UNIFORMIZERS FOR ELLIPTIC SHEAVES

A. Álvarez

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1. Introduction.

In this paper we define $k$-elliptic sheaves, $A$-motives and $t$-modules over $A$, which are obvious generalizations of elliptic sheaves, $t$-motives and $t$-modules. Following results of [An1], [D], [LRSt], [Mu], [St],... we shall obtain the equivalence of this objects. Bearing in mind [Al] we also describe a correspondence between $k$-elliptic sheaves with formal level structures (2.9) and discrete subspaces. The moduli scheme for these objects shall be a subscheme of Sato’s infinite Grassmannian, in this setting the moduli for classical $t$-motives will be a closed subscheme of this last subscheme. In the section 6 for these discrete subspaces, we show a result analogous to [An1] for the behaviour of the determinant.

In the same way $A$-motives with formal level structures have associated locally dense subspaces (3.1) these subspaces are determined by a subspace of formal series of dimension $k$, called subspace of uniformizers (3.5). With this we see that a $k$-elliptic sheaf with formal level structure could be seen as an orbit of the action of $\text{Gl}_n(F_q[[t_x]])$ over the set of these subspaces of uniformizers. These subspaces allow us to get an injective morphism between the moduli scheme of $k$-elliptic sheaves with formal level structures and a Grassmannian of finite $k$-dimensional subspaces of formal series, Definition 3.3. An explicit computation can be made for these subspaces of uniformizers for $k = 1$ and one can check certain analogy between these subspaces of uniformizers and Dirichlet series for Drinfeld modules [3]. These uniformizers allow us to obtain the arithmetic counterpart of the Baker function, [SW], defined in the setting of the theory of soliton equations (see [BIS] for elliptic sheaves together an introduction of soliton theory, an intensive study for 1-elliptic modules is done in [An2]). We can also make explicit the action of the classical
arithmetics groups, $\text{Gl}_n(\mathbb{F}_q((t_x)))$, over these subspaces of uniformizers, with this result we shall obtain the "elliptic" Baker function in an explicit way. Furthermore we can translate the results of the anti-equivalence of the Krichever functor $[\text{MR}, \mathbb{Q}, \mathbb{MW}]$, to these 1-dimensional subspaces and gain similar objects to Schur pairs for elliptic sheaves, that we will call elliptic sheaf triples. As immediate consequence of these last results, the subspaces of uniformizers also determine the curve where the elliptic sheaf is defined. We can get a moduli scheme for these triples.

As for discrete subspaces, we shall study the determinant for locally dense subspaces associated to $A$-motives of $\sigma$-rank 1 and rank $n$ (Drinfeld modules) with formal level structures. We give an open condition in the moduli scheme of $A$-motives of $\sigma$-rank 1 and rank $n$ with formal level structures, which implies that the determinant of a certain "scattering" matrix, $[\text{An}]$, obtained from the uniformizers, generates as $A \otimes \mathcal{O}_S\{\sigma\}$-module, the determinant of the locally dense subspace associated to the $A$-motive. When $A = \mathbb{F}_q[t]$, we see that this "scattering" determinant gives the subspace of uniformizers associated to the $A$-motive determinant. It would be very interesting to obtain a similar result for arbitrary $A$.

Let us now briefly state the contents of the different sections of this paper. In the second section we define $k$-elliptic sheaves, $A$-motives and $t$-modules over $A$ with theirs level structures and we recall the "equivalences" among these objects, after this we are recalling some facts about Sato’s infinite Grassmannian scheme and vector bundles, we see $k$-elliptic sheaves with formal level structures could be considered as discrete subspaces and the moduli functor for this objects is representable by a closed subscheme of Sato’s infinite Grassmannian scheme.

In the third section we prove $A$-motives with formal level structures give in a unique way that subspaces of uniformizers and we construct a moduli scheme for these objects.

In the fourth section we consider discrete subspaces attached to elliptic sheaves and the elliptic Baker function.

In the fifth section we study an analogous structure to classical Schur pairs in the set of Drinfeld modules, elliptic sheaf triples.

In the sixth section we settle some results about the determinant in our setting.

**List of notation**

- $\mathbb{F}_q$ a finite field of $q$ elements ($q = p^m$, $p$ prime)
- $X$ a smooth, proper and geometrically irreducible curve over $\mathbb{F}_q$
- $\otimes$ denotes $\otimes_{\mathbb{F}_q}$
- $g$ is the genus of $X$
- $\infty$ a rational point of $X$
- $A = H^0(X - \{\infty\}, \mathcal{O}_X)$
- $\Omega_X, \Omega_A$ the sheaves of differentials on $X$ and $A$ respectively
- $x \neq \infty$ a point of $\text{Spec}(A)$
- $m_x$ is the maximal ideal of $x$
$t_x$ a uniformizer at $x$

$k(x)$ residue field of $x$

deg$(x) = \dim_{\mathbb{F}_q}(k(x))$

$v_x$ is the valuation associated to $x$

$\Omega_x$ is the completion of $\Omega_X$ at $x$

$D$ is an effective divisor over $\text{Spec}(A)$

$I_D$ is the ideal in $A$ associated to $D$

$S$ an arbitrary scheme over $\mathbb{F}_q$

$e$ the structure map $S \to \mathbb{F}_q$

$\pi$ the natural projection $X \times S \to S$

$\sigma$ Frobenius endomorphism over $S$ ($\sigma(\lambda) = \lambda^q$)

$\sigma^\ast(\cdot)$ is the inverse image by $\sigma$ of the sections on $S$ of a sheaf defined in the category of $\mathbb{F}_q$-schemes.

$F = Id \times \sigma : X \times S \to X \times S$

$(\mathbb{G}_a)_S$ the additive line group over $S$

2. $k$-Elliptic sheaves, $A$-motives and $t$-modules over $A$.

The results of the beginning of this section are essentially obtained from \cite{An1}, \cite{E}, \cite{LRSt}, \cite{Mu}. Moreover, we can get, in a easy way, the same results of \cite{Al} for these objects. Some times to easy notation we are going to consider $S = \text{Spec}(B)$,

$\text{End}_B(\mathbb{G}_a)_B = B\{\sigma\}$, where $B\{\sigma\}$ is the non-commutative polynomial ring with the commutation rule $(\sigma.b = b^q\sigma)$.

Definition 2.1. An $A$-motive of rank $n$ and $\sigma$-rank $k$ over $B$ is an $A \otimes B\{\sigma\}$-module, $N$, locally free of rank $n$ as $A \otimes B$-module, and free of rank $k$ as $B\{\sigma\}$-module.

From this definition we obtain $N = \bigcup N_i$, where

$N_i = \{p_1(\sigma)e_1 + \cdots + p_k(\sigma)e_k : \ deg_\sigma(p_1(\sigma)) \leq i \cdots deg_\sigma(p_k(\sigma)) \leq i\}$

being $e_1, \cdots, e_k$ a $B\{\sigma\}$-basis of $N$.

Definition 2.2. An $A$-motive $N$ of rank $n$ and $\sigma$-rank $k$ over $B$ is said pure if there exists natural numbers $\bar{k}$, $\bar{n}$ such that

$N = \bigcup_{r \in \mathbb{N}} N_{r\bar{n}}$

is a graded $\bigcup_{r \in \mathbb{N}} H^0(X, \mathcal{O}_X(r\bar{k}\infty)) \otimes B$-module.
In (2.3.1) we prove \( n/k = \bar{n}/\bar{k} \). Recall that \( \bigcup_{r \in \mathbb{N}} H^0(\mathcal{O}_X(r\bar{k}\infty)) = A \).

