Attractor-repeller pair, Morse decomposition and Lyapunov function for random dynamical systems

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Abstract

In the stability theory of dynamical systems, Lyapunov functions play a fundamental role. In this paper, we study the attractor-repeller pair decomposition and Morse decomposition for compact metric space in the random setting. In contrast to [8], by introducing slightly stronger definitions of random attractor and repeller, we characterize attractor-repeller pair decompositions and Morse decompositions for random dynamical systems through the existence of Lyapunov functions. These characterizations, we think, deserve to be known widely.

Key words: Random dynamical systems; Attractor-repeller pair; Morse set; Morse decomposition; Lyapunov function

1 Introduction and main result

In the stability theory of dynamical systems, Lyapunov functions have been playing a fundamental role ever since first introduced in Lyapunov’s 1892 thesis [11]. The simple idea of linking dynamics and topology by means of functions decreasing along trajectories has subsequently been instrumental in the development of sophisticated tools such as e.g. Morse decompositions and Floer homology. Attractor-repeller pairs and Morse sets are special invariant sets that play an important role in understanding the asymptotic behavior of a topological dynamical system defined on a compact metric state space. A complete treatment of Morse theory for deterministic case can be found in the monograph of Conley [6]. Among interesting results in [6] is a proposition which claims that mutually disjoint invariant sets are an attractor-repeller pair if and only if there

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exists a Lyapunov function for those sets. The result is then developed in Huang [9] as a criterion to detect Morse sets and Morse decomposition once a Lyapunov function for those sets can be constructed.

Random dynamical systems (RDS) arise in the modeling of many phenomena in physics, biology, economics, climatology, etc, and the random effects often reflect intrinsic properties of these phenomena rather than just to compensate for the defects in deterministic models. The history of study of random dynamical systems goes back to Ulam and von Neumann [15], and it has flourished since the 1980s due to the discovery that the solutions of stochastic ordinary differential equations yield a cocycle over a metric dynamical system which models randomness, i.e., a random dynamical system. In developing a comprehensive theory of random dynamical systems, members of the Bremen Group started establishing analogous notions, techniques and results for the stochastic setting. Lyapunov functions for RDS were introduced by Arnold and Schmalfuss [2], and Crauel et al. [8] established Morse decompositions and studied some of their basic properties for RDS.

The present paper contributes to this ongoing process. We study the attractor-repeller pair decomposition and Morse decomposition for compact metric space in the random setting. In contrast to [8], by introducing slightly stronger definitions of random attractor and repeller, we can construct measurable Lyapunov functions for attractor-repeller pair and Morse decomposition for RDS. And moreover we also prove that the existence of continuous Lyapunov functions is also the sufficient condition to conclude that two (or finite) mutually disjoint invariant random compact sets constitute an attractor-repeller pair (or a Morse decomposition) for RDS.

The paper is organized as follows. In Section 2, we recall some basic definitions and results for RDS. In Section 3, we give the definitions of limit set (omega-limit set and alpha-limit set), attractor and repeller for RDS. In Section 4, we study the attractor-repeller pair decomposition on compact metric space in the random setting and characterize it by Lyapunov function. And at last we characterize the Morse decompositions on compact metric space through the Lyapunov function and give a simple example in Section 5.

2 Random dynamical systems

Throughout the paper all assertions about $\omega$ are assumed to hold on a $\theta$ invariant set of full measure unless otherwise stated. First we give the definition of continuous random dynamical systems (cf. Arnold [1]).

**Definition 2.1** Let $X$ be a metric space with a metric $d$. A (continuous) random dynamical system (RDS), shortly denoted by $\varphi$, consists of two ingredients:

- A model of the noise, namely a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$, where $(\Omega, \mathcal{F}, P)$ is a probability space and $(t, \omega) \mapsto \theta_t \omega$ is a measurable flow which leaves $P$ invariant, i.e., $\theta_t P = P$ for all $t \in \mathbb{R}$. For simplicity
also assume that \( \theta \) is ergodic under \( P \), meaning that a \( \theta \)-invariant set has probability 0 or 1.

- A model of the system perturbed by noise, namely a cocycle \( \varphi \) over \( \theta \), i.e. a measurable mapping \( \varphi : \mathbb{R} \times \Omega \times X \rightarrow X, (t, \omega, x) \mapsto \varphi(t, \omega, x) \), such that \( (t, x) \mapsto \varphi(t, \omega, x) \) is continuous for all \( \omega \in \Omega \) and the family \( \varphi(t, \cdot, \cdot) = \varphi(t, \cdot) : X \rightarrow X \) of random self-mappings of \( X \) satisfies the cocycle property:

\[
\varphi(0, \omega) = \text{id}_X, \quad \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \quad \text{for all} \quad t, s \in \mathbb{R}, \omega \in \Omega.
\]

(1)

It follows from (1) that \( \varphi(t, \omega) \) is a homeomorphism of \( X \), and the fact \( \varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega) \) is very useful in the following.

Any mapping from \( \Omega \) into the collection of all subsets of \( X \) is said to be a multifunction (or a set valued mapping) from \( \Omega \) into \( X \). We now give the definition of random set, which is a fundamental concept for RDS.

**Definition 2.2** Let \( X \) be a metric space with a metric \( d \). The multifunction \( \omega \mapsto D(\omega) \) taking values in the closed/compact subsets of \( X \) is said to be a random closed/compact set if the mapping \( \omega \mapsto \text{dist}_X(x, D(\omega)) \) is measurable for any \( x \in X \), where \( \text{dist}_X(x, B) := \inf_{y \in B} d(x, y) \). The multifunction \( \omega \mapsto U(\omega) \) taking values in the open subsets of \( X \) is said to be a random open set if \( \omega \mapsto U^c(\omega) \) is a random closed set, where \( U^c \) denotes the complement of \( U \).

Afterwards, we also call a multifunction \( D(\omega) \) measurable for convenience if the mapping \( \omega \mapsto \text{dist}_X(x, D(\omega)) \) is measurable for any \( x \in X \).

**Definition 2.3** A random set \( D(\omega) \) is said to be forward invariant under the RDS \( \varphi \) if \( \varphi(t, \omega)D(\omega) \subset D(\theta_t \omega) \) for all \( t \geq 0 \); It is said to be backward invariant if \( \varphi(t, \omega)D(\omega) \supset D(\theta_t \omega) \) for all \( t \geq 0 \); It is said to be invariant if \( \varphi(t, \omega)D(\omega) = D(\theta_t \omega) \) for all \( t \in \mathbb{R} \).

