solution of Eq.(2.3) has been reduced by two, as a naive power counting shows. Since the bare amplitude is logarithmically divergent, it would have been sufficient to consider a first order differential equation. The degree of divergence would then have been reduced by one, yielding a correct renormalized factor. However, the use of the laplacian instead of a linear derivative allows to easily impose manifest Euclidean invariance on the solution of Eq.(2.3) just by requiring that $A$ is a function of $x^2$. This increasing of the natural order of the differential equation yields an additive integration constant, dropped to ensure sensible power damping of amplitudes at infinity. In any case, the Fourier transform of this constant is a delta function in momentum space which vanishes as the laplacian produces powers of momenta when it acts by parts. The other integration constant, $M$, is a mass scale that plays a central role in the method: it is the renormalization scale of the amplitude. We will come back to this issue later on.

It is important to note that equality Eq.(2.4) is exact for all values of $x$ except for $x = 0$, where it is undefined. The fact is that the right-hand-side of (2.4) is a well-defined object in the sense of distributions. We can define thus a DR subtraction operator $T$ that allows to eliminate the strong singular behavior at short distances of bare amplitudes, without altering them in any other region, giving

$$\Gamma^{ren} = (1 - T)\Gamma^{bare}, \quad (2.5)$$

where $\Gamma^{ren}$ is now a well-defined distribution. Following this formulation, the operation that selects the singular part of the $1/x^4$ factor is

$$T_{x,0} \frac{1}{x^4} \equiv \frac{1}{x^4} - \left( \frac{1}{4} \ln \frac{x^2 M^2}{x^2} \right). \quad (2.6)$$

\[2\] Throughout this paper we use the notation $\delta(x) \equiv \delta^{(4)}(x)$ and $x^2 = x_\mu x^\mu$. 

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so that using Eq.(2.5) the correct DR renormalized amplitude can be obtained. Introducing a regulator in Eq.(2.6), as done in ref. [20], we can give an explicit meaning to the formal operation carried out here. Recall that the regulated propagator is

$$\Delta_\epsilon(x) = \frac{1}{4\pi^2} \frac{1}{x^2 + \epsilon^2},$$  \hspace{1cm} (2.7)

and we have

$$\frac{1}{x^4} \rightarrow \frac{1}{(x^2 + \epsilon^2)^2} = -\frac{1}{4} \frac{\ln(x^2 + \epsilon^2)/\epsilon^2}{x^2} = -\frac{1}{4} \frac{\ln(x^2 + \epsilon^2)M^2}{x^2} - \pi^2 \ln \epsilon^2 M^2 \delta(x).$$ \hspace{1cm} (2.8)

Therefore, the subtraction is

$$T_{x,0} \frac{1}{(x^2 + \epsilon^2)^2} = -\pi^2 \ln \epsilon^2 M^2 \delta(x).$$ \hspace{1cm} (2.9)

The use of a regulator is, however, not necessary. The $T$ operator isolates the singular part of the factor $1/x^4$, which only has support in $x = 0$. It can be subtracted using Eq.(2.5) and render the amplitude renormalized. The requirement of locality is in this way preserved. We also note that the operator $T$ is obviously shaped as a replacement. It substitutes a singular expression by its renormalized form.

The DR renormalized amplitude corresponding to Eq.(2.1) can be finally written as

$$\Gamma_{\text{ren}}(x,y,z,w) = -\frac{\lambda^2}{128\pi^4} \left( \begin{array}{c} \delta(x-z)\delta(x-w) \left( \ln \frac{(x-y)^2 M^2}{(x-y)^2} \right) \\ \delta(x-z)\delta(x-w) \left( \ln \frac{(x-y)^2 M^2}{(x-y)^2} \right) \end{array} \right).$$ \hspace{1cm} (2.10)

Notice that the dependence of Eq.(2.10) on the mass scale $M$ is such that

$$M \frac{\partial}{\partial M} \Gamma_{\text{ren}}(x,y,z,w) = \frac{3\lambda^2}{16\pi^2} \delta(x-z)\delta(x-w)\delta(x-y),$$ \hspace{1cm} (2.11)

so a change in $M$ can be reabsorbed in a change of the coupling constant $\lambda$. This is an important property of DR. DR renormalized amplitudes automatically satisfy renormalization group equations with $M$ playing the role of renormalization scale. From them we extract the $\beta$ function, obtaining

$$\beta = \frac{3\lambda}{16\pi^2},$$ \hspace{1cm} (2.12)

which is the correct result.

We center now our attention on the study of a two-loop bare amplitude, (Fig.2) given by

$$\Gamma_{\text{bare}}(x,y,z,w) = -\frac{\lambda^3}{2} \left( \begin{array}{c} \delta(z-w)\Delta(x-z)\Delta(y-z) \left( \Delta(x-y) \right)^2 \\ \delta(z-w)\Delta(x-z)\Delta(y-z) \left( \Delta(x-y) \right)^2 \end{array} \right).$$ \hspace{1cm} (2.13)

The singular factor of Eq.(2.13) is

$$f(x,y,0) = \frac{1}{x^2 y^2 (x-y)^4},$$ \hspace{1cm} (2.14)

(we have set $z = 0$). One first has to treat the subfactor $1/(x-y)^4$ in the same way that it was done in Eq.(2.4), so

$$(1 - T_{x,y}) f(x,y,0) = -\frac{1}{4} \frac{1}{x^2 y^2} \ln \frac{(x-y)^2 M^2}{(x-y)^2}.$$ \hspace{1cm} (2.15)

