Biorthogonal polynomials for 2-matrix models with semiclassical potentials

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Abstract

We consider the biorthogonal polynomials associated to the two–matrix model where the eigenvalue distribution has potentials $V_1, V_2$ with arbitrary rational derivative and whose supports are constrained on an arbitrary union of intervals (hard-edges). We show that these polynomials satisfy certain recurrence relations with a number of terms $d_i$ depending on the number of hard-edges and on the degree of the rational functions $V_i$. Using these relations we derive Christoffel–Darboux identities satisfied by the biorthogonal polynomials: this enables us to give explicit formulæ for the differential equation satisfied by $d_i + 1$ consecutive polynomials. We also define certain integral transforms of the polynomials and use them to formulate a Riemann–Hilbert problem for $(d_i + 1) \times (d_i + 1)$ matrices constructed out of the polynomials and these transforms. Moreover we prove that the Christoffel–Darboux pairing can be interpreted as a pairing between two dual Riemann–Hilbert problems.
1 Introduction and setting

In this paper we consider the biorthogonal polynomials associated to the two–matrix model. The model is defined by a measure on the space of pairs of Hermitian matrices $M_1, M_2$ of the form

$$d\mu(M_1, M_2) := dM_1dM_2e^{-\text{Tr}(V_1(M_1) + V_2(M_2) + M_1M_2)}.$$  (1-1)
Using Itzykson–Zuber/Harish-Chandra’s formula the model can be reduced to the study of biorthogonal polynomials [14] (BOPs for short), namely two sequences of polynomials \( \{ \pi_n(x) \}, \{ \sigma_m(y) \} \)

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} dx dy e^{-V_1(x)-V_2(y)+xy} \pi_n(x) \sigma_m(y) = \delta_{nm} . \tag{1-2}
\]

For the model to have a probabilistic interpretation, the potentials should be real and satisfy certain growth conditions to ensure the convergence of the integrals. In order to introduce the setting of this paper we consider the following situation (which is strictly included in the more general setting to be expounded later)

1. There is a finite collection of disjoint intervals \( I = \bigcup I_j \subseteq \mathbb{R}_x \) and \( J = \bigcup J_j \subseteq \mathbb{R}_y \) (\( \mathbb{R}_x \) denotes the real axis of the \( x \)-variable), in the complement of which the potentials are \( +\infty \): in other words the matrices \( M_1, M_2 \) have spectrum confined to these multi-intervals, so that the associated BOPs satisfy

\[
\int_I \int_J dx dy e^{-V_1(x)-V_2(y)+xy} \pi_n(x) \sigma_m(y) = \delta_{nm} . \tag{1-3}
\]

2. The two potentials \( V_1(x) \) and \( V_2(y) \) are the restriction to \( I, J \) (respectively) of real-analytic functions with rational derivative (with poles symmetrically placed off the real axis, or on the complement of the intervals on the real axis) together with the necessary growth condition if the intervals are unbounded.

This situation has been addressed in [3] within the general context of bilinear moment functionals. Indeed it is convenient to recast the orthogonality condition in a more abstract setting where one considers a biorthogonal functional \( \mathcal{L} : \mathbb{C}[x] \otimes \mathbb{C}[y] \to \mathbb{C} \) defined by

\[
\mathcal{L}(x^i | y^j) := \int_I \int_J dx dy x^i y^j e^{-V_1(x)-V_2(y)+xy} = \mu_{ij} . \tag{1-4}
\]

and then extended by linearity to arbitrary polynomials. The biorthogonality condition then reads

\[
\mathcal{L}(\pi_n | \sigma_m) = \delta_{nm} . \tag{1-5}
\]

The properties of the potentials \( V_1, V_2 \) and the supports of integration can be dealt with on the same footing by purely algebraic methods: to this end one introduces four polynomials \( A_i, B_i, i = 1, 2 \) according to the strategy outlined hereafter. Let \( (x_j, m_j) \) be the location of the poles of \( V'_i(x) \) with their order (we include all of the poles, in this case also the complex conjugates, which clearly come in with the same multiplicities) and let \( a_j \) be the endpoints of \( I \). We define then \( A_1, B_1 \) (and similar expressions for \( A_2, B_2 \)) as follows

\[
B_1(x) = \prod (x - x_j)^{m_j} \prod (x - a_j) , \quad A_1 := V'_1 B_1 - B'_1 , \tag{1-6}
\]

so that now \( V'_1 = \frac{A_1 + B'_1}{B_1} \). It is a straightforward exercise to verify (using integration by parts) that the bilinear functional satisfies the following distributional identities for arbitrary \( p(x) \in \mathbb{C}[x], s(y) \in \mathbb{C}[y] \)

\[
\mathcal{L} \left( -B_1(x)p'(x) + A_1(x)p(x) \bigg| s(y) \right) = \mathcal{L} \left( B_1(x)p(x) \bigg| y s(y) \right) , \tag{1-7}
\]

\[
\mathcal{L} \left( p(x) - B_2(y)s'(y) + A_2(y)s(y) \bigg| y \right) = \mathcal{L} \left( x p(x) \bigg| B_2(y) s(y) \right) . \tag{1-8}
\]
Abstracting formulæ (1-7,1-8) from the specific context, we will say that a bimoment functional \( \mathcal{L} \) is semi-classical if it satisfies those same relations (1-7,1-8) for some given (and fixed) polynomials \( A_i, B_i \). The name comes from a similar usage in the context of ordinary orthogonal polynomials [13].

Such functionals have been studied in [3], where it was shown that

**Proposition 1.1** For given \( A_i, B_i \), \( i = 1, 2 \), a semiclassical moment functional \( \mathcal{L} \) is the linear combination of \( s_1 s_2 \) independent functionals \( \mathcal{L}_{\nu,\mu} \), \( \mu = 1, \ldots, s_1 \), \( \nu = 1, \ldots, s_2 \), where \( s_i = \max(\deg A_i, \deg B_i + 1) \)

More importantly (at least in the case \( \deg A_i \geq \deg B_i + 1 \)) all of these moment functionals \( \mathcal{L}_{\mu,\nu} \) can be given an integral representation completely analog to (1-4), but without any restriction on the reality of the potentials or of the contours of integration: this is the setting of the present paper.

1.1 Connection to other orthogonal polynomials

The algebraic properties of semiclassical bilinear moment functionals apply to a slightly different class of orthogonal polynomials. Let us consider in fact orthogonal polynomials in the complex plane with respect to a measure of the form

\[
d\mu(z, \overline{z}) := e^{-|z|^2 + 2RV(z)} d^2z
\]

where \( V(z) \) is a holomorphic function such that \( V'(z) \) is rational. The convergence of the measure mandates that the residues of \( V'(z)dz \) must have real part greater than \(-\frac{1}{2}\) and that the behavior at \( \infty \) of \( V \) cannot exceed the second power (and also a certain open condition on the coefficient of this quadratic term which we do not specify here). Orthogonal polynomials are defined as a holomorphic basis of \( L^2(\mathbb{C}, d\mu) \). It is amusing to note that the moment functional

\[
\mathcal{L}(z^j \overline{z}^k) := \int_{\mathbb{C}} z^j \overline{z}^k d\mu(z, \overline{z}) =: \mu_{j,k}
\]

is a semiclassical moment functional (using Stokes’ thm. in place of integration by parts) with just some (obvious) reality constraint on the bimoments. Therefore all the algebraic manipulations that rely on the semiclassicity alone carry out verbatim to this case and confirm certain manipulations used in [21]. With very minor and trivial modifications, Section 2 almost entirely generalizes (in particular Thm. 2.1). Significant differences (sufficient to require a different analysis to appear elsewhere) arise in the construction of the fundamental systems and the Riemann–Hilbert problem.

1.2 Connection to 2-Toda equations

The framework of this paper is connected to the general theory of 2-Toda equations [22, 1, 2]. This is the theory of a pair of (semi)-infinite matrices \( P, Q \) (in our notation) where \( Q \) is lower-Hessenberg and \( P \) is upper–Hessenberg\(^3\) which evolve under a bi-infinite set of commuting flows \( \{t_j, \tilde{t}_j\}_{j \in \mathbb{N}} \)

\[
\partial_{t_j} Q = -\frac{1}{2} [Q, (Q^j)_-] , \quad \partial_{\tilde{t}_j} Q = -\frac{1}{2} [Q, (P^j)_-] \]

\(^3\)We say that a matrix is lower Hessenberg if \( (i, i + 1 + k) \) entries vanish (\( \forall k = 1, 2, \ldots \)) and also all \( (i, i + 1) \)-entries are nonzero. A matrix is upper Hessenberg if its transposed is lower-Hessenberg.
\[ \partial_t P = -\frac{1}{\mathcal{J}} [P, (Q^\dagger)_{+0}], \quad \partial_{\bar{t}} P = -\frac{1}{\mathcal{J}} [P, (P^\dagger)_{+0}] \tag{1-12} \]

where the subscript \( \pm_0 \) denotes the upper/lower triangular part plus half of the diagonal (we are assuming the normalization such that the upper triangular part of \( Q \) coincides with the transposed of the lower-triangular part of \( P \)).

Let now \( Q, P \) be semi-infinite matrices. We can use \( Q, P \) to denote the matrices expressing the multiplicative recurrence relations of a sequence of polynomials,

\[ x_n = \sum_{j=0}^{n+1} Q_{nj} x_j, \quad y_n = \sum_{j=0}^{n+1} P_{nj} y_j, \tag{1-13} \]

where the polynomials are recursively defined by this relation. Using the generalization of Favard’s theorem proved in our \([3]\) we prove the existence of (unique) a bimoment functional \( \mathcal{L} : \mathbb{C}[x] \otimes \mathbb{C}[y] \to \mathbb{C} \) such that

\[ \mathcal{L}(x_n | y_m) = \delta_{nm}. \tag{1-14} \]

It then follows easily that the 2-Toda flows are linearized by this moment map, in the sense that the solutions \( Q(t, \bar{t}), P(t, \bar{t}) \) are simply the multiplication matrices for the biorthogonal polynomials of the moment functional

\[ \mathcal{L}_t(x | y) := \mathcal{L} \left( e^{-\sum \frac{t j x_j}{\mathcal{J}}} x | e^{-\sum \frac{t j y_j}{\mathcal{J}}} y \right). \tag{1-15} \]

The moment functionals of semiclassical type (eqs. 1-7, 1-8) that we are going to analyze form a particular class of reductions of the above-mentioned 2-Toda hierarchy. The simplest situation is the one of bimoment semiclassical functionals with polynomial potentials as the ones considered in \([4, 5, 6]\), where the matrices \( P, Q \) are also finite band. Moreover the solutions which arise in the context of semiclassical bilinear functionals also satisfy the (compatible) constraint of the string equation

\[ [P, Q] = \hbar 1 \tag{1-16} \]

(the constant \( \hbar \) can be disposed of by a rescaling). The parameters of the (finite–dimensional) reduction are the coefficients of the potentials: for more general semiclassical moment functionals as the ones considered in this paper, the parameters involve not only the coefficients of the partial fraction expansions of the (derivatives of the) potentials, but also the position of the poles and the position of the end-points of the supports of the measure (the hard–edge endpoints).

The paper is organized as follows

1. In Section 2 we derive the recurrence relation satisfied by the biorthogonal polynomials of a semiclassical moment functional. There are two types of recurrence relations: one which involves the multiplication by the spectral parameter (and plays the rôle of the more standard three–term recurrence relation for orthogonal polynomials) and one which involves a differential operator acting on the polynomials.

2. In Section 3 we recall some possibly not well known facts about a certain class of linear homogeneous ODEs. These equations are next in simplicity to the class of constant coefficients ODEs, inasmuch as the
coefficients are allowed to be linear functions of the independent variable. When considering the formal adjoint equation then the classical bilinear concomitant provides a nondegenerate pairing between the solution spaces of the pair of mutually adjoint ODEs. In this case we give an interpretation of it in terms of an intersection pairing between certain contours used in the representation of the solutions as contour-integrals. This part of the paper is logically quite independent on the rest but it is nevertheless necessary in order to understand certain constructs of the following section.

3. In Section 4 we define the auxiliary wave vectors for our functionals, using a certain multiple integral transform which relies upon the form of the bilinear concomitant associated to our semiclassical moment functional (extending some of the results of [8]). These expression will prove crucial in the formulation of a first order ODE of rank $d_i + 1 = 1 + \deg(A_i)$ satisfied by the biorthogonal polynomials. We also derive the analog of the Christoffel–Darboux identities satisfied by standard orthogonal polynomials to our case of biorthogonal polynomials: similar expression were extensively used in [5, 8] for the case where the potentials $V_i$ are polynomials (which is a subcase strictly included in our present setting) and in absence of hard-edge endpoints. The novel feature is that these new identities involve not only the biorthogonal polynomials of the moment functional $\mathcal{L}$ itself, but also those of the associated bilinear semiclassical moment functionals

$$\tilde{\mathcal{L}} := \mathcal{L}(B_1 \bullet | \bullet) ; \quad \tilde{\mathcal{L}} := \mathcal{L}(\bullet | B_2 \bullet).$$

(1-17)

This feature appears prominently in the perfect duality of the Riemann–Hilbert problems appearing in the next section.

4. In Section 5 we define a pair\footnote{In fact there are two such pairs, the other being obtained by interchanging the rôles of $x, y, B_1, B_2$ etc.} of piecewise–analytic matrices constructed out of the entries of the wave-vectors and their auxiliary wave-vectors. They satisfy certain jump conditions on contours in the complex plane and some asymptotic behavior at the zeroes of $B_1$. Moreover they satisfy rational first order ODEs with poles at the zeroes of $B_1$. The Christoffel–Darboux identity, when written as a bilinear expression for these matrices becomes a perfect pairing (Thm. 5.1) in the sense that establishes a nondegenerate constant (in $x$) duality-pairing between the two solution spaces. This pairing should be thought of as the “dressed” form of the bilinear concomitant pairing introduced in Sect. 3. Similar Riemann–Hilbert problems have appeared elsewhere in the literature, e.g. [19, 18, 8, 4].

In order to facilitate the navigation through the paper all proofs of more technical nature are collected in the appendix and only those that may help the understanding are left in the main body of the paper.

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2 Semiclassical bilinear moment functionals of type $BB$

We consider an arbitrary bilinear semiclassical moment functional (as defined in the introduction) \[ \text{[3]}, \text{i.e. satisfying (1-7, 1-8)]. Let } q_i = \deg(B_i) \text{ and } d_i = \deg(A_i): \text{ we assume that } d_i \geq q_i + 1 \text{ ("type BB" in the terminology of [3]). We also make the assumption that the two pairs of polynomials } A_i, B_i \text{ are reduced in the sense that the only common zeroes of } A_i \text{ and } B_i \text{ (} \forall i = 1, 2 \text{) are amongst the simple zeroes of } B_i. \text{ Any moment functional coming from a representation like the one in the introduction (1-4) has this property of reducedness. In [3] the case of non-reduced moment functional is also considered, and it corresponds to functionals which may be expressed as delta functions (or derivatives thereof): we refer ibidem for details.}

It is known \[ \text{[3]} \text{ that any such reduced moment functional can be expressed in integral form}

\[
\mu_{ij} := \mathcal{L}(x^i | y^j) = \sum_{\mu=1}^{d_1} \sum_{\nu=1}^{d_2} \kappa_{\mu,\nu} \mathcal{L}_{\mu,\nu}(x^i | y^j) \tag{2-1}
\]

\[
\mathcal{L}_{\mu,\nu}(x^i | y^j) = \int_{\Gamma_{x,\mu}} \int_{\Gamma_{y,\nu}} e^{-V_1(x) - V_2(y) + xy} dx dy \tag{2-2}
\]

\[
V_i'(y) = \frac{A_i + B_i'}{B_i} \tag{2-3}
\]

\[
\mathcal{L} = \int_{\kappa} x^i y^j e^{-V_1(x) - V_2(y) + xy} dx dy \tag{2-4}
\]

\[
\int_{\kappa} := \sum_{\mu=1}^{d_1} \sum_{\nu=1}^{d_2} \kappa_{\mu,\nu} \int_{\Gamma_{x,\mu}} \int_{\Gamma_{y,\nu}} \tag{2-5}
\]

The two sets of contours of integration $\Gamma_{x,\mu}$ and $\Gamma_{y,\nu}$ are defined in the $x$ and $y$ complex planes respectively and in completely parallel fashion: we will define them in Section 3.1. We have also introduced the short-hand notation

Note that the case of hard-edges is included: the hard-edges are the zeroes of $B_i$ that cancel with the zeroes of the denominator defining $V_i'$ in eq. (2-3).

The constants $\kappa_{\mu,\nu} \in \mathbb{C}$ are arbitrary (not all zero). In the paper we will often invoke "genericity" conditions for the moment functional $\mathcal{L}$: by this we mean that the genericity is in the choice of the $\kappa$-constants and not in the choice of $A_i, B_i$ which we consider as given once and for all. All of the genericity conditions that we will use can be translated into the nonvanishing of certain infinite sequences of minors of the matrix of bimoments $M = [\mu_{ij}]$: since the moments $\mu_{ij}$ are linear in $\kappa$ as per (2-1), this genericity boils down to avoiding an at-most-denumerable collection of divisors of homogeneous polynomials in the $\kappa$-space.

