Stratified sampling is a fast and simple method to generate point sets with super-uniform distribution in hypercubes. However, it has the prominent drawback that the number of sampled points in $n$ dimensions has to be an $n$-th power of an integer number. This forced exponential growth makes its application unattractive or even infeasible in high dimensions. We present a stratification procedure that eliminates this problem by a recursive binary partitioning of the hypercube. To enable the arbitrary number of points, we accept a stratification that is not invariant under rotation or reflection. The algorithm runs in linear time and tries to minimize the hyperboxes’ deviation from the square shape. With another extension, we obtain the Sukharev grid as a special case. We analyze the properties of the algorithm using discrepancy and covering radius.

1 Introduction

Stratified sampling is a classic strategy for generating point sets with super-uniform distribution in hypercubes. The general idea of the algorithm is to partition the space into disjunct strata and then to sample one point per stratum. With an appropriate choice of strata, e.g., as cells of a paraxial grid, a linear run time can be achieved for this procedure. Thus, it is among the fastest methods available for the sampling task and especially suited for large sample sizes. It has potential applications, e.g., in rendering of computer graphics [16, pp. 302-315] or in motion-planning of robots [11, pp. 185-209].

It can be shown that stratified sampling gives certain advantages over random uniform sampling, e.g., when regarding the variance of Monte Carlo estimators [13]. However, in the seminal paper of McKay et al. [13], also Latin hypercube sampling (LHS) was introduced and obtained a clearly more favorable evaluation. Additionally, and perhaps
more importantly, the number of points \( N \) in stratified sampling cannot be chosen freely due to the grid structure. If the resolution (the number of bins) of the grid is the same in each dimension, \( N \) has to be an \( n \)-th power of this integer number. Likewise, if the number of bins is allowed to vary across dimensions, we still have to find a factorization of \( N \) into \( n \) integers, and ideally we would need problem knowledge to decide which dimensions to resolve finer or coarser. In this work, we will show a possible way for giving up the grid structure, and thus enabling an arbitrary number of points without extra requirements. Additionally, we show how sampling each stratum with a non-uniform distribution can improve certain aspects of the resulting point set.

Without loss of generality, we will assume that our region of interest is \( X = [0, 1]^n \subset \mathbb{R}^n \), and we want to uniformly sample \( N \) points in it. We also limit our investigations to algorithms with a linear run time in the number of points. To begin with a graphic example, Fig. 1 compares stratified sampling with Latin hypercube sampling and the Halton sequence \[6\]. The distributions of the Halton sequence and stratified sampling are more uniform than Latin hypercube sampling, but the Halton sequence and Latin hypercube sampling have more uniform one-dimensional projections than the stratified points.

The outline of the paper is as follows. In Sect. 2 we will describe the proposed algorithm. In Sect. 3 we explain why the two summary characteristics discrepancy and covering radius are relevant for the assessment of (uniform) point sets, and how the stratification can be used to efficiently calculate upper bounds for the covering radius. In Sect. 4 we experimentally investigate an implementation of the proposed methodology, before we draw conclusions in Sect. 5.
Algorithm 1 Stratification of a hypercube with arbitrary number of points

**Input:** number of strata \( N \), number of variables \( n \), lower bounds \( \ell = (\ell_1, \ldots, \ell_n)\top \), upper bounds \( u = (u_1, \ldots, u_n)\top \)

