The geometry of spherical random fields

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Introduction

Siamo liberi di sceglierci ogni volta
invece che lasciare troppa cosa già decisa
a scegliere per noi. (Tiromancino)

This Ph.D. thesis The geometry of spherical random fields collects research results obtained in these last three years. The main purpose is the study of random fields indexed by the two-dimensional unit sphere $S^2 \subset \mathbb{R}^3$.

Let us first fix some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 0.0.1. A random field $T$ on $S^2$ [40] is a (possibly complex-valued) measurable map

$$T : (\Omega \times S^2, \mathcal{F} \otimes \mathcal{B}(S^2)) \to (\mathbb{C}, \mathcal{B}(\mathbb{C})) ; \quad (\omega, x) \mapsto T_x(\omega) ,$$

where $\mathcal{B}(S^2)$ (resp. $\mathcal{B}(\mathbb{C})$) denotes, as usual, the Borel $\sigma$-field on the sphere (resp. the field of complex numbers).

Often in this work we will write $T(\cdot, x)$ instead of $T_x(\cdot)$. Loosely speaking, $T$ is a collection of r.v.'s $(T_x)_{x \in S^2}$ indexed by the points of the sphere or, equivalently, it can be seen as a r.v. $x \mapsto T_x$ taking values in some space of functions on $S^2$.

In particular, we are interested in rotationally invariant or isotropic random fields (e.g. see [40, 9, 8, 39]): briefly we mean that the random field $T$ and the “rotated” $T^g := (T_{gx})_{x \in S^2}$, have the same law for every $g \in SO(3)$ (for details see Definition VI)
1.3.6. $SO(3)$, as usual, denotes the group of all rotations of $\mathbb{R}^3$ about the origin, under the operation of composition.

Spherical random fields naturally arise in a wide class of instances in geophysics, atmospheric sciences, medical imaging and cosmology. The application domain we are interested in concerns the latter, mainly in connection with the analysis of Cosmic Microwave Background (CMB) radiation.

We can image that physical experiments for CMB measure, for each point $x \in S^2$, an ellipse on $T_xS^2$ - the tangent plane to the sphere at $x$ ([40, 38]). The “width” of this ellipse is related to the temperature of this radiation whereas the other features (elongation and orientation) are collected in complex polarization data.

Indeed, the modern random model for the absolute temperature of CMB is an isotropic random field on the sphere, according to Definition 0.0.1 (see also Part 1). Instead, to model the polarization of this radiation we need a more complex structure, namely an invariant random field on the sphere taking values in some space of algebraic curves (the so-called spin random fields - see Part 3).

To test some features of the CMB – such as whether it is a realization of a Gaussian field, is a question that has attracted a lot of attention in last years: asymptotic theory must hence be developed in the high-frequency sense (see Part 2).

Although our attention has been mostly attracted by the spherical case, in this work we decided to treat more general situations whenever it is possible to extend our results from the sphere to other structures. Actually the interplay between the probabilistic aspects and the geometric ones produces sometimes fascinating insights. We shall deal for instance with homogeneous spaces of a compact group (Part 1) as well as vector bundles (Part 3).

This thesis can be split into three strongly correlated parts: namely Part 1: Gaussian fields, Part 2: High-energy Gaussian eigenfunctions and Part 3: Spin random fields. It is nice to note that this work will turn out to have a “circulant” structure, in a sense to make clear below (see Theorem 0.0.3 and Theorem 0.0.13).
Related works

Throughout the whole thesis, we refer to the following:

- P. Baldi, M. Rossi. *On Lévy’s Brownian motion indexed by elements of compact groups*, Colloq. Math. 2013 ([7]);
- P. Baldi, M. Rossi. *Representation of Gaussian isotropic spin random fields*, Stoch. Processes Appl. 2014 ([8]);
- D. Marinucci, M. Rossi. *Stein-Malliavin approximations for nonlinear functionals of random eigenfunctions on $S^d$*, J. Funct. Anal. 2015 ([44]);
- D. Marinucci, G. Peccati, M. Rossi, I. Wigman. (2015+) *Non-Universality of nodal length distribution for arithmetic random waves*, Preprint arXiv:1508.00353 ([43]).

However some of the results presented here are works still in progress, and should appear in forthcoming papers:

- M. Rossi. (2015+) *The Defect of random hyperspherical harmonics*, in preparation ([60]);
- M. Rossi (2015) *Level curves of spherical Gaussian eigenfunctions*, Preprint.

Moreover, we decided not to include some other works: for brevity [18] written with S. Campese and D. Marinucci, and to avoid heterogeneity [4, 5], both joint works with P. Baldi and L. Caramellino.

Part 1: Gaussian fields

Chapters 1 & 2

Our investigation starts from a “typical” example of random field on the sphere, i.e. P. Lévy’s spherical Brownian motion. We mean a centered Gaussian field $W =$
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$(W_x)_{x \in \mathbb{S}^2}$ whose covariance kernel $K$ is given by

$$K(x, y) := \frac{1}{2} (d(x, o) + d(y, o) - d(x, y)) \ , \quad x, y \in \mathbb{S}^2 ,$$  \hspace{1cm} (0.0.1)

where $o$ is some fixed point on the sphere – say the “north pole”, and $d$ denotes the usual geodesic distance. Note that (0.0.1) implies $W_o = 0$ (a.s.).

In particular, we recall P. Lévy’s idea [37] for constructing $W$ (see Example 2.1.2). Consider a Gaussian white noise $S$ on the sphere, i.e. an isometry between the space of square integrable functions on $\mathbb{S}^2$ and finite-variance r.v.’s, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. P. Lévy defines a spherical Gaussian field $T$ as

$$T_x := \sqrt{\pi} S(1_{H_x}) \ , \quad x \in \mathbb{S}^2 ,$$  \hspace{1cm} (0.0.2)

where $1_{H_x}$ denotes the indicator function of the half-sphere centered at $x$. It turns out that $T$ is isotropic and

$$\mathbb{E}[|T_x - T_y|^2] = d(x, y) \ , \quad x, y \in \mathbb{S}^2 .$$

From now on, $\mathbb{E}$ denotes the expectation under the probability measure $\mathbb{P}$.

P. Lévy’s spherical Brownian motion is hence the Gaussian field $W$ defined as

$$W_x := T_x - T_o \ , \quad x \in \mathbb{S}^2 .$$  \hspace{1cm} (0.0.3)

It is worth remarking that the Brownian motion on the $m$-dimensional unit sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ ($m > 2$) is analogously defined and P. Lévy itself extended the previous construction to the higher dimensional situation.

Our first question is the following.

- Can we extend this technique to construct isotropic Gaussian fields $T$ on $\mathbb{S}^2$?

We answered this question in the first part of [8]. We note that (0.0.2) can be rewritten as

$$T_x := \sqrt{\pi} S(L_g 1_{H_x}) \ , \quad x \in \mathbb{S}^2 ,$$
where $g = g_x$ is any rotation matrix $\in SO(3)$ mapping the north pole $o$ to the point $x$ and $L_g 1_{H_o}$ is the function defined as $L_g 1_{H_o}(y) := 1_{H_o}(g^{-1}y)$, $y \in S^2$. Actually,

$$L_g 1_{H_o}(y) = 1_{H_o}(g^{-1}y) = 1_{gH_o}(y) = 1_{H_o}(y), \quad y \in S^2.$$

$L$ coincides with the left regular representation (1.2.4) of $SO(3)$.

Consider now some homogeneous space $X$ (see Definition 1.1.1) of a compact group $G$ (e.g. $X = S^2$ and $G = SO(3)$). As for the spherical case, we have the following.

**Definition 0.0.2.** A random field $T$ on $X$ is a (possibly complex-valued) measurable map

$$T : (\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X)) \rightarrow (C, \mathcal{B}(C)) ; \quad (\omega, x) \mapsto T_x(\omega),$$

where $\mathcal{B}(X)$ denotes, as usual, the Borel $\sigma$-field on $X$.

We develop P.Lévy’s construction to obtain isotropic Gaussian fields on $X$. First we consider a Gaussian white noise $S$ on $X$, extended to the space $L^2(X)$ of square integrable complex functions $f$. $S$ respects the real character of $f$, i.e. $f$ is real if and only if $S(f)$ is real. Let us fix once forever some point $x_0 \in X$ and denote $K$ the isotropy group of $x_0$, i.e. the closed subgroup of elements $g \in G$ fixing $x_0$. Recall that $X$ is isomorphic to the quotient space $G/K$.

To each $f \in L^2(X)$ which is moreover left invariant w.r.t. the action of $K$, we associate an isotropic complex-valued Gaussian field $T^f$ on $X$ as

$$T^f_x := S(L_g f), \quad x \in X,$$

where the function $L_g f$ is defined as $L_g f(y) := f(g^{-1}y)$, $y \in X$ and $g \in G$ is any element that maps the point $x_0$ to the point $x = gx_0$. $L$ coincides with the left regular representation of $G$ (see (1.2.4)).

The law of the field $T^f$ is completely characterized by the associated positive definite function $\phi^f$ which is defined, for $g \in G$, as

$$\phi^f(g) := \text{Cov} \left(T^f_x, T^f_{x_0}\right) = \langle L_g f, f \rangle,$$

where

$$\left(0.0.5\right)$$
where $x$ is such that $gx_0 = x$. As usual, $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathcal{X})$.

Moreover we need the “relation” function of $T_f$

$$\zeta_f (g) := \text{Cov} \left( T^f_x, T^f_{x_0} \right) = \langle Lg f, \overline{f} \rangle, \quad (0.0.6)$$

where $T^f_{x_0}$ (and $\overline{f}$) denotes complex conjugation.

• Now we ask whether every isotropic, complex-valued Gaussian random field on $\mathcal{X}$ can be obtained with this construction.

The answer is no in general (see Remark 2.1.1 and (2.2.18) for some counterexample). It is however positive if we consider isotropic real Gaussian fields $T$ on $\mathcal{X}$. Our first result is the following (Theorem 2.2.3).

**Theorem 0.0.3.** Let $T$ be a real isotropic Gaussian field on $\mathcal{X}$. Then there exists a real left-$K$-invariant function $f \in L^2(\mathcal{X})$ such that $T$ and $T_f$ have the same law

$$T \overset{\mathcal{L}}{=} T_f, \quad (0.0.4)$$

where $T^f$ is defined as (0.0.4).

Actually, we prove that the associated positive definite function $\phi$ on the group $G$ of $T$ is of the form (0.0.5). Precisely, if $\phi$ is defined as before as $\phi(g) := \text{Cov} \left( T^f_x, T^f_{x_0} \right)$, where $x = gx_0$, then we show (see Proposition 2.2.2) that there exists a real function $f \in L^2(\mathcal{X})$ such that

$$\phi(g) = \langle Lg f, f \rangle, \quad g \in G,$$

i.e. $T$ and $T_f$ have the same distribution.

**Chapter 3**

Assume now that $\mathcal{X}$ is in addition endowed with some metric $d$. Analogously for the spherical case, P.Lévy’s Brownian motion on the metric space $(\mathcal{X}, d)$ is defined as a real centered Gaussian field on $\mathcal{X}$ which vanishes at some point $x_0 \in \mathcal{X}$ and such that $\mathbb{E}[|X_x - X_y|^2] = d(x,y)$. By polarization, its covariance function is

$$K(x,y) = \frac{1}{2} (d(x,x_0) + d(y,x_0) - d(x,y)), \quad (0.0.7)$$

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Note that it is not obvious that the Brownian motion exists on \((\mathcal{X}, d)\), equivalently that the kernel (0.0.7) is positive definite on \(\mathcal{X}\).

Positive definiteness of \(K\) for \(\mathcal{X} = \mathbb{R}^{m+1}\) and \(d\) the Euclidean metric had been proved by Schoenberg [62] in 1938 and, as recalled above, P. Lévy itself constructed the Brownian motion on \(\mathcal{X} = S^m\), here \(d\) being the spherical distance. Later Gangolli [31] gave an analytical proof of the positive definiteness of the kernel (0.0.7) for the same metric space \((S^m, d)\), in a paper that dealt with this question for a large class of homogeneous spaces.

Finally Takenaka in [65] proved the positive definiteness of the kernel (0.0.7) for the Riemannian metric spaces of constant sectional curvature equal to \(-1, 0\) or 1, therefore adding the hyperbolic disk to the list. To be precise in the case of the hyperbolic space \(\mathcal{H}_m = \{(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} : x_1^2 + \ldots x_m^2 - x_0^2 = 1\}\), the distance under consideration is the unique, up to multiplicative constants, Riemannian distance that is invariant with respect to the action of \(G = L_m\), the Lorentz group.

- Now we ask the question of the existence of P. Lévy’s Brownian motion on \(\mathcal{X} = SO(3)\), endowed with the Riemannian metric induced by the embedding \(SO(3) \hookrightarrow \mathbb{R}^9\).

There are deep motivations for this choice, connected to the spin theory, which will be clearer in Part 3.

We answer this question in [7] (Proposition 3.3.3).

**Proposition 0.0.4.** The kernel \(K\) in (0.0.7) is not positive definite on \(SO(3)\), endowed with the Riemannian metric induced by the embedding \(SO(3) \hookrightarrow \mathbb{R}^9\).

This is somehow surprising as, in particular, \(SO(3)\) is locally isometric to \(SU(2)\), where positive definiteness of \(K\) is immediate since isomorphic to the unit hypersphere \(S^3\).

Proposition 0.0.4 moreover allows to prove the non existence of P. Lévy’s Brownian motion on the group \(SO(n)\) of all rotations of \(\mathbb{R}^{n+1}\) for \(n > 3\). Actually, \(SO(n)\) contains a closed subgroup that is isomorphic to \(SO(3)\). Indeed the same argument holds also on e.g. the group \(SU(n)\) of all \(n \times n\) unitary matrices with determinant one, for \(n \geq 3\).
(see Corollary 3.3.4). Our method could be applied to investigate positive definiteness of the Brownian kernel on other compact Lie groups.

**Part 2: High-energy Gaussian eigenfunctions**

**Chapters 4, 5 & 6**

As already briefly stated, the investigation of spherical random fields has been strongly motivated by cosmological applications (e.g. concerning CMB): the asymptotic analysis in this setting must be hence developed in the high-energy sense, as follows.

First recall that the eigenvalues of the Laplace-Beltrami operator $\Delta_{S^2}$ on $S^2$ are integers of the form $-\ell(\ell + 1)$, $\ell \in \mathbb{N}$.

Under Gaussianity, an isotropic random field $T$ on $S^2$ can be decomposed in terms of its random Fourier components $T_\ell$, $\ell \in \mathbb{N}$. The latter are independent and isotropic centered Gaussian fields, whose covariance kernel is

$$E[T_\ell(x)T_\ell(y)] = P_\ell(\cos d(x, y)) , \quad x, y \in S^2 , \quad (0.0.8)$$

where $P_\ell$ is the $\ell$-th Legendre polynomial [64, 40] and $d(x, y)$ denotes the spherical distance between $x$ and $y$.

The following spectral representation holds [40, Propositions 5.13]

$$T_x = \sum_{\ell \in \mathbb{N}} c_\ell T_\ell(x) ,$$

where the series converges in $L^2(\Omega \times S^2)$ and the nonnegative sequence $(c_\ell)_\ell$ is the power spectrum of the field [40].

$T_\ell$ is known as the $\ell$-th Gaussian spherical eigenfunction or random spherical harmonic (see (4.2.15) for a precise definition), indeed “pathwise” satisfies

$$\Delta_{S^2} T_\ell + \ell(\ell + 1)T_\ell = 0 .$$

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In this second part, we investigate the high-energy behavior (i.e., as $\ell \to +\infty$) of $T_\ell$. We are interested in the geometry of the $z$-excursion set ([1] e.g.), which is defined for $z \in \mathbb{R}$ as

$$A_\ell(z) := \{ x \in S^2 : T_\ell(x) > z \}.$$

(0.0.9)

For instance, one can investigate the area of $A_\ell(z)$, the length of the boundary $\partial A_\ell(z)$ -- that is the length of level curves $T_\ell^{-1}(z)$, and the Euler-Poincaré characteristic of these domains. For completeness, we recall that these three quantities correspond to the so-called Lipschitz-Killing curvatures on the sphere [1].

Many authors have studied properties of excursion sets of random fields on the sphere or other manifolds: for instance, one can investigate the behavior of the excursion probability [23], i.e. as $z \to +\infty$

$$\mathbb{P}\left( \sup_{x \in S^2} T_x > z \right),$$

where $T$ is some random field on the sphere; (see also e.g. [1, 22, 20, 19, 21, 22, 16]).

It is worth remarking that random spherical harmonics have attracted great interest also in other disciplines, such as Mathematical Physics. Indeed Berry’s Random Wave Model ([10]) allows to compare - at least for “generic” chaotic Riemannian surfaces $\mathcal{M}$ - a deterministic Laplace eigenfunction $f$ on $\mathcal{M}$ of some large eigenvalue $E$ to a “typical” instance of an isotropic, monochromatic random wave with wavenumber $\sqrt{E}$ (see also [69]). In view of this conjecture, much effort has been devoted to 2-dimensional manifolds such as the torus $\mathbb{T}$ (see e.g. [34]) and the sphere $S^2$ (see e.g. [15], [14], [47], [68]), as stated just above. In this setting, the nodal case corresponding to $z = 0$ has received the greatest attention. Indeed nodal domains (the complement of the set where eigenfunctions are equal to zero) appear in many problems in engineering, physics and the natural sciences: they describe the sets that remain stationary during vibrations, hence their importance in such areas as musical instruments industry, earthquake study and astrophysics (for further motivating details see [69]).

- In this thesis we want to investigate the geometry of excursion sets (0.0.9) of high-energy Gaussian eigenfunctions $T_\ell$ on the sphere. The geometric features we are
interested in can be written as nonlinear functionals of the random field itself (and its spatial derivatives).

**Excursion area**

The area $S_\ell(z)$ of $z$-excursion sets (0.0.9) can be written as

$$S_\ell(z) = \int_{S^2} 1_{(z,+\infty)}(T_\ell(x)) \, dx,$$

where $1_{(z,+\infty)}$ is the indicator function of the interval $(z,+\infty)$. The expected value is simply computed to be $\mathbb{E}[S_\ell(z)] = 4\pi(1 - \Phi(z))$, where $\Phi$ denotes the cumulative distribution function of a standard Gaussian r.v. The variance has been studied in [46, 47, 45]: we have, as $\ell \to +\infty$,

$$\text{Var}(S_\ell(z)) = z^2 \phi(z)^2 \cdot \frac{1}{\ell} + O\left(\frac{\log \ell}{\ell^2}\right) , \quad (0.0.10)$$

where $\phi$ is the standard Gaussian probability density function. In particular, for $z \neq 0$, (0.0.10) gives the exact asymptotic form of the variance.

The nodal case corresponds to the Defect $D_\ell$, which is defined as

$$D_\ell := \int_{S^2} 1_{(0,+\infty)}(T_\ell(x)) \, dx - \int_{S^2} 1_{(-\infty,0)}(T_\ell(x)) \, dx ,$$

i.e. the difference between the measure of the positive and negative regions. Note that $D_\ell = 2S_\ell(0) - 4\pi$. We have $\mathbb{E}[D_\ell] = 0$ and from [45]

$$\text{Var}(D_\ell) = \frac{C}{\ell^2}(1 + o(1)) , \quad \ell \to +\infty , \quad (0.0.11)$$

for some $C > \frac{32}{\sqrt{2}}$.

It is worth remarking that the Defect variance is of smaller order than the non-nodal case. This situation is similar to the *cancellation phenomenon* observed by Berry in a different setting ([10]).

In [47] Central Limit Theorems are proved for the excursion area:

$$\frac{S_\ell(z) - \mathbb{E}[S_\ell(z)]}{\sqrt{\text{Var}(S_\ell(z))}} \overset{\ell}{\to} Z , \quad z \neq 0 ,$$

$$\frac{D_\ell}{\sqrt{\text{Var}(D_\ell)}} \overset{\ell}{\to} Z ,$$

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$Z \sim \mathcal{N}(0, 1)$ being a standard Gaussian r.v. and $\rightarrow^\mathcal{L}$ denoting the convergence in distribution from now on. Often we will write $\rightarrow^d$ instead of $\rightarrow^\mathcal{L}$.

A CLT result is “only” an asymptotic result with no information on the speed of convergence to the limiting distribution. More refined results indeed aim at the investigation of the asymptotic behaviour for various probability metrics, such as Wasserstein, Kolmogorov and total variation distances, see (4.1.11). In this respect, a major development in the last few years has been provided by the so-called fourth-moment literature, which is summarized in the recent monograph [53]. In short, a rapidly growing family of results is showing how to establish bounds on probability distances between multiple stochastic integrals and the Gaussian distribution analyzing the fourth-moments/fourth cumulants alone ([55, 3, 54, 52, 17] e.g.).

- We establish a quantitative CLT for the excursion area of random spherical harmonics.

In [44] we consider a more general situation, i.e. nonlinear functionals of Gaussian eigenfunctions $(T_\ell)_{\ell \in \mathbb{N}}$ on the $m$-dimensional unit sphere $S^m$, $m \geq 2$. The eigenvalues of the Laplace-Beltrami operator $\Delta_{S^m}$ on $S^m$ are integers of the form $-\ell(\ell + m - 1)$, $\ell \in \mathbb{N}$. The $\ell$-th Gaussian eigenfunction $T_\ell$ on $S^m$ (4.2.15)

$$\Delta_{S^m} T_\ell + \ell(\ell + m - 1) T_\ell = 0 \, , \text{ a.s.}$$

is a centered isotropic Gaussian field with covariance function

$$\mathbb{E}[T_\ell(x)T_\ell(y)] = G_{\ell,m}(\cos d(x, y)) \, , \quad (0.0.12)$$

where $G_{\ell,m}$ denotes the normalized Gegenbauer polynomial [64] and $d$ the usual distance on the $m$-sphere.

Precisely, we consider sequences of r.v.’s of the form

$$S_\ell(M) := \int_{S^m} M(T_\ell(x)) \, dx \, ,$$

where $M : \mathbb{R} \to \mathbb{R}$ is some measurable function such that $\mathbb{E}[M(Z)^2] < +\infty$, $Z \sim \mathcal{N}(0, 1)$. Note that if we choose $M = 1_{(z, +\infty)}$, then $S_\ell(M) = S_\ell(z)$ the excursion

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volume of random hyperspherical harmonics, i.e. the empirical measure of the set where eigenfunctions lie upon the level $z$.

The main idea for our proof is first to develop $S_\ell(M)$ into Wiener chaoses, i.e. as a series in $L^2(\mathbb{P})$ of the type (4.1.5)

$$S_\ell(M) = \sum_{q=0}^{+\infty} \frac{J_q(M)}{q!} \int_{\mathbb{S}^m} H_q(T_\ell(x)) \, dx,$$

where $H_q$ denotes the $q$-th Hermite polynomial (4.1.3) (see also [64, 53]) and $J_q(M) := \mathbb{E}[M(Z)H_q(Z)]$. Then, we study the asymptotic behavior of each summand $h_{\ell,q,m}$ of the previous series by means of a careful investigation of asymptotic variances (see Proposition 5.1.2) and the Fourth Moment Theorem (4.1.14): we are hence able to prove a quantitative CLT for $h_{\ell,q,m}$ (Proposition 5.1.3) in Wasserstein distance (4.1.11). To be more precise, we can restrict ourselves to even integers $\ell$ (see the related discussion in Chapter 5).

It turns out that, if the projection of $M(Z)$ onto the second order Wiener chaos is not zero ($J_2(M) \neq 0$), then this component dominates the whole series, i.e., as $\ell \to +\infty$

$$\frac{S_\ell(M) - \mathbb{E}[S_\ell(M)]}{\sqrt{\text{Var}(S_\ell(M))}} = \frac{J_2(M)}{2} h_{\ell,2,m} \frac{1}{\sqrt{\text{Var}(S_\ell(M))}} + o_p(1).$$

We can therefore prove the following (Theorem 5.1.8).

**Theorem 0.0.5.** If $J_2(M) \neq 0$, then

$$d_W \left( \frac{S_\ell(M) - \mathbb{E}[S_\ell(M)]}{\sqrt{\text{Var}(S_\ell(M))}}, Z \right) = O(\ell^{-1/2}),$$

that gives the rate of convergence, as $\ell \to +\infty$, to the standard Gaussian distribution in Wasserstein distance $d_W$.

Moreover if $M = 1_{(z,+\infty)}$, then it easy to compute that $J_2(M) \neq 0 \iff z \neq 0$. The following corollary is therefore immediate (Theorem 5.1.1).
Corollary 0.0.6. If $z \neq 0$, then
\[
d_W \left( \frac{S_\ell(z) - \mathbb{E}[S_\ell(z)]}{\sqrt{\text{Var}(S_\ell(z))}}, Z \right) = O(\ell^{-1/2}) , \quad \ell \to \infty .
\]

We have just obtained a quantitative CLT for the excursion volume of random hyperspherical eigenfunctions in the non-nodal case.

The nodal case which correspond to the Defect $D_\ell$ requires harder work, since it is no longer true that the second chaotic component dominates. In [60] we show first the exact rate for the Defect variance (Theorem 6.2.2).

**Theorem 0.0.7.** For $m > 2$, as $\ell \to +\infty$
\[
\text{Var}(D_\ell) = \frac{C_m}{\ell^m} (1 + o(1)) ,
\]
where $C_m > 0$ is some constant depending on the dimension $m$.

Remark that the case $m = 2$ has been solved in [45]. Moreover we prove CLT results for the Defect, in the high-energy limit (Theorem 6.3.1).

**Theorem 0.0.8.** For $m \neq 3, 4, 5$ we have, as $\ell \to +\infty$,\[
\frac{D_\ell}{\sqrt{\text{Var}(D_\ell)}} \xrightarrow{\mathcal{L}} Z ,
\]
where as before $Z \sim \mathcal{N}(0,1)$.

The remaining cases ($m = 3, 4, 5$) require a precise investigation of fourth-order cumulant of r.v.’s $h_{\ell,3,m}$ and are still work in progress in [60], where moreover the quantitative CLT for the Defect in the Wasserstein distance will be proved:
\[
d_W \left( \frac{D_\ell}{\sqrt{\text{Var}(D_\ell)}}, Z \right) = O(\ell^{-1/4}) .
\]

XVIII
Chapters 7 & 8

Length of level curves

The length \( \mathcal{L}_\ell(z) \) of level curves \( T^{-1}_\ell(z) \) can be formally written as

\[
\mathcal{L}_\ell(z) = \int_{S^2} \delta_z(T_\ell(x)) \| \nabla T_\ell(x) \| \, dx,
\]

where \( \delta_z \) denotes the Dirac mass in \( z \), \( \| \cdot \| \) the norm in \( \mathbb{R}^2 \) and \( \nabla \) the gradient. The expected value is [68, 69]

\[
\mathbb{E}[\mathcal{L}_\ell(z)] = 4\pi \cdot \frac{e^{-z^2/2}}{2\sqrt{2}} \sqrt{\ell(\ell+1)}
\]

and for the variance we have [69] if \( z \neq 0 \)

\[
\operatorname{Var}(\mathcal{L}_\ell(z)) \sim C e^{-z^2} z^4 \cdot \ell , \quad \ell \to +\infty ,
\]

for some \( C > 0 \). I.Wigman computed (private calculations) moreover the exact constant

\[
C = \frac{\pi^2}{2}.
\]

For the nodal length \( \mathcal{L}_\ell := \mathcal{L}_\ell(0) \) we have [68]

\[
\operatorname{Var}(\mathcal{L}_\ell) \sim \frac{1}{32} \cdot \log \ell , \quad \ell \to +\infty . \tag{0.0.13}
\]

Here also we observe the different behavior for asymptotic variances: in the nodal case it is of smaller order (logarithmic) rather than what should be the “natural” scaling \( \approx \ell \). This is due to some analytic cancellation in the asymptotic expansion of the length variance which occurs only for \( z = 0 \). This phenomenon has been called “obscure” Berry’s cancellation (see [11, 68]).

- We investigate the asymptotic distribution of the length of level curves.

We try to answer this question in [42]: here also we first compute the chaotic expansion of \( \mathcal{L}_\ell(z) \) (Proposition 7.2.2). Let us denote \((\partial_1 \tilde{T}_\ell, \partial_2 \tilde{T}_\ell)\) the normalized gradient, i.e. \( \sqrt{\frac{2}{\ell(\ell+1)}} \nabla T_\ell \).
Proposition 0.0.9. The chaotic expansion of $\mathcal{L}_\ell(z)$ is

$$\mathcal{L}_\ell(z) = \mathbb{E}[\mathcal{L}_\ell(z)] + \sqrt{\frac{\ell(\ell + 1)}{2}} \sum_{q=2}^{+\infty} \sum_{u=0}^{q} \sum_{k=0}^{u} \frac{\alpha_{k,u-k}\beta_{q-u}(z)}{(k)!(u-k)!(q-u)!} \times (0.0.14) \times \int_{S^2} H_{q-u}(T_\ell(x)) H_k(\partial_1 \tilde{T}_\ell(x)) H_{u-k}(\partial_2 \tilde{T}_\ell(x)) \, dx,$$

where the series converges in $L^2(\mathbb{P})$ and $(\beta_l(z))_l$, $(\alpha_{n,m})_{n,m}$ are respectively the chaotic coefficients of the Dirac mass in $z$ and the norm in $\mathbb{R}^2$.

By some computations, it is possible to give an exact formula for the second chaotic component (Proposition 7.3.1)

$$\text{proj}(\mathcal{L}_\ell(z)|C_2) = \sqrt{\frac{\ell(\ell + 1)}{2}} \frac{1}{4} e^{-z^2/2} z^2 \int_{S^2} H_2(T_\ell(x)) \, dx = \sqrt{\frac{\ell(\ell + 1)}{2}} \frac{1}{4} e^{-z^2/2} z^2 h_{\ell,2;2}.$$

Note that, as for the excursion area, the second component vanishes if and only if $z = 0$.

Computing the exact variance of $\text{proj}(\mathcal{L}_\ell(z)|C_2)$, it turns out that, for $z \neq 0$

$$\lim_{\ell \to +\infty} \frac{\text{Var}(\mathcal{L}_\ell(z))}{\text{Var}(\text{proj}(\mathcal{L}_\ell(z)|C_2))} = 1,$$

so that, as $\ell \to \infty$,

$$\frac{\mathcal{L}_\ell(z) - \mathbb{E}[\mathcal{L}_\ell(z)]}{\sqrt{\text{Var}(\mathcal{L}_\ell(z))}} \overset{\text{d}}{\to} \frac{\text{proj}(\mathcal{L}_\ell(z)|C_2)}{\sqrt{\text{Var}(\mathcal{L}_\ell(z))}} + o_p(1).$$

This implies that the total length has the same asymptotic distribution of the second chaotic projection (Theorem 7.1.1).

Theorem 0.0.10. As $\ell \to +\infty$, if $z \neq 0$, we have

$$\frac{\mathcal{L}_\ell(z) - \mathbb{E}[\mathcal{L}_\ell(z)]}{\sqrt{\text{Var}(\mathcal{L}_\ell(z))}} \overset{\text{d}}{\to} Z,$$

where $Z \sim \mathcal{N}(0,1)$.  

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The nodal case requires harder work, indeed it is not a simple task to derive an explicit expression for the fourth-order chaos and is still a work in progress. We can anticipate that the fourth-order chaotic projection dominates the whole nodal length and limit theorems will come hopefully soon ([42]).

Furthermore we decided to investigate the nodal case on the standard 2-torus $T$: in [43] we prove a Non-Central Limit Theorem for nodal lengths of arithmetic random waves (Theorem 8.1.1). The situation is analogous to the sphere: indeed the second chaotic component disappears and the fourth-order chaos dominates. The limit distribution is unexpectedly non-Gaussian, indeed it is a linear combination of $H_2(Z_1)$ and $H_2(Z_2)$, where $H_2$ is the second Hermite polynomial and $Z_1, Z_2$ are i.i.d. standard Gaussian r.v.’s.

Euler-Poincaré characteristic

The Euler-Poincaré characteristic of $z$-excursion set $\chi(A_\ell(z))$ for random spherical harmonics has been investigated in [16]. In the quoted paper, a precise expression for the asymptotic variance is proven, moreover the Gaussian Kinematic Formula [1] gives immediately the expected value:

$$E[\chi(A_\ell(z))] = \sqrt{\frac{2}{\pi}} e^{-z^2/2} z \frac{\ell(\ell + 1)}{2} + 2(1 - \Phi(z)),$$

and

$$\lim_{\ell \to +\infty} \ell^{-3}\text{Var}(\chi(A_\ell(z))) = \frac{1}{4}(z^3 - z)^2 \phi(z)^2.$$

The same phenomenon happens here: i.e. the nodal variance is of smaller order than the case $z \neq 0$.

For a unified formula for asymptotic variances of excursion area, length of level curves and Euler-Poincaré characteristic see (1.11) in [16].

Quantitative CLTs for $\chi(A_\ell(z))$ will be treated in forthcoming papers by the same authors V.Cammarota, D.Marinucci and I.Wigman.

Remark 0.0.11. A careful investigation of previous results on asymptotic variances, suggests that there is a strict connection between Berry’s cancellation phenomenon
for Lipschitz-Killing curvatures and chaotic expansions. Indeed the unique case which shows a different scaling for the variance as well as a zero second order chaotic component is the nodal one. Moreover, in situations analyzed above, there is always a single dominating chaotic projection: the second one at non-zero level and the fourth one in the nodal case.

We conjecture that this qualitative behaviour should be universal somehow: we mean that it should hold for all Lipschitz-Killing curvatures on every “nice” compact manifold.

**Part 3: Spin random fields**

**Chapter 9**

As briefly stated above, in cosmology and astrophysics spherical random fields are used to model CMB data [40]. More precisely, the *temperature* of this radiation is seen as a single realization of an isotropic random field on $S^2$, whereas to model its *polarization* we need to introduce random fields which do not take ordinary scalar values but have a more complex geometrical structure, the so-called spin random fields [40, 38, 8, 28]. Roughly speaking, they can be seen as random structures that at each point of the sphere take as a value a random “curve”.

This family of random models is indexed by integers $s \in \mathbb{Z}$: for instance, a spin-0 random field is simply a spherical random field, whereas a spin-1 random field takes as a value at the point $x \in S^2$ a random tangent vector to the sphere at $x$. The polarization of CMB is seen as a realization of a spin-2 random field.

From a mathematical point of view, spin random fields are random sections of particular complex-line bundles on the sphere $S^2$, whose construction can be interpreted in terms of group representation concepts. These are special cases of so-called homogeneous vector bundle, which we handle in the second part of [8]. Briefly, given a compact topological group $G$ and an irreducible representation $\tau$ of its closed subgroup $K$, we can construct the $\tau$-homogeneous vector bundle as follows. Let $H$ be the
(finite-dimensional) Hilbert space of the representation $\tau$. Consider the action of $K$ on $G \times K$ defined as

$$k(g, h) := (gk, \tau(k^{-1})h) ,$$

and denote by $G \times_{\tau} H$ the quotient space. The $\tau$-homogeneous vector bundle is the triple $\xi := (G \times_{\tau} H, \pi_{\tau}, G/K)$ where the bundle projection is

$$\pi_{\tau} : G \times_{\tau} H \to G/K; \quad \theta(g, h) \mapsto gK ,$$

$\theta(g, h)$ denoting the orbit of the pair $(g, h)$ and $gK$ the lateral class of $g$.

**Definition 0.0.12.** A random field $T$ on the $\tau$-homogeneous vector bundle is a random section of $\xi$, i.e. a measurable map

$$T : \Omega \times G/K \to G \times_{\tau} H; \quad (\omega, x) \mapsto T_x(\omega) ,$$

where for each $\omega \in \Omega$, the sample path is a section of $\xi$, that is $\pi_{\tau} \circ T(\omega) = id_{G/K}$.

Of course this means that for each $x = gK \in G/K$, $T_x$ takes as a value a random element in $\pi_{\tau}^{-1}(gK)$.

In the quoted paper, we first introduce a new approach to study random fields in $\tau$-homogeneous vector bundles: the “pullback” random field. The latter is a (complex-valued) random field $X$ on $G$ whose paths satisfy the following invariance property

$$X_{gk} = \tau(k^{-1})X_g , \quad g \in G, k \in K . \quad (0.0.15)$$

There is one to one correspondence between (random) sections of $\xi$ and (random) functions on $G$ satisfying $(0.0.15)$ (Proposition 9.1.2) and we prove that $T$ is equivalent to its pullback $X$ (see Proposition 9.1.4).

Now our attention is devoted to the spherical case. Here $G = SO(3)$, $K \cong SO(2)$: the latter being abelian, its irreducible representations are all one-dimensional and coincide with its linear characters $\chi_s$, $s \in \mathbb{Z}$. Each $k \in SO(2)$ is a counterclockwise
rotation of the complex plane $\mathbb{C}$ by an angle $\theta(k)$. The action of $k$ through $\chi_s$ can be seen as a clockwise rotation by the angle $s \cdot \theta(k)$.

