Abstract
In this paper, a new class of hemivariational inequalities is introduced. It concerns Laplace operator on locally finite graphs together with multivalued nonmonotone nonlinearities expressed in terms of Clarke’s subdifferential. First of all, we state and prove some results on the subdifferentiability of nonconvex functionals defined on graphs. Thereafter, an elliptic hemivariational inequality on locally finite graphs is considered and the existence and uniqueness of its weak solutions are proved by means of the well-known surjectivity result for pseudomonotone mappings. In the end of this paper, we tackle the problem of hemivariational inequalities of parabolic type on locally finite graphs and we prove the existence of its weak solutions.

Keywords Locally finite graphs · Nonconvex sum functionals · Hemivariational inequality · Clarke’s subdifferential

Mathematics Subject Classification 49J40 · 49J52 · 49J53 · 05C63 · 35J60 · 35K55 · 49A70
1 Introduction

Discrete calculus incorporates the various research works that focus on developing a proper theory for differential operators on discrete spaces with a net separation from the classical continuous calculus. From this perspective, discrete calculus should be differentiated from discretized calculus which concerns the discretization of the continuous framework for numerical and algorithmic purposes. Difference calculus, as a particular case of discrete calculus, is performed generally on the $d$–dimensional lattice graph (or grid) $\mathbb{Z}^d$ for some $d \geq 1$ and plays the role of an intermediate discipline. Discrete calculus aims then to establish a distinct and coherent core of calculus that operates purely in the discrete space without any reference to an underlying continuous counterpart. The philosophy behind this, is the fact that there is a solid connection between dynamics and the mathematical description of the space where they occur Tonti (1976); Grady and Polimeni (2010).

The spaces of predilection for the discrete calculus are graphs and networks, from which cell complexes arise as general space structures Biggs et al. (1986). The first application of graph theory to the modelling of physical systems came from Kirchhoff, who both developed the basic laws of circuit theory and also made fundamental contributions to graph theory Kirchhoff (1847). Among applications of modern graph theory one can mention manifold learning, filtering (denoising), content extraction, ranking, clustering, and network characterization. The main technique here is to use the data to define weights on the network and then methods are used to formulate content extraction problems as convex energy minimization problems Grady and Alvino (2009); Sethian (1999). Nonconvex energy models appears also in data filtering on graphs with explicit discontinuities (rapid data change) Mumford and Shah (1989) and in nonsmooth nonconvex Regularizer in variational models for image restoration and segmentation Geman and Reynolds (1992); Jung and Kang (2014); Nikolova (2005).

After a decade from the development of the theory of nonlinear circuit networks in the 60s, operator theory on infinite graphs and the underlying Sobolev spaces began to be systematically developed as a theoretical core for studying elliptic and parabolic problems on graphs and networks. An embryonic study was initiated by M. Yamasaki and co-authors in Nakamura and Yamasaki (1976); Yamasaki (1975, 1977) and more elaborated work is exposed by M.I. Ostrovskii in Ostrovskii (2005). For some important use of the discrete version of Sobolev spaces we refer to Grady and Polimeni (2010) and references therein. The most modern expository on discrete operators and Sobolev spaces is the book of D. Mugnolo Mugnolo (2014) where the central topic is the interplay of differential, difference operators and subdifferentials of convex functionals with the functional analytic theory of evolution equations together with combinatorial methods.

In many physical and social phenomena, the Laplacian operator arises naturally in the mathematical description of diffusion through discrete and continuous media. Discrete diffusion theory Kelly (1964) based on discrete Fick’s law Ha and Levy (2009), can be certainly considered as an approximation of its continuous counterpart, nonetheless problems still frequently arise where it would be advantageous to have access to a diffusion theory valid specifically for discrete media Kelly (1964). The starting point for this theory is the formulation of Laplace operator on graphs and its associated discrete energy functional. Nakamura and Yamasaki introduced (for $\gamma \equiv 1$ and $\kappa \equiv 0$) in Nakamura and Yamasaki (1976) on an infinite graph $G$ with node set $V$ the convex functional
$E^p_{\gamma, \kappa} : \mathbb{R}^V \ni \phi \mapsto \frac{1}{p} \sum_{v, w \in V \atop w \sim v} \gamma(v, w)|\phi(v) - \phi(w)|^p + \frac{1}{p} \sum_{v \in V} \kappa(v)|\phi(v)|^p \in [0, \infty].$

The associated operator $(L^G_{\gamma, \kappa} := \partial E^p_{\gamma, \kappa})$ is nothing but the discrete $p-$Laplace operator. Let us note, parenthetically, the remark in Mugnolo (2013), that the development of the theory of nonlinear electric circuits and the theory of monotone operators and subdifferentials of convex functionals was simultaneous by Minty and Rockafellar among others Minty (1960). Different aspects of the discrete $p-$Laplacian are studied in the literature Galewski and Wieteska (2016); Keller and Lenz (2010); Mugnolo (2013) and found applications in nonlinear circuit theory, spectral clustering and image processing, sphere packing problem and with the tug-of-war-theory Elmoataz et al. (2008); Elmoataz and Buysens (2017); Grady and Alvino (2009); Ha et al. (2010); Ha and Levy (2009); Ta et al. (2007); Tamás and Anna (2012) or emerging phenomena of a population of dynamically interacting units Ha et al. (2010); Tamás and Anna (2012). A systematic study of the Laplacian operators on graphs is achieved with means of discrete Dirichlet forms by D. Lenz and co-authors Keller and Lenz (2010); Haeseler et al. (2012) and references therein. For the Laplacian on finite weighted graphs with nonlinear terms, we refer to Huo et al. (2018) and references therein.

In the development of functional analysis on graphs, the finite difference operator plays a fundamental role. It started with an intuition that goes back to G. Boole Boole (1860) revealing that the operator $I_T \phi(v, w) := \phi(v) - \phi(w), \phi \in \mathbb{R}^V,$
can be looked as a discretized version of the first derivative of the function $\phi$ defined in all points of the underlying graph $G.$ In the definition of Sobolev spaces on graphs, the operator $I_T \phi$ is an equivalent of the gradient. The parallel between discrete functional calculus and the classical continuous settings can be made by taking into account the following rules: scalar functions replaces vectors of the node set, vector fields replaces vectors of the edge set and gradient of scalar functions at some point replace evaluation of the difference operator at an edge.

The goal of this paper is to formulate a new class of variational-type inequalities consisting of nonmonotone multivalued perturbation of the discrete Laplacian on a locally finite graph. The pseudomonotone term is brought by a nonconvex functional defined by an integral. Let $j : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function whose Clarke’s subdifferential satisfies a growth condition. The following sum functional

$$J(\phi) = \sum_{v \in V} \mu(v)j(\phi(v)),$$
can be looked as a discretized version of the well-studied integral functionals of the form

$$J(\phi) = \int_G j(\phi(x)) \, d\mu(x).$$

The primary question in such situation is the relation between the subdifferential of $j$ and $J.$ In the integral functional case, this is what we commonly call Aubin–Clarke theorem. In this paper, we prove its discrete counterpart, that is,

$$\partial J(\phi) \subset \sum_{v \in V} \mu(v)\partial j(\phi(v)).$$
Having this discrete version of Aubin–Clarke theorem at one’s disposal one may formulate elliptic problem as follows

\[ \langle \mathcal{L}^G_{\gamma, \kappa} \phi, \psi - \phi \rangle + \sum_{v \in V} \mu(v) j^\circ(\phi(v); \psi(v) - \phi(v)) \geq \langle f, \psi - \phi \rangle, \quad (1.1) \]

where \( j^\circ \) is the generalized Clarke directional derivative of \( j \) and its parabolic counterpart

\[ \langle \phi' + \mathcal{L}^G_{\gamma, \kappa} \phi, \psi - \phi \rangle + \sum_{v \in V} \mu(v) j^\circ(t, \phi(v); \psi(v) - \phi(v)) \geq \langle f, \psi - \phi \rangle. \quad (1.2) \]

Such problems will be called discrete hemivariational inequalities or hemivariational inequalities on graphs.

The general theory of hemivariational inequalities is a natural generalization of the classical variational theory where convex energy functionals are involved. Mathematical formulation of many engineering problems reveals cases that lack of monotonicity and corresponds to nonconvex superpotentials which cannot be formulated by the classical variational tools. By applying the mathematical notion of generalized gradient of Clarke (1990), Panagiotopoulos (1993) introduced for the first time the so-called hemivariational inequalities. Since then, such formulation found applications in many fields, i.e. Navier–Stokes equations Mahdioui et al. (2020), Mahdioui et al. (2020), boundary value problems Aayadi et al. (2021); Migórski (2004), frictional contact Migórski and Zeng (2018), history-dependent problems Sofonea et al. (2018), nonlocal problems Liu and Tan (2017) to name a few. Different methods are applied for the solvability of hemivariational inequalities, we can mention Galerkin approximation method, critical point theory, surjectivity theorems, extremal solutions method, Rothe approximation method, equilibrium problem method, penalty method, etc. The main assumption on the locally Lipschitz function include Rauch condition, growth condition or unilateral growth condition.

One of the main applications of hemivariational inequality theory is related to semipermeability problems. Such problems were first studied in Duvaut and Lions (1972) for monotone regulation of some physical quantities on the boundary or in the interior of the media. The formulation of such problems lead to a variational inequality. In Panagiotopoulos (1985), the same problem was studied but without assuming monotonicity and leads to an hemivariational inequality. On the other hand, synaptic depression dynamic model Bobrowski and Morawska (2012) suppose that each vertex is a semipermeable membrane Bobrowski (2012); Gregosiewicz (2020). As in this case the semipermeability occurs in the interior of the media, we can analogously to Panagiotopoulos (1985), suppose that the exterior force \( f = f_1 + f_2 \) is such that \( -f_2(v) \in \partial j(\varphi(v)) \) for any vertex \( v \in V \), the set of vertices modelling the neural network. This can be a good application of the theory developed in this paper.

