Quantum Observable Generalized Orthoalgebras

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Abstract. Let $S(\mathcal{H})$ denote the set of all self-adjoint operators (not necessarily bounded) on a Hilbert space $\mathcal{H}$, which is the set of all physical quantities on a quantum system $\mathcal{H}$. We introduce a binary relation $\perp$ on $S(\mathcal{H})$. We show that if $A \perp B$, then $A$ and $B$ are affiliated with some abelian von Neumann algebra. The relation $\perp$ induces a partial algebraic operation $\oplus$ on $S(\mathcal{H})$. We prove that $(S(\mathcal{H}), \perp, \oplus, 0)$ is a generalized orthoalgebra. This algebra is a generalization of the famous Birkhoff–von Neumann quantum logic model. It establishes a mathematical structure on all physical quantities on $\mathcal{H}$. In particular, we note that $(S(\mathcal{H}), \perp, \oplus, 0)$ has a partial order $\preceq$, and prove that $A \preceq B$ if and only if $A$ has a value in $\Delta$ implies that $B$ has a value in $\Delta$ for every Borel set $\Delta$ not containing 0. Moreover, the existence of the infimum $A \wedge B$ and supremum $A \vee B$ for $A, B \in S(\mathcal{H})$ (with respect to $\preceq$) is studied, and it is shown at the end that the position operator $Q$ and momentum operator $P$ in the Heisenberg commutation relation satisfy $Q \wedge P = 0$.

Key Words. Quantum observable, Self-adjoint operator, Generalized orthoalgebra, Order.

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1 Introduction

The sixth problem of Hilbert’s Mathematical Problems (Hilbert outlined 23 major mathematical problems in 1900 to be studied in the coming century) is about mathematical treatment of the axioms of physics – “The investigations on the foundations of geometry suggest the problem: To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.” In 1933, Kolmogorov axiomatized modern probability theory ([1]). In Kolmogorov’s theory, the set \( \mathcal{L} \) of experimentally verifiable events form a Boolean \( \sigma \)-algebra; therefore, Boolean algebra theory can be used to describe classical logic. However, Kolmogorov’s theory does not describe situations that arise from quantum mechanics, e.g. the Heisenberg uncertainty principle ([2]). One of the most fundamental problems in quantum theory is to find a mathematical description for the structure of random events in a quantum system. This was originally studied in 1930s by Birkhoff and von Neumann ([3]). In von Neumann’s approach, a quantum system is represented by a separable complex Hilbert space \( \mathcal{H} \); each physical quantity is represented by a self-adjoint operator on \( \mathcal{H} \), and is called a quantum observable. The set of all quantum observables is denoted by \( \mathcal{S}(\mathcal{H}) \). Since the spectrum \( \sigma(P) \) of a projection operator \( P \) is contained in \( \{0, 1\} \), if the truth values (true and false) for two-valued propositions about the quantum system are encoded by 0 and 1, then these propositions can be represented by projections on \( \mathcal{H} \). Birkhoff and von Neumann considered the set \( \mathcal{P}(\mathcal{H}) \) of all projections on \( \mathcal{H} \) as the logic of the quantum system ([3]). If \( A \in \mathcal{S}(\mathcal{H}) \) and \( P^A \) is the spectrum measure of \( A \), then for each real Borel set \( \Delta \), \( P^A(\Delta) \) represents the event that the values of physical quantity \( A \) are contained in \( \Delta \).

Let \( \mathcal{L} \) be a lattice with two binary operations the supremum \( \lor \) and the infimum \( \land \). If there are elements 0 and \( I \) in \( \mathcal{L} \) and a unary operation \( ' : \mathcal{L} \to \mathcal{L} \) such that \( x'' = x, x \lor x' = I, x \land x' = 0, \) and \( 0 \leq x \leq I \) for each \( x \in \mathcal{L} \), then \( (\mathcal{L}, \lor, \land, ' , 0, I) \) is said to be an ortholattice, and \( ' \) is said to be an orthocomplementation operation.

We say that an ortholattice \( (\mathcal{L}, \lor, \land, ' , 0, I) \) satisfies the orthomodular law if for
$x, y \in \mathcal{L},$

$$x \leq y \Rightarrow y = x \lor (y \land x').$$

The name of orthomodular was suggested by Kaplansky. An ortholattice satisfying the orthomodular law is said to be an orthomodular lattice ([4]). We say an ortholattice $(\mathcal{L}, \lor, \land, ', 0, I)$ satisfies the modular law if for $x, y, z \in \mathcal{L},$

$$x \leq y \Rightarrow y \land (x \lor z) = x \lor (z \land y).$$

An ortholattice satisfying the modular law is said to be a modular lattice ([4]).

Let $p, q \in \mathcal{P}(\mathcal{H})$. We say that $p \leq q$ if $\langle px, x \rangle \leq \langle qx, x \rangle$ for all $x \in \mathcal{H}$. Then $(\mathcal{P}(\mathcal{H}), \leq)$ is a lattice with respect to the partial order “$\leq$” with the minimal element 0 and the maximal element $I$. If we define $p' = I - p$, then $(\mathcal{P}(\mathcal{H}), \lor, \land, ', 0, I)$ is an ortholattice. Husimi ([5]) showed that $(\mathcal{P}(\mathcal{H}), \lor, \land, ', 0, I)$ is an orthomodular lattice. Also see [4] for the use of orthomodular lattices in quantum logic. Birkhoff and von Neumann ([3]) showed that if $\mathcal{H}$ is finite dimensional, then $(\mathcal{P}(\mathcal{H}), \lor, \land, ', 0, I)$ is a modular lattice.

There are properties that clearly distinguish quantum logic from classical logic. Note that each Boolean algebra $\mathcal{A}$ (for classical logic) is a distributive ortholattice, that is, for $x, y, z \in \mathcal{A},$

$$x \land (y \lor z) = (x \land y) \lor (x \land z) \quad \text{and} \quad x \lor (y \land z) = (x \lor y) \land (x \lor z),$$

while the distributive law does not hold in $(\mathcal{P}(\mathcal{H}), \lor, \land, ', 0, I)$ (for quantum logic).

Let $(\mathcal{L}, \lor, \land, ', 0, I)$ be an orthomodular lattice. We say that $x$ and $y$ satisfy the binary relation $\bot$ if $x \leq y'$. We define a partial operation $\oplus$ on $\mathcal{L}$ by $x \oplus y = x \lor y$ if $x \bot y$. Then, we obtain a new algebraic structure $(\mathcal{L}, \bot, \oplus, 0, I)$ with the following properties:

(OA1) If $x \bot y$, then $y \bot x$ and $x \oplus y = y \oplus x$.

(OA2) If $y \bot z$ and $x \bot (y \oplus z)$, then $x \bot y$, $(x \oplus y) \bot z$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

(OA3) For each $x \in \mathcal{L}$, there exists a unique $y \in \mathcal{L}$ such that $x \bot y$ and $x \oplus y = I$.

(OA4) If $x \bot x$, then $x = 0$. 

