From forward integrals to Wick-Itô integrals: the fractional Brownian motion and the Rosenblatt process cases

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Abstract

In this paper, we combine Hida distribution theory and Sobolev-Watanabe-Kree spaces in order to study finely the link between forward integrals obtained by regularization and Wick-Itô integrals with respect to fractional Brownian motion and the Rosenblatt process. The new methodology developed in this paper allows to retrieve results for fractional Brownian motion and to obtain new results regarding the Rosenblatt process. In particular, an Itô formula for functionals of the Rosenblatt process is obtained which holds in the space of square-integrable random variables.

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Introduction

Context

Since the construction of the Itô integral, there have been many approaches to extend stochastic integration for integrand and integrator processes which are not covered by the classical theory. Forward integration is a natural generalization of Itô integration allowing for anticipating integrands and for more general integrator processes. There are basically two approaches in order to define a forward integral with respect to Brownian motion: by means of Wiener analysis (and/or white noise analysis) as done in [5, 12, 9, 3] and by regularization techniques first introduced in [15]. The Brownian forward integral allows for anticipating integrands, is well approximated by forward Riemann sums and is linear when the integrand is a constant random variable.

The regularization technique developed by F. Russo and P. Vallois in [15] is an almost pathwise method allowing to define forward, backward and symmetric integrals with respect to general integrators. This forward integral coincides essentially with the classical Itô integral when the integrator is a semi-martingale. Moreover, in the Brownian motion case, thanks to a Wiener analysis point of view, it is possible to link the forward integral defined by regularization techniques and the forward integral introduced thanks to Malliavin calculus tools (see Theorem 2.1 of [15]). They are the same when the limiting procedure of the regularization is strengthened (see Remark 2.2 of [15]).

Regarding fractional Brownian motion (fBm), the regularization techniques can be applied readily when $H > 1/2$. Indeed, in this case, fBm is a zero quadratic variation process which admits a modification whose sample path are almost surely $\eta$-Hölder continuous, for any $1/2 < \eta < H$. Therefore, the forward, backward and symmetric integrals with respect to fBm exist and essentially coincide (for regular enough integrands). In [1], the explicit link is made between the symmetric integral and the divergence operator associated with fractional Brownian motion for $H > 1/2$. In this

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Gaussian framework, a term involving the stochastic derivative of the integrand appears. Moreover, when \( H > 1/2 \), the Young type integral with respect to fractional Brownian motion can be defined and coincide with the forward integral with respect to fBm obtained by regularization (see Proposition 3 page 155 of [16]). In this framework \( (H > 1/2) \), there is therefore an unequivocal stochastic integral with respect to fBm defined by pathwise methods.

Moreover, fBm belongs to the family of Hermite processes. These processes appear in non-central limit theorems (see e.g. [6, 18, 19]). They are defined, for \( d \geq 1 \), by:

\[
\forall t > 0 \quad X^{H,d}_t = c(H_0) \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \left( \int_0^t \prod_{j=1}^d (s-x_j)^{H_0-1} ds \right) dB_{x_1} \ldots dB_{x_d},
\]

where \( \{B_x : x \in \mathbb{R}\} \) is a two-sided Brownian motion, \( c(H_0) \) is a normalizing constant such that \( \mathbb{E}[|X^{H,d}_1|^2] = 1 \) and \( H_0 = \frac{1}{2} + \frac{d}{2H} \) with \( H \in (\frac{1}{2}, 1) \). For \( d = 1 \), one recovers fBm denoted by \( \{B^{H}_t\} \) and for \( d = 2 \), the process is named the Rosenblatt process, denoted by \( \{X^{H}_t\} \) in the sequel. Hermite processes share many properties in common with fBm. Indeed, they are \( H \)-self-similar processes with stationary increments, have the same covariance structure and so their sample paths are almost-surely \( \eta \)-Hölder continuous, for every \( 1/2 < \eta < H \). In particular, regarding their stochastic calculus, the regularization techniques of F. Russo and P. Vallois apply readily (as well as Young integration) so that the forward, backward, symmetric and Young integrals are well defined and coincide for smooth enough integrands.

However, since their stochastic natures are very different (they live in different Wiener chaoses), one expects that the stochastic properties of the stochastic integrals with respect to Hermite processes of different orders would be quite different. This phenomenon does not seem reachable by purely pathwise integration methods and has been first observed partially in [20] for the Rosenblatt process. Indeed, in [20], based on another representation of the Rosenblatt process, the author studies the link between the forward integral obtained by regularization techniques and the divergence integral with respect to the Rosenblatt process. He notes the appearance of two trace terms which differs significantly from the Gaussian case (see Theorem 2 of Section 7 in [20]). However, existence of these two trace terms is not fully studied in [20] even in the case where the integrand process is a smooth functional of the Rosenblatt process (see Theorem 3 of Section 8 in [20]). Therefore, there is still room for improvements.

In this paper, we study the link between forward integration by regularization techniques and Wick-Itô integration with respect to fractional Brownian motion and with respect to the Rosenblatt process. We obtain explicit decompositions of the forward integrals in both cases when the integrand processes are smooth functionals of the integrator processes (see Theorems 1 and 3 below). In particular, we obtain existence and explicit simple formulae for the two trace terms appearing in the Rosenblatt process case. The methodology we develop is based on Hida distribution theory and on Sobolev-Watanabe-Kree spaces. We comment briefly on it.

- The first step in our procedure is to compute the \( S \)-transform of \( F(X^{H,d}_t)(X^{H,d}_{t+\epsilon} - X^{H,d}_t) \), for \( \epsilon > 0 \), and to identify each terms thanks to Hida distribution theory. The formulae hold true in \( (S)^* \), the Hida distributions space.

- Then, we prove that each term of the decomposition is a real random variable by using appropriate stochastic gradient operators (and their adjoints) naturally linked to the integrator process. The regularity of the integrand process plays a role in these representations.

- Finally, we prove convergence in \( (L^2) \), the space of square-integrable random variables, for each term appearing in the decompositions. This last step ensures that the forward integrals obtained coincide with the forward integral defined by regularization.
This methodology is applied to the fBm and the Rosenblatt process cases \((d = 1, 2)\). Nevertheless, it seems robust enough to possibly handle the following extensions:

- Any Hermite processes of any order \(d\), the difficulty being the increasing number of terms appearing in the decomposition of \(F(X^H_{t+\epsilon})(X^H_{t+\epsilon} - X^H_t)\) to analyse.

- Any self-similar processes with stationary increments with \(H \in (1/2, 1)\) represented by:

\[
\forall t > 0 \quad Y^H_{t,d} = c(H_0) \int \ldots \int_R \left( \int_0^t q_{H,d}(s - x_1, ..., s - x_d)ds \right) dB_{x_1}...dB_{x_d},
\]

where \(q\) is a symmetric function on \(\mathbb{R}^d\) verifying appropriate conditions (see e.g. [11]).

Due to the prominent roles of the fBm and of the Rosenblatt process, we only study these cases. The generalizations will be done in subsequent papers.

Main results and some notations

Before stating the main results of this paper we introduce some notations. We denote by \(I^d\) the multiple Wiener-Itô integral of order \(d\). Moreover, for any random variable \(X\) and any \(k \geq 1\), we denote by \(\kappa_k(X)\) the \(k\)-th cumulant of \(X\) when it exists. We denote by \((\nabla^{H-1/2})^*\) the adjoint of the stochastic gradient operator naturally associated with fractional noise (see Propositions 18 and 19). We denote by \((\nabla^{(2)})^*\) and by \((\nabla)^*\) the adjoints of the first and second order stochastic gradients associated with the white noise (see Proposition 12). We denote by \(F, F', F''_n, F^{(3)}_n, ...\) the functional and its derivatives. Finally, we define the forward integral by regularization as the following limit in probability (for \(F\) smooth enough):

\[
\lim_{\epsilon \to 0^+} \int_a^b F(X^H_{t,d}) \frac{(X^H_{(t+\epsilon)\wedge b} - X^H_t)}{\epsilon} dt \overset{P}{=} \int_a^b F(X^H_{t,d})d^-X^H_t
\]

**Theorem 1.** Let \((a, b) \subset \mathbb{R}_+\). Let \(F\) be a continuously differentiable function on \(\mathbb{R}\) such that:

\[
\forall x \in \mathbb{R}, \quad \max\{F(x), F'(x)\} \leq Ce^{\lambda x^2},
\]

for some \(C > 0\) and \(\lambda > 0\) with \(\lambda < 1/(4b^{2H})\). Then, we have, in \((L^2)\):

\[
\int_a^b F(B^H_t)dB^H_t = (\nabla^{H-1/2})^*(F(B^H)) + H \int_a^b t^{2H-1}F''_n(B^H_t)dt.
\]

**Remark 2.**

- This result should be compared with Proposition 3 of [1] where a similar result holds true. The authors use the representation of the fBm as a Wiener integral on a compact interval and the intrinsic Malliavin calculus with respect to it whereas we use its representation as a Wiener integral on \(\mathbb{R}\) and stochastic gradient operators on the white noise space.

- As a straightforward corollary, we obtain the following well-known Itô formula for \(F \in C^2(\mathbb{R})\) with appropriate growth conditions:

\[
F(B^H_b) - F(B^H_a) = (\nabla^{H-1/2})^*(F'(B^H)) + H \int_a^b t^{2H-1}F'''_n(B^H_t)dt.
\]
Theorem 3. Let \((a, b) \subset \mathbb{R}_+\). Let \(F\) be an infinitely differentiable function with polynomial growth at most (together with its derivative). Then, we have, in \((L^2)\):
\[
\int_a^b F(X_t^H) d^{-} X_t^H = d(H)(\nabla^{(2)})^* \left( \int_a^b F'(X_t^H) \frac{(t-\cdot)^{\frac{H}{2}-1}(t - \#)^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})^2} dt \right) + B(H) \nabla^* \left( \int_a^b F'(X_t^H)(t - \cdot)^{\frac{H}{2}-1} \right) \times I_1(t^H) dt + H \int_a^b t^{2H-1} F'(X_t^H) dt + \frac{H}{2} \kappa_3(X_1^H) \int_a^b t^{3H-1} F^{(2)}(X_t^H) dt + C(H) \int_a^b I_2(c_{t,t}^H)F^{(3)}(X_t^H) dt,
\]
with,
\[
l^H_t(x) = \int_0^t (u-x)^{\frac{H}{2}-1} |t-u|^{H-1} du,
\]
\[
c^H_{t,t}(x_1, x_2) = \int_0^t \int_0^t (u-x_1)^{\frac{H}{2}-1}(v-x_2)^{\frac{H}{2}-1} |u-t|^{H-1} |v-t|^{H-1} dudv,
\]
\[
B(H) = \frac{4d(H)}{\Gamma\left(\frac{H}{2}\right)} \sqrt{\frac{H(2H-1)}{2}},
\]
\[
C(H) = \frac{2d(H)}{\Gamma\left(\frac{H}{2}\right)} \sqrt{H(2H-1)}.
\]

Remark 4. • This result should be compared with Theorem 2 of Section 8 in [20] where a similar result is obtained under the assumption of existences of the trace terms. The author use the representation of the Rosenblatt process on a compact interval and the Malliavin calculus with respect to Brownian motion whereas we use its representation as a double Wiener integral on \(\mathbb{R}^2\) and stochastic gradient operators on the white noise space.

• As a straightforward corollary, we have the following new Itô formula:
\[
F(X_t^H) - F(X_a^H) = d(H)(\nabla^{(2)})^* \left( \int_a^b F'(X_t^H) \frac{(t-\cdot)^{\frac{H}{2}-1}(t - \#)^{\frac{H}{2}-1}}{\Gamma(\frac{H}{2})^2} dt \right) + B(H) \nabla^* \left( \int_a^b F'(X_t^H)(t - \cdot)^{\frac{H}{2}-1} \right) \times I_1(t^H) dt + H \int_a^b t^{2H-1} F'(X_t^H) dt + \frac{H}{2} \kappa_3(X_1^H) \int_a^b t^{3H-1} F^{(3)}(X_t^H) dt + C(H) \int_a^b I_2(c_{t,t}^H)F^{(3)}(X_t^H) dt.
\]

• The previous Itô formula should be compared with Theorem 3.16 of [2] where we obtain an Itô formula in the white noise sense for entire analytic functionals with growth conditions of the Rosenblatt process. The link between these two formulæ can be made by using iterated integration by parts on the white noise space thanks to the pointwise multiplications with \(I_1(t^H_{t,t})\) and \(I_2(c_{t,t}^H)\) for smooth enough functionals.

Organisation

This paper is organized as follows. In the first section, we introduce the relevant tools from Hida distribution theory and we define the Sobolev-Watanabe-Kreée spaces on the white noise space. In the second section, we define the fractional and the Rosenblatt noises, the stochastic integrals with respect to them and the associated stochastic gradient operators. In the third section, we start by analyzing the fractional brownian motion case and we end with the Rosenblatt process case. In particular, for the Rosenblatt process, we separate the studies of the \((L^2)\)-convergences for each term appearing in the decomposition of \(F(X_t^H)(X_t^H - X_{t+t}^H)\).
1 Hida distribution and Sobolev-Watanabe-Kree spaces.

In this section, we briefly remind the white noise analysis introduced by Hida and al. in [7]. For a good introduction to the theory of white noise, we refer the reader to the book of Kuo [8]. The underlying probability space \((\Omega, \mathcal{F}, P)\) is the space of tempered distributions endowed with the \(\sigma\)-field generated by the open sets with respect to the weak* topology in \(S'(\mathbb{R})\) and with the infinite dimensional Gaussian measure \(\mu\) whose existence is ensured by the Bochner-Minlos theorem.

For all \((\phi_1, \ldots, \phi_n) \in S(\mathbb{R})\), the space of \(C^\infty(\mathbb{R})\) functions with rapid decrease at infinity, the vector \(< \cdot; \phi_1 >, \ldots, < \cdot; \phi_n >\) is a centered Gaussian random vectors with covariance matrix \(< \phi_i; \phi_j >\)\((i,j)\). As it is written in Kuo [8], for any function \(f \in L^2(\mathbb{R})\), we can define \(< f >\) as the random variable in \(L^2(\Omega, \mathcal{F}, P)\) obtained by a classical approximation argument and the following isometry:

\[
\forall (\phi, \psi) \in S(\mathbb{R})^2 \quad \mathbb{E}[< \psi; < \cdot; \phi >] = < \psi; \phi >_{L^2(\mathbb{R})}
\]

Thus, for any \(t \in \mathbb{R}\), we define \((\mu\text{-almost everywhere}):\)

\[
B_t(.) = \begin{cases} < ; \mathbb{I}_{[0,t]} > & t \geq 0 \\ -< ; \mathbb{I}_{(t,\infty]} > & t < 0 \\ \end{cases}
\]

From the isometry property, it follows immediately that \(B_t\) is a Brownian motion on the white noise space and, by the Kolmogorov-Centsov theorem, it admits a continuous modification. Moreover, using the approximation of any function \(f \in L^2(\mathbb{R})\) by step functions, we obtain:

\[
<f> = \int_{\mathbb{R}} f(s)dB_s
\]

We note \(\mathcal{G}\), the sigma field generated by Brownian motion and \((L^2) = L^2(\Omega, \mathcal{G}, P)\). By the Wiener-Itô theorem, any functionals \(\Phi \in L^2\) can be expanded uniquely into a series of multiple Wiener-Itô integrals:

\[
\Phi = \sum_{n=0}^{\infty} I_n(\phi_n)
\]

where \(\phi_n \in L^2(\mathbb{R}^n)\), the space of square-summable symmetric functions. Using this theorem and the second quantization operator of the harmonic oscillator operator, \(A = -\frac{d^2}{dx^2} + x^2 + 1\), Hida and al. introduced the stochastic space of test functions \((S)\) and its dual, the space of generalized functions \((S)^*\) or Hida distributions. We refer the reader to pages 18-20 of [8] for an explicit construction. We have the following Gel’fand triple:

\[
(S) \subset (L^2) \subset (S)^*
\]

We denote by \(\langle \cdot; \cdot \rangle\) the duality bracket between elements of \((S)\) and \((S)^*\) which reduces to the classical inner product on \((L^2)\) for two elements in \((L^2)\).

