Nonlinear gravitons from the initial value constraints of GR in Ashtekar variables

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Abstract. In this paper we provide a possible realization of Penrose’s idea of nonlinear gravitons by constructing a solution to the initial value constraints in Ashtekar variables. The solution inputs are a spatial SU(2) connection and two free functions of position, and can be constructed as a formal operatorial expansion in powers of the cosmological constant about spacetimes of Petrov Type O. We first present the linear case, and then provide a simple nonlinear example to first order using a spatially homogeneous connection.

1. Introduction: The linear graviton
In [1] Roger Penrose takes issue with the standard view of the graviton as a weak-field perturbation of a background spacetime. He proposes the idea that each graviton should carry its measure of curvature, corresponding to a solution of the full nonlinear Einstein equations. Motivated by Penrose’s idea, in this paper we will provide a possible realization of the concept of nonlinear gravitons using the initial value constraints of general relativity (GR). The initial value constraints of GR with cosmological constant $\Lambda$ in the Ashtekar variables [2] are the Gauss’ law, diffeomorphism and Hamiltonian constraints ($G_a, H_i, H$) given respectively by

$$D_i \tilde{\sigma}_a^i = \partial_i \tilde{\sigma}_a^i + f_{abc} A_b^i \tilde{\sigma}_a^i = 0; \quad \epsilon_{ijk} \tilde{\sigma}_a^j B_a^k = 0; \quad \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \left( \frac{\Lambda}{3} \tilde{\sigma}_c^k + B_c^k \right) = 0,$$

where $A_a^i$ is a left-handed $SU(2)$ gauge connection and $\tilde{\sigma}_a^i$ is a densitized triad.\(^1\) We have also defined $B_a^i = \frac{1}{2} \epsilon^{ijk} (\partial_j A_a^k - \partial_k A_a^j + f_{abc} A_b^i A_c^k)$ as the magnetic field of $A_a^i$, and $D_i v_a = \partial_i v_a + f_{abc} A_b^i v_c$ is the $SU(2)$-covariant derivative with structure constants $f_{abc} = \epsilon_{abc}$. Define the Ansatz

$$\tilde{\sigma}_a^i = -\left( \frac{3}{\Lambda} \delta_{ae} + \epsilon_{ae} \right) B_e^i, \quad (\det B) \neq 0,$$

where $\epsilon_{ae} \in SU(2) \otimes SU(2)$. Substitution of (2) into (1) yields

$$B_e^i D_i \{\epsilon_{ae}\} = 0; \quad \epsilon_{dae} \epsilon_{ae} = 0; \quad \text{tr} \epsilon + \frac{\Lambda}{3} \text{Var} \epsilon + \frac{\Lambda^2}{3} \text{det} \epsilon = 0,$$

\(^1\) Symbols from the beginning part of the Latin alphabet $a, b, c, \ldots$ denote internal indices, and from the middle $i, j, k, \ldots$ denote spatial indices.
where we have defined $\text{Var} M = (\text{tr} M)^2 - \text{tr} M^2$. To obtain the third equation of (3) we have cancelled off a factor of $\det B \neq 0$ as well as any numerical pre-factors. Defining vector fields $v_\epsilon = B^i_i \partial_i$, and the magnetic ‘helicity density matrix’ $C_{be} = A^b_i B^i_e$, then the Gauss’ law constraint $G_a$ can be written as

$$B^i_e D_i \epsilon_{ae} = v_\epsilon \{ \epsilon_{ae} \} + (f_{abf} \delta_{ge} + f_{ebg} \delta_{af}) C_{be} \Psi fg \equiv w_\epsilon \{ \epsilon_{ae} \} = 0. \quad (4)$$

