Modulation-induced long-range magnon-magnon bound states in a quantum spin chain

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We study two magnons in a spin chain under a gradient magnetic field with nearest-neighbor interaction and periodically driven hopping. In the resonant condition where the driven frequency matches and smoothes the potential bias, the system respects the translational invariance in both space and time in the rotating frame, and thus we can develop a Floquet-Bloch band theory for two magnons. We find a new kind of bound states with relative distance at two sites, apart from the conventional bound state with relative distance at one site. Such novel bound state indicates the modulation-induced next-nearest-neighbor interaction, which can be analytically explained by an effective Hamiltonian via the many-body perturbation theory. The effective interaction can be tuned as positive or negative one. When the nearest-neighbor interaction equals to the driven frequency, the two kinds of bound state become mixed with each other. We also propose to use quantum walks to probe the effective long-range magnon-magnon bound states.

I. INTRODUCTION

Periodic modulations in quantum systems have attracted tremendous interests and attentions in recent years [1–8]. Periodic modulation not only provides a versatile tool to manipulate the quantum particles, but also brings novel states of matter into the quantum systems [9]. It has already been utilized to control the hopping [10, 11], band structure [12, 13], and quantized transport of a single particle [14–19]. Remarkably, artificial gauge field [20] and Floquet topological insulator [21–23] have been realized by well-designed modulation protocols. In principle, Floquet-Bloch theory of a single particle has been well developed to analyze the properties of periodically modulated systems [24]. However, many-body effects induced by time modulations are more challenging and appealing.

There are interesting novel states of matter in time modulation of many-body systems, such as collective emission of matter-wave [25], discrete time crystal [26], etc. It becomes possible to periodically modulate many-body systems in various methods. For example, periodically driven interactions may induce density-dependent correlated tunneling [27], density-dependent synthetic gauge fields [28] and tunable three-body interactions which support fractional quantum hall states [29]. Periodically driven gradient magnetic field in a spin chain makes it possible to tune long-range interaction to short-range interaction [30]. It provides a route to create the nearest-neighbor (NN) interaction and density-assisted tunneling via both the time-dependent on-site energy and interaction [31]. Considering the Hubbard model, periodically driven tunneling strength is an alternate way for engineering the interaction [32, 33]. But it is still unclear what the correlation properties of the states arising from the engineered interaction are and how to probe the engineered interaction. Moreover, magnon excitations well describe the elementary properties of quantum magnetism and it is interesting to explore the influence of effective interactions in a spin chain.

In this paper, we study two-magnon excitation in a spin chain under a gradient magnetic field with NN interaction, and the hopping strength is modulated via an ac field. When the potential bias between NN sites equals to the modulation frequency, a single magnon can resonantly tunnel onto its neighbors. We analyze a Floquet-Bloch band theory for two magnons, since the system in the rotating frame is invariant by shifting the two magnons as a whole along the real space and along one time period. Apart from conventional bounds states, we find that there are two magnons bound with two-site distance by calculating the magnon-magnon correlation, indicating the effective next-nearest-neighbor-interaction bound states. An effective two-magnon model is obtained using the many-body perturbation theory, which interprets the physical mechanism of modulation-induced long-range bound states. The effective next-nearest-neighbor (NNN) interaction can be tuned to be positive or negative. When the NN interaction equals to the modulation frequency, there exists a resonance between the bound states with relative one-site and two-site distances. We also propose the two-magnon quantum walks to probe the effective long-range magnon-magnon bound states.

This paper is organized as follows. In Sec. II, we briefly describe the driven two-magnon model. In Sec. III, we calculate the quasienergy spectrum as a function of NN interaction, by using the Floquet spectrum analysis in...
both time and frequency domains. In Sec. IV, we analyze the Floquet-Bloch band, derive an effective two-magnon model for the modulation-induced long-range bound states, and show the resonance between two kinds of bound states. In Sec. V, we verify the long-range magnon-magnon bound states via the quantum walks. In Sec. VI, we give a brief summary and discussion.

II. MODEL

We consider a periodically driven spin-1/2 Heisenberg XXZ chain under a gradient magnetic field, which is governed by the Hamiltonian,

$$ \hat{H} = \sum_{l=1}^{L} \left( J(t) \hat{S}_l^x \hat{S}_{l+1}^x + \text{H.c.} + \Delta \hat{S}_l^z \hat{S}_{l+1}^z + iB \hat{S}_l^z \right). $$

