Caged Black Holes:  
Black Holes in Compactified Spacetimes I – Theory

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ABSTRACT: In backgrounds with compact dimensions there may exist several phases of black objects including the black-hole and the black-string. The phase transition between them raises puzzles and touches fundamental issues such as topology change, uniqueness and Cosmic Censorship. No analytic solution is known for the black hole, and moreover, one can expect approximate solutions only for very small black holes, while the phase transition physics happens when the black hole is large. Hence we turn to numerical solutions. Here some theoretical background to the numerical analysis is given, while the results will appear in a forthcoming paper. Goals for a numerical analysis are set. The scalar charge and tension along the compact dimension are defined and used as improved order parameters which put both the black hole and the black string at finite values on the phase diagram. Predictions for small black holes are presented. The differential and the integrated forms of the first law are derived, and the latter (Smarr’s formula) can be used to estimate the “overall numerical error”. Field asymptotics and expressions for physical quantities in terms of the numerical ones are supplied. Techniques include “method of equivalent charges”, free energy, dimensional reduction, and analytic perturbation for small black holes.
1. Introduction

In the presence of extra compact dimensions, there may exist several phases of massive solutions of General Relativity, depending on the relative size of the object and the relevant length scales in the compact dimensions. For concreteness, we consider a background with a single compact dimension \(-\mathbb{R}^{d-2,1} \times S^1\). We denote the coordinate along the compact dimension by \(z\) and the period by \(\hat{L}\). The problem is characterized by a single dimensionless parameter, for instance\(^1\) the dimensionless mass \(\mu = G_N M / \hat{L}^{d-3}\) where \(G_N\) is the \(d\) dimensional Newton constant and \(M\) is the mass (measured at infinity).

The relevant phases are

\(^1\)Later we will define an alternative parameter \(x\).
• The uniform black string.

A string solution is one which has an $S^{d-3} \times S^1$ horizon topology. The uniform string is $z$-independent and it is described by the $d - 1$ Schwarzschild - Tangherlini metric \cite{4} with the addition of $+dz^2$ – a spectating $z$ coordinate. This solution was shown to be unstable to gravitational perturbation below a certain critical $\mu$ by Gregory and Laflamme (GL) \cite{5, 6}.

• The non-uniform string.

These are solutions with the string horizon topology, which are no longer $z$-independent. The marginally unstable GL - mode implies that such a branch of solutions is emanating from the GL critical point (for instance due to Morse theory \cite{4}), and a priori there could be other branches as well. Gubser determined that the phase transition at the GL point is first order by an analytic perturbation analysis \cite{5}. Morse theory implies that the emanating branch would be unstable, at least at the beginning \cite{4}. Finally these solutions were found numerically by Wiseman in 6d \cite{6}, who managed to formulate axially-symmetric gravitostatics (namely, essentially 2d) in a “relaxation” form (a procedure familiar from electrostatics) while presenting the constraints through “Cauchy-Riemann - like” relations.

• The black hole.

These are solutions with $S^{d-2}$ horizon topology. For small (dimensionless) mass we expect these solutions to resemble a $d$ dimensional black hole near the horizon. No analytic solution is known, and we consider our numerical solutions in 5d \cite{7} as strong evidence for their existence (indications appeared already in \cite{4}).

• The stable non-uniform string.

Horowitz and Maeda [9] postulated the existence of an additional stable phase, to serve as the end-point for the GL decay. This phase should be distinguished from the unstable non-uniform string which is presumably a consequence of Gubser’s analysis (or from strings which are too massive to serve as the end-point for decay – clearly the stability of the solution is critical for the physical interpretation). Horowitz and Maeda argued, based on theorems for the increasing area of the event horizon, that horizon pinching is impossible. By now there is mounting evidence against the existence of this phase: in \cite{4} it was shown that a continuous topology change (as a function of a parameter rather than time) is actually possible, moreover Morse theory implies an obstacle, namely that the addition of this phase must be accompanied by some other one, in \cite{4} it was shown numerically that the whole branch of static non-uniform strings originating from the GL point is too massive to serve as an end-point, finally a full time-dependent numerical simulation had to stop at $R_{\min}/R_{\max} \sim 1/13$ (due to grid stretching) without finding clear evidence for stabilization towards this phase \cite{10} \cite{10} ($R_{\max}$, $R_{\min}$ refer to the $z$-dependent radius of the horizon). Additional

$^2$Note however that the authors of this paper did not interpret their results either as supporting or as countering the conjecture.
circumstantial evidence is supplied by Gubser’s frustrated hope to find a second order transition \cite{Gubser2004} and by the failed attempt to find the non-uniform string analytically by looking for 5d algebraically - special solutions \cite{Gubser2005}.

Our interpretation is that the argument of \cite{Gubser2004} fails at a deep and interesting level – the horizon theorems being used there require the singularity not to leave the horizon, while it seems to us plausible that this system constitutes an example for violation of Cosmic Censorship.

In addition to the research described above we would like to mention: an ansatz that reduces the number of unknown functions due to Harmark and Obers \cite{Harmark2005}, speculations for a generalization of uniqueness to \(d > 4\) \cite{Harmark2006}, some phenomenological implications for huge explosions accompanying the transition \cite{Harmark2007}, numerical evidence for the ideas of \cite{Gubser2004} including the approach to a cone \cite{Harmark2008}, a time-symmetric initial data study \cite{Harmark2009}, a numerical study of black holes on a brane which have a similar geometry \cite{Harmark2010} a recently discovered, and a related instability in highly rotating black holes \cite{Harmark2011}. Other related research included a relation between the GL transition and gauge theories through Matrix M-Theory \cite{Harmark2012}, a prediction for a non-uniform near-extremal black string \cite{Harmark2013}, a useful review of numerical relativity \cite{Harmark2014}, a relation between the GL instability and the negative heat capacity of black holes \cite{Harmark2015}, a discussion of implication of the GL instability to more general spacetimes with horizons \cite{Harmark2016}, a discussion of the explosion expected during these phase transition \cite{Harmark2017}, a recent analysis of 4d metrics with a compact dimension but different asymptotic conditions and a study of non-uniform strings in the two-brane system \cite{Harmark2018}. See also \cite{Harmark2019, Harmark2020, Harmark2021} for a short and non-representative list of papers on black holes and large extra dimensions/brane worlds/ accelerator prospects.

Here, motivated by \cite{Gubser2004} we take the route of approaching the phase transition by increasing the mass of the black hole, rather than through the GL instability. Since no analytic solution is known for the black hole we turn to numerical solutions. Moreover, even though one can expect approximate analytic solutions to exist for very small black holes, the phase transition physics happens when the black hole is of the same size as the compact dimension, and so a numerical simulation is essential. It is important to list the goals of such a numerical study, even if currently we cannot reach all of them

- Establish the existence of the black hole solution.

- Determine the maximal (dimensionless) mass for a black hole (it is expected not to fit in the compact circle beyond a certain size \cite{Gubser2004}). Identify the direct reason for that.

- Establish another failure of generalized uniqueness, by demonstrating a black hole solution with the same mass as some (stable) black string.

- Test the description of approach to topology change proposed in \cite{Gubser2004}, namely that the geometry in the region between the two poles approaches a cone and that at least for \(5 \leq d < 10\) the geometry destabilizes (perturbatively) before reaching transition.
• To determine whether the black string decay violates Cosmic Censorship is one of the most interesting issues pertaining to the system. However, it is not clear if the study of black holes would contribute directly.

This paper contains theoretical background and results related to the black hole solutions. After describing our ansatz for the metric in section 2, we turn to an analysis of the asymptotics and charges at infinity. We start in section 3 (inspired by [30, 31]) with what we term “a method of equivalent sources” where one pretends that gravity is weak throughout and the source is large and smeared. We identify the numerically measured asymptotics both in harmonic and in conformal gauge and find their relation with the physical charges, the mass and the tension. We promote a physical picture of the tension being a tension of an imaginary string, which acts to decrease the size of the extra dimension counteracting the effect of the mass which “wants to make space” for itself. We also stress the importance of the tension as a natural coordinate for the system’s phase diagram instead of the previously used \( \lambda = \frac{1}{2}[(R_{\text{max}}/R_{\text{min}}) - 1] \) one that has the advantage of putting both the black hole and the black string on the same phase diagram, being always finite. This section ends by making some predictions for small black holes.

