Blow-up and strong instability of standing waves for the NLS-$\delta$ equation on a star graph

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Abstract

We study strong instability (by blow-up) of the standing waves for the nonlinear Schrödinger equation with $\delta$-interaction on a star graph $\Gamma$. The key ingredient is a novel variational technique applied to the standing wave solutions being minimizers of a specific variational problem. We also show well-posedness of the corresponding Cauchy problem in the domain of the self-adjoint operator which defines $\delta$-interaction. This permits to prove virial identity for the $H^1$-solutions to the Cauchy problem. We also prove certain strong instability results for the standing waves of the NLS-$\delta'$ equation on the line.

Keywords: $\delta$- and $\delta'$-interaction, Nonlinear Schrödinger equation, strong instability, standing wave, star graph, virial identity.

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1. Introduction

Let $\Gamma$ be a star graph, i.e. $N$ half-lines $(0,\infty)$ joined at the vertex $\nu = 0$. On $\Gamma$ we consider the following nonlinear Schrödinger equation with $\delta$-interaction (NLS-$\delta$)

$$i\partial_t U(t,x) - HU(t,x) + |U(t,x)|^{p-1}U(t,x) = 0,$$

where $p > 1$, $U(t,x) = (u_j(t,x))_{j=1}^N : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}^N$, nonlinearity acts componentwise, i.e. $(|U|^{p-1} U)_j = |u_j|^{p-1} u_j$ and $H$ is the self-adjoint operator on $L^2(\Gamma)$ defined by

$$(HV)(x) = (-v_j''(x))_{j=1}^N, \quad x > 0, \quad V = (v_j)_{j=1}^N,$$

$$\text{dom}(H) = \left\{ V \in H^2(\Gamma) : v_1(0) = \cdots = v_N(0), \sum_{j=1}^N v_j'(0) = \alpha v_1(0) \right\}.$$

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Condition (1.2) is an analog of δ-interaction condition for the Schrödinger operator on the line (see [3]). On each edge of the graph (i.e. on each half-line) we have

\[
i\partial_{t}u_{j}(t, x) + \partial_{x}^{2}u_{j}(t, x) + |u_{j}(t, x)|^{p-1}u_{j}(t, x) = 0, \quad x > 0, \quad j \in \{1, \ldots, N\},
\]

moreover, the vectors \(U(t, 0) = (u_{j}(t, 0))_{j=1}^{N}\) and \(U'(t, 0) = (u_{j}'(t, 0))_{j=1}^{N}\) satisfy conditions in (1.2).

In the present paper we are aimed to study the strong instability of the standing wave solutions \(U(t, x) = e^{i\omega t}\Phi(x)\) to (1.1). It is easily seen that \(\Phi(x)\) satisfies the following stationary equation

\[
H\Phi + \omega \Phi - |\Phi|^{p-1}\Phi = 0.
\]

In [2] the following description of the real-valued solutions to (1.3) was obtained.

**Theorem 1.1.** Let \([s]\) denote the integer part of \(s \in \mathbb{R}, \alpha \neq 0\). Then equation (1.3) has \([\frac{N-1}{2}] + 1\) (up to permutations of the edges of \(\Gamma\)) vector solutions \(\Phi_{k}^{\alpha} = (\varphi_{k,j}^{\alpha})_{j=1}^{N}, \quad k = 0, \ldots, [\frac{N-1}{2}]\), which are given by

\[
\varphi_{k,j}^{\alpha}(x) = \begin{cases} 
\frac{(p+1)}{2} \text{sech}^{2} \left( \frac{(p-1)}{2} \sqrt{\frac{\omega}{\alpha}} x - a_{k} \right) \frac{1}{p-1}, & j = 1, \ldots, k; \\
\frac{(p+1)}{2} \text{sech}^{2} \left( \frac{(p-1)}{2} \sqrt{\frac{\omega}{\alpha}} x + a_{k} \right) \frac{1}{p-1}, & j = k+1, \ldots, N,
\end{cases}
\]

where \(a_{k} = \tanh^{-1} \left( \frac{\alpha}{(2k-N)\sqrt{\omega}} \right)\), and \(\omega > \frac{\alpha^{2}}{(N-2k)^{2}}\).

**Definition 1.2.** We say that \(e^{i\omega t}\Phi_{k}^{\alpha}\) is strongly unstable if for any \(\varepsilon > 0\) there exists \(U_{0} \in \mathcal{E}(\Gamma)\) such that \(||U_{0} - \Phi_{k}^{\alpha}||_{H^{1}(\Gamma)} < \varepsilon\) and the solution \(U(t)\) of (1.1) with \(U(0) = U_{0}\) blows up in finite time (see definition of \(\mathcal{E}(\Gamma)\) in Notation section).

Study of the orbital stability of the profiles \(\Phi_{k}^{\alpha}\) was initiated in [2, 3]. In particular, the authors considered the case \(\alpha < 0, k = 0\). They proved that for \(1 < p \leq 5\) and \(\omega \in (\frac{\alpha^{2}}{N^{2}}, \omega^{*})\) one gets orbital stability in \(\mathcal{E}(\Gamma)\), while for \(p > 5\) and \(\omega > \omega^{*}\) the standing wave is orbitally unstable. The case of \(k = 0\) and \(\alpha > 0\) was considered in [3, 13]. Essentially it had been proven that the standing wave is orbitally unstable for any \(p > 1\) and \(\omega > \frac{\alpha^{2}}{N^{2}}\). The case of \(\alpha \neq 0, k \neq 0\) was studied in [7, 13].

The main results of this paper are the following two strong instability theorems for \(k = 0\).

**Theorem 1.3.** Let \(\alpha > 0, \omega > \frac{\alpha^{2}}{N^{2}}, \text{and} \ p \geq 5\), then the standing wave \(e^{i\omega t}\Phi_{0}^{\alpha}(x)\) is strongly unstable.

Observe that in [3, Theorem 1.1] the authors obtained orbital instability results only for \(1 < p < 5\). Namely, the above theorem completes instability results for \(p \geq 5\).

**Theorem 1.4.** Let \(\alpha < 0, \ p > 5, \omega > \frac{\alpha^{2}}{N^{2}}\). Let \(\xi_{1}(p) \in (0, 1)\) be a unique solution of

\[
\int_{\xi}^{1} \frac{p-5}{2} (1-s^{2})^{\frac{p}{2}-1} ds = \xi(1-\xi^{2})^{\frac{2}{p-1}}, \quad (0 < \xi < 1),
\]
and define $\omega_1 = \omega_1(p, \alpha) = \frac{a^2}{N^2 \xi_1(p)}$. Then the standing wave solution $e^{i\omega t} \Phi_\alpha^0$ is strongly unstable for all $\omega \in [\omega_1, \infty)$.

To prove the above theorems we use the ideas by [12, 18]. It is worth mentioning that recently a lot of strong instability results have been obtained for different models based on the NLS equation (see [16, 19, 20, 21] and references therein).

Classically the essential ingredient in the proofs of blow-up results is the virial identity for the solution to the Cauchy problem with the initial data from the $L^2$-weighted space of the quadratic weight (see [11, Chapter 6]). In Subsection 2.2 we prove the virial identity for the NLS-$\delta$ equation on $\Gamma$ using classical approach based on the approximation of $H^1$-initial data by the sequence of initial data functions with higher regularity. In particular, to do this we first prove the well-posedness of (1.1) in $\text{dom}(H)$ with the norm $\| \cdot \|_H = \| (H + m) \cdot \|_{L^2}$ (here $H + m > 0$).

Another important ingredient in the strong instability proofs is the variational characterization of the profile $\Phi_\alpha^0$. In particular, this profile is the minimizer of the action functional $S_{\omega}$ in the space $E_{\text{eq}}(\Gamma)$ restricted to the Nehari manifold. This characterization follows from the results obtained in [13, 14] for the NLS equation with $\delta$-interaction on the line.

In Section 5 we apply our technique to show strong instability Theorems 5.4 and 5.5 for the standing waves of the NLS equation with attractive $\delta'$-interaction on the line. Their variational characterization has been obtained in [1].

The paper is organized as follows. In Section 2 we prove well-posedness of the NLS-$\delta$ equation in $\text{dom}(H)$ and show the virial identity as well. The Section 3 is devoted to the variational characterization of the profile $\Phi_\alpha^0$, while in Section 4 we prove Theorems 1.3 and 1.4. In Section 5 we consider the NLS-$\delta'$ equation on the line. In Section 6 we show so-called “product rule” for the derivative of the unitary group $e^{iHt}$, which is strongly used in the proof of the well-posedness.

**Notation.**

The domain and the spectrum of the operator $H$ are denoted by $\text{dom}(H)$ and $\sigma(H)$ respectively.

By $H^1_0(\mathbb{R})$ we denote the subspace of even functions in the Sobolev space $H^1(\mathbb{R})$. The dual space for $H^1(\mathbb{R} \setminus \{0\})$ is denoted by $H^{-1}(\mathbb{R} \setminus \{0\})$.

On the star graph $\Gamma$ we define

$$L^q(\Gamma) = \bigoplus_{j=1}^N L^q(\mathbb{R}_+), \quad q \geq 1, \quad H^1(\Gamma) = \bigoplus_{j=1}^N H^1(\mathbb{R}_+), \quad H^2(\Gamma) = \bigoplus_{j=1}^N H^2(\mathbb{R}_+).$$

For instance, the norm in $L^q(\Gamma)$ is

$$\| V \|_{L^q(\Gamma)}^q = \sum_{j=1}^N \| v_j \|_{L^q(\mathbb{R}_+)}^q, \quad V = (v_j)_{j=1}^N.$$

By $\| \cdot \|_q$ we will denote the norm in $L^q(\cdot)$ (for the function on $\Gamma$, or $\mathbb{R}$, or $\mathbb{R}_+$).
We also define the spaces
\[ E(\Gamma) = \{ \mathbf{V} \in H^1(\Gamma) : v_1(0) = \cdots = v_N(0) \}, \]
\[ E_{eq}(\Gamma) = \{ \mathbf{V} \in E(\Gamma) : v_1(x) = \cdots = v_k(x), v_{k+1}(x) = \cdots = v_N(x), x \in \mathbb{R}_+ \}. \]

Moreover, the dual space for \( E(\Gamma) \) is denoted by \( E'(\Gamma) \). Finally, we set \( \Sigma(\Gamma) \) for the following weighted Hilbert space
\[ \Sigma(\Gamma) = \{ \mathbf{V} \in E(\Gamma) : x \mathbf{V} \in L^2(\Gamma) \}. \]

For \( W = (w_j)_{j=1}^N \) on \( \Gamma \), we will abbreviate
\[ \int_\Gamma W \, dx = \sum_{j=1}^N \int_{\mathbb{R}_+} w_j \, dx. \]

Given the quantity
\[ 0 < m := 1 - 2 \inf \sigma(H) < \infty, \]
we introduce the norm \( ||\Psi||_H := ||(H + m)\Psi||_2 \) that endows \( \text{dom}(H) \) with the structure of a Hilbert space. Observe that this norm for any real \( \alpha \) is equivalent to \( H^2 \)-norm on the graph. Indeed
\[ ||\Psi||_H^2 = ||\Psi'||_2^2 + m^2||\Psi||_2^2 + 2m||\Psi'||_2^2 + 2m\alpha|\psi_1(0)|^2. \]

Due to the choice of \( m \) and the Sobolev embedding we get
\[ C_1||\Psi||_{H^2(\Gamma)}^2 \leq ||\Psi'||_2^2 + m||\Psi||_2^2 \leq ||\Psi||_H^2 \leq C_2||\Psi||_{H^2(\Gamma)}^2. \]

In what follows we will use the notation \( D_H = (\text{dom}(H), ||\cdot||_H) \).

By \( C_j, C_j(\cdot), j \in \mathbb{N} \) and \( C(\cdot) \) we will denote some positive constants.

2. Well-posedness

2.1. Well-posedness in \( H^1(\Gamma) \).

It is known (see \([2, 6, 10]\)) that the Cauchy problem for equation (1.1) is well-posed. In particular, the following result holds.

