THE $C_p$-STABLE CLOSURE OF THE CLASS OF SEPARABLE METRIZABLE SPACES

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Abstract. Denote by $C_p[\mathcal{M}_0]$ the $C_p$-stable closure of the class $\mathcal{M}_0$ of all separable metrizable spaces, i.e., $C_p[\mathcal{M}_0]$ is the smallest class of topological spaces that contains $\mathcal{M}_0$ and is closed under taking subspaces, homeomorphic images, countable topological sums, countable Tychonoff products, and function spaces $C_p(X,Y)$. Using a recent deep result of Chernikov and Shelah (2014), we prove that $C_p[\mathcal{M}_0]$ coincides with the class of all Tychonoff spaces of cardinality strictly less than $\mathbf{\Sigma}_{\omega_1}$. Being motivated by the theory of Generalized Metric Spaces, we characterize also other natural $C_p$-type stable closures of the class $\mathcal{M}_0$.

1. Introduction

All topological spaces considered in this paper are assumed to be Tychonoff. Many natural and important classes $\mathcal{X}$ of topological spaces usually are preserved by basic topological operations, which motivated us in [3] to introduce the following stability property of $\mathcal{X}$. Following [3], we say that a class $\mathcal{X}$ of topological spaces is stable if $\mathcal{X}$ is closed under taking subspaces, homeomorphic images, countable topological sums, and countable Tychonoff products. For two classes $\mathcal{X}$ and $\mathcal{X}'$ of topological spaces, we shall say that $\mathcal{X}'$ is an extension of $\mathcal{X}$ if $\mathcal{X} \subset \mathcal{X}'$. Among all stable extensions of $\mathcal{X}$ there is the smallest one denoted by $[\mathcal{X}]$ and called the stable closure of the class $\mathcal{X}$.

It is clear that the class $\mathcal{M}_0$ of all separable metrizable spaces is stable. One of the most natural generalizations of the class $\mathcal{M}_0$ is the class of all cosmic spaces. Recall (see [9]) that a topological space $X$ is called cosmic (a $\sigma$-space) if it is regular and possesses a countable (respectively, a $\sigma$-locally finite) network. A family $\mathcal{N}$ of subsets of $X$ is a network for $X$ if for any $x \in X$ and any open neighborhood $U$ of $x$ there is a set $N \in \mathcal{N}$ such that $x \in N \subset U$. The class $\mathcal{C}$ of all cosmic spaces is stable [11] and so are many other classes of generalized metric spaces, in particular, the classes of $\sigma$-spaces, $\mathcal{N}_0$-spaces, $\mathcal{N}$-spaces, $\mathcal{P}_0$-spaces and $\mathcal{P}$-spaces (see, [11] [2] [2] [4]). Clearly, the class $\mathcal{Z}$ of all topological spaces is stable, but the class $\mathcal{S}$ of all Lindelöf spaces is not stable. Note that any cosmic space is Lindelöf.

For two topological spaces $X$ and $Y$, we denote by $C_k(X,Y)$ and $C_p(X,Y)$ the space $C(X,Y)$ of all continuous functions from $X$ into $Y$ endowed with the compact-open topology and the topology of pointwise convergence, respectively. Having in mind the stability of the classes of $\mathcal{N}_0$-spaces and $\mathcal{P}_0$-spaces under taking function spaces $C_k(X,Y)$ (see [11] [2]), we defined in [3] a class $\mathcal{Z}$ of topological spaces to be $C_k$-stable if $\mathcal{Z}$ is stable and for any Lindelöf space $X \in \mathcal{Z}$ and any space $Y \in \mathcal{Z}$ the function space $C_k(X,Y)$ belongs to the class $\mathcal{Z}$. In [3] we proved that the smallest $C_k$-stable extension of the class $\mathcal{M}_0$ coincides with the class $C_k(\mathcal{M}_0,\mathcal{M}_0)$ of all topological spaces which can be embedded into the function spaces $C_k(X,Y)$ for suitable separable metric spaces $X$ and $Y$. These results motivate the following definition.

Definition 1.1. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be classes of topological spaces. The class $\mathcal{Z}$ is called $C_p^X,Y$-stable if $\mathcal{Z}$ is stable and for any spaces $X \in \mathcal{Z} \cap \mathcal{X}$ and $Y \in \mathcal{Z} \cap \mathcal{Y}$ the function space $C_p(X,Y)$ belongs to the class $\mathcal{Z}$. The smallest $C_p^X,Y$-stable extension of the class $\mathcal{Z}$ is denoted by $C_p^X,Y[\mathcal{Z}]$ and called the $C_p^X,Y$-stable closure of the class $\mathcal{Z}$.

In this paper we shall describe the $C_p^X,Y$-stable closures of the class $\mathcal{M}_0$ for the cases $\mathcal{X}$ and $\mathcal{Y}$ equal to $\mathcal{M}_0$ or $\mathcal{S}$, the class of all topological spaces.

In case $\mathcal{X} = \mathcal{Y} = \mathcal{S}$ the class $C_p^{\mathcal{S},\mathcal{S}}[\mathcal{Z}]$ will be denoted by $C_p[\mathcal{Z}]$ and called the $C_p$-stable closure of $\mathcal{Z}$.

Our first theorem (which will be proved in Section 2) essentially follows from Michael’s results [11].

