A SERRE RELATION IN THE $K$-THEORETIC HALL ALGEBRA OF SURFACES

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Abstract. We prove a Serre relation in the $K$-theoretic Hall algebra of surfaces constructed by Kapranov-Vasserot [3] and the second author [11].

1. Introduction

Given a smooth quasi-projective surface over $\mathbb{C}$, an associative algebra structure on

$$K(\text{Quot}) = \bigoplus_{n \geq 1} K_{\text{GL}_n}(\text{Quot}_n),$$

was constructed by Kapranov-Vasserot [3] and the second author [11] for the case $S = \mathbb{A}^2$ and constructed by Sala-Schiffmann [8] and Alexander Minets [5] when $S$ is the cotangent bundle of an algebra curve $C$. The $K$-theoretic Hall algebra was categorified by Porta-Sala [7], and the two-dimensional categorified Hall algebra was studied by Diaconescu-Porta-Sala [2] when $S$ is the crepant resolution of type A singularities.

When $S = \mathbb{A}^2$ with equivariant $\mathbb{G}_m$ action, the $K$-theoretic Hall algebra could be identified by the positive part of the elliptic Hall algebra, i.e. $\mathbb{Z}[q_1, q_2]$-algebra with generators $\{E_k\}_{k \in \mathbb{Z}}$ modulo the following relations:

$$(1) \quad (z - w q_1)(z - w q_2)(z - w q_3)E(z)E(w) = (z - w q_1)(z - w q_2)(z - w q_3)E(w)E(z)$$

$$(2) \quad [[E_{k+1}, E_{k-1}], E_k] = 0 \quad \forall k \in \mathbb{Z}$$

where

$$(3) \quad E(z) = \sum_{k \in \mathbb{Z}} \frac{E_k}{z^k}$$

The purpose of this note is to show that the Serre relations also exist in the $K$-theoretic Hall algebra of any surface. We prove that

**Theorem 1.1.** Given an integer $k$, let $e_k = [z^k O_S] \in K_{\text{GL}_1}(\text{Quot}_1)$ where $z$ is the standard representation of $\text{GL}_1$. Then we have

$$[[e_{k+1}, e_{k-1}], e_k] = 0$$

The main idea is inherited from Negut [6], while we study the stack case instead of the moduli space of stable sheaves. Thus we could generalize the Serre relations to other settings, like the PT categories of local surfaces [10]. We could also remove the Assumption A and Assumption S in [6].
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2. Moduli spaces of sheaves and their geometry

2.1. Quot Schemes and Flag Schemes. We follow the notation of [11] for Quot and flag schemes. Given a non-negative integer \( d \), we denote the Grothendieck’s Quot scheme \( \text{Quot}^d \) as the moduli scheme of quotients of coherent sheaves \( \{ \phi_d : \mathcal{O}_d \to \mathcal{E}_d \} \), such that

1. The sheaf \( \mathcal{E}_d \) has dimension 0 and length \( d \);
2. The morphism \( H^0(\phi) : k^d \to H^0(\mathcal{E}_d) \) is an isomorphism.

Over \( \text{Quot}^d \times \mathcal{S} \), there is a universal quotient of coherent sheaves with kernel denoted by \( \phi_d : \mathcal{O}^d \to \mathcal{E}_d \).

Given a sequence of non-decreasing integers \( d_\bullet = (d_0, d_1, \ldots, d_l) \), such that \( d_0 = 0 \) and \( d_l = d \), we fix a flag of inclusion maps \( F = \{ \mathcal{O}_{d_1} \subset \cdots \subset \mathcal{O}_{d_l} \} \) and denote \( \text{Flag}^d \) the moduli space of coherent sheaves \( \{ \mathcal{E}_{d_1} \subset \cdots \mathcal{E}_{d_l} \} \) with quotient maps \( \phi_i : \mathcal{O}_{d_i} \to \mathcal{E}_{d_i} \), such that

1. \( \mathcal{E}_{d_i} \) has dimensional 0 and length \( d_i \);
2. \( \phi_i \) are commutative with the inclusion maps;
3. \( H^0(\phi_i) : k^{d_i} \to H^0(\mathcal{E}_{d_i}, k) \) are isomorphisms.

We also denote

\[
\text{Quot}^d \mathbf{d}_\bullet = \prod_{i=1}^{k} \text{Quot}^{d_i-d_{i-1}} \mathbf{d}_\bullet.
\]

For each \( i \), over \( \text{Flag}^d \mathbf{d}_\bullet \times \mathcal{S} \) there are universal quotients of coherent sheaves \( \phi_i : \mathcal{O}_{d_i} \to \mathcal{E}_{d_i} \).

Fixing an isomorphism \( \mathcal{O}^{d_i-d_{i-1}} = \mathcal{O}^{d_i}/\mathcal{O}^{d_{i-1}} \) and defining \( \mathcal{E}_{d_i,d_{i-1}} := \mathcal{E}_{d_i}/\mathcal{E}_{d_{i-1}} \), we have

\[
\phi_{i,i-1} : \mathcal{O}^{d_i-d_{i-1}} \to \mathcal{E}_{d_i,d_{i-1}}
\]

is also surjective. It induces a morphism

\[
p_{d_\bullet} : \text{Flag}^d \mathbf{d}_\bullet \to \text{Quot}^d \mathbf{d}_\bullet.
\]

We will also consider the group actions on Quot schemes and flag schemes, with the following notations:

