Higher order Josephson effects

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Received 8 September 2009
Published 11 May 2010
Online at stacks.iop.org/JPhysA/43/225301

Abstract
Gaussian linking of superconducting loops containing Josephson junctions with enclosed magnetic fields gives rise to interference shifts in the phase that modulates the current carried through the loop, proportional to the magnitude of the enclosed flux. We generalize these results to higher order linking of a superconducting loop with several magnetic solenoids, and show that there may be interference shifts proportional to the product of two or more fluxes.

PACS numbers: 03.65.−w, 85.25.Dq, 85.25.Cp, 03.75.Lm

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Interference effects, both constructive and destructive, are mainstays for distinguishing between quantum and classical phenomena. Examples include interfering scattering amplitudes, both bosonic and fermionic, formation of condensates, entanglement, etc. The Aharonov–Bohm effect [1] describes the self-interference of a charged particle that can travel along two semiclassical paths whose combined path is Gaussian linked with a magnetic solenoid carrying the flux \( \Phi \). The measurable phase shift is \( \phi \propto \Phi \). We have argued in [2] that there could exist generalizations to cases of higher order linkings. The simplest example is a Borromean ring arrangement where the semiclassical path corresponds to one ring, which has higher order linking with two flux tubes carrying fluxes \( \Phi_1 \) and \( \Phi_2 \), which make up the other two rings. We found that the phase shift in this system is \( \phi \propto \Phi_1 \Phi_2 \). Higher order cases were explored in [3, 4] and shown to be related to commutator algebras of homotopy generators of the configuration space \( \mathbb{R}^3 \setminus \left\{ T_1 \cup T_2 \right\} \), where \( T_1 \) and \( T_2 \) are the tubes containing the fluxes. The same general logic can be applied to systems of superconductors, Josephson junctions and magnetic fluxes where the Josephson effect can arise [5]. Here we will study...
interference in a symmetric arrangement of two identical semicircular superconductors joined by two identical Josephson junctions and derive the response of such systems. We conclude with a discussion of possible applications.

In the case of Gaussian linking of a loop of superconductor with a magnetic solenoid, the Mercereau effect [6] is due to the phase change in the macroscopic wavefunction, which is in turn related to the currents in the superconducting components. The effect is due to the presence of a vector potential $A$, which is the fundamental object responsible for the phase change. Exploration of higher order linking is again due to the presence of a vector potential but in these instances it requires careful choices of gauge.

2. The Josephson effect

It will be sufficient for our purposes to consider a macroscopic model of superconductors. Following Feynman [7], we approximate the superconductors coupled via a Josephson junction as a two-level system. Let $\psi_1$ and $\psi_2$ be the states, and $E_1$ and $E_2$ the energy levels of the superconductors. The Schrödinger equation for the coupled system of the two superconductors becomes

$$i\hbar(\partial/\partial t) = E_1 \psi_1 + K \exp(i\phi) \psi_2,$$

$$i\hbar(\partial/\partial t) = E_2 \psi_2 + K \exp(-i\phi) \psi_1,$$

where $K$ is the coupling energy and $\phi$ is a phase, which arises from the most general Hermitian Hamiltonian $2 \times 2$ matrix. The dependence of $\phi$ on the vector potential $A$ can be found from gauge invariance considerations. For a gauge transformation $A \mapsto A + \nabla f$, $\psi_1 \mapsto \psi_1 \exp(iq f / 2\hbar c)$, $\psi_2 \mapsto \psi_2 \exp(-iq f / 2\hbar c)$ with an arbitrary function $f$ of space coordinates, we find $\phi \mapsto \phi + q f / \hbar c$, from which it follows that $\phi = (q / \hbar c) \int [A \cdot d\mathbf{x}]$. Here $q = 2e$ is the charge of an electron pair.

After the substitutions $\psi_1 = |\psi_1| \exp(i\theta_1)$ and $\psi_2 = |\psi_2| \exp(i\theta_2)$, the Schrödinger equation becomes

$$\hbar(\partial/\partial t) |\psi_1|^2 / \partial t = 2K |\psi_1||\psi_2| \sin \theta,$$

$$\hbar(\partial/\partial t) |\psi_2|^2 / \partial t = -2K |\psi_1||\psi_2| \sin \theta,$$

$$-\hbar |\psi_1|(\partial \theta_1 / \partial t) = E_1 |\psi_1| + K |\psi_2| \cos \theta,$$

$$-\hbar |\psi_2|(\partial \theta_2 / \partial t) = E_2 |\psi_2| + K |\psi_1| \cos \theta,$$