After curing the subdivergence $x \sim y$, $f$ is still logarithmically divergent when the three points, $x,y,0$, approach each other simultaneously. However, notice that now every factor of (2.15), $1/x^2$, $1/y^2$ and $\ln(x-y)^2 M^2/(x-y)^2$, is a well-defined object in the sense of distributions. Only their product is not. In
the spirit of DR, one should define the renormalized factor corresponding to Eq. (2.15) as derivatives of a less singular function. A priori it is not clear how one should proceed here, since there are two independent variables, and it is not obvious how derivatives should be extracted. The strategy followed in ref.[11] was to move the laplacian to the left, by some exact manipulations, in order to use

\[
\frac{1}{x^2} = -4\pi^2 \delta(x), \tag{2.16}
\]

The problem was then reduced basically to that of one variable, and the principle of extracting derivatives could be easily implemented. However, the use of (2.16) is very much dependent on the propagator factors appearing in the amplitude. Eq. (2.16) cannot be applied to some higher loop graphs (see the example of III.3). We will generalize the subtraction procedure that we used in the previous case, where problems arose when two points came close, to this case, where we deal with a three-point problem.

Notice, first, that Eq.(2.15) and

\[
\delta(y) \int d^4 y \frac{1}{x^2 y^2} \frac{\ln(x - y)^2 M^2}{(x - y)^2} = -4\pi^2 \delta(y) \frac{\ln x^2 M^2}{x^4}, \tag{2.17}
\]

have the same divergent behavior when \( x \sim y \sim 0 \), therefore they will need the same subtraction. To prove it, notice that (2.15) is a well-defined distribution in \( y \) and study

\[
\int d^4 y \varphi(y) \frac{1}{x^2 y^2} \frac{\ln(x - y)^2 M^2}{(x - y)^2},
\]

where \( \varphi(y) \) is a test function. Let us write,

\[
\int d^4 y (\varphi(y) - \varphi(0)) \frac{1}{x^2 y^2} \frac{\ln(x - y)^2 M^2}{(x - y)^2} + \varphi(0) \int d^4 y \frac{1}{x^2 y^2} \frac{\ln(x - y)^2 M^2}{(x - y)^2}. \tag{2.18}
\]

The second term corresponds to the action of (2.17) onto the test function. The factor \((\varphi(y) - \varphi(0))\) is a test function which vanishes in the origin. We can write it as \(|y| \psi(y)\), being \( \psi(y) \) a test function. The first term will therefore yield a well defined distribution in \( x \) since the factor

\[
\frac{|y|}{x^2 y^2} \frac{\ln(x - y)^2 M^2}{(x - y)^2}
\]

is now power counting finite. Thus, we have concentrated the divergent behavior in \( x \) around the point 0 in the second term. Therefore the divergent behavior of Eq.(2.15) when \( x \sim y \sim 0 \) is in effect given by Eq.(2.17). In appendix A an alternative proof of this statement can be found. Let us remark that all the integrals considered are well behaved in the infrared region.

We want to find a subtraction that renormalizes (2.15) and is local, in other words, with support only on \( x = y = 0 \). Since (2.15) and (2.17) have the same behavior in the conflictive region \( x \sim y \sim 0 \), they will both need the same subtraction. In this way, we have reduced a three-point problem to that of a two-point function. We can now use the principle of extracting derivatives described in the previous example, so we have to solve the following differential equation

\[
\frac{\ln x^2 M^2}{x^4} = \Box B(x). \tag{2.19}
\]
The solution of Eq.(2.19) is

\[
\frac{\ln x^2 M^2}{x^4} = -\frac{1}{8} \ln^2 x^2 M^2 + \frac{2 \ln x^2 M^2}{x^2},
\]

(2.20)

where \(M'\) is an integration constant that has dimensions of mass. The other integration constant present in the general solution of Eq.(2.19) has been dropped for the same reasons as in Eq.(2.4).

The \(T\)-operator that subtracts the singular part of Eq.(2.15) is in this case

\[
T_{x,y,0} (1 - T_{x,y}) f(x, y, 0) \equiv \pi^2 \delta(y) \left( \frac{\ln x^2 M^2}{x^4} + \frac{1}{8} \ln^2 x^2 M^2 + \frac{2 \ln x^2 M^2}{x^2} \right),
\]

(2.21)

so that

\[
f^{\text{ren}}(x, y, 0) = (1 - T_{x,y,0}) (1 - T_{x,y}) f(x, y, 0)
\]

\[
= -\frac{1}{4} \frac{\partial}{\partial y^\mu} \left( \frac{\ln(x - y)^2 M^2}{(x - y)^2} \right) - \pi^2 \delta(y) \left( \frac{\ln x^2 M^2}{x^4} + \frac{1}{8} \ln^2 x^2 M^2 + \frac{2 \ln x^2 M^2}{x^2} \right).
\]

(2.22)

The sum of these three terms is a well-defined distribution, even though the first two are not. One can now perform exact manipulations in the first term of Eq.(2.22), taking into account the identity

\[
A\Box B = \partial_\nu \left( A \frac{\partial}{\partial y^\nu} B \right) + B\Box A.
\]

(2.23)

Then, using formula Eq.(2.16), we have

\[
f^{\text{ren}}(x, y, 0) =
\]

\[
= -\frac{1}{4} \frac{\partial}{\partial y^\mu} \left( \frac{\ln(x - y)^2 M^2}{(x - y)^2} \right) + \pi^2 \delta(y) \frac{\ln x^2 M^2}{x^4}
\]

\[
- \pi^2 \delta(y) \left( \frac{\ln x^2 M^2}{x^4} + \frac{1}{8} \ln^2 x^2 M^2 + \frac{2 \ln x^2 M^2}{x^2} \right)
\]

\[
= -\frac{1}{4} \frac{\partial}{\partial y^\mu} \left( \frac{\ln(x - y)^2 M^2}{(x - y)^2} \right) - \pi^2 \delta(y) \frac{\ln x^2 M^2}{x^4} + \frac{\pi^2}{8} \delta(y) \ln^2 x^2 M^2 + \frac{2 \ln x^2 M^2}{x^2}.
\]