**Theorem 2.1.** Let \( p > 1 \). Then for any \( U_0 \in E(\Gamma) \) there exists \( T > 0 \) such that equation (1.1) has a unique solution \( U(t) \in C([0, T], E(\Gamma)) \cap C^1([0, T], E'(\Gamma)) \) satisfying \( U(0) = U_0 \).

For each \( T_0 \in (0, T) \) the mapping \( U_0 \in E(\Gamma) \mapsto U(t) \in C([0, T_0], E(\Gamma)) \) is continuous. Moreover, equation (1.1) has a maximal solution defined on an interval of the form \([0, T_{H^1})\), and the following blow-up alternative holds: either \( T_{H^1} = \infty \) or \( T_{H^1} < \infty \) and
\[ \lim_{t \to T_{H^1}} ||U(t)||_{H^1(\Gamma)} = \infty. \]
Furthermore, the solution $U(t)$ satisfies

$$E(U(t)) = E(U_0), \quad ||U(t)||_H^2 = ||U_0||_H^2 \quad (2.1)$$

for all $t \in [0, T_H)$, where the energy is defined by

$$E(V) = \frac{1}{2}||V'||_2^2 + \frac{\alpha}{2}|v_1(0)|^2 - \frac{1}{p+1}||V||_{p+1}^{p+1}. \quad (2.2)$$

Remark 2.2. Observe that for $1 < p < 5$ the global well-posedness holds due to the above conservation laws and Gagliardo-Nirenberg inequality (2.5).

2.2. Well-posedness in $D_H$ and virial identity

Theorem 2.3. Let $p \geq 4$ and $U_0 \in \operatorname{dom}(H)$. Then there exists $T > 0$ such that equation (1.1) has a unique solution $U(t) \in C([0, T], D_H) \cap C^1([0, T], L^2(\Gamma))$ satisfying $U(0) = U_0$. Moreover, equation (1.1) has a maximal solution defined on an interval of the form $[0, T_H)$, and the following blow-up alternative holds: either $T_H = \infty$ or $T_H < \infty$ and

$$\lim_{t \to T_H} ||U(t)||_H = \infty.$$

Proof. Let $T > 0$ to be chosen later. We will use the notation

$$X_H = C([0, T], D_H) \cap C^1([0, T], L^2(\Gamma)),$$

and equip the space $X_H$ with the norm

$$||U(t)||_{X_H} = \sup_{t \in [0, T]} ||U(t)||_H + \sup_{t \in [0, T]} ||\partial_t U(t)||_2.$$

Consider

$$E = \{U(t) \in X_H : U(0) = U_0, \quad ||U(t)||_{X_H} \leq M\},$$

where $M$ is a positive constant that will be chosen later as well. It is easily seen that $(E, d)$ is a complete metric space with the metric $d(U, V) = ||U - V||_{X_H}$. Now we consider the mapping defined by

$$\mathcal{H}(U)(t) = \mathcal{T}(t)U_0 + i\mathcal{G}(U)(t),$$

where $\mathcal{T}(t) = e^{-iHt}$, $\mathcal{G}(U)(t) = \int_0^t e^{-iH(t-s)}|U(s)|^{p-1}U(s)ds$, and $U \in E, \quad t \in [0, T]$.

Our aim is to show that $\mathcal{H}$ is a contraction of $E$, and then to apply Banach’s fixed point theorem.

Step 1. We will show that $\mathcal{H} : E \to X_H$.

1. Recall that $\operatorname{dom}(H) = \{\Psi \in L^2(\Gamma) : \lim_{h \to 0} h^{-1}(\mathcal{T}(h) - I)\Psi \text{ exists}\}$. It is easily seen that $W(t) := \mathcal{T}(t)U_0 \in \operatorname{dom}(H)$. Hence $\partial_t W(t) = -iHe^{-iH}U_0 = -iHW(t)$. Obviously $\partial_t W(t) \in C([0, T], L^2(\Gamma))$ (due to the continuity of the unitary group $\mathcal{T}(t)$). The latter implies

$$||H(W(t_n) - W(t))||_2 \to 0,$$
where $t_n, t \in [0, T]$, and consequently $W(t) \in X_H$.

2. The inclusion $G(U)(t) \in C^1([0, T], L^2(\Gamma))$ follows rapidly. Indeed, [11, Lemma 4.8.4] implies that $\partial_t(|U(t)|^{p-1}U(t)) \in L^1([0, T], L^2(\Gamma))$, and the formula

$$\partial_t G(U)(t) = i e^{-iHt}|U(0)|^{p-1}U(0) + i \int_0^t e^{-iH(t-s)} \partial_s(|U(s)|^{p-1}U(s)) ds,$$  \hspace{1cm} (2.3)

from the proof of [11, Lemma 4.8.5] induces $G(U)(t) \in C^1([0, T], L^2(\Gamma))$.

3. Below we will show that $G'(U)(t) \in C([0, T], D_H)$. First we need to prove that $G'(U)(t) \in \text{dom}(H)$. Note that

$$|||U||^{p-1}U - |V|^{p-1}V||_2 \leq C_1(p)(||U||^{p-1}_\infty + ||V||^{p-1}_\infty)||U - V||_2,$$

which implies

$$|||U||^{p-1}U - |V|^{p-1}V||_2 \leq C_1(p)(||U||^{p-1}_\infty + ||V||^{p-1}_\infty)||U - V||_2.$$ 

Therefore, by the Gagliardo-Nirenberg inequality

$$||\Psi||_p \leq C||\Psi'||^{\frac{1}{2} - \frac{1}{p}}_2 ||\Psi||^{\frac{1}{2} + \frac{1}{p}}_2, \quad p \in [2, \infty], \quad \Psi \in H^1(\Gamma),$$  \hspace{1cm} (2.5)

for $U, V \in E$ we have

$$|||U||^{p-1}U - |V|^{p-1}V||_2 \leq C(M)||U - V||_2,$$  \hspace{1cm} (2.6)

where $C(M)$ is a positive constant depending on $M$. This implies $|U(t)|^{p-1}U(t) \in C([0, T], L^2(\Gamma))$.

For $t \in [0, T)$ and $h \in [0, T - t]$ we get

$$\frac{T(h) - I}{h} G(U)(t) = \frac{1}{h} \int_0^t T(t + h - s)|U(s)|^{p-1}U(s) ds - \frac{1}{h} \int_0^t T(t - s)|U(s)|^{p-1}U(s) ds$$

$$= \frac{G(U)(t + h) - G(U)(t)}{h} - \frac{1}{h} \int_t^{t+h} T(t + h - s)|U(s)|^{p-1}U(s) ds.$$  \hspace{1cm} (2.7)

Letting $h \to 0$, by the Mean Value Theorem, we arrive at $H G(U)(t) = G(U)'(t) - |U(t)|^{p-1}U(t)$, i.e we obtain the existence of the limit in (2.7), and therefore $G(U)(t) \in \text{dom}(H)$. This is still true for $t = T$ since operator $H$ is closed. Note that we have used differentiability of $G(U)(t)$ proved above.

It remains to prove the continuity of $G(U)(t)$ in $H$-norm. We will use the integration by
parts formula (it follows from Proposition 6.1)

\[
\mathcal{G}(U)(t) = \int_0^t e^{-iH(t-s)}|U(s)|^{p-1}U(s)ds \\
= -i(H + m)^{-1}|U(t)|^{p-1}U(t) + ie^{-iHt}(H + m)^{-1}|U(0)|^{p-1}U(0) \\
+ m(H + m)^{-1} \int_0^t e^{-iH(t-s)}|U(s)|^{p-1}U(s)ds + \frac{(p + 1)i}{2}(H + m)^{-1} \int_0^t e^{-iH(t-s)}|U(s)|^{p-1}\partial_s U(s)ds \\
+ \frac{(p - 1)i}{2}(H + m)^{-1} \int_0^t e^{-iH(t-s)}U^2(s)|U(s)|^{p-3}\partial_s U(s)ds.
\]

Above we have used the formula

\[
\partial_t(|U(t)|^{p-1}U(t)) = |U(t)|^{p-1}\partial_t U(t) + (p - 1)U(t)|U(t)|^{p-3}\text{Re}(|U(t)\partial_t U(t))
\]

Let \( t_n, t \in [0, T] \), and \( t_n \to t \). By (2.8) we deduce

\[
||\mathcal{G}(U)(t) - \mathcal{G}(U)(t_n)||_H \leq ||U(t)|^{p-1}U(t) - |U(t_n)|^{p-1}U(t_n)||_2 \\
+ m \int_0^t ||(e^{-iH(t-s)} - e^{-iH(t_n-s)})|U(s)|^{p-1}U(s)||_2ds + m \int_0^t ||e^{-iH(t_n-s)}|U(s)|^{p-1}U(s)||_2ds \\
+ \frac{p+1}{2} \int_0^t ||(e^{-iH(t-s)} - e^{-iH(t_n-s)})|U(s)|^{p-1}\partial_s U(s)||_2ds + \frac{p+1}{2} \int_0^t ||e^{-iH(t_n-s)}|U(s)|^{p-1}\partial_s U(s)||_2ds \\
+ \frac{p-1}{2} \int_0^t ||(e^{-iH(t-s)} - e^{-iH(t_n-s)})U^2(s)|U(s)|^{p-3}\partial_s U(s)||_2ds \\
+ \frac{p-1}{2} \int_0^t ||e^{-iH(t_n-s)}U^2(s)|U(s)|^{p-3}\partial_s U(s)||_2ds.
\]

(2.10)

Therefore, using (2.6), (2.10), unitarity and continuity properties of \( e^{-iHt} \), we obtain continuity of \( \mathcal{G}(U)(t) \) in \( D_H \).

**Step 2.** Now our aim is to choose \( T \) in order to guarantee invariance of \( E \) for the mapping \( \mathcal{H} \), i.e. \( \mathcal{H} : E \to E \).
1. Using (2.9), we obtain

\[ |U(t)|^{p-1}U(t) = \int_0^t \partial_s \left( |U(s)|^{p-1}U(s) \right) ds + |U(0)|^{p-1}U(0) \]

\[ = \int_0^t \left\{ \frac{p+1}{2} |U(s)|^{p-1} \partial_s U(s) + \frac{p-1}{2} U^2(s) |U(s)|^{p-3} \partial_s U(s) \right\} ds + |U(0)|^{p-1}U(0). \]

Using (2.3), (2.5), (2.9), (2.13), we obtain the estimate

\[ \|H + U\| \leq \|U\| + \int_0^t |U(s)|^{p-1} \partial_s U(s) ds \]

\[ \|U\| \leq \|U_0\| + \int_0^t \frac{p+1}{2} \|U(s)\|^{p-1} \| \partial_s U(s) \| ds + |U(0)|^{p-1}U(0)\|_{L^2} \]

\[ + \|U(0)\|^{p-1}U(0)\|_{L^2} + m \int_0^t \|U(s)\|^{p-1}U(s)\|_{L^2} ds + \frac{p+1}{2} \int_0^t \|U(s)\|^{p-1} \| \partial_s U(s) \| ds \]

\[ + \frac{p-1}{2} \int_0^t \|U^2(s)\|^{p-3} \| \partial_s U(s) \| ds \]

\[ \leq \|U_0\| + C_1 \|U_0\|_H + C_2 \int_0^t \|U\|_H^{p-1} \| \partial_s U(s) \| ds + C_3 \int_0^t \|U\|_\infty^{p-1} \|U(s)\|_H \]

\[ \leq \|U_0\| + C_1 \|U_0\|_H + C_1(M)TM^p. \]

(2.11)

2. Below we will estimate \( \|\partial_t H(U)(t)\|_2 \). Observe that

\[ \|\partial_t e^{-iHt}U_0\|_2 = \|HU_0\|_2 = \|U_0'\|_2 \leq \|U_0\|_H. \]

(2.13)

Using (2.3), (2.5), (2.9), (2.13), we obtain the estimate

\[ \|\partial_t H(U)(t)\|_2 \leq \|U_0\|_H + \|U(0)\|^{p-1}U(0)\|_2 + \frac{p+1}{2} \int_0^t \|U(s)\|^{p-1} \| \partial_s U(s) \| ds \]

\[ + \frac{p-1}{2} \int_0^t \|U^2(s)\|^{p-3} \| \partial_s U(s) \| ds \leq \|U_0\|_H + C_4 \|U_0\|_H^{p} + C_5 \int_0^t \|U\|_\infty^{p-1} \| \partial_s U(s) \| ds \]

\[ \leq \|U_0\|_H + C_4 \|U_0\|_H^{p} + C_2(M)TM^p. \]

(2.14)
Finally, combining (2.12) and (2.14), we arrive at
\[ ||\mathcal{H}(U)(t)||_{X_H} \leq 2||U_0||_H + (C_1 + C_4)||U_0||^p_H + (C_1(M) + C_2(M))TM^p. \]
We now let
\[ M = \frac{1}{2} \left( 2||U_0||_H + (C_1 + C_4)||U_0||^p_H \right). \]
By choosing \( T \leq \frac{1}{2(C_1(M) + C_2(M))M^p} \), we get
\[ ||\mathcal{H}(U)(t)||_{X_H} \leq M, \]
and therefore \( \mathcal{H} : E \to E \).

