Quantifying Rapid Variability in Accreting Compact Objects

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ABSTRACT
I discuss some practical aspects of the analysis of millisecond time variability X-ray data obtained from accreting neutron stars and black holes. First, I give an account of the statistical methods that are at present commonly applied in this field. These are mostly based on Fourier techniques. To a large extent these methods work well: they give astronomers the answers they need. Then I discuss a number of statistical questions that astronomers don’t really know how to solve properly and that statisticians may have ideas about. These questions have to do with the highest and the lowest frequency ranges accessible in the Fourier analysis: how do you determine the shortest time scale present in the variability, how do you measure steep low-frequency noise. The point is stressed that in order for any method that resolves these issues to become popular, it is necessary to retain the capabilities the current methods already have in quantifying the complex, concurrent variability processes characteristic of accreting neutron stars and black holes.

1 Introduction

The purpose of this talk is to explain to statisticians how astrophysicists, mostly using Fourier transform techniques, go about analyzing X-ray time-series data obtained from accreting compact objects (neutron stars and black holes), and to point out a few problems with the usual approaches. The point will be made, that the conglomerate of statistical methods that is being applied in this branch of high-energy astrophysics, even though most definitely not always rigorous, on the whole serves its purpose well and is providing astronomers with the quantitative answers they require. This talk will aim, however, at a few areas where we run into problems, and where more statistical expertise might help. In thinking about how to solve the problems that I shall outline, it will be important to keep an eye on what the capabilities of the existing methods are, as those capabilities will need to be preserved in whatever new approach one would like to propose.

Mostly, accreting neutron stars and black holes occur in double star systems known as X-ray binary stars, where a normal star and the compact object are in a close orbit around each other (Fig. 1). Matter flows from the normal star to the compact object by way of a flat, spiraling flow called
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An accretion disk, and finally accretes onto the compact object. A large amount of energy is released in this process (typically $10^{36}$ to $10^{38}$ erg/sec), and is emitted, mostly in the form of X-rays. The characteristic variability timescale for the X-ray emitting regions is predicted (and observed) to be very short (less than a millisecond). By studying the properties of this rapid X-ray variability it is possible to extract a great deal of information about the flow of matter onto the compact object, and, indirectly, about the object itself. See van der Klis (1995) for a recent review of the results of studies of this type.

![Figure 1. X-ray binary star.](image)

2. The data

In order to understand the character of the data we are dealing with, I shall follow the flow of information from star to computer. An X-ray binary star emits X-ray photons at a very large rate (say, $10^{46}$ photons/sec). For all practical purposes, the X-ray photon rate produced by the star can be considered as a continuous function of time $I(t)$. These photons are emitted isotropically, or at least over a solid angle of order $4\pi$ sterad. Because the X-ray detector onboard the X-ray satellite spans only a very small solid angle as seen from the X-ray star, only a very small fraction $\epsilon$ (say, $10^{-43}$) of the photons is detected in the instrument. The time series of photon arrival times $t_i$, $i = 1, \ldots, N_{\text{phtot}}$ (with $N_{\text{phtot}}$ the total number of detected photons) is the information that, ideally, we would like to have available for analysis. However, typically, instrumental limitations (and the maximum telemetry rate) prevent the registration and transmission of all these arrival times. Therefore, onboard the satellite, the data are binned into equidistant time bins. The information that is finally telemetered to the ground station consists of a sequence of numbers $x_m, m = 0, \ldots, N_{\text{tot}} - 1$, where $N_{\text{tot}}$ is
the total number of time bins, \( x_m \) representing the number of photons that was detected during time bin \( m \). In most cases, and unlike the usual case in astronomy, the time bins are equidistant and contiguous (no gaps). The statistical problem facing us can be summarized as follows: “Given \( x_m, m = 0, \ldots, N_{\text{tot}} - 1 \), reconstruct as much as possible about \( I(t) \).

We shall assume that for the bright X-ray binaries that I am discussing here, the rate of background photons can be considered to be negligible, and that there are no other relevant effects affecting the photon arrival time series than the huge geometrical dilution factor just described. Therefore, if the X-ray star does not vary intrinsically, the \( t_i \) will to a high degree of precision be uniformly and randomly distributed, so that the \( x_k \) follow Poisson statistics appropriate to a rate \( \mu = \langle x_m \rangle \), with a standard deviation \( \sigma_{x_m} \) equal to \( \sqrt{\mu} \).