(2.24)

Hence, the complete renormalized amplitude is

\[
\Gamma^{\text{ren}}(x, y, z, w) =
\]

\[
= \frac{\lambda^3}{8(2\pi)^8} \left[ \delta(x - w) \frac{\partial}{\partial y^\mu} \left( \frac{1}{(x - z)^2 (y - z)^2} \frac{\ln(x - y)^2 M^2}{(x - y)^2} \right) \right.
\]

\[
+ \pi^2 \delta(x - y) \delta(y - z) \ln^2(x - z)^2 M^2 + \frac{2 \ln(x - z)^2 M^2}{(x - z)^2} + (5 - \text{perm}) \right).
\]

(2.25)

This is the result obtained in ref.[11]. It should be noted that this amplitude in momentum space has a far more complicated expression than the one given in Eq.(2.25), since in momentum space it contains a dilogarithm function. Notice also that we find two different mass scales in Eq.(2.25), \(M\) and \(M'\). The first one, \(M\), comes from the renormalization of the subdivergence, and is the same that appears in the one-loop renormalized amplitude Eq.(2.10). Here it has been promoted from \(\ln x^2 M^2\) to \(\ln^2 x^2 M^2\); this is nothing
but the action of the renormalization group. The new mass scale $M'$ has been introduced to renormalize the overall divergence. One could set to any arbitrary value the relation between $M$ and $M'$, the final selection of this relation being fixed by the choice of a particular renormalization scheme.

Up to now we have carefully analyzed two amplitudes in the way DR processes them. We have defined a subtraction operation which replaces by its renormalized form the singular part of those bare amplitudes for a two-point and a three point problem, and we have shown that this last case can be reduced to the former. This is a very important feature that will be used to study $n$-point primitively divergent graphs in the following subsection. The keystone of this reduction is the locality of the divergence occurring in such $n$-point functions.

We would like to point out here an important issue. Formulas such as Eq.(2.23) were used in ref.[11] in order to easily locate and identify divergent factors that needed renormalization. These kind of manipulations are exact and thus were correctly used in ref.[11]. They are totally harmless since they do not alter the singular behavior of bare amplitudes. Such manipulations of derivatives should not be confused with the real core of DR, which lies in the use of differential formulas, as Eq.(2.4) and Eq.(2.20). DR promotes these differential formulas to be the definitions of renormalized amplitudes in the sense of distributions. The integration by parts rule is naturally implemented in this framework and so reduces the degree of divergences of bare amplitudes.

3.-General procedure

We present in this subsection the general rule that DR prescribes to process an arbitrary Feynman graph. For the sake of simplicity, we are going to consider massless scalar theories without derivative couplings, working always in Euclidean space.

We start by recalling some basic definitions concerning Feynman graphs to set up the notations that are going to be used here. A Feynman graph $G$ is a collection of vertices $V_G = \{V_1, \ldots, V_m\}$ and lines $L_G = \{l_1, \ldots, l_\ell\}$ associated with a specific term in the perturbation series of a Green function, and it maps the arguments of the Green function into points or simple vertices on a plane, and each propagator into a line (an oriented line in the case of fermion theories) connecting them. The form of the amplitude corresponding to the Feynman graph is

\[
G \sim \prod_{V_i \in V_G} \chi(V_i) \prod_{l_j \in L_G} \Delta_{l_j}, \tag{2.26}
\]

where $\chi(V_i)$ is the vertex part that carries information about the interaction, and $\Delta_{l_j}$ is the propagator of the theory associated to the line $l_j$.

A generalized vertex $U$ of a Feynman graph $G$ is a subgraph of $G$ containing a set of vertices of $G$ together with all the lines connecting them. Given a specific generalized vertex $U$ of a scalar theory its overall degree of divergence is given through the following formula

\[
\omega(U) = \sum_{\text{conn.}} \left( 2 \left( \frac{d}{2} - 1 \right) \right) - d(m - 1), \tag{2.27}
\]

where $d$ is the space-time dimension, $m$ the number of simple vertices that $U$ contains, and the sum is extended over all the lines which connect two simple vertices. If $\omega(U) \geq 0$, then $U$ is superficially divergent.
A Feynman graph is connected if it is not the union of two disjoint diagrams, and is one-particle and irreducible (1PI), if it is connected and stays so after the removal of any one line. If it is not, it is called one-particle and reducible (1PR).

We first analyze 1PI graphs which are primitively divergent, that is, with no subdivergences. Consider then a primitively divergent 1PI graph which has two external vertices and \( \ell \) lines connecting them. The propagator part of the amplitude is then

\[
f(x, y) = \prod_{\ell} \Delta(x - y).
\] (2.28)

One can set \( y = 0 \) due to translational invariance. DR prescribes to write \( f \) as derivatives, more specifically laplacians in the case of a scalar theory, of a less singular function. In general

\[
f(x) = \Box^{(N)} A(x),
\] (2.29)

where \( N \) specifies the number laplacians in the differential operator, and it is given by the minimum value that satisfies

\[
\omega(A) = \omega(G) - 2N < 0,
\] (2.30)

so that the degree of divergence of \( A \) is negative. Then \( A \) has a well-defined Fourier transform. For any sensible renormalizable quantum field theory this number is not bigger than 2.