**Output:** partition of the space \( A \)

1: \( S_1 \leftarrow (\ell, u) \) // initialize first stratum with hypercube
2: \( N_1 \leftarrow N \) // assign all points to it
3: \( A = \{S_1\} \) // init data structure with currently existing strata
4: **while** \( \exists S_j \in A : N_j > 1 \) **do** // as long as stratum with more than 1 point exists
5: \( N_a \leftarrow \left\lfloor \frac{N_j}{2} \right\rfloor \) // divide number of points as evenly as possible
6: **if** \( N_j \geq 6 \land N_a \mod 2 \neq 0 \land N_j \mod 2 = 0 \) **then**
7: \( N_a \leftarrow N_a - 1 \) // avoid odd numbers of points (optional)
8: **end if**
9: \( N_b \leftarrow N_j - N_a \)
10: \( s \leftarrow \arg \max \{u_{j,i} - \ell_{j,i} \mid i = 1, \ldots, n\} \) // identify the longest side of the stratum
11: \( p_s \leftarrow \ell_{j,s} + (u_{j,s} - \ell_{j,s}) \cdot N_a/N_j \) // calculate the split position
12: \( u_a \leftarrow (u_{j,1}, \ldots, u_{j,s-1}, u_{s}, u_{j,s+1}, \ldots, u_{j,n})\top \)
13: \( \ell_b \leftarrow (\ell_{j,1}, \ldots, \ell_{j,s-1}, p_s, \ell_{j,s+1}, \ldots, \ell_{j,n})\top \)
14: \( S_a \leftarrow (\ell_j, u_a) \) // create stratum with updated upper bounds
15: \( S_b \leftarrow (\ell_b, u_j) \) // create stratum with updated lower bounds
16: \( A \leftarrow A \setminus \{S_j\} \) // remove current stratum
17: \( A \leftarrow A \cup \{S_a, S_b\} \) // add the two smaller ones
18: **end while**
19: **return** \( A \)

2 Algorithm Description

Our algorithm is inspired by the part-and-select algorithm (PSA) of Salomon et al. [19], which is geared to the efficient selection of a uniform subset of a given point set. PSA itself is closely related to several algorithms for vector quantization, as the median-cut algorithm by Heckbert [7], the mean-split algorithm by Wu and Witten [27], and the method of Wan et al. [23]. These approaches are all divisive clustering algorithms using hyperboxes to describe the clusters. The main difference between them is the criterion determining where a cluster is split in two. In contrast to the other algorithms, PSA does not aim to minimize the quantization error, but simply to obtain a uniform subset of the original data. It was developed originally for subset selection in multiobjective optimization, but there are no special assumptions that prevent a universal application.

In [23, pp. 63–68], it was noticed that subset selection algorithms can be easily abused for sampling, by generating random uniform points and selecting from them. However, the resulting distributions may exhibit subtle deviations from uniformity, especially in the boundary region of the hypercube, and the resulting run time is super-linear. Thus, the stratified sampling as presented in Alg. 1 was developed based on a central aspect of PSA and its ancestors, namely the splitting of a hyperbox in half at its longest side. Contrarily to PSA, the algorithm does not have to keep track to which hyperbox each
point belongs, as the points are not yet existing. Instead, each stratum is assigned
a number of points to be later sampled in it. During the partitioning, the algorithm
maintains the invariant that the relative volume of the strata is proportional to the
number of points assigned to them. Thus, in case a stratum contains an odd number
of points, the side lengths of the resulting strata after the split cannot be exactly half
of the previous one, but have to correspond to the integer numbers of points assigned
to each stratum. The partitioning is continued until each stratum is assigned exactly one point.

At this stage, each stratum has a volume of $1/N$ of the initial hypercube. Finally, the
strata are returned and a random uniform point can be sampled from each one. Given
that the used strata are saved, the $N$ sampled points can even be augmented in linear
time by $kN$ new points later, while maintaining the super-uniform distribution.

Started on a hypercube, the algorithm also maintains the invariant that the ratio
between the shortest and longest side of each stratum does not fall below $1/3$. This is
caused by the fact that the most uneven split can appear with three points left. For all
splits involving more points, the split ratio is closer to $1/2$, and because we always split
the longest side, a lower ratio cannot have appeared earlier either. In many cases, it will
be advisable to avoid splits resulting in an odd number of points in both new strata, to
ultimately avoid the extreme case of a $1/3$ ratio (see Fig. 2). Thus, we include lines 6–7
in the algorithm, to detect this situation and move one point to the other new stratum,
if we have at least 6 points left for the current hyperbox.

