THE COMPLEX-SYMPLECTIC GEOMETRY OF
SL(2, C)-CHARACTERS OVER SURFACES

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Dedicated to M.S. Raghunathan on his sixtieth birthday

Abstract. The SL(2, C)-character variety \( X \) of a closed surface \( M \) enjoys a natural complex-symplectic structure invariant under the mapping class group \( \Gamma \) of \( M \). Using the ergodicity of \( \Gamma \) on the SU(2)-character variety, we deduce that every \( \Gamma \)-invariant meromorphic function on \( X \) is constant. The trace functions of closed curves on \( M \) determine regular functions which generate complex Hamiltonian flows. For simple closed curves, these complex Hamiltonian flows arise from holomorphic flows on the representation variety generalizing the Fenchel-Nielsen twist flows on Teichmüller space and the complex quakebend flows on quasi-Fuchsian space. Closed curves in the complex trajectories of these flows lift to paths in the deformation space \( \mathbb{CP}^1(M) \) of complex-projective structures with the same holonomy (grafting). If \( P \) is a pants decomposition, then the trace map \( \tau_P : X \to \mathbb{CP}^P \) defines a holomorphic completely integrable system. Furthermore, if \( \Gamma_P \) is the subgroup of \( \Gamma \) preserving \( P \), then every \( \Gamma_P \)-invariant holomorphic function \( X \to \mathbb{C} \) factors through \( \tau_P \). This holomorphic integrable system is related to the complex Fenchel-Nielsen coordinates on quasi-Fuchsian space \( QF(M) \) developed by Tan and Kourouniotis, and relate to recent formulas of Platis and Series on complex-length functions and complex twist flows on \( QF(M) \).

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Date: March 29, 2022.

The author gratefully acknowledges partial support from National Science Foundation grant DMS-0103889.
Introduction

The $\text{SL}(2, \mathbb{C})$-character variety of a closed surface $M$ with fundamental group $\pi$ is the space of equivalence classes of representations $\phi$ of the fundamental group $\pi$ of $M$ into $\text{SL}(2, \mathbb{C})$. The Teichmüller space $\mathcal{T}(M)$, the moduli space $X_U$ of irreducible flat unitary $\text{SU}(2)$-bundles, the deformation space $\mathbb{C}P^1(M)$ of $\mathbb{C}P^1$-structures on $M$, and the quasi-Fuchsian space $\mathcal{QF}(M)$ all lie in the smooth stratum $X$ of the $\text{SL}(2, \mathbb{C})$-character variety. These spaces all share several common features: a symplectic geometry derived from the topology of $M$, and a compatible action of the mapping class group $\Gamma$ of $M$. This paper investigates these structures in terms of the $\Gamma$-invariant complex-symplectic structure on $X$.

A complex-symplectic structure on a complex manifold is a nondegenerate closed holomorphic exterior 2-form. The $\text{SL}(2, \mathbb{C})$-character variety is an affine variety defined over $\mathbb{C}$ whose set of smooth $\mathbb{C}$-points is a complex-symplectic manifold of complex dimension $-3\chi(M)$ whose complex-symplectic structure is invariant under $\Gamma$.

In [46], Fenchel and Nielsen develop a set of coordinates for $\mathcal{T}(M)$ based on hyperbolic geometry. This set of coordinates is based on a pants decomposition $P = \{\alpha_1, \ldots, \alpha_N\}$ on $M$, that is a set of disjoint simple closed curves $\alpha_i$ cutting $M$ into three-holed spheres (“pants”). In a hyperbolic structure on $M$, the curves $\alpha_i$ are represented by disjoint simple closed geodesics and their lengths define a function $l_P : \mathcal{T}(M) \rightarrow (\mathbb{R}_+)^N$ which consistute half of the Fenchel-Nielsen coordinates on $\mathcal{T}(M)$. According to Wolpert [48], these functions Poisson-commute and define a completely integrable Hamiltonian system. In the language of classical mechanics, these are the action variables, for which the Fenchel-Nielsen twist vector fields define angle variables. In particular $l_P$ is the moment map for a Hamiltonian $\mathbb{R}^N$-action. Choosing a section $\sigma$ to $l_P$ determines a symplectomorphism

$$(\mathbb{R}_+)^N \times \mathbb{R}^N \rightarrow \mathcal{T}(M)$$

$$(\lambda; t) \mapsto \xi_t \sigma(\lambda)$$

where $\xi_t$ is the Hamiltonian $\mathbb{R}^N$-action defined by $l_P$. See Wolpert [46, 47, 48] for details.

This picture motivated the study of a symplectic geometry of moduli spaces $\text{Hom}(\pi, G)/G$ developed in [9, 11]. The results there extend directly to the complex-symplectic geometry of deformation spaces $\text{Hom}(\pi, G)/G$ where $G$ is a complex Lie group with an $\text{Ad}$-invariant
complex-orthogonal structure $\mathbb{B}$ on its Lie algebra. In this paper we consider only the special case $G = \text{SL}(2, \mathbb{C})$ where $\mathbb{B}$ is the trace form. Corresponding to an element $\alpha \in \pi$ is the function $f_\alpha : X \rightarrow \mathbb{C}$ associating to the equivalence class of $\phi$ the trace of $\phi(\alpha)$. When $\alpha$ is represented by a simple closed curve $A$, then the complex-Hamiltonian vector field $\text{Ham}(f_\alpha)$ generates a flow which is covered by a complex twist flow on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$. We study this flow, computing its periods, and relating it to the action on $X$ of the Dehn twist about $A$. Following [9, 11], we relate the complex-symplectic geometry and the Hamiltonian actions to the Fenchel-Nielsen twist flows on $\mathbb{F}(M)$ and the complex earthquakes and bending deformations on $\mathbb{C}P^1(M)$ and $Q_{\mathbb{C}}F(M)$.

In the presence of a conformal structure on $M$, the moduli spaces $\text{Hom}(\pi, G)/G$ admit stronger structures (a Kähler structure when $G$ is compact, and hyper-Kähler when $G$ is complex), but these stronger structures fail to be $\Gamma$-invariant. However the symplectic (and complex-symplectic) structures are $\Gamma$-invariant. The symplectic structure defines a $\Gamma$-invariant measure. By [13] and Pickrell-Xia [38], the resulting measure is ergodic under $\Gamma$ when $G$ is a compact Lie group. This has the following consequence for the holomorphic geometry when $G$ is complex:

**Theorem.** There are no nonconstant $\Gamma$-invariant meromorphic functions on $X$.

The proof uses the inclusion of the set $X_U$ of irreducible unitary characters in $X$, and the ergodicity of the action of $\Gamma$ on $X_U$ ([15]). A key idea in the proof is the action of the subgroup $\Gamma_P$ preserving a pants decomposition $P$ of $M$. The group $\Gamma_P$ is a free abelian group freely generated by the Dehn twists about the curves in $P$. This $\mathbb{Z}^N$-action lies in a Hamiltonian $\mathbb{R}^N$-action. The map which associates to $[\phi]$ the collection of traces

$$X_U \rightarrow \mathbb{R}^N$$

is a moment map for the $\mathbb{R}^N$-action and is also the ergodic decomposition for the $\mathbb{Z}^N$-action. The holomorphic analog is:

**Theorem.** Every $\Gamma_P$-invariant meromorphic function on $X$ factors through the map

$$X \rightarrow \mathbb{C}^N$$

which associates to a character its values on the curves in $P$. 
The complex twist flows have been extensively studied in the \textit{quasi-Fuchsian space} $\mathcal{QF}(M)$, which is the open subset of the $\mathrm{SL}(2, \mathbb{C})$-character variety comprising equivalence classes of quasi-Fuchsian representations. In this case, the complex twist flows correspond geometrically to quake-bending pleated surfaces in quasi-Fuchsian hyperbolic 3-manifolds, that is, composing Fenchel-Nielsen twist flows (earthquakes) with bending deformations (Epstein-Marden \cite{EM}). These quakebends are defined more generally for geodesic laminations, although we only consider deformations supported on simple closed curves here. The complex Fenchel-Nielsen coordinates of Kourouniotis \cite{Kourouniotis, Kourouniotis2, Kourouniotis3} and Tan \cite{Tan} are holomorphic Darboux coordinates for the complex-symplectic structure. We recover results of Platis \cite{Platis} expressing the symplectic duality between the complex twist flows and the complex length functions, and the formula of Series \cite{Series} for the derivative of a complex length function under a twist flow in terms of the complex-symplectic geometry of $X$.

More generally the complex twist flows are defined for $\mathbb{CP}^1$-structures, that is, geometric structures with coordinates modelled on $\mathbb{CP}^1$ with coordinate changes in $\mathrm{PSL}(2, \mathbb{C})$. Let $\mathbb{CP}^1(M)$ denote the deformation space of $\mathbb{CP}^1$-structures on $M$. Using the local biholomorphicity of the holonomy mapping

$$\text{hol} : \mathbb{CP}^1(M) \to \text{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))//\mathrm{SL}(2, \mathbb{C}).$$

one obtains complex-symplectic structures, complex length functions, and Hamiltonian complex twist flows on $\mathbb{CP}^1(M)$. The complex twist flows on $\mathbb{CP}^1$-structures can be described geometrically by inserting annuli into a $\mathbb{CP}^1$-manifold split along a simple closed curve which is locally circular. This is a special case of the \textit{grafting} construction considered in Tanigawa \cite{Tanigawa} and McMullen \cite{McMullen}. In particular closed curves in the complex trajectory of a complex twist flow lift to paths between different $\mathbb{CP}^1$-structures with the same holonomy (Maskit and Hejhal, see Goldman \cite{Goldman}). The holomorphic properties of the grafting construction are discussed in McMullen \cite{McMullen}, Tanigawa \cite{Tanigawa} and Scannell-Wolf \cite{Scannell-Wolf}.

1. Representation varieties and character varieties

Let $M$ be a closed oriented surface with fundamental group $\pi$. Let $\text{Hom}(\pi, \mathrm{SL}(2, \mathbb{C}))$ denote the complex affine variety consisting of homomorphisms $\pi \to \mathrm{SL}(2, \mathbb{C})$. (It is irreducible, by Goldman \cite{Goldman}; see Benyash-Krivets -Chernousov -Rapinchuk \cite{BCR} for stronger more general results, and also Li \cite{Li}.)
The group $\text{SL}(2, \mathbb{C})$ acts by conjugation on the affine algebraic set $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$. Let $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$ denote the set of $\text{SL}(2, \mathbb{C})$-orbits of $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$. Denote the orbit of a representation $\phi$ by 
$$[\phi] \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C})).$$

1.1. Stable and semistable points. A homomorphism 
$$\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$$
is irreducible if it leaves invariant no proper linear subspace of $\mathbb{C}^2$. Equivalently, $\phi$ is irreducible if the corresponding projective action fixes no point in $\mathbb{CP}^1$. Irreducible homomorphisms are the stable points of $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ (with respect to the $\text{SL}(2, \mathbb{C})$-action). The subset of $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ consisting of irreducible homomorphisms is Zariski-open and nonsingular. $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ is a complex manifold of complex dimension $6g - 3$ (see [9]).

