Kaon semileptonic vector form factor with $N_f=2+1+1$
Twisted Mass fermions

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Motivation

Precise determination of the CKM matrix element is important to test the SM

Semileptonic decay rate:
\[ \Gamma(K \rightarrow \pi l \nu) \propto \left( |V_{us}| f_+(0) \right)^2 \]

Experimental average
\[ |V_{us}| f_+(0) = 0.2163(5) \]

Extracting \( f_+(0) \) from Lattice QCD allows us to estimate \( |V_{us}| \)
Outline

- Simulation Details
- General strategy
- Plateaux
- Interpolation of $f_+$ and $f_0$ to $q^2=0$
- Chiral and continuum extrapolation of $f(0)$
- Results and CKM unitarity tests
- Summary and Conclusions
Something on the action:

- Wilson Twisted Mass action at maximal twist with \( N_f=2+1+1 \) sea quarks
- Osterwalder-Seiler valence quark action
- Iwasaki gluon action
Simulation Details

Details of the ensembles used in this $N_f = 2+1+1$ analysis

The valence light quark mass is put equal to the sea quark mass

| ensemble | $\beta$ | $V/a^4$ | $a\mu_{sea} = a\mu_t$ | $a\mu_\sigma$ | $a\mu_\delta$ | $N_{c_{fg}}$ | $a\mu_\sigma$ | $a\mu_\delta$ | $a\mu_\epsilon$ |
|----------|--------|---------|-------------------|--------|--------|--------|--------|--------|--------|
| A30.32   | 1.90   | $32^3 \times 64$ | 0.0030 | 0.15 | 0.19 | 150   | 0.0145 | 0.1750 | 0.1750 |
| A40.32   |        |         | 0.0040 |        |        | 150   | 0.0185 | 0.2140 | 0.2140 |
| A50.32   |        |         | 0.0050 |        |        | 90    | 0.0225 | 0.2530 | 0.2530 |
| A40.24   | 1.90   | $24^3 \times 48$ | 0.0040 | 0.15 | 0.19 | 150   | 0.0141 | 0.2920 | 0.2920 |
| A60.24   |        |         | 0.0060 |        |        | 150   | 0.0180 | 0.3500 | 0.3500 |
| A80.24   |        |         | 0.0080 |        |        | 150   | 0.0219 | 0.3510 | 0.3510 |
| A100.24  |        |         | 0.0100 |        |        | 150   |        |        |        |
| B25.32   | 1.95   | $32^3 \times 64$ | 0.0025 | 0.135 | 0.170 | 150   | 0.0141 | 0.1750 | 0.1750 |
| B35.32   |        |         | 0.0035 |        |        | 150   | 0.0180 | 0.2140 | 0.2140 |
| B55.32   |        |         | 0.0055 |        |        | 150   | 0.0219 | 0.2530 | 0.2530 |
| B75.32   |        |         | 0.0075 |        | 75     | 0.0219 | 0.3500 | 0.3500 |
| B85.24   | 1.95   | $24^3 \times 48$ | 0.0085 | 0.135 | 0.170 | 150   | 0.0141 | 0.2920 | 0.2920 |
| D15.48   | 2.10   | $48^3 \times 96$ | 0.0015 | 0.12  | 0.1385 | 60    | 0.0118 | 0.1795 | 0.1795 |
| D20.48   |        |         | 0.0020 |        | 90     | 0.0151 | 0.2120 | 0.2120 |
| D30.48   |        |         | 0.0030 |        | 90     | 0.0184 | 0.2450 | 0.2450 |

Lattice Spacings

- $a(\beta = 1.90) = 0.0885(36)$ fm
- $a(\beta = 1.95) = 0.0815(30)$ fm
- $a(\beta = 2.10) = 0.0619(18)$ fm

Range of the simulated pion masses

| $\beta$ | $L$(fm) | $M_\pi$(MeV) | $M_\pi L$ |
|--------|--------|------------|--------|
| 1.90   | 2.84   | 245.41     | 3.53   |
|        |        | 282.13     | 4.06   |
|        |        | 314.43     | 4.53   |
| 1.90   | 2.13   | 282.13     | 3.05   |
|        |        | 343.68     | 3.71   |
|        |        | 396.04     | 4.27   |
|        |        | 442.99     | 4.78   |
| 1.95   | 2.61   | 238.67     | 3.16   |
|        |        | 280.95     | 3.72   |
|        |        | 350.12     | 4.64   |
|        |        | 408.13     | 5.41   |
| 1.95   | 1.96   | 434.63     | 4.32   |
| 2.10   | 2.97   | 211.18     | 3.19   |
|        |        | 242.80     | 3.66   |
|        |        | 295.55     | 4.46   |