The $\chi_s$-homogeneous vector bundle $\xi_s := \xi_{\chi_s}$ on the sphere is called spin $s$ line bundle and a random field in $\xi_s$ is known as spin $s$ random field.

Our aim is to extend the representation formulas for isotropic Gaussian fields on homogeneous spaces of a compact group in Part 1 to the spin case. The pullback approach allows to deal with isotropic Gaussian fields $X$ on $SO(3)$ whose sample paths satisfy the invariance property (0.0.15), that is

$$X_{gk} = \chi_s(k^{-1})X_g, \quad g \in SO(3), k \in SO(2).$$

Indeed we prove, with an analogous construction to the one developed in Chapter 2, that for each function $f \in L^2(SO(3))$ bi-s-associated, i.e. such that

$$f(k_1gk_2) = \chi_s(k_1)f(g)\chi_s(k_2), \quad g \in SO(3), k_1, k_2 \in SO(2),$$

an isotropic complex Gaussian spin $s$ random field $X^f$ is associated (Proposition 9.3.1). Moreover also the converse is true (Theorem 9.3.2).

**Theorem 0.0.13.** Let $T$ be an isotropic complex-Gaussian section of the spin $s$ line bundle and $X$ its pullback random field on $SO(3)$. Then there exists a bi-s-associated function $f \in L^2(SO(3))$ such that $X$ and $X^f$ have the same law.

Finally, we prove that our approach is equivalent to the existing ones: by Malyarenko [38] (Proposition 9.1.9) and Marinucci & Geller [32] (§9.4, especially Lemma 9.4.2).

The anticipated “circulant” structure can be found in Theorem 0.0.3 and Theorem 0.0.13: this connection is the starting point in the analysis of spin random fields. Indeed open questions concern how to extend results presented in Chapters 1–8 to the spin case. We leave it as a possible topic for future research.
Part 1
Gaussian fields
Chapter 1

Background: isotropic random fields

In this first chapter we recall basic results concerning both, Fourier analysis for a topological compact group $G$, and the structure of isotropic random fields indexed by elements of $G$-homogeneous spaces.

The plan is as follows. In §1.1, we give main definitions and fix some notations, whereas in §1.2 we investigate Fourier developments for square integrable functions on $G$-homogeneous spaces [29]. In §1.3 we recollect some useful properties of isotropic random fields from several works ([8, 9, 40] e.g.). Finally, the last section is devoted to the connection between isotropy and positive definite functions on compact groups - highlighting the main features we will deeply need in the sequel.

A great attention is devoted to the case of the 2-dimensional unit sphere $S^2$, to whom we particularize the results recalled in each section of this chapter.

1.1 Preliminaries

Throughout this work $G$ denotes a topological compact group (e.g. [29]). Let us recall the notion of homogeneous space.
Definition 1.1.1. A topological space $\mathcal{X}$ is said to be a $G$-homogeneous space if $G$ acts on $\mathcal{X}$ with a continuous and transitive action which we shall denote 

$$G \times \mathcal{X} \longrightarrow \mathcal{X} ; \quad (g, x) \mapsto gx .$$

Remark that $G$ itself is a $G$-homogeneous space, indeed the left multiplication $(g, h) \mapsto g^{-1}h$ is a continuous and transitive action.

$\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(G)$ stand for the Borel $\sigma$-fields of $\mathcal{X}$ and $G$ respectively and $dg$ for the Haar measure (see [29] e.g.) of $G$. The latter induces on $\mathcal{X}$ a $G$-invariant measure which we denote $dx$ abusing notation, given by

$$dx := \int_G \delta_{gx} \, dg ,$$

where $\delta_{gx}$ as usual stands for the Dirac mass at the singleton $\{gx\}$. For $G$-invariant we mean that for each integrable function $f$ on $\mathcal{X}$ we have for any $g \in G$

$$\int_{\mathcal{X}} f(gx) \, dx = \int_{\mathcal{X}} f(x) \, dx .$$

We assume that both these measures have total mass equal to 1, unless explicitly stated. For instance, in the case $\mathcal{X} = S^d$ the unit $d$-dimensional sphere and $G = SO(d + 1)$ the special orthogonal group of order $d + 1$, we have $\int_{S^d} dx = \mu_d$ where

$$\mu_d := \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} . \quad (1.1.1)$$

We set $L^2(G) := L^2(G, dg)$ and similarly $L^2(\mathcal{X}) := L^2(\mathcal{X}, dx)$; the $L^2$-spaces are spaces of complex-valued square integrable functions, unless otherwise stated.

Let us fix a point $x_0 \in \mathcal{X}$ once forever and denote by $K$ the isotropy group of $x_0$

$$K = \{ g \in G : gx_0 = x_0 \} ,$$

i.e. the (closed) subgroup of the elements $g \in G$ fixing $x_0$. Then it is immediate to check that $\mathcal{X} \cong G/K$ i.e., there exists a $G$-invariant isomorphism $\phi : \mathcal{X} \longrightarrow G/K$. Actually
the morphism $\tilde{\phi} : G \rightarrow \mathcal{X}$ defined as $\tilde{\phi}(g) := gx_0$ is surjective and $K = \tilde{\phi}^{-1}(x_0)$. For instance, in the case $G = SO(3)$ the group of all rotations about the origin of three-dimensional Euclidean space $\mathbb{R}^3$ (under the operation of composition), and $\mathcal{X} = \mathbb{S}^2$ the two-dimensional unit sphere, $x_0$ will be the north pole and the subgroup $K$ of matrices that leave $x_0$ fixed will be isomorphic to $SO(2)$, the special orthogonal group of order two.

The $G$-invariant isomorphism $\mathcal{X} \cong G/K$ suggests that it is possible to identify functions defined on $\mathcal{X}$ with particular functions on the group $G$.

**Definition 1.1.2.** The pullback on $G$ of a function $f : \mathcal{X} \rightarrow \mathbb{C}$ is the function $\tilde{f}$ defined as

$$\tilde{f}(g) := f(gx_0) , \quad g \in G . \quad (1.1.2)$$

Note that $\tilde{f}$ is a right-$K$-invariant function on $G$, i.e. constant on left closets of $K$. Actually for $g \in G, k \in K$ it is immediate that $\tilde{f}(gk) = f(gkx_0) = f(gx_0) = \tilde{f}(g)$. We have

$$\int_{\mathcal{X}} f(x) \, dx = \int_{G} \tilde{f}(g) \, dg , \quad (1.1.3)$$

by the integration rule of image measures, whenever one of the above integrals has sense.

**Remark 1.1.3.** In particular, from (1.1.2) and (1.1.3), the map $f \mapsto \tilde{f}$ is an isometry between $L^2(\mathcal{X})$ and the (closed) subspace of right-$K$-invariant functions in $L^2(G)$.

## 1.2 Fourier expansions

In this section we briefly recall Fourier expansions on compact groups (for further details see [29]).

The left regular representation $L$ of $G$ is given, for $g \in G$ and $f \in L^2(G)$, by

$$L_g f(h) = f(g^{-1}h) , \quad h \in G . \quad (1.2.4)$$
Let $\hat{G}$ be the dual of $G$, i.e., the set of the equivalence classes of irreducible unitary representations of $G$. The compactness of $G$ implies that $\hat{G}$ is at most countable.

In what follows, we use the same approach as in [9, 8, 29]). Let us choose, for every $\sigma \in \hat{G}$, a representative $(D^\sigma, H_\sigma)$ where $D^\sigma$ is a unitary operator acting irreducibly on $H_\sigma$ (a complex finite dimensional Hilbert space).

As usual, $\langle \cdot, \cdot \rangle$ denotes the inner product of $H_\sigma$ and $\dim \sigma := \dim H_\sigma$ its dimension. Recall that the character of $\sigma$ is the (continuous) function on $G$ defined as
\[ \chi_\sigma(g) := \text{tr}D^\sigma(g) \, , \quad g \in G , \]
where $\text{tr}D^\sigma(g)$ denotes the trace of $D^\sigma(g)$. Given $f \in L^2(G)$, for every $\sigma \in \hat{G}$ we define the Fourier operator coefficient
\[ \hat{f}(\sigma) := \sqrt{\dim \sigma} \int_G f(g) D^\sigma(g^{-1}) \, dg \tag{1.2.5} \]
which is a linear endomorphism of $H_\sigma$. Let us denote for $g \in G$
\[ f^\sigma(g) := \sqrt{\dim \sigma} \text{tr}(\hat{f}(\sigma)D^\sigma(g)) ; \tag{1.2.6} \]
by standard arguments in Representation Theory [63], $f^\sigma$ is a continuous function on $G$. Let us denote $\ast$ the convolution operator on $G$, defined for $f_1, f_2 \in L^2(G)$ as
\[ f_1 \ast f_2(g) := \int_G f_1(h) f_2(h^{-1}g) \, dh \, , \]
so that
\[ \hat{f}_1 \ast \hat{f}_2(\sigma) = \frac{1}{\sqrt{\dim \sigma}} \hat{f}_2(\sigma) \hat{f}_1(\sigma) . \tag{1.2.7} \]
Actually by a Fubini argument and the $G$-invariance property of Haar measure
\[
\begin{align*}
\hat{f}_1 \ast \hat{f}_2(\sigma) & = \sqrt{\dim \sigma} \int_G f_1 \ast f_2(\sigma) D^\sigma(g^{-1}) \, dg = \\
& = \sqrt{\dim \sigma} \int_G \left( \int_G f_1(h) f_2(h^{-1}g) \, dh \right) D^\sigma(g^{-1}) \, dg = \\
& = \sqrt{\dim \sigma} \int_G f_1(h) \left( \int_G f_2(g) D^\sigma(g^{-1}) \, dg \right) D^\sigma(h^{-1}) \, dh = \frac{1}{\sqrt{\dim \sigma}} \hat{f}_2(\sigma) \hat{f}_1(\sigma) .
\end{align*}
\]
Moreover note that

\[ f^\sigma = \dim \sigma \cdot f \ast \chi_\sigma . \]

The Peter-Weyl Theorem and Schur’s orthogonality relations (see [40] or [63] e.g.) imply the following, known as Plancherel’s Theorem.

**Theorem 1.2.1.** Let \( f \in L^2(G) \). Then

\[ f(g) = \sum_{\sigma \in \hat{G}} f^\sigma(g) , \tag{1.2.8} \]

the convergence of the series taking place in \( L^2(G) \);

\[ \| f \|^2_{L^2(G)} = \sum_{\sigma \in \hat{G}} \| \hat{f}(\sigma) \|^2_{H.S.} , \]

where \( \| \cdot \|_{H.S.} \) denotes the Hilbert-Schmidt norm [29]. If \( f_1, f_2 \in L^2(G) \), then

\[ \langle f_1, f_2 \rangle_{L^2(G)} = \sum_{\sigma \in \hat{G}} \operatorname{tr} \hat{f}_1(\sigma) \hat{f}_2(\sigma)^* . \]

Recall that a function \( f \) is said to be a *central* (or class) function if for every \( g \in G \)

\[ f(hgh^{-1}) = f(g), \quad h \in G . \]

**Proposition 1.2.2.** The set of characters \( \{ \chi_\sigma : \sigma \in \hat{G} \} \) is an orthonormal basis of the space of square integrable central functions on \( G \).

Now fix any orthonormal basis \( v_1, v_2, \ldots, v_{\dim \sigma} \) of \( H_\sigma \) and for \( i, j = 1, \ldots, \dim \sigma \) denote \( D^\sigma_{ij}(g) := \langle D^\sigma(g)v_j, v_i \rangle \) the \((i,j)\)-th coefficient of the matrix representation for \( D^\sigma(g) \) with respect to this basis. The matrix representation for \( \hat{f}(\sigma) \) has entries

\[
\hat{f}(\sigma)_{i,j} = \langle \hat{f}(\sigma)v_j, v_i \rangle = \sqrt{\dim \sigma} \int_G f(g) \langle D^\sigma(g^{-1})v_j, v_i \rangle \, dg = \\
= \sqrt{\dim \sigma} \int_G f(g) D^\sigma_{ij}(g^{-1}) \, dg = \sqrt{\dim \sigma} \int_G f(g) \overline{D^\sigma_{ji}(g)} \, dg ,
\]
and Theorem 1.2.1 becomes

\[
    f(g) = \sum_{\sigma \in \hat{G}} \sqrt{\dim \sigma} \sum_{i,j=1}^{\dim \sigma} \hat{f}(\sigma)_{j,i} D^\sigma_{i,j}(g),
\]

the above series still converging in \( L^2(G) \). The Peter-Weyl Theorem also states that the set of functions \( \{ \sqrt{\dim \sigma} D^\sigma_{i,j}, \sigma \in \hat{G}, i,j = 1, \ldots, \dim \sigma \} \) is a complete orthonormal basis for \( L^2(G) \). Therefore (1.2.9) is just the corresponding Fourier development and \( \hat{f}(\sigma)_{j,i} \) is the coefficient corresponding to the element \( \sqrt{\dim \sigma} D^\sigma_{i,j}(g) \) of this basis.

Let \( \mathcal{L}^2_{\sigma}(G) \subset L^2(G) \) be the \( \sigma \)-isotypical subspace, i.e. the subspace generated by the functions \( D^\sigma_{i,j}, i,j = 1, \ldots, \dim \sigma \); it is a \( G \)-module that can be decomposed into the orthogonal direct sum of \( \dim \sigma \) irreducible and equivalent \( G \)-modules \( (\mathcal{L}^2_{\sigma,j}(G))_{j=1,\ldots,\dim \sigma} \) where each \( \mathcal{L}^2_{\sigma,j}(G) \) is spanned by the functions \( D^\sigma_{i,j} \) for \( i = 1, \ldots, \dim \sigma \), loosely speaking by the \( j \)-th column of the matrix \( D^\sigma \). Note that \( f^\sigma \) is the component (i.e. the orthogonal projection) of \( f \) in \( \mathcal{L}^2_{\sigma}(G) \).

Equivalently the Peter-Weyl Theorem can be stated as

\[
    L^2(G) = \bigoplus_{\sigma \in \hat{G}} \bigoplus_{j=1}^{\dim \sigma} \mathcal{L}^2_{\sigma,j}(G),
\]

the direct sums being orthogonal.

Let us now deduce the Fourier expansion of functions \( f \in L^2(\mathcal{X}) \). It can be easily obtained from Theorem 1.2.1 and Remark 1.1.3, indeed their pullbacks \( \tilde{f} \) belong to \( L^2(G) \) and form a \( G \)-invariant closed subspace of \( L^2(G) \). We can therefore associate to \( f \in L^2(\mathcal{X}) \) the family of operators \( (\tilde{f}(\sigma))_{\sigma \in \hat{G}} \). Let \( H_{\sigma,0} \) denote the subspace of \( H_{\sigma} \) (possibly reduced to \{0\}) formed by the vectors that remain fixed under the action of \( K \), i.e. for every \( k \in K, v \in H_{\sigma,0}, D^\sigma(k)v = v \). Right-\( K \)-invariance implies that the image of \( \tilde{f}(\sigma) \) is contained in \( H_{\sigma,0} \):

\[
    \tilde{f}(\sigma) = \sqrt{\dim \sigma} \int_G \tilde{f}(g) D^\sigma(g^{-1}) \, dg =
    \sqrt{\dim \sigma} \int_G \tilde{f}(k) D^\sigma(g^{-1}) \, dg = \sqrt{\dim \sigma} \int_G \tilde{f}(h) D^\sigma(h^{-1}) \, dh =
    D^\sigma(k) \sqrt{\dim \sigma} \int_G \tilde{f}(h) D^\sigma(h^{-1}) \, dh = D^\sigma(k) \tilde{f}(\sigma).
\]

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Sec. 1.2 - Fourier expansions

Let us denote by \( P_{\sigma,0} \) the projection of \( H_{\sigma} \) onto \( H_{\sigma,0} \), so that \( \hat{\tilde{f}}(\sigma) = P_{\sigma,0}\hat{\tilde{f}}(\sigma) \), and \( \hat{G}_0 \) the set of irreducible unitary representations of \( G \) whose restriction to \( K \) contains the trivial representation. If \( \sigma \in \hat{G}_0 \) let us consider a basis of \( H_{\sigma} \) such that the elements \( \{v_{p+1}, \ldots, v_{\dim\sigma}\} \), for some integer \( p = p(\sigma) \geq 0 \), span \( H_{\sigma,0} \). Then the first \( p \) rows of the representative matrix of \( \hat{\tilde{f}}(\sigma) \) in this basis contain only zeros. Actually, by (1.2.11) and \( P_{\sigma,0} \) being self-adjoint, for \( i \leq p \)

\[
\hat{f}_{i,j}(\sigma) = \langle \hat{f}(\sigma)v_j, v_i \rangle = \langle P_{\sigma,0}\hat{\tilde{f}}(\sigma)v_j, v_i \rangle = \langle \hat{\tilde{f}}(\sigma)v_j, P_{\sigma,0}v_i \rangle = 0 .
\]

Identifying \( L^2(\mathcal{X}) \) as the closed subspace of right-\( K \)-invariant functions in \( L^2(G) \), the Peter-Weyl Theorem entails that

\[
L^2(\mathcal{X}) = \bigoplus_{\sigma \in \hat{G}_0} \oplus_{j=p+1}^{\dim\sigma} L^2_{\sigma,j}(G) ,
\]

the direct sums being orthogonal.

Now we consider an important class of functions we shall need in the sequel.

**Definition 1.2.3.** A function \( f : G \rightarrow \mathbb{C} \) is said to be bi-\( K \)-invariant if for every \( g \in G, k_1, k_2 \in K \)

\[
f(k_1gk_2) = f(g) .
\]

(1.2.12)

If moreover \( f \in L^2(G) \), the equality in (1.2.12) entails that, for every \( k_1, k_2 \in K, \sigma \in \hat{G} \),

\[
\hat{f}(\sigma) = D^\sigma(k_1)\hat{f}(\sigma)D^\sigma(k_2)
\]

and therefore a function \( f \in L^2(G) \) is bi-\( K \)-invariant if and only if for every \( \sigma \in \hat{G} \)

\[
\hat{f}(\sigma) = P_{\sigma,0}\hat{f}(\sigma)P_{\sigma,0} .
\]

(1.2.13)

Note that we can identify of course bi-\( K \)-invariant functions in \( L^2(G) \) with left-\( K \)-invariant functions in \( L^2(\mathcal{X}) \).
1.2 Fourier expansions

1.2.1 Spherical harmonics

Now we focus on the case of $X = S^2$ under the action of $G = SO(3)$, first specializing previous results and then recalling basic facts we will need in the rest of this work (see [29], [40] e.g. for further details). The isotropy group $K \cong SO(2)$ of the north pole is abelian, therefore its unitary irreducible representations are unitarily equivalent to its linear characters which we shall denote $\chi_s, s \in \mathbb{Z}$, throughout the whole work.

A complete set of unitary irreducible matrix representations of $SO(3)$ is given by the so-called Wigner’s $D$ matrices $\{D^\ell, \ell \geq 0\}$, where each $D^\ell(g)$ has dimension $(2\ell + 1) \times (2\ell + 1)$ and acts on a representative space that we shall denote $H_\ell$. The restriction to $K$ of each $D^\ell$ being unitarily equivalent to the direct sum of the representations $\chi_m, m = -\ell, \ldots, \ell$, we can suppose $v_{-\ell}, v_{-\ell+1}, \ldots, v_\ell$ to be an orthonormal basis for $H_\ell$ such that for every $m : |m| \leq \ell$

$$D^\ell(k)v_m = \chi_m(k)v_m, \quad k \in K. \tag{1.2.14}$$

Let $D^\ell_{m,n} = \langle D^\ell v_n, v_m \rangle$ be the $(m,n)$-th entry of $D^\ell$ with respect to the basis fixed above. It follows from (1.2.14) that for every $g \in SO(3), k_1, k_2 \in K$,

$$D^\ell_{m,n}(k_1 g k_2) = \chi_m(k_1)D^\ell_{m,n}(g)\chi_n(k_2). \tag{1.2.15}$$

The functions $D^\ell_{m,n} : SO(3) \rightarrow \mathbb{C}, \ell \geq 0, m, n = -\ell, \ldots, \ell$ are usually called Wigner’s $D$ functions.

Given $f \in L^2(SO(3))$, its $\ell$-th Fourier coefficient (1.2.5) is

$$\hat{f}(\ell) := \sqrt{2\ell + 1} \int_{SO(3)} f(g)D^\ell(g^{-1})\,dg \tag{1.2.16}$$

and its Fourier development (1.2.9) becomes

$$f(g) = \sum_{\ell \geq 0} \sqrt{2\ell + 1} \sum_{m,n=-\ell}^\ell \hat{f}(\ell)_{n,m}D^\ell_{m,n}(g). \tag{1.2.17}$$

If $\tilde{f}$ is the pullback of $f \in L^2(S^2)$, (1.2.14) entails that for every $\ell \geq 0$

$$\tilde{f}(\ell)_{n,m} \neq 0 \iff n = 0.$$
Moreover if $f$ is left-$K$-invariant, then
\[ \hat{f}(n,m) \neq 0 \iff n, m = 0 . \]

In words, an orthogonal basis for the space of the square integrable right-$K$-invariant functions on $SO(3)$ is given by the central columns of the matrices $D^\ell$, $\ell \geq 0$. Furthermore the subspace of the bi-$K$-invariant functions is spanned by the central functions $D^\ell_{0,0}(\cdot)$, $\ell \geq 0$, which are real-valued.

The important role of the other columns of Wigner’s $D$ matrices will appear further in this work.

**Definition 1.2.4.** For every $\ell \geq 0$, $m = -\ell, \ldots, \ell$, let us define the spherical harmonic $Y_{\ell,m}$ as
\[
Y_{\ell,m}(x) := \sqrt{\frac{2\ell + 1}{4\pi}} D^\ell_{m,0}(g_x) , \quad x \in S^2 ,
\] (1.2.18)
where $g_x$ is any rotation mapping the north pole of the sphere to $x$.

Remark that this is a good definition thanks to the invariance of each $D^\ell_{m,0}(\cdot)$ under the right action of $K$. The functions in (1.2.18) form an orthonormal basis of the space $L^2(S^2)$ considering the sphere with total mass equal to $4\pi$.

Often, e.g. in the second part “High-energy eigenfunctions”, we work with real spherical harmonics, i.e. the orthonormal set of functions given by
\[
\frac{1}{\sqrt{2}} (Y_{\ell,m} + \overline{Y_{\ell,m}}) , \quad \frac{1}{i\sqrt{2}} (Y_{\ell,m} - \overline{Y_{\ell,m}})
\] (1.2.19)
which abusing notation we will again denote by $Y_{\ell,m}$ for $\ell \geq 0$, $m = 1, \ldots, 2\ell + 1$.

Every $f \in L^2(S^2)$ admits the Fourier development of the form
\[
f(x) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(x) ,
\] (1.2.20)
where the above series converges in $L^2(S^2)$ and
\[
a_{\ell,m} = \int_{S^2} f(x) \overline{Y_{\ell,m}(x)} \, dx .
\]
Moreover the Fourier expansion of a left-$K$-invariant function $f \in L^2(S^2)$ is

$$f = \sum_{\ell=0}^{+\infty} \beta_\ell Y_{\ell,0},$$

(1.2.21)

where $\beta_\ell := \int_{S^2} f(x)Y_{\ell,0}(x)\,dx$. The functions $Y_{\ell,0}$, $\ell \geq 0$ are called central spherical harmonics.

We stress that there exists an alternative characterization of spherical harmonics, as eigenfunctions of the spherical Laplacian $\Delta_{S^2}$ (see [40] e.g.). We shall deeply use this formulation in Part 2: High-energy Gaussian eigenfunctions.

Recall that the spherical Laplacian is the Laplace-Beltrami operator on $S^2$ with its canonical metric of constant sectional curvature $1$, moreover its (totally discrete) spectrum is given by the set of eigenvalues $\{-\ell(\ell + 1) =: -E_\ell, \ell \in \mathbb{N}\}$.

It can be proved that for $\ell \geq 0$, $m = -\ell, \ldots, \ell$

$$\Delta_{S^2}Y_{\ell,m} + E_\ell Y_{\ell,m} = 0,$$

and the subset of spherical harmonics $\{Y_{\ell,m}, m = -\ell, \ldots, \ell\}$ is an orthonormal basis for the eigenspace $\mathcal{H}_\ell$ corresponding to the $\ell$-th eigenvalue. The Spectral Theorem for self-adjoint compact operators then entails that $\mathcal{H}_\ell$ and $\mathcal{H}_{\ell'}$ are orthogonal whenever $\ell \neq \ell'$ and moreover

$$L^2(S^2) = \bigoplus_{\ell \geq 0} \mathcal{H}_\ell,$$

which coincides with the Peter-Weyl decomposition for the sphere.

### 1.3 Isootropic random fields

Let us recall main definitions and facts about isotropic random fields on homogeneous spaces (see [40, 8, 9] e.g.). First fix some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote $L^2(\mathbb{P}) := L^2(\Omega, \mathbb{P})$ the space of finite-variance random variables.
Definition 1.3.1. A (complex-valued) random field $T = (T_x)_{x \in \mathcal{X}}$ on the $G$-homogeneous space $\mathcal{X}$ is a collection of (complex-valued) random variables indexed by elements of $\mathcal{X}$ such that the map

$$T : \Omega \times \mathcal{X} \rightarrow \mathbb{C} ; \quad (\omega, x) \mapsto T_x(\omega)$$

is $\mathcal{F} \otimes \mathcal{B}(\mathcal{X})$-measurable.

Note that often we shall write $T(x)$ instead of $T_x$.

Definition 1.3.2. We say that the random field $T$ on the $G$-homogeneous space $\mathcal{X}$ is second order if $T_x \in L^2(\mathbb{P})$ for every $x \in \mathcal{X}$.

Definition 1.3.3. We say that the random field $T$ on the $G$-homogeneous space $\mathcal{X}$ is a.s. continuous if the functions $\mathcal{X} \ni x \mapsto T_x$ are a.s. continuous.

In this work the minimal regularity assumption for the paths of a random field $T$ is the a.s. square integrability. From now on, $\mathbb{E}$ shall stand for the expectation under the probability measure $\mathbb{P}$.

Definition 1.3.4. We say that the random field $T$ on the $G$-homogeneous space $\mathcal{X}$ is

(i) a.s. square integrable if

$$\int_{\mathcal{X}} |T_x|^2 \, dx < +\infty \text{ a.s.}$$

(ii) mean square integrable if

$$\mathbb{E} \left[ \int_{\mathcal{X}} |T_x|^2 \, dx \right] < +\infty .$$

Note that the mean square integrability implies the a.s. square integrability.

Remark 1.3.5. If $T$ is a.s. square integrable, it can be regarded to as a random variable taking a.s. its values in $L^2(\mathcal{X})$ i.e. $T(\cdot) = (x \mapsto T_x(\cdot))$. 

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We define now the notion of isotropy. Let $T$ be a.s. square integrable. For every $f \in L^2(\mathcal{X})$, we can consider the integral

$$T(f) := \int_{\mathcal{X}} T_{x} f(x) \, dx$$

which defines a r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$. For every $g \in G$, let $T^g$ be the rotated field defined as

$$T^g_x := T_{gx}, \quad x \in \mathcal{X}.$$ 

Losely speaking, $T$ is isotropic if its law and the law of the rotated field $T^g$ coincide for every $g \in G$. We give the following formal definition of isotropy (see [41, 8, 9])

**Definition 1.3.6.** An a.s. square integrable random field $T$ on the homogeneous space $\mathcal{X}$ is said to be (strict sense) $G$-invariant or isotropic if the joint laws of

$$(T(f_1), \ldots, T(f_m)) \quad \text{and} \quad (T(L_g f_1), \ldots, T(L_g f_m)) = (T^g(f_1), \ldots, T^g(f_m)) \quad (1.3.24)$$

coincide for every $g \in G$ and $f_1, f_2, \ldots, f_m \in L^2(\mathcal{X})$.

This definition is somehow different from the one usually considered in the literature, where the requirement is the equality of the finite dimensional distributions, i.e. that the random vectors

$$(T_{x_1}, \ldots, T_{x_m}) \quad \text{and} \quad (T_{gx_1}, \ldots, T_{gx_m}) \quad (1.3.25)$$

have the same law for every choice of $g \in G$ and $x_1, \ldots, x_m \in \mathcal{X}$. Remark that (1.3.24) implies (1.3.25) (see [41]) and that, conversely, by standard approximation arguments (1.3.25) implies (1.3.24) if $T$ is continuous.

We see now how the Peter-Weyl decomposition naturally applies to random fields. It is worth remarking that every a.s. square integrable random field $T$ on $\mathcal{X}$ uniquely defines an a.s. square integrable random field on $G$ (whose paths are the pullback functions of the paths $x \mapsto T_x$). Therefore w.l.o. we can investigate the case $\mathcal{X} = G$. 

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Sec. 1.3 - Isotropic random fields

To every a.s. square integrable random field $T$ on $G$ we can associate the set of operator-valued r.v.’s $(\hat{T}(\sigma))_{\sigma \in \hat{G}}$ defined “pathwise” as

$$\hat{T}(\sigma) = \sqrt{\dim \sigma} \int_G T_g D^\sigma(g^{-1}) \, dg . \quad (1.3.26)$$

From (1.2.8) therefore

$$T_g = \sum_{\sigma \in \hat{G}} T^\sigma_g \quad (1.3.27)$$

where the convergence takes place in $L^2(G)$ a.s. Remark that

$$T^\sigma_g := \sqrt{\dim \sigma \tr(\hat{T}(\sigma)D^\sigma(g))} = \sqrt{\dim \sigma} \sum_{i,j=1}^{\dim \sigma} \hat{T}(\sigma)_{j,i} D^\sigma_{i,j}(g) , \quad (1.3.28)$$

is the projection of $T$ on $H_\sigma$ and $T^\sigma$ is continuous.

For the proof of the following see [9].

**Proposition 1.3.7.** Let $T$ be an a.s. square integrable random field on $G$. Then $T$ is isotropic if and only if, for every $g \in G$, the two families of r.v’s

$$(\hat{T}(\sigma))_{\sigma \in \hat{G}} \quad \text{and} \quad (\hat{T}(\sigma)D^\sigma(g))_{\sigma \in \hat{G}}$$

are equi-distributed.

If the random field $T$ is second order and isotropic (so that (1.3.23) holds by a standard Fubini argument), it is possible to say more about the convergence of the series in (1.3.27).

Indeed we have the following result, that we reproduce here from [40].

**Theorem 1.3.8.** Let $T$ be a second order and isotropic random field on the compact group $G$. Then

$$T = \sum_{\sigma \in \hat{G}} T^\sigma . \quad (1.3.29)$$
The convergence of the infinite series is both in the sense of $L^2(\Omega \times G, \mathbb{P} \otimes dg)$ and $L^2(\mathbb{P})$ for every fixed $g$, that is, for any enumeration $\{\sigma_k : k \geq 1\}$ of $\hat{G}$, we have both

$$\lim_{N \to +\infty} E\left[ \int_G \left| T_g - \sum_{k=1}^{N} T_{g}^{\sigma_k} \right|^2 dg \right] = 0 , \quad (1.3.30)$$

$$\lim_{N \to +\infty} E\left[ \left| T_g - \sum_{k=1}^{N} T_{g}^{\sigma_k} \right|^2 \right] = 0 . \quad (1.3.31)$$

The previous theorem has the following interesting consequence (for a proof see [41]).

**Proposition 1.3.9.** Every second order and isotropic random field $T$ on the homogeneous space $\mathcal{X}$ of a compact group is mean square continuous, i.e.

$$\lim_{y \to x} E[|T_y - T_x|^2] = 0 . \quad (1.3.32)$$

It is worth remarking some features of Fourier coefficients of a second order and isotropic random field $T$ (see [9]).

**Theorem 1.3.10.** If $\sigma \in \hat{G}$ is not the trivial representation, then

$$E[\hat{T}(\sigma)] = 0 ,$$

moreover for $\sigma_1, \sigma_2 \in \hat{G}$, we have

(i) if $\sigma_1, \sigma_2$ are not equivalent, the r.v.’s $\hat{T}(\sigma_1)_{i,j}$ and $\hat{T}(\sigma_2)_{k,l}$ are orthogonal for $i, j = 1, \ldots, \dim \sigma_1$ and $k, l = 1, \ldots, \dim \sigma_2$;

(ii) if $\sigma_1 = \sigma_2 = \sigma$, and $\Gamma(\sigma) = E[\hat{T}(\sigma)\hat{T}(\sigma)^*]$, then Cov $(\hat{T}(\sigma)_{i,j}, \hat{T}(\sigma)_{k,l}) = \delta^j_l \Gamma(\sigma)_{i,k}$.

In particular coefficients belonging to different columns are orthogonal and the covariance between entries in different rows of a same column does not depend on the column.

Theorem 1.3.10 states that the entries of $\hat{T}(\sigma)$ might not be pairwise orthogonal and this happens when the matrix $\Gamma$ is not diagonal. This phenomenon is actually already
been remarked by other authors (see [39] e.g.). Of course there are situations in which orthogonality is still guaranteed: when the dimension of $H_{\sigma,0}$ is one at most (i.e. in every irreducible $G$-module the dimension of the space $H_{\sigma,0}$ of the $K$-invariant vectors in one at most) as is the case for $G = SO(m + 1)$, the special orthogonal group of dimension $m + 1$, $K = SO(m)$ and $G/K \cong \mathbb{S}^m$ the $m$-dimensional unit sphere. In this case actually the matrix $\hat{T}(\sigma)$ has just one row that does not vanish and $\Gamma(\sigma)$ is all zeros, but one entry in the diagonal.

Let us now focus on Gaussian fields, which will receive the greatest attention in this work. First it is useful to recall the following.

**Definition 1.3.11.** Let $Z = Z_1 + iZ_2$ be a complex random variable (we mean that $Z_1, Z_2$ are real random variables). We say that

- $Z$ is a complex-valued Gaussian random variable if $(Z_1, Z_2)$ are jointly Gaussian;
- $Z$ is a complex Gaussian random variable if $Z_1, Z_2$ are independent Gaussian random variables with the same variance.

Furthermore we say that the random vector $(Y_1, Y_2, \ldots, Y_m)$ is a complex (resp. complex-valued) Gaussian vector if

$$\sum_i a_i Y_i$$

is a complex (resp. complex-valued) Gaussian random variable for every choice of $a_1, a_2, \ldots, a_m \in \mathbb{C}$.

From this definition it follows that if $T$ is complex-valued Gaussian, meaning that the r.v. $T(f)$ is complex-valued Gaussian for every $f \in L^2(\mathcal{X})$, then its Fourier coefficients are complex-valued Gaussian r.v.’s. Furthermore, if each representation of $G$ occurs at most once in the Peter-Weyl decomposition of $L^2(\mathcal{X})$ and $T$ is Gaussian and isotropic, we have that these Fourier coefficients are pairwise independent from Theorem 1.3.10. This is the case for instance for $G = SO(m + 1)$ and $\mathcal{X} = \mathbb{S}^m$. 
In [6] a characterization of isotropic real Gaussian fields on homogeneous spaces of compact groups is given: under some mild additional assumption also the converse is true, namely that if a random field is isotropic and its Fourier coefficients are independent, then it is necessarily Gaussian. For more discussions on this topic see also [9].

**Isotropic spherical random fields**

Let us consider a random field $T = (T_x)_{x \in \mathbb{S}^2}$ on $\mathbb{S}^2$ according to Definition (1.3.1). We assume that $T$ is a.s. square integrable. From previous sections, $T$ admits the following stochastic Fourier expansion

$$T_x = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}(x)$$

(1.3.33)

where $a_{\ell,m} = \int_{\mathbb{S}^2} T_x \overline{Y_{\ell,m}(x)} \, dx$ are the Fourier coefficients w.r.t. the basis of spherical harmonics and the convergence is in the sense of $L^2(\mathbb{S}^2)$ a.s.

If the random field $T$ is in addition second order and isotropic, Theorem (1.3.8) states that the convergence of the series in (1.3.33) holds both in the sense of $L^2(\Omega \times \mathbb{S}^2, \mathbb{P} \otimes dx)$ and $L^2(\mathbb{P})$ for every fixed $x$, and furthermore, Corollary 1.3.9 states that $T$ is mean square continuous.

