Conditional Statistics of Temperature Fluctuations in Turbulent Convection

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We find that the conditional statistics of temperature difference at fixed values of the locally averaged temperature dissipation rate in turbulent convection become Gaussian in the regime where the mixing dynamics is expected to be driven by buoyancy. Hence, intermittency of the temperature fluctuations in this buoyancy-driven regime can be solely attributed to the variation of the locally averaged temperature dissipation rate. We further obtain the functional behavior of these conditional temperature structure functions. This functional form demonstrates explicitly the failure of dimensional arguments and enhances the understanding of the temperature structure functions.

Extensive efforts have been devoted to the understanding of the problem of intermittency or anomalous scaling. In his Refined Similarity Hypothesis (RSH), Kolmogorov attributed this intermittent nature of the velocity fluctuations to the spatial variations of the energy dissipation rate. Various models have been put forth for the statistics of the locally averaged energy dissipation rate. The most recent model of She and Leveque proposed a hierarchical structure for the moments, which leads to predictions that are in good agreement with experiments. This moment hierarchy was later shown to be naturally satisfied by log-Poisson statistics.

High Rayleigh number convection has been a well-studied model system for investigating turbulence. Fluid motion is driven by an applied temperature difference across the top and the bottom plates of a closed experimental cell filled with fluid. The temperature field in convection is thus an so-called active scalar. The flow state is characterized by the geometry of the cell and two dimensionless parameters: the Rayleigh number $Ra = \alpha g \Delta / (\nu \kappa)$ and the Prandtl number $Pr = \nu / \kappa$, where $L$ is the height of the cell, $\Delta$ is the applied temperature difference, $g$ the acceleration due to gravity, and $\alpha, \nu, \kappa$ respectively the volume expansion coefficient, the kinematic viscosity and the thermal diffusivity of the fluid. When $Ra$ is large enough, the convection becomes turbulent.

In turbulent convection, the temperature fluctuations are also intermittent. As for velocity fluctuations in high Reynolds number Navier-Stokes turbulence, it is of interest to understand the intermittency of temperature fluctuations in high Rayleigh number convection. Turbulent convection poses additional interesting questions of its own. There is the issue of whether and how the characteristics of turbulence are affected by the presence of buoyancy. One expects the mixing dynamics to be driven by buoyancy at scales larger than the Bolgiano scale, $l_B \equiv \bar{c}^{5/4}/[\bar{\chi}^{3/4}(\alpha g)^{3/2}]$, where $\bar{c}$ and $\bar{\chi}$ are respectively the average energy and temperature (variance) dissipation rates. On the other hand, for length scales smaller than $l_B$, the mixing dynamics is expected to be driven by the inertial force of the fluid motion and the temperature field is effectively passive. Recently, one of us (Ching) has analyzed the intermittency of temperature field in turbulent convection. The normalized temperature structure functions have indeed been found to have different scaling exponents in the buoyancy-driven and in the inertia-driven regimes.

In our present project, we attempt to understand the intermittency problem of temperature by separating it into two parts: the understanding of the conditional statistics of temperature fluctuations at fixed values of the locally averaged temperature dissipation rate and the understanding of the statistics of the local temperature dissipation. In this paper, we report our study of the first part. The second part of our study is reported elsewhere. This separation allows us to especially address whether RSH type ideas would be fruitful. We shall see that the intermittent
nature of the temperature fluctuations in the buoyancy-driven regime can indeed be attributed to the variations of the locally averaged temperature dissipation rate. Moreover, a change in the statistical features of the temperature fluctuations is again observed when the Bolgiano scale $l_B$ is crossed. This change manifests itself as a change in the behavior of the conditional PDFs of the temperature difference at fixed value of the locally averaged temperature dissipation rate.