PRA027
"QCD simulations for flavor physics in the Standard Model and beyond" (35 millions of core-hours at the BG/P system in Julich from December 2010 to March 2011)
Simulation Details

Details of the ensembles used in this N_f = 2+1+1 analysis

The valence light quark mass is put equal to the sea quark mass

| ensemble  | β   | V/a^4     | a\mu_{sea} = a\mu_{l} | a\mu_{σ} | a\mu_{δ} | N_{c,f,g} | a\mu_{s}           | a\mu_{c}               |
|-----------|-----|-----------|------------------------|----------|----------|-----------|------------------|------------------------|
| A30.32    | 1.90 | 32^3 × 64 | 0.0030                 | 0.15     | 0.19     | 150       | 0.0145,          | 0.1800, 0.2200,         |
| A40.32    |     |           | 0.0040                 |          |          |           | 0.0185,          | 0.2600, 0.3000,         |
| A50.32    |     |           | 0.0050                 |          |          |           | 0.0225,          | 0.3600, 0.4400          |
| A40.24    | 1.90 | 24^3 × 48 | 0.0040                 | 0.15     | 0.19     | 150       | 0.1750,          | 0.2140,                |
| A60.24    |     |           | 0.0060                 |          |          |           | 0.2180,          | 0.2530, 0.2920,         |
| A80.24    |     |           | 0.0080                 |          |          |           | 0.0219,          | 0.3510, 0.4290          |
| A100.24   |     |           | 0.0100                 |          |          |           | 0.0219,          | 0.3510, 0.4290          |
| B25.32    | 1.95 | 32^3 × 64 | 0.0025                 | 0.135    | 0.170    | 150       | 0.1410,          | 0.1750, 0.2140,         |
| B35.32    |     |           | 0.0035                 |          |          |           | 0.0180,          | 0.2530, 0.2920,         |
| B55.32    |     |           | 0.0055                 |          |          |           | 0.0219,          | 0.3510, 0.4290          |
| B75.32    |     |           | 0.0075                 |          |          | 75        | 0.0219,          | 0.3510, 0.4290          |
| B85.24    | 1.95 | 24^3 × 48 | 0.0085                 | 0.135    | 0.170    | 150       | 0.0118,          | 0.1790, 0.2190,         |
| D15.48    | 2.10 | 48^3 × 96 | 0.0015                 | 0.12     | 0.1385   | 60        | 0.0118,          | 0.1470, 0.1795,         |
| D20.48    |     |           | 0.0020                 |          |          | 90        | 0.0151,          | 0.2120, 0.2450,         |
| D30.48    |     |           | 0.0030                 |          |          | 90        | 0.0184,          | 0.2945, 0.3595          |

Range of the simulated pion masses

| β   | L(fm) | M_π(MeV) | M_πL  |
|-----|-------|----------|-------|
| 1.90| 2.84  | 245.41   | 3.53  |
|     |       | 282.13   | 4.06  |
|     |       | 314.43   | 4.53  |
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| 2.10| 2.97  | 211.18   | 3.19  |
|     |       | 242.80   | 3.66  |
|     |       | 295.55   | 4.46  |

Lattice Spacings

| a(β = 1.90) | 0.0885(36)fm |
| a(β = 1.95) | 0.0815(30)fm |
| a(β = 2.10) | 0.0619(18)fm |

Three different values of the lattice spacing: 0.06 fm ÷ 0.09 fm
Details of the ensembles used in this $N_f = 2+1+1$ analysis

