On stability of diagonal actions and tensor invariants

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Abstract. For a connected simply connected semisimple algebraic group $G$ we prove the existence of invariant tensors in certain tensor powers of rational $G$-modules and establish relations between the existence of such invariant tensors and stability of diagonal actions of $G$ on affine algebraic varieties.

Bibliography: 12 titles.

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§ 1. Introduction

Recall that an action of a reductive algebraic group $G$ on an affine algebraic variety $X$ is called stable [1] if generic orbits of it are closed. Many actions $G : X$ do not have this property, but in [2] it was proved that if $G$ is semisimple and $X$ is affine and normal, then every action $G : X$ can be made stable by considering a diagonal action of $G$ on a sufficiently large number of copies of $X$. Let us consider an example of a diagonal action of $\text{SL}_n(k)$, $k$ being an algebraically closed field, on the product of $k$ copies of its standard representation $k^n \times \cdots \times k^n$. For small values of $k$, namely for $k < n$, such an action is not stable because it has a dense orbit that does not coincide with $k^n \times \cdots \times k^n$. For $k = n$ generic orbits of this action are level surfaces of the determinant

$$O_c = \{(v_1, \ldots, v_n) \in k^n \times \cdots \times k^n \mid \det(v_1, \ldots, v_n) = c\},$$

and are therefore closed. For $k > n$ generic orbits are closed, too.

Stability of diagonal actions of $G$ is closely related to the existence of nonzero $G$-invariant elements in tensor powers of rational $G$-modules. Let us consider the standard representation of $\text{SL}_n$ from this point of view. Representations in tensor powers $\text{SL}_n : (k^n)^{\otimes k}$ with $k < n$ have no nonzero invariant elements, while the action on the $n$th tensor power does have nonzero invariants. In this example we observe that the minimal tensor power that contains nonzero invariants is the same as the minimal number of copies of $k^n$ necessary to get a stable diagonal action. This fact is no coincidence; as we shall show later in Theorem 3, the absence of invariants in low tensor powers implies the existence of unstable diagonal actions with a small number of copies.

Relations between stability of actions and tensor invariants have been revealed in [3], Theorem 10 and have later been used in [2] to prove that every effective action of a semisimple group can be made stable by passing to an appropriate diagonal action.

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In this article we continue the investigation of relations between stability of diagonal actions and the existence of nonzero invariant elements in tensor powers of rational modules. We provide lower and upper bounds on the number of copies needed to obtain a stable diagonal action and explicitly calculate the diagonal of the weight semigroup of the action $G^n$. These results extend the results of [2] and prove that the number of copies of a $G$-variety $X$ required to obtain a stable action is bounded by a number depending only on $G$.

Let us pass to the formulation of the main results. Below the ground field $k$ is assumed to be an algebraically closed field of characteristic zero; when no explicit characterization of a group $G$ is given, it is assumed to be connected simply connected and semisimple; weights of group $G$ are taken with respect to a fixed maximal torus $T \subseteq G$; simple roots and fundamental weights are numbered in the same way as in [4].

**Definition 1.** Let $G$ be a connected algebraic group. Denote

$$M(G) := \{ n \in \mathbb{N} \mid (V^{\otimes n})^G \neq \{0\} \text{ for every nonzero rational } G\text{-module } V \}.$$ 

It is clear that $M(G)$ is an additive semigroup. Denote by $m(G)$ the minimal element of the semigroup $M(G)$, or $+\infty$ if it is empty.

One does not have to verify that $(V^{\otimes n})^G \neq \{0\}$ for all nonzero rational $G$-modules; it suffices to prove that all simple modules have this property. Indeed, if we have $(V^{\otimes n})^G \neq \{0\}$ for all nonzero $G$-modules $V$, then, a fortiori, all nonzero simple $G$-modules have this property. Conversely, fix a nonzero module $V$ and a simple submodule $U \subset V$; we have $(V^{\otimes n})^G \supset (U^{\otimes n})^G \neq \{0\}$.

**Theorem 1.** Semigroups $M(G)$ with $G$ simple are listed in the table below:

| Group $G$ | $M(G)$ | Group $G$ | $M(G)$ |
|-----------|--------|-----------|--------|
| $SL_n$    | $n\mathbb{N}$ | $G_2$     | $n \in \mathbb{N} \mid n \geq 2$ |
| $Spin_{2n+1}$ | $2\mathbb{N}$ | $F_4$     | $n \in \mathbb{N} \mid n \geq 2$ |
| $Spin_{4n+2}$ | $4\mathbb{N}$ | $E_6$     | $3\mathbb{N}$ |
| $Spin_{4n+4}$ | $2\mathbb{N}$ | $E_7$     | $2\mathbb{N}$ |
| $Sp_{2n}$  | $2\mathbb{N}$ | $E_8$     | $n \in \mathbb{N} \mid n \geq 2$ |

Calculation of $M(G)$ for an arbitrary (not necessarily reductive) group $G$ can be reduced to the cases listed in the table above by applying the following two propositions.

**Proposition 1.** Let $G$ be a connected affine algebraic group, let $F$ be the unipotent radical of it and $H = G/F$. Then $M(G) = M(H)$.

In fact this proposition shows that $M(G)$ is to be calculated only for semisimple groups $H$, not for reductive groups. Indeed, if $Z \subseteq H$ is a nontrivial central torus, then $M(H) = \emptyset$; it follows from the fact that $Z$ can act nontrivially by multiplications on all tensor powers of $k^1$. 
Proposition 2. Let $G = G_1 \times G_2$ be a product of two reductive groups $G_1$ and $G_2$. Then $M(G) = M(G_1) \cap M(G_2)$.

Applying this proposition one can easily find $M(G)$ if $G$ is a connected simply connected semisimple group, that is, if $G$ is a product of simply connected simple groups. Considering groups $G$ that are not simply connected is a more involved problem and it seems probable that every group $G$ that is not simply connected requires an ad hoc approach. However, it is clear that if $G_1$ and $G_2$ are semisimple groups of the same type and $G_1$ is simply connected, then $M(G_1) \subseteq M(G_2)$.

It turns out that calculation of semigroups $M(G)$ is tightly related to describing balanced collections of elements of the Weyl group of $G$.

