INTERACTIONS IN QUASICRYSTALS

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Although the effects of interactions in solid state systems still remains a widely open subject, some limiting cases such as the three dimensional Fermi liquid or the one-dimensional Luttinger liquid are by now well understood when one is dealing with interacting electrons in periodic crystalline structures. This problem is much more fascinating when periodicity is lacking as it is the case in quasicrystalline structures. Here, we discuss the influence of the interactions in quasicrystals and show, on a controlled one-dimensional model, that they lead to anomalous transport properties, intermediate between those of an interacting electron gas in a periodic and in a disordered potential.

1 Introduction

Our progress in solid state physics have been drastically dependent on our understanding of the physical properties of periodic structures. The knowledge of the eigenstates (Bloch waves) in such structures, enabled us to tackle other physical effects such as electron-electron interactions and transport properties. Some studies have also been made to understand systems totally lacking periodicity such as disordered ones. In this case also the knowledge of the diffusive, or exponentially localized nature of the wavefunctions has been a good starting point. An intriguing situation occurs in the case of quasicrystals, for which the system is neither periodic nor disordered. On the experimental side the transport exhibits many unusual properties. These metallic alloys are notably characterized by a low electrical conductivity $\sigma$ which increases when either temperature or disorder increases. The very low temperature behaviour of $\sigma$ is still an open question and strongly depends on the materials. For example, in AlCuFe and AlCuRu, a finite conductivity at zero temperature is expected whereas recent results seem to confirm a Mott’s variable range hopping mechanism ($\sigma(T) \sim \exp(-T_0/T)^{1/4}$) for $i\text{AlPdRe}$ icosahedral phase down to 20 mK. The optical conductivity is also unusual since there is no Drude Peak for icosahedral quasicrystals.

From a theoretical point of view, the case of independent electrons in one-
dimensional (1D) systems has been deeply investigated for different quasiperiodic structures (Harper model, Fibonacci chain,...), giving rise to singular continuous spectra with an infinite number of gaps. Moreover, the corresponding eigenstates are neither extended nor localized but critical, and are known to be responsible of anomalous diffusion \cite{10,11}. For higher dimensional systems (Penrose tiling, Octagonal tiling, icosahedral structure...), similar studies had also displayed complex and intricated spectra, with analogous characteristics of the electronic states \cite{12,13}. Finally, studies of transport properties \cite{14,15,16,17,18} has also displayed anomalous behaviour such as a power-law decreasing of the conductance with the system size \cite{19,20}.

Given the difficulty of the full quasiperiodic problem, and the relatively poor knowledge of the single particle wavefunctions, adding interactions is a very difficult task. Even in 1D incommensurate structures, few results have been obtained \cite{21,22,23,24,25}. In order to treat interactions in quasicrystals we thus follow a different route, used with success for disordered systems. We first solve the periodic system in presence of interactions. This is relatively easy, either in the one-dimensional case which is considered in this paper, for which exact solutions exist, or even in higher dimensions through approximate (Fermi liquid) solutions. Then, we study the effect of a weak perturbative quasiperiodic potential on this solution via a renormalization group approach. The main idea consists in analyzing the dependence of the low-energy physics with respect to the Fourier spectrum of the potential. The influence of the renormalization group procedure is discussed in term of an effective elimination of the various potential harmonics around the Fermi level, leading to unusual properties in the case of a self-similar quasiperiodic structure \cite{26}.

2 The model

Let us consider a system of interacting spinless fermions on a lattice embedded in a general on-site (diagonal) potential \(W\), described by the following Hamiltonian:

\[
H = -t \sum_{\langle i,j \rangle} c_i^\dagger c_j + V \sum_i n_i n_{i+1} + \sum_i W_i n_i
\]

\[
= H_0(t, V) + H_W
\]

where \(c_i^\dagger\) (resp. \(c_i\)) denotes the creation (resp. annihilation) fermion operator, \(n_i = c_i^\dagger c_i\) represents the fermion density on site \(i\), and \(\langle \ldots \rangle\) stands for nearest neighbors pairs.

