AUTOMORPHISMS ACTING ON THE LEFT-ORDERINGS OF A BI-ORDERABLE GROUP

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Abstract. We generalize a result of Koberda [9], by showing that the natural action of the automorphism group on the space of left-orderings is faithful for all nonabelian bi-orderable groups \(G\), as well as for a certain class of left-orderable groups that includes the braid groups. As a corollary we show that the action of \(\text{Aut}(G)\) on \(\partial G\) is faithful whenever \(G\) is bi-orderable and hyperbolic, following the approach of [9]. We also analyze the action of the commensurator of \(G\) on its space of virtual left-orderings.

October 25, 2016

1. Introduction

Let \(G\) be a group. We call a strict total ordering \(<\) of the elements of \(G\) a left-ordering if \(g < h\) implies \(fg < fh\) for all \(f, g, h \in G\). If \(G\) admits a left-ordering \(<\) that is also right-invariant, in the sense that \(g < h\) implies \(gf < hf\) for all \(f, g, h \in G\), then \(<\) is a bi-ordering of \(G\).

Each of these concepts can equivalently be defined in terms of positive cones. That is, given a left-ordering \(<\) of \(G\), we can identify \(<\) with its positive cone \(P = \{g \in G \mid g > 1\}\) which is a subset of \(G\) satisfying:

1. \(P \cdot P \subset P\)
2. \(P \sqcup P^{-1} \sqcup \{1\} = G\).

Conversely, given a subset \(P \subset G\) satisfying (1) and (2), it determines a positive cone according to the prescription \(g < h\) if and only if \(g^{-1}h \in P\) for all \(g, h \in G\). Bi-orderings may be similarly defined in terms of positive cones, but the positive cone of any bi-ordering must also satisfy a third condition, namely \(gPg^{-1} \subset P\) for all \(g \in G\).

We write \(\text{LO}(G)\) for the set of all positive cones \(P \subset G\) satisfying (i) and (ii) above, and, thinking of it as a subset of \(2^G\) (equipped with the product topology) we endow \(\text{LO}(G)\) with the subspace topology. Thus the open sets of \(\text{LO}(G)\) are finite intersections of sets of the form

\[ U_g = \{P \in \text{LO}(G) \mid g \in P\}\] and \(U_g^c = \{P \in \text{LO}(G) \mid g^{-1} \in P\}\).

We call \(\text{LO}(G)\) the space of left-orderings of the group \(G\). We similarly can define the space of bi-orderings of \(G\), \(\text{BiO}(G)\), by taking all positive cones \(P\) that satisfy the additional third...
condition of $gPg^{-1} \subset P$ for all $g \in G$. Topologizing $\text{BiO}(G)$ in the same way, we evidently have $\text{BiO}(G) \subset \text{LO}(G)$. Endowed with these topologies, both $\text{LO}(G)$ and $\text{BiO}(G)$ are compact spaces.

There is an action of $G$ on $\text{LO}(G)$ defined by $g(P) = gPg^{-1}$. More generally, there is an action of $\text{Aut}(G)$ on $\text{LO}(G)$ by observing that $\phi(P)$ is again a positive cone for all $P \in \text{LO}(G)$ and $\phi \in \text{Aut}(G)$. The action of $\text{Aut}(G)$ on $\text{LO}(G)$ is an action by homeomorphisms. Since the positive cones which are fixed under conjugation correspond to the bi-orderings of $G$, there is also an action of $\text{Out}(G)$ on $\text{BiO}(G)$.

With the topological structure and group actions as above, $\text{LO}(G)$ has found many applications within the study of orderable groups (for example, it was used to show that every left-orderable group has finitely many or uncountably many left-orderings [10], and was used to demonstrate a connection between orderability and amenability [12]), though applications beyond the realm of orderability are few. In recent work Koberda provided an example of such an application, by showing that whenever $G$ is a residually torsion-free nilpotent hyperbolic group, the natural action of $\text{Aut}(G)$ on $\partial G$ is faithful [9]. This application relies on the following theorem, which was also extended in [13] by replacing $\text{Aut}(G)$ with the commensurator of $G$:

**Theorem 1.1.** [9, Theorem 1.1] If $G$ is a finitely generated residually torsion-free nilpotent group, then the natural action of $\text{Aut}(G)$ on $\text{LO}(G)$ is faithful.

In this paper, we characterize the action of $\text{Aut}(G)$ on $\text{LO}(G)$ when $G$ is a bi-orderable group. Recall that finitely-generated residually torsion-free nilpotent groups are bi-orderable, though the converse is not true. For example, Thompson’s group $F$ is bi-orderable, but not residually nilpotent since $[F, F]$ is a simple group [6, Section 1.2.4].

Note that for some bi-orderable groups, like $\mathbb{Q}^k$ for all $k > 0$, we should not expect the action of $\text{Aut}(G)$ on $\text{LO}(G)$ to be faithful. For if $G = \mathbb{Q}^k$ then multiplication by a positive rational $p/q$ in each coordinate of $\mathbb{Q}^k$ can easily be seen to preserve all orderings of $\mathbb{Q}^k$. However, it turns out that these automorphisms of abelian groups are the only nontrivial automorphisms of bi-orderable groups which act trivially on the space of left-orderings. For an abelian group $G$ and a fixed $p/q \in \mathbb{Q}$, we denote by $\tau_{p/q} : G \rightarrow G$ the automorphism satisfying $\tau_{p/q}(g^p) = g^p$ for all $g \in G$, when it exists. We prove:

**Theorem 1.2.** Let $G$ be a bi-orderable group.

(i) If $G$ is nonabelian then $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$.