Moreover from Theorem 1.3.10 we obtain easily

$$\mathbb{E}(a_{\ell,m}) = 0 \text{ for every } m = -\ell, \ldots, \ell \text{ and } \ell > 0$$

(1.3.34)

so that $\mathbb{E}(T_x) = \mathbb{E}(a_{0,0})/\sqrt{4\pi}$, as $Y_{0,0} = 1/\sqrt{4\pi}$, according to the fact that the mean of an isotropic random field is constant. If $c$ is any additive constant, the random field $T^c := T + c$ has the same Fourier expansion as $T$, except for the term $a_{0,0}^c Y_{0,0} = c + a_{0,0} Y_{0,0}$ because for every $\ell > 1$ the spherical harmonics $Y_{\ell,m}$ are orthogonal to the constants. In what follows we often consider *centered* isotropic random fields, this is generally done by ensuring that also the trivial coefficient $a_{0,0}$ is a centered random
variable. However we will often require that $a_{0,0} = 0$, i.e., the average of the random field vanishes on $\mathbb{S}^2$:

$$\int_{\mathbb{S}^2} T_x \, dx = 0 \quad \text{.} \quad \text{(1.3.35)}$$

As in the Peter-Weyl decomposition of $L^2(\mathbb{S}^2)$ two irreducible representations with $\ell \neq \ell'$ are not equivalent, the random coefficients $a_{\ell,m}, m = -\ell, \ldots, \ell$ are pairwise orthogonal and moreover the variance of $a_{\ell,m}$ does not depend on $m$. We denote

$$c_\ell := \mathbb{E}[|a_{\ell,m}|^2]$$

the variance of $a_{\ell,m}$. The (nonnegative) sequence $(c_\ell)_\ell$ is known as the \textit{angular power spectrum} of the field.

It turns out that $T$ is Gaussian and isotropic if and only $a_{\ell,m}$ are Gaussian independent random variables.

In this case, from (1.3.33), setting

$$T_\ell(x) := \sum_{m=-\ell}^{\ell} \frac{a_{\ell,m}}{\sqrt{c_\ell}} Y_{\ell,m}(x)$$

we can write

$$T_x = \sum_\ell \sqrt{c_\ell} T_\ell(x) \quad ,$$

where $T_\ell$ is known as the $\ell$-th Gaussian eigenfunctions on $\mathbb{S}^2$ or random spherical harmonics (see (4.2.15) for further details).

\section*{1.4 Positive definite functions}

To every second order random field $T$ one can associate the \textit{covariance kernel} $R : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{C}$ defined as

$$R(x,y) = \text{Cov} (T_x, T_y) \quad .$$
This kernel is positive definite, as, for every choice of \( x_1, \ldots, x_m \in \mathcal{X} \) and of \( \xi \in \mathbb{C}^m \) we have
\[
\sum_{i,j=1}^{m} R(x_i, x_j) \xi_i \xi_j = \sum_{i,j=1}^{m} \text{Cov} (T_{x_i}, T_{x_j}) \xi_i \xi_j = \text{Var} \left( \sum_i T_{x_i} \xi_i \right) \geq 0 .
\]
If in addition \( T \) is isotropic we have, for every \( g \in G \),
\[
R(gx, gy) = R(x, y)
\]
and, in this case, \( R \) turns out to be continuous, thanks to proposition (1.3.9). Moreover to this kernel one can associate the function on \( G \)
\[
\phi(g) := R(gx_0, x_0) . \tag{1.4.36}
\]
This function \( \phi \) is

- continuous, as a consequence of the continuity of \( R \).
- bi-\( K \)-invariant i.e. for every \( k_1, k_2 \in K \) and \( g \in G \) we have
  \[
  \phi(k_1 g k_2) = R(k_1 g k_2 x_0, x_0) = R(k_1 g x_0, x_0) = R(g x_0, k_1^{-1} x_0) = R(g x_0, x_0) = R(g x_0, x_0) = \phi(g)
  \]
  \( \text{where} \ k_2 x_0 = x_0 \quad \text{if} \quad k_2 \in K \quad \text{and} \quad R \text{ is } G \)-invariant.
- positive definite, actually as \( R \) is a positive definite kernel, for every \( g_1, \ldots, g_m \in G \) and \( \xi_1, \ldots, \xi_m \in \mathbb{C} \) we have
  \[
  \sum_{i,j} \phi(g_i^{-1} g_j) \xi_i \xi_j = \sum_{i,j} R(g_i^{-1} g_j x_0, x_0) \xi_i \xi_j = \sum_{i,j} R(g_j x_0, g_i x_0) \xi_i \xi_j \geq 0 . \tag{1.4.37}
  \]
By standard approximation arguments (1.4.37) imply that for every continuous function \( f \) we have
\[
\int_G \int_G \phi(h^{-1} g) f(h) \overline{f(g)} \, dg \, dh \geq 0 . \tag{1.4.38}
\]
Moreover $\phi$ determines the covariance kernel $R$ by observing that if $g_x x_0 = x, g_y x_0 = y$, then
\[ R(x, y) = R(g_x x_0, g_y x_0) = R(g_y^{-1} g_x x_0, x_0) = \phi(g_y^{-1} g_x) . \]

Now it is useful to introduce the following functions and their properties.

**Definition 1.4.1.** Let $\zeta$ be a function defined on $G$. We denote by $\hat{\zeta}$ the function
\[ \hat{\zeta}(g) := \overline{\zeta(g^{-1})} \]

**Remark 1.4.2.** We have just defined a map
\[ \zeta \mapsto \hat{\zeta} \] (1.4.39)
that is an *involution* of the convolution algebra $L^2(G)$ that becomes an $H^*$-algebra. $L^2(G)$ is known as the *group algebra* of $G$.

**Remark 1.4.3.** If $\zeta \in L^2(G)$, then for every $\sigma \in \hat{G}$ we have
\[ \hat{\zeta}(\sigma) = \hat{\zeta}(\sigma)^* . \]

Actually,
\[ \hat{\zeta}(\sigma) = \int_G \zeta(g) D^\sigma(g^{-1}) \, dg = \int_G \overline{\zeta(g^{-1})} D^\sigma(g^{-1}) \, dg . \]

Thus, for every $v \in H_\sigma$,
\[ \langle \hat{\zeta}(\sigma) v, v \rangle = \int_G \overline{\zeta(g^{-1})} \langle D^\sigma(g^{-1}) v, v \rangle \, dg = \]
\[ = \int_G \overline{\zeta(g^{-1})} \langle v, D^\sigma(g) v \rangle \, dg = \int_G \overline{\zeta(g^{-1})} \langle D^\sigma(g) v, v \rangle \, dg = \]
\[ = \langle \zeta(\sigma) v, v \rangle = \langle v, \zeta(\sigma) v \rangle . \]

Remark that every positive definite function $\phi$ on $G$ (see [63] p.123) satisfies
\[ \hat{\phi} = \phi . \]

The following proposition states some (not really unexpected) properties of continuous positive definite functions that we shall need later.
Proposition 1.4.4. Let $\phi$ a continuous positive function and $\sigma \in \hat{G}$.

a) Let, $\phi(\sigma) : H_\sigma \to H_\sigma$ the operator coefficient $\phi(\sigma) = \int_G \phi(g)D^\sigma(g^{-1})\,dg$. Then $\phi(\sigma)$ Hermitian positive definite.

b) Let $\phi^\sigma : G \to \mathbb{C}$ the component of $\phi$ corresponding to $\sigma$. Then $\phi^\sigma$ is also a positive definite function.

Proof. a) Let us fix a basis $v_1, \ldots, v_{d_\sigma}$ of $H_\sigma$, we have

$$
\langle \hat{\phi}(\sigma)v, v \rangle = \int_G \phi(g)\langle D^\sigma(g^{-1})v, v \rangle \,dg.
$$

(1.4.40)

By the invariance of the Haar measure

$$
\int_G \phi(g)\langle D^\sigma(g^{-1})v, v \rangle \,dg = \int_G \int_G \phi(h^{-1}g)\langle D^\sigma(g^{-1}h)v, v \rangle \,dg \,dh =
$$

$$
= \int_G \int_G \phi(h^{-1}g)\langle D^\sigma(h)v, D^\sigma(g)v \rangle \,dg \,dh =
$$

$$
= \int_G \int_G \phi(h^{-1}g) \sum_k \langle D^\sigma(h)v_k, D^\sigma(g)v_k \rangle \,dg \,dh =
$$

$$
\sum_k \int_G \int_G \phi(h^{-1}g)f_k(h)\overline{f_k(g)} \,dg \,dh \geq 0,
$$

where we have set, for every $k$, $f_k(g) = (D^\sigma(g)v)_k$ and (1.4.38) allows to conclude. Let $\phi^\sigma$ be the projection of $\phi$ onto the $\sigma$-isotypical subspace $L^2_\sigma(G) \subset L^2(G)$.

b) The Peter-Weyl theorem states that

$$
\phi = \sum_{\sigma \in \hat{G}} \phi^\sigma,
$$

(1.4.41)

the convergence of the series taking place in $L^2(G)$.

Let $f \in L^2_\sigma(G)$ in 1.4.38 and replace $\phi$ with its Fourier series. Recall that $f$ is a continuous function. We have

$$
0 \leq \int_G \int_G \sum_{\sigma'} \phi'^\sigma(h^{-1}g)f(h)f(g) \,dg \,dh = \int_G \sum_{\sigma'} \int_G \phi'^\sigma(h^{-1}g)f(h)\overline{f(g)} \,dg \,dh.
$$

= \int_G \phi'^\sigma \overline{f \ast \phi'^\sigma} \,dg.
Now recalling that the subspaces $L^2_{\sigma'}(G)$ are pairwise orthogonal under the product of convolution, we obtain

$$f * \phi_{\sigma'} \neq 0 \iff \sigma' = \sigma.$$ 

Therefore for every $\sigma \in \hat{G}$

$$\int_G \int_G \phi_{\sigma}(h^{-1}g)f(h)\overline{f(g)}\,dg\,dh = \int_G \int_G \phi(h^{-1}g)f(h)\overline{f(g)}\,dg\,dh \geq 0 \quad (1.4.42)$$

for every $f \in L^2_{\sigma}(G)$. Let now $f \in L^2(G)$ and let $f = \sum_{\sigma'} f_{\sigma'}$ its Fourier series. The same argument as above gives

$$\int_G \int_G \phi_{\sigma}(h^{-1}g)f(h)\overline{f(g)}\,dg\,dh = \int_G \int_G \phi_{\sigma}(h^{-1}g)f_{\sigma}(h)\overline{f_{\sigma}(g)}\,dg\,dh \geq 0,$$

so that $\phi_{\sigma}$ is a positive definite function.

Another important property enjoyed by positive definite and continuous functions on $G$ is shown in the following classical theorem (see [31], Theorem 3.20, p.151).

**Theorem 1.4.5.** Let $\zeta$ be a continuous positive definite function on $G$. Let $\zeta_{\sigma}$ be the component of $\zeta$ on the $\sigma$-isotypical subspace $L^2_{\sigma}(G)$, then

$$\zeta = \sum_{\sigma \in \hat{G}} \sqrt{\dim \sigma} \text{tr} \hat{\zeta}(\sigma) < +\infty,$$

and the Fourier series

$$\zeta = \sum_{\sigma \in \hat{G}} \zeta_{\sigma}$$

converges uniformly on $G$.

**Remark 1.4.6.** This theorem is an extension of a classical result for trigonometric series: *every continuous function on the unit circle with all nonnegative Fourier coefficients has its Fourier series converging uniformly on the unit circle.*
Chapter 2

Representation of isotropic Gaussian fields

In this chapter we recollect the first part of [8]: as stated in the Introduction, starting from P. Lévy’s construction of his spherical Brownian motion, we prove a representation formula for isotropic Gaussian fields on homogeneous spaces $\mathcal{X}$ of a compact group $G$ (§2.1 and §2.2).

In particular, we show that to every square integrable bi-$K$-invariant function $f$ on $G$ a Gaussian isotropic random field on $\mathcal{X}$ can be associated and also that every real Gaussian isotropic random field on $\mathcal{X}$ can be obtained in this way.

This kind of result is extended to the case of random fields in the spin-line bundles of the sphere in the second part of [8] and will be presented in the last chapter of this thesis.

2.1 Construction of isotropic Gaussian fields

In this section we point out the method for Gaussian isotropic random fields on the homogeneous space $\mathcal{X}$ of a compact group $G$. We start with the construction of a white noise on $\mathcal{X}$. 

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Let \((X_n)_n\) be a sequence of i.i.d. standard Gaussian r.v.’s on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and denote by \(\mathcal{H} \subset L^2(\mathbb{P})\) the real Hilbert space generated by \((X_n)_n\). Let \((e_n)_n\) be an orthonormal basis of \(L^2_\mathbb{R}(\mathcal{X})\), the space of real square integrable functions on \(\mathcal{X}\). We define an isometry \(S : L^2_\mathbb{R}(\mathcal{X}) \to \mathcal{H}\) by

\[
L^2_\mathbb{R}(\mathcal{X}) \ni \sum_k \alpha_k e_k \leftrightarrow \sum_k \alpha_k X_k \in \mathcal{H}.
\]

It is easy to extend \(S\) to an isometry on \(L^2(\mathcal{X})\), indeed if \(f \in L^2(\mathcal{X})\), then

\[
f = f_1 + if_2,
\]

with \(f_1, f_2 \in L^2_\mathbb{R}(\mathcal{X})\), hence just set \(S(f) = S(f_1) + iS(f_2)\). Such an isometry respects the real character of the function \(f \in L^2(\mathcal{X})\) (i.e. if \(f\) is real then \(S(f)\) is a real r.v.).

Let \(f\) be a left \(K\)-invariant function in \(L^2(\mathcal{X})\). We then define a random field \((T^f_x)_{x \in \mathcal{X}}\) associated to \(f\) as follows: set \(T^f_x = S(L_g f)\) and, for every \(x \in \mathcal{X}\),

\[
T^f_x = S(L_g f), \quad (2.1.1)
\]

where \(g \in G\) is such that \(gx_0 = x\) (\(L\) still denotes the left regular action of \(G\)). This is a good definition: in fact if also \(\tilde{g} \in G\) is such that \(\tilde{g}x_0 = x\), then \(\tilde{g} = gk\) for some \(k \in K\) and therefore \(L_{\tilde{g}} f(x) = f(k^{-1}g^{-1}x) = f(g^{-1}x) = L_g f(x)\) so that

\[
S(L_{\tilde{g}} f) = S(L_g f).
\]

The random field \(T^f\) is mean square integrable, indeed

\[
\mathbb{E} \left[ \int_{\mathcal{X}} |T^f_x|^2 \, dx \right] < +\infty.
\]

Actually, if \(g_x\) is any element of \(G\) such that \(g_xx_0 = x\) (chosen in some measurable way), then, as \(\mathbb{E}[|T^f_x|^2] = \mathbb{E}[|S(L_{g_x} f)|^2] = \|L_{g_x} f\|^2_{L^2(\mathcal{X})} = \|f\|^2_{L^2(\mathcal{X})}\), we have

\[
\mathbb{E} \int_{\mathcal{X}} |T^f_x|^2 \, dx = \|f\|^2_{L^2(\mathcal{X})}.
\]

\(T^f\) is a centered and complex-valued Gaussian random field. Let us now check that \(T^f\) is isotropic. Recall that the law of a complex-valued Gaussian random vector \(Z = (Z_1, Z_2)\) is completely characterized by its mean value \(\mathbb{E}[Z]\), its covariance matrix \(\mathbb{E}[(Z - \mathbb{E}[Z])(Z - \mathbb{E}[Z])^*]\) and the pseudocovariance or relation matrix \(\mathbb{E}[(Z - \mathbb{E}[Z])(Z - \mathbb{E}[Z])^*]\). We have
(i) as $S$ is an isometry

\[
\mathbb{E}[T_{gx}^f T_{gy}^f] = \mathbb{E}[S(L_{gg_x} f)\overline{S(L_{gg_y} f)}] = \langle L_{gg_x} f, L_{gg_y} f \rangle_{L^2(X)} = \\
= \langle L_{g_x} f, L_{g_y} f \rangle_{L^2(X)} = \mathbb{E}[T_{x}^f T_{y}^f].
\]

(ii) Moreover, as complex conjugation commutes both with $S$ and the left regular representation of $G$,

\[
\mathbb{E}[T_{gx}^f T_{gy}^f] = \mathbb{E}[S(L_{gg_x} f)\overline{S(L_{gg_y} f)}] = \langle L_{gg_x} f, L_{gg_y} f \rangle_{L^2(X)} = \\
= \langle L_{g_x} f, L_{g_y} f \rangle_{L^2(X)} = \mathbb{E}[T_{x}^f T_{y}^f].
\]

Therefore $T^f$ is isotropic because it has the same covariance and relation kernels as the rotated field $(T^f)^g$ for every $g \in G$.

If $R^f(x, y) = \mathbb{E}[T_{x}^f T_{y}^f]$ denotes its covariance kernel, then the associated positive definite function $\phi^f(g) := R(gx_0, x_0)$ satisfies

\[
\phi^f(g) = \mathbb{E}[S(L_{g} f)\overline{S(f)}] = \langle L_{g} f, f \rangle = \\
= \int_G \tilde{f}(g^{-1} h) \overline{\tilde{f}(h)} \, dh = \int_G \tilde{f}(g^{-1} h) \tilde{\zeta}(h^{-1}) \, dh = \tilde{f} * \tilde{\zeta}(g^{-1}),
\]

where $\tilde{f}$ is the pullback on $G$ of $f$ and the convolution $*$ is in $G$. Moreover the relation function of $T^f$, that is $\zeta^f(g) := \mathbb{E}[T_{gx_0}^f T_{x_0}^f]$ satisfies

\[
\zeta^f(g) = \mathbb{E}[S(L_{g} f)S(f)] = \langle L_{g} f, f \rangle.
\]

One may ask whether every a.s. square integrable, isotropic, complex-valued Gaussian centered random field on $\mathcal{X}$ can be obtained with this construction: the answer is no in general. It is however positive if we consider real isotropic Gaussian random fields (see Theorem 2.2.3 below). Before considering the case of a general homogeneous space $\mathcal{X}$, let us look first at the case of the sphere, where things are particularly simple.

**Remark 2.1.1.** (Representation of real Gaussian isotropic random fields on $S^2$) If $\mathcal{X} = S^2$ under the action of $SO(3)$, every isotropic, real Gaussian and centered random
field is of the form (2.1.1) for some left-$K$-invariant function $f : S^2 \to \mathbb{R}$. Indeed let us consider on $L^2(S^2)$ the Fourier basis $Y_{\ell,m}$, $\ell = 1, 2, \ldots$, $m = -\ell, \ldots, \ell$, given by the spherical harmonics (1.2.18). Every continuous positive definite left-$K$-invariant function $\phi$ on $S^2$ has a Fourier expansion of the form (1.2.21)

$$\phi = \sum_{\ell \geq 0} \alpha_\ell Y_{\ell,0},$$

(2.1.4)

where (Proposition 1.4.4) $\alpha_\ell \geq 0$ and

$$\sum_{\ell \geq 0} \sqrt{2\ell + 1} \alpha_\ell < +\infty$$

(Theorem 1.4.5). The $Y_{\ell,0}$’s being real, the function $\phi$ in (2.1.4) is real, so that, $\phi(g) = \phi(g^{-1})$ (in this remark and in the next example we identify functions on $S^2$ with their pullbacks on $SO(3)$ for simplicity of notations).

If $\phi$ is the positive definite left-$K$-invariant function associated to $T$, then, keeping in mind that $Y_{\ell,0} * Y_{\ell',0} = (2\ell + 1)^{-1/2} Y_{\ell,0} \delta_{\ell,\ell'}$, a “square root” $f$ of $\phi$ is given by

$$f = \sum_{\ell \geq 0} \beta_\ell Y_{\ell,0},$$

(2.1.5)

where $\beta_\ell$ is a complex number such that

$$\frac{|\beta_\ell|^2}{\sqrt{2\ell + 1}} = \sqrt{\alpha_\ell}.$$  

Therefore there exist infinitely many real functions $f$ such that $\phi(g) = \phi(g^{-1}) = f * f(g)$, corresponding to the choices $\beta_\ell = \pm((2\ell + 1)\alpha_\ell)^{1/4}$. For each of these, the random field $T^f$ has the same distribution as $T$, being real and having the same associated positive definite function.

As stated in the Introduction, this method generalizes P. Lévy’s construction of his spherical Brownian motion. In the following example, we show the connection between this construction and our method. Moreover, it is easy to extend the following to the case of the hyperspherical Brownian motion.
Example 2.1.2. (P. Lévy’s spherical Brownian field). Let us choose as a particular instance of the previous construction $f = c1_H$, where $H$ is the half-sphere centered at the north pole $x_0$ of $S^2$ and $c$ is some constant to be chosen later.

Still denoting by $S$ a white noise on $S^2$, from (2.1.1) we have

$$T^f_x = cS(1_{H_x}),$$

(2.1.6)

where $1_{H_x}$ is the half-sphere centered at $x \in S^2$. Now, let $x, y \in S^2$ and denote by $d(x, y) = \theta$ their distance, then, $S$ being an isometry,

$$\text{Var}(T^f_x - T^f_y) = c^2 \|1_{H_x \triangle H_y}\|^2 .$$

(2.1.7)

The symmetric difference $H_x \triangle H_y$ is formed by the union of two wedges whose total surface is equal to $\frac{\theta}{\pi}$ (recall that we consider the surface of $S^2$ normalized with total mass = 1). Therefore, choosing $c = \sqrt{\pi}$, we have

$$\text{Var}(T^f_x - T^f_y) = d(x, y)$$

(2.1.8)

and furthermore $\text{Var}(T^f_x) = \frac{\pi}{2}$. Thus

$$\text{Cov}(T^f_x, T^f_y) = \frac{1}{2} \left( \text{Var}(T^f_x) + \text{Var}(T^f_y) - \text{Var}(T^f_x - T^f_y) \right) = \frac{\pi}{2} - \frac{1}{2}d(x, y) .$$

(2.1.9)

Note that the positive definiteness of (2.1.9) implies that the distance $d$ is a Schoenberg restricted negative definite kernel on $S^2$ (see (3.1.3)). The random field $W$

$$W_x := T^f_x - T^f_o ,$$

(2.1.10)

where $o$ denotes the north pole of the sphere is $P. Lévy’s spherical Brownian field$, as $W_o = 0$ and its covariance kernel is

$$\text{Cov}(W_x, W_y) = \frac{1}{2} \left( d(x, o) + d(y, o) - d(x, y) \right) .$$

(2.1.11)

In particular the kernel at the r.h.s. of (2.1.11) is positive definite (see also [31]). Let us compute the expansion into spherical harmonics of the positive definite function $\phi$ associated to the random field $T^f$ and to $f$. We have $\phi(x) = \frac{\pi}{2} - \frac{1}{2}d(x, o)$, i.e. $\phi(x) = \frac{\pi}{2} - \frac{1}{2} \vartheta$ in spherical coordinates, $\vartheta$ being the colatitude of $x$, whereas $Y_{\ell,0}(x) = \frac{\pi}{2} - \frac{1}{2} \vartheta$. 

(2.1.11)
\[ \sqrt{2\ell + 1} P_\ell(\cos \vartheta) \] where \( P_\ell \) is the \( \ell \)-th Legendre polynomial. This formula for the central spherical harmonics differs slightly from the usual one, as we consider the total measure of \( \mathbb{S}^2 \) to be \( = 1 \). Then, recalling the normalized measure of the sphere is \( \frac{1}{4\pi} \sin \vartheta \, d\vartheta \, d\varphi \) and that \( Y_{\ell,0} \) is orthogonal to the constants

\[ \int_{\mathbb{S}^2} \phi(x) Y_{\ell,0}(x) \, dx = -\frac{1}{4} \sqrt{2\ell + 1} \int_0^\pi \vartheta P_\ell(\cos \vartheta) \sin \vartheta \, d\vartheta = -\frac{1}{4} \sqrt{2\ell + 1} \int_{-1}^1 \arccos t P_\ell(t) \, dt = \frac{1}{4} \sqrt{2\ell + 1} c_\ell , \]

where

\[ c_\ell = \pi \left\{ \frac{3 \cdot 5 \cdots (\ell - 2)}{2 \cdot 4 \cdots (\ell + 1)} \right\}^2 \quad \ell = 1, 3, \ldots \]

and \( c_\ell = 0 \) for \( \ell \) even (see [67], p. 325). As for the function \( f = \sqrt{\pi} 1_H \), we have

\[ \int_{\mathbb{S}^2} f(x) Y_{\ell,0}(x) \, dx = \frac{\sqrt{\pi}}{2} \sqrt{2\ell + 1} \int_0^{\pi/2} P_\ell(\cos \vartheta) \sin \vartheta \, d\vartheta = \frac{\sqrt{\pi}}{2} \sqrt{2\ell + 1} \int_{-1}^1 P_\ell(t) \, dt . \]

The r.h.s. can be computed using Rodriguez formula for the Legendre polynomials (see again [67], p. 297) giving that it vanishes for \( \ell \) even and equal to

\[ (-1)^{m+1} \frac{\sqrt{\pi}}{2} \sqrt{2\ell + 1} \frac{(2m)!(2m+1)}{2^{2m+1}(2m+1)!} \quad (2.1.12) \]

for \( \ell = 2m + 1 \). Details of this computation are given in Remark 2.1.3. Simplifying the factorials the previous expression becomes

\[ (-1)^m \frac{\sqrt{\pi}}{2} \sqrt{2\ell + 1} \frac{(2m)!}{2^{2m+1}m!(m+1)!} = (-1)^m \frac{\sqrt{\pi}}{2} \sqrt{2\ell + 1} \frac{3 \cdots (2m-1)}{2 \cdots (2m+2)} =
\]

\[ = (-1)^m \frac{1}{2} \sqrt{2\ell + 1} \sqrt{c_\ell} . \]

Therefore the choice \( f = \sqrt{\pi} 1_H \) corresponds to taking alternating signs when taking the square roots. Note that the choice \( f' = \sum \beta_\ell Y_{\ell,0} \) with \( \beta_\ell = \frac{1}{2} \sqrt{2\ell + 1} \sqrt{c_\ell} \) would have given a function diverging at the north pole \( o \). Actually it is elementary to check that the series \( \sum \beta_\ell (2\ell + 1) \sqrt{c_\ell} \) diverges so that \( f' \) cannot be continuous by Theorem 1.4.5.\]
Remark 2.1.3. Rodriguez formula for the Legendre polynomials states that

\[ P_\ell(x) = \frac{1}{2\ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)\ell. \]

Therefore

\[ \int_0^1 P_\ell(x) \, dx = \frac{1}{2\ell!} \frac{d^{\ell-1}}{dx^{\ell-1}} (x^2 - 1)\ell \bigg|_0^1. \quad (2.1.13) \]

The primitive vanishes at 1, as the polynomial \((x^2 - 1)^\ell\) has a zero of order \(\ell\) at \(x = 1\) and all its derivatives up to the order \(\ell - 1\) vanish at \(x = 1\). In order to compute the primitive at 0 we make the binomial expansion of \((x^2 - 1)^\ell\) and take the result of the \((\ell - 1)\)-th derivative of the term of order \(\ell - 1\) of the expansion. This is actually the term of order 0 of the primitive. If \(\ell\) is even then \(\ell - 1\) is odd so that this term of order \(\ell - 1\) does not exist (in the expansion only even powers of \(x\) can appear). If \(\ell = 2m + 1\), then the term of order \(\ell - 1 = 2m\) in the expansion is

\[ (-1)^m \binom{2m + 1}{m} z^{2m} \]

and the result of the integral in (2.1.13) is actually, as given in (2.1.12),

\[ (-1)^{m+1} \frac{(2m)!}{2^{2m+1}(2m + 1)!} \binom{2m + 1}{m}. \]

\[ \square \]

2.2 Representation formula

The result of Remark 2.1.1 concerning \(\mathbb{S}^2\) can be extended to the case of a general homogeneous space \(\mathcal{X}\). We shall need the following “square root” theorem in the proof of the representation formula of Gaussian isotropic random fields on \(\mathcal{X}\).

Theorem 2.2.1. Let \(\phi\) be a bi-\(K\)-invariant positive definite continuous function on \(G\). Then there exists a bi-\(K\)-invariant function \(f \in L^2(G)\) such that \(\phi = f * \tilde{f}\). Moreover, if \(\phi\) is real valued then \(f\) also can be chosen to be real valued.
Proof. For every \( \sigma \in \hat{G} \), \( \hat{\phi}(\sigma) \) is Hermitian positive definite. Therefore there exist matrices \( \Lambda(\sigma) \) such that \( \Lambda(\sigma)\Lambda(\sigma)^* = \hat{\phi}(\sigma) \). Let

\[
  f = \sum_{\sigma \in \hat{G}} \sqrt{\dim \sigma} \text{tr}(\Lambda(\sigma)D^\sigma) .
\]

This actually defines a function \( f \in L^2(G) \) as it is easy to see that

\[
  \|f^\sigma\|^2 = \sum_{i,j=1}^{\dim \sigma} (\Lambda(\sigma)_{ij})^2 = \text{tr}(\Lambda(\sigma)\Lambda(\sigma)^*) = \text{tr}(\hat{\phi}(\sigma))
\]

so that

\[
  \|f\|^2 = \sum_{\sigma \in \hat{G}} \|f^\sigma\|^2 = \sum_{\sigma \in \hat{G}} \text{tr}(\hat{\phi}(\sigma)) < +\infty
\]

thanks to (1.4.5). By Remark 1.4.3 and (1.2.7), we have

\[
  \phi = f^* \hat{f} .
\]

Finally the matrix \( \Lambda(\sigma) \) can be chosen to be Hermitian and with this choice \( f \) is bi-\( K \)-invariant as the relation (1.2.13) \( \hat{f}(\sigma) = P_{\sigma,0}f(\sigma)P_{\sigma,0} \) still holds. The last statement follows from next proposition.

**Proposition 2.2.2.** Let \( \phi \) be a real positive definite function on a compact group \( G \), then there exists a real function \( f \) such that \( \phi = f^* \hat{f} \).

**Proof.** Let

\[
  \phi(g) = \sum_{\sigma \in \hat{G}} \phi^\sigma(g) = \sum_{\sigma \in \hat{G}} \sqrt{\dim \sigma} \text{tr}(\hat{\phi}(\sigma)D^\sigma(g))
\]

be the Peter-Weyl decomposition of \( \phi \) into isotypical components. We know that the Hermitian matrices \( \hat{\phi}(\sigma) \) are positive definite, so that there exist square roots \( \hat{\phi}(\sigma)^{1/2} \) i.e. matrices such that \( \hat{\phi}(\sigma)^{1/2}\hat{\phi}(\sigma)^{1/2*} = \hat{\phi}(\sigma) \) and the functions

\[
  f(g) = \sum_{\sigma \in \hat{G}} \sqrt{\dim \sigma} \text{tr}(\hat{\phi}(\sigma)^{1/2}D^\sigma(g))
\]
are such that \( \phi = f \ast \tilde{f} \). We need to prove that these square roots can be chosen in such a way that \( f \) is also real. Recall that a representation of a compact group \( G \) can be classified as being of real, complex or quaternionic type (see [66], p. 93 e.g. for details).

a) If \( \sigma \) is of real type then there exists a conjugation \( J \) of \( H_\sigma \subset L^2(G) \) such that \( J^2 = 1 \). A conjugation is a \( G \)-equivariant antilinear endomorphism. It is well known that in this case one can choose a basis \( v_1, \ldots, v_{d_\sigma} \) of \( H_\sigma \) formed of “real” vectors, i.e. such that \( Jv_i = v_i \). It is then immediate that the representative matrix \( D^\sigma \) of the action of \( G \) on \( H_\sigma \) is real. Actually, as \( J \) is equivariant and \( Jv_i = v_i \),

\[
D^\sigma_{ij}(g) = \langle gv_j, v_i \rangle = \langle Jgv_j, Jv_i \rangle = \langle gv_j, v_i \rangle = D^\sigma_{ij}(g).
\]

With this choice of the basis, the matrix \( \hat{\phi}(\sigma) \) is real and also \( \hat{\phi}(\sigma)^{1/2} \) can be chosen to be real and \( g \mapsto \sqrt{\dim \sigma \text{ tr}(\hat{\phi}(\sigma)^{1/2}D^\sigma(g))} \) turns out to be real itself.

b) If \( \sigma \) is of complex type, then it is not isomorphic to its dual representation \( \sigma^* \). As \( D^{\sigma^*}(g) := D^\sigma(g^{-1})^t = D^\sigma(g) \) and \( \phi \) is real-valued, we have

\[
\hat{\phi}(\sigma^*) = \overline{\hat{\phi}(\sigma)},
\]

so that we can choose \( \hat{\phi}(\sigma^*)^{1/2} = \overline{\hat{\phi}(\sigma)^{1/2}} \) and, as \( \sigma \) and \( \sigma^* \) have the same dimension, the function

\[
g \mapsto \sqrt{\dim \sigma \text{ tr}(\hat{\phi}(\sigma)^{1/2}D^\sigma(g))} + \sqrt{\dim \sigma^* \text{ tr}(\hat{\phi}(\sigma^*)^{1/2}D^{\sigma^*}(g))}
\]

turns out to be real.

c) If \( \sigma \) is quaternionic, let \( J \) be the corresponding conjugation. It is immediate that the vectors \( v \) and \( Jv \) are orthogonal and from this it follows that \( \dim \sigma = 2k \) and that there exists an orthogonal basis for \( H_\sigma \) of the form

\[
v_1, \ldots, v_k, w_1 = J(v_1), \ldots, w_k = J(v_k).
\]

(2.2.14)

In such a basis the representation matrix of any linear transformation \( U : H_\sigma \to H_\sigma \) which commutes with \( J \) has the form

\[
\begin{pmatrix}
A & B \\
-B & A
\end{pmatrix}
\]

(2.2.15)
and in particular $D^\sigma(g)$ takes the form
\[
D^\sigma(g) = \begin{pmatrix} A(g) & B(g) \\ -B(g) & A(g) \end{pmatrix}.
\] (2.2.16)

By (2.2.16) we have also, $\phi$ being real valued,
\[
\hat{\phi}(\sigma) = \begin{pmatrix} \int_G \phi(g) A(g^{-1}) \, dg & \int_G \phi(g) B(g^{-1}) \, dg \\ -\int_G \phi(g) B(g^{-1}) \, dg & \int_G \phi(g) A(g^{-1}) \, dg \end{pmatrix} := \begin{pmatrix} \phi_A & \phi_B \\ -\phi_B & \phi_A \end{pmatrix}.
\] (2.2.17)

More interestingly, if $\phi$ is any function such that, with respect to the basis above, $\hat{\phi}(\sigma)$ is of the form (2.2.17), then the corresponding component $\phi^\sigma$ is necessarily a real valued function: actually
\[
\phi^\sigma(g) = \text{tr}(\hat{\phi}(\sigma) D^\sigma(g)) = \text{tr}(\phi_A A(g) - \phi_B B(g) + \phi_B B(g) + \phi_A A(g)) = \\
= \text{tr}(\phi_A A(g) + \overline{\phi_A A(g)}) - \text{tr}(\phi_B B(g) + \overline{\phi_B B(g)}).
\]

We now prove that the Hermitian square root, $U$ say, of $\hat{\phi}(\sigma)$ is of the form (2.2.17). Actually note that $\hat{\phi}(\sigma)$ is self-adjoint, so that it can be diagonalized and all its eigenvalues are real (and positive by Proposition 1.4.4 a)). Let $\lambda$ be an eigenvalue and $v$ a corresponding eigenvector. Then, as $\hat{\phi}(\sigma) Jv = J \hat{\phi}(\sigma) = J \lambda v = \lambda v$, $Jv$ is also an eigenvector associated to $\lambda$. Therefore there exists a basis as in (2.2.14) that is formed of eigenvectors, i.e. of the form $v_1, \ldots, v_k, w_1, \ldots, w_k$ with $Jv_j = w_j$ and $v_j$ and $w_j$ associated to the same positive eigenvalue $\lambda_j$. In this basis $\hat{\phi}(\sigma)$ is of course diagonal with the (positive) eigenvalues on the diagonal. Its Hermitian square root $U$ is also diagonal, with the square roots of the eigenvalues on the diagonal. Therefore $U$ is also the form (2.2.17) and the corresponding function $\psi(g) = \text{tr}(UD(g))$ is real valued and such that $\psi * \tilde{\psi} = \phi^\sigma$.

Note that the decomposition of Theorem 2.2.1 is not unique, as the Hermitian square root of the positive definite operator $\hat{\phi}(\sigma)$ is not unique itself. Now we prove the main result of this chapter.
Theorem 2.2.3. Let \( \mathcal{X} \) be the homogeneous space of a compact group \( G \) and let \( T \) be an a.s. square integrable isotropic Gaussian real random field on \( \mathcal{X} \). Then there exists a left-\( K \)-invariant function \( f \in L^2(\mathcal{X}) \) such that \( Tf \) has the same distribution as \( T \).

**Proof.** Let \( \phi \) be the invariant positive definite function associated to \( T \). Thanks to (2.1.2) it is sufficient to prove that there exists a real \( K \)-invariant function \( f \in L^2(\mathcal{X}) \) such that \( \phi(g) = \tilde{f} \ast \tilde{f}(g^{-1}) \). Keeping in mind that \( \phi(g) = \tilde{\phi}(g^{-1}) \), as \( \phi \) is real, this follows from Theorem 2.2.1. \( \square \)

As remarked above \( f \) is not unique.

