A Note on the KP hierarchy

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Abstract: Given the two boson representation of the conformal algebra $\hat{W}_\infty$, the second Hamiltonian structure of the KP hierarchy, I construct a bi-Hamiltonian hierarchy for the two associated currents. The KP hierarchy appears as a composite of this new and simpler system. The bi-Hamiltonian structure of the new hierarchy gives naturally all the Hamiltonian structures of the KP system.
There are deep relations between conformal field theory and non-linear integrable systems. For instance, the classical Virasoro algebra forms the second Hamiltonian structure of the usual KdV equation \[1\]. More generally, the extended conformal algebras $W_N$ were shown \[2\] to form the second Hamiltonian structure of the generalized equations of KdV type \[4\]. Later, it was demonstrated \[3\] that $W_{1+\infty}$, a linear deformation of the $W_N$ algebras in the large $N$ limit \[7\], is related to the first Hamiltonian structure of the KP hierarchy \[8\]. The second Hamiltonian structure of the KP hierarchy \[9\] was then shown to be isomorphic to $\hat{W}_\infty$ \[10\], a centerless deformation of $W_\infty$.

In \[11\], a two boson representation of $W_\infty$ was given. Yu and Wu then presented the corresponding realization of $\hat{W}_\infty$, in terms of two bosons $\phi$ and $\bar{\phi}$ \[12\] (see also \[13\]). With this representation of the conformal algebra $\hat{W}_\infty$ (the Hamiltonian structure for the integrable system formed by the KP hierarchy), it is shown in this letter that it is possible to write down a hierarchy for the currents $j(x) = \partial_x \phi(x)$ and $\bar{j}(x) = \partial_x \bar{\phi}(x)$. Given this new hierarchy and the expression of the fields entering the KP hierarchy in terms of $j$ and $\bar{j}$, it is shown that the flows of the KP hierarchy themselves can be written as composites of the flows of the $j$, $\bar{j}$ hierarchy. This new hierarchy is proved to be bi-Hamiltonian and to possess an infinite number of local conserved quantities. The two simplest of the possible Hamiltonian structures for the $j$, $\bar{j}$ hierarchy generate $W_{1+\infty}$ and $\hat{W}_\infty$ when expressed in terms of the KP fields; the use of the recursion operator of the $j$, $\bar{j}$ hierarchy allows one to construct an infinite number of Hamiltonian structures for the KP hierarchy. However, this construction shows that these new Hamiltonian structures are all related.

Through examples, the existence of a zero-curvature or matrix formulation for the new hierarchy is then conjectured. A complete zero-curvature formulation would help understand some aspects of this new hierarchy, and in particular, the results presented here give some hopes to obtain the KP hierarchy as a reduction of the self-dual Yang-Mills equations in four dimensions. Work on this is in progress.

Some open problems: The results presented in this letter and the existence of a two fermion $\psi$ and $\bar{\psi}$ realization of $W_{1+\infty}$ \[14\], the first Hamiltonian structure of the KP hierarchy, suggest the existence of a “$\psi$, $\bar{\psi}$ hierarchy”, the KP hierarchy appearing as a composite of that $\psi$, $\bar{\psi}$ hierarchy. Also, two boson realizations of $W_N$ algebras exist \[15\], and it should
be possible to translate the present work to the framework of the generalized KdV equations, thus allowing one to rewrite the generalized KdV hierarchies \[4\] and maybe even the fractional ones \[15\] \[17\] \[18\] as composites of much simpler hierarchies. Preliminary calculations show that such is indeed the case. Finally, this work might also help understand the symmetries of the \(c = 1, d = 2\) string.

