DEFINABLE TRANSFORMATION TO NORMAL CROSSINGS OVER HENSELIAN FIELDS WITH SEPARATED ANALYTIC STRUCTURE

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Abstract. Consider a non-trivially valued field $K$ with separated analytic structure. Canonical desingularization due to Bierstone–Milman cannot be directly applied on the space $(K^\circ)^k \times (K^{\circ\circ})^l$ because the rings $A^I_{k,l}(K)$ of analytic functions seem to suffer from lack of good algebraic properties. The main purpose is to achieve a definable adaptation of this algorithm (hypersurface case), which will be carried out within a category of definable, strong analytic manifolds and maps. Also given are some applications to the problem of definable retractions and extending definable continuous functions.

1. Introduction

Fix a Henselian, non-trivially valued field $K$ of equicharacteristic zero and not necessarily algebraically closed. Denote by $v$, $\Gamma = \Gamma_K$, $K^\circ$, $K^{\circ\circ}$ and $\bar{K}$ the valuation, its value group, the valuation ring, maximal ideal and residue field, respectively. The multiplicative norm corresponding to $v$ will be denoted by $| \cdot |$. The $K$-topology on $K^n$ is the one induced by the valuation $v$. The ground field $K$ will carry a separated analytic $\mathcal{A}$-structure over a separated Weierstrass system $\mathcal{A} = \{ A_{m,n} \}_{m,n \in \mathbb{N}}$, which is a collection of homomorphisms $\sigma_{m,n}$ from $A_{m,n}$ to the ring of $K^\circ$-valued functions on $(K^\circ)^m \times (K^{\circ\circ})^n$, $m, n \in \mathbb{N}$. It will be treated in an analytic Denef–Pas language $\mathcal{L} = \mathcal{L}_\mathcal{A}$, which is the two sorted, semialgebraic language $\mathcal{L}_{Hen}$ (with the main valued field sort $K$ and the auxiliary sort $RV$), augmented on the valued field sort $K$ by the multiplicative inverse $1/x$ (with $1/0 := 0$) and the names of all functions of the collection $\mathcal{A}$, together with the induced language on the auxiliary sort $RV$. Power series $f$ from $A_{m,n}$ are construed as $f^a$ via the analytic $\mathcal{A}$-structure on their natural domains and as

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zero outside them. The theory of analytic structures, developed by Cluckers–Lipshitz–Robinson [1, 4, 5], unifies several earlier approaches to non-Archimedean analytic geometry.

Without changing the family of definable sets, we can assume that the homomorphism $\sigma_{0,0}$ from $A_{0,0}$ into $K^\circ$ is injective (whence so are the homomorphisms $\sigma_{m,n}$), which will be adopted in the sequel. Recall further that the field $K$ has $\mathcal{A}(K)$ structure by extension of parameters (cf. [5, Section 4.5] and also [2, Section 2]); more generally, $K$ has $\mathcal{A}(F)$ structure for any subfield $F$ of $K$. Let $T_\mathcal{A}$ be the $\mathcal{L}_\mathcal{A}$-theory of all Henselian, non-trivially valued fields of equicharacteristic zero with analytic $\mathcal{A}$-structure. The field $K$ is embeddable into every model $L$ of the $L_{\mathcal{A}(K)}$-theory of $K$. Under the above assumption of injectivity, the field $L$ has $\mathcal{A}(F)$-structure for any subfield $F$ of $L$. By abuse of notation, we shall also identify the series $f \in A_{m,n}^\dagger(K)$ with their interpretations $f^\sigma$ on their natural domains.

This paper is concerned with the canonical algorithm for resolution of singularities due to Bierstone–Milman [1], which provides a local invariant such that blowing up its maximum strata leads to desingularization or transformation to normal crossings (op.cit., Theorems 1.6 and 1.10). But that algorithm cannot be directly applied on the space $M_0 = (K^\circ)^k \times (K^\circ)^l$ because it seems that the rings $A_{k,l}(K)$ of analytic functions suffer from lack of good algebraic properties. Only the rings $A_{n,0}(K)$ and $A_{0,n}(K)$ of analytic functions with one kind of variables enjoy good algebraic properties being, for instance, Noetherian, factorial, normal and excellent (as they fall under the Weierstrass–Rückert theory; cf. [5, Section 5.2] and [4]).

The main purpose is to adapt to the definable settings resolution of singularities in the hypersurface case (where the concepts of strict and weak transforms coincide) and transforming an ideal to normal crossings. This will be carried out within a category of definable, strong analytic manifolds and maps, which is a strengthening of the weak concept of analyticity determined by a given separated Weierstrass system (treated in the classical case e.g. by Serre [15]), which works well within the definable settings. By strong analytic functions and manifolds, we mean those analytic functions and manifolds definable in the structure $K$ which remain analytic in each field $L$ elementarily equivalent to $K$ in the language $L_{\mathcal{A}(K)}$. Examples of such functions and manifolds are those obtained by means of the implicit function theorem and the zero loci of definable strong analytic submersions, respectively. In Section 2, we introduce the concepts of a blowup along smooth strong analytic center and of (weak) transform.
The definable version of the desingularization algorithm for separated power series is of significance because those series enable elimination of valued field quantifiers in the analytic Denef–Pas language. And quantifier elimination together with resolution of singularities are powerful tools of geometry.

It is essential that the output data of the algorithm are strong analytic whenever so are the input data. We shall show that the desingularization invariant takes only finitely many values and its equimultiple loci are definable. Our approach, pursued in Section 3, is based on analysis of the data from which the invariant is built, which in turn relies principally on the following four crucial points:

1. The functions and submanifolds involved in the resolution process, are definable and strong analytic. Consequently, via a model-theoretic compactness argument, the orders of those functions are definable, i.e. their equimultiple loci are finite in number and definable. This enables analysis of the entries $\nu_r(a)$ of the invariant being a kind of higher order, rational multiplicities of certain definable, strong analytic functions. Hence and by the canonical character of the process, the successive centers of blowups, being the maximum strata of the desingularization invariant, are definable and strong analytic.