Here, $J(t) = (J_0 + J_1 \cos(\omega t))/2$, is the dc-ac field with $J_0$, $J_1$ as the amplitudes and $\omega$ as the modulation frequency. In the following, we set $J_0 = \hbar = 1$ by default. $\hat{S}_l^z(i = x, y, z)$ are spin-1/2 operators and $\hat{S}_l^z = \hat{S}_l^x \pm i\hat{S}_l^y$ are the spin raising and lowering operators at site $l$. $\Delta$ denotes the NN longitudinal spin-exchange coupling, which is set as $\Delta \geq 0$ by default. $B$ is the gradient magnetic field, which break the translational symmetry of the system. And the total chain length is $L = 2L + 1$. The ground state is the fully ferromagnetic state $|\downarrow \downarrow \ldots \downarrow \rangle$ for a positive and sufficiently large $B$. By flipping spins over the ground state $|\downarrow \downarrow \ldots \downarrow \rangle$, we obtain the excited states. Once one considers the ground state $|\downarrow \downarrow \ldots \downarrow \rangle$ as a vacuum state, magnon can be regarded as the basic excitation around the ferromagnetic ground state. Considering the mapping relations $|\downarrow \rangle \leftrightarrow |0\rangle$, $|\uparrow \rangle \leftrightarrow |1\rangle$, $\hat{S}_l^z \leftrightarrow \hat{a}_l^\dagger \hat{a}_l$, $\hat{S}_l^+ \leftrightarrow \hat{a}_l$, $\hat{S}_l^- \leftrightarrow \hat{n}_l - \frac{1}{2}$, it amounts to load magnons into the titled optical lattice with periodically driven hopping rate and NN interaction, i.e.,

$$ \hat{H} = \sum_{l=1}^{L} \left( \frac{1}{2} \hat{a}_l^\dagger \hat{a}_{l+1} + \text{H.c.} + \Delta \hat{n}_l \hat{n}_{l+1} + Bl\hat{n}_l \right). $$

$\hat{a}_l^\dagger $ ($\hat{a}_l$) creates (annihilates) a magnon at site $l$ and satisfies the commutation relations of the hard-core bosons. $\hat{n}_l = \hat{a}_l^\dagger \hat{a}_l$ is the number operator.

By a unitary treatment $\hat{H}' = \hat{U} \hat{H} \hat{U}^\dagger - i\hat{U} \frac{\partial}{\partial t} \hat{U}^\dagger$ with $\hat{U} = \exp(i \sum_{l} B l \hat{n}_l) [34, 35]$, the driven magnon Hamiltonian in the rotating frame is given as,

$$ \hat{H}' = \sum_{l=-L}^{L} \left( Je^{-ibt} \hat{a}_l^\dagger \hat{a}_{l+1} + \text{H.c.} + \Delta \hat{n}_l \hat{n}_{l+1} \right). $$

Different from the three-color modulation in the Fermi-Hubbard model [33], our driven magnon model (2) only involves the commensurable dual frequencies. If we impose periodic boundary condition, Hamiltonian (2) preserves translational invariance, which is slightly different from Hamiltonian (1) at the boundary. We only consider the bulk properties where the tiny difference between Hamiltonian (1) and (2) takes no effect. For convenience, we focus on Hamiltonian (2) with periodic boundary condition in the following paper. Since $[\hat{H}', \hat{n}] = 0$ with $\hat{n} = \sum_l \hat{n}_l$, the total number of magnons is conserved. It indicates that subspaces with different numbers of magnons are decoupled. For convenience, we focus our analysis on the subspace of two-magnon excitation. Such a driven two-magnon model can be schematically described in Fig.1(a).

In the absence of the gradient magnetic field and periodic driving ($B = J_1 = 0$), the spin-1/2 XXZ chain occurs a quantum phase transition at $\Delta = J$. For the two-magnon system, an isolated bound-state band and a continuum scattering band appear in the energy spectrum for $\Delta > J$. While the bound-state band merges into the continuum one for $0 < \Delta < J$. After flipping two spins over the ground state with all spins downward, two magnons undergo Bloch oscillations in the presence of the gradient magnetic field [36]. Coherent delocalization occurs by introducing the resonant modulation $\omega = B$, which is potentially applied for measuring the magnetic field gradient [37, 38]. In our paper, we only consider the resonant driven condition, i.e., $\omega = B$, where photon-assisted tunneling resonances may happen [39, 40].

![FIG. 1. (Color online) (a) The driven two-magnon model (2) with NN interaction and time-modulated hopping rate. (b) The effective two-magnon model (16) with modulation-induced NNN interaction, modified NN interaction and reduced hopping rate.](image)

III. FLOQUET SPECTRUM ANALYSIS

There is only NN interaction in the Hamiltonian (2). The interplay between NN interaction and time modulation makes it possible to induce long-range bound states. To understand how the long-range bound states come from, we need to analyze the Floquet spectrum and the quasieigenstates. There are two equivalent ways to calculate the Floquet-Bloch spectrum, one is in the time domain and the other is in the frequency domain. In
the subsection III A and III B, we will respectively analyze how the the Floquet spectrum changes with $\Delta$ in the time and frequency domains in the high-frequency region.