In section 4 we use a rigorous technique based on the (Gibbons-Hawking) free energy to revisit some of the issues of the previous section and to derive the first law. Together one arrives at a form of the first law, \( dm = T dS + \tau d\hat{L} \) which is completely analogous to the familiar \( dE = T dS - P dV \) (for gas) (see also [12]). This (differential) form of the first law is then readily combined with scale invariance to yield the integrated form of the first law (also known as “Smarr’s formula”). The latter is an important test for numerics since it ties quantities at the horizon with those at infinity and relies on the satisfaction of the equations of motion everywhere (since the derivation uses integration by parts) and so it gives a measure for the “overall numerical error”.

Next, in section 5 we use the fact that in the asymptotic region the \( z \) dependence is lost to perform a dimensional reduction. The mass defined in section 3 is seen to agree with the standard definition of mass in the lower dimension. The asymptotic constant for the decay of the size of the extra dimension is interpreted in the lower dimension as a scalar charge. The scalar charge, like the tension is a natural parameter for the system’s phase diagram and it is especially useful for the Gregory-Laflamme transition since it vanishes for the string and is non-zero for all other known solutions. It shares another interesting property with the tension: the tension, much like the mass, is always positive [33, 34] and it seems that so is the scalar charge – it would be interesting to prove/refute this.

In section 6 we turn to the horizon, identify the measurable quantities, and give predictions for small black holes. We conclude by summarizing our main results in section 7. The appendices include details of a coordinate transformation, yet another confirmation of the expression for the mass (this time using the Hawking-Horowitz expression), the details of action evaluation in the harmonic gauge and a different proof of Smarr’s formula.

Very recently the paper [35] appeared which overlaps with some of these theoretical considerations. Our 5d numerical results will follow in a sequel paper [7]. Since the appearance of the first version of this paper, another interesting paper appeared [36].
2. Ansatz for metric

The system under study, a static (no angular momentum) black object in \( \mathbb{R}^{d-2,1} \times S^1 \), is characterized by three dimensionful constants: \( \hat{L} \) the size of the extra dimension such that \( z \) and \( z + \hat{L} \) are identified, \( M \) the mass of the system measured at infinity of \( \mathbb{R}^{d-2} \), and \( G_N \) the Newton constant in \( d \) dimensions. From these a single dimensionless parameter can be constructed

\[
\mu = \frac{(G_N M)}{\hat{L}^{d-3}},
\]  

while the \((d-1)\) dimensional effective Newton constant is given by \( G_{d-1} = G_N / \hat{L} \).

The isometries of these solutions are \( O(d-2) \times U(1) \times (\mathbb{Z}_2)^2 \), where the \( O(d-2) \) comes from the spherical symmetry in \( \mathbb{R}^{d-2} \) which we assume and the \( U(1) \) comes from time independence. The discrete factors represents reflections with respect to time and the \( z \) coordinate \( t \rightarrow -t, z \rightarrow -z \). The most general metric with these isometries is

\[
d s^2 = -e^{2 \hat{A}} \, d t^2 + d s^2_{(r,z)} + e^{2 \hat{C}} \, d \Omega^2,
\]  

which is a general metric in the \((r,z)\) plane together with two functions on the plane \( \hat{A} = \hat{A}(r,z), \hat{C} = \hat{C}(r,z) \). The horizon is a line determined by \( e^{2 \hat{A}} = 0 \) and \( d \Omega^2 \) is the line element for \( S^{d-3} \).

There are several alternatives to fix the gauge and choose coordinates in the \((r,z)\) plane, though for most of the current theoretical analysis this does not matter. For the numerical convenience we choose the ansatz

\[
d s^2 = -A^2 \, d t^2 + e^{2 B} (d r^2 + d z^2) + r^2 e^{2 C} \, d \Omega^2
\]

where the conformal gauge is chosen for the 2d metric and the new variables \( A, C \) are defined in terms of the variables in eq. (2.2)

\[
A := e^{\hat{A}} \\
C := \hat{C} - \log(r)
\]

so that \( A \) is regular at the horizon \( (A)_{\text{hor}} = 0 \) and such that \( C \rightarrow 0 \) at infinity.

This ansatz still contains a gauge freedom for conformal transformations which we finally fix by setting the domain of definition for the three functions to be \( \{(r, z) : |z| \leq L, \, r^2 + z^2 \geq \rho_h^2 \} \), where we define \( L := \hat{L}/2 \) for numerical convenience. It was shown in [17] that transforming to this domain is always possible by writing down elliptic equations for the coordinate transformation. Thus each domain is characterized by a dimensionless constant (the conformal invariant)

\[
x := 2 \rho_h / \hat{L}.
\]

The normalization was chosen so that \( x = 1 \) is the maximum possible value. Since the problem has only one dimensionless parameter all quantities will be a function of \( x \) (and in this sense it replaces the dimensionless mass \( \mu \)).

We still need to state the boundary conditions. There are four boundaries

\[^3\text{In the Lorentzian solutions it is the non-compact version of } U(1).\]
\( r \to \infty \) infinity.

The metric is asymptotically flat

\[
g_{\mu\nu} \sim \eta_{\mu\nu} := \text{diag}(-1,1,\ldots,1) \Leftrightarrow A = 1, \quad B = C = 0
\] (2.6)

- Horizon – a regular horizon.
- \( r = 0 \) axis – regularity on the axis.
- \(|z| = \hat{L}/2\) – periodicity (and reflection symmetry at \( z = 0 \)).

By comparing with the Schwarzschild metric one can find the relation \( \rho = \rho(\rho_S) \) where \( \rho_S \) is the Schwarzschild coordinate (see appendix A)

\[
\frac{\rho}{\rho_h} = \left[ \left( \frac{\rho_S}{\rho_0} \right)^{\frac{d-3}{2}} + \sqrt{\left( \frac{\rho_S}{\rho_0} \right)^{d-3} - 1} \right]^{\frac{2}{d-3}}
\] (2.7)

where \( \rho_S = \rho_0 \) is the location of the horizon in Schwarzschild coordinates. Asymptotic flatness chooses a natural radial coordinate at infinity, hence we require that asymptotically \( \rho \sim \rho_S \), and we find the relation between \( \rho_h \) and \( \rho_0 \)

\[
\rho_h = \rho_0 / 2^{\frac{2}{d-3}}.
\] (2.8)

3. Charges and the method of equivalent sources

We start by analyzing the system at \( r \to \infty \). In this limit all the metric functions become independent of \( z \) (since the mass of the \( z \)-dependent modes is proportional to \( 1/\hat{L} \) all \( z \) dependence drops like \( \exp(-r/\hat{L}) \) as one gets away from the black hole). Moreover, at infinity we are in the weak field limit, namely the Newtonian limit.

It is possible to derive the properties of the asymptotics by pretending that the black hole source is smeared over a much larger area such that the gravitational field is everywhere weak, while the energy-momentum tensor is non-vanishing (recall that there are no sources in the sought-for black hole solutions). This method is inspired by the analysis of \([30, 31]\), and should be valid since far away from the source it should not matter whether the source is a black hole or a low density star. We term this approach “a method of equivalent sources”. After we derive the expression for the charges in terms of the constants of asymptotics within this method, we return in the next section and prove it rigorously using the free energy and the first law.

