Resummation of small $x$ contributions to hard-scattering amplitudes

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The summation of the small $x$ corrections to hard scattering QCD amplitudes by collinear factorisation method is reconsidered and the K-factor is derived in leading ln $x$ approximation. The corresponding expression by Catani and Hautmann (1994) has to be corrected. The significance of the correction is demonstrated in the examples of structure function $F_L$ and of exclusive electroproduction.

1. Small $x$ resummation

Semi-hard processes are characterized by two essentially different large momentum scales, the hard-scattering scale $Q^2$ and the large c.m.s. energy squared $s$, $x$ being the small ratio of these scales. The QCD calculation of the hard processes involving the factorization of collinear singularities has to be improved by including the corrections enhanced by the large logarithm of $x$. The results of the QCD Regge asymptotics [1] provide the basis for the resummation of these large corrections. The method of fitting the BFKL solution consistently into the collinear factorization, called also $k_T$ factorization, has been developed by M. Ciafaloni and collaborators starting in 1990 [2–4].

The resummed small $x$ corrections affect the hard-scale evolution of the parton distributions in terms of the anomalous dimension of two-gluon composite operators and generate a K-factor that can be viewed as an improvement of the coefficient function. Quite a number of papers is relying on this scheme in general and on the results given in [3] in particular, e.g [7–12].

Small $x$ resummation was of great importance for physics at HERA and it will be even more important for LHC physics. Motivated by this and by the idea of extending the applications to exclusive semi-hard production [13] we have reconsidered the small $x$ resummation. We confirm the known factorization scheme but disagree with the expression for the K-factor given in [3] and approved in [4]. Unlike the latter our expression has the angular momentum singularity of the BFKL solution. In examples of the structure function $F_L$ and vector meson electroproduction we demonstrate the significance of the discrepancy. Clearly, the K-factor has to be corrected on the leading log level before advancing to the next-to-leading log level.

1.1. Scheme of resummation

The amplitude of a hard scattering process calculated in the collinear factorization scheme has the structure of the convolution of the coefficient function with the (generalized) parton distribution,

$$ A = C_A^{(0)} \otimes GPD, \quad (1) $$

where the hard-scale dependence of the GPD is calculated from the DGLAP/ERBL equation [5,6], summing the contributions enhanced by logs of the hard scale $Q^2$. At large energy squared $s$ or small $x \approx Q^2/s$ the gluon contribution dominates; the corresponding anomalous dimension is denoted by $\gamma_\omega^{(0)}$. In the calculation the parton distribution is emerging formally from a bare distribution by absorbing the factorized collinear singularities.

At small $x$ corrections enhanced by logs of $x$ have to be summed, improving both the coefficient function and the evolution kernel. This resummation is done relying on the BFKL ap-
where the considered amplitude is calculated from the convolution of the two reggeized-gluon Green function $g$ with impact factors $\Phi$,

$$A_{BFKL} = \Phi_A \otimes g \otimes \Phi_B.$$  

(2)

The consistent small $x$ improvement of the collinear factorization result (1) requires the identification of collinear singularities in the BFKL amplitude and their factorization according to the adopted scheme (e.g., $\overline{MS}$). The corrections improve the evolution kernel or the anomalous dimensions, $\gamma(0) \rightarrow \gamma$, and result in a correction factor $R_\omega$ which appears as the improvement of the coefficient function, $C_A(\omega) = C_A(0)(\omega)R_\omega$.

Actually there are no collinear singularities in the BFKL amplitude (2) as long as the impact factors refer to the scattering of colourless particles. It takes to replace $\Phi_B$ by a partonic impact factor in convolution with a (bare) parton distribution. The projection of this partonic impact factor onto the channel isotropic in the azimuthal angle is simply constant in transverse momentum and its convolution with the BFKL Green function $g$ results in the collinear singularities which are factorized to all order of the strong coupling constant into an universal (but renormalization scheme dependent) transition function $\Gamma$, according to

$$F(0) = g \otimes \Phi^{part} = F \cdot \Gamma,$$

(3)

where the K-factor $R_\omega$ appears in

$$F(\omega, \kappa, \mu_F) = \gamma_\omega R_\omega \left( \frac{\kappa^2}{\mu_F^2} \right)^\gamma$$

(4)

and then improves the coefficient function as seen above, whereas the transition function $\Gamma$ will be absorbed into the renormalized (generalized) parton distribution.