**Definition 2.3.** A \( k \)-elliptic sheaf, \((\mathcal{E}_j, i_j, \tau)\), is a commutative diagram of vector bundles of rank \( n \) over \( X \times S \), and injective morphisms of modules \( \{i_h\}_{h \in \mathbb{N}}, \tau \):

\[
\cdots \xrightarrow{\tau} \mathcal{E}_-k \xrightarrow{i_{-k}} \mathcal{E}_0 \xrightarrow{i_0} \cdots \xrightarrow{i_{k(n-1)}} \mathcal{E}_{kn} \xrightarrow{i_{kn}} \cdots
\]

\[
\cdots \xrightarrow{\tau} F^*\mathcal{E}_-k \xrightarrow{F^*i_{-k}} F^*\mathcal{E}_0 \xrightarrow{F^*i_0} \cdots \xrightarrow{F^*i_{kn}} \cdots
\]

satisfying:

a) For any \( s \in S \), \( \deg((\mathcal{E}_{hk})_s) = n(g - 1) + (h + 1)k \). \( \deg \) denotes the degree.

b) For all \( i \in \mathcal{E} \), \( \mathcal{E}_{(i+n)k} = \mathcal{E}_{ik}(k\infty) = \mathcal{E}_{ik} \otimes \mathcal{O}_X(k\infty) \otimes \mathcal{O}_S \).

c) \( \mathcal{E}_{ik} + \tau(F^*\mathcal{E}_{ik}) = \mathcal{E}_{(i+1)k} \).

d) \( R^1\pi_*(\mathcal{E}_{-k}) = 0 \). (As consequence \( \pi_*(\mathcal{E}_0) \) is a locally free \( \mathcal{O}_S \)-module of rank \( k \), in the case of elliptic sheaves this property is redundant).

(2.3.1) Following [D], [Mu], we obtain an equivalence between the categories of pure \( A \)-motives of rank \( n \) and \( \sigma \)-rank \( k \) and \( k \)-elliptic sheaves of rank \( n \), just by considering \( \mathcal{E}_{j\bar{k}} = \bigoplus_{r \in \mathbb{N}} N_{j+r\bar{k}} \) as \( \text{Proj}(\bigoplus_{r \in \mathbb{N}} H^0(\mathcal{O}_X(r\bar{k}\infty)) \otimes \mathcal{O}_S) \)-graded module. \( i_h \) (for each \( h \in \mathbb{Z} \)) and \( \tau \) are obtained from the natural inclusions among \( \{\mathcal{E}_{j\bar{k}}\}_j \) and the multiplication on the left by \( \sigma \) respectively. We can observe that for \( r > 0 \) \( H^0(\mathcal{E}_0((r+1)\bar{k}\infty)) - H^0(\mathcal{E}_0(r\bar{k}\infty)) = \text{rank}_{\mathcal{B}}(N_{r+1}\bar{n}/N_r\bar{n}) = n\bar{k} \) moreover using the Riemann-Roch theorem \( h^0(\mathcal{E}_0((r+1)\bar{k}\infty)) - h^0(\mathcal{E}_0(r\bar{k}\infty)) = nk \) therefore \( n/k = \bar{n}/\bar{k} \).

**Definition 2.4.** \( E = \bigoplus_{1 \leq j \leq k} \mathbb{G}_a \) is said to be an abelian \( t \)-module over \( A \) of rank \( n \) and \( \sigma \)-rank \( k \) if \( E \) is an \( A \)-module valued functor in the category of \( \mathbb{F}_q \)-schemes verifying: \( \text{Hom}_{\text{End}_A(\mathbb{G}_a)_S}(E, (\mathbb{G}_a)_S) \) is a locally free \( A \otimes \mathcal{O}_S \)-module of rank \( n \), for each \( S \).

Again, paraphrasing [An1] we know that the categories of abelian \( t \)-modules over \( A \) and \( A \)-motives are antiequivalent, where

\[ \mathcal{N} = \text{Hom}_{\text{End}_A(\mathbb{G}_a)_S}(E, (\mathbb{G}_a)_S) \]

An \( t \)-module over \( A \) is called pure when the associated \( A \)-motive is it. From now, we suppose \( A \)-motives and \( t \)-modules over \( A \) are pure.
Remark 1. When \( A = \mathbb{F}_q[t] \) and \( B \) is a perfect field \( K \), it can be added another condition to the already considered objects: there exists an element \( \theta \in K \) and \( N \in \mathbb{N} \) such that \((t - \theta)^N \cdot \text{Lie}(E) = 0\), that is \((t - \theta)^N (N/\sigma N) = 0\) for \( A \)-motives, so we recover the usual definitions of \( t \)-motives and \( t \)-modules. Over an arbitrary ring \( B \), this condition means that the characteristic polinomyal of \( t : N/\sigma N \to N/\sigma N \) is \( (x^{p'} - b)^m \) \((k = p'm \text{ and } (p, m) = 1)\) for some \( b \in B \), \( t \) is the multiplication by \( t \).

Now we are recalling the definitions of level structures for \( A \)-motives, \( k \)-elliptic sheaves and \( t \)-modules over \( A \).

**Definition 2.5.** A twisted \( \hat{D} \)-level structure on a vector bundle of rank \( n \) over \( X \times S \) is a pair, \((M, f_D)\), with \( f_D \) a surjective morphism of sheaves of \( \mathcal{O}_{X \times S} \)-modules

\[
f_D : M \to (\Omega_X/\Omega_X(-\hat{D}))^n \otimes \mathcal{O}_S
\]

Two \( \hat{D} \)-level structures, \((M, f_D)\) and \((M', f_{D'})\), are said to be equivalent if there exists an isomorphism of sheaves of \( \mathcal{O}_{X \times S} \)-modules, \( \phi : M \to M' \), compatible with the twisted level structures.

**Definition 2.6.** A twisted \( \hat{D} \)-level structure, \((E_{jk}, i_{jk}, \tau, f_{\hat{D}})\), for a \( k \)-elliptic sheaf of rank \( n \) over \( S \), is a twisted \( \hat{D} \)-level structure for each vector bundle \( E_{jk} \) compatible with the morphisms \( \{i_{jk}, \tau\} \).

**Definition 2.7.** A \( \hat{D} \)-level structure in a \( \hat{A} \)-motive of rank \( n \) and \( \sigma \)-rank \( k \) over \( B, N \), is a pair \((N, h_{\hat{D}})\) where

\[
h_{\hat{D}} : N \to (\Omega_A/\Omega_A(-\hat{D}))^n \otimes B
\]

is a surjective morphism of \( A \otimes B\{\sigma\} \)-modules. \( (\Omega_A/\Omega_A(-\hat{D}))^n \otimes B \) is \( B\{\sigma\} \)-module by defining \( \sigma(\omega \otimes b) = \omega \otimes b^\theta \)

**Definition 2.8.** If \( E \) is a \( t \)-module over \( A \) of rank \( n \) and \( \sigma \)-rank \( k \) over \( B \), a \( \hat{D} \)-module structure for \( E \) is a pair \((E, i_{\hat{D}})\). \( i_{\hat{D}} \) is an isomorphism of \( A \)-modules

\[
i_{\hat{D}} : E_{I_{\hat{D}}}(B) \to (\hat{I}_{\hat{D}}/A)^n(B)
\]

where \( E_{I_{\hat{D}}} \) is the subgroup scheme of \( I_{\hat{D}} \) division points of \( E \) and \( (\hat{I}_{\hat{D}}/A)^n \) is the constant sheaf of stalk \( (I_{\hat{D}}/A)^n \).