The linear run time is achieved by proper bookkeeping of the final and the unfinished
strata. This can either be done by using a double-ended queue and appending final and
unfinished strata to different ends, or (preferably) by using two separate data structures.
In the implementation, care should be taken to randomize certain decisions, i.e., $N_a$ and
$N_b$ should be swapped randomly, and in line 10 ties should be broken randomly as well.
An implementation of this algorithm is provided in the Python package diversipy [26].

At values of $N = 2^{ni}, i \in \mathbb{N}$, the result of the algorithm is identical to conventional
stratified sampling. To obtain the conventional stratification also for other $n$-th powers
of arbitrary bases $x \in \mathbb{N}$, the usual algorithm has to be used instead. Figure 2 shows
example outputs of the two algorithms for $N = 12^2$. In a real-world application, Alg. 1
would of course be rather employed for values of $N$ where the conventional algorithm is
not available.

### 2.1 Sampling with Bates Distribution

Alternatively to random uniform sampling, we could use the Bates distribution [10,
p. 297] to obtain a larger expected separation distance between the points. Random
numbers of this distribution are obtained by taking the mean of $b \in \mathbb{N}$ independent
uniformly distributed random numbers. Thus, we can obtain a point $x = (x_1, \ldots, x_n)^T$
by drawing

$$x_i = \frac{1}{b} \sum_{j=1}^{b} U(\ell_i, u_i),$$

4
with \( \ell = (\ell_1, \ldots, \ell_n)^\top \) being the lower and \( u = (u_1, \ldots, u_n)^\top \) the upper bounds of a stratum. \( U(\ell_i, u_i) \) is the uniform distribution, so it is guaranteed by construction that the point \( x \) does not exceed these boundaries. For \( b = 1 \), we retain random uniform sampling, while \( b = \infty \) can be interpreted as deterministically taking the centroid of the stratum. This choice leads us to a sample identical to the Sukharev grid \( [22] \) in the case of conventional stratified sampling. Figure 3 shows some examples with slightly larger point sets than in Fig. 2 and different \( b \) values, created with Alg. 1. It is obvious that with increasing \( b \), the distribution becomes more grid-like, which reduces the uniformity of the low-dimensional projections.

The idea could also be applied to Latin hypercube sampling to obtain intermediate solutions between its perturbed and centered deterministic variants. However, we do not follow this path here.

3 Summary Characteristics

3.1 Covering Radius

The covering radius (CR) is an important measure for global optimization, because a worst-case error bound for the approximation of the global optimum can be given based on it \([15, \text{p. 149}]\). To keep this worst-case bound low, the covering radius should be minimized.

**Definition 1** (Covering radius, Niederreiter \([15, \text{p. 148}]\)). If \((X, d)\) is a bounded metric space and the point set \( P \) consists of \( x_1, \ldots, x_N \in X \), then the covering radius of \( P \) in \( X \) is defined by

\[
\text{d}_{\text{cr}}(P, X) = \sup_{x \in X} \left\{ \min_{1 \leq i \leq N} \{d(x, x_i)\} \right\}.
\]

Niederreiter \([15, \text{p. 148}]\) coined the term dispersion for \( \text{d}_{\text{cr}} \), which sounds antithetic to the necessary minimization of this measure. Thus, we will use the name covering radius,
Figure 3: Examples of stratified sampling with different settings of $N$ and $b$. One-dimensional projections are indicated at the upper and right sides, respectively.

which is used for example by Damelin et al. [2], because $d_{cr}$ is the smallest radius for which closed balls around the points of $P$ completely cover $\mathcal{X}$. Note that in the area of computer experiments, Def. 1 is also known as minimax distance design criterion [9].

Unfortunately, it is quite difficult to calculate $d_{cr}$ in general, due to the involvement of the uncountable $\mathcal{X}$. But Pronzato and Müller [17] give an algorithm for calculating $d_{cr}(P, [0, 1]^n)$ regarding Euclidean distance, based on Delaunay tessellation. It eludes us why the Delaunay tessellation is used in their description, as the Voronoi tessellation is a much more natural fit [11, p. 202]. The algorithm simply consists of mirroring the point set at all lower and upper boundaries of $\mathcal{X}$ and then computing the Voronoi tessellation of the multiplied point set. The covering radius is then obtained by calculating the maximal distance of any Voronoi vertex to its nearest neighbor in $P$. This algorithm has run time $O((nN)^{\left\lfloor n/2 \right\rfloor})$, due to the Voronoi tessellation.

