A new prescription towards thermodynamic phase transition

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One of the serious open problems of theoretical physics is to find a consistent quantum gravity theory. Recent developments in thermodynamic phase transition of black holes and their van der Waals like behavior may provide an interesting quantum interpretation of classical gravity. Studying different methods for investigating phase transition can deepen our insight into the nature of quantum gravity. In this paper, we present a new theoretical prescription for finding out thermodynamic phase transition in the extended phase space. Unlike standard methods of the usual equation of state which is according to temperature, our new prescription is based on a new typical equation of state which is originating from the slope of temperature versus entropy. This new prescription addresses some of the shortcomings of other usual methods and provides a simple powerful way for studying critical behavior of a thermodynamical system in more details. Among the achievements of the new prescription, we emphasize to addressing the problem of ensemble dependency, analytical demonstration of possible phase transition points and denoting the nonphysical range of horizon radius for the black holes.

I. INTRODUCTION

Phase transition is one of the interesting empirical/theoretical challenging topics in various aspects of sciences; from biological \textsuperscript{1} to physical systems \textsuperscript{2}. Regarding the physical systems, one may find that phase transitions play important roles in elementary particle \textsuperscript{3}, condensed matter \textsuperscript{4}, usual thermodynamics \textsuperscript{5}, cosmology \textsuperscript{6}, black holes \textsuperscript{7} and other branches of physics. In this regard, we focus on thermodynamic phase transition of black holes and other matters leave aside. There are several approaches towards studying the critical behavior of black holes. One of these approaches is using heat capacity in the canonical ensemble. The discontinuities of heat capacity are places in which the phase transitions take place. Another method is based on studying the van der Waals like behavior of black holes in the extended phase space by considering a proportionality between cosmological constant and a dynamical pressure. This method is based on an equation of state which is free of temperature, and therefore, it is not a usual equation of state. But in our new prescription, we use \(\left(\frac{\partial T}{\partial S}\right) = \left(\frac{\partial^2 M}{\partial S^2}\right)\) to obtain a new relation for the pressure. This new relation is free of temperature, and therefore, it is not a usual equation of state. One can find that this new relation and other related calculations and phase diagrams are completely different from those obtained in the usual extended phase space. Our new method provides the possibility of mapping all the phase transition points that system can acquire. In addition, it can point out the nonphysical range of horizon radius in which black hole solutions do not exist. Furthermore, the ratio of the incremental of nonphysical range of horizon radius for different critical values could be obtained by this method.

Here is a layout of the paper: In Sec. \textsuperscript{III} we give a brief review of the canonical ensemble approach and the van der waals approach to black hole phase transitions. In Sec. \textsuperscript{IV} we introduce a new scheme for calculating the phase transition parameters. In Sec. \textsuperscript{V} we do a case study of a d-dimensional Reissner-Nordström AdS black hole to elaborate the new method. Lastly, we make several remarks in the conclusion.

II. DESCRIPTION OF VARIOUS METHODS

A. Usual prescription: Heat capacity and extended phase space

In the canonical ensemble, discontinuities of the heat capacity are indicated as the phase transition points. The
heat capacity in the canonical ensemble is given by
\[
C_Q = \left( \frac{T}{\partial (M_v)} \right)_Q = T \left( \frac{\partial S}{\partial T} \right)_Q. \tag{1}
\]

The main application of heat capacity is studying thermal stability. Positivity of \( C_Q \) can guarantee thermal stability of a system while its negative sign is regarded as instability.

On the other hand, in order to study the critical behavior of a thermodynamical system, one is required to obtain equation of state, \( P = P(T, V) \). In the context of black hole thermodynamics, regarding the temperature of a typical black hole in the presence of cosmological constant, one may find \( T = T(m, r_+, \Lambda, Q, \text{other hairs}) \). We can consider the cosmological constant as a dynamical pressure and take into account the relation between the event horizon radius \( r_+ \) with the volume to find an equation of state, \( P = P(T, V) \). Applying the properties of a critical point in isothermal \( P - V \) diagrams (inflection point), one finds \( \left( \frac{\partial^2 P}{\partial v^2} \right)_T = \left( \frac{\partial P}{\partial v} \right)_T = 0 \). The mentioned relations help us to find the critical points, hence possible phase transitions. This method is highly depending on the values of temperature.It is worth to mention that this method is not practical for black holes with non-spherical horizons in most of gravitational theories. In order to avoid such restriction, we have to use a new prescription for obtaining the critical values in the extended phase space.

