THE POSITIVE ENERGY CONJECTURE FOR A CLASS OF AHM METRICS ON $\mathbb{R}^2 \times \mathbb{T}^{n-2}$

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ABSTRACT. We prove the positive energy conjecture for a class of asymptotically Horowitz-Myers (AHM) metrics on $\mathbb{R}^2 \times \mathbb{T}^{n-2}$. This generalizes the previous results of Barzegar-Chruściel-Hörzinger-Maliborski-Nguyen [1] as well as the authors [5].

1. Introduction

In [4], Horowitz-Myers constructed so-called AdS solitons with toroidal topology

$$-r^2 dt^2 + \frac{1}{r^2 (1 - \frac{\tilde{r}^n}{r^n})} dr^2 + r^2 \left(1 - \frac{\tilde{r}^n}{r^n}\right) d\xi^2 + r^2 n \sum_{i=3}^n (d\phi^i)^2$$

which are globally static vacuum $(n+1)$-dimensional spacetime with cosmological constant

$$\Lambda = -\frac{n(n-1)}{2\ell^2}.$$ 

For convenience, we assume $\ell = 1$ throughout the paper. These metrics are geodesically complete when

$$r \in [\tilde{r}_0, \infty), \quad \xi \in [0, \beta_0], \quad \phi^i \in [0, \lambda_i]$$

for constants $\tilde{r}_0 > 0, \lambda_i > 0, i = 3, \ldots, n$ and

$$\beta_0 = \frac{4\pi}{n\tilde{r}_0}.$$ (1.1)

The induced Riemannian metrics $g_{\text{HM}}$ on the constant time slices

$$g_{\text{HM}} = \frac{1}{r^2 (1 - \frac{\tilde{r}^n}{r^n})} dr^2 + r^2 \left(1 - \frac{\tilde{r}^n}{r^n}\right) d\xi^2 + r^2 n \sum_{i=3}^n (d\phi^i)^2$$ (1.2)

are asymptotically locally hyperbolic (ALH) and are referred as Horowitz-Myers metrics. Horowitz-Myers also verified that the Hawking-Horowitz mass of $g_{\text{HM}}$ is negative and conjectured that, among all metrics which are asymptotic to $g_{\text{HM}}$ with the same period $\beta_0$ for $\xi$ and with the scalar curvature

$$R \geq -n(n-1),$$ (1.3)

g_{\text{HM}}$ is the unique metric with the least Hawking-Horowitz mass [4].
Throughout the paper, we misuse the notation and refer a function $f$ type $O(\frac{1}{r^m})$ if for all $k \geq 0$, its $k$th partial derivative with respect to $r$ satisfies
\[ \partial^k f \sim \frac{1}{r^{m+k}}, \quad r \to \infty. \] (1.4)
Furthermore, a tensor is referred as type $O(\frac{1}{r^m})$ if its components belong to $O(\frac{1}{r^m})$ in coordinates $\{r, \xi, \phi^i\}$.

As both Schoen-Yau’s method and Witten’s method are hard to apply in this situation, there is less progress towards proof of the conjecture. In 2020, Barzegar-Chruściel-Hörzinger-Maliborski-Nguyen studied the following asymptotically Horowitz-Myers metrics on $\mathbb{R}^2 \times T^{n-2}$
\[ g_1 = e^{2u(r)} dr^2 + e^{2v(r)} d\xi^2 + e^{2w(r)} \sum_{i=3}^{n} (d\phi^i)^2, \] (1.5)
where $u, v, w$ are functions of $r$ which satisfy
\[ u(r) = -\ln r - \frac{1}{2} \ln \left( 1 - \frac{r_0^n}{r} \right) + u_n + O\left( \frac{1}{r^{n+1}} \right), \]
\[ v(r) = \ln r + \frac{1}{2} \ln \left( 1 - \frac{r_0^n}{r} \right) + v_n + O\left( \frac{1}{r^{n+1}} \right), \] (1.6)
\[ w(r) = \ln r + \frac{w_n}{r^n} + O\left( \frac{1}{r^{n+1}} \right), \]
for constants $u_n, v_n$ and $w_n$. The metric $g_1$ is asymptotic to $g_{HM}$ up to order $O(\frac{1}{r^n})$. In [1], they verified Horowitz-Myers conjecture for $g_1$ if its scalar curvature satisfies (1.3) with $\ell = 1$.

Let $a$ be certain constant and $r_+ \geq 0$ be the largest positive root of the equation
\[ 1 + \frac{a}{r^{n-1}} - \frac{r_0^n}{r^n} = 0 \]
where constant $r_0 > 0$. Denote
\[ \beta = \frac{4\pi}{r_+ \left( n - 1 + \frac{a}{r_+^{n-1}} \right)}. \] (1.7)
In [5], we construct metrics of Horowitz-Myers type with the negative constant scalar curvature
\[ \hat{g} = \frac{1}{r^2 \left( 1 + \frac{a}{r^{n-1}} - \frac{r_0^n}{r^n} \right)} dr^2 + r^2 \left( 1 + \frac{a}{r^{n-1}} - \frac{r_0^n}{r^n} \right) d\xi^2 + r^2 \sum_{i=3}^{n} (d\phi^i)^2, \] (1.8)
where
\[ r \in [r_+, \infty), \quad \xi \in [0, \beta], \quad \phi^i \in [0, \lambda_i]. \]
This is a geodesically complete metric on $\mathbb{R}^2 \times T^{n-2}$ of constant scalar curvature $-n(n-1)$. And it is asymptotic to $g_{HM}$ up to order $O(\frac{1}{r^{n-1}})$ provided $\beta_0 = \beta$. As $\hat{g}$ decays weaker than $g_1$, we can not apply their positive energy.
Horowitz-Myers Conjecture

Theorem to \( \hat{g} \) directly. However, we can still verify that the Horowitz-Myers conjecture holds for \( \hat{g} \) [5].