We use temperature data obtained in the well-documented Chicago experiment of low-temperature helium gas [11][3] for our analyses. The experimental cell heated from below is cylindrical with a diameter of 20 cm and a height of 40 cm. A mean circulating flow is present for $Ra \geq 10^4$. The temperature at the center of the cell, $T(t)$, was measured as a function of time $t$. We evaluate the temperature difference between two times: $T_r(t) = T(t + \tau) - T(t)$. The intermittency of the temperature fluctuations is manifested as a change in the shape of the PDF of $T_r$ as $\tau$ varies. In our earlier study of this $\tau$-dependence [3], the dissipative and the circulation time scales, $\tau_d$ and $\tau_c$, were identified. A time scale corresponding to $l_B$ is naturally defined by $\tau_B = \tau_d l_B/L$. It was shown [3] that $l_B$ can be written as

$$ l_B = \frac{Nu^4 L}{(Ra Pr)^{1/4}} \tag{1} $$

where the Nusselt number (Nu) is the heat flux normalized by that when there was only conduction. Thus, $\tau_B$ can be easily evaluated using the measured values of Nu, Ra, and Pr.

The locally averaged temperature dissipation rate $\chi_r$ is the spatial average of $\kappa |\nabla T|^2$ over a ball of radius $r$. We estimate it by $\chi_r$, which is defined as

$$ \chi_r(t) = \frac{1}{\tau} \int_t^{t+\tau} \frac{\kappa}{\langle u_r^2 \rangle} \left( \frac{\partial T}{\partial t} \right)^2 dt' \tag{2} $$

and can be calculated using the one-point temperature measurements. Here, $\langle u_r^2 \rangle$ is the mean square velocity fluctuations at the center of the cell.

We start by investigating the conditional PDF of $T_r$, at fixed values of $\chi_r$. We consider those $T_r(t)$ whose corresponding $ln \chi_r(t)$ assumes a certain value within a small range $\delta$, and calculate the conditional PDFs $P(Y_r | \chi_r)$ where

$$ Y_r = \frac{T_r}{\sqrt{\langle T_r^2 | \chi_r \rangle}} \tag{3} $$

As the conditional mean $\langle T_r | \chi_r \rangle$ is approximately zero, $P(Y_r | \chi_r)$ is standardized with zero mean and unit standard deviation. For a given $\tau$, $P(Y_r | \chi_r)$ is found to be independent of $\chi_r$ for a range of $\chi_r$ that contains most of the data. The conditional PDFs for different values of $\tau$ are plotted in Fig. 1. We measure the value of $\chi_r$ in units of $\chi \equiv \kappa (\partial T/\partial t)^2 / \langle u_r^2 \rangle$. In the limit $\tau \to 0$, $\chi_r \sim T_r^2$, therefore, the conditional PDF is bimodal for small $\tau$, as seen in the figure. As $\tau$ increases, $P(Y_r | \chi_r)$ changes from bimodal to a function with one maximum and varies with $\tau$. But for larger $\tau$, it becomes a standardized Gaussian distribution and is thus independent of $\tau$. Such a change in behavior occurs at $\tau \approx \tau_B$.

Hence, a change in the statistical features of the temperature fluctuations is again observed as the Bolgiano scale is crossed, demonstrating that buoyancy does have an effect on the characteristics of turbulence in convection. Moreover, the physical nature of the presently observed change is clear. We have the interesting result that the temperature fluctuations at fixed values of $\chi_r$ become self-similar and thus non-intermittent in the regime where the mixing dynamics is expected to be driven by buoyancy. In other words, intermittency of the temperature fluctuations in this buoyancy-driven regime can be solely attributed to the variations of $\chi_r$.

In the remaining of this paper, we shall obtain the functional dependence of the conditional temperature structure functions $\langle |T_r|^p \rangle |\chi_r|$ on $p$, $\tau$, and $\chi_r$.

