Evidence of Odderon-exchange from scaling properties of elastic scattering at TeV energies

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Abstract We study the scaling properties of the differential cross section of elastic proton-proton (pp) and proton-antiproton (p̅p) collisions at high energies. We introduce a new scaling function, that scales – within the experimental errors – all the ISR data on elastic pp scattering from \( \sqrt{s} = 23.5 \) to 62.5 GeV to the same universal curve. We explore the scaling properties of the differential cross-sections of the elastic pp and p̅p collisions in a limited TeV energy range. Rescaling the TOTEM pp data from \( \sqrt{s} = 7 \) TeV to 2.76 and 1.96 TeV, and comparing it to D0 p̅p data at 1.96 TeV, our results provide an evidence for a t-channel Odderon exchange at TeV energies, with a significance of at least 6.55\( \sigma \).

1 Introduction

One of the most important and critical tests of quantum chromodynamics (QCD) in the infrared regime is provided by the ongoing studies of elastic differential hadron-hadron scattering cross section at various energies and momentum transfers. The characteristics of the elastic amplitude, its both real and imaginary parts, carry a plenty of information about the inner proton structure, the proton profile in the impact parameter space and its energy dependence, as well as about the properties of QCD exchange interaction at low momentum transfers.

The first and most precise measurements of the total, elastic and differential cross sections of elastic pp collisions, together with the p-parameter, has recently been performed by the TOTEM Collaboration at the Large Hadron Collider (LHC) at CERN at the highest energy frontier of \( \sqrt{s} = 13 \) TeV (for the corresponding recent TOTEM publications, see Refs. [1–4]). A correct theoretical interpretation of the LHC data, together with the lower-energy Tevatron and ISR data, is a subject of intense debates and ongoing research development in the literature, see e.g. Refs. [5, 6]. Among the important recent advances, the recent data by the TOTEM Collaboration for the first time have indicated the presence of an odd-under-crossing (or C-odd) contribution to the elastic scattering amplitude known as the Odderon [7]. In particular, a comparison of the differential cross-section of elastic proton-proton pp scattering obtained by the TOTEM Collaboration at \( \sqrt{s} = 2.76 \) TeV with D0 results on elastic proton-antiproton p̅p scattering at 1.96 TeV [8] indicates important qualitative differences that can be attributed to the Odderon effect [4, 9]. In more rigorous language of QCD, an Odderon exchange is usually associated with a quarkless odd-gluon (e.g. three-gluon, to the lowest order) bound state such as a vector glueball, and a vast literature is devoted to theoretical understanding of its implications (for recent developments and claims, see e.g. Refs. [6, 10, 11]).

In earlier studies of Refs. [12, 13], the Odderon signatures have been identified and qualitatively described in a model-independent way using the power of the so-called Lévy imaging technique [9]. One of such signatures concern the presence of a dip-and-bump structure in the differential cross section of elastic pp collisions and the lack of such a structure in elastic p̅p collisions. The latter effectively emerges in the t-dependence of the elastic slope \( B(t) \), that crosses zero for elastic pp collisions and remains non-negative for all values of t in elastic p̅p collisions. Besides, Ref. [9] noted that the position of the node of the nuclear phase \( \phi(t) \), as reconstructed with the help of the Lévy expansion method, is characteristically and qualita-
tively different for elastic $pp$ from $p\bar{p}$ collisions, thus, indicating the Odderon exchange. In addition, the presence of a smaller substructure of the proton has been revealed in the data that is imprinted in the behaviour of the $t$-dependent elastic slope $B(t)$, apparent at large values of $t$. In particular, in Refs. [9, 12–14] a substructure of the two distinct sizes has been identified in the low (a few tens of GeV) and high (a few TeV) energy domains, respectively. Besides, a new statistically significant feature in the $b$-dependent shadow (or inelasticity) profile has been found at the maximal available energy $\sqrt{s} = 13$ TeV and represents a long-debated hollowness, or “black-ring” effect that emerges instead of the conventionally anticipated “black-disk” regime [12, 14].

In this paper, in order to further unveil the important characteristics of elastic hadron-hadron scattering we study the scaling properties of the existing data sets available from the ISR and Tevatron colliders as well as those provided by the TOTEM Collaboration at a TeV energy range [1–4, 15]. We investigate a generic scaling behavior of elastic differential proton-(anti)proton scattering cross section, with the goal of transforming out the trivial colliding energy dependent variation of the key observables like that of the total and elastic cross-sections $\sigma_{\text{tot}}(s)$ and $\sigma_{\text{el}}(s)$, the elastic slope $B(s)$ and the real-to-imaginary ratio $\rho(s)$. We search for a universal scaling function and the associated data-collapsing behaviour that is valid not only in the low-$|t|$ domain, but also in the dip-and-bump region. We then discuss the physics implications of such a scaling behaviour and explore its consequences for understanding of the Odderon effect as well as the high-energy behaviour of the proton structure.

The paper is organised as follows. In section 2, we recapitulate the formalism that is utilized for evaluation of the observables of elastic proton-(anti)proton scattering in the TeV energy range. In section 3, we connect this formalism to a more general strategy of the experimental Odderon search, namely, to the search for a crossing-odd component in the differential cross-section of elastic proton-(anti)proton scattering. In section 4, we study some of the scaling functions of elastic scattering already existing in the literature as well as propose a new scaling function denoted as $H(s)$ that is readily measurable in $pp$ and $p\bar{p}$ collisions, and present a first test of the $H(s)$ scaling in the ISR energy range of 23.5 – 62.5 GeV. Subsequently, in section 5 we extend these studies to the TeV (Tevatron and LHC) energy range, where the possible residual effects of Reggeon exchange are expected to be below the scale of the experimental errors, Ref. [16]. In section 6, we present a method of how to quantify the significance of our findings, giving the formulas that are used to evaluate $\chi^2$, confidence level (CL), and significance in terms of the standard deviation, $\sigma$. In section 7, we discuss how to employ the newly found scaling behavior of the differential cross-section in the search for an Odderon effect. In section 8, we present further, more detailed results of our studies with the help of $H(s)$ and compare such a scaling function for $pp$ differential cross-sections at the LHC energies with the $p\bar{p}$ scaling function at the Tevatron energy. In section 9 we evaluate the significance of the Odderon-effect, and find that it is at least a 6.55$\sigma$-significant effect. Subsequently, we present several cross-checks in section 10 and discuss the main results in section 11. Finally, we summarize and conclude our work in section 12.

2 Formalism

For the sake of completeness and clarity, let us start first with recapitulating the connection between the scattering amplitude and the key observables of elastic scattering, following the conventions of Refs. [17–20].

The Mandelstam variables $s$ and $t$ are defined as usual $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ for an elastic scattering of particles $a$ and $b$ with incoming four-momenta $p_1$ and $p_2$, and outgoing four-momenta $p_3$ and $p_4$, respectively.

The elastic cross-section is given as integral of the differential cross-section of elastic scattering:

$$\sigma_{\text{el}}(s) = \int_0^\infty d|t| \frac{d\sigma(s,t)}{dt}$$

(1)

The elastic differential cross section is

$$\frac{d\sigma(s,t)}{dt} = \frac{1}{4\pi} |T_{\text{el}}(s,\Delta)|^2, \quad \Delta = \sqrt{|t|}.$$  

(2)

The $t$-dependent slope parameter $B(s,t)$ is defined as

$$B(s,t) = \frac{d}{dt} \ln \frac{d\sigma(s,t)}{dt}$$

(3)

and in the experimentally accessible low-$t$ region this function is frequently assumed or found within errors to be a constant. In this case, a $t$-independent slope parameter $B(s)$ is introduced as

$$B(s) \equiv B_0(s) = \lim_{t \to 0} B(s,t),$$

(4)

where the $t \to 0$ limit is taken within the experimentally probed region. Actually, experimentally the optical $t = 0$ point can only be approached by extrapolations from the measurements in the $-t > 0$ kinematically accessible regions.

According to the optical theorem, the total cross section is also found by a similar extrapolation. Its value is given by

$$\sigma_{\text{tot}}(s) \equiv 2 \Im T_{\text{el}}(\Delta = 0, s),$$

(5)

while the ratio of the real to imaginary parts of the elastic amplitude is found as

$$\rho(s,t) \equiv \frac{\Re T_{\text{el}}(s,\Delta)}{\Im T_{\text{el}}(s,\Delta)}$$

(6)
3 Looking for Odderon effects in the differential cross-section of elastic scattering

As noted in Refs. [16, 27], the only direct way to see the Odderon is by comparing the particle and antiparticle scattering at sufficiently high energies provided that the high-energy \( pp \) or \( pp \) elastic scattering amplitude is a sum or a difference of even and odd C-parity contributions, respectively,

\[
T_{el}^{pp}(s,t) = T_{el}^{s}(s,t) + T_{el}^{o}(s,t),
\]

\[
T_{el}^{pp}(s,t) = T_{el}^{+}(s,t) - T_{el}^{-}(s,t),
\]

\[
T_{el}^{s}(s,t) = T_{el}^{+}(s,t) + T_{el}^{-}(s,t),
\]

\[
T_{el}^{o}(s,t) = T_{el}^{+}(s,t) + T_{el}^{-}(s,t).
\]

where the even-under-crossing part consists of the Pomeron and the \( f \) Reggeon trajectory, while the odd-under-crossing part contains the Odderon and a contribution from the \( \omega \) Reggeon. It is clear from the above formulae that the odd component of the amplitude can be extracted from the difference of the \( pp \) and \( p\bar{p} \) scattering amplitudes.

At sufficiently high energies, the relative contributions from secondary Regge trajectories are suppressed, as they decay as negative powers of the colliding energy \( \sqrt{s} \). In Ref. [27], the authors argued that the LHC energy scale is already sufficiently large to suppress the Reggeon contributions, and they presented the \((s,t)\)-dependent contributions of an Odderon exchange to the differential and total cross-sections at typical LHC energies. More recently, this observation was confirmed in Ref. [16], suggesting that indeed the relative contribution of the Reggeon trajectories is well below the experimental precision in elastic \( pp \) scattering in the TeV energy range. The analysis of Ref. [27] relies on a model-dependent, phenomenological picture formulated in the framework of the Phillips-Barger model [28] and is focused primarily on fitting the dip region of elastic \( pp \) scattering, but without a detailed analysis of the tail and cone regions. In Ref. [16], a phenomenological Reggeon + Pomeron + Odderon exchange model is employed to study, in particular, the possible hollowness effect in the high-energy elastic \( pp \) collisions. More recently, a similar study of the Philips-Barger model was performed in Ref. [29] using the most recent TOTEM data on elastic \( pp \) scattering. Similarly, Ref. [30] has also argued that the currently highest LHC energy of \( \sqrt{s} = 13 \) TeV is sufficiently high to see the Odderon contribution.

In this paper, we follow Refs. [16, 27, 30] and assume that the Reggeon contributions to the elastic scattering amplitudes for \( \sqrt{s} \geq 1.96 \) TeV and at higher energies are negligibly small. We search for an odd-under-crossing contribution to the scattering amplitude, in a model independent way, and find that such a non-vanishing contribution is present at a TeV scale that is recognised as an Odderon effect. The vanishing nature of the Reggeon contributions offers a direct way of extracting the Odderon as well as the Pomeron contributions, \( T_{el}^{p}(s,t) \) and \( T_{el}^{pp}(s,t) \), respectively, from the elastic \( pp \) and \( p\bar{p} \) scattering data at sufficiently high colliding energies as follows

\[
T_{el}^{p}(s,t) = \frac{1}{2} \left( T_{el}^{pp}(s,t) + T_{el}^{pp}(s,t) \right) \quad \text{for} \ \sqrt{s} \geq 1 \ \text{TeV},
\]

\[
T_{el}^{p}(s,t) = \frac{1}{2} \left( T_{el}^{pp}(s,t) - T_{el}^{pp}(s,t) \right) \quad \text{for} \ \sqrt{s} \geq 1 \ \text{TeV}.
\]

These kind of studies rely on the extrapolation of the fitted model parameters of \( pp \) and \( p\bar{p} \) reactions to an exactly the same energy, given that the elastic \( pp \) and \( p\bar{p} \) scattering data have not been measured at the same (or close enough) energies in the TeV region so far. Another problem is a lack of precision data at the low- and high-\( |t| \), pri-
marily, in $p\bar{p}$ collisions. Recently, the TOTEM Collaboration noted in Ref. [4] that "Under the condition that the effects due to the energy difference between TOTEM and D0 can be neglected, the result" (namely the differential cross-section measured by TOTEM at $\sqrt{s} = 2.76$ TeV) "provides evidence for a colourless 3-gluon bound state exchange in the $t$-channel of the proton-proton elastic scattering". In other words, if the effects due to the energy difference between TOTEM and D0 measurements can be neglected, the direct comparison of the differential cross section of elastic $pp$ scattering at $\sqrt{s} = 2.76$ with that of $p\bar{p}$ scattering at $\sqrt{s} = 1.96$ TeV provides a conditional evidence for a colourless three-gluon state exchange in the $t$-channel.

In this paper, we show that the conditional evidence stated by TOTEM can be turned into an evidence, i.e. a discovery of the Odderon, by closing the energy gap as much as possible at present, without a direct measurement, based on a re-analysis of already published TOTEM and D0 data. Here we take the data at a face value as given in published sources and do not attempt to extrapolate any model or model parameters towards their unmeasured values (in unexplored energy domains). Instead, we discuss a new kind of scaling relations, that we test on the experimental data and show their data-collapsing behaviour in a limited energy range. We demonstrate that such a data-collapsing behaviour can be used to close the small energy gap between the highest-energy elastic $pp$ collisions, $\sqrt{s} = 1.96$ TeV and the lowest-energy elastic $pp$ collisions at the LHC where the public data are available, $\sqrt{s} = 2.76$ TeV. We then look for even-under-crossing and odd-under-crossing contributions by comparing the scaling functions of $pp$ and $p\bar{p}$ collisions in the TeV energy range. In other words, we look for a robust Odderon signature in the difference of the scaling functions of the elastic differential cross-section between $pp$ and $p\bar{p}$ collisions. We thus discuss the Odderon features that can be extracted in a model-independent manner directly by comparing the corresponding data sets to one another.

