Quantum Pontryagin Principle under Continuous Measurements and Feedback

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Abstract

In this note we develop the theory of the quantum Pontryagin principle for continuous measurements and feedback. The analysis is carried out under the assumption of compatible events in the output channel. The plant is a quantum system, which generally is in a mixed state, coupled to a continuous measurement channel. The Pontryagin Maximum Principle is derived in both the Schrödinger picture and Heisenberg picture, in particular in statistical moment coordinates. To avoid solving stochastic equations we derive a LQG scheme which is more suitable for control purposes.

I. INTRODUCTION

Quantum optimal control (QOC, for short) is a powerful tool for achieving quantum control objectives in many practical problems of interest for the emerging field of Quantum Technologies (see [1], [2], and the references therein). It has been successfully used in a vast area of quantum applications from physical chemistry, [3], to multi-dimensional nuclear magnetic resonance experiments, [4], through time-optimal control problems, [5], [6].

In constrained optimization, the Pontryagin Maximum Principle (PMP) is an attractive technique based on the variational method of Lagrange multipliers. Few results have appeared in recent years based on the explicit calculation of the extremal solutions and those are applicable only to discrete low-dimensional systems: fast generation of a given-structure wave package, [7]; adiabatic population transfer in Λ-systems via intermediate states that are subject to decay, [11]; fast adiabatic cooling in harmonic traps and non-interacting collection of harmonic oscillators with a shared frequency, [12], [13], [14], [15], [16]; optimal cooling power of a reciprocating quantum refrigerator, [17]; time-optimal control for one or two spins (or qubits), [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]; minimum-time adiabatic-like paths for the expansion of a quantum piston, [28].

In those applications the external controls were determined via a deterministic Pontryagin principle based on the controlled Schrödinger equation:

$$\frac{d}{dt} \psi = i H(u) \psi$$

where $H(u)$ is the Hamiltonian driven by a control action $u$ and $\psi$ is a state vector in the Hilbert space of the system to be controlled. In most of these aforementioned approaches, the feedback is absent [29]. In other words, the system is not subjected to any measurement and a subsequent quantum filtering, which reduces the Pontryagin principle to its deterministic version. On the other hand, a few results promote ad-hoc model-dependent solutions, [7], [17].

Motivated by these previous studies, here we address the task of stating both a firm theoretical ground and a formalization of the quantum PMP for systems with quantum feedback provided by continuous weak measurements [8], [9]. This may help the subsequent verification procedures to check if the resulting control is optimal [10].

In the Schrödinger picture the conditional dynamics of the system evolving under these weak measurements is described by quantum filtering theory. Filtering concerns the processing of the information yielded by the measurement process. This information is generally incomplete because the state is not
fully accessible by the measurement setting, and is inherently corrupted by noise. In this context, optimal control problems are solved by using a cost function expressed in terms of the state given by the filter, which is often called an information state. The quantum Belavkin filter, or stochastic master equation, computes this information state [30].

The importance of a quantum Pontryagin principle stems from the fact that allows us to tackle with state constraints. Indeed this stands for a future borderline and challenge by itself where a formulation in the Pontryagin’s style is mandatory.

This work is organized as follows. In Section II the QOC problem is analyzed in a quantum probability space. There, measurement operators are considered for the abelian subalgebra generated by commuting projectors. As a result, filtrations in the quantum probability space as well as adapted processes are defined in terms of von Neumann subalgebras. This allows us to define feasible and admissible control action in terms of operators. Section III is devoted to the derivation of the quantum PMP in a global form from the Hamilton-Jacobi-Bellman equation in the Schrödinger picture. The result is a system of coupled forward-backward stochastic differential equations that replaces the HJB partial differential equation.

In addition, as a very natural way of defining state-space realizations is in the Heisenberg picture, particularly in coordinates of expectation and variance, we derive in section IV a Hamilton-Jacobi-Bellman equation in these coordinates as a preceding step to build a quantum PMP. One drawback of stochastic PMP is that the control problem cannot be solved in closed form, and one needs to resort to numerical optimization. To overcome this difficulty, for quantum linear models [31] we derive a LQG from the quantum PMP based on the statistical moment coordinates. This adds a novel result to the literature on measurement based optimal control of quantum linear systems. In the last decade or so, there has been a significant body of work on the systems theory and control for the quantum linear models, including measurement-based and coherent feedback control of this class of systems (theory and experiments), as well as their various applications (see [32], [33] and references therein). Finally, the concluding remarks are given in Section V.

II. OPTIMAL QUANTUM CONTROL PROBLEM

Any quantum system can be associated to a separable complex Hilbert space $\mathcal{H}$ on which a von Neumann algebra of linear operators $\mathcal{A}$ is defined (a Banach algebra of bounded operators on $\mathcal{H}$). Unbounded operators are assumed to have spectral projectors in $\mathcal{A}$. For a system in a mixed state, i.e. a quantum system in the presence of classical uncertainty, the predual space is needed $\mathcal{A}_\ast$, and in particular those positive elements of $\mathcal{A}_\ast$ with unit trace, which are called density operators or normal states. Then a quantum state $\rho \in S(\mathcal{H})$ is such that $\rho = \rho^\dagger \geq 0$ and $Tr(\rho) = 1$. The convex set of normal states is denoted as $S(\mathcal{H}) \subset \mathcal{A}_\ast$.

Here we consider a quantum system evolving with a controlled-Hamiltonian $H$ while coupled to a measurement apparatus in a Markovian manner via an auxiliary field [35]. Let $\mathcal{F}$ be the Hilbert space of that field, initially in its vacuum state $\phi$. The operator describing the coupling of the auxiliary field to the measurement channel is given by $L$ which does not generally need to be self-adjoint.

For a given time horizon $T$, let $\Omega_T$ be the set of outcomes $\omega_T$ of a continuous measurement on the quadrature operator $L + L^\dagger$ (sometimes this sample set is called space of measurement trajectories):

$$\Omega_T = \{\omega_T : 0 \leq \tau \leq T\}$$

where, yet even when $L$ is finite dimensional, and thus has a finite discrete spectrum, a continuous-measurement of $L + L^\dagger$ can take on continuous values for $\omega_T$ (e.g., when homodyne or heterodyne detection is performed), not only values in the spectrum of $L + L^\dagger$.

