LOWER REGULARITY SOLUTIONS OF NON-HOMOGENEOUS BOUNDARY VALUE PROBLEMS OF THE SIXTH ORDER BOUSSINESQ EQUATION IN A QUARTER PLANE

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Abstract. In this article, we study an initial-boundary-value problem of the sixth order Boussinesq equation on a half line with nonhomogeneous boundary conditions:

\[
\begin{align*}
    u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} + (u^2)_{xx} &= 0, \quad x > 0, \ t > 0, \\
    u(x, 0) &= \varphi(x), \ u_t(x, 0) = \psi''(x), \\
    u(0, t) &= h_1(t), \ u_{xx}(0, t) = h_2(t), \ u_{xxxx}(0, t) = h_3(t),
\end{align*}
\]

where \( \beta = \pm 1 \). It is shown that the problem is locally well-posed in \( H^s(\mathbb{R}^+) \) for \( -\frac{1}{2} < s \leq 0 \) with initial condition \( (\varphi, \psi) \in H^s(\mathbb{R}^+) \times H^{s-1}(\mathbb{R}^+) \) and boundary condition \( (h_1, h_2, h_3) \) in the product space \( H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s-3}{3}}(\mathbb{R}^+) \).

1. Introduction

The Boussinesq equation,

\[
    u_{tt} - u_{xx} + (u^2)_{xx} - u_{xxxx} = 0,
\]

was originally derived by J. Boussinesq (1871) \[9\] in his study on propagation of small amplitude, long waves on the surface of water. It possesses some special traveling wave solutions called solitary waves, and it is the first equation that gives a mathematical explanation to the phenomenon of solitary waves that was discovered and reported by John Scott Russell in the 1830s. The original Boussinesq equation has been used in a considerable range of applications such as coast and harbor engineering, simulation of tides and tsunamis.

However, the original Boussinesq equation (1.1) has a drawback, it is ill-posed for its initial-value problem in the sense that a slight difference in initial data might evolve into a large change in solutions. This can be seen, for example, by considering its linear equation as

\[
    u_{tt} - u_{xxxx} := (\partial_t + \partial_{xx})(\partial_t - \partial_{xx})u = 0.
\]

The “\( \partial_t - \partial_{xx} \)" can be treated as the heat equation and it is well-posed, but “\( \partial_t + \partial_{xx} \)”, the backward heat equation, is ill-posed. One way to correct this
ill-posedness issue is to alter the sign of the fourth order derivative term, which leads to the “good” Boussinesq equation

\[ u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0. \]  

Similarly, its well-posedness can be seen by considering its linearized equation as

\[ u_{tt} + u_{xxxx} := (\partial_t + i\partial_{xx})(\partial_t - i\partial_{xx})u = 0, \]

a combination of the Schrödinger and the reversed Schrödinger equations whose initial-value problems are both well-posed. The mathematical study on the “good” Boussinesq equation has been well-developed in the past decades [1, 2, 3, 4, 12, 17, 18, 19, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45].

But, due to the change of the sign to the Boussinesq equation, the “good” Boussinesq equation cannot be well justified as a physical modeling of water waves as the original Boussinesq equation. To remedy the case, Christov, Maugin and Velarde [10] modified the original Boussinesq’s physical modeling and derived the sixth order Boussinesq equation,

\[ u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} + (u^2)_{xx} = 0, \quad \beta = \pm 1. \]

Its well-posedness can be seen from the linearized equation:

\[ u_{tt} - u_{xxxxxx} := (\partial_t + \partial_{xxx})(\partial_t - \partial_{xxx})u = 0, \]

which can be considered as a coupled linear KdV equations, where the KdV and the reversed KdV equations are both well-posed for their initial value problems.

The sixth order Boussinesq equation was also proposed in modeling the nonlinear lattice dynamics in elastic crystals by Maugin [35]. In addition, Feng et al [20] studied the solitary waves and their interactions of the sixth order Boussinesq equation. Kamenov [21] obtained an exact periodic solution through the Hirota’s bilinear transform method. The mathematical study of the initial-value problem (IVP) for the sixth order Boussinesq equation posed on \( \mathbb{R} \),

\[
\begin{aligned}
&u_{tt} - u_{xx} + (u^2)_{xx} + \beta u_{xxxx} - u_{xxxxxx} = 0, \quad x \in \mathbb{R}, \\
u(x, 0) = \varphi(x), u_t(x, 0) = \psi'(x),
\end{aligned}
\]  

was started by Esfahani, Farah and Wang [15]. By applying the Strichartz type smoothing inherited from Linares’ work [29] for the Boussinesq equation, they established the local well-posedness in \( L^2(\mathbb{R}) \) and \( H^1(\mathbb{R}) \) for the IVP (1.3) for initial data \((\varphi, \psi) \in L^2(\mathbb{R}) \times H^{-1}(\mathbb{R}) \) and \((\varphi, \psi) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \), respectively. Esfahani and Farah [14] proved the local well-posedness in the space \( H^s(\mathbb{R}) \) for \( s > -\frac{1}{2} \) by using the corresponding Bourgain space, \( X^{s,b} \), inherited from Kenig, Ponce and Vega’s work [25] on the KdV equation. Later on, they improved their result to \( s > -\frac{3}{4} \) (c.f. [16]) by using the \([k; Z]-multiplier norm method. In addition, for the IVP posed on a periodic
domain $\mathbb{T}$, it was shown to be locally well-posed in the space $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$ by Wang and Esfahani [40].

This paper studies another physically relevant problem that arises in simulations and predictions of water waves, where the spatial variable $x$ is taken from $\mathbb{R}^+$ and the boundary data is specified at the boundary $x = 0$, that is, an initial-boundary-value problem (IBVP) for $x > 0$, such as:

\begin{align}
\begin{cases}
    u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} + (u^2)_{xx} = 0, & \text{for } x > 0, t > 0, \\
    u(x, 0) = \varphi(x), u_t(x, 0) = \psi''(x), \\
    u(0, t) = h_1(t), u_{xx}(0, t) = h_2(t), u_{xxxx}(0, t) = h_3(t).
\end{cases}
\end{align}

Unlike the study for its Cauchy problem, this non-homogeneous boundary-value problem for the sixth order Boussinesq has not yet been well-studied compared to its Cauchy problem. We have studied its IBVP posed on the half plane, $\mathbb{R}^+$, and finite domain, $[0, L]$, in [27, 28].

The figure shows the model when the sixth order Boussinesq equation is applied to a physical problem. It is a model of unidirectional wave in an open uniform channel with a wavemaker at one end. The $x$-axis denotes the distance from the wavemaker, the $t$-axis denotes the elapsed time and the dependent function $u(x, t)$ denotes the velocity of fluid along the $x$-direction at certain layer of the channel at position $x$ and on time $t$.

![Sketch of the half line problem](image)

We consider the well-posedness issue of the IBVP (1.4) and aim to establish the local well-posedness for the IBVP (1.4) in the classic $L^2$-based Sobolev spaces, $H^s(\mathbb{R}^+)$, when the initial data lies in $H^s(\mathbb{R}^+) \times H^{s-1}(\mathbb{R}^+)$ and the boundary data $(h_1, h_2, h_3)$ is drawn from the product space

\begin{equation}
H^{s_1}(0, T) \times H^{s_2}(0, T) \times H^{s_3}(0, T)
\end{equation}

or some appropriate $s_1$, $s_2$ and $s_3$ depending on $s$. The following two questions arise naturally.

1. While the initial data $\varphi$ and $\psi$ are required to be in the spaces $H^s(\mathbb{R}^+)$ and $H^{s-1}(\mathbb{R}^+)$, respectively, for the well-posedness of the
IBVP (1.4) in the space $H^s(\mathbb{R}^+)$, what are the optimal (minimum) regularity requirements on the boundary data $(h_1, h_2, h_3)$ for the IBVP (1.4)?

(2) What is the smallest value of $s$ such that the IBVP (1.4) is well-posed in the space $H^s(\mathbb{R}^+)$?

As the solution $u$ of the Cauchy problem

\[
\begin{cases}
  u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} = 0, & x \in \mathbb{R}, \\
  u(x, 0) = \varphi(x), u_t(x, 0) = \psi'(x),
\end{cases}
\]

possesses the sharp Kato smoothing property for any $s$:

\[
\sup_{x \in \mathbb{R}} \| \partial_j^2 u(x, \cdot) \|_{H^\frac{s-1}{3}(\mathbb{R}^+)} \lesssim \| \varphi \|_{H^s(\mathbb{R})} + \| \psi \|_{H^{s-1}(\mathbb{R})}, \quad j = 0, 1, \ldots, 5,
\]

it is therefore necessary to have (or the smallest possible choices in numbers for the $s_1, s_2$ and $s_3$ in (1.5)):

\[
(1.6) \quad h_1 \in H^\frac{s+1}{3}_{loc}(\mathbb{R}^+), \quad h_2 \in H^\frac{s-1}{3}_{loc}(\mathbb{R}^+), \quad h_3 \in H^\frac{s-3}{3}_{loc}(\mathbb{R}^+)
\]

for the IBVP (1.4) being well-posed in the space $H^s(\mathbb{R}^+)$. It has already been shown (c.f. [11]) that (1.6) is also sufficient and the related IBVP is indeed well-posed for $s \geq 0$. It is then natural to expect a similar conclusion for the IBVP (1.4) for $s \leq 0$. In this article, we will use the Bourgain space (c.f. [7, 8, 22, 23, 24, 25, 26]) aiming to show the local well-posedness for $-\frac{1}{2} < s \leq 0$.

Before stating our main theorem, we define $\langle x \rangle = \sqrt{1 + x^2}$ and a related Bourgain space, $X^{s,b}$, for the sixth order Boussinesq equation.

**Definition 1.1.** For $s, b \in \mathbb{R}$, $X^{s,b}$ denotes the completion of the Schwartz class $S(\mathbb{R}^2)$ with

\[
\| u \|_{X^{s,b}(\mathbb{R}^2)} = \left\| \langle \xi \rangle^s \langle \tau \rangle^b \hat{u}(\xi, \tau) \right\|_{L^2_\xi,\tau(\mathbb{R}^2)},
\]

where $\phi(\xi) = \sqrt{\xi^6 + \beta \xi^4 + \xi^2}$ and " $\wedge$ " denotes the Fourier transform on both time and space.

According to this definition, $X^{s,b}$ is imbedded in the $C^0_t H^s_x$ for $b > \frac{1}{2}$. Moreover, inspired by Farah’s work [14, 18], we will show that the norm of the $X^{s,b}$ space we defined for the sixth order Boussinesq equation has an equivalence to the one for the KdV-type equations (c.f. [22, 23, 24, 25]). In addition, we shall point out that the Bourgain space defined here is for functions on the whole plane, $(x, t) \in \mathbb{R} \times \mathbb{R}$. However, the IBVP (1.4) is posed on the quarter plane, $\mathbb{R}^+ \times \mathbb{R}^+$. It is therefore necessary to consider a restricted Bourgain space,

\[
X^{s,b}(\mathbb{R}^+ \times \Omega) := X^{s,b}|_{\mathbb{R}^+ \times \Omega}, \quad \Omega \subset \mathbb{R}^+,
\]

with the quotient norm,

\[
\| u \|_{X^{s,b}(\mathbb{R}^+ \times \Omega)} := \inf_{w \in X^{s,b}(\mathbb{R}^2)} \{ \| w \|_{X^{s,b}(\mathbb{R}^2)} : w(x, t) = u(x, t) \text{ on } \mathbb{R}^+ \times \Omega \},
\]
for the wellposedness issue. For the sake of simplicity on notation, we consider \( X^{s,b} \) as \( X^{s,b}(\mathbb{R}^+ \times \mathbb{R}^+) \) if no domain are imposed. We now state the main result for this article.

**Theorem 1.1.** Let \( -\frac{1}{2} < s \leq 0 \) be given, there exists a \( T > 0 \) such that for any

\[
(\varphi, \psi, h_1, h_2, h_3) \in H^{s}(\mathbb{R}^+) \times H^{s-1}(\mathbb{R}^+) \times H^{s+\frac{1}{2}}(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s-3}{3}}(\mathbb{R}^+),
\]

the IBVP (1.4) admits a unique solution

\[
u \in C(0,T; H^{s}(\mathbb{R}^+)) \cap X^{s,b}(\mathbb{R}^+ \times (0,T))
\]

for \( \frac{1}{2} - b > 0 \) sufficient small. Moreover, the corresponding solution map is Lipschitz continuous.

It is notable that the theorem shows no difference in the local well-posedness for different \( \beta \) (consider \( \beta = \pm 1 \)). However, without the sixth order term (i.e. \( u_{xxxxxxx} \) in (1.4)), theories for the “bad” (1.1) and “good” (1.2) Boussinesq equations are quite distinct. In addition, the conclusion on \( s > -\frac{1}{2} \) for the half line problem of the sixth order Boussinesq equation is better when comparing to \( s > -\frac{1}{4} \) (c.f. [11]) which is by far the best result for the related problem of the “good” Boussinesq equation. Moreover, based on the proof of the theorem, one can consider the IBVPs for other types of boundary conditions, such as

\[
u(0,t) = h_4(t), \quad u_x(0,t) = h_5(t), \quad u_{xx}(0,t) = h_6(t),
\]
or

\[
u_x(0,t) = h_7(t), \quad u_{xx}(0,t) = h_8(t), \quad u_{xxx}(0,t) = h_9(t),
\]

without compromising the local well-posedness.

