A solution to the 2/3 conjecture

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Abstract

We prove a vertex domination conjecture of Erd˝ os, Faudree, Gould, Gy´ arf´ as, Rousseau, and Schelp, that for every n-vertex complete graph with edges coloured using three colours there exists a set of at most three vertices which have at least 2n/3 neighbours in one of the colours. Our proof makes extensive use of the ideas presented in “A New Bound for the 2/3 Conjecture” by Kr´ al’, Liu, Sereni, Whalen, and Yilma.

1 Introduction

In this paper we prove the 2/3 conjecture of Erd˝ os, Faudree, Gould, Gy´ arf´ as, Rousseau, and Schelp [6]. Before we discuss this problem we first require some definitions.

A graph is a pair of sets \( G = (V(G), E(G)) \) where \( V(G) \) is the set of vertices and \( E(G) \) is a family of 2-subsets of \( V(G) \) called edges. A complete graph is a graph containing all possible edges.

An \( r \)-colouring of the edges of a graph \( G \) is a map from \( E(G) \) to a set of size \( r \). Given an \( r \)-colouring of the edges of a complete graph \( G = (V(G), E(G)) \), a colour \( c \), and \( A, B \subseteq V(G) \), we say that \( A \) \( c \)-dominates \( B \) if for every \( b \in B \setminus A \) there exists \( a \in A \) such that the edge \( ab \) is coloured \( c \). We say that \( A \) strongly \( c \)-dominates \( B \) if for every \( b \in B \) there exists an \( a \in A \) such that the edge \( ab \) is coloured \( c \). Note that if \( A \) strongly \( c \)-dominates \( B \) then it also \( c \)-dominates \( B \).

Erd˝ os and Hajnal [8] showed that given a positive integer \( t \), a real value \( \epsilon > 0 \), and a 2-coloured complete graph on \( n \) vertices, there exists a colour \( c \) and a set of \( t \) vertices that \( c \)-dominate at least \( (1 - (1 - \epsilon)(2/3)^t)n \) vertices for some colour \( c \). They asked whether 2/3 could be replaced by 1/2, which was answered by Erd˝ os, Faudree, Gy´ arf´ as, and Schelp [7].

Theorem 1.1 (Erd˝ os, Faudree, Gy´ arf´ as, and Schelp [7]). For any positive integer \( t \), and any 2-coloured complete graph on \( n \) vertices, there exists a set of \( t \) vertices that \( c \)-dominate at least \( (1 - (1 - (1/2)^t))n \) of the vertices.

Erd˝ os, Faudree, Gy´ arf´ as, and Schelp went on to ask whether their result could be generalised to say that all \( r \)-coloured complete graphs contain a set of \( t \) vertices that \( c \)-dominate at least \( (1 - (1 - 1/r)^t)n \) of the vertices. However, in

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they presented a construction by Kierstead showing this to be false even for $r = 3$ and $t = 3$. Simply partition the vertices of a complete graph into 3 equal classes $V_0, V_1, V_2$, and colour the edges such that an edge $xy$ with $x \in V_i$ and $y \in V_j$ is coloured $i$ if $i = j$ or $i \equiv j + 1 \mod 3$, see Figure 1. The construction shows that for 3-colourings it is impossible for a small set of vertices to monochromatically dominate significantly more than $2/3$ of the vertices. (It also shows that regardless of the size of our dominating set we cannot guarantee more than $\lceil 2n/3 \rceil$ vertices will be strongly $c$-dominated.) Motivated by this example Erdős, Faudree, Gould, Győrfi, Rousseau, and Schelp [6], made the following conjecture.

**Conjecture 1.2** (Erdős, Faudree, Gould, Győrfi, Rousseau, and Schelp [6]). For any 3-coloured complete graph, there exists a colour $c$ and a set of at most 3 vertices that $c$-dominates at least $2/3$ of the vertices.

They were able to show that the conjecture holds true when it was relaxed to asking for a dominating set of at most 22 vertices, but were unable to reduce 22 to 3. We note that 3 is best possible because in a typical random 3-colouring of a complete graph of order $n$ no pair of vertices will monochromatically dominate more than $5n/9 + o(n)$ vertices (this follows simply from Chernoff’s bound). For completeness we should also mention that in [6] the authors showed there always exist 2 vertices that monochromatically dominate at least $5(n-1)/9$ vertices in a 3-coloured complete graph.

Král’, Liu, Sereni, Whalen and Yilma [11], made significant progress with Conjecture 1.2 by proving that there exists a colour $c$ and set of size at most 4 which not only $c$-dominates but strongly $c$-dominates at least $2/3$ of the vertices in a 3-coloured complete graph. Their proof makes use of Razborov’s semidefinite flag algebra method [12] to show that Kierstead’s construction is essentially extremal. We will discuss flag algebras in more detail in Section 2.1.