One defines

\[
g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}
\]

\[
\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} h^{\alpha}_{\alpha} \eta_{\mu\nu}
\] (3.1)
where $\eta_{\mu\nu}$ is the flat space metric, which is used for raising and lowering indices, and $h \ll 1$ is the perturbation. Choosing the harmonic (or “Lorentz”) gauge
\[ \partial^{\mu} \bar{h}_{\mu\nu} = 0, \tag{3.2} \]
the linearized field equations become
\[ \triangle \bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu}(x) \tag{3.3} \]
where $T_{\mu\nu}(x)$ is the stress energy tensor. The solution may be written in a multipole expansion:\footnote{where the constant of proportionality is read as usual from $\triangle(1/r^{d-4}) = -\Omega_{d-3} (d-4) \delta(\vec{x})$.} The leading term at infinity comes from the monopoles (dipoles, including angular momentum effects vanish here due to the reflection symmetries, so the next correction will come from quadruples), and in our geometry\footnote{In order to get the solution in uncompactified spacetime replace $(d-4)\Omega_{d-3} \hat{L} r^{d-4} \rightarrow (d-3)\Omega_{d-2} r^{d-3}$ in the following formulae.} it is
\[ \bar{h}_{\mu\nu} = 2 k_d \frac{G_N}{(d-4) \hat{L} r^{d-4}} T_{\mu\nu}^Q \tag{3.4} \]
where
\[ T_{\mu\nu}^Q := \int d^{d-1}x T_{\mu\nu} \tag{3.5} \]
are “total mass-energy charges” at $t = \text{const}$, and
\[ k_d := \frac{8 \pi}{\Omega_{d-3}} \tag{3.6} \]
\[ \Omega_{d-1} := d \frac{\pi^{d/2}}{\Gamma(d/2 + 1)} = d \frac{\pi^{d/2}}{(d/2)!}. \tag{3.7} \]
$\Omega_{d-1}$ is the area of the unit $S^{d-1}$ sphere, and for odd $d$ we define $(d/2)! := \sqrt{\pi} \frac{3 \cdot 5 \cdot \ldots \cdot d}{2 \cdot 4 \cdot \ldots \cdot (d-1)}$.

From now till the end of this section we will use “asymptotics adapted units” such that $G_{d-1} = G_N / \hat{L} = 1$.

Finally one inverts the transformation (3.1) to retrieve $h$
\[ h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{d-2} \bar{h}^\alpha_{\alpha\mu\nu} = \frac{2 k_d}{(d-4) r^{d-4}} \left( T_{\mu\nu}^Q - \frac{T^Q}{d-2} \eta_{\mu\nu} \right), \tag{3.8} \]
where $T^Q := T_{\alpha\beta}^Q \eta^{\alpha\beta}$.

### 3.1 Mass and tension

The symmetries of the problem restrict the form of the energy-momentum charges $T_{\mu\nu}^Q$: time reflection implies $T_{00}^Q = T_{20}^Q = 0$. Similarly, $z$ axis reflection implies $T_{2i}^Q = 0$. Spherical symmetry implies that $T_{ij}^Q = T_{rr}^Q \delta_{ij}$. Here $i, j$ are any index other than $0 \equiv t, z$. 

It is important that not all three energy-momentum charges $T^Q_{00}$, $T^Q_{zz}$ $T^Q_{rr}$ are independent, as a result of GR’s constraint equations. The relation we are about to derive is a consequence of the constraint $G_{rr} = 0$ (where $G$ is the Einstein tensor), but here we show how to get the same result from conservation of energy-momentum

$$\partial^\mu T_{\mu\nu} = 0$$

(3.9)

where we used $D_\mu = \partial_\mu + O(h)$. In particular

$$\partial^r T_{rr} = 0 \Rightarrow T_{rr}(r) = \text{const}$$

(3.10)

but

$$T_{rr}(r \gg \hat{L}) = 0 \Rightarrow T_{rr}(r) = 0$$

(3.11)

and hence

$$T^Q_{rr} = 0.$$ 

(3.12)

We denote

$$m := T^Q_{00},$$

(3.13)

the mass of the black hole. $T_{zz}$ is pressure and since the conservation law tells us it is constant along $z$: $\partial_z T_{zz} = 0$ it makes sense to define

$$\tau := -\frac{T^Q_{zz}}{\hat{L}}$$

(3.14)

to be the tension along the periodic direction. This is the tension measured by an asymptotic observer, which interprets it as the tension of an imaginary string stretched along the compact circle. The choice of sign reflects the fact that the tension is always positive \[33, 34\]. Altogether we write the energy-momentum charges of the black hole as

$$T^Q_{\mu\nu} = \text{diag}(m, -\tau \hat{L}, 0, \ldots, 0)$$

(3.15)

### 3.2 From harmonic to conformal coordinates

The most general ansatz for the asymptotic region in harmonic coordinates \[3.2\] is

$$ds^2 = -e^{-2A_H} dt^2 + e^{2B_H} dz^2 + e^{2C_H} (dr_H^2 + r_H^2 d\Omega_{d-3}^2)$$

(3.16)

where $A_H, B_H, C_H$ depend only on the harmonic radial coordinate $r_H$, and the sign for $A_H$ was defined for later convenience.

Define the constants $a_H, b_H, c_H$ (with dimension length$^{d-4}$) from the asymptotic form of the functions

$$A_H = \frac{a_H}{r^{d-4}} + O\left(\frac{1}{r^{d-3}}\right)$$

$$B_H = \frac{b_H}{r^{d-4}} + O\left(\frac{1}{r^{d-3}}\right)$$

$$C_H = \frac{c_H}{r^{d-4}} + O\left(\frac{1}{r^{d-3}}\right).$$

(3.17)
The harmonic gauge condition (3.2) sets some relations between these constants. Actually, only $\partial_r \tilde{h}_{rr}$ is not identically zero and gives

$$- a_H + b_H + (d - 4) c_H = 0.$$  \hspace{1cm} (3.18)

Eqs. (3.8, 3.15) tell us that

$$\begin{bmatrix} a_H \\ b_H \\ c_H \end{bmatrix} = \frac{k_d}{(d - 4)(d - 2)} \begin{bmatrix} d - 3 & -1 & m \\ 1 & -(d - 3) & \tau \hat{L} \end{bmatrix} \begin{bmatrix} d - 3 \\ -1 \\ 1 \end{bmatrix} (3.19)$$

Note that owing to this relation the constraint (3.18) is satisfied automatically, which can be traced to the conservation of energy-momentum (3.10).

Comparing with the conformal ansatz (2.3), where this time the functions $A, B, C$ depend only on $r$, one gets

$$A = e^{-A_H}$$  \hspace{1cm} (3.20)

$$B = B_H$$  \hspace{1cm} (3.21)

$$e^B dr = e^{C_H} dr_H$$  \hspace{1cm} (3.22)

$$C = C_H - \log(r/r_H)$$  \hspace{1cm} (3.23)

The equation for $r$ from (3.22, 3.21)

$$dr = e^{C_H - B_H} dr_H =$$

$$= (1 + \frac{c_H - b_H}{r_H^{d-4}} + O(\frac{1}{r_H^{d-5}})) dr_H$$  \hspace{1cm} (3.24)

can be integrated to give

$$r = \begin{cases} r_H + k_5 \tau \hat{L} \log(\frac{r_H}{r_H^0}) + O(\frac{1}{r_H}) & \text{for } d = 5; \\ r_H + r_0 - \frac{k_d \tau \hat{L}}{(d-4)(d-5)r_H^{d-5}} + O(\frac{1}{r_H^{d-4}}) & \text{for } d > 5, \end{cases}$$  \hspace{1cm} (3.25)

where $r_0, r_H^0$ are constants of integration which are needed in the numerics in order to keep the location of the horizon fixed. Note that the case $d = 5$ is somewhat special and needs to be treated separately.