Now we study the following more general perturbation of Horowitz-Myers metrics on \( \mathbb{R}^2 \times \mathbb{T}^{n-2} \)

\[
g = e^{2u(r)} dr^2 + e^{2v(r, \xi)} d\xi^2 + r^2 \left( \sum_{i=3}^{n} (d\phi^i)^2 + \hat{w}(r, \xi, \phi^i) \right),
\]

where

\[
u(r) = - \ln r - \frac{1}{2} \ln \left( 1 + \frac{a}{r^{n-1}} - \frac{r_0^n}{r^n} \right) + \hat{\nu}(r),
\]

\[
w(r, \xi) = \ln r + \frac{1}{2} \ln \left( 1 + \frac{a}{r^{n-1}} - \frac{r_0^n}{r^n} \right) + \hat{\nu}(r, \xi),
\]

\[
e^{\hat{u}(r)} = 1 + \frac{u_{n-1}}{r^{n-1}} + \frac{u_n}{r^n} + O \left( \frac{1}{r^{n+1}} \right),
\]

\[
e^{\hat{v}(r, \xi)} = 1 + \frac{v_{n-1}(\xi)}{r^{n-1}} + \frac{v_n(\xi)}{r^n} + O \left( \frac{1}{r^{n+1}} \right),
\]

\[
\hat{w}(r, \xi, \phi^i) = \frac{2w_{n-1}(\xi, \phi^i)}{r^{n-1}} + \frac{2w_n(\xi, \phi^i)}{r^n} + O \left( \frac{1}{r^{n+1}} \right),
\]

and \( u_{n-1}, u_n \) are real constants, \( v_{n-1}(\xi), v_n(\xi) \) are smooth functions of \( \xi \), \( w_{n-1}(\xi, \phi^i) \) and \( w_n(\xi, \phi^i) \) are 2-tensors on \( \mathbb{T}^{n-2} \) which depend on \( \xi, \phi^i \). The restriction of \( g \) on \( \mathbb{T}^{n-2} \) is denoted as

\[
\gamma = r^2 \left( \sum_{i=3}^{n} (d\phi^i)^2 + \hat{w}(r, \xi, \phi^i) \right).
\]

In this paper, we show that the Horowitz-Myers conjecture is true for the above AHM metrics.

**Theorem 1.1.** Let \( n \geq 3 \) and let \( g \) be the smooth metric (1.9) on \( \mathbb{R}^2 \times \mathbb{T}^{n-2} \), which satisfies (1.10). Suppose the scalar curvature \( R(g) \) of \( g \) satisfies

\[
R(g) + n(n-1) \geq 0,
\]

\[
\left[ R(g) + n(n-1) \right] r \in L^1(\mathbb{R}^2 \times \mathbb{T}^{n-2}, g),
\]

and suppose

\[
\beta_0 = \beta.
\]

If \( n \geq 4 \), we assume in addition that the scalar curvature \( R(\gamma) \) of \( \gamma \) satisfies

\[
\int_{\mathbb{T}^{n-2}} R(\gamma) d\phi^3 \cdots d\phi^n \leq 0
\]

for all \( r \) and \( \xi \). Then the total energy

\[
E(g) - E(g_{HM}) \geq 0.
\]

The equality holds if and only if \( g \) is isometric to \( g_{HM} \).

**Remark 1.1.** The assumption that (1.4) holds for all \( k \geq 0 \) is only for the sake of simplicity of notations appeared in the paper. Indeed, the proof of Theorem 1.1 needs only \( k = 0, 1, 2 \).
Remark 1.2. Condition (1.13) is trivial when \( n = 3 \), since in this case, \( \gamma \) is a one dimensional metric with \( R(\gamma) \equiv 0 \). If \( n = 4 \), then the Gauss-Bonnet theorem states that
\[
\int_{\gamma} R(\gamma)dV_\gamma = 0,
\]
which can not implies (1.13).

Remark 1.3. In [1], the components of \( \gamma \) are assumed to be independent of \( \phi^i \) (e.g. (1.3)). In this case, \( \gamma \) is a flat metric on \( \mathbb{T}^{n-2} \) and (1.13) holds for all \( n \).

The paper is organized as follows. In Section 2, we show that \( g \) can be written as an asymptotically Poincaré-Einstein (APE) metric. Then the total energy can be computed in terms of the definitions of Wang [9] and Chruściel-Herzlich [2]. In Section 3, we prove the Horowitz-Myers conjecture for \( g \).

2. APE metrics

In this section, we show that \( g \) is actually an APE metric. The boundary-at-infinity \( \partial_\infty M \) of \( \mathbb{R}^2 \times \mathbb{T}^{n-2} \) is \( \mathbb{T}^{n-1} \). Let \( x \) be a special defining function for \( \partial_\infty M \). That is, on a collar neighborhood of \( \partial_\infty M \),

\begin{enumerate}
\item \( x \geq 0 \) with \( \{x = 0\} = \partial_\infty M \),
\item \( |dx|_{x^2g} \equiv 1 \).
\end{enumerate}

On this collar neighborhood, the metric \( g \) can be written in the form
\[
g = \frac{1}{x^2}(dx^2 \oplus h_x), \quad \lim_{r \to \infty} x r = 1
\]
where \( h_x \) are the induced metrics of \( x^2g \) on the constant \( x \) slices,
\[
\frac{1}{x}dx = -\frac{e^{\tilde{u}(r)}}{r \sqrt{1 + \frac{a}{s^{n-1}} - \frac{r_0^n}{r^n}}}dr.
\]
It follows that
\[
\ln x = -\int_{r_+}^{r} \frac{e^{\tilde{u}(s)}}{s \sqrt{1 + \frac{a}{s^{n-1}} - \frac{r_0^n}{s^n}}}ds + C
\]
\[
= -\ln r - \int_{r_+}^{r} \left( \frac{u_{n-1} - \frac{1}{2}a}{s^n} + \frac{u_n + \frac{1}{2}r_0^n}{s^{n+1}} + O(s^{-n-2}) \right) ds
\]
\[
+ \ln r_+ + C,
\]
where
\[ C = - \ln r_+ + \int_{r_+}^\infty \frac{1}{s} \left( \frac{e^{u(s)}}{\sqrt{1 + \frac{a}{s^n} - \frac{r_0^n}{s^n}}} - 1 \right) ds. \]

