On zero dimensional sequential spaces

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Abstract

We develop tools to recognize sequential spaces with large inductive
dimension zero. We show the Hawaiian earring group $G$ is 0 dimensional,
when endowed with the quotient topology, inherited from the space of
based loops with the compact open topology. In particular $G$ is $T_4$ and
hence inclusion $G \hookrightarrow F_M(G)$ is a topological embedding into the free
topological group $F_M(G)$ in the sense of Markov.

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1 Introduction

When is a Hausdorff sequential space zero dimensional? The fundamental
group of the Hawaiian earring serves as catalyst for such an inquiry in the
context of the following three questions. We answer the first question
affirmatively via partial answers to second and third.

1) Is the large inductive dimension of the Hawaiian earring group zero,
if $G$ enjoys the quotient topology inherited from the space of based loops?
2) If a sequential space $G$ continuously injects into a countable inverse
of limit of discrete spaces, what conditions ensure $G$ is zero dimensional?
3) If a sequential space $G$ is a quotient of a countable product of
discrete spaces, what conditions ensure $G$ is zero dimensional?

Wild algebraic topology is loosely described as the study of locally
complicated spaces, and their attendant homotopy/homology groups. The
motive to impose a topology on the latter objects might come from func-
torality of the fundamental group \cite{9}, from a canonical bijection between
$\pi_1(X,p)$ and the fibres of a semicovering $E \to X$ \cite{25} \cite{26} \cite{27}, or to measure
the extent to which $\pi_1$ might act continuously on a space \cite{8}.

At center stage \cite{17} \cite{10} \cite{18} \cite{14} \cite{11} \cite{13} is the Hawaiian earring $HE$, a
null sequence of loops joined at a common point, the inverse limit of
nested sequence of bouquets on $n$ loops, under retraction bonding maps,
collapsing the $n$th loop to the special point $p$.

The induced homomorphism $\phi : \pi_1(HE,p) \to \lim_{\leftarrow} F_n$, with $F_n$ the
discrete free group on $n$ generators, is one to one \cite{15} \cite{24} \cite{22}. Thus the
what is allowed to converge. For example the sequence $(x_1 x_n^{-1} x_n)^n \to 0 \in \pi_1(HE, p)$, but all corresponding lifts diverge, with the topology of uniform convergence.

At another extreme, invoking a construction similar to the familiar universal cover, the group $\pi_1(HE, p)$ acts freely and isometrically on a corresponding generalized universal cover of $HE$, a uniquely arcwise connected, locally path connected metric space, i.e. a topological R-tree. The trade off here is that it is difficult for our isometries to converge. Treated as a group of isometries of $(x_1 x_n^{-1} x_n)^n \to 0 \in \pi_1(HE, p)$, but all corresponding lifts diverge, with the topology of uniform convergence.

Continuity of the injection $\phi : \pi_1(HE, p) \to \lim_\leftarrow F_0$ ensures $\pi_1(HE, p)$ is $T_2$, since the codomain is $T_2$. Moreover $\pi_1(HE, p)$ is a quotient of a separable metric space and hence $\pi_1(HE, p)$ is Lindelöf. Unfortunately $\phi$ is not a topological embedding, and $\pi_1(HE, p)$ is not a topological group in TOP. This calls into question whether $\pi_1(HE, p)$ is at least $T_3$, and hence $T_4$, since $\pi_1(HE, p)$ is a Lindelöf space. To prove $\pi_1(HE, p)$ is $T_4$ it suffices to prove $\pi_1(HE, p)$ has large inductive dimensions zero, that disjoint closed sets can be thicken down into disjoint clopens.

While the class of contractible spaces shows dimension is generally not an invariant of the homotopy type of an underlying space, functorality ensures $\pi_1(X)$ and $\pi_1(Y)$ have the same dimension if $X$ and $Y$ are homotopy equivalent. More esoterically, the knowledge that $\pi_1(HE, p)$ is zero dimensional will ensure for example, that $\pi_1(HE, p)$ cannot contain a copy of the totally disconnected 1 dimensional Erdős space.

To prove $\pi_1(HE, p)$ is zero dimensional we establish Theorems 2.15 and 3.1 applicable to suitably well behaved quotients of the inverse limit of countably many discrete nested retracts.

The potential difficulty of such an inquiry is highlighted by the familiar dimension raising closed quotient of the Cantor set $\{0, 1\} \times \{0, 1\} \ldots \to [0, 1]$ mapping each binary sequence onto the corresponding real number. What goes wrong? All finite approximations to $(0, 1, 1, \ldots)$ and $(1, 0, 0, \ldots)$ are distinct, yet the points are identified in the limit. This is analogous to a failure of $\pi_1$ injectivity in shape theory. To avoid this, in this paper we only consider quotients where the above phenomenon does not happen, and in particular the above does not happen in the Hawaiian earring.

Most of the paper is devoted to a proof of the following. Suppose
$X_1 \subset X_2$... is a nested sequence of discrete retracts $X_{n+1} \to X_n$ and $q_n : X_n \to G_n$ is a quotient map. This data induces a quotient map $q : \lim_{\leftarrow} X_n \to G$, and the following two extra assumptions ultimately ensure that $q$ does not raise dimension. 1) The retraction $X_{n+1} \to X_n$ induce a map $G_{n+1} \to G_n$ and 2) The inclusion maps $X_n \to X_{n+1}$ induce a map $G_n \to G_{n+1}$. Our main result (Theorems 2.15 and 3.1) is that a space $G$ constructed in this manner has large inductive dimension zero. The proof uses the well ordering principle, and we now indicate why well ordering is useful to circumvent the failure of a less sophisticated approach.

The proof idea for Theorems 2.15 and 3.1 stems from an easier but still complicated construction designed to prove that the Hawaiian earring group $\pi_1(HE, p)$ is a $T_3$ space. To prove that $\pi_1(HE, p)$ is a $T_3$ space it suffices to prove that $\pi_1(HE, p)$ has a basis of clopen sets, i.e. that $\pi_1(HE, p)$ has small inductive dimension 0. In turn, since $\pi_1(HE, p)$ is homogeneous, it suffices, to start with a closed set $B \subset \pi_1(HE, p)$ so that the identity $e \notin B$, and thicken $B$ into a clopen set $U(B)$ so that $B \subset U(B)$ and $e \notin U(B)$. We now outline the overall strategy for building $U(B)$ and point out a potentially fatal pitfall.

By construction $\pi_1(HE, p) = G$ is equipped with a canonical countable collection of clopen sets $U(g_n) \subset \pi_1(HE, p)$, the preimage of $g_n \in G_n$ under the retraction $G \to G_n$, here $G_n$ is the free group on $n$ generators. The overall idea is to somehow thicken $B \subset \pi_1(HE, p)$ into a union $U(B)$ of our special clopens, so that $U$ is clopen and $e \notin B$.

Given a closed set $B \subset \pi_1(HE, p)$, the simplest idea to construct $U(B)$ is to exploit the God given retraction $R : G \setminus \{e\} \to (\bigcup G_n) \setminus \{e\}$. Unfortunately this does not work, as indicated below, suggesting why more refined methods for building $U(B)$ are needed.

Consider the retraction $R : G \setminus \{e\} \to (\bigcup G_n) \setminus \{e\}$ taking the nontrivial word $g \in G$ to $R(g) = g_n$ (deleting all letters greater than $n$ from $g$) with $n$ minimal so that $g_n$ is nontrivial. Given $B \subset G$ closed with $e \notin B$ it is at least plausible that $\bigcup_{g \in B} R(g)$ is clopen, but the following example shows this is false, $e$ can be a missing limit point. Suppose for $k > 1$ the closed set $B$ is the sequence of finite words $\{w(k)\}$ with $w(k) = (x_1x_{k+1}^{-1}x_k^{-1}x_{k+1})^k x_k$. Thus $R(w(k)) = x_k$ and hence $e$ is a missing limit point of the union of the sets $U(x_k)$.

For the latter example the reader might notice that we could get an acceptable thickening $U(B)$ using the union of the sets $U(x_1x_{k+1}^{-1}x_k^{-1}x_{k+1})^k x_k$, i.e. given $b \in B$ we should look for a large index approximation of $b$, rather than a small index approximation of $b$. This is indeed the right idea, but comes at the cost of no obvious best method to approximate $b$, we have too many choices as shown by the example $b = x_1x_2x_3 \ldots \in \pi_1(HE, p)$. In other words the best way to approximate $b \in B$ is context dependent, depending both on $B$ and also previously made choices when attempting to thicken some of $B$.

To make the previous sentence more precise, we indicate a more systematic way to thicken $B \subset \pi_1(HE, p)$ into a clopen. The overarching idea is to impose a linear order on $\pi_1(HE, p)$ (not compatible with its topology, and not well ordered), but so that nevertheless each closed set $A \subset \pi_1(HE, p)$ has a minimal element. Given such an ordering, we start with the minimal $b_0 \in B$, then thicken $b_0$ into $U(R(b_0))$, then let
Then we select $U(R(b_1))$ and so on. Crucially we hope to ensure the union of the sequence $U(R(b_0)) \cup U(R(b_1)) \cup \ldots$ is clopen.

If we are suitably careful with the definition of our linear order on $G$, Lemma 2.11 ensures that, proceeding by transfinite recursion, the union of our selected sets $U(R(b_i))$ is indeed the desired clopen set $U(B)$. This idea is the key to the paper, and we hope it will remain undisguised by the superficially technical appearance of its implementation.

Corollary 3.2 shows $\pi_1(He, p)$ is indeed 0 dimensional and in particular $\pi_1(He, p)$ is a Tychonoff space (completely regular). This pays off categorically both in TOPGRP [2] and the (compactly generated groups) $k\text{-GRP}$ [25].

On the one hand we might ingressively blame the failure of $\pi_1(He, p)$ to be a topological group in TOP on an abundance of closed sets (Theorem 1 [3]). As summarized in section 3 [6], can $\pi_1(He, p)$ be repaired categorically, by coarsening $\pi_1(He, p)$ to be the canonical quotient of the free Markov topological group $FM(\pi_1(He, p))$. The payoff here is that our new knowledge that $\pi_1(He, p)$ is a Tychonoff space contributes to our understanding of the latter construction. We are assured that inclusion $\pi_1(He, p) \to FM(\pi_1(He, p))$ is a topological embedding, as noted in Lemma 3.1 [6]. See also [2].

