HORO-CONVEX HYPERSURFACES WITH PRESCRIBED SHIFTED
GAUSS CURVATURES IN $\mathbb{H}^{n+1}$

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Abstract. In this paper, we consider prescribed shifted Gauss curvature equations for horo-convex hypersurfaces in $\mathbb{H}^{n+1}$. Under some sufficient condition, we obtain an existence result by the standard degree theory based on the a prior estimates for the solutions to the equations. Different from the prescribed Weingarten curvature problem in space forms, we do not impose a sign condition for radial derivative of the functions in the right-hand side of the equations to prove the existence due to the horo-convexity of hypersurfaces in $\mathbb{H}^{n+1}$.

1. Introduction

Different from hypersurfaces in $\mathbb{R}^{n+1}$, there are four different kinds of convexity for hypersurfaces in $\mathbb{H}^{n+1}$ [1, 2]. One of them is the horo-convexity which is defined as follows.

Definition 1.1. A smooth hypersurface $M \subset \mathbb{H}^{n+1}$ is called horo-convex if $\kappa_i(p) > 1$ for all $p \in M$ and $1 \leq i \leq n$, where $\kappa = (\kappa_1, ..., \kappa_n)$ are the principal curvatures of $M \subset \mathbb{H}^{n+1}$ which are defined by eigenvalues of the Weingarten matrix $\mathcal{W} = (h^i_j)$.

Geometrically, a hypersurface $M \subset \mathbb{H}^{n+1}$ is called horo-convex if and only if it is convex by horospheres, where the horospheres in hyperbolic space are hypersurfaces with constant principal curvatures equal to 1 everywhere. In [13], the authors suggest that horospheres can be naturally regarded in many ways as hyperplanes in the hyperbolic space $\mathbb{H}^{n+1}$. This fact implies the similarity between the horo-convexity in $\mathbb{H}^{n+1}$ and the convexity in $\mathbb{R}^{n+1}$ from geometric aspect.

Furthermore, this interesting formal similarities, or the general similarities between the geometry of horo-convex regions in $\mathbb{H}^{n+1}$ (that is, regions which are given by the

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intersection of a collection of horo-balls), and that of convex Euclidean bodies have been deeply explored in \cite{13} and \cite{5}. In particular, Andrews-Chen-Wei \cite{5} introduce the shifted Weingarten matrix

$$\mathcal{W} - I$$

for the hypersurface $M \subset \mathbb{H}^{n+1}$ along the line that the convexity of hypersurfaces in $\mathbb{R}^{n+1}$ can be describes by the positive definite of their Weingarten matrix. Clearly, $M \subset \mathbb{H}^{n+1}$ is horo-convex if and only if the shifted Weingarten matrix $\mathcal{W} - I$ is positive definite. Moreover, they \cite{5} define the shifted principal curvatures by

$$(\lambda_1, ..., \lambda_n) := (\kappa_1 - 1, ..., \kappa_n - 1),$$

which are eigenvalues of the shifted Weingarten matrix $\mathcal{W} - I$. Thus, similar to $k$-mean curvature, we can define $k$-th shifted mean curvature for $1 \leq k \leq n$ by

$$\sigma_k(\kappa - 1) := \sigma_k(\kappa_1 - 1, ..., \kappa_n - 1) = \sum_{i_1 < i_2 < ... < i_k} (\kappa_{i_1} - 1) \cdots (\kappa_{i_k} - 1).$$

Since

$$\sigma_k(\kappa - 1) = \sum_{i=0}^{k} (-1)^{k-i} C_{n-i}^{n-k} \sigma_i(\kappa),$$

the $k$-th shifted mean curvature can also be regarded as the linear combination of the $i$-mean curvature $\sigma_i(\kappa)$ of $M$ for $1 \leq i \leq k$, where $C_{n-i}^{n-k} = \frac{(n-i)!}{(n-k)!(k-i)!}$.

In \cite{5, 23}, curvature flows for horo-convex hypersurfaces in hyperbolic space with speed given by the function $f$ of the shifted principal curvatures $\lambda_i$ have been extensively studied by Andrews-Chen-Wei and Hu-Li-Wei respectively. As applications, they prove some new geometric inequalities involving the weighted integral of $k$-th shifted mean curvature for horo-convex hypersurfaces. Later, Wang-Wei-Zhou \cite{41} study inverse shifted curvature flow in hyperbolic space.