B. Review: van der Waals liquid-gas system

Van der Waals system is one of the particular models for describing a real liquid–gas system and its critical behavior. The equation of state in this model is a modification of the ideal gas with consideration of the nonzero size of molecules and the attraction between them. The van der Waals equation of state is given by
\[
(P + \frac{a}{v^2})(v - b) = kT(P, v), \tag{2}
\]
in which \( P \) and \( T \) are the pressure and temperature, respectively. Also, \( v \) is the specific volume which is read as \( v = \frac{V}{M} \), \( b \) is a free parameter related to the nonzero size of the molecules of a fluid and \( a \) represents the strength of attraction between the molecules. Here \( k \) is a constant of proportionality can be set to one. Note that setting \( a = b = 0 \) yields a familiar ideal gas law. Further, the black hole thermodynamic parameters can be expressed in terms of \( a \) and \( b \) solely. Due to van der Waals-black hole correspondence, the temperature (and hence equation of state) of the fluid will be related with temperature of the black hole. The existence of critical behavior could be examined through the properties of inflection point which satisfy
\[
\left( \frac{\partial P}{\partial v} \right)_T = \left( \frac{\partial^2 P}{\partial v^2} \right)_T = 0. \tag{3}
\]

Using the property of inflection point with the equation of state of van der Waals system \( P \), it is straightforward to find the following critical values
\[
v_c = 3b, \quad P_c = \frac{a}{27b^2}, \quad T_c = \frac{8a}{27bk}. \tag{4}
\]

Inserting \( T = T_c \) in the equation of state, we find two insepable phases of liquid-gas in which phase transition takes place between them. For the case \( T < T_c \), similar to previous case, we have a phase transition between two phases of liquid and gas. But contrary to previous case, here, there exists a region of specific volume in which no physical system exits and phase transition takes place over it. In other words, for this case, two specific volumes with the same pressure exist in which phase transition takes place between them. In usual methods, in order to obtain all of these critical behaviors and their specific critical values, one has to take into account all the temperatures smaller than \( T_c \), which practically is not possible. Our new method, introduced in this paper, provides the possibility of obtaining all of these critical points analytically. We will demonstrate this possibility in what follows.

The Gibbs free energy of this system could be acquired in the following form
\[
G = -kT \left( 1 + \ln \left[ \frac{v - b}{\Phi T^\frac{3}{2}} \right] \right) - \frac{a}{v} + Pv, \tag{5}
\]
in which \( \Phi \) is a constant characterizing the gas. The entropy of system could be obtained through the differential equation of \( dG = -SdT + v dP \), which leads to entropy being
\[
S = k \left( \frac{5}{2} + \ln \left[ \frac{v - b}{\Phi T^\frac{3}{2}} \right] \right). \tag{6}
\]

In order to use the method introduced in this paper, one should obtain enthalpy of the system. This quantity could be calculated through different methods where the following relation is one of them
\[
H = G - TS = -kT \left( \frac{7}{2} + 2 \ln \left[ \frac{v - b}{\Phi T^\frac{3}{2}} \right] \right) - \frac{a}{v} + Pv, \tag{6}
\]
which is known as the enthalpy of van der Waals system.