We verify Conjecture 1.2 by proving the following theorem.

**Theorem 1.3.** For any 3-colouring of the edges of a complete graph on $n \geq 3$ vertices, there exists a colour $c$ and a set of 3 vertices that strongly $c$-dominate at least $2n/3$ vertices.
Our proof builds on the work of Král’, Liu, Sereni, Whalen, and Yilma [11]. The main difference is that by using an idea of Hladky, Král’, and Norine [10] we have additional constraints to encode the 2/3 condition when applying the semidefinite flag algebra method (see Lemma 2.4). Another difference is that we conduct our computations on 6 vertex graphs, whereas in [11] they only look at 5 vertex graphs.

2 Proof of Theorem 1.3

For the remainder of this paper we will let \( \hat{G} \) be a fixed counterexample to Theorem 1.3 with \( |V(\hat{G})| = k \). So \( \hat{G} \) is a 3-coloured complete graph on \( k \geq 3 \) vertices such that every set of 3 vertices strongly \( c \)-dominates strictly less than \( 2k/3 \) vertices for each colour \( c \). We will show that \( \hat{G} \) cannot exist by proving that it would have to satisfy two contradicting properties.

Given a 3-coloured complete graph and a vertex \( v \), let \( A_v \) denote the set of colours of the edges incident to \( v \).

The following lemma is implicitly given in the paper by Král’, Liu, Sereni, Whalen, and Yilma [11].

**Lemma 2.1** (Král’, Liu, Sereni, Whalen, and Yilma [11]). Our counterexample \( \hat{G} \) must contain a vertex \( v \in V(\hat{G}) \) with \( |A_v| = 3 \).

**Proof.** Since \( \hat{G} \) is a counterexample it cannot contain a vertex \( v \) with \( |A_v| = 1 \) otherwise any set of 3 vertices containing \( v \) will strongly \( c \)-dominate all the vertices, for \( c \in A_v \). So it is enough to show that if \( |A_v| = 2 \) for every vertex then \( \hat{G} \) is not a counterexample.

Let the set of colours be \( \{1, 2, 3\} \). If every vertex has \( |A_v| = 2 \) we can partition the vertices into three disjoint classes \( V_1, V_2, V_3 \) where \( v \in V_i \) if \( i \notin A_v \). Without loss of generality we can assume \( |V_1| \geq |V_2| \geq |V_3| \). Note that the colour of all edges \( uv \) with \( u \in V_1 \) and \( v \in V_2 \) is 3 because \( A_u \cap A_v = \{2, 3\} \cap \{1, 3\} = \{3\} \). Consequently any set of 3 vertices containing a vertex from \( V_1 \) and a vertex from \( V_2 \) must strongly 3-dominate \( V_1 \cup V_2 \) which is at least 2/3 of the vertices.

To complete the proof we need to consider what happens if we cannot choose a vertex from \( V_1 \) and \( V_2 \). This can only occur if \( V_2 = \emptyset \) which implies \( V_3 = \emptyset \) and \( V_1 = V(\hat{G}) \), i.e. \( \hat{G} \) is 2-coloured. In this case we can apply the result of Erdős, Faudree, Gyárfás, and Schelp, Theorem 1.1. Although technically the theorem is not stated in terms of strongly \( c \)-dominating a set, its proof given in [7] is constructive and it can be easily checked that the dominating set it finds is strongly \( c \)-dominating (for \( t \geq 2 \) and \( n \geq 2 \)).

We will show via the semidefinite flag algebra method that \( \hat{G} \) cannot contain a vertex \( v \) with \( |A_v| = 3 \) contradicting Lemma 2.1. The flag algebra method is primarily used to study the limit of densities in sequences of graphs. As such we will not apply it directly to \( \hat{G} \) but to a sequence of graphs \( (G_n)_{n \in \mathbb{N}} \) where \( G_n \) is constructed from \( \hat{G} \) as follows. \( G_n \) is a 3-coloured complete graph on \( nk \) vertices where each vertex \( u \in V(\hat{G}) \) has been replaced by a class of \( n \) vertices \( V_u \). The edges of \( G_n \) are coloured as follows: edges between two classes \( V_u \) and \( V_v \) have the same colour as \( uv \) in \( \hat{G} \), while edges within a class, \( V_u \) say, are coloured independently and uniformly at random with the colours from \( A_u \).
We would like to claim that $G_n$ is also a counterexample, but this may not be true. However, there exist particular types of 3 vertex sets which with high probability strongly c-dominate at most $2/3 + o(1)$ of the vertices in $G_n$ for some colour $c$. (Unless otherwise stated $o(1)$ will denote a quantity that tends to zero as $n \to \infty$.)