where $\theta = \phi + \theta_2 - \theta_1$. The current from superconductor 1 to superconductor 2, which is equal to minus the current from superconductor 2 to superconductor 1, is thus $I = (2K / \hbar)|\psi_1||\psi_2| \sin \theta$. (In a self-consistent computation, a current from a battery which connects the two superconductors is also included. The result for the superconducting current is precisely $I$; see, for example, [8].) The electron densities in the two superconductors are approximately equal and independent of time; let $\rho$ be this common constant. This gives $I = I_0 \sin \theta$, where $I_0 = 2K \rho / \hbar$. Integrating the phase equations, we find

$$\theta(t) = \phi + \theta(0) + \hbar^{-1} \int_0^t (E_1(t') - E_2(t')).$$

The quantity $(E_1(t) - E_2(t)) / q$ represents an electric potential applied to the junction. The dc and ac Josephson effects [5] arise for $|E_1(t) - E_2(t)| \ll K$ and $|E_1(t) - E_2(t)| \gg K$, respectively.
Figure 1. A diagram of an experimental setup for the detection of the Josephson effect. $C'$ and $C''$ are the paths from the point $P$ to the point $Q$ through the superconductors with the Josephson junctions $J'$ and $J''$ and the total current $I$ from $P$ to $Q$. $C_1$ is the magnetic solenoid carrying the flux $\Phi_1$. The Josephson effect (for a review see [9]) is due to the first-order (Gaussian) linking of the closed curves $C = C'C'' - C_1$.

Our interest here is in the Josephson effect with zero potential across the junction, $E_1(t) - E_2(t) = 0$, and nonzero magnetic field constrained to the opening of the superconducting ring with two Josephson junctions, see figure 1. Let $\theta'$ and $\theta''$ be the phase changes due to the vector potential $A_1$ of the currents through the junctions $J'$ and $J''$. The phase changes from the point $P$ to the point $Q$ along the paths $C'$ and $C''$ are

$$\phi' = \theta' + (q/\hbar c) \int_{C'} A_1 \cdot dx,$$

$$\phi'' = \theta'' + (q/\hbar c) \int_{C''} A_1 \cdot dx.$$

Since the wavefunction is single valued, this requires $\phi' = \phi''$, and so we find that $\theta'' - \theta' = 2\pi \Phi_1/\Phi_0$. Here $\Phi_1 = \oint_{C_1} A_1 \cdot dx$ is the flux due to the solenoid along $C_1$ passing through a surface spanned by a closed curve $C = C'C'' - C_1$ and $\Phi_0 = 2\pi \hbar c/q$ is the flux quantum. The total current from the point $P$ to the point $Q$ is

$$I = I_0 \sin \left( \frac{1}{2} (\theta' + \theta'') \right) \cos \left( \frac{\pi \Phi_1}{\Phi_0} \right).$$

For a fixed value of $\Phi_1$, the corresponding maximal total current is

$$I_{\text{max}} = I_0 \left| \cos \left( \frac{\pi \Phi_1}{\Phi_0} \right) \right|,$$

which itself has maxima when $\Phi_1 = n\Phi_0$, $n \in \mathbb{Z}$.

The flux is actually $\Phi_1 = \Phi_{1,\text{ext}} + LI$ where $\Phi_{1,\text{ext}}$ is the external flux through the loop, $L$ is the self-inductance, but here and in what follows we assume that $L$ is negligible. (We have made a number of simplifying assumptions, for example, that self-inductance of SQUID components are negligible, none of which, if relaxed, affect our basic conclusions.)

We will call the phenomena reviewed in this section the first-order Josephson effects to distinguish them from their generalizations which we now proceed to describe.

3. The second and higher order Josephson effects

Now consider the case where we have two solenoids carrying the magnetic fluxes $\Phi_1$ and $\Phi_2$ and whose center lines run along $C_1$ and $C_2$, and a superconducting ring along the closed curve $C = C'C'' - C_1$ with two Josephson junctions $J'$ and $J''$ in parallel as shown.
Figure 2. A diagram of an experimental setup for the detection of the second-order Josephson effect. \(C_1\) and \(C_2\) are the magnetic solenoids. \(C'\) and \(C''\) are the paths from the point \(P\) to the point \(Q\) through the superconductors connected by Josephson junctions \(J'\) and \(J''\). The total current from \(P\) to \(Q\) is \(I\). \(C_1\) and \(C_2\) are the magnetic solenoids carrying the fluxes \(\Phi_1\) and \(\Phi_2\). The second-order Josephson effect is due to the second-order linking of the set of three closed curves \(C = C' C'' - 1\), \(C_1\) and \(C_2\).

Figure 3. Classic form of the Borromean rings.