The subtraction T-operator for the two-point graph is defined as

\[
T_{x,0} f(x, 0) \equiv f(x, 0) - \Box^{(N)} A(x),
\] (2.31)

so the renormalized factor is

\[
f^\text{ren}(x, 0) = (1 - T_{x,0}) f(x, 0).
\] (2.32)

Let us now consider a primitively divergent graph depending on three vertices such that the divergence arises when the three points coincide and there is no other partial subdvergence. We denote by \( f(x, y, 0) \) this general three point function. We can use the same strategy used in the three-point example we studied but let us present a more rigorous way to proceed. First, one should analyze

\[
\lim_{x \to 0} \int d^4 y \varphi(y) \vartheta(y) f(x, y, 0),
\] (2.33)

where \( \varphi(y) \) is a test function and \( \vartheta \) is defined by

if \( |y| \leq R \) then \( \vartheta(y) = 1 \),
if \( |y| > R \) then \( \vartheta(y) = 0 \),
and \( R \) is an arbitrary vanishing distance. Let us rescale the integration variable, \( y = |x| s \) and expand \( \varphi(|x| s) \) in Taylor series around \( s = 0 \). The degree of divergence of \( f \) will tell us how many orders of the expansion should be kept to obtain its divergent behavior. If \( f \) is logarithmically divergent, the only term of the Taylor expansion that we need is the first, so

\[
\varphi(0) \int d^4 y \vartheta(y) f(x, y, 0) \equiv \varphi(0) F(x).
\] (2.34)
These manipulations are possible since distributions are linear and continuous mappings on the space of test functions. That is, if a sequence of test functions \( \varphi_k \) converges to \( \varphi \) so that the support of \( \varphi \) is included in the union of the supports of the set of \( \varphi_k \), then the sequence given by the action of a distribution \( f \) on the \( \varphi_k \) converges to the action of \( f \) on \( \varphi \). The limit \( x \to 0 \) is only used to truncate the series in order to isolate only the power counting divergent terms. This process of regularization is well known in the mathematical theory of distributions (see for instance ref. [26]) and can always be applied to tempered singularities, such as the ones we encounter in the ultraviolet regime of quantum field theories.

In the limit \( x \to 0 \) we find that \( f(x,y,0) \) is a distribution in \( y \) with support in \( y = 0 \) that is, a delta function or derivatives of the delta function, whose coefficient \( F(x) \) has no Fourier transform. The function \( F(x) \) can now be treated using the two-point procedure described above. The three-point problem has been reduced to a two-point one. We can now define the \( T \)-subtraction operator for the three-point graph, which is

\[
T_{x,y,0}f(x,y,0) = \delta(y) \left( F(x) - \square^{(N)} B(x) \right).
\]

(2.35)

Therefore, the renormalized factor is obtained carrying out the operation

\[
f^{\text{ren}}(x,y,0) = (1 - T_{x,y,0}) f(x,y,0).
\]

(2.36)

This procedure can be easily systematized to treat any \( n \)-point primatively divergent graph, so in general

\[
T_{x_1,\ldots,x_{n-1},0}f(x_1,\ldots,x_{n-1},0) = \delta(x_2)\ldots\delta(x_{n-1}) \left( F(x_1) - \square^{(N)} A(x_1) \right).
\]

(2.37)

where the subtraction has been reduced to the one of a two-point function. Note that this property only holds for theories with only local primitive divergences. Here \( F(x_1) \) is obtained by studying the divergent behavior of \( f(x_1,\ldots,x_{n-1},0) \) when \( x_1 \to 0 \), proceeding in the way that was explained above. The renormalized factor is

\[
f^{\text{ren}}(x_1,\ldots,x_{n-1},0) = (1 - T_{x_1,\ldots,x_{n-1},0}) f(x_1,\ldots,x_{n-1},0).
\]

(2.38)

Let us point out that we wrote Eq.(2.35) and Eq.(2.37) in a compact form considering that in both cases the graph was logarithmically divergent. In general, for more divergent graphs, \( T_{x_1,\ldots,x_{n-1},0}f(x_1,\ldots,x_{n-1},0) \) will also include terms containing derivatives of delta functions (\( T f \) is said to be a quasilocal operator). We present in III.4 the renormalization of a quadratically divergent correlation function of a composite operator to illustrate this fact. Notice that we have cast the systematic version of differential renormalization in the language of subtraction operators which is the most familiar to the renormalization community. However, the true spirit of differential renormalization is closer to the “replacement operation” defined by Bogoliubov and Shirkov [1]. Nevertheless, we have decided to use the more standard language of the \( T \) subtraction for the sake of clarity.

Up to now we have studied how to deal with a primatively divergent 1PI graph. However, in general, one must handle the case where apart from the overall divergence there are also subdivergences, which can be disjoint, nested or overlapping. Tadpoles are not going to be considered, since DR chooses to renormalize them to zero in the massless scalar theories we are presenting.
the overall divergence. The exact way to do it is dictated by the Bogoliubov’s recursion formula. We are
going to recall it here.

Remember that the criteria to find a Feynman graph \( G \) free of UV singularities is given by the Weinberg
convergence theorem: If \( \omega(U) < 0 \) for every generalized vertex \( U \) of \( G \), including \( G \) itself, then \( G \) is absolutely convergent in Euclidean space [7]. Given a Feynman graph \( G \) that contains certain divergent subgraphs
\( (\omega(U) \geq 0) \), the renormalized graph \( RG \) is given by

\[
\begin{align*}
\text{if } \omega(G) < 0 & \quad \text{then } RG = RRG \\
\text{if } \omega(G) \geq 0 & \quad \text{then } RG = (1 - T_G) RRG
\end{align*}
\]

where

\[
RRG = 1 + \sum_{\mathcal{P}} \prod_{U \in \mathcal{P}} (-T_U R U) \prod_{\text{conn}} \Delta,
\]

and the sum extends over all possible partitions \( \mathcal{P} \) of \( G \) into generalized vertices; \( U \) is a generalized vertex
belonging to a certain partition \( \mathcal{P} \) of \( G \); \( \prod_{\text{conn.}} \) is taken over all lines which connect the different sets of the
partition, and \( \Delta \) is the propagator; \( T \) is the DR subtraction operator which acts on \( U \) as follows,

- if \( U \) is simple vertex, \( T_U = U \),
- if \( U \) is 1PR, \( T_U = 0 \),
- if \( U \) is 1PI, \( T_U \) acts on the propagator part of \( U \) as it has been described.

