On Concentration and Revisited Large Deviations Analysis of Binary Hypothesis Testing

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Abstract—This paper first introduces a refined version of the Azuma-Hoeffding inequality for discrete-parameter martingales with uniformly bounded jumps. The refined inequality is used to revisit the large deviations analysis of binary hypothesis testing.

Index Terms—Fisher information, hypothesis testing, large deviations, relative entropy.

I. INTRODUCTION

An analysis of binary hypothesis testing from an information-theoretic point of view, and a derivation of its related error exponents in analogy to optimum channel codes was provided in [4]. A nice exposition of the subject is also provided in [6] Chapter 11] where the exact error exponents for the large deviation analysis of binary hypothesis testing are provided in terms of relative entropies.

The Azuma-Hoeffding inequality is by now a well-known methodology that has been often used to prove concentration of measure phenomena. It is due to Hoeffding [9] who proved it first for a sum of independent and bounded RVs, and Azuma [2] who later extended it to bounded-difference martingales. For a nice exposition of the martingale approach, used for establishing concentration inequalities, the reader is referred to e.g. [5] and [11]. The starting point of this work is an analysis of binary hypothesis testing from an information-theoretic point of view, and a derivation of its related error exponents in analogy to optimum channel codes. Since this inequality is referred several times in this paper, it will be named Azuma’s inequality for the sake of brevity.

II. PRELIMINARIES AND A NEW CONCENTRATION INEQUALITY

In the following, we present briefly essential background on the martingale approach that is used in this paper to derive concentration inequalities. A refined version of Azuma’s inequality is then introduced. This concentration inequality is applied in the next section for revising the large deviations analysis of binary hypothesis testing.

A. Doob’s Martingales

This sub-section provides a short background on martingales to set definitions and notation. For a more thorough study of martingales, the reader it referred to, e.g., [3].

Definition 1: [Doob’s Martingale] Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. A Doob’s martingale sequence is a sequence \(X_0, X_1, \ldots\) of random variables (RVs) and corresponding sub \(\sigma\)-algebras \(\mathcal{F}_0, \mathcal{F}_1, \ldots\) (also denoted by \(\{X_i, \mathcal{F}_i\}\)) that satisfy the following conditions:

1) \(X_i \in L^1(\Omega, \mathcal{F}_i, \mathbb{P})\) for every \(i\), i.e., each \(X_i\) is defined on the same sample space \(\Omega\), it is measurable with respect to the corresponding \(\sigma\)-algebra \(\mathcal{F}_i\) (i.e., \(X_i\) is \(\mathcal{F}_i\)-measurable) and \(\mathbb{E}[|X_i|] = \int_{\Omega} |X_i(\omega)| d\mathbb{P}(\omega) < \infty\).

2) \(\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots\) (where this sequence of \(\sigma\)-algebras is called a filtration).

3) \(X_i = \mathbb{E}[X_{i+1} | \mathcal{F}_i]\) holds almost surely (a.s.) for every \(i\).

For preliminary material on the construction of discrete-time martingales, see Appendix A which is relevant to the analysis in Section III.

B. Azuma’s Inequality

Azuma’s inequality\(^1\) forms a useful concentration inequality for bounded-difference martingales [2]. In the following, this inequality is introduced. The reader is referred to, e.g., [1], Chapter 11, [5] and [11] for surveys on concentration inequalities for (sub/ super) martingales.

Theorem 1: [Azuma’s inequality] Let \(\{X_k, \mathcal{F}_k\}_{k=0}^{\infty}\) be a discrete-parameter real-valued martingale sequence such that

\[^1\text{Azuma’s inequality is also known as the Azuma-Hoeffding inequality. Since this inequality is referred several times in this paper, it will be named from this point as Azuma’s inequality for the sake of brevity.}\]
for every \( k \in \mathbb{N} \), the condition \( |X_k - X_{k-1}| \leq d_k \) holds a.s. for some non-negative constants \( \{d_k\}_{k=1}^\infty \). Then

\[
P(|X_n - X_0| \geq r) \leq 2 \exp\left(-\frac{r^2}{2 \sum_{k=1}^n d_k^2}\right) \quad \forall r \geq 0. \quad (1)
\]

The concentration inequality stated in Theorem 1 was proved in [9] for independent bounded random variables, followed by a discussion on sums of dependent random variables; this inequality was later derived in [2] for bounded-difference martingales. For a proof of Theorem 1 see, e.g., [3] and [8] Chapter 2.4.

C. A Refined Version of Azuma’s Inequality

Theorem 2: Let \( \{X_k, \mathcal{F}_k\}_{k=0}^\infty \) be a discrete-parameter real-valued martingale. Assume that, for some constants \( d, \sigma > 0 \), the following two requirements are satisfied a.s.

\[
|X_k - X_{k-1}| \leq d, \quad \text{Var}(X_k|\mathcal{F}_{k-1}) = \mathbb{E}[(X_k - X_{k-1})^2|\mathcal{F}_{k-1}] \leq \sigma^2
\]

for every \( k \in \{1, \ldots, n\} \). Then, for every \( \alpha \geq 0 \),

\[
P(|X_n - X_0| \geq \alpha n) \leq 2 \exp\left(-n D\left(\frac{\gamma}{1 + \gamma} \parallel \frac{\gamma}{1 + \gamma}\right)\right) \quad (2)
\]

where

\[
\gamma \triangleq \frac{\sigma^2}{d^2}, \quad \delta \triangleq \frac{\alpha}{d}
\]

and

\[
D(p||q) \triangleq p \ln\left(\frac{p}{q}\right) + (1-p) \ln\left(\frac{1-p}{1-q}\right), \quad \forall p, q \in [0, 1] \quad (4)
\]

is the divergence (a.k.a. relative entropy or Kullback-Leibler distance) between the two probability distributions \((p, 1-p)\) and \((q, 1-q)\). If \( \delta > 1 \), then the probability on the left-hand side of (2) is equal to zero.