Let us now give some notations. We will be using differential \(\partial\) and pseudo-differential \(\partial^{-1}\) operators, where \(\partial^{-1}\) is an integration symbol defined by \(\partial\partial^{-1}a = \partial^{-1}\partial a = a\). Where needed, we will use square brackets to make clear how \(\partial\) or \(\partial^{-1}\) act. For instance, \(\partial_x a(x) = a(x)\partial_x + a'(x)\) whereas \([\partial_x a(x)] = a'(x)\) (with the same convention for \(\partial_x^{-1}\)). Also, to shorten the expressions of the Poisson brackets, we write \(\{a, b\} = c\delta'\) for \(\{a(x), b(x')\} = c(x)\partial_x\delta(x, x')\). Finally, commas in indices usually denote differentiation so that \(u_{i,t_r}\) means 

\[\partial_t u_i\].

1. Reformulating the KP hierarchy.

The KP hierarchy: The KP hierarchy is an integrable system consisting of an infinite number of non-linear differential equations. It is usually formulated in terms of a Lax operator \(L\), spanned by an infinite number of fields, with an infinite tail of integration or pseudo-differential symbols:

\[
L = \partial + u_0\partial^{-1} + u_1\partial^{-2} + \ldots = \partial + \sum_{i=0}^{\infty} u_i\partial^{-i-1}, \quad \partial = \partial/\partial_x .
\] (1.1)

The fields \(u_i\) depend on \(x\) as well as on an infinite number of time coordinates \(t_2, t_3, \ldots\). The \(r^{th}\) member or \(r^{th}\) flow of the KP hierarchy is given by

\[
\frac{\partial}{\partial t_r} L = [(L^r)_+, L] ,
\] (1.2)

where as usual, \((L^r)_+\) denotes the differential part of \(L^r\). Each value of \(r \geq 2\) gives rise to a non-trivial flow, in \((1+1)\) dimension (i.e. for the variables \(x\) and \(t_r\)). It is always possible to think of \(u_1, u_2, \ldots\) as auxiliary fields and to eliminate them by simple integration to give rise to an equation in \(u_0\) only. The first few flows for the lowest values of \(r\) are

\[
r = 2:\quad u_{0,t_2} = 2u_{1,x} + u_{0,xx} ,
\]

\[
u_{1,t_2} = 2u_{2,x} + u_{1,xx} + 2u_0u_{0,x} ,
\]

\[
u_{2,t_2} = 2u_{3,x} + u_{2,xx} + 4u_{0,x}u_1 - 2u_0u_{0,xx} .
\] (1.3)
\[ r = 3 : \quad u_{0,t_3} = 3u_{2,x} + 3u_{1,xx} + u_{0,xxx} + 6u_0u_{0,x} , \]
\[ u_{1,t_3} = 3u_{3,x} + 3u_{2,xx} + u_{1,xxx} + 6(u_0u_1)_x , \]
\[ u_{2,t_3} = 3u_{4,x} + 3u_{3,xx} + u_{2,xxx} + 3u_0u_{2,x} + 9u_0,u_1 + 6u_1u_{1,x} - 3u_0u_1,xx - 3u_0,xxu_1. \] (1.4)

Eliminating \( u_1 \) and \( u_2 \) by (1.3) and plugging in the first equation of (1.4) gives the usual KP equation: \( 4u_{0,xt_3} = (u_{0,xxx} + 12u_0u_{0,x})_x + 3u_{0,t_2t_2} . \)

From now on, we restrict ourselves to the KP hierarchy (1.2). Its flows commute with each other and are Hamiltonian, i.e. they can always be written as \( u_{i,t_r} = \{ u_i, \int \mathcal{H}_r \} \) where \( \{ , \} \) is a definite Poisson bracket and \( \int \mathcal{H}_r \) some Hamiltonian. There are two such Hamiltonian structures: the first one was given by Watanabe [8]. It was recently shown [5] to correspond to the conformal algebra \( W_{1+\infty} \) with \( c = 0 \). A second Hamiltonian structure was given by Dickey [9], which was then related to \( \hat{W}_{\infty} \) [10], a centerless deformation of the usual conformal algebra \( W_{\infty} \). The Hamiltonian densities for the KP hierarchy take a particularly simple form, namely

\[ \mathcal{H}_r = \frac{1}{r} \text{res } L^r , \] (1.5)

where the residue \( \text{res} \) of a pseudo-differential operator is the coefficient of its \( \partial^{-1} \) term. These are the same densities for either Hamiltonian structure, that is we can rewrite (1.2) as \( u_{i,t_r} = \{ u_i, \int \mathcal{H}_{r+1} \}_1 = \{ u_i, \int \mathcal{H}_r \}_2 \), where \( \{ , \}_1,2 \) denote the first and second Hamiltonian structures respectively.

