Analysis of Nonlinear Oscillation Systems using He’s Variational Approach

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Abstract. This paper applies a new variational approach proposed by Ji-Huan He for nonlinear oscillators. Two examples are given to illustrate the effectiveness and convenience of the method. The obtained results are valid for the whole solution domain with high accuracy. The method can be easily extended to other strongly nonlinear oscillations and it can be found widely applicable in engineering and science.

1. Introduction
Nonlinear oscillations systems are such phenomena that mostly occur nonlinearly; hence solving of governing equations have been one of the most time-consuming and difficult affairs among researchers of vibrations. Therefore, many researchers and scientist of both vibrations and mathematics have recently paid much attention to find and develop approximate solutions. If there is no small parameter in the equation, the traditional perturbation methods cannot be applied directly. Recently, considerable attention has been directed towards the analytical solutions for nonlinear equations without possible small parameters. The traditional perturbation methods have many shortcomings, and they are not valid for strongly nonlinear equations. To overcome the shortcomings, many new techniques have appeared in open literature [1–5]. Variational methods have been, and continue to be, popular tools for nonlinear analysis. When contrasted with other approximate analytical methods, variational methods combine the following two advantages: (1) they provide physical insight into the nature of the solution of the problem; (2) the obtained solutions are the best among all the possible trial-functions. Recently, some approximate variational methods, including approximate energy method [6, 7] and variational iteration method [8–12], to soliton solution, bifurcation, limit cycle and period solutions of nonlinear equations have been given much attention. The approximate energy approach [6] can be applied not only to weakly nonlinear equations, but also to strongly nonlinear ones. The so obtained results are valid for the whole solution domain [7]. Variational iteration method is based on a general Lagrange multiplier, and it can be applied to various nonlinear equations [13, 14]. In the present paper, we use a new variational method proposed by J.H. He for nonlinear oscillators [15].
2. He’s Variational Method

3. Two examples are given to illustrate the effectiveness and convenience of the method.

Example 1

In dimensionless form, a mass attached to the center of a stretched elastic wire has the equation of motion [16]

\[ u'' + u - \lambda u \left(1 + u^2\right)^{1/2} = 0, \]  

(1)

This is an example of a conservative nonlinear oscillatory system having an irrational elastic item. In general, oscillations systems contain two important physical parameters, i.e. the frequency \( \omega \) and the amplitude \( A \) of oscillation. All the motions corresponding to Eq. (1) are periodic [18], the system will oscillate between symmetric bounds \([-A, A]\), and its angular frequency \( \omega \) and corresponding periodic solution are dependent on the amplitude \( A \). Its variational principle can be easily established using the semi-inverse method [3]

\[ J(u) = \int_0^{T/4} \left( -\frac{1}{2} u'^2 + F(u) \right) \, dt \]  

(2)

where \( T = 2\pi/\omega \) is period of the nonlinear oscillator.

Using Eq. (2) and \( F(u) = \int \left[ u - \left( \frac{\lambda}{2} u \left(1 + u^2\right)^{1/2} \right) \right] \, du \), yield;

\[ J(u) = \int_0^{T/4} \left( -\frac{1}{2} u'^2 + \frac{1}{2} u^2 - \lambda \left(1 + u^2\right)^{1/2} \right) \, dt \]  

(3)

Without loss of any generality, consider such initial conditions:

\[ u(0) = A, \quad u'(0) = 0, \]  

(4)

Assume that its solution can be expressed as

\[ u(t) = A \cos \omega t \]  

(5)

Substituting (5) into (3) results in

\[ J(A, \omega) = \int_0^{\pi/4} \left( -\frac{1}{2} A^2 \omega^2 \sin^2 \omega t + \frac{1}{2} A^2 \cos^2 \omega t - \lambda \left(1 + A^2 \cos^2 \omega t\right)^{1/2} \right) \, dt \]  

(6)

Applying the Ritz method, we require

\[ \frac{\partial J}{\partial A} = 0 \]  

