Jung’s Theorem and fixed points for $p$-uniformly convex spaces

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1 Introduction

Definition 1.1. [BCL][K] A metric space $(X, d)$ is called a $p$-uniformly convex space with parameter $k > 0$, if $(X, d)$ is a geodesic space and for any three points $x, y, z \in X$, any minimal geodesic $\gamma := (\gamma_t)_{t \in [0, 1]}$ in $X$ with $\gamma_0 = x, \gamma_1 = y$, and the midpoint $m$ of $x$ and $y$,

$$d^p(z, m) \leq \frac{1}{2} d^p(z, x) + \frac{1}{2} d^p(z, y) - \frac{k}{8} d^p(x, y).$$

By definition, putting $z = \gamma_0$ or $z = m$, we see $k \in (0, 2]$ and $p \in [1, \infty)$. The inequality yields the strict convexity of $Y \ni x \rightarrow d^p(z, x)$ for a fixed $z \in Y$. Any closed convex subset of a $p$-uniformly convex space is again a $p$-uniformly convex space with the same parameter. Any $L^p$ space over a measurable space is $p$-uniformly convex with parameter $k = 2^{3-p}$ provided $p > 2$, and it is 2-uniformly convex with parameter $k = 2(p - 1)$ provided $1 < p \leq 2$. More details are provided in [BCL]. A geodesic space is CAT(0) space if and only if it is a 2-uniformly convex space with parameter $k = 2$. Ohta [O] proved that for $\kappa > 0$ any CAT($\kappa$)-space $Y$ with $\text{diam}(Y) < R_\kappa/2$ is a 2-uniformly convex space with parameter $\{(\pi - 2\sqrt{\varepsilon}) \tan \sqrt{\varepsilon}\}$ for any $\varepsilon \in (0, R_\kappa/2 - \text{diam}(Y)]$.

The classical Jung theorem gives an optimal upper estimate for the radius of a bounded subset of $\mathbb{R}^n$ in terms of its diameter and dimension. In [LS], Lang and Schroeder also proved the similar Jung’s theorem for CAT($\kappa$) spaces. Here we will give an upper bound for the radius of a bounded subset of $p$-uniformly convex spaces.

Theorem 1.2. Let $X$ be a complete $p$-uniformly convex space and $S$ be a nonempty bounded subset of $X$. Then there exists a unique closed circumball $B(z, \text{rad}(S))$ of $S$ and

$$\text{rad}(S) \leq (1 + \frac{2^{p-3}k}{2^{p-1}-1})^{-\frac{1}{p}} \text{diam}(S).$$
Remark 1.3. For \( p = 2, k = 2 \), our result coincides with the classical Jung theorem for CAT(0) spaces. Using a similar method, we can give a shorter proof of the Jung theorem of CAT(\( \kappa \)) spaces [LS].

Theorem 1.4. [LS] Let \( X \) be a complete CAT(\( \kappa \)) space and \( S \) a nonempty bounded subset of \( X \). In case \( \kappa > 0 \) assume that \( \text{rad}(S) < \pi/(2\sqrt{\kappa}) \). Then there exists a unique closed circumball \( B(z, \text{rad}(S)) \) of \( S \) and

\[
\text{sn}_\kappa \text{ rad}(S) \leq \sqrt{2} \text{sn}_\kappa (\text{diam}(S)/2),
\]

where \( \text{sn}_\kappa \) is the function

\[
\text{sn}_\kappa(x) = \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}x) & \text{if } \kappa > 0, \\
x & \text{if } \kappa = 0, \\
\frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}x) & \text{if } \kappa < 0.
\end{cases}
\]

A mapping \( T : M \to M \) of a metric space \( (M, d) \) is said to be uniformly \( L \)-lipschitzian if there exists a constant \( L \) such that \( d(T^n x, T^n y) \leq L d(x, y) \), for all \( x, y \in M \) and \( n \in \mathbb{N} \). In [DKS] there is a following result for CAT(0) spaces

Theorem 1.5. [DKS] Let \( (X, d) \) be a bounded complete CAT(0) space. Then every uniformly \( L \)-lipschitzian mapping \( T : X \to X \) with \( L < \sqrt{2} \) has a fixed point.

Remark 1.6. In [GK], Baillon gave a uniformly \( \frac{\pi}{2} \)-lipschitzian mapping of Hilbert spaces which is fixed point free.

