Eigenvalue Bounds for Dirac and Fractional Schrödinger Operators with Complex Potentials

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Mathematical Challenges in Quantum Mechanics
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Motivation

- Lieb-Thirring Inequalities (S.A.)
- Lieb-Thirring Inequalities (N.S.A.)

New Results

- Dirac and Fractional Schrödinger Operators
- Method of Proof
Outline

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**Lieb-Thirring Inequalities (S.A.)**

- $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$, with $D(H_0) = H^2(\mathbb{R}^d)$. Then $H_0^* = H_0$ and $\sigma(H_0) = \sigma_{ess}(H_0) = [0, \infty)$.
- $H = H_0 + V$, where $V \in L^{d/2+\gamma}(\mathbb{R}^d; \mathbb{R})$.

**Lieb-Thirring and CLR inequalities**

$$
\sum_{E \in \sigma_d(H)} |E|^\gamma \leq L_{d,\gamma} \int_{\mathbb{R}^d} V_-(x)^{d/2+\gamma} \, dx,
$$

where $V_-(x) = \min\{V(x), 0\}$.

- $\gamma \geq 0$ if $d \geq 3$,
- $\gamma > 0$ if $d = 2$,
- $\gamma \geq 1/2$ if $d = 1$. 

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Preliminaries: Non-Selfadjoint Operators

$X$ a Banach space, $T$ a closed operator in $X$.

**Definition**

- $\rho(T) := \{ z \in \mathbb{C} : T - z \text{ is bijective} \}$, $\sigma(T) := \mathbb{C} \setminus \rho(T)$.
- $\sigma_d(T) := \{ z \in \mathbb{C} : z \text{ isolated e.v. of finite algebraic mult.} \}$.
- $\sigma_{\text{ess}}(T) := \{ z \in \mathbb{C} : T - z \text{ is not Fredholm} \}$.

**Fact 1:** $(T - z)^{-1} - (S - z)^{-1}$ compact $\implies$ $\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(S)$.

**Fact 2:** If each connected component of $\mathbb{C} \setminus \sigma_{\text{ess}}(T)$ contains a point in $\rho(T)$, then

$$\sigma(T) = \sigma_d(T) \cup \sigma_{\text{ess}}(T).$$
N.S.A. Schrödinger Operators: Single Eigenvalues

- $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$, with $D(H_0) = H^2(\mathbb{R}^d)$.
- $H = H_0 + V$, where $V \in L^{d/2+\gamma}(\mathbb{R}^d; \mathbb{C})$.
- Assume $z \in \mathbb{C} \setminus [0, \infty)$ is an eigenvalue of $H$. 

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Lieb-Thirring Inequalities (N.S.A.)

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Theorem (Abramov, Aslanyan, Davies 2001 [1])

Assume $d = 1$. Then

$$|z|^{1/2} \leq \frac{1}{2} \int_{\mathbb{R}} |V(x)| dx.$$

Proof.

$$1 \leq \|V^{1/2}R_0(z)\|V^{1/2}\|^2 \leq \int \int \frac{|V(x)|e^{-2\Im \sqrt{z}|x-y|}|V(y)|}{4|z|} \leq \frac{\|V\|_1^2}{4|z|}.$$
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Conjecture (Laptev, Safronov 2009 [10])

Assume $d \geq 1$ and $0 < \gamma \leq d/2$. Then

$$|z|^{\gamma} \leq C_{d,p} \int_{\mathbb{R}} |V(x)|^{d/2+\gamma} dx.$$

- Case $0 < \gamma \leq 1/2$ proved by R. Frank [6].
  Proof relies on uniform Sobolev inequality of C. Kenig, A. Ruiz, C. Sogge [9]:

$$\|R_0(z)\|_{L^p \rightarrow L^{p'}} \leq C|z|^{d(1/p-1/2)-1}, \quad \frac{2d}{d+2} \leq p \leq \frac{2(d+1)}{d+3}.$$
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- Assume $z \in \mathbb{C} \setminus [0, \infty)$ is an eigenvalue of $H$.

