We consider the harmonic-superspace formalism in the $N = 4$ supersymmetry using the $SU(4)/SU(2) \times SU(2) \times U(1)$ harmonics which was earlier applied to the abelian gauge theory. The $N = 4$ non-abelian constraints in a standard superspace are reformulated as the harmonic-superspace equations for two basic analytic superfields: the independent superfield strength $W$ of a dimension 1 and the dimensionless harmonic gauge 4-prepotential $V$ having the $U(1)$ charge 2. These constraint equations manifestly depend on the Grassmann coordinates $\theta$, although they are covariant under the unusual $N = 4$ supersymmetry transformations. We analyze an alternative harmonic formalism of the supergauge theory for two unconstrained nonabelian analytic superfields $W$ and $V$. The gauge-invariant action $A(W,V)$ in this formalism contains $\theta$ factors in each term, it is invariant under the $SU(4)$ automorphism group. In this model, the interaction of two infinite-dimensional $N = 4$ supermultiplets with the physical and auxiliary fields arises at the level of component fields. The action $A(W,V)$ generate analytic equations of motion II alternative to the harmonic-superspace superfield constraints I. Both sets of equations give us the equivalent equations for the physical component fields of the $N = 4$ gauge supermultiplet, they connect auxiliary and physical fields of two superfields. The nonlinear effective interaction of the abelian harmonic superfield $W$ is constructed.

Keywords: Harmonic superspace, extended supersymmetry, Yang-Mills theory
1 Introduction

The superfield constraints for the $N = 3, 4, D = 4$ gauge theories were considered in the corresponding superspaces with the coordinates $x^m, \theta^a_k, \bar{\theta}^{\dot{a}}_k$ [1], where $k$ describes spinor representations of the $SU(3)$ or $SU(4)$ automorphism groups of these supersymmetries. It was shown that the constraints for superfield strengths yield the equations of motion.

We use the harmonic superspace (HS) approach to the supersymmetric gauge theory and supergravity described in the book [2]. The important achievement of this approach is the off-shell superfield formalism of the $N = 3$ supersymmetric Yang-Mills theory using three conjugated pairs of the $SU(3)$ harmonics $u_1^i, u_2^i, u_3^i$ and $\bar{u}_1^i, \bar{u}_2^i, \bar{u}_3^i$ describing the coset $SU(3)/U(1) \times U(1)$ [3]. The basic Grassmann-analytic gauge superfields of this formalism $V_2^1, V_3^2, V_3^1$ contain an infinite number of the component off-shell fields. The Chern-Simons-type action in the $N = 3$ harmonic superspace corresponds to the zero-curvature equations of motion for the superfield strengths. The HS transform in this theory connects equations of motion in the harmonic superspace with the superfield constraints in the ordinary $N = 3$ superspace. The further development of the $N = 3$ theory was considered in [4]-[8]. The classical equations of motion of the $N = 3$ gauge theory possess the $N = 4$ supersymmetry [7], however, it is not preserved off-shell.

The harmonic superspaces for the $N = 4$ supersymmetry were studied in many papers [9]-[19]. The paper [10] describes the twistor-harmonic superspace based on the bosonic spinor harmonics of the Lorentz group by analogy with the twistor integrability conditions for $N \geq 3$ Yang-Mills theory [11]. It is not clear how to connect this nonstandard superfield description with the off-shell field interactions in the ordinary Minkowski space. As a rule, different harmonic superfield constructions are used for the description of the short abelian $N = 4$ supermultiplet on the mass shell [12, 13, 14, 15]. The harmonic superspace of ref. [17] uses the $USp(4)$ harmonics which do not guarantee the manifest $SU(4)$ symmetry. Constrained short superfields of different harmonic formalisms are used in [18] for a description of effective nonlinear interactions. The action of the $N = 4$ supersymmetric theory with central charges in the harmonic $USp(4)$ formalism was expressed via the constrained superfield strength [19].

The coset space $\mathcal{U}_8 = SU(4)/H, \quad H = SU(2) \times SU(2) \times U(1)$ and corresponding harmonic coordinates $u_k^{a}, \bar{u}_k^a$ are described in Appendix A. We consider in detail the $SU(4)$ invariant harmonic derivatives and irreducible harmonic combinations in $\mathcal{U}_8$ having indices of the $SU(4)$ and $H$ groups. The Appendix B is devoted to the analysis of the $\mathcal{U}_8$ harmonic analytic superspace $\mathcal{H}(4+8|8)$. We analyze conjugation rules for harmonics, spinor coordinates, harmonic and spinor derivatives and study the superconformal transformations in the harmonic superspace. The conventions and formulas from these appendices are widely used in the paper.

The harmonic superspace $\mathcal{H}(4+8|8)$ was applied in [12, 13, 14] to the description of the abelian superfield strength $W^{++}$ in the $N = 4$ gauge theory on the mass shell. We analyze the non-abelian generalization of this construction in sect. 2. Our formalism includes the dimensionless analytic gauge $V_1^{++a}$-superfields (4-prepotential) and independent gauge covariant superfield $W^{++}$ of a dimension 1. The $SU(4)$ self-duality condition for the superfield strength is solved automatically in this harmonic formalism. We formulate the superfield constraints I in the harmonic superspace. The solutions of these constraints connect $W^{++}$ and $V_1^{++a}$ su-
The corresponding gauge covariant equations depend manifestly on the Grassmann coordinates. Nevertheless, these equations are covariant under the unusual $N = 4$ supersymmetry transformations. Expanding the constraint equations I in terms of the field components yields the known field equations of the $N = 4$ Yang-Mills theory, all auxiliary fields vanish or are expressed via the physical fields.

In sect. 3, we consider the alternative harmonic superfield formalism of the nonabelian $N = 4$ gauge theory and construct the action $A$ of independent unconstrained analytic superfields $W^{++}$ and $V^{++}_\hat{a}$ which includes only first harmonic derivatives of the gauge superfields and is invariant under the nonabelian gauge group and the automorphism group $SU(4)$. The action $A$ is an integral on the analytic harmonic superspace, it manifestly depends on the Grassmann coordinates $\theta$ and breaks the $N = 4$ supersymmetry. The equations of motion II of the $A$-model are derived by varying of the action in the independent analytic superfields $W^{++}$ and $V^{++}_\hat{a}$. The superfield equations II are formally equivalent to some combinations of the superfield constraints I from sect. 2 with nilpotent $\theta$ multipliers.

Component fields of the infinite-dimensional $N = 4$ supermultiplets $W^{++}$ and $V^{++}_\hat{a}$ arise in the Grassmann and harmonic expansions of superfields. The component version of the action $A$ contains an infinite number of fields. The equations of motion of the $A$-model connect the physical fields of the $N = 4$ gauge supermultiplet with some set of auxiliary fields, an infinite number of additional auxiliary fields vanish on the mass shell. The $N = 4$ supersymmetry is restored after the exclusion of auxiliary component fields, the corresponding equations for the physical fields are covariant under the $N = 4$ supersymmetry.

Section 4 is devoted to the construction of the $N = 4$ supersymmetric nonlinear interactions of the abelian superfield $W^{++}$ by analogy with the $N = 3$ abelian effective self-interaction [6]. We show that the auxiliary fields of the $W^{++}$ superfield play an important role in the construction of nonlinear effective interactions of the physical $N = 4$ fields which describe the possible quantum corrections in this model.

In sect. 5, we study the manifestly supersymmetric interaction $S$ of the gauge prepotential $V^{++}_\hat{a}$ in the harmonic superspace and try to use $S$ as the action of some gauge model. We find inconsistencies in this $S$-model which contains interactions of the standard $N = 4$ gauge supermultiplet with additional scalar, vector, tensor and spinor fields without necessary additional gauge symmetries.

## 2 On-shell harmonic superfields in $N = 4$ gauge theory

### 2.1 $N = 4$ superfield constraints in the central basis

The superfield constraints of the $N = 4$ Yang-Mills theory are described by the following equations [1]:

\[
\begin{align*}
\{\nabla^k_{\alpha}, \nabla^j_\beta\} &= \varepsilon_{\alpha\beta} W^{kj}, & \{\nabla_{k\hat{\alpha}}, \nabla_{j\hat{\beta}}\} &= \varepsilon_{\hat{\alpha}\hat{\beta}} \bar{W}_{kj}, \\
\{\nabla^k_{\alpha}, \nabla_{j\hat{\beta}}\} &= -2i \delta^k_j \nabla_{\alpha\hat{\beta}},
\end{align*}
\] (2.1)
where $\nabla$ are the spinor and vector covariant derivatives in the the central basis (CB)

$$
\nabla^k_\alpha = D^k_\alpha + A^k_\alpha(z), \quad \nabla_{k\dot{a}} = \bar{D}_{k\dot{a}} + \bar{A}_{k\dot{a}}(z),
$$

$$
\nabla_{\alpha\dot{a}} = \partial_{\alpha\dot{a}} + A_{\alpha\dot{a}}(z)
$$

and the $N = 4$ spinor derivatives $D^k_\alpha$ and $\bar{D}_{k\dot{a}}$ are defined in appendix B. We use the superfield strength $W^{kj}$

$$
W^{kj} = \frac{1}{2} \varepsilon^{\beta\alpha}(D^k_\alpha A^j_\beta + D^j_\beta A^k_\alpha + \{A^k_\alpha, A^j_\beta\})
$$

and the conjugated CB superfield strength $\bar{W}_{kj}$. The subsidiary $SU(4)$ self-duality condition has the form

$$
\bar{W}_{ij} = \frac{1}{2} \varepsilon_{ijkl} W^{kl}.
$$

The CB gauge transformations use the anti-Hermitian superfield parameters $C(z)$

$$
\delta A^k_\alpha = -\nabla^k_\alpha C, \quad \delta \bar{A}_{k\dot{a}} = -\bar{\nabla}_{k\dot{a}} C, \quad \delta W^{kl} = [C, W^{kl}].
$$

The Bianchi identities of a dimension 3/2 yield the CB equations of motion for $W^{ij}$

$$
2 \nabla^k_\gamma W^{ij} - \nabla^i W^{jk} - \nabla^j W^{ki} = 0,
$$

$$
\bar{\nabla}_{k\dot{a}} W^{ij} + \frac{1}{3} \delta^i_k \nabla_{l\dot{a}} W^{li} - \frac{1}{3} \delta^j_k \nabla_{l\dot{a}} W^{lj} = 0
$$

and the conjugated equations for $\bar{W}_{kj}$.

