Relation between strength of interaction and accuracy of measurement for a quantum measurement

Takayuki Miyadera
Research Center for Information Security (RCIS),
National Institute of Advanced Industrial Science and Technology (AIST).
Daibiru building 1003, Sotokanda,
Chiyoda-ku, Tokyo, 101-0021, Japan.
(E-mail: miyadera-takayuki@aist.go.jp)

The process of measuring a two-level quantum system was examined by applying Hamiltonian formalism. For the measurement of an observable that does not commute with the system Hamiltonian, a non-trivial relationship among the strength of interaction, the time interval of the process, and the accuracy of the measurement was obtained. Particularly, to achieve an error-free measurement of such an observable, a condition stating that the interaction Hamiltonian does not commute with the system Hamiltonian needs to be satisfied.

PACS numbers: 03.65.Ta

I. INTRODUCTION

Von Neumann formulated the measurement process as the dynamics of a compound system that comprises a system and an apparatus [1]. This theory is now widely accepted and has been extensively investigated by researchers. In the formalism, a measurement process is determined by identifying the following variables: the Hilbert space of an apparatus, the initial state of the apparatus, and the unitary operator acting on a composite system. A good measurement process is obtained by cleverly choosing these variables. Because the process must be realized by physical devices at least in principle, its dynamics is often identified by a Hamiltonian operator. Several previous models have identified the dynamics in this manner. They include measurement processes of various physical quantities and approximated joint measurements of noncommutative observables [2]. In contrast, another direction of research investigates the limitations of this process. In this case, because the process does not have to be realized by realistic physical devices, its dynamics are often identified by giving possible unitary operators of the composite system. A typical result in this direction takes the form of an impossibility theorem: some operations are not achievable even if one may use unitary operators. For instance, the uncertainty principle states the impossibility of jointly measuring noncommutative observables [3–9], and the Wigner-Araki-Yanase theorem states the impossibility of performing precise measurements in the presence of an additive conserved quantity [10–15].

In the present study, we examine a problem that takes an intermediate position between the above two research directions. We use a Hamiltonian to identify a measurement process and discuss its limitations. Let us specifically describe the problem. Assume that there exists a system whose dynamics is governed by a system Hamiltonian $H_S$. To measure an observable $Q$ of this system, one needs to prepare an apparatus and introduce an interaction between the system and the apparatus. This interaction is described by an interaction Hamiltonian $V$. We want to determine the strength of interaction $V$ and the time interval $\tau$ required for measuring $Q$. We investigate the measurement of the simplest system, a two-level quantum system, and obtain a non-trivial relationship among the strength of interaction $V$, the time interval $\tau$, and the accuracy of the measurement of $Q$ that does not commute with $H_S$. The conclusion contains a simple interpretation of this result by using the uncertainty principle.

II. FORMULATION AND RESULTS

A. Formulation

In the following section, we study the dynamics of quantum measurement by applying Hamiltonian formalism (See [2] for a general treatment of measurement). The dynamics of quantum measurement is described by an interaction process between a system and an apparatus. Suppose that the system is described by a Hilbert space $\mathcal{H}_S$, and the apparatus is described by a Hilbert space $\mathcal{H}_A$. The observable to be measured is denoted by a self-adjoint operator $Q$ on $\mathcal{H}_S$. $Q$ is diagonalized as $Q = \sum_{q \in \mathcal{S}_Q} q P_q$, where $\{P_q\}_{q \in \mathcal{S}_Q}$ forms a projection-valued measure (PVM). That is, each $P_q$ is a projection operator and $\sum_{q \in \mathcal{S}_Q} P_q = 1_S$ holds. Measurement is a physical process that transfers the value of $Q$ at time $t = 0$ to an observable of the apparatus at a certain time $t = \tau$. The entire state of the composite
system evolves from time \( t = 0 \) to \( t = \tau \) following the Schrödinger (or von Neumann) equation. The total Hamiltonian is written as

\[
H = H_S \otimes 1_A + 1_S \otimes H_A + V,
\]

where \( H_S \) (resp. \( H_A \)) represents the Hamiltonian acting only on the system (resp. the apparatus), and \( V \) represents the interaction Hamiltonian. At time \( t = 0 \), the initial state has a product form such as:

\[
\rho(0) = \rho_S \otimes |\Omega\rangle \langle \Omega|,
\]

where the unit vector \( |\Omega\rangle \in \mathcal{H}_A \) does not depend on \( \rho_S \). At time \( t = \tau \), the state of the composite system becomes \( \rho(\tau) = U(\tau) \rho(0) U(\tau)^* \), where \( U(\tau) := \exp(-i\frac{tH}{\hbar}) \). Without the interaction term \( V \), the state keeps its product form; therefore, no information transfer from the system to the apparatus occurs. In this study, we investigate how large \( V \) and \( \tau \) should be in order to describe a measurement process.