In the context of white noise analysis, the main tool is the \(S\)-transform. It is a functional on \(S(\mathbb{R})\) which characterizes completely the elements in \((S)^*\) (as well as the strong convergence in \((S)^*\)).

**Definition 1.** Let \(\Phi \in (S)^*\). For every function \(\xi \in S(\mathbb{R})\), we define the \(S\)-transform of \(\Phi\) by:

\[
S(\Phi)(\xi) = \langle \Phi; \exp(< \cdot; \xi >) \rangle
\]

where \(\exp(< \cdot; \xi >) := \exp(< \cdot; \xi > - \frac{||\xi||_{L^2(\mathbb{R})}^2}{2}) = \sum_{n=0}^{\infty} \frac{I_n(\xi^\otimes n)}{n!} \in (S)\).

**Remark 5.** For every \(\Phi \in (L^2)\), we have:

\[
S(\Phi)(\xi) = \mathbb{E}[\Phi : \exp(< \cdot; \xi >)] = \mathbb{E}^{\mu_\xi}[\Phi]
\]

where \(\mu_\xi\) is the translated infinite dimensional measure defined by:

\[
\mu_\xi(dx) = \exp(< x; \xi > - \frac{||\xi||_{L^2(\mathbb{R})}^2}{2})\mu(dx).
\]
Regarding the S-transform, we have the following properties and results:

**Theorem 6.** 1. The S-transform is injective. If ∀ξ ∈ S(ℝ), S(Φ)(ξ) = S(Ψ)(ξ) then Φ = Ψ in (S)*.

2. Let Ψ ∈ (S)* such that Ψ = ∑∞n=0 In(ψn) with ψn ∈  S'(ℝn):
   \[ \forall ξ ∈ S(ℝ), S(Ψ)(ξ) = \sum_{n=0}^{∞} \langle ψ_n; ξ^n \rangle. \]

3. For Φ, Ψ ∈ (S)* there is a unique element Φ⊙Ψ ∈ (S)* such that for all ξ ∈ S(ℝ), S(Ψ)(ξ)S(Φ)(ξ) = S(Φ ⊙ Ψ)(ξ). It is called the Wick product of Φ and Ψ.

4. Let Φn ∈ (S)* and F = S(Φn). Then Φn converges strongly in (S)* if and only if the following conditions are satisfied:
   - \( \lim_{n→∞} F_n(ξ) \) exists for each \( ξ ∈ S(ℝ) \).
   - There exists strictly positive constants \( K, a \) and \( p \) independent of \( n \) such that:
   \[ \forall n ∈ \mathbb{N}, ξ ∈ S(ℝ) \quad |F_n(ξ)| ≤ K \exp(a||A^pξ||_{L^2(ℝ)}) \]

In the sequel, we introduce the differential calculus and Sobolev-Watanabe-Kree spaces on the white noise probability space. For further details, we refer the reader to chapter 9 of [8] and chapter 5 of [7]. First, we define the Gâteaux derivative of elements in (S) for direction in S'(ℝ).

**Theorem 7.** Let \( y ∈ S'(ℝ) \) and \( Φ ∈ (S) \). The operator \( D_y \) is continuous from (S) into itself and we have:
   \[ \forall ξ ∈ S(ℝ) \quad D_y(Φ)(ξ) = \sum_{n=1}^{∞} nI_{n-1}(y ⊗_1 φ_n)(ξ), \]
   where we denote by \( ⊗_1 \) the contraction of order 1 (see [7]).

*Proof.* See Theorem 9.1 of [8].

The next result states that every test random variable is actually infinitely often differentiable in Gâteaux and in Fréchet senses.

**Theorem 8.** Let \( Φ ∈ (S) \). \( Φ \) is infinitely often Gâteaux differentiable in every direction of S'(ℝ) and infinitely often differentiable in Fréchet sense. Moreover, for every \( k ∈ \mathbb{N}^* \) and for every \( y_1, ..., y_k ∈ (S'(ℝ))^k \), we have:
   \[ D_{y_1} ⊙ D_{y_2} ⊙ ... ⊙ D_{y_k}(Φ) = \langle y_1 ⊗ y_2 ⊗ ... ⊗ y_k; \nabla^{(k)}(Φ) \rangle, \]
where \( \nabla^{(k)}(Φ) \) is the \( k \)-th Fréchet derivative of \( Φ \) and the equality stands in (S). In particular, \( \nabla^{(k)}(Φ) ∈  S'(ℝ^k) ⊗ (S) \).

*Proof.* See Theorems 5.7 and 5.14 of [7].

We introduce as well the number operator, \( N \), on \( S \) in order to define Sobolev-Watanabe-Kree spaces.

**Definition 2.** Let \( r ≥ 0 \) and \( Φ ∈ (S) \) given by \( Φ = \sum_{n=0}^{∞} I_n(φ_n) \). We have:
   \[ N^r Φ = \sum_{n=0}^{∞} n^r I_n(φ_n). \]
Moreover, \( N^r \) is a linear and continuous operator from (S) into itself.
We denote by $(\mathcal{W}^r, \nabla)$ Proposition 12. Let $r \geq 0$ and $(\mathcal{P}) \subset (\mathcal{S})$ the algebra of polynomial random variables generated by elements of the form $I_1(\xi)$ with $\xi \in \mathcal{S}(\mathbb{R})$. We denote by $(\mathcal{W}^{r,2})$ the completion of $(\mathcal{P})$ with respect to the norm:

$$\forall \Phi \in (\mathcal{P}), \|\Phi\|_{r,2} = \|(N + E)^k \Phi\|_{L^2},$$

where $E$ is the identity operator.

**Theorem 9.** Let $k \in \mathbb{N}^*$ and $r \geq k$. $\nabla^{(k)}$ extends to a continuous operator from $(\mathcal{W}^{r,2})$ into $L^2(\mathbb{R}^k) \otimes (\mathcal{W}^{r-k,2})$.

**Proof.** See Theorem 5.24 and Corollary 5.25 of [12].

We introduce a space of test random variables which is useful when considering functionals of fractional Brownian motion as well as functionals of the Rosenblatt process. For this purpose, we define, for any $r \in \mathbb{R}$ and any $p > 1$, $(\mathcal{W}^{r,p})$ by the completion of $(\mathcal{P})$ with respect to the norm $\|(N + E)^{r/2}\|_{L^p}$. We denote by $(\mathcal{W}^{\infty,\infty})$ the projective limit of the family $\{(\mathcal{W}^{r,p}), p > 1, r \in \mathbb{R}\}$. Due to Meyer inequality (see chapter 1.5 of [11]), for every $k \geq 1$, the operator $\nabla^{(k)}$ is continuous from $(\mathcal{W}^{\infty,\infty})$ into itself, this space is stable under pointwise multiplication and the following version of the product and chain rules hold.

**Proposition 10.** Let $\Phi, \Psi \in (\mathcal{W}^{\infty,\infty})$ and let $F$ be an infinitely continuously differentiable function on $\mathbb{R}$ such that $F$ and its derivatives have polynomial growth. Then, $F(\Phi) \in (\mathcal{W}^{\infty,\infty})$ and:

$$\nabla(F(\Phi)) = F'(\Phi) \nabla(\Phi) + \Psi \nabla(\Phi),$$

$$\nabla(I_n \Phi) = \sum_{i=0}^{\infty} I_{n+1} \langle y \hat{\otimes} \psi_n \rangle,$$

where $I_n$ is a generalized Wiener-Itô integral in $(\mathcal{S})^*$. Moreover, we have the following generalized Wiener-Itô decomposition for $D^*_y(\Psi)$:

**Proof.** This proposition is a consequence of the product and chain rules on $(\mathcal{P})$ for $\nabla$ as well as Remark 1 page 78, Proposition 1.5.1 and Proposition 1.5.6 of [11].

We end this section by the definition of the adjoint of the Gâteaux derivative $D_y$ for every $y \in \mathcal{S}'(\mathbb{R})$ and by continuity results regarding the adjoint operators $\nabla^*$ and $(\nabla^{(2)})^*$.

**Theorem 11.** Let $y \in \mathcal{S}'(\mathbb{R})$ and $\Psi \in (\mathcal{S})^*$. The adjoint operator $D^*_y$ is continuous from $(\mathcal{S})^*$ into itself and we have:

$$\forall \xi \in \mathcal{S}(\mathbb{R}), S(D^*_y(\Psi))(\xi) = \langle y, \xi \rangle.$$
Proof. The first part of the proposition comes from Theorem 5.27 of [7]. Let us prove the second part. Let \( k \geq 2 \). Let \( X \in \tilde{L}^2(\mathbb{R}^2) \otimes (P) \). We have, by duality:
\[
\| (\nabla^{(2)})^\ast (X) \|_{k-2,2} = \| (E + N)^{\frac{k-2}{2}} (\nabla^{(2)})^\ast (X) \|_{(L^2)},
\]
\[
= \sup_{\| \Phi \|_{(L^2)} = 1} \| \langle \Phi; (E + N)^{\frac{k-2}{2}} (\nabla^{(2)})^\ast (X) \rangle \|,
\]
\[
= \sup_{\| \Phi \|_{(L^2)} = 1} \| \langle (\nabla^{(2)}(((E + N)^{\frac{k-2}{2}} \Phi)); X \rangle_{L^2(\mathbb{R}^2) \otimes (L^2)} \|,
\]
\[
= \sup_{\| \Phi \|_{(L^2)} = 1} \| \langle ((3E + N)^{\frac{k}{2}} \nabla^{(2)}(((E + N)^{-1} \Phi)); X \rangle_{L^2(\mathbb{R}^2) \otimes (L^2)} \|,
\]
\[
\leq \sup_{\| \Phi \|_{(L^2)} = 1} \| \nabla^{(2)}(((E + N)^{-1} \Phi)) \|_{L^2(\mathbb{R}^2) \otimes (L^2)} \| (3E + N)^{\frac{k}{2}} X \|_{L^2(\mathbb{R}^2) \otimes (L^2)},
\]
\[
\leq \| (3E + N)^{\frac{k}{2}} X \|_{L^2(\mathbb{R}^2) \otimes (L^2)},
\]

since, by continuity,
\[
\sup_{\| \Phi \|_{(L^2)} = 1} \| \nabla^{(2)}(((E + N)^{-1} \Phi)) \|_{L^2(\mathbb{R}^2) \otimes (L^2)} \leq 1.
\]

Thus, we have:
\[
\| (\nabla^{(2)})^\ast (X) \|_{k-2,2} \leq 3^2 \| (E + N)^{\frac{k}{2}} X \|_{L^2(\mathbb{R}^2) \otimes (L^2)},
\]
which concludes the proof. \(\square\)

2 Stochastic analysis of fractional Brownian motion and of the Rosenblatt process.

In this section, we state the definition of fractional Brownian motion and of the Rosenblatt process. Following [3] and [2], we remind that these processes are \((S)^\ast\)-differentiable and compute their \((S)^\ast\) derivatives. Then, we define stochastic derivative operators of first and second orders which play a significant role in the trace terms appearing in the relationship between Wick-Itô integral and forward integral with respect to these two processes. Moreover, we compute explicitly the Hilbert space adjoint of the first order stochastic gradient which is linked to Wick-Itô integral with respect to fractional Brownian motion. Therefore, we give a brief introduction to the stochastic integrals with respect to the fractional and the Rosenblatt noises in the Wick-Itô sense and make explicit the aforementioned link with the adjoint operator. In the rest of the article, we fix an interval \((a, b)\) included in \(\mathbb{R}_+\).

Definition 3. For \( H > 1/2 \), we define fractional Brownian motion and the Rosenblatt process by:
\[
\forall t \in (a, b), \quad B_t^H = A(H) \int_{\mathbb{R}} \left( \int_0^t \frac{(s-x)^{H-\frac{1}{2}}}{\Gamma(H-\frac{1}{2})} ds \right) dB_X,
\]
\[
X_t^H = d(H) \int_{\mathbb{R}^2} \left( \int_0^t \frac{(s-x_1)^{H-1}}{\Gamma(H)} \frac{(s-x_2)^{H-1}}{\Gamma(H)} ds \right) dB_{X_1} dB_{X_2},
\]

where \( A(H) \) and \( d(H) \) are positive constants such that \( \mathbb{E}[|B_t^H|^2] = \mathbb{E}[|X_t^H|^2] = 1 \) and defined by:
\[
A(H) = \left( \frac{\Gamma(H-\frac{1}{2})H(2H-1)\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)} \right)^{\frac{1}{2}},
\]
\[
d(H) = \sqrt{\frac{H(2H-1)}{2} \frac{(\Gamma(H))^2}{\beta(1-H;\frac{H}{2})}}.
\]
Definition 4. Fractional Brownian motion and the Rosenblatt process are \((S)^∗\)-differentiable and their derivatives, the fractional noise, \(\{B^H_t\}\), and the Rosenblatt noise, \(\{\dot{X}^H_t\}\), admit the following \((S)\)-transforms:

\[
\forall \xi \in S(\mathbb{R}), \quad S(\dot{B}^H_t)(\xi) = A(H)I^H_+\frac{1}{2}(\xi)(t),
\]

\[
S(\dot{X}^H_t)(\xi) = d(H)(I^H_+ (\xi)(t))².
\]

where for every \(0 < \alpha < 1\), \(I^\alpha_0(\xi)(t) = 1/(\Gamma(\alpha)) \int_\mathbb{R}(t - s)^{\alpha-1}\xi(s)ds\) is the fractional integral of order \(\alpha\) of \(\xi\) on the real line ([17] chapter 2).

Proof. See the proof of Lemma 2.15, Theorem 2.17 and Definition 2.18 of [11] for fractional Brownian motion and Lemma 3.4 of [2] for the Rosenblatt process.

The next Lemma is a technical one allowing us to define the first order stochastic derivative operators associated with fractional Brownian motion and the Rosenblatt process.

Lemma 13. For every \(\alpha \in (0,1/2)\) and every \(r \geq 0\), the operator \(I^\alpha_+ \otimes E\) admits a continuous extension from \(L^2(\mathbb{R}) \otimes (\mathcal{W}^r,2)\) to \(L^2((a,b)) \otimes (\mathcal{W}^r,2)\).

Proof. We define \(I^\alpha_+ \otimes E\) on simple element of \(L^2(\mathbb{R}) \otimes (\mathcal{W}^r,2)\) by:

\[
\forall \phi, \Phi \in L^2(\mathbb{R}) \times (\mathcal{W}^r,2), \quad (I^\alpha_+ \otimes E)(\phi \otimes \Phi) = I^\alpha_+(\phi) \otimes \Phi.
\]

and we extend it by linearity. Moreover, we have:

\[
\|I^\alpha_+(\phi) \otimes \Phi\|_{L^2((a,b)) \otimes (\mathcal{W}^r,2)} = \|I^\alpha_+(\phi)\|_{L^2((a,b))} \|\Phi\|_{(\mathcal{W}^r,2)},
\]

\[
\leq C_{a,b,\alpha} \|I^\alpha_+(\phi)\|_{L^\frac{2r}{2r-2}(\mathbb{R})} \|\Phi\|_{(\mathcal{W}^r,2)},
\]

\[
\leq C_{a,b,\alpha} \|\phi\|_{L^2(\mathbb{R})} \|\Phi\|_{(\mathcal{W}^r,2)}.
\]

since \(I^\alpha_+\) is a continuous operator from \(L^2(\mathbb{R})\) to \(L^\frac{2r}{2r-2}(\mathbb{R})\) and \(2/(1-2\alpha) > 2\) (Theorem 5.3 of [17]).