Proof: The idea of the proof of Theorem 2 is essentially similar to the proof of [8] Corollary 2.4.7. The full proof is provided in [12] Section III.

Proposition 1: Let \( \{X_k, \mathcal{F}_k\}_{k=0}^\infty \) be a discrete-parameter real-valued martingale. Then, for every \( \alpha \geq 0 \),

\[
P(|X_n - X_0| \geq \alpha \sqrt{n}) \leq 2 \exp\left(-\frac{\delta^2}{2 \gamma}\left(1 + O(n^{-\frac{1}{2}})\right)\right).
\]

Proof: This inequality follows from Theorem 2 (see [12] Appendix G).

III. Binary Hypothesis Testing

Binary hypothesis testing for finite alphabet models was analyzed via the method of types, e.g., in [6] Chapter 11 and [7]. It is assumed that the data sequence is of a fixed length \( (n) \), and one wishes to make the optimal decision (based on the Neyman-Pearson ratio test) based on the received sequence.

Let the RVs \( X_1, X_2, \ldots \) be i.i.d. \( \sim Q \), and consider two hypotheses:

\begin{itemize}
  \item \( H_1 : Q = P_1 \).
  \item \( H_2 : Q = P_2 \).
\end{itemize}

For the simplicity of the analysis, let us assume that the RVs are discrete, and take their values on a finite alphabet \( \mathcal{X} \) where \( P_1(x), P_2(x) > 0 \) for every \( x \in \mathcal{X} \).

In the following, let

\[
L(X_1, \ldots, X_n) \triangleq \ln \frac{P_n^n(X_1, \ldots, X_n)}{P_2^n(X_1, \ldots, X_n)} = \sum_{i=1}^n \ln \frac{P_1(X_i)}{P_2(X_i)}
\]

designate the log-likelihood ratio. By the strong law of large numbers (SLLN), if hypothesis \( H_1 \) is true, then a.s.

\[
\lim_{n \to \infty} \frac{L(X_1, \ldots, X_n)}{n} = D(P_1||P_2)
\]

and otherwise, if hypothesis \( H_2 \) is true, then a.s.

\[
\lim_{n \to \infty} \frac{L(X_1, \ldots, X_n)}{n} = -D(P_2||P_1)
\]

where the above assumptions on the probability mass functions \( P_1 \) and \( P_2 \) imply that the relative entropies, \( D(P_1||P_2) \) and \( D(P_2||P_1) \), are both finite. Consider the case where for some fixed constants \( \bar{\lambda}, \underline{\lambda} \in \mathbb{R} \) where

\[
-D(P_2||P_1) < \underline{\lambda} \leq \bar{\lambda} < D(P_1||P_2)
\]

one decides on hypothesis \( H_1 \) if

\[
L(X_1, \ldots, X_n) > n\bar{\lambda}
\]

and on hypothesis \( H_2 \) if

\[
L(X_1, \ldots, X_n) < n\underline{\lambda}
\]

Note that if \( \bar{\lambda} = \underline{\lambda} = \lambda \) then a decision on the two hypotheses is based on comparing the normalized log-likelihood ratio (w.r.t. \( n \)) to a single threshold \( (\lambda) \), and deciding on hypothesis \( H_1 \) or \( H_2 \) if this normalized log-likelihood ratio is, respectively, above or below \( \lambda \). If \( \bar{\lambda} < \lambda \) then one decides on \( H_1 \) or \( H_2 \) if the normalized log-likelihood ratio is, respectively, above the upper threshold \( \lambda \) or below the lower threshold \( \lambda \). Otherwise, if the normalized log-likelihood ratio is between the upper and lower thresholds, then an erasure is declared and no decision is taken in this case.

Let

\[
\alpha_n^{(1)} \triangleq P_n^n\left( L(X_1, \ldots, X_n) \leq n\bar{\lambda} \right)
\]

\[
\alpha_n^{(2)} \triangleq P_n^n\left( L(X_1, \ldots, X_n) \leq n\underline{\lambda} \right)
\]

\[
\beta_n^{(1)} \triangleq P_n^n\left( L(X_1, \ldots, X_n) \geq n\bar{\lambda} \right)
\]

\[
\beta_n^{(2)} \triangleq P_n^n\left( L(X_1, \ldots, X_n) \geq n\underline{\lambda} \right)
\]

then \( \alpha_n^{(1)} \) and \( \beta_n^{(1)} \) are the probabilities of either making an error or declaring an erasure under, respectively, hypotheses \( H_1 \) and \( H_2 \); similarly \( \alpha_n^{(2)} \) and \( \beta_n^{(2)} \) are the probabilities of making an error under hypotheses \( H_1 \) and \( H_2 \), respectively.