More congressively [12], instead of coarsening $\pi_1(He, p)$, we might instead accept $\pi_1(He, p)$ as a 0-dimensional first class citizen in $k\text{-SEQGROUP}$, the category of sequential spaces with group structure, so that the group operations are sequentially continuous, and acknowledge that the familiar product topology is categorically not always the most useful way to multiply spaces [9] [5].

Looking ahead the hope is that the tools developed in this paper will prove useful to answer more general questions such as “If $X$ is a planar continuum, with the quotient topology must $\pi_1(X, p)$ be zero dimensional?” We conjecture the answer is “yes”.

2 Three axioms ensure $G$ is zero dimensional

The purpose of this section is to prove Theorem 2.15, every space $G$ satisfying the three axioms below has large inductive dimension zero.

We do not assume $G$ is the Hawaiian earring group, but the reader may find it helpful to assume otherwise, to assume $G_n$ is the free group of maximally reduced words on $n$ generators, and to treat a nontrivial element $g \in G$ as an “irreducible” countably infinite word in the letters $x_1, x_1^{-1}, x_2, \ldots$ so that each letter appears finitely many times, and so that each subinterval of letters in $g$ represents a nontrivial loop in the Hawaiian earring.

Constructing a useful linear order on $G$ is a complicated affair carried out in detail in section 3 but the rough idea is the following. Assuming $G$ is the Hawaiian earring group, to compare $g \in G$ and $h \in G$, first delete all letters except $x_1$ and $x_1^{-1}$, but don’t reduce. If the surviving
unreduced words are different this is adequate to tell \( g \) and \( h \) apart. We can also arrange that homotopy classes of words in \( x_1, x_1^{-1} \) determine non interlaced subsets of \( G \). Having well ordered the unreduced words in \( x_1, x_1^{-1} \), we then extend the well ordering to the unreduced words in \( \{ x_1, x_1^{-1}, x_2, x_2^{-1} \} \) so as to ensure axioms 2.2 and 2.3 are destined to hold. The important point is that to compare \( g \) and \( h \) we look, successively at their unreduced approximations, until we find the minimal index where they differ as unreduced words. In particular \( G \) does not have the lexicographic order determined by \( G_n \).

**axiom 2.1.** We assume \( G \) is a sequential space (a space so that if \( A \subset G \) is not closed, there exists a convergent sequence \( a_n \to x \) so that \( a_n \in A \) and \( x \notin A \)). We assume \( G_1 \subset G_2, \ldots \) is a nested sequence of closed discrete subspaces and for each \( n \in \{1, 2, 3, \ldots \} \) the map \( \Pi_n : G \to G_n \) is a retraction. We assume the canonical map \( \phi : G \to \Pi G_n \) is one to one, defined as \( \phi(g) = \{ \Pi_n(g) \} \). We do NOT require that \( \phi \) is a topological embedding and we do NOT require that \( \phi \) is a surjection. We assume \( \Pi_n^{-1} = \Pi_{n-1} \Pi_n \) and we assume the sequence \( \Pi_n(g) \to g \) pointwise for each \( g \in G \).

**axiom 2.2.** We assume \( G \) admits a linear order \( < \) so that every closed set \( B \subset G \) has a minimal element, so that \( \Pi_1(g) \leq \Pi_2(g) \leq \ldots \leq g \), so that each subspace \( G_n \) is well ordered, and for each strictly increasing sequence \( g_1 < g_2 < \ldots \) in \( G \), either every subsequence of \( \{ g_n \} \) diverges, or \( \lim g_n = \sup \{ g_n \} \). (We do NOT require that \( G, < \) has the order topology, and we do NOT assume that the discrete subspace \( G_n \) has the order topology).

**axiom 2.3.** Define \( G_\infty = G_1 \cup G_2, \ldots \) and given \( k \in G_\infty \) obtain \( N \) minimal so that \( k \in G_N \) and define Blowup \((k) = \Pi_N^{-1} \Pi N(k) \). We assume if \( k_1 < k_2 < k_3 \) with each \( k_n \in G_\infty \) then if \( \text{Blowup}(k_3) \subset \text{Blowup}(k_1) \) then \( \text{Blowup}(k_2) \subset \text{Blowup}(k_1) \).

### 2.1 Basic Lemmas

**Remark 2.4.** Since each space \( G_n \) is discrete the countable product \( \Pi_{n=1}^\infty G_n \) is \( T_2 \). Hence, \( G \) is also \( T_2 \) since \( \phi : G_\infty \to \lim_\infty G_n \) is continuous and one to one (although typically NOT a topological embedding). In particular convergent sequences in \( G \) have unique limits.

**Lemma 2.5.** The restriction \( \phi|G_\infty \) maps \( G_\infty \) bijectively onto the eventually constant sequences in \( \lim_\infty G_n \). (In general \( \phi|G_\infty \) is NOT a topological embedding.)

**Proof.** Given \( g \in G_\infty \) obtain \( M \) minimal so that \( g \in G_M \). If \( M \leq M + n \) then \( G_M \subset G_{M+n} \), and since \( \Pi_{M+n} \) is a retraction we have \( \Pi_{M+n}(g) = g \). Thus \( \phi(g) \) is eventually constant.

Conversely suppose \( g \in G \) and \( \phi(g) \) is eventually constant. Obtain \( N \) minimal so that \( \Pi_{N+n}(g) = \Pi_{N}(g) \) if \( N \leq N+n \). By axiom 2.2 \( \Pi_{n}(g) \to g \) and by Remark 2.4 \( \Pi_{N}(g) = g \). Thus since \( \Pi_{N} \) is a retraction \( g \in G_{N} \) and hence \( g \in G_\infty \).
Lemma 2.6. Suppose \( \{a, b\} \subset G \) and \( a < b \). Then there exists \( N \) so that if \( N \leq n \), then \( a < \Pi_n(b) \).

Proof. By axioms 2.1 we know \( \Pi_n(b) \to b \) and also \( \Pi_1(b) \leq \Pi_2(b) \ldots \leq b \). Thus if the result were false, and since \( \prec \) is a linear order, we would have \( \Pi_n(b) \leq a \) for all \( n \). Hence since \( \sup b_n = b \) by axiom 2.2 we would obtain the contradiction \( b \leq a \).

Lemma 2.7. Suppose \( V \subset G \) is clopen and \( V < b \). Then there exists \( N \) so that \( V < \Pi_n(b) \) if \( N \leq n \).

Proof. Since the sequence \( \Pi_n\{ \} \) is nondecreasing (axiom 2.2) it suffices to find \( N \) so that \( V < \Pi_N(b) \). To get a contradiction suppose no such \( N \) exists. For each \( n \) obtain \( k_n \in V \) so that \( \Pi_n(b) \leq k_n \). If there exists \( k_N \) so that \( \Pi_n(b) \leq k_n \) for all \( n \), then \( b \leq k_N \) by axiom 2.2. But since \( k_N \in V \), this would contradict the assumption that \( V < b \). Thus no such \( k_N \) exists and hence for each \( N \) the inequality \( \Pi_N(b) \leq k_N \) as only finitely many solutions. Hence, starting at \( k_1 \), we can recursively manufacture interleaved subsequences \( k_1 < b_2 \leq k_3 < k_4 \leq \ldots \). Thus by axiom 2.2 both subsequences converge to the same limit, and in particular \( \{k_n\} \) has a subsequence converging to \( b \). Since \( V \) is closed, we get the contradiction \( b \in V \).

Lemma 2.8. If \( U \subset G \) is nonempty and clopen in \( G \), then \( \min U \in G_\infty \).

If the convergent strictly increasing sequence \( g_1 < g_2 \ldots \to g \) then \( g \notin G_\infty \) and in particular there exists \( N \) so that if \( N \leq n < m \) then \( \Pi_n(g) < \Pi_m(g) < g \).

Proof. Suppose \( U \subset G \) is nonempty and clopen. By axiom 2.2 \( \min U \) exists. Suppose \( b \in U \setminus G_\infty \). Since \( U \) is open and since \( \Pi_n(b) \to b \) (axiom 2.1), obtain \( N \) so that \( \Pi_N(b) \in U \). By axiom 2.2 \( \Pi_n(b) \leq b \) and since \( \Pi_N(b) \in G_N \subset G_\infty \) and since \( b \notin G_\infty \) we have \( \Pi_N(b) < b \). Hence \( b \) is not minimal in \( U \).

If \( g \in G_N \), then since \( \Pi_N \leq id|G_N \), \( g \) is minimal in the clopen set \( V = \Pi_N^{-1}\Pi_N(g) \). Since \( V \) is open and since \( g \) is minimal in \( V \) it is impossible that there exists a convergent sequence \( g_1 < g_2 \ldots \to g \).

Definition 2.9. If \( L \) and \( H \) are linealry ordered sets a function \( f : L \to 2^H \) is strictly increasing if, given \( s < t \) in \( L \), if \( x \in f(s) \) and \( y \in f(t) \), then \( x < y \).

Lemma 2.10. Let \( S \subset 2^G \) denote the collection of clopen sets in \( G \) and suppose \( \{0, i\} \) is an initial segment of the well ordered set \( J \). Suppose \( \gamma : \{0, i\} \to S \) is strictly increasing and suppose \( V(j) = \cup_{k \leq j} \gamma(k) \) is clopen for each \( j < i \). Then \( V(i) = \cup_{k \leq i} \gamma(k) \) is missing at most one limit point \( x \). If so, there exists an increasing \( s_1 < s_2 \ldots \) sequence terminal in \( [0, i] \).

Moreover for any terminal increasing sequence \( t_1 < t_2 \ldots \) in \( [0, i] \), if \( x_n \in \gamma(t_n) \) then \( x_n \to x \) with \( x = \sup(V(i)) \).
Proof. Note \( V(i) \) is open in \( G \). If \( V(i) \) is not closed in \( G \), then, since \( G \) is a sequential space (axiom 2.1) suppose \( \{k_n\} \subset V(i) \) is a convergent sequence so that \( k_n \to x \notin V(i) \). Suppose \( a_n \) is also a convergent sequence in \( V_i \) with \( a_n \to y \notin V(i) \). To prove \( V(i) \) is missing precisely one limit point it suffices to show that \( x = y \), since this will ensure \( V(i) \cup \{x\} \) is sequentially closed in the sequential space \( T_2 \) space \( G \) (axiom 2.1), and hence that \( V(i) \cup \{x\} \) is closed in \( G \).