### B. Measurement of a two-level system

Throughout this paper, we assume that the system is a two-level quantum system and that the observable \( Q \) has only two outcomes: 1 and 0. That is, \( Q \) is a projection operator on \( \mathcal{H}_S \). We write the eigenstates as \( |q_1\rangle \) and \( |q_0\rangle \), where \( Q |q_1\rangle = |q_1\rangle \) and \( Q |q_0\rangle = 0 \) hold. We consider two initial states of the system, \( \rho_S^0 = |q_0\rangle \langle q_0| \) and \( \rho_S^1 = |q_1\rangle \langle q_1| \). From the viewpoint of information transfer, the quality of measurement is characterized by its ability to distinguish between the states of the apparatus at \( t = \tau \). The time evolution of the two initial states results in the following two final states of the apparatus:

\[
\begin{align*}
\rho_A^0(\tau) &= \text{tr}_{\mathcal{H}_S}(U(\tau)(\rho_S^0 \otimes |\Omega\rangle \langle \Omega|)U(\tau)^*) \\
\rho_A^1(\tau) &= \text{tr}_{\mathcal{H}_S}(U(\tau)(\rho_S^1 \otimes |\Omega\rangle \langle \Omega|)U(\tau)^*),
\end{align*}
\]

where \( \text{tr}_{\mathcal{H}_S} \) represents a partial trace with respect to \( \mathcal{H}_S \). The process can only have an error-free measurement when these two final states of the apparatus are perfectly distinguishable. Note that we do not impose any condition on the states of the system after the measurement, while a repeatability condition is often imposed in literatures that discuss measurement. A measurement that satisfies the repeatability condition is a special kind of measurement called an ideal measurement. We employ a quantity called fidelity as a measure of the distinguishability between states of the apparatus after the interaction. The fidelity between states \( \rho \) and \( \sigma \) on a Hilbert space \( \mathcal{H} \) is defined by

\[
F(\rho, \sigma) := \text{tr}(\sqrt{\sigma^{1/2} \rho \sigma^{1/2}}),
\]

which is symmetric with respect to \( \rho \) and \( \sigma \) and satisfies the condition \( 0 \leq F(\rho, \sigma) \leq 1 \). The following theorem is useful for understanding its operational meaning.

**Lemma 1** [71] Suppose that \( \rho \) and \( \sigma \) are states on a Hilbert space \( \mathcal{H} \). The fidelity between these states can be represented as:

\[
F(\rho, \sigma) = \min_{E \in \{E_i\}_{i=1}^M} \sum_i \text{tr}(\rho E_i)^{1/2} \text{tr}(\sigma E_i)^{1/2},
\]

where the minimum is taken with respect to all PVMs on \( \mathcal{H} \).

The right-hand side of the above lemma represents the degree of overlap between the probability distributions \( \{\text{tr}(\rho E_i)\}_i \) and \( \{\text{tr}(\sigma E_i)\}_i \). It is 0 if there is no overlap and 1 if the probability distributions coincide with each other. Thus, if a process describes an error-free measurement, \( F(\rho_A^0(\tau), \rho_A^1(\tau)) = 0 \) must be satisfied.