Let \( \pi_1, \pi_2 \in (0, 1) \) denote the a-priori probabilities of the hypotheses \( H_1 \) and \( H_2 \), respectively, so

\[
P_e^{(1)} = \pi_1 \alpha_n^{(1)} + \pi_2 \beta_n^{(1)}
\]

is the probability of having either an error or an erasure, and

\[
P_e^{(2)} = \pi_1 \alpha_n^{(2)} + \pi_2 \beta_n^{(2)}
\]

is the probability of error.
A. Exact Exponents

When we let \( n \) tend to infinity, the exact exponents of \( \alpha_n^{(j)} \) and \( \beta_n^{(j)} \) (\( j = 1, 2 \)) are derived via Cramér’s theorem. The resulting exponents form a straightforward generalization of, e.g., [8] Theorem 3.4.3 and [10] Theorem 6.4] that addresses the case where the decision is made based on a single threshold of the log-likelihood ratio. In this particular case where \( \bar{\lambda} = \lambda \), the option of erasures does not exist, and \( P_{e,n}^{(1)} = P_{e,n}^{(2)} = P_{e,n} \) is the error probability.

In the considered general case with erasures, let

\[
\lambda_1 \triangleq \bar{\lambda}, \quad \lambda_2 \triangleq -\bar{\lambda}
\]

then Cramér’s theorem on \( \mathbb{R} \) yields that the exact exponents of \( \alpha_n^{(1)}, \alpha_n^{(2)}, \beta_n^{(1)} \) and \( \beta_n^{(2)} \) are given by

\[
\begin{align*}
\lim_{n \to \infty} -\frac{\ln \alpha_n^{(1)}}{n} &= I(\lambda_1) \quad (14) \\
\lim_{n \to \infty} -\frac{\ln \alpha_n^{(2)}}{n} &= I(\lambda_2) \quad (15) \\
\lim_{n \to \infty} -\frac{\ln \beta_n^{(1)}}{n} &= I(\lambda_2) - \lambda_2 \quad (16) \\
\lim_{n \to \infty} -\frac{\ln \beta_n^{(2)}}{n} &= I(\lambda_1) - \lambda_1 \quad (17)
\end{align*}
\]

where the rate function \( I \) is given by

\[
I(r) \triangleq \sup_{t \in \mathbb{R}} (tr - H(t)) \quad (18)
\]

and

\[
H(t) = \ln \left( \sum_{x \in \mathcal{X}} P_1(x)^{1-t} P_2(x)^{t} \right), \quad \forall t \in \mathbb{R}. \quad (19)
\]

The rate function \( I \) is convex, lower semi-continuous (l.s.c.) and non-negative (see, e.g., [8] and [10]). Note that

\[
H(t) = (t-1)D_t(P_2 || P_1)
\]

where \( D_t(P||Q) \) designates Rényi’s information divergence of order \( t \), and \( I \) in (13) is the Fenchel-Legendre transform of \( H \) (see, e.g., [8] Definition 2.2.2).

From (12)–(17), the exact exponents of \( P_{e,n}^{(1)} \) and \( P_{e,n}^{(2)} \) are equal to

\[
\lim_{n \to \infty} -\frac{\ln P_{e,n}^{(1)}}{n} = \min \{ I(\lambda_1), I(\lambda_2) - \lambda_2 \} \quad (20)
\]

and

\[
\lim_{n \to \infty} -\frac{\ln P_{e,n}^{(2)}}{n} = \min \{ I(\lambda_2), I(\lambda_1) - \lambda_1 \}. \quad (21)
\]

For the case where the decision is based on a single threshold for the log-likelihood ratio (i.e., \( \lambda_1 = \lambda_2 \triangleq \lambda \)), then \( P_{e,n}^{(1)} = P_{e,n}^{(2)} = P_{e,n} \), and its error exponent is equal to

\[
\lim_{n \to \infty} -\frac{\ln P_{e,n}}{n} = \min \{ I(\lambda), I(\lambda) - \lambda \} \quad (22)
\]

which coincides with the error exponent in [8] Theorem 3.4.3] (or [10] Theorem 6.4). The optimal threshold for obtaining the best error exponent of the error probability \( P_{e,n} \) is equal to zero (i.e., \( \lambda = 0 \)); in this case, the exact error exponent is equal to

\[
I(0) = -\min_{0 \leq t \leq 1} \ln \left( \sum_{x \in \mathcal{X}} P_1(x)^{1-t} P_2(x)^{t} \right) \leq C(P_1, P_2) \quad (23)
\]

which is the Chernoff information of the probability measures \( P_1 \) and \( P_2 \) (see [6, Eq. (11.239)]) and is symmetric (i.e., \( C(P_1, P_2) = C(P_2, P_1) \)). Note that, from (15), \( I(0) = \sup_{t \in \mathbb{R}} (-H(t)) = -\inf_{t \in \mathbb{R}} (H(t)) \); the minimization in (23) over the interval \([0, 1]\) (instead of taking the infimum of \( H \) over \( \mathbb{R} \)) is due to the fact that \( H(0) = H(1) = 0 \) and the function \( H \) in (19) is convex, so it is enough to restrict the infimum of \( H \) to the closed interval \([0, 1]\) for which it turns to be a minimum.

B. Lower Bound on the Exponents via Theorem 2

In the following, the tightness of Theorem 2 is examined by using it for the derivation of lower bounds on the error exponent and the exponent of the event of having either an error or an erasure. These results will be compared in the next sub-section to the exact exponents from the previous sub-section.