The remainder of the paper is structured as follows. In Sect. 2, we recall the functional setting on graphs and some concepts from nonsmooth analysis. In Sect. 3, we extend, to the framework of locally finite graphs, the Aubin–Clarke Theorem concerning the subdifferentiability of sum functionals. This may serve as a building block for developing variational methods for elliptic and parabolic problems involving the discrete Laplace operator and nonsmooth corresponding energy functional. In Sect. 4, we prove the existence and uniqueness of the elliptic hemivariational problem on locally finite graphs. The main tool is the well-known surjectivity result for pseudomonotone mappings. Section 4 is devoted to the discrete parabolic hemivariational inequality. The existence of a solution is reached by the use of a surjectivity result for the sum of maximal monotone and pseudomonotone mappings. In the last section, we provide some extensions to the problems discussed in previous sections.
It concerns Galerkin scheme for discrete hemivariational inequalities, discrete variational–hemivariational inequalities and discrete quasi-hemivariational inequalities.

2 Preliminaries

2.1 Sobolev spaces on graphs

For the concepts on graphs used in this paper and the underlying functional analysis which is the theoretical core we deploy in our formulations, we refer to the complete and self-contained book Mugnolo (2014).

Let $G = (V, E)$ be a direct graph, where $V$ is the set of nodes, which is finite or countable set and $E$ the set of edges, which is a subset of $V \times V$. A weighted graph is a quadruple $G = (E, V, \rho, \mu)$ where $(V, E)$ is a direct graph, $\mu : V \rightarrow (0, \infty)$ is some given function and $\rho : E \rightarrow (0, \infty)$ is some other given function such that $\rho(e) = \rho(\bar{e})$ whenever $e, \bar{e} \in E$ ($\bar{e} = (w, v)$ when $e = (v, w)$). For $e = (v, w)$, we note $e^- := v$ the initial endpoint of $e$ and $e^+ := w$ the terminal endpoint of $e$ and we say that they are adjacent (shortly $v \sim w$).

Define

$$
\eta^+_ve = \begin{cases} 
1 & \text{if } v \text{ is initial endpoint of } e \\
0 & \text{otherwise}
\end{cases}, \quad \eta^-ve = \begin{cases} 
1 & \text{if } v \text{ is terminal endpoint of } e \\
0 & \text{otherwise}
\end{cases}
$$

Definition 2.1 A weighted graph $G := (V, E, \rho, \mu)$ is called outward locally finite if its outdegree function satisfies

$$
\deg^+(v) := \sum_{e \in E} \eta^+_ve \rho(e) \leq M^+_v \quad \text{for all } v \in V \text{ and some } M^+_v > 0.
$$

It is called inward locally finite if its indegree function satisfies

$$
\deg^-(v) := \sum_{e \in E} \eta^-ve \rho(e) \leq M^-_v \quad \text{for all } v \in V \text{ and some } M^-_v > 0.
$$

It is locally finite if it is both inward and outward locally finite, i.e., if its degree function satisfies

$$
\deg(v) := \deg^+(v) + \deg^-(v) \leq M_v \quad \text{for all } v \in V \text{ and some } M_v.
$$

Example 2.2 If $G$ is unweighted, then it is locally finite if and only if each node has only finitely incident edges.

Throughout this paper we suppose that $G := (V, E, \rho, \mu)$ is a locally finite graph such that $\rho(e) > 0$ for all $e \in E$, $\mu(v) > 0$ for all $v \in V$ and $\mu(V) < \infty$. Let $p \in [1, +\infty)$, we denote by $\ell^p(E, \rho)$ the space of all functions $\varphi : E \rightarrow \mathbb{R}$ such that

$$
\|\varphi\|_{\ell^p(E, \rho)} := \left( \sum_{e \in E} |\varphi(e)|^p \rho(e) \right)^{1/p} < \infty,
$$

or else

$$
\|\varphi\|_{\ell^\infty(E, \rho)} := \sup_{e \in E} |\varphi(e)| \rho(e) < \infty.
$$
For $p = 2$, $\ell^2(E, \rho)$ is a Hilbert space endowed with the inner product

$$\langle \varphi_1, \varphi_2 \rangle_\rho = \sum_{e \in E} \rho(e) \varphi_1(e) \varphi_2(e).$$

Similarly, we denote by $\ell^p(V, \mu)$ the space of all functions $\phi : V \to \mathbb{R}$ such that

$$\|\phi\|_{\ell^p(V, \mu)} := \left( \sum_{v \in V} |\phi(v)|^p \mu(v) \right)^{1/p} < \infty,$$

or else

$$\|\phi\|_{\ell^\infty(V, \mu)} := \sup_{v \in V} |\phi(v)| \mu(v) < \infty.$$

For $p = 2$, $\ell^2(V, \mu)$ is a Hilbert space endowed with the inner product

$$\langle \phi_1, \phi_2 \rangle_\mu = \sum_{v \in V} \mu(v) \phi_1(v) \phi_2(v).$$

We simply write $\ell^p(E)$ if $\rho \equiv 1$ and $\ell^p(V)$ if $\mu \equiv 1$. Define

$$(I^T \phi)(e) = \phi(e_+) - \phi(e_-), \quad \phi \in \mathbb{R}^V, \quad e \in E.$$

The difference operator $I^T \phi$ can be looked as a discretized version of the first derivative of a function $\phi$ defined in all points of $G$. This plays a relevant role in the development of functional analysis on graphs. For $p \in [0, +\infty[$, we define the discrete Sobolev spaces of order one by

$$W^{1, p}_{\rho, \mu}(V) = \{ \phi \in \ell^p(V, \mu) : I^T \phi \in \ell^p(E, \rho) \}.$$

The space $W^{1, p}_{\rho, \mu}(V)$ is a Banach space endowed with the norm

$$\|\phi\|_{W^{1, p}_{\rho, \mu}} = \|\phi\|_{\ell^p(V, \mu)} + \|I^T \phi\|_{\ell^p(E, \rho)},$$

and a Hilbert space for $p = 2$ endowed with the inner product

$$\langle \phi, \psi \rangle_{\rho, \mu} = \sum_{v \in V} \mu(v) \phi(v) \psi(v) + \sum_{e \in E} \rho(e) (\phi(e_+) - \phi(e_-)) (\psi(e_+) - \psi(e_-)).$$

Recall that the distance $\text{dist}_\rho(v, w)$ of two nodes $v, w$ is defined as the infimum of the lengths of all paths from $v$ to $w$. In this way, $G$, to be more precise $V$, becomes a metric space which is not complete in general unless $\rho$ is uniformly bounded away from 0, i.e., $1/\rho \in \ell^\infty$. The ball of radius $r > 0$ and center $v_0$ with respect to $\text{dist}_\rho$ is defined by

$$B_\rho(v_0, r) := \{ w \in V : \text{dist}_\rho(v_0, w) < r \}.$$

If $G$ is connected, then by Proposition 38 (1) in Mugnolo (2014), the space $W^{1, p}_{\rho, \mu}(V)$ is densely and continuously embedded in $\ell^p(V, \mu)$ for all $1 \leq p \leq \infty$. Let additionally $p < \infty$, then by Proposition 38 (2) in Mugnolo (2014) this embedding is compact if for $\epsilon > 0$ there are $v \in V$ and $r > 0$ such that
(i) \( B_\rho(v, r) \) is a finite set
(ii) there holds
\[
\sum_{w \in B_\rho(v, r)} |\phi(w)|^p \mu(w) < \epsilon^p, \tag{2.1}
\]
for all \( \phi \) in the unit ball of \( W_{\rho, \mu}^{1,p}(V) \).

**Remark 2.3**

1. Condition (i) is satisfied if \( \rho \) is uniformly bounded from below away from 0, and in particular (2.1) holds.
2. For all \( \epsilon > 0 \), condition (ii) is satisfied for \( r > \text{vol}_\rho(G) := \sum_{e \in E} \rho(e) \) if \( \text{vol}_\rho(G) \) is finite.

The space \( W_{0, \rho, \mu}^{1,p}(V) \) is the closure of the space \( C_0(V) \) of finitely supported functions on \( V \) in the norm of \( W_{\rho, \mu}^{1,p}(V) \) (\( C_0(V) \) plays the role of test functions). For \( 1 \leq p \leq \infty \), \( W_{0, \rho, \mu}^{1,p}(V) \) is a Banach space with respect to the norm of \( W_{\rho, \mu}^{1,p}(V) \), and a Hilbert space for \( p = 2 \). For \( 1 \leq p < \infty \) it is continuously and densely embedded into \( \ell^p(V, \mu) \). If \( 1 \leq p < \infty \), then it is separable in \( \ell^p(V, \mu) \) and if \( 1 < p < \infty \) it is uniformly convex and hence reflexive (Mugnolo 2014).

### 2.2 Abstract surjectivity result

Let \( E \) be a reflexive Banach space with its dual \( E^* \) and \( A : D(A) \subset E \to 2^{E^*} \) be a multivalued function, where \( D(A) = \{ u \in E : Au \neq \emptyset \} \), stands for the domain of \( A \). We say that \( A \) is monotone if \( \langle u^* - v^*, u - v \rangle_{E^* \times E} \geq 0 \) for all \( u^* \in Au, \ v^* \in Av \) and \( u, v \in D(A) \).

If, moreover, \( A \) has a maximal graph in the sense of inclusion among all monotone operators, then we say that \( A \) is maximal monotone. We say that \( A \) is pseudomonotone operators if it satisfies the following properties,

(a) for each \( u \in E \), the set \( Au \) is nonempty, closed and convex in \( E^* \).

(b) \( A \) is upper semicontinuous from each finite dimensional subspace of \( E \) into \( E^* \) endowed with its weak topology;

(c) if \( u_n \rightharpoonup u \) weakly in \( E \), \( u_n^* \in Au_n \) and \( \limsup_{n \to \infty} \langle u_n^*, u_n - u \rangle_{E^* \times E} \leq 0 \), then for each \( \nu \in E \) there exists \( v^* \in Au \) such that \( \langle v^*, u - v \rangle_{E^* \times E} \leq \liminf_{n \to \infty} \langle u_n^*, u_n - v \rangle_{E^* \times E} \).