1.2. BFKL and collinear singularities

Consider the BFKL equation in the forward limit in $2 + 2\varepsilon$ dimensions,

$$\omega g(\omega, \vec{\kappa}, \vec{k}^0) = \delta^{2+2\varepsilon}(\vec{k} - \vec{k}_0) + \bar{\alpha}_S \hat{K} \cdot g(\omega, \vec{\kappa}, \vec{k}_0)$$

(5)

In one-loop approximation the operator $\hat{K}$ acts as

$$\hat{K} \cdot g(\vec{\kappa}, \vec{k}_0) = \frac{1}{\pi} \int \frac{d^2 k'}{(2\pi)^2} < \vec{\kappa} \middle| \hat{K} \middle| \vec{k}' > g(\vec{k}', \vec{k}_0),$$

where

$$< \vec{\kappa} \middle| \hat{K} \middle| \vec{k}' > = \frac{1}{(\vec{k} - \vec{k}')^2} - \delta(\vec{k} - \vec{k}') \alpha_g(\kappa),$$

and the gluon trajectory reads

$$\alpha_g(\kappa) = \frac{1}{2} \int \frac{d^2 k''(\vec{k}''^2)}{2(\vec{k}'' - \vec{k}')^2}.$$  

(7)

We have introduced $\bar{\alpha}_S = \frac{\alpha_S}{\mu^2}$. For simplicity we restrict ourselves here to the exchange channel $n = 0$ represented by functions invariant under transverse rotations.

We calculate the action of the kernel on functions that would be eigenfunctions at $\varepsilon = 0$.

$$\hat{K} \cdot (\vec{k}'')^{\gamma-1} = \lambda(\gamma, \varepsilon)(\vec{k}'')^{\gamma-1+\varepsilon},$$

(8)

$$\lambda(\gamma, \varepsilon) = \frac{1}{(4\pi)^\varepsilon} \left[ b(\gamma, \varepsilon) - \frac{1}{2} b(0, \varepsilon) \right].$$

(9)

$$b(\gamma, \varepsilon) = \Gamma^{-1}(\varepsilon) B(\varepsilon, 1 - \gamma - \varepsilon) B(\varepsilon, \gamma + \varepsilon).$$

(10)

The solution for the Green function in the $n = 0$ channel can be formulated using the quasi-eigenvalues $\lambda(\gamma, \varepsilon)$ and the operator of shifts in $\gamma$ by $\varepsilon$,

$$g_0(\kappa, \kappa_0) = \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi i} (\vec{k}'')^\gamma$$

$$\times \frac{1}{\omega - \bar{\alpha}_S e^{-\varepsilon \partial_\gamma} \lambda(\gamma, \varepsilon)} (\kappa_0^2)^{-\gamma}.$$  

(11)

Now $g_0$ is convoluted with the partonic impact factor:

$$F^{(0)}(\omega, \kappa, \varepsilon) = \frac{d^2 k''}{2\pi i} g_0(\kappa, \kappa_0, \varepsilon) \Phi^{part}$$

(12)

with $\Phi^{part} = \frac{\bar{\alpha}_S}{\mu^2} \Omega(\mu_F^2 - \kappa_0^2)$, $\mu$ being the dimensional regularization scale. The singularity in the integration over $\kappa_0$ is regularized in the infrared by $\varepsilon$ with a positive real part and the cut-off at the factorization scale $\mu_F$ is unavoidable to prevent the divergence at large $\kappa_0$.  

$$F^{(0)}(\omega, \kappa, \varepsilon) = \bar{\alpha}_S \left( \frac{\mu_F^2}{\mu^2} \right)^\varepsilon \int \frac{d\gamma}{2\pi i} \left( \frac{\kappa_0^2}{\mu^2} \right)^\gamma$$

$$\times \frac{1}{\omega - \bar{\alpha}_S \lambda_1(\gamma, \varepsilon) e^{-\varepsilon \partial_\gamma} (\varepsilon - \gamma)}.$$  