Consequently, we have the following result:

Proposition 14. Let \(r \geq 1\), \(\alpha \in (0,1/2)\). There exists a continuous operator, denoted \(\nabla^\alpha\), from \((\mathcal{W}^r,2)\) into \(L^2((a,b)) \otimes (\mathcal{W}^r,1,2)\) such that:

\[
\forall \Phi \in (\mathcal{W}^r,2), \quad \nabla^\alpha(\Phi)(t,\omega) = \sum_{n=1}^\infty nI_{n-1}(\delta_t \circ I^\alpha_+; \phi_n)(\omega),
\]

with \(\Phi = \sum_{n=1}^\infty I_n(\phi_n)\).

Proof. From Lemma 13 and Theorem 14 the operator \(\nabla^\alpha = (I^\alpha_+ \otimes E) \circ \nabla\) is continuous from \((\mathcal{W}^r,2)\) into \(L^2((a,b)) \otimes (\mathcal{W}^r,1,2)\). We only have to prove that the previous equality holds. First of all, notice that, for \(n \geq 1\), by Theorem 24.1 of [17]:

\[
\int_a^b \|< \delta_t \circ I^\alpha_+; \phi_n >\|^2_{L^2(\mathbb{R}^{n-1})} dt \leq C_{a,b,\alpha} \|I^\alpha_+; \phi_n\|^2_{L^2(\mathbb{R}^{n-1})} < +\infty.
\]

Then, \(I_{n-1}(\delta_t \circ I^\alpha_+; \phi_n)(\cdot)\) is an element of \(L^2((a,b)) \otimes (\mathcal{W}^r,1,2)\). By a standard argument, one can show that \(\sum_{n=1}^N nI_{n-1}(\delta_t \circ I^\alpha_+; \phi_n)(\cdot)\) converges in \(L^2((a,b)) \otimes (\mathcal{W}^r,1,2)\), since \(\Phi \in (\mathcal{W}^r,2)\), to an element which we denote by \(\sum_{n=1}^\infty nI_{n-1}(\delta_t \circ I^\alpha_+; \phi_n)(\cdot)\). Since \((S)\) is dense in \((\mathcal{W}^r,2)\), there
exists a sequence $\{\Phi_n\} \in (S)_{[0]}$ such that $\Phi_n \xRightarrow{n \to +\infty} \Phi$ in $L^2((a, b)) \otimes (\mathcal{W}^{r,2})$. By continuity, $\nabla^a(\Phi_n) \xRightarrow{n \to +\infty} \nabla^a(\Phi)$ in $L^2((a, b)) \otimes (\mathcal{W}^{r-1,2})$. Moreover, for all $n \in \mathbb{N}$, we have:

$$\forall (t, \omega) \in \mathbb{R} \times S'(\mathbb{R}), \; \nabla^a(\Phi_n)(t, \omega) = \left( (I_n^\alpha \otimes E) \circ \nabla \right) (\Phi_n)(t, \omega),$$

where we use Theorem 8 for the next to last equality. Since $\Phi$ of Proposition 14 noting that, for $n$ the proof is similar to one of Lemma 13.

**Proof.** The proof is similar to one of Lemma 13. □

Consequently, we have the following result:

**Proposition 16.** Let $r \geq 2$. There exists a continuous operator, denoted $\nabla^{(2),H/2}$, from $(\mathcal{W}^{r,2})$ to $L^2((a, b) \times (a, b)) \otimes (\mathcal{W}^{r-2,2})$ such that, for every $\Phi \in (\mathcal{W}^{r,2})$:

$$\lambda^{\otimes 2} \otimes \mu - a.e. (s, t, \omega) \in (a, b)^2 \times S' (\mathbb{R}), \; \nabla^{(2),H/2}(\Phi)(s, t, \omega) = \sum_{n=2}^{\infty} n(n-1) I_{n-2} < \delta_n \circ I^H_+ \otimes \delta_t \circ I^H_+; \phi_n > (\omega),$$

with $\Phi = \sum_{n=0}^{\infty} I_n (\phi_n)$.

**Proof.** The operator $\nabla^{(2),H/2} = I^H_+ \otimes I^H_+ \otimes E \circ \nabla^{(2)}$ is a continuous operator from $(\mathcal{W}^{r,2})$ to $L^2((a, b) \times (a, b); (\mathcal{W}^{r-2,2}))$ by the previous Lemma and Theorem 3. The equality is proved similarly to the one of Proposition 13 noting that, for $n \geq 2$:

$$\left( \int_{(a, b) \times (a, b)} \| \delta_n \circ I^H_+ \otimes \delta_t \circ I^H_+; \phi_n \|_{L^2(\mathbb{R}^{r-2})} dsdt \right)^{\frac{1}{2}} \leq C_{H,a,b} \| I^H_+ \otimes I^H_+ \otimes E \circ \nabla^{(2)}(\Phi_n) \|_{L^{(a, b) \otimes (a, b) \otimes (\mathcal{W}^{r,2})}} < \infty.$$
2. There is a $p \in \mathbb{N}$, a strictly positive constant $a$ and a non-negative function $L \in L^1((a, b))$ such that:

$$\forall \xi \in S(\mathbb{R}), \ |S(\Phi_t)(\xi)| \leq L(t) \exp (a\|A\xi\|_2^2)$$

Then, $\Phi_t \circ \dot{B}^H_t$ and $\Phi_t \circ \dot{X}^H_t$ are $(S)^*$ integrable over $(a, b)$ and we define the fractional noise integral and the Rosenblatt noise integral of $\{\Phi_t\}$ by:

$$\int_{(a,b)} \Phi_t dB^H_t = \int_{(a,b)} \Phi_t \circ \dot{B}^H_t dt,$$

$$\int_{(a,b)} \Phi_t dX^H_t = \int_{(a,b)} \Phi_t \circ \dot{X}^H_t dt.$$

Moreover, we have the following representation:

$$\int_{(a,b)} \Phi_t dB^H_t = \int_{(a,b)} (D^*_{A(H)\delta_0 I^H_{-\frac{1}{2}}})(\Phi_t) dt,$$

$$\int_{(a,b)} \Phi_t dX^H_t = \int_{(a,b)} (D^*_{\sqrt{d(H)\delta_0 I^H_{-\frac{1}{2}}}})^2(\Phi_t) dt.$$

**Proof.** See Definition-Theorem 3.10 of [2] for the Rosenblatt noise integral and Section 3.3 of [4] for the fractional noise integral. \[\square\]

Finally, we compute the Hilbert space adjoint of $\nabla^\alpha$, for $\alpha \in (0; 1/2)$, and link this operator with the fractional noise integral for a certain class of stochastic integrand processes.

**Proposition 18.** Let $\alpha \in (0, 1/2)$ and $r \geq 1$. The operator $(\nabla^\alpha)^* = \nabla^* \circ (I^\alpha_+(I(a,b)) \otimes E)$ is a linear and continuous operator from $L^2((a, b)) \otimes (W^{r-2})$ into $(W^{r-1,2})$, where $I^\alpha_+$ is defined by:

$$\forall \xi \in S(\mathbb{R}), \ I^\alpha_+(\xi)(t) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} (s-t)^{\alpha-1}\xi(s)ds.$$

**Proof.** By Proposition 14, $\nabla^\alpha = (I^\alpha_+ \otimes E) \circ \nabla$ is a linear and continuous operator from $(W^{r-2})$ into $L^2((a, b)) \otimes (W^{r-1,2})$. Thus, by definition, $(\nabla^\alpha)^*$ is a linear and continuous operator from $L^2((a, b)) \otimes (W^{r-1,2})^*$ into $(W^{r,2})^*$. Moreover, $(\nabla^\alpha)^*$ is equal to $\nabla^* \circ (I^\alpha_+ \otimes E)^*$. Let us compute $(I^\alpha_+ \otimes E)^*$. We have, for every $s \geq 0$, $f \in L^2((a, b)) \otimes (W^{r,2})^*$ and $g \in L^2(\mathbb{R}) \otimes (W^{s,2})$:

$$< (I^\alpha_+ \otimes E)^*(f); g >_{(L^2(\mathbb{R})\otimes(W^{s,2})^*,L^2(\mathbb{R})\otimes(W^{s,2}))} = < f, (I^\alpha_+ \otimes E)(g) >_{(L^2((a,b))\otimes(W^{r,2})^*,L^2((a,b))\otimes(W^{r,2}))}.$$

Assume $f = f_1 \otimes F_1$ and $g = g_1 \otimes G_1$. Then, by relation 5.16 of [17]:

$$< (I^\alpha_+ \otimes E)^*(f); g >_{(L^2(\mathbb{R})\otimes(W^{s,2})^*,L^2(\mathbb{R})\otimes(W^{s,2}))} = < f_1; I^\alpha_+ (g_1) >_{L^2((a,b))} < F_1; G_1 >_{((W^{s,2})^*,(W^{s,2}))}.$$

Thus, $(I^\alpha_+ (I(a,b)) \otimes E)$ and $(I^\alpha_+ \otimes E)^*$ coincide on simple elements of $L^2((a, b)) \otimes (W^{r-2})^*$. Since, both operators are linear and bounded operators on $L^2((a, b)) \otimes (W^{r-2})^*$, they agree on $L^2((a, b)) \otimes (W^{r-2})^*$. Consequently, $(\nabla^\alpha)^*$ is equal to $\nabla^* \circ (I^\alpha_+ (I(a,b)) \otimes E)$ on $L^2((a, b)) \otimes (W^{r-1,2})^*$. Moreover, it is well known that the operator $\nabla^*$ is a linear and continuous operator from $L^2(\mathbb{R}) \otimes (W^{r,2})$ into $(W^{r-1,2})$ for any $r \geq 1$ (see Proposition 5.27 of [7]). This concludes the proof. \[\square\]

**Proposition 19.** Let $\alpha \in (0, 1/2)$ and $\Phi \in L^2((a, b)) \otimes (W^{1,2})$. We have:

$$(\nabla^\alpha)^*(\Phi) = \int_a^b D^*_{\dot{\alpha}}(\Phi_t) dt.$$
Proof. Let $\xi \in S(\mathbb{R})$. $(\nabla^\alpha)^*(\Phi) \in (L^2) \subset (\mathcal{W}^{1,2})^*$ and $e^{<\xi,\cdot>}$ is in $(S) \subset (\mathcal{W}^{1,2})$ for any $s \geq 0$. Thus, we have:

$$S((\nabla^\alpha)(\Phi)^*) = e^{<\xi,\cdot>} \in (L^2),$$

$$= e^{<\Phi, \nabla^\alpha>* (\xi)>} \in (\mathcal{W}^{1,2})^*,$$

$$= e^{<\Phi, \nabla^\alpha>* (\xi)>} \in L^2((a,b) \cap (L^2)),$$

$$= e^{<\Phi, \nabla^\alpha>* (\xi)>} \in L^2((a,b) \cap (L^2)),$$

$$= \int_a^b I_n^\alpha(\xi)(t) < \Phi_{t_0} : e^{<\xi,\cdot>} > (L^2) dt,$$

$$= \int_a^b S(\Phi_t)(\xi) I_n^\alpha(\xi)(t) dt,$$

$$= \left( \int_a^b D_n^\alpha (\Phi_t) dt \right) (\xi).$$

\[ \square \]

3 From forward integrals to Wick-Itô integrals.

3.1 Fractional Brownian motion.

Proposition 20. Let $\{\Phi_t : t \in (a,b)\}$ be a stochastic process such that for all $t \in (a,b)$, $\Phi_t \in (\mathcal{W}^{1,2})$. Then, we have, for every $\epsilon > 0$ and $t \in (a,b)$:

$$\Phi_t \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} \in \Phi_t \circ \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} + A(H) \int_t^{t+\epsilon} \nabla^{H-\frac{3}{2}}(\Phi_t)(s) ds.$$

Proof. Let $\epsilon > 0$ be small enough such that $t + \epsilon \in (a,b)$. Since $\Phi_t \in (\mathcal{W}^{1,2}) \subset (L^2)$, $\Phi_t = \sum_{n=0}^{\infty} I_n(\phi_n(t))$ with $\phi_n(t) \in \hat{L}^2(\mathbb{R}^n)$. Let us fix $n \geq 1$. Using the multiplication formula from Malliavin calculus (see Proposition 1.1.3 of [14]), we obtain:

$$I_n(\phi_n(t))(B_{t+\epsilon}^H - B_t^H) = A(H)I_n(\phi_n(t))I_1\int_t^{t+\epsilon} \frac{(s - \cdot)^{(H-\frac{3}{2})}}{\Gamma(H-\frac{3}{2})} ds,$$

$$= I_{n+1}(\phi_n(t) \otimes g_{t+\epsilon}^H) + nI_{n-1}(\phi_n(t) \otimes g_{t+\epsilon}^H),$$

where $g_{t+\epsilon}(s) = A(H)/\Gamma(H - 1/2) \int_t^{t+\epsilon} (s - \cdot)^{(H-3/2)} ds$. Then, for any $\xi \in S(\mathbb{R})$, we have:

$$S(I_n(\phi_n(t))(B_{t+\epsilon}^H - B_t^H))(\xi) = \phi_n(t); \xi^{\otimes n} > g_{t+\epsilon}^H \xi > + n \phi_n(t) \otimes g_{t+\epsilon}^H \xi^{\otimes n-1}.$$  

By continuity of the scalar product in $(L^2)$, we have:

$$S(\Phi_t(B_{t+\epsilon}^H - B_t^H))(\xi) = \sum_{n=0}^{\infty} (\phi_n(t); \xi^{\otimes n} > g_{t+\epsilon}^H \xi > + n \phi_n(t) \otimes g_{t+\epsilon}^H \xi^{\otimes n-1}).$$

We want to compute separately the two infinite series appearing in the right hand side of the previous equality. First of all, we have:

$$\sum_{n=0}^{\infty} \phi_n(t); \xi^{\otimes n} > g_{t+\epsilon}^H \xi > = S(\Phi_t)(\xi)S(B_{t+\epsilon}^H - B_t^H)(\xi),$$

$$= S(\Phi_t \circ (B_{t+\epsilon}^H - B_t^H))(\xi).$$
Moreover, for the second term, we have, by Fubini theorem:

\[ n < \phi_n(t) \otimes_I g_{t,t+\epsilon}; \xi^{\otimes n - 1} > = A(H) n \int_t^{t+\epsilon} \left< I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s); \xi^{\otimes n - 1} \right> ds, \]

\[ = A(H) \int_t^{t+\epsilon} S(nI_{n-1}(I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s)))(\xi) ds, \]

\[ = A(H) \int_t^{t+\epsilon} S(\nabla^{H-\frac{1}{2}} (I_n(\phi_n(t)))(s))(\xi) ds. \]

We want to invert the infinite series and the integral over \((t, t+\epsilon)\). For any \( n \geq 1 \) and \( t \in (a, b) \), we have:

\[ \left| \int_t^{t+\epsilon} < I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s); \xi^{\otimes n - 1} > ds \right| \leq \int_t^{t+\epsilon} \left| < I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s); \xi^{\otimes n - 1} > \right| ds, \]

\[ \leq \|\xi\|_{L^2(\mathbb{R})}^{n-1} \int_t^{t+\epsilon} \| I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s) \|_{L^2(\mathbb{R}^{n-1})} ds, \]

\[ \leq \|\xi\|_{L^2(\mathbb{R})}^{n-1} \int_a^b \| I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s) \|_{L^2(\mathbb{R}^{n-1})} ds < +\infty, \]

since \( I_{+,\ldots,+}^{(0,\ldots,0,H-1/2)}(\phi_n(t)) \in L^{(2\ldots,2,1/(1-H))}(\mathbb{R}^n) \) and \( 1/(1-H) > 2 \) (see Theorem 24.1 of [17]).