We first derive a lower bound on the exponent of \( \alpha_n^{(1)} \). Under hypothesis \( H_1 \), let us construct the martingale sequence \( \{U_k, \mathcal{F}_k\}_{k=0}^n \) where \( \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \mathcal{F}_n \) is the filtration

\[
\mathcal{F}_0 = \{ \emptyset, \Omega \}, \quad \mathcal{F}_k = \sigma(X_1, \ldots, X_k), \quad \forall k \in \{1, \ldots, n\}
\]

and

\[
U_k = \mathbb{E}_{P_1^n} \left[ L(X_1, \ldots, X_n) \mid \mathcal{F}_k \right]. \quad (24)
\]

For every \( k \in \{0, \ldots, n\} \)

\[
U_k = \mathbb{E}_{P_1^n} \left[ \sum_{i=1}^n \ln \frac{P_1(X_i)}{P_2(X_i)} \mid \mathcal{F}_k \right]
\]

\[
= \sum_{i=1}^k \ln \frac{P_1(X_i)}{P_2(X_i)} + \sum_{i=k+1}^n \mathbb{E}_{P_1^n} \left[ \ln \frac{P_1(X_i)}{P_2(X_i)} \right]
\]

\[
= \sum_{i=1}^k \ln \frac{P_1(X_i)}{P_2(X_i)} + (n-k)D(P_1 || P_2). \quad (25)
\]

In particular

\[
U_0 = nD(P_1 || P_2), \quad (26)
\]

and, for every \( k \in \{1, \ldots, n\} \),

\[
U_k - U_{k-1} = \ln \frac{P_1(X_k)}{P_2(X_k)} - D(P_1 || P_2). \quad (27)
\]

Let

\[
d_1 \triangleq \max_{x \in \mathcal{X}} \left| \ln \frac{P_1(x)}{P_2(x)} - D(P_1 || P_2) \right| \quad (28)
\]

so \( d_1 < \infty \) since by assumption the alphabet set \( \mathcal{X} \) is finite, and \( P_1(x), P_2(x) > 0 \) for every \( x \in \mathcal{X} \). From (27) and (28)

\[
|U_k - U_{k-1}| \leq d_1
\]
holds a.s. for every $k \in \{1, \ldots, n\}$, and
\[
\mathbb{E} P_{n}^{(1)} [(U_{k} - U_{k-1})^2 | \mathcal{F}_{k-1}] \\
= \mathbb{E} P_{n} \left[ \left( \ln \frac{P_{1}(X_k)}{P_{2}(X_k)} - D(P_1 || P_2) \right)^2 \right] \\
= \sum_{x \in \mathcal{X}} \left\{ P_{1}(x) \left( \ln \frac{P_{1}(x)}{P_{2}(x)} - D(P_1 || P_2) \right)^2 \right\} \\
\triangleq \sigma_n^2.
\] (29)

Let
\[
\epsilon_{1,1} = D(P_1 || P_2) - \bar{\lambda}, \quad \epsilon_{2,1} = D(P_2 || P_1) + \bar{\lambda} \\
\epsilon_{1,2} = D(P_1 || P_2) - \bar{\lambda}, \quad \epsilon_{2,2} = D(P_2 || P_1) + \bar{\lambda}
\] (30) (31)

The probability of making an erroneous decision on hypothesis $H_2$ or declaring an erasure under the hypothesis $H_1$ is equal to $\alpha_n^{(1)}$, and from Theorem 2
\[
\alpha_n^{(1)} \triangleq P_{n}^{(1)} (L(X_1, \ldots, X_n) \leq n \bar{\lambda})
\]
\[
\overset{(a)}{=} P_{n}^{(1)} (U_n - U_0 \leq -\epsilon_{1,1} n) \\
\overset{(b)}{\leq} \exp \left( -n D \left( \frac{\delta_{1,1} + \gamma_{1}}{1 + \gamma_{1}} \left\lVert \gamma_{1} \right\rVert \right) \right)
\] (32) (33)

where equality (a) follows from (25), (26) and (30), and inequality (b) follows from Theorem 2 with
\[
\gamma_{1} \triangleq \frac{\sigma_{n}^2}{\delta_{1,1}}, \quad \delta_{1,1} \triangleq \frac{\epsilon_{1,1}}{\delta_{1,1}}.
\] (34)

Note that if $\epsilon_{1,1} > \delta_{1}$ then it follows from (27) and (28) that $\alpha_n^{(1)}$ is zero; in this case $\delta_{1,1} > 1$, so the divergence in (33) is infinity and the upper bound is also equal to zero. Hence, it is assumed without loss of generality that $\delta_{1,1} \in [0, 1]$.

Similarly to (24), under hypothesis $H_2$, let us define the martingale sequence $\{U_k, \mathcal{F}_k\}_{k=0}^{n} = k$ with the same filtration and $U_k = \mathbb{E} P_{n}^{(2)} [L(X_1, \ldots, X_n) | \mathcal{F}_k]$, $\forall k \in \{0, \ldots, n\}$. (35)

For every $k \in \{0, \ldots, n\}$
\[
U_k = \sum_{i=1}^{k} \ln \frac{P_{1}(X_i)}{P_{2}(X_i)} - (n - k) D(P_2 || P_1)
\]
and in particular
\[
U_0 = -n D(P_2 || P_1), \quad U_n = L(X_1, \ldots, X_n).
\] (36)

For every $k \in \{1, \ldots, n\}$,
\[
U_k - U_{k-1} = \ln \frac{P_{1}(X_k)}{P_{2}(X_k)} + D(P_2 || P_1).
\] (37)

Let
\[
d_2 \triangleq \max_{x \in \mathcal{X}} \ln \frac{P_{2}(x)}{P_{1}(x)} - D(P_2 || P_1)
\] (38)

then, the jumps of the latter martingale sequence are uniformly bounded by $d_2$ and, similarly to (29), for every $k \in \{1, \ldots, n\}$
\[
\mathbb{E} P_{n}^{(2)} [(U_k - U_{k-1})^2 | \mathcal{F}_{k-1}] \\
= \sum_{x \in \mathcal{X}} \left\{ P_{2}(x) \left( \ln \frac{P_{2}(x)}{P_{1}(x)} - D(P_2 || P_1) \right)^2 \right\} \\
\triangleq \sigma_n^2.
\] (39)

