Research Article

The First Solution for the Helical Flow of a Generalized Maxwell Fluid within Annulus of Cylinders by New Definition of Transcendental Function $B_N(rr_n)$

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Most articles choose the transcendental function $B_1(rr_n)$ to define the finite Hankel transform, and very few articles choose $B_0(rr_n)$. The derivations of $B_0(rr_n)$ and $B_1(rr_n)$ are also considered the same. In this paper, we find that the derivative formulas for the transcend function $B_N(rr_n)$ are different and prove the derivative formulas for $B_0(rr_n)$ and $B_1(rr_n)$. Based on the exact formulas of $B_0(rr_n)$ and $B_1(rr_n)$, we keep on studying the helical flow of a generalized Maxwell fluid between two boundless coaxial cylinders. In this case, the inner and outer cylinders start to rotate around their axis of symmetry at different angular frequencies and slide at different linear velocities at time $t = 0^+$. We deduced the velocity field and shear stress via Laplace transform and finite Hankel transform and their inverse transforms. According to generalized $G$ and $R$ functions, the solutions we obtained are given in the form of integrals and series. The solution of ordinary Maxwell fluid has been also obtained by solving the limit of the general solution of fractional Maxwell fluid.

1. Introduction

Most industry and academic workers are interested in the motion of fluids on plates or cylinders. Regarding the flow of Newtonian fluid in cylinders, we can find the transient velocity distribution in [1]. With regard to the flow of non-Newtonian fluid in the cylindrical domain, the first exact solutions are those of Srivastava [2] with Maxwell fluids and Ting [3] with second-grade fluids. Casarella and Laura [4] obtained an analytical expression on a smooth circular cylindrical rod. They studied rods with longitudinal and torsional motion. For flow in the same case, Rajagopal [5] found two solutions. Rajagopal and Bhatnagar [6] obtained these solutions for Oldroyd-B fluid. Khan et al. [7] gave an exact analytical solution for the flow of a Burger’s fluid via Fourier transform. The oscillating pressure gradient they considered was the main driving force for the motion. With regard to second-order fluids, Hayat et al. [8, 9] gave an exact analytical solution for unsteady unidirectional flows. They also considered pressure gradients and the flows induced by the motion of one or two plates or the pressure gradient. In terms of unsteady unidirectional flows, Rajagopal [10] also established two types of exact solutions. Among these two types of flows, one is owing to the rigid plate that oscillates in its own direction and the other is the time-periodic Poiseuille flow on account of the oscillating pressure gradient. Recently, many papers of this type have been obtained by Yang and Wang [11–13].

Fractional calculus has made great progress in describing the motion of viscoelastic fluids in recent years. Tong et al. [14, 15] discussed an exact analytic solution for the unsteady transient rotational flow of an Oldroyd-B fluid in terms of the fractional calculus definition. Many researchers have studied the rotational flow of a generalized Maxwell fluid between two boundless coaxial cylinders. For example, Fetecau and Fetecau [16] established analytic solutions for this motion. The longitudinal vibration shear stresses that were imposed by the inner cylinder were the main driving
force for the motion. Shaowei and Mingyu [17] considered the unsteady Couette flow with fractional derivative. When the outer cylinder remained immobile, the inner cylinder started to move around their axis of symmetry at a fixed angular velocity. They used the integral transform and the generalized Mittag-Leffler function to get the first solution. Zheng et al. [18] discussed the unsteady rotating flow with a fractional derivative model, where the flow is induced by the rotation of the inner cylinder and the oscillation of the pressure gradient. The first solution is given by integral and series via Laplace transform and Hankel transform. Mahmod et al. [19] also discussed the torsional oscillatory motion. The movement of the fluid was produced when the inner and outer cylinders began to oscillate around their symmetry axis. Chaudry et al. [20] established the first analytic solution for the oscillatory flow, when the outer cylinder oscillated along the axis of symmetry and the inner cylinder was stationary.

In the problem of using Hankel transform to solve the unsteady flow in a coaxial cylinder, we need to derive the transcendental function \( B_N(r r_n) \) in the Hankel transform. Most articles [17–19] select the transcendental function \( B_1(r r_n) \) when defining the finite Hankel transform, and very few articles [21] choose \( B_0(r r_n) \). However, these documents use the same derivation conclusions when calculating the derivatives of \( B_0(r r_n) \) and \( B_1(r r_n) \); it is proved that the derivative formula is dependent on \( N \); in fact, their derivations conclusions are different. Motivated by this problem, in Section 2 of this paper, we study the derivative of the transcendental function \( B_N(r r_n) \), when \( N \) takes different values, and use the mathematical induction method to prove the derivative formula of the transcendental function \( B_N(r r_n) \). Then, we introduce the basic governing equation in Section 3. Based on the exact formula of \( B_0(r r_n) \) and \( B_1(r r_n) \), we calculate the velocity field and shear stress via Laplace transform and finite Hankel transform and their inverse transforms in Sections 4 and 5. According to generalized \( G \) and \( R \) functions, the solutions we obtained are given in the form of integrals and series. In Section 6, we obtained the solution of Maxwell fluid by solving the limit of the general solution.

### 2. Derivative of the Transcendental Function \( B_N(r r_n) \)

According to the Bessel functions, Eldabe et al. [22] defined the finite Hankel transformation of \( f(r) \) in \((a, b)\) as follows:

\[
F(r_n) = \int_a^b f(r) B_N(r r_n) \, dr, \quad b > a,
\]

where the transcendental function is

\[
B_N(r r_n) = J_N(r r_n) Y_N(b r_n) - J_N(b r_n) Y_N(r r_n), \quad N = 0, 1, 2, \ldots \tag{2}
\]

\( r_n \) are the positive roots of the transcendental equation:

\[
J_N(b r_n) Y_N(a r_n) - J_N(a r_n) Y_N(b r_n) = 0, \tag{3}
\]

and \( J_N \) is the first-type Bessel function of the \( N \)-order and \( Y_N \) is the second-type Bessel function of the \( N \)-order [23]. In this part, we will correct the derivative of the transcendental function \( B_N(r r_n) \) and obtain the following conclusions.