2.1 Biorthogonal polynomials

Let us consider the biorthogonal polynomials associated to this bilinear moment functional, namely two sequences of monic polynomials satisfying the following conditions

\[
\{\pi_n(x), \sigma_n(y)\}_{n \in \mathbb{N}}
\]

6
The existence of these BOPs is guaranteed provided that the principal minors of the matrix of bimoments do not vanish
\[
\Delta_n[\mathcal{L}] := \det[\mu_{ij}]_{0 \leq i,j \leq n-1} \neq 0 \quad \forall n \in \mathbb{N},
\]
which also guarantees that \( h_n \neq 0 \), \( \forall n \in \mathbb{N} \) \((3)\). We find it more convenient to deal with the normalized BOPs:
\[
p_n := \frac{\pi_n}{\sqrt{h_n}}, \quad s_n := \frac{\sigma_n}{\sqrt{h_n}}
\]
We will use the following quasipolynomials
\[
\psi_n := p_ne^{-V_1(x)}, \quad \phi_n := s_ne^{-V_2(y)}
\]
and the following semi-infinite vectors (wave vectors)
\[
p(x) := [p_0, p_1, \ldots, p_n, \ldots]^t, \quad s(y) := [s_0, s_1, \ldots, s_n, \ldots]^t
\]
\[
\Psi := p(x)e^{-V_1(x)}, \quad \Phi := s(y)e^{-V_2(y)}
\]
It will become necessary to consider the following associated semiclassical functionals defined by the relations
\[
\hat{\mathcal{L}}(p) := \mathcal{L}(p|B_2 s), \quad \tilde{\mathcal{L}}(p|s) := \mathcal{L}(B_1 p|s).
\]
We leave to the reader the simple check that these are also semiclassical moment functionals where the potentials are replaced—respectively—by
\[
\hat{\mathcal{L}} \quad \longleftrightarrow \quad \begin{cases}
\hat{V}_1(x) = V_1(x) \\
\hat{V}_2(y) := V_2(y) - \ln B_2(y)
\end{cases}
\]
\[
\tilde{\mathcal{L}} \quad \longleftrightarrow \quad \begin{cases}
\tilde{V}_1(x) := V_1(x) - \ln B_1(x) \\
\tilde{V}_2(y) = V_2(y)
\end{cases}
\]
These definitions amount to \( \hat{A}_2 = A_2 - B'_2, \hat{B}_2 = B_2 \) so that \( V_2' = \frac{\hat{A}_2 + B'_2}{\hat{B}_2} = \frac{\hat{A}_2}{\hat{B}_2} \).

Note, however, that they are defined along the same contours as \( \mathcal{L} \) and with the same coefficients \( \kappa \)'s.

### 2.2 Multiplicative recurrence relations

We now prove
Theorem 2.1 The BOPs satisfy the following finite-term recurrence relations:

\[
x \left( p_n + \sum_{j=1}^{q_2} \ell_j(n)p_{n-j} \right) - \sum_{j=1}^{d_2} \alpha_j(n)p_{n-j} \quad (2-16)
\]

\[
y \left( s_n + \sum_{j=1}^{q_1} m_j(n)s_{n-j} \right) - \sum_{j=1}^{d_1} \beta_j(n)s_{n-j} \quad (2-17)
\]

where \( \ell_j(n) = 0 \) for \( n \leq d_2 \) and \( m_j(n) = 0 \) for \( n \leq d_1 \), under the genericity assumption (to be further discussed in Remark 2.1).

\[
\Delta_{n,2} := \det \begin{bmatrix}
\mu_{10} & \ldots & \mu_{1,q_2-1} & \mu_{00} & \ldots & \mu_{0,n-q_2-1} \\
\mu_{20} & \ldots & \mu_{2,q_2-1} & \mu_{10} & \ldots & \mu_{1,n-q_2-1} \\
\vdots & & \vdots & \vdots & \ddots & \vdots \\
\mu_{n,0} & \ldots & \mu_{n,q_2-1} & \mu_{n-1,0} & \ldots & \mu_{n-1,n-q_2-1}
\end{bmatrix} \neq 0, \quad \forall n > q_2 \in \mathbb{N}. \quad (2-18)
\]

The coefficients \( \alpha_{-1}(n) \) and \( \beta_{-1}(n) \) are nonzero for any \( n \); furthermore, under the same genericity assumptions letting \( a_i, b_i \) be the leading coefficients of \( A_i, B_i \) we have

\[
\begin{align*}
b_2 a_2 d_2 (n) & \sqrt{h_{n-d_2}} - a_2 \ell_{q_2} (n) \sqrt{h_{n-q_2}} \neq 0, \quad n \geq d_2 \\
b_1 b_1 d_1 (n) & \sqrt{h_{n-d_1}} = a_1 m_{q_1} (n) \sqrt{h_{n-q_1}} \neq 0, \quad n \geq d_1
\end{align*} \quad (2-19)
\]

Proof. We prove only one relation, the other being proved by interchanging the rôles.

The statement \( \alpha_{-1}(n) \neq 0 \) follows from the form of the recurrence relation by comparison of the leading coefficients, which gives

\[
\alpha_{-1}(n) = \sqrt{\frac{h_{n+1}}{h_n}} \neq 0. \quad (2-20)
\]

The fact that \( \ell_j(n) = 0 \) for \( n \leq d_2 \) is a choice of convenience: indeed, since \( d_2 > q_2 \) any \( xp_n \) can be written as a linear combination of the same BOPs of degrees \( m = 0, \ldots, n+1 \) for \( n \leq d_2 \).

Consider \( xp_n(x) \); by ”integration by parts” (i.e. using relation 1-8 from right to left), we immediately conclude that

\[
xp_n(x) \perp B_2(y) \mathbb{C}\{1, y, \ldots, y^{n-d_2-1}\} =: V_n^{(2)} \quad (2-21)
\]

Therefore \( V_n^{(2)} \) is in the common annihilator of \( xp_n(x), \ldots, xp_{n-q_2}(x) \). We now show that it is generically possible to fix the coefficients \( \ell_j(n) \) of a linear combination as the left hand side of eq. (2-16) such that the result is perpendicular to any polynomial \( q(y) \) of degree \( \deg(q) < n - d_2 \). Let

\[
q(y) = B_2(y)a(y) + b(y) \quad (2-22)
\]

be the long division of \( q \) by \( B_2 \) with remainder \( b \); then

\[
\mathcal{L} \left( xp_n(x) \right| q(y) \right) = \mathcal{L} \left( xp_n(x) \right| B_2(y)a(y) + b(y) \right) = \mathcal{L} \left( xp_n(x) \right| b(y) \right) \quad (2-23)
\]
Let us now check that this genericity assumption is actually equivalent to requiring solving the system

$$0 = \mathcal{L} \left( x \left( p_n + \sum_{j=1}^{q_2} \ell_j(n)p_{n-j} \right) | y^k \right), \quad k = 0, \ldots, q_2 - 1 .$$

After doing so we have that a suitable linear combination in $x^C \{p_n, \ldots, p_{n-q_1}\}$ is perpendicular to any $q = B_2a + b$ with $\deg(a) < n - d_2 - q_2$, $\deg(b) \leq q_2 - 1$, or - in other words - to any $q(y)$ of degree less than $n - d_2$, thus proving the shape of the recurrence relation.

In order to clarify the genericity assumption we are imposing we express the above condition as a non-vanishing condition of certain submatrices of the matrix of moments. Indeed the polynomials $\bar{p}_n := p_n + \sum_{j=1}^{q_2} \ell_j(n)p_{n-j}$ are uniquely determined by the condition that (for $n > q_2$)

1. The degree of $\bar{p}_n$ is $n$;
2. The polynomial $\bar{p}_n$ is $\mathcal{L}$-orthogonal to $1, y, \ldots, y^{n-q_2-1}$
3. The polynomial $x\bar{p}_n$ is $\mathcal{L}$-orthogonal to $1, y, \ldots, y^{q_2-1}$.

This determines them (for $n > q_2$) as the following determinants (up to a nonzero multiplicative constant)

$$\bar{p}_n := c_n \det \begin{bmatrix}
\mu_{10} & \cdots & \mu_{1,q_2-1} & 1 \\
\mu_{20} & \cdots & \mu_{2,q_2-1} & x \\
\vdots & & \vdots & \vdots \\
\mu_{n,0} & \cdots & \mu_{n,q_2-1} & x^n \\
\mu_{n+1,0} & \cdots & \mu_{n+1,q_2-1} \\
\end{bmatrix}$$

The genericity condition is then the nonvanishing of the principal minor of size $n$ of the above expression, namely the nonvanishing of the determinants advocated in the statement of the theorem (eq. 2-18).

The normalization that $\bar{p}_n = p_n + \text{(lower degree)}$ gives for the $c_n$ of eq. (2-25)

$$c_n = \frac{1}{\Delta_{n,2} \sqrt{h_n}}$$

Let us now check that this genericity assumption is actually equivalent to requiring $a_{d_2}(n) \neq 0, \forall n$. Denoting by $a_2, b_2$ the leading coefficients of $A_2(y), B_2(y)$ we find

$$b_2a_{d_2}(n) \sqrt{h_n - d_2} = \mathcal{L}(x \bar{p}_n | B_2 y^{n-d_2-q_2}) = \mathcal{L}(\bar{p}_n | A_2 y^{n-d_2-q_2} - O(y^{n-d_2-1})) = \mathcal{L}(\bar{p}_n | a_2 y^{n-q_2}) = \ell_{q_2}(n) a_2 \sqrt{h_n - q_2}$$

(2-27)
This proves the identity (2.19): to prove that it does not vanish under our genericity conditions we compute

\[
\mathcal{L}(p_n|a_2y^{n-q_2}) = \frac{a_2}{\Delta_{n,2}\sqrt{h_n}} \det \begin{pmatrix}
\mu_{10} & \cdots & \mu_{1,q_2-1} & \mu_{0,0} & \cdots & \mu_{0,n-q_2} \\
\mu_{20} & \cdots & \mu_{2,q_2-1} & \mu_{10} & \cdots & \mu_{1,n-q_2} \\
\vdots & & \vdots & \vdots & & \vdots \\
\mu_{n+1,0} & \cdots & \mu_{n+1,q_2-1} & \mu_{n,0} & \cdots & \mu_{n,n-q_2}
\end{pmatrix} = \frac{a_2\Delta_{n+1,2}}{\Delta_{n,2}\sqrt{h_n}} \neq 0
\]

Q.E.D.

We can represent the previous recurrence relations in matrix form as follows

**Proposition 2.1** The wave vectors satisfy the following recurrence relations

\[
x(1 + L)\Psi = A\Psi , \quad y(1 + M)\Phi = B\Phi
\]

where \(L\) is the lower triangular matrix with \(q_2\) subdiagonals whose matrix entries are \(L_{nm} = \ell_n(n - m)\) and \(A\) is a lower Hessenberg matrix with entries \(A_{nm} = \alpha_{n}(m - n)\) (similarly for \(M, B\)). The entries in the lowest and highest diagonals in \(1 + L, A\) are non-vanishing.

### 2.3 Differential recurrence relations

**Proposition 2.2** Under the genericity assumption\(^5\) that the principal minors of the associated moment functionals \(\hat{L}, \hat{\Psi}\) are all non-vanishing (or –which is the same– the existence of biorthogonal polynomials for \(\hat{L}, \hat{\Psi}\)), the BOPs satisfy the following differential finite–term recurrence relations

\[
\nabla_x \left( p_n + \sum_{j=1}^{q_1} \hat{m}_j(n + j)p_{n+j} \right) = -\sum_{j=1}^{d_1} \hat{\beta}_j(n + j)p_{n+j} \\
\nabla_y \left( s_n + \sum_{j=1}^{q_2} \hat{r}_j(n + j)s_{n+j} \right) = -\sum_{j=1}^{d_2} \hat{\alpha}_j(n + j)s_{n+j} \\
\n\nabla_x := \partial_x - V_1'(x) , \quad \nabla_y := \partial_y - V_2'(y) .
\]

In matrix form we have

\[
\nabla_x(1 + \hat{M}^t)p = -\hat{B}^t p \\
\nabla_y(1 + \hat{L}^t)s = -\hat{\Lambda}^t s ,
\]

where the matrices above are defined by

\[
\hat{M}_{nk} = \hat{m}_{n-k}(n) , \quad \hat{B}_{nk} = \hat{\beta}_{n-k}(n) \\
\hat{L}_{nk} = \hat{\ell}_{n-k}(n) , \quad \hat{\Lambda}_{nk} = \hat{\alpha}_{n-k}(n) .
\]

Note that they have the same shape as \(M, B, L, A\) respectively (whence the mnemonics of the symbols). \(^5\)See Remark 2.1.
This implies that the degree of the matrices requiring that the vectors (the superscript \(Q\)) of \(B_0\) for Proposition 2.3 are linearly independent: indeed this is rational.

Moreover

\[
\mathcal{L}\left(-\partial_x + V'_1 - \partial_t\right)p_n = \mathcal{L}\left(-B_1q'_n + A_1q_n\right)y^k = \mathcal{L}\left(B_1q_n\right)y^{k+1} = 0 \quad \text{for} \quad k < n - 1
\]  

Note that the above relation (2-36) is implicitly an assumption on the existence of polynomials \(q_n\) of exact degree \(n\) which are \(\mathcal{L}\)-orthogonal to all lower powers of \(y\): this is equivalent to saying that there must exist the BOPs for \(\mathcal{L}\), whence our genericity assumption in the statement of the theorem. \textbf{Q.E.D.}

For later convenience we also remark that the genericity condition we are invoking now is also equivalent to requiring that the vectors (the superscript \(r\)) denoting the \(r\)-th derivative)

\[
p_n^{(r)}(x), \ldots, p_{n+q_1-1}^{(r)}(x)
\]

be linearly independent: indeed

\[
p_n + \sum_{j=1}^{q_1} m_j(n + j)p_{n+j} = c_n \begin{bmatrix} p_n(x_1) & p_{n+q_1}(x_1) \\ \vdots & \vdots \\ p_n^{(r_j)}(x_1) & p_{n+q_1}^{(r_j)}(x_1) \\ \vdots & \vdots \\ p_n^{(r_s)}(x_s) & p_{n+q_1}^{(r_s)}(x_s) \\ p_n(x) & p_{n+1}(x) & \ldots & p_{n+q_1}(x) \end{bmatrix}
\]  

where \(c_n\) is the inverse of the \((q_1 + 1, 1)\)-cofactor of the above matrix. The proposition can be rewritten for the wave vectors as follows

**Proposition 2.3** The wave vectors satisfy the following differential equations

\[
\partial_x(1 + \tilde{M}^t)\psi = -\tilde{B}^t\psi, \quad \partial_y(1 + \tilde{\Lambda}^t)\Phi = -\tilde{A}^t\Phi,
\]  

where \(\tilde{M}_{nk} = \tilde{m}_{k-n}(n)\) and \(\tilde{A}_{nk} = \tilde{\alpha}_{k-n}(n)\) (and similar expressions for \(\tilde{M}, \tilde{B}\)).

The matrices \(\tilde{M}, \tilde{B}, \tilde{L}, \tilde{A}\) play the same role of \(M, B\) and \(L, A\) for the moment functionals \(\mathcal{L}\) and \(\mathcal{L}\) respectively.
**Proposition 2.4** The vectors of polynomials

\[ \tilde{p}(x) := (1 + \hat{\Lambda})^{-1}p, \quad \tilde{s}(y) := \frac{1}{B_2(y)}(1 + \hat{\Lambda})s(y) \]  

(2-41)

(where \( \hat{\Lambda} \) and \( \hat{\Lambda} \) are defined by eqs. (2-33) of Prop. 2.2) are the biorthogonal polynomials for \( \hat{\Lambda} \). Similarly, the vectors of polynomials

\[ \tilde{p}(x) := \frac{1}{B_1(x)}(1 + \hat{\Lambda}^t)p, \quad \tilde{s}(y) := (1 + \hat{\Lambda})^{-1}s \]  

are the biorthogonal polynomials for \( \hat{\Lambda} \).

**Proof.** The two statements are completely parallel and hence we prove only the first.

By definition of the matrix \( \hat{\Lambda} \) in Prop. 2.2 the polynomial entries of \((1 + \hat{\Lambda})^t\) are all divisible by \( B_2 \), therefore \( \tilde{s} \) is indeed a vector of polynomials. Next we have (using an obvious matrix notation)

\[ \hat{\Lambda} \left( \tilde{p}\big| s^t \right) = \hat{\Lambda} \left( (1 + \hat{\Lambda})^{-1}p\big| s^t (1 + \hat{\Lambda}) \right) = (1 + \hat{\Lambda})^{-1}\hat{\Lambda} \left( p\big| s^t \right) (1 + \hat{\Lambda}) = 1 \quad \text{Q.E.D.} \]  

(2-43)

We also have

**Lemma 2.1** The matrices \( L, A, M, B \) and the matrices \( \hat{\Lambda}, \hat{A}, \hat{M}, \hat{B} \) are related by

\[ A(1 + \hat{\Lambda}) = (1 + L)\hat{A}, \quad B(1 + \hat{\Lambda}) = (1 + M)\hat{B}. \]  

(2-44)

**Proof.** Once more we prove only the first.

\[ A(1 + \hat{\Lambda}) = \hat{\Lambda} \left( A\big| s^t (1 + \hat{\Lambda}) \right) = \hat{\Lambda} \left( x(1 + L)p\big| s^t (1 + \hat{\Lambda}) \right) = \]  

\[ = \hat{\Lambda} \left( x(1 + L)p\big| -B_2 e_y + A_2 s^t \right) = \hat{\Lambda} \left( (1 + L)p\big| -\nabla_y B_2 s^t \right) = \]  

\[ = \hat{\Lambda} \left( (1 + L)p\big| -\nabla_y s^t (1 + \hat{\Lambda}) \right) = \hat{\Lambda} \left( (1 + L)p\big| s^t A \right) = (1 + L)\hat{A} \quad \text{Q.E.D.} \]  

(2-45)

**Lemma 2.2** The associated wave vectors \( \tilde{p}, \tilde{s} \) and \( \tilde{p}, \tilde{s} \) satisfy

\[ x(1 + \hat{\Lambda})\tilde{p} = \hat{A}\tilde{p} \]
\[ y(1 + \hat{\Lambda})\tilde{s} = \hat{B}\tilde{s} \]  

(2-46)

Moreover, under the same genericity assumptions

\[ \hat{n}_{q_1}(n) \neq 0 \neq \hat{n}_{q_2}(n) \quad \forall n \]  

(2-47)

\[ b_2 \hat{\alpha}_{q_2}(n) \sqrt{\hat{n}_{n-q_2}} = a_2 \hat{\alpha}_{q_2}(n) \sqrt{\hat{n}_{n-q_2}} \neq 0 \]  

(2-48)

\[^6\text{The expressions } (1 + \hat{\Lambda})^{-1} \text{ etc. are defined by the geometric series; since } \hat{\Lambda} \text{ is strictly lower triangular, such geometric series is entry-wise well defined.}\]
| Functional | BOPs | Mult. rec. | Diff. Rec. |
|------------|------|------------|------------|
| $\mathcal{L}(\bullet|\bullet)$ | \(p, s\) | \(x(1 + L)p = Ap\) | \(\nabla_x(1 + \tilde{M}^t)p = -\tilde{B}^t p\) |
| $\tilde{\mathcal{L}}(\bullet|\bullet) = \mathcal{L}(\bullet|B_2|\bullet)$ | $\tilde{p} = (1 + L)^{-1}p$ | $x(1 + \tilde{L})\tilde{p} = \tilde{A}\tilde{p}$ | $\nabla_x(1 + \tilde{M}^t)\tilde{p} = -\tilde{B}^t \tilde{p}$ |
| $\hat{\mathcal{L}}(\bullet|\bullet) = \mathcal{L}(B_1 \bullet|\bullet)$ | $\tilde{s} = (1 + \tilde{M})^{-1} \tilde{s}$ | $y(1 + \tilde{M})\tilde{s} = \tilde{B}\tilde{s}$ | $\nabla_y(1 + \tilde{L}^t)\tilde{s} = -\tilde{A}^t\tilde{s}$ |
| $\hat{\mathcal{L}}(\bullet|\bullet) = \mathcal{L}(B_1 \bullet|B_2|\bullet)$ | $\tilde{p} = \frac{(1 + \tilde{M})^t}{\pi_1} \tilde{p}$ | $x(1 + \tilde{L})\tilde{p} = \tilde{A}\tilde{p}$ | $\nabla_x(1 + \tilde{M})\tilde{s} = \tilde{B}\tilde{s}$ |
| \((1 + L)\tilde{A} = A (1 + \tilde{L})\) | \((1 + \tilde{L})\tilde{A} = \tilde{A} (1 + \tilde{L})\) | \((1 + \tilde{M})\tilde{B} = B (1 + \tilde{M})\) | \((1 + \tilde{L})\tilde{B} = \tilde{B} (1 + \tilde{M})\) |

Table 1: the various recurrence relations.