Note \( \{k_n\} \) admits no constant subsequence since otherwise we get the contradiction \( x \in \gamma(s) \) for some \( s \). Thus we may refine so that the terms of \( \{k_n\} \) and \( \{a_n\} \) are distinct. Moreover since \( [0, i) \) is well ordered, and since \( \gamma_i \) is increasing, we may further refine so that both sequences are strictly increasing. Thus wolog \( k_n \in \gamma_i(s_n) \) and \( a_n \in \gamma_i(t_n) \) with \( s_1 < s_2 \ldots \) and \( t_1 < t_2 \ldots \).

Note \( \{s_n\} \) is unbounded in \( [0, i) \) since otherwise we get the following contradiction. Let \( \sup \{s_n\} = s < i \). Then \( V(s) \) is a closed subspace of \( G \) and hence \( x \in V(s) \subset V(i) \). Thus both sequences \( \{s_n\} \) and \( \{t_n\} \) are unbounded in the well ordered set \( [0, i) \) and hence the sequences are interlaced. It follows from axiom 2.2 that \( x = y \) and \( x = \sup(V(i)) \).  

\[ \text{Lemma 2.11. Suppose } [0, i) \text{ is an initial segment of the well ordered set } I, \text{ suppose } S \text{ denotes the clopen subsets of } G. \text{ Suppose the functions } \kappa : [0, i) \to G \text{ and } \gamma : [0, i) \to S \text{ are strictly increasing. Suppose is a function } K : [0, i) \to S. \text{ Suppose } V(j) = \cup_{k \leq j} \gamma(k) \text{ is clopen in } G \text{ for each } j < i. \text{ Suppose for each } j < i \text{ we have } \gamma(j) = (\Pi^{-1}_{\gamma(j)}(\Pi_{\gamma(j)}(\kappa(\gamma(j))))/K(j) \text{ with } K(j) \text{ clopen in } G \text{ and } \kappa(j) < K(j). \text{ Suppose given } j, \kappa(j) \text{ and } K(j), \text{ the index } \eta(j) \text{ is minimal to ensure that } \gamma([0, j)) \text{ is increasing. Suppose } K(j) \text{ is eventually constant, i.e. there is a clopen set } K \subset G \text{ and } \lambda_1 \in [0, i) \text{ so that } K(j) = K \text{ if } \lambda_1 \leq j < i. \text{ Then } V(i) = \cup_{j < i} \gamma(j) \text{ is clopen in } G. \]

Proof. Note \( V(i) \) is open in \( G \). If \( [0, i) \) is bounded then \( i \) has a predecessor \( j = i - 1 \) and thus \( V(i) = V(j) \) is clopen by hypothesis. Hence assume \( [0, i) \) is unbounded. To obtain a contradiction suppose \( V(i) \) is not closed in \( G \).

Recall Lemma 2.10 let \( \{c\} = \overline{\Pi} \setminus V_i \) with \( c = \sup(V(i)) \). By Lemma 2.10 obtain a convergent sequence \( s_1 < s_2 \ldots \) terminal in \( [0, i) \), and note \( \kappa(s_n) \to c \). Since \( \kappa(s_n) \notin K \) and since \( K \) is clopen, \( c \notin K \). By axiom 2.2 \( c = \sup(\kappa(s_n)) \) and since eventually \( \kappa(s_n) < K \) we have \( c < K \).

Since \( \gamma(\lambda_1) < c \) we apply Lemma 2.7 and Lemma 2.8 to obtain \( N \) minimal so that if \( N \leq n < m \) then, \( \gamma(\lambda_1) < \Pi_N(c) \leq \Pi_n(c) < \Pi_m(c) < c \).

Minimality of \( N \) ensures \( \Pi_{\lambda_1-n}(c) < \Pi_N(c) \) if \( n \geq 1 \) and in particular \( \Pi_{\lambda_1-n}(c) \neq \Pi_N(c) \). Thus, recalling axiom 2.4 we have \( \text{Blowup}(\Pi_N(c)) = \Pi^{-1}_N(\Pi_N(c)). \)

We next show \( \Pi_N(c) \notin \gamma(s) \) for all \( s \). Note by definition \( \gamma(\lambda_1) < \Pi_N(c) < c < K \). Thus, since \( \gamma \) is increasing, \( \Pi_N(c) \notin \gamma(s) \) if \( s \leq \lambda_1 \). If \( \lambda_1 < s \) then \( \gamma(s) = \Pi^{-1}_{\lambda(s)}(\Pi_{\lambda(s)}(\kappa(s))) \setminus K \). Thus, since \( \Pi_N(c) < c < K \) if \( \Pi_N(c) \notin \gamma(s) \) we would get the contradiction \( c \in \gamma(s) \).

Since \( \kappa(s_1) < \kappa(s_2) \ldots \to c \), since \( \gamma(s_1) < \gamma(s_2) \ldots \) and since \( \Pi_N(c) < c \), obtain \( M \) so that \( \Pi_N(c) < \min(\gamma(s_M)) \) and also so that if \( M \leq n \) then (by continuity of the map \( \Pi_N \) with image in the discrete space \( G_N \)) we have \( \Pi_N(\kappa(s_n)) = \Pi_N(c) \).
In particular for all \( n \leq N \) we have \( \Pi_n(\kappa(s_M)) = \Pi_n(c) \). Consequently \( N < n(s_M) \) since otherwise we get the contradiction \( c \in \gamma(s_M) \).

Recalling axiom 2.21 minimality of \( \eta(s_M) \) ensures \( \text{Blowup}(\Pi_n(s_M)) = \Pi_n(\kappa(s_M)) \) and thus, since \( N < \eta(s_M) \) and since \( \Pi_N(\Pi_n(s_M)(\kappa(s_M))) = \Pi_N(c) \), we have \( \text{Blowup}(\kappa(s_M)) \subseteq \text{Blowup}(\Pi_N(c)) \).

Recall we have shown \( \Pi_N(c) \notin \gamma(s) \) for all \( s \), and also that \( \Pi_N(c) < \min \gamma(s_M) \), and hence \( \emptyset \neq \{ \lambda \in [0,i] | \Pi_N(c) < \min \gamma(\lambda) \} \). Thus, since \( [0,i] \) is a well ordered set, obtain \( \lambda \in [0,i] \) minimal so that \( \Pi_N(c) < \min \gamma(\lambda_2) \). Hence, since \( \text{Blowup}(\kappa(s_M)) \subseteq \text{Blowup}(\Pi_N(c)) \), and since \( \Pi_N(c) < \min \gamma(\lambda_2) \leq \min \gamma(s_M) \), it follows Lemma 2.8 and axiom 2.2 that \( \text{Blowup}(\min \gamma(\lambda_2)) \subseteq \text{Blowup}(\Pi_N(c)) \).

Recall the injection \( \phi : G \rightarrow \lim_{n}G_n \) from axiom 2.1 and note by Lemma 2.21, \( G_\infty \) maps \( G_\infty \) precisely onto the eventually constant sequences in \( \lim_{n}G_n \). Hence, following the proof of Lemma 2.21, \( g \in G_\infty \), to obtain \( \text{Blowup}(g) = \Pi_M g \) defined in axiom 2.2 we use the minimal constant \( M \) so \( \Pi_M(g) = \Pi_M g \) for all \( n \). Thus, our knowledge that \( \text{Blowup}(\min \gamma(\lambda_2)) \subseteq \text{Blowup}(\Pi_N(c)) \) ensures \( \Pi_N(c) = \Pi_n(\kappa(\lambda_2)) \) for all \( N \). However, since \( \Pi_N(c) < \min \gamma(\lambda_2) = \Pi_n(\kappa(\lambda_2)) \) the latter set inclusion is proper and hence \( N < \eta(\kappa(\lambda_2)) \).

The contradiction is as follows. Given \( \lambda_2, \kappa(\lambda_2) \) and \( K \), the index \( \eta(\lambda_2) \) was chosen minimal to ensure \( \gamma([0, \lambda_2]) \) is increasing. On the order hand \( N < \eta(\kappa(\lambda_2)) \) would have been an admissible choice since \( \gamma(s) < \Pi_N(\kappa(\lambda_2)) \) for all \( s < \lambda_2 \).

**Remark 2.12.** Suppose \( [0,i] \) is a linearly ordered set and \( \gamma : [0,i] \rightarrow 2^G \) is increasing and suppose \( \gamma(j) \) is clopen for each \( j < i \). Suppose \( V(j) = \cup_{k \leq j} \gamma(k) \) is clopen for each \( j < i \). Then the set \( W(j) = \cup_{k < j} \gamma(j) \) is clopen since \( W(j) = V(j) \setminus \gamma(j) \), the difference of two clopens.

## 2.2 Thickening \( B \) when \( a < B \)

**Theorem 2.13.** Suppose \( a \in G \) and \( a < B \) with \( B \subseteq G \) a nonempty closed set. The following proof constructs a clopen set \( V(a,B) \subseteq G \) so that \( a < V(a,B) \) and \( B \subseteq V(a,B) \). Moreover if the convergent increasing sequence \( a_1 < a_2 \ldots \rightarrow a \) there exists \( M \) so that if \( M \leq n \) then \( V(a_n,B) = V(a,B) \).

**Proof.** Let \( B \) be a well ordered set with minimal element 0 so that \( G_\infty < |J| \). By axiom 2.12 let \( b_0 = \min B \). Apply Lemma 2.11 and obtain \( N \) minimal so that \( a < \Pi_N(b_0) \). Define \( \gamma(0) = \Pi^{-1}_N(\Pi_N(b_0)) \). We will use Lemma 2.11 repeatedly, in the special case \( K_j = \emptyset \) for all \( j \).

Let \( S \) denote the clopen subsets of \( G \). Suppose \( i \in I \) and \( \gamma([0,i]) \rightarrow S \) is strictly increasing, suppose for each \( j \leq i \) the set \( V(j) = \cup_{k \leq j} \gamma(k) \) is clopen in \( G \). Suppose for each \( j < i \) we have \( \gamma(j) = \Pi_{n_j}^{-1}\Pi_{n_j}(c_j) \). Suppose given \( j \) and \( c_j \) the index \( n_j \) is minimal to ensure that \( \gamma([0,j]) \) is increasing. Thus by Lemma 2.11 the set \( W(i) = \cup_{j < i} V(j) \) is clopen in \( G \). If \( B \subseteq W(i) \) let \( V(a,B) = W(i) \) and we are done.