The valence light quark mass is put equal to the sea quark mass

| ensemble  | $\beta$ | $V/a^4$ | $a\mu_{\text{sea}} = a\mu_l$ | $a\mu_\sigma$ | $a\mu_\delta$ | $N_{\text{cfg}}$ | $a\mu_s$ | $a\mu_c$ |
|-----------|---------|---------|-------------------------------|----------------|----------------|-----------------|----------|---------|
| $A30.32$  | 1.90    | $32^3 \times 64$ | 0.0030 | 0.15 | 0.19 | 150 | 0.0145, 0.1800, 0.2200, 0.2600, 0.3000, 0.3600, 0.4400 |
| $A40.32$  | 1.90    | $32^3 \times 64$ | 0.0040 | 0.0050 | 0.19 | 90 | 0.0185, 0.2600, 0.3000, 0.3600, 0.4400 |
| $A50.32$  | 1.90    | $32^3 \times 64$ | 0.0040 | 0.0050 | 0.19 | 150 | 0.0225, 0.3600, 0.4400 |
| $A40.24$  | 1.90    | $24^3 \times 48$ | 0.0040 | 0.0060 | 0.15 | 150 | 0.1800, 0.2200, 0.2600, 0.3000, 0.3600, 0.4400 |
| $A60.24$  | 1.90    | $24^3 \times 48$ | 0.0040 | 0.0060 | 0.15 | 150 | 0.1800, 0.2200, 0.2600, 0.3000, 0.3600, 0.4400 |
| $A80.24$  | 1.90    | $24^3 \times 48$ | 0.0040 | 0.0060 | 0.15 | 150 | 0.1800, 0.2200, 0.2600, 0.3000, 0.3600, 0.4400 |
| $A100.24$ | 1.90    | $24^3 \times 48$ | 0.0040 | 0.0060 | 0.15 | 150 | 0.1800, 0.2200, 0.2600, 0.3000, 0.3600, 0.4400 |
| $B25.32$  | 1.95    | $32^3 \times 64$ | 0.0025 | 0.0035 | 0.0075 | 0.135 | 0.170 | 150 | 0.0141, 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| $B35.32$  | 1.95    | $32^3 \times 64$ | 0.0025 | 0.0035 | 0.0075 | 0.135 | 0.170 | 150 | 0.0141, 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| $B55.32$  | 1.95    | $32^3 \times 64$ | 0.0025 | 0.0035 | 0.0075 | 0.135 | 0.170 | 150 | 0.0141, 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| $B75.32$  | 1.95    | $32^3 \times 64$ | 0.0025 | 0.0035 | 0.0075 | 0.135 | 0.170 | 150 | 0.0141, 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| $B85.24$  | 1.95    | $24^3 \times 48$ | 0.0085 | 0.135 | 0.170 | 150 | 0.0141, 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| $D15.48$  | 2.10    | $48^3 \times 96$ | 0.0015 | 0.12 | 0.1385 | 60 | 0.0118, 0.1470, 0.1795, 0.2120, 0.2450, 0.2945, 0.3595 |
| $D20.48$  | 2.10    | $48^3 \times 96$ | 0.0020 | 0.0030 | 0.1385 | 90 | 0.0151, 0.2450, 0.2945, 0.3595 |
| $D30.48$  | 2.10    | $48^3 \times 96$ | 0.0015 | 0.12 | 0.1385 | 90 | 0.0184, 0.2945, 0.3595 |

Lattice Spacings

| $a(\beta = 1.90)$ | 0.0885(36)fm |
| $a(\beta = 1.95)$ | 0.0815(30)fm |
| $a(\beta = 2.10)$ | 0.0619(18)fm |

Different volumes: $2\text{ fm} \div 3\text{ fm}$

Range of the simulated pion masses

| $\beta$ | $L(\text{fm})$ | $M_\pi(\text{MeV})$ | $M_\pi L$ |
|---------|----------------|-----------------|---------|
| 1.90    | 2.84           | 245.41          | 3.53    |
| 1.90    | 2.13           | 282.13          | 4.06    |
| 1.90    | 2.13           | 314.43          | 4.53    |
| 1.90    | 2.61           | 280.95          | 3.16    |
| 1.90    | 2.61           | 350.12          | 3.72    |
| 1.90    | 2.61           | 408.13          | 4.64    |
| 1.90    | 1.96           | 434.63          | 4.32    |
| 2.10    | 2.97           | 211.18          | 3.19    |
| 2.10    | 2.97           | 242.80          | 3.66    |
| 2.10    | 2.97           | 295.55          | 4.46    |

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The valence light quark mass is put equal to the sea quark mass