The shifted curvatures can also be well be interpreted in the work \cite{13} by Espinar-Gálvez-Mirain on the extension of the Christoffel problem \cite{11, 14} to space forms. They show that the Christoffel problem can be naturally formulated in the context of hypersurfaces $M^n \subset \mathbb{H}^{n+1}$ by introducing the hyperbolic curvature radii of $M^n \subset \mathbb{H}^{n+1}$, defined as

$$\mathcal{R}_i := \frac{1}{|\kappa_i - 1|}.$$
Clearly, the hyperbolic curvature radii of the horo-convex hypersurface $M^n \subset \mathbb{H}^{n+1}$ are in fact the shifted principle curvature radii. Thus, the shifted Gauss curvature can be interpreted as

$$\sigma_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{R_1 \cdots R_n}$$

in a very simple way to the Gauss curvature of hypersurfaces in Euclidean.

All the work above motivate us to study furtherly the geometry and analysis of horo-convex hypersurfaces with the shifted curvatures in $\mathbb{H}^{n+1}$. In this paper, we consider the the problem of prescribed shifted curvatures for horo-convex hypersurfaces in $\mathbb{H}^{n+1}$.

Let $M$ be a horo-convex hypersurface in $\mathbb{H}^{n+1}$. We can choose a point $o$ inside $M$ such that $M$ is star-shaped with respect to $o$, thus $M$ can be parametrized as a radial graph over $S^n$. So we consider geodesic polar coordinates centered at $o$, the hyperbolic space $\mathbb{H}^{n+1}$ can be regarded as a warped product space $[0, +\infty) \times S^n$ with metric

$$\overline{g} = dr^2 + sinh^2 r g_{S^n},$$

where $g_{S^n}$ is the standard sphere metric. Thus, $M$ can be represented by

$$M = \{ (x, r(x)) : x \in S^n \}.$$

Let $h_{ij}$ be the second fundamental form of $M$ and $g$ be the induced metric of $M$. Denote by $\tilde{h}_{ij} = h_{ij} - g_{ij}$. Thus

$$\det(\tilde{h}_{ij}(X)) = (\kappa_1 - 1)(\kappa_2 - 1) \cdots (\kappa_n - 1)$$

is the shifted Gauss curvature of $M$, $\kappa(X) = (\kappa_1(X), \ldots, \kappa_n(X))$ are the principle curvatures of hypersurface $M$ at $X$. In this paper, we study the problem of prescribed shifted Gauss curvature

(1.2) \quad \det(\tilde{h}_{ij}(X)) = f(x, r(x)),

on a horo-convex hypersurface $M \subset \mathbb{H}^{n+1}$, where $X = (x, r(x)) \in M$ and $f$ is given smooth functions in $S^n \times [0, +\infty)$. Clearly, the problem (1.2) can also be regarded as the problem of the prescribed linear combination of the Weingarten curvatures in view of (1.1).

We mainly get the following theorem.
Theorem 1.1. Let $n \geq 3$ and $f(x,r) \in C^\infty(S^n \times [0, +\infty))$ be a positive function, assume that

$$\coth r - 1 \geq f(x,r) \quad \text{for} \quad r \geq r_2,$$

(1.3)

$$\coth r - 1 \leq f(x,r) \quad \text{for} \quad r \leq r_1.$$  

Then there exists at least a smooth horo-convex, closed hypersurface $M$ in $\{(x, r) \in \mathbb{H}^{n+1} : r_1 \leq r \leq r_2, x \in S^n\}$ satisfies equation (1.2).

Remark 1.2. For the prescribed Weingarten curvatures problem, it is need usually to impose a sign condition for radial derivative of $f$ in order to derive a prior gradient estimate. However, we do not need such condition in Theorem 1.1 due to the horo-convexity of the hypersurface in $\mathbb{H}^{n+1}$.