Therefore, we obtain
\[ r = x^{-1} + \frac{u_{n-1} - \frac{1}{2} a}{n-1} x^{n-2} + \frac{u_n + \frac{1}{2} r_0^n}{n} x^{n-1} + O(x^n). \]

Putting this into (1.9) and applying the asymptotic conditions (1.10), we obtain
\[ g = \frac{1}{x^2} \left\{ dx^2 + d\xi^2 + \sum_{i=3}^{n} (d\phi^i)^2 \right. \\
+ \frac{x^{n-1}}{n-1} \left[ ((n-2)a + 2u_{n-1} + 2(n-1)v_{n-1})d\xi^2 \right. \\
+ (2u_{n-1} - a) \sum_{i=3}^{n} (d\phi^i)^2 + 2(n-1)w_{n-1} \right] \\
+ \frac{x^n}{n} \left[ -(n-1)r_0^n + 2u_n + 2nv_n \right] d\xi^2 \\
+ \left. (r_0^n + 2u_n) \sum_{i=3}^{n} (d\phi^i)^2 + 2nw_n \right\} + O(x^{n+1}). \]

Denote
\[ h_0 = d\xi^2 + \sum_{i=3}^{n} (d\phi^i)^2, \]
\[ \theta = \left((n-2)a + 2u_{n-1} + 2(n-1)v_{n-1}\right)d\xi^2 \]
\[ + (2u_{n-1} - a) \sum_{i=3}^{n} (d\phi^i)^2 + 2(n-1)w_{n-1}, \]
\[ \kappa = \left(-(n-1)r_0^n + 2u_n + 2nv_n\right)d\xi^2 \]
\[ + (r_0^n + 2u_n) \sum_{i=3}^{n} (d\phi^i)^2 + 2nw_n. \]

The conformal infinity of \( g \) is defined as the conformal class \([h_0]\) on \( \partial_\infty M \) (cf. [3]). The mass aspect function, the total energy are defined as
\[ \text{tr}_{h_0} \kappa = -r_0^n + 2(n - 1)u_n + 2nv_n + 2n\text{tr}_{h_0}w_n, \]
\[ E(g) = \int_{\partial_\infty M} \text{tr}_{h_0} \kappa dV_{h_0}. \]
respectively \[6, 2\]. When \(u_n = v_n = w_n = 0\), we obtain

\[
E(g_{\text{HM}}) = -\tau_0^n \text{Vol}(\partial_{\infty} M, h_0).
\]

Therefore

\[
E(g) - E(g_{\text{HM}}) = \int_{\partial_{\infty} M} \left[ \tau_0^n - \tau_0^n + 2(n-1)u_n + 2nv_n + 2ntr_{h_0} w_n \right] dV_{h_0}. \tag{2.4}
\]

Now we express the scalar curvature \(R(g)\) in some appropriate form. Denote

\[
W^r = \frac{1}{2} \text{tr}_\gamma (\partial_r \gamma) = \frac{1}{2} (\partial_r \gamma_{ij}) \gamma^{ij}
\]

and let \(\tilde{W}^r\) be the trace-less part of \(\partial_r \gamma\). It is straightforward that

\[
\partial_r \gamma = \frac{2W^r}{n - 2} \gamma + \tilde{W}^r. \tag{2.5}
\]

By the asymptotic condition of \(\gamma\), we obtain

\[
W^r = \frac{n - 2}{r} - \frac{(n-1)tr_{h_0} w_{n-1}}{r^n} - \frac{ntr_{h_0} w_n}{r^{n+1}} + O(r^{-n-2}). \tag{2.6}
\]

Similarly, denote

\[
W^\xi = \frac{1}{2} \text{tr}_\gamma (\partial_\xi \gamma) = \frac{1}{2} (\partial_\xi \gamma_{ij}) \gamma^{ij}
\]

and let \(\tilde{W}^\xi\) be the trace-less part of \(\partial_\xi \gamma\). We can also get

\[
\partial_\xi \gamma = \frac{2W^\xi}{n - 2} \gamma + \tilde{W}^\xi. \tag{2.7}
\]

Set

\[
\hat{W}^r = W^r - \frac{n - 2}{r}. \tag{2.8}
\]

**Lemma 2.1.** With the above notations, the scalar curvature \(R(g)\) of \(g\) is

\[
R(g) = -2e^{-u} e^v \partial_r \left[ e^{v-u} (\partial_r e^u + W^r) \right] - 2e^{-v} \partial_\xi (e^{-v} W^\xi) + R(\gamma) - 2e^{-2u} \left[ \frac{n-1}{2(n-2)} (W^r)^2 + \frac{1}{8} |\tilde{W}^r|^2 \right] \tag{2.9}
\]

\[
- 2e^{-2v} \left[ \frac{n-1}{2(n-2)} (W^\xi)^2 + \frac{1}{8} |\tilde{W}^\xi|^2 \right].
\]

**Proof.** For the following metrics

\[
N(s)^2 ds^2 + \eta, \quad \eta = \eta_{ab}(s, y^1, \cdots, y^n) dy^a dy^b,
\]

it is well-known that their scalar curvature is

\[
R(\eta) - N^{-1} (N^{-1} tr_{\eta} \eta')' - \frac{1}{4N^2} |\eta'|_\eta^2 - \frac{1}{4N^2} (tr_{\eta} \eta')^2, \tag{2.10}
\]
where $R(\eta)$ is the scalar curvature of $\eta$, $'$ is the partial derivative with respect to $s$,

$$\eta' = \eta'^a dy^a dy^b.$$ 

Now we apply formula (2.10) to the following part of $g$

$$\rho = e^{2v} d\xi^2 + \gamma,$$

and obtain

$$R(\rho) = R(\gamma) - 2e^{-v} \partial_\xi(e^{-v} \text{tr} \gamma \partial_\xi \gamma) - \frac{1}{4} e^{-2v} (\partial_\xi \gamma)_{\gamma}^2 - \frac{1}{4} e^{-2v} (\text{tr} \gamma \partial_\xi \gamma)^2$$

$$= R(\gamma) - 2e^{-v} \partial_\xi(e^{-v} W \xi) - 2e^{-2v} \left( \frac{n-1}{2(n-2)} (W \xi)^2 + \frac{1}{8} |W \xi |_\gamma^2 \right).$$

