Renormalization of the Twist-3 Flavor Singlet Operators in a Covariant Gauge

J. Kodaira\textsuperscript{a}, T. Nasuno\textsuperscript{a}, H. Tochimura\textsuperscript{a}, K. Tanaka\textsuperscript{b}, and Y. Yasui\textsuperscript{c}

\textsuperscript{a}Department of Physics, Hiroshima University, Higashi-Hiroshima 739, Japan
\textsuperscript{b}Department of Physics, Juntendo University, Inba-gun, Chiba 270-16, Japan
\textsuperscript{c}RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, NY, 11973, USA

Abstract

We investigate the nucleon’s transverse spin-dependent structure function $g_2(x, Q^2)$ in the framework of the operator product expansion and the renormalization group. We construct the complete set of the twist-3 operators for the flavor singlet channel, and give the relations among them. We develop an efficient, covariant approach to derive the anomalous dimension matrix of the twist-3 singlet operators by computing the off-shell Green’s functions. As an application, we investigate the $Q^2$-evolution of $g_2(x, Q^2)$ for the lowest moment case, and discuss its experimental implication.

1 Introduction

The nucleon spin structure observed in deep inelastic scattering (DIS) is described by the two structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$. $g_1$ and $g_2$ are present for the longitudinally and transversely polarized targets, respectively. In the framework of the operator product expansion, the leading contribution for $g_1$ comes only from the twist-2 operators, while for $g_2$ the twist-3 operators also contribute in the leading order of $1/Q^2$ in addition to the twist-2 operators\textsuperscript{[1, 2, 3, 4]}. $g_2(x, Q^2)$ is unique, and plays a distinguished role in spin physics\textsuperscript{[1, 2, 4]}. Firstly, it is a “measurable” higher-twist structure function: In general, it is difficult to extract the higher-twist structure functions by experiments because they usually constitute small corrections to the leading twist-2 term. However, $g_2(x, Q^2)$ is somewhat immune to this difficulty, and contributes as a leading term to the asymmetry in the DIS using the transversely polarized target. Secondly, $g_2(x, Q^2)$ is related to an interesting sum rule, referred to as the Burkhardt-Cottingham sum rule: $\int_0^1 dx g_2(x, Q^2) = 0$. Thirdly, $g_2(x, Q^2)$ contains information inaccessible by the more familiar spin structure function $g_1(x, Q^2)$: $g_2(x, Q^2)$ is related to the nucleon’s transverse polarization and to the twist-3 operators describing the quark-gluon correlation in the nucleon.

Recently, the first data of $g_2(x, Q^2)$ have been reported\textsuperscript{[5]}. Extensive study is expected to be performed at DESY, CERN, and SLAC. These future measurements of $g_2(x, Q^2)$ will give the first opportunity to obtain new information about the dynamics of QCD and the nucleon structure beyond those from the twist-2 structure functions. Theoretically, the determination of the $Q^2$-dependence of $g_2(x, Q^2)$ is very important to extract these physical information from
experimental data. Furthermore, the comparison of the $Q^2$-evolution itself between theory and experiment will provide a deeper test of QCD beyond the conventional twist-2 level.

The $Q^2$-evolution of $g_2(x, Q^2)$ is governed by the anomalous dimensions, which are determined by the renormalization of the relevant twist-3 operators as well as the twist-2 ones. A characteristic feature of the higher-twist operators is the occurrence of the complicated operator mixing under renormalization: Many gauge-invariant operators, the number of which increases with spin (moment of the structure function), mix with each other. Furthermore, the operators which are proportional to the equation of motion ("EOM operators"), as well as the ones which are not gauge invariant ("alien operators"), also participate through the renormalization mixing.

There have been a lot of works on the $Q^2$-evolution of $g_2(x, Q^2)$. Most of them discussed the flavor nonsinglet case [7, 10, 11]. Only a few works treated the singlet case [8]: Bukhvostov, Kuraev and Lipatov derived evolution equations for the twist-3 quasi-partonic operators. Recently, Müller computed evolution kernel based on the nonlocal light-ray operator technique, and obtained the identical results. However, both of these two works employ a similar framework based on the renormalization of the nonlocal operators in the (light-like) axial gauge. Balitsky and Braun also treated the nonlocal operators although they employed the background field method. On the other hand, covariant approach based on the local composite operators is missing. Furthermore, some subtle infrared problem occurring in the renormalization of the generic flavor singlet operators has been emphasized in [9]. Therefore, the computation of the anomalous dimensions for the flavor singlet part in a covariant and fully consistent scheme is desirable and should provide useful framework.

In this work, we develop a covariant framework to investigate the flavor singlet part of $g_2(x, Q^2)$ based on the operator product expansion (OPE) and the renormalization group (RG), by extending our recent work on the flavor nonsinglet part [10, 11]. Sect.2 is devoted to a detailed OPE analysis for the flavor singlet part of $g_2(x, Q^2)$. We list up all relevant twist-3 flavor singlet operators appearing in QCD. We derive the relations satisfied by these operators. In particular, we obtain a new operator identity, which relates the gluon bilinear operator with the trilinear ones. Based on these developments, we give a basis of the independent operators for the renormalization. In sect.3, we discuss a general framework to perform the renormalization of the twist-3 flavor singlet operators in a covariant gauge. We compute the off-shell Green’s functions inserting the relevant local composite operators. Infrared cut-off is provided by the external off-shell momenta. In this case, the EOM operators as well as the alien operators should be included as independent operators. We introduce convenient projection technique [12, 11] to avoid the complexity stemmed from the renormalization mixing of many gauge-noninvariant operators. Sect.4 contains an application of our framework to investigate the $Q^2$-evolution of the lowest ($n = 3$) moment. We discuss experimental as well as theoretical implication of our results. The final sect.5 concludes with a brief summary.

