D = 1 supergravity as a constrained system

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Abstract.
I review the classical and quantum dynamics of systems with local world-line supersymmetry. The Hamiltonian formulation, in particular the covariant Hamiltonian approach, is emphasized. Anomalous behaviour of local quantum supersymmetry is investigated and illustrated by supersymmetric dynamics on the sphere $S^2$.

1. Introduction
Standard dynamical systems are described in terms of a set of $r$ real or complex generalized coordinates $\{q_i(\tau)\}$, $i = 1, \ldots, r$, defining the manifold of possible configurations of the system. The evolution of a classical dynamical system then traces out a continuous curve in this configuration space, and solving the dynamics amounts to finding a prescription to construct this curve for given initial data. Borrowing a term from the theory of relativity, this curve is sometimes referred to as the world line of the system. The evolution of a quantum dynamical system amounts to finding a probability amplitude for the system to develop from the initial configuration into some specific final configuration; a heuristic way to construct such amplitudes is in terms of the path integral, computed by summing complex phases defined in terms of paths in the classical configuration space.

Many systems of interest also possess discrete degrees of freedom, like spin or bit variables. Such discrete quantities may conveniently be represented by anti-commuting Grassmann variables $\{\psi_a(\tau)\}$, $a = 1, \ldots, s$, with $\psi_a \psi_b = -\psi_b \psi_a$ [1]. For Grassmann variables a complete calculus has been developed [2] defining operations like differentiation and integration of Grassmann-valued functions, as a result of which they can be treated as pseudo-classical continuous degrees of freedom. The configuration space of dynamical systems involving discrete variables can then be mapped to a graded manifold with both classical and pseudo-classical coordinates $\{q_i, \psi_a\}$.

A special class of systems with grassmannian degrees of freedom is formed by systems with a symmetry relating the dynamics of the classical and pseudo-classical variables [3]-[8]. This symmetry is known as world-line supersymmetry. In the following the dynamics of such supersymmetric systems is developed.

2. D = 1 superfields
One starting point for the construction of systems with world-line supersymmetry is to complement the continuous real time parameter $\tau$ with an grassmannian parameter $\theta$, $\theta^2 = 0$,
to form a one-dimensional graded base space spanned by \((\tau, \theta)\). The most common degrees of freedom of the system can then be represented by real Grassmann-even functions

\[
\Phi(\tau, \theta) = a(\tau) + i\theta \psi(\tau),
\]  
(1)

representing a graded pair \((a(\tau), \psi(\tau))\) of even and odd dynamical degrees of freedom. The factor of \(i\) has been introduced in (1) as neither \(\theta\) nor \(\psi\) are affected by complex conjugation, but this operation does reverse the order of the variables.

We consider graded translations of the world-line parameters defined by the graded pair of shift parameters \((\xi, \epsilon)\) and acting on the base-space co-ordinates as

\[
\tau' = \tau + \xi + i\theta \epsilon, \quad \theta' = \theta - \epsilon.
\]
(2)

On the function \(\Phi(\tau, \theta)\) these translations act as

\[
\Phi(\tau', \theta') = \Phi(\tau, \theta) + \xi \partial_\tau a - i\epsilon \psi + i\theta (\xi \partial_\tau \psi + \epsilon \partial_\tau a) + \mathcal{O}[\xi^2, \epsilon \xi],
\]
(3)

where the graded translation operators are defined by

\[
P = \partial_\tau, \quad Q = i\partial_\theta - \theta \partial_\tau.
\]
(4)

They have the algebraic properties

\[
iQ^2 = P, \quad PQ - QP = 0.
\]
(5)

The operator \(P\) is the usual time-translation operator, whilst the Grassmann-odd operator \(Q\) is the supersymmetry operator. By itself it acts on the components in such a way that to first order in the shift parameters

\[
\delta a = -i\epsilon \psi, \quad \delta \psi = \epsilon \partial_\tau a.
\]
(6)

These transformations define infinitesimal supersymmetry transformations on the graded pair of world-line variables \((a, \psi)\). Now observe that the superderivative operator

\[
D = i\partial_\theta + \theta \partial_\tau
\]
(7)

has the properties

\[
-iD^2 = P, \quad DQ + QD = 0, \quad DP - PD = 0.
\]
(8)

A brief calculations then shows that

\[
I = -\frac{i}{2} \int d\tau d\theta D\Phi D^2\Phi = \frac{1}{2} \int d\tau \left( (\partial_\tau a)^2 + i\psi \partial_\tau \psi \right)
\]
(9)

defines an action which is invariant under the supersymmetry transformations (6) modulo boundary terms.
3. Local world-line supersymmetry

The supersymmetry transformations in the previous section were defined as the Grassmann-odd part of the constant graded world-line shifts (2). In this section we introduce a set of graded local world-line transformations including a local realization of supersymmetry [4, 5, 9, 11]. Such a formalism is most conveniently developed in terms of graded pairs of variables, rather than using the $D = 1$ superfield formalism described in section 2.

