QUALITATIVE PROPERTIES OF STATIONARY SOLUTIONS OF THE NLS ON THE HYPERBOLIC SPACE WITHOUT AND WITH EXTERNAL POTENTIALS

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Abstract. In this paper, we prove some qualitative properties of stationary solutions of the NLS on the Hyperbolic space. First, we prove a variational characterization of the ground state and give a complete characterization of the spectrum of the linearized operator around the ground state. Then we prove some rigidity theorems and necessary conditions for the existence of solutions in weighted spaces. Finally, we add a slowly varying potential to the homogeneous equation and prove the existence of non-trivial solutions concentrating on the critical points of a reduced functional. The results are the natural counterparts of the corresponding theorems on the Euclidean space. We produce also the natural virial identity on the Hyperbolic space for the complete evolution, which however requires the introduction of a weighted energy, which is not conserved and so does not lead directly to finite time blow-up as in the Euclidean case.

1. Introduction. Consider the following Nonlinear Schrödinger Equation on the Hyperbolic space (\(\mathbb{H}^d\)-NLS)

\[
\begin{cases}
  i\phi_t + \Delta_{\mathbb{H}^d} \phi + (1 + \epsilon h(x)) |\phi|^{p-1} \phi = 0, & \Omega \in \mathbb{H}^d \\
  \phi(0, x) = \phi_0(x).
\end{cases}
\]

(1)

Here \(\Delta_{\mathbb{H}^d}\) is the Laplace-Beltrami operator on the Hyperbolic space, \(d \geq 2\) is the spatial dimension, the nonlinearity has \(H^1\)-subcritical exponent (\(1 < p < 1 + \frac{4}{d-2}\) for \(d \geq 3\) and \(1 < p < +\infty\) for \(d = 2\)), the perturbation parameter \(\epsilon\) is such that \(\epsilon \geq 0\), the external potential \(h\) is bounded \(h \in L^\infty(\mathbb{H}^d)\) and the wave function is defined by \(\phi : \mathbb{R} \times \mathbb{H}^d \to \mathbb{C}\) and satisfies (1).

The study of the homogeneous (\(\epsilon = 0\)) nonlinear Schrödinger equation on the Hyperbolic space has received increasing interest in the recent years. People have tried to extend the theorems true in the flat space to the case of negatively curved space and results on local existence, blow-up, scattering and properties of standing waves have been proved. We refer to [6, 4, 5, 7, 9] and the references therein for an extended list of results on those important issues.

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The focus of this paper is on the qualitative properties of standing waves, namely solutions of the form
\[ \phi(t, x) = e^{i\lambda t}u(x), \]
with \( u : \mathbb{R}^d \to \mathbb{C} \) being a solution of
\[ -\Delta_{\mathbb{R}^d} u + \lambda u - (1 + ch(x)) |u|^{p-1} u = 0. \]
These solutions represent particles at rest and are usually present in the case of focusing nonlinearity, as it is our case. They play an important role for the dynamics as well, in particular for the dichotomy between scattering and blow-up of solutions.

Equation (2) is variational and it is the Euler-Lagrange equation of the associated energy functional
\[ E[u] = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^d} |u|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^d} (1 + ch(x)) |u|^{p+1} dx, \]
with \( dx \) the natural measure on \( \mathbb{R}^d \). Therefore, the natural space in which this functional is well defined is \( H^1(\mathbb{R}^d) \), endowed with the norm
\[ \|u\|_{H^1(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} (|u(x)|^2 + |\nabla u(x)|^2) dx \right)^{1/2}. \]

As noted in [11], the lower bound on the spectrum
\[ \lambda_1(-\Delta_{\mathbb{R}^d}) := \inf_{u \neq 0, u \in H^1(\mathbb{R}^d)} \frac{\int_{\mathbb{R}^d} |\nabla_{\mathbb{R}^d} u|^2 dx}{\int_{\mathbb{R}^d} |u|^2 dx} = \frac{(d-1)^2}{4} \]
implies that, if \( \lambda > -\frac{(d-1)^2}{4} \), the norm \( \| \cdot \|_\lambda \) defined by
\[ \|u\|_{\lambda} := \sqrt{\|\nabla_{\mathbb{R}^d} u\|^2_{L^2(\mathbb{R}^d)} + \lambda \|u\|^2_{L^2(\mathbb{R}^d)}}, \]
which appears in \( E[u] \), is equivalent to \( \| \cdot \|_{H^1(\mathbb{R}^d)} \). Since \( E \in C^2(H^1(\mathbb{R}^d), \mathbb{R}) \), solutions of equation (2) can be found by means of critical point theory. Least energy solutions are commonly called ground states and they minimize the energy, namely they satisfy
\[ E[u] = g, \]
where
\[ g := \inf\{E[u] : u \text{ is a non-trivial weak solution of (2)}\}. \]

We denote by \( \mathcal{G}_\lambda \) the set of weak solutions to (2) satisfying (4).

Christianson and Marzuola [9] (see also Banica-Duyckaerts [6] for some corrections and theorem [13]) proved that for \( \epsilon = 0 \), in dimensions \( d \geq 2 \), for all \( \lambda > -(d-1)^2/4 \), for any exponent \( p \) which is \( H^1 \)-subcritical (\( 1 < p < +\infty \) for \( d = 2 \) and \( 1 < p < 1 + \frac{4}{d-2} \) for \( d \geq 3 \)), there exists a positive, decreasing, spherically symmetric ground state solving (2).

Mancini and Sandeep [11] proved uniqueness of the ground state in dimensions \( d \geq 3 \) for \( 1 < p < 1 + \frac{4}{d-2} \) and \( \lambda > -(d-1)^2/4 \) and in dimension \( d = 2 \), for \( 1 < p < +\infty \), but only for \( \lambda \geq -\frac{2(p+1)}{(p+3+\frac{4}{d-2})} \), leaving open the case \( -\frac{2(p+1)}{(p+3)\frac{4}{d-2}} > \lambda > -\frac{1}{2} \). See also Selvitella [15] and Wang [17].