**Step 3.** Now we will choose \( T \) to guarantee that \( \mathcal{H} \) is a strict contraction on \( (E, d) \). Let \( U, V \in E \).

1. First, observe that (2.4) induces
\[ |||U|^{p-1}U - |V|^{p-1}V||_\infty \leq C_2(p)(|||U|||^{p-1}_\infty + |||V|||^{p-1}_\infty)||U - V||_\infty. \quad (2.15) \]
From (2.8), (2.11) it follows that
\[
||\mathcal{H}(U)(t) - \mathcal{H}(V)(t)||_H = || \int_0^t e^{-iH(t-s)} (|||U(s)|||^{p-1}U(s) - |||V|||^{p-1}V(s)) \) ds\)||_H \\
\leq m \int_0^t |||U(s)|||^{p-1}U(s) - |||V|||^{p-1}V(s)||_2 ds + (p + 1) \int_0^t |||U(s)|||^{p-1}\partial_s U(s) - |||V|||^{p-1}\partial_s V(s)||_2 ds \\
+ (p - 1) \int_0^t |||U^2(s)|||^{p-3}\partial_s U(s) - V^2(s)|||V|||^{p-3}\partial_s V(s)||_2 ds. \quad (2.16) \]
To obtain the contraction property we need to estimate two last members of inequality (2.16). Using convexity of the function \( f(x) = x^\alpha, \alpha > 1, x > 0 \), one gets
\[ |u|^{p-1} - |v|^{p-1} \leq (p - 1)|u|^{p-2}|u - v|, |u| \geq |v|, \]
and therefore
\[ ||u|^{p-1} - |v|^{p-1}|| \leq (p - 1)(||u|^{p-2} + ||u|^{p-2})|u - v|. \quad (2.17) \]
Using (2.17), we obtain
\[
|||U(s)|||^{p-1}\partial_s U(s) - |||V(s)|||^{p-1}\partial_s V(s)||_2 \leq |||U|||^{p-1}(\partial_s U - \partial_s V)||_2 + ||\partial_s V(|||U|||^{p-1} - |||V|||^{p-1})||_2 \\
\leq |||U|||^{p-1}\partial_s U - \partial_s V||_2 + ||\partial_s V||_2||U||^{p-1} - |||V|||^{p-1}||_\infty \\
\leq C_1M^{p-1}||U - V||_X_H + C_2M(|||U|||^{p-2} + |||V|||^{p-2})||U - V||_\infty \leq C_3M^{p-1}||U - V||_{X_H}. \quad (2.18) \]
Let us estimate the last term of (2.16). Using (2.15) and (2.17), we get
\[
||U^2(s)U(s)||_{p-3}^p \partial_s U(s) - V^2(s)V(s)|_{p-3} \partial_s V(s)||_2 \\
\leq ||U^2||^p ||(\partial_s U - \partial_s V)||_2 + ||\partial_s V(U^2 - V^2)||_{p-3}||V^p - V||_2 \\
\leq ||U||_{p-1}^p ||\partial_s U - \partial_s V||_2 + ||(U||_{p-3}^3 - V^3)V(U + V)\partial_s V||_2 \\
+ ||UV(U||_{p-3} - |V||_{p-3})\partial_s V||_2 \\
\leq C_4 M^{p-1}||U - V||_{X_H} + ||\partial_s V||_2||U + V||_{X_H}||U||_{p-3}^3 - ||V||_{p-3}^3||V||_\infty \\
+ C_5||U||_{X_H}||V||_{X_H}^p (||U||_{p-4}^p + ||V||_{p-4}^p) ||U - V||_{X_H}||\partial_s V||_2 \leq C_6 M^{p-1}||U - V||_{X_H}.
\]
Finally, combining (2.6), (2.16), (2.18), (2.19), we obtain
\[
||H(U) - H(V)||_H \leq C_7 M^{p-1} T ||U - V||_{X_H} (2.20)
\]

2. To get the contraction property of $H$ we need to estimate $L^2$-part of $X_H$-norm of $H(U)(t) - H(V)(t)$. From (2.3), we deduce
\[
||\partial_t H(U)(t) - \partial_t H(V)(t)||_2 \leq \int_0^t ||\partial_s (||U(s)||_{p-1}^p U(s)) - \partial_s (||V(s)||_{p-1}^p V(s))||_2 ds. (2.21)
\]
Using (2.9), (2.18), (2.19), from (2.21) we get
\[
||\partial_t H(U)(t) - \partial_t H(V)(t)||_2 \leq C_8 M^{p-1} T ||U - V||_{X_H} (2.22)
\]
and finally from (2.20), (2.22), we obtain
\[
||H(U)(t) - H(V)(t)||_{X_H} \leq (C_7 + C_8) M^{p-1} T ||U - V||_{X_H}.
\]
Thus, for
\[
T < \min \left\{ \frac{1}{(C_7 + C_8) M^{p-1}}, \frac{1}{2(C_1(M) + C_2(M)) M^{p-1}} \right\}
\]
the mapping $H$ is the strict contraction of $(E, d)$. Therefore, by the Banach fixed point theorem, $H$ has a unique fixed point $U \in E$ which is a solution of (1.1).

Uniqueness of the solution follows standardly. Suppose that $U_1(t)$ and $U_2(t)$ are two solutions to (1.1), and $\tilde{M} = \sup_{t \in [0, T]} \{||U_1(t)||_{X_H}, ||U_2(t)||_{X_H} \}$. Then
\[
||U_1(t) - U_2(t)||_2 = || \int_0^t e^{-iH(t-s)}((|U_1(s)||_{p-1}^p U_1(s) - |U_2(s)||_{p-1}^p U_2(s))ds||_2 \\
\leq C(\tilde{M}) \int_0^t ||U_1(s) - U_2(s)||_2 ds,
\]
and the result follows from Gronwall’s lemma. The blow-up alternative can be shown by bootstrap.
Remark 2.4. (i) The assumption $p \geq 4$ is technical. We believe that the result also holds for the smaller values of $p$ (see [11, Subsection 4.12]).

(ii) The idea of the proof of the above theorem was given in [10] (see Proposition 2.5) without details.

Below we will show the virial identity which is crucial for the proof of the strong instability. Define

$$
P(V) = \|V'\|^2 + \frac{\alpha}{2}|v_1(0)|^2 - \frac{p-1}{2(p+1)}\|V\|^{p+1}_{p+1}, \quad V \in \mathcal{E}(\Gamma) .
$$

(2.23)

Proposition 2.5. Let $U_0 \in \Sigma(\Gamma)$, and let $U(t)$ be the corresponding maximal solution to (1.1). Then $U(t) \in C([0,T^{H_1}), \Sigma(\Gamma))$, moreover, the function

$$
f(t) := \int_{\Gamma} x^2|U(t,x)|^2 dx
$$

belongs to $C^2[0,T^{H_1})$, $f'(t) = 4\text{Im} \int_{\Gamma} xU\partial_x U dx$, (2.24)

and

$$
f''(t) = 8P(U(t)) \quad \text{(virial identity)}
$$

(2.25)

for all $t \in [0,T^{H_1})$.

Proof. The proof is similar the one of [11, Proposition 6.5.1]. We give it for convenience of the reader.

Step 1. Let $\varepsilon > 0$, define $f_\varepsilon(t) = \|e^{-\varepsilon x^2}U(t)\|_2^2$, for $t \in [0,T]$, $T \in (0,T^{H_1})$. Then, noting that $e^{-2\varepsilon x^2}x^2U(t) \in H^1(\Gamma)$, we get

$$
f'_\varepsilon(t) = 2\text{Re} \int_{\Gamma} e^{-2\varepsilon x^2}x^2\partial_t U dx = 2\text{Re} \int_{\Gamma} e^{-2\varepsilon x^2}x^2U (i\partial_x^2 U + i|U|^{p-1}U) dx
$$

$$
= -2\text{Im} \int_{\Gamma} e^{-2\varepsilon x^2}x^2\partial_x^2 U dx = 4\text{Im} \int_{\Gamma} \left\{e^{-\varepsilon x^2}(1 - 2\varepsilon x^2)\right\} U x e^{-\varepsilon x^2} \partial_x U dx.
$$

(2.26)

Observe that $|e^{-\varepsilon x^2}(1 - 2\varepsilon x^2)| \leq C(\varepsilon)$ for any $x$. From (2.26), by the Cauchy-Schwarz inequality, we obtain

$$
|f'_\varepsilon(t)| \leq 4 \int_{\Gamma} \left\{e^{-\varepsilon x^2}(1 - 2\varepsilon x^2)\right\} U x e^{-\varepsilon x^2} \partial_x U dx \leq 4C(\varepsilon) \int_{\Gamma} \left|e^{-\varepsilon x^2}xU\partial_x U\right| dx
$$

$$
\leq 4C(\varepsilon) \sum_{j=1}^{N} \|\partial_x u_j\|_2 \|e^{-\varepsilon x^2}xu_j\|_2 \leq C(\varepsilon, N)\|U\|_{H^1(\Gamma)} \sqrt{f_\varepsilon(t)} .
$$

(2.27)
From (2.27) one implies

\[ \int_{0}^{t} \frac{f_{\varepsilon}'(s)}{f_{\varepsilon}(s)} ds \leq C(\varepsilon, N) \int_{0}^{t} ||U(s)||_{H^1(\Gamma)} ds, \]

and therefore

\[ \sqrt{f_{\varepsilon}(t)} \leq ||xU_0||_2 + \frac{C(\varepsilon, N)}{2} \int_{0}^{t} ||U(s)||_{H^1(\Gamma)} ds, \quad t \in [0, T]. \]

Letting \( \varepsilon \to 0 \) and applying Fatou’s lemma, we get that \( xU(t) \in L^2(\Gamma) \) and \( f(t) \) is bounded in \([0, T]\). Observe that from (2.26) one induces

\[ f_{\varepsilon}(t) = f_{\varepsilon}(0) + 4 \text{Im} \int_{\Gamma} \left\{ e^{-\varepsilon x^2} (1 - 2\varepsilon x^2) \right\} \bar{U} x e^{-\varepsilon x^2} \partial_x U dx. \]  

We have the following estimates for any positive \( x \) and \( \varepsilon \):

\begin{align*}
    e^{-2\varepsilon x^2} x^2 |U(t)|^2 &\leq x^2 |U(t)|^2; \\
    e^{-2\varepsilon x^2} x^2 |U_0|^2 &\leq x^2 |U_0|^2, \\
    |e^{-\varepsilon x^2} (1 - 2\varepsilon x^2) \bar{U} x e^{-\varepsilon x^2} \partial_x U| &\leq C(\varepsilon) |\partial_x U| |xU|.
\end{align*}

Having pointwise convergence, and using (2.29), by the Dominated Convergence Theorem we get from (2.28)

\[ f(t) = ||xU(t)||_2 = ||xU_0||_2 + 4 \text{Im} \int_{0}^{t} \int_{\Gamma} x \bar{U} \partial_x U dx. \]

Since \( U(t) \) is strong \( H^1 \)-solution, \( f(t) \) is \( C^1 \)-function, and (2.24) holds for any \( t \in [0, T_{H^1}] \).