We note that Chernoff’s bound implies that for all $u \in V(\hat{G})$, $c \in A_u$ and $v \in V_u \subset V(G_n)$ we have

$$|\{w \in V_u : vw \text{ is coloured } c\}| = \frac{n}{|A_u|} + o(n),$$

with probability $1 - o(1)$.

Given a 3-coloured complete graph and a colour $c$ we define a good set for $c$ to be a set of 3 vertices $\{x, y, z\}$ such that either

(i) at least two of the edges $xy, xz, yz$ are coloured $c$, or

(ii) one of the edges, $xy$ say, is coloured $c$ and the remaining vertex $z$ satisfies $|A_z \cup \{c\}| = 3$.

(Although this definition does not appear particularly natural, it has the advantage of being easily encoded by the semidefinite flag algebra method.)

**Lemma 2.2.** Any good set for $c$ in $G_n$ strongly c-dominates at most $2/3 + o(1)$ of the vertices with probability $1 - o(1)$.

**Proof.** For $u \in V(\hat{G})$ recall that $V_u$ is the corresponding class of $n$ vertices in $G_n$. Given $S = \{x, y, z\}$ a good set for $c$ in $G_n$, we will consider its “preimage” in $\hat{G}$ which we denote $S'$, i.e. $S' \subseteq V(\hat{G})$ is minimal such that $S \subseteq \bigcup_{u \in S'} V_u$.

Let the strongly c-dominated sets be $D_S$ and $D_{S'}$ for $S$ in $G_n$ and $S'$ in $\hat{G}$ respectively. For $v \in S$ there exists some $u \in V(\hat{G})$ such that $v \in V_u$, let us define $W_v$ to be the vertices that lie within $V_u$ that are strongly c-dominated by $v$.

We first consider the case where $|S'| = 3$ (which occurs when no two members of $S$ lie in the same vertex class). Since $\hat{G}$ is a counterexample we have $|D_{S'}| < 2k/3$ or equivalently $|D_{S'}| \leq 2k/3 - 1/3$. It is easy to check that

$$D_S = \left( \bigcup_{d \in D_{S'}} V_d \right) \cup \left( \bigcup_{v \in S} W_v \right).$$

We can split the problem into two cases depending on which type of good set $S$ is. If $S$ is of type (i) then it is easy to see that $W_v \subseteq \bigcup_{d \in D_{S'}} V_d$ for every $v \in S$. Consequently $D_S = \bigcup_{d \in D_{S'}} V_d$ implying $|D_S| = n|D_{S'}| \leq 2nk/3 - n/3$, hence $D_S$ contains at most $2/3$ of the vertices in $G_n$, as required.

If $S$ is of type (ii), with $xy$ coloured $c$, then $W_x, W_y \subseteq \bigcup_{d \in D_{S'}} V_d$. If $c \in A_z$ then $|A_z| = 3$ and Chernoff’s bound implies that $|W_z| \leq n/3 + o(n)$ holds with probability $1 - o(1)$. Note that this also holds when $c \notin A_z$ as $W_z = \emptyset$. So $D_S = W_z \cup \bigcup_{d \in D_{S'}} V_d$ and

$$|D_S| \leq |W_z| + n|D_{S'}| \leq n/3 + o(n) + n(2k/3 - 1/3) = (2/3 + o(1))nk$$

with probability $1 - o(1)$ as claimed.

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To complete the proof we need to consider what happens when $|S'| < 3$. This can only occur if in $G_n$ there exists a vertex class, $V_m$ say (with $m \in V(\hat{G})$), that contains two or three members of $S$. By the definition of a good set at least two of the vertices in $S$ are incident with an edge of colour $c$, so at least one such vertex is present in $V_m$ implying $c \in A_m$. This shows that we can always choose a set $T$ of three vertices in $\hat{G}$ that contains both $S'$ and a vertex $u$ (possibly contained in $S'$) with the property that $um$ is coloured $c$. The fact that $S'$ is a subset of $T$ gives us

$$D_S \subseteq \left( \bigcup_{d \in D_T} V_d \right) \cup \left( \bigcup_{v \in S} W_v \right),$$

where $D_T$ is the set strongly $c$-dominated by $T$ in $\hat{G}$, and having $u \in T$ ensures $W_v \subseteq \bigcup_{d \in D_T} V_d$ for every $v \in S \cap V_m$. Consequently we can apply the same argument we used for $|S'| = 3$. We note that (since $\hat{G}$ is a counterexample) $|D_T| \leq 2k/3 - 1/3$, and that if $S$ is of type (i) we have $D_S \subseteq \bigcup_{d \in D_T} V_d$ otherwise $S$ is of type (ii) and $D_S \subseteq W_z \cup \bigcup_{d \in D_T} V_d$. In either case we get the desired result that $D_S$ contains at most $2/3 + o(1)$ of the vertices with high probability.

