EMBEDDING PUNCTURED $n$-MANIFOLDS IN EUCLIDEAN $(2n-1)$-SPACE

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ABSTRACT. Let $N$ be a closed orientable connected $n$-manifold, $n \geq 4$. We classify embeddings of the punctured manifold $N_0$ into $\mathbb{R}^{2n-1}$ up to isotopy. Our result in some sense extends results of J.C. Becker – H.H. Glover (1971) and O. Saeki (1999).

1. INTRODUCTION AND MAIN RESULTS

This paper is on the classical Knotting Problem: for a manifold $N$ and a number $m$ describe the set $\text{Emb}^m(N)$ of isotopy classes of embeddings $N \to \mathbb{R}^m$. For recent surveys, see [Sk06, HCEC].

We classify embeddings of the punctured $n$-manifold $N_0$ into $\mathbb{R}^{2n-1}$ up to isotopy.

Unless otherwise stated, we work in the PL (piecewise linear) or DIFF (smooth) category and the results are valid in both categories. For a manifold $X$ we denote by $\text{Emb}^sX$ the set of isotopy classes of embeddings $X \to \mathbb{R}^s$, and by $X_0$ we denote $X$ minus the interior of a codimension 0 open ball.

Let $N$ be a closed orientable connected $n$-manifold. For $n \geq 6$ the set $\text{Emb}^{2n-1}N$ was described in [Ya84], see also [Sk10]. For $n = 3$ and $H_1(N)$ torsion-free, $\text{Emb}^{2n-1}N_0 = \text{Emb}^5N_0$ was described by Saeki [Sa99], see below. The method of [Sa99] used classification of normal bundles of embeddings $N_0 \to \mathbb{R}^5$ and could not be directly generalized on higher dimensions. Our main result is Theorem 1 below which implies a description of $\text{Emb}^{2n-1}N_0$ for $n \geq 4$, see Corollary 1.

If (co)homology coefficients are omitted, they are assumed to be $\mathbb{Z}$. We denote $\mathbb{Z}_{(n)} := \mathbb{Z}$ when $n$ is even and $\mathbb{Z}_{(n)} := \mathbb{Z}_2$ when $n$ is odd. For an abelian group $G$ denote by $B_s(G)$ the group of symmetric (for $s$ even) or antisymmetric (for $s$ odd) elements of $G^\otimes 2 := G \otimes_{\mathbb{Z}} G$.

**Corollary 1.** Let $N$ be a closed connected orientable $n$-manifold, $n \geq 4$. Then, as sets,

$$\text{Emb}^{2n-1}N_0 = A \times H_1(N; \mathbb{Z}_{(n-1)})$$

where $A$ is a quotient set of $H_1(N)^\otimes 2$.

We also present a sketch of proof of the following conjecture, a detailed proof of which will appear in a subsequent version of the paper.

**Conjecture 1.** Let $N$ be a closed connected orientable $n$-manifold, $n \geq 4$. Then, as sets,

$$\text{Emb}^{2n-1}N_0 = B_n(H_1(N)) \times H_1(N; \mathbb{Z}_{(n-1)})$$

Define the cone map

$$\Lambda : \text{Emb}^m(N_0) \to \text{Emb}^{m+1}(N)$$

which adds a cone over $\partial N_0$, see figure 1. This map is well-defined in the PL category and for $m \geq 3n/2 + 1$ in the smooth category.

**Theorem 1.** Let $N$ be a closed homologically $k$-connected orientable $n$-manifold, $n \geq 4k + 4$, $k \geq 0$. Then the cone map $\Lambda : \text{Emb}^{2n-2k-1}N_0 \to \text{Emb}^{2n-2k}N$ is surjective and the preimage $\lambda^{-1}x$ of each element is in 1-1 correspondence with a quotient set of $H_{k+1}(N)^\otimes 2$.

In other words, there is the following exact sequence of sets with an action $a$.

$$H_{k+1}(N)^\otimes 2 \xrightarrow{a} \text{Emb}^{2n-2k-1}N_0 \xrightarrow{\lambda} \text{Emb}^{2n-2k}N \longrightarrow 0.$$

\footnote{The sphere $\partial N_0$ is unknotted in $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$ for $m \geq 3n/2 + 1$, so we can smoothen the cone by changing a neighborhood of the cone’s vertex.}
Corollary 3]). Our main result in Theorem 1 is the estimation of the 'kernel' of the cone map $\Lambda$.