Definition 2. Let $\mathcal{W}$ be the Weyl group of $G$. A collection of elements $w_1, \ldots, w_k \in \mathcal{W}$ is called balanced if $w_1 + \cdots + w_k = 0$ (the sum is considered as a sum of endomorphisms of the $\mathbb{Q}$-linear span of roots of $G$).

Theorem 2. Let $\mathcal{W}$ be the Weyl group of a simple simply connected group $G$. There exists a balanced collection of $m$ elements of $\mathcal{W}$ if and only if $m \in M(G)$.

Now we pass to relations between semigroups $M(G)$ and stability of diagonal actions of groups $G$.

Definition 3. Let $G$ be a connected semisimple algebraic group which is not necessarily simply connected. Denote

- $s_m(G)$ the smallest natural number such that for every affine variety $X$ with an effective action of $G$ the diagonal action on the product of $s_m(G)$ copies of $X$ is stable;
- $s(G)$ the smallest natural number such that for every affine variety $X$ with an effective action of $G$ and for every $k \geq s(G)$ the diagonal action of $G$ on the product of $k$ copies of $X$ is stable.

Let us call the numbers $s_m(G)$ and $s(G)$ metastability index and stability index respectively.

The existence of $s(G)$ for a semisimple group $G$ will be shown in Theorem 4. Note that $s_m(G) \leq s(G)$. The reason for separating metastability and stability indices is that stability of the diagonal action on $k$ copies of a $G$-variety $X$ does not imply stability of the action on $r$ copies of $X$ with $r > k$. Such a phenomenon is exhibited by symplectic groups $\text{Sp}_{2m}$. Indeed, consider the standard representation $\text{Sp}_{2m} : \mathbb{k}^{2m}$; it is easy to see that if $k$ is even then the stabilizer of a generic point in $(\mathbb{k}^{2m})^k$ is isomorphic to $\text{Sp}_{2m-k}$ and therefore reductive; by [5], Theorem 1 we have that the action on $(\mathbb{k}^{2m})^k$ is stable. If $k$ is odd then the stabilizer in general position contains a nontrivial normal unipotent subgroup; hence this action is not stable.

The following two statements give bounds of stability indices in terms of $m(G)$.

Theorem 3. Let $G$ be a simple simply connected group. Then $m(G) \leq s_m(G)$.

Theorem 4. Let $e(G)$ be the smallest natural number such that for every affine variety $X$ with effective action of $G$ the diagonal action $G : X^e(G)$ has finite stabilizer in general position. Then $s(G) \leq e(G)m(G)$.
Theorem 4 is proved by a simple modification of the argument in [2], Theorem 1. Note that the number $e(G)$ exists and is not greater than the dimension of $G$. It would be interesting to calculate $e(G)$ for semisimple groups $G$.

The result of Theorem 3 can be substantially improved for groups that have only self-conjugate linear representations. This improvement can be made by applying results of [6].

**Theorem 5.** Let $G$ be a connected semisimple algebraic group which is not necessarily simply connected. Suppose additionally that all linear representations of $G$ are self-conjugate. Then $s_m(G) = m(G) = 2$.

The author has considered several examples of actions of groups $G$ that have linear representations which are not self-conjugate. These examples suggest that if $G$ is simple then it is superfluous to suppose that all linear representations of $G$ are self-conjugate.

**Conjecture.** If $G$ is a simple group then $s_m(G) = m(G)$.

§ 2. Calculation of semigroups $M(G)$

2.1. Auxiliary statements. The demonstration of Theorem 1 relies on the PRV-theorem on extremal weights of submodules in tensor product of simple modules. Let us recall the necessary definitions and facts.

Denote by $W$ the Weyl group of $G$ and let $V(\lambda)$ be the irreducible $G$-module with highest weight $\lambda$. Let $\tau$ be a weight occurring in $V(\lambda)$. The weight $\tau$ is said to be *extremal* if it is $W$-equivalent to $\lambda$. Since every weight is $W$-equivalent to a unique dominant weight, the module $V(\lambda)$ is uniquely determined by any of its extremal weights. This observation permits us to define $V(\tau)$ with $\tau$ not necessarily dominant. The following statement is called the PRV-theorem; it partially describes the decomposition of tensor product of two irreducible modules.

**Theorem 6** (see [7], [8]). Let $\lambda$ and $\mu$ be arbitrary weights. Then the tensor product $V(\lambda) \otimes V(\mu)$ contains the irreducible submodule $V(\lambda + \mu)$.

The PRV-theorem establishes the following relation between lengths of balanced collections in $W$ and elements of $M(G)$.

**Lemma 1.** Let $w_1, \ldots, w_m \in W$ be a balanced collection of $m$ elements. Then $M(G) \supseteq m\mathbb{N}$.

**Proof.** Take any dominant weight $\lambda$ and the irreducible module $V(\lambda)$ which corresponds to it. We have $V(\lambda)^{\otimes m} = V(w_1\lambda) \otimes V(w_2\lambda) \otimes \cdots \otimes V(w_m\lambda)$. It follows from the PRV-theorem that this module contains a submodule with extremal weight

$$w_1\lambda + w_2\lambda + \cdots + w_m\lambda = (w_1 + w_2 + \cdots + w_m)\lambda = 0,$$

therefore $(V(\lambda)^{\otimes m})^G \neq \{0\}$ and $M(G) \supseteq m\mathbb{N}$. 

The author would like to thank I. V. Arzhantsev for stating the problem and for many helpful discussions. The idea of applying the PRV-theorem to the calculation of semigroups $M(G)$ is due to D. A. Timashev. The author would also like to thank V. L. Popov for his comments.
The above lemma proves one of the implications of Theorem 2. The other implication, namely the existence of balanced collections of \( m \) elements with \( m \in M(G) \) will be derived from the proof of Theorem 2.

The following statement is sufficient to prove in most of the cases that a given number \( m \) does not belong to \( M(G) \).

**Lemma 2.** Let \( Z(G) \) be the centre of \( G \) and let \( H \subseteq Z(G) \) be a cyclic subgroup of order \( m \). Then \( M(G) \subseteq m\mathbb{N} \).