We are now ready to find the constants in the conformal gauge asymptotics. For $A, B$ the work is done already due to (3.19, 3.21). We define $a, b$ by

$$1 - A = \frac{a}{r^{d-4}} + o(\frac{1}{r^{d-4}})$$

$$B = \frac{b}{r^{d-4}} + o(\frac{1}{r^{d-4}})$$  \hspace{1cm} (3.26)

and we get

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{k_d}{(d - 4)(d - 2)} \begin{bmatrix} d - 3 & -1 & m \\ 1 & -(d - 3) & \tau \hat{L} \end{bmatrix} (3.27)$$
which can be inverted to give

\[
\begin{bmatrix}
m \\
\tau \hat{L}
\end{bmatrix} = \frac{1}{k_d} \begin{bmatrix}
d - 3 & -1 \\
1 & -(d - 3)
\end{bmatrix} \begin{bmatrix}
a \\
b
\end{bmatrix}
\]

(reminder: \(k_d\) is defined in (3.6)).

The last equation is one of our main results – it tells us the physical charges in terms of the numerical asymptotics.

We now turn to the function \(C(r)\) which we treat separately for \(d = 5\) and for \(d > 5\). For \(d = 5\) an expansion of (3.23) using the change of variables (3.25) gives

\[
C = c_H + k_5 \tau \hat{L} \log(r_{H0}) - \frac{k_5 \tau \hat{L}}{r} \log(r) + O\left(\frac{\log^2 r}{r^2}\right).
\]

The leading term at infinity is \(\log(r)/r\) and so we may define \(c_5\) by

\[
C = c_5 \frac{\log(r)}{r} + \ldots,
\]

and using (3.29, 3.28) we get

\[
c_5 = b_{H5} - c_{H5} = 2b - a.
\]

This constraint is the 5d manifestation of (3.18). Note that the next term \(O(1/r)\) is hard to distinguish from the leading term and it contains an arbitrary constant \(r_{H0}\).

For \(d > 5\) (3.23) yields

\[
C = \frac{c_H}{r^{d-4}} - \log\left(1 + \frac{r_0}{r_H}\right) - \frac{k_d \tau \hat{L}}{r^{d-4}} + O\left(\frac{1}{r^{d-4}}\right)
\]

which together with the inversion of (3.25)

\[
r_H = r - r_0 + \frac{k_d \tau \hat{L}}{(d-4)(d-5)r^{d-5}} + O\left(\frac{1}{r^{d-4}}\right)
\]

tells us that the leading term in \(C(r)\) is \(-\frac{r_0}{r}\) while the charge \(\tau\) enters only at order \(O(1/r)\) and is mixed with \(r_0\).

3.3 Small black holes

Myers [31] argues that for small black holes \((x \ll 1)\) one should take zero tension, \(\tau = 0\), namely the equivalent source is dust. In this case we use (3.27) to compute the asymptotics

\[
\begin{bmatrix}
a \\
b
\end{bmatrix} \simeq \frac{k_d}{(d-2)(d-3)} m \begin{bmatrix}
d - 3 \\
1
\end{bmatrix},
\]

where the \(\simeq\) sign denotes that the ratio of the two sides approaches 1 as \(x \to 0\).

Moreover, in this limit the mass is determined by the horizon size

\[
\frac{2k_{d+1}}{(d-2)L^{d-3}} G_N m \simeq \left(\frac{\rho_0}{L}\right)^{d-3} = 4 \left(\frac{x}{L}\right)^{d-3}.
\]

This can be seen by using the method of equivalent sources to compute the mass of a Schwarzschild black hole [30] and then noting that after compactification the mass does not change in the weak gravity region since the Newtonian potential satisfies Gauss’ law.
4. Charges, free energy and the first law

4.1 Charges from free energy

Let us derive the form of the first law (conservation of energy) using rigorous free energy techniques. We will see that the results agree with those of the previous section.

The (Gibbons-Hawking [37]) free energy action integral is defined for the Euclidean continuation of static metrics and is given by

\[ I = -\beta F = \frac{1}{16\pi G_N} \int dV_d R_d + \frac{1}{8\pi G_N} \int_{\partial} dV_{d-1} [K - K^0] \] (4.1)

where \( \beta = 1/T \) is the inverse temperature (the period of the Euclidean time), \( F \) is the free energy, \( R_d \) is the \( d \) dimensional Ricci scalar, \( dV_d, dV_{d-1} \) are the \( d, (d-1) \) invariant volume elements, the second term is integrated over the boundary of the manifold \( \partial \), \( K \) is the trace of the second fundamental form, while \( K^0 \) is the same quantity for a reference geometry with the same boundary.

Consider the action to be a function of the boundary conditions

\[ I = I(\beta, \hat{L}) \] (4.2)

evaluated after extremization over metrics. In order to get the first law we need an explicit expression for the action, find the total differential

\[ dI = \left( \frac{\partial I}{\partial \beta} \right) d\beta + \left( \frac{\partial I}{\partial \hat{L}} \right) d\hat{L} \]

and then use thermodynamic identities to transform it into the standard form.

The action in terms of the harmonic fields (3.16) reads (see appendix C for the derivation)

\[ I = \frac{\beta a}{2k_4 G_N} \int dr_H r_H^{d-3} e^{-A_H + B_H + (d-4)C_H} K_{ij} X_i^t X^t_j \] (4.3)

where we use a short-hand notation \( X_i = [A_H, B_H, C_H] \) and the matrix \( K_{ij} \) is

\[ K = \begin{bmatrix}
0 & -1 & -(d-3) \\
-1 & 0 & (d-3) \\
-(d-3) & (d-3) & (d-3)(d-4)
\end{bmatrix} \] (4.4)

Moreover, there is no boundary term.

Having the expression for the action we wish to compute its differential. By a general theorem in mechanics \(^6\) it is given by the conjugate momentum at the boundary

\[ \partial_{X_i} I = \frac{\partial L}{\partial \dot{X}_i} \bigg|_{r_H=R} \] (4.5)

where \( L \) is the Lagrangian density.

Using the constants of asymptotics (3.17) we may compute

\[ \begin{bmatrix}
\partial_{A_H} \\
\partial_{B_H} \\
\partial_{C_H}
\end{bmatrix} I = -\frac{\beta a}{k_4 G_N} (d-4) K \begin{bmatrix}
a_H \\
b_H \\
c_H
\end{bmatrix} \] (4.6)

\(^6\)See for example [38]: \( \delta I = \left[ \frac{\delta L}{\delta q} \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial q} \right) \delta q \, dt \) and since the second term vanishes as we are evaluating on solutions of the equations of motion, only the first term contributes.
where (1.5) was evaluated at the $R \to \infty$ limit.

$\partial C_H$ is precisely proportional to the constraint (3.18) (derived from $G_{rr} = 0$) and thus vanishes identically. This is to be expected since a shift in $C$ is equivalent to a shift of the boundary in $r_H$ without changing the asymptotic sizes $\beta, \hat{L}$, and since we are evaluating $I$ in the $R \to \infty$ limit, the action should not change. Now we may use the constraint (3.18) to eliminate $c_H$ from the expressions (4.6) and we may drop the $H$ subscript from $a,b$ according to (3.27) and get

$$\begin{bmatrix} \partial A_H \\ \partial B_H \end{bmatrix} I = \frac{\beta_0 \hat{L}}{k_d G_N} \begin{bmatrix} d - 3 & -1 \\ -1 & d - 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

(4.7)

Finally the relations $d\beta = -\beta_0 dA_H, d\hat{L} = \hat{L}_0 dB_H$ valid asymptotically where $A_H = B_H = 0$ yield the differential of the action

$$\frac{k_d G_N}{\hat{L}} dI = -((d - 3) a - b) d\beta - \frac{\beta}{\hat{L}} (a - (d - 3) b) d\hat{L}$$

(4.8)

Now let us compare with standard thermodynamics. In analogy with gas thermodynamics where one has

$$dE = T dS - P dV$$

(4.9)

the first law must take the following form

$$dm = T dS + \tau d\hat{L}$$

(4.10)

This can be considered the proper definition of the mass $m$ and the tension $\tau$. We will now see that this definition coincides with the definition in terms of asymptotics (3.28) and hence all is consistent. The free energy is related to $m$ by $F = m - TS$ and the action is $I = -\beta F$. Putting the last two relations together one finds

$$dI = -m d\beta - \beta \tau d\hat{L}$$

(4.11)

Comparing with (4.8) we re-obtain the expressions for the mass and tension (3.28) as anticipated above. Alternatively, if one takes (3.28) to be the definition of mass and tension, then (4.8) proves the form of the first law (4.10) and suggest the analogy with gas thermodynamics.