The Becker–Glover Theorem 2. [BG71] Let $N$ be a closed homologically $k$-connected $n$-manifold and $m \geq 3n/2 + 2$. The cone map $\Lambda : \text{Emb}^m(N_0) \to \text{Emb}^{m+1}(N)$ is one-to-one for $m \geq 2n - 2k$ and is surjective for $m = 2n - 2k - 1$.

Analogously to Theorem 1 we could prove that there is an exact sequence with an action $a$

$$B_3(H_1(N)) \xrightarrow{a} \text{Emb}^5N_0 \longrightarrow H_1(N, \mathbb{Z}) \longrightarrow 0.$$ 

in the DIFF category. This result for $H_1(N)$ torsion free is known [Sa99]. Moreover, in this case each stabilizer of the action $a$ is in 1-1 correspondence with $I_N := \{i_a(\mu) := \cdot \cap \cdot \cap a \mid a \in H_1(M)\} \subset B_3(H_1(N))$ where $\mu : H_1(N) \times H_1(N) \times H_1(N) \to \mathbb{Z}$ is the intersection form. In other words, the preimage of $\Lambda$ is in 1-1 correspondence with $H_1(N) \times B_3(H_1(N))/I_N$.

We conjecture there is a geometric construction of the invariant $\text{Emb}^{2n-1}N_0 \to H_1(N)^{\otimes 2}$ implicitly obtained in Corollary 1. If $N$ is a connected sum of several $(S^1 \times S^{n-1})$’s then the pairwise linking coefficients $lk(\ast \times S_i^{n-1}, \ast \times S_j^{n-1})$ provide such an invariant if we take into account that $H_1(N) \cong H_{n-1}(N)$ in this case (the invariant will actually be in $B_n(H_1(N))$). In the general case, even if $H_1(N)$ is free, the linking coefficient is harder to define since $(n - 1)$-cycles of $N_0$ may intersect each other.

Proof of Corollary 1. Corollary 1 follows from Theorem 1(a) and the fact there is a bijection $\text{Emb}^{2n}N \to H_1(N, \mathbb{Z}_{(n-1)})$, the Whitney invariant [EBSR].

2. PROOF OF THEOREM 1

Proof of Theorem 1 for $k = 0$. Denote $m = 2n - 2k$. Consider the following commutative diagram of sets, in which the horizontal maps are bijections.

$$\begin{array}{ccc}
\text{Emb}^m(N) & \xrightarrow{\alpha} & \pi_{eq}^{m-1}(\widetilde{N}) \\
\Lambda \uparrow & & \lambda \uparrow \\
\text{Emb}^{m-1}(N_0) & \xrightarrow{\alpha_0} & \pi_{eq}^{m-2}(\widetilde{N}_0) \xrightarrow{\Sigma} \pi_{eq}^{m-1}(\Sigma\widetilde{N}_0)
\end{array}$$

Here,

- $\widetilde{N}$ is the deleted product of $N$, i.e. $N^2$ minus an open tubular neighborhood of the diagonal, with standard involution;
- $\pi_i^X$ is the set of equivariant maps $X \to S^i$ up to equivariant homotopy.
- maps $\alpha$ and $\alpha_0$ are the Haefliger-Wu invariants [Sk06, 5.2].
- $\Sigma$ is the suspension.
- $\lambda^*$ is induced by an equivariant map $\lambda : \widetilde{N} \to \Sigma\widetilde{N}_0$ defined below.

Construction of $\lambda$. We repeat the construction of [BG71]. Represent

$$\Sigma\widetilde{N}_0 = \frac{\widetilde{N}_0 \times [-1; 1]}{\widetilde{N}_0 \times \{-1\}, \widetilde{N}_0 \times \{1\}}.$$

Figure 1. The cone map $\Lambda$ which adds a cone to an embedding of $N_0$. The surjectivity of $\Lambda$ in Main Theorem 1 was known, see Theorem 2 below (compare [Vr89, Corollary 3.3]).
For $x \in \tilde{N}_0$ set $\lambda(x) := (x, 0)$. We identify $U_\varepsilon(P)$ with the unit ball in $\mathbb{R}^n$, with $P$ corresponding to $0 \in \mathbb{R}^n$. Now set (see figure 2)

$\lambda(x) := ((x_1, v), t - 1)$ for $x = (x_1, tv) \in N_0 \times U_\varepsilon(P)$ where $x_1 \in N_0$; $v \in \partial U_\varepsilon(P)$; $t \in [0, 1]$.