Recall that a complex valued Gaussian r.v. \( Z = X + iY \) is said to be complex Gaussian if the r.v.’s \( X, Y \) are jointly Gaussian, are independent and have the same variance. A \( \mathbb{C}^m \)-valued r.v. \( Z = (Z_1, \ldots, Z_m) \) is said to be complex-Gaussian if the r.v. \( \alpha_1 Z_1 + \cdots + \alpha_m Z_m \) is complex-Gaussian for every choice of \( \alpha_1, \ldots, \alpha_m \in \mathbb{C} \).

**Remark 2.2.4.** An a.s. square integrable random field \( T \) on \( \mathcal{X} \) is complex Gaussian if and only if the complex valued r.v.’s

\[
\int_{\mathcal{X}} T_x f(x) \, dx
\]

are complex Gaussian for every choice of \( f \in L^2(\mathcal{X}) \).

Complex Gaussian random fields will play an important role in the last chapter of this work. By now let us remark that, in general, it is not possible to obtain a complex Gaussian random field by the procedure (2.1.1).

**Proposition 2.2.5.** Let \( \zeta(x,y) = \mathbb{E}[T_x T_y] \) be the relation kernel of a centered complex Gaussian random field \( T \). Then \( \zeta \equiv 0 \).

**Proof.** It easy to check that a centered complex valued r.v. \( Z \) is complex Gaussian if and only if \( \mathbb{E}[Z^2] = 0 \). As for every \( f \in L^2(\mathcal{X}) \)

\[
\int_{\mathcal{X}} \int_{\mathcal{X}} \zeta(x,y) f(x) f(y) \, dx \, dy = \mathbb{E} \left[ \left( \int_{\mathcal{X}} T_x f(x) \, dx \right)^2 \right] = 0,
\]

it is easy to derive that \( \zeta \equiv 0 \). \( \square \)
Going back to the situation of Remark 2.1.1, the relation function $\zeta$ of the random field $T^f$ is easily found to be

$$\zeta^f = \sum_{\ell \geq 0} \beta^2 \ell \ Y_{\ell,0},$$

and cannot vanish unless $f \equiv 0$ and $T^f$ vanishes itself. Therefore no isotropic complex Gaussian random field on the sphere can be obtained by the construction (2.1.1).
Chapter 3

On Lévy’s Brownian fields

In 1959 P. Lévy [37] asked the question of the existence of a random field $X$ indexed by the points of a metric space $(\mathcal{X}, d)$ and generalizing the Brownian motion, i.e. of a real Gaussian process which would be centered, vanishing at some point $x_0 \in \mathcal{X}$ and such that $\mathbb{E}(|X_x - X_y|^2) = d(x, y)$. By polarization, the covariance function of such a process would be

$$K(x, y) = \frac{1}{2} (d(x, x_0) + d(y, x_0) - d(x, y)) \quad (3.0.1)$$

so that this question is equivalent to the fact that the kernel $K$ is positive definite. As anticipated in the Introduction, $X$ is called P. Lévy’s Brownian field on $(\mathcal{X}, d)$. Positive definiteness of $K$ for $\mathcal{X} = \mathbb{R}^{m+1}$ and $d$ the Euclidean metric had been proved by [62] in 1938 and P. Lévy itself constructed the Brownian field on $\mathcal{X} = \mathbb{S}^m$, the euclidean sphere of $\mathbb{R}^{m+1}$, $d$ being the distance along the geodesics (Example 2.1.2). Later Gangolli [31] gave an analytical proof of the positive definiteness of the kernel (3.0.1) for the same metric space $(\mathbb{S}^m, d)$, in a paper that dealt with this question for a large class of homogeneous spaces.

Finally Takenaka in [65] proved the positive definiteness of the kernel (3.0.1) for the Riemannian metric spaces of constant sectional curvature equal to $-1, 0$ or $1$, therefore adding the hyperbolic disk to the list. To be precise in the case of the hyperbolic space $\mathcal{H}_m = \{(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} : x_1^2 + \ldots x_m^2 - x_0^2 = 1\}$, the distance under
consideration is the unique, up to multiplicative constants, Riemannian distance that is invariant with respect to the action of $G = L_m$, the Lorentz group.

In [7] we investigate this question for the cases $\mathcal{X} = SO(n)$. The answer is that the kernel (3.0.1) is not positive definite on $SO(n)$ for $n > 2$. This is somehow surprising as, in particular, $SO(3)$ is locally isometric to $SU(2)$, where positive definiteness of the kernel $K$ is immediate as shown below.

It is immediate that this imply the non existence on $SU(n), n > 2$.

The plan of this chapter is as follows. In §3.1 we recall some elementary facts about invariant distances and positive definite kernels. In §3.2 we treat the case $G = SU(2)$, recalling well known facts about the invariant distance and Haar measure of this group. Positive definiteness of $K$ for $SU(2)$ is just a simple remark, but these facts are needed in §3.3 where we treat the case $SO(3)$ and deduce from the case $SO(n), n \geq 3$.

### 3.1 Some elementary facts

In this section we recall some well known facts about Lie groups (see mainly [29] and also [30, 63]).

#### 3.1.1 Invariant distance of a compact Lie group

In this chapter $G$ denotes a compact Lie group. It is well known that $G$ admits at least a bi-invariant Riemannian metric (see [30] p.66 e.g.), that we shall denote $\{\langle \cdot, \cdot \rangle_g \}_{g \in G}$ where of course $\langle \cdot, \cdot \rangle_g$ is the inner product defined on the tangent space $T_g G$ to the manifold $G$ at $g$ and the family $\{\langle \cdot, \cdot \rangle_g \}_{g \in G}$ smoothly depends on $g$. By the bi-invariance property, for $g \in G$ the diffeomorphisms $L_g$ and $R_g$ (resp. the left multiplication and the right multiplication of the group) are isometries. Since the tangent space $T_g G$ at any point $g$ can be translated to the tangent space $T_e G$ at the identity element $e$ of the group, the metric $\{\langle \cdot, \cdot \rangle_g \}_{g \in G}$ is completely characterized by $\langle \cdot, \cdot \rangle_e$. Moreover, $T_e G$ being the Lie algebra $\mathfrak{g}$ of $G$, the bi-invariant metric corresponds to an inner product
\langle \cdot , \cdot \rangle$ on $\mathfrak{g}$ which is invariant under the adjoint representation $Ad$ of $G$. Indeed there is a one-to-one correspondence between bi-invariant Riemannian metrics on $G$ and $Ad$-invariant inner products on $\mathfrak{g}$. If in addition $\mathfrak{g}$ is semisimple, then the negative Killing form of $G$ is an $Ad$-invariant inner product on $\mathfrak{g}$ itself.

If there exists a unique (up to a multiplicative factor) bi-invariant metric on $G$ (for a sufficient condition see [30], Th. 2.43) and $\mathfrak{g}$ is semisimple, then this metric is necessarily proportional to the negative Killing form of $\mathfrak{g}$. It is well known that this is the case for $SO(n), (n \neq 4)$ and $SU(n)$; furthermore, the (natural) Riemannian metric on $SO(n)$ induced by the embedding $SO(n) \hookrightarrow \mathbb{R}^{n^2}$ corresponds to the negative Killing form of $so(n)$.

Endowed with this bi-invariant Riemannian metric, $G$ becomes a metric space, with a distance $d$ which is bi-invariant. Therefore the function $g \in G \rightarrow d(g, e)$ is a class function as

$$d(g, e) = d(hg, h) = d(hgh^{-1}, hh^{-1}) = d(hgh^{-1}, e), \quad g, h \in G.$$  

(3.1.2)

It is well known that geodesics on $G$ through the identity $e$ are exactly the one parameter subgroups of $G$ (see [49] p.113 e.g.), thus a geodesic from $e$ is the curve on $G$

$$\gamma_X(t) : t \in [0, 1] \rightarrow \exp(tX)$$

for some $X \in \mathfrak{g}$. The length of this geodesic is

$$L(\gamma_X) = \|X\| = \sqrt{\langle X, X \rangle}.$$

Therefore

$$d(g, e) = \inf_{X \in \mathfrak{g} : \exp X = g} \|X\|.$$

### 3.1.2 Brownian kernels on a metric space

Let $(\mathcal{X}, d)$ be a metric space.
Lemma 3.1.1. The kernel $K$ in (3.0.1) is positive definite on $\mathcal{X}$ if and only if $d$ is a restricted negative definite kernel, i.e., for every choice of elements $x_1, \ldots, x_n \in \mathcal{X}$ and of complex numbers $\xi_1, \ldots, \xi_n$ with $\sum_{i=1}^{n} \xi_i = 0$

$$\sum_{i,j=1}^{n} d(x_i, x_j) \xi_i \bar{\xi}_j \leq 0.$$ (3.1.3)

Proof. For every $x_1, \ldots, x_n \in \mathcal{X}$ and complex numbers $\xi_1, \ldots, \xi_n$

$$\sum_{i,j} K(x_i, x_j) \xi_i \bar{\xi}_j = \frac{1}{2} \left( a \sum_{i} d(x_i, x_0) \xi_i + a \sum_{j} d(x_j, x_0) \bar{\xi}_j - \sum_{i,j} d(x_i, x_j) \xi_i \bar{\xi}_j \right)$$ (3.1.4)

where $a := \sum_i \xi_i$. If $a = 0$ then it is immediate that in (3.1.4) the l.h.s. is $\geq 0$ if and only if the r.h.s. is $\leq 0$. Otherwise set $\xi_0 := -a$ so that $\sum_{i=0}^{n} \xi_i = 0$. The following equality

$$\sum_{i,j=0}^{n} K(x_i, x_j) \xi_i \bar{\xi}_j = \sum_{i,j=1}^{n} K(x_i, x_j) \xi_i \bar{\xi}_j$$ (3.1.5)

is then easy to check, keeping in mind that $K(x_i, x_0) = K(x_0, x_j) = 0$, which finishes the proof. \qed

For a more general proof see [31] p. 127 in the proof of Lemma 2.5.

If $\mathcal{X}$ is the homogeneous space of some topological group $G$, and $d$ is a $G$-invariant distance, then (3.1.3) is satisfied if and only if for every choice of elements $g_1, \ldots, g_n \in G$ and of complex numbers $\xi_1, \ldots, \xi_n$ with $\sum_{i=1}^{n} \xi_i = 0$

$$\sum_{i,j=1}^{n} d(g_i g_j^{-1} x_0, x_0) \xi_i \bar{\xi}_j \leq 0$$ (3.1.6)

where $x_0 \in \mathcal{X}$ is a fixed point. We shall say that the function $g \in G \rightarrow d(g x_0, x_0)$ is restricted negative definite on $G$ if it satisfies (3.1.6).

In our case of interest $\mathcal{X} = G$ a compact (Lie) group and $d$ is a bi-invariant distance as in §3.1.
The Peter-Weyl development for the class function $d(\cdot, e)$ on $G$ (see Theorem 1.2.2) is

$$d(g, e) = \sum_{\ell \in \hat{G}} \alpha_{\ell} \chi_{\ell}(g), \quad (3.1.7)$$

where $\hat{G}$ denotes the family of equivalence classes of irreducible representations of $G$ and $\chi_{\ell}$ the character of the $\ell$-th irreducible representation of $G$.

**Remark 3.1.2.** A function $\phi$ with a development as in (3.1.7) is restricted negative definite if and only if $\alpha_{\ell} \leq 0$ but for the trivial representation.

Actually note first that, by standard approximation arguments, $\phi$ is restricted negative definite if and only if for every continuous function $f : G \to \mathbb{C}$ with 0-mean (i.e. orthogonal to the constants)

$$\int_G \int_G \phi(gh^{-1})f(g)\overline{f(h)} \, dg \, dh \leq 0 \quad (3.1.8)$$

dg denoting the Haar measure of $G$. Choosing $f = \chi_{\ell}$ in the l.h.s. of (3.1.8) and denoting $d_{\ell}$ the dimension of the corresponding representation, a straightforward computation gives

$$\int_G \int_G \phi(gh^{-1})\chi_{\ell}(g)\overline{\chi_{\ell}(h)} \, dg \, dh = \frac{\alpha_{\ell}}{d_{\ell}} \quad (3.1.9)$$

so that if $\phi$ restricted negative definite, $\alpha_{\ell} \leq 0$ necessarily.

Conversely, if $\alpha_{\ell} \leq 0$ but for the trivial representation, then $\phi$ is restricted negative definite, as the characters $\chi_{\ell}$’s are positive definite and orthogonal to the constants.

### 3.2 $SU(2)$

The special unitary group $SU(2)$ consists of the complex unitary $2 \times 2$-matrices $g$ such that $\det(g) = 1$. Every $g \in SU(2)$ has the form

$$g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1. \quad (3.2.10)$$
If \( a = a_1 + ia_2 \) and \( b = b_1 + ib_2 \), then the map

\[
\Phi(g) = (a_1, a_2, b_1, b_2)
\]

is an homeomorphism (see [29], [63] e.g.) between \( SU(2) \) and the unit sphere \( S^3 \) of \( \mathbb{R}^4 \). Moreover the right translation

\[
R_g : h \mapsto hg, \quad h, g \in SU(2)
\]

of \( SU(2) \) is a rotation (an element of \( SO(4) \)) of \( S^3 \) (identified with \( SU(2) \)). The homeomorphism (3.2.11) preserves the invariant measure, i.e., if \( dg \) is the normalized Haar measure on \( SU(2) \), then \( \Phi(dg) \) is the normalized Lebesgue measure on \( S^3 \). As the 3-dimensional polar coordinates on \( S^3 \) are

\[
a_1 = \cos \theta, \\
a_2 = \sin \theta \cos \varphi, \\
b_1 = \sin \theta \sin \varphi \cos \psi, \\
b_2 = \sin \theta \sin \varphi \sin \psi,
\]

\((\theta, \varphi, \psi) \in [0, \pi] \times [0, \pi] \times [0, 2\pi]\), the normalized Haar integral of \( SU(2) \) for an integrable function \( f \) is

\[
\int_{SU(2)} f(g) \, dg = \frac{1}{2\pi^2} \int_0^\pi \sin \varphi \, d\varphi \int_0^\pi \sin^2 \theta \, d\theta \int_0^{2\pi} f(\theta, \varphi, \psi) \, d\psi
\]

(3.2.12)

The bi-invariant Riemannian metric on \( SU(2) \) is necessarily proportional to the negative Killing form of its Lie algebra \( su(2) \) (the real vector space of \( 2 \times 2 \) anti-hermitian complex matrices). We consider the bi-invariant metric corresponding to the \( Ad \)-invariant inner product on \( su(2) \)

\[
\langle X, Y \rangle = -\frac{1}{2} \text{tr}(XY), \quad X, Y \in su(2).
\]

Therefore as an orthonormal basis of \( su(2) \) we can consider the matrices

\[
X_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]
The homeomorphism (3.2.11) is actually an isometry between $SU(2)$ endowed with this distance and $S^3$. Hence the restricted negative definiteness of the kernel $d$ on $SU(2)$ is an immediate consequence of this property on $S^3$ which is known to be true as mentioned in the introduction ([31], [37], [65]). In order to develop a comparison with $SO(3)$, we shall give a different proof of this fact in §3.4.

3.3 $SO(n)$

We first investigate the case $n = 3$. The group $SO(3)$ can also be realized as a quotient of $SU(2)$. Actually the adjoint representation $Ad$ of $SU(2)$ is a surjective morphism from $SU(2)$ onto $SO(3)$ with kernel $\{\pm e\}$ (see [29] e.g.). Hence the well known result

$$SO(3) \cong SU(2)/\{\pm e\}. \quad (3.3.14)$$

Let us explicitly recall this morphism: if $a = a_1 + ia_2, b = b_1 + ib_2$ with $|a|^2 + |b|^2 = 1$ and

$$\tilde{g} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

then the orthogonal matrix $Ad(\tilde{g})$ is given by

$$g = \begin{pmatrix} a_1^2 - a_2^2 - (b_1^2 - b_2^2) & -2a_1a_2 - 2b_1b_2 & -2(a_1b_1 - a_2b_2) \\ 2a_1a_2 - 2b_1b_2 & (a_1^2 - a_2^2) + (b_1^2 - b_2^2) & -2(a_1b_2 + a_2b_1) \\ 2(a_1b_1 + a_2b_2) & -2(-a_1b_2 + a_2b_1) & |a|^2 - |b|^2 \end{pmatrix}. \quad (3.3.15)$$

The isomorphism in (3.3.14) might suggest that the positive definiteness of the Brownian kernel on $SU(2)$ implies a similar result for $SO(3)$. This is not true and actually it turns out that the distance $(g, h) \rightarrow d(g, h)$ on $SO(3)$ induced by its bi-invariant Riemannian metric is not a restricted negative definite kernel (see Lemma 3.1.1).

As for $SU(2)$, the bi-invariant Riemannian metric on $SO(3)$ is proportional to the negative Killing form of its Lie algebra $so(3)$ (the real $3 \times 3$ antisymmetric real matrices).
We shall consider the \textit{Ad}-invariant inner product on \( so(3) \) defined as
\[
\langle A, B \rangle = -\frac{1}{2} \text{tr}(AB), \quad A, B \in so(3).
\]
An orthonormal basis for \( so(3) \) is therefore given by the matrices
\[
A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
Similarly to the case of \( SU(2) \), it is easy to compute the distance from \( g \in SO(3) \) to the identity. Actually \( g \) is conjugated to the matrix of the form
\[
\Delta(t) = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp(tA_1)
\]
where \( t \in [0, \pi] \) is the rotation angle of \( g \). Therefore if \( d \) still denotes the distance induced by the bi-invariant metric,
\[
d(g, e) = d(\Delta(t), e) = t
\]
i.e. the distance from \( g \) to \( e \) is the rotation angle of \( g \).

Let us denote \( \{\chi_{\ell}\}_{\ell \geq 0} \) the set of characters for \( SO(3) \). It is easy to compute the Peter-Weyl development in (3.1.7) for \( d(\cdot, e) \) as the characters \( \chi_{\ell} \) are also simple functions of the rotation angle. More precisely, if \( t \) is the rotation angle of \( g \) (see [40] e.g.),
\[
\chi_{\ell}(g) = \frac{\sin \left( \frac{(2\ell+1)t}{2} \right)}{\sin \frac{t}{2}} = 1 + 2 \sum_{m=1}^{\ell} \cos(mt).
\]
We shall prove that the coefficient
\[
\alpha_{\ell} = \int_{SO(3)} d(g, e) \chi_{\ell}(g) \, dg
\]
is positive for some $\ell \geq 1$. As both $d(\cdot, e)$ and $\chi_\ell$ are functions of the rotation angle $t$, we have
\[
\alpha_\ell = \int_0^\pi t \left( 1 + 2 \sum_{j=1}^\ell \cos(jt) \right) p_T(t) \, dt
\]
where $p_T$ is the density of $t = t(g)$, considered as a r.v. on the probability space $(SO(3), dg)$. The next statements are devoted to the computation of the density $p_T$. This is certainly well known but we were unable to find a reference in the literature.

We first compute the density of the trace of $g$.

**Proposition 3.3.1.** The distribution of the trace of a matrix in $SO(3)$ with respect to the normalized Haar measure is given by the density
\[
f(y) = \frac{1}{2\pi} (3 - y)^{1/2} (y + 1)^{-1/2} 1_{[-1,3]}(y) .
\]  \hspace{1cm} (3.3.16)

*Proof.* The trace of the matrix (3.3.15) is equal to
\[
\text{tr}(g) = 3a_1^2 - a_2^2 - b_1^2 - b_2^2 .
\]
Under the normalized Haar measure of $SU(2)$ the vector $(a_1, a_2, b_1, b_2)$ is uniformly distributed on the sphere $S^3$. Recall the normalized Haar integral (3.2.13) so that, taking the corresponding marginal, $\theta$ has density
\[
f_1(\theta) = \frac{2}{\pi} \sin^2(\theta) \, d\theta .
\]  \hspace{1cm} (3.3.17)

Now
\[
3a_1^2 - a_2^2 - b_1^2 - b_2^2 = 4\cos^2 \theta - 1 .
\]
Let us first compute the density of $Y = \cos^2 X$, where $X$ is distributed according to the density (3.3.17). This is elementary as
\[
F_Y(t) = \mathbb{P}(\cos^2 X \leq t) = \mathbb{P}(\arccos(\sqrt{t}) \leq X \leq \arccos(-\sqrt{t})) = \\
\int_{\arccos(-\sqrt{t})}^{\arccos(\sqrt{t})} \sin^2(\theta) \, d\theta .
\]

Taking the derivative it is easily found that the density of $Y$ is, for $0 < t < 1$,

$$F_Y'(t) = \frac{2}{\pi} (1 - t)^{1/2} t^{-1/2}.$$ 

By an elementary change of variable the distribution of the trace $4Y - 1$ is therefore given by (3.3.16).

**Corollary 3.3.2.** The distribution of the rotation angle of a matrix in $SO(3)$ is

$$p_T(t) = \frac{1}{\pi} (1 - \cos t) 1_{[0,\pi]}(t).$$

**Proof.** It suffices to remark that if $t$ is the rotation angle of $g$, then its trace is equal to $2 \cos t + 1$. $p_T$ is therefore the distribution of $W = \arccos \left( \frac{Y - 1}{2} \right)$, $Y$ being distributed as (3.3.16). The elementary details are left to the reader.

Now it is easy to compute the Fourier development of the function $d(\cdot, e)$.

**Proposition 3.3.3.** The kernel $d$ on $SO(3)$ is not restricted negative definite.

**Proof.** It is enough to show that in the Fourier development

$$d(g, e) = \sum_{\ell \geq 0} \alpha\ell \chi\ell(g)$$

$\alpha\ell > 0$ for some $\ell \geq 1$ (see Remark 3.1.2). We have

$$\alpha\ell = \int_{SO(3)} d(g, e) \chi\ell(g) dg = \frac{1}{\pi} \int_0^\pi t \left( 1 + 2 \sum_{m=1}^\ell \cos(mt) \right) (1 - \cos t) dt =$$

$$= \frac{1}{\pi} \int_0^\pi t(1 - \cos t) dt + \frac{2}{\pi} \sum_{m=1}^\ell \int_0^\pi t \cos(mt) dt - \frac{2}{\pi} \sum_{m=1}^\ell \int_0^\pi t \cos(mt) \cos t dt.$$

Now integration by parts gives

$$I_1 = \frac{\pi^2}{2} + 2, \quad I_2 = \frac{(-1)^m - 1}{m^2},$$
whereas, if $m \neq 1$, we have

$$I_3 = \int_0^{\pi} t \cos(mt) \cos t \, dt = \frac{m^2 + 1}{(m^2 - 1)^2}((-1)^m + 1)$$

and for $m = 1$,

$$I_3 = \int_0^{\pi} t \cos^2 t \, dt = \frac{\pi^2}{4}.$$

Putting things together we find

$$\alpha_\ell = \frac{2}{\pi} \left( 1 + \sum_{m=1}^\ell \frac{(-1)^m - 1}{m^2} + \sum_{m=2}^\ell \frac{m^2 + 1}{(m^2 - 1)^2}((-1)^m + 1) \right).$$

If $\ell = 2$, for instance, we find $\alpha_2 = \frac{2}{9\pi} > 0$, but it is easy to see that $\alpha_\ell > 0$ for every $\ell$ even.

Consider now the case $n > 3$. $SO(n)$ (resp. $SU(n)$) contains a closed subgroup $H$ that is isomorphic to $SO(3)$ and the restriction to $H$ of any bi-invariant distance $d$ on $SO(n)$ (resp. $SU(n)$) is a bi-invariant distance $\tilde{d}$ on $SO(3)$. By Proposition 3.3.3, $\tilde{d}$ is not restricted negative definite, therefore there exist $g_1, g_2, \ldots, g_m \in H$, $\xi_1, \xi_2, \ldots, \xi_m \in \mathbb{R}$ with $\sum_{i=1}^m \xi_i = 0$ such that

$$\sum_{i,j} d(g_i, g_j) \xi_i \xi_j = \sum_{i,j} \tilde{d}(g_i, g_j) \xi_i \xi_j > 0. \quad (3.3.18)$$

We have therefore

**Corollary 3.3.4.** Any bi-invariant distance $d$ on $SO(n)$ and $SU(n)$, $n \geq 3$ is not a restricted negative definite kernel.

Remark that the same argument applies to other compact groups. Moreover the bi-invariant Riemannian metric on $SO(4)$ is not unique, meaning that it is not necessarily proportional to the negative Killing form of $so(4)$. In this case Corollary 3.3.4 states that every such bi-invariant distance cannot be restricted negative definite.
3.4 Final remarks

We were intrigued by the different behavior of the invariant distance of $SU(2)$ and $SO(3)$ despite these groups are locally isometric and decided to compute also for $SU(2)$ the development

$$d(g,e) = \sum_{\ell} \alpha_{\ell} \chi_{\ell}(g). \quad (3.4.19)$$

This is not difficult as, denoting by $t$ the distance of $g$ from $e$, the characters of $SU(2)$ are

$$\chi_{\ell}(g) = \frac{\sin((\ell + 1)t)}{\sin t}, \quad t \neq k\pi$$

and $\chi_{\ell}(e) = \ell + 1$ if $t = 0$, $\chi_{\ell}(-) = (-1)^{\ell}(\ell + 1)$ if $t = \pi$. Then it is elementary to compute, for $\ell > 0,$

$$\alpha_{\ell} = \frac{1}{\pi} \int_0^\pi t \sin((\ell + 1)t) \sin t \, dt = \begin{cases} -\frac{8}{\pi} \frac{m+1}{m^2(m+2)^2} & \ell \text{ odd} \\ 0 & \ell \text{ even} \end{cases}$$

thus confirming the restricted negative definiteness of $d$ (see Remark 3.1.2). Remark also that the coefficients corresponding to the even numbered representations, that are also representations of $SO(3)$, here vanish.
Part 2
High-energy Gaussian eigenfunctions
Chapter 4

Background: Fourth-Moment phenomenon and Gaussian eigenfunctions

As made clear by the title, this chapter is first devoted to the so-called Fourth Moment phenomenon. Main results in this area are summarized in the recent monograph [53]: a beautiful connection has been established between Malliavin calculus and Stein’s method for normal approximations to prove Berry-Esseen bounds and quantitative Central Limit Theorems for functionals of a Gaussian random field.

Finally we recall definitions and fix some notation for Gaussian eigenfunctions on the $d$-dimensional unit sphere $\mathbb{S}^d$ ($d \geq 2$) whose properties we will deeply investigate in the sequel of this work.

4.1 Fourth-moment theorems

4.1.1 Isonormal Gaussian fields

Let $H$ be a (real) separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and $(\Omega, \mathcal{F}, \mathbb{P})$ some probability space.
Definition 4.1.1. The isonormal Gaussian field $T$ on $H$ is a centered Gaussian random field $(T(h))_{h \in H}$ whose covariance kernel is given by

$$\text{Cov}(T(h), T(h')) = \langle h, h' \rangle_H, \quad h, h' \in H.$$  \hfill (4.1.1)

Consider, from now on, the case $H = L^2(X, \mathcal{X}, \mu)$ the space of square integrable functions on the measure space $(X, \mathcal{X}, \mu)$, where $X$ is a Polish space, $\mathcal{X}$ is the $\sigma$-field on $X$ and $\mu$ is a positive, $\sigma$-finite and non-atomic measure on $(X, \mathcal{X})$. As usual the inner product is given by $\langle f, g \rangle_H = \int_X f(x)g(x) \, d\mu(x)$.

Let us recall the construction of an isonormal Gaussian field on $H$. Consider a (real) Gaussian measure over $(X, \mathcal{X})$, i.e. a centered Gaussian family $W$

$$W = \{W(A) : A \in \mathcal{X}, \mu(A) < +\infty\}$$

such that for $A, B \in \mathcal{X}$ of $\mu$-finite measure, we have

$$\mathbb{E}[W(A)W(B)] = \mu(A \cap B).$$

We define a random field $T = (T(f))_{f \in H}$ on $H$ as follows. For each $f \in H$, let

$$T(f) = \int_X f(x) \, dW(x)$$  \hfill (4.1.2)

be the Wiener-Itô integral of $f$ with respect to $W$. The random field $T$ is the isonormal Gaussian field on $H$: indeed it is centered Gaussian and by construction

$$\text{Cov}(T(f), T(g)) = \langle f, g \rangle_H.$$

4.1.2 Wiener chaos and contractions

Let us recall now the notion of Wiener chaos. Define the space of constants $C_0 := \mathbb{R} \subseteq L^2(\mathbb{P})$, and for $q \geq 1$, let $C_q$ be the closure in $L^2(\mathbb{P}) := L^2(\Omega, \mathcal{F}, \mathbb{P})$ of the linear subspace generated by random variables of the form

$$H_q(T(f)), \quad f \in H, \|f\|_H = 1,$$
where $H_q$ denotes the $q$-th Hermite polynomial, i.e.

$$H_q(t) := (-1)^q \phi^{-1}(t) \frac{d^q}{dt^q} \phi(t) , \quad t \in \mathbb{R} ,$$

(4.1.3)

$\phi$ being the density function of a standard Gaussian r.v. $Z \sim \mathcal{N}(0, 1)$. $C_q$ is called the $q$-th Wiener chaos.

The following, well-known property is very important: let $Z_1, Z_2 \sim \mathcal{N}(0, 1)$ be jointly Gaussian; then, for all $q_1, q_2 \geq 0$

$$E[H_{q_1}(Z_1)H_{q_2}(Z_2)] = q_1! E[Z_1Z_2]^{q_1} \delta_{q_1}^{q_2} .$$

(4.1.4)

**Theorem 4.1.2.** The Wiener-Itô chaos expansion holds

$$L^2(\mathbb{P}) = \bigoplus_{q=0}^{+\infty} C_q ,$$

the above sum being orthogonal from (4.1.4). Equivalently, each random variable $F \in L^2(\mathbb{P})$ admits a unique decomposition in the $L^2(\mathbb{P})$-sense of the form

$$F = \sum_{q=0}^{\infty} J_q(F) ,$$

(4.1.5)

where $J_q : L^2(\mathbb{P}) \rightarrow C_q$ is the orthogonal projection operator onto the $q$-th Wiener chaos. Remark that $J_0(F) = E[F]$.

Often we will use the symbols $\text{proj}(F|C_q)$ or $F_q$ instead of $J_q(F)$.

We denote by $H^\otimes q$ and $H^\odot q$ the $q$-th tensor product and the $q$-th symmetric tensor product of $H$ respectively. Therefore $H^\otimes q = L^2(X^q, \mathcal{X}^q, \mu^q)$ and $H^\odot q = L^2_s(X^q, \mathcal{X}^q, \mu^q)$, where by $L^2_s$ we mean the symmetric and square integrable functions w.r.t. $\mu^q$. Note that for $(x_1, x_2, \ldots, x_q) \in X^q$ and $f \in H$, we have

$$f^\otimes q(x_1, x_2, \ldots, x_q) = f(x_1)f(x_2)\ldots f(x_q) .$$

Now for $q \geq 1$, let us define the map $I_q$ as

$$I_q(f^\otimes q) := H_q(T(f)) , \quad f \in H ,$$

(4.1.6)
which can be extended to a linear isometry between $H^q$ equipped with the modified norm $\sqrt{q!} \| \cdot \|_{H^q}$ and the $q$-th Wiener chaos $C_q$. Moreover for $q = 0$, set $I_0(c) = c \in \mathbb{R}$. Hence (4.1.5) becomes

$$F = \sum_{q=0}^{\infty} I_q(f_q),$$

(4.1.7)

where the kernels $f_q, q \geq 0$ are uniquely determined, $f_0 = \mathbb{E}[F]$ and for $q \geq 1 f \in H^q$.

In our setting, it is well known that for $h \in H^q$, $I_q(h)$ coincides with the multiple Wiener-Ito integral of order $q$ of $h$ with respect to the Gaussian measure $W$, i.e.

$$I_q(h) = \int_{X^q} h(x_1, x_2, \ldots x_q) dW(x_1) dW(x_2) \ldots dW(x_q)$$

(4.1.8)

and, loosely speaking, $F$ in (4.1.7) can be seen as a series of (multiple) stochastic integrals.

For every $p, q \geq 1$, $f \in H^p, g \in H^q$ and $r = 1, 2, \ldots, p \wedge q$, the so-called contraction of $f$ and $g$ of order $r$ is the element $f \otimes_r g \in H^{p+q-2r}$ given by

$$f \otimes_r g (x_1, \ldots, x_{p+q-2r}) =$$

$$= \int_{X^r} f(x_1, \ldots, x_{p-r}, y_1, \ldots, y_r) g(x_{p-r+1}, \ldots, x_{p+q-2r}, y_1, \ldots, y_r) d\mu(y),$$

(4.1.9)

where we set $d\mu(y) := d\mu(y_1) \ldots d\mu(y_r)$.

For $p = q = r$, we have $f \otimes_r g = \langle f, g \rangle_{H^r}$ and for $r = 0$, $f \otimes_0 g = f \otimes g$. Note that $f \otimes_r g$ is not necessarily symmetric, let us denote by $f \tilde{\otimes}_r g$ its canonical symmetrization.

The following multiplication formula is well-known: for $p, q = 1, 2, \ldots, f \in H^p, g \in H^q$, we have

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \tilde{\otimes}_r g).$$

### 4.1.3 Some language of Malliavin calculus

Let $S$ be the set of all cylindrical r.v.'s of the type $F = f(T(h_1), \ldots, T(h_m))$, where $m \geq 1, f : \mathbb{R}^m \rightarrow \mathbb{R}$ is an infinitely differentiable function with compact support and
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\( h_i \in H \). The Malliavin derivative \( DF \) (or \( D^1 F \)) of \( F \) w.r.t. \( T \) is the element \( \in L^2(\Omega, H) \) defined as

\[
DF = \sum_i \frac{\partial f}{\partial x_i} (T(h_1), \ldots, T(h_m)) h_i .
\]

We can define, by iteration, the \( r \)-th derivative \( D^r F \) which is an element of \( L^2(\Omega, H^{\otimes r}) \) for every \( r \geq 2 \). Recall that \( \mathcal{S} \) is dense in \( L^q(\mathbb{P}) \) for each \( q \geq 1 \).

For \( r \geq 1 \) and \( q \geq 1 \), let us denote by \( \mathbb{D}^{r,q} \) the closure of \( \mathcal{S} \) w.r.t. the norm \( \| \cdot \|_{\mathbb{D}^{r,q}} \) defined by the relation

\[
\| F \|_{\mathbb{D}^{r,q}} := \left( \mathbb{E}[|F|^q] + \cdots + \mathbb{E}[\|D^r F\|_{H^{\otimes r}}^q] \right)^{\frac{1}{q}} .
\]

For \( q, r \geq 1 \), the \( r \)-th Malliavin derivative of the random variable \( F = I_q(f) \) \( \in C_q \) where \( f \in H^{\otimes q} \), is given by

\[
D^r F = \frac{q!}{(q-r)!} I_{q-r}(f) , \quad (4.1.10)
\]

for \( r \leq q \), and \( D^r F = 0 \) for \( r > q \).

It is possible to show that if we consider the chaotic representation (4.1.7), then \( F \in \mathbb{D}^{r,2} \) if and only if

\[
\sum_{q=r}^{+\infty} q^r q! \| f_q \|^2_{H^{\otimes q}} < +\infty
\]

and in this case

\[
D^r F = \sum_{q=r}^{+\infty} \frac{q!}{(q-r)!} I_{q-r}(f_q) .
\]

We need to introduce also the generator of the Ornstein-Uhlenbeck semigroup, defined as

\[
L := -\sum_{q=1}^{\infty} q J_q ,
\]

where \( J_q \) is the orthogonal projection operator on \( C_q \), as in (4.1.5). The domain of \( L \) consists of \( F \in L^2(\mathbb{P}) \) such that

\[
\sum_{q=1}^{+\infty} q^2 \| J_q(F) \|^2_{L^2(\mathbb{P})} < +\infty .
\]
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The pseudo-inverse operator of $L$ is defined as

$$L^{-1} = -\sum_{q=1}^{\infty} \frac{1}{q} J_q$$

and satisfies for each $F \in L^2(\mathbb{P})$

$$LL^{-1}F = F - \mathbb{E}[F],$$

equality that justifies its name.

### 4.1.4 Main theorems

We will need the following definition throughout the rest of this thesis.