The main idea of this paper is inspired by Bona, Sun and Zhang’s work for the IBVP of the KdV equation [5, 6] and Tzirakis’s work [11, 13] on the Schrödinger and Boussinesq equations. Similar to our previous work [27] on the IBVP of the sixth order Boussinesq equation with \( s \geq 0 \), the key for the problem with \( s \leq 0 \) is to explore an explicit integral formula for the associated linear problem:

\[
\begin{cases}
u_{tt} - \nu_{xx} + \beta \nu_{xxxx} - \nu_{xxxxxx} = 0, & x > 0, t > 0, \\
u(x,0) = 0, \quad u_t(x,0) = 0,
\end{cases}
\]

(1.7)

However, for \( s \leq 0 \), the Strichartz estimates that we established in [27] won’t help to address the IBVP (1.4). We need to study its estimates on \( X^{s,b} \) and show the solution for the problem (1.7) satisfying:

\[
\sup_{t \geq 0} \|u(\cdot,t)\|_{H_x^s(\mathbb{R}^+)} + \sup_{x \geq 0} \|\partial_x^3 u(x,\cdot)\|_{H_x^{s+\frac{1}{3}}(\mathbb{R}^+)} + \|\eta(t)u(x,t)\|_{X^{s,b}} \lesssim \|(h_1, h_2, h_3)\|_{H_x^{s+\frac{1}{3}}(\mathbb{R}^+) \times H_x^{s-\frac{1}{3}}(\mathbb{R}^+) \times H_x^{s-3}(\mathbb{R}^+)'.
\]
for $j = 0, 2, 4, -\frac{1}{2} < s \leq 0$ and $b$ close to $\frac{1}{2}$ with $\eta$ being a cut-off function. Such regularities of the boundary data perfectly match the sharp Kato smoothing
\[
\sup_{x \in \mathbb{R}} \| \partial^j_x u(x, \cdot) \|_{H^{s-j+1/4}(\mathbb{R}^+)} \lesssim \| (p, q) \|_{H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})}, \quad j = 0, 2, 4,
\]
for the pure initial-value problem,
\[
\begin{aligned}
&u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} = 0, \quad x \in \mathbb{R}, t > 0, \\
&u(x, 0) = p(x), \quad u_t(x, 0) = q''(x).
\end{aligned}
\]
This sharp Kato smoothing provides us a necessary condition for the regularities of the boundary data which also happens to be a sufficient condition on its wellposedness issue. Then, combining with the conclusions for its Cauchy problem (c.f. [14]), we can study the IBVP,
\[
\begin{aligned}
&u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} = 0, \quad x > 0, t > 0, \\
&u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi''(x), \\
&u(0, t) = 0, \quad u_{xx}(0, t) = 0, \quad u_{xxxx}(0, t) = 0,
\end{aligned}
\]
and show that for $\varphi \in H^s(\mathbb{R}^+)$ and $\psi \in H^{s-1}(\mathbb{R}^+)$,
\[
\sup_{t \geq 0} \| u(\cdot, t) \|_{H^s(\mathbb{R}^+)} + \sup_{x \geq 0} \| \partial^j_x u(x, \cdot) \|_{H^{s-j+1/4}(\mathbb{R}^+)} + \| \eta(t) u(x, t) \|_{X_{s,b}} \lesssim \| \varphi \|_{H^s(\mathbb{R}^+)} + \| \psi \|_{H^{s-1}(\mathbb{R}^+)},
\]
with given $-\frac{1}{2} < s \leq 0$, $b$ close to $\frac{1}{2}$ and $j = 0, 2, 4$. Similar argument can be applied to the forced linear problem,
\[
\begin{aligned}
&u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} = g_{xx}(x, t), \quad x > 0, t > 0, \\
&u(x, 0) = 0, \quad u_t(x, 0) = 0, \\
&u(0, t) = 0, \quad u_{xx}(0, t) = 0, \quad u_{xxxx}(0, t) = 0,
\end{aligned}
\]
via the Duhamel’s principle. All of those estimates help us to establish a contraction mapping for the well-posedness problem. We shall point out that some of our estimates for the forced linear problem are only valid for $b < \frac{1}{2}$, while the bilinear estimate for the Cauchy problem of the sixth order Boussinesq equation in [14] only valid for $b > \frac{1}{2}$. Therefore, we will need to derive a proper bilinear estimate for $b < \frac{1}{2}$.

The paper is organized as follows: In section 2, some important notations, basic lemmas of inequality and some existing results on the Cauchy problem of the sixth order Boussinesq equation will be provided. Section 3 is devoted to consider the associated linear problems and it is divided into two subsections. Explicit solution formulas for the associated linear problems will be derived in subsection 3.1. The sharp Kato smoothing property, $X_{s,b}$ estimates and other related estimates will be provided for the linear problems in subsection 3.2. Section 4 is devoted to show a valid bilinear estimate for $b < \frac{1}{2}$. The local well-posedness will be established in Section 5 through contraction mapping principle.
2. Preliminary

This section introduces some important notations and lemmas that will be used in the later study. Followings are some important lemmas of elementary calculus inequalities and the reader can refer both of the proof to \[13, 14\].

**Lemma 2.1.** If $\alpha, \beta \geq 0$, $\alpha + \beta > 1$ and $\gamma = \min\{\alpha, \beta, \alpha + \beta - 1\}$, then,

$$\int_{\mathbb{R}} \frac{1}{(x-a)^{\alpha}(x-b)^{\beta}} \, dx \lesssim \frac{1}{(a-b)^{1-\gamma}}.$$

**Lemma 2.2.** For $l \in \left(\frac{1}{2}, 1\right)$,

$$\int_{\mathbb{R}} \frac{1}{(x)^{l}\sqrt{|x-a|}} \lesssim \frac{1}{(a)^{l-\frac{1}{2}}}.$$

Next, the lemma below comes from \[28\] which is inspired by Farah \[14, 18\].

**Lemma 2.3.** There exists a $c > 0$ such that

$$\frac{1}{c} \leq \sup_{x,y \geq 0} \frac{1 + |x - \sqrt{y^3} - \frac{\beta}{2}\sqrt{y^3}|}{1 + |x - \sqrt{y^3 + \beta y^2 + y}|} \leq c, \quad \beta = \pm 1.$$

It then can lead to an equivalence between the norm of the Bourgain space for the sixth order Boussinesq and the one for the KdV-type equation (c.f. \[22, 23, 24, 25\]), that is,

$$\|u\|_{X^{s,b} (\mathbb{R}^2)} \sim \|\langle \xi \rangle^s (|\tau| - |\xi|^3 + \frac{\beta}{2}|\xi|)^b \hat{u}(\xi, \tau)\|_{L^2_{\xi,\tau} (\mathbb{R}^2)}.$$

For convenience, we will denote $\eta$ to be a cut-off function all through this article satisfying $\eta \in \mathcal{C}^\infty(\mathbb{R})$ with

$$\eta = \begin{cases} 1, & x \in [-1,1], \\ 0, & x < -2 \text{ or } x > 2. \end{cases}$$

We define $\eta_T(t) = \eta(t/T)$ for $0 < T < 1$. The following lemma is introduced in \[13\] and will be useful in the later proof.

**Lemma 2.4.** For any $-\frac{1}{2} < b' < b < \frac{1}{2}$, $\|\eta_T(t)u\|_{X^{s,b}} \lesssim T^{b-b'}\|u\|_{X^{s,b'}}$.

Finally, we introduce the operator $W_{\mathbb{R}}$ such that $u = [W_{\mathbb{R}}(f_1, f_2)](x,t)$ solve the linear problem,

$$\begin{cases} u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = f_1(x), u_t(x,0) = f_2''(x). \end{cases}$$

and write $[W_{\mathbb{R}}(f_1, f_2)](x,t) := [V_1(f_1)](x,t) + [V_2(f_2)](x,t)$ with

$$[V_1(f_1)](x,t) := \frac{1}{2} \int_{\mathbb{R}} \left( e^{i(t\phi(\xi)+x\xi)} + e^{i(-t\phi(\xi)+x\xi)} \right) \hat{f}_1(\xi) \, d\xi$$

and

$$[V_2(f_2)](x,t) := \frac{1}{2i} \int_{\mathbb{R}} \left( e^{i(t\phi(\xi)+x\xi)} - e^{i(-t\phi(\xi)+x\xi)} \right) \frac{\xi^2 \hat{f}_2(\xi)}{\phi(\xi)} \, d\xi,$$
The lemma below are some estimates of $W_{\mathbb{R}}$ that comes from [14, 15, 27].

**Lemma 2.5.** Given $f_1 \in H^s(\mathbb{R})$ and $f_2 \in H^{s-1}(\mathbb{R})$, for any $s$ and $b$, the solution $u$ of the IVP (2.1) satisfies
\[
\|\eta(t)u(x,t)\|_{X^{s,b}(\mathbb{R}^2)} \lesssim \|f_1\|_{H^s(\mathbb{R})} + \|f_2\|_{H^{s-1}(\mathbb{R})},
\]
\[
\sup_{t>0}\|u(\cdot,t)\|_{H^s(\mathbb{R})} \lesssim \|f_1\|_{H^s(\mathbb{R})} + \|f_2\|_{H^{s-1}(\mathbb{R})},
\]
\[
\sup_{x>0}\|\partial_x^k u(x,\cdot)\|_{H^{s-k+1}(\mathbb{R})} \lesssim \|f_1\|_{H^s(\mathbb{R})} + \|f_2\|_{H^{s-1}(\mathbb{R})},
\]
for $k = 0, 1, 2, \ldots, 5$.

According to the Duhamel’s principal, the pure IVP for the forced linear equation,
\[
\begin{align*}
\begin{cases}
  u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} = g_{xx}(x,t), & x \in \mathbb{R}, \ t > 0, \\
  u(x,0) = 0, \ u_t(x,0) = 0,
\end{cases}
\end{align*}
\]
has its solution $u$ in the form of
\[
u(x,t) = \int_0^t [W_{\mathbb{R}}(0,g)](x,t - \tau)d\tau.
\]

The following lemma comes from [14].

**Lemma 2.6.** Let $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$, then for any $s$,\[
\left\| \eta_1(t) \int_0^t [W_{\mathbb{R}}(0,g)](x,t - \tau)d\tau \right\|_{X^{s,b}(\mathbb{R}^2)} \lesssim T^{1-b+b'} \left\| \left( \frac{\xi^2 \hat{g}(\xi,\tau)}{2i\phi(\xi)} \right)^\vee \right\|_{X^{s,b'}(\mathbb{R}^2)},
\]
where $\phi(\xi) = \sqrt{\xi^6 + \beta \xi^4 + \xi^2}$, $\wedge$ and $\vee$ denote the Fourier transform and inverse Fourier transform in both time and space.

3. Linear Problems

This section concerns on looking for an explicit formula along with some related estimates for the IBVP of the associated forced linear equation:

\[
\begin{align*}
\begin{cases}
  u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} = g_{xx}(x,t), & x > 0, \ t > 0, \\
  u(x,0) = \varphi(x), \ u_t(x,0) = \psi'(x), \\
  u(0,t) = h_1(t), \ u_x(0,t) = h_2(t), \ u_{xxx}(0,t) = h_3(t).
\end{cases}
\end{align*}
\]

3.1. Boundary integral operators and solution formulas. First, we consider the linear problem with homogeneous initial conditions,

\[
\begin{align*}
\begin{cases}
  u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} = 0, & x > 0, \ t > 0, \\
  u(x,0) = 0, \ u_t(x,0) = 0, \\
  u(0,t) = h_1(t), \ u_x(0,t) = h_2(t), \ u_{xxx}(0,t) = h_3(t).
\end{cases}
\end{align*}
\]
Applying the Laplace transform with respect to $t$, the IBVP (3.2) is converted to
\begin{align*}
\rho^2 \tilde{u} - \tilde{u}_{xx} + \beta \tilde{u}_{xxxx} - \tilde{u}_{xxxxxx} &= 0, \\
\tilde{u}(0,\rho) &= \tilde{h}_1(\rho), \quad \tilde{u}_{xx}(0,\rho) = \tilde{h}_2(\rho), \quad \tilde{u}_{xxxx}(0,\rho) = \tilde{h}_3(\rho), \\
\tilde{u}(+\infty,\rho) &= 0, \quad \tilde{u}_{xx}(+\infty,\rho) = 0, \quad \tilde{u}_{xxxx}(+\infty,\rho) = 0,
\end{align*}
(3.3)
where
\[ \tilde{u}(x,\rho) = \int_0^{+\infty} e^{-\rho t} u(x,t) dt, \quad \tilde{h}_j(\rho) = \int_0^{+\infty} e^{-\rho t} h_j(t) dt, \quad j = 1, 2, 3. \]

For any $\rho$ with $\text{Re}(\rho) > 0$ satisfying $\rho \neq \frac{2\sqrt{2} \pm i}{3\sqrt{3}}$, the solution $\tilde{u}(x,\rho)$ of (3.3) can be written in the form
\[ \tilde{u}(x,\rho) = \sum_{j=1}^3 c_j(\rho) e^{\gamma_j(\rho)x}, \]
(3.4)
where $\gamma_1 = \gamma_1(\rho)$, $\gamma_2 = \gamma_2(\rho)$ and $\gamma_3 = \gamma_3(\rho)$, $\text{Re}(\gamma_j) < 0$ for $j = 1, 2, 3$, are three solutions of the characteristic equation
\[ \gamma^6 - \gamma^4 + \gamma^2 - \rho^2 = 0; \]
and $c_j(\rho)$, for $j = 1, 2, 3$, solve the linear system
\begin{align*}
\begin{cases}
  c_1 + c_2 + c_3 &= \tilde{h}_1(\rho), \\
  \gamma_1^2 c_1 + \gamma_2^2 c_2 + \gamma_3^2 c_3 &= \tilde{h}_2(\rho), \\
  \gamma_1^4 c_1 + \gamma_2^4 c_2 + \gamma_3^4 c_3 &= \tilde{h}_3(\rho).
\end{cases}
\end{align*}
(3.6)
Let $\Delta(\rho)$ be the determinant of the coefficient matrix of (3.6) and $\Delta_j(\rho)$ be the determinants of the matrices with the $j$-th column replacing by $(\tilde{h}_1(\rho), \tilde{h}_2(\rho), \tilde{h}_3(\rho))^T$ for $j = 1, 2, 3$. In addition, for $j, m = 1, 2, 3$, we set $\Delta_{j,m}(\rho)$ obtained from $\Delta_j(\rho)$ by letting $\tilde{h}_m(\rho) = 1$ and $\tilde{h}_j(\rho) = 0$ for $j \neq m$. Moreover, $\gamma_j^\pm(\mu)$ for $j = 1, 2, 3$ denote the three distinct roots of the characteristic equation (3.5) by changing the variable $\rho = i\phi(\mu)$ where
\[ \phi(\mu) = \mu\sqrt{\mu^4 + \mu^2 + 1}, \quad \text{for} \quad \mu > 0. \]
Then, one can solve,
\[ \gamma_1^+ = i\mu, \quad \gamma_2^+ = -p(\mu) - iq(\mu), \quad \gamma_3^+ = -p(\mu) + iq(\mu), \]
(3.8)
where
\[ p(\mu) = \frac{1}{\sqrt{2}} \left( \sqrt{\mu^2 + 1 + \sqrt{4\mu^4 + 4\mu^2 + 4}} \right), \]
\[ q(\mu) = \frac{1}{\sqrt{2}} \left( \sqrt{\sqrt{4\mu^4 + 4\mu^2 + 4} - \mu^2 - 1} \right). \]
Taking inverse Laplace transform of (3.4), we can obtain the formula for the solution of (3.2) presented in the following lemma.
Lemma 3.1. Given \( \vec{h} = (h_1, h_2, h_3) \), the solution \( u \) of the IBVP \((3.2)\) can be written in the form

\[
    u(x, t) = [W_{bdr}(\vec{h})](x, t) := \sum_{m=1}^{3} [W_{bdr,m}(h_m)](x, t)
\]

with

\[
    [W_{bdr,m}(h_m)](x, t) = I_m(x, t) + I_m(x, t), \quad m = 1, 2, 3,
\]

where

\[
    I_m(x, t) = \frac{1}{2\pi} \sum_{j=1}^{3} \int_{0}^{+\infty} e^{i\mu \sqrt{\mu^2 + 1}} e^{\gamma_j^+ (\mu) x} \frac{\Delta_{i,m}(\mu) 3\mu^4 + 2\mu^2 + 1}{\Delta_j^+(\mu) \sqrt{\mu^2 + 1}} \overline{h}^+_m(\mu) d\mu,
\]

and

\[
    \overline{h}^+_m(\mu) = \int_{0}^{\infty} e^{i\phi(\mu)t} h_m(t) dt.
\]