## 2 OPE analysis of twist-3 flavor singlet operators for $g_2(x, Q^2)$

Let us start from a quick look at the factorization theorem in QCD [1, 13]. As is well known, the cross section for the DIS is given by the “cut diagram” corresponding to the discontinuity of the forward virtual Compton amplitude between the virtual photon with the momentum $q_\mu$.*

---

*Recently, it has been proved that the twist-3 structure functions obey simple DGLAP evolution equation for $N_c \to \infty$, and that this simplification is universal for all twist-3 nonsinglet structure functions [16].
\[ (q^2 = -Q^2) \text{ and the nucleon with the momentum } P_\mu (P^2 = M^2 \text{ with } M \text{ the nucleon mass}). \]

In general kinematics, this virtual Compton amplitude contains all the complicated interactions between the virtual photon and the nucleon, possibly including the “soft interactions” where the soft momenta are exchanged. However, drastic simplification occurs if one goes to the Bjorken limit \( Q^2 \to \infty \) with \( x = Q^2 / 2P \cdot q \) fixed: The amplitude is dominated by the contribution which is factorized into the hard (short distance) and the soft (long distance) parts, and the other complicated contributions are suppressed by the powers of \( 1/Q \). The factorized amplitude corresponds to the process where a parton carrying the momentum \( k = \xi P \) (\( 0 \leq \xi \leq 1 \)) comes out from the soft part, followed by the hard hitting by the virtual photon, and then goes back to the soft part. For the case of the polarized target, the structure functions \( g_1(x, Q^2) \) and \( g_2(x, Q^2) \) appear in the cross section, and they are given by, corresponding to the factorized amplitude,

\[
\begin{align*}
g_1(x, Q^2) &= \sum_i \int_x^1 \frac{d\xi}{\xi} \phi^i_1(\xi, \mu^2) H_{1i}(\frac{x}{\xi}, \frac{Q}{\mu}, \alpha_s(\mu)), \\
g_2(x, Q^2) &= \sum_i \int_x^1 \frac{d\xi}{\xi} \phi^i_2(\xi, \mu^2) H_{2i}(\frac{x}{\xi}, \frac{Q}{\mu}, \alpha_s(\mu)).
\end{align*}
\]

where \( \phi^i_1, \phi^i_2 \) are the parton distribution functions corresponding to the soft part. As is well known, the twist-2 distribution \( \phi^i_1(\xi, \mu^2) \) is interpreted as the probability density to find a parton of type \( i (= \text{gluon, } u, d, \bar{u}, \bar{d}, \cdots) \) in the nucleon, carrying a fraction \( \xi \) of the nucleon’s momentum. The summation of (1), (2) is over all the possible types of parton, \( i \). \( H_{1i}, H_{2i} \) denote the hard parts; they correspond to the hard scattering coefficients for the scattering between the virtual photon and a parton \( i \). They are calculable systematically by perturbation theory, and the dependence on the strong coupling constant \( \alpha_s = g^2 / 4\pi \) is explicitly shown. In (1), (2), the soft and the hard parts are divided at the renormalization scale \( \mu \).

The soft part (parton distribution function) is a process-independent quantity which appears universally in various hard processes. Following [14] this is written as Fourier transform of nucleon matrix elements of nonlocal light-cone operators, reflecting the light-cone dominance in hard processes. For our case, the relevant quantity is

\[
\Phi^i_\mu(\xi, \mu^2) = P^+ \int \frac{dz^-}{2\pi} e^{i\xi P^z} \langle PS|\bar{\psi}_i(0)\gamma_\mu\gamma_5[0, z]\psi_i(z)|PS\rangle.
\]

Here \( z \) is a light-like vector \( z^2 = 0, |PS\rangle \) is the nucleon state with momentum \( P \) and spin \( S \) \( (P \cdot S = 0, S^2 = -M^2) \), and \( \psi_i \) is the quark field. Our Lorentz frame is chosen as \( P \cdot z = P^+z^- \). The bilocal operator is renormalized at the scale \( \mu \). \( [0, z] = P \exp \left[ ig \int_0^1 d\alpha z^\mu A_\mu(\alpha z) \right] \) is the path-ordered gauge factor, and therefore (3) is gauge-invariant. We considered the case where the quark of flavor \( i \) comes out of the nucleon, followed by the coupling with the vertex \( \gamma_\mu\gamma_5 \), and then goes back to the nucleon. The quark has \( \xi P^+ \) as the plus component of the momentum.