For a linear, maximal monotone operator \( L : D(L) \subset E \to E^* \), an operator \( A \) is said to be pseudomonotone with respect to \( D(L) \)(or \( L \)-pseudomonotone) if (a) and (b) are satisfied and

(c') for each sequences \( \{ u_n \} \subset D(L) \) and \( \{ u_n^* \} \subset E^* \) with \( u_n \rightharpoonup u \) weakly in \( E \), \( Lu_n \rightharpoonup Lu \) weakly in \( E^* \), \( u_n^* \in Au_n \) for all \( n \in \mathbb{N} \), \( u_n^* \rightharpoonup u^* \) weakly in \( E^* \) and \( \limsup_{n \to +\infty} \langle u_n^*, u_n - u \rangle_{E^* \times E} \leq 0 \), we have \( u^* \in Au \) and \( \lim_{n \to +\infty} \langle u_n^*, u_n \rangle_{E^* \times E} = \langle u^*, u \rangle_{E^* \times E} \).

A is coercive if there exists a function \( c : \mathbb{R}^+ \to \mathbb{R} \) with \( c(r) \to \infty \) as \( r \to \infty \) such that \( \langle u^*, u \rangle_{E^* \times E} \geq c(\|u\|_E)\|u\|_E \) for every \( u, u^* \in \text{Graph}(A) \).

Now let \( f : E \to \overline{\mathbb{R}} := \mathbb{R} \cup \{ +\infty \} \) be a proper, convex and lower semicontinuous functional. The mapping \( \partial_{c} f : E \to 2^{E^*} \) defined by
\[
\partial_{c} f(u) = \{ u^* \in E^* : \langle u^*, v - u \rangle_{E^* \times E} \leq f(v) - f(u) \text{ for all } v \in E \},
\]
is called the subdifferential of \( f \). Any element \( u^* \in \partial c f(u) \) is called a subgradient of \( f \) at \( u \). It is a well-know fact that \( \partial c f \) is a maximal monotone operator.

Let \( F : E \to \mathbb{R} \) be a locally Lipschitz continuous functional and \( u, v \in E \). We denote by \( F^\circ(u; v) \) the generalized Clarke directional derivative of \( F \) at the point \( u \) in the direction \( v \) defined by

\[
F^\circ(u; v) = \limsup_{w \to u, t \downarrow 0} \frac{F(w + tv) - F(w)}{t}.
\]

The generalized Clarke gradient \( \partial F : E \to 2^{E^*} \) of \( F \) at \( u \in E \) is defined by

\[
\partial F(u) = \{ \xi \in E^* : \langle \xi, v \rangle_{E^* \times E} \leq F^\circ(u; v) \text{ for all } v \in E \}.
\]

We collect the following properties

(a) the function \( v \mapsto F^\circ(u; v) \) is positively homogeneous, subadditive and satisfies

\[
|F^\circ(u; v)| \leq L_u \|v\|_E \quad \text{for all } v \in E,
\]

where \( L_u > 0 \) is the rank of \( F \) near \( u \).

(b) \( (u, v) \mapsto F^\circ(u; v) \) is upper semicontinuous.

(c) \( \partial F(u) \) is a nonempty, convex and weakly* compact subset of \( E^* \) with \( \|\xi\|_{E^*} \leq L_u \) for all \( \xi \in \partial F(u) \).

(d) for all \( v \in E \), we have \( F^\circ(u; v) = \max \{ \langle \xi, v \rangle_{E^* \times E} : \xi \in \partial F(u) \} \).

We say that a function \( F : E \to \mathbb{R} \) is regular at \( x \), if for all \( v \), the usual one-sided directional derivative

\[
F'(x, v) := \lim_{h \downarrow 0} \frac{F(x + hv) - F(x)}{h}
\]

exists and is equal to the generalized directional derivative \( F^\circ(x; v) \). By Proposition 2.3.6 in Clarke (1990), if \( F \) is locally Lipschitz and convex, then it is regular at any \( x \).

The following surjectivity result for operators which are \( L \)–pseudomonotone will be used in our existence theorems in Sects. 3 and 4 (cf. (Papageorgiou et al. 1999, Theorem 2.1)).

**Theorem 2.4** If \( E \) is a reflexive strictly convex Banach space, \( L : D(L) \subset E \to E^* \) is a linear maximal monotone operator, and \( A : E \to 2^{E^*} \) is a multivalued operator, which is bounded, coercive and \( L \)–pseudomonotone. Then \( L + A \) is a surjective operator, i.e. for all \( f \in E^* \), there exists \( u \in E \) such that \( Lu + Au \ni f \).

It is worth to mention that one can drop the strict convexity of the reflexive Banach space \( E \). It suffices to invoke the Troyanski renorming theorem to get an equivalent norm so that the space itself and its dual are strictly convex (cf. (Zeidler 1990, Proposition 32.23, p. 862)).

### 3 Nonconvex sum functionals on graphs

In this section, we will prove the discrete counterpart of the Aubin–Clarke theorem concerning the subdifferentiability of nonconvex sum functionals. We consider a function \( j : \mathbb{R} \to \mathbb{R} \) which satisfies the following hypothesis \( H(j) \):
H(j)\textsubscript{1} \; j : \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz.}

H(j)\textsubscript{2} \; \text{there exists } \alpha_j > 0 \text{ such that}

\[ |z| \leq \alpha_j(1 + |s|), \quad \forall z \in \partial j(s). \]

Next we define the superpotential \( J : \ell^2(V, \mu) \to \mathbb{R} \) defined by

\[ J(\phi) = \sum_{v \in V} \mu(v) j(\phi(v)). \]

for all \( \phi \in \ell^2(V, \mu) \). The sum functional \( J \) can be seen as the discrete version of the classical integral functionals. The following result is the discrete version of the Aubin–Clarke theorem Clarke (1990).

**Proposition 3.1** Under the assumption \( H(j) \):

(1) The functional \( J \) is well defined and finite on \( \ell^2(V, \mu) \).

(2) \( J \) is locally Lipschitz.

(3) For all \( \phi, \psi \in \ell^2(V, \mu) \), we have

\[ J(\phi) - J(\psi) \leq \sum_{v \in V} \mu(v) j(\phi(v)) - j(\psi(v)). \] (3.1)

(4) For all \( \phi \in \ell^2(V, \mu) \), we have

\[ \partial J(\phi) \subseteq \sum_{v \in V} \mu(v) \partial j(\phi(v)). \]

This inclusion is understood in the sense that for each \( \phi^* \in \partial J(\phi) \subseteq \ell^2(V, \mu) \), there exists a mapping \( V \ni v \mapsto \xi(v) \) such that \( \xi(v) \in \partial j(\phi(v)) \) and

\[ \langle \phi^*, \psi \rangle = \sum_{v \in V} \mu(v) \xi(v) \psi(v). \]

for all \( \psi \in \ell^2(V, \mu) \).

**Proof** By \( H(j) \), Lebourg’s mean value theorem and Hölder’s inequality, we have

\[
|J(\phi_1) - J(\phi_2)| \leq \sum_{v \in V} \mu(v) |j(\phi_1(v)) - j(\phi_2(v))|
\leq \sum_{v \in V} \mu(v) |\phi_1(v) - \phi_2(v)| (\xi \in \partial j(s) \text{ with } s \in [\phi_1(v), \phi_2(v)])
\leq \alpha_j \sum_{v \in V} \mu(v) (1 + |\phi_1(v)| + |\phi_2(v)|) |\phi_1(v) - \phi_2(v)|
\leq \alpha_j \left( \sum_{v \in V} \mu(v)(1 + |\phi_1(v)| + |\phi_2(v)|)^2 \right)^{1/2} \|\phi_1 - \phi_2\|_{\ell^2(V, \mu)}
\leq \alpha_j (1 + \|\phi_1\|_{\ell^2(V, \mu)} + \|\phi_2\|_{\ell^2(V, \mu)}) \|\phi_1 - \phi_2\|_{\ell^2(V, \mu)}
\leq \alpha_j' \|\phi_1 - \phi_2\|_{\ell^2(V, \mu)},
\]

where \( \alpha_j \) depends only on \( \alpha_j, \mu \) and \( m \) where \( m \) is such that \( \|\phi_1\|_{\ell^2(V, \mu)}, \|\phi_2\|_{\ell^2(V, \mu)} \leq m \). Consequently, the functional \( J \) is well-defined, finite and locally Lipschitz.
Let $\phi, \psi \in \ell^2(V, \mu)$, by Fatou’s Lemma with counting measure,

$$J^0(\phi; \psi) = \limsup_{\theta \to \phi, \lambda \to 0} \frac{J(\theta + \lambda \psi) - J(\theta)}{\lambda}$$

$$= \limsup_{\theta \to \phi, \lambda \to 0} \frac{1}{\lambda} \sum_{v \in V} \mu(v) \left( j(\theta(v) + \lambda \psi(v)) - j(\theta(v)) \right)$$

$$\leq \sum_{v \in V} \mu(v) \limsup_{\theta \to \phi, \lambda \to 0} \frac{j(\theta(v) + \lambda \psi(v)) - j(\theta(v))}{\lambda}$$

$$\leq \sum_{v \in V} \mu(v) j^0(\phi(v); \psi(v)).$$

Now, define $\hat{j}$ and $\hat{J}$ as follows:

$$\hat{j}(\psi) = j^0(\phi; \psi), \quad \hat{J}(\psi) = \sum_{v \in V} \mu(v) \hat{j}(\psi(v)).$$

It is clear that $\hat{j}$ is convex and thus, so is $\hat{J}$. If we observe that $\hat{J}(0) = \hat{J}(0) = 0$, we have $\hat{J}(\psi) - \hat{J}(0) \geq \langle \xi, \psi \rangle_{\ell^2(V, \mu)}$ for all $\psi$ and $\xi \in \partial \hat{J}(0)$. Since $\partial \hat{J}(0) \subset \sum_{v \in V} \mu(v) \partial j(\phi)$, for convex functions, see Ioffe and Levin (1972). Then there exists a map $v \mapsto \xi(v)$ with $\xi(v) \in \partial j(\phi)$ such that for every $\theta \in \ell^2(V, \mu)$

$$\langle \xi, \theta \rangle_{\ell^2(V, \mu)} = \sum_{v \in V} \mu(v) \xi(v) \theta(v).$$

However, $\partial \hat{J}(0) = \partial j(\phi)$, so the result would follow.