3
Foulis, Greechie and Rüttimann called this structure \((\mathcal{L}, \perp, \oplus, 0, I)\) an orthoalgebra ([6]). Kalmbach, Riečanová, Hedlíková, Pulmannová and Dvurečenskij introduced the following definition ([7, 8, 9]):

**Definition 1.** A generalized orthoalgebra \((\mathcal{E}, \perp, \oplus, 0)\) is a set \(\mathcal{E}\) with an element 0, a binary relation \(\perp\), and a partial operation \(\oplus\), such that if \(x \perp y\), then \(x \oplus y\) is defined and satisfies the following conditions:

- **(OA1).** If \(x \perp y\), then \(y \perp x\) and \(x \oplus y = y \oplus x\).
- **(OA2).** If \(y \perp z\) and \(x \perp (y \oplus z)\), then \(x \perp y\), \((x \oplus y) \perp z\) and \((x \oplus y) \oplus z = x \oplus (y \oplus z)\).
- **(OA4).** If \(x \perp x\), then \(x = 0\).
- **(GOA1).** If \(x \perp y\), \(x \perp z\) and \(x \oplus y = x \oplus z\), then \(y = z\).
- **(GOA2).** \(x \perp 0\) and \(x \oplus 0 = x\) for all \(x \in \mathcal{E}\).
- **(GOA3).** If \(x \perp y\) and \(x \oplus y = 0\), then \(x = y = 0\).

Let \((\mathcal{E}, \perp, \oplus, 0)\) be a generalized orthoalgebra. For \(a, b \in \mathcal{E}\), if there is a \(c \in \mathcal{E}\) such that \(a \perp c\) and \(a \oplus c = b\), then we say that \(a \preceq b\). It can be shown that \(\preceq\) is a partial order. Moreover, \(x \perp y\) if and only if \(x \leq y'\) ([9]). Generalized orthoalgebras are very important models of quantum logic ([9]).

In [10], Gudder defined a binary relation \(\perp\) on the set \(\mathcal{S}_b(\mathcal{H})\) of all bounded self-adjoint operators on \(\mathcal{H}\) by \(A \perp B\) once \(AB = 0\), and then define \(A \oplus B = A + B\). However, many of the operators that arise naturally in physics are not bounded. For example, in Heisenberg’s commutation relation, a fundamental relation in quantum mechanics, \(QP - PQ = -i\hbar I\), the position operator \(Q\) and the momentum operator \(P\) cannot be realized by bounded operators (see [11] for a full account on this). Therefore, it is necessary to study unbounded operators and, in particular, the set \(\mathcal{S}(\mathcal{H})\) of all self-adjoint (possibly unbounded) operators on \(\mathcal{H}\).

In this paper, we introduce a binary relation \(\perp\) on \(\mathcal{S}(\mathcal{H})\). For \(A, B \in \mathcal{S}(\mathcal{H})\), we say \(A \perp B\) if and only if \(\overline{\text{ran}}(A)\) is orthogonal to \(\overline{\text{ran}}(B)\), where \(\overline{\text{ran}}(\cdot)\) denotes the closure of the range of an operator. If \(A \perp B\), define \(A \oplus B = A + B\). We show that if \(A \perp B\), then \(A\) and \(B\) are affiliated with some abelian von Neumann algebra. Moreover, we show that \((\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)\) is a generalized orthoalgebra. In
this way, we establish a new quantum logic structure on all physics quantities of the quantum system $\mathcal{H}$. Note that the generalized orthoalgebra $(\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)$ has a nature partial order $\preceq$. We show that $A \preceq B$ if and only if $A$ has a value in $\Delta$ implies that $B$ has a value in $\Delta$ for every Borel set $\Delta$ not containing $0$. The existence of the infimum $A \wedge B$ and supremum $A \vee B$ for $A, B \in \mathcal{S}(\mathcal{H})$ with respect to $\preceq$ is also studied. At the end, we show that the position operator $Q$ and momentum operator $P$ satisfy $Q \wedge P = 0$ with respect to $\preceq$.

2 Definitions and Facts of Self-adjoint Operators

We first recall some elementary concepts and facts of unbounded linear operators (see Section 4 of [11] for a brief summary, and Sections 2.7 and 5.6 of [12] and Section 6.1 of [13] for more details). A linear operator $A$ we consider will have a domain $\mathcal{D}(A)$ that is dense in $\mathcal{H}$. Given two linear operators $A : \mathcal{D}(A) \to \mathcal{H}$ and $B : \mathcal{D}(B) \to \mathcal{H}$, we write $A \subseteq B$ and say that $B$ is an extension of $A$, if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Ax = Bx$ for all $x \in \mathcal{D}(A)$. For a linear operator $A : \mathcal{D}(A) \to \mathcal{H}$, the adjoint of $A$, denoted by $A^*$, is defined as follows. Its domain consists of those vectors $y$ in $\mathcal{H}$ such that for some $y^*$ in $\mathcal{H}$, $\langle x, y^* \rangle = \langle Ax, y \rangle$ for all $x \in \mathcal{D}(A)$, and $A^*y = y^*$ for each $y \in \mathcal{D}(A^*)$. We say that $A$ is symmetric if $A \subseteq A^*$, and self-adjoint if $A = A^*$. If $A \subseteq B$, then $B^* \subseteq A^*$. If $A \subseteq B$ with $A$ self-adjoint and $B$ symmetric, then $A = B$ (in this case $A$ has no proper symmetric extension, that is, a self-adjoint operator is maximal symmetric). We say that $A$ is closed if its graph $G(A) = \{(x, Tx)| x \in \mathcal{D}(A)\}$ is closed, and $A$ is closable (or preclosed) if there exists a closed linear operator $B$ such that its graph is the closure of the graph of $A$, $\overline{G(A)} = G(B)$. In this case, $B$ is called the closure of $A$, denoted by $\overline{A}$. Self-adjoint operator are closed and symmetric operators are closable. If $A$ is closable, then $(\overline{A})^* = A^*$. If $A$ is closed and $\overline{G(A|\mathcal{D}_0)} = G(A)$, where $\mathcal{D}_0$ is a dense linear subspace of $\mathcal{D}(A)$, we say that $\mathcal{D}_0$ is a core for $A$.

A family $\{E_\lambda\}$ of projections indexed by $\mathbb{R}$, satisfying

(i) $\bigwedge_{\lambda \in \mathbb{R}} E_\lambda = 0$ and $\bigvee_{\lambda \in \mathbb{R}} E_\lambda = I$,
(ii) $E_{\lambda_1} \leq E_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$;
(iii) $\wedge_{\lambda \geq \lambda_1} E_\lambda = E_{\lambda_1},$

is said to be a resolution of the identity. The following is a spectral theorem for self-adjoint operators.

**Lemma 1.** If $A$ is a self-adjoint operator on $\mathcal{H}$, then there is a unique (projection-valued) spectral measure $P^A$ defined on all Borel subsets of $\mathbb{R}$ such that

$$A = \int_{\mathbb{R}} \lambda dP^A(\lambda).$$

If we denote $E^A_\lambda = P^A((-\infty, \lambda])$, then $\{E^A_\lambda\}$ is a resolution of the identity, and it is said to be the resolution of the identity for $A$. Let $F^A_n = E^A_n - E^A_{-n}$. Then $\bigcup_{n=1}^{\infty} F^A_n(\mathcal{H})$ is a core for $A$, and for each $x \in F^A_n(\mathcal{H})$ and $n \in \mathbb{N}$,

$$Ax = \int_{-n}^{n} \lambda dE^A_\lambda x$$

in the sense of norm convergence of approximating Riemann sums. In addition, for $m, n \in \mathbb{N}$, $F^A_n F^A_m = F^A_n F^A_m$, $F^A_n A \subseteq AF^A_n$, and $AF^A_n x \to Ax$ for each $x \in D(A)$.

**Lemma 2.** If $A$ is a closed, then the null space of $A$, $\text{null}(A) = \{x \in D(A) : Ax = 0\}$, is a closed subspace of $\mathcal{H}$. Moreover, $(\text{ran}(A))^\perp = \text{null}(A^*)$, $(\text{ran}(A^*))^\perp = \text{null}(A)$, $\overline{\text{ran}}(A^*A) = \overline{\text{ran}}(A^*)$, $\text{null}(A^*A) = \text{null}(A)$.