Remark that \( A \)-motives, \( k \)-elliptic sheaves and \( t \)-modules over \( A \) have a level structures on \( \hat{D} \), if the restrictions to \( I_{\hat{D}} \times S \) of \( N/\sigma N \) and \( \mathcal{E}_0/\tau(\mathcal{F}^* \mathcal{E}_{-k}) \) are zero and \( \text{Ann} \text{lie} \) is an \( (\hat{D} \times S) \)-module such that \( \text{supp} \text{lie} \) \( (\hat{I}_{\hat{D}}/A)^n \) is the constant sheaf of stalk \( (I_{\hat{D}}/A)^n \).

Remark 2. One can check in a standard way that the moduli categories of \( \hat{D} \)-level structures for \( k \)-elliptic sheaves and \( A \)-motives are equivalent. On other hand if \( I_{\hat{D}} = (a) \) with \( a \in A \), one obtains a similar result for \( A \)-motives and \( t \)-modules over \( A \). Following \textit{[An1]} we are going to recall how is settled this last "equivalence".
Let us consider an $a$-level structure (means $(a)_0$-level structure), $(E, i_a)$, on a $t$-module over $A$, and $\mathcal{N}$ the associated $A$-motive to the $t$-module over $A$, $E$, then $i_a$ gives an $a$-level structure on $\mathcal{N}$:

let $\phi$ be the surjective morphism of $A \otimes B\{\sigma\}$-modules defined by

$$\phi : \mathcal{N} = B\{\sigma\}e_1 \oplus \cdots \oplus B\{\sigma\}e_k \to \text{Hom}_{\mathcal{N}}(E_a(B), B)$$

where $\phi(p(\sigma)e_i)(\alpha_1, \cdots, \alpha_k) = p(\sigma)(\alpha_i)$ and

$$(\alpha_1, \cdots, \alpha_k) \in E_a(B) \subset E(B) = \mathbb{G}_a(B) \oplus \cdots \oplus \mathbb{G}_a(B)$$

then if $\text{Spec}(B)$ is connected from the isomorphism of $A \otimes B$-modules $i_a : E_a(B) \xrightarrow{\sim} (a^{-1}A/A)^n$ we get

$$\text{Hom}_{\mathcal{N}}(E_a(B), B) \xrightarrow{(i_a)'} \text{Hom}_{\mathcal{N}}((a^{-1}A/A)^n, B) = \text{Hom}_{\mathcal{N}}((a^{-1}A/A)^n, \mathbb{F}_q) \otimes B$$

and by using the residue pairing (for simplicity $aA = m_x^n$ with $x \in X$)

$$\Omega_A/m_x^n\Omega_A \times m_x^{-r}/A \to k(x) \xrightarrow{tr} \mathbb{F}_q$$

we obtain a $rx$-level structure for $\mathcal{N}$ in a direct way. $tr$ is the trace map in $x \in \text{Spec}(A)$.

Now we are going to sketch some facts about Sato’s infinite Grassmannian and vector bundles. We can suppose $x$ rational to easy notation.

Let $\mathbb{F}_q[[t_x]], \mathbb{F}_q((t_x))$ be the functors in the category of $\mathbb{F}_q$-schemes associated to the ring and field of formal series, namely

$$\mathbb{F}_q[[t_x]](S) = H^0(S)[[t_x]] \text{ and } \mathbb{F}_q((t_x))(S) = H^0(S)[[t_x]][t_x^{-1}]$$

respectively, for each scheme $S$. We denote by $\mathcal{O}_S((t_x))$ and $\mathcal{O}_S[[t_x]]$ to the sheaves in the category of $S$-schemes $e^*(\mathbb{F}_q((t_x)))$ and $e^*(\mathbb{F}_q[[t_x]])$ respectively, recall that $e$ is the structure map $S \to \mathbb{F}_q$.

There exists a scheme

$$\text{Gr}(\mathbb{F}_q[[t_x]]^n, \mathbb{F}_q((t_x))^n)$$

(Sato’s infinite Grassmannian scheme)

which points over an arbitrary scheme $S$, are quasicoherent $\mathcal{O}_S$-submodules, $\mathcal{L}$, of $\mathcal{O}_S((t_x))$, such that in the category of $S$-schemes is

$$\gamma^*\mathcal{L} \subset \mathcal{O}_{S'}((t_x))^n$$

where $\gamma : S' \to S$, also must exist some $r \in \mathbb{N}$ verifying

$$\mathcal{L} + t_x^r\mathcal{O}_S[[t_x]]^n = \mathcal{O}_S((t_x))^n$$

and $\mathcal{L} \cap t_x^r\mathcal{O}_S[[t_x]]^n$ is a coherent free $\mathcal{O}_S$-module (this last rule if $S$ is compact). These submodules are called discrete submodules.

Let $\mathcal{M}_x^n$ be the moduli functor of vector bundles of rank $n$ over $X$ with level structures on $rx$ and

$$\mathcal{M}_x^n = \lim_{r \in \mathbb{N}} \mathcal{M}_{rx}^n$$

If one fix an uniformizer for $x, t_x$, we obtain by the relative Krichever’s morphism, $\mathbb{M}, [2]$, that $\mathcal{M}_x^n$ is the subfunctor of the functor of points of Sato’s infinite grassmannian scheme, of discrete submodules which are $H^0(X - \{x\}, \mathcal{O}_X)$-submodules. Moreover, it is representable by a closed subscheme of the infinite grassmannian $\mathbb{A}$.
Theorem 2.10. The elements of \((M, f_z) \in \mathcal{A}_D^{n,k}(S)\) are called formal \(x\)-level structures over \(X \times S\) in an analogous way one can define twisted \(x\)-formal level structure. If \(A_M\) is the discrete submodule associated to the formal \(x\)-level structure over \(X \times S\), \((M, f_z)\), then \(\pi_*(M(rx)) = A_M \cap t_x^*\mathcal{O}_S([t_x]^n)\) and \(R^1\pi_*(M(rx)) = \mathcal{O}_S((t_x))^n/A_M + t_x^*\mathcal{O}_S([t_x]^n)\).

To see more details about these stamens, one can consult \([\text{BeL}], [\text{Al}]\).

To study the moduli problem for \(k\)-elliptic sheaves we have to settle some definitions.

**Definition 2.9.** A twisted \(x\)-formal level structure in a \(k\)-elliptic sheaf of rank \(n\) over \(S\), \((\mathcal{E}_{jk}, i_{jk}, \tau, f_{tw}^i)\), is a twisted \(x\)-formal level structure, \(f_{tw}^i\) in each \(\mathcal{E}_{jk}\), compatible with the morphisms \(\{i_{jk}\}\).

Therefore, fixing an isomorphism of \(\mathbb{F}_q[[t_x]]\)-modules \(\mu_x : \Omega_x \rightarrow \mathbb{F}_q[[t_x]]\) we can obtain from a twisted \(x\)-formal level structures a \(x\)-formal level structure for the \(k\)-elliptic sheaves, \(f_z\), by considering \(f_z = (\oplus \mu_x)(f_{tw}^i)\).

By copying the results of \([\text{Al}]\), we can obtain:

**Theorem 2.10.** The moduli functor, \(D^{n,k}_x\), of \(k\)-elliptic sheaves of rank \(n\) with \(x\)-formal level structures, \((\mathcal{E}_{jk}, i_{jk}, \tau, f_z)\), is representable by a locally closed subscheme, \(D^{n,k}_x\), of \(\text{Gr}(\mathbb{F}_q[[t_x]]^n, \mathbb{F}_q((t_x))^n)\).

**Proof.** A discrete submodule \(L_0\) gives a \(k\)-elliptic sheaf with formal level structure if it verifies:

- \(L_0\) is a \(H^0(X - \{x\}, \mathcal{O}_X)\)-module.
- For all \(1 \leq i \leq n\), \((\sigma^*)^iL_0 \subseteq L_0(k\infty)\) and \(L_0 + \cdots + (\sigma^*)^iL_0 + \cdots + (\sigma^*)^nL_0 = L_0(k\infty)\).
- For all \(1 \leq i \leq n\), \(L_0(k\infty)/L_0 + \cdots + (\sigma^*)^iL_0\) is locally free of rank \(k(n-i)\).