Analogously, for $x = (tv, x_1) \in B^n \times N_0$ set $\lambda(x) := ((x_1, v), 1 - t)$.

**Proof of commutativity of the diagram above.** Consider an embedding $f : N_0 \subset \mathbb{R}^{m-1}$. It induces an equivariant map $f_\ast : \tilde{N}_0 \to S^{m-2}$. By definition of the Haefliger-Wu invariant, $[f_\ast] = \alpha_0[f]$. Next, $Af$ induces an equivariant map $(Af)_\ast : \tilde{N} \to S^{m-1}$, $[(Af)_\ast] = \alpha_1[f]$. The commutativity of the diagram above is equivalent to the following fact: the map $(Af)_\ast$ is equivariantly homotopic to the composition

$$\tilde{N} \xrightarrow{\lambda} \Sigma \tilde{N}_0 \xrightarrow{\Sigma f_\ast} S^{m-1}.$$ 

The maps $(Af)_\ast$ and $(\Sigma f_\ast)\lambda$ coincide on $\tilde{N}_0$, both send the ‘vertical’ component $B^n \times N_0$ of $\tilde{N} \setminus \tilde{N}_0$ to the upper hemisphere of $S^{m-1}$ and the ‘horizontal’ component $N_0 \times B^n$ to the lower hemisphere. Thus $((\Sigma f_\ast)\lambda)(x) \in S^{m-1}$ and $(Af)_\ast(x) \in S^{m-1}$ are not antipodal for each $x \in \tilde{N}$, meaning that $(\Sigma f_\ast)\lambda$ and $(Af)_\ast$ are equivariantly homotopic.

The map $\alpha$ is one-to-one by the Haefliger-Weber theorem [Ha63, We67], [Sk06, 5.2 and 5.4]. The map $\alpha_0$ is one-to-one by the the Haefliger theorem for manifolds with boundary (see [Ha63, 6.4], [Sk02, Theorem 1.1.α∅] for the DIFF case and [Sk02, Theorem 1.3.α∅] for the PL case). Next, $\Sigma$ is one-to-one by the equivariant version of Freudenthal suspension theorem [CF60, Theorem 2.5].

It remains to prove that $\lambda^\ast$ is surjective and each preimage $\lambda^\ast f_0$ is in 1-1 correspondence with a quotient set of $H_{k+1}(N)\otimes\mathbb{Z}$ for each $f_0 \in \pi^{m-1}_{eq}(\tilde{N})$. We will need the following assertion which is proved below.

**Assertion 1.** For a $k$-connected $n$-manifold and the constructed $\lambda$ consider the equivariant cohomology groups $H^\ast_{eq}(\Sigma \tilde{N}_0, \lambda \tilde{N})$, with respect to the trivial action of $\mathbb{Z}_2$ on $\mathbb{Z}$-coefficients. Then

(a) for $i \leq 2k$ we get $H^i_{eq}(\Sigma \tilde{N}_0, \lambda \tilde{N}) = 0$;

(b) $H^{2n-2k-1}_{eq}(\Sigma \tilde{N}_0, \lambda \tilde{N}) \cong H_{k+1}(N)\otimes\mathbb{Z}$.