Let $P_A$ and $N_A$ denote the projections whose ranges are $\overline{\text{ran}}(A)$ and $\text{null}(A)$, respectively.

**Lemma 3.** Let $A, B \in \mathcal{S}(\mathcal{H})$. Then $P^A(\{0\}) = N_A$, $P^A(\mathbb{R} \setminus \{0\}) = P_A$, $P_A + N_A = I$ and $P_A \lor P_B = I - N_A \land N_B$.

**Lemma 4** ([14]). Let $A \in \mathcal{S}(\mathcal{H})$. If $B$ is a bounded linear operator on $\mathcal{H}$ and $BA \subseteq AB$, then for a Borel set $\Delta \subseteq \mathbb{R}$, $P^A(\Delta)B = BP^A(\Delta)$.

**Lemma 5.** Suppose that $A$ and $B$ are densely defined on $\mathcal{H}$. Then $A^* + B^* \subseteq (A + B)^*$ if $A + B$ is densely defined, and $B^* A^* \subseteq (AB)^*$ if $AB$ is densely defined.

**Lemma 6.** If $A$ and $C$ are densely defined closable operators on $\mathcal{H}$ and $B$ is a bounded operator on $\mathcal{H}$ such that $A = BC$, then $A^* = C^* B^*$. 

6
Lemma 7. Let $A, B \in S(H)$. Then the following statements are equivalent:

(i) $A \perp B$, that is, $\text{ran}(A)$ is orthogonal to $\text{ran}(B)$.

(ii) $\text{ran}(A) \subseteq \text{null}(B)$.

(iii) $\text{ran}(B) \subseteq \text{null}(A)$.

(iv) $AB \subseteq 0$ and $\mathcal{D}(AB) = \mathcal{D}(B)$.

(v) $BA \subseteq 0$ and $\mathcal{D}(BA) = \mathcal{D}(A)$.

Proof. Clearly, $(i) \iff (ii) \iff (iii)$.

$(ii) \iff (v)$: Suppose $BA \subseteq 0$ and $\mathcal{D}(BA) = \mathcal{D}(A)$. For each $x \in \mathcal{D}(A)$, $Ax \in \mathcal{D}(B)$ and $BAx = 0$. That is $Ax \in \text{null}(B)$. So $\text{ran}(A) \subseteq \text{null}(B)$. Since $\text{null}(B)$ is closed, $\text{ran}(A) \subseteq \text{null}(B)$. Conversely, suppose that $\text{ran}(A) \subseteq \text{null}(B)$. Then, for each $x \in \mathcal{D}(A)$, $Ax \in \text{null}(B)$, we have $x \in \mathcal{D}(BA)$ and $BAx = 0$. Therefore, $\mathcal{D}(BA) = \mathcal{D}(A)$ and $BA \subseteq 0$.

Similarly, $(iii) \iff (iv)$.

3 The Affiliate Relationship

We say that a closed operator $T$ is affiliated with a von Neumann algebra $\mathcal{R}$ and write $T \eta \mathcal{R}$ when $UT = TU$ for each unitary operator $U$ commuting with $\mathcal{R}$. (Note that the equality $UT = TU$ means that $\mathcal{D}(UT) = \mathcal{D}(T)$ and $UTx = TUx$ for each $x \in \mathcal{D}(T)$ and $U$ maps $\mathcal{D}(T)$ onto itself.)

Murray and von Neumann showed ([15]) that the family of operators affiliated with a factor of type $\Pi_1$ (or, more generally, affiliated with a finite von Neumann algebra, those in which the identity operator is finite) admits surprising operations of addition and multiplication that suit the formal algebraic manipulations used by the founders of quantum mechanics in their mathematical model. See Section 6 of [11] for fundamental properties of affiliated operators.

Lemma 8 ([12]). If $A$ is a self-adjoint operator, and $A$ is affiliated with some abelian von Neumann algebra $\mathcal{R}$, then $\{E^A_\lambda\} \subseteq \mathcal{R}$.

Lemma 9 ([12]). If $\{E_\lambda\}$ is a resolution of the identity, $\mathcal{R}$ is an abelian von Neumann algebra containing $\{E_\lambda\}$, then there is a self-adjoint operator $A$ affiliated
with \( R \), and

\[
Ax = \int_{-n}^{n} \lambda dE_{\lambda}x
\]

for each \( x \in F_{n}(\mathcal{H}) \) and \( n \in \mathbb{N} \), where \( F_{n} = E_{n} - E_{-n} \); and \( \{ E_{\lambda} \} \) is the resolution of the identity for \( A \).

**Example** ([12]). If \( (S, \varphi, m) \) is a \( \sigma \)-finite measure space and \( \mathcal{A} \) is its multiplication algebra acting on \( L^{2}(S, \varphi, m) \), then \( \mathcal{A} \) is a closed densely defined operator affiliated with \( \mathcal{A} \) if and only if \( \mathcal{A} = M_{g} \) (multiplication by \( g \)) for some measurable function \( g \) finite almost everywhere on \( S \). In this case, \( \mathcal{A} \) is self-adjoint if and only if \( g \) is real-valued almost everywhere.

Kadison and Liu ([11]) showed that the Heisenberg relation \( QP - PQ = -i\hbar I \) cannot be satisfied with self-adjoint operators affiliated with any finite von Neumann algebra.

**Theorem 1.** Let \( A, B \in \mathcal{S}(\mathcal{H}) \). If \( A \perp B \), then there exists an abelian von Neumann algebra \( \mathcal{R} \) such that \( A\eta \mathcal{R} \) and \( B\eta \mathcal{R} \). Moreover, \( \bigcup_{n=1}^{\infty} F_{n}^{A}F_{n}^{B}(\mathcal{H}) \) is a common core for \( A \) and \( B \).

**Proof.** Suppose that \( A \perp B \), that is, \( \text{ran}(A) \) is orthogonal to \( \overline{\text{ran}}(B) \). It follows that \( AF_{n}^{A}BF_{m}^{B} = BF_{m}^{B}AF_{n}^{A} = 0 \) for \( m, n \in \mathbb{N} \). For each \( x \in \mathcal{D}(B) \), as \( BF_{m}^{B}x \to Bx \), we have \( AF_{n}^{A}B \subseteq BAF_{n}^{A} \). From Lemma 4, \( F_{m}^{B}AF_{n}^{A} = AF_{n}^{A}F_{m}^{B} \) for \( m, n \in \mathbb{N} \). Similarly, we have \( F_{n}^{A}BF_{m}^{B} = BF_{m}^{B}F_{n}^{A} \). Also, we note that \( F_{n}F_{m} = F_{m}F_{n} \) (see Lemma 18 and Proposition 32 of [16]). Moreover, \( \bigcup_{n=1}^{\infty} F_{n}^{A}F_{n}^{B}(\mathcal{H}) \) is a common core for \( A \) and \( B \).

Let \( \mathcal{R} \) be the von Neumann algebra generated by \( \{ F_{n}^{A}, AF_{n}^{A}, F_{n}^{B}, BF_{n}^{B} : n = 1, 2, \ldots \} \). Since the elements in \( \{ F_{n}^{A}, AF_{n}^{A}, F_{n}^{B}, BF_{n}^{B} : n = 1, 2, \ldots \} \) are commuting, \( \mathcal{R} \) is abelian. If \( U \) is a unitary operator in \( \mathcal{R}' \) and \( x \in \bigcup_{n=1}^{\infty} F_{n}^{A}(\mathcal{H}) \) (a core of \( A \)), then \( AUx = AF_{n}^{A}Ux = UAF_{n}^{A}x = UAx \) for some \( n \). So \( A\eta \mathcal{R} \). Similarly, \( B\eta \mathcal{R} \). ■
4 Generalized Orthoalgebra \((\mathcal{S}(\mathcal{H}), \bot, \oplus, 0)\)

In this section, we show that \((\mathcal{S}(\mathcal{H}), \bot, \oplus, 0)\) is a generalized orthoalgebra.