### 4.2 Smarr’s formula

In this section we describe Smarr’s formula, also known as the integrated form of the first law, for the geometry under study. It is a relation between thermodynamic quantities at the horizon with those at infinity, relying on the generalized Stokes formula and the validity of the equations of motion in the interior, and as such it estimates the “overall numerical error” in our numerical implementation.

In general Smarr’s law can be gotten from the first law (4.10) after taking account of the scaling dimensions. Performing a scaling transformation $\hat{L} \to (1 + \epsilon)\hat{L}$, recalling the
relations \( T = \kappa/(2\pi) \), \( S = A/(4G_N) \), and that the area has dimension \( d - 2 \) while the mass has \( d - 3 \), we get by we expanding (4.10) to first order in \( \epsilon \)

\[
(d - 3) m = (d - 2) \frac{\kappa A}{8\pi G_N} + \tau \hat{L}.
\]

(4.12)

In order to put (4.12) into a “numerically adapted form”, that is one that uses only the numerical quantities we use (3.28) to get

\[
\frac{1}{8\pi G_N} A \kappa = \frac{(d - 4) \hat{L}}{k_d G_N} a,
\]

(4.13)

where on LHS we have horizon quantities and on the RHS asymptotic ones. Finally the LHS can be written also as \( TS \).

5. Dimensional reduction and the scalar charge

At infinity, since all \( z \) dependence is lost, it is natural to look at the system from a \((d - 1)\) point of view. We will see that this point of view offers a natural interpretation for the mass formula (3.28) – the mass can be read from the metric in the usual way. In addition, this perspective suggests the importance of the scalar charge to be discussed next.

The \( d \) dimensional metric can be written as

\[
ds^2 = \tilde{g}_{ab}^{(d-1)} dx^a dx^b + e^{2\phi} dz^2
\]

(5.1)

where \( a, b \) run over the \( d - 1 \) dimensions. Thus after reduction we get a metric and a scalar field \( \phi \) (for a general dimensional reduction one gets a vector as well, but it vanishes in our static ansatz).

Computing the action after reduction we have (C.1)

\[
R^{(d)} = \tilde{R}^{(d-1)} + \ldots
\]

\[
\sqrt{-g^{(d)}} = e^\phi \sqrt{-\tilde{g}^{(d-1)}}
\]

(5.2)

where the \( R \) is the Ricci scalar and \( \ldots \) could include other terms such as \((\partial\phi)^2, \Delta(\phi)\). One notices that the gravitational action behaves as \( e^\phi \sqrt{-\tilde{g}^{(d-1)}} \tilde{R}^{(d-1)} \) and is not canonically normalized. In order to mend that a Weyl rescaling is required

\[
\tilde{g}^{(d-1)} = e^{2w} g^{(d-1)}.
\]

(5.3)

The changes are (C.3)

\[
\tilde{R}^{(d-1)} = e^{-2w} \tilde{R}^{(d-1)} + \ldots
\]

\[
\sqrt{-\tilde{g}^{(d-1)}} = e^{(d-1)w} \sqrt{-g^{(d-1)}}
\]

(5.4)

where the \( \ldots \) denote this time terms with \((\partial w)^2 \) and \( \Delta(w) \). We see that in order to get the Einstein action we should choose

\[
w = -\frac{\phi}{d - 3}.
\]

(5.5)
Let us look at \( h^{(d-1)}_{00} \)
\[
h^{(d-1)}_{00} = e^{-2w} g^{(d-1)}_{00} + 1 \simeq \\
\simeq (1 + \frac{2b}{(d-3) \, r^{d-4}}) (-1 + \frac{2a}{r^{d-4}}) + 1 \simeq \\
\simeq \frac{2}{r^{d-4}} (a - \frac{1}{d-3} \, b)
\] (5.6)

But this should equal \((\rho_0 / r)^{d-4}\) hence
\[
2 (a - \frac{1}{d-3} \, b) = (\rho_0)^{d-4} = \frac{2 \, k_d \, G_N}{(d-3) \, L} \, m
\] (5.7)

where the second equality comes from (3.35) and altogether we reproduced (3.28).

Let us reinterpret the asymptotic constant \( b \) in light of the dimensional reduction. Since by definition \( B_H = \phi \), \( b \) describes now the fall-off of the scalar field \( \phi \)
\[
\phi \simeq \frac{b}{r^{d-4}}
\] (5.8)

Such a quantity is usually called the scalar charge (of \( \phi \)). In [40] it was shown that the scalar charge appears in the first law as \((dm/d\phi)_{\infty} = G_{ij} \Sigma^j\), where \(\Sigma^j\) are scalar charges defined in analogy with \( b \) in (5.8), the indices \( i, j \) allow for several scalar fields with a non-trivial metric read from the kinetic term \( G_{ij}(\phi) \partial \phi^i \partial \phi^j \). Indeed after the Weyl rescaling \( b \) replaces \( \tau \) in the expression for \( dI \) (as in (4.11), while at the same time we must use the transformed \( \beta \).

We can now interpret the equation for \( b \) (3.27) as telling us that the mass increases \( b \) and tries to open up the extra dimension, while the tension counteracts. For the uniform string these two effects exactly cancel each other. Hence \( b \) is a useful order parameter to describe the departure from the uniform string. Moreover, it has the advantage of being finite also for the black hole and hence one can naturally put all the phases on the same phase diagram as advocated in [4]. We notice that \( b \) is always positive as far as we know. It would be interesting to understand/ refute this.

6. At the horizon

It turns out that all the measurable quantities which we define reside either at infinity or at the horizon. After discussing the former in the previous sections, we now turn to the latter. In the vicinity of the horizon it is convenient to transform the \((r, z)\) plane into polar \((\rho, \chi)\) coordinates through \( z = \rho \cos(\chi), \ r = \rho \sin(\chi) \).

6.1 Definition of measurables

There are two thermodynamic quantities defined at the horizon

- The area, \( A \).
In the conformal gauge (2.3) it is given by

\[
A = \int_0^\pi \rho_h e^B \, d\chi \cdot \Omega_{d-3} (\rho_h e^C \sin(\chi))^{d-3} = \Omega_{d-3} \rho_h^{d-2} \int_0^\pi e^{B+(d-3)C} \sin(\chi)^{d-3} \, d\chi
\] (6.1)

• Surface gravity, \( \kappa \).

Since the geometry is static we may use the definition of kappa by taking the Euclidean continuation of the metric and finding the period \( \beta \) of Euclidean time such that there is no conical singularity at the horizon. Then \( \kappa = 2\pi/\beta \). In conformal coordinates it is given by

\[
\kappa = e^{-B} \partial_\rho A
\] (6.2)

The same expression is gotten from the equivalent definition \( \kappa^2 = -(1/2)(\partial^\mu \xi^\nu)(\partial_\nu \xi_\mu) \) where \( \xi^\mu \) is the Killing field and the expression is evaluated on the horizon (see for example [41]).

In addition we monitor two quantities that measure the geometry of the horizon

• The eccentricity, \( \epsilon \).

The spherical symmetry of a small black hole, \( SO(d - 1) \), is broken to \( SO(d - 2) \), as it grows. In the \( \{r, z\} \) plane we define the eccentricity \(^7\)

\[
\epsilon := \frac{A_{\perp}}{A_{\parallel}} - 1
\]

\[
A_{\parallel} := \Omega_{d-3} e^{(d-3)C} \rho_h^{d-3}, \text{ at } z = 0,
\]

\[
A_{\perp} := \int_0^\pi \rho_h e^B \, d\chi \cdot \Omega_{d-4} (e^C \rho_h \sin(\chi))^{d-4} = \Omega_{d-4} \rho_h^{d-3} \int_0^\pi e^{B+(d-4)C} \sin(\chi)^{d-4} \, d\chi
\] (6.3)

\( A_{\parallel}, A_{\perp} \) are sections of the horizon: \( A_{\parallel} \) is the area of the equatorial sphere at \( z = 0 \), while \( A_{\perp} \) is the area of the axial (or "polar") sphere at \( \chi = 0, \pi \).