More generally, a homomorphism is reductive if every invariant subspace possesses an invariant complement. Equivalently, $\phi$ is reductive if it is either reducible or fixes a pair of distinct points on $\mathbb{CP}^1$ or is central. A central homomorphism maps into the center $\{\pm I\}$ of $\text{SL}(2, \mathbb{C})$ and acts trivially on $\mathbb{CP}^1$. Letting $\text{Fix}(\phi)$ denote the subset of $\mathbb{CP}^1$ fixed by $\phi$, a homomorphism $\phi$ is reductive if and only if $\text{Fix}(\phi)$ equals $\emptyset$, a pair of distinct points, or all of $\mathbb{CP}^1$. Reductive homomorphisms are the semistable points of $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ (with respect to the $\text{SL}(2, \mathbb{C})$-action), comprising the subset
$$\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^{ss} \subset \text{Hom}(\pi, \text{SL}(2, \mathbb{C})).$$

A semistable (respectively stable) orbit is the orbit of a semistable (respectively stable) point.

Let $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$ denote the set of $\mathbb{C}$-points of the categorical quotient of the $\text{SL}(2, \mathbb{C})$-action on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$, that is, the variety whose coordinate ring is the ring of invariants of $\text{SL}(2, \mathbb{C})$ acting on the coordinate ring $\mathbb{C}[\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))]$ of $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$. The natural map
$$\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C}) \longrightarrow \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$$
is surjective, but not injective. However $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$ may be identified with the set of semistable orbits. Thus the inclusion
$$\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^{ss} \hookrightarrow \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$$
induces a bijection
$$\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^{ss}/\text{SL}(2, \mathbb{C}) \longleftrightarrow \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C}).$$
The group $\text{SL}(2, \mathbb{C})$ acts freely and properly on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s$. The quotient $X := \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s / \text{SL}(2, \mathbb{C})$ is thus a $(6g-6)$-dimensional complex manifold which embeds as a Zariski open subset of the categorical quotient $\text{Hom}(\pi, \text{SL}(2, \mathbb{C})) / \text{SL}(2, \mathbb{C})$. Thus the map

$$X = \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s / \text{SL}(2, \mathbb{C}) \rightarrow \text{Hom}(\pi, \text{SL}(2, \mathbb{C})) / \text{SL}(2, \mathbb{C})$$

induced by $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s \rightarrow \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ is an embedding onto an open subset. Thus $X$ is a smooth irreducible complex quasi-affine variety which is dense in the quotient $\text{Hom}(\pi, \text{SL}(2, \mathbb{C})) / \text{SL}(2, \mathbb{C})$.

1.2. Symplectic geometry of deformation spaces. By the general construction of [9], $X$ has a natural complex-symplectic structure $\Omega$, which is $\Gamma$-invariant. Furthermore $\Gamma$ is algebraic in the sense that there exists an algebraic tensor field on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ inducing $\Omega$. (See [9] for an explicit formula.)

The Zariski tangent space $T_\phi \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ is the space

$$Z^1(\pi, \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\phi})$$

of 1-cocycles $\pi \rightarrow \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\phi}$ and the tangent space $T_\phi(G \cdot \phi)$ of the $G$-orbit equals the subspace

$$B^1(\pi, \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\phi}) \subset Z^1(\pi, \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\phi})$$

of 1-coboundaries. (Compare Raghunathan [40].) The quotient vector space is the cohomology

$$H^1(\pi, \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\phi})$$

which, under the de Rham isomorphism is isomorphic to the cohomology $H^1(M; V_\phi)$ where $V_\phi$ denotes the flat vector bundle over $M$ corresponding to the $\pi$-module $\mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\phi}$.

Let $\mathbb{B} : \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathbb{C}$ be the trace form of the standard representation on $\mathbb{C}^2$: if $\alpha, \beta \in \mathfrak{sl}(2, \mathbb{C})$, then the inner product is defined as

$$\mathbb{B}(\alpha, \beta) := \text{tr}(\alpha \beta).$$

Since $\mathbb{B}$ is $\text{Ad}$-invariant, it defines a bilinear pairing of $\pi$-modules

$$\mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\phi} \times \mathfrak{sl}(2, \mathbb{C})_{\text{Ad}_\phi} \rightarrow \mathbb{C}$$

or equivalently flat vector bundles

$$V_\phi \times V_\phi \rightarrow \mathbb{C}.$$ 

Cup-product defines a bilinear pairing

$$\Omega_\phi : H^1(M; V_\phi) \times H^1(M; V_\phi) \rightarrow \mathbb{C}$$

with coefficients paired by $\mathbb{B}$. Symmetry of $\mathbb{B}$ implies that $\Omega_\phi$ is skew-symmetric. Since $\mathbb{B}$ is nondegenerate, $\Omega_\phi$ is nondegenerate. It
follows from [3] that $\Omega_\phi$ can be expressed as an algebraic tensor on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$, and thus is a holomorphic exterior 2-form. By arguments of [9] or Karshon [29], Weinstein [45], Guruprasad-Huebschmann-Jeffrey-Weinstein [19], $\Omega$ is closed. Thus $\Omega$ is a nondegenerate closed holomorphic $(2,0)$-form, that is a complex-symplectic structure.

1.3. The mapping class group. The automorphism group $\text{Aut}(\pi)$ of $\pi$ acts algebraically on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ commuting with the action of $\text{SL}(2, \mathbb{C})$. The normal subgroup $\text{inn}(\pi)$ of inner automorphisms acts trivially on the quotient $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$. The quotient $\text{Out}(\pi) = \text{Aut}(\pi)/\text{inn}(\pi)$, acts on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$, leaving invariant the subset $X$. Furthermore the mapping class group $\Gamma := \pi_0(\text{Diff}^+(M))$ of $M$ is isomorphic to $\text{Out}(\pi)$ by Nielsen [37]. The algebraic complex-symplectic structure on $X$ is $\Gamma$-invariant.

1.4. Ergodicity and its holomorphic analog. The subset $X_U$ of unitary characters is invariant under $\Gamma$. Furthermore the complex-symplectic structure restricts to a (real) symplectic structure on $X_U$ which is the Kähler form for a Kähler structure on $X_U$. In particular the Lebesgue measure class on $X_U$ is invariant under $\Gamma$. The main result of [15] is that this action is ergodic, that is every $\Gamma$-invariant measurable function on $X_U$ is constant almost everywhere.

Ergodicity no longer holds for the $\text{SL}(2, \mathbb{C})$-character variety $X$ (see §4.3). However ergodicity on $X_U$ does imply the following property of holomorphic functions on $X$:

**Theorem 1.4.1.** A $\Gamma$-invariant meromorphic function $X \xrightarrow{h} \mathbb{C}P^1$ is constant.

**Proof.** The restriction of $h$ to $X_U$ is a $\Gamma$-invariant measurable function, and by the main result of Goldman [15], $h$ must be constant almost everywhere. Since $h$ is continuous, it is constant.

Now we argue that in local holomorphic coordinates, $X_U$ is equivalent to $\mathbb{R}^n \subset \mathbb{C}^n$ and a holomorphic function constant on $X_U$ must be globally constant on $X$. (Compare, for example, Lemma 1 of §2.3 of Platis [39].) For the reader’s convenience, we supply a brief proof. Let $U$ be a nonempty open coordinate neighborhood with local holomorphic coordinates $z = (z^1, \ldots, z^n)$. In local holomorphic coordinates, $X_U \cap U$ is described by

$$z \in \mathbb{R}^n \subset \mathbb{C}^n$$

and $h$ is given by a power series

$$h(z) = \sum_{k=0}^{\infty} a_k z^k$$
which converges in the nonempty open set $z(\mathcal{U}) \in \mathbb{C}^n$. Since the restriction of $h$ to a $z(\mathcal{U}) \cap \mathbb{R}^n$ is constant, $a_k = 0$ for $k > 0$ and thus $h$ is constant on $\mathcal{U}$. Since $X$ is connected, analytic continuation implies that $h$ must be constant. \hfill \square

We do not know whether $X - X_U$ admits $\Gamma$-invariant nonconstant meromorphic functions.

2. The Hamiltonian vector field of a character function

Corresponding to free homotopy classes $\alpha$ of closed curves on $M$ are complex regular functions $f_\alpha : X \rightarrow \mathbb{C}$. The complex-symplectic structure associates to these functions complex Hamiltonian vector fields $\text{Ham}(f_\alpha)$, which generate holomorphic local flows on $X$. When $\alpha$ corresponds to a simple closed curve, we define complex twist flows on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ which cover these holomorphic local flows on $X$. We begin with an preliminary section on the traces in $\text{SL}(2, \mathbb{C})$.

2.1. The variation of the trace function on $\text{SL}(2, \mathbb{C})$. Let

$$f : \text{SL}(2, \mathbb{C}) \rightarrow \mathbb{C}$$

be the trace function $f(P) = \text{tr}(P)$ and $\mathbb{B}$ the trace form

$$\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathbb{C}
\quad (X,Y) \mapsto \text{tr}(XY).$$

As in Goldman [11], the differential of $f$ and the orthogonal structure $\mathbb{B}$ determines a variation function

$$F : \text{SL}(2, \mathbb{C}) \rightarrow \mathfrak{sl}(2, \mathbb{C})$$

characterized by the identity

$$\left. \frac{d}{dt} \right|_{t=0} f(P \exp(tX)) = \mathbb{B}(F(P), X).$$

This function is defined by:

$$F(P) = P - \frac{\text{tr}(P)}{2} \mathbb{I}$$

and corresponds to the composition of the inclusion

$$\text{SL}(2, \mathbb{C}) \hookrightarrow \mathfrak{gl}(2, \mathbb{C})$$

with orthogonal projection

$$\mathfrak{gl}(2, \mathbb{C}) \twoheadrightarrow \mathfrak{sl}(2, \mathbb{C})$$
(orthogonal with respect to the trace form $\mathbb{B}$ on $\mathfrak{sl}(2, \mathbb{C})$). Invariance of the trace
\[ f(QPQ^{-1}) = f(P) \]
and $\text{Ad}$-invariance of the orthogonal structure $\mathbb{B}$
\[ \mathbb{B}(\text{Ad}(Q)X, \text{Ad}(Q)Y) = \mathbb{B}(X, Y) \]
implies $\text{Ad}$-equivariance of its variation:
\[ F(QPQ^{-1}) = \text{Ad}(Q)F(P). \]
Taking $Q = P$ shows that $F(P)$ lies in the centralizer of $P$ in $\mathfrak{sl}(2, \mathbb{C})$. In particular the complex one-parameter subgroup of $\text{SL}(2, \mathbb{C})$
\[ \zeta_t := \exp(tF(P)) \]
centralizes $A$ in $\text{SL}(2, \mathbb{C})$.