**Proof.** The group \( G \) has a faithful irreducible representation, therefore there exists a simple \( G \)-module \( U \) such that \( H \) is faithfully represented in \( U \). Since the actions of \( G \) and \( H \) on \( U \) commute and \( U \) is simple with respect to \( G \), the group \( H \) acts on \( U \) by multiplications by powers of an \( m \)th root of unity. Faithfulness of a representation of \( H \) implies that one of its generators \( x_0 \) acts by multiplication by an \( m \)th root of unity; denote this root \( \varepsilon \). In every tensor power \( U \otimes k \) the generator \( x_0 \) acts by multiplication by \( \varepsilon^k \). Therefore if \( k \) is not divisible by \( m \) then \( H \) acts in \( U \otimes k \) by nontrivial multiplications and \( U \otimes k \) has no \( G \)-invariant elements. It implies that \( M(G) \subseteq m\mathbb{N} \).

While proving Theorem 1 we will construct balanced collections in Weyl groups. Their construction in cases of Weyl groups of types \( F_4 \), \( E_6 \) and \( E_8 \) relies heavily on properties of Coxeter elements of these Weyl groups. Let us recall the definition of Coxeter element. Let \( \mathcal{W} \) be the Weyl group corresponding to an irreducible essential root system \( \Phi \). The product of reflections corresponding to all simple roots in \( \Phi \) is called a Coxeter element of \( \mathcal{W} \). This definition depends on ordering of simple reflections, but all elements obtained in such a way are conjugate in \( \mathcal{W} \); therefore they all have the same order and the same eigenvalues. Later on by Coxeter element we mean any Coxeter element of \( \mathcal{W} \).

**Theorem 7** ([9], Proposition 3.18 and Theorem 3.19). Let \( \Phi \) be an irreducible essential root system. Then

1) the order of a Coxeter element of \( \Phi \) is \( h = |\Phi|/\text{rk } \Phi \);

2) if \( \exp(2\pi im_1/h), \ldots, \exp(2\pi im_r/h) \) are all eigenvalues of a Coxeter element \( (r = \text{rk } \Phi, 0 \leq m_i < h) \), then order \( |\mathcal{W}| \) of the Weyl group of \( \Phi \) is \( \prod_i(m_i+1) \).

The numbers \( m_i \) are called exponents of the Weyl group \( \mathcal{W} \). In cases that we consider the exponents can be calculated by applying the following statement.

**Lemma 3** (see [9], Proposition 3.20). Let \( \Phi \) be an irreducible essential root system and let \( h \) be the order of a Coxeter element of \( \Phi \) and let \( m \) be any natural number that is not greater than \( h \). Suppose additionally that \( m \) and \( h \) are coprime. Then \( m \) is one of the exponents of the Weyl group corresponding to \( \Phi \).

In many cases the following statement can be used to prove that specific powers of Coxeter elements make up a balanced collection.

**Lemma 4.** Let \( w \in \mathcal{W} \) be an element of order 3 such that 1 is not an eigenvalue of \( w \). Then the elements \( w, w^2, w^3 \) are a balanced collection and \( M(G) \supseteq 3\mathbb{N} \).

**Proof.** Note that for every \( x \in \mathbb{R}^{\text{rk } G} \) the vector \((\text{Id} + w + w^2)x\) is \( w \)-invariant, hence zero. Thus, \( \text{Id} + w + w^2 = 0 \) and by Lemma 1, \( M(G) \supseteq 3\mathbb{N} \).
2.2. Calculation of $M(G)$ for simple groups $G$.

Proof of theorem 1. Case $G = \text{SL}_n$. Let $e_i$ be the vectors of the standard basis of $\mathbb{R}^n$. Simple roots of the system $A_{n-1}$ are the vectors $e_1 - e_2$, $e_2 - e_3$, $\ldots$, $e_{n-1} - e_n$; the Weyl group of $A_{n-1}$ is the symmetric group $S_n$ and it acts in $\mathbb{R}^n$ by permuting the coordinates. Denote by $\varepsilon \in \mathcal{W}$ the cyclic permutation $(1 \ 2 \ 3 \ \ldots \ n)$. We have

$$
\varepsilon + \varepsilon^2 + \cdots + \varepsilon^n = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix},
$$

(the above sum is considered as a sum in $\text{End}(\mathbb{R}^n)$).

The restriction of this operator to the span of simple roots is zero. Indeed, the span of simple roots is the subspace $\{x_1 + x_2 + \cdots + x_n = 0\}$ and all such vectors are taken to zero by $\varepsilon + \varepsilon^2 + \cdots + \varepsilon^n$. By Lemma 1, $M(\text{SL}_n) \supseteq n\mathbb{N}$. The centre of $\text{SL}_n$ is isomorphic to the group of $n$th roots of unity, thus by applying Lemma 2 we get the reverse inclusion $M(\text{SL}_n) \subseteq n\mathbb{N}$.

It is useful to remark that the cyclic permutation $\varepsilon$ is a Coxeter element of the root system $A_{n-1}$.

Case $G = \text{Spin}_{2n+1}$ or $G = \text{Sp}_{2n}$. In these two cases the Weyl group is $S_n \wr \{\pm 1\}^n$ and it acts in $\mathbb{R}^n$ by permuting the coordinates and changing signs of coordinates. This means that $- \text{Id} \in \mathcal{W}$ and by Lemma 1 we have $M(\text{Spin}_{2n+1}), M(\text{Sp}_{2n}) \supseteq 2\mathbb{N}$. Both $\text{Spin}_{2n+1}$ and $\text{Sp}_{2n}$ have centres isomorphic to $\mathbb{Z}_2 \simeq \mathbb{Z}/2\mathbb{Z}$ [4], Table 3; by Lemma 2 we get the reverse inclusion $M(\text{Spin}_{2n+1}), M(\text{Sp}_{2n}) \subseteq 2\mathbb{N}$.

Case $G = \text{Spin}_{2n}$. The Weyl group of type $D_n$ is isomorphic to $S_n \wr \{\pm 1\}^{n-1}$ and it acts in $\mathbb{R}^n$ by permuting the coordinates and changing signs of the coordinates in an even number of positions. It is necessary to consider two subcases.

If $n$ is even then $-\text{Id}$ is in $\mathcal{W}$ and the centre $Z(\text{Spin}_{2n})$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This yields $M(\text{Spin}_{2n}) = 2\mathbb{N}$.