(ii) If $G$ is abelian then the kernel of the action of $\text{Aut}(G)$ on $\text{LO}(G)$ contains precisely the automorphisms $\tau_{p/q}$, if any such automorphisms exist.

Note that part (ii) of Theorem 1.2 already appears as [13, Proposition 4.3(2)]. We are also able to analyze the behaviour of the action of $\text{Aut}(G)$ on $\text{LO}(G)$ with respect to certain kinds of extensions.
Theorem 1.3. Suppose that $G$ is left-orderable and that
\[ 1 \to K \to G \to Z \to 1 \]
is a short exact sequence of groups. Suppose that $\text{Aut}(K)$ acts faithfully on $\text{LO}(K)$. If conjugation by the generator of $Z$ preserves a left-ordering of $K$, then $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$.

Since bi-orderability is not preserved under extensions (even under extensions such as those in the statement of the theorem above), this allows us to create non-bi-orderable groups $G$ for which $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$. See also Proposition 3.1.

As a corollary of Theorem 1.2 we can extend Koberda’s result concerning the action of $\text{Aut}(G)$ on $\partial G$ to all bi-orderable hyperbolic groups.

Corollary 1.4. If $G$ is a bi-orderable hyperbolic group, then $\text{Aut}(G)$ acts faithfully on $\partial G$.

The proof of Corollary 1.4 is a combination of Theorem 1.2 and Proposition 4.1.

The paper is organized as follows. In Section 2 we provide additional background on left-orderings and bi-orderings of groups, and prove Theorem 1.2. In Section 3 we prove Theorem 1.3 and also study the braid groups $B_n$. In Section 4 show that the action of $\text{Aut}(G)$ on $\partial G$ is faithful when $G$ is hyperbolic and bi-orderable, and describe the action of $\text{Comm}(G)$ on $\text{VLO}(G)$ for all bi-orderable groups.

2. Automorphisms of bi-orderable groups acting on the space of orderings

By insisting that the group $G$ be bi-orderable, we allow ourselves some flexibility in creating new left-orderings of $G$. The orderings that we will create arise from considering the action of $G$ on itself by conjugation, which is an order-preserving action if $G$ is bi-ordered (see Lemma 2.2). With this line of reasoning we will create sufficiently many left-orderings to show that whenever $\phi \in \text{Aut}(G)$ and $\phi(g) \neq g$ for some $g \in G$, then there exists $P \in \text{LO}(G)$ that contains $g$ but not $\phi(g)$. It follows that the action of $\phi$ on $\text{LO}(G)$ is nontrivial, because the positive cone $P$ satisfies $\phi(P) \neq P$.

Recall that a subset $S \subset G$ is called isolated if $g^k \in S$ for some $k \in \mathbb{Z}$ implies that $g \in S$. The isolator of a subgroup $H$ of $G$ is the set
\[ I(H) = \{ g \in G \mid \text{there exists } k \in \mathbb{Z} \text{ such that } g^k \in H \}. \]

In general, $I(H)$ is not a subgroup. However, when $H$ is abelian and $G$ is bi-orderable, then $I(H)$ is an abelian subgroup. Essential in proving this fact is the following property of bi-orderable groups: In a bi-orderable group, when $g^k$ and $h^\ell$ commute for some $k, \ell \in \mathbb{Z}$, then so do $g$ and $h$. This fact will also be used several times in the proofs of this section.

When $H$ is a rank one abelian subgroup of $G$, so is $I(H)$. If $g$ is a nonidentity element of a bi-orderable group $G$, then we will denote the isolator of the cyclic subgroup $\langle g \rangle$ by $I(g)$ for
short. Thus \( I(g) \) is always a rank one abelian group. We record the following fact for future use:

**Lemma 2.1.** Let \( G \) be a group. If \( g, h \) are distinct elements of \( G \), then either \( I(g) = I(h) \) or \( I(g) \cap I(h) = \{1\} \).

**Proof.** Suppose there exists \( f \in I(h) \cap I(g) \) where \( f \neq 1 \). Since \( f \in I(g) \), there exist \( n, m \in \mathbb{Z} \) such that \( f^n = g^m \). But now \( g^n \in I(h) \) and since \( I(h) \) is isolated, \( g \) is also in \( I(h) \) and \( I(h) = I(g) \). \qed

Recall that a subset \( S \) in a left-ordered group \( G \) is called convex with respect to a given left-ordering \( < \) if \( g, h \in S \) and \( g < f < h \) implies \( f \in S \). Of particular importance is the case when a subgroup \( C \) of a left-ordered group \( G \) is convex, as the convex subgroups of a left-ordering determine its structure in a sense described below. The convex subgroups of a left-ordered group \( G \) are ordered by inclusion. A subgroup is relatively convex if there exists a left-ordering relative to which it is convex.

Given a subgroup \( C \) of a left-ordered group \( G \), the natural quotient ordering of the left cosets \( G/C \) is well-defined if and only if \( C \) is convex, in this case the natural left-action of \( G/C \) preserves the quotient ordering. Therefore we can think of the ordering of \( G \) as lexicographic: it is constructed via inclusion of the left-ordered subgroup \( C \) and via pullback of the natural ordering on the cosets \( G/C \).

Consequently, if \( C \) is a convex subgroup of a left-ordered group, then the left-ordering of \( G \) may be altered by replacing the left-ordering of \( C \) with any left-ordering that we please. It follows that relative convexity is transitive, in the sense that if \( K \) is relatively convex in \( H \), and \( H \) is relatively convex in \( G \), then \( K \) is relatively convex in \( G \). This fact is needed in the proof of the following lemma.