As a first step consider local time reparametrizations $\tau \rightarrow \tau' = \tau - \xi(\tau)$. As common in differential geometry we distinguish between world-line scalars $A(\tau)$ and world-line 1-forms $N = N(\tau)d\tau$ transforming as

$$A'(\tau') = A(\tau), \quad N'(\tau')d\tau' = N(\tau)d\tau.$$  \hfill (10)

Following the conventions of general relativity for time-reparametrizations we introduce a specific 1-form referred to as the lapse function. In terms of this a reparametrization-invariant derivative and an invariant integral are defined for scalars by

$$\mathcal{D}A = \frac{1}{N} \frac{dA}{d\tau}, \quad I = \int d\tau N(\tau)A(\tau).$$  \hfill (11)

To first order in $\xi(\tau)$ the scalar and lapse function transform as

$$\delta A = \xi \partial_\tau A, \quad \delta N = \xi \partial_\tau N + N \partial_\tau \xi = \partial_\tau (\xi N).$$  \hfill (12)

We now also introduce time-dependent supersymmetry transformations in terms of a Grassmann-odd function $\epsilon(\tau)$, and generalize the previous discussion to consider combined time- and super-reparametrizations. First we define a graded pair of variables $G = (N, \chi)$ involving the lapse function and a Grassmann-odd scalar transforming as

$$\delta N = \partial_\tau (\xi N) - 2i\epsilon \chi N, \quad \delta \chi = \xi \partial_\tau \chi + \mathcal{D}\epsilon.$$  \hfill (13)

These transformations obey the commutation rules

$$[\delta(\xi_2, \epsilon_2), \delta(\xi_1, \epsilon_1)] = \delta(\xi_3, \epsilon_3),$$  \hfill (14)

with

$$\xi_3 = \xi_1 \partial_\tau \xi_2 - \xi_2 \partial_\tau \xi_1 - \frac{2i\epsilon_1 \epsilon_2}{N}, \quad \epsilon_3 = \xi_1 \partial_\tau \epsilon_2 - \xi_2 \partial_\tau \epsilon_1 + 2i\epsilon_1 \epsilon_2 \chi.$$  \hfill (15)

As the commutator algebra of the transformations closes they form a well-defined infinitesimal graded transformation group. Two more realizations of this infinitesimal group will be introduced here. The first is one is in terms of a graded pair $\Sigma = (a, \psi)$, where $a(\tau)$ is Grassmann-even and $\psi(\tau)$ is Grassmann-odd. On this pair the superparametrizations are defined by

$$\delta a = \xi \partial_\tau a - i\epsilon \psi, \quad \delta \psi = \xi \partial_\tau \psi + \epsilon (\mathcal{D}a + i\chi \psi).$$  \hfill (16)

The second one is an inversely graded pair $\Phi = (\eta, f)$ where $\eta(\tau)$ is odd and $f(\tau)$ is even, with transformations defined by

$$\delta \eta = \xi \partial_\tau \eta + \epsilon f, \quad \delta f = \xi \partial_\tau f - i\epsilon (\mathcal{D}\eta - f \chi).$$  \hfill (17)

1 Equivalently one can introduce a graded pair of 1-forms $(N, \omega)$ where $\omega = N\chi$: in applications the use of $\chi$ is more convenient.
In both cases the transformations satisfy the commutation rules (14), (15). Inversely graded pairs \( \Phi \) are useful to construct super-invariant integrals, as

\[
\delta \left[ d \tau N (f - i \chi \eta) \right] = d \left[ -i \epsilon \eta + \xi N (f - i \chi \eta) \right],
\]

and therefore

\[
I = \int d \tau N (f - i \chi \eta)
\]

is invariant modulo boundary terms.

It is possible to compose graded pairs by various simple rules. Scalar multiplication with a number \( \lambda \) is obvious:

\[
\lambda \Sigma = (\lambda a, \lambda \psi), \quad \lambda \Phi = (\lambda \eta, \lambda f),
\]

and addition and linear combination of graded pairs of same type is straightforward:

\[
\lambda \Sigma_1 + \mu \Sigma_2 = (\lambda a_1 + \mu a_2, \lambda \psi_1 + \mu \psi_2),
\]

\[
\lambda \Phi_1 + \mu \Phi_2 = (\lambda \eta_1 + \mu \eta_2, \lambda f_1 + \mu f_2).
\]

Multiplication of (inversely) graded pairs labeled by \( i,j, (a,b) \) is defined by the following rules:

\[
\Sigma_i \times \Sigma_j = \Sigma_{ij} = (a_i a_j, a_i \psi_j + a_j \psi_i),
\]

\[
- \iota \Phi_a \times \Phi_b = \Sigma_{ab} = (-i \eta_a \eta_b, f_a \eta_b + f_b \eta_a),
\]

\[
\Sigma_i \times \Phi_a = \Phi_{ia} = (a_i \eta_a, a_i f_a - i \psi_i \eta_a).
\]

The above rules allow the construction of arbitrary polynomial functions of graded pairs, e.g.

\[
F(\Sigma_i) = (F(a_i), \psi_j \partial_j F(a_i)).
\]

Finally one can introduce the superderivative \( D \) acting on pairs as

\[
D : \Sigma \xrightarrow{D} \Phi \xrightarrow{D} \Sigma' \xrightarrow{D} \Phi' \xrightarrow{D} ...
\]

by the rules

\[
\Phi = D \Sigma = (\psi, D a + i \chi \psi), \quad D \Phi = (f, D \eta - \chi f).
\]

It then follows that

\[
\Sigma' = D^2 \Sigma = (D a + i \chi \psi, D - \chi D a), \quad \Phi' = D^2 \Phi = (D \eta - \chi f, D f + i \chi D \eta).
\]

These components actually define supercovariant derivatives:

\[
\nabla a = D a + i \chi \psi, \quad \nabla \psi = D \psi - \chi \nabla a, \quad \nabla^2 a - D \nabla a + i \chi \nabla \psi, \quad ..., \nabla \eta = D \eta - \chi f, \quad \nabla f = D f + i \chi \nabla \eta, \quad ..., \]

with the property that \((\nabla a, \nabla \psi)\) and \((\nabla \eta, \nabla f)\) are (inversely) graded pairs if this holds for \((a, \psi)\) and \((\eta, f)\), respectively. Clearly there is a rule \(D^2 = \nabla\), keeping in mind that \(\nabla\) is defined on individual components.
4. Dynamics