The nondegeneracy of the ground state was originally proved by Ganguly and Sandeep [14], where the authors considered only real-valued functions, used the Poincaré disk model of \( \mathbb{H}^d \) and did not discuss the 0-Fredholm property of the
linearized operator. These properties were proved by Selvitella in [15] using the polar model and extending the nondegeneracy to complex valued functions.

In this manuscript, we continue the study of qualitative properties of ground states of (1). Consider the case $\epsilon = 0$. We first prove that the ground state is a mountain pass critical point of the energy functional $E$ and that the Morse Index of the ground state is 1. Indeed, we have the following theorem.

**Theorem 1.1.** Suppose $\epsilon = 0$. We have the following properties.

- (Existence) There exists a mountain pass level $c$ of $E$ which is a critical level for $E$ and a critical point $u_c$ of $E$ such that $u_c > 0$ and $u_c \in C^2(\mathbb{H}^d, \mathbb{R})$.
- (MP=GS) Consider the mountain pass critical level $c$, the Nehari critical level $d$ (defined in Section 3 below) and the ground state level $g$. Then $c = d = g$.

In particular, the mountain pass critical point $u_c$ is a ground state and any ground state is a mountain pass critical point.

- (Morse Index=1) Let $m(u_c)$ be the Morse index of the ground state $u_c$. Then $m(u_c) = 1$.

Theorem 1.1 is the natural counterpart on the Hyperbolic space of the well known characterization of the ground state of the corresponding problem on the Euclidean space. See [2] for more details.

Now, we pose ourselves in the weighted space $\cosh(r)u \in L^2, \cosh(r)\nabla_{\mathbb{H}^d}u \in L^2, \cosh^{1/(p+1)}(r)u \in L^{p+1}$. Here $(x_0, x) = (\cosh(r), \sinh(r)\omega) \in \mathbb{H}^d$, with $\omega \in \mathbb{S}^{d-1}$. We have the following necessary conditions for existence of solutions to equation (2).

**Theorem 1.2.** Suppose that there exists a solution $0 \neq u \in H^1(\mathbb{H}^d)$ to equation (2) with $\epsilon = 0$ such that

$$\cosh^{1/2}(r)u, \cosh^{1/2}(r)\nabla_{\mathbb{H}^d}u \in L^2(\mathbb{H}^d) \text{ and } \cosh^{1/(p+1)}(r)u \in L^{p+1}(\mathbb{H}^d).$$

Then, one of the following conditions need to be satisfied. For $d \geq 3$, then:

- if $-\frac{(d-1)^2}{4} < \lambda < -\frac{d(d-2)}{4}$, then $p > 2^* - 1$;
- if $\lambda = -\frac{d(d-2)}{4}$, then $p = 2^* - 1$;
- if $\lambda > -\frac{d(d-2)}{4}$, then $p < 2^* - 1$.

For $d = 2$, then:

- $\lambda > 0$.

**Remark 1.** We should compare this theorem to the results of Mancini-Sandeep [11]. As proved in [11], there is no positive solution for $\lambda < -\frac{(d-1)^2}{4}$. For $\lambda = -\frac{(d-1)^2}{4}$, there is no positive solution in the energy space. So we do no treat those cases. For $-\frac{(d-1)^2}{4} < \lambda < -\frac{(d-3)(d+1)}{4}$, the ground state $u_c$ does not satisfy the integrability conditions $\cosh^{1/2}(r)u, \cosh^{1/2}(r)\nabla_{\mathbb{H}^d}u \in L^2(\mathbb{H}^d)$ and $\cosh^{1/(p+1)}(r)u \in L^{p+1}(\mathbb{H}^d)$, so our result is not in contradiction with [11]. When $\lambda > -\frac{(d-3)(d+1)}{4}$, the ground state is integrable and our result is in agreement with [11]. Nothing in [11] is said for what concerns the supercritical case $p > 2^* - 1$. Analogous argument for $d = 2$, where the ground state $u_c$ satisfies the integrability conditions only for $\lambda > 3/4$. 
Then, we consider the perturbed case, namely we take the parameter $\epsilon$ such that $0 < \epsilon \ll 1$. The presence of the inhomogeneous term $h(x) \neq 0$ breaks the symmetry of equation (2) and recovers compactness. Consider again the following problem:

$$
\begin{cases}
-\Delta_{H^d} u + u - (1 + \epsilon h(x)) |u|^{p-1} u = 0, \\
u \in H^1(\mathbb{H}^d), \quad u > 0,
\end{cases}
$$

where $d \geq 3$, $1 < p < \frac{d+2}{d-2}$ and $h \in L^\infty(\mathbb{H}^d)$. This type of problems is very well understood in the Euclidean setting (see for example Ambrosetti-Malchiodi [1]). We have the following theorem.

**Theorem 1.3.** Suppose $h \in L^\infty(\mathbb{H}^d)$ is such that $\int_{\mathbb{H}^d} h(x) u^{p+1} dx \neq 0$ and smooth. Then problem (1) has a solution, provided $\epsilon$ is small enough.

We conclude the paper with a Virial Identity which formally mimics the one in the Euclidean case and some remarks on how that identity does not easily imply blow-up as in the Euclidean case. See Section 6 for more details.

The remaining part of this paper is organized as follows. In Section 2, we recall the polar model representation of the Hyperbolic space. In Section 3, we prove Theorem 1.1 on the variational characterization of the ground state. In Section 4, we prove Theorem 1.2 about necessary conditions for existence of solutions in weighted spaces. In Section 5, we prove Theorem 1.3 on concentration of solutions for the perturbed equation. In Section 6, we discuss the virial identity.

2. **The polar model representation of the hyperbolic space.** Recall the polar model representation of the Hyperbolic space:

$$\mathbb{H}^d := \{(x_0, x) \in \mathbb{R}^{d+1} | (x_0, x) = (\cosh(r), \sinh(r) \omega), r \geq 0, \omega \in S^{d-1}\}.$$ 

We endow $\mathbb{H}^d$ with the Riemannian metric

$$ds^2 := dr^2 + \sinh^2 r d\omega^2$$

induced by the restriction to $\mathbb{H}^d$ of the Lorentz metric $ds^2_L = -x_0^2 + |x|^2$ on $\mathbb{R}^{d+1}$. The Laplace-Beltrami operator on the Hyperbolic space in polar coordinates can be rewritten as

$$\Delta_{\mathbb{H}^d} := \Delta_r + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{d-1}} + \frac{d-1}{\tanh r} \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{d-1}}.$$  \hspace{0.5cm} (5)

3. **Variational characterization of the ground state.** In this section, we give the full proof of Theorem 1.1. We divide the argument into several steps. We recall the Embedding theorems, basic facts about the Mountain Pass theorem, the Nehari manifold and Palais’ Symmetric Criticality principle. We collect these results to prove Theorem 1.1.