There is a 1-1 correspondence between $\pi^{m-1}_{eq} \tilde{N}$ and $\pi^{m-1}_{eq} \lambda \tilde{N}$ since $\lambda$ is not injective only on some cells of dimension $n < m - 2$. We will thus work with $\pi^{m-1}_{eq} \lambda \tilde{N}$ and $\pi^{m-1}_{eq} \tilde{N}$ interchangeably. Take an equivariant map $f_0 : \lambda \tilde{N} \to S^{m-1}$. It can be extended to an equivariant map $f_1 : \Sigma \tilde{N}_0 \to S^{m-1}$ since by Assertion 1(a),

$$H^i_{eq}(\Sigma \tilde{N}_0, \lambda \tilde{N}; \pi_{i-1} S^{m-1}) = 0$$

for each $i$.\footnote{Square brackets denote a natural class of equivalence which is clear from context. Here these equivalences are: the existence of an equivariant homotopy between two equivariant maps and of an isotopy between two embeddings.}

\footnote{The cone maps from $\text{Emb}^{2m-1} N_0$ and from the set of individual embeddings are both denoted by $\Lambda$.}

\footnote{The Haefliger-Weber theorem says that the Haefliger-Wu invariant $\alpha : \text{Emb}^m N \to \pi^{m-1}_{eq}(N)$ is one-to-one for $2m \geq 3n + 4$ and the Haefliger theorem for manifolds with boundary says that $\alpha$ is one-to-one if $N$ has $(n - d - 1)$-dimensional spine for $2m \geq 3n + 1 - d$ in the DIFF category and for $2m \geq 3n + 2 - d$ in the PL category, $d \geq 0$.}
This proves that \( \lambda^* \) is surjective.

Fix an extension \( f_1 : \Sigma \widetilde{N}_0 \to S^{m-1} \) of \( f_0 \). Denote by \( \pi_{eq}^{m-1}(\Sigma \widetilde{N}_0, \lambda \widetilde{N}) \) the set of equivariant extensions of \( f_0 \) on \( \Sigma \widetilde{N}_0 \) up to equivariant homotopy fixed on \( \lambda \widetilde{N} \). Consider the following diagram.

\[
\begin{array}{ccc}
  0 & \to & \pi_{eq}^{m-1}(\widetilde{N}) \\
  \uparrow & & \uparrow \\
  \pi_{eq}^{m-1}(\Sigma \widetilde{N}_0) & \xrightarrow{\lambda^*} & \pi_{eq}^{m-1}(\Sigma \widetilde{N}_0, \lambda \widetilde{N}) \\
  \downarrow j & & \downarrow \\
  \pi_{eq}^{m-1}(\Sigma \widetilde{N}_0, \lambda \widetilde{N}) & \xrightarrow{d_2} & H_{eq}^{m-1}(\Sigma \widetilde{N}_0, \lambda \widetilde{N})
\end{array}
\]

Here

- \( j \) is the natural map;
- \( d_2 \) is the ‘degree’ map, well-defined and bijective by the relative equivariant version of the Hopf–Whitney Theorem since \( H^r_{eq}(\Sigma \widetilde{N}_0, \lambda \widetilde{N}) = 0 \) for \( r \geq 2m \) by Assertion 1(a).

We obtain \( \lambda^* f_0 = \text{Im } j \) is in 1-1 correspondence with a quotient of \( \pi_{eq}^{m-1}(\Sigma \widetilde{N}_0, \lambda \widetilde{N}) \). Main Theorem 2(a) now follows from Assertion 1(b).

**Proof of Assertion 1(a),(b).** Let us introduce some notation. Take \( P \in N \setminus N_0 \). We denote by \( \begin{array}{c} \square \end{array}, \begin{array}{c} \blacksquare \end{array} \) the following submanifolds of \( N \times N \), respectively: \( N_0 \times \{ P \}, \{ P \} \times N_0, \{ P \} \times N_0 \cup \text{diag } N_0 \), where diag \( N_0 \) is the diagonal embedding. Let \( UZ \) denote a regular neighborhood of an embedded \( Z \subset \Sigma \widetilde{N}_0 \). Let \( \text{con}_{+}(\partial U \begin{array}{c} \square \end{array}) \subset \Sigma \widetilde{N}_0 \) be the upper cone over \( \partial U \begin{array}{c} \square \end{array} \), and analogously denote the lower cone.