Figure 4. Borromean rings altered by fixing \(C_1\) and \(C_2\) as hard rings, allowing \(C\) to be flexible and then pulling on \(C_1\) and \(C_2\).

in figure 2. The two solenoids and the superconducting ring are in a Borromean ring [10] configuration. Note that in this arrangement neither \(C_1\) nor \(C_2\) has Gaussian linking with the superconducting ring \(C\), nor do \(C_1\) and \(C_2\) link with each other. However, the set of three rings \(C, C_1, C_2\) is indeed linked. This second-order linking and its higher order generalizations are what will lead to our results. We will find that even though our system lacks first-order (Gaussian) linking, a phase difference can still be nonzero upon traveling around the superconductor.

To begin the analysis of the Borromean rings, we first need to show the precise form of the path \(C\). In particular, we will show that \(C\) has no Gaussian linking with \(C_1\) or \(C_2\), yet the three paths are still inextricably linked. First, we draw the three rings, see figure 3. Now assume that \(C\) is flexible, but \(C_1\) and \(C_2\) are not, and pull on \(C_1\) and \(C_2\) to arrive at the configuration shown in figure 4. Now pinch \(C\) as shown in figure 5 and note that we have labeled \(C\) as a set of four components. Following the full circuit around \(C\), we first travel along \(a_1\), followed by
This is precisely the case for the Boromean rings, and hence over a closed path is path independent if and only if the path can be written as a commutator, \( a_1a_2a_1^{-1}a_2^{-1} \).

\[ \Phi_{12}(C) = k_2 \frac{e^2}{\hbar c^2} \Phi_1 \Phi_2 \]  

Once physical units have been restored. Note that this result is independent of the initial point \((\delta_1, \delta_2)\) where we choose to start the path \(C\). It is straightforward to check that an integral of \(A_{12}\) over a closed path is path independent if and only if the path can be written as a commutator. This is precisely the case for the Boromean rings, and hence \(A_{12}\) is an appropriately chosen gauge potential.
Now returning to our specific generalized Josephson configuration in the form of Borromean rings, we need only replace \( A_1 \) in the usual Josephson configuration with \( A_{12} \) in the phase integrals (8) and (9) so that \( \theta'' - \theta' \) becomes \( 4\pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2 \). Hence, the total current from the point \( P \) to the point \( Q \) is
\[
I = I_0 \sin \left( \frac{1}{2} (\theta' + \theta'') \right) \cos \left( 2\pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2 \right).
\]
For fixed values of \( \Phi_1 \) and \( \Phi_2 \), the maximal total current flowing in the superconductor is
\[
I_{\text{max}} = I_0 | \cos \left( 2\pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2 \right) |.
\]

The smallest value of the constant \( k_2 > 0 \) for which the fluxes \( \Phi_1 = m_1 \Phi_0 \), \( \Phi_2 = m_2 \Phi_0 \), where \( m_1, m_2 \in \mathbb{Z} \), lead to maxima of the quantity \( I_{\text{max}} \) is \( k_2 = (2\pi)^{-1} \). This is precisely the value we obtained in [2] by imposing an analog of the Dirac string condition on the second-order phase for the Aharonov–Bohm effect. Nevertheless, the value of \( k_2 \) must ultimately be determined by experiment.

Also, for the value \( k_2 = (2\pi)^{-1} \), if either \( \Phi_1 \) or \( \Phi_2 \) is equal to \( \Phi_0 \) or \( -\Phi_0 \), then in terms of the other flux, appropriately relabeled, expression (18) for the second-order \( I_{\text{max}} \) reduces to expression (11) for the first-order \( I_{\text{max}} \).

An essential feature of the above result is that the current \( I \) is a periodic function of the quantity \( \pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2 \) with period 1. We can also derive this property by modifying the method used by Block [12] for the first-order Josephson effect as follows.

The total gauge potential includes internal and external parts, \( A = A_{\text{in}} + A_{\text{ext}} \), the external magnetic field being due to the external sources. Assuming that the external field \( \nabla \times A_{\text{ext}} \) vanishes inside the superconductors, we can write \( A_{\text{ext}} = \nabla \gamma_{\text{ext}} \). As a result, \( A \) is a gauge transformation of \( A_{\text{in}} \), and so
\[
\psi(A_{\text{in}} + A_{\text{ext}}) = \psi(A_{\text{in}}) \exp(i q \gamma_{\text{ext}} / \hbar c).
\]
Since \( \psi(A) \) is single valued, we find that \( \psi(A_{\text{in}}) \) is multiplied by the factor \( \exp(-4i\pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2) \) after the charge \( q \) travels around a closed curve \( C \). This factor is a periodic function of \( \pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2 \) with period 2. This implies the same periodicity property for the wavefunction \( \psi(A_{\text{in}}) \) and the energy \( E \). Assuming time reversal symmetry as in [12], we find that the free energy and thus the current, which is given by minus the derivative of the free energy with respect to the external flux, are both periodic functions of \( \pi^2 k_2 \Phi_1 \Phi_2 / \Phi_0^2 \) with period 1, in agreement with the result proved earlier.