We note that there are other types of 3 vertex sets that we could potentially utilize other than the “good sets”, however our proof does not require them and so we will not discuss them here. Also similar results can be proven for sets larger than 3 which may be of use in other problems. For a more general treatment we refer the reader to Král’, Liu, Sereni, Whalen, and Yilma [11].

Given two 3-coloured complete graphs $F, G$ with $|V(F)| \leq |V(G)|$, we define $d_F(G)$, the density of $F$ in $G$, to be the proportion of sets of size $|V(F)|$ in $G$ that induce a 3-coloured complete graph that is identical to $F$ up to a re-ordering of vertices.

In [11] the authors bound the density in $G_n$ of a family of graphs in order to contradict Lemma 2.1. In particular they chose their family to consist of all 3-coloured complete graphs on 5 vertices that contain a vertex $v$ with $|A_v| = 3$. We will instead bound the density of a single 6 vertex graph $X$, whose coloured
edge sets are given by
\{14, 23, 35, 56, 62\}, \{25, 34, 46, 61, 13\}, \{36, 45, 51, 12, 24\},
see Figure 2.

Observe that for our counterexample \( \hat{G} \), Lemma 2.1 implies that there exists a vertex \( u \) with \( |A_u| = 3 \), which in turn implies there exists a class of \( n \) vertices \( V_u \) in \( G_n \) with the edges coloured uniformly at random. By considering the probability of finding \( X \) in \( V_u \) we have the following simple bound for \( d_X(G_n) \).

**Corollary 2.3.** With probability \( 1 - o(1) \),
\[
d_X(G_n) \geq k^{-|V(X)|3^{-|E(X)|}} + o(1).
\]

By encoding Lemma 2.2 using flag algebras we will show that with high probability, \( d_X(G_n) = o(1) \), a contradiction proving that no counterexample exists.

### 2.1 Flag algebras

Razborov’s semidefinite flag algebra method introduced in [12] and [13] has proven to be an invaluable tool in extremal graph theory. Many results have been found through its application, see for example [1], [3], [4], [9], [10], [11]. We also refer interested readers to [2] for a minor improvement to the general method. Our notation and description of the method for 3-coloured graphs is largely adapted from the explanation given by Baber and Talbot in [3].

We will say that two 3-coloured complete graphs are isomorphic if they can be made identical by permuting their vertices. Let \( \mathcal{H} \) be the family of all 3-coloured complete graphs on \( l \) vertices, up to isomorphism. If \( l \) is sufficiently small we can explicitly determine \( \mathcal{H} \) (by computer search if necessary). For \( H \in \mathcal{H} \) and a large 3-coloured complete graph \( K \), we define \( p(H; K) \) to be the probability that a random set of \( l \) vertices from \( K \) induces a 3-coloured complete graph isomorphic to \( H \).

Using this notation and averaging over \( l \) vertex sets in \( G_n \) (with \( l \geq |V(X)| \)), we can show
\[
d_X(G_n) = \sum_{H \in \mathcal{H}} d_X(H)p(H; G_n),
\]
and hence \( d_X(G_n) \leq \max_{H \in \mathcal{H}} d_X(H) \). This bound is unsurprisingly extremely poor. We will rectify this by creating a series of inequalities from Lemma 2.2 that we can use to improve (1). To do this we first need to consider how small pairs of 3-coloured complete graphs can intersect. We will use Razborov’s method and his notion of flags and types to formally do this.

A flag, \( F = (K', \theta) \), is a 3-coloured complete graph \( K' \) together with an injective map \( \theta : \{1, \ldots, s\} \rightarrow V(K') \). If \( \theta \) is bijective (and so \( |V(K')| = s \)) we call the flag a type. For ease of notation given a flag \( F = (K', \theta) \) we define its order \( |F| \) to be \( |V(K')| \). Given a type \( \sigma \) we call a flag \( F = (K', \theta) \) a \( \sigma \)-flag if the induced labelled 3-coloured subgraph of \( K' \) given by \( \theta \) is \( \sigma \).

For a type \( \sigma \) and an integer \( m \geq |\sigma| \), let \( \mathcal{F}_m^\sigma \) be the set of all \( \sigma \)-flags of order \( m \), up to isomorphism. For a non-negative integer \( s \) and 3-coloured complete graph \( K \), let \( \Theta(s, K) \) be the set of all injective functions from \( \{1, \ldots, s\} \) to \( V(K) \). Given \( F \in \mathcal{F}_m^\sigma \) and \( \theta \in \Theta(|\sigma|, K) \) we define \( p(F, \theta; K) \) to be the probability that
an $m$-set $V'$ chosen uniformly at random from $V(K)$ subject to $\text{im}(\theta) \subseteq V'$, induces a $\sigma$-flag $(K[V'], \theta)$ that is isomorphic to $F$.