Where \(L_0(k\infty) = L_0 \otimes \mathcal{O}_{X - \{x\}}(k\infty)\)

this gives a diagram of discrete subspaces:

\[
\cdots \longrightarrow L_{-k} \longrightarrow L_0 \longrightarrow \cdots \longrightarrow L_{nk} \longrightarrow \cdots
\]

\[
\cdots \sigma^* L_{-k} \longrightarrow \sigma^* L_0 \longrightarrow \cdots \longrightarrow \sigma^* L_{nk} \longrightarrow \cdots
\]

where the morphisms are the natural inclusions and \(L_{ik} = L_0 + \cdots + (\sigma^*)^iL_0\) if \(i \geq 0\) and if \(i < 0\) and \(ik + \lambda kn \geq 0\), \(L_{ik} = L_{ik + \lambda kn}(-\lambda k\infty)\). Recall that \(\sigma^* L_0\) is the pull back of \(L_0\) by the Frobenius morphism \(\sigma\).

- If \((\mathcal{E}_{-k}, f_z)\) is the formal level associated to \(L_{-k}\), \(\text{deg}(\mathcal{E}_{-k}) = n(g - 1)\) and \(R^1\pi_*(\mathcal{E}_{-k}) = 0\) therefore \(\pi_*(\mathcal{E}_0)\) is locally free of rank \(k\). In the case of elliptic sheaves it is enough to impose the condition over the \(\text{deg}\).

Conversely if we have \((\mathcal{E}_{jk}, i_{jk}, \tau, f_z)\) over \(S\), \(L_{ik}\) is

\[
L_{ik} = f_z(\mathcal{E}_{ik}[X - \{x\} \times S] \subset \mathcal{O}_S([t_x]^n)_{|X - \{x\} \times S} = \mathcal{O}_S((t_x))^n
\]

When \(A = \mathbb{F}_q[t]\) the condition of the Remark \([\text{Al}]\) is verified over a closed subscheme of \(D^{n,k}_x\).

Actually the \(S\)-points of \(D^{n,k}_x\) are not \(A\)-motives of \(\sigma\)-rank \(k\) but \(A\)-motives of locally (over \(S\)) \(\sigma\)-rank \(k\), it depends on \(\pi_*(\mathcal{E}_0)\), in the case of elliptic sheaves there are no problems with this because \(\pi_*(\mathcal{E}_0)\) is free of rank 1.
3. Subspace of uniformizers and locally dense subspaces for $k$-elliptic sheaves.

We shall denote by $\mathcal{L}_0^\xi$ to the discrete subspace $\mathcal{L}_0$ associated to a $k$-elliptic sheaf, $(\mathcal{E}_{jk}, i_{jk}, \tau, f_z)$, with a $x$-formal level structure.

**Definition 3.1.** A cuasicoherent $\mathcal{O}_S$-submodule $D$ of $\mathcal{O}_S[[t_x]]^n$, such that for each $S$-scheme $\gamma : S' \rightarrow S$, $\gamma^* D \subset \mathcal{O}_S[[t_x]]^n$, is said a locally dense subspace over $S$ if for some $m \geq 0$ and for every $h \geq 0$, $D \cap t_x^{m+h} \mathcal{O}_S[[t_x]]^n / D \cap t_x^m \mathcal{O}_S[[t_x]]^n$ is a locally free $\mathcal{O}_S$-module of rank $nh$.

**Proposition 3.2.** If $(\mathcal{E}_{jk}, i_{jk}, \tau, f_z)$ is a $k$-elliptic sheaf of rank $n$ with a $x$-formal level structure over $S$, then its associated $A$-motive of $\sigma$-rank $k$ and rank $n$ with a formal level structure on $x$, $(N, g_z)$ can be characterized as a locally dense subspace, $D_N$, of $\mathcal{O}_S[[t_x]]^n$.

**Proof.** The locally dense subspace, $D_N$, is $g_z(N)$, where $g_z$ is the level structure $g_z : N \rightarrow \mathcal{O}_S[[t_x]]^n$, (we have fixed $\mu_x : \Omega_x \xrightarrow{\sim} \mathbb{F}_q[[t_x]]$). Moreover, since $N = \bigcup_{j \geq 0} \pi_* (\mathcal{E}_j)$, $D_N$ is $\bigcup_{i \geq 0} (\mathcal{L}_j^\xi \cap \mathcal{O}_S[[t_x]]^n) \quad \text{and as} \quad f_z(\pi_*(\mathcal{E}_j)) = \mathcal{L}_j^\xi \cap \mathcal{O}_S[[t_x]]^n$ we conclude. \(\square\)

Now we are going to characterize $k$-elliptic sheaves of rank $n$ with formal level structures as $k$-dimensional vector subspaces of $\mathbb{F}_q[[t_x]]^n$. For this we need to precise the relative definition of these subspaces:

**Definition 3.3.** $H$ is said a $k$-dimensional vector subspaces of $\mathcal{O}_S[[t_x]]^n$ over $S$ if $H$ is a coherent locally free submodule of rank $k$ of $\mathcal{O}_S[[t_x]]^n$, in the sense explained before, i.e: $\gamma^* H \subset \mathcal{O}_S[[t_x]]^n$, for each $S$-scheme $S'$ ($\gamma : S' \rightarrow S$).

It is not hard to prove that there exists a grassmannian scheme (not finite dimensional), $Gr^k(\mathbb{F}_q[[t_x]]^n)$, for these objects.

In the next theorem, we are going to show $D_{x}^{n,k}$ is a subscheme of this Grassmannian.

**Theorem 3.4.** There exist an injective morphism of schemes

$$\psi : D_{x}^{n,k} \rightarrow Gr^k(\mathbb{F}_q[[t_x]]^n).$$

**Proof.** $\psi$ is defined in the functors of points of the schemes $D_{x}^{n,k}$ and $Gr^k(\mathbb{F}_q[[t_x]]^n)$ by

$$\psi(\mathcal{E}_{jk}, i_{jk}, \tau, f_z) = \mathcal{L}_0^{\xi} \cap \mathcal{O}_S[[t_x]]^n = f_z(\pi_* \mathcal{E}_0) = H$$

this is a subspace in the sense of the last definition, because of standard properties for discrete subspaces and vector bundles. $\psi$ is injective since $H$ generates the locally dense subspace $D_N$, as $A \otimes \mathcal{O}_S \{\sigma\}$-module. \(\square\)

When $k = 1$ and $n = 1$ (i.e classical elliptic sheaves of rank 1) $\text{Im}(\psi)$ takes values on the subfunctor in the category of $\mathbb{F}_q$-schemes $H^0(S)[[t_x]]^* / H^0(S)^* \subset Gr^1(\mathbb{F}_q[[t_x]])(S)$, where $()^*$ means units of rings, because the characteristic of the elliptic sheaves is away from $x$. 

**Definition 3.5.** \(s_1, \ldots, s_k \in H^0(S)[[t_x]]^n\) are uniformizers for a \(k\)-elliptic sheaf of rank \(n\) (or \(A\)-motives or \(t\)-modules over \(A\)) over \(S\) if it is a basis for the associated \(\mathcal{O}_S\)-module \(H\).