1. The group \( GL_d = GL_d(k) \) has a natural action on \( \text{Quot}^d \mathbf{d}_\bullet \) by acting on \( \mathcal{O}^n \).
2. Let \( P_{d_\bullet} \) be the parabolic group of \( GL_d \) which preserves the flag \( F \). \( P_{d_\bullet} \) has a natural action on \( \text{Flag}^d \mathbf{d}_\bullet \). Let \( B_{d_\bullet} \) be the parabolic Lie subalgebra of \( gl_n \) which preserves the flag \( F \).
(3) By \( \overline{\text{Flag}}_{d \ast}^0 \), is a closed subscheme of \( \text{Quot}_d^0 \) with an inclusion map
\[
i_{d \ast} : \overline{\text{Flag}}_{d \ast}^0 \to \text{Quot}_d^0.
\]
The morphism \( i_{d \ast} \) is \( P_{d \ast} \)-equivariant. Let \( \overline{\text{Flag}}_{d \ast}^0 = \text{Flag}_{d \ast}^0 \times_{P_{d \ast}} G_{d \ast} \). \( i_{d \ast} \) induces a proper \( G_{d \ast} \)-equivariant morphism
\[
q_{d \ast} : \overline{\text{Flag}}_{d \ast}^0 \to \text{Quot}_d^0.
\]

(4) We will use the notation \( \text{Quot}_{n,m}^\lambda \) for \( \text{Quot}_{d \ast}^0 \), where \( d \ast = (0, n, n \choose m) \) and \( \text{Quot}_{n,m,l}^\lambda = \text{Quot}_{d \ast}^0 \), for \( d \ast = (0, n, n, n \choose m + l) \). The same principle holds for other notations, like \( \text{Flag}_{d \ast}^0, P_{d \ast}, q_{d \ast} \) and so on.

(5) Given a matrix \( X \) (or other notations like \( Y, g, \) etc.) we will always denote \( X_{ij} \) (or \( Y_{ij}, g_{ij} \), etc.) the \( i \)-th row and \( j \)-th column of the matrix.

**Lemma 2.1.** \( \overline{\text{Flag}}_{d \ast}^0 \) is the moduli space of coherent sheaves
\[
\{ \mathcal{E}_{d_1} \subset \cdots \mathcal{E}_{d_l} \}
\]
with a quotient map
\[
\phi_d : \mathcal{O}^d \to \mathcal{E}_d
\]
such that \( H^0(\phi_d) : k^d \to H^0(\mathcal{E}, k) \) is an isomorphism.

**Proof.** It is obvious as \( [\overline{\text{Flag}}_{d \ast}^0/GL_d] = [\text{Flag}_{d \ast}^0/P_{d \ast}] \) which is the moduli stack of coherent sheaves
\[
\{ \mathcal{E}_{d_1} \subset \cdots \mathcal{E}_{d_l} \}
\]
where \( \mathcal{E}_{d_i} \) is a dimension 0, length \( d_i \), coherent sheaf. \( \square \)

We follow the notation of \( [6] \) for the set partition, i.e. an equivalence relation on a finite ordered set. We represent partitions suggestively, for example \( (x, y, x) \) will refer to the partition of a 3-element set into distinct 1-element subsets, while \( (x, y, x) \) (respectively \( (x, x, x) \)) refers to the equivalence relation which sets the first and the last element (respectively all elements) equivalent to each other. The size of a partition \( \lambda \), which is denoted by \(|\lambda|\), is the number of elements of the underlying set.

**Definition 2.2.** Given a positive integer \( n \), let \( d \ast = (0, 1, \ldots, n) \). For a set partition \( \lambda \) of size \( n \), we will consider the schemes \( \text{Flag}_\lambda^0 \) (or \( \overline{\text{Flag}}_\lambda^0 \)) which consist of elements in \( \text{Flag}_\lambda^0 \) such that
\[
\{ \mathcal{E}_0 = 0 \subset x_1 \mathcal{E}_1 \subset x_2 \cdots \subset x_n \mathcal{E}_n \}
\]
for some \( x_1, \ldots, x_n \in S \) such that \( x_i = x_j \) if \( i \sim j \in \lambda \), where \( \mathcal{F}' \subset_x \mathcal{F} \) means that \( \mathcal{F}' \subset \mathcal{F} \) and \( \mathcal{F}/\mathcal{F}' \cong k_x \).

**Remark 2.3.** On \( \text{Flag}_{1,1}^0 \) (or \( \overline{\text{Flag}}_{1,1}^0 \)), there are tautological vector bundles \( \mathcal{U}_1 \subset \mathcal{U}_2 \) such that the fiber at each closed point is the vector space of global sections of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) respectively. Moreover, we define line bundles \( \mathcal{L}_2 = \mathcal{U}_2/\mathcal{U}_1 = z_2 \mathcal{O} \) and \( \mathcal{L}_1 = \mathcal{U}_1 = z_1 \mathcal{O} \), where \( z_1 \) (or \( z_2 \)) is a character of \( P_{1,1} \) by mapping the matrix \( X \in P_{11} \) to \( X_{11} \) (or \( X_{22} \)).

On \( \overline{\text{Flag}}_{1,1}^0 \), there are universal vector bundles \( \mathcal{U}_i \) whose fiber at each closed point is the vector space of global sections of \( \mathcal{E}_i \) and we define line bundles \( \mathcal{L}_i = \mathcal{U}_i/\mathcal{U}_{i-1} \). \( \mathcal{L}_i = z_i \mathcal{O} \) is \( P_{1,1} \) equivariant on \( \text{Flag}_{1,1}^0 \) and \( G_3 \) equivariant on \( \overline{\text{Flag}}_{1,1}^0 \), where \( z_i \) are characters of \( P_{1,1} \) which map the matrix \( X \) to \( X_{ii} \).
The following lemmas state the local geometric properties of Flag schemes. While those lemmas hold for all smooth quasi-projective surfaces, we only need to prove the case when \( S = \mathbb{A}^2 \), as the problem is local.