![Quasienergy spectrum](image)

**FIG. 2.** (Color online) (a) Quasienergy spectrum $E$ as a function of $\Delta$ given by the effective statistic Hamiltonian $\hat{H}_F$ (black solid lines). The red and blue dashed lines respectively represent $\Delta$ and $\Delta - \omega$ as a function of $\Delta$. (b) The quasienergy spectrums of the Floquet-Bloch lattice model (10) (black solid lines) and the effective two-magnon model (16) (red dotted lines) for $F = 5$. The other parameters are chosen as $\omega = B = 10$, $J_0 = 1$ and $J_1 = 0.01$. There is continuum flat band around zero energy, with eigenstates regarded as the scattering states. When $J_0$, $J_1 \ll \Delta$, two magnons at the NN sites approximate the bound states with energy $\sim \Delta$. The red dashed line $\Delta$ as a function of $\Delta$ is added in Fig. 2(a), which is well fit with an isolated band in quasienergy spectrum (black solid lines) of $\hat{H}_F$. Thus, there is one isolated band $\sim \Delta$ corresponding to bound state with relative distance as 1 in quasienergy spectrum of $\hat{H}_F$. It becomes clearer that $\Delta - \omega$ (blue dashed line) is the replica of $\Delta$ (red dashed line). Around the resonant condition $\Delta - \omega \approx 0$, the bound-state band $\Delta - \omega$ (blue dashed line) is completely different from the quasienergy spectrum (black solid lines) of $\hat{H}_F$ and mixes with the its continuum band, and two isolated bands appear in the quasienergy spectrum of $\hat{H}_F$, see Fig. 2(a). Thus, the two isolated bands around $\Delta - \omega \approx 0$ must be a modulation-induced effect.

### B. Frequency-domain analysis

We can equivalently analyze the quasienergy spectrum in the frequency domain. The arbitrary two-magnon states can be expanded as $|\Psi\rangle = \sum_{l_1<l_2} \psi_{l_1,l_2}|l_1,l_2\rangle$, with probability amplitudes $\psi_{l_1,l_2} = \langle 0|\hat{U}_{l_1,l_2}|\psi\rangle$ for one magnon at the $l_1$-th site and the other at the $l_2$-th site. After substituting the two-magnon state in the Fock basis into the Schrödinger equation

$$i \frac{d}{dt}|\Psi(t)\rangle = \hat{H}'|\psi(t)\rangle,$$

the amplitude probability $\psi_{l_1,l_2}(t)$ at instantaneous time $t$ satisfies

$$i \frac{\partial}{\partial t} \psi_{l_1,l_2}(t) = M(t) \left( \psi_{l_1,l_2+1} + \psi_{l_1+1,l_2}(t) \right) + M^*(t) \left( \psi_{l_1,l_2-1}(t) + \psi_{l_1-1,l_2}(t) \right) + \Delta \delta_{l_1,l_2+1} \psi_{l_1+1,l_2}(t)$$

with $M(t) = J_A/4 + J_B/2e^{-i\omega t} + J_1/4e^{-i\omega t}$. Considering the periodic boundary condition, we have $\psi_{l_1,l_1+L_1} = \psi_{l_1,l_2}$. Due to the time periodicity, we can express the probability amplitudes $\psi_{l_1,l_2}(t)$ by the Fourier components [11, 24]

$$\psi_{l_1,l_2}(t) = e^{-iEt} \sum_{\chi=-\infty}^{\infty} e^{-i\chi\omega t} U_{l_1,l_2,\chi}.$$

Then we have

$$|\Psi\rangle = e^{-iEt} \sum_{l_1,l_2,\chi} e^{-i\chi\omega t} U_{l_1,l_2,\chi}|l_1,l_2,\chi\rangle \tag{8}$$

where $|U_{l_1,l_2,\chi}|^2$ is the probability distributing in the Floquet state $|l_1,l_2,\chi\rangle$. $\chi$ is the Floquet index taking values from $-\infty$ to $\infty$. The Floquet state $|l_1,l_2,\chi\rangle$ intuitively represents two particles living in real space and Floquet

A. Time-domain analysis

The driven two-magnon Hamiltonian (2) satisfies a discrete time translation symmetry, $\hat{H}(t + T) = \hat{H}(t)$ with a Floquet period $T = 2\pi/\omega$. We can define a time-evolution operator in one period as

$$\hat{U}_T = \mathcal{T} \exp(-i \int_0^T \hat{H}'(t)dt) \equiv \exp(-i\hat{H}_F T), \tag{3}$$

where the static effective Hamiltonian

$$\hat{H}_F = \frac{i}{T} \log \hat{U}_T \tag{4}$$

governs the dynamics at stroboscopic time $nT$ ($n = 1, 2, 3, ...$). Before analyzing the dynamics, it is helpful to solve the eigenvalue problem, $\hat{H}_F |u_n\rangle = E_n |u_n\rangle$, where $E_n$ and $|u_n\rangle$ are quasienergy and Floquet eigenstate, respectively. $E_n$ can be restricted in the interval $[-\omega/2, \omega/2)$, which we term as the first Floquet-Brillouin zone. The quasienergy has a period $\omega$ and consists of replicas of that in the first Floquet-Brillouin zone. We calculate the quasienergy spectrum as a function of the NN interaction regarding the static effective Hamiltonian (4), see Fig. 2(a). The parameters are chosen as...
space. Due to the particle conservation, the two particles always share a common $\chi$, that is, the two particles transfer from $\chi$ to $\chi'$ as a bounded pair. Replacing the state (8) into the Eq. (5) and averaging it over one period, we can obtain