Theorem 1.2. The classes $C_p^{\mathcal{M}_0,\mathcal{M}_0}[\mathcal{M}_0]$ and $C_p^{\mathcal{M}_0,\mathcal{N}_0}[\mathcal{M}_0]$ coincide with the class of all cosmic spaces.

The class $C_p^{\mathcal{N}_0,\mathcal{N}_0}[\mathcal{M}_0]$ is strictly larger than $C_p^{\mathcal{N}_0,\mathcal{N}_0}[\mathcal{M}_0] = C_p^{\mathcal{S},\mathcal{S}}[\mathcal{M}_0]$ and admits the following description. For a topological space $X$, let $X^{<\omega} = \bigoplus_{n \in \omega} X^n$ be the topological sum of all its finite powers. By $SC_p(X^{<\omega})$...
Theorem 1.3. The class $C_p^{\mathfrak{m}_0,\mathbb{T}}[\mathfrak{m}_0]$ coincides with the class $SC_p(\mathfrak{m}_0^{<\omega})$.

Theorem 1.5 motivates studying the class $SC_p(\mathfrak{m}_0^{<\omega})$ in more details. It turns out that this class contains all cosmic spaces, all metrizable spaces of weight $\leq \kappa$ and also all generalized ordered spaces whose order topology is second countable. Let us recall the necessary definitions related to generalized ordered spaces.

By a linearly ordered space we understand a linearly ordered set $(X, \leq)$ endowed with the order topology generated by the subbase consisting of open half-intervals $(-\infty, a] = \{x \in X : x < a\}$ and $[a, \infty) = \{x \in X : x > a\}$ where $a \in X$. By [6, 3.12.4(d)], the weight $w(X)$ of a linearly ordered space $X$ coincides with its network weight $nw(X)$. A subset $C \subseteq X$ is order convex if for any points $x \leq y$ in $C$ the order interval $[x, y] = \{z \in X : x \leq z \leq y\}$ is contained in $C$.

A generalized ordered space (briefly, a GO-space) is a topological space $X$ endowed with a linear order $\leq$ such that open order convex subsets of $X$ form a base of the topology of $X$. Standard examples of GO-spaces are the Sorgenfrey line and the Michael line (see [6, 1.2.2 and 5.1.32]). Also the real line $\mathbb{R}_d$ endowed with the discrete topology is a GO-space.

By the order topology on a GO-space $X$ we understand the topology on $X$ generated by the linear order. The linear weight $lw(X)$ of a GO-space $X$ is the weight of its order topology. In particular, $lw(\mathbb{R}_d) = \omega$. It follows that $w(X) \geq lw(X) \geq nw(X)$ for each GO-space $X$. Note that $w(\mathbb{R}_d) = \kappa > lw(\mathbb{R}_d) = \aleph_0$.

Theorem 1.4. The class $SC_p(\mathfrak{m}_0^{<\omega})$ contains all cosmic spaces, all metrizable spaces of weight $\leq \kappa$, and all generalized ordered spaces of countable linear weight.

An upper bound on the class $SC_p(\mathfrak{m}_0^{<\omega})$ is given by the class of spaces with countable $i$-weight. We recall that the $i$-weight $iw(X)$ of a Tychonoff space $X$ is the smallest cardinal $\kappa$ for which there is an injective continuous map $i : X \to Y$ into a Tychonoff space $Y$ of weight $w(Y) \leq \kappa$. It is clear that $iw(X) \leq w(X)$, and if $iw(X) \leq \aleph_0$ then $|X| \leq \kappa$. The following theorem, proved in Section 5, gives an upper bound on the class $SC_p(\mathfrak{m}_0^{<\omega})$.

Theorem 1.5. Each space $X \in C_p^{\mathfrak{m}_0,\mathbb{T}}[\mathfrak{m}_0] = SC_p(\mathfrak{m}_0^{<\omega})$ has weight $w(X) \leq \kappa$, cardinality $|X| \leq \kappa$ and $i$-weight $iw(X) \leq \aleph_0$.

Example 1.6. By [8], the Banach space $X = \ell_1(\mathbb{N})$ endowed with the weak topology is an $\aleph_1$-space of cardinality $|X| = \kappa$, weight $w(X) = 2^\kappa$ and $i$-weight $iw(X) = \aleph_0$. By Theorem 1.5 the space $X$ does not belong to the class $C_p^{\mathfrak{m}_0,\mathbb{T}}[\mathfrak{m}_0] = SC_p(\mathfrak{m}_0^{<\omega})$. Let us recall [9, §11] that a regular topological space $X$ is an $\aleph_1$-space if $X$ has a $\sigma$-locally finite $k$-network. A family $\mathcal{N}$ of subsets of $X$ is called a $k$-network in $X$ if for any open set $U \subseteq X$ and compact subset $K \subseteq U$ there is a finite subfamily $\mathcal{F} \subseteq \mathcal{N}$ such that $K \subseteq \bigcup \mathcal{F} \subseteq U$.

Since each compact space of countable $i$-weight is metrizable, Theorem 1.5 implies:

Corollary 1.7. Each compact space in the class $C_p^{\mathfrak{m}_0,\mathbb{T}}[\mathfrak{m}_0] = SC_p(\mathfrak{m}_0^{<\omega})$ is metrizable.

This corollary shows that the class $C_p^{\mathfrak{m}_0,\mathbb{T}}[\mathfrak{m}_0]$ can be considered as a new class of generalized metric spaces.