Now we enumerate some basic results about random sets in the following proposition, for details the reader can refer to Castaing and Valadier [1], Crauel [7] and Arnold [3] for instance.

**Proposition 2.1** Let \( X \) be a Polish space, then the following assertions hold:

(i) if \( D \) is a random closed set, then so is the closure of \( D^c \);

(ii) if \( D \) is a random open set, then the closure \( \overline{D} \) of \( D \) is a random closed set;

(iii) if \( D \) is a random closed set, then \( \text{int} D \), the interior of \( D \), is a random open set;

(iv) if \( \{D_n, n \in \mathbb{N}\} \) is a sequence of random closed sets and there exists \( n_0 \in \mathbb{N} \) such that \( D_{n_0} \) is a random compact set, then \( \bigcap_{n \in \mathbb{N}} D_n \) is a random compact set;

(v) if \( f : \Omega \times X \rightarrow X \) is a function such that \( f(\cdot, \cdot) \) is continuous for all \( \omega \) and \( f(\cdot, x) \) is measurable for all \( x \), then \( \omega \mapsto f(\omega, D(\omega)) \) is a random compact set provided that \( D(\omega) \) is a random compact set.
3 Limit set, attractor and repeller

Definition 3.1 For any given random set $D(\omega)$, we denote $\Omega_D(\omega)$ the omega-limit set of $D(\omega)$, which is determined as follows:

$$\Omega_D(\omega) := \bigcap_{T \geq 0} \bigcup_{t \geq T} \phi(t, \theta_{-t} \omega) D(\theta_{-t} \omega);$$

and we denote $\alpha_D(\omega)$ the alpha-limit set of $D(\omega)$, which is determined as follows:

$$\alpha_D(\omega) := \bigcap_{T \geq 0} \bigcup_{t \geq T} \phi(-t, \theta_t \omega) D(\theta_t \omega).$$

Definition 3.2 For given two random sets $D(\omega), A(\omega)$, we say $A(\omega)$ (pull-back) attracts (repels) $D(\omega)$ if

$$\lim_{t \to \infty} d(\phi(t, \theta_{-t} \omega) D(\theta_{-t} \omega) | A(\omega)) = 0 \quad (\lim_{t \to -\infty} d(\phi(t, \theta_{-t} \omega) D(\theta_{-t} \omega) | A(\omega)) = 0)$$

holds almost surely, where $d(A | B)$ stands for the Hausdorff semi-metric between two sets $A, B$, i.e. $d(A | B) := \sup_{x \in A} \inf_{y \in B} d(x, y)$.

Remark 3.1 It is well-known that $x \in \Omega_D(\omega)$ if and only if $\exists t_n \to \infty, x_n \in D(\theta_{-t_n} \omega)$ such that $\phi(t_n, \theta_{-t_n} \omega) x_n \to x, n \to \infty$. If a non-void random set $D(\omega)$ is attracted by a random compact set $K(\omega)$, then $\Omega_D(\omega) \neq \emptyset$ almost surely and it is invariant. Moreover, $\Omega_D(\omega)$ pull-back attracts $D(\omega)$.

Definition 3.3 (i) An invariant random compact set $A(\omega)$ is called an (local) attractor if there exists a random closed neighborhood $N(\omega)$ of $A(\omega)$ such that $A(\omega) = \Omega_N(\omega)$. The closed neighborhood $N(\omega)$ is called a fundamental neighborhood of $A(\omega)$.

(ii) An invariant random compact set $R(\omega)$ is called a (local) repeller if there exists a random closed neighborhood $N(\omega)$ of $R(\omega)$ such that $R(\omega) = \alpha_D(\omega)$. The closed neighborhood $N(\omega)$ is called a fundamental neighborhood of $R(\omega)$.

The following definition of basin is adopted in [8, 12].

Definition 3.4 (i) Assume $A(\omega)$ is an attractor with a fundamental neighborhood $N(\omega)$. Then we call

$$B(A)(\omega) := \{ x | \phi(t, \omega) x \in \text{int} N(\theta_t \omega) \text{ for some } t \geq 0 \}$$

the basin of attraction of $A(\omega)$;

(ii) Assume $R(\omega)$ is a repeller with a fundamental neighborhood $N(\omega)$. Then we call

$$B(R)(\omega) := \{ x | \phi(t, \omega) x \in \text{int} N(\theta_t \omega) \text{ for some } t \leq 0 \}$$

the basin of repulsion of $R(\omega)$.
Remark 3.2 The basins of attractor and repeller are well defined, i.e. they do not depend on the choice of their fundamental neighborhoods. The readers can refer to [3, 12] for details.

Lemma 3.1 Assume $N(\omega)$ is a random closed set and an invariant random compact set $A(\omega) \subset \text{int} N(\omega)$ satisfying that $\Omega_N(\omega) = A(\omega)$, then there exists a forward invariant random closed set $\hat{N}(\omega)$ with the same properties as $N(\omega)$.

Proof. Let

$$\hat{N}(\omega) := \bigcup_{t \geq 0} \varphi(t, \theta_{-t}\omega)N(\theta_{-t}\omega),$$

then by Proposition 1.5.1 of [5] we have $\hat{N}(\omega)$ is a universally measurable forward invariant random closed set and $A(\omega) \subset \text{int} \hat{N}(\omega)$ (note that $N(\omega) \subset \hat{N}(\omega)$).

Now we show that $\Omega_{\hat{N}}(\omega) = A(\omega)$.

$$\Omega_{\hat{N}}(\omega) = \bigcap_{T \geq 0} \bigcup_{s \geq T} \varphi(s, \theta_{-s}\omega)\hat{N}(\theta_{-s}\omega)$$

$$= \bigcap_{T \geq 0} \bigcup_{s \geq T} \left[ \varphi(s, \theta_{-s}\omega) \bigcup_{t \geq 0} \varphi(t, \theta_{-t} \circ \theta_{-s}\omega)N(\theta_{-t} \circ \theta_{-s}\omega) \right]$$

$$= \bigcap_{T \geq 0} \bigcup_{s \geq T} \bigcup_{t \geq 0} \varphi(s, \theta_{-s}\omega) \circ \varphi(t, \theta_{-t} \circ \theta_{-s}\omega)N(\theta_{-t} \circ \theta_{-s}\omega)$$

$$= \bigcap_{T \geq 0} \bigcup_{s \geq T} \bigcup_{t \geq 0} \varphi(s+t, \theta_{-s-t}\omega)N(\theta_{-s-t}\omega)$$

$$= \bigcap_{T \geq 0} \bigcup_{s \geq T} \varphi(s, \theta_{-s}\omega)N(\theta_{-s}\omega)$$

$$= \Omega_N(\omega) = A(\omega),$$

where the second “$=$” holds since for any random set $D(\omega)$ we have

$$\bigcup_{t \geq T} \varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) = \bigcup_{t \geq T} \varphi(t, \theta_{-t}\omega)\overline{D(\theta_{-t}\omega)}.$$ 

By Lemma 2.7 in [7], there exists an $\mathcal{F}$-measurable random closed set $\hat{N}(\omega) = \hat{N}(\omega)$ almost surely. This completes the proof of the lemma.