**Definition 4.1.3.** Denote by $\mathcal{P}$ the collection of all probability measures on $\mathbb{R}$, and let $d : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ be a distance on $\mathcal{P}$. We say that the $d$ metrizes weak convergence on $\mathcal{P}$ if the following double implication holds for every collection $\{\mathbb{P}, \mathbb{P}_n : n \geq 1\} \subset \mathcal{P}$, as $n \to \infty$:

$$\mathbb{P}_n \text{ converges weakly to } \mathbb{P} \text{ if and only if } d(\mathbb{P}_n, \mathbb{P}) \to 0.$$ 

Given two random variables $X_1, X_2$ and a distance $d$ on $\mathcal{P}$, by an abuse of notation we shall write $d(X_1, X_2)$ to indicate the quantity $d(D(X_1), D(X_2))$, where $D(X_i)$ indicates the distribution of $X_i$, $i = 1, 2$. Recall that, given random variables $\{X, X_n : n \geq 1\}$, one has that $D(X_n)$ converges weakly to $D(X)$ if and only if $X_n$ converges in distribution to $X$. In this case, we write

$$X_n \xrightarrow{d} X \text{ or } X_n \xrightarrow{L} X,$$

whereas $X \overset{d}{=} Y$ or $X \overset{L}{=} Y$ indicates that $D(X) = D(Y)$.

Outstanding examples of distances metrizing weak convergence are the **Prokhorov distance** (usually denoted by $\rho$) and the **Fortet-Mourier distance** (or bounded Wasserstein distance, usually denoted by $\beta$). These are given by

$$\rho(\mathbb{P}, \mathbb{Q}) = \inf \{\epsilon > 0 : \mathbb{P}(A) \leq \epsilon + \mathbb{Q}(A^c) \}, \text{ for every Borel set } A \subset \mathbb{R},$$
where \(A^\epsilon := \{x : |x - y| < \epsilon, \text{ for some } y \in A\}\), and
\[
\beta (\mathbb{P}, \mathbb{Q}) = \sup \left\{ \left| \int_{\mathbb{R}} f \, d(\mathbb{P} - \mathbb{Q}) \right| : \|f\|_{BL} \leq 1 \right\}.
\]
where \(\| \cdot \|_{BL} = \| \cdot \|_L + \| \cdot \|_\infty\), and \(\| \cdot \|_L\) is the usual Lipschitz seminorm (see e.g. [27, Section 11.3] for further details on these notions).

Let us recall moreover the usual Kolmogorov \(d_K\), total variation \(d_{TV}\) and Wasserstein \(d_W\) distances between r.v.’s \(X, Y\): for \(\mathcal{D} \in \{K, TV, W\}\)
\[
d_{\mathcal{D}}(X, Y) := \sup_{h \in H_\mathcal{D}} \left| \mathbb{E}[h(X)] - \mathbb{E}[h(Y)] \right|, \tag{4.1.11}
\]
where \(H_K = \{1(\cdot \leq z), z \in \mathbb{R}\}\), \(H_{TV} = \{1_A(\cdot), A \in \mathcal{B}(\mathbb{R})\}\) and \(H_W\) is the set of Lipschitz functions with Lipschitz constant one.

It is a standard fact (see e.g. [53, Proposition C.3.1]) that \(d_K\) does not metrize, in general, weak convergence on \(\mathcal{P}\).

The connection between stochastic calculus and probability metrics is summarized in the following result (see e.g. [53], Theorem 5.1.3), which will provide the basis for most of our results to follow.

From now on, \(\mathcal{N}(\mu, \sigma^2)\) shall denote the Gaussian law with mean \(\mu\) and variance \(\sigma^2\).

**Proposition 4.1.4.** Let \(F \in \mathcal{D}^{1,2}\) such that \(\mathbb{E}[F] = 0, \mathbb{E}[F^2] = \sigma^2 < +\infty\). Then we have for \(Z \sim \mathcal{N}(0, \sigma^2)\)
\[
d_W(F, Z) \leq \sqrt{\frac{2}{\sigma^2 \pi} \mathbb{E}[\|\sigma^2 - \langle DF, -DL^{-1}F \rangle_H\|^2]}.
\]
Also, assuming in addition that \(F\) has a density
\[
d_{TV}(F, Z) \leq \frac{2}{\sigma^2} \mathbb{E}[\|\sigma^2 - \langle DF, -DL^{-1}F \rangle_H\|],
\]
\[
d_K(F, Z) \leq \frac{1}{\sigma^2} \mathbb{E}[\|\sigma^2 - \langle DF, -DL^{-1}F \rangle_H\|].
\]
Moreover if \(F \in \mathcal{D}^{1,4}\), we have also
\[
\mathbb{E}[\|\sigma^2 - \langle DF, -DL^{-1}F \rangle_H\|] \leq \sqrt{\text{Var}[\langle DF, -DL^{-1}F \rangle_H]}.
\]
Furthermore, in the special case where \( F = I_q(f) \) for \( f \in H^{\otimes q}, q \geq 2 \) then from \([53]\), Theorem 5.2.6

\[
\mathbb{E}[\|\sigma^2 - \langle DF, -DL^{-1}F \rangle_H \|] \leq \sqrt{\text{Var} \left( \frac{1}{q} \|DF\|^2_H \right)},
\]

(4.1.12)

and Lemma 5.2.4 gives

\[
\text{Var} \left( \frac{1}{q} \|DF\|^2_H \right) = \frac{1}{q^2} \sum_{r=1}^{q-1} r^2 r!^2 \left( \frac{q}{r} \right)^4 (2q - 2r)! \|f_\otimes r f\|^2_{H^{\otimes 2q-2r}}.
\]

(4.1.13)

In addition it is possible to show the powerful chain of inequalities: for \( q \geq 2 \)

\[
\text{Var} \left( \frac{1}{q} \|DF\|^2_H \right) \leq \frac{q - 1}{3q} k_4(F) \leq (q - 1) \text{Var} \left( \frac{1}{q} \|DF\|^2_H \right),
\]

where

\[ k_4(F) := \mathbb{E}[F^4] - 3(\sigma^2)^2 \]

is the fourth cumulant of \( F \).

**Remark 4.1.5.** Note that in (4.1.13) we can replace \( \|f_\otimes r f\|^2_{H^{\otimes 2q-2r}} \) with the norm of the unsymmetrized contraction \( \|f \otimes r f\|^2_{H^{\otimes 2q-2r}} \) for the upper bound, since

\[ \|f_\otimes r f\|^2_{H^{\otimes 2q-2r}} \leq \|f \otimes r f\|^2_{H^{\otimes 2q-2r}} \]

by the triangular inequality.

We shall make an extensive use of the following.

**Corollary 4.1.6.** Let \( F_n, n \geq 1, \) be a sequence of random variables belonging to the \( q \)-th Wiener chaos, for some fixed integer \( q \geq 2 \). Then we have the following bound: for \( D \in \{K, TV, W\} \)

\[
d_D \left( \frac{F_n}{\sqrt{\text{Var}(F_n)}}, Z \right) \leq C_D(q) \sqrt{\frac{k_4(F_n)}{\text{Var}(F_n)^2}}.
\]

(4.1.14)

where \( Z \sim \mathcal{N}(0, 1) \), for some constant \( C_D(q) > 0 \). In particular, if the right hand side in (4.1.14) vanishes for \( n \to +\infty \), then the following convergence in distribution holds

\[
\frac{F_n}{\sqrt{\text{Var}(F_n)}} \xrightarrow{L} Z.
\]
4.2 Gaussian eigenfunctions on the $d$-sphere

4.2.1 Some more notation

For any two positive sequences $a_n, b_n$, we shall write $a_n \sim b_n$ if $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ and $a_n \ll b_n$ or $a_n = O(b_n)$ if the sequence $\frac{a_n}{b_n}$ is bounded; moreover $a_n = o(b_n)$ if $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$.

Also, we write as usual $dx$ for the Lebesgue measure on the unit $d$-dimensional sphere $S^d \subset \mathbb{R}^{d+1}$, so that $\int_{S^d} dx = \mu_d$ where $\mu_d := \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$, as already stated in (1.1.1). Recall that the triple $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space and $\mathbb{E}$ stands for the expectation w.r.t $\mathbb{P}$; convergence (resp. equality) in law is denoted by $\to^L$ or equivalently $\to^d$ (resp. $=^L$ or $=^d$) and finally, as usual, $\mathcal{N}(\mu, \sigma^2)$ stands for a Gaussian random variable with mean $\mu$ and variance $\sigma^2$.

Let $\Delta_{S^d}$ ($d \geq 2$) denote the Laplace-Beltrami operator on $S^d$ and $(Y_{\ell,m};d)_{\ell,m}$ the orthonormal system of (real-valued) spherical harmonics, i.e. for $\ell \in \mathbb{N}$ the set of eigenfunctions

$$\Delta_{S^d}Y_{\ell,m;d} + \ell(\ell + d - 1)Y_{\ell,m;d} = 0 \ , \ m = 1, 2, \ldots, n_{\ell,d} \ .$$

For $d = 2$ compare with (1.2.19). As well-known, the spherical harmonics $(Y_{\ell,m;d})_{m=1}^{n_{\ell,d}}$ represent a family of linearly independent homogeneous polynomials of degree $\ell$ in $d+1$ variables restricted to $S^d$ of size

$$n_{\ell,d} := \frac{2\ell + d - 1}{\ell} \binom{\ell + d - 2}{\ell - 1} \sim \frac{2}{(d-1)!} \ell^{d-1}, \ as \ \ell \to +\infty \ ,$$

see e.g. [2] for further details.

4.2.2 Definition and properties

Definition 4.2.1. For $\ell \in \mathbb{N}$, the Gaussian eigenfunction $T_\ell$ on $S^d$ is defined as

$$T_\ell(x) := \sum_{m=1}^{n_{\ell,d}} a_{\ell,m} Y_{\ell,m;d}(x) \ , \ x \in S^d , \quad (4.2.15)$$
with the random coefficients \( (a_{\ell,m})_{m=1}^{n_{\ell,d}} \) Gaussian i.i.d. random variables, satisfying the relation

\[
\mathbb{E}[a_{\ell,m}a_{\ell,m'}] = \frac{\mu_d}{n_{\ell,d}} \delta_{m,m'},
\]

where \( \delta_a^b \) denotes the Kronecker delta function and \( \mu_d = \frac{2^{d+1}}{\Gamma\left(\frac{d+1}{2}\right)} \) the hypersurface volume of \( S^d \), as in (1.1.1).

It is then readily checked that \( (T_\ell)_{\ell \in \mathbb{N}} \) represents a sequence of isotropic, zero-mean Gaussian random fields on \( S^d \), according to Definition (1.3.1) and moreover

\[
\mathbb{E}[T_\ell(x)^2] = 1, \quad x \in S^d.
\]

\( T_\ell \) is a continuous random field (Definition 1.3.3) and its isotropy simply means that the probability laws of the two random fields \( T_\ell(\cdot) \) and \( T_\ell^{\#}(\cdot) := T_\ell(g \cdot) \) are equal (in the sense of finite dimensional distributions) for every \( g \in SO(d+1) \) (see (1.3.25)).

As briefly states in the Introduction of this thesis, it is also well-known that every Gaussian and isotropic random field \( T \) on \( S^d \) satisfies in the \( L^2(\Omega \times S^d) \)-sense the spectral representation (see [38, 1, 40] e.g.)

\[
T(x) = \sum_{\ell=1}^{\infty} c_\ell T_\ell(x), \quad x \in S^d,
\]

where for every \( x \in S^d \), \( \mathbb{E}[T(x)^2] = \sum_{\ell=1}^{\infty} c_\ell^2 < \infty \); hence the spherical Gaussian eigenfunctions \((T_\ell)_{\ell \in \mathbb{N}}\) can be viewed as the Fourier components (Chapter 1) of the field \( T \) (note that w.l.o.g. we are implicitly assuming that \( T \) is centered). Equivalently these random eigenfunctions (4.2.15) could be defined by their covariance function, which equals

\[
\mathbb{E}[T_\ell(x)T_\ell(y)] = G_{\ell,d}(\cos d(x,y)), \quad x, y \in S^d.
\]

Here and in the sequel, \( d(x,y) \) is the spherical distance between \( x, y \in S^d \), and \( G_{\ell,d} : [-1,1] \rightarrow \mathbb{R} \) is the \( \ell \)-th normalized Gegenbauer polynomial, i.e.

\[
G_{\ell,d} \equiv \frac{P_{\ell + \frac{d}{2} - 1}^{d-1}}{\Gamma\left(\ell + \frac{d}{2} - 1\right)}.
\]
where $P_{\ell}^{(\alpha,\beta)}$ are the Jacobi polynomials; throughout the whole thesis therefore $G_{\ell;1}(1) = 1$. As a special case, for $d = 2$, it equals $G_{\ell;2} \equiv P_{\ell}$, the degree-$\ell$ Legendre polynomial. Remark that the Jacobi polynomials $P_{\ell}^{(\alpha,\beta)}$ are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(t) = (1-t)^{\alpha}(1+t)^{\beta}$ and satisfy $P_{\ell}^{(\alpha,\beta)}(1) = \binom{\ell+\alpha}{\ell}$, see e.g. [64] for more details.

### 4.2.3 Isonormal representation

Let us give the isonormal representation (4.1.2) on $L^2(S^d)$ for the Gaussian random eigenfunctions $T_{\ell}$, $\ell \geq 1$. We shall show that the following identity in law holds:

$$T_{\ell}(x) \overset{\mathcal{L}}{=} \int_{S^d} \sqrt{\frac{n_{\ell,d}}{\mu_d}} G_{\ell,d}(\cos d(x,y)) \, dW(y), \quad x \in S^d,$$

where $W$ is a Gaussian white noise on $S^d$. To compare with (4.1.2),

$$T_{\ell}(x) = T(f_x),$$

where $T$ is the isonormal Gaussian field on $L^2(S^d)$ and

$$f_x(\cdot) := \sqrt{\frac{n_{\ell,d}}{\mu_d}} G_{\ell,d}(\cos d(x, \cdot)).$$

Moreover we have immediately that

$$\mathbb{E}\left[ \int_{S^d} \sqrt{\frac{n_{\ell,d}}{\mu_d}} G_{\ell,d}(\cos d(x,y)) \, dW(y) \right] = 0,$$

and by the reproducing formula for Gegenbauer polynomials ([64])

$$\mathbb{E}\left[ \int_{S^d} \sqrt{\frac{n_{\ell,d}}{\mu_d}} G_{\ell,d}(\cos d(x_1,y_1)) \, dW(y_1) \int_{S^d} \sqrt{\frac{n_{\ell,d}}{\mu_d}} G_{\ell,d}(\cos d(x_2,y_2)) \, dW(y_2) \right] = \frac{n_{\ell,d}}{\mu_d} \int_{S^d} G_{\ell,d}(\cos d(x_1,y)) G_{\ell,d}(\cos d(x_2,y)) \, dy = G_{\ell,d}(\cos d(x_1,x_2)).$$

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Chapter 5

Empirical measure of excursion sets

The asymptotic behavior (i.e. for growing eigenvalues) of Gaussian eigenfunctions on a compact Riemannian manifold is a topic which has recently drawn considerable attention, see e.g. [?, 34, 12].

In particular, in view of Berry’s Random Wave model [69] much effort has been devoted to the case of the sphere $S^2$ (see [50, 68, 45, 47]).

As anticipated in the Introduction, the aim of this chapter is the investigation of the asymptotic distribution of the empirical measure $S_{\ell}(z)$ of $z$-excursion sets of random spherical harmonics. A Central Limit Theorem has already been proved [47], but it can provide little guidance to the actual distribution of random functionals, as it is only an asymptotic result with no information on the speed of convergence to the limiting distribution.

In [44] therefore we exploit the results about fourth-moments phenomenon (see [53] and Chapter 4) to establish quantitative Central Limit Theorems for the excursion volume of Gaussian eigenfunctions on the $d$-dimensional unit sphere $S^d$, $d \geq 2$ (see also [59]).

We note that there are already results in the literature giving rates of convergence in CLTs for value distributions of eigenfunctions of the spherical Laplacian, see in particular [48], which investigates however the complementary situation to the one considered
here, i.e. the limit for eigenfunctions of fixed degree $\ell$ and increasing dimension $d$.

To achieve our goal, we will provide a number of intermediate results of independent interest, namely the asymptotic analysis for the variance of moments of Gaussian eigenfunctions, the rates of convergence for various probability metrics for so-called Hermite subordinated processes, the analysis of arbitrary polynomials of finite order and square integrable nonlinear transforms. All these results could be useful to attack other problems, for instance quantitative Central Limit Theorems for Lipschitz-Killing curvatures of arbitrary order. A more precise statement of our results and a short outline of the proof is given in §5.1.

\section{Main results and outline of the proofs}

The excursion volume of $T_\ell$ (4.2.15), for any fixed $z \in \mathbb{R}$ can be written as

$$S_\ell(z) = \int_{\mathbb{R}^d} 1(T_\ell(x) > z) \, dx,$$

where $1(\cdot > z)$ denotes the indicator function of the interval $(z, \infty)$; note that $E[S_\ell(z)] = \mu_d(1 - \Phi(z))$, where $\Phi(z)$ is the standard Gaussian cdf and $\mu_d$ as in (1.1.1).

The variance of this excursion volume will be shown below to have the following asymptotic behavior (as $\ell \to +\infty$)

$$\text{Var}(S_\ell(z)) = \frac{z^2 \phi(z)^2}{2} \frac{\mu_d^2}{n_{\ell,d}} + o(\ell^{-d}),$$

where $\phi$ denotes the standard Gaussian density and $n_{\ell,d} \sim \frac{2}{(d-1)!} \ell^{d-1}$ is the dimension of the eigenspace related to the eigenvalue $-\ell(\ell + d - 1)$, as in Chapter 4.

Note that the variance is of order smaller than $\ell^{-(d-1)}$ if and only if $z = 0$.

The main result of this chapter is then as follows.

\textbf{Theorem 5.1.1.} \textit{The excursion volume $S_\ell(z)$ in (5.1.1) of Gaussian eigenfunctions $T_\ell$ on $\mathbb{S}^d$, $d \geq 2$, satisfies a quantitative CLT as $\ell \to +\infty$, with rate of convergence in}
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the Wasserstein distance (4.1.11) given by, for \( z \neq 0 \) and \( Z \sim \mathcal{N}(0,1) \)

\[
d_W \left( \frac{S_\ell(z) - \mu_d(1 - \Phi(z))}{\sqrt{\text{Var}[S_\ell(z)]}}, Z \right) = O \left( \ell^{-1/2} \right)
\]

An outline of the main steps and auxiliary results to prove this theorem is given in the following subsection.

5.1.1 Steps of the proofs

The first tool to investigate quantitative CLTs for the excursion volume of Gaussian eigenfunctions on \( S^d \) (compare for \( d = 2 \) with [47]) is to study the asymptotic behavior, as \( \ell \to \infty \), of the random variables \( h_{\ell,q,d} \) defined for \( \ell = 1,2, \ldots \) and \( q = 0,1, \ldots \) as

\[
h_{\ell,q,d} = \int_{S^d} H_q(T_\ell(x)) \, dx , \tag{5.1.3}
\]

where \( H_q \) represent the family of Hermite polynomials (4.1.3) (see also [53]). The rationale to investigate these sequences is the fact that the excursion volume, and more generally any square integrable transform of \( T_\ell \), admits the Wiener-Ito chaos decomposition (4.1.5) (for more details e.g. [53], §2.2), i.e. a series expansion in the \( L^2(\mathbb{P}) \)-sense of the form

\[
S_\ell(z) = \sum_{q=0}^{+\infty} \frac{J_q(z)}{q!} h_{\ell,q,d}, \tag{5.1.4}
\]

where \( J_0(z) = 1 - \Phi(z) \) and for \( q \geq 1 \), \( J_q(z) = \mathbb{E}[1(1 > z)H_q(Z)] \), \( \Phi \) and \( \phi \) denoting again respectively the cdf and the density of \( Z \sim \mathcal{N}(0,1) \).

The main idea in our argument will then be to establish first a CLT for each of the summands in the series, and then to deduce from this a CLT for the excursion volume. The starting point will then be the analysis of the asymptotic variances for \( h_{\ell,q,d} \), as \( \ell \to +\infty \).

To this aim, note first that, for all \( d \)

\[
h_{\ell,0,d} = \mu_d , \quad h_{\ell,1,d} = 0
\]
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a.s., and therefore it is enough to restrict our discussion to \( q \geq 2 \). Moreover \( \mathbb{E}[h_{\ell,q,d}] = 0 \) and

\[
\text{Var}[h_{\ell,q,d}] = q! \mu_d \mu_{d-1} \int_0^\pi G_{\ell,d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta
\]

(5.1.5)

(see §5.6 for more details). Gegenbauer polynomials satisfy the symmetry relationships [64]

\[
G_{\ell,d}(t) = (-1)^\ell G_{\ell,d}(-t)
\]

whence the r.h.s. integral in (5.1.5) vanishes identically when both \( \ell \) and \( q \) are odd; therefore in these cases \( h_{\ell,q,d} = 0 \) a.s. For the remaining cases we have

\[
\text{Var}[h_{\ell,q,d}] = 2q! \mu_d \mu_{d-1} \int_0^\pi G_{\ell,d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta
\]

(5.1.6)

We have hence the following asymptotic result for these variances, whose proof is given in §5.6.1.

**Proposition 5.1.2.** As \( \ell \to \infty \), for \( q,d \geq 3 \),

\[
\int_0^\pi G_{\ell,d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta = \frac{c_{q,d}}{\ell^d} (1 + o_{q,d}(1))
\]

(5.1.7)

The constants \( c_{q,d} \) are given by the formula

\[
c_{q,d} = \left(2^{\frac{q}{2}-1} \left(\frac{d}{2} - 1\right) \right)^q \int_0^{+\infty} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q\left(\frac{d}{2}-1\right)+d-1} \, d\psi
\]

(5.1.8)

where \( J_{\frac{d}{2}-1} \) is the Bessel function of order \( \frac{d}{2} - 1 \). The r.h.s. integral in (5.1.8) is absolutely convergent for any pair \( (d,q) \neq (3,3) \) and conditionally convergent for \( d = q = 3 \).

It is well known that for \( d \geq 2 \), the second moment of the Gegenbauer polynomials is given by

\[
\int_0^\pi G_{\ell,d}(\cos \vartheta)^2 (\sin \vartheta)^{d-1} \, d\vartheta = \frac{\mu_d}{\mu_{d-1} n_{\ell,d}},
\]

whence

\[
\text{Var}(h_{\ell,2,d}) = 2\frac{\mu_d^2}{n_{\ell,d}} \sim 4\mu_d \mu_{d-1} c_{2,d} \frac{c_{2,d}}{\ell^{d-1}}
\]

as \( \ell \to +\infty \),

(5.1.10)
where $c_{2,d} := \frac{(d-1)\mu_d}{q_{d-1}}$. For $d = 2$ and every $q$, the asymptotic behavior of these integrals was resolved in [45]. In particular, it was shown that for $q = 3$ or $q \geq 5$

\[
\text{Var}(h_{\ell,q,2}) = (4\pi)^2 q! \int_0^{\frac{\pi}{2}} P_\ell(\cos \theta)^q \sin \theta \, d\theta = (4\pi)^2 q! \frac{c_{q,2}}{\ell^2} (1 + o_q(1)), \quad (5.1.11)
\]

where

\[
c_{q,2} = \int_0^{+\infty} J_0(\psi)^q \psi \, d\psi, \quad (5.1.12)
\]

$J_0$ being the Bessel function of order 0 and the above integral being absolutely convergent for $q \geq 5$ and conditionally convergent for $q = 3$. On the other hand, for $q = 4$, as $\ell \to \infty$,

\[
\text{Var}[h_{\ell,q,2}] \sim 24^2 \frac{\log \ell}{\ell^2} = (4\pi)^2 4! \frac{c_{4,2}}{\ell^2} \log \ell, \quad (5.1.13)
\]

where we set $c_{4,2} := \frac{3}{2\pi^2}$. Clearly for any $d, q \geq 2$, the constants $c_{q,d}$ are nonnegative and it is obvious that $c_{q,d} > 0$ for all even $q$. Moreover, as we will recall in the next chapter, there exists an explicit formula in the case $q = 3$.

We conjecture that this strict inequality holds for every $(d, q)$, but leave this issue as an open question for future research; also, in view of the previous discussion on the symmetry properties of Gegenbauer polynomials, to simplify the discussion in the sequel we restrict ourselves to even multipoles $\ell$.

As argued earlier, the following step is to establish quantitative CLTs for $h_{\ell,q,d}$ (see §5.3) in various probability metrics (4.1.11). Here the crucial point to stress is that the Gaussian eigenfunctions $(T_\ell)_\ell$ can be always expressed as stochastic integrals with respect to a Gaussian white noise measure on $S^d$, as seen in Chapter 4. As a consequence, the random sequences $h_{\ell,q,d}$ can themselves be represented as multiple Wiener-Ito integrals, and therefore fall inside the domain of quantitative CLTs by means of the Nourdin-Peccati approach (Chapter 4). It is thus sufficient to investigate the so-called circular components of their normalized fourth-order cumulants (Proposition 4.1.4) to establish the following Proposition 5.1.3.
Proposition 5.1.3. For all \( d, q \geq 2 \) and \( D \in \{ K, TV, W \} \) we have, as \( \ell \to +\infty \),

\[
d_D \left( \frac{h_{\ell,q,d}}{\sqrt{\Var[h_{\ell,q,d}]}} Z \right) = O \left( \ell^{-\delta(q,d)} (\log \ell)^{-\eta(q,d)} \right),
\]

where for \( d = 2 \)

\[
\delta(2; 2) = \delta(3; 2) = 1/2, \quad \delta(4; 2) = 0, \quad \delta(q; 2) = 1/4 \quad \text{for } q \geq 5;
\]
\[
\eta(4; 2) = 1, \quad \eta(5; 2) = \eta(6; 2) = -1, \quad \delta(q; 2) = 0 \quad \text{for } q = 2, 3 \text{ and for } q \geq 7;
\]

whereas for \( d \geq 3 \) we have

\[
\eta(q; d) = 0 \quad \text{for } q \geq 2;
\]

\[
\delta(2; d) = (d - 1)/2, \quad \delta(3; d) = (d - 5)/2, \quad \delta(4; d) = (d - 3)/2
\]

\[
\delta(q; d) = (d - 1)/4 \quad \text{for } q \geq 5.
\]

Let us set \( R(\ell; q, d) := \ell^{-\delta(q,d)} (\log \ell)^{-\eta(q,d)} \). The following corollary is immediate.

Corollary 5.1.4. For all \( q \) such that \( (d, q) \neq (3,3), (3,4), (4,3), (5,3) \) and \( c_{q,d} > 0, \ d \geq 2, \)

\[
\frac{h_{2\ell,q,d}}{\sqrt{\Var[h_{2\ell,q,d}]}} \xrightarrow{\mathcal{L}} Z, \quad \text{as } \ell \to +\infty,
\]

where \( Z \sim N(0,1) \).

Remark 5.1.5. For \( d = 2 \), the CLT in (5.1.16) was already provided by [47]; nevertheless Theorem 5.1.3 improves the existing bounds on the speed of convergence to the asymptotic Gaussian distribution. More precisely, for \( d = 2, q = 2, 3, 4 \) the same rate of convergence as in (5.1.15) was given in their Proposition 3.4; however for arbitrary \( q \) the total variation rate was only shown to satisfy (up to logarithmic terms)

\[
d_{TV} = O(\ell^{-\delta_4}),
\]

where \( \delta_4 = \frac{1}{10}, \delta_5 = \frac{1}{7}, \) and \( \delta_q = \frac{q - 6}{4q - 6} < \frac{1}{4} \) for \( q \geq 7 \).

Remark 5.1.6. The cases not included in Corollary 5.1.4 correspond to the pairs where \( q = 4 \) and \( d = 3, \) or \( q = 3 \) and \( d = 3, 4, 5; \) in these circumstances the bounds we establish on fourth-order cumulants in Proposition 5.3.3 are not sufficient to ensure that the CLT holds. We leave these computations as a topic for future research.
As briefly anticipated earlier in this subsection, the random variables $h_{\ell q,d}$ defined in (5.1.3) are the basic building blocks for the analysis of any square integrable nonlinear transforms of Gaussian eigenfunctions on $S^d$. Indeed, let us first consider generic polynomial functionals of the form

$$Z_{\ell} = \sum_{q=0}^{Q} b_q \int_{S^d} T_\ell(x)^q \, dx , \quad Q \in \mathbb{N}, \ b_q \in \mathbb{R}, \quad (5.1.17)$$

which include, for instance, the so-called polyspectra (see e.g. [40], p.148) of isotropic random fields defined on $S^d$. Note

$$Z_{\ell} = \sum_{q=0}^{Q} \beta_q h_{2\ell q,d}$$

for some $\beta_q \in \mathbb{R}$. It is easy to establish CLTs for generic polynomials (5.1.18) from convergence results on $h_{2\ell q,d}$, see e.g. [57]. It is more difficult to investigate the speed of convergence in the CLT in terms of the probability metrics we introduced earlier; indeed, in §5.4 we establish the following.

**Proposition 5.1.7.** As $\ell \to \infty$, for $Z \sim \mathcal{N}(0,1)$

$$d_D \left( \frac{Z_{\ell} - \mathbb{E}[Z_{\ell}]}{\sqrt{\text{Var}[Z_{\ell}]}} , Z \right) = O(R(Z_{\ell};d)) ,$$

where $d_D = d_{TV}, d_W, d_K$ and for $d \geq 2$

$$R(Z_{\ell};d) = \begin{cases} \ell^{-\left(\frac{d}{2}\right)} & \text{if } \beta_2 \neq 0 , \\ \max_{q = 3, \ldots , Q} \beta_q c_q d, \beta_q c_q, d \neq 0 \ R(\ell ;q,d) & \text{if } \beta_2 = 0 . \end{cases}$$

The previous results can be summarized as follows: for polynomials of *Hermite rank* 2, i.e. such that $\beta_2 \neq 0$ (more details later on the notion of Hermite rank) the asymptotic behavior of $Z_{\ell}$ is dominated by the single term $h_{\ell 2,d}$, whose variance is of order $\ell^{-d-1}$ rather than $\ell^{-d}$ as for the other terms. On the other hand, when $\beta_2 = 0$, the convergence rate to the asymptotic Gaussian distribution for a generic
polynomial is the slowest among the rates for the Hermite components into which \( Z_\ell \) can be decomposed, i.e. the terms \( \beta_q h_{2\ell, q, d} \) in (5.1.18).

The fact that the bound for generic polynomials is of the same order as for the Hermite case (and not slower) is indeed rather unexpected; it can be shown to be due to the cancelation of some cross-product terms, which are dominant in the general Nourdin-Peccati framework, while they vanish for spherical eigenfunctions of arbitrary dimension (see (5.4.46) and Remark 5.4.1). An inspection of our proof will reveal that this result is a by-product of the orthogonality of eigenfunctions corresponding to different eigenvalues; it is plausible that similar ideas may be exploited in many related circumstances, for instance random eigenfunction on generic compact Riemannian manifolds.

Proposition 5.1.7 shows that the asymptotic behavior of arbitrary polynomials of Hermite rank 2 is of particularly simple nature. Our result below will show that this feature holds in much greater generality, at least as far as the Wasserstein distance \( d_W \) is concerned. Indeed, we shall consider the case of functionals of the form

\[
S_\ell(M) = \int_{\mathbb{R}^d} M(T_\ell(x)) \, dx ,
\]

where \( M : \mathbb{R} \to \mathbb{R} \) is some measurable function such that \( \mathbb{E}[M(Z)^2] < +\infty \), where \( Z \sim \mathcal{N}(0, 1) \). As in Chapter 4, for such transforms the following chaos expansion holds in the \( L^2(\mathbb{P}) \)-sense (4.1.5)

\[
M(T_\ell) = \sum_{q=0}^{\infty} \frac{J_q(M)}{q!} H_q(T_\ell) , \quad J_q(M) := \mathbb{E}[M(T_\ell) H_q(T_\ell)] .
\]

Therefore the asymptotic analysis, as \( \ell \to \infty \), of \( S_\ell(M) \) in (5.1.19) directly follows from the Gaussian approximation for \( h_{\ell, q, d} \) and their polynomial transforms \( Z_\ell \). More precisely, in §5.5 we prove the following result.

**Proposition 5.1.8.** Let \( Z \sim \mathcal{N}(0, 1) \). For functions \( M \) in (5.1.19) such that

\[
\mathbb{E}[M(Z) H_2(Z)] = J_2(M) \neq 0 ,
\]
we have
\[
d_W \left( \frac{S_{2\ell}(M) - \mathbb{E}[S_{2\ell}(M)]}{\sqrt{\text{Var}[S_{2\ell}(M)]}}, Z \right) = O(\ell^{-1/2}) , \quad \text{as } \ell \to \infty , \tag{5.1.21}
\]
in particular
\[
\frac{S_{2\ell}(M) - \mathbb{E}[S_{2\ell}(M)]}{\sqrt{\text{Var}[S_{2\ell}(M)]}} \xrightarrow{\mathcal{L}} Z . \tag{5.1.22}
\]

Proposition 5.1.8 provides a Breuer-Major like result on nonlinear functionals, in the high-frequency limit (compare for instance [?]). While the CLT in (5.1.22) is somewhat expected, the square-root speed of convergence (5.1.21) to the limiting distribution may be considered quite remarkable; it is mainly due to some specific features in the chaos expansion of Gaussian eigenfunctions, which is dominated by a single term at \( q = 2 \). Note that the function \( M \) need not be smooth in any meaningful sense; indeed, as explained above, our main motivating rationale here is the analysis of the asymptotic behavior of the excursion volume in (5.1.1) \( S_{\ell}(z) = S_{\ell}(M) \), where \( M(\cdot) = M_z(\cdot) = 1(\cdot > z) \) is again the indicator function of the interval \((z, +\infty)\). An application of Proposition 5.1.8 (compare (5.1.4) to (5.1.20)) provides a quantitative CLT for \( S_{\ell}(z) \), \( z \neq 0 \), thus completing the proof of our main result.

The plan of the rest of this chapter is as follows: in §5.2 we specialize results in Chapter 4 to the hypersphere, in §5.3 we establish the quantitative CLT for the sequences \( h_{\ell,q,d} \), while §5.4 extends these results to generic finite-order polynomials. The results for general nonlinear transforms and excursion volumes are given in §5.5; most technical proofs and (hard) estimates, including in particular evaluations of asymptotic variances, are collected in §5.6.

### 5.2 Polynomial transforms in Wiener chaoses

As mentioned earlier in §5.1, we shall be concerned first with random variables \( h_{\ell,q,d} \), \( \ell \geq 1, q,d \geq 2 \)
\[
h_{\ell,q,d} = \int_{\mathbb{S}^d} H_q(T_{\ell}(x)) \, dx ,
\]
and their (finite) linear combinations

\[ Z_\ell = \sum_{q=2}^{Q} \beta_q h_{\ell,q,d}, \quad \beta_q \in \mathbb{R}, Q \in \mathbb{N}. \] (5.2.23)

Our first objective is to represent (5.2.23) as a (finite) sum of (multiple) stochastic integrals as in (4.1.7), in order to apply the results recalled in Chapter 4. Note that by (4.1.6), we have

\[ H_q(T_\ell(x)) = I_q(f_{x}^{\otimes q}) = \int_{(S^d)^q} \left( \frac{n_{\ell,d}}{\mu_d} \right)^{q/2} G_{\ell,d}(\cos d(x, y_1)) \cdots G_{\ell,d}(\cos d(x, y_q)) \, dW(y_1) \cdots dW(y_q), \]

so that

\[ h_{\ell,q,d} \equiv \int_{(S^d)^q} g_{\ell,q}(y_1, \ldots, y_q) \, dW(y_1) \cdots dW(y_q), \]

where

\[ g_{\ell,q}(y_1, \ldots, y_q) := \int_{S^d} \left( \frac{n_{\ell,d}}{\mu_d} \right)^{q/2} G_{\ell,d}(\cos d(x, y_1)) \cdots G_{\ell,d}(\cos d(x, y_q)) \, dx. \] (5.2.24)

Thus we just established that \( h_{\ell,q,d} \equiv I_q(g_{\ell,q}) \) and therefore

\[ Z_\ell \equiv \sum_{q=2}^{Q} I_q(\beta_q g_{\ell,q}), \] (5.2.25)

as required. It should be noted that for such random variables \( Z_\ell \), the conditions of the Proposition 4.1.4 are trivially satisfied.

### 5.3 The quantitative CLT for Hermite transforms

In this section we prove Proposition 5.1.3 with the help of Proposition 4.1.4 and (4.1.13) in particular. The identifications of §5.2 lead to some very explicit expressions for the contractions (4.1.9), as detailed in the following result.