### 2.2 Harmonic interpretation of non-abelian superfield constraints

We use the $U_8$ harmonics $u^{+a}_k$, $u^{-\dot{a}}_k$, $\bar{u}^{-k}_a$, $\bar{u}^{+k}_\dot{a}$ from Appendix A and consider harmonic projections of the $N = 4$ constraints (2.1)

$$
\{D^{+a}_\alpha, D^{+b}_\beta\} = \frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon^{ba} W^{++},
$$

$$
\{\bar{D}^{+a}_{\dot{a}}, \bar{D}^{+\dot{b}}_{\dot{b}}\} = \frac{1}{2} \varepsilon_{\dot{a}\dot{b}} \varepsilon_{ba} \bar{W}^{++} = -\frac{1}{2} \varepsilon_{\dot{a}\dot{b}} \varepsilon_{ba} (W^{++})^\dagger = \frac{1}{2} \varepsilon_{\dot{a}\dot{b}} \varepsilon_{ba} W^{++},
$$

$$
\{D^{+a}_\alpha, \bar{D}^{+\dot{a}}_{\dot{a}}\} = 0,
$$

where

$$
D^{+a}_\alpha = u^{+a}_k \nabla^k_\alpha = D^{+a}_\alpha + A^{+a}_\alpha,
$$

$$
\bar{D}^{+\dot{a}}_{\dot{a}} = \bar{u}^{+\dot{a}}_\dot{a} \nabla_{\dot{a}} = \bar{D}^{+\dot{a}}_{\dot{a}} + A^{+\dot{a}}_{\dot{a}},
$$

$$
W^{++} = \varepsilon_{ab} u^{+a}_k u^{+b}_l W^{kl}, \quad \bar{W}^{++} = \varepsilon^{\dot{a}\dot{b}} \bar{u}^{-k}_a \bar{u}^{-l}_\dot{b} \bar{W}_{kl}.
$$

Using the harmonic properties (A.28) and (A.29) we connect the self-duality condition (2.4)

with the harmonic condition of anti-Hermiticity

$$
[(W^{++})^A_B]^\sim = - (W^{++})^B_A, \quad A, B = 1, 2, \ldots, n.
$$
We analyze the superfield equations for the superfields $A_\alpha^+$, $\bar{A}_{\dot{a}\dot{a}}^+$ and $W^{++}$ using additional harmonic equations

\[
[\partial_{a}^{++}, D_{a}^{+}] = 0, \quad [\partial_{\dot{a}}^{++}, D_{\dot{a}}^{\dot{+}}] = 0, \quad [\partial_{\dot{a}}^{++}, \partial_{b}^{++}] = 0, \\
\partial_{a}^{++}W^{++} = 0, \quad \partial_{\dot{a}}^{++}W^{++} = 0, \quad (2.11)
\]

which are evident in the central basis.

The harmonized $N = 4$ constraints (2.7) have more complicated form than the corresponding Grassmann integrability conditions in the $N = 3$ gauge harmonic-superspace theory [2,3] which do not include superfield strengths

\[
u^{i}_{1}u^{j}_{1}\{\nabla^{i}_{a}, \nabla^{j}_{\dot{b}}\} = 0, \quad \nu^{i}_{3}u^{j}_{3}\{\nabla^{i}_{a}, \nabla^{j}_{\dot{b}}\} = 0, \quad \nu^{k}_{1}u^{j}_{3}\{\nabla^{k}_{a}, \nabla^{j}_{\dot{b}}\} = 0. \quad (2.13)
\]

The AB spinor derivatives $D_{a}^{++}$, $\bar{D}_{\dot{a}}^{\dot{+}}$ and the harmonic derivative $D_{\dot{a}}^{++}$ of the analytic basis are defined in Appendix B. In the non-abelian theory, we define the Hermitian analytic gauge 4-prepotential

\[
(V_{b}^{++})_{A} = [(V_{b}^{++})_{B}^{A}], \quad D_{a}^{+c}V_{b}^{++} = \bar{D}_{\dot{c}\dot{a}}V_{b}^{++} = 0 \quad (2.14)
\]

in the adjoint representation of the gauge group $SU(n)$

\[
\delta_{A}V_{b}^{++} = -D_{b}^{++}\lambda + [\lambda, V_{b}^{++}] \quad (2.15)
\]

where $\lambda_{A}^{B}$ are the analytical superfield gauge parameters

\[
\lambda_{A}^{B} = -(\lambda_{B}^{A})^{\sim} \quad (2.16)
\]

The pure analytic harmonic covariant derivative is defined off mass shell

\[
D_{b}^{++} = D_{b}^{++} + V_{b}^{++}. \quad (2.17)
\]

The commutator of pure analytic covariant harmonic derivatives

\[
[D_{a}^{++}, D_{\dot{b}}^{++}] = \frac{1}{2}F^{(+)ab}\varepsilon_{\dot{a}\dot{b}} + \frac{1}{2}\tilde{F}_{\dot{a}\dot{b}}^{(+)ab}\varepsilon_{\dot{a}\dot{b}} \quad (2.18)
\]

is expressed via two dimensionless gauge covariant superfields

\[
\tilde{F}_{\dot{a}\dot{b}}^{(+)ab} = \varepsilon_{ab}(D_{a}^{++}V_{b}^{++} - D_{b}^{++}V_{a}^{++} + [V_{a}^{++}, V_{b}^{++}]) = \tilde{F}_{\dot{a}\dot{b}}^{(+)ab}, \\
F^{(+)ab} = \varepsilon_{\dot{b}\dot{a}}(D_{a}^{++}V_{b}^{++} - D_{b}^{++}V_{a}^{++} + [V_{a}^{++}, V_{b}^{++}]) = -(\tilde{F}_{\dot{a}\dot{b}}^{(+)ab})^{\dagger}, \\
(D_{a}^{++}V_{b}^{++})^{\dagger} = -D_{a}^{++}V_{b}^{++}. \quad (2.19)
\]

We consider the HS transform from the central basis to the analytic basis

\[
D_{a}^{++} = e^{-v}\nabla_{a}^{+}e^{v}, \quad \bar{D}_{\dot{a}\dot{a}}^{+} = e^{-v}\bar{\nabla}_{\dot{a}\dot{a}}^{+}e^{v}, \quad W^{++} = e^{-v}W^{++}e^{v} \quad (2.20)
\]
where $v(z,u)$ is an anti-Hermitian superfield matrix on the mass shell, and $W^{++}$ is a harmonic superfield in AB. We use the $N = 4$ noncovariant on-shell representation of the spinor covariant derivatives in the analytic basis

$$\nabla^+_{\alpha} = D^+_{\alpha} - \frac{1}{4} \theta^-_{\alpha} W^{++}, \quad \hat{\nabla}^+_{\hat{\alpha}} = \hat{D}^+_{\hat{\alpha}} + \frac{1}{4} \hat{\theta}^-_{\hat{\alpha}} W^{++} = (\nabla^+_{\alpha})^\dagger \quad (2.21)$$

where $W^{++}$ is an independent analytic anti-Hermitian covariant superfield

$$\delta_\lambda W^{++} = [\lambda, W^{++}], \quad (W^{++})^\dagger = -W^{++}. \quad (2.22)$$

By analogy with the CB constraints (2.7), these spinor AB covariant derivatives satisfy the constraints

$$\{\nabla^+_{\alpha}, \nabla^+_{\beta}\} = \frac{1}{2} \varepsilon_{\alpha\beta\varepsilon^b a} W^{++}, \quad \{\hat{\nabla}^+_{\hat{\alpha}}, \nabla^+_{\beta}\} = \frac{1}{2} \varepsilon_{\hat{\alpha}\beta\varepsilon^b a} W^{++},$$

$$\{\nabla^+_{\alpha}, \hat{\nabla}^+_{\beta}\} = 0 \quad (2.23)$$

where we use the linear relations

$$D^b_{\beta} W^{++} = D^b_{\hat{b}\hat{\alpha}} W^{++} = 0, \quad D^+_{\alpha} \theta^-_{\beta} = -\varepsilon_{\alpha\beta\varepsilon^b a}, \quad D^+_{\hat{\alpha}} \hat{\theta}^-_{\hat{\beta}} = \varepsilon_{\alpha\hat{\beta}\varepsilon^b \hat{a}}. \quad (2.24)$$

We define the on-shell harmonic covariant AB derivative

$$e^v \partial^+ a e^{-v} = \nabla^+ a = D^+ a + V^+ a - \frac{1}{4} \left(\left(\theta^- a \theta^+ a\right) + \left(\hat{\theta}^-_{\hat{a}} \hat{\theta}^+_{\hat{a}}\right)\right) W^{++} \quad (2.25)$$

which contains the analytic prepotential and the Hermitian non-analytic term manifestly depending on Grassmann coordinates $\theta$ in accordance with representation (2.21). We find that of the harmonic transform in the $N = 4$ gauge theory and the corresponding harmonic transform in the $N = 3$ gauge theory (2) have essentially different structures.