**Proof.** Recalling that $\tilde{p} = (1 + L)^{-1}p$ (by definition), we find

$$x(1 + \tilde{L})\tilde{p} = x p = (1 + L)^{-1}Ap = \tilde{A}(1 + \tilde{L})^{-1}p = \tilde{A}\tilde{p} .$$

(2-49)

The relations (2-48) for the moment functionals $\hat{\mathcal{L}}, \hat{\mathcal{L}}$ are proved in exactly the same way relations (2-19) are proved for $\mathcal{L}$. Q.E.D.

We can summarize all the relations collected so far in Table 1. Here the matrices $A, \tilde{A}, \tilde{A}$ are lower–Hessenberg matrices with $d_2$ nontrivial sub–diagonals, $B, \tilde{B}, \tilde{B}$ are lower–Hessenberg with $d_1$ nontrivial sub–diagonals. The matrices $L, \tilde{L}, \tilde{L}$ and $M, \tilde{M}, \tilde{M}$ are strictly lower triangular matrices with $q_2$ or $q_1$ nontrivial subdiagonals respectively.

The $*$’s mean that there are (possibly under similar genericity requirements for the corresponding functional) similar relations as in the corresponding box on the first line, for which however we do not need to define symbols for our purposes.

**Remark 2.1** We now address the genericity assumptions invoked in Thm. 2.1 and Prop. 2.2 in the case of real potentials and support on the real axes as discussed in the Introduction. For Prop. 2.2 the assumption is simply the existence of the BOPs for the associated moment–functional $\hat{\mathcal{L}}$; in the case of real potentials with supports on the real line one can follow [12] and show that BOPs do exist. Since in $\mathcal{L}$ and $\hat{\mathcal{L}}$ have the same supports and $B_2(y)$ would be positive on the support (provided that none of the higher-multiplicity zeroes lie within the support) then one can conclude that for the case of relevance to the Hermitean two–matrix model the Prop. 2.2) is always valid.

Less transparent is the extent of the limitation imposed by the genericity assumption (2-18) used in Thm. 2.1; the polynomials $\tilde{p}_n$, appearing in the proof of said theorem play the same role in respect to $p_n$’s as
the \( p_n \)'s play regarding the \( \tilde{p}_n \)'s (see Table 1); this means that they could be obtained from a functional \( \mathcal{L}'(\bullet|\bullet) = \mathcal{L}(\bullet|B_{2}^{-1}\bullet) \).

Note, however, that since the matrices \( L \) and \( A \) are not uniquely defined in the \( d_2 \times d_2 \) principal minor (in the Theorem we fixed the ambiguity by setting the corresponding block of \( L \) to zero), the above possibility is not the sole choice.

Moreover, if some of the hard-edges zeroes of \( B_2(y) \) belong to the real axis (i.e. if there are hard-edges on \( \mathbb{R} \)) then such choice is not viable because the integral defining \( \mathcal{L}' \) would be divergent at the hard-edges. In this case this simply means that \( \tilde{p}_n \)'s do not belong to a biorthogonal pair for some semiclassical functional but are just defined (for \( n > q_2 \)) by eq. (2-25) and the genericity issue cannot be resolved easily.

Vice-versa, in the case none of the hard-edge zeroes of \( B_2 \) belong to \( \mathbb{R} \) then \( \mathcal{L}' \) indeed exists and is a semiclassical functional (with \( V_2 \) replaced by \( V_2 + \ln B_2 \)). The existence of the corresponding BOPs (hence the verification of the genericity assumption) then follows again from the result in [12].

3 Adjoint differential equations and the bilinear concomitant

In this section we recall some results which –although simple– I was not able to find in the literature. We consider a \( n \)th order differential equations of the form

\[
\left( A(\partial_x) - x B(\partial_x) \right) f(x) = 0
\]

where \( A(D) \) and \( B(D) \) are polynomials and \( n = \max(\deg(A), \deg(B)) \): the reader should keep in mind the polynomials \( A_i, B_i \) of our matrix model. If we look for solutions written as “Fourier–Laplace” transforms

\[
f_{\Gamma}(x) := \int_{\Gamma} dy e^{\gamma y - V(y)},
\]

–where the contour of integration is so far unspecified–, formal manipulations involving integration by parts show that

\[
V'(y) = \frac{A(y) + B'(y)}{B(y)}.
\]

We point out that the relation between \( V \) and \( A, B \) in these formulæ is exactly the same as the relations between the \( V_i \)'s and \( A_i, B_i \)'s of the first part of the paper.

In the situation of interest to us we will have \( A, B \) reduced

Definition 3.1 Two polynomials \( A, B \) are called reduced if the only zeroes that they share (if any) are amongst the simple zeroes of \( B \).

Lemma 3.1 Two polynomials \( A, B \) are reduced if and only if \( A \pm B' \) and \( B \) are.

Proof Suppose \( A, B \) are reduced. If \( c \) is a common zero of \( A \) and \( B \) (hence simple for \( B \)) then \( B'(c) \neq 0 \); therefore, \( A(c) \pm B'(c) \neq 0 \) (because \( A(c) = 0 \) and \( B'(c) \neq 0 \)). So \( A \pm B' \) and \( B \) do not share this particular zero.
Now let \( c \) be a common zero of \( A \pm B' \) and \( B \): if it were not a simple zero of \( B \) then \( B'(c) = 0 \) and hence also \( A(c) = 0 \). But this contradicts that \( A \) and \( B \) are reduced because they share a zero which is non simple for \( B \).

Vice versa: suppose \( \hat{A} := A \pm B' \) and \( B \) are reduced. Then by the above \( \hat{A} \mp B' = A \) and \( B \) are reduced. Q.E.D.

This "duality" of the notion of reducedness will be important when considering the adjoint differential operator.

We now remark that \( V' \) is a rational function with poles at a subset of the zeroes of \( B \)

\[
B(y) = c \prod_{j=1}^{r}(y - b_j)^{m_j+1}, \; c \neq 0, \; \deg B = \sum_{j=1}^{r} m_j, \; m_j \in \mathbb{N}.
\]

\[
V'(y) = \sum_{\ell=0}^{d} v_{\ell+1} y^\ell - \sum_{j \in J, k} \sum_{k=0}^{m_j} \frac{t_{k,j}}{(y - b_j)^{k+1}}
\]

\[
e^{-V}(y) = \prod_{j \in J}(y - b_j)^{a_{0,j}} \exp \left[ \sum_{\ell=0}^{d} \frac{v_{\ell+1}}{\ell+1} y^\ell + \sum_{j \in J} \sum_{k=1}^{m_j} \frac{k t_{k,j}}{k(y - b_j)^{k}} \right]
\]

\[
W(y) := e^{-V}(y)
\]

\[
d := \deg(A) - \deg(B).
\]

[Here it is understood that if \( \deg(A) < \deg(B) \) then the first sum in \( V' \) is absent.]

Some of the zeroes of \( B(y) \) may appear also as zeroes of \( A(y) + B'(y) \) and hence in the partial fraction expansion of \( V' \) those points do not appear. Since \( A, B \) are reduced, all multiple zeroes of \( B \) are not shared with \( A + B' \). We will call the zeroes of \( B \) which are common with \( A + B' \) the hard-edge points (note that not all simple zeroes of \( B \) are hard-edge points, but all hard-edge points are simple zeroes).

We now define some sectors \( S^{(j)}_{k} \), \( j = 1, \ldots p_1 \), \( k = 0, \ldots m_j - 1 \). around the multiple zeroes of \( B \) \( (b_j \) for which \( m_j > 0 \) \) in such a way that

\[
\Re(V(y)) \xrightarrow{y \to b_j, \; y \in S^{(j)}_{k}} +\infty.
\]

The number of sectors for each pole is the degree of that pole in the exponential part of \( W(x) \), that is \( d + 1 \) for the pole at infinity and \( g_j \) for the \( j \)-th pole. Explicitly

\[
S^{(0)}_{k} := \left\{ y \in \mathbb{C}; \frac{2k\pi - \frac{\pi}{2} + \epsilon}{d+1} < \arg(y) + \frac{\arg(v_{d+1})}{d+1} < \frac{2k\pi + \frac{\pi}{2} - \epsilon}{d+1} \right\},
\]

\[
S^{(j)}_{k} := \left\{ y \in \mathbb{C}; \frac{2k\pi - \frac{\pi}{2} + \epsilon}{m_j} < \arg(y - b_j) + \frac{\arg(t_{m_j,j})}{m_j} < \frac{2k\pi + \frac{\pi}{2} - \epsilon}{m_j} \right\},
\]

These sectors are defined precisely in such a way that approaching any of the essential singularities (i.e. an \( b_j \) such that \( m_j > 0 \)) the function \( W(y) \) tends to zero faster than any power.
3.1 Definition of the contours

The contours we are going to define are precisely the type of contours $\Gamma_{x,\mu}, \Gamma_{y,\nu}$ entering the definition of the bimoment functional $L$. Let $A, B$ be reduced: we then define $n = \max\{\deg(A), \deg(B)\}$ contours. The definition of the contours follows directly [3, 20]. We first remark that the weight $W(y)$ is –in general– multi-valued since it contains powers like $(y - c)^t$ with non-integer $t$; the multivaluedness is multiplicative and in fact is not very important which branch one chooses in the definition of the integrals (3-2) since different choices correspond to multiplying the same function by a nonzero constant. Nonetheless it will be convenient at some point to have a reference normalization for the integrals and hence we define some cuts so as to have a simply connected domain where $W(y)$ is single-valued. We do so by removing semi-infinite arcs extending from each branch-point of $W(y)$ to infinity: for convenience we choose the cuts approaching each singularity in one of the sectors, for example $S_{0}^{(j)}$, and approaching infinity within $S_{0}^{(0)}$. If $\deg(A) \leq \deg(B) - 1$ then no sector is defined at $\infty$ and then we just choose arbitrarily an asymptotic direction for these cuts. Note that if $\deg(A) \leq \deg(B) - 2$ then the sum of the finite residues of $V' dy$ is zero, hence we could define the cuts as finite arcs joining in a chain the finite branch-points of $W(y)$: the resulting domain is not simply connected, however $W(y)$ is single valued in such domain precisely because of the vanishing of the sum of the residues of its logarithmic derivative. We will denote by $D$ the connected domain obtained after such surgery.

In the following our primary focus is on the case $\deg(A) \geq \deg(B) + 1$ and we leave to the reader to check the literature [20] for the remaining cases (only minor modifications are needed).

1. For any zero $b_j$ of $B$ for which there is no essential singularity in $W$ we have two cases

   (a) If $b_j$ is a branch point (i.e. $t_{0,j} \in \mathbb{C}\setminus\mathbb{Z}$) we take a loop (referred to as a lasso) starting at infinity in some fixed sector (e.g. $S_{0}^{(0)}$) encircling the singularity and going back to infinity in the same sector.

   (b) If $b_j$ is a pole of $W$ (i.e. $t_{0,j} \in \{-1, -2, -3, \ldots\}$) then we take a small circle around it.

   (c) If $b_j$ is a regular point (namely $t_{0,j} \in \{0, 1, 2, \ldots\}$) we take a line joining $b_j$ to infinity and approaching $\infty$ in the same sector $S_{0}^{(0)}$ as before (this case includes the hard-edge points for which we may say that $t_{0,j} = 0$).

2. For any multiple zero $b_j$ for which there is an essential singularity (i.e. for which $m_j > 0$) we define $m_j$ contours (which we call the petals) starting from $b_j$ in the sector $S_{0}^{(j)}$ and returning to $b_j$ in the next (counterclockwise) sector. Finally we join the singularity $b_j$ to $\infty$ by a path (called the stem) approaching $\infty$ within the sector $S_{0}^{(0)}$ chosen at point 1(a).

3. If $\deg(A) \geq \deg(B) + 1$ we define $b_0 := \infty$ and we take $d := \deg(A) - \deg(B)$ contours starting at $X_0$ in the sector $S_{k}^{(0)}$ and returning at $X_0$ in the sector $S_{k+1}^{(0)}$.

The reasons for the “floral” names should be clear by looking at an example like the one in Fig. 1. Cauchy’s theorem grants us large freedom in the choices of such contours; we use this freedom so that the contours do not intersect each other in $\mathbb{C}\setminus\{b_j\}_{j=1,\ldots,\deg(B)}$ and do not cross the chosen cuts.
Figure 1: An example of contours $\Gamma$ and $\hat{\Gamma}$ for a pair of reduced adjoint differential operators. The thick contours are the admissible ones for $L$ while the thick dashed ones are the admissible ones for $L^\ast$. Also shown in the picture are the cuts for $W(y)$ and $\hat{W}(s)$ (line-dotted thin lines).
We will refer to these contours collectively as admissible contours for the differential $W(y)dy$. Note that we have defined exactly $n = \max(\deg(A), \deg(B))$ contours.

It is a straightforward check to see that

$$f_\Gamma(x) := \int_\Gamma dy e^{xy - V(y)} - \int_\Gamma e^{xy} W(y)dy ,$$

(3.11)

all satisfy the differential equation (3.1): in these checks one is always allowed to perform integration by parts discarding all boundary terms because of the properties of the contours. We leave this check to the reader.

The content of [20] (and of the fix contained in [3]) was to show that these functions are also linearly independent, hence providing a basis for the solution space.

### 3.2 Adjoint differential operators and the bilinear concomitant

In general, given a $n$-th order linear operator with polynomial coefficients

$$L := \sum_{j}^n a_j(x) \partial_x^j ,$$

(3.12)

its classical adjoint is defined as

$$L^* := \sum_{j}^n (-\partial_x)^j a_j(x) .$$

(3.13)

Between the solution spaces of a pair of adjoint such operators Legendre defined a nondegenerate pairing called the bilinear concomitant [16]. We will show that this pairing for our class of reduced operators admits a natural interpretation as intersection pairing.

We begin by noticing that in our case the pair of adjoint operators is written

$$L := A(\partial_x) - xB(\partial_x) , \quad L^* := A(-\partial_x) - B(-\partial_x)x .$$

(3.14)

Since $A, B$ are reduced then $L^*$ is also reduced since

$$L^* = A(-\partial_x) - B'(-\partial_x) - xB(-\partial_x)$$

(3.15)

in view of Lemma 3.1 (here the polynomials are $A(-y) - B'(-y)$ and $B(-y)$ which are clearly reduced iff $A(z) - B'(z)$ and $B(z)$ are). Therefore $L^*$ is in the same class of operators as $L$ and can be solved by contour integrals in the same way. The solutions of $L^*g = 0$ are of the form

$$g = \int_\Gamma e^{-xs + \hat{V}(s)} ds ,$$

(3.16)

$$\hat{V}(s)' := \frac{A(s)}{B(s)} = V'(s) - (\ln B(s))' .$$

(3.17)

An inspection shows that the sectors around the multiple zeroes of $B(s)$ where $\Re(\hat{V}(s)) \to -\infty$ are precisely the complementary sectors defined in (3.10) for $V$. We normalize $\hat{V}(s)$ by choosing the integration constant in such a way that

$$\hat{W}(s) := e^{\hat{V}(s)} = \frac{1}{B(s)} e^{\hat{V}(s)}$$

(3.18)
(here $e^V$ is supposed to be defined on the simply connected domain $\tilde{D}$). One then proceeds in the definition of the admissible contours $\tilde{\Gamma}$ for the weight $\tilde{W}(s)$ and of the simply connected domain $\tilde{D}$ in exactly the same way used for $W(y)$. We make the following important remarks:

1. If $b_j$ is a hard-edge point for $W(y)$ (i.e. it is a zero of $B(y)$ but a regular point for $W(y)$ where $W$ does not vanish) then $b_j$ is a simple pole of $\tilde{W}(s)$.