**Theorem 1.** Derivative formula of transcendental function \( B_N(r r_n) \):

\[
\frac{\partial}{\partial r} \left( B_N(r r_n) \right) = r_n B_N(r r_n) - \frac{N}{r} B_N(r r_n), \quad N = 1, 2, \ldots .
\]

(4)

where

\[
B_N(r r_n) = J_N(r r_n) Y_N(b r_n) - J_N(b r_n) Y_N(r r_n),
\]

\[
B_N(r r_n) = J_{N-1}(r r_n) Y_N(b r_n) - J_{N-1}(b r_n) Y_{N-1}(r r_n).
\]

(5)

**Proof.** When \( N = 1 \),

\[
\begin{align*}
B_1(r r_n) & = J_1(r r_n) Y_1(b r_n) - J_1(b r_n) Y_1(r r_n), \\
B_1(r r_n) & = J_0(r r_n) Y_1(b r_n) - J_1(b r_n) Y_0(r r_n).
\end{align*}
\]

(6)

To prove

\[
\frac{\partial}{\partial r} \left( B_1(r r_n) \right) = r_n B_1(r r_n) - \frac{1}{r} B_1(r r_n)
\]

\[
= \left( r_n J_0(r r_n) - \frac{1}{r} J_1(r r_n) \right) Y_1(b r_n)
\]

\[
- J_1(b r_n) \left( r_n Y_0(r r_n) - \frac{1}{r} Y_1(r r_n) \right),
\]

(7)

we need to obtain

\[
\frac{\partial}{\partial r} \left( J_1(r r_n) \right) = r_n J_0(r r_n) - \frac{1}{r} J_1(r r_n),
\]

\[
\frac{\partial}{\partial r} \left( Y_1(r r_n) \right) = r_n Y_0(r r_n) - \frac{1}{r} Y_1(r r_n).
\]

(8)

Since

\[
\frac{\partial}{\partial r} \left( J_1(r r_n) \right) = \frac{\partial}{\partial r} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k + 1)!} \left( \frac{r r_n}{2} \right)^{2k+1} \right]
\]

\[
\frac{\partial}{\partial r} \left( J_1(r r_n) \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k + 1)!} \left( \frac{r r_n}{2} \right)^{2k+1}
\]

\[
= r_n \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k + 1)!} \left( \frac{r r_n}{2} \right)^{2k+1}
\]

\[
= r_n J_0(r r_n) - \frac{1}{r} J_1(r r_n),
\]

(9)

we get

\[
\frac{\partial}{\partial r} \left( J_1(r r_n) \right) = r_n J_0(r r_n) - \frac{1}{r} J_1(r r_n).
\]

(10)
Similarly,

\[
\frac{\partial}{\partial r} (Y_1(rr_n)) = \frac{2}{\pi} \left( r_n f_0(rr_n) - \frac{1}{r} f_1(rr_n) \right) \ln \frac{rr_n}{2} + \frac{2}{\pi} \frac{r}{r^2} f_1(rr_n) + \frac{2}{\pi} \frac{r}{r^2} f_0(rr_n) - \frac{1}{\pi \cdot 2} \sum_{k=0}^{\infty} (2k + 1) \cdot \frac{r_n}{2} \cdot \frac{(-1)^k}{k!(k+1)!} \left[ \psi(k+1) + \psi(k+2) \right] \left( \frac{rr_n}{2} \right)^{2k} 
\]

\[
= r_n \cdot \frac{2}{\pi} f_0(rr_n) \ln \frac{rr_n}{2} - \frac{2}{\pi} \frac{r}{r^2} f_1(rr_n) \ln \frac{rr_n}{2} + \frac{2}{\pi} \frac{r}{r^2} f_0(rr_n) \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left( \frac{rr_n}{2} \right)^{2k+1} \right] 
\]

\[
+ \left( -1 \right)^{k+1} \frac{1}{k!(k+1)!} \left[ \psi(k+1) + \psi(k+2) \right] \left( \frac{rr_n}{2} \right)^{2k} 
\]

\[
= r_n \cdot \frac{2}{\pi} f_0(rr_n) \ln \frac{rr_n}{2} - r_n \cdot \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+1)!} \left[ \psi(k+1) + \psi(k+2) \right] \left( \frac{rr_n}{2} \right)^{2k} - \frac{1}{r} Y_1(rr_n), 
\]

where

\[
\psi(k+1) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k}, 
\]

\[
\psi(k+2) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \frac{1}{k+1}. 
\]

Therefore, we establish the following:

\[
\frac{\partial}{\partial r} (Y_1(rr_n)) = r_n \left[ \frac{2}{\pi} f_0(rr_n) \ln \frac{rr_n}{2} - \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+1)!} \left[ 2\psi(k+1) \right] \left( \frac{rr_n}{2} \right)^{2k} \right] - \frac{1}{r} Y_1(rr_n) 
\]

\[
= r_n Y_0(rr_n) - \frac{1}{r} Y_1(rr_n), 
\]

\[
\frac{\partial}{\partial r}(B_1(rr_n)) = \left( r_n f_0(rr_n) - \frac{1}{r} f_1(rr_n) \right) Y_1(br_n) - f_1(br_n) \left( r_n Y_0(rr_n) - \frac{1}{r} Y_1(rr_n) \right) 
\]

\[
= r_n B_1(rr_n) - \frac{1}{r} B_1(rr_n). 
\]

Assume that when \( N = m \), the conclusion is true. Now, we prove that when \( N = m + 1 \),

\[
B_{m+1}(rr_n) = J_{m+1}(rr_n) Y_{m+1}(br_n) - J_{m+1}(br_n) Y_{m+1}(rr_n), 
\]

\[
\overline{B}_{m+1}(rr_n) = J_m(rr_n) Y_{m+1}(br_n) - J_{m+1}(br_n) Y_m(rr_n). 
\]