In the continuum limit, \(H_0(t, V)\) writes \cite{27}:

\[
H_0(t, V) = \frac{1}{2\pi} \int dx \left[ (uK)(\pi \Pi)^2 + \left( \frac{u}{K} \right) (\partial_x \phi)^2 \right]
\]

where \(\phi\) is a boson field related to the long wave length part of the fermionic density by \(\rho(x) = -\nabla \phi(x)/\pi\), and \(\Pi\) is its canonically conjugate field. This approximation provides the correct description of the low energy physics. All the interactions are absorbed in the two constants \(u\) and \(K\). \(u\) is the renormalized Fermi velocity (in the non interacting case: \(u = v_F = 2ta \sin(k_F a)\)). \(K\) is the Luttinger parameter controlling the decay of various correlation functions: \(K=1\) in the non-interacting
case, $K > 1$ for attractive interactions ($V < 0$) and $K < 1$ for repulsive interactions ($V > 0$). Note that, for many models, these parameters can be explicitly computed in terms of the microscopic details.

The perturbative part of $H$, can also be expressed in terms of these boson fields:

$$H_W = \frac{1}{\pi \alpha} \int dx W(x) \cos [2k_F x - 2\phi(x)]$$

where $\alpha$ is a short distance cut-off of the order of the lattice constant $a$.

## 3 Renormalization group analysis

To determine the long distance physics, we use a standard perturbative renormalization group approach. Full details can be found in and we focus here on the main equations for the potential and interaction parameter:

$$\frac{dK}{dl} = -\frac{K^2}{2} G(l)$$

$$\frac{dy_q}{dl} = (2 - K) y_q$$

with

$$G(l) = \sum_{\varepsilon = \pm 1} \sum_q y_q^2 J [(q + \varepsilon 2k_F) \alpha(l)].$$

$y_q = \alpha \tilde{W}(q)/u$ is the dimensionless Fourier components of $W$ and $\alpha(l) = \alpha(0) e^l$ is the renormalized short distance cutoff. In Eq. (5), $J$ is an ultraviolet regulator whose precise form depends on the procedure used to eliminate the high energy degrees of freedom. Typically one has:

$$J(x) \simeq 1 \quad \text{if} \quad x < 1$$

$$= 0 \quad \text{otherwise}$$

Physically, this renormalization is equivalent to an investigation of the low-energy properties in a window around $2k_F$ in the reciprocal space whose width is proportional to $e^{-l}$. Thus, the full Fourier landscape of the potential determine the scaling of the parameters $u$ and $K$, and one has to distinguish several situations.

If the Fourier spectrum of $W$ has only a single peak or few well separated peaks, then the system is analogous to a periodic system. If the Fermi level is not on one of the peaks then at some lengthscale the regulator $J$ gives always zero (see figure 1). The potential $W$ is irrelevant and one recovers a Luttinger liquid metallic phase. On the contrary, if the Fermi level is right on one of the peaks (e.g. $2k_F = q_0$), all physical properties are dominated by this single harmonic (the regulator cuts all the others) and the system feel a periodic potential $W(x) = \lambda \cos(q_0 x)$. The case of the single harmonic is well known: the periodic potential is relevant for $K < 2$ and opens a gap

$$\Delta \sim y_q^{\frac{1}{1-K}}$$

For $K > 2$ the periodic potential is irrelevant and system remains a Luttinger liquid with gapless excitations. Our equations clearly show that novel physics occurs when
Figure 1. Fourier spectrum of the potential $W$ for the periodic (a), disordered (b) or quasiperiodic (c) cases. All the peaks within a window of width $\sim e^{-l}$ (see text) around the Fermi energy $E_F$ (shown as a box) control the physics of the problem. This is crucial for the disordered (b) and quasiperiodic (c) systems for which it is impossible to really isolate a single peak, contrarily to the case of periodic systems (a).
the spectrum of the potential $W$ is dense. This is the case both for disordered systems and for the quasiperiodic ones (see figure 1). Indeed in that case it is never possible to really isolate a single peak. The physics is thus not controlled by the presence or not of a peak at the Fermi level but by the forest of peaks in the (shrinking) window of width $e^{-l}$. In this respect, the properties of a quasicrystal are much closer (with proper differences discussed below) to the one of a disordered system than the one of a “periodic” (i.e. with a limited number of harmonics) one.

