Weyl Pair, Current Algebra and Shift Operator

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Abstract

The Abelian current algebra on the lattice is given from a series of the independent Weyl pairs and the shift operator is constructed by this algebra. So the realization of the operators of the braid group is obtained. For $|q| \neq 1$ the shift operator is the product of the theta functions of the generators $w_n$ of the current algebra. For $|q| = 1$ it can be expressed by the quantum dilogarithm of $w_n$.

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1 Introduction

Owing to the discrete space-time picture much benefit to the lattice sine-Gordon model ones try to discretize the conformal field theory where the lattice Virasoro algebra plays an important role [1, 2, 3]. One of the approaches is given by Faddeev and Volkov [2]. They start with the periodic free field (Abelian current) which can be gotten from the continuous limit of the following current algebra:

\[ w_{n-1}w_n = q^2 w_n w_{n-1}, \quad n = 2, 3, \cdots, 2N, \]

\[ w_{2N}w_1 = q^2 w_1 w_{2N}, \]

\[ w_m w_n = w_n w_m, \quad 1 < |m - n| < 2N - 1, \quad (1) \]

for the operators \( w_n (n = 1, 2, \cdots, 2N) \) where \( q \) is a complex constant. The key step in this approach is to construct the shift operator and Ref. [2] has discussed it with \( |q| < 1 \). What is about the case of \( |q| \geq 1 \) is just the aim of this letter.

In section 2 the \( w \) algebra mentioned above is denoted by a set of independent Weyl pairs and the shift operator is given for \( |q| < 1 \). The different initial values of the products in the shift operator mean some factors times \( w_n \). The case for \( |q| > 1 \) is discussed in section 3. We construct the shift operator by the inverse of it. In the above two cases the shift operator can be always denoted in the form of the theta functions [4]. In section 4 we constructed the shift operator by using the quantum dilogarithm [5, 6] which appeared in two and three-dimensional lattice models [7, 8, 9].

2 Weyl Pair and Shift Operator with \(|q| < 1\)

From \( N - 1 \) independent Weyl pairs \( x_i, y_i (i = 1, 2, \cdots, N - 1) \) with the relations

\[ x_i y_j = q^{2\delta_{ij}} y_j x_i, \quad x_i x_j = x_j x_i, \]

\[ y_i y_j = y_j y_i, \quad 1 \leq i, j \leq N - 1, \quad (2) \]

where \( \delta_{ij} \) is the Kronecker symbol and \( q \) is a complex constant, we can get the current algebra (1) by setting

\[ w_{2i+1} = x_{i+1}^{-1} x_i, \quad i = 0, 1, 2, \cdots, N - 1, \]

\[ w_{2j} = y_j, \quad j = 1, 2, \cdots, N - 1, \]

\[ w_{2N} = \prod_{l=1}^{N-1} y_l^{-1}. \quad (3) \]

Let

\[ C_1 = \prod w_1 w_3 \cdots w_{2N-1}, \quad C_2 = \prod w_2 w_4 \cdots w_{2N}. \quad (4) \]
They are the two central elements of the \(w\) algebra. If \(C_1 \neq C_2\), by making the transformations of
\[
 w'_1 = C_2 w_1, \quad w'_{2N} = C_1 w_{2N}, \quad w'_i = w_i, \quad 1 < i < 2N,
\]
we can get the \(w'\) algebra with the equal central elements \(C'_1 = C'_2 = C_1 C_2\). So we always assume that the two central elements of the \(w\) algebra are equal, \(i.e.C_1 = C_2\).