(7)

\[ \frac{\partial J}{\partial \omega} = 0 \]  

(8)

But by a careful inspection, for most cases we find that

\[ \frac{\partial J}{\partial \omega} < 0 \]  

(9)

Thus, in this method, we using the conditions (7) and (8) into a more simply form:
\[ \frac{\partial J}{\partial A} = 0 \]  

The stationary condition with respect to \( A \) reads

\[ \frac{\partial J}{\partial A} = \int_0^{T/4} \left( -A \omega^2 \sin^2 \omega t + A \cos^2 \omega t - \lambda A \cos^2 \omega t \left[ 1 + \lambda^2 \cos^2 \omega t \right] \right) dt \]

\[ = \int_0^{\pi/2} \left( -A \omega^2 \sin^2 t + A \cos^2 t - \lambda A \cos^2 t \left[ 1 + \lambda^2 \cos^2 t \right] \right) dt \]

This leads to the following result;

\[ \omega = \left( \frac{\pi}{4} \left( 4 \lambda \left( \text{EllipticK} \left( -A^2 \right) \right) - 4 \lambda \left( \text{EllipticE} \left( -A^2 \right) \right) + A^2 \pi \right) \right)^{1/2} / A \pi \]  

Where the incomplete elliptic integral \( \text{EllipticE} \) and \( \text{EllipticK} \) are defined by [17].

Hence, the approximate period is

\[ T = 2\pi / \omega = 2A \pi^2 \left( \frac{\pi}{4} \left( 4 \lambda \left( \text{EllipticK} \left( -A^2 \right) \right) - 4 \lambda \left( \text{EllipticE} \left( -A^2 \right) \right) + A^2 \pi \right) \right)^{1/2} \]  

Its exact period [16] is

\[ T_e = 4 \int_0^{\pi/2} \left( 1 - 2\lambda \right) \left( 1 + A^2 \sin^2 t \right)^{1/2} + \left( 1 + A^2 \right)^{1/2} \right] dt. \]

for \( \lambda = 0.1, \lambda = 0.5, \lambda = 0.75 \) and \( \lambda = 0.95 \), comparison of the approximate periods \( T \) with exact periods \( T_e \) are tabulated in Tables 1 and 2, respectively.

According to Eqs. (5 and 12), we can obtain the following approximate solution

\[ u(t) = A \cos \left( \frac{\pi}{4} \left( 4 \lambda \left( \text{EllipticK} \left( -A^2 \right) \right) - 4 \lambda \left( \text{EllipticE} \left( -A^2 \right) \right) + A^2 \pi \right) \right)^{1/2} / A \pi \]  

Which agrees very well with the exact solution as illustrated in Fig. 1 and Fig. 2.

**Table 1.** Comparison of the approximate period with exact period when \( \lambda = 0.1 \) and \( \lambda = 0.5 \)

| A   | \( \lambda = 0.1 \) | \( \lambda = 0.5 \) |
|-----|--------------------|--------------------|
|     | \( T \)            | \( T_e \)          | Relative error | \( T \)            | \( T_e \)          | Relative error |
| 0.1 | 6.62168803         | 6.62168805         | 2.0000E-08     | 8.86925461       | 8.86925711       | 2.50000E-06   |
| 1   | 6.53745555         | 6.53750789         | 5.2340E-05     | 7.98854799       | 7.99213348       | 3.58549E-03   |
| 10  | 6.32292330         | 6.32293875         | 5.7500E-06     | 6.48976514       | 6.49020840       | 4.43260E-04   |
| 100 | 6.28718803         | 6.28718821         | 1.8000E-07     | 6.30327576       | 6.30328033       | 4.57000E-06   |

