Two- and Many-Dimensional Quasi-Exactly Solvable Models

With An

Inhomogeneous Magnetic Field

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Let group generators having finite-dimensional representation be realized as Hermitian linear differential operators without inhomogeneous terms as takes place, for example, for the SO(n) group. Then corresponding group Hamiltonians containing terms linear in generators (along with quadratic ones) give rise to quasi-exactly solvable models with a magnetic field in a curved space. In particular, in the two-dimensional case such models are generated by quantum tops. In the three-dimensional one for the SO(4) Hamiltonian with an isotropic quadratic part the manifold within which a quantum particle moves has the geometry of the Einstein universe.

Exact solutions with a magnetic field are extremely rare in quantum mechanics. One can find only two examples of such a kind in textbooks: a free electron or an harmonic oscillator, a magnetic field being homogeneous. Below we show how that in fact there exists a variety of systems with exact solutions of the Schrödinger equation with an inhomogeneous magnetic field among quasi-exactly solvable models (QESM). In so doing, not only a potential and magnetic field are present but also the analog of a gravitational one in that a corresponding particle is moving on a curved surface [1], [2].

First let us consider the two-dimensional case following a general procedure [3]. We construct Hamiltonian which contains quadratic and linear terms in generators of a Lie-algebra having a finite-dimensional representation and choose the algebra of the SO(3) group,
i.e. algebra of an angular momentum operators:

\[ H = \alpha L_x^2 + \beta L_y^2 + \gamma L_z^2 + C_1 L_x + C_2 L_x + C_3 L_z, \]  

(1)

\[ L_x = i(\sin \phi \frac{\partial}{\partial \theta} + \cos \phi \cot \theta \frac{\partial}{\partial \phi}), \quad L_y = i(-\cos \phi \frac{\partial}{\partial \theta} + \sin \phi \cot \theta \frac{\partial}{\partial \phi}), \quad L_z = -i \frac{\partial}{\partial \phi}, \]  

\[ \theta \text{ and } \phi \text{ are the angles of the spherical coordinate system.} \]

Substitute these expressions into the Schrödinger equation

\[ H \Phi = E \Phi. \]  

(2)

It can be represented in the form

\[ -g^{\mu \nu} \frac{\partial^2 \Phi}{\partial X^\mu \partial X^\nu} + T^\mu \frac{\partial \Phi}{\partial X^\mu} = E \Phi, \]  

(3)

\[ \mu, \nu = \theta, \phi. \]

This can be rewritten as follows,

\[ -g^{\mu \nu}(\nabla_\mu - A_\mu)(\nabla_\nu - A_\nu)\Phi + U \Phi = E \Phi \]  

(4)

where \( \nabla \) is the covariant derivative operator. Comparing (3) and (4) one can find the explicit expression for \( A_\mu \). In order for the equation (4) to take this form it is necessary that in the expression for \( A_\mu \),

\[ A_\mu = a_\mu + ib_\mu, \]  

(5)

its real part be a pure gradient: \( a_\mu = \rho,\mu \). Then, by substitution \( \Psi = \Phi e^{-\rho} \) eq. (4) is reduced to the form

\[ -g^{\mu \nu}(\nabla_\mu - ib_\mu)(\nabla_\nu - ib_\nu)\Psi + U \Psi = E \Psi. \]  

(6)

Imaginary terms in (6) can be attributed in a natural way to the two-dimensional analog of a magnetic field (or its component orthogonal to the surface if the model is thought of as embedded into three-dimensional space). The value of this field, \( B = \frac{b_\nu,\mu - b_\mu,\nu}{\sqrt{g}} \) (\( g = \det g_{\mu \nu} \)) determines the value of the two-dimensional field invariant \( B^2 = \frac{1}{2} F_{\mu \nu} F^{\mu \nu}, \quad F_{\mu \nu} = b_{\nu,\mu} - b_{\mu,\nu}. \)
Comparing (3) and (4) one can find the potential $U = g_{\mu\nu} A^\mu A^\nu - g^{-1/2} (g^{1/2} A^\mu)_{,\mu}$ which is in general complex: $U = U_1 + iU_2,$

$$U_2 = 2a_\mu b^\mu - g^{-1/2} (g^{1/2} a^\mu)_{,\mu}. \tag{7}$$

If $b_\mu = 0$ we return to the situation discussed in detail in [3]. In that case the condition of integrability

$$a_\mu = \rho_{,\mu} \tag{8}$$

is sufficient for the equation under consideration to take the Schrödinger form. However, if $b_\mu \neq 0$ the second condition $U_2 = 0$ is called, otherwise the Hamiltonian would become non-Hermitian. In general, one could expect that the imaginary part of the potential $U_2 \neq 0$ which is the obstacle for obtaining a physically reasonable QESM with a magnetic field. A nontrivial point in the problem under consideration is that for the S)(3) Hamiltonian the potential turns out to be purely real as it follows after some direct but lengthy calculations. In general the expression for the potential is rather cumbersome, so we list only some simplest examples.