In [L2], Lim proved a general theorem for \( L^p \) spaces

Theorem 1.7. [L2] Let \( K \) be a closed convex bounded nonempty subset of \( L^p, 2 < p < \infty \), then every uniformly \( L \)-lipschitzian mapping \( T : K \to K \) with \( L < L_0 \) has a fixed point. Here

\[
L_0 \geq (1 + \frac{1}{2p-1})^\frac{1}{p} (p - 1)^{-\frac{1}{p}} (p - 2)^{\frac{1}{p} - 1} (1 + \frac{1}{2p-1})^\frac{1}{p}.
\]

We prove similar results for \( p \)-uniformly convex spaces,

Theorem 1.8. Let \( (X, d) \) be a bounded complete \( p \)-uniformly convex space with parameter \( k > 0 \). Then there exists a constant \( C = (1 + \frac{2p-3}{2p-1})^\frac{1}{p} (1 + \frac{1}{2p-1})^\frac{1}{p} \) such that for every uniformly \( L \)-lipschitzian mapping \( T : X \to X \) with \( L < C \) has a fixed point.

Remark 1.9. For CAT(0) spaces we have \( p = 2, k = 2 \), hence the Lifschitz constant \( L(X) \geq \sqrt{2} \) which is coincide with the result in [DKS]. For \( L^p \) spaces we have \( k = \frac{1}{2p-1} \), hence \( L(X) \geq (1 + \frac{1}{2p-1})^\frac{1}{p} \).
This paper is organized as follows. In Section 2 we introduce the classical Jung theorem and prove a similar one for $p$-uniformly convex spaces. Moreover using the same method, we can give a shorter proof for CAT($\kappa$) spaces. In Section 3 we show a general fixed point theorem for $p$-uniformly convex spaces which generalize the results in [EF][KP]. In Section 4 we prove that $p$-uniformly convex spaces enjoy the Property (P) which is defined by Lim and Xu. In Section 5, we generalize the result about $\Delta$-convergence from [EF][KP] for CAT($\kappa$) spaces.

2 Jung’s Theorem for $p$-uniformly convex space

Let $(X, d)$ be a metric space. For a nonempty bounded subset $D \subset X$, set

\[
  r_x(D) = \sup\{d(x, y) : y \in D\}, x \in X;
\]
\[
  \text{rad}(D) = \inf\{r_x(D) : x \in X\};
\]
\[
  \text{diam}(D) = \sup\{d(x, y) : x, y \in D\}.
\]

Clearly $\text{rad}(D) \leq \text{diam}(D) \leq 2\text{rad}(D)$. Jung’s theorem states that each bounded subset $D$ of $\mathbb{R}^n$ is contained in a unique closed ball with $\text{rad}(D)$, where

\[
  \text{rad}(D) \leq \sqrt{\frac{n}{2(n+1)}}\text{diam}(D).
\]

**Theorem 2.1.** Let $X$ be a complete geodesic $p$-uniformly convex space and $S$ be a nonempty bounded subset of $X$. Then there exists a unique closed circumball $B(z, \text{rad}(S))$ of $S$ and

\[
  \text{rad}(S) \leq (1 + \frac{2p^{-3}k}{2p^{-1} - 1})^{-\frac{1}{p}}\text{diam}(S).
\]

**Proof.** For any bounded closed subset $S \subset X$, choose $\{x_n\} \in S$ such that $\max\{\lim\sup_{n \to \infty} d(x, x_n), x \in S\} = \text{rad}(S)$. Now we want to show that $\{x_n\}$ is a Cauchy sequence. Suppose not, then there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$ there exist $i, j \geq N$ such that $d(x_i, x_j) \geq \varepsilon$. Choose $m_i$ as the midpoint of the segment $[x_i, x_j]$ and according to the $p$-uniformly convexity, we have

\[
  d^p(y, m_i) \leq \frac{1}{2}d^p(y, x_i) + \frac{1}{2}d^p(y, x_j) - \frac{k}{8}d^p(x_i, x_j)
\]

for all $i \geq N$ and $\forall y \in S$. Choose $N \in \mathbb{N}$ large enough such that $d^p(y, x_i) < \text{rad}^p(S) + \frac{k}{16}\varepsilon^p$ for all $i \geq N$ and $\forall y \in S$. Then we have

\[
  d^p(y, m_i) \leq \text{rad}^p(S) - \frac{k}{16}\varepsilon^p
\]

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which means $d(y, m_i) < \text{rad}(S)$ for all $i \geq N$ and $\forall y \in S$. Contradicts with the definition of $\text{rad}(S)$.