Note for $\epsilon_{ae} = 0$ that (3) is identically satisfied, which corresponds to a particular class of spacetimes of Petrov Type O since all three eigenvalues of $\epsilon_{ae}$ are equal. Let us first consider the case of a linear gravitational wave propagating on these background spacetimes. Choose $|\epsilon_{ae}| << \frac{1}{3 \lambda}$ as a perturbation and choose a connection $A^0_i = \delta^0_i \alpha + a^0_i$, where $|a^0_i| << \alpha$. Then we substitute this Ansatz into (3) and expand to linear order in $\epsilon_{ae}$ and $a^0_i$. The magnetic field to linearised order is given by $B^i_e = \epsilon^{ijk} \partial_j a^0_k - \alpha a^0_i + \delta^i_j (\alpha^2 + \text{otra})$. Since (3) is already of linearised order in $\epsilon_{ae}$, then we need only expand $B^i_e$ and $C_{ae}$ to zeroth order in $a^0_i$, since all terms of the form $\epsilon \epsilon$, $\epsilon a$ and $aa$ are of second order. So we only need $v_\epsilon = B^i_i \partial_i \sim \alpha^2 \partial_i$ and $C_{ae} \sim \delta_{ae} \alpha^3$. Note that using $C_{ae} \propto \delta_{be}$ causes the terms in brackets in (4) to drop out due to antisymmetry of the structure constants $f_{abc}$. Then the linearized counterparts to (3) to zeroth order in $\lambda$ reduce to

$$\partial_i \epsilon_{ae} = 0; \quad \epsilon_{ae} \epsilon_{ae} = 0; \quad \text{tr} \epsilon = 0. \quad (5)$$

Equation (5) states that the perturbation $\epsilon_{ae}$ is transverse, traceless and symmetric, namely that it corresponds to a linearized gravitational wave.

2. Generalization to the full nonlinear case

Having shown that the initial value constraints can produce linearized gravitons, we will now progress to the full nonlinear case. We will define the full Hamiltonian constraint can be written in the form

$$\text{tr} \epsilon + \Lambda f^{abc} \epsilon_{ae} \epsilon_{bf} + \Lambda^2 E^{abc} \epsilon_{ae} \epsilon_{bf} \epsilon_{cf} = 0, \quad (6)$$

where we have defined $f^{abc} = \frac{1}{3} (\delta^a_b \delta^c_f - \delta^a_c \delta^b_f)$ and $E^{abc} = \frac{1}{3} \epsilon^{abc} \epsilon_{fg}$. Defining $\epsilon^f_{ae}$ and $E^f_{ae}$ as an orthogonal basis of diagonal and off-diagonal symmetric matrix elements, let us write $\epsilon_{ae}$ in the matrix form

$$\epsilon_{ae} = \epsilon^f_{ae} \varphi_f + E^f_{ae} \Psi_f = \begin{pmatrix} \varphi_1 & \Psi_3 & \Psi_2 \\ \Psi_3 & \varphi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \varphi_3 \end{pmatrix}.$$

Then defining $C_{[ae]} = C_{ae} - C_{ea}$, the Gauss’ law constraint (4) is given by $\epsilon^f_{ae} w_\epsilon \{ \varphi_f \} + E^f_{ae} w_\epsilon \{ \Psi_f \} = 0$ with matrix form

$$\begin{pmatrix} v_1 - C_{[23]} & -c_{32} & c_{23} \\ c_{31} & v_2 - C_{[31]} & -c_{13} \\ -c_{21} & c_{12} & v_3 - C_{[12]} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} + \begin{pmatrix} c_{22} - c_{33} & v_3 - C_{12} + 2C_{21} & v_2 - 2C_{31} + C_{13} \\ v_3 - 2C_{12} + C_{21} & c_{33} - C_{11} & v_1 - C_{23} + 2C_{32} \\ v_2 - C_{31} + 2C_{13} & v_1 - 2C_{23} + C_{32} & c_{11} - C_{22} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = 0.$$