In the isotropic case $\alpha = \beta = \gamma$ one may choose the coordinate system in such a way that $C_1 = C_2 = 0, C_3 = C$. Then $U = -\frac{C^2 \sin^2 \theta}{4}, B = C \cos \theta, g_{\theta\theta} = \alpha^{-1}, g_{\phi\phi} = \alpha^{-1} \sin^2 \theta$. The manifold now is nothing but a sphere. If $\alpha = \beta, \gamma = 0, C_2 = 0$ the Riemannian curvature $R = -4\alpha \cos^{-2} \theta, B = C_3 \cos^{-2} \theta - \frac{C_1}{2} \cos \phi tg \theta$.

The manifold within which a quantum particle moves is compact if $\alpha, \beta, \gamma > 0$ and noncompact if one of these constants is zero. It turns out that the wave function is normalizable even in cases (like in the example above) when curvature or potential can contain singularities.

Note that since the square of the angular momentum commutes with the Hamiltonian (1), all the space of states can be divided into subspaces with fixed values of angular momentum $l$. In turn, in any such a subspace the Hamiltonian (1) in a matrix representation is equivalent to a spin Hamiltonian with spin $l$ and generates a one-dimensional QESM with a potential.
composed of elliptic functions [1]. This is rather interesting correspondence between the one-dimensional QESM and two-dimensional ones defined on curved surfaces and with a magnetic field. For small values of \(l\) the expressions for energy levels and wave functions can be found explicitly. For an arbitrary \(l\) they enter into the algebraic equation of a finite degree that is typically for QESM.

It is the generalization of the obtained results to the many-dimensional case that we now turn to. Consider group Hamiltonian

\[
H = C_{ab}L^a L^b + C_a L^a
\]

with real coefficients, \(C_{ab} = C_{ba}\). In fact, we use, as the starting point, approach of [3] where it was shown that the choice \(C_a = 0\) leads to both hermiticity of (9) with a certain measure in Riemannian space and an absence of a magnetic field. Below we show that taking \(C_a \neq 0\) also preserves hermiticity and corresponds to the appearance of a certain magnetic field of the Schrödinger operator. Let the generators have the form

\[
L^a = i h^{a\mu} \frac{\partial}{\partial x^\mu}
\]

with real \(h^{a\mu}\). Substituting (10) into the Schrödinger equation we obtain the second order differential equation which can be rewritten in the form (4) with \(g_{\mu\nu} = C_{ab} h^{a\mu} h^{b\nu}, T^\mu = -C_{ab} h^{a\nu} h^{b\mu} + h^{a\mu} C_a, A^\mu = \frac{1}{2} (P^\mu + T^\mu), P^\mu \equiv (\sqrt{g} g^{\mu\nu}) / \sqrt{g}, U_2\) having the same form as in the two-dimensional case. The crucial point is whether or not \(U_2 = 0\). The real part of the above expression for \(A^\mu\) gives us

\[
C_{ab} h^{a\mu} (2 h^{b\nu} a_\nu - h^{b\nu} \sqrt{g} \, A_{\nu} - h^{b\nu}) = 0. \tag{11}
\]

Now we invoke an additional assumption [4]: let operators \(L_a\) be Hermitian in some metric \(g^{(0)}_{\mu\nu}\) in which the scalar product is determined in a standard way:

\[
(\phi_2, \phi_1) = \int d^{n-1} x \sqrt{g^{(0)}} \phi_2^* \phi_1. \tag{12}
\]

For example, for the \(SO(n)\) group \(g^{(0)}_{\mu\nu}\) is the metric of a \(n - 1\) dimensional hypersphere. Then the hermiticity condition
\[ (\phi_2 L^\mu \phi_1) = (L^\mu \phi_2, \phi_1) \]  