$$EU_{l_1,l_2,\chi} = (\Delta\delta_{l_1,l_2\pm1} - \chi\omega)U_{l_1,l_2,\chi} + \sum_q J_q (U_{l_1,l_2+1,\chi-q} + U_{l_1+1,l_2,\chi-q})$$

$$+ U_{l_1,l_2-1,\chi+q} + U_{l_1-1,l_2,\chi+q})$$

(9)

with $J_0 = J_2 = J_1/4$ and $J_1 = J_0/2$. It equals to a Floquet-Bloch lattice model,

$$\hat{H}_{FB} = \hat{H}_{FB}^0 + \hat{H}_{FB}^1,$$

(10)

with

$$\hat{H}_{FB}^0 = \sum_{l_1,l_2,\chi} (\Delta\delta_{l_1,l_2\pm1} - \chi\omega)|l_1,l_2,\chi⟩⟨l_1,l_2,\chi|$$

(11)

and

$$\hat{H}_{FB}^1 = \sum_{l_1,l_2,\chi,q} (J_q |l_1,l_2,\chi⟩⟨l_1,l_2+1,\chi-q| + H.c.)$$

$$+ \sum_{l_1,l_2,\chi,q} (J_q |l_1,l_2,\chi⟩⟨l_1+1,l_2,\chi-q| + H.c.).$$

The Floquet index $\chi$ labels the extra dimension. We establish an equivalence between a periodically driven two-magnon model (2) and the Floquet-Bloch lattice model (10). The motion of two magnons on the periodically driven one-dimensional lattice in (a) is equivalent to the time-independent hopping dynamics of two magnons on a two-dimensional lattice model with a potential energy gradient along the $\chi$ direction. Here, $\hat{H}_{FB}^0$ consists of the NN interaction and the potential gradient $-\chi\omega$ along $\chi$ direction. $\hat{H}_{FB}^1$ consists of individual hopping of magnons in the same $\chi$ and pair-hopping between $\chi$ and $\chi'$ with $\chi' = \chi \pm 1$, $\chi \pm 2$.

Numerically, we can calculate the quasienergy spectrum by truncating the Floquet spaces ranging from $\chi = -f$ to $f$, and the total truncation number $F = 2f + 1$. Generally, $f$ can be ranging from $-\infty$ to $\infty$. In the high-frequency case, $J_1/4 \ll [\omega, \sqrt{J_0\omega/2}]$, the energy barrier between $\chi$ and $\chi \pm 1$ are so large that the two magnons tends to localize at a few $\chi$. A small truncation number gives sufficiently exact results and enables us to apply the perturbation theory to obtain an effective Hamiltonian. However, as the modulation frequency decreases, the wave function become spread over a larger range of $\chi$, and a larger truncation number is needed. By choosing the same parameters as Fig. 2(a) and $F = 5$, we also calculate the quasienergy spectrum with the change of $\Delta$ given by the Floquet-Bloch lattice model (10) (black solid lines), see Fig. 2(b). Compared with the quasienergy spectrum of $\hat{H}_{FB}$ (black solid lines) in Fig. 2(a) and $\hat{H}_{FB}$ (black solid lines) in Fig. 2(b), they are almost the same. It means that these two methods are equivalent. However, the analysis in the frequency domain is a more powerful method to deeper understand the emergence of modulation-induced isolated bands. Based on the analysis in the frequency domain, we will analyze the Bloch states under a periodic boundary condition far away and around the resonant parameter.

IV. TWO-BODY FLOQUET-BLOCH BAND

In this section, we derive a Floquet-Bloch lattice model which has cotranslational symmetry along the real space but violates such symmetry along the Floquet space due to the effective potential gradient $-\chi\omega$. Thus, we can apply the many-body Bloch theorem in the real space and leave alone the Floquet index $\chi$. Remarkably, we obtain two-body Floquet-Bloch band, a significant analysis of the many-body Bloch theorem to the periodically modulated system. Away from the resonant condition $\Delta - \omega = 0$, We analyze the isolated Floquet-Bloch bands and reveal the modulation-induced long-range magnon-magnon bound states. Such process can also be perfectly captured by an effective Hamiltonian via the many-body perturbation theory. Near the resonant condition, we find the hybrid of two kinds of bound states.