Of course these stars do vary. The variability time scales of interest are \( \lesssim 1 \) millisecond, and the detected photon rates \( 10^2 \)–\( 10^5 \) photons/sec, which means that the time bins must be chosen such that typically there are on average only a few (sometimes \( \ll 1 \)) photons per time bin, and \( \sigma_{x_m} \) is of order \( \mu \) (sometimes \( \sigma_{x_m} \gg \mu \)). As the intrinsic variability of the star often has a relatively small amplitude, only a few percent of the total flux, it is clear that we are in a low signal-to-noise regime (with the “signal” the intrinsic stellar variability and the “noise” the Poisson fluctuations). Fortunately, typical observations span \( 10^3 \) to \( 10^5 \) sec, so that the number of time bins \( N_{\text{tot}} \) (the number of “measurements”) is \( 10^6 \) to \( 10^8 \), which allows us to recover the signal from the noise. The techniques used for this are described in the next section.

3 Standard analysis

The standard approach (see van der Klis 1989) is to divide the time series into \( M \) equal-length segments of \( N \) time bins \( x_k, k = 0, \ldots, N - 1 \) each (so, ideally \( N_{\text{tot}} = MN \)), to calculate the discrete Fourier transform of each segment:

\[
a_j \equiv \sum_{k=0}^{N-1} x_k e^{2\pi ijk/N} \quad j = 0, \ldots, N/2,
\]

\(^1\)This is of course not exactly true. The background does not pose large problems in practice, as it just constitutes an additional source of detections (of photons as well as charged particles) that is not strongly variable on the time scales we are interested in and uncorrelated to the fluctuations due to the star. Detector deadtime processes leading to “missed” photons do constitute a serious complication (e.g., van der Klis 1989, Mitsuda and Dotani 1989, Vikhlinin et al. 1994, W. Zhang 1995) that will be ignored here.
to convert this into a power spectrum for each segment

\[ P_j \equiv \frac{2}{a_0} |a_j|^2 \quad j = 0, \ldots, N/2, \]

and then to average these power spectra (see below). Note that in our application

\[ a_0 \equiv \sum_{k=0}^{N-1} x_k = N_{ph}, \]

the number of photons detected in the segment. With this power spectral normalization, due to Leahy et al. (1983), it is true that if the \( x_k \) are distributed according to the Poisson distribution, then the \( P_j \) follow the \( \chi^2 \) distribution with 2 degrees of freedom, so \( \langle P_j \rangle = 2 \) and \( \sigma_{P_j} = 2 \).

This white noise component with mean level 2 and standard deviation 2, induced in the power spectrum due to the Poisson fluctuations in the time series, is called the “Poisson level”. It can be considered as “background” in the power spectrum, against which we are trying to observe the other power spectral features, which are caused by the intrinsic variability of the X-ray binary. The physical dimension of the thus defined powers is the same as that of the time series: \( [P_j] = [a_j] = [x_k] \). Often the Y-axis of plots of power spectra in this normalization is just labeled “POWER”, which reflects the fact that the physical interpretation of the \( P_j \) in terms of properties of the star is inconvenient. For this reason, in recent years another power spectral normalization has become popular where the powers are reported as \( Q_j \equiv P_j/\lambda \), with \( \lambda \) the “count rate”, the number of detected photons per second: \( \lambda = N_{ph}/T \), where \( T \) is the duration of a segment. The \( Q_j \) are dimensionless, and can be interpreted as estimates of the power density \( Q(\nu_j) \) near the frequency \( \nu_j \equiv j/T \), where \( Q(\nu) \) is a function of frequency whose integral gives the fractional root-mean-square amplitude \( r \) of the variability. This latter quantity is defined as

\[
 r \equiv \sqrt{\frac{1}{N} \sum_{k=0}^{N-1} (x_k - \overline{x})^2} \quad \text{where} \quad \overline{x} = \frac{1}{N} \sum_{k=0}^{N-1} x_k.
\]

(This follows directly from Parseval’s theorem.) The fractional rms amplitude \( r_{12} \) due to fluctuations in a given frequency range \((\nu_1, \nu_2)\) is given by

\[
 r_{12} = \int_{\nu_1}^{\nu_2} Q(\nu) d\nu.
\]

So, the physical interpretation of \( Q(\nu) \) is easy: it is the function whose integral gives you the square of the fractional rms amplitude of the variability in the original time series. The physical unit used for \( Q(\nu) \) is \((\text{rms/mean})^2/\text{Hz}\), where “rms” and “mean” both refer to the time series; “rms/mean” is just the dimensionless quantity \( r \).