Otherwise by axioms 2.12 let \( b_i = \min(B \setminus W(i)) \) and by Lemma 2.11 obtain \( N \) minimal so that \( W(i) < \Pi_N^{-1}(\Pi_N(b_i)) \). Define \( \gamma([0,i]) = \gamma([0,i]) \cup \{ i, \Pi_N^{-1}(\Pi_N(b_i)) \} \). If \( \gamma(j) \) has been defined we have \( \min \gamma(j) \in G_\infty \) by
Lemma 2.8. Thus, since \(|G_\infty| < |J|\) the transfinite recursive construction eventually terminates in the desired clopen set \(V(a, B)\).

Suppose the increasing sequence \(a_1 < a_2\ldots \to a\). Recall \(\gamma(0) = \Pi_N^{-1}\Pi_N(b_0)\) with \(b_0\) minimal in \(B\) and \(N\) minimal so that \(a < \Pi_N^{-1}\Pi_N(b)\). Note for \(0 < j\) the definition of \(\gamma(j)\) only depends on \(\gamma(k)\) for \(k < j\), and in particular the definition of \(\gamma(j)\) does not depend on \(a\). Thus \(V(a, B)\) is determined by the data \(B\) and \(\Pi_N^{-1}\Pi_N(b_0)\). Hence \(V(a_m, B) = V(a, B)\) provided \(N\) is minimal so that \(a_m < \Pi_N(b)\). Since \(a < b_0\), by axiom 2.2 the convergent nondecreasing sequences \(\{\Pi_m(a)\}\) and \(\{\Pi_n(b_0)\}\) can only be interleaved for finitely many terms, and hence there exists \(M\) so that if \(M \leq m\) then \(N\) is also the minimal solution to \(a_m < \Pi_N(b_0)\). Thus \(V(a_m, B) = V(a, B)\) if \(M \leq m\).

Lemma 2.14. Suppose \(X\) is a space and the set \([0, i]\) is well ordered with the order topology and suppose there exists a terminal sequence \(s_1 < s_2\ldots\) in \([0, i]\) and suppose \(f : [0, i] \to X\) is a (not necessarily continuous) function. Then the following are equivalent 1) \(f\) is at continuous at \(i\), (i.e. for each open \(U \subset X\) with \(\kappa(i) \in U\), there exists \(j_U\), \(j_U \in f^{-1}(U)\)) 2) If the sequence \(t_1 < t_2\ldots\) is terminal in \([0, i]\) then \(f(t_n) \to f(i)\). As a special case, if \(\{t_n\}\) is eventually constant for each terminal increasing sequence \(t_1 < t_2\ldots\) then \(f([0, i])\) is eventually constant.

Proof. 1 \(\Rightarrow\) 2. Given \(\kappa(i) \in U\), get the mentioned \(j_U\), and note with finitely many exceptions we have \(j_U < t_n\). 2 \(\Rightarrow\) 1 To obtain a contradiction suppose not 1. Get an open \(U \subset X\) with \(\kappa(i) \in U\) but so that \(V = f^{-1}(U)\) contains no open right ray. Starting with \(n = 1\) and proceeding recursively, for each \(n\) obtain \(t_n > s_n\) and \(t_n > t_{n-1}\) so that \(t_n \notin V\). Thus \(t_1 < t_2\), and \(\{t_n\}\) is terminal in \([0, i]\). Thus with finitely many exceptions \(f(t_n) \in U\) and hence \(t_n \in V\) eventually. This contradicts \(t_n \notin V\).

2.3 \(G\) has large inductive dimension 0

Theorem 2.15. \(G\) has large inductive dimension 0.

Proof. Let \(J\) be a well ordered set with minimal element 0 so that \(|J| \geq |G_\infty|\). Suppose \(A\) and \(B\) are disjoint nonempty closed sets in \(G\). Define \(\kappa(0) = \min(A \cup B)\). If \(\kappa(0) \in A\) define \(B(0) = B\), and define \(K(0) = V(\kappa(0), B(0))\), the clopen set from Theorem 2.14. If \(\kappa(0) \in B\) define \(A(0) = A\) and, again using the construction from Theorem 2.14 define \(K(0) = V(\kappa(0), A)\). Define \(\gamma(0) = \Pi_1^{-1}\Pi_1(\kappa(0)) \setminus K(0)\).

Transfinite induction hypothesis: Suppose \(i \in J\) and \(\gamma : [0, i) \to S\) satisfies, with one possible exception, all of the hypotheses of Lemma 2.11 but not necessarily the requirement that \(K(j)\) is eventually constant.

To be precise suppose \(\kappa : [0, i) \to A \cup B\) is a function. Suppose \(V(j) = \cup_{k \leq j} \gamma(k)\) is clopen in \(G\) for each \(j < i\). Suppose for each \(j < i\) we have \(\gamma(j) = (\Pi_N^{-1}\Pi_N)(\kappa(j))\setminus K(j)\) with \(K(j)\) clopen in \(G\) and
\(\kappa(j) < K(j)\). Suppose given \(j, \kappa(j)\) and \(K(j)\) the index \(\eta(j)\) is minimal to ensure that \(\gamma([0, j])\) is increasing. Let \(U(j) = (\prod_{n \in j} \Pi_{\eta(n)}(\kappa(n)))\). We also assume for all \(j < i\) that \(\gamma(j) \cap A = \emptyset\) or \(\gamma(j) \cap B = \emptyset\). If \(j < i\) \(W(j) = U_{k < j} \gamma(k)\) and we note by Remark 2.12 that \(W(j)\) is clopen.

Suppose \(\gamma\) also satisfies the following 2 conditions.

1) Suppose if \(j < i\) then \(\kappa(j) = \min((A \cup B) \setminus W(j))\), if \(\kappa(j) \in A\) then \(K(j) = V(\kappa(j), B(j))\) with \(B(j) = B \setminus W(j)\), and if \(\kappa(j) \in B\) then \(K(j) = V(\kappa(j), A(j))\) with \(A(j) = A \setminus W(j)\).

2) The sets \(V(\kappa(j), B(j))\) and \(V(\kappa(j), A(j))\) are defined as in Theorem 2.13.

Now the heart of the matter is understand why \(\cup_{j < i} \gamma(j)\) is clopen, and for this we argue by contradiction. If \(\cup_{j < i} \gamma(j)\) is not closed apply Lemma 2.10 and let \(c = \sup(\cup_{j < i} \gamma(j))\). Our plan is to ultimately show there exists \(j_A\) so that if \(j_A \leq j\) then wolog \(\kappa(j) \in A\), and \(V(\kappa(j), B(j)) = V(\kappa(j_A), B(j_A)) = K(j_A)\). It will then follow directly from Lemma 2.11 that \(\cup_{j < i} \gamma(j)\) is clopen, contradicting our assumption that \(\cup_{j < i} \gamma(j)\) is not closed.

By Lemma 2.10 obtain an increasing sequence \(s_1 < s_2 \ldots\) terminal in \([0, i]\). By Lemma 2.10 \(\kappa(s_n) \to c\) and hence, since \(im(\kappa) \subset A \cup B\) and since \(A\) and \(B\) are disjoint closed sets, eventually either \(\kappa(s_n) \in A\) or \(\kappa(s_n) \in B\). Thus wolog \(c(s_n) \in A\) eventually. Observe if the increasing sequence \(t_1 < t_2 \ldots\) is also terminal in \([0, i]\) then \(\{s_n\}\) and \(\{t_n\}\) are interlaced and hence by Lemma 2.10 \(\kappa(t_n) \to c\). Thus by Lemma 2.14 the extended function \(\kappa([0, i]) = \kappa([0, i] \cup \{i, c\})\) is continuous at \(i\) with the order topology on \([0, i]\). In particular there must exist \(M\) so that if \(s_M \leq j\) then \(\kappa(j) \in A\), since otherwise we could manufacture a pair of interlaced increasing terminal sequences \(\{s_n\}\) \(\{t_n\}\) with \(\kappa(s_n) \in A\) and \(\kappa(t_n) \in B\) yielding the contradiction \(c \in A \cap B\).

Next we will show if \(s_m \leq j\) then \(B(s_m) = B(j)\). Note if \(j = s_m\) then \(B(s_m) = B(j)\). Proceeding by transfinite induction, suppose \(s_m < j < i\) and \(B(s_m) = B(t)\) whenever \(s_m \leq t < j\). By definition \(B(j) = B \setminus W(j)\) and if \(s_m \leq t < j\) then \(B(t) = B \setminus W(t)\) and thus (since \(W(t) \subset W(j)\)) we have \(B(j) \subset B(t)\), and in particular \(B(j) \subset B(s_m)\). Conversely note \(W(j) = W(s_m) \cup \{s_m \leq t < j\} \gamma(t) = W(s_m) \cup (\cup_{s_m \leq t < j}(U(t) \setminus B(s_m)))\). By definition \(B(s_m) = B \setminus W(s_m)\). Thus if \(x \in B(s_m)\), then \(x \notin W(j)\) and hence \(x \in B(j)\).

We have established that if \(s_m \leq j < i\) then \(K(j) = V(\kappa(j), B(s_M))\) with \(\kappa(j) \in A\). Thus by Theorem 2.13 for each increasing terminal sequence \(t_1 < t_2 \ldots \leq i\) there exists \(M\) so that if \(M \leq n < m\) then \(V(c(t_n), B(t_n)) = V(\kappa(t_m), B(t_m))\). Thus if we define \(f : [s_m, i) \to S\) as \(f(j) = V(\kappa(j), B(j))\) then \(f|[s_m, j)\) is eventually constant by Lemma 2.11. Hence there exists \(j_A \in [s_m, i)\) so that if \(j_A \leq j\) then \(\gamma(j) = U(j) \setminus V(\kappa(j_A), B(s_M))\). It now follows from Lemma 2.11 that \(\cup_{j < i} \gamma(j)\) is clopen after all.

Now there are two cases.

Case 1. If the the clopen set \(W(i) = \cup_{j < i} \gamma(j)\) covers neither \(A\) nor \(B\), define \(\kappa(i) = \min((A \cup B) \setminus W(i))\). If \(\kappa(i) \in A\) define \(K(i) = V(\kappa(i), B \setminus W(i))\) and if \(\kappa(i) \in B\) define \(K(i) = V(\kappa(i), A \setminus W(i))\).
$A \setminus W(i)$.