**Lemma 2.4.** The schemes \( \text{Flag}^{2}_{x,y} \), \( \text{Flag}^{2}_{x,x} \), \( \text{Flag}^{2}_{x,y,z} \), \( \text{Flag}^{2}_{x,y,x} \), \( \text{Flag}^{2}_{y,x,x} \) are Gorenstein schemes of 5, 4, 9, 8, 8 dimensions respectively.

**Proof.** When \( S = \mathbb{A}^2 \),
- \( \text{Flag}^{2}_{x,y} = \{ X, Y \in B_{1,1} : [X, Y] = 0 \} \) is a subscheme of \( \mathbb{A}^6 = \mathbb{A}[X_{ij}, Y_{ij}]_{1 \leq i \leq j \leq 2} \) cut out by the following equation
  \[ X_{12}(Y_{11} - Y_{22}) = Y_{12}(X_{11} - X_{22}) \]
  and hence has dimension 5.
- \( \text{Flag}^{2}_{x,x} = \{ X, Y \in B_{1,1} : [X, Y] = 0 \} \) is a subscheme of \( \mathbb{A}^6 = \mathbb{A}[X_{ij}, Y_{ij}]_{1 \leq i \leq j \leq 2} \) cut out by the following equation
  \[ X_{11} = X_{22} \quad Y_{11} = Y_{22} \]
  and hence has dimension 4.
- \( \text{Flag}^{2}_{x,y,z} = \{ X, Y \in B_{1,1,1} : [X, Y] = 0 \} \) is a subscheme of \( \mathbb{A}^{12} = \mathbb{A}[X_{ij}, Y_{ij}]_{1 \leq i \leq j \leq 3} \) cut out by the following equations
  \[ X_{12}(Y_{11} - Y_{22}) = Y_{12}(X_{11} - X_{22}) \]
  \[ X_{23}(Y_{22} - Y_{33}) = Y_{23}(X_{22} - X_{33}) \]
  \[ X_{13}(Y_{11} - Y_{33}) - Y_{13}(X_{11} - X_{33}) = X_{12}Y_{23} - X_{23}Y_{12} \]
  and hence has dimension 9.
- \( \text{Flag}^{2}_{x,y,x} = \{ X, Y \in B_{1,1,1} : [X, Y] = 0, X_{22} = X_{33}, Y_{22} = Y_{33} \} \) is a subscheme of \( \mathbb{A}^{12} \) cut out by equations
  \[ X_{22} = X_{33}, Y_{22} = Y_{33} \]
  \[ X_{12}(Y_{11} - Y_{22}) = Y_{12}(X_{11} - X_{22}) \]
  \[ X_{13}(Y_{11} - Y_{22}) - Y_{13}(X_{11} - X_{22}) = X_{12}Y_{23} - X_{23}Y_{12} \]

Thus, it has dimension 8. The similar computation also holds for \( \text{Flag}^{2}_{y,x,x} \).

**Lemma 2.5.** \( \text{Flag}^{2}_{x,y,x} \) is Cohen-Macaulay of dimension 8.

**Proof.** When \( S = \mathbb{A}^2 \), \( \text{Flag}^{2}_{x,y,x} = \{ X, Y \in B_{1,1,1} : [X, Y] = 0, X_{11} = X_{33}, Y_{11} = Y_{33} \} \) is a subscheme of \( \mathbb{A}^{12} \) cut out by equations
\[ X_{11} = X_{33} \quad Y_{11} = Y_{33} \]
\[ X_{12}(Y_{11} - Y_{22}) = Y_{12}(X_{11} - X_{22}) \]
\[ X_{23}(Y_{22} - Y_{33}) = Y_{23}(X_{22} - X_{33}) \]
\[ X_{13}(Y_{11} - Y_{33}) - Y_{13}(X_{11} - X_{33}) = X_{12}Y_{23} - X_{23}Y_{12} \]

Let \( x_1 = X_{12}, x_2 = X_{11} - X_{22}, y_1 = Y_{12}, y_2 = Y_{11} - Y_{22}, x_3 = X_{23}, y_3 = Y_{23} \), then we only need to prove that
\[ k[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1y_2 - x_2y_1, x_2y_3 - x_3y_2, x_3y_1 - x_1y_3) \]
is Cohen-Macaulay of dimension 4, which follows from Claim 5.22 of [6].
Lemma 2.6. The schemes \( \text{Flag}_{x,y}^2, \text{Flag}_{x,x}^2, \text{Flag}_{x,y,x}^2, \text{Flag}_{x,x,y}^2, \text{Flag}_{x,y,x}^2, \text{Flag}_{y,x,x}^2 \) are normal.

Proof. By Lemma 2.4, \( \text{Flag}_{x,y}^2 \) is cut out by the equation

\[
X_{21}(Y_{11} - Y_{22}) - Y_{21}(X_{11} - X_{22}).
\]

By taking partial derivatives, we can compute that the singular locus is locally given by

\[
X_{21} = Y_{21} = X_{11} - X_{22} = Y_{11} - Y_{22} = 0,
\]

which is of codimension 3 and thus smooth.

Similarly, for the other cases, it suffices to show that the singular loci of \( \text{Flag}_{x,x,y}^2, \text{Flag}_{x,y,x}^2, \text{and Flag}_{y,x,x}^2 \) have codimension \( \geq 2 \). Here we only prove the \( \text{Flag}_{x,y,x}^2 \) case. In the proof of Lemma 2.4, we computed that \( \text{Flag}_{x,y,x}^2 \) is isomorphic to

\[
\mathbb{A}^4 \times \mathbb{A}^6 / (x_1 y_2 - x_2 y_1, x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3).
\]

The Jacobian matrix of this ideal is

\[
J = \begin{pmatrix}
y_2 & -y_1 & -x_2 & x_1 \\
y_3 & -y_1 & -x_3 & x_1 \\
y_3 & -y_2 & -x_3 & x_2
\end{pmatrix}
\]

At general points, the rank of \( J \) is 2. The singular locus is the set of points where \( \text{rank}(J) \leq 1 \). This only happens when \( x_i = y_i = 0 \) for all \( i \). Thus, the singular locus of \( \text{Flag}_{x,y,x}^2 \) has codimension 4, as desired.