| ensemble     | \( \beta \) | \( V/a^4 \) | \( a\mu_{\text{sea}} = a\mu_{\text{t}} \) | \( a\mu_\sigma \) | \( a\mu_\delta \) | \( N_{c_{fg}} \) | \( a\mu_s \) | \( a\mu_c \)       |
|--------------|-------------|-------------|-----------------|-----------------|-----------------|----------------|-----------------|-----------------|
| A30.32       | 1.90        | \( 32^3 \times 64 \) | 0.0030           | 0.15            | 0.19            | 150            | 0.0145,         | 0.1800, 0.2200, |
|              |             |             | 0.0040           |                 |                 | 90             | 0.0185,         | 0.2600, 0.3000, |
|              |             |             | 0.0050           |                 |                 | 150            | 0.0225          | 0.3600, 0.4400  |
| A40.32       |             |             | 0.0040           | 0.15            | 0.19            | 150            | 0.0141,         | 0.1750, 0.2140, |
| A50.32       |             |             | 0.0060           |                 |                 | 150            | 0.0180,         | 0.2530, 0.2920, |
| A40.24       | 1.90        | \( 24^3 \times 48 \) | 0.0025           | 0.135           | 0.170           | 150            | 0.0141,         | 0.1750, 0.2140, |
| A60.24       |             |             | 0.0035           |                 |                 | 150            | 0.0180,         | 0.2530, 0.2920, |
| A80.24       |             |             | 0.0055           |                 |                 | 150            | 0.0219          | 0.3510, 0.4290  |
| A100.24      |             |             | 0.0075           |                 |                 | 75             |                 |                 |
| B25.32       | 1.95        | \( 32^3 \times 64 \) | 0.0085           | 0.135           | 0.170           | 150            | 0.1750,         | 0.1750, 0.2140, |
| B35.32       |             |             | 0.0025           |                 |                 | 150            | 0.2140,         | 0.2530, 0.2920, |
| B55.32       |             |             | 0.0035           |                 |                 | 150            | 0.2920,         | 0.3510, 0.4290  |
| B75.32       |             |             | 0.0075           |                 |                 | 75             |                 |                 |
| B85.24       | 1.95        | \( 24^3 \times 48 \) | 0.0015           | 0.12            | 0.1385          | 60             | 0.0118,         | 0.1470, 0.1795, |
| D15.48       |             |             | 0.0015           |                 |                 | 90             | 0.0151,         | 0.2120, 0.2450, |
| D20.48       | 2.10        | \( 48^3 \times 96 \) | 0.0020           | 0.12            | 0.1385          | 90             | 0.0184          | 0.2945, 0.3595  |
| D30.48       |             |             | 0.0030           |                 |                 | 90             |                 |                 |

#### Range of the simulated pion masses

| \( \beta \) | \( L (\text{fm}) \) | \( M_\pi (\text{MeV}) \) | \( M_\pi L \) |
|--------------|------------------|------------------|---------------|
| 1.90         | 2.84             | 245.41           | 3.53          |
|              | 3.53             | 282.13           | 4.06          |
|              |                  | 314.43           | 4.53          |
| 1.90         | 2.13             | 282.13           | 3.05          |
|              |                  | 343.68           | 4.06          |
|              |                  | 396.04           | 4.27          |
|              |                  | 442.99           | 4.78          |
| 1.95         | 2.61             | 238.67           | 3.16          |
|              |                  | 280.95           | 3.72          |
|              |                  | 350.12           | 4.64          |
|              |                  | 408.13           | 5.41          |
| 1.95         | 1.96             | 434.63           | 4.32          |
| 2.10         | 2.97             | 211.18           | 3.19          |
|              |                  | 242.80           | 3.66          |
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#### Lattice Spacings

| \( a(\beta = 1.90) \) | 0.0885(36)fm |
|-----------------------|--------------|
| \( a(\beta = 1.95) \) | 0.0815(30)fm |
| \( a(\beta = 2.10) \) | 0.0619(18)fm |

Pion masses in range 210 ÷ 440 MeV
Details of the ensembles used in this $N_f = 2+1+1$ analysis