Remark 3.3 By Lemmas 3.1 for a given attractor, we can always choose a forward invariant random closed set as its fundamental neighborhood. Hence from now on when we talk about fundamental neighborhood we mean a forward invariant one; when we say “strong” fundamental neighborhood $N(\omega)$ of $A(\omega)$ we mean that $N(\omega)$ is a forward invariant fundamental neighborhood and it satisfies that $\varphi(t, \omega)x \in \text{int} N(\theta t \omega)$ for arbitrary $x \in N(\omega)$ and $t > 0$. 

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4 Attractor-repeller pair and Lyapunov function

From now on we assume \( X \) is a compact metric space, i.e. we will study the attractor-repeller pair decomposition and Morse decomposition on compact metric space.

In this section, we mainly consider the relation between attractor-repeller pair and Lyapunov function for RDS.

Lemma 4.1 Assume \( \varphi \) is an RDS on a compact metric space \( X \), \( A(\omega) \) is an attractor of \( \varphi \) with a fundamental neighborhood \( N(\omega) \) and the basin of attraction \( B(A)(\omega) \). Then \( R(\omega) := X \setminus B(A)(\omega) \) is a random repeller with the basin of repulsion \( X \setminus A(\omega) \) and a fundamental neighborhood \( X \setminus \text{int} N(\omega) \).

Proof. Since \( N(\omega) \) is a forward invariant random compact set, we have that \( \hat{N}(\omega):=X \setminus \text{int} N(\omega) \) is a backward invariant random compact set (see page 35 of [1]). Denote \( \hat{R}(\omega):=\alpha_{\hat{N}}(\omega) \). Then \( \hat{R}(\omega) \) is a random repeller with a fundamental neighborhood \( \hat{N}(\omega) \). By the definition of alpha-limit set, the facts \( R(\omega) \subset \hat{R}(\omega) \) and the invariance of \( R(\omega) \) we have \( R(\omega) \subset \hat{R}(\omega) \). If there exists \( x_0 \in \hat{R}(\omega) \setminus R(\omega) \), then \( x_0 \in B(A)(\omega) \). Therefore there exists some \( t_0 \geq 0 \) such that \( \varphi(t_0, \omega)x_0 \in \text{int} N(\theta_{t_0}\omega) \). Noting that \( \hat{R}(\omega) \) is an invariant random compact set, we have \( \varphi(t_0, \omega)x_0 \in \hat{R}(\theta_{t_0}\omega) \). This is a contradiction to the fact \( \hat{R}(\omega) \cap \text{int} N(\omega) = \emptyset \) for each \( \omega \). Therefore we have obtained \( R(\omega) = \hat{R}(\omega) \), i.e. \( R(\omega) \) is a random repeller with a fundamental neighborhood \( \hat{N}(\omega) \). We now show \( B(R)(\omega) = X - A(\omega) \).

\[
\begin{align*}
B(R)(\omega) &= \bigcup_{n \in \mathbb{N}} \varphi(n, \theta_{-n}\omega)\hat{N}(\theta_{-n}\omega) \\
&= \lim_{n \to \infty} \varphi(n, \theta_{-n}\omega)\hat{N}(\theta_{-n}\omega) \\
&= \lim_{n \to \infty} \varphi(n, \theta_{-n}\omega)[\text{int} N(\theta_{-n}\omega)]^c \\
&= \lim_{n \to \infty} [\varphi(n, \theta_{-n}\omega)\text{int} N(\theta_{-n}\omega)]^c \\
&= X - A(\omega),
\end{align*}
\]

where the 4th “=” follows from the fact that \( \varphi(n, \omega) \) is a homeomorphism on \( X \). This completes the proof of the lemma. \( \square \)

Now we can give the definition of attractor-repeller pair of \( \varphi \).

Definition 4.1 Assume \( \varphi \) is an RDS on a compact metric space \( X \), \( A(\omega) \) is an attractor of \( \varphi \) with a fundamental neighborhood \( N(\omega) \) and the basin of attraction \( B(A)(\omega) \). Then the random set given by

\[
R(\omega) = X \setminus B(A)(\omega)
\]

is called the repeller corresponding to \( A(\omega) \) with the basin of repulsion \( X \setminus A(\omega) \) and a fundamental neighborhood \( X \setminus \text{int} N(\omega) \). And we call \( (A, R) \) an attractor-repeller pair of \( \varphi \).
Lemma 4.2 Assume $A(\omega)$ is an attractor with a fundamental neighborhood $N(\omega)$ and the basin of attraction $B(A)(\omega)$. Then for arbitrary random closed set $K(\omega) \subset B(A)(\omega)$, there exists $T(K, \omega) \geq 0$ such that

$$\varphi(t, \omega)K(\omega) \subset \text{int}N(\theta_{t}\omega), \ \forall t \geq T(K, \omega).$$

(2)

Proof. For given $K(\omega) \subset B(A)(\omega)$ and $\forall x \in K(\omega)$, there exists a $t(x) \geq 0$ such that

$$\varphi(s, \omega)x \in \text{int}N(\theta_{s}\omega), \ \forall s \geq t(x)$$

by the definition of basin of attraction and the forward invariance of $\text{int}N$ (the forward invariance of $\text{int}N$ follows from the fact that $N$ is forward invariant, see page 35 of [11]). Since $\varphi(s, \omega)$ is a homeomorphism of $X$, there exists an open neighborhood $U(x)$ of $x$ such that

$$\varphi(s, \omega)U(x) \subset \text{int}N(\theta_{s}\omega).$$

(3)

By the compactness of $K(\omega)$, there exists a finite collection of such neighborhoods $\{U_{i}\}_{i=1}^{n}$ which constitutes a finite open covering of $K(\omega)$ such that [3] hold with $U_{i}$ instead of $U(x)$. Denote $t_{i}$ the entrance time of $U_{i}$ into $N$ and let $T(K, \omega) = \max\{t_{i} | i = 1, \ldots, n\}$, then we obtain that [2] holds by the forward invariance of $\text{int}N$. This completes the proof of the lemma.