Now we show that $C_p[\mathfrak{m}_0] := C_p^{\mathbb{T}}[\mathfrak{m}_0]$ of the class $\mathfrak{m}_0$. It turns out that the class $C_p[\mathfrak{m}_0]$ is huge and coincides with the class of all Tychonoff spaces of cardinality (or weight) strictly smaller than $\beth_1$. Here $\beth_0 = \aleph_0$ and $\beth_\alpha = \sup\{2^{\beth_\beta} : \beta < \alpha\}$ for any ordinal $\alpha > 0$.

Theorem 1.8. The $C_p$-stable closure $C_p[\mathfrak{m}_0]$ of the class $\mathfrak{m}_0$ of separable metrizable spaces coincides with the class of all Tychonoff spaces of cardinality strictly less than $\beth_1$.

We prove this theorem in Section 6 using a recent deep result of Chernikov and Shelah [4].

2. Proof of Theorem 1.2

We shall deduce Theorem 1.2 from the following characterization of cosmic spaces due to Michael [11].

Fact 2.1 ([11]). For a Tychonoff space $X$ the following conditions are equivalent:

1. $X$ is cosmic;
2. $X$ is a continuous image of a separable metrizable space;
3. the function space $C_p(X) := C_p(X; \mathbb{R})$ is cosmic;
4. for any metrizable separable space $Y$ the function space $C_p(X, Y)$ is cosmic.
For a Tychonoff space $X$, we denote by $\delta : X \to C_p(C_p(X))$ the canonical map assigning to each point $x \in X$ the Dirac measure $\delta_x$ concentrated at $x$. The Dirac measure $\delta_x : C_p(X) \to \mathbb{R}$ assigns to each function $f \in C_p(X)$ its value $f(x)$ at the point $x$. The following important fact is well-known and can be found in [1, 0.5].

**Fact 2.2.** For any Tychonoff space $X$, the canonical map $\delta : X \to C_p(C_p(X))$ is a topological embedding.

Now we present a proof of Theorem 1.2

**Proof of Theorem 1.2** Let $X$ be a cosmic space. By Fact 2.1, the space $C_p(X)$ is the image of a separable metrizable space $M$ under a continuous map $\xi : M \to C_p(X)$. It follows that the dual map $\xi^* : C_p(C_p(X)) \to C_p(M)$, $\xi^*(f) = f \circ \xi$, is a topological embedding of $C_p(C_p(X))$ into $C_p(M) \in C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0]$. Now Fact 2.2 implies that $X \in C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0] \subset C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0]$. Therefore, the class $C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0]$ contains all cosmic spaces.

To complete the proof of the theorem it is enough to show that the class $C$ of all cosmic spaces is $C_p^{\mathfrak{m}_0,\mathfrak{m}_0}$-stable. But this immediately follows from the stability of the class $C$ and Fact 2.1. $\square$

3. **Proof of Theorem 1.3**

For topological spaces $X_1, \ldots, X_n, Y, Z$, we denote by $SC_p(X_1 \times \cdots \times X_n, Y)$ the space of all separately continuous functions $f : X_1 \times \cdots \times X_n \to Y$ endowed with the topology of pointwise convergence. It is well-known that the function space $SC_p(X_1 \times \cdots \times X_n, Y)$ is canonically homeomorphic to $C_p(X_1, SC_p(X_2 \times \cdots \times X_n, Y))$. In particular, $SC_p(X \times Y, Z)$ is canonically homeomorphic to $C_p(X, C_p(Y, Z))$. In the sequel the function space $SC_p(X_1 \times \cdots \times X_n, \mathbb{R})$ will be denoted by $SC_p(X_1 \times \cdots \times X_n)$. For $n = 1$ the space $SC_p(X_1)$ coincides with $C_p(X_1)$.

We shall prove by induction that for every $n \in \mathbb{N}$ and every space $X \in \mathfrak{m}_0$ the function space $SC_p(X^n)$ belongs to the class $C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0]$. For $n = 1$ the space $SC_p(X) = C_p(X)$ belongs to $C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0]$ by definition. Assume that for some number $n \in \mathbb{N}$ we have proved that the space $SC_p(X^n)$ belongs to $C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0]$. Taking into account that $SC_p(X^{n+1})$ is (canonically) homeomorphic to $C_p(X, SC_p(X^n))$, we conclude that $SC_p(X^{n+1}) \in C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0]$. Taking into account that the function space $SC_p(X^{<\omega})$ is homeomorphic to $\prod_{n \in \mathbb{N}} SC_p(X^n)$, we conclude that $SC_p(X^{<\omega}) \in C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0]$ and hence $SC_p(\mathfrak{m}_0^{<\omega}) \in C_p^{\mathfrak{m}_0,\mathfrak{m}_0}[\mathfrak{m}_0]$.

To prove the reverse inclusion it is enough to check that the class $SC_p(\mathfrak{m}_0^{<\omega})$ is closed under taking countable topological sums, countable Tychonoff products and taking function spaces with metrizable separable domain.