The prescribed Weingarten curvature equation $\sigma_k(\kappa(X)) = f(X)$ has been widely studied in the past two decades. Such results were obtained for case of prescribed mean curvature by Bakelman-Kantor [6, 7] and Treibergs-Wei [40]. For the case of prescribed Gaussian curvature by Oliker [34]. For general Weingarten curvatures by Aleksandrov [3], Firey [14], Caffarelli-Nirenberg-Spruck [9] for a general class of fully nonlinear operators $F$, including $F = \sigma_k$ and $F = \frac{\sigma_k}{\sigma_l}$. Some results have been obtained by Li-Oliker [32] on unit sphere, Barbosa-de Lira-Oliker [8] on space forms, Jin-Li [26] on hyperbolic space, Andrade-Barbosa-de Lira [4] on warped product manifolds, Li-Sheng [30] for Riemannain manifold equipped with a global normal Gaussian coordinate system.

For prescribed curvature problems in the case $f$ also depends on the normal vector field $\nu$ along the hypersurface $M$, see Caffarelli-Nirenberg-Spruck [9], Ivochkina [24, 25], Guan-Li-Li [21], Guan-Lin-Ma [20], Guan-Guan [18], Guan-Ren-Wang [22], Li-Ren-Wang [29], Chen-Li-Wang [10, 36] and Ren-Wang [36, 37].

The organization of the paper is as follows. In Sect. 2 we start with some preliminaries. $C^0$, $C^1$ and $C^2$ estimates are given in Sect. 3. In Sect. 4 we prove theorem 1.1.

2. Preliminaries

2.1. Setting and General facts. For later convenience, we first state our conventions on Riemann Curvature tensor and derivative notation. Let $M$ be a smooth manifold
and \( g \) be a Riemannian metric on \( M \) with Levi-Civita connection \( \nabla \). For a \((s, r)\)-tensor field \( \alpha \) on \( M \), its covariant derivative \( \nabla \alpha \) is a \((s, r + 1)\)-tensor field given by

\[
\nabla \alpha(Y^1, \ldots, Y^s, X_1, \ldots, X_r, X) \\
= \nabla_X \alpha(Y^1, \ldots, Y^s, X_1, \ldots, X_r) \\
= X(\alpha(Y^1, \ldots, Y^s, X_1, \ldots, X_r)) - \alpha(\nabla_X Y^1, \ldots, Y^s, X_1, \ldots, X_r) \\
- \ldots - \alpha(Y^1, \ldots, Y^s, X_1, \ldots, \nabla_X X_r).
\]

The coordinate expression of which is denoted by

\[
\nabla \alpha = (\alpha_{l_1 \cdots l_s}^{k_1 \cdots k_r, k_{r+1}}).
\]

We can continue to define the second covariant derivative of \( \alpha \) as follows:

\[
\nabla^2 \alpha(Y^1, \ldots, Y^s, X_1, \ldots, X_r, X, Y) = (\nabla_Y (\nabla \alpha))(Y^1, \ldots, Y^s, X_1, \ldots, X_r, X).
\]

The coordinate expression of which is denoted by

\[
\nabla^2 \alpha = (\alpha_{l_1 \cdots l_s}^{k_1 \cdots k_r, k_{r+1}, k_{r+2}}).
\]

Similarly, we can also define the higher order covariant derivative of \( \alpha \):

\[
\nabla^3 \alpha = \nabla(\nabla^2 \alpha), \nabla^4 \alpha = \nabla(\nabla^3 \alpha), \ldots,
\]

and so on. For simplicity, the coordinate expression of the covariant differentiation will usually be denoted by indices without semicolons, e.g.,

\[
u_i, \ u_{ij} \text{ or } u_{ijk}
\]

for a function \( u : M \to \mathbb{R} \).

Let \( X : M \to \mathbb{H}^{n+1} \) be an immersed hypersurface with the standard metric \( \overline{g} \) and Levi-Civita connection \( \overline{\nabla} \). Then \( M \) can get the induced metric \( g \) and the Levi-Civita connection \( \nabla \) of \( g \). Pick a local coordinate chart \( \{x^i\}_{i=1}^n \) on \( M \), then \( e_i = \frac{\partial X}{\partial x_i} \) form a local frame field of \( X(M) \). Let \( \nu \) be a given unit normal and \( h_{ij} \) be the second fundamental form \( A \) of the hypersurface with respect to \( \nu \), that is

\[
h_{ij} = -\langle \nabla e_i, \nabla e_j, X, \nu \rangle_{\overline{g}}.
\]

Recalling the Codazzi equation

\[
(2.1) \quad \nabla_k h_{ij} = \nabla_j h_{ik}
\]

and Simons’ identity (see also [5])

\[
(2.2) \quad \nabla (i \nabla_j) h_{kl} = \nabla (k \nabla_l) h_{ij} + h_{ij} h_{km} h_{l}^{m} - h_{kl} h_{i}^{m} h_{mj} - g_{ij} h_{kl} + g_{kl} h_{ij},
\]
where the brackets denote symmetrisation.