Let us start with three general remarks as direct consequences of Eqs. (17,18):

- If the Odderon exchange effect is negligibly small (within errors, equal to zero) or if it does not interfere with that of the Pomeron at a given energy, then the differential cross sections of the elastic $pp$ and $p\bar{p}$ scattering have to be equal:

$$T_{el}^{O}(s,t) = 0 \implies \frac{d\sigma^{pp}}{dt} = \frac{d\sigma^{p\bar{p}}}{dt} \quad \text{for } \sqrt{s} \geq 1 \text{ TeV}. \quad (19)$$

- If the differential cross sections of elastic $pp$ and $p\bar{p}$ collisions are equal within the experimental errors, this does not imply that the Odderon contribution has to be equal to zero. Indeed, the equality of cross sections does not require the equality of complex amplitudes:

$$\frac{d\sigma^{pp}}{dt} = \frac{d\sigma^{p\bar{p}}}{dt} \quad \text{for } \sqrt{s} \geq 1 \text{ TeV} \implies T_{el}^{O}(s,t) = 0. \quad (20)$$

- If the $pp$ differential cross sections differ from that of $p\bar{p}$ scattering at the same value of $s$ in a TeV energy domain, then the Odderon contribution to the scattering amplitude cannot be equal to zero, i.e.

$$\frac{d\sigma^{pp}}{dt} \neq \frac{d\sigma^{p\bar{p}}}{dt} \quad \text{for } \sqrt{s} \geq 1 \text{ TeV} \implies T_{el}^{O}(s,t) \neq 0. \quad (21)$$

Our research strategy in this paper is thus to scale out the $s$-dependence of the differential cross section by factoring out its dependencies on $\sigma_{el}(s)$, $\sigma_{el}(s)$, $B(s)$ and $\rho(s)$ functions. The residual scaling functions will be compared for the $pp$ and $p\bar{p}$ elastic scattering to see if any difference remains. Such residual difference is considered to be a clear-cut signal for the Odderon-exchange, if the differential cross sections were measured at exactly the same energies. However, currently such data are lacking in the TeV energy range. So we may expect that after scaling out the trivial $s$-dependencies, only small scaling violating terms remain that depend on $s$, which can be estimated by the scaling violations of the differential cross sections measured at various nearby energies. We look for significant differences between the scaling functions of $pp$ and $p\bar{p}$ collisions as compared to these possible $s$-dependent scaling violating terms, as such observations provide a significant signal of the Odderon effect.

In what follows, we introduce and discuss the newly found scaling function $H(x)$ in section 4 and subsequently evaluate the significance of these observations as detailed in sections 6 and 9.

### 4 Possible scaling relations at low values of $|t|$}

In this section, let us first investigate the scaling properties of the experimental data based on a simple Gaussian model elaborating on the discussion presented in Ref. [31]. The motivation for this investigation is that we would like to work out a scaling law that works at least in the simplest, exponential diffractive cone approximation, and scales out the trivial $s$-dependencies of $\sigma_{el}(s)$, $\sigma_{el}(s)$, $\rho(s)$, and $B(s)$. Based on the results of such a frequently used exponential approximation, we gain some intuition and experience on how to generalize such scaling laws for realistic non-exponential differential cross sections.

Experimentally, the low-$|t|$ part of the measured distribution is usually approximated with an exponential,

$$\frac{d\sigma}{dt} = A(s) \exp \left[B(s)|t|\right], \quad (22)$$
where it is explicitly indicated that both the normalization parameter $A \equiv A(s)$ and the slope parameter $B \equiv B(s)$ are the functions of the center-of-mass energy squared $s$. If the data deviate from such an exponential shape, that can be described if one allows for a $t$-dependence of the slope parameter $B \equiv B(s,t)$ as defined in Eq. (3). For simplicity, we would like to scale out the energy dependence of the elastic slope $B(s) \equiv B(s,t=0)$ from the differential cross section of elastic scattering, together with the energy dependence of the elastic and total cross sections, $σ_{el}(s)$ and $σ_{tot}(s)$, as detailed below. For this purpose, let us follow the lines of a similar derivation in Refs. [16, 31].

It is clear that Eq. (22) corresponds to an exponential “diffractive cone” approximation, that may be valid in the low-$t$ domain only. This equation corresponds to the so called “Grey Gaussian” approximation that suggests a relationship between the nuclear slope parameter $B(s)$, the real-to-imaginary ratio $ρ_0(s)$, the total cross section $σ_{tot}(s)$, and the elastic cross section $σ_{el}(s)$ as follows [16, 32, 33]:

$$A(s) = B(s) σ_{el}(s) = \frac{1 + ρ_0^2(s)}{16 \pi} σ_{tot}^2(s), \quad (23)$$

$$B(s) = \frac{1 + ρ_0^2(s)}{16 \pi} \frac{σ_{tot}^2(s)}{σ_{el}(s)}. \quad (24)$$

Such relations for $A$ and $B$ parameters in terms of the elastic and total cross sections are particularly useful when studying the shadow profile function as detailed below. The above relationships, in a slightly modified form, have been utilized by TOTEM to measure the total cross section at $\sqrt{s} = 2.76, 7, 8$ and $13$ TeV in Refs. [1, 34–36], using the luminosity independent method. In what follows, we do not suppress the $s$-dependence of the observables, i.e. $σ_{tot} \equiv σ_{tot}(s), \quad σ_{el} \equiv σ_{el}(s)$.

4.1 Scaling properties of the shadow profiles

In the exponential approximation given by Eqs. (22,23,24), the shadow profile function introduced in Eq. (12) has a remarkable and very interesting scaling behaviour, as anticipated in Ref. [16]:

$$P(b,s) = 1 - \left[1 - r(s) \exp \left(-\frac{b^2}{2B(s)}\right)^2 - \frac{b^2}{B(s)}\right],$$

$$r(s) \equiv \frac{4 σ_{el}(s)}{σ_{tot}(s)}. \quad (25)$$

Thus, the shadow profile at the center, $P_0(s) \equiv P(b = 0, s)$ reads as

$$P_0(s) = \frac{1}{1 + ρ_0^2(s)} - \left[1 + ρ_0^2(s)\right] r(s) - \frac{1}{1 + ρ_0^2(s)}, \quad (27)$$

which cannot become maximally absorptive (or black), i.e. $P_0(s) = 1$ is not reached at those colliding energies, where $ρ_0$ is not negligibly small. The maximal absorption corresponds to $P_0(s) = \frac{1}{1 + ρ_0^2(s)}$, which is rather independent of the detailed $b$-dependent shape of the inelastic collisions [16]. It is achieved when the ratio of the elastic to total cross sections approaches the value $r(s) = \frac{1}{1 + ρ_0^2(s)}$. Thus, at such a threshold, we have the following critical value of the ratio $\frac{σ_{el}(s)}{σ_{tot}(s)}$:

$$\frac{σ_{el}(s)}{σ_{tot}(s)} \mid_{\text{threshold}} = \frac{1}{4 \left[1 + ρ_0^2(s)\right]} \quad (28)$$

As $ρ_0 ≤ 0.15$ for the existing measurements and $ρ_0(s)$ decreases to zero with increasing energies at least in the $8 \leq \sqrt{s} \leq 13$ TeV region, the critical value of the elastic-to-total cross section ratio (28) corresponds to, roughly, $σ_{el}/σ_{tot} ≈ 24.5 − 25.0 \%$. Evaluating the second derivative of $P(b,s)$ at $b = 0$, one may also observe that it changes sign from a negative to a positive one exactly at the same threshold given by Eq. (28). Such a change of sign can be interpreted as an onset of the hollowness effect [16]. The investigation of such a hollowness at $b = 0$ is a hotly debated topic in the literature. For early papers on this fundamental feature of $pp$ scattering at the LHC and asymptotic energies, see Refs. [22, 23, 33, 37–40], as well as Refs. [16, 21, 24, 25, 41–48] for more recent theoretical discussions.

As pointed out in Ref. [31], the threshold (28), within errors, is reached approximately already at $\sqrt{s} = 2.76$ TeV. The threshold behavior saturates somewhere between 2.76 and 7 TeV and a transition may happen around the threshold energy of $\sqrt{s} = 2.76 - 4$ TeV. The elastic-to-total cross section ratio becomes significantly larger than the threshold value at $\sqrt{s} = 13$ TeV colliding energies. As a result, the shadow profile function of the proton undergoes a qualitative change in the region of $2.76 < \sqrt{s} < 7$ TeV energies. At high energies, with $σ_{el} ≥ σ_{tot}/4$, the hollowness effect may become a generic property of the impact parameter distribution of inelastic scatterings. However, the expansion at low impact parameters corresponds to the large-$|t|$ region of elastic scattering, where the diffractive cone approximation of Eqs. (22,23,24) technically breaks down, and more refined studies are necessary (see below). For the most recent, significant and model-independent analysis of the hollowness effect at the LHC and its extraction directly from the TOTEM data, see Ref. [14].

4.2 Scaling functions for testing the black-disc limit

When discussing the scaling properties of the differential cross section of elastic scattering, let us mention that various scaling laws have been proposed to describe certain features and data-collapsing behaviour of elastic scattering proton-proton scattering already in the 1970-s. One of the
early proposals was the so called geometric scaling property of the inelastic overlap function [49, 50]. The concept of geometric scaling was based on a negligibly small ratio of the real-to-imaginary part of the scattering amplitude at $t = 0$, $R_0 \leq 0.01$ and resulted in an $s$-independent ratio of the elastic-to-total cross-sections, $\sigma_{el}/\sigma_{tot} \approx \text{const}(s)$, while at the LHC energies, $R_0$ is not negligibly small and the elastic-to-total cross-section ratio is a strongly rising function of $s$. Here, we just note about the geometric scaling as one of the earliest proposals to have a data-collapsing behavior in elastic scattering, but we look in detail for other kind of scaling laws that are more in harmony and consistency with the recent LHC measurements [31].

Let us first detail the following two dimensionless scaling functions proposed in Ref. [19] and denoted as $F(y)$ and $G(z)$ in what follows. These scaling functions were introduced in order to cross-check if elastic $pp$ collisions at the LHC energies approach the so-called black-disc limit, expected at ultra-high energies, or not. In a strong sense, the black disc limit corresponds to the shadow profile $P(b) = \theta(R_b - b)$ that results in $\sigma_{el} / \sigma_{tot} = 1/2$, independently of the black disc radius $R_b$. This limit is clearly not yet approached at LHC energies, but in a weak sense, a black-disc limit is considered to be reached also if the shadow profile function at $b = 0$ reaches unity, i.e. $P(b = 0) = 1$, corresponding to black disc scattering at zero impact parameter. This kind of black disc scattering might have been approached at $\sqrt{s} = 7$ TeV LHC energies [20].

The first scaling function of the differential cross-section is defined as follows:

$$F(y) = \frac{|t|}{\sigma_{tot}} \frac{d\sigma}{dt}, \quad y = t/\sigma_{tot}, \quad (29)$$

In the diffractive cone approximation, the $s$-dependence in $F(y)$ does not cancel out but can be approximately written as

$$F(y) \simeq y B(s) \frac{\sigma_{el}(s)}{\sigma_{tot}(s)} \frac{d\sigma}{dt} \bigg|_{t = y/\sigma_{tot}(s)}, \quad (31)$$

$$B(s)t = \frac{B(s)}{\sigma_{tot}(s)}. \quad (32)$$

This result clearly indicates that in the diffractive cone, the $F(y)$ scaling is strongly violated by the energy-dependent factors, while for a black-disc scattering, the $F(y)$ scaling has to be valid, see Ref. [19] for more details. Indeed, the aim to introduce the scaling function $F(y)$ was to clarify that even at the highest LHC energies we do not reach the black-disk limit (in the strong sense). As discussed in the previous section, the deviations from the black-disk limit might be due to the effects of the real part and the hollowness, i.e. reaching a black-ring limit instead of a black-disc one at the top LHC energies.

Since in the $F(y)$ scaling function the position of the diffractive minimum (dip) remains $s$-dependent, yet another scaling function denoted as $G(z)$ was proposed to transform out such $s$-dependence of the dip. This function was introduced also in Ref. [19] as follows:

$$G(z) = \frac{z|l_{ap}(s)|}{\sigma_{tot}(s)} \frac{d\sigma}{dt} \bigg|_{t = z|l_{ap}(s)|}, \quad (33)$$

$$z = \frac{t}{|l_{ap}(s)|}. \quad (34)$$

In principle, all black-disc scatterings, regardless of the value of the total cross section, should show a data-collapsing behaviour to the same $G(z)$ scaling function. As observed in Ref. [19], such an asymptotic form of the $G(z)$ scaling function is somewhat better approached at the LHC energies as compared to the lower ISR energies but still not reproduced it exactly. This is one the key indications the black-disc limit in the elastic $pp$ scattering is not achieved at the LHC, up to $\sqrt{s} = 13$ TeV. This may have several other important implications. For example, this result indicates that in simulations of relativistic heavy-ion collisions at the LHC energies, more realistic profile functions have to be used to describe the impact parameter dependence of the inelastic $pp$ collisions: a simple gray or black-disc approximation for the inelastic interactions neglects the key features of elastic $pp$ collisions at the TeV energy scales.

One advantage of the scaling variables $y$ and $z$ mentioned above is that they are dimensionless. Numerically, $G(z)$ corresponds to the $F(y)$ function if the scaling variable $y$ is rescaled to $z$. As indicated in Fig. 23 of Ref. [19], indeed the main difference between $F(y)$ and $G(z)$ is that the diffractive minimum is rescaled in $G(z)$ to the $z = 1$ position, so $G(z)$ has less evolution with $s$ as compared to $F(y)$. However, as it is clear from the above discussion, the function

$$G(z) \simeq \frac{\sigma_{el}(s)}{\sigma_{tot}(s)} B(s) z|l_{ap}(s)| \frac{d\sigma}{dt} \bigg|_{t = z|l_{ap}(s)|}, \quad (35)$$

$$B(s)t = B(s)|l_{ap}(s)|z, \quad (36)$$

is well-defined only for $pp$ elastic scattering, where a unique dip structure is observed experimentally.

Even the dip region is not always measurable in $pp$ reactions if the experimental acceptance is limited to the cone region, which is a sufficient condition for the total cross section measurements. If the acceptance is not large enough in $|t|$ to observe the diffractive minimum, or, in the case when the diffractive minimum does not clearly exist, then the both $F(y)$ and $G(z)$ scaling functions cannot be used. So, the major disadvantage of these scaling functions for extracting the Odderon signatures from the data is that in $pp$ collisions no significant diffractive minimum is found by the D0 collaboration at 1.96 TeV [8]. Besides, even if $z$ variable were defined, the above expressions indicate, in agreement with Fig. 23 of Ref. [19], that the $G(z)$ scaling function has a
non-trivial energy-dependent evolution in the cone ($z \ll 1$) region. Due to these reasons, variables $z$ and $y$ are not appropriate scaling variables for a scale-invariant analysis of the crossing-symmetry violations at high energies.

Having recapitulated the considerations in Ref. [31], with an emphasis on the $s$-dependence of the parameters, let us now consider, how these $s$-dependencies can be scaled out at low values of $|t|$, where the diffraction cone approximation is valid, by evaluating the scaling properties of the experimental data on the differential elastic $pp$ and $p\bar{p}$ cross sections. For this purpose, let us look into the scaling properties of the differential cross sections and their implications related to the Odderon discovery in a new way.