**Lemma 2.2.** [2, Lemma 2.4] Suppose that \( G \) is a bi-orderable group, and that \( g \in G \) is not the identity. Then \( I(g) \) is relatively convex.

**Proof.** Let \( G_i, i = 1, 2 \) denote two copies of the group \( G \), and equip each copy with a given bi-ordering \( < \). Create a total ordering of \( G_1 \cup G_2 \) using \( < \) to order each \( G_i \), and declare the elements of \( G_1 \) smaller than those of \( G_2 \).

Now consider the action of \( G \) on \( G_1 \cup G_2 \) defined by conjugation on the elements of \( G_1 \), and by left-multiplication on the elements of \( G_2 \). This defines an effective, order-preserving action of \( G \) on the totally ordered set \( G_1 \cup G_2 \). Fix a nonidentity element \( g \in G_1 \) and well-order \( G_1 \cup G_2 \) so that \( g \) is smallest. Then using the action of \( G \) on \( G_1 \cup G_2 \) one may create a left-ordering of \( G \) in the standard way, relative to which \( Stab_G(g) = C_G(g) \) is convex. Here, \( C_G(g) \) denotes the centralizer of \( g \) in \( G \) (See [2, Proposition 2.3] or [3, Example 1.11 and Problem 2.16] for details of this construction). Now as \( C_G(g) \) is bi-orderable, the centre \( Z(C_G(g)) \) is relatively convex in \( C_G(g) \) by [1, Theorem 2.4]. Moreover, \( I(g) \subset Z(C_G(g)) \) since every element of \( I(g) \) has some power which lies in \( \langle g \rangle \), and thus commutes with all elements of \( C_G(g) \). Since \( I(g) \)
is an isolated subgroup and \(Z(C_G(g))\) is abelian, \(I(g)\) is relatively convex in \(Z(C_G(g))\). Thus \(I(g)\) is relatively convex in \(G\).

**Proposition 2.3.** Suppose that \(G\) is a bi-orderable group, and that \(\phi \in \text{Aut}(G)\). If there exists \(g \in G\) such that \(\phi(I(g)) \neq I(g)\), or if there exists \(g \in G\) such that \(\phi(g)^n = g^{-m}\) for some \(m, n > 0\), then the action of \(\phi\) on \(\text{LO}(G)\) is nontrivial.

**Proof.** Suppose there exists \(g \in G\) such that \(\phi(g)^n = g^{-m}\) for some \(m, n > 0\). Consider an arbitrary positive cone \(P \in \text{LO}(G)\). We can assume \(g \in P\), if not we replace \(P\) by \(P^{-1}\). Then \(g \in P\) and \(\phi(g) \notin P\), so we have \(P \neq \phi(P)\).

Now suppose there exists \(g\) such that \(I(g) \neq \phi(I(g))\), and note that \(\phi(I(g)) = I(\phi(g))\). By Lemma 2.2 \(I(g)\) is convex in some left-ordering of \(G\) with positive cone \(P \in \text{LO}(G)\). Applying \(\phi\), one checks that \(\phi(I(g)) = I(\phi(g))\) is convex relative to the ordering of \(G\) determined by \(\phi(P)\).

To show that \(\phi(P) \neq P\), we need only show that \(I(g)\) is not convex relative to the ordering of \(G\) determined by \(\phi(P)\). If it were, we would have either \(I(g) \subset I(\phi(g))\) or \(I(\phi(g)) \subset I(g)\), since convex subgroups are ordered by inclusion. By Lemma 2.1, either inclusion forces \(I(g) = I(\phi(g)) = \phi(I(g))\), a contradiction. □

Therefore, by Proposition 2.3, when \(G\) is a bi-orderable group and \(\phi \in \text{Aut}(G)\) we know that \(\phi\) acts nontrivially on \(\text{LO}(G)\) unless \(\phi\) satisfies:

\[(*) \quad \forall g \in \text{domain}(\phi) \exists n, m > 0 \text{ such that } \phi(g)^n = g^m.\]

We therefore investigate the existence of such automorphisms of bi-orderable groups.

Our lemmas below are stated in a slightly more general setting than needed in this section, as we will also be using them in our investigation of the action of \(\text{Comm}(G)\) on \(\text{VLO}(G)\) in Section 4.

Recall that when \(G\) is abelian, we denote by \(\tau_{p/q} : G \to G\) the automorphism satisfying \(\tau_{p/q}(g^q) = g^p\) for all \(g \in G\), when it exists. More generally, if \(H_1, H_2\) are finite index abelian subgroups of a group \(G\), we denote by \(\tau_{p/q} : H_1 \to H_2\) the isomorphism satisfying \(\tau_{p/q}(g^q) = g^p\) for all \(g \in H_1\), when it exists.

**Lemma 2.4.** Suppose \(G\) is a bi-orderable group with finite index torsion-free abelian subgroups \(H_1, H_2\), and \(\phi : H_1 \to H_2\) is an isomorphism satisfying \((*)\). Then there exist \(p, q > 0\) such that \(\phi(g)^q = g^p\) for all \(g \in H_1\), so that \(\phi = \tau_{p/q}\).