Remark that if either $j$ or $-j$ is regular, then $J$ is also regular and equality in (3.1) holds true. In fact, one have

$$J^0(\phi; \psi) = \limsup_{\theta \to \phi, h \downarrow 0} \frac{J(\theta + h \psi) - J(\psi)}{h}$$

$$\geq \lim_{h \downarrow 0} \frac{j(\phi + h \psi) - j(\phi)}{h}$$

$$= \lim_{h \downarrow 0} \sum_{v \in V} \mu(v) \frac{j(\phi(v) + h \psi(v)) - j(\phi(v))}{h}$$

$$= \sum_{v \in V} \mu(v) \lim_{h \downarrow 0} \frac{j(\phi(v) + h \psi(v)) - j(\phi(v))}{h}$$

$$= \sum_{v \in V} \mu(v) j'(\phi(v); \psi(v))$$

$$= \sum_{v \in V} \mu(v) j^0(\phi(v); \psi(v)).$$

which, by Proposition 3.1, leads to

$$J^0(\phi; \psi) = \sum_{v \in V} \mu(v) j^0(\phi(v); \psi(v)).$$
Moreover, $J^\circ = J'$ since
\[
J'(\phi; \psi) = \lim_{h \downarrow 0} \frac{\sum_{v \in V} \mu(v) j(\phi(v) + h \psi(v)) - j(\phi(v))}{h} \\
= \sum_{v \in V} \mu(v) j'(\phi(v); \psi(v)) \\
= J^\circ(\phi; \psi).
\]

**Proposition 3.2** Under hypothesis $H(j)$, the following inequalities hold
\[
J^0(\phi; \psi) \leq \alpha_j \left(1 + \|\phi\|_{\ell^2(V, \mu)}\right) \|\psi\|_{\ell^2(V)}, \quad \forall \phi, \psi \in \ell^2(V, \mu),
\]
and
\[
\|\theta\|_{\ell^2(V, \mu)} \leq \alpha_j \left(1 + \|\phi\|_{\ell^2(V, \mu)}\right), \quad \forall \theta \in \partial J_{|\ell^2(V, \mu)}(\phi), \phi \in \ell^2(V, \mu).
\]

**Proof** Let $\phi, \psi \in \ell^2(V, \mu)$, we have
\[
J^0(\phi; \psi) \leq \sum_{v \in V} \mu(v) j^0(\phi(v); \psi(v)) \\
= \sum_{v \in V} \mu(v) \max[\theta \cdot \psi(v) \mid \theta \in \partial j(\phi(v))] \\
= \sum_{v \in V} \mu(v) \max[|\theta| \cdot |\psi(v)| \mid \theta \in \partial j(\phi(v))].
\]

By $H(j)$, we have
\[
J^0(\phi; \psi) \leq \alpha_j \sum_{v \in V} \mu(v) (1 + |\phi(v)|) |\psi(v)| \\
\leq \alpha_j \sum_{v \in V} \sqrt{\mu(v)} (1 + |\phi(v)|) \sqrt{\mu(v)} |\psi(v)| \\
\leq \alpha_j \left(\sum_{v \in V} \mu(v) (1 + |\phi(v)|)^2\right)^{1/2} \|\psi(v)\|_{\ell^2(V, \mu)} \\
\leq \alpha'_j \left(1 + \|\phi\|_{\ell^2(V, \mu)}\right) \|\psi(v)\|_{\ell^2(V, \mu)}.
\]

This, together with (Clarke 1990, Proposition 2.1.2), yields
\[
\|\theta\|_{\ell^2(V, \mu)} = \sup\{\langle \theta, \psi \rangle_{\ell^2(V, \mu)} \mid \|\psi\|_{\ell^2(V, \mu)} \leq 1\} \\
\leq \sup\{J^0(\phi; \psi) \mid \|\psi\|_{\ell^2(V, \mu)} \leq 1\} \\
\leq \alpha'_j \left(1 + \|\phi\|_{\ell^2(V, \mu)}\right), \quad \forall \theta \in \partial J_{|\ell^2(V, \mu)}(\phi), \phi \in \ell^2(V, \mu).
\]

In what follows, we consider a superpotential $j$ which subdifferential is obtained by "filling in the gaps" procedure Rauch (1977). For $(v, t) \in V \times \mathbb{R}$, define
\[
j(v, t) = \int_0^t \beta(v, s) \, ds,
\]
where $\beta : V \times \mathbb{R} \to \mathbb{R}$ is a given function.
where $\beta : V \times \mathbb{R} \to \mathbb{R}$ is a function such that $\beta(v, \cdot)$ is measurable for all $v \in V$ and satisfies the following growth condition:

$$|\beta(v, t)| \leq \alpha \beta(1 + |t|),$$

for all $v \in V$ and a.e $t \in \mathbb{R}$. Note that $j(v, \cdot)$ is locally Lipschitz and satisfies a growth condition with eventually different constant.

We present the discrete version of the functional in Sect. 2 of Chang (1981) which is in the following form:

$$J(\phi) = \sum_{v \in V} \mu(v) \int_{0}^{\phi(v)} \beta(v, t) \, dt.$$ 

Then $J$ is a locally Lipschitz function defined on $\ell^2(V, \mu)$.

Let’s first describe the “filling in the gaps” procedure. Let $\theta \in L^\infty_{\text{loc}}(\mathbb{R})$, for $\varepsilon > 0$ and $t \in \mathbb{R}$, we define:

$$\underline{\theta}_\varepsilon(t) = \text{ess inf}_{|t-s| \leq \varepsilon} \theta(s), \quad \overline{\theta}_\varepsilon(t) = \text{ess sup}_{|t-s| \leq \varepsilon} \theta(s).$$

For a fixed $t \in \mathbb{R}$, the functions $\underline{\theta}_\varepsilon$, $\overline{\theta}_\varepsilon$ are decreasing and increasing in $\varepsilon$, respectively. Let

$$\underline{\theta}(t) = \lim_{\varepsilon \to 0^+} \underline{\theta}_\varepsilon(t), \quad \overline{\theta}(t) = \lim_{\varepsilon \to 0^+} \overline{\theta}_\varepsilon(t),$$

and let $\hat{\theta}(t) : \mathbb{R} \to 2^\mathbb{R}$ be a multifunction defined by

$$\hat{\theta}(t) = [\underline{\theta}(t), \overline{\theta}(t)].$$

From Chang Chang (1981) we know that a locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ can be determined up to an additive constant by the relation

$$j(t) = \int_{0}^{t} \theta(s) \, ds,$$

such that $\partial j(t) \subset \hat{\theta}(t)$ for all $t \in \mathbb{R}$. If moreover, the limits $\theta(t \pm 0)$ exist for every $t \in \mathbb{R}$, then $\partial j(t) = \hat{\theta}(t)$.

Now by Proposition 3.1, we have

$$\partial J(\phi) \subset \sum_{v \in V} \mu(v) \hat{\beta}(v, \phi(v)).$$

If we note $j(\phi) = \sum_{v \in V} \mu(v) \hat{\beta}(v, \phi(v))$ and $J(\phi) = \sum_{v \in V} \mu(v) \bar{\beta}(v, \phi(v))$ we can write

$$\partial J(\phi) \subset [j(\phi), J(\phi)], \quad \text{for all } \phi \in \ell^2(V, \mu). \quad (3.2)$$

If $J$ is convex, then it is regular and equality in (3.2) holds true.

4 Existence result for elliptic problem

Let $G = (V, E, \rho, \mu)$ be a weighted direct graph. Let $j : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function and $j^0(\cdot, \cdot)$ denotes its generalized directional derivative. Consider a function $\gamma : E \to (0, \infty)$ such that $\gamma(e) = \gamma(\bar{e})$ and a function $\kappa : V \to (0, \infty)$ such that the following assumption H(G) hold:

$$\text{(H(G))}$$
(G1) There exist $\alpha, \alpha_\gamma > 0$ such that
$$\alpha \gamma \rho(e) \leq \gamma(e) \leq \alpha_\gamma \rho(e), \quad \text{for all } e \in E.$$  

(G2) There exist $\alpha, \alpha_\mu > 0$ such that
$$\alpha \mu \kappa(v) \leq \mu(v) \leq \alpha_\mu \kappa(v), \quad \text{for all } v \in V.$$ 

We denote $W_0 := W_{1,2}^{1,2}(V)$ and we define the operator $\mathcal{L}_{\gamma,\kappa}^G : W_0 \to W_0$ by
$$(\mathcal{L}_{\gamma,\kappa}^G \phi)(v) := \frac{1}{\mu(v)} \sum_{w \sim v} \gamma(v, w)(\phi(v) - \phi(w)) + \frac{\kappa(v)}{\mu(v)} \phi(v), \quad v \in V, \phi \in W_0.$$ 

The aim of this paper is to prove the existence of solutions to the problem of finding $\phi$ such that
$$\mathcal{L}_{\gamma,\kappa}^G \phi + \partial J(\phi) \ni f, \quad \phi \in W_0,$$  

which is equivalent to finding $\phi \in W_0$ such that
$$\sum_{v \in V} \mu(v)(\mathcal{L}_{\gamma,\kappa}^G \phi)(v)(\psi(v) - \phi(v)) + \sum_{v \in V} \mu(v) j_0^0(\phi(v); \psi(v) - \phi(v)) \geq 0.$$ 

Note that problems (4.1) and (4.3) are not equivalent. This will be the case if, for example, either $j$ or $-j$ is regular. Generally, if $\phi$ is a solution of Problem (5.6), then it is a solution of Problem (4.3).

**Definition 4.1** We say that $\phi \in W_0$ is a weak solution to problem (4.1), if
$$\sum_{v \in V} \mu(v)(\mathcal{L}_{\gamma,\kappa}^G \phi)(v)(\psi(v) - \phi(v)) + \sum_{v \in V} \mu(v) j_0^0(\phi(v); \psi(v) - \phi(v)) \geq \sum_{v \in V} \mu(v) f(v)(\psi(v) - \phi(v)).$$ 

for all $\psi \in W_0$.

The next theorem gives an existence result in the elliptic case. The proof relies on a surjectivity result for pseudomonotone operators. The smallness condition (4.4) ensures the coercivity of the problem.