To see that the class $SC_p(\mathfrak{m}_0^{<\omega})$ is closed under taking countable topological sums, it is enough to prove that for any non-empty spaces $X_n \in \mathfrak{m}_0$, $n \in \omega$, the topological sum $\bigoplus_{n \in \omega} SC_p(X_n^{<\omega})$ embeds into the function space $SC_p(X^{<\omega})$ for some space $X \in \mathfrak{m}_0$. Consider the topological sum $X = \bigoplus_{n \in \omega} X_n$ and the topological embedding $e : \bigoplus_{n \in \omega} SC_p(X_n^{<\omega}) \to SC_p(X^{<\omega})$ assigning to each function $f \in SC_p(X_n^{<\omega})$, $n \in \omega$, the function $\hat{f} \in SC_p(X^{<\omega})$ defined by

$$\hat{f}(x) = \begin{cases} f(x), & \text{if } x \in X_n^{<\omega}; \\ 0, & \text{if } x \in X^{<\omega} \setminus X_n^{<\omega}. \end{cases}$$

As $X \in \mathfrak{m}_0$, we conclude that $SC_p(X^{<\omega})$ and hence $\bigoplus_{n \in \omega} SC_p(X_n^{<\omega})$ belong to the class $SC_p(\mathfrak{m}_0^{<\omega})$.

Next we prove that for any metrizable space $X$ and any space $Y \in SC_p(\mathfrak{m}_0^{<\omega})$ we get $C_p(X, Y) \in SC_p(\mathfrak{m}_0^{<\omega})$. The space $Y \in SC_p(\mathfrak{m}_0^{<\omega})$ can be identified with a subspace of the function space $SC_p(Z^{<\omega})$ for some separable metrizable space $Z$. Then the space $C_p(X, Y)$ can be identified with the subspace of $C_p(X, SC_p(Z^{<\omega}))$, which is canonically homeomorphic to

$$C_p\left(X, \prod_{n \in \omega} SC_p(Z^n)\right) = \prod_{n \in \omega} C_p(X, SC_p(Z^n)) = \prod_{n \in \omega} SC_p(X \times Z^n).$$

As $X \times Z^n$ is canonically homeomorphic to a retract $X \times \{x_0\}^{n-1} \times Z^n$ of $X \times Z^n = (X 
 \times Z)^n$, we obtain

$$C_p(X, SC_p(Z^{<\omega})) = \prod_{n \in \omega} SC_p(X \times Z^n) \to \prod_{n \in \omega} SC_p((X \times Z)^n) = SC_p((X \times Z)^{<\omega}).$$

Since $X \times Z \in \mathfrak{m}_0$, we finally get $C_p(X, Y) \in SC_p((X \times Z)^{<\omega}) \in SC_p(\mathfrak{m}_0^{<\omega})$.

Finally we show that the class $SC_p(\mathfrak{m}_0^{<\omega})$ is closed under taking countable Tychonoff products. Fix any spaces $X_n \in SC_p(\mathfrak{m}_0^{<\omega})$, $n \in \omega$. As the class $SC_p(\mathfrak{m}_0^{<\omega})$ is closed under taking countable topological sums,
the topological sum \( X = \bigoplus_{n \in \omega} X_n \) belongs to the class \( \text{SC}_p(\mathcal{M}_0^{\leq \omega}) \). Since the class \( \text{SC}_p(\mathcal{M}_0^{\leq \omega}) \) is closed also under taking function spaces with separable metrizable domain, the function space \( C_p(\omega, X) \) belongs to the class \( \text{SC}_p(\mathcal{M}_0^{\leq \omega}) \). Taking into account that \( \prod_{n \in \omega} X_n \subset X^{\omega} = C_p(\omega, X) \in \text{SC}_p(\mathcal{M}_0^{\leq \omega}) \), we conclude that \( \prod_{n \in \omega} X_n \subset \text{SC}_p(\mathcal{M}_0^{\leq \omega}) \).

4. Proof of Theorem 1.4

We divide the proof of Theorem 1.4 into three lemmas. The first of them follows from Theorem 1.2 and the obvious inclusion \( C_p^{\omega, \omega}(\mathcal{M}_0) \subset C_p^{\omega, \omega}(\mathcal{M}_0) \).

Lemma 4.1. The class \( C_p^{\omega, \omega}(\mathcal{M}_0) \) contains all cosmic spaces.

The next lemma will play an important role also in the proof of Theorem 1.8. In this lemma an ordered field is endowed with the topology generated by the linear order.

Lemma 4.2. For every ordered field \( F \), the space \( SC_p(F \times F, F) \cong C_p(F, C_p(F, F)) \) contains a discrete subspace \( D \) of cardinality \(|F|\).

Proof. Consider the discontinuous separately continuous function \( sp^F : F \times F \to F \) defined by

\[
sp^F(x, y) = \begin{cases} 
0, & \text{if } (x, y) = (0, 0), \\
\frac{2xy}{x^2+y^2}, & \text{otherwise}.
\end{cases}
\]

For every \( a, b \in F \), consider the shifted function \( sp^{F, b}_a : F \times F \to F \), \( sp^{F, b}_a(x, y) := sp^F(x - a, y - b) \), and observe that the subspace \( D = \{ sp^{F, a}_0 : a \in F \} \) is discrete in the space \( SC_p(F \times F, F) \cong C_p(F, C_p(F, F)) \). \( \square \)

Lemma 4.3. Each metrizable space \( X \) of weight \( \leq \kappa \) belongs to the class \( C_p^{\omega, \omega}(\mathcal{M}_0) \).