It is illuminating to first work out what functional form is predicted by simple phenomenology and dimensional arguments. One expects $T_r$, the temperature difference across a scale $r$, depends on $r$, $\chi_r$, and $u_r$, the velocity difference across the same scale $r$. In the inertia-driven regime, $u_r$ is related to the locally averaged energy dissipation rate $\epsilon_r$ by $u_r \sim (\epsilon_r r)^{1/5}$ while in the buoyancy-driven regime, $u_r$ is generated by buoyancy: $u_r^2/r \sim \alpha g T_r$. Hence, we have

$$ T_r \sim \begin{cases} \frac{r^{1/3} \epsilon_r^{-1/6} \chi_r^{1/2}}{\epsilon_r^{1/5} \chi_r^{2/5} (\alpha g)^{-1/5}} & r < l_B \\ \frac{r^{1/3} \epsilon_r^{-1/6} \chi_r^{1/2}}{\epsilon_r^{1/5} \chi_r^{2/5} (\alpha g)^{-1/5}} & r > l_B \end{cases} \tag{4} $$

Equation (4) implies that
\begin{equation}
\langle |T_\tau|^p |\chi_\tau \rangle \sim \begin{cases} 
\langle u_2^p \rangle^{p/6} \tau^{p/3} \chi_\tau^{p/12} (\epsilon_\tau^{-p/6})^\chi_\tau & \tau < \tau_B \\
\langle u_2^p \rangle^{p/10} \tau^{p/5} \chi_\tau^{2p/5} (\alpha g)^{-p/5} & \tau > \tau_B
\end{cases}
\end{equation}

if \( T_\tau, \chi_\tau \), and \( \epsilon_\tau \) have the same scaling behavior in \( \tau \) as the corresponding quantities with subscript \( r \) in \( r \) with \( \tau = \langle u_2^p \rangle^{1/2} \). 

If the variations of \( \chi_\tau \) and \( \epsilon_\tau \) are both ignored, (5) implies that the temperature frequency power spectrum has a scaling \( \omega^{-7/3} \) for frequency \( \omega < \omega_B \) and \( \omega^{-5/3} \) for \( \omega > \omega_B \), where \( \omega_B = 2\pi/\tau_B \). The former scaling behavior was reported for the temperature frequency power spectra measured in water [13] and helium [14], while the latter one was reported for that measured in low Pr mercury [13].

Now we proceed with the analyses. From the result that \( P(Y_\tau | \chi_\tau) \) is independent of \( \chi_\tau \), we get

\begin{equation}
\langle |T_\tau|^p |\chi_\tau \rangle = F_p(\tau)\sigma^p(\tau,\chi_\tau)
\end{equation}

where

\begin{equation}
\sigma(\tau,\chi_\tau) \equiv \sqrt{\langle T_\tau^2 |\chi_\tau \rangle}
\end{equation}

By definition, \( F_2(\tau) = 1 \). For \( \tau > \tau_B \), \( P(Y_\tau | \chi_\tau) \) becomes a standard Gaussian, thus

\begin{equation}
F_p(\tau > \tau_B) = \left( \frac{2\sqrt{p}}{\pi} \right) \Gamma\left( \frac{p+1}{2} \right)
\end{equation}

is independent of \( \tau \). For \( \tau_d < \tau < \tau_B \), we find that \( F_p(\tau) \) can be fitted by a power law (see Fig. 2), that is

\begin{equation}
F_p(\tau) \approx C_p \tau^{\alpha_p} \quad \tau_d < \tau < \tau_B
\end{equation}

This \( \tau \) dependence of \( F_p \) echoes that of \( P(Y_\tau | \chi_\tau) \) for \( \tau < \tau_B \). Using (5), such dependence can be attributed to the additional variation of the local energy dissipation rate \( \epsilon_\tau \), even when the local temperature dissipation rate \( \chi_\tau \) is held fixed. The scaling exponents \( \alpha_p \) are plotted in Fig. 3. Since \( \alpha_0 = \alpha_2 = 0 \) by definition, \( \alpha_p \) has to be a nonlinear function of \( p \), as is found.