For \( \ell \geq 1, q \geq 2 \), let \( g_{\ell,q} \) be defined as in (5.2.24).
Lemma 5.3.1. For all $q_1, q_2 \geq 2$, $r = 1, \ldots, q_1 \wedge q_2 - 1$, we have the identities
\[
\|g_{\ell,q_1} \otimes_r g_{\ell,q_2}\|_{H^{\otimes n}}^2 = \int_{(\mathbb{S}^d)^n} G^r_{\ell,d} \cos d(x_1, x_2) G^{q_1, q_2 - r}_{\ell,d}(\cos d(x_2, x_3)) G^r_{\ell,d}(\cos d(x_3, x_4)) \times G^{q_1, q_2 - r}_{\ell,d}(\cos d(x_1, x_4)) \, dx,
\]
where we set $dx := dx_1 dx_2 dx_3 dx_4$ and $n := q_1 + q_2 - 2r$.

Proof. Assume w.l.o.g. $q_1 \leq q_2$ and set for simplicity of notation $dt := dt_1 \ldots dt_r$. The contraction (4.1.9) here takes the form
\[
(g_{\ell,q_1} \otimes_r g_{\ell,q_2})(y_1, \ldots, y_n) = \int_{(\mathbb{S}^d)^r} g_{\ell,q_1}(y_1, \ldots, y_{q_1 - r}, t_1, \ldots, t_r) g_{\ell,q_2}(y_{q_1 - r + 1}, \ldots, y_n, t_1, \ldots, t_r) \, dt = \int_{(\mathbb{S}^d)^r} \left( \frac{n_{\ell,d}}{\mu_d} \right)^{q_1/2} G_{\ell,d}(\cos d(x_1, y_1)) \ldots G_{\ell,d}(\cos d(x_1, t_r)) dx_1 \times \int_{\mathbb{S}^d} \left( \frac{n_{\ell,d}}{\mu_d} \right)^{q_2/2} G_{\ell,d}(\cos d(x_2, y_{q_1 - r + 1})) \ldots G_{\ell,d}(\cos d(x_2, t_r)) dx_2 \, dt = \int_{(\mathbb{S}^d)^2} \left( \frac{n_{\ell,d}}{\mu_d} \right)^{n/2} G_{\ell,d}(\cos d(x_1, y_1)) \ldots G_{\ell,d}(\cos d(x_1, y_{q_1 - r})) \times G_{\ell,d}(\cos d(x_2, y_{q_1 - r + 1})) \ldots G_{\ell,d}(\cos d(x_2, y_n)) G^r_{\ell,d}(\cos d(x_1, x_2)) \, dx_1 dx_2,
\]
where in the last equality we have repeatedly used the reproducing property of Gegenbauer polynomials ([64]). Now set $dy := dy_1 \ldots dy_n$. It follows at once that
\[
\|g_{\ell,q_1} \otimes_r g_{\ell,q_2}\|_{H^{\otimes n}}^2 = \int_{(\mathbb{S}^d)^n} (g_{\ell,q_1} \otimes_r g_{\ell,q_2})^2(y_1, \ldots, y_n) \, dy = \int_{(\mathbb{S}^d)^n} \int_{(\mathbb{S}^d)^2} \left( \frac{n_{\ell,d}}{\mu_d} \right)^n G_{\ell,d}(\cos d(x_1, y_1)) \ldots G_{\ell,d}(\cos d(x_2, y_n)) G^r_{\ell,d}(\cos d(x_1, x_2)) dx_1 dx_2 \times \int_{\mathbb{S}^d} G^r_{\ell,d}(\cos d(x_4, y_1)) \ldots G^r_{\ell,d}(\cos d(x_3, y_n)) G_{\ell,d}(\cos d(x_3, x_4)) \, dx_3 dx_4 \, dy = \int_{(\mathbb{S}^d)^4} G^r_{\ell,d}(\cos d(x_1, x_2)) G^{q_1 - r}_{\ell,d}(\cos d(x_2, x_3)) G_{\ell,d}(\cos d(x_3, x_4)) G^{q_2 - r}_{\ell,d}(\cos d(x_1, x_4)) \, dx,
\]
as claimed. \qed
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We need now to introduce some further notation, i.e. for \( q \geq 2 \) and \( r = 1, \ldots, q - 1 \)
\[
K_{\ell}(q; r) := \int_{(S^d)^4} G_{\ell,d}^r(\cos d(x_1, x_2)) G_{\ell,d}^{q-r}(\cos d(x_2, x_3)) \times \\
G_{\ell,d}^r(\cos d(x_3, x_4)) G_{\ell,d}^{q-r}(\cos d(x_1, x_4)) \, dx_1 dx_2 dx_3 dx_4,
\]
Lemma 5.3.1 asserts that
\[
K_{\ell}(q; r) = \| g_{\ell,q} \otimes r g_{\ell,q} \|^2_{H^{|q-2r|}} ; \tag{5.3.26}
\]
it is immediate to check that
\[
K_{\ell}(q; r) = K_{\ell}(q; q - r) . \tag{5.3.27}
\]
In the following two propositions we bound each term of the form \( K(q; r) \) (from (5.3.27) it is enough to consider \( r = 1, \ldots, \left[ \frac{q}{2} \right] \)). As noted in §5.1.1, these bounds improve the existing literature even for the case \( d = 2 \), from which we start our analysis.

For \( d = 2 \), as previously recalled, Gegenbauer polynomials become standard Legendre polynomials \( P_\ell \), for which it is well-known that (see (5.1.9))
\[
\int_{S^2} P_\ell(\cos d(x_1, x_2))^2 \, dx_1 = O \left( \frac{1}{\ell} \right) ; \tag{5.3.28}
\]
also, from [47], Lemma 3.2 we have that
\[
\int_{S^2} P_\ell(\cos d(x_1, x_2))^4 \, dx_1 = O \left( \frac{\log \ell}{\ell^2} \right) . \tag{5.3.29}
\]
Finally, it is trivial to show that
\[
\int_{S^2} |P_\ell(\cos d(x_1, x_2))| \, dx_1 \leq \sqrt{\int_{S^2} P_\ell(\cos d(x_1, x_2))^2 \, dx_1} = O \left( \frac{1}{\sqrt{\ell}} \right) \tag{5.3.30}
\]
and by Cauchy-Schwartz inequality
\[
\int_{S^2} |P_\ell(\cos d(x_1, x_2))^3 | \, dx_2 = O \left( \sqrt{\frac{\log \ell}{\ell^3}} \right) . \tag{5.3.31}
\]
These results will be the main tools to establish the upper bounds which are collected in the following Proposition, whose proof is deferred to the last section.
Proposition 5.3.2. For all \( r = 1, 2, \ldots, q - 1 \), we have
\[
\mathcal{K}_\ell(q; r) = O\left(\frac{1}{\ell^5}\right) \text{ for } q = 3 ,
\]
\[
\mathcal{K}_\ell(q; r) = O\left(\frac{1}{\ell^4}\right) \text{ for } q = 4 ,
\]
\[
\mathcal{K}_\ell(q; r) = O\left(\frac{\log \ell}{\ell^{9/2}}\right) \text{ for } q = 5, 6
\]
and
\[
\mathcal{K}_\ell(q; 1) = \mathcal{K}_\ell(q; q - 1) = O\left(\frac{1}{\ell^{9/2}}\right) , \mathcal{K}_\ell(q; r) = O\left(\frac{1}{\ell^5}\right) , \text{ } r = 2, \ldots, q - 2, \text{ for } q \geq 7 .
\]

We can now move to the higher-dimensional case, as follows. Let us start with the bounds for all order moments of Gegenbauer polynomials. From (5.1.9)
\[
\int_{S^d} G_{\ell,d}(\cos d(x_1, x_2))^2 dx_1 = O\left(\frac{1}{\ell^{d-1}}\right) ;
\]
also, from Proposition 5.1.2, we have that if \( q = 2p, p = 2, 3, 4..., \)
\[
\int_{S^d} G_{\ell,d}(\cos d(x_1, x_2))^q dx_1 = O\left(\frac{1}{\ell^d}\right) .
\]

Finally, it is trivial to show that
\[
\int_{S^d} |G_{\ell,d}(\cos d(x_2, x_3))| dx_2 \leq 
\]
\[
\leq \sqrt{\int_{S^d} G_{\ell,d}(\cos d(x_2, x_3))^2 dx_2} = O\left(\frac{1}{\ell^{d-1}}\right) ,
\]
\[
\int_{S^d} |G_{\ell,d}(\cos d(x_2, x_3))|^3 dx_2 \leq 
\]
\[
\leq \sqrt{\int_{S^d} G_{\ell,d}(\cos d(x_2, x_3))^2 dx_2} \sqrt{\int_{S^d} G_{\ell,d}(\cos d(x_1, x_2))^4 dx_1} = O\left(\frac{1}{\ell^{d-1/2}}\right)
\]
Sec. 5.3 - The quantitative CLT for Hermite transforms

and for \( q \geq 5 \) odd,

\[
\int_{S^d} |G_{\ell,d}(\cos d(x_2, x_3))|^q \, dx_2 \leq \sqrt{\int_{S^d} G_{\ell,d}(\cos d(x_2, x_3))^4 \, dx_2} \sqrt{\int_{S^d} G_{\ell,d}(\cos d(x_1, x_2))^{2(q-2)} \, dx_1} = O \left( \frac{1}{\ell^3} \right). \quad (5.3.40)
\]

Analogously to the 2-dimensional case, we can exploit the previous results to obtain the following bounds, whose proof is again collected in §5.6.

**Proposition 5.3.3.** For all \( r = 1, 2, \ldots, q - 1 \)

\[
\mathcal{K}_\ell(q; r) = O \left( \frac{1}{\ell^{2d + \frac{d+1}{2}}} \right) \text{ for } q = 3, \quad (5.3.41)
\]

\[
\mathcal{K}_\ell(q; r) = O \left( \frac{1}{\ell^{2d + \frac{d-1}{2}}} \right) \text{ for } q = 4, \quad (5.3.42)
\]

and for \( r = 2, \ldots, q - 2 \)

\[
\mathcal{K}_\ell(q; 1) = \mathcal{K}_\ell(q; q - 1) = O \left( \frac{1}{\ell^{2d + \frac{d+1}{2}}} \right), \quad \mathcal{K}_\ell(q; r) = O \left( \frac{1}{\ell^{3d-1}} \right) \text{ for } q \geq 5. \quad (5.3.43)
\]

Exploiting the results in this section and the variance evaluation in Proposition 5.1.2 in §5.6, we have the proof of our first quantitative CLT.

**Proof Proposition 5.1.3.** By Parseval’s identity, the case \( q = 2 \) can be treated as a sum of independent random variables and the proof follows from standard Berry-Esseen arguments, as in Lemma 8.3 of [40] for the case \( d = 2 \). For \( q \geq 3 \), from Proposition 4.1.4 and (4.1.13), for \( d_D = d_K, d_{TV}, d_W \)

\[
d_D \left( \frac{h_{\ell,q}}{\sqrt{\text{Var}[h_{\ell,q,d}]}} Z \right) = O \left( \sup_r \sqrt{\frac{\mathcal{K}_\ell(q; r)}{\text{Var}[h_{\ell,q,d}]}} \right). \quad (5.3.44)
\]

The proof is thus an immediate consequence of the previous equality and the results in Proposition 5.1.2, Proposition 5.3.2 and Proposition 5.3.3. \qed
5.4 General polynomials

In this section, we show how the previous results can be extended to establish quantitative CLTs, for the case of general, non Hermite polynomials. To this aim, we need to introduce some more notation, namely (for $Z_\ell$ defined as in (5.2.23))

$$K(\ell; d) := \max_{q: \beta_q \neq 0} \max_{r=1, \ldots, q-1} K(\ell; q; r),$$

and as in Proposition 5.1.7

$$R(\ell; q, d) = \begin{cases} \ell^{-\frac{d+1}{2}}, & \text{for } \beta_2 \neq 0, \\ \max_{q=3, \ldots, Q} R(\ell; q, d), & \text{for } \beta_2 = 0. \end{cases}$$

In words, $K(\ell; d)$ is the largest contraction term among those emerging from the analysis of the different Hermite components, and $R(\ell; d)$ is the slowest convergence rate of the same components. The next result is stating that these are the only quantities to look at when considering the general case.

**Proof Theorem 5.1.7.** We apply Proposition 4.1.4. In our case $H = L^2(\mathbb{S}^d)$ and

$$\text{Var}[\langle D Z_\ell, -DL^{-1} Z_\ell \rangle_H] = \text{Var} \left[ \sum_{q_1=2}^{Q} \beta_{q_1} Dh_{\ell; q_1, d}, - \sum_{q_2=2}^{Q} \beta_{q_2} DL^{-1} h_{\ell; q_2, d} \right] =$$

$$= \text{Var} \left[ \sum_{q_1=2}^{Q} \sum_{q_2=2}^{Q} \beta_{q_1} \beta_{q_2} \langle Dh_{\ell; q_1, d}, DL^{-1} h_{\ell; q_2, d} \rangle_H \right].$$

From Chapter 4 recall that for $q_1 \neq q_2$

$$E[\langle Dh_{\ell; q_1, d}, DL^{-1} h_{\ell; q_2, d} \rangle_H] = 0.$$
Sec. 5.4 - General polynomials

whence we write

\[
\text{Var} \left[ \sum_{q_1=2}^{Q} \sum_{q_2=2}^{Q} \beta_{q_1} \beta_{q_2} \langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d} \rangle_H \right] = \\
= \sum_{q_1=2}^{Q} \sum_{q_2=2}^{Q} \beta_{q_1}^2 \beta_{q_2}^2 \text{Cov} \left( \langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d} \rangle_H, \langle Dh_{\ell;q_2,d}, -DL^{-1}h_{\ell;q_2,d} \rangle_H \right) + \\
+ \sum_{q_1,q_3=2}^{Q} \sum_{q_2 \neq q_1}^{Q} \sum_{q_4 \neq q_3}^{Q} \beta_{q_1} \beta_{q_2} \beta_{q_3} \beta_{q_4} \times \\
\times \text{Cov} \left( \langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d} \rangle_H, \langle Dh_{\ell;q_3,d}, -DL^{-1}h_{\ell;q_4,d} \rangle_H \right).
\]

Now of course we have

\[
\text{Cov} \left( \langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d} \rangle_H, \langle Dh_{\ell;q_2,d}, -DL^{-1}h_{\ell;q_2,d} \rangle_H \right) \leq \\
\leq \left( \text{Var} \left[ \langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d} \rangle_H \right] \text{Var} \left[ \langle Dh_{\ell;q_2,d}, -DL^{-1}h_{\ell;q_2,d} \rangle_H \right] \right)^{1/2},
\]

\[
\text{Cov} \left( \langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d} \rangle_H, \langle Dh_{\ell;q_3,d}, -DL^{-1}h_{\ell;q_4,d} \rangle_H \right) \leq \\
\leq \left( \text{Var} \left[ \langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_2,d} \rangle_H \right] \text{Var} \left[ \langle Dh_{\ell;q_3,d}, -DL^{-1}h_{\ell;q_4,d} \rangle_H \right] \right)^{1/2}.
\]

Applying [53], Lemma 6.2.1 it is immediate to show that

\[
\text{Var} \left[ \langle Dh_{\ell;q_1,d}, -DL^{-1}h_{\ell;q_1,d} \rangle_H \right] \leq \\
\leq q_1^2 \sum_{r=1}^{q_1-1} ((r-1)!)^2 \left( \frac{q_1 - 1}{r - 1} \right)^4 (2q_1 - 2r)! \| g_{\ell;q_1} \otimes_r g_{\ell;q_1} \|_{H^2 \otimes_{q_1^2} -2r} = \\
= q_1^2 \sum_{r=1}^{q_1-1} ((r-1)!)^2 \left( \frac{q_1 - 1}{r - 1} \right)^4 (2q_1 - 2r)! \mathcal{K}_\ell(q_1; r).
\]

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Also, for \( q_1 < q_2 \)
\[
\text{Var} \left[ \langle Dh_{\ell,q_1,d}, -DL^{-1}h_{\ell,q_2,d} \rangle_H \right] = \]
\[
= q_1^2 \sum_{r=1}^{q_1} ((r-1)!)^2 \left( \frac{q_1 - 1}{r - 1} \right)^2 \left( \frac{q_2 - 1}{r - 1} \right)^2 (q_1 + q_2 - 2r)! \left\| g_{\ell,q_1} \otimes_r g_{\ell,q_2} \right\|^2_{H^{\otimes (q_1 + q_2 - 2r)}} = \]
\[
= q_1^2 ((q_1 - 1)!)^2 \left( \frac{q_2 - 1}{q_1 - 1} \right)^2 (2q_2 - 2r)! \left\| g_{\ell,q_1} \otimes_{q_1} g_{\ell,q_2} \right\|^2_{H^{\otimes (q_2 - q_1)}} + \]
\[
+ q_1 \sum_{r=1}^{q_1-1} ((r-1)!)^2 \left( \frac{q_1 - 1}{r - 1} \right)^2 \left( \frac{q_2 - 1}{r - 1} \right)^2 (q_1 + q_2 - 2r)! \left\| g_{\ell,q_1} \otimes_r g_{\ell,q_2} \right\|^2_{H^{\otimes (q_1 + q_2 - 2r)}} =: A + B .
\]

Let us focus on the first summand \( A \), which includes terms that, from Lemma 5.3.1, take the form
\[
\left\| g_{\ell,q_1} \otimes_{q_1} g_{\ell,q_2} \right\|^2_{H^{\otimes (q_2 - q_1)}} \leq \left\| g_{\ell,q_1} \otimes_{q_1} g_{\ell,q_2} \right\|^2_{H^{\otimes (q_2 - q_1)}} = \]
\[
= \int_{(S^d)^{q_2-q_1}} \int_{(S^d)^2} \left( \frac{n_{\ell,d}}{\mu_d} \right)^{q_2-q_1} G_{\ell,d}(\cos d(x_2, y_1)) \cdots \times \]
\[
\times \cdots G_{\ell,d}(\cos d(x_2, y_{q_2-q_1})) G_{\ell,d}(\cos d(x_1, x_2))^{q_1} \, dx_1 \, dx_2 \times \]
\[
\times \int_{(S^d)^2} G_{\ell,d}(\cos d(x_3, y_1)) \cdots G_{\ell,d}(\cos d(x_3, y_{q_2-q_1})) G_{\ell,d}(\cos d(x_3, x_4))^{q_1} \, dx_3 \, dx_4 \, dy =: I ,
\]
where for the sake of simplicity we have set \( dy := dy_1 \cdots dy_{q_2-q_1} \). Applying \( q_2 - q_1 \) times the reproducing formula for Gegenbauer polynomials ([64]) we get
\[
I = \int_{(S^d)^4} G_{\ell,d}(\cos d(x_1, x_2))^{q_1} G_{\ell,d}(\cos d(x_2, x_3))^{q_2-q_1} G_{\ell,d}(\cos d(x_3, x_4))^{q_1} \, dx_1 \, dx_2 \, dx_3 \, dx_4 . \quad (5.4.45)
\]

In graphical terms, these contractions correspond to the diagrams such that all \( q_1 \) edges corresponding to vertex 1 are linked to vertex 2, vertex 2 and 3 are connected by \( q_2 - q_1 \) edges, vertex 3 and 4 by \( q_1 \) edges, and no edges exist between 1 and 4, i.e. the diagram has no proper loop. Now immediately we write
\[
(5.4.45) = \int_{S^d} G_{\ell,d}(\cos d(x_1, x_2))^{q_1} \, dx_1 \int_{S^d} G_{\ell,d}(\cos d(x_3, x_4))^{q_1} \, dx_4 \times \]
\[
\times \int_{(S^d)^2} G_{\ell,d}(\cos d(x_2, x_3))^{q_2-q_1} \, dx_2 \, dx_3 = \]
\[
= \frac{1}{(q_1!)^2} \text{Var} \left[ h_{\ell,q_1,d} \right] \int_{(S^d)^2} G_{\ell,d}(\cos d(x_2, x_3))^{q_2-q_1} \, dx_2 \, dx_3 .
\]
Moreover we have
\[ \int_{(\mathbb{S}^d)^2} G_{\ell,d}(\cos d(x_2, x_3))^{q_2 - q_1} dx_2 dx_3 = 0, \quad \text{if } q_2 - q_1 = 1 \tag{5.4.46} \]
and from (5.1.9) if \( q_2 - q_1 \geq 2 \)
\[ \int_{(\mathbb{S}^d)^2} G_{\ell,d}(\cos d(x_2, x_3))^{q_2 - q_1} dx_2 dx_3 \leq \mu_d \int_{\mathbb{S}^d} G_{\ell,d}(\cos d(x,y))^2 dx = O \left( \frac{1}{\ell^{d-1}} \right). \]
It follows that
\[ \| g_{\ell,q_1} \otimes q_1 g_{\ell,q_2} \|_{H^\otimes(q_2 - q_1)}^2 = O \left( \var[h_{\ell,q_1,d}]^2 \frac{1}{\ell^{d-1}} \right) \tag{5.4.47} \]
always. For the second term, still from [53], Lemma 6.2.1 we have
\[ B \leq \frac{q_1^2}{2} \sum_{r=1}^{q_1-1} ((r-1)!)^2 \left( \begin{array}{c} q_1 - 1 \\ r - 1 \end{array} \right) \left( \begin{array}{c} q_2 - 1 \\ r - 1 \end{array} \right) (q_1 + q_2 - 2r)! \times \]
\[ \times \left( \| g_{\ell,q_1} \otimes q_1 - r g_{\ell,q_1} \|_{H^\otimes2r}^2 + \| g_{\ell,q_2} \otimes q_2 - r g_{\ell,q_2} \|_{H^\otimes2r}^2 \right) = \]
\[ = \frac{q_1^2}{2} \sum_{r=1}^{q_1-1} ((r-1)!)^2 \left( \begin{array}{c} q_1 - 1 \\ r - 1 \end{array} \right) \left( \begin{array}{c} q_2 - 1 \\ r - 1 \end{array} \right) (q_1 + q_2 - 2r)! (K_{\ell}(q_1; r) + K_{\ell}(q_2; r)), \tag{5.4.48} \]
where the last step follows from Lemma 5.3.1.

Let us first investigate the case \( d = 2 \). From (5.1.10), (5.1.11) and (5.1.13) it is immediate that
\[ \var[Z_\ell] = \sum_{q=2}^{Q} \beta_q^2 \var[h_{\ell,q;2}] = \begin{cases} O(\ell^{-1}), & \text{for } \beta_2 \neq 0 \\ O(\ell^{-2} \log \ell), & \text{for } \beta_2 = 0, \beta_4 \neq 0 \\ O(\ell^{-2}), & \text{otherwise}. \end{cases} \tag{5.4.49} \]
Hence we have that for \( \beta_2 \neq 0 \) and \( Z \sim \mathcal{N}(0,1) \)
\[ d_{TV} \left( \frac{Z_\ell - EZ_\ell}{\sqrt{\var[Z_\ell]}}, Z \right) = O \left( \frac{\sqrt{K_{\ell}(2; r)}}{\sqrt{\var[Z_\ell]}} \right) = O \left( \ell^{-1/2} \right); \]
for $\beta_2 = 0$, $\beta_4 \neq 0$,

$$d_{TV} \left( \frac{Z_\ell - E Z_\ell}{\sqrt{\text{Var}[Z_\ell]}}, Z \right) = O \left( \frac{\sqrt{K_\ell(4; r)}}{\text{Var}[Z_\ell]} \right) = O \left( \frac{1}{\log \ell} \right)$$

and for $\beta_2 = \beta_4 = 0$, $\beta_5 \neq 0$ and $c_5 > 0$

$$d_{TV} \left( \frac{Z_\ell - E Z_\ell}{\sqrt{\text{Var}[Z_\ell]}}, Z \right) = O \left( \frac{\sqrt{K_\ell(5; r)}}{\text{Var}[Z_\ell]} \right) = O \left( \frac{\log \ell}{\ell^{1/4}} \right),$$

and analogously we deal with the remaining cases, so that we obtain the claimed result for $d = 2$.

For $d \geq 3$ from (5.1.9) and Proposition 5.1.2, it holds

$$\text{Var}[Z_\ell] = \sum_{q=2}^{Q} \beta_q^2 \text{Var}[h_{\ell q, d}] = \begin{cases} O(\ell^{-(d-1)}) , & \text{for } \beta_2 \neq 0 , \\ O(\ell^{-d}) , & \text{otherwise} . \end{cases}$$

Hence we have for $\beta_2 \neq 0$

$$d_{TV} \left( \frac{Z_\ell - E Z_\ell}{\sqrt{\text{Var}[Z_\ell]}}, Z \right) = O \left( \frac{\sqrt{K_\ell(2; r)}}{\text{Var}[Z_\ell]} \right) = O \left( \frac{1}{\ell^{d-2}} \right).$$

Likewise for $\beta_2 = 0$, $\beta_3, c_{3,d} \neq 0$,

$$d_{TV} \left( \frac{Z_\ell - E Z_\ell}{\sqrt{\text{Var}[Z_\ell]}}, Z \right) = O \left( \frac{\sqrt{K_\ell(3; r)}}{\text{Var}[Z_\ell]} \right) = O \left( \frac{1}{\ell^{d-2}} \right)$$

and for $\beta_2 = \beta_3 = 0$, $\beta_4 \neq 0$

$$d_{TV} \left( \frac{Z_\ell - E Z_\ell}{\sqrt{\text{Var}[Z_\ell]}}, Z \right) = O \left( \frac{\sqrt{K_\ell(4; r)}}{\text{Var}[Z_\ell]} \right) = O \left( \frac{1}{\ell^{d-2}} \right).$$

Finally if $\beta_2 = \beta_3 = \beta_4 = 0$, $\beta_q, c_{q,d} \neq 0$ for some $q$, then

$$d_{TV} \left( \frac{Z_\ell - E Z_\ell}{\sqrt{\text{Var}[Z_\ell]}}, Z \right) = O \left( \frac{\sqrt{K_\ell(q; r)}}{\text{Var}[Z_\ell]} \right) = O \left( \sqrt{\frac{\ell^{2d}}{\ell^{2d+4} - 2}} \right) = O \left( \frac{1}{\ell^{d+2}} \right).$$
Remark 5.4.1. To compare our result in these specific circumstances with the general bound obtained by Nourdin and Peccati, we note that for (5.4.45), these authors are exploiting the inequality

\[ \| g_{\ell; q_1} \otimes g_{\ell; q_2} \|_{H^{\otimes (q_2 - q_1)}}^2 \leq \| g_{\ell; q_1} \|_{H^{\otimes q_1}}^2 \| g_{\ell; q_2} \otimes g_{\ell; q_2} \|_{H^{\otimes q_1}} \],

see [53], Lemma 6.2.1. In the special framework we consider here (i.e., orthogonal eigenfunctions), this provides, however, a less efficient bound than (5.4.47): indeed from (5.4.45), repeating the same argument as in Lemma 5.3.1, one obtains

\[
\begin{align*}
\| g_{\ell; q_1} \otimes g_{\ell; q_2} \|_{H^{\otimes (q_2 - q_1)}}^2 &= \int_{(S^d)^4} G_{\ell,d}(\cos d(x_1, x_2))^{q_1} G_{\ell,d}(\cos d(x_2, x_3))^{q_2 - q_1} \times \\
&\quad \times G_{\ell,d}(\cos d(x_3, x_4))^{q_1} d\overline{x} \leq \int_{(S^d)^2} G_{\ell,d}(\cos d(x_1, x_2))^{q_1} dx_1 dx_2 \times \\
&\quad \times \left( \int_{(S^d)^4} G_{\ell,d}(\cos d(x_1, x_2))^{q_1} G_{\ell,d}(\cos d(x_2, x_3))^{q_2 - q_1} \times \\
&\quad \times G_{\ell,d}(\cos d(x_3, x_4))^{q_1} G_{\ell,d}(\cos d(x_1, x_4))^{q_2 - q_1} \right)^{1/2} d\overline{x} = \\
&= O \left( \frac{\text{Var}[h_{\ell,q_1,d}] \sqrt{\mathcal{K}_\ell(q_2, q_1)}}{\text{Var}[h_{\ell,q_1,d}]} \right),
\end{align*}
\]

yielding a bound of order

\[ O \left( \frac{\text{Var}[h_{\ell,q_1,d}] \sqrt{\mathcal{K}_\ell(q_2, q_1)}}{\text{Var}[h_{\ell,q_1,d}]} \right) \quad (5.4.50) \]

rather than

\[ O \left( \sqrt{\frac{\mathcal{K}_\ell(q_2, q_1)}}{\text{Var}[h_{\ell,q_1,d}]} \right) ; \quad (5.4.51) \]

for instance, for \( d = 2 \) note that (5.4.50) is typically \( = O(\ell \times \ell^{-9/8}) = O(\ell^{-1/8}) \), while we have established for (5.4.51) bounds of order \( O(\ell^{-1/4}) \).

Remark 5.4.2. Clearly the fact that \( \| g_{\ell; q_1} \otimes g_{\ell; q_2} \|_{H^{\otimes (q_2 - q_1)}}^2 = 0 \) for \( q_2 = q_1 + 1 \) entails that the contraction \( g_{\ell; q_1} \otimes g_{\ell; q_2} \) is identically null. Indeed repeating the same
Sec. 5.5 - Nonlinear functionals and excursion volumes

The techniques and results developed previously are restricted to finite-order polynomials. In the special case of the Wasserstein distance, we shall show below how they can indeed be extended to general nonlinear functionals of the form (5.1.19)

\[ S_\ell(M) = \int_{\mathbb{S}^d} M(T_\ell(x)) \, dx ; \]

here \( M : \mathbb{R} \to \mathbb{R} \) is a measurable function such that \( \mathbb{E}[M(Z)^2] < \infty \), \( Z \sim \mathcal{N}(0, 1) \) as in §5.1.1, and \( J_2(M) \neq 0 \), where we recall that \( J_q(M) := \mathbb{E}[M(Z)H_q(Z)] \).

**Remark 5.5.1.** Without loss of generality, the first two coefficients \( J_0(M), J_1(M) \) can always be taken to be zero in the present framework. Indeed, \( J_0(M) := \mathbb{E}[M(Z)] = 0 \), assuming we work with centered variables and moreover as we noted earlier \( h_{\ell,1,d} = \int_{\mathbb{S}^d} T_\ell(x) \, dx = 0 \).

**Proof Proposition 5.1.8.** As in [46], from (5.1.20) we write the expansion

\[ S_\ell(M) = \int_{\mathbb{S}^d} \sum_{q=2}^{\infty} \frac{J_q(M)H_q(T_\ell(x))}{q!} \, dx , \]

Precisely, we write for \( d = 2 \)

\[ S_\ell(M) = \frac{J_2(M)}{2} h_{\ell,2,2} + \frac{J_3(M)}{3!} h_{\ell,3,2} + \frac{J_4(M)}{4!} h_{\ell,4,2} + \int_{\mathbb{S}^2} \sum_{q=5}^{\infty} \frac{J_q(M)H_q(T_\ell(x))}{q!} \, dx , \]

(5.5.52)
whereas for $d \geq 3$

$$S_{\ell}(M) = \frac{J_2(M)}{2} h_{\ell;2,2} + \int_{S^d} \sum_{q=3}^{\infty} \frac{J_q(M) H_q(T_{\ell}(x))}{q!} dx .$$

(5.5.53)

Let us first investigate the case $d = 2$. Set for the sake of simplicity

$$S_{\ell}(M; 1) := \frac{J_2(M)}{2} h_{\ell;2,2} + \frac{J_3(M)}{3!} h_{\ell;3,2} + \frac{J_4(M)}{4!} h_{\ell;4,2} ,$$

$$S_{\ell}(M; 2) := \int_{S^2} \sum_{q=3}^{\infty} \frac{J_q(M) H_q(T_{\ell}(x))}{q!} dx .$$

Consider $Z \sim N(0, 1)$ and $Z_{\ell} \sim N\left(0, \frac{\text{Var}[S_{\ell}(M;1)]}{\text{Var}[S_{\ell}(M)]}\right)$. Hence from (5.5.52) and the triangular inequality

$$d_W \left( \frac{S_{\ell}(M)}{\sqrt{\text{Var}[S_{\ell}(M)]}}, Z \right) \leq$$

$$d_W \left( \frac{S_{\ell}(M)}{\sqrt{\text{Var}[S_{\ell}(M)]}}, \frac{S_{\ell}(M; 1)}{\sqrt{\text{Var}[S_{\ell}(M)]}} \right) + d_W \left( \frac{S_{\ell}(M; 1)}{\sqrt{\text{Var}[S_{\ell}(M)]}}, Z_{\ell} \right) + d_W (Z_{\ell}, Z) \leq$$

$$\leq \frac{1}{\sqrt{\text{Var}[S_{\ell}(M)]}} \mathbb{E} \left[ \left( \int_{S^2} \sum_{q=5}^{\infty} \frac{J_q(M) H_q(T_{\ell}(x))}{q!} dx \right)^2 \right]^{\frac{1}{2}} +$$

$$+ d_W \left( \frac{S_{\ell}(M; 1)}{\sqrt{\text{Var}[S_{\ell}(M)]}}, Z_{\ell} \right) + d_W (Z_{\ell}, Z) .$$

Let us bound the first term of the previous summation. Of course

$$\text{Var}[S_{\ell}(M)] = \text{Var}[S_{\ell}(M; 1)] + \text{Var}[S_{\ell}(M; 2)] ;$$

now we have (see [46])

$$\text{Var}[S_{\ell}(M; 1)] = \frac{J_2^2(M)}{2^2} \text{Var}[h_{\ell;2,2}] + \frac{J_3^2(M)}{6^2} \text{Var}[h_{\ell;3,2}] + \frac{J_4^2(M)}{(4!)^2} \text{Var}[h_{\ell;4,2}]$$

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and moreover
\[
\text{Var}\left[ S_\ell(M; 2) \right] = \mathbb{E} \left[ \left( \int_{\mathbb{S}^2} \sum_{q=5}^\infty \frac{J_q(M)H_q(T_\ell(x))}{q!} \, dx \right)^2 \right] = \sum_{q=5}^\infty \frac{J_q^2(M)}{(q!)^2} \text{Var}[h_{\ell,q,2}] \ll \frac{1}{\ell^2} \sum_{q=5}^\infty \frac{J_q^2(M)}{q!} \ll \frac{1}{\ell^2},
\]
where the last bounds follows from (5.1.11) and (5.1.12). Remark that
\[
\text{Var}(S_\ell(M)) = \sum_{q=0}^\infty \frac{J_q^2(M)}{q!} < +\infty.
\]

Therefore recalling also (5.1.10) and (5.1.13)
\[
\frac{1}{\text{Var}[S_\ell(M)]} \mathbb{E} \left[ \left( \int_{\mathbb{S}^2} \sum_{q=5}^\infty \frac{J_q(M)H_q(T_\ell(x))}{q!} \, dx \right)^2 \right] \ll \frac{1}{\ell}.\]

On the other hand, from Proposition 5.1.7
\[
d_W \left( \frac{S_\ell(M; 1)}{\sqrt{\text{Var}[S_\ell(M)]}}, Z_\ell \right) = O \left( \frac{1}{\sqrt{\ell}} \right)
\]
and finally, using Proposition 3.6.1 in [53],
\[
d_W \left( Z_\ell, Z \right) \leq \sqrt{\frac{2}{\pi}} \left| \frac{\text{Var}[S_\ell(M; 1)]}{\text{Var}[S_\ell(M)]} - 1 \right| = O \left( \frac{1}{\ell} \right),
\]
so that the proof for \( d = 2 \) is completed.

The proof in the general case \( d \geq 3 \) is indeed analogous, just setting
\[
S_\ell(M; 1) := \frac{J_2(M)}{2} h_{\ell;2,d},
\]
\[
S_\ell(M; 2) := \int_{\mathbb{S}^2} \sum_{q=3}^\infty \frac{J_q(M)H_q(T_\ell(x))}{q!} \, dx
\]
and recalling from (5.1.9) that \( \text{Var}[h_{\ell;2,d}] = O(\frac{1}{\ell^{d-1}}) \) whereas for \( q \geq 3 \), \( \text{Var}[h_{\ell;q,d}] = O(\frac{1}{\ell^{q-1}}) \) from Proposition 5.1.2. \(\)
We are now in the position to establish our main result, concerning the volume of
the excursion sets, which we recall for any fixed $z \in \mathbb{R}$ is given by

$$S_t(z) := S_t(\mathbb{I}(\cdot > z)) = \int_{S^d} \mathbb{I}(T_t(x) > z)dx.$$ 

Again, $\mathbb{E}[S_t(z)] = \mu_d(1 - \Phi(z))$, where $\Phi(z)$ is the cdf of the standard Gaussian law,
and in this case we have $M = M_z := \mathbb{I}(\cdot > z)$, $J_2(M_z) = z\phi(z)$, $\phi$ denoting the standard
Gaussian density. The proof of Theorem 5.1.1 is then just an immediate consequence
of Proposition 5.1.8.