**Theorem 4.2** Under assumptions $H(G)$ and $H(j)$ with
$$\alpha_j < \frac{1}{2} \alpha_\gamma \wedge \alpha_\mu,$$  

the problem (4.3) has at least one weak solution.
Proof We observe first that
\[
\langle \mathcal{L}^G_{y,k} \phi, \psi \rangle = \sum_{v \in V} \mu(v) \langle \mathcal{L}^G_{y,k} \phi(v) \rangle(v) \psi(v)
\]
\[
= \sum_{v \in V} \sum_{w \in V} \gamma(v, w)(\phi(v) - \phi(w)) \psi(v) + \sum_{v \in V} \kappa(v) \phi(v) \psi(v)
\]
\[
= \frac{1}{2} \sum_{(v, w) \in E} \gamma(v, w)(\phi(v) - \phi(w))(\psi(v) - \psi(w)) + \sum_{v \in V} \kappa(v) \phi(v) \psi(v).
\]
From one side, we have
\[
\langle \mathcal{L}^G_{y,k} \phi, \phi \rangle = \frac{1}{2} \sum_{(v, w) \in E} \gamma(v, w)(\phi(v) - \phi(w))^2 + \sum_{v \in V} \kappa(v) \phi^2(v)
\]
\[
= \frac{1}{2} \sum_{(v, w) \in E} \gamma(v, w) \rho(v, w)(\phi(v) - \phi(w))^2 + \sum_{v \in V} \frac{\kappa(v)}{\mu(v)} \rho(v, w) \phi^2(v)
\]
\[
\geq \frac{1}{2} \alpha_y \wedge \alpha_{\mu} \| \phi \|^2_{W_0}.
\]
Thus, \( \mathcal{L}^G_{y,k} \) is strongly monotone and coercive. From another side
\[
\langle \mathcal{L}^G_{y,k} \phi, \psi \rangle = \frac{1}{2} \sum_{(v, w) \in E} \gamma(v, w) \rho(v, w)(\phi(v) - \phi(w))(\psi(v) - \psi(w))
\]
\[
+ \sum_{v \in V} \frac{\kappa(v)}{\mu(v)} \rho(v, w) \phi(v) \psi(v)
\]
\[
\leq \frac{1}{2} \alpha_y \| \mathcal{I}^T \phi \|_{\ell^2(E, \rho)} \| \mathcal{I}^T \psi \|_{\ell^2(E, \rho)} + \alpha_{\mu} \| \phi \|_{\ell^2(V, \mu)} \| \psi \|_{\ell^2(V, \mu)}
\]
\[
\leq \frac{1}{2} \alpha_y \| \phi \|_{W_0} \| \psi \|_{W_0}.
\]
Thus, \( \mathcal{L}^G_{y,k} \) is continuous.

Claim 1: The operator \( \mathcal{L}^G_{y,k} + \partial J \) is coercive.

We have
\[
\inf \{ \langle \mathcal{L}^G_{y,k} \phi + \xi, \phi \rangle | \xi \in \partial J(\phi) \} = \langle \mathcal{L}^G_{y,k} \phi, \phi \rangle + \inf \{ \langle \xi, \phi \rangle_{\ell^2(V, \mu)} | \xi \in \partial J(\phi) \}
\]
\[
\geq \frac{1}{2} \alpha_y \wedge \alpha_{\mu} \| \phi \|^2_{W_0} - \sup \{ \| \xi \|_{\ell^2(V, \mu)} | \xi \in \partial J(\phi) \} \| \phi \|_{\ell^2(V, \mu)}
\]
\[
\geq \frac{1}{2} \alpha_y \wedge \alpha_{\mu} \| \phi \|^2_{W_0} - \alpha_J \| \phi \|_{\ell^2(V, \mu)} - \alpha_J \| \phi \|^2_{W_0}
\]
\[
\geq \frac{1}{2} \alpha_y \wedge \alpha_{\mu} \| \phi \|^2_{W_0} - \alpha_J \| \phi \|_{W_0} - \alpha_J \| \phi \|^2_{W_0}
\]
\[
\geq \left( \frac{1}{2} \alpha_y \wedge \alpha_{\mu} - \alpha_J \right) \| \phi \|^2_{W_0} - \alpha_J \| \phi \|_{W_0}.
\]
If \( \frac{1}{2} \alpha_y \wedge \alpha_{\mu} > \alpha_J \), the above inequality implies that \( \mathcal{L}^G_{y,k} + \partial J : W_0 \rightarrow W_0^* \) is coercive.

Claim 2: The operator \( \mathcal{L}^G_{y,k} + \partial J \) is pseudomonotone.

We know that \( \partial J \) is nonempty, convex, weak-compact subset of \( W_0 \). Then for each \( \phi \in W_0 \), \( \mathcal{L}^G_{y,k} \phi + \partial J(\phi) \) is nonempty, bounded, closed and convex subset of \( W_0 \). Moreover, \( \mathcal{L}^G_{y,k} \phi + \partial J(\phi) \) is upper semicontinuous from \( W_0 \) to \( w - W_0 \).
Let $\phi_k$ be a sequence in $W_0$ converging weakly to $\phi$, and $\xi_k \in \partial J(\phi_k)$ such that

$$\limsup_{k \to \infty} \langle L_{\gamma,k}^G (\phi_k) + \xi_k, \phi_k - \phi \rangle \leq 0,$$

which implies

$$\limsup_{k \to \infty} \langle L_{\gamma,k}^G (\phi_k), \phi_k - \phi \rangle + \liminf_{k \to \infty} \langle \xi_k, \phi_k - \phi \rangle \leq 0. \quad (4.7)$$

The embedding $W_0 \hookrightarrow \ell^2(V, \mu)$ is compact. Therefore, $\phi_k$ converges strongly in $\ell^2(V, \mu)$ to $\phi$. By applying Theorem 2.2 in Chang (1981), we have

$$\partial (J|_{W_0}) (\phi) \subset \partial (J|_{\ell^2(V, \mu)}) (\phi), \quad \forall \phi \in W_0.$$

Therefore,

$$|\langle \theta_k, \phi_k - \phi \rangle|_{W_0} | \leq \text{const} \| \theta_k \|_{\ell^2(V, \mu)} \| \phi_k - \phi \|_{\ell^2(V, \mu)}.$$

Thus,

$$|\langle \theta_k, \phi_k - \phi \rangle|_{W_0} \to 0, \quad \text{as } k \to \infty.$$

Then, from (4.7) we have

$$\limsup_{k \to \infty} \langle L_{\gamma,k}^G (\phi_k), \phi_k - \phi \rangle \leq 0.$$

We have from $\phi_k \to \phi$ weakly in $W_0$

$$\limsup_{k \to \infty} \langle L_{\gamma,k}^G (\phi_k) - L_{\gamma,k}^G (\phi), \phi_k - \phi \rangle \leq 0.$$

By the coercivity of $L_{\gamma,k}^G$, we get

$$\limsup_{k \to \infty} \| \phi_k - \phi \|_{W_0} \leq 0.$$

Therefore, we obtain

$$\phi_k \to \phi \quad \text{strongly in } W_0,$$

$$L_{\gamma,k}^G \phi_k \to L_{\gamma,k}^G \phi \quad \text{strongly in } W_0.$$

It follows that there exists $\xi$ such that $\xi_k \to \xi$ weakly* in $W_0$ and

$$\lim_{k \to \infty} \langle L_{\gamma,k}^G (\phi_k) + \xi_k, \phi_k - \psi \rangle = \langle L_{\gamma,k}^G (\phi) + \xi, \phi - \psi \rangle$$

This implies that $L_{\gamma,k}^G + \partial J : W_0 \to W_0$ is pseudomonotone, which completes the proof. $\square$

Let us consider the following additional assumption

$H(j^0)$ There exists $\alpha_{j^0} > 0$ such that

$$j^0(s; t - s) + j^0(t; s - t) \leq \alpha_{j^0} |t - s|^2,$$

for all $s, t \in \mathbb{R}$.

The next theorem gives a uniqueness result of the hemivariational inequality in the elliptic case. The proof is elementary and the condition $H(j^0)$ is a standard one.
Theorem 4.3 Under assumptions $H(G)$, $H(j)$ and $H(j^0)$ with
\[ \alpha_j^0 \lor \alpha_j < \frac{1}{2} \alpha_{\gamma} \land \alpha_{\mu}, \]
the weak solution of the Problem (4.1) is unique.

Proof Let $\phi_1$ and $\phi_2$ be two weak solutions of Problem (4.1). It follows then that
\[
\langle L^G_{y,k} \phi_1, \psi - \phi_1 \rangle + \sum_{v \in V} \mu(v)^0(\phi_1(v); \psi(v) - \phi_1(v)) \geq (f, \psi - \phi_1), \text{ for all } \phi , \tag{4.8}
\]
\[
\langle L^G_{y,k} \phi_2, \psi - \phi_2 \rangle + \sum_{v \in V} \mu(v)^0(\phi_2(v); \psi(v) - \phi_2(v)) \geq (f, \psi - \phi_2), \text{ for all } \phi . \tag{4.9}
\]
We replace $\phi$ in (4.8) by $\phi_2$ and in (4.9) by $\phi_1$ and we sum up the two inequalities. We obtain that
\[
\langle L^G_{y,k} (\phi_1 - \phi_2), \phi_2 - \phi_1 \rangle + \sum_{v \in V} \mu(v)[j^0(\phi_1(v); \phi_2(v) - \phi_1(v)) + j^0(\phi_2(v); \phi_1(v) - \phi_2(v))] \geq 0.
\]
It follows that
\[-\frac{1}{2} \alpha_{\gamma} \land \alpha_{\mu} \| \phi_2 - \phi_1 \|^2_{W_0} + \alpha_j^0 \sum_{v \in V} \mu(v) | \phi_2(v) - \phi_1(v) |^2 \geq 0.\]
Thus,
\[
(\alpha_j^0 - \frac{1}{2} \alpha_{\gamma} \land \alpha_{\mu}) \| \phi_2 - \phi_1 \|^2_{W_0} \geq 0.
\]
If $\alpha_j^0 < \frac{1}{2} \alpha_{\gamma} \land \alpha_{\mu}$, we have $\phi_1 = \phi_2$, which completes the proof. \hfill \Box