(13)
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The notation $\lambda_1(\gamma, \varepsilon) = \gamma \lambda(\gamma - \varepsilon, \varepsilon)$ is introduced. In order to understand how the asymptotics $\varepsilon \to 0$ is extracted it is instructive to consider first the case where $\lambda_1(\gamma, \varepsilon)$ is substituted simply by 1, which corresponds to the double-log approximation. The shift operator can be treated easily after decomposition into geometric series and this results without further approximations in

$$F_{d.t.}^{(0)} = \frac{\alpha_S}{\omega} \left( \frac{\mu_F^2}{\mu^2} \right)^{\gamma(0)} \left( \exp \left( \frac{1}{\varepsilon} \gamma(0) \right) - 1 \right).$$

(14)

In this crude approximation we obtain $R_\omega = 1$ and also no corrections to the anomalous dimension $\gamma(0)$. The case without the simplification in $\lambda_1(\gamma, \varepsilon)$ gets closer to the considered one by the substitution $\gamma' = \frac{\gamma}{\lambda_1(\gamma, \varepsilon)}$. We obtain

$$F^{(0)}(\omega, \kappa, \varepsilon) = \frac{\alpha_S}{\omega} \left( \frac{\mu_F^2}{\mu^2} \right)^{\gamma(0)} \int \frac{d\gamma'}{\pi} \frac{\lambda_1(\gamma, \varepsilon)}{1 - \gamma' \lambda_1'(\gamma, \varepsilon)} \times \left( \frac{\mu_F^2}{\mu^2} \right)^{\gamma'} \frac{1}{\gamma' - \gamma} \gamma' \frac{1}{\omega - \gamma} \gamma''$$

(15)

where we have defined the following operator

$$\gamma'' = \frac{\alpha_S}{\omega} e^{-\varepsilon \partial_\gamma}. \tag{16}$$

The factor arising by this change of integration variable involves $\lambda_1(\gamma, \varepsilon) = \partial_\gamma \lambda_1(\gamma, \varepsilon)$ and it becomes the essential ingredient in the result for the K-factor $R_\omega$. Now the operator ordering takes more care and results in

$$F^{(0)}(\omega, \kappa, \varepsilon) = \frac{\alpha_S}{\omega} \left[ 1 - \frac{\lambda(0) \lambda'(0)}{\mu_F^2} \right] \left( \frac{\mu_F^2}{\mu^2} \right)^{\gamma(0)} \exp \left( \frac{1}{\varepsilon} \int_0^1 \frac{d\alpha}{\alpha} \gamma(0, \alpha, \varepsilon) \right) \times \exp \left( \frac{1}{\varepsilon} \int_0^1 \frac{d\alpha}{\alpha} \gamma(0, \alpha, \varepsilon) \right). \tag{17}$$

We expand the solution in $\varepsilon$,

$$\gamma(0, \varepsilon) = \gamma(0) + \varepsilon \gamma'(0) + \ldots \tag{18}$$

From this equation we obtain $\gamma_\omega$ by iteration as a power series in $\gamma(0)$ and we show the first terms of the well known result of the all order expansion of the BFKL anomalous dimension

$$\gamma(0) = \gamma(0) + 2\zeta(3) \left( \gamma(0)^3 \right) + 2\zeta(5) \left( \gamma(0)^5 \right) + \ldots$$

and also of its $\varepsilon$-correction term

$$\gamma'(0) = 2\zeta(3) \left( \gamma(0)^3 \right) - 3\zeta(4) \left( \gamma(0)^4 \right) + \ldots \tag{19}$$

1.3. K-factor

We can now write the final result as

$$F^{(0)}(\omega, \kappa, \varepsilon) = F(\omega, \kappa, \mu_F) \Gamma \left( \frac{\mu_F^2}{\mu^2} \right)^{\gamma(0), \varepsilon} \tag{20}$$

where

$$F(\omega, \kappa, \mu_F) = \gamma_\omega R_\omega \left( \frac{\mu_F^2}{\mu^2} \right)^{\gamma(0)} \tag{21}$$