By the analogy with the trivial CB constraints (2.11), we analyze the following AB constraints:

$$[\nabla^+ a, \nabla^b a] = 0, \quad [\nabla^+ a, \nabla^+ b] = 0,$$

$$[\nabla^+ a, \hat{\nabla}^+ b] = 0 \quad (2.26)$$

using the relations

$$D^+ a \theta^- b = \varepsilon^{ab} \theta^+ a, \quad D^+ a \hat{\theta}^- b = -\varepsilon^{ab} \hat{\theta}^+ a. \quad (2.28)$$

The nonlinear consistency equations for these AB constraints have the form

$$I) \quad \text{dim. 1 : } \quad D^+ a W^{++} = D^+ a W^{++} + [V^+ a, W^{++}] = 0, \quad (2.29)$$

$$\text{dim. 0 : } \quad E^{(+4)ab} = E^{(+4)ab} + \left(\theta^+ a \hat{\theta}^+ b\right) W^{++} = 0,$$

$$E^{(+4)}_{\hat{a} \hat{b}} = \hat{E}^{(+4)}_{\hat{a} \hat{b}} + \left(\theta^+ a \hat{\theta}^+ b\right) W^{++} = 0, \quad (2.30)$$
and the last equations explicitly depend on the Grassmann coordinates. These equations connect two analytic superfield strengths of different dimensions on the mass shell. The constraint equations are covariant under the nonstandard $N = 4$ supersymmetry transformations

$$
\delta W^{++} = [(\epsilon^k Q^k) + (\epsilon^k \bar{Q}_k)] W^{++},
$$
(2.31)
$$
\delta_e V_\alpha^{++a} = [(\epsilon^k Q^k) + (\epsilon^k \bar{Q}_k)] V_\alpha^{++a} + \frac{1}{2}[(\epsilon^{-a} \theta^+_a) + (\epsilon^{-a} \bar{\theta}^+_a)] W^{++},
$$
(2.32)
$$
\delta_e \hat{F}^{(4)}_{\alpha\beta} = \varepsilon_{\alpha\beta} (D^{++a} \delta_e V^{++b} - D^{++b} \delta_e V^{++a}),
$$
$$
\delta_e \hat{E}^{(4)}_{\alpha\beta} = [\{\epsilon^k \bar{Q}_k\}] \hat{E}^{(4)}_{\alpha\beta} + \frac{1}{2} \varepsilon_{\alpha\beta} \{[(\epsilon^{-a} \theta^+_a) + (\epsilon^{-a} \bar{\theta}^+_a)] D^{++b} - [(\epsilon^{-a} \theta^+_a) + (\epsilon^{-a} \bar{\theta}^+_a)] D^{++b}\} W^{++}.
$$
(2.33)

Using the component decompositions of the superfields $W^{++}$ and $V_\alpha^{++a}$ from the next section we can analyze equations (2.29) and (2.30) on the physical and auxiliary fields. In particular, the field equations of the $N = 4$ supermultiplet are equivalent to the CB superfield equations (2.6). All auxiliary fields in the superfields $W^{++}$ and $V_\alpha^{++a}$ vanish on shell or are expressed in terms of physical fields.

In the abelian case, the on-shell superfield constraints for the superfield $W^{kl}$ (2.6) are equivalent to the following linear harmonic-superspace equations:

$$
D_\alpha^{++} W^{++} = \bar{D}_{\alpha\alpha}^{++} W^{++} = 0, \quad D_\alpha^{++a} W^{++} = 0,
$$
(2.34)
$$
(W^{++})^\sim = -W^{++}.
$$
(2.35)

The simple HS reality constraint (2.35) uses the harmonic conditions

$$
\frac{1}{2} \varepsilon^{ijkl} \varepsilon_{ab} u_i^{++a} u_j^{++b} = \varepsilon^{ab} u_{\alpha+k}^{++a} u_{\bar{l}}^{++b} = - (\varepsilon_{ab} u_{\alpha+k}^{++a} u_{\bar{l}}^{++b})^\sim
$$
(2.36)

and guarantees the self-duality condition (2.4).

The AB superfield equation $D_\alpha^{++a} W^{++} = 0$ gives free equations for the physical component fields of the on-shell analytic superfield solution which contains the following structures of the $N = 4$ gauge supermultiplet:

$$
W^{++} \to w^{++} \sim U_{{|kl|}}^{++} \phi^{kl} - 4i \bar{\theta}_b^{++a} \bar{\theta}_b^{++b} U_{{|kl|}}^{++} \partial_{a\alpha}\phi^{kl}
$$
$$
+[\varepsilon^{b\beta} \phi_\beta^{++}} u_{\alpha+k}^{++a} \lambda_{\alpha\beta} + i \Theta^{++a\beta} \bar{\theta}_b^{++b} \lambda_{\alpha\beta} + i \Theta^{++a\beta} \partial_{a\alpha} A_\beta^{++a} - c.c.] + \ldots
$$
(2.37)

where $\phi^{kl}$ is the real 6-component scalar field, $A_m$ is the abelian gauge field, and $\lambda_{\alpha\beta}$ is the spinor field. Here we use the notation for harmonic and Grassmann polynomials from appendices. Similar equations and solution for the abelian real superfield $W^{++}$ were considered earlier in [12, 13, 14].

We note that the AB constraint equations of this section are not derived from the action principle, the harmonic $N = 4$ transform (2.25) connects the central and analytic bases on the mass shell.
3 Action of supergauge model in the harmonic $N = 4$ superspace and component representation of equations of motion

3.1 Superfield action with $\theta$ depending terms

We consider an alternative harmonic formalism of the $N = 4$ gauge theory off mass shell and construct a gauge invariant and $SU(4)$ invariant action ($A$-model) including the independent analytic superfield $W^{++}$ and the prepotential $V_{a}^{++}$

$$A(W, V) \sim \frac{1}{g^2} \int d^{(8)} \zeta d u \text{Tr} \{ W^{++}[(\theta^{+a} \bar{\theta}^{b}) F_{ab}^{(+4)} + (\bar{\theta}^{+a} \theta^{b}) F_{ab}^{(+4)} + \frac{1}{2} W^{++}(\Theta^{+4} + \bar{\Theta}^{+4})] \} \ (3.1)$$

where $g$ is a coupling constant. The action $A$ is also invariant under the Poincare group transformations and the scale transformations. Each term of $A$ explicitly depends on the Grassmann coordinates $\theta^+_a$, $\bar{\theta}^{+a}$, so the $N = 4$ supersymmetry is broken.

Unconstrained analytic $SU(n)$ superfields satisfy the conditions

$$[(W^{++})^A_B] = - (W^{++})^B_A, \quad [(V^{++})^A_B] = (V^{++})^B_A. \quad (3.2)$$

Varying the action in the superfield $V_{a}^{++}$ gives us the equation

$$II) \ \text{dim.0} : \ [\varepsilon_{ab}(\theta^{+a} \theta^{+b}) + \varepsilon^{ab}(\bar{\theta}^{+a} \bar{\theta}^{+b})] D_{b}^{+b} W^{++} = 0 \quad (3.3)$$

which is proportional to the covariant equation (2.29) and contains the $\theta$ depending multiplier in the brackets.

Varying $A$ in $W^{++}$ we obtain the equation

$$II) \ \text{dim.} -1 : \ (\theta^{+a} \theta^{+b}) \hat{F}_{ab}^{(+4)} + (\bar{\theta}^{+a} \bar{\theta}^{+b}) \bar{F}_{ab}^{(+4)} + W^{++}(\Theta^{+4} + \bar{\Theta}^{+4})$$

$$= (\theta^{+a} \theta^{+b}) \hat{F}_{ab}^{(+4)} + (\bar{\theta}^{+a} \bar{\theta}^{+b}) \bar{F}_{ab}^{(+4)} = 0 \quad (3.4)$$

which is a combination of two equations (2.30) with nilpotent $\theta$ coefficients.

Thus, these $A$-induced equations II are not equivalent but compatible with equations I (2.29), (2.30). The component equations for physical fields are equivalent in both cases.

3.2 Component analysis of the $A$-model

We analyze the component decomposition of the imaginary analytic abelian gauge parameter

$$\lambda(\zeta, u) = i a(x_A) + U^{k}_l a^k_l(x_A) + \theta^{+a} u^a_k \beta^k_\alpha(x_A) + \bar{\theta}^{+a} \bar{u}^{-k}_c \beta^l_{k\alpha}(x_A)$$
$$+ (\theta^{+b} \theta^{-\tilde{b}}) U^{(kl)}_{(bc)} \bar{a}^{(kl)}_l(x_A) - (\bar{\theta}^{+b} \theta^{-\tilde{b}}) U^{(kl)}_{(bc)} \bar{d}^{(kl)}_l(x_A) + i \bar{\theta}^{+\alpha} \bar{\theta}^{+\tilde{c}} U^{-\bar{c}k}_{l\alpha a}(x_A)$$
$$+ \Theta^{++\alpha\beta} U^{l^l}_{[kl]} c^{[kl]}_{(\alpha\beta)}(x_A) - \Theta^{++\tilde{\alpha}\tilde{\beta}} U^{l^l}_{[\alpha\beta]} c^{[kl]}_{(\alpha\beta)}(x_A) + \ldots \quad (3.5)$$
where all higher Grassmann and harmonic terms are omitted and the notation from appendices is used. It is not difficult to consider the component decomposition of the harmonic derivative term $D_{\bar{a}a}^{++a}\lambda$ (2.15) and construct the $WZ$-type gauge condition for the prepotential

$$(V_{\bar{a}a}^{++a})_{WZ} = v_{\bar{a}a}^{++a} + \mathcal{V}_{\bar{a}a}^{++a}$$

(3.6)

where $v_{\bar{a}a}^{++a}$ contains the standard $N = 4$ supermultiplet $\phi^{[kl]}$, $A_m$, $\lambda_{ka}$, $\bar{\lambda}_{\bar{a}}^k$

$$v_{\bar{a}a}^{++a} = -2\theta_{\bar{a}a}^\alpha \tilde{\theta}^{\alpha\bar{a}}A_{\alpha\bar{a}} + \phi^{[kl]}[(\theta_{\bar{a}a}^\beta \tilde{\theta}^{\alpha\bar{a}})U_{[kl]}^{\alpha\bar{a}} + (\tilde{\theta}^{\alpha\bar{a}} \tilde{\theta}^{\beta\bar{a}})U_{\alpha\bar{a}[kl]}]
\quad + (\theta_{\bar{a}a}^\beta \tilde{\theta}^{\alpha\bar{a}})\bar{u}_k^\bar{e} \bar{\lambda}_{\bar{a}}^k - (\tilde{\theta}^{\alpha\bar{a}} \tilde{\theta}^{\beta\bar{a}})\bar{u}_c^\bar{e} \lambda_{\alpha a}$$