For example, if $P$ is the diagonal matrix
\[ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \]
then $f(P) = \lambda + \lambda^{-1}$ and
\[ F(P) = \frac{\lambda - \lambda^{-1}}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]
with corresponding complex one-parameter subgroup:
\[ \zeta_t = \begin{bmatrix} e^{t(\lambda-\lambda^{-1})/2} & 0 \\ 0 & e^{-t(\lambda-\lambda^{-1})/2} \end{bmatrix}. \]
If $P$ is $\pm$ the unipotent matrix
\[ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]
then $f(P) = \pm 2$ and
\[ F(P) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \]
with corresponding complex one-parameter subgroup:
\[ \zeta_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

The following lemma, whose proof is immediate from the above calculations, will be needed in the sequel. Recall that for any subset $S \subset \text{SL}(2, \mathbb{C})$, $\text{Fix}(S)$ denotes the subset of $\mathbb{CP}^1$ fixed by $S$.

**Lemma 2.1.1.** If $P \in \text{SL}(2, \mathbb{C})$, then $\text{Fix}(\zeta_t) = \text{Fix}(P)$ or $\zeta_t = \pm \mathbb{I}$
2.2. Character functions. Let $\alpha \in \pi$. The character function

$$\text{Hom}(\pi, \text{SL}(2, \mathbb{C})) \xrightarrow{\tilde{f}_\alpha} \mathbb{C}$$

$$\phi \mapsto f(\phi(\alpha))$$

is a regular function. Since $f$ is a class function, $\tilde{f}_\alpha$ is invariant under $\text{SL}(2, \mathbb{C})$ and defines a regular function

$$f_\alpha : X \rightarrow \mathbb{C}.$$ 

Let $[\phi] \in X$. Then the tangent space to $X$ equals $H^1(M; V_\phi)$, so the Hamiltonian vector field $\text{Ham}(f_\alpha)$ associates to $[\phi] \in X$ a tangent vector in $H^1(M; V_\phi)$.

Cap product with the fundamental homology class $[M] \in H_2(M; \mathbb{C})$ defines the Poincaré duality isomorphism:

$$\cap [M] : H^1(M; V_\phi) \xrightarrow{\cong} H_1(M; V_\phi).$$

Choose a basepoint $x_0 \in M$, an isomorphism of the fiber of $V_\phi$ over $x_0$ with $\mathbb{C}^2$, and a representative holonomy homomorphism

$$\phi_0 : \pi_1(M; x_0) \rightarrow \text{SL}(2, \mathbb{C}).$$

Let $s_0 \in S^1$ be a basepoint. Let $\alpha_0 : (S^1, s_0) \rightarrow (M, x_0)$ be a based loop in $M$ corresponding to $\alpha$. Let $\sigma$ be the parallel section of the flat vector bundle $\alpha_0^* V_\phi$ over $S^1$ which equals $F(\phi_0(\alpha_0))$ at $s_0$. Then $\sigma$ defines a $V_\phi$-valued 1-cycle in $M$, with homology class

$$[\sigma] \in H_1(M; V_\phi).$$

**Lemma 2.2.1.** The value of the Hamiltonian vector field $\text{Ham}(f_\alpha)$ at a point $[\phi] \in X$ is the vector in

$$T_{[\phi]} X \cong H^1(M; V_\phi)$$

corresponding to the Poincaré dual of the homology class of the $V_\phi$-valued cycle $\sigma$:

$$\cap [M] : H^1(M; V_\phi) \rightarrow H_1(M; V_\phi)$$

$$\text{Ham}(f_\alpha) \mapsto [\sigma].$$

Using the above formula and the duality between cup-product and intersections of cycles, we obtain a formula for the Poisson bracket of trace functions ([11]):

**Proposition 2.2.2.** Let $\alpha, \beta$ be oriented closed curves meeting transversely in double points $p_1, \ldots, p_k$. For each $p_i$, choose representatives

$$\phi_i : \pi_1(M; p_i) \rightarrow \text{SL}(2, \mathbb{C}).$$
Let $\alpha_i$ and $\beta_i$ be the elements of $\pi_1(M; p_i)$ representing $\alpha, \beta$ respectively. Then the Poisson bracket of functions $f_{\alpha}, f_{\beta}$ is:

$$\{f_{\alpha}, f_{\beta}\} = \Omega(\Ham(f_{\alpha}), \Ham(f_{\beta}))$$

$$= \sum_{i=1}^{k} \epsilon(p_i; \alpha, \beta) B(F(\phi_i(\alpha_i)), F(\phi_i(\beta_i)))$$

where $\epsilon(p_i; \alpha, \beta)$ denotes oriented intersection number.

Corollary 2.2.3. If $\alpha, \beta$ are disjoint, then $f_{\alpha}$ and $f_{\beta}$ Poisson-commute.

By arguments in [11] (based on a suggestion of S. Wolpert), the converse holds if one of $\alpha$ or $\beta$ is simple. A purely topological proof of this fact has recently been given by Chas [5].

2.3. Complex twist flows. Suppose that $\alpha$ is represented by a simple closed curve $A$. Then a holomorphic $\mathbb{C}$-action on the representation variety $\Hom(\pi, \SL(2, \mathbb{C}))$ covers the complex Hamiltonian flow on $X$. The complex twist flow $\tau_\alpha$ is the holomorphic action of $\mathbb{C}$ of $\Hom(\pi, \SL(2, \mathbb{C}))$ defined as follows.

There are two cases, depending on whether $A$ separates $M$ or not. Denote by $M|A$ the compact surface with boundary whose interior is homeomorphic to the complement $M - A$. Denote the two components of $\partial(M|A)$ by $A_+$ and $A_-$. The quotient map $q : M|A \rightarrow A$ results from identifying these components by a homeomorphism $\eta : A_+ \rightarrow A_-.$

2.3.1. The nonseparating case. If $A$ is nonseparating, then $M|A$ is connected, and choosing a basepoint in $A_+$ we express $\pi$ as an HNN-extension

$$\pi_1(M|A) *_t$$

as follows. Choose a basepoint $x_0 \in A$, and lift it to a basepoint $\tilde{x}_0 \in A_+ \subset M|A$. Let $\alpha_+$ denote the element of $\pi_1(M|A; \tilde{x}_0)$ corresponding to $A_+$. Let $\tilde{\beta}$ denote a simple arc joining $\tilde{x}_0$ to $\eta(\tilde{x}_0)$ in $M|A$, and $\alpha_-$ the based loop $\tilde{\beta}^{-1} \cdot \alpha_+ \cdot \tilde{\beta}$ at the basepoint $\tilde{x}_0$. Then $\eta$ corresponds to the isomorphism

$$t : \langle \alpha_+ \rangle \rightarrow \langle \alpha_- \rangle$$

between the two subgroups of $\pi_1(M|A)$. Let $N$ be the normal closure of the element

$$\beta \alpha_+ \beta^{-1} (\alpha_-)^{-1}$$
of $\pi_1(M|A) \ast \langle \beta \rangle$. Then the HNN-extension $\pi_1(M|A) \ast_t$ is defined as the quotient

$$\pi \cong \left( \pi_1(M|A) \ast \langle \beta \rangle \right)/N.$$  

As in [11], for any $\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$, and $t \in \mathbb{C}$, define

$$(2.5) \quad \xi_t(\phi) : \gamma \mapsto \begin{cases} 
\phi(\gamma) & \text{if } \gamma \in \pi_1(M|A) \\
\phi(\beta) \zeta_t & \text{if } \gamma = \beta.
\end{cases}$$

where

$$\zeta_t = \exp \left( tF(\phi(\alpha_+)) \right)$$

is the one-parameter subgroup corresponding to $\phi(\alpha_+)$.  

2.3.2. The separating case. If $A$ separates $M$, then let $M_+$ and $M_-$ denote the two components of $M|A$. Then $\pi$ is the free product of subgroups corresponding to $\pi_+(M_+)$ and $\pi_-(M_-)$ respectively, amalgamated over the images $\langle \alpha \rangle \rightarrow \pi_1(M_{\pm})$. Then as in [11], for any $\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$, $t \in \mathbb{C}$, define

$$\xi_t^\alpha(\phi) : \gamma \mapsto \begin{cases} 
\phi(\gamma) & \text{if } \gamma \in \pi_1(M_+) \\
\zeta_t \phi(\gamma) \zeta_{-t} & \text{if } \gamma \in \pi_1(M_-)
\end{cases}$$

where

$$\zeta_t = \exp \left( tF(\phi(\alpha)) \right)$$

is the one-parameter subgroup corresponding to $\phi(\alpha)$.  

2.3.3. Irreducibility. To show that the Hamiltonian flows act on $X$, we need to show that the complex twist flows preserve the set of irreducible representations. 

**Lemma 2.3.1.** Suppose that $\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s$. Then

$$\xi_t^\alpha(\phi) \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s.$$  

**Proof.** Let $M'$ be a component of $M|A$. As above, choose basepoints and identifications to express $\pi$ as an amalgamated free product or HNN construction with $\pi_1(M') \hookrightarrow \pi_1(M)$. Then

$$\xi_t^\alpha(\phi(\pi)) \supset \phi(\pi_1(M'))$$

implies that if $\phi_{\pi_1(M')}^\alpha$ is irreducible, then so is $\xi_t^\alpha(\phi)$. Thus we may assume that $\phi_{\pi_1(M')}^\alpha$ is reducible but $\phi$ is irreducible.  

Suppose first that $A$ is nonseparating. We assume $\xi_t^\alpha$ is reducible and derive a contradiction. Let $\Phi = \text{Fix}(\phi_{\pi_1(M|A)})$. Since

$$\phi(\alpha_+) \in \phi(\pi_1(M|A)),$$
the element $\phi(\alpha_+)\text{ fixes } \Phi$, and Lemma 2.1.1 implies that $\zeta_t$ fixes $\Phi$. Since $\xi^t_\alpha \phi(\pi)$ is reducible and is generated by $\phi(\pi_1(M|A))$ and $\phi(\beta)$, it follows that

$$
\xi^t_\alpha \phi(\beta) = \phi(\beta) \zeta_t
$$

fixes no element of $\Phi$. Since $\zeta_t$ fixes $\Phi$, this implies that $\phi(\beta)$ fixes $\Phi$, a contradiction.

Suppose finally that $A$ separates $M$ into two components $M_\pm$. By the remark above, we may assume that $\phi(\pi_1(M_\pm))$ is reducible. Let $\Phi_\pm = \text{Fix}(\phi(\pi_1(M_\pm)))$.