Next, we analyze the functional dependence of \( \sigma \). We fix \( \tau \) and study its dependence on \( \chi_\tau \). When \( \tau \) is not too large, \( \sigma(\tau,\chi_\tau) \) indeed scales with \( \chi_\tau \) for a range of \( \chi_\tau \) that contains most of the data. The scaling exponent \( b(\tau) \), however, varies with \( \tau \). When \( \tau \) is large, the data scatter. Thus, we have

\begin{equation}
\sigma(\tau,\chi_\tau) = G(\tau)\chi_\tau^{b(\tau)}
\end{equation}

From the relation \( \chi_\tau \sim T_\tau^2 \) in the limit of \( \tau \to 0 \), one gets \( b(\tau) \to 1/2 \) as \( \tau \to 0 \). Indeed, as shown in Fig. 4, \( b(\tau) \) is about 1/2 for \( \tau \leq \tau_d \). It then crosses over to an approximately linear function of \( \ln \tau \), and has a value of 2/5 at \( \tau \approx \tau_B \). This is, therefore, in contrary to the behavior of \( \sigma(\tau,\chi_\tau) \sim \tau^{1/3} \chi_\tau^{1/2} \) and \( \sigma(\tau,\chi_\tau) \sim \tau^{1/5} \chi_\tau^{2/5} \) respectively in the inertia-driven \( (\tau_B < \tau < \tau_B) \) and buoyancy-driven \( (\tau_B < \tau < \tau_B) \) regimes that simple phenomenology and dimensional arguments would predict [see (3)]. In Fig. 5, we plot \( \sigma(\tau,\chi_\tau)(\chi_\tau/\chi_\tau)^{-b(\tau)} \) for various values of \( \chi_\tau \). The linear fit of \( b(\tau) \) in \( \ln \tau \) is used for \( \tau > \tau_B \). The data for different values of \( \chi_\tau \) collapse to one single curve, thus confirming (10). We take the average of the data to get an estimate of \( G(\tau)\chi_\tau^{b(\tau)} \), which is shown in the inset. It can be fitted by a power law for \( \tau > \tau_B \) with an exponent about 0.27.

The temperature structure functions \( \langle |T_\tau|^p \rangle \) are related to the conditional ones at fixed values of \( \chi_\tau \) as follows:

\begin{equation}
\langle |T_\tau|^p \rangle = \int_0^\infty \langle |T_\tau|^p |\chi_\tau \rangle P_\tau(\chi_\tau)d\chi_\tau
\end{equation}

where \( P_\tau(\chi_\tau) \) is the PDF of \( \chi_\tau \). Using (3) and (10), we thus get

\begin{equation}
\langle |T_\tau|^p \rangle = F_p(\tau)G_p(\tau)\langle \chi_\tau^{b(\tau)} \rangle
\end{equation}

Equation (12) implies that the change in the scaling exponent of the normalized structure functions \( \langle |T_\tau|^p \rangle / \langle T_\tau^2 \rangle^{p/2} \) observed, when \( \tau_B \) is crossed, is the combined effect of the change in the \( \tau \) dependence of \( F_p(\tau) \) and \( \langle \chi_\tau^{b(\tau)} \rangle \). The comparison of (12) with data will be presented elsewhere.

In summary, we have studied systematically the conditional statistics of the temperature fluctuations at fixed values of local temperature dissipation \( \chi_\tau \) in turbulent convection. We have found that such conditional statistics become
self-similar in the buoyancy-driven regime, demonstrating that the intermittency of the temperature field in this
regime can be attributed solely to the variations of $\chi_{\tau}$. We have worked out the functional behavior of the conditional
structure functions $\langle |T_\tau|^p \chi_{\tau} \rangle$. There is indeed scaling behavior in $\chi_{\tau}$ but the scaling exponent $b(\tau)$ depends on
$\tau$, in contrary to what simple phenomenology and dimensional arguments might predict. We emphasize that this $\tau$
dependence demonstrates explicitly the failure of dimensional arguments. Together with the knowledge of the
statistical properties of $\chi_{\tau}$, this functional behavior would enable us to better understand the temperature structure
functions.