Now suppose that $n$ is odd. In this case $Z(\text{Spin}_{2n}) \cong \mathbb{Z}_4$, and we have $M(\text{Spin}_{2n}) \subseteq 4\mathbb{N}$. Consider the following four elements of $\mathcal{W}$:

$$
w_1 = \text{diag}(1, -1, -1, -1, \ldots, -1),
$$

$$
w_2 = \text{diag}(-1, 1, -1, -1, \ldots, -1),
$$

$$
w_3 = \text{diag}(-1, -1, 1, 1, \ldots, 1),
$$

$$
w_4 = \text{diag}(1, 1, 1, 1, \ldots, 1).
$$

These elements add up to zero, hence $M(\text{Spin}_{2n}) \supseteq 4\mathbb{N}$.

Case $G = G_2$. In this case the Weyl group is the dihedral group of order 12. We have $-\text{Id} \in \mathcal{W}$ and $M(G_2) \supseteq 2\mathbb{N}$. Let $\varepsilon \in \mathcal{W}$ be a rotation by $2\pi/3$. We have $\text{Id} + \varepsilon + \varepsilon^2 = 0$ and $M(G_2) \supseteq 3\mathbb{N}$. As a result we get $M(G_2) = \{n \in \mathbb{N} \mid n \geq 2\}$.

Case $G = F_4$. The Weyl group of type $F_4$ contains the element $-\text{Id}$ [4], Table 1, hence $M(F_4) \supseteq 2\mathbb{N}$. Let $\varepsilon$ be a Coxeter element of $\mathcal{W}$. According to Theorem 7 the element $\varepsilon$ has order 12 and, according to Lemma 3, it has 1, 5, 7, 11 for exponents. As a result, the element $\varepsilon^4$ has no real eigenvalues. Applying Lemma 4 to $\varepsilon^4$ we obtain the inclusion $M(G_2) \supseteq 3\mathbb{N}$. As we can see, $M(F_4) = \{n \in \mathbb{N} \mid n \geq 2\}$. 

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Case $G = E_6$. Consider a Coxeter element $\varepsilon$. It has order 12. Unlike the previous case Lemma 3 yields only four exponents: 1, 5, 7 and 11. Eigenvalues of Coxeter elements come in pairs $\lambda$ and $\bar{\lambda}$, therefore the remaining two exponents are $m$ and $12 - m$. Theorem 7 states that order of the Weyl group $|W| = 2^7 \cdot 3^4 \cdot 5$ coincides with the product $2 \cdot 12 \cdot 6 \cdot 8 \cdot (m + 1) \cdot (12 - m + 1)$. From this equality we find the remaining exponents. They are 4 and 8. Thus the element $\varepsilon^4$ has no real eigenvalues and Lemma 4 yields the inclusion $M(E_6) \supseteq 3\mathbb{N}$. Since $Z(E_6) \cong \mathbb{Z}_3$, we get $M(E_6) = 3\mathbb{N}$.

Case $G = E_7$. The Weyl group of type $E_7$ contains the mapping $-\text{Id}$, therefore $M(E_7) \supseteq 2\mathbb{N}$. Since $Z(E_7) \cong \mathbb{Z}_2$, we get $M(E_7) = 2\mathbb{N}$.

Case $G = E_8$. The Weyl group of type $E_8$ contains the mapping $-\text{Id}$, therefore $M(E_8) \supseteq 2\mathbb{N}$. Consider a Coxeter element $\varepsilon$. Its order is 30 and Lemma 3 yields its eight exponents which are coprime with 30. As a result, the element $\varepsilon^{10}$ has no real eigenvalues and Lemma 4 is applicable to it. As we can see, $M(E_8) \supseteq 3\mathbb{N}$, therefore $M(E_8) = \{ n \in \mathbb{N} \mid n \geq 2 \}$.

Proof of Theorem 2. In view of Lemma 1 it remains to prove that if $m \in M(G)$ then there exists a balanced collection containing $m$ elements. Such balanced collections have been constructed in the above proof.

2.3. Calculation of $M(G)$ for an arbitrary group $G$. Let us first prove Proposition 1 which asserts that semigroups $M(G)$ need to be calculated only for reductive groups.

Proof of Proposition 1. The inclusion $M(G) \subseteq M(H)$ is obvious. Indeed, let $\pi : G \to H$ be the natural map. Every $H$-module $V$ can be considered as a $G$-module with multiplication $g \cdot x = \pi(g)x$ and we have $(V^\otimes m)^H = (V^\otimes m)^G$.

Take $m \in M(H)$ and a $G$-module $V$. Its submodule $W = V^F$ is nonzero. The unipotent radical $F$ is a normal subgroup in $G$ thus the action $G : W$ gives rise to the action $H : W$; since $m \in M(H)$, we have $(W^\otimes m)^H \neq \{0\}$. So, we have $(V^\otimes m)^G \supseteq (W^\otimes m)^G = (W^\otimes m)^{G/F} \neq \{0\}$, and $M(G) \supseteq M(H)$.

It has already been remarked that a reductive group $G$ with nontrivial central torus has empty semigroup $M(G)$. It suffices therefore to calculate $M(G)$ for semisimple groups $G$. If $G$ is semisimple and simply connected then it is a product of several simply connected simple groups and Proposition 2 yields $M(G)$.

Proof of Proposition 2. Take $m \in M(G_1 \times G_2)$. Let $V$ and $W$ be arbitrary modules over $G_1$ and $G_2$ respectively. Each of them can be considered as a module over $G_1 \times G_2$ with trivial action of one of the factors. By the choice of $m$ we have $(V^\otimes m)^{G_1} = (V^\otimes m)^{G_1 \times G_2} \neq \{0\}$ and $(W^\otimes m)^{G_2} = (W^\otimes m)^{G_1 \times G_2} \neq \{0\}$. Thus, $m \in M(G_1)$ and $m \in M(G_2)$ and we obtain the inclusion

$$M(G_1 \times G_2) \subseteq M(G_1) \cap M(G_2).$$

Conversely, take $m \in M(G_1) \cap M(G_2)$ and a simple $G_1 \times G_2$-module $U$. Both groups $G_1$ and $G_2$ are reductive, hence $U = V \otimes W$ for appropriate simple modules $V$ and $W$ over $G_1$ and $G_2$ respectively. We have $U^\otimes m \cong V^\otimes m \otimes W^\otimes m$. In view of the choice of $m$ we have $(V^\otimes m)^{G_1} \neq \{0\}$ and $(W^\otimes m)^{G_2} \neq \{0\}$. As a result, $(V^\otimes m \otimes W^\otimes m)^{G_1 \times G_2} \neq \{0\}$. It proves that $m \in M(G_1 \times G_2)$. 