**Table 2.** Comparison of the approximate period with exact period when \( \lambda = 0.75 \) and \( \lambda = 0.95 \)

| A   | \( \lambda = 0.75 \) | \( \lambda = 0.95 \) |
|-----|--------------------|--------------------|
|     | \( T \)            | \( T_e \)          | Relative error | \( T \)            | \( T_e \)          | Relative error |
| 0.1 | 12.49670709        | 12.49673546        | 3.06600E-05    | 27.15435026      | 27.15678532      | 2.435060E-03  |
| 1   | 9.604912470        | 9.625404980        | 2.049251E-02   | 11.97333327      | 12.07527692      | 10.194365E-02 |
| 10  | 6.601000770        | 6.602092470        | 1.091700E-03   | 6.694228750      | 6.696116520      | 1.887770E-03  |
| 100 | 6.313393530        | 6.313403890        | 1.03600E-05    | 6.321522920      | 6.321539660      | 1.674000E-05  |
Example 2

As a last example, we consider the following nonlinear Duffing–harmonic oscillation [18];

\[ u^2 + u^4 / (1 + u^2) = 0, \quad u(0) = A, \quad u'(0) = 0. \] (15)

Which \( f(u) = u^3 / (1 + u^2) \).

Its variational formulation is

\[ J(u) = \int_0^{T/4} \left( -\frac{1}{2} u^2 + \frac{1}{2} u^2 - \frac{1}{2} \ln (1 + u^2) \right) dt \] (16)

Similar of previous examples, we have

\[ J(A) = \int_0^{T/4} \left( -\frac{1}{2} A^2 \omega^2 \sin^2 \omega t + \frac{1}{2} A^2 \cos^2 \omega t - \frac{1}{2} \ln (1 + A^2 \cos^2 \omega t) \right) dt \] (17)

\[ \frac{\partial J}{\partial A} = \int_0^{T/4} \left( -A \omega^2 \sin^2 \omega t + A \cos^2 \omega t - \left( A \cos^2 \omega t \right) / \left( 1 + A^2 \cos^2 \omega t \right) \right) dt \] (18)

From (18) we have

\[ \omega = \left( (A^2 + 1)^{1/2} \left( 2 \ \text{csgn} \left( (A^2 + 1)^{1/2} \right) + A^2 (A^2 + 1)^{1/2} - 2 (A^2 + 1)^{1/2} \right) / (A (A^2 + 1)) \right)^{1/2} \] (19)

The \text{csgn} is defined in Maple Package Software

In the example, we assume \( A = 0.01, 0.05, 0.1, 0.5, 1, 5, 10, 50, 100 \). The obtained exact results are expressed in [19].

The computed results for the approximate frequency \( \omega \) with exact frequency \( \omega_e \) listed in Table 3. From there results we see that the maximum error is 0.022%.
Table 3. Comparison results for the angular frequencies of approximation and exact solution with $A = 0.01, 0.05, 0.1, 0.5, 1, 5, 10, 50, 100$

| $A$ | 0.01 | 0.05 | 0.1 | 0.5 | 1.0 | 5.0 | 10.0 | 50.0 | 100.0 |
|-----|------|------|-----|-----|-----|-----|------|------|-------|
| $\omega_c$ | 0.00847 | 0.04232 | 0.08439 | 0.38737 | 0.63678 | 0.96698 | 0.99092 | 0.99961 | 0.99990 |
| $\omega$ | 0.00837 | 0.04325 | 0.08624 | 0.39423 | 0.64360 | 0.96731 | 0.99099 | 0.99952 | 0.99980 |
| Relative error | 1E–04 | 9.3E–04 | 1.85E–03 | 6.86E–03 | 6.82E–03 | 3.3E–04 | 7E–05 | 9E–05 | 1E–04 |

4. Conclusions
We used a very simple but effective new method for nonlinear oscillators. The first–order approximate solutions are of a high accuracy. Of course the accuracy can be improved upon using a higher order approximate solution. These examples have shown that the approximate analytical solutions are in excellent agreement with the corresponding exact solutions. The method can be easily extended to any nonlinear oscillator without any difficulty. Moreover, the present paper can be used as paradigms for many other applications in searching for periodic solutions of nonlinear oscillations.

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