3.1. **Embeddings.** In this subsection, we collect the Embedding theorems that we will use to prove Theorem 1.1. In Mancini-Sandeep [11], we can find the following Poincaré-Sobolev inequality.

**Lemma 3.1** (Poincaré-Sobolev inequality, [11]). For any $d \geq 3$ and every $p \in \left(1, \frac{d+2}{d-2}\right]$ or for $d = 2$ and any $p > 1$, there is a constant $S_{d,p}$ such that

$$S_{d,p} \left(\int_{\mathbb{H}^d} |u|^{p+1} dx_{\mathbb{H}^d}\right)^{\frac{2}{p+1}} \leq \int_{\mathbb{H}^d} \left[|\nabla_{\mathbb{H}^d} u|^2 - \frac{(d-1)^2}{4} u^2\right] dx_{\mathbb{H}^d}$$

for every $u \in H^1(\mathbb{H}^d)$. 

As in the Euclidean space, the embedding for radial functions is compact, as shown in Bhakta-Sandeep [8].

**Lemma 3.2** (Compact Embedding, [8]). The embedding of $H^1_r(\mathbb{H}^d)$ in $L^p(\mathbb{H}^d)$ is compact if $d \geq 3$ and $p \in \left(1, \frac{d+2}{d-2}\right)$ or if $d = 2$ and $p > 1$.

3.2. **Critical point theory.** Let $B$ be a Banach space and $E \in C^1(B, \mathbb{R})$ a real valued smooth functional defined on $B$. If $E$ satisfies the conditions

**MP-1** $E(0) = 0$ and there exists $r, \rho$ such that $E(u) \geq \rho$ for all $u \in S_r := \{u \in B, \|u\| = r\};$

**MP-2** there exists $e \in B$ such that $E(e) \leq 0$;

then $E$ is said to possess the *mountain pass geometry*. Consider the class of paths:

$$\Gamma := \{\gamma \in C([0,1],B) : \gamma(0) = 0 \in B, \gamma(1) = e \in B\}$$

and the *mountain pass level*

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)).$$

A sequence $u_n \in B$ is called *Palais-Smale sequence on $B$, if $E(u_n)$ is bounded and $E'(u_n) \to 0$, as $n \to +\infty$. If $E(u_n) \to c$, as $n \to +\infty$, we will call $u_n \in B$ a Palais-Smale sequence at level $c$. We say that $E$ satisfies the Palais-Smale condition at level $c$ if every Palais-Smale sequence at level $c$ has a converging subsequence. We have the celebrated Ambrosetti-Rabinowitz Theorem [3].

**Theorem 3.3** (Mountain Pass, [3]). Suppose $E \in C^1(B, \mathbb{R})$ satisfies **MP-1** and **MP-2** and the Palais-Smale condition at the mountain pass level $c$. Then there exists $u \in B$, such that $u \neq 0$, and $u \neq e \in B$ satisfying $E(u) = c$ and $E'(u) = 0$.

We know the Morse Index of a Mountain Pass critical point (see [2]).

**Theorem 3.4.** If in addition to the assumptions of the Mountain Pass theorem, $E \in C^2(B, \mathbb{R})$ and $Z_c := \{u \in B : E(u) = c$ and $E'(u) = 0\}$ is discrete, then there exists $z \in Z_c$ such that the Morse Index $m(z) \leq 1$.

3.3. **The Nehari manifold.** In this subsection, we construct the Nehari manifold.

**Lemma 3.5** (Nehari Identity 1). Suppose $u \in H^1(\mathbb{H}^d)$ is a solution to (4), then $u$ satisfies the following identity:

$$\|\nabla_{\mathbb{H}^d} u\|_{L^2(\mathbb{H}^d)}^2 + \lambda \|u\|_{L^2(\mathbb{H}^d)}^2 - \|u\|_{L^{p+1}(\mathbb{H}^d)}^{p+1} = 0.$$

**Proof.** Rewrite the equation using the polar model representation and multiply by $u(r) \sinh^{d-1}(r)$. Then integrate in $\theta \in S^{d-1}, r \in [0, +\infty)$ by parts and use that

$$\|\nabla_{\mathbb{H}^d} u\|_{L^2(\mathbb{H}^d)}^2 = \int_{S^{d-1}} d\theta \int_0^{+\infty} \left[|u'|^2 + \frac{1}{\sinh^2(r)}|\nabla_{S^{d-1}} u|^2\right] \sinh^{d-1}(r) dr.$$

We define the *Nehari critical level $d$* for the functional $E$ on $H^1(\mathbb{H}^d)$ as:

$$d := \inf\{E[u] : u \in H^1(\mathbb{H}^d)$ and $E'(u)u = 0\}.$$
3.4. Palais’ symmetric criticality principle. The following is a result originally proved by Palais [12], which reduces the dimensionality of a problem in the case of symmetries.

Proposition 1. Suppose $\mathcal{H}$ is a Hilbert Space, $G$ a topological group acting on $\mathcal{H}$ through isometries and $E \in C^1(\mathcal{H}, \mathbb{R})$. Let $\text{Fix}(G) := \{ u \in \mathcal{H} : gu = u, \forall g \in G \}$. Then $E'(u) \perp \text{Fix}(G)$ if and only if $E'(u) = 0$.

3.5. Proof of Theorem 1.1. In this subsection we give the proof of Theorem 1.1. We work for now in $\mathcal{H} = H^1_r(\mathbb{H}^d)$. Note that the functional $E$ is also smooth in $H^1_r(\mathbb{H}^d)$.