For elliptic sheaves with \(x\)-formal level structures there always exists uniformizers since that \(H\) is free:

\[
H = \mathcal{L}_0^\xi \cap \mathcal{O}_S[[t_x]]^n / \mathcal{L}_0^\xi \cap \Gamma \mathcal{O}_S[[t_x]]^n = \mathcal{O}_S
\]

because

\[
\mathcal{L}_0^\xi \cap \Gamma \mathcal{O}_S[[t_x]]^n = \mathcal{O}_S((t_x))^n / \mathcal{L}_0^\xi + \Gamma \mathcal{O}_S[[t_x]]^n = \mathcal{O}_S((t_x))^n / \mathcal{L}_0^\xi + \mathcal{O}_S[[t_x]]^n = 0
\]

where \(\Gamma\) is the diagonal matrix \((t_x, 1, \ldots, 1)\). For general \(k\)-elliptic sheaves there just exists uniformizers locally over \(S\).

**Remark 3.** For \(A = \mathbb{F}_q[t], x = (t)\) and \(n = k = 1\) (Carlitz's modules), this last theorem has a connection with Serre's class field theory:

If we denote by \(J_{\mathbb{F}_q, \hat{x}}\) (local Jacobian) the representant scheme of the functor

\[
S \to H^0(S)[[t_x]]^*/H^0(S)^*
\]

\(J_{\mathbb{F}_q, \hat{x}}\) is obviously an open subscheme of \(\text{Gr}^1(\mathbb{F}_q[[t_x]])\). Moreover, by Serre's class field theory

\[
\psi : D_{\hat{x}}^1 \to J_{\mathbb{F}_q, \hat{x}}\]

is the blowing up of the Albanese morphism

\[
\text{Spec}(\mathbb{F}_q[t^{-1}]) \to J_{\mathbb{F}_q, \hat{x}}
\]

by the isogeny

\[
\text{Id}/F^* : J_{\mathbb{F}_q, \hat{x}} \to J_{\mathbb{F}_q, \hat{x}}
\]

here the Albanese morphism is given by the formal serie \((1 - t^{-1}t_x)^{-1}\). Therefore, one universal uniformizer \(s_1\) (unique up to units) can be obtained of the relation \(s_1/\sigma s_1 = (1 - t^{-1}t_x)^{-1}\). The coefficients of \(s_1\) are the roots of the Carlitz's polynomials \(\mathbb{F}_q\). For details about the local Jacobian, \(J_{\mathbb{F}_q, \hat{x}}\) to see [C], [KSU], [AMP].

**Remark 4.** Following the Remark 3 we can do an explicit calculation of \(H\) (here \(x\) is not necessarily rational), we suppose \(S = \text{Spec}(B)\) and connected. If \((E, i_{\hat{x}})\) is a \(t\)-module over \(A\) of rank \(n\) and \(\sigma\)-rank \(k\) with a \(x\)-formal level structure:

\[
i_{\hat{x}} : E_{\hat{x}}(B) = \liminf_{r \in \mathbb{N}} E_{mr_x}(B) \xrightarrow{\sim} (k(x)((t_x))/k(x)[[t_x]])^n
\]

and \((N = \sum_{i=1}^n B(\sigma) e_i, g_{\hat{x}})\) the associated \(A\)-motive with a \(x\)-formal level structure then \(H = \langle s_1, \ldots, s_k \rangle\), where \(s_i\) is the image of \(e_i\) by the level structure morphism \(g_{\hat{x}}\). By direct calculation we get

\[
(s_i)_x = \sum_{h\geq 0} \left( \sum_{1 \leq c \leq \deg(x)} \phi(c)(\alpha_{c,h}) \omega_{c,h} t_x^h \right)
\]
(similar to Theorem 5) where \( \{ \omega_c \}_{1 \leq c \leq \text{deg}(x)} \) is a \( \mathbb{F}_q \)-orthonormal basis of \( k(x) \) for the pairing trace. \((s_i)_c \) is the \( c^\text{th} \)-component of \( s_i \in (k(x) \otimes B)[[t_x]]^n \) and \( \alpha_{c,h}^x \in \lim_{r \to \infty} E_{m_{c,h}}(B) \) verifying
\[
i_x(\alpha_{c,h}^x) = (0, \ldots, \omega_c t_x^{-1 - h}, \ldots, 0) \in (k(x)((t_x))/k(x)[[t_x]])^n
\]
in this last equality we have used the fixed isomorphism of \( k(x)[[t_x]] \)-modules: \( \mu_x : \Omega_x \to k(x)[[t_x]] \).

From now we are studying the case of classical elliptic sheaves in a more precise way.

4. Discrete subspaces and elliptic Baker function.

In this section we answer the following questions: How acts the classical arithmetic groups on \( H = \langle s_1 \rangle \)? (unfortunately, the classical arithmetic groups do not act on general \( A \)-motives, because d) of the Definition 2.3). Is it possible to describe basis for \( D_N \) and \( \mathcal{L}_0^\xi \) in terms of \( \langle s_1 \rangle \)? Which is the counterpart of the Baker function in this context?
Let \( \beta \) be an element of \( \text{GL}_n(k(x)((t_x))) \), with \( v_x(\text{det}(\beta)) = l \), then the discrete subspace associated to the classical action, \( (\mathcal{E}_j, i_j, \tau, f_j)^\beta \), of \( \beta \), in the elliptic sheaves of rank \( n \) with a formal level structure over \( S \) is:
\[\beta \mathcal{L}_{-l} \cap e^*(k(x)[[t_x]]^n)\]
and if \( l \geq 0 \), \( \beta^{-1} s_1 \in \mathcal{L}_0^\xi \) because
\[\mathcal{L}_{-l} \cap e^*(k(x)[[t_x]]^n) \subseteq \mathcal{L}_0^\xi \cap e^*(\beta^{-1}(k(x)[[t_x]]^n))\]

In the next theorem, we are going to get a basis for \( \mathcal{L}_0^\xi \), by using the action of \( \text{GL}_n(k(x)((t_x))) \) on \( \langle s_1 \rangle \). Again to do this we suppose \( x \) rational.

**Lemma 4.1.** \( \{ \beta_{(i,r)}^{-1} s_1^{\beta_{(i,r)}} \}_{r \geq 0, 1 \leq i \leq n} \) is a basis of \( \mathcal{L}_0^\xi \) as \( \mathcal{O}_S \)-module, where \( \beta_{(i,r)} \) is the diagonal matrix \( \text{diag}(1, \ldots, 1, t_x^{-r}, 1, \ldots, 1) \).
(The elliptic sheaf associated to \( s_1 \) is defined over \( S \).

**Proof.** By the before observation
\[\beta_{(i,r)}^{-1} s_1^{\beta_{(i,r)}} \in \mathcal{L}_0^\xi \cap (\mathcal{O}_S[[t_x]] \oplus \cdots \oplus t_x^{-r} \mathcal{O}_S[[t_x]]) \]
however we deduce that
\[\{ \beta_{(r,i)}^{-1} s_1^{\beta_{(r,i)}} \}_{r \geq 0, 1 \leq i \leq n}\]
are linearly independent, since that the \( k^\text{th} \)-components of \( s_1^{\beta_{(i,r)}} \) are units in \( \mathcal{O}_S[[t_x]] \) because if \( \beta \in \text{GL}_n(\mathbb{F}_q((t_x))) \) with \( \text{deg}(\beta) > 0 \) is \( \mathcal{L}_0^\beta \cap \beta \mathcal{O}_S[[t_x]]^n = 0 \). On the other hand, as
\[\bigcup_{h \in \mathbb{N}} (\mathcal{L}_0^\xi \cap t_x^{-h} \mathcal{O}_S[[t_x]]^n) = \mathcal{L}_0^\xi \]
and \( \text{rank} \mathcal{O}_x(\mathcal{L}_0^\xi \cap t_x^{-h} \mathcal{O}_S[[t_x]]) = nh + 1 \) we conclude that \( \{ \beta_{(i,r)}^{n-1} s_{1^{\beta(i,r)}} \}_{r \geq 0, 1 \leq i \leq n} \) are generators, therefore it is a basis.