2.2. Star-shaped hypersurfaces in $\mathbb{H}^{n+1}$. Let $M$ be a closed hypersurface containing the origin in $\mathbb{H}^{n+1} = [0, +\infty) \times S^n$ with metric

$$\bar{g} = dr^2 + \sinh^2 r g_{S^n},$$

where $g_{S^n}$ is the standard sphere metric. Then, we have the following lemma (See [23]).

**Lemma 2.1.** Assume $M$ can be parametrized as a radial graph over $S^n$

$$M = \{(x, r(x)) : x \in S^n\}.$$ Let $\{x^1, ..., x^n\}$ be a local coordinate on $S^n$, $\{\partial_1, ..., \partial_n\}$ be the corresponding tangent vector filed and $\partial_r$ be the radial vector field in $\mathbb{R}^{n+1}$. $D_i \varphi = D_{\partial_i} \varphi$, $D_i D_j \varphi = D^2 \varphi(\partial_i, \partial_j)$ denote the covariant derivatives of $\varphi$ with respect to the round metric $\sigma$ of $S^n$. Then, the tangential vector takes the form

$$e_i = \partial_i + r_i \partial_r.$$

The induced metric on $M$ has

$$g_{ij} = D_i r D_j r + \sinh^2 r \sigma_{ij}$$

We also have the outward unit normal vector of $\Sigma$

$$\nu = \frac{1}{v} \left( \partial_r - \sinh^{-2} r D^j r \partial_j \right),$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and $D^i r = \sigma^{ij} D_j r$. Define a new function $u : S^n \to \mathbb{R}$ by

$$u(\theta) = \int_c^{r(\theta)} \frac{1}{\sinh s} ds.$$ (2.3)

Then the induced metric on $M$ has

$$g_{ij} = \sinh^2 r (D_i u D_j u + \sigma_{ij})$$

with the inverse

$$g^{ij} = \sinh^{-2} r (\sigma^{ij} - \frac{D^i u D^j u}{v^2}),$$

where

$$v^2 = 1 + \sigma^{ij} D_i u D_j u = 1 + | Du |^2.$$ (2.4)
Let $h_{ij}$ be the second fundamental form of $M \subset \mathbb{H}^{n+1}$ in term of the tangential vector fields $\{e_i, ..., e_n\}$. Then,

$$h_{ij} = \frac{\sinh r}{v} \left( \cosh r (D_i u D_j u + \sigma_{ij}) - D_i D_j u \right)$$

and

$$(2.5) \quad h^i_j = \frac{1}{v \sinh r} (\cosh r \delta^i_j - \tilde{g}^{ik} D_k D_j u),$$

where $\tilde{g}^{ij} = \sigma^{ij} - \frac{D^i u D^j u}{v^2}$.

Consider the function

$$\Lambda(r) = \int_0^r \sinh s ds$$

and the vector field

$$V = \sinh r \partial_r,$$

which is a conformal killing field in $\mathbb{H}^{n+1}$. Then, we need the following lemma for $\Lambda$ and the support function $\langle V, \nu \rangle$ of the hypersurface $M \subset \mathbb{H}^{n+1}$.

**Lemma 2.2.**

$$\nabla_i \nabla_j \Lambda = \cosh g_{ij} - h_{ij} \langle V, \nu \rangle$$

and

$$\langle V, \nu \rangle_{ij} = \cosh h_{ij} + \sinh r \tilde{g}_{pq} h_{ijp} \nabla_q r - h_{im} h^m_j \langle V, \nu \rangle.$$ 

See Lemma 2.2 and Lemma 2.6 in [17] or [26] for the proof.