In addition, we only observe compatible events in this channel by commuting projectors $\{P_\omega\}_{\omega \in \Omega_T}$. The change of the state of the observed system induced by the measurement leads to posterior states...
which are described in mathematical terms as stochastic processes. Specifically, the reduced conditional evolution is described by a nonlinear map, [35],

$$
\rho (t) \mapsto \rho_w (t) = \frac{\operatorname{Tr}_\mathcal{F} (U (t) (\rho \otimes \phi) U^* (t) P_\omega)}{\operatorname{Tr} (U (t) (\rho \otimes \phi) U^* (t) P_\omega)}
$$

where \( \operatorname{Tr}_\mathcal{F} (\cdot) \) is the partial trace over \( \mathcal{F} \) and \( U (t) \) is the unitary evolution operator of the composite system quantum field. We denote by \( \rho_w \) the posterior state after a measurement with an outcome given by \( \omega \in \Omega_T \). In this regard, the state of a continuously measured quantum system is a \( \mathcal{S} \)-valued stochastic process on \( \Omega_T \), i.e. \( \rho_* : \Omega_T \times \mathbb{R}^+ \to \mathcal{S} \). We use the symbol \( \bullet \) as a subscript when the kernel symbol describes a random variable and we are not interested in displaying the variable \( \omega \).

We consider a measurement system characterized at the output of a quantum noisy channel by a white noise process \( \xi = (\xi_t)_{t \geq 0} \). The quantum white noise is usually written as an integrated process \( W (t) = \int_0^t \xi (s) \, ds \). Given the space \( \Omega_T \) we denote by \( \Omega_{[0,T]} \) the space of all functions \( \gamma : [0,T] \to \Omega_T \), \( t \mapsto \omega_t \). We also introduce the coordinate process \( W = (W_t)_{t \geq 0} \) on \( \Omega_{[0,T]} \) by setting \( W_t (\gamma) \equiv \omega_t \) for each \( \gamma \in \Omega_{[0,T]} \) and \( t \in [0,T] \). On the basis of the integrated process \( W (t) \) a natural choice of probability space is the canonical one \( (\mathcal{C} [0,T] , \mathcal{F}, \mathbb{P}) \) where \( \mathcal{F} = B(\mathcal{C} [0,T]) \) (the Borel \( \sigma \)-algebra of the space of continuous functions on \( [0,T] \)), and \( \mathbb{P} \) being the Wiener measure. On this canonical probability space, the coordinate process \( W \) is a Wiener process which represents the innovations process.

A quantum probability space can be defined as the pair \((\mathcal{A}, \rho)\) where \( \mathcal{A} \) is a von Neumann algebra which describes quantum random variables and observables and the quantum state \( \rho \) is the probability law that specifies the statistical properties of the system variables whose dynamics are governed by quantum stochastic differential equations.

Given a filtered quantum probability space \( \{ (\mathcal{A}, \rho, \mathcal{A}_t) , t \geq 0 \} \) satisfying the usual condition (i.e. \( \mathcal{A}, \rho \) is complete, \( \mathcal{I} \in \mathcal{A}_0 \) and \( \{ \mathcal{A}_t \} , t \geq 0 \) is right continuous), on which a standard Wiener process \( W \) is defined, the conditioned state of the system, \( \rho_* (t) \) which is continuously updated using the output of the measurement apparatus, will then satisfy a stochastic Schrödinger equation of the type

$$
d\rho_* (t) = w (t, u (t) , \rho_* (t)) \, dt + \sigma (\rho_* (t)) \, dW_t \tag{1}
$$

where according to Ito calculus \( dW_t \, dW_t = dt \) holds. If one denotes by \( y_t \) the integrated measurement process from the apparatus, then the corresponding innovation process is determined through \( dW_t = dy_t - \pi_t (L + L^\dagger) \, dt \) with \( \pi_t (X) = \operatorname{tr} (\rho_w (t) X) = \langle \rho_w (t) , X \rangle \). Roughly speaking, the innovation process give the difference between the observations and our expectations. In Eq.\( (1) \) \( w : [0,T] \times U \times \mathcal{S} \to \mathcal{S} \) refers to the unconditional evolution of states (the dual of a Lindblad generator \( \mathcal{L} \)):

$$
w (t, u, \rho) = \mathcal{L}^\dagger (\rho) = - \frac{i}{\hbar} [H, \rho] + \frac{1}{2} (L [\rho, L^*] + [L, \rho] L^*)
$$

Here the space of control actions \( U \) is Hausdorff. The fluctuation operator \( \sigma : \mathcal{S} \to \mathcal{S} \) is assumed to be independent of the control action \( u \) and nondegenerate, [36]:

$$
\sigma (\rho) = L \rho + \rho L^\dagger - \langle \rho, L + L^\dagger \rangle \rho
$$

\( U \) is a given separable metric space, and \( T \in (0, \infty) \) is fixed.

The stochastic equation \((1)\) describing the evolution of \( \rho_* (t) \) under a continuous measurement process is referred to as the Belavkin quantum filter. Furthermore, the stochastic differential equation for the conditional expectation of any operator \( X \) (the conditional expectation of \( X \) with respect to a measurement output \( \omega_t \) up to time \( t \)) \( \pi_t (X) \) is given by

$$
d\pi_t (X) = \pi_t (\mathcal{L} (X)) \, dt + \langle \pi_t (XL + L^\dagger X) - \pi_t (L + L^\dagger) \pi_t (X) \rangle dW_t
$$

where \( \mathcal{L}^\dagger (X) \) is the adjoint of a generator in Lindblad form, [37].
By the information field \(\{A_i, t \geq 0\}\), the controller is well-informed of what has happened in the past but, because of the uncertainty of the system, it is not able to predict the future. As a consequence there exists a non-anticipative restriction on the controller: for any instant the controller cannot decide its control action before the instant occurs. This restriction is expressed in mathematical terms as \("u(\cdot) is \{A_i\}_{t \geq 0} \text{ adapted}\), and the control is taken from the set

\[
U_{[0,T]} = \left\{ u : \Omega_T \times [0, T] \to U : u(\cdot) \text{ is } \{A_i\}_{t \geq 0} \text{ adapted} \right\}
\]

Any \(u_*(\cdot) \in U_{[0,T]}\) is called a feasible control. The process \(u_*(\cdot)\) transforms the master equation into a SDE with random coefficients. In particular fixing a control the drift and the dispersion depend on \(\Omega_T\), i.e. \(w, \sigma : \Omega_T \times \mathbb{R}^+ \to A_\ast\). Given a \(A_0\)-measurable density matrix \(\rho\) for every \(u_*(\cdot)\) the master equation admits a unique solution \(\rho(t;u)\) (seen as a continuous \(\{A_i, t \geq 0\}\) -adapted process) if

(i) \(\rho(0) = \rho\) almost sure in probability.
(ii) \(\int_0^T (|w(\tau, \rho_\omega(\tau), u_\omega(\tau))| + ||\sigma(\tau, \rho_\omega(\tau))||^2) d\tau < \infty\) for \(t \geq 0\), almost sure in probability for \(\omega \in \Omega_T\).

