Magnetic Monopoles and the Topology of Gauge Fields*†

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Lattice calculations performed in Abelian gauges give strong evidence that confinement is realized as a dual Meissner effect, implying that the Yang–Mills vacuum consists of a condensate of magnetic monopoles. We show in Polyakov gauge how the Pontrjagin index of the gauge field is related to the magnetic monopole charges.

1. Introduction

There are two basic low energy features of strong interactions, which should be explained by QCD: confinement and spontaneous breaking of chiral symmetry. While the latter can be more or less explained in terms of instantons [1], confinement is much less understood. Recently, however, much evidence has been accumulated for the idea that confinement is realized as a dual Meissner effect [2], at least in the so-called Abelian gauges [3]. In these gauges magnetic monopoles arise, which have to condense in order to form the dual superconducting ground state.

From the conceptional point of view it is unsatisfactory that spontaneous breaking of chiral symmetry and confinement is attributed to different types of field configurations. This is because lattice calculations show that both phenomena occur at about the same energy scale. In fact, lattice calculations indicate that the deconfinement phase transition and the restoration of chiral symmetry occur at about the same temperature.

Furthermore, for the spontaneous breaking of chiral symmetry the topological properties of the instantons are crucial, giving rise to zero modes of the quarks, which in turn lead to a non-zero level density at zero virtuality. The latter is sufficient to ensure a non-zero quark condensate, which is the order parameter of chiral symmetry breaking [1]. On the other hand, magnetic monopoles are long-range fields, which should also be relevant for the global topological properties of gauge fields. We can therefore expect an intimate relation between magnetic monopoles and the topology of gauge fields. It is the aim of my talk to explain how the non-trivial topology of gauge fields is generated in Abelian gauges by magnetic monopoles [4].

2. Emergence of Magnetic Monopoles in Abelian Gauges

Magnetic monopoles arise in the so-called Abelian gauges, which fix the coset $G/H$ of the gauge group $G$, but leave the Abelian gauge invariance with respect to the Cartan subgroup $H$ [3]. Recent lattice calculations show that the dual Meissner effect is equally well realized in all Abelian gauges studied [5].

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while the Abelian and monopole dominance is more pronounced in the so-called maximum Abelian gauge [2]. Here, for simplicity, we will use the Polyakov gauge, defined by a diagonalization of the Polyakov loop

\[
\Omega(\vec{x}) = P \exp \left( - \int_0^T dx_0 A_0 \right) \tag{1}
\]

\[
= V^\dagger \omega V \rightarrow \omega, \tag{2}
\]

where \( \omega \in H \) is the diagonal Polyakov loop and the matrix \( V \in G/H \) is obviously defined only up to an Abelian gauge transformation \( V \rightarrow gV, g \in H \). This gauge is equivalent to the condition

\[
A^\text{ch}_0 = 0, \quad \partial_0 A^n_0 = 0, \tag{3}
\]

where \( A^n_0 \) and \( A^\text{ch}_0 \) denote the diagonal (neutral with respect to \( H \)) and off-diagonal (charged) parts of the gauge field, respectively.

When the Polyakov loop (2) becomes a center element of the gauge group at some isolated point \( \bar{x}_i \) in 3-space there is a topological obstruction in the diagonalization and the induced gauge field

\[
A = V \partial V^\dagger, \tag{4}
\]

arising from the gauge transformation \( V \in G/H \) which makes the Polyakov loop diagonal develops a magnetic monopole [3,4].

Let us illustrate this for the gauge group \( G = SU(2) \), where the Cartan subgroup is \( H = U(1) \) and the Polyakov loop can be parametrized as

\[
\Omega(\vec{x}) = \exp(i\vec{\chi}(\vec{x}) \vec{\tau})
= \cos \chi + i\vec{\chi} \vec{\tau} \sin \chi,
\]

\[
(\chi = |\vec{\chi}|, \vec{\chi} = \vec{x}/\chi), \tag{5}
\]

with \( \tau_a, a = 1, 2, 3 \) being the Pauli matrices. In this case the matrix \( V \in G/H \) can be chosen as [4]

\[
V(\vec{x}) = e^{i\Theta \tau_\phi/2},
(\tau_\phi = \sin \phi \tau_1 + \cos \phi \tau_2), \tag{6}
\]

where \( \Theta \) and \( \phi \) denote the polar and azimuthal angles of the color unit vector \( \hat{\chi} \), respectively, and the induced gauge field (4) acquires the form

\[
A_\mu = -i/2 \tau_\phi \partial_\mu \Theta + i/2 \sin \Theta \tau_\rho \partial_\mu \phi
+ i/2 (1 - \cos \Theta) \partial_\mu \phi \tau_3 \tag{7}
\]

\[
(\tau_\rho = -\partial_\phi \tau_\phi).
\]