A. Modulation-induced long-range magnon-magnon bound states

Imposing periodic boundary condition, the Floquet-Bloch lattice model (10) is invariant by shifting the two magnons as a whole in the real space. Naturally, we introduce the center-of-mass and relative position are $R = (l_1 + l_2)/2$ and $r = l_1 - l_2$, respectively. The center-of-mass quasimomentum $K$ is a conserved quantity. According the many-body Bloch theorem, the wavefunction is a Bloch wave along the coordinate of center-of-mass position, i.e., $U_{l_1,l_2,\chi} = e^{iK\phi_\chi(r)}$ where $\phi_\chi(r)$ is the amplitude depending on the relative position $r$ and the Floquet position $\chi$. Substituting the above ansatz into Eq. (10), the amplitude $\phi_\chi(r)$ satisfies the following eigenequation in the quasimomentum space,

$$E\phi_\chi(r) = \sum_q J^K_q (\phi_{\chi-q}(r-1) + \phi_{\chi-q}(r+1))$$

$$+ (\Delta\delta_{\chi,\pm1} - \chi\omega)\phi_\chi(r)$$

(12)

with $J^K_0 = J_1/2\cos(K/2)$, $J^K_{\pm1} = J_0/2e^{\pm iK/2}$ and $J^K_{\pm2} = J_1/4e^{\pm iK/2}$. Under the periodic boundary condition, we find $e^{iKL_1} = 1$ and $\phi_\chi(r + L_1) = e^{iKL_1/2}\phi_\chi(r)$ with $K = 2\pi\alpha/L_1$ for $\alpha = -L_1, -L + 1, \ldots, L$. Moreover, we have $\phi_\chi(0) = 0$ and $\phi_\chi(r) = \phi_\chi(-r)$ since the magnons are hard-core bosons.

Solving the above eigenequation (12), we can obtain the two-body Floquet-Bloch bands vs. $K$, see Fig. 3. The parameters are chosen as $\omega = B = 10$, $J_0 = 1$, $J_1 = 0.01$, $L_1 = 5$, and $\chi$.
and $\Delta$ are chosen at the left- and right-hand sides of the resonant point as $\Delta = 7$, 13 for Figs. 3(a) and (b), respectively. At the left-hand side of the resonant point, there are two isolated bands above a continuum band which ranges from $-|J_1|$ to $|J_1|$, see Fig. 3(a). This is qualitatively different from a conventional two-band energy spectrum of two-magnon excitations in NN interaction quantum spin chains without the periodic driving [41]. The extra band appears with a quite similar structure to the bound-state band arising from the NN interaction. While at the right-hand side of the resonant point, the two isolated bands sandwich the continuum band, see Fig. 3(b). It will be clear later that the difference between Figs. 3(a) and (b) relates to the positive and negative values of the effective interaction. Suitable choice of parameters results in a negative modulation-induced bound band below the continuum band.

To better understand the band structure, it is instructive to calculate the magnon-magnon correlations of corresponding Floquet states labeled I and II in Fig. 3. The magnon-magnon correlation is defined as

$$C_{xy} = \langle \Psi(T) | \hat{a}_x^+ \hat{a}_y^+ \hat{a}_y \hat{a}_x | \Psi(T) \rangle$$  (13)

where $|\Psi(T)\rangle$ is the Floquet state with a given quasi-momentum $K$. $x$ and $y$ take values from $-L$ to $L$. The magnon-magnon correlations at two specific lines $x = y \pm 1$ in the $(x, y)$ planes serve as a sensitive evidence of the two-magnon bound states, where $d$ depends on the specific magnon-magnon interactions. For example, due to the NN interaction, the Floquet state marked with I is well distributed among the minor-diagonal lines $x = y \pm 1$ of the magnon-magnon correlations, see the inset I of Fig. 3. While for the Floquet state marked with II, the magnon-magnon correlations are mainly distributed in the next-minor-diagonal lines, see the inset II of Fig. 3. The type II Floquet state indicates the existence of effective long-range bound states. The correlation properties of the other Floquet states in these two bands are similar to their Floquet states with quasimomentum $K = 0$. They are characteristic signatures of two types of bound-state bands, respectively derived from the NN interaction and NNN interaction. Thus, we can claim the existence of modulation-induced long-range bound states.