While we introduced the fidelity to characterize the distinguishability of states, in the problem of measurement one usually discusses the error probability for a fixed observable \( Z \) of the apparatus, which is called a pointer observable or a meter observable. \( Z \) has two outcomes 0 and 1, and it is a projection operator acting on \( \mathcal{H}_A \). We define the following quantities for \( j = 0, 1 \):

\[
\begin{align*}
p(1|j) &:= \text{tr}(\rho_A^j(\tau)Z) \\
p(0|j) &:= \text{tr}(\rho_A^j(\tau)(1_A - Z)).
\end{align*}
\]
That is, \( p(i|j) \) represents the conditional probability to obtain an outcome \( i \) with respect to the initial state \( |q_j\rangle \) of the system. In the error-free case, \( p(i|j) = \delta_{ij} \) holds. In general, none of the \( p(i|j) \)'s are vanishing. Let us consider an average error defined by

\[
P_{\text{error}} := \frac{1}{2}(p(1|0) + p(0|1)).
\]

This quantity is related to the fidelity by the following lemma.

**Lemma 2** \( P_{\text{error}} \) is related to the fidelity as,

\[
F(\rho_0^A(\tau), \rho_1^A(\tau)) \leq 2\sqrt{P_{\text{error}} - P_{\text{error}}^2}.
\]

**Proof:** Thanks to Lemma 1 we obtain

\[
F(\rho_0^A(\tau), \rho_1^A(\tau))^2 \leq \left( \sqrt{p(0|0)p(0|1) + p(1|0)p(1|1)} + 2\sqrt{p(0|0)p(0|1)p(1|0)p(1|1)} \right)^2 \\
= p(0|0)p(0|1) + p(1|0)p(1|1) + 2\sqrt{p(0|0)p(0|1)p(1|0)p(1|1)} \\
\leq 2((1 - p(1|0))p(0|1) + p(1|0)(1 - p(0|1))) \\
= 4\left(P_{\text{error}} - p(1|0)p(0|1)\right) \\
= 4\left(P_{\text{error}} - p(1|0)(2P_{\text{error}} - p(0|1))\right) \\
= 4\left(p(1|0)^2 - 2P_{\text{error}}p(1|0) + P_{\text{error}}\right) \leq 4(P_{\text{error}} - P_{\text{error}}^2).
\]

\[\square\]

The following is our main theorem.

**Theorem 1** Let us consider a measurement process on a two-level system, as introduced above. That is, \( \mathcal{H}_S \) denotes the two-dimensional Hilbert space of the system, \( \mathcal{H}_A \) denotes the Hilbert space of the apparatus, and \( H = H_S \otimes 1_A + 1_S \otimes H_A + V \) denotes the Hamiltonian describing the interaction process. We consider the measurement process for the observable \( Q \) that has a pair of eigenstates: \( Q|q_1\rangle = |q_1\rangle \) and \( Q|q_0\rangle = 0 \). For any initial state \( |\Omega\rangle \in \mathcal{H}_A \) of the apparatus and time interval \( \tau \) for the process, the following inequality holds:

\[
\|\|Q, H_S\|\| \leq \|H_S\|F(\rho_0^A(\tau), \rho_1^A(\tau)) + \frac{\tau}{\hbar}\|\|V, H_S \otimes 1_A\|\|,
\]

where \( \|\cdot\| \) is the operator norm defined by \( \|A\| := \sup_{\phi|\phi\rangle \neq 0, |\phi\rangle \in \mathcal{H}} \frac{\|A|\phi\rangle\|}{\|\phi\|} \) for an operator \( A \) on \( \mathcal{H} \), and \( F(\rho_0^A(\tau), \rho_1^A(\tau)) \) represents the fidelity between a pair of states on the apparatus after the interaction.

**Proof:** Note that time evolution preserves the total Hamiltonian. That is, \( H = U(\tau)^*HU(\tau) \) holds. We operate on it with \( \langle q_0, \Omega| |q_1, \Omega\rangle \) to obtain,

\[
\langle q_0|H_S|q_1\rangle + \langle q_0, \Omega|1_S \otimes H_A|q_1, \Omega\rangle + \langle q_0, \Omega|V|q_1, \Omega\rangle \\
= \langle q_0, \Omega|U(\tau)^*(H_S \otimes 1_A)U(\tau)|q_1, \Omega\rangle + \langle q_0, \Omega|U(\tau)^*(1_S \otimes H_A)U(\tau)|q_1, \Omega\rangle + \langle q_0, \Omega|U(\tau)^*VU(\tau)|q_1, \Omega\rangle,
\]

where we keep the second term of the left-hand side although \( \langle q_0, \Omega|1_S \otimes H_A|q_1, \Omega\rangle = 0 \) holds owing to the orthogonality of \( |q_0\rangle \) and \( |q_1\rangle \). Equation 1 can be further reduced as