If $F_1 \in F^\sigma_{m_1}$, $F_2 \in F^\sigma_{m_2}$, and $\theta \in \Theta(|\sigma|, K)$ then $p(F_1, \theta; K)p(F_2, \theta; K)$ is the probability that two sets $V_1, V_2 \subseteq V(K)$ with $|V_1| = m_1, |V_2| = m_2$, chosen independently at random subject to $\text{im}(\theta) \subseteq V_1 \cap V_2$, induce $\sigma$-flags $(K[V_1], \theta)$, $(K[V_2], \theta)$ that are isomorphic to $F_1, F_2$ respectively. We define the related probability, $p(F_1, F_2, \theta; K)$, to be the probability that two sets $V_1, V_2 \subseteq V(K)$ with $|V_1| = m_1, |V_2| = m_2$, chosen independently at random subject to $\text{im}(\theta) = V_1 \cap V_2$, induce $\sigma$-flags $(K[V_1], \theta)$, $(K[V_2], \theta)$ that are isomorphic to $F_1, F_2$ respectively.

Note that the difference in definitions between $p(F_1, \theta; K)p(F_2, \theta; K)$ and $p(F_1, F_2, \theta; K)$ is that of choosing the two sets with or without replacement. It is easy to show that $p(F_1, \theta; K)p(F_2, \theta; K) = p(F_1, F_2, \theta; K) + o(1)$ where the $o(1)$ term vanishes as $|V(K)|$ tends to infinity.

Taking the expectation over a uniformly random choice of $\theta \in \Theta(|\sigma|, K)$ gives

$$E_{\theta \in \Theta(|\sigma|, K)} [p(F_1, \theta; K)p(F_2, \theta; K)] = E_{\theta \in \Theta(|\sigma|, K)} [p(F_1, F_2, \theta; K)] + o(1).$$

Furthermore the expectation on the right hand side can be rewritten in terms of $p(H; K)$ by averaging over $l$-vertex subgraphs of $K$, provided $m_1 + m_2 - |\sigma| \leq l$ (i.e. $F_1$ and $F_2$ intersecting on $\sigma$ fits inside an $l$ vertex graph). Hence

$$E_{\theta \in \Theta(|\sigma|, K)} [p(F_1, \theta; K)p(F_2, \theta; K)] = \sum_{H \in \mathcal{H}} E_{\theta \in \Theta(|\sigma|, H)} [p(F_1, F_2, \theta; H)] p(H; K) + o(1). \quad (2)$$

Observe that the right hand side of (2) is a linear combination of $p(H; K)$ terms whose coefficients can be explicitly calculated using just $\mathcal{H}$, this will prove useful as (1) is of a similar form.

Given $F^\sigma_m$ with $2m - |\sigma| \leq l$ and a positive semidefinite matrix $Q = (q_{ab})$ of dimension $|F^\sigma_m|$, let $p_\theta = (p(F, \theta; K) : F \in F^\sigma_m)$ for $\theta \in \Theta(|\sigma|, K)$. Using (2) and the linearity of expectation we have

$$0 \leq E_{\theta \in \Theta(|\sigma|, K)} [p_\theta^T Q p_\theta] = \sum_{H \in \mathcal{H}} a_H(\sigma, m, Q) p(H; K) + o(1) \quad (3)$$

where

$$a_H(\sigma, m, Q) = \sum_{F_a, F_b \in F^\sigma_m} q_{ab} E_{\theta \in \Theta(|\sigma|, H)} [p(F_a, F_b, \theta; H)].$$

Note that $a_H(\sigma, m, Q)$ is independent of $K$ and can be explicitly calculated. Combining (3) when $K = G_n$ with (1) gives

$$d_X(G_n) \leq \sum_{H \in \mathcal{H}} (d_X(H) + a_H(\sigma, m, Q)) p(H; G_n) + o(1) \leq \max_{H \in \mathcal{H}} (d_X(H) + a_H(\sigma, m, Q)) + o(1).$$

Since some of the $a_H(\sigma, m, Q)$ values may be negative (for a careful choice of $Q$) this may be a better bound (asymptotically) for $d_X(G_n)$. To help us further reduce the bound we can of course create multiple inequalities of the form given by (3) by choosing different types $\sigma$, orders of flags $m$, and positive
semidefinite matrices \(Q_i\). Let \(\alpha_H = \sum_i a_H(\sigma_i, m_i, Q_i)\) and hence we can say 
\[d_X(G_n) \leq \max_{H \in \mathcal{H}} (d_X(H) + \alpha_H) + o(1).\]
Finding the optimal choice of matrices \(Q_i\), which lowers the bound as much as possible is a convex optimization problem, in particular a semidefinite programming problem. As such we can use freely available software such as CSDP [5] to find the \(Q_i\).