**B. Effective two-magnon model**

Below particular attention is paid to attain an effective two-magnon model for understanding and interpreting the origination of the modulation-induced long-range bound states. Given the perturbation conditions $|\Delta - \omega| \gg J_0/2$ and $|\Delta - 2\omega| \gg J_1/4$, we divide the Floquet-Bloch lattice model (10) into two parts, $\hat{H}^0_{FB}$ as a dominate term and $\hat{H}^1_{FB}$ as a perturbed term. In the high-frequency region $J_1/4 \ll \max[\omega, \sqrt{\omega J_0}/2]$, it is sufficient to just take into account the five Floquet sites with $\chi = 0, \pm 1, \pm 2$. The unperturbed term $\hat{H}^0_{FB}$ is separated into two subspaces $\mathcal{U}$ and $\mathcal{V}$. The subspace $\mathcal{U}$ includes two kinds of states: (i) $E_P = \Delta$ for the states $\{|l, l + 1, 0\rangle\}$ with $-L \leq l \leq L$ and (ii) $E_P = 0$ for the states $\{|l_1, l_2, 0\rangle\}$ with $l_1 \neq l_2 - 1$, $-L \leq l_1 < l_2 \leq L$. The complementary subspace $\mathcal{V}$ consists of (iii) $E_S = \Delta - \chi\omega$ for the states $\{|l, l + 1, \chi\rangle\}$ with $-L \leq l \leq L$ and (iv) $E_S = -\chi\omega$ for the states $\{|l_1, l_2, \chi\rangle\}$ with $l_1 \neq l_2 \pm 1$, $-L \leq l_1 < l_2 \leq L$ for $\chi = \pm 1, \pm 2$. The project operators are defined as $\hat{P} = \sum_{l_1, l_2} |l_1, l_2, 0\rangle\langle l_1, l_2, 0|$ onto $\mathcal{U}$ and $\hat{S} = 1 - \hat{P}$ onto $\mathcal{V}$. We calculate the effective two-magnon model $\hat{H}_{eff} = \hat{h}_0 + \hat{h}_1 + \hat{h}_2$ via a perturbative expansion up to the second order. In the lowest order and first order, we have

$$\hat{h}_0 = E_P \hat{P} = \Delta \sum_l |l, l + 1, 0\rangle\langle l, l + 1, 0|$$  (14)

and

$$\hat{h}_1 = \hat{P} \hat{H}^1_{FB} \hat{P} = \frac{J_1}{4} \sum_{l_1, l_2} |l_1, l_2, 0\rangle\langle l_1, l_2 + 1, 0| + |l_1 + 1, l_2, 0\rangle\rangle + \text{H.c.},$$  (15)

which respectively retains the original NN interaction $\Delta$ and NN tunneling in $\chi = 0$. For simplicity, we respec-
tively label the eigenstates and eigenvalues of the unperturbed term $\hat{H}_\text{FB}$ as $|i\rangle$ and $E_i$, and $|i_p\rangle$ represent states in the subspace $\mathcal{U}$. The second-order effective Hamiltonian reads

$$\hat{h}_2 = \sum_{l_1 \neq l_2 \in P \cap \bar{P}} \frac{J_{l_1 l_2}^{\text{Eff}}}{2} |i_1\rangle \langle i_1| \hat{H}_\text{FB}^1 |i_2\rangle \langle i_2| \hat{H}_\text{FB}^1 |i_3\rangle \langle i_3|$$

$$= -\Delta_2 \sum_{l} (|l, l+1, 0\rangle \langle l, l+1, 0| - |l, l+2, 0\rangle \langle l, l+2, 0|)$$

with $J_{l_1 l_2}^{\text{Eff}} = (1/(E_{i_1} - E_{i_2}) + 1/(E_{i_1} - E_{i_3}))$ and $\Delta_2 = J_0^2 \Delta / [2(\omega^2 - \Delta^2)] + J_0^2 \Delta / [8(4\omega^2 - \Delta^2)]$. Interestingly, the second-order process not only contributes to the NNN interaction, but modifies the original NN interaction. Since the high-order term is much smaller than the second-order one in the perturbation parameter region, it allows us to ignore the results beyond the second-order term.

By means of the perturbation theory [42-44], the effective two-magnon model up to second order in $\chi = 0$ is given as

$$\hat{H}_\text{Eff} = \frac{J_1}{4} \sum_{l} (\hat{a}_l^\dagger \hat{a}_{l+1} + \text{H.c.}) + \sum_{n=1,2,l} \Delta_n \hat{n}_l \hat{n}_{l+1}\hat{n}_n$$

with $\Delta_1 = \Delta - \Delta_2$. $\sum_l \hat{n}_l = 2$ restricts $\hat{H}_\text{Eff}$ to the two-magnon sector. The effective two-magnon model (16) can be schematically described in Fig. 1(b) with the renormalized parameters. The effective NNN interaction comes from the asymmetric pathways between absorbing and consequently emitting phonons, $|l_2 - l_1 = 2, 0\rangle \rightarrow |l_2 - l_1 = 1, +\rangle \rightarrow |l_2 - l_1 = 2, 0\rangle$, $\chi = 0$ and the inverse process, $|l_2 - l_1 = 2, 0\rangle \rightarrow |l_2 - l_1 = 1, -\rangle \rightarrow |l_2 - l_1 = 2, 0\rangle$, $\chi = 0$. This is the origin of NNN interactions in a periodically modulated interacting systems. The hopping rate is reduced to $J_1/4$. It must be pointed out that modulation amplitude $J_1$ determines the width of the continuum band $[\pm |J_1|, |J_1|]$. Once $J_1$ is sufficiently large, the NNN-interaction bound-state band may be not enough to completely separate from the continuum band. With the effective two-magnon model (16), we can systematically create and control the NNN interaction. The long-range magnon-magnon bound states arising from the modulation-induced NNN interaction constitutes the central idea of our paper.