\[
|\langle q_0|H_S|q_1\rangle| \leq |\langle q_0, \Omega|U(\tau)^*(H_S \otimes 1_A)U(\tau)|q_1, \Omega\rangle| + |\langle q_0, \Omega|U(\tau)^*(1_S \otimes H_A + V)U(\tau) - (1_S \otimes H_A + V)|q_1, \Omega\rangle|,
\]

where the triangular inequality was used. The first term of the right-hand side can be bounded as follows. Let us consider an arbitrary PVM \( E = \{E_i\} \) on \( \mathcal{H}_A \). Because \( \sum_i E_i = 1_A \) holds, it follows that

\[
|\langle q_0, \Omega|U(\tau)^*(H_S \otimes 1_A)U(\tau)|q_1, \Omega\rangle| = |\sum_i \langle q_0, \Omega|U(\tau)^*(1_S \otimes E_i)(H_S \otimes 1_A)U(\tau)|q_1, \Omega\rangle|.
\]
The commutativity between $1_S \otimes E_i$ and $H_S \otimes 1_A$ allows the further derivation
\[
|\sum_i \langle q_0, \Omega | U(\tau)^*(1_S \otimes E_i)(H_S \otimes 1_A)U(\tau)|q_1, \Omega \rangle| \\
= |\sum_i \langle q_0, \Omega | U(\tau)^*(1_S \otimes E_i)(H_S \otimes 1_A)(1_S \otimes E_i)U(\tau)|q_1, \Omega \rangle| \\
\leq \sum_i \|H_S\| \langle q_0, \Omega | U(\tau)^*(1_S \otimes E_i)U(\tau)|q_0, \Omega \rangle^{1/2} \langle q_1, \Omega | U(\tau)^*(1_S \otimes E_i)U(\tau)|q_1, \Omega \rangle^{1/2} \\
= \|H_S\| \sum_i \text{tr}(|\rho_0^A(\tau)E_i|^{1/2}) \text{tr}(|\rho_1^A(\tau)E_i|^{1/2}),
\]
where the Cauchy-Schwarz inequality was used. Because the choice of a PVM $\{E_i\}$ is arbitrary, applying Lemma\[\[\ we obtain
\[
|\langle q_0, \Omega | U(\tau)^*(H_S \otimes 1_A)U(\tau)|q_1, \Omega \rangle| \leq \|H_S\| F(\rho_0^A(\tau), \rho_1^A(\tau)).
\]
The second term of \[\[\ can be bounded as follows. Applying the conservation of the total Hamiltonian we obtain
\[
U(\tau)^*(1_S \otimes H_A + V)U(\tau) - (1_S \otimes H_A + V) = H_S \otimes 1_A - U(\tau)^*(H_S \otimes 1_A)U(\tau).
\]
Its right-hand side is bounded by using the Heisenberg equation. Because $U(t)^*(H_S \otimes 1_A)U(t)$ satisfies
\[
ith \frac{d}{dt}U(t)^*(H_S \otimes 1_A)U(t) = U(t)^*[H_S \otimes 1_A, H]U(t),
\]
we obtain
\[
U(\tau)^*(H_S \otimes 1_A)U(\tau) - H_S \otimes 1_A = \frac{1}{ith} \int_0^\tau dtU(t)^*[H_S \otimes 1_A, V]U(t),
\]
which derives
\[
|\langle q_0, \Omega | U(\tau)^*(1_S \otimes H_A + V)U(\tau) - (1_S \otimes H_A + V)|q_1, \Omega \rangle| = |\langle q_0, \Omega | H_S \otimes 1_A - U(\tau)^*(H_S \otimes 1_A)U(\tau)|q_1, \Omega \rangle| \\
\leq \|H_S \otimes 1_A - U(\tau)^*(H_S \otimes 1_A)U(\tau)\| \\
\leq \frac{1}{h} \int_0^\tau dt\|U(t)^*[H_S \otimes 1_A, V]U(t)\| \\
= \frac{\tau}{h} \|H_S \otimes 1_A, V\|.
\]
Finally, we analyze the left-hand side of Equation \[\[. We derive the equality $|\langle q_0 | H_S | q_1 \rangle| = \|i[Q, H_S]\|$ in the following. Because $i[Q, H_S]$ is a self-adjoint operator, $\|i[Q, H_S]\| = \max_{|\psi\rangle = \|\psi\| = 1} |\langle \psi | i[Q, H_S] | \psi \rangle|$ holds. For any unit vector $|\psi\rangle = c_0|q_0\rangle + c_1|q_1\rangle$, we have
\[
|\langle \psi | i[Q, H_S] | \psi \rangle| = 2|\text{Im}(\overline{c_0} c_1 \langle q_0 | H_S | q_1 \rangle)|.
\]
Its right-hand side can be bounded as
\[
2|\text{Im}(\overline{c_0} c_1 \langle q_0 | H_S | q_1 \rangle)| \leq 2|\overline{c_0}| c_1 |\langle q_0 | H_S | q_1 \rangle| \\
\leq |c_0|^2 + |c_1|^2 |\langle q_0 | H_S | q_1 \rangle| \\
= |\langle q_0 | H_S | q_1 \rangle|.
\]
In the above inequality, if we insert $c_0 = \frac{\langle q_0 | H_S | q_1 \rangle}{\sqrt{|\langle q_0 | H_S | q_1 \rangle|}}$ and $c_1 = \frac{1}{\sqrt{2}}$, we obtain an equality. Thus, we proved
\[
\|i[Q, H_S]\| = \|i[Q, H_S]\| = |\langle q_0 | H_S | q_1 \rangle|.
\]
It ends the proof.