Now apply Lemma 2.4 and let $\eta(i)$ be minimal so that $W(i) < \prod_{\eta(i)}(\kappa(i))$. Define $U(i) = \prod_{\eta(i)}^{-1} \prod_{\eta(i)}(\kappa(i))$ and define $\gamma(i) = U(i) \setminus K(i)$. Note, by definition, if $\kappa(i) \in A$ and $b \in B$ satisfies $b < \kappa(i)$ then $b \in W(i)$. If $b \in B$ satisfies $\kappa(i) < b$ then $b \in K(i)$. Thus $\gamma(j) \cap B = \emptyset$. By a symmetric argument if $\kappa(i) \in B$ then $\gamma(j) \cap A = X\emptyset$. Thus replacing the index $i$ with $i+1$ and defining $V(i) = W(i) \cup \gamma(i)$, the transfinite induction hypothesis is preserved. Proceeding via transfinite induction we continue the construction in Case 1 until case 2 is achieved.

Case 2. If the clopen set $W(i) = \cup_{j < i} \gamma(j)$ covers $A$ or $B$ then wolog $A \subset W(i)$. Let $T_A = \{ j < i \mid \kappa(j) \in A \}$. Let $U(A) = \cup_{j \in T_A} \gamma(j)$. We must show $U(A)$ is a clopen set such that $A \subset U(A)$ and $B \cap U(A) = \emptyset$, it will then follow that $U(A)$ and $G \setminus U(A)$ are disjoint clopen sets covering $A$ and $B$ respectively.

That $U(A)$ is clopen follows from basic topology. Given any collection $R$ of pairwise disjoint clopen sets in the space $X$, if the union is clopen in $X$, then the union taken over any subset $H \subset R$ will also be clopen in $X$. By hypothesis if $j < i$ then $j \in T_A$ iff $\gamma(j) \cap B = \emptyset$. Thus $U(A)$ is a clopen set covering $A$ such that $U(A) \cap B = \emptyset$.

3 Applications

The motivation for this paper is to prove (with the quotient topology inherited from the space of based loops), that the Hawaiian earring group has large inductive dimension zero. Starting from a combination of first principles, the basics of finite free groups, and the knowledge that $HE$ is $\tau_1$ shape injective, we will ultimately reduce the question of calculating the dimension of $\tau_1(HE, p)$ to Theorem 3.1.

**Theorem 3.1.** Suppose for each $n \in \{1, 2, 3, \ldots \}$, $X_n$ is a discrete space. We assume $X_1 \subset X_2 \subset \ldots$ and for each $n$ the map $R_n : X_{n+1} \to X_n$ is a retraction. Let the space $X_\infty = \lim_{\leftarrow n} X_n$ denote the topological inverse limit, i.e. $X_\infty$ is the subspace of the countable product $X_1 \times X_2 \times \ldots$ so that $(x_1, x_2, \ldots) \in X_\infty$ iff $R_n(x_{n+1}) = x_n$ for each $n$.

Suppose furthermore we have a sequence of quotient maps $q_n : X_n \to G_n$ so that the formula $q_nR_nq_n^{-1} = r_n$ induces a map such that $r_nq_n^{-1} = q_nR_n$, i.e. there is an induced map $r_n : G_{n+1} \to G_n$, commuting with the retraction $X_{n+1} \to X_n$. Finally suppose the formula $q_n^{-1}r_nq_n^{-1}$ induces an embedding $j_n : G_n \to G_{n+1}$, commuting with inclusion $X_n \to X_{n+1}$.

By definition the maps $\{q_n\}$ induce an equivalence relation on $X_\infty$ such that $(x_1, x_2, \ldots) \sim (y_1, y_2, \ldots)$ iff $q_n(x_n) = q_n(y_n)$ for each $n$. Define $G$ as the corresponding topological quotient $q : X_\infty \to G$. Then $G$ has large inductive dimension 0.

**Proof.** Our strategy is to apply Theorem 2.15 by first showing $G$ can be made to satisfy axioms 2.1, 2.2 and 2.3. For axioms 2.2 and 2.3 we must build a linear order on $X_\infty$ which induces a suitable linear order on $G$. This is ultimately straightforward, but with a few restrictions imposed by the starting data $\{q_n\}$, the need to ensure axiom 2.8, and the need to...
ensure that $R_n \leq \text{id}|_{X_{n+1}}$. To define a lexical order on $X_\infty$ it suffices to proceed recursively, first defining a well ordering of $X_1$, then extending to a well ordering of $X_2$ and so on.

To impose a well ordering on $X_1$, first arbitrarily well order $G_1$, and then arbitrarily well order each point prime under $q_1 : X_1 \to G_1$.

Now we define on $X_1$ a kind of local lexical order as follows. To compare two points $\{x_1, y_1\} \subseteq X_1$, if $q_1(x_1) \neq q_1(y_1)$ let the order in $G_1$ determine which is bigger. If $q_1(x_1) = q_1(y_1)$ let the order on point preimages of $q_1$ decide which is bigger. Crucially if $q_1(y_1) \neq q_1(x_1)$ and $q_1(x_1) = q_1(z_1)$ then $y_1 < \{x_1, z_1\}$ or $y_1 > \{x_1, z_1\}$.

To extend the well ordering of $X_1$ to $X_2$ we begin as follows. First, for each $x_1 \in X_1$ define $X_2(x_1) = X_2 \cap R_1^1(x_1)$ and note $x_1 \in X_2(x_1)$ since $X_1 \subseteq X_2$ and $R_1(x_1) = \{z_1\}$. Next, define $G_2(x_1) = q_2(X_2(x_1))$ and note $G_2(x_1) \subseteq G_2$. Now well order $G_2(x_1)$ to have minimal element $q_2(x_1)$ and otherwise the well ordering of $G_2(x_1)$ is arbitrary. Next, well order each point preimage of the map $q_2|_{X_2(x_1)} : X_2(x_1) \to G_2(x_1)$ subject only to the constraint that $x_1 = \min q_2(x_1)$. Thus $x_1 = \min \{X_2(x_1)\}$.

To complete the definition of the well ordering on $X_2$ suppose we are given distinct points $\{x_2, y_2\} \subseteq X_2$. If $R_1(x_2) \neq R_1(y_2)$ we require the order of $X_1$ to dictate which is bigger. If $R_1(x_2) = R_1(y_2)$ and $q_2(x_2) \neq q_2(y_2)$ we require the order of $G_2(R_2(x_2))$ to dictate which is bigger. If $R_1(x_2) = R_1(y_2)$ and $q_2(x_2) = q_2(y_2)$ we require the order on $q_2(x_2)$ to dictate which is bigger. In summary, lexical inspection of the ordered triples $(R_1(x_2), q_2(x_2), x_2)$ and $(R_1(y_2), q_2(y_2), y_2)$ determines which of $\{x_2, y_2\}$ is bigger.

Crucially, if $\{x_2, y_2, z_2\} \subseteq X_2$ and $x_2 < y_2 < z_2$ and $q_1R_1(x_2) = q_1R_1(y_2)$, then $q_1R_1(x_2) = q_1R_1(y_2)$, and we argue the contrapositive as follows. Suppose $q_1R_1(x_2) \neq q_1R_1(y_2)$ and $q_1R_1(x_2) = q_1R_1(y_2)$ with $\{x_2, y_2, z_2\} \subseteq X_2$. Let $x_1 = q_1(x_2)$ and $y_1 = q_1(y_2)$ and $z_1 = q_1(z_2)$. As noted we must have $y_1 < \{x_1, z_1\}$ or $y_1 > \{x_1, z_1\}$, and hence by definition $y_2 < \{x_2, z_2\}$ or $y_2 > \{x_2, z_2\}$. Finally note $R_1 \leq \text{id}|_{X_2}$.

Proceeding recursively, suppose $X_{n-1}$ has been well ordered so that if $q_{n-1}(x_{n-1}) \neq q_{n-1}(y_{n-1})$ and $q_{n-1}(x_{n-1}) = q_{n-1}(y_{n-1})$ then $y_{n-1} < \{x_{n-1}, z_{n-1}\}$ or $y_{n-1} > \{x_{n-1}, z_{n-1}\}$. Suppose $R_{n-1} \leq \text{id}|_{X_{n-1}}$.

For each $x_{n-1} \in X_{n-1}$ define $X_n(x_{n-1}) = X_n \cap R_{n-1}^1(x_{n-1})$, and note $x_{n-1} \in X_n(x_{n-1})$ since $X_{n-1} \subseteq X_n$ and $R_{n-1}(x_{n-1}) = x_{n-1}$. Next, define $G_n(x_{n-1}) = q_n(X_n(x_{n-1}))$ and note $G_n(x_{n-1}) \subseteq G_n$. Now well order $G_n(x_{n-1})$ to have minimal element $q_n(x_{n-1})$ and otherwise the well ordering of $G_n(x_{n-1})$ is arbitrary. Next, well order each point preimage of the map $q_n|_{X_n(x_{n-1})} : X_n(x_{n-1}) \to G_n(x_{n-1})$ subject only to the constraint that $x_{n-1} = \min q_n(x_{n-1})$.

To complete the definition of the well ordering on $X_n$ suppose we are given distinct points $\{x_n, y_n\} \subseteq X_n$. If $R_{n-1}(x_n) \neq R_{n-1}(y_n)$ we require the order of $X_{n-1}$ to dictate which is bigger. If $R_{n-1}(x_n) = R_{n-1}(y_n)$ and $q_n(x_n) \neq q_n(y_n)$ we require the order of $G_n(R_{n-1}(x_n))$ to dictate which is bigger. If $R_{n-1}(x_n) = R_{n-1}(y_n)$ and $q_n(x_n) = q_n(y_n)$ we require the order on $q_n(x_n)$ to dictate which is bigger. In summary, lexical inspection of the ordered triples $(R_{n-1}(x_n), q_n(x_n), x_n)$ and $(R_{n-1}(y_n), q_n(y_n), y_n)$ determines which of $\{x_n, y_n\}$ is bigger.