Now we construct the shift operator with \(|q| < 1\). Letting
\[
 h_n = \psi_a(w_n, \alpha) \psi_b(w_n^{-1}, \beta), \quad n = 1, 2, \cdots, 2N,
\]
where
\[
 \psi_a(w_n, \alpha) = \prod_{j=a}^{\infty} (1 + w_n \alpha q^{2j-1}), \quad \psi_b(w_n^{-1}, \beta) = \prod_{j=b}^{\infty} (1 + w_n^{-1} \beta q^{2j-1}),
\]
we get that
\[
 w_n h_{n-1} h_n = h_{n-1} h_n w_{n-1}, \quad n = 2, 3, \cdots, N,
\]
under the restrict condition of
\[
 \alpha \beta q^{2(a+b-2)} = 1
\]
for the integers \(a, b\) and the parameters \(\alpha, \beta\). Then by setting
\[
 U = h_1 h_2 \cdots h_{2N-1}
\]
we have that
\[
 w_n U = U w_{n-1}
\]
for \(n = 1, 2, \cdots, 2N\) with \(w_0 = w_{2N}\) where \(C_1 = C_2\) has been used. So \(U\) is a shift operator. By considering the relation (4) we have
\[
 h_n = G^{-1} \vartheta_3(w_n \alpha q^{2(a-1)})
\]
where [4]
\[
 \vartheta_3(v) = G \prod_{n=1}^{\infty} (1 + q^{2n-1} v) \prod_{n=1}^{\infty} (1 + q^{2n-1} v^{-1}), \quad G = \prod_{n=1}^{\infty} (1 - q^{2n}).
\]
From Eq. (8) we have
\[
 h_n h_{n-1} h_n = h_{n-1} h_n h_{n-1}, \quad n = 1, 2, \cdots, 2N
\]
where \(h_0 = h_{2N}\). This gives the representation of the operators of the braid group. By making the notation of
\[
 \psi(A) = \prod_{j=1}^{\infty} (1 + A q^{2j-1}),
\]
\( \psi_a(x, \alpha) \) and \( \psi_b(y, \beta) \) have the property
\[
\psi_b(y, \beta) \psi_a(x, \alpha) = \psi(x \alpha q^{2(a-1)} + y \beta q^{2(b-1)} + qyx)
\] (16)
for the Weyl pair \( x, y \) with the relation \( xy = q^2yx \). Then the shift operator can be expressed as
\[
U = \psi(w_n^{-1} \beta q^{2(b-1)} + w_2 \alpha q^{2(a-1)} + qw_1^{-1} w_2) \psi(w_2^{-1} \beta q^{2(b-1)} + w_3 \alpha q^{2(a-1)} + qw_2^{-1} w_3)
\]
\[
\cdots \psi(w_{2N-1}^{-1} \beta q^{2(b-1)} + w_{2N} \alpha q^{2(a-1)} + qw_{2N-1}^{-1} w_{2N})
\]
\[
= \psi(w_1 \alpha q^{2(a-1)} + w_2^{-1} \beta q^{2(b-1)} + qw_1 w_2^{-1}) \psi(w_2 \alpha q^{2(a-1)} + w_3^{-1} \beta q^{2(b-1)} + qw_2 w_3^{-1})
\]
\[
\cdots \psi(w_{2N-1} \alpha q^{2(a-1)} + w_{2N}^{-1} \beta q^{2(b-1)} + qw_{2N-1} w_{2N}^{-1})
\] (17)
It is just the expression given by Faddeev and Volkov. This result means that the initial values \( a, b, \alpha \) and \( \beta \) with the relation (9) denote the factor \( \alpha q^{2(a-1)} \) times \( w_n \) in the shift operator. And the operator \( U \) constructed by Eq. (10) with the relation (6) is the product of the theta functions of the \( w_n \). Similarly as the Eq. (16), using the mathematical reduction we can prove further that
\[
\prod_{j=1}^{n} (\beta + yq^{2j-1}) \prod_{j=1}^{n} (\alpha + xq^{2j-1}) = \prod_{j=1}^{n} (\alpha \beta + Y q^{2j-1}), \quad n \geq 1,
\] (18)
for the arbitrary \( \alpha, \beta \) and \( q \) where \( Y = \beta x + \alpha y + qyx \).