Next we apply (2.10) to the metric

$$g = e^{2u} dr^2 + \rho$$

and obtain

$$R(g) = R(\rho) - e^{-u} \partial_r(e^{-u} \text{tr} \rho \partial_r \rho) - \frac{1}{4} e^{-2u} (\partial_r \rho)_{\rho}^2 - \frac{1}{4} e^{-2u} (\text{tr} \rho \partial_r \rho)^2.$$

Since

$$\partial_r \rho = 2(\partial_r v) e^{2v} d\xi^2 + \partial_r \gamma,$$

$$\text{tr} \rho \partial_r \rho = 2(\partial_r v) + \text{tr} \gamma \partial_r \gamma = 2(\partial_r v) + 2W \gamma,$$

$$|\partial_r \rho|_{\rho}^2 = 4(\partial_r v)^2 + |\partial_r \gamma|_{\gamma}^2 = 4(\partial_r v)^2 + \frac{4}{n-2} (W \gamma)^2 + |W \gamma|_{\gamma},$$

we obtain

$$R(g) = R(\rho) - 2e^{-u} \partial_r [e^{-u} (\partial_r v + W \gamma)]$$

$$- e^{-2u} \left( 2(\partial_r v)^2 + \frac{n-1}{n-2} (W \gamma)^2 + 2\partial_r v W \gamma + \frac{1}{4} |W \gamma|_{\gamma}^2 \right)$$

$$= R(\rho) - 2e^{-u} \partial_r [e^{-u} (\partial_r v + W \gamma)]$$

$$- 2e^{-2u} \left[ \frac{n-1}{2(n-2)} (W \gamma)^2 + \frac{1}{8} |W \gamma|_{\gamma}^2 \right].$$

Therefore the lemma follows.

**Q.E.D.**

**Proposition 2.1.** For the metric (1.9), the condition

$$[R(g) + n(n-1)] \in L^1(\mathbb{R}^2 \times \mathbb{T}^{n-2}, g)$$

holds if and only if

$$\text{tr}_{\eta_0} \theta = 0 \iff u_{n-1} + v_{n-1} + \text{tr}_{\eta_0} w_{n-1} = 0.$$

(2.11)

In this case,

$$R + n(n-1) = O\left( \frac{1}{r^{n+1}} \right).$$

(2.12)
Proof. Note if all \( \hat{u}, \hat{v}, \hat{W}^r, \hat{W}^\xi \) vanish, then \( g \) reduces to
\[
e^{-2f}dr^2 + e^{2f}d\xi^2 + r^2 \sum_{i,j=3}^{n} \gamma_{ij}(\phi^3, \ldots, \phi^n)d\phi^i d\phi^j,
\]
where
\[
f = \ln r + \frac{1}{2} \ln \left( 1 + \frac{a}{r^{n-1}} - \frac{v_0}{r} \right).
\]
By (2.9), the scalar curvature of this metric is
\[-n(n-1) + R(\gamma).
\]
Putting (1.10) and (2.8) into (2.9), we obtain
\[
R(g) = -n(n-1)e^{-2u} + 2e^{2f-2u} \left[ -e^{-\partial_r(\hat{v} + \hat{W}^r)} \right.
\]
\[
- \partial_r(\hat{v} - \hat{u} + 2f + (n-1)\ln r) \cdot (\partial_r\hat{v} + \hat{W}^r)
\]
\[
+ \left( \frac{1}{r} - \partial_r f \right) \partial_r\hat{v} + (\partial_r f + \frac{n-2}{r})\partial_r\hat{u}
\]
\[
- \frac{(n-1)}{2(n-2)} (\hat{W}^r)^2 - \frac{1}{8} |\hat{W}^r|^2
\]
\[- 2e^{-v} \partial_\xi (e^{-v}W^\xi) + R(\gamma)
\]
\[- 2e^{-2v} \left[ \frac{n-1}{2(n-2)} (W^\xi)^2 + \frac{1}{8} |W^\xi|^2 \right]
\]
\[= -n(n-1) + (tr_{h_0} \theta)r^{1-n} + O(r^{-n-1}).
\]
Here we use that
\[R(\gamma) = O(r^{-n-1}),
\]
which could be confirmed by direct computation. Since
\[\sqrt{\det g} = O(r^{n-2}),
\]
and
\[tr_{h_0} \theta = 2(n-1)(v_{n-1} + u_{n-1} + tr_{h_0} w_{n-1}),
\]
so the lemma follows. Q.E.D.

By (2.1), we have
\[g = x^{-2} \bar{g}, \quad \bar{g} = dx^2 + d\xi^2 + \sum_{i=3}^{n} (d\phi^i)^2 + \frac{x^{n-1}}{n-1} \theta + O(x^n).
\]
We denote \( \phi^0 = x \) and \( \phi^1 = \xi \). Let \( a, b \) and \( c \) be indices ranging from 1 to \( n \). Under the coordinates \( \{\phi^0, \phi^1, \ldots, \phi^n\} \), it holds
\[
\bar{g}_{00} = 1, \quad \bar{g}_{0a} = 0, \quad \bar{g}_{ab} = \delta_{ab} + \frac{x^{n-1}}{n-1} \theta_{ab} + O(x^n).
\]
It is direct to show that the Christoffel symbols of $\bar{g}$ satisfy

$$\Gamma^c_{ab} = O(x^{n-1}),$$
$$\Gamma^0_{ab} = -\frac{1}{2}x^{n-2}\theta_{ab} + O(x^{n-1}),$$
$$\Gamma^b_a0 = \Gamma^b_0a = \frac{1}{2}x^{n-2}\theta_{ab} + O(x^{2n-3}),$$

and the others vanish. So the Ricci curvature of $\bar{g}$ satisfies

$$\bar{R}_{00} = -\frac{1}{2}(n-3)x^{n-3}\text{tr}_{\bar{g}}\theta + O(x^{2n-4}),$$
$$\bar{R}_{0a} = O(x^{n-2}),$$
$$\bar{R}_{ab} = O(x^{n-2}).$$