The simplest procedure to develop the dynamics of supersymmetric systems is to construct invariant actions. This can be done by generalization and extension of the action (9) to representations of local supersymmetry, making use of the construction (19). In this context we take a set of pairs \( \Sigma^i, i = 1, ..., r \), interpreted as the co-ordinates of some graded manifold which will become the configuration space of our dynamical system. The first step is to form the inversely graded pairs

\[
D \Sigma^i \times D^2 \Sigma^j = \left( \psi^i \nabla a^j, \nabla a^i \nabla a^j + i \psi^i \nabla \psi^j \right).
\]

The second step is to complete this expression by contraction with a function \( G_{ij}(\Sigma) \) acting as a metric on the graded manifold:

\[
\Phi = G_{ij}(\Sigma) \times D \Sigma^i \times D^2 \Sigma^j = \left( G_{ij} \psi^i \nabla a^j, G_{ij} \nabla a^i \nabla a^j + i G_{ij} \psi^j \left( \nabla \psi^j + \nabla a^k \Gamma^j_{kl} \psi^l \right) \right),
\]

where \( \Gamma^j_{kl}(a) \) is the connection constructed from the metric \( G_{ij}(a) \). If we now substitute the components of \( \Phi \) into the expression (19) and normalize we get a supersymmetric action

\[
I = \frac{1}{2} \int d \tau N \left[ G_{ij} \nabla a^i \nabla a^j + i G_{ij} \psi^i \left( \nabla \psi^j + \nabla a^k \Gamma^j_{kl} \psi^l \right) - i \chi G_{ij} \psi^i \nabla a^j \right]
\]

transforming under local supersymmetry into

\[
\delta I = \int d \left( - \frac{i \epsilon}{2} G_{ij} \psi^i \nabla a^j \right).
\]

By varying the action (29) with respect to the dynamical degrees of freedom \((a^i, \psi^j)\), keeping the boundary values fixed, one derives the equations of motion. They can be written in manifestly supersymmetric form as

\[
\nabla^2 a^i + \Gamma^i_{jk} \nabla a^j \nabla a^k - \frac{i}{2} \psi^k \psi^j R_{kij}^i \nabla a^j = 0,
\]

\[
\nabla \psi^j + \nabla a^k \Gamma^j_{kl} \psi^l = 0,
\]

where \( R_{kij}^l(a) \) is the Riemann tensor of the configuration manifold with metric \( G_{ij}(a) \). Furthermore, varying the action with respect to the non-dynamical variables \((N, \chi)\) one finds two first-class constraints related to the local time- and super-reparametrization invariance:

\[
Q = G_{ij} Da^i \psi^j = 0, \quad \mathcal{H} = \frac{1}{2} G_{ij} Da^i Da^j = 0.
\]

They are referred to as the supercharge and hamiltonian constraint, respectively [12, 13, 14]. Observe that for positive definite metrics \( G_{ij} \) the hamiltonian constraint is extremely restrictive, essentially freezing all degrees of freedom. To get non-trivial dynamics an indefinite metric is strongly favored.
5. Hamiltonian formulation

The aim of the hamiltonian formalism is to replace a set of \( r \) second-order differential equations by \( 2r \) first-order differential equations. In that spirit we develop a procedure to recast the dynamics of the supersymmetric systems introduced in the previous section entirely in terms of first-order differential equations \([10]\). We do this by a Legendre transform of the action for the Grassmann-even variables \( a^i \) only, as the odd variables \( \psi^i \) already obey first-order equations of motion \([11, 15]\). To this effect we define the Grassmann-even momenta \( p_i \) by

\[
p_i = \frac{\delta I}{\delta \dot{a}^i} = G_{ij} \left( \dot{a}^j + i \chi \psi^j \right) + \frac{i}{2} G_{ij,k} \psi^j \psi^k,
\]

where as usual the overdot denotes a derivative with respect to time \( \tau \) and we use the comma notation for a partial derivative w.r.t. any of the \( a^i \).

Replacing the dynamical velocities \( \dot{a}^i \) in the action by the momenta it takes the form

\[
I_c = \int d\tau N \left( p_i \dot{a}^i + \frac{i}{2} G_{ij} \psi^i \psi^j - H_c \right),
\]

where the hamiltonian \( H_c \) now is a function of the generalized co-ordinates and momenta:

\[
H_c = \frac{1}{2} G^{ij} \left( p_i - \frac{i}{2} G_{ik,j} \psi^k \psi^j + i G_{ik} \chi \psi^k \right) \left( p_j - \frac{i}{2} G_{jm,n} \psi^m \psi^n + i G_{jn} \chi \psi^n \right).
\]

It is easy to verify that a similar substitution of momenta in the supercharge results in

\[
Q_c = p_i \psi^i.
\]

The action (34) with the hamiltonian (35) is guaranteed to produce equations of motion for the dynamical degrees of freedom which are first-order differential equations in time:

\[
\begin{bmatrix}
0 & -\delta^j_i & \frac{i}{2} G_{kj,i} \psi^k \\
\delta^j_i & 0 & 0 \\
\frac{i}{2} G_{ik,j} \psi^k & 0 & i G_{ij}
\end{bmatrix}
\begin{bmatrix}
\dot{a}^i \\
\dot{p}_i \\
\dot{\psi}^i
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial H}{\partial a^i} \\
\frac{\partial H}{\partial p_i} \\
\frac{\partial H}{\partial \psi^i}
\end{bmatrix}.
\]