In contrast to the weak attraction in its basin in [8], our attractor pull-back attracts random closed sets in its basin. See the following lemma. We remark that the similar result also holds for repellers and the proof is completely similar.

Lemma 4.3 Assume $A(\omega)$ is a random attractor and $B(A)(\omega)$ is the corresponding basin of attraction, then for any random closed set $D(\omega) \subset B(A)(\omega)$ we have $A(\omega)$ pull-back attracts $D(\omega)$.

Proof. Assume $N(\omega)$ is a fundamental neighborhood of $A(\omega)$, then by Lemma [12] we know that for any random closed (hence compact, for $X$ being compact) set $D(\omega) \subset B(A)(\omega)$ there exists a $T_{D}(\omega) \geq 0$ such that

$$\varphi(t, \omega)D(\omega) \subset \text{int}N(\theta_{t}\omega), \ \forall t \geq T_{D}(\omega).$$

For arbitrary non-random $k \in \mathbb{N}$, by the measure preserving of $\theta_{t}$ we obtain that

$$\mathbb{P}\{\omega| \varphi(t, \theta_{-t} \circ \theta_{-k}\omega)D(\theta_{-t} \circ \theta_{-k}\omega) \subset N(\theta_{-k}\omega), t \geq T_{D}(\theta_{-k}\omega)\}$$

$$= \mathbb{P}\{\omega| \varphi(t, \theta_{-k}\omega)D(\theta_{-k}\omega) \subset N(\theta_{t} \circ \theta_{-k}\omega), t \geq T_{D}(\theta_{-k}\omega)\}$$

$$= 1.$$

Hence

$$\mathbb{P}\{\omega| \varphi(k, \theta_{-k}\omega) \circ \varphi(t, \theta_{-t} \circ \theta_{-k}\omega)D(\theta_{-t} \circ \theta_{-k}\omega)$$

$$\subset \varphi(k, \theta_{-k}\omega)N(\theta_{-k}\omega), t \geq T_{D}(\theta_{-k}\omega)\} = 1$$
Therefore we have
\[ \bigcup_{t \geq T_D(\theta - k \omega)} \phi(t + k, \theta_{{t-k}\omega})D(\theta_{{t-k}\omega}) \subset \phi(k, \theta_{{k}\omega})N(\theta_{{k}\omega}) \]
almost surely. By the definition of omega-limit sets we then obtain that
\[ \Omega_D(\omega) \subset \phi(k, \theta_{{k}\omega})N(\theta_{{k}\omega}), \forall k \in \mathbb{N} \]
aalmost surely and hence
\[ \Omega_D(\omega) \subset \bigcap_{k \in \mathbb{N}} \phi(k, \theta_{{k}\omega})N(\theta_{{k}\omega}) = \Omega_N(\omega) = A(\omega) \]
almost surely. Where \[ \bigcap_{k \in \mathbb{N}} \phi(k, \theta_{{k}\omega})N(\theta_{{k}\omega}) = \Omega_N(\omega) \] holds because \[ N(\omega) \] is forward invariant, which follows that
\[ \phi(t, \theta_{{t}\omega})N(\theta_{{t}\omega}) \subset \phi(s, \theta_{{s}\omega})N(\theta_{{s}\omega}), \forall t > s. \]
Hence we have \[ A(\omega) \] pull-back attracts \[ D(\omega) \] by Remark 3.1. This completes the proof of the lemma. \[ \square \]

Remark 4.1 With respect to the relation between our definition of attractor and that of [8], it seems that our definition is stronger. But if in their Definition 4.1, the fundamental neighborhood is not exactly the basin of attraction (note that in their definition, basin of attraction is a special fundamental neighborhood) and the basin contains the closure of a fundamental neighborhood, then their definition is equivalent to ours. Besides this, we do not know how to construct a Lyapunov function for their attractor if no further condition is assumed.

Remark 4.2 By Lemma 4.3 we know that an attractor pull-back attracts any random closed sets inside its basin, but it can not pull-back attracts its basin itself, for \[ \Omega_{B(A)}(\omega) = B(A)(\omega) \] by the invariance of \[ B(A)(\omega) \]. Hence given an attractor \[ A(\omega) \] (here to distinguish we call “attractor” in our definition and call “weak attractor” in [8]) and an invariant random open neighborhood \[ U(\omega) \] of \[ A(\omega) \] with the property that \[ A(\omega) \] pull-back attracts any random closed set inside \[ U(\omega) \], then \[ U(\omega) \] must be the basin of attraction of \[ A(\omega) \]. In fact, if we only know that the attractor \[ A(\omega) \] attracts any random closed set inside \[ U(\omega) \] in probability, the result also holds. Since in this case, \[ A(\omega) \] is also a weak attractor defined in [8] and \[ U(\omega) \] is the basin of it, see Lemma 4.2 and Proposition 5.1 of [8]. But when \[ A(\omega) \] is regarded as an attractor, the basin of it should be the same as when it is regarded as a weak attractor, for the basin being unique.

Lemma 4.4 Assume \( (A, R) \) is an attractor-repeller pair of \( \varphi \), then there exists an \( F \times \mathcal{B}(X) \)-measurable Lyapunov function \( L \) for \( (A, R) \) such that:
(i) \( L(\omega, x) = 0 \) when \( x \in A(\omega) \), and \( L(\omega, x) = 1 \) when \( x \in R(\omega) \);
(ii) for \( x \in X \setminus (A(\omega) \cup R(\omega)) \) and \( t > 0 \), \( 1 > L(\omega, x) > L(\theta_{{t}\omega}, \varphi(t, \omega)x) > 0 \).
Proof. The idea of the proof is originated from [3, 2]. Assume $N(\omega)$ is a fundamental neighborhood of $A(\omega)$, and we define the first entrance time of $\varphi(t, \omega)x$ into $N(\theta_t \omega)$ as follows:

$$
\tau(\omega, x) := \begin{cases} 
-\infty, & x \in A(\omega); \\
\inf \{ t \in \mathbb{R} \mid \varphi(t, \omega)x \in N(\theta_t \omega) \}, & x \in X \setminus (A(\omega) \cup R(\omega)); \\
+\infty, & x \in R(\omega).
\end{cases}
$$

(4)

Since $\omega \mapsto d(x, N(\omega))$ is measurable, $x \mapsto d(x, N(\omega))$ is continuous, we have $(\omega, x) \mapsto d(x, N(\omega))$ is measurable. Hence for arbitrary $t \in \mathbb{R}$, $(\omega, x) \mapsto d(\varphi(t, \omega)x, N(\theta_t \omega))$ is measurable. For $\forall a \in \mathbb{R}$, 

$$\{(\omega, x) \mid \tau(\omega, x) \geq a\} = \bigcap_{t < a, t \in \mathbb{Q}} \{ (\omega, x) \mid d(\varphi(t, \omega)x, N(\theta_t \omega)) > 0 \},$$

which verifies that $(\omega, x) \mapsto \tau(\omega, x)$ is measurable.