We obtain the following chain of isomorphisms for each \( i < n - 2 \).

\[
\begin{align*}
H_{eq}^{2n-i}(\Sigma \widetilde{N}_0, \lambda \widetilde{N}_0) & \cong H_{eq}^{2n-i}(\Sigma \widetilde{N}_0, \widetilde{N}_0) \\
& \cong H_{eq}^{2n-i}(\Sigma \widetilde{N}_0, \widetilde{N}_0 \cup \text{con}_+(\partial U \begin{array}{c} \square \end{array}) \cup \text{con}_-(\partial U \begin{array}{c} \blacksquare \end{array})) \\
& \cong H_{eq}^{2n-i}(\Sigma \widetilde{N}_0, \widetilde{N}_0 \cup \text{con}_+(\partial U \begin{array}{c} \square \end{array}) \cup \text{con}_-(\partial U \begin{array}{c} \blacksquare \end{array})) \\
& \cong H_{eq}^{2n-i-1}(\widetilde{N}_0 \cup \widetilde{N}_0, \partial U \begin{array}{c} \square \end{array} \cup \partial U \begin{array}{c} \blacksquare \end{array}) \\
& \cong H_{eq}^{2n-i-1}(N_0 \cup N_0, \partial U \begin{array}{c} \square \end{array} \cup \partial U \begin{array}{c} \blacksquare \end{array}) \cong \cdots
\end{align*}
\]

As follows from the definition of \( \lambda \)

By equivariant homeomorphism between the two pairs; the induced involution \( g' \) on \( (\Sigma \widetilde{N}_0) \lor (\Sigma \widetilde{N}_0) \) is described below

Desuspension isomorphism

Because \( g' \) is a diffeomorphism from one component of \( (\Sigma \widetilde{N}_0) \lor (\Sigma \widetilde{N}_0) \) onto another

Excision

By exact sequence of pair, \( 2n - i - 2 > n \)

Poincaré duality

Excision

This pair has homological type of the smash product, see below

By Künneth formula
The induced involution $g'$ on $(\Sigma \tilde{N}_0) \vee (\Sigma \tilde{N}_0)$ is the composition of the map changing the two components of $(\Sigma \tilde{N}_0) \vee (\Sigma \tilde{N}_0)$ and of the involution $g$ on $\Sigma \tilde{N}_0$ applied componentwise.

Here $N \cap N := N^2/\langle N \cap N \rangle$, where $N \cap N \subset N^2$ is given by the vertical and horizontal embeddings. The isomorphism $H_{i+1}(N^2, \mathbb{Z}) \cong H_{i+1}(N \cap N)$ is implied by the following easy fact. The map $H_{i+1}N \to H_{i+1}N^2$ induced by diagonal embedding coincides with the composition

$$H_{i+1}N \xrightarrow{id \oplus id} H_{i+1}N \oplus H_{i+1}N \xrightarrow{v \oplus h} H_{i+1}N^2$$

where $v, h$ are induced by vertical and horizontal embeddings, respectively. ■

Proof of Conjecture 1 (sketch). We continue from the place where the proof of Theorem 1 ended. Inside this proof we set $k = 0$. The diagram from above can be completed up to the following commutative diagram.

Here the new map $d_1$ is the ‘degree’ map (i.e. the first obstruction for a given map to be equivariantly homotopic to $f_1$), well-defined and bijective by the equivariant version of the Hopf–Whitney Theorem [Pr06, p. 103 Theorem 10.5] since $H_r^e(\Sigma \tilde{N}_0) = 0$ for $r \geq 2m$ [Sk10, Deleted Product Lemma].

The preimage $\lambda^{*-1} f_0 = \text{Im} j$ is in 1-1 correspondence with $\text{Im} j'$, and Conjecture 1 follows from the following Assertion (we need only the case $k = 0$). ■

**Assertion 1 (c).** For a $k$-connected $n$-manifold $N$ and the constructed $\lambda$ consider the equivariant cohomology groups $H^*_e(\Sigma \tilde{N}_0, \lambda \tilde{N})$, $H^*_e(\Sigma \tilde{N}_0)$ with respect to the trivial action of $\mathbb{Z}_2$ on $\mathbb{Z}$-coefficients. Then the image of $j'$: $H^{2n-2k-1}_e(\Sigma \tilde{N}_0, \lambda \tilde{N}) \to H^{2n-2k-1}_e(\Sigma \tilde{N}_0)$ is isomorphic to $B_{n+k}(H_{k+1}(N))$.