We may assume that each $\Phi_\pm$ is nonempty but $\Phi_+ \cap \Phi_- = \emptyset$. Now $\phi(\alpha)$ fixes each $\Pi_\pm$, and Lemma 2.1.1 implies that $\zeta_t$ fixes each $\Pi_\pm$. Since $\phi(\pi_1(M_+))$ and $\zeta_t \phi(\pi_1(M_-)) \zeta_t^{-1}$ generate $\xi^t_\alpha \phi(\pi)$,

$$
\text{Fix}(\xi^t_\alpha \phi(\pi)) = \text{Fix}(\phi(\pi_1(M_+))) \cap \text{Fix}(\zeta_t \phi(\pi_1(M_-)) \zeta_t^{-1})
$$

$$
= \text{Fix}(\phi(\pi_1(M_+))) \cap \zeta_t \text{Fix}(\phi(\pi_1(M_-)))
$$

$$
= \Phi_+ \cap \zeta_t \Phi_-
$$

$$
= \Phi_+ \cap \Phi_- = \emptyset
$$

as desired. \qed

2.4. {f Gauge-Theoretic Interpretation.} These twist flows admit an interpretation in terms of flat connections. The character variety $X$ identifies with the quotient of the space $\mathcal{F}^{irr}(E)$ of irreducible flat $\text{SL}(2,\mathbb{C})$-connections on a principal $\text{SL}(2,\mathbb{C})$-bundle $E$ over $M$ by the group $\mathcal{G}(E)$ of gauge transformations of $E$. (Since a principal $\text{SL}(2,\mathbb{C})$-bundle over $M$ which admits a flat connection is necessarily trivial, we may assume $E$ is the product bundle.)

Let $A \subset M$ be a simple closed curve, and choose a basepoint $a_0 \in A$. Choose an orientation on $A$ and let $\alpha$ be a based loop on $M$ corresponding to the orientation on $A$.

Pull $E$ back to a principal $\text{SL}(2,\mathbb{C})$-bundle $q^*E$ over $M|A$ by the quotient map $q : M \rightarrow M|A$. Choose a point $e_0$ in the fiber of $E$ over $a_0$. Let $a_\pm$ be the two elements of $q^{-1}(a_0)$ in $A_\pm$ respectively and $e_\pm$ the elements in the fibers of $q^*E$ over $a_\pm$ corresponding to $e_0$.

Define the twist flow $\tilde{\xi}_t$ on $\mathcal{F}^{irr}(E)$ as follows. Let $\nabla \in \mathcal{F}^{irr}(E)$ be a flat connection on $E$. Parallel transport of $e_0$ along $\alpha$ with respect to $\alpha$ defines a holonomy transformation $\phi_0(\alpha_0)$ as above. Let $\zeta_t$ denote the corresponding one-parameter subgroup of $\text{SL}(2,\mathbb{C})$. There is a one-parameter family of gauge-transformations $g_t$ of $q^*E$ supported in a collar neighborhood $N$ of $A_+ \subset M|A$ assuming the “value” $\zeta_t$ on $A_+$. 

Explicitly, choose a smooth embedding
\[ \psi : [0, 1] \times S^1 \rightarrow N \hookrightarrow M|A \]
mapping \( \{1\} \times S^1 \) diffeomorphically to \( A_+ \). Let \( r : [0, 1] \rightarrow S^1 \) denote the quotient mapping identifying \( 0, 1 \in [0, 1] \). Parallel transport of \( e_+ \) defines a trivialization of the pullback of \( E \) to \([0, 1] \times [0, 1]\) so that \( \psi^*E \) identifies with the quotient of
\[ [0, 1] \times [0, 1] \times \text{SL}(2, \mathbb{C}) \]
by the identification
\[ (s, 0, h) \leftrightarrow (s, 1, \phi_0(\alpha_0)h) \]
Define \( g_t \) as the gauge transformation which is the identity map on the complement of \( N \) in \( M|A \), and equals
\[ (s, \theta, h) \rightarrow (s, \theta, \zeta_{s,t}h) \]
in this trivialization.
Since \( \zeta_t \) centralizes the holonomy of \( q^*\nabla \) along \( A_+ \), the identification map \( \eta : A_- \rightarrow A_+ \) identifies the restrictions of \((g_t)^*(q^*\nabla)\) to \( A_\pm \). Thus a unique flat connection \( \tilde{\xi}_t(\nabla) \) exists, satisfying
\[ q^*(\tilde{\xi}_t(\nabla)) = (g_t)^*(q^*\nabla). \]
This is the orbit of the twist flow on flat connections. Clearly the gauge transformation \( g_t \) does not arise from a gauge transformation of \( E \), unless the holonomy along \( A \) is \( \pm I \). The orbit covers the orbit of the holonomy of \( \nabla \) under the twist flow in \( X \cong \mathcal{F}^{\text{irr}}(E)/\mathcal{G}(E) \).

The flow \( \tilde{\xi} \) on \( \mathcal{F}^{\text{irr}}(E) \) depends only on the choice of the collar neighborhood \( N \subset M|A \) and \( \psi : [0, 1] \times S^1 \rightarrow N \).

2.5. Periods. The subspace \( \text{Hom}(\pi, \text{SU}(2))^s \) maps to a (real) symplectic submanifold \( X_U \subset X \), and the corresponding real flows define Hamiltonian systems which have been studied in [11], and are closely related to periodic flows studied by Jeffrey-Weitsman [21, 25, 26, 27]. The orbits of these flows are all closed, although the period of the orbit varies with the value of \( f \).

Proposition 2.5.1. Let \( [\phi] \in X_U \) and let \( \alpha \in \pi \) be represented by a simple loop \( A \). If \( f_\alpha([\phi]) \neq \pm 2 \), the period of the trajectory of the flow of \( \text{Ham}(f_\alpha) \) at \( [\phi] \) equals
\[ \frac{4\pi}{\sqrt{4 - f_\alpha([\phi])^2}} \]
if $A$ is nonseparating and

$$\frac{2\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

if $A$ separates. If $f_\alpha([\phi]) = \pm 2$, then $[\phi]$ is fixed under the flow of $\text{Ham}(f_\alpha)$.

Proof. Let $P \in \text{SU}(2)$. Apply an inner automorphism of $\text{SU}(2)$ to assume that $P$ is diagonal:

$$P = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$ (2.6)

Then

$$f(P) = 2 \cos(\theta), \quad F(P) = \begin{bmatrix} i \sin(\theta) & 0 \\ 0 & -i \sin(\theta) \end{bmatrix}$$

with corresponding one-parameter subgroup

$$\zeta_t = \exp(tF(P)) = \begin{bmatrix} e^{i \sin(\theta)t} & 0 \\ 0 & e^{-i \sin(\theta)t} \end{bmatrix}$$

Since the $\text{PSL}(2, \mathbb{C})$-action on $\text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s$ is proper and free, the quotient map is a principal $\text{PSL}(2, \mathbb{C})$-fibration:

$$\text{PSL}(2, \mathbb{C}) \longrightarrow \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))^s \longrightarrow X.$$  

Thus the trajectory of the flow $\xi^\alpha_t$ projects diffeomorphically to the trajectory of the flow of $\text{Ham}(f_\alpha)$ on $X$. The period equals the infimum of all $t > 0$ such that

$$\xi^\alpha_t(\phi) = \phi.$$

If $\alpha$ is nonseparating, then (2.5) implies that $T$ equals the infimum of all $t > 0$ such that

$$\zeta_t \phi(\beta) = \phi(\beta).$$

By (2.2),

$$T = \frac{2\pi}{\sin(\theta)} = \frac{4\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

If $\alpha$ separates, then (2.6) implies that $T$ equals the infimum of all $t > 0$ such that

$$\zeta_t \phi(\beta) \zeta_{-t} = \phi(\beta).$$

for all $\beta \in \pi_1(M_-)$. If $\phi(\beta) \in \{\pm I\} = \text{center(SU}(2))$
for all $\beta \in \pi_1(M_\pm)$, then $\phi(\alpha) = 1$. Since $\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$, there exists $\beta \in \pi$ such that $\phi(\beta) \neq 1$. The period $T$ is the infimum of all $t > 0$ such that $\zeta_t = \pm 1$. By (2.2),

$$T = \frac{\pi}{\sin(\theta)} = \frac{2\pi}{\sqrt{4 - f_\alpha([\phi])^2}}$$

\[\square\]

2.6. **Dehn twists.** Closely related to the twist flows are the maps of $X_U$ induced by Dehn twists. Recall that the Dehn twist about a simple loop $A \subset M$ is the diffeomorphism supported in a tubular neighborhood of $A$. If

$$\psi : [0, 1] \times S^1 \hookrightarrow M$$

$$\{\frac{1}{2}\} \times S^1 \hookrightarrow A$$

is such a tubular neighborhood, then the Dehn twist $\tau_A$ about $A$ is $\psi$-related to the diffeomorphism of $[0, 1] \times S^1$ (restricting to the identity on the boundary) defined by:

$$(s, e^{i\theta}) \mapsto (s, e^{2\pi is}e^{i\theta}).$$

The action of $\tau_A$ on the fundamental group is given by:

$$(\tau_A)_* (\phi) : \gamma \mapsto \begin{cases} 
\phi(\gamma) & \text{if } \gamma \in \pi_1(M\mid A) \\
\phi(\beta) \phi(\alpha) & \text{if } \gamma = \beta.
\end{cases}$$

if $A$ is nonseparating, and by:

$$(\tau_A)_* (\phi) : \gamma \mapsto \begin{cases} 
\phi(\gamma) & \text{if } \gamma \in \pi_1(M_+) \\
\phi(\alpha)\phi(\gamma)\phi(\alpha)^{-1} & \text{if } \gamma \in \pi_1(M_-)
\end{cases}$$

if $A$ separates.

Let $P$ be as in (2.6). Comparing (2.6) with (2.2), $\zeta_t = P$ for

$$\theta = \sin(\theta)t,$$

that is,

$$t = \frac{\theta}{\sin(\theta)} = \frac{\cos^{-1}(f(P)/2)}{2\pi \sqrt{4 - f(P)^2}}$$

(2.7)

Combining (2.1), (2.7) with (2.7) implies:
Proposition 2.6.1. Let \( \alpha \in \pi \) correspond to a simple closed curve \( \gamma \subset M \). The map \((\tau_\gamma)_*\) on \( \text{Hom}(\pi, \text{SL}(2, \mathbb{C})) \) induced by Dehn twist about \( \gamma \) equals the time-\( t \) map of the flow generated by \( \text{Ham}(f_\alpha) \), where

\[
t = \frac{2 \cos^{-1}(f_\alpha([\phi])/2)}{\sqrt{4 - f_\alpha([\phi])^2}}.
\]

In terms of a parametrization \( \mathbb{R}/\mathbb{Z} \to X_U \) of this trajectory, \((\tau_\gamma)_*\) acts by translation of

\[
\frac{\theta}{2\pi} = \frac{\cos^{-1}(f(P)/2)}{2\pi}
\]

which has infinite order for almost every value of \( f(A) \).

