A NEW AND SIMPLE PROOF OF THE FALSE CENTRE THEOREM

LUIS MONTEJANO AND EFREN MORALES-AMAYA

1. Introduction and preliminaries

We say that a set $A \subset \mathbb{R}^n$ is symmetric if and only if there is a translated copy $A'$ of $A$ such that $A' = -A'$. In this case, if $A' = A - x_0$, we say that $x_0$ is the center of symmetry of $A$. By convention, the empty set $\emptyset$ is symmetric.

The purpose of this paper is to give a new and simple proof of the following theorem:

**Theorem 1.1.** [False Centre Theorem] Let $K$ be a convex body in euclidean $3$-space and let $p$ be a point of $\mathbb{R}^3$. Suppose that for every plane $H$ through $p$, the section $H \cap K$ is symmetric. Then either $p$ is a centre of symmetry of $K$ or $K$ is an ellipsoid.

For the proof we will use the following two known theorems.

**Theorem 1.2.** Let $K$ be a convex body in euclidean $3$-space. Suppose that for every plane $H$, the section $H \cap K$ is symmetric. Then $K$ is an ellipsoid.

**Theorem 1.3.** Let $K$ be a convex body in euclidean $3$-space and let $p$ be point in $\mathbb{R}^3$. Suppose that for every plane $H$ through $p$, the section $H \cap K$ is symmetric. Then $K$ is symmetric.

Theorem 1.1 was first proved in all its generality by D. G. Larman [4]. Theorem 1.3 was proved by C. A. Rogers [7], when $p \in \text{int} K$ and by G. R. Burton in general (Theorem 2 of [2]). Theorem 1.2 was first proved by H. Brunn [1] under the hypothesis of regularity and in general by G. R. Burton [3] (see (3.3) and (3.6) of Petty’s survey [6]). For more about characterization of ellipsoids see [8] and Section 2.12 of [5].

We need some notation. Let $K$ be a convex body in euclidean $3$-space $\mathbb{R}^3$, let $p \in \mathbb{R}^3$ and let $L$ be a directed line. We denote by $p_L$ the directed chord $(p + L) \cap K$ of $K$, by $|p_L|$ the length of the chord $p_L$ and by $\frac{p}{L} \in \mathbb{R} \cup \{\infty\}$, the radio in which the point $p$ divides the directed interval $p_L$. That is, if $p_L = [a, b]$, then $\frac{p}{L} = \frac{a}{b}$, where $pa$ and $bp$ denotes the signed length of the directed chords $[p, a]$ and $[b, p]$ in the directed line $p + L$, and by convention, if $p_L = [p, p]$, then $\frac{p}{L} = 1$. If $p_L = (p + L) \cap K = \emptyset$, then by convention $|p_L| = -1$ and $\frac{p}{L} = -1$.

Suppose now $B$ is a convex figure in the plane and let $p$ be a point of $\mathbb{R}^2$. If $B$ is symmetric with center the origin, $L$ is a directed line through the origin and $q = -p$, then:

1. $|p_L| = |q_L|$, and
2. $\frac{p}{L} \cdot \frac{q}{L} = 1$,

where by convention $0\infty = \infty 0 = 1$.

Conversely, if $B$ is a convex figure and $p, q \in \mathbb{R}^2$ are two points for which (1) and (2) holds, for every directed line $L$ through the origin, then $B$ is symmetric with center at the midpoint of $p$ and $q$. Essentially, this is so because, if $p = -q$ and $p_L = [a, b]$, then $q_L = [-b, -a]$. 

1
Let $K$ be a convex body in euclidean 3-space and let $p$ be point of $\mathbb{R}^3$. Suppose that for every plane through $p$, the corresponding section is symmetric. By Theorem 1.3, we may assume that $K$ is symmetric with the center at the origin. Suppose $p$ is not the origin 0. We shall prove that $K$ is an ellipsoid. Let $H$ be a plane through the origin that does not contain the point $p$. By hypothesis, the plane $(p + H) \cap K$ is symmetric. Suppose $v \neq p$ is the center of $(p + H) \cap K$, and $w = tv$, for some $t \in \mathbb{R}$. We shall prove first that $(w + H) \cap K$ is symmetric with the center at $w$. For that purpose, it will be enough to prove that for every line $L \subset H$ through the origin:

1. $|p - v + w|_L = |v - (p + w)|_L$, and  
2. $\frac{v - p + w}{L} \cdot \frac{p - v + w}{L} = 1$.

Note that the points $p - v + w$ and $v - p + w$ lie in $(w + H) \setminus \partial K$ and if $L \subset H$, then $(p - v + w)_L$ and $(v - p + w)_L$ are chords of $(w + H) \cap K$.

Let $\Gamma$ be the plane through the origin generated by $L$ and $v$. Hence, by hypothesis, $(p + \Gamma) \cap K$ is symmetric. Suppose first that $K$ is strictly convex. In order to prove (1) and (2), we shall prove first:

a) $|p_L| = |(p - 2v)_L|$,  
b) the center of $(p + \Gamma) \cap K$ lies in $(p - v)_L$, and  
c) $\frac{p}{L} = \frac{p - 2v}{L}$.