2. The entries $\nu_r(a)$ can be defined by computations involving orders of vanishing in suitable local coordinates (independently of their choice) induced by generic affine coordinates of the ambient affine space. Therefore, such computations can be performed through suitable definable families of coordinates induced by affine coordinates. This is of importance, especially in absence of definable Skolem functions. Hence $\nu_r(a)$ are definable, i.e. their equimultiple loci are finite in number and definable.

3. Making use of the closedness theorem, it is possible to partition each ambient manifold, achieved by blowing up, into a finite number of definable clopen pieces so that, on each of them, both the exceptional hypersurfaces (which reflect the history of the process and enable the construction of the desingularization invariant) and next the successive blowup, can be described in a definable geometric way. This geometric bypass compensate for inability to globally describe the centers of the successive blowups in a purely analytic way, which is caused by lack of good algebraic properties of the rings of global analytic functions.

4. The canonical algorithm depends only on the completions of the local rings of analytic function germs at the points on the ambient manifolds. Therefore, finite partitions of those manifold into definable clopen pieces do not affect its output data, although quasi-affinoid
structure may change. This legitimizes partitions indicated above. More generally, the resolution process is available even for definable, strong analytic functions. (And along with elimination of valued field quantifiers and term structure of definable functions, it may be useful for topological problems of definable sets and functions.)

In Section 4, we outline some applications of definable resolution of singularities to the problems of definable retractions and extending continuous definable functions.

Note finally that the collections of $A_{m,n}$ and $A^\dagger_{m,n}$ correspond to the collections of $S_{m,n}$ and $S_{m,n}$, respectively, which were earlier studied in the paper [8] in the case of complete non-Archimedean fields $K$.

2. STRONG ANALYTICITY: BLOWUPS AND (WEAK) TRANSFORMS

For simplicity, the words "$L_{A(K)}$-definable" and "definable, strong analytic" will be shortened to "definable" and "analytic"; the latter because all analytic functions and manifolds occurring in the resolution process turn out to be strong analytic.

Let $f : M \to K$ be a (definable) analytic function on an analytic manifold $M$. By $\text{supp } f$, the support of $f$, we mean the closure (in the $K$-topology) of the complement of its zero locus $V(f)$. It is not difficult to check that $\text{supp } f$ is a clopen definable subset and that the order of vanishing $\mu_a(f) \in \mathbb{N}$ of $f$ at a point $a \in \text{supp } f$ is finite; obviously, $\mu_a(f) = \infty$ iff $a \in \text{supp } f$. The following basic result on order of vanishing will often be used in the sequel.

**Proposition 2.1.** Under the above assumptions, the set of orders of vanishing $\{\mu_x(f) : a \in M\}$ is finite. Moreover, the conclusion remains true for definable families of analytic functions.

**Proof.** The assertion follows directly, via a routine model-theoretic compactness argument, from the assumption of strong analyticity. □

**Remark 2.2.** It follows immediately from the proposition that the sets of orders of vanishing over the fields $L$ elementarily equivalent to $K$ in the language $L_{A(K)}$ coincide with one another.

Let $C$ be a closed analytic submanifold of $M$. Likewise in the classical case, we can define the order of the analytic function $f$ along $C$ at a point $a \in C$ by putting

$$\mu_{C,a}(f) := \min \{\mu_x(f) : x \in C \text{ near } a\}.$$
Then $\mu_{C,a}(f)$ takes only finitely many values and is constant on clopen definable subsets $F_k$. Using the closedness theorem, it is not difficult to show the following

**Proposition 2.3.** Under the above assumptions, there are a finite number of pairwise disjoint, clopen definable subsets $\Omega_k$ of $M$ covering $C$ such that the order of vanishing $\mu_{C,a}(f)$ is constant on $C \cap \Omega_k$. Further, we can ensure that $f$ vanishes on $\Omega_k$ if $\mu_{C,a}(f) = \infty$ on $C \cap \Omega_k$.

A definable construction of the blowup along $C$. Suppose $C$ is a closed (definable) analytic submanifold of $M_0 = (K^\circ)^k \times (K^{\circ\circ})^l$ of dimension $p \leq n - 1$, $n = k + l$, and consider the finite subsets $I$ of $\{1, \ldots, n\}$ of cardinality $p$. Define the canonical projections $\phi_I$ onto the $x_I$ variables by putting

$$(\phi_I(x))_i = \begin{cases} x_i & \text{if } i \in I, \\ 0 & \text{otherwise} \end{cases}$$

Let $U_I$ be the set of all points $a \in C$ at which the restriction of $\phi_I$ to $C$ is an immersion. Obviously, the sets $U_I$ are a finite open definable covering of $C$. By [R, Corollary 2.4] (being a consequence of the closedness theorem), that there exists a finite clopen definable partition $\Omega_I$ of $C$ such that $\Omega_I \subset U_I$. For a fixed $I$, denote the restriction of $\phi_I$ to $\Omega_I$ by $\phi$. The fibres of $\phi$ are of course finite. It follows from the closedness theorem that $\phi$ is a definably closed map. Therefore, for any $a \in \Omega_I$ and a neighbourhood $U$ of $\phi^{-1}(\phi(a))$, there exists a neighbourhood $V$ of $\phi(a)$ such that $\phi^{-1}(V) \subset U$. Hence and by the implicit function theorem, the set

$$F := \{(a, z) \in \Omega_I \times M_0 : \exists x \in \Omega_I, x \neq a, \, \phi(x) = \phi(a), \, z = x - a\}$$

is a closed definable subset of $\Omega_I \times M_0$. Again by the closedness theorem, there is an $\epsilon \in |K^\times|$, $\epsilon > 0$, for which $|a - x| > \epsilon$ for every $a, x \in \Omega_I$ such that $\phi(a) = \phi(x)$ and $a \neq x$. Then the restriction of $\phi_I$ to the clopen subset

$$B(a, \epsilon) \cap \phi^{-1}(\phi(\Omega_I \cap B(a, \epsilon)))$$

is injective whence a bianalytic map onto the clopen image. Therefore we can define the blowup of the clopen definable neighbourhood of $\Omega_I$ below

$$\Omega_I^* := \bigcup_{a \in \Omega_I} (B(a, \epsilon) \cap \phi_I^{-1}(\phi_I(\Omega_I \cap B(a, \epsilon))))$$

in an obvious, purely geometric way.