This concludes the abstract presentation of our general procedure. The reader may find the above
formulae too arid. We lead them to a down to earth application of the forest construction equipped with
our subtraction prescription in the examples of Sect. III.

III.- Examples

In this section we present some more involved examples to illustrate the DR systematic procedure. For
simplicity, we bound our study to graphs occurring in massless euclidean \( \phi^4 \) theory in four dimensions.

1.-The Cateye

Our first graph (Fig 3.a) is a paradigm of overlapping divergences. We informally call it “cateye”. Its
bare amplitude is

\[
\Gamma(x_1, x_2, x_3, x_4) = \lambda^4 \frac{1}{4(4\pi)^2} \delta(x_1 - x_2) \delta(x_3 - x_4) f(x_1 - x_3) + \text{2-\,perm},
\]

where

\[
f(x) = \int d^4 u d^4 v \frac{1}{u^2 v^2 (x - u)^2 (x - v)^2 (u - v)^4}.
\]

The forest of this graph is depicted in Fig 3.b. Therefore, its associated forest formula states on \( f \) that

\[
f^{ren}(x) = (1 - T_{x,v}) (1 - T_{x,u,v} - T_{0,u,v} - T_{u,v}) f(x).
\]
The second factor tells how to subtract the divergences corresponding to the two overlapping regions, \( x \sim u \sim v \) and \( 0 \sim u \sim v \).

The action of the operator \((1 - T_{u,v})\) on \( f(x)\) is to substitute in (3.2) the \(1/(u - v)^4\) factor by its renormalized value. We have

\[
(1 - T_{u,v})f(x) = -\frac{1}{4} \int d^4u d^4v \frac{1}{u^2 v^2 (x - u)^2 (x - v)^2} \ln(u - v)^2 M^2 (u - v)^2. \tag{3.4}
\]

According to what is prescribed in the previous section, the subtraction corresponding to each subdivergent region is

\[
T_{x,u,v}(1 - T_{u,v})f(x) = -\pi^2 \int d^4u d^4v \frac{1}{u^2 v^2} \delta(x - u) \left[ \ln(x - v)^2 M^2 + \frac{1}{8} \ln^2(x - v)^2 M^2 + \frac{(v^2 - u^2)^2 M^2}{v^2} \right], \tag{3.5}
\]

\[
T_{0,u,v}(1 - T_{u,v})f(x) = -\pi^2 \int d^4u d^4v \frac{1}{(x - u)^2 (x - v)^2} \delta(u) \left[ \ln(v^2 M^2) + \frac{(u^2 - v^2)^2 M^2}{u^2 \ln(u^2 M^2)} \right]. \tag{3.6}
\]

We can now integrate by parts the \( \square \) in (3.4) , in order to obtain a more suitable expression. Notice that this is an exact operation. From the distribution point of view, it consists in computing a sort of convolution of three well defined distributions, namely \(1/u^2\), \(1/(x - u)^2\) and \(\ln(u - v)^2 M^2/(u - v)^2\), which is well-defined. We obtain

\[
(1 - T_{u,v})f(x) = \\
\pi^2 \int d^4v \frac{\ln v^2 M^2}{x^2 (x - v)^2 v^4} + \pi^2 \int d^4v \frac{\ln(x - v)^2 M^2}{x^2 v^2 (x - v)^4} \\
-\frac{1}{4} \int d^4u d^4v \frac{1}{v^2 (x - v)^2} \partial_{\mu} \left( \frac{1}{u^2} \right) \partial_{\mu} \left( \frac{1}{(x - u)^2} \right) \ln(u - v)^2 M^2. \tag{3.7}
\]

The first two integrals display logarithmic divergences in the regions \( v \sim 0 \) and \( v \sim x \), respectively. However, the third term has no subdivergences whatsoever. We perform the subtraction of (3.5) and (3.6) , which clearly amounts to replacing the divergent factors in the two first terms of (3.7) . The remaining integrals, which are all well-defined, can be computed with more or less technical difficulties (see ref [11]) yielding

\[
\pi^2 \int d^4v \frac{\ln v^2 M^2}{x^2 (x - v)^2 v^4} = \pi^2 \int d^4v \frac{\ln(x - v)^2 M^2}{x^2 v^2 (x - v)^4} = \frac{1}{2} \pi^4 \ln x^2 M^2 + 2 \ln x^2 M^2
\]

and

\[
\frac{1}{4} \int d^4 u d^4 v \frac{1}{v^2 (x - v)^2} \partial_{\mu} \left( \frac{1}{u^2} \right) \partial_{\mu} \left( \frac{1}{(x - u)^2} \right) \ln(u - v)^2 M^2 = 2 \pi^4 \ln x^2 M^2 + 2,
\]

which imply that

\[
(1 - T_{x,u,v} - T_{0,u,v})(1 - T_{u,v})f(x) = \pi^4 \ln x^2 M^2 - 4. \tag{3.8}
\]

Actually, the difference between (3.4) and (3.5) and (3.6) has no subdivergences. The manipulations that we perform on (3.4) just help to exhibit that, in effect, the three point divergence is equivalent to a two point one. From the final expression we obtain, (3.8) , it is straightforward to deal with the remaining overall divergence. Using the general expression (B.1), we write

\[
f_{\text{ren}}(x) = \\
(1 - T_{x,0})(1 - T_{x,u,v} - T_{0,u,v})(1 - T_{u,v})f(x) =
\]

12
\[-\frac{\pi^4}{12} \ln^3 x^2M^2 + 3 \ln^2 x^2M^2 - 6 \ln x^2M^2 \]

(3.9)

We have therefore shown that the original reference [11] was treating the overall divergence problem of this graph in the right manner. The original result was correct because the manipulations performed on (3.4) happened to split correctly the two overlapping regions of divergence.