Crucially, if $\{x_n, y_n, z_n\} \subseteq X_n$ and $x_n < y_n < z_n$ and $q_{n-1}R_{n-1}(x_n) = \cdots$
contrapositive as follows. Suppose \( q_{n-1}R_{n-1}(x_n) \neq q_{n-1}R_{n-1}(y_n) \) and \( q_{n-1}R_{n-1}(x_n) = q_{n-1}R_{n-1}(z_n) \) with \( \{x_n,y_n,z_n\} \subset X_n \). Let \( x_{n-1} = q_{n-1}(x_n) \) and \( y_{n-1} = q_{n-1}(y_n) \) and \( z_{n-1} = q_{n-1}(z_n) \). By the induction hypothesis we must have \( y_{n-1} < \{x_{n-1},z_{n-1}\} \) or \( y_{n-1} > \{x_{n-1},z_{n-1}\} \), and hence, appealing to our local definition, \( y_n < \{x_n,z_n\} \) or \( y_n > \{x_n,z_n\} \). Finally note \( R_{n-1} \leq id|X_n \).

To check that axiom \( \text{(2.1)} \) holds, note the map \( X_n \to X_{\infty} \) sending \( x_n \) to \( (x_1, \ldots, x_n, x_{n+1}, \ldots) \) is a topological embedding, henceforth we conflate the discrete space \( X_n \) with the corresponding discrete subspace of eventually constant sequences in \( X_{\infty} \), the sequences whose terms coincide from index \( n \) upward. Thus, with moderate abuse of notation, we extend the map \( R_n|X_{n+1} \) canonically to \( R_n : X_{\infty} \to X_N \), so that \( R_n(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \), and note \( R_n = R_n R_{n+1} \).

By definition a point \( g \in G \) is a subspace \( g \subset X_{\infty} \) so that distinct points \( \{x, y\} \subset g \) have \( q_n \) equivalent coordinates for each \( n \), and a point \( g_0 \in G_0 \) is a subspace \( g_n \subset X_n \). Our starting assumptions ensure we have a well defined function \( G_n \hookrightarrow G \) sending \( g_n \in G_n \) to \( (\ldots, x_{n-1}(g_n), x_n, j_n(g_n), j_{n+1}(g_n), \ldots) \in G \). This is a topological embedding, and henceforth we conflate \( G_n \) with the corresponding subspace of \( G \). The map \( q_n R_n : X_{\infty} \to G_n \) is constant on sets of the form \( q^{-1}(g) \) and thus there is a unique induced map \( \Pi_n : G \to G_n \) such that \( \Pi_n = \Pi_{n-1} \Pi_{n+1} \). The maps \( \{\Pi_n\} \) determine a continuous injection \( \phi : G \to \lim_{\leftarrow} G_n \).

If \( \phi \) were a topological embedding then it would follow easily that \( \text{dim}G = 0 \), since \( \phi \) embeds \( G \) into the 0 dimensional metric space \( G_1 \times G_2 \ldots \) However in general \( \phi \) is NOT a topological embedding \( \mathbf{[13]} \).

Note each space \( G_n \) is discrete, since topological quotients of discrete spaces are discrete. Moreover \( G_n \) is a quotient of a metrizable space and hence \( G_n \) sequential. Thus axiom \( \text{(2.1)} \) will hold provided we can show pointwise convergence \( \Pi_n \to id|G \). The latter claim holds, shown as follows. Given \( g \in G \) and an open \( U \subset G \) so that \( g \in U \), lift \( g \) to some \( (x_1, x_2, \ldots) \in q^{-1}(g) \subset X_{\infty} \) and obtain a basic open \( V \subset q^{-1}(U) \subset X_{\infty} \) with \( (x_1, x_2, \ldots) \in V \) so that \( V = \{x_1\} \times \{x_2\} \times \{x_N\} \times X_{N+1} \ldots \). Now given \( n \geq N \) to check \( \Pi_n(g) \in U \) it suffices to check, since \( q : X_{\infty} \to G \) is a quotient map, that some lift of \( \Pi_n(g) \in U \). Our special point \( (x_1, x_2, \ldots) \in V \) suffices. Thus axiom \( \text{(2.1)} \) holds.

The space \( \lim_{\leftarrow} X_n = X_{\infty} \) is a topological inverse limit of discrete spaces under retraction bonding maps \( R_n|X_{n+1} \). We have well ordered each set \( X_n \) so that \( R_n(x_{n+1}) \leq x_n \) for \( x_{n+1} \in X_{n+1} \). Thus with the induced lexicographic order on \( X_{\infty} \) we have \( R_n(x) \leq x \) for \( x \in X_{\infty} \).

The space \( X_{\infty} \) has the topology of pointwise convergence, and thus, since \( X_n \) is a discrete space, a sequence \( \{s_n\} \subset X_{\infty} \) converges iff \( \Pi_n(s_n) \) is eventually constant for each \( N \). Thus, if the strictly increasing sequence \( s_1 < s_2 \ldots \subset X_{\infty} \) diverges, then every subsequence of \( \{s_n\} \) diverges, since \( \{\Pi_n(s_n)\} \) is nondecreasing for each \( N \). Conversely, if the increasing sequence \( \{s_n\} \subset X_{\infty} \) converges, then, since \( \{\Pi_n(s_n)\} \) is nondecreasing and eventually constant for each \( N \), \( \lim \{s_n\} = \sup \{s_n\} \).

Since each set \( X_n \) is well ordered, and since no well ordered set admits
a strictly decreasing sequence with infinitely many terms, each strictly decreasing sequence \( s_1 > s_2 \ldots \subset X_\infty \) converges. Consequently if \( B \subset X_\infty \) is nonempty and closed, then \( \min(B) \) exists. To see why, let \( b_1 = \min \Pi_1(B) \). Let \( b_2 = \min \Pi_2 \Pi_1^{-1}(b_1) \). Let \( b_3 = \min \Pi_3 \Pi_2^{-1}(b_2) \) and so on. Note \( b_1 \geq b_2 \ldots \) and let \( b = \lim \{ b_n \} \). Thus \( b \in B \) since \( B \) is closed .Note \( b \leq a \) for each \( a \in B \) and thus \( b = \min(B) \).

Note \( G \) is \( T_2 \) by Remark 2.3 and in particular \( G \) is \( T_1 \). Thus point preimages are closed under the map \( q : X_\infty \rightarrow G \). Define \( \sigma : G \rightarrow X_\infty \) as \( \sigma(g) = \min(q^{-1}(g)) \). Since \( \sigma \) is one to one and since subsets of linearly ordered sets are linearly ordered, we obtain a linear order on \( G \) defined so that \( g < h \) iff \( \sigma(g) < \sigma(h) \). In particular the sequence \( \{g_n\} \) is strictly increasing in \( G \) iff \( \{\sigma(g_n)\} \) is strictly increasing in \( X_\infty \). In the process of checking axiom 2.2 we will show \( \sigma \) is left continuous but not right continuous.

First we observe some basic properties of our definition of the linear order on the set \( X_\infty \), conflating \( X_N \) with the subspace of \( X_\infty \), all of whose terms are constant from index \( N \) and above. If \( x = (x_1, x_2, \ldots) \in X_\infty \) then \( x_n \leq x_{n+1} \leq x \), and \( x_n \rightarrow x \), and \( x = \sup \{x_n\} \).

Next we observe a basic property of \( \sigma \). If \( (y_1, y_2, \ldots) = y < z = \sigma(g) = (z_1, z_2, \ldots) \in X_\infty \) and if \( N \) is minimal so that \( y_N \neq z_N \), then \( y_N < z_N \), \( \min(q_N(y_N)) < z_N = \min(q_N(z_N)) \), and hence \( q_N(y_N) < q_N(z_N) \).

To check axiom 2.2 if \( A \subset G \) is closed, then \( q^{-1}(A) = B \) is closed in \( X_\infty \) and thus \( \min A = q(\min B) \) and in particular \( A \) has a minimal point. Suppose \( \{g_n\} \) is a strictly increasing sequence in \( G \).

Case 1. Suppose the corresponding increasing sequence \( \{\sigma(g_n)\} \) converges to some \( x = (x_1, x_2, \ldots) \in X_\infty \). Note the sequences \( \{\sigma(g_n)\} \) and \( \{x_n\} \) are interlaced.

Sequential continuity of \( q \) at \( x \) shows \( \{g_n\} \) converges to some \( q(x) = g \in G \). We will show \( x = \sigma(g) \) and \( g = \sup \{g_n\} \).

Suppose \( y = (y_1, y_2, \ldots) \in X_\infty \) with \( y < x \). Obtain \( N \) minimal so that \( y_N < x_N \). Obtain \( M \) minimal so that \( R_N(\sigma(g_M)) = x_N \). Note \( R_n(y) = R_n(x) = R_n(\sigma(g_M)) \) if \( n < M \). Thus \( q_N(y_N) < q_N(\sigma(g_M)) = q_N(x_N) \) and hence \( q(y) < q(x) \). This shows \( x = \min(q(x)) \), i.e. \( x = \sigma(h) \) for some \( h \in G \). However, by definition \( q \sigma = \id(G) \), and thus \( q = q(x) = q(h) = h \). Hence \( x = \sigma(g) \). This argument also shows if \( \sigma(k) = y < x = \sigma(g) \) then \( q(y) < q(g) \) and hence \( g = \sup \{g_n\} \).

Case 2. If the increasing sequence \( \{\sigma(g_n)\} \) diverges in \( X_\infty \) then every subsequence of \( \{\sigma(g_n)\} \) diverges in \( X_\infty \) and hence, since \( g \) is a quotient map, every subsequence of \( \{g_n\} \) diverges in \( G \).

The injection \( \sigma(G_n) \) determines that \( G_n \) is well ordered. Now we must check that \( \Pi_n \leq \Pi_{n+1} \leq \id(G) \) while keeping in mind that \( G \) does NOT in general have the lexicographic order. That is to say, it can happen that \( g < h \) in \( G \) but \( \Pi_n(g) \geq \Pi_n(h) \) with \( n \) minimal so that \( \Pi_n(g) \neq \Pi_n(h) \).

Given \( g \in G \) let \( x = \sigma(g) \in X_\infty \) and let \( x_n = R_n(x) = R_n(R_{n+1}(x)) = R_n(x_{n+1}) \) with \( x_{n+1} = R_{n+1}(x_n) \). Our recursive definition of the linear order on \( X_{n+1} \) ensures \( \min(q_n(x_n)) \leq \min(q_{n+1}(x_{n+1})) \) and inclusion \( G_n \rightarrow G \) ensures \( \min(q_{n+1}(x_{n+1})) \leq \min(q(x)) \) and hence \( \Pi_n \leq \Pi_{n+1} \leq \id(G) \). Thus axiom 2.2 holds.