The representation of \( \hat{W}_{\infty} \): In [12], Yu and Wu constructed a free field realization of \( \hat{W}_{\infty} \) in terms of two scalar fields. To this end, one considers two currents, \( j(x) \) and \( \bar{j}(x) \), with Poisson brackets

\[ \{ j(x), j(x') \}_2 = 0 , \quad \{ j(x), \bar{j}(x') \}_2 = \delta'(x, x') , \quad \{ \bar{j}(x), j(x') \}_2 = 0 . \] (1.6)

Introducing the generating functional

\[ L = \partial + \sum_{i=0}^{\infty} u_i \partial^{-i-1} = \partial + \bar{j} \frac{1}{\partial - (j + \bar{j})} j , \] (1.7)
one finds the expression of the $u_i$'s in terms of $j$ and $\bar{j}$. The first few explicit expressions are

\[
\begin{align*}
  u_0 &= j\bar{j} , \\
  u_1 &= -j'\bar{j} + j^2\bar{j} + jj^2 , \\
  u_2 &= j''\bar{j} - 3jj'\bar{j} - 2j'j^2 - j\bar{j}j' + 3\bar{j}j + 2j^2j^2 + j\bar{j}j^3 , \\
  u_3 &= -j'''\bar{j} + 4jj''\bar{j} + 3j'^2\bar{j} + 3j'\bar{j}j' + jj\bar{j}j'' - 6j'j^2\bar{j} - 9jj'\bar{j}^2 - 3j^2\bar{j}j' - 3j^2j^3 + j^4 + 3j^3j^2 + 3j^2j^3 + jj^4 .
\end{align*}
\] (1.8)

Yu and Wu have shown that given (1.6) and (1.8), one obtains a realization of $\hat{W}_\infty$ in terms of $j$ and $\bar{j}$, that is the brackets $\{u_i, u_j\}_2$ as calculated from (1.6) and (1.8) form the $\hat{W}_\infty$-algebra.

A reformulation of KP: Spurred by these results, I tried to determine whether the KP hierarchy itself could be written in terms of a hierarchy for the $j$ and $\bar{j}$ fields. Let us first observe that (1.7) gives the simple formula

\[
u_i = j[\left(-\partial + j + \bar{j}\right)^i j] .
\] (1.9)

Using the expression (1.3) for the Hamiltonian, we can generate flows for the $j$ and $\bar{j}$ fields, by using

\[
\mathcal{J}_t = \left(\begin{array}{c} j \\ \bar{j} \end{array}\right) = P_2\nabla_j \int \mathcal{H}_r ,
\] (1.10)

where $\nabla_j = (\delta/\delta j, \delta/\delta \bar{j})$ and $P_2$ is the Hamiltonian structure corresponding to (1.6),

\[
P_2 = \frac{j}{\bar{j}} \left(\begin{array}{cc} 0 & \partial \\ \partial & 0 \end{array}\right) .
\] (1.11)

The first few flows with their Hamiltonian densities are given by

\[
r = 1 : \mathcal{H}_1 = j\bar{j} , \\
  j, t_1 = j' , \\
  \bar{j}, t_1 = \bar{j}' .
\] (1.12)

\[
r = 2 : \mathcal{H}_2 = -j'\bar{j} + j^2\bar{j} + jj^2 ,
\]

\[
  j, t_2 = (-j' + j^2 + 2jj)' ,
\]

\[
  \bar{j}, t_2 = ( \bar{j}' + j^2 + 2j\bar{j})' .
\] (1.13)