• The polar distance, \( L_{\text{poles}} \).

This is the proper distance between the "north" and "south" poles along the \( r = 0 \) axis and it is given by

\[
L_{\text{poles}} := 2 \int_{\rho_h}^{L/2} dz \, e^B, \text{ at } r = 0.
\] (6.4)

\(^7\)This definition differs from the standard definition of an ellipse's eccentricity. It is analogous to \( a/b - 1 = (1 - e^2)^{-1/2} \approx 1 + 0.5 \epsilon^2 \), where \( \epsilon \) is the conventionally defined eccentricity.
6.2 Predictions for small black holes

Out of the two thermodynamic quantities $A, \kappa$ one can form a dimensionless quantity

$$A^{(\kappa)} := A \kappa^{d-2}$$  \hspace{1cm} (6.5)

For small black holes we may take the Schwarzschild metric as an approximation, where $A = \Omega_{d-2} \rho_0^{d-2}$, $\kappa = (d - 3)/(2 \rho_0)$, and hence

$$A^{(\kappa)} = A_0^{(\kappa)} := \Omega_{d-2} \left( \frac{d - 3}{2} \right)^{d-2}. \hspace{1cm} (6.6)$$

More generally we may wish to expand $A^{(\kappa)} = A^{(\kappa)}(x)$ in a Taylor series

$$A^{(\kappa)}(x) = \sum_{j=0}^{\infty} A_j^{(\kappa)} x^j \hspace{1cm} (6.7)$$

Note that $A_j^{(\kappa)}$ depends both on $j$ and on the dimension $d$ which is suppressed in this notation.

It can be shown \[42\] that the next non-vanishing term is $A_{d-3}^{(\kappa)}$ and it is given by

$$A_{d-3}^{(\kappa)} = - A_0^{(\kappa)} \left( d - 2 \right) \zeta(d - 3) \left( \frac{\rho_0}{L} \right)^{d-3} \Rightarrow$$

$$\frac{A_{d-3}^{(\kappa)}}{A_0^{(\kappa)}} = - \frac{4 (d-2)}{2d-3} \zeta(d - 3), \hspace{1cm} (6.8)$$

where $\zeta$ is Riemann’s zeta function, and we used eq. (2.8).

For the eccentricity the prediction is \[42\]

$$\epsilon = c_\epsilon x^{d-1} \hspace{1cm} (6.9)$$

while the constant of proportionality is currently computed only in 5d

$$c_\epsilon (d=5) = \frac{8}{3} \zeta(4) = \frac{4 \pi^4}{15} \approx 2.89 \hspace{1cm} (6.10)$$

All 5d constants stated in this subsection are confirmed numerically \[4\].

7. Summary

We summarize our main results:

- Asymptotics.

There are two constants of asymptotics. In the harmonic ansatz \[3.16\] there are 3 of them defined by \[3.17\] and they satisfy the constraint \[3.18\]. In the conformal ansatz \[2.3\] the constants of asymptotics, $a, b$, are defined in \[3.26\], related to the harmonic constants in \[3.27\], and expressed in terms of the system’s mass and tension (around the $z$ coordinate) in \[3.28\]. We stress the physical interpretation of the tension.
5d is somewhat special and we can furthermore define the asymptotic quantity $c_5$ in (3.30), and express it in terms of $a, b$ in (3.31).

For small black holes we have a prediction for all quantities in terms of $x$ in (3.34, 3.35).

- **Thermodynamics.**
  
  We derive the (differential form of the) first law in (4.10) and the integrated form in (4.12).

- **Horizon region.**
  
  We make predictions for small black holes for two dimensionless quantities: $A \kappa^{d-2}$ (6.8) and the horizon eccentricity (6.10).

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**A. Schwarzschild in conformal coordinates**

Let us transform the Schwarzschild metric into conformal coordinates and find the relation between the radii $\rho_0$ and $\rho_h$. The Schwarzschild solution is

$$ds^2 = -f \, dt^2 + f^{-1} \, d\rho_S^2 + \rho_S^2 \, d\Omega_{d-2}^2$$

$$f = 1 - \left( \frac{\rho_0}{\rho_S} \right)^{d-3}$$ (A.1)

One considers a new radial variable $\rho = \rho(\rho_S)$ such that

$$f^{-1} \, d\rho_S^2 = e^{2B} \, d\rho^2$$

$$\rho_S^2 = e^{2B} \, \rho^2$$ (A.2)

in terms of which the metric becomes

$$ds^2 = -f \, dt^2 + e^{2B} (d\rho^2 + \rho^2 \, d\Omega_{d-2}) =$$

$$= -f \, dt^2 + e^{2B} [d\rho^2 + \rho^2 (d\chi^2 + \sin^2(\chi) \, d\Omega_{d-3})]$$ (A.3)

From (A.2) one extracts the equation for $\rho$

$$\frac{d\rho}{\rho} = f^{-1/2} \frac{d\rho_S}{\rho_S}$$ (A.4)

The integral on the right can be solved by a change of variables $\cosh(\beta) = (\rho_S/\rho_0)^{(d-3)/2}$ and we get

$$\rho/\rho_h = \exp \left[ \frac{2}{d-3} \arccosh((\rho_S/\rho_0)^{(d-3)/2}) \right] =$$

$$= \left[ \frac{\rho_S}{\rho_0} \right]^{d-3} + \sqrt{\left( \frac{\rho_S}{\rho_0} \right)^{d-3} - 1} \right]^{\frac{d-3}{d-3}}.$$ (A.5)
The metric in the new conformal (or isotropic) coordinates takes the following simple form

\[ ds^2 = - \left( \frac{1 - \psi}{1 + \psi} \right)^2 dt^2 + (1 + \psi)^{4/(d-3)} \left[ d\rho^2 + \rho^2 d\Omega^2_{d-2} \right] \]

\[ \psi := \left( \frac{\rho_h}{\rho} \right)^{d-3}. \]  

(A.6)

B. Hawking-Horowitz mass

Let us compute the mass directly from the Hawking-Horowitz formula as an additional “independent” derivation.\(^8\) The Hawking-Horowitz formula \([43]\) for a static vacuum solution is

\[ m_{HH} = - \frac{1}{8 \pi G_N} \partial_\hat{n} [\Sigma^{(d-2)} - 0 \Sigma^{(d-2)}] \]  

(B.1)

where \(\Sigma^{(d-2)}\) is the \((d-2)\) area of the boundary of a \(t = \text{const}\) slice, \(\Sigma^{(d-2)}\) is the same quantity for a reference geometry, and \(\partial_\hat{n}\) is a derivative with respect to orthogonal motion of this boundary.