The adapted solution exists and it is unique if the drift and the dispersion are continuous measurable functions and they satisfy a Lipschitz condition.

Let \(\rho_*(\cdot)\) be the solution to the filtering equation with initial condition \(\rho_*(0) = \rho\). The cost \(J\) for a feasible control action \(u_*(\cdot)\) is a random variable on \(\Omega_T\), i.e. \(J : \Omega_T \times U_{[0,T]} \times \mathcal{S} \times [0, T] \to \mathbb{R}\), defined by

\[
J_*(u_*(\cdot); \rho_*(\cdot), t) = \int_0^T C(\tau, u_*(\tau), \rho_*(\tau)) d\tau + M(\rho_*(T))
\]

where the cost density \(C\) and the terminal cost \(M\) are linear; for each \(\omega \in \Omega_T\) and each instant \(\tau \in (0, T]\):

\[
C(\rho_\omega(\tau), \rho_\omega(\tau)) = \langle \rho_\omega(\tau), C(\tau, u_\omega(\tau)) \rangle
\]

\[
M(\rho_\omega(T)) = \langle \rho_\omega, M \rangle
\]

with \(C\) and \(M\) positive self-adjoint operators. Note that this cost may be different according to the requirements of the QOC problem, such as minimizing the control time, the control energy, the error between the final state and target state, or a combination of these. The goal is to minimize the criterion by selecting a nonanticipative decision among the ones satisfying all the quantum state constraints:

**Definition 1:** Let \(\{(A, \rho, A_i), t \geq 0\}\) be a filtered quantum probability space with the usual condition and let \(W\) be a given standard \(\{A_i, t \geq 0\}\) -Wiener process. A control \(u_*(\cdot)\) is called q-admissible, and \((\rho_*(\cdot), u_*(\cdot))\) is a q-admissible pair if

(i) \(u_*(\cdot) \in U_{[0,T]}\).
(ii) \(u_*(\cdot)\) is the unique solution to the master equation \(\Pi\).
(iii) \(\rho_*(\cdot)\) satisfies a prescribed state constraint.
(iv) For each \(\omega \in \Omega_T\), \(C(\cdot, u_\omega(\tau), \rho_\omega(\tau)) \in L_\mathcal{A}(0, T; \mathbb{R})\) and \(S(\rho_\omega(T)) \in L_{\mathcal{A}_T}^1(\Omega_T; \mathbb{R})\).

Here the spaces \(L_\mathcal{A}(0, T; \mathbb{R})\) and \(L_{\mathcal{A}_T}^1(\Omega_T; \mathbb{R})\) are defined on the filtered probability space \(\{(A, \rho, A_i), t \geq 0\}: L_\mathcal{A}(0, T; \mathbb{R})\) is the set of all \(\{A_i\}_{t \geq 0}\)-adapted \(\mathbb{R}\)-valued processes \(X(\cdot)\) such that \(E \left[ \int_0^T |X(t)| dt \right] < \infty\) and \(L_{\mathcal{A}_T}^1(\Omega_T; \mathbb{R})\) is the set of \(\mathbb{R}\)-valued \(\mathcal{A}_T\)-measurable random variables \(X\) such that \(E[|X|] < \infty\).

The filtration \(\{A_i, t \geq 0\}\) as well as the Wiener process \(W\) are fixed and independent of the control. The set of all q-admissible controls will be denoted by \(U_{ad}[0, T]\). The quantum optimal control (QOC) problem can be stated as follows:

**Problem QOC:** Minimize \(E[J_*(u_*(\cdot); \rho_*(\cdot), t)]\) over \(U_{ad}[0, T]\), where \(E[\cdot]\) denotes the expectation value on \(\Omega_T\), i.e. the expectation over all the possible continuous trajectories \(\omega_t\) with \(0 \leq t \leq T\).

The goal is to find \(u_*(\cdot) \in U_{ad}[0, T]\) such that

\[
E[J_*(u_*(\cdot))] = \min_{u_*(\cdot) \in U_{ad}[0, T]} E[J_*(u_*(\cdot))]
\]
The control task can be formulated as a problem of searching for a set of admissible controls satisfying the system dynamic equations while simultaneously minimizing a cost functional. Any \( u^*_\text{(·)} \in \mathcal{U}_{ad}[0, T] \) satisfying (3) is called a q-optimal control and if it is unique the problem QOC is said to be q-
solvable. The corresponding state process \( \rho^*_\text{\textbullet}(\cdot) \) and the state-control pair \( (u^*_\text{\textbullet}(\cdot), \rho^*_\text{\textbullet}(\cdot)) \) will be called a q-optimal state process and a q-optimal pair, respectively.

III. QUANTUM PONTRYAGIN PRINCIPLE IN THE SCHröDINGER PICTURE

The objective of this section is to derive the quantum Pontryagin’s Maximum Principle in a global form from the Hamilton-Jacobi-Bellman equation (HJB) for quantum optimal control. The result is a system of coupled forward-backward stochastic differential equations that replaces the HJB partial differential equation.