To prove

\[
\frac{\partial}{\partial r} (B_{m+1}(rr_n)) = r_n \overline{B}_{m+1}(rr_n) - \frac{m+1}{r} B_{m+1}(rr_n) 
\]

\[
= \left( r_n J_m(rr_n) - \frac{m+1}{r} J_{m+1}(rr_n) \right) Y_{m+1}(br_n) - J_{m+1}(br_n) \left( r_n Y_m(rr_n) - \frac{m+1}{r} Y_{m+1}(rr_n) \right). 
\]
we need to prove
\[
\frac{\partial}{\partial r} (J_{m+1}(rr_n)) = r_n J_m(rr_n) - \frac{m + 1}{r} J_{m+1}(rr_n),
\]
\[
\frac{\partial}{\partial r} (Y_{m+1}(rr_n)) = r_n Y_m(rr_n) - \frac{m + 1}{r} Y_{m+1}(rr_n),
\]
where
\[
J_m(rr_n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left( \frac{rr_n}{2} \right)^{2k+m},
\]
\[
J_{m+1}(rr_n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + m + 1)!} \left( \frac{rr_n}{2} \right)^{2k+m+1},
\]
\[
Y_m(rr_n) = \frac{2}{\pi} J_m(rr_n) \ln \frac{rr_n}{2} - \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left( \frac{rr_n}{2} \right)^{2k-m} - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left[ \psi(k + m + 1) + \psi(k + 1) \right] \left( \frac{rr_n}{2} \right)^{2k+m},
\]
\[
Y_{m+1}(rr_n) = \frac{2}{\pi} J_{m+1}(rr_n) \ln \frac{rr_n}{2} - \frac{1}{\pi} \sum_{k=0}^{m} \frac{(m-k)!}{k!} \left( \frac{rr_n}{2} \right)^{2k-m-1} - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + m + 1)!} \left[ \psi(k + m + 2) + \psi(k + 1) \right] \left( \frac{rr_n}{2} \right)^{2k+m+1}.
\]
(16)

Since
\[
\frac{\partial}{\partial r} (J_{m+1}(rr_n)) = \sum_{k=0}^{\infty} \left( 2k + m + 1 \right) \cdot \frac{r_n}{2} \cdot \frac{(-1)^k}{k!(k + m + 1)!} \left( \frac{rr_n}{2} \right)^{2k+m}
\]
\[
= \sum_{k=0}^{\infty} \left[ 2(k + m + 1) - (m + 1) \right] \cdot \frac{r_n}{2} \cdot \frac{(-1)^k}{k!(k + m + 1)!} \left( \frac{rr_n}{2} \right)^{2k+m}
\]
\[
= r_n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + m)!} \left( \frac{rr_n}{2} \right)^{2k+m} - \frac{m + 1}{r} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k + m + 1)!} \left( \frac{rr_n}{2} \right)^{2k+m+1}
\]
\[
= r_n J_m(rr_n) - \frac{m + 1}{r} J_{m+1}(rr_n),
\]
(17)

then
\[
\frac{\partial}{\partial r} (J_{m+1}(rr_n)) = r_n J_m(rr_n) - \frac{m + 1}{r} J_{m+1}(rr_n).
\]
(19)
Similarly, 

\[
\frac{\partial}{\partial r} (Y_{m+1}(rr_n)) = \frac{2}{\pi} \left( r_n J_m(rr_n) - \frac{m+1}{r} J_{m+1}(rr_n) \right) \ln \frac{rr_n}{2} + \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{rr_n}{2} \right)^{2k+m+1} \\
- \frac{r_n}{\pi} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left( \frac{rr_n}{2} \right)^{2k-m} + \frac{m+1}{r} \sum_{k=0}^{m-1} \frac{(m-k)!}{k!} \left( \frac{rr_n}{2} \right)^{2k-(m+1)} \\
- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(2k+m+1)}{k!(k+m)!} \cdot \frac{r_n}{2} \cdot \frac{(-1)^k}{k!} \left[ (k+m+2) + (k+1) \right] \left( \frac{rr_n}{2} \right)^{2k+m} \\
- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(2k+m+1)}{k!(k+m)!} \cdot \left[ (k+m+2) + (k+1) \right] \left( \frac{rr_n}{2} \right)^{2k+m+1} \\
= r_n \left\{ \frac{2}{\pi} J_m(rr_n) \ln \frac{rr_n}{2} - \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left( \frac{rr_n}{2} \right)^{2k-m} \\
- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(2k+m+1)}{k!(k+m)!} \cdot \left[ (k+m+2) + (k+1) \right] \left( \frac{rr_n}{2} \right)^{2k+m} \\
- \frac{m+1}{r} Y_{m+1}(rr_n) \right\} - \frac{m+1}{r} Y_{m+1}(rr_n).
\]

So we obtain 

\[
\frac{\partial}{\partial r} (B_{m+1}(rr_n)) = \left( r_n J_m(rr_n) - \frac{m+1}{r} J_{m+1}(rr_n) \right) Y_{m+1}(br_n) \\
- J_{m+1}(br_n) \left( r_n Y_{m+1}(rr_n) - \frac{m+1}{r} Y_{m+1}(rr_n) \right) \\
= r_n B_{m+1}(rr_n) - \frac{m+1}{r} B_{m+1}(rr_n).
\]

(2) Derivative formula of transcendental function 

B_1(rr_n) is as follows:

\[
\frac{\partial}{\partial r} (B_1(rr_n)) = \left( r_n J_0(rr_n) - \frac{1}{r} J_1(rr_n) \right) Y_1(br_n) - J_1(br_n) \\
= \left( r_n Y_0 (rr_n) - \frac{1}{r} Y_1(rr_n) \right).
\]

Hence, the conclusion is true. □

**Corollary 1**

(1) Derivative formula of transcendental function B_0(rr_n) is as follows:

\[
\frac{\partial}{\partial r} (B_0(rr_n)) = -r_n [J_1(rr_n) Y_0(br_n) - J_0(br_n) Y_1(rr_n)].
\]
Proof

(1) Since

\[
\frac{\partial}{\partial r} (J_0(rr_n)) = \frac{\partial}{\partial r} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{k!k!} \left( \frac{r r_n}{2} \right)^{2k} \right]
\]

\[= \sum_{k=1}^{\infty} 2k \cdot \frac{r_n}{2} \cdot \frac{(-1)^k}{k!} \cdot \left( \frac{r r_n}{2} \right)^{2k-1} \]

\[= -r_n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)k!} \left( \frac{r r_n}{2} \right)^{(2k-1)+1} \]

\[= -r_n J_1(r r_n). \]

\[
\frac{\partial}{\partial r} (Y_0(rr_n)) = \frac{\partial}{\partial r} \left[ 2J_0(r r_n) \ln \frac{r r_n}{2} - \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!k!} \left( 2\psi(k+1) \right) \left( \frac{r r_n}{2} \right)^{2k} \right]
\]

\[= -r_n \cdot \frac{2}{\pi} J_1(r r_n) \ln \frac{r r_n}{2} + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k-1)!} \left( 2\psi(k+1) \right) \left( \frac{r r_n}{2} \right)^{2k-1} \]

\[= -r_n \cdot \frac{2}{\pi} J_1(r r_n) \ln \frac{r r_n}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(k-1)!} \left( 2\psi(k+1) \right) \left( \frac{r r_n}{2} \right)^{2k-1} \]