These corollaries immediately follow:
Corollary 1: Based on the above theorem, the following holds for any pointer observable $Z$:

$$||[Q, H_S]|| \leq 2\|H_S\|\sqrt{P_{\text{error}}^2 - P_{\text{error}}^2} + \frac{\tau}{\hbar}||[V, H_S \otimes 1_A]||.$$ 

Corollary 2: In order to attain an error-free measurement, the interaction Hamiltonian $V$ and the time interval $\tau$ must satisfy

$$\tau \cdot ||[V, H_S \otimes 1_A]|| \geq \hbar ||[Q, H_S]||.$$ 

The above theorem and corollaries show that if a measured observable does not commute with the system Hamiltonian, then there exists a non-trivial trade-off relationship among the strength of interaction, the time interval, and the accuracy of the measurement. Particularly, according to Corollary 2, in order to achieve an error-free measurement process for such an observable, the interaction Hamiltonian must be noncommutative with the system Hamiltonian. Note that the inequalities do not contain the Hamiltonian of the apparatus. In the discussion, we give a brief interpretation of this result.

**C. When the observable commutes with the system Hamiltonian**

When the observable $Q$ commutes with $H_S$, the above theorem becomes trivial. The following example shows that in such a case $V$ can commute with $H_S$ and $\tau$ can be arbitrarily small even for an error-free measurement.

Let us consider the standard model of the measurement process [17]. Assume that the system is a two-level system described by the two-dimensional Hilbert space $\mathcal{H}_S$, and assume that the apparatus is a particle moving in one degree of freedom so that $H_A = L^2(\mathbb{R})$. The observable to be measured is denoted by $Q$, which is a projection operator with eigenstates $|q_1\rangle$ and $|q_0\rangle$. Assume that the total Hamiltonian is defined by

$$H = H_S + H_A + V = 0 + 0 + Q \otimes P_A,$$

where $P_A$ is the momentum operator of the apparatus. Because $H_S$ is trivial, both $[Q, H_S] = 0$ and $[V, H_S \otimes 1_A] = 0$ hold. The initial states $|q_0\rangle \otimes |\Omega\rangle$ and $|q_1\rangle \otimes |\Omega\rangle$ for any $|\Omega\rangle \in \mathcal{H}_A$ evolve as

$$|q_0\rangle \otimes |\Omega\rangle \rightarrow |q_0\rangle \otimes |\Omega_0\rangle,$$

$$|q_1\rangle \otimes |\Omega\rangle \rightarrow |q_1\rangle \otimes |\Omega_\lambda\rangle,$$

where $|\Omega_\lambda\rangle$ is defined by $\langle x|\Omega_\lambda\rangle := \langle x - \lambda|\Omega\rangle$ for any $\lambda \in \mathbb{R}$ in the position representation. Therefore, for any $\tau > 0$, if we prepare $|\Omega\rangle$ so that it is sharply localized in the position representation, $|\Omega_\xi\rangle$ and $|\Omega_0\rangle$ become disjoint in the position representation, and are perfectly distinguishable.