Conjecture (Laptev, Safronov 2009 [10])

Assume $d \geq 1$ and $d/2 \leq p \leq d$. Then

$$|z|^{p-d/2} \leq C_{d,p} \int_{\mathbb{R}} |V(x)|^p \, dx.$$ 

Case $0 < \gamma \leq 1/2$ proved by R. Frank [6]. Proof relies on uniform Sobolev inequality of C. Kenig, A. Ruiz, C. Sogge [9]:

$$\|R_0(z)\|_{L^p \to L^{p'}} \leq C |z|^{d(1/p-1/2)-1}, \quad \frac{2d}{d+2} \leq p \leq \frac{2(d+1)}{d+3}.$$ 

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Complex Eigenvalue Bounds
**N.S.A. Schrödinger Operators: Sums of Eigenvalues**

- \( H_0 = -\Delta \) in \( L^2(\mathbb{R}^d) \), with \( D(H_0) = H^2(\mathbb{R}^d) \).
- \( H = H_0 + V \), where \( V \in L^p(\mathbb{R}^d; \mathbb{C}) \).

**Theorem (Frank, Laptev, Lieb, Seiringer 2006 [7])**

For \( \theta \in (0, \pi/2) \) and \( p \geq d/2 + 1 \):

\[
\sum_{z \in \sigma_d(H) \setminus \Omega_\theta} |z|^{p-d/2} \leq C_{d,p}(1 + 2/\tan(\theta))^{p} \| V \|_{p}^{p}.
\]

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Complex Eigenvalue Bounds
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- $H = H_0 + V$, where $V \in L^p(\mathbb{R}^d; \mathbb{C})$.

**Theorem (Demuth, Hansmann, Katriel 2009 [2])**

For $p \geq d/2 + 1$, $d \geq 0$, $\epsilon > 0$ and $\Re(V) \geq 0$:

$$\sum_{z \in \sigma_d(H)} \frac{\text{dist}(z, [0, \infty))^{p+\epsilon}}{|z|^{d/2}(1 + |z|)^{2\epsilon}} \leq C_{d,p} \|V\|_p^p.$$
N.S.A. Schrödinger Operators: Sums of Eigenvalues

- $H_0 = -\Delta$ in $L^2(\mathbb{R}^d)$, with $D(H_0) = H^2(\mathbb{R}^d)$.
- $H = H_0 + V$, where $V \in L^p(\mathbb{R}^d; \mathbb{C})$.

**Conjecture (Demuth, Hansmann, Katriel [3])**

For $p > d/2$:

$$
\sum_{z \in \sigma_d(H)} \frac{\text{dist}(z, [0, \infty))^p}{|z|^{d/2}} \leq C_{d,p} \|V\|_p^p.
$$

- In part. $\sigma_d(H) \ni z_n \to z^* \in (0, \infty) \implies (\Im z_n)_{n \in \mathbb{N}} \subset l^p(\mathbb{N})$.
- What is the lowest possible $p$?

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### Fractional Schrödinger Operator

- \[ H_0 = (m^2 - \Delta)^{s/2} \text{ in } L^2(\mathbb{R}^d), \text{ with } 0 < s < d \] and 
  \[ D(H_0) = H^s(\mathbb{R}^d). \] Then \( \sigma(H_0) = \sigma_{\text{ess}}(H_0) = [m, \infty) \)

- \( H = H_0 + V, \) where \( V \in L^p(\mathbb{R}^d; \mathbb{C}) \) or \( V \in (L^p \cap L^q)(\mathbb{R}^d; \mathbb{C}). \)

### Dirac Operator

- \( H_0 = \alpha \cdot D + m\beta \text{ in } L^2(\mathbb{R}^d, \mathbb{C}^N), \) w. \( D(H_0) = H^1(\mathbb{R}^d, \mathbb{C}^N). \) Then \( \sigma(H_0) = \sigma_{\text{ess}}(H_0) = (-\infty, m] \cup [m, \infty). \)

- \( H = H_0 + V, \) where \( V \in (L^p \cap L^q)(\mathbb{R}^d; \mathbb{C}^{N \times N}). \)

- We define \( s = 1 \) in this case.