Using continuity of \( ||xU(t)||_2 \) and the inclusion \( U(t) \in C([0, T_{H^1}], \mathcal{E}(\Gamma)) \), we get \( U(t) \in C([0, T_{H^1}], \Sigma(\Gamma)) \).

**Step 2.** Let \( U_0 \in \text{dom}(H) \). By Theorem 2.3 the solution \( U(t) \) to the corresponding Cauchy problem belongs to \( C([0, T_{H^1}], D_H) \cap C^1([0, T_{H^1}], L^2(\Gamma)) \). Following the ideas of proofs of [11], Theorem 5.3.1, Theorem 5.7.1 and using Strichartz estimate from [8, Theorem 1.3], one can show that \( T_{H^1} = T_H \).

Let \( \varepsilon > 0 \) and \( \theta_{\varepsilon}(x) = e^{-\varepsilon x^2} \). Define

\[ h_{\varepsilon}(t) = \text{Im} \int_{\Gamma} \theta_{\varepsilon x} \bar{U} \partial_x U dx \quad \text{for} \quad t \in [0, T], \; T \in (0, T_H). \]  

(2.30)
First, let us show that

\[ h'_\varepsilon(t) = - \text{Im} \int \partial_t U \left\{ 2 \theta \varepsilon \overline{x} \partial_x U + (\theta + x\theta'_\varepsilon) \overline{U} \right\} \, dx \]  

(2.31)

or equivalently

\[ h_\varepsilon(t) = h_\varepsilon(0) - \text{Im} \int_0^t \int \partial_t U \left\{ 2 \theta \varepsilon \overline{x} \partial_x U + (\theta + x\theta'_\varepsilon) \overline{U} \right\} \, dx. \]  

(2.32)

Let us prove that identity (2.32) holds for \( U(t) \in C([0, T], H^1(\Gamma)) \cap C^1([0, T], L^2(\Gamma)) \). Note that by density argument it is sufficient to show (2.32) for \( U(t) \in C^1([0, T], H^1(\Gamma)) \cap C^1([0, T], L^2(\Gamma)) \). From (2.30), it follows

\[ h'_\varepsilon(t) = - \text{Im} \int \left\{ \theta \varepsilon \partial_t U \overline{\partial_x U} + \theta \varepsilon x \partial_x U \overline{\partial_t U} \right\} \, dx. \]  

(2.33)

Note that

\[ \theta \varepsilon x \partial_x U \overline{\partial_t U} = \theta \varepsilon x \partial_x U \overline{\partial_t U} - \theta \varepsilon U \overline{\partial_t U} - \theta \varepsilon x \partial_x U \overline{\partial_t U} - x\theta'_\varepsilon \partial_t U, \]

which induces

\[ \int \theta \varepsilon x \partial_x U \overline{\partial_t U} \, dx = - \int \partial_t U \left\{ \theta \varepsilon (U + x \partial_x U) + x\theta'_\varepsilon U \right\} \, dx. \]

Therefore, from (2.33), we get

\[ h'_\varepsilon(t) = - \text{Im} \int \left\{ \theta \varepsilon x \partial_t U \overline{\partial_x U} + \partial_t U \left( \theta \varepsilon (U + x \partial_x U) + x\theta'_\varepsilon U \right) \right\} \, dx. \]

Consequently we obtain (2.32) for \( U(t) \in C^1([0, T], H^1(\Gamma)) \cap C^1([0, T], L^2(\Gamma)) \) and hence for \( U(t) \in C([0, T], H^1(\Gamma)) \cap C^1([0, T], L^2(\Gamma)) \) which implies (2.31).

Since \( U(t) \in C([0, T_H], D_H) \), from (2.31) we get

\[ h'_\varepsilon(t) = - \text{Re} \int \left\{ -HU + |U|^{p-1}U \right\} \left\{ 2 \theta \varepsilon x \overline{\partial_x U} + (x\theta'_\varepsilon) \overline{U} \right\} \, dx. \]  

(2.34)

Below we will consider separately linear and nonlinear part of identity (2.34). Integrating by parts, we obtain

\[ - \text{Re} \int \left\{ -HU + |U|^{p-1}U \right\} \left\{ 2 \theta \varepsilon x \overline{\partial_x U} + (x\theta'_\varepsilon) \overline{U} \right\} \, dx \]

\[ = \alpha |u_1(0)|^2 + \int \left\{ 2x\theta'_\varepsilon |\partial_x U|^2 + (\theta + x\theta''_\varepsilon) \text{Re}(\overline{U} \partial_x U) \right\} \, dx \]

(2.35)
and
\[-\text{Re} \int |U|^{p-1} U \left\{ 2\theta_\varepsilon x \partial_x U + (x\theta_\varepsilon)'U \right\} dx\]
\[= - \int |U|^{p+1} \theta_\varepsilon dx - \int |U|^{p+1} x\theta_\varepsilon' dx - \int (|U|^2)^{p-1} \partial_x (|U|^2) dx\]
\[= - \frac{p-1}{p+1} \int |U|^{p+1} \theta_\varepsilon dx - \frac{p-1}{p+1} \int |U|^{p+1} x\theta_\varepsilon' dx.\]

Finally, from (2.34)-(2.36) we get
\[
h_\varepsilon'(t) = \left[ 2 \int \theta_\varepsilon |\partial_x U|^2 dx + \alpha |u_1(0)|^2 - \frac{p-1}{p+1} \int |U|^{p+1} \theta_\varepsilon dx \right]
+ \left[ \int 2x\theta_\varepsilon' |\partial_x U|^2 dx + \int (2\theta_\varepsilon' + x\theta_\varepsilon'') \text{Re}(\overline{U} \partial_x U) dx \right] - \frac{p-1}{p+1} \int |U|^{p+1} x\theta_\varepsilon' dx.\]

Since \(\theta_\varepsilon, x\theta_\varepsilon', x\theta_\varepsilon''\) are bounded with respect to \(x\) and \(\varepsilon\), and
\[\theta_\varepsilon \rightarrow 0, \quad x\theta_\varepsilon' \rightarrow 0, \quad x\theta_\varepsilon'' \rightarrow 0\] pointwise as \(\varepsilon \rightarrow 0\),
by the Dominated Convergence Theorem we have
\[\lim_{\varepsilon \to 0} h_\varepsilon'(t) = 2||U'||^2 + \alpha |u_1(0)|^2 - \frac{p-1}{p+1}||U||^{p+1}_p =: g(t).\]

Moreover, again by the Dominated Convergence Theorem,
\[\lim_{\varepsilon \to 0} h_\varepsilon(t) = \text{Im} \int xU \partial_x U dx =: h(t).\]

Using continuity of \(g(t)\) and the fact that operator \(T = \frac{d}{dt}\) with \(\text{dom}(T) = H^1[0, T]\) is closed,
we arrive at \(h'(t) = g(t), \quad t \in [0, T]\), i.e.
\[h'(t) = 2||U'||^2 + \alpha |u_1(0)|^2 - \frac{p-1}{p+1}||U||^{p+1}_p,\]
and \(h(t)\) is \(C^1\) function. Finally, (2.25) holds for \(U_0 \in \text{dom}(H)\).

To conclude the proof consider \(\{U_0^n\}_{n \in \mathbb{N}} \subset \text{dom}(H)\) such that \(U_0^n \rightarrow U_0\) in \(H^1(\Gamma)\) and
\(xU_0^n \rightarrow xU_0\) in \(L^2(\Gamma)\) as \(n \to \infty\). Let \(U^n(t)\) be the maximal solutions of the corresponding Cauchy problem associated with (1.1). From (2.24) and (2.25) we obtain
\[||xU^n(t)||^2_2 = ||xU^n_0||^2_2 + 4t \text{Im} \int xU^n_0 \partial_x U^n_0 dx + \int_0^t \int_0^s 8P(U^n(t))dsdt.\]
Using continuous dependence and repeating the arguments from [11, Corollary 6.5.3], we obtain as $n \to \infty$

$$||xU(t)||_2^2 = ||xU_0||_2^2 + 4t \text{ Im} \int_{\Gamma} xU_0 \partial_x U_0 dx + \int_0^t \int_0^s 8P(U(t))dsdt,$$

that is (2.25) holds for $U_0 \in \mathcal{E}(\Gamma)$. □

**Remark 2.6.** In [17] the authors proved the virial identity for the NLS equation with $\delta$-potential on the line using approximation of $\delta$-potential by smooth potentials $V_\epsilon(x) = \frac{1}{\epsilon}e^{-\pi/1+\epsilon^2 x^2}$, $\epsilon \to 0$, and applying the virial identity to the NLS equation on $\mathbb{R}$ with the smooth potential (which is classical). Observe that in the present paper we overcome this procedure by proving the well-posedness in $D_H$. Obviously our proof can be repeated for the NLS equation with $\delta$-potential on the line.

### 3. Variational analysis

Define the following action functional

$$S_\omega(V) = \frac{1}{2}||V'||_2^2 + \omega||V||_2^2 - \frac{1}{p+1}||V||_{p+1}^{p+1} + \frac{\alpha}{2}|v_1(0)|^2. \quad (3.1)$$

We also introduce

$$I_\omega(V) = ||V'||_2^2 + \omega||V||_2^2 - ||V||_{p+1}^{p+1} + \alpha|v_1(0)|^2.$$

Observe that

$$I_\omega(V) = \partial_\lambda S_\omega(\lambda V)|_{\lambda=1} = \langle S'_\omega(V), V \rangle,$$

and

$$S_\omega(V) = \frac{1}{2}I_\omega(V) + \frac{p-1}{2(p+1)}||V||_{p+1}^{p+1}. \quad (3.2)$$

In [2] it was shown that for any $p > 1$ there is $\alpha^* < 0$ such that for $-N\sqrt{\omega} < \alpha < \alpha^*$ the profile $\Phi_0^\alpha$ defined by (1.4) minimizes the action functional $S_\omega$ on the Nehari manifold

$$\mathcal{N} = \{V \in \mathcal{E}(\Gamma) \setminus \{0\} : I_\omega(V) = 0\}.$$

Namely, the profile $\Phi_0^\alpha$ is the ground state for the action $S_\omega$ on the manifold $\mathcal{N}$. In [3] the authors showed that $\Phi_0^\alpha$ is a local minimizer of the energy functional $E$ defined by (2.2) among functions with equal fixed mass.

Note that $\Phi_k^\alpha \in \mathcal{N}$ for all $k$. In [2] it was proved that for $k \neq 0$ and $\alpha < 0$ we have $S_\omega(\Phi_0^\alpha) < S_\omega(\Phi_k^\alpha) < S_\omega(\Phi_{k+1}^\alpha)$.

Until now nothing is known about variational properties of the profiles $\Phi_k^\alpha$ for $\alpha > 0$. Anyway, one can easily verify that $S_\omega(\Phi_0^\alpha) > S_\omega(\Phi_k^\alpha) > S_\omega(\Phi_{k+1}^\alpha)$, $k \neq 0$.  