**Lemma 4.2.** For \( r < 0, 1 \leq \xi \leq n \) is

\[
S_{1}^{\beta_{(\xi,r)}} = \beta_{(\xi,r)} \det \begin{pmatrix}
\alpha_0^\xi & \alpha_1^\xi & \cdots & \alpha_{r-1}^\xi & s_1 \\
(\alpha_0^t)^q & (\alpha_1^t)^q & \cdots & (\alpha_{r-1}^t)^q & \sigma^*(s_1) \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
(\alpha_0^q)^{q-r} & (\alpha_1^q)^{q-r} & \cdots & (\alpha_{r-1}^q)^{q-r} & (\sigma^*)^{-r}(s_1)
\end{pmatrix}
\]

up to units of \( H^0(S) \). We have used the notation of the Remark \(^4\) recall that in this case as \( k = 1 \), \( \phi(e_1) = Id \).

**Proof.** The proof follows \(< S_{1}^{\beta_{(\xi,r)}} > = \beta_{(\xi,r)} \mathcal{L}_0^\xi \cap \mathcal{O}_S[[t_x]]^n \) and

\[
\beta_{(\xi,r)} \det \begin{pmatrix}
\alpha_0^\xi & \alpha_1^\xi & \cdots & \alpha_{r-1}^\xi & s_1 \\
(\alpha_0^t)^q & (\alpha_1^t)^q & \cdots & (\alpha_{r-1}^t)^q & \sigma^*(s_1) \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
(\alpha_0^q)^{q-r} & (\alpha_1^q)^{q-r} & \cdots & (\alpha_{r-1}^q)^{q-r} & (\sigma^*)^{-r}(s_1)
\end{pmatrix} \in \beta_{(\xi,r)} \mathcal{L}_0^\xi
\]

Note that if \( d \in \mathbb{N} \) is such that \( m^d_{\infty} = bA_x \) is principal and

\[
H^0(X - \{ x \}, \mathcal{O}_X) = A_x \subset \mathbb{F}_q((t_x))
\]

then \( b = ut_x^{-d} \) where \( u \) is a unit in \( \mathbb{F}_q((t_x)) \), therefore \( s_1^{\text{diag}(t_x^{-d}, \ldots, t_x^{-s})} = u^{-1}s_1 \) since that \( s_1^{u t_x^{-d}} = s_1 \) (up to units).

This remark together Lemma 4.1 and

\[
\beta_{(\xi,1)} = t_x^{d_{\xi-1,1-d}} \cdots \beta_{(\xi,-d+1)} \cdots \beta_{(\xi,-d)}
\]

gives us an explicit basis of \( \mathcal{L}_0^\xi (S_1^t = us_1) \), therefore \( H =< s_1 > \) determines \( \mathcal{L}_0^\xi \).

The Proposition 5.1 of [SW] about the Baker function gives here an analogous result. Before we are going to state some notation

\[
\text{GL}_n^0(\mathbb{F}_q((t_x)))(S) = \{ g \in \text{GL}_n(H^0(S)^n((t_x))) / \text{v}_x(\text{det}(g)) = 0 \text{ for all } s \in \text{Spec}(H^0(S)) \mid H^0(S)^n = \{ c \in H^0(S)/c^d = c \} \}
\]

**Proposition 4.3.** For each \( \mathcal{L}_0^\xi \), with the usual notation in this paper, there exists a unique, up to units of \( H^0(S) \), (algebraic) function:

\[
\Psi_{\mathcal{L}_0^\xi} : \text{GL}_n^0(\mathbb{F}_q((t_x)))(S) \to H^0(S)((t_x))^n
\]

such that:

1) \( \Psi_{\mathcal{L}_0^\xi}(g) \in \mathcal{L}_0^\xi \) for every \( g \in \text{GL}_n^0(\mathbb{F}_q((t_x)))(S) \).

2) \( \Psi_{\mathcal{L}_0^\xi}(g) = g^{-1}(\sum_{i=0}^{\infty} a_i(t_x)^i) \) with \( a_i \) (algebraic) functions from \( \text{GL}_n^0(\mathbb{F}_q((t_x)))(S) \) to \( H^0(S)^n \), \( a_0 \) takes values on \( H^0(S)^{n^n} \).

**Proof.** This function is defined by \( \Psi_{\mathcal{L}_0^\xi}(g) = g^{-1}S_1^t \). It is unique by the properties of the elliptic sheaves. Obviously it extends on the whole of \( \text{GL}_n^0(\mathbb{F}_q((t_x)))(S) \).
This function is algebraic since it can be interpreted as a morphism of functors in the category of $S$-schemes.

From the Lemma 4.2 we can get an explicit calculation for $\Psi_{\mathcal{L}}(g)$, with $g \in \text{Gl}_n(\mathbb{F}_q[[t_x]])$, since diagonal matrixs and $\text{Gl}_n(\mathbb{F}_q[[t]])$ generate $\text{Gl}_n(\mathbb{F}_q[[t]])$ and $\Psi_{\mathcal{L}}(g) = s_1$ for all $g \in \text{Gl}_n(\mathbb{F}_q[[t]])$.

In an obvious way we can translate these results to Krichever $D$-modules [3].

5. Elliptic sheaf triples.

In this section, we are going to translate the results of the antiequivalence of the super Krichever functor, [MR], [SW], into our setting of 1-dimensional subspaces of uniformizers:

Paraphrasing the classic case for Schur pairs, data $(\infty, X, x, g_x, (\mathcal{E}, i_j, \tau, f_z))$ have associated triples $(\infty, A_x, \mathcal{L}^\xi_0)$, where $x$ is a rational point of the nonsingular projective curve $X$, $\infty$ is a point of $X - \{x\}$, $(\mathcal{E}, i_j, \tau, f_z)$ an elliptic sheaf of rank $n$ on $A = H^0(X - \{\infty\}, \mathcal{O}_X)$ with a $x$-formal level structure. $g_x : \mathcal{O}_X \to \mathbb{F}_q[[t_x]]$ a $x$-formal level structure on $\mathcal{O}_X$ which is a ring morphism, compatible with $f_z$ and the $\mathcal{O}_x$-module structure of $\mathcal{E}$, $A_x$ the subspace discrete of $\mathbb{F}_q((t_x))$ defined by $A_x = g_x(H^0(X - \{x\}, \mathcal{O}_X))$. $\mathcal{L}^\xi_0$ is the discrete subspace associated to $(\mathcal{E}, i_j, \tau, f_z)$ in the last section. We name elliptic sheaf triples to triples $(\infty, A_x, \mathcal{L}^\xi_0)$.

Bearing in mind [MR], [SW], it is not hard to set up an one to one correspondence between isomorphisms of data

$$(\infty, X, x, g_x, (\mathcal{E}, i_j, \tau, f_z))$$

and isomorphisms of elliptic sheaf triples.

Remark 5. If we fix $d \in \mathbb{N}$ with $m^d_\infty = bA_x$, $< s_1 >$ determines $(\infty, A_x, \mathcal{L}^\xi_0)$; because $< s_1 >$ and the action of $\text{Gl}_n(\mathbb{F}_q((t_x)))$ gives a basis of $\mathcal{L}^\xi_0$ (Lemma 4.2) moreover since $A_x$ is non singular

$$A_x = \{ c \in \mathbb{F}_q((t_x))/c.\mathcal{L}^\xi_0 \subseteq \mathcal{L}^\xi_0 \}$$

and $\infty = supp(\mathcal{L}^\xi_0 + \sigma^*(\mathcal{L}^\xi_0)/\mathcal{L}^\xi_0)$. $d$ is said to be the exponent of the elliptic sheaf triple $(\infty, A_x, \mathcal{L}^\xi_0)$. Theorem 3.1 shows this result by assuming $\infty$ and $A_x$ fixed.