By the definition of $\tau(\omega, x)$, we have

$$
\tau(\theta_t \omega, \varphi(t, \omega)x) = \inf \{ s \in \mathbb{R} \mid \varphi(s, \theta_t \omega) \circ \varphi(t, \omega)x \in N(\theta_{t+s} \omega) \}
= \inf \{ s \in \mathbb{R} \mid \varphi(t+s, \omega)x \in N(\theta_{t+s} \omega) \}
= \tau(\omega, x) - t.
$$

Define

$$L(\omega, x) = \begin{cases} 
\frac{1}{2} e^{\tau(\omega, x)}, & -\infty \leq \tau(\omega, x) < 0; \\
\frac{1}{2} (1 + \frac{1}{2} \arctan \tau(\omega, x)), & 0 \leq \tau(\omega, x) \leq +\infty.
\end{cases}
$$

Since $\tau(\omega, x)$ is $\mathcal{F} \times \mathcal{B}(X)$-measurable, hence $L(\omega, x)$ is. It is obvious that the so defined $L(\omega, x)$ satisfies (i) of Lemma 4.4 and (ii) follows from the fact $\tau(\theta_t \omega, \varphi(t, \omega)x) = \tau(\omega, x) - t$. This terminates the proof of the lemma.

Lemma 4.5 Assume $(A, R)$ is an attractor-repeller pair of $\varphi$ and there exists a strong fundamental neighborhood $N(\omega)$ of $A(\omega)$, then there exists a Lyapunov function $L$ for $(A, R)$ with properties stated in Lemma 4.4 and that $x \mapsto L(\omega, x)$ is continuous for each $\omega \in \Omega$.

Proof. We only need to prove the continuity of $x \mapsto \tau(\omega, x)$. The proof is completely similar to Proposition 6.6 of [2], which is in turn originated from its deterministic case, see page 71 of [3]. So we omit details here.

To distinguish, we call the Lyapunov function obtained in Lemma 4.4 measurable Lyapunov function and the one in Lemma 4.5 continuous Lyapunov function. In contrast to Lemma 4.5 we have the following result.

Lemma 4.6 Assume $A(\omega), R(\omega)$ are two disjoint invariant random compact sets and $L$ is a continuous Lyapunov function for $(A, R)$ with properties stated in Lemma 4.5. Then $(A, R)$ is an attractor-repeller pair of $\varphi$.
Proof. Denote

\[ M(\omega) := \{ x \mid L(\omega, x) < 1 \}, \]

then it is easy to see that \( R(\omega) = M^c(\omega) \) and hence \( M(\omega) \) is an invariant random open set. For \( \forall 0 < \alpha < 1 \), denote

\[ M_\alpha(\omega) = \{ x \mid L(\omega, x) \leq \alpha \}. \]

Since for any \((x, \omega) \in X \times \Omega\), we have

\[ L(\omega, x) \geq \inf_{t \geq t_0} L(\theta_t \omega, \varphi(t, \omega)x), \quad t \geq 0, \]

and hence \( x \in M_\alpha(\omega) \) implies \( \varphi(t, \omega, x) \in M_\alpha(\theta_t \omega) \), i.e. \( M_\alpha(\omega) \) is a forward invariant random compact set and it is a random neighborhood of \( A(\omega) \). Define

\[ A_\alpha(\omega) := \Omega_{M_\alpha(\omega)} = \bigcap_{T \geq t_0} \bigcup_{t \geq T} \varphi(t, \theta^{-t}\omega)M_\alpha(\theta^{-t}\omega), \]

then by the forward invariance of \( M_\alpha \) we have

\[ A_\alpha(\omega) = \bigcap_{t \geq 0} \varphi(t, \theta^{-t}\omega)M_\alpha(\theta^{-t}\omega). \]

On one hand, we have

\[ A(\omega) = \bigcap_{t \geq 0} \varphi(t, \theta^{-t}\omega)A(\theta^{-t}\omega) \subset \bigcap_{t \geq 0} \varphi(t, \theta^{-t}\omega)M_\alpha(\theta^{-t}\omega) = A_\alpha(\omega). \]

On the other hand we also have \( A_\alpha(\omega) \subset A(\omega) \). In fact, consider

\[ L(\omega) := \sup_{x \in A_\alpha(\omega)} L(\omega, x). \]

If the assertion is false, similar to the argument of Proposition 6.2 in [2], then we have \( L(\cdot) > 0 \) with positive probability and hence

\[ L(\cdot) > L(\theta_t), \quad \forall t > 0 \]

with positive probability, a contradiction to the invariance of \( \mathbb{P} \). Hence we have got that \( A = A_\alpha \). Therefore we obtain that \( A(\omega) \) pull-back attracts \( M_\alpha(\omega) \) (since \( A_\alpha(\omega) \) does so by its definition), i.e. \( A(\omega) \) is an attractor with \( M_\alpha(\omega) \) a fundamental neighborhood. We now only need to show that \( M(\omega) \) is in fact the basin of attraction of \( A(\omega) \), i.e. \( B(A)(\omega) = M(\omega) \).

For any random closed set \( D(\omega) \subset M(\omega) \) and \( \forall \epsilon > 0 \), there exists \( \alpha < 1 \) such that

\[ \mathbb{P}\{\omega \mid D(\omega) \subset M_\alpha(\omega)\} \geq 1 - \epsilon. \quad (5) \]

By the triangle inequality, we have

\[ d(\varphi(t, \omega)D(\omega)|A(\theta_t\omega)) \leq d(\varphi(t, \omega)D(\omega)|\varphi(t, \omega)M_\alpha(\omega)) \]
This together with (5), the facts $A_\alpha$ attracts $M_\alpha$ and $A = A_\alpha$ verifies that

$$P - \lim_{t \to \infty} d(\varphi(t, \cdot)D(\cdot)|A(\theta_t \cdot)) = 0,$$

i.e. $A(\omega)$ attracts $D(\omega)$ in probability.