Denote $z$ as the circumcenter of $S$ and choose $w \in S$ such that $d(z, w) = \text{rad}(S)$. Choose $z_i \in S$ as the the midpoint of segment $[z, z_{i-1}]$, where $z_0 = w$ and $w_i \in S$ such that $d(z_i, w_i) \geq \text{rad}(S)$. Applying to the $p$-uniformly convexity we have

$$d^p(z_i, w_i) \leq \left(\frac{2^{i+1} - 1}{2^{i+1}} + \frac{1}{2^{i+1}}\right)\text{diam}^p(S) - \frac{k}{2^{i+1}} \sum_{j=0}^{i} \frac{1}{2^{j(p-1)}} l^p \text{diam}^p(S).$$

i.e.

$$l^p \text{diam}^p(S) \leq \left(\frac{2^{i+1} - 1}{2^{i+1}} l^p + \frac{1}{2^{i+1}}\right)\text{diam}^p(S) - \frac{k}{2^{i+1}} \sum_{j=0}^{i} \frac{1}{2^{j(p-1)}} l^p \text{diam}^p(S)$$

where $l = \frac{\text{rad}(S)}{\text{diam}(S)}$ i.e.

$$l^p \left(1 + \frac{k}{4} \sum_{j=0}^{i} \frac{1}{2^{j(p-1)}}\right) \leq 1.$$ 

Let $i \to \infty$, we obtain

$$l^p \leq \left(1 + \frac{k}{4} \frac{2^{p-1}}{2^{p-1} - 1}\right)^{-1} = \left(1 + \frac{2^{p-3}k}{2^{p-1} - 1}\right)^{-1}.$$ 

Hence we obtain

$$\text{rad}(S) \leq \left(1 + \frac{2^{p-3}k}{2^{p-1} - 1}\right)^{-\frac{1}{p}} \text{diam}(S).$$

Now we give a shorter proof of Jung’s theorem for CAT($\kappa$) spaces.

**Theorem 2.2.** [LS] Let $X$ be a complete CAT($\kappa$) space and $S$ a nonempty bounded subset of $X$. In case $\kappa > 0$ assume that $\text{rad}(S) < \pi/(2\sqrt{\kappa})$. Then there exists a unique closed circumball $B(z, \text{rad}(S))$ of $S$ and

$$\text{sn}_\kappa \text{rad}(S) \leq \sqrt{2} \text{sn}_\kappa (\text{diam}(S)/2),$$

Here we give the prove for $\kappa = 1, -1$.

**Lemma 2.3.** Let $X$ be a CAT($1$) space and $x, y, z \in X$ such that $d(x, y) + d(y, z) + d(z, x) < 2\pi$. Let $t \in [0, 1]$ and $u$ is on the segment $[y, z]$ such that $d(y, u) = td(y, z)$. Then

$$\cos d(x, u) \sin d(y, z) \geq \cos d(x, y) \sin(td(y, z)) + \cos d(x, z) \sin((1-t)d(y, z)).$$
Lemma 2.4. Let $X$ be a CAT(−1) space and $x, y, z \in X$. Let $t \in [0, 1]$ and $u$ is on the segment $[y, z]$ such that $d(y, u) = td(y, z)$. Then
\[
\cosh d(x, u) \sinh d(y, z) \leq \cosh d(x, y) \sinh( td(y, z) ) + \cosh d(x, z) \sinh((1 - t)d(y, z)).
\]

Proof. Consider a comparison triangle $\triangle(x, y, z)$ and apply the CAT(κ) inequality, we got the results.

Now we prove the Theorem 2.2

Proof. The uniqueness and existence of the circumball are directly from the result of $p$-uniformly convexity. Case $\kappa = 1$: for any bounded closed subset $S \subset X$, denote $z$ as the circumcenter of $S$ and choose $w \in S$ such that $d(z, w) = \text{rad}(S)$. $u_t$ is on the segment $[z, w]$ such that $d(z, u_t) = td(z, w)$ and $w_t \in S$ such that $d(u_t, w_t) \geq \text{rad}(S)$. According to the Lemma 2.3 we have
\[
\cos d(w_t, u_t) \sin d(z, w) \geq \cos d(w_t, w) \sin (td(z, w)) + \cos d(w_t, z) \sin((1-t)d(z, w))
\]
i.e.
\[
\cos \text{rad}(S) \sin \text{rad}(S) \geq \cos \text{diam}(S) \sin(\text{trad}(S)) + \cos \text{rad}(S) \sin((1-t)\text{rad}(S))
\]
\[
2 \cos \text{rad}(S) \sin \frac{t}{2} \text{rad}(S) \cos(1 - \frac{t}{2})\text{rad}(S) \geq \cos \text{diam}(S) \sin(\text{trad}(S)).
\]
Hence
\[
\frac{\cos \text{rad}(S) \cos(1 - \frac{t}{2})\text{rad}(S)}{\cos \frac{t}{2} \text{rad}(S)} \geq \cos \text{diam}(S) \text{ for all } t \in (0, 1). \text{ Let } t \to 0, \text{ we obtain}
\]
i.e.
\[
\cos^2 \text{rad}(S) \geq \cos \text{diam}(S)
\]
\[
1 - \cos^2 \text{rad}(S) = \sin^2 \text{rad}(S) \leq 1 - \cos \text{diam}(S) = 2 \sin^2 \text{diam}(S)/2.
\]
Thus
\[
\sin \text{rad}(S) \leq \sqrt{2} \sin \text{diam}(S)/2.
\]
Case $\kappa = -1$: it is similar as the case of $\kappa = 1$.