By reparametrizing the flow, we obtain Hamiltonian flows whose time-one map is the identity map or the Dehn twist. Jeffrey-Weitsman [25] (§5.1), consider flows of Hamiltonians given by invariant functions

\[
\theta(P) = \cos^{-1}\left(\frac{f(P)}{2}\right)
\]

which define \( S^1 \)-actions, but are undefined at \( P = \pm I \). Thus their flows are only defined on the on the dense open subset where \( \phi(P) \) is not central. On the other hand their flows are periodic with period \( 2\pi \) if \( A \) is nonseparating, and period \( \pi \) if \( A \) separates. (Actually they work with the symplectic form which is \( 1/(4\pi^2) \) of ours, to obtain a 2-form with integral cohomology class. Thus their periods are \( 1/(2\pi) \) if \( A \) is nonseparating and \( 1/(4\pi) \) if \( A \) separates.)

2.6.1. \textit{Orbits of complex twist flows.} On the \( \text{SL}(2, \mathbb{C}) \)-character variety \( X \), the Hamiltonian vector field \( \text{Ham}(f_\alpha) \) generates a holomorphic \( \mathbb{C} \)-action. (2.4) implies that if \( f(A) = \pm 2 \) and \( A \neq \pm \mathbb{I} \) (that is, \( A \) is \( \pm \)-unipotent), then the trajectory defines an injective holomorphic map

\[
(2.8) \quad \mathbb{C} \to X
\quad t \mapsto \xi_t^\alpha([\phi]).
\]

If \( f(A) \neq \pm 2 \), then (2.3) implies that \( \zeta_t = 1 \) whenever \( t \in t_0\mathbb{Z} \) where

\[
(2.9) \quad t_0 := \frac{4\pi i}{\lambda - \lambda^{-1}} = \frac{4\pi i}{(f(A)^2 - 4)^{1/2}}.
\]
(which is well-defined only up to sign). In this case the trajectory of $[\phi]$ is the image of the holomorphic embedding

$$e^z \mapsto \xi^{\left(\frac{2z}{\lambda-\lambda^{-1}}\right)} \left([\phi]\right).$$

2.6.2. **Dehn twists.** At a point $[\phi] \in X$ where $f_\alpha(\phi) = \pm 2$, then either

- $\phi(\alpha) = \pm \mathbb{I}$ is a central element and $[\phi]$ is fixed under the entire $\mathbb{C}$-action, or
- $\phi(\alpha)$ is $\pm$ a parabolic element and (2.14) implies $\phi(\alpha) = \pm \exp F(\alpha)$. In the latter case the embedding (2.10) is equivariant with respect to the action of $\mathbb{Z}$ by translation on $\mathbb{C}$ and the $\mathbb{Z}$-action generated by $(\tau_\alpha)_*$ on $X$.

Suppose that $f_\alpha([\phi]) \neq \pm 2$. In that case $\zeta_t = P$ precisely when $t \equiv t_1 \pmod{t_0}$, where

$$t_1 = \frac{\log(\lambda)}{\lambda - \lambda^{-1}} = \frac{\log \left( (f(A) \pm (f(A)^2 - 4)^{1/2})/2 \right)}{(f(A)^2 - 2)^{1/2}}.$$  

(The two choices for $(f(A)^2 - 4)^{1/2}$ differ by sign but determine equal values for $t_1$.) In that case the embedding (2.10) is equivariant with respect to the actions generated by multiplication by

$$\lambda = \frac{(f(A) \pm (f(A)^2 - 4)^{1/2})}{2}$$

on $\mathbb{C}^*$ and by $(\tau_\alpha)_*$ on $X$.

2.6.3. **Fenchel-Nielsen twist flows for $G = \text{SL}(2, \mathbb{R})$.** Since the complex trace form $\mathcal{B}$ on $\mathfrak{sl}(2, \mathbb{C})$ restricts to the trace form on $\mathfrak{sl}(2, \mathbb{R})$, the complex-symplectic structure $\Omega$ on $X$ restricts to the symplectic structure on $\mathfrak{F}(M)$ defined by the trace form of $\text{SL}(2, \mathbb{R})$. By (2) this symplectic structure equals (-2) the Weil-Petersson Kähler form.

Inside the complex twist flows are the **Fenchel-Nielsen twist flows** when the holonomy is hyperbolic. Suppose that $\phi \in \text{Hom}(\pi, \text{SL}(2, \mathbb{R}))$ is a discrete embedding (that is, a Fuchsian representation). The subset of $X$ consisting of equivalence classes of Fuchsian representations corresponds bijectively to the space of marked hyperbolic structures on $M$, that is, the Teichmüller space $\mathfrak{T}(M)$. In that case, for every $\mathbb{I} \neq \alpha \in \pi$, the element $\phi(\alpha) \in \text{SL}(2, \mathbb{R})$ is hyperbolic. Geometrically it is a *transvection* along a geodesic $\gamma$, and is conjugate to a diagonal matrix

$$P = \pm \begin{bmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{bmatrix}.$$
$l$ is the distance $\phi(\alpha)$ moves points along $\gamma$. In the hyperbolic structure on $M$, the element $\alpha$ corresponds to a homotopy class of closed loops; $\gamma$ corresponds to the unique closed geodesic $A$ freely homotopic to a free loop in $\alpha$ and $l$ is the length of this geodesic.

The character function is related to the length function by:

$$f = 2 \cosh(l/2).$$

Thus the invariant function $l$ (which is only defined on the subset of $\text{SL}(2, \mathbb{R})$ consisting of hyperbolic elements) defines functions

$$l_\alpha : \mathfrak{T}(M) \rightarrow \mathbb{R}_+.$$

As in [11], the corresponding variation is

$$L(P) = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

with corresponding one-parameter subgroup

$$\zeta_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix},$$

the one-parameter subgroup of transvections moving at speed 2 along the geodesic.

The inner product of these infinitesimal transvections can be computed as follows. The matrices

$$L_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} -\sin(\psi) & \cos(\psi) \\ \cos(\psi) & \sin(\psi) \end{bmatrix}$$

represent infinitesimal transvections about axes which intersect with angle $\psi$ and

$$\mathbb{B}(L_1, L_2) = \text{tr}(L_1 L_2) = 2 \cos(\psi).$$

Thus (as in §3.9 of [11]), Wolpert’s cosine formula follows from Proposition 2.2.2.

If $\alpha$ corresponds to a simple closed curve $A$, then the corresponding twist flow $\eta^\alpha_t$ corresponds to the *Fenchel-Nielsen twist flow* on $\mathfrak{T}(M)$ defined geometrically as follows. (Compare Abikoff [11], Buser [11], and Harvey [20].) Represent $A$ by the (necessarily simple) closed geodesic. Split $M$ along $A$, and identify the boundary components $A_+, A_-$ of $M|A$ by a translation of length $t$ in the positively oriented direction. This is a well-defined path of marked hyperbolic structures in $\mathfrak{T}(M)$; see Wolpert [16] for details. In [16], Wolpert proves that the Fenchel-Nielsen twist flow is Hamiltonian with respect to the Weil-Petersson Kähler form on $\mathfrak{T}(M)$, with Hamiltonian potential the length function $l_\alpha$ [17].
The Fenchel-Nielsen twist flow \( \eta^\alpha \) and the twist flow \( \xi^\alpha \) are re-parametrizations of one another. Specifically,

\[
\xi^\alpha_t = \eta^\alpha_{\sinh(l^\alpha)}.
\]

As in [9] (see also [11]), the Weil-Petersson Kähler form equals \(-\frac{1}{2}\) the symplectic structure defined by the trace form \( B \) restricted to \( \text{SL}(2, \mathbb{R}) \), implying the Fenchel-Nielsen flow is Hamiltonian. Since the Fenchel-Nielsen twist flow is Hamiltonian for the length function \( l^\alpha \), the derivative of the length function along the twist vector field equals the Poisson bracket \( \{ l^\alpha, l^\beta \} \), which can be computed by Proposition 2.2.2.

We thus obtain Wolpert’s Derivative Formula:

**Theorem 2.6.2** (Wolpert [17, 16]). Let \( \alpha, \beta \in \pi \) where \( \alpha \) is represented by a simple closed curve. Then the derivative of the length function \( l^\beta \) with respect to the Fenchel-Nielsen twist flow equals the sum

\[
\sum_{i=1}^{k} \cos(\theta_i)
\]

where \( \alpha \) and \( \beta \) are represented by closed geodesics, \( p_1, \ldots, p_k \) are their intersection points, and \( \theta_i \) is the angle from \( \alpha \) to \( \beta \) at \( p_i \).

In particular the action of the Dehn twist \( \tau_A \) equals the time-\( t \) map of the Fenchel-Nielsen twist flow at time \( t = l/2 \).

### 3. Abelian Hamiltonian Actions

The classical Fenchel-Nielsen coordinates on \( \mathfrak{T}(M) \) can be interpreted as a moment map arising from a pants decomposition, that is, a maximal collection of disjoint simple closed curves \( P = \{ \alpha_1, \ldots, \alpha_N \} \) on \( M \) which are each homotopically nontrivial and mutually nonhomotopic. If \( M \) has genus \( g \), then \( N = 3g - 3 \) and the length functions define a map

\[
l_P = (l_{\alpha_1}, \ldots, l_{\alpha_N}) : \mathfrak{T}(M) \longrightarrow (\mathbb{R}^+)^N
\]

which is the moment map for a Hamiltonian \( \mathbb{R}^N \)-action. This \( \mathbb{R}^N \)-action is proper and free, and choosing a cross-section (a left-inverse to \( l_P \))

\[
\sigma : (\mathbb{R}^+)^N \longrightarrow \mathfrak{T}(M)
\]

defines a diffeomorphism

\[
(\mathbb{R}^+)^N \times \mathbb{R}^N \longrightarrow \mathfrak{T}(M)
\]

(3.1)

\[
(\lambda, t) \longmapsto \eta^\alpha_{l_1} \cdots \eta^\alpha_{l_N}(\sigma(\lambda))
\]
The map $l_P$ defines the action variables $\lambda$ while the coordinates $t \in \mathbb{R}^N$ are the angle variables. Indeed, Wolpert shows [18] that this completely integrable system defines a symplectomorphism with $(\mathbb{R}_+)^N \times \mathbb{R}^N$: the Fenchel-Nielsen coordinates are Darboux coordinates with respect to the symplectic structure $\omega_{WP}$ defined by the Weil-Petersson Kähler form:

$$\omega_{WP} = \sum_{i=1}^N d\lambda_i \wedge dt_i$$

This section extends this theory from $\text{SL}(2, \mathbb{R})$ to $\text{SL}(2, \mathbb{C})$.