We evaluate (B.1) in the harmonic ansatz (3.16) for a boundary at \(r_H = R\) and take the \(R \to \infty\) limit. The first term is

\[ \partial_\hat{n} \Sigma^{(d-2)} = e^{-C_H} \partial_r \left[ (\hat{L} e^{B_H}) \Omega_{d-3} (r_H e^{C_H})^{d-3} \right] |_{r_H = R} = \]

\[ = \hat{L} \Omega_{d-3} e^{-C_H} \partial_r [r_H^{d-3} e^{B_H + (d-3) C_H}] |_{r_H = R} \approx \]

\[ \approx \hat{L} \Omega_{d-3} [(1 - \frac{C_H}{R^{d-4}}) (d-3) R^{d-4} + (b_H + (d-3) c_H)] = \]

\[ = \hat{L} \Omega_{d-3} [(d-3) R^{d-4} + b_H]. \]  

(B.2)

The reference \(t = \text{const}\) geometry is flat: \(ds^2 = dz^2 + dr^2 + r^2 d\Omega_{d-3}\) with period \(\hat{L}_R = \hat{L} e^{B_H(R)}\) and \(r = R e^{C_H(R)}\) to match with the boundary. The second term is

\[ \partial_\hat{n} \Sigma^{(d-2)} = \partial_r [\hat{L}_R \Omega_{d-3} r^{d-3}] |_{r = R e^{C_H(R)}} = \]

\[ = (\hat{L} e^{B_H(R)}) \Omega_{d-3} (d-3) R^{d-4} e^{(d-4) C_H(R)} = \]

\[ \approx \hat{L} \Omega_{d-3} (d-3) [R^{d-4} + (b_H + (d-4) c_H)] = \]

\[ = \hat{L} \Omega_{d-3} (d-3) [R^{d-4} + a_H]. \]  

(B.3)

where in the last equality we used the relation (3.18).

\(^8\)This mass definition coincides with the ADM mass when both are applicable.
Combining (B.2, B.3) according to (B.1) we get
\[ m_{HH} = \frac{\hat{L}}{k_d G_N} ((d - 3) a - b) = m \]  
which agrees with (3.28), and we omitted the superfluous \( H \) subscripts (3.27).

C. The action in the harmonic ansatz

Here we derive the form of the action integral (4.1) the action in terms of the harmonic fields (3.16). Actually the asymptotic region will suffice and there the fields depend only on \( r_H \). In order to evaluate Ricci’s scalar it is useful to have the formula for it in the presence of a general fibration

\[ ds^2 = ds^2_X + \sum_i e^{2F_i} ds^2_{Y_i} \Rightarrow \\ R = R_X + \sum_i \left[ e^{-2F_i} R_{Y_i} - 2 d_i \triangle(F_i) - d_i (\partial F_i)^2 \right] - \sum_{i,j} d_i d_j (\partial F_i \cdot \partial F_j) \]  
where the fibration fields depend only on the \( x \) coordinates, \( R_X, R_{Y_i} \) are the Ricci scalars of the spaces \( X, Y_i, F_i = F_i(x) \), and the Laplacian (\( \triangle \)) and grad-squared (\( \partial \cdot \partial \)) are evaluated in the \( X \) space.

Reducing over \( t, z \) one has

\[ R_d = R_{d-2} - 2 (\partial A_H)^2 - 2 (\partial B_H)^2 + 2 (\partial A_H \cdot \partial B_H) + 2 \triangle A_H - 2 \triangle B_H \]  
Note that this equation should be (and is) symmetric under the exchange \( B_H \leftrightarrow -A_H \).

Since the \( d - 2 \) metric is conformal to flat space we may use the expression for the Ricci scalar of a conformally transformed metric (see for example [38])

\[ \tilde{ds}^2 = e^{2w} ds^2 \Rightarrow \tilde{R} = e^{-2w} \left[ R - 2 (\tilde{d} - 1) \triangle w - (\tilde{d} - 2) (\partial w)^2 \right] \]  
where \( \tilde{d} \) is the dimension of the space and the Laplacian and grad-squared are evaluated in the non-tilded metric. In our case \( w = C_H, \tilde{d} = d - 2 \) and \( R = 0 \), hence

\[ R_{d-2} = -e^{-2C_H} [-2 (d - 3) \triangle C_H - (d - 3)(d - 4) (\partial C_H)^2] \]  
where the flat space Laplacian can be written as \( \triangle_{d-2} = \partial_{r_H}^2 + (d - 4)/r \partial_{r_H} \).

Putting (C.2, C.4) together one has

\[ d^d x \sqrt{g_d} R_d = -2 d r_H r_H^d - 3 \beta_0 \hat{L}_0 \Omega_{d-3} e^{-A_H + B_H + (d - 4) C_H} \times \]

\[ \times [(\partial_{r_H}^2 + \frac{d - 3}{r_H} \partial_{r_H}) [-A_H + B_H + (d - 3)C_H] + \]

\[ + (d - 4) C_H' (-A_H' + B_H') + \]

\[ + \frac{(d - 3)(d - 4)}{2} C_H'^2 + A_H'^2 + B_H'^2 - A_H' B_H'] \]  

- 19 –
where $\beta_0 \hat{L}_0$ are the periods of the $t$ and $z$ coordinates, the prime denotes a derivative with respect to $r_H$ and we used $\Delta = \partial_{r_H}^2 + (d-4)/r_H \partial_{r_H} + (d-4) C'_H \partial_{r_H}$.

Performing integration by parts we end with

$$I = \frac{\beta_0 \hat{L}_0}{2 k d G_N} \int dr_H r_H^{d-3} e^{-A_H+B_H+(d-4)C_H} K_{ij} X_i' X_j'$$

where we use a short-hand notation $X_i = [A_H, B_H, C_H]$ and the matrix $K_{ij}$ is

$$K = \begin{bmatrix} 0 & -1 & -(d-3) \\ -1 & 0 & (d-3) \\ -(d-3) & (d-3) & (d-3)(d-4) \end{bmatrix}$$

Moreover, the original boundary term in (4.1) is exactly such that it cancels against the boundary term generated by the integration by parts so that (C.6) is the full action.

**D. Another derivation of Smarr’s formula**

Here we give a more direct derivation of the integrated form of the first law (Smarr’s formula). The free energy is evaluated in terms of the asymptotics, and the derivation is completed by combined it with the analogous expression for the mass, and the general thermodynamic relation $F = E - TS$. The latter identity relies on integration by parts to relate the horizon with infinity, and thus saves us from performing this step explicitly in our derivation.

The free energy is defined in (4.1). Since we discuss vacuum black hole solutions we have $R_d = 0$, and so only the second term contributes. It can be rewritten as

$$I = -\beta F = \frac{1}{8 \pi G_N} \partial_h [\Sigma^{(d-1)} - 0 \Sigma^{(d-1)}]$$

where $\Sigma^{(d-1)}$ is the $(d-1)$ area of the boundary ($0 \Sigma^{(d-1)}$ is the same for the reference geometry), and $\partial_h$ is a derivative with respect to orthogonal motion of this boundary.

Let us compute the free energy in the harmonic ansatz (3.16) for a boundary at $r_H = R$ and take the $R \to \infty$ limit. The first term is

$$\partial_h \Sigma^{(d-1)} = e^{-C_H} \partial_{r_H} [(\beta e^{-A_H}) (\hat{L} e^{B_H}) \Omega_{d-3} (r_H e^{C_H})^{d-3}]_{r_H=R} =$$

$$= \beta \hat{L} \Omega_{d-3} e^{-C_H} \partial_{r_H} [r_H^{d-3} e^{-A_H+B_H+(d-3)C_H}]_{r_H=R} =$$

$$\simeq \beta \hat{L} \Omega_{d-3} [(1 - \frac{C_H}{R^{(d-4)}}) (d-3) R^{d-4} + (-a_H + b_H + (d-3) c_H)] =$$

$$= \beta \hat{L} \Omega_{d-3} [(d-3) R^{d-4} + (b_H - a_H)].$$

The reference geometry is flat: $ds^2 = -dt^2 + dz^2 + dr^2 + r^2 d\Omega_{d-3}$ with periods $\beta_R = \beta e^{-A_H(R)}$, $\hat{L}_R = \hat{L} e^{B_H(R)}$ and $r = R e^{C_H(R)}$ to match with the boundary. The second
The term is
\[
\partial_R \Sigma^{(d-1)} = \partial_r \left[ \beta_R \hat{\mathcal{L}}_R \Omega_{d-3} r^{d-3} \right] \bigg|_{r=R e^{C_H(R)}} =
\]
\[
= (\beta e^{-A_H(R)}) (\hat{L} e^{B_H(R)}) \Omega_{d-3} (d-3) R^{d-4} e^{(d-4) C_H(R)}
\]
\[
\simeq \beta \hat{L} \Omega_{d-3} (d-3) \left[ R^{d-4} + (-a_H + b_H + (d-4) c_H) \right] =
\]
\[
= \beta \hat{L} \Omega_{d-3} (d-3) R^{d-4}. \quad (D.3)
\]
where in the last equality we used the constraint (3.18).