In principle, one can insert various gamma matrices in between the quark fields. In fact, one can generate all quark distribution functions up to twist-4 by inserting all the possible gamma matrices [14]. For example, if \( \gamma_\mu \) were inserted in place of \( \gamma_\mu\gamma_5 \), we would obtain the spin-independent distribution functions of twist-2 and twist-4, corresponding to the structure functions \( F_1(x, Q^2), F_2(x, Q^2) \). In our case of (3), additional \( \gamma_5 \) distinguishes the helicity of the quark. Therefore, (3) gives the spin-dependent distribution functions \( \phi^i_1, \phi^i_2 \) relevant to the structure functions \( g_1(x, Q^2) \) and \( g_2(x, Q^2) \). Explicitly, we have [14]

\[
\Phi^i_\mu(\xi, \mu^2) = 2 \left[ \phi^i_1(\xi, \mu^2) P_\mu \frac{(S \cdot z)}{(P \cdot z)} + \left( \phi^i_1(\xi, \mu^2) + \phi^i_2(\xi, \mu^2) \right) S_{\perp \mu} \right],
\]

(4)
up to the twist-4 corrections. Here $S_{\perp\mu} = S_\mu - P_\mu (S \cdot z)/(P \cdot z) + M^2 z_\mu (S \cdot z)/(P \cdot z)^2$ is the projection of the spin vector into transverse direction. By comparing both sides of (4), one can obtain explicit operator definitions of the relevant distribution functions.

For further development, it is convenient to employ the moment given by $\int d^4\xi \xi^{n-1} \Phi_{\mu, \nu}^f (\xi, \mu^2)$, which is related to the moments of the structure functions $M_n [g_1 (Q^2) ] = \int_0^1 dx x^{n-1} g_1 (x, Q^2)$, $M_n [g_2 (Q^2) ] = \int_0^1 dx x^{n-1} g_2 (x, Q^2)$ (see (1), (4)): As seen from (3), going over to the moment space corresponds to expanding the nonlocal operator at small quark-antiquark separation. This gives the OPE. We obtain (we consider the case of odd $n$ which is relevant for the DIS)

$$\int_1^{-1} d\xi^{n-1} \Phi_{\mu}^f (\xi, \mu^2) = \int_0^1 d\xi^{n-1} \left( \Phi_{\mu}^l (\xi, \mu^2) + \Phi_{\mu}^r (\xi, \mu^2) \right)$$

$$= \Delta^{\mu_1} \cdots \Delta^{\mu_{n-1}} (PS) \bar{\psi}_1 (0) \gamma_\mu \gamma_5 i D_{\mu_1} \cdots i D_{\mu_{n-1}} \psi_1 (0) |PS\rangle,$$

where $\Delta^{\mu} = z^{\mu} / (P \cdot z)$, and $D_\mu = \partial_\mu - ig A_\mu$ is the covariant derivative. The local composite operators in the r.h.s. are symmetric and traceless for the indices $\mu_1, \cdots, \mu_{n-1}$, but have one free index $\mu$. If one recalls that the twist of the local composite operator is defined as “dimension minus spin”, one immediately concludes the following: The symmetrization of the index $\mu$ with the other indices and the subtraction of the trace terms corresponding to $g_{\mu \nu k}$ gives the twist-2 operators; these twist-2 operators correspond to $g_1 (x, Q^2)$. On the other hand, the antisymmetrization of $\mu$ with the other indices introduces one unit of twist, giving the twist-3 operators; the matrix element of these operators, having antisymmetric pair of indices, should involve $S_{\perp \mu}$, and therefore contributes to $g_2 (x, Q^2)$ (see (3)).

From (4) and (3), one sees that the twist-2 operators also contribute to $g_2 (x, Q^2)$. As is well known, these contributions can be conveniently extracted by (4)

$$g_2 (x, Q^2) = -g_1 (x, Q^2) + \int_x^1 dy g_1 (y, Q^2) \cdot y + \bar{g}_2 (x, Q^2),$$

where we denote the twist-3 part by $\bar{g}_2 (x, Q^2)$. Now $M_n [\bar{g}_2 (Q^2) ]$ is given by matrix element of

$$R_{n,E}^{\mu_1 \cdots \mu_{n-1}} = i^{n+2} \left( \frac{n-1}{n} \right) S A \bar{\psi}\gamma_5 D^{\mu_1} \cdots D^{\mu_{n-1}} \psi \right] - \text{traces},$$

where $S$ symmetrizes $\mu_1, \cdots, \mu_{n-1}$, and $A$ antisymmetrizes $\sigma$ and $\mu_1$ as $A f^{\sigma \mu_1} = \frac{1}{2} (f^{\sigma \mu_1} - f^{\mu_1 \sigma})$. “- traces” stands for the subtraction of the trace terms to make the operators traceless. $n = 3, 5, 7, \cdots$, because only the charge-conjugation even operators contribute to the DIS. Here and below we suppress the flavor index of the quark field: For the flavor nonsinglet part, the flavor matrices $\lambda_i$ should be inserted between the quark fields. On the other hand, for the present case of the singlet part, the flavor indices have to be summed over.