3 Fixed points in $p$-uniformly convex space

We now turn to the definition of the Lifshic character of a metric space $X$. Balls in $X$ are said to be $c$-regular if the following holds: For each $k < c$ there exist $\mu, \alpha \in (0, 1)$ such that for each $x, y \in X$ and $r > 0$ with $d(x, y) \geq (1 - \mu)r$, there exists $z \in X$ such that
\[
B(x; (1 + \mu)r) \cap B(y; k(1 + \mu)r) \subset B(z; \alpha r)
\]
The Lifshitz character $L(X)$ of $X$ is defined as follows:
\[
L(X) = \sup \{ c \geq 1 : \text{balls in } X \text{ are } c\text{-regular} \}. 
\]
Theorem 3.1. Let \((X,d)\) be a bounded complete metric space. Then every uniformly \(L\)-lipschitzian mapping \(T : X \to X\) with \(L < L(X)\) has a fixed point.

Theorem 3.2. Let \((X,d)\) be a bounded complete \(p\)-uniformly convex space with parameter \(k > 0\). Then there exists a constant \(C = (1 + 2p^{-3k})^{\frac{1}{p}}\) such that for every uniformly \(L\)-lipschitzian mapping \(T : X \to X\) with \(L < C\) has a fixed point.

Proof. We just have to show the Lifshitz character of \(X\)

\[
L(X) \geq (1 + \frac{2p^{-3k}}{2^{p-1} - 1})^{\frac{1}{p}}.
\]

For each \(x, y \in X\) and \(r > 0\) with \(d(x, y) \geq (1 - \mu)r\), denote

\[
A := B(x; (1 + \mu)r) \cap B(y; l(1 + \mu)r).
\]

Choose the midpoint \(m_0\) between \(x\) and \(y\), for any \(z \in A\) applying the \(p\)-uniformly convexity, we have

\[
d(z, m_0)^p \leq \left(\frac{1}{2} + \frac{1}{2}(1 + \mu)^p r^p - \frac{k}{8}(1 - \mu)^p r^p\right).
\]

Let \(\mu\) small enough such that \((1 + \mu)^p < 1 + \varepsilon\). Since \((1 + \mu)^p + (1 - \mu)^p \geq 2\), we obtain \((1 - \mu)^p \geq 1 - \varepsilon\). Hence

\[
d(z, m_0)^p \leq \left(\frac{1}{2} + \frac{1}{2}(1 + \mu)^p r^p - \frac{k}{8}(1 - \mu)^p r^p\right).
\]

Choose \(m_1\) be the midpoint between \(x\) and \(m_0\), for any \(z \in A\) applying the \(p\)-uniformly convexity again, we have

\[
d(z, m_1)^p \leq \left(\frac{1}{2} + \frac{1}{2}(1 + \mu)^p r^p - \frac{k}{8}(1 - \mu)^p r^p\right).
\]

i.e.

\[
d(z, m_1)^p \leq \left(\frac{3}{4} + \frac{1}{4}(1 + 2p^{-3k})^p r^p - \frac{k}{8}(1 - \mu)^p r^p + \frac{1}{2^{p-1}} M \varepsilon r^p\right).
\]

Inductively, choose \(m_i\) as the midpoint of \(x\) and \(m_{i-1}\). Therefore we have

\[
d(z, m_i)^p \leq \left(\frac{2^{i+1} - 1}{2^{i+1}} + \frac{1}{2^{i+1}}(1 + \mu)^p r^p - \frac{k}{2^{i+3}} \sum_{j=0}^{i} \frac{1}{2^{j(p-1)}} r^p + \frac{1}{2^{i-2}} M \varepsilon r^p\right).
\]