An admissible control \( u\text{\textbullet}(\cdot) \) is \( \{A_t, t \geq 0\} \)-adapted implies that the quantum state \( \rho \) is actually not uncertain for the controller at time \( t \), and this means that \( \rho \) is almost surely deterministic under an appropriate probability measure. Given a q-optimal control \( u^*_\text{\textbullet}(\cdot) \), let us denote \( S(t, \rho) \) as a minimum posterior cost-to-go (sometimes called the value function):

\[
S(t, \rho) = \min_{u \text{\textbullet}(\cdot) \in \mathcal{U}_{ad}[t, T]} \mathbb{E}[J_t(t, \rho; u\text{\textbullet}(\cdot))] \quad \forall t \in [0, T) \times \mathcal{S}
\]

which solves the quantum HJB equation derived from the quantum filtering equation for all \( (t, \rho) \in [0, T) \times \mathcal{S} \):

\[
-\frac{\partial S(t, \rho)}{\partial t} = \min_{u \in \mathcal{U}} \mathcal{H} \left( t, u, \rho, \frac{\partial S(t, \rho)}{\partial \rho}, \left( \frac{\partial}{\partial \rho} \otimes \frac{\partial}{\partial \rho} \right) S(t, \rho) \right)
\]

where \( \mathcal{H} \) is the generalized Hamiltonian

\[
\mathcal{H}(t, u, \rho, P) = \frac{1}{2} \langle \sigma(\rho) \otimes \sigma(\rho), P \rangle
- \langle w(t, u, \rho), P \rangle
+ C(t, u, \rho)
\]

The operators \( p(t, \rho) \) and \( P(t, \rho) \) are defined as follows

\[
p(t, \rho) = \frac{\partial}{\partial \rho} S(t, \rho)
\]

\[
P(t, \rho) = \left( \frac{\partial}{\partial \rho} \otimes \frac{\partial}{\partial \rho} \right) (S(t, \rho))
\]

The operational differentiation of \( S: \mathbb{R}^+ \times \mathcal{A}_\star \rightarrow \mathbb{R} \) with respect to the density matrix \( \rho \) is computed in a natural way:

\[
\frac{\partial S}{\partial \rho} = \begin{pmatrix}
\frac{\partial S}{\partial \rho_{11}} & \ldots & \frac{\partial S}{\partial \rho_{1n}} \\
\vdots & \ddots & \vdots \\
\frac{\partial S}{\partial \rho_{n1}} & \ldots & \frac{\partial S}{\partial \rho_{nn}}
\end{pmatrix}
\]

and

\[
\left( \frac{\partial}{\partial \rho} \otimes \frac{\partial}{\partial \rho} \right) (S(t, \rho)) = \begin{pmatrix}
\frac{\partial^2 S}{\partial \rho \rho_{11}} & \ldots & \frac{\partial^2 S}{\partial \rho \rho_{1n}} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 S}{\partial \rho \rho_{n1}} & \ldots & \frac{\partial^2 S}{\partial \rho \rho_{nn}}
\end{pmatrix}
\]

In the master equation the fluctuation operator \( \sigma \) does not depend on the control action and is not degenerated. At any time instant the controller is knowledgeable about some information (as specified by the information field \( \{A_t, t \geq 0\} \) of what has occurred up to that moment, but not able to predict what is going to happen afterwards due to the uncertainty of the system. It is remarkable that \( \{A_t, t \geq 0\} \) is the natural filtration generated by \( W \) so that the unique source of uncertainty proceeds from the noise of
the measurement and all the past information around the noise is available to the controller. Let \( u^* (\cdot) \) be the optimal control action in the convex control domain \( \mathcal{U} \).

From the definition of \( p \) and \( P \) it is obvious that

\[
\mathcal{H} (t, u^*, \rho, p, q) = \inf_{u(\cdot) \in \mathcal{U}} \mathcal{H} (t, u, \rho, p, P)
\]

The quantum filtering equation together with the optimal control can be written in terms of \( \mathcal{H} \):

\[
d\rho (t) = -\frac{\partial \mathcal{H}}{\partial \rho} (t, u^*, \rho, p, P) \, dt + \sigma (\rho) \, dW, \tag{6}
\]

where from here in advance for short \( dW_t \equiv dW \) is used. Given that the adjoint operator \( p \) depends both on time and the density operator \( \rho \), we can compute its differential resorting to Itô’s lemma:

\[
dp = \frac{\partial p}{\partial t} \, dt + \left\langle \frac{\partial }{\partial \rho}, d\rho \right\rangle (p) + \frac{1}{2} \left\langle \frac{\partial }{\partial \rho} \otimes \frac{\partial }{\partial \rho}, d\rho \otimes d\rho \right\rangle (p)
\]

where the inner products are differential operators applied to the adjoint variable \( p \). In particular, the tensor product \( d\rho \otimes d\rho \) can be rewritten from (6) by noting that \( \frac{\partial \mathcal{H}}{\partial \rho} (t, u^*, \rho, p, P) = -w (t, u^*, \rho) \):

\[
d\rho \otimes d\rho = (w \otimes w) \, dt^2 + (w \otimes \sigma (\rho) \, dW) \, dt
\]

\[
+ (\sigma (\rho) \, dW \otimes w) \, dt + (\sigma (\rho) \otimes \sigma (\rho)) \, dW^2
\]

For \( dt \to 0 \) we use the Itô’s rules: \( dt^2 \to 0, dW dt \to 0, dW^2 \to dt \). Thus \( d\rho \otimes d\rho = (\sigma (\rho) \otimes \sigma (\rho)) \, dt \) and

\[
dp = \frac{\partial p}{\partial t} \, dt + \left\langle \frac{\partial }{\partial \rho}, d\rho \right\rangle (p) + \frac{1}{2} \left\langle \frac{\partial }{\partial \rho} \otimes \frac{\partial }{\partial \rho}, \sigma (\rho) \otimes \sigma (\rho) \right\rangle (p) \, dt \tag{7}
\]

Inserting \( d\rho \) into (7) yields

\[
dp = \frac{\partial p}{\partial t} \, dt + \left\langle \frac{\partial }{\partial \rho}, \frac{\partial \mathcal{H}}{\partial \rho} (t, u, \rho, p, P) \right\rangle (p) \, dt
\]

\[
+ \left\langle \frac{\partial }{\partial \rho}, \sigma (\rho) \, dW \right\rangle (p)
\]

\[
+ \frac{1}{2} \left\langle \frac{\partial }{\partial \rho} \otimes \frac{\partial }{\partial \rho}, \sigma (\rho) \otimes \sigma (\rho) \right\rangle (p) \, dt
\]

Given that \( p (t, \rho) = \frac{\partial S(t, \rho)}{\partial \rho} \) it follows that \( \frac{\partial p}{\partial t} = \frac{\partial}{\partial \rho} \frac{\partial S(t, \rho)}{\partial \rho} \), and from (1):

\[
-\frac{\partial p}{\partial t} = \frac{\partial}{\partial \rho} \mathcal{H} (t, u^*, \rho, p, P) = \frac{\partial \mathcal{H}}{\partial \rho} + \left\langle \frac{\partial \mathcal{H}}{\partial q}, \frac{\partial }{\partial \rho} (\cdot) \right\rangle (p) +
\]

\[
+ \left\langle \frac{\partial \mathcal{H}}{\partial \rho}, \frac{\partial }{\partial \rho} (\cdot) \otimes \frac{\partial }{\partial \rho} (\cdot) \right\rangle (p)
\]