We rewrite the preexponential factor of (17) in terms of the standard notation $\chi(\gamma) = 2\psi(1) - \psi(1) - \psi(1 - \gamma) - \psi'(\gamma)$ for the BFKL eigenvalue function (at $\varepsilon = 0, n = 0$) and its derivative. The collinear singularities, appearing from the dimensional regularization analysis as a series of pole in $1/\varepsilon$, are now factorized into the $\overline{MS}$ scheme gluon transition function

$$\Gamma(\gamma(0), \varepsilon) = \exp \left( \frac{1}{\varepsilon} \int_0^1 \frac{d\alpha}{\alpha} \gamma(0, \alpha) \right) \tag{22}$$

with the corresponding usual factor $S_\varepsilon = \exp\{-\varepsilon[\psi(1) + \ln 4\pi]\}$. In eq.(22), we have written explicitly the factor $\left( \frac{\mu_F^2}{\mu^2} \right)^{\gamma(0)}$ in front of $\gamma(0)$ to make clear that $F^{(0)}$ is $\mu_F$-independent, as we can see from the previous expression of $\Gamma$. We obtain the K-factor as

$$R_\omega(\gamma(0), \varepsilon) = \exp \left( \frac{1}{\varepsilon} \int_0^1 \frac{d\alpha}{\alpha} \gamma(0, \alpha) \right) \tag{23}$$

$$L(\gamma) = \frac{1}{2} \int_0^{\gamma} \frac{2\psi'(1) - \psi(1 - \gamma) - \psi'(\gamma)}{\chi(\gamma)} \exp \left( \frac{1}{\varepsilon} \int_0^1 \frac{d\alpha}{\alpha} \gamma(0, \alpha) \right) \tag{24}$$

In this form the result can be compared with the one by Catani and Hautmann [3,4]. There the
exponential factor is the same whereas the preexponential one is the square root of ours. We plot these K-factors as a function of $\gamma$ in the physical range $[0, 1/2]$:

Our K-factor is clearly enhanced in comparison to the one of Catani-Hautmann, in particular when going towards the saturation value of the anomalous dimension.

With our preexponential the dependence on $\omega$ has the square root branch point characteristic for the BFKL solutions.

We have checked that the equation used in [3] is equivalent to BFKL without modifications and therefore the singularity structure of the solution has to be the one of standard BFKL. The solution given primarily in that paper in terms of sum can be shown to lead to our result. The discrepancy is caused by restricting to a distinguished anomalous dimension in solving BFKL equation. In [4] the argument involves an incorrectness in the analysis of the $\varepsilon \to 0$ asymptotics.

\section{Structure function $F_L$}

As an application of the previous discussion, we can write the factorized expression of the longitudinal structure function $F_L$ in the Mellin space, in agreement with [3]

$$F_L^\omega(Q^2) = C_L^\omega(\bar{\alpha}_S, Q^2/\mu_F^2) f_2^\omega(\mu_F^2),$$

where $f_2^\omega(\mu^2)$ is the $\omega-$moment of the renormalized gluon distribution function and we have defined the gluonic (improved) coefficient function

$$C_L^\omega(\bar{\alpha}_S, Q^2/\mu_F^2) = h_L, \omega(\gamma_\omega) \ R_\omega(Q^2/\mu_F^2)^\gamma_\omega(27)$$

with the Mellin transform of the corresponding hard cross-section

$$h_L, \omega(\gamma_\omega) = \frac{\bar{\alpha}_S}{2\pi} N_f T_R \frac{4}{3} \frac{1 - \frac{1}{3} \gamma_\omega + \left\{ \frac{34}{9} - \zeta(2) \right\}}{\Gamma(2 - 2 \gamma_\omega)}$$

(28)