Existence+Positivity+Smoothness

Now, we show that our functional $E$ satisfies MP-1 and MP-2 with $\mathcal{H} = H^1_r(\mathbb{H}^d)$. Consider:

$$\Gamma := \{ \gamma \in C([0,1], H^1_r(\mathbb{H}^d)) : \gamma(0) = 0 \in H^1_r(\mathbb{H}^d), \gamma(1) = e \in H^1_r(\mathbb{H}^d) \}$$

and the corresponding mountain pass level

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E(\gamma(t)).$$

We have the following

- **MP-1** Fix $v \in H^1_r(\mathbb{H}^d)$. Since, if $\lambda > -\frac{(d-1)^2}{4}$ the norm $\| \cdot \|_\lambda$ is equivalent to $\| \cdot \|_{H^1(\mathbb{H}^d)}$, then there exists $C > 0$, such that $\| v \|_\lambda \geq C \| v \|_{H^1(\mathbb{H}^d)}$. By Poincaré-Sobolev inequality in Lemma (3.1), there exists $K > 0$ such that

$$E[v] \geq C \| v \|_{H^1(\mathbb{H}^d)}^2 - \frac{K}{p+1} \| v \|_{H^1(\mathbb{H}^d)}^{p+1}.$$ 

So, there exists $r > 0$ as small as we want such that for $\| v \|_{H^1(\mathbb{H}^d)}^2 = r$, we have $E[v] = \delta > 0$. Therefore, for every $\gamma \in \Gamma$ there exists a point $s, \gamma \in [0,1]$ such that $E[\gamma(s)] \geq \delta > 0$. Therefore, we have:

$$\max_{s \in [0,1]} E[\gamma(s)] \geq E[\gamma(s_{\gamma})] \geq \delta > 0.$$

By definition of mountain pass level:

$$c \geq \delta > 0.$$

This implies that $E$ satisfies MP-1.

- **MP-2** Take $v \in H^1_r(\mathbb{H}^d)$ not identically null. Take $s > 0$, we get:

$$E[s v] = \frac{1}{2} s^2 \int_{\mathbb{H}^d} |\nabla v|^2 dx + \frac{s^2}{2} \int_{\mathbb{H}^d} |v|^2 dx - \frac{s^{p+1}}{p+1} \int_{\mathbb{H}^d} |u|^{p+1} dx.$$

Note that

$$\lim_{s \to +\infty} E[s v] = -\infty.$$

Therefore, there exists $\tilde{s} > 0$ such that

$$E[\tilde{s} v] < 0.$$

Therefore, we can build a continuous path $\gamma : [0,1] \to H^1_r(\mathbb{H}^d)$ given by $\gamma(s) = \tilde{s} s v$, such that $E[\gamma(0)] = 0$ and $E[\gamma(1)] < 0$. Define $e := \gamma(1) = \tilde{s} v$. Then this $\gamma$ is such that $\gamma \in \Gamma$. 
If we take \( r < \|\gamma(1)\|_{H^1(\mathbb{H}^d)} \), then \( E \) satisfies \textbf{MP-2}. Since \( E \) satisfies both \textbf{MP-1} and \textbf{MP-2}, then \( E \) has the mountain pass geometry.

Now, consider a Palais-Smale sequence at level \( c \), namely take \( u_n \in H^1_r(\mathbb{H}^d) \) such that \( E[u_n] \to c \) and \( E'[u_n] \to 0 \). By Theorem 3.2, the MP level \( c \) of \( E \) on \( u \in H^1_r(\mathbb{H}^d) \) is a critical level and so there exists a critical point \( u_c \) such that \( E[u_c] = c \) and \( E'[u_c] = 0 \) and so there exists a solution \( u \in H^1_r(\mathbb{H}^d) \) of equation (2) for any \( \lambda > -\frac{(d-1)^2}{4} \). By Theorem 1, \( u_c \) is also a critical point in \( H^1(\mathbb{H}^d) \) and it is a mountain pass critical point by the previous discussion. By elliptic regularity, \( u \in C^2 \) and by the maximum principle \( u > 0 \) on \( \mathbb{H}^d \).

\[ c = d = g \]

Consider again the path \( \gamma(s) = \dd s v \in \Gamma \), this time taking \( v \) in the Nehari manifold. Note that \( E[\gamma(s)] \) as a function of \( s \) is a polynomial which reaches its maximum at \( s = 1 \) and so when \( \gamma(s) = v \) (just compute the derivative in \( s \) of \( E[\gamma(s)] \)). This implies that \( c \leq d \).

By Lemma 3.5,

\[ d \leq g. \]

By definition of least energy level

\[ g \leq c. \]

Therefore \( c \leq d \leq g \leq c \)

and so

\[ c = d = g. \]

Morse Index \( m(u_c) = 1 \)

By Theorem 3.4 and since \( u_c \) is a Mountain Pass critical point, \( u_c \) has Morse Index at most 1. Since \( E''[u_c]u_c = -(p-1)u_c^p \), then \( m(u_c) = 1 \).

This completes the proof of Theorem 1.1.

4. Rigidity theorems. In this subsection, we give a complete proof of Theorem 1.2. To proof this theorem, we need a couple of identities that can be proved using carefully chosen multipliers.

4.1. Weighted Nehari manifold. We have a version of the Nehari Manifold in weighted spaces.

\textbf{Lemma 4.1 (Nehari Identity II).} Suppose that there exists a solution \( u \in H^1(\mathbb{H}^d) \) to equation (2) with \( \epsilon = 0 \) such that \( \cosh^{1/2}(r)u, \cosh^{1/2}(r)\nabla_{\mathbb{H}^d}u \in L^2(\mathbb{H}^d) \) and \( \cosh^{1/(p+1)}(r)u \in L^{p+1}(\mathbb{H}^d) \). Then \( u \) satisfies the following identity:

\[
\| \cosh^{1/2}(r)\nabla_{\mathbb{H}^d}u \|_{L^2(\mathbb{H}^d)}^2 \wedge \left( \lambda - \frac{d}{2} \right) \| \cosh^{1/2}(r)u \|_{L^2(\mathbb{H}^d)}^2 - \| \cosh^{1/(p+1)}(r)u \|_{L^{p+1}(\mathbb{H}^d)}^{p+1} = 0.
\]

\textbf{Proof.} By density, we can assume that \( u \in C_0^\infty(\mathbb{H}^d, \mathbb{R}) \). Rewrite the equation using the polar model and multiply by \( u(r) \cosh(r) \sinh^{d-1}(r) \) and integrate in \( \theta \in S^{d-1}, r \in [0, +\infty) \). The result comes by integration by parts.