3.1.1 Linear-Time Upper Bound

A lower bound can be obtained for any $L_p$ norm by using a Monte Carlo approach, because if $\mathcal{X}$ is finite with $|\mathcal{X}| = M$, calculation of the measure becomes straightforward.
with run time $O(nMN)$ \cite{[15]}. However, this lower bound is of limited use, because the covering radius is to be minimized. In the following we will show that for stratified sampling, an upper bound can be computed in $O(nN)$, by calculating the distance of each point to the furthest corner of its stratum.

**Proposition 1.** Let $S = [ℓ_1, u_1] \times \cdots \times [ℓ_n, u_n]$. The distance between a given point $x \in S$ and the furthest other point $y^* \in S$, i.e., $\forall y \in S : d(x, y) \leq d(x, y^*)$, is under every $L_p$ norm

$$d(x, y^*) = \left( \sum_{i=1}^{n} \max\{x_i - ℓ_i, u_i - x_i\}^p \right)^{1/p}.$$ 

*Proof.* Suppose there exists a $y^* \in S$ for which (1) does not hold. Then there must be a dimension $i$ with $|x_i - y_i^*| \neq \max\{x_i - ℓ_i, u_i - x_i\}$. If $|x_i - y_i^*| > \max\{x_i - ℓ_i, u_i - x_i\}$, the point would not be in $S$. If $|x_i - y_i^*| < \max\{x_i - ℓ_i, u_i - x_i\}$, there exists a $y' \in S$ with $|x_i - y_i^*| < |x_i - y_i'|$ and subsequently $d(x, y^*) < d(x, y')$, so that $y^*$ is not optimal, in contradiction to our assumption. \hfill \Box

Note that $d(x, y^*)$ is essentially the covering radius $d_{cr}(x, S)$ of a single point $x \in S$ for this closed set $S$. To obtain the upper bound for $d_{cr}$, it is sufficient to calculate

$$\overline{d_{cr}}(P, X) := \max\{d_{cr}(x_i, S_i) \mid i = 1, \ldots, N\},$$

where $S_i$ are the strata with $\bigcup_{i=1}^{N} S_i = X$ and $x_i \in S_i$ are the corresponding sample points. For the correctness of the upper bound the strata are not necessarily required to be disjunct. It suffices that the whole $X$ is covered and only one point is sampled per stratum. In Sect. \cite{[3]}, we will show experimentally that this bound is often tight.

### 3.2 Discrepancy

In the area of quasi-Monte Carlo methods, a lot of theory has been developed regarding error bounds of estimated integrals, depending on the point sequences used for the numerical integration. To achieve low error bounds, point sets must possess a low discrepancy. Niederreiter \cite{[15], p. 13] states that “discrepancy can be viewed as a quantitative measure for the deviation from uniform distribution.” Several types of discrepancy can be defined by changing the aggregation of individual deviations or by considering differently shaped subsets of the region of interest. The first theoretical results have been obtained using an $L_\infty$ norm for aggregation and considering the family $J^*$ of all subsets $J = [0, u_1] \times \cdots \times [0, u_n]$ of the unit hypercube \cite{[15], p. 14]. This way, the discrepancy

$$D_N^* = \sup_{J \in J^*} \left| \frac{N_J}{N} - \text{vol}(J) \right|$$

was defined, where $\text{vol}(J) = \prod_{i=1}^{n} (u_i - ℓ_i)$ is the volume of the respective subset and $N_J = |\{x \mid x \in P \land x \in J\}|$ is the number of points of the set that fall into it. Using
$D_N^*$ and an appropriate definition for the variation $V(f)$ of $f$, the integration error can be bounded by the Koksma-Hlawka inequality [15, p. 20]

$$\left| \frac{1}{N} \sum_{i=1}^{N} f(x_i) - \int_{X} f(x) \, dx \right| \leq V(f)D_N^*(P). \tag{4}$$