2.- The non-planar three-loop graph

Our next example is the non-planar three-loop graph occurring in the four point amplitude (Fig 4). This graph is primitively divergent so its forest is trivial. It only exhibits an overall divergence. This can be checked on its bare expression for the propagator factors,

\[ f(x, y, z, 0) = \frac{1}{x^2y^2z^2(x-y)^2(x-z)^2(y-z)^2}. \]

(3.10)

The subtraction will be found by studying the behavior of (3.10) in the limit \( x \to 0 \), as a distribution in \( y \) and \( z \). Let \( \varphi(y, z) \) be a test function. By performing the analysis that was described in the previous section, we can see that

\[ \lim_{x \to 0} \int d^4y d^4z \varphi(y, z) \frac{1}{x^2y^2z^2(x-y)^2(x-z)^2(y-z)^2} \sim \varphi(0, 0) \int d^4y d^4z \frac{1}{x^2y^2z^2(x-y)^2(x-z)^2(y-z)^2}. \]

(3.11)

The integral can be computed using Gegenbauer polynomial techniques. It yields

\[ \int d^4y d^4z \frac{1}{x^2y^2z^2(x-y)^2(x-z)^2(y-z)^2} = \frac{6\pi^4}{x^4} \zeta(3), \]

(3.12)

where \( \zeta \) is the Riemann zeta function. Remark that none of this integrals has infrared problems. The subtraction necessary to render finite this graph is therefore,

\[ T_{x,y,z,0} f(x, y, z, 0) = 6\pi^4 \zeta(3) \delta(y) \delta(z) \left[ \frac{1}{x^4} + \frac{1}{4} \ln \frac{M^2}{x^2} \right]. \]

(3.13)

The difference between (3.10) and (3.13) is a well-defined distribution in the three variables \( x, y, z \). However, we do not attain in this case a closed expression for the renormalized amplitude.

3.- A six-loop graph

The following example is the six-loop graph in Fig 5. After dealing trivially with subdivergences, the product of propagators yields

\[ f(x, y) = \frac{\ln x^2M^2 \ln y^2M^2}{x^2} \frac{\ln(x-y)^2M^2}{y^2} \frac{\ln(x-y)^2M^2}{(x-y)^2}. \]

(3.14)

The overall divergence of the graph, in the region \( x \sim y \sim 0 \), has still to be cured. The use of the antisymmetric derivative formula (2.23) to reduce the problem to that of a two-point function is no longer useful here since the \( \Box \) would hit now a \( \ln y^2M^2/y^2 \) factor which does not produce a delta. We resort to the systematic DR procedure and study the behavior of \( f(x, y) \) in the limit \( x \to 0 \). We find

\[ \lim_{x \to 0} f(x, y) \sim -4\pi^2 \delta(y) \frac{\ln^3 x^2M^2}{x^4} \]

(3.15)
The subtraction is therefore,

\[
T_{x,y,0}f(x,y) = -4\pi^2\delta(y) \left[ \frac{\ln^2 x^2 M^2}{x^2} + \frac{1}{16} \frac{\ln^4 x^2 M^2 + 4 \ln^3 x^2 M^2 + 12 \ln^2 x^2 M^2 + 24 \ln x^2 M^2}{x^2} \right]
\]  (3.16)

Now, the renormalized function, \( f^{\text{ren}}(x,y) = (1 - T_{x,y,0})f(x,y) \) is, by definition, a distribution.

We can perform some exact manipulations in \( f(x,y) \) in order to obtain a closed expression for \( f^{\text{ren}} \). Let us add and subtract from \( f(x,y) \), the following expression,

\[
f_1(x,y) = \frac{\ln^2 x^2 M^2}{x^2} \frac{1}{y^2} \frac{\ln(x-y)^2 M^2}{(x-y)^2}
\]  (3.17)

It can be seen that the difference between \( f(x,y) \) and \( f_1(x,y) \) is well-defined since both terms need the same subtraction. The antisymmetric derivative formula (2.23) can now be applied to \( f_1(x,y) \). The promised closed expression for \( f^{\text{ren}}(x,y) \) is therefore,

\[
f^{\text{ren}}(x,y) = -\frac{\ln x^2 M^2 \ln \frac{x^2}{y^2}}{x^2} \frac{\ln(x-y)^2 M^2}{(x-y)^2} + \partial_\mu \left( \frac{\ln^2 x^2 M^2}{x^2} \frac{1}{y^2} \partial_\mu \frac{\ln(x-y)^2 M^2}{(x-y)^2} \right)
\]

\[
+ \frac{1}{4} \pi^2 \delta(y) \frac{\ln^4 x^2 M^2 + 4 \ln^3 x^2 M^2 + 12 \ln^2 x^2 M^2 + 24 \ln x^2 M^2}{x^2}.
\]  (3.18)

4.-Composite operators

The last example illustrates the case of a three point overall quadratic divergence. Such divergence comes out in the computation of the three-point function,

\[
\langle \cdot : \phi^4(x) : : \phi^2(0) : \rangle,
\]

whose Feynman diagram is shown in Fig.6. Once the subdivergence is cured, we face the overall divergence given by

\[
f(x,y,0) = \frac{1}{x^2} \frac{1}{y^2} \frac{\ln(x-y)^2 M^2}{(x-y)^2}. \]  (3.19)