The disordered case can easily be obtained from our equations. Let us assume that $W$ is uncorrelated with averages given by

$$\langle W^*(q)W(q') \rangle = \lambda^2 \delta_{qq'}$$

In the limit of weak disorder, (5) can be integrated neglecting the renormalization of $K$: $y(l) = y(0) e^{(2-K)l}$. Then, (5) simply becomes:

$$\frac{dK}{dl} = -K^2 Ce^{(3-2K)l}$$

where $C$ is a constant. Eq. (11) provides a critical value $K_c = 3/2$, separating an Anderson-like insulating phase ($K < K_c$) from a metallic state ($K > K_c$).

Note that contrary to the periodic case, this metal-insulator transition occurs for any value of $k_F$. Indeed, as shown figure 1, the Fourier spectrum is uniform (after averaging) so that all the positions of the Fermi level are equivalent.

4 The quasiperiodic case

The quasiperiodic case provides an interesting intermediate situation. Let us consider for simplicity a Fibonacci potential although our results can be easily extended to more general situations.

The quasiperiodicity is provided by the $W_i$’s that take two discrete values $W_A = \lambda/2$ or $W_B = -\lambda/2$ given by the spatial modulation of the Fibonacci chain. In fact, we consider a periodic approximant of this structure with $F_j$ sites per unit cell that can be obtained by $j$ iterations of the substitution rules: $A \rightarrow AB$, $B \rightarrow A$, where $F_j$ is the $j^{th}$ element of the Fibonacci sequence defined by:

$$F_1 = F_2 = 1, \quad F_{j+1} = F_j + F_{j-1}$$

We denote $p = F_{j-2}, s = F_{j-1}, n = F_j$ and $n' = s$ (resp. $n' = p$) if $j$ is even (resp. odd). In the quasiperiodic limit ($j \rightarrow \infty$), the ratio $s/p$ converges toward the golden mean $\tau = \frac{1+\sqrt{5}}{2}$. The Fourier transform of $W$ can be straightforwardly using the conumbering scheme:

$$\tilde{W} \left( q = \frac{2\pi m}{na} \right) = \frac{\lambda e^{i2\pi n'/(n-1)}}{n \sin (\pi m/n)} \sin \left( \frac{\pi mn'}{n} \right)$$

for $m = 1$ to $n-1$ ($a$ is the lattice spacing). A global shift of the $W_i$ allows us to deal with a zero-averaged potential so that we can set $\tilde{W}(0) = 0$. Moreover, the underlying substitution rule provides a self-similar structure that can be readily
seen in Fig. 1. In the non interacting case, each Fourier component of the potential opens a gap whose width is given, at first order in perturbation, by the amplitude of the corresponding component. In the quasiperiodic limit, the Fourier spectrum of the potential become dense so that the spectral measure goes to zero. Therefore, for any filling factor, the Fermi velocity vanishes and the system is an insulator (at zero temperature). In contrast, given the complexity of the Fourier spectrum, one can expect the long distance physics of the interacting system to depend on the position of the Fermi level. Indeed, the flow equations show that the behaviour of $G$ is strongly sensitive to the position of the window in the spectrum. For a given maximum renormalization length scale $l_{\text{max}}$, corresponding to accessible physical range, two different cases must be distinguished.

First, if the Fermi momentum $2k_F$ is close to a main peak of the Fourier spectrum, i.e. $(q \pm 2k_F)^{-1} > l_{\text{max}}$ then, at long distance, i.e. up to $l \sim l_{\text{max}}$, the flow of $K$ is controlled by this harmonic (see Fig. 1 (upper curve)). A metal-insulator transition can occurs with $K_c = 2$. This case is very similar to the simple periodic one.