This result suggests that it is generally advisable to minimize the discrepancy to obtain low integration errors, if $f$ shall be treated as a black box. Unfortunately, several problems are associated with $D_N^*$. First of all, calculating $L_\infty$ discrepancies is an NP-hard problem [4], which makes it infeasible in most practical situations. Secondly, Santner et al. [20, pp. 146–148] give an example where $D_N^*$ favors points along the diagonal of the region of interest, and thus does not reflect the human intuition of uniformity. They further argue that discrepancy’s relation to integration error is not necessarily relevant in the context of computer experiments [20, p. 144]. To avoid the run time problem, the $L_2$ norm of the deviations from uniformity is usually taken, leading to the discrepancy

$$T_N^* = \left( \int_{X} \left( \frac{N_J}{N} - \text{vol}(J) \right)^2 \, dx \, dy \right)^{1/2}. \tag{4}$$

It should be noted that integration error can in principle also be bounded by $L_2$ discrepancy [14] [21] [24]. However, also $T_N^*$ is disputed. Morokoff and Caflisch [14] write: “While useful in theoretical discussions due to its relationship with $D_N^*$, $T_N^*$ suffers as a means of comparing sequences and predicting performance because of the strong emphasis it puts on points near 0.” In the example of Santner et al., this means that $D_N^*$ regards the diagonal as better than the antidiagonal, which also seems dubious. Even worse, Matoušek [12] shows that $T_N^*$ generally gives unreliable results for $N < 2^n$. As a workaround, the more general family $J$ of subsets $J = [\ell_1, u_1) \times \ldots \times [\ell_n, u_n)$ can be considered, yielding

$$T_N = \left( \int_{X \times X, x_i < y_i} \left( \frac{N_J}{N} - \text{vol}(J) \right)^2 \, dx \, dy \right)^{1/2}. \tag{4}$$

This unanchored discrepancy formula at least gets rid of the origin’s special role, but no information could be found on its effect on the problem addressed by Matoušek. The analogous formula for $D_N$ is obtained by simply replacing $J^*$ with $J$ in (3).

While a lot of the literature deals with theoretical bounds on the discrepancy of quasirandom sequences, we are interested in computing the discrepancy of arbitrary point sets here. Morokoff and Caflisch [14] give the following explicit formula for $T_N$, which can be computed in $O(N^2 n)$:

$$(T_N)^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \prod_{k=1}^{n} \left( 1 - \max\{x_{i,k}, x_{j,k}\} \right) \cdot \min\{x_{i,k}, x_{j,k}\}$$

$$- \frac{2^{1-n}}{N} \sum_{i=1}^{N} \prod_{k=1}^{n} x_{i,k}(1 - x_{i,k}) + 12^{-n}. \tag{4}$$
A useful property of discrepancy is the possibility to compute its expected value for a random uniform point set. For \((T_N)^2\) the formula is \[14\]
\[
E((T_N)^2) = \frac{6^{-n}(1 - 2^{-n})}{N}.
\]

Hickernell \[8\] proposes several other variants of discrepancy that possess certain additional invariance properties. However, it seems that the corresponding expected values are unknown, so we will keep using \(T_N\) instead.

4 Experimental Analysis

With its linear run time of \(O(nN)\), Alg. 1 is in principle a competitor to quasirandom sequences, such as the Sobol’ sequence \[21\] or the generalized Halton sequence \[5\]. Thus, we compare stratified sampling to such algorithms in terms of discrepancy and covering radius.