Next, for the linear problems with homogeneous boundary conditions,

\[
    \begin{align*}
        u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} &= 0, & x > 0, \ t > 0, \\
        u(x, 0) &= \varphi(x), \ u_t(x, 0) = \psi'(x), \\
        u(0, t) &= 0, \ u_{xx}(0, t) = 0, \ u_{xxxx}(0, t) = 0,
    \end{align*}
\]

we have the following two lemmas based on Lemma 3.1.

Lemma 3.2. For any \( s \), given \( \varphi \in H^s(\mathbb{R}^+) \) and \( \psi \in H^{s-1}(\mathbb{R}^+) \), let \( \varphi^*, \ \psi^* \) be extensions of \( \varphi \) and \( \psi \) from \( \mathbb{R}^+ \to \mathbb{R} \), respectively. For

\[
    p(x, t) = [W_{R}(\varphi^*, \psi^*)](x, t),
\]

set \( \vec{p} = (p_1, p_2, p_3) \) and

\[
    p_1(t) = p(0, t), \quad p_2(t) = p_{xx}(0, t), \quad p_3(t) = p_{xxxx}(0, t).
\]

Then the solution \( u \) of the IBVP \((3.9)\) can be written in the form

\[
    u(x, t) = [W_{R}(\varphi^*, \psi^*)](x, t) - [W_{bdr}(\vec{p})](x, t).
\]

Lemma 3.3. For \( g \in C^\infty([0, T] \times \mathbb{R}^+) \), let \( g^* \in C^\infty([0, T] \times \mathbb{R}) \) be an extension of \( g \) from \( \mathbb{R}^+ \) to \( \mathbb{R} \). Then the corresponding solution \( u \) of \((3.10)\) can be written as

\[
    u(x, t) = \int_{0}^{t} [W_{R}(0, g^*)](x, t - \tau) d\tau - [W_{bdr}(\vec{q})](x, t)
\]

where \( \vec{q} = (q_1, q_2, q_3) \) and

\[
    q_1(t) = q(0, t), \quad q_2(t) = q_{xx}(0, t), \quad q_3(t) = q_{xxxx}(0, t).
\]
with
\[ q(x,t) = \int_0^t [W_\mathbb{R}(0,g^*)](x,t-\tau)d\tau. \]

For more details on the generation of these explicit solutions of the IBVPs, the reader can refer to [27].

3.2. Linear estimate. In this subsection, we consider the estimates of the solutions that describe in Lemma 3.1, 3.2 and 3.3.

3.2.1. Linear problem with homogeneous initial conditions. First, we consider the estimates for the solution of the IBVP
\[ \begin{cases} u_{tt} - u_{xx} + \beta u_{xxxx} - u_{xxxxxx} = 0, & x > 0, t > 0, \\ u(x,0) = 0, u_t(x,0) = 0, \\ u(0,t) = h_1(t), u_{xx}(0,t) = h_2(t), u_{xxxx}(0,t) = h_3(t). \end{cases} \]

(3.11)

We denote \( \tilde{h} = (h_1, h_2, h_3) \) and state the following conclusions for \( W_{\text{bdr}} \) introduced in Lemma 3.1.

**Proposition 3.4.** For \( s \leq 0 \),
\[ \sup_{t>0} \| [W_{\text{bdr}}(\tilde{h})](\cdot,t) \|_{H^s(\mathbb{R}^+)} \lesssim \| h_1 \|_{H^{s+1}(\mathbb{R}^+)} + \| h_2 \|_{H^{s+1}(\mathbb{R}^+)} + \| h_3 \|_{H^{s+3}(\mathbb{R}^+)}, \]
\[ \sup_{x>0} \| \partial_x^k [W_{\text{bdr}}(\tilde{h})](x,\cdot) \|_{H^{s-k+1/2}(\mathbb{R}^+)} \lesssim \| h_1 \|_{H^{s+1/2}(\mathbb{R}^+)} + \| h_2 \|_{H^{s+1/2}(\mathbb{R}^+)} + \| h_3 \|_{H^{s+3/2}(\mathbb{R}^+)}, \]

with \( k = 0, 1, 2, ..., 5. \)

**Proposition 3.5.** For \( s \leq 0 \) and \( |1/2 - b| \) sufficient small,
\[ \| \eta(t) W_{\text{bdr}}(\tilde{h}) \|_{X^{s,b}} \lesssim \| h_1 \|_{H^{s+1}(\mathbb{R}^+)} + \| h_2 \|_{H^{s+1}(\mathbb{R}^+)} + \| h_3 \|_{H^{s+3}(\mathbb{R}^+)}. \]

**Proof.** (Proposition 3.4) The proof for the second estimate is same as for \( s \geq 0 \) in [27]. For the first estimate, it suffices to prove for \( \tilde{h} = (h_1,0,0) \). According to Lemma 3.1,
\[ [W_{\text{bdr}}(h_1,0,0)](x,t) = I_1(x,t) + \overline{I_1(x,t)}, \]

where
\[ I_1(x,t) = \frac{1}{2\pi} \sum_{j=1}^3 \int_0^{+\infty} e^{i\phi(\mu)t} e^{i\gamma_j^+(\mu)x} \frac{\Delta_{j,m}^+(\mu)}{\Delta^+(\mu)} \frac{3\mu^4 + 2\mu^2 + 1}{\sqrt{\mu^4 + \mu^2 + 1}} h_j^+(\mu)d\mu := \frac{1}{2\pi} \sum_{j=1}^3 I_{1,j}, \]

with \( h_j^+(\mu) = \int_0^{+\infty} e^{i\phi(\mu)t} h_j(t)dt \) and \( \gamma_1^+, \gamma_2^+, \gamma_3^+ \) given in (3.8). It then suffices to show the estimate for \( I_1 \).

For \( j = 1 \), we have \( \gamma_j^+(\mu) = i\mu \). Through direct computation,
\[ \frac{\Delta_{1,1}^+(\mu)}{\Delta^+(\mu)} \sim 1 \quad \text{and} \quad \frac{3\mu^4 + 2\mu^2 + 1}{\sqrt{\mu^4 + \mu^2 + 1}} \sim \mu^2 \quad \text{as} \quad \mu \rightarrow \infty, \]
then,
\[
\|I_{1,1}(\cdot, t)\|_{H^s(\mathbb{R}^+)} \lesssim \left\| \int_0^{+\infty} e^{i\mu x} \mu^2 \tilde{h}_1^+(\mu) d\mu \right\|_{H^s(\mathbb{R}^+)} \\
\lesssim \left\| \mu^2 \tilde{h}_1^+(\mu) \right\|_{L^2_s(\mathbb{R}^+)} \\
\lesssim \zeta \int_0^{+\infty} e^{i\zeta \tau} h_1(\tau) d\tau \right\|_{L^2_s(\mathbb{R}^+)} \lesssim \|h_1\|_{H^{s+1}(\mathbb{R}^+)}.
\]
For \( j = 2 \), we have \( \gamma_2^+(\mu) = -p(\mu) - iq(\mu) \). Through direct computation,
\[
\frac{\Delta_{2,1}^+(\mu)}{\Delta^+(\mu)} \sim 1 \quad \text{and} \quad \frac{3\mu^4 + 2\mu^2 + 1}{\sqrt{\mu^4 + \mu^2 + 1}} \sim \mu^2 \quad \text{as} \quad \mu \to \infty,
\]
then,
\[
\|I_{1,2}(\cdot, t)\|_{H^s(\mathbb{R}^+)} \lesssim \left\| \int_0^{+\infty} e^{-iq(\mu)x} \mu^2 \tilde{h}_1^+(\mu) d\mu \right\|_{H^s(\mathbb{R}^+)} \\
\lesssim \left\| \int_0^{+\infty} e^{-iqx} \left( \mu(q) \right)^2 \frac{d\mu}{dq} \tilde{h}_1^+(q) dq \right\|_{H^s(\mathbb{R}^+)} \\
\lesssim \left\| q^s \left( \mu(q) \right)^2 \frac{d\mu}{dq} \tilde{h}_1^+(q) \right\|_{L^2_s(\mathbb{R}^+)} \\
\lesssim \left\| \left( q(\mu) \right)^s \mu^2 \frac{d\mu}{dq} \tilde{h}_1^+(\mu) \right\|_{L^2_s(\mathbb{R}^+)} \lesssim \|h_1\|_{H^{s+1}(\mathbb{R}^+)}.
\]
since \( q(\mu) \sim \mu \) as \( \mu \to \infty \). For \( j = 3 \), similar process can be applied. The proof is now complete. \( \square \)

**Proof.** (Proposition 3.5) Similar to Proposition 3.4, it suffices to prove for \( h = (h_1, 0, 0) \). We write
\[
|W_{bdr}(h_1, 0, 0)|(x, t) = I_1(x, t) + \bar{I_1}(x, t),
\]
with
\[
I_1(x, t) = \frac{1}{2\pi} \sum_{j=1}^3 \int_0^{+\infty} e^{i\phi(\mu)t} e^{i\gamma_j^+(\mu)x} \frac{\Delta_{j,m}^+(\mu) \Delta^+(\mu)}{\Delta^+(\mu)} \frac{3\mu^4 + 2\mu^2 + 1}{\sqrt{\mu^4 + \mu^2 + 1}} \tilde{h}_1^+(\mu) d\mu
\]
\[
:= I_{1,1} + I_{1,2} + I_{1,3}.
\]
It then suffices to show the estimate for \( I_1 \).

For \( j = 1 \), one has \( \gamma_1^+(\mu) = i\mu \), then according to Lemma 2.5,
\[
\|\eta I_{1,1}\|_{X^{s,b}} \lesssim \|\eta W_{bdr}(\Phi, 0)\|_{X^{s,b}(\mathbb{R}^2)} \lesssim \|\Phi\|_{H^s(\mathbb{R})} \lesssim \|h_1\|_{H^{s+1}(\mathbb{R}^+)}
\]
where \( W_{bdr} \) is defined by (2.2) and (2.3) and the Fourier transform of \( \Phi(x) \) defined by:
\[
\hat{\Phi}(\mu) = \frac{\Delta_{j,m}^+(\mu) \Delta^+(\mu)}{\Delta^+(\mu)} \frac{3\mu^4 + 2\mu^2 + 1}{\sqrt{\mu^4 + \mu^2 + 1}} \tilde{h}_1^+(\mu) \chi_{(0,\infty)}(\mu).
\]
For \( j = 2 \), one has \( \gamma_j^+ (\mu) = -p(\mu) - iq(\mu) \). Let \( f(x) = e^{-x}\theta(x) \) where \( \theta \) is a cut-off function such that \( \theta(x) = 1 \) for \( x \geq 0 \) and \( \theta(x) = 0 \) for \( x < -1 \). Then, for \((x, t) \in \mathbb{R}^+ \times \mathbb{R}^+ \), we can write

\[
\eta(t)I_{1,2}(x, t) = \int_{\mathbb{R}} \left( \eta(t)e^{i\phi(\mu)t} \right) \left( f(xp(x)) e^{-i\eta(x)x} \right) \frac{\Delta_{j,m}(\mu) 3\mu^4 + 2\mu^2 + 1}{\Delta^+(\mu)} \frac{1}{\sqrt{\mu^4 + \mu^2 + 1}} \tilde{h}_1(\mu) d\mu.
\]

Since \( \eta \) and \( \theta \) are cut-off functions, we now can extend \( \eta(t)I_{1,2}(x, t) \) to the entire plane, \( \mathbb{R} \times \mathbb{R} \), with

\[
\tilde{\eta}I_{1,2}(\xi, \tau) = \int_{\mathbb{R}} \tilde{\eta}(\tau - \phi(\mu)) \tilde{f} \left( \frac{\xi + q}{p} \right) \frac{1}{p} \frac{\Delta_{j,m}(\mu) 3\mu^4 + 2\mu^2 + 1}{\Delta^+(\mu)} \frac{1}{\sqrt{\mu^4 + \mu^2 + 1}} \tilde{h}_1(\mu) d\mu.
\]