2.2. Quadruple and Triple moduli space of sheaves.

Definition 2.7. We define the quadruple moduli space \( \mathcal{Y} \) which parameterizes the following commutative diagram

\[
\begin{array}{ccc}
x & \to & \mathcal{E}_1 \\
\downarrow y & & \downarrow \phi \\
0 & \to & \mathcal{E}_2
\end{array}
\]

of coherent sheaves where each successive inclusion is colength 1 and supported at the point indicated on the diagram, with a surjective morphism \( \phi : \mathcal{O}^2 \to \mathcal{E}_2 \) such that \( h^0(\phi) \) is an isomorphism.

We define \( \mathcal{Y}_+, \mathcal{Y}_- \) to be the moduli space which parameterize the following commutative diagrams:

\[
\begin{array}{ccc}
x & \to & \mathcal{E}_1 \\
\downarrow y & & \downarrow \phi \\
0 & \to & \mathcal{E}_2
\end{array}
\]
respectively, of coherent sheaves where each successive inclusion is colength 1 and supported at the point indicated on the diagram, with a surjective morphism $\phi : O^3 \to E_3$ such that $h^0(\phi)$ is an isomorphism.

For the above moduli spaces $\mathcal{Y}, \mathcal{Y}_+, \mathcal{Y}_-$, we denote $E_i = E_i'$ if $E_i'$ is not in diagrams (6)-(8). We denote $U_i$ (or $U_i'$) to be the locally free sheaves with fibers the vector space of global sections of $E_i$ (or $E_i'$) and denote

$$L_i = U_i / U_{i-1},$$

$$L_i' = U_i' / U_{i-1}' .$$

**Lemma 2.8.** $\mathcal{Y}, \mathcal{Y}_\pm$ are reduced.

**Proof.** We only prove the case $S = \mathbb{A}^2$, as it is still a local question. Let $\mathcal{Y}'$ be the fiber product

$$\mathcal{Y}' \longrightarrow Flag_{1,1} \times GL_2$$

$$\downarrow\phi'$$

$$Flag_{1,1} \times GL_2 \longrightarrow Quot_2 \times S \times S$$

Then $\mathcal{Y} = \mathcal{Y}'/(P_{1,1} \times P_{1,1})$ and to prove $\mathcal{Y}$ is reduced, we only need prove that $\mathcal{Y}'$ is reduced.

By (9), $\mathcal{Y}'$ contains

$$\{(X, Y, g; X', Y', g') : X, Y, X', Y' \in B_{1,1}, g, g' \in GL_2\}$$

such that

$$\begin{cases}
(X_{11}, Y_{11}) = (X_{22}', Y_{22}'), (X_{22}, Y_{22}) = (X_{11}', Y_{11}'), \\
gXg^{-1} = g'X'g'^{-1}, gYg^{-1} = g'Y'g'^{-1}, \\
XY = YX, X'Y' = Y'X'
\end{cases}$$

We replace $h = gg'^{-1}$, then $\mathcal{Y}'$ contains elements in

$$\{(X, Y, h; X', Y') : X, Y, X', Y' \in B_{1,1}, g \in GL_2\} \times GL_2$$

such that

$$\begin{cases}
(X_{11}, Y_{11}) = (X_{22}', Y_{22}'), (X_{22}, Y_{22}) = (X_{11}', Y_{11}') \\
hXh^{-1} = X', hYh^{-1} = Y', \\
XY = YX, X'Y' = Y'X'
\end{cases}$$

The condition $X'h = hX$ is equivalent to

$$\begin{cases}
h_{11}(X_{22} - X_{11}) = h_{12}X_{21}, \\
h_{11}X_{21}' = h_{22}X_{21}, \\
h_{22}(X_{22} - X_{11}) = h_{12}X_{21}'
\end{cases}$$
and similar for $Y'' h = h Y$. The condition $XY = YX$ is equivalent to

$$X_{21} (Y_{22} - Y_{11}) = Y_{21} (X_{22} - X_{11}).$$

We have two cases (since $h \in \text{GL}_2$):

- $h_{11} \neq 0$ (i.e., in the open subset $\{h_{11} \neq 0\} \cap \text{GL}_2$). We obtain

  $$\begin{cases}
  X_{22} - X_{11} = \frac{h_{22}}{h_{11}} X_{21}, \\
  X'_{21} = \frac{h_{22}}{h_{11}} X_{21},
  \end{cases} \quad \text{and} \quad
  \begin{cases}
  Y_{22} - Y_{11} = \frac{h_{22}}{h_{11}} Y_{21}, \\
  Y'_{21} = \frac{h_{22}}{h_{11}} Y_{21}.
  \end{cases}$$

  These equations cut out an affine space.

- $h_{12} \neq 0$. We obtain

  $$\begin{cases}
  X'_{21} = \frac{h_{22}}{h_{12}} (X_{22} - X_{11}), \\
  X_{21} = \frac{h_{22}}{h_{12}} (X_{22} - X_{11}),
  \end{cases} \quad \text{and} \quad
  \begin{cases}
  Y'_{21} = \frac{h_{22}}{h_{12}} (Y_{22} - Y_{11}), \\
  Y_{21} = \frac{h_{22}}{h_{12}} (Y_{22} - Y_{11}).
  \end{cases}$$

  These equations cut out an affine space.