4.3 A new scaling function for the elastic cone

In the elastic cone region, all the $pp$ and $p\bar{p}$ differential cross sections can be rescaled to a straight line in a linear-logarithmic plot, when the horizontal axis is scaled by the slope parameter to $-tB(s)$ while the vertical axis is simultaneously rescaled by $B(s)\sigma_{el}(s)$, namely,

$$\frac{1}{B(s)\sigma_{el}(s)} \frac{d\sigma}{dt} = \exp[-tB(s)] \text{ versus } x = -tB(s).$$

(37)

This representation, in the diffractive cone, scales out the $s$-dependencies of the total and elastic cross section, $\sigma_{tot}(s)$ and $\sigma_{el}(s)$, and also that of the slope parameter, $B(s)$. As a function of the scaling variable $x = -tB$, it will correspond to the plot of $\exp(-x)$ i.e. a straight line with slope $-1$ on a linear-logarithmic plot. It is well-known that the elastic scattering is only approximately exponential in the diffractive cone, but by scaling out this exponential feature one may more clearly see the scaling violations on this simple scaling plot. We will argue that such a scaling out of the trivial energy-dependent terms can be used as a powerful method in the search for the elusive Odderon effects in the comparison of elastic $pp$ and $p\bar{p}$ data in the TeV energy range.

In what follows, we investigate the scaling properties of the new scaling function,

$$H(x) \equiv \frac{1}{B(s)\sigma_{el}(s)} \frac{d\sigma}{dt} ,$$

(38)

$$x = -tB(s).$$

(39)

This simple function has four further advantages summarized as follows:

1. First of all, it satisfies a sum-rule or normalization condition rather trivially, $\int dx H(x) = 1$, as follows from the definition of the elastic cross section.
2. Secondly, if almost all of the elastically scattered particles belong to the diffractive cone, the differential cross-section at the optical point is also given by $\frac{d\sigma}{dt}\bigg|_{t=0} = A(s) = B(s)\sigma_{el}(s)$, and in these experimentally realized cases we have another (approximate) normalization condition, namely, $H(0) = 1$.
3. Third, in the diffractive cone, all the energy dependence is scaled out from this function, i.e., $H(x) = \exp(-x)$ that shows up as a straight line on a linear-logarithmic plot with a trivial slope $-1$.
4. Last, but not least, the slope parameter $B(s)$ is readily measurable not only for $pp$ but also for $p\bar{p}$ collisions, hence the $pp$ and the $p\bar{p}$ data can be scaled to the same curve without any experimental difficulties.

Let us first test these ideas by using the ISR data in the energy range of $\sqrt{s} = 23.5 - 62.5$ GeV. The results are shown in Fig. 1 which indicates that the ISR data indeed show a data-collapsing behaviour.

At low values of $x$, the scaling function is indeed, approximately, $H(x) \approx \exp(-x)$, that remains a valid approximation over, at least, five orders of magnitude in the decrease of the differential cross section. However, at the ISR energies, the scaling seems to be valid, within the experimental uncertainties, not only at low values of $x = -Bt$, but extended to the whole four-momentum transfer region, including the dip and bump region ($15 \leq x \leq 30$) as well. Even at large-$|t|$ after the bump region, corresponding to $x \geq 30$, the data can approximately be scaled to the same, non-exponential scaling function ($H(x) \neq \exp(-x)$ in the tails of the distribution). Thus, Fig. 1 indeed indicates a non-trivial data-collapsing behaviour to the same, non-trivial scaling function at the ISR energy range of $\sqrt{s} = 23.5 - 62.5$ GeV.

This observation motivated us to generalize the derivation presented above in this section, to arbitrary positively definite non-exponential scaling functions $H(x)$. Such a generalization is performed in the next section in order to explain the data-collapsing behaviour in Fig. 1.

4.4 Generalized scaling functions for non-exponential differential cross-sections

In this section, we search for a novel type of scaling functions of $pp$ elastic data that may be valid not only in the diffractive cone, but also in the crucial dip and bump region, as well. In Fig. 1, we have noticed that the data-collapsing behaviour may extend well above the small $x = -Bt$ region significantly beyond the diffractive maximum, indicating a clear deviation of the scaling function $H(x)$ from the exponential shape.

In addition, a recent detailed study of the low-$|t|$ behaviour of the differential elastic $pp$ cross section at $\sqrt{s} = 8$ TeV observed a more than 7σ-significant deviation from the exponential shape [52, 53], which also corresponds to a non-exponentiality in the scaling function $H(x)$ even in the low-$|t|$, or small $x$, range.
In this section, we thus further generalize the derivation of the $H(x) = \exp(-x)$ scaling function, in order to allow for arbitrary positively definite functions with $H(x = 0) = 1$ normalisation, and to develop a physical interpretation of the experimental observations.

Let us start the derivation from the relation of the elastic scattering profile $\tilde{\sigma}(s, b)$ and the complex opacity function $\Omega(s, b)$ based on Eq. (11), using the same notation as in Ref. [20]:

$$t_{el}(s, b) = i \left[ 1 - \exp(-i \Im \Omega(s, b)) \sqrt{1 - \tilde{\sigma}_{el}(s, b)} \right].$$

The shadow profile function $P(s, b)$ is equal to the inelastic scattering profile $\tilde{\sigma}_{in}(s, b)$ as follows from Eq. (12), $P(s, b) = \tilde{\sigma}_{in}(s, b)$. The imaginary part of the opacity function $\Omega$ is generally not known or less constrained by the data, but it is experimentally known that $\rho_0(s)$ is relatively small at high energies: at all the measured LHC energies and below, $\rho_0 \leq 0.15$, hence, $\rho^2 \leq 2.3 \%$.

Here, we thus follow the choice of Ref. [20], that has demonstrated that the ansatz

$$\Im \Omega(s, b) \propto \tilde{\sigma}(s, b)$$

(41)
gives a satisfactory description of the experimental data in the $-t \leq 2.5$ GeV$^2$ region, with a small coefficient of proportionality that was denoted in Ref. [20] by $\alpha$ parameter.

This ansatz assumes that the inelastic collisions at low four-momentum transfers correspond to the cases when the parts of proton suffer elastic scattering but these parts are scattered to different directions, not parallel to one another. Soon we shall see that the physical interpretation of this parameter being close to unity is actually due to $\rho_0 \ll 1$. So we will use this approximation in our analysis below.

Based on the results of the previous section obtained in the diffractive cone in the $\rho_0 \ll 1$ limit, we have the following scaling property of the opacity function:

$$\Re \exp[-\Omega(s, b)] = 1 - r(s)E(\tilde{x}),$$

(42)

$$\Im \exp[-\Omega(s, b)] = \rho_0(s) r(s)E(\tilde{x}),$$

(43)

$$\tilde{x} = \frac{b}{R(s)},$$

(44)

$$R(s) = \sqrt{B(s)},$$

(45)

where $r(s)$ is four times the ratio of the elastic to the total cross section, as given in Eq. (26), and $E(\tilde{x})$ describes the distribution of the inelastic collisions as a function of the dimensionless impact parameter $b$ normalised to $\sqrt{B(s)}$. 

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**Fig. 1** Scaling behaviour of the differential cross section $d\sigma/dt$ of elastic $pp$ collisions in the ISR energy range of $\sqrt{s} = 23.5 - 62.5$ GeV. The measured differential cross section data are taken from Ref. [51] and references therein. These data are rescaled to $H(x) = \frac{1}{\sigma_{el}} \frac{d\sigma}{dt}$ as a function of $x = -tB$. This figure indicates a clear, better than expected data-collapsing behaviour.
the characteristic length-scale of the $pp$ collisions at a given value of the center-of-mass energy $\sqrt{s}$.

This ansatz allows for a general shape of the impact parameter $b$-dependent scattering amplitude, under the assumption that the $b$-dependence may occur only through the two-dimensional scaling variable $\tilde{x}$, as described by the scaling function $E(\tilde{x})$,

$$t_{el}(s, b) = (i + \rho_0(s)) r(s) E(\tilde{x}).$$

(46)

Here we assume that $E(\tilde{x})$ is a real function that depends on the modulus of the dimensionless impact parameter $\tilde{x} = b/R(s)$. For normalization, we choose that the Fourier-transform $\mathcal{F}(\tilde{x}) = 1$, which also corresponds to the condition

$$\int d^2\tilde{x} E(\tilde{x}) = 1,$$

(47)

keeping in mind that we have two-dimensional Fourier-transform which at zero is equal to the integral over the two different directions in the impact-parameter space.

Let us investigate first the consequences of this scaling ansatz for the shadow profile function $P(s, b)$. The algebra is really very similar to that of the exponential cone approximation that was implemented above. We obtain the following result:

$$P(s, b) = \frac{1}{1 + \rho_0^2(s)} - (1 + \rho_0^2(s)) \left[ r(s) \frac{b}{R(s)} - \frac{1}{1 + \rho_0^2(s)} \right]^2 \equiv \rho(b).$$

(48)

Evaluating the above relation at $b = 0$ and using the normalization condition $E(0) = 1$, we obtain again that the shadow profile at zero impact parameter value has a maximum that is slightly less than unity: $P(0, s) \leq 1/(1 + \rho_0^2)$. It is interesting to note that the maximum in the profile function is reached at the same threshold (28) as in the case of the exponential cone approximation, corresponding to

$$r(s)_{\text{threshold}} = \frac{1}{1 + \rho_0^2(s)},$$

(49)

$$\frac{\sigma_{el}}{\sigma_{el, \text{threshold}}} = \frac{1}{4(1 + \rho_0^2(s))}.$$ 

(50)

Thus a threshold-crossing behaviour seems to happen if the elastic-to-total cross-section ratio exceeds 0.25. Remarkably, in the domain of validity of our derivation, this threshold crossing point is independent of the detailed shape of the $H(x)$ scaling function for a broad class of models. However, it is also clear from Eq. (48) that the shape of $E(\tilde{x})$ function plays an important role in determining the hollowness effect, so a detailed precision shape analysis is necessary to obtain the significance of this effect.

Starting from the definition, Eq. (2), the scattering amplitude in the $b$-space (46) yields the following form of the differential cross section in the momentum space:

$$\frac{d\sigma}{dt} = \frac{1 + \rho_0^2(s)}{4\pi} r^2(s) R^4(s) E(R(s)\Delta)^2.$$ 

(51)

Utilizing Eq. (45), we find that this form of the differential cross section is dependent on the four-momentum transfer squared, $t$, indeed only through the variable $x = B(s) t = R^2 s \Delta^2$, so it is a promising candidate to be a scaling variable.

Now, if we consider the function (51) at the optical point, $t = 0$, we find

$$A(s) = \left. \frac{d\sigma}{dt} \right|_{t=0} = \frac{1 + \rho_0^2(s)}{4\pi} r^2(s) R^4(s) |E(0)|^2.$$ 

(52)

If the impact parameter dependent elastic amplitude has an $x$-dependent internal scale and $s$-dependent strength, we thus obtain the following generalized scaling relation for arbitrary elastic scattering amplitudes that satisfy Eq. (46):

$$1 \frac{d\sigma}{A(s) \frac{dt}{dx}} \equiv H(x) = \frac{|E(\sqrt{x})|^2}{[E(x=0)]^2}.$$ 

(53)

is satisfied. This scaling function clarifies that, in general, the normalization of $H(x)$ scaling function on the left hand side of Eq. (53) should be made by the value of the differential cross section at the optical ($t = 0$) point as given by Eq. (52). This value for differential cross sections with nearly exponential diffractive cone is indeed approximately equal to $A(s) = B(s) \sigma_{el}(s)$. In this case, the normalization condition $H(0) = 1$ is maintained, while the integral of $H(x)$ becomes unity only for differential cross sections dominated by the exponential cone (i.e. when the integral contribution from the non-exponential tails is several orders of magnitude smaller as compared to the integral of the cone region).

For the total cross section, we find from Eq. (5)

$$\sigma_{tot}(s) = 2 r(s) R^2(s) \tilde{E}(0) = \sqrt{\frac{16\pi A(s)}{1 + \rho_0^4(s)}}.$$ 

(54)

Note that here we have indicated the normalization just for clarity, but one should keep in mind that in our normalization, $\tilde{E}(0) = 1$, and correspondingly, $H(x = 0) = 1$ by definition.

As clarified by Eq. (53), the scaling function $H(x)$ coincides with the modulus squared of the normalized Fourier-transform of the scaling function $E(\tilde{x})$, if the elastic amplitude depends on the impact parameter $b$ only through its scale invariant combination $x = \frac{b}{R(s)}$. This way, the $H(x)$ scaling is directly connected to the impact parameter dependence of the elastic amplitude and transforms out the trivial $s$-dependencies coming from $\sigma_{tot}(s), \sigma_{el}(s), B(s)$, and $\rho_0(s)$ functions.

The above derivation also indicates that it is a promising possibility to evaluate the $H(x)$ scaling function directly
from the experimental data. It has a clear normalization condition, \( H(0) = 1 \). Furthermore, in the diffractive cone, for nearly exponential cone distributions, \( H(x) \approx \exp(-x) \).

We have shown above in this section that even if one neglects the possible \( t \) dependence of \( p(s, t) \), the arbitrary \( H(x) \) profile functions can be introduced if the elastic amplitude is a product of \( s \)-dependent functions, and its impact parameter dependence originates only through an \( s \)-dependent scaling variable which can be conveniently defined as \( \frac{\Delta}{\rho_0} \).

Thus, the violations of the \( H(x) \) scaling may happen if not only the slope parameter \( B(s) \), the real-to-imaginary ratio \( \rho_0(s) \) and the integrated elastic and total cross sections \( \sigma_{el}(s) \) and \( \sigma_{tot}(s) \) depend on \( s \), but also the \( b \)-dependence of the elastic scattering amplitude starts to change noticeably. Namely, the \( H(x) \) scaling breaks down if and only if the scaling relation \( I_{el}(b, s) = C(s)E(b/R(s)) \) gets violated.

Finally, let us note that the exponential shape of \( H(x) \approx \exp(-x) \) can be derived as a consequence of the analyticity of \( T(s, \Delta) \) at \( \Delta = 0 \) corresponding to the \( t = 0 \) optical point. However, our recent analysis of the differential elastic cross sections in the LHC energy range \([9, 12]\) suggests that this approximation breaks down since the TOTEM experiment observed a significant non-exponential behaviour already in the diffractive cone. In this case, at low values of \( |t| \), nearly Lévy stable source distributions can be introduced, that lead to an approximate \( H(x) \approx \exp(-x^\alpha) \) behaviour, where \( \alpha = \alpha_{\text{asy}}/2 \leq 1 \). For example, as we have shown in Refs. \([9, 12]\), at low \( |t| \), a stretched exponential form with \( \alpha \simeq 0.9 \) describes the elastic scattering data from ISR to LHC energies reasonably well in a very broad energy range from 23.5 GeV to 13 TeV.