Let \(\alpha \to 1, \mu \to 0\), we get

\[
\left(\frac{2^{i+1} - 1}{2^{i+1}} + \frac{1}{2^{i+1}}(1 + \mu)^p r^p - \frac{k}{2^{i+3}} \sum_{j=0}^{i} \frac{1}{2^{j(p-1)}} r^p \right) \leq r^p.
\]
\[ l^p \leq 1 + \frac{k}{4} \sum_{j=0}^{i} \frac{1}{2^{j(p-1)}}. \]

Let \( i \to \infty \), we obtain
\[ l^p \leq 1 + \frac{k}{4} \frac{2^{p-1} - 1}{2^{p-1} - 1} = 1 + \frac{2^{p-3}k}{2^{p-1} - 1}. \]

Hence \( L(X) \geq (1 + \frac{2^{p-3}k}{2^{p-1} - 1})^\frac{1}{p} \). \( \square \)

4 \( p \)-uniformly convex spaces and Property (P)

A subset \( A \) of \( X \) is said to be admissible if \( \text{cov}(A) = A \) here
\[ \text{cov}(A) = \cap \{ B : B \text{ is a closed ball and } A \subset B \}. \]

The number
\[ \hat{N}(X) := \sup \left\{ \frac{\text{rad}(A)}{\text{diam}(A)} \right\}, \]

where the supremum is taken over all nonempty bounded admissible subsets \( A \) of \( X \) for which \( \delta(A) > 0 \), is called the normal structure coefficient of \( X \). If \( \hat{N}(X) \leq c \) for some constant \( c < 1 \) then \( X \) is said to have uniform normal structure.

Lim and Xu introduced the so-called property (P) for metric spaces. A metric space \( (X, d) \) is said to have property (P) if given two bounded sequences \( \{x_n\} \) and \( \{z_n\} \) in \( X \), there exists \( z \in \cap_{n \geq 1} \text{cov}(\{z_j : j \geq n\}) \) such that
\[ \limsup_{n} d(z, x_n) \leq \limsup_{j} \limsup_{n} d(z_j, x_n). \]

The following theorem is the main result of \cite{LX}

**Theorem 4.1.** \cite{LX} Let \( (X, d) \) be a complete bounded metric space with both property (P) and uniform normal structure. Then every uniformly \( L \)-lipschitzian mapping \( T : X \to X \) with \( L < \hat{N}(X)^{-\frac{1}{p}} \) has a fixed point.

From Theorem 2.1, for any \( p \)-uniformly convex space \( X \) we have \( \hat{N}(X) \leq (1 + \frac{2^{p-3}k}{2^{p-1} - 1})^{-\frac{1}{p}} < 1 \). Hence \( X \) has uniform normal structure. In this section we show that every complete geodesic \( p \)-uniformly convex spaces have property (P).

Let \( \{x_n\} \) be bounded sequence in a complete geodesic \( p \)-uniformly convex space and let \( K \) be a closed and convex subset of \( X \). Define \( \phi : X \to \mathbb{R} \) by setting \( \phi(x) = \limsup_{n \to \infty} d(x, x_n), \ x \in X \).
Proposition 4.2. There exists a unique point \( u \in K \) such that
\[
\phi(u) = \inf_{x \in K} \phi(x)
\]

Proof. Let \( r = \inf_{x \in K} \phi(x) \) and let \( \epsilon > 0 \). Then by assumption there exists \( x \in K \) such that \( \phi(x) < r + \epsilon \); thus for \( n \) sufficiently large \( d(x,x_n) < r + \epsilon \), i.e., for \( n \) sufficiently large \( x \in B(x_n, r + \epsilon) \). Thus
\[
C_\epsilon := \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} B(x_i, r + \epsilon) \cap K \neq \emptyset.
\]
As the ascending union of convex sets, clearly \( C_\epsilon \) is convex. Also the closure \( \bar{C}_\epsilon \) is also convex. Therefore
\[
C := \cap_{\epsilon > 0} \bar{C}_\epsilon \neq \emptyset.
\]
Clearly for \( u \in C \), \( \phi \leq r \). Uniqueness of such a \( u \) follows from the \( p \)-uniformly convexity.

In the view of the above, \( X \) has property (P) if given two bounded sequences \( \{x_n\} \) and \( \{z_n\} \) in \( X \), there exists \( z \in \cap_{n=1}^{\infty} \text{cov}\{z_j : j \geq n\} \) such that
\[
\phi(z) \leq \limsup_{j \to \infty} \phi(z_j),
\]
where \( \phi \) is defined as above.