By the above argument and Lemma 4.2 and Proposition 5.1 of [8], we know that $A(\omega)$ is a weak attractor (defined in [8]) and $M(\omega)$ is the corresponding basin of attraction. Hence by Remark 4.2 we obtain that $M(\omega)$ is also the basin of $A(\omega)$ when $A(\omega)$ is regarded as an attractor (defined in present paper). Therefore $R(\omega) = M^c(\omega)$ is the repeller corresponding to $A(\omega)$. Hence $(A, R)$ is an attractor-repeller pair of $\varphi$. This completes the proof of the lemma. □

By Lemmas 4.5 and 4.6, we obtain the following result.

**Theorem 4.1** Assume $\varphi$ is an RDS on a compact metric space $X$ and $A, R$ are two disjoint invariant random compact sets. Then $(A, R)$ is an attractor-repeller pair with strong fundamental neighborhood if and only if there exists a Lyapunov function $L: \Omega \times X \to [0, 1]$ such that:

(i) $\omega \mapsto L(\omega, x)$ is measurable for each $x \in X$, and $x \mapsto L(\omega, x)$ is continuous for each $\omega \in \Omega$;

(ii) $L(\omega, x) = 0$ when $x \in A(\omega)$, and $L(\omega, x) = 1$ when $x \in R(\omega)$;

(iii) for $x \in X \setminus (A(\omega) \cup R(\omega))$ and $t > 0$, $1 > L(\omega, x) > L(\theta_t \omega, \varphi(t, \omega)x) > 0$.

**5 Morse decomposition and Lyapunov function**

In this section, we mainly consider the relation between Morse decomposition and Lyapunov function for RDS.

First, we give the definition of Morse decomposition for random dynamical systems, which was introduced in [8]. For the deterministic case of Morse decomposition, one can refer to [8].

**Definition 5.1** (Morse decomposition) Let $\varphi$ be an RDS on a compact metric space $X$. Assume that $(A_i, R_i)$ are attractor-repeller pairs of $\varphi$ with

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = X \text{ and } X = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset.$$

Then the family $D = \{M_i\}_{i=1}^n$ of invariant random compact sets of $X$, defined by

$$M_i = A_i \bigcap R_{i-1}, \quad 1 \leq i \leq n$$

is called a Morse decomposition for $\varphi$ on $X$, and each $M_i$ is called Morse set. If $D$ is a Morse decomposition, $M(D)$ is defined to be $\bigcup_{i=1}^n M_i$.

**Remark 5.1** By the definitions of attractor-repeller pair and Morse decomposition, it is easy to see that $\{\emptyset, X\}$ and $\{X, \emptyset\}$ are two trivial attractor-repeller pairs and hence $\{X\}$ is a trivial Morse decomposition for $\varphi$ on $X$. Moreover,
attractor-repeller pair decomposition is a special case of Morse decomposition. That is, if \((A, R)\) is an attractor-repeller pair, then \(\{A_1(= A), M_2(= R)\}\) is a Morse decomposition. Conversely, if \(D = \{M_i\}_{i=1}^n\) is a Morse decomposition, then we can easily obtain attractor-repeller pairs from it. Moreover, Morse decompositions can be coarsened. For example, assume \(D = \{M_i\}_{i=1}^n\) is a Morse decomposition, where \(M_i = A_i \cap R_{i-1}\). Let \(i_j^{k-1} \subset \{i\}_{i=1}^{n-1}\) and denote
\[
\tilde{A}_0 = \emptyset, \quad \tilde{A}_j = A_{i_j}, \quad \tilde{A}_k = X, \quad \text{where} \quad j = 1, \ldots, k - 1,
\]
then we obtain a coarsened Morse decomposition
\[
\tilde{D} = \{\tilde{M}_j\}_{j=1}^k, \quad \text{where} \quad \tilde{M}_j = \tilde{A}_j \cap \tilde{R}_{j-1}.
\]
In particular, when \(k = 2\), the coarsened Morse decomposition \((\tilde{M}_1, \tilde{M}_2)\) is in fact a non-trivial attractor-repeller pair. It is obvious that we have
\[
M(D) \subset M(\tilde{D}).
\]
Clearly \(D = \{X\}\) is the coarsest Morse decomposition (not decomposing at all), but there is no finest Morse decomposition, see Example 2.16 of [13] for a deterministic example.

Similar to the deterministic case, we have the following result about Morse decomposition for RDS.

**Lemma 5.1** Assume \(D = \{M_i\}_{i=1}^n\) is a Morse decomposition for \(\varphi\) on \(X\), then we have
\[
M(D) = \bigcap_{i=0}^n (A_i \cup R_i).
\]

**Proof.** The proof is completely similar to that of Lemma 6 in [9], so we omit the details here. \(\square\)

**Lemma 5.2** Assume \(D = \{M_1, M_2, \ldots, M_n\}\) is a Morse decomposition for \(\varphi\) on \(X\). Then there exists an \(\mathcal{F} \times \mathcal{B}(X)\)-measurable Lyapunov function \(L : \Omega \times X \to [0, 1]\) such that:
(i) \(L\) is constant on each \(M_i\), i.e. for \(\forall x, y \in M_i(\omega)\), \(L(\omega, x) = L(\omega, y) = \alpha_i\), and \(\alpha_i\) is independent of \(\omega\), \(i = 1, \ldots, n\);
(ii) \(\alpha_1 < \alpha_2 < \cdots < \alpha_n\), i.e. \(L(\cdot, M_1(\cdot)) < L(\cdot, M_2(\cdot)) < \cdots < L(\cdot, M_n(\cdot))\);
(iii) for \(x \in X \setminus (\bigcup_{i=1}^n M_i(\omega))\) and \(t > 0\), \(L(\omega, x) > L(\theta_t(\omega), \varphi(t, \omega)x)\).