5 Results in the TeV energy range

Keeping in mind that the \( H(x) \) scaling holds within experimental errors at the ISR energies, where the center-of-mass energies vary from 23.5 to 62.5 GeV, that is less then by a factor of three, let us also investigate the same scaling function at the LHC energies, where the TOTEM measurements span, on a logarithmic scale, a similar energy range, from 2.76 TeV to 13 TeV, i.e. slightly more than by a factor of four. The TOTEM data at 13, 7 and 2.76 TeV are collected from Refs. \([1, 15]\), and Ref. \([4]\), respectively, and plotted in Fig. 2. Note that the possible scaling violating terms are small in the \( \sqrt{s} = 2.76 - 7 \) TeV region: they are within the statistical errors, when increasing \( \sqrt{s} \) from 2.76 to 7 TeV, i.e. by about a factor of 2.5, by 2.5 starts to be significantly violated at higher energies. Let us look into this effect in more detail.

This plot indicates that the \( H(x) \) scaling is approximately valid in the diffraction cone also in the LHC energy range, however, the range of validity of this scaling is more limited. Instead of being approximately valid in the whole measurable \( x \) region, at the LHC this scaling remains valid only through about 3-4 orders of magnitude drop in the differential cross-section. The so called “swing” effect becomes clear at \( \sqrt{s} = 13 \) TeV: the scaling function starts to decrease faster than exponential before the diffractive minimum, and also the diffractive minimum moves to lower values in \( x \) as compared to its position at lower LHC energies. This swing effect, apparent in Fig. 2, can be interpreted in terms of changes in the shadow profile of protons at the LHC energies as the energy range increases from 2.76 through 7 to 13 TeV. Indeed, such small \( s \)-dependent scaling violations in the \( H(x) \) scaling function show the same qualitative picture as what has been observed by the direct reconstruction of the \( P(s, b) \) shadow profiles in the TeV energy range in several earlier papers, see for example Refs. \([24, 25, 54]\) or our Refs. \([9, 12, 20]\).

Inspecting directly Fig. 2, we find, that the \( H(x) \) scaling functions agree within statistical errors if the colliding energy is increased from \( \sqrt{s} = 2.76 \) TeV to 7 TeV, and change significantly if the colliding energy increases further to \( \sqrt{s} = 13 \) TeV. This implies that the possible scaling violating terms are small in the \( \sqrt{s} = 2.76 - 7 \) TeV region as they are within the statistical errors, when increasing \( \sqrt{s} \) from 2.76 to 7 TeV, by about a factor of 2.5. However, the \( H(x) \) scaling is violated by \( s \)-dependent terms when increasing \( \sqrt{s} \) from 7 to 13 TeV, and such a scaling violation is larger than the quadratically added statistical and \( t \)-dependent systematic errors.

This behaviour may happen due to approaching a new domain, where the shadow profile function of \( pp \) scattering changes from a nearly Gaussian form to a saturated shape, that in turn may develop hollowness at 13 TeV and higher energies. The experimental indications of such a threshold-crossing behaviour were summarized recently in Ref. \([31]\), and are also described above: a new domain may be indicated by a sudden change of \( B(s) \) in between 2.76 and 7 TeV and, similarly, the crossing of the critical \( \sigma_{el}(s)/\sigma_{tot}(s) = 1/4 \) line in multi-TeV range of energies, somewhere between 2.76 and 7 TeV. From the theoretical side, we have previously noted such as drastic change in the size of the proton substructure between the ISR and LHC energy domains from a dressed quark-like to a dressed di-quark type of a substructure \([9, 12]\) which may be, in principle, connected to such a dramatic change in the scaling behaviour of the elastic cross section. However, in this work we are focused on the scaling properties of the experimental data, and we are not intended to draw any model-dependent conclusions here.

Instead, in Fig. 3 we directly compare the scaling properties of the differential cross sections in the form of the \( H(x) \) scaling function, using the same data sets at the ISR energies, as in Figs. 1 and 2. This range of data now spans nearly a factor of about 500, about a three orders of mag-
the Odderon search, and a precise quantification of the differential cross sections in the TeV energy range is a promising method for studying these data sets, we will search for a significant difference between the $H(x)$ scaling function of elastic $pp$ collisions at $\sqrt{s} = 2.76$ and $7$ TeV LHC energies on Fig. 5. These plots are similar to the panels of Fig. 4. The $H(x)$ scaling functions are remarkably similar, in fact, they are the same within the statistical errors of these measurements. Due to their great similarity, it is important to quantify precisely how statistically significant their difference is.

We stress in particular that the possible scaling violations are small, apparently within the statistical errors, when $pp$ results are compared at LHC energies and $\sqrt{s}$ is increased from $2.76$ to $7$ TeV, by about a factor of $2.5$. This makes it very interesting to compare the differential cross-sections of $pp$ and $p\bar{p}$ elastic scattering at the nearest measured energies in the TeV range, where crossing-odd components are associated with Odderon effects given that all Reggeon contributions are expected to be negligibly small in the TeV energy range. Actually, the largest $\sqrt{s}$ of $p\bar{p}$ elastic scattering data is $1.96$ TeV, a measurement by the D0 collaboration [8] while at the LHC the public data set on the elastic $pp$ scattering is available at $\sqrt{s} = 2.76$ TeV [4], corresponding a change in $\sqrt{s}$ by a factor of $2.76/1.96 \approx 1.4$. This is a rather small multiplicative factor on the logarithmic scale, relevant to describe changes both in high energy $pp$ and $p\bar{p}$ collisions. Given that the $H(x)$ scaling function is nearly constant between $2.76$ TeV and $7$ TeV within the statistical errors of these data sets, we will search for a significant difference between the $H(x)$ scaling function of elastic $pp$ collisions at $\sqrt{s} = 2.76$ and $7$ TeV as well as that of the elastic $p\bar{p}$ scattering at $\sqrt{s} = 1.96$ TeV. If such a difference is observed, then there must be a crossing-odd (Odderon) component in the scattering amplitude of elastic $pp$ and $p\bar{p}$ scatterings.

Let us now consider Fig. 6. This plot compares the $H(x)$ scaling functions for $p\bar{p}$ collisions at various energies from

![Fig. 2](image)

Fig. 2 Scaling behaviour of the differential cross section $d\sigma/dt$ of elastic $pp$ collisions at LHC energies. Elastic scattering data are measured by the TOTEM Collaboration at $\sqrt{s} = 13$ TeV [1], at $\sqrt{s} = 7$ TeV [15], and at $\sqrt{s} = 2.76$ TeV [4]. Left panel shows the data points with statistical errors only, while the right panel shows the same data with statistical and $t$-dependent systematic errors added in quadrature. The left panel indicates, that the $H(x)$ scaling is within statistical errors valid between $\sqrt{s} = 2.76$ TeV and $7$ TeV, while the right panel indicates that the $H(x)$ scaling is violated at $\sqrt{s} = 13$ TeV, with scaling violations that go beyond the combined statistical and systematic errors.

Fig. 6 compares the $H(x)$ scaling functions of elastic $pp$ collisions at the ISR energy range of $23.5$ GeV to $13$ TeV. As can be seen in the corresponding Fig. 3, the scaling works approximately in the diffractive cone, however, the $H(x)$ scaling function cannot be considered as an approximately constant if such a huge change in the colliding energies is considered.

Comparing Figs. 1, 2 and 3, we find that the $s$-dependence of the $H(x)$ scaling functions is rather weak if $s$ changes within a factor of two, however, there are very significant changes if the range of energies is changing by a factor of a few hundred, from the ISR energy range of $\sqrt{s} = 23.5 - 62.5$ GeV to the LHC energy range of $2.76 - 7.0 - 13.0$ TeV.

In the left panel of Fig. 4, the $H(x)$ function of the $\sqrt{s} = 2.76$ TeV TOTEM data set of Ref. [4] is compared with that of the $p\bar{p}$ collisions measured by the D0 collaboration at $\sqrt{s} = 1.96$ TeV Tevatron energy [8]. The right panel of Fig. 4 compares the $H(x)$ scaling functions of elastic $pp$ collision at $\sqrt{s} = 7.0$ TeV LHC energy [15, 55] to that of the elastic $p\bar{p}$ collisions at the Tevatron energy, $\sqrt{s} = 1.96$ TeV. On both panels, the statistical errors and $t$-dependent systematic errors are added in quadrature. Lines are shown to guide the eye corresponding to fits with the model-independent Lévy series studied in Refs. [9, 12]. These plots suggest that the comparison of the $H(x)$ scaling functions or elastic $pp$ collisions in the TeV energy range is a promising method for the Odderon search, and a precise quantification of the difference between the $H(x)$ scaling functions for $pp$ to $p\bar{p}$ collisions data sets is important. But how big is the difference between the $H(x)$ scaling functions of elastic $pp$ collisions at similar energies?

The $H(x)$ scaling of the differential cross section $d\sigma/dt$ of elastic $pp$ collisions is compared at the nearby $\sqrt{s} = 2.76$ and $7$ TeV LHC energies on Fig. 5. These plots are similar to the panels of Fig. 4. The $H(x)$ scaling functions are remarkably similar, in fact, they are the same within the statistical errors of these measurements. Due to their great similarity, it is important to quantify precisely how statistically significant their difference is.
we have demonstrated in Figs. 1, 2 that in a limited energy at each measured energies, while for larger values of \( x \) the scaling law breaks down in an energy dependent manner. At lower energies, the exponential region extends to larger values of \( x \approx 13 \), and the tail regions are apparently changing with varying colliding energies. Due to this reason, in this paper we do not scale the differential cross section of elastic \( p\bar{p} \) collisions to different values of \( \sqrt{s} \) as this cannot be done model-independently. This property of elastic \( p\bar{p} \) collisions is in contrast to that of the elastic \( pp \) collisions, where we have demonstrated in Figs. 1, 2 that in a limited energy range between \( \sqrt{s} = 23.5 \) and 62.5 GeV, as well as at the LHC in the energy range between \( \sqrt{s} = 2.76 \) and 7 TeV, the \( H(x) \) scaling works well. Due to these experimental facts and the apparent violations of the \( H(x) \) scaling for \( p\bar{p} \) collisions in the \( x = -tB \geq 10 \) region, in this paper we do not attempt to evaluate the energy dependence of the differential cross sections for \( p\bar{p} \) collisions. However, based on the observed \( H(x) \) scaling in \( pp \) collisions, we do find a model-independent possibility to rescale the differential cross sections of elastic \( pp \) collisions in limited energy ranges.

After the above qualitative discussion of \( H(x) \) scaling for both \( pp \) and \( p\bar{p} \) elastic collisions, let us work out the details of the possibility of rescaling the measured differential cross sections to other energies in the domain where \( H(x) \) indicates a scaling behaviour within experimental errors.

The left panel of Fig. 7 indicates the result of rescaling of the differential cross sections of elastic \( pp \) scattering from the lowest \( \sqrt{s} = 23.5 \) GeV to the highest 62.5 GeV ISR energy, using Eq. (66). We have evaluated the level of agreement of the rescaled 23.5 GeV \( pp \) data with the measured 62.5 GeV \( pp \) data with the help of Eq. (59). The result indicates that the data measured at \( \sqrt{s} = 25.5 \) GeV and duly rescaled to 62.5 GeV are, within the errors of the measurements, consistent with the differential cross section of elastic \( pp \) collisions as measured at \( \sqrt{s} = 62.5 \) GeV. This demonstrates that our method can also be used to extrapolate the differential cross sections at other energies by rescaling, provided that the \( H(x) \) scaling is not violated in that energy range and that the nuclear slope and the elastic cross sections are known at a new energy as well as at the energy from where such a rescaling starts.

A similar method is applied at the LHC energies in the middle panel of Fig. 7. This plot also indicates a clear agreement between the 2.76 TeV data and the rescaled 7 TeV data, which corresponds to a \( \chi^2/NDF = 39.3/63 \) and a CL of 99.2 % and a deviation on the 0.01 \( \sigma \) level only. This suggests that indeed the rescaling of the differential cross section of elastic scattering can be utilized not only in the few tens of GeV range but also in the few TeV energy range. Most importantly, this plot indicates that there is a scaling regime in elastic \( pp \) collisions, that includes the energies of \( \sqrt{s} = 2.76 \) and 7 TeV at LHC, where the \( H(x) \) scaling is within errors, not violated. This is in a qualitative contrast to the elastic \( p\bar{p} \) collisions at TeV energies, where the validity of the \( H(x) \) scaling is limited only to the diffractive cone region with \( x \leq 10 \), while at larger values of \( x \), the \( H(x) \) scaling is violated.

The right panel of Fig. 7 indicates a surprising agreement: after rescaling of the differential cross section of elastic \( pp \) collisions from 2.76 TeV to 1.96 TeV, we find no significant difference between the rescaled 2.76 TeV \( pp \) data with the \( p\bar{p} \) data at the same energy, \( \sqrt{s} = 1.96 \) TeV. The agreement between the extrapolated \( pp \) and the measured \( p\bar{p} \) differential cross sections correspond to an agreement at

\( \sqrt{s} = 546 \) GeV to 1.96 TeV. Within experimental errors, an exponential cone is seen that extends to \( x = -tB \approx 10 \) at each measured energies, while for larger values of \( x \) the scaling law breaks down in an energy dependent manner.

The left panel of Fig. 7 indicates the result of rescaling of the differential cross sections of elastic \( pp \) scattering from the lowest \( \sqrt{s} = 23.5 \) GeV to the highest 62.5 GeV ISR energy, using Eq. (66). We have evaluated the level of agreement of the rescaled 23.5 GeV \( pp \) data with the measured 62.5 GeV \( pp \) data with the help of Eq. (59). The result indicates that the data measured at \( \sqrt{s} = 25.5 \) GeV and duly rescaled to 62.5 GeV are, within the errors of the measurements, consistent with the differential cross section of elastic \( pp \) collisions at energy as measured at \( \sqrt{s} = 62.5 \) GeV. This demonstrates that our method can also be used to extrapolate the differential cross sections at other energies by rescaling, provided that the \( H(x) \) scaling is not violated in that energy range and that the nuclear slope and the elastic cross sections are known at a new energy as well as at the energy from where such a rescaling starts.

A similar method is applied at the LHC energies in the middle panel of Fig. 7. This plot also indicates a clear agreement between the 2.76 TeV data and the rescaled 7 TeV data, which corresponds to a \( \chi^2/NDF = 39.3/63 \) and a CL of 99.2 % and a deviation on the 0.01 \( \sigma \) level only. This suggests that indeed the rescaling of the differential cross section of elastic scattering can be utilized not only in the few tens of GeV range but also in the few TeV energy range. Most importantly, this plot indicates that there is a scaling regime in elastic \( pp \) collisions, that includes the energies of \( \sqrt{s} = 2.76 \) and 7 TeV at LHC, where the \( H(x) \) scaling is within errors, not violated. This is in a qualitative contrast to the elastic \( p\bar{p} \) collisions at TeV energies, where the validity of the \( H(x) \) scaling is limited only to the diffractive cone region with \( x \leq 10 \), while at larger values of \( x \), the \( H(x) \) scaling is violated.