Proposition 4.3. A complete geodesic \( p \)-uniformly convexity has property (P).

Proof. Let \( \{x_n\} \) and \( \{z_n\} \) be two bounded sequences in \( X \) and define \( \phi : X \to \mathbb{R} \) by setting \( \phi(x) = \limsup_{n \to \infty} d(x, x_n) \), \( x \in X \). For each \( n \), let
\[
C_n := \text{cov}\{z_j : j \geq n\}.
\]
By Proposition 4.2 there exists a unique point \( u_n \in C_n \) such that
\[
\phi(u_n) = \inf_{x \in C_n} \phi(x).
\]
Moreover, since \( z_j \in C_n \) for \( j \geq n \), \( \phi(u_n) \leq \phi(z_j) \) for all \( j \geq n \). Thus \( \phi(u_n) \leq \limsup_{j \to \infty} \phi(z_j) \) for all \( n \). We assert that \( \{u_n\} \) is a Cauchy sequence.

To see this, suppose not. Then there exists \( \epsilon > 0 \) such that for any \( N \in \mathbb{N} \) there exist \( i, j \geq N \) such that \( d(u_i, u_j) \geq \epsilon \). Also, since the sets \( \{C_n\} \) are descending, the sequence \( \{\phi(u_n)\} \) is increasing. Let \( d := \lim_{n \to \infty} \phi(u_n) \geq \frac{\epsilon}{2} \).

Choose \( \xi > 0 \) so small that \( \xi < (\cosh \frac{\xi}{4} - 1) \sinh \frac{\xi}{4} \), and choose \( N \) so large that \( \sinh \frac{\xi}{2} < \sinh \phi(u_j) - \xi \leq \sinh \phi(u_i) \leq \sinh \phi(u_j) \leq \sinh d \) if \( j \geq i \geq N \). Let \( m_{ij} \) denote the midpoint of the geodesic joining \( u_i \) and \( u_j \), and let \( n \in \mathbb{N} \). Then by the \( p \)-uniformly convexity
\[
d^p(m_{ij}, x_n) \leq \frac{1}{2}d^p(u_i, x_n) + \frac{1}{2}d^p(u_j, x_n) - \frac{k}{8}d^p(u_i, u_j).
\]
This implies
\[ \phi^p(m_j) \leq \phi^p(u_j) - \frac{k}{8} \varepsilon^p. \]
Since \( m_j \in C_j \), this contradicts the definition of \( u_j \).

This proves that \( \{u_n\} \) is a Cauchy sequence. Consequently there exists a \( z \in \bigcap_{n=1}^{\infty} C_n \) such that \( \lim_{n \to \infty} u_n = z \) and, since \( \phi \) is continuous, \( \lim_{n \to \infty} \phi(u_n) = \phi(z) \). Hence we conclude that
\[ \phi(z) \leq \limsup_{j \to \infty} \phi(z_j). \]
\[ \square \]

5 Basic properties of \( \Delta \)-convergence

In this section we show that \( \Delta \)-convergence can be used in \( p \)-uniformly convex spaces in a similar way as it is used in [KP] for \( \text{CAT}(0) \) spaces, obtaining a collection of similar results. To show this we begin with the definition of \( \Delta \)-convergence.

Let \( X \) be a complete \( p \)-uniformly convex space and \( (x_n) \) a bounded sequence in \( X \). For \( x \in X \) set
\[ r(x, (x_n)) = \limsup_{n \to \infty} d(x, x_n). \]

The asymptotic radius \( r((x_n)) \) of \( (x_n) \) is given by
\[ r((x_n)) = \inf \{r(x, (x_n)) : x \in X\}, \]
the asymptotic radius \( r_C((x_n)) \) with respect to \( C \subset X \) of \( (x_n) \) is given by
\[ r_C((x_n)) = \inf \{r(x, (x_n)) : x \in C\}, \]
the asymptotic center \( A((x_n)) \) of \( (x_n) \) is given by the set
\[ A((x_n)) = \{x \in X : r(x, (x_n)) = r((x_n))\}, \]
and the asymptotic center \( A_C((x_n)) \) with respect to \( C \subset X \) of \( (x_n) \) is given by the set
\[ A_C((x_n)) = \{x \in C : r(x, (x_n)) = r_C((x_n))\}. \]

From Proposition 4.2, we have the following

**Proposition 5.1.** Let \( X \) be a complete \( p \)-uniformly convex space, \( C \subset X \) nonempty bounded, closed and convex, and \( (x_n) \) a bounded sequence in \( X \). Then \( A_C((x_n)) \) consists of exactly one point.
Definition 5.2. A sequence \((x_n)\) in \(X\) is said to \(\Delta\)-converge to \(x \in X\) if \(x\) is the unique asymptotic center of \((u_n)\) for every subsequence \((u_n)\) of \((x_n)\). In this case we write \(\Delta\)-\(\lim_{n \to \infty} x_n = x\) and call \(x\) the \(\Delta\)-limit of \((x_n)\).