**Proof.** This lemma is essentially Case 2 of the proof of [13, Proposition 4.3]. Here is an alternative proof. Assume \(\phi : H_1 \to H_2\) satisfies \((*)\) and that \(H_1\) is torsion free abelian. Let \(g, h \in H_1\) and suppose \(\phi(g)^m = g^n\) and \(\phi(h)\ell = h^k\) for some \(k, \ell, m, n > 0\). By uniqueness of roots, we may assume that \(\gcd(m, n) = \gcd(k, \ell) = 1\), we wish to show that \(m = \ell\) and \(n = k\). If \(I(g) = I(h)\) then the result follows by applying \(\phi\) to a common power of \(g\) and \(h\) which lies in
Lemma 2.5. Suppose $G$ is a bi-orderable group with finite index subgroups $H_1, H_2$, that $\phi : H_1 \to H_2$ is an isomorphism satisfying (*), and that $\phi$ is not the identity. Then for every $g \in H_1$ there exist $p, q > 0$ such that $\phi(g)^q = g^p$ where $p \neq q$.

Proof. Since $\phi$ is not the identity there exists $g \in H_1$ with $\phi(g) \neq g$, say $\phi(g)^s = g^t$ with $s \neq t$ (necessarily $s \neq t$ since $G$ is bi-orderable). Now let $h \in G$ be given. By (*) there exists $n, m > 0$ such that $\phi(h)^n = h^m$. If $n = m$ then $\phi(h) = h$ since $G$ is bi-orderable. But $\phi(g^s h) = g^t h$, so $g^t h \in I(g^s h)$. But then $(g^t h)(h^{-1} g^{-s}) = g^{t-s} I(g^s h)$. Therefore $g \in I(g^s h)$, and so $h \in I(g^s h)$, and $I(g) = I(h)$. Now since $I(g)$ is abelian we may apply Lemma 2.4 to the restriction isomorphism $\phi|_{I(g)} : I(g) \to I(g)$ arising from $\phi$. We conclude that $n = s$ and $m = t$, contradicting the fact that $n \neq m$. Thus $n \neq m$.

Note that we can improve the conclusion of the previous lemma, by using uniqueness of roots in a bi-orderable group to show that $p, q$ exist with $\gcd(p, q) = 1$. However this is not needed for our purposes.

Lemma 2.6. Suppose $G$ is a bi-orderable group with finite index subgroups $H_1, H_2$ and that $\phi : H_1 \to H_2$ is an isomorphism satisfying (*). Let $g, h \in H_1$ be given and suppose that $\phi(g)^m = g^n$ and $\phi(h)^\ell = h^k$. Then

$$g^{n-m} h g^{m-n} \in I(h) \text{ and } h^{k-\ell} g h^{\ell-k} \in I(g).$$

Proof. By symmetry, it suffices to show only $g^{n-m} h g^{m-n} \in I(h)$. First, notice that $\phi(f) \in I(f)$ for all $f \in H_1$ by (*). Therefore $\phi(g^m h^\ell g^{-m}) = g^n h^k g^{-n} \in I(g^m h g^{-m})$, and since $I(g^m h g^{-m})$ is isolated we conclude $g^{n-m} h g^{-m} \in I(g^m h g^{-m})$. Next, notice that if $x \in I(h)$ then $g^i x g^{-i} \in I(g h g^{-1})$ for all $i \in \mathbb{Z}$, and thus $g^{n-m} h g^{-m} \in I(h)$.

Lemma 2.7. Suppose $G$ is a bi-orderable group with finite index subgroups $H_1, H_2$, that $\phi : H_1 \to H_2$ is an isomorphism satisfying (*), and that $\phi$ is not the identity. Then $H_1$ is abelian.

Proof. Let $g, h \in H_1$ be given. If $I(g) = I(h)$ then $g$ and $h$ commute. Thus we assume $I(g) \neq I(h)$. By Lemma 2.5 there exist $m, n > 0$ and $k, \ell > 0$ with $m \neq n$ and $k \neq \ell$ such that $\phi(g)^m = g^n$ and $\phi(h)^\ell = h^k$. Consider $h^{k-\ell} g^{n-m} h^{\ell-k} g^{m-n}$. On one hand, we have $h^{k-\ell} g^{n-m} h^{\ell-k} g^{m-n} = (h^{k-\ell} g^{n-m} h^{\ell-k}) \cdot g^{m-n} \in I(g)$, since it is a product of elements of $I(g)$ (here we use Lemma 2.6). On the other hand, $h^{k-\ell} (g^{n-m} h^{\ell-k} g^{m-n}) \in I(h)$ by similar reasoning. By Lemma 2.1 $I(g) \cap I(h) = \{1\}$ and so $h^{k-\ell} g^{n-m} h^{\ell-k} g^{m-n} = 1$. But this means the nontrivial powers $h^{k-\ell}$ and $g^{n-m}$ commute, so $h$ and $g$ commute since $G$ is bi-orderable. Thus $H_1$ is abelian.

$H_1$, such a common power exists since $|G : H_1|$ is finite. So suppose $I(g) \neq I(h)$, and therefore $I(g) \cap I(h) = \{1\}$ by Lemma 2.1.

Considering $g^m h^\ell$, we see that $\phi(g^m h^\ell) = g^n h^k \in I(g^m h^\ell)$, so there exist relatively prime $s, t > 0$ such that $(g^m h^\ell)^s = (g^n h^k)^t$. Since $H_1$ is abelian $g^{ms-nt} = h^{tk-st}$, and since both are in $I(g) \cap I(h)$, both are equal to 1. Since $\gcd(m, n) = \gcd(s, t) = 1$, from $ms - nt = 0$ we find $m = t$ and $s = n$. Similarly from $tk - st$ we find $t = \ell$ and $k = s$, so we are done. □
Proof of Theorem 1.2. Let $G$ be a bi-orderable group and let $\phi \in \text{Aut}(G)$ be nontrivial. If $G$ is nonabelian, then by Lemma 2.7 $\phi$ cannot satisfy (*). By Proposition 2.3 $\phi$ acts nontrivially on $\text{LO}(G)$, so the action of $\text{Aut}(G)$ on $\text{LO}(G)$ is faithful.