**Proposition 1.** Let \(A, B \in \mathcal{S}(\mathcal{H})\) with \(A^2 = BA\). Then

(i) \(\bigcup_{n=1}^{\infty} F_n^{B}(\mathcal{H})\) is a common core for \(A\) and \(B\).

(ii) \(\mathcal{D}(B) \subseteq \mathcal{D}(A)\).

**Proof.** (i) Since \(A^2\) is self-adjoint and \(A^2 = BA\), \(BA\) is self-adjoint and \(AB = A^{\ast}B^{\ast} \subseteq (BA)^{\ast} = BA\) (Lemma 5). Now, with \(A^2 = BA\) and \(AB \subseteq BA\), we have \((AF_m^A)^2 = BAF_m^A \supseteq AF_m^A B\) for each \(m \in \mathbb{N}\). From Lemma 4, \(F_n^{B}AF_m^A = AF_m^{A}F_n^{B}\). For each \(x \in \mathcal{D}(A)\) and \(n \in \mathbb{N}\), since \(AF_m^{A}x \to Ax\) as \(m \to \infty\), \(F_n^{B}Ax = F_n^{B}(\lim_n AF_m^{A}x) = \lim_n F_n^{B}AF_m^{A}x = \lim_n AF_m^{A}F_n^{B}x\). Since \(A\) is closed and \(F_n^{B}F_n^{B}x \to F_n^{B}x\) as \(m \to \infty\), we have \(F_n^{B}x \in \mathcal{D}(A)\) and \(AF_n^{B}x = F_n^{B}Ax\). So \(F_n^{B}A \subseteq AF_n^{B}\) for each \(n \in \mathbb{N}\). It follows that \(\bigcup_{n=1}^{\infty} F_n^{B}(\mathcal{H})\) is a core for \(A\), hence a common core for \(A\) and \(B\).

(ii) Since \(F_n^{B}A \subseteq AF_n^{B}\) (note that \(AF_n^{B}\) is closed since \(A\) is closed and \(F_n^{B}\) is bounded), \(F_n^{B}A\) is closable. From Lemma 18 of [17], \(F_n^{B}A = AF_n^{B}\). Thus \((F_n^{B}A)^{\ast} = (F_n^{B}A)^{\ast} = (AF_n^{B})^{\ast}\) (recall that if \(T\) is closable, then \(T^{\ast} = T^{\ast}\)) and it follows from Lemma 5 and Lemma 6 that \(F_n^{B}A \subseteq (AF_n^{B})^{\ast} = (F_n^{B}A)^{\ast} = AF_n^{B}\). So \(AF_n^{B}\) is self-adjoint for each \(n \in \mathbb{N}\). Since \(A^2 = BA\) and \((AF_n^{B}F_m^{B})^2 = BF_n^{B}AF_m^{B} = AF_m^{B}BF_n^{B}\) for \(m, n \in \mathbb{N}\), we have \(BF_n^{B}A \subseteq ABF_n^{B}\). By Lemma 4, \(P_AB BF_n^{B} = BF_n^{B}P_A\). For each \(x \in \mathcal{D}(B), BF_n^{B}x \to Bx, BF_n^{B}P_Ax = P_A BF_n^{B}x \to P_A Bx\). Since \(F_n^{B}P_Ax \to P_A x\) and \(B\) is closed, we have \(P_A x \in \mathcal{D}(B)\) and \(BP_Ax = P_A Bx\). That is \(P_A B \subseteq BP_A\).

Since \(AB \subseteq BA = A^2\), for each \(x \in \mathcal{D}(B)\) with \(Bx = 0\), it follows that \(x \in \mathcal{D}(A^2)\) and \(A^2 x = 0\). Since \(null(A^2) = null(A^{\ast}A) = null(A)\) (Lemma 2), we have \(Ax = 0\) and \(null(B) \subseteq null(A)\). Then

\[ \mathcal{H} = \overline{\text{ran}}(A) \oplus (\text{null}(A) \cap \overline{\text{ran}}(B)) \oplus \text{null}(B). \]

For each \(x \in \mathcal{D}(B), x = x_1 + x_2 + x_3\) where \(x_1 \in \overline{\text{ran}}(A), x_2 \in (\text{null}(A) \cap \overline{\text{ran}}(B))\) and \(x_3 \in \text{null}(B)\). Then \(P_A B(x_1 + x_2) = B P_A (x_1 + x_2) = B P_A x_1 = Bx_1\). Thus \(x_1 \in \mathcal{D}(B)\). Since \(x_1 \in \overline{\text{ran}}(A)\), there exists a sequence \(\{y_m\} \subseteq \mathcal{D}(A)\) such that \(A y_m \to x_1\). For each \(n \in \mathbb{N}, AF_n^{B}x_1 = AF_n^{B}(\lim_m A y_m) = \lim_m AF_n^{B} A y_m = \)
\[ \lim_m BAF_n^B y_m = \lim_m BF_n^B AF_n^B y_m = BF_n^B (\lim_m F_n^B A y_m) = BF_n^B x_1 \rightarrow Bx_1. \]

Since \( A \) is closed, we have \( x_1 \in D(A) \) and \( Ax_1 = Bx_1 \). We have \( x_i \in D(A) \) for \( i = 1, 2, 3 \) and hence \( x \in D(A) \). Therefore, \( D(B) \subseteq D(A) \). \( \blacksquare \)

**Proposition 2.** Let \( A, B \in S(\mathcal{H}) \) with \( A \perp B \). Then \( A + B \) is densely defined and self-adjoint. (cf. Lemma 5)

**Proof.** From Theorem 1, \( \bigcup_{n=1}^\infty F_n^A F_n^B(\mathcal{H}) \) is a common core for \( A \) and \( B \). So \( D(A + B) = D(A) \cap D(B) \) is dense.

To see that \( A + B \) is closed, let \( \{x_n\} \subseteq D(A + B) \) with \( x_n \rightarrow x \) and \( (A + B)x_n \rightarrow y \). Since \( \mathcal{H} = \overline{\text{ran}}(A) \oplus \text{null}(A) \), we have \( x_n = x_n^{(1)} + x_n^{(2)} \) where \( \{x_n^{(1)}\} \subseteq \overline{\text{ran}}(A) \) and \( \{x_n^{(2)}\} \subseteq \text{null}(A) \). Since \( \overline{\text{ran}}(A) \subseteq \text{null}(B) \) (Lemma 7), we have \( (A + B)x_n = Ax_n^{(1)} + Bx_n^{(2)} \rightarrow y \). Now \( NB(A + B)x_n = N_B Ax_n^{(1)} + N_B Bx_n^{(2)} = Ax_n^{(1)} + 0 \rightarrow N_By \), then \( Bx_n^{(2)} = y - Ax_n^{(1)} \rightarrow y - N_By \). Since \( A \) is closed and \( Ax_n = Ax_n^{(1)} \rightarrow N_By \), it follows that \( x \in D(A) \) and \( Ax = N_By \). Similarly, since \( B \) is closed and \( Bx_n = Bx_n^{(2)} \rightarrow y - N_By \), we have \( x \in D(B) \) and \( Bx = y - N_By \). Therefore, \( x \in D(A) \cap D(B) \) and \( (A + B)x = y \), which implies that \( A + B \) is closed.