We are going to use this last result to study the relative elliptic sheaf triples and its moduli problem.

Let $f : S \to \bar{S}$ be a morphism of schemes with $\bar{S}$ a scheme where the Frobenius morphism $\sigma$ is the identity.

Definition 5.1. An elliptic sheaf triple of rank $n$ and exponent $d$ over $f : S \to \bar{S}$, is a triple $(I, A_x, \mathcal{L}_0)$ where $A_x$ is a ring discrete subspace of $\mathcal{O}_{\bar{S}}((t_x))$ together a discrete ideal $I$ of $\mathcal{O}_{\bar{S}}((t_x))$ such that $A_x/I = 0_{\bar{S}}$. $I/I^2$ is locally free of rank 1 over $\mathcal{O}_{\bar{S}}$ and $f^*I^d = \text{ut}^d f^*A_x$ with $u$ a unit in $H^0(S)^\sigma [[t_x]] (H^0(S)^\sigma = \{ c \in H^0(S)/c^d = c \}). \mathcal{L}_0$ is a discrete subspace of $\mathcal{O}_{\bar{S}}((t_x))^n$ which is an $f^*A_x$-module fulfilling the following conditions:

1) $f^*I(\mathcal{L}_0 + \sigma^*\mathcal{L}_0 + \cdots + (\sigma^*)^n\mathcal{L}_0) \subseteq \mathcal{L}_0$.

2) For every $1 \leq i \leq n$

$$\mathcal{L}_0/f^*I(\mathcal{L}_0 + \sigma^*\mathcal{L}_0 + \cdots + (\sigma^*)^i\mathcal{L}_0)$$

is locally free of rank $n - i$. 
3) For every $s \in S$
\[
dim_{k(s)}(\mathcal{L}_0, k(s)[[t_x]]^n) - \dim_{k(s)}(k(s)((t_x)))/\mathcal{L}_0 + k(s)[[t_x]]^n = 1.
\]

4) $(A_x)_s$ is a Dedekind ring for each $s \in S$.

We denote by $F^d_n$ to the functor of elliptic sheaf triples of rank $n$ and exponent $d$.

As in the last section, $(I, A_x, \mathcal{L}_0)$ has associated a 1-dimensional subspace of uniformizers, $H = < s_1 > = \mathcal{L}_0 \cap \mathcal{O}_S[[t_x]]^n$, furthermore this subspace determines the triple $(I, A_x, \mathcal{L}_0)$:

**Proposition 5.2.** If $(I, A_x, \mathcal{L}_0)$ and $(I', A'_x, \mathcal{L}'_0)$ are two elliptic sheaf triples of rank $n$ and exponent $d$, over a scheme $S$ with subspace of uniformizers $H = < s_1 >$, then $(I, A_x, \mathcal{L}_0) = (I', A'_x, \mathcal{L}'_0)$.

**Proof.** Since $k(s) = \mathbb{F}_q$ for each $s \in S$, $(\sigma = Id)$, Remark 3 shows that
\[
(I, A_x, \mathcal{L}_0) = (I', A'_x, \mathcal{L}'_0)
\]
where $f$ is the morphism from $S$ to $\bar{S}$, thus the closed subscheme $C$ where $\mathcal{L}'_0 = \mathcal{L}_0$ (Theorem 3.7) contains $f^{-1}(s)$ for each $s \in S$, we have to prove that $C = S$. Locally $S = \text{Spec}(B)$, $C = \text{Spec}(B/J)$ and $J \subseteq \bigcap s_m B$, with $m_s$ the prime ideal associated to $s \in \bar{S}$, bearing in mind that $\sigma$ is the identity on $\bar{S}$ is $J_{m_s} = 0$ for each $s \in \bar{S}$, so $J = 0$.

On the other hand a similar argument proves that, $I = I'$ and $A_x = A'_x$. □

This last proposition gives an injective morphism of functors between $F^d_n$ and the functor of points of $Gr^1((\mathbb{F}_q[[t_x]])^n)$.

**Theorem 5.3.** $F^d_n$ is representable.

**Proof.** We are going to sketch the proof. Bearing in mind Sato’s infinite Grassmannian scheme and (Theorem 3.7), the set of points which define discrete subspaces $(L', L)$ verifying $L'L = L$, $\mathbb{F}_q \subset L$ and $L'L = L'$ is a subspace of $Gr(\mathbb{F}_q[[t_x]], \mathbb{F}_q((t_x))) \times Gr(\mathbb{F}_q[[t_x]], \mathbb{F}_q((t_x)))$

so the functor of pairs $(I, A_x)$ where $A_x$ is a discrete subring over $S$ and $I$ is an discrete ideal of $A_x$ with the added conditions $A_x/I = \mathcal{O}_S$ and $(A_x)_s$ is a Dedekind ring for each $s \in S$ is representable by a scheme $Z$.

Let $Z_\sigma$ be the subscheme of $Z$ where the Frobenius morphism $\sigma$ is the identity.

Let $\Sigma_0$ and $(I, A_x)$ be the pull back of universal objects for discrete subspaces and pairs $\{(I, A_x)\}$ over $Z_\sigma$, respectively, on the scheme $W = Gr(\mathbb{F}_q[[t_x]]^n, \mathbb{F}_q((t_x)))^n \times Z_\sigma$ by the natural projections, then to get a representant for $F^d_n$ we must impose to $\Sigma_0$ and $(I, A_x)$ similar conditions to the Definition 5.1 ones:

- $\Sigma_0$ is an $A_{x}$-module.
- For every $s \in W$ $\dim_{k(s)}(\Sigma_0, k(s)[[t_x]]^n) = 1$ and
\[
\dim_{k(s)}(k(s)((t_x)))/\Sigma_0 + (k(s)[[t_x]]^n) = 0.
\]
Proposition 6.2. 

If locally \( s_1 > 0 \) \( \mathfrak{L}_0 \cap (H^0(W')[[t_x]])^n \), \( s_1^{-d} = u^{-1}s_1 \) where \( u \) is a unit in \( H^0(W') \), the scheme where the last two conditions are verified, the definition of \( s_1^{-d} \) is obtained from the Lemma [12].

\( \mathcal{I}^d = u t_x^{-d} \mathcal{A}_x \).

\( \mathcal{I}/\mathcal{I}^2 \) is locally free of rank 1.

\( \mathcal{I} (\mathfrak{L}_0 + \sigma^* \mathfrak{L}_0 + \cdots + (\sigma^*)^n \mathfrak{L}_0) \subseteq \mathfrak{L}_0 \).

For every \( 1 \leq i \leq n \),

\[
\mathfrak{L}_0/\mathcal{I}(\mathfrak{L}_0 + \sigma^* \mathfrak{L}_0 + \cdots + (\sigma^*)^i \mathfrak{L}_0)
\]

is locally free of rank \( n - i \).

\[ \square \]

6. Determinants for elliptic sheaves.

In this section we are going to study the behaviour of the determinant for discrete and locally dense subspaces associated to elliptic sheaves. In our setting, for discrete subspaces, we obtain the same results that [An1].