If $G$ is abelian, and if $\phi$ does not satisfy (*), then Proposition 2.3 tells us that $\phi$ acts nontrivially on $\text{LO}(G)$. If $\phi$ does satisfy (*), then Lemma 2.4 tells us that $\phi = \tau_{p/q}$ for some $p/q \in \mathbb{Q}$. It is easy to see that in this case, $\phi$ acts trivially on $\text{LO}(G)$. Thus the kernel of the action of $\text{Aut}(G)$ on $\text{LO}(G)$ consists exactly of the automorphisms $\tau_{p/q}$. □

3. Non-bi-orderable groups

For certain classes of left-orderable groups, it is sometimes sufficient to examine the action of $\text{Aut}(G)$ on a small subset of $\text{LO}(G)$ (perhaps even a finite subset) in order to determine that the action is faithful.

Recall that a left-ordering of $G$ is discrete if there is a smallest positive element. If $\phi : G \to G$ is an automorphism, and if $P$ is the positive cone of a discrete left-ordering with smallest positive element $g \in G$, then $\phi(P)$ is the positive cone of a discrete left-ordering whose smallest positive element is $\phi(g)$. Thus if $g \neq \phi(g)$, then $P \neq \phi(P)$. We apply this idea in the following proposition.

Proposition 3.1. Suppose that $G$ is a left-orderable group with generators $\{g_i\}_{i \in I}$, and that for each $i \in I$ there exists $P_i \in \text{LO}(G)$ which is the positive cone of a discrete left-ordering with $g_i$ as smallest positive element. Then $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$.

Proof. If $\phi : G \to G$ is a nontrivial automorphism, then there exists a generator $g_i$ such that $\phi(g_i) \neq g_i$. But then $\phi(P_i) \neq P_i$, so that $\phi$ acts nontrivially on $\text{LO}(G)$. □

Example 3.2. Recall the Artin presentation of braid group $B_n$ is given by

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \begin{array}{c}
s_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \\
s_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i - j| = 1
\end{array} \right\rangle.$$ 

By Dehornoy, the braid groups $B_n$ are left orderable for all $n$, as is the braid group $B_\infty$ [4]. The Dehornoy ordering of $B_n$ is a left-ordering that is defined in terms of representative words of braids as follows: A word $w$ in the generators $\sigma_1, \ldots, \sigma_{n-1}$ is called $i$-positive (respectively $i$-negative) if $w$ contains at least one occurrence $\sigma_i$, no occurrence of $\sigma_1, \ldots, \sigma_{i-1}$, and every occurrence of $\sigma_i$ has positive (respectively negative) exponent. A braid $\beta \in B_n$ is called $i$-positive (respectively $i$-negative) if it admits a representative word $w$ in the generators $\sigma_1, \ldots, \sigma_{n-1}$ that is $i$-positive (respectively $i$-negative). The Dehornoy ordering of the braid group $B_n$ is the ordering whose positive cone $P_\beta$ is the set of all braids $\beta \in B_n$ that are $i$-positive for some $i$. Using $sh^{n-j} : B_j \to B_n$ to denote the shift homomorphism sending $\sigma_i$ to $\sigma_{i+j}$, the convex subgroups of $B_n$ are $sh^{n-j}(B_j) = \langle \sigma_{n-j+1}, \ldots, \sigma_{n-1} \rangle \subset B_n$ [5], in particular the Dehornoy ordering is discrete with smallest positive element $\sigma_{n-1}$.

We can also define a related left-ordering as follows: a word $w$ in generators $\sigma_1, \ldots, \sigma_{n-1}$ is called $i$-reverse positive, if it has no occurrence of $\sigma_{i+1}, \ldots, \sigma_{n-1}$, and every occurrence of $\sigma_i$ has
positive exponent. Now similar to Dehornoy ordering, define an ordering \(<_D'\) on \(B_n\), whose positive cone \(P'_D\) is consists of all braids \(\beta \in B_n\) that are \(i\)-reverse positive for some \(i\).

It a straightforward check that \(<_D'\) is a also a discrete ordering of \(B_n\), with \(\sigma_1\) as its least positive element. Moreover, the convex subgroups of \(B_n\) with respect to \(<_D'\) are exactly the subgroups \(B_j = \langle \sigma_1, \ldots, \sigma_{j-1}\rangle \subset B_n\) for \(1 \leq j \leq n\).

Now given any \(i\) where \(1 \leq i \leq n - 1\), we can construct a left ordering \(<_i\) on \(B_n\) with \(\sigma_i\) as its least positive element. First, we left-order \(B_n\) with \(<_D'\). Since \(B_{i+1}\) is convex with respect to \(P'_D\), we can replace the left ordering \(<_D'\) on \(B_{i+1}\) with the left ordering of \(<_D\). Denote the resulting ordering of \(B_n\) by \(<_i\). By construction, \(<_i\) is a discrete ordering with \(\sigma_i\) as its least positive element. Based on this construction and Proposition 3.1, \(\text{Aut}(B_n)\) acts faithfully on \(\text{LO}(B_n)\).