To show the concreteness, we compare the quasienergy spectrum given by the effective two-magnon model (16) and that given by $\hat{H}_\text{FB}$, respectively see the red dotted and black solid lines in Fig. 2(b). It is clear that the isolated bands are well consistent in the perturbation parameter regime $|\Delta - \omega| \gg J_0/2$ and $|\Delta - 2\omega| \gg J_1/4$. The perturbation conditions mean the energy gap between subspaces $\mathcal{P}$ and $\mathcal{S}$ should be much larger than their tunneling rate. This energy gap decreases as $\Delta$ approaches to $\omega$, so that the perturbation condition is no longer satisfied and the effective two-magnon model (16) is invalid. As shown in Fig. 2(b) (red dotted lines), the isolated bound band of $\hat{H}_\text{Eff}$ is no longer fitting with the modulation-induced band of $\hat{H}_\text{FB}$ when the NNI interaction $\Delta$ nears the modulation frequency $\omega$. But their continuum bands are always well consistent. In order to clearly clarify the modulation-induced bound band, we just plot the continuum band of $\hat{H}_\text{FB}$ in Fig. 2(b).

C. Resonance between two types of bound states

FIG. 4. (Color online) Quasienergy spectrum $E$ vs. $K$ in the resonant condition $\Delta = \omega$. The insets I and II respectively describe the normalized magnon-magnon correlations $C_{xy} = C_{xy}/C_{xy}^{\text{max}}$ of Floquet states labeled in the top and bottom band in the quasienergy spectrum. The other parameters are chosen as $\omega = B = 10$, $J_0 = 1$ and $J_1 = 0.01$.

However, the effective Hamiltonian (16) becomes invalid around the resonant point $\Delta = \omega$. To understand what happens in the resonant condition, we calculate the band structure and magnon-magnon correlations via the Floquet spectrum analysis in the frequency domain, see Fig. 4. The truncated Floquet space is limited in $\chi = 0, \pm 1, \pm 2$.

For $\hat{H}_\text{FB}$, its eigenstates $|l_1, l_1 + 1, \chi\rangle : -L \leq l \leq L$ and $|l_1, l_2, \chi - 1\rangle$ $l_1 \neq l_2 - 1, -L \leq l_1 < l_2 \leq L$ become degenerate with the energy difference $\Delta - \omega = 0$. Once $\hat{H}_\text{FB}$ is added, this energy degeneracy will be broken, and these states are not longer eigenstates of the full Hamiltonian (10). Two types of bound states $|l_2 - l_1 =
1, \chi) and |l_2 - l_1 = 2, \chi - 1) are coupled by first-order process with the tunneling rate J_0/2, while state |l_2 - l_1 = 1, \chi) couples state \{|l_2 - l_1 > 2, \chi - 1) : l_1 \neq l_2 - 1, l_2 - 2\} by second-order or even higher-order term, which can be neglected in the high-frequency region. Here we are able to just consider the first-order tunneling process between two types of bound states |l, l + 1, \chi) and |l, l + 2, \chi - 1), then the corresponding states of the system (10) can be written as a superposition \(|l, l + 1, \chi \pm |l, l + 2, \chi - 1)\)/\sqrt{2} and the corresponding energies are E = E_0 ± J_0/2.

Now we focus on the zeroth Floquet-Brillouin zone E ∈ [-π/2, π/2], there is no energy gap between bound states |1) = |l, l + 1, 1) and |2) = |l, l + 2, 0) with energy E_0 = 0. In terms of a basis \{|1), |2)\}, we build a two-level system with a degenerate energy E_0. Once the first-order tunneling process J_0/2 is added, states |1) and |2) are not eigenstates of the full system whose eigenstates follow |±) = (|1) ± |2))/\sqrt{2} and eigenvalues are E_0 ± J_0/2. The energy bands at the top and at the bottom are almost symmetrical with respect to E_0 = 0, and the energy gap is J_0, see Fig. 4. The parameters are chosen as J_0 = 1, J_1 = 0.01 and Δ = \omega = 10, in the high-frequency region J_1/4 ≪ max (\omega, \sqrt{\omega J_0/2}). We respectively mark the two Floquet states as I and II in the quasienergy spectrum and analyze their correlated properties, see the insets I and II of Fig. 4. The top band and bottom one behave almost the same, that is, the magnon-magnon correlation is basically equal probability distribution on the minor-diagonal and next-minor-diagonal lines. These numerical results completely follow the related theoretical analysis.

There exists a process in which, with the increasing of Δ, a bound-state band caused by the NNN interaction in \chi = 0 emerges from the continuum band. When Δ nears \omega, it gradually mixes with the bound-state band caused by the NN interaction in \chi = 1. The NN-interaction bound-state band in \chi = 1 plays a dominant role until Δ is large enough. Similarly, with the increasing of Δ, an isolated NN-interaction bound-state band of \chi = 1 appears below the continuum band, is mixed with the NNN-interaction one in \chi = 0 around the resonant point Δ = \omega, and finally turns to the one of the NNN interaction in \chi = 0 for a sufficiently large Δ.