The right panel of Fig. 7 indicates a surprising agreement: after rescaling of the differential cross section of elastic \( pp \) collisions from 2.76 TeV to 1.96 TeV, we find no significant difference between the rescaled 2.76 TeV \( pp \) data with the \( p\bar{p} \) data at the same energy, \( \sqrt{s} = 1.96 \) TeV. The agreement between the extrapolated \( pp \) and the measured \( p\bar{p} \) differential cross sections correspond to an agreement at
dependent terms, that arise from the energy-dependent that the \( H \) important to quantify how significant is this difference, given candidate for an Odderon search. Due to this reason, it is im-
portant to contrast with the evolution of the elastic collisions at 2.76 and 7.0 TeV colliding ener-
gies, we see no qualitative difference. By extrapolation, we expect that the \( H(x) \) scaling function may be approximately energy independent in a bit broader interval, that extends down to 1.96 TeV. Such a lack of energy evolution of the \( H(x) \) scaling function of the pp collisions is in a qualitative contrast with the evolution of the \( H(x) \) scaling functions of \( pp \) collisions at energies of \( \sqrt{s} = 0.546 - 1.96 \) TeV, where a qualitative and significant energy evolution is seen in the \( x = -tB > 10 \) kinematic range. Thus, our aim is to quantify the Odderon effect in particular in this kinematic range of \( x = -tB > 10 \) in order to evaluate the significance of this qualitative difference between elastic \( pp \) and \( pp \) collisions.

6 Quantification

In this section, we investigate the question of how to compare the two different scaling functions \( H(x) = \frac{1}{\Delta x} \frac{d\sigma}{dx} \) with \( x = -tB \) introduced above measured at two distinct energies. We would like to determine if two different measurements correspond to significantly different scaling functions \( H(x) \), or not. In what follows, we introduce and describe a model-independent, simple and robust method, that enables us to quantify the difference of datasets or \( H(x) \) measurements. The proposed method takes into account the fact that the two distinct measurements may have partially overlapping acceptance in \( x \) and their binning might be different, so the datasets may correspond to two different sets of \( x \) values.

Let us first consider two different datasets denoted as \( D_i \), with \( i = 1, 2 \). In the considered case, \( D_i = \{ x_i(j), H_i(j), e_i(j) \} \), \( j = 1, \ldots, n_i \) consists of a set of data points located on the horizontal axis at \( n_i \) different values of \( x_i \), ordered as \( x_i(1) < x_i(2) < \ldots < x_i(n_i) \). \( H_i(j) = H(x_i(j)) \) are the measured values of \( H(x) \) at \( x = x_i(j) \) points, and \( e_i(j) = e_i(x_i(j)) \) is the corresponding error found at \( x_i(j) \) point.

In general, two different measurements have data points at different values of \( x \). Let us denote as \( X_1 = \{ x_1(1), \ldots, x_1(n_1) \} \) the domain of \( D_1 \), and similarly \( X_2 = \{ x_2(1), \ldots, x_2(n_2) \} \) stands for the domain of \( D_2 \). Let us choose the dataset \( D_i \) which corresponds to \( x_1(1) < x_2(1) \). In other words, \( D_1 \) is the dataset that starts at a smaller value of the scaling variable \( x \) as compared to the second dataset \( D_2 \). If the first dataset ends before the second one starts, i.e. when \( x_1(n_1) < x_2(1) \), their acceptances would not overlap. In the latter limiting case the two datasets cannot be compared using our method. Fortunately, however, the relevant cases e.g. the D0 data on elastic \( pp \) collisions at \( \sqrt{s} = 1.96 \) TeV have an overlapping acceptance in \( x \) with the elastic \( pp \) collisions of TOTEM at \( \sqrt{s} = 2.76, 7 \) and 13 TeV. So from now on we consider the case with \( x_1(n_1) > x_2(1) \).

If the last datapoint in \( D_2 \) satisfies \( x_2(n_2) < x_1(n_1) \), then \( D_2 \) is within the acceptance of \( D_1 \). In this case, let us introduce \( f_2 = n_2 \) as the final point with the largest value of \( x_f \) from \( D_2 \). If \( D_2 \) has \( x_2(n_2) > x_1(n_1) \), then the overlapping acceptance ends at the largest (final) value of index \( f_2 \) such that \( x_2(f_2) < x_1(n_1) < x_2(f_2 + 1) \). This means that the point \( f_2 \) of \( D_2 \) is below the largest value of \( x \) in \( D_1 \), but the next point in \( D_2 \) is already above the final, largest value of \( x(n_1) \) in \( D_1 \).

The beginning of the overlapping acceptance can be found in a similar manner. Due to our choice of \( D_1 \) as being a dataset that starts at a lower value, \( x_1(1) < x_2(1) \), let us determine the initial point \( i_1 \) in \( D_1 \) that already belongs to the acceptance domain of \( D_2 \). This is imposed by the criterion that \( x_1(i_1 - 1) < x_2(1) < x_1(i_1) \).

We compare the \( D_1 \) and \( D_2 \) datasets in the region of their overlapping acceptance, defined above, either in a one-way or in a two-way projection method. The projection \( 1 \rightarrow 2 \) has the number of degrees of freedom NDF(1 → 2) equal to the number of points of \( D_2 \) in the overlapping acceptance. For any of such a point \( x(1,2) \), we used linear interpolation of the nearest points from \( D_1 \) so that \( x_1(1) < x_1(2) \leq x_1(i_1 + 1) \) to evaluate the data and the errors of \( D_1 \) at this particular value of \( x = x_i(2) \). We used a linear interpolation using as a default a (linear, exponential) scales in the \((x, H(x))\) plane, that is expected to work well in the diffraction cone, where the exponential cone is a straight line. However, for safety and due to the unknown exact structure at the dip and bump region, we have also tested the linear interpolation utilizing the (linear, linear) scales in the \((x, H(x))\) plane.

Similarly, the projection \( 2 \rightarrow 1 \) has the number of degrees of freedom NDF(2 → 1) as the number of points of dataset \( D_1 \) that fell into the overlapping common accep-
as the left panel, but now using elastic pp panels, statistical errors and compared to that of the elastic pp collisions at the Tevatron energy of $\sqrt{s} = 1.96$ TeV (blue), shown as a function of $x = -tB$. Right panel: Same as the left panel, but now using elastic pp data at $\sqrt{s} = 7$ TeV (red), as compared to elastic $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV (blue). On both panels, statistical errors and $t$-dependent systematic errors are added in quadrature. Lines are shown to guide the eye, corresponding to fits with the model-independent Lévy series from Refs. [9, 12].

![Fig. 4](image)

**Fig. 4** Left panel: Scaling function $H(x) = \frac{1}{s} \frac{d\sigma}{dxt}$ of the differential cross section of elastic pp collisions at $\sqrt{s} = 2.76$ TeV LHC (red), as compared to that of the elastic $p\bar{p}$ collisions at the Tevatron energy of $\sqrt{s} = 1.96$ TeV (blue), shown as a function of $x = -tB$. Right panel: Same as Fig. 4, but now the $H(x)$ scaling of the differential cross section $d\sigma/dx$ of elastic pp collisions is compared at the nearby $\sqrt{s} = 2.76$ and 7 TeV LHC energies. Left panel shows the data with statistical errors only, while on the right panel, statistical errors and $t$-dependent systematic errors are added in quadrature. The two scaling $H(x)$ scaling functions are, within statistical errors, apparently the same.

A linear extrapolation was used for each $x_i(1)$ points in this overlapping acceptance, so that $x_i(2) \leq x_{i+1}(2)$, using both (linear, exponential) and (linear, linear) scales in the $(x, H(x))$ planes.

For the two-way projections, for example using $1 \leftrightarrow 2$ has the number of degrees of freedom is the sum of the points of $D_1$ and $D_2$ in the overlapping acceptance, defined as $NDF(1 \leftrightarrow 2) = NDF(1 \rightarrow 2) + NDF(2 \rightarrow 1)$

Let us describe the two-way projections a bit more details, as the one-way projections can be considered as a special cases of this method.

A common domain $X_{12} = \{x_{12}(1),...,x_{12}(n_{12})\}$ in the region of the overlap of the $X_1$ and $X_2$ domains can be introduced as follows.

Take the data points in the interval $[i_1...n_{12}]$ from the $D_1$ set and the data points in the interval $[1...f_2]$ from the $D_2$ set. This selection procedure provides a total of $n_{12} = n_{1} + f_2 - i_1 + 1$ points. Let us order this new set of points and denote such a united domain as $X_{12}$. This domain corresponds to a common acceptance region which has $n_{12}$ data points on the horizontal axis denoted as $\{x_{12}(1),...,x_{12}(n_{12})\}$.

In order to compare the datasets $D_1$ and $D_2$, one needs to build two analog datasets that are both extrapolated to the
same common domain $X_{12}$ starting from $D_1$ and $D_2$ as if the data in both analog datasets were measured at the same values of $x$. So far, either $D_1$ or $D_2$ has some data value on any element of the domain $X_{12}$, but only one of them is determined.

Let us take first those points from $X_{12}$ that belong to $D_1$, and label them with $j$ index. There are $n_1 - i_1 + 1$ such points. For such points, the data and error-bars of the extrapolated data set $D_{12}$ will be taken from $D_1$: $d_{12}(x_{12}(j)) = d_1(x_1(j))$, $e_{12}(x_{12}(j)) = e_1(x_1(j))$. However, for the same points, $D_2$ has no measured value. But we need to compare the data of $D_1$ and $D_2$ at common values of $x$. So $D_2$ data and errors can be interpolated using linear or more sophisticated interpolation methods. If the binning is fine enough, linear interpolation between the neighbouring datapoints can be used.

At this point, let us consider that in the diffractive cone, when an exponential approximation to the differential cross section can be validated, the shape of the scaling function is known to be $H(x) \approx \exp(-x)$. This function is linear on a (linear, logarithmic) plot of $(x, H(x))$. In what follows, we will test both a (linear, exponential) interpolation in the $(x, H(x))$ plots (that is expected to give the best results in the diffractive cone) and a (linear, linear) interpolation that has the least assumptions and that may work better than the (linear, exponential) interpolation technique around the diffractive minimum. These two different interpolation methods also allow us to estimate the systematic error that comes from the interpolation procedure itself. If the data points are measured densely enough in the $(x, H(x))$ plot, both methods are expected to yield similar results. We present our final results using both techniques and note that indeed we find similar results with both these methods.

Suppose that for the $j$-th point of data set $D_{12}$ and for some $i$ value of $D_2$, $x_{12}(i) < x_{12}(j) < x_{2}(i+1)$. Then a linear interpolation between the $i$-th and $i+1$-th point of $D_2$ yields

---

**Fig. 6** Approximate $H(x) = \frac{1}{\sqrt{s}} \frac{d\sigma}{dt}$ scaling of the differential cross section $d\sigma/dt$ of elastic $p\bar{p}$ collisions at $\sqrt{s} = 0.546$ to 1.96 TeV. The scaling behaviour is valid in the exponential cone region, with the scaling function $H(x) = \exp(-x)$. The scaling domain starts at $x = 0$ and extends up to $x = -tB \simeq 10$. Scaling violations are evident in the $-tB \geq 10$ region, when the colliding energy increases from 546 GeV to 1.96 TeV, nearly by a factor of four.
the following formula:

\[
d_{12}(j) = d_2(i) + (d_2(i+1) - d_2(i)) \frac{x_{12}(j) - x_2(i)}{x_2(i+1) - x_2(i)}. \tag{55}
\]

Similarly, the errors can also be determined by linear interpolation as

\[
e_{12}(j) = e_2(i) + (e_2(i+1) - e_2(i)) \frac{x_{12}(j) - x_2(i)}{x_2(i+1) - x_2(i)}. \tag{56}
\]

This way, one extends \( D_2 \) to the domain \( X_{12} \), corresponding to the overlapping acceptance of two measurements. If there is a measured value in \( D_2 \), we use that value and its error bar. If there is no measurement in \( D_2 \) precisely at that given value of \( x \) that is part of the overlapping acceptance (corresponding to a value \( x \) from \( D_1 \) then we use the two neighbouring points from \( D_2 \) and use a (linear) interpolation to estimate the value at this intermediate point. This method works if the binning of both data sets is sufficiently fine so that non-linear structures are well resolved.

This way, for those \( j = 1, \ldots, n_2 - t_1 + 1 \) points from \( X_{12} \) that belonged to \( D_1 \), we have defined the data values from \( D_1 \) by identity and defined the data points from \( D_2 \) by linear interpolation from the neighbouring bins, so for these points both data sets are defined.

A similar procedure works for the remaining points in \( D_{12} \) that originate from \( D_2 \). There are \( f_2 \) number of such points. Let us index them with \( k = 1, \ldots, f_2 \). For these points, data and error-bars of the extrapolated data set \( D_{12} \) will be taken from \( D_2: d_{12}(x_{12}(k)) = d_2(x_2(k)) \), while the errors are given as \( e_{12}(x_{12}(k)) = e_2(x_2(k)) \). However, for the same points, \( D_1 \) has no measured value. As we need to compare the data of \( D_1 \) and \( D_2 \) at common values of \( x \), for these points, \( D_1 \) data and errors can be extrapolated using the linear or more sophisticated interpolation methods based on the nearest measured points. If the binning is fine enough, linear interpolation between the neighbouring data-points can be appropriately used. For broader bins, more sophisticated interpolation techniques may also be used that take into account non-linear interpolations based on more than two nearby bins, for example interpolations using Levy series expansion techniques of Ref. [9]. However, in the present manuscript such refinements are not necessary as the (linear, linear) and the (linear, exponential) interpolations in \((x, H(x))\) give similar results.

Consider now that for the \( k \)-th point of data set \( D_{12} \) and for some \( l \)-th value of \( D_2 \), \( x_1(l) < x_{12}(k) < x_1(l + 1) \). Then linear interpolation between the \( l \)-th and \( l + 1 \)-th point of \( D_2 \) yields the following formula:

\[
d_{21}(k) = d_1(l) + (d_1(l + 1) - d_1(l)) \frac{x_{12}(k) - x_1(l)}{x_1(l+1) - x_1(l)}. \tag{57}
\]

Similarly, the errors can also be determined by linear interpolation as

\[
e_{21}(k) = e_1(l) + (e_1(l + 1) - e_1(l)) \frac{x_{12}(k) - x_1(l)}{x_1(l+1) - x_1(l)}. \tag{58}
\]

This way, using the linear interpolation techniques between the neighbouring data points, we can now compare the extended \( D_1 \) and \( D_2 \) to their common kinematic range: \( D_1 \) was embedded and extrapolated to data points and errors denoted as \( d_{12}(x_{12}) \) and \( e_{12}(x_{12}) \) while \( D_2 \) was embedded and extrapolated to data points and errors denoted as \( d_{21}(x_{12}) \) and \( e_{21}(x_{12}) \), respectively. Note that the domain of both of these extended data sets is the same \( X_{12} \) domain.
The index “12” indicates that $D_1$ was extended to $X_{12}$, while index “21” indicates that $D_2$ was extended to domain $X_{21}$.