The general lower bound for any covering radius of \(N\) points in the \(n\)-dimensional unit hypercube is
\[
\frac{1}{2^\left\lfloor N^{1/n} \right\rfloor},
\]
according to Sukharev \[22\]. In Fig. 4, we evaluate sets of up to 1024 points, generated by Alg. 1 (including lines 6–7). We plot the upper bound \(2\), a Monte Carlo lower bound using \(M = 10N\) samples, and the exact value, as far as it can be computed in a timely manner. The exact algorithm as described in Sect. 3.1 is realized with the Qhull library \[1\], via its SciPy interface. For a better visualization, the values are divided by the general lower bound \(6\), which is a step function that changes only at \(n\)-th powers of integers. These positions are indicated by vertical dotted gray lines. The plots show that the upper bound is quite tight and often exact (29%, 21%, and 47% of the cases where we computed the exact values in two, three, and five dimensions, respectively), while the lower bound with so few points is naturally quite noisy. However, already in this setting the lower bound is much more expensive than the upper bound, and its quality deteriorates with increasing dimension.

Event plots at the bottom of each subfigure indicate the cases where Alg. 1 without lines 6–7 achieves a lower covering radius. Interestingly, these cases are clustered together. Quite naturally, the avoidance of odd splits can only considerably reduce the covering radius for even numbers of points, which is the reason why we see high-frequency oscillations in the upper bound, especially in low dimensions. With increasing dimension, however, this effect vanishes together with the usefulness of lines 6–7. So, in ten dimensions the two algorithm variants almost always lead to the same covering radius. Table 1 in the appendix contains the actual numbers of wins and losses for the two algorithm variants.

Asymptotically, every low-discrepancy point set also has a low covering radius (but not vice versa) \[15\], p. 152]. However, for concrete point sets, quite a trade-off can be detected between the two, as our experiments show. This is illustrated in Fig. 5 where
Figure 4: Covering radius for the new stratified sampling algorithm with \( b = \infty \). The event plots at the bottom of the subfigures indicate the cases where Alg. 1 without lines 6–7 obtains a lower covering radius than with lines 6–7.

We investigate sets of sizes 100 and 4900 in dimensions 2, 5, and 10. Results of stratified sampling with different values of \( b \) are shown as colored dots. For a comparison, we add random uniform points, Latin hypercube designs, generalized Halton sequences\(^1\) and, where possible, a Sukharev grid. Each configuration is sampled 30 times. (For the generalized Halton sequence, only two possible permutations exist in two dimensions.) The vertical line marks \( 6 \), the horizontal line \( 5 \). Where it is feasible we plot the exact covering radius, otherwise we use the upper bound \( 2 \) and a Monte Carlo lower bound with \( M = 10^4 \cdot 2n \). For the point sets that do not result from stratified sampling, we first have to find a partition of the unit hypercube to compute the upper bound for CR. We do

\(^1\) Implemented in the ghalton Python library at https://github.com/fmder/ghalton\(^3\). While Sobol’ sequences may be more popular overall, we choose this software for its ease of use, under the assumption that other quasirandom sequences would give similar results in our analysis.
Figure 5: Discrepancy and covering radius of stratified sampling, depending on parameter $b$ of the Bates distribution. In case of approximate CR values, the marker is drawn at the upper bound and an error bar extends to the lower bound.
this with a variation of the mean-split algorithm \cite{27}, which needs \(O(nN \log_2 N)\) average run time. This algorithm variant chooses the longest side of a hyperbox for a split, as the original (ties are broken randomly). However, the heuristic for the determination of the split position differs. Ours works as follows: we calculate the mean value of the points in the chosen dimension as a preliminary split position. Then, we identify the closest neighbor points on either side of the preliminary position. The actual position is then chosen as the mean of these two points. Once we have partitioned the whole space, with each hyperbox containing one point, the upper bound can be computed just as for stratified sampling. We iterate this process 10 times, due to the stochasticity in the partitioning, and take the best upper bound we find.

The figures show that stratified sampling fills a gap between the Sukharev grid and the generalized Halton sequence. In low dimensions, i.e., at least for \(n = 2\), it is easily possible to keep both discrepancy and covering radius below the values of random uniform points. In higher dimensions, larger sample sizes would be required to obtain the same effect. Otherwise, the difference between stratified sampling with \(b = 1\) and non-stratified samplings vanishes for the investigated sample sizes. However, it is always possible to improve the covering radius, albeit at the cost of a worse discrepancy.