2. If $b_j$ is a zero of multiplicity $m$ of $W(y)$ (i.e. a simple zero of $B(y)$ such that the residue of $(A + B')/Bdy$ is a negative integer) then it is a pole of order $m + 1$ for $\tilde{W}(s)$.

3. In all other cases, the type of singularity of $W$ and $\tilde{W}$ is the same (logarithmic branch-points or essential singularities of the same exponential type).

4. The intersection $\mathcal{D} \cap \tilde{\mathcal{D}}$ is the disjoint union of simply connected domains where $W(y)\tilde{W}(y)B(y)$ is constant. These constants depend only on the residues of $V'(y)dy \mod \mathbb{Z}$.

These observations and the fact that $B(y)W(y)\tilde{W}(y)$ is locally constant (where they are both defined) follows immediately from their definition and eq. (3-18).

From the definitions of the contours it is not difficult to realize that dual contours can be chosen such that

1. For each flower (petal + stem) one can choose a dual flower whose elements intersect only the arcs of the given flower. (This includes the petals at $\infty$, in the case $\deg(A) \geq \deg(B) + 1$).

2. For each pole $c$ of $W(y)$ (whose corresponding admissible contours $\Gamma$ is a small circle) the dual admissible contour for $\tilde{W}(s)$ is a semi-infinite arc starting at $c$ and going to $\infty$ and can be chosen so that it intersects only its dual.

3. For each zero or hard-edge point $a$ of $W(y)$ (whose corresponding admissible contour is a semi-infinite arc starting at $a$) the dual admissible contours for $\tilde{W}(s)$ (which is a small circle around $a$) intersects only $\Gamma$.

4. For each non-essential other singularity of $W(y)$ (i.e. a simple zero $c$ of $B(y)$ such that the residue of $(A + B')/Bdy$ is in $\mathbb{C}\setminus\mathbb{Z}$, where the admissible contour $\Gamma$ is a lasso around $c$, the dual loop $\tilde{\Gamma}$ (also a lasso around $c$) is also chosen so that it intersects only the dual lasso (at two points).

**Lemma 3.2** Consider the two adjoint differential equations

$$
\left( A(\partial_x) - xB(\partial_x) \right) f(x) = 0 \quad (3-19)
$$

$$
\left( A(-\partial_x) - B(-\partial_x)x \right) g(x) = 0 . \quad (3-20)
$$

19
The solutions are of the form\footnote{The formula depends on the integration constant in $V$, namely these solutions are defined up to multiplicative constants since they are solutions of a homogeneous linear ODE.}

\[ f(x) = f_\Gamma(x) := \int_\Gamma e^{-V(y)+xy}dy, \quad V := \int \frac{A(y) + B'(y)}{B(y)}dy \tag{3-21} \]

\[ g(x) = g_\Gamma(x) := \int_\Gamma e^{\tilde{V}(s)-xs}ds, \quad \tilde{V}(s) := \int \frac{A(s)}{B(s)}ds \tag{3-22} \]

Then the following expression is constant and defines a nondegenerate bilinear pairing \emph{(the bilinear concomitant)} between the solutions spaces of the two adjoint equations:

\[ \mathcal{B}(f, g) := \int_\Gamma \int_\Gamma \left[ (B(y) - B(s)) \left( \frac{x}{y-s} - \frac{1}{(y-s)^2} \right) - \frac{A(y) - A(s) - B'(s)}{y-s} \right] e^{x(y-s)-V(y)\tilde{V}(s)}dyds \tag{3-23} \]

\textbf{Proof.} The integral representation of the solution is easily verified. We now write

\[ 0 \equiv g(x) \int_\Gamma (xB(y) - A(y)) e^{-V(y)+xy}dy \tag{3-24} \]

\[ 0 \equiv f(x) \int_\Gamma (xB(s) - A(s) - B'(s)) e^{\tilde{V}(s)-xs}ds \tag{3-25} \]

We take the difference and obtain

\[ 0 \equiv \int_\Gamma \int_\Gamma (xB(y) - B(s)) - (A(y) - B'(s) - A(s))) e^{x(y-s)-V(y)\tilde{V}(s)}dyds \tag{3-26} \]

It is promptly seen that the integrand of this double integral is absolutely summable w.r.t. the arc-length parameters along $\Gamma$ and $\hat{\Gamma}$, hence we can integrate w.r.t. $x$ under the integral sign, thus obtaining the bilinear concomitant;

\[ \int_\Gamma \int_\Gamma \left( (B(y) - B(s)) \left( \frac{x}{y-s} - \frac{1}{(y-s)^2} \right) - \frac{A(y) - B'(s) - A(s)}{y-s} \right) e^{x(y-s)-V(y)\tilde{V}(s)}dyds \tag{3-27} \]

Note that the expression under integration is regular at $y = s$, and is –in fact– a polynomial in $y,s$

\[ (B(y) - B(s)) \left( \frac{x}{y-s} - \frac{1}{(y-s)^2} \right) - \frac{A(y) - B'(s) - A(s)}{y-s} \sim_{y\to s} xB'(s) - \frac{1}{2} B''(s) - A'(s) + \mathcal{O}(y-s) \]

In particular the integrand is absolutely integrable w.r.t. the arc-length parameters and hence the order of integrations is irrelevant. This concludes the proof. \textbf{Q.E.D.}

The bilinear concomitant is –in a certain sense– an integral representation of the intersection pairing of the contours of integration. To make this statement more precise we first prove the following standard

\textbf{Lemma 3.3} \textit{Let} $\Omega(y,s)$ \textit{be a meromorphic function} $\mathcal{D} \times \hat{\mathcal{D}}$ \textit{where} $\mathcal{D}$ \textit{and} $\hat{\mathcal{D}}$ \textit{are simply connected domains and with the only singularities being a double pole as} $y \to s$ \textit{(in} $\mathcal{D} \cap \hat{\mathcal{D}}$). \textit{Suppose that in each connected component of} $\mathcal{D} \cap \hat{\mathcal{D}}$ \textit{there is a constant} $c$ \textit{such that}

\[ \Omega(y,s) = \frac{c}{(y-s)^2} + \mathcal{O}(1) \tag{3-28} \]
as $y \to s$ within the intersection domain. Let $\Gamma \subset \mathcal{D}$ be a smooth curve such that

$$\int_{\Gamma} \Omega(y, s) dy \equiv 0 \quad (3.29)$$

Let $\hat{\Gamma} \subset \hat{\mathcal{D}}$ be a curve of finite length intersecting once $\Gamma$ at $p$ and oriented positively w.r.t. $\Gamma$: then

$$\int_{\Gamma} dy \int_{\Gamma} ds \Omega(y, s) - 2i\pi c(p) \quad (3.30)$$

**Proof.** The integral

$$f(s) := \int_{\Gamma} \Omega dy \quad (3.31)$$

defines –in principle– different holomorphic functions in the connected components of $\hat{\mathcal{D}} \backslash \Gamma$: the difference among them -however- is the residue

$$\text{res}_{y=s} \Omega(y, s) dy \quad (3.32)$$

which is zero by the assumption on $\Omega$. Hence the analytic continuations of $f(s)$ from one component to the other all coincide. In our case they are all zero. The key fact is that, since $\Omega$ is singular on the diagonal, the orders of integration matters (otherwise (3.30) would give zero by interchanging the order of integration).

We compute the integral as a limit of regular integrals where we can interchange the order of integration

$$(3.30) = \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} dy \int_{\Gamma} ds \Omega(y, s), \quad (3.33)$$

where $\Gamma_\epsilon$ is the curve (or union of curves) obtained by removing a small $\epsilon$-arc (which we denote by $\Gamma^\epsilon$, i.e. an arc from $p - \epsilon$ to $p + \epsilon$, where these two points lie on the curve $\Gamma$ at distance $|\epsilon|$ from the intersection and the direction of $\epsilon$ is the same as the orientation of $\Gamma$) around the intersection point $p$. This allows us to interchange the order of integration under the limit sign

$$\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} dy \int_{\Gamma} ds \Omega(y, s) = \lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} ds \int_{\Gamma} dy \Omega(y, s) = - \lim_{\epsilon \to 0} \int_{\Gamma} ds \int_{\Gamma^\epsilon} dy \Omega(y, s) = - \lim_{\epsilon \to 0} \int_{\Gamma} ds \int_{\Gamma^\epsilon} dy \left( \frac{c(p)}{(y-s)^2} + \mathcal{O}(1) \right) = - \lim_{\epsilon \to 0} \int_{\Gamma} ds \int_{\Gamma^\epsilon} dy \frac{c(p)}{(y-s)^2} \quad (3.34)$$

where we have dropped the $\mathcal{O}(1)$ part since the length of $\hat{\Gamma}$ is finite and that of $\Gamma^\epsilon$ tends to zero. In the last expression the inner integral is –strictly speaking– defined only for $s \neq p$: however on the "left" and "right" the result is the same and gives

$$- \lim_{\epsilon \to 0} \int_{\Gamma} ds \int_{\Gamma^\epsilon} dy \frac{c(p)}{(y-s)^2} = c(p) \lim_{\epsilon \to 0} \int_{\Gamma} ds \left( \frac{1}{b-p-\epsilon} - \frac{1}{b-p+\epsilon} \right) = - c(p) \lim_{\epsilon \to 0} \ln \left( \frac{b-p-\epsilon}{a-p-\epsilon} \right) - \ln \left( \frac{b-p+\epsilon}{a-p+\epsilon} \right) \quad (3.35)$$

In this last limit the logarithms appearing have different branches: in particular the second differ by $2i\pi$ from the first, hence the result follows by taking the limit. Q.E.D.
We now come back to the computation of the concomitant: first of all, since we know that the result is independent of \( x \) we set \( x = 0 \), so that we have to compute

\[
B(f, g) := \int_{\Gamma} \int_{\Gamma} \left[ -\frac{B(y) - B(s)}{(y - s)^2} - \frac{A(y) - A(s) - B'(s)}{y - s} \right] e^{-V(y) + \tilde{V}(s)} dy ds
\]  

(3-36)

We have already remarked that this integral can be computed in either orders and gives the same result. We express it in terms of

\[
B(f, g) - (2) - (1) = \int_{\Gamma} dy \int_{\Gamma} ds \left[ \frac{B(y)}{(y - s)^2} - \frac{A(y)}{y - s} \right] W(y) \tilde{W}(s) \]  

(3-37)

\[
(1) := \int_{\Gamma} dy \int_{\Gamma} ds \left[ \frac{B(y)}{(y - s)^2} - \frac{A(y)}{y - s} \right] W(y) \tilde{W}(s) \]  

(3-38)

\[
(2) := \int_{\Gamma} dy \int_{\Gamma} ds \left[ \frac{B(s)}{(y - s)^2} - \frac{A(s) + B'(s)}{y - s} \right] W(y) \tilde{W}(s) \]  

(3-39)

The integral (2) is zero because the inner integral w.r.t. \( s \) defines (for \( y \notin \hat{\Gamma} \)) the identically zero function, as it is easily seen after an integration by parts. The integral (1) is computed using Lemma 3.3 after noticing that

\[
\Omega(y, s) := \left[ \frac{B(y)}{(y - s)^2} - \frac{A(y)}{y - s} \right] W(y) \tilde{W}(s) = \frac{B(s)W(s)\tilde{W}(s)}{(y - s)^2} + O(1) \]  

(3-40)

and hence satisfies the condition of the Lemma for \( \Omega \). The contour \( \Gamma \) satisfies the condition of the Lemma. The contour \( \hat{\Gamma} \) is not necessarily of finite length, but we can take only a small arc around the point of intersection and the remainder will be computed to be zero by interchanging the order of the integrals. To rigor one should also consider the common endpoints of contours like the petals and dual petals: it is easily seen, however that those points do not correspond to a singularity of the integrals (w.r.t. the arclength parameters) because of the fast decay of the weights \( W \) and \( \tilde{W} \).

For example, if the two contours \( \Gamma, \hat{\Gamma} \) form an angle \( \theta \in \left[ \theta_0 + \epsilon, \pi - \epsilon \right] \) (asymptotically) near a point \( b \) (where \( W, \tilde{W} \) have an essential singularity) then

\[
\left| \frac{W(y)\tilde{W}(s)}{(y - s)^2} \right| \leq \left| \frac{W(y)\tilde{W}(s)}{\sin^2 \theta |y - b|^2} \right| \quad \text{[see fig. 2].} 
\]  

(3-41)

which is still jointly integrable w.r.t. the arc lengths (recall that the directions of approach of \( \Gamma \) and \( \hat{\Gamma} \) are such that the weights tend to zero faster than any power of the local coordinate).

It is then clear that if \( \Gamma, \hat{\Gamma} \) are a circle and a semi-infinite arc (or vice-versa) the bilinear concomitant for the corresponding dual solutions is a nonzero constant (which depends on the choices of the branches of \( W \) and \( \tilde{W} \)). This is immediate for a pair of contours which intersect only once. For a pair of lassos (which intersect twice and with opposite orientations), calling \( p_1, p_2 \) the points of intersection we have

\[
B(f_{\Gamma}, g_{\hat{\Gamma}}) = \pm (W(p_1)\tilde{W}(p_1)B(p_1) - W(p_2)\tilde{W}(p_2)B(p_2)) \]  

(3-42)

Since the local behavior at the singularity embraced by the lassos is a noninteger power, let’s say \((y - c)^\theta\), then the values of \( BW\tilde{W} \) on the two intersection points (which lie on different sizes of the union of the cuts
for $W$ and $\hat{W}$ satisfies

$$W(p_1)\hat{W}(p_1)B(p_1) - e^{2i\pi t}W(p_2)\hat{W}(p_2)B(p_2)$$

so that

$$B(f_\Gamma, g_\hat{\Gamma}) = \pm(W(p_1)\hat{W}(p_1)B(p_1)(1 - e^{2i\pi t}) \neq 0$$

For dual flowers it is convenient to choose different paths for the dual contours as shown in Fig. 3, where the petals have been replaced by stems using a linear combination of the contour-integrals of the same petals and stem. The sub-block of the concomitant involving these contours is nondegenerate, since it can be given a diagonal form with nonzero entries on the diagonal. The precise values are not important since we are free to re-scale each solution $f_\Gamma$ and $g_\hat{\Gamma}$. Summarizing we have proved that

**Proposition 3.1** There is a normalization of the integrals $f_\Gamma$ and $g_\hat{\Gamma}$ such that the bilinear concomitant is precisely the intersection pairing of the contours $\Gamma$ and $\hat{\Gamma}$. With appropriate choice and labeling of the contours the pairing is represented by the identity matrix.

**Remark 3.1** The content of Prop. 3.1 is that if the solutions $f_\Gamma$ and $g_\hat{\Gamma}$ correspond to contours that can be deformed (by Cauchy’s theorem and without changing the analyticity properties of the functions $f_\Gamma, g_\hat{\Gamma}$ respectively) in such a way that they do not intersect, then the bilinear concomitant of this pair is zero.

Vice versa, if this cannot be done, the bilinear concomitant is nonzero; we can always choose the contours and dual contours in such a way that each contour intersects one and only one dual contour. For example the equivalent choice of contours to Fig. 1 is given by the arrangement in Fig. 4.

Note that two dual lassoes intersect at two points but that —by virtue of (3-44) their “weighted” intersection number is nonzero (whereas the usually defined intersection number would be zero).

## 4 Auxiliary wave vectors

**Caveat** In this section we will make statements concerning the biorthogonal polynomials $p_n, s_n$ and the
Figure 3: The equivalent choice of contours for the dual admissible petals.

Figure 4: The arrangement of dual contours for the same example as in Fig. 1: in evidence only the different choice of admissible dual contours at the “flower”.
corresponding quasipolynomials $\psi_n, \phi_n$. It will be understood that

1. Any statement made on the $\psi_n$’s and the Fourier–Laplace transforms of the $\phi_n$’s admits a specular statement for the $\phi_n$’s and the F-L transforms of the $\psi_n$’s.

2. Any statement made on the $\psi_n$’s admits an analog statement for the $\widehat{\psi}_n$’s and $\check{\psi}_n$’s by replacing the moment functional $\mathcal{L}$ with $\check{\mathcal{L}}$ or $\tilde{\mathcal{L}}$, and specular statements for $\phi_n, \check{\phi}_n$.

Consider the functions

$$B_2(x; y, s) := \left( \frac{B_2(y) - B_2(s)}{y - s} \right) \left( x - \frac{1}{y - s} \right) - \frac{A_2(y) - B_2(s) - A_2(s)}{y - s} \left( x - \frac{1}{y - s} \right)$$  \hspace{1cm} (4-1)

$$\psi_n^{(\Gamma)} := \frac{1}{2i\pi} \int_\Gamma ds \int_\mathcal{D} d\xi dy B_2(x; y, s)e^{\xi y - x s - V_2(y) + V_2(s) / x - \xi} \hspace{1cm} (4-2)$$

If $x$ belongs to a contour $\Gamma_{x, \mu}$ of the integration $\int_\mathcal{D}$ we obtain

$$\psi_n^{(\Gamma)}(x)_+ = \psi_n^{(\Gamma)}(x)_- + \sum_\nu B_2(\Gamma, \Gamma_{y, \nu}) \chi_{\mu, \nu} \psi_n(x) \hspace{1cm} (4-3)$$

where the subscript $x_{\pm}$ denotes the boundary values from the left/right and $B_2(\Gamma, \Gamma_{y, \nu})$ stands for the constant (in $x$) bilinear concomitant

$$B_2(\Gamma, \Gamma_{y, \nu}) := \frac{1}{2i\pi} \int_\mathcal{D} ds \int_{\Gamma_{y, \nu}} dy B_2(x; y, s)e^{V_2(s) - V_2(y + x - y - s)} \hspace{1cm} (4-4)$$

Therefore their jump across the contours of discontinuity is a constant multiple of $\psi_n(x)$.