More generally, it is straightforward to arrange \( n \) solenoids with the fluxes \( \Phi_1, \ldots, \Phi_n \) and a superconducting ring in such a way that they are linked with nonzero \( n \)-th-order linking [10]. We similarly find the phase difference
\[
\theta'' - \theta' = (2\pi)^n k_n \Phi_1 \cdots \Phi_n / \Phi_0^n,
\]
the current
\[
I = I_0 \sin \left( \frac{1}{2} (\theta' + \theta'') \right) \cos \left( \frac{1}{2} (2\pi)^n k_n \Phi_1 \cdots \Phi_n / \Phi_0^n \right)
\]
and its maximal value for fixed values of \( \Phi_1, \ldots, \Phi_n \)
\[
I_{\text{max}} = I_0 | \cos \left( \frac{1}{2} (2\pi)^n k_n \Phi_1 \cdots \Phi_n / \Phi_0^n \right) |.
\]
Other properties of these systems can be investigated.

The smallest value of the constant \( k_n > 0 \) for which the fluxes \( \Phi_j = m_j \Phi_0 \), where \( m_j \in \mathbb{Z} \), lead to the maxima of the quantity \( I_{\text{max}} \) is \( k_n = (2\pi)^{1-n} \). Again this is precisely the value we obtained in [2, 3] by imposing an analog of the Dirac string condition on the phase for the \( n \)-th-order Aharonov–Bohm effect. Nevertheless, as pointed out with the \( k_2 \) case above, the value of \( k_n \) must ultimately be determined by experiment.
Also, for the value \( k_n = (2\pi)^{1-n} \), if one of the fluxes is equal to \( \Phi_0 \) or \(-\Phi_0\), then in terms of the remaining fluxes, appropriately relabeled, expression (22) for the \( n \)th-order \( I_{\text{max}} \) reduces to the analogous expression (22) for the \((n - 1)\)st-order \( I_{\text{max}} \).

Similar to the case \( n = 2 \) above, we can modify the method used by Block and prove that for any \( n \) the current \( I \) is a periodic function of the quantity \( \frac{1}{2} (2\pi)^{n-1} k_n \Phi_1 \cdots \Phi_n / \Phi_0^n \) with period 1.

4. Discussion and conclusion

Our previous work on the subject of higher order phases [2, 3] focused on generalizations of the Aharonov–Bohm experiment because we believe higher order interference effects are most easily understood in the language of the Aharonov–Bohm effect where wavefunction self-interference is most evident as measured by electron holography. In contrast, this paper, while being intimately related to the previous work, is focused on higher order phase effects in superconducting quantum interference experiments. Specifically, we have generalized the Josephson effect from the first-order (Gaussian) linking case to second-order linking of the Borromean ring type and have shown that the second-order phase in the current depends on the product of the magnetic fluxes in the two solenoids as shown in figure 2. We also discussed higher order analogs where a superconducting loop containing Josephson junctions links with several magnetic solenoids and where the resulting interference shifts are proportional to the product of multiple fluxes. We hope that the present paper is useful to the low-temperature community since it translates and extends our previous results into an alternate form that suggests how one would measure higher order interference effects via currents flowing through SQUIDs. The above results should be useful for providing a broader understanding of higher order interference effects.

The Josephson junction experimental setup we suggest for measuring higher order phases is perhaps somewhat over simplified but still sufficiently realistic that it could be used as a starting point for an experimentalist who wants to make a practically designed experiment that incorporates contingencies to deal with potential issues, such as stray fields modifying the Hamiltonian, experimental separation of first and higher order effects, etc. We should mention that if higher order interference effects are found in one system, then it would suggest that there could be many other systems with similar behavior, from neutron beams to atomic Bose–Einstein condensates.

One can conceive of a number of applications for devices built to take advantage of higher order linking. Such a system could be less invasive than first-order devices because it could keep the SQUID some distance from an experimental sample. Possible applications include both rf and dc SQUIDs that measure higher order linking of multiple fluxes. Under some circumstances such devices could be useful in measurements of complex biological systems, or any systems where direct Gaussian linking of a magnetic flux with a SQUID is impractical, but where higher order linking is possible. For example, one could have a system of (i) a fixed but adjustable flux tube, i.e. a solenoid; (ii) an unknown flux to be measured; and (iii) a SQUID. If the three components can be arranged to have higher order linking, then the unknown flux could be measured, even though it has no Gaussian linking with the SQUID.

Acknowledgments

The work of RVB was supported by DOE grant number DE-FG02-91ER40661 and that of TWK by DOE grant number DE-FG05-85ER40226.
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