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that question that we now turn. We already know the Hamiltonian structure of a multi-Hamiltonian structure for that new hierarchy is much less obvious and it is to

\[ \mathcal{H}_3 = j^{''}j - 3jj'j - 2j'j^2 - j\tilde{j}j' + j^3\tilde{j} + 3j^2\tilde{j}^2 + j\tilde{j}^3, \]

\[ j,_{t_3} = (j^{''} - 3jj' + j^3 + 6j^2\tilde{j} + 3j^2j')', \]

\[ \tilde{j},_{t_3} = (j^{''} + 3jj' + j^3 + 6j\tilde{j}^2 + 3j^2\tilde{j}'). \quad (1.14) \]

One can easily check that, for instance, the third flow of the KP hierarchy given in (1.4) is given by computing the \( t_3 \) derivative of the \( u_i \) fields, using the expressions (1.8) and the flows (1.14). The same holds true for the other flows. It would be interesting to find a formula analogous to (1.3) directly in terms of some scalar operator for the fields \( j \) and \( \tilde{j} \).

**The Hamiltonian structures:** the possibility to reformulate the KP hierarchy in terms of \( j \) and \( \tilde{j} \) is fairly clear, given the two boson realization of \( \hat{W}_\infty \). However, the existence of a multi-Hamiltonian structure for that new hierarchy is much less obvious and it is to that question that we now turn. We already know the Hamiltonian structure \( P_2 \) given in (1.11). Ascribing a weight of 1 to \( j, \tilde{j} \) and \( \partial \), and writing down the most general self-adjoint operator of a given weight, one finds by brute force a unique Hamiltonian structure for each weight:

\[
\begin{pmatrix}
\frac{\partial + j - \tilde{j}}{j} \\
\frac{\partial+j-j}{j}
\end{pmatrix}
\]

\[ P_1 = \begin{pmatrix}
(\partial + j - \tilde{j})^{-1}j + & -1 - (\partial + j - \tilde{j})^{-1}j - \\
\frac{1}{j(\partial + j - \tilde{j})^{-1}} & \frac{1}{j(\partial + j - \tilde{j})^{-1}}
\end{pmatrix}, \quad (1.15)
\]

\[ P_2 = \begin{pmatrix}
0 & \partial \\
\partial & 0
\end{pmatrix}, \quad (1.16)
\]

\[ P_3 = \begin{pmatrix}
\partial j + j\partial & -\partial^2 + \partial j + \tilde{j}\partial \\
\partial^2 + j\partial + \tilde{j}\partial & \partial \tilde{j} + \tilde{j}\partial
\end{pmatrix}, \quad (1.17)
\]

\[ P_4 = \begin{pmatrix}
-\partial^2 j + j\partial^2 + 2j\partial j + 2j\tilde{j}\partial j + & \partial^3 - \partial^2 j - \partial j\partial - \tilde{j}\partial^2 - \partial \tilde{j}\partial + \\
\frac{2\partial j\tilde{j} + 2\partial j\partial^{-1} j\partial}{j^2} & \frac{\partial j^2 + j^2\partial + 2\partial j\tilde{j} + 2j\tilde{j}\partial +}{2\partial j\partial^{-1} j\partial} \\
\partial^3 + j\partial^2 + \partial j\partial + \partial^2 \tilde{j} + \partial \tilde{j}\partial + & \partial^2 \tilde{j} - j\partial^2 + 2j\partial \tilde{j} + 2j\tilde{j}\partial + \\
\frac{j^2 \partial + j\partial^2 + 2\partial j\tilde{j} + 2j\tilde{j}\partial +}{2\partial \tilde{j}\partial^{-1} j\partial} & \frac{2\partial \tilde{j}\partial + 2\partial \tilde{j}\partial^{-1} j\partial}{j^2}
\end{pmatrix}. \quad (1.18)
\]

These structures are such that \( \tilde{j},_{t_i} = P_i \nabla j \int \mathcal{H}_{r+2-i}, \quad i = 1, 2, 3, 4. \) As we will see shortly, there exists an infinite number of such Hamiltonian structures, although only two of them are independent. In spite of the non-locality of the Hamiltonian structures \( P_1 \) and \( P_4 \), the
corresponding Hamiltonian densities are local, since they are polynomial in the \( u_i \)'s as seen from (1.3), and hence polynomial in the fields \( j \) and \( \bar{j} \), as seen from (1.9).