Finally, we note that, using Cauchy-Schwarz and Jensen inequalities and Proposition [14]

\[ \sum_{n=1}^{+\infty} \left[ n \|\xi\|_{L^2(\mathbb{R})}^{n-1} \int_a^b \| I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s) \|_{L^2(\mathbb{R}^{n-1})} ds \right] \leq \exp(\frac{1}{2} \|\xi\|_{L^2(\mathbb{R})}^2) \left( \sum_{n=1}^{+\infty} \frac{n!}{2} \left( \int_a^b \| I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s) \|_{L^2(\mathbb{R}^{n-1})}^2 ds \right)^{\frac{1}{2}}, \right. \]

\[ \left. \sum_{n=1}^{+\infty} \left[ n \left( \int_a^b \| I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s) \|_{L^2(\mathbb{R}^{n-1})} ds \right) \|\xi\|_{L^2(\mathbb{R})}^{n-1} \right] \leq (b-a)^{\frac{1}{2}} \exp(\frac{1}{2} \|\xi\|_{L^2(\mathbb{R})}^2) \times \right. \]

\[ \left. \left( \sum_{n=1}^{\infty} \frac{n!}{2} \int_a^b \| I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s) \|_{L^2(\mathbb{R}^{n-1})}^2 ds \right)^{\frac{1}{2}}, \right. \]

which is finite since \( \Phi_t \in (W^{1,2}) \). Thus, we have:

\[ \sum_{n=1}^{+\infty} n < \phi_n(t) \otimes_I g_{t,t+\epsilon}; \xi^{\otimes n - 1} > = A(H) \int_t^{t+\epsilon} \sum_{n=1}^{+\infty} [S(nI_{n-1}(I_{+,\ldots,+}^{(0,\ldots,0,H-\frac{1}{2})} (\phi_n(t))(s)))(\xi)] ds, \]

\[ = A(H) \int_t^{t+\epsilon} S(\nabla^{H-\frac{1}{2}} (\Phi_t)(s))(\xi) ds, \]

\[ = S(A(H) \int_t^{t+\epsilon} \nabla^{H-\frac{1}{2}} (\Phi_t)(s) ds)(\xi), \]

where we use the square-integrability of \( \|\nabla^{H-\frac{1}{2}} (\Phi_t)(s)\|_{L^2} \) on \((a, b)\) to justify the last equality. This concludes the proof.
Proposition 21. Let \( \{ \Phi_t : t \in (a, b) \} \) be a stochastic process such that for all \( t \in (a, b) \), \( \Phi_t \in (W^{1,2}) \).
Moreover, assume that:
\[
\int_a^b \| \Phi_t \|_{L^2}^2 dt < +\infty, \\
\int_a^b \| \nabla^{H-\frac{1}{2}}(\Phi_t) \|_{L^2(a,b)\otimes(L^2)}^2 dt < +\infty.
\]
Then, we have:
\[
\int_a^b \Phi_t \frac{B_{t+\epsilon} - B_t}{\epsilon} dt \overset{(S)^*}{=} \int_a^b \Phi_t \frac{B_{t+\epsilon} - B_t}{\epsilon} dt + \text{A}(H) \int_a^b \int_t^{t+\epsilon} \nabla^{H-\frac{1}{2}}(\Phi_t)(s) \frac{ds}{\epsilon} dt.
\]
Proof. In order to prove the \((S)^*\)-integrability of \( \Phi_t \frac{B_{t+\epsilon} - B_t}{\epsilon} \) over \((a, b)\), we are going to prove the \((S)^*\)-integrability of the two terms appearing in Proposition 20.
Let us start with the first term. For every \( \xi \in \mathcal{S}(\mathbb{R}) \), we have:
\[
S(\Phi_t \frac{B_{t+\epsilon} - B_t}{\epsilon})(\xi) = S(\Phi_t)(\xi) S(\frac{B_{t+\epsilon} - B_t}{\epsilon})(\xi),
\]
\[
= \mathbb{E}[\Phi_t \exp(\xi^2 \frac{B_{t+\epsilon} - B_t}{\epsilon})] \mathcal{L}^{1/2}(\xi) \int_t^{t+\epsilon} \frac{I^H_{t+\frac{1}{2}}(\xi)(s)}{\epsilon} ds.
\]
By Cauchy-Schwarz inequality and \( \| L^H_{t+\frac{1}{2}}(\xi) \|_{\mathbb{R}} \leq C_H(\| \xi \|_{\mathbb{R}} + \| \xi^{(1)} \|_{\mathbb{R}} + \| \xi \|_{L^1(\mathbb{R})}) \), we have:
\[
|S(\Phi_t \frac{B_{t+\epsilon} - B_t}{\epsilon})(\xi)| \leq C_H(\| \xi \|_{\mathbb{R}} + \| \xi^{(1)} \|_{\mathbb{R}} + \| \xi \|_{L^1(\mathbb{R})}) \exp(\frac{1}{2}\| \xi \|_{\mathbb{R}}^2) \mathcal{L}^{1/2}(\xi)^{\frac{1}{2}}.
\]
Using the fact that \( \int_a^b \| \Phi_t \|_{L^2}^2 dt < +\infty \), we have the \((S)^*\)-integrability of the first term. Let us deal with the second term now. For every \( \xi \in \mathcal{S}(\mathbb{R}) \), we have:
\[
|S(\int_t^{t+\epsilon} \nabla^{H-\frac{1}{2}}(\Phi_t)(s, \omega) ds)(\xi)| \leq \int_t^{t+\epsilon} |S(\nabla^{H-\frac{1}{2}}(\Phi_t)(s))(\xi)| ds,
\]
\[
\leq \int_t^{t+\epsilon} \exp(\frac{1}{2}\| \xi \|_{\mathbb{R}}^2) \int_t^{t+\epsilon} \| \nabla^{H-\frac{1}{2}}(\Phi_t)(s) \|_{L^2} ds,
\]
\[
\leq \int_t^{t+\epsilon} \exp(\frac{1}{2}\| \xi \|_{\mathbb{R}}^2) \int_a^b \| \nabla^{H-\frac{1}{2}}(\Phi_t)(s) \|_{L^2} ds,
\]
\[
\leq (b-a)^\frac{1}{2} \exp(\frac{1}{2}\| \xi \|_{\mathbb{R}}^2) \mathcal{L}^{1/2}(\xi) \mathcal{L}^{1/2}(\Phi_t)^{\frac{1}{2}}.
\]
Using the fact that \( \int_a^b \| \nabla^{H-\frac{1}{2}}(\Phi_t) \|_{L^2(a,b)\otimes(L^2)}^2 dt < +\infty \), we have the \((S)^*\)-integrability of the second term. This ends the proof.

The next proposition examines the chain rule property for certain functionals of fractional Brownian motion.

Proposition 22. Let \( F \) be in \( C^1(\mathbb{R}) \). Assume that, for every \( t \in (a, b) \), \( F(B^H_t) \in (W^{1,2}) \). Then, we have, in \( L^2((a, b)) \otimes (L^2) \):
\[
\nabla^{H-\frac{1}{2}}(F(B^H_t))(s, \omega) = \frac{A(H)\beta(H - \frac{1}{2}, 2 - 2H)}{(\Gamma(H - \frac{1}{2}))^2} \left( \int_0^t |s-r|^{2H-2} dr \right) F'(B^H_t)(\omega).
\]
Proof. We denote by \( \{H_{n,t}^{2H} : n \in \mathbb{N} \} \) the family of Hermite polynomials of parameter \( t^{2H} \). Then, \( \{H_{n,t}^{2H}/(\sqrt{n!}H_n) : n \in \mathbb{N} \} \) is an orthonormal family of \( L^2(\mathbb{R}, \gamma_{t^{2H}}(dx)) \) where \( \gamma_{t^{2H}} \) is the centered Gaussian probability measure over \( \mathbb{R} \) with variance \( t^{2H} \). Since \( \mathbb{E}[(F(B_t^{H}))^2] < \infty \), we have:

\[
F = \sum_{n=0}^{\infty} c_{n,t^{2H}} \frac{H_{n,t}^{2H}}{\sqrt{n!}H_n},
\]

where \( c_{n,t^{2H}} \) are the Hermite coefficients of \( F \) with respect to \( \gamma_{t^{2H}} \). Using Wiener-Itô Theorem, we have:

\[
F(B_t^{H}) = \sum_{n=0}^{\infty} c_{n,t^{2H}} \frac{I_n((g_0^{H})^\otimes n)}{\sqrt{n!}H_n}.
\]

Let \( n \geq 0 \). Denote by \( d_{n,t^{2H}} \) the Hermite coefficients of \( F' \) with respect to \( \gamma_{t^{2H}} \). We have:

\[
d_{n,t^{2H}} = \int_{\mathbb{R}} F'(x) \frac{H_{n,t}^{2H}(x)}{\sqrt{n!}H_n} \gamma_{t^{2H}}(dx).
\]

Integrating by part, we obtain:

\[
d_{n,t^{2H}} = -\int_{\mathbb{R}} F(x)[nH_{n-1,t}^{2H}(x) - \frac{x}{t^{2H}} H_{n,t}^{2H}(x)] \frac{\gamma_{t^{2H}}(dx)}{\sqrt{n!}H_n},
\]

\[
d_{n,t^{2H}} = \int_{\mathbb{R}} F(x) H_{n+1,t}^{2H}(x) \frac{\gamma_{t^{2H}}(dx)}{\sqrt{n!}H_n(n+2)},
\]

\[
d_{n,t^{2H}} = \frac{\sqrt{n+1}}{t^{H}} c_{n+1,t^{2H}}.
\]

Since \( \mathbb{E}[(N+E)^{1/2}F(B_t^{H})^2] < +\infty \), \( F' \in L^2(\mathbb{R}, \gamma_{t^{2H}}(dx)) \) and we have:

\[
F' = \sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{t^{H}} c_{n+1,t^{2H}} \frac{H_{n,t}^{2H}}{\sqrt{n!}H_n},
\]

\[
F' = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{t^{H}} c_{n,t^{2H}} \frac{H_{n-1,t}^{2H}}{\sqrt{(n-1)!}H(n-1)}.
\]

Thus, by Wiener-Itô Theorem, we obtain:

\[
F'(B_t^{H}) = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{t^{H}} c_{n,t^{2H}} \frac{I_n-1((g_0^{H})^\otimes n-1)}{(n-1)!H(n-1)},
\]

\[
F'(B_t^{H}) = \sum_{n=1}^{\infty} \frac{c_{n,t^{2H}}}{\sqrt{n!}H_n} I_{n-1}((g_0^{H})^\otimes n-1).
\]

Moreover, by assumption, \( F(B_t^{H}) \in (W^{1,2}) \). Consequently, by Proposition \[14\] we have, \( \lambda \otimes \mu \)-a.e:

\[
\nabla^{H-\frac{1}{2}}(F(B_t^{H}))(s,\omega) = \sum_{n=1}^{\infty} \frac{c_{n,t^{2H}}}{\sqrt{n!}H_n} I_{n-1}(<\delta_s \circ I_{t}^{-\frac{1}{2}}; (g_0^{H})^\otimes n>)(\omega),
\]

\[
\nabla^{H-\frac{1}{2}}(F(B_t^{H}))(s,\omega) = I_{t}^{H-\frac{1}{2}}(g_0^{H})(s) \sum_{n=1}^{\infty} \frac{c_{n,t^{2H}}}{\sqrt{n!}H_n} I_{n-1}((g_0^{H})^\otimes n-1)(\omega),
\]

\[
\nabla^{H-\frac{1}{2}}(F(B_t^{H}))(s,\omega) = A(H)\bar{\beta}(H-\frac{1}{2}, 2-2H) \left( \int_{0}^{t} |s-r|^{2H-2}dr \right) F'(B_t^{H})(\omega).
\]

\[\square\]
Proposition 23. Let $F$ be in $C^1(\mathbb{R})$. Assume that, for every $t \in (a,b)$, $F(B_t^H) \in (W^{1,2})$. Then, we have:

$$A(H) \int_a^b \left( \int_t^{t+\epsilon} \nabla^{\frac{1}{2}}(F(B_t^H))(s) \frac{ds}{\epsilon} \right) dt \to_e \int_a^b t^{2H-1} F'(B_t^H) dt,$$

Proof. Using Proposition 22 as well as standard calculations lead to the convergence of this trace term.

Remark 24. Let $F$ be as in Proposition 23. In order to apply Proposition 22 to $\Phi_t = F(B_t^H)$, it is sufficient to prove that:

$$\int_a^b \|F(B_t^H)\|_{(L^2)}^2 dt < +\infty,$$

$$\int_a^b \|\nabla^{\frac{1}{2}}(F(B_t^H))\|_{L^2((a,b);(L^2))}^2 dt < +\infty.$$

If we assume that there exist $C > 0$ and $\lambda > 0$ with $\lambda < 1/(4b^2H)$ such that:

$$\forall x \in \mathbb{R}, \max\{F(x), F'(x)\} \leq Ce^{\lambda x^2}.$$

Then, we have:

$$\int_a^b \|F(B_t^H)\|_{(L^2)}^2 dt \leq C \int_a^b \frac{dt}{\sqrt{1-4\lambda t^2H}} < \infty,$$

$$\int_a^b \|\nabla^{\frac{1}{2}}(F(B_t^H))\|_{L^2((a,b);(L^2))}^2 dt = \int_a^b \|\nabla^{\frac{1}{2}}(F(B_t^H))\|_{L^2((a,b);(L^2))}^2 dt,$$

$$\leq C_{a,b,H} \int_a^b \frac{t^{2H}}{\sqrt{1-4\lambda t^2H}} dt < \infty.$$

We conclude this subsection by the proof of Theorem 1.

Proof. We are going to prove that the term $\int_a^b F(B_t^H) \circ (B_{t+\epsilon}^H - B_t^H) dt/\epsilon$ converges in $(L^2)$ to the following random variable:

$$(\nabla^{\frac{1}{2}})^*(F(B_t^H)).$$

First of all, for every $t \in (a,b)$, since $F(B_t^H) \in (W^{1,2})$:

$$F(B_t^H) \circ \frac{B_{t+\epsilon}^H - B_t^H}{\epsilon} = A(H)F(B_t^H) \circ I_1\left( \int_t^{t+\epsilon} \frac{(s - \cdot )^{\frac{H-3}{2}} ds}{\Gamma(H - \frac{1}{2})} \right),$$

$$= \frac{A(H)}{\Gamma(H - \frac{1}{2})} \nabla^* \left( F(B_t^H) \int_t^{t+\epsilon} (s - \cdot )^{\frac{H-3}{2}} ds \right).$$

Indeed, using the $S$-transform, we have, for every $\xi \in S(\mathbb{R})$:

$$S(\nabla^* \left( F(B_t^H) \int_t^{t+\epsilon} (s - \cdot )^{\frac{H-3}{2}} ds \right))(\xi) = E[\nabla^* \left( F(B_t^H) \int_t^{t+\epsilon} (s - \cdot )^{\frac{H-3}{2}} ds \right) : e(\xi)],$$

$$= E[F(B_t^H) \int_t^{t+\epsilon} (s - \cdot )^{\frac{H-3}{2}} ds : e(\xi)],$$

$$= \Gamma(H - \frac{1}{2}) A(H) S(F(B_t^H))(\xi) S(\frac{B_{t+\epsilon}^H - B_t^H}{\epsilon})(\xi).$$
Therefore, we are left to prove that:

\[
\frac{1}{\Gamma(H - \frac{1}{2})} \nabla^* \left( \int_a^b F(B(t)^H) \left( \int_t^{t+\epsilon} (s-x)_+^{H-\frac{3}{2}} ds \, dt \right) \frac{(L_1^2)}{\epsilon} \right) \nabla^* \left( I_i^{-\frac{1}{2}} (\mathbf{n}_i, (.). F(B(t)^H)) \right).
\]