(3.7)

and $\mathcal{V}_{\bar{a}a}^{++a}$ includes an infinite number of additional bosonic and fermionic component fields. Note that the condition $(v_{\bar{a}a}^{++a})_{WZ} = v_{\bar{a}a}^{++a}$ gives us the reality of the vector field $A_m$ and the self-duality of the scalar field $\phi^{[kl]} = \frac{1}{2} \epsilon_{kl} \phi_{ij}^{[kl]}$ using the self-duality of the neutral irreducible harmonic $U_{ij}^{\alpha\bar{a}}$ [A.25]

$$\phi^{[kl]}[(\theta_{\bar{a}a}^\beta \tilde{\theta}^{\alpha\bar{a}})U_{[kl]}^{\alpha\bar{a}} + (\tilde{\theta}^{\alpha\bar{a}} \tilde{\theta}^{\beta\bar{a}})U_{\alpha\bar{a}[kl]}] = \phi^{[kl]}(\theta_{\bar{a}a}^\beta \tilde{\theta}^{\alpha\bar{a}})U_{[kl]}^{\alpha\bar{a}} + \bar{\phi}_{[kl]}(\tilde{\theta}^{\alpha\bar{a}} \tilde{\theta}^{\beta\bar{a}})U_{\alpha\bar{a}[kl]}^*.$$  

(3.8)

We calculate the standard part of the analytic superfield strength $\tilde{F}_{\bar{a}a}^{++4} = \varepsilon_{ab}(D_{\bar{a}a}^{++a}v_{\bar{a}a}^{++b} + D_{\bar{b}b}^{++a}v_{\bar{a}a}^{++b})$

$$\tilde{f}_{\bar{a}a}^{++4} = \varepsilon_{ab}(D_{\bar{a}a}^{++a}v_{\bar{a}a}^{++b} + D_{\bar{b}b}^{++a}v_{\bar{a}a}^{++b}) = 2i\varepsilon_{ab}\varepsilon^{\beta\alpha}(\theta_{\bar{a}c}^\beta \tilde{\theta}^{\alpha}\theta_{\bar{a}d}^\alpha \tilde{\theta}^{\beta} \tilde{\theta}^{\alpha\bar{a}}(\partial_{\beta\bar{a}}A_{\alpha\bar{a}} + \partial_{\alpha\bar{a}}A_{\beta\bar{a}}))
\quad + 2\varepsilon_{ab}(\theta_{\bar{a}c}^\beta \tilde{\theta}^{\alpha}\theta_{\bar{a}d}^\alpha \tilde{\theta}^{\beta} \tilde{\theta}^{\alpha\bar{a}}u_k^\bar{e} \bar{\lambda}_{\bar{a}}^k - 2\varepsilon_{ab}(\theta_{\bar{a}c}^\beta \tilde{\theta}^{\alpha}\theta_{\bar{a}d}^\alpha \tilde{\theta}^{\beta} \tilde{\theta}^{\alpha\bar{a}}u_k^\bar{e} \bar{\lambda}_{\bar{a}}^k)
\quad + 2\varepsilon_{ab}(\theta_{\bar{a}c}^\beta \tilde{\theta}^{\alpha}\theta_{\bar{a}d}^\alpha \tilde{\theta}^{\beta} \tilde{\theta}^{\alpha\bar{a}}u_k^\bar{e} \bar{\lambda}_{\bar{a}}^k - 2\varepsilon_{ab}(\theta_{\bar{a}c}^\beta \tilde{\theta}^{\alpha}\theta_{\bar{a}d}^\alpha \tilde{\theta}^{\beta} \tilde{\theta}^{\alpha\bar{a}}u_k^\bar{e} \bar{\lambda}_{\bar{a}}^k) + 2\varepsilon_{ab}(\theta_{\bar{a}c}^\beta \tilde{\theta}^{\alpha}\theta_{\bar{a}d}^\alpha \tilde{\theta}^{\beta} \tilde{\theta}^{\alpha\bar{a}}u_k^\bar{e} \bar{\lambda}_{\bar{a}}^k)$$

(3.9)

The conjugated quantity $f_{(n+4)ab}$ includes $\bar{\phi}_{[ij]} = \frac{1}{2}\varepsilon_{ijkl}\phi^{[kl]}$, $\bar{\lambda}_{\bar{a}}^k$ and the conjugated bispinor field-strength $F_{a\beta}(A)$.

The prepotential contains additional terms with the lowest vector and tensor dimension-1 fields having pairs of $SU(4)$ indices

$$\theta^{\alpha\beta} \tilde{\theta}^{\alpha\bar{a}}U_{[kl]}^{\alpha\bar{a}}A_{[k\bar{a}\bar{a}]} + \theta^{\alpha\beta} \tilde{\theta}^{\alpha\bar{a}}U_{[kl]}^{\alpha\bar{a}}A_{[k\bar{a}\bar{a}]}$$

(3.10)

and additional (dimension-3/2) fermionic fields

$$\Theta_{\bar{a}}^{(+3)\alpha} \bar{u}_{-ak} \xi_{ka} + \Theta_{\bar{a}}^{(+3)\alpha} \bar{u}_{ak} \xi_{ka} + c.c.$$  

(3.11)

We study the lowest terms with the (dimension-2) auxiliary fields

$$\Theta_{\bar{a}}^{(+3)\alpha} \bar{u}_{-ak} \xi_{ka} + \Theta_{\bar{a}}^{(+3)\alpha} \bar{u}_{ak} \xi_{ka} + c.c.$$  

(3.12)
The off-shell prepotential includes also an infinite number of auxiliary fields of different dimensions with more than two SU(4) indices, for instance, the nontrivial dimensionless term
\[ A^{(i,j)}_{[kl]} U_{\bar{a}(ij)}^{++a[kl]} + \text{c.c.} \] (3.13)

The off-shell decomposition of the independent \sim-imaginary abelian superfield strength contains independent component fields

\[ W^{++} = U_{[kl]}^{++} F^{[kl]} + \varepsilon^{bc} \theta_b^{+\beta} \bar{u}_c \Lambda_{k\beta} - \varepsilon_b \bar{\theta}^{+\alpha} \bar{u}_k \Lambda_{\alpha}^k + i\Theta_{\alpha\beta}^{++} F^{\alpha\beta} + i\bar{\Theta}_{\dot{\alpha}\dot{\beta}}^{++} \bar{F}^{\dot{\alpha}\dot{\beta}} \]

\[ + \Theta_{\alpha\beta}^{+\alpha} \bar{u}_b \Lambda_{\alpha}^b - \Theta_{\dot{b}}^{+\dot{\alpha}} \bar{u}_b \Lambda_{\dot{\alpha}}^b + \Theta_{\alpha\beta}^{+\alpha} u_k \Lambda_{\alpha}^k + \Theta_{\dot{b}}^{+\dot{\alpha}} \bar{u}_b \Lambda_{\dot{\alpha}}^b \]

\[ + (\Theta_{++}^{+\alpha}) T^{[kl]} + \Theta_{\alpha\beta}^{+\alpha} \bar{u}_b \Lambda_{\alpha}^b \]

\[ + \Theta_{\dot{b}}^{+\dot{\alpha}} \bar{u}_b \Lambda_{\dot{\alpha}}^b \]

\[ \sim (3.14) \]

where higher Grassmann and harmonic terms are omitted. The fields \( F^{[kl]} \), \( W^{[kl]} \) and \( T^{[kl]} \) are self-dual by construction.

We stress that the nilpotency of the Grassmann coordinates yields the cancellation of some higher component terms in the combinations \( W^{++}(\theta^{+\dot{a}} \theta^{+\dot{b}}) \bar{F}^{+4}_{\dot{a}\dot{b}} \), \( \Theta^{+4}(W^{++})^2 \), \ldots from the action \( A \).

The abelian version of eq. (3.3) gives us algebraic restrictions for the infinite supermultiplet
\[ V_t^k = 0, \quad T^{[kl]} = 0, \quad P^k_\alpha = 0, \ldots \] (3.15)

and differential constraints
\[ \partial^{+\beta} F_{\alpha\beta} = 0, \quad \partial^{+\beta} \Lambda_{k\beta} = 0, \quad \Lambda_{[\alpha\dot{\alpha}] = -4i\partial_{\alpha\dot{\alpha}} F^{[kl]}, \]
\[ \partial^{+\alpha} W_{\alpha\dot{\alpha}}^{[kl]} = 0, \quad R_{k(\alpha\beta)\dot{\alpha}} = i\partial_{(\alpha\dot{\alpha})} \Lambda_{k\beta}, \ldots \] (3.16)

All auxiliary fields in \( W^{++} \) vanish or are expressed via the basic fields \( F_{\alpha\beta}, \bar{F}^{[kl]}, \Lambda_{k\beta} \) on the mass shell, and these basic fields form the \( N = 4 \) multiplet of auxiliary field-strengths and satisfy the free equations of motion.

In the abelian case, eq. (3.4) yields relations between the physical and auxiliary field-strengths and also the restrictions for auxiliary fields in \( V^{++a}_a \) and \( W^{++} \)
\[ F^{[kl]} \sim \phi^{[kl]}, \quad F_{\alpha\beta} \sim \partial_{\alpha\dot{a}} A_{\dot{a}}^{\dot{\alpha} \alpha} + \partial_{\beta \dot{\alpha}} A_{\dot{a}}^{\dot{\alpha} \alpha}, \]
\[ A^{(i,j)}_{[kl]} = 0, \quad A_{l\alpha} = 0, \quad B_{(\alpha\beta)}^{(kl)} = 0, \]
\[ \rho^{(i,j)}_{(\alpha\beta)\dot{a}} = 0, \ldots \] (3.18)

Comparing two equations we obtain the free equations for the physical fields \( \phi^{[kl]} \), \( A_m \) and \( \lambda_{k\alpha} \) in the abelian case, and all auxiliary fields vanish or are expressed via the physical fields on the
mass shell. The non-abelian nonlinear equations (3.3) and (3.4) preserve gauge covariance of the algebraic and differential constraints. Excluding the auxiliary fields we obtain the standard $N = 4$ Yang-Mills equations for the physical component fields.

The proof of consistency of the abelian component action in the $A$-model and its gauge invariance are sufficient for the proof of consistency of the corresponding nonabelian component action. The possible superfield quantization of the $A$-model will be discussed elsewhere.