This same construction can also be used to produce a left-ordering of \(B_\infty\) with \(\sigma_i\) as smallest positive element for all \(i \geq 1\). Thus \(\text{Aut}(B_\infty)\) acts faithfully on \(\text{LO}(B_\infty)\) as well. \(\square\)

If \(K\) and \(H\) are bi-orderable groups and

\[
1 \to K \to G \to H \to 1
\]

is a short exact sequence, then \(G\) can be lexicographically bi-ordered if and only if there exists a bi-ordering of \(K\) whose positive cone is invariant under the conjugation action of \(H\). By relaxing this condition, we are able to create groups which are not bi-orderable, but for which the automorphism group acts faithfully on the space of left-orderings.

**Theorem 3.3.** Suppose that \(G\) is left-orderable and that

\[
1 \to K \to G \to \mathbb{Z} \to 1
\]

is a short exact sequence of groups. Suppose that \(\text{Aut}(K)\) acts faithfully on \(\text{LO}(K)\). If conjugation by the generator of \(\mathbb{Z}\) preserves a left-ordering of \(K\), then \(\text{Aut}(G)\) acts faithfully on \(\text{LO}(G)\).

**Proof.** Suppose that \(\phi : G \to G\) is a nontrivial automorphism. If \(\phi(K) \neq K\), choose \(g \in K\) with \(\phi(g) \notin K\). Then by choosing signs appropriately, we may use the given short exact sequence to construct a positive cone \(P \subset G\) for which \(g \in P\) while \(\phi(g) \notin P\). Thus \(\phi(P) \neq P\).

On the other hand, suppose that \(\phi(K) = K\). If there exists \(k \in K\) for which \(\phi(k) \neq k\), then we know there is a positive cone \(P_K \in \text{LO}(K)\) for which \(\phi(P_K) \neq P_K\) since \(\text{Aut}(K)\) acts faithfully on \(\text{LO}(K)\). Using the given short exact sequence we may extend \(P_K\) to a positive cone \(P \subset G\) satisfying \(\phi(P) \neq P\).

Last, suppose that \(\phi(k) = k\) for all \(k \in K\), and choose \(t \in G\) which maps to the generator of \(\mathbb{Z}\). Equip \(K\) with a positive cone \(P_K\) that is preserved by conjugation by \(t\), and proceed as in [16, Lemma 3.4]. Note that every \(g \in G\) can be written uniquely as \(kt^n\) for some \(n \in \mathbb{Z}\) and \(k \in K\), and since \(\phi\) is nontrivial and satisfies \(\phi(k) = k\) for all \(k \in K\) it follows that \(\phi(t) \neq t\). Construct a positive cone \(P \subset G\) as follows: an element \(kt^n\) is in \(P\) if \(k \in P_K\) or \(k = 1\) and \(n > 0\). Then
P clearly satisfies $P \cup P^{-1} = G \setminus \{1\}$ and $P \cap P^{-1} = \emptyset$. Moreover if $kt^n$ and $k't^m$ are both in $P$, then so is $kt^nk't^m = k(t^n k'^{-n})t^{m+n}$ since conjugation by $t$ preserves $P_K$. One can easily verify that the subgroup $(t)$ is convex relative to the ordering of $G$ determined by $P$, so that $P$ determines a discrete ordering of $G$ with $t$ as smallest positive element. The positive cone $\phi(P)$ will determine a left-ordering of $G$ with $\phi(t)$ as smallest positive element. As $\phi(t) \neq t$, we conclude that $\phi(P) \neq P$. \hfill $\square$

If $K$ is a bi-orderable group, automorphisms $\phi : K \to K$ which preserve a left-ordering of $K$ but not a bi-ordering are likely quite common. However, there is little in the literature dealing with automorphism-invariant left-orderings, as the focus has primarily been on automorphism-invariant bi-orderings [14, 15, 11].

Here is one example of how an automorphism-invariant left-ordering (which is not a bi-ordering) may arise, which we use to illustrate an application of Theorem 3.3.

**Example 3.4.** Set $K = \mathbb{Q}^2 \rtimes \mathbb{Z}$ where the conjugation action of $\mathbb{Z}$ on $\mathbb{Q}^2$ is by the matrix $A = (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})$. Then $K$ is bi-orderable, since the action of $A$ preserves the bi-ordering of $\mathbb{Z}^2$ defined by $(a, b) > (0, 0)$ if and only if $(a, b) \cdot (\sqrt{2}, 1) = b + \sqrt{2}a > 0$. In fact, since the eigenvectors of $A$ are $\left(\begin{smallmatrix} \sqrt{2} \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -\sqrt{2} \\ 1 \end{smallmatrix}\right)$ with positive and negative eigenvalues respectively, the ordering described above (and its opposite) are the only orderings of $\mathbb{Q}^2$ preserved by $A$. Thus $K$ is bi-orderable and nonabelian, so by Theorem 1.2 $\text{Aut}(K)$ acts faithfully on $\text{LO}(K)$.

Now we define $G = K \rtimes \mathbb{Z}$ where the action of the generator of $\mathbb{Z}$ on an element of $K$ is $((a, b), c) \mapsto (-A(a, b)^T, c)$. The action of $-A$ on the subgroup $\mathbb{Q}^2 \subset K$, having the same eigenvectors as $A$ but with eigenvalues of opposite sign, preserves only the ordering defined by $(a, b) > (0, 0)$ if and only if $(a, b) \cdot (-\sqrt{2}, 1) = b - \sqrt{2}a > 0$, and its opposite. Using this ordering on $\mathbb{Q}^2$, and lexicographically ordering $K$ using the short exact sequence $1 \to \mathbb{Q}^2 \to K \to \mathbb{Z} \to 1$, we arrive at a left-ordering of $K$ preserved by the action of the generator of $\mathbb{Z}$.