V. PROBING LONG-RANGE MAGNON-MAGNON BOUND STATES

Two-magnon quantum walks provides an excellent method to probe the modulation-induced long-range magnon-magnon bound states. It is worth to numerically simulate the dynamics of two strongly correlated magnons, initially localizing on the sites l = -1 and l = 1, which can be prepared by flipping two NNN spins from a saturated ferromagnetic state with all spins downward |↓↓ ... ↓). To investigate the dynamics of magnons initially locating on the spin chain in the bulk, we resort to numerically solve the time-dependent Schrödinger equation. Starting from the driven spin-1/2 Heisenberg XXZ chain under a gradient magnetic field (1), the time evolution of an arbitrary two-magnon state |ψ(t)⟩ obeys the time-dependent Schrödinger equation \(i\frac{d}{dt}|ψ(t)⟩ = \hat{H}|ψ(t)⟩\), where |ψ(t)⟩ = \(\sum_{l_1 < l_2} \psi_{l_1 l_2}(t)|l_1 l_2⟩\) and the probability amplitudes \(\psi_{l_1 l_2}(t) = \langle 0|\hat{S}_{l_1}^x \hat{S}_{l_2}^x |ψ(t)⟩\).

FIG. 5. (Color online) Quantum walks of two magnons in the driven Heisenberg XXZ chain under a gradient magnetic field. (a) is the time evolution of spin distributions \(S_l^x(t)\). (b), (c) and (d) are the normalized magnon-magnon correlations \(C_{xy} = C_{xy}/C_{xx}\) for different moments marked as A, B and C in (a). At the initial moment A (t = 0), two magnons are prepared in \(-1, 1\), so one can see two points in (-1, 1) and (1, -1) in (b). In the course of the time evolution of two magnons moves along the next-minor-diagonal lines in (c) and (d). The parameters are chosen as Δ = 7, \omega = B = 10, J_0 = 1 and J_1 = 0.01.

the instantaneous wave function |ψ(t)⟩, we trace out the time-dependent spin distributions

\[ S_l^x(t) = \langle ψ(t)|\hat{S}_l^x|ψ(t)⟩ = \frac{1}{2} (\langle \hat{S}_l^x \rangle + \langle \hat{S}_l^− \rangle) \]

and the instantaneous magnon-magnon correlations \(C_{xy}(t) = \langle ψ(t)|\hat{S}_l^x \hat{S}_y^x \hat{S}_y^− \hat{S}_l^−|ψ(t)⟩\) for different moments, where l, x and y take values from \(-L\) to \(L\).

Fig. 5 displays the time evolution of spin distributions and the corresponding instantaneous magnon-magnon correlations at three different moments for \(J_0 = 1\), \(J_1 = 0.01\), \(Δ = 7\), \(ω = B = 10\), and \(L_l = 21\), with the periodic boundary condition. Fig. 5(a) takes on a light cone in the time evolution of spin distributions before the two magnons collide with the boundaries. We timely cut off the evolution of density distributions to avoid the boundary effects and analyze its magnon-magnon correlations, marking A, B and C in Fig. 5(a). As shown in Figs. 5(b), (c) and (d), almost all of the magnons are distributed among the next-minor-diagonal lines in the magnon-magnon correlation. These results follow from the circumstance that two strongly NNN interacting magnons initially occupying the NNN sites form a bound pair and tunnel together on the spin chain. These time evolved results are in agreement with our analytical predictions. The quantum walks of two magnons...
paves a way for experimentally verifying whether the modulation-induced long-range bound states exist or not.

VI. SUMMARY AND DISCUSSIONS

Based on the magnon excitations in a spin chain, we have presented a scheme for generating tunable long-range bound states via time-periodic hopping rate. Taking advantage of the flexible tunability, desired long-range bound states can be engineered by tuning the NN interaction $\Delta$ and modulated parameters. The prerequisite lies in the modulation frequency $\omega$ equals to the magnetic field gradient $B$, which enables a single magnon to tunnel resonantly onto its neighbors. It can be understood as the phonon-assisted tunneling among the spin chain. We not only analyze the correlation properties of modulation-induced long-range bound states, but also study the interplay between the original and modulation-induced bound states. Further, we obtain an effective two-magnon model via a many-body perturbation theory to interpret the origin and condition for the remarkable long-range bound states. The effective two-magnon model becomes invalid near the resonant point $\Delta = \omega$, where resonance between the two types of bound states happens. In addition, two-magnon quantum walks can be used as an experimental verification of modulation-induced long-range bound states.

The fractional topological states exist in the one-dimensional superlattice with the dipole-dipole interactions [45]. This proposed scheme has potential to realize the fractional topological states. By introducing the modulation-induced NNN interaction, the competitive relation between two kinds of bound states may bring different topological effects in periodically modulated NN-interaction spin chain [46]. In addition, we focus on the analysis in the high-frequency region, while more abundant phenomena may arise in the low-frequency one.

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