4 Nonlinear interactions in the abelian $N = 4$ gauge theory

We consider the bilinear component electromagnetic terms from the abelian version $A_0$ of the action (3.1)

$$
\frac{1}{64}[F^\alpha\beta F^\alpha\beta + 4F^\alpha\beta (\partial^\alpha A^\beta + \partial^\beta A^\alpha) + c.c],
$$

(4.1)

$$
F^2 = F^\alpha\beta F^\alpha\beta, \quad \bar{F}^2 = \bar{F}^\dot{\alpha}\dot{\beta} \bar{F}^\dot{\alpha}\dot{\beta}.
$$

(4.2)

Excluding auxiliary fields $F^\alpha\beta$ and $\bar{F}^\dot{\alpha}\dot{\beta}$ we obtain the standard electromagnetic Lagrangian $-\frac{1}{4}(\partial_m A^n - \partial_n A_m)^2$. Excluding of other auxiliary fields in the component version of the action $A_0$ yields the standard abelian action of the physical $N = 4$ fields.

The $N = 4$ supersymmetric fourth-order interaction of the abelian harmonic superfield

$$
S_4 = N_1 f^2 \int d^{(-8)}\zeta du (W^{++})^4
$$

(4.3)

contains the coupling constant $f$ of a dimension $-2$ and some normalization factor $N_1$. This term describes the simplest effective interaction by analogy with the nonlinear abelian $N = 3$ gauge model [6]. In particular, this superfield term yields the component interaction

$$
L_4 \sim f^2 F^2 \bar{F}^2
$$

(4.4)

and similar fourth-order interactions for other auxiliary component fields. The simplest $N = 4$ effective action $A_0 + S_4$ describes nonpolynomial interactions of physical fields, if we exclude auxiliary fields. We see that auxiliary fields play the important role in the study of effective interactions of $N = 4$ supermultiplets.

By analogy with [6], we can construct the $U(1)$ neutral analytic superfield $A$ of a dimension 8

$$
A = N_2 D^4 \bar{D}^4 [(D^{-a} D^{-a})^4 (W^{++})^4].
$$

(4.5)

We choose the constant $N_2$ by the normalization condition of the component decomposition

$$
A = F^2 \bar{F}^2 + \ldots
$$

(4.6)
where all other terms are omitted. The nonlinear interaction of the superfield $W^{++}$ is defined by an arbitrary dimensionless function of this analytic superfield

$$S(W) = f^2 \int d^{(8)} \zeta du (W^{++})^4 E(f^4 A)$$

$$= f^2 \int d^{(8)} \zeta du (W^{++})^4 [N_1 + e_1 f^4 A + e_2 f^8 A^2 + \ldots]$$

(4.7)

where $e_1$, $e_2$, ... are some constants.

We stress that the superfield interaction $S(W)$ generates consistent equations in combination with the bilinear abelian action $A_0$ (3.1). Varying $V^{++\dot{a}}$ gives the linear eq. (3.3), and varying $W^{++}$ leads to a nonlinear generalization of eq. (3.4). Excluding the auxiliary fields of $W^{++}$ from the component decomposition of these superfield interactions we obtain nonlinear effective interactions of the physical $N = 4$ supermultiplet. These interactions describe possible quantum corrections to the $N = 4$ classical action.

5 Inconsistencies of the manifestly covariant model in harmonic superspace

We analyze the simplest superfield invariant of the gauge and superconformal groups using the gauge prepotential $V^{++\dot{a}}$

$$S = \int d^{(8)} \zeta du \text{Tr}[F^{(+4)ab} F^{(+4)\dot{a}\dot{b}} + \hat{F}^{(+4)\dot{a}\dot{b}} \hat{F}^{(+4)ab}]$$

(5.1)

as an action of some gauge theory (S-model). Varying the prepotential $V^{++\dot{a}}$ we obtain the equation of motion of the S-model

$$D^{++\dot{a}} \hat{F}^{+4\dot{a}} + D^{++ab} F^{+4ab} = 0.$$  

(5.2)

The abelian version of this equation has the form

$$\varepsilon_{ce} D^{++\dot{a}\dot{b}} (D^{++ae} V^{++c} - D^{++\dot{b}c} V^{++\dot{a}}) + \varepsilon^{\dot{c}\dot{e}} D^{++\dot{a}} (D^{++\dot{c}a} V^{++b} - D^{++\dot{b}c} V^{++\dot{a}})$$

$$= 2 D^{++\dot{a}} D^{++\dot{b}} V^{++c} - D^{++\dot{a}} D^{++\dot{c}} V^{++\dot{b}} = 0.$$  

(5.3)

It is easy to analyze the component decomposition of the superfield abelian action and equation of motion. The standard component fields from $v^{++\dot{a}}$ (3.7) have the reasonable interactions. The additional dimension-1 fields (3.10) satisfy dynamical differential equations of motion, however, we do not find additional gauge symmetries for interactions of unusual vector fields $A^{kk}_{\alpha\dot{a}}$ and tensor fields $B^{(kl)}_{(\alpha\beta)}$ in this S-model. We know that the gauge invariance $\delta A^{kk}_{\alpha\dot{a}} = \partial_{\alpha\dot{a}} A^{k}_l$ is necessary for a consistent description of the free massless vector field with a positive energy, and similar gauge invariance is important for tensor fields. Thus, the component decomposition of the covariant superfield action is inconsistent at the bilinear level for these unusual fields. We conclude that the manifestly covariant superfield action $S$ describes the inconsistent gauge field interaction of the standard $N = 4$ supermultiplet with additional bosonic and fermionic fields.

1 Similar invariant actions were considered in the $USp(4)$ harmonic superspace formalism [17].
6 Conclusions

We reformulate the superfield constraints of the $N = 4$ gauge theory in the formalism of the $U_8$ harmonic superspace $H(4 + 8|8)$. In the nonabelian case, the harmonic-superspace equations connect the independent dimension-1 harmonic superfield $W^{++}$ and the dimensionless gauge prepotential $V^{++a}$. These superfield equations explicitly depend on the Grassmann coordinates, although they are covariant with respect to deformed $N = 4$ supersymmetry transformations of prepotentials.

We use the unconstrained superfields $W^{++}$ and $V^{++a}$ in the nonabelian superfield action $A$. This action explicitly depends on the Grassmann coordinates, although the $SU(4)$ automorphism symmetry is preserved. At the field-component level, the action $A$ describes interactions of two infinite-dimensional $N = 4$ supermultiplets. The $A$-model equations of motion are compatible with the harmonic-superspace constraint equations. On mass shell, all auxiliary fields vanish or are expressed in terms of the physical $N = 4$ fields $\phi^{[kl]}$, $A_m$ and $\lambda_{k\alpha}$.

Possible quantum corrections in the $N = 4$ theory are described by the nonlinear effective interaction of the superfield $W^{++}$.

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Appendix A. $SU(4)/SU(2) \times SU(2) \times U(1)$ harmonic coset space

A.1 Harmonic variables and harmonic derivatives

We use the harmonics parameterizing the 8 dimensional coset space $U_8 = G/H$

$$u_k^{+a}, \quad u_k^{-\hat{a}}$$

where $G = SU(4)$, $H = SU_L(2) \times SU_R(2) \times U(1)$, $k = 1, 2, 3, 4$ is the spinor index of $SU(4)$, $a = 1, 2$ describe the $SU_L(2)$ doublet and $\hat{a} = \hat{1}, \hat{2}$ corresponds to the $SU_R(2)$ doublet, and $\pm$ are charges of $U(1)$. They form an $SU(4)$ matrix and are covariant under the independent $G$ and $H$ transformations.

The Hermitian conjugation of the harmonic matrix $SU(4)$ gives us the conjugated harmonics

$$\bar{u}_k^{+a} = \bar{u}_a^{-k}, \quad \bar{u}_k^{-\hat{a}} = \bar{u}_{\hat{a}}^{+k}.$$ (A.2)

The harmonics satisfy the following basic relations:

$$u_i^{+a} \bar{u}_b^{-i} = \delta_a^b, \quad u_i^{+a} \bar{u}_b^{-i} = 0.$$ (A.3)
These neutral harmonic derivatives satisfy the SU(4) Lie algebra
\[ \delta_b^a = u_i^{+a} \partial_i^b - \bar{u}_b^{-i} \partial_i^{+a} - \frac{1}{2} \delta_b^a (u_i^{+f} \partial_i^{-j} - \bar{u}_i^{-f} \partial_i^{+a}), \]
\[ \delta_b^{-a} = u_i^{-a} \partial_i^b - \bar{u}_b^{+i} \partial_i^{-a} - \frac{1}{2} \delta_b^{-a} (u_i^{-f} \partial_i^{+j} - \bar{u}_i^{-f} \partial_i^{-a}), \]
\[ \delta^0 = u_i^{+f} \partial_i^{-j} - \bar{u}_i^{-j} \partial_i^{+f} + \partial_i^{-j} \partial_i^{+f} - u_i^{-f} \partial_i^{+j}. \]

The special \( \sim \)-conjugation for these harmonics and other quantities preserves \( U(1) \) charge and changes indices of two SU(2) subgroups
\[ (u_i^{+a})^\sim = -\bar{u}_i^{+i}, \quad (u_i^{-a})^\sim = -\bar{u}_i^{-i}, \quad (\varepsilon^{ab}) = \varepsilon_{ba}. \]
\[ (\bar{u}_a^{-i})^\sim = -u_a^{-\hat{a}}, \quad (\bar{u}_a^{+i})^\sim = -u_a^{+\hat{a}}. \]

It is consistent with the basic harmonic relations. Note that our convention of \( \sim \)-conjugation uses an additional sign — in comparison with [9].

The \( U(1) \) neutral SU(4)-invariant harmonic derivatives
\[ \partial_b^{-i} = \frac{\partial}{\partial u_i^b}, \quad \partial_b^{+i} = \frac{\partial}{\partial u_i^b}, \quad \partial_i^{-b} = \frac{\partial}{\partial u_i^b}, \quad \partial_i^{+b} = \frac{\partial}{\partial u_i^b}. \]

These neutral harmonic derivatives satisfy the \( SU_L(2) \times SU_R(2) \times U(1) \) Lie algebra.