In addition to (7), one can construct other types of gauge-invariant operators of twist-3 [3, 7]:

$$R_{n,l}^{\mu_1 \cdots \mu_{n-1}} = \frac{1}{2n} (V_l - V_{n-1-l} + U_l + U_{n-1-l}) \quad (l = 1, \cdots, n-2),$$

$$R_{n,m}^{\mu_1 \cdots \mu_{n-1}} = i^n S m \bar{\psi}\gamma_5 D^{\mu_1} \cdots D^{\mu_{n-2}} \gamma^{\mu_{n-1}} \psi \right] - \text{traces},$$

$$R_{n,E}^{\mu_1 \cdots \mu_{n-1}} = i^n \frac{n-1}{2n} S \left( \bar{\psi} (i \gamma_5 - m) \gamma_\mu \gamma_5 D^{\mu_1} \cdots D^{\mu_{n-2}} \gamma^{\mu_{n-1}} \psi \right)$$

$$+ \bar{\psi} \gamma_5 D^{\mu_1} \cdots D^{\mu_{n-2}} \gamma^{\mu_{n-1}} (i \gamma_5 - m) \psi \right] - \text{traces},$$

where

$$V_l = i^n g S \bar{\psi}\gamma_5 D^{\mu_1} \cdots G^{\mu_1} \cdots D^{\mu_{n-2}} \gamma^{\mu_{n-1}} \psi \right] - \text{traces},$$

$$U_l = i^{n-3} g S \bar{\psi} D^{\mu_1} \cdots \bar{G}^{\mu_1} \cdots D^{\mu_{n-2}} \gamma^{\mu_{n-1}} \psi \right] - \text{traces},$$

4
Here $m$ represents the quark mass (matrix). The combination in the r.h.s. of (8) constitutes an independent set of charge-conjugation even operators. The operators in (8) contain the gluon field strength $G_{\mu\nu}$ or the dual tensor $\tilde{G}_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}$ explicitly; this implies that they represent the effect of quark-gluon correlations. $R_{n,E}$ is the EOM operator; it vanishes by the use of the QCD equations of motion, although it is in general a nonzero operator due to quantum effects. $R_{n,m}$ is due to the quark mass effect.

The above operators are not all independent. By using $D_\mu = \{\gamma_\mu, \varphi\}/2$ and $[D_\mu, D_\nu] = -igG_{\mu\nu}$, it is straightforward to derive the following equation\[6, 2\]:

\[
R^{\mu_1 \cdots \mu_{n-1}}_{n,E} = \frac{n-1}{n} R^{\mu_1 \cdots \mu_{n-1}}_{n,m} + \sum_{l=1}^{n-2} (n-1-l) R^{\mu_1 \cdots \mu_{n-1}}_{n,l} + R^{\mu_1 \cdots \mu_{n-1}}_{n,E}.
\] (13)

Therefore we can exclude one operator among (7)-(10) to form an independent basis. A convenient choice of the independent operators will be (8), (9) and (10).

For the flavor nonsinglet part of $g_2(x, Q^2)$, the operators (8)-(10) form the complete set of the twist-3 gauge-invariant operators. For the singlet part, however, this is not the whole story: The QCD radiative corrections could replace the quark fields involved in (7), (8), (11) and (12) by the gluon fields, and generate the operators which are bilinear or trilinear in the gluon fields. In addition, because the QCD lagrangian in a covariant gauge involves the ghost field as a dynamical variable, the alien operators involving these unphysical degrees of freedom could also participate. These considerations lead to the following “new” operators:

\[
T^{\mu_1 \cdots \mu_{n-1}}_{n,G} = i^{n-1} S \mathcal{A} \mathcal{S} \left[ G^{-\mu_1} D^{\mu_2} \cdots D^{\mu_{n-1}} G^\sigma \right] - \text{traces},
\] (14)

\[
T^{\mu_1 \cdots \mu_{n-1}}_{n,l} = i^{n-2} g S \left[ G^{-\mu_1} D^{\mu_2} \cdots \tilde{G}^{-\mu_l} \cdots D^{\mu_{n-2}} G^\nu \right] - \text{traces} \quad (l = 2, \ldots, n-1),
\] (15)

\[
\tilde{T}^{\mu_1 \cdots \mu_{n-1}}_{n,l} = i^{n-2} g S \left[ \tilde{G}^{-\mu_1} D^{\mu_2} \cdots G^{-\mu_l} \cdots D^{\mu_{n-2}} G^\nu \right] - \text{traces} \quad (l = 2, \ldots, n-1),
\] (16)

\[
T^{\mu_1 \cdots \mu_{n-1}}_{n,B} = \left. i^{n-1} S \{ G^{-\mu_1} D^{\mu_2} \cdots D^{\mu_{n-2}} \} \left[ \left( \frac{1}{\alpha} \partial^{\mu_{n-1}} (\partial^\nu A^a_\nu) + g f^{abc} (\partial^{\mu_{n-1}} \chi^b) \right) \right] - \text{traces} \right|_{(17)}
\]

\[
T^{\mu_1 \cdots \mu_{n-1}}_{n,E} = i^{n-1} S \left\{ G^{-\mu_1} D^{\mu_2} \cdots D^{\mu_{n-2}} \right\} \left[ \left( \frac{1}{\alpha} \partial^{\mu_{n-1}} (\partial^\nu A^a_\nu) - g f^{abc} (\partial^{\mu_{n-1}} \chi^b) \right) \right] - \text{traces} \cdot \left(18\right)
\]

Here the gluon field $A_\mu$ and the covariant derivative $D_\mu$ are in the adjoint representation. $\chi$ and $\bar{\chi}$ are the ghost fields, and $\alpha$ is the gauge parameter. $t^a$ is the color matrix as $[t^a, t^b] = i f^{abc} t^c$, $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$. $T_{n,l}$ and $\tilde{T}_{n,l}$ are trilinear in the gluon field strength and its dual tensor, and thus represent the effect of three gluon correlations. In contrast to the quark-gluon operators (8), (11) and (12), $T_{n,l}$ ($\tilde{T}_{n,l}$) is charge-conjugation even by itself. $T_{n,E}$ is the EOM operator, and vanishes by the naive use of the equations of motion for the gluon. $T_{n,B}$ is the BRST-exact operator which is the BRST variation of the operator $i T_{n} S \{ G^{-\mu_1} D^{\mu_2} \cdots D^{\mu_{n-2}} \} \partial^{\mu_{n-1}} \chi^a - \text{traces}$.