The valence light quark mass is put equal to the sea quark mass

| ensemble   | $\beta$ | $V/a^4$ | $a\mu_{sea} = a\mu_l$ | $a\mu_\sigma$ | $N_{cfg}$ | $a\mu_s$   | $a\mu_c$       |
|------------|--------|---------|------------------------|---------------|-----------|------------|----------------|
| A30.32     | 1.90   | $32^3 \times 64$ | 0.0030, 0.0040, 0.0050 | 0.15          | 150       | 0.0145, 0.1800, 0.2200, 0.2600, 0.3000, 0.3600, 0.4400 |
| A40.32     | 1.90   | $24^3 \times 48$ | 0.0040, 0.0060, 0.0080, 0.0100 | 0.15          | 150       | 0.0145, 0.1800, 0.2200, 0.2600, 0.3000, 0.3600, 0.4400 |
| A50.32     | 1.90   | $24^3 \times 48$ | 0.0040, 0.0060, 0.0080, 0.0100 | 0.15          | 150       | 0.0145, 0.1800, 0.2200, 0.2600, 0.3000, 0.3600, 0.4400 |
| A40.24     | 1.90   | $32^3 \times 64$ | 0.0055, 0.0075 | 0.135          | 150       | 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| B25.32     | 1.95   | $32^3 \times 64$ | 0.0025, 0.0035, 0.0040 | 0.135          | 150       | 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| B35.32     | 1.95   | $32^3 \times 64$ | 0.0025, 0.0035, 0.0040 | 0.135          | 150       | 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| B55.32     | 1.95   | $32^3 \times 64$ | 0.0025, 0.0035, 0.0040 | 0.135          | 150       | 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| B75.32     | 1.95   | $32^3 \times 64$ | 0.0025, 0.0035, 0.0040 | 0.135          | 150       | 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| B85.24     | 1.95   | $24^3 \times 48$ | 0.0085   | 0.135          | 150       | 0.1750, 0.2140, 0.2530, 0.2920, 0.3510, 0.4290 |
| D15.48     | 2.10   | $48^3 \times 96$ | 0.0015, 0.0020, 0.0030 | 0.12          | 60        | 0.1470, 0.1795, 0.2120, 0.2450, 0.2945, 0.3595 |
| D20.48     | 2.10   | $48^3 \times 96$ | 0.0015, 0.0020, 0.0030 | 0.12          | 60        | 0.1470, 0.1795, 0.2120, 0.2450, 0.2945, 0.3595 |
| D30.48     | 2.10   | $48^3 \times 96$ | 0.0015, 0.0020, 0.0030 | 0.12          | 60        | 0.1470, 0.1795, 0.2120, 0.2450, 0.2945, 0.3595 |

To inject momenta we used non-periodic boundary conditions

| $\beta$ | $V/a^4$ | $\theta$ |
|---------|---------|----------|
| 1.90    | $32^3 \times 64$ | 0.0, ±0.400, ±0.933, ±1.733 |
|         | $24^3 \times 48$ | 0.0, ±0.300, ±0.700, ±1.300 |
| 1.95    | $32^3 \times 64$ | 0.0, ±0.366, ±0.854, ±1.588 |
|         | $24^3 \times 48$ | 0.0, ±0.275, ±0.641, ±1.191 |
| 2.10    | $48^3 \times 96$ | 0.0, ±0.424, ±0.986, ±1.832 |
General strategy

\[ \langle \pi (p') | V_\mu | K (p) \rangle = (p_\mu + p'_\mu) f_+ (q^2) + (p_\mu - p'_\mu) f_- (q^2) \]

extract the matrix element \( \langle \pi (p') | V_\mu | K (p) \rangle \) from appropriate ratio of three-points correlation function to build \( f_0(q^2) \) and \( f_+(q^2) \)

- we used the ratio
  \[ R_\mu = \frac{C^K_{\mu} (t, \bar{p}, \bar{p}')}{C^K_{\mu} (t, \bar{p}', \bar{p})} \]

Fit simultaneously \( f_0(q^2) \) and \( f_+(q^2) \) to get \( f_0(0) = f_+(0) \)

- \( z \) expansion
- Polynomial fit

Perform the Chiral and continuum extrapolation of \( f(0) \)

- \( SU(2) \) ChPT
- \( SU(3) \) ChPT
General strategy

The matrix element of the vector current between two PS mesons decomposes into two form factors

$$\langle \pi (p') | V_\mu | K (p) \rangle = (p_\mu + p'_\mu) f_+ (q^2) + (p_\mu - p'_\mu) f_- (q^2)$$

depending on the momentum transfer

$$q^2 = (E - E')^2 - (p_i - p'_i)^2$$

The matrix element can be derived in lattice QCD from a combination of Euclidean three-point functions

We define the ratio:

$$R_\mu = \frac{C_\mu^{K\pi} (t, \bar{p}, \bar{p}')}{C_\mu^{\pi\pi} (t, \bar{p}', \bar{p})} \frac{C_\mu^{\pi K} (t, \bar{p}', \bar{p})}{C_\mu^{KK} (t, \bar{p}, \bar{p})}$$

in which the renormalization $Z_V$ and $Z_K$ and $Z_\pi$ cancels

$$C_\mu^{K\pi} (t_x, t_y, \bar{p}, \bar{p}') \frac{1}{(t_x - t_y)} \Rightarrow \frac{\sqrt{Z_K Z_\pi}}{4E_K E_\pi} \langle \pi (p') | \hat{V}_\mu | K (p) \rangle e^{-E_K t_x - E_\pi (t_x - t_y)}$$

$$\hat{R}_\mu = \frac{\left( \langle \pi (p') | V_\mu | K (p) \rangle \right)^2}{4p_\mu p'_\mu}$$
General strategy