3.1. Pants decompositions. By Corollary 2.2.3, the complex Hamiltonian vector fields of $f_\alpha$ and $f_\beta$ Poisson commute if $\alpha$ and $\beta$ are represented by disjoint curves. Thus for a family of disjoint simple closed curves

$$\mathcal{P} = \{A_1, \ldots, A_N\}$$

the corresponding complex twist flows of $\text{Ham}f_\alpha$ generate a $\mathbb{C}^n$-action. Suppose these curves are each homotopically nontrivial and mutually nonhomotopic. The resulting map

$$\tau_\mathcal{P} : X \rightarrow \mathbb{C}^N$$

is a moment map for a complex-Hamiltonian action of $\mathbb{C}^N$ on the complex-symplectic manifold $X$.

Suppose that $\mathcal{P}$ is maximal, that is $N = 3g - 3$, in which case each component

$$P \subset M|\bigcup_{i=1}^N A_i$$

is homeomorphic to a 3-holed sphere. In that case Fricke’s theorem [13, 16] implies that the generic inverse image $(\tau_\mathcal{P})^{-1}(z)$ is a single $\mathbb{C}^N$-orbit. Specifically, let $(\mathbb{C}^N)^s$ denote the subset consisting of $z \in \mathbb{C}^N$ such that, for every $(i, j, k)$ for which $\alpha_i, \alpha_j, \alpha_k$ bound a 3-holed sphere $P_{ijk}$,

$$z^2_{\alpha_i} + z^2_{\alpha_j} + z^2_{\alpha_k} - z_{\alpha_i}z_{\alpha_j}z_{\alpha_k} \neq 4.$$

This expresses the condition that the restriction $\phi|_{\pi_1(P_{ijk})}$ is irreducible; Fricke’s theorem asserts the triple

$$\left(f(\phi(\alpha_i)), f(\phi(\alpha_j)), f(\phi(\alpha_k))\right) \in \mathbb{C}^3$$

determines the equivalence class of such an irreducible representation. If $z \in (\mathbb{C}^N)^s$, the fiber $(\tau_\mathcal{P})^{-1}(z)$ is a single $\mathbb{C}^N$-orbit.

The subgroup $\Gamma_\mathcal{P}$ of $\Gamma$ preserving the pants decomposition is the free abelian group generated by the Dehn twists $\tau_{\alpha_i}$ for $i = 1, \ldots, N$. 
The resulting $\mathbb{Z}^n$-action lies in the $\mathbb{C}^N$-action by the formulas in §2.6. Namely, at a point $[\phi] \in X$ the orbit is an embedded product
\[
\prod_{i=1}^N G_i
\]
where
\[
G_i = \begin{cases} 
\{1\} & \text{if } \alpha_i([\phi]) = \pm I \\
\mathbb{C} & \text{if } f_{\alpha_i}([\phi]) = \pm 2 \text{ and } \alpha_i([\phi]) \neq \pm I \\
\mathbb{C}^* & \text{if } f_{\alpha_i}([\phi]) \neq \pm 2
\end{cases}
\]
For each $i$ with $f_{\alpha_i}(\phi) \neq 2$, let
\[
\lambda_i = \frac{f_{\alpha_i}(\phi) \pm (f_{\alpha_i}(\phi)^2 - 4)^{1/2}}{2}.
\]
Then $(k_1, \ldots, k_N) \in \mathbb{Z}^N$ acts by translation by $k_i$ on the $i$-th factor $G_i \approx \mathbb{C}$ for each $i$ with $f_{\alpha_i}(\phi) = \pm 2$ and by multiplication by $\lambda_i$ on each factor $G_i \approx \mathbb{C}^*$ with $f_{\alpha_i}(\phi) \neq \pm 2$.

3.2. The moment map as an ergodic decomposition and its holomorphic extension. The action of $\Gamma_P$ on $X_U$ is discussed in [15]. In this case the restriction
\[
f_P : X_U \longrightarrow [-2, 2]^N
\]
is a moment map for a Hamiltonian $\mathbb{R}^n$-action whose orbits are all closed. These orbits are tori, invariant under the $\Gamma_P \cong \mathbb{Z}^N$-action. Almost every fiber has dimension $N$, and the action is equivalent to an $\mathbb{Z}^N$-action on $T^N$ by translation. For almost all level sets, this action is ergodic. It follows that the moment map $f_P$ for the Hamiltonian $\mathbb{R}^N$-action is also the ergodic decomposition for the $\mathbb{Z}^N$-action ([15], Theorem 2.2):

**Proposition 3.2.1.** Let $h : X_U \longrightarrow \mathbb{R}$ be a $\Gamma_P$-invariant measurable function. Then there exists a measurable function $\psi : [-2, 2]^N \longrightarrow \mathbb{R}$ such that $h = \psi \circ f_P$ almost everywhere.

**Theorem 3.2.2.** Let $h : X \longrightarrow \mathbb{C}P^1$ be a meromorphic function which is invariant under $\Gamma_P$. Then there exists a meromorphic function $H : \mathbb{C}^P \longrightarrow \mathbb{C}P^1$ such that $h = H \circ \tau_P$.

**Proof.** Let $G$ denote the complex-Hamiltonian $\mathbb{C}^P$-action on $X$ and let $G_U$ denote the Hamiltonian $\mathbb{R}^P$-action on $X_U$. By Proposition 3.2.1, the measurable function $h$ must be almost everywhere constant on the $G_U$-orbits on $X_U$. Thus for each $t \in \tau_P(X_U)$, the function $h$ is constant on the preimage $\tau_P^{-1}(t) \cap X_U$ and:
• The preimages of the restriction of $\tau_P$ to $X_U$ are the $G_U$-orbits in $X_U$;
• Each $G_U$-orbit in $\tau_P^{-1}(t)$ is $\mathbb{C}$-Zariski dense in its $G$-orbit;
• $G$ acts transitively on $\tau_P^{-1}(t)$.

Hence $h$ is constant on each $\tau_P^{-1}(t)$. for $t \in \tau_P(X_U)$.

Now we find a holomorphic section of $\tau_P$. There exists a connected open neighborhood $W$ of $X_U$, a Zariski-closed subset $Z \subset X$ of positive codimension, and a holomorphic map

$$W - Z \xrightarrow{\sigma} \tau_p p^{-1}(W - Z)$$

so that the composition

$$W - Z \xrightarrow{\sigma} \tau_p p^{-1}(W - Z) \xrightarrow{\tau_P} W - Z$$

equals the identity. Then the meromorphic function

$$h - h \circ \sigma \circ \tau_P$$

on $\tau_p p^{-1}(W - Z)$ vanishes on each fiber $\tau_p p^{-1}(t)$ for $t \in W - Z \cap X_U$. Since $W - Z \cap X_U$ is totally real, it follows that $h - h \circ \sigma \circ \tau_P$ vanishes on all of $W - Z$. Since $W - Z$ is dense in $W$, this function vanishes on $W$. Since $W$ is nonempty and open, this function vanishes on all of $X$. Thus $h$ is constant on the fibers of $\tau_P$ as desired.

\[ \square \]

4. DEFORMATION SPACES OF $\mathbb{CP}^1$-STRUCTURES

A $\mathbb{CP}^1$-structure on $M$ is a geometric structure with local coordinate charts mapping to $\mathbb{CP}^1$ with coordinate changes in the group $\text{PSL}(2, \mathbb{C})$. The holonomy mapping $\text{hol}$ maps the deformation space $\mathbb{CP}^1(M)$ locally biholomorphically to the character variety $X - X_U$. Therefore $\Omega$ induces a complex-symplectic structure $\text{hol}^* \Omega$ on $\mathbb{CP}^1(M)$.

On the other hand, $\mathbb{CP}^1(M)$ is classically the total space of an affine bundle over $\mathcal{F}(M)$ whose associated vector bundle is the cotangent bundle $T^* \mathcal{F}(M)$. The canonical exact complex-symplectic structure on $T^* \mathcal{F}(M)$ defines a complex-symplectic structure on $\mathbb{CP}^1(M)$. Kawai [30] has proved this complex-symplectic structure equals $\text{hol}^* \Omega$:

**Theorem.** (Kawai) The complex-symplectic structure on $\mathbb{CP}^1(M)$ induced from the holomorphic cotangent bundle structure on $T^* \mathcal{F}(M)$ equals the complex-symplectic structure induced from the complex-symplectic structure $\Omega$ on $X$ by

$$\mathbb{CP}^1(M) \xrightarrow{\text{hol}} \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C}).$$
The complex twist flows on $X$ lift to complex twist flows on $\mathbb{CP}^1(M)$. We relate the complex twist geometry on the open subset $\mathcal{Q}(M) \subset \mathbb{CP}^1(M)$ comprising quasi-Fuchsian structures to the quakebends studied by Epstein-Marden [7], Kourouniotis [32, 33, 34], McMullen [36], Platis [39], Series [42], and Tanigawa [44].

4.1. $\mathbb{CP}^1$-structures. Let $M$ be a smooth surface. A $\mathbb{CP}^1$-atlas on $M$ consists of

- An open covering $U$ of $M$ by coordinate patches;
- For each coordinate patch $U \in U$, a coordinate chart $\psi_U : U \rightarrow \mathbb{CP}^1$ which is a diffeomorphism of $U$ onto its image $\psi_U(U)$;
- For each connected open subset $C \subset U \cap V$ of the intersection of coordinate patches $U, V \in U$, a transformation $g_C \in \text{PSL}(2, \mathbb{C})$ such that

$$g_C \circ \psi_U|_{U \cap V} = \psi_V|_{U \cap V}.$$ 

The coordinate change $g_C$ is unique.

Such atlases are partially ordered by inclusion. A $\mathbb{CP}^1$-structure on $M$ is a maximal $\mathbb{CP}^1$-atlas. A $\mathbb{CP}^1$-manifold is a manifold with a $\mathbb{CP}^1$-structure.

Suppose $M$ and $M'$ are $\mathbb{CP}^1$-manifolds. A mapping $\phi : M \rightarrow M'$ is a $\mathbb{CP}^1$-mapping if and only if for each coordinate chart $(U, \psi_U)$ in $M$ and each coordinate chart $(U', \psi'_{U'})$ in $M'$ the composition

$$(\psi'_{U'}) \circ \phi \circ (\psi_U)^{-1} : \psi_U(U \cap \phi^{-1}(U')) \rightarrow \psi'_{U'}(\phi(U) \cap U')$$

extends to an element of $\text{PSL}(2, \mathbb{C})$ on each component of $\psi_U(U \cap \phi^{-1}(U'))$. A $\mathbb{CP}^1$-mapping is necessarily a local diffeomorphism.

If $f : M \rightarrow M'$ is a local diffeomorphism, then every $\mathbb{CP}^1$-structure on $M'$ determines a unique $\mathbb{CP}^1$-structure on $M$ such that $f$ is a $\mathbb{CP}^1$-mapping. In particular every covering space of a $\mathbb{CP}^1$-manifold is a $\mathbb{CP}^1$-manifold, and its group of covering transformations is realized by a group of $\mathbb{CP}^1$-automorphisms.