So far the bound on \(d_X(G_n)\) is valid for any 3-coloured complete graph; we have not yet made any use of the fact that \(G_n\) comes from our counterexample \(\hat{G}\). Král’, Liu, Sereni, Whalen, and Yilma remedy this, see Lemma 3.3 in [11], by constructing a small set of constraints that \(G_n\) must satisfy but a general 3-coloured complete graph may not. By using an idea of Hladky, Král’, and Norine [10] we can significantly increase the number of such constraints.

We say that a \(\sigma\)-flag \(F\) is \(c\)-good if the colouring of \(F\) and the size of \(\sigma\) imply that \(\sigma\) is a good set for \(c\) in \(F\).

**Lemma 2.4.** Given a colour \(c\) and a \(c\)-good \(\sigma\)-flag \(F\), the following holds with probability \(1 - o(1)\).

\[
E_{\theta \in \Theta(|\sigma|, G_n)} \left[ p(F; \theta; G_n) \left( \frac{2}{3} p(\sigma; \theta; G_n) - \sum_{F' \in \mathcal{D}} p(F'; \theta; G_n) \right) \right] + o(1) \geq 0,
\]

where \(\mathcal{D} \subseteq F'_{|\sigma|+1}\) is the set of all \(\sigma\)-flags on \(|\sigma| + 1\) vertices where the vertex not in \(\sigma\) is \(c\)-dominated by the type.

We note that when \(F = \sigma\) Lemma 2.4 is equivalent to Lemma 3.3 in [11].

**Proof.** For a fixed \(\theta \in \Theta(|\sigma|, G_n)\) if \(p(F; \theta; G_n) = 0\) then trivially we get

\[
p(F; \theta; G_n) \left( \frac{2}{3} p(\sigma; \theta; G_n) - \sum_{F' \in \mathcal{D}} p(F'; \theta; G_n) \right) \geq 0.
\]

If \(p(F; \theta; G_n) > 0\) then there exists a copy of \(F\) in \(G_n\) and so the image of \(\theta\) is \(\sigma\) (or equivalently \(p(\sigma; \theta; G_n) = 1\)) and \(\sigma\) must be a good set for \(c\). By Lemma 2.2 we know that with probability \(1 - o(1)\),

\[
\frac{2}{3} + o(1) \geq \sum_{F' \in \mathcal{D}} p(F'; \theta; G_n),
\]

which implies

\[
p(F; \theta; G_n) \left( \frac{2}{3} p(\sigma; \theta; G_n) - \sum_{F' \in \mathcal{D}} p(F'; \theta; G_n) \right) + o(1) \geq 0.
\]

Taking the expectation completes the proof.

Given a \(c\)-good flag \(F\), equation (2) tells us that provided \(|F| + 1 \leq l\) we can express the inequality given in Lemma 2.4 as

\[
\sum_{H \in \mathcal{H}} b_H(c, F) p(H; G_n) + o(1) \geq 0,
\]

where \(b_H(c, F)\) can be explicitly calculated from \(c, F, \) and \(H\). Equation (4) is of the same form as (3) and as such we can use it in a similar way to improve
the bound on $d_X(G_n)$. Moreover observe that we can multiply by any non-negative real value without changing its form.

Let $C$ be a set of pairs of colours $c$ and $c$-good flags $F$ satisfying $|F| + 1 \leq l$. For $(c,F) \in C$ let $\mu(c,F) \geq 0$ be a real number (whose value we will choose later to help us improve the bound on $d_X(G_n)$). To ease notation we define $\beta_H = \sum_{(c,F) \in C} \mu(c,F) b_H(c,F)$. It is easy to check $\sum_{H \in \mathcal{H}} \beta_H p(H;G_n) + o(1) \geq 0$, thus combining it with (1) and terms such as (3) gives
\begin{equation*}
\begin{aligned}
d_X(G_n) &\leq \sum_{H \in \mathcal{H}} (d_X(H) + \alpha_H + \beta_H) p(H;G_n) + o(1) \\
&\leq \max_{H \in \mathcal{H}} (d_X(H) + \alpha_H + \beta_H) + o(1)
\end{aligned}
\end{equation*}

Finding an optimal set of non-negative coefficients $\mu(c,F)$ and semidefinite matrices $Q_i$ can still be posed as a semidefinite programming problem.

We complete the proof of Theorem 1.3 with the following lemma which contradicts Corollary 2.3.