Denote by $\text{Ric}$ the Ricci curvature of $g$. Let

$$\Omega = \text{Ric} + (n-1)g.$$  

By the formulas of conformal transformation of the Ricci and scalar curvatures, we finally obtain

$$\Omega(\partial_x, \partial_x) = \frac{1}{2}(n-3)x^{n-3}\text{tr}_{\bar{g}}\theta + O(x^{n-2}),$$
$$\Omega(\partial_a, \partial_b) = \frac{1}{2}x^{n-3}(\text{tr}_{\bar{g}}\theta)\delta_{ab} + O(x^{n-2}),$$
$$\Omega(\partial_x, \partial_a) = O(x^{n-2}).$$

Recall Definition 2.1 in [7] that the metric $g$ is asymptotically Poincaré–Einstein (APE) if

$$\lim_{x \to 0} |\Omega|_g = O(x^n).$$

By Lemma 2.2 [7], $g$ is APE if and only if $\text{tr}_{\bar{g}}\theta = 0$. (This can also be seen from the above computation.) Therefore we have

**Proposition 2.2.** For the metric $(1.9)$, the condition

$$[R(g) + n(n-1)]r \in L^1(\mathbb{R}^2 \times \mathbb{T}^{n-2}, g)$$

holds if and only if $g$ is APE.

Finally, we study the consequence of smoothness of $g$ at $r = r_+$. 

**Proposition 2.3.** If the metric $g$ is smooth at $r = r_+$, then

$$\hat{v}(r_+, \xi) = \hat{u}(r_+) \quad \text{for all } \xi.$$  

(2.13)

**Proof.** We follow the proof of Theorem II.1 in [5] to construct a regular coordinate system around $r = r_+$. Set

$$A(r) = r^2\left(1 + \frac{a}{r^{n-1}} - \frac{r_0^n}{r^n}\right)$$
For $r > 0$. Differentiating $A$ at $r = r_+$ and using (1.7), we find

$$A'(r_+) = \frac{4\pi}{\beta} > 0.$$  

This implies that $A$ has a smooth inverse, which we denote by $B$, round $r_+$. Clearly $B(0) = r_+$. Now we can introduce new coordinates $(\rho, \Xi)$ given by

$$\rho = A^\frac{1}{2}(r), \quad \Xi = \frac{2\pi \xi}{\beta}$$

for $r_+ < r < r_+ + \varepsilon$, $0 \leq \Xi < 2\pi$, with some $\varepsilon > 0$. So $(\rho, \Xi)$ can be viewed as the standard polar coordinates around the origin in $\mathbb{R}^2$. Set

$$z^1 = \rho \cos \Xi, \quad z^2 = \rho \sin \Xi.$$  

Obviously, $\{r = r_+\} = \{z^1 = z^2 = 0\}$. It is shown in [5] that $(z^1, z^2, \phi^3, \cdots, \phi^n)$ is a regular coordinate system around $r = r_+$ for $\hat{\gamma}$. Since $g$ is smooth, the components in this coordinate system are smooth around $(z^1, z^2) = (0, 0)$. For our purpose, we only rewrite the following part

$$g(0) := e^{2\hat{u}(r)}A^{-1}(r)d\rho^2 + e^{2\hat{v}(r, \xi)}A(r)d\xi^2$$

of $g$ in the new coordinate system. Since

$$A^{-1}(r)d\rho^2 = \frac{4}{A'(r)^2}d\rho^2, \quad A(r)d\xi^2 = \frac{4\rho^2}{A'(r_+)^2}d\Xi^2,$$

we have

$$g(0) = \frac{4e^{2\hat{u}(r)}}{A'(r)^2}d\rho^2 + \frac{4e^{2\hat{v}(r, \xi)}}{A'(r_+)^2}\rho^2d\Xi^2 = g(\rho) + g(\Xi) + g_{\Xi^2}$$

where

$$g(\rho) = \left[ \frac{4e^{2\hat{u}(r)}}{A'(r)^2} - \frac{4e^{2\hat{u}(r_+)}}{A'(r_+)^2} \right]d\rho^2,$$

$$g(\Xi) = \frac{4e^{2\hat{u}(r_+)}}{A'(r_+)^2} e^{2\hat{v}(r, \xi) - \hat{u}(r_+)} - 1 \rho^2d\Xi^2,$$

$$g_{\Xi^2} = \frac{4e^{2\hat{u}(r_+)}}{A'(r_+)^2}(d\rho^2 + \rho^2d\Xi^2).$$

Since $\frac{4e^{2\hat{u}(r)}}{A'(r)^2}$ is smooth near $r = r_+$, there exists a smooth function $\varphi(r)$ so that

$$\frac{4e^{2\hat{u}(r)}}{A'(r)^2} - \frac{4e^{2\hat{u}(r_+)}}{A'(r_+)^2} = (r - r_+)^2\varphi(r)$$

for $r$ near $r_+$. Note that

$$(r - r_+)^2\varphi(r) = \left[ B(\rho^2) - B(0) \right] (\varphi \circ B)(\rho^2).$$

Since $B$ is smooth near $0$, there also exists a smooth function $\psi(\rho)$ such that

$$B(\rho^2) - B(0) = \rho^2\psi(\rho^2).$$
So
\[ g(\rho) = (\varphi \circ B)(\rho^2)\psi(\rho^2)\rho^2 d\rho^2 \]
\[ = ((\varphi \circ B) \cdot \psi)((z^1)^2 + (z^2)^2)(z^1 dz^1 + z^2 dz^2)^2. \]

This implies that \( g(\rho) \) is smooth around \((z^1, z^2) = (0, 0)\). As a result, \( g(\Xi) \) is also smooth around \((z^1, z^2) = (0, 0)\). However,
\[ \rho^2 d\Xi^2 = (dz^1)^2 + (dz^2)^2 - d\rho^2 \]
\[ = (dz^1)^2 + (dz^2)^2 - \frac{1}{\rho^2}(z^1 dz^1 + z^2 dz^2)^2 \]
is singular at \((z^1, z^2) = (0, 0)\). So it must hold that
\[ \lim_{r \to r^+} \hat{v}(r, \xi) = \hat{u}(r^+) \]
for all \( \xi \). This proves (2.13). Q.E.D.