§ 3. Relation between stability indices and $m(G)$

3.1. Auxiliary facts about HV-varieties. In order to prove Theorem 3 we need to give examples of actions $G : X$ such that diagonal actions $G : X^{m(G)−1}$ are not stable. Necessary examples are given by actions on so-called HV-varieties. All facts that we need about these varieties can be found in [10] and [11].

Let $\lambda$ be a dominant weight of $G$ and let $v_\lambda$ be the highest weight vector in $V(\lambda)$. Consider the $G$-orbit of $v_\lambda$. Its closure is denoted $X(\lambda)$ and called a HV-variety corresponding to the dominant weight $\lambda$ [10], Definition 1.

Theorem 8 ([10], Theorem 1). Let $\lambda$ be a dominant weight of $G$ and $v_\lambda$ be the highest weight vector in $V(\lambda)$. Then $X(\lambda) = G \cdot v_\lambda \cup \{0\}$.

A collection $(\lambda_1, \ldots, \lambda_s)$ of dominant weights of $G$ is said to be invariant-free [11], Definition 2 if $(V(n_1\lambda_1) \otimes \cdots \otimes V(n_s\lambda_s))^G = \{0\}$ for every tuple of natural numbers $n_1, \ldots, n_s$.

Theorem 9 ([11], Theorem 10). Let $(\lambda_1, \ldots, \lambda_s)$ be a collection of dominant weights of $G$. The following properties are equivalent:

- the collection $(\lambda_1, \ldots, \lambda_s)$ is invariant-free;
- the closure of every $G$-orbit in $X(\lambda_1) \times \cdots \times X(\lambda_s)$ contains $(0, \ldots, 0) \in V(\lambda_1) \oplus \cdots \oplus V(\lambda_s)$;
- $k[X(\lambda_1) \times \cdots \times X(\lambda_s)]^G = k$.

3.2. Auxiliary facts about tensor products of Spin$_{2r}$-modules. In order to prove Theorem 3 for $G = \text{Spin}_{4n+2}$ we need to find explicitly the decomposition of a certain tensor product. To this end we employ the generalized Littlewood-Richardson rule. Necessary facts about this generalization can be found in [12].

Definition 4 ([12], Appendix A.3). Let $\varpi_i$, $1 \leq i \leq r$ be the fundamental weights of Spin$_{2r}$ and let $\lambda = \sum_{i=1}^r a_i \varpi_i$, $a_i \geq 0$, be a dominant weight. A Young diagram of shape $\lambda$ is a Young diagram corresponding to the partition $(c_1, \ldots, c_r)$ with $c_p$ defined as:

$$c_p = \begin{cases} 2 \sum_{i=p}^{r-2} a_i + a_{r-1} + a_r & \text{if } p \leq r - 2, \\ a_{r-1} + a_r & \text{if } p = r - 1, \\ a_r & \text{if } p = r. \end{cases}$$

Remark 1. We treat the numbers $c_p$ as lengths of rows ($c_1$ being the length of the bottom row) and draw the rows left-aligned and from bottom to top.

Definition 5 ([12], Appendix A.4). Let $T$ be a Young diagram of shape $a\varpi_{2r}$ and suppose that cells of $T$ are filled with natural numbers. The diagram $T$ with filled cells is said to be a Spin$_{2r}$-standard Young tableau if it satisfies the following requirements:

- all cells of $T$ contain natural numbers that are not greater than $2r$,
- entries in rows are strictly ascending (the rows are oriented left-to-right),
- entries in columns are ascending (the columns are oriented bottom-to-top),
- no row contains $i$ and $2r + 1 - i$ simultaneously,
- every row has an even number of entries that are greater than $r$. 


Remark 2. The definition of standard Young tableau $T$ of arbitrary shape $\mu$ is more involved and imposes more constraints on entries of $T$. We will not provide this definition in full detail for the additional constraints are automatically satisfied in the case that we consider. An interested reader is encouraged to consult [12], Appendix A.4 and see the definition in its full generality.

Definition 6 ([12], Appendix A.4). Let $T$ be a standard Young tableau. Denote by $C_T(i)$ the number of entries of $T$ that are equal to $i$. Define the weight of the tableau $T$ as

$$v(T) = \frac{1}{2} [(C_T(1) - C_T(2r))\varepsilon_1 + (C_T(2) - C_T(2r - 1))\varepsilon_2 + \cdots].$$

Denote by $v_m(T)$ the weight of tableau $T_m$ obtained from $T$ by removing all rows below the $m$th.

Definition 7 ([12], Appendix A.4). Let $\mu$ be a dominant weight. A standard Young tableau $T$ is called $\mu$-dominant if the weights $2\mu + 2v_m(T)$ are dominant for every $m$.

Theorem 10 ([12], Appendix A.4). Let $\lambda$ and $\mu$ be dominant weights of $\text{Spin}_{2r}$. Then

$$V(\lambda) \otimes V(\mu) = \bigoplus_T V(\lambda + v(T)),$$

the sum on the right-hand side runs over all $\lambda$-dominant standard Young tableaux of shape $\mu$.

3.3. An example of an invariant-free triple of weights of the group $\text{Spin}_{4n+2}$. In [11] it has been proved that the collection $(\varpi_{2n+1}, \varpi_{2n+1}, \varpi_{2n+1})$ of weights of $\text{Spin}_{4n+2}$ is primitive, that is

$$\dim(V(n_1 \varpi_{2n+1}) \otimes V(n_2 \varpi_{2n+1}) \otimes V(n_3 \varpi_{2n+1}))^{\text{Spin}_{4n+2}} \leq 1$$

for all natural numbers $n_1, n_2, n_3$. We need more accurate information about this collection. Precisely, we need to prove that it is invariant-free.

Lemma 5. Let $p$ and $q$ be two natural numbers such that $p \geq q$. Then we have the following decomposition:

$$V(p \varpi_{2n+1}) \otimes V(q \varpi_{2n+1}) = \bigoplus V((p + q - 2r)\varpi_{2n+1} + \varpi_{i_1} + \cdots + \varpi_{i_r}),$$

the sum on the right-hand side runs over all $r$ in $0, \ldots, q$ and over all collections of odd natural numbers $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq 2n - 1$.