(13)

along with (10) and (12) entails

\[ h^{\alpha\mu} \partial_\mu \ln \sqrt{g(0)} \]  

(14)

It follows from (11), (14) that

\[ C_{ab} h^{\alpha\mu} h^{b\nu} (2a_{\nu} - \sqrt{g}_{\nu} + \sqrt{g(0)}_{\nu}) = 0. \]  

(15)

It is clear that irrespective of \( C_{ab} \) there exists the solution \( a_{\mu} = \rho_{\mu} \) with \( \rho = \frac{1}{2} \ln \sqrt{g/g(0)} \). After the substitution \( \Psi = \Phi e^{-\rho} \) the Schrödinger equation takes the form (6), \( \Psi \) being normalized according to \( (\Psi, \Psi) = \int d^{n-1}x \sqrt{g} |\Psi|^2 = \int d^{n-1}\sqrt{g(0)} |\Phi|^2 \). Thus, normalizability of \( \Phi \) entails normalizability of \( \Psi \).

Until now our treatment has run almost along the same lines as in [5] where it was assumed \( C_a = 0 \). The key new moment which makes our problem non-trivial is that for \( C_a \neq 0 \) the potential contains the imaginary part \( iU_2 \) and one must elucidate whether or not \( U_2 = 0 \). Using explicit formulae for the coefficients of the differential equations listed above one can show that

\[ 2a_{\mu} b^{\mu} = \frac{1}{2} h^{\alpha\mu} C_a (\sqrt{g}_{\mu} - \sqrt{g(0)}_{\mu}), \]  

(16)

\[ \frac{(b^{\mu} \sqrt{g})_{\mu}}{\sqrt{g}} = b^{\mu} + \sqrt{g}_{\mu} b^{\mu} = \frac{C_a}{2} (h^{\alpha\mu}_{\mu} + \sqrt{g}_{\mu} h^{\alpha\mu}) \]  

(17)

Making use of (14), we see that expressions (16) and (17) coincide completely, so according to (7) \( U_2 = 0 \)!

Thus, the general form of generators (10) along with the hermiticity condition (12) and (13) entail the integrability condition (8) and, simultaneously, ensure that the potential is real. In other words, if generators (10) Hermitian in a space with the metric \( g_{\mu\nu}^{(0)} \), Hamiltonian of the Schrödinger equation is Hermitian in a space with the metric \( g_{\mu\nu} \).
The result obtained shows that there is essential difference between QESM based on SO(n) groups and, say, SU(n) ones. In the latter case even when a magnetic field is absent it is rather difficult task to find coefficients $C_{ab}$ for which the integrability condition (8) is satisfied [3]. In the second case we obtain at once QESM with well defined Hamiltonian for which this condition is satisfied automatically and, moreover, the effective magnetic field $F_{\mu\nu} = b_{\nu,\mu} - b_{\mu,\nu}$ is present.

Consider briefly an example of QESM of such a kind in three-dimensional space based on generators of SO(4) $L_{ik} = -i(x^i \partial/\partial x^k - x^k \partial/\partial x^i)$. For Hamiltonian $H = \sum_{i<k} L_{ik}^2 + CL_{12}$ direct calculations show that the potential and field tensor are equal to $U = -\frac{C^2}{2} \sin^2 \xi \sin^2 \theta$, $F_{\theta\phi} = -C \sin^2 \xi \sin \theta \cos \theta$, $F_{\xi\phi} = -C \sin^2 \xi \sin \xi \cos \xi$, $F_{\xi\theta} = 0$. The metric reads $ds^2 = d\theta^2 + \sin^2 \theta d\xi^2 + \sin^2 \theta \sin^2 \xi d\phi^2$ where $\theta, \xi, \phi$ are angles of the hyperspherical coordinate systems. The obtained metric is nothing else than that of the Einstein universe.

It is worth noting that the approach outlined above enables one to obtain QESM with a magnetic field which are not reduced to exactly solvable ones. Whereas for exactly solvable cases such a charged particle or an harmonic oscillator in an homogeneous magnetic field finding exact solutions implies separation of variables, such separation is not needed for solutions under discussion. In so doing, we obtain many-parametric classes of solutions at once, a magnetic field being inhomogeneous. The essential feature of QESM in question is that a manifold on which a quantum particle moves is inevitable curved. This can be of interest, for example, for applications in relativistic cosmology.

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