3. Proof of the theorem

In this section, we adopt the idea in [1] to prove Theorem 1.1. We will replace \( r \) by a new coordinate function \( \tilde{r} \) to change \( e^{2u(r)} dr^2 \) in (1.9) to \( \tilde{r}^{-2} \left( 1 - \frac{\tilde{r}^n}{\tilde{r}_0^n} \right)^{-1} dr^2 \).

**Lemma 3.1.** There exist a positive number \( \tilde{r}_0 \) and a smooth increasing function \( r \mapsto \tilde{r}(r) \) from \([r^+, \infty)\) to \([\tilde{r}_0, \infty), \) with \( \tilde{r}(r^+) = \tilde{r}_0 \), such that
\[ \frac{1}{\tilde{r}} \sqrt{1 - \frac{\tilde{r}^n}{\tilde{r}_0^n}} d\tilde{r} = \frac{1}{r} \sqrt{1 + \frac{a}{\tilde{r}_0^n} - \frac{\tilde{r}^n}{\tilde{r}_0^n}} e^{u(r)} dr, \]
(3.1)

and
\[ \tilde{r}(r) = r + \frac{a - 2un - 1}{2(n - 1)} r^{2-n} + \frac{\tilde{r}_0^n - \tilde{r}_0^n - 2un}{2n} r^{1-n} + O(r^{-n}). \]
(3.2)

**Proof.** Define
\[ F(r) = \int_1^r \frac{1}{s \sqrt{1 - \frac{1}{s^n}}} ds, \quad r \geq 1. \]

For large \( r \), we have
\[ F(r) = F_0 + \ln r - \frac{1}{2n} r^{-n} + O(r^{-n-1}), \]
(3.3)

where
\[ F_0 = \int_1^\infty \frac{1}{s} \left( \frac{1}{\sqrt{1 - \frac{1}{s^n}}} - 1 \right) ds. \]
Clearly, $F$ is a smooth increasing function. Define $\tilde{r}(r)$ on $[r_+, \infty)$ by

$$
F\left(\frac{\tilde{r}(r)}{\tilde{r}_0}\right) = \int_{r_+}^{r} \frac{e^{\hat{v}(s)}}{s \sqrt{1 + \frac{a}{s^{n-1}} - \frac{r^n_0}{s^n}}} ds.
$$

(3.4)

Here $\tilde{r}_0$ is a positive number to be specified later. The integral on the right hand side has the following asymptotic expansion as $r \to \infty$:

$$
C + \ln r + \frac{a - 2u_{n-1}}{2(n-1)} r^{1-n} - \frac{2u_n + r^n_0}{2n} r^{-n} + O(r^{-n-1}),
$$

(3.5)

where

$$
C = \lim_{r \to \infty} \left( \int_{r_+}^{r} \frac{1}{s \sqrt{1 + \frac{a}{s^{n-1}} - \frac{r^n_0}{s^n}}} ds - \ln r \right)
$$

$$
= - \ln r_+ + \int_{r_+}^{\infty} \frac{1}{s} \left( \frac{1}{\sqrt{1 + \frac{a}{s^{n-1}} - \frac{r^n_0}{s^n}}} - 1 \right) ds.
$$

Putting (3.3) and (3.5) into (3.4), we obtain

$$
F_0 - \ln \tilde{r}_0 + \ln \tilde{r} - \frac{\tilde{r}_0^n}{2n} \tilde{r}^{-n} + O(\tilde{r}^{-n-1})
$$

$$
= C + \ln r + \frac{a - 2u_{n-1}}{2(n-1)} r^{1-n} - \frac{2u_n + r^n_0}{2n} r^{-n} + O(r^{-n-1}).
$$

Now we set

$$
\tilde{r}_0 = e^{F_0 - C}.
$$

The above equation implies the asymptotic expansion (3.2). The equation (3.1) follows directly by the definition of $\tilde{r}$. Q.E.D.

With the function $\tilde{r}$, we rewrite the metric

$$
g = e^{2\tilde{u}(\tilde{r})} d\tilde{r}^2 + e^{2\tilde{v}(\tilde{r}, \xi)} d\xi^2 + \tilde{r}^2 \left( \sum_{i=3}^{n} (d\phi_i)^2 + \hat{w}(\tilde{r}, \xi, \phi_i) \right),
$$

(3.6)

where

$$
\tilde{u}(\tilde{r}) = - \ln \tilde{r} - \frac{1}{2} \ln \left( 1 - \frac{\tilde{r}^n_0}{\tilde{r}^n} \right),
$$

$$
\tilde{v}(\tilde{r}, \xi) = - \tilde{u}(\tilde{r}) + \tilde{\nu}(\tilde{r}, \xi),
$$

$$
\tilde{e}(\tilde{r}, \xi) = 1 + \frac{\tilde{v}_{n-1}(\xi)}{\tilde{r}^{n-1}} + \frac{\tilde{\nu}_n(\xi)}{\tilde{r}^n} + O\left( \frac{1}{\tilde{r}^{n+1}} \right),
$$

$$
\tilde{w}(\tilde{r}, \xi, \phi_i) = \frac{2\tilde{w}_{n-1}(\xi)}{\tilde{r}^{n-1}} + \frac{2\tilde{\nu}_n(\xi)}{\tilde{r}^n} + O\left( \frac{1}{\tilde{r}^{n+1}} \right).
$$

(3.7)