To check axiom 2.3 we first decode the notion of blowups in the context of \( X_\infty \). By definition \( G_\infty = G_1 \cup G_2 \ldots \) and \( \sigma(G_\infty) \subset X_\infty \). Given
$x \in \sigma(G_\infty)$ let $x = \sigma(g)$, and obtain $N$ minimal so that $g \in G_N$. Thus $x \in X_N$ with $R_{N-1}(x) < R_N(x) = R_{N+k}(x)$. Thus $x \in X_N \setminus X_{N-1}$ and hence by definition of our linear order on $X_N$ we have $x = \inf(q_N(x))$. By definition $\text{blowup}(g) = \Pi_N^1 \Pi_N(g) = \Pi_N^1(g) \subset G$. The pullback in $X_\infty$ is precisely $q^{-1}(g)$.

The lexicographic order on $X_\infty$ ensures the following lifted version of axiom 2.3 holds. Suppose $x_1 < x_2 < x_3$ with $x_n \in X_{N_n}$ and $N_n$ minimal so that the mentioned membership holds, but also so that $R_{N_2}^{-1}(x_3) \subset R_{N_1}^{-1}(x_1)$. Then $R_{N_2}^{-1}(x_2) \subset R_{N_1}^{-1}(x_2)$. To see why, note $R_{N_1}(x_3) = x_1$ and hence by definition of lexicographic order we have $R_{N_1}(x_2) \neq x_1$.

Thus given points $k_1 < k_2 < k_3$ in $G_\infty$ so that $\text{blowup}(k_3) \subset \text{blowup}(k_1)$, let $x_n = \sigma(k_n)$ and deduce $\text{blowup}(k_2) \subset \text{blowup}(k_1)$. Hence axiom 2.3 holds.

**Corollary 3.2.** The Hawaiian earring $HE$ is a subspace of the plane, the union of a sequence circles centered at $(0,1/n)$ with radius $1/n$ and sharing the common point $(0,0)$. The Hawaiian earring group $G$ is the fundamental group of $HE$, the set of path components of the space of based loops in $HE$, with group operation concatenation. Endowed with the quotient topology inherited from the space of based loops in $HE$, the Hawaiian earring group has large inductive dimension 0.

**Proof.** Let $L(S^1, 1, HE, p)$ denote the space of based loops in $HE$. Note $L(S^1, 1, HE, p)$ is a separable metric space with the uniform metric topology (equivalent in this case to the compact open topology), since both $S^1$ and $HE$ are compact metric spaces. Our plan is to manufacture a space $G$ as in the hypothesis of Theorem 3.1 and build a quotient map $F : L(S^1, 1, HE, p) \to G$ whose point preimages are precisely the path components of $L(S^1, 1, HE, p)$. Consequently, by basic general topology, there is an induced homeomorphism $h : \pi_1(HE, p) \to G$, and hence both spaces have the same dimension.

If $p \in HE$ is the interesting point, given an arbitrary based loop $f \in L(S^1, 1, HE, p)$, for each component $J \subset f^{-1}(HE \setminus p)$, we can homotopically tighten $f|J$ within its image to a linearly parameterized loop or to a constant. Since $f$ is uniformly continuous, the union of the tightenings $f_{n\ell}$ is continuous and path homotopic to $f$, and we call the resulting map $f_{n\ell}$ weak tight. Thus, a loop $f_{n\ell} \in L(S^1, 1, HE, p)$ is weak tight provided $f|J$ is linear and one to one, for each component $J \subset f_{n\ell}^{-1}(HE \setminus p)$. Notice the weak tight loops $WT(S^1, 1, HE, p)$ comprise a closed subspace of $L(S^1, 1, HE, p)$, since being not weak tight is an open property for loops in $L(S^1, 1, HE, p)$.

If $H$ denotes the group of orientation preserving homeomorphisms of $S^1$ which fix 1, then $H$ acts isometrically on $WT(S^1, 1, HE, p)$ via right composition, i.e. $h \in H$ sends $f_w \in WT(S^1, 1, HE, p)$ to the map $f_w h$. Thus by Lemma 1.1 the quotient $WT(S^1, 1, HE, p)/H(WT(S^1, 1, HE, p))$ is metrizable. For convenience rename the mentioned quotient space $X_\infty$ and the quotient map $Q_w : WT(S^1, 1, HE, p) \to X_\infty$. A typical point of $X_\infty$
is a weak tight path up to monotone orientation preserving reparameter- 
ization, i.e. two weak tight paths are equivalent if they pass through the 
same points in the same order.

Let $HE_n \subset HE$ denote the bouquet of the first $n$ loops. Thus if we 
define $X_n \subset X_\infty$ as the subspace with image in $HE_n$ we have an 
induced retraction $R_n : X_\infty \to X_n$ deleting all large index loops. Crucially notice 
$X_n$ is the discrete monoid on $n$ letters $(x_1, x_1^{-1}, \ldots, x_n^{-1})$, with the empty 
word corresponding to the constant loop at $p$, and $X_\infty = \lim_\leftarrow X_n$, with bonding map $R_n|_{X_{n+1}}$. Thus we can think of points of $X_\infty$ as unreduced 
infinite words in $\{x_1, x_1^{-1}, x_2, \ldots\}$ so that each letter appears finitely many 
times.

Now let $G_n$ denote with the discrete topology, the free group on $n$ 
letters $\{x_1, \ldots, x_n\}$ and let $q_n : X_n \to G_n$ denote the canonical quotient map. Note the maps $\{q_n\}$ induce an equivalence relation on $X_\infty$: 
two infinite words $w \in X_\infty$ and $v \in X_\infty$ are equivalent iff for all $n$ 
$q_n R_n (w)=q_n R_n (v) \in G_n$. Let $q : X_\infty \to G$ denote the corresponding 
quotient map determined by this equivalence relation.

The previous paragraphs establish a composition of functions $L(S^1, 1, HE, p) \to 
WT(S^1, 1, HE, p) \to X_\infty \to G$. The first arrow is a discontinuous retraction, the second and third arrows are continuous quotient maps, and we let $F$ denote the composition $L(S^1, 1, HE, p) \to G$. By definition $\pi_1(HE, p)$ is the quotient of $L(S^1, 1, HE, p)$ modding out by the path 
components. Thus, to prove the existence of an induced homeomorphism 
h : $\pi_1(HE, p) \to G$, it suffices, by basic general topology, to show that 
$F$ is a quotient map whose point preimages are the path components of 
$L(S^1, 1, HE, p)$. Let $W : L(S_1, 1, HE, p) \to WT(S^1, 1, HE, p)$ denote the 
discontinuous retraction described previously.

To check continuity of $F$ suppose $f_n \to f$ uniformly in $L(S^1, 1, HE, p)$. 
We apply Lemma 4.4 to the sequence $\{F(f_n)\}$ and first show $\{\Pi_N(F(f_n))\}$ 
is eventually constant for each $N$. Let $\kappa_N : HE \to HE_N$ denote the canonical retraction, notice locally at $f$, the composition $\{\kappa_N(f)\}$ eventually 
preserves the homotopy path class of $\{\kappa_N(f)\}$ in $HE_N$. Thus $\{\Pi_N(F(f_n))\}$ is eventually constant. To check that $\{\sigma(F(f_n))\}$ has 
compact closure we will apply Ascoli’s Theorem. First note the map $W$ pre-
serves or improves equicontinuity data (and the image of $1 \in S^1$ is constant 
and thus convergent), and hence $\{W(f_n)\}$ has compact closure in 
$WT(S^1, 1, HE, p)$.

The following definition has an algebraic analogue, the different ways 
that one might start with an unreduced word in $X_N$ and then cancel 
inverse pairs to create the irreducible representative. Given a weak tight 
loop $\beta \in WT(S^1, 1, HE, p)$ and a natural number $N$, define $\Sigma(\beta, N) \subset 
WT(S^1, 1, HE, p)$ as the subspace of irreducible loops in $HE_N$, obtained 
by starting with $\kappa_N(\beta)$ and deleting successive nonconstant $p$ based inessential 
loops, replacing each with the constant map $p$.

The important observation is that each loop in $\Sigma(\beta, N)$ has equi-
continuity data no worse than that of $\beta$. Thus, since $\{W(f_n)\}$ has compact 
closure, the union over $N$ and not the subspaces $\Sigma(W(f_n), N)$ has compact 
closure in $WT(S^1, 1, HE, p)$. Call the latter compactum $C$, recall 
Lemma 4.4 and observe $\{\sigma(F(f_n))\} \subset C$. Thus $F$ is continuous by Lemma 
4.4. Since $qQ_w$ is a quotient map, and since the (discontinuous) map $W$ is
a retraction, it follows from Lemma 4.2 that $F$ is a quotient map.

To see that point preimages of $F$ are precisely the path components of $L(S^2,1,HE,p)$, first note $W(β)$ is path homotopic in $HE$ to $β$. Thus if $α$ and $β$ are path homotopic in $HE$ then $W(α)$ and $W(β)$ are path homotopic in $HE$. Thus $q_nR_n(Q_uW(α)) = q_nR_n(Q_uW(β))$ for all $n$ and hence $F(α) = F(β)$. Conversely, since the Hawaiian earring is $π_1$ shape injective, if $α$ and $β$ are not path homotopic in $HE$ then $q_nR_n(Q_uW(α))$ or $q_nR_n(Q_uW(β))$ for some $n$ and hence $F(α) ≠ F(β)$. Thus $π_1(HE,p)$ with the quotient topology, is homeomorphic to $G$. It follows from Theorem 3.1 that $G$ has large inductive dimension zero, and hence $π_1(HE,p)$ has large inductive dimension zero.

4 Miscellaneous

Lemma 4.1. Suppose $(X,d)$ is a metric space and $H$ is a group (under function composition) of isometries of $X$. Then the orbit closures under the action forms a partition of $X$ and the Hausdorff metric is compatible with the quotient topology. (It is not necessary to assume $X$ is complete, that the orbits are bounded, or that the action is free.) Given $x ∈ X$ define $C(x) = \{H(x)\}$. Thus $C(x)$ is a typical element of the quotient space. With moderate abuse of notation we denote the quotient space $X/\overline{H(X)}$.