Hence $\mathcal{Y}'$ is smooth and thus reduced.

For the scheme $\mathcal{Y}'_+$, by applying the similar argument, we define $\mathcal{Y}'_+$ through the Cartesian diagram:

$$\mathcal{Y}'_+ \longrightarrow \text{Flag}^c_{x,y,z} \times P_{21}$$

$$\downarrow \downarrow$$

$$\text{Flag}^c_{y,z,x} \longrightarrow \text{Flag}^{c}_{2,1} \times S \times S$$

and we only need to prove that $\mathcal{Y}'_+$ is reduced. We have $\mathcal{Y}'_+$ contains the elements in

$$\{(g, X, Y; X', Y') \in P_{21} \times D_{111}^1\}$$

which satisfy:

$$\begin{cases}
  X_{11} = X_{33} = X'_{11} = X'_{22}, X_{22} = X_{33}, \\
  Y_{11} = Y_{33} = Y'_{11} = Y'_{22}, Y_{22} = Y_{33}, \\
  X' = gXg^{-1}, Y' = gYg^{-1}, \\
  XY' - YX = X'Y' - Y'X' = 0.
  \end{cases}$$

The condition $X'g = gX$ is equivalent to

$$\left(\begin{array}{cccc}
  g_{11} X_{11} & g_{11} X'_{21} + g_{21} X_{11} \\
  g_{11} X'_{31} + g_{21} X'_{32} + g_{31} X_{22} & g_{22} X_{11} & g_{22} X'_{31} + g_{32} X_{22} & g_{23} X_{11} \\
  g_{21} X_{11} & g_{22} X_{21} + g_{23} X_{31} & g_{22} X_{22} + g_{32} X_{32} & g_{23} X_{11} \\
  g_{31} X_{11} + g_{32} X_{21} + g_{33} X_{31} & g_{32} X_{21} + g_{33} X_{32} & g_{32} X_{22} + g_{33} X_{32} & g_{33} X_{11}
\end{array}\right)$$

Since $g_{11} \neq 0$, we can solve $X'_{21}, X'_{31}$:

$$X'_{21} = \frac{1}{g_{11}} (g_{22} X_{21} + g_{32} X_{31})$$

$$X'_{31} = \frac{1}{g_{11}} (g_{31} X_{11} + g_{32} X_{21} + g_{33} X_{31} - g_{21} X'_{32} - g_{31} X_{22})$$
The other equations are:

\[
\begin{align*}
g_{22}X_{32} &= g_{33}X_{32}, \\
g_{22}(X_{11} - X_{22}) &= g_{23}X_{32}, \\
g_{33}(X_{11} - X_{22}) &= g_{23}X_{32}'.
\end{align*}
\]

The condition \(XY - YX = 0\) gives

\[
\begin{align*}
X_{21}(Y_{11} - Y_{22}) &= Y_{21}(X_{11} - X_{22}), \\
X_{32}(Y_{11} - Y_{22}) &= Y_{32}(X_{11} - X_{22}), \\
X_{21}Y_{32} &= X_{32}Y_{21}.
\end{align*}
\]

Denote \(X_0 = X_{11} - X_{22}\) and \(Y_0 = Y_{11} - Y_{22}\). Then \(Y'_+\) is

\[
\text{Spec } \mathbb{C}[X_{11}, X_{21}, X_{31}, X_{32}, X_0, X'_3, Y_{11}, Y_{21}, Y_{31}, Y_{32}, Y_0, Y'_3, g_{ij}]
\]

cut out by the equations

\[
\begin{align*}
g_{22}X'_{32} &= g_{33}X_{32}, \\
g_{22}X_0 &= g_{23}X_{32}, \\
g_{33}X_0 &= g_{23}X'_3, \\
X_{21}Y_0 &= Y_{21}X_0, \\
X_{32}Y_0 &= Y_{32}X_0,
\end{align*}
\]

On the open subset such that \(g_{23} \neq 0\), we have \(X_{32} = \frac{g_{23}X_0}{g_{23}}\) and \(X'_{32} = \frac{g_{23}X_0}{g_{23}}\)
and similarly for \(Y\)'s. The remaining equation is

\[
X_{21}Y_0 = Y_{21}X_0,
\]

and hence reduced.

On the open subset that \(g_{22}g_{33} \neq 0\), we have \(X'_{32} = \frac{g_{22}X_0}{g_{22}}\) and \(X_0 = \frac{g_{22}X_{32}}{g_{22}}\) and similarly for \(Y\)'s. Thus, the remaining equation is

\[
X_{21}Y_{32} = X_{32}Y_{21},
\]

and also reduced. Since \(\det g \neq 0\), we have either \(g_{23} \neq 0\) or \(g_{22}g_{33} \neq 0\), thus \(Y'_+\) and \(Y\) are reduced. The similar arguments hold for \(Y'_-\). □

### 2.3. A Vanishing Argument.

**Definition 2.9.** We define the following morphisms:

\[
\begin{align*}
\mathcal{Y} & \xrightarrow{\pi} \mathcal{Y}_+ & \mathcal{Y} & \xrightarrow{\pi} \mathcal{Y}_- \\
\overline{\text{Flag}}^{\alpha}_{y,x} & \mathcal{Y}_+ & \overline{\text{Flag}}^{\alpha}_{y,x} & \mathcal{Y}_- \\
\overline{\text{Flag}}^{\beta}_{y,x} & \mathcal{Y}_+ & \overline{\text{Flag}}^{\beta}_{y,x} & \mathcal{Y}_-
\end{align*}
\]

obtained by remembering only the top part of the square in (6)-(8). We also define the following morphisms:

\[
\begin{align*}
\mathcal{Y} & \xleftarrow{\pi} \mathcal{Y}_+ & \mathcal{Y} & \xleftarrow{\pi} \mathcal{Y}_- \\
\overline{\text{Flag}}^{\alpha}_{y,x} & \mathcal{Y}_+ & \overline{\text{Flag}}^{\alpha}_{y,x} & \mathcal{Y}_- \\
\overline{\text{Flag}}^{\beta}_{y,x} & \mathcal{Y}_+ & \overline{\text{Flag}}^{\beta}_{y,x} & \mathcal{Y}_-
\end{align*}
\]

obtained by remembering only the bottom part of the square in (6)-(8).
Proposition 2.10. We have the following commutative diagram:

\[
\begin{array}{c}
\mathcal{Y} \\
\downarrow \iota^\uparrow \quad \downarrow \\
\mathbb{P}_{\text{Flag}_{1,1}}(U_2) \\
\downarrow \\
\text{Flag}_{1,1}^0
\end{array}
\]

where \(\iota^\uparrow\) and \(\iota^\downarrow\) are closed embeddings. The same property holds for the spaces \(\mathcal{Y}_+, \mathcal{Y}_-\) while replacing \(\text{Flag}_{x,y}^0\) by the corresponding moduli spaces in \([10]\) and \([11]\) and replacing \(U_2\) by \(U_2\) or \(U_3/\!\!/U_1\) respectively.

Proof. We will only prove the statement above for \(\mathcal{Y}\), as the cases of \(\mathcal{Y}_-\) and \(\mathcal{Y}_+\) hold analogously. The inclusion of locally free sheaves \(L_1 \subset U_2\) induces the morphism \(\iota : \mathcal{Y} \to \mathbb{P}_{\text{Flag}_{1,1}}\). We claim that corresponding map \(\iota\) is a closed embedding.

By Theorem 1.7.8 of [4], it suffices to show that for each closed point \(p \in \text{Flag}_{1,1}^0\), the map between the fibers \(\iota_p : \mathcal{Y}_p \to \mathbb{P}_{\text{Flag}_{1,1}}(U_2)_p\) is a closed embedding. When \(E_2\) are supported in two different points or supported in a single point but not semi-simple, \(\mathcal{Y}_p\) is a closed point. When \(E_2\) is semi-simple and supported in a single point, i.e. \(E_2 \cong kx^2\) for \(x \in S\), then \(\mathcal{Y}_p = \text{Flag}_{1,1}^0\) and \(\iota_p\) is an isomorphism. \(\square\)

Proposition 2.11. The morphism \(\pi^\uparrow : \mathcal{Y} \to \text{Flag}_{1,1}^0\) is proper and satisfies

\[
R^i\pi^\uparrow_! \mathcal{O}_\mathcal{Y} = \begin{cases} 
\mathcal{O}_{\text{Flag}_{1,1}^0}, & \text{if } i = 0, \\
0, & \text{if } i > 0.
\end{cases}
\]

The analogous properties hold for \(\pi^\downarrow\). Moreover, the analogous properties hold with the scheme \(\mathcal{Y}\) replaced by the schemes \(\mathcal{Y}_-\) and \(\mathcal{Y}_+\).

Proof. By Proposition 2.10 we can embed \(\mathcal{Y}\) into a \(\mathbb{P}^1\)-bundle \(\mathbb{P}(U_2)\) over \(\text{Flag}_{1,1}^0\). Denote the projection \(\pi : \mathbb{P}(U_2) \to \text{Flag}_{1,1}^0\), then

\[
R^i\pi_*(\mathcal{O}_{\mathbb{P}(U_2)}) = 0 \text{ for all } i \geq 1,
\]

and for any coherent sheaf \(\mathcal{F}\) on \(\mathbb{P}(U_2)\),

\[
R^i\pi_*(\mathcal{F}) = 0 \text{ for all } i \geq 2.
\]

Now, from the exact sequence

\[
0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}(U_2)} \to \iota_* \mathcal{O}_\mathcal{Y} \to 0,
\]

(where \(\mathcal{K}\) is the kernel sheaf) we obtain the long exact sequence

\[
\cdots \to R^i\pi_*(\mathcal{O}_{\mathbb{P}(U_2)}) \to R^i\pi_*(\mathcal{O}_\mathcal{Y}) \to R^{i+1}\pi_*(\mathcal{K}) \to \cdots
\]

This implies that \(R^i\pi_*(\mathcal{O}_\mathcal{Y}) = 0\) for \(i \geq 1\). The \(i = 0\) case follows from Stein factorization and the following facts:

- \(\text{Flag}_{1,1}^0\) is normal. (Lemma 2.10)
- \(\mathcal{Y}\) is reduced. (Lemma 2.8)
- \(\pi^\uparrow\) is proper and all its fibers are either a point or \(\mathbb{P}^1\).

\(\square\)
Proposition 2.12. On \( Y \) or \( Y_+ \), the natural map 
\[
\mathcal{L}_1 = \mathcal{U}_1 \subset \mathcal{U}_2 \rightarrow \mathcal{U}_2/\mathcal{U}_1' = \mathcal{L}_2'
\]
induces a global section of \( \mathcal{L}_2' \otimes \mathcal{L}_1^{-1} \), with the zero section consisting of the data 
\[
\{ (\mathcal{E}_1,x) = (\mathcal{E}_1',y) \} \subset Y,
\]
which is isomorphic to \( \widetilde{\text{Flag}}_{2,x} \) or \( \widetilde{\text{Flag}}_{2,x,x} \).

Analogously, on \( Y_- \), there is a global section of \( \mathcal{L}_3' \otimes \mathcal{L}_2^{-1} \) such that the zero section is isomorphic to \( \widetilde{\text{Flag}}_{2,x,x} \).