We conclude $\text{Aut}(G)$ will act faithfully on $\text{LO}(G)$ by Theorem 3.3.

Note that $G$ is left-orderable by a straightforward short exact sequence argument, but is not bi-orderable since the actions of $A$ and $-A$ on $\mathbb{Q}^2 \subset K$ do not preserve a common ordering, so Theorem 1.2 does not apply. Proposition 3.1 also cannot apply to $G$ since any generator of $\mathbb{Q}^2 \subset G$ cannot be the smallest positive element of a left-ordering of $G$. \hfill $\square$

Despite these extensions and examples, one cannot hope to replace “bi-orderable” in Theorem 1.2 with either the weaker condition of local indicability or the condition that $G$ admit an ordering that is recurrent for every cyclic subgroup (See [12] for more information on recurrent orderings). Koberda points out that for the Klein bottle group, $K = \langle x, y \mid xyx^{-1} = y^{-1} \rangle$, the action of $\text{Aut}(K)$ on $\text{LO}(K)$ is not faithful. Yet $K$ is both locally indicable and admits recurrent orderings, as it only has four left-orderings.
4. Applications and Generalizations

The action of $\text{Aut}(G)$ on $\text{LO}(G)$ is connected to the action of $\text{Aut}(G)$ on $\partial G$ by the following theorem. Though not stated in full generality in [9], the proof below appears there as part of the proof of [9, Theorem 1.2]. As it is relatively short, we repeat it here for the reader’s convenience. For background and further information on hyperbolic groups, see [9, 7, 8].

**Proposition 4.1.** If $G$ is a left-orderable hyperbolic group and $\text{Aut}(G)$ acts faithfully on $\text{LO}(G)$, then it acts faithfully on $\partial G$.

*Proof.* Recall that for each element $g \in G$, there are two distinct points in the boundary $\partial G$ defined by $x_g = \lim_{n \to \infty} g^n$ and $y_g = \lim_{n \to \infty} g^{-n}$. Moreover, given $g, h \in G$ if $\langle g, h \rangle$ is not a virtually cyclic group, then $g$ and $h$ determine distinct points on the boundary.

Choose a nontrivial automorphism $\phi \in \text{Aut}(G)$, $g \in G$ and $P \in \text{LO}(G)$ such that $g \in P$ and $\phi(g) \notin P$ (and thus $\phi(P) \neq P$). Since we cannot have $g^k = \phi(g)^\ell$ for some $k, \ell > 0$, there are two cases. Recall we defined $I(g)$ in Section 2 to be the isolator of the cyclic subgroup $\langle g \rangle$.

**Case 1.** $\phi(g) \in I(g)$ and there exists $k, \ell > 0$ such that $g^{-k} = \phi(g)^\ell$. In this case, observe that $x_g = \lim_{n \to \infty} (g^\ell)^n$, so that $\phi(x_g) = \lim_{n \to \infty} \phi(g^\ell)^n = \lim_{n \to \infty} (g^{-k})^n = y_g$, so that $\phi$ acts nontrivially on $\partial G$.

**Case 2.** $\phi(g) \notin I(g)$. Then $\phi(g)$ and $g$ do not generate a virtually cyclic subgroup, so $x_g$ and $\phi(x_g) = x_{\phi(g)}$ are distinct. Thus $\phi$ acts nontrivially on $\partial G$. \qed

Consequently, by applying Theorem 1.2, we arrive at Corollary 1.4. If $G$ is hyperbolic and satisfies the hypotheses of Theorem 3.3 or Proposition 3.1, then $\text{Aut}(G)$ acts faithfully on $\partial G$, too. However it seems difficult to construct a hyperbolic group $G$ satisfying the hypotheses of either result.

There are also two natural generalizations one may consider, both developed by Witte Morris in [13]. First, one may replace the automorphism group with the commensurator group $\text{Comm}(G)$ of $G$. Recall that a commensuration of a group $G$ is an isomorphism $\phi : H_1 \to H_2$ of finite index subgroups $H_i \subset G$. Two commensurations $\phi : H_1 \to H_2$ and $\phi' : H'_1 \to H'_2$ are equivalent if there exists a finite index subgroup $H \subset H_1 \cap H'_1$ such that $\phi|_H = \phi'|_H$. The set of equivalence classes of commensurations forms the commensurator group $\text{Comm}(G)$ of $G$.

Witte Morris points out that for torsion free locally nilpotent groups, $\text{Comm}(G)$ acts naturally on $\text{LO}(G)$. This follows from an application of Koberda’s theorem (Theorem 1.1), and the fact that for every subgroup $H$ of a torsion-free locally nilpotent group $G$, the restriction map $r : \text{LO}(G) \to \text{LO}(H)$ is surjective. When $G$ is a bi-orderable group, the restriction $r : \text{LO}(G) \to \text{LO}(H)$ is not a surjective map in general, so this generalization is not possible in our setting.
However, using the restriction map $r : \text{LO}(G) \to \text{LO}(H)$ for each finite index subgroup $H \subset G$, one can define the space of virtual left-orderings of $G$ as the limit

$$VLO(G) = \varinjlim \text{LO}(H),$$

where the limit is over all finite-index subgroups $H$ of $G$ \cite{13}. When $P \in \text{LO}(H)$ and $H$ is a finite index subgroup of $G$, we will denote the corresponding element of $VLO(G)$ by $[P]$. Then $\text{Comm}(G)$ naturally acts on $VLO(G)$: for each commensuration $\phi : H_1 \to H_2$ and each positive cone $P \in \text{LO}(H)$, set $\phi([P]) = [\phi(P \cap H_1)]$. It is straightforward to check that this definition respects the necessary equivalence relations.