By comparing (1.9) and (3.6), we have

$$
\tilde{\nu} = \hat{v} + \hat{u} - \ln \frac{d\tilde{r}}{dr}
$$

(3.8)
and
\[
\hat{w} = \frac{r^2}{\tilde{r}^2} \tilde{w} + \left( \frac{r^2}{\tilde{r}^2} - 1 \right) \sum_{i=3}^{n} (d\phi^i)^2. 
\tag{3.9}
\]

In particular, we obtain
\[
\begin{align*}
\tilde{v}_{n-1} &= v_{n-1} + \frac{(n-2)a + 2u_{n-1}}{2(n-1)}, \\
\tilde{v}_n &= v_n + \frac{(n-1)(\tilde{r}_0^n - r_0^n) + 2u_n}{2n}, \\
\tilde{w}_{n-1} &= w_{n-1} + \frac{2u_{n-1} - a}{2(n-1)} \sum_{i=3}^{n} (d\phi^i)^2, \\
\tilde{w}_n &= w_n + \left( \tilde{r}_0^n - r_0^n \right) + 2u_n \sum_{i=3}^{n} (d\phi^i)^2.
\end{align*}
\tag{3.10}
\]

As a result, (2.11) becomes
\[
\tilde{v}_{n-1} + \text{tr}_{h_0} \tilde{w}_{n-1} = 0, 
\tag{3.11}
\]
and (2.4) becomes
\[
E(g) - E(g_{HM}) = (\tilde{r}_0^n - \tilde{r}_0^n) \text{Vol}(\partial_\infty M, h_0) + 2n \int_{\partial_\infty M} (\tilde{v}_n + \text{tr}_{h_0} \tilde{w}_n) dV_{h_0}. 
\tag{3.12}
\]

**Lemma 3.2.** Under the above condition, it holds that
\[
e^{\hat{\phi}(\tilde{r}_0, \xi)} = \frac{\tilde{r}_0}{r_0} 
\tag{3.13}
\]
for all \(\xi\).

**Proof.** By (3.8) and (2.13), we have
\[
e^{\hat{\phi}(\tilde{r}_0, \xi)} = e^{2\hat{u}(r_+)} \frac{dr}{dr}(r_+). 
\]

Since (3.1) gives that
\[
\left( \frac{dr}{dr} \right)^2 = \frac{\hat{r}^2 \left( 1 - \hat{r}^n \right)}{r^2 \left( 1 + \frac{a}{r_{n-1}} - \frac{\hat{r}_0^n}{r_+} \right)} e^{2\hat{u}(r)},
\]
it follows
\[
\left( \frac{dr}{dr} \right)^2 (r_+) = \frac{\tilde{r}_0^2}{r_+^2} e^{2\hat{u}(r_+)} \lim_{r \to r_+} \frac{1 - \frac{\tilde{r}_0^n}{r_+^n}}{1 + \frac{a}{r_{n-1}} - \frac{\tilde{r}_0^n}{r_+^n}} = \frac{n \tilde{r}_0 e^{2\hat{u}(r_+)} \frac{dr}{dr}(r_+)}{r_+^{n-2} + \frac{\tilde{r}_0^n}{r_+^{n-1}}}. 
\]
Recalling that \( r_+ \) is a root of \( 1 + \frac{a}{r_+^n} - \frac{r_+^n}{r} = 0 \), this implies
\[
\frac{d\tilde{r}}{dr}(r_+) = e^{2\hat{a}(r_+)} \frac{n\tilde{r}_0}{r_+(n - 1 + \frac{1}{r_+^n})}.
\]

By (1.12), we obtain
\[
\frac{d\tilde{r}}{dr}(r_+) = e^{2\hat{a}(r_+)} \frac{\tilde{r}_0}{\tilde{r}_0}.
\]

So the lemma follows. Q.E.D.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Denote \( \tilde{W}^r, \tilde{W}^r, \tilde{W}^r \) for (3.6) the analogous quantities \( W^r, W^r, W^r \) for (1.9). In the new coordinate function \( \tilde{r} \), we have
\[
R(g) = -n(n-1) + R(\gamma)
- 2e^{-\hat{a}-(n-1)\ln \hat{e}^r} \partial_r e \left[ e^{-2\hat{a} + \hat{e}^r n-1} \left( \partial_r \hat{e} + \tilde{W}^r \right) \right]
- \frac{n\tilde{r}_0}{r_+^{n-1}} \partial_r \hat{e}^r - 2e^{-\hat{a} + \hat{e}^r} \partial_r \xi (e^{-\hat{e}^r} \tilde{W}^r) - 2e^{-\hat{a} + \hat{e}^r} \frac{n-1}{2(n-2)} (\tilde{W}^r)^2 + \frac{1}{8} |\tilde{W}^r|^2
- 2e^{-\hat{a} + \hat{e}^r} \frac{n-1}{2(n-2)} (\tilde{W}^r)^2 + \frac{1}{8} |\tilde{W}^r|^2.
\]

This equality can be rewritten as
\[
-2\partial_r e^{-2\hat{a} + \hat{e}^r n-1} \left( \partial_r \hat{e} + \tilde{W}^r \right) = n\tilde{r}_0 \partial_r (e^\hat{a}^r) + 2e^{2\hat{a} + \hat{e}^r n-1} \partial_r \xi (e^{-\hat{e}^r} \tilde{W}^r)
- \hat{e}^r n-1 R(\gamma) + A
\]
where
\[
A = e^\hat{a}^r n-1 \left[ R(g) + n(n - 1) \right]
+ 2e^{-2\hat{a} + \hat{e}^r n-1} \left[ \frac{n-1}{2(n-2)} (\tilde{W}^r)^2 + \frac{1}{8} |\tilde{W}^r|^2 \right]
+ 2e^{2\hat{a} - \hat{e}^r n-1} \left[ \frac{n-1}{2(n-2)} (\tilde{W}^r)^2 + \frac{1}{8} |\tilde{W}^r|^2 \right] \geq 0.
\]