Proof. A standard Young tableau $T$ of shape $r \varpi_{2n+1}$ is a rectangle with $2n + 1$ columns and $r$ rows. Since a row of $T$ has $2n + 1$ entries, it is uniquely defined by those of its entries that are not greater than $2n + 1$. Let $I = \{i_1 < i_2 < \cdots < i_p\}$ and $J = \{j_1 < \cdots < j_s\}$ be two sets of natural numbers such that $I \cup J = \{1, \ldots, 2n + 1\}$. If a row of $T$ starts with $I$ then the remaining numbers are necessarily $4n + 3 - j_s, \ldots, 4n + 3 - j_1$. The weight of such a row is $(\sum_{i=1}^{2n+1} a_i \varepsilon_i)/2$ with $a_i = +1$ if $i \in I$ and $a_i = -1$ if $i \in J$. 
Denote for brevity \( i' = (4n + 3 - i) \). In what follows we say that elements of \( J \) are removed from the interval \( 1, \ldots, (2n + 1) \) and a row of tableau \( T \) that corresponds to \( I \) and \( J \) (that is, one that starts with \( I \)) is said to be obtained from the interval \( 1, \ldots, (2n + 1) \) by removing elements of \( J \).

Let us describe all \( \varpi_{2n+1} \)-dominant Young tableaux \( T \). First consider two adjacent rows of \( T \). Let \( p \) and \( q \) be the smallest numbers removed from the top and bottom line respectively. Then these rows end with numbers \( p' \) and \( q' \) respectively. Tableau \( T \) is standard, hence \( p' \geq q' \), and \( p \leq q \). Now let us show that every row of \( T \) is obtained by removing trailing numbers from the interval \( 1, \ldots, (2n + 1) \).

Combined with the previous statement, it shows that \( T \) looks like the tableau below (\( k_1 \leq k_2 \leq \cdots \leq k_r \)):

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
1 & \ldots & k_1 & (2n + 1)' & \ldots & (k_1 + 1)' \\
1 & \ldots & k_2 & (2n + 1)' & \ldots & (k_2 + 1)' \\
1 & \ldots & k_3 & (2n + 1)' & \ldots & (k_3 + 1)' \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \ldots & & & & 2n + 1
\end{array}
\]

Note that all numbers \( k_i \) are odd because every row has an even number of entries that are greater than \( 2n + 1 \).

Consider the topmost row of the tableau \( T \). If it is obtained from \( 1 \ldots 2n + 1 \) by removing any set other than a trailing interval \( s \ldots 2n + 1 \) then there are two numbers \( 1 \leq x < y \leq 2n + 1 \) such that \( x \) is removed while \( y \) is not. It implies that the weight \( 2p\varpi_{2n+1} + 2v_1(T) \) equals \( (p + 1)\varepsilon_1 + \cdots + (p - 1)\varepsilon_x + \cdots + (p + 1)\varepsilon_y + \cdots \). Since the coefficient of \( \varepsilon_x \) is smaller than the coefficient of \( \varepsilon_y \), the weight \( 2p\varpi_{2n+1} + 2v_1(T) \) is not dominant. This contradicts the assumption of \( p\varpi_{2n+1} \)-dominance of the tableau \( T \). Therefore the topmost row of \( T \) is obtained from \( 1 \ldots 2n + 1 \) by removing some trailing part of this interval.

Now we proceed by induction. Suppose that the top \( l \) rows of the tableau \( T \) are obtained by removing trailing intervals. Let \( k_1 + 1, \ldots, k_l + 1 \) be the smallest numbers removed from the top rows of \( T \). We assume inductively that we have inequalities \( k_1 \leq \cdots \leq k_l \). For such a table we have

\[
2p\varpi_{2n+1} + 2v_1(T) = 2(p + q - 2l)\varpi_{2n+1} + 2\varpi_{k_1 + \cdots + \varpi_{k_l}}.
\]

Without loss of generality we may assume that \( k_l < 2n + 1 \). Let us apply to the \((l+1)\)-st row the same argument that we have applied to the topmost row of \( T \). The smallest number removed from the \((l + 1)\)th row is not smaller than the smallest number removed from the \( l \)th row. Therefore the numbers \( x \) and \( y \) yielded by the argument will be greater than \( k_l \). The fundamental weights \( \varpi_{k_i} \) are sums of \( \varepsilon_i \) with \( i \leq k_l < x \), hence they do not influence the coefficients of \( \varepsilon_x \) and \( \varepsilon_y \). From this fact it follows that the reasoning based on the comparison of the coefficients of \( \varepsilon_x \) and of \( \varepsilon_y \) stays valid and proves that the weight \( 2p\varpi_{2n+1} + 2v_{l+1}(T) \) is not dominant.

As we can see, every standard \( p\varpi_{2n+1} \)-dominant Young tableau looks like one on the picture above. From the proof it follows that, conversely, every Young tableau depicted above is \( p\varpi_{2n+1} \)-dominant if \( q \leq r \). Indeed, such a tableau is standard and all partial weights \( 2p\varpi_{2n+1} + 2v_l(T) \) are dominant because \( p + q - 2l \geq p - q \geq 0 \).

The weight of the tableau \( T \) depicted above is \( (p + q - 2s) + \varpi_{k_1} + \cdots + \varpi_{k_s} \), with \( s \) being the number of rows that have some of the numbers removed and
(k_i + 1) being the smallest number removed in the ith row. This means that all simple modules contained in the decomposition of \( V(p \varpi_{2n+1}) \otimes V(q \varpi_{2n+1}) \) equal \( V((p + q - 2s) \varpi_{2n+1} + \varpi_{k_1} + \cdots + \varpi_{k_s}) \) for an appropriate collection of \( k_i \).

To complete the demonstration we have to show that every weight

\[
\lambda = (p + q - 2l) \varpi_{2n+1} + \varpi_{k_1} + \cdots + \varpi_{k_l},
\]

with \( k_i \) being odd natural numbers not greater than \( 2n - 1 \), can be obtained as \( p \varpi_{2n+1} + v(T) \) for an appropriately chosen \( p \varpi_{2n+1} \)-dominant Young tableau \( T \).