Remark 5.5.2. It should be noted that the rate obtained here is much sharper than
the one provided by [58] for the Euclidean case with $d = 2$. The asymptotic setting
we consider is rather different from his, in that we consider the case of spherical eigen-
function with diverging eigenvalues, whereas he focusses on functionals evaluated on
increasing domains $[0, T]^d$ for $T \to \infty$. However the contrast in the converging rates
is not due to these different settings, indeed [14] establish rates of convergence analo-
gous to those by [58] for spherical random fields with more rapidly decaying covariance
structure than the one we are considering here. The main point to notice is that the
slow decay of Gegenbauer polynomials entails some form of long range dependent be-
haviour on random spherical harmonics; in this sense, hence, our results may be closer
in spirit to the work by [25] on empirical processes for long range dependent stationary
processes on $\mathbb{R}$.

5.6 Technical proofs

5.6.1 On the variance of $h_{t,q,d}$

In this section we study the variance of $h_{t,q,d}$ defined in (5.1.3). By (4.1.4) and the
definition of Gaussian random eigenfunctions (4.2.16), it follows that (5.1.5) hold at
once:

\[
\text{Var}[h_{q,d}] = \mathbb{E} \left( \left( \int_{\mathbb{S}^d} H_q(T_\ell(x)) \, dx \right)^2 \right) = \int_{(\mathbb{S}^d)^2} \mathbb{E}[H_q(T_\ell(x_1))H_q(T_\ell(x_2))] \, dx_1 dx_2 =
\]

\[
= q! \int_{(\mathbb{S}^d)^2} \mathbb{E}[T_\ell(x_1)T_\ell(x_2)]^q \, dx_1 dx_2 = q! \int_{(\mathbb{S}^d)^2} G_{\ell,d}(\cos d(x_1,x_2))^q \, dx_1 dx_2 =
\]

\[
= q! \mu_{d-1} \int_0^\pi G_{\ell,d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} \, d\vartheta.
\]

Now we prove Proposition 5.1.2, inspired by the proof of [45], Lemma 5.2.

**Proof of Proposition 5.1.2.** By the Hilb’s asymptotic formula for Jacobi polynomials (see [64], Theorem 8.21.12), we have uniformly for \( \ell \geq 1, \vartheta \in [0, \frac{\pi}{2}] \)

\[
(sin \vartheta)^{\frac{d}{2}-1} G_{\ell,d}(\cos \vartheta) = \frac{2^{\frac{d}{2}-1}}{(\ell + \frac{d}{2}-1)} \left( a_{\ell,d} \left( \frac{\vartheta}{\sin \vartheta} \right)^{\frac{1}{2}} \right) J_{\frac{d}{2}-1}(L \vartheta) + \delta(\vartheta),
\]

where \( L = \ell + \frac{d-1}{2} \),

\[
a_{\ell,d} = \frac{\Gamma(\ell + \frac{d}{2})}{(\ell + \frac{d-1}{2})^{\frac{d-1}{2}} \ell!} \sim 1 \quad \text{as} \ \ell \rightarrow \infty, \quad (5.6.54)
\]

and the remainder is

\[
\delta(\vartheta) \ll \begin{cases} 
\sqrt{\vartheta} \ell^{-\frac{3}{2}} & \ell^{-1} < \vartheta < \frac{\pi}{2}, \\
\vartheta \left( \frac{d}{2}-1 \right)^{-2} \ell^{-\frac{d}{2}-1} & 0 < \vartheta < \ell^{-1}.
\end{cases}
\]

Therefore, in the light of (5.6.54) and \( \vartheta \rightarrow \frac{\vartheta}{\sin \vartheta} \) being bounded,

\[
= \left( \frac{2^{\frac{d}{2}-1}}{(\ell + \frac{d}{2}-1)} \right)^q a_{\ell,d} \int_0^{\frac{\pi}{2}} (\sin \vartheta)^{-q(\frac{d}{2}-1)} \left( \frac{\vartheta}{\sin \vartheta} \right)^{\frac{1}{2}} J_{\frac{d}{2}-1}(L \vartheta)(\sin \vartheta)^{d-1} \, d\vartheta +
\]

\[
+ O \left( \frac{1}{\ell^{q(\frac{d}{2}-1)}} \int_0^{\frac{\pi}{2}} (\sin \vartheta)^{-q(\frac{d}{2}-1)} |J_{\frac{d}{2}-1}(L \vartheta)|^{q-1} \delta(\vartheta)(\sin \vartheta)^{d-1} \, d\vartheta \right),
\]

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where we used 
\[
\left( \frac{\ell + \frac{d}{2} - 1}{\ell} \right) \ll \frac{1}{\ell^{\frac{d}{2} - 1}}
\]
(note that we readily neglected the smaller terms, corresponding to higher powers of \(\delta(\vartheta)\)). We rewrite (5.6.55) as
\[
\int_0^{\pi/2} G_{\ell,d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} d\vartheta = N + E,
\]
where
\[
N = N(d, q; \ell) := \left( \frac{2^d - 1}{(2^d + 1)} \right)^q a^q_{\ell,d} \int_0^{\pi/2} (\sin \vartheta)^{-q(\frac{d}{2} - 1)} \left( \frac{\vartheta}{\sin \vartheta} \right)^{\frac{q}{2}} J_{\frac{d}{2} - 1}(L\vartheta)^q (\sin \vartheta)^{d-1} d\vartheta
\]
and
\[
E = E(d, q; \ell) \ll \frac{1}{\ell^{q(\frac{d}{2} - 1)}} \int_0^{\pi/2} (\sin \vartheta)^{-q(\frac{d}{2} - 1)} |J_{\frac{d}{2} - 1}(L\vartheta)|^{q-1} \delta(\vartheta)(\sin \vartheta)^{d-1} d\vartheta.
\]
To bound the error term \(E\) we split the range of the integration in (5.6.58) and write
\[
E \ll \frac{1}{\ell^{q(\frac{d}{2} - 1)}} \int_0^{\frac{\pi}{2}} (\sin \vartheta)^{-q(\frac{d}{2} - 1)} |J_{\frac{d}{2} - 1}(L\vartheta)|^{q-1} \vartheta^{d+1} d\vartheta + \frac{1}{\ell^{q(\frac{d}{2} - 1)}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin \vartheta)^{-q(\frac{d}{2} - 1)} |J_{\frac{d}{2} - 1}(L\vartheta)|^{q-1} \sqrt{\vartheta} \ell^{-\frac{3}{2}} (\sin \vartheta)^{d-1} d\vartheta.
\]
For the first integral in (5.6.59) recall that \(J_{\frac{d}{2} - 1}(z) \sim z^{\frac{d}{2} - 1}\) as \(z \to 0\), so that as \(\ell \to \infty\),
\[
\frac{1}{\ell^{(q-1)(\frac{d}{2} - 1)}} \int_0^{\frac{\pi}{4}} \left( \frac{\vartheta}{\sin \vartheta} \right)^{q(\frac{d}{2} - 1) - d + 1} |J_{\frac{d}{2} - 1}(L\vartheta)|^{q-1} \vartheta^{-(q-1)(\frac{d}{2} - 1)+d+1} d\vartheta \ll \int_0^{\frac{\pi}{4}} \vartheta^{d+1} d\vartheta = \frac{1}{\ell^{d+2}},
\]
which is enough for our purposes. Furthermore, since for \(z\) big \(|J_{\frac{d}{2} - 1}(z)| = O(z^{-\frac{1}{2}})\)
(and keeping in mind that \(L\) is of the same order of magnitude as \(\ell\)), we may bound
the second integral in (5.6.59) as
\[
\ll \frac{1}{\ell^{q(q-1)+\frac{d}{2}}} \int_0^{\frac{\pi}{2}} \left( \frac{\vartheta}{\sin \vartheta} \right)^{q(q-1)-d+1} |J_{\frac{q}{2}-1}(L \vartheta)|^{q-1} \vartheta^{\frac{d}{2}+d-\frac{1}{2}} d\vartheta \ll
\]
\[
\ll \frac{1}{\ell^{q(q-1)+\frac{d}{2}}} \int_0^{\frac{\pi}{2}} (L \vartheta)^{-q(q-1)+d-\frac{1}{2}} d\vartheta = \frac{1}{\ell^{q(q-1)+2}} \int_0^{\frac{\pi}{2}} \vartheta^{-q(q-1)+d} d\vartheta \ll
\]
\[
\ll \frac{1}{\ell^{(d+2)(q-1)+1}} = o(\ell^{-d}) , \quad (5.6.61)
\]
where the last equality in (5.6.61) holds for \( q \geq 3 \). From (5.6.60) (bounding the first integral in (5.6.59)) and (5.6.61) (bounding the second integral in (5.6.59)) we finally find that the error term in (5.6.56) is
\[
E = o(\ell^{-d}) \quad (5.6.62)
\]
for \( q \geq 3 \), admissible for our purposes.

Therefore, substituting (5.6.62) into (5.6.56) we have
\[
\int_0^{\frac{\pi}{2}} G_{\ell,d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} d\vartheta =
\]
\[
= \left( \frac{2^{\frac{d}{2}-1}}{(\ell+\frac{d}{2} - 1)} \right)^q a_{\ell,d}^{L} \int_0^{\frac{\pi}{2}} (\sin \vartheta)^{q(q-1)} \left( \frac{\vartheta}{\sin \vartheta} \right)^{\frac{d}{2}} J_{\frac{q}{2}-1}(L \vartheta)^q (\sin \vartheta)^{d-1} d\vartheta + o(\ell^{-d}) =
\]
\[
= \left( \frac{2^{\frac{d}{2}-1}}{(\ell+\frac{d}{2} - 1)} \right)^q a_{\ell,d}^{L} \frac{1}{L} \int_0^{\frac{\pi}{2}} (\sin \frac{\psi}{L})^{q(q-1)} \left( \frac{\psi/L}{\sin \frac{\psi}{L}} \right)^{\frac{d}{2}} \times
\]
\[
\times J_{\frac{q}{2}-1}(\psi)^q (\sin \frac{\psi}{L})^{d-1} d\psi + o(\ell^{-d}) , \quad (5.6.63)
\]
where in the last equality we transformed \( \psi/L = \vartheta \); it then remains to evaluate the first term in (5.6.63), which we denote by
\[
N_L := \left( \frac{2^{\frac{d}{2}-1}}{(\ell+\frac{d}{2} - 1)} \right)^q a_{\ell,d}^{L} \frac{1}{L} \int_0^{\frac{\pi}{2}} (\sin \psi/L)^{-q(q-1)} \left( \frac{\psi/L}{\sin \psi/L} \right)^{\frac{d}{2}} J_{\frac{q}{2}-1}(\psi)^q (\sin \psi/L)^{d-1} d\psi .
\]

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Now recall that as $\ell \to \infty$
\[
\left( \ell + \frac{d}{2} - 1 \right) \sim \frac{\ell^{d-1}}{(\frac{d}{2} - 1)!};
\]
moreover (5.6.54) holds, therefore we find of course that as $L \to \infty$
\[
N_L \sim \frac{(2^{d-1}(\frac{d}{2} - 1)!)^q}{L^q(\frac{d}{2}-1)+1} \int_0^{L^\frac{d}{2}} (\sin \psi/L)^{-q(\frac{d}{2}-1)} \left( \frac{\psi/L}{\sin \psi/L} \right)^{\frac{d}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin \psi/L)^{d-1} d\psi.
\]
In order to finish the proof of Proposition 5.1.2, it is enough to check that, as $L \to \infty$
\[
L^d \frac{(2^{d-1}(\frac{d}{2} - 1)!)^q}{L^q(\frac{d}{2}-1)+1} \int_0^{L^\frac{d}{2}} (\sin \psi/L)^{-q(\frac{d}{2}-1)} \left( \frac{\psi/L}{\sin \psi/L} \right)^{\frac{d}{2}} J_{\frac{d}{2}-1}(\psi)^q \left( \frac{\psi}{L} \right)^{d-1} d\psi \to c_{q,d},
\]
actually from (5.6.63) and (5.6.64), we have
\[
\lim_{\ell \to +\infty} \ell^d \int_0^{L^\frac{d}{2}} G_{\ell,d}(\cos \vartheta)^q (\sin \vartheta)^{d-1} d\vartheta =
\]
\[
= \lim_{L \to +\infty} L^d \frac{(2^{d-1}(\frac{d}{2} - 1)!)^q}{L^q(\frac{d}{2}-1)+1} \int_0^{L^\frac{d}{2}} (\sin \psi/L)^{-q(\frac{d}{2}-1)} \left( \frac{\psi/L}{\sin \psi/L} \right)^{\frac{d}{2}} J_{\frac{d}{2}-1}(\psi)^q \left( \frac{\psi}{L} \right)^{d-1} d\psi.
\]
Now we write
\[
\frac{\psi/L}{\sin \psi/L} = 1 + O \left( \psi^2/L^2 \right),
\]
so that
\[
L^d \frac{(2^{d-1}(\frac{d}{2} - 1)!)^q}{L^q(\frac{d}{2}-1)+1} \int_0^{L^\frac{d}{2}} (\sin \psi/L)^{-q(\frac{d}{2}-1)} \left( \frac{\psi/L}{\sin \psi/L} \right)^{\frac{d}{2}} J_{\frac{d}{2}-1}(\psi)^q (\sin \psi/L)^{d-1} d\psi =
\]
\[
= \left( 2^{d-1}(\frac{d}{2} - 1) ! \right)^q \int_0^{L^\frac{d}{2}} \left( \frac{\psi/L}{\sin \psi/L} \right)^q \left( \frac{\psi}{L} \right)^{d-1} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1} d\psi =
\]
\[
= \left( 2^{d-1}(\frac{d}{2} - 1) ! \right)^q \int_0^{L^\frac{d}{2}} \left( 1 + O \left( \psi^2/L^2 \right) \right)^q \left( \frac{\psi}{L} \right)^{d-1} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1} d\psi =
\]
\[
= \left( 2^{d-1}(\frac{d}{2} - 1) ! \right)^q \int_0^{L^\frac{d}{2}} \left( \psi^2 \right) \left( \frac{\psi}{L} \right)^{d-1} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1} d\psi +
\]
\[
+ O \left( \frac{1}{L^2} \int_0^{L^\frac{d}{2}} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1} d\psi \right).
\]
Note that as \( L \to +\infty \), the first term of the previous summation converges to \( c_{q,d} \) defined in (5.1.8), i.e.

\[
\left( 2^{\frac{d}{2}-1} \left( \frac{d}{2} - 1 \right)! \right) q \int_0^{L_2} J_{\frac{d}{2} - 1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1} \, d\psi \to c_{q,d}.
\]

(5.6.65)

It remains to bound the remainder

\[
\frac{1}{L^2} \int_0^{L_2} |J_{\frac{d}{2} - 1}(\psi)|^q \psi^{-q(\frac{d}{2}-1)+d+1} \, d\psi = O(1) + \frac{1}{L^2} \int_1^{L_2} |J_{\frac{d}{2} - 1}(\psi)|^q \psi^{-q(\frac{d}{2}-1)+d+1} \, d\psi.
\]

Now for the second term on the r.h.s.

\[
\int_1^{L_2} |J_{\frac{d}{2} - 1}(\psi)|^q \psi^{-q(\frac{d}{2}-1)+d+1} \, d\psi \ll \int_1^{L_2} \psi^{-q(\frac{d}{2}-\frac{1}{2})+d+1} \, d\psi = O(1 + L^{-q(\frac{d}{2}-\frac{1}{2})+d+2}).
\]

Therefore we obtain

\[
\left( 2^{\frac{d}{2}-1} \left( \frac{d}{2} - 1 \right)! \right)^q \int_0^{L_2} J_{\frac{d}{2} - 1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1} \, d\psi + O\left( \frac{1}{L^2} \int_0^{L_2} J_{\frac{d}{2} - 1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d+1} \, d\psi \right) = \\
= \left( 2^{\frac{d}{2}-1} \left( \frac{d}{2} - 1 \right)! \right)^q \int_0^{L_2} J_{\frac{d}{2} - 1}(\psi)^q \psi^{-q(\frac{d}{2}-1)+d-1} \, d\psi + O(L^{-2} + L^{-q(\frac{d}{2}-\frac{1}{2})+d})
\]

so that we have just checked the statement of the present proposition for \( q > \frac{2d}{d-1} \). This is indeed enough for each \( q \geq 3 \) when \( d \geq 4 \).

It remains to investigate separately just the case \( d = q = 3 \). Recall that for \( d = 3 \) we have an explicit formula for the Bessel function of order \( \frac{d}{2} - 1 \) ([64]), that is

\[
J_{\frac{d}{2}-1}(z) = \sqrt{\frac{2}{\pi z}} \sin(z),
\]

and hence the integral in (5.1.8) is indeed convergent for \( q = d = 3 \) by integrations by parts.
We have hence to study the convergence of the following integral

$$\frac{8}{\pi^2} \int_0^{\pi/2} \left( \frac{\psi/L}{\sin \psi/L} \right) \frac{\sin^3 \psi}{\psi} d\psi.$$ 

To this aim, let us consider a large parameter $K \gg 1$ and divide the integration range into $[0, K]$ and $[K, \pi/2]$; the main contribution comes from the first term, whence we have to prove that the latter vanishes. Note that

$$\int_K^{\pi/2} \left( \frac{\psi/L}{\sin \psi/L} \right) \frac{\sin^3 \psi}{\psi} d\psi \ll \frac{1}{K}, \quad (5.6.66)$$

where we use integration by part with the bounded function $I(T) = \int_0^T \sin^3 z d\psi$. On $[0, K]$, we write

$$\frac{8}{\pi^2} \int_0^K \left( \frac{\psi/L}{\sin \psi/L} \right) \frac{\sin^3 \psi}{\psi} d\psi = \frac{8}{\pi^2} \int_0^K \frac{\sin^3 \psi}{\psi} d\psi + O \left( \frac{1}{L^2} \int_0^K \psi \sin^3 \psi d\psi \right) = \frac{8}{\pi^2} \int_0^K \frac{\sin^3 \psi}{\psi} d\psi + O \left( \frac{K^2}{L^2} \right).$$

Consolidating the latter with (5.6.66) we find that

$$\frac{8}{\pi^2} \int_0^{\pi/2} \left( \frac{\psi/L}{\sin \psi/L} \right) \frac{\sin^3 \psi}{\psi} d\psi = \frac{8}{\pi^2} \int_0^K \frac{\sin^3 \psi}{\psi} d\psi + O \left( \frac{1}{K} + \frac{K^2}{L^2} \right).$$

Now as $K \to +\infty$, 

$$\frac{8}{\pi^2} \int_0^K \frac{\sin^3 \psi}{\psi} d\psi \to c_{3,3};$$

to conclude the proof, it is then enough to choose $K = K(L) \to \infty$ sufficiently slowly, i.e. $K = \sqrt{L}$.

Proof (5.3.2). The bounds (5.3.32), (5.3.33) are known and indeed the corresponding integrals can be evaluated explicitly in terms of Wigner’s 3j and 6j coefficients, see [40] e.g. The bounds in (5.3.34),(5.3.35) derives from a simple improvement in the proof.

5.6.2 Proofs of Propositions 5.3.2 and 5.3.3

Proof (5.3.2). The bounds (5.3.32), (5.3.33) are known and indeed the corresponding integrals can be evaluated explicitly in terms of Wigner’s 3j and 6j coefficients, see [40] e.g. The bounds in (5.3.34),(5.3.35) derives from a simple improvement in the proof.
of Proposition 2.2 in [47], which can be obtained when focusing only on a subset of the terms (the circulant ones) considered in that reference. In the proof to follow, we exploit repeatedly (5.3.28), (5.3.29), (5.3.30) and (5.3.31).

Let us start investigating the case $q = 5$:

$$
\mathcal{K}_\ell(5; 1) = \int_{(S^2)^4} |P_\ell(\cos d(x_1, x_2))|^4 |P_\ell(\cos d(x_2, x_3))| \times
\times |P_\ell(\cos d(x_3, x_4))|^4 |P_\ell(\cos d(x_4, x_1))| \ dx_1 dx_2 dx_3 dx_4 \\
\leq \int_{(S^2)^4} |P_\ell(\cos d(x_1, x_2))|^4 |P_\ell(\cos d(x_2, x_3))| |P_\ell(\cos d(x_3, x_4))|^4 \ dx_1 dx_2 dx_3 dx_4 \\
\leq \int_{(S^2)^3} |P_\ell(\cos d(x_1, x_2))|^4 |P_\ell(\cos d(x_2, x_3))| \int_{S^2} |P_\ell(\cos d(x_3, x_4))|^4 \ dx_4 \ dx_1 dx_2 dx_3 \\
\leq O \left( \frac{\log \ell}{\ell^2} \right) \times \int_{S^2 \times S^2} |P_\ell(\cos d(x_1, x_2))|^4 \left\{ \int_{S^2} |P_\ell(\cos d(x_2, x_3))| \ dx_3 \right\} \ dx_1 dx_2 \\
\leq O \left( \frac{\log \ell}{\ell^2} \right) \times O \left( \frac{1}{\sqrt{\ell}} \right) \times \int_{S^2 \times S^2} |P_\ell(\cos d(x_1, x_2))|^4 \ dx_1 dx_2 \\
\leq O \left( \frac{\log \ell}{\ell^2} \right) \times O \left( \frac{1}{\sqrt{\ell}} \right) \times O \left( \frac{\log \ell}{\ell^2} \right) = O \left( \frac{\log^2 \ell}{\ell^9/2} \right);
$$

$$
\mathcal{K}_\ell(5; 2) = \int_{(S^2)^4} |P_\ell(\cos d(x_1, x_2))|^3 |P_\ell(\cos d(x_2, x_3))|^2 \times
\times |P_\ell(\cos d(x_3, x_4))|^3 |P_\ell(\cos d(x_4, x_1))|^2 \ dx_1 dx_2 dx_3 dx_4 \\
\leq \int_{(S^2)^4} |P_\ell(\cos d(x_1, x_2))|^3 |P_\ell(\cos d(x_2, x_3))|^2 |P_\ell(\cos d(x_3, x_4))|^3 \ dx_1 dx_2 dx_3 dx_4 \\
\leq \int_{(S^2)^3} |P_\ell(\cos d(x_1, x_2))|^3 |P_\ell(\cos d(x_2, x_3))|^2 \int_{S^2} |P_\ell(\cos d(x_3, x_4))|^3 \ dx_4 \ dx_1 dx_2 dx_3 \\
\leq O \left( \frac{\sqrt{\log \ell}}{\ell^{3/2}} \right) \times \int_{S^2 \times S^2} |P_\ell(\cos d(x_1, x_2))|^3 \left\{ \int_{S^2} |P_\ell(\cos d(x_2, x_3))|^2 \ dx_3 \right\} \ dx_1 dx_2 \\
\leq O \left( \frac{\sqrt{\log \ell}}{\ell^{3/2}} \right) \times O \left( \frac{1}{\ell} \right) \times \int_{S^2 \times S^2} |P_\ell(\cos d(x_1, x_2))|^3 \ dx_1 dx_2 \\
\leq O \left( \frac{\sqrt{\log \ell}}{\ell^{3/2}} \right) \times O \left( \frac{1}{\ell} \right) \times O \left( \frac{\sqrt{\log \ell}}{\ell^{3/2}} \right) = O \left( \frac{\log \ell}{\ell^4} \right).
$$
For $q = 6$ and $r = 1$ we simply note that $K_\ell(6; 1) \leq K_\ell(5; 1)$, actually

$$K_\ell(6; 1) = \int_{(S^2)^4} |P_\ell(\cos d(x_1, x_2))|^5 |P_\ell(\cos d(x_2, x_3))| \times$$

$$\times |P_\ell(\cos d(x_3, x_4))|^5 |P_\ell(\cos d(x_4, x_1))| \, dx_1 dx_2 dx_3 dx_4 \leq$$

$$\leq \int_{(S^2)^4} |P_\ell(\cos d(x_1, x_2))|^4 |P_\ell(\cos d(x_2, x_3))|^2 \times$$

$$\times |P_\ell(\cos d(x_3, x_4))|^4 |P_\ell(\cos d(x_4, x_1))|^2 \, dx_1 dx_2 dx_3 dx_4 \leq$$

$$\leq \int_{(S^2)^2} |P_\ell(\cos d(x_1, x_2))|^4 dx_1 \int_{S^2} |P_\ell(\cos d(x_2, x_3))|^2 dx_2 \times$$

$$\times \int_{S^2} |P_\ell(\cos d(x_3, x_4))|^4 dx_4 dx_3 =$$

$$= O \left( \frac{\log \ell}{\ell^2} \right) \times O \left( \frac{1}{\ell} \right) \times O \left( \frac{\log \ell}{\ell^2} \right) = O \left( \frac{\log^2 \ell}{\ell^5} \right)$$

and likewise

$$K_\ell(6; 3) = \int_{(S^2)^4} |P_\ell(\cos d(x_1, x_2))|^3 |P_\ell(\cos d(x_2, x_3))|^3 \times$$

$$\times |P_\ell(\cos d(x_3, x_4))|^3 |P_\ell(\cos d(x_4, x_1))|^3 \, dx_1 dx_2 dx_3 dx_4 \leq$$

$$\leq \int_{(S^2)^4} |P_\ell(\cos d(x_1, x_2))|^3 |P_\ell(\cos d(x_2, x_3))|^3 |P_\ell(\cos d(x_3, x_4))|^3 \, dx_1 dx_2 dx_3 dx_4 =$$

$$= O \left( \frac{\sqrt{\log \ell}}{\ell^{3/2}} \right) \times O \left( \frac{\sqrt{\log \ell}}{\ell^{3/2}} \right) \times O \left( \frac{\sqrt{\log \ell}}{\ell^{3/2}} \right) = O \left( \frac{\log^{3/2} \ell}{\ell^{9/2}} \right).$$
Finally for \( q = 7 \)

\[
K_\ell(7; 1) = \int_{S^2 \times \ldots \times S^2} |P_\ell(\cos d(x_1, x_2))|^6 |P_\ell(\cos d(x_2, x_3))| \times \\
\times |P_\ell(\cos d(x_3, x_4))|^6 |P_\ell(\cos d(x_4, x_1))| dx_1 dx_2 dx_3 dx_4 \leq \\
\leq \int_{S^2 \times S^2} |P_\ell(\cos d(x_1, x_2))|^6 \int_{S^2} |P_\ell(\cos d(x_2, x_3))| dx_3 \times \\
\times \int_{S^2} |P_\ell(\cos d(x_3, x_4))|^6 dx_4 dx_2 = O \left( \frac{1}{\ell^2} \right) \times O \left( \frac{1}{\ell^{1/2}} \right) \times O \left( \frac{1}{\ell^2} \right) = O \left( \frac{1}{\ell^{9/2}} \right)
\]

and repeating the same argument we obtain

\[
K_\ell(7; 2) = O \left( \frac{1}{\ell^5} \right) \quad \text{and} \quad K_\ell(7; 3) = O \left( \frac{\log^{9/2} \ell}{\ell^{11/2}} \right).
\]

From (5.3.27), we have indeed computed the bounds for \( K_\ell(q; r) \), \( q = 1, \ldots, 7 \) and \( r = 1, \ldots, q-1 \).

To conclude the proof we note that, for \( q > 7 \)

\[
\max_{r=1,\ldots,q-1} K_\ell(q; r) = \max_{r=1,\ldots,\left[\frac{q}{2}\right]} K_\ell(q; r) \leq \max_{r=1,\ldots,3} K_\ell(6; r) = O \left( \frac{1}{\ell^{9/2}} \right).
\]

Moreover in particular

\[
\max_{r=2,\ldots,\left[\frac{q}{2}\right]} K_\ell(q; r) \leq K_\ell(7; 2) \lor K_\ell(7; 3) = O \left( \frac{1}{\ell^5} \right),
\]

so that the dominant terms are of the form \( K_\ell(q; 1) \). \( \square \)

**Proof (5.3.3).** The proof relies on the same argument of the proof of Proposition 5.3.2, therefore we shall omit some calculations. In what follows we exploit repeatedly the inequalities (5.3.37), (5.3.38), (5.3.39) and (5.3.40).
For $q = 3$ we immediately have

$$K_{\ell}(3; 1) = \int_{(S^d)^4} |G_{\ell,d}(\cos d(x_1, x_2))|^2 |G_{\ell,d}(\cos d(x_2, x_3))| \times$$

$$\times |G_{\ell,d}(\cos d(x_3, x_4))|^2 |G_{\ell,d}(\cos d(x_4, x_1))| \, dx_1 \, dx_2 \, dx_3 \, dx_4 \leq$$

$$\leq \int_{(S^d)^4} |G_{\ell,d}(\cos d(x_1, x_2))|^2 |G_{\ell,d}(\cos d(x_2, x_3))| |G_{\ell,d}(\cos d(x_3, x_4))|^2 \, dx_1 \, dx_2 \, dx_3 \, dx_4 =$$

$$= O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}+\frac{d}{2}}/d}\right).$$

Likewise for $q = 4$

$$K_{\ell}(4; 1) = \int_{(S^d)^4} |G_{\ell,d}(\cos d(x_1, x_2))|^3 |G_{\ell,d}(\cos d(x_2, x_3))| \times$$

$$\times |G_{\ell,d}(\cos d(x_3, x_4))|^3 |G_{\ell,d}(\cos d(x_4, x_1))| \, dx_1 \, dx_2 \, dx_3 \, dx_4 \leq$$

$$\leq \int_{(S^d)^4} |G_{\ell,d}(\cos d(x_1, x_2))|^3 |G_{\ell,d}(\cos d(x_2, x_3))| |G_{\ell,d}(\cos d(x_3, x_4))|^3 \, dx_1 \, dx_2 \, dx_3 \, dx_4 =$$

$$= O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) = O\left(\frac{1}{\ell^{2d+\frac{d}{2}+\frac{d}{2}}/d}\right)$$

and moreover

$$K_{\ell}(4; 2) = \int_{(S^d)^4} |G_{\ell,d}(\cos d(x_1, x_2))|^2 \times$$

$$\times |G_{\ell,d}(\cos d(x_2, x_3))|^2 |G_{\ell,d}(\cos d(x_3, x_4))|^2 |G_{\ell,d}(\cos d(x_4, x_1))|^2 \, dx_1 \, dx_2 \, dx_3 \, dx_4 \leq$$

$$\leq \int_{(S^d)^4} |G_{\ell,d}(\cos d(x_1, x_2))|^2 |G_{\ell,d}(\cos d(x_2, x_3))|^2 |G_{\ell,d}(\cos d(x_3, x_4))|^2 \, dx_1 \, dx_2 \, dx_3 \, dx_4 =$$

$$= O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) \times O\left(\frac{1}{\ell^{d-1}}\right) = O\left(\frac{1}{\ell^{3d-3}}\right),$$

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where we set $dx := dx_1 dx_2 dx_3 dx_4$. Similarly, for $q = 5$ we get the bounds

$$K_\ell(5; 1) = \int_{S^d \times \ldots \times S^d} |G_\ell d(\cos d(x_1, x_2))|^4 \times |G_\ell d(\cos d(x_2, x_3))| |G_\ell d(\cos d(x_3, x_4))| |G_\ell d(\cos d(x_4, x_1))| dx_1 dx_2 dx_3 dx_4 \leq \int_{S^d \times \ldots \times S^d} |G_\ell d(\cos d(x_1, x_2))|^4 |G_\ell d(\cos d(x_2, x_3))| |G_\ell d(\cos d(x_3, x_4))| \, d_\ell =$$

$$= O \left( \frac{1}{\ell^d} \right) \times O \left( \frac{1}{\ell^{d - \frac{1}{2}}} \right) \times O \left( \frac{1}{\ell^d} \right) = O \left( \frac{1}{\ell^{2d + \frac{d}{2} - \frac{1}{2}}} \right)$$

and

$$K_\ell(5; 2) = O \left( \frac{1}{\ell^{3d - 2}} \right).$$

It is immediate to check that

$$K_\ell(6; 1) = K_\ell(7; 1) = O \left( \frac{1}{\ell^{2d + \frac{d}{2} - \frac{1}{2}}} \right), \quad K_\ell(6; 2) = K_\ell(7; 2) = O \left( \frac{1}{\ell^{2d + d - 1}} \right),$$

whereas

$$K_\ell(6; 3) = O \left( \frac{1}{\ell^{2d + d - \frac{3}{2}}} \right) \quad \text{and} \quad K_\ell(7; 3) = O \left( \frac{1}{\ell^{2d + d - \frac{5}{2}}} \right).$$

The remaining terms are indeed bounded thanks to (5.3.27).

In order to finish the proof, it is enough to note, as for that for $q > 7$

$$\max_{r=1, \ldots, [\frac{q}{2}]} K_\ell(q; r) = \max_{r=1, \ldots, [\frac{q}{2}]} K_\ell(q; r) \leq \max_{r=1, \ldots, 3} K_\ell(6; r) = O \left( \frac{1}{\ell^{2d + \frac{d}{2} - \frac{1}{2}}} \right). \quad (5.6.67)$$

In particular we have

$$\max_{r=2, \ldots, [\frac{q}{2}]} K_\ell(q; r) \leq K_\ell(7; 2) \lor K_\ell(7; 3) = O \left( \frac{1}{\ell^{3d - 1}} \right), \quad (5.6.68)$$

so that the dominant terms are again of the form $K_\ell(q; 1)$. 

\[\square\]
Chapter 6

On the Defect distribution

In this chapter we refer to [60], where the high-energy limit distribution of the Defect of random hyperspherical harmonics is investigated. Indeed in the previous chapter quantitative Central Limit Theorems for the empirical measure of $z$-excursion sets has been shown but for the case $z = 0$.

We find the exact asymptotic rate for the Defect variance and a CLT for the case of the $d$-sphere, $d > 5$. The CLT in the 2-dimensional case has been already proved in [47] whereas the variance has been investigated in [45].

The remaining cases ($d = 3, 4, 5$) will be investigated in [60], where moreover quantitative CLTs will be proved (work still in progress).

6.1 Preliminaries

Consider the sequence of random eigenfunctions $T_\ell$, $\ell \in \mathbb{N}$ (4.2.15) on $\mathbb{S}^d$, $d \geq 2$. As in the previous chapter, the empirical measure of excursion sets (0.0.9) can be written, for $z \in \mathbb{R}$, as

$$S_\ell(z) := \int_{\mathbb{S}^d} 1(T_\ell(x) > z) \, dx ,$$

(6.1.1)

where $1_{(z, +\infty)}$ is the indicator function of the interval $(z, +\infty)$. The case $z \neq 0$ has been treated in [47, 44] (see also Chapter 5).
Now consider the Defect, i.e. the difference between “cold” and “warm” regions

\[ D_\ell := \int_{S^d} 1(T_\ell(x) > 0) \, dx - \int_{S^d} 1(T_\ell(x) < 0) \, dx \; ; \quad (6.1.2) \]

note that

\[ D_\ell = 2S_\ell(0) - \mu_d , \]

\( \mu_d \) denoting the hyperspherical volume (1.1.1).

Recall that the Heaviside function is defined for \( t \in \mathbb{R} \) as

\[
H(t) := \begin{cases} 
1 & t > 0 \\
0 & t = 0 \\
-1 & t < 0 
\end{cases}
\]

thus (6.1.2) can be rewritten simply as

\[ D_\ell = \int_{S^d} H(T_\ell(x)) \, dx . \]

Note that exchanging expectation and integration on \( S^d \) we have

\[
E[D_\ell] = \int_{S^d} E[H(T_\ell(x))] \, dx = 0 ,
\]

since \( E[H(T_\ell(x))] = 0 \) for every \( x \), by the symmetry of the Gaussian distribution.

6.2 The Defect variance

The proofs in this section are inspired by [45], where the case \( d = 2 \) has been investigated.

Lemma 6.2.1. For \( \ell \) even we have

\[
\text{Var}(D_\ell) = \frac{4}{\pi} \mu_d \mu_{d-1} \int_0^\frac{\pi}{2} \arcsin(G_{\ell,d}(\cos \theta))(\sin \theta)^{d-1} \, d\theta , \quad (6.2.3)
\]

where \( \mu_d \) is the hyperspherical volume (1.1.1) and \( G_{\ell,d} \) the \( \ell \)-th normalized Gegenbauer polynomial (Chapter 4 or [64]).
Proof. The proof is indeed analogous to the proof of Lemma 4.1 in [45]. First note that
\[
\text{Var}(D_\ell) = \int_{S^d} \int_{S^d} \mathbb{E}[\mathcal{H}(T_\ell(x))\mathcal{H}(T_\ell(y))] \, dx \, dy = \mu_d \int_{S^d} \mathbb{E}[\mathcal{H}(T_\ell(x))\mathcal{H}(T_\ell(N))] \, dx,
\]
by the isotropic property of the random field $T_\ell$, where $N$ denotes some fixed point in $S^d$. As explained in the proof of Lemma 4.1, we have
\[
\mathbb{E}[\mathcal{H}(T_\ell(x))\mathcal{H}(T_\ell(N))] = \frac{2}{\pi} \arcsin(G_{\ell,d}(\cos \vartheta)) ,
\]
where $\vartheta$ is the geodesic distance between $x$ and $N$. Moreover evaluating the last integral in hyperspherical coordinates we get
\[
\text{Var}(D_\ell) = \mu_d\mu_{d-1} \int_0^{\pi/2} \frac{2}{\pi} \arcsin(G_{\ell,d}(\cos \vartheta))(\sin \vartheta)^{d-1} d\vartheta. \quad (6.2.4)
\]
For $\ell$ even, we can hence write
\[
\text{Var}(D_\ell) = \frac{4}{\pi} \mu_d\mu_{d-1} \int_0^{\pi/2} \arcsin(G_{\ell,d}(\cos \vartheta))(\sin \vartheta)^{d-1} d\vartheta. \quad (6.2.5)
\]
\[\square\]

Note that, by (6.2.4), if $\ell$ is odd, then $D_\ell = 0$, therefore we can restrict ourselves to even $\ell$ only.