We have

**Proposition 4.1** The sequences of functions $\{\psi_n^{(\Gamma)}\}_{n \in \mathbb{N}}$ satisfy the same recurrence relations (for $n$ large enough) as the quasipolynomials $\psi_n$

$$x \left( \psi_n^{(\Gamma)} + \sum_{j=1}^{q_2} \ell_j(n) \psi_{n-j}^{(\Gamma)} \right) = \sum_{j=1}^{d_2} \alpha_j(n) \psi_{n-j}^{(\Gamma)}, \quad n \geq d_2 + q_2 \hspace{1cm} (4-5)$$

$$\partial_x \left( \psi_n^{(\Gamma)} + \sum_{j=1}^{q_1} \tau_j(n + j) \psi_{n+j}^{(\Gamma)} \right) = - \sum_{j=1}^{d_3} \beta_j(n + j) \psi_{n+j}^{(\Gamma)}, \quad n \geq 1 \hspace{1cm} (4-6)$$

(For the proof see App. A.1)

**Definition 4.1** Beside the wave vector $\Psi$ we define the following $d_2$ auxiliary wave-vectors

$$\Psi_{x, \nu}^{(\nu)}(x) := \frac{1}{2i\pi} \int_{\Gamma_{y, \nu}} ds \int_\mathcal{D} d\xi dy B_2(x; y, s)e^{\xi y - x s - V_2(y) + V_2(s) / x - \xi} \Psi(x) \hspace{1cm} \nu = 1, \ldots, d_2$$

$$\Psi_{x}^{(0)}(x) := \Psi(x) \hspace{1cm} (4-7)$$

$$\Psi_{x, \nu}^{(\nu)}(x) := \Psi_{x}^{(\nu)}(x) \hspace{1cm} (4-8)$$
We also define the dual wave vectors
\[
\Phi^{(0)}_x(x) := e^{V_1(x)} \int_{x}^{\infty} e^{\xi y - V_1(\xi)} \frac{1}{x - \xi} \Phi(y) d\xi dy
\] (4-9)
\[
\Phi^{(\nu)}_x(x) := \int_{\Gamma_{y,\nu}} dy e^{x y} \Phi(y) , \nu = 1, \ldots, d_2
\] (4-10)

**Proposition 4.2** The components of the dual wave vectors satisfy the recurrence relations
\[
x \left( \dot{\phi}^{(\nu)}_n + \sum_{j=1}^{q_2} \hat{\epsilon}_j (n + j) \phi^{(\nu)}_{n+j} \right) = \sum_{j=1}^{d_2} \hat{\alpha}_j (n + j) \phi^{(\nu)}_{n+j} + \delta_{\nu 0} \delta_{n 0} \sqrt{h_0} e^{V_1(x)} , \nu = 0, \ldots, d_2
\] (4-11)
\[
\hat{\beta}_x \left( \phi^{(\nu)}_n + \sum_{j=1}^{q_1} m_j(n) \phi^{(\nu)}_{n-j} \right) = \sum_{j=1}^{d_1} \beta_j(n) \phi^{(\nu)}_{n-j} , \nu = 1, \ldots, d_2
\] (4-12)

**Remark 4.1** The wave vector \( \Phi^{(0)}_x \) does not satisfy a finite-term differential recurrence relation: a formula can be derived but it is not useful for our purposes.

**Proof** The formulæ for the Fourier–Laplace transforms follow from integration by parts from the relations satisfied by \( \phi_n(y) \) (Prop. 2.3). We only point out that integration by parts does not give any boundary contribution because \( s_n + \sum \hat{\epsilon}_j (n + j) s_{n+j}(y) \) is divisible by \( B_2(y) \) and hence vanishes at the hard-edge end-points.

The only relation that needs to be checked is the multiplicative relation for \( \nu = 0 \). Denoting temporarily by a tilde the linear combination
\[
\tilde{\phi}_n := \phi_n + \sum_{j=1}^{q_1} \hat{\epsilon}_j (n + j) \phi_{n+j}
\] (4-13)
we have
\[
x \tilde{\phi}^{(0)}_n(x) = e^{V_1(x)} \int_{x}^{\infty} e^{\xi y - V_1(\xi)} \frac{x}{x - \xi} \tilde{\phi}_n(y) d\xi dy =
\]
\[
= e^{V_1(x)} \int_{x}^{\infty} e^{\xi y - V_1(\xi)} \tilde{\phi}_n(y) d\xi dy + e^{V_1(x)} \int_{x}^{\infty} e^{\xi y - V_1(\xi)} \frac{\xi}{x - \xi} \tilde{\phi}_n(y) d\xi dy
\]
\[
= e^{V_1(x)} \delta_{0 0} \sqrt{h_0} + e^{V_1(x)} \int_{x}^{\infty} e^{\xi y - V_1(\xi)} \frac{\xi}{x - \xi} \tilde{\phi}_n(y) d\xi dy
\]
\[
= e^{V_1(x)} \delta_{0 0} \sqrt{h_0} + \sum_{j=1}^{d_2} \hat{\alpha}_j (n + j) \phi^{(0)}_{n+j} . \text{ Q.E.D.}
\] (4-14)

**4.1 Christoffel–Darboux identities**

In the general theory of the two–matrix model the following kernel plays an essential rôle in the computation of statistical correlation functions
\[
K_{12}^N(x, y) := \sum_{j=0}^{N-1} p_j(x) s_j(y) e^{-V_1(x) - V_2(y)} = \sum_{j=0}^{N-1} \psi_j(x) \phi_j(y)
\] (4-15)
In a previous paper by the author and collaborators [4, 8] the case of polynomial potentials $V_j$ was considered (without hard-edges) and it was of capital importance the existence of a Christoffel–Darboux identity allowing to express $K_{12}^N$ (or rather some transform of it) in terms of bilinear combinations of the BOPs involving only a number of BOPs depending only on the degrees of the potentials.

We look for a similar bilinear expression in this model.

**Definition 4.2** We define the windows of the wave vectors $\Psi^{(i)}_x$ and $\Phi^{(i)}_x$, $\mu = 0, \ldots, d_2$

\[
\begin{align*}
\Phi^{(i)}_x(x) &= \begin{bmatrix} \phi^{(i)}_{n-d_2}, \ldots, \phi^{(i)}_{n-1} \end{bmatrix}, \\
\Psi^{(i)}_x(x) &= \begin{bmatrix} \psi^{(i)}_{n-d_2}, \ldots, \psi^{(i)}_{n} \end{bmatrix}^t.
\end{align*}
\] (4-16)

We rewrite (4-15) in terms of the wave vectors

\[
K_{12}^N = \Phi^i(y)\Pi_N(x), \quad \Pi_N := \begin{cases} 
\delta_{ij}, & 0 \leq i \leq N - 1 \\
0, & \text{otherwise}.
\end{cases}
\] (4-17)

Recall the multiplicative and differential recurrence relations in Prop. 2.1 and Prop. 2.3 (which we rewrite here for the reader’s convenience)

\[
\partial_y \Phi^i(x + \hat{L}) = -\Phi^i \hat{A}, \quad A(1 + L)\Psi = A\Psi
\]

\[
(1 + L)^{-1}A - \hat{A}(1 + \hat{L})^{-1} = Q.
\]

Consider now the following expressions

\[
(x + \partial_y)\Phi^i(y)(1 + \hat{L})\Pi(1 + \hat{L})^{-1}\Psi(x) = \\
= \Phi^i(y)\Pi(1 + \hat{L})^{-1}\hat{A}(1 + \hat{L})^{-1}\Psi(x) - \Phi^i(y)\Pi(1 + \hat{L})^{-1}\Psi(x) = \\
-\Phi^i\Pi\hat{A}(1 + \hat{L})^{-1}\Psi + \Phi^i\Pi\hat{A}(1 + \hat{L})^{-1}\Psi = \\
= \Phi^i\hat{A}(1 + \hat{L})^{-1}\Psi = \Phi^i\hat{A}(1 + \hat{L})^{-1}\Psi = \\
-\Phi^i\hat{A}(1 + \hat{L})^{-1}\Psi.
\] (4-18)

where we have set $\tilde{Q} := (1 + \hat{L})^{-1}\hat{A}$. We now use the fact that $\tilde{Q}$ is the recurrence matrix for the associated $\hat{\Psi}$ wave vector (see Prop. 2.2 where $\hat{\Psi} := \hat{\Phi}e^{-V_1(x)}$) and obtain

\[
(x + \partial_y)\Phi^i(y)(1 + \hat{L})\Pi(1 + \hat{L})^{-1}\Psi(x) = \Phi^i(y)[x\hat{L} - \hat{A}, \Pi]\hat{\Psi}(x)
\] (4-19)

\[
\hat{\Psi} = (1 + \hat{L})^{-1}\Psi = \hat{\Phi}(x)e^{-V_1(x)}
\] (4-20)

\[
\hat{\Phi} = (1 + \hat{L})\Phi = \hat{s}(y)B_2e^{-V_2(y)} = \hat{s}(y)e^{-\hat{V}_2(y)}
\] (4-21)

With these notation we have
Theorem 4.1 (Christoffel–Darboux identity) For the kernels
\[ K^{N,j}_{11}(x,x') = \int_{\Gamma_{N,j}} e^{\imath \eta \Phi^t_{\infty}(y)} \bar{\Pi}_N \tilde{\Psi}^t(x') = \frac{\Phi(x)}{\Pi \tilde{\Psi}(x')} , \quad j = 1, \ldots, d_2 \]
we have the identities
\[ (x' - x)K^{N,j}_{11}(x', x) = \Phi(x')(x')^t \left( \Pi \tilde{\Psi}(x) \right) \]
(4-24)
\[ (x' - x)K^{N,j}_{11}(x',x) = \Phi(x')(x')^t \left( \Pi \tilde{\Psi}(x) \right) . \]
(4-25)

(note the argument \( \hat{A}_N \) in the two formulae) where \( \hat{A}_N(x) := \left[ \hat{A} - x \hat{L}, \Pi_N \right] \).

Proof The identity for \( K^{N,j}_{11}(x,x') \) follows by performing integration by parts on (4-19) and noticing that the boundary contributions vanish since \( \hat{\Phi}(y) = B_2(y)\bar{\Phi}(y)e^{-\imath \sqrt{2}(y)} \) and \( B_2(y) \) vanishes at the hard-edges. The identity for \( K^{N,j}_{11}(x,x') \) follows from the one for \( \hat{K}^{N,j}_{11} \) and this manipulation
\[ (x' - x)\hat{\Phi}^t(x') \Pi \tilde{\Psi}(x) = (x' - x)\hat{\Phi}^t(x') \left( \Pi \tilde{\Psi}(x) \right) = \]
\[ = (x' - x) \left[ \hat{\Phi}^t(x') \left[ \hat{L}, \Pi \right] \tilde{\Psi}(x) + \hat{\Phi}^t(x') \Pi \tilde{\Psi}(x) \right] - \]
\[ = (x' - x) \hat{K}^{N,j}_{11}(x',x) + (x' - x)\hat{\Phi}^t(x') \left[ \hat{L}, \Pi \right] \tilde{\Psi}(x) \]
(4-26)
so that
\[ (x' - x)\hat{K}^{N,j}_{11}(x',x) = \]
\[ = (x' - x)\hat{K}^{N,j}_{11}(x',x) - (x' - x)\hat{\Phi}^t(x') \left[ \hat{L}, \Pi \right] \tilde{\Psi}(x) = \hat{\Phi}^t(x') \left[ \hat{L}, \Pi \right] \tilde{\Psi}(x) \]
(4-27)
Q.E.D.

Note that—with a slight abuse of notation—in the RHS of the CDIs we can replace the wave vectors \( \hat{\Phi} \) by the corresponding window \( \hat{\Phi} \), since the matrix \( \hat{A}_N \) has a nonzero square block of size \( d_2 + 1 \) with top-right corner in the \( (n - 1, n) \) entry, and hence the bilinear expression \( \hat{\Phi} \hat{A} \hat{\Psi} \) only involves the terms in the dual windows \( \hat{\Phi} \) and \( \hat{\Psi} \). We will denote from now on by \( \hat{A} \) only the \( d_2 + 1 \) square matrix which is relevant to the pairing.

The importance of the theorem is that we can express the kernel \( K_{11} \) in terms of the dual quantities \( \hat{\phi}_n(x) \) and \( \hat{\psi}_n(x') \) involving only the indexes \( N - d_2 \leq n \leq N \).

Note, however, that we must introduce the orthogonal polynomials \( \hat{p} \) for the associated moment functional \( \hat{L} \) in order to find a Christoffel–Darboux relation similar to the standard one for orthogonal polynomials.
Theorem 4.2 (Auxiliary CDIs) The auxiliary wave vectors enter in the following auxiliary Christoffel–Darboux identities

\[(a) \ (z - x) \frac{\Phi^{(0)}_x(z)}{\Phi^{(0)}_x} \Pi_n \psi^{(0)}_x(x) = \frac{\Phi^{(0)}_x(z)}{\Phi^{(0)}_x} \hat{\Pi}_n \hat{\psi}_n(x) + e^{V_i(z) - V_i(x)} \]
\[(z - x) \frac{\Phi^{(0)}_x(z)}{\Phi^{(0)}_x} \Pi_n \psi^{(0)}_x(x) = \frac{\Phi^{(0)}_x(z)}{\Phi^{(0)}_x} \hat{\Pi}_n \hat{\psi}_n(x) + e^{V_i(z) - V_i(x)} \ (4-28)\]

\[(b) \ (z - x) \frac{\Phi^{(j)}_x(z)}{\Phi^{(j)}_x} \Pi_n \psi^{(k)}_x(x) = \frac{\Phi^{(j)}_x(z)}{\Phi^{(j)}_x} \hat{\Pi}_n \hat{\psi}_n(x) - \frac{1}{2i\pi} \int_{y_v} \int_{y_u} B_2(x; y, s) e^{y z - x s + V_2(s) - V_2(y)} , \]
\[(z - x) \frac{\Phi^{(j)}_x(z)}{\Phi^{(j)}_x} \Pi_n \psi^{(k)}_x(x) = \frac{\Phi^{(j)}_x(z)}{\Phi^{(j)}_x} \hat{\Pi}_n \hat{\psi}_n(x) - \frac{1}{2i\pi} \int_{y_v} \int_{y_u} B_2(x; y, s) e^{y z - x s + V_2(s) - V_2(y)} , \]
\[j, k = 1, \ldots, d_2 . \ (4-29)\]

(For the proof see App.

4.2 Ladder matrices

In this section we derive an expression for the ODE satisfied by the polynomials in terms of the so-called "folding" (see [4]). This will have certain advantages when explaining the relations between the various ODEs that naturally appear in the problem: a different explicit representation of the ODE will be given in the next section as well, using a completely different approach based upon the explicit integral representations of the wave vectors and on the duality provided by the Christoffel–Darboux pairing.

We first have the simple lemma

Lemma 4.1 (Ladder matrices) The multiplicative recurrence relations for the wave vectors \( \hat{\psi}^{(0)}_x, \hat{\psi}^{(j)}_x \) \( (j = 1, \ldots, d_2) \)

\[x(1 + L)\hat{\psi}^{(0)}_x = A\hat{\psi}^{(0)}_x , \quad x(1 + \hat{L}^t)\hat{\psi}^{(j)}_x = \hat{A}^t\hat{\psi}^{(j)}_x \ (4-30)\]

are equivalent to the relations

\[\psi^{(0)}_{n+1}(x) = a_n(x) \psi^{(0)}_n(x) , \ (4-31)\]
\[\psi^{(j)}_{n+1}(x) = \frac{\psi^{(j)}_n(x)}{a_n(x)} \psi^{(j)}_{n+1}(x) \ (4-32)\]

where

\[a_n(x) = \Lambda - \frac{1}{\alpha_{-1}(n)} \begin{bmatrix} 0 \\ \vdots \\ \alpha_d(n), \ldots, \alpha_0(n) + \frac{x}{\alpha_{-1}(n)} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} [0, \ldots, \ell_{q_2}(n), \ldots, \ell_1(n), 1]}^{4-33} \]
Lemma 4.2 (Folded recursion relations) The differential recurrence relations for the wave-vector
\[ \tilde{\Psi}_n(x) = \Lambda - \frac{1}{\hat{\alpha}_{-1}(n-1)} \begin{bmatrix} \hat{\alpha}_0(n) \\ \hat{\alpha}_1(n+1) \\ \vdots \\ \hat{\alpha}_{d_2}(n+d_2) \end{bmatrix} \begin{bmatrix} 1 \\ 0, \ldots, 0 \\ \frac{x}{\hat{\alpha}_{-1}(n-1)} \end{bmatrix} \begin{bmatrix} \hat{\ell}_1(n+1) \\ \vdots \\ \hat{\ell}_{q_2}(n+q_2) \end{bmatrix} \begin{bmatrix} 1 \\ 0, \ldots, 0 \end{bmatrix} \] (4-34)

and \( \Lambda \) denotes the upper shift matrix (of size \( d_2 + 1 \)). The relations (4-31) and (4-32) hold also for the other sequences of windows \( \Psi_n^{(j)} \) and \( \Phi_n^{(j)} \) provided that \( n \geq d_2 + q_2 \) (\( n \geq 1 \) respectively).

Proof. The proof follows immediately from the recurrence relations for the wave vectors \( \Psi_n^{(j)} \) (the quasipolynomials) and \( \Phi_n^{(j)} \) (the Fourier–Laplace transforms) by solving for \( \psi_{n+1}(x) \) (or \( \phi_n^{(j)} \)) in terms of \( \psi_{n-d_2}, \ldots, \psi_n \) \( (\phi_{n-d_2}, \ldots, \phi_{n+d_2}) \) and rewriting the relation in matrix form. The statement for the other sequences of windows follows from the fact that the corresponding wave vectors satisfy the same finite-term recurrence relations in the specified range (see Prop. 4.1 and Prop. 4.2). Q.E.D.