The next result follows as a consequence of the previous proposition.

Corollary 5.3. Let \(X\) be a complete bounded \(p\)-uniformly convex space and \((x_n)\) a sequence in \(X\). Then \((x_n)\) has a \(\Delta\)-convergent subsequence.

Next we show that we can give analogs in 2-uniformly convex spaces to those other results in Section 3 of \([KP]\) for CAT(0) spaces. Notice that this generalizes these results. In all the next definitions \(X\) is a 2-uniformly convex space and \(K \subset X\) bounded and convex.

Definition 5.4. A mapping \(T : K \to X\) is said to be of type \(\Gamma\) if there exists a continuous strictly increasing convex function \(\gamma : \mathbb{R}^+ \to \mathbb{R}^+\) with \(\gamma(0) = 0\) such that, if \(x, y \in K\) and if \(m\) and \(m'\) are the mid-points of the segments \([x, y]\) and \([T(x), T(y)]\) respectively, then
\[
\gamma(d(m, T(m))) \leq |d(x, y) - d(T(x), T(y))|.
\]

Definition 5.5. A mapping \(T : K \to X\) is called \(\alpha\)-almost convex for \(\alpha : \mathbb{R}^+ \to \mathbb{R}^+\) continuous, strictly increasing, and \(\alpha(0) = 0\), if for \(x, y \in K\),
\[
J_T(m) \leq \alpha(\max\{J_T(x), J_T(y)\}),
\]
where \(m\) is the mid-point of the segment \([x, y]\), and \(J_T(x) := d(x, T(x))\).

Definition 5.6. A mapping \(T : K \to X\) is said to be of convex type on \(K\) if for \((x_n), (y_n)\) two sequences in \(K\) and \((m_n)\) the sequence of the mid-points of the segments \([x_n, y_n]\),
\[
\begin{align*}
\lim_{n \to \infty} d(x_n, T(x_n)) &= 0, \\
\lim_{n \to \infty} d(y_n, T(y_n)) &= 0 \\
\lim_{n \to \infty} d(m_n, T(m_n)) &= \lim_{n \to \infty} d(m_n, T(m_n)) = 0.
\end{align*}
\]

Proposition 5.7. Let \(K\) be a nonempty bounded closed convex subset of a 2-uniformly convex space \(X\) and let \(T : K \to X\), then the following implications hold:
\[T\text{ is nonexpansive} \Rightarrow T\text{ is of type } \Gamma \Rightarrow T\text{ is } \alpha\text{-almost convex} \Rightarrow T\text{ is of convex type.}\]

Lemma 5.8. \([O]\) Let \(X\) be a 2-uniformly convex geodesic space with some parameter \(k > 0\). For any \(x, y, z \in X\), denote \(m\) as the midpoint of the segments \([y, z]\). Then, we have
\[
d^2(x, m) \leq \frac{4}{k}\left\{\frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z)\right\}
\]

Now we prove the above proposition.
Proof. For the first implication, let $m$ denote the midpoint of the segment $[x, y]$ for $x, y \in K$, and let $m'$ denote the midpoint of the segment $[T(x), T(y)]$. From the lemma, we have

\[
d^2(m', T(m)) \leq \frac{4}{k} \left( \frac{1}{2} d^2(T(m), T(x)) + \frac{1}{2} d^2(T(m), T(y)) - \frac{1}{4} d^2(T(x), T(y)) \right)
\]

\[
\leq \frac{1}{k} (d^2(x, y) - d^2(T(x), T(y)))
\]

\[
\leq \frac{2 \text{diam}(K)}{k} (d(x, y) - d(T(x), T(y)))
\]

Thus it suffices to take $\gamma(t) = \frac{k}{2 \text{diam}(K)} t^2$ to complete the first implication.