The eight charged coset harmonic derivatives
\[ \partial_b^{+a} = u_i^{+a} \partial_i^b - \bar{u}_b^{-i} \partial_i^{+a}, \]
\[ \partial_b^{-\hat{a}} = u_i^{-\hat{a}} \partial_i^b - \bar{u}_b^{+i} \partial_i^{-\hat{a}}. \]

satisfy the SU(4) Lie algebra
\[ [\partial_b^{+a}, \partial_b^{-\hat{b}}] = \partial_b^{a\hat{b}} \delta_a^\hat{b} - \delta_a^\hat{b} \partial_b^{a\hat{b}} + \frac{1}{2} \delta_a^\hat{b} \partial_b^{a\hat{b}}, \]
\[ [\partial_b^b, \partial_a^{+a}] = \delta_c^a \partial_c^{+b} - \frac{1}{2} \delta_c^a \partial_c^{+a}, \quad [\partial_b^{+a}, \partial_a^{+a}] = -\delta_a^{+a} \partial_{\hat{a}}^{+\hat{b}} + \frac{1}{2} \delta_a^{+a} \partial_{\hat{a}}^{+\hat{b}}, \]
\[ [\partial_b^{-\hat{a}}, \partial_a^{+a}] = -\delta_a^{+a} \partial_{\hat{a}}^{+\hat{b}} + \frac{1}{2} \delta_a^{+a} \partial_{\hat{a}}^{+\hat{b}}, \quad [\partial_b^{+a}, \partial_a^{-\hat{a}}] = \delta_a^{-\hat{a}} \partial_{\hat{a}}^{-\hat{b}} - \frac{1}{2} \delta_a^{-\hat{a}} \partial_{\hat{a}}^{-\hat{b}}, \]
\[ [\partial_b^{+a}, \partial_a^{+a}] = 2 \partial_a^{+a}, \quad [\partial_b^{+a}, \partial_a^{-\hat{a}}] = -2 \partial_a^{-\hat{a}}. \]

The special Hermitian conjugation is defined for the harmonic derivatives
\[ (\partial_b^{+i})^\dagger = \bar{\partial}_b^{+i}, \quad (\partial_b^{-i})^\dagger = \bar{\partial}_b^{-i}, \]
\[ (\partial_b^b)^\dagger = \bar{\partial}_b^b, \quad (\partial^0)^\dagger = -\partial^0, \quad (\partial_b^{+a})^\dagger = \partial_b^{+a}, \quad (\partial_b^{-\hat{a}})^\dagger = \partial_b^{-\hat{a}}, \]
\[ (\partial_b^{+a} f)^\sim = [\partial_b^{+a}, f]^\dagger = -\partial_b^{+a} \bar{f}, \quad (\partial_b^b f)^\sim = [\partial_b^b, f]^\dagger = -\partial_b^b \bar{f}. \]
A.2 Irreducible harmonic combinations

We study irreducible in $SU(4)$ and $SU_L(2) \times SU_R(2)$ group indices combinations of the harmonic coordinates with different $U(1)$ charge $q$

- $q = 0$ harmonics

We consider the bilinear traceless neutral combinations of harmonics

$$U^k_l = u^{+b}_i \bar{u}^{-k}_b - \bar{u}^{+b}_b u^{-k}_i = -(U^l_k)^\sim, \quad U^k_l U^l_j = \delta^k_j. \quad (A.21)$$

The completely traceless Hermitian neutral combination with four $SU(4)$ indices

$$U^{ik}_{jl} = \frac{1}{4} U^k_l U^j_i + \frac{1}{60} \delta^k_i \delta^l_j - \frac{1}{15} \delta^k_j \delta^l_i = (U^{ij}_{kl})^\sim \quad (A.22)$$

has the combined symmetry $U^{ik}_{jl} = U^{kj}_{il}$.

The simplest bilinear combinations with $SU_L(2) \times SU_R(2)$ indices have the form

$$U^{a\bar{a}}_{[ij]} = \frac{1}{2} u^{+a}_i \bar{u}^{-\bar{a}}_j - \frac{1}{2} u^{+a}_j \bar{u}^{-\bar{a}}_i, \quad U^{a\bar{a}}_{(ij)} = \frac{1}{2} u^{+a}_i \bar{u}^{-\bar{a}}_j + \frac{1}{2} u^{+a}_j \bar{u}^{-\bar{a}}_i, \quad (A.23)$$

$$\bar{U}^{ij}_{[a\bar{a}]} = (U^{a\bar{a}}_{[ij]})^\sim, \quad \bar{U}^{ij}_{(a\bar{a})} = (U^{a\bar{a}}_{(ij)})^\sim. \quad (A.24)$$

We consider the important $U(1)$ neutral self-duality relation connecting conjugated harmonics

$$U^{a\bar{a}}_{[ij]} = \frac{1}{2} \varepsilon_{ijkl} \varepsilon^{ab} \varepsilon^{\bar{a}b} \bar{U}^{[kl]}_{bb}. \quad (A.25)$$

Note that the determinant condition $(A.6)$ follows from this relation.

The double traceless bilinear neutral combinations

$$U^{ak}_{bl} = u^{+a}_i \bar{u}^{-k}_b - \frac{1}{2} \delta^a_b u^{+e}_i \bar{u}^{-k}_e, \quad \bar{U}^{bl}_{ak} = (U^{ak}_{bl})^\sim. \quad (A.26)$$

satisfy the relations $U^{ak}_{bl} U^l_k = U^{aj}_{bl} = U^{aj}_{k l}$. We can construct the traceless combination with four $SU(4)$ indices.
• \( q = 1 \) harmonics

We construct the traceless combination with three \( SU(4) \) indices

\[
U_{jl}^{+ai} = \frac{1}{2} u_j^{+a} u_i^{+b} \bar{u}^l_{-b} - \frac{2}{3} \delta_j^{+a} u_l^{+a} - \frac{7}{30} \delta_l^{+a} u_j^{+a} = U_{ij}^{ik} u_k^{+a} \tag{A.27}
\]

and the conjugated combination \( \bar{U}_{ai}^{+jl} = (U_{jl}^{+ai})^\sim \).

• \( q = 2 \) harmonics

The \( q = 2 \) charged self-duality condition

\[
\frac{1}{2} \varepsilon^{kl} U_{ij}^{++} = \bar{U}_{ij}^{++[kl]} \tag{A.28}
\]

connects the corresponding combinations of harmonics

\[
U_{ij}^{++} = \varepsilon_{ab} u_i^{+a} u_j^{+b}, \quad \bar{U}_{ij}^{++} = \varepsilon^{\tilde{e}\tilde{e}'} \bar{u}_{\tilde{e}}^{+i} \bar{u}_{\tilde{e}'}^{+j} = -(U_{ij}^{++})^\sim. \tag{A.29}
\]

We consider the bilinear combinations with two \( SU(4) \) indices

\[
U_{(ij)}^{++(ab)} = \frac{1}{2} (u_i^{+a} u_j^{+b} + u_i^{+b} u_j^{+a}), \quad \bar{U}_{(ab)}^{++(ij)} = \frac{1}{2} (\bar{u}_a^{+i} \bar{u}_b^{+j} + \bar{u}_b^{+i} \bar{u}_a^{+j}), \tag{A.30}
\]

\[
U_{bi}^{++bj} = u_i^{+b} u_b^{+j}. \tag{A.31}
\]

We calculate the harmonic derivative of neutral harmonics

\[
\partial_{a}^{++a} U_{l}^{k} = -2 U_{at}^{++ak}, \tag{A.32}
\]

\[
\partial_{a}^{++a} U_{ij}^{++} = \frac{1}{2} \delta_{a}^{++a} U_{ij}^{++}, \tag{A.33}
\]

\[
\partial_{a}^{++a} U_{(ij)}^{++(ab)} = \delta_{a}^{++a} U_{(ij)}^{++(ab)}, \quad \partial_{a}^{++a} \bar{U}_{(ij)}^{++(ab)} = -\delta_{a}^{++a} \bar{U}_{(ab)}^{++(ij)}, \tag{A.34}
\]

\[
\partial_{a}^{++a} U_{cl}^{bk} = -\delta_{a}^{++a} u_{l}^{+b} \bar{u}_{a}^{+k} + \frac{1}{2} \delta_{c}^{++c} u_{l}^{+a} \bar{u}_{b}^{+k}, \tag{A.35}
\]

The traceless combinations with four \( SU(4) \) indices have the form

\[
U_{a(ik)}^{++a(jl)} = \partial_{a}^{++a} U_{(ik)}^{(jl)}, \quad U_{a[ik]}^{++a[jl]} = \partial_{a}^{++a} U_{[ik]}^{[jl]}, \tag{A.36}
\]

\[
U_{a(ik)}^{++a[jl]} = \frac{1}{4} (U_{a[ik]}^{++aj} U_{l}^{i} + U_{a[kl]}^{++aj} U_{i}^{j} - U_{a[kl]}^{++aj} U_{i}^{j} - U_{a[kl]}^{++aj} U_{i}^{j}), \tag{A.37}
\]

\[
+ \frac{1}{8} (\delta_{j}^{++j} U_{a[kl]}^{++aj} + \delta_{j}^{++j} U_{a[kl]}^{++aj} - \delta_{k}^{++k} U_{a[ik]}^{++aj} - \delta_{k}^{++k} U_{a[ik]}^{++aj}).
\]

Note that the harmonic \( U_{a(ik)}^{++a[jl]} \) cannot be presented as a total \( \partial_{a}^{++a} \) derivative of a neutral harmonic.
• $q = -1$ harmonics

We construct the traceless combination with three $SU(4)$ indices

$$U_{ijkl}^{-} = \frac{1}{2} \hat{u}_a^{-} u_i^{-} u_j^{-} + u_i^{+} \hat{u}_a^{+}, \quad \tilde{U}_{ijkl}^{-} = \frac{1}{2} (\bar{u}_a^{-} \hat{u}_b^{+} + \hat{u}_b^{+} \bar{u}_a^{-}),$$

(A.38)

and the conjugated combination.