Lemma 4.2 (Folded recursion relations) The differential recurrence relations for the wave-vector \( \Psi_n \)
\[ \partial_x (1 + \tilde{M}_x) \Psi_n = - \tilde{B}_x^i \Psi_n \] (4-35)
are equivalent to the relations
\[ \partial_x \left( \tilde{M}_n(x) \Psi_n \right) = - \tilde{B}_n(x) \Psi_n \] (4-36)
\[ \tilde{M}_n := 1 + \sum_{j=1}^n \tilde{m}_j(n) a_n \cdots a_{n+j-1} \] (4-37)
\[ \tilde{m}_j(n) := \text{diag}(\tilde{m}_j(n + j - d_2), \ldots, \tilde{m}_j(n + j)) \] (4-38)
\[ \tilde{B}_n := \tilde{B}_{n-1}(n)(a_{n-1})^{-1} + \tilde{B}_0(n) + \sum_{j=1}^{d_1} \tilde{B}_j(n) a_n \cdots a_{n+j-1} \] (4-39)
\[ \tilde{B}_j(n) := \text{diag}(\tilde{B}_j(n + j - d_2), \ldots, \tilde{B}_j(n + j)) \] (4-40)

Proof. The formula is an iterated application of the ladder recurrence relations (on a window of consecutive elements with indexes \( n - d_2, \ldots, n \)) to the differential recurrence relation for the wave vector (see [4] for more details). Q.E.D.

Remark 4.2 A completely analogous statement can be derived for the windows of the dual vector \( \Phi_n^{(j)}, \) \( j = 1, \ldots, d_2. \)

Remark 4.3 The matrices \( a_n \) have a companion-form and are invertible since the determinant is \( -\frac{\alpha_{d_2}^{(n)}}{\alpha_{-1}^{(n)}} \)
which has been proved nonvanishing in Thm. 2.1. Moreover the inverse is also linear in \( x \) (the details are left to the reader).
Remark 4.4 By the very definition $\tilde{M}_n(x)\Psi_n = \tilde{\Psi}_n$ is the window of quasipolynomials (and associated functions) for the moment functional $\tilde{\mathcal{L}}$.

Corollary 4.1 The $d_2 + 1$ columns provided by the windows of the auxiliary wave vectors $\psi^{(j)}(x)$ provide a fundamental system for the ODE (4-36) for $n \geq d_2 + q_2$.

Proof. From Prop. 4.1 we know that the components of the auxiliary wave vectors satisfy the same recurrence relations (both multiplicative and differential) as the quasipolynomials provided $n$ is large enough. Moreover the recurrence relations always involve a fixed number of terms with indexes "around $n$": since the derivation of the ODE is entirely based on the recurrence relations the statement follows. Q.E.D.

Proposition 4.3 The determinant of $\tilde{M}_n(x)$ is proportional to $B_1(x)$ by a nonzero constant.

Proof. Consider the window of polynomials $p_n := [p_{n-d_2}, \ldots, p_n]^t$: from the definition of the matrix $\tilde{M}$ it follows that

$$\tilde{M}_n(x)p_n(x) = B_1(x)\tilde{p}_n(x) \quad (4-41)$$

We first prove that $\det \tilde{M}_n$ (which is a fortiors a polynomial) is divisible by $B_1$. Let $c$ be a zero of $B_1$ of multiplicity $r$: at least one component (say the $\ell$-th) of $p_n(c)$ is nonzero because of the very genericity assumption which guarantees the existence of $\tilde{M}$ (2-38). Let $E(x)$ be the matrix obtained by replacing the $\ell$-th column of the identity with $p_n(x)$. Clearly $\det E(x)$ is nonzero in a neighborhood of $x = c$ by our definition of $\ell$. It follows that the $\ell$-th column of $\tilde{M}_nE$ is precisely $B_1\tilde{p}_n$ and hence each component vanishes at $c$ of order $r$. Also

$$\det \tilde{M}_nE = p_{n-d_2+\ell+1}(x)\det \tilde{M}_n \quad (4-42)$$

and $p_{n-d_2+\ell+1}(c) \neq 0$. On the other hand $\det \tilde{M}_nE$ must vanish at $x = c$ of order $r$ since the whole $\ell$-th column does. Repeating this for all roots of $B_1$ we find the assertion of divisibility of $\det \tilde{M}_n$ by $B_1(x)$.

On the other hand, using a technique of evaluation of determinants used in [4],

$$\det \tilde{M}_n = \det \begin{bmatrix} \begin{bmatrix} a_{n+q_1} \\ \vdots \\ \tilde{m}_{q_1}(n) \end{bmatrix} \begin{bmatrix} a_n \\ \tilde{m}_{1}(n) \end{bmatrix} \end{bmatrix} 
- \begin{bmatrix} \begin{bmatrix} a_{n+q_1} \\ \vdots \\ \tilde{m}_{q_1}(n) \end{bmatrix} \begin{bmatrix} a_n \end{bmatrix} \end{bmatrix} \quad (4-43)$$

Considering carefully the structure of the sparse matrix in the last identity, one realizes that the highest power in $x$ is

$$\det \tilde{M}_n = x^{q_1} \frac{\tilde{m}_{q_1}(n + q_1)}{\prod_{j=1}^{q_1} (n + j)} + O(x^{n-1}) \quad (4-44)$$

This shows that (since the coefficient does not vanish as per (2-19,2-48)) the determinant is of degree $q_1 = \deg B_1$; since it must be also divisible by $B_1$, this concludes the proof. Q.E.D.
Corollary 4.2 The windows $\Psi_n$, $\tilde{\Psi}_n$ satisfy

\begin{align}
\partial_x\Psi_n &= -\tilde{\mathcal{M}}^{-1}_n \left( \tilde{B}_n + \partial_x \tilde{\mathcal{M}}_n \right) \Psi_n \quad (4-45) \\
\partial_x \tilde{\Psi}_n &= -\tilde{B}_n \tilde{\mathcal{M}}^{-1}_n \tilde{\Psi}_n \quad (4-46)
\end{align}

where $\tilde{B}_n, \tilde{\mathcal{M}}_n$ are defined in ((4-37)–(4-40)). The ODEs have the same singularity structure as $V'_1$.

The first relation follows from (4-36) and the second from the fact that $\tilde{\mathcal{M}}_n(x) \Psi_n(x) = \tilde{\Psi}_n(x)$.

This shows that the ODE’s for $\Psi_n$ and $\tilde{\Psi}_n$ are gauge-equivalent, the gauge being provided by the (polynomial) matrix $\tilde{\mathcal{M}}_n$. Moreover formula (4-46) together with Prop. 4.3 shows that the singularities of the differential equation are at the zeroes of $B_1(x)$.

4.3 Differential equations for the dual pair of systems

In this section we present an explicit formula for the ODE satisfied by the dual pair of fundamental systems, in particular the polynomials $\psi_n$ and the Fourier–Laplace transforms $\phi_n$’s. The result generalizes those of [7] but the method of derivation is similar to the one adopted in [8], with additional complications deriving from the presence of boundary contributions in the integration by parts at various steps of the derivation.

Notation. In the proof of this and the following theorems we will encounter integrations by parts that yield nonzero boundary contributions. Typically we will encounter integrals of the form

\[
\int_{-\infty}^{\infty} ye^{yV_1(\rho)} F(\rho) \phi_m(y) dy d\rho ,
\]

where $F(\rho)$ is some expression (typically polynomial or rational in $\rho$) possibly depending on “external” variables. If we attempt an integration by parts on the term $ye^{y\rho} = \partial_{\rho} e^{y\rho}$, we obtain a certain number of boundary terms. In all cases they will be boundary evaluation on the various contours $\Gamma_{x,\mu}$; it is the nature of all these integrals that only the contours emanating from a hard–edge point give a contribution, due to the fast decay of $e^{-V_1(\rho)}$ at all the boundary points of the other contours. In the above example and in all minute detail, we have

\[
\int_{-\infty}^{\infty} ye^{yV_1(\rho)} F(\rho) \phi_m(y) dy = - \int_{-\infty}^{\infty} e^{yV_1(\rho)} (-\partial_{\rho} + V'_1(\rho)) F(\rho) \phi_m(y) + \text{(Boundary terms)}
\]

(4-48)

The evaluation at the boundary points of the various contours $\Gamma_{x,\mu}$ is clearly to be understood as limits along the contours; the decay of $e^{-V_1(\rho)}$ along the contours gives zero contributions except for the hard–edge contours, at the (finite) boundary of which $V_1(\rho)$ is regular. In order to economize on space, we introduce the following shorthand notation for the above boundary terms

\[
F(\rho) e^{-V_1(\rho)} \phi_m^{(\infty)}(\rho) \bigg|_{\rho \in \partial \Sigma, x} := \text{(Boundary terms)}
\]

(4-49)
Theorem 4.3  The dual fundamental system.

\[
\Phi_n(x) := \left[ \begin{array}{c} \phi_n^{(0)} \\ \vdots \\ \phi_n^{(d_2)} \end{array} \right] = \left[ \begin{array}{cccc} \phi_{n-1}^{(0)} & \phi_n^{(0)} & \cdots & \phi_{n+d_2-1}^{(0)} \\ \phi_{n-1}^{(1)} & \phi_n^{(1)} & \cdots & \phi_{n+d_2-1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1}^{(d_2)} & \phi_n^{(d_2)} & \cdots & \phi_{n+d_2-1}^{(d_2)} \end{array} \right]  \tag{4-50} \]

satisfies the ODE

\[
\Phi_n^{-1}(x) \Phi_n'(x) = \begin{bmatrix} V_1'(x) & 0 & \cdots & 0 \\ P_{n,n-1} & P_{n,n} & \cdots & P_{n,n+d_2-1} \\ 0 & P_{n+1,n} & \ddots & \vdots \\ 0 & 0 & \ddots & P_{n+d_2,n+d_2-1} + P_{n+d_2,n+d_2-1} \end{bmatrix} + \text{diag}(P_{n+d_2,n-1}, \ldots, P_{n+d_2,n+d_2-1}) \Phi_n^{-1}(x) + \hat{\Lambda}(x) \left[ \frac{\hat{\phi}_n(p)\Phi_n(x)}{x-p} \right]_{p \in \mathbb{R}^+} - \hat{\Lambda}(x) W(x) + W_{ab}(x) = \mathcal{L}\left(\hat{\beta}_n + \frac{V_1(x) - V_1(p)}{x-p} s_{n-1+b}(y)\right), \quad a, b = 0, 1, \ldots, d_2 \tag{4-51} \]

\[
P_{j,k} := \mathcal{L}(p_j) y_{s_k}. \tag{4-52} \]

where \( n \) is the ladder matrix for the dual wave vector (Note that \( P = (1 + M)^{-1}B \)).

(For the proof see App. A.3).

Theorem 4.4  The direct system

\[
\Psi_n(x) := \left[ \begin{array}{c} \psi_n^{(0)} \\ \psi_n^{(1)} \\ \vdots \\ \psi_n^{(d_2)} \end{array} \right] = \left[ \begin{array}{cccc} \hat{\psi}_{n-d_2}^{(0)} & \hat{\psi}_{n-d_2}^{(1)} & \cdots & \hat{\psi}_{n-d_2}^{(d_2)} \\ \vdots & \vdots & & \vdots \\ \hat{\psi}_{n-1}^{(0)} & \hat{\psi}_{n-1}^{(1)} & \cdots & \hat{\psi}_{n-1}^{(d_2)} \\ \hat{\psi}_n^{(0)} & \hat{\psi}_n^{(1)} & \cdots & \hat{\psi}_n^{(d_2)} \end{array} \right]  \tag{4-53} \]

satisfies the ODE

\[
\Psi_n' \Psi_n^{-1} = -\begin{bmatrix} \hat{P}_{n-d_2,n-d_2} & \cdots & \hat{P}_{n-d_2,n-1} & 0 \\ \hat{P}_{n-d_2+1,n-d_2} & \cdots & 0 & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & \hat{P}_{n-1,n-2} & \hat{P}_{n-1,n-1} & 0 \end{bmatrix} + \text{diag}(\hat{P}_{n+1,n-d_2}, \ldots, \hat{P}_{n+1,n}) \hat{\alpha}_{n-1}^{-1} + \text{diag}(\hat{P}_{n+1,n-d_2}, \ldots, \hat{P}_{n+1,n}) \hat{\alpha}_{n-1}^{-1} + \hat{\Lambda}(x) + W(x) \hat{\Lambda}(x) + W(x) \hat{\Lambda}(x) \tag{4-54} \]

\[
P_{j,k} := \mathcal{L}(p_j) y_{s_k}, \tag{4-54} \]

33
where $W(x)$ was defined in the previous theorem and $\hat{a}_{n-1}$ is the ladder matrix implementing the multiplicative recurrence relations $\hat{\Psi}_n = \hat{a}_{n-1}\Psi_{n-1}$ as per Lemma 4.1 (in particular eq. (4-31)) specified to the hat-wave vectors.

(For the proof see App. A.4).

5 Dual Riemann–Hilbert problems

The shape of the Christoffel–Darboux identity (Thm. 4.1) suggests that the duality of the Riemann–Hilbert problems (and of the differential equations) involves naturally the dual pair of fundamental systems $\Phi_n(x), \Psi_n(x)$ defined in Thm. 4.3 and Thm. 4.4. Recall (from Section 3) that we can choose a basis in the relative homology of contours $\Gamma_{y,\nu}$ and $\hat{\Gamma}_{y,\nu}$ (and a rescaling of the $\hat{\psi}_j^{(j)}$ wave vectors depending only on the residues of $V_2'(y)dy$) which span the solution space of the two adjoint equations and with bilinear concomitant

$$B_2(\Gamma_{y,\nu}, \hat{\Gamma}_{y,\mu}) := \Gamma_{y,\nu} \hat{\Psi}_{y,\mu} = \delta_{\mu\nu}. \tag{5-1}$$

We can rewrite (Thm. 4.1) as

$$\begin{align*}
(x-x') \sum_{j=0}^{n-1} \hat{\phi}_j^{(\nu)}(x) \hat{\psi}_j^{(0)}(x') &= \hat{\Phi}_n(x) \hat{\Psi}_n(x) \hat{\Lambda}(x'') \hat{\Psi}_n(x') \tag{5-2} \\
(x-x') \sum_{j=0}^{n-1} \phi_j^{(\nu)}(x) \psi_j^{(0)}(x') &= \Phi_n(x) \Psi_n(x) \Lambda(x'') \Psi_n(x') \tag{5-3}
\end{align*}$$

$\nu = 1, \ldots, d_2$, where we stress the fact that on the LHS we have the quasipolynomials $\psi_n$ whereas on the RHS we have the $\hat{\psi}_n$’s.

**Theorem 5.1** The fundamental dual pair is put in perfect duality by the Christoffel–Darboux matrix $\hat{\Lambda}$

$$\begin{bmatrix}
1 & 0 \\
0 & B_2(\bullet, \bullet)
\end{bmatrix} \tag{5-4}$$

where $B_2(\bullet, \bullet)$ represents the (constant in $x$) bilinear concomitant for the solutions of the adjoint ODEs along the contours $\Gamma_{y,\nu}, \hat{\Gamma}_{y,\mu}$, $\mu, \nu = 1, \ldots, d_2$. By suitable choice of the homology classes we have seen that we can always assume it to be diagonal. The entries on the diagonal are nonzero and may be set to 1 by suitable rescaling of the $d_2$ left-most columns of $\Psi_n$: these re-scalings depend on the way we have performed the cuts in the definitions of $V_2$ and $\hat{V}_2$ but depend only on the residues of $V_2'(y)dy \mod \mathbb{Z}$.

(For the proof see App. A.5).

5.1 Riemann–Hilbert data

In this section we summarily indicate how to obtain the data of the Riemann–Hilbert problems solved by the dual fundamental systems. The details are considerably involved and not strictly necessary in this paper. They will appear in a different publication.
Since the two matrices $\Phi_n$ and $\hat{\Psi}_n$ are put in perfect duality by the Christoffel–Darboux pairing, it is -in principle- sufficient to describe the Riemann–Hilbert data of one of the two members of the pair, the data for its partner being completely determined by duality.

It is significantly simpler to analyze the RH data for the matrix $\Phi_n$. We recall that this means controlling the jump discontinuities and the asymptotic behaviors near the singularities.

**Jump discontinuities.** They are uniquely due to the first row in the definition of $\Phi_n$ and occur at the contours $\Gamma_{x,\nu}$:

$$
\Phi_n(x_+) - \Phi_n(x_-) = \begin{bmatrix}
1 & 2i\pi \kappa_{\nu,1} & 2i\pi \kappa_{\nu,2} & \ldots & 2i\pi \kappa_{\nu,d_2} \\
& & & & \\
& & & & \\
& & & & \\
& & & & 1
\end{bmatrix}
$$

where $x_{\pm}$ denote the boundary values on the left/right of the point $x \in \Gamma_{x,\nu}$.

Note that the fundamental matrix $\hat{\Psi}_n(x)$ satisfies a similar jump condition which can be read off eq. (4-3) (specified to the $\hat{\psi}_n$ quasipolynomials).

**Singularities** The bottom $d_2$ rows (the Fourier–Laplace transforms) are entire functions. The only singularities in the finite part of the plane arise from the first row $\Phi_n^{(0)}(x)$: apart from the jump discontinuities (discussed above) we have all the singularities of $e^{V_1(x)}$ and the logarithmic branching singularities around the hard-edge endpoints. Note that the (piecewise analytic) function

$$
F_n(x) := \int_{\mathbb{R}} \hat{\Phi}_n(y) \frac{e^{-V_1(\xi) + \xi y}}{x - \xi} = e^{-V_1(x)} \hat{\Phi}_n(x)
$$

has a well defined limit as $x$ approaches any of the non hard-edge endpoints (where it is understood that the approach occurs within one connected component of its domain of analyticity). Indeed, if $c$ is such a point one finds

$$
F_n(c) = \int_{\mathbb{R}} \hat{\Phi}_n(y) \frac{e^{-V_1(\xi) + \xi y}}{c - \xi}
$$

which is a well-defined value. In other words, near a non hard-edge singularity one has

$$
\Phi_n(x) - \text{diag}\left(e^{V_{1,\text{sing}}(x)}, 1, \ldots, 1\right) Y_0(1 + O(x - c)).
$$

where $Y_0$ is just the evaluation of the Fourier–Laplace rows and the $F_n(x)$ defined above at the point $c$, and $V_{1,\text{sing}}$ denotes the singular part of $V_1$ at $c$.