3. **The A Priori Estimates**

In order to prove Theorem 1.1, we use the degree theory for nonlinear elliptic equation developed in [31] and the proof here is similar to [32]. First, we consider the family of equations for $0 \leq t \leq 1$

$$det^{\pm}(\tilde{h}_{ij}) = tf(x, r) + (1 - t) \varphi(r)(\coth r - 1)$$

and $\varphi$ is a positive function which satisfies the following conditions:

(a) $\varphi(r) > 0$;

(b) $\varphi(r) > 1$ for $r \leq r_1$;
(c) $\varphi(r) < 1$ for $r \geq r_2$;
(d) $\varphi'(r) < 0$.

3.1. $C^0$ Estimates. Now, we can prove the following proposition which asserts that the solution of the equation (1.2) have uniform $C^0$ bound.

**Proposition 3.1.** Under the assumptions (1.3) and (1.4) mentioned in Theorem 1.1, if the horo-convex hypersurface $M = \{(x, r(x)) : x \in S^n\} \subset H^{n+1}$ satisfies the equation (3.1) for a given $t \in [0, 1]$, then
\[ r_1 < r(x) < r_2, \quad \forall \ x \in S^n. \]

**Proof.** Assume $r(x)$ attains its maximum at $x_0 \in S^n$ and $r(x_0) \geq r_2$, then recalling (2.5)
\[ h^i_j = \frac{1}{v \sinh r}(\cosh r \delta^i_j - \tilde{g}^{ik} D_k D_j u), \]
which implies together with the fact the matrix $D_i D_j u$ is non-positive definite at $x_0$
\[ h^i_j(x_0) - \delta^i_j = \frac{1}{\sinh r}(\cosh r \delta^i_j - D^i D_j u) - \delta^i_j \geq (\coth r - 1) \delta^i_j. \]
Thus, we have at $x_0$
\[ \det \tilde{h}_{ij}(x_0) \geq (\coth r - 1). \]
So, we arrive at $x_0$
\[ tf(x, r) + (1 - t) \varphi(r) (\coth r - 1) \geq (\coth r - 1). \]
Thus, we obtain at $x_0$
\[ f(x, r) > (\coth r - 1), \]
which is in contradiction with (1.3). Thus, we have $r(x) < r_2$ for $x \in S^n$. Similarly, we can obtain $r(x) > r_1$ for $x \in S^n$. $\square$

Now, we prove the following uniqueness result.

**Proposition 3.2.** For $t = 0$, there exists an unique horo-convex solution of the equation (3.1), namely $M = \{(x, r(x)) \in H^{n+1} : r(x) = r_0\}$, where $r_0$ satisfies $\varphi(r_0) = 1$.

**Proof.** Let $M$ be a solution of (3.1), for $t = 0$
\[ \det \tilde{h}_{ij} - \varphi(r) (\coth r - 1) = 0. \]
Assume $r(x)$ attains its maximum $r_{\text{max}}$ at $x_0 \in \mathbb{S}^n$, then we have at $x_0$

$$h^i_j = \frac{1}{\sinh r} (\cosh r \delta^i_j - D^i_i u),$$

which implies together with the fact the matrix $D_i D_j u$ is non-positive definite at $x_0$

$$\det(\tilde{h}_{ij}) \geq (\coth r - 1)^n.$$

Thus, we have by the equation (3.1)

$$\varphi(r_{\text{max}}) \geq 1.$$

Similarly,

$$\varphi(r_{\text{min}}) \leq 1.$$

Thus, since $\varphi$ is a decreasing function, we obtain

$$\varphi(r_{\text{min}}) = \varphi(r_{\text{max}}) = 1.$$

We conclude

$$r(x) = r_0$$

for any $(x, r(x)) \in M$, where $r_0$ is the unique solution of $\varphi(r_0) = 1$. \hfill \Box

3.2. $C^1$ Estimates. For the prescribed Weingarten curvature problem, it is need to impose a sign condition for radial derivative of $f$ to derived the a prior gradient estimate. However, such condition is not need here, since the horo-convexity of the hypersurface in $\mathbb{H}^{n+1}$ automatically yields the bounded of the gradient of the radial function.