Again, the operators (14)-(18) are not all independent: Firstly, due to the Bose statistics of the gluons, we obtain the symmetry relations $T^{\mu_1 \cdots \mu_{n-1}}_{n,n-l} = T^{\mu_1 \cdots \mu_{n-1}}_{n,n-l}$ and $\tilde{T}^{\mu_1 \cdots \mu_{n-1}}_{n,n-l} = -\tilde{T}^{\mu_1 \cdots \mu_{n-1}}_{n,n-l}$. Therefore, we can choose $T_{n,l}$ for $l = 2, \ldots, \frac{n-1}{2}$ as independent operators, as indicated in (15) and (16). Secondly, one can derive the relation between $T_{n,l}$ and $\tilde{T}_{n,l}$:

\[
\tilde{T}^{\mu_1 \cdots \mu_{n-1}}_{n,j} = \sum_{l=2}^{j-1} \left( C^i_{l-2} - C^j_{l-2} \right) (-1)^l \left( T^{\mu_1 \cdots \mu_{n-1}}_{n,l} + \left( C^{n-2-j}_j - 2 \right) (-1)^j T^{\mu_1 \cdots \mu_{n-1}}_{n,j} \right).
\]
function as

\[ \langle D \rangle \text{ Derators are given by replacing some of the covariant derivatives as BRST-noninvariant EOM operators. As explained in [10, 15], gauge -noninvariant EOM operators as well as the BRST-exact operators mix through r enormalization as nonzero infrared singularities[9], we keep the quark and gluon external lines o ff-shell; in this case the 1-loop corrections. We employ the Feynman gauge (} \]  

\[ D_{\mu} \text{ the quark EOM operator } R_g \text{ the relation } \]

\[ \text{Note that the quark-gluon-quark operator } C_n \text{, as well as the operators (15), (17) and (18). Choose a set of independent operators as (15), (17) and (18). As a result of (19), (20), we can conveniently}
\]

\[ \text{to symmetrize the Lorentz indices } g_{\mu\nu} \varepsilon_{\alpha\beta} = g_{\mu\alpha} \varepsilon_{\nu\beta} + g_{\mu\beta} \varepsilon_{\nu\alpha} + g_{\alpha\beta} \varepsilon_{\mu\nu}. \text{ The operator identity (20) is new, and is one of the main results of this work. As a result of [13], (20), we can conveniently}
\]

\[ \text{To summarize our independent basis of the twist-3 flavor singlet operators for the } n\text{-th moment (see [8]-[10], [13], (17) and (18)): The quark-gluon-quark operators } R_{n,l} \text{ (} n \text{= 1, } \ldots, n-2\};
\]

\[ \text{the quark EOM operator } R_{n,E}; \text{ the quark mass operator } R_{n,m}; \text{ the three-gluon operators } T_{n,l} \text{ (} l = 2, \ldots, \frac{n-1}{2}\); \text{ the gluon EOM operator } T_{n,E}; \text{ the BRST-exact operator } T_{n,B}. \text{ Among them,}
\]

\[ R_{n,l}, R_{n,E}, R_{n,m}, \text{ and } T_{n,l} \text{ are gauge-invariant, while } T_{n,E} \text{ and } T_{n,B} \text{ are not gauge-invariant but}
\]

\[ \text{BRST-invariant. These } (3n+1)/2 \text{ operators will mix with each other under renormalization.}
\]

3 Renormalization of twist-3 singlet operators

The } Q^2\text{-evolution of the flavor singlet part of } \bar{g}_s(x, Q^2) \text{ is governed by the anomalous dimensions, which enter into the RG equation for the relevant twist-3 singlet operators obtained in sect.2. In this section we discuss the renormalization of these operators to obtain the anomalous dimensions.

We follow the standard method to renormalize the local composite operators[8]. We multiply the operators discussed in sect.2 by a light-like vector } \Delta_{\mu_i} \text{ to symmetrize the Lorentz indices and to eliminate the trace terms: } \Delta_{\mu_1} \cdots \Delta_{\mu_{n-1}} R_{n,l}^{\mu_1 \cdots \mu_{n-1}} \equiv \Delta \cdot R_{n,l}, \Delta_{\mu_2} \cdots \Delta_{\mu_{n-1}} T_{n,l}^{\mu_1 \cdots \mu_{n-1}} \equiv \Delta \cdot T_{n,l}, \text{ etc. We then embed the operators } \bar{O}_j = \Delta \cdot R_{n,l}, \Delta \cdot T_{n,l}, \text{ etc. into the three-point function as } \langle 0 | T \bar{O}_j(0) A_{\mu}(x) \psi(y) \bar{\psi}(z) | 0 \rangle, \langle 0 | T \bar{O}_j(0) A_{\mu}(x) A_{\nu}(y) A_{\alpha}(z) | 0 \rangle, \text{ etc., and compute the 1-loop corrections. We employ the Feynman gauge } (\alpha = 1) \text{ and renormalize the operators in the MS scheme. To perform the renormalization in a consistent manner without subtle infrared singularities[8], we keep the quark and gluon external lines off-shell; in this case the EOM operators as well as the BRST-exact operators mix through renormalization as nonzero operators.}