Recall

\[-u'' - \frac{d-1}{\tanh(r)} u' - \frac{1}{\sinh^2(r)} \Delta_{\mathbb{S}^{d-1}} u + \lambda u - |u|^{p-1} u = 0.\]
Define
\[ E_1 := -u'' - \frac{d - 1}{\tanh(r)} u', \quad E_2 := -\frac{1}{\sinh^2(r)} \Delta_{\mathbb{S}^{d-1}} u, \quad E_3 := \lambda u(r) - |u|^{p-1} u, \]
and integrate each term separately.

Integrate \( E_3 \)
\[
\int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \, u(r) \, \cosh(r) \, \sinh^{d-1}(r) E_3 \\
= \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \, u(r) \, \cosh(r) \, \sinh^{d-1}(r) \left[ \lambda u - |u(r)|^{p-1} u \right] \\
= \lambda \| \cosh^{1/2}(r) u \|_{L^2(\mathbb{S}^d)}^2 - \| \cosh^{1/(p+1)}(r) u \|_{L^{p+1}(\mathbb{H}^d)}^2.
\]

Integrate \( E_2 \)
\[
\int_0^{+\infty} dr \int_{\mathbb{S}^{d-1}} u(r) \, \cosh(r) \, \sinh^{d-1}(r) E_2 \\
= - \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \, u(r) \, \cosh(r) \, \sinh^{d-1}(r) \frac{1}{\sinh^2(r)} \Delta_{\mathbb{S}^{d-1}} u \\
= \int_0^{+\infty} dr \left[ u(r) \, \cosh(r) \, \sinh^{d-1}(r) \frac{1}{\sinh^2(r)} \nabla_{\mathbb{S}^{d-1}} u \right]_{\partial\mathbb{S}^{d-1}} + \| \cosh^{1/2}(r) \nabla_{\mathbb{S}^{d-1}} u \|_{L^2(\mathbb{H}^d)}^2 \\
= \int_0^{+\infty} dr \left[ \cosh(r) \, \sinh^{d-1}(r) \frac{1}{\sinh^2(r)} |\nabla_{\mathbb{S}^{d-1}} u|^2 \right].
\]

Integrate \( E_1 \)
\[
\int_0^{+\infty} dr \int_{\mathbb{S}^{d-1}} u(r) \, \cosh(r) \, \sinh^{d-1}(r) E_1 \\
= - \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \, u(r) \, \cosh(r) \, \sinh^{d-1}(r) \left[ +u'' + \frac{d - 1}{\tanh(r)} u' \right] \\
= - \int_{\mathbb{S}^{d-1}} \left[ u(r) \, \cosh(r) \, \sinh^{d-1}(r) u' \right]_{r=0}^{r=+\infty} \\
+ \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \, u(r) \, \cosh(r) \, \sinh^{d-1}(r) u' \\
- \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \, \frac{d - 1}{\tanh(r)} u' u \, \cosh(r) \, \sinh^{d-1}(r) \\
= \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \left[ u'^2 \, \cosh(r) \, \sinh^{d-1}(r) + u' u \, \sinh^2(r) \right] \\
+ \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \left[ (d - 1) u \, \cosh^2(r) \, \sinh^{d-2}(r) - (d - 1) u \, \cosh^2(r) \, \sinh^{d-2}(r) \right] \\
= \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \left[ u'^2 \, \cosh(r) \, \sinh^{d-1}(r) + u' u \, \sinh^2(r) \right] \\
= \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \left[ u'^2 \, \cosh(r) \, \sinh^{d-1}(r) + \sinh^2(r) \frac{d}{2} \frac{d}{dr} u^2 \right] \\
= \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} dr \left[ u'^2 \, \cosh(r) \, \sinh^{d-1}(r) - \sinh^{d-1}(r) \, \cosh(r) \frac{d}{2} u^2 \right] 
\]
This concludes the proof of the lemma.

4.2. Weighted Pohozaev Manifold. We have a version of the Pohozaev Manifold in weighted spaces.

**Lemma 4.2 (Pohozaev Identity).** Suppose that there exists a solution \( u \in H^1(\mathbb{H}^d) \) to equation (2) with \( \epsilon = 0 \) such that \( \cosh^{1/2}(r)u, \cosh^{1/2}(r)\nabla_{\mathbb{H}^d}u \in L^2(\mathbb{H}^d) \) and \( \cosh^{1/(p+1)}(r)u \in L^{p+1}(\mathbb{H}^d) \). Then \( u \) satisfies the following identity:

\[
\frac{d}{2} \left( \frac{d}{2} \right) \| \cosh^{1/2}(r)\nabla_{\mathbb{H}^d}u \|_{L^2(\mathbb{H}^d)}^2 + \frac{d}{2} \| \cosh^{1/2}(r)u \|_{L^2(\mathbb{H}^d)}^2 - \frac{d}{p+1} \| \cosh^{1/(p+1)}(r)u \|_{L^{p+1}(\mathbb{H}^d)}^{p+1} = 0.
\]

**Proof.** By density, we can assume that \( u \in C_0^\infty(\mathbb{H}^d, \mathbb{R}) \). Rewrite the equation using the polar model and multiply by \( u'(r) \sinh^d(r) \) and integrate in \( \theta \in S^{d-1}, r \in [0, +\infty) \). The result comes by integration by parts. We use a similar decomposition as in the proof of the Nehari Identity II in Lemma 4.1.