Proof. Let \( sp := sp^\mathbb{R} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be the classical discontinuous separately continuous function. By Lemma 4.2, the subspace \( D = \{ sp_{a, a} : a \in \mathbb{R} \} \) is discrete in the space \( SC_p(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \). Moreover, the subspace \( H = \{ t \cdot sp_{a, a} : a \in \mathbb{R}, t \in [0, 1] \} \) of \( SC_p(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \) is homeomorphic to a hedgehog with \( \kappa \) spines. It follows from \( SC_p(\mathbb{R} \times \mathbb{R}, \mathbb{R}) \cong C_p(\mathbb{R}, C_p(\mathbb{R}, \mathbb{R})) \in C_p^{\omega, \omega}(\mathcal{M}_0) \) that \( H \in C_p^{\omega, \omega}(\mathcal{M}_0) \), and hence \( H^\omega \in C_p^{\omega, \omega}(\mathcal{M}_0) \). By [6, 4.4.9], each metrizable space of weight \( \leq \kappa \) embeds into \( H^\omega \). Consequently, the class \( C_p^{\omega, \omega}(\mathcal{M}_0) \) contains all metrizable spaces of weight \( \leq \kappa \). \( \square \)

Lemma 4.4. The class \( C_p^{\omega, \omega}(\mathcal{M}_0) \) contains all generalized ordered spaces of countable linear weight.

Proof. Let \((X, \leq, \tau)\) be a generalized ordered space with countable linear weight. Consider the following two subsets of \( X \):

\[
X_\ell = \{ x \in X : (\ell, x) \text{ is open in } X \}, \quad \text{and } X_\tau = \{ x \in X : [x, \to) \text{ is open in } X \}.
\]

Let \( \tau_\leq \) be the topology on \( X \) generated by the linear order. By our assumption, the topology \( \tau_\leq \) on \( X \) is second countable, so, by [6, 6.3.2(e)], the linearly ordered space \((X, \tau_\leq)\) is order homeomorphic to a subspace of the real line \( \mathbb{R} \). Therefore we can assume that \( X \subset \mathbb{R} \) and the topology \( \tau \) on \( X \) is generated by the subbase

\[
\tau_\leq \cup \{(\ell, x) : x \in X_\ell \} \cup \{[x, \to) : x \in X_\tau \}.
\]

Denote by \( X_\ell \) and \( X_\tau \), the sets \( X_\ell \) and \( X_\tau \), respectively endowed with the separable metrizable topology inherited from the real line \( \mathbb{R} \).

Besides the Euclidean topology \( \tau_\leq \) generated by the linear order on the real line \( \mathbb{R} \), we shall consider the following three GO-topologies on \( \mathbb{R} \):

- the topology \( \tau_\ell \) generated by the subbase \( \tau_\leq \cup \{(-\infty, x) : x \in X_\ell \} \);
- the topology \( \tau_\tau \) generated by the subbase \( \tau_\leq \cup \{[x, +\infty) : x \in X_\tau \} \);
- the topology \( \tau \) generated by the subbase \( \tau_\ell \cup \tau_\tau \).

It is easy to see that the identity map \( \text{id} : (X, \tau) \to (\mathbb{R}, \tau) \) is a topological embedding. So, it suffices to show that \((\mathbb{R}, \tau) \in \text{SC}_p(\mathcal{M}_0^{\leq \omega})\).

For this purpose we consider the following two separately continuous functions \( L, R : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by the formulas

\[
L(x, y) := \begin{cases} 
\frac{2xy}{x^2+y^2}, & \text{if } x, y < 0, \\
0, & \text{otherwise},
\end{cases} \quad R(x, y) := \begin{cases} 
\frac{2xy}{x^2+y^2}, & \text{if } x, y > 0, \\
0, & \text{otherwise}.
\end{cases}
\]
For every $a \in \mathbb{R}$, consider the shifted functions $L_a, R_a : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$L_a(x, y) := L(x - a, y - a) \quad \text{and} \quad R_a(x, y) := R(x - a, y - a), \quad \forall x, y \in \mathbb{R}.$$ 

Let $a \in \mathbb{R}, \varepsilon > 0$ and $(x, y) \in \tilde{X}_\ell \times \tilde{X}_\ell$. If $x < a$ and $y < a$, we can find $\delta > 0$ such that

$$|L_a(x, y) - L_a(y, x)| < \varepsilon, \quad \forall b \in (a - \delta, a + \delta) \in \tilde{\tau}_\leq.$$ 

If $x > a$ or $y > a$, we can find $\delta > 0$ such that

$$L_a(x, y) = L_b(x, y) = 0, \quad \forall b \in (a - \delta, a + \delta) \in \tilde{\tau}_\leq.$$ 

Assume that $x = y = a$. Then $a \in \tilde{X}_\ell$ and

$$L_a(x, y) = L_b(x, y) = 0, \quad \forall b \in (-\infty, a) \in \tilde{\tau}_\leq.$$ 

Since $SC_p(\tilde{X}_\ell \times \tilde{X}_\ell)$ carries the topology of pointwise convergence, (1)–(3) imply that the map

$$\mathcal{L} : (\mathbb{R}, \tilde{\tau}) \to SC_p(\tilde{X}_\ell \times \tilde{X}_\ell), \quad \mathcal{L} : a \mapsto L_a|_{\tilde{X}_\ell \times \tilde{X}_\ell},$$

is continuous. Note also that if $a \in \tilde{X}_\ell$ and $b > a$, then $L_b(a, a) = 1$ and (see (3))

$$\mathcal{L}((-\infty, a]) = \mathcal{L}(\mathbb{R}) \cap U,$$

where $U = \{ f \in SC_p(\tilde{X}_\ell \times \tilde{X}_\ell) : |f(a, a)| < 1/2 \}$ is open in $SC_p(\tilde{X}_\ell \times \tilde{X}_\ell)$.