As a result,

\[
dp = -\frac{\partial \mathcal{H}}{\partial \rho} \, dt + \left\langle \frac{\partial }{\partial \rho}, \sigma (\rho) \, dW \right\rangle (p) \tag{8}
\]

The last equation can be rewritten in a compact form by defining the first order adjoint operator

\[
q (t, \rho) = \left\langle \frac{\partial }{\partial \rho}, \sigma (\rho) \right\rangle \left( \frac{\partial S(t, \rho)}{\partial \rho} \right)
\]

The evolution of the adjoint operator \( p (t, \rho) \) is given by

\[
dp = -\frac{\partial \mathcal{H}}{\partial \rho} \, dt + q (t, \rho) \, dW \tag{9}
\]
The first order adjoint system $(p(t,\rho), q(t,\rho))$ is a pair of $\{A_t, t \geq 0\}$-adapted processes which give a solution of the backward stochastic differential equation (9). Furthermore every pair satisfying equation (9) is an adapted solution. The adjoint operators $p, q \in L^2(0,T;A_\star)$ now live in the same space and have the same dimension. We are in order to write the systems of forward-backward quantum differential equations for the Quantum Pontryagin principle:

$$d\rho^* = -\frac{\partial H^*}{\partial p} dt + \sigma^*(\rho^*) dW$$

$$dp^* = -\frac{\partial H^*}{\partial \rho} dt + q^*(\rho^*) dW$$

$$\rho^*(0) = \rho_0$$

$$p^*(T) = \frac{\partial S(T,\rho^*(T))}{\partial \rho}$$

where the superscript ‘*’ indicates that the terms are optimal.

Here an explicit equation with boundary conditions for the first order adjoint operator $q$ is not necessary since $q$ is connected to $p$ in a differential way.

IV. QUANTUM PONTRYAGIN PRINCIPLE IN THE HEISENBERG PICTURE

A. Quantum Linear Model

We make the following standard assumptions, [31], [38]:

(A1) The environment (or thermal bath) is described by a quantized electromagnetic field, i.e. a collection of quantum harmonic fields each of them corresponding to a mode of the field at a given angular frequency. The interaction between the system and the environment admits a field interpretation as a transmission line.

(A2) We assume the rotating wave approximation i.e. the neglect of highly-oscillating terms in the energy flowing between the system and the free field.

(A3) The system operators coupled to the environment have a strength independent of the frequency (this is due to a first Markov approximation).

Assumptions (A2) and (A3) are necessary to obtain a quantum stochastic differential equations and an idealized “white noise”.

(A4) The measurement process is indirect by sensing the effect of the system on the environment via a radiated field. If $b_{in}(t)$ and $b_{out}(t)$ are an input field and an output field respectively, the integrals $B_{in}(t) = \int_{t_0}^t b_{in}(\tau) d\tau$ and $B_{out}(t) = \int_{t_0}^t b_{out}(\tau) d\tau$ are interpreted as a noise (a quantum Wiener process) whenever the state of the field is incoherent, e.g. a thermal equilibrium state or when the field is in vacuum.

(A5) We couple the open quantum system to $d$ measurement channels (independent noise field inputs) via coupling operators $L_i$ (for the i-th channel). The indirect measurement is developed through a coupled measurement channel playing the role of a quantum noise bath.

For a system of annihilation operators $\{X_k : k = 1, \ldots, m\}$ and a system of creation operators $\{X_{k+m} = X_k^\dagger : k = 1, \ldots, m\}$, let $X_-$ be the stacking of annihilation operators and $X_+$ the stacking of creation operators. We define the state $X$ in the Heisenberg picture as $X = (X_-^T X_+^T)^T$. In a multiple-boson system the operators $X_k$ and $X_k^\dagger$ are not Hermitian and satisfy the canonical commutation relations:

$$[X_j, X_k] = \delta_{j+m,k} I$$

$$[X_j, X_k] = \left[X_j^\dagger, X_k^\dagger\right] = 0$$

where $\delta_{j,k}$ is the Dirac delta. Defining a matrix of state commutation $[X, X]$ with $[X_i, X_j]$ as the entry $(i,j)$, it is easy to verify that $[X, X] = (\mathbb{S} \otimes I)$ where $\mathbb{S}$ stands for the symplectic matrix

$$\mathbb{S} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$
Also we define the system Hamiltonian

\[ H_{sys} = \frac{1}{2} X^\top (R \otimes I) X \]

where \( R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \in \mathbb{R}^{2m \times 2m} \) is selected in such way that \( H_{sys} \) is Hermitian. The adjoint of the system Hamiltonian operator is \( H_{sys}^\dagger = \frac{1}{2} X^\top (\mathbb{J} R^T \mathbb{J} \otimes I) X \) where \( \mathbb{J} \) is the antidiagonal matrix

\[ \mathbb{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \]

From this fact it follows that \( R_{11}^T = R_{22}, R_{12}^T = R_{21} \) and \( R_{12} = R_{21} \).

For each measurement channel we define a control action and these actions are collected in a vector \( u \in \mathbb{R}^d \). The control can be carried out by the coupling of one or more tunable electromagnetic fields. We define the controlled Hamiltonian \( H(u) \) as

\[ H(u) = \frac{1}{2} \left( X^\top (K u \otimes I) + (u^T K^\dagger \otimes I) X \right) = \frac{1}{2} \left( u^T (K^T + K^\dagger) \otimes I \right) X \]

where \( K \in \mathbb{R}^{2m \times d} \) stands for a complex matrix of gains. Let us write \( K = \begin{pmatrix} K_- \\ K_+ \end{pmatrix} \), for \( H(u) \) to be Hermitian, \( H(u) = H(u)^\dagger = \frac{1}{2} \left( u^T (K^T + K^\dagger) \mathbb{J} \otimes I \right) X \), and it is necessary that \( \text{Re}(K_-) = \text{Re}(K_+) \).

We couple the open quantum system with internal Hamiltonian \( H_{sys} \) and controlled Hamiltonian \( H(u) \) to \( d \) measurement channels (independent noise field inputs) via the vector operator \( L = (\Gamma \otimes I) X \); this is an open quantum system with multiple field channels where \( \Gamma \) is an appropriate operator.