**Proof.** Assume the Morse decomposition \(D = \{M_i\}_{i=1}^n\) is determined by attractor-repeller pairs \((A_i, R_i)\), \(i = 0, 1, \ldots, n\) and assume \(l_i(\omega, x)\) is the Lyapunov function constructed in Lemma 4.4 for the attractor-repeller pair \((A_i, R_i)\). Let
\[
L(\omega, x) = \sum_{i=0}^n \frac{2l_i(\omega, x)}{3^{i+1}}, \quad (7)
\]
It is obvious that $M_i(\omega), \ 1 \leq i \leq n$, it is easy to see that

$$M_i(\omega) \subset A_j(\omega), \ j \geq i \ 	ext{and} \ M_i(\omega) \subset R_j(\omega), \ j \leq i - 1.$$ 

Hence by the definition of $l_i(\omega, x)$, we have $L(\omega, M_i(\omega)) = \sum_{j=0}^{i-1} \frac{1}{\alpha_j}$, which verifies (i)–(ii) in Lemma 5.2. For $x \in X \setminus M_D(\omega)$, by Lemma 5.1 we know that there exists an $0 \leq i \leq n$ such that $x \notin A_i(\omega) \cup R_i(\omega)$. Therefore we have $l_i(\omega, x) > l_i(\theta_{i\omega}, \varphi(t, \omega)x)$ for $\forall t > 0$, which together with the fact $l_j(\omega, x) \geq l_j(\theta_{i\omega}, \varphi(t, \omega)x)$ for each $0 \leq j \leq n$ verify (iii). □

**Remark 5.2** If for each Morse set $M_i = A_i \cap R_{i-1}$ in Lemma 5.2, $A_i$ has a strong fundamental neighborhood $N_i$, then by Lemma 5.3 and (7) the Lyapunov function obtained in Lemma 5.2 is continuous, i.e. $x \mapsto L(\omega, x) = L(\omega, x)$ is continuous for each $\omega \in \Omega$.

**Lemma 5.3** Let $D = \{M_1, M_2, \ldots, M_n\}$ be a finite collection of mutually disjoint invariant random compact sets and assume there exists a continuous Lyapunov function for D with properties stated in Remark 5.2, then $D$ is a Morse decomposition for $\varphi$ on $X$.

**Proof.** Assume $L(\omega, x)$ is a Lyapunov function for $D$. For definiteness, let $L(\omega, M_i(\omega)) = \alpha_i$. By property (i), (ii) of Lemma 5.2, $\alpha_i$ are non-random constants and $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. Let $A_1 := M_1$. For arbitrary $\alpha_{1,2}$ with $\alpha_1 < \alpha_{1,2} < \alpha_2$, define

$$N_{1,2}(\omega) = \{x| \alpha_1 \leq L(\omega, x) \leq \alpha_{1,2}\}.$$ 

Then completely similar to the proof of Lemma 4.6, we know that $A_1 (= M_1)$ is an attractor with a fundamental neighborhood $N_{1,2}(\omega)$ and the corresponding basin of attraction is

$$B(A_1)(\omega) = \{x| \alpha_1 \leq L(\omega, x) < \alpha_2\}.$$ 

Therefore the repeller $R_1$ corresponding to $A_1$ is

$$R_1(\omega) = \{x| L(\omega, x) \geq \alpha_2\}.$$ 

Hence $M_2, \ldots, M_n \subset R_1$. For $\forall \alpha_{2,3} \in (\alpha_2, \alpha_3)$, define

$$N_{2,3}(\omega) = \{x| \alpha_1 \leq L(\omega, x) \leq \alpha_{2,3}\}.$$ 

It is obvious that $M_1 \cup M_2 \subset N_{2,3}$ and $N_{2,3}$ is a fundamental neighborhood. Assume $A_2$ is the attractor inside $N_{2,3}$, i.e.

$$A_2(\omega) = \bigcap_{t \geq 0} \varphi(t, \theta_{-t\omega})N_{2,3}(\theta_{-t\omega}). \quad (8)$$
Hence we have $M_1 \cup M_2 \subset A_2$. Therefore we have obtained $A_2 \cap R_1 \supset M_2$, next we show that $A_2 \cap R_1 \subset M_2$. Since for any $x \in N_{2,3}(\omega) \setminus (M_1(\omega) \cup M_2(\omega))$ and $\forall t > 0$, we have

$$L(\theta_1 \omega, \varphi(t, \omega)x) < L(\omega, x).$$

Therefore, by the proof of Lemma 4.6, for $\forall \alpha \in (\alpha_2, \alpha_3)$, the forward invariant random compact set

$$N_\alpha(\omega) = \{x| \alpha_1 \leq L(\omega, x) \leq \alpha\}$$

is always a fundamental neighborhood of $A_2(\omega)$. Hence we have

$$A_2(\omega) \subset \bigcap_{n \in \mathbb{N}} N_{\alpha_2 + \frac{1}{n}}(\omega),$$

and similarly we also have

$$R_1(\omega) \subset \bigcap_{n \in \mathbb{N}} \tilde{N}_{\alpha_2 - \frac{1}{n}}(\omega),$$

where

$$N_{\alpha_2 + \frac{1}{n}}(\omega) = \{x| \alpha_1 \leq L(\omega, x) \leq \alpha_2 + \frac{1}{n}\}, \quad \tilde{N}_{\alpha_2 - \frac{1}{n}}(\omega) = \{x| L(\omega, x) \geq \alpha_2 - \frac{1}{n}\}.$$ 

Thus

$$A_2(\omega) \cap R_1(\omega) \subset \left( \bigcap_{n \in \mathbb{N}} N_{\alpha_2 + \frac{1}{n}}(\omega) \right) \cap \left( \bigcap_{n \in \mathbb{N}} \tilde{N}_{\alpha_2 - \frac{1}{n}}(\omega) \right)$$

$$= \{x| L(\omega, x) = \alpha_2\} = M_2(\omega),$$

i.e. we have obtained $A_2 \cap R_1 = M_2$. Then we can obtain $R_2$ from $A_2$, i.e.

$$R_2(\omega) = \{x| L(\omega, x) \geq \alpha_3\}.$$

Similar to the above arguments, let

$$N_{3,4}(\omega) = \{x| \alpha_1 \leq L(\omega, x) \leq \alpha_{3,4}\}, \text{ where } \alpha_{3,4} \in (\alpha_3, \alpha_4),$$

and we immediately obtain $A_3$ similar to $N$. Hence we at once obtain the repeller $R_3$ corresponding to $A_3$. Inductively, we can obtain $A_4, R_4, \ldots, A_{n-1}, R_{n-1}$ in the same way. Let $A_0 = R_n = \emptyset, A_n = R_0 = X$. Therefore we have obtained

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = X$$

and $X = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset$

from $M_i, i = 1, \ldots, n$ satisfying

$$M_i = A_i \cap R_{i-1}, \ 1 \leq i \leq n.$$ 

This shows that $D$ is a Morse decomposition for $\varphi$ on $X$ and hence completes the proof of the lemma.