At those isolated points \( \bar{x}_i \) in space where \( \chi(\bar{x}_i) = \pi n_i, n_i \) being an integer, and consequently the Polyakov loop becomes a center element

\[
\Omega(\bar{x}_i) = (-1)^{n_1}, \tag{8}
\]

the Abelian part of the induced gauge potential \( A \) (7) has the form of a magnetic monopole. To be more precise the Abelian part of the magnetic field, \( \vec{B} = \vec{\nabla} \times \vec{A} \) develops an ordinary Dirac monopole with a Dirac string along the negative 3-axis. Note that the \( U(1) \) gauge freedom in the definition of \( V \in G/H \) can be exploited to move the Dirac string around. The Abelian component of the magnetic field \( \vec{b} = (\vec{A} \times \vec{A})^3 \), contains an oppositely directed monopole field, but without a Dirac string. In the Abelian component of the total magnetic field \( \vec{B}_3 = \frac{1}{2} \epsilon_{ijk} F_{ijk}[A]^3 \) the monopole field cancels and only the Dirac string survives [6]. This is easily understood by noticing that the field \( A \) (4) is a pure gauge and its field strengths \( F_{ij}[A] \) should hence vanish unless there are singularities, which here occur on the Dirac string.

An important property of the magnetic monopoles arising in the Abelian gauges is
that their magnetic charge

\[ m[V] = \frac{1}{4\pi} \int_{\bar{S}_2} d\Sigma \vec{B} \cdot \vec{A} \]

\[ = -\frac{1}{8\pi} \int_{\bar{S}_2} d\Sigma \epsilon_{ijk}[A_i, A_j] \]  

(9)
is topologically quantized and given by the winding number \( m[V] \in \Pi_2(SU(2)/U(1)) \) of the mapping \( V(\vec{x}) \in SU(2)/U(1) \) [7,4,6]. In the above equation \( S_2 \) is an infinitesimal 2-sphere around the monopole position and \( \bar{S}_2 \) arises from \( S_2 \) by leaving out the position of the Dirac string.

3. Magnetic Monopoles as Sources of Non-Trivial Topology

Since the magnetic monopoles are long-range fields, we expect that they are relevant for the topological properties of gauge fields. As is well known, the gauge fields \( A_\mu(x) \) are topologically classified by the Pontrjagin index

\[ \nu = -\frac{1}{16\pi^2} \int d^4 x Tr F_{\mu\nu} \tilde{F}_{\mu\nu} \]  

(10)

\( (\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} F_{\rho\lambda}) \).

In the gauge \( \partial_0 A_0 = 0 \) (which is satisfied by the Polyakov gauge) and for temporally periodic spatial components of the gauge field \( \vec{A}(\vec{x},T) = \vec{A}(\vec{x},0) \), the Pontrjagin index \( \nu[A] \) (10) equals the winding number of the Polyakov loop \( \Omega(\vec{x}) \) [4]:

\[ n[\Omega] = \frac{1}{24\pi^2} \int d^3 x \epsilon_{ijk} Tr L_i L_j L_k \]  

(11)

\( (L_i = \partial_0 \Omega^i) \).