Hence, it follows from Theorem 2 that
\[
\beta_n^{(1)} \triangleq P_{n}^{(2)} (L(X_1, \ldots, X_n) \geq n \bar{\lambda}) \\
= P_{n}^{(2)} (U_n - U_0 \geq \epsilon_{2,1} n) \\
\leq \exp \left( -n D \left( \frac{\delta_{2,1} + \gamma_{2}}{1 + \gamma_{2}} \left\lVert \gamma_{2} \right\rVert \right) \right)
\] (40) (41)

where the equality in (40) holds due to (36) and (30), and (41) follows from Theorem 2 with
\[
\gamma_{2} \triangleq \frac{\sigma_n^2}{d_2^2}, \quad \delta_{2,1} \triangleq \frac{\epsilon_{2,1}}{d_2^2}
\] (42)
and $d_2$, $\sigma_n^2$ are introduced, respectively, in (38) and (39).

From (12), (33) and (41), the exponent of the probability of either having an error or an erasure is lower bounded by
\[
\lim_{n \to \infty} - \frac{\ln P_{n}^{(2)}}{n} \geq \min_{i=1,2} D \left( \frac{\delta_{i,1} + \gamma_{i}}{1 + \gamma_{i}} \left\lVert \gamma_{i} \right\rVert \right).
\] (43)

Similarly to the above analysis, one gets from (13) and (31) that the error exponent is lower bounded by
\[
\lim_{n \to \infty} - \frac{\ln P_{n}^{(2)}}{n} \geq \min_{i=1,2} D \left( \frac{\delta_{i,2} + \gamma_{i}}{1 + \gamma_{i}} \left\lVert \gamma_{i} \right\rVert \right)
\] (44)
where
\[
\delta_{1,2} \triangleq \frac{\epsilon_{1,2}}{d_1}, \quad \delta_{2,2} \triangleq \frac{\epsilon_{2,2}}{d_2}.
\] (45)

For the case of a single threshold (i.e., $\bar{\lambda} = \lambda \triangleq \lambda$) then (43) and (44) coincide, and one obtains that the error exponent satisfies
\[
\lim_{n \to \infty} - \frac{\ln P_{n}^{(2)}}{n} \geq \min_{i=1,2} D \left( \frac{\delta_{i} + \gamma_{i}}{1 + \gamma_{i}} \left\lVert \gamma_{i} \right\rVert \right)
\] (46)
where $\delta_{i}$ is the common value of $\delta_{i,1}$ and $\delta_{i,2}$ for $i = 1, 2$.

In this special case, the zero threshold is optimal (see, e.g., [8, p. 93]), which then yields that (46) is satisfied with
\[
\delta_{1} = \frac{D(P_1 || P_2)}{d_1}, \quad \delta_{2} = \frac{D(P_2 || P_1)}{d_2}
\] (47)
with $d_1$ and $d_2$ from (28) and (33), respectively. The right-hand side of (46) forms a lower bound on Chernoff information which is the exact error exponent for this special case.

C. Comparison of the Lower Bounds on the Exponents with those that Follow from Azuma’s Inequality

The lower bounds on the error exponent and the exponent of the probability of having either errors or erasures, that were derived in the previous sub-section via Theorem 2 are compared in the following to the loosened lower bounds on these exponents that follow from Azuma’s inequality.

We first obtain upper bounds on $\alpha_n^{(1)}$, $\alpha_n^{(2)}$, $\beta_n^{(1)}$ and $\beta_n^{(2)}$ via Azuma’s inequality, and then use them to derive lower bounds on the exponents of $P_{n}^{(1)}$ and $P_{n}^{(2)}$.

From (27), (28), (32), (33), and Azuma’s inequality
\[
\alpha_n^{(1)} \leq \exp \left( -\frac{\delta_{1,1} n}{2} \right)
\] (48)
and, similarly, from (37), (38), (40), (42), and Azuma’s inequality
\[
\beta_n^{(1)} \leq \exp \left( -\frac{\delta_{2,1} n}{2} \right).
\] (49)
From (9), (11), (51), (45) and Azuma’s inequality
\[ a_n^{(2)} \leq \exp \left( -\frac{\delta^2_{1,2} n}{2} \right) \] (50)
\[ \beta_n^{(2)} \leq \exp \left( -\frac{\delta^2_{1,2} n}{2} \right). \] (51)

Therefore, it follows from (12), (13) and (48)–(51) that the resulting lower bounds on the exponents of \( P_{c,n}^{(1)} \) and \( P_{c,n}^{(2)} \) are
\[ \lim_{n \to \infty} \frac{\ln P_{c,n}^{(j)}}{n} \geq \min_{i=1,2} \frac{\delta_{i,j}}{2}, \quad j = 1, 2 \] (52)
as compared to (43) and (44) which give, for \( j = 1, 2, \)
\[ \lim_{n \to \infty} \frac{\ln P_{c,n}^{(j)}}{n} \geq \min_{i=1,2} D \left( \frac{\delta_{i,j} + \gamma_i}{1 + \gamma_i} \bigg| \frac{\gamma_i}{1 + \gamma_i} \right). \] (53)

For the specific case of a zero threshold, the lower bound on the error exponent which follows from Azuma’s inequality is given by
\[ \lim_{n \to \infty} \frac{\ln P_{c,n}^{(j)}}{n} \geq \min_{i=1,2} \frac{\delta_{i,j}}{2}. \] (54)
with the values of \( \delta_1 \) and \( \delta_2 \) in (47).