Without loss of generality we may assume that \( k_1 \leq \cdots \leq k_l \). Fill the topmost row of \( T \) in the following way: write the numbers \( 1, \ldots, k_1 \) into \( k_1 \) leading cells and pad them with \( (2n + 1)' \), \( \ldots, (k_1 + 1)' \); the next \( l - 1 \) rows are filled analogously and the last \( q - l \) rows are filled with \( 1, \ldots, 2n + 1 \). It is clear that for the tableau \( T \) constructed by this process we have \( \lambda = p \varpi_{2n+1} + v(T) \). Applying the rule of Littlewood-Richardson we get that \( V(\lambda) \) is indeed a submodule of \( V(p \varpi_{2n+1}) \otimes V(q \varpi_{2n+1}) \). It is obvious that \( \lambda \) can uniquely be represented as \( p \varpi_{2n+1} + v(T) \), hence \( V(\lambda) \) is contained in the tensor product with multiplicity one.

For brevity we will employ the multi-index notation. Let \( I = (i_1, \ldots, i_s) \) be a multi-index with all components \( i_j \) being odd natural numbers not greater than \( 2n - 1 \). Denote by \( |I| \) the number of components of \( I \) and define \( \varpi_I \) as the sum \( \sum_{i \in I} \varpi_i \). Using this notation one can rewrite the decomposition of \( V(p \varpi_{2n+1}) \otimes V(q \varpi_{2n+1}) \) in this way:

\[
\bigoplus V((p + q - 2|I|) \varpi_{2n+1} + \varpi_I).
\]

**Lemma 6.** The triple \((\varpi_{2n+1}, \varpi_{2n+1}, \varpi_{2n+1})\) is invariant-free.

**Proof.** Take three natural numbers \( p \geq q \geq r \). According to the previous lemma we have

\[
V(p \varpi_{2n+1}) \otimes V(q \varpi_{2n+1}) \otimes V(r \varpi_{2n+1})
= \bigoplus_I \left[ V((p + q - 2|I|) \varpi_{2n+1} + \varpi_I) \otimes V(r \varpi_{2n+1}) \right].
\]

Let us show that no module in the right-hand side contains \( G \)-invariant elements. The tensor product \( V((p + q - 2|I|) \varpi_{2n+1} + \varpi_I) \otimes V(r \varpi_{2n+1}) \) decomposes into a direct sum of modules with highest weights \( (p + q - 2|I|) \varpi_{2n+1} + \varpi_I + v(T) \) for appropriately chosen Young tableaux \( T \) of shape \( r \varpi_{2n+1} \). Let us show that \(-(p + q - 2s) \varpi_{2n+1} + \varpi_i + \cdots + \varpi_j \) cannot be equal to the weight of any standard tableau \( T \). To this end, fix an arbitrary standard Young tableau \( T \) of shape \( r \varpi_{2n+1} \), that is, a rectangle with \( (2n + 1) \) columns and \( r \) rows. The tableau \( T \) is standard and therefore its bottom \( t \) rows start with 1 and the other \( r - t \) rows start with numbers that are greater than 1, hence \( T \) has weight \( v(T) = 1/2((t - (r - t))\varepsilon_1 + \cdots) \). Therefore we have

\[
(p + q - 2|I|) \varpi_{2n+1} + \varpi_I + v(T) = \left(\frac{p + q - r}{2} + t\right)\varepsilon_1 + \cdots.
\]

If \( (p + q - 2|I|) \varpi_{2n+1} + \varpi_I + v(T) = 0 \), then \( (p + q - r)/2 + t = 0 \). The last equality is absurd because \( (p + q - r)/2 \geq p/2 > 0 \).
3.4. An example of a diagonal unstable action of the group $E_6$ on two copies of variety $X$.

Lemma 7. Let $G$ be a semisimple algebraic group, let $\mathfrak{g}$ be the Lie algebra of $G$ and $T$ a maximal torus in $G$. Let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the weight decomposition of $\mathfrak{g}$ with respect to $T$. Finally, let $V$ be a module over $G$ and $v \in V$ a weight vector with respect to $T$. Then the Lie algebra $\mathfrak{h}$ of the stabilizer of $v$ is regular, that is, it equals $\mathfrak{h} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$, with $\mathfrak{h}_0 \subseteq \mathfrak{t}$ and $\Gamma \subset \Delta$.

Proof. Consider an arbitrary element $\xi \in \mathfrak{h}$. Let $\xi = \sum_{\mu \in \Delta} \xi_\mu$ be its weight decomposition. Since $0 = \xi \cdot v = \sum_{\mu \in \Delta} \xi_\mu \cdot v$ and every summand is a weight vector, we conclude that every summand is zero, that is, $\xi_\mu \in \mathfrak{h}$ for all $\mu$. This proves the regularity of $\mathfrak{h}$.

Lemma 8. The action of $E_6$ on $X(\varpi_1) \times X(\varpi_1)$ has a dense orbit.

Proof. Let us start by calculating the dimension of $X(\varpi_1)$. Denote by $P(\varpi_1)$ the set of weights that occur in the module $V(\varpi_1)$. The module $V(\varpi_1)$ has the following property: if $\xi \in \text{Lie} E_6$ and $v \in V(\varpi_1)$ are nonzero weight elements with weights $\mu$ and $\nu$ such that $\mu + \nu \in P(\varpi_1)$, then $\xi \cdot v \neq 0$. In view of the regularity of the stabilizer of $v$, we conclude that this stabilizer is the direct sum of a subspace in $\mathfrak{t}$ of codimension 1 and weight spaces $\mathfrak{g}_\alpha$, with $\varpi_1 + \alpha \notin P(\varpi_1)$. This reasoning shows that the dimension of $X(\varpi_1)$ equals 1 plus the number of roots $\alpha$ of Lie $E_6$ such that $\varpi_1 + \alpha \in P(\varpi_1)$. Using this fact one easily finds that $\dim X(\varpi_1) = 17$.

Note that $X(\varpi_1)$ has a vector of weight $\varepsilon_6 - \varepsilon$. Indeed, the Weyl group of $E_6$ contains all permutations of $\varepsilon_i$ and the mapping $\varepsilon_i \mapsto \varepsilon_i$, $\varepsilon \mapsto -\varepsilon$; these mappings can be used to obtain the necessary vector from the highest weight vector of $V(\varpi_1)$ by applying an appropriate element of the Weyl group.