Now, we are done with the preparations to compare the two data sets, using the following $\chi^2$ definition:

$$
\chi^2 \equiv \chi^2_A = \sum_{j=1}^{n_{12}} \frac{(d_{12}(j) - d_{21}(j))^2}{\sigma_{12}(j) + \sigma_{21}(j)}.
$$

In this comparison, there are no free parameters, so the number of degrees of freedom is $NDF = n_{12} = n_1 + f_2 - i_1 + 1$, the number of data points in the unified data sample.

Based on the above Eq. (59) we get the value of $\chi^2$ and NDF, which can be used to evaluate the $p$-value, or the confidence level (CL), of the hypothesis that the two data sets represent the same $H(x)$ scaling function. If CL satisfies the criteria that $CL > 0.1\%$, the two data sets do not differ significantly. In the opposite case, if CL $< 0.1\%$ the hypothesis that the two different measurements correspond to the same a priori $H(x)$ scaling function, can be rejected.

The advantage of the above $\chi^2$ definition by Eq. (59) is that it is straightforward to implement it, however, it has a drawback that it does not specify how to deal with the correlated $t$ or $x = -tB$ dependent errors, and horizontal or $x$ errors. The $t$ measurements at $\sqrt{s} = 7$ TeV are published with their horizontal errors according to Table 5 of Ref. [15]. These errors should be combined with the published errors on the nuclear slope parameter $B$ to get a horizontal error on $x$ indicated as $\delta x$. Such a horizontal error has to be taken into account in the final calculations of the significance of the Odderon observation.

Regarding the correlations among the measured values, and the measured errors, the best method would be to use the full covariance matrix of the measured differential cross section data. However, this covariance matrix is typically unknown or unpublished, with an exception of the $\sqrt{s} = 13$ TeV elastic $pp$ measurement by TOTEM [3]. Given that this TOTEM measurement of $d\sigma/dt$ at 13 TeV indicates already the presence of small scaling violating terms in $H(x)$ according to Fig. 2, this 13 TeV dataset cannot be used directly in our Odderon analysis, that is based on the $s$-independence of the scaling function of the differential elastic $pp$ cross section $H(x) \neq H(x,s)$ in a limited range that includes $\sqrt{s} = 2.76$ and 7 TeV, but does not extend up to 13 TeV. However, we can utilize this TOTEM measurement of $d\sigma/dt$ at 13 TeV, to test the method of diagonalization of the covariance matrix that we apply in our final analysis of the Odderon significance.

Our analysis of the covariance matrix relies on a method developed by the PHENIX Collaboration and described in detail in Appendix A of Ref. [56]. This method is based on the following separation of the various types of experimental uncertainties:

![Fig. 8](image-url)
Type A) errors are point-to-point uncorrelated systematic uncertainties.

Type B) errors are point-to-point varying but correlated systematic uncertainties, for which the point-to-point correlation is 100 %, as the uncorrelated part is separated and added to type A) errors in quadrature.

Type C) systematic errors are point-independent, overall systematic uncertainties, that scale all the data points up and down by exactly the same, point-to-point independent factor.

Type D) errors are point-to-point varying statistical errors. These type D) errors are uncorrelated hence can be added to type A) errors in quadrature.

In this paper, where we apply this method to compare two different \( H(x) \) scaling functions, we also consider a fifth kind of error, type E) that corresponds to the theoretical uncertainty, which we identify with the error of the interpolation of one of the (projected) data sets to the \( x \) values that are compared at some (measured) values of \( a \) to a certain measured data point at a measured \( x \) value. This type E) error is identified with the value calculated from the linear interpolation, described above, as given for each A), B), C) and D) type of errors similarly by Eq. (58). Type D) errors are added in quadrature to type A) errors, and in what follows we index these errors with the index of the data point as well as with subscripts \( a, b, c \) and \( e \), respectively.

Using this notation, Eq. (A16) of Ref. [56] yields the following \( \chi^2 \) definition, suitable for the projection of dataset \( D_2 \) to \( D_1 \), or \( 2 \rightarrow 1 \):

\[
\chi^2(2 \rightarrow 1) = \sum_{j=1}^{NDF_1} \left( \frac{d_1(j) - d_2(j) + e_{b,1}e_b(j) + e_{c,1}d_1(j)e_c}{d_1(j)} \right)^2 + e_{b,1}^2 + e_{c,1}^2,
\]

where \( \delta x_{12}(j) \) is the type A) uncertainty of the data point \( j \) of the united data set \( D_{12} \) scaled by a multiplicative factor such that the fractional uncertainty is unchanged under multiplication by a point-to-point varying factor:

\[
\delta a_{12}(j) = e_{a,1}(j) \left( \frac{d_1(j) + e_{b,1}e_b(j) + e_{c,1}d_1(j)e_c}{d_1(j)} \right).
\]

In these sums, there are NDF\(_1 = f_1 - i_1 - 1 \) number of data points in the overlapping acceptance from dataset \( D_1 \). A similar sum describes the one-way projection \( 1 \rightarrow 2 \), but there are NDF\(_2 = f_2 \) points in the common acceptance. For the two-way projections, not only the number of degrees of freedom add up, NDF\(_{12} = \text{NDF}_1 + \text{NDF}_2 \), but also the \( \chi^2 \) values are added as \( \chi^2(1 \leftrightarrow 2) = \chi^2(1 \rightarrow 2) + \chi^2(2 \rightarrow 1) \).

Let us note at this point, that \( H(x) \) is a scaling function that is proportional to the differential cross section normalized by the integrated cross section. In this ratio, the overall, type C) point-independent normalization errors multiply both the numerator and the denominator, hence these type C) errors cancel out in \( H(x) \). Given that these type C) errors are typically rather large, for example, 14.4 % for the D0 measurement of Ref. [8], it is an important advantage in the significance computation that we use a normalized scaling function \( H(x) \). So in what follows, we set \( e_{c,1} = 0 \) and rewrite the equation for the \( \chi^2 \) definition accordingly. This effect increases the significance of a \( H(x) \)-scaling test.

The price we have to pay for this advantage is that we have to take into account the horizontal errors on \( x \) in order to not overestimate the significance of our \( \chi^2 \) test. In this step, we follow the propagation of the horizontal error to the \( \chi^2 \) as utilized by the so-called effective variance method of the CERN data analysis programme ROOT. This yields the final \( \chi^2 \) definition that we have utilized in our significance analysis for the case of symmetric errors in \( x \):

\[
\chi^2(2 \rightarrow 1) = \sum_{j=1}^{NDF_1} \left( \frac{d_1(j) - d_2(j) + e_{b,1}e_b(j) + e_{c,1}d_1(j)e_c}{d_1(j)} \right)^2 + e_{b,1}^2 + e_{c,1}^2, \tag{62}
\]

where \( \delta x_{12}(j) \) is the (symmetric) error of \( x \) in the \( j \)-th data point of the data set \( D_1 \), and \( d_1(j)e_c \) is the numerically evaluated derivative of the extrapolated value of the projected data point obtained with the help of a linear interpolation using Eq. (57). Such definition is valid when the type B) errors are known and are symmetric for the data set \( D_1 \) and the errors on \( x \) are also symmetric. When the data set \( D_1 \) corresponds to the D0 measurement of elastic \( pp \) collisions, Ref. [8], we have to take into account that D0 did not publish the separated statistical and \( |t| \)-dependent systematic errors, but decided to publish their values added in quadrature. So we use these errors as type A) errors and with this method, we underestimate the significance of the results as we neglect the correlations among the errors of the data points in the D0 dataset. The TOTEM published the \( |t| \)-dependent statistical type D) errors and the \( |t| \)-dependent systematic errors both for the 2.76 TeV and 7 TeV measurements of the differential cross sections [4, 15, 55], with the note that the \( |t| \)-dependent systematic errors are almost fully correlated. In these works, TOTEM did not separate the point-to-point varying uncorrelated part of the \( |t| \)-dependent systematic errors. We thus estimate the type A) errors by the statistical errors of these TOTEM measurements, we then slightly underestimate them, hence overestimate the \( \chi^2 \) and the difference between the compared data sets. Given that they are almost fully correlated, we estimate the type B) errors by the point-to-point varying almost fully correlated systematic errors published by the TOTEM. We have tested this scheme by evaluating the \( \chi^2 \) from a full covariance matrix fit and from the PHENIX method of diagonalizing the covariance matrix at \( \sqrt{s} = 13 \) TeV, using the Lévy expansion method of Ref. [9]. We find that the fit with the full covariance matrix results the same minimum within one standard deviation of the fit parameters, hence the same significance as the fit with the PHENIX method of Appendix A of Ref. [56].

We have thus validated the PHENIX method of Ref. [56] for the application of the analysis of differential cross sec-
tion at $\sqrt{s} = 13$ TeV, together with the effective variance method of the ROOT package. Now, we can employ our final $\chi^2$ definition of Eq. (62) to estimate the significance of the Odderon signal in comparison of the $H(x)$ scaling functions for elastic $pp$ and $p\bar{p}$ collisions. This validation is important as the full covariance matrix of the $\sqrt{s} = 2.76$ TeV and 7 TeV measurements by TOTEM is not published, but the PHENIX method appended with the ROOT method of effective variances can be used to effectively diagonalize the covariance matrix and to get similar results within the errors of the analysis.

7 Extrapolations

In this section, we discuss how to extrapolate the data points to energies where measurements are missing. As we have found, for example, in the ISR energy range of $\sqrt{s} = 23.5$ – 62.5 GeV the $H(x)$ scaling function is independent of $\sqrt{s}$ within errors. We show how to extrapolate data points to unmeasured energies, under the condition that in a given energy range, $H(x)$ is independent of the collision energy, $H(x) \neq H(x,s)$. In general, such a feature has to be established or cross-checked experimentally. This case is important, given that we have shown before, for example in Fig. 5, that $H(x)$ for $pp$ collisions stays energy-independent within the experimental errors between the LHC energies of $2.76 \text{ TeV} \leq \sqrt{s} \leq 7 \text{ TeV}$. Furthermore, we have already shown that for $p\bar{p}$ collisions, $H(x) = H(x,s)$ in the energy range of $0.546 \leq \sqrt{s} \leq 1.96 \text{ TeV}$, as indicated in Fig. 6.

Let us denote two different center-of-mass energies between which $H(x) = \text{const} (\sqrt{s})$ within the experimental errors as $\sqrt{s_1}$ and $\sqrt{s_2}$. Analogically, we denote various observables as $B_1 \equiv B(x_1), \sigma_i \equiv \sigma_{i,t_1} \equiv \sigma_{i,t}(x_1), x_i \equiv B_{i,t}$.

The energy independence of the $H(x)$ scaling function formally can be written as

$$H_1(x_1) = H_2(x_2) = H(x) \quad \text{if} \quad x_1 = x_2.$$  (63)

This simple statement has tremendous experimental implications. The equality $x_1 = x_2$ means that the scaling function is the same, if at center-of-mass energy $\sqrt{s_1}$ it is measured at $t_1$ and at energy $\sqrt{s_2}$ it is measured at $t_2$, so that

$$t_1 B_1 = t_2 B_2 \quad \text{if} \quad x_1 = x_2.$$  (64)

The equality $H_1(x_1) = H_2(x_2) = H(x)$ is expressed as

$$\left. \frac{1}{B_1 \sigma_1} \frac{d\sigma}{dt} \right|_{t_1 = x/B_1} = \left. \frac{1}{B_2 \sigma_2} \frac{d\sigma}{dt} \right|_{t_2 = x/B_2}.$$  (65)

Putting these equations together, this implies that the experimental data can be scaled to other energies in an energy range where $H(x)$ is found to be independent of $\sqrt{s}$ as follows:

$$d\sigma \bigg|_{t_1} = \frac{B_1 \sigma_1}{B_2 \sigma_2} \frac{d\sigma}{dt} \bigg|_{t_2 = x/B_2}.$$  (66)

With the help of this equation, the data points on differential cross sections can be scaled to various different colliding energies, if in a certain energy region the $H(x)$ scaling holds within the experimental errors. In other words, the differential cross section can be rescaled from $\sqrt{s_1}$ to $\sqrt{s_2}$ by rescaling the $|t|$-variable using the ratio of $B_1/B_2 = B(s_1)/B(s_2)$, and by multiplying the cross section with the ratio $B_1/\sigma_1$.  

8 Results

In this section, we present our results and close the energy gap, as much as possible without a direct measurement, between the TOTEM data on elastic $pp$ collisions at $\sqrt{s} = 2.76$ and 7.0 TeV and D0 data on elastic $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV. This section is based on the application of Eq. (66) in this energy range. After the rescaling procedure, the resulting data set at the new energy is compared with the measured data quantitatively with the help of Eq. (59).

We have used the rescaling equation, Eq. (66) first to test and to cross-check, if the rescaling of the $\sqrt{s} = 23.5$ GeV ISR data to other ISR energies works, or not. The left panel of Fig. 7 indicates that such a rescaling of the differential cross sections from the lowest ISR energy of $\sqrt{s} = 23.5$ to the highest ISR energy of $\sqrt{s} = 62.5$ GeV actually works well. The level of agreement of the rescaled 23.5 GeV $pp$ data with the measured 62.5 GeV $pp$ data has been evaluated with the help of Eq. (59). We found an agreement with a $\chi^2/NDF = 111/100$, corresponding to a CL = 21.3 % and a difference is at the level of 1.25 $\sigma$ only. This result demonstrates that our rescaling method can also be used to get the differential cross sections at other energies, provided that the nuclear slope and the elastic cross sections are known at the new energy as well as at the energy from where we start the rescaling procedure.