\[= -r_n \cdot \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(k-1)!} \left( 2\psi(k+1) \right) \left( \frac{r r_n}{2} \right)^{2k-1} \]

\[= -r_n \left\{ \frac{2}{\pi} J_1(r r_n) \ln \frac{r r_n}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(k-1)!} \left( 2\psi(k+1) \right) \left( \frac{r r_n}{2} \right)^{2k-1} \right\} \]

\[= -r_n \left\{ \frac{2}{\pi} J_1(r r_n) \ln \frac{r r_n}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(k-1)!} \left( 2\psi(k+1) \right) \left( \frac{r r_n}{2} \right)^{2k-1} \right\} \]

we get

\[
\frac{\partial}{\partial r} (B_0(\rho(r))) = \frac{\partial}{\partial r} \left[ J_0(rr_n)Y_0(br_n) - J_0(br_n)Y_0(rr_n) \right]
\]

\[= (-r_n J_1(r r_n)) Y_0(br_n) - J_0(br_n) (-r_n Y_1(r r_n)) \]

\[= -r_n J_1(r r_n) Y_0(br_n) - J_0(br_n) Y_1(r r_n). \]

(2) It can be directly obtained from Theorem 1 conclusion. □

3. Basic Governing Equations

In this paper, based on the exact formula of $B_0(\rho(r))$ and $B_1(\rho(r))$, we keep on studying the helical flow of a generalized Maxwell fluid between two boundless coaxial cylinders. We research the velocity of the helical flow of the following form:

\[
V = V(r, t) = \omega(r, t)e_\theta + v(r, t)e_z, \tag{26}
\]

where $\omega(r, t)$ is the rotating velocity and $v(r, t)$ is the sliding velocity. The inner and outer cylinders start to rotate around their axis of symmetry at different angular frequencies $\omega_1$.
and \( \omega_2 \) and slide along the same axis of symmetry at different linear velocities \( A_1 t \) and \( A_2 t (A_1 \neq A_2) \) at time \( t = 0^+ \).

The initial and boundary conditions we gave are

\[
\begin{align*}
\nu(r, 0) &= \omega(r, 0) = 0, \quad \frac{\partial \nu(r, 0)}{\partial t} = \frac{\partial \omega(r, 0)}{\partial t} = 0, \quad r \in [R_1, R_2], \\
\omega(R_1, t) &= \omega_1 \sin \beta_1 t, \\
\omega(R_2, t) &= \omega_2 \sin \beta_2 t, \\
\nu(R_1, t) &= A_1 t, \\
\nu(R_2, t) &= A_2 t,
\end{align*}
\tag{27}
\]

where \( \lambda \) is the material constant, \( \mu \) is the dynamic viscosity of the fluid, and \( S_{r\theta} \) and \( S_{rz} \) are the shear stresses. Without considering the body forces and pressure gradient, the momentum conservation equation is

\[
\begin{align*}
\rho \frac{\partial \omega(r, t)}{\partial t} &= \frac{\partial S_{r\theta}}{\partial r} + \frac{2}{r} S_{r\theta}, \\
\rho \frac{\partial \nu(r, t)}{\partial t} &= \frac{\partial S_{rz}}{\partial r} + \frac{S_{rz}}{r},
\end{align*}
\tag{32, 33}
\]

where \( \rho \) is the constant density of the fluid.

### 4. Calculation of the Velocity Field

Applying the Laplace transform [26, 27] to (30)–(33), we obtain

\[
\begin{align*}
(1 + \lambda \delta^\alpha) S_{r\theta} &= \mu \left( \frac{\partial \omega(r, s)}{\partial r} - \frac{\omega(r, s)}{r} \right), \\
(1 + \lambda \delta^\alpha) S_{rz} &= \mu \frac{\partial \nu(r, s)}{\partial r}, \\
\rho s \omega(r, s) &= \frac{\partial S_{r\theta}}{\partial r} + \frac{2}{r} S_{r\theta},
\end{align*}
\tag{34, 35, 36}
\]

According to (34)–(37), we obtain the following ordinary differential equation:

\[
\begin{align*}
\frac{\partial^2 \omega(r, s)}{\partial r^2} + \frac{1}{r} \frac{\partial \omega(r, s)}{\partial r} - \frac{\omega(r, s)}{r^2} - \frac{\rho s (1 + \lambda \delta^\alpha)}{\mu} \omega(r, s) &= 0, \\
\frac{\partial^2 \nu(r, s)}{\partial r^2} + \frac{1}{r} \frac{\partial \nu(r, s)}{\partial r} - \frac{\rho s (1 + \lambda \delta^\alpha)}{\mu} \nu(r, s) &= 0,
\end{align*}
\tag{38, 39}
\]

where the functions \( \omega(r, s) \) and \( \nu(r, s) \) are the Laplace transformations of \( \omega(r, t) \) and \( \nu(r, t) \). Similarly, applying the Laplace transforms (28) and (29), we obtain

\[
\begin{align*}
\overline{\omega}(R_i, s) &= \frac{\omega_i \beta_i}{s^2 + \beta_i^2}, \\
\overline{\nu}(R_i, s) &= \frac{A_i}{s^2}, \quad i = 1, 2.
\end{align*}
\tag{40}
\]

According to Section 2, we know that

\[
\begin{align*}
\overline{\omega}_H(s) &= \int_{R_1}^{R_2} r \overline{\omega}(r, s) B_1 (r r_m) dr, \quad n = 1, 2, 3, \ldots, \\
\overline{\nu}_H(s) &= \int_{R_1}^{R_2} r \overline{\nu}(r, s) B_0 (r r_m) dr, \quad m = 1, 2, 3, \ldots,
\end{align*}
\tag{41}
\]

are the finite Hankel transforms of \( \overline{\omega}(r, s) \) and \( \overline{\nu}(r, s) \), where \( r_n \) and \( r_m \) are the positive roots of equations \( B_1 (R_1 r) = 0 \) and \( B_0 (R_1 r) = 0 \), and

\[
\begin{align*}
B_0 (r r_m) &= J_0 (r r_m) Y_0 (R_2 r_m) - J_0 (R_2 r_m) Y_0 (r r_m), \\
B_1 (r r_m) &= J_1 (r r_m) Y_1 (R_2 r_m) - J_1 (R_2 r_m) Y_1 (r r_m).
\end{align*}
\tag{42}
\]