A more interesting behaviour is encountered when the Fermi level is far from a dominant harmonic of the quasiperiodic potential, for example at half-filling $(2k_F = \pi/a)$. Indeed, the low energy properties up to $l_{\text{max}}$ are no more dominated by the ultimate presence of a gap or not but by the precise dependence of $G$ with the scale. As can be seen from figure 2, $G$ has an exponential scaling $G \sim e^{(2-2K)l}$. Such a

$$\ln(G(l) / \exp (4-2K)l)$$

Figure 2. Scaling of the function $G$. The presence of many peaks gives a scaling dimension different both from disordered or periodic systems.
behaviour has several consequences. First it provides a critical value of \( K = 1 \) for the metal-insulator transition. Let us point out that this value is the smallest critical value that one can expect for such a transition. Indeed it is clear that repulsive interactions \( (K < 1) \) enforce the insulating character of the system. To know whether this analysis is asymptotically correct or not, one would clearly need an analytical calculation of \( G \), a rather complicated task. Note however that our predictions seem to be confirmed by numerical work based on the Density Matrix Renormalization Group method on the XXZ chain with quasiperiodic exchange \[39\]. This work provides evidence for different behaviours for the scaling of the gap with the size for various \( K \). In particular, a transition is observed for \( K = 1 \) at half-filling even in the strong coupling regime. An interesting, and yet unanswered question is how the spectrum, or more generally the density of states is modified when \( K \) varies. It is very natural to assume that the small gaps close up when \( K \) increases, given that the system is always gapless for \( K = 2 \). The spectrum would thus evolve from a set of zero measure for \( K = 1 \) to a set of finite measure for \( K > 1 \) as indicated in Figure 3. It would be interesting to check this scenario on the XXZ chains under a magnetic field. So, the quasicrystal differs from a periodic one, for which the gap

![Figure 3. A possible evolution of the density of states as a function of the Luttinger liquid parameter \( K \). For \( K = 1 \) the spectrum is a set of measure zero and gaps are present almost everywhere. For \( 1 < K < 2 \) the small gaps close as \( K \) increases. For \( K = 2 \) even the largest gaps are closed since even a periodic potential is irrelevant above \( K = 2 \).](image)

only acts for a given position of the Fermi level with \( K_c = 2 \), and a disordered system for which the potential is relevant regardless of the position of the Fermi level, but below a constant critical value \( K_c = 3/2 \). This important modification of \( K_c \) is reminiscent of a correlated disorder with long range correlations in space for which the averaged disorder potential \( \bar{W}(q)\bar{W}(q') = \delta_{qq'}\Delta(q) \) is not constant.

The other important consequences of such a behaviour are of course exhibited by the transport properties. A straightforward calculation would give a power law for the frequency dependence of the conductivity \( \sigma(\omega) \sim (1/\omega)^{5-2K-\mu} \) with \( \mu = 2 \) (at variance with \( \mu = 0 \) for periodic and \( \mu = 1 \) for disordered). This anomalous power law depending both on the interactions and the spectrum properties would also appear in a size dependence of the Landauer conductance \( R(L) \sim L^{4-2K-\mu} \).
The temperature dependence of the conductivity raises an interesting question, similar to the one occurring for periodic systems. Cutting the flow at the natural lengthscale $l = \log(v_F/T)$ would lead to a power law dependence of the conductivity $\sigma(T) \sim T^{3-2K-\mu}$, and thus to a finite conductivity. In order to obtain such a behaviour it is probably necessary to introduce dephasing processes (like a coupling to a thermal bath). If such a temperature dependence of the conductivity occurs it would be again a demonstration of the close connection between quasiperiodic systems and disordered one (for which it is quite natural to expect dissipation at finite temperature).

Let us note that it is possible to investigate the effects of disorder on the quasiperiodic potential using (5). Disorder in $d=1$ leads in itself to localization, which is to be avoided if one wants to make connection with higher dimensions. One way to achieve this is simply to introduce forward scattering on impurities which cannot lead to localization (3). Introducing such a disorder cuts the flow in (5) at a characteristic lengthscale $\xi_F$ inversely proportional to the disorder. Since it prevents the (quasi-)periodic potential to grow it makes thus the system more conducting (more details can be found in [3]). One thus recovers on this simple model the physical effects leading to the inverse Mathiessen law experimentally observed.

Exploring these issues, as well as investigating the consequences in higher dimension is a challenging problem left for the future.

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This point was missed in previous studies of a quasiperiodic potential, that considered a Harper like potential $\cos(Qx)$. Indeed, as was already known for the noninteracting case, such a potential is perturbatively very similar to a simple periodic one, and quasiperiodic effects only occur markedly for large enough potential, making it unsuitable for an RG study. The Fibonacci potential studied here does not suffer from such problems and exhibit the quasiperiodic effects even at small coupling.