One serious problem in the calculation is the mixing of the many gauge-noninvariant as well as BRST-noninvariant EOM operators. As explained in [10, 15], gauge-noninvariant EOM operators are given by replacing some of the covariant derivatives } D^\mu_i \text{ by the ordinary derivatives.
The results can be summarized in the following matrix form:

\[
\begin{pmatrix}
\Delta \cdot R^\sigma_{n,i} \\
\Delta \cdot R^\sigma_{n,m} \\
\Delta \cdot T^\sigma_{n,j} \\
\Delta \cdot T^\sigma_{n,B} \\
\Delta \cdot R^\sigma_{n,E} \\
\Delta \cdot T^\sigma_{n,E} \end{pmatrix}_{bare} = \begin{pmatrix}
Z^{FF}_{ik} & Z^{FF}_{im} & Z^{FG}_{ij} & Z^{FG}_{iB} & Z^{FG}_{iE} & Z^{FG}_{iE} \\
0 & Z^{FF}_{mm} & 0 & 0 & 0 & 0 \\
Z^{GF}_{ik} & Z^{GF}_{im} & Z^{GG}_{ij} & Z^{GG}_{iE} & Z^{GG}_{iE} & Z^{GG}_{iE} \\
0 & 0 & Z^{GF}_{nm} & Z^{GF}_{BE} & Z^{GF}_{BE} & Z^{GF}_{BE} \\
0 & 0 & 0 & Z^{GG}_{EE} & Z^{GG}_{EE} & Z^{GG}_{EE} \\
0 & 0 & 0 & 0 & Z^{GG}_{EE} & Z^{GG}_{EE} \\
\end{pmatrix},
\]

(22)

where the operators with (without) the suffix “bare” are the bare (renormalized) quantities. In the MS scheme we express the renormalization constants \(Z^{AA'}_{pp'}\) as

\[
Z^{AA'}_{pp'} = \delta_{pp'}\delta_{AA'} + \frac{g^2}{8\pi^2(4 - D)} X^{AA'}_{pp'}
\]

(23)

with \(D\) the space-time dimension. Here \(p = 1, \ldots, n-2, m, E\) for \(A = F\), while \(p = 2, \ldots, (n - 1)/2, B, E\) for \(A = G\). The renormalization constant matrix is triangular because the physical matrix elements of the EOM operators and of the BRST-exact operators vanish.

We here concentrate on diagrams (e)-(k) which are relevant for an example discussed in sect.4: Diagrams (g)-(k) have already been computed in the present scheme, and the results have been obtained as \(Z_{ik}\) \((i, k = 1, \ldots, n - 2)\) of eqs.(14)-(17) in [11]. In the singlet case, however, these renormalization constants \(Z_{ik}\), which we denote by \((Z_{ik})_{NS}\) here and in the following, should be identified as

\[
(Z_{ik})_{NS} = Z^{FF}_{ik} + (-1)^k n C^{n-3}_{k-1} Z^{FG}_{iE},
\]

(24)

because \(T_{n,E}\) of (18) contains the quark-gluon-quark term \((i^{i-1} S \{ \bar{C}^{\sigma_1} D^{\mu_2} \cdots D^{\mu_{n-2}} \}^{\alpha} \bar{\psi}_i \gamma^{\mu_n-1} \psi_j\)

which can be reexpressed by \(R_{n,k}\) with the coefficient in the second term of (24). On the other hand, \(Z_{im}, Z_{iE}\) and \(Z_{mm}\) given by eq.(18) of [11] should be identified as \(Z^{FF}_{im}, Z^{FF}_{iE}\) and \(Z^{FF}_{mm}\). From the computation of diagrams (e) and (f), we obtain

\[
X^{FG}_{iE} = (-1)^f \frac{N_f}{n^2} \frac{4}{C_{i-1}},
\]

(25)
where $N_f$ is the number of quark flavors, and $l=1,\ldots, (n-1)/2$. For $l=(n-1)/2+1,\ldots, n-2$, the replacement $l\to l'=n-l-1$ should be understood. (24) and (25) determine $Z_{ik}^{\text{FF}}$ for general $n$, and the result agrees with that of [3]. Similar computation including other diagrams (a)-(d) gives other renormalization constants. These results will be published elsewhere.