We note that \( \bigcup_{n=1}^\infty F_n^A F_n^B(\mathcal{H}) \) is also a core for \( (A + B)^* \). To see this, since \( F_n^A F_n^B(A + B) \subseteq (A + B)F_n^A F_n^B \) and \( F_n^A F_n^B = F_n^B F_n^A \), we have \( F_n^A F_n^B(A + B)^* \subseteq (A + B)^* F_n^A F_n^B \). For each \( x \in D((A + B)^*) \), \( F_n^A F_n^B x \rightarrow x \) and \( (A + B)^* F_n^A F_n^B x = F_n^A F_n^B (A + B)^* x \rightarrow (A + B)^* x \). So \( \bigcup_{n=1}^\infty F_n^A F_n^B(\mathcal{H}) \) is a core for \( (A + B)^* \).

Since \( A + B = A^* + B^* \subseteq (A + B)^* \) and they have the same common core, we have \( A + B = (A + B)^* \) and \( A + B \) is self-adjoint. \( \blacksquare \)

Now, for \( A, B \in S(\mathcal{H}) \), we define \( A \oplus B = A + B \) when \( A \perp B \).

**Theorem 2.** \( (\mathcal{S}(\mathcal{H}), \perp, \oplus, 0) \) is a generalized orthoalgebra.

**Proof.** Clearly, the conditions (OA1) and (GOA2) hold in \( (\mathcal{S}(\mathcal{H}), \perp, \oplus, 0) \).

(OA2): Let \( A \perp B \) and \( C \perp (A \oplus B) \). We first show that \( C \perp B \). For each \( x \in D(C) \), since \( (A + B) \perp C \), we have \( (A + B)Cx = ACx + BCx = 0 \). Then \( \langle ACx, BCx \rangle + \langle BCx, BCx \rangle = 0 \). Since \( A \perp B \), \( \langle ACx, BCx \rangle = 0 \). So \( \langle BCx, BCx \rangle = 0 \) and \( BCx = 0 \). Thus \( \text{ran}(C) \subseteq \text{null}(B) \). Since \( \text{null}(B) \) is closed, we have \( \overline{\text{ran}}(C) \subseteq \text{null}(B) \). It follows from Lemma 7 that \( C \perp B \). Similarly, we have \( C \perp A \). Next,
we show that \((B + C) \perp A\). For each \(x \in \mathcal{D}(B + C) = \mathcal{D}(B) \cap \mathcal{D}(C)\), the fact
that \(A \perp B\) and \(A \perp C\) implies that \(Bx \in \text{null}(A)\) and \(Cx \in \text{null}(A)\). Thus
\((B + C)x \in \text{null}(A)\). Then \(\text{ran}(B + C) \subseteq \text{null}(A)\). By Lemma 7 again, we obtain
\((B + C) \perp A\), that is, \((B \oplus C) \perp A\). It is obvious that \((A \oplus B) \oplus C = A \oplus (B \oplus C)\).

(OA4). Let \(A \perp A\). Then \(\text{ran}(A)\) is orthogonal to \(\text{ran}(A)\). For each \(x \in \mathcal{D}(A)\),
\(\langle Ax, Ax \rangle = 0\) and \(Ax = 0\). Since \(\mathcal{D}(A)\) is dense in \(\mathcal{H}\), \(A = 0\).

(GOA1). Let \(A \oplus B = A \oplus C\). We have proved that \(\bigcup F_n^A F_n^B(\mathcal{H}), \bigcup F_n^A F_n^C(\mathcal{H})\)
are the common cores for \(A, B\) and \(A, C\), respectively. Obviously, \(\{F_n^A F_n^B F_n^C\}\) has
strong operator limit \(I\), the identity operator, and \(\bigcup_{n=1}^{\infty} F_n^A F_n^B F_n^C(\mathcal{H})\) is dense in \(\mathcal{H}\).
It follows that \(\bigcup_{n=1}^{\infty} F_n^A F_n^B F_n^C(\mathcal{H})\) is a common core for \(A, B\), and \(C\). Since \(A \oplus B = A \oplus C\), we have \((A + B)|_{\bigcup F_n^A F_n^B F_n^C(\mathcal{H})} = (A + C)|_{\bigcup F_n^A F_n^B F_n^C(\mathcal{H})}\). Thus \(B|_{\bigcup F_n^A F_n^B F_n^C(\mathcal{H})} = C|_{\bigcup F_n^A F_n^B F_n^C(\mathcal{H})}\) and \(B = C\) (since \(B = C\) on their common core).

(GOA3). Suppose \(A \perp B\) and \(A \perp B = 0\). Then \(\text{ran}(A)\) is orthogonal to \(\text{ran}(B)\)
and \(A + B = 0\). For each \(x \in \mathcal{D}(A) \cap \mathcal{D}(B)\), \(Ax + Bx = 0\), \(\langle Ax + Bx, Ax \rangle = 0\). Since
\(\langle Ax, Bx \rangle = 0\), \(\langle Ax, Ax \rangle = 0\) and \(Ax = 0\). Then \(Ax = 0\) on \(\mathcal{D}(A) \cap \mathcal{D}(B)\) (dense in
\(\mathcal{H}\)). Thus, \(A = 0\). Similarly, We have \(B = 0\).

Therefore, \((\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)\) is a generalized orhtoalgebra. ■

5 The order properties of \((\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)\)

In this section, we study order properties of \((\mathcal{S}(\mathcal{H}), \perp, \oplus, 0)\). For \(A, B \in \mathcal{S}(\mathcal{H})\), we
define \(A \preceq B\) if there exists a \(C \in \mathcal{S}(\mathcal{H})\) such that \(A \perp C\) and \(A \oplus C = B\). It is
clear \(0 \preceq A\) for each \(A \in \mathcal{S}(\mathcal{H})\).

**Proposition 3.** Let \(A, B \in \mathcal{S}(\mathcal{H})\). Then \(A \preceq B\) if and only if \(A^2 = BA\).

**Proof.** Suppose \(A \preceq B\). There is a \(C \in \mathcal{S}(\mathcal{H})\) such that \(A \perp C\) and \(A \oplus C = B\),
namely, \(A + C = B\). Let \(x \in \mathcal{D}(A^2)\), which implies \(x \in \mathcal{D}(A)\) and \(Ax \in \mathcal{D}(A)\).
Since \(A \perp C\), \(\text{ran}(A) \subseteq \text{null}(C)\). So \(Ax \in \mathcal{D}(C)\). Then \(Ax \in \mathcal{D}(A) \cap \mathcal{D}(C) = \mathcal{D}(B)\)
and \(B Ax = A^2 x + C Ax\). Since \(C Ax = 0\), \(B Ax = A^2 x\) for each \(x \in \mathcal{D}(A^2)\). Thus
\(A^2 \subseteq BA\). Now, suppose \(x \in \mathcal{D}(BA)\). Then \(x \in \mathcal{D}(A)\) and \(Ax \in \mathcal{D}(B)\). Since
\(A \perp C\), \(\bigcup_{n=1}^{\infty} F_n^A F_n^C(\mathcal{H})\) is a common core for \(A\) and \(C\) and so it is also a core for
\[ B(= A + C). \] Since \( BF_n^A F_n^C Ax = AF_n^A F_n^C Ax + CF_n^A F_n^C Ax \) and \( BF_n^A F_n^C Ax \to B Ax \), \( CF_n^A F_n^C Ax \to CAx = 0 \), we have \( AF_n^A F_n^C Ax \to B Ax \). Since \( F_n^A F_n^C Ax \to Ax \) and \( A \) is closed, we obtain \( Ax \in \mathcal{D}(A) \) and \( A^2 x = B Ax \). It follows that \( BA \subseteq A^2 \). Hence \( A^2 = BA \).