Then a quantum linear model in the state space representation based on annihilators has the form

\[
\begin{align*}
dX_t &= (AX_t + Bu_t) dt + FdV_t \\
dY_t &= (CX_t + Dv_t) dt + GdW_t
\end{align*}
\]

where \( V_t \) and \( W_t \) represent quantum noise in the form of Wiener process, and

\[
\begin{align*}
A &= \frac{i}{2\hbar} \left( -\mathbb{S} \left( R^T + R + \mathfrak{F} \left( \mathbb{J} \Gamma^\dagger \Gamma \right) \right) \otimes I \right) \\
B &= -\frac{i}{2\hbar} (\mathbb{S} (K + K^*) \otimes I) \\
v_t &= u_t \otimes I
\end{align*}
\]

with \( \mathfrak{F} \left( \mathbb{J} \Gamma^\dagger \Gamma \right) = \mathbb{J} \Gamma^\dagger \Gamma - \Gamma^T \Gamma^* \mathbb{J} \). In most of the cases \( \Gamma \) is real so that \( \mathfrak{F} \left( \mathbb{J} \Gamma^\dagger \Gamma \right) = 0 \).

**B. Hamilton-Jacobi-Bellman Equation in Expectation and Variance Coordinates**

Let us define a complete set of coordinates describing the total probability distribution given by \( \rho \), \([39] \):

\[
\begin{align*}
\hat{X}_i &= \langle \rho, X_i \rangle \\
\Sigma_{ij} &= \langle \rho, X_i X_j \rangle - \hat{X}_i \hat{X}_j
\end{align*}
\]

with dynamics, \([40] \),

\[
\begin{align*}
d\hat{X}_t &= \left( A\hat{X}_t + Bu_t \right) dt + \hat{K}_t d\hat{Y}_t \\
\hat{K}_t &= (\Sigma C^T + M)
\end{align*}
\]
where $d\tilde{Y}_t$ is the innovation martingale and $M$ is a covariance matrix of noise increments.

Let us assume that we have achieved an optimal control action $u^*(\cdot)$ in the interval $[t + \Delta t, T]$. Applying the optimality principle the problem is reduced to search for an optimal solution $u(\cdot)$ in the interval $[t, t + \Delta t]$: 

$$u(s) = \begin{cases} 
  u(s) & s \in [t, t + \Delta t] \\
  u^*(s) & s \in [t + \Delta t, T]
\end{cases}$$

Under these conditions the cost-to-go function $S(t, \hat{X}, \Sigma)$ can be divided in two parts:

$$S(t, \hat{X}, \Sigma) = \min_{u_*(\cdot) \in U_{ad}[t, T]} \mathbb{E} \left[ J_\bullet \left( t, \hat{X}, \Sigma; u_*(\cdot) \right) \right] = S(t, \hat{X}, \Sigma) = \min_{u_*(\cdot) \in U_{ad}[t, T]} \mathbb{E} \left[ \int_t^{t+\Delta t} C(\tau, u_*(\tau), \hat{X}_*(\tau), \Sigma_*(\tau)) d\tau \right] + S(t + \Delta t, \hat{X}, \Sigma)$$

Note that $\hat{X}$ and $\Sigma$ are stochastic processes so that we can apply the Itô’s calculus:

$$dS(t, \hat{X}, \Sigma) = \frac{\partial S}{\partial t} + \left\langle \frac{\partial}{\partial \hat{X}}, d\hat{X} \right\rangle (S) + \frac{1}{2} \left\langle \left( \frac{\partial}{\partial \hat{X}} \otimes \frac{\partial}{\partial \hat{X}} \right) d\hat{X} \otimes d\hat{X} \right\rangle (S)$$

Let us observe that the term $d\Sigma \otimes d\Sigma$ is not included in (12) since the differential equation for the covariance matrix does not include uncertainty. As $dt \to 0$ the Itô’s rules lead to $dt^2 \to 0$, $d\tilde{Y} dt \to 0$, and $(d\tilde{Y} \otimes d\tilde{Y}) \to dt$, resulting in $d\hat{X} \otimes d\hat{X} = (\tilde{K} \otimes \tilde{K}) dt$, and

$$dS(t, \hat{X}, \Sigma) = \frac{\partial S}{\partial t} + D S(t, \hat{X}, \Sigma) dt + \left\langle \frac{\partial}{\partial \hat{X}}, \tilde{K} d\tilde{Y} \right\rangle (S) dt$$

where the stochastic differential operator $D$ is now defined in the following terms:

$$D := \left\langle \frac{\partial}{\partial \hat{X}}, A \hat{X} + Bu \right\rangle + \left\langle \left( \frac{\partial}{\partial \Sigma} \right) (\cdot), G(\Sigma) \right\rangle$$

with $G(\Sigma) = A\Sigma + \Sigma A^T - \tilde{K}_t^T \tilde{K}_t$. Writing (13) in integral form yields

$$S(t + \Delta t, \hat{X}, \Sigma) = S(t, \hat{X}, \Sigma)$$

$$+ \int_t^{t+\Delta t} \left( \frac{\partial S}{\partial \tau} + D S(\tau, \hat{X}, \Sigma) \right) d\tau$$

$$+ \int_t^{t+\Delta t} \left\langle \frac{\partial}{\partial \hat{X}}, \tilde{K} d\tilde{Y} \right\rangle (S(\tau, \hat{X}, \Sigma)) d\tau$$

$$=S(t, \hat{X}, \Sigma) + \int_t^{t+\Delta t} \left( \frac{\partial S}{\partial \tau} + D S(\tau, \hat{X}, \Sigma) \right) d\tau$$

$$+ \int_t^{t+\Delta t} \left\langle \frac{\partial}{\partial \hat{X}}, \tilde{K} d\tilde{Y} \right\rangle (S(\tau, \hat{X}, \Sigma)) d\tau$$

Note that $\hat{X}$ and $\Sigma$ are stochastic processes so that we can apply the Itô’s calculus:
Folding (14) into (11),

\[ S\left( t, \hat{X}, \Sigma \right) = \min_{u^{*}(\cdot) \in U_{ad}[t,T]} \mathbb{E}[S\left( t, \hat{X}, \Sigma \right)] \]

\[ + \int_{t}^{t+\Delta t} C\left( \tau, u^{*}(\tau), \hat{X}^{*}(\tau), \Sigma^{*}(\tau) \right) d\tau \]

\[ + \int_{t}^{t+\Delta t} \left( \frac{\partial S\left( \tau, \hat{X}, \Sigma \right)}{\partial \tau} + D S\left( \tau, \hat{X}, \Sigma \right) \right) d\tau \]

where we have account for

\[ \mathbb{E} \left[ \int_{t}^{t+\Delta t} \left\langle \frac{\partial}{\partial \hat{X}}, \tilde{K}d\tilde{Y} \right\rangle \left( S\left( \tau, \hat{X}, \Sigma \right) \right) d\tau \right] = 0 \]

for being \( \tilde{Y} \) a Gaussian process.