5 Existence result for parabolic problem

For $0 < T < \infty$, we denote by $\mathcal{V} := L^2(0, T; \ell^2(V, \mu))$ the usual time Sobolev space endowed with the norm
\[
\| \phi \|_{\mathcal{V}} = \left( \int_0^T \| \phi(t) \|_{\ell^2(V, \mu)}^2 \, dt \right)^{1/2},
\]
and consider a function $j : (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ which satisfies the following hypothesis $H(j)$:

$H(j)_1$ for each $r \in \mathbb{R}$, the function $t \mapsto j(t, r)$ is measurable on $(0, T)$,

$H(j)_2$ for a.e. $t \in (0, T)$, the functional $r \mapsto j(t, r)$ is locally Lipschitz,

$H(j)_3$ there exists $\alpha_j > 0$ such that
\[
|z| \leq \alpha_j (1 + |r|), \quad \text{for all } z \in \partial j(t, r) \text{ and a.e. } t \in (0, T).
\]

We define the superpotential $J : \mathcal{V} \rightarrow \mathbb{R}$ defined by
\[
J(\phi) = \sum_{v \in V} \mu(v) \int_0^T j(t, \phi(t, v)) \, dt,
\]
for all $\phi \in \mathcal{V}$. Similarly to Proposition 3.2, we have the following result
Proposition 5.1 Assume the hypothesis $H(j)$ is fulfilled. Then the functional $J$ is locally Lipschitz and there exists $c_J > 0$ such that the following inequalities hold:

$$J^0(\phi; \psi) \leq c_J (1 + \|\phi\|_V) \|\psi\|_V, \quad \forall \phi, \psi \in V,$$

and

$$\|\theta\|_V \leq c_J (1 + \|\phi\|_V), \quad \forall \theta \in \partial(J|_V)(\phi), \phi \in V.$$

We introduce the function spaces

$$W_0 = L^2(0, T; W_0), \quad M_0 = \{\phi \in W_0 \mid \frac{\partial \phi}{\partial t} \in W_0\},$$

where the time derivative $\frac{\partial \phi}{\partial t}$ is understood in the sense of vector-valued distributions. The norm

$$\|\phi\|_{M_0} := \|\phi\|_{W_0} + \|\frac{\partial \phi}{\partial t}\|_{W_0},$$

make the space $M_0$ a Banach space. Moreover, the embeddings $M_0 \subset L^2(0, T; \ell^2(\mu))$ and $M_0 \subset C(0, T; \ell^2(\mu))$ are compact and continuous, respectively.

The purpose of this section is to prove the existence of solutions for the parabolic hemivariational inequalities on graphs, which can be stated as follows: Find $\phi \in M_0$ such that

$$\begin{cases}
\phi' + L_{G,\gamma,\kappa}(\phi) + \partial J(\phi) \ni f \quad &\text{in } V \times (0, T), \\
\phi(0, v) = \phi_0 \quad &\text{in } V.
\end{cases} \tag{5.1}$$

Definition 5.2 We say that $\phi \in M_0$ is a weak solution to problem (5.1), if $\phi(0, v) = \phi_0(v)$ in $V$, and the following inequality holds

$$\int_0^T \sum_{v \in V} \mu(v) \frac{\partial \phi(t, v)}{\partial t} (\psi(t, v) - \phi(t, v)) \, dt + \int_0^T \sum_{v \in V} \mu(v) \frac{\partial \phi(t, v)}{\partial t} (\psi(t, v) - \phi(t, v)) \, dt + \int_0^T \sum_{v \in V} f(t, v) (\psi(t, v) - \phi(t, v)) \, dt \geq \int_0^T \sum_{v \in V} f(t, v) (\psi(t, v) - \phi(t, v)) \, dt,$$

for all $\psi \in W_0$.

The following theorem is the main result of this section. The proof relies on a variant of the surjectivity result used in the previous section. It concerns the sum of a maximal monotone operator and a pseudomonotone one.

Theorem 5.3 Let $f \in W_0$ and assume that the hypotheses $H(G)$ and $H(j)$ are fulfilled. If $\alpha_j < \frac{1}{2} \alpha_{\gamma,\kappa} \wedge \alpha_{\mu}$, then the problem (5.1) admits a weak solution.

Proof First, we define the operator $\Lambda : W_0 \to W_0$ by

$$\Lambda(\phi)(\psi) := \int_0^T \sum_{v \in V} \mu(v) \psi(t, v) L_{G,\gamma,\kappa}(\phi(t, v) + \phi_0(t, v)) \, dt,$$

for all $\phi, \psi \in W_0$ and where $\phi_0$ is such that $\phi_0(t, v) = \phi_0(v)$ for all $(t, v) \in (0, T) \times V$. From (4.6) we have
\[ \Lambda(\phi)(\psi) = \int_0^T \sum_{v \in V} \mu(v) \psi(t, v) \mathcal{L}^G_{\gamma, k}(\phi(t, v) + \tilde{\phi}_0(t, v)) \, dt \]

\[ \leq \frac{1}{2} \alpha_y \vee \alpha_{\mu} \int_0^T \| \psi \|_{w_0} \| \phi + \tilde{\phi}_0 \|_{w_0} \, dt \]

\[ \leq \frac{1}{2} \alpha_y \vee \alpha_{\mu} \| \psi \|_{w_0} \| \phi + \tilde{\phi}_0 \|_{w_0}. \]

It follows that

\[ \| \Lambda(\phi) \|_{w_0} \leq \frac{1}{2} \alpha_y \vee \alpha_{\mu} \left( \| \phi \|_{w_0} + \| \tilde{\phi}_0 \|_{w_0} \right), \quad \text{for all } \phi \in w_0. \]

Thus, the operator \( \Lambda \) is continuous. Moreover, the operator \( \Lambda \) is strongly monotone. In fact, from (4.5), one can obtain

\[ \langle \Lambda(\phi) - \Lambda(\psi), \phi - \psi \rangle_{w_0} = \langle \Lambda(\phi), \phi - \psi \rangle_{w_0} - \langle \Lambda(\psi), \phi - \psi \rangle_{w_0} \]

\[ = \int_0^T \sum_{v \in V} \mu(v) (\phi(t, v) - \psi(t, v)) \mathcal{L}^G_{\gamma, k}(\phi(t, v) + \tilde{\phi}_0(t, v)) \, dt \]

\[ - \int_0^T \sum_{v \in V} \mu(v) (\phi(t, v) - \psi(t, v)) \mathcal{L}^G_{\gamma, k}(\psi(t, v) + \tilde{\phi}_0(t, v)) \, dt \]

\[ = \int_0^T \sum_{v \in V} \mu(v) (\phi(t, v) - \psi(t, v)) \mathcal{L}^G_{\gamma, k}(\phi(t, v) - \psi(t, v)) \, dt \]

\[ \geq \frac{1}{2} \alpha_y \wedge \alpha_{\mu} \int_0^T \| \phi(t, .) - \psi(t, .) \|_{w_0}^2 \, dt \]

\[ = \frac{1}{2} \alpha_y \wedge \alpha_{\mu} \| \phi - \psi \|_{w_0}^2, \]

for all \( \phi, \psi \in w_0 \).

Define the operator \( L : D(L) \subset w_0 \to w_0 \) by

\[ L\phi = \frac{\partial \phi}{\partial t}, \quad D(L) := \{ \phi \in M_0 \mid \phi(0) = 0 \}, \]

which is closed, linear, densely defined and maximal monotone operator Zeidler (1990).

Now, we shall prove the hypotheses of the surjectivity theorem.

Claim 1: The multivalued operator \( \Lambda + \partial J(\cdot + \tilde{\phi}_0) : w_0 \to 2^{w_0} \) is bounded and pseudomonotone with respect to \( D(L) \).

In fact, by the properties of Clarke’s subdifferential, we deduce that the set \( \Lambda(\phi) + \partial J(\phi + \tilde{\phi}_0) \) is nonempty, closed and convex in \( w_0 \) for all \( \phi \in w_0 \). By Proposition 5.1 and the continuity of \( \Lambda \), we obtain

\[ \| \Lambda \phi + \xi \|_{w_0} \leq \| \Lambda \phi \|_{w_0} + \| \xi \|_{w_0} \]

\[ \leq \frac{1}{2} \alpha_y \vee \alpha_{\mu} \left( \| \phi \|_{w_0} + \| \tilde{\phi}_0 \|_{w_0} \right) + \alpha_j \left( 1 + \| \phi \|_{w} + \| \tilde{\phi}_0 \|_{w} \right) \]

\[ \leq \alpha_j + (\alpha_j + \frac{1}{2} \alpha_y \vee \alpha_{\mu})(\| \phi \|_{w_0} + \| \tilde{\phi}_0 \|_{w_0}), \]

which implies that \( \Lambda + \partial J(\cdot + \tilde{\phi}_0) : w_0 \to 2^{w_0} \) is bounded. Moreover, since \( \Lambda \) is linear and continuous (hence demicontinuous) and \( \partial J \) is uppersemicontinuous from \( w_0 \) to \( w - w_0 \), then \( \Lambda + \partial J(\cdot + \tilde{\phi}_0) : w_0 \to 2^{w_0} \) is also uppersemicontinuous from \( w_0 \) to \( w - w_0 \).
It remains to verify the last condition. Let \( \{\phi_n\} \subset D(L) \) and \( \{\phi_n^*\} \subset \mathcal{W}_0 \) be such that \( \phi_n \to \phi \) weakly in \( \mathcal{W}_0 \), \( L\phi_n \rightharpoonup L\phi \) weakly in \( \mathcal{W}_0 \), \( \phi_n^* \in \Lambda \phi_n + \partial J(\phi_n + \tilde{\phi}_0) \) with \( \phi_n^* \to \phi^* \) weakly in \( \mathcal{W}_0 \), and

\[
\limsup_{n \to \infty} \langle \phi_n^*, \phi_n - \phi \rangle_{\mathcal{W}_0} \leq 0. \quad (5.2)
\]

Then, we are able to find a sequence \( \{\xi_n\} \subset \mathcal{W}_0 \) such that \( \xi_n \in \partial(\phi_n + \tilde{\phi}_0) \) and

\[
\phi_n^* = \Lambda \phi_n + \xi_n, \quad \text{for each } n \in \mathbb{N}.
\]