Multiplying both sides of (38) and (39) by \( r B_1 (r r_m) \) and \( r B_0 (r r_m) \), integrating with respect to \( r \) from \( R_1 \) to \( R_2 \), and considering conditions (40), we can obtain [28]

\[
\begin{align*}
\int_{R_1}^{R_2} r \left( \frac{\partial^2 \overline{\omega}(r, s)}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{\omega}(r, s)}{\partial r} - \frac{\overline{\omega}(r, s)}{r^2} \right) B_1 (r r_m) dr &= 0, \\
&= \frac{2}{\pi} \frac{\omega_2 \beta_2}{s^2 + \beta_2^2} - \frac{2}{\pi} \frac{\omega_1 \beta_1}{s^2 + \beta_1^2} J_1 (r R_2) - r^2 \overline{\omega}_H(s), \\
\int_{R_1}^{R_2} r \frac{\rho s (1 + \lambda \delta^\alpha)}{\mu} \overline{\nu}(r, s) B_0 (r r_m) dr &= \frac{\rho s (1 + \lambda \delta^\alpha)}{\mu} \overline{\nu}_H(s).
\end{align*}
\tag{43, 44}
\]
Since
\[
\frac{\partial}{\partial r} (J_0(rr_m)) = -r_m J_1(rr_m),
\]
\[
\frac{\partial}{\partial r} (Y_0(rr_m)) = -r_m Y_1(rr_m),
\]
\[
J_{m+1}(x)Y_m(x) - J_m(x)Y_{m+1}(x) = \frac{2}{\pi x},
\]
\[
\int_{R_1}^{R_2} r \left( \frac{\partial^2 \mathcal{V}(r,s)}{\partial r^2} + \frac{1}{r} \frac{\partial \mathcal{V}(r,s)}{\partial r} \right) B_0(rr_m) dr = \int_{R_1}^{R_2} \frac{\partial \mathcal{V}(r,s)}{\partial r} r B_0(rr_m) dr + \int_{R_1}^{R_2} \frac{\partial^2 \mathcal{V}(r,s)}{\partial r^2} B_0(rr_m) dr
\]
\[
= \frac{\partial \mathcal{V}(r,s)}{\partial r} r B_0(rr_m) \bigg|_{R_1}^{R_2} - \int_{R_1}^{R_2} \frac{\partial \mathcal{V}(r,s)}{\partial r} \frac{\partial}{\partial r} [rB_0(rr_m)] dr + \int_{R_1}^{R_2} \frac{\partial^2 \mathcal{V}(r,s)}{\partial r^2} B_0(rr_m) dr
\]
\[
= \left[ r B_0(rr_m) \right]_{J_1(rr_m) Y_0(R_2r_m) - J_0(R_2r_m) Y_1(rr_m)}^{J_1(rr_m) Y_0(R_1r_m) - J_0(R_1r_m) Y_1(rr_m)} - \int_{R_1}^{R_2} \frac{\partial \mathcal{V}(r,s)}{\partial r} r^2 B_0(rr_m) dr
\]
\[
= \frac{2}{n} \mathcal{V}(R_2,s) - \mathcal{V}(R_1,s) R_1 r_m J_1(R_1r_m) J_0(R_2r_m) Y_0(R_2r_m) - J_0(R_2r_m) J_0(R_1r_m) Y_1(R_1r_m)
\]
\[
- r_m^2 \mathcal{V}_H(s)
\]
\[
= \frac{2}{n} \mathcal{V}(R_2,s) - \frac{2}{n} \mathcal{V}(R_1,s) \frac{J_0(R_2r_m)}{J_0(R_1r_m)} - r_m^2 \mathcal{V}_H(s)
\]
\[
= \frac{2}{n} \mathcal{V}(R_2,s) - \frac{2}{n} \mathcal{V}(R_1,s) \frac{J_0(R_2r_m)}{J_0(R_1r_m)} - r_m^2 \mathcal{V}_H(s)
\]
\[
= \frac{2 A_2}{n s^2} - \frac{2 A_1}{n s^2} \frac{J_0(R_2r_m)}{J_0(R_1r_m)} - r_m^2 \mathcal{V}_H(s),
\]
we can get
\[
\int_{R_1}^{R_2} \frac{\partial^2 \mathcal{V}(r,s)}{\partial r^2} B_0(rr_m) dr = \frac{2}{n} \mathcal{V}(R_2,s) - \frac{2}{n} \mathcal{V}(R_1,s) \frac{J_0(R_2r_m)}{J_0(R_1r_m)} - r_m^2 \mathcal{V}_H(s) = 0,
\]
\[
\int_{R_1}^{R_2} \frac{\partial \mathcal{V}(r,s)}{\partial r} r B_0(rr_m) dr = \frac{2 A_2}{n s^2} - \frac{2 A_1}{n s^2} \frac{J_0(R_2r_m)}{J_0(R_1r_m)} - r_m^2 \mathcal{V}_H(s) = 0,
\]
\[
\int_{R_1}^{R_2} \frac{\partial \mathcal{V}(r,s)}{\partial r} \frac{\partial}{\partial r} [rB_0(rr_m)] dr = \frac{2 A_2}{n s^2} - \frac{2 A_1}{n s^2} \frac{J_0(R_2r_m)}{J_0(R_1r_m)} - r_m^2 \mathcal{V}_H(s) = 0,
\]
According to (43)–(47), or equivalently
\[
\frac{\partial}{\partial r} (J_0(rr_m)) = -r_m J_1(rr_m),
\]
\[
\frac{\partial}{\partial r} (Y_0(rr_m)) = -r_m Y_1(rr_m),
\]
\[
J_{m+1}(x)Y_m(x) - J_m(x)Y_{m+1}(x) = \frac{2}{\pi x},
\]
\[
\omega_H(s) = \frac{2}{\pi} \frac{\omega_2 \beta_2}{s^2 + \beta_2^2} \frac{\mu}{\rho s + \rho \lambda s^{\alpha+1} + \mu r_n^2} - \frac{2}{\pi} \frac{\omega_1 \beta_1}{s^2 + \beta_1^2} \frac{J_1(R_2 r_n)}{J_1(R_1 r_n)} \frac{\mu}{\rho s + \rho \lambda s^{\alpha+1} + \mu r_n^2}
\] (49)

\[
\nu_H(s) = \frac{2}{\pi} \frac{A_2}{s^2} \frac{\mu}{\rho s + \rho \lambda s^{\alpha+1} + \mu r_m^2} - \frac{2}{\pi} \frac{A_1}{s^2} \frac{J_0(R_2 r_m)}{J_0(R_1 r_m)} \frac{\mu}{\rho s + \rho \lambda s^{\alpha+1} + \mu r_m^2}
\] (50)