### 3 The Shift Operator with \( |q| > 1 \)

Setting
\[
h_n = \tilde{\psi}_a(w_n, \alpha) \tilde{\psi}_b(w_n^{-1}, \beta)
\] (19)
where
\[
\tilde{\psi}_a(w_n, \alpha) = \prod_{j=a}^{\infty} (1 + w_n \alpha \bar{q}^{2j-1}), \quad \tilde{\psi}_b(w_n^{-1}, \beta) = \prod_{j=b}^{\infty} (1 + w_n^{-1} \beta \bar{q}^{2j-1})
\] (20)
with the notation \( \bar{q} = 1/q \), from Eq. (1), we have
\[
w_n h_{n+1} h_n = h_{n+1} h_n w_{n+1}, \quad n = 1, 2, \cdots, 2N - 1,
\] (21)
when the integers \( a, b \) and the parameters \( \alpha, \beta \) satisfy the relation: \( \alpha \beta q^{2(a+b-2)} = 1 \). In this way, \( h_n \) is given also by the theta function
\[
h_n = G^{-1} \partial_3(w_n \alpha q^{2(a-1)})
\] (22)
where \( \vartheta_3 \) is defined by relation (13) with the substitution of \( \bar{q} \) for \( q \). And the representation of the operators of the braid group is also given. Set

\[
U^{-1} = h_{2N} h_{2N-1} \cdots h_2
\]  

(23)

By considering that the two central elements \( C_1, C_2 \), of the current algebra (1), are equal, from Eq. (21), we have

\[
w_n U^{-1} = U^{-1} w_{n+1}, \quad n = 1, 2, \ldots, 2N
\]  

(24)

with \( w_{2N+1} \equiv w_1 \). So operator \( U \) expressed by Eq. (23) is indeed the shift operator for \( |q| > 1 \). Furthermore, it can be expressed as

\[
U^{-1} = \tilde{\psi}(w_{2N}^{-1} \beta q^{2(1-b)} + w_{2N-1} \alpha q^{2(1-a)} + qw_{2N-1} w_{2N}^{-1})
\]

\[
\cdots \tilde{\psi}(w_{2N}^{-1} \beta q^{2(1-b)} + w_{2N-1} \alpha q^{2(1-a)} + qw_{2N-1} w_{2N}^{-1})
\]

\[
\cdots \tilde{\psi}(w_{2N} \alpha q^{2(1-a)} + w_{2N-1} \beta q^{2(1-b)} + qw_{2N-1} w_{2N})
\]

\[
\cdots \tilde{\psi}(w_{2N} \alpha q^{2(1-a)} + w_{2N-1} \beta q^{2(1-b)} + qw_{2N-1} w_{2N-1})
\]

\[
\cdots \tilde{\psi}(w_{2N}^{-1} \beta q^{2(1-b)} + w_{2N-1} \alpha q^{2(1-a)} + qw_{2N-1} w_{2N}^{-1})
\]

\[
\cdots \tilde{\psi}(w_{2N} \alpha q^{2(1-a)} + w_{2N-1} \beta q^{2(1-b)} + qw_{2N-1} w_{2N})
\]

\[
\cdots \tilde{\psi}(w_{2N}^{-1} \beta q^{2(1-b)} + w_{2N-1} \alpha q^{2(1-a)} + qw_{2N-1} w_{2N}^{-1})
\]

\[
\cdots \tilde{\psi}(w_{2N} \alpha q^{2(1-a)} + w_{2N-1} \beta q^{2(1-b)} + qw_{2N-1} w_{2N})
\]

(25)

where \( \tilde{\psi} \) is defined as \( \tilde{\psi}(A) = \prod_{j=1}^{\infty} (1 + A q^{1-2j}) \) for operator \( A \). Then we get the shift operator with \( |q| > 1 \) by considering the inverse of it. It should be noted that the above relation can be obtained also by defining that \( U^{-1} = h_{2N-1} h_{2N-2} \cdots h_1 \).

4 The Shift Operator with \( |q| = 1 \)

Set

\[
q^2 = \omega = \exp(2\pi i/L), \quad \omega^{1/2} = \exp(\pi i/L).
\]  

(26)

From Eq. (18) we get that

\[
(x + y)^N = x^N + y^N, \quad xy = \omega yx,
\]  

(27)

by considering the relation \([10, 11]\)

\[
\prod_{j=1}^{L} (1 - x \omega^j) = 1 - x^N.
\]  

(28)

Now we fix the \( L \)-th powers of the generators \( w_n \) of the current algebra (1) are the identity operator:

\[
w_n^L = 1, \quad n = 1, 2, \ldots, 2N
\]  