Then we easily obtain from (25) and (20) the all order perturbative expansion in power of $\gamma_\omega(0)$ of this coefficient function for the simpler case $\mu_F^2 = Q^2$

$$C_L^\omega(\bar{\alpha}_S, 1) = \frac{\bar{\alpha}_S}{2\pi} T_R N_f \frac{4}{3} \left( 1 - \frac{1}{3} \gamma_\omega + \left\{ \frac{34}{9} - \zeta(2) \right\} \gamma_\omega^2 \right.$$

$$× \left( \gamma_\omega^2 \right)^2 + \left( \frac{1}{3} \zeta(2) - \frac{40}{27} + \frac{14}{3} \zeta(3) \right) \gamma_\omega^3$$

$$+ \left( \frac{1216}{81} - \frac{34}{9} \zeta(2) - \frac{20}{9} \zeta(3) - 6 \zeta(4) \right) \gamma_\omega^4$$

$$+ O \left( \left( \gamma_\omega^4 \right)^5 \right) \right)$$

(29)

$$\approx \frac{\bar{\alpha}_S}{2\pi} T_R N_f \frac{4}{3} \left( 1 - 0.33 \gamma_\omega + 2.13 \left( \gamma_\omega^2 \right)^2$$

$$+ 4.68 \left( \gamma_\omega^3 \right)^3 - 0.37 \left( \gamma_\omega^4 \right)^4 + O \left( \left( \gamma_\omega^5 \right)^5 \right) \right)$$

which has to be compared with eq.(5.24) of Catani-Hautmann’s paper [3]. Indeed, the results start to deviate at the fourth loop in the perturbative expansion. The calculation made by Moch, Vermaseren and Vogt [15] on the same coefficient function derived by complete loop calculations in pure collinear factorization scheme extends to
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three loops and is, therefore, still not sufficient to discriminate the small $x$ resummation results.

Going from $(\omega, \gamma)$ to the $(x, Q^2)$ space we can now show the structure function $F^L_\omega(Q^2)$ as a function of $x$ for different values of the hard scale $Q^2$. We simply used for the convolution with the soft part the following parametrization of the gluon PDF $xg(x, Q^2_0)$ at $Q^2_0 = 30 GeV^2$ (cf. [15])

$$xg(x, Q^2_0) = 1.6 x^{-0.3} (1 - x)^{4.5} (1 - 0.6 x^{0.3})(30)$$

and we have made it evolve through the DGLAP equation in the small $x$ limit at the 1 loop accuracy to obtain its $Q^2$ dependance. We then obtain the following curves.

![Figure 2. $F^L_\omega, Q^2 = 30 GeV^2$](image)
scattering amplitude with the leading twist non-perturbative Distribution Amplitude (DA) [14], where both photon and vector meson are longitudinally polarized.

\[ h_{VM}(k^2_t) = \int_0^1 dz \frac{Q^2}{k^2_t + z(1-z)Q^2} \phi_{VM}(z) \]  

(33)

and its Mellin transform reads for an asymptotic vector meson DA \( \phi_{VM}(z) = 6z(1-z) \).

\[ h_{VM,\omega}(\gamma_\omega) = \gamma_\omega \int_0^\infty dk_{t}^2 \left( \frac{k^2_t}{Q^2} \right)^{\gamma_\omega} h_V(k^2_t) = \frac{\Gamma^3[1 + \gamma_\omega] \Gamma[1 - \gamma_\omega]}{\Gamma[2 + 2\gamma_\omega]} \]  

(34)

We replace in the high energy term the gluon GPD by its forward limit \( H^g(x,\xi) \rightarrow xg(x) \). Contrarily to the previous study for \( F_L \), we keep this expression for the soft part without doing any \( Q^2 \)-evolution. We replace the Born term by a very simple model \( H^g(\xi,\xi) = 1.2 \xi g(\xi) \). We obtain the following curves, in the same spirit as previously by doing the high energy expansion.

Figure 4. VM electroproduction, \( Q^2 = 30 GeV^2 \)

The solid curve corresponds to the Born term, the dotted curve to the Born term with the NLO corrections. Note that the numerical value of the \( C^g_{VM,1} \) coefficient is much larger than in the \( F_L \) case, giving these stronger and very negative NLO corrections. This appears as an instability in the perturbative prediction for this process. Here the small \( x \) resummation is really needed to obtain a reasonable and stable prediction [13]. After twelve iterations we get the dashed curve corresponding to our result and the dashed-dotted one corresponding to the same analysis with the Catani-Hautmann K-factor. Also here the convergence of the series expansion after twelve iterations is very good. Although the curves are only plotted here with the factorization scale \( \mu^2_F = Q^2 \), we observe that the factorization scale dependence is reduced when taking into account the high energy resummations compared to the Born or even NLO case. We also note that the sensitivity of this choice is equivalent for both K-factor expressions.