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We consider three minimization problems

\[
d_{\text{eq}}(\omega) = \inf \{ S_\omega(V) : V \in \mathcal{E}_{\text{eq}}(\Gamma) \setminus \{0\}, L_\omega(V) = 0 \},
\]

(3.3)

and

\[
d_{\text{line}}^t(\omega) = \inf \left\{ \frac{1}{2} \left[ \frac{1}{2} \| v' \|^2 + \omega \| v \|^2 - \frac{1}{p+1} \| v \|^{p+1} + \frac{\alpha}{N} \| v(0) \|^2 : \| v' \|^2 + \omega \| v \|^2 - \| v \|^{p+1} + \frac{\alpha}{N} \| v(0) \|^2 = 0, v \in H_1^t(\mathbb{R}) \setminus \{0\} \right\},
\]

(3.4)

\[
d^{\text{half}}(\omega) = \inf \left\{ \frac{1}{2} \left[ \frac{1}{2} \| v' \|^2 + \omega \| v \|^2 - \frac{1}{p+1} \| v \|^{p+1} + \frac{\alpha}{N} \| v(0) \|^2 : \| v' \|^2 + \omega \| v \|^2 - \| v \|^{p+1} + \frac{\alpha}{N} \| v(0) \|^2 = 0, v \in H^1(\mathbb{R}) \setminus \{0\} \right\}.
\]

It is easily seen that

\[
d_{\text{eq}}(\omega) = N d^{\text{half}}(\omega) = \frac{N}{2} d_{\text{line}}^t(\omega).
\]

From the results by [13, 14] one gets

\[
d_{\text{eq}}(\omega) = S_\omega(\Phi_0^\alpha) = \frac{N}{2} d_{\text{line}}^t(\omega) = \frac{N}{2} \left( \frac{1}{2} \| \phi' \|^2 + \alpha \| \phi \|^2 - \frac{1}{p+1} \| \phi \|^{p+1} + \frac{\alpha}{N} \| \phi(0) \|^2 \right),
\]

(3.5)

where

\[
\phi_\omega(x) = \left\{ \frac{p+1}{2} \omega \text{sech}^2 \left( \frac{p-1}{2} \sqrt{\omega} |x| - \tanh^{-1} \left( \frac{\alpha}{N \sqrt{\omega}} \right) \right) \right\}^{\frac{1}{p+1}}.
\]

Using (3.2), we obtain the following useful formula

\[
d_{\text{eq}}(\omega) = S_\omega(\Phi_0^\alpha) = \inf \left\{ \frac{p-1}{2(p+1)} \| V \|^{p+1} : V \in \mathcal{E}_{\text{eq}}(\Gamma) \setminus \{0\}, L_\omega(V) = 0 \right\}.
\]

(3.6)

In the sequel for simplicity we will always use the notation \( \Phi(x) := \Phi_0^\alpha(x) \).

**Remark 3.1.** Note that in the case \( \alpha = 0 \) one arrives at analogous result, that is

\[
d^0_{\text{eq}}(\omega) = S^0_{\omega}(\Phi_0^0) = \frac{N}{2} d_{\text{line},0}^t(\omega) = \frac{N}{2} \left( \frac{1}{2} \| \phi_{\omega,0}' \|^2 + \alpha \| \phi_{\omega,0} \|^2 - \frac{1}{p+1} \| \phi_{\omega,0} \|^{p+1} \right),
\]

where

\[
\phi_{\omega,0}(x) = \left\{ \frac{p+1}{2} \omega \text{sech}^2 \left( \frac{p-1}{2} \sqrt{\omega} x \right) \right\}^{\frac{1}{p+1}}, x \in \mathbb{R}, \quad \Phi_0^0(x) = (\phi_{\omega,0}(x))_{j=1}^N, x \in \mathbb{R},
\]

and \( d^0_{\text{eq}}(\omega), S^0_{\omega}, d_{\text{line},0}^t(\omega) \) correspond to the case \( \alpha = 0 \) in (3.1), (3.4), (3.5).

4. Proof of strong instability results

**4.1. Proof of Theorem 1.3**

The proof of theorem relies on the following three lemmas. Recall the functional \( P(V) \) defined by (2.23).
Lemma 4.1. If $V \in \mathcal{E}_{eq}(\Gamma) \setminus \{0\}$ satisfies $P(V) \leq 0$, then

$$d_{eq}(\omega) \leq S_\omega(V) - \frac{1}{2}P(V).$$

Proof. Let $V \in \mathcal{E}_{eq}(\Gamma) \setminus \{0\}$ satisfy $P(V) \leq 0$. Define $V^\lambda(x) = \lambda^{1/2}V(\lambda x)$ for $\lambda > 0$, and consider the function

$$(0, \infty) \ni \lambda \mapsto I_\omega(V^\lambda) = \lambda^2 ||V'||_2^2 + \alpha \lambda ||v_1(0)||^2 - \lambda^\beta ||V||_{p+1}^p + \omega ||V||_2^2,$$

where we put $\beta = \frac{p-1}{2} \geq 2$. Then, we have

$$\lim_{\lambda \to +0} I_\omega(V^\lambda) = \omega ||V||_2^2 > 0, \quad \lim_{\lambda \to +\infty} I_\omega(V^\lambda) = -\infty. \hspace{1cm} (4.1)$$

By (4.1), there exists $\lambda_0 \in (0, \infty)$ such that $I_\omega(V^{\lambda_0}) = 0$. Then, by definition (3.3), we have $d_{eq}(\omega) \leq S_\omega(V^{\lambda_0})$. Moreover, since $\beta \geq 2$, the function

$$(0, \infty) \ni \lambda \mapsto S_\omega(V^\lambda) - \frac{\lambda^2}{2}P(V) = \frac{2\lambda - \lambda^2}{4}\alpha ||v_1(0)||^2 + \frac{\beta \lambda^2 - 2\lambda^\beta}{2(p+1)}||V||_{p+1}^p + \frac{\omega}{2}||V||_2^2$$

attains its maximum at $\lambda = 1$. Indeed, to show this it is sufficient to study the derivative of the function $f(\lambda) := S_\omega(V^\lambda) - \frac{\lambda^2}{2}P(V)$. Thus, by using $P(V) \leq 0$, we have

$$d_{eq}(\omega) \leq S_\omega(V^{\lambda_0}) \leq S_\omega(V^{\lambda_0}) - \frac{\lambda^2}{2}P(V) \leq S_\omega(V) - \frac{1}{2}P(V).$$

This completes the proof. \qed

We introduce

$$B^+_\omega := \{V \in \mathcal{E}_{eq}(\Gamma) : S_\omega(V) < d_{eq}(\omega), P(V) < 0\}.$$

Upper index $+$ means that we consider the case of positive $\alpha$.

Lemma 4.2. The set $B^+_\omega$ is invariant under the flow of $[1.1]$. That is, if $U_0 \in B^+_\omega$, then the solution $U(t)$ to $[1.1]$ with $U(0) = U_0$ belongs to $B^+_\omega$ for all $t \in [0, T_{H^1})$.

Proof. First, by [6, Theorem 3.4], we have $U(t) \in \mathcal{E}_{eq}(\Gamma)$ for all $t \in [0, T_{H^1})$. Further, by conservation laws $[2.1]$, for all $t \in [0, T_{H^1})$, we have

$$S_\omega(U(t)) = E(U(t)) + \frac{\omega}{2}||U(t)||_2^2 = S_\omega(U_0) < d_{eq}(\omega).$$

Next, we prove that $P(U(t)) < 0$ for all $t \in [0, T_{H^1})$. Suppose that this were not true. Then, there exists $t_0 \in (0, T_{H^1})$ such that $P(U(t_0)) = 0$. Moreover, since $U(t_0) \neq 0$, it follows from Lemma $[1.1]$ that

$$d_{eq}(\omega) \leq S_\omega(U(t_0)) - \frac{1}{2}P(U(t_0)) = S_\omega(U(t_0)).$$

This contradicts the fact that $S_\omega(U(t)) < d_{eq}(\omega)$ for all $t \in [0, T_{H^1})$. Hence, we have $P(U(t)) < 0$ for all $t \in [0, T_{H^1})$. \qed
Lemma 4.3. If $U_0 \in B^+_\omega \cap \Sigma(\Gamma)$, then the solution $U(t)$ to (1.1) with $U(0) = U_0$ blows up in finite time.

Proof. By Lemma 4.2 and Proposition 2.5 we have $U(t) \in B^+_\omega \cap \Sigma(\Gamma)$ for all $t \in [0, T_{H^1})$. Moreover, by virial identity (2.25), conservation laws (2.1) and Lemma 4.1 we have

$$\frac{1}{16} \frac{d^2}{dt^2} \|xU(t)\|^2 = \frac{1}{2} P(U(t)) \leq S_\omega(U(t)) - d_{eq}(\omega) = S_\omega(U_0) - d_{eq}(\omega) < 0$$

for all $t \in [0, T_{H^1})$. Denoting $-m := S_\omega(U_0) - d_{eq}(\omega) < 0$ we get

$$\|xU(t)\|^2 \leq -16mt^2 + Ct + \|xU_0\|,$$

from which we conclude $T_{H^1} < \infty$.

Remark 4.4. Observe that for $\alpha = 0$ one can prove analogously the result: Let $\omega > 0$, and $p \geq 5$, then the standing wave $e^{i\omega t}\Phi_0(x)$ is strongly unstable.

Remark 4.5. (i) In [9] the authors studied the strong instability of the standing wave solution (ground state) to the NLS equation

$$i \partial_t u = -\Delta u - |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

They have used the fact that the ground state is the minimizer of the problem

$$d(\omega) = \inf \{ S_\omega(v) : v \in H^1(\mathbb{R}^n) \setminus \{0\}, \ P(v) = 0 \},$$

where $S_\omega$ is the corresponding action functional, and $P$ is from the virial identity. Similarly to the proof of Theorem 1.3 the authors use invariance of the set

$$B_\omega = \{ v \in H^1(\mathbb{R}^n) : S_\omega(v) < d(\omega), \ P(v) < 0 \}$$

under the flow of the NLS equation.
(ii) In [17] the authors considered the particular case \( n = 2 \), i.e. the NLS-\( \delta \) equation on the line. Namely, the strong instability of the standing wave \( \varphi_{\omega, \gamma} \) was proved for \( \gamma < 0 \) and \( p \geq 5 \). The authors used the fact that \( \varphi_{\omega, \gamma} \) is the minimizer of the problem

\[
  d_M = \inf\{ S_{\omega, \gamma}(v) : H^1_\omega(\mathbb{R}) \setminus \{0\}, \ P_\gamma(v) = 0, \ I_{\omega, \gamma}(v) \leq 0 \}.
\]

Moreover, the invariance of the set

\[
  \mathcal{B}_{\omega, \gamma} = \{ v \in H^1_\omega(\mathbb{R}) : S_{\omega, \gamma}(v) < S_{\omega, \gamma}(\varphi_{\omega, \gamma}), \ P_\gamma(v) < 0, \ I_{\omega, \gamma}(v) < 0 \}
\]

under the flow of the NLS-\( \delta \) equation was used.

(iii) The proof by [17] mentioned above can be generalized to the case of \( \Gamma \) and \( \alpha > 0 \). Namely, one needs to prove that \( \Phi^\alpha_0 \) is the minimizer of

\[
  d_M(\omega) = \inf\{ S_\omega(V) : V \in \mathcal{E}_{eq}(\Gamma) \setminus \{0\} : P(V) = 0, \ I_\omega(V) \leq 0 \},
\]

and to substitute \( \mathcal{B}_{\omega, \gamma} \) by

\[
  \mathcal{B}_{\omega, \gamma} = \{ V \in \mathcal{E}_{eq}(\Gamma) : S_\omega(V) < S_\omega(\Phi^\alpha_0), \ I_\omega(V) < 0, \ P(V) < 0 \}.
\]

4.2. Proof of Theorem 1.4

As in the previous case the proof can be divided into series of lemmas.

Lemma 4.6. Let \( \alpha < 0, p > 5 \) and \( \omega > \frac{\alpha^2}{N^2} \). Let \( \omega_1 \) be the number defined in Theorem 1.4. Then \( \partial_\lambda^2 \mathcal{E}(\Phi^\lambda)|_{\lambda=1} \leq 0 \) if and only if \( \omega \geq \omega_1 \).