These equations can be inverted to give

\[
\begin{bmatrix}
\dot{a}^i \\
\dot{p}_i \\
\dot{\psi}^i
\end{bmatrix}
=
\begin{bmatrix}
0 & \delta^j_i & 0 \\
-\delta^j_i & 0 & \frac{i}{2} G_{mn} G_{mk,j} G_{nj} \psi^k \psi^j \\
0 & -\frac{1}{2} G_{kl,i} G_{lk,j} \psi^k & -i G_{ij}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial a^i} \\
\frac{\partial H}{\partial p_i} \\
\frac{\partial H}{\partial \psi^i}
\end{bmatrix}.
\]

The canonical hamiltonian formulation of the dynamics now amounts to the following: there exists a graded continuum of (gauge-equivalent) phase spaces labeled by the graded pair \((N, \chi)\) of which the dynamical variables \((a^i, p_i, \psi^i)\) are the co-ordinates. For any specific choice of \((N, \chi)\) the dynamics on that representative phase space is defined by a bracket

\[
\mathcal{D} F = \{ F, H_c \},
\]
generating the evolution equations of a phase-space function \( F(a^i, p_i, \psi^i) \) for fixed \((N, \chi)\) by application of the canonical brackets

\[
\begin{align*}
\{a^i, p_j\} &= -\{p_j, a^i\} = \delta^i_j, \\
\{p_i, p_j\} &= -\{p_j, p_i\} = -\frac{i}{4} G^{mn} G_{mk,i} G_{nl,j} \psi^k \psi^l, \\
\{\psi^i, \psi^j\} &= \{\psi^j, \psi^i\} = -i G^{ij}, \\
\{p_i, \psi^j\} &= -\{\psi^j, p_i\} = \frac{1}{2} G_{kl,i} G^{lj} \psi^k.
\end{align*}
\]

(40)

The dynamical equations (39) thus represent the appropriate generalization of the Hamilton equations to locally supersymmetric systems. An obvious choice of representative phase space is the one labeled by \((N, \chi) = (1, 0)\); however these values are to be substituted only after imposing the first-class constraints (32) in the form

\[
Q_c = 0, \quad H_c = 0.
\]

(41)

We close this section by observing that

\[
\{Q_c, Q_c\} = 2i (H_c - i\chi Q_c), \quad \{Q_c, H_c\} = 2\chi H_c.
\]

(42)

in agreement with the first-class nature of the constraints.

### 6. Hamilton-Jacobi equation

Consider the solutions of the equations of motion tracing out curves in phase space between a fixed initial point at time \(\tau_1\) and various end points \((a^i, p_i, \psi^i)\) at time \(\tau_2\). Then one can define a function of the end point \(S(a^i, \psi^i)\) by integrating the action \(I_c\) over the corresponding curve:

\[
S = \int_{\tau_1}^{\tau_2} d\tau N \left( p_i \frac{D}{Da^i} + \frac{i}{2} G^{ij} \psi^i \psi^j - H_c \right).
\]

(43)

To see that this is a function only of \((a^i, \psi^i)\) it suffices to consider variations of the end point with different corresponding paths in phase space. As along each path the equations of motion (37) are satisfied the corresponding variation of \(S\) comes from the variation of the boundary point and is given (for fixed initial point) by

\[
\delta S = \left[ \delta a^i p_i - \frac{i}{2} \delta \psi^j G_{ij} \psi^j \right]_{\tau_2}.
\]

(44)

There is no contribution from variations \(\delta p_i\), as the time-derivatives of the momenta do not appear in the action or in the integral (43). As the variations \(\delta a^i\) and \(\delta \psi^i\) at the end point are independent, it also follows that the dependence of \(S\) on the end points results in

\[
\frac{\partial S}{\partial a^i} = p_i, \quad \frac{\partial S}{\partial \psi^i} = -\frac{i}{2} G_{ij} \psi^j.
\]

(45)

In view of this the first-class constraint for the supercharge, which is satisfied for all true solutions, implies that

\[
G^{ij} \frac{\partial S}{\partial a^i} \frac{\partial S}{\partial a^j} = 0.
\]

(46)

This is the supersymmetric variant of the Hamilton-Jacobi equation [11].
7. Covariant Hamiltonian Formalism

The equations of motion (31) have a purely geometric form, describing the world line of a spinning particle in \( r \) dimensions on which the spin variables \( \psi^i \) move by parallel transport. In the canonical Hamiltonian formulation, obtained in section 5 by Legendre transform of the same action (29), the manifest geometric structure is lost. In particular the canonical brackets (40) do not have a direct geometric representation. Manifestly covariant formulations of the dynamics do however exist [16]. In this section we develop actually two such formulations.

The first one is obtained by replacing the canonical momenta by covariant momenta:

\[
p_i \rightarrow P_i = p_i - \frac{i}{2} G_{ij,k} \psi^j \psi^k.
\]

They are covariant as after reparametrizing phase space in terms of \((a^i, P_i, \psi^i)\) the brackets (40) are replaced by the manifestly covariant expressions

\[
\{a^i, P_j\} = -\{P_j, a^i\} = \delta^i_j, \quad \{P_i, P_j\} = -\{P_j, P_i\} = -\frac{i}{2} \psi^k \psi^l R_{klij},
\]

\[
\{\psi^i, \psi^j\} = \{\psi^j, \psi^i\} = -iG^{ij}, \quad \{P_i, \psi^j\} = -\{\psi^j, P_i\} = \Gamma_{ik}^j \psi^k,
\]

where the structure functions of the various brackets are defined by the metric, the connection and the Riemann curvature of the manifold [17].