We first write (49) and (50) under the suitable form as follows:

\[
\omega_H(s) = \frac{2}{\pi r_n^2} \frac{\omega_2 \beta_2}{s^2 + \beta_2^2} - \frac{2}{\pi r_n^2} \frac{\omega_2 \beta_2}{s^2 + \beta_2^2} \frac{\rho s(1 + \lambda s^\alpha)}{\rho s + \rho \lambda s^{\alpha+1} + \mu r_n^2} - \left( \frac{2}{\pi r_n^2} \frac{\omega_1 \beta_1}{s^2 + \beta_1^2} \frac{J_1(R_2 r_n)}{J_1(R_1 r_n)} \frac{\rho s(1 + \lambda s^\alpha)}{\rho s + \rho \lambda s^{\alpha+1} + \mu r_n^2} \right)
\]

\[
\nu_H(s) = \frac{2}{\pi r_m^2} \frac{A_2}{s^2} - \left( \frac{2}{\pi r_m^2} \frac{A_1}{s^2} \frac{J_0(R_2 r_m)}{J_0(R_1 r_m)} \frac{\rho s(1 + \lambda s^\alpha)}{\rho s + \rho \lambda s^{\alpha+1} + \mu r_m^2} \right)
\]

and we use the inverse Hankel transform formula [29, 30]:

\[
\bar{w}(r, s) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^2 J_1^2(R_1 r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \bar{\omega}_H(s),
\] (52)

\[
\bar{v}(r, s) = \frac{\pi^2}{2} \sum_{m=1}^{\infty} \frac{r_m^2 J_0^2(R_1 r_m)}{J_0^2(R_1 r_m) - J_0^2(R_2 r_m)} \times \bar{\nu}_H(s).
\]

As we know if

\[
A(r) = \frac{A R_1 (R_2^2 - r^2) + B R_2 (r^2 - R_1^2)}{(R_2^2 - R_1^2)r}
\] (53)

then

\[
a_n = \int_{R_2}^{R_1} r a(r) B_1(r r_n) dr = \frac{2}{\pi r_n^2} B - \frac{2}{\pi r_n^2} \frac{J_1(R_2 r_n)}{J_1(R_1 r_n)} A.
\] (54)

If

\[
b(r) = \frac{C (\ln R_2 - \ln r) + D (\ln r - \ln R_1)}{\ln R_2 - \ln R_1},
\] (55)

then
\[
\int_{r_1}^{r_2} r \ln \frac{r_2}{r_1} B_0(r r_m) \, dr = \frac{1}{\ln R_2 - \ln R_1} \left\{ \frac{R_2 \ln R_2}{r_m} J_1(R_2 r_m) Y_0(R_2 r_m) - \frac{R_1 \ln R_1}{r_m} J_1(R_1 r_m) Y_0(R_2 r_m) \right. \\
- \frac{R_2 \ln R_2}{r_m} J_1(R_2 r_m) Y_1(R_2 r_m) + \frac{R_1 \ln R_1}{r_m} J_1(R_1 r_m) Y_1(R_1 r_m) \\
- \frac{R_2 \ln R_2}{r_m} J_1(R_2 r_m) Y_0(R_2 r_m) + \frac{R_1 \ln R_1}{r_m} J_1(R_1 r_m) Y_0(R_2 r_m) - \frac{J_0(R_2 r_m) Y_0(R_2 r_m)}{r_m^2} \\
+ \frac{J_0(R_2 r_m) Y_0(R_2 r_m)}{r_m^2} - \frac{J_0(R_2 r_m) Y_0(R_1 r_m)}{r_m^2} \left\} \\
= \frac{1}{\ln R_2 - \ln R_1} \left\{ \frac{R_1 \ln R_1}{r_m} \left[ J_0(R_2 r_m) Y_1(R_1 r_m) - J_1(R_1 r_m) Y_0(R_2 r_m) \right] \\
+ \frac{R_1 \ln R_1}{r_m} \left[ J_1(R_1 r_m) Y_0(R_2 r_m) - J_0(R_2 r_m) Y_1(R_1 r_m) \right] \right\} \\
= - \frac{2}{\pi} \frac{1}{r_m^2} J_0(R_1 r_m),
\]

(56)
so
\[ b_n = \int_{R_1}^{R_2} ra(r)B_0(rr_m)dr = \frac{2}{n\pi r_m^2} - \frac{2}{n\pi r_m^2} J_0(R_1r_m)C. \] (57)

\[ \bar{\omega}(r, s) = \frac{\omega_1\beta_1}{s^2 + \beta_1^2} \left( R_1^2 - r^2 \right) + \frac{\omega_2\beta_2}{s^2 + \beta_2^2} \left( R_2^2 - r^2 \right) - \frac{\pi}{\rho s} \sum_{n=1}^{\infty} J_1(R_1r_n)B_n(rr_n) \]
\[ \times \frac{\rho s (1 + \lambda s^\alpha)}{\rho s + \rho \lambda s^{\alpha+1} + \rho r_n^2} \left( J_1(R_1r_n) \frac{\omega_2\beta_2}{s^2 + \beta_2^2} - J_1(R_2r_n) \frac{\omega_1\beta_1}{s^2 + \beta_1^2} \right). \] (58)

\[ \gamma(r, s) = A_1 \frac{\ln R_2 - \ln r}{s^2 + \ln R_2 - \ln R_1} + A_2 \frac{\ln r - \ln R_1}{s^2 + \ln R_2 - \ln R_1} - \frac{\pi}{\rho s} \sum_{n=1}^{\infty} J_0(R_1r_n)B_0(rr_n) \]
\[ \times \frac{\rho s (1 + \lambda s^\alpha)}{\rho s + \rho \lambda s^{\alpha+1} + \rho r_n^2} \left( J_0(R_1r_n) A_2 \frac{1}{s^2} - J_0(R_2r_n) A_1 \frac{1}{s^2} \right). \]