Notice that \( f \) is a Schwartz function with

\[
p = \sqrt{\sqrt{\mu^4 + \mu^2 + 1} + \frac{1}{2} \mu^2 + \frac{1}{2}}, \quad q = \sqrt{\sqrt{\mu^4 + \mu^2 + 1} - \frac{1}{2} \mu^2 - \frac{1}{2}},
\]

one has

\[
\left| \tilde{f} \left( \frac{\xi + q}{p} \right) \right| \lesssim \frac{1}{1 + (\xi + q)^3/p^3} \lesssim \frac{1 + \mu^3}{1 + |\xi|^3 + k|\mu|^3},
\]

for some \( k > 1 \). Since \( p \sim |\mu| \) as \( \mu \to +\infty \) and \( \langle \tau - \phi(\xi) \rangle \lesssim \langle \tau - \phi(\mu) \rangle \langle \phi(\mu) - \phi(\xi) \rangle \), we apply Young’s inequality and it follows

\[
\| \eta \|_{X_{s,b}(\mathbb{R}^2)}^b = \left\| \left( \xi \right)^s \langle \tau - \phi(\xi) \rangle \int_{\mathbb{R}} \tilde{\eta}(\tau - \phi(\mu)) \tilde{f} \left( \frac{\xi + q}{p} \right) \frac{1}{p} \frac{\Delta_{j,m}(\mu) 3\mu^4 + 2\mu^2 + 1}{\Delta^+(\mu)} \frac{1}{\sqrt{\mu^4 + \mu^2 + 1}} \tilde{h}_1(\mu) d\mu \right\|_{L_x^2 L_t^p}
\]

\[
\lesssim \| \langle \tau - \phi(\mu) \rangle \langle \tau - \phi(\mu) \rangle \|_{L_x^\infty L_t^\infty} \langle \phi(\mu) - \phi(\xi) \rangle \langle \xi \rangle \cdot \left\| \tilde{\eta}(\tau - \phi(\mu)) \right\|_{L_x^2 L_t^p} \lesssim \| \tilde{\eta}(\tau - \phi(\mu)) \right\|_{L_x^2 L_t^p} \lesssim \| h_1 \|_{H_{s+1}^b(\mathbb{R}^+)}.
\]

The above bound holds because

\[
\int_{\mathbb{R}} \langle \phi(\mu) - \phi(\xi) \rangle \frac{2b}{\langle \xi \rangle} \frac{2s}{\langle \xi \rangle} d\xi \lesssim \int_{\mathbb{R}} \left( \frac{1}{1 + |\xi|^3 + k|\mu|^3} \right)^2 \frac{2b}{\langle \xi \rangle} \frac{2s}{\langle \xi \rangle} d\xi \lesssim \int_{\mathbb{R}} \left( \frac{1}{1 + |\xi|^3 + k|\mu|^3} \right)^{2b - 2b} \langle \xi \rangle d\xi,
\]

and

\[
\int_{|\xi| < |\mu|} \left( \frac{1}{1 + |\xi|^3 + k|\mu|^3} \right)^{2b - 2b} d\xi \lesssim |\mu|^{-3(2 - 2b)} \int_{|\xi| < |\mu|} \langle \xi \rangle d\xi \lesssim |\mu|^{2s - 6b - 5},
\]

\[
\int_{|\xi| > |\mu|} \left( \frac{1}{1 + |\xi|^3 + k|\mu|^3} \right)^{2b - 2b} d\xi \lesssim \int_{|\xi| > |\mu|} |\xi|^{2s + 6b - 6} d\xi \lesssim |\mu|^{2s + 6b - 5},
\]

and

\[
\int_{|\xi| < |\mu|} \left( \frac{1}{1 + |\xi|^3 + k|\mu|^3} \right)^{2b - 2b} d\xi \lesssim \int_{|\xi| < |\mu|} \langle \xi \rangle d\xi \lesssim |\mu|^{2s - 6b - 5},
\]

and

\[
\int_{|\xi| > |\mu|} \left( \frac{1}{1 + |\xi|^3 + k|\mu|^3} \right)^{2b - 2b} d\xi \lesssim \int_{|\xi| > |\mu|} |\xi|^{2s + 6b - 6} d\xi \lesssim |\mu|^{2s + 6b - 5},
\]

and

\[
\int_{|\xi| < |\mu|} \left( \frac{1}{1 + |\xi|^3 + k|\mu|^3} \right)^{2b - 2b} d\xi \lesssim \int_{|\xi| < |\mu|} \langle \xi \rangle d\xi \lesssim |\mu|^{2s - 6b - 5},
\]

and

\[
\int_{|\xi| > |\mu|} \left( \frac{1}{1 + |\xi|^3 + k|\mu|^3} \right)^{2b - 2b} d\xi \lesssim \int_{|\xi| > |\mu|} |\xi|^{2s + 6b - 6} d\xi \lesssim |\mu|^{2s + 6b - 5}.
\]
for $s < 1$ and $|\frac{1}{2} - b|$ sufficient small. The proof for $j = 3$ is similar to $j = 2$, therefore omitted. The proof is now complete.

3.2.2. Linear problem with homogeneous boundary conditions. Next, we consider estimates of the solution for the IBVP,

$$
\begin{align*}
\left\{ \begin{array}{l}
    \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^4 u}{\partial x^4} - \frac{\partial^6 u}{\partial x^6} = 0, \quad x > 0, t > 0, \\
    u(x, 0) = \varphi(x), u_t(x, 0) = \psi''(x), \\
    u(0, t) = 0, u_{xx}(0, t) = 0, u_{xxxx}(0, t) = 0.
\end{array} \right.
\end{align*}
$$

(3.12)

According to Lemma 3.2, the solution of IBVP (3.12) can be written as

$$
u(x, t) = [WR(\varphi^*, \psi^*)](x, t) - [W_{bd}(\bar{p})](x, t),$$

where $\bar{p}(t) = (p_1(t), p_2(t), p_3(t))$ with

$$
p_1(t) = [WR(\varphi^*, \psi^*)](0, t), \quad v_2(t) = \partial_x^2 [WR(\varphi^*, \psi^*)](0, t), \quad v_3(t) = \partial_x^4 [WR(\varphi^*, \psi^*)](0, t).
$$

Hence, combining Propositions 3.4, 3.5 and Lemmas 2.5, 3.2, one has the following statement.

**Proposition 3.6.** Given $s \leq 0$ and $|\frac{1}{2} - b|$ sufficient small, for any $\varphi \in H^s(\mathbb{R}^+)$ and $\psi \in H^{s-1}(\mathbb{R}^+)$, the corresponding solution $u$ of the IBVP (3.12) satisfies

$$
sup_{t > 0} \left\| u(\cdot, t) \right\|_{H^s(\mathbb{R}^+)} + sup_{x > 0} \left\| \partial_x^j u(x, \cdot) \right\|_{H_{2j+1}^{\frac{3}{4}}(\mathbb{R}^+)} + \left\| \eta(t) u(x, t) \right\|_{X_{s,b}} \lesssim \left\| \varphi \right\|_{H^s(\mathbb{R}^+)} + \left\| \psi \right\|_{H^{s-1}(\mathbb{R}^+)},$$

with $j = 0, 1, 2, \ldots, 5$.

3.2.3. Linear problem with forcing. Finally, we consider the estimates for the solution of the IBVP,

$$
\begin{align*}
\left\{ \begin{array}{l}
    \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^4 u}{\partial x^4} - \frac{\partial^6 u}{\partial x^6} = g(x, t), \quad x > 0, t > 0, \\
    u(x, 0) = 0, u_t(x, 0) = 0, \\
    u(0, t) = 0, u_{xx}(0, t) = 0, u_{xxxx}(0, t) = 0.
\end{array} \right.
\end{align*}

(3.13)

To study the estimate of IBVP (3.13), based on Lemma 2.6, 3.3 and Proposition 3.4, 3.5, 3.6, it suffices to establish the sharp Kato smoothing of the solution for the forced IVP.

$$
\begin{align*}
\left\{ \begin{array}{l}
    \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^4 u}{\partial x^4} - \frac{\partial^6 u}{\partial x^6} = g(x, t), \quad x \in \mathbb{R}, t > 0, \\
    u(x, 0) = 0, u_t(x, 0) = 0.
\end{array} \right.
\end{align*}

(3.14)

**Proposition 3.7.** For $s \leq 0$ and $|\frac{1}{2} - b| > 0$ sufficient small,

$$
sup_{x \geq 0} \left\| \eta(t) \int_0^t \partial_x^2 [WR(0, g)](x, t - \tau) d\tau \right\|_{H_{2j+1}^{\frac{3}{4}}(\mathbb{R})} \lesssim \left\| \varphi \right\|_{H^s(\mathbb{R}^+)} + \left\| \psi \right\|_{H^{s-1}(\mathbb{R}^+)}. $$
Proof. We start by verify the estimate for (2.2) and (2.3), we write, at $x \in D$ where $A$ is the region

\[
\left\| \begin{array}{l}
\left( \frac{\partial^j}{\partial t^j} \right) \hat{g} \\
\left( \frac{\partial^j}{\partial t^j} \right) \hat{g}
\end{array} \right\|_{L^2}^{X, \lambda - b}, \quad j = 0, 3,
\]

\[
\left\| \begin{array}{l}
\left( \frac{\partial^j}{\partial t^j} \right) \hat{g} \\
\left( \frac{\partial^j}{\partial t^j} \right) \hat{g}
\end{array} \right\|_{L^2}^{X, \lambda - a}, \quad j = 1, 4,
\]

\[
\left\| \begin{array}{l}
\left( \frac{\partial^j}{\partial t^j} \right) \hat{g} \\
\left( \frac{\partial^j}{\partial t^j} \right) \hat{g}
\end{array} \right\|_{L^2}^{X, \lambda - b}, \quad j = 2, 5,
\]

where $D$ is the region $\{ \xi \in \mathbb{R} : |\xi|^3 \gg |\tau|, |\xi| \geq 1 \}$.

Let $\theta$ be a cut-off function with $\theta = 1$ on $[-1, 1]$ and $\text{supp}(\theta) \subset (-2, 2)$, and denote $\theta^c = 1 - \theta$. Then,

\[
V_1 = \int_{\mathbb{R}^2} \frac{\xi^2 e^{it\phi(\xi)} \hat{g}(\xi, \lambda)}{\phi(\xi)} \left( \int_0^t e^{i\tau(\lambda - \phi(\xi))} d\tau \right) d\lambda d\xi,
\]

\[
V_2 = \int_{\mathbb{R}^2} \frac{e^{it\xi} - e^{-it\phi(\xi)}}{\lambda + \phi(\xi)} F(\xi, \lambda) d\lambda d\xi,
\]

with

\[
F(\xi, \lambda) := \frac{\xi^2 \hat{g}(\xi, \lambda)}{\phi(\xi)}.
\]

It suffices to consider the estimate for $V_1$. Let $\theta$ be a cut-off function with $\theta = 1$ on $[-1, 1]$ and $\text{supp}(\theta) \subset (-2, 2)$, and denote $\theta^c = 1 - \theta$. Then,

\[
V_1 = \int_{\mathbb{R}^2} \frac{e^{it\xi} - e^{-it\phi(\xi)}}{\lambda - \phi(\xi)} \theta(\lambda - \phi(\xi)) F(\xi, \lambda) d\lambda d\xi + \int_{\mathbb{R}^2} \frac{e^{it\xi}}{\lambda - \phi(\xi)} \theta^c(\lambda - \phi(\xi)) F(\xi, \lambda) d\lambda d\xi
\]

\[
= A_1 + A_2 - A_3.
\]

For $A_1$, using Taylor expansion, we have

\[
A_1 = \int_{\mathbb{R}^2} -e^{i\lambda \xi} \sum_{k=0}^{\infty} \frac{(it)^k (\lambda - \phi(\xi))^{k-1}}{k!} \theta(\lambda - \phi(\xi)) F(\xi, \lambda) d\lambda d\xi.
\]
Hence,
\[
\| \eta(t) A_1 \|_{\mathcal{H}^{s+1}_t (\mathbb{R})} \lesssim \sum_{k=0}^{\infty} \frac{\| t^k \eta(t) \|_{\mathcal{H}^1(\mathbb{R})}}{k!} \left\| \int_{\mathbb{R}^2} e^{i\lambda t} (\lambda - \phi(\xi))^{k-1} \theta(\lambda - \phi(\xi)) F(\xi, \lambda) d\xi d\lambda \right\|_{\mathcal{H}^s_t (\mathbb{R})}
\]
\[
\lesssim \sum_{k=0}^{\infty} \frac{1}{k!} \left\| (\lambda)^{\frac{s+1}{2}} \int_{|\lambda - \phi(\xi)| \leq 1} F(\xi, \lambda) d\xi d\lambda \right\|_{L^2_{\lambda}}
\]
\[
\lesssim \left[ \int_{\mathbb{R}} \langle \lambda \rangle^{\frac{2s+2}{3}} \left( \int_{|\lambda - \phi(\xi)| \leq 1} \langle \xi \rangle^{-2s} d\xi \right) \left( \int_{|\lambda - \phi(\xi)| \leq 1} \langle \xi \rangle^{2s} (F(\xi, \lambda))^2 d\lambda \right) \right]^{\frac{1}{2}}
\]
\[
\lesssim \sup_{\lambda} \left( \langle \lambda \rangle^{\frac{2s+2}{3}} \int_{|\lambda - \phi(\xi)| \leq 1} \langle \xi \rangle^{-2s} d\xi \right)^{\frac{1}{2}} \| F \|_{L^2_{\lambda}} := K_1 \| F \|_{X^{s,b}}.
\]

It then suffices to bound the term \( K_1 \). To handle that, we consider the domain of the integral, \(|\lambda - \phi(\xi)| \leq 1\), in either \(|\xi|, |\lambda| \lesssim 1\) or \(|\xi|, |\lambda| \gg 1\). The former case is easy to check. The later case can be verified by substituting \( \mu = \phi(\xi) \sim |\xi|^3 \). Therefore, we can have the bound,
\[
\langle \lambda \rangle^{\frac{2s+2}{3}} \int_{|\lambda - \phi(\xi)| \leq 1} \langle \mu \rangle^{-\frac{2s+2}{3}} d\mu \lesssim \langle \lambda \rangle^{\frac{2s+2}{3}} \langle \lambda \rangle^{-\frac{2s+2}{3}} \lesssim 1, \quad \text{for } |\lambda| \gg 1.
\]