**Proposition 3.3.** If the horo-convex hypersurface $M = \{(x, r(x)) : x \in \mathbb{S}^n\} \subset \mathbb{H}^{n+1}$ satisfies (3.1), then there exists a constant $C$ depending on the minimum and maximum values of $r$ such that

$$|Dr(x)| \leq C, \quad \forall \ x \in \mathbb{S}^n.$$

**Proof.** Assume $x_0$ is the maximum value point of $|Du|^2(x)$. Thus, we arrive at $x_0$,

$$\sigma^{kl} D_l u D_k D_j u = 0$$

for all $1 \leq j \leq n$, which implies at $x_0$

$$D_i u h^i_k g^{jk} D_j u = \frac{D_i u}{v \sinh r} (\cosh r \delta^i_k - \tilde{g}^{il} D_l D_k u) \sinh^{-2} r \frac{D^k u}{v^2} = \frac{1}{v^3 \sinh^3 r} |Du|^2.$$
in view of
\[ g^{jk} D_j u = \sinh^{-2} r \left( \sigma^{jk} - \frac{D_j u D_k u}{v^2} \right) D_j u = \sinh^{-2} r \frac{D_k u}{v^2}. \]
Since \( M \) is horo-convex, so \( h^i_j > \delta^i_j \). Thus, we obtain at \( x_0 \)
\[ \frac{1}{v^3} \frac{\coth r}{\sinh^3 r} |Du|^2 > \sinh^{-2} r \frac{|Du|^2}{v^2}, \]
which implies
\[ \frac{\coth r}{\sinh r} > \sqrt{1 + |Du|^2}. \]
So, our proof is completed. \( \square \)

3.3. \( C^2 \) Estimates. For convenience, we denote by
\[ G(\tilde{h}_{ij}) = \det^{\frac{1}{2}}(\tilde{h}_{ij}), \quad \tilde{f}(x, r) = tf(x, r) + (1 - t)\varphi(r)(\coth r - 1), \]
and
\[ G_{ij}^{\tilde{h}_{ij}} = \frac{\partial G}{\partial h_{ij}}, \quad G_{ij,rs}^{\tilde{h}_{ij}} = \frac{\partial^2 G}{\partial h_{ij} \partial h_{rs}}. \]
To estimate the second fundamental form of \( M \), we need the following two lemmas.

**Lemma 3.4.** Let \( M = \{(x, r(x)) : x \in S^n \} \subset \mathbb{H}^{n+1} \) be a horo-convex solution of \( \text{(3.1)} \), then we have the following equality
\[ G^{ij}(V, \nu)_{ij} + (V, \nu)G^{ij}h_{im}h_{mj}^{\nu} = \sinh r \nabla_p \tilde{f} \nabla^p r + \cosh r(\tilde{f} + G^{ij}g_{ij}). \]

**Proof.** We have by Lemma 2.2
\[ (V, \nu)_{ij} = \cosh rh_{ij} + \sinh r g^{pq} h_{ij,p} \nabla_q r - h_{im}h_{mj}^{\nu}(V, \nu), \]
which results in
\[ G^{ij}(V, \nu)_{ij} = \sinh r G^{ij}h_{ij,p} \nabla^p r + \cosh r G^{ij}h_{ij} - (V, \nu)G^{ij}h_{im}h_{mj}^{\nu}. \]
Differentiating the equation \( \text{(3.1)} \) once, we have
\[ G^{ij}h_{ij,p} = \nabla_p \tilde{f}, \]
which implies together with \( G^{ij}h_{ij} = G + G^{ij}g_{ij} \)
\[ G^{ij}(V, \nu)_{ij} + (V, \nu)G^{ij}h_{im}h_{mj}^{\nu} = \sinh r \nabla_p \tilde{f} \nabla^p r + \cosh r(G + G^{ij}g_{ij}). \]
Therefore we complete the proof. \( \square \)
Lemma 3.5. Let \( M = \{(x, r(x)) : x \in \mathbb{S}^n\} \subset \mathbb{H}^{n+1} \) be a horo-convex solution of (3.1), then we have the following equalities for \( \tilde{f} \)

\[
|\nabla \tilde{f}| \leq C|\tilde{f}|_{C^1}(1 + |Dr|)
\]

and

\[
|\nabla^2 \tilde{f}| \leq C|\tilde{f}|_{C^2}(1 + |Dr|^2 + H),
\]

where the constant \( C \) depends on the minimum and maximum values of \( r \).