In addition to a transformation of the phase-space co-ordinates and brackets, to reobtain the covariant equations of motion (31) we also recast the supercharge and Hamiltonian in covariant form:

\[
Q_{\text{cov}} = p_i \psi^i, \quad H_{\text{cov}} = \frac{1}{2} G^{ij} P_i P_j.
\]

These are obtained from \(Q_c\) and \(H_c\) by the substitution (47) and subsequently taking

\[
Q_{\text{cov}} = Q_c, \quad H_{\text{cov}} = H_c + i\chi Q_c.
\]

Note that such a recombination of \(H_c\) and \(Q_c\) does not alter the first-class constraints, as they are equivalent with

\[
Q_{\text{cov}} = 0, \quad H_{\text{cov}} = 0.
\]

However, the canonical equations of motion (39) are now replaced by the covariant Hamilton equations

\[
\nabla F = D F + i\chi \{F, Q_{\text{cov}}\} = \{F, H_{\text{cov}}\},
\]

for any function \(F(a, P, \psi)\) on the phase space. To evaluate these expressions we collect here the phase-space supersymmetry transformations generated by the supercharge

\[
\{a^i, Q_{\text{cov}}\} = \psi^i, \quad \{P_i, Q_{\text{cov}}\} = \Gamma_{ij}^k P_k \psi^j, \quad \{\psi^i, Q_{\text{cov}}\} = -iG^{ij} P_j.
\]

It then follows in particular that

\[
P_i = G_{ij} \nabla a^j.
\]

Using this expression in the equations of motion for \(P_i\) and \(\psi^i\) returns the covariant equations (31). As concerns the algebra of constraints, with the help of the relations (53) it is straightforward to establish that

\[
\{Q_{\text{cov}}, Q_{\text{cov}}\} = -2iH_{\text{cov}}, \quad \{Q_{\text{cov}}, H_{\text{cov}}\} = 0.
\]
Clearly these relations are in full agreement with the constraints (51), confirming again their first-class character.

Another covariant formulation, fully equivalent at the level of classical dynamics, can be constructed in terms of local tangent frames. These are spanned at any point of the manifold by an orthonormal set of \( r \) vectors \( e^a_i(a) \):

\[
G_{ij} e^i_a e^j_b = \eta_{ab}, \quad a, b = (1, \ldots, r),
\]

(56)

where \( \eta_{ab} \) is the tangent-space euclidean or pseudo-euclidean metric with all eigenvalues \( \pm 1 \); in locally (pseudo-)cartesian co-ordinates \( \eta = \text{diag}(\pm 1, \ldots, \pm 1) \). Being non-singular there exist dual 1-forms \( e^i_a = e^{i}_{a} (a) \) such that

\[
e^i_a e^b_j = \delta^b_a,
\]

(57)

from which it follows that

\[
e^a_i = G_{ij} e^j_b \eta^{ba}, \quad \eta^{ab} e^a_i e^b_j = G_{ij}.
\]

(58)

Parallel transport of the frames now involves local \( r \)-dimensional (pseudo-)rotations in the tangent space, which are kept track of by the spin connection \( \omega^a_{\dot{b}} = \omega^a_{\dot{b}}(a) \):

\[
e^a_j \dot{i} = \partial_i e^a_j - \Gamma^k_{ij} e^a_k = \omega^a_{\dot{b}} e^b_j.
\]

(59)

Representing a (pseudo-)rotation the spin connection has the anti-symmetry property

\[
\omega_{ab} = \eta_{ac} \omega^c_{\dot{b}} = -\omega_{ba}.
\]

(60)

In the context of supersymmetric dynamical systems we use the tangent frames to redefine the Grassmann-odd co-ordinates as taking values in the local tangent frame:

\[
\psi^i \rightarrow \phi^a = e^a_i \psi^i.
\]

(61)

The brackets (48) then are replaced by

\[
\{ a^i, P_j \} = - \{ P_j, a^i \} = \delta^i_j, \quad \{ P_i, P_j \} = - \{ P_j, P_i \} = -i \phi^a \phi^b R_{abij},
\]

(62)

\[
\{ \phi^a, \phi^b \} = \{ \phi^b, \phi^a \} = -i \eta^{ab}, \quad \{ P_i, \phi^a \} = - \{ \phi^a, P_i \} = -i \omega^a_{\dot{b}} \phi^b.
\]

(63)

After the redefinition the supercharge becomes

\[
Q_{\text{cov}} = P_i e^i_a \phi^a.
\]

(64)

The hamiltonian (49) and the algebra of constraints (55) remain unchanged. The equations of motion now read

\[
P_i = G_{ij} \nabla a^j, \quad \nabla^2 a^i + \Gamma^k_{ij} \nabla a^j \nabla a^k = \frac{i}{2} \phi^a \phi^b R_{abij} \nabla a^i,
\]

(65)

\[
\nabla \phi^a + \nabla a^i \omega^a_{\dot{b}} \phi^b = 0.
\]
8. Quantum theory

The local tangent-frame formulation of the pseudo-classical dynamics of locally supersymmetric systems is most convenient for the purpose of canonical quantization [11]. In this procedure the phase-space variables are replaced by self-adjoint operators in Hilbert space, and the correspondence principle is used to determine the commutation or anti-commutation relations between these operators from the classical brackets. For the systems at hand the first step is to introduce operators corresponding to the physical degrees of freedom

\[ a^i \rightarrow \xi^i, \quad p_i \rightarrow \pi_i, \quad \phi^a \rightarrow \frac{1}{\sqrt{2}} \gamma^a. \]  

(65)

Following the rule

\[ i(\text{phase space brackets}) \rightarrow [\text{(quantum operators)}], \]

with square brackets [ , ] denoting commutators and accolades { , } now anti-commutators, the fundamental operator commutation relations are postulated to be

\[ [\xi^i, \pi_j] = i\delta^i_j, \quad \{\gamma^a, \gamma^b\} = 2\eta^{ab}, \]

\[ [\gamma^a, \pi_i] = i\omega^a_{\ b}\gamma^b, \quad [\pi_i, \pi_j] = \frac{1}{2} \sigma^{ab} R_{abij}, \]  