For \( A_2 \), one has,
\[
\| \eta(t) A_2 \|_{\mathcal{H}^{s+1}_t (\mathbb{R})} \lesssim \int_{\mathbb{R}^2} e^{i\lambda \theta(\lambda - \phi(\xi))} F(\xi, \lambda) d\xi d\lambda \left\| (\lambda)^{\frac{s+1}{2}} F(\xi, \lambda) \right\|_{L^2_{\lambda}}
\]
\[
\lesssim \left[ \int_{\mathbb{R}} \langle \lambda \rangle^{\frac{2s+2}{3}} \left( \int_{\mathbb{R}} \frac{1}{\langle \xi \rangle^{2s} (\lambda - \phi(\xi))^{2 - 2b}} d\xi \right) \left( \int_{\mathbb{R}} \langle \xi \rangle^{2s} (\lambda - \phi(\xi))^{-2b} \right) \right]^{\frac{1}{2}}
\]
\[
\lesssim \sup_{\lambda} \left( \langle \lambda \rangle^{\frac{2s+2}{3}} \int_{\mathbb{R}} \frac{1}{\langle \xi \rangle^{2s} (\lambda - \phi(\xi))^{2 - 2b}} d\xi \right)^{\frac{1}{2}} \| F \|_{X^{s,b}} := K_2 \| F \|_{X^{s,b}}.
\]

It then suffices to bound the term \( K_2 \). We set \( z = \phi(\xi) \) and \( \frac{dz}{d\xi} \sim |\xi|^{-2} \sim |z|^{-\frac{3}{4}}, \) then according to Lemma 2.1 with \( s \leq 0 \) and \( b < \frac{1}{2} \),
\[
K_2^2 \lesssim \langle \lambda \rangle^{\frac{2s+2}{3}} \int_{\mathbb{R}} \frac{1}{\langle z \rangle^{\frac{4s+2}{3}} (\lambda \pm z)^{2 - 2b}} dz \lesssim \langle \lambda \rangle^{\frac{2s+2}{3}} \langle \lambda \rangle^{-\frac{2s+2}{3}} \lesssim 1.
\]
For $A_3$, substituting $\mu = \phi(\xi)$, then, for $b < \frac{1}{2}$, one has
\[
\left\| \eta(t)A_3 \right\|_{L_t^{\frac{6}{5}}(\mathbb{R})} \lesssim \left\| \int_{\mathbb{R}^2} e^{it\mu} \frac{\theta(\lambda - \mu) F(\mu, \lambda) \mu^{-\frac{2}{3}} d\mu d\lambda}{\lambda - \mu} \right\|_{L_t^{\frac{6}{5}}(\mathbb{R})}
\lesssim \left\| \left\langle \mu \right\rangle \frac{\theta(\lambda - \mu) F(\mu, \lambda) \mu^{-\frac{2}{3}} d\lambda}{\lambda - \mu} \right\|_{L_\mu^2}
\lesssim \left( \int_{\mathbb{R}} \langle \mu \rangle^{\frac{2s+2}{3}} \mu^{-\frac{4}{3}} \int_{\mathbb{R}} \frac{1}{(\lambda - \mu)^{2b}} F^2(\mu, \lambda) d\lambda \int_{\mathbb{R}} \frac{1}{(\lambda - \mu)^{2-2b}} d\lambda d\mu \right)^{\frac{1}{2}}
\lesssim \left( \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} \frac{1}{(\lambda - \phi(\xi))^{2b}} F^2(\xi, \lambda) d\lambda d\xi \right)^{\frac{1}{2}} \lesssim \| \hat{F} \|_{L_{x,-b}^s}.
\]

Next, we turn to show the estimates holds for $j = 1$. Similar to the previous proof, with the same definition of $F(\xi, \lambda)$, let us write
\[
\left. \int_0^t \partial_x [W_\mathbb{R}(0, g)](x, t - \tau) d\tau \right|_{x=0}
= \int_0^t \int_{\mathbb{R}} \frac{\xi (e^{it(\xi - \phi(\xi))} - e^{-it(\xi - \phi(\xi))})}{2i\phi(\xi)} \xi^2 \hat{g}(\xi, \tau) d\xi d\tau
= \int_0^t \int_{\mathbb{R}} \frac{\xi (e^{it(\xi - \phi(\xi))} - e^{-it(\xi - \phi(\xi))})}{2i\phi(\xi)} \xi^2 \left( \int e^{i\lambda \tau} \hat{g}(\xi, \lambda) d\lambda \right) d\xi d\tau
=: \frac{1}{2t} (W_1 + W_2),
\]
where
\[
W_1 = \int_{\mathbb{R}^2} \frac{\xi (e^{i\xi \lambda} - e^{i\phi(\xi)})}{\lambda - \phi(\xi)} F(\xi, \lambda) d\lambda d\xi.
\]
and
\[
W_2 = \int_{\mathbb{R}^2} \frac{\xi (e^{i\xi \lambda} - e^{i\phi(\xi)})}{\lambda - \phi(\xi)} F(\xi, \lambda) d\lambda d\xi.
\]
Again, it suffices to verify the estimate for $W_1$, and we write
\[
W_1 = \int_{\mathbb{R}^2} \frac{\xi (e^{i\xi \lambda} - e^{i\phi(\xi)})}{\lambda - \phi(\xi)} \theta(\lambda - \phi(\xi)) F(\xi, \lambda) d\lambda d\xi
+ \int_{\mathbb{R}^2} \frac{\xi e^{i\xi \lambda}}{\lambda - \phi(\xi)} \theta(\lambda - \phi(\xi)) F(\xi, \lambda) d\lambda d\xi
- \int_{\mathbb{R}^2} \frac{\xi e^{i\phi(\xi)}}{\lambda - \phi(\xi)} \theta(\lambda - \phi(\xi)) F(\xi, \lambda) d\lambda d\xi
=: B_1 + B_2 - B_3.
\]
The proof for $B_1$ and $B_3$ are similar to the ones for $A_1$ and $A_3$. For $B_2$, it follows from similar process as in (3.15) that
\[
\left\| \eta(t)B_2 \right\|_{L_t^{\frac{6}{5}}(\mathbb{R})} \lesssim \left\| \left\langle \lambda \right\rangle^{-\frac{2}{3}} \int_{\mathbb{R}} \frac{\xi}{(\lambda - \phi(\xi))} F(\xi, \lambda) d\xi \right\|_{L_\lambda^2}.
\]
For $\xi \in D$, since $|\xi|^3 \gg |\lambda|$ and $|\xi| \gtrsim 1$, $\langle \lambda - \phi(\xi) \rangle \sim |\xi|^3$, then the result follows. For $\xi \notin D$, that is, either $|\xi|^3 \lesssim |\lambda|$ or $|\xi| \lesssim 1$. We then can repeat the process in $A_2$ (c.f. (3.16)) and arrive at the bound,

$$
\|\eta(t)B_2\|_{H^{s+1}_t(L^2(\mathbb{R}))} \lesssim \sup_{\lambda} \left( \langle \lambda \rangle^{2s} \int_\mathbb{R} \frac{|\xi|^2}{(\langle \xi \rangle^{2s} (\lambda - \phi(\xi)))^{2-2b}} d\xi \right)^{\frac{1}{2}} \|\tilde{F}\|_{X^{s,-b}}.
$$

If $|\xi|^3 \lesssim |\lambda|$, then $|\xi|^2 \gtrsim |\lambda|^2$ and

$$
\langle \lambda \rangle^{2s} \int_\mathbb{R} \frac{|\xi|^2}{(\langle \xi \rangle^{2s} (\lambda - \phi(\xi)))^{2-2b}} d\xi \lesssim \langle \lambda \rangle^{2s+2} \int_\mathbb{R} \frac{1}{(\langle \xi \rangle^{2s} (\lambda - \phi(\xi)))^{2-2b}} d\xi.
$$

The bound follows exact same as (3.16). If $|\xi| \lesssim 1$ then $|\xi|^2 \lesssim 1$ and

$$
\langle \lambda \rangle^{2s} \int_\mathbb{R} \frac{|\xi|^2}{(\langle \xi \rangle^{2s} (\lambda - \phi(\xi)))^{2-2b}} d\xi \lesssim \langle \lambda \rangle^{2s+2} \int_\mathbb{R} \frac{1}{(\langle \xi \rangle^{2s} (\lambda - \phi(\xi)))^{2-2b}} d\xi.
$$

The bound is obtained similar to (3.16). Similar proof can show the estimate for $j = 2$.

Finally, we consider the proof for $j = 3, 4, 5$. It has shown (c.f. Lemma 2.12 in [27]) that

$$
\partial_x^3 \left( \int_0^t [W_\mathbb{R}(0, g)](x, t - \tau) d\tau \right) = \partial_t \left( \int_0^t [W_\mathbb{R}(0, g)](x, t - \tau) d\tau \right),
$$

then

$$
\partial_x^4 \left( \int_0^t [W_\mathbb{R}(0, g)](x, t - \tau) d\tau \right) = \partial_t \left( \int_0^t [\partial_x W_\mathbb{R}(0, g)](x, t - \tau) d\tau \right),
$$

$$
\partial_x^5 \left( \int_0^t [W_\mathbb{R}(0, g)](x, t - \tau) d\tau \right) = \partial_t \left( \int_0^t [\partial_x^2 W_\mathbb{R}(0, g)](x, t - \tau) d\tau \right).
$$

Since $\eta(t)$ is a cut-off function with $\eta \in C^\infty(\mathbb{R})$, then, for $j = 3, 4, 5$,

$$
\left\| \eta(t) \int_0^t \partial_x^j W_\mathbb{R}(0, g)(x, t - \tau) d\tau \right\|_{H^{s+j+1}_t(L^2(\mathbb{R}))} \lesssim \left\| \eta(t) \partial_t \left( \int_0^t \partial_x^{j-3} W_\mathbb{R}(0, g)(x, t - \tau) d\tau \right) \right\|_{H^{s+j+1}_t(L^2(\mathbb{R}))},
$$

these will reduce the proof to the case $j = 0, 1, 2$.

\section{Bilinear Estimates}

As we pointed out in the introduction as well as in Proposition 3.7, some of the estimates on $X^{s,b}$ only works for $b < \frac{1}{2}$. However, the bilinear estimate for the IVP of the sixth order Boussinesq equation on $X^{s,b}$ [13] is only valid for $\frac{1}{2} < b < 1$. Therefore, we need to show a proper bilinear estimate for $b < \frac{1}{2}$ in order to establish a contraction mapping.
Proposition 4.1. For $-\frac{1}{2} < s \leq 0$, $\frac{1}{2} - b > 0$ sufficient small,

$$\left\| \frac{\xi^2 \hat{u}}{\phi(\xi)} \right\|_{X_s^{a,b}} \lesssim \|u\|_{X^{a,b}} \|v\|_{X^{a,b}}.$$ 

Proposition 4.2. For $-\frac{1}{2} < s \leq 0$, $\frac{1}{2} - b > 0$ sufficient small,

$$\left\| (\tau)^{\frac{s-1}{2}} \int_D (\xi)^{j-3} \frac{\xi^2 \hat{u}}{\phi(\xi)} d\xi \right\|_{L_x^2} \lesssim \|u\|_{X^{a,b}} \|v\|_{X^{a,b}}, \quad j = 1, 2,$$

where $D := \{\xi \in \mathbb{R} : |\xi|^3 \gg |\tau|, |\xi| \gtrsim 1\}$.

Proof. (Proposition 4.1) For simplicity, we only consider $\beta = 1$. Let $u, v \in X^{s,b}$, $w \in X^{-s,b}$ and define

$$f(\xi, \tau) = \langle \xi \rangle^s (|\tau| - |\xi|^3 - \frac{1}{2} |\xi|)^2 \hat{u}(\xi, \tau), \quad g(\xi, \tau) = \langle \xi \rangle^s (|\tau| - |\xi|^3 - \frac{1}{2} |\xi|)^2 \hat{u}(\xi, \tau),$$

$$h(\xi, \tau) = \langle \xi \rangle^{-s} (|\tau| - |\xi|^3 - \frac{1}{2} |\xi|)^2 \hat{u}(\xi, \tau).$$

Then, according to duality, it suffices to show

$$|W(f, g, h)| \lesssim \|f\|_{L_{\xi,\tau}^2} \|g\|_{L_{\xi,\tau}^2} \|h\|_{L_{\xi,\tau}^2},$$

where

$$W(f, g, h) = \int_{\mathbb{R}^4} \frac{|\xi|^2 \langle \xi \rangle^s (\xi - \xi_1)^{-s} f(\xi, \tau_1) g(\xi - \xi_1, \tau - \tau_1) h(\xi, \tau) d\xi d\tau d\xi_1 d\tau_1}{\phi(\xi)(|\tau| - |\xi|^3 - \frac{1}{2} |\xi|)^b(|\tau_1| - |\xi_1|^3 - \frac{1}{2} |\xi_1|)^b(|\tau - \tau_1| - |\xi - \xi_1|^3 - \frac{1}{2} |\xi - \xi_1|)^b}$$

(4.2)

$$:= \int_{\mathbb{R}^4} M(\xi, \xi_1, \tau, \tau_1) f(\xi_1, \tau_1) g(\xi - \xi_1, \tau - \tau_1) h(\xi, \tau) d\xi d\tau d\xi_1 d\tau_1.$$

To achieve the estimate, we apply a similar argument as Farah [14] and Tzirakis [11, 13] for the Boussinesq-type equations and analysis the six possible cases for the sign of $\tau$, $\tau_1$ and $\tau - \tau_1$,

1. I, $\tau_1 \geq 0, \tau - \tau_1 \geq 0$;
2. II, $\tau_1 \geq 0, \tau - \tau_1 \leq 0, \tau \geq 0$;
3. III, $\tau_1 \geq 0, \tau - \tau_1 \leq 0, \tau \leq 0$;
4. IV, $\tau_1 \leq 0, \tau - \tau_1 \leq 0$;
5. V, $\tau_1 \leq 0, \tau - \tau_1 \leq 0, \tau \leq 0$;
6. VI, $\tau_1 \leq 0, \tau - \tau_1 \leq 0, \tau \geq 0$.