Analogously it can be shown that the function

$$\mathcal{R} : (\mathbb{R}, \tilde{\tau}) \to SC_p(\tilde{X}_r \times \tilde{X}_r), \quad \mathcal{R} : b \mapsto R_b|_{\tilde{X}_r \times \tilde{X}_r},$$

is continuous, and if $b \in \tilde{X}_r$ and $c < b$, then $R_c(b, b) = 1$ and

$$\mathcal{R}([b, +\infty)) = \mathcal{R}(\mathbb{R}) \cap V,$$

where $V = \{ f \in SC_p(\tilde{X}_r \times \tilde{X}_r) : |f(b, b)| < 1/2 \}$ is open in $SC_p(\tilde{X}_r \times \tilde{X}_r)$.

Set $T := \mathbb{R} \times SC_p(\tilde{X}_\ell \times \tilde{X}_\ell) \times SC_p(\tilde{X}_r \times \tilde{X}_r) \in SC_p(\mathfrak{M}_0^{<\omega})$ and define the following map

$$\mathcal{S} := (\text{id}, \mathcal{L}, \mathcal{R}) : (\mathbb{R}, \tilde{\tau}) \to T.$$ 

Clearly, the map $\mathcal{S}$ is continuous and injective. If $(a, b) \in \tilde{\tau}_\leq$, then

$$\mathcal{S}((a, b)) = \mathcal{S}(\mathbb{R}) \cap \left[ (a, b) \times SC_p(\tilde{X}_\ell \times \tilde{X}_\ell) \times SC_p(\tilde{X}_r \times \tilde{X}_r) \right].$$

If $a \in X_\ell$ and $b \in X_r$, then (4) and (3) imply

$$\mathcal{S}((-\infty, a]) = \mathcal{S}(\mathbb{R}) \cap [\mathbb{R} \times U \times SC_p(\tilde{X}_r \times \tilde{X}_r)]$$

and

$$\mathcal{S}([b, +\infty)) = \mathcal{S}(\mathbb{R}) \cap [\mathbb{R} \times SC_p(\tilde{X}_r \times \tilde{X}_\ell) \times V].$$

The equalities (4)–(8) imply that the map $\mathcal{S}$ is a topological embedding of the GO-space $(\mathbb{R}, \tilde{\tau})$ into the space $T \in SC_p(\mathfrak{M}_0^{<\omega})$, which implies that the GO-space $(\mathbb{R}, \tilde{\tau})$ and its subspace $(X, \tau)$ both belong to the class $SC_p(\mathfrak{M}_0^{<\omega})$. \hfill \Box

5. Proof of Theorem 1.5

Fix any space $X \in C_p^{\omega_1}(\mathfrak{M}_0)$. By Theorem 1.3 the space $X$ embeds into the function space $SC_p(Z^{<\omega})$ for some separable metrizable space $Z$. Then the space $X \subset SC_p(Z^{<\omega}) \subset \mathbb{R}Z^{<\omega}$ has weight $w(X) \leq w(SC_p(Z^{<\omega})) \leq w(\mathbb{R}Z^{<\omega}) \leq |Z^{<\omega}| \leq \mathfrak{c}.$

To see that $iw(X) \leq \mathfrak{c}$, choose a countable dense subset $D$ of $Z$ and observe that the restriction operator $R : SC_p(Z^{<\omega}) \to SC_p(D^{<\omega}) \subset \mathbb{R}D^{<\omega}, \ R : f \mapsto f|_{D^{<\omega}},$ is continuous.

Let us show that $R$ is also injective. Since $R$ is a linear operator, it is enough to show that a function $f \in SC_p(X^n)$ equals zero if $f|_{D^n} = 0$. Since $f$ is separately continuous and $D$ is dense in $X$, we have

$$f|_{X \times D^{n-1}} = f|_{X^n \times D^{n-2}} = \cdots = f|_{X^n} = 0.$$

Therefore, the operator $R : X \to \mathbb{R}D^{<\omega}$ is continuous and injective, which implies that $iw(X) \leq w(\mathbb{R}D^{<\omega}) \leq |D^{<\omega}| \leq \mathfrak{R}_0$ and $|X| = |\mathbb{R}D^{<\omega}| \leq |\mathbb{R}^\omega| = \mathfrak{c}.$
6. Proof of Theorem [LS]

For a linearly ordered set $X$ by a cut of $L$ we understand a pair $(A, B)$ consisting of two subsets $A, B$ of $X$ such that $X = A \cup B$ and $a < b$ for any $a \in A$ and $b \in B$. Each point $x \in X$ determines two cuts $x_-(\overline{a}, (x, \rightarrow))$ and $x_+(\overline{a}, (x, \rightarrow))$ of $X$. The set of all cuts of $X$ will be denoted by $X$. The set $X$ is linearly ordered by the relation $(A, B) \leq (C, D)$ if for each $a \in A$ there is $c \in C$ such that $a \leq c$. It is easy to see that the set $X^+ = \{x_- : x \in X\}$ is order dense in the linearly ordered set $X$.