(66)

with \( \gamma^a \) acting as the generators of an \( r \)-dimensional Clifford algebra and

\[ \sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b] \]  

(67)

as the generators of the associated (pseudo) rotation group \( SO(r - s, s) \). Here \( s \) denotes the number of non-compact (time-like) dimensions in tangent space. The consistency of the commutation relations (66) may be checked from the graded Jacobi identities, using the Clifford and Bianchi identities

\[ [\sigma^{bc}, \gamma^a] = \eta^{ac} \gamma^b - \eta^{ab} \gamma^c, \quad R_{ab[ijk]} = 0, \]  

(68)

which guarantee that for all \( (A, B, C) \in \{\xi^i, \pi_i, \gamma^a\} \)

\[ (-1)^{CA}[[A, B], C] + (-1)^{AB}[[B, C], A] + (-1)^{BC}[[C, A], B] = 0. \]  

(69)

It now remains to find a realization of these operators in some representation of Hilbert space. The commutation relation of the \( \xi^i \) and \( \pi_i \) implies that the \( \pi_i \) act as a set of non-commutative derivatives on functions of the \( \xi^i \). Next considering representations of the Clifford algebra spanned by the operators \( \gamma^a \), we observe that there is an irreducible representation of their anti-commutation relations in terms of Dirac matrices of dimension \( 2^{(r/2)} \times 2^{(r/2)} \), which suggests to take the elements of Hilbert space to be \( 2^{(r/2)} \)-component spinors \( \Psi \). However, to guarantee the self-adjointness of the operators some additional steps are necessary. The precise steps depend on the number \( s \) of non-compact directions in tangent space. We restrict our discussion to the cases \( s = 0 \) and \( s = 1 \) corresponding to euclidean and minkowskian tangent-space geometries, respectively.

For euclidean tangent spaces with positive-definite metric \( \eta_{ab} = \delta_{ab} = \text{diag}(+1, ..., +1) \) the irreducible representations of the Dirac matrices are hermitean; therefore one can define an invariant scalar product on the Hilbert space of spinors by

\[ (\Phi, \Psi) = \int d^r \xi \, \Phi^\dagger \Psi, \]  

(70)
where \( e = \det e^i_a = \sqrt{G} \) is included to make the integration measure invariant under co-ordinate transformations. Owing to the hermitean property if the Dirac matrices they represent self-adjoint operators:

\[
(\Phi, \gamma^a \Psi) = (\gamma^a \Phi, \Psi).
\]  

(71)

For minkowskian tangent spaces with indefinite metric \( \eta_{ab} = \text{diag}(-1,+1,\ldots,+1) \) only the compact (space-like) components of \( \gamma^a, a = 1,\ldots,r-1 \), are represented by hermitean matrices, whilst \( \gamma^0 \) is anti-hermitean. However, it then follows that one can construct a complete set of hermitean matrices \( \gamma^0, \gamma^a \) for all \( a = 0, 1,\ldots,r-1 \):

\[
(\gamma^0 \gamma^a)^\dagger = \gamma^0 \gamma^a.
\]  

(72)

Following Dirac we therefore define a modified invariant scalar product

\[
(\Phi, \Psi) = \int d^r \xi \bar{\Phi} \Psi, \quad \bar{\Phi} = \Phi^\dagger \gamma^0,
\]  

(73)

where now \( e = \sqrt{-G} \). In view of (72) it is then again guaranteed that the operators \( \gamma^a \) are self-adjoint as in (71) with respect to this modified scalar product.

Having established the conditions for the operators \( \gamma^a \) to be self-adjoint we can also provide a self-adjoint representation of the momentum operators \( \pi_i \) in terms of the spin connection:

\[
\pi_i = -\frac{i}{\sqrt{e}} D_i \sqrt{e}, \quad D_i = \partial_i - \frac{1}{2} \omega_{iab} \sigma^{ab},
\]  

(74)

as in euclidean space \( \sigma^{ab}_+ = -\sigma_{ab} \), whilst in the minkowskian case eq. (72) implies that

\[
\gamma^0 \sigma^{ab} = -\left(\gamma^0 \sigma^{ab}\right)^\dagger.
\]  

(75)

The first equation (68) then establishes the commutation relation between \( \gamma^a \) and \( \pi_i \), whilst the commutator of two momenta holds because of the Ricci identity:

\[
[\pi_i, \pi_j] = -\frac{1}{\sqrt{e}} [D_i, D_j] \sqrt{e} = -\frac{1}{2} \sigma^{ab} R_{abij}.
\]  

(76)

With the operator representations defined above we can construct a self-adjoint supercharge operator:

\[
Q = \frac{1}{2} \left( \gamma^a e^i_a \pi_i + \pi_i e^i_a \gamma^a \right) = -i \gamma^a e^i_a D_i,
\]  

(77)

where in the last step we have used the definition of the spin-connection (59). The supersymmetry constraint then reduces to the condition

\[
-i \gamma^a e^i_a D_i \Psi = 0,
\]  

(78)

which is the massless Dirac equation on the curved manifold with metric \( G_{ij}[\xi] \). The corresponding hamiltonian constraint is

\[
\mathcal{H} = \frac{1}{2} \left\{ Q, \mathcal{Q} \right\} = -\frac{1}{2} D_i^2 - \frac{1}{8} R,
\]  

(79)

where \( R \) is the Riemann scalar. Note that any operator-ordering ambiguities in the definition of the hamiltonian have been solved by its relation to the supercharge and the condition of self-adjointness of the latter [8]. As in the classical models the consistency of the constraints follows from (79) and its consequence

\[
[Q, \mathcal{H}] = 0.
\]  