Consider a point of $X(\varpi_1) \times X(\varpi_1)$ that has a vector $v$ of weight $\varpi_1 = \varepsilon_1 + \varepsilon$ as its first component and a vector $w$ of weight $\varepsilon_6 - \varepsilon$ as its second component. One can easily find the stabilizer of this point using the argument employed for the calculation of $\dim X(\varpi_1)$. This argument shows that the dimension of the orbit of $(v, w)$ is 34. Thus, the dimension of this orbit coincides with $\dim X(\varpi_1) \times X(\varpi_1)$. Therefore the orbit of $(v, w)$ is dense.

Remark 3. It is clear that the action described in the above lemma is not stable for it is not transitive. Indeed, all points in the described orbit have both components nonzero, so the point $(0, 0) \in X(\varpi_1) \times X(\varpi_1)$ is not contained in the orbit of $(v, w)$.

3.5. Proof of Theorem 3. Case $G = \text{SL}_n$. If $k < n$ then the action $\text{SL}_n : (k^n)^k$ is not transitive and has a dense orbit, hence $s_m(\text{SL}_n) \geq n = m(\text{SL}_n)$.

Case when $G$ is one of $\text{Spin}_{2n+1}$, $\text{Spin}_{4n+4}$, $\text{Sp}_{2n}$, $G_2$, $F_4$, $E_7$, $E_8$. In all these cases the statement of the theorem is trivial because these groups have $m(G) = 2$ and every action on an HV-variety is not stable.

Case $G = \text{Spin}_{4n+2}$. The triple of dominant weights $(\varpi_{2n+1}, \varpi_{2n+1}, \varpi_{2n+1})$ is invariant-free according to Lemma 6. In view of Theorem 9 the action $\text{Spin}_{4n+2} : X^3$ with $X = X(\varpi_{2n+1})$ is not stable, therefore $s_m(\text{Spin}_{4n+2}) \geq 4 = m(\text{Spin}_{4n+2})$.

Case $G = E_6$. Lemma 8 gives an example of a diagonal action $E_6 : X^2$ that is not stable. Therefore $s_m(E_6) \geq 3 = m(E_6)$. 

3.6. More on bounds of stability indices. Theorem 3 gives a lower bound of \( s_m(G) \) and \( s(G) \). The lower bound of \( s(G) \) obtained in this way is not exact, that is, there are groups \( G \) that have \( s(G) > m(G) \). One example of such a group is the symplectic group \( \text{Sp}_{2m} \) because its standard representation requires passing to the diagonal action with \( 2m \) copies in order to stabilize. On the other hand, Theorem 5 shows that the lower bound of \( s_m(G) \) is in many cases exact.

Proof of Theorem 5. Let \( \theta \) be the Weyl involution of \( G \), that is, an involutive automorphism of \( G \) that acts as inversion on some maximal torus in \( G \). It is well-known that for every linear representation \( R \) of the group \( G \) its \( \theta \)-twisting \( R \circ \theta \) is isomorphic to the conjugate representation \( R^* \). For an affine irreducible \( G \)-variety \( X \) set \( X^{\theta} \) to be \( X \) with \( \theta \)-twisted action of \( G \).

Every affine \( G \)-variety \( X \) admits an equivariant closed immersion into an appropriate \( G \)-module \( V \). Since all linear representations of \( G \) are self-conjugate we have an equivariant isomorphism \( \varphi : V \to V^\theta \) which can be used to construct an isomorphism of \( X \times X \) with \( X \times X^\theta \). From this fact it follows that the stability of the action of \( G \) on \( X \times X \) is equivalent to the stability of the action of \( G \) on \( X \times X^\theta \). The latter action is stable in view of [6], Proposition 1.6 so we have \( s_m(G) = 2 \).

To complete the proof we need to show that \( m(G) = 2 \) for any semisimple group \( G \) that has only self-conjugate linear representations. This is obvious because for every \( G \)-module \( V \) we have \( (V \otimes 2)^G \cong (V \otimes V^*)^G \cong (\text{End}(V))^G \) and \( \text{End}(V) \) contains a nonzero \( G \)-invariant element, for example, the identity map \( \text{Id}_V \). Thus, \( 2 \in M(G) \) and \( m(G) = 2 \).

Bibliography

[1] V. L. Popov, “On the stability of the action of an algebraic group on an algebraic variety”, Izv. Akad. Nauk SSSR Ser. Mat. 36:2 (1972), 371–385; English transl. in Math. USSR-Izv. 6:2 (1972), 367–379.

[2] I. V. Arzhantsev, “On the stability of diagonal actions”, Mat. Zametki 71:6 (2002), 803–806; English transl. in Math. Notes 71:5–6 (2002), 735–738.

[3] E. B. Vinberg, “On stability of actions of reductive algebraic groups”, Lie algebras, rings and related topics (Tainan, Taiwan 1997), Springer-Verlag, Hong-Kong 2000, pp. 188–202.

[4] A. L. Onishchik and È. B. Vinberg, Lie groups and algebraic groups, Nauka, Moscow 1988; English transl., Springer Ser. Soviet Math., Springer-Verlag, Berlin 1990.

[5] V. L. Popov, “Stability criteria for the action of a semisimple group on a factorial manifold”, Izv. Akad. Nauk SSSR Ser. Mat. 34:3 (1970), 523–531; English transl. in Math. USSR-Izv. 4:3 (1970), 527–535.

[6] D. I. Panyushev, “A restriction theorem and the Poincaré series for \( U \)-invariants”, Math. Ann. 301:1 (1995), 655–675.

[7] S. Kumar, “Proof of the Parthasarathy–Rao–Varadarajan conjecture”, Invent. Math. 93:1 (1998), 117–130.

[8] O. Mathieu, “Construction d’un groupe de Kac-Moody et applications”, Compositio Math. 69:1 (1989), 37–60.

[9] J. E. Humphreys, Reflection groups and Coxeter groups, Cambridge Stud. Adv. Math., vol. 29, Cambridge Univ. Press, Cambridge 1990.
[10] È. B. Vinberg and V. L. Popov, “On a class of quasihomogeneous affine varieties”, Izv. Akad. Nauk SSSR Ser. Mat. 36:4 (1972), 749–764; English transl. in Math. USSR-Izv. 6:4 (1972), 743–758.

[11] V. L. Popov, “Tensor product decompositions and open orbits in multiple flag varieties”, J. Algebra 313:1 (2007), 392–416.

[12] P. Littelmann, “A generalization of the Littlewood–Richardson rule”, J. Algebra 130:2 (1990), 328–368.

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