For this purpose, we are going to prove that:

\[
\frac{1}{\Gamma(H - \frac{1}{2})} \int_a^b F(B(t)^H) \left( \int_t^{t+\epsilon} (s-x)_+^{H-\frac{3}{2}} ds \, dt \right) \frac{(L_1^2)}{\epsilon} \rightarrow \int_a^b \frac{(t-s)_+^{H-\frac{3}{2}}}{\Gamma(H - \frac{1}{2})} F(B(t)^H)dt,
\]

where the convergence holds in \( L^2(\mathbb{R}) \otimes (W^{1,2}) \). We denote by \((I)\) the square of the \( L^2(\mathbb{R}) \otimes (W^{1,2})\)-norm of the difference between the left-hand side and the right-hand side. We have:

\[
(I) = \int_\mathbb{R} \int_a^b F(B(t)^H) \left( \int_t^{t+\epsilon} ((s-x)_+^{H-\frac{3}{2}} - (t-x)_+^{H-\frac{3}{2}}) \frac{ds}{\epsilon} \right) dt \|_{(W^{1,2})}^2 dx,
\]

\[
(I) = \int_\mathbb{R} \int_a^b F(B(t)^H) \left( \int_t^{t+\epsilon} ((s-x)_+^{H-\frac{3}{2}} - (t-x)_+^{H-\frac{3}{2}}) \frac{ds}{\epsilon} \right) dt_1; \int_a^b F(B(t)^H) \left( \int_t^{t+\epsilon} ((s-x)_+^{H-\frac{3}{2}} - (t-x)_+^{H-\frac{3}{2}}) \frac{ds}{\epsilon} \right) dt_2; \|_{(W^{1,2})}^2 dx,
\]

\[
(I) = \int_{(a,b) \times (a,b)} \langle F(B(t_1)^H), F(B(t_2)^H) \rangle_{(W^{1,2})} \left( \int_\mathbb{R} \int_{(t_1,t_1+\epsilon) \times (t_2,t_2+\epsilon)} \left( (s-x)_+^{H-\frac{3}{2}} - (t-x)_+^{H-\frac{3}{2}} \right) \right) \times \left( (s-x)_+^{H-\frac{3}{2}} - (t-x)_+^{H-\frac{3}{2}} \right) \frac{ds dr}{\epsilon^2} dx dt_1 dt_2,
\]

\[
(I) = \beta(H - \frac{1}{2}, 2 - 2H) \int_{(a,b) \times (a,b)} \langle F(B(t_1)^H), F(B(t_2)^H) \rangle_{(W^{1,2})} \left( \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} |s-r|^{2H-2} - |t_2 - s|^{2H-2} - |t_1 - r|^{2H-2} + |t_1 - t_2|^{2H-2} \frac{ds dr}{\epsilon^2} \right) dt_1 dt_2.
\]

For the last equality, we have used the following relation which holds for every \( \gamma \in (-1, -1/2) \):

\[
\int_\mathbb{R} (s_1 - u)^\gamma (s_2 - u)^\gamma du = \beta(\gamma + 1, 1 - 2\gamma - 1) |s_2 - s_1|^{2\gamma + 1}.
\]

Moreover, by Cauchy-Schwarz inequality, we have:

\[
\|\langle F(B(t_1)^H), F(B(t_2)^H) \rangle_{(W^{1,2})} \|_{(W^{1,2})} \leq \| F(B(t_1)^H) \|_{(W^{1,2})} \| F(B(t_2)^H) \|_{(W^{1,2})},
\]

And, by Meyer inequality, we have, for every \( i \in \{1, 2\} \):

\[
\| F(B(t)^H) \|_{(W^{1,2})} \leq C \left( \| F(B(t)^H) \|_{(L^2)}^2 + \| \nabla (F(B(t)^H)) \|_{(L^2)}^2 \right)^{\frac{1}{2}} \leq C \left( \frac{1}{\sqrt{1 - 4\lambda_i^{2H}}} + \frac{t_i^{2H}}{\sqrt{1 - 4\lambda_i^{2H}}} \right)^{\frac{1}{2}} \leq C(1 + t_i^{2H})\left( \frac{1}{1 - 4\lambda_i^{2H}} \right)^{\frac{1}{2}}.
\]

Thus:

\[
(I) \leq C \int_{(a,b) \times (a,b)} \prod_{i=1}^2 \left( (1 + t_i^{2H})\left( \frac{1}{1 - 4\lambda_i^{2H}} \right)^{\frac{1}{2}} \right) \left( \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} |s-r|^{2H-2} - |t_2 - s|^{2H-2} - |t_1 - r|^{2H-2} + |t_1 - t_2|^{2H-2} \frac{ds dr}{\epsilon^2} \right) dt_1 dt_2.
\]
For every $t_1, t_2$ in $(a, b) \times (a, b)$ such that $t_1 \neq t_2$, we have:

$$\left( \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} \left[ |s-r|^{2H-2} - |t_2-s|^{2H-2} - |t_1-r|^{2H-2} + |t_1-t_2|^{2H-2} \right] \frac{dsdr}{\epsilon^2} \right) \rightarrow 0.$$ 

To pursue, we need to provide an upper bound for:

$$(II) = \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} \left[ |s-r|^{2H-2} - |t_2-s|^{2H-2} - |t_1-r|^{2H-2} + |t_1-t_2|^{2H-2} \right] \frac{dsdr}{\epsilon^2}.$$ 

We assume without loss of generality that $t_1 \neq t_2$, $t_1 > t_2$ and $\epsilon < (t_1 - t_2)/2$. Note that $t_2 + \epsilon < t_1$. We have, for every $(s, r) \in (t_1, t_1 + \epsilon) \times (t_2, t_2 + \epsilon)$, since $2H - 2 < 0$:

$$(t_1 + \epsilon - t_2)^{2H-2} \leq (t_1 - t_2 - \epsilon)^{2H-2},$$

$$(t_1 + \epsilon - t_2)^{2H-2} \leq (s - t_2)^{2H-2} \leq (t_1 - t_2)^{2H-2},$$

$$(t_1 - t_2)^{2H-2} \leq (t_1 - r)^{2H-2} \leq (t_1 - t_2 - \epsilon)^{2H-2}.$$ 

Thus,

$$(II) \leq (t_1 - t_2 - \epsilon)^{2H-2} - (t_1 + \epsilon - t_2)^{2H-2},$$

$$(t_1 + \epsilon - t_2)^{2H-2} \leq (t_1 - t_2)^{2H-2} \leq (t_1 - t_2)^{2H-2} - \epsilon \left( 1 + \frac{\epsilon}{t_1 - t_2} \right)^{2H-2},$$

$$\leq (t_1 - t_2)^{2H-2} \max_{\epsilon \in [0, \frac{1}{2}]} [(1 - \epsilon)^{2H-2} - (1 + \epsilon)^{2H-2}].$$

Consequently, we have the following upper bound:

$$\left| \left( \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} \left[ |s-r|^{2H-2} - |t_2-s|^{2H-2} - |t_1-r|^{2H-2} + |t_1-t_2|^{2H-2} \right] \frac{dsdr}{\epsilon^2} \right) \right| \leq C'|t_1 - t_2|^{2H-2}.$$ 

To conclude, we need to prove the finiteness of the following integral:

$$\int_{(a,b)\times(a,b)} \prod_{i=1}^{2} \left( 1 + t_i^{2H} \right)^{H} \left( \frac{1}{1 - 4\lambda t_i^{2H}} \right)^{\frac{1}{2}} |t_1 - t_2|^{2H-2} dt_1 dt_2.$$ 

Denoting by $h(t) = (1 + t^{2H})^{1/2} (1/(1 - 4\lambda t^{2H}))^{1/4}$ and using Hardy-Littlewood-Sobolev inequality (Theorem 4.3 page 106 of [10]), we obtain:

$$\int_{(a,b)\times(a,b)} h(t_1)h(t_2) |t_1 - t_2|^{2H-2} dt_1 dt_2 \leq C_H \|h\|_{L^{1/H}((a,b))}^2.$$ 

But $1/H \in (1, 2)$, we have:

$$\int_{(a,b)\times(a,b)} \prod_{i=1}^{2} \left( 1 + t_i^{2H} \right)^{H} \left( \frac{1}{1 - 4\lambda t_i^{2H}} \right)^{\frac{1}{2}} |t_1 - t_2|^{2H-2} dt_1 dt_2 \leq C_{H,a,b} \|h\|_{L^2((a,b))}^2 < +\infty.$$ 

By Lebesgue dominated convergence theorem, we obtain the desired convergence. Applying Proposition 23 and Remark 23 leads to the result.

### 3.2 The Rosenblatt process.

Before stating the first result of this section, we introduce a useful notation. For any $t \in (a, b)$ and for any $\epsilon > 0$, we define:

$$h^H_{t,t+\epsilon}(x_1, x_2) = d(H) \int_{t}^{t+\epsilon} \frac{(s-x_1)^{H-1}}{\Gamma(H/2)} \frac{(s-x_2)^{H-1}}{\Gamma(H/2)} ds$$
Proposition 25. Let \( t \in (a, b) \). Let \( \epsilon > 0 \) such that \( t + \epsilon < b \). Let \( F \) be in \( C^\infty(\mathbb{R}) \) with polynomial growth (as well as its derivatives). Then, we have:

\[
F(X_t^H) \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon} = F(X_t^H) \circ \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon} + 2d(H) \int_t^{t+\epsilon} D^*_{\delta_o I_+^H} \left( \nabla^H(F(X_t^H))(s,.) \right) \frac{ds}{\epsilon} + \langle \nabla^{(2)}(F(X_t^H)); h_{t,t+\epsilon}^H \rangle_{L^2(\mathbb{R}^2)}.
\]

Proof. Let \( \xi \in S(\mathbb{R}) \). We have:

\[
S(F(X_t^H)(X_{t+\epsilon}^H - X_t^H))(\xi) = \mathbb{E} \left[ : e^{(\xi_1)} : F(X_t^H)I_2(h_{t,t+\epsilon}^H) \right],
\]

\[
= \mathbb{E} \left[ : e^{(\xi_1)} : F(X_t^H)(\nabla^{(2)})^*(h_{t,t+\epsilon}^H) \right],
\]

\[
= \langle \nabla^{(2)}( : e^{(\xi_1)} : F(X_t^H)); h_{t,t+\epsilon}^H \rangle_{L^2(\mathbb{R}^2) \otimes (L^2)},
\]

\[
= \langle \nabla^{(2)} : e^{(\xi_1)} : F(X_t^H) + 2\xi : \nabla(F(X_t^H)); h_{t,t+\epsilon}^H \rangle_{L^2(\mathbb{R}^2) \otimes (L^2)}
\]

Now we compute separately the three terms which appear in the previous sum. For the first term, we have:

\[
\langle \nabla^{(2)} : e^{(\xi_1)} : F(X_t^H); h_{t,t+\epsilon}^H \rangle_{L^2(\mathbb{R}^2) \otimes (L^2)} = S(F(X_t^H)(\xi)S(I_2(h_{t,t+\epsilon}^H))(\xi),
\]

\[
= S(F(X_t^H) \circ (X_{t+\epsilon}^H - X_t^H))(\xi).
\]

For the second term, we have:

\[
\langle \xi : e^{(\xi_1)} : \nabla(F(X_t^H)); h_{t,t+\epsilon}^H \rangle_{L^2(\mathbb{R}^2) \otimes (L^2)} = \langle \xi \otimes \mathbb{E} \left[ : e^{(\xi_1)} : \nabla(F(X_t^H)) \right]; h_{t,t+\epsilon}^H \rangle_{L^2(\mathbb{R}^2)},
\]

\[
= \int_{\mathbb{R}^2} \xi(x_1) \mathbb{E} \left[ : e^{(\xi_1)} : \nabla(F(X_t^H)(x_2,.)) \right] h_{t,t+\epsilon}^H(x_1, x_2) dx_1 \otimes dx_2,
\]

\[
= d(H) \int_{\mathbb{R}^2} \xi(x_1) G(t, x_2) \left( \int_t^{t+\epsilon} \frac{(s - x_1)^{H-1}}{\Gamma(H/2)} \frac{(s - x_2)^{H-1}}{\Gamma(H/2)} ds \right) dx_1 \otimes dx_2,
\]

\[
= d(H) \int_t^{t+\epsilon} I_+^H(\xi)(s) I_+^H(G(t,.))(s) ds,
\]

\[
= d(H) \int_t^{t+\epsilon} I_+^H(\xi)(s) S(\nabla^H(F(X_t^H)))(s,.)(\xi) ds,
\]

\[
= d(H) S \left( \int_t^{t+\epsilon} D^*_{\delta_o I_+^H} \left( \nabla^H(F(X_t^H))(s,.)) \right) ds \right)(\xi),
\]

where we have set,

\[
G(t, x_2) = \mathbb{E} \left[ : e^{(\xi_1)} : \nabla(F(X_t^H)(x_2,.)) \right],
\]

and used successively Fubini Theorem twice and the \((S)^*\)-integrability of,

\[
s \to D^*_{\delta_o I_+^H} \left( \nabla^H(F(X_t^H))(s,.)) \right),
\]

over \((t, t + \epsilon)\). We note in particular that the key fact is that \((I_+^H \otimes E) \circ | \nabla(F(X_t^H)) | \) is in \( L^2((a,b)) \otimes (L^2) \). For the third term, we have:

\[
\langle : e^{(\xi_1)} : \nabla^{(2)}(F(X_t^H)); h_{t,t+\epsilon}^H \rangle_{L^2(\mathbb{R}^2) \otimes (L^2)} = S \left( \langle \nabla^{(2)}(F(X_t^H)); h_{t,t+\epsilon}^H \rangle_{L^2(\mathbb{R}^2)} \right)(\xi).
\]
In the subsequent subsections, we analyse independently the three terms in the previous decomposition. In particular, we will prove the following strong convergence result which allows us to obtain the explicit decomposition of the forward integral of $F(X^H_t)$ with respect to $X^H_t$.

**Proposition 26.** We have, in $(L^2)$:

$$
\int_a^b F(X^H_t) \circ \frac{X^H_{t+\epsilon} - X^H_t}{\epsilon} dt \xrightarrow{\epsilon \to 0^+} d(H)(\nabla^{(2)})^* \left( \int_a^b F(X^H_t) \frac{(t - \#)^{H-1}(t - \#)^{H-1}}{\Gamma(H/2)^2} dt \right),
$$

$$
2d(H) \int_a^b 2 \left( \int_t^{t+\epsilon} D_{\delta(x,t)} \frac{\nabla^2}{\epsilon} (F(X^H_t))(s,\cdot) \frac{ds}{\epsilon} \right) dt \xrightarrow{\epsilon \to 0^+} B(H) \nabla^2 \left( \int_a^b F'(X^H_t)(t - \#)^{H-1} I_1(l^H_{H,t}) dt \right),
$$

$$
\langle \nabla^{(2)}(F(X^H_t)); h_t^H(t+\epsilon) \rangle_{l^2(R^2)} \xrightarrow{\epsilon \to 0^+} H \int_a^b t^{2H-1} F'(X^H_t) dt + \frac{H}{2}\eta_3(X^H_t) \int_a^b t^{3H-1} F^{(2)}(X^H_t) dt + C(H) \int_a^b I_2(e^{(H)}_{t^2}) F^{(2)}(X^H_t) dt,
$$

with,

$$
l^H_{t,t}(x) = \int_0^t (u - x)^{H/2-1} |t - u|^{H-1} du,
$$

$$
e^{(H)}_{t,t}(x_1, x_2) = \int_0^t \int_0^t (u - x_1)^{H/2-1} (v - x_2)^{H/2-1} \left| u - t \right|^{H-1} \left| v - t \right|^{H-1} dudv,
$$

$$
B(H) = \frac{4d(H)}{(\Gamma(H/2))^2} \sqrt{\frac{H(2H-1)}{2}},
$$

$$
C(H) = \frac{2d(H)}{(\Gamma(H/2))^2} H(2H-1).
$$

**Proof.** This a combination of Propositions 28 and 32 and Lemmas 34 and 36.

As a direct application of Propositions 25 and 26, we obtain the proof of Theorem 3.