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FIGURE CAPTIONS

FIG 1. The conditional PDFs $P(Y|\chi)$ versus $Y$ for $Ra = 6.0 \times 10^{11}$ and $\chi_{\tau}/\chi = 0.18$ for various values of $\tau$. $\tau = 8$ (dotted line), $\tau = 16$ (dashed line), $\tau = 32$ (dot-dashed line), $\tau = 64$ (circles), $\tau = 128$ (squares), and $\tau = 256$ (triangles). It can be seen that $P(Y|\chi)$ becomes a standard Gaussian distribution (solid line) for $\tau > \tau_B \approx 70$. All times are in units of the sampling time = 1/409.6 s. The conditional PDFs are found to be independent of $\chi_{\tau}$.

FIG. 2. The logarithm of the normalized conditional temperature structure functions $F_p(\tau) \equiv \langle |T_{\tau}|^p \rangle/\langle T_{\tau}^2 \rangle^{p/2}$ versus $\ln \tau$ for $Ra = 7.3 \times 10^{10}$ and $\chi_{\tau}/\chi = 0.43$ for various values of $p$. The three time scales $\tau_d$, $\tau_B$ and $\tau_c$ are approximately 6, 60 and 1750 respectively, and are indicated by the dashed lines. All times are in units of the sampling time = 1/320 s. $p = 0.5$ (circles), $p = 1.5$ (diamonds), $p = 1.75$ (triangles), $p = 2.25$ (crosses), $p = 2.5$ (squares), and $p = 2.75$ (pluses). For $\tau_d < \tau < \tau_B$, $F_p(\tau)$ can be fitted by a power-law $C_p \tau^{a_p}$ (solid lines) and for $\tau > \tau_B$, it becomes $\sqrt{2p/\pi} \Gamma((p+1)/2)$ (dot-dashed lines) and is thus independent of $\tau$.

FIG. 3. The scaling exponent $\alpha_p$ versus $p$ for $Ra = 4.0 \times 10^9$ (circles), $Ra = 7.3 \times 10^{10}$ (squares), and $Ra = 6.0 \times 10^{11}$ (diamonds).

FIG. 4. The scaling exponent $b(\tau)$ versus $\ln \tau$ for $Ra = 4.0 \times 10^9$. The time scales $\tau_d$ and $\tau_B$ are approximately 8 and 50 respectively, and are indicated by the dashed lines. All times are in units of the sampling time = 1/160.8 s. It can be seen that $b(\tau)$ is close to $1/2$ for $\tau \leq \tau_d$ and can be fitted by a linear function in $\ln \tau$ (solid line) for $\tau > \tau_d$. Moreover, $b(\tau) \approx 2/5$ at $\tau \approx \tau_B$.

FIG. 5. $\ln \sigma(\tau, \chi_{\tau})(\chi_{\tau}/\chi)^{-b(\tau)}$ versus $\ln \tau$ for $Ra = 7.3 \times 10^{10}$ for $\chi_{\tau}/\chi = 0.13$ (circles), $\chi_{\tau}/\chi = 0.35$ (squares), and $\chi_{\tau}/\chi \approx 0.96$ (triangles). The three sets of data collapse into a single function of $\tau (= G(\tau)^{b(\tau)})$ confirming (10). The times are in units of the sampling time = 1/320 s while $\sigma$ is in units of the standard deviation of the temperature fluctuations. Shown in the inset is the average of the three sets of data (solid line), which can be fitted by a power law (dot-dashed line) for $\tau > \tau_B$ (indicated by dashed line).
\[ \ln \left[ \sigma(\tau, \chi_\tau)(\chi_\tau/\chi)^{-b(\tau)} \right] \]