For this winding number to be well defined, \( \Omega(\vec{x}) \) has to approach at spatial infinity \( |\vec{x}| \rightarrow \infty \) a value independent of the direction \( \hat{x} \), so that the 3-space \( \mathbb{R}^3 \) can be topologically compactified to the 3-sphere \( S^3 \). Without loss of generality we can assume that \( \Omega(\vec{x}) \) approaches a center element

\[ \Omega(\vec{x}) \xrightarrow{|\vec{x}|\rightarrow \infty} (-1)^{n_0} \]  

(12)

with \( n_0 \) integer. In [4] the following exact relation between the winding number \( n[\Omega] \) and the magnetic monopole charges has been derived

\[ n[\Omega] = \sum_i l_i m_i. \]  

(13)

Here the summation runs over all magnetic monopoles, \( m_i \) being the monopole charges, and the integer

\[ l_i = n_i - n_0 \]  

(14)
is defined by the center element \( \Omega(\vec{x}_i) = (-1)^{n_i} \) which the Polyakov loop becomes at the monopole position \( \vec{x}_i \) (cf. eq. 8), and by the boundary condition eq. 12. For an isolated monopole with a Dirac string running to infinity the quantity \( l_i \) represents the length of the Dirac string in group space.

To illustrate the above relation (13) consider the familiar instanton, for which the temporal component of the gauge field is given by

\[ A_0^a(\vec{x}) = \frac{2x^a}{x_0^2 + r^2 + \rho^2}, \quad r = |\vec{x}|, \]  

(15)

with \( \rho \) being the instanton size. The Polyakov loop for the instanton field acquires the form of a hedgehog

\[ \Omega(\vec{x}) = \exp(i\chi(r)\hat{x}\tau), \]  

(16)

\[ \chi(r) = \pi \frac{r}{\sqrt{r^2 + \rho^2}} \]

In the center of the hedgehog there is a magnetic monopole \( \Omega(\vec{x} = 0) = 1 \), for which \( n_{i=1} = 0 \), and which carries a magnetic charge \( m_{i=1} = 1 \). Furthermore, the hedgehog, eq. 16, satisfies the boundary condition
eq. 12 with \( n_0 = 1 \). In this case the length of the Dirac string in group space becomes \( l = 0 - 1 = -1 \) and the winding number becomes \( n[Ω] = -1 \), which is the correct result.

For the sake of illustration let us also consider a monopole-anti-monopole pair with opposite magnetic charges connected by a Dirac string. We denote the value of the Polyakov loop at the monopole and anti-monopole positions by \( Ω = (-1)^n \) and \( Ω = (-1)^{\bar{n}} \), respectively. From eq. 13 we then find for the winding number

\[
n[Ω] = (n - n_0)m - (\bar{n} - n_0)m = (n - \bar{n})m.
\] (17)

Again the winding number of the Polyakov loop field is given by the product of the magnetic charge \( m \) times the length of the Dirac string \( n - \bar{n} \) connecting monopole and anti-monopole. The asymptotic value of the Polyakov loop field at spatial infinity (12) drops out from this expression.

Finally let us replace our spatial manifold \( \mathbb{R}^3 \) by the compact space \( S_3 \). In this case the point at spatial infinity becomes part of the manifold, and there is an extra monopole due to our boundary condition at spatial infinity, eq. 12, collecting all the Dirac monopoles running to infinity. Noticing that on a compact manifold the net charge has to vanish, \( \sum_{i=0} m_i = 0 \), the monopole at infinity must carry the opposite net charge \( m_0 \) of all the remaining magnetic monopoles \( m_0 = -\sum_{i\neq 0} m_i \). Eq. 13 then simplifies to

\[
n[Ω] = \sum_{i=0} n_i m_i,
\] (18)

where the summation over the monopoles now also includes the monopole at infinity \( (i = 0) \).

Eq. 13 shows that in the Polyakov gauge the topology of gauge fields is exclusively determined by the magnetic monopoles. Given the fact that all Abelian gauges considered so far show the dual Meissner effect for the Yang–Mills vacuum we conjecture that this result is not restricted to the Polyakov gauge but applies to all Abelian gauges. The above considerations show that magnetic monopoles are entirely sufficient to generate a non-trivial topology and hence a non-zero topological susceptibility. Therefore, magnetic monopoles are also sufficient to trigger spontaneous breaking of chiral symmetry, which is usually attributed to instantons. Instantons in Abelian gauges give rise to magnetic monopoles as we have seen above, but monopoles can exist in the absence of instantons. The above considerations show that in the Abelian gauges both confinement and spontaneous breaking of chiral symmetry can be generated by the same field configurations: magnetic monopoles.

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