The lower bounds on the exponents in (52) and (53) are compared in the following. Note that the lower bounds in (52) are loosened as compared to those in (53) since they follow, respectively, from Azuma’s inequality and its improvement in Theorem 2.

The divergence in the exponent of (53) is equal to
\[ D \left( \frac{\delta_{i,j} + \gamma_i}{1 + \gamma_i} \bigg| \frac{\gamma_i}{1 + \gamma_i} \right) = \left( \frac{\delta_{i,j} + \gamma_i}{1 + \gamma_i} \right) \ln \left( 1 + \frac{\delta_{i,j}}{\gamma_i} \right) + \left( 1 - \frac{\delta_{i,j}}{\gamma_i} \right) \ln (1 - \delta_{i,j}) \]
\[ = \frac{\gamma_i}{1 + \gamma_i} \left[ \left( 1 + \frac{\delta_{i,j}}{\gamma_i} \right) \ln (1 + \frac{\delta_{i,j}}{\gamma_i}) + \left( 1 - \frac{\delta_{i,j}}{\gamma_i} \right) \ln (1 - \delta_{i,j}) \right]. \] (55)

Lemma 1:
\[ (1 + u) \ln (1 + u) \geq \begin{cases} u + \frac{u^2}{2}, & u \in [-1, 0) \\ u + \frac{u^2}{2} - \frac{u^3}{6}, & u \geq 0 \end{cases} \] (56)
where at \( u = -1 \), the left-hand side is defined to be zero (it is the limit of this function when \( u \rightarrow -1 \) from above).

Proof: The proof follows by elementary calculus. Since \( \delta_{i,j} \in [0, 1] \), then (55) and Lemma 1 imply that
\[ D \left( \frac{\delta_{i,j} + \gamma_i}{1 + \gamma_i} \bigg| \frac{\gamma_i}{1 + \gamma_i} \right) \geq \frac{\delta^2_{i,j}}{2\gamma_i} - \frac{\delta^3_{i,j}}{6\gamma^2_i(1 + \gamma_i)}. \] (57)
Hence, by comparing (52) with the combination of (53) and (57), then it follows that (up to a second-order approximation) the lower bounds on the exponents that were derived via Theorem 2 are improved by at least a factor of \( (\max \gamma_i)^{-1} \) as compared to those that follow from Azuma’s inequality.

Example 1: Consider two probability measures \( P_1 \) and \( P_2 \) where
\[ P_1(0) = P_2(1) = 0.4, \quad P_1(1) = P_2(0) = 0.6, \] and the case of a single threshold of the log-likelihood ratio that is set to zero (i.e., \( \lambda = 0 \)). The exact error exponent in this case is Chernoff information that is equal to
\[ C(P_1, P_2) = 2.04 \cdot 10^{-2}. \]
The improved lower bound on the error exponent in (46) and (47) is equal to \( 1.77 \cdot 10^{-2} \), whereas the loosened lower bound in (54) is equal to \( 1.39 \cdot 10^{-2} \). In this case \( \gamma_1 = \frac{2}{4} \) and \( \gamma_2 = \frac{2}{5} \), so the improvement in the lower bound on the error exponent is indeed by a factor of approximately \( (\max \gamma_i)^{-1} \). Note that, from (33), (31) and (48)–(51), these are lower bounds on the error exponents for any finite block length \( n \), and not only asymptotically in the limit where \( n \to \infty \). The operational meaning of this example is that the improved lower bound on the error exponent assures that a fixed error probability can be obtained based on a sequence of i.i.d. RVs whose length is reduced by \( 22.2\% \) as compared to the loosened bound which follows from Azuma’s inequality.

D. Comparison of the Exact and Lower Bounds on the Error Exponents, Followed by a Relation to Fisher Information

In the following, we compare the exact and lower bounds on the error exponents. Consider the case where there is a single threshold on the log-likelihood ratio (i.e., referring to the case where the erasure option is not provided) that is set to zero. The exact error exponent in this case is given by the Chernoff information (see (23)), and it will be compared to the two lower bounds on the error exponents that were derived in the previous two subsections.

Let \( \{P_\theta\}_{\theta \in \Theta} \), denote an indexed family of probability mass functions where \( \Theta \) denotes the parameter set. Assume that \( P_\theta \) is differentiable in the parameter \( \theta \). Then, the Fisher information is defined as
\[ \mathcal{J}(\theta) \triangleq \mathbb{E}_\theta \left[ \frac{\partial}{\partial \theta} \ln P_\theta(x) \right]^2 \] (58)
where the expectation is w.r.t. the probability mass function \( P_\theta \). The divergence and Fisher information are two related information measures, satisfying the equality
\[ \lim_{\theta' \to \theta} \frac{D(P_{\theta'} || P_{\theta'})}{(\theta - \theta')^2} = \frac{\mathcal{J}(\theta)}{2} \] (59)
(note that if it was a relative entropy to base 2 then the right-hand side of (59) would have been divided by \( \ln 2 \), and be equal to \( \mathcal{I}(\theta) \) as \( \ln 2 \) in [6, Eq. (12.364)]).