The matrix element of the vector current between two PS mesons decomposes into two form factors

\[
\langle \pi (p') | V_\mu | K (p) \rangle = (p_\mu + p'_\mu) f_+ (q^2) + (p_\mu - p'_\mu) f_- (q^2)
\]

\[
\hat{R}_\mu = \frac{\langle \pi (p') | V_\mu | K (p) \rangle^2}{4 p_\mu p'_\mu}
\]

We can define \( V_0 \) and \( V_1 \) related to the form factors by the relations

\[
V_0 = 2 \sqrt{R_0} \sqrt{EE'} = ((E + E') f_+ (q^2) + (E - E') f_- (q^2))
\]

\[
V_i = 2 \sqrt{R_i} \sqrt{p_i p'_i} = ((p_i + p'_i) f_+ (q^2) + (p_i - p'_i) f_- (q^2))
\]

and resolving the system we obtain:

\[
f_+ (q^2) = \frac{(E - E') V_i - (p_i - p'_i) V_0}{2 E p'_i - 2 E' p_i}
\]

\[
f_- (q^2) = \frac{(p_i + p'_i) V_0 - (E + E') V_i}{2 E p'_i - 2 E' p_i}
\]

\[
f_0 (q^2) = f_+ (q^2) + \frac{q^2}{m_K^2 - m_\pi^2} f_- (q^2)
\]

- \( V_0 \) and \( V_1 \) are extracted from the double ratio of the three-points correlation function
- momenta are fixed by the non-periodic boundary conditions
- energies are extracted from the dispersion relation with the masses obtained fitting the two-points correlation function at rest
example of plateaux of $V_0$ and $V_1$ for all the selected kinematics

$\beta = 1.90$
$\mu^{(sea)} = 0.0050$
$\mu_s = 0.0145$
Plateaux

example of plateaux of the effective mass of a two-points correlation function at zero momentum

The fit intervals for the two-point correlation functions at rest are the one reported in [arXiv:1403.4504]

| β  | V/a^4 | [t_{min}, t_{max}]/(ℓ, ℓ, s)/a |
|----|------|-------------------------------|
| 1.90 | 24^3 × 48 | [12, 23] |
| 1.90 | 32^3 × 64 | [12, 31] |
| 1.95 | 24^3 × 48 | [13, 23] |
| 1.95 | 32^3 × 64 | [13, 31] |
| 2.10 | 48^3 × 96 | [18, 40] |
We fitted simultaneously $f_+(q^2)$ and $f_0(q^2)$ to extract $f(0)$ using the $z$ expansion.*

To interpolate at $q^2=0$ we neglect the points corresponding to large negative $q^2$.

\[
\begin{align*}
    f_+(q^2) &= \frac{a_0 + a_1 \left( z + \frac{1}{2} z^2 \right)}{1 - \frac{q^2}{M_V^2}} \\
    f_0(q^2) &= \frac{b_0 + b_1 \left( z + \frac{1}{2} z^2 \right)}{1 - \frac{q^2}{M_S^2}}
\end{align*}
\]

with

\[
    z = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}
\]

\[
    t_+ = (m_K + m_\pi)^2 \quad t_0 = (m_K + m_\pi) \left( \sqrt{m_K} - \sqrt{m_\pi} \right)^2
\]

* Bourrely Caprini and Lellouch [Phys.Rev. D79 (2009) 013008]
Extracting $f(0)$

We fitted simultaneously $f_+(q^2)$ and $f_0(q^2)$ to extract $f(0)$ using the $z$ expansion to interpolate at $q^2=0$ we neglect the points corresponding to large negative $q^2$.

$$f_+(q^2) = \frac{a_0 + a_1 \left( z + \frac{1}{2} z^2 \right)}{1 - \frac{q^2}{M^2_v}}$$

$$f_0(q^2) = \frac{b_0 + b_1 \left( z + \frac{1}{2} z^2 \right)}{1 - \frac{q^2}{M^2_s}}$$

The fit works well even for a larger range in $q^2$.
We fitted simultaneously $f_{+}(q^{2})$ and $f_{0}(q^{2})$ to extract $f(0)$ using the z expansion to interpolate at $q^{2}=0$ we neglect the points corresponding to large negative $q^{2}$.

$$f_{+}(q^{2}) = a_{0} + a_{1} \left( z + \frac{1}{2} z^{2} \right) \frac{q^{2}}{1-M_{V}^{2}}$$

$$f_{0}(q^{2}) = b_{0} + b_{1} \left( z + \frac{1}{2} z^{2} \right) \frac{q^{2}}{1-M_{s}^{2}}$$