Power counting analysis reveals that (3.19) is quadratically divergent. To find an equivalent divergence, we follow the usual strategy and compute

\[
\lim_{x \to 0} \int d^4 y \varphi(y) \frac{1}{x^2} \frac{1}{y^2} \frac{\ln(x-y)^2 M^2}{(x-y)^2},
\]

where \( \varphi(y) \) is a test function. Because of the quadratic divergence, we have to keep three terms in the Taylor expansion of the test function. Finally, we find that \( f(x,y,0) \) and

\[
\delta(y) \frac{1}{x^2} \int d^4 y \frac{1}{y^2} \frac{\ln(x-y)^2 M^2}{(x-y)^2} - \partial_\mu \delta(y) \frac{1}{x^2} \int d^4 y \frac{y^\mu}{y^2} \frac{\ln(x-y)^2 M^2}{(x-y)^2}
\]

\[
+ \partial_\mu \partial_\nu \delta(y) \frac{1}{x^2} \int d^4 y \frac{y^\mu y^\nu}{y^2} \frac{\ln(x-y)^2 M^2}{(x-y)^2}
\]

have the same divergent behavior in the limit \( x \to 0 \). Again a three point problem can be reduced to a two point one, thanks to the locality of the subtraction to be performed. To evaluate its exact form, we
need to compute the \( y \)-integrals. Such integrals are understood as being the convolution of two well-defined distributions. Therefore, they are also well-defined. The calculation yields

\[
\int d^4 y \frac{1}{y^2} \ln\left(\frac{(x-y)^2 M^2}{(x-y)^2}\right) = -4\pi^2 \frac{\ln x^2 M^2}{x^2}. \tag{3.20}
\]

\[
\int d^4 y \frac{y^\mu}{y^2} \ln\left(\frac{(x-y)^2 M^2}{(x-y)^2}\right) = 8\pi^2 \partial^\mu \frac{\ln x^2 M^2}{x^2}, \tag{3.21}
\]

\[
\int d^4 y \frac{y^\mu y^\nu}{y^2} \ln\left(\frac{(x-y)^2 M^2}{(x-y)^2}\right) = 8\pi^2 \left(\delta^{\mu\nu} \frac{\ln x^2 M^2}{x^2} - 2 \frac{x^\mu x^\nu}{x^4} - 4x^\mu x^\nu \frac{\ln x^2 M^2}{x^2} - 2 \right). \tag{3.22}
\]

We just have to find the proper differential identities to construct the subtraction. For each term, we have,

\[
S_1(x, y, 0) = -4\pi^2 \delta(y) \left[ \frac{1}{x^2} \ln \frac{x^2 M^2}{x^2} - \frac{1}{8} \frac{\ln x^2 M^2}{x^2} \right], \tag{3.23}
\]

\[
S_2(x, y, 0) = -8\pi^2 \partial_\mu \delta(y) \left[ \frac{1}{x^2} \partial_\mu \ln \frac{x^2 M^2}{x^2} + \frac{1}{16} \partial_\mu \frac{\ln^2 x^2 M^2 + \ln x^2 M^2}{x^2} \right], \tag{3.24}
\]

\[
S_3(x, y, 0) = 8\pi^2 \partial_\mu \partial_\nu \delta(y) \left[ \left(\delta^{\mu\nu} \ln \frac{x^2 M^2}{x^4} - 2 \frac{x^\mu x^\nu}{x^6} \frac{\ln x^2 M^2}{x^2} - 2 \right) \right.
\]

\[
- \frac{1}{8} \partial_\mu \partial_\nu \left[ 1 - 2 \frac{\ln x^2 M^2}{x^2} \right] - \frac{1}{16} \delta^{\mu\nu} \frac{\ln x^2 M^2}{x^2}. \tag{3.25}
\]

The renormalized amplitude is thus found by subtracting the three contributions above,

\[
f_{ren}^\ell(x, y, 0) = (1 - T_{x, y, 0}) f(x, y, 0) = f(x, y, 0) - S_1(x, y, 0) - S_2(x, y, 0) - S_3(x, y, 0). \tag{3.26}
\]

### IV.- Discussion

Most of the quantum field theory community is used to renormalizing Feynman amplitudes using dimensional regularization and minimal subtraction. The common procedure is to solve UV problems by redefining the theory in \( d \) dimensions, rather than 4. This analytical continuation gives meaning to the Feynman amplitudes, and physical results are obtained once poles are removed. The price to pay for this success is that, at an intermediate step, all amplitudes have been changed everywhere often destroying some symmetries which should hold in the final answers. When the limit to 4 dimensions is taken after subtracting the poles, one expects to recover the original amplitudes away from the singularities plus a smooth extension to all points of space-time. The advantage of this method is that formal unitarity is maintained since infinities are absorbed into the bare parameters of the lagrangian.

Let us emphasize that, in principle, no regularization step should be needed to correct for the few points in space-time where amplitudes are ill-defined. This is indeed the main philosophy of the standard BPHZ renormalization procedure as well as ours. Both methods guarantee unitarity through the correct combination of subtractions when subdivergences are present, which is achieved through Bogoliubov’s formula.

The main difference between the two methods is the way the subtraction is performed. In BPHZ, the core of each singularity is isolated by expanding in Taylor series the integrand of loop integrals around some
external momenta. Then, one plainly subtracts this core singularity from the initial amplitude. A finite result is obtained upon computation of the subtracted integral. Differential renormalization, instead, produces a subtraction that replaces the core of the singularity by its renormalized version. The subtraction is done at the level of the amplitude rather than in an integrand and the answer naturally carries a renormalization scale, reflecting the different ways a function singular at one point can be extended into a distribution. The natural scale in BPHZ comes from the external momenta, whereas in DR scale invariance is necessarily broken by the integration constant that comes from writing a singular function as a derivative of less singular functions. As a consequence of avoiding subtraction in integrands at zero external momenta, DR seems to bypass infrared problems as compared to BPHZ where the treatment of massless theories becomes much more involved [27].