Since $(p + H) \cap K$ is a symmetric section with center at $v$, then $|p_L| = |(2v - p)_L|$ and $\frac{2v - p}{L} = 1$. By the symmetry of $K$, $|(p - 2v)_L| = |(2v - p)_L|$ and $\frac{p - 2v}{L} \cdot \frac{2v - p}{L} = 1$. Consequently, $|p_L| = |(p - 2v)_L|$. So both chords, $p_L$ and $(p - 2v)_L$, of the symmetric section $(p + \Gamma) \cap K$ have the same length. By the strictly convexity of $K$, the parallel mid chord contains the center, that is, the center of $(p + \Gamma) \cap K$ lies in $(p - v)_L$. Furthermore, $\frac{p}{L} = \frac{p - 2v}{L}$. This proves a), b) and c).
Let us prove now that a), b) and c) implies (1) and (2). The parallel chords \((p - v - w)_L\) and \((p - v + w)_L\) of the symmetric section \((p + \Gamma) \cap K\) have as a mid chord \((p - v)_L\). The fact that the center of \((p + \Gamma) \cap K\) lies in \((p - v)_L\) implies that \(|(p - v + w)_L| = |(p - v - w)_L|\). Since \(K\) is symmetric with center at the origin, then (1) holds, that is \(|(p - v + w)_L| = |(v - p + w)_L|\). Since \(\frac{\mathbf{v}}{\mathbf{L}} = \mathbf{L} - 2\mathbf{v}\), then \((p - 2\mathbf{v})_L + 2\mathbf{v} = \mathbf{p}_L\). By symmetry of \((p + \Gamma) \cap K\), \((p - v - w)_L + 2w = (p - v + w)_L\). Hence \(\frac{v - p - w}{L} = \frac{w - p + w}{L}\). On the other hand, by the symmetry of \(K\), \(\frac{v - p - w}{L} = 1\). Consequently, \(\frac{v - p + w}{L} = 1\), thus proving 1) and 2) and hence that \((w + H) \cap K\) is symmetric with the center at \(w\).

This proves that every section of \(K\) parallel to \(H\) is symmetric. Let us prove now that the collection of planes through the origin such that the center of the section \((p + H) \cap K\) is not \(p\), is dense. Let \(\Omega\) be the collection of planes through the origin such that the center of the section \((p + H) \cap K\) is \(p\) and let \(H \in \text{int}\Omega\). If this is the case, by symmetry of \(K\), the center of the section \((-p + H) \cap K\) is \(-p\) and therefore, \((-p + H) \cap K\). Since the same hold for every section sufficiently close, we conclude that \((H \cap H) + \{tp \in \mathbb{R}^3 \mid |t| \leq 1\} \subset \partial K\), contradicting the strictly convexity assumption. Consequently, the collection of symmetric sections of \(K\) is dense and since the limit of a sequence of symmetric sections is a symmetric section, then every section of \(K\) is symmetric. By Brunn’s Theorem 1.2, \(K\) is an ellipsoid. In the non strictly convex case, our arguments for the case in which the center of \((p + H) \cap K\) is not \(p\), only showed that \((w + H) \cap K\) is symmetric, when \(w = tv\) from \(|t| < 1\), thus proving that every section of \(K\) sufficiently close to the origin and parallel to \(H\) is symmetric and hence that \(H \cap \partial K\) is contained in a shadow boundary. On the other side, if \(H \in \text{int}\Omega\) and \((H \cap \partial K) + \{tp \in \mathbb{R}^3 \mid |t| \leq 1\} \subset \partial K\), then clearly \(H \cap \partial K\) is contained in a shadow boundary. Consequently, by Blaschke’s Theorem 2.12.8 of [5], \(K\) is an ellipsoid.

The version of Theorem 1.1 for dimensions \(n \geq 3\) and any codimension less than \(n - 1\) is true. The proof follows from our Theorem 1.1 using standard arguments in the literature.

Acknowledgments. L. Montejano acknowledges support from CONACyT under project 166306 and from PAPIIT-UNAM under project IN112614. E. Morales-Amaya acknowledges support from CONACyT, SNI 21120.

References

[1] Brunn, H., Über Kurven ohne Wendepunkte. Habilitationsschrift, Ackermann, München, 1889.
[2] Burton G.R., Sections of Convex Bodies, J. London Math. Society 12(1976), 331-336
[3] Burton G.R., Some characterizations of the ellipsoid, Israel J. of Math, 28 (1977),339-
[4] Larman, D.G., A note on the false center problem, Mathematika 21 (1974), 216-217.
[5] Martini, H., Montejano, L., Oliveros, D., Bodies of Constant Width; An introduction to convex geometry with applications. Birkhäuser, Boston, Basel, Stuttgart, 2019.
[6] Petty, C. M., Ellipsoids, in: Convexity and its Applications, Eds. P. M. Gruber and J. M. Wills, pp. 264–276, Birkh"auer, Basel, 1983.
[7] Rogers, C.A., Sections and projections of convex bodies, Portugal Math. 24 (1965), 99-103
[8] Soltan, V., Characteristic properties of ellipsoids and convex quadrics, Aequat. Math., 93 (2019),371-413

UNAM at Querétaro. Mexico

Facultad de matemáticas at Acapulco, UAGro. Guerrero, Mexico.