We start to consider the estimate in (I). According to the Cauchy-Schwartz inequality, we only need to verify,

$$\left\| \int_{\mathbb{R}^2} M(\xi, \xi_1, \tau, \tau_1) f(\xi_1, \tau_1) g(\xi - \xi_1, \tau - \tau_1) d\xi_1 d\tau_1 \right\|_{L_{\xi,\tau}^2} \lesssim \|f\|_{L_{\xi,\tau}^2} \|g\|_{L_{\xi,\tau}^2}.$$
Moreover, Using the Cauchy-Schwartz and Young's inequalities, the left-hand side of (4.3) is bounded by

\[
\left\| M \right\|_{L^2_{\xi_1, r_1}} \left\| f(\xi_1, r_1) g(\xi - \xi_1, r - r_1) \right\|_{L^2_{\xi_1, r_1}} \left\| f \right\|_{L^2_{\xi_1, r_1}} \left\| g \right\|_{L^2_{\xi_1, r}}.
\]

Therefore, it suffices to bound the term \( \left\| M \right\|_{L^2_{\xi_1, r_1}} \). Set \( r = -s \geq 0 \), we have

\[
\left\| M \right\|_{L^2_{\xi_1, r_1}}^2 = \int_{\mathbb{R}^2} \phi^2(\xi)(\tau - |\xi|^2 - \frac{1}{2} |\xi|) d\xi d\tau = \int_{\mathbb{R}^2} \frac{\langle \xi \rangle^{-2r-2} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r} d\xi_1 d\tau}{\langle \xi \rangle^{-2r} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r}}.
\]

Since \( \langle p \rangle \langle q \rangle \geq \langle p + q \rangle \), \( \xi^4/\phi^2(\xi) \lesssim \langle \xi \rangle^{-2} \), \( \frac{1}{2} - b > 0 \) sufficient small and Lemma 2.1,

\[
\left\| M \right\|_{L^2_{\xi_1, r_1}}^2 \lesssim \int_{\mathbb{R}^2} \frac{\langle \xi \rangle^{-2r-2} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r} d\xi_1 d\tau}{\langle \xi \rangle^{-2r} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r}} \lesssim \int_{\mathbb{R}^2} \frac{1}{|\xi - \xi_1|^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2} |\xi - \xi_1| + \frac{1}{2} |\xi_1| - \frac{1}{2} |\xi| |(|\xi - \xi_1|^3 + \frac{3|\xi|^2}{3} |\xi_1|^3 + \frac{1}{2}(|\xi_1|^3 + \frac{1}{2} |\xi_1| - \frac{1}{2} |\xi|)^2 d\xi_1,
\]

where

\[
A_1 = \{ \xi_1 \in \mathbb{R} : \langle \xi_1 \rangle \langle \xi - \xi_1 \rangle \lesssim \langle \xi \rangle \} \quad \text{and} \quad A_2 = \{ \xi_1 \in \mathbb{R} : \langle \xi_1 \rangle \langle \xi - \xi_1 \rangle \gg \langle \xi \rangle \}.
\]

In region \( A_1 \), one has

\[
\int_{A_1} \frac{1}{|\xi - \xi_1|^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2} |\xi - \xi_1| + \frac{1}{2} |\xi_1| - \frac{1}{2} |\xi| |(|\xi - \xi_1|^3 + \frac{3|\xi|^2}{3} |\xi_1|^3 + \frac{1}{2}(|\xi_1|^3 + \frac{1}{2} |\xi_1| - \frac{1}{2} |\xi|)^2 d\xi_1,
\]

since \( \xi^2 - 3\xi_1 + 3\xi_1^2 \geq 0 \). Set \( x = 3\xi_1(\xi - \xi_1) \), then

\[
\xi_1 = \frac{3\xi^2 \pm \sqrt{9\xi^4 - 12\xi x}}{6\xi}, \quad \text{and} \quad dx = \pm 2|\xi|^3 \sqrt{9\xi^4 - 12\xi x} d\xi_1.
\]
According to Lemma 2.2, one has the following bound,
\[ \|M\|_{L^2_{\xi_1,r_1}}^2 \lesssim \langle \xi \rangle^{-\frac{3}{2}} \int \frac{1}{(x)^{1-\sqrt{9\xi^2-12x}}}dx \lesssim \langle \xi \rangle^{-\frac{3}{2}} \langle \xi^3 \rangle^{-\frac{1}{2}+} \lesssim \langle \xi \rangle^{-4+}. \]

In region $A_2$, we write
\[ N := \int \frac{\langle \xi \rangle^{-2r-2} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r}}{\langle \xi - \xi_1 \rangle^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2} |\xi - \xi_1| + \frac{1}{2} |\xi_1| - \frac{1}{2} |\xi|} \cdot B^{2b} d\xi_1, \]
and consider the domain of the integral in following cases:

- $|\xi_1| \gg |\xi|$,  
- $|\xi| \gg |\xi_1| \gg 1$,  
- $|\xi| \sim |\xi_1| \gg 1$.

Furthermore, for each case above, we consider the estimates within different regions below,

(a): $B_1 := \{ (\xi, \xi_1) : \xi - \xi_1 \geq 0, \xi_1 \geq 0, \xi \geq 0 \}$;  
(b): $B_2 := \{ (\xi, \xi_1) : \xi - \xi_1 \geq 0, \xi_1 \leq 0, \xi \geq 0 \}$;  
(c): $B_3 := \{ (\xi, \xi_1) : \xi - \xi_1 \geq 0, \xi_1 \leq 0, \xi \leq 0 \}$;  
(d): $B_4 := \{ (\xi, \xi_1) : \xi - \xi_1 \leq 0, \xi_1 \leq 0, \xi \leq 0 \}$;  
(e): $B_5 := \{ (\xi, \xi_1) : \xi - \xi_1 \leq 0, \xi_1 \geq 0, \xi \leq 0 \}$;  
(f): $B_6 := \{ (\xi, \xi_1) : \xi - \xi_1 \leq 0, \xi_1 \geq 0, \xi \geq 0 \}$.

Notice that cases for $\xi - \xi_1 \geq 0$, $\xi_1 \geq 0$ and $\xi \leq 0$ and $\xi - \xi_1 \leq 0$, $\xi_1 \leq 0$ and $\xi \geq 0$ do not exist for $\xi$ and $\xi_1$ nonzero. In addition, we only need to consider the estimate in regions $B_1$, $B_2$ and $B_3$ since the rest cases are equivalence.

**Case 1.** For $|\xi_1| \gg |\xi|$, one has $\langle \xi - \xi_1 \rangle \sim \langle \xi_1 \rangle$. Notice that $B_1$ cannot contribute to this domain. For both $B_2$ and $B_3$, we have
\[ \langle \xi - \xi_1 \rangle^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2} |\xi - \xi_1| + \frac{1}{2} |\xi_1| - \frac{1}{2} |\xi| \cdot B^{2b} \]
\[ = \langle \xi - \xi_1 \rangle^3 - \xi_3^3 \pm \xi^3 + \frac{1}{2} \langle \xi - \xi_1 \rangle - \frac{1}{2} \xi_1 \pm \frac{1}{2} \xi_1 \cdot B^{2b} \sim \langle \xi_3^3 + \xi_1 \rangle^{1-}. \]

If $|\xi_1| \leq 1$, the results follows. If $|\xi_1| > 1$, then,
\[ N \lesssim \langle \xi \rangle^{-2r-2} \int_{|\xi_1| \gg |\xi|} \langle \xi_1 \rangle^{4r-3+} d\xi_1 \lesssim \langle \xi \rangle^{2r-4+}, \]
which is bounded if $r < \frac{1}{2}$.

**Case 2.** For $|\xi| \gg |\xi_1| \gg 1$, one has $\langle \xi - \xi_1 \rangle \sim \langle \xi \rangle$. For both $B_1$ and $B_2$, we have
\[ \langle \xi - \xi_1 \rangle^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2} |\xi - \xi_1| + \frac{1}{2} |\xi_1| - \frac{1}{2} |\xi| \cdot B^{2b} \]
\[ = \langle \xi - \xi_1 \rangle^3 \pm \xi_3^3 - \xi^3 + \frac{1}{2} \langle \xi - \xi_1 \rangle \pm \frac{1}{2} \xi_1 \pm \frac{1}{2} \xi_1 \cdot B^{2b} \sim \langle \xi_3^3 \xi_1 \rangle^{1-} \sim \langle \xi \rangle - \langle \xi_1 \rangle^{1-}. \]

Thus,
\[ N \lesssim \langle \xi \rangle^{-4+} \int_{|\xi_1| \ll |\xi|} \langle \xi_1 \rangle^{2r-1+} d\xi_1 \lesssim \langle \xi \rangle^{2r-4+}, \]
which is bounded if \( r < 2 \). While for \( B_3 \), we have

\[
\langle |\xi - \xi_1|^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2}|\xi - \xi_1| + \frac{1}{2}|\xi_1| - \frac{1}{2}(|\xi|)^2b \rangle
\]

\[
= \langle |\xi - \xi_1|^3 - \xi_1^3 + \xi_1^3 + \frac{1}{2}(\xi - \xi_1) - \frac{1}{2}\xi_1 + \frac{1}{2}(\xi)^2b \rangle \sim \langle \xi^3 \rangle^1 - \langle \xi^3 \rangle^2.
\]

Thus, one has the bound if \( r < 2 \), since

\[
N \lesssim \langle \xi \rangle^{-5+} \int_{|\xi_1| < |\xi|} \langle \xi_1 \rangle^{2r} d\xi_1 \lesssim |\xi|^{2r-4+}.
\]

**Case 3.** For \( |\xi| \sim |\xi_1| \gg 1 \), since we are addressing integral in region \( A_2 \), that is \( \langle \xi_1 \rangle \langle \xi - \xi_1 \rangle \gg \langle \xi \rangle \), this gives \( |\xi - \xi_1| \gg 1 \). For all of \( B_1, B_2 \) and \( B_3 \), we have

\[
\langle |\xi - \xi_1|^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2}|\xi - \xi_1| + \frac{1}{2}|\xi_1| - \frac{1}{2}(|\xi|)^2b \rangle
\]

\[
\gtrsim \langle (|\xi - \xi_1|^3 + \xi_1^3 - |\xi|^3)^2b \rangle \sim \langle \xi_1 \rangle^{2-}(\xi - \xi_1)^1.
\]

Thus, for \( r < \frac{1}{2} \), then \( 2r - 1 < 0 \), and the bound follows,

\[
N \lesssim \langle \xi \rangle^{-2r-2} \int_{|\xi_1| < |\xi|} \langle \xi_1 \rangle^{2r-2+} \langle \xi - \xi_1 \rangle^{2r-1+} d\xi_1 \lesssim \langle \xi \rangle^{-3+}.
\]

Therefore, we have shown the desired estimate in (I).

Next, we use similar idea to process (II). By exchange the role of \( (\xi, \tau) \) and \( (\xi_1, \tau_1) \), we intend to verify,

\[
\left\| \int_{\mathbb{R}^2} M(\xi, \xi_1, \tau, \tau_1)g(\xi - \xi_1, \tau - \tau_1)h(\xi, \tau)d\xi d\tau \right\|_{L^2_{\xi_1, \tau_1}} \lesssim \|g\|_{L^2_{\xi_1, \tau_1}} \|h\|_{L^2_{\xi_1, \tau_1}},
\]

with \( M \) defined in (4.2). Then, similar argument for (I) follows that it suffices to bound the term \( L \) in the following inequality,

\[
\|M\|_{L^2_{\xi, \tau}}^2 \lesssim \int_{A_3 + A_4} \langle |\xi - \xi_1|^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2}|\xi - \xi_1| + \frac{1}{2}|\xi_1| - \frac{1}{2}(|\xi|)^2b \rangle d\xi := L,
\]

where

\[ A_3 = \{ \xi \in \mathbb{R} : \langle \xi_1 \rangle |\langle \xi - \xi_1 \rangle \lesssim \langle \xi \rangle \} \quad \text{and} \quad A_4 = \{ \xi \in \mathbb{R} : \langle \xi_1 \rangle |\langle \xi - \xi_1 \rangle \gg \langle \xi \rangle \}. \]

In \( A_3 \), one has

\[
\int_{A_3} \langle |\xi - \xi_1|^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2}|\xi - \xi_1| + \frac{1}{2}|\xi_1| - \frac{1}{2}(|\xi|)^2b \rangle d\xi \lesssim \int_{A_3} \frac{1}{(3\xi_1(\xi - \xi_1))^1-1} d\xi,
\]

then the rest of the proof is similar to the one for \( A_1 \) in (I).