Proof. We will only prove the case on \( Y \). The morphism is identity when \( \mathcal{E}_2 \) is supported in two different points and always zero when \( \mathcal{E}_2 \) is supported in one point and not semisimple. When \( \mathcal{E}_2 \) is supported in one point and semisimple, then the morphism vanishes if and only if \( \mathcal{E}_1 = \mathcal{E}_1' \). \( \square \)

3. \( K \)-theoretic Hall algebra on Surfaces

In this section, we review the \( K \)-theoretic Hall algebra on surfaces in [11]. Given a reductive group \( G \) acting on \( X \), we denote \( K_G(X) \) the \( G \)-equivariant Grothendieck groups of coherent sheaves on \( X \).

3.1. Refined Gysin maps and Derived Fiber Squares. A morphism \( f : X \rightarrow Y \) is called a local complete intersection (l.c.i.) morphism if \( f \) is the composition of a regular embedding and a smooth morphism. In this case, \( f \) has finite Tor dimension.

Definition 3.1. Suppose we have a Cartesian diagram
\[
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow f' & & \downarrow f \\
Y' & \longrightarrow & Y \\
\end{array}
\]
where \( f \) is a l.c.i. morphism. The refined Gysin map \( f^! : K(Y') \rightarrow K(X') \) is defined by
\[
f^!(\mathcal{F}) = \sum_i (-1)^i \text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}).
\]

Consider a Cartesian diagram
\[
\begin{array}{ccc}
X' & \overset{g}{\longrightarrow} & X \\
\downarrow f' & & \downarrow f \\
Y' & \overset{g'}{\longrightarrow} & Y \\
\end{array}
\]
where \( f : X \rightarrow Y \) is a l.c.i. morphism and \( g' : Y' \rightarrow Y \) is a closed embedding.

Definition 3.2. In diagram (12), if \( X' \) has the expected dimension, i.e., if 
\[
\dim X' - \dim Y' = \dim X - \dim Y,
\]
then we say that (12) is a derived fiber square.

A derived fiber square has the following property:
Proof. Without the loss of generality, we assume that \( f \) is a regular embedding. The problem is local, and thus we assume that \( Y = SpecA \) for a local ring \( A \), and \( X = SpecA/(f_1, \ldots, f_k) \) for some \( f_1, \ldots, f_k \) in \( A \) such that \( k = \dim Y - \dim X \).

Moreover, by Lemma 2.8 of [11], we can assume that \( k = 1 \). Let \( Y' = SpecB \) for another Cohen-Macaulay local ring. By Krull’s principal ideal theorem, \( f_1 \) is a non-zero divisor in \( B \) and thus \( Tor^i_A(\mathcal{F}, A/(f_1)) = 0 \) if \( i > 0 \) when \( \mathcal{F} \) is locally free.

3.2. \( K \)-theoretic Hall Algebra on Surfaces. The \( K \)-theoretic Hall algebra on \( S \) [11] is an associative algebra structure on

\[
K(Coh) = \bigoplus_{n=0}^{\infty} K_{GL_n}(Quot^n_{n,m}).
\]

The algebra structure is given as follows:

- For any two non-negative integers \( n, m \), there is a linear morphism of locally free sheaves \( \psi_{n,m} : W_{n,m} \to V_{n,m} \) on \( Quot^n_{n,m} \) with the Cartesian diagram:

\[
\begin{array}{ccc}
\text{Flag}^o_{n,m} & \longrightarrow & W_{n,m} \\
\downarrow & & \downarrow \psi_{n,m} \\
Quot^n_{n,m} \times Quot^m_{n,m} & \longrightarrow & V_{n,m}
\end{array}
\]

which induces the refined Gysin map:

\[
\psi_{n,m} : K_{GL_n}(Quot^n_{n,m}) \times K_{GL_m}(Quot^m_{n,m}) \to K_{p_{n,m}}(\text{Flag}^o_{n,m}).
\]

Moreover,

\[
dim W_{n,m} - dim V_{n,m} = hom(\mathcal{I}^n, \mathcal{E}_m) - ext^1(\mathcal{I}^n, \mathcal{E}_m) = nm
\]

by the Grothendieck-Riemann-Roch formula.

- Formula (5.2.17) of [11] induces the isomorphism:

\[
\text{ind}^{GL_{n+m}}_{p_{n,m}} : K_{p_{n,m}}(\text{Flag}^o_{n,m}) \cong K_{GL_{n+m}}(\text{Flag}^o_{n,m})
\]

- The morphism \( q_{n,m} : \text{Flag}^o_{n,m} \to Quot^o_{n+m} \) is proper and induces a push forward morphism: \( q_{n,m} : K_{GL_{n+m}}(\text{Flag}^o_{n,m}) \to K_{GL_{n+m}}(Quot^o_{n+m}) \).

- We define the algebraic structure \( * : K_{GL_n}(Quot^n_{n,m}) \times K_{GL_m}(Quot^m_{n,m}) \to K_{GL_{n+m}}(Quot^o_{n+m}) \) as the composition of \( \psi_{n,m} \), \( \text{ind}^{GL_{n+m}}_{p_{n,m}} \) and \( q_{n,m} \).