Consequently, from (5.2), we get

\[
\limsup_{n \to \infty} \langle \Lambda \phi_n, \phi_n - \phi \rangle_{\mathcal{W}_0} + \liminf_{n \to \infty} \langle \xi_n, \phi_n - \phi \rangle_{\mathcal{W}_0} \leq 0. \quad (5.3)
\]

Since \( \mathcal{W}_0 \subset \ell^2(V, \mu) \) and the embedding of \( \mathcal{W}_0 \) in \( \ell^2(V, \mu) \) is compact, we have that \( \phi_n \) strongly converges to \( \phi \) in \( \mathcal{V} \). Furthermore, one has

\[
\partial(J|_{\mathcal{W}_0})(\phi) \subset \partial(J|_{\mathcal{V}})(\phi), \quad (5.4)
\]

which means that

\[
\langle \xi_n, \phi_n - \phi \rangle_{\mathcal{W}_0} = \langle \xi_n, \phi_n - \phi \rangle_{\mathcal{V}}. \quad (5.5)
\]

Further, from the boundedness of \( \{u_n\} \) in \( \mathcal{W}_0 \), we have that \( \{\xi_n\} \) is bounded both in \( \mathcal{V} \) and in \( \mathcal{W}_0 \). Then, by (5.5), we pass to the limit as \( n \to \infty \) to get

\[
\lim_{n \to \infty} \langle \xi_n, \phi_n - \phi \rangle_{\mathcal{W}_0} = \lim_{n \to \infty} \langle \xi_n, \phi_n - \phi \rangle_{\mathcal{V}} = 0.
\]

This convergence combined with (5.3) and the monotonicity of \( \Lambda \) implies

\[
\limsup_{n \to \infty} \|\phi_n - \phi\|^2_{\mathcal{W}_0} \leq A^{-1} \limsup_{n \to \infty} \langle \Lambda \phi_n, \phi_n - \phi \rangle_{\mathcal{W}_0} + A^{-1} \lim_{n \to \infty} \langle \Lambda \phi, \phi - \phi_n \rangle_{\mathcal{W}_0} \leq 0,
\]

where \( A = \frac{1}{2} \alpha^\gamma \wedge \alpha^\mu \). Hence \( \phi_n \to \phi \) strongly in \( \mathcal{W}_0 \). On the other side, the reflexivity of \( \mathcal{W}_0 \) and boundedness of \( \{\xi_n\} \subset \mathcal{W}_0 \) allow to assume, at least for a subsequence, that \( \xi_n \) converges weakly in \( \mathcal{W}_0 \) to some \( \xi \in \mathcal{W}_0 \). Since \( \partial J \) is upper semicontinuous from \( \mathcal{W}_0 \) to \( w - \mathcal{W}_0 \) and it has convex and closed values, it is closed from \( \mathcal{W}_0 \) to \( w - \mathcal{W}_0 \) (see (Kamenskii et al. 2001, Theorem 1.1.4)). Therefore, we obtain \( \xi \in \partial\phi + \tilde{\phi}_0 \).

To conclude, we have \( \phi^* = \xi + \Lambda \phi \in \Lambda \phi + \partial J(\phi + \tilde{\phi}_0) \) and

\[
\langle \phi_n^*, \phi_n \rangle_{\mathcal{W}_0} = \langle \xi_n + \Lambda \phi_n, \phi_n \rangle_{\mathcal{W}_0} \to \langle \xi + \Lambda \phi, \phi \rangle_{\mathcal{W}_0} = \langle \phi^*, \phi \rangle_{\mathcal{W}_0},
\]

which means that the operator \( \Lambda + \partial J(\phi + \tilde{\phi}_0) : \mathcal{W}_0 \to 2^{\mathcal{W}_0} \) is pseudomonotone with respect to \( D(L) \).

**Claim 2:** The operator \( \Lambda + \partial J(\phi + \tilde{\phi}_0) : \mathcal{W}_0 \to 2^{\mathcal{W}_0} \) is coercive.

For all \( \phi \in \mathcal{W}_0 \) one has

\[
\langle \Lambda \phi + \partial J(\phi + \tilde{\phi}_0), \phi \rangle_{\mathcal{W}_0} = \langle \Lambda \phi, \phi \rangle_{\mathcal{W}_0} + \langle \partial J(\phi + \tilde{\phi}_0), \phi \rangle_{\mathcal{V}}
\]

\[
\geq A \|\phi\|^2_{\mathcal{W}_0} - A \|\tilde{\phi}_0\|_{\mathcal{W}_0} \|\phi\|_{\mathcal{W}_0} + \langle \partial J(\phi + \tilde{\phi}_0), \phi \rangle_{\mathcal{V}}
\]

\[
\geq A \|\phi\|^2_{\mathcal{W}_0} - A \|\tilde{\phi}_0\|_{\mathcal{W}_0} \|\phi\|_{\mathcal{W}_0} - \|\partial J(\phi + \tilde{\phi}_0)\|_{\mathcal{V}} \|\phi\|_{\mathcal{V}}
\]

\[
\geq A \|\phi\|^2_{\mathcal{W}_0} - A \|\tilde{\phi}_0\|_{\mathcal{W}_0} \|\phi\|_{\mathcal{W}_0} - \alpha_J \left(1 + \|\tilde{\phi}_0\|_{\mathcal{V}}\right) \|\phi\|_{\mathcal{V}}
\]

\[
\geq A \|\phi\|^2_{\mathcal{W}_0} - A \|\tilde{\phi}_0\|_{\mathcal{W}_0} \|\phi\|_{\mathcal{W}_0} - \alpha_J \|\phi\|_{\mathcal{W}_0} - \alpha_J \|\phi\|^2_{\mathcal{W}_0} - \alpha_J \|\phi\|_{\mathcal{W}_0} \|\tilde{\phi}_0\|_{\mathcal{W}_0}
\]

\[
\geq A \|\phi\|^2_{\mathcal{W}_0} - A \|\tilde{\phi}_0\|_{\mathcal{W}_0} \|\phi\|_{\mathcal{W}_0} - \alpha_J \|\phi\|_{\mathcal{W}_0} - \alpha_J \|\phi\|^2_{\mathcal{W}_0} - \alpha_J \|\phi\|_{\mathcal{W}_0} \|\tilde{\phi}_0\|_{\mathcal{W}_0}
\]
\[
\begin{aligned}
\geq \left( (A - \alpha_j) \| \phi \|_{W_0} - (A + \alpha_j) \| \tilde{\phi}_0 \|_{W_0} + \alpha_j \right) \| \phi \|_{W_0} \\
\geq c(\| \phi \|_{W_0}) \| \phi \|_{W_0},
\end{aligned}
\]

where \( c : \mathbb{R}^+ \to \mathbb{R} \) with \( c(r) = (A - \alpha_j) r - (A + \alpha_j) \| \tilde{\phi}_0 \|_{W_0} + \alpha_j. \) It is clear that \( c(r) \to \infty \) as \( r \to \infty, \) thus the operator \( \Lambda + \partial J(\cdot + \tilde{\phi}_0) : W_0 \to 2^{W_0} \) is coercive.

We are now in a position to apply the surjectivity result. We deduce that there exists a function \( \chi \in W_0 \) with \( \chi(0) = 0 \) solving the following inclusion problem

\[
\begin{cases}
Lx + \Lambda x + \partial J(x + \tilde{\phi}_0) \ni f, \quad \text{in } W_0 \\
\chi(0) = 0.
\end{cases}
\]

(5.6)

**Claim 3:** If \( \chi \in M_0 \) is a solution to problem (5.6), then \( \phi = \chi + \tilde{\phi}_0 \) is a weak solution to problem (5.1).

Let \( \chi \in M_0 \) be a solution to problem (5.6). Hence, \( \phi = \chi + \tilde{\phi}_0 \) solves the following problem

\[
\begin{cases}
\frac{\partial \phi}{\partial t} + \Lambda (\phi - \tilde{\phi}_0) + \partial J(\phi) \ni f, \quad \text{in } W_0, \\
\phi(0) = \phi_0.
\end{cases}
\]

(5.7)

By the definition of generalized Clarke subdifferential we obtain (5.1). This completes the proof.

\[\square\]

### 6 Concluding remarks

In this section, we give some remarks and extensions of the results proved in previous sections.

(1) Let \( \Phi : W_0 \to \overline{\mathbb{R}} \) be a proper, convex and lower semicontinuous functional such that \( 0 \in \partial C \Phi(\phi_0), \) where \( \partial C \) is the subdifferential in the sense of convex analysis. Suppose additionally that \( \phi_0 \in \text{int} D(\Phi). \) Then, the variational-hemivariational inequality: Find \( \phi \in M_0 \) such that

\[
\begin{cases}
\phi' + \mathcal{L}_{G, \gamma, \kappa} \phi + \partial J(\phi) + \partial C \Phi(\phi) \ni f \quad \text{in } V \times (0, T), \\
\phi(v, 0) = \phi_0 \quad \text{in } V,
\end{cases}
\]

(6.1)

admits a solution. To prove the existence of Problem (6.1), let us consider the functional \( \Psi : W_0 \to \overline{\mathbb{R}} \) defined by

\[
\Psi(\phi) = \int_0^T \Phi(\phi + \tilde{\phi}_0) \, dt
\]

Now, it suffices to continue on the proof of Theorem 5.3 and prove, additionally, that the operator \( \partial C \Psi \) is maximal monotone and strongly quasi-bounded with \( 0 \in \partial \Psi(0). \) The existence follows by the surjectivity result stated by Theorem 3.1 in Gasinski et al. (2015). Let us mention that the conditions on \( \Phi \) are fulfilled, for example, for the indicator function of \( K, \) a nonempty closed and convex subset of \( W_0 \) such that \( \phi_0 \in \text{int} K, \) see (Gasinski et al. 2015, Example 5.1.).