(80)
9. Examples: supersymmetric $S^2$

The evolution of dynamical quantum operators $F(\xi, \pi, \gamma)$ are obtained by straightforward application of the hamiltonian:

$$D F + i \chi [F, Q] = [F, \mathcal{H}].$$

Here the auxiliary gauge variables $(N, \chi)$ are to be considered external parameters fixing the representation of the Hilbert space. As alluded to above, it is perfectly allowed to chose a particular representative, the one with $N = 1$ and $\chi = 0$ being the simplest one to work with. This choice is employed below. It should however be remembered that their role was to impose the first-class constraints, in the quantum theory on the states in Hilbert space:

$$Q \Psi = 0, \quad \mathcal{H} \Psi = 0.$$

As the second constraint is an automatic consequence of the first one, the supercharge constraint is the fundamental constraint, the hamiltonian one is used for mathematical simplifications. Thus the physical states of a quantum system with world-line supersymmetry are characterized by the generalized Dirac equation (78); the space of solutions consists of the set of zero modes, the kernel, of the Dirac operator.

The existence of physical states is not guaranteed; if the kernel of the Dirac operators is empty there is no supersymmetric system in which the dynamical constraint can be realized. This might be interpreted as an anomaly of the local world-line supersymmetry at the quantum level. In fact we have already noticed in section 4 that in the classical theory for positive-definite metrics $G_{ij}$, the system is frozen, and this is manifested by the absence of normalizable zero-modes in the quantum theory.

An explicit example of such a situation is provided by the supersymmetric sphere $S^2$ [18]. Its tangent space is euclidean $R^2$ and in polar co-ordinates $\xi^i = (\theta, \varphi)$ the metric and corresponding tangent frame vectors are

$$G_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \quad e^a_i = \begin{pmatrix} 1 & 0 \\ 0 & \sin \theta \end{pmatrix}.$$

The tangent-space components of the spin connection 1-form are

$$\omega_i d\xi^i = \begin{pmatrix} 0 & \cos \theta \\ -\cos \theta & 0 \end{pmatrix} d\varphi,$$

In two dimensions the Dirac matrices are identical with the Pauli matrices: $\gamma^a = \sigma^a, a = (1, 2)$. The Dirac operator in this representation is then found to be

$$-i \gamma^a e^a_i D_i = \begin{pmatrix} 0 & -iD_- \\ -iD_+ & 0 \end{pmatrix}, \quad D_\pm = \partial_\theta + \frac{1}{2} \cot \theta \pm \frac{i}{\sin \theta} \partial_\varphi.$$

To find the eigenvalues of this operator solve the equations

$$-i \gamma^a e^a_i D_i \Psi = \kappa \Psi, \quad \Psi(\theta, \varphi) = e^{im\varphi} \begin{bmatrix} \psi_+(\theta) \\ \psi_-(\theta) \end{bmatrix},$$

where $m$ is an integer; the components satisfy

$$-i \left( \partial_\theta + \frac{1}{2} \cot \theta + \frac{m}{\sin \theta} \right) \psi_- = \kappa \psi_-, \quad -i \left( \partial_\theta + \frac{1}{2} \cot \theta - \frac{m}{\sin \theta} \right) \psi_+ = \kappa \psi_-.$$
The Hamiltonian form is more practical as it diagonalizes the equations:

\[
\left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{m^2 + \frac{1}{4}}{\sin^2 \theta} \pm \frac{m \cos \theta}{\sin^2 \theta} - \frac{1}{4}\right) \psi_\pm = -\kappa^2 \psi_\pm.
\]  

(88)

After a change of variable \( z = \cos \theta \) this equation takes the form

\[
\left[(1 - z^2) \frac{\partial^2}{\partial z^2} - 2\xi \frac{\partial}{\partial z} - \frac{m^2 + \frac{1}{4}}{1 - z^2} \pm \frac{mz}{1 - z^2} - \frac{1}{4}\right] \psi_\pm = -\kappa^2 \psi_\pm.
\]  

(89)

In the case of \( m \geq 0 \) we first take the upper component; the solutions are of the form

\[
\psi_+ = (1 - z)^\frac{n-m}{4} (1 + z)^\frac{n+m}{4} J_n^{m-1/2,m+1/2}(z),
\]  

(90)

where \( J_n^{p,q} \) is a Jacobi polynomial of degree \( n \). For \( p = m - 1/2 \) and \( q = m + 1/2 \) the corresponding eigenvalues are

\[
\kappa_{nm}^2 = \left(n + m + \frac{1}{2}\right)^2.
\]  

(91)

The corresponding solutions for the lower component in (89) are obtained by computing

\[
\kappa \psi_- = \left(\sqrt{1 - z^2} \frac{\partial}{\partial z} - \frac{1}{2} \frac{z}{\sqrt{1 - z^2}} + \frac{m}{\sqrt{1 - z^2}}\right) \psi_+,
\]  

(92)

and have the same eigenvalue spectrum. For negative \( m \) the situation is reversed, the lower components \( \psi_- \) being described by (90) and the upper components by (92). As both \( n \) and \( m \) are integers it follows that there are no solutions of eq. (86) with \( \kappa = 0 \).