One can easily show that the determinant for an elliptic sheaf of rank \( n \), with the obvious definition of determinant, is an elliptic sheaf of rank 1, but when we work with level structures, we have to twist the natural definition of determinant for elliptic sheaves with \( x \)-formal level structures to get again elliptic sheaves with \( x \)-formal level structures:

Recall from definitions that if \( (\mathcal{E}_j, i_j, \tau, f_\hat{x}) \) is an elliptic sheaf of rank \( n \) over \( S \) with a \( x \)-formal level structure, then \( f_\hat{x} \) is \( (\hat{\otimes}_{x}\mu_x)(((f_\hat{x})^1)_{x}) \) where

\[
f_\hat{x} : \mathfrak{E}_0 \rightarrow \lim_{m \in \mathbb{N}} \left( \Omega_X / \Omega_X (-mx) \right)^n
\]

is a twisted \( x \)-formal level structure on the elliptic sheaf and \( \mu_x \) is a fixed isomorphism from \( \Omega_x \) to \( \mathbb{F}_q[[t_x]] \).

**Definition 6.1.** We define the elliptic sheaf determinant with \( x \)-formal level structure, \( \det(\mathcal{E}_j, i_j, \tau, f_\hat{x}) \), of an elliptic sheaf of rank \( n \) over \( S \) with a \( x \)-formal level structure, \( (\mathcal{E}_j, i_j, \tau, f_\hat{x}) \). As

\[
\left( (\bigwedge_{n} \mathcal{E}_j) \otimes \Omega_X^{1-n} \right)_{x} \Lambda_{\bigwedge_{n} i_j, \Lambda_{\bigwedge_{n} \tau, \mu_x (f_\hat{x})}}
\]

being \( f_\hat{x} \) the formal level structure obtained from

\[
\bigwedge_{n} f_\hat{x} : \bigwedge_{n} \mathcal{E}_0 \rightarrow \lim_{m \in \mathbb{N}} \Omega_X^{1-n} / \Omega_X (-mx)
\]

by tensoring by \( \Omega_X^{1-n} \). Recall that a level structure over \( \mathfrak{E}_0 \) (with some extra conditions) determines the level structure in the \( k \)-elliptic sheaf, \( (\mathcal{E}_j, i_j, \tau, f_\hat{x}) \), since characteristic is away from \( x \).

From this definition we can obtain the analogous result of [An1] for discrete subspaces in a straightforward way:

**Proposition 6.2.** If \( \mathfrak{L}_0^d \) is the associated discrete subspace to \( (\mathcal{E}_j, i_j, \tau, f_\hat{x}) \) then

\[
\left( (\bigwedge_{n} \mathfrak{L}_0^d) \otimes \mu_x (\Omega_X (-x))^{1-n} (n-1)(g-1) \right)
\]
is the discrete subspace for \( \det(\mathcal{E}_j, i_j, \tau, f_z) \).

Now we are describing the determinant for locally dense subspaces, \( D_N \), associated to elliptic sheaves. With the notation of the last definition we must study the \( A \otimes O_S \{ \sigma \} \)-module
\[
\mu_x(f_z^*)((\bigwedge^n_{\mathcal{O}_X \otimes O_S} \mathcal{E}_0) \otimes O_X^{1-n})_{\text{Spec}(A) \times S}) = (\bigwedge^n_{A \otimes O_S} D_N) \otimes A \mu_x(\Omega_A)^{1-n} \subset O_S[[t_x]]^n
\]
for it we are going to consider \( \bigwedge^n_{A \otimes O_S} D_N \).

**Lemma 6.3.** If \( O_S \{ \sigma \}(= \mathcal{N}) \) is the \( A \)-motive associated to an elliptic sheaf of rank \( n \) then \( \bigwedge^n_{A \otimes O_S} O_S \{ \sigma \} \) is generated by \( 1 \wedge \sigma \wedge \cdots \wedge \sigma^{n-1} \) as \( A \otimes O_S \{ \sigma \} \)-module if and only if \( \{ 1, \sigma, \cdots, \sigma^{n-1} \} \) are generators of the \( O_S \)-module \( O_S \{ \sigma \}/O_S \{ \sigma \} J \), where \( J \) is the ideal characteristic for the elliptic sheaf. (\( J = \text{Ann}(\mathcal{E}_1/\mathcal{E}_0) \)) and \( O_X \otimes O_S / J \otimes O_S \) with the standard notation.

**Proof.** To prove the lemma it suffices to suppose that \( S = \text{Spec}(K) \) is a field.
\[
\bigwedge^n_{A \otimes K} K \{ \sigma \}
\]
are a \( A \otimes K \)-basis of \( K \{ \sigma \} = \text{Spec}(A \otimes K) \). If we denote by \( D_m \) the effective divisor on \( X_K \) where \( \{ \sigma^m, \cdots, \sigma^{m+n-1} \} \) is not a basis, we have to show that \( \bigcap_{m} \text{supp}(D_m) = \phi \) if and only if \( q \) is not in \( \text{supp}(D_0) \), where \( q \in X_K \) is the characteristic of the elliptic sheaf. Since \( D_{m+1} = F^* D_m + q \) and therefore \( D_{m+1} = F^* D_0 + q + F^* q + \cdots + (F^*)^m q \) to conclude it is enough to prove \( \bigcap_{m} \text{supp}((F^*)^m D_0) = \phi \). But if \( y \in \bigcap_{m} \text{supp}((F^*)^m D_0) \) then there exists \( z \in \text{Spec}(A) \) such that \( \{ 1, \sigma, \cdots, \sigma^{n-1} \} \) are not linearly independent in \( K \{ \sigma \}/K \{ \sigma \} m_z \) as \( A/m_z \otimes K \)-module. But by induction over \( r \leq n - 1 \) one can check that \( \{ 1, \sigma, \cdots, \sigma^r \} \) are linearly independent for all \( r \leq n - 1 \) in \( K \{ \sigma \}/K \{ \sigma \} m_z \) as \( A/m_z \otimes K \)-module.

One can observe that in an elliptic sheaf over \( S \), the set of points where the lemma is verified is an open subset of \( S \). One can also check through a similar reasoning as in the foregoing lemma that if the characteristic of an elliptic sheaf is a rational point of \( \text{Spec}(A) \), this elliptic sheaf verifies the conditions of the lemma. So we can hope that the open subset of \( D^*_N \) where the condition of the lemma is verified is not empty.

From these lemmas we obtain:

**Theorem 6.4.** If \( (\mathcal{E}_j, i_j, \tau, f_z) \) is an elliptic sheaf of rank \( n \) over \( S \) in the conditions of the previous lemma, with a \( x \)-formal level structure and locally dense subspace \( D_N \) then \( \bigwedge^n(D_N) \) is generated as \( A \otimes O_S \{ \sigma \} \)-module by the determinant of the "scattering" matrix
\[
(s_1, \sigma^* s_1, \cdots, (\sigma^*)^{n-1} s_1)
\]
where \( < s_1 > \) is the subspace of uniformizers associated to \((\mathcal{E}_j, i_j, \tau, f_\tilde{x})\) in the last section.

Remark 6. It would be very interesting to obtain the subspace of uniformizers associated to \(\det(\mathcal{E}_j, i_j, \tau, f_\tilde{x})\) in terms of \( < s_1 > \leftrightarrow (\mathcal{E}_j, i_j, \tau, f_\tilde{x})\). When \(X = \mathbb{P}_1\) this subspace is \( < \det(s_1, \sigma s_1, \cdots, (\sigma^*)^{n-1}s_1) >\) because \((1, \sigma, \cdots, \sigma^{n-1})\) is a \(\mathbb{F}_q[t] \otimes \mathcal{O}_S\)-basis of \(\mathcal{O}_S\{\sigma\}\). Unfortunately, one can check that this happens only when \(g = 0\).

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Álvarez Vázquez, Arturo
e-mail: aalvarez@gugu.usal.es

DEPARTAMENTO DE MATEMÁTICA PURA Y APLICADA, UNIVERSIDAD DE SALAMANCA, PLAZA DE LA MERCEDE 1-4, SALAMANCA 37008. SPAIN.