**Lemma 2.4.** $\Omega_I^*$ is a closed subset of $M_0$. 
Proof. Indeed, take a point \( b \) from the closure of \( \Omega_I^* \). Then \( B(b, \delta) \cap \Omega_I^* \neq \emptyset \) for every \( \delta < \epsilon \). Further,
\[
B(b, \delta) \cap B(a, \delta) \cap \phi_I^{-1}(\phi_I(\Omega_I \cap B(a, \epsilon))) \neq \emptyset
\]
for some \( a \in \Omega_I \). Hence
\[
B(b, \delta) \cap \phi_I^{-1}(\phi_I(\Omega_I \cap B(b, \epsilon))) \neq \emptyset
\]
and
\[
\phi_I(B(b, \delta) \cap \Omega_I \cap B(a, \epsilon)) \neq \emptyset, \quad B(\phi_I(b), \delta) \cap \phi_I(\Omega_I \cap B(a, \epsilon)) \neq \emptyset.
\]
Thus \( \phi_I(b) \) lies in the closure of \( \phi_I(\Omega_I \cap B(a, \epsilon)) \). Since, by the closedness theorem, this is a closed subset, we get \( \phi_I(b) \in \phi_I(\Omega_I \cap B(a, \epsilon)) \). Therefore \( \phi_I(b) = \phi_I(c) \) for some \( c \in \Omega_I \cap B(b, \epsilon) \). Then
\[
b \in B(c, \epsilon) \cap \phi_I^{-1}(\phi_I(\Omega_I \cap B(c, \epsilon))),
\]
and thus \( b \in \Omega_I^* \), as required. \( \square \)

Construction 2.5. Taking \( \epsilon > 0 \) small enough, we get pairwise disjoint, clopen definable neighbourhoods \( \Omega_I^* \) of the clopen pieces \( \Omega_I \) of the submanifold \( C \) and the blowups of those neighbourhoods along \( C \). On the complement
\[
M_0 \setminus \bigcup_I \Omega_I^*,
\]
which is a clopen subset of \( M_0 \), it suffices to define the blowup to be the identity. It is clear that the blowup \( \sigma_1 : M_1 \to M_0 \) constructed in this way is a (strong) analytic map and \( M_1 \) is an analytic submanifold of \( M_0 \times \mathbb{P}^{q-1}(K) \), where \( q = n - p \) is the codimension of \( C \) in \( M_0 \). We should once again emphasize that this construction is performed in the category of definable strong analytic manifolds and maps.

Remark 2.6. In the foregoing manner, the blowup \( M_1 \) can be further analyzed by using the \( q \) standard affine clopen charts on \( \mathbb{P}^{q-1}(K) \).

We can regard the restrictions of \( \phi_I \) to \( \Omega_I^* \) as coordinate charts on \( C \) in analogy to regular coordinate charts considered in [\textsc{I}, Section 3]. What will play an important role is that every \( \Omega_I \) is the zero locus of the analytic submersion \( \theta : \Omega_I^* \to K^q \) defined as follows. Let \( J := \{1, \ldots, n\} \setminus I \) and \( \psi_J \) be the canonical projection onto the variables \( x_J \). Any point \( x \in \Omega_I^* \) lies in
\[
B(a, \epsilon) \cap \phi_I^{-1}(\phi_I(\Omega_I \cap B(a, \epsilon)))
\]
for some \( a \in \Omega_I \). Take a unique \( y \in \Omega_I \cap B(a, \epsilon) \) such that \( \phi_I(x) = \phi_I(y) \) and set \( \theta(x) := \psi_J(x) - \psi_J(y) \). Note finally that the above enables the
standard definition of the blowup of $\Omega^*_i$ along $C$ as well. This leads to the following definition.

**Definition 2.7.** Let $M$ be a closed analytic submanifold of dimension $m$ of $M_0 := (K^\circ)^k \times (K^\circ)^l$, $\phi_1, \ldots, \phi_n$ $k + l = n$, be affine coordinates on $M_0$, $\Omega^*$ a clopen definable subset of $M_0$ and $\Omega := \Omega^* \cap M$. We say that $\phi_1, \ldots, \phi_m$ are a definable coordinate system for $M$ on $\Omega^*$ if the restriction of $(\phi_1, \ldots, \phi_m)$ is an immersion of $\Omega$ such that for each point $x \in \Omega^*$ there is a unique point $y \in \Omega$ that is closest to $x$ from among $(\phi_1, \ldots, \phi_m)^{-1}(x) \cap \Omega$. We then call $\Omega$ a definable chart with coordinates $\phi_1, \ldots, \phi_m$ on $\Omega^*$. As demonstrated above, $\Omega$ is then the zero locus of an analytic submersion $\theta : \Omega^* \to K^{n-m}$.

We have thus proven the following

**Proposition 2.8.** Every closed analytic submanifold $M$ of dimension $m$ in $M_0 = (K^\circ)^k \times (K^\circ)^l$ can be partitioned into a finite number of pairwise disjoint, clopen definable charts $\Omega$ with coordinates on clopen subsets $\Omega^*$ of $M_0$. Moreover, $M \cap \Omega^*$ is the zero locus of an analytic submersion $\theta : \Omega^* \to K^{n-m}$. $\square$

We can easily deduce the following two results.