Integrate \( E_3 \):

\[
\int_{S^{d-1}} \int_0^{+\infty} dr' \sinh^d(r) E_3
= \int_{S^{d-1}} \int_0^{+\infty} dr' \sinh^d(r) \left[ \lambda u - |u(r)|^{p-1}u \right]
= \int_{S^{d-1}} \int_0^{+\infty} dr \sinh^d(r) \frac{d}{dr} \left[ \lambda \frac{1}{2} u^2 - \frac{1}{p+1} |u(r)|^{p+1} \right]
= -\frac{d}{p+1} \| \cosh^{1/2}(r)u \|_{L^2(\mathbb{H}^d)}^2 + \frac{d}{p+1} \| \cosh^{1/(p+1)}(r)u \|_{L^{p+1}(\mathbb{H}^d)}^{p+1}.
\]

Integrate \( E_2 \):

\[
\int_{S^{d-1}} \int_0^{+\infty} dr' \sinh^d(r) E_2 = -\int_{S^{d-1}} \int_0^{+\infty} dr' \sinh^d(r) \left[ \frac{1}{\sinh^2(r)} \Delta_{S^{d-1}} u \right]
= -\int_{S^{d-1}} \int_0^{+\infty} dr' \sinh^{d-2}(r) \Delta_{S^{d-1}} u
= \int_{S^{d-1}} \int_0^{+\infty} dr \sinh^{d-2}(r) \nabla_{S^{d-1}} u' \nabla_{S^{d-1}} u
\]
\[
\int_{S^{d-1}} \int_0^{+\infty} dr \sinh^{d-2}(r) \frac{1}{2} \frac{d}{dr} \left| \nabla_{S^{d-1}} u \right|^2 \\
= - \frac{d-2}{2} \int_{S^{d-1}} \int_0^{+\infty} dr \sinh^{d-3}(r) \cosh(r) \left| \nabla_{S^{d-1}} u \right|^2 \\
= - \frac{d-2}{2} \int_{S^{d-1}} \int_0^{+\infty} dr \sinh^{d-1}(r) \cosh(r) \frac{\left| \nabla_{S^{d-1}} u \right|^2}{\sinh^2(r)}.
\]

Integrate \( E_1 \)

\[
\int_{S^{d-1}} \int_0^{+\infty} dr u' \sinh^d(r) E_1 \\
= - \int_{S^{d-1}} \int_0^{+\infty} dr u' \sinh^d(r) \left[ u'' + \frac{d-1}{\tanh(r)} u' \right] \\
= - \int_{S^{d-1}} \int_0^{+\infty} dr \sinh^d(r) u'' u' - \int_{S^{d-1}} \int_0^{+\infty} dr \sinh^d(r) \frac{d-1}{\tanh(r)} u'^2 \\
= - \frac{1}{2} \int_{S^{d-1}} \int_0^{+\infty} dr \sinh^d(r) \frac{d}{dr} u'^2 - \int_{S^{d-1}} \int_0^{+\infty} dr \sinh^d(r) \frac{d-1}{\tanh(r)} u'^2 \\
= \frac{1}{2} \int_{S^{d-1}} \int_0^{+\infty} dr u'^2 \frac{d}{dr} \sinh^d(r) - \int_{S^{d-1}} \int_0^{+\infty} dr \sinh^d(r) \frac{d-1}{\tanh(r)} u'^2 \\
= \int_{S^{d-1}} \int_0^{+\infty} dr u'^2 \left[ \frac{d}{2} \sinh^{d-1}(r) \cosh(r) - (d-1) \sinh^{d-1}(r) \cosh(r) \right] \\
= \left( \frac{d}{2} - d + 1 \right) \int_{S^{d-1}} \int_0^{+\infty} dr u'^2 \sinh^{d-1}(r) \cosh(r) \\
= - \frac{d-2}{2} \int_{S^{d-1}} \int_0^{+\infty} dr u'^2 \sinh^{d-1}(r) \cosh(r).
\]

Using \( E_1 + E_2 + E_3 = 0 \), we get:

\[
0 = - \frac{d-2}{2} \int_{S^{d-1}} \int_0^{+\infty} dr \left[ u'^2 + \frac{\left| \nabla_{S^{d-1}} u \right|^2}{\sinh^2(r)} \right] \sinh^{d-1}(r) \cosh(r) \\
- \frac{d\lambda}{2} \left\| \cosh^{1/2}(r) u \right\|_{L^2(\mathbb{H}^d)}^2 + \frac{d}{p+1} \left\| \cosh^{1/(p+1)}(r) u \right\|_{L^{p+1}(\mathbb{H}^d)}^{p+1} \\
= - \left( \frac{d-2}{2} \right) \left\| \cosh^{1/2}(r) \nabla_{\mathbb{H}^d} u \right\|_{L^2(\mathbb{H}^d)}^2 - \frac{d\lambda}{2} \left\| \cosh^{1/2}(r) u \right\|_{L^2(\mathbb{H}^d)}^2 \\
+ \frac{d}{p+1} \left\| \cosh^{1/(p+1)}(r) u \right\|_{L^{p+1}(\mathbb{H}^d)}^{p+1}.
\]

This concludes the proof of the lemma. \( \square \)

4.3. Proof of Theorem 1.2. In this section, we use Lemma 4.1 and Lemma 4.2 to prove Theorem 1.2.

Proof of Theorem 1.2. Combining Lemma 4.1 and Lemma 4.2, we get the three following identities:

\[
\left[ \left( \frac{d-2}{2} \right) - \frac{d\lambda}{\lambda - \frac{q}{2}} \right] \left\| \cosh^{1/2}(r) \nabla_{\mathbb{H}^d} u \right\|_{L^2(\mathbb{H}^d)}^2
\]
\[ + \left[ \frac{d \lambda}{\lambda - d} - \frac{d}{p + 1} \right] \| \cosh^{1/(p+1)}(r)u\|_{L^{p+1}(\mathbb{H}^d)}^{p+1} = 0, \]

(the case \( \lambda = d/2 \) can be treated separately)

\[ \left[ \lambda + \frac{d(d-2)}{4} \right] \| \cosh^{1/2}(r)u\|_{L^2(\mathbb{H}^d)}^2 \]

\[ + \left[ \frac{d-2}{2} - \frac{d}{p+1} \right] \| \cosh^{1/(p+1)}(r)u\|_{L^{p+1}(\mathbb{H}^d)}^{p+1} = 0 \]

and

\[ \left[ \frac{d-2}{2} - \frac{d}{p+1} \right] \| \cosh^{1/2}(r)\nabla_{\mathbb{H}^d}u\|_{L^2(\mathbb{H}^d)}^2 \]

\[ + \left[ \frac{d \lambda}{2} - \frac{d}{p+1} \left( \lambda - \frac{d}{2} \right) \right] \| \cosh^{1/2}(r)u\|_{L^2(\mathbb{H}^d)}^2 = 0. \]

The result follows by joining the necessary conditions for the above identities to be satisfied.