While the interaction Hamiltonian in the above example is unbounded in norm, the following example shows that the norm of the interaction Hamiltonian can be made arbitrarily small. We consider a two-level system interacting with a two-level apparatus, whose dynamics is governed by $H = V = \lambda(|q_1\rangle \langle q_1| \otimes |1\rangle \langle 1| + |q_0\rangle \langle q_0| \otimes |0\rangle \langle 0|)$ for $\lambda > 0$, where $|q_0\rangle$ and $|q_1\rangle$ are eigenstates of $Q$. If we put the initial state of the apparatus as $|\Omega\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, the initial states $|q_1\rangle \otimes |\Omega\rangle$ and $|q_0\rangle \otimes |\Omega\rangle$ evolve as

$$|q_1\rangle \otimes |\Omega\rangle \rightarrow |q_1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$|q_0\rangle \otimes |\Omega\rangle \rightarrow |q_0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

in time $\tau := \frac{\pi \hbar}{2\lambda}$. Note that $||V|| = \lambda$ can be arbitrarily small.

**III. DISCUSSION**

In this paper, we applied Hamiltonian formalism to the measurement process of a two-level system. For a measured observable that does not commute with the system Hamiltonian, a non-trivial trade-off relationship among the strength of interaction, the time interval, and the accuracy of the measurement has been obtained. In particular, in order to achieve an error-free measurement process for such an observable, the interaction Hamiltonian must be noncommutative with the system Hamiltonian. We show that this impossibility result can be derived by the uncertainty
principle for joint measurement. Let us consider a two-level quantum system $H_S$ interacting with an apparatus $H_A$. We denote the total Hamiltonian by $H = H_S \otimes 1_A + 1_S \otimes H_A + V$. The system Hamiltonian $H_S$ is diagonalized as $H_S = \epsilon_1|\epsilon_1\rangle\langle\epsilon_1| + \epsilon_0|\epsilon_0\rangle\langle\epsilon_0| \neq \epsilon_1$. The unitary evolution $U(\tau) = \exp(-i\frac{H\tau}{\hbar})$ and the initial state of the apparatus $|\Omega\rangle$ define an isometry $W: H_S \to H_S \otimes H_A$ by $W|\psi\rangle := U(\tau)|\psi\rangle \otimes |\Omega\rangle$. If the process describes an error-free measurement of $Q$, there exists a PVM $M = \{M_j, M_f\}$ on $H_A$ satisfying $|q_j\rangle\langle q_j| = W^\dagger (1_S \otimes M_j)W$ for $j = 0, 1$. In addition, if $V$ commutes with $H_S$, one obtains $|\epsilon_n\rangle\langle\epsilon_n| = W^\dagger (|\epsilon_n\rangle\langle\epsilon_n| \otimes 1_A)W$ for $n = 0, 1$. Thus we can introduce a positive-operator-valued measure $Y = \{Y_{n,j}\}$ by $Y_{n,j} := W^\dagger (|\epsilon_n\rangle\langle\epsilon_n| \otimes M_j)W$ which jointly measures $Q$ and $H_S$. That is, $Y_{n0} + Y_{n1} = |\epsilon_n\rangle\langle\epsilon_n|$ and $Y_{0j} + Y_{1j} = |q_j\rangle\langle q_j|$ hold for $n, j = 0, 1$. According to the uncertainty principle for joint measurement, these relations can be true only for commutative PVMs $\{|\epsilon_0\rangle\langle\epsilon_0|, |\epsilon_1\rangle\langle\epsilon_1|\}$ and $\{|q_0\rangle\langle q_0|, |q_1\rangle\langle q_1|\}$ (see for e.g. [2]).