The RG equation for the twist-3 singlet operators can be obtained in a standard way from (22). As is well known, the RG equation decouples for each $n$. In the leading-logarithmic approximation (LLA), this equation is solved to give

$$\langle\langle R_n,i(Q^2)\rangle\rangle = \sum_{k=1}^{n-2} \langle\langle R_{n,k}(\mu^2)\rangle\rangle \left[ L^{X}_{i,k}^{FF} + \langle\langle R_{n,m}(\mu^2)\rangle\rangle \left[ L^{X}_{i,m}^{FF} \right] \sum_{j=2}^{n-1} \sum_{l=2}^{n-1} \langle\langle T_{n,j}(\mu^2)\rangle\rangle \left[ L^{X}_{i,j}^{FF} \right] \right]$$

where $i=1,\ldots, n-2$, $l=2,\ldots, n-2$, $L=\alpha_s(\mu^2)/\alpha_s(Q^2)$, and $b=(11N_c-2N_f)/3$ for $N_c$ color. Here we introduced the reduced matrix element $\langle\langle O(\mu^2)\rangle\rangle$ by

$$\langle P|O^{\sigma_{\mu_1}\cdots \mu_{n-1}}(\mu^2)\rangle PS = -2\langle\langle O(\mu^2)\rangle\rangle S\mathcal{A}[S^\sigma P^{\mu_1} \cdots P^{\mu_{n-1}}] - \text{traces},$$

with $O = R_{n,i}, R_{n,m}$, and $T_{n,l}$. (26)-(28) exhibit the complicated mixing characteristic of the higher-twist operators. On the other hand, the EOM operators $R_{n,E}, T_{n,E}$ as well as the BRST-exact operator $T_{n,B}$ decouple from the result because their matrix elements vanish [4].

Before ending this section, we give explicit formula for the $Q^2$-evolution of $g_2(x, Q^2)$ in the LLA: In the LLA, we substitute, in (2), $\mu^2 = Q^2$ and $H_2i\left(\frac{2}{3}, 1, \alpha_s(Q^2)\right) = (e^2)i\delta \left(\frac{2}{3} - 1\right) + O(\alpha_s)$ for $i = u, \bar{u}, d, \bar{d}, \ldots$, with $e$ the charge matrix of the quarks. Therefore, the $Q^2$-evolution of $g_2(x, Q^2)$ is governed directly by the scale dependence of the local composite operators corresponding to the moment of $\phi^2_2(\xi, Q^2)$. As shown in (1), the twist-2 contributions contained in $g_2(x, Q^2)$ are determined completely by $g_1(x, Q^2)$ whose $Q^2$-evolution is well known. On the other hand, for the twist-3 part $g_3(x, Q^2)$, we have already determined an independent basis of relevant operators and their scale dependence. Corresponding to the decomposition of the charge matrix squared by $e^2 = (e^2)1 + A_3 \lambda_3 + A_8 \lambda_8$ with $\lambda_i$ the flavor matrices, we decompose $g_2(x, Q^2)$ into the singlet and the nonsinglet parts by $g_2(x, Q^2) = g_2^S(x, Q^2) + g_2^NS(x, Q^2)$. Then, from (2)-(13) and (29), it is straightforward to see that the moment $M_n[g_2^S(Q^2)] = \int_1^\infty dx x^{n-1}g_2^S(x, Q^2)$ is given by the matrix elements of our independent operators as (see also [10])

$$M_n[g_2^S(Q^2)] = \frac{e^2}{2} \left[ \sum_{k=1}^{n-2} (n-1-k) \langle\langle R_{n,k}(Q^2)\rangle\rangle + \sum_{k=1}^{n-1} \langle\langle R_{n,m}(Q^2)\rangle\rangle \right].$$

The equations (26)-(31) completely determine the $Q^2$-evolution of the singlet part.

The moment of the flavor nonsinglet part $M_n[g_2^NS(Q^2)]$ is given by the equation similar to (30) with $R_{n,k}(Q^2), R_{n,m}(Q^2)$ and $\langle\langle R_{n,m}(Q^2)\rangle\rangle$ replaced by the corresponding nonsinglet quantities. In this case, the relevant scale dependence is given by (26)-(27) with the replacement $X^{FF} \to (X)^{NS}$ and $\langle\langle T_{n,j}(\mu^2)\rangle\rangle \to 0$. For the detail of the nonsinglet part, we refer the readers to e.g. [10, 11].

### 4 Application to the $n = 3$ case

We now present an example of the $Q^2$-evolution for the lowest $(n = 3)$ moment. In this case there exists no three-gluon operator $\Delta \cdot T_{n,l}^3$, while there exists one quark-gluon-quark operator
\[ \Delta \cdot R_{3,1} = \frac{1}{3} g \bar{\psi} G^{\sigma \mu} \Delta \psi \text{ (see (13), (8))}. \] If we neglect the contribution of the quark mass operator \( \Delta \cdot R_{3,m} \), only one operator \( \Delta \cdot R_{3,1}^\sigma \) contributes to the \( Q^2 \)-evolution of the physical matrix elements as given by (24). Therefore, the \( Q^2 \)-evolution is completely determined from our results (24), (25). From (26), the relevant renormalization constant is \( Z^{FF}_{11} \). We obtain, from (24), (25) and eqs.(14)-(17) of [11],

\[
Z^{FF}_{11} = (Z^{NS}_{11}) + 3 g^2 \left( C_F/3 - 3C_G - 2N_f/3 \right) / [8 \pi^2 (4 - D)],
\]

where \( C_F = (N_c^2 - 1)/2N_c \), \( C_G = N_c \). This result combined with (23), (26) and (30) gives the \( Q^2 \)-evolution:

\[
M_3 \left[ \tilde{g}_2^{S}(Q^2) \right] = \left( \frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \right)^{3C_G - C_F/3 + 2N_f/3} M_3 \left[ \tilde{g}_2^{S} (\mu^2) \right].
\] (31)