Conversely, suppose \( A^2 = BA \). By Proposition 1, \( \mathcal{D}(B) \subseteq \mathcal{D}(A) \) and \( \mathcal{D}(B - A) \) is sense in \( \mathcal{H} \). Since \( B - A = B^* - A^* \subseteq (B - A)^* \), \( B - A \) is symmetric which implies that \( B - A \) is closable. Define \( C = \overline{B - A} \). From Proposition 1, \( \bigcup_{n=1}^{\infty} F_n^B(\mathcal{H}) \) is a common core for \( A \) and \( B \). So \( \bigcup_{n=1}^{\infty} F_n^B(\mathcal{H}) \) is a core for \( C \), and \( F_n^B C \subseteq CF_n^B \). Then \( F_n^B C^* \subseteq C^* F_n^B \), and \( \bigcup_{n=1}^{\infty} F_n^B(\mathcal{H}) \) is a core for \( C^* = (B - A)^* \). Since \( C = \overline{B - A} \subseteq (B - A)^* = C^* \), we have \( C = C^* (Cx = C^* x \text{ for each } x \text{ in the common core } \bigcup_{n=1}^{\infty} F_n^B(\mathcal{H}) \text{ for } C \text{ and } C^*) \) and \( C \) is self-adjoint. For each \( x \in \mathcal{D}(A) \), \( F_n^B x \to x \), \( AF_n^B x = F_n^B Ax \to Ax \), and \( A^2 F_n^B x = BAF_n^B x \). It follows that \( CAF_n^B x = BAF_n^B x - A^2 F_n^B x = 0 \to 0 \). Since \( C \) is closed, \( Ax \in \mathcal{D}(C) \) and \( CAx = 0 \). Thus \( \text{ran}(A) \subseteq \text{null}(C) \) and \( \text{ran}(A) = \text{null}(C) \). Then \( C \perp A \) and \( A \oplus C = A + C \). Since \( B \subseteq A + (B - A) = A + C \) and \( A + C \) is self-adjoint (Proposition 2), from the fact that a self-adjoint operator is maximal symmetric, we have \( B = A + (B - A) = A + C \). By definition, \( A \leq B \).

Note that from Proposition 1 and Proposition 3, \( A \leq B \) implies \( \mathcal{D}(B) \subseteq \mathcal{D}(A) \).

**Proposition 4.** For each \( A \in S(\mathcal{H}) \), if \( B, C \in S(\mathcal{H}) \) with \( B, C \preceq A \) and \( B \perp C \), then \( B \oplus C \preceq A \). In this case, we say that \( A \) is *principal*.

**Proof.** Suppose \( B, C \in S(\mathcal{H}) \) with \( B, C \preceq A \) and \( B \perp C \). It follows from Proposition 3 that \( B^2 = AB \) and \( C^2 = AC \). From Proposition 1, \( \bigcup_{n=1}^{\infty} F_n^A(\mathcal{H}) \) is a common core for \( A, B \) and \( C \), and therefore \( A - (B + C) \) is densely defined. Since \( A - (B + C) \subseteq (A - (B + C))^* \), \( A - (B + C) \) is closable. Define \( H = \overline{A - (B + C)} \). Just as in the proof of Proposition 3, one can prove that \( H \) is self-adjoint. For each \( x \in \mathcal{D}(B + C) \), \( F_n^A x \to x \) and \( (B + C) F_n^A x = F_n^A(B + C) x \to (B + C) x \). Since \( B \perp C \), it follows that \( H(B + C) F_n^A x = (A - (B + C))(B + C) F_n^A x = A(B + C) F_n^A x - (B^2 + C^2) F_n^A x = 0 \). Since \( H \) is closed, \( (B + C) x \in \mathcal{D}(H) \) and \( H(B + C) x = 0 \). From Lemma 7, \( H \perp (B + C) \) and \( H \oplus (B + C) = \overline{A - (B + C) + (B + C)} = A \). Hence,
\[ B \oplus C \preceq A \text{ and } A \text{ is principle.} \]

Recall the canonical ordering on \( S(\mathcal{H}) \). We say that \( A \preceq B \) if \( \mathcal{D}(B) \subseteq \mathcal{D}(A) \) and \( \langle Ax, x \rangle \leq \langle Bx, x \rangle \) for each \( x \in \mathcal{D}(B) \). Regarding \( \preceq \) and the newly defined ordering \( \preceq \), we have the following results:

**Proposition 5.** If \( A \preceq B \) and \( B \geq 0 \), then \( A \leq B \).

**Proof.** Suppose \( A \preceq B \). Then \( \mathcal{D}(B) \subseteq \mathcal{D}(A) \) and there exists a \( C \in S(\mathcal{H}) \) such that \( A \perp C \) and \( A \oplus C = B \). For \( x \in \mathcal{D}(B) \), \( x = y + z \) where \( y \in \overline{\mathcal{M}}(A) \) and \( z \in \text{null}(A) \). Then \( x, z \in \mathcal{D}(A) \) implies \( y \in \mathcal{D}(A) \), and \( \overline{\mathcal{M}}(A) \subseteq \text{null}(C) \) implies \( y \in \mathcal{D}(C) \). Hence \( y \in \mathcal{D}(A) \cap \mathcal{D}(C) \). If follows that \( y \in \mathcal{D}(B) \) and \( z \in \mathcal{D}(B) \). Then
\[
\langle (B - A)x, x \rangle = \langle (B - A)(y + z), y + z \rangle = \langle (B - A)x, y \rangle + \langle (B - A)x, z \rangle = \langle z, (B - A)(y + z) \rangle = \langle z, Bz \rangle \geq 0.
\]
So \( \langle Ax, x \rangle \leq \langle Bx, x \rangle \) for each \( x \in \mathcal{D}(B) \). Hence, \( A \leq B \).

Let \( B(\mathbb{R}) \) be the set of all Borel subsets of \( \mathbb{R} \). We now characterize the ordering \( \preceq \) on \( S(\mathcal{H}) \) in terms of the spectral measure of self-adjoint operators.

**Lemma 10.** ([10]) For \( A, B \in S_b(\mathcal{H}) \), \( A \preceq B \) (that is, there exists a \( C \in S_b(\mathcal{H}) \) such that \( A \perp C \) and \( A \oplus C = B \)) if and only if \( P^A(\Delta) \leq P^B(\Delta) \) for every \( \Delta \in B(\mathbb{R}) \) with \( 0 \notin \Delta \).

**Theorem 3.** Let \( A, B \in S(\mathcal{H}) \). Then \( A \preceq B \) if and only if \( E_{\Delta \lambda_j}^A \leq E_{\Delta \lambda_j}^B \), where \( E_{\Delta \lambda_j}^A = E_{\lambda_j}^A - E_{\lambda_{j-1}}^A \), \( 0 \notin (\lambda_{j-1}, \lambda_j] \), \( j = 1, 2, 3 \cdots \), and \( \{E_\lambda^A\} \) is the resolution of the identity for \( A \).