**Corollary 2.9.** Let $C \subset M$ be two closed analytic submanifolds of $M_0$ of dimension $p$ and $m$, respectively. Then there exist a finite number of pairwise disjoint, clopen definable subsets $U_s$ of $M$ which cover $C$ and such that $C \cap U_s$ are the zero loci of some analytic submersions $\theta_s : U_s \to K^{m-p}$. In particular, if $C$ is a hypersurface in $M$, then $C \cap U_s = V(\theta_s)$ for some analytic submersions $\theta_s : U_s \to K$. $\square$

**Corollary 2.10.** Let $f : M \to K$ be a (definable) analytic function on an analytic submanifold of $M_0$. If $f$ is a normal crossing divisor (in the usual sense), then there exists a finite partition of $M$ into clopen definable subsets $\Omega_s$ and, for each $s$, analytic submersions $\theta_{s_1}, \ldots, \theta_{s_l} : \Omega_s \to K$ and $k_1, \ldots, k_l \in \mathbb{N}$ such that

$$f \sim \theta_{s_1}^{k_1} \cdots \theta_{s_l}^{k_l};$$

here $\sim$ means equal up to an analytic unit. $\square$

In view of the foregoing, we can readily construct in a definable way the transform of an analytic hypersurface.

**Construction 2.11.** Consider a blowup $\sigma_1 : M_1 \to M_0$ along smooth analytic center with exceptional hypersurface $E$ and an analytic hypersurface $X$ of $M_0$ determined by an analytic function $f : M_0 \to K$;
put \( f_1 \) := \( f \circ \sigma_1 \). By Corollary 2.9 and Proposition 2.3, there exist a finite number of pairwise disjoint, clopen subsets \( U_s \) of \( M_1 \) which cover \( E \) and such that \( E \cap U_s = V(\theta_s) \) for an analytic submersion \( \theta_s \), and the order of vanishing \( \mu_{E,a}(f_1) = d_s \) is constant on \( E \cap U_s \). Then the transform \( X_1 \) of \( X \) is determined on \( U_s \) by the analytic function
\[
\theta_s^{-d_s} \cdot f \circ \sigma_1;
\]
actually, \( d_s \) is the largest power of \( \theta_s \) that factors from \( g \circ \sigma_1 \).

3. Definable adaptation of the desingularization algorithm

In this section, the desingularization algorithm by Bierstone–Milman [1], Chapter II] is adapted to the definable settings. To be brief, for majority of details the reader is referred to their paper. We give a concise outline of the process of transforming an analytic function \( g \in A_{k,l}^1(K) \) to normal crossings or, equivalently, resolving singularities of a hypersurface \( X = X_0 = V(g) \) of the manifold \( M_0 = (K^\times)^k \times (K^{\infty})^l \) of dimension \( n = k + l \). Actually, \( g \) may be a definable, strong analytic function on \( M_0 \). We begin with some notation related to the local invariant for desingularization.

Consider a sequence of admissible blowups \( \sigma_j : M_j \to M_{j-1} \) along admissible smooth centers \( C_j, j = 1, 2, \ldots; \) let \( E_j \) denote the set of exceptional hypersurfaces in \( M_j \) (op.cit., p. 212). Let \( X_1, X_2, X_3, \ldots \) denote the successive transforms of the given hypersurface \( X \); here the strict and weak transforms coincide. Admissible means that \( C_j \) and \( E_j \) simultaneously have only normal crossings and that \( \text{inv}_X(\cdot) \) is locally constant on \( C_j \) for all \( j \). We can now state the main result, being a definable version of op.cit., Theorem 1.6.

**Theorem 3.1.** Under the above assumptions, there exists a finite sequence of blowups with smooth admissible centers \( C_j \) such that:

1) For each \( j \), either \( C_j \subset \text{Sing} X_j \) or \( X_j \) is smooth and \( C_j \subset X_j \cap E_j \);

2) the final transform \( X' \) of \( X \) is smooth (unless empty), and \( X' \) and the final exceptional hypersurface \( E' \) simultaneously have only normal crossings.

We proceed with the necessary notation:

\[
E(a) := \{ H \in E_j : a \in H \}.
\]

For a point \( a = a_j \in M_j \), let \( a_{j-1} \in M_{j-1}, \ldots, a_0 \in M_0 \) be the images of \( a \) under the successive blowups.
DEFINABLE TRANSFORMATION TO NORMAL CROSSINGS

The order of vanishing of an analytic function germ $f$ at $a$ is $\mu_a(f)$.

In each year $j$, the local invariant $\text{inv}_X(a)$ at a point $a \in M_j$ is the word:

$$\text{inv}_X(a) = (\nu_1(a), s_1(a); \nu_2(a), s_2(a); \ldots; \nu_t(a), s_t(a); \nu_{t+1}(a)),$$

where $0 < \nu_1(a), \ldots, \nu_t(a) \in \mathbb{Q}$, $s_1(a), \ldots, s_t(a) \in \mathbb{N}$ and $\nu_{t+1}(a) = 0$ or $\infty$; note that $t \leq n$ (op.cit., p. 213); $\nu_1(a) = \mu_a(g)$ where $g$ is a local equation at $a$ of $X$. We consider such words with the lexicographic ordering. The inductive resolution process terminates unless $0 < \nu_r(a) < \infty$.

The invariant $\text{inv}_X(\cdot)$ is upper semicontinuous (i.e. each point $a \in X_j$ admits an open neighbourhood $U$ such that $\text{inv}_X(x) \leq \text{inv}_X(a)$ for all $x \in U$) and infinitesimally upper-semicontinuous (i.e. $\text{inv}_X(a) \leq \text{inv}_X(\sigma_j(a)$ for all $j \geq 1$); op.cit., Theorem 1.14.