Assuming invertibility of the matrix of differential operators $E^f_{ae} w_\epsilon$, then we can define $J^f_j = -(E^f_{aj} w_\epsilon)^{-1} (\epsilon^g_{ae} w_\epsilon g) f$ as a propagator from the diagonal to the off-diagonal elements of $\epsilon_{ae}$. This enables one formally to rewrite the Gauss’ law constraint as an embedding map

$$\Psi_f = J^f_j \varphi_g \rightarrow \epsilon_{ae} = (\epsilon^g_{ae} + E^f_{ae} J^f_j) \varphi_f \equiv T^f_{ae} \varphi_f = T^f_{ae} [A] \varphi_f. \quad (7)$$
Then defining $Q_{fg} \equiv I^{abfc} T^f_{ab} T^g_{bc}$ and $Q_{fgh} \equiv E^{abf} g I T^a_{ab} T^g_{be} T^h_{ec}$ the Hamiltonian constraint on solutions to the Gauss’ law constraint becomes

$$\varphi_3 = -\varphi_1 - \varphi_2 - \Lambda Q_{fg} \varphi_f \varphi_g - \Lambda^2 Q_{fgh} \varphi_f \varphi_g \varphi_h \equiv \Lambda Q(\varphi; A). \quad (8)$$

This can be solved by fixed point iteration by defining a sequence $(\varphi_3)_0 = -(\varphi_1 - \varphi_2)$ and the recursion relation $(\varphi_3)_{n+1} = Q[\varphi_1, \varphi_2, (\varphi_3)_n; A]$. Then the full solution, if convergent, is given by $\varphi_3 = \lim_{n \to \infty} (\varphi_3)_n$. So the prescription for writing down a solution starts by making a choice of connection $A_\mu^a$. This defines the Gauss’ law propagator $J_f^g = \mathcal{J}_{fg}[A]$ and the integrodifferential matrix operator for the embedding map $\hat{T}_{ef}^a = \hat{T}_{ef}^a[A]$. Then one chooses two free functions $\varphi_1$ and $\varphi_2$, and constructs the solution

$$e_{ae} = \hat{T}^a_{ae} \varphi_1 + \hat{T}^a_{ae} \varphi_2 + \hat{T}^3_{ae} \varphi_3[\varphi_1, \varphi_2; A] = e_{ae}[\varphi_1, \varphi_2; A]. \quad (9)$$

In (9) $\varphi_1$ and $\varphi_2$ constitute the two physical degrees of freedom, and the connection $A_\mu^a$ forms an input into $\hat{T}_{ae}^a$ and into $\varphi_3$ through the latter.

### 3. Example: spatially homogeneous connection

We will demonstrate using a spatially homogeneous connection $A_\mu^a$, with $\varphi_f = n_f e^{\beta r}$ and $\Psi_f = m_f e^{\beta r}$. For simplicity, we have chosen $m_f, n_f$ and $k_f$ as numerical constants. The following objects follow from $A_\mu^a$

$$B_\mu^a = (\det A)(A^{-1})_a^i; \quad C_{ae} = (\det A) \delta_{ae}; \quad v_a = (\det A)(A^{-1})_a^i \partial_i, \quad (10)$$

and the vector fields $v_a$ have the following action

$$v_a \{ \varphi_f \} = (\det A) e_a \varphi_f; \quad v_a \{ \Psi_f \} = (\det A) e_a \Psi_f, \quad (11)$$

where we have defined $e_a \equiv (A^{-1})_a^i k_i$. After dividing by $(\det A) \neq 0$, then the Gauss’ law constraint propagator $\mathcal{J}_f^g$ consists of c-numbers, given by

$$\mathcal{J}_f^g = \frac{1}{2} \begin{pmatrix} (e_1)^2 (e_2 e_3)^{-1} & -e_2 (e_1 e_3)^{-1} & -e_3 (e_2 e_1)^{-1} \\ -e_1 (e_3 e_2)^{-1} & (e_2)^2 (e_3 e_1)^{-1} & -e_3 (e_1 e_2)^{-1} \\ -e_2 (e_1 e_3)^{-1} & -e_3 (e_1 e_2)^{-1} & (e_3)^2 (e_1 e_2)^{-1} \end{pmatrix}.$$  