III. NEW PROPOSAL TO PHASE TRANSITION

Taking into account the postulates of usual thermodynamic systems, it seems that all complete differentiations can be written as functions of three thermodynamic coordinates. It is determined that these three coordinates are not independent, but there is an equation of state which reduces the number of degrees of freedom to two parameters. On the other hand, the equations regarding
thermodynamical properties of the systems are combination of different factors. For example, in most of the cases, pressure, temperature, internal energy, Gibbs free energy and etc. of thermodynamical systems are not independent of each other. Pressure and temperature are related to each other by equation of state. Internal and Gibbs free energies are also pressure and temperature dependent. It is very important to see if it is possible to obtain relations in which these quantities are independent of each other. These new relations should preserve information regarding thermodynamical behavior of the system, hence phase transitions and critical behavior.

In practice, obtaining all the possible critical points of a system and the ranges in which phase transition takes place is not possible through the usual methods. The main reason is that one has to take into account the value of a particular thermodynamical quantity less than its critical value to obtain the critical points and their corresponding range of phase transition. Mathematically, it is not possible to overcome such problem, analytically, through the usual methods. Here, we would like to introduce a new method for providing such possibility which results into new relations for different thermodynamical quantities. These new relations provide information regarding phase transitions of the system and their corresponding range of phase transition.

The method’s prescription is as follows: instead of considering the usual equation of state (which is temperature dependent), we use the slope of $T$ versus $S$, meaning $\left( \frac{\partial T}{\partial S} \right)$. The mentioned equation should be solved with respect to the pressure which exists due to its proportionality with the cosmological constant in the extended phase space. This leads to a new relation for the pressure which is only volume dependent. This new relation differs from the other relation which could be obtained through the use of equation of state. The possibility of existence of maximum(s) for this new relation of the pressure must be examined. The maximum(s) of this relation is(are) the critical point(s) in which the phase transition takes place. In other words, maximum of this new relation is where the system goes under a phase transition.

It is evident that by obtaining such maximum, one is able to extract the critical pressure and the horizon radius (volume) at the same time. Therefore, it helps to save more times for calculations. It is worthwhile to mention that for obtaining the new relation for pressure, one can use $\left( \frac{\partial^2 H}{\partial S^2} \right)$ instead of $\left( \frac{\partial^2 H}{\partial v^2} \right)$, where $H$ is enthalpy of the system. The maximum is where system acquires phase transition (the maximum indicates the critical values of system). On the other hand, the thermodynamical concepts indicate that for the pressures smaller than the critical one, two critical volumes exist in which phase transition takes place between them. As for the pressures larger than the critical pressure, no phase transition exists. We see that due to the existence of critical pressure at the maximum of this relation, such property is preserved in this method and all the possible critical points and their corresponding ranges are obtained. Furthermore, using the new relation for the pressure and replacing it in the usual equation of state, one can obtain a new relation for the temperature which is free of pressure. The same could be done regarding the enthalpy, internal energy and Gibbs free energy which leads to new relations which are only volume dependent. In order to elaborate the efficiency of the new method, in what follows, we give two typical but general examples in the context of usual thermodynamics and black hole thermodynamics.

A. The Model

Now, we are in a position to use the new method. Since both the entropy $S(v)$ and enthalpy $H(v)$ are volume dependent, we can use the following relation

$$\frac{d^2 H}{dS^2} = \left( \frac{dS}{dv} \right)^{-1} \frac{d}{dv} \left[ \left( \frac{dH}{dv} \right) \left( \frac{dS}{dv} \right)^{-1} \right], \quad (7)$$

which leads to following equation for van der Waals liquid/gas systems

$$\frac{d^2 H}{dS^2} = \frac{b - v}{kT} \left[ a (2b - v) + P v^3 \right]. \quad (8)$$

By solving this relation with respect to $P$, one can find following new relation for pressure which differs from equation of state

$$P_{new} = \frac{v - 2b}{v^3} a. \quad (9)$$

Using the concept of extremum of this relation being the critical point, one can find following critical volume and pressure

$$v_c = 3b, \quad \& \quad P_c = \frac{a}{27b^2},$$

which are identical to those obtained previously [4]. By replacing the new pressure with its corresponding in equation of state, Gibbs free energy and enthalpy, one can find new relations for temperature, enthalpy and Gibbs free energy (also other thermodynamical quantities) which are pressure independent

$$T_{new} = \frac{2a (b - v)^2}{k_v^2},$$

$$G_{new} = -kT \left( 1 + 2 \ln \left[ \frac{v - b}{\Phi} T^2 \right] \right) - 2ab \frac{v}{v^3},$$

$$H_{new} = -kT \left( \frac{7}{2} + 2 \ln \left[ \frac{v - b}{\Phi} T^2 \right] \right) - 2ab \frac{v}{v^3}.$$