An infinitesimal presentation (of codimension $p$) is the following data (op.cit., p. 222):

$$(N(a), \mathcal{H}(a), \mathcal{E}(a)) \text{ where }$$

$N_p(a)$ is a germ at $a$ of a regular submanifold of codimension $p$;
$\mathcal{H}(a) = \{(h, \mu_h)\}$ is a finite collection of pairs with $h \in \mathcal{O}_{N,a}$, $\mu_h \in \mathbb{Q}$, $0 \leq \mu_h \leq \mu_a(h)$;
$\mathcal{E}(a)$ is a collection of smooth hypersurfaces $H \ni a$ such that $N$ and $\mathcal{E}(a)$ simultaneously have only normal crossings, and $N \not\subset H$ for all $H \in \mathcal{E}(a)$.

The equimultiple locus of the infinitesimal presentation is

$$S_{\mathcal{H}(a)} := \{x \in N : \mu_x(h) \geq \mu_h \ \forall (h, \mu_h) \in \mathcal{H}(a)\}.$$  

$$\mu_{\mathcal{H}(a)} := \min \left\{ \frac{\mu_a(h)}{\mu_h} \right\}.$$  

Remark 3.2. In view of the canonical character of the resolution process, the maximum loci of the desingularization invariant (being at the same time the centers of the successive blowups) are strong analytic because locally they are constructed within rigid analytic geometry.

At this stage we can readily pass to the resolution process. The easiest is the initial year zero before any blowup.

Year zero. For each $a \in M_0$, we start with the following codimension 0 presentation for the equation $g$:

$$(N_0(a), \mathcal{G}_1(a), \mathcal{E}_1(a)), \ N_0(a) = M_0, \ \mathcal{G}_1(a) = \{(g, \mu_a(g))\}, \ \mathcal{E}_1(a) = \emptyset.$$
Put \( d := \nu_1(a) = \mu_1(a)(g), \) \( s_1 := 0 \) and \( F_1(a) = G_1(a) \). The further definable constructions should take into account the equimultiple strata of the entry \( \nu_1 \), and so on. Apply Construction 4.18, op.cit., to get a codimension 1 presentation \( H_1(a) \) as explained below.

First, consider the family of (all, for the sake of definability) suitable affine coordinates \( x_1, \ldots, x_n, \) \( n = k + l \), at \( a \in M_0 \), i.e. such affine coordinates that \( \partial^d g/\partial x_n^d(a) \neq 0 \) with \( d := \mu_a(g) \). More precisely, two kinds of variables: \( \xi_1, \ldots, \xi_k \) and \( \rho_1, \ldots, \rho_l \) occur here. We can thus consider, among others, the family of affine coordinates of the form

\[
\xi_1' = \xi_1 + u_1 \rho_1, \ldots, \xi_k' = \xi_k + u_k \rho_1, \rho_1' = \rho_1 + v_1 \rho_1, \ldots, \rho_l' = \rho_l + v_1 \rho_1,
\]

with \( u_1, \ldots, u_k, v_1, \ldots, v_l \in K^\circ \). For simplicity, we shall further write the coordinates \( x_1, \ldots, x_n \), considering the definable family of all suitable coordinates (coming from the affine ones in the ambient space), which of course depend on the point \( a \). Finally, set

\[
N_1(a) = V(\partial^d g/\partial x_n^d), \quad \mathcal{E}_1(a) = \emptyset,
\]

and

\[
\nu_2(a) = \mu_{\mathcal{H}_1(a)} := \min \left\{ \frac{\mu_a(\partial^d g/\partial x_n^d|N_1(a))}{d - q}, \ q = 0, \ldots, d - 2 \right\}.
\]

Notice that \( N_1(a) \) can be regarded both as a codimension 1 submanifold in the open subset \( M_0 \setminus V(\partial^d g/\partial x_n^d) \) (which is beneficial for the analysis of definability) or as its germ (which is the case treated originally in the theory of infinitesimal presentations, op.cit.). It follows directly from Proposition 2.1 that the entry \( \nu_2 \) takes only finitely many values and hence its equimultiple strata are definable. Again, further definable constructions should take into account those strata. The same holds over each field \( L \) elementarily equivalent to \( K \) in the language \( L_A(K) \) (with the same set of orders of vanishing).

Now, applying Construction 4.23, op.cit., we get the codimension 1 presentation:

\[
F_2(a) = G_2(a) := \left\{ (\partial^d g/\partial x_n^d|N_1(a), (d - q) \cdot \nu_2(a)), \ q = 0, 1, \ldots, d - 2 \right\},
\]

which satisfies the conditions of Proposition 4.12, op.cit.; in particular, \( \mu_{F_2(a)} = 1 \). Why the construction falls into the three stages \( G, F \) and \( H \) will be clear in the next years of the process.

Next, repeat Construction 4.18, op.cit. To this end, consider, as before, the family of suitable coordinates on \( N_1(a) \) induced by generic affine coordinates on the ambient space, taking also into account the
DEFINABLE TRANSFORMATION TO NORMAL CROSSINGS

strata on which given pairs \((h, \mu_h) \in \mathcal{F}_2(a)\) satisfy the condition \(\mu_a(h) = \mu_h\). In this way, we get a codimension 2 presentation \(\mathcal{H}_2(a)\) determined by some definable data expressed in terms partial derivatives with respect to the definable family of suitable coordinates.

The resolution process will be continued until \(\nu_{t+1} = 0\) or \(\infty\), which must happen for a \(t \leq n\). In year zero, however, we eventually get the invariant \(\text{inv}_X(a)\) of the form \((\ldots; \infty)\), whose maximum stratum \(S = C_0\) is an analytic submanifold of \(M_0\). After blowing up the stratum \(S = C_0\), we pass to the next years.