**Proof.** Suppose \( A \preceq B \). Then there exists a \( C \in S(\mathcal{H}) \) such that \( A \perp C \) and \( A \oplus C = B \), and \( \bigcup_{n=1}^\infty F^A_n F^C_n(\mathcal{H}) \) is a common core for \( A \) and \( C \) (Theorem 1), and therefore a core for \( B \). From Proposition 3, we have \( A^2 = BA \). Then \( (AF^A_n F^C_n)^2 = (BF^A_n F^C_n)(AF^A_n F^C_n) \) for each \( n \in \mathbb{N} \). Again, by Proposition 3, \( AF^A_n F^C_n \preceq BF^A_n F^C_n \) for each \( n \in \mathbb{N} \). Since \( \{E_\lambda^A F^A_n F^C_n\} \) and \( \{E_\lambda B F^A_n F^C_n\} \) are the resolutions of the identity for \( AF^A_n F^C_n|_{F^A_n F^C_n(\mathcal{H})} \) and \( BF^A_n F^C_n|_{F^A_n F^C_n(\mathcal{H})} \), respectively. By Lemma 10, we have \( E_{\Delta \lambda_j}^A F^A_n F^C_n \leq E_{\Delta \lambda_j}^B F^A_n F^C_n \) for each \( n \in \mathbb{N} \) and \( 0 \notin (\lambda_{j-1}, \lambda_j] \). Since \( \{F^A_n F^C_n\} \) has strong operator limit \( I \), it follows that \( E_{\Delta \lambda_j}^A \leq E_{\Delta \lambda_j}^B \), \( 0 \notin (\lambda_{j-1}, \lambda_j] \).

Suppose that \( E_{\Delta \lambda_j}^A \leq E_{\Delta \lambda_j}^B \), \( 0 \notin (\lambda_{j-1}, \lambda_j] \). For each \( n \in \mathbb{N} \), \( \{E_\lambda^A F^A_n\} \)
and \( \{E^B_n\} \) are the resolutions of the identity for \( AF^A_n|_{F^A_n(\mathcal{H})} \) and \( BF^B_n|_{F^B_n(\mathcal{H})} \), respectively. For a fixed \( n \in \mathbb{N} \), for each \( (\lambda_{j-1}, \lambda_j) \) not containing 0, either \( E^A_{\Delta \lambda_j}F^A_n = 0 \) and \( E^B_{\Delta \lambda_j}F^B_n = 0 \), or \( E^A_{\Delta \lambda_j}F^A_n \leq E^B_{\Delta \lambda_j}F^B_n \). Then \( AF^A_n \preceq BF^B_n \) and \( (AF^A_n)^2 = (BF^B_n)(AF^A_n) \). For each \( x \in D(A^2), x \in D(A) \) and \( Ax \in D(A) \). As \( F^B_nF^A_nAx \to Ax \), we have \( BF^B_nF^A_nAx = BF^B_nAF^A_nx = (AF^A_n)^2x = F^A_nA^2x \to A^2x \).

Since \( B \) is closable, \( Ax \in D(B) \) and \( BAx = A^2x \). So \( A^2 \subseteq BA \). Conversely, for each \( x \in D(BA), x \in D(A) \) and \( Ax \in D(A) \). As \( (AF^A_n)^2 \) is self-adjoint, we have \( (BF^B_n)(AF^A_n) = (AF^A_n)(BF^B_n) \). By Lemma 4, we have \( BF^B_nF^A_n = F^A_nBF^B_n \). Since \( F^A_nAx \to Ax \), we have \( AF^A_nAx = (AF^A_n)^2x = (BF^B_n)(AF^A_n)x = F^B_nBF^A_nAx = F^B_nF^A_nBAx \to BAx \). As \( A \) is closable, \( Ax \in D(A) \) and \( A^2x = BAx \). So \( BA \subseteq A^2 \).

Therefore, \( A^2 = BA \) which implies \( A \preceq B \).

**Corollary 1.** Let \( A, B \in S(\mathcal{H}) \). Then \( A \preceq B \) if and only if \( P^A(\Delta) \leq P^B(\Delta) \) for every \( \Delta \in \mathcal{B}(\mathbb{R}) \) with \( 0 \notin \Delta \).

Next, we study the existence of \( A \land B \) and \( A \lor B \) for \( A, B \in S(\mathcal{H}) \). For each \( \Delta \in \mathcal{B}(\mathbb{R}) \), if \( \Delta = \bigcup_{i=1}^n \Delta_i \), where \( \{\Delta_i\}_{i=1}^n \) are pairwise disjoint sets in \( \mathcal{B}(\mathbb{R}) \), then we say \( \gamma = \{\Delta_i\}_{i=1}^n \) is a partition of \( \Delta \). We denote all the partitions of \( \Delta \) by \( \Gamma(\Delta) \).

Let \( A, B \in S(\mathcal{H}) \). Define \( P : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H}) \) as follows. Let \( P(\emptyset) = 0 \), and for each nonempty \( \Delta \in \mathcal{B}(\mathbb{R}) \) and \( \gamma \in \Gamma(\Delta) \),

\[
P(\Delta) = \begin{cases} \land_{\gamma \in \Gamma(\Delta)} \sum_{\Delta_i \in \gamma} (P^A(\Delta_i) \land P^B(\Delta_i)), & 0 \notin \Delta \\ I - P(\mathbb{R} \setminus \Delta), & 0 \in \Delta \end{cases}
\]

**Lemma 11 ([18]).** As defined above, \( P : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H}) \) is a spectral measure.

**Theorem 4.** Let \( A, B \in S(\mathcal{H}) \). Then \( A \land B \) exists in \( S(\mathcal{H}) \) with respect to \( \preceq \).

**Proof.** Let \( \{E^A_\lambda\}, \{E^B_\lambda\} \) be the resolutions of identity for \( A \) and \( B \), \( P^A \) and \( P^B \)
be the spectral measures for \( A \) and \( B \), respectively. Define \( P(\Delta) \) as above for each Borel set \( \Delta \in \mathcal{B}(\mathbb{R}) \) and then \( P \) is a spectral measure. Define \( E_\lambda = P((\infty, \lambda]) \) and \( \{E_\lambda\} \) is a resolution of identity. By Lemma 9, there exists a self-adjoint operator \( C \) such that

\[
Cx = \int_{-\infty}^{\infty} \lambda dE_\lambda x,
\]

14
where \( x \in F_n(H) \), \( n \in \mathbb{N} \), \( F_n = E_n - E_{-n} \), and \( \{E_\lambda\} \) is the resolution of the identity for \( C \). Let \( \Delta \in \mathcal{B}(\mathbb{R}) \) with \( 0 \notin \Delta \). For each \( \gamma \in \Gamma(\Delta) \),

\[
P_\gamma = \sum_{\Delta_i \in \gamma} (P^A(\Delta_i) \land P^B(\Delta_i)) 
\leq (\sum_{\Delta_i \in \gamma} P^A(\Delta_i)) \land (\sum_{\Delta_i \in \gamma} P^B(\Delta_i)) 
= P^A(\Delta) \land P^B(\Delta).
\]

Then \( P(\Delta) = \land_{\gamma \in \Gamma(\Delta)} P_\gamma \leq P^A(\Delta) \land P^B(\Delta) \). From Corollary 1, \( C \preceq A \) and \( C \preceq B \).