In this case the action of the Gauss’ law propagator $\mathcal{J}_f^g$ is algebraic, due to the functional form of $e_{ae}$. So (4) is the same as

$$m_1 = \frac{1}{2} \left[ \left( \frac{e_1}{e_2 e_3} \right)^2 n_1 - \left( \frac{e_2}{e_3} \right)^2 n_2 - \left( \frac{e_3}{e_2} \right) n_3 \right];$$

$$m_2 = \frac{1}{2} \left[ -\left( \frac{e_1}{e_3} \right)^2 n_1 + \left( \frac{e_2}{e_3 e_1} \right)^2 n_2 - \left( \frac{e_3}{e_1} \right) n_3 \right];$$

$$m_3 = \frac{1}{2} \left[ -\left( \frac{e_1}{e_2} \right)^2 n_1 - \left( \frac{e_2}{e_1} \right)^2 n_2 + \left( \frac{e_3}{e_1 e_2} \right)^2 n_3 \right]. \quad (12)$$

Using (12), after some algebra we have the following relations

$$V_{are} = 2 (n_1 n_2 + n_2 n_3 + n_3 n_1) - (m_1)^2 - (m_2)^2 - (m_3)^2;$$

$$\det e = n_1 n_2 n_3 + 2 m_1 m_2 n_3 - n_1 (m_1)^2 - n_2 (m_2)^2 - n_3 (m_3)^3 = 0 \quad (13)$$
where we have defined $|e|^2 = (e_1)^2 + (e_2)^2 + (e_3)^2$. Note that the determinant of $\epsilon_{ae}$ vanishes. Substitution of (13) into (6) yields

$$n_3 = -n_1 - n_2 - \Lambda Q[\vec{m}; \vec{e}]; \quad Q = \frac{1}{3} \sum_{f=1}^{3} I^{fh}[\left(\frac{|e|}{e_f}\right)^2 n_g n_h - \frac{1}{2} \left(\frac{|e|}{e_f n_f}\right)^2],$$

Equation (14) is a quadratic algebraic equation, which can be solved in closed form for $n_3$ as a function of $n_1$ and $n_2$. But the whole point is to use the fixed point iteration procedure, which applies in the general case where $A^a_i$ is not constant. To first order in $\Lambda$ this is given by

$$(n_3)_{(1)} = -(n_1 + n_2) + \frac{\Lambda |e|}{3e_1 e_2 e_3} \left[ -\frac{1}{2} ((e_3)^2 + (e_1)^2)n_1 n_1 
- \frac{1}{2} ((e_2)^2 + (e_3)^2)n_2 n_2 + ((e_1 e_2)^2 - (e_3)^2)|e|^2 n_1 n_2 \right].$$

This can be repeated to any desired order, producing an expansion in powers of $\Lambda$ whose coefficients depend on $(n_1, n_2)$. The full solution is

$$\tilde{\delta}_a = -\left(\frac{\Lambda}{3} \delta_{ae} + (\tilde{T}^{1}_{ae} n_1 + \tilde{T}^{2}_{ae} n_2 + \tilde{T}^{3}_{ae} n_3[n_1, n_2; \vec{e}])(A^{-1})_e^a (\det A)\right).$$

4. Discussion

We have presented a formula for nonlinear gravitons by expansion about Type O spacetimes where $\det B \neq 0$. The solution is fixed by two free functions $\varphi_1$ and $\varphi_2$ and a connection $A^a_i$, and can be developed as an formal operatorial expansion by fixed point iteration. We propose this as a possible realization of Penrose’s idea [1] at a formal level.

References

[1] Roger Penrose. ‘The nonlinear graviton’ First Award winning essay of the Gravity research Foundation (1975)’

[2] Abhay Ashtekar. ‘New perspectives in canonical gravity’, (Bibliopolis, Napoli, 1988).