These new relations enable us to extract all the possible critical points that system can acquire. In other words, using these new relations, one can find all the critical temperature, Gibbs free energy and enthalpy that system can acquire. In order to highlight the validation of new method, we plot Fig. [4].
FIG. 1: Top panel: $P_{\text{new}}$ (dash-dotted line) and $P$ curves versus $v$ for $T = 0.9T_c$ (continuous line), $T = T_c$ (dotted line) and $T = 1.1T_c$ (dashed line).
Bottom panel: $T_{\text{new}}$ (dash-dotted line) and $T$ curves versus $v$ for $P = 0.9P_c$ (continuous line), $P = P_c$ (dotted line) and $P = 1.1P_c$ (dashed line). In both diagrams, we chose $b = 1$, $a = 4$ and $k = 1$.

It is evident that the maximum of new relations for the temperature $T_{\text{new}}$ and pressure $P_{\text{new}}$ are where system goes under phase transition. Note that in the $P - v$ picture, temperature is kept constant while in the $T - v$ picture, pressure is kept constant. In addition, for the pressures (temperatures) smaller than the critical pressure (temperature), new relation enjoys a single pressure (temperature) with two related volumes. Phase transition takes place between these two volumes at a specific pressure. As one can see, all the possible critical points and their corresponding ranges of phase transitions are included in this new method. This is one of the important features of our new method which was not possible in previous methods. It is interesting to note that the minima of $T$ ($P$) coincides with the maxima of the $T_{\text{new}}$ ($P_{\text{new}}$). We will continue with another example in the context of black holes.

IV. PHASE TRANSITION IN HIGHER DIMENSIONAL REISSNER-NORDSTRÖM ADS BLACK HOLE

The main motivation to study asymptotically AdS black holes stems from the hypothesis of AdS/CFT correspondence. Using the thermal field theoretic approaches, it has been deduced that AdS black holes undergo certain phase transitions. A first sign of such phase transitions was observed by Page and Hawking for the Schwarzschild AdS black hole [11]. Further with the addition of parameters such as electric charge and spin, the phase transition process is more elaborated and enhanced. It is quite interesting to note that the pressure-volume or P-V picture of ideal gas fluid for constant temperature is also mimicked by the AdS black holes (see Fig.1 top panel). This analogy between a gravitational system (an AdS black hole) and a non-gravitational thermal system (such as a ideal gas or a van der waal’s fluid) is established by identifying a correspondence between their parameters i.e. mass with enthalpy; temperature with surface gravity; entropy with area and cosmological constant with the pressure of fluid. Thus in the first law of thermodynamics, the cosmological constant appears as pressure which is conjugate to the volume of black hole [12]. Besides, using the Reverse Isoperimetric Inequality, it has been deduced that entropy inside a horizon of a given volume is maximized for a Schwarzschild AdS BH [13].

In the black hole systems, it was shown that one can assume the negative cosmological constant as a thermodynamical pressure [14] with following relation

$$P = -\frac{\Lambda}{8\pi}. \quad (10)$$

As a complementary note, we should mention that for specific black holes in modified general relativity such as dilatonic gravity and gravity’s rainbow, one has to use a modified proportionality relation instead of Eq. [10] [17, 18]. Although in this paper we regard a well-known Reissner–Nordström AdS black holes, our technique is consistent for other black holes in modified theories of gravity. Replacing the cosmological constant with the thermodynamical pressure (working in the extended phase space thermodynamics) leads to the following important results:

I) The resulting temperature for the black hole is the equation of state.