The averaging of the power spectra mentioned above is usually performed both by averaging individual power spectra (from different segments) together (averaging the \( P_j \)’s with the same \( j \) from the \( M \) different segments)
FIGURE 2. Top: An average of 6166 power spectra of EXOSAT ME data on the X-ray binary GX 5−1 showing power-law noise and QPO. Bottom: The standard deviation of the 6166 power values averaged in each frequency bin. Inset: the ratio of standard deviation to mean. The standard deviation equals the mean power as expected for $\chi^2$ distributed powers. From Van der Klis (1989).

and by averaging powers at adjacent frequencies ($P_{j+1}$ to $P_{j+W}$, say). The main purpose of this is of course to decrease the standard deviation of the power estimates, which in the raw spectra is equal to the mean power. The reason to calculate many power spectra of segments of the data rather than one very large power spectrum of the whole data set, apart from computational difficulties with this approach, is that in this way it is possible to study the variations in the power spectrum as a function of time.

The final step in the analysis is to fit various functional shapes $f(\nu)$ to the power spectra using the method of $\chi^2$ minimization that is also used in X-ray spectroscopy (the Levenberg-Marquardt method described in Press et al. 1992, Chapter 15). Because many individual power estimates have been averaged in the analysis process, the central limit theorem ensures
that the uncertainties of the final power estimates $\overline{Q}$ are approximately normally distributed, as required for this method to work well. A problem is what “uncertainties” $\sigma_{\overline{Q}}$ to assign to these power estimates. Usually $\sigma_{\overline{Q}} = \overline{Q}/\sqrt{MW}$ is assumed, where $MW$ is the number of individual powers averaged to obtain $\overline{Q}$ ($W$ stands for the “width”, the number of adjacent powers averaged; $M$ for the number of averaged power spectra). This is approximately correct, and was experimentally verified (Fig. 2), in the case that the dynamic range of the power spectrum is dominated by the intrinsic differences in the mean amplitude of the star’s variability as a function of frequency rather than by the stochastic fluctuations in power. If instead the stochastic variations dominate then this procedure for estimating the uncertainties can lead to severe underestimation of the power, as accidentally low powers will get high weights in the fitting procedure and vice versa. A solution to this problem that is sometimes adopted is to estimate the uncertainty in $\overline{Q}$ as $f(\nu)/\sqrt{MW}$, where $f(\nu)$ is the fit function and $\nu$ the frequency corresponding to $\overline{Q}$.

4 Applications and results

The method described in the previous section works. It allows astronomers to quickly characterize the variability properties of large amounts of data, to study the changes in the properties of the variability as a function of time and other source characteristics, and to measure amplitudes and characteristic time scales of the variability. The method straightforwardly extends to simultaneous multiple time series, for example time series obtained in different photon energy bands. For $N_{\text{band}}$ simultaneous time series, it is possible to calculate $N_{\text{band}}(N_{\text{band}} - 1)$ different cross-spectra between them, and to look for systematic time delays in the variability as a function of energy. A comprehensive description of the methods used for this can be found in the paper by Vaughan et al. (1994).

A very important aspect is the possibility to identify different “power spectral components” in the variability. This is done by studying the changes in the shape of the power spectra as a function of time and other source properties, such as brightness or photon energy spectrum. It usually turns out that the simplest way to describe the changes in the power spectrum as a function of time is in terms of the sum of a number of components whose properties (strength, characteristic frequency) depend in a smooth, systematic and repeatible way on, for example, brightness. If this is the case, then it is natural to interpret these power spectral components as due to different physical processes (or different aspects of the same process) that are all affecting the count rates at the same time, and that have been disentangled from each other in the analysis just described.