### 3.2.1 Second order divergence term

In this section, we prove the following strong convergence result:

$$
\int_a^b F(X^H_t) \circ \frac{X^H_{t+\epsilon} - X^H_t}{\epsilon} dt \xrightarrow{\epsilon \to 0^+} d(H)(\nabla^{(2)})^* \left( \int_a^b F(X^H_t) \frac{(t - \#)^{H-1}(t - \#)^{H-1}}{\Gamma(H/2)^2} dt \right).
$$

For this purpose, we proceed in the following way:

- We find a new representation for $\int_a^b F(X^H_t) \circ \frac{X^H_{t+\epsilon} - X^H_t}{\epsilon} dt$ ensuring that it is an element in $(L^2)$
- Then, we prove the wanted convergence result

First, we have the following lemma.

**Lemma 27.** Let $\epsilon > 0$. We have:

$$
\int_a^b F(X^H_t) \circ \frac{X^H_{t+\epsilon} - X^H_t}{\epsilon} dt = (\nabla^{(2)})^* \left( \int_a^b F(X^H_t) h^H_{t,t+\epsilon} dt \right).
$$
Proof. Let $\xi \in S(\mathbb{R})$. Then, we have:

$$S\left(\left(\nabla^{(2)}\right)^* \left( \int_a^b F(X_t^H) \frac{h^{H+\epsilon}}{\epsilon} dt \right) \right)(\xi) = \langle \left(\nabla^{(2)}\right)^* \left( \int_a^b F(X_t^H) \frac{h^{H+\epsilon}}{\epsilon} dt \right) : e^{(\xi)} \rangle_{(L^2)},$$

$$= \langle \int_a^b F(X_t^H) \frac{h^{H+\epsilon}}{\epsilon} dt : \nabla^{(2)}(e^{(\xi)}) \rangle_{L^2(\mathbb{R}^2) \otimes (L^2)},$$

$$= \langle \int_a^b F(X_t^H) \frac{h^{H+\epsilon}}{\epsilon} dt : \xi \otimes e^{(\xi)} \rangle_{L^2(\mathbb{R}^2) \otimes (L^2)},$$

$$= \int_a^b S(F(X_t^H)) (\xi) S(X_{t+\epsilon}^H - X_t^H)(\xi) \frac{dt}{\epsilon},$$

$$= S\left( \int_a^b F(X_t^H) \left( \frac{X_{t+\epsilon}^H - X_t^H}{\epsilon} \right) dt \right)(\xi).$$

Proposition 28. We have in $(L^2)$:

$$\left(\nabla^{(2)}\right)^* \left( \int_a^b F(X_t^H) \frac{h^{H+\epsilon}}{\epsilon} dt \right) \frac{dt}{\epsilon \rightarrow 0^+} \left( \frac{d(H)\left(\nabla^{(2)}\right)^* \left( \int_a^b F(X_t^H) \frac{(t - \cdot)^{\#_1^{-1}}(t - \cdot)^{\#_2^{-1}}}{\Gamma(H/2)^2} dt \right) \right).$$

Proof. By the second part of Proposition 12 we have to prove the following convergence in $\hat{L}^2(\mathbb{R}^2) \otimes (\mathcal{W}^{2,2})$:

$$\int_a^b F(X_t^H) \frac{h^{H+\epsilon}}{\epsilon} dt \frac{dt}{\epsilon \rightarrow 0^+} \int_a^b F(X_t^H) \frac{(t - \cdot)^{\#_1^{-1}}(t - \cdot)^{\#_2^{-1}}}{\Gamma(H/2)^2} dt.$$

For this purpose, we proceed as in the proof of Theorem 1. Let us consider the following quantity:

$$J_\epsilon = \int_{\mathbb{R}^2} \left\| \int_a^b F(X_t^H) \left( \frac{h^{H+\epsilon}}{\epsilon} - \frac{(t - x_1)^{\#_1^{-1}}(t - x_2)^{\#_2^{-1}}}{\Gamma(H/2)^2} \right) dt \right\|_{(\mathcal{W}^{2,2})}^2 dx_1 dx_2.$$

Then, we have:

$$J_\epsilon = \int_{\mathbb{R}^2} \left( \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} \left( \frac{(s - x_1)^{\#_1^{-1}}}{\Gamma(H/2)^2} - \frac{(t_2 - t_1)^{\#_2^{-1}}}{\Gamma(H/2)^2} \right) ds \right) dt_1 \int_a^b F(X_t^H)$$

$$\times \left( \int_{t_2}^{t_2+\epsilon} \left( \frac{(s - x_1)^{\#_1^{-1}}}{\Gamma(H/2)^2} - \frac{(t_2 - t_1)^{\#_2^{-1}}}{\Gamma(H/2)^2} \right) ds \right) dt_2 \left( \int_{\mathbb{R}^2} \left( \frac{(s - x_1)^{\#_1^{-1}}}{\Gamma(H/2)^2} - \frac{(t_2 - t_1)^{\#_2^{-1}}}{\Gamma(H/2)^2} \right) ds \right) dt_1 \frac{dt_1 dt_2}{\epsilon^2}.$$

Straightforward computations as in the proof of Theorem 1 lead to the following equality:

$$J_\epsilon = \frac{\beta(H/2, 1 - H)^2}{\Gamma(H/2)^4} \int_{(a,b) \times (a,b)} \langle F(X_{t_1}^H); F(X_{t_2}^H) \rangle_{(\mathcal{W}^{2,2})} \left( \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} \left[ |s - r|^{2H-2} - |t_1 - r|^{2H-2} - |t_2 - s|^{2H-2} + |t_2 - t_1|^{2H-2} \right] \frac{ds dr}{\epsilon^2} \right) dt_1 dt_2.$$
To conclude, we proceed as in the proof of Theorem 1. First of all, by Cauchy-Schwarz inequality, we have:

$$\| \langle F(X_{t_1}^H); F(X_{t_2}^H) \rangle_{(W^{2,2})} \| \leq \| F(X_{t_1}^H) \|_{(W^{2,2})} \| F(X_{t_2}^H) \|_{(W^{2,2})}$$

Now, by Meyer inequality, we have:

$$\| F(X_{t_1}^H) \|_{(W^{2,2})}^2 \leq C \left( \| F(X_{t_1}^H) \|_{L^2}^2 + \| \nabla (F(X_{t_1}^H)) \|_{L^2(\mathbb{R}) \otimes (L^2)}^2 + \| \nabla^2 (F(X_{t_1}^H)) \|_{L^2(\mathbb{R}^2) \otimes (L^2)}^2 \right).$$

Since $F$ is infinitely differentiable with polynomial growth (and its derivatives as well) we have the following bounds:

$$\| F(X_{t_1}^H) \|_{L^2}^2 \leq C E \left[ (1 + | X_t^H |)^{2N_1} \right],$$

$$\leq C_1 P_{2N_1} (t^H),$$

$$\| \nabla (F(X_{t_1}^H)) \|_{L^2(\mathbb{R}) \otimes (L^2)} \leq 4 E \left[ F'(X_{t_1}^H)^2 I_2 \left( \int_\mathbb{R} h_t^H (x, x_1) dx \right) \right] + 4 \| h_t^H \|_{L^2(\mathbb{R})}^2 \| F'(X_{t_1}^H) \|_{L^2(\mathbb{R})},$$

$$\leq 4 E \left[ F'(X_{t_1}^H)^2 I_2 \left( \int_\mathbb{R} h_t^H (x, x_1) dx \right) \right] + C_2 t^{2H} Q_{2N_2} (t^H),$$

$$\leq 4 E \left[ F'(X_{t_1}^H)^2 \right] \left( \int_\mathbb{R} h_t^H (x, x_1) dx \right) + C_2 t^{2H} Q_{2N_2} (t^H),$$

$$\leq C_3 (R_{4N_2} (t^H)) \left( \int_0^t \left( \int_0^t h_t^H (x, x_1) h_t^H (x, x_2) dx \right) dx_1 dx_2 \right) + C_2 t^{2H} Q_{2N_2} (t^H),$$

$$\leq C_3 (H) \left( R_{4N_2} (t^H) \right) \left( \int_0^t \left( \int_0^t h_t^H (x, x_1) h_t^H (x, x_2) dx \right) dx_1 dx_2 \right) + C_2 t^{2H} Q_{2N_2} (t^H),$$

$$\leq C_3 (H) \left( R_{4N_2} (t^H) \right) \left( \int_0^t \left( \int_0^t h_t^H (x, x_1) h_t^H (x, x_2) dx \right) dx_1 dx_2 \right) + C_2 t^{2H} Q_{2N_2} (t^H),$$

$$\leq C_3 (H) \left( R_{4N_2} (t^H) \right) \left( \int_0^t \left( \int_0^t h_t^H (x, x_1) h_t^H (x, x_2) dx \right) dx_1 dx_2 \right) + C_2 t^{2H} Q_{2N_2} (t^H),$$

with $N_1, N_2 \geq 1$ and $P_{2N_1}, Q_{2N_2}$ and $R_{4N_2}$ are polynomials of respective degrees $2N_1 2N_2$ and $4N_2$. We obtain a similar bound for $\| \nabla^2 (F(X_{t_1}^H)) \|_{L^2(\mathbb{R}) \otimes (L^2)}^2$. Thus, we have the following bound:

$$\| \langle F(X_{t_1}^H); F(X_{t_2}^H) \rangle_{(W^{2,2})} \| \leq C_H G(t^H) G(t^H),$$

where $G$ is a positive valued function with power growth at most. This implies, the following estimate on $| J_s |$:

$$| J_s | \leq C_H \int_{(a,b) \times (a,b)} G(t_1^H) G(t_2^H) \int_{t_1}^{t_{1+\epsilon}} \int_{t_2}^{t_{2+\epsilon}} \left[ | s-r|^{2H-2} - | t_1-r|^{2H-2} - | t_2-\epsilon|^{2H-2} + | t_2-\epsilon^{2H-2} \right] ds \frac{dr}{\epsilon^2} dt_1 dt_2.$$

Thanks to Lebesgue dominated convergence theorem and similar arguments as in the proof of Theorem 1 we obtain the result.

### 3.2.2 Trace term of order 1

In this section, we want to prove that:

$$2d(H) \int_a^b \left( \int_t^{t+\epsilon} D_s^\delta \nabla \frac{B(t)}{\epsilon} (F(X_{t_1}^H)(s, *)) \frac{ds}{\epsilon} \right) \left( \frac{B(t)}{\epsilon^2} \right) \left( I_s^H \right) dt \left( L^2 \right) \epsilon \rightarrow 0^+ B(H) \nabla^* \left( \int_a^b F'(X_{t_1}^H)(t-.) \frac{| H - 1 |}{| H - 1 |} I_s^H dt \right).$$

For this purpose, we proceed in the following way:
First, we introduce a sequence of kernels \((K^k_t(s,r))_k\) defined by
\[
\forall s,r \in (t, +\infty), K^0_t(s,r) = |s-r|^{H-1}, K^1_t(s,r) = \int_0^t |s-u|^{H-1}|r-u|^{H-1}du
\]
\[
\forall k \geq 3, K^{k-2}_t(s,r) = \int_0^t \cdots \int_0^t |s-x_1|^{H-1}|x_2-x_1|^{H-1} \cdots |x_{k-2}-x_{k-3}|^{H-1}|r-x_{k-2}|^{H-1}dx_1 \cdots dx_{k-2}.
\]

**Remark 29.**
- We note that for any \(t \in (a,b)\), we have:
  \[
  K^1_t(t,t) = \int_0^t |t-u|^{H-1}|t-u|^{H-1}du = \frac{t^{2H-1}}{2H-1} < \infty,
  \]
  \[
  K^2_t(t,t) \leq \left( \int_0^t \int_0^t |x_1-x_2|^{2H-2}dx_1dx_2 \right) \left( \int_0^t \int_0^t |x_1-x_2|^{2H-2}dx_1dx_2 \right) \leq \frac{t^{2H-1}}{H(2H-1)(2H-1)} < \infty.
  \]
- Moreover, by Lebesgue dominated convergence theorem, one can show that \(K^1_t(\cdot,\cdot)\) and \(K^2_t(\cdot,\cdot)\) are continuous on \([t, +\infty) \times [t, +\infty)\).
- Finally, for any \(s \in [t, +\infty)\), we have:
  \[
  K^2_t(s,s) \leq K^2(t,t).
  \]

Then, we have the following technical lemma.

**Lemma 30.** We have:
\[
\nabla^H_T(F(X^H_t))(s,\omega) = \frac{2}{\Gamma(\frac{H}{2})} \sqrt{\frac{H(2H-1)}{2}} I_1(\int_0^t (u-)^{\frac{H}{2}-1}|s-u|^{H-1}du)F'(X^H_t).
\]

**Proof.** By Proposition [10] and by definition of the operator \(\nabla^H_T\), we have:
\[
\nabla^H_T(F(X^H_t))(s,\omega) = \nabla^H_T(X^H_t)(s,\omega)F'(X^H_t)(\omega).
\]
Moreover, by Proposition [13] with \(\alpha = H/2\), we obtain:
\[
\nabla^H_T(F(X^H_t))(s,\omega) = 2I_1(0,\frac{H}{2}) (f^H_t(\cdot,\cdot)) F'(X^H_t)(\omega),
\]
\[
\nabla^H_T(F(X^H_t))(s,\omega) = \frac{2d(H)}{(\Gamma(\frac{H}{2}))^3} I_1(\int_0^t (u-)^{\frac{H}{2}-1}\left( \int_R (s-x)^{\frac{H}{2}-1}dx \right)du)F'(X^H_t)(\omega),
\]
\[
\nabla^H_T(F(X^H_t))(s,\omega) = \frac{2d(H)\beta(1-H, \frac{H}{2})}{(\Gamma(\frac{H}{2}))^3} I_1(\int_0^t (u-)^{\frac{H}{2}-1}|u-s|^{H-1}du)F'(X^H_t)(\omega),
\]
\[
\nabla^H_T(F(X^H_t))(s,\omega) = \frac{2}{\Gamma(\frac{H}{2})} \sqrt{\frac{H(2H-1)}{2}} I_1(\int_0^t (u-)^{\frac{H}{2}-1}|s-u|^{H-1}du)F'(X^H_t)(\omega).
\]
\[
\square
\]
Thus, we obtain the following representation.