II) Total mass of the black hole is no longer internal energy. In fact, it is replaced by the enthalpy under such consideration. This results into the following relation for Gibbs free energy

$$G = M - TS. \quad (11)$$

Now, we would like to give an example regarding the validity of our new prescription and its consistency with previous methods in the context of black hole systems. For this purpose, we study the critical behavior
of $d$-dimensional Reissner–Nordström AdS black holes. Previously, the results regarding this specific black holes were derived through usual method in Ref. 10. The metric of these black holes in spherically symmetric spacetime is given as

$$ds^2 = -\psi(r) dt^2 + \frac{dr^2}{\psi(r)} + r^2 d\Omega^2_{d-2},$$

(12)

where we used the notation $\Omega_{d-2} = d - i$. In Eq. (12), $d\Omega^2_{d-2}$ denotes the metric of $(d-2)$-dimensional unit sphere and

$$\psi(r) = 1 - \frac{m}{r} - \frac{2\Lambda}{d-2} r^2 + \frac{2d_3 q^2}{d_2 r^{d_3/2}}.$$

(13)

Temperature, entropy and total finite mass of these black holes are calculated by using surface gravity, the area law and ADM approach, respectively, which leads to

$$T = \frac{d_3}{4\pi r^+} - \frac{\Lambda r^+}{2\pi d_2} - \frac{d_3 q^2}{2\pi d_2 r^{d_3/2} r^+},$$

(14)

$$S = \frac{r^d}{4},$$

(15)

$$M = \frac{d_2}{16\pi} m,$$

(16)

in which $r^+$ is the outer horizon of the black holes. By evaluating metric function on the outer horizon ($\psi(r = r^+) = 0$), we can obtain

$$M = \frac{d_2}{16\pi} \left( r^d \Lambda - \frac{2r^d_1}{d_1 d_2} + \frac{2d_3 q^2}{d_2 r^{d_3/2}} \right),$$

(17)

with the following relation for the Gibbs free energy of these black holes

$$G = \frac{r^d}{16\pi} + \frac{d_3 (2d - 5) q^2}{8\pi d_2 r^{d_3/2}} - \frac{r^d_1}{d_1 d_2} P.$$

(18)

Now, we are in a position to calculate the critical values through the usual method. First, we calculate the volume conjugating to the pressure as

$$V = \left( \frac{\partial H}{\partial P} \right)_{S,Q} = \left( \frac{\partial M}{\partial P} \right)_{S,Q} = \frac{r^d_1}{d_1}.$$

(19)

Since the volume depends on the horizon radius, one can use the horizon radius for investigating thermodynamic behaviors which is proportional (linearly) to the specific volume of black holes 10. Using Eqs. 10 and 14, one can obtain the equation of state as

$$P = \frac{d_2}{16\pi r^+} \left( 4\pi T r^d + 3q^2 r^4 - 3q^2 r^4 \right) + \frac{2d_3 q^2 r^2}{16\pi r^+}.$$  

(20)

Now, we employ the properties of inflection point, $\left( \frac{\partial P}{\partial r^+} \right)_T = \left( \frac{\partial^2 P}{\partial r^+} \right)_T = 0$, to obtain the critical horizon radius (volume), $r_c$ which leads to

$$r_c^2 - 4d_3 d_5/2 q^2 r_c^{-2d_4} = 0$$

with the following solution 10

$$r_c = \left( 4q^2 d_3 d_5/2 q^2 \right)^{1/5}.$$  

(21)

It is a matter of calculation to obtain the critical temperature and pressure as

$$T_c = \frac{d_3}{2\pi (4q^2 d_3 d_5/2 q^2)^{5/2}} - \frac{d_3 q^2}{\pi},$$

(22)

$$P_c = \frac{d_3 d_4}{16\pi (4q^2 d_3 d_5/2 q^2)^{5/2}} - \frac{d_3 q^2}{4\pi (4q^2 d_3 d_5/2 q^2)^{3/2}}.$$  

(23)