**Lemma 2.5.** With probability $1 - o(1)$ we have $d_X(G_n) = o(1)$.

**Proof.** By setting $l = 6$ (the order of the graphs $H$) and solving a semidefinite program we can find coefficients $\mu(c,F)$ and semidefinite matrices $Q_i$ such that $\max_{H \in \mathcal{H}} (d_X(H) + \alpha_H + \beta_H) = 0$. The relevant data needed to check this claim can be found in the data file 2-3.txt. There is too much data to check by hand so we also provide the C++ program DominatingDensityChecker to check the data file. Both data file and proof checker may be downloaded from the arXiv [http://arxiv.org/e-print/1306.6202v1](http://arxiv.org/e-print/1306.6202v1).

It is worth noting that in order to get a tight bound we used the methods described in Section 2.4.2 of [1] to remove the rounding errors from the output of the semidefinite program solvers.

We end by mentioning that when $l = 6$ the computation has to consider 25506 non-isomorphic graphs which form $\mathcal{H}$, and as a result solving the semidefinite program is very time consuming. However, our method for proving the result makes no preferences between the colours. Consequently it is quite easy to see that if there exists a solution then there must also exist a solution which is invariant under the permutations of the colours. So if $H_1, H_2 \in \mathcal{H}$ are isomorphic after a permutation of their colours then in an “invariant solution” $d_X(H_1) + \alpha_{H_1} + \beta_{H_1} = d_X(H_2) + \alpha_{H_2} + \beta_{H_2}$ must necessarily hold. Therefore by restricting our search to invariant solutions we need only worry about those $H$ in $\mathcal{H}$ the set of 3-coloured complete graphs on $l$ vertices that are non-isomorphic even after a permutation of colours. For $l = 6$, $|\mathcal{H}| = 4300$ which results in a significantly easier computation.

### 3 Open problems

Erdős, Faudree, Gould, Gyárfás, Rousseau, and Schelp ask in [6] whether every 4-coloured complete graph always contains a small set of vertices that monochromatically dominate at least $3/5$ of the vertices. The value of $3/5$ comes from considering the 4 colour equivalent of Kierstead’s construction given in Figure 3. It shows that we cannot hope to find a small set of vertices that monochromatically dominate significantly more than $3/5$ of the vertices. Also regardless
of the size of the dominating set we can at most guarantee that \( \lceil 3n/5 \rceil \) vertices will be strongly monochromatically dominated in an \( n \) vertex graph.

By applying Chernoff’s bound it is easy to see that a typical random 4-colouring on an \( n \) vertex graph contains no 3-sets that monochromatically dominate more than \((1 - (3/4)^3)n + o(n)\) vertices, which is less than \(3n/5\) when \(n\) is large. So the minimal possible dominating set size is 4.

We could not prove that there always exists a 4-set that strongly monochromatically dominates \(3/5\) of the vertices in a complete 4-coloured graph. However, by generalising the method given in Section 2 and replacing \(X\) with a specific family of graphs, we were able to show that there exist 4-sets that strongly monochromatically dominate 0.5711 of the vertices. The family of graphs we chose to bound instead of \(X\) are the 48 graphs on 5 vertices that are, up to a permutation of colours, isomorphic to one of those given in Figure 4. It is not immediately obvious why such a family should have a positive density in a
counterexample, so we will outline why this is the case.

**Lemma 3.1.** Any 4-coloured complete graph $\hat{G}$ on $k$ vertices that has the property that every set of 3 vertices strongly $c$-dominates strictly less than $3/5$ of the vertices for every colour $c$, must contain either

(i) a vertex $u$ with $|A_u| = 4$, or

(ii) two vertices $v, w$ with $|A_v| = |A_w| = 3$ and $A_v \neq A_w$.

**Proof.** Let the set of colours be $\{1, 2, 3, 4\}$. Trivially $\hat{G}$ cannot contain a vertex $u$ with $|A_u| = 1$, otherwise any set containing $u$ will strongly dominate all the vertices. It is therefore sufficient to show that no $\hat{G}$ can exist in which every vertex $u$ satisfies $|A_u| = 2$, or $A_u = \{1, 2, 3\}$.

We can take our vertices and partition them into disjoint classes based on their value of $A_v$. For ease of notation we will refer to these classes by $V_S$ where $S$ is a subset of the colours, for example $V_{13}$ contains all the vertices $v$ which have $A_v = \{1, 3\}$. We will split our argument into multiple cases depending on whether or not a class is empty. Throughout we will make use of the fact that $\hat{G}$ cannot be 3-coloured otherwise by Theorem 1.3 we can find a 3-set which dominates over $3/5$ of the vertices. Also note that if $S,T \subseteq \{1, 2, 3, 4\}$ and $S \cap T = \emptyset$ then either $V_S$ or $V_T$ must be empty as any edge that goes between the two classes must have a colour in $S \cap T$.