Subsequently, one can also rescale the TOTEM data at $\sqrt{s} = 2.76$ or 7 TeV to 1.96 TeV, given that $H(x)$ is (within errors) energy independent in the range of $2.76 – 7 \text{ TeV}$, corresponding to nearly a factor of 2.5 change in $\sqrt{s}$, while the change in $\sqrt{s}$ from 1.96 to 2.76 TeV is only a factor of 1.4. The right panel of Fig. 7 indicates that rescaling the differential elastic $pp$ cross section from $\sqrt{s} = 2.76$ to 1.96 TeV also gives valuable results. We have evaluated the confidence level of the comparison of the rescaled 2.76 TeV $pp$ data with the 1.96 TeV $p\bar{p}$ data with the help of Eq. (59). As was already mentioned above, we have found a surprising agreement with a $\chi^2/NDF = 18.1/11$, corresponding to a CL = 7.93 %, and a difference at the level of 1.75 $\sigma$ only.
Another important result is illustrated in Fig. 8. This comparison indicates a difference between the rescaled \( \sqrt{s} = 7 \) TeV elastic \( pp \) differential cross-section \([15, 55]\) to the \( \sqrt{s} = 1.96 \) TeV energy and to the corresponding \( p\bar{p} \) data measured at \( \sqrt{s} = 1.96 \) TeV \([8]\). To obtain a first estimate, this difference is quantified with the help of Eq. (59) yielding a CL of \( 5.13 \times 10^{-7} \% \). As this method adds the statistical and systematic errors in quadrature, it underestimates the actual significance of the difference between the two data sets. Although this estimate already provides a significant, greater than \( 5\sigma \) effect for the Odderon observation, corresponding to a significant, \( 5.84\sigma \) difference between the \( pp \) dataset and the \( 1.96 \) TeV \( p\bar{p} \) dataset, however, the evaluation of this significance does not yet take into account the rather large overall normalization error of 14.4 \% that has been published by the D0 collaboration. The comparison of the differential cross-sections is sensitive to such type C errors, hence this effect has to be taken into account in the final significance analysis, or the significance has to be finalized using the \( H(x) \) scaling function, where the type C errors of the absolute normalization cancel.

It can be seen in Fig. 8 that in the swing region, before the dip, the rescaled \( pp \) differential cross section differs significantly from that of \( p\bar{p} \) collisions. This \( \chi^2 \) analysis also took into account the horizontal errors of the TOTEM data discussed above. Although the estimates of statistical significances given in this Section are based on a \( \chi^2 \) test that includes the \(|t|\)-dependent statistical errors and the \(|t|\)-dependent systematic errors added in quadrature, the values of \( \chi^2 /\text{NDF} \) and significances given above can still be only considered as estimates. Indeed, although the \(|t|\)-dependent systematic errors on these \( \sqrt{s} = 7 \) TeV data are known to be almost fully correlated, the covariance matrix is not publicly available at the time of closing this manuscript from the TOTEM measurement at \( \sqrt{s} = 7 \) TeV. It is clear that the \( \chi^2 \) is expected to increase if the covariance matrix is taken into account, and this effect would increase the disagreement between the measured \( p\bar{p} \) and the extrapolated \( pp \) differential cross sections at \( \sqrt{s} = 1.96 \) TeV. Note, the above estimate of significances does not yet take into account the overall correlated, \(|t|\)-independent vertical uncertainty in the differential cross section measurements. This uncertainty shifts all the data points up or down by a common, \(|t|\)-independent factor and may also decrease the significance of the difference between the measured \( p\bar{p} \) and the extrapolated \( pp \) cross sections at \( \sqrt{s} = 1.96 \) TeV.

So this indicates that we have to consider the proposed rescaling method as conservative as possible, that allows us to take into account the statistical and \(|t|\)-dependent correlated systematic errors, as well as the \(|t|\)-independent correlated systematic errors. Such an analysis is presented in the next section, where we quantify the differences between the scaling functions \( H(x) \) of elastic \( pp \) and \( p\bar{p} \) collisions using the fact that \( H(x) \) is free of \(|t|\)-independent normalisation errors.

9 A significant Odderon signal

In this section, we summarize our discovery of at least \( 6.55\sigma \) Odderon signal that we demonstrate below when comparing the \( H(x) \) scaling functions of \( pp \) and \( p\bar{p} \) collisions. We have found a significant Odderon signal by comparing the \( H(x) \) scaling functions of the differential cross section of elastic \( pp \) collisions with \( \sqrt{s} = 7 \) TeV to that of \( p\bar{p} \) collisions with \( \sqrt{s} = 1.96 \) TeV, as indicated in Fig. 10. The comparison is made in both possible ways, by comparing the \( pp \) data to the \( p\bar{p} \) data, and vice versa. The difference between these two datasets corresponds to at least a \( \chi^2 /\text{NDF} = 84.6/17 \), giving rise to a CL of \( 5.78 \times 10^{-9} \% \) and to a \( 6.55\sigma \) significance. The overall, \(|t|\)-independent normalization error of 14.4 \% on the D0 data set cancels from this \( H(x) \), and does not propagate to our conclusions.

These results are obtained for the \( \sigma_{el} = 17.6 \pm 1.1 \) mb value of the elastic \( p\bar{p} \) cross section at \( \sqrt{s} = 1.96 \) TeV, and for the linear-exponential interpolation in \((x, H(x))\). Using this method of interpolation, the nearest points were connected with a linear-exponential line, that corresponds to a straight line on a linear-logarithmic plot in \((x, H(x))\). We have used the published values of the differential cross sections \( \frac{d^2 \sigma}{d x} \) that of the nuclear slope parameter \( B \) and the measured value of the elastic cross section \( \sigma_{el} \) for \( 7 \) TeV \( pp \) elastic collisions. For the elastic cross section of \( p\bar{p} \) collisions at \( \sqrt{s} = 1.96 \) TeV, we have numerically integrated the differential cross section with an exponential approximation at very low-\(|t|\) that provided us with \( \sigma_{el} = 17.6 \pm 1.1 \) mb.

We have systematically checked the effect of variations in our interpolation method by switching from the (linear-exponential) in \((x, H(x))\) interpolation to a linear-linear one and by changing the value of the elastic \( pp \) collisions from the numerically integrated differential cross-section value of \( \sigma_{el} = 20.2 \pm 1.4 \) mb, which is an unusually large value, but equals within the quoted 14.4 \% systematic error to the \( \sigma_{el} = 17.6 \pm 1.1 \) mb value, that corresponds to the trend published by the Particle Data Group, see the Fig. 51.6, bottom panel, yellow line of Ref. [57]. The input values of the nuclear slope parameter \( B \) and the elastic cross-section \( \sigma_{el} \) are summarized in Table 1.

As part of our systematic studies, we have also changed the direction of the projection. The results are summarized in Table 2. They indicate that the final version of Fig. 8, shown as the top left panel of Fig. 8 and evaluated with the help of our final \( \chi^2 \) definition of Eq. (62) corresponds to the most conservative case of Odderon observation based on the \( \sqrt{s} = 7 \) TeV TOTEM and the \( \sqrt{s} = 1.96 \) TeV D0 data sets. This panel indicates that the Odderon signal is observed in
this comparison with at least a 6.55σ significance, corresponding to a statistically significant Odderon observation.

The detailed figures, that show the $\chi^2(\epsilon_b)$ functions for each of these cases are summarized in the left and right panels of Fig. 9 for the comparison of the 7 TeV TOTEM data set with the 1.96 TeV D0 data set. Each plot indicates a clear, nearly quadratic minimum. The values of $\chi^2$ at the minima are summarized in Table 2, together with other characteristics of significance, like the confidence level and the significance in terms of standard variations. Similarly, the $\chi^2(\epsilon_b)$ functions for the comparison of the 2.76 TeV TOTEM data set with the 1.96 TeV D0 data set are summarized in Fig. 11. The values of $\chi^2$ at the minima are given in Table 3, together with other relevant characteristics.

As summarized in Fig. 10, a significant Odderon signal is found in the comparison of the $H(x)$ scaling functions of the differential elastic $pp$ (at $\sqrt{s} = 7.0$ TeV) vs $p\bar{p}$ ($\sqrt{s} = 1.96$ TeV) cross sections. The horizontal error bars are indicated by a properly scaled horizontal line or $|$ at the data point. The statistical (type A, point-to-point fluctuating) errors are indicated by the size of the vertical error bars ($|$), while shaded boxes indicate the size of the (asymmetric) type B (point-to-point varying, correlated) systematic errors. The overall normalization errors ($|t|$-independent, type C errors) cancel from the $H(x)$ scaling functions since they multiply both the numerator and the denominator of $H(x)$ in the same way. The correlation coefficient of the $|t|$-dependent systematic errors, $\epsilon_b$, is optimized to minimize the $\chi^2$ based on Eq. (62), and the values indicated in Fig. 10 correspond to the minimum of the $\chi^2(\epsilon_b)$. These $\chi^2$ values, as well as the numbers of degrees of freedom (NDFs) and the corresponding confidence levels (CLs) are indicated on both panels of Fig. 10, for both projections. The $\chi^2(\epsilon_b)$ functions are summarized in Fig. 9. The 1.96 TeV $\to$ 7 TeV projection has a statistical significance of 6.55$\sigma$ of an Odderon signal, corresponding to a $\chi^2$/NDF $= 84.6/17$ and CL = $5.78 \times 10^{-9}$ %. Thus the probability of Odderon observation in this analysis is $P = 1 - CL = 0.999999999422$.

Fig. 10 summarizes the results of our systematic studies in four different panels described as follows. The top-left panel of this figure uses a linear-exponential interpolation in the $(x, H(x))$ plane and uses the value of 17.6 ± 1.1 mb for the elastic $p\bar{p}$ cross section at $\sqrt{s} = 1.96$ TeV. This case gives the lowest (6.55$\sigma$) significance for the Odderon observation from among the possible cases that we have considered. The top-right panel is similar but for a linear-linear interpolation in the $(x, H(x))$. The bottom-left panel is similar to the top-left panel, but now using 20.2 ± 1.4 mb for the elastic $p\bar{p}$ cross section at $\sqrt{s} = 1.96$ TeV and also using a linear-exponential interpolation in $(x, H(x))$. The bottom-right panel is similar to the bottom-left panel, but using a linear-linear interpolation method.

The results of the scaling studies for a comparison of elastic $pp$ collisions at $\sqrt{s} = 2.76$ TeV, measured by the TOTEM experiment at the LHC [4] to that of $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV, measured by D0 at the Tevatron [8] are summarized in Fig. 12. The top-left panel uses $\sigma_d = 17.6 \pm 1.7$ mb and a linear-exponential interpolation method in $(x, H(x))$. The top-right panel is the same as the top-left panel, but for a linear-linear interpolation in $(x, H(x))$. The bottom-left panel is nearly the same as the top-right panel, but for $\sigma_d = 20.2 \pm 1.4$ mb. The bottom-right panel is the same as the bottom-left panel, but for a linear-linear interpolation in $(x, H(x))$. Neither of these comparisons shows a significant difference between the $H(x)$ scaling function of elastic $pp$ collisions at $\sqrt{s} = 2.76$ TeV as compared to that of $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV. It seems that the main reason for such a lack of significance is the acceptance limitation of the TOTEM dataset at $\sqrt{s} = 2.76$ TeV, which extends up to $x = -tB \approx 13$, in contrast to the acceptance of the 7 TeV TOTEM measurement that extends up to $x = -tB \approx 20$. We have cross-checked this by limiting the 7 TeV data set also to the same acceptance region of $4.4 < -Bt < 12.7$ as that of the 2.76 TeV data set. This artificial acceptance limitation has resulted in a profound loss of significance, down a to $\chi^2$/NDF $= 25.7/11$, that corresponds to a CL = 0.71% and to a deviation at the 2.69 σ level only. This result indicates that if we limit the acceptance of the 7 TeV TOTEM measurement to the acceptance of the 2.76 TeV TOTEM measurement, the significance of the Odderon observation decreases well below the 5σ discovery threshold.

10 A summary of cross-checks

In this section, we summarize some of the most important cross-checks that we performed using our methods and results.

We have cross-checked what happens if one rescales the differential cross section of elastic $pp$ scattering form the lowest ISR energy of $\sqrt{s} = 23.5$ GeV to the top ISR energy of $\sqrt{s} = 62.5$ GeV. As can be expected based on the approximate equality of all the $H(x)$ scaling functions at the ISR energies, as indicated on the left panel of Fig. 7, the rescaled

| Energy (GeV) | $\sigma_d$ (mb) | $B$ (GeV$^{-2}$) | Reference |
|-------------|----------------|----------------|-----------|
| 1960 ($p\bar{p}$) | 17.6 ± 1.1 | 20.2 ± 1.4 | 16.86 ± 0.2 | Fig. 51.6 of Ref. [57] from low $-t$ fit to data [8] |
| 2760 ($p\bar{p}$) | 21.8 ± 1.4 | 17.1 ± 0.3 | [34] |
| 7000 ($p\bar{p}$) | 25.4±1.02 | 19.89 ± 0.272 | [35] |

Table 1 Summary of cross-sections $\sigma_d$, the nuclear slope parameters $B$, and their sources or references.
Fig. 9  Dependence of $\chi^2$ on the coefficient of the correlated but point-to-point varying systematic errors, $\varepsilon_b$, for the comparison of the $H(x)$ scaling functions of elastic $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV with that of $pp$ collisions at $\sqrt{s} = 7.0$ TeV. Each of the four cases are shown together corresponding to the direction of the projection. **Left panel** indicates the results of the $1.96 \rightarrow 7.0$ TeV projection. **Right panel** indicates the results of the $7.0 \rightarrow 1.96$ TeV projection. Both cases indicate four $\chi^2(\varepsilon_b)$ curves corresponding to the choice of linear-linear or linear-exponential interpolations in $(x,H(x))$, as well as to the choice of the elastic $p\bar{p}$ cross section at $\sqrt{s} = 1.96$ TeV (20.2 $\pm$ 1.4 mb vs 17.6 $\pm$ 1.1 mb). A parabolic structure is seen in each case with a clear minimum, and the fit quality corresponding to these minima in $\varepsilon_b$ is summarized in Table 2.

| $\sigma_{el}$ (mb) | interpolation | direction of comparison | $\chi^2$ | NDF | CL [%] | Odderon significance in units of $\sigma$ |
|---------------------|---------------|-------------------------|----------|-----|--------|-----------------------------------------|
| 17.6 $\pm$ 1.1      | lin-exp       | 7 $\rightarrow$ 1.96 TeV | 84.6     | 17  | 5.8E-09 | 6.55                                    |
|                     |               | 1.96 $\rightarrow$ 7 TeV  | 289      | 65  | 5.3E-28 | 11.38                                   |
| 20.2 $\pm$ 1.4      | lin-lin       | 7 $\rightarrow$ 1.96 TeV  | 91.1     | 17  | 3.8E-10 | 6.94                                    |
|                     |               | 1.96 $\rightarrow$ 7 TeV  | 314      | 65  | 2.6E-32 | 12.22                                   |
|                     | lin-exp       | 7 $\rightarrow$ 1.96 TeV  | 90       | 17  | 6.1E-10 | 6.88                                    |
|                     |               | 1.96 $\rightarrow$ 7 TeV  | 309      | 65  | 1.9E-11 | 12.05                                   |
|                     | lin-lin       | 7 $\rightarrow$ 1.96 TeV  | 96.2     | 17  | 4.5E-11 | 7.24                                    |
|                     |               | 1.96 $\rightarrow$ 7 TeV  | 335      | 65  | 5.4E-36 | 12.89                                   |

Table 2  Summary table of the significant Odderon signal in the one-way comparison of the $H(x)$ scaling functions of $pp$ collisions at $\sqrt{s} = 7$ TeV measured by the TOTEM experiment at the LHC, and $p\bar{p}$ elastic collisions at $\sqrt{s} = 1.96$ TeV measured by the D0 experiment at Tevatron. This table indicates that the Odderon signal is observed in this comparison with at least a 6.55 $\sigma$ significance corresponding to an Odderon discovery.