Now, we obtain the critical values through the new prescription. Using Eqs. 15 and 17 and replacing the cosmological constant with the pressure 10, one can obtain

$$\left( \frac{\partial^2 M}{\partial S^2} \right)_{Q} = \frac{16P}{d_3^5 r^d} - \frac{d_3}{2\pi d_2 r^d} + \frac{4d_3 q^2}{\pi d_1 r^d},$$

(24)

Using Eq. 24 and solving it with respect to $P$, we obtain a new relation for the pressure

$$P_{new} = \frac{d_3 d_4}{16\pi r^d} - \frac{d_3 q^2}{4\pi r^{d_4}}.$$  

(25)

Regarding this relation and replacing corresponding pressure in the temperature 14, mass 17 and Gibbs free energy 18 with the new pressure relation 24, one can obtain new relations for the mentioned thermodynamical quantities in following forms

$$T_{new} = \frac{d_3}{2\pi r^+} - \frac{d_3 q^2}{\pi} $$

(26)

$$M_{new} = \frac{d_3 d_4}{8\pi d_1} + \frac{d_3 d_5 q^2}{4\pi d_1 r^d},$$

(27)

$$G_{new} = \frac{r^d}{8\pi d_1} + \frac{d_3 d_5 q^2}{2\pi d_1 r^d}.$$  

(28)

It is evident that obtained relation for the pressure 24 is different from the usual equation of state 20. In order to obtain the maximum of this relation, we use the mathematical nature of extremum points

$$\left( \frac{dP_{new}}{dr^+} \right)_{r^+ = r_{NC}} = \frac{d_3 d_4}{2\pi r_{NC}^{d_3/2}} - \frac{d_3 d_4}{8\pi r_{NC}^d} = 0$$

with the following solution for the new critical (NC) horizon radius, $r_{NC}$

$$r_{NC} = \left( 4d_3 d_5/2 q^2 \right)^{1/5}.$$  

(29)
FIG. 2: Top panel: $P_{\text{new}}$ (dash-dotted line) and $P$ curves versus $r_+$, for sub-critical case $T = 0.9T_c$ (continuous line), critical case $T = T_c$ (dotted line) and super-critical case $T = 1.1T_c$ (dashed line).

Middle panel: $T_{\text{new}}$ (dash-dotted line) and $T$ curves against $r_+$, for sub-critical $P = 0.9P_c$ (continuous line), critical $P = P_c$ (dotted line) and super-critical $P = 1.1P_c$ (dashed line).

Bottom panel: $P_{\text{new}}$ (dash-dotted line) and $C_Q$ curves versus $r_+$, for sub-critical $P = 0.9P_c$ (continuous line), critical $P = P_c$ (dotted line) and super-critical $P = 1.1P_c$ (dashed line). In all three figures, we kept $q = 1$ and $d = 4$.

a matter of calculation to find

$$T_{NC} = \frac{d_3}{2 \pi (4q^2d_3d_5/2)^{\frac{d_5}{d_3}}} - \frac{d_5/2}{d_3} d_3^2 q^2$$

(30)

$$P_{NC} = \frac{d_2 d_3}{16 \pi (4q^2d_3d_5/2)^{\frac{d_5}{d_3}}} - \frac{d_5/2}{4 \pi (4q^2d_3d_5/2)^{\frac{d_5}{d_3}}}$$

(31)

which are exactly the same as previously calculated the critical temperature (22) and the pressure (23). These results show that the critical values calculated here are consistent with those calculated before with the usual method of extended phase space.

Using the obtained critical radius (29) with the new relations for the mass (27) and Gibbs free energy (28), we obtain the critical mass (enthalpy) and Gibbs free energy as well

$$M_{NC} = \frac{d_2 d_3 (d^2 - 5d + 7) q}{4 \pi d_1 \sqrt{d_3 d_5/2}},$$

$$G_{NC} = \frac{\sqrt{d_3 d_5/2} q}{2 \pi d_1}.$$
observed in other two methods. For the pressure being the critical pressure, like the other two methods, only one discontinuity is observed for the heat capacity. For $P > P_c$, no discontinuity is observed for the heat capacity. This behavior indicates that these three pictures are uniform and have consistent results.