\( P_2 \) and \( P_3 \) satisfy the Jacobi identities. We observe that \( P_4 = P_3 P^{-1}_2 P_3 \). Also, \( P_3 = P_2 P^{-1}_1 P_2 \) (which, given the complicated form of \( P_1 \) and \( P_3 \), is most easily checked in the form \( P_2^{-1} P_1 P_2^{-1} P_3 = 1 \)). Hence, just as in the KdV case \([12]\), the potential Poisson brackets turn out to be related, and only two of them are independent. The Jacobi identities are satisfied for all the \( P_i \)'s, since as we mentioned they are satisfied by \( P_2 \) and \( P_3 \).

Note that there is no room for an independent third Hamiltonian structure, since the \( P_i \)'s given in (1.15)–(1.18) are uniquely defined.

Let us now see what Hamiltonian structures the \( P_i \)'s correspond to in terms of the original KP fields \( u_0, u_1, u_2, \ldots \) With respect to the first structure \( P_1 \), we find

\[
\begin{align*}
\{u_0, u_0\}_1 &= 0, \quad \{u_1, u_0\}_1 = u_0 \delta' , \quad \{u_1, u_1\}_1 = 2u_1 \delta' + u_1^' \delta , \\
\{u_0, u_2\}_1 &= u_0 \delta'' + 2(u_1 + u_0') \delta' + (2u_1 + u_0')' \delta .
\end{align*}
\]

(1.19)

With the field redefinitions

\[
W_1 = u_0 , \quad W_2 = u_1 + \frac{1}{2} u_0' , \quad W_3 = u_2 + u_1' + \text{terms in } u_0 ,
\]

(1.20)

these become

\[
\begin{align*}
\{W_1, W_1\}_1 &= 0, \quad \{W_2, W_1\}_1 = W_1 \delta' , \\
\{W_2, W_2\}_1 &= 2W_2 \delta' + W_2' \delta , \quad \{W_1, W_3\}_1 = 2W_2 \delta' + 2W_2' \delta .
\end{align*}
\]

(1.21)

which are the first few brackets of \( W_{1+\infty} \) with central charge \( c = 0 \) [3]. This was expected simply by looking at the dimensions, or from the fact that \( W_{1+\infty} \) has already been shown to be the first Hamiltonian structure of KP in its usual formulation. Note that \( W_2 \) is the stress tensor. Alternatively, (1.13) can be thought of as a new realization of \( W_{1+\infty} \).

\( P_2 \) was already shown to give \( \hat{W}_\infty \). In this case, the field redefinitions that yield the \( \hat{W}_\infty \) algebra, as given in [10], are

\[
\begin{align*}
W_2 &= u_0 , \quad W_3 = u_1 + \frac{1}{2} u_0' , \quad W_4 = u_2 + u_1' + \frac{1}{2} u_0^2 + \frac{1}{5} u_0'' , \\
W_5 &= u_3 + \frac{3}{2} u_2' + u_1'' + \frac{1}{4} u_0''' + 3u_0 u_1 + \frac{3}{2} u_0 u_0' .
\end{align*}
\]