For the diagonalization of the covariance matrix on fits to the $\sqrt{s} = 13$ TeV TOTEM data of ref. [3]. This PHENIX method resulted, within one standard deviation, the same minimum, hence the same significances, as the use of the full covariance matrix at $\sqrt{s} = 13$ TeV elastic $pp$ collisions. At the lower LHC energies of $\sqrt{s} = 2.76$ and 7.0 TeV, due to the lack of publicly available information on the covariance matrix, only the PHENIX method of Ref. [56] was available for our final significance analysis.
same time, we have also seen that the comparison of the 2.76 TeV $pp$ dataset to the 1.96 TeV $p\bar{p}$ dataset does not indicate a significant Odderon effect. We have found that the Odderon signal vanishes from the comparison of the 7 TeV $pp$ and the 1.96 TeV $p\bar{p}$ datasets too, if we limit the acceptance of the 7 TeV dataset to the acceptance in $x = -tB$ as that of the 2.76 TeV $pp$ dataset: the significance of the Odderon observation decreased from a 6.55$\sigma$ discovery effect to a 2.69$\sigma$ level agreement. We may note that a similar observation was made already in Ref. [27] that pointed out a strong $|t|$ dependence of the Odderon contribution.

11 Discussion

We have explored the scaling properties of the elastic differential cross sections at various energies, from the ISR up to the highest LHC energy. We have recalled that the earlier proposals for the $F(y)$ and $G(z)$ scaling functions were useful to explore if elastic scattering of protons in the LHC energy range is already close to the black-disc limit or not. After investigating several possible new dimensionless scaling variables and scaling function candidates, we have realized that in order to look for scaling violations in the low $|t|$ kinematic range, corresponding to the diffractive cone it is advisable to scale all the diffractive cones to the same dimensionless scaling function, $H(x) \approx \exp(-x)$. This function can be obtained as the differential cross section normal-
Fig. 11  Dependence of $\chi^2$ on the coefficient of the correlated but point-to-point varying systematic errors, $\varepsilon_b$, for the comparison of the $H(x)$ scaling functions of elastic $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV, measured by the D0 experiment at Tevatron [8], with that of elastic $pp$ collisions at $\sqrt{s} = 2.76$ TeV, measured by the TOTEM experiment at the LHC [4]. All the eight cases are shown together corresponding to the choice of linear-linear or linear-exponential interpolations in $H(x)$, to a different choice of the elastic cross section of $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV ($20.2 \pm 1.4$ mb vs $17.6 \pm 1.1$ mb), and to the direction of the projection ($1.96 \rightarrow 2.76$ TeV, or $2.76$ TeV $\rightarrow 1.96$ TeV). A clear parabolic structure is seen in each case and the fit quality of the results that belong to these minima in $\varepsilon_b$ is summarized in Table 3.

The effects due to the energy-induced difference between TOTEM and D0 data sets can be estimated by the change of the $H(x)$ scaling function for $pp$ scattering between 2.76 TeV and 7 TeV, which are within the systematic errors of the TOTEM data sets. However, the $H(x)$ scaling function of elastic $pp$ scattering at $\sqrt{s} = 7.0$ TeV is significantly different from the corresponding result of elastic $p\bar{p}$ scattering at $\sqrt{s} = 1.96$ TeV. These qualitative and quantitative differences, first, show up below the diffractive minimum of the $pp$ elastic scattering, namely, the $H(x)$ function for $pp$ collisions indicates a strong "swing" or faster than exponential decrease effect, before developing a characteristic diffractive minimum. In contrast, the D0 data on $p\bar{p}$ elastic scattering features a structureless exponential decrease that in turn changes to a plateau or a shoulder-like structure at higher values of the scaling variable $x$. No clear indication of a diffractive maximum is seen in the $p\bar{p}$ elastic scattering data [8], while the TOTEM data sets at each LHC energies of 2.76, 7 and 13 TeV clearly indicate a diffractive minimum followed by an increasing part of the differential cross section before the edge of the TOTEM acceptance is reached, respectively [3, 4, 55].
These qualitative and quantitative differences between the $H(x)$ scaling functions of elastic $pp$ and $p\bar{p}$ scatterings provide a clear-cut and statistically significant evidence for a crossing-odd component in the scattering amplitude in the TeV energy range. This corresponds to the observation of the Odderon exchange in the $t$-channel of the elastic scattering. The Odderon in this context is a trajectory that at $J = 1$ contains a $J^{PC} = 1^{-}$ vector glueball as well as other glueball states with higher angular momentum. Hence, one of the implication of our result is that not only one but several glueball states should exist in Nature.

Due to the presence of the faster-than exponentially decreasing (swing) region in elastic $pp$ scatterings, the high-statistic $pp$ elastic scattering data at $\sqrt{s} = 1.96$ TeV may be taken as an additional measurement clearly closing the energy gap. However, the aperture limitation of the LHC accelerator is already resulting in a loss of significance of the comparison of the $H(x)$ scaling functions at $2.76$ TeV with that of the D0 data at $1.96$ TeV. Due to this reason, we propose an additional measurement of the dip and bump region of elastic $pp$ collisions in the domain where the $H(x)$ scaling was shown to work, in between $2.76$ TeV and $7$ TeV, if that can be harmonized with the LHC running schedule and scenarios.

The current TOTEM acceptance ends at $-tB \approx 12$ at $\sqrt{s} = 2.76$ TeV. Although more detailed acceptance studies are necessary, it seems that reaching $x = -tB \approx 8 - 9$ seems to be a sufficient acceptance, as the swing effect in this range is already making a substantial and qualitative difference between the $H(x) = (1/A)dA/dx$ scaling functions of elastic $pp$ and $p\bar{p}$ collisions. New elastic $pp$ scattering data around $\sqrt{s} \approx 4 - 5$ TeV could be particularly useful to
Table 3 Summary table of the search for an Odderon signal in the one-way comparison of the $H(x)$ scaling functions of $pp$ collisions at $\sqrt{s} = 2.76$ TeV measured by the TOTEM experiment at the LHC, and $p\bar{p}$ elastic collisions at $\sqrt{s} = 1.96$ TeV measured by the D0 experiment at Tevatron.

| $\sigma_{el}$ (mb) | interpolation | direction of comparison | $\chi^2$ | NDF | CL [%] | Odderon significance in units of $\sigma$ |
|-------------------|---------------|-------------------------|---------|-----|--------|----------------------------------------|
| $17.6 \pm 1.1$    | lin-exp       | 2.76 $\rightarrow$ 1.96 TeV | 7.64    | 11  | 74.5   | 0.33                                   |
|                   |               | 1.96 $\rightarrow$ 2.76 TeV | 20.30   | 27  | 81.8   | 0.23                                   |
|                   | lin-lin       | 2.76 $\rightarrow$ 1.96 TeV | 7.90    | 11  | 72.2   | 0.36                                   |
|                   |               | 1.96 $\rightarrow$ 2.76 TeV | 24.50   | 27  | 60.2   | 0.52                                   |
| $20.2 \pm 1.4$    | lin-exp       | 2.76 $\rightarrow$ 1.96 TeV | 3.85    | 11  | 97.4   | 0.03                                   |
|                   |               | 1.96 $\rightarrow$ 2.76 TeV | 15.40   | 27  | 96.3   | 0.05                                   |
|                   | lin-lin       | 2.76 $\rightarrow$ 1.96 TeV | 4.32    | 11  | 96.0   | 0.05                                   |
|                   |               | 1.96 $\rightarrow$ 2.76 TeV | 18.20   | 27  | 89.7   | 0.13                                   |

Table 4 Summary table of the search for an Odderon signal in the two-way comparison, for the significance of an Odderon signal in the comparison of the $H(x)$ scaling functions of $pp$ collisions at $\sqrt{s} = 7$ TeV, measured by the TOTEM experiment at the LHC, and $p\bar{p}$ elastic collisions at $\sqrt{s} = 1.96$ TeV, measured by the D0 experiment at Tevatron. This table indicates that the Odderon signal is observed with at least a 14$\sigma$ significance, when both projections are combined from the previous Table 2, by adding the $\chi^2$ and the NDF values of both directions of the comparisons. These results are remarkably stable with respect to the choice of the unknown integrated elastic cross section at $\sqrt{s} = 1.96$ TeV, and also with respect to the choice of the linear-exponential or linear-linear interpolations. This effectively indicates that the combined significance of the Odderon discovery is at least a 13$\sigma$ effect.

| $\sigma_{el}$ (mb) | interpolation | direction of comparison | $\chi^2$ | NDF | CL [%] | Odderon significance in units of $\sigma$ |
|-------------------|---------------|-------------------------|---------|-----|--------|----------------------------------------|
| $17.6 \pm 1.1$    | lin-exp       | 7 $\leftrightarrow$ 1.96 TeV | 373.6   | 82  | 8.3E-37 | 13.03                                 |
|                   |               | 7 $\leftrightarrow$ 1.96 TeV | 405.1   | 82  | 3.0E-42 | 13.95                                 |
|                   | lin-lin       | 7 $\leftrightarrow$ 1.96 TeV | 399     | 82  | 3.5E-41 | 13.78                                 |
| $20.2 \pm 1.4$    | lin-exp       | 7 $\leftrightarrow$ 1.96 TeV | 431.2   | 82  | 7.7E-47 | 14.69                                 |
|                   | lin-lin       | 7 $\leftrightarrow$ 1.96 TeV | 431.2   | 82  | 7.7E-47 | 14.69                                 |

determine more precisely any possible residual dependence of these Odderon effects as a function of $\sqrt{s}$.

The current significance of the Odderon observation can be further increased from the 6.55$\sigma$ effect by a tedious experimental re-analysis of some of the already published data, for example, by separating the point-to-point uncorrelated statistical and systematic errors (type A errors) from the point-to-point correlated systematic errors in elastic $p\bar{p}$ collisions by D0, as well as by determining the covariance matrix of the elastic cross section measurement of $pp$ collisions at 2.76 and 7 TeV colliding energies by TOTEM.

12 Summary and conclusions

We have introduced a new, straightforwardly measurable scaling function $H(x)$ of elastic proton-(anti)proton scattering. This scaling function transforms out the trivial energy-dependent factors, in particular, the effects due to the $s$-dependencies stemming from the elastic slope $B(s)$, from the real-to-imaginary ratio $\rho_0(s)$, as well as from the total and elastic cross sections, $\sigma_{tot}(s)$ and $\sigma_{el}(s)$, respectively.

Figs. 8 and 10 clearly indicate a difference between the scaling properties of the elastic $pp$ and $p\bar{p}$ collisions, corresponding to a crossing-odd component of the elastic scattering amplitude at the TeV energy scale. As in this kinematic region the Reggeon contributions to the scattering amplitude
Table 5  Summary table of the search for an Odderon signal in the two-way comparison of the $H(x)$ scaling functions of $pp$ collisions at $\sqrt{s} = 2.76$ TeV, measured by the TOTEM experiment at the LHC, and $p\bar{p}$ elastic collisions at $\sqrt{s} = 1.96$ TeV, measured by the D0 experiment at Tevatron. The lowest value of significance in this comparison is found to be 0.01$\sigma$, which means that the $H(x)$ scaling functions of 1.96 TeV $p\bar{p}$ and 2.76 TeV $pp$ elastic collisions are nearly the same within errors. The level of maximal difference is much less than a 3$\sigma$ effect which does not reach the statistical significance of a discovery effect in this comparison.

| $\sigma_{el}$ (mb) | interpolation | direction of comparison | $\chi^2$ | NDF | CL [%] | Odderon significance in units of $\sigma$ |
|-------------------|--------------|------------------------|--------|-----|-------|----------------------------------------|
| 17.6 ± 1.1        | lin-exp      | 2.76 <--> 1.96 TeV     | 27.9   | 38  | 88.4  | 0.15                                   |
|                   | lin-lin      | 2.76 <--> 1.96 TeV     | 32.4   | 38  | 72.6  | 0.35                                   |
| 20.2 ± 1.4        | lin-exp      | 2.76 <--> 1.96 TeV     | 19.3   | 38  | 99.5  | 0.01                                   |
|                   | lin-lin      | 2.76 <--> 1.96 TeV     | 22.5   | 38  | 97.8  | 0.03                                   |

are suppressed by their power-law decays, a significant characteristic difference between the $H(x)$ scaling functions of elastic $pp$ and $p\bar{p}$ collisions at the logarithmically similar energies of 7, 2.76 and 1.96 TeV is considered as a clear-cut Odderon effect, because the trivial energy dependences of $\sigma_{el}(s)$ and $B(s)$ as well as that of $\rho(s)$ and $\sigma_{tot}(s)$ are scaled out from $H(x)$ by definition.

A comparison in Fig. 10 indicates a significant difference between the rescaled 7 TeV $pp$ data set down to 1.96 TeV with the corresponding $p\bar{p}$ data measured at $\sqrt{s} = 1.96$ TeV. In the swing region, i.e. before the dip, this difference is quantified with a CL of $5 \times 10^{-9}$ %. These re-analyzed D0 and TOTEM data, taken together with the verified energy independence of the $H(x)$ scaling function in the $\sqrt{s} = 2.76 - 7.0$ TeV energy range amount to the closing of the energy gap between 2.76 and 1.96 TeV in a model-independent way, as much as reasonably possible without a direct measurement.

At the same time, Fig. 5 indicates that the same 7 TeV data rescaled down to $\sqrt{s} = 2.76$ TeV do not significantly differ from the TOTEM data measured at the same energy of 2.76 TeV, which is logarithmically close to 1.96 TeV, the highest available colliding energy of $p\bar{p}$ elastic collisions. So, we have utilized the observed energy independence of the $H(x)$ scaling function of elastic $pp$ collisions in the few TeV energy range. One of the new, qualitative Odderon effects that we have identified was the approximate energy independence of the $H(x)$ scaling function for elastic $pp$ collisions in the few TeV energy range, in contrast to a stronger energy dependence of the $H(x)$ scaling function for elastic $p\bar{p}$ collisions.

In conclusion, we find from a model-independent re-analysis of the scaling properties of the differential cross sections of already published D0 and TOTEM data sets a statistically significant, more than a 6.55$\sigma$ effect of $t$-channel Odderon exchange.

Elastic $pp$ scattering data in a vicinity of $\sqrt{s} \approx 2$ TeV as well as in between 2.76 and 7 TeV would definitively be most useful for confirming our significant Odderon signal. Our analysis indicates that the statistically significant Odderon signal is in the kinematic range of $10 \leq x = -tB \leq 20$, hence it is important to measure these elastic scattering cross-sections well beyond the kinematic domain of the diffractive cone.

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