Regarding top panel of Fig. 2 one finds the so-called saturation curve with dash-dotted style. Taking into account $P - V$ isothermal diagram with $T < T_c$ (continuous line), we can decrease the horizon radius to find two points of intersection with saturation curve ($r_{+1}$ and $r_{+2}$ with $r_{+1} < r_{+2}$). The black hole systems are unstable for $r_{+1} < r_{+} < r_{+2}$. In other words, there is a phase transition between small and large black holes (between $r_{+1}$ and $r_{+2}$). This phase transition may occur by a sudden burst of thermal Hawking radiations from the black hole i.e. the size of black hole suddenly shrinks from size $r_{+2}$ to $r_{+1}$ without changing the black hole temperature. Black hole solutions are not physical between these two points. This could be explained by the fact that the heat capacity is negative (see bottom panel of Fig. 2 for more details), and also, speed of sound is larger than speed of light [19]. Notice that similar discontinuities in specific heat capacity also occur in Born-Infeld black holes [9]. It is worthwhile to note that for $T = T_c$, both intersection points meet to obtain the critical horizon radius $r_{+1} = r_{+2} = r_c$. The same statement could be said about the middle panel of Fig. 2 regarding the temperature.

Before we finish our paper, it is worth pointing out the significants of the new prescription.

First of all, as one can see, the new prescription provides the possibility of obtaining different thermodynamical quantities independent of each other. In other words, as one can see, Eqs. (26)-(28) only depend on the properties such as dimensions, electric charge and horizon radius. If we generalize the action to other models of gravitational theories or including other matter fields, by using this method, the resultant new temperature, pressure, mass and Gibbs free energy enjoy the same properties (being only a function of black holes properties).

Second, these new relations include only the critical points that black holes could acquire in different conditions. In usual methods, in order to have all the points which phase transitions take place between them, one has to consider pressures which are equal or smaller than critical pressure. Technically, such task is impossible to be done for all pressures. Here, using the new method, one can find all the possible phase transition horizon radii and their corresponding pressures that system can acquire. The same could be said about new relations for the temperature, mass and Gibbs free energy. In other words, by using the new prescription, one can obtain all the phase transition points and their corresponding critical temperature, pressure, mass and Gibbs free energy that system can acquire in analytical forms.

Third, using this new method, one can recognize the range of horizon radius in which black holes solutions do not exist depending on the critical values. For more clarifications, one should take a look at the diagram for new pressure in up panel of Fig. 2 (dashed-dotted line). Evidently, the phase transition takes place between two points with the same pressure. The prohibited range of horizon radius for the black holes is between these points. By taking a closer look, one can see that by using the new prescription, one can obtain the maximum range of the horizon radius in which the black hole solutions do not exist. Such maximum could not be obtained through the usual methods easily. In addition, by using the new prescription, one can measure the rate of increscent for the prohibited range of horizon radius by studying the behavior of its diagram. Such measurement may encounter with significant problems for the usual method. Finally, we should point out that these three points are also valid for usual thermodynamical systems as well.

V. CLOSING REMARKS

Motivated by interests in van der Waals like behavior and recent progresses in thermodynamic phase transition of black holes, we introduced a new prescription for studying the phase transition points in both usual thermodynamical systems and black holes.

Although the usual method of studying phase transition is originated from the temperature as the equation of state, our new prescription is based on the slope of temperature versus entropy. This new prescription is a powerful method to address the critical behavior of a thermodynamical system in more details.

The results of new prescription are in agreement with other methods, but it also provides us with further information regarding critical behavior of thermodynamical systems which could not be derived through other methods. As for the highlights of this method one can name: I) Providing new relations for different thermodynamical quantities which were independent of each other. II) Mapping all the possible critical point and region in which phase transition takes place between them.

The method introduced here is applicable for both usual thermodynamical systems and black holes. This shows that one can also employ the general structure of this method in the context of other physical systems such superconductors, condense matter systems, gague/gravity duality or even in the context of quantum systems.

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