Suppose there exists another \( C_1 \in \mathcal{S}(H) \) such that \( C_1 \preceq A \) and \( C_1 \preceq B \). For each \( \Delta \in \mathcal{B}(\mathbb{R}) \) with \( 0 \notin \Delta \) and \( \gamma \in \Gamma(\Delta) \), since \( P^{C_1}(\Delta_i) \leq P^A(\Delta_i) \) and \( P^{C_1}(\Delta_i) \leq P^B(\Delta_i) \) for each Borel subsets \( \Delta_i \in \gamma \), we have

\[
P^{C_1}(\Delta) = \sum_{\Delta_i \in \gamma} P^{C_1}(\Delta_i) \leq \sum_{\Delta_i \in \gamma} P^A(\Delta_i) \land P^B(\Delta_i).
\]

So we obtain

\[
P^{C_1}(\Delta) \leq \land_{\gamma \in \Gamma(\Delta)} \sum_{\Delta_i \in \gamma} P^A(\Delta_i) \land P^B(\Delta_i) = P(\Delta).
\]

Therefore, \( C_1 \preceq C \) and \( C = A \land B \). \( \blacksquare \)

**Remark 1.** If \( \{A_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{S}(H) \), then \( A = \land_\alpha A_\alpha \) exists in \( \mathcal{S}(H) \). In fact, define \( P(\emptyset) = 0 \) and for each nonempty \( \Delta \in \mathcal{B}(\mathbb{R}) \),

\[
P(\Delta) = \begin{cases} 
\land_{\gamma \in \Gamma(\Delta)} \sum_{\Delta_i \in \gamma} \left( \land_\alpha P^{A_\alpha}(\Delta_i) \right) & 0 \notin \Delta \\
I - P(\mathbb{R} \setminus \Delta) & 0 \in \Delta.
\end{cases}
\]

It can be proved that \( P : \mathcal{B}(\mathbb{R}) \to P(\mathcal{H}) \) is a spectral measure ([18]). Let \( E_\lambda = P((-\infty, \lambda]) \) and \( \{E_\lambda\} \) is a resolution of the identity. Then we have \( A = \land_\alpha A_\alpha \), where \( Ax = \int_{-n}^{n} \lambda dE_\lambda x \), for each \( x \in F_n(H) \) and \( n \in \mathbb{N} \).

With \( A, B \in \mathcal{S}(H) \), now we know that \( A \preceq B \) implies \( P^A(\Delta) \leq P^B(\Delta) \) for each \( \Delta \in \mathcal{B}(\mathbb{R}) \) with \( 0 \notin \Delta \). We have \( P^A(\Delta) = P^A(\Delta)P^B(\Delta) = P^B(\Delta)P^A(\Delta) \) and \( P^A(\Delta_1)P^A(\Delta_2) = 0 \) for \( \Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R}) \) with \( \Delta_1 \cap \Delta_2 = \emptyset \). The following result is straightforward.

\[15\]
Lemma 12. Let $A, B \in \mathcal{S}(\mathcal{H})$. Suppose that $H \in \mathcal{S}(\mathcal{H})$ is an upper bound of $A$ and $B$ with respect to $\preceq$. Then, for any $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$ with $\Delta_1 \cap \Delta_2 = \emptyset$ and $0 \notin \Delta_1 \cup \Delta_2$, we have

$$P^A(\Delta_1)P^B(\Delta_2) = P^A(\Delta_1)P^H(\Delta_1)P^H(\Delta_2)P^B(\Delta_2) = 0.$$ 

Lemma 13 ([19]). Let $A, B \in \mathcal{S}(\mathcal{H})$. Suppose that $P^A(\Delta_1)P^B(\Delta_2) = 0$ for each pair $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$ with $\Delta_1 \cap \Delta_2 = \emptyset$ and $0 \notin \Delta_1 \cup \Delta_2$. Then the following mapping $P : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H})$ defines a spectral measure:

$$P(\Delta) = \begin{cases} P^A(\Delta) \lor P^B(\Delta), & 0 \notin \Delta \\ P^A(\Delta \setminus \{0\}) \lor P^B(\Delta \setminus \{0\}) + N_A \land N_B, & 0 \in \Delta. \end{cases}$$

Theorem 5. Let $A, B \in \mathcal{S}(\mathcal{H})$. If there exists a $C \in \mathcal{S}(\mathcal{H})$ such that $A \preceq C$ and $B \preceq C$, then $A \lor B$ exists in $\mathcal{S}(\mathcal{H})$ with respect to $\preceq$.

Proof. Define $P$ as in Lemma 13. Then $P$ is a spectral measure and $\{E_\lambda\}$ is a resolution of the identity, where $E_\lambda = P((-\infty, \lambda])$. By Lemma 9, there exists a self-adjoint operator $C$ such that

$$Cx = \int_{-n}^n \lambda dE_\lambda x,$$

where $x \in F_n(\mathcal{H}), n \in \mathbb{N}, F_n = E_n - E_{-n}$, and $\{E_\lambda\}$ is the resolution of the identity for $C$. Clearly, $P^A(\Delta) \leq P(\Delta)$ and $P^B(\Delta) \leq P(\Delta)$ for each $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. It follows from Corollary 1, $A \preceq C$ and $B \preceq C$. If there exists another $C_1 \in \mathcal{S}(\mathcal{H})$ such that $A \preceq C_1$ and $B \preceq C_1$. Then $P^A(\Delta) \leq P^{C_1}(\Delta), P^B(\Delta) \leq P^{C_1}(\Delta)$ and $P^A(\Delta) \lor P^B(\Delta) \leq P^{C_1}(\Delta)$ for each $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. Then $P^C(\Delta) \leq P^{C_1}(\Delta)$ for each $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. Therefore, by Corollary 1, $C \preceq C_1$ and $C = A \lor B$. \hfill \blacksquare

Remark 2. Let $\{A_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{S}(\mathcal{H})$ and $A_\alpha \preceq H$ for each $\alpha \in \Lambda$. Then $A = \lor_{\alpha} A_\alpha$ exists in $\mathcal{S}(\mathcal{H})$. In fact, define

$$P(\Delta) = \begin{cases} \lor_{\alpha} P^A_\alpha(\Delta), & 0 \notin \Delta \\ \lor_{\alpha} P^A_\alpha(\Delta \setminus \{0\}) \lor N_\alpha \land N_{A_\alpha}, & 0 \in \Delta. \end{cases}$$
It can be proved that $P : \mathcal{B}(\mathcal{R}) \to \mathcal{P}(\mathcal{H})$ defines a spectral measure. Then $\{E_\lambda\}$, where $E_\lambda = P((\infty, \lambda])$, is a resolution of the identity. There exists a self-adjoint operator $A$ such that $Ax = \int_{-\infty}^{\lambda} \lambda dE_\lambda x$ for each $x \in F_n(\mathcal{H})$ and $n \in \mathbb{N}$. Then $\bigvee_{\alpha} A_{\alpha} = A$.

**Theorem 6.** Let $\mathcal{H} = L^2(-\infty, +\infty)$. Then $Q \land P = 0$ with respect to the order $\preceq$, where $Q$ and $P$ are the position operator $Q$ and momentum operator $P$ satisfying the Heisenberg’s commutation relation $QP - PQ = -i\hbar I$.

**Proof.** Suppose that there exists an $A \in \mathcal{S}(\mathcal{H})$ such that $A \preceq P$ and $A \preceq Q$. By Proposition 3, $A^2 = PA$ and $A^2 = QA$. It follows that $A^3 = PA^2 = PQA A^2 = QA^2 = QPA$, and therefore $PQA = QPA$. Applying Heisenberg’s commutation relation $QP - PQ = -i\hbar I$, we have

$$QPA - PQA = (QP - PQ)A = -i\hbar IA.$$  

Since $\bigcup_{n=1}^{\infty} F_n^A(\mathcal{H}) \subseteq \mathcal{D}(A^3) = \mathcal{D}(PQA) = \mathcal{D}(QPA)$, $QPAX - PQAX = (QP - PQ)AX = -i\hbar IAX$ for each $x \in \bigcup_{n=1}^{\infty} F_n^A(\mathcal{H})$. So $Ax = 0$ for each $x \in \bigcup_{n=1}^{\infty} F_n^A(\mathcal{H})$, which implies that $A = 0$. Therefore, we have $Q \land P = 0$.

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