Examples of power spectral components that are distinguished in practice
FIGURE 3. Power spectra of black hole candidates. The spectrum labeled LS (low state) is of Cygnus X-1; the other two are of GS 1124−68. From Van der Klis (1995).

are “power law noise”, “band-limited noise” and “quasi-periodic oscillations (QPO)”. All of these are all presumed to be stochastic processes in the time series that cause, respectively, a component in the power spectra that fits a function $f_{PL}(\nu) = C_{PL}\nu^{-\alpha}$, one that fits $f_{BLN}(\nu) = C_{BLN}\nu^{-\alpha}e^{-\nu/\nu_{\text{cut}}}$ and one that fits a Lorentzian $f_{QPO}(\nu) = C_{QPO}/(\nu - \nu_{QPO})^2 + (\Gamma/2)^2)$. For a component to be called QPO it is sufficient that the corresponding power spectral component has a shape approximately described by $f_{QPO}(\nu)$ with the peak full-width-at-half-maximum (FWHM) $\Gamma$ less than half its centroid frequency $\nu_{QPO}$; of course this definition is arbitrary.

Figure 3 shows a number of actually observed power spectra, plus the functions that were fit to them in order to describe them in terms of the power spectral components just mentioned. QPO peaks, band-limited noise and power law noise can all be seen.
Sometimes the shape of the observed power spectrum is ambiguous with respect to the decomposition into power spectral components as described here. An advantage of the $\chi^2$ minimization method is that it allows to quantify the degree of this ambiguity by comparing the $\chi^2$ values of the different possible combinations of fit parameters.

5 Problems at the low-frequency end: low-frequency leakage

At the low frequency end (typically, below 0.01 Hz) of the power spectra a number of problems occurs in an analysis along the lines described above that is usually not really satisfactorily dealt with.

One problem is, that in order to reach the lowest frequencies in the first place, it is necessary to choose the length of the time segments $T$ relatively large and therefore $M$ relatively small. This means that at the lowest frequencies the statement made in Section 3, that a large number of individual powers has been averaged and therefore the average power is approximately normally distributed is no longer true. Other (e.g., maximum-likelihood) fitting procedures are required to take the true distribution of the average powers into account, but such methods are not usually applied. See Papadakis and Lawrence (1993) for a discussion of a method to remedy some of these problems.

A more serious problem is, that the true power spectrum at these frequencies often seems to be a quite steep power law. The finite time window $T$ in those situations leads to so-called “low-frequency leakage” (see Deeter 1984 and references therein): power shows up at a higher frequency in the power spectrum than where it belongs. One way to see this is by noting that the lowest frequency accessible in the discrete Fourier transform is $1/T$. If there is a lot of variability at lower frequencies than this, then due to these slow variations the time series of an individual time segment will usually have a large overall trend. The Fourier transform of this trend produces a power law with index $-2$ in the power spectrum. Another way to describe the effect of low-frequency leakage is by noting that according to the Fourier convolution theorem the Fourier transform calculated in the finite time window $T$ is related to the true Fourier transform by a convolution of the true transform with the transform of the window function. As the window function is a boxcar, its Fourier transform is the well-known sinc function, with a big central lobe and upper and lower sidelobes that gradually decrease in amplitude. For true power spectra steeper than a power law of index $-2$, the contribution to the convolution of the lower side lobes overwhelms that of the central lobe, and the result is a power spectrum that is a power law with index $-2$ (e.g., Bracewell 1965). So, for any true power spectrum steeper than $-2$, the actually measured power spectrum
will have slope of $\sim -2$.

There are well-known solutions to this well-known problem. The most famous one is data tapering: instead of a boxcar window a tapered window is used, i.e., a window that makes the data go to zero near its end points more gradually than by a sudden drop to zero. Other methods are polynomial detrending (fitting a polynomial to the data and subtracting it) and end-matching (subtracting a linear function that passes through the first and the last data point). Deeter et al. (1982) and Deeter (1984) have explored a number of non-Fourier methods. All these methods work in the sense that to some extent they suppress the side lobes of the response function and therefore they are able to recover power laws steeper than with index $-2$ (the value of the power law index where the method breaks down is different in each case). However, typically these methods have only been evaluated for the case where the time series is pure power-law noise, and in many cases even only with respect to their effectiveness in recovering the power law index, not even the noise strength. Some methods require that the index of the power law is known in advance! Nearly nothing is known about the way in which these methods affect the results of fits to power spectra with complicated shapes such as those described above. These methods may recover the correct power law index for the low-frequency part of the power spectrum, but what will be the effect on the fractional rms values and time scales of all the other components? For this reason, these methods are not usually applied.