Now we can take expectations on both sides of (15) and put \( S\left( t, \hat{X}, \Sigma \right) \) out of the minimization since it does not depend on \( u \):

\[ 0 = \min_{u^{*}(\cdot) \in U_{ad}[t,T]} \mathbb{E}\left[ \int_{t}^{t+\Delta t} \left( C\left( \tau, u^{*}(\tau), \hat{X}^{*}(\tau), \Sigma^{*}(\tau) \right) \right. \]

\[ + \left. \frac{\partial S\left( \tau, \hat{X}, \Sigma \right)}{\partial \tau} + D S\left( \tau, \hat{X}, \Sigma \right) \right) d\tau \]

In order to satisfy this expression the integrand should be zero and interchanging the minimum operation with the integral the HJB equation in the Heisenberg picture is finally derived:

\[ \min_{u(\cdot) \in U_{ad}[t,T]} \left\{ C\left( t, u, \hat{X}, \Sigma \right) + \frac{\partial S\left( t, \hat{X}, \Sigma \right)}{\partial t} + D S\left( t, \hat{X}, \Sigma \right) \right\} = 0 \]

or equivalently

\[ -\frac{\partial S\left( t, \hat{X}, \Sigma \right)}{\partial t} = \min_{u(\cdot) \in U_{ad}[t,T]} \mathcal{H}\left( t, u, \hat{X}, \Sigma \right) \]

where

\[ \mathcal{H}\left( t, u, \hat{X}, \Sigma, \frac{\partial S}{\partial \hat{X}}, \left( \frac{\partial}{\partial \hat{X}} \otimes \frac{\partial}{\partial \hat{X}} \right) (S), \Sigma, \frac{\partial S}{\partial \Sigma} \right) = \]

\[ C\left( t, u, \hat{X}, \Sigma \right) + D S\left( t, \hat{X}, \Sigma \right) \]

Henceforth, the control action \( u(\cdot) \) minimizing \( S \) can be found in terms of \( t, \hat{X}, \Sigma, \frac{\partial S}{\partial \hat{X}}, \left( \frac{\partial}{\partial \hat{X}} \otimes \frac{\partial}{\partial \hat{X}} \right) (S), \Sigma, \) and \( \frac{\partial S}{\partial \Sigma} \).
C. Pontryagin Principle in Coordinates  \( \hat{X} \) and  \( \Sigma \)

On the basis of the HJB formulation in the Heisenberg picture we can derive the Pontryagin’s maximum principle. To this end we previously define the adjoint variables for the quantum optimal control problem:

\[
\begin{align*}
p_X &= \frac{\partial S(t, \hat{X}, \Sigma)}{\partial \hat{X}} \\
q_X &= \left( \frac{\partial}{\partial \hat{X}} \otimes \frac{\partial}{\partial \hat{X}} \right) \left( S(t, \hat{X}, \Sigma) \right) \\
p_\Sigma &= \frac{\partial S(t, \hat{X}, \Sigma)}{\partial \Sigma}
\end{align*}
\]

Note that the covariance matrix  \( \Sigma \) has a deterministic dynamics (without Gaussian noise) so that  \( q_\Sigma = 0 \).

Also we define the Hamiltonian function  \( H \) in the Heisenberg picture as

\[
H(t, u, \hat{X}, p_X, q_X, \Sigma, p_\Sigma) = C(t, u, \hat{X}, \Sigma) + \left( p_X, A\hat{X} + Bu \right) + \frac{1}{2} \left( q_X, \left( \tilde{K} \otimes \tilde{K} \right) \right) + \langle p_\Sigma, G(\Sigma) \rangle
\]

With these definitions the quantum version of (18) is as follows

\[
- \frac{\partial S(t, \hat{X}, \Sigma)}{\partial t} = \min_{u(\cdot) \in U[t, T]} H(t, u, \hat{X}, p_X, q_X, \Sigma, p_\Sigma)
\]

At this point it is assumed that there exists a unique minimizing control law  \( u^*(t, \hat{X}, p_X, q_X, \Sigma, p_\Sigma) \) such that

\[
H(t, u^*, \hat{X}, p_X, q_X, \Sigma, p_\Sigma) = \inf_{u(\cdot) \in U} H(t, \hat{X}, p_X, q_X, \Sigma, p_\Sigma)
\]

The dynamics of the expectation  \( \hat{X} \) can be derived directly from the definition of  \( H \):

\[
d\hat{X} = \frac{\partial H}{\partial p_X} dt + dV_t
\]

On the other hand the dynamics of the adjoint operators  \( p_X(t, \hat{X}, \Sigma) \) and  \( p_\Sigma(t, \hat{X}, \Sigma) \) can be determined by resorting to the Itô’s Lemma:

\[
\begin{align*}
dp_X &= \frac{\partial p_X}{\partial t} dt + Dp_X + \left( \frac{\partial}{\partial \hat{X}} (\cdot), dV \right) (p_X) \\
dp_\Sigma &= \frac{\partial p_\Sigma}{\partial t} dt + Dp_\Sigma
\end{align*}
\]

In view of (18) it follows that

\[
\begin{align*}
\frac{\partial p_X}{\partial t} &= \frac{\partial}{\partial \hat{X}} \frac{\partial S(t, \hat{X}, \Sigma)}{\partial t} \\
\frac{\partial p_\Sigma}{\partial t} &= \frac{\partial}{\partial \Sigma} \frac{\partial S(t, \hat{X}, \Sigma)}{\partial t}
\end{align*}
\]

and from the chain rule of the differentiation:

\[
\begin{align*}
- \frac{\partial p_X}{\partial t} &= \frac{\partial H}{\partial \hat{X}} + Dp_X \\
- \frac{\partial p_\Sigma}{\partial t} &= \frac{\partial H}{\partial \Sigma} + Dp_\Sigma
\end{align*}
\]
Henceforth,

\[
d p_X = - \frac{\partial H}{\partial X} + \left( \frac{\partial}{\partial X} \cdot , dV \right) p_X
\]

\[
d p_\Sigma = - \frac{\partial H}{\partial \Sigma}
\]