**Remark 3.3.** The further analysis on the successive spaces \(M_j, j \geq 1\), comes down to the case of affine ambient spaces with affine coordinates via the standard charts on the projective spaces involved when blowing up.

Suppose now that the process has been carried out in the years \(0, 1, 2, \ldots, j\).

**Year \((j + 1)\).** We have thus constructed the following sequence of blowups (op.cit., Section 1):

\[
\sigma_j : M_j \to M_{j-1}, \quad \sigma_{j-1} : M_{j-1} \to M_{j-2}, \ldots, \quad \sigma_1 : M_1 \to M_0;
\]

the centers \(C_{k-1}\) of \(\sigma_k\) are admissible and the exceptional hypersurfaces \(E_k\) on \(M_k\) and \(C_k\) simultaneously have only normal crossings.

As before, for each \(a \in M_j\), we start with the following codimension 0 presentation for the transform \(g_1\) of \(g\) under \(\sigma_1\):

\[
(N_0(a), G_1(a), E_1(a)), \quad N_0(a) = M_1, \quad G_1(a) = \{(g_1, \mu_a(g_1))\},
\]

where \(N_0 = M_1\) and \(E_1(a)\) is defined as follows: let \(\nu_1 := \mu_a(g_1)\), \(i = i(a) \leq j\) be the smallest \(k\) such that \(\nu_1(a) = \nu_1(a_k)\),

\[
E_1^1(a) := \{H \in E(a) : H \text{ is the transform of some element of } E(a_i)\},
\]

\[
s_1(a) := \sharp E_1^1(a) \text{ and } E_1(a) := E(a) \setminus E_1^1(a). \quad \text{Since the invariant } \text{inv}_X(\cdot) \text{ takes only finitely many values and is upper-semicontinuous and infinitesimally upper-semicontinuous, it is not difficult to check that the equimultiple strata of the invariant } i(\cdot) \text{ are definable, whence so are the families } E_1^1(\cdot) \text{ and } E_1(\cdot).
\]

Next, let \(\mathcal{F}_1(a)\) be \(G_1(a)\) together with all pairs \((f, \mu_f) = (\theta_H, 1)\) with \(H \in E_1^1(a)\), where \(\theta_H\) is an analytic equation of \(H\). By Corollary 2.9, \(\mathcal{F}_1(a)\) is determined by definable data. Now, apply Construction 4.18, op.cit., as in the year zero, to get a codimension 1 presentation

\[
(N_1(a), H_1(a), E_1(a)),
\]
which is determined by definable data as well. Then \( \mu_2(a) := \mu_{H_1(a)} = \infty \) iff \( H_1(a) = 0 \). If \( \mu_2(a) < \infty \), set

\[
\mu_{2,H} := \min \left\{ \frac{\mu_{H,a}(h)}{\mu_h} : (h, \mu_h) \in H_1(a) \right\}, \quad H \in E(a)
\]

and

\[
\mu_2(a) := \mu_{H_1(a)}, \quad \nu_2(a) := \mu_2(a) - \sum_H \mu_{2,H}(a).
\]

By Proposition 2.23, the invariant \( \nu_2(\cdot) \) takes only finitely many values and its equimultiple loci are definable.

If \( \nu_2(a) = 0 \) or \( \infty \), set \( \text{inv}_X(a) := (\nu_1(a), s_1(a) ; \nu_2(a)) \). Otherwise, apply Construction 4.23, op.cit., to get a codimension 1 presentation

\[
(N_1(a), G_2(a), E_1(a)) \quad \text{with} \quad \mu_{G_2(a)} = 1.
\]

The construction consists in dividing the \( h \in H_1(a) \), previously scaled so that the \( \mu_h \) are equal, by their greatest common divisor that is a monomial in the equations \( \theta_H \) of \( H \in E(a) \). Hence and again by Proposition 2.23, \( G_2(a) \) is determined by definable data.

Now, let \( i = i(a) \leq j \) be the smallest \( k \) such that

\[
(\nu_1(a), s_1(a) ; \nu_2(a)) = (\nu_1(a_k), s_1(a_k) ; \nu_2(a_k)),
\]

\[
E^2(a) := \{ H \in E_1(a) : H \text{ is the transform of some element of } E_1(a_i) \},
\]

\[
s_2(a) := \sharp E^2(a) \text{ and } E_2(a) := E_1(a) \setminus E^1(a).
\]

Then

\[
(N_1(a), G_2(a), E_2(a))
\]

is a codimension 1 presentation determined by definable data as well.

Next, let \( F_2(a) \) be \( G_2(a) \) together with all pairs \( (f, \mu_f) = (\theta_H, 1) \) with \( H \in E^2(a) \), where \( \theta_H \) is an analytic equation of \( H \). The process continues inductively until \( \nu_{t+1} = 0 \) or \( \infty \) for a \( t \leq n \), and eventually yields the invariant \( \text{inv}_X(\cdot) \) on \( M_j \) which takes only finitely many values and whose equimultiple loci

\[
S_X(a) := \{ x \in M_j : \text{inv}_X(x) = \text{inv}_X(a) \}, \quad a \in M_j,
\]

are definable; \( S_X(a) \) will also be regarded as a germ at \( a \). Its maximum stratum \( S \) is an analytic submanifold or a normal crossing submanifold according as its maximum value is \((\ldots ; \infty)\) or \((\ldots ; 0)\). In the latter case, for any \( a \in S \), the irreducible components \( Z \) of \( S_X(a) \) are of the form (op.cit., Theorem 1.14):

\[
Z = S_X(a) \cap \bigcap \{ H \in E(a) : Z \subset H \}.
\]

Then, in order to eventually achieve a smooth maximum stratum, the invariant should be extended as outlined below.
Consider any total ordering on the collection of all subsets $I$ of $E_j$. Observe that whether the intersection $S_X(a) \cap \bigcap I$ is an analytic submanifold at $a$ is a definable property with respect to the points $a$ (which is expressed, in view of equality [3.1], by means of suitable coordinate projections). Therefore the components $Z$ of $S_X(a)$ can be defined by the following formula (*):

$$Z = S_X(a) \cap \bigcap I$$

is an analytic submanifold at $a$ for some $I$ and, for every $J$, if $Z \subset S_X(a) \cap \bigcap J$ is an analytic submanifold, then $Z = S_X(a) \cap \bigcap J$.