• $q = -2$ harmonics

The $q = -2$ charged self-duality condition

$$\frac{1}{2} \varepsilon^{ijkl} U_{[ij]}^{-} = \tilde{U}_{[kl]}^{-}$$

(A.39)

connects the corresponding combinations of harmonics

$$U_{[ij]}^{-} = \varepsilon_{ab} u_i^{-} u_j^{-}, \quad \tilde{U}_{[ij]}^{-} = \varepsilon^{ce} \bar{u}_e^{-} i u_j^{-} = -(U_{[ij]}^{-})^\sim,$$

$$\varepsilon^{ijkl} U_{[ij]}^{-} U_{[kl]}^{++} = 4.$$

(A.40)

We use the relation

$$\partial_{\hat{a}}^{++a} U_{[ij]}^{-} = 2 \varepsilon_{ab} U_{[ij]}^{ab}.$$ (A.41)

We consider the bilinear combinations with two $SU(4)$ indices and two doublet indices

$$U_{(ij)}^{-} = \frac{1}{2} (u_i^{-} a u_j^{-} + u_i^{+} b u_j^{+}), \quad \tilde{U}_{(ab)}^{-} = \frac{1}{2} (\bar{u}_a^{-} i \bar{u}_b^{-} + \bar{u}_b^{-} i \bar{u}_a^{-}),$$

(A.42)

$$U_{ai}^{-} = u_i^{-} a \bar{u}_a^{-}.$$ (A.43)

satisfying the relations

$$\partial_{\hat{a}}^{++a} U_{(ij)}^{-} = \frac{1}{2} \delta_{\hat{b}}^{b}(u_i^{+a} u_i^{-} u_j^{-} + u_i^{-a} u_i^{+} u_j^{+}),$$

(A.44)

$$\partial_{\hat{a}}^{++a} U_{bi}^{-} = \delta_{\hat{b}}^{b}(u_i^{+a} \bar{u}_b^{-} - \frac{1}{2} \delta_{\hat{a}}^{a} u_i^{+a} \bar{u}_c^{-} - \delta_{\hat{a}}^{a} u_i^{-a} \bar{u}_b^{-} + \frac{1}{2} \delta_{\hat{a}}^{a} u_i^{-a} \bar{u}_c^{-} + \frac{1}{2} \delta_{\hat{a}}^{a} u_i^{+a} \bar{u}_c^{-} - \delta_{\hat{a}}^{a} u_i^{-a} \bar{u}_b^{-} + \frac{1}{2} \delta_{\hat{a}}^{a} u_i^{+a} \bar{u}_c^{-}).$$

(A.45)

**Appendix B. $SU(4)/SU(2) \times SU(2) \times U(1)$ harmonic superspace**

**B.1 Analytic basis**

We use the $N = 4$ superspace $R(4|16)$ with 4 space-time and 8+8 spinor coordinates

$$z = (x^m, \theta^a_k, \bar{\theta}^{\hat{a}}_{\hat{k}})$$

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where \( k = 1, 2, 3, 4 \) is the \( SU(4) \) index, \( m = 0, 1, 2, 3 \) is the vector index and \( \alpha, \dot{\alpha} \) are the \( SL(2, C) \) spinor indices. The supersymmetry transformations have the form

\[
\delta x^m = -i(\epsilon_k \sigma^m \bar{\theta}^k) + i(\theta_k \sigma^m \bar{e}^k), \quad \delta \theta^\alpha_k = \epsilon^\alpha_k, \quad \delta \bar{\theta}^{\dot{\alpha}}_k = \bar{\epsilon}^{\dot{\alpha}}_k. \tag{B.1}
\]

Spinor derivatives satisfy the relations

\[
\{ D^\alpha_a, D^\beta_b \} = 0, \quad \{ \bar{D}_{\bar{\alpha}} \dot{a}, \bar{D}_{\bar{\beta}} \dot{b} \} = 0,
\]

\[
\{ D^\alpha_a, \bar{D}_{\bar{\alpha}} \dot{a} \} = -2i\delta^\alpha_i (\sigma^m)_{\alpha \dot{\alpha}} \partial_m = -2i\delta^\alpha_i \partial_{\alpha \dot{\alpha}}. \tag{B.2}
\]

We construct the analytic coordinates \( \zeta = (x^m_A, \theta^+_{\alpha \dot{a}}, \bar{\theta}^{++ \alpha \dot{\alpha}}) \) in the \( U_8 \) harmonic superspace \( \mathcal{H}(4 + 8|8) \)

\[
x^m_A = x^m - i(\theta^\alpha \sigma^m \bar{\theta}^{\dot{\alpha}}) + i(\bar{\theta}^\alpha \sigma^m \theta^{\dot{\alpha}}),
\]

\[
\delta x^m_A = 2i(\bar{\theta}^\alpha \sigma^m \epsilon^\dot{\alpha} - 2i(\epsilon^\alpha \sigma^m \bar{\theta}^{\dot{\alpha}}),
\]

\[
\theta^+_{\alpha \dot{a}} = \theta^+_{\alpha \dot{a}}, \quad \bar{\theta}^{++ \alpha \dot{\alpha}} = \bar{\theta}^{++ \alpha \dot{\alpha}} = -(\theta^\alpha)^\sim = \bar{\theta}^{\dot{\alpha}}. \tag{B.6}
\]

We consider bilinear products of spinor coordinates

\[
\theta^+_{\alpha \dot{a}} \theta^+_{\beta \dot{b}} = \frac{1}{2} \varepsilon^{\beta \alpha} (\theta^+_{\alpha \dot{a}} \theta^+_{\beta \dot{b}}) + \frac{1}{2} \varepsilon_{\alpha \beta} \Theta^{++ \alpha \beta}, \tag{B.7}
\]

\[
\Theta^{++ \alpha \beta} = \varepsilon^{\dot{b} \dot{a}} \theta^+_{\dot{a} \dot{b}} \theta^+_{\dot{b} \dot{a}}, \tag{B.8}
\]

\[
\Theta^{++ \alpha \beta} (\theta^+_{\dot{a} \dot{b}}) = 0 \tag{B.9}
\]

and analogous relations for conjugated quantities

\[
\bar{\theta}^{++ \alpha \dot{a}} \bar{\theta}^{++ \beta \dot{b}} = \frac{1}{2} \varepsilon^{\alpha \dot{\beta}} (\bar{\theta}^{++ \alpha \dot{a}} \bar{\theta}^{++ \beta \dot{b}}) + \frac{1}{2} \varepsilon_{\alpha \beta} \bar{\Theta}^{++ \alpha \beta}, \tag{B.10}
\]

\[
\bar{\Theta}^{++ \alpha \beta} = (\Theta^{++ \alpha \beta})^\sim = \varepsilon_{\dot{b} \dot{a}} \bar{\theta}^{++ \alpha \dot{a}} \bar{\theta}^{++ \beta \dot{b}}. \tag{B.11}
\]

The third degree relations read

\[
\Theta^{++3 \alpha}_{\dot{a}} = (\theta^+_{\dot{a}} \theta^+_{\dot{b}} \theta^+_{\dot{c}}) \Theta^{++3 \alpha}_{\dot{a}} = \varepsilon^{\dot{c}} \theta^+_{\dot{a}} \theta^+_{\dot{b}} \theta^+_{\dot{c}} \Theta^{++3 \alpha}_{\dot{a}} = \theta^+_{\dot{a}} \Theta^{++3 \alpha}_{\dot{a}}, \tag{B.12}
\]

\[
(\theta^+_{\dot{a}} \theta^+_{\dot{b}} \theta^+_{\dot{c}}) \theta^+_{\dot{a}} = \frac{1}{2} \varepsilon^{\dot{c}} \varepsilon^\alpha (\theta^+_{\dot{a}} \theta^+_{\dot{b}} \theta^+_{\dot{c}}) = \frac{1}{3} \varepsilon^\alpha \Theta^{+(3) \alpha}_{\dot{a}}, \tag{B.13}
\]

\[
\theta^+_{\dot{a}} \Theta^{++3 \rho}_{\dot{a}} = \frac{1}{3} \varepsilon^{\dot{a}} \varepsilon^\beta \Theta^{+(3) \beta}_{\dot{a}} = \frac{1}{3} \varepsilon^{\dot{a}} \varepsilon^\alpha \Theta^{+(3) \alpha}_{\dot{a}}, \tag{B.14}
\]

\[
\bar{\Theta}^{+(3) \alpha}_{\dot{a}} = -(\Theta^{+(3) \alpha}_{\dot{a}})^\sim = (\bar{\theta}^{++ \alpha \dot{a}} \bar{\theta}^{++ \beta \dot{b}})^\sim = \bar{\theta}^{++ \alpha \dot{a}}. \tag{B.15}
\]

\[
\theta^+_{\dot{a}} \bar{\theta}^{++ \alpha \dot{a}} \Theta^{++ \beta \dot{b}} = \frac{1}{2} \varepsilon_{\dot{a}} \varepsilon^\beta (\theta^+_{\dot{a}} \theta^+_{\dot{b}}) = \varepsilon_{\dot{a} \dot{b}} \Theta^{++ \alpha \beta} \bar{\theta}^{++ \alpha \dot{a}}. \tag{B.16}
\]
We consider the 4th degree relations

\[ \Theta^{4} = (\theta^{\dagger^{\beta}}\theta^{\dagger^{\gamma}})(\theta^{\dagger^{\alpha}}\theta^{\dagger^{\beta}}) = \Theta^{\dagger^{\gamma\alpha}}\Theta^{\dagger^{\beta}}, \]  
(B.17)

\[ (\theta^{\dagger^{\beta}}\theta^{\dagger^{\gamma}})(\theta^{\dagger^{\alpha}}\theta^{\dagger^{\gamma}}) = \frac{1}{6} (\varepsilon_{\alpha\beta}\varepsilon_{\alpha\delta} + \varepsilon_{\beta\delta}\varepsilon_{\alpha\epsilon}) \Theta^{4}, \]  
(B.18)

\[ \Theta^{(\dagger)} = \frac{1}{2} \varepsilon_{\beta\alpha} \left( (\theta^{\dagger^{\alpha}}\theta^{\dagger^{\gamma}})(\theta^{\dagger^{\alpha}}\theta^{\dagger^{\beta}}) = \frac{1}{2} \varepsilon_{\alpha\beta} \varepsilon_{\alpha\epsilon} \Theta^{4}, \]  
(B.19)

\[ \Theta^{(\dagger)}(\theta^{\dagger^{\alpha}}\theta^{\dagger^{\beta}}) = \Theta^{(\dagger^{\alpha})\beta}(\theta^{\dagger^{\alpha}}\theta^{\dagger^{\beta}}). \]  
(B.20)