Combining (D.2, D.3) according to (D.1) we get
\[
-\beta F = \frac{\beta \hat{L}}{k_d G_N} (b_H - a_H), \quad (D.4)
\]
amely
\[
F = \frac{\hat{L}}{k_d G_N} (a - b) \quad (D.5)
\]
where we omitted the superfluous \( H \) subscripts (3.27).

Now we may use the relation
\[
F = m - T S \quad (D.6)
\]
where \( T = \kappa/(2 \pi) \) is the temperature, and \( S = A/(4 G_N) \) is the entropy. This relation holds for black hole thermodynamics and is proven using integration by parts. Substituting \( m \) from (3.28) we get
\[
T S = \frac{(d-4) \hat{L}}{k_d G_N} a. \quad (D.7)
\]
which coincides with the “numerically adapted form” of Smarr’s formula (4.13).

References

[1] F. R. Tangherlini, Nuovo Cimento 27, 636 (1963).
[2] R. Gregory and R. Laflamme, “Black Strings And P-Branes Are Unstable,” Phys. Rev. Lett. 70, 2837 (1993) [arXiv:hep-th/9301052].
[3] R. Gregory and R. Laflamme, “The Instability of charged black strings and p-branes,” Nucl. Phys. B 428, 399 (1994) [arXiv:hep-th/9404071].
[4] B. Kol, “Topology change in general relativity and the black-hole black-string transition,” arXiv:hep-th/0206220.
[5] S. S. Gubser, “On non-uniform black branes,” arXiv:hep-th/0110193.
[6] T. Wiseman, “Static axisymmetric vacuum solutions and non-uniform black strings,” Class. Quant. Grav. 20, 1137 (2003) [arXiv:hep-th/0209051].
[7] E. Sorkin, B. Kol, and T. Piran, “Caged black holes: black holes in compactified spacetimes II – 5d numerical implementation,” to appear.
[8] E. Sorkin and T. Piran, “Initial data for black holes and black strings in 5d,” Phys. Rev. Lett. 90, 171301 (2003) [arXiv:hep-th/0211210].

[9] G. T. Horowitz and K. Maeda, “Fate of the black string instability,” Phys. Rev. Lett. 87, 131301 (2001) [arXiv:hep-th/0105111].

[10] M. W. Choptuik, L. Lehner, I. Olabarrieta, R. Petryk, F. Pretorius and H. Villegas, “Towards the final fate of an unstable black string,” Phys. Rev. D 68, 044001 (2003) [arXiv:gr-qc/0304085].

[11] P. J. Smet, “Black holes on cylinders are not algebraically special,” arXiv:hep-th/0206106.

[12] T. Harmark and N. A. Obers, “Black holes on cylinders,” JHEP 0205, 032 (2002) [arXiv:hep-th/0204047].

[13] B. Kol, “Speculative generalization of black hole uniqueness to higher dimensions,” arXiv:hep-th/0208056.

[14] B. Kol, “Explosive black hole fission and fusion in large extra dimensions,” arXiv:hep-ph/0207037.

[15] T. Wiseman, “From black strings to black holes,” Class. Quant. Grav. 20, 1177 (2003) [arXiv:hep-th/0211028].

[16] B. Kol and T. Wiseman, “Evidence that highly non-uniform black strings have a conical waist,” Class. Quant. Grav. 20, 3493 (2003) [arXiv:hep-th/0304070].

[17] H. Kudoh, T. Tanaka and T. Nakamura, “Small localized black holes in braneworld: Formulation and numerical method,” Phys. Rev. D 68, 024035 (2003) [arXiv:gr-qc/0301089].

[18] R. Emparan and R. C. Myers, “Instability of ultra-spinning black holes,” JHEP 0309, 025 (2003) [arXiv:hep-th/0308056].

[19] L. Susskind, “Matrix theory black holes and the Gross Witten transition,” arXiv:hep-th/9805115.

[20] G. T. Horowitz and K. Maeda, “Inhomogeneous near-extremal black branes,” arXiv:hep-th/0201241.

[21] L. Lehner, “Numerical relativity: A review,” Class. Quant. Grav. 18, R25 (2001) [arXiv:gr-qc/0106072].

[22] H. S. Reall, “Classical and thermodynamic stability of black branes,” Phys. Rev. D 64, 044005 (2001) [arXiv:hep-th/0104071].

[23] V. E. Hubeny and M. Rangamani, “Unstable horizons,” JHEP 0205, 027 (2002) [arXiv:hep-th/0202189].

[24] R. Casadio and B. Harms, “Black hole evaporation and large extra dimensions,” Phys. Lett. B 487, 209 (2000) [arXiv:hep-th/0004004].

[25] A. V. Frolov and V. P. Frolov, “Black holes in a compactified spacetime,” Phys. Rev. D 67, 124025 (2003) [arXiv:hep-th/0302085].

[26] T. Tamaki, S. Kanno and J. Soda, “Radionic non-uniform black strings,” arXiv:hep-th/0307278.

[27] P. C. Argyres, S. Dimopoulos and J. March-Russell, “Black holes and sub-millimeter dimensions,” Phys. Lett. B 441, 96 (1998) [arXiv:hep-th/9808138].
[28] R. Emparan, G. T. Horowitz and R. C. Myers, “Black holes radiate mainly on the brane,” Phys. Rev. Lett. 85, 499 (2000) [arXiv:hep-th/0003118].

[29] S. B. Giddings and S. Thomas, “High energy colliders as black hole factories: The end of short distance physics,” Phys. Rev. D 65, 056010 (2002) [arXiv:hep-ph/0106219].

[30] R. C. Myers and M. J. Perry, “Black Holes In Higher Dimensional Space-Times,” Annals Phys. 172, 304 (1986).

[31] R. C. Myers, “Higher Dimensional Black Holes In Compactified Space-Times,” Phys. Rev. D 35, 455 (1987).

[32] P. K. Townsend and M. Zamaklar, “The first law of black brane mechanics,” Class. Quant. Grav. 18, 5269 (2001) [arXiv:hep-th/0107228].

[33] J. Traschen, “A positivity theorem for gravitational tension in brane spacetimes,” arXiv:hep-th/0308173.

[34] T. Shiromizu, D. Ida and S. Tomizawa, “Kinematical bound in asymptotically translationally invariant spacetimes,” arXiv:gr-qc/0309061.

[35] T. Harmark and N. A. Obers, “New Phase Diagram for Black Holes and Strings on Cylinders,” arXiv:hep-th/0309116.

[36] T. Harmark and N. A. Obers, “Phase Structure of Black Holes and Strings on Cylinders,” arXiv:hep-th/0309230.

[37] G. W. Gibbons and S. W. Hawking, “Action Integrals And Partition Functions In Quantum Gravity,” Phys. Rev. D 15, 2752 (1977).

[38] R. M. Wald, “General Relativity,” appendix D, The University of Chicago Press, 1984.

[39] L. D. Landau and E. M. Lifshitz, “Mechanics” par. 43, Pergamon Press, 1976.

[40] G. W. Gibbons, R. Kallosh and B. Kol, “Moduli, scalar charges, and the first law of black hole thermodynamics,” Phys. Rev. Lett. 77, 4992 (1996) [arXiv:hep-th/9607108].

[41] R. M. Wald, “General Relativity,” eq. (12.5.14), The University of Chicago Press, 1984.

[42] D. Gorbonos and B. Kol, to be published.

[43] S. W. Hawking and G. T. Horowitz, “The Gravitational Hamiltonian, action, entropy and surface terms,” Class. Quant. Grav. 13, 1487 (1996) [arXiv:gr-qc/9501014].