In order to prove the second implication, we have first

\[
J_T(m) = d(m, T(m)) \leq d(m, m') + d(m', T(m))
\]

\[
\leq d(m, m') + \gamma^{-1}(|d(x, y) - d(T(x), T(y))|)
\]

\[
\leq d(m, m') + \gamma^{-1}(d(x, T(x)) + d(y, T(y))]
\]

Choose $p$ as the midpoint of the segment $[m, m']$, applying the 2-uniformly convexity, we have

\[
\frac{k}{8} d^2(m, m') \leq \frac{1}{2} d^2(x, m) + \frac{1}{2} d^2(x, m') - d^2(x, p),
\]

similarly

\[
\frac{k}{8} d^2(m, m') \leq \frac{1}{2} d^2(T(x), m) + \frac{1}{2} d^2(T(x), m') - d^2(T(x), p),
\]

\[
\frac{k}{8} d^2(m, m') \leq \frac{1}{2} d^2(y, m) + \frac{1}{2} d^2(y, m') - d^2(y, p),
\]

\[
\frac{k}{8} d^2(m, m') \leq \frac{1}{2} d^2(T(y), m) + \frac{1}{2} d^2(T(y), m') - d^2(T(y), p),
\]

Since $d^2(x, p) + d^2(y, p) \geq d^2(x, m) + d^2(y, m)$ and $d^2(T(x), p) + d^2(T(y), p) \geq d^2(T(x), m') + d^2(T(y), m')$, we could obtain the following

\[
\frac{k}{2} d^2(m, m') \leq \frac{1}{2} d^2(x, m') - \frac{1}{2} d^2(T(x), m') + \frac{1}{2} d^2(T(x), m) - \frac{1}{2} d^2(x, m) + \frac{1}{2} d^2(y, m') - \frac{1}{2} d^2(T(y), m') + \frac{1}{2} d^2(T(y), m) - \frac{1}{2} d^2(y, m)
\]

\[
\leq 2D(d(x, T(x)) + d(y, T(y))]
\]

where $D = \text{diam}(K)$. Thus

\[
J_T(m) \leq \sqrt{\frac{4D}{k} (d(x, T(x)) + d(y, T(y)) + \gamma^{-1}(d(x, T(x)) + d(y, T(y)))}
\]

\[
\leq \alpha(\max\{J_T(x), J_T(y)\}),
\]

where $\alpha(t) = \sqrt{\frac{8D}{k} t + \gamma^{-1}(2t)}$.

The third implication is immediate. $\square$
We finish this section with the equivalent result of Theorem 3.14 in [KP] and [?] for $p$-uniformly convex spaces.

**Theorem 5.9.** Let $K$ be a bounded closed convex subset of $X$ a complete $p$-uniformly convex space, and let $T : K \to X$ be continuous and of convex type. Suppose

$$\inf\{d(x, T(x)) : x \in K\} = 0.$$  

Then $T$ has a fixed point in $K$.

**Proof.** Let $x_0 \in X$ be fixed and define

$$\rho_0 = \inf\{\rho > 0 : \inf\{d(x, T(x)) : x \in B(x_0, \rho) \cap K\} = 0\}.$$  

Since $K$ is bounded, $\rho_0 < \infty$. Moreover if $\rho_0 = 0$ then $x_0 \in K$ and $T(x_0) = x_0$ by the continuity of $T$. So assume that $\rho_0 > 0$. Choose $(x_n) \subset K$ such that $d(x_n, T(x_n)) \to 0$ and $d(x_n, x_0) \to \rho_0$. It suffices to show that $(x_n)$ is convergent to prove the theorem. If not, then there exists $\varepsilon > 0$ and subsequences $(u_k)$ and $(v_k)$ of $(x_n)$ such that $d(u_k, v_k) \geq \varepsilon$. Again, if necessary we may suppose $d(u_k, x_0) \leq \rho_0 + \frac{1}{k}$ and $d(v_k, x_0) \leq \rho_0 + \frac{1}{k}$. Denote $m_k$ as the midpoint of the segment $[u_k, v_k]$. Then applying the $p$-uniformly convexity to triangle $\triangle(x_0, u_k, v_k)$ we have

$$d^p(x_0, m_k) \leq \frac{1}{2} d^p(x_0, u_k) + \frac{1}{2} d^p(x_0, v_k) - C d^p(u_k, v_k)$$

$$\leq (\rho_0 + \frac{1}{k})^p - \frac{C}{8} \varepsilon^p.$$  

We consider $k$ large enough, such that

$$d(x_0, m_k) \leq \bar{\rho} < \rho_0.$$  

On the other hand, since $T$ is of convex type, $\lim_{k \to \infty} d(m_k, T(m_k)) = 0$. This contradicts the definition of $\rho_0$. \qed

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