(1.22)
As for $P_3$, with the same field redefinitions as above (but with the index of the $W$’s shifted as $W_i \rightarrow W_{i+1}$), we find

$$\{W_3, W_3\}_3 = 4W_4\delta' + 2W_4'\delta',$$

$$\{W_3, W_4\}_3 = W_3\delta''' + 2W_3\delta''' + (5W_5 + \frac{3}{2}W_3' + \frac{3}{2}W_3)\delta' + (3W_5 + \frac{3}{2}W_3 + \frac{3}{2}W_3'')\delta',$$

$$\{W_3, W_5\}_3 = \frac{14}{5}W_4\delta'' + \frac{32}{5}W_4'\delta'' + (6W_6 + 3W_4'' - 6W_3W_4)\delta' + (4W_6' - 4W_3W_4 - 2W_3W_4')\delta',$$

$$\{W_4, W_3\}_3 = 3W_4\delta'' + \frac{9}{2}W_4'\delta'' + (6W_6 + \frac{1}{2}W_4'' - 6W_3W_4)\delta' + (3W_6' - \frac{1}{2}W_4'' - 3W_3W_4)'\delta'. \quad (1.23)$$

This is a new Hamiltonian structure for the KP hierarchy.

In the case of the $\text{sl}(N)$ KdV hierarchies, we know that the so-called second Hamiltonian structure corresponds to the classical $W_N$ algebra. The first structure is then obtained by shifting the highest spin field by a constant, that is under a shift of the highest spin field $u_N$, $u_N \rightarrow u_N + \lambda$, the second Hamiltonian structure gets transformed as $\{.,.\}_2 \rightarrow \{.,.\}_2 + \lambda\{.,.\}_1$. It would be interesting to know what the relation between the Hamiltonian structures is, both for the KP and the $\jmath$, $\bar{\jmath}$ hierarchy. Of course we know that it is possible to find field redefinitions so that we can go from $W_{1+\infty}$ to $W_{\infty}$ (see first work of [10]); similarly, one can decouple the spin 2 field in $W_{\infty}$ to obtain an algebra $W_{\infty-2}$ that contains fields of spins 3,4,\ldots This procedure can be repeated $ad$ $infinitum$ to derive an algebra with fields of spins $N$, $N+1$, \ldots The full structure of these algebras is given in [20]. Here $W_{\infty}$ is a non-linear deformation of $W_{\infty}$, so it is natural to expect that the algebra (1.23) is a deformation of $W_{\infty-2}$. The first commutator of (1.23) supports this hypothesis.

It would therefore seem that the KP hierarchy in its usual formulation (1.2) possesses an infinite number of Hamiltonian structures $W_{1+\infty}$, $\hat{W}_{\infty}$, $\hat{W}_{\infty-2}$, \ldots When considered from the point of view of the $\jmath$, $\bar{\jmath}$ formulation however, only two of these structures turn out to be independent, all other structures being generated by a recursion operator $(P_{i+1}P_i^{-1})$. Also, it is not clear which of $P_1$, $P_2$ or $P_2$, $P_3$ should be taken as the fundamental Hamiltonian structures. $P_1$, $P_2$ correspond to the most natural structures from the point of view of $W$-algebras, whereas the pair $P_2$, $P_3$ has the advantage of being local. Let
us note however that this locality of $P_2$, $P_3$ is linked to the choice of our starting point, namely the choice of $j$ and $\bar{j}$ as fundamental fields. In terms of a $\psi$, $\bar{\psi}$ hierarchy of the type mentioned in the introduction, we expect the Hamiltonian structures corresponding to $W_{1+\infty}$ and $\tilde{W}_{\infty}$ to be the local ones.

We see that we have been able to reformulate the KP hierarchy in terms of a very simple one; the relation between the KP fields and the fields $j$ and $\bar{j}$ takes the form (1.3), and the Hamiltonian structures of KP are summarized by $P_2$ and $P_3$.

2. Towards a zero-curvature formulation.

The KP hierarchy is usually written in the form (1.2), i.e. in terms of a scalar pseudo-differential operator, but it is possible to write it in terms of $\text{sl}(\infty)$ matrices [21]. In terms of $j$ and $\bar{j}$, the matrix formulation should require only $2 \times 2$ matrices.