- Here: \( m \geq 0. \)

- The range of admissible \( p, q \) will depend on \( d \) and \( s. \)
**Motivation**

**New Results**

**Dirac and Fractional Schrödinger Operators**

**Method of Proof**

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**Main result**

**Assumptions:**

\[
\Lambda_{\text{crit}}(H_0) = \begin{cases} 
\{m\} & \text{if } H_0 = (m^2 - \Delta)^{s/2}, \\
\{-m, m\} & \text{if } H_0 = \alpha \cdot D + m\beta.
\end{cases}
\]

**Assume**

\[
\begin{align*}
V &\in L^{p \in [d/s, (d+1)/2]} & \text{if } s \geq 2d/(d+1), \\
V &\in L^{d/s} \cap L^{(d+1)/2} & \text{if } s < 2d/(d+1).
\end{align*}
\]

**Theorem (Bounds for single eigenvalues)**

Assume \(H_0 = (m^2 - \Delta)^{s/2}\) with \(s \geq 2d/(d+1), d \geq 2\). Then all complex eigenvalues of \(H\) lie in a compact neighborhood of \(\Lambda_{\text{crit}}(H_0)\). In particular, for \(m = 0 \Rightarrow |z|^{p - \frac{d}{s}} \leq \|V\|_p^p\).

In \(d = 1\) the Theorem is true for Dirac, but not for \((m^2 - \Delta)^{1/2}\).
Main result

Assumptions:

- \( \Lambda_{\text{crit}}(H_0) = \begin{cases} \{m\} & \text{if } H_0 = (m^2 - \Delta)^{s/2}, \\ \{-m, m\} & \text{if } H_0 = \alpha \cdot D + m\beta. \end{cases} \)

- Assume \( V \in L^{p \in \left[\frac{d}{s}, \frac{(d+1)/2}{d}\right]} \text{ if } s \geq 2d/(d+1), \)
  \( V \in L^{d/s} \cap L^{(d+1)/2} \text{ if } s < 2d/(d+1). \)

Theorem (Bound on distribution of eigenvalues)

Let \((z_n)_n \subset \sigma_d(H)\) such that \(z_n \to z^\ast \in \sigma(H_0) \setminus \Lambda_{\text{crit}}(H_0).\) Then \((\text{dist}(z_n, \sigma(H_0)))_{n \in \mathbb{N}} \in l^1(\mathbb{N}).\)

- The case \(s = 2\) is due to Frank and Sabin [8].
- Dubuisson [4]–[5] proved \((\text{dist}(z_n, \sigma(H_0)))_{n \in \mathbb{N}} \in l^p(\mathbb{N})\) for larger \(p.\)
Main result

Assumptions:

\[ \Lambda_{\text{crit}}(H_0) = \begin{cases} 
\{m\} & \text{if } H_0 = (m^2 - \Delta)^{s/2}, \\
\{-m, m\} & \text{if } H_0 = \alpha \cdot D + m\beta.
\end{cases} \]

Assume

\[ V \in L^{p \in [d/s, (d+1)/2]} \] if \( s \geq 2d/(d + 1), \]

\[ V \in L^{d/s} \cap L^{(d+1)/2} \] if \( s < 2d/(d + 1). \]

Corollary

\[ \#\{z \in \sigma_d : |\Im z| \geq s\} \leq \frac{C}{s}. \]

If \( d \geq 2d/(d + 1) \) and \( \| V \|_{L^{d/s}} \ll 1 \), then \( \# \) complex spectrum; in fact, \( H \) is similar to \( H_0 \) (\( |V|^{1/2} \) is Kato-smooth w.r.t \( H_0 \)).
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Perturbation Determinant