Figure 5. VM electroproduction, \( Q^2 = 10 GeV^2 \)

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REFERENCES

1. L.N. Lipatov, Sov.J.Nucl.Phys. 23(1976)338
V.S. Fadin, E.A. Kuraev and L.N. Lipatov, Phys. Lett 60B(1975)50; Sov.Phys. JETP 44(1976)443; *ibid* 45(1977)199
Small $x$ resummation

Y.Y. Balitski and L.N. Lipatov, Sov.J.Nucl.Phys. 28(1978)882. L. N. Lipatov, Sov. Phys. JETP 63 (1986) 904 [Zh. Eksp. Teor. Fiz. 90 (1986) 1536].

2. S. Catani, M. Ciafaloni and F. Hautmann, Phys. Lett. B 242 (1990) 97. Nucl. Phys. B 366 (1991) 135. Phys. Lett. B 307 (1993) 147.

3. S. Catani and F. Hautmann, Nucl. Phys. B 427 (1994) 475 [arXiv:hep-ph/9405388].

4. M. Ciafaloni and D. Colferai, JHEP 0509 (2005) 069 [arXiv:hep-ph/0507106].

5. V.G. Gribov and L.N. Lipatov, Sov. J. Nucl. Phys. 15(1972)438; L.N. Lipatov, Yad. Fiz. 20(1974)532; G. Altarelli and G. Parisi, Nucl. Phys.B126(1977)298; Yu.L. Dokshitzer, ZhETF 71(1977)1216

6. G.P. Lepage and S.J. Brodsky, Phys. Lett. B87, 359 (1979); A.V. Efremov and A.V. Radyushkin, Phys. Lett. B94, 245 (1980).

7. M. Ciafaloni, D. Colferai, G. P. Salam and A. M. Stasto, Phys. Lett. B 576 (2003) 143 [arXiv:hep-ph/0305254]. Phys. Lett. B 587 (2004) 87 [arXiv:hep-ph/0311325]. Phys. Lett. B 635 (2006) 320 [arXiv:hep-ph/0601200]. JHEP 0708 (2007) 046 [arXiv:0707.1453 [hep-ph]].

8. G. Altarelli, R. D. Ball and S. Forte, Nucl. Phys. B 575 (2000) 313 [arXiv:hep-ph/9911273]. Nucl. Phys. B 599 (2001) 383 [arXiv:hep-ph/0011270]. Nucl. Phys. B 621 (2002) 359 [arXiv:hep-ph/0109178]. Nucl. Phys. B 674 (2003) 459 [arXiv:hep-ph/0306156]. Nucl. Phys. Proc. Suppl. 191 (2009) 64 [arXiv:0901.1294 [hep-ph]].

9. R. D. Ball and R. K. Ellis, JHEP 0105 (2001) 053 [arXiv:hep-ph/0101199].

10. S. Marzani and R. D. Ball, Nucl. Phys. B 814 (2009) 246 [arXiv:0812.3602 [hep-ph]].

11. S. Marzani, R. D. Ball, V. Del Duca, S. Forte and A. Vicini, Nucl. Phys. B 800 (2008) 127 [arXiv:0801.2544 [hep-ph]].

12. G. Diana, arXiv:0906.4159 [hep-ph].

13. D. Y. Ivanov, “Exclusive vector meson electroproduction," arXiv:0712.3193 [hep-ph], contribution to Blois EDS07

14. S. J. Brodsky and G. P. Lepage, Phys. Rev. D 24 (1981) 1808.

15. S. Moch, J. A. M. Vermaseren and A. Vogt, Phys. Lett. B 606, 123 (2005), Nucl. Phys. B 724, 3 (2005). [arXiv:hep-ph/0504242].