To find this matrix formulation, recall that in the formalism of Drinfeld and Sokolov, the relation between the scalar Lax operator $L$ and the matrix operator $\partial + Q$ is found for instance by writing $\Psi^T = (\psi_1, \psi_2, \ldots, \psi_N)$. Then in the $N$ equations

$$ (\partial + Q)\Psi = 0, $$

one eliminates $\psi_2, \ldots, \psi_N$ in terms of $\psi_1$. The resulting equation is of the form $L\psi_1 = 0$, where $L$ is the scalar Lax operator of the $\text{SL}(N)$ KdV hierarchy. Assuming the same holds true here, the most natural matrix operator is

$$ Q = \begin{pmatrix} 0 & \bar{j} \\ -j & -(j + \bar{j}) \end{pmatrix} $$

so that $(\partial + Q)\Psi = 0$ is seen to reduce to $L\psi_1 = 0$, where $L$ corresponds to (1.7). With the form (2.2) of $Q$ and imposing that the zero-curvature condition

$$ [\partial_x + Q, \partial_{t_r} + H_r] = 0 $$

(2.3)

gives the $r^{th}$ flow of the $j$, $\bar{j}$ hierarchy, one finds that the Hamiltonians $H_r$ are uniquely defined, and are explicitly given by

$$ H_1 = \begin{pmatrix} 0 & \bar{j} \\ -j & -(j + \bar{j}) \end{pmatrix} $$

(2.4)

$$ H_2 = \begin{pmatrix} -j\bar{j} & \bar{j}^2 + j\bar{j} + \bar{j}^2 \\ j' - j\bar{j} - j^2 & j' - j\bar{j} - 3j\bar{j} - j^2 - \bar{j}^2 \end{pmatrix} $$

(2.5)
whereas for $H_3$, the matrix elements are given by

\begin{align*}
(H_3)_{11} &= j'' - j'j'' - 2j\bar{j} - 2j\bar{j}^2, \\
(H_3)_{12} &= j'' + j'\bar{j} + 2j\bar{j}' + 3j\bar{j}' + j^2\bar{j} + 4j\bar{j}^2 + j^3, \\
(H_3)_{21} &= -j'' + j\bar{j}' + 2j'\bar{j} + 3j\bar{j}' - j\bar{j}^2 - 4j\bar{j} - j^3, \\
(H_3)_{22} &= (H_3)_{21} - (H_3)_{12} + (H_3)_{11}. \quad (2.6)
\end{align*}

Let us stress that (2.4)–(2.6) were found by requiring (2.3) to reproduce the flows (1.13)–(1.14), for lack of a guiding principle. The existence and uniqueness of $H$ shows it should be possible to derive the flows in a self-consistent way from (2.3) alone. To that end, one would like to find an equivalent to the expansion in a spectral parameter which is used in the zero-curvature formulation of KdV flows; alternatively, one could start from the self-dual Yang-Mills equations in four dimensions, reduced by two Killing symmetries (see [16] [22] for details). Starting in a space with metric $ds^2 = 2dx\,dy + 2dz\,dt$, the self-duality conditions on the field strength $F$ of Yang-Mills theory, $F = *F$, become

\begin{align*}
F_{xt} &= 0, \quad F_{xy} = F_{zt}, \quad F_{yz} = 0. \quad (2.7)
\end{align*}

After reducing these equations with respect to $y$ and $z$, and making the identification $A_x = Q$, $A_t = H_r$, one finds, after imposing for instance the equations of the second flow (1.13), that $A_y$ and $A_z$ are not uniquely defined, but rather depend on 4 numerical factors. This indicates that it should be possible to obtain the KP hierarchy in the $j, \bar{j}$ formulation as a reduction of the self-dual Yang-Mills equations; the Ansatz to be imposed on the connections is not clear yet. In all likelihood, the right Ansatz for $Q$ is not the one given in (2.2), but it must somehow be related to that form.

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