Proof. Given $x ∈ X$ define $C(x) = \{H(x)\}$. To check the orbit closures are disjoint, given $y ∈ C(x)$ let $y = \lim h_n(x)$ for some sequence $\{h_n\} ⊂ H$. Suppose $ε > 0$ and $z ∈ H(y)$. Let $z = h(y)$, get $n$ so that $d(y, h_n(x)) < ε$. Then $d(z, hh_n(x)) < ε$. Thus $z ∈ C(x)$ and hence $H(y) ⊂ C(x)$. Thus $C(y) ⊂ C(x)$ since $C(x)$ is closed. By a symmetric argument $C(x) ⊂ C(y)$ and thus $C(x) = C(y)$. Thus the sets of the form $C(x)$ determine a partition of $X$ into pairwise disjoint closed sets.

Given orbit closures $C(x)$ and $C(y)$ let $ε$ denote inf $\{d(x,y)\}$ taken over all $z ∈ H(y)$. Define $D(C(x), C(y)) = ε$. It is straight forward to check this is the Hausdorff metric, and the canonical map $X → X/\overline{H(X)}$ is a contraction. To check it’s a quotient map. Suppose $A ⊂ X/\overline{H(X)}$ is not closed with $C(a_n) → C(x)$ with $C(a_n) ∈ A$ and $C(x) ∉ A$. Obtain $x_n ∈ C(a_n)$ with $x_n → x$. Thus the preimage of $A$ is not closed in $X$.

Lemma 4.2. Suppose $X$ is a space and $r : X → Y$ is a (possibly discontinuous) retraction onto the subspace $Y$. Suppose $q : Y → Z$ is a (continuous) quotient map and $A ⊂ Z$ is not closed. Then $(qr)^{-1}(A)$ is not closed in $X$.

Proof. Pullback $A$ to $B = q^{-1}(A) ⊂ Y$. Note $B$ is not closed in $Y$ since $q$ is a quotient map. Thus $B = (qr)^{-1}(A)$ is not closed in $X$.

Lemma 4.3. Suppose the $p$ based loops $f_n$ and $g_n$ are path homotopic in the Hawaiian earring $HE$ with $p$ the special point. Suppose $f_n → f$ uniformly and $g_n → g$ uniformly. Then $f$ and $g$ are path homotopic.
Proof. Concatenating with the reverse path, the inessential loops \( f_n g_n^{-1} \to f g^{-1} \) uniformly. Since the bouquet of \( n \) loops \( HE_n \) is locally contractible, \( fg^{-1} \) is inessential in \( HE_n \) for each \( n \) and hence, since \( HE \) is \( \pi_1 \) shape injective \( fg^{-1} \) is inessential. See also a direct proof in [15].

Lemma 4.4. Suppose \( X_n \) is the discrete free monoid on letters \( \{x_1, x_1^{-1}, x_2, \ldots, x_n^{-1}\} \) with empty identity. Suppose \( R_n \mid X_{n+1} \to X_n \) is the forgetful retraction, deleting all occurrences of \( x_n \) with the subspace of \( X_n \) from index \( n \) onward. Let \( R_n : X_\infty \to X_n \) denote the canonical retraction. Let \( q_n \colon X_n \to G_n \) denote the canonical quotient onto the free group \( G_n \) on \( n \) letters. Let \( \sigma_n : G_n \to X_n \) denote the embedding mapping \( g_n \in G_n \) to its maximally reduced representative. Let \( q \colon X_\infty \to G \) denote the quotient map under the equivalence relation \( u \sim v \) iff \( q_n R_n(u) = q_n R_n(v) \) for all \( n \).

By definition \( q_n R_n \) descends to the quotient inducing a map \( \Pi_n : G \to G_n \).

Claim 1. There is a well defined (discontinuous) injection \( \sigma : G \to X_\infty \) with \( \sigma(g) = \lim_{n \to \infty} \sigma_n \Pi_n(g) \) and \( \sigma(g) \in g \). Claim 2. The sequence \( \{g_n\} \) converges in the space \( G \) iff for all \( N \) the sequence \( \Pi_N(g_n) \) is eventually constant, and also if the sequence \( \{\sigma(g_n)\} \) has compact closure.

Proof. We have a canonical partial order on \( X_\infty \) defined as follows. Given \( w = (w_1, w_2, \ldots) \in X_\infty \) let \( T(w_1, w_2, \ldots) = (N_1(w), N_2(w), \ldots) \) with \( N_k(w) \geq 0 \) the combined number of occurrences of \( \{x_k, x_k^{-1}\} \) in \( w_k \). The function \( T \) determines a partial lexicographic order on \( X_\infty \) with \( T(v) < T(w) \) if \( N_k(v) < N_k(w) \) with \( k \) minimal so that \( N_k(v) \neq N_k(w) \).

Given \( x \in X_\infty \) let \( c(x, N) \) denote the total number of occurrences of \( \{x_N, x_N^{-1}\} \) in the word \( R_N(x) \). Define \( \phi : G \to G_1 \times G_2 \times \ldots \) via \( \phi(g) = \Pi_1(g), \Pi_2(g), \ldots \) and note \( \phi \) is one to one. Thus \( G \) is \( T_2 \) since the codomain is \( T_2 \). Consequently convergent sequences in \( G \) have unique limits.

Proof of Claim 1. Note for each \( g_n \in G_n \) the corresponding subset \( g_n \subset X \) has a unique minimal element \( \sigma_n(g_n) \in X_n \). To obtain a contradiction suppose \( g \in G \) and \( N \notin \{1, 2, 3, \ldots\} \) is minimal so that \( \{R_N(\sigma_n \Pi_n(g))\} \) is not eventually constant. Obtain \( x \in g \). Obtain \( M \) so that if \( M \leq n \) and \( 1 \leq k \leq N - 1 \) then \( \{R_N(\sigma_n \Pi_n(g))\} \) is constant. Thus for each \( n \geq M \) we have \( R_N(\sigma_n \Pi_n(g)) = R_N(\sigma_n \Pi_n(1)) \) or \( R_N(\sigma_n \Pi_n(g)) = R_N(\sigma_n \Pi_n(1)) = R_N(\sigma_n \Pi_n(1)) \). Thus, if \( R_N(\sigma_n \Pi_n(g)) \) is not eventually constant we have \( c(R_N(\sigma_n \Pi_n(g)), N) \to \infty \), contradicting the fact that \( c(x, N) \geq c(R_N(\sigma_n \Pi_n(g)), N) \) for all \( n \).

By definition if \( g \in G \) and \( x \in g \) then \( R_N(x) \) and \( R_N(\sigma(g)) \) are \( q_n \) equivalent and hence \( g \in \sigma(g) \). To see that \( \sigma \) is one to one, note if \( \{g, h\} \subset G \) with \( g \neq h \) get \( N \) minimal so that \( \Pi_N(g) \neq \Pi_N(h) \), and note \( \sigma_n(\Pi_N(g)) \neq \sigma_n(\Pi_N(h)) \). Note, in \( X_N \), \( R_N(\sigma(g)) \) reduces to \( \sigma_n(\Pi_N(g)) \) and \( R_N(\sigma(h)) \) reduces to \( \sigma_n(\Pi_N(h)) \). Thus \( R_N(\sigma(g)) \neq R_N(\sigma(h)) \) and hence \( \sigma(h) \neq \sigma(g) \). Note \( \sigma(x_1 x_0 x_1^{-1}) \rightarrow \emptyset \in \text{Gand}(\emptyset) = \emptyset \). However \( \sigma(x_1 x_0 x_1^{-1}) \to (x_1 x_1^{-1}, x_1 x_1^{-1}, \ldots) \neq \emptyset \). Thus \( \sigma \) is not continuous. This proves claim 1.

To prove claim 2 suppose \( \{g_n\} \) is a convergent sequence in \( G \). Since \( \Pi_n \) is continuous and \( G_N \) is discrete, the sequence \( \Pi_N(g_n) \) is eventually constant. Since \( X_\infty \) is metrizable to prove \( \{\sigma(g_n)\} \) has compact closure
it suffices to prove \( \{\sigma(g_n)\} \) is sequentially compact. Thus it suffices to prove, for each \( N \), allowing \( n \) to vary, the set \( \{R_N(\sigma g_n)\} \) is finite. Fix \( N \). To seek a contradiction, if \( \{R_N(\sigma g_n)\} \) were infinite then some subsequence is comprised of infinitely many distinct terms in the discrete space \( X_N \). Wolog we assume \( R_N(\sigma g_n) \) itself is comprised of infinitely many distinct terms. Note \( c(R_N(\sigma g_n), N) \to \infty \). On the other hand since \( q \) is a quotient map, some subsequence of \( \{g_n\} \) lifts to a convergent sequence in \( X_\infty \). Thus \( c(\ast, N) \) of the subsequential lifts is bounded. This contradicts the general fact that if \( q(x) = g \) then \( c(x, N) \geq c(\sigma(y), N) \), the latter inequality argued as follows. If we let \( c(x, N, M) \) denote the total number of occurrences of \( \{x_N, x_N^{-1}\} \) in \( R_M(x) \), then by definition of \( X_\infty \) we have \( c(x, N, M) = 0 \) if \( M < N \) and \( c(x, N, M) = c(x, N) \) if \( N \leq M \). By definition of \( \sigma \) obtain \( M \geq N \) so that \( R_N(\sigma_M(q_M(R_Mx))) = R_N(\sigma q(x)) \). Thus, \( c(x, N) = c(x, N, M) \geq c(\sigma_M(q_M(R_Mx)), N, M) = c(\sigma q(x), N, M) = c(\sigma q(x), N) \).

For the converse of claim 2, suppose \( \{g_n\} \) is a sequence in \( G \) such that \( \{\Pi_N(g_n)\} \) converges for each \( N \) and such that \( \{\sigma(g_n)\} \) has compact closure. Since \( X_\infty \) is metrizable, \( \{\sigma(g_n)\} \) is sequentially compact. Let \( x \in X_\infty \) and \( y \in X_\infty \) be subsequential limits of \( \{\sigma(g_n)\} \). With subsequences \( v_n \to x \) and \( w_n \to y \). It suffices to prove \( q(x) = q(y) \), i.e. to prove \( \Pi_N(q(x)) = \Pi_N(q(y)) \) for all \( N \). Exploiting our hypothesis, continuity of \( R_N \), and the fact that \( X_N \) is a discrete space, obtain \( M \) so that \( \Pi_N(q(x)) = q_N(R_N x) = q_N(R_N(v_M)) = \Pi_N(g_M) = q_N(R_N(w_M)) = q_N(R_N(y)) = \Pi_N(q(y)) \). This proves claim 2. \( \square \)

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