By Lemmas 5.2, 5.3 and Remark 5.2 we obtain the following theorem.
Theorem 5.1 Assume $\varphi$ is an RDS on a compact metric space $X$ and let $D = \{M_1, M_2, \ldots, M_n\}$ be a finite collection of mutually disjoint invariant random compact sets. Then $D$ is a Morse decomposition for $\varphi$ on $X$ with each $A_i$ having a strong fundamental neighborhood if and only if there exists a Lyapunov function $L : \Omega \times X \to [0, 1]$ such that:

(i) $\omega \mapsto L(\omega, x)$ is measurable for each $x \in X$, and $x \mapsto L(\omega, x)$ is continuous for each $\omega \in \Omega$;

(ii) $L$ is constant on each $M_i$, i.e. for $\forall x, y \in M_i(\omega)$, $L(\omega, x) = L(\omega, y) = \alpha_i$, and $\alpha_i$ is independent of $\omega$, $i = 1, \ldots, n$;

(iii) $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, i.e. $L(\cdot, M_1(\cdot)) < L(\cdot, M_2(\cdot)) < \cdots < L(\cdot, M_n(\cdot))$;

(iv) for $x \in X \setminus \bigcup_{i=1}^n M_i(\omega)$ and $t > 0$, $L(\omega, x) > L(\theta_t \omega, \varphi(t, \omega) x)$.

Remark 5.3 By Lemmas 4.4 and 5.2, we can construct measurable Lyapunov functions for attractor-repeller pairs and Morse decompositions. But to construct continuous Lyapunov functions, by Theorems 4.1 and 5.1 we see that we must find a strong fundamental neighborhood for a given attractor, which is not an easy thing. Note that the construction in [2], which follows from [3], is not applicable when $A$ is not globally attracting. For deterministic case, an invariant compact set $A$ is called an attractor if there exists a fundamental neighborhood $U$ of $A$ such that the omega-limit set of $U$, $\Omega_U = A$, see [6]. This implies that there exists a strong fundamental neighborhood $U$ of $A$ such that $\Omega_U = A$, see Proposition 1.9 on page 409 of [14] for details. But for random case, we do not know whether or not similar result holds. That is, we do not know generally how to construct a strong fundamental neighborhood. Therefore we do not request that the fundamental neighborhood of an attractor be a strong one in Definition 3.3. Note also that the construction of Proposition 1.10 on page 409 of [14] does not hold in the random setting. The main difficulty for these stems from the non-uniformity and the non-autonomy of RDS. This non-uniformity is one of the essential features of RDS.

Now we give a simple example to illustrate our results. The example is borrowed from [8], which is also used in [12].

Example 5.1 Consider the Stratonovich stochastic differential equation (SDE)

$$dX_t = (X_t - X_t^3)dt + (X_t - X_t^3) \circ dW_t$$

(9)

on the interval $[-1, 1]$. To put a stochastic differential equation in the framework of RDS, we model white noise as a metric dynamical system as follows: Let $\Omega$ be the space of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ satisfying that $\omega(0) = 0$, let $\mathcal{F}$ be the Borel sigma-algebra induced by the compact-open topology of $\Omega$, and let $\mathbb{P}$ be the Wiener measure on $(\Omega, \mathcal{F})$, i.e. the distribution on $\mathcal{F}$ of a standard Wiener process with two-sided time. The shift $\theta_t$ is defined by $\theta_t \omega(s) = \omega(t+s) - \omega(t)$. Then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system driving the SDE (9), and $W_t(\omega) = \omega(t)$. See Appendix A.3 of [1] for details.
From p.123 of [10] we know that the RDS $\varphi : \mathbb{R} \times \Omega \times [-1, 1] \mapsto [-1, 1]$ generated by SDE (9) can be expressed by

$$\varphi(t, \omega) x = xe^{t+W_t(\omega)} \quad \left(1 - x^2 + x^2e^{2t+2W_t(\omega)}\right)^{\frac{1}{2}}.$$ 

Hence

$$\varphi(t, \theta - t \omega) x = xe^{t-W_t(\omega)} \quad \left(1 - x^2 + x^2e^{2t-2W_t(\omega)}\right)^{\frac{1}{2}}.$$ 

Consider the interval $N := [1/2, 1]$, then we can easily see that $\Omega_N(\omega) \equiv \{1\}$ by the fact that $\lim_{t \to \infty} W_t = 0$ almost surely. Hence $\{1\}$ is an attractor with a fundamental neighborhood $[1/2, 1]$. And we can obtain a forward invariant fundamental neighborhood of $\{1\}$ as

$$\tilde{N}(\omega) = \bigcup_{t \geq 0} \varphi(t, \theta - t \omega)[1/2, 1].$$

Clearly the basin of attraction of $\{1\}$ is $(0, 1]$ and hence the corresponding repeller to it is $[-1, 0]$. By Lemma 4.4 there exists a Lyapunov function $L(\omega, x)$ for the attractor-repeller pair which is 0 when $x = 1$, is strictly decreasing when $x \in (0, 1)$, is 1 when $x \in [-1, 0]$. Similarly $\{-1\}$ is an attractor with basin of attraction $[-1, 0]$ and the corresponding repeller is $[0, 1]$. Therefore $\{-1, 1\}$ is also an attractor with basin of attraction $[-1, 0] \cup (0, 1]$ and the corresponding repeller is $\{0\}$. If we set $A_0 = \emptyset, A_1 = \{-1\}, A_2 = \{-1, 1\}, A_3 = X$, then the corresponding repellers are $R_0 = X, R_1 = [0, 1], R_2 = \{0\}, R_3 = \emptyset$. Consequently the corresponding Morse sets are $M_1 = \{-1\}, M_2 = \{1\}, M_3 = \{0\}$. Thus by Lemma 5.2 there exists a Lyapunov function for this Morse decomposition. If we initially set $A_0 = \emptyset, A_1 = \{1\}, A_2 = \{-1, 1\}, A_3 = X$, then the corresponding Morse sets are $M_1 = \{1\}, M_2 = \{-1\}, M_3 = \{0\}$ and we can obtain the similar result.

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