We use the spinor derivatives in the analytic basis

\[ D^{\alpha}_{b} = \partial^{\alpha}_{b}, \quad \bar{D}^{\alpha}_{b} = -\bar{\partial}^{\alpha}_{b}, \]  
(B.21)

\[ D^{\alpha}_{a} = \partial^{\alpha}_{a} + 2i\tilde{\theta}^{\dagger^{\alpha\beta}}\partial^{\beta}_{a}, \quad \bar{D}^{\alpha}_{a} = -\bar{\partial}^{\alpha}_{a} - 2i\tilde{\theta}^{\dagger^{\alpha}}\partial^{\alpha}_{a}, \]  
(B.22)

\[ \partial^{\alpha}_{a} = \frac{\partial}{\partial \theta^{\dagger^{\alpha}}}, \quad \partial^{\alpha}_{a} = \frac{\partial}{\partial \theta^{\dagger^{\alpha}}}, \quad \bar{\partial}^{\alpha}_{a} = \frac{\partial}{\partial \bar{\theta}^{\dagger^{\alpha}}}, \quad \bar{\partial}^{\alpha}_{a} = \frac{\partial}{\partial \bar{\theta}^{\dagger^{\alpha}}}, \]  
(B.23)

The CR harmonic derivative in AB

\[ D^{\dagger^{\alpha_{b}}}_{b} = \partial^{\dagger^{\alpha_{b}}}_{b} + 2i\tilde{\theta}^{\dagger^{\alpha\beta}}\theta^{\dagger^{\beta}}\partial^{\alpha}_{b} - \theta^{\dagger^{\beta}}\partial^{\dagger^{\beta}_{b}} + \bar{\theta}^{\dagger^{\alpha_{b}}}\bar{\partial}^{\dagger^{\alpha_{b}}}_{b} = (D^{\dagger^{\alpha_{b}}}_{b})^{\dagger}, \]  
(B.24)

preserves analyticity

\[ [D^{\dagger^{\alpha_{b}}}_{b}, D^{\dagger^{\alpha_{a}}}_{a}] = 0, \quad [D^{\dagger^{\alpha_{b}}}_{b}, \bar{D}^{\alpha}_{a}] = 0. \]  
(B.25)

We also define the nonanalytic and neutral harmonic derivatives in AB

\[ D^{\dagger}_{b} = \partial^{\alpha}_{b} - 2i(\tilde{\theta}^{\dagger^{\alpha}}\sigma^{m}\tilde{\theta}^{\dagger^{\alpha}})\partial^{\alpha}_{m} - \theta^{\dagger^{\alpha}}\partial^{\dagger^{\alpha}}_{b} + \bar{\theta}^{\dagger^{\alpha}}\bar{\partial}^{\dagger^{\alpha}}_{b}, \]  
(B.26)

\[ D^{0} = \partial^{0} + \theta^{\dagger^{\alpha}}\partial^{\dagger^{\alpha}} + \tilde{\theta}^{\dagger^{\alpha\beta}}\partial^{\alpha\beta} - \theta^{\dagger^{\beta}}\partial^{\dagger^{\beta}} + \bar{\theta}^{\dagger^{\alpha}}\bar{\partial}^{\dagger^{\alpha}}_{a}, \]  
(B.27)

\[ D^{a}_{b} = \partial^{a}_{b} + \tilde{\theta}^{\dagger^{a\alpha}}\partial^{\dagger^{a}_{b}} - \frac{1}{2} \delta^{a}_{b} \theta^{\dagger^{a\beta}}\partial^{\dagger^{a}}_{c} - \theta^{\dagger^{a}}\partial^{\dagger^{a}_{b}} + \frac{1}{2} \delta^{a}_{b} \theta^{\dagger^{a}}\partial^{\dagger^{c}_{a}} + \frac{1}{2} \delta^{a}_{b} \theta^{\dagger^{c}}\partial^{\dagger^{a}_{b}} + \frac{1}{2} \delta^{a}_{b} \theta^{\dagger^{c}}\partial^{\dagger^{a}_{b}}. \]  
(B.28)

The integral measure in the analytic superspace has the form

\[ d\zeta^{(-8)} = d^{4}x_{A}D^{-4}\bar{D}^{-4}, \]  
(B.30)

\[ D^{-4} = \frac{1}{24} (D^{0} D)^{-1}_{b} (D^{0} D)^{-1}_{b}, \quad \bar{D}^{-4} = \frac{1}{24} (\bar{D}^{0} \bar{D})^{-1}_{b} (\bar{D}^{0} \bar{D})^{-1}_{b}, \]  
(B.31)
B.2 Superconformal transformations in analytic basis

The even and odd parameters of the $N = 4$ superconformal group $SU(2, 2|4)$ are

$$ b, \quad \omega^a, \quad \chi^k, \quad \epsilon^a, \quad \hat{e}^{\alpha a}, \quad \eta^{\alpha}, \quad \hat{n}^{a}. \quad (B.32) $$

We start from the superconformal transformations of harmonics

$$ \delta_{sc} u^{+a}_i = -\Lambda^{++a}_b u_i^-, \quad \delta_{sc} u^{-b}_k = -((\delta_{sc} u^{+a}_i)^\sim) = \Lambda^{++a}_b u_b^- \quad (B.33) $$

$$ \delta u^{-a}_i = 0, \quad \delta u^{--b}_k = 0 \quad (B.34) $$

where the composite parameter

$$ \Lambda^{++a}_b = 2i \theta^{+b}_a \bar{\theta}^{+a}_b k_{\beta \bar{\beta}} + 2i \bar{u}^{+a}_k u_i^+ \lambda_k - 2i \bar{\theta}^{+a}_b \bar{u}^+_k \eta_{k \beta} + 2i \theta^{+a}_b \bar{u}^{-b}_k \eta_{\beta \bar{b}} = -((\Lambda^{++a}_b)^\sim) \quad (B.35) $$

satisfies the conditions

$$ D^{++b}_b \Lambda^{++a}_a = 0, \quad D^{c c}_b \Lambda^{++a}_a = \delta^c_a \Lambda^{++a}_c - \frac{1}{2} \delta^c_a \Lambda^{++a}_a, \quad \bar{D}^{\pm b}_b \Lambda^{++a}_a = -\delta^{\pm b}_a \Lambda^{++a}_a + \frac{1}{2} \delta^c_a \Lambda^{++a}_a \quad (B.36) $$

These transformations preserve the basic relations for harmonics (A.3)-(A.6). The superconformal transformations of the analytic coordinates have the form

$$ \delta_{sc} \eta^{+a}_i = \frac{1}{2} b \theta^{+a}_i + \omega^a \theta^{+a}_i + \theta^{+a}_i \bar{x}^{\beta}_A - 2i \bar{u}^{+a}_k u_j^{-b} \theta^{+a}_i \chi_j + \epsilon^a \bar{u}^{+a}_i - \bar{u}_i^{+a} x^A_+ \eta_{\beta j} + 2i \theta^{+a}_b \bar{u}_b^{-a} \eta_{j \beta} \quad (B.37) $$

$$ \delta_{sc} \bar{\eta}^{-a}_i = \frac{1}{2} b \theta^{-a}_i + \omega^a \theta^{-a}_i + \theta^{-a}_i \bar{x}^{\beta}_A - 2i \bar{u}^{-a}_k u_j^{+b} \theta^{-a}_i \chi_j + \epsilon^a \bar{u}^{-a}_i - \bar{u}_i^{-a} x^A_+ \eta_{\beta j} + 2i \theta^{-a}_b \bar{u}_b^{+a} \eta_{j \beta} \quad (B.38) $$

$$ \delta_{sc} \bar{x}^{\alpha a}_A = c^{\alpha a} + b x^{\alpha a}_A + \omega^a x^{\beta}_A + \theta^{\alpha a} \bar{x}^{\beta a}_A + \bar{x}^{\alpha a} \lambda_k - 2i \theta^{+a}_b \bar{u}^{-b}_j \bar{u}_j^{-a} \chi_k + 2i \theta^{+a}_b \bar{u}^{+b}_j \bar{u}_j^{+a} \chi_k \quad (B.39) $$

Using the complex conjugation of harmonics (A.2) and the corresponding conjugation of other AB coordinates

$$ \bar{\theta}^{+a}_i = \bar{\theta}^{-a}_i, \quad \bar{\theta}^{+a}_i = \theta^{-a}_i, \quad \bar{x}^{\alpha a}_A = x^m A \pm 2i (\theta^{-} \sigma^m \bar{\theta}^{+a}_i) - 2i (\theta^{+a}_i \sigma^m \bar{\theta}^{-a}_i) \quad (B.40) $$

we can obtain superconformal transformations of the spinor coordinates with negative charges, for instance,

$$ \delta_{sc} \eta^{-a}_i = \frac{1}{2} b \theta^{-a}_i + \omega^a \theta^{-a}_i + \theta^{-a}_i \bar{x}^{\beta}_A - 2i \bar{u}^{-a}_k u_j^{+b} \theta^{-a}_i \chi_j + \epsilon^a \bar{u}^{-a}_i - \bar{u}_i^{-a} x^A_+ \eta_{\beta j} + 2i \theta^{-a}_b \bar{u}^{+b}_j \bar{u}_j^{+a} \chi_k \quad (B.41) $$

We obtain the superconformal transformations of the harmonic derivatives

$$ \delta_{sc} D^{++a}_a = \Lambda^{+-a}_b D^{b}_a - \Lambda^{++a}_b D^{-b}_a - \frac{1}{2} \Lambda^{++a}_b D^0 \quad (B.42) $$

$$ \delta_{sc} D^{-b}_b = -(D^{+b}_b \Lambda^{--c}_e D^{-c}_e) D^{-b}_b \quad (B.43) $$

$$ \delta_{sc} [D^{++a}_a, D^{-b}_b] = 0. \quad (B.44) $$
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