Integrating (3.14) on \([\tilde{r}_0, \infty)\), noting that \( e^{-2\hat{a}} \) vanishes at \( \tilde{r} = \tilde{r}_0 \), and using (3.13), we obtain
\[
\lim_{\tilde{r} \to \infty} -2e^{-2\hat{a} + \hat{e}^r n-1} \left( \partial_r \hat{e} + \tilde{W}^r \right) = n\tilde{r}_0 \left( 1 - \frac{\tilde{r}_0}{\tilde{r}_0} \right)
+ \int_{\tilde{r}_0}^{\infty} \left[ 2e^{2\hat{a} + \hat{e}^r n-1} \partial_r \xi (e^{-\hat{e}^r} \tilde{W}^r) - \hat{e}^r n-1 R(\gamma) + A \right] d\tilde{r}.
\]

(3.15)
Since
\[ \hat{W}^r = \frac{1}{2} \text{tr}_\gamma (\partial_\gamma \gamma) - \frac{n-2}{\hat{r}} \]
\[ = - \frac{n-1}{\hat{r}^n} \text{tr}_{h_0} \hat{w}_{n-1} - \frac{n}{\hat{r}^{n+1}} \text{tr}_{h_0} \hat{w}_n + O(\hat{r}^{-2-n}). \]

Combining this with (3.7) and (3.11), we obtain
\[ \partial_\gamma \hat{v} + \hat{W}^r = - \frac{n-1}{\hat{r}^n} (\hat{v}_{n-1} + \text{tr}_{h_0} \hat{w}_{n-1}) - \frac{n}{\hat{r}^{n+1}} (\hat{v}_n + \text{tr}_{h_0} \hat{w}_n) + O(\hat{r}^{-2-n}) \]
\[ = - \frac{n}{\hat{r}^{n+1}} (\hat{v}_n + \text{tr}_{h_0} \hat{w}_n) + O(\hat{r}^{-2-n}), \]

which implies
\[ \lim_{\hat{r} \to \infty} -2e^{-2\hat{v}} \hat{W}^{n-1} (\partial_\gamma \hat{v} + \hat{W}^r) = 2n (\hat{v}_n + \text{tr}_{h_0} \hat{w}_n). \]

Put this into (3.15), and then integrate over \((\partial_\infty M, h_0)\). Noting that the term containing \(\partial_\xi (e^{-\hat{v}} \hat{W}_\xi)\) cancels away after integrating over \(\xi\), and recalling the assumption (1.13), we obtain
\[ 2n \int_{\partial_\infty M} (\hat{v}_n + \text{tr}_{h_0} \hat{w}_n) dV_{h_0} \geq n \hat{r}_0^n \left( 1 - \frac{\hat{r}_0}{\hat{r}} \right) \text{Vol}(\partial_\infty M, h_0) \]
\[ + \int_{\partial_\infty M} \int_{\hat{r}_0}^{\hat{r}} A d\hat{r} dV_{h_0}. \]

Inserting this into (3.12), we obtain
\[ E(g) - E(g_{HM}) \geq \hat{r}_0^n \left( n - 1 + \left( \frac{\hat{r}_0}{\hat{r}} \right)^n - n \frac{\hat{r}_0}{\hat{r}} \right) \text{Vol}(\partial_\infty M, h_0) \]
\[ + \int_{\partial_\infty M} \int_{\hat{r}_0}^{\hat{r}} A d\hat{r} dV_{h_0}. \]

It is easy to see that
\[ n - 1 + s^n - ns = (1 - s)(n - 1 - s - \cdots - s^{n-1}) \geq 0 \]
for \(s > 0\), and the equality occurs if and only if \(s = 1\). Taking \(s = \frac{\hat{r}_0}{\hat{r}}\), we obtain
\[ E(g) - E(g_{HM}) \geq 0. \]

For the case of equality, we have
\[ \hat{r}_0 = \hat{r}_0, \quad A \equiv 0, \]
which implies
\[ R + n(n-1) = 0, \quad \hat{W}^r = \hat{W}_\xi = 0, \quad \hat{v}_n = 0, \quad \hat{w}_n = 0. \]

It follows that
\[ \hat{W}^r = \frac{n-2}{\hat{r}} + \hat{W}^r = \frac{n-2}{\hat{r}}. \]
Hence
\[ \partial_r (\tilde{\gamma}_{ij}) = \frac{2\tilde{W}^r}{n-2} \tilde{\gamma}_{ij} + \tilde{W}^r = \frac{2}{r} \tilde{\gamma}_{ij}. \]

Therefore
\[ \partial_r \left( \frac{1}{r^2} \tilde{\gamma}_{ij} \right) = 0. \]

It implies that
\[ \frac{1}{r^2} \tilde{\gamma}_{ij} = \frac{1}{r^2} \tilde{\gamma}_{ij} \bigg|_{r=\infty} = \delta_{ij}, \quad \gamma = r^2 \sum_{i=3}^{n} (d\phi^i)^2. \]

Now we can simplify (3.11) and find that
\[ \partial_r^2 \left( e^\hat{\nu} \right) + \left( \frac{n+1}{r} + \frac{1}{r^2} - 1 \right) \frac{\hat{\nu}^n_0}{\hat{\nu}^n_0 - \hat{\nu}^n} \left( \partial_r e^\hat{\nu} \right)^2 = 0. \]

Since
\[ e^{\hat{\nu}(\hat{r}_0, \xi)} = \frac{\hat{r}_0}{\hat{r}_0} = 1 = \lim_{\hat{r} \to \infty} e^{\hat{\nu}(\hat{r})}, \]

we can show \( \hat{\nu} \equiv 0 \) by the maximum principle. Finally, we obtain
\[ g = \frac{1}{r^2 \left( 1 - \frac{\hat{r}^n}{\hat{r}_0^n} \right)} dr^2 + r^2 \left( 1 - \frac{1}{r^n} \right) d\xi^2 + r^2 \sum_{i=3}^{n} (d\phi^i)^2. \]

Thus \( g \) is the Horowitz–Myers metric \( g_{\text{HM}} \). Q.E.D.

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