(2) One can think about an alternative proof of existence in both the elliptic and parabolic problems using the Galerkin scheme adapted to graph theory context. Let \( ( \mathcal{G}_n )_{n \geq 0} \)
be a growing family of finite graphs that exhaust $G$ in the sense of (Mohar 1982, Definition 4.1) and (Mugnolo 2013, Definition 3.3) and consider the problem of finding $\phi_n$ such that

$$\phi_n' + \mathcal{L}_{G_n}^{n} \phi_n + \sum_{v \in V_n} \mu(v) j_n'(\phi_n(v)) = f,$$

(6.2)

where $j_n$ is a mollification of $j$. Using techniques from the proof of Theorem 3.6 in Hua and Mugnolo (2015) and some standard calculation on the nonlinear term, one can prove that the sequence $(\phi_n)_n$ on $G_n$ is bounded in $H^1(0, T; \ell^2(V, \mu))$ and weak* in $L^\infty(0, T; W^0_0)$. By taking a subsequence, if necessary, it is possible to prove that $(\phi_n)_n$ converges to some $\phi$ and $j_n'(\phi_n)$ converges in $\ell^2(V, \mu)$ to some $\xi$. By the convergence theorem of Aubin and Cellina Aubin and Cellina (1984), it is clear that $\xi \in \partial j(\phi)$ and by taking the limit in (6.2), one can see that $\phi$ resolves Problem (5.3).

(3) Let $h$ be some nonnegative continuous function which satisfies with $j$ the following growth condition

$$|h(\xi_1)\xi| \leq c(1 + |\xi_1| + |\xi_2|), \text{ for all } \xi_1, \xi_2 \in \mathbb{R}, \text{ with } \xi \in \partial j(\xi_2),$$

where $c$ is some nonnegative constant. One can prove a version of Aubin-Clarke theorem for discrete functionals in the form:

$$J(\phi) = \sum_{v \in V} \mu(v) h(\phi(v)) j(\phi(v)).$$

With the above hypotheses we have for $\phi, \psi$ in $\ell^2(V, \mu)$ that

$$J^0(\phi; \psi) \leq \sum_{v \in V} \mu(v) h(v) j^0(\phi(v); \psi(v)).$$

With some modifications, one can prove that the quasi-hemivariational versions of Problems (5.1) and (5.3) admit weak solutions.

(4) Using Theorem 3.6 in Hua and Mugnolo (2015), the theory in this paper can be applied for the operator $\mathcal{L}_{G_0}^G$ if we assume that $G$ is uniformly locally finite and satisfy the $d-$isoperimetric inequality for some $d \geq 2$.

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**References**

Aayadi K, Akhlil K, Ben Aadi S, El Ouali M (2021) *Multivalued nonmonotone dynamic boundary condition*. Bound Value Probl 2021, 43 https://doi.org/10.1186s13661-021-01517-6

Aubin JP, Cellina A (1984) Differential inclusions. Set-Valued Maps and Viability Theory, Springer, Berlin, New York, Tokyo

Boole G (1860) A treatise on the calculus of finite differences. Macmillan, Cambridge

Bobrowski Adam, Morawska Katarzyna (2012) From a PDE model to an ODE model of dynamics of synaptic depression. Discrete Continuous Dyn Syst B 17(7):2313

Biggs N, Lloyd E, Wilson R (1986) Graph Theory, 1736–1936. Clarendon, Oxford

Bobrowski A (2012) From diffusions on graphs to Markov Chains via asymptotic state lumping. Ann. Henri Poincaré 13:1501–1510

Chang KC (1981) Variational methods for non-differentiable functionals and their applications to partial differential equations. J Math Anal Appl 80:102–129
Clarke F H (1990) Optimization and Nonsmooth Analysis, SIAM
Duvaut G, Lions JL (1972) Les inéquations en mécanique et en physique. Dunod, Paris
Elmoataz A, Lezoray O, Bougeleux S (2008) Nonlocal discrete regularization on weighted graphs: a framework for image and manifold processing. IEEE Trans Image Process 17(7):1047–1060
Elmoataz A, Buysens P (2017) On the connection between tug-of-war games and nonlocal PDEs on graphs. Comptes Rendus Mécanique 345(3):177–183
Grady LJ, Polimeni JR (2010) Discrete calculus: applied analysis on graphs for computational science. Springer, New York
Galewski M, Wieteska R (2016) Existence and multiplicity results for boundary value problems connected with the discrete p(.)−Laplacian on weighted finite graphs. Appl Math Comput 290:376–391
Gasinski L, Migórski S, Ochal A (2015) Existence results for evolution inclusions and variational-hemivariational inequalities. Appl Anal 94:1670–1694
Geman D, Reynolds G (1992) Constrained restoration and recovery of discontinuities. IEEE Trans Pattern Anal Mach Intell 14(3):367–383
Grady L, Alvino C (2009) The piecewise smooth Mumford-Shah functional on an arbitrary graph. IEEE Trans Image Process 18(11):2547–2561
Gregosiewicz A (2020) Asymptotic behaviour of fast diffusions on graphs. Semigroup Forum 101:619–653
Ha S-Y, Ha T, Kim J-H (2010) Emergent behavior of a Cucker-Smale type particle model with nonlinear velocity couplings. IEEE Trans Autotmat Control 55(7):1679–1683
Ha S-Y, Levy D (2009) Particle, kinetic and fluid models for phototaxis. Discrete Contin 12(1):77–108
Hua B, Mugnolo D (2015) Time regularity and long-time behavior of parabolic p−Laplace equations on infinite graphs. Journal of Differential Equations 259(11):6162–6190
Huo Q, Tian Y, Ma T (2018) Critical point theory to isotropic discrete boundary value problems on weighted finite graphs. J Differ Equ Appl 24(4):503–519
Haeseler S, Keller M, Lenz D, Wojciechowski R (2012) Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions. J Spectr Theory 2:397–432
Ioffe AD, Levin VL (1972) Subdifferentials of convex functions. Trudy Mosk Mat Obshch 26:497–508
Jung M, Kang M (2014) Efficient Nonsmooth Nonconvex Optimization for Image Restoration and Segmentation. Journal of Scientific Computing, 62(2),
Kamenskii M, Obukhovskii V, Zecca P (2001) Condensing multivalued maps and semilinear differential inclusions in banach space. Walter de Gruyter, Berlin
Keller M, Lenz D (2010) Unbounded Laplacians on graphs: basic spectral properties and the heat equation. Math Model Nat Phenom 5:198–224
Kelly R (1964) Theory of diffusion for discrete media-part I simple one-dimensional motion. Acta Metallurgica 12(2):123–127
Kirchhoff G (1847) Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. Annalen der Physik und Chemie 72:497–508
Mahdioui H, Ben Aadi S, Akhili K (2020) Hemivariational Inequality for Navier-Stokes Equations: Existence, Dependence, and Optimal Control. Bull. Iran. Math. Soc. (2020). https://doi.org/10.1007/s41980-020-00470-x
Mahdioui H, Ben Aadi S, Akhili K (2020) Weak Solutions and Optimal Control of Hemivariational Evolutionary Navier-Stokes Equations under Rauch Condition. Journal of Function Spaces, vol. 2020, Article ID 6573219, 14 pages, 2020. https://doi.org/10.1155/2020/6573219
Migórski S (2004) Boundary hemivariational inequality of parabolic type. Nonlinear Anal 57(4):579–596. https://doi.org/10.1016/s0362-546x(04)00071-9
Migórski S, Nguyen VT, Zeng S-D (2020) Solvability of parabolic variational-hemivariational inequalities involving space-fractional Laplacian, Applied Mathematics and Computation, Volume 364, 2020. https://doi.org/10.1016/j.amc.2019.124668
Migórski S, Zeng S (2018) Penalty and regularization method for variational-hemivariational inequalities with application to frictional contact. ZAMM-Journal of Applied Mathematics and Mechanics / Zeitschrift Für Angewandte Mathematik Und Mechanik 98(8):1503–1520
Minty G. (1960) Monotone networks. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 257, 194–212 (1960)
Mohar B (1982) The spectrum of an infinite graph. Linear Algebra Appl. 48:245–256
Mugnolo D (2014) Semigroups methods for Evolution Equations on Networks, Springer 2014
Mugnolo D (2013) Parabolic theory of the discrete -Laplace operator. Nonlinear Anal 87:33–60
Mumford D, Shah J (1989) Optimal approximations by piecewise smooth functions and associated variational problems. Commun Pure Appl Math 42:577–685
Nakamura T, Yamasaki M (1976) Generalized extremal length of an infinite network. Hiroshima Math J 6:95–111
Nikolova M (2005) Analysis of the recovery of edges in images and signals by minimizing nonconvex regularized least-squares. SIAM J Multiscale Model Simul 4(3):960–991
Ostrovskii MI (2005) Sobolev spaces on graphs. Quaest Math 28:501–523
Panagiotopoulos PD (1985) Nonconvex problems of semi-permeable media and related topics. ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift Für Angewandte Mathematik Und Mechanik 65(1):29–36
Panagiotopoulos PD (1993) Hemivariational Inequalities, Applications in Mechanics and Engineering, Springer-Verlag
Papageorgiou NS, Papalini F, Renzacci F (1999) Existence of solutions and periodic solutions for nonlinear evolution inclusions. Rend Circ Mat Palermo 48:341–364
Rauch J (1977) Discontinuous semilinear differential equations and multiple valued maps, Proceedings of the American Mathematical Society. 64(2),277-282
Sethian J (1999) Level Set Methods and Fast Marching Methods. Cambridge University Press, Cambridge
Sofonea M, Migórski S, Han W (2018) A penalty method for history-dependent variational-hemivariational inequalities. Comput Math Appl 75(7):2561–2573
Ta V, Bougleux S, Elmoataz A, Lezoray O (2007) Nonlocal anisotropic discrete regularization for image. Tech. Rep., Univ. Caen, Caen, France, Data Filtering and Clustering
Tamás V, Anna Z (2012) Collective motion. Phys Rep 517:71–140
Tonti E (1976) The reason for analogies between physical theories. Applied Mathematical Modelling I, 37–50
Yamasaki M (1975) Extremum problems on an infinite network. Hiroshima Math J 5:223–250
Yamasaki M (1977) Parabolic and hyperbolic infinite networks. Hiroshima Math J 7:135–146
Zeidler E (1990) Nonlinear functional analysis and applications II A/B. Springer, New York (NY)
Liu Zhenhai, Tan Jinggang (2017) Nonlocal elliptic hemivariational inequalities. Electron J Qualitative Theory Differ Equ 66:1–7

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