Proof. Since \( P(\Phi) = (|\Phi'|^2 + \frac{\omega}{2} |\varphi(0)|^2 - \frac{p-1}{2(p+1)} ||\Phi||_{p+1}^2 = 0 \), the condition \( \partial_\lambda^2 \mathcal{E}(\Phi^\lambda)|_{\lambda=1} = (|\Phi'|^2 - \frac{(p-1)(p-3)}{4(p+1)} ||\Phi||_{p+1}^2 \leq 0 \) is equivalent to

\[
  -\alpha |\varphi(0)|^2 \leq \frac{(p-1)(p-5)}{2(p+1)} ||\Phi||_{p+1}^2.
\]

Denoting \( \xi = \frac{-N}{N \sqrt{\omega}} \), we obtain

\[
  |\varphi(0)|^2 = \left[ \frac{(p+1)\omega}{2} \text{sech}^2(\text{tanh}^{-1} \xi) \right]^{\frac{2}{p+1}} = \left[ \frac{(p+1)\omega}{2} (1 - \xi^2) \right]^{\frac{2}{p+1}},
\]

and

\[
  ||\Phi||_{p+1}^2 = N \int_{\mathbb{R}^+} \left[ \frac{(p+1)\omega}{2} \text{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} x + \text{tanh}^{-1} \xi \right) \right]^{\frac{p+1}{p+1}} dx
\]

\[
  = \frac{2N}{(p-1)\sqrt{\omega}} \left[ \frac{(p+1)\omega}{2} \right]^{\frac{p+1}{p+1}} \int_{\text{tanh}^{-1} \xi}^\infty \left( \text{sech}^2 y \right)^{\frac{p+1}{p+1}} dy
\]

\[
  = \frac{2N}{(p-1)\sqrt{\omega}} \left[ \frac{(p+1)\omega}{2} \right]^{\frac{p+1}{p+1}} \int_\xi^1 (1 - s^2)^{\frac{2}{p+1}} ds.
\]
Using (4.3) and (4.4), we see that (4.2) is equivalent to

$$\frac{p-5}{2} \int_{\xi}^{1} (1-s^2)^{\frac{p-2}{2}} ds \geq \xi (1-\xi^2)^{\frac{p-2}{2}}.$$  \hspace{1cm} (4.5)

Consider the function $f(\xi) = \frac{p-5}{2} \int_{\xi}^{1} (1-s^2)^{\frac{p-2}{2}} ds - \xi (1-\xi^2)^{\frac{p-2}{2}}, \xi \in [0,1]$. Observing that $f(0) > 1$, $f(1) = 0$, and the derivative $f'(\xi)$ has a unique zero in $(0,1)$, the function $f$ has a unique zero $\xi_1$ in $(0,1)$. Hence $f(\xi) \geq 0$ for $\xi \in [0,\xi_1]$, and therefore, recalling that $\xi = \frac{-\alpha}{\sqrt{N^2}}$, inequality (4.5) holds for $\omega \geq \omega_1 = \frac{\alpha^2}{N^2}$. 

Throughout this Section we impose the assumption $\omega \geq \omega_1$ or equivalently, by the above Lemma, we assume that $\partial^2_{\lambda} E(\Phi^\lambda)|_{\lambda=1} \leq 0$.

**Lemma 4.7.** If $V \in \mathcal{E}_{eq}(\Gamma)$ and $\|V\|_{p+1} = \|\Phi\|_{p+1}$, then $S_\omega(V) \geq d_{eq}(\omega)$.

**Proof.** First, we prove $I_\omega(V) \geq 0$ by contradiction. Suppose that $I_\omega(V) < 0$. Let

$$\lambda_1 = \left( \frac{\|V\|^2 + \omega \|V\|^2 + \alpha |v_1(0)|^2}{\|V\|^p} \right)^{1/(p-1)}.$$  

Then, $0 < \lambda_1 < 1$ and $I_\omega(\lambda_1 V) = 0$. Moreover, since $\lambda_1 V \in \mathcal{E}_{eq}(\Gamma) \setminus \{0\}$, it follows from (3.6) and (3.2) and that

$$\frac{p-1}{2(p+1)} \|\Phi\|_{p+1}^p \leq d_{eq}(\omega) = S_\omega(\lambda_1 V) = S_\omega(\lambda_1 V) - \frac{1}{2} I_\omega(\lambda_1 V) = \frac{p-1}{2(p+1)} \|\lambda_1 V\|^p_{p+1}.$$  

This contradicts the assumption $\|V\|_{p+1} = \|\Phi\|_{p+1}$. Thus, we have $I_\omega(V) \geq 0$.

Finally, we arrive at

$$d_{eq}(\omega) = \frac{p-1}{2(p+1)} \|\Phi\|_{p+1}^p \leq \frac{p-1}{2(p+1)} \|V\|^p_{p+1} + \frac{1}{2} I_\omega(V) = S_\omega(V).$$

This completes the proof. \hfill \Box

**Lemma 4.8.** If $V \in \mathcal{E}_{eq}(\Gamma)$ satisfies

$$\|V\|_2 \leq \|\Phi\|_2, \quad \|V\|_{p+1} > \|\Phi\|_{p+1}, \quad P(V) \leq 0,$$

then

$$d_{eq}(\omega) \leq S_\omega(V) - \frac{1}{2} P(V).$$
Proof. Define

$$
\lambda_0 = \left( \frac{||\Phi||^{p+1}}{||V||^{p+1}} \right)^{\frac{1}{p+1}},
$$

then $0 < \lambda_0 < 1$, moreover, $||V^{\lambda_0}||^{p+1} = \lambda_0^2||V||^{p+1} = ||\Phi||^{p+1}$.

The key ingredient of the proof is the inequality $S_\omega(\Phi) \leq S_\omega(V^{\lambda_0})$. It follows by Lemma [4.7] since $d_{eq}(\omega) = S_\omega(\Phi)$ and $||\Phi||^{p+1} = ||V^{\lambda_0}||^{p+1}$.

Define $f(\lambda) = S_\omega(V^{\lambda}) - \frac{\lambda^2}{2} P(V)$, $\lambda \in (0, 1)$. Suppose that $f(\lambda_0) \leq f(1)$. Using $P(V) \leq 0$, one gets

$$S_\omega(\Phi) \leq S_\omega(V^{\lambda_0}) \leq S_\omega(V^{\lambda_0}) - \frac{\lambda^2}{2} P(V) \leq S_\omega(V) - \frac{1}{2} P(V),$$

and we are done. Thus, it is sufficient to prove $f(\lambda_0) \leq f(1)$. The proof is analogous to the proofs of [12, Lemma 3.2] and [18, Lemma 3.1]. Denote $\beta = \frac{p-1}{2}$. Observe that

$$f(\lambda_0) \leq f(1) \iff -\alpha |v_3(0)|^2 \leq \frac{2}{p+1} \frac{2\alpha^2 - \beta^2 - 2 + \beta}{(\lambda_0 - 1)^2} ||V||^{p+1}. \quad (4.6)$$

Thus, one should be aimed to prove the second inequality in (4.6). Note that the condition $\Phi^{(1)} = ||\Phi||^2 - \frac{(p-1)(p-3)}{4(p+1)} ||\Phi||^{p+1} \leq 0$ is equivalent to

$$||\Phi'||^2 \leq \frac{\beta(\beta - 1)}{p+1} ||\Phi||^{p+1}. \quad (4.7)$$

Using Pohozaev-type equality

$$||\Phi'||^2 - \omega ||\Phi||^2 + \frac{2}{p+1} ||\Phi||^{p+1} = 0$$

and estimate (4.7), we deduce

$$\omega ||\Phi||^2 = ||\Phi'||^2 + \frac{2}{p+1} ||\Phi||^{p+1} \leq \frac{\beta^2 - \beta + 2}{p+1} ||\Phi||^{p+1}. \quad (4.8)$$

Combining $||V||^2 \leq ||\Phi||^2$ and $||\Phi||^{p+1} = \lambda_0^2 ||V||^{p+1}$, we obtain from (4.8)

$$\omega ||\Phi||^2 \leq \frac{\beta^2 - \beta + 2}{p+1} \lambda_0^2 ||V||^{p+1}. \quad (4.9)$$

By the proof of Lemma [4.7], we have

$$I_\omega(V^{\lambda_0}) = \lambda_0^2 ||V'||^2 + \omega ||V||^2 + \lambda_0 \alpha |v_3(0)|^2 - \lambda_0^\beta ||V||^{p+1} \geq 0,$$

and therefore

$$- \lambda_0 \alpha |v_3(0)|^2 \leq \lambda_0^2 ||V'||^2 + \omega ||V||^2 - \lambda_0^\beta ||V||^{p+1}. \quad (4.10)$$

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The condition $P(V) = ||V||_2^2 + \frac{\alpha}{2} |v_1(0)|^2 - \frac{\beta}{p+1} ||V||_{p+1}^{p+1} \leq 0$ implies

$$||V'||_2^2 \leq -\frac{\alpha}{2} |v_1(0)|^2 + \frac{\beta}{p+1} ||V||_{p+1}^{p+1}.$$  \hspace{1cm} (4.11)

Combining (4.9)-(4.11) we get

$$-\alpha |v_1(0)|^2 \leq \frac{2}{p+1} \beta (\lambda_0^2 + (\beta - 3) \lambda_0^3) \lambda_0 (2 - \lambda_0) ||V||_{p+1}^{p+1}.$$  \hspace{1cm} (4.12)

By (4.6) and (4.12), we conclude that $f(\lambda_0) \leq f(1)$ holds if

$$\frac{\beta (\lambda^2 + (\beta - 3) \lambda^3)}{\lambda (2 - \lambda)} \leq \frac{2 \lambda^2 - \beta \lambda^2 - 2 + \beta}{(\lambda - 1)^2}$$

for $\lambda \in (0, 1)$. \hspace{1cm} (4.13)

Inequality (4.13) can be verified by proving that the derivative of the function

$$g(\lambda) = \frac{\beta (\lambda^2 + (\beta - 3) \lambda^3)}{\lambda (2 - \lambda)} - \frac{2 \lambda^2 - \beta \lambda^2 - 2 + \beta}{(\lambda - 1)^2}$$

is nonpositive for $\lambda \in (0, 1)$. This can be done similarly to the second part of the proof of [12, Lemma 3.2].

**Remark 4.9.** Observe that the condition $\partial^2 \mathcal{E}(\Phi^\lambda)|_{\lambda=1} \leq 0$ is crucial for the proof of the key inequality $d_{eq}(\omega) \leq S_\omega(V) - \frac{1}{2}P(V)$.

We introduce

$$\mathcal{B}_\omega^- := \left\{ V \in \mathcal{E}_{eq}(\Gamma) : S_\omega(V) < d_{eq}(\omega), P(V) < 0, ||V||_2 \leq ||\Phi||_2, ||V||_{p+1} > ||\Phi||_{p+1} \right\}.$$

**Lemma 4.10.** The set $\mathcal{B}_\omega^-$ is invariant under the flow of (1.1). That is, if $U_0 \in \mathcal{B}_\omega^-$, then the solution $U(t)$ to (1.1) with $U(0) = U_0$ belongs to $\mathcal{B}_\omega^-$ for all $t \in [0, T_{H^1})$.

**Proof.** First, by [3, Theorem 3.4], we have $U(t) \in \mathcal{E}_{eq}(\Gamma)$ for all $t \in [0, T_{H^1})$. Further, by conservation laws (2.1), for all $t \in [0, T_{H^1})$, we have

$$S_\omega(U(t)) = E(U(t)) + \frac{\omega}{2} ||U(t)||_2^2 = S_\omega(U_0) < d_{eq}(\omega), ||U(t)||_2 = ||U_0||_2 \geq ||\Phi||_2.$$

Next, we prove that $P(U(t)) < 0$ for all $t \in [0, T_{H^1})$. Suppose that this were not true. Then, there exists $t_0 \in (0, T_{H^1})$ such that $P(U(t_0)) = 0$. Moreover, since $U(t_0) \neq 0$, it follows from Lemma 4.8 that

$$d_{eq}(\omega) \leq S_\omega(U(t_0)) - \frac{1}{2}P(U(t_0)) = S_\omega(U(t_0)).$$

This contradicts the fact that $S_\omega(U(t)) < d_{eq}(\omega)$ for all $t \in [0, T_{H^1})$. Thus, we have $P(U(t)) < 0$ for all $t \in [0, T_{H^1})$.