**Lemma 4.2.** Let $G$ be a left-orderable group and $\phi : H_1 \to H_2$ a commensuration of $G$ where $H_1$ is abelian. If $\phi = \tau_{p/q}$ for some $p/q \in \mathbb{Q}$ then the element of $\text{Comm}(G)$ represented by $\phi$ acts trivially on $VLO(G)$.

**Proof.** Suppose that $H$ is a finite index subgroup and $P \subset H$ is the positive cone of a left-ordering. Consider $P \cap H_1$ and $\phi(P \cap H_1)$. The first is the positive cone of a left-ordering of $H \cap H_1$, the second is the positive cone of a left-ordering of $H \cap H_2$. Using the fact that $\phi$ satisfies (*) one can show that these orderings agree on the finite index subgroup $H \cap H_1 \cap H_2$ so that $[P] = [\phi(P \cap H_1)]$, and thus $\phi$ acts trivially on $VLO(G)$. \hfill \Box

**Theorem 4.3.** Let $G$ be a bi-orderable group.

(i) If $G$ is not virtually abelian then $\text{Comm}(G)$ acts faithfully on $VLO(G)$.

(ii) If $G$ is virtually abelian then the kernel of the action of $\text{Comm}(G)$ on $VLO(G)$ contains precisely the elements represented by commensurations $\tau_{p/q} : H_1 \to H_2$, if any such commensurations exist.

**Proof.** First suppose that $G$ is not virtually abelian, and let $\phi : H_1 \to H_2$ be a nontrivial commensuration of $G$. By Lemma 2.7, $\phi$ cannot satisfy (*) since $H_1$ is not abelian. Thus there exists $g \in H_1$ such that $\phi(g) \neq g$ and either $\phi(g)^n = g^{-m}$ for some $m, n > 0$ or $I(g) \neq I(\phi(g))$. In either case we can construct a left-ordering of $G$ with positive cone $P$ satisfying $g \in P$ and $\phi(g) \notin P$ using arguments identical to those in the proof of Lemma 2.3. Then $[P] \neq [\phi(P \cap H_1)]$, so (the class of) $\phi : H_1 \to H_2$ acts nontrivially on $VLO(G)$.

On the other hand, suppose $G$ is virtually abelian, and let $\phi : H_1 \to H_2$ be a nontrivial commensuration of $G$. If $\phi$ does not satisfy (*), then an argument identical to the previous paragraph shows that the class of $\phi$ acts nontrivially on $VLO(G)$. On the other hand, if $\phi$ does satisfy (*), then $\phi = \tau_{p/q}$ for some $p/q \in \mathbb{Q}$ by Lemma 2.4. In this case, $\phi = \tau_{p/q}$ acts trivially on $VLO(G)$ by Lemma 4.2. \hfill \Box

**References**

[1] Roberta Botto Mura and Akbar Rhemtulla. *Orderable groups*. Marcel Dekker Inc., New York, 1977. Lecture Notes in Pure and Applied Mathematics, Vol. 27.

[2] Adam Clay. Left orderings and quotients of the braid groups. *J. Knot Theory Ramifications*, 21(14):1250130, 9, 2012.
[3] Adam Clay and Dale Rolfsen. Ordered groups and topology. 2015.
[4] Patrick Dehornoy. Braid groups and left distributive operations. *Trans. Amer. Math. Soc.*, 345(1):115–150, 1994.
[5] Patrick Dehornoy, Ivan Dynnikov, Dale Rolfsen, and Bert Wiest. *Ordering Braids*, volume 148 of *Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.
[6] B. Deroin, A. Navas, and C. Rivas. Groups, orders, and dynamics. 2014.
[7] M. Gromov. *Hyperbolic Groups*, pages 75–263. Springer New York, New York, NY, 1987.
[8] Ilya Kapovich and Nadia Benakli. Boundaries of hyperbolic groups. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, volume 296 of *Contemp. Math.*, pages 39–93. Amer. Math. Soc., Providence, RI, 2002.
[9] Thomas Koberda. Faithful actions of automorphisms on the space of orderings of a group. *New York J. Math.*, 17:783–798, 2011.
[10] Peter A. Linnell. The space of left orders of a group is either finite or uncountable. *Bull. Lond. Math. Soc.*, 43(1):200–202, 2011.
[11] Peter A. Linnell, Akbar H. Rhemtulla, and Dale P. O. Rolfsen. Invariant group orderings and Galois conjugates. *Journal of Algebra*, 319(12):4891–4898, 2008.
[12] D. Morris. Amenable groups that act on the line. *Algebr. Geom. Topol.*, 6:2509–2518, 2006.
[13] Dave Witte Morris. The space of bi-invariant orders on a nilpotent group. *New York J. Math.*, 18:261–273, 2012.
[14] Bernard Perron and Dale Rolfsen. On orderability of fibred knot groups. *Math. Proc. Cambridge Philos. Soc.*, 135(1):147–153, 2003.
[15] Bernard Perron and Dale Rolfsen. Invariant ordering of surface groups and 3-manifolds which fibre over $S^1$. *Math. Proc. Cambridge Philos. Soc.*, 141(2):273–280, 2006.
[16] Dale P. O. Rolfsen Peter A. Linnell, Akbar H. Rhemtulla. Discretely ordered groups. *Algebra & Number Theory*, 3(7):797–807, 2009.

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