Several comments are in order here: (i) The result (31) coincides with that of [6, 8], though our approach is quite different from those works. This fact confirms the theoretical prediction (31), and also demonstrates the efficiency of our method. (ii) The term \( 2N_f/3 \) would be absent from the exponent of (31), if we considered the nonsinglet case [10, 11]. Our derivation shows that this difference of \( 2N_f/3 \) comes from \( Z^{FG}_{11} \), which describes the mixing between \( \Delta \cdot R_{3,1}^\sigma \) and the gluon EOM operator \( \Delta \cdot T_{3,E}^\sigma \). This reveals a peculiar role played by the EOM operator: It produces observable effects although its physical matrix elements vanish. (iii) For \( N_c = 3 \) and \( N_f = 4 \), the exponent of (31) is 101/9b, while the corresponding exponent for the nonsinglet case is 77/9b. Thus the singlet case obeys rather stronger \( Q^2 \)-evolution compared to the nonsinglet case. Fig.2 shows the behavior of (31) for the case of \( N_c = 3, N_f = 4 \), and \( \Lambda_{QCD} = 0.5 \text{GeV} \).

![Figure 2: Q²-evolution of \( \tilde{g}_2(x, Q^2) \) for the lowest \( n = 3 \) moment](image)

5 Conclusions

In the present study, we have developed a manifestly covariant approach to investigate the flavor singlet part of \( g_2(x, Q^2) \). We gave a thorough OPE analysis of the twist-3 flavor singlet operators which contribute to \( g_2(x, Q^2) \). We derived the operator identities which relate the two-particle operators with the three-particle ones. We have chosen the three-particle operators as an independent operator’s basis. To identify the renormalization constants correctly, the off-shell Green’s functions are considered. We have shown that the EOM operators as well as the BRST-exact operators play an important role. As an application, we computed the \( Q^2 \)-evolution of the lowest \( (n = 3) \) moment. Our result confirmed the prediction obtained by different methods [3, 8].
The off-shell Green’s functions are free from the infrared singularity coming from the collinear configuration, as well as from any unphysical singularities. We believe that calculating the off-shell Green’s functions is the safest method to obtain the anomalous dimensions.

Acknowledgment
The work of J.K. was supported in part by the Monbusho Grant-in-Aid for Scientific Research No. C-09640364. The work of K.T. was supported in part by the Monbusho Grant-in-Aid for Scientific Research No. 09740215.

References
[1] A.J.G. Hey and J.E. Mandula, Phys. Rev. D5 (1972) 2610.
[2] R.L. Jaffe, Comments Nucl. Part. Phys. 19 (1990) 239, and references therein.
[3] The Spin Muon Collaboration (D. Adams et al.), Phys. Lett. B336 (1994) 125; E143 Collaboration (K. Abe et al.), Phys. Rev. Lett. 76 (1996) 587.
[4] H.D. Politzer, Nucl. Phys. B172 (1980) 349.
[5] J. C. Collins, Renormalization, (Cambridge Univ. Press, 1984), and references therein.
[6] E.V. Shuryak and A.I. Vainshtein, Nucl. Phys. B199 (1982) 951; B201 (1982) 141.
[7] P.G. Ratcliffe, Nucl. Phys. B264 (1986) 493; X. Ji and C. Chou, Phys. Rev. D42 (1990) 3637.
[8] A. P. Bukhvostov, E. A. Kuraev and L. N. Lipatov, Sov. Phys. JETP 60 (1984) 22; D. Müller, Phys. Lett. B407 (1997) 314; I.I. Balitsky and V.M. Braun, Nucl. Phys. B311 (1988/89) 541.
[9] J. C. Collins and R. J. Scalise, Phys. Rev. D50 (1994) 4117.
[10] J. Kodaira, Y. Yasui and T. Uematsu, Phys. Lett. B344 (1995) 348.
[11] J. Kodaira, Y. Yasui, K. Tanaka and T. Uematsu, Phys. Lett. B387 (1996) 855.
[12] Y. Koike and K. Tanaka, Phys. Rev. D51 (1995) 6125.
[13] J.C. Collins, D. Soper and G. Sterman, Factorization of hard processes in QCD, in Perturbative Quantum Chromodynamics, (World Scientific, Singapore, 1989, ed. A.H. Mueller).
[14] J.C. Collins and D.E. Soper, Nucl. Phys. B194 (1982) 445; R.L. Jaffe and X. Ji, Nucl. Phys. B375 (1992) 527.
[15] H. Kawamura, T. Uematsu, J. Kodaira and Y. Yasui, Mod. Phys. Lett. A12 (1997) 135; H. Kawamura, Z. Phys. C75 (1997) 27.
[16] A. Ali, V.M. Braun and G. Hiller, Phys. Lett. B266 (1991) 117; I.I. Balitsky, V.M. Braun, Y. Koike and K. Tanaka, Phys. Rev. Lett. 77 (1996) 3078; Y. Koike and N. Nishiyama, Phys. Rev. D55 (1997) 3068.