**Proposition 31.** Let \( t \in (a, b) \) and \( \epsilon > 0 \) such that \( t + \epsilon < b \). We have:

\[
\int_t^{t+\epsilon} D_s \frac{\partial}{\partial s} \left( \nabla^H \left( F(X^H_t)(s) \right) \right) \frac{ds}{\epsilon} = \frac{1}{\epsilon} \left( \nabla^H \right)^*(I_{(t,t+\epsilon)} \nabla^H \left( F(X^H_t) \right)).
\]

**Proof.** First of all, we have:

\[
\nabla^H \left( F(X^H_t)(s) \right) = 2\frac{C}{(1+\epsilon)^2} \int_{(0, 1]} \frac{H(2H-1)}{2} I_t \left( \int_0^t (u-s)|s-u|H^{-1} du \right) F'(X^H_t).
\]

Thus, we obtain the following representation.

\[
\int_t^{t+\epsilon} D_s \frac{\partial}{\partial s} \left( \nabla^H \left( F(X^H_t)(s) \right) \right) \frac{ds}{\epsilon} = \frac{1}{\epsilon} \left( \nabla^H \right)^*(I_{(t,t+\epsilon)} \nabla^H \left( F(X^H_t) \right)),
\]

where \( t_{s,t}(\cdot) = \int_0^t (u-s)^{1/2} |s-u|^{H-1} du \). We want to apply Proposition 19 in order to obtain the result. For this purpose, we have to prove that:

\[
\int_t^{t+\epsilon} \|I_1(t_{s,t}(\cdot)) \|_{\mathcal{W}_{1,2}}^2 ds < +\infty.
\]

By Proposition 1.5 of [14], we have:

\[
\|I_1(t_{s,t}(\cdot)) F'(X^H_t)\|_{\mathcal{W}_{1,2}} \leq C \|I_1(t_{s,t}(\cdot)) \|_{\mathcal{W}_{1,4}} \|F'(X^H_t)\|_{\mathcal{W}_{1,4}}
\]

Let us estimate \( \|I_1(t_{s,t}(\cdot)) \|_{\mathcal{W}_{1,4}} \). We have:

\[
\|I_1(t_{s,t}(\cdot)) \|_{\mathcal{W}_{1,4}} = \mathbb{E}[\|I_1(t_{s,t}(\cdot))\|^4] + \mathbb{E}[\|\nabla(I_1(t_{s,t}(\cdot))\|^4_{L^2(\mathbb{R})}]
\]

Since \( I_1(t_{s,t}(\cdot)) \) is a Gaussian random variable, we have:

\[
\mathbb{E}[\|I_1(t_{s,t}(\cdot))\|^4] = 3\mathbb{E}[\|I_1(t_{s,t}(\cdot))\|^2]^2,
\]

\[
= 3(1-H, \frac{H}{2}) \int_0^t \int_0^t |s-u|^{H-1} |s-v|^{H-1} |u-v|^{H-1} du dv)^2,
\]

\[
= 3(1-H, \frac{H}{2}) K^2_t(s, s)^2.
\]

Moreover, we have,

\[
\|\nabla(I_1(t_{s,t}(\cdot))\|^2_{L^2(\mathbb{R})} = \|t_{s,t}\|_{L^2(\mathbb{R})} = (1 - H, \frac{H}{2}) K^2_t(s, s)^2.
\]

Thus,

\[
\mathbb{E}[\|\nabla(I_1(t_{s,t}(\cdot))\|^4_{L^2(\mathbb{R})}] = \beta(1-H, \frac{H}{2})^2 K^2_t(s, s)^2.
\]

Therefore, we obtain the following equality:

\[
\|I_1(t_{s,t}(\cdot))\|_{\mathcal{W}_{1,4}} = \left( 4\beta(1-H, \frac{H}{2})^2 K^2_t(s, s)^2 \right)^{1/4},
\]

\[
\|I_1(t_{s,t}(\cdot))\|_{\mathcal{W}_{1,4}} = \left( \beta(1-H, \frac{H}{2}) K^2_t(s, s) \right)^{1/2}.
\]

Moreover, since \( F \) is an infinitely continuously differentiable function on \( \mathbb{R} \) such that \( F \) and its derivatives have polynomial growth, \( F'(X^H_t) \in (\mathcal{W}^{\infty,\infty}) \). Thus, \( \|F'(X^H_t)\|_{\mathcal{W}_{1,4}} \) is finite and independent of \( s \). Finally, the kernel \( K^2_t \) is continuous on \([t, +\infty) \times [t, +\infty) \). Therefore,

\[
\int_t^{t+\epsilon} \|I_1(t_{s,t}(\cdot)) F'(X^H_t)\|_{\mathcal{W}_{1,2}}^2 ds \leq C(1-H, \frac{H}{2}) \|F'(X^H_t)\|_{\mathcal{W}_{1,4}}^2 \int_t^{t+\epsilon} K^2_t(s, s) ds < +\infty.
\]

This concludes the proof.
A direct application of Proposition 18 leads to the following equality:

\[
\frac{1}{\epsilon}(\nabla \frac{H}{2}^*) (t_{r,t+}) \nabla \frac{H}{2} (F(X_t^H))) = \sqrt{\frac{H(2H-1)}{2} \frac{2}{\Gamma(\frac{H}{2})^2} \nabla^* \left( F^r(X_t^H) \int_t^{t+\epsilon} (r-s)^{\frac{H}{2}} - 1 \left( I_1(t_{r,t}^H) \frac{dr}{\epsilon} \right) \right).}
\]

**Proposition 32.** We have in \((L^2)\):

\[
\nabla^* \left( \int_a^b F^r(X_t^H) \left( \int_t^{t+\epsilon} (r-.)^\frac{H}{2} - 1 \left( I_1(t_{r,t}^H) \frac{dr}{\epsilon} \right) dt \right) \right) \to \nabla^* \left( \int_a^b F^r(X_t^H) (t-.)^\frac{H}{2} - 1 \left( I_1(t_{r,t}^H) \right) dt \right).
\]

**Proof.** In order to prove the proposition, since the operator \(\nabla^*\) is continuous from \(L^2(\mathbb{R}) \otimes (W^{1,2})\) to \((L^2)\), we will show that, in \(L^2(\mathbb{R}) \otimes (W^{1,2})\):

\[
\int_a^b F^r(X_t^H) \left( \int_t^{t+\epsilon} (r-.)^\frac{H}{2} - 1 \left( I_1(t_{r,t}^H) \frac{dr}{\epsilon} \right) dt \right) \to \int_a^b F^r(X_t^H) (t-.)^\frac{H}{2} - 1 \left( I_1(t_{r,t}^H) \right) dt.
\]

For this purpose, we denote them respectively by \(I_r(\cdot)\) and \(I(\cdot)\). We have:

\[
\int_R \| I_r(x) - I(x) \|^2_{W^{1,2}} dx = \int_R \int_R \int_{(a,b) \times (a,b)} \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} (F^r(X_{t_1}^H) (|r-s|^{H-1} \langle F^r(X_{t_1}^H) I_1(l_{s,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{s,t}) \rangle)) dsdr dt_1 dt_2 dx.
\]

Integrating with respect to \(x\), we obtain:

\[
\int_R \| I_r(x) - I(x) \|^2_{W^{1,2}} dx = \beta(1-H, \frac{H}{2}) \int_a^b \int_a^b \int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} (|r-t_2|^{H-1} \langle F^r(X_{t_1}^H) I_1(l_{t_1,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{t_1,t}) \rangle) dsdr dt_1 dt_2.
\]

To prove the result, we want to apply Lebesgue dominated convergence theorem. For this purpose, we have to show, at first, that for almost every \((t_1, t_2) \in (a, b) \times (a, b)\):

\[
\int_{t_1}^{t_1+\epsilon} \int_{t_2}^{t_2+\epsilon} (|r-s|^{H-1} \langle F^r(X_{t_1}^H) I_1(l_{s,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{s,t}) \rangle) dsdr dt_1 dt_2 = 0
\]

In order to do so, we will show that \((r, s) \to \langle F^r(X_{t_1}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle\) is continuous in \((t_1^+, t_2^+)\). Let \((r, s) \in [t_1, +\infty) \times [t_2, +\infty)\), we have:

\[
(I) = |\langle F^r(X_{t_1}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2} \leq |\langle F^r(X_{t_1}^H) \rangle - \langle F^r(X_{t_2}^H) \rangle|_{L^2} + |\langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2} + |\langle F^r(X_{t_2}^H) \rangle - \langle F^r(X_{t_2}^H) \rangle|_{L^2} + |\langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2}.
\]

Using Proposition 1.5.6 of [14], we obtain for the first term on the right-hand side of the previous inequality:

\[
|\langle F^r(X_{t_1}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2} \leq C |\langle F^r(X_{t_1}^H) \rangle - \langle F^r(X_{t_2}^H) \rangle|_{L^2} + |\langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2}.
\]

Using Proposition 1.5.6 of [14], we obtain for the first term on the right-hand side of the previous inequality:

\[
||F^r(X_{t_1}^H) I_1(l_{r,t})|_{L^2} \leq C |\langle F^r(X_{t_1}^H) \rangle - \langle F^r(X_{t_2}^H) \rangle|_{L^2} + |\langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2}.
\]

Using Proposition 1.5.6 of [14], we obtain for the first term on the right-hand side of the previous inequality:

\[
||F^r(X_{t_1}^H) I_1(l_{r,t})|_{L^2} \leq C |\langle F^r(X_{t_1}^H) \rangle - \langle F^r(X_{t_2}^H) \rangle|_{L^2} + |\langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2}.
\]

Using Proposition 1.5.6 of [14], we obtain for the first term on the right-hand side of the previous inequality:

\[
||F^r(X_{t_1}^H) I_1(l_{r,t})|_{L^2} \leq C |\langle F^r(X_{t_1}^H) \rangle - \langle F^r(X_{t_2}^H) \rangle|_{L^2} + |\langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2}.
\]

Using Proposition 1.5.6 of [14], we obtain for the first term on the right-hand side of the previous inequality:

\[
||F^r(X_{t_1}^H) I_1(l_{r,t})|_{L^2} \leq C |\langle F^r(X_{t_1}^H) \rangle - \langle F^r(X_{t_2}^H) \rangle|_{L^2} + |\langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2}.
\]

Using Proposition 1.5.6 of [14], we obtain for the first term on the right-hand side of the previous inequality:

\[
||F^r(X_{t_1}^H) I_1(l_{r,t})|_{L^2} \leq C |\langle F^r(X_{t_1}^H) \rangle - \langle F^r(X_{t_2}^H) \rangle|_{L^2} + |\langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle - \langle F^r(X_{t_2}^H) I_1(l_{r,t}) \rangle|_{L^2}.
\]
Moreover, as in the proof of the previous proposition and since $K_2^2(s, s) \leq K_2^2(t_2, t_2)$ for any $s \in [t_2, +\infty)$, we obtain:

$$
\|I_1(t_{s,t_2}^H)\|_{1,4} = \sqrt{2}\|I_{s,t_2}^H\|_{L^2(\mathbb{R})} = \sqrt{2\beta(1 - \frac{H}{2})K_2^2(s, s)},
$$

$$
\leq C_H (K_2^2(t_2, t_2))^\frac{1}{2} = C_H (t_2)^{\frac{2H-1}{2}}(K_1^2(1, 1))^\frac{1}{2}.
$$

Similarly, we have,

$$
\|I_1(l_{r,t_1}^H) - I_1(l_{t_{s,t_1}}^H)\|_{1,4} = \sqrt{2}\|l_{r,t_1}^H - l_{t_{s,t_1}}^H\|_{L^2(\mathbb{R})} = \sqrt{2\beta(1 - \frac{H}{2})}[K_2^2(r, r) + K_2^2(t_1, t_1) - 2K_2^2(r, t_1)].
$$

Thus, we have:

$$
(I) \leq C_H \|F'(X_t^H)\|_{1,4}\|F'(X_t^H)\|_{1,4}(t_2)\left(1/(1 + \sqrt{2H-1}) (K_1^2(1, 1))^{1/2} \sqrt{[K_2^2(r, r) + K_2^2(t_1, t_1) - 2K_2^2(r, t_1)]} + C_H \|F'(X_t^H)\|_{1,4}\|F'(X_t^H)\|_{1,4}(t_1)\left(1/(1 + \sqrt{2H-1}) (K_1^2(1, 1))^{1/2} \sqrt{[K_2^2(s, s) + K_2^2(t_2, t_2) - 2K_2^2(s, t_2)]}.
$$

Since, the kernel $K_2^2(s, r)$ is continuous on $[t, +\infty) \times [t, +\infty)$, we obtain the continuity of $(r, s) \rightarrow \langle F'(X_t^H)I_1(l_{r,t_1}^H); F'(X_t^H)I_1(l_{s,t_2}^H)\rangle_{(\mathbb{W}^1)^2}$ in $(t_1^+, t_2^+)$. Consequently, we obtain that for any $t_1 \neq t_2$ in $(a, b) \times (a, b)$:

$$
\int_{t_1}^{t_1+} \int_{t_2}^{t_2+} \left(\left|r - s\right|^{H-1}(F'(X_t^H)I_1(l_{r,t_1}^H); F'(X_t^H)I_1(l_{s,t_2}^H)\right)_{(\mathbb{W}^1)^2} + \left|t_1 - t_2\right|^{H-1}(F'(X_t^H)I_1(l_{r,t_1}^H); F'(X_t^H)I_1(l_{s,t_2}^H)\right)_{(\mathbb{W}^1)^2} + \left|t_1 - t_2\right|^{H-1}(F'(X_t^H)I_1(l_{r,t_1}^H); F'(X_t^H)I_1(l_{s,t_2}^H)\right)_{(\mathbb{W}^1)^2}) \frac{dsdr}{t_2 - t_1} \rightarrow 0
$$

To conclude, we need to check the dominating condition by bounding the previous integral which we denote by (II). To this end, we have to bound, at first:

- $m_{r,s} = \langle F'(X_t^H)I_1(l_{r,t_1}^H); F'(X_t^H)I_1(l_{s,t_2}^H)\rangle_{(\mathbb{W}^1)^2}$,
- $m_{r,t_2} = \langle F'(X_t^H)I_1(l_{r,t_1}^H); F'(X_t^H)I_1(l_{s,t_2}^H)\rangle_{(\mathbb{W}^1)^2}$,
- $m_{t_1,s} = \langle F'(X_t^H)I_1(l_{r,t_1}^H); F'(X_t^H)I_1(l_{s,t_2}^H)\rangle_{(\mathbb{W}^1)^2}$,
- $m_{t_1,t_2} = \langle F'(X_t^H)I_1(l_{r,t_1}^H); F'(X_t^H)I_1(l_{s,t_2}^H)\rangle_{(\mathbb{W}^1)^2}$.

Then, we have:

$$
|m_{r,s}| \leq C\|F'(X_t^H)\|_{1,4}\|F'(X_t^H)\|_{1,4}\|I_1(l_{r,t_1}^H)\|_{1,4}\|I_1(l_{s,t_2}^H)\|_{1,4},
$$

$$
|m_{r,s}| \leq C_H \|F'(X_t^H)\|_{1,4}\|F'(X_t^H)\|_{1,4}\sqrt{K_2^2(r, r)K_2^2(s, s)},
$$

$$
|m_{r,s}| \leq C_H \|F'(X_t^H)\|_{1,4}\|F'(X_t^H)\|_{1,4}\sqrt{K_2^2(t_1, t_1)K_2^2(t_2, t_2)},
$$

$$
|m_{r,s}| \leq C_H \|F'(X_t^H)\|_{1,4}\|F'(X_t^H)\|_{1,4}\left(1/(1 + \sqrt{2}) (K_1^2(1, 1))^{1/2} \sqrt{[K_2^2(s, s) + K_2^2(t_2, t_2) - 2K_2^2(s, t_2)]}ight).
$$

Moreover, for any $i \in \{1, 2\}$, we have, by definition:

$$
\|F'(X_t^H)\|_{1,4} = \left(\|F'(X_t^H)\|^4 + \|\nabla(F'(X_t^H))\|^4\right)^{1/4}
$$

26
By hypothesis, $F$ and all its derivatives have polynomial growth. Then, using Hypercontractivity,

$$
\mathbb{E}[|F''(X_{t_i}^H)|^4] \leq C \left( 1 + 4N_1 t_i^H + C_2^{4N_1} t_i^{2H} + \sum_{p=3}^{4N_1} C_p^{4N_1} t_i^{Hp}(p-1)^p \right),
$$

$$
\mathbb{E}[|F''(X_{t_i}^H)|^4] \leq CP_{N_1}(t_i^H),
$$

for some $C > 0$, $N_1 \geq 1$ and $P_{N_1}(\cdot)$ a polynomial of degree $4N_1$ with strictly positive coefficients. Moreover, we have:

$$
\|\nabla(F'(X_{t_i}^H))\|_{L^2(\mathbb{R})} = 2|F^{(2)}(X_{t_i}^H)| \left( \int_{\mathbb{R}} (I_2(f_{t_i}^H(x,.)))^2 \, dx \right)^{\frac{1}{2}},
$$

$$
= 2|F^{(2)}(X_{t_i}^H)| \left( I_2(\int_{\mathbb{R}} f_{t_i}^H(x,.)) \otimes f_{t_i}^H(x,*) \, dx + \|f_{t_i}^H\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}},
$$

$$
= 2|F^{(2)}(X_{t_i}^H)| \left( I_2(\int_{\mathbb{R}} f_{t_i}^H(x,.)) \otimes f_{t_i}^H(x,*) \, dx + \frac{t_i^{2H}}{2} \right)^{\frac{1}{2}}.
$$