Near a hard–edge point $x = a$, if $\Gamma_{x,\nu_a}$ is the the hard-edge contour originating from $a$, we find that the
matrix

$$Y(x) := \begin{pmatrix}
    1 & \ln(x - a)\gamma_{n,1} & \ldots & \ln(x - a)\gamma_{n,d_2} \\
    \vdots & \ddots & \ddots \\
    & & & 1
\end{pmatrix}$$

has a removable singularity at $x = a$ and from this we can obtain the asymptotic behavior near the hard–edge endpoints.

**Stokes Phenomenon.** Possibly the most intricate part is the description of the Stokes' phenomenon at $x = \infty$.

Indeed, apart from the aforementioned jump-discontinuities of $\Phi_n^{(0)}$ in a neighborhood of $\infty$ (which may be interpreted as part of the Stokes data), the first row displays no Stokes’ phenomenon, and has an asymptotic behavior which encodes the orthogonality

$$\phi_n^{(0)}(x) = e^{V_i(x)} \int_{\gamma} e^{-V_i(\zeta) + \xi y} \frac{\phi_n(y)}{x - \zeta} \sim \sqrt{h_n e^{V_i(x)} x^{-n-1}} (1 + O(1/x))$$

The remaining part of the Stokes phenomenon is given by the asymptotic behavior of the $d_2$ Fourier–Laplace transforms: this is precisely the same Stokes’ phenomenon displayed by the solutions of the ODE

$$(A_2(\partial_x) - x B_2(\partial_x)) f = 0$$

These solutions are described by contour integrals of the same kind as the ones appearing in the expressions for $\Phi_n^{(n)}$; a standard steepest descent formal argument shows that the leading asymptotic is determined by the saddle-point equation

$$\frac{A_2(y) + B_2(y)}{B_2(y)} = V_2(y) = x$$

($x \to \infty$) which has $d_2 - H$ solutions ($H$ being the number of hard-edge contours, i.e. the number of (simple) zeroes of $B_2$ which cancel against corresponding zeroes of the numerator in (5-12)).

Whereas it is not very difficult to analyze the formal properties of the asymptotic, it is considerably harder and outside of the intents of the present paper to present the Stokes matrices associated to this Stokes’ phenomenon. We leave this topic to a different publication.

### 5.1.1 Isomonodromic deformations

The (generalized) 2-Toda equations for this reduction as explained in the introduction, determine the evolution of the biorthogonal polynomials under infinitesimal deformations of the parameters entering the semiclassical data $A_i, B_i$. It is more convenient to parametrize the polynomials $A_i, B_i$ not by their coefficients but by the location of the zeroes of $B_i$ and the coefficients in the partial fraction expansions of the derivative potentials $V_i'$. Following the strategy in our [4, 9, 11] one could easily write the pertinent 2-Toda flows corresponding to these infinitesimal deformations.
At the level of the pair of fundamental systems the flows will generate isomonodromic deformations for the ODEs satisfied by $\Phi_n$ and $\Psi_n$, provided that the exponents of formal monodromy at the singularities remain unchanged. In this case these are precisely the residues of $V'_1(x)\,dx$ and $V'_2(y)\,dy$ at the various singularities.

The reason why the deformations are isomonodromic is that—by their very definition—the fundamental systems are functions of these deformation parameters and the matrices $\hat{\Phi}, \Phi_n^{-1}$ (and $\hat{\Psi}, \Psi_n^{-1}$), the dot representing a derivative w.r.t. one of the monodromy-preserving parameters) are rational (or polynomial) functions of $x$, which follows from the analysis of their behavior at the various singularities ([17, 10] for details on the general properties of isomonodromic deformations).

The details of this isomonodromic system could be derived from the complete Riemann–Hilbert characterization of the fundamental systems and are beyond the scope of this paper, although their derivation is—in principle—a straightforward computation.

A.1 Proof of Proposition 4.1

We temporarily denote by a tilde the following linear combination

$$\tilde{\psi}_n = \psi_n + \sum_{j=1}^{q_2} \ell_j(n)\psi_{n-j} \quad (1-1)$$

and notice that

$$x\tilde{\psi}_n = \sum_{j=1}^{d_2} \alpha_j(n)\psi_{n-j} \quad (1-2)$$

For the transformed functions $\psi_n^{(\tilde{F})}$ (denoting by a tilde the same linear combination)

$$x\tilde{\psi}_n^{(\tilde{F})} = \frac{x}{2i\pi} \int_{\Gamma} \int_{\infty} B_2(2; y, s) e^{\xi y-x\xi - V_2(y) + \tilde{V}_2(s)} \frac{\tilde{\psi}_n(\xi)}{x - \xi} = \quad (1-3)$$

$$= \frac{1}{2i\pi} \int_{\Gamma} \int_{\infty} B_2(2; y, s) e^{\xi y-x\xi - V_2(y) + \tilde{V}_2(s)} \left(\frac{\tilde{\psi}_n(\xi)}{x - \xi} + \frac{\xi\tilde{\psi}_n(\xi)}{x - \xi}\right) = \quad (1-4)$$

$$= \sum_{j=1}^{d_2} \alpha_j(n)\psi_{n-j}^{(\tilde{F})} + \int_{\Gamma} \int_{\infty} B_2(2; y, s) e^{\xi y-x\xi - V_2(y) + \tilde{V}_2(s)} \frac{\tilde{\psi}_n(\xi)}{x - \xi} \quad (1-5)$$
where the last term vanishes for \( n \geq q_2 + d_2 \) because the bilinear concomitant kernel \( B_2(x; y, s) \) is a polynomial in \( y \) of degree \( d_2 - 1 \) and the linear combination \( \tilde{\psi}_n \) contains the orthogonal function \( \psi_{n-q_2} \).

For the differential equation we have (by definition of the \( \psi_n \)’s)

\[
\tilde{\psi}_n := \psi_n + \sum_{j=1}^{q_1} \eta_j (n + j) \psi_{n+j}
\]

(1-6)

We then have

\[
\partial_x \tilde{\psi}_n^{(\tilde{\psi})} = \int_\infty \int_\infty e^{-xs} (\partial_x - s) \frac{B_2(x; y, s) e^{\xi y - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi)}{x - \xi} =
\]

\[
= \int_\infty \int_\infty e^{-xs} \frac{B_2(y) - B_2(s)}{y - s} e^{\xi y - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi) \frac{2}{x - \xi} + \int_\infty \int_\infty \frac{B_2(y)}{x - \xi} \tilde{\psi}_n(\xi) -
\]

\[
= \int_\infty \int_\infty \frac{B_2(y) - B_2(s)}{y - s} e^{\xi y - x s - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi) \frac{2}{x - \xi} + \int_\infty \int_\infty \frac{B_2(x; y, s)}{x - \xi} (\partial_x - s) e^{\xi y - x s - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi) -
\]

\[
- \sum_{j=1}^{d_1} \beta_j (n + j) \psi_{n+j}^{(\tilde{\psi})} +
\]

\[
+ \int_\infty \int_\infty \frac{B_2(y) - B_2(s)}{y - s} e^{\xi y - x s - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi) \frac{2}{x - \xi} + \int_\infty \int_\infty \frac{B_2(x; y, s)}{x - \xi} e^{\xi y - x s - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi) -
\]

\[
= \sum_{j=1}^{d_1} \beta_j (n + j) \psi_{n+j}^{(\tilde{\psi})} +
\]

\[
= \sum_{j=1}^{d_1} \beta_j (n + j) \psi_{n+j}^{(\tilde{\psi})} + \int_\infty \int_\infty (x B_2(y) - A_2(y)) e^{\xi y - x s - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi) \frac{2}{x - \xi} =
\]

\[
= \sum_{j=1}^{d_1} \beta_j (n + j) \psi_{n+j}^{(\tilde{\psi})} + \int_\infty \int_\infty B_2(y) e^{\xi y - x s - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi) +
\]

\[
\text{total derivative in } y
\]

\[
+ \int_\infty \int_\infty (\xi B_2(y) - A_2(y)) e^{\xi y - x s - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi) \frac{2}{x - \xi} =
\]

\[
= \sum_{j=1}^{d_1} \beta_j (n + j) \psi_{n+j}^{(\tilde{\psi})} + \int_\infty \int_\infty B_2(y) e^{\xi y - x s - V_2(y)} + \tilde{V}_2(s) \tilde{\psi}_n(\xi)
\]

(1-7)

In the step marked with * we have performed an integration by parts: in this integration we do not get any boundary contributions because the quasipolynomials \( \tilde{\psi}_n \) by definition are divisible by \( B_1 \) (which vanishes at all endpoints and in particular at the hard-edge ones). This concludes the proof. Q.E.D.

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A.2 Proof of Theorem 4.2

During this and following proofs we use the notation

\[ \Phi_n(y) := [\phi_{n-1}, \ldots, \phi_{n+d-1}] , \tag{1-8} \]

for the row-vector of quasipolynomials in \( y \). Moreover, at the risk of marginal confusion, we omit all differentials of the integration variables since which variables are integrated and on which contour should be always uniquely determined by the context (the formulas become significantly longer otherwise). For eq. (4.1) we have (recall that \( \hat{A}(\xi) \) is linear in \( \xi \))

\[
\text{(LHS of 4-28)} = \sum_{j=0}^{n-1} e^{V_1(z)} \int_{\mathbb{R}} e^{-V_1(\xi) + \xi y} \hat{P}_n(y) \hat{A}^*(\xi) \hat{P}_n(x) \left( \frac{z - x}{\xi} \right) = \\
- e^{V_1(z)} \int_{\mathbb{R}} e^{-V_1(\xi) + \xi y} \hat{P}_n(y) \hat{A}^*(\xi) \hat{P}_n(x) \left( \frac{z - x}{\xi} \right) - \\
- e^{V_1(z)} \int_{\mathbb{R}} e^{-V_1(\xi) + \xi y} \hat{P}_n(y) \hat{A}^*(\xi) \hat{P}_n(x) \left( \frac{1}{z - \xi} \right) = \\
- e^{V_1(z)} \int_{\mathbb{R}} e^{-V_1(\xi) + \xi y} \hat{P}_n(y) \hat{A}^*(\xi) \hat{P}_n(x) \left( \frac{z - x}{\xi} \right) - \\
- e^{V_1(z)} \int_{\mathbb{R}} e^{-V_1(\xi) + \xi y} \hat{P}_n(y) \hat{A}^*(\xi) \hat{P}_n(x) \left( \frac{1}{z - \xi} \right) = \\
\Phi_n^{(0)}(z) \hat{A}(\xi) \hat{P}_n(x) + e^{V_1(z)} \sum_{j=0}^{n-1} \int_{\mathbb{R}} e^{-V_1(\xi) + \xi y} \hat{P}_n(y) \hat{A}^*(\xi) \hat{P}_n(x) = \\
\Phi_n^{(0)}(z) \hat{A}(\xi) \hat{P}_n(x) + e^{V_1(z)} - V_1(x) , \tag{1-9} \]

where in the identity marked \( \ast \) we have used the linearity of \( \hat{A} \) which implies the following identity

\[ \frac{\hat{A}(\xi)}{z - \xi} - \frac{\hat{A}(\xi)}{x - \xi} = \frac{\hat{A}(z)}{z - \xi} - \frac{\hat{A}(x)}{x - \xi} . \tag{1-10} \]

The second form of (a) is proved along the same lines using the principal CDI for the kernel \( \hat{K}_{11} \) (in Thm. 4.1). For the remaining CDI's we have

\[
\text{(LHS of 4-29)} = \frac{z - x}{2\pi i} \sum_{j=0}^{n-1} \int_{\Gamma_j} e^{y \phi_r(y)} \int_{\mathbb{R}} \int_{\mathbb{R}} B_2(x, \eta, \rho) e^{\eta y - z \xi + \hat{V}_2(s) - V_2(y) \frac{\hat{A}(\rho)}{x - \rho}} = \\
= \frac{1}{2\pi i} \int_{\Gamma_j} e^{y \phi_r(y)} \int_{\mathbb{R}} \int_{\mathbb{R}} B_2(x, \eta, \rho) e^{\eta y - z \xi + \hat{V}_2(s) - V_2(y) \frac{\hat{A}(\rho)}{x - \rho}} \hat{P}_n(\rho) = \\
= \frac{1}{2\pi i} \int_{\Gamma_j} e^{y \phi_r(y)} \int_{\mathbb{R}} \int_{\mathbb{R}} B_2(x, \eta, \rho) e^{\eta y - z \xi + \hat{V}_2(s) - V_2(y) \hat{A}(\rho)} \left( \frac{1}{x - \rho} - \frac{1}{z - \rho} \right) \hat{P}_n(\rho) = \\
= \Phi_n^{(j)}(z) \hat{A}(\xi) \hat{P}_n(x) - \frac{1}{2\pi i} \sum_{j=0}^{n-1} \int_{\Gamma_j} e^{y \phi_r(y)} \int_{\mathbb{R}} \int_{\mathbb{R}} B_2(x, \eta, \rho) e^{\eta y - z \xi + \hat{V}_2(s) - V_2(y) \hat{P}_n(\rho)} = \\
- \Phi_n^{(j)}(z) \hat{A}(\xi) \hat{P}_n(x) - \frac{1}{2\pi i} \sum_{j=0}^{n-1} \int_{\Gamma_j} e^{y \phi_r(y)} \int_{\mathbb{R}} \int_{\mathbb{R}} B_2(x, \eta, \rho) e^{\eta y - z \xi + \hat{V}_2(s) - V_2(y)} \hat{P}_n(\rho) , \tag{1-11} \]

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where the identity marked ★ is valid for \( n \geq d_2 \) (so that the kernel reproduces the polynomial \( B_2(x; \eta, s) \) of degree \( d_2 - 1 \)).

The proof of the second form of (b) is only marginally different in that we have to use the second form of the principal CDI for the kernel \( \hat{K}_{11} \) (in Thm. 4.1). Q.E.D.

A.3 Proof of Theorem 4.3

Let \( n - 1 \leq m \leq n + d_2 - 1 \): in the following chain of equalities all the steps are “elementary” and hence the computation is straightforward. For reader’s convenience we have tried to make annotations on the formula in order to highlight less obvious steps.

\[
\frac{\partial_x \phi_m^{(0)}}{\partial_m} = V_1(x) \phi_m^{(0)} + e^{V_1(x)} \int_{\mathbb{R}} e^{F_1(\xi)} \frac{e^{-V_1(\xi)}}{x - \xi} \phi_m(y) =
\]

\[
= -\frac{e^{V_1(x) - V_1(\xi)}}{x - \xi} \phi_m^{(0)}(\xi) + e^{V_1(x)} \int_{\mathbb{R}} \frac{\left( V_1(x) - V_1(\xi) \right) e^{-V_1(\xi) + \xi \phi_m(y)}}{x - \xi} + e^{V_1(x)} \int_{\mathbb{R}} \frac{y e^{-V_1(\xi) + \xi \phi_m(y)}}{x - \xi} =
\]

\[
= (-B + C) + \sum_{j=0}^{n-1} \phi_j^{(0)}(x) \int_{\mathbb{R}} \psi_j(\rho) \eta \phi_m(\eta) e^{\rho \eta} =
\]

\[
= (-B + C) + \sum_{j=0}^{n-1} \phi_j^{(0)}(x) \int_{\mathbb{R}} \psi_j(\rho) \eta \phi_m(\eta) e^{\rho \eta} + \sum_{j=0}^{n+d_2} \phi_j^{(0)}(x) \int_{\mathbb{R}} \psi_j(\rho) \eta \phi_m(\eta) e^{\rho \eta} =
\]

\[
= (-B + C) + \sum_{j=0}^{n-1} \phi_j^{(0)}(x) P_j m + \sum_{j=n}^{n+d_2} \phi_j^{(0)}(x) \left[ \psi_j(\rho) \frac{\phi_m(\rho)}{\rho \phi_m(\rho)} \right]_{\rho \in \mathbb{R}} +
\]

\[
= (-B + C) + \sum_{j=0}^{n+d_2} \phi_j^{(0)}(x) P_j m + \left[ \frac{\Phi_m^{(0)}(x) \hat{A}(x) \hat{\Psi}_m(\rho) + e^{V_1(x) - V_1(\xi)}}{x - \rho} \phi_m^{(0)}(\xi) \right]_{\rho \in \mathbb{R}} +
\]

\[
= (C) + \sum_{j=n}^{n+d_2} \phi_j^{(0)}(x) P_j m + \left[ \frac{\Phi_m^{(0)}(x) \hat{A}(x) \hat{\Psi}_m(\rho) + e^{V_1(x) - V_1(\xi)}}{x - \rho} \phi_m^{(0)}(\xi) \right]_{\rho \in \mathbb{R}} +
\]

\[
= \int_{\mathbb{R}} \phi_m(\eta) e^{\rho \eta} V_1(\phi) \frac{\phi_m^{(0)}(x) \hat{A}(x) \hat{\Psi}_m(\rho) + e^{V_1(x) - V_1(\xi)}}{x - \rho} + V_1(x) e^{V_1(x)} \int_{\mathbb{R}} \frac{e^{-V_1(\xi) + \xi \phi_m(y)}}{x - \xi} =
\]
\[
= \sum_{j=n}^{n+d_2} \phi^{(0)}(x) P_{jm} + \left[ \frac{\Phi^{(0)}_n(x) \hat{\Psi}_n(\rho)}{x - \rho} \phi^{(0)}(\rho) \right]_{\rho \in \mathbb{R}} + \int_{\mathbb{R}} \phi_m(\eta) e^{\eta \gamma} \frac{V'_i(\rho) - V'_i(x)}{x - \rho} \Phi^{(0)}_n(x) \hat{\Phi}(x) \hat{\Psi}_n(\rho) \\
+ V'_i(x) \int_{\mathbb{R}} \phi_m(\eta) e^{\eta \gamma} \frac{\Phi^{(0)}_n(x) \hat{\Phi}(x) \hat{\Psi}_n(\rho)}{x - \rho} + e^{\eta \gamma} (x - V'_i(\rho)) = \\
= \sum_{j=n}^{n+d_2} \phi^{(0)}(x) P_{jm} + \left[ \frac{\Phi^{(0)}_n(x) \hat{\Phi}(x) \hat{\Psi}_n(\rho)}{x - \rho} \phi^{(0)}(\rho) \right]_{\rho \in \mathbb{R}} - \Phi^{(0)}_n(x) \hat{\Phi}(x) \int_{\mathbb{R}} \hat{\Psi}_n(\rho) \phi_m(\eta) e^{\eta \gamma} \frac{V'_i(\rho) - V'_i(x)}{x - \rho} \\
+ V'_i(x) \sum_{j=0}^{n-1} \phi^{(0)}(x) \int_{\mathbb{R}} \phi_m(\eta) e^{\eta \gamma} \psi_j(\rho) = \\
= \sum_{j=n}^{n+d_2} \phi^{(0)}(x) P_{jm} + \left[ \frac{\Phi^{(0)}_n(x) \hat{\Phi}(x) \hat{\Psi}_n(\rho)}{x - \rho} \phi^{(0)}(\rho) \right]_{\rho \in \mathbb{R}} - \Phi^{(0)}_n(x) \hat{\Phi}(x) \int_{\mathbb{R}} \hat{\Psi}_n(\rho) \phi_m(\eta) e^{\eta \gamma} \frac{V'_i(\rho) - V'_i(x)}{x - \rho} \\
+ V'_i(x) \phi^{(0)}(x) \delta_{m,n-1}
\]

We note that in this last expression we have \( \hat{\psi}_m^{(0)}(x) \) expressed purely in terms of \( \phi^{(0)}(x) \) for \( \ell = n - 1, \ldots, n + d_2 \), the value \( \ell = n + d_2 \) entering only in the first expression. Given that \( \phi^{(0)}(x) \) satisfies the same multiplicative recurrence relations as the Fourier–Laplace transforms for \( n \geq 1 \), we can re-express \( \phi^{(0)}_{n+d_2} \) in terms of the elements of the window \( \Phi^{(0)}_n(x) \), obtaining the result.