Proposition 2: Under the above assumptions,
- The Chernoff information and Fisher information are related information measures that satisfy the equality
\[ \lim_{\theta' \to \theta} \frac{C(P_{\theta}, P_{\theta'})}{(\theta - \theta')^2} = \frac{\mathcal{J}(\theta)}{8}. \] (60)
- Let
\[ E_L(P_\theta, P_{\theta'}) \triangleq \min_{i=1,2} D \left( \frac{\delta_i + \gamma_i}{1 + \gamma_i} \bigg| \frac{\gamma_i}{1 + \gamma_i} \right) \] (61)
be the lower bound on the error exponent in (46) which corresponds to \( P_1 \triangleq P_\theta \) and \( P_2 \triangleq P_{\theta'} \), then also
\[ \lim_{\theta' \to \theta} \frac{E_L(P_{\theta}, P_{\theta'})}{(\theta - \theta')^2} = \frac{\mathcal{J}(\theta)}{8}. \] (62)
forms a refined version of Azuma’s inequality. The tightness of parameter martingales with uniformly bounded jumps, which this concentration inequality is studied via a large deviations Hoeffding inequality in information-theoretic aspects.

Indeed, the exact values of \( C(P_{\theta_1}, P_{\theta_2}) \) and \( E_L(P_{\theta_1}, P_{\theta_2}) \) are \( 2.00 \cdot 10^{-4} \) and \( 1.997 \cdot 10^{-4} \), respectively.

**IV. Summary**

This work introduces a concentration inequality for discrete-parameter martingales with uniformly bounded jumps, which forms a refined version of Azuma’s inequality. The tightness of this concentration inequality is studied via a large deviations analysis of binary hypothesis testing, and the demonstration of its improved tightness over Azuma’s inequality is revisited in this context. Some links of the derived lower bounds on the error exponents to some information measures (e.g., the relative entropy and Fisher information) are obtained along the way. This paper presents in part the work in [12] where further concentration inequalities that form a refinement of Azuma’s inequality were derived, followed by some further applications of these concentration inequalities in information theory, communication, and coding theory. It is meant to stimulate the use of some refined versions of the Azuma-Hoeffding inequality in information-theoretic aspects.

**APPENDIX A**

**SOME COMPLEMENTARY REMARKS CONCERNING THE CONSTRUCTION OF DOOB’S MARTINGALES**

This appendix is relevant to the analysis in Section [III].

**Remark 1:** Let \( \{X_i, F_i\} \) be a martingale sequence. For every \( i \), \( \mathbb{E}[X_{i+1}] = \mathbb{E}[\mathbb{E}[X_{i+1} | F_i]] = \mathbb{E}[X_i] \), so the expectation of a martingale stays constant.

**Remark 2:** One can generate martingale sequences by the following procedure: Given a RV \( X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \) and an arbitrary filtration of sub-\( \sigma \)-algebras \( \{F_i\} \), let\[
X_i = \mathbb{E}[X | F_i], \quad i = 0, 1, \ldots \n\]
Then, the sequence \( X_0, X_1, \ldots \) forms a martingale since
1) The RV \( X_i = \mathbb{E}[X | F_i] \) is \( F_i \)-measurable, and also \( \mathbb{E}[|X_i|] \leq \mathbb{E}[|X|] < \infty \) (since conditioning reduces the expectation of the absolute value).
2) By construction \( \{F_i\} \) is a filtration.
3) From the tower principle for conditional expectations, since \( \{F_i\} \) is a filtration, then for every \( i \)
\[\mathbb{E}[X_{i+1} | F_i] = \mathbb{E}[\mathbb{E}[X_{i+1} | F_i] | F_i] = \mathbb{E}[X | F_i] \quad \text{a.s.}\]

**Remark 3:** In continuation to Remark 2 one can choose \( F_0 = \{\Omega, \emptyset\} \) and \( F_n = \mathcal{F} \). Hence, \( X_0, X_1, \ldots, X_n \) is a martingale sequence where
\[X_n = \mathbb{E}[X | F_n] = \mathbb{E}[X] \quad \text{(since } X \text{ is independent of } F_0)\]
\[X_n = \mathbb{E}[X | F_n] = X \quad \text{a.s.} \quad \text{(since } X \text{ is } \mathcal{F} \text{-measurable).}\]
This has the following interpretation: At the beginning, we don’t know anything about \( X \), so it is initially estimated by its expectation. We then reveal at each step more and more information about \( X \) until we can specify it completely (a.s.).