5. Concentration for the perturbed problem. In this section, we give a complete proof of Theorem 1.3. To do this, we first set our problem in the abstract setting of [1] and apply Theorem 2.16 from [1] (that we recall below).

5.1. The abstract setting. Consider the following functional

\[ I_\epsilon[u] = I_0[u] + \epsilon G[u], \]

with \( 0 < \epsilon \ll 1 \), \( I_0, G : \mathcal{H} \to \mathbb{R} \) and \( \mathcal{H} \) a Hilbert space. Suppose that \( I_0 \) has a smooth finite dimensional critical manifold \( Z \) (every \( z \in Z \) is a critical point of \( I_0 \), and \( \text{dim}(Z) < +\infty \)). We have the following theorem.

**Theorem 5.1** (Thm. 2.16 from [1]). Let \( I_0, G \in C^2(\mathcal{H}, \mathbb{R}) \) and suppose that \( I_0 \) has a smooth critical manifold \( Z \) which is non-degenerate. Let \( z \in Z \) be a strict local maximum or minimum of \( \Gamma := G|_Z \). Then for \( \epsilon \), such that \( |\epsilon| \ll 1 \), the functional \( I_\epsilon \) has a critical point \( u_\epsilon \) and if \( z \) is isolated, then \( u_\epsilon \to z \) as \( \epsilon \to 0 \).

Note that equation (2) with \( \epsilon = 0 \) is invariant under phase shifts and hyperbolic rotations. Therefore, the orbit of the ground state \( u_c \) of equation (2) with \( \epsilon = 0 \) forms the manifold \( Z_{\theta,d} \) given by

\[ Z_{\theta,d} := \{ e^{i\theta}L_d u_c(x), \quad \theta \in [0,2\pi) \text{ and } L_dQ(x) \text{ an hyperbolic isometry} \} \]

of dimension \( d + 1 \). The definition of nondegeneracy for the ground state \( u_c \) and so for the manifold \( Z_{\theta,d} \) is the following.

**Definition 5.2** (Nondegeneracy, [15]). The ground state \( u_c \) of (2) with \( \epsilon = 0 \) is nondegenerate if the following properties hold:

- (ND) \( \ker \{ E''[u_c] \} = \{ iu_c(x_0,x) ; x_0 \frac{\partial}{\partial x_j} u_c(x_0,x) + x_j \frac{\partial}{\partial x_0} u(x_0,x), \text{ for any } j = 1, \ldots, d \} \);

- (Fr) \( E''[u_c] \) is an index 0 Fredholm map.
5.2. **Proof of Theorem 1.3.** First, we set our problem in the framework of Theorem 5.1. Recall, we are studying equation (2)

$$-\Delta_{\mathbb{H}^d} u + \lambda u - (1 + \epsilon h(x)) |u|^{p-1} u = 0$$

with $d \geq 3$, $1 < p < 1 + \frac{4}{d-2}$ and also $\epsilon > 0$ and $h \in L^\infty(\mathbb{H}^d)$. The associated energy functional (3) is

$$E[u] = \frac{1}{2} \int_{\mathbb{H}^d} |\nabla u|^2 dx + \frac{\lambda}{2} \int_{\mathbb{H}^d} |u|^2 dx - \frac{1}{p+1} \int_{\mathbb{H}^d} (1 + \epsilon h(x)) |u|^{p+1} dx. \quad (6)$$

Therefore, in our case the Hilbert space is

$$\mathcal{H} = H^1(\mathbb{H}^d),$$

the unperturbed functional is

$$I_0[u] = \frac{1}{2} \int_{\mathbb{H}^d} |\nabla u|^2 dx + \frac{\lambda}{2} \int_{\mathbb{H}^d} |u|^2 dx - \frac{1}{p+1} \int_{\mathbb{H}^d} |u|^{p+1} dx$$

and

$$G[u] = -\frac{1}{p+1} \int_{\mathbb{H}^d} h(x)|u|^{p+1}. \quad (7)$$

We will assume that $h$ satisfies hypothesis such that $G[u]$ is well defined for any $u \in \mathcal{H}$. With this notation, we have $I_\epsilon[u] = I_0[u] + \epsilon G[u]$. The unperturbed problem $I_0'[u] = 0$ is therefore

$$-\Delta_{\mathbb{H}^d} u + \lambda u - |u|^{p-1} u = 0 \quad (7)$$

and has the smooth critical manifold of solutions given by

$$Z_{\theta,d} := \{ e^{i\theta} L_d u_c(x), \quad \theta \in [0,2\pi) \text{ and } L_d u_c(x) \text{ an hyperbolic isometry} \}$$

The infinitesimal generators of $L_d$ are in Definition 5.2. This manifold is nondegenerate by the following theorem proved in [15].

**Theorem 5.3** (Nondegeneracy, [15]). For any exponent $p$ which is $H^1$-subcritical ($1 < p < +\infty$ for $d = 2$ and $1 < p < 1 + \frac{4}{d-2}$ for $d \geq 3$) and $\lambda > 0$, the ground state $u_c$ of equation (7) is nondegenerate in the sense of Definition 5.2 above.