(29)
since they are the central elements of the w algebra in this case. Introducing the operators

\[ W_n^{(1)} = k(w_n^{-1} + w_{n+1} - \omega^{1/2}w_n^{-1}w_{n+1}), \quad W_n^{(2)} = k(w_n + w_{n+1}^{-1} - \omega^{1/2}w_n w_{n+1}^{-1}), \]

where \( n = 1, 2, \ldots, 2N - 1 \) and \( k^L = 1/3 \) we have that

\[ W_n^{(1) L} = W_n^{(2) L} = 1 \]

by taking account of the relations (27) and (29). The spectrum of any operator, A, whose \( L \)-th power is the identity operator, is given by \( L \) distinct numbers

\[ \omega^l, \quad l = 0, 1, \ldots, L - 1. \]

Then all of the operators \( w_n, W_n^{(1)}, W_n^{(2)}, (n = 1, 2, \ldots, 2N - 1) \) and \( w_{2N} \) have one and the same spectrum (32). Define \( w(a,b,c|l) = \prod_{j=1}^{l} b/(c - a\omega^j), \quad a^L + b^L = c^L, \quad n \geq 0, \)

with \( w(a,b,c|0) = 1 \). The quantum dilogarithm \( \Psi(A) \) depending on the operator \( A \) which has the spectrum (32) can be defined as an operator commuting with \( A \) and has the spectrum

\[ \Psi(\omega^l) = \Psi(1)w(a,b,c|l) \]

where \( \Psi(1) \) is a non-zero complex factor. So the ”functional” relation of the quantum dilogarithm \( \Psi(A) \) has the following form:

\[ \Psi(\omega^{-1}A)\Psi(A)^{-1} = (c - aA)/b. \]

It determines the operator \( \Psi(A) \) up to a complex factor. By setting

\[ c^2 = \omega a^2 \]

it can be proved easily that the operator \( U \) defined by Eq. (10) with

\[ h_n = \Psi(w_n)\Psi(w_n^{-1}), \quad n = 1, 2, \ldots, 2N - 1, \]

is the shift operator for \( |q| = 1 \). And \( h_n \) gives also the representation of the braid group. If we make the limit of:

\[ L \to \infty, \quad b = 2^{1/L} \to 1, \quad k = 3^{-1/L} \to 1, \]

the ”functional” relation of the quantum dilogarithm \( \Psi(A) \) can be written as

\[ \Psi(\omega^{-1}A)\Psi(A) = 1 - \omega^{-1/2}A \]
where

\[ A = w_n, \quad w_n^{-1} + w_{n+1} - \omega^{1/2} w_n^{-1} w_{n+1}, \quad w_n + w_{n+1}^{-1} - \omega^{1/2} w_n w_{n+1}, \quad w_{2N}, \]

with \( n = 1, 2, \ldots, 2N - 1 \). Then in this limiting the shift operator \( U \) for \( |q| = 1 \) can be expressed also in the form \( (17) \) with substituting \( \psi(w_{n-1}^{-1} \beta q^{2(b-1)} + w_{n+1} \alpha q^{2(a-1)} + qw_n^{-1} w_{n+1}) \) and \( \psi(w_n \alpha q^{2(a-1)} + w_{n+1}^{-1} \beta q^{2(b-1)} + qw_n^{-1} w_{n+1}) \) by \( \Psi(w_n^{-1} + w_{n+1} + \omega^{1/2} w_n w_{n+1}) \) and \( \Psi(w_n + w_{n+1}^{-1} + \omega^{1/2} w_n w_{n+1}) \), \( n = 1, 2, \ldots, 2N - 1 \), respectively.

As the conclusion, we get the shift operator with \( q \) on a complex plane from the current algebra (1) which can be constructed from a series of independent Weyl pairs and can be reduced to the periodic free field in a continues limit. So the representation of the operators of the braid group is obtained. The integers \( a, b \) and the parameters \( \alpha, \beta \) in \( h_n \) mean some factors appearing for \( w_n \) when it denotes the shift operator which is the product of the theta functions of the operators \( w_n \) with \(|q| \neq 1\). When \(|q| > 1\) we get the shift operator by considering the inverse of it. The shift operator with \(|q| = 1\) is expressed by the quantum dilogarithm \( 6 \) which appeared also in Baxter-Bazhanov model \( 5, 7 \). So it is an interesting subject to discuss the lattice Virasoro algebra and construct the shift operator from the respect of the chiral Potts model \( 9 \).

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