The same eigenvalue problem in the context of a space with indefinite metric appears in a cosmological setting [11] if we consider a homogeneous space-time of Friedmann-Lemaitre type with scale factor \( a(t) = e^{\xi_0 \sqrt{\kappa}} \) and two spatially homogeneous scalar fields \( \xi^m(t), \; m = (1,2) \) taking values on the sphere \( S^2 \). The non-supersymmetric classical action for such a system is

\[
S = \frac{1}{2} \int d^4 \tau N g_{ij} D\xi^i D\xi^j, \quad g_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{G_{mn}} \end{pmatrix},
\]  

(93)

where \( i, j = (0,1,2) \), \( m, n = (1,2) \) and \( G_{mn} \) is the scalar-field metric on \( S^2 \). Introducing a cosmological time \( \sigma \) defined in terms of the scale factor and polar co-ordinates such that \( \xi^i = (\sigma, \theta, \varphi) \) the metric reads

\[
g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 & \sin^2 \theta \end{pmatrix}.
\]  

(94)

The supersymmetric version of the corresponding quantum cosmology in which \( \gamma^0 = i\sigma^3, \; \gamma^{1,2} = \sigma^{1,2} \), is defined by the Dirac operator

\[
-i\gamma^a e_a^i D_i = \begin{pmatrix} \partial_\sigma & -iD_- \\ -iD_+ & -\partial_\sigma \end{pmatrix},
\]  

(95)

where the covariant derivatives \( D_\pm \) on the sphere are defined as before, eq. (85). The eigenfunctions are 2-component spinors

\[
\Psi(\sigma, \theta, \varphi) = e^{i\kappa \sigma + i m \varphi} \begin{pmatrix} \psi_+ (\theta) \\ \psi_- (\theta) \end{pmatrix},
\]  

(96)

and now the zero-modes are the solutions of (89) for all allowed values of \( \kappa \). Thus we have an infinite set of solutions of then form (90), (92) labeled by integers \( (n,m) \) with the spectrum of allowed values \( \kappa_{nm} \) given by (91).
10. Potentials

In the previous sections we have focussed on the description of supersymmetric dynamical systems with purely geometrical hamiltonians (32). In this section we describe how to extend the dynamics with interactions of potential type [9]. From sect. 3 we recall that supersymmetric actions are constructed from elementary or composite inversely graded superpairs \((\eta, f)\) using expression (19). A straightforward generalization of this construction is to take an elementary pair \(\Phi = (\eta, f)\) and multiply it with some function \(W[\Sigma]\) of the scalar pairs \(\Sigma_i = (a_i, \psi_i)\) according to the last rule (22):

\[
\Phi_W \equiv W[\Sigma] \times \Phi = \left(W(a)\eta, W(a)f - i\partial_i W(a)\psi^i\eta\right).
\]  

(97)

Another contribution is constructed by taking

\[
D\Phi \times \Phi = \left(f\eta, f^2 + i\eta\nabla\eta\right).
\]  

(98)

Combine these terms to form

\[
\Phi_{pot} = \frac{1}{2} D\Phi \times \Phi - \Phi_W,
\]  

(99)

and insert the components of \(\Phi_{pot}\) into expression (19) to get

\[
I_{pot} = \int d\tau N\left[\frac{i}{2}\eta D\eta + \frac{1}{2} f^2 - W f - i\eta\chi W - i\eta\psi^i\partial_i W\right].
\]  

(100)

Now the auxiliary variable \(f\) can be eliminated by its algebraic equation of motion \(f = W\), equivalent to completing the square, to get

\[
I_{pot} \simeq \int d\tau N\left[\frac{i}{2}\eta D\eta - i\eta\chi W - i\eta\psi^i\partial_i W - \frac{1}{2} W^2\right].
\]  

(101)

Adding this to the action (29) it supplies a scalar potential \(W^2(a)/2\) plus supersymmetric completion terms, at the price of having an extra Grassmann-odd degree of freedom. Of course an arbitrary number of such potentials can be added in principle, each carrying its own Grassmann-odd variable along. With the single contribution of the potential terms (101) the equations of motion are modified to

\[
\nabla^2 a^i + \Gamma^i_{jk}\nabla a^j\nabla a^k - \frac{i}{2} \psi^k\psi R_{kij} a^j = -G^{ij} \left(W_j W + i\eta\psi^k W_{jk} + i\eta\chi W_{ji}\right),
\]  

\[
\nabla \psi^i + \nabla a^k \Gamma^i_{kj} \psi^j = G^{ij} W_j \eta,
\]  

\[
\nabla \eta = D\eta - \chi W = \psi^i W_{;i},
\]  

(102)

where as before we use the comma notation to denote partial derivatives w.r.t. the \(a^i\). In addition we get modified constraints from variations w.r.t. \(N\) and \(\chi\):

\[
\mathcal{H} = \frac{1}{2} G_{ij} Da^i Da^j + \frac{1}{2} W^2 + i\eta \left(\chi W + \psi^i W_{;i}\right) = 0,
\]  

(103)

\[
Q = G_{ij} Da^i \psi^j + \eta W = 0.
\]

Upon quantization the additional Grassmann-odd variables will become operators \(\eta \rightarrow \alpha/\sqrt{2}\) extending the Clifford algebra (66) with one or more extra generators \(\alpha\):

\[
\alpha^2 = 1, \quad \alpha\gamma^a + \gamma^a\alpha = 0.
\]  

(104)
The supercharge operator (77) is accordingly generalized to

\[ Q = -i\gamma^a e^a_i D_i + \alpha W. \]  \hspace{1cm} (105)

This allows for turning the supersymmetry constraint into version of the Dirac equation extended by a mass term. In the example of the sphere \( S^2 \) the introduction of a mass term in (85) would be equivalent to consider a single mode of the cosmological model (95), which we have seen to have normalizable solutions. This amounts to a form of reduction from 2+1 to 2 dimensions, though not by compactification but by mode selection. Equivalently, the full set of solutions of the cosmological model corresponds to a full tower of Dirac equations for massive states on \( S^2 \), with quantized masses (91). Thus the anomalous behaviour of local world-line supersymmetry is cured by the introduction of a mass term accompanied by an extension of the Clifford algebra, signifying an extra Grassmann-odd degree of freedom in the corresponding pseudo-classical model.

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