We can write the systems of forward-backward quantum differential equations for the Quantum Pontryagin principle in the Heisenberg picture:

\[
\begin{aligned}
d \dot{X}^* &= \frac{\partial H^*}{\partial X} dt + dV_i \\
d p_X^* &= - \frac{\partial H^*}{\partial X} +\left( \frac{\partial}{\partial X} \cdot , dV \right) p_X^* \\
d p_\Sigma^* &= - \frac{\partial H^*}{\partial \Sigma} \\
p_X^*(0) &= X_0 \\
p_X^*(T) &= \frac{\partial s(t, X^*(T), \Sigma^*(T))}{\partial X} \\
p_\Sigma^*(T) &= \frac{\partial s(t, X^*(T), \Sigma^*(T))}{\partial \Sigma}
\end{aligned}
\]

(19)

As a matter of fact, \( \Sigma \) is is governed by a deterministic Riccati differential equation so it is not necessary to compute its costate.

D. Stochastic LQG control from the Quantum Pontryagin Principle

The forward-backward quantum differential equations in (19) cannot be solved in closed form so it is mandatory resorting to numerical solutions. As an attempt to avoid this numerical computation in this section a LQG scheme will be derived from the quantum PMP. Let us consider the following dynamics and cost functional:

\[
d \dot{X} = \left( A \dot{X} + Bu \right) dt + dV \\
X(0) = X_0 \\
J(u) = E[\dot{X}^T(T) F \dot{X}(T)] + \frac{1}{2} \int_0^T \left( \dot{X}^T(t) Q(t) \dot{X}(t) + u^T(t) R(t) u(t) \right) dt
\]

with \( F \succeq 0, Q(t), R(t) \succ 0 \). The Hamiltonian function \( H \) is defined as

\[
H(t, \dot{X}, u, p_X, q_X) = \frac{1}{2} \left( \dot{X}^T Q \dot{X} + u^T R u \right) + \left( p_X, A \dot{X} + Bu \right) + \frac{1}{2} \left( q_X, \left( \hat{K} \otimes \hat{K} \right) \right)
\]

and the Pontryagin’s necessary conditions are:

\[
\begin{aligned}
d \dot{X}^* &= \left( A \dot{X}^* + Bu^* \right) dt + dV \\
d p_X^* &= - \left( Q \dot{X}^* + A^T p_X^* \right) dt + \left( \frac{\partial}{\partial X} \cdot , dV \right) p_X^* \\
X^*(0) &= \hat{X}_0 \\
p_X^*(T) &= F \dot{X}^*(T)
\end{aligned}
\]

From the inequality \( H(t, u^*, \dot{X}^*, p_X^*, q_X) \leq H(t, u, \dot{X}^*, p_X^*, q_X) \) we can obtain an optimal control

\[
u^*(t) = \arg \min_u \frac{1}{2} \left( u^T(t) R(t) u(t) + u^T(t) B^T p_X^*(t) \right) = - R^{-1}(t) B^T(t) p_X^*(t)
\]
Folding this control minimizing $\mathcal{H}$ into the differential equations for $\dot{X}^*$ and $p^*$ yields:

$$
\begin{align*}
\frac{d}{dt} \dot{X}^* &= \left( A\dot{X}^* - BR^{-1}(t) B^T(t) p_X^*(t) \right) dt + dV \\
X^*(0) &= \dot{X}_0
\end{align*}
$$

Given that $p_X^*$ linearly depends on $\dot{X}^*$ we can use the ansatz $p_X^*(t) = K(t)\dot{X} + \varphi(t)$, and applying the rules of the stochastic differential calculus along with the fact that $\frac{\partial p_X(t)}{\partial \dot{X}} = \left\langle \frac{\partial}{\partial \dot{X}} (\cdot), dV \right\rangle (p_X^*) = K(t)$ it is concluded that

$$
\begin{align*}
\frac{dp_X^*}{dt} &= (K'(t)\dot{X}^* + \varphi'(t)) \\
&+ K(t) \left[ A\dot{X}^* - BR^{-1}(t) B^T(t) \left( K(t)\dot{X}^* + \varphi(t) \right) \right] dt \\
&+ \left\langle \frac{\partial}{\partial \dot{X}} (\cdot), dV \right\rangle (p_X^*)
\end{align*}
$$

On the other hand, from the backward-forward system,

$$
\begin{align*}
\frac{dp_X^*}{dt} &= - \left( Q\dot{X}^* + A^T \left( K(t)\dot{X}^* + \varphi(t) \right) \right) dt \\
&+ \left\langle \frac{\partial}{\partial \dot{X}} (\cdot), dV \right\rangle (p_X^*)
\end{align*}
$$

Matching (20) and (21),

$$
\begin{align*}
K'(t)\dot{X}^* + \varphi'(t) + K(t) \left[ A\dot{X}^* - BR^{-1}(t) B^T(t) \left( K(t)\dot{X}^* + \varphi(t) \right) \right] \\
&= - \left( Q(t)\dot{X}^* + A^T \left( K(t)\dot{X}^* + \varphi(t) \right) \right)
\end{align*}
$$

This finally leads to the differentials equations for $K(t)$ and $\varphi$:

$$
\begin{align*}
K'(t) &= -K(t)A - A^T K(t) \\
+ K(t) BR^{-1}(t) B^T(t) K(t) - Q(t) \\
K(T) &= F \\
\varphi'(t) &= - \left[ A - BR^{-1}(t) B^T K(t) \right]^T \varphi(t) \\
\varphi(T) &= 0
\end{align*}
$$

V. Discussion

A quantum maximum principle has been addressed for continuous-time measurements. This has been derived from the Hamilton-Jacobi-Bellman equation in the Schrödinger picture. Then the scheme has been extended to the Heisenberg picture in statistical moment coordinates. Since a stochastic Pontryagin principle requires a numerical solution to solve the equations we have derived a LQG scheme which is more suitable for control purposes.

The Pontryagin principle taps its roots in the method of Lagrange multipliers applied in constrained optimization. A constraint on the state essentially introduces infinitely many additional constraints compared with a deterministic state constrained control problem; PMP approach allows to transform an infinite dimensional optimization problem (the search over a set of functions) into a finite dimensional optimization problem (the search over a set of parameters). In the future it should be interesting to explore QOC problems under state constraints.
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