The results obtained with polynomial fit formulae are also compatible

$$f_{+}(q^{2}) = f(0) (1 + P_{1} q^{2} + P_{2} q^{4})$$

$$f_{0}(q^{2}) = f(0) (1 + P_{3} q^{2} + P_{4} q^{4})$$
f+(0) Chiral and continuum extrapolation

Two different approaches for the chiral extrapolation

- SU(2) ChPT
  \[ f_+(0) = F_0^+ \left( 1 - \frac{3}{4} \xi \log \xi + P_2 \xi + P_3 a^2 \right) \]

- SU(3) ChPT (at fixed \( m_s \))
  \[ f_+(0) = 1 + f_2 + \Delta f \]

  \[ f_2^{\text{fullQCD}} = \frac{3}{2} H_{\pi K} + \frac{3}{2} H_{\eta K} \]

  \[ H_{PQ} = -\frac{1}{64\pi^2 f_\pi^2} \left[ M_P^2 + M_Q^2 + \frac{2M_P^2 M_Q^2}{M_P^2 - M_Q^2} \log \frac{M_Q^2}{M_P^2}\right] \]

  \[ \Delta f = (m_s - m_l)^2 \left[ \Delta_0 + \Delta_1 m_l \right] + \Delta_2 a^2 \]
**f+(0): SU(2) Chiral and continuum extrapolation**

**f+(0)**

we performed a small interpolation in the data to arrive at $m_{s \text{phys}}$

**SU(2) Chiral fit**

![Graph showing SU(2) Chiral fit with data points and curves representing different lattice spacings.]

**SU(2) Chiral fit formula:**

$$f+(0) = F_0^+ \left(1 - \frac{3}{4} \xi \log \xi + P_2 \xi + P_3 a^2\right)$$

$$\xi = \frac{2B_0 m_t}{(4\pi f_0)^2}$$

The compatibility between ensemble A40.32 and A40.24 shows that FSE are small.

To calculate $f+(0)$ we used $m_s = 99.6(4.1)$ MeV and $m_{ud} = 3.70(17)$ MeV from our previous work [arXiv:1403.4504]

our result:

$$f+(0) = 0.9641(58)$$
f+(0): SU(3) Chiral and continuum extrapolation

we performed a small interpolation in the data to arrive at \( m_{s,\text{phys}} \)

SU(3) Chiral fit

fit formula:

\[
f_{+}(0) = 1 + f_{2} + \Delta f
\]

\[
\Delta f = (m_{s} - m_{l})^{2} \left[ \Delta_{0} + \Delta_{1} m_{l} \right] + \Delta_{2} a^{2}
\]

\( M_{K}^{2}, M_{\pi}^{2} \) and \( f_{\pi} \) appearing in \( f_{2} \) are expressed at LO.

We also tried the same fit using \( f_{\pi} \) instead of \( f_{0} \) in the definition of \( f_{2} \) obtaining consistent results:

\( f_{+}(0) = 0.9734(40) \)

The compatibility between ensemble A40.32 and A40.24 shows that FSE are small.

To calculate \( f_{+}(0) \) we used \( m_{s} = 99.6(4.1) \) MeV and \( m_{w/a} = 3.70(17) \) MeV extracted in our previous work [arXiv:1403.4504]

our result:

\[
f_{+}(0) = 0.9725(41)
\]
The results from our analysis:

\[ f_+(0) = 0.9683(65) \]

in particular:

\[ f_+(0) = 0.9683(50)_{\text{stat+fit}} (42)_{\text{chiral}} \]

\textbf{stat+fit} is referred to both the statistical uncertainties (including the total error on the light and strange quark mass determination) and the uncertainties due to the fitting procedure.

\textbf{Chiral} extrapolation systematic uncertainties have been evaluated comparing the results obtained from two different fit formulae i.e. SU(2) ChPT and SU(3) ChPT.