Finally, we prove that $||U(t)||_{p+1} > ||\Phi||_{p+1}$ for all $t \in [0, T_{H^1})$. Again suppose that this were not true. Then, there exists $t_1 \in (0, T_{H^1})$ such that $||U(t_1)||_{p+1} = ||\Phi||_{p+1}$. By Lemma 4.7, we have $d_{eq}(\omega) \leq S_\omega(U(t_1))$. This contradicts the fact that $S_\omega(U(t)) < d_{eq}(\omega)$ for all $t \in [0, T_{H^1})$. Hence, we have $||U(t)||_{p+1} > ||\Phi||_{p+1}$ for all $t \in [0, T_{H^1})$. \hfill \Box
Lemma 4.11. If $U_0 \in B_\omega^- \cap \Sigma(\Gamma)$, then the solution $U(t)$ to (1.1) with $U(0) = U_0$ blows up in finite time.

Proof. By Lemma 4.10 and Proposition 2.5, we have $U(t) \in B_\omega^- \cap \Sigma(\Gamma)$ for all $t \in [0, T_{H^1})$. Moreover, by virial identity (2.25), conservation laws (2.1) and Lemma 4.8, we have

$$\frac{1}{16} \frac{d^2}{dt^2} \|x U(t)\|^2_2 = \frac{1}{2} \mathbf{P}(U(t)) \leq S_\omega(U(t)) - d_{eq}(\omega) = S_\omega(U_0) - d_{eq}(\omega) < 0$$

for all $t \in [0, T_{H^1})$, from which we conclude $T_{H^1} < \infty$. \hfill \qed

Finally, we give the proof of Theorem 1.4.

Proof of Theorem 1.4. First, we note that $\Phi = (\varphi, \ldots, \varphi) \in \mathcal{E}_{eq}(\Gamma) \cap \Sigma(\Gamma)$. Let $\omega \geq \omega_1$, then, by Lemma 4.6, $\partial_\lambda^2 \mathbf{E}(\Phi^\lambda)|_{\lambda = 1} \leq 0$.

Since $S_\omega(\Phi) = 0$ and $\beta = \frac{\omega - 1}{2} > 2$, the function

$$(0, \infty) \ni \lambda \mapsto S_\omega(\Phi^\lambda) = \frac{\lambda^2}{2} \|\Phi^\lambda\|^2_2 + \frac{\alpha}{2} \lambda|\varphi(0)|^2 + \frac{\omega}{2} \|\Phi^\lambda\|^2_2 - \frac{\lambda^2}{p + 1} \|\Phi^\lambda\|_{p+1}^{p+1}$$

attains its maximum at $\lambda = 1$, and we see that

$$S_\omega(\Phi^1) < S_\omega(\Phi) = d_{eq}(\omega), \quad \mathbf{P}(\Phi^1) = \lambda \partial_\lambda S_\omega(\Phi^1) < 0,$$

$$\|\Phi^\lambda\|^2_2 = \|\Phi\|^2_2, \quad \|\Phi^\lambda\|_{p+1} = \lambda \|\Phi\|_{p+1} > \|\Phi\|_{p+1}$$

for all $\lambda > 1$. Thus, for $\lambda > 1$, $\Phi^\lambda \in B_\omega^- \cap \Sigma(\Gamma)$, and it follows from Lemma 4.11 that the solution $U(t)$ of (1.1) with $U(0) = \Phi^\lambda$ blows up in finite time.

Finally, since $\lim_{\lambda \to 1} \|\Phi^\lambda - \Phi\|_{H^1} = 0$, the proof is completed. \hfill \qed

Remark 4.12. In [21], the authors considered the strong instability of the standing wave $\varphi_{\omega, \gamma}$ to the NLS-$\delta$ equation on the line for $\gamma > 0, p > 5$. In particular, it was shown that the condition $E(\varphi_{\omega, \gamma}) > 0$ guarantees strong instability of $\varphi_{\omega, \gamma}$. Here $E$ is the corresponding energy functional. The proof by [21] can be easily adapted to the case of the NLS-$\delta$ equation on $\Gamma$, that is, the condition $E(\Phi) > 0$ guarantees the strong instability of $\Phi$ for $\alpha < 0, p > 5$.

In [18], it was noted that the condition $E(\Phi) > 0$ implies $\partial_\lambda^2 \mathbf{E}(\Phi^\lambda)|_{\lambda = 1} \leq 0$, and therefore Theorem 1.4 is slightly better than an analogous result with the condition $E(\Phi) > 0$.

5. NLS-$\delta'$ equation on the line

In this section we consider strong instability of the standing wave solution $u(t, x) = e^{i\omega t} \varphi(x)$ to the NLS-$\delta'$ equation on the line

$$i \partial_t u(t, x) - H_\gamma u(t, x) + |u|^{p-1} u = 0,$$

where $u(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, and $H_\gamma$ is the self-adjoint operator on $L^2(\mathbb{R})$ defined by

$$(H_\gamma v)(x) = -v''(x), \quad x \neq 0,$$

$$\text{dom}(H_\gamma) = \{v \in H^2(\mathbb{R} \setminus \{0\}) : v'(0-) = v'(0+), \, v(0+) - v(0-) = -\gamma v'(0)\}.$$
The corresponding stationary equation has the form

\[ H_\gamma \varphi + \omega \varphi - |\varphi|^{p-1} \varphi = 0. \]  

(5.2)

From [1, Proposition 5.1] it follows that for \( \gamma > 0 \) two functions below (odd and asymmetric) are the solutions to (5.2).

\[ \varphi^{\text{odd}}_{\omega, \gamma}(x) = \text{sign}(x) \left[ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} (|x| + y_0) \right) \right]^{\frac{1}{p-1}}, \quad x \neq 0; \quad \frac{4}{\gamma^2} < \omega, \]  

(5.3)

\[ \varphi^{\text{as}}_{\omega, \gamma}(x) = \begin{cases} \left[ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} (x + y_1) \right) \right]^{\frac{1}{p-1}}, & x > 0; \\ -\left[ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left( \frac{(p-1)\sqrt{\omega}}{2} (x - y_2) \right) \right]^{\frac{1}{p-1}}, & x < 0, \end{cases} \]  

(5.4)

where \( y_0 = \frac{2}{(p-1)\sqrt{\omega}} \tanh^{-1}(\frac{2}{\gamma^2}) \) and \( y_j = \frac{2}{(p-1)\sqrt{\omega}} \tanh^{-1}(t_j), \quad j \in \{1, 2\} \). Here \( 0 < t_1 < t_2 \) are constants satisfying the system (see formula (5.2) in [1]):

\[ \begin{cases} t_1^{p-1} - t_1^{p+1} = t_2^{p+1} - t_2^{p-1}, \\ t_1^{2} + t_2^{2} = \gamma \sqrt{\omega}. \end{cases} \]  

(5.5)

Note that when transposing \( y_1 \) and \( y_2 \) in (5.4), one gets the second asymmetric solution to (5.1). In [1, Theorem 5.3] it had been proven that \( \varphi^{\text{odd}}_{\omega, \gamma}(x) \) and \( \varphi^{\text{as}}_{\omega, \gamma}(x) \) are the minimizers (for \( \frac{1}{\gamma^2} < \omega < \frac{4}{\gamma^2} \) and \( \omega > \frac{4}{\gamma^2} \)) respectively) of the problem

\[ d_\gamma(\omega) = \inf \{ S_{\omega, \gamma}(v) : v \in H^1(\mathbb{R} \setminus \{0\}) \setminus \{0\}, \quad I_{\omega, \gamma}(v) = 0 \}, \]

where

\[ S_{\omega, \gamma}(v) = \frac{1}{2} \|v'\|^2 + \widehat{\omega} \|v\|^2 - \frac{1}{p+1} \|v\|^{p+1} - \frac{1}{2\gamma} |v(0^+) - v(0^-)|^2, \]

and

\[ I_{\omega, \gamma}(v) = \|v'\|^2 + \widehat{\omega} \|v\|^2 - \|v\|^{p+1} - \frac{1}{\gamma} |v(0^+) - v(0^-)|^2. \]

Moreover, for \( \omega > \frac{4}{\gamma^2} \) the odd profile \( \varphi^{\text{odd}}_{\omega, \gamma} \) is the minimizer of the problem (see the proof of Theorem 6.13 in [1])

\[ d_{\gamma, \text{odd}}(\omega) = \inf \{ S_{\omega, \gamma}(v) : v \in H^1_{\text{odd}}(\mathbb{R} \setminus \{0\}) \setminus \{0\}, \quad I_{\omega, \gamma}(v) = 0 \}. \]

The well-posedness result (in \( H^1(\mathbb{R} \setminus \{0\}) \)) analogous to Theorem 2.1 was affirmed in [1, Proposition 3.3 and 3.4]. Namely, the next proposition holds.

**Proposition 5.1.** Let \( p > 1 \). Then for any \( u_0 \in H^1(\mathbb{R} \setminus \{0\}) \) there exists \( T > 0 \) such that equation (5.1) has a unique solution \( u(t) \in C([0, T], H^1(\mathbb{R} \setminus \{0\})) \cap C^1([0, T], H^{-1}(\mathbb{R} \setminus \{0\})) \) satisfying \( u(0) = u_0 \). For each \( T_0 \in (0, T) \) the mapping \( u_0 \in H^1(\mathbb{R} \setminus \{0\}) \mapsto u(t) \in C([0, T_0], H^1(\mathbb{R} \setminus \{0\})) \) is continuous. Moreover, equation (5.1) has a maximal solution
defined on an interval of the form \([0, T_{H^1})\), and the following ”blow-up alternative” holds: either \(T_{H^1} = \infty\) or \(T_{H^1} < \infty\) and

\[
\lim_{t \to T_{H^1}} \|u(t)\|_{H^1(\mathbb{R}\setminus\{0\})} = \infty.
\]

Furthermore, the charge and the energy are conserved

\[
E_\gamma(u(t)) = E_\gamma(u_0), \quad \|u(t)\|_2^2 = \|u_0\|_2^2
\]

for all \(t \in [0, T_{H^1})\), where the energy is defined by

\[
E_\gamma(v) = \frac{1}{2}||v'||_2^2 - \frac{1}{2\gamma}|v(0+) - v(0-)|^2 - \frac{1}{p + 1}||v|^{p+1}_{p+1}.
\]

Remark 5.2. The well-posedness in \(H^1_{\text{odd}}(\mathbb{R}\setminus\{0\})\) was shown in the proof of [1, Theorem 6.11] using the explicit form of the integral kernel for the unitary group \(e^{-itH_2}\).

Observing that \(\inf \sigma(H_\gamma) = \left\{\begin{array}{ll}
-\frac{4}{\gamma}, & \gamma < 0 \\
0, & \gamma \geq 0,
\end{array}\right.\) and repeating the proof of Theorem 2.3 and Proposition 2.5, one gets the well-posedness in \(D_{H_\gamma}\), and the following virial identity for the solution \(u(t)\) to the Cauchy problem with the initial data \(u_0 \in H^1(\mathbb{R}\setminus\{0\}) \cap L^2(x^2, \mathbb{R})\)

\[
\frac{d^2}{dt^2}\|xu(t)\|_2^2 = 8P_\gamma(u(t)), \quad t \in [0, T_{H^1}).
\]

(5.6)

Here

\[
P_\gamma(v) = ||v'||_2^2 - \frac{1}{2\gamma}|v(0+) - v(0-)|^2 - \frac{1}{2(p+1)}||v|^{p+1}_{p+1}, \quad v \in H^1(\mathbb{R}\setminus\{0\}).
\]

Remark 5.3. Observe that Strichartz estimates for \(e^{-itH_2}\) analogous to estimates from [8, Theorem 1.3] might be obtained using the explicit formula (3.6) in [4]. In particular, the case of \(A_+ = 0 > A_\omega\) takes place in formula (3.6).