The computation for the Fourier-Laplace transforms gives also the same differential equation, indeed

\[
\hat{\psi}_m^{(r)}(x) = \int_{\mathbb{R}} e^{\rho x} \Phi_m(\rho) = \sum_{j=0}^{n} \phi^{(r)}(x) \int_{\mathbb{R}} e^{\rho \gamma} \phi_m(\rho) \psi_j(\rho) = \\
= \sum_{j=n}^{n+d_2} \phi^{(r)}(x) P_{jm} + \sum_{j=0}^{n-1} \phi^{(r)}(x) \int_{\mathbb{R}} e^{\rho \gamma} \Phi_m(\rho) \psi_j(\rho) = \\
+ \sum_{j=0}^{n-1} \phi^{(r)}(x) \left[ \psi_j(\rho) \phi^{(0)}_m(\rho) \right]_{\rho \in \mathbb{R}} = \\
= \sum_{j=n}^{n+d_2} \phi^{(r)}(x) P_{jm} + \Phi^{(r)}_n(x) \hat{\Phi}(x) \int_{\mathbb{R}} e^{\rho \gamma} \Phi_m(\rho) \frac{V'_i(\rho) - V'_i(x)}{x - \rho} \\
+ \Phi^{(r)}_n(x) \hat{\Phi}(x) \left[ \frac{\hat{\Psi}_n(\rho) \phi^{(0)}_m(\rho)}{x - \rho} \right]_{\rho \in \mathbb{R}} = \\
= \sum_{j=n}^{n+d_2} \phi^{(r)}(x) P_{jm} + \Phi^{(r)}_n(x) \hat{\Phi}(x) \int_{\mathbb{R}} \hat{\Psi}_n(\rho) e^{\rho \gamma} \Phi_m(\rho) \frac{V'_i(\rho) - V'_i(x)}{x - \rho} \\
+ V'_i(x) \Phi^{(r)}_n(x) \hat{\Phi}(x) \int_{\mathbb{R}} \hat{\Psi}_n(\rho) e^{\rho \gamma} \Phi_m(\rho) \frac{V'_i(\rho) - V'_i(x)}{x - \rho} \\
+ V'_i(x) \sum_{j=0}^{n-1} \phi^{(r)}(x) \int_{\mathbb{R}} \psi_j(\rho) e^{\rho \gamma} \Phi_m(\rho) + \Phi^{(r)}_n(x) \hat{\Phi}(x) \left[ \frac{\hat{\Psi}_n(\rho) \phi^{(0)}_m(\rho)}{x - \rho} \right]_{\rho \in \mathbb{R}} = \\

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\[= \sum_{j=n}^{n+d_2} \phi^{(r)}_{n-j}(x) P_{jm} + \Phi^{(r)}_{n-j}(x) \hat{A}(x) \int_{\mathbb{R}} \tilde{\Psi}_n(\rho) e^{\eta \rho} \phi_m(\eta) \frac{V'_i(\rho) - V'_i(x)}{x - \rho} + \]
\[+ V'_i(x) \delta_{m,n-1} \phi^{(r)}_{n-j}(x) + \Phi^{(r)}_{n-j}(x) \hat{A}(x) \frac{\tilde{\Psi}_n(\rho) \phi^{(r)}_m(\rho)}{x - \rho} \right]_{\rho \in \mathbb{C}_{\infty}}. \quad (1-13)\]

The coefficients of these expressions in terms of \(\phi^{(r)}_{n-1}, \ldots, \phi^{(r)}_{n+d_2-1}\) are precisely the same as for the previous computation, hence completing the proof. Q.E.D.

### A.4 Proof of Theorem 4.4

Let \(n - d_2 \leq m \leq n\) and let us compute

\[\hat{\psi}_m(x) = e^{-V_1(x)} (\hat{\psi}_m(x)) \hat{\theta}_m(x) = -V'_i(x) \hat{\psi}_m(x) + \sum_{j=0}^{n-1} \hat{\psi}_j(x) \int_{\mathbb{R}} \tilde{\Psi}_m(\xi) e^{\xi y} - V'_i(x) \hat{\psi}_j(y) = \]
\[- V'_i(x) \hat{\psi}_m(x) + \sum_{j=0}^{n-1} \hat{\psi}_j(x) \left[ \tilde{\Psi}_m(\xi) \hat{\theta}_j(\xi) \right]_{\xi \in \mathbb{C}_{\infty}} - \sum_{j=0}^{n-1} \hat{\psi}_j(x) \int_{\mathbb{R}} \tilde{\Psi}_m(\xi) e^{\xi y} \hat{\theta}_j(y) = \]
\[- V'_i(x) \hat{\psi}_m(x) + \sum_{j=0}^{n-1} \hat{\psi}_j(x) \int_{\mathbb{R}} \tilde{\Psi}_m(\xi) e^{\xi y} \hat{\theta}_j(y) = \]
\[- V'_i(x) \delta_{mn} \hat{\psi}_n(x) + \sum_{j=0}^{n-1} \hat{\psi}_j(x) \int_{\mathbb{R}} \tilde{\Psi}_m(\xi) e^{\xi y} \hat{\theta}_j(y) = \]
\[- V'_i(x) \delta_{mn} \hat{\psi}_n(x) + \sum_{j=0}^{n-1} \hat{\psi}_j(x) \int_{\mathbb{R}} \tilde{\Psi}_m(\xi) e^{\xi y} (V'_i(\rho) - V'_i(x)) \hat{\theta}_j(y) - \sum_{j=0}^{n-1} \hat{\psi}_j(x) \hat{P}_mj - \]
\[- V'_i(x) \delta_{mn} \hat{\psi}_n(x) + \sum_{j=0}^{n-1} \hat{\psi}_j(x) \int_{\mathbb{R}} \tilde{\Psi}_m(\xi) e^{\xi y} \hat{P}_mj. \quad (1-14)\]

The last term contains \(\hat{\psi}_{n-d_2-1}\) for \(m = n - d_2\) which is "outside" of the window of the quasipolynomials. Using the recurrence relations and re-expressing it in terms of elements in the window (using the ladder matrices) we obtain the formula.

For completeness one should also consider the other columns of the fundamental system \(\tilde{\Psi}_n\) and show that they satisfy the same differential relation as the quasipolynomials. Let \(n - d_2 \leq m \leq n\), then

\[\hat{\psi}_m(x) = \frac{1}{2i\pi} \int_{\mathbb{C}_{\infty}} \int_{\mathbb{R}} B_2(x; y, s) e^{\xi y - x s + \xi y - V_2(s) - V_2(y)} \hat{\psi}_m(\xi) \frac{1}{x - \xi}\]

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\[- \frac{1}{2\pi} \int_{\gamma} \int_{-\infty}^{\infty} e^{ixy - xs} \psi_2(s) - V_z(y) \hat{\psi}_m(\xi) (\partial_x - s) \frac{B_2(x; y, s)}{x - \xi} \]

\[- \frac{1}{2\pi} \int_{\gamma} \int_{-\infty}^{\infty} e^{ixy - xs} \hat{\psi}_2(s) - V_z(y) \hat{\psi}_m(\xi) \left[ \frac{B_2(y) - B_2(s)}{(y - s)(x - \xi)} - \frac{B_2(x; y, s)}{x - \xi} \right] \]

\[- \frac{1}{2\pi} \int_{\gamma} \int_{-\infty}^{\infty} \hat{\psi}_m(\xi) e^{ixy - xs} \frac{\hat{\psi}_m(\xi)}{x - \xi} \]
\[ + V_i'(x) \left( 1 - \delta_{m,n} \psi_m^{(r)}(x) \right) \right) - \sum_{j=m-1}^{n-1} \hat{P}_{mj} \hat{\psi}_j^{(r)}(x) = \\
= \frac{\hat{\psi}_m(\xi) \hat{\Phi}_n^{(m)}(\xi)}{\xi - x} \hat{A}(x) \hat{\Psi}_n(x) + \int_x^{\xi} \frac{V'_i(\xi) - V'_i(x)}{\xi - x} e^{\xi y} \hat{\psi}_m(\xi) \hat{\Phi}_n(y) \hat{A}(x) \hat{\Psi}_n(x) \\
- \delta_{mn} V_i^2(x) \psi_m^{(r)}(x) - \sum_{j=0}^{n-1} \hat{\psi}_j^{(r)}(x) \int_x^{\xi} \hat{\psi}_m(\xi) y \hat{\phi}_j(y) e^{\xi y} \]

This is the same expression as for the quasipolynomials: since the auxiliary wave functions \( \hat{\psi}_j^{(r)}(x) \) satisfy the same multiplicative recurrence relation (for \( n \) large enough) as the quasipolynomials, re-expressing \( \hat{\psi}_n^{(r)}(x) \) in terms of the elements of the window yields the same differential equation. Q.E.D.

### A.5 Proof of Theorem 5.1

For brevity we denote \( \hat{A}_n(x) \) simply by \( \hat{A}(x) \) during this proof. Since the rows (columns) of \( \Phi_n \) (\( \hat{\Phi}_n \)) are of two types, we need to carry out four types of computations

\[
\text{(a)} = \Phi_n^{(0)}(x) \hat{A}_n(x) \hat{\Phi}_n^{(0)}(x), \quad \text{(b)} = \Phi_n^{(0)}(x) \hat{A}_n(x) \hat{\Phi}_n^{(j)}(x), \quad j = 1 \ldots d_2 \\
\text{(c)} = \Phi_n^{(j)}(x) \hat{A}_n(x) \hat{\Phi}_n^{(0)}(x), \quad j = 1 \ldots d_2 \quad \text{(d)} = \Phi_n^{(j)}(x) \hat{A}_n(x) \hat{\Phi}_n^{(m)}(x), \quad \ell, m = 1 \ldots d_2 
\]

It follows trivially from (5-3) that \( \text{(c)} = 0 \) (set \( x = x' \) in the LHS). For \( \text{(a)} \) we have

\[
\text{(a)} = e^{V_i(x)} \int_x^{\xi} \Phi_n^{(0)}(\xi) e^{-V_i(\xi) + \xi \xi} \hat{A}(x) \hat{\Phi}_n(x) = \\
= e^{V_i(x)} \int_x^{\xi} d\xi e^{-V_i(\xi) + \xi \xi} \sum_{j=0}^{n-1} \hat{\phi}_j(\xi) \hat{\psi}_j(x) = e^{V_i(x)} \int_x^{\xi} d\xi e^{-V_i(\xi) + \xi \xi} \hat{\phi}_0(\xi) \hat{\psi}_0(x) = 1
\]

where we have used that \( \hat{\phi}_j(\xi), \ j \geq 1 \) are orthogonal to \( p(\xi) \equiv 1 \). Note also that we had to use the CDI in the form (5-2). Then we have to compute for \( 1 \leq \ell, m \leq d_2 \) (we suppress explicit reference to the variables of integration because there is no possibility of ambiguity)

\[
\text{(d)} = \frac{1}{2 \pi i} \Phi_n^{(\ell)}(x) \int_x^{\xi} \int_{F_m} B_2(x; \eta, s) \frac{\hat{A}(x) \hat{\Phi}_n(\xi)}{x - \xi} e^{\xi \eta - x \eta - V_2(\eta) + V_2(s)} = \\
= \frac{1}{2 \pi i} \sum_{j=0}^{n-1} \hat{\phi}_j(\xi) \int_x^{\xi} \int_{F_m} B_2(x; \eta, s) \hat{\psi}_j(\xi) e^{\xi \eta - x \eta - V_2(\eta) + V_2(s)} = \\
= \frac{1}{2 \pi i} \sum_{j=0}^{n-1} \int_{F_m} ds \int_x^{\xi} d\phi \hat{\phi}_j(\eta) e^{\xi \eta - x \eta - V_2(\eta) + V_2(s)} = \\
= \frac{1}{2 \pi i} \int_{F_m} ds \int_x^{\xi} d\phi B_2(x; y, s) e^{x(y - s) - V_2(y) + V_2(s)} = \\
- B_2(\Gamma_{y, \ell}, \Gamma_{y, m}) - \delta_{\ell m},
\]
where in the step marked with a star we have used that for the polynomial of $\eta$ $P(\eta) := B_2(x; \eta, s)$ is reproduced by the kernel

$$P(y) = \sum_{j=0}^{n-1} s_j(y) \int_x d\eta d\xi \psi_j(\xi) e^{-V_2(\eta)+\xi y} P(\eta)$$

(1-23)

provided that $n - 1 \geq \deg P = d_2 - 1$. Note also that in this latter computation we are forced to use the other form of the CDI (5-3). Finally we need to compute (b), which involves quintuple integrals

$$(b) = \frac{e^{V_1(x)}}{2\pi i} \int_x \Phi_n(\rho) \int_x e^{-V_1(\xi) + \rho \xi} \int_x B_2(x; \eta, s) \frac{\hat{A}(x) \hat{\Psi}_n(\xi)}{x - \xi} e^{\xi \eta - x s - V_2(\eta) + V_2(s)} =$$

(1-24)

$$= \frac{e^{V_1(x)}}{2\pi i} \int_x \Phi_n(\rho) \int_x e^{-V_1(\xi) + \rho \xi} \int_x B_2(x; \eta, s) \frac{\hat{A}(\xi) \hat{\Psi}_n(\xi)}{x - \xi} e^{\xi \eta - x s - V_2(\eta) + V_2(s)} +$$

$$+ \frac{e^{V_1(x)}}{2\pi i} \int_x \Phi_n(\rho)[\hat{L}, \hat{p}_n] e^{-V_1(\xi) + \rho \xi} \int_x B_2(x; \eta, s) \frac{\hat{\Psi}_n(\xi)}{x - \xi} e^{\xi \eta - x s - V_2(\eta) + V_2(s)} =$$

$$= \frac{e^{V_1(x)}}{2\pi i} \sum_{j=0}^{n-1} \int_x \int_x \int_x ds e^{-V_1(\xi) + \xi p + \xi q - x s - V_2(\eta) + V_2(s)} \phi_j(\rho) \psi_j(\xi) B_2(x; \eta, s) \frac{\zeta - \xi}{(x - \xi)(x - \zeta)} =$$

$$= \frac{e^{V_1(x)}}{2\pi i} \sum_{j=0}^{n-1} \int_x \int_x \int_x ds e^{-V_1(\xi) + \xi q - x s - V_2(\eta) + V_2(s)} \phi_j(\rho) \psi_j(\xi) B_2(x; \eta, s) \frac{1}{x - \xi} =$$

$$= \frac{e^{V_1(x)}}{2\pi i} \sum_{j=0}^{n-1} \int_x \int_x \int_x ds e^{-V_1(\xi) + \xi q - x s - V_2(\eta) + V_2(s)} \phi_j(\rho) \psi_j(\xi) B_2(x; \eta, s) \frac{1}{x - \xi} +$$

$$= \frac{e^{V_1(x)}}{2\pi i} \int_x \int_x \int_x ds e^{-V_1(\xi) + \xi q - x s - V_2(\eta) + V_2(s)} B_2(x; \rho, s) \frac{1}{x - \xi} =$$

$$= \frac{e^{V_1(x)}}{2\pi i} \int_x \int_x \int_x ds e^{-V_1(\xi) + \xi q - x s - V_2(\eta) + V_2(s)} B_2(x; \eta, s) \frac{1}{x - \xi} =$$

(1-25)

Once more, we are forced to use the CDI in the form (5-3). This concludes the proof. Q.E.D.

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