**Proof of Assertion 1(c) (sketch).** We continue from the place where the proof of Assertion 1(a),(b) ended. Let $\phi : H_{k+1}(N) \otimes^2 \to H^{2n-2k-1}_e(\Sigma \tilde{N}_0, \lambda \tilde{N})$ be the inverse to the composition the long chain of isomorphisms above. Then $\phi = l \sigma m$ on the following commutative diagram.

Here $m, \sigma, l$ are the isomorphisms from the chain of isomorphisms above, horizontal maps are natural $\sigma$ and $l$ are the non-relative analogues of $\sigma$ and $l$. Let $g : N^2 \to N^2$ be the standard involution. By $g$ we will also denote induced maps in (co)homology groups induced by this involution. Clearly,
σ(x) = Σx ∨ Σgx The analogous formula holds for ̄σ. We also get ̄l(x ∨ y) = x + (−1)^{n+1}y since the map g has degree (−1)^n.

Let us show that r is monomorphic. Indeed, Ker r is the image of H^{2n−2k−1}(N̄_0) under the map from exact sequence of pair. But H^{2n−2k−1}(N̄_0) ∼= H_{2k+1}(N^2, [N]) ∼= 0 where [N] = N_0 × {P} ⊔ {P} × N_0 ⊔ diag N_0 and the last isomorphism is analogous to the isomorphism H_{2k+1}(N^2, [N]) ∼= 0 proved above.

Suppose z ∈ H^k+1(N)^{⊗2}. Then z ∈ Ker φ if and only if Σm(z + (−1)^{n+1}gz) = 0. Since r is monomorphic, this is equivalent to z = (−1)^ngz. Finally, for a, b ∈ H^k+1(N)^{⊗2} we get g(a ⊗ b) = (−1)^k+1(b ⊗ a), so z = (−1)^ngz if and only if z ∈ B_{n+k+1}(H^{k+1}(N)). Thus Ker j′ = B_{n+k+1}(H^{k+1}(N)) and so Im j′ = B_{n+k}(H^{k+1}(N)). Part (b) of Assertion 1 is proved.

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References

[Ba75] D. R. Bausum. Embeddings and immersions of manifolds in Euclidean space, Trans. Amer. Math. Soc. 213, 263–303 (1975).
[BG71] J.C. Becker, H.H. Glover. Note on the embedding of manifolds in euclidean space. Proc. AMS, Vol. 27 No. 2 (1971).
[CF60] P.E. Conner, E.E. Floyd. Fixed points free involutions and equivariant maps, Bull. AMS 66, 416–441 (1960).
[EBSR] http://www.map.him.uni-bonn.de/index.php/Embeddings just below the stable range: classification (Manifold Atlas).
[Ha63] A. Haefliger. Plongements différentiables dans le domaine stable, Comment. Math. Helv. 36, 155–176 (1963).
[HCEC] http://www.map.him.uni-bonn.de/index.php/High codimension embeddings: classification (Manifold Atlas).
[Pr06] V. Prasolov. Elements of homology theory (in Russian), Moscow, MCCME (2006).
[Sa99] O. Saeki. On punctured 3-manifolds in 5-sphere, Hiroshima Math. J. 29, 255–272 (1999).
[Sk02] A. Skopenkov. On the Haefliger-Hirsch-Wu invariants for embeddings and immersions, Comment. Math. Helv. 77, 78–124 (2002).
[Sk06] A. Skopenkov. Embedding and knotting of manifolds in Euclidean spaces, in: Surveys in Contemporary Mathematics, Ed. N. Young and Y. Choi, London Math. Soc. Lect. Notes, 347 (2008) 248–342. Available at the arXiv:math/0604045v1.
[Sk10] A. Skopenkov. Embeddings of k-connected n-manifolds into $\mathbb{R}^{2n−k−1}$, to appear in Proc. AMS (2010). Available at the arXiv:0812.0263.
[We67] C. Weber. Plongements de polyèdres dans le domaine metastable, Comment. Math. Helv. 42, 1–27 (1967).
[Ya84] T. Yasui. Enumerating embeddings of n-manifolds in Euclidean (2n − 1)-space, J. Math. Soc. Japan 36:4, 555–57 (1984).

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5Note that σ, ̄σ, l are isomorphisms ̄l is epimorphic. We will now prove r is monomorphic and it will follow that Ker ̄l = Ker j′.