Proof. A direct calculation implies

\[
\nabla_p \tilde{f} = \tilde{f}_{p} + \tilde{f}_r r_p.
\]

Thus, we have by noticing that \( g^{ij} = \sinh^{-2} r(\sigma ^{ij} - D^i u D^j u) \)

\[
|\nabla \tilde{f}|^2 = g^{pq} \tilde{f}_p \tilde{f}_q + \tilde{f}_r^2 r_p r_q g^{pq} \leq C|\tilde{f}|_{C^1}^2(1 + |Dr|^2).
\]

Moreover, we have

\[
\nabla_p \nabla_p \tilde{f} = \tilde{f}_{pp} + 2 \tilde{f}_{pr} r_p + \tilde{f}_{rr} (r_p)^2 + \tilde{f}_r r_{pp},
\]

and we know from Lemma 2.2

\[
\sinh r \nabla_i \nabla_j r + \cosh r \nabla_i r \nabla_j r = \cosh r g_{ij} - h_{ij} \langle V, \nu \rangle.
\]

Thus, we arrive

\[
|\nabla^2 r| \leq C|f|_{C^2}(1 + |Dr|^2 + H).
\]

Now we begin to estimate the second fundamental form.

Proposition 3.6. If the horo-convex hypersurface \( M = \{(x, r(x)) : x \in \mathbb{S}^n\} \subset \mathbb{H}^{n+1} \) satisfies (3.1) and \( f(x, r) \in C^\infty(\mathbb{S}^n \times [0, +\infty)) \) is a positive function, then there exists a constant \( C \), depending on \( n \), \( |f|_{C^2} \) and \( |r|_{C^1(\mathbb{S}^n)} \) such that

\[
|\kappa_i(p)| \leq C, \quad \forall \ p \in M, 1 \leq i \leq n,
\]

where \( \kappa_i \) is the principal curvature of \( M \).

Proof. Since \( M \) is horo-convex, we only need to estimate the mean curvature \( H \) of \( M \). Taking the auxillary function

\[
W(X) = \log H - \log \langle V, \nu \rangle.
\]
Assume that \( p_0 \) is the maximum point of \( W \). Then at \( p_0 \),

\[
0 = W_i = \frac{H_i}{H} - \frac{\langle V, \nu \rangle_i}{\langle V, \nu \rangle}
\]

and

\[
0 \geq W_{ij} = \frac{H_{ij}}{H} - \frac{\langle V, \nu \rangle_{ij}}{\langle V, \nu \rangle}.
\]

Choosing a suitable coordinate \( \{x^1, x^2, ..., x^n\} \) on the neighborhood of \( p_0 \in M \) such that the matrix \( g_{ij}(p_0) = \delta_{ij} \) and \( \{h_{ij}\} \) is diagonal at \( p_0 \). This implies at \( p_0 \)

\[
0 \geq G^{ij}W_{ij} = \sum_{l=1}^{n} \frac{1}{H} G^{ii}h_{ll;ii} - \frac{G^{ii}\langle V, \nu \rangle_{ii}}{\langle V, \nu \rangle}.
\]

From (2.2), we obtain

\[
H_{ii} = \Delta h_{ii} - (h_{ii}^2 + 1)H + (|A|^2 + n)h_{ii},
\]

which results in at \( p_0 \)

\[
0 \geq \frac{1}{H} \sum_{l=1}^{n} G^{ii}h_{ii;ll} - G^{ii}(h_{ii}^2 + 1) + G^{ii}h_{ii} \frac{|A|^2 + n}{H} - \frac{G^{ii}\langle V, \nu \rangle_{ii}}{\langle V, \nu \rangle}.
\]

Differentiating the equation (3.1) twice, we have

\[
G^{ij}h_{ij;l} = \nabla_l \tilde{f}
\]

and

\[
G^{ij,rs}h_{ij;ll} + G^{ij}h_{ij;ll} = \nabla_l \nabla_l \tilde{f}.
\]