**APPENDIX B**

**PROOF OF PROPOSITION 2**

The proof of (60) is based on calculus, and it is similar to the proof of the limit in (59) that relates the divergence and Fisher information. For the proof of (62), note that
\[C(P_{\theta}, P_{\theta'}) \geq E_L(P_{\theta}, P_{\theta'}) \geq \min_{i=1,2} \left\{ \frac{\delta_i^2}{2\gamma_i} - \frac{\delta_i^2}{6\gamma_i^2(1 + \gamma_i)} \right\}. \quad (65)\]
The left-hand side of (65) holds since \( E_L \) is a lower bound on the error exponent, and the exact value of this error exponent is the Chernoff information. The right-hand side of (65) follows from Lemma [1] (see (57)) and the definition of \( E_L \) in (61). By definition \( \gamma_i \triangleq \frac{\sigma_i^2}{\delta_i^2} \) and \( \delta_i \triangleq \frac{\sigma_i^2}{\gamma_i} \) where, based on (47),
\[\varepsilon_i \triangleq D(P_{\theta} || P_{\theta'}), \quad \varepsilon_i \triangleq D(P_{\theta'} || P_{\theta}). \quad (66)\]
The term on the left-hand side of (65) therefore satisfies
\[\frac{\delta_i^2}{2\gamma_i} - \frac{\delta_i^2}{6\gamma_i^2(1 + \gamma_i)} = \frac{\varepsilon_i^2}{2\sigma_i^2} - \frac{\varepsilon_i^2}{6\sigma_i^2(\sigma_i^2 + d_i^2)} \geq \varepsilon_i^2 \frac{1}{2\sigma_i^2} \left( 1 - \frac{\varepsilon_i d_i}{3} \right) \]
so it follows from (65) and the last inequality that
\[C(P_{\theta}, P_{\theta'}) \geq E_L(P_{\theta}, P_{\theta'}) \geq \min_{i=1,2} \left\{ \frac{\varepsilon_i^2}{2\sigma_i^2} \left( 1 - \frac{\varepsilon_i d_i}{3} \right) \right\}. \quad (67)\]
Based on the continuity assumption of the indexed family \( \{ P_\theta \}_{\theta \in \Theta} \), then it follows from (66) that
\[
\lim_{\theta' \to \theta} \varepsilon_i = 0, \quad \forall i \in \{1, 2\}
\]
and also, from (23) and (38) with \( P_1 \) and \( P_2 \) replaced by \( P_\theta \) and \( P_\theta' \) respectively, then
\[
\lim_{\theta' \to \theta} d_i = 0, \quad \forall i \in \{1, 2\}.
\]
It therefore follows from (69) and (67) that
\[
\frac{J(\theta)}{8} \geq \lim_{\theta' \to \theta} \frac{E_L(P_\theta, P_\theta')}{(\theta - \theta')^2} \geq \lim_{\theta' \to \theta} \min_{i=1,2} \left( \frac{\varepsilon_i^2}{2\sigma_i^2 (\theta - \theta')^2} \right).
\]
(68)
The idea is to show that the limit on the right-hand side of this inequality is \( \frac{J(\theta)}{8} \) (same as the left-hand side), and hence, the limit of the middle term is also \( \frac{J(\theta)}{8} \).
\[
\lim_{\theta' \to \theta} \frac{\varepsilon_i^2}{2\sigma_i^2 (\theta - \theta')^2} = \frac{J(\theta)}{8}
\]
(69)
where equality (a) follows from (66), equalities (b), (e) and (f) follow from (59), equality (c) follows from (29) with \( P_1 = P_\theta \) and \( P_2 = P_\theta' \), equality (d) follows from the definition of the divergence, and equality (g) follows by calculus (the required limit is calculated by using L'Hôpital's rule twice) and from the definition of Fisher information in (53). Similarly, also
\[
\lim_{\theta' \to \theta} \frac{\varepsilon_i^2}{2\sigma_i^2 (\theta - \theta')^2} = \frac{J(\theta)}{8}
\]
so
\[
\lim_{\theta' \to \theta} \min_{i=1,2} \left( \frac{\varepsilon_i^2}{2\sigma_i^2 (\theta - \theta')^2} \right) = \frac{J(\theta)}{8}.
\]
Hence, it follows from (68) that
\[
\lim_{\theta' \to \theta} E_L(P_\theta, P_\theta') = \lim_{\theta' \to \theta} \frac{\varepsilon_i^2}{2\sigma_i^2 (\theta - \theta')^2} = \frac{J(\theta)}{8}.
\]
This completes the proof of (62).

We prove now Eq. (64). From (23), (38), (47) and (63)
\[
\bar{E}_L(P_\theta, P_\theta') = \min_{i=1,2} \frac{\varepsilon_i^2}{2d_i^2}
\]
with \( \varepsilon_1 \) and \( \varepsilon_2 \) in (66). Hence,
\[
\lim_{\theta' \to \theta} \frac{\bar{E}_L(P_\theta, P_\theta')}{(\theta - \theta')^2} \leq \lim_{\theta' \to \theta} \frac{\varepsilon_i^2}{2d_i^2 (\theta - \theta')^2} \]
and from (69) and last inequality then it follows that
\[
\frac{J(\theta)}{8} \geq \lim_{\theta' \to \theta} \frac{\sigma_i^2}{d_i^2 (\theta - \theta')^2} \leq \frac{J(\theta)}{8} \lim_{\theta' \to \theta} \frac{\varepsilon_i^2}{2d_i^2 (\theta - \theta')^2} \]
(70)
It is clear that the second term on the right-hand side of (70) is bounded between zero and one (if the limit exists). This limit can be made arbitrarily small, i.e., there exists an indexed family of probability mass functions \( \{ P_\theta \}_{\theta \in \Theta} \) for which the second term on the right-hand side of (70) can be made arbitrarily close to zero. For a concrete example, let \( \alpha \in (0, 1) \) be fixed, and \( \theta \in \mathbb{R}^+ \) be a parameter that defines the following indexed family of probability mass functions over the ternary alphabet \( X = \{0, 1, 2\} \):
\[
P_\theta(0) = \frac{\theta(1 - \alpha)}{1 + \theta}, \quad P_\theta(1) = \alpha, \quad P_\theta(2) = \frac{1 - \alpha}{1 + \theta}.
\]

Then, it follows by calculus that for this indexed family
\[
\lim_{\theta' \to \theta} \sum_{x \in X} P_\theta(x) \left( \ln \frac{P_\theta(x)}{P_\theta'(x)} - \frac{\varepsilon_i^2}{2\sigma_i^2 (\theta - \theta')^2} \right)^2 = (1 - \alpha)\theta
\]
so, for any \( \theta \in \mathbb{R}^+ \), the above limit can be made arbitrarily close to zero by choosing \( \alpha \) close enough to 1. This completes the proof of (64), and also the proof of Proposition 2.

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