In \( A_4 \), we still estimate the integral by splitting the domain into the same cases as for \( A_2 \) in (I):

**Case 1.** For \( |\xi_1| \gg |\xi| \), one has \( \langle \xi - \xi_1 \rangle \sim \langle \xi_1 \rangle \). Notice that \( B_1 \) cannot contribute to this domain. For both \( B_2 \) and \( B_3 \), we have

\[
\langle |\xi - \xi_1|^3 + |\xi_1|^3 - |\xi|^3 + \frac{1}{2}|\xi - \xi_1| + \frac{1}{2}|\xi_1| - \frac{1}{2}(|\xi|)^2b \rangle
\]
For Case 2. have the following bound, we have since for \(|\xi_1| \leq 1\) the bound can be obtained directly. Thus, for \(r < 2\), we have the following bound,

\[
L \lesssim \langle \xi_1 \rangle^{4r-3+} \int_{|\xi| \ll |\xi_1|} \langle \xi \rangle^{-2r-2} d\xi \lesssim \langle \xi_1 \rangle^{2r-4+}.
\]

**Case 2.** For \(|\xi| \gg |\xi_1| \gg 1\), one has \(\langle \xi - \xi_1 \rangle \sim \langle \xi \rangle\). For both \(B_1\) and \(B_2\), we have

\[
\langle |\xi - \xi_1|^3 + |\xi|^3 - |\xi^3 - 2|\xi_1^2| + \frac{1}{2}|\xi - \xi_1| - \frac{1}{2}|\xi| \rangle^{2b} = \langle (\xi - \xi_1)^3 + \xi^3 - \xi^3 - \frac{1}{2}(\xi - \xi_1) + \frac{1}{2} \xi_1 - \frac{1}{2} \xi \rangle^{2b} \sim \langle \xi^2 \xi_1 \rangle^{1-} \sim \langle \xi^2 - \xi_1 \rangle^{1-}.
\]

Thus,

\[
L \lesssim \langle \xi_1 \rangle^{2r-1+} \int_{|\xi| \gg |\xi_1|} \langle \xi \rangle^{-4+} d\xi \lesssim \langle \xi_1 \rangle^{2r-4+},
\]

which is bounded if \(r < 2\). While for \(B_3\), we have

\[
\langle |\xi - \xi_1|^3 + |\xi_1|^3 - |\xi^3 + \frac{1}{2}|\xi - \xi_1| + \frac{1}{2}|\xi_1| - \frac{1}{2}|\xi| \rangle^{2b} \gtrsim \langle (\xi - \xi_1)^3 - \xi^3 + \xi_1^3 \rangle^{2b} \sim \langle \xi^3 \rangle^{1-} \sim \langle \xi^3 \rangle^{-}.
\]

Thus,

\[
L \lesssim \langle \xi_1 \rangle^{2r} \int_{|\xi| \gg |\xi_1|} \langle \xi \rangle^{-5+} d\xi \lesssim \langle \xi_1 \rangle^{2r-4+},
\]

which is bounded if \(r < 2\).

**Case 3.** For \(|\xi| \sim |\xi_1| \gg 1\), since we are addressing integral in region \(A_2\), \(\langle \xi_1 \rangle \langle \xi - \xi_1 \rangle \gg \langle \xi \rangle\), this gives \(\langle |\xi - \xi_1|^3 + |\xi|^3 - |\xi^3 + \frac{1}{2}|\xi - \xi_1| + \frac{1}{2}|\xi_1| - \frac{1}{2}|\xi| \rangle^{2b} \gtrsim \langle (\xi - \xi_1)^3 - \xi^3 + \xi_1^3 \rangle^{2b} \sim \langle \xi_1 (\xi - \xi_1) \rangle^{2b} \sim \langle \xi \rangle^{2-} \langle \xi - \xi_1 \rangle^{1-}.
\]

Thus, for \(r < \frac{1}{2}\), we have \(2r - 1 < 0\), and the bound follows,

\[
L \lesssim \int_{|\xi| \sim |\xi_1|} \langle \xi \rangle^{-4+} \langle \xi - \xi_1 \rangle^{2r-1+} d\xi \lesssim \langle \xi_1 \rangle^{-3+}.
\]

Finally, similar technique applied in Farah [13] and Tzirakis [11, 13] to handle III, it can be reduced to the estimate of \(M\) by performing the change of variables \((\xi_1, \tau_1) \rightarrow (\xi - \xi_1, \tau - \tau_1)\). The proof is now complete. \(\square\)

**Proof.** (Proposition 4.2) Similar to Proposition 4.1 by introducing

\[
f(\xi, \tau) = \langle \xi \rangle^s \langle |\tau| - |\xi|^3 + \frac{1}{2}|\xi| \rangle^{b} \hat{u}(\xi, \tau), \quad g(\xi, \tau) = \langle \xi \rangle^s \langle |\tau| - \frac{1}{2}|\xi| \rangle^{b} \hat{v}(\xi, \tau),
\]

and setting \(r = -s \geq 0\), the desired estimate becomes

\[
\left\| \tau^{-r-j+1 \over 3} \int_{\Omega} \langle |\tau| - |\xi|^3 + \frac{1}{2}|\xi_1| \rangle^{b} \langle |\tau - \tau_1| - |\xi - \xi_1|^3 + \frac{1}{2}|\xi - \xi_1| \rangle^{b} d\xi_1 d\tau_1 d\xi \right\|_{L^2}^2
\]
(4.4) 
\[ \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}}, \]

where \( \Omega = \mathbb{R}^2 \times D \). Applying the Cauchy-Schwartz inequality to integral in \((\xi, \xi_1, \tau_1)\) space and the Young's inequality arrives at a bound,

\[
\|M\|_{L^2(\Omega)} \|f(\xi_1, \tau)g(\xi - \xi_1, \tau - \tau_1)\|_{L^2_{\xi,\xi_1,\tau_1}} \leq \sup_{\tau} \|M\|_{L^2(\Omega)} \|f^2 \ast g^2\|_{L^2_{\xi,\tau}},
\]

where

\[
M := \frac{\langle \tau \rangle^{-r-j+1} \langle \xi_1 \rangle^{4(j-4)(\xi_1)^r} \langle \xi - \xi_1 \rangle^r}{\langle |\tau_1| - |\xi_1|^3 - \frac{1}{2}|\xi_1|^1 \rangle^{b} \langle |\tau - \tau_1| - |\xi - \xi_1|^3 - \frac{1}{2}|\xi - \xi_1|^1 \rangle^{b}}.
\]

In the following proof, we will justify the proof for \( j = 2 \), and the one for \( j = 1 \) can be obtained in the same way. In order to bound,

(4.6) 
\[ \|M\|_{L^2(\Omega)}^2 = \int_{\Omega} \frac{\langle \xi \rangle^{-2r-2} \langle \xi_1 \rangle^{4(j-4)(\xi_1)^r} \langle \xi - \xi_1 \rangle^{2r}}{\langle |\tau_1| - |\xi_1|^3 - \frac{1}{2}|\xi_1|^1 \rangle^{2b} \langle |\tau - \tau_1| - |\xi - \xi_1|^3 - \frac{1}{2}|\xi - \xi_1|^1 \rangle^{2b}} d\xi_1 d\tau_1, \]

we split the integral domain into following cases:

**Case 1.** Consider \( \tau_1(\tau - \tau_1) \geq 0 \). Since \( r \geq 0 \), we can drop the \( \langle \tau \rangle \) term and apply Lemma [2.1]. Then, (4.6) is bounded by the following,

\[
\|M\|_{L^2(\Omega)}^2 \lesssim \int_{\mathbb{R} \times D} \frac{\langle \xi \rangle^{-4(j-4)(\xi_1)^r} \langle \xi - \xi_1 \rangle^{2r}}{\langle |\xi_1|^3 + |\xi - \xi_1|^3 + \frac{1}{2}|\xi_1|^1 + \frac{1}{2}|\xi - \xi_1|^1 \rangle^{4b-1}} d\xi d\xi_1.
\]

In addition, for either \( |\xi| \lesssim |\xi_1| \) or \( |\xi| \gg |\xi_1| \), we always have \( |\xi_1|^3 + |\xi - \xi_1|^3 \gtrsim \max \{ |\xi|^3, |\xi_1|^3 \} \) and \( |\xi_1| + |\xi - \xi_1| \gtrsim \max \{ |\xi_1|, |\xi_1| \} \). Since \( |\xi| \gg \tau_1, |\xi_1| \gtrsim 1 \), we can reduce the bound to

\[
\|M\|_{L^2(\Omega)}^2 \lesssim \int_{\mathbb{R} \times D} \frac{\langle \xi \rangle^{-4(j-4)(\xi_1)^r} \langle \xi - \xi_1 \rangle^{2r}}{\langle \max \{ |\xi_1|, |\xi_1| \} \rangle^{12b-3}} d\xi d\xi_1.
\]

Now, we split the domain of the integral into two regions

\[ D_1 = \{ (\xi, \xi_1) \in D \times \mathbb{R} : |\xi_1| \lesssim |\xi| \}, \quad D_2 = \{ (\xi, \xi_1) \in D \times \mathbb{R} : |\xi| \gg |\xi_1| \} . \]

For the former case, one has \( \langle \xi \rangle \lesssim \langle \xi_1 \rangle \) and \( \langle \xi - \xi_1 \rangle \lesssim \langle \xi \rangle \), then

\[
\int_{D_1} \frac{\langle \xi \rangle^{-4(j-4)(\xi_1)^r} \langle \xi - \xi_1 \rangle^{2r}}{\langle \max \{ |\xi_1|, |\xi_1| \} \rangle^{12b-3}} d\xi d\xi_1 \lesssim \int_{|\xi_1| \lesssim |\xi|} \langle \xi \rangle^{4(r-4)} \int_{|\xi_1| \lesssim |\xi|} \langle \xi_1 \rangle^{-12b+3} d\xi_1 d\xi \lesssim \langle \xi \rangle^{4(r-12b+1)},
\]

which is bounded if \( r < \frac{5}{2} \) and \( \frac{1}{2} - b > 0 \) sufficient small. For the latter case, one has \( \langle \xi - \xi_1 \rangle \sim \langle \xi_1 \rangle \), then

\[
\int_{D_2} \frac{\langle \xi \rangle^{-4(j-4)(\xi_1)^r} \langle \xi - \xi_1 \rangle^{2r}}{\langle \max \{ |\xi_1|, |\xi_1| \} \rangle^{12b-3}} d\xi d\xi_1 \lesssim \int_{|\xi_1| \gg |\xi|} \langle \xi \rangle^{4(r-4)} \int_{|\xi_1| \gg |\xi|} \langle \xi_1 \rangle^{4(r-12b+3)} d\xi_1 d\xi \lesssim \langle \xi \rangle^{4(r-12b+1)},
\]
which is bounded if $r < \frac{1}{2}$ and $\frac{1}{2} - b > 0$ sufficient small.

**Case 2.** Consider $\tau_1(\tau - \tau_1) < 0$ and $|\xi_1| \lesssim |\xi|$. Similar to the Case 1, by dropping the $\langle \tau \rangle$ term, it leads to

$$
\|M\|_{L^2(\Omega)}^2 \lesssim \int_{\mathbb{R} \times D} \frac{\langle \xi \rangle^{-4} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r}}{\langle \tau \pm (|\xi_1|^3 - |\xi - \xi_1|^3 + \frac{1}{2}|\xi_1| - \frac{1}{2}|\xi - \xi_1|) \rangle^{4b-1}} d\xi d\xi_1.
$$

We then estimate the integral in the following two regions

$D_3 = \{ (\xi, \xi_1) \in D \times \mathbb{R} : \xi_1(\xi - \xi_1) \geq 0 \}, \quad D_4 = \{ (\xi, \xi_1) \in D \times \mathbb{R} : \xi_1(\xi - \xi_1) < 0 \}.$

For the former case, by changing the variable $z = 2\xi_1^3 - 3\xi_1^2 \xi + 3\xi_1 \xi^2 - \xi^3 + \xi_1 - \frac{1}{2} \xi$ with $|z| \lesssim \max\{ |\xi_1|^3, |\xi| \}$ and

$$
dz = (6\xi_1^2 - 6\xi_1 \xi + 3\xi^2 + 1) d\xi_1 = \left(3(\xi_1 - \xi)^2 + 3\xi_1^2 + 1\right) d\xi_1,
$$

one has,

$$
\int_{D_3} \frac{\langle \xi \rangle^{-4} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r}}{\langle \tau \pm (|\xi_1|^3 - |\xi - \xi_1|^3 + \frac{1}{2}|\xi_1| - \frac{1}{2}|\xi - \xi_1|) \rangle^{4b-1}} d\xi_1 d\xi 
\lesssim \int_{D_3} \frac{\langle \xi \rangle^{-4} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r}}{\langle \tau \pm (2\xi_1^3 - 3\xi_1^2 \xi + 3\xi_1 \xi^2 - \xi^3 + \xi_1 - \frac{1}{2} \xi) \rangle^{4b-1}} d\xi_1 d\xi 
\lesssim \int_{D_3 \times \{|z| \leq \max\{ |\xi_1|^3, |\xi| \}\}} \frac{1}{\langle \xi \rangle^{-4} \langle \xi_1 \rangle^{2r} (\tau + z)^{4b-1}} dz d\xi,
$$

since for $r < \frac{1}{2}$, $|\xi| \gtrsim 1$ and $|\xi_1| \lesssim |\xi|$, $\frac{\langle \xi_1 \rangle^{2r} (\tau - \xi_1)^{2r}}{\xi_1^2 + (\xi_1 - \xi)^2} \lesssim 1$.

Recall that $|\tau| \ll |\xi|^3$ in $D$, then,

$$
\int_{D \times \{|z| \leq \max\{ |\xi_1|^3, |\xi| \}\}} \frac{1}{\langle \xi \rangle^{-4} \langle \tau + z \rangle^{4b-1}} dz d\xi 
\lesssim \int_{D} \langle \xi \rangle^{-4} \int_{|z| \leq \max\{ |\xi_1|^3, |\xi| \}} \frac{1}{\langle \tau + z \rangle^{4b-1}} dz d\xi 
\lesssim \langle \xi \rangle^{-4} \langle \tau \rangle^{4b-1} d\xi \lesssim \int \langle \xi \rangle^{-4} \langle \tau \rangle^{4b-1} d\xi,
$$

which is bounded for $\frac{1}{2} - b > 0$ sufficient small. For the later case, by changing variable $z = 3\xi_1^2 \xi - 3\xi_1^2 \xi_1 + 3\xi_1 \xi^2 + \frac{1}{2} \xi$ with $|z| \lesssim \max\{ |\xi_1|^3, |\xi| \}$

$$
dz = 3\xi(2\xi_1 - \xi) d\xi_1,
$$

one has,

$$
\int_{D_4} \frac{\langle \xi \rangle^{-4} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r}}{\langle \tau \pm (|\xi_1|^3 - |\xi - \xi_1|^3 + \frac{1}{2}|\xi_1| - \frac{1}{2}|\xi - \xi_1|) \rangle^{4b-1}} d\xi_1 d\xi 
\lesssim \int_{D_4} \frac{\langle \xi \rangle^{-4} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r}}{\langle \tau \pm (|\xi_1|^3 - |\xi - \xi_1|^3 + \frac{1}{2}|\xi_1| - \frac{1}{2}|\xi - \xi_1|) \rangle^{4b-1}} d\xi_1 d\xi 
\lesssim \int_{D_4 \times \{|z| \leq \max\{ |\xi_1|^3, |\xi| \}\}} \frac{1}{\langle \xi \rangle^{-4} \langle \xi_1 \rangle^{2r} (\tau + z)^{4b-1}} dz d\xi.
$$
One shall notice that if \( \xi_1(\xi - \xi_1) < 0 \) then \( 2\xi_1 - \xi > |\xi| \). In addition, if \( |\xi_1| \lesssim |\xi| \) then \( |\xi - \xi_1| \lesssim |\xi| \), thus \( \langle \xi - \xi_1 \rangle \lesssim \langle \xi \rangle \). Then, one has the bound

\[
\int_D \frac{\langle \xi \rangle^{-4} \langle \xi \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r}}{|\xi(2\xi_1 - \xi)|^{4b-1}} d\xi d\tau \\
\lesssim \int_D \frac{\langle \xi \rangle^{-4} \langle \xi \rangle^{2r}}{|\xi|^2 (\tau \pm z)^{4b-1}} d\xi d\tau \\
\lesssim \int_D \langle \xi \rangle^{4r-6} \int_{|\xi| \leq \max \{|\xi|^3, |\xi|\}} \frac{1}{(\tau \pm z)^{4b-1}} d\xi d\tau \lesssim \int_{\mathbb{R}} \langle \xi \rangle^{4r-6} (\max \{|\xi|^3, |\xi|\})^{2-4b} d\xi,
\]

which is again bounded if \( r < \frac{1}{2} \) and \( b - \frac{1}{2} < 0 \) sufficient small.