To make the calculation of calculus more convenient, we can write [31, 32] the following:
\[ \frac{\rho s (1 + \lambda s^\alpha)}{\rho s + \rho \lambda s^{\alpha+1} + \rho r_n^2} = \frac{1}{1 + \left( \frac{\rho r_n^2}{\rho s + \rho \lambda s^{\alpha+1} + \rho r_n^2} \right)} \]
\[ = \sum_{k=0}^{\infty} \left( \frac{\rho r_n^2}{\rho s + \rho \lambda s^{\alpha+1} + \rho r_n^2} \right)^k \cdot s^{-k} \frac{\left( \frac{\rho r_n^2}{\rho s + \rho \lambda s^{\alpha+1} + \rho r_n^2} \right)^k}{(1 + \lambda s^\alpha)^k} \]
\[ = \sum_{k=0}^{\infty} -\left( \frac{\rho r_n^2}{\rho s + \rho \lambda s^{\alpha+1} + \rho r_n^2} \right)^k \cdot s^{-k} \frac{\left( \frac{\rho r_n^2}{\rho s + \rho \lambda s^{\alpha+1} + \rho r_n^2} \right)^k}{(1 + \lambda s^\alpha)^k} \] (59)

and we use the Laplace transform of the functions \( G_{a,b,c} (d, t) \):
\[ \mathcal{L}^{-1} \left[ \frac{s^b}{(s^a - d^c)} \right] = G_{a,b,c} (d, t), \] (60)

where
\[ G_{a,b,c} (d, t) = \sum_{j=0}^{\infty} \frac{(c)_j d^j t^{j+c} a^{-b-1}}{j!\Gamma[(j+c) a-b]} \] (61)

is the generalized \( G \) function and \( (c)_j \) is the Pochhammer polynomial [33].

If \( f(t) = \mathcal{L}^{-1} \left( f(s) \right) \) and \( g(t) = \mathcal{L}^{-1} \left( g(s) \right) \), then
\[ \mathcal{L}^{-1} \left( f(s) \cdot g(s) \right) = \mathcal{L}^{-1} \left( f \ast g \right) (t) = \int_0^t f(t - \tau) g(\tau) d\tau \]
\[ = \int_0^t f(t) g(t - t) d\tau. \] (62)

We attain the following velocity field:

\[ \omega(r, t) = \frac{R_1}{R_2 - R_1^2} \sin(\beta_1 t) + \frac{R_2^2 - R_1^2}{R_2 - R_1^2} \sin(\beta_2 t) \]
\[ - \frac{\pi}{\rho s} \sum_{n=1}^{\infty} J_1(R_1r_n)B_n(rr_n) \]
\[ \times \left( \frac{\mu r_n^2}{\lambda \rho} \right)^k \int_0^\tau [J_1(R_1r_n) \omega_2 \sin(\beta_2 (t - \tau)) \right]. \] (63)

\[ \nu(r, t) = \frac{\ln R_2 - \ln r}{\ln R_2 - \ln R_1} - \frac{\ln r - \ln R_1}{\ln R_2 - \ln R_1} \]
\[ - \frac{\pi}{\rho s} \sum_{n=1}^{\infty} J_0(R_1r_n)B_0(rr_n) \]
\[ \times \left( \frac{\mu r_n^2}{\lambda \rho} \right)^k \int_0^\tau [J_1(R_1r_n) \omega_2 \sin(\beta_2 (t - \tau)) \right]. \] (64)

We let \( t = 0, r = R_1 \), and \( r = R_2 \) in (63) and (64) and found that
\[ \nu(r, 0) = \omega(r, 0) = 0, \]
\[ \omega(R_1, t) = \omega_1 \sin(\beta_1 t), \]
\[ \omega(R_2, t) = \omega_2 \sin(\beta_2 t), \] (65)
\[ \nu(R_1, t) = A_1 t, \]
\[ \nu(R_2, t) = A_2 t. \]

So the velocity field satisfies our initial and boundary conditions.
5. Calculation of the Shear Stress

According to (34) and (35), we can obtain that

\[ \mathfrak{g}_{r \theta} = \frac{\mu}{1 + \lambda s^a} \left( \frac{\partial \mathfrak{m}(r,s)}{\partial r} - \frac{\mathfrak{m}(r,s)}{r} \right), \]

\[ \mathfrak{g}_{r \varphi} = \frac{\mu}{1 + \lambda s^a} \frac{\partial \mathfrak{m}(r,s)}{\partial r}. \]

where

\[ \mathfrak{B}_1(r_{rn}) = f_0(r_{rn})Y_1(R_{rn}) - f_1(r_{rn})Y_0(r_{rn}), \]

\[ \mathfrak{B}_0(r_{rm}) = f_1(r_{rm})Y_0(R_{rn}) - f_0(R_{rn})Y_1(r_{rn}). \]

By using Corollary 1, so

\[ \frac{\partial \mathfrak{m}(r,s)}{\partial r} - \frac{\mathfrak{m}(r,s)}{r} = \frac{2\mu R_1^2 R_2}{(R_2^2 - R_1^2)^2} \frac{\omega_2 \beta_2}{s^2 + \beta_2^2} + \frac{2\mu R_2^2 R_1}{(R_1^2 - R_2^2)^2} \frac{\omega_1 \beta_1}{s^2 + \beta_1^2} - \frac{\mu \pi}{r} \sum_{n=1}^{\infty} \frac{J_1(R_{rn})}{J_1^2(R_{rn})} \]

\[ \times \left( \frac{r_{rn} \mathfrak{B}_1(r_{rn}) - 2/r f_1(r_{rn})}{\rho s + \rho \lambda s^{a+1} + \mu r_{rn}^2} \right) \left( \frac{J_1(R_{rn})}{s^2 + \beta_2^2} - \frac{J_1(R_{rn})}{s^2 + \beta_1^2} \right). \]

Introducing (70) and (71) into (66) and (67), we can obtain

\[ \mathfrak{g}_{r \theta} = \frac{2\mu R_1^2 R_2}{(R_2^2 - R_1^2)^2} \frac{1}{1 + \lambda s^a} \frac{\omega_2 \beta_2}{s^2 + \beta_2^2} - \frac{2\mu R_2^2 R_1}{(R_1^2 - R_2^2)^2} \frac{1}{1 + \lambda s^a} \frac{\omega_1 \beta_1}{s^2 + \beta_1^2} - \frac{\mu \pi}{r} \sum_{n=1}^{\infty} \frac{J_1(R_{rn})}{J_1^2(R_{rn})} \]

\[ \times \left( \frac{r_{rn} \mathfrak{B}_1(r_{rn}) - 2/r f_1(r_{rn})}{\rho s + \rho \lambda s^{a+1} + \mu r_{rn}^2} \right) \left( \frac{J_1(R_{rn})}{s^2 + \beta_2^2} - \frac{J_1(R_{rn})}{s^2 + \beta_1^2} \right). \]