Definition 3.4. We define the \( K \)-theoretic class \( e_{(d_1, \ldots, d_n)} \in K_{GL_n}(Quot^n_{n,m}) \) when \( n \leq 3 \) by \( e_k = [z^k \mathcal{O}_S] \) where \( z \) is the standard character of \( \mathbb{G}_m \) and

\[
e_{k_1,k_2} = q_{1,1,1}(\mathcal{L}^{k_1} \mathcal{L}^{k_2} \mathcal{O}_{\text{Flag}^o_{n,m}});
\]

\[
e_{k_1,k_2,k_3} = q_{1,1,1,1}(\mathcal{L}^{k_1} \mathcal{L}^{k_2} \mathcal{L}^{k_3} \mathcal{O}_{\text{Flag}^o_{n+1,m}})
\]

Proposition 3.5. Given two integers \( d \geq k \),

\[
[e_d, e_k] := e_d \ast e_k - e_k \ast e_d = \sum_{a=k}^{d-1} e_{a,d+k-a} \in K_{GL_2}(Quot^o_2).
\]
Proof. By Lemma 2.14, Flag\textsuperscript{0} \textsubscript{x,y} is Cohen-Macaulay of expected dimensions. By Proposition 3.3, we have
\[ e_d \cdot e_k = q_{1,1} (L_1^d L_2^k) = q_{1,1} \cdot \pi_1^* (L_1^d L_2^k) \]
\[ e_k e_d = q_{1,1} (L_1^d L_2^k) = q_{1,1} \cdot \pi_1^* (L_1^k L_2^d) = q_{1,1} \cdot \pi_2^* (L_1^k L_2^d) \]
\[ e_{i,j} = q_{1,1} \cdot \pi_i^* (L_1^d L_2^i L_3^j O_{\widetilde{\text{Flag}}_{x,y}^\circ}) \]
Hence we only need to prove that
\[ [L_1^d L_2^k] - [L_1^k L_2^d] = \sum_{i=0}^{d-k-1} [L_1^{k+i} L_2^{d-i} O_{\widetilde{\text{Flag}}_{x,y}^\circ}] \]
on \text{K}_{GL_2}(\mathcal{Y}).
By Proposition 2.12, there is an exact sequence of GL\textsubscript{2}-equivariant coherent sheaves on \mathcal{Y}:
\[ 0 \to L_1^{-1} \otimes L_2 \to O_\mathcal{Y} \to O_{\text{Flag}_{x,y}^\circ} \to 0 \]
Since \( L_1 L_2 \cong L_1 L'_2 \), on \text{K}_{GL_2}(\mathcal{Y}) we have
\[ [L_1] - [L_2] = [O_{\text{Flag}_{x,y}^\circ} \otimes L_2^2] = [O_{\text{Flag}_{x,y}^\circ}] [L_2] \]
From this, we can calculate on \text{K}_{GL_2}(\mathcal{Y}) that
\[ [L_1^d L_2^k] - [L_1^k L_2^d] = [L_1^d L_2^k ([L_1] - [L_2])] \sum_{i=0}^{d-k-1} [L_1^{k+i} L_2^{d-i}] \]
\[ = [L_1^d L_2^k] [L_2] \sum_{i=0}^{d-k-1} [L_1^{k+i} L_2^{d-i}] \sum_{i=0}^{d-k-1} [L_1^{k+i} L_2^{d-i}] \]
as on \( \text{Flag}_{x,y}^\circ \), \( L_1' \cong L_1 \) and \( L_2' \cong L_2 \). \qed

Proposition 3.6. Suppose \( d_1, d_2, k \) are integers, then
\[ [e_{d_1, d_2}, e_k] = \begin{cases} \sum_{a=d_1}^{k-1} e_{a, d_1+k-a, d_2}, & \text{if } k \geq d_1, \\ \sum_{a=k}^{d_1} e_{a, d_1+k-a, d_2}, & \text{if } k < d_1 \end{cases} + \begin{cases} \sum_{a=d_2}^{k-1} e_{d_1, a, d_2+k-a}, & \text{if } k \geq d_2, \\ \sum_{a=k}^{d_2} e_{d_1, a, d_2+k-a}, & \text{if } k < d_2. \end{cases} \]

Proof. As \( \text{Flag}_{x,y}^\circ, \text{Flag}_{x,y}^\circ, \text{Flag}_{x,y}^\circ, \text{Flag}_{x,y}^\circ, \text{Flag}_{x,y}^\circ, \text{Flag}_{x,y}^\circ \) are all Cohen-Macaulay of expected dimension, we have
\[ e_{d_1, d_2} \cdot e_k = p_{1,1,1} ([L_1^d L_2^k L_3^2 O_{\text{Flag}_{x,y}^\circ}]) \]
\[ e_k \cdot e_{d_1, d_2} = p_{1,1,1} ([L_1^k L_2^d L_3^2 O_{\text{Flag}_{x,y}^\circ}]) \]
We define \( f_{d_1,k,d_2} = p_{1,1,1} ([L_1^d L_2^k L_3^2 O_{\text{Flag}_{x,y}^\circ}]) \), then by the similar argument as Proposition 3.5, we have
\[ e_{d_1, d_2} \cdot e_k - f_{d_1,k,d_2} = \begin{cases} \sum_{a=d_1}^{k-1} e_{a, d_1+k-a, d_2}, & \text{if } k \geq d_1, \\ \sum_{a=k}^{d_1} e_{a, d_1+k-a, d_2}, & \text{if } k < d_1, \end{cases} \]

\[ f_{d_1,k,d_2} - e_k \cdot e_{d_1,d_2} = \begin{cases} -\sum_{a=k}^{d_2-1} e_{d_1,a,d_2+k-a}, & \text{if } k \geq d_2, \\ \sum_{a=k}^{d_2-1} e_{d_1,a,d_2+k-a}, & \text{if } k < d_2. \end{cases} \]

## 3.3. The proof of Theorem 1.1

Theorem 1.1 directly follows from Proposition 3.5 and Proposition 3.6 as

\[
\begin{align*}
[e_{k+1}, e_{k-1}] & = [e_{k-1,k+1} + e_{k,k}, e_k] \\
& = e_{k-1,k+1} - e_{k-1,k,k+1} \\
& = 0.
\end{align*}
\]

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