If we reduce our study to real valued functions, namely to $\mathcal{H} \cap \{ \bar{u} = u \}$, then the critical manifold reduces to

$$Z_d := \{ L_d u_c(x), L_d u_c(x) \text{ an hyperbolic isometry} \}$$

According to Theorem 5.1, we are reduced to find critical points of the reduced functional

$$G|_{Z_d}(u_c) = -\frac{1}{p+1} \int_{\mathbb{H}^d} h(x)L_d u_c^{p+1}(x) dx.$$ 

Consider one of the isometries $L_d$, namely the isometry $L_d(\alpha_1)$ moving only the first and second coordinates:

$$L_d(\alpha_1) = \begin{bmatrix} \cosh \alpha_1 & \sinh \alpha_1 & 0_{1\times(n-1)} \\ \sinh \alpha_1 & \cosh \alpha_1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times1} & 0_{(n-1)\times1} & \text{Id}_{(n-1)\times(n-1)} \end{bmatrix}.$$ 

We have:

$$L_d(\alpha_1) u_c(x) = u_c(x_0 \cosh \alpha_1 + x_1 \sinh \alpha_1, x_0 \sinh \alpha_1 + x_1 \cosh \alpha_1, x_2, \ldots, x_d).$$
All the fundamental isometries $L_d(\alpha_i)$ for $i = 1, \ldots, d$ are analogous but with $(x_0, x_i)$ instead of $(x_0, x_1)$ for $i = 1, \ldots, d$. Therefore, a general $L_d$ is of the form:

$$L_d = L_d(\alpha) = L_d(\alpha_1) \circ \cdots \circ L_d(\alpha_d)$$

with $\alpha = (\alpha_1, \ldots, \alpha_d)$. This means that

$$\Gamma(\alpha) := G|_{Z_d}(u_c) = -\frac{1}{p+1} \int_{\mathbb{H}^d} h(x)L_d(\alpha)u_c^{p+1}(x)dx.$$ 

Since the ground state $u_c$ is exponentially decaying, then

$$\lim_{\alpha \to \pm \infty} \Gamma(\alpha) = 0.$$ 

By hypothesis, $\int_{\mathbb{H}^d} h(x)u_c^{p+1}(x)dx \neq 0$ and note that $\Gamma$ is smooth in $\alpha$, since $u_c$ is smooth. Therefore, $\Gamma(\alpha)$ is not identically zero and so it must have either a maximum or a minimum. By Theorem 5.1, then (1) has a non-negative solution. This completes the proof of Theorem 1.3.

**Remark 2.** Note that this is possibly the simplest result of concentration of solutions in the presence of external potentials. We believe that thanks to Theorem 5.3, the perturbation results proved in the Euclidean case can be extended to the Hyperbolic case. For a comprehensive treatment of extensions of Theorem 1.3 in the Euclidean space and so ideas of results in the Hyperbolic space, we refer to [1].

6. **Blow-up?** In this section, we prove a virial identity for the NLS on the Hyperbolic space which takes the exact same form of its Euclidean counterpart originally proved by Glassey [10]. However, instead of the energy functional, we need to identify a weighted energy functional which is neither monotone and nor conserved and so, we cannot directly deduce finite time blow-up. We have the following lemma

**Lemma 6.1 (Virial Identity).** Suppose that $u(t)$ solves equation (1) with $\epsilon = 0$ for any $t \in [0, T)$ with $T$ the maximal time of existence and $u_0 \in H^1(\mathbb{H}^d)$ as initial datum. Suppose also that $\sinh(r) \nabla_{\mathbb{H}^d} u(t) \in L^2(\mathbb{H}^d)$ and $\sinh^{1/(p+1)}(r)u(t) \in L^{p+1}(\mathbb{H}^d)$ for every $t \in [0, T)$. Let

$$E_w[u(t)] = \frac{1}{2} \int_{\mathbb{H}^d} |\nabla_{\mathbb{H}^d} u(t)|^2 \sinh(r)dx - \frac{1}{p+1} \int_{\mathbb{H}^d} |u(t)|^{p+1} \sinh(r)dx,$$  

(8)

with $dx$ the natural measure on $\mathbb{H}^d$ and suppose that $u(t)$ is such that $E_w[u(t)]$ is finite for every $t \in [0, T)$. Then, $\sinh(r)u(t) \in L^2(\mathbb{H}^d)$ for any $t \in [0, T)$, the function $V(t) := \|\sinh^{1/2}(r)u(t)\|_{L^2(\mathbb{H}^d)}^2$ is $C^2$ and satisfies the following identities:

$$\dot{V}(t) = 4Im \int_{\mathbb{H}^d} \sinh(r) \bar{u} \frac{d}{dr} u(t)$$

$$\dot{V}(t) = 2E_w[u(t)] + \frac{4 - d(p - 1)}{2(p + 1)} \|u(t) \sinh^{1/(p+1)}(r)\|_{L^{p+1}(\mathbb{H}^d)}^{p+1}.$$ 

Here $r$ is such that $x = (\cosh(r), \sinh(r)\omega) \in \mathbb{H}^d$ with $r \in [0, +\infty)$ and $\omega \in S^{d-1}$.

**Proof.** By density, we can assume that $u \in C_0^\infty(\mathbb{H}^d, \mathbb{C})$. Rewrite the equation using the polar model and multiply by $u(r) \sinh^{1/2}(r)$; then integrate in $\theta \in S^{d-1}, r \in [0, +\infty)$. The result comes by integration by parts. $\square$
Note that in the Euclidean space this identity proves directly the existence of blow-up solutions in the case $p \geq 1 + \frac{4}{d}$ if $u_0$ has negative energy. In this case it does not, because
\[
\frac{d}{dt} E_w[u(t)] = -Re \int_{\mathbb{R}^d} \frac{d}{dr} u(t) \dot{\bar{u}}(t) \sinh(r) dx
\]
and so the weighted energy $E_w$ is not conserved and the Glassey argument with the multiplier chosen is inconclusive for what concerns blow-up. Unless we assume $E_w[u(t)] < 0$ to be negative for every $t \in [0, T)$, which however seems not natural, but would give the following.

**Theorem 6.2.** Suppose that $p \geq 1 + \frac{4}{d}$ and $u(t)$ satisfies the hypothesis of Lemma 6.1. Moreover, assume that there exists $C > 0$ such that $E_w[u(t)] \leq -C < 0$ for every $t \in [0, T)$. Then the maximal time of existence is finite, namely $T < +\infty$.

**Proof.** Follows the classical argument of Glassey [10], using concavity of $V(t)$. □

**Remark 3.** The multiplier $\sinh(r)$ seemed to us the most natural. Even if our argument is inconclusive for what concerns blow-up, we decided to propose the result of Lemma 6.1, because all the computations match the ones in the Euclidean case. We do not exclude the possibility of the existence of a more successful multiplier which gives directly finite time blow up from the virial identity.

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