The family of the components $Z$ of $S_X$ at the points $a$ is thus definable (consider the product of $\sharp E_j$ copies of $M_j$). For a component $Z$ at $a$, let $J(Z)$ be the set of all $H \in E_j$ containing $Z$. Set

$$J(a) := \max \{J(Z) : Z \text{ a component of } S_X(a)\}.$$ 

Then the index $J(a)$ is definable: $J(a) = I^*$ iff formula (*) holds for $I^*$ at $a$ and, for every $I \supseteq I^*$ and $J > I^*$, formula (*) holds at $a$ neither for $I$ nor for $J$. Extend the invariant on $M_j$ by putting

$$\text{inv}_X^e(a) := (\text{inv}_X(a); J(a)).$$

Then the maximum locus of $\text{inv}_X^e(\cdot)$ is smooth (op.cit., Remark 1.15). Actually, for any component $Z$ of the maximum locus of $\text{inv}_X(\cdot)$ at a point $a$, one can choose an ordering above so that $J(Z) = J(a) = \max E_j$. Therefore the component $Z$ extends to an analytic submanifold of $M_j$, the maximum locus of $\text{inv}_X^e(\cdot)$. Furthermore, by choosing a suitable ordering on the subsets of each $E_j$, one can achieve the extended invariant $\text{inv}_X^e(\cdot)$ with the property that every germ $S_{\text{inv}_X^e(a)}$ is smooth (op.cit., Remark 1.16). Hence the maximum locus of $\text{inv}_X^e(\cdot)$ is an analytic submanifold of the ambient space.

*Sketch of resolution of singularities.* Now we briefly outline the desingularization algorithm in the hypersurface case (op.cit., Theorem 1.6), which in this case immediately yields transformation to normal crossings (op.cit., Theorem 1.10). The proof, given in op.cit., Section 10, carries over verbatim to the definable settings treated here. It essentially relies on that the (extended) desingularization invariant takes only finitely many values and, though those values $\nu_\cdot(\cdot)$ are merely rational numbers, it behaves as if those values were integers (unless $\infty$). This follows directly from the infinitesimal upper-semicontinuity of the invariant and from that it takes only finitely many values in each year of the process along with the estimates below (op.cit., p. 214).
In each year of the process, the entries $\nu_r(a)$, $r = 2, \ldots, t \leq n$, are quotients of positive integers whose denominators are bounded in terms of the previous part of the invariant $\text{inv}_X(a)$. More precisely, define recursively $e_2(a) := \nu_1(a)$ and $e_{r+1}(a) := \max\{e_r(a)!, e_r(a) \cdot \nu_r(a)\}$. Then $e_r(a) \cdot \nu_r(a) \in \mathbb{N}$ and $e_{t+1}(a) = \mu_{t+1}(a) \in \mathbb{N}$.

In each year $j$ of the process, the maximum locus $C_j$ of the invariant $\text{inv}_X$ (or the extended invariant $\text{inv}^e_X$ if $\nu_{t+1}(a) = 0$ on the maximum locus of $\text{inv}_X$) is smooth so that one can blow it up. For each point $a \in C_j$, if $\text{inv}_X(a) = (\ldots; \infty)$, then

$$\text{inv}_X(a') < \text{inv}_X(a) \quad \text{for all} \quad a' \in \sigma_{j+1}^{-1}(a).$$

Otherwise (op.cit., Theorem 1.14), we get

$$(\text{inv}_X(a'), \mu_X(a')) < (\text{inv}_X(a), \mu_X(a)) \quad \text{for all} \quad a' \in \sigma_{j+1}^{-1}(a),$$

where $\mu_X(a) = \mu_{t+1}(a)$ if $\nu_{t+1}(a) = 0$. Hence the maximum value of the invariant must decrease after a finite number of blowups. Eventually, after finitely many successive admissible blowups, the transform $X_j$ is smooth. But some successive admissible blowups are needed so that the final transform $X_k$ and $E_k$ may simultaneously have only normal crossings. To this end, we must blow up the successive maximum strata of the invariant $\text{inv}_X$ until its parameter $s_1$ has decreased to zero everywhere on $X_k$ (op.cit., p. 285). Then we attain the final step of the desingularization process. \qed

In a similar manner, we can achieve a definable version of transforming to normal crossings a sheaf of ideals $\mathcal{I} = \mathcal{I}_0 \subset \mathcal{O}_{M_0}$, generated by a finite number of definable, strong analytic functions $f_1, \ldots, f_s$ on $M_0$. This process uses the successive weak transforms $\mathcal{I}_j$ of the ideal $\mathcal{I}$ when blowing up the maximal strata of the desingularization invariant (op.cit., Theorem 1.10). We adopt the previous notation and, for convenience, remind the reader the statement.