Equality (5.6) is the key ingredient of the proof of subsequent strong instability results.

Theorem 5.4. Let \(\gamma > 0, p > 5\). There exists \(\omega_2 > \frac{4}{\gamma} \frac{p+1}{p-1}\) such that \(e^{i\omega t} \varphi^\omega(x)\) is strongly unstable in \(H^1(\mathbb{R}\setminus\{0\})\) for \(\omega \geq \omega_2\).

Theorem 5.5. Let \(\gamma > 0, p > 5, \omega > \frac{4}{\gamma} \frac{p+1}{p-1}\). Let \(\xi_3(p) \in (0, 1)\) be a unique solution of

\[
\frac{p - 5}{2} \int_0^1 (1 - s^2)^{\frac{3}{2} - \frac{5}{p+1}} ds = \xi(1 - \xi^2)^{\frac{3}{2} - \frac{5}{p+1}}, \quad (0 < \xi < 1),
\]

and define \(\omega_3 = \omega_3(p, \gamma) = \frac{4}{\gamma \xi^3_3(p)}\). Then the standing wave solution \(e^{i\omega t} \varphi^\omega(x)\) is strongly unstable for all \(\omega \in [\omega_3, \infty)\).

Remark 5.6. Observe that \(\omega_3 > \frac{4}{\gamma} \frac{p+1}{p-1}\) since by [1, Proposition 6.11] \(e^{i\omega t} \varphi^\omega(x)\) is orbitally stable for \(\frac{4}{\gamma} < \omega < \frac{4}{\gamma} \frac{p+1}{p-1}\).
Key steps of the proofs of Theorem 5.4 and 5.5. Basically one needs to repeat the proof of Theorem 1.4. The only step which should be checked carefully is Lemma 4.6.

1. Consider the case of $\varphi_{\omega, \gamma}^a(\cdot)$. Denote $\varphi_\gamma := \varphi_{\omega, \gamma}^a$. We need to show that $\partial_\lambda^2 E_\gamma(\varphi_\lambda^\gamma) |_{\lambda=1} \leq 0$ for $\omega \in [\omega_2, \infty)$, where $\omega_2$ is sufficiently large. Using, $P_\gamma(\varphi_\gamma) = 0$, it is easily seen that the condition $\partial_\lambda^2 E_\gamma(\varphi_\lambda^\gamma) |_{\lambda=1} \leq 0$ is equivalent to

$$\frac{1}{\gamma} |\varphi_\gamma(0^+) - \varphi_\gamma(0^-)|^2 < \frac{(p-5)(p-1)}{2(p+1)} ||\varphi_\gamma||_{p+1}^{p+1}. \quad (5.7)$$

From (5.4) and (5.5) one gets

$$|\varphi_\gamma(0^+) - \varphi_\gamma(0^-)|^2 = \left(\frac{p+1}{2} \omega\right)^{p-1} \left((1-t_1^2)^{p-1} + (1-t_2^2)^{p-1}\right)^2, \quad (5.8)$$

and

$$||\varphi_\gamma||_{p+1}^{p+1} = \frac{2}{(p-1)\sqrt{\omega}} \left(\frac{p+1}{2} \omega\right)^{p-1} \left[\int_{t_1}^{1} (1-s^2)^{p-1} ds + \int_{t_2}^{1} (1-s^2)^{p-1} ds \right]. \quad (5.9)$$

Combining (5.8) and (5.9), we deduce from (5.7) that the condition $\partial_\lambda^2 E_\gamma(\varphi_\lambda^\gamma) |_{\lambda=1} \leq 0$ is equivalent to

$$\frac{p-5}{2} \left[\int_{t_1}^{1} (1-s^2)^{\frac{2}{p-1}} ds + \int_{t_2}^{1} (1-s^2)^{\frac{2}{p-1}} ds \right] - \frac{1}{\beta \sqrt{\omega}} \left[(1-t_1^2)^{\frac{1}{p-1}} + (1-t_2^2)^{\frac{1}{p-1}}\right]^2 > 0. \quad (5.10)$$

Observe that $t_1$ and $t_2$ have the following asymptotics as $\omega \to \infty$ (see formula (6.34) in [1])

$$t_1 = \frac{1}{\gamma \sqrt{\omega}} + o(\omega^{-\frac{1}{2}}), \quad t_2 = 1 - \frac{1}{2\gamma^{p-1} \omega^{\frac{p-1}{2}}} + o(\omega^{-\frac{p-1}{2}}).$$

From the above asymptotics, sending $\omega$ to infinity, one gets that the limit of the expression in (5.10) is positive and equals $\frac{p-5}{2} \int_{0}^{1} (1-s^2)^{\frac{2}{p-1}} ds$. Hence the expression in (5.10) is positive for $\omega$ large enough. This ensures the existence of $\omega_2$ such that $\partial_\lambda^2 E_\gamma(\varphi_\lambda^\gamma) |_{\lambda=1} \leq 0$ for $\omega \in [\omega_2, \infty)$.

2. Let now $\varphi_\gamma := \varphi_{\omega, \gamma}^o$. We need to show that $\partial_\lambda^2 E_\gamma(\varphi_\lambda^\gamma) |_{\lambda=1} \leq 0$ for $\omega \in [\omega_3, \infty)$. The proof repeats the one of Lemma 4.6. The only difference is that inequality (4.2) has to be substituted by

$$\frac{1}{\gamma} |\varphi_\gamma(0^+) - \varphi_\gamma(0^-)|^2 < \frac{(p-5)(p-1)}{2(p+1)} ||\varphi_\gamma||_{p+1}^{p+1}$$

and $\xi(\omega, \gamma) = \frac{2}{\gamma \sqrt{\omega}}$.
6. Appendix

Let \( \mathcal{I} \subseteq \mathbb{R} \) be an open interval. We say that the function \( g(s) : \mathcal{I} \to L^2(\Gamma) \) is \( L^2 \)-differentiable on \( \mathcal{I} \) if the limit \( \frac{d}{ds}g(s) := \lim_{h \to 0} \frac{g(s+h)-g(s)}{h} \) exists in \( L^2(\Gamma) \) for any \( s \in \mathcal{I} \). Below we give a sketch of the proof of the following "product rule".

**Proposition 6.1.** Let operator \( H \) be defined by

\[
\frac{d}{ds} \left[ -i(H + m)^{-1}e^{ihs}g(s) \right] = e^{ihs}g(s) - m(H + m)^{-1}e^{ihs}g(s) - i(H + m)^{-1}e^{ihs} \frac{d}{ds}g(s). \quad (6.1)
\]

**Proof.** Denote \( F(s) = -i(H + m)^{-1}e^{ihs}g(s) \), then \( \frac{d}{ds}F(s) = \lim_{h \to 0} \frac{F(s+h)-F(s)}{h}. \) We have

\[
\frac{F(s+h) - F(s)}{h} = \frac{1}{h} \left\{ -i(H + m)^{-1}e^{i(H+m)(s+h)}e^{-im(s+h)}g(s + h) + i(H + m)^{-1}e^{i(H+m)s}e^{-im}s\right\}
\]

\[
= -i(H + m)\frac{1}{h} \left( e^{i(H+m)(s+h)} - e^{i(H+m)s} \right)e^{-im(s+h)}g(s + h)
\]

\[
- i(H + m)^{-1}e^{i(H+m)s}\frac{1}{h} \left\{ (e^{-im(s+h)} - e^{-ims})g(s + h) + e^{-ims}(g(s + h) - g(s)) \right\}.
\]

To prove the assertion we need to analyze three last terms of (6.2), that is we are aimed to prove that

\[
- i(H + m)^{-1}e^{i(H+m)(s+h)} - e^{i(H+m)s}e^{-im(s+h)}g(s + h) \rightarrow e^{ihs}g(s),
\]

\[
- i(H + m)^{-1}e^{i(H+m)s}\frac{1}{h} \left( e^{-im(s+h)} - e^{-ims} \right)g(s + h) \rightarrow -m(H + m)^{-1}e^{ihs}g(s),
\]

\[
- i(H + m)^{-1}e^{i(H+m)s}\frac{1}{h} \left( e^{-ims}(g(s + h) - g(s)) \right) \rightarrow -i(H + m)^{-1}e^{ihs} \frac{d}{ds}g(s)
\]

as \( h \to 0 \).

- By the Spectral Theorem for the self-adjoint operator \( H \) we have:

\[
\| - i(H + m)^{-1}e^{i(H+m)(s+h)} - e^{i(H+m)s}e^{-im(s+h)}g(s + h) - e^{ihs}g(s) \|^2_2
\]

\[
\leq 2\| - i(H + m)^{-1}e^{i(H+m)(s+h)} - e^{i(H+m)s}e^{-im(s+h)}g(s + h) - e^{ihs}g(s + h) \|^2_2
\]

\[
+ 2\|g(s + h) - g(s)\|^2_2
\]

\[
\leq 2 \int_{\mathbb{R}} \left| - i(z + m)^{-1}e^{i(z+m)(s+h)} - e^{i(z+m)s}e^{-im(s+h)} - e^{izs} \right|^2 d(E_H(z)g(s + h), g(s + h))
\]

\[
+ 2\|g(s + h) - g(s)\|^2_2,
\]

\[
(6.3)
\]
where $E_H(z)$ is the spectral measure associated with $H$. Denote by $f_h(z)$ the function under the integral in the above inequality. Making trivial manipulations one may show that $f_h(z) = e^{izs}e^{i\pi e^{im(h-h)}} - 1$, where $h$ lies between 0 and $h$. It is obvious that $f_h(z)$ is bounded and converges to zero pointwise as $h \to 0$. Observing that

$$(E_H(M)g(s+h), g(s+h)) \to_{h \to 0} (E_H(M)g(s), g(s))$$

for any Borel set $M$, and using the Dominated Convergence Theorem, we conclude that expression (6.3) tends to zero as $h \to 0$. Finally, $-i(H+m)^{-1}\frac{1}{h}(e^{i(H+m)(s+h)} - e^{i(H+m)s})e^{-im(s+h)}g(s+h)$ tends to $e^{iHs}g(s)$.

- Using boundedness of the resolvent $(H + m)^{-1}$ we get

$$|| - i(H + m)^{-1}e^{i(H+m)s}\frac{1}{h}(e^{-im(s+h)} - e^{-ims})g(s+h) + m(H + m)^{-1}e^{i(H+m)s}e^{-ims}g(s)||_2 \leq ||(H + m)^{-1}e^{i(H+m)s}\left\{ -\frac{1}{h}(e^{-im(s+h)} - e^{-ims})g(s+h) + me^{-ims}g(s) \right\} ||_2$$

$$\leq C\left| -\frac{1}{h}(e^{-im(s+h)} - e^{-ims}) + me^{-ims} \right||g(s+h)||_2 + C|me^{-ims}| ||g(s+h) - g(s)||_2$$

The expression above obviously tends to zero and therefore $-i(H+m)^{-1}e^{i(H+m)s}\frac{1}{h}(e^{-im(s+h)} - e^{-ims})g(s+h)$ tends to $-m(H + m)^{-1}e^{iHs}g(s)$.

- Finally, estimating the last term in (6.2)

$$|| - i(H + m)^{-1}e^{i(H+m)s}e^{-ims}\left( \frac{1}{h}(g(s+h) - g(s)) - \frac{d}{ds}g(s) \right)||_2$$

$$\leq C||\frac{1}{h}(g(s+h) - g(s)) - \frac{d}{ds}g(s)||_2,$$

we get that $-i(H + m)^{-1}e^{i(H+m)s}e^{-ims}\frac{1}{h}(g(s+h) - g(s))$ tends to $-i(H + m)^{-1}e^{iHs}\frac{d}{ds}g(s)$. Summarizing the estimates of the three last terms in (6.2), we finally obtain formula (6.1).

\[\square\]

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