To each linearly ordered set $L$ we can assign the ordered group $H(L)$ consisting of all functions $f : L \to \mathbb{Q}$ with finite support $\text{supp}(f) = \{x \in L : f(x) \neq 0\}$ and called the Hahn group (see [5, 1.25], where this group is denoted by $\mathfrak{y}(\mathbb{Q}, L)$). The order on the Hahn group $H(L)$ is defined by letting $f > g$ if $\min(\text{supp}(f)) > 0$. The Hahn group $H^2(L) = H(L(H(L)))$ has the structure of an ordered field. For any two functions $f, g : H(L) \to \mathbb{Q}$ their product in $H^2(L)$ is defined as the convolution $f * g : z \mapsto \sum_{x+y=z} f(x)g(y)$. By Theorem 2.15 of [5], $H^2(L)$ is a real-closed ordered field of cardinality $\max\{|L|, \aleph_0\}$.

For any (order dense) subspace $D \subset L$ we can identify the Hahn group $H(D)$ with the (order dense) subspace $\{f \in H(L) : \text{supp}(f) \subset D\}$ of $H(L)$ and the field $H^2(D)$ with the (order dense) subfield $\{f \in H^2(L) : \text{supp}(f) \subset H(D)\}$ of the field $H^2(L)$.

The proof of Theorem [LS] is based on a recent deep result of Chernikov and Shelah [3] Corollary 2.11.

**Fact 6.1 (Chernikov-Shelah).** For any infinite cardinal $\kappa$ there are linearly ordered spaces $L_0, \ldots, L_6$ such that $|L_0| \leq \kappa$, $|L_0| \geq 2^\kappa$ and $|L_{k+1}| \leq |L_k|$ for every $k \in \{0, 1, 2, 3, 4, 5\}$.

This result of Chernikov and Shelah will be applied in the proof of the following lemma.

**Lemma 6.2.** For every ordinal $\alpha < \omega_1$, each Tychonoff space of weight $\leq \beth_\alpha$ belongs to the class $C_p[\mathfrak{m}_\alpha]$.

**Proof.** We prove the lemma by transfinite induction. For $\alpha = 0$ the statement is trivially true. Assume that for some countable ordinal $\alpha$ and all ordinals $\beta < \alpha$ we have proved that all Tychonoff spaces of weight $\leq \beth_\beta$ belong to the class $C_p[\mathfrak{m}_\beta]$. If the ordinal $\alpha$ is limit, then $\beth_\alpha = \sup_{\beta < \alpha} \beth_\beta$. By the inductive assumption, the cardinals $\beth_\beta$, $\beta < \alpha$, endowed with the discrete topology belong to the class $C_p[\mathfrak{m}_\beta]$. Since this class is closed under taking countable topological sums, the cardinal $\beth_\alpha$ belongs to the class $C_p[\mathfrak{m}_\alpha]$. Then the Tychonoff power $\mathbb{R}^{\beth_\alpha}$, being homeomorphic to the function space $C_p(\beth_\alpha)$, belongs to the class $C_p[\mathfrak{m}_\alpha]$ too. Since each Tychonoff space of weight $\leq \beth_\alpha$ embeds into $\mathbb{R}^{\beth_\alpha} \in C_p[\mathfrak{m}_\alpha]$, the class $C_p[\mathfrak{m}_\alpha]$ contains all Tychonoff spaces of weight $\leq \beth_\alpha$. This completes the inductive step in the case $\alpha$ is a limit ordinal.

Now assume that $\alpha = \beta + 1$ is a successor ordinal. Then $\beth_\alpha = 2^{\beth_\beta}$. By Fact 6.1 there are infinite linearly ordered sets $L_0, \ldots, L_6$ such that $|L_0| \leq \beth_\beta$, $|L_6| \leq |L_0|$, and $|L_{k+1}| \leq |L_k|$ for all $k \in \{0, \ldots, 5\}$. By induction, for every $k \in \{0, \ldots, 6\}$ we shall prove that each Tychonoff space of weight $\leq |L_k|$ belongs to the class $C_p[\mathfrak{m}_0]$. For $k = 0$ this follows from the inequality $|L_0| \leq \beth_\beta$ and the inductive assumption. Assume that for some number $k \in \{0, \ldots, 6\}$ we have proved that every Tychonoff space of weight $\leq |L_k|$ belongs to the class $C_p[\mathfrak{m}_0]$. Consider the linearly ordered space $L_k$, and observe that $L_k^+ = \{x \in L_k : x < x_0\}$ is an order dense subset of $L_k$ and

$$\{(\overline{a}, x) : x \in L_k^+\} \cup \{(x, \overline{a}) : x \in L_k^+\}$$

is a base of the order topology of $L_k$. So, the linearly ordered space $L_k$ has topological weight $\leq |L_k|$. Now consider the Hahn field $H^2(L_k)$ over the linearly ordered set $L_k$ and its order dense subfield $H^2(L_k)$. Taking into account that the weight of an ordered field is equal to its density, we obtain that the ordered field $F_k := H^2(L_k)$ has topological weight

$$w(F_k) \leq |H^2(L_k)| = |L_k^+| = |L_k|.$$

By the inductive assumption, the space $F_k$ belongs to the class $C_p[\mathfrak{m}_0]$ and so does the function space $C_p(F_k, C_p(F_k, F_k))$. This function space is canonically homeomorphic to the subspace $SC_p(F_k \times F_k, F_k)$ of $F_k \times F_k$ consisting of separately continuous functions.