Suppose $V_{123} = \emptyset$. Without loss of generality we can assume $V_{12} \neq \emptyset$ (implying $V_{34} = \emptyset$). There must be another non-empty class otherwise $\hat{G}$ is 3-coloured. Without loss of generality we may assume $V_{13} \neq \emptyset$ (implying $V_{24} = \emptyset$). To avoid being 3-coloured we must have $V_{14} \neq \emptyset$ (implying $V_{23} = \emptyset$). There are no more classes we could add and $\hat{G}$ has all its vertices strongly 1-dominated by a 2-set containing a vertex from $V_{12}$ and a vertex from $V_{13}$ which is a contradiction.

Suppose $V_{121} \neq \emptyset$. To avoid being 3-coloured at least one of $V_{14}, V_{24}, V_{34}$ must be non-empty. Without loss of generality assume $V_{14} \neq \emptyset$ (implying $V_{23} = \emptyset$). To avoid having every vertex strongly 1-dominated by a 2-set containing a vertex from $V_{123}$ and a vertex from $V_{14}$, either $V_{24}$ or $V_{34}$ must be non-empty. Without loss of generality assume $V_{24} \neq \emptyset$ (implying $V_{13} = \emptyset$). If $V_{34} = \emptyset$, then the vertices in $\hat{G}$ are partitioned into three disjoint classes $V_{123}, V_{124}, V_{134},$ and $V_{24}$. We can strongly $c$-dominate at least $2/3$ of the vertices by choosing $c$ to be the colour of the edges that go between the largest two of the classes and by choosing our dominating set to contain a vertex from each of the largest two classes.

The only case left to consider is when $V_{123} \neq \emptyset$, $V_{14} \neq \emptyset$, $V_{24} \neq \emptyset$, and $V_{34} \neq \emptyset$. (Note that $\hat{G}$ resembles Figure 3.) We may suppose that $|V_{14}| \geq |V_{24}| \geq |V_{34}|$ and so either $|V_{14} \cup V_{24} \cup V_{34}| \geq 3k/5$ (where $k$ is the order of $\hat{G}$) and a 2-set containing a vertex from each of $V_{14}$ and $V_{24}$ will strongly 4-dominate at least $3/5$ of the vertices or $|V_{121} \cup V_{14}| \geq 3k/5$ and a 2-set containing a vertex from each of $V_{123}$ and $V_{14}$ will strongly 1-dominate at least $3/5$ of the vertices.

**Corollary 3.2.** Let $\hat{G}$ be a 4-coloured complete graph on $k$ vertices that has the property that every set of 3 vertices strongly $c$-dominates strictly less than $3/5$ of the vertices for every colour $c$. If $G_n$ is constructed from $\hat{G}$ as before then, with probability $1 - o(1)$,

$$d^*_F(G_n) \geq k^{-|V(G)|}4^{-|E(G)|} + o(1).$$
holds for some graph $F$ that is, up to a permutation of colours, isomorphic to one of the graphs given in Figure 4.

Proof. By Lemma 3.1 $\hat{G}$ must have a vertex $u$ with $|A_u| = 4$ or two vertices $v, w$ with $|A_v| = |A_w| = 3$ and $A_v \neq A_w$. If we have a vertex $u$ with $|A_u| = 4$ there will be a vertex class of size $n$ in $G_n$ with all its edges coloured uniformly at random. The result trivially holds by considering the density of any 5 vertex graph inside this vertex class.

Suppose instead there exist two vertices $v, w$ in $\hat{G}$ with $|A_v| = |A_w| = 3$ and $A_v \neq A_w$. To ease notation let $c_{xy}$ be the colour of the edge $xy$. Note that by the definition of $\hat{G}$ we know that every vertex is not strongly $c_{vw}$-dominated by the set $\{v, w\}$. Consequently there must exist a vertex $z$ such that $c_{vz} \neq c_{vw}$ and $c_{wz} \neq c_{vw}$. Now consider the vertex classes $V_v, V_w, \text{ and } V_z$ in $G_n$. There are 9 possible 5 vertex graphs that could be formed from taking one vertex in $V_z$ and two vertices in $V_v$ and $V_w$. Only one of these 9 graphs has the property that the two vertices $v_1, v_2$ chosen from $V_v$ satisfy $A_{v_1} = A_{v_2} = A_v$ and the two vertices $w_1, w_2$ chosen from $V_w$ satisfy $A_{w_1} = A_{w_2} = A_w$. This graph is, up to a permutation of colours, isomorphic to one of those given in Figure 4. The result trivially follows by considering its density in $V_v \cup V_w \cup V_z$. 

Although our discussion has centred on 4-colourings, it would also be interesting to know what happens for complete graphs which are $r$-coloured for $r \geq 5$.

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