6 Problems at the high frequency end: what is the shortest time scale?

The high-frequency end of the power spectra, near the Nyquist frequency, is of particular interest to astrophysicists, as it is there that we expect to find the signatures of the fastest accessible physical processes going on in the star. A question that is often asked in this context is: “what is the shortest variability time scale $\tau$ we can detect in the data?” Generally, what is observed at high frequency is that the power spectrum more or less gradually slopes down towards the Poisson level. The problem is to decide out to which frequency intrinsic power is observed above the Poisson background, and what this number means.

An, I think, decidedly misleading way to answer the question about the shortest time scale is by determining the shortest time interval $\Delta t$ within which significant variations can be detected. It is obvious, that for a slowly varying source, by zooming in on some gradual slope in its light curve $I(t)$, this shortest time interval can be made as short as one wishes, just by improving the quality of the data (by increasing $\epsilon$ for example). Yet, most of the work on “shortest time scales” seems to aim for measuring
It seems clear that one must also take into account the amplitude of the variations, not just the time within which they occur to do something that is physically useful. Following Fabian (1987) one can for example define the variability time scale as

\[ \tau(t) = \frac{\dot{I}(t)}{\dot{I}(t)} \]

where the dot denotes the time derivative. Defined this way, \( \tau \) is a measure of how steeply \( I \) changes with \( t \), and depends itself on \( t \). Using a power spectrum, one would measure some average of this quantity by determining the fractional rms amplitude \( r(\nu_{hi}) \) near some high frequency \( \nu_{hi} \) in the way described in Section 3, and then write

\[ \tau = C \times \frac{1}{\nu_{hi} r(\nu_{hi})} \]

where \( C \) is a constant of order unity depending on assumptions on exactly how the variations causing the power in the spectrum took place. However, for precise work one has to worry about low-frequency leakage, too. Exactly the same problem as described in Section 5 can occur here: power from lower frequencies can leak up to higher frequencies due to the sinc response associated with the power estimators. That this problem is serious is apparent from the fact that the observed power spectrum of for example the famous black-hole candidate Cyg X-1 has an index of \( \sim -2 \) for frequencies above \( \sim 10 \) Hz (see Fig. 3), just the value at which low-frequency leakage begins to worry us. It would be of great interest to have a foolproof way to subtract power that has leaked up from lower frequencies, or even to have a way to make a conservative estimate of (obtain a lower limit on) the true high-frequency power. It is not clear to what extent this can be accomplished. Obviously, as in any convolution problem, some information has been lost, but how much can be recovered is a problem X-ray astronomers do not know the answer to. I wonder to what extent standard deconvolution procedures might be useful here.

I note parenthetically that a method that has been applied for determining the shortest variability in a time series by Meekins et al. (1984) seems to suffer from the same problem of low-frequency leakage. In this method, the time series is divided up in very short segments of, for example, \( N = 10 \) bins each. In each segment a quantity called “chi-squared” is calculated as follows:

\[ \chi^2_{Meekins} = \sum_{k=0}^{N-1} \frac{(x_k - N_{ph}/N)^2}{N_{ph}/N}. \]

Here \( x_k \) is the number of photons detected in bin \( k \) and \( N_{ph} \) is the total number of photons in the segment. One recognizes an “observed over
expected” variance ratio for an expected Poisson distribution. The distribution of this quantity is then compared to that expected if all variability in the time series would be due to Poisson fluctuations and if a significant excess is found, this is interpreted as detection of variability on time scales between the length $T$ of a segment and the duration of one bin. One would expect low-frequency leakage to be as much of a problem here as in the power spectral method. It is easy to see that if there are variations in the time series on time scales much longer than the segment length, the data points in the segments will usually follow steep trends, causing large values of $\chi^2_{\text{Meekins}}$ that are not related to variability on time scales shorter than $T$. Indeed, Meekins et al. show that $\chi^2_{\text{Meekins}}$ is closely related to the Fourier power in the segment, so from the point of view of low-frequency leakage the Meekins et al. method is less effective than the standard power spectral approach, as it requires $T$ to be quite short and therefore increases the probability of steep trends in the data segments.

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