**Case 3.** Consider \( \tau_1(\tau - \tau_1) < 0 \) and \( |\xi_1| \gg |\xi| \). We establish (4.4) directly. According to duality, it suffices to show that

\[
\int_{\Omega_1} M f(\xi_1, \tau_1) g(\xi - \xi_1, \tau - \tau_1) h(\tau) d\xi d\tau d\xi_1 d\tau_1 \lesssim \|f\|_{L^2_{\xi_1, \tau_1}} \|g\|_{L^2_{\xi_1, \tau_1}} \|h\|_{L^2_{\xi_1}},
\]

where \( M \) is defined in (4.5) for \( j = 2 \) and

\[\Omega_1 = D \times \{\xi_1 \in \mathbb{R} : |\xi_1| \gg |\xi|\} \times \{(\tau, \tau_1) \in \mathbb{R}^2 : \tau_1(\tau - \tau_1) < 0\}\].

Using the Cauchy-Schwartz, Holder and Young’s inequalities, we arrive at

\[
\int_{\Omega_1} M f(\xi_1, \tau_1) g(\xi - \xi_1, \tau - \tau_1) h(\tau) d\xi d\tau d\xi_1 d\tau_1 \\
\lesssim \left\| M g(\xi - \xi_1, \tau - \tau_1) h(\tau) d\xi d\tau \right\|_{L^2_{\xi_1, \tau_1}} \|f\|_{L^2_{\xi_1, \tau_1}} \\
\lesssim \left\| M(\xi)^{\frac{3}{2}} \right\|_{L^2(\Omega_2)} \|g(\xi - \xi_1, \tau - \tau_1) h(\tau) \langle \xi \rangle^{-\frac{3}{2}} \|_{L^2_{\xi_1, \tau_1}} \|f\|_{L^2_{\xi_1, \tau_1}} \\
\lesssim \sup_{\xi_1, \tau_1} \|M(\xi)^{\frac{3}{2}} \|_{L^2(\Omega_2)} \|\langle \xi \rangle^{-\frac{3}{2}}\|_{L^2_{\xi_1, \tau_1}} \|g\|_{L^2_{\xi_1, \tau_1}} \|h\|_{L^2_{\xi_1}},
\]

where \( \Omega_2 = \{(\xi, \tau) : |\xi|^3 \gg |\tau|, |\xi| \gtrsim 1, |\xi_1| \gg |\xi|, \tau_1(\tau - \tau_1) < 0\} \). It remains to show that for \( \tau_1 > 0 \) and \( \tau - \tau_1 < 0 \),

\[
\|M(\xi)^{\frac{3}{2}}\|_{L^2(\Omega_2)} = \int_{\Omega_2} \langle \tau \rangle^{\frac{3}{2} - 2b} \langle \xi \rangle^{3 - 3b} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r} d\xi d\tau
\]

is bounded, since \( \tau_1 < 0 \) and \( \tau - \tau_1 > 0 \) can be established similarly. Notice that in \( \Omega_2 \), one has \( \langle \xi - \xi_1 \rangle \sim \langle \xi_1 \rangle \), \( |\xi - \xi_1|^3 - |\xi_1|^3 + \frac{1}{2}|\xi - \xi_1| - \frac{1}{2}|\xi_1| = \pm |\xi|(\xi^2 - 3\xi \xi_1 + 3\xi_1^2 + 1) \) and \( \langle \tau \rangle \pm |\xi|(\xi^2 - 3\xi \xi_1 + 3\xi_1^2 + 1)) \sim \langle \xi_1^2 \rangle \sim \langle \xi \rangle \langle \xi_1 \rangle^2 \). This leads to

\[
\int_{\Omega_2} \langle \tau \rangle^{\frac{3}{2} - 2b} \langle \xi \rangle^{3 - 3b} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r} d\xi d\tau \\
\lesssim \int_{\Omega_2} \langle \tau \rangle^{\frac{3}{2} - 2b} \langle \xi \rangle^{3 - 3b} \langle \xi_1 \rangle^{2r} \langle \xi - \xi_1 \rangle^{2r} d\xi d\tau
\]
with its usual product topology. Let \( Y(5.1) \)
where \( g \)
which is bounded if \( r < \frac{1}{2} \) and \( b < \frac{1}{2} \). The proof is now complete. \( \square \)

5. Local Well-Posedness

Now, we consider the nonlinear problem:
\[
(5.1) \begin{cases}
    u_t - u_{xx} + \beta u_{xxxxx} - u_{xxxxx} + (u^2)_{xx} = 0, & \text{for } x > 0, t > 0, \\
    u(x, 0) = \varphi(x), u_t(x, 0) = \psi'(x), \\
    u(0, t) = h_1(t), u_{xx}(0, t) = h_2(t), u_{xxxx}(0, t) = h_3(t).
\end{cases}
\]

Let \( Q_s \) be defined as below
\[ Q_s = H^s(\mathbb{R}^+) \times H^{s-1}(\mathbb{R}^+) \times \mathcal{H}^s(\mathbb{R}^+) \]
where
\[ \mathcal{H}^s(\mathbb{R}^+) := H^{\frac{s+2}{2}}(\mathbb{R}^+) \times H^{\frac{s}{2}}(\mathbb{R}^+) \times H^{\frac{s-2}{2}}(\mathbb{R}^+) \]
with its usual product topology. Let \( Y_{s,T} \) be the collection of
\[ v \in C(0, T; H^s(\mathbb{R}^+)) \cap X^{s,b}(\mathbb{R}^+ \times (0, T)) \]
with its norm \( \| \cdot \|_{Y_{s,T}} \) defined by
\[ \| v \|_{Y_{s,T}} = \sup_{t < T} \| v(\cdot, t) \|_{H^s(\mathbb{R}^+)} + \| v(x, \cdot) \|_{X^{s,b}(\mathbb{R}^+ \times (0, T))}. \]

Proof. (Theorem 1.1) For \( T > 0 \), we write
\[ u(x, t) = \eta(t) W(\varphi^*, \psi^*) + \eta(t) W_{bdr}(\vec{h} - \vec{p}) \]
\[ + \eta_T(t) \left( \int_0^t [W(0, g)](x, t - \tau) d\tau - W_{bdr}(\vec{q}) \right), \]
where \( g = \eta_T(t) u^2, \vec{p} \) as defined in Lemma 3.2 and \( \vec{q} = \eta_T(t)(q_1, q_2, q_3) \)
with \( q_j \) for \( j = 1, 2, 3 \) defined in Lemma 3.3. It is then easy to verify that \( u = u(x, t) \) solves the IBVP in \([0, T] \) for \( T < 1 \). For given \( (\varphi, \psi, \vec{h}) \in Q_s \), let \( r > 0 \) and \( T > 0 \) be constants to be determined later. Let
\[ S_{r,T} = \{ u \in Y_{s,T}, \| u \|_{Y_{s,T}} \leq r \}, \]
then the set \( S_{r,T} \) is a convex, closed and bounded subset of \( Y_{s,T} \). Define a map \( \Gamma \) on \( S_{r,T} \) by
\[ \Gamma(u) = u(t), \]
for \( u \in S_{r,T} \). We will show that \( \Gamma \) is a contraction map from \( S_{r,T} \) to \( S_{r,T} \) for proper \( r \) and \( T \). Applying Lemma 3.1, 3.2 and 3.3, Proposition 3.4 and 3.6
yields that,
\[ \| \Gamma(u) \|_{H^s(\mathbb{R}^+)} \leq \| \eta(t) W(\varphi^*, \psi^*) \|_{H^s(\mathbb{R}^+)} + \| \eta(t) W_{bdr}(\vec{h} - \vec{p}) \|_{H^s(\mathbb{R}^+)} \]
Thus, 

$$
\|\eta_T^n(t) \left( \int_0^t [W_R(0, u^2)](x, t - \tau)d\tau - W_{\text{bdr}}(\vec{q}) \right) \|_{H^2_x(\mathbb{R}^+)} \lesssim \|((\varphi, \psi, \vec{h})\|_{Q_s} + \|\eta_T^n(t) \left( \int_0^t [W_R(0, u^2)](x, t - \tau)d\tau - W_{\text{bdr}}(\vec{q}) \right) \|_{X^{s,a_1}},
$$

with $a_1 = \frac{3-2b}{4}$. This is because $X^{s,a_1} \subseteq C^0_t H^s_x$ for $a_1 > \frac{1}{2}$ (c.f. [22] [23] [24] [25]). Then, according to Lemma 2.4, 2.6 and Proposition 4.1

$$
\left\| \eta_T(t) \int_0^t [W_R(0, g)](x, t - \tau)d\tau \right\|_{X^{s,a}} \lesssim T^{1-(a_1+b)} \left\| \frac{x^2 \tilde{g}}{\phi(\xi)} \right\|_{X^{s,-a_2}} \lesssim T^{\frac{1}{4} - \frac{b}{2}} \|u\|_{X^{s,b}}^2.
$$

Moreover, according to Lemma 2.4, Proposition 3.5, 3.7 and set $a_2 = \frac{6b-1}{4} < b$,

$$
\|\eta_T^n(t)[W_{\text{bdr}}(\vec{q})](x, t)\|_{X^{s,a}} \lesssim \|\vec{q}\|_{H^s} \lesssim \left\| \frac{x^2 \tilde{g}}{\phi(\xi)} \right\|_{X^{s,-a_2}} \lesssim \|\eta_T^n(t)\|_{X^{s,a_2}} \|\vec{u}\|_{X^{s,a_2}} \lesssim T^{\frac{b-1}{2}} \|u\|_{X^{s,b}}^2.
$$

Hence,

$$
\|\Gamma(u)\|_{H^s(\mathbb{R}^+)} \leq C \|((\varphi, \psi, \vec{h})\|_{Q_s} + 2CT^{\frac{1}{4} - \frac{b}{2}}r^2.
$$

Furthermore, we have,

$$
\|\Gamma(u)\|_{X^{s,b}} \lesssim \|\eta(T)W_R(\varphi^*, \psi^*)\|_{X^{s,b}} + \|\eta(T)W_{\text{bdr}}(\vec{h} - \vec{p})\|_{X^{s,b}}
$$

$$
 + \left\| \eta_T^n(t) \left( \int_0^t [W_R(0, g)](x, t - \tau)d\tau - W_{\text{bdr}}(\vec{q}) \right) \right\|_{X^{s,b}} \lesssim \|((\varphi, \psi, \vec{h})\|_{Q_s} + \left\| \eta_T^n(t) \left( \int_0^t [W_R(0, g)](x, t - \tau)d\tau - W_{\text{bdr}}(\vec{q}) \right) \right\|_{X^{s,b}}.
$$

Again, according to Proposition 3.7, 4.1 and 4.2, Lemma 2.6 one has,

$$
\left\| \eta_T^n(t) \int_0^t [W_R(0, g)](x, t - \tau)d\tau \right\|_{X^{s,b}} \lesssim T^{1-b-a_2} \left\| \frac{x^2 \tilde{g}}{\phi(\xi)} \right\|_{X^{s,-a_2}} \lesssim T^\frac{1}{4} - \frac{b}{2} T^\frac{1}{4} - \frac{b}{2} \|u\|_{X^{s,b}}^2,
$$

and

$$
\left\| \eta_T^n(t)W_{\text{bdr}}(\vec{q}) \|_{X^{s,b}} \lesssim \|\vec{q}\|_{H^s} \lesssim T^\frac{1}{4} - \frac{b}{2} \|u\|_{X^{s,b}}^2.
$$

Thus,

$$
\|\Gamma(u)\|_{X^{s,b}} \lesssim C \|((\varphi, \psi, \vec{h})\|_{Q_s} + C(T^\frac{3}{4} - 3b + T^\frac{1}{4} - \frac{b}{2})r^2,
$$

and we can choose $T$ small enough depending on $r$ such that,

$$
r = 2C \|((\varphi, \psi, \vec{h})\|_{Q_s}, \ (T^\frac{3}{4} - 3b + T^\frac{1}{4} - \frac{b}{2})r \leq \frac{1}{2}, \ 2T^\frac{1}{4} - \frac{b}{2}r \leq \frac{1}{2}
$$

then,

$$
\|\Gamma(u)\|_{C(0, T; H^s(\mathbb{R}^+)) \cap X^{s,b}(\mathbb{R}^+ \times (0, T))} \leq r.
$$

Similar estimates can be drawn for $\Gamma(u) - \Gamma(v)$, then for such $r$ and $T$ the map $u = \Gamma(u)$ is contraction. \qed
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