By Lemma 4.2 the space $SC_p(F_k \times F_k, F_k) \in C_p[\mathfrak{m}_0]$ contains a discrete subspace $D$ of cardinality $|F_k|$. Observe that

$$|D| = |F_k| = |H^2(L_k)| = |L_k| \geq |L_{k+1}|.$$

Hence $C_p[\mathfrak{m}_0]$ contains also the function space $C_p(D)$, which is homeomorphic to $\mathbb{R}^D$. Since each Tychonoff space of cardinality $\leq |L_{k+1}| \leq |D|$ embeds into $\mathbb{R}^D$, the class $C_p[\mathfrak{m}_0]$ contains all Tychonoff spaces of cardinality $\leq |L_{k+1}|$. This completes the inductive step. For $k = 6$ we get $|L_6| \geq \beth_\alpha$, which implies that the class $C_p[\mathfrak{m}_0]$ contains all Tychonoff spaces of weight $\leq \beth_\alpha$. □

Now we are ready to prove Theorem [LS].
7. Open Problems

By Theorem 1.3, the class $\text{SC}_p(M_0^\omega)$ contains all cosmic spaces and all metrizable spaces of weight $\leq \omega_1$. The classes of cosmic spaces and of metrizable spaces are contained in the class of $\sigma$-spaces. It can be shown that each $\sigma$-space $X$ of weight $\leq \omega_1$ admits a continuous injective map onto a metrizable space of weight $\leq \omega_1$ and hence has countable $i$-weight.

**Problem 7.1.** Does the class $\text{SC}_p(M_0^\omega)$ contain all $\sigma$-spaces of weight $\leq \omega_1$?

It is easy to see that for every metrizable separable space $X$ the space $SC_p(X^\omega)$ is canonically homeomorphic to the countable product $\prod_{n \in \mathbb{N}} SC_p(X^n)$. We can ask how different are the function spaces $SC_p(X^n)$ for various $n \in \mathbb{N}$.

**Problem 7.2.** Is it true that for every positive integers $n < m$ the space $SC_p([0,1]^m)$ does not embed into the space $SC_p([0,1]^n)$? (The answer is affirmative if $n = 1$ as $SC_p([0,1]) = C_p([0,1])$ is cosmic while $SC_p([0,1]^m)$ is not).

The affirmative answer to the following problem would simplify the proof of Theorem 1.3. Observe that this problem has affirmative answer under Generalized Continuum Hypothesis.

**Problem 7.3.** Is it true that for every infinite cardinal $\kappa$ there exists a Tychonoff space $X$ of weight $w(X) \leq \kappa$ such that the function space $SC_p(X \times X)$ contains a discrete subspace of cardinality $2^\kappa$?

Using Fact 2.1 and (the proof of) Theorem 1.2 one can show for the class $\mathcal{C}$ of all cosmic spaces we get $C_{p,\ell^\infty}[M_0] = C_{p,\ell^\infty}[M_0] = SC_p(M_0^\omega)$. It would be interesting to determine the extensions $C_{p,\ell^\infty}[M_0]$ of the class $M_0$ for some other classes $\mathcal{X}$ of topological spaces, in particular, for the class $\mathcal{E}$ of Lindelöf spaces.

**Problem 7.4.** Describe the class $C_{p,\ell^\infty}[M_0]$. Does it contain non-metrizable compacta?

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**References**

1. A. V. Arhangel’skii, *Topological function spaces (translated from Russian)*, Kluwer Academic, Dordrecht, 1992.
2. T. Banakh, $\Psi_0$-spaces, preprint [http://arxiv.org/abs/1311.1468](http://arxiv.org/abs/1311.1468).
3. T. Banakh, S. Gabriyelyan, On the $C_k$-stable closure of the class of (separable) metrizable spaces, preprint [http://arxiv.org/abs/1412.2216](http://arxiv.org/abs/1412.2216).
4. A. Chernikov, S. Shelah, On the number of Dedekind cuts and two-cardinal models of dependent theories, preprint [http://arxiv.org/abs/1308.3099](http://arxiv.org/abs/1308.3099).
5. H.G. Dales, W.H. Woodin, *Super-real fields. Totally ordered fields with additional structure*, London Mathematical Society Monographs. New Series, 14. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1996.
6. R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.
7. S. Gabriyelyan, J. Kąkol, On $\Psi$-spaces and related concepts, preprint, [http://arxiv.org/abs/1412.1494](http://arxiv.org/abs/1412.1494).
8. S. Gabriyelyan, J. Kąkol, W. Kubis, W. Marciszewski, Networks for the weak topology of Banach and Fréchet spaces, preprint [http://arxiv.org/abs/1412.1748](http://arxiv.org/abs/1412.1748).
9. G. Gruenhage, *Generalized Metric Spaces*, in: Handbook of set-theoretic topology, 423–501, North-Holland, Amsterdam, 1984.
10. A. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.
11. E. Michael, $\aleph_0$-spaces, J. Math. Mech. 15 (1966), 983–1002.

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