As an example showing that an interaction Hamiltonian satisfying $[V, H_S] \neq 0$ helps reducing error, we consider a modified version of the standard model. Setting $|\Omega\rangle$ sharply located, we denote by $\tau > 0$ the time interval required to accomplish error-free measurement in the standard model discussed in Sec. II C. Let us consider a modified model described by $H = H_S \otimes 1 + Q \otimes P_A$, where $[H_S, Q] \neq 0$ holds. Because the time evolution $U(\tau) = \exp(-i\frac{H\tau}{\hbar})$ satisfies $\|U(\tau) - \exp(-i\frac{Q+P_A}{\hbar}\tau)\| \leq \frac{\tau}{\hbar}\|H_S\|$, we obtain an estimate $F(\rho^A_0(\tau), \rho^A_\tau(\tau)) \leq 2\sqrt{2\epsilon}\|H_S\|$. Note that it is possible to make $\tau$ arbitrarily small by making the initial state $|\Omega\rangle$ sufficiently sharp.

Similarly, it is possible to treat a modified version of the second example in Sec. II C. If we take $\lambda > 0$ sufficiently large for $H = H_S \otimes 1_A + V = H_S \otimes 1_A + \lambda(|q_1\rangle\langle q_1| \otimes |1\rangle \langle 1| + |q_0\rangle\langle q_0| \otimes |0\rangle \langle 0|)$, $\tau = \frac{\sqrt{\epsilon}}{2\lambda}$ becomes small and the fidelity between final states on the apparatus can be made arbitrarily small.

While the obtained inequality is non-trivial, it may not always be strong. In fact, although in the proof of theorem III we have used the inequality

$$\|\langle q_0, \Omega|H_S \otimes 1_A - U(\tau)^* (H_S \otimes 1_A)U(\tau)|q_1, \Omega\rangle\| \leq \frac{\|H_S \otimes 1_A - U(\tau)^* (H_S \otimes 1_A)U(\tau)\|}{\hbar}\|H_S \otimes 1_A, V\|,$$

which was the origin of the linear term with respect to $\tau$, this bound is not strong because it does not take into consideration the dynamics in detail. The left-hand side of the above inequality can be written as

$$\|\langle q_0, \Omega|H_S \otimes 1_A - U(\tau)^* (H_S \otimes 1_A)U(\tau)|q_1, \Omega\rangle\| \leq \frac{\hbar}{\lambda} \int_0^\tau dt |\langle q_0, \Omega|U(t)^* (H_S \otimes 1_A, V)U(t)|q_1, \Omega\rangle|.$$

The last term contains the correlation function $\langle q_0, \Omega|U(t)^* (H_S \otimes 1_A, V)U(t)|q_1, \Omega\rangle$ that, in general, decays rapidly for a large apparatus. Therefore, the term may not grow proportionally to $\tau$ in physically realistic models. We hope to investigate this problem in the future.

Acknowledgments: I would like to thank an anonymous referee for helpful comments.

[1] J. von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton University Press, Princeton, 1955.
[2] P. Busch, M. Grabowski, and P. Lahti, Operational Quantum Physics, Springer-Verlag, Berlin, 1995.
[3] P. Busch, T. Heinonen, and P. Lahti, Physics Reports 311, 350 (2004).
[4] P. Busch and D. B. Pearson, J. Math. Phys. 48, 082103 (2007).
[5] R. F. Werner, Quantum Inform. Comput. 4, 546 (2004).
[6] D. M. Appleby, Int. J. Theor. Phys. 37, 1491 (1998).
[7] M. Ozawa, Ann. Phys. 311, 350 (2004).
[8] B. Janssens, e-print arXiv:quant-ph/0606093.
[9] T. Miyadera and H. Imai, Phys. Rev. A 78, 052119 (2008).
[10] E. P. Wigner, Zeitschrift für Physik, 133, 101 (1952).
[11] H. Araki and M. M. Yanase, Phys. Rev. 120, 622 (1960).
[12] M. Ozawa, Phys. Rev. Lett. 86, 054002 (2002).
[13] E. G. Beltrametti, G. Cassinelli, and P. Lahti, J. Math. Phys. 31, 91 (1990).
[14] T. Miyadera and H. Imai, Phys. Rev. A 74, 024101 (2006).
[15] L. Loveridge and P. Busch, arXiv:1012.4362 to appear in Eur. Phys. J. D (2011).
[16] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. 76, 2818 (1996).
[17] P. Busch and P. Lahti, Found. Phys. 26, 875 (1996).