- Assume \( V^{1/2} R_0(z) | V |^{1/2} \in \mathcal{G}^\alpha(L^2(\mathbb{R}^d)) \).
- Regularized determinant

\[
f : \rho(H_0) \to \mathbb{C}, \quad f(z) := \det(I + V^{1/2} R_0(z) | V |^{1/2}),
\]

where for \( A \in \mathcal{G}^n(L^2(\mathbb{R}^d)) \):

\[
\det(I + A) := \prod_k [1 + \lambda_k(A)] \exp \left( \sum_{j=1}^{n-1} (-1)^{j-1} \lambda_k(A)^j \right).
\]

- \( f \) is holomorphic, and

\[
f(z) = 0 \iff z \in \sigma_d(H).
\]

- \( \ln |f(z)| \leq C_\alpha \| V^{1/2} R_0(z) | V |^{1/2} \| \mathcal{G}^\alpha. \)
Jensen’s Identity

- $h : \mathbb{D} \to \mathbb{C}$ holomorphic, $h(0) = 1$.
- $N(h; s)$ number of zeros of $h$ in $B(0, s)$.

**Jensen’s identity: $\forall r \in (0, 1)$**

$$\int_0^r \frac{N(h; s)}{s} \, ds = \sum_{\{w \in B(0, r) : h(w) = 0\}} \ln \left| \frac{r}{w} \right| = \frac{1}{2\pi} \int_{0}^{2\pi} \ln \left| h(re^{i\theta}) \right| \, d\theta$$

- In particular, if $\sup_{|w| = 1} |\ln h(w)| \leq M$, then

$$\sum_{\{z \in B(0, r) : h(z) = 0\}} (1 - |z|) \leq \sum_{\{z \in B(0, r) : h(z) = 0\}} \ln \left| \frac{1}{z} \right| \leq M.$$
Conformal map

- \( \psi : \mathbb{D} \rightarrow \rho(H_0) \) conformal map s.t. \( \psi(0) \in \rho(H_0) \).
- \( h : \mathbb{D} \rightarrow \mathbb{D}, \)

\[
h(w) := \frac{\det[\alpha](I + V^{1/2}R_0(\psi(w))|V|^{1/2})}{\det[\alpha](I + V^{1/2}R_0(\psi(0))|V|^{1/2})}.
\]

- \( \psi^{-1} \) extends diffeomorphically to \( \mathbb{C} \setminus \Lambda_{\text{crit}}(H_0) \).
- Koebe distortion theorem: \( z = \psi(w) \)

\[
\implies (1 - |w|) \approx \left| \frac{dw}{dz} \right| \text{dist}(z, \sigma(H_0)).
\]

- \( \exists \mu_j \geq 0:\)

\[
|\ln h(w)| \leq C(V) \prod_{z_j \in \Lambda_{\text{crit}}(H_0) \cup \{\infty\}} |w - \psi^{-1}(z_j)|^{-\mu_j}.
\]
Motivation

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Uniform resolvent bounds in Schatten spaces

Theorem

Let $H_0 \in \{(m^2 - \Delta)^{s/2}, \alpha \cdot D + m\beta\}$. There exists $N : \rho(H_0) \to \mathbb{R}_+$ with continuous extension to $\mathbb{C} \setminus \Lambda_{\text{crit}}(H_0)$ s.t.

a) If $s \geq 2d/(d + 1)$ and $V \in L^{p \in [d/s, (d + 1)/2]}$, then

$$\| V^{1/2} R_0(z) |V|^{1/2} \|_{\mathcal{S}^p(d-1)/(d-p)} \leq N(z) \| V \|_{L^p}$$

b) If $s < 2d/(d + 1)$ and $V \in L^{d/s} \cap L^{(d+1)/2}$, then $\forall \epsilon > 0$

$$\| V^{1/2} R_0(z) |V|^{1/2} \|_{\mathcal{S}^{\max\{d+1,d/s+\epsilon\}}} \leq N(z) \| V \|_{L^d \cap L^{(d+1)/2}}$$

- The case $s = 2$ is due to Frank and Sabin [8].
- Proof uses Stein’s interpolation theorem for analytic families of operators.
- Theorem is valid for more general operators.

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Complex Eigenvalue Bounds
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