To make the calculation of calculus more comfortable, we write

\[ \frac{\rho s}{\rho s + \rho \lambda s^{a+1} + \mu r_{rn}^2} = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left( \frac{\mu r_{rn}^2}{\lambda \rho} \right)^k \lambda^{-k} (\lambda^{1+s} + \lambda^{s+1})^{-k}. \]

and using the Laplace transform of the generalized \( G_{a,b,c}(d,t) \) function and \( R_{a,b}(c,d,t) \) function [33], we obtain

\[ L^{-1} \left\{ \frac{\beta^b}{(s^a - \gamma^b)} \right\} = G_{a,b,c}(d,t), \]

\[ L^{-1} \left\{ e^{-\gamma s} \beta^b \right\} = R_{a,b}(c,d,t), \]

where
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6. Limiting Case $\alpha \to 1$

(1) When $\alpha \to 1$, (63) and (64) can be reduced to

$$
\omega(r, t) = \frac{R_1(R_2^2 - r^2)}{(R_2^2 - R_1^2)r} \omega_1 \sin(\beta_1 t) + \frac{R_2(r^2 - R_1^2)}{(R_2^2 - R_1^2)r} \omega_2 \sin(\beta_2 t) - \pi \sum_{n=1}^{\infty} \frac{J_1(R_n r)B_1(rr_n)}{f_1^2(R_n r) - f_2^2(R_n r)} \left( - \frac{\mu \rho_n^2}{\lambda} \right)^k \int_0^t [J_1(R_n r)\omega_1 \sin(\beta_1 (t - \tau)) - J_1(R_n r)\omega_1 \sin(\beta_1 (t - \tau))] G_{1-k,k}(-\lambda^{-1}, \tau) d\tau,
$$

$$
\nu(r, t) = \frac{\ln R_2 - \ln r}{\ln R_2 - \ln R_1} A_1(t) + \frac{\ln r - \ln R_1}{\ln R_2 - \ln R_1} A_2(t) - \pi \sum_{n=1}^{\infty} \frac{J_0(R_n r)B_0(rr_n)}{f_0^2(R_n r) - f_0^2(R_n r)} \left( - \frac{\mu \rho_n^2}{\lambda} \right)^k \int_0^t [J_0(R_n r)\omega_1 \sin(\beta_1 (t - \tau)) - J_0(R_n r)\omega_1 \sin(\beta_1 (t - \tau))] G_{1-k,k}(-\lambda^{-1}, \tau) d\tau.
$$
(2) When $\alpha \to 1$, (77) and (78) can be reduced to

$$S_{\theta \theta} = \frac{2\mu R_{1} R_{2}}{\lambda (R_{1}^2 - R_{2}^2)} \int_{0}^{t} \left[ R_{1} \omega_{2} \sin(\beta_{2} (t - \tau)) - R_{2} \omega_{1} \sin(\beta_{1} (t - \tau)) \right] \left[ R_{1,0} (-\lambda^{-1}, 0, \tau) \right] d\tau$$

$$- \frac{\mu \pi}{\lambda} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{f_{1}(R_{1} r_{n})}{f_{2}^{*}(R_{1} r_{n})} \frac{r_{m} l_{0}(R_{1} r_{m})}{J_{0}^{*}(R_{2} r_{m})} \left( -\frac{\mu r_{n}^{2}}{\lambda \rho} \right)^{k} \times \int_{0}^{t} \left[ f_{1}(R_{1} r_{n}) \omega_{2} \sin(\beta_{2} (t - \tau)) - f_{1}(R_{2} r_{n}) \omega_{1} \sin(\beta_{1} (t - \tau)) \right] G_{1-k, k+1} (-\lambda^{-1}, \tau) d\tau,$$

$$S_{\theta z} = \frac{\mu}{r (\ln R_{2} - \ln R_{1})} \int_{0}^{t} \left[ A_{2} (t - \tau) - A_{1} (t - \tau) \right] R_{1,0} (-\lambda^{-1}, 0, \tau) d\tau$$

$$+ \frac{\mu \pi}{\lambda} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{r_{m} l_{0}(R_{1} r_{m})}{J_{0}^{*}(R_{2} r_{m})} \left( -\frac{\mu r_{n}^{2}}{\lambda \rho} \right)^{k} \times \int_{0}^{t} \left[ J_{0}(R_{1} r_{m}) A_{2} (t - \tau) - J_{0}(R_{2} r_{m}) A_{1} (t - \tau) \right] G_{1-k, k+1} (-\lambda^{-1}, \tau) d\tau. \quad (81)$$

$$\times \int_{0}^{t} \left[ f_{1}(R_{1} r_{n}) \omega_{2} \sin(\beta_{2} (t - \tau)) - f_{1}(R_{2} r_{n}) \omega_{1} \sin(\beta_{1} (t - \tau)) \right] G_{1-k, k+1} (-\lambda^{-1}, \tau) d\tau, \quad (82)$$

When $\alpha = 1$, the fractional Maxwell fluid model is an ordinary Maxwell fluid model. Therefore, (79)–(82) are the analytical solutions for the ordinary Maxwell fluid.

7. Conclusion

As we know, most articles choose the transcendental function $B_{1}(rr_{n})$ and few articles choose $B_{0}(rr_{n})$ to define the finite Hankel transform; moreover, the derivations of $B_{0}(rr_{n})$ and $B_{1}(rr_{n})$ are often considered the same. Actually, in this paper, by use of the mathematical induction method, we prove the derivative formula for the transcendental function $B_{0}(rr_{n})$. Then, according to the momentum conservation law, the relevant and meaningful equation and constitutive equation are given. Based on the exact formula of $B_{0}(rr_{n})$ and $B_{1}(rr_{n})$, we calculate the velocity field and shear stress via Laplace transform and finite Hankel transform and their inverse transforms in Sections 4 and 5. In order to present our solution in a simpler and more comfortable form, we combine the generalized $G$ function and $R$ function. The solutions we obtained are given in the form of integrals and series containing the generalized $G$ function and $R$ function. At the end of Section 4, we explain that these solutions satisfy initial and boundary conditions that we gave. When $\alpha = 1$, the fractional Maxwell fluid model is an ordinary Maxwell fluid model. As before, we obtained the solution of Maxwell fluid by solving the limit of the general solution.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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