If $M$ is a simply-connected $\mathbb{CP}^1$-manifold, then any chart on $M$ extends to a globally defined developing map

$$\text{dev} : M \rightarrow \mathbb{CP}^1$$

which is a $\mathbb{CP}^1$-mapping. This mapping is unique up to left-composition with transformations in $\text{PSL}(2, \mathbb{C})$.

We denote by $\Sigma$ the Riemann surface whose underlying manifold is $M$ for which $\text{dev}$ is holomorphic.
In general the universal covering space $\tilde{M}$ of a $\mathbb{CP}^1$-manifold admits a developing map into $\mathbb{CP}^1$. The group $\pi_1(M)$ of covering transformations of $\tilde{M} \to M$ admits a representation $\rho : \pi_1(M) \to \text{PSL}(2, \mathbb{C})$ such that the diagram

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\text{dev}} & \mathbb{CP}^1 \\
\gamma \downarrow & & \downarrow \rho(\gamma) \\
\tilde{M} & \xrightarrow{\text{dev}} & \mathbb{CP}^1
\end{array}
\]

commutes for every $\gamma \in \pi_1(M)$. The representation $\rho$ is called the holonomy representation. The developing pair $(\text{dev}, \rho)$ is unique up to the action of $\text{PSL}(2, \mathbb{C})$ (by left-composition with $\text{dev}$ and conjugation on $\rho$). The developing map globalizes the coordinate charts and the holonomy representation globalizes the coordinate changes.

4.1.1. Deformation spaces. Let $S$ be a fixed topological surface with fundamental group $\pi = \pi(S)$. A marked $\mathbb{CP}^1$-structure on $S$ consists of a $\mathbb{CP}^1$-manifold $M$ and a homeomorphism $f_M : S \to M$. Two marked $\mathbb{CP}^1$-structures $(M, f_M)$ and $(M', f_{M'})$ are equivalent if and only if there exists a $\mathbb{CP}^1$-isomorphism $\phi : M \to M'$ such that

$$\phi \circ f_M \simeq f_{M'}.$$

Let $\mathbb{CP}^1(S)$ denote the set of equivalence classes of marked $\mathbb{CP}^1$-structures on $S$. Holonomy defines a mapping

$$\text{hol} : \mathbb{CP}^1(S) \to \text{Hom}(\pi, \text{PSL}(2, \mathbb{C}))/\text{PSL}(2, \mathbb{C}).$$

As in [14] (see also Earle [6], Hubbard [22] and Kapovich [28]), there is a natural topology on $\mathbb{CP}^1(S)$ such that $\text{hol}$ is a local homeomorphism. Furthermore Gallo-Kapovich-Marden [8] proved that the image of $\text{hol}$ is $X - X_U - X_D$, where $X_D \subset X - X_U$ is the closed subset consisting of equivalence classes $[\phi]$ of representations leaving invariant a pair of points of $\mathbb{CP}^1$ (these correspond to irreducible representations for which an index-two subgroup acts reducibly).

4.2. Holonomy and the conformal structure. Since $\text{PSL}(2, \mathbb{C})$ acts holomorphically on $\mathbb{CP}^1$, a $\mathbb{CP}^1$-atlas on $M$ is a holomorphic atlas, that is, an atlas for a complex structure on $M$. Thus underlying every $\mathbb{CP}^1$-manifold is a Riemann surface. Recording the complex structure underlying a $\mathbb{CP}^1$-structure is a map

$$\mathbb{CP}^1(S) \xrightarrow{\Pi} \mathcal{F}(M).$$

A projective structure on a Riemann surface $\Sigma$ is a $\mathbb{CP}^1$-structure whose underlying complex structure is $\Sigma$. 
Lemma 4.2.1. $\Pi$ is holomorphic.

Proof. We recall the classical description of $\mathbb{CP}^1(S)$ as an affine bundle $Q(M)$ over $\mathcal{X}(M)$ whose associated vector bundle equals the cotangent bundle $T^*(\mathcal{X}(M))$. (See for example §2.3 of Earle [6], Hubbard [22] or Gunning [18]). Fix a Riemann surface $\Sigma$ homeomorphic to $M$ and a projective structure $s_0$ on $\Sigma$. Given any other projective structure on $\Sigma$, its developing map is a holomorphic map from the universal covering $\tilde{\Sigma}$ to $\mathbb{CP}^1$. Its Schwarzian derivative (in the local projective coordinates defined by $s_0$) is a holomorphic quadratic differential which is invariant under $\pi$, and therefore defines a holomorphic quadratic differential on $\Sigma$. Furthermore this quadratic differential completely determines the developing map up to an element of $\text{PSL}(2, \mathbb{C})$ of $\mathbb{CP}^1$. Thus the fiber $\Pi^{-1}(\Sigma)$ admits a simply transitive action of the complex vector space $H^0(\Sigma; \mathcal{O}(\kappa^2))$ comprising holomorphic quadratic differentials on $\Sigma$. The composition law for the Schwarzian derivative implies that changing the origin $s_0$ effects a translation of $Q(\Sigma)$, so $\Pi^{-1}(\Sigma)$ is an affine space modelled on $H^0(\Sigma; \mathcal{O}(\kappa^2))$, which is the cotangent space $T^*\mathcal{X}(M)$ at $[\Sigma]$.

In particular $\mathbb{CP}^1(S)$ inherits a complex structure, making it a holomorphic affine bundle over $\mathcal{X}(M)$. In particular the projection $\Pi$ is holomorphic in this complex structure.

By Lemma 2 of Earle [6] or Hubbard [22], the holonomy mapping $\text{hol}$ is a local biholomorphism, and therefore this complex structure on $\mathbb{CP}^1$ induced by $\text{hol}$ is the above complex structure. Thus $\Pi$ is holomorphic with respect to the complex structure induced by $\text{hol}$. $\square$

Kawai’s theorem implies that $\Pi$ is a holomorphic Lagrangian fibration. As described in Hubbard [22], the differential $d\Pi$ identifies with the map

$$T_{s_0}\mathbb{CP}^1(S) \cong H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C})) \longrightarrow H^1(\Sigma; \mathcal{O}(\kappa^{-1})) \cong T_{[\Sigma]}\mathcal{X}(M)$$

induced by the short exact sequence of sheaves

$$(4.1) \quad 0 \longrightarrow \mathfrak{sl}(2, \mathbb{C}) \overset{i}{\longrightarrow} \mathcal{O}(\kappa^{-1}) \overset{D_3}{\longrightarrow} \mathcal{O}(\kappa^2) \longrightarrow 0.$$  

Here $D_3$ is the map of holomorphic sheaves given in local holomorphic coordinates by

$$h(z) \frac{\partial}{\partial z} \longmapsto h'''(z) \, dz^2.$$ 

Its kernel $\mathfrak{sl}(2, \mathbb{C})$ is the locally constant sheaf of locally projective vector fields (whose coefficients in affine coordinates are quadratic). The
monomorphism $i$ regards a locally projective vector field as a holomorphic vector field. Then (4.1) induces the exact sequence

$$0 \longrightarrow H^0(\Sigma, \mathcal{O}(\kappa^2)) \xrightarrow{\delta} H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C})) \xrightarrow{i_*} H^1(\Sigma; \mathcal{O}(\kappa^{-1})) \longrightarrow 0.$$ 

If $\alpha \in H^0(\Sigma, \mathcal{O}(\kappa^2))$ is a holomorphic quadratic differential and $\beta \in H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C}))$, then

$$\langle \delta(\alpha), \beta \rangle = \alpha \cdot i_*(\beta)$$

where the first pairing $\langle \cdot, \cdot \rangle$ is the pairing on $H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C}))$ induced by the complex Killing form $\mathcal{B}$ on $\mathfrak{sl}(2, \mathbb{C})$ and the second pairing is the Serre duality pairing

$$H^0(\Sigma, \mathcal{O}(\kappa^2)) \times H^1(\Sigma; \mathcal{O}(\kappa^{-1})) \longrightarrow H^1(\Sigma, \mathcal{O}(\kappa)) \cong \mathbb{C}.$$ 

This discussion implies that $\Pi$ is a Lagrangian fibration. Suppose $\beta, \gamma \in \ker(d\Pi) = \delta(H^0(\Sigma, \mathcal{O}(\kappa^2))) = \ker(i_*)$.

Then $\gamma = \delta(\alpha)$ for some $\alpha \in H^0(\Sigma, \mathcal{O}(\kappa^2))$, and (4.2) implies $\langle \gamma, \beta \rangle = 0$ since $i_*\beta = 0$. Therefore $\ker(d\Pi)$ is isotropic. Since its dimension is half of that of $H^1(\Sigma; \mathfrak{sl}(2, \mathbb{C}))$, the fibers of $\Pi$ are Lagrangian. For further details, see Kawai [30].

4.2.1. Meromorphic functions on $\mathbb{CP}^1(S)$.

**Proposition 4.2.2.** There exist nonconstant $\Gamma$-invariant meromorphic functions on $\mathbb{CP}^1(S)$.

**Proof.** The Riemann moduli space of curves is the quotient $\mathcal{M}(M)/\Gamma$, which by Knudsen [31], is a quasiprojective variety. Thus $\mathcal{M}(M)/\Gamma$ admits nonconstant meromorphic functions and $\mathcal{M}(M)$ admits nonconstant $\Gamma$-invariant meromorphic functions.

Let $\psi$ be a nonconstant $\Gamma$-invariant meromorphic function on $\mathcal{M}(M)$. Composing with the projection

$$\Pi : \mathbb{CP}^1(S) \longrightarrow \mathcal{M}(M)$$

provides a nonconstant $\Gamma$-invariant meromorphic function $\psi \circ \Pi$ on $\mathbb{CP}^1(S)$. □

4.3. Quasi-Fuchsian space. A $\mathbb{CP}^1$-structure is Fuchsian if and only if a developing map embeds $\tilde{M}$ in a hyperbolic plane $H^2_\mathbb{C} \subset \mathbb{CP}^1$. Necessarily the holonomy representation is conjugate to a Fuchsian representation, that is, a discrete embedding $\pi \hookrightarrow \text{PU}(1, 1)$ (where $\text{PU}(1, 1)$ is the stabilizer in $\text{PSL}(2, \mathbb{C})$ of the Poincaré disc $H^2_\mathbb{C}$). However by a construction of Maskit and Hejhal (see also Goldman [12]) there exist $\mathbb{CP}^1$-structures with Fuchsian holonomy with surjective developing
maps. A quasi-Fuchsian $\mathbb{CP}^1$-structure is a $\mathbb{CP}^1$-structure topologically conjugate to a Fuchsian structure, that is, there exists a homeomorphism $h$ of $\mathbb{CP}^1$ such that

$$(h \circ \text{dev}, h \circ \rho \circ h^{-1})$$

is the developing pair for a Fuchsian structure. The limit set $\Lambda$ of $\rho(\pi)$ is the Jordan curve $h(\partial H_C)$. The developing image $\text{dev}(\tilde{M})$ is one component of the complement $\mathbb{CP}^1 - \Lambda$.