**Theorem 3.4.** Under the above assumptions, there exists a finite sequence of blowups with smooth admissible centers $C_j$ such that the final weak transform of $\mathcal{I}$ is $\mathcal{I}_k = \mathcal{O}_{M_k}$ and the pull-back $\sigma^{-1}(\mathcal{I}) \cdot \mathcal{O}_{M_k} = E_k$ of the sheaf of ideals $\mathcal{I}$ is a normal crossing divisor; here $\sigma$ is the composite of the $\sigma_j$. \qed

**Remark 3.5.** The process of resolution of singularities can be, in fact, carried over on every closed, definable, strong analytic submanifold $M$ of $M_0$ (or of $M_0 \times \mathbb{P}^N(K)$, etc.).
4. APPLICATION TO THE PROBLEM OF DEFINABLE RETRACTIONS

In this section, we outline some applications of definable resolution of singularities to the problems of definable retractions and extending continuous definable functions. The main aim is here the following theorem on the existence of $\mathcal{L}$-definable retractions onto an arbitrary closed $\mathcal{L}$-definable subset, whereby definable non-Archimedean versions of the extension theorems by Tietze–Urysohn and Dugundji follow directly.

**Theorem 4.1.** Let $Z \subset W$ be closed subvarieties of the unit balls $(K^\circ)^N$ or $(K^{\circ\circ})^N$, of the projective space $\mathbb{P}^n(K)$ or of the products $(K^\circ)^N \times \mathbb{P}^n(K)$ or $(K^{\circ\circ})^N \times \mathbb{P}^n(K)$, and let $X := W \setminus Z$. Then, for each closed $\mathcal{L}$-definable subset $A$ of $X$, there exists an $\mathcal{L}$-definable retraction $X \to A$.

We immediately obtain

**Corollary 4.2.** For each closed $\mathcal{L}$-definable subset $A$ of $K^n$, there exists an $\mathcal{L}$-definable retraction $K^n \to A$. $\square$

The case of analytic structures, determined on complete rank one valued fields $K$ by separated power series, was established in our previous paper [1], Theorem 1. The proof was based on the following tools: elimination of valued field quantifiers (due to Cluckers–Lipshitz–Robinson [14–16]), transforming an ideal to normal crossings and embedded resolution of singularities by blowing up (due to Bierstone–Milman [1] or Temkin [16]), the technique of quasi-rational subdomains (due to Lipshitz–Robinson) and our closedness theorem [10, 11, 12]. We should emphasize that the advantage of working here with the strong analytic settings lies also in that we do not need to appeal to the theory of quasi-affinoid subdomains.

We can now repeat that previous proof almost verbatim, using the definable version of transformation to normal crossings treated here, but op.cit., Lemma 3.1, because the full version of resolution of singularities seems to be unavailable in the definable settings. Below we state and prove this lemma, whereby the proof of Theorem 4.1 is complete over Henselian fields with analytic structure.

**Lemma 4.3.** Let $Z \subsetneq X$ be two closed subvarieties of the product $(K^\circ)^N \times \mathbb{P}^n(K)$ and $A$ a closed $\mathcal{L}$-definable subset of $Z$. Suppose that $X$ is non-singular of dimension $N$ and Theorem 4.1 holds for closed $\mathcal{L}$-definable subsets of every non-singular variety of this kind of dimension $< N$. Then there exists an $\mathcal{L}$-definable retraction $r : Z \to A$. 
Remark 4.4. In our paper [14], this lemma holds in the above generality. In fact it was involved in induction in the proof of Theorem [14], and used only when \( Z \) was the zero locus of one analytic function

\[
\psi_j = (\psi_{j-1} \circ \sigma_j) \cdot \chi_j \quad \text{with} \quad j = 0, \ldots, k,
\]

thus an analytic hypersurface of \( M \) in the algebro-geometric sens. By abuse of notation, we often use the same letter for an analytic subvariety and its support, i.e. underlying topological space, which does not lead to confusion. Here Lemma [13] holds if \( Z \) is the zero locus of finitely many definable, strong analytic functions \( f_1, \ldots, f_s \).

**Proof.** Apply Theorem [3.4] to transform to normal crossings the sheaf of ideals \( I \) generated by \( f_1, \ldots, f_s \) on the analytic manifold \( X \). Set

\[
\tau_j := \sigma_1 \circ \ldots \sigma_j, \quad j = 1, \ldots, k, \quad \text{and} \quad A^\tau := \tau^{-1}(A).
\]

Then \( Z^{\tau_k} = \bigcup E_k \). Considering the canonical map from the disjoint union \( \coprod E_k \) onto \( \bigcup E_k \) and using the assumption of the lemma, it is not difficult to check that there is a definable retraction \( \rho_k : Z^{\tau_k} \to A^{\tau_k} \).

Therefore, by op.cit., Corollary 2.13, there is a definable retraction \( r_{k-1} : Z^{\tau_{k-1}} \to (C_{k-1} \cup A^{\tau_{k-1}}) \). Again by the assumption, there is a definable retraction \( C_{k-1} \to (C_{k-1} \cap A^{\tau_{k-1}}) \), and hence a definable retraction \( \rho_{k-1} : Z^{\tau_{k-1}} \to A^{\tau_{k-1}} \).

As before, by op.cit., Corollary 2.13, there is a definable retraction \( r_{k-2} : Z^{\tau_{k-2}} \to C_{k-2} \cup A^{\tau_{k-2}} \). Again by the assumption, there is a definable retraction \( C_{k-2} \to (C_{k-2} \cap A^{\tau_{k-2}}) \), and hence a definable retraction \( \rho_{k-2} : Z^{\tau_{k-2}} \to A^{\tau_{k-2}} \).

Proceeding recursively, we eventually achieve a definable retraction \( \rho_0 : Z \to (A) \), we are looking for. \( \square \)

We conclude the paper with the following comment.

**Remark 4.5.** The theorems on the existence of definable retractions onto closed definable subsets and on extending continuous definable functions are valid over Henselian valued fields with strictly convergent analytic structure, because every such a structure can be extended in a definitional way (extension by Henselian functions) to a separated analytic structure (cf. [14]). It is plausible that these theorems will also hold in more general settings of certain tame non-Archimedean geometries considered in the papers [7] and [3].
DEFINABLE TRANSFORMATION TO NORMAL CROSSINGS

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