We know from Lemma 3.5

\[
|\nabla_l \tilde{f}| \leq C, \quad |\nabla_l \nabla_l \tilde{f}| \leq C(1 + H).
\]
Combining (3.4), (3.5) and (3.6), we arrive at $p_0$ by Lemma 3.4 and the inequality $|A|^2 \geq \frac{H^2}{n}$

\[
0 \geq \frac{1}{H} \sum_{i=1}^{n} \nabla_i \nabla_i \tilde{f} - G_{ii}(h_{ii}^2 + 1) + G_{ii} h_{ii} \frac{|A|^2 + n}{H} - \frac{G_{ii} \langle V, \nu \rangle_{ii}}{\langle V, \nu \rangle} \\
\geq -C \frac{H + 1}{H} - G_{ij} g_{ij} + G_{ij} h_{ij} \frac{|A|^2 + n}{H} \\
- \frac{\sinh r}{\langle V, \nu \rangle} \nabla_i \tilde{f} \nabla_i r - \frac{\cosh r}{\langle V, \nu \rangle} G_{ij} h_{ij} \\
\geq -C \frac{H + 1}{H} - G_{ij} g_{ij} + (\tilde{f} + G_{ij} g_{ij}) \frac{|H|^2 + n^2}{nH} \\
- C - CG_{ij} g_{ij} \\
\geq CH - C,
\]

if we assume $H$ is big enough, otherwise our proposition holds true. So we can derive $H \leq C$ at $p_0$. So, our proof is completed. □

4. The proof of Theorem 1.1

In this section, we can use the degree theory for nonlinear elliptic equation developed in [31] to prove Theorem 1.1. We only sketch will be given below, see [32, 4, 26, 30] for details.

After establishing the a priori estimates Proposition 3.1, Proposition 3.3 and Proposition 3.6, we know that the equation (3.1) is uniformly elliptic. From [12, 27], and Schauder estimates, we have

\[ |r|_{C^{4, \alpha}(S^n)} \leq C \]

for any horo-convex solution $M$ to the equation (1.2). We define

\[ C_{0}^{4, \alpha}(S^n) = \left\{ r \in C^{4, \alpha}(S^n) : M = \{(x, r(x)) : x \in S^n \} \text{ is horo-convex} \right\}. \]

Let us consider

\[ F(\cdot, t) : C_{0}^{4, \alpha}(S^n) \to C^{2, \alpha}(S^n), \]

which is defined by

\[ F(r, t) = det^{\frac{1}{2}}(\tilde{h}_{ij}) - tf(x, r) - (1 - t) \varphi(r)(\coth r - 1). \]

Let

\[ \mathcal{O}_R = \{ r \in C_{0}^{4, \alpha}(S^n) : |r|_{C^{4, \alpha}(S^n)} < R \}, \]
which clearly is an open set of $C^4_{0}(S^n)$. Moreover, if $R$ is sufficiently large, $F(r, t) = 0$ has no solution on $\partial O_R$ by the a priori estimate established in (4.1). Therefore the degree $\deg(F(\cdot, t), O_R, 0)$ is well-defined for $0 \leq t \leq 1$. Using the homotopic invariance of the degree, we have

$$\deg(F(\cdot, 1), O_R, 0) = \deg(F(\cdot, 0), O_R, 0).$$

Proposition 3.2 shows that $r = r_0$ is the unique solution to the above equation for $t = 0$. Direct calculation show that

$$F(sr_0, 0) = [1 - \varphi(sr_0)](\coth sr_0 - 1).$$

Using the fact $\varphi(r_0) = 1$, we have

$$\delta w F(r_0, 0) = \frac{d}{ds} |_{s=1} F(sr_0, 0) = -\varphi'(r_0)(\coth r_0 - 1) > 0,$$

where $\delta F(r_0, 0)$ is the linearized operator of $F$ at $r_0$. Clearly, $\delta F(r_0, 0)$ takes the form

$$\delta w F(r_0, 0) = -a^{ij}w_{ij} + b^iw_i - \varphi'(r_0)(\coth r_0 - 1)w,$$

where $a^{ij}$ is a positive definite matrix. Since $-\varphi'(r_0)(\coth r_0 - 1) > 0$, thus $\delta F(r_0, 0)$ is an invertible operator. Therefore,

$$\deg(F(\cdot, 1), O_R, 0) = \deg(F(\cdot, 0), O_R, 0) = \pm 1.$$

So, we obtain at least a solution at $t = 1$. This completes the proof of Theorem 1.1.

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