Thus,

$$
\mathbb{E}[\|\nabla(F'(X_{t_i}^H))\|^4_{L^2(\mathbb{R})}] = 16\mathbb{E} \left[ |F^{(2)}(X_{t_i}^H)|^4 \left( I_2(\int_{\mathbb{R}} f_{t_i}^H(x,.)) \otimes f_{t_i}^H(x,*) \, dx + \frac{t_i^{2H}}{2} \right)^2 \right],
$$

$$
\leq 32\mathbb{E} \left[ |F^{(2)}(X_{t_i}^H)|^4 \left( I_2(\int_{\mathbb{R}} f_{t_i}^H(x,.)) \otimes f_{t_i}^H(x,*) \, dx \right)^2 \right],
$$

$$
+ 8t_i^{4H} \mathbb{E} \left[ |F^{(2)}(X_{t_i}^H)|^4 \right].
$$

As previously, we obtain, for some $C > 0$, $N_2 \geq 1$ and $Q_{N_2}$ a polynomial of degree $4N_2$ with strictly positive coefficients:

$$
\mathbb{E} \left[ |F^{(2)}(X_{t_i}^H)|^4 \right] \leq CQ_{N_2}(t_i^H)
$$

Moreover, by Cauchy-Schwarz inequality, we have:

$$
\mathbb{E} \left[ |F^{(2)}(X_{t_i}^H)|^4 \left( I_2(\int_{\mathbb{R}} f_{t_i}^H(x,.)) \otimes f_{t_i}^H(x,*) \, dx \right)^2 \right] \leq \mathbb{E} \left[ |F^{(2)}(X_{t_i}^H)|^4 \right] \mathbb{E} \left[ \left( I_2(\int_{\mathbb{R}} f_{t_i}^H(x,.)) \otimes f_{t_i}^H(x,*) \, dx \right)^4 \right]^{\frac{1}{2}},
$$

$$
\leq C(R_{N_3}(t_i^H)) \mathbb{E} \left[ \left( I_2(\int_{\mathbb{R}} f_{t_i}^H(x,.)) \otimes f_{t_i}^H(x,*) \, dx \right)^4 \right]^{\frac{1}{2}},
$$

$$
\leq C(R_{N_3}(t_i^H)) \mathbb{E} \left[ \left( I_2(\int_{\mathbb{R}} f_{t_i}^H(x,.)) \otimes f_{t_i}^H(x,*) \, dx \right)^2 \right],
$$

$$
\leq C(R_{N_3}(t_i^H)) \left( \int_{\mathbb{R}^2} \left( \int_{0}^{t_i} \int_{0}^{t_i} f_{t_i}^H(y,x_1) f_{t_i}^H(y,x_2) \, dy \, dx_1 \, dx_2 \right) \right),
$$

$$
\leq C_H(R_{N_3}(t_i^H)) \left( \int_{\mathbb{R}^2} \left( \int_{0}^{t_i} \int_{0}^{t_i} \int_{0}^{t_i} (r_1 - r_2)^{H-1} |r_1 - r_2|^2 |r_1 - r_3|^2 \, dr_1 \, dr_2 \, dr_3 \right) \right),
$$

$$
\leq C_H(R_{N_3}(t_i^H)) \left( \int_{0}^{t_i} \int_{0}^{t_i} \int_{0}^{t_i} |r_1 - r_2 |^{H-1} |r_2 - r_3 |^{H-1} \times |r_3 - r_4 |^{H-1} |r_4 - r_1 |^{H-1} \right),
$$

$$
\leq C_H(R_{N_3}(t_i^H)) \left( t_i^{4H} \right).
$$
where $N_3 \geq 1$ and $R_{N_3}$ is a polynomial of degree $8N_3$ with strictly positive coefficients. Consequently, we obtain:

$$|m_{r,s}| \leq C_H \prod_{i=1}^{2} (t_i)^{3H-1} |P_{N_1}(t_i^H) + t_i^{AH}Q_{N_2}(t_i^H) + t_i^{AH}(R_{N_3}(t_i^H))^{1/2}|^{1/2}. $$

Similar bounds hold for $|m_{t_1,s}|$, $|m_{r,t_2}|$ and $|m_{t_1,t_2}|$. Therefore, using the fact that $H - 1 < 0$, we have:

$$|(II)| \leq C_H p(t_1, t_2)|t_1 - t_2|^{H-1},$$

where $p(t_1, t_2) = \prod_{i=1}^{2} (t_i)^{(3H-1)/2}|P_{N_1}(t_i^H) + t_i^{AH}Q_{N_2}(t_i^H) + t_i^{AH}(R_{N_3}(t_i^H))^{1/2}|^{1/2}$. Standard computations lead to the following bound:

$$\int_{(a,b)\times(a,b)} p(t_1, t_2)|t_1 - t_2|^{H-1}dt_1dt_2 \leq C_H \|p\|_{\infty,[a,b] \times [a,b]}(b-a)^{H+1} < +\infty.$$  

This concludes the proof of the proposition.

\[ \square \]

### 3.2.3 Trace term of order 2

The strong convergence of the trace term of order 2 is easier to handle with. As we will see from the next computations, it admits the following representation:

$$\forall \epsilon > 0, \int_{a}^{b} \langle \nabla^{(2)} (F(X_t^H)); h_{t,\epsilon}^H \rangle_{L^2(\mathbb{R}^2)} dt = d(H) \int_{a}^{b} \left( \int_{t}^{t+\epsilon} \nabla^{(2)} (F(X_t^H))(s, s) \frac{ds}{\epsilon} \right) dt,$$

Then, the first step is to compute $\nabla^{(2), H/2}(F(X_t^H))$ as done in the next technical lemma.

**Lemma 33.** We have:

$$d(H)\nabla^{(2), H/2}(F(X_t^H))(s_1, s_2, \omega) = H(2H - 1)K^1_t(s_1, s_2)F'(X_t^H) + 4\left( \frac{H(2H - 1)}{2} \right)^3 K^2_t(s_1, s_2)F^{(2)}(X_t^H) + \frac{2d(H)}{(\Gamma(\frac{H}{2}))^2} H(2H - 1)I_2(t) \int_{0}^{t} \int_{0}^{t} (u - s)^{-H+1} \, du \, dv F^{(2)}(X_t^H).$$

**Proof.** Using Proposition [13] and the definition of the operator $\nabla^{(2), H/2}$, we have:

$$\nabla^{(2), H/2}(F(X_t^H))(s_1, s_2, \omega) = \nabla^{(2), H/2}(F(X_t^H))(s_1, s_2, \omega)F'(X_t^H) + \nabla^{(2), H/2}(F(X_t^H))(s_1, \omega)\nabla^{(2), H/2}(F(X_t^H))(s_2, \omega)F^{(2)}(X_t^H),$$

Using Proposition [14] and Proposition [16] we have:

$$\nabla^{(2), H/2}(F(X_t^H))(s_1, s_2, \omega) = 2I_{++}^{(H/2)}(f_{t}^H)(s_1, s_2)F'(X_t^H) + 4I_1(1_{++}^{(0, H/2)}(f_{t}^H)(, s_1))I_1(1_{++}^{(0, H/2)}(f_{t}^H)(, s_2))F^{(2)}(X_t^H).$$
By the multiplication formula from Malliavin calculus (Proposition 1.1.3 of [14]), we get:

\[ \nabla^{(2), \frac{dH}{d\omega}}(F(X^H_t))(s_1, s_2, \omega) = 24I_{29}^{(2)}(f^H_t)(s_1, s_2)F'(X^H_t) + 4I_{29}^{(2)}(f^H_t)(s_1, s_2)\otimes I_{29}^{(2)}(f^H_t)(s_1, s_2) \]

\[ + \left< I_{29}^{(0, \frac{dH}{d\omega}}(f^H_t)(s_1); I_{29}^{(0, \frac{dH}{d\omega}}(f^H_t)(s_2) \right> F^{(2)}(X^H_t) \]

\[ = 2\frac{d(H)}{(\Gamma(\frac{H}{2}))^2} H(2H-1) I_2(\int_0^H\int_0^H (u-x)^{\frac{H-1}{2}}(v-x)^{\frac{H-1}{2}}|u-s_1|^{H-1}|v-s_2|^{H-1}dudv)F^{(2)}(X^H_t) \]

\[ + \frac{2}{(\Gamma(\frac{H}{2}))^2} H(2H-1) I_2(\int_0^H\int_0^H (u-x)^{\frac{H-1}{2}}(v-x)^{\frac{H-1}{2}}|u-s_1|^{H-1}|v-s_2|^{H-1}dudv)F^{(2)}(X^H_t) \]

Thus,

\[ d(H)\nabla^{(2), \frac{dH}{d\omega}}(F(X^H_t))(s_1, s_2, \omega) = H(2H-1)K^2_{1}(s_1, s_2)F'(X^H_t) + 4\left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 K^2_{1}(s_1, s_2)F^{(2)}(X^H_t) \]

\[ + \frac{2d(H)}{(\Gamma(\frac{H}{2}))^2} H(2H-1) I_2(\int_0^H\int_0^H (u-x)^{\frac{H-1}{2}}(v-x)^{\frac{H-1}{2}}|u-s_1|^{H-1}|v-s_2|^{H-1}dudv)F^{(2)}(X^H_t) \]

Then, we consider separately the strong convergence of the appropriate terms coming from the previous decomposition.

**Lemma 34.** We have a.s. and in \((L^2)\):

\[ H(2H-1)\int_a^b \left( \int_t^{t+\epsilon} K^1_t(s, s)\frac{ds}{\epsilon} \right) F'(X^H_t)dt \to H^{2H-1} F'(X^H_t)dt, \]

\[ 4\left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 \int_a^b \left( \int_t^{t+\epsilon} K^2_t(s, s)\frac{ds}{\epsilon} \right) F^{(2)}(X^H_t)dt \to \frac{H}{2} K^2_{3}K^1_t(X^H_t)\int_a^b \epsilon^{2H-1} F^{(2)}(X^H_t)dt. \]

**Proof.** By Remark [29] and Lebesgue dominated convergence theorem, we have:

\[ H(2H-1)\int_a^b \left( \int_t^{t+\epsilon} K^1_t(s, s)\frac{ds}{\epsilon} \right) F'(X^H_t)dt \to H(2H-1) \int_a^b K^1_t(t, t)F'(X^H_t)dt, \]

\[ 4\left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 \int_a^b \left( \int_t^{t+\epsilon} K^2_t(s, s)\frac{ds}{\epsilon} \right) F^{(2)}(X^H_t)dt \to 4\left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 \int_a^b K^2_t(t, t)F^{(2)}(X^H_t)dt. \]

By scaling property of the kernels, we have:

\[ H(2H-1)\int_a^b K^1_t(t, t)F'(X^H_t)dt = H(2H-1) \int_0^1 (1-x)^{2H-2} dx \int_a^b \epsilon^{2H-1} F'(X^H_t)dt, \]

\[ 4\left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 \int_a^b K^2_t(t, t)F^{(2)}(X^H_t)dt = 4\left( \sqrt{\frac{H(2H-1)}{2}} \right)^3 K^2_{1}(1, 1) \int_a^b \epsilon^{2H-1} F^{(2)}(X^H_t)dt. \]
Moreover, we note that:

\[ K_1^2(1, 1) = H \int_0^1 \int_0^1 \int_0^1 |x_1 - x_2|^{|H-1|} |x_2 - x_3|^{|H-1|} |x_3 - x_1|^{|H-1|} dx_1 dx_2 dx_3. \]

Thus,

\[ H(2H - 1) \int_a^b K_1^2(t, t) F'(X_t^H) dt = H \int_a^b t^{2H-1} F'(X_t^H) dt, \]

\[ 4\left( \frac{H(2H - 1)}{2} \right)^2 \int_a^b K_1^2(t, t) F'^2(X_t^H) dt = \frac{H}{2} \kappa_3(X_1^H) \int_a^b t^{3H-1} F'^2(X_t^H) dt. \]

Finally, we conclude by two lemmas regarding the strong convergence of the last term of the decomposition of \( \nabla'^{(2)H/2}(X_t^H) \) from Lemma 33.

**Lemma 35.** Let \( t \in (a, b) \). We have:

\[
\int_0^t \int_0^t \left( u - \cdot \right)^{H-1} \left( v - \cdot \right)^{H-1} |u - s|^{H-1} |v - s|^{H-1} dudv \frac{ds}{\epsilon} \\
\to \frac{L^2}{\epsilon} \to \int_0^t \int_0^t \left( u - \cdot \right)^{H-1} \left( v - \cdot \right)^{H-1} |u - t|^{H-1} |v - t|^{H-1} dudv.
\]

**Proof.** First of all, note that:

\[
\int_0^t \int_0^t \left( u - \cdot \right)^{H-1} \left( v - \cdot \right)^{H-1} |u - t|^{H-1} |v - t|^{H-1} dudv \in L^2(\mathbb{R}^2).
\]

Indeed, we have:

\[
\int_0^t \int_0^t \int_0^t |u - t|^{H-1} |v - t|^{H-1} |u' - t|^{H-1} |v' - t|^{H-1} |u - u'|^{H-1} |v - v'|^{H-1} dudvdu' dv' = (K_1^2(t, t))^2 < \infty.
\]

Let \( \epsilon > 0 \) have:

\[
\mathbb{E} \left[ \left( \int_t^{t+\epsilon} I_2(e_t^H(\cdot, \cdot)) \frac{ds}{\epsilon} - I_2(e_t^H(\cdot, \cdot)) \right)^2 \right] = \mathbb{E} \left[ \left( \int_t^{t+\epsilon} (I_2(e_t^H(\cdot, \cdot)) - I_2(e_t^H(\cdot, \cdot))) \frac{ds}{\epsilon} \right)^2 \right],
\]

\[
= \frac{2}{\epsilon^2} \int_t^{t+\epsilon} \int_t^{t+\epsilon} \left( e_{t,s}^H(\cdot, \cdot) - e_{t,t}^H(\cdot, \cdot); e_{s,s}^H(\cdot, \cdot) - e_{t,t}^H(\cdot, \cdot) > L^2(\mathbb{R}^2) \right) dsds',
\]

where \( e_{t,s}^H(\cdot, \cdot) = \int_0^s u - \cdot \left( u - \cdot \right)^{H-1} |u - s|^{|H-1|} |s|^{|H-1|} dudv \). Moreover, we have:

\[
\frac{1}{\epsilon^2} \int_t^{t+\epsilon} \int_t^{t+\epsilon} \left( e_{t,s}^H(\cdot, \cdot); e_{s,s}^H(\cdot, \cdot) > L^2(\mathbb{R}^2) \right) dsds' = \frac{2(\beta(1 + \frac{H}{2}))^2}{\epsilon^2} \int_t^{t+\epsilon} \int_t^{t+\epsilon} (K_1^2(s, s')) dsds' \]

Thus,

\[
\mathbb{E} \left[ \left( \int_t^{t+\epsilon} I_2(e_t^H(\cdot, \cdot)) \frac{ds}{\epsilon} - I_2(e_t^H(\cdot, \cdot)) \right)^2 \right] = \frac{2(\beta(1 + \frac{H}{2}))^2}{\epsilon^2} \int_t^{t+\epsilon} \int_t^{t+\epsilon} (K_1^2(s, s')) dsds' \]

\[
- \frac{4(\beta(1 + \frac{H}{2}))^2}{\epsilon} \int_t^{t+\epsilon} (K_1^2(t, s))^2 ds + 2(\beta(1 + \frac{H}{2}))^2 (K_1^2(t, t))^2.
\]

The continuity of \( K_1^2(\cdot, \cdot) \) on \( [t, +\infty) \times [t, +\infty) \) concludes the proof.

**Lemma 36.** We have:

\[
\int_a^b \left( \int_t^{t+\epsilon} I_2(e_t^H(\cdot, \cdot)) \frac{ds}{\epsilon} \right) F'^2(X_t^H) dt \to \frac{(L^1)}{\epsilon} \to \int_a^b I_2(e_t^H(\cdot, \cdot)) F'^2(X_t^H) dt.
\]

**Proof.** This follows from standard estimates, the previous Lemma, properties of the kernel \( K_1^2(\cdot, \cdot) \) on \( [t, +\infty) \times [t, +\infty) \) and the Lebesgue dominated convergence theorem.
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