Figure 6 gives another view of the trade-off between discrepancy and covering radius in two and three dimensions. Here, we plot the measures against the sample size, with \(\lfloor 400/n \rfloor\) stochastic replications to filter out the noise. Solid lines mark the median and dotted lines pointwise 95% confidence intervals for the median. The figure confirms our previous observations: we can see that the generalized Halton sequence has the best convergence order for discrepancy, followed by stratified sampling with \(b = 1\). LHS acts very similarly to random uniform sampling, and stratified sampling with \(b > 1\) exhibits spikes at \(N = 2^ni, i \in \mathbb{N}\), corresponding to the cases where the conventional stratified sampling is recovered. As already seen in Fig. 5 the discrepancy of stratified sampling with \(b \gg 1\) is much worse than with random uniform sampling (for \(n \geq 3\) and \(N \leq 2^{12}\)). For the covering radius, the ordering of the algorithms is more or less reversed, with the exception of the Halton sequence, which is still better than LHS and random uniform sampling.

5 Conclusion

We presented a novel stratified sampling algorithm, which uses binary space partitioning to enable an arbitrary number of points to be sampled. Uniformity is guaranteed by ensuring that the final partition of the space consists of equal-sized hyperboxes. Furthermore, we proposed to sample each hyperbox with the Bates distribution. By varying the distribution's parameter, a changeover between low discrepancy and low covering radius can be obtained. Regarding these two measures, the resulting samples fill a gap between the Sukharev grid and quasirandom sequences. Additionally, we demonstrated how to compute lower and upper bounds for the covering radius, because the exact computation has a prohibitive time complexity in high dimensions due to a necessary Voronoi tessellation. To our best knowledge, the upper bound has not been considered before,
(a) $n = 2$

(b) $n = 3$

Figure 6: Discrepancy and covering radius of various point sets, depending on sample size. The horizontal black line marks the expected discrepancy for random uniform points.

although it is important for minimizing the covering radius. The upper bound only needs linear time and reuses the hyperboxes of the stratified sampling. It can also be obtained for arbitrary point sets, if we build a partition of the space around the points retroactively. In the future, it should be investigated if any theoretical results can be obtained regarding its tightness.

Regarding applications, it remains to be seen which one benefits most from the compromise offered by Alg. 1. We shall certainly investigate global optimization, because it has the most obvious (and yet only little explored practically) connection to the covering radius, and probably only requires sample sizes $< 10^6$. Monte Carlo-applications may be less suitable, because sample sizes are often in the millions and quasirandom sequences have the advantage over LHS and stratified sampling that they can generate points on-
Table 1: Comparing the two variants of Alg. 1 regarding covering radius.

| Dimension | w/ Lines 6–7 Wins | Ties | w/o Lines 6–7 Wins | Measure |
|-----------|-------------------|------|--------------------|---------|
| 2         | 822               | 96   | 102                | exact CR|
| 3         | 610               | 96   | 314                | exact CR|
| 4         | 800               | 94   | 126                | exact CR|
| 5         | 698               | 120  | 202                | exact CR|
| 6         | 720               | 167  | 133                | UB      |
| 7         | 712               | 271  | 37                 | UB      |
| 8         | 517               | 466  | 37                 | UB      |
| 9         | 136               | 807  | 77                 | UB      |
| 10        | 142               | 778  | 100                | UB      |

line, i.e., without keeping all previous points in memory. We could also devise some kind of an online variant for stratified sampling by interpreting Alg. 1 as a depth-first approach, but not with all the benefits of the quasirandom sequences, such as uniformity guarantees for subsequences and the ability to continue indefinitely.

Finally, Alg. 1 may be also interesting for aesthetic reasons. E.g., computer graphics applications such as procedural texture generation may benefit from it, because the random uniform variant does not exhibit the same subtle patterns as quasirandom sequences. Also the strata themselves might be interesting structures for texture generation (see Fig. 2).

Appendix

Table 1 compares the two variants of Alg. 1 regarding the covering radius. Point sets from 4 to 1024 points were tested in two to ten dimensions. The variant containing lines 6–7 has an advantage in the majority of cases.

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