From a practical point of view, it is remarkable that using differential renormalization one can compute explicitly complicated renormalized amplitudes, e.g. diagram in Fig.2. Using BPHZ, one encounters a finite two-loop integral which is rather involved. Essentially, coordinate space computations postpone inner integrations to higher loops and produce more compact expressions for the amplitudes at low orders in perturbation theory.

Let us briefly mention some few issues to complete our presentation of systematic DR. Even though we restricted our study to massless scalar theories without derivative couplings, the extension to more complicated cases is rather straightforward. Fermion theories and theories with derivative couplings only differ from the case we studied in the change of the computation of the overall degree of divergence Eq.(2.27) for a generalized vertex, and thus in the order of the differential equations that have to be solved. The extension to massive theories can be easily done, since the presence of masses does not alter the UV behavior of amplitudes. If ones chooses to work in a mass independent renormalization scheme, then one should take the massless limit of the amplitudes and proceed in the exact way that has been explained to locate and cure the UV divergences, using the same kind of simple differential formulas of the massless theory. In a mass dependent renormalization scheme some more complicated differential formulas, involving the presence of Bessel functions, should be used [14]. In any case, the systematic procedure we have set up holds for all these theories. In the presence of symmetries, such as gauge symmetries, these are kept after the renormalization by imposing that amplitudes fulfill Ward identities. These identities are seen in DR as relations among some of the subtraction scales appearing in different amplitudes. The case of QED has been thoroughly analyzed to two loops in ref. [16], with also a study of the chiral anomaly. In this framework, anomalies result when the Ward identities overconstraint the values of the renormalization scales [11].

We would like to finish by asserting one more time that, as presented here, differential renormalization is tailored as a minimal procedure to make sense out of a field theory. It i) locates and isolates the core of the singularity of a bare amplitude, ii) replaces it with a renormalized version, which only differs from the bare one by a local term, and carries an inherent scale, and iii) keeps unitarity by organizing the subtractions in Bogoliubov’s formula which leads to the fulfillment of RG equations. Furthermore, the complications due to the use of a regulator are avoided. A crucial test still ahead of DR is its application to the weak sector of the Standard Model. So far, only a Yukawa model with $\gamma_5$ has been investigated [17] but, there, $\gamma_5$ plays a passive role.

**Acknowledgments**

This work has been partially supported by CICYT, EEC through the Science Twinning Grant SC1000337 and NATO under contract # CRG-910890. C.M. and X.V.C. acknowledge the Ministerio de Educación y Ciencia for an FPI grant. We would like to thank D.Z. Freedman, P.E. Haagensen, J. Soto and R. Tarrach for useful discussions.
Appendix A

In this appendix we present an alternative way to prove that eq. (2.15) and (2.17) have the same divergent behavior in the region $x \sim y \sim 0$. We simply check that their difference is ultraviolet finite by going to momentum space. The respective Fourier transforms are

\[
\int d^4 y d^4 x e^{-iyP} e^{-ixQ} \left(-\frac{1}{4} \frac{1}{x^2} \frac{1}{y^2} \ln(x - y)^2 M^2 \right) = -\pi^2 \int d^4 p \frac{1}{(p - P)^2} \frac{1}{(p - Q)^2} \ln p^2 / \bar{M}^2,
\]

\[
\int d^4 y d^4 x e^{-iyP} e^{-ixQ} \left(\pi^2 \delta(y) \frac{1}{x^2} \frac{1}{x^2} \ln x^2 M^2 \right) = -\pi^2 \int d^4 p \frac{1}{(p - Q)^2} \ln p^2 / \bar{M}^2 \frac{p^2}{p^2},
\]

where $\bar{M} = 2M / \gamma$, and $\gamma = 1.781072...$ is the Euler constant. The difference is

\[
\pi^2 \int d^4 p \frac{\ln p^2 / \bar{M}^2}{(p - Q)^2} \frac{p^2 - 2p \cdot P}{p^2(p - P)^2}
\]

and it shows to be power counting finite.

Appendix B

We include in this appendix a useful general differential identity.

\[
\ln^n \frac{x^2 M^2}{x^4} = -\frac{1}{4(n+1)} \sum_{k=1}^{n+1} \frac{(n+1)! \ln^k x^2 M^2}{k!} x^2. \tag{B.1}
\]

This equation reproduces the results from [11],

\[
\frac{1}{x^4} = -\frac{1}{4} \ln \frac{x^2 M^2}{x^2},
\]

\[
\frac{\ln x^2 M^2}{x^4} = -\frac{1}{8} \ln^2 \frac{x^2 M^2 + 2 \ln x^2 M^2}{x^2},
\]

\[
\frac{\ln^2 x^2 M^2}{x^4} = -\frac{1}{12} \ln^3 \frac{x^2 M^2 + 3 \ln x^2 M^2 + 6 \ln x^2 M^2}{x^2}.
\]
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Figure Captions

Fig.1 The one-loop four-point amplitude in $\lambda\phi^4$.

Fig.2 A contribution to the two-loop four-point amplitude in $\lambda\phi^4$.

Fig.3.a The cateye.

Fig.3.b The forest of the cateye.

Fig.4 The non-planar three-loop graph in $\lambda\phi^4$.

Fig.5 A six-loop graph.

Fig.6 A composite operator three-point function, $<\phi^4(x) : \phi^4(y) : \phi^2(0) :>$.