Quasi-Fuchsian space $QF(M)$ is the subset of $\mathbb{CP}^1(S)$ corresponding to quasi-Fuchsian structures. The holonomy representation $\rho$ is a quasi-Fuchsian representation. The holonomy mapping

$$\text{hol}: QF(M) \rightarrow X$$

embeds $QF(M)$ as the open subset of $X$ consisting of conjugacy classes of quasi-Fuchsian representations. Quasi-Fuchsian space $QF(M)$ is both the open subset of $\mathbb{CP}^1(S)$ consisting of quasi-Fuchsian structures and the open subset of $X - X_U$ consisting of characters of quasi-Fuchsian representations.

The conformal structures of the quotients of $\mathbb{CP}^1 - \Lambda$ by $\phi(\pi)$ determine an ordered pair $\mathfrak{F}(M) \times \bar{\mathfrak{F}}(M)$; the first parameter is the marked conformal structure underlying the $\mathbb{CP}^1$-structure. The simultaneous uniformization theorem of Bers [3] asserts that the corresponding map

$$QF(M) \xrightarrow{\cong} \mathfrak{F}(M) \times \bar{\mathfrak{F}}(M)$$

is a biholomorphism. It is evidently $\Gamma$-equivariant.

Since $QF(M) \hookrightarrow \mathbb{CP}^1(S)$, Corollary [12.2] implies that nonconstant $\Gamma$-invariant meromorphic functions exist on $QF(M)$.

4.4. Twist flows of $\mathbb{CP}^1$-structures and grafting. Let $\alpha \in \pi$. Then the composition

$$\mathbb{CP}^1(S) \xrightarrow{\text{hol}} X \xrightarrow{f_\alpha} \mathbb{C}$$

defines a holomorphic function on $\mathbb{CP}^1(S)$ and its complex-Hamiltonian $\text{Ham}(f_\alpha \circ \text{hol})$ is a holomorphic vector field on $\mathbb{CP}^1(S)$. If $\alpha$ corresponds to a simple closed curve $A$, then there is a complex twist flow $\tilde{\xi}^\alpha_t$ on $\mathbb{CP}^1$ covering the complex twist flow generated by the vector field $\text{Ham}(f_\alpha)$ on $X$.

This twist flow is defined geometrically on Fuchsian structures as follows (see [12], Kouroumovitis [32, 33, 34]). Let $s_0$ be a Fuchsian $\mathbb{CP}^1$-structure, so a developing map embeds $\tilde{M}$ as a geometric disc $\Delta \subset \mathbb{CP}^1$. Then the simple closed curve $A \subset M$ can be represented by a simple closed geodesic with respect the Poincaré metric induced from that of $\Delta$. Let $\ell(A)$ denote the Poincaré length of this geodesic. A lift $\tilde{A}$ of $A$ to
\( \hat{M} \) develops to a circular arc orthogonal to \( \partial \text{dev}(\hat{M}) \). The split surface \( M|A \) inherits a \( \mathbb{C}\mathbb{P}^1 \)-structure whose boundary components develop to circular arcs. To define \( s_t := \xi_\alpha (s_0) \), insert an annulus \( A_\theta \) with \( \mathbb{C}\mathbb{P}^1 \)-structure into \( M|A \). The angular parameter \( \theta \) is the imaginary part \( \text{Im}(t) \). The annulus is a \( \theta \)-annulus in the sense of §2.12 of [12]. Choose a holomorphic universal covering map

\[
E : \mathbb{C} \rightarrow \mathbb{C}\mathbb{P}^1 - \text{Fix}(\phi(\alpha))
\]

which is periodic with period \( 2\pi i \). (When \( \text{Fix}(\phi(\alpha)) = \{0, \infty\} \), then this is just the exponential map \( \exp : \mathbb{C} \rightarrow \mathbb{C}^* \). For general \( A \), we compose \( \exp \) with a projective transformation taking \( \{0, \infty\} \) to \( \text{Fix}(\phi(\alpha)) \).

Define the \( \theta \)-strip

\[
S_\theta := \mathbb{R} + i[0, \theta] \subset \mathbb{C}
\]

with \( \mathbb{C}\mathbb{P}^1 \)-structure induced by \( E \). The developing map \( E \) for \( S_\theta \) is equivariant with respect to the \( \mathbb{Z} \)-action on \( S_\theta \) generated by translation by \( \ell(A) \) and the \( \mathbb{Z} \)-action on \( \mathbb{C}\mathbb{P}^1 - \text{Fix}(\phi(\alpha)) \). The \( \theta \)-annulus \( A_\theta \) is defined as the quotient \( \mathbb{C}\mathbb{P}^1 \)-manifold of \( S_\theta \).

The grafted manifold \( M(t) \) is obtained by inserting a \( \theta \)-annulus into \( M|A \). Choose one component of \( A_- \subset \partial(M|A) \) and attach \( A_\theta \) to \( A_- \) to obtain a \( \mathbb{C}\mathbb{P}^1 \)-manifold homeomorphic to \( M|A \) with one boundary component \( A'_- \) corresponding to the component \( A_+ \) of \( \partial(M|A) \) and another boundary component \( A' \) corresponding to the other component of \( \partial(A_\theta) \). Identify \( A'_- \) to \( A' \) by a transvection of displacement \( \text{Re}(t) \).

When \( \theta = 0 \), this is just the Fenchel-Nielsen twist deformation, obtained from \( M|A \) by identifying the two components of \( \partial(M|A) \) by a transvection of displacement \( \text{Re}(t) \).

Let \( P \in \text{SL}(2, \mathbb{C}) \). These holomorphic flows on \( \text{Hom}(\pi, \text{SL}(2, \mathbb{C})) \) cover holomorphic flows on \( X \) which have been extensively studied on quasi-Fuchsian space \( \mathcal{QF}(M) \subset X \). On \( \mathcal{QF}(M) \) these flows geometrically correspond to the quakebends or complex earthquakes discussed by Epstein-Marden [7], Kouremiotis [32, 33, 34], McMullen [36], Platis [39], Series [42], and Tanigawa [44]. In particular we obtain the following result of Platis [39]:

**Theorem 4.4.1** ([Platis [39], Theorem 7, §2.2). The complex twist vector field \( \tilde{\xi} \) on \( \mathcal{QF}(M) \) is Hamiltonian with respect to the complex length function \( l^C \).

Just as the geodesic length function \( l \) on the hyperbolic subset of \( \text{SL}(2, \mathbb{R}) \) and the angle function \( \theta \) on \( \text{SU}(2) \setminus \{ \pm \mathbb{I} \} \) are more geometrically natural than the trace functions, we consider the complex length
“function” on $\text{SL}(2, \mathbb{C})$ defined by:

$$2 \cosh \left( \frac{l^C(P)}{2} \right) = f(P)$$

or equivalently

$$l^C(P) = 2 \log \left( \frac{(f(P) \pm (f(P)^2 - 4)^{1/2})}{2} \right).$$

(This differs from Tan [43] whose complex length is half of ours. Our definition is consistent with Kourouniotis [32, 33, 34], Platis [39] and Series [42].) This function takes values in $(\mathbb{C}/4\pi i \mathbb{Z})\setminus\{\pm 1\}$, since the logarithm is well-defined only modulo $4\pi i$ and the choice of square root introduces an ambiguity of sign.

Choose the branch $\tilde{l}_C$ of the complex length which is positive on the subset of hyperbolic elements in $\text{SL}(2, \mathbb{R})$; since $\mathcal{QF}(M)$ is simply connected

$$[\phi] \mapsto (\tilde{l}_C(\phi(\alpha_1)), \ldots, \tilde{l}_C(\phi(\alpha_N)))$$

defines a single-valued function

$$l^C|_P : \mathcal{QF}(M) \longrightarrow \mathbb{C}^N.$$ The restriction of $l^C$ to $\mathcal{Q}(M)$ is the length function $l_P : \mathcal{Q}(M) \longrightarrow (\mathbb{R}_+)^N$ defined by (3.1). The fibers of $l^C|_P$ are orbits of the complex-Hamiltonian $\mathbb{C}^N$-action having $l^C|_P$ as moment map. Tan [43] and Kourouniotis [33] show that there exists a section $\sigma : \mathbb{C}^N \longrightarrow \mathcal{QF}(M)$ and a neighborhood of $(\mathbb{R}_+)^N \times \{0\}$ in $\mathbb{C}^N \times \mathbb{C}^N$ which maps biholomorphically to $\mathcal{QF}(M)$.

Kourouniotis [33] and Series [42] derive formulas for the derivative of the complex length functions along quakebend flows. We briefly sketch how their formulas can be derived from Proposition 2.2.2 as a Poisson bracket, referring to [33, 42] for details.

Let $P \in \text{SL}(2, \mathbb{C})$ be loxodromic with invariant axis $a_P$. Then the variation function $L^C$ associated to the invariant function $l^C$ and the complex-orthogonal structure $\mathbb{B}(X,Y) = \text{tr}(XY)$ maps a diagonal matrix to

$$D_0 := \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(which is not uniquely defined without restrictions to make the function $l^C$ single-valued). By (2.1), the element of $\text{PSL}(2, \mathbb{C})$ corresponding to $L^C(P)$ is the involution fixing $a_P$. If $P_1, P_2 \in \text{PSL}(2, \mathbb{C})$ are loxodromic, with complex distance $z$ between their invariant axes, then a simple calculation implies that $\mathbb{B}(L^C(P_1), L^C(P_2))$ equals $\cosh(z)$. Applying Proposition 2.2.2 to the complex length functions $l^C_\beta$ and $l^C_\alpha$, where $\alpha$
corresponds to a simple closed curve, we obtain the following extension of Theorem 2.6.2.

**Theorem 4.4.2** ([Korouniotis 33, Series 42]). Let $\alpha, \beta \in \pi$ where $\alpha$ is represented by a simple closed curve $A$. Then the derivative of the complex length function $l^C_\beta$ on $QF(M)$ with respect to the quakebend flow with respect to $A$ equals the sum

$$\sum_{i=1}^k \cosh(d_i)$$

where $\alpha$ and $\beta$ are represented by closed geodesics, $p_1, \ldots, p_k$ are their intersection points, and $d_i$ the complex distance between the axes of $\phi_i(\alpha_i)$ to $\phi_i(\beta_i)$ at $p_i$, where $\phi_i$ is a representative homomorphism from $\pi_1(M; p_i)$ and $\alpha_i, \beta_i$ are elements of $\pi_1(M; p_i)$ representing $\alpha$ and $\beta$.

See Korouniotis [33] and Series [42] for details.

Tan [43] and Kourouniotis [32, 33, 34] extend Fenchel-Nielsen coordinates to complex Fenchel-Nielsen coordinates on $QF(M)$. (See also Series [42] for another exposition.)
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