An estimate of the systematic effects associated to the \textbf{FSE} has not been performed yet. However comparing ensembles A40.24 and A40.32 we expect these effect to be small compared to the other uncertainties.
Testing the CKM unitarity

Testing the first row

\[ |V_u|^2 = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \]

\[ V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \]

| Experimental input \(^{(1)}\) | our result \(^{(2)}\) | determination of \(|V_{us}|\) |
|-----------------------------|----------------|------------------|
| \( K_{\ell 2} \) | \( \frac{|V_{us}|}{|V_{ud}|} \frac{f_K^+}{f_{\pi^+}} = 0.2758(5) \) | \( \frac{f_{K^+}}{f_{\pi^+}} = 1.183(17) \) | \( |V_{us}| = 0.2271(33) \) |
| \( K_{\ell 3} \) | \( |V_{us}| f_+(0) = 0.2163(5) \) | \( f_+(0) = 0.9683(65) \) | \( |V_{us}| = 0.2234(16) \) |

\[ |V_{ud}| = 0.97425(22) \quad \text{from \(\beta\)-decay}\(^{(3)}\) \]

\[ K_{\ell 2} \quad |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.0007(16) \]

\[ K_{\ell 3} \quad |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9991(8) \]

\(^{(1)}\) Eur.Phys.J. C69 (2010) 399-424

\(^{(2)}\) PoS LATTICE 2013 (Carrasco et al.)

\(^{(3)}\) Phys.Rev. C79 (2009) 055502
Testing the CKM unitarity

Our result for $|V_{us}|$ is

Our result for $|V_{ud}|/|V_{ud}|$

FLAG average for $N_f=2+1+1$

FLAG average for $N_f=2$ + nuclear $\beta$ decay

Our result for $|V_{us}|$

Unitarity
Conclusions

❁ We presented $N_f=2+1+1$ preliminary results for the semileptonic form factor $f_+(0)$

| Summary of the results in comparison with FLAG averages: |
|--------------------------------------------------------|
|                                               | Our results | FLAG$_{N_f=2}$ | FLAG$_{N_f=2+1}$ | FNAL/MILC$^{*}_{N_f=2+1+1}$ |
| $f_+(0)$                                        | 0.9683(65)  | 0.9560(84)     | 0.9661(32)       | 0.9704(32)                |

Future plans:

❁ compare with an estimate of $f_+(0)$ coming from the scalar density

❁ A more detailed analysis of the $q^2$ dependence of the form factor and a comparison with the experimental data

* PRL 112 (2014) (Bazavov et. al.)
Backup
Extrapolating the vector pole mass $M_V$ obtained from the fit in $q^2$ of $f_+$ we should get a rough estimate of the $K^*$ mass

Our result:

$$M_{V}^{\text{Phys}} = 937(42)\text{MeV}$$

while $K^*$ has a mass of 892 MeV
Excluding the ensemble A30.32 from the chiral and continuum extrapolation of $f(0)$ we get

SU(2) result:

$f_+(0) = 0.9649(61)$

while with all the points we get

$f_+(0) = 0.9641(58)$

So even if A30.32 seems to be off is overall effect is less than 0.1% and therefore marginal
Ademollo Gatto Theorem

The AG theorem states that in SU(3) limits $f_+(0)=1$ and deviation from unity are of the order of $(m_s-m_l)^2$

$$f_+(0) = 1 + \propto (m_s - m_l)^2$$

We can test AG theorem plotting $\Delta f = f_+(0) - 1 - f_2$ divided by $(m_s - m_l)^2$ as a function $m_l$ notice the the data as been extrapolated at $m_s^{\text{phys}}$
On the action

- Wilson Twisted Mass action at maximal twist with Nf=2+1+1 sea quarks

Light degenerate doublet

\[ S_{lm}^{l} = a^4 \sum_x \bar{\psi}(x) \left\{ \frac{1}{2} \gamma_{\mu}(\nabla_{\mu} + \nabla_{\mu}^*) - i \gamma_5 \tau^3 \left[ M_0 - \frac{a}{2} \nabla_{\mu} \nabla_{\mu}^* \right] + \mu_l \right\} \psi(x) \]

Heavy non degenerate doublet

\[ S_{lm}^{h} = a^4 \sum_x \bar{\psi}(x) \left\{ \frac{1}{2} \gamma_{\mu}(\nabla_{\mu} + \nabla_{\mu}^*) - i \gamma_5 \tau^1 \left[ M_0 - \frac{a}{2} \nabla_{\mu} \nabla_{\mu}^* \right] + \mu_\sigma + \mu_\delta \tau^3 \right\} \psi(x) \]

- Osterwalder-Seiler valence quark action

\[ S_{OS}^{f} = a^4 \sum_x \bar{q}_f(x) \left\{ \frac{1}{2} \gamma_{\mu}(\nabla_{\mu} + \nabla_{\mu}^*) - i \gamma_5 r_f \left[ M_0 - \frac{a}{2} \nabla_{\mu} \nabla_{\mu}^* \right] + \mu_f \right\} q_f(x) \]