The main result of this section is the following.

**Theorem 6.2.2.** The defect variance is asymptotic to, as $\ell \to +\infty$ along even integers
\[
\text{Var}(D_\ell) = \frac{C_d}{\ell^d}(1 + o(1)) ,
\]
where $C_d$ is a strictly positive constant depending on $d$, that can be expressed by the formula
\[
C_d = \frac{4}{\pi} \mu_d\mu_{d-1} \int_0^{+\infty} \psi^{d-1} \left( \arcsin \left( \frac{d}{2} \right)^{d-1} \binom{d}{2} \psi \right) \psi^{-\left( \frac{d}{2} - 1 \right)} \psi J_{\frac{d}{2}-1}(\psi) \psi^{-\left( \frac{d}{2} - 1 \right)} d\psi.
\]
Proof. Here we are inspired by [45, Proposition 4.2, Theorem 1.2]. Since [64]
\[ \int_0^{\pi/2} G_{\ell,d}(\cos \theta)(\sin \theta)^{d-1} d\theta = 0 , \]
from Lemma 6.2.1 we can write
\[ \text{Var}(D_\ell) = \frac{4}{\pi} \mu_d \mu_{d-1} \int_0^{\pi/2} (\arcsin(G_{\ell,d}(\cos \theta)) - G_{\ell,d}(\cos \theta)) (\sin \theta)^{d-1} d\theta . \]

Let now
\[ \arcsin(t) - t = \sum_{k=1}^{+\infty} a_k t^{2k+1} \]
be the Taylor expansion of the arcsine, where
\[ a_k = \frac{(2k)!}{4^k(k!)^2(2k+1)} \sim \frac{1}{2\sqrt{\pi}k^{3/2}} , \quad k \to +\infty . \]

Since the Taylor series is uniformly absolutely convergent, we may write
\[ \text{Var}(D_\ell) = \frac{4}{\pi} \mu_d \mu_{d-1} \sum_{k=1}^{+\infty} a_k \int_0^{\pi/2} G_{\ell,d}(\cos \theta)^{2k+1} (\sin \theta)^{d-1} d\theta . \]

Now from Proposition 5.1.2 we have
\[ \lim_{\ell \to +\infty} \ell^d \int_0^{\pi/2} G_{\ell,d}(\cos \theta)^{2k+1} (\sin \theta)^{d-1} d\theta = c_{2k+1,d} , \quad (6.2.6) \]
where
\[ c_{2k+1,d} = \left( \frac{2^{d-1}}{2^2} \left( \frac{d}{2} - 1 \right)! \right)^{2k+1} \int_0^{+\infty} J_{d-1}(\psi)^{2k+1} \psi^{-(2k+1)} \left( \frac{d}{2}-1 \right)^{d-1} d\psi \]

Therefore we would expect that
\[ \text{Var}(D_\ell) \sim \frac{C_d}{\ell^d} , \quad (6.2.7) \]
where
\[ C_d = \frac{4}{\pi} \mu_d \mu_{d-1} \sum_{k=1}^{+\infty} a_k c_{2k+1,d} . \quad (6.2.8) \]
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Before proving (6.2.7) that is the statement of this theorem, let us check that \( C_d > 0 \), assuming (6.2.8) true. This is easy since the r.h.s. of (6.2.8) is a series of nonnegative terms and from [2, p. 217] we have

\[
c_{3; d} = (2^{\frac{d}{2} - 1} \left( \frac{d}{2} - 1 \right)!)^3 \frac{3^{\frac{d}{2} - \frac{3}{2}}}{2^3(\frac{d}{2} - 1)^{-1} \sqrt{\pi \Gamma \left( \frac{d}{2} - \frac{1}{2} \right)}} > 0.
\] (6.2.9)

Moreover, assuming (6.2.8) true, we get

\[
C_d = \frac{4}{\pi} \mu_d \mu_{d-1} \sum_{k=1}^{+\infty} a_k \left( 2^{\frac{d}{2} - 1} \left( \frac{d}{2} - 1 \right)! \right)^{2k+1} \int_0^{+\infty} J_{\frac{d}{2} - 1}(\psi) (d - 1)^{-1} d\psi =
\]

\[
= \frac{4}{\pi} \mu_d \mu_{d-1} \int_0^{+\infty} \sum_{k=1}^{+\infty} a_k \left( 2^{\frac{d}{2} - 1} \left( \frac{d}{2} - 1 \right)! J_{\frac{d}{2} - 1}(\psi) - \psi^{-1} \right)^{2k+1} \psi^{d-1} d\psi =
\]

\[
= \frac{4}{\pi} \mu_d \mu_{d-1} \int_0^{+\infty} \arcsin \left( 2^{\frac{d}{2} - 1} \left( \frac{d}{2} - 1 \right)! J_{\frac{d}{2} - 1}(\psi) - \psi^{-1} \right)^{2k+1} \psi^{d-1} d\psi ,
\]

which is the second statement of this theorem. To justify the exchange of the integration and summation order, we consider the finite summation

\[
\sum_{k=1}^{m} a_k \int_0^{+\infty} \left( 2^{\frac{d}{2} - 1} \left( \frac{d}{2} - 1 \right)! J_{\frac{d}{2} - 1}(\psi) - \psi^{-1} \right)^{2k+1} \psi^{d-1} d\psi
\]

using \( a_k \sim \frac{c}{k^{3/2}} \) (\( c > 0 \)) and the asymptotic behavior of Bessel functions for large argument [64] to bound the contributions of tails, and take the limit \( m \to +\infty \).

Let us now formally prove the asymptotic result for the variance (6.2.7). Note that

\[
\sum_{k=m+1}^{+\infty} a_k \int_0^{+\infty} |G_{\ell; d}(\cos \theta)|^{2k+1} (\sin \theta)^{d-1} d\theta \leq
\]

\[
\leq \sum_{k=m+1}^{+\infty} a_k \int_0^{+\infty} |G_{\ell; d}(\cos \theta)|^{2k+1} (\sin \theta)^{d-1} d\theta \ll \frac{1}{\ell^{\alpha}} \sum_{k=m+1}^{+\infty} \frac{1}{k^{3/2}} \ll \frac{1}{\sqrt{m} \ell^{\alpha}}.
\]

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Therefore we have for \( m = m(\ell) \) to be chosen
\[
\text{Var}(D_\ell) = \frac{4}{\pi} \mu_d \mu_{d-1} \sum_{k=1}^{m} a_k \int_0^{\frac{\pi}{2}} G_{\ell,d}(\cos \theta)^{2k+1}(\sin \theta)^{d-1} d\theta + O \left( \frac{1}{\sqrt{m \ell^d}} \right).
\]

From (6.2.6), we can write
\[
\text{Var}(D_\ell) = C_{d,m} \cdot \frac{1}{\ell^d} + o(\ell^{-d}) + \frac{1}{\sqrt{m \ell^d}},
\]
where
\[
C_{d,m} := \frac{4}{\pi} \mu_d \mu_{d-1} \sum_{k=1}^{m} a_k c_{2k+1,d}.
\]

Now since \( C_{d,m} \to C_d \) as \( m \to +\infty \), we can conclude. □

### 6.3 The CLT

In this section we prove a CLT for the Defect of random eigenfunctions on the \( d \)-sphere, \( d \neq 3, 4, 5 \), whose proof is inspired by the proof of Corollary 4.2 in [47].

**Theorem 6.3.1.** As \( \ell \to +\infty \) along even integers, we have
\[
\frac{D_\ell}{\sqrt{\text{Var}(D_\ell)}} \xrightarrow{\mathcal{L}} Z,
\]
where \( Z \sim \mathcal{N}(0,1) \).

Since our aim for the next future is to find a quantitative CLT, we will first compute its chaotic expansion.

#### 6.3.1 Chaotic expansions

Let us write the chaotic expansion (4.1.5) for the Defect in the form
\[
D_\ell = \sum_{q=0}^{+\infty} \frac{J_q(D_\ell)}{q!} \int_{S^d} H_q(T_\ell(x)) \, dx.
\]
Recalling that $D_\ell = 2S_\ell(0) - \mu_d$, let us find the chaotic expansion for $S_\ell(0)$. Note that $\mathbb{E}[S_\ell(0)] = \frac{1}{2}\mu_d$. For $q \geq 1$

\[ J_q(S_\ell(0)) = \int_{\mathbb{R}} 1(z > 0)(-1)^q \phi^{-1}(z) \frac{d^q}{dz^q} \phi(z) \phi(z) \, dz = \]

\[ = (-1)^q \int_0^{+\infty} \frac{d^q}{dz^q} \phi(z) \, dz = \]

\[ = -(-1)^{q-1} \phi(z) \phi^{-1}(z) \frac{d^{(q-1)}}{dz^{(q-1)}} \phi(z) \bigg|_0^{+\infty} = \]

\[ = -\phi(z) H_{q-1}(z) \bigg|_0^{+\infty} = \phi(0) H_{q-1}(0) = \begin{cases} 0 & \text{if } q \text{ even} \\ \frac{(-1)^{q-1}}{\sqrt{2\pi(2q-1)!}} & \text{if } q \text{ odd} \end{cases}. \]

Therefore the Wiener-Itô chaos decomposition for the Defect is

\[ D_\ell = 2S_\ell(0) - \mu_d = \sum_{k=1}^{+\infty} \sqrt{\frac{2}{\pi}} \frac{(-1)^k}{(2k+1)!(2k)!!} \int_{\mathbb{R}^d} H_{2k+1}(T_\ell(x)) \, dx, \]

with

\[ J_{2k}(D_\ell) = 0 , \quad J_{2k+1}(D_\ell) = \sqrt{\frac{2}{\pi}} \frac{(-1)^k}{(2k)!!}. \]

### 6.3.2 Proof of Theorem 6.3.1

Let $m \in \mathbb{N}, m \geq 2$ to be chosen later and set

\[ D_{\ell,m} := \sum_{k=1}^{m-1} \sqrt{\frac{2}{\pi}} \frac{(-1)^k}{(2k+1)!(2k)!!} \int_{\mathbb{R}^d} H_{2k+1}(T_\ell(x)) \, dx. \]
Simple estimates give

\[
\mathbb{E} \left[ \left( \frac{D_\ell}{\sqrt{\text{Var}(D_\ell)}} - \frac{D_{\ell,m}}{\sqrt{\text{Var}(D_{\ell,m})}} \right)^2 \right] \leq 2 \mathbb{E} \left[ \left( \frac{D_\ell}{\sqrt{\text{Var}(D_\ell)}} - \frac{D_{\ell,m}}{\sqrt{\text{Var}(D_{\ell,m})}} \right)^2 \right] + 2 \mathbb{E} \left[ \left( \frac{D_{\ell,m}}{\sqrt{\text{Var}(D_\ell)}} - \frac{D_{\ell,m}}{\sqrt{\text{Var}(D_{\ell,m})}} \right)^2 \right] \leq 2 \mathbb{E} \left[ \left( \frac{D_\ell}{\sqrt{\text{Var}(D_\ell)}} - \frac{D_{\ell,m}}{\sqrt{\text{Var}(D_\ell)}} \right)^2 \right] + 2 \mathbb{E} \left[ \left( \frac{D_{\ell,m}}{\sqrt{\text{Var}(D_\ell)}} + 1 - 2 \frac{\text{Var}(D_{\ell,m})}{\text{Var}(D_\ell)} \right)^2 \right].
\]

The first term to control is therefore

\[
\frac{D_\ell}{\sqrt{\text{Var}(D_\ell)}} - \frac{D_{\ell,m}}{\sqrt{\text{Var}(D_{\ell,m})}}.
\]

We have, repeating the same argument as in [47]

\[
\mathbb{E} \left[ \left( \frac{D_\ell}{\sqrt{\text{Var}(D_\ell)}} - \frac{D_{\ell,m}}{\sqrt{\text{Var}(D_{\ell,m})}} \right)^2 \right] = \frac{1}{\text{Var}[D_\ell]} \mathbb{E} \left[ (D_\ell - D_{\ell,m})^2 \right] = \frac{1}{\text{Var}[D_\ell]} \sum_{k=m}^{+\infty} \frac{2}{\pi} \frac{(-1)^k}{(2k+1)!(2k)!!} \int_{\mathbb{S}^d} H_{2k+1}(T_\ell(x)) \, dx \leq \frac{1}{\text{Var}[D_\ell]} \sum_{k=m}^{+\infty} \frac{2}{\pi} \frac{1}{((2k+1)!(2k)!!)^2} \text{Var} \left( \int_{\mathbb{S}^d} H_{2k+1}(T_\ell(x)) \, dx \right) \leq \frac{1}{\text{Var}[D_\ell]} \left( \frac{1}{\ell^d} \sum_{q=m}^{+\infty} a_q c_{2q+1,d} + o(\ell^{-d}) \right) \leq \frac{1}{\text{Var}[D_\ell]} \left( \frac{1}{2\sqrt{\pi}} \frac{1}{\ell^d} \sum_{q=m}^{+\infty} \frac{c_{5,d}}{q^2} + o(\ell^{-d}) \right) = \frac{1}{\text{Var}[D_\ell]} \times O \left( \frac{1}{\ell^d \sqrt{m}} \right) = O \left( \frac{1}{\sqrt{m}} \right),
\]

where we used Theorem 6.2.2.
Moreover for the second term we have, from (6.2.6) and Theorem 6.2.2
\[
\frac{\text{Var}(D_{\ell,m})}{\text{Var}(D_{\ell})} + 1 - 2 \sqrt{\frac{\text{Var}(D_{\ell,m})}{\text{Var}(D_{\ell})}} = 2 + O \left( \frac{1}{\sqrt{m}} \right) - 2 \sqrt{1 + O \left( \frac{1}{\sqrt{m}} \right)} = O \left( \frac{1}{\sqrt{m}} \right).
\]
Putting things together we immediately get
\[
\mathbb{E} \left[ \left( \frac{D_{\ell}}{\sqrt{\text{Var}(D_{\ell})}} - \frac{D_{\ell,m}}{\sqrt{\text{Var}(D_{\ell,m})}} \right)^2 \right] = O \left( \frac{1}{\sqrt{m}} \right).
\]
For every fixed \( m \), Corollary 5.1.4 and §5.4 gives, if \( d \neq 3, 4, 5 \)
\[
\frac{D_{\ell,m}}{\sqrt{\text{Var}(D_{\ell,m})}} \to Z,
\]
where \( Z \sim \mathcal{N}(0,1) \), so that since \( m \) can be chosen arbitrarily large we must have
\[
\frac{D_{\ell}}{\sqrt{\text{Var}(D_{\ell})}} \to Z.
\]

### 6.4 Final remarks

For the remaining cases \( d = 3, 4, 5 \), we need to prove a CLT for the random variables \( h_{\ell,3:d} \), as \( \ell \to +\infty \). Indeed bounds on fourth order cumulants obtained in Theorem 5.1.3 are not enough to guarantee the convergence to the standard Gaussian distribution. In [60] we will investigate the exact rate for fourth order cumulants of \( h_{\ell,3:d} \), which will allow to extend Theorem 6.3.1 to dimensions \( d = 3, 4, 5 \).

Moreover, we will prove quantitative CLTs for the Defect in the high-energy limit which should be of the form
\[
d_W \left( \frac{D_{\ell}}{\sqrt{\text{Var}(D_{\ell})}}, Z \right) = O \left( \ell^{-1/4} \right),
\]
where \( d_W \) denotes Wasserstein distance (4.1.11).
Chapter 7

Random length of level curves

In this chapter, our aim is to investigate the high-energy behavior for the length of level curves of Gaussian spherical eigenfunctions $T_{\ell}$, $\ell \in \mathbb{N}$ (4.2.15) on $S^2$.

7.1 Preliminaries

7.1.1 Length: mean and variance

Consider the total length of level curves of random eigenfunctions, i.e. the sequence the random variables $\{L_{\ell}(z)\}_{\ell \in \mathbb{N}}$ given by, for $z \in \mathbb{R}$,

$$L_{\ell}(z) := \text{length}(T_{\ell}^{-1}(z)).$$

(7.1.1)

As already anticipated in the Introduction of this thesis, the expected value of $L_{\ell}(z)$ was computed (see e.g. [69]) to be

$$E[L_{\ell}(z)] = 4\pi \frac{e^{-z^2/2}}{2\sqrt{2}} \sqrt{\ell(\ell + 1)},$$

(7.1.2)

consistent to Yau's conjecture [71, 70]. The asymptotic behaviour of the variance $\text{Var}(L_{\ell}(z))$ of $L_{\ell}(z)$ was resolved in [69, 68] as follows.

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For $z \neq 0$, we have
\[ \Var(\mathcal{L}_\ell(z)) \sim \ell \cdot C z^4 e^{-z^2}, \quad \ell \to +\infty, \quad (7.1.3) \]
for some $C > 0$. Moreover, I. Wigman computed the exact constant (private computations)
\[ C = \frac{\pi^2}{2}. \]
For the nodal case ($z = 0$) we have
\[ \Var(\mathcal{L}_\ell(0)) \sim \frac{1}{32} \cdot \log \ell, \quad \ell \to +\infty. \]
The order of magnitude of $\Var(\mathcal{L}_\ell(0))$ is smaller than what would be a natural guess (i.e., $\ell$ as for the non-nodal case); this situation is due to some cancelation in the asymptotic expansion of the nodal variance (“obscure Berry’s cancellation” – see [69, 68]) and is similar to the cancellation phenomenon observed by Berry in a different setting [11].

### 7.1.2 Main result

Our principal aim is to study the asymptotic behaviour, as $\ell \to \infty$, of the distribution of the sequence of normalized random variables
\[ \tilde{\mathcal{L}}_\ell(z) := \frac{\mathcal{L}_\ell(z) - \mathbb{E}[\mathcal{L}_\ell(z)]}{\sqrt{\Var(\mathcal{L}_\ell(z))}}, \quad \ell \geq 1. \quad (7.1.4) \]

The following statement is the main result of this chapter.

**Theorem 7.1.1.** For $z \neq 0$ the sequence $\{\tilde{\mathcal{L}}_\ell(z) : \ell \geq 1\}$ converges in distribution to a standard Gaussian r.v. $Z$. In particular
\[ \lim_{\ell \to +\infty} d(\tilde{\mathcal{L}}_\ell(z), Z) = 0, \quad (7.1.5) \]
where $d$ denotes either the Kolmogorov distance (4.1.11), or an arbitrary distance metrizing the weak convergence on $\mathcal{P}$ the space of all probability measures on $\mathbb{R}$ (see Chapter 4).
7.1.3 Wiener chaos and Berry’s cancellation

The proof of our result rely on a pervasive use of Wiener-Itô chaotic expansions (see Chapter 4 e.g.) and the reader is referred to the two monographs [53, 56] for an exhaustive discussion.

According to (4.2.15), the Gaussian spherical eigenfunctions considered in this work are built starting from a family of i.i.d. Gaussian r.v.’s $\{a_{\ell,m} : \ell \geq 1, m = 1, \ldots, 2\ell + 1\}$, defined on some probability space $(\Omega, \mathcal{F}, P)$ and verifying the following properties:

$$E[a_{\ell,m}] = 0, \quad E[a_{\ell,m}a_{\ell',m'}] = \frac{4\pi}{2\ell + 1}\delta_{m,m'}\delta_{\ell,\ell'}.$$  

We define $A$ to be the closure in $L^2(P)$ of all real finite linear combinations of $\{a_{\ell,m} : \ell \geq 1, m = 1, \ldots, 2\ell + 1\}$. $A$ is a real centered Gaussian space (that is, a linear space of jointly Gaussian centered real-valued random variables, that is stable under convergence in $L^2(P)$) (compare to Chapter 4).

**Definition 7.1.2.** For every $q = 0, 1, 2, \ldots$ the $q$th Wiener chaos associated with $A$ (compare to Chapter 4), written $C_q$, is the closure in $L^2(P)$ of all real finite linear combinations of random variables with the form

$$H_{p_1}(\xi_1)H_{p_2}(\xi_2)\cdots H_{p_k}(\xi_k),$$

where the integers $p_1, \ldots, p_k \geq 0$ verify $p_1 + \cdots + p_k = q$, and $(\xi_1, \ldots, \xi_k)$ is a real centered Gaussian vector with identity covariance matrix extracted from $A$ (note that, in particular, $C_0 = \mathbb{R}$).

$C_q \perp C_m$ (where the orthogonality holds in the sense of $L^2(P)$) for every $q \neq m$, and moreover

$$L^2(\Omega, \sigma(A), P) = \bigoplus_{q=0}^{\infty} C_q,$$  \hspace{1cm} (7.1.6)

that is: each real-valued functional $F$ of $A$ can be (uniquely) represented in the form

$$F = \sum_{q=0}^{\infty} \text{proj}(F \mid C_q),$$  \hspace{1cm} (7.1.7)
where \( \text{proj}(\bullet | C_q) \) stands for the projection operator onto \( C_q \), and the series converges in \( L^2(\mathbb{P}) \). Plainly, \( \text{proj}(F | C_0) = \mathbb{E}[F] \).

Now recall the definition of \( T_\ell \) given in (4.2.15): the following elementary statement shows that the Gaussian field

\[
\left\{ T_\ell(\theta), \frac{\partial}{\partial \theta_1} T_\ell(\theta), \frac{\partial}{\partial \theta_2} T_\ell(\theta) : \theta = (\theta_1, \theta_2) \in \mathbb{S}^2 \right\}
\]

is a subset of \( \mathbf{A} \), for every \( \ell \in \mathbb{N} \).

**Proposition 7.1.3.** Fix \( \ell \in \mathbb{N} \), let the above notation and conventions prevail. Then, for every \( j = 1, 2 \) one has that

\[
\partial_j T_\ell(\theta) := \frac{\partial}{\partial \theta_j} T_\ell(\theta) = \sum_m a_{\ell,m} \frac{\partial}{\partial \theta_j} Y_{\ell,m}(\theta), \quad (7.1.8)
\]

and therefore \( T_\ell(\theta), \partial_1 T_\ell(\theta), \partial_2 T_\ell(\theta) \in \mathbf{A} \), for every \( \theta \in \mathbb{S}^2 \). Moreover, for every fixed \( \theta \in \mathbb{S}^2 \), one has that \( T_\ell(\theta), \partial_1 T_\ell(\theta), \partial_2 T_\ell(\theta) \) are stochastically independent (see e.g. [68]).

We shall often use the fact that

\[
\text{Var}[\partial_j T_\ell(\theta)] = \frac{\ell(\ell + 1)}{2},
\]

and, accordingly, for \( \theta = (\theta_1, \theta_2) \in \mathbb{S}^2 \) and \( j = 1, 2 \), we will denote by \( \partial_j \tilde{T}_\ell(\theta) \) the normalized derivative

\[
\partial_j \tilde{T}_\ell(\theta) := \sqrt{\frac{2}{\ell(\ell + 1)}} \frac{\partial}{\partial \theta_j} T_\ell(\theta) . \quad (7.1.9)
\]

The next statement gathers together some of the main technical achievements of the present chapter. It shows in particular that the already evoked ‘arithmetic Berry cancellation phenomenon’ (see [11]) – according to which the variance of the nodal length \( L_\ell := L_\ell(0) \) (as defined in (7.1.1)) has asymptotically the same order as \( \log \ell \) (rather than the expected order \( \ell \)) – should be a consequence of the following:
Sec. 7.2 - Chaotic expansions

(i) The projection of $L_\ell(z)$ on the second Wiener chaos $C_2$ is exactly equal to zero for every $\ell \in \mathbb{N}$ if and only if $z = 0$ (and so holds for the projection of $L_\ell(0)$ onto any chaos of odd order $q \geq 3$).

(ii) For $z \neq 0$, the variance of $\text{proj}(L_\ell(z) | C_2)$ has the order $\ell$, as $\ell \to +\infty$, and one has moreover that

$$\text{Var}(L_\ell(z)) \sim \text{Var}(\text{proj}(L_\ell(z) | C_2)).$$

7.1.4 Plan

The rest of the chapter is organized as follows: §7.2 contains a study of the chaotic representation of nodal lengths, §7.3 focuses on the projection of nodal lengths on the second Wiener chaos, whereas §7.4 contains a proof of our main result.

7.2 Chaotic expansions

The aim of this section is to derive an explicit expression for each projection of the type $\text{proj}(L_\ell(z) | C_q)$, $q \geq 1$. In order to accomplish this task, we first focus on a family of auxiliary random variables $\{L_\varepsilon^\ell(z,\omega) : \varepsilon > 0\}$ that approximate $L_\ell(z)$ in the sense of the $L^2(\mathbb{P})$-norm.

7.2.1 Preliminary results

- For each $z \in \mathbb{R}$, $L_\varepsilon(z)$ is the $\omega$-a.s. limit, for $\varepsilon \to 0$, of the $\varepsilon$-approximating r.v.

$$L_\varepsilon^\ell(z,\omega) := \frac{1}{2\varepsilon} \int_{S^2} 1_{[z-\varepsilon,z+\varepsilon]}(T_\ell(\theta,\omega)) \| \nabla T_\ell(\theta,\omega) \| d\theta,$$  \hspace{1cm} (7.2.10)

where

$$\nabla T_\ell := (\partial_1 T_\ell, \partial_2 T_\ell),$$

$\| \cdot \|$ is the norm in $\mathbb{R}^2$, and we have used the notation (7.1.8).
Sec. 7.2 - Chaotic expansions

We know that

- \( \mathcal{L}_\ell(z) \in L^2(\mathbb{P}) \), for every \( z \in \mathbb{R} \) ([69, 68]) .

Now we want to prove that \( \mathcal{L}_\ell(z) \) is the \( L^2(\Omega) \)-limit, for \( \varepsilon \to 0 \) of \( \mathcal{L}_\ell^\varepsilon(z) \). Remark first that analogous arguments as in [68], prove that the function \( z \mapsto \mathbb{E}[\mathcal{L}_\ell(z)^2] \) is continuous (further details will appear in [42]).

**Lemma 7.2.1.** It holds that

\[
\lim_{\varepsilon \to 0} \mathbb{E}[(\mathcal{L}_\ell^\varepsilon(z) - \mathcal{L}_\ell(z))^2] = 0 .
\]

**Proof.** Since \( \mathcal{L}_\ell^\varepsilon(z) \to \mathcal{L}_\ell(z) \) a.s., it is enough to show that

\[
\mathbb{E}[\mathcal{L}_\ell^\varepsilon(z)^2] \to \mathbb{E}[\mathcal{L}_\ell(z)^2] ,
\]

and then use the well-known fact that convergence a.s. plus convergence of the norms implies convergence in mean square [13, Proposition 3.39].

By Fatou's Lemma (for the first inequality) we have

\[
\mathbb{E}[\mathcal{L}_\ell(z)^2] \leq \lim \inf_{\varepsilon} \mathbb{E}[\mathcal{L}_\ell^\varepsilon(z)^2] \leq \lim \sup_{\varepsilon} \mathbb{E}[\mathcal{L}_\ell^\varepsilon(z)^2] ^* 
\]

\[
\overset{\overset{\ast}{\ast}}{=} \lim \sup_{\varepsilon} \mathbb{E} \left[ \left( \int_{\mathbb{R}} \mathcal{L}_\ell(u) \frac{1}{2\varepsilon} 1_{[-\varepsilon,\varepsilon]}(u - z) \, du \right)^2 \right] ,
\]

where to establish the equality \( ^* \) we have used the co-area formula [1, (7.14.13)], which in our case gives

\[
\mathcal{L}_\ell^\varepsilon(z) = \frac{1}{2\varepsilon} \int_{\mathbb{R}^2} 1_{[z-\varepsilon,z+\varepsilon]}(T_\ell(\theta)) \| \nabla T_\ell(\theta) \| \, d\theta = \int_{\mathbb{R}} du \int_{T_\ell^{-1}(u)} \frac{1}{2\varepsilon} 1_{[-\varepsilon,\varepsilon]}(u - z) \, d\theta = \frac{1}{2\varepsilon} \int_{\mathbb{R}} \mathcal{L}_\ell(u) 1_{[-\varepsilon,\varepsilon]}(u - z) \, du .
\]

Now by Jensen inequality we find that

\[
\lim \sup_{\varepsilon} \mathbb{E} \left[ \left( \int_{\mathbb{R}} \mathcal{L}_\ell(u) \frac{1}{2\varepsilon} 1_{[-\varepsilon,\varepsilon]}(u - z) \, du \right)^2 \right] \leq
\]

\[
\leq \lim \sup_{\varepsilon} \int_{\mathbb{R}} \mathbb{E} \left[ \mathcal{L}_\ell(u)^2 \right] \frac{1}{2\varepsilon} 1_{[-\varepsilon,\varepsilon]}(u - z) \, du =
\]

\[
= \mathbb{E} \left[ \mathcal{L}_\ell(z)^2 \right] ,
\]

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the last step following by continuity of the map $u \mapsto \mathbb{E}[\mathcal{L}_\ell(u)^2]$. □

To conclude the section, we observe that the previous result suggests that the random variable $\mathcal{L}_\ell(z)$ can be formally written as

$$\mathcal{L}_\ell(z) = \int_{\mathbb{R}^2} \delta_z(T_\ell(\theta)) \|\nabla T_\ell\| d\theta ,$$

where $\delta_z$ denotes the Dirac mass in $z$.

### 7.2.2 The chaotic expansion for $\mathcal{L}_\ell(z)$

In view of the convention (7.1.9), throughout the section we will rewrite (7.2.10) as

$$\mathcal{L}_\ell(z) = \sqrt{\frac{\ell(\ell + 1)}{2}} \frac{1}{2z} \int_{\mathbb{R}^2} 1_{[z-\varepsilon,z+\varepsilon]}(T_\ell(\theta)) \sqrt{\partial_1 \tilde{T}_\ell(\theta)^2 + \partial_2 \tilde{T}_\ell(\theta)^2} d\theta .$$

(7.2.12)

We also need to introduce two collection of coefficients $\{\alpha_{n,m} : n, m \geq 1\}$ and $\{\beta_l(z) : l \geq 0\}$, that are connected to the (formal) Hermite expansions of the norm $\|\cdot\|$ in $\mathbb{R}^2$ and the Dirac mass $\delta_z(\cdot)$ respectively. These are given by

$$\beta_l(z) := \phi(z) H_l(z) ,$$

(7.2.13)

where $\phi$ is the standard Gaussian pdf, $H_l$ denotes the $l$-th Hermite polynomial (4.1.3) and $\alpha_{n,m} = 0$ but for the case $n, m$ even

$$\alpha_{2n,2m} = \sqrt{\frac{\pi}{2}} \frac{(2n)!(2m)!}{n!m!} \frac{1}{2^{n+m} p_{n+m} \left(\frac{1}{4}\right)} ,$$

(7.2.14)

where for $N = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$

$$p_N(x) := \sum_{j=0}^{N} (-1)^j (-1)^N \binom{N}{j} \frac{(2j+1)!}{(j!)^2} x^j ,$$

(7.2.15)

$\frac{(2j+1)!}{(j!)^2}$ being the so-called swinging factorial restricted to odd indices.
Proposition 7.2.2 (Chaotic expansion of $\mathcal{L}_\ell(z)$). For every $\ell \in \mathbb{N}$ and $q \geq 2$,\allowbreak
\begin{equation}
\text{proj}(\mathcal{L}_\ell(z) | C_q) = \sqrt{\frac{\ell(\ell+1)}{2}} \sum_{Q=2}^{+\infty} \sum_{u=0}^{Q} \sum_{k=0}^{u} \frac{\alpha_{k,u-k}\beta_{q-u}(z)}{(k)!(u-k)!(q-u)!} \int_{S^2} H_{q-u}(T_\ell(\theta)) H_k(\partial_1 \tilde{T}_\ell(\theta)) H_{u-k}(\partial_2 \tilde{T}_\ell(\theta)) d\theta. \tag{7.2.16}
\end{equation}
As a consequence, one has the representation \allowbreak
\begin{equation}
\mathcal{L}_\ell(z) = \mathbb{E}\mathcal{L}_\ell(z) + \sqrt{\frac{\ell(\ell+1)}{2}} \sum_{Q=2}^{+\infty} \sum_{u=0}^{Q} \sum_{k=0}^{u} \frac{\alpha_{k,u-k}\beta_{q-u}(z)}{(k)!(u-k)!(q-u)!} \times \int_{S^2} H_{q-u}(T_\ell(\theta)) H_k(\partial_1 \tilde{T}_\ell(\theta)) H_{u-k}(\partial_2 \tilde{T}_\ell(\theta)) d\theta, \tag{7.2.17}
\end{equation}
where the series converges in $L^2(\mathbb{P})$.

Proof of Proposition 7.2.2. The proof is divided into four steps.

Step 1: dealing with indicators. We start by expanding the function $\frac{1}{2\varepsilon}1_{[z-\varepsilon,z+\varepsilon]}(\cdot)$ into Hermite polynomials, as defined in §7.1.3: $\frac{1}{2\varepsilon}1_{[z-\varepsilon,z+\varepsilon]}(\cdot) = \sum_{l=0}^{+\infty} \frac{1}{l!} \beta^\varepsilon_l(z) H_l(\cdot).$ \allowbreak
One has that $\beta^\varepsilon_0(z) = \frac{1}{2\varepsilon} \int_{z-\varepsilon}^{z+\varepsilon} \phi(x) \, dx$, and, for $l \geq 1$, $\begin{align*}
\beta^\varepsilon_l(z) &= \frac{1}{2\varepsilon} \int_{z-\varepsilon}^{z+\varepsilon} \phi(x) H_l(x) \, dx = \frac{1}{2\varepsilon} \int_{z-\varepsilon}^{z+\varepsilon} \phi(x)(-1)^l \phi^{-1}(x) \frac{d^l}{dx^l} \phi(x) \, dx = \\
&= (-1)^l \frac{1}{2\varepsilon} \int_{z-\varepsilon}^{z+\varepsilon} \frac{d^l}{dx^l} \phi(x) \, dx .
\end{align*}$
Using the notation (7.2.13), we have that $\lim_{\varepsilon \to 0} \beta^\varepsilon_0(z) = \phi(z) = \phi(z) H_0(z) = \beta_0(z)$, and for all $l \geq 1$, $\lim_{\varepsilon \to 0} \beta^\varepsilon_l(z) = (-1)^l \frac{d^l}{dx^l} \phi(x) \big|_{z=z} = \phi(z) H_l(z) = \beta_l(z).$ \allowbreak
\begin{equation}
\tag{7.2.18}
\end{equation}
Step 2: dealing with the Euclidean norm. Fix \( x \in \mathbb{S}^2 \), and recall that, according to Proposition 7.1.3, the vector

\[
\nabla \tilde{T}_\ell := (\partial_1 \tilde{T}_\ell, \partial_2 \tilde{T}_\ell)
\]

is composed of centered independent Gaussian random variables with variance one. Now, since the random variable \( \|\nabla \tilde{T}_\ell(\theta)\| \) is square-integrable, it can be expanded into the following infinite series of Hermite polynomials:

\[
\|\nabla \tilde{T}_\ell(\theta)\| = \sum_{u=0}^{+\infty} \sum_{m=0}^{u} \frac{\alpha_{u,n-m}}{u!(u-m)!} H_u(\partial_1 \tilde{T}_\ell(\theta)) H_{u-m}(\partial_2 \tilde{T}_\ell(\theta)),
\]

where

\[
\alpha_{n,n-m} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} H_n(y) H_{n-m}(z) e^{-\frac{y^2 + z^2}{2}} dydz.
\] (7.2.19)

Our aim is to compute \( \alpha_{n,n-m} \) as explicitly as possible. First of all, we observe that, if \( n \) or \( n-m \) is odd, then the above integral vanishes (since the two mappings \( z \mapsto \sqrt{y^2 + z^2} \) and \( y \mapsto \sqrt{y^2 + z^2} \) are even). It follows therefore that

\[
\|\nabla \tilde{T}_\ell(\theta)\| = \sum_{n=0}^{+\infty} \sum_{m=0}^{n} \frac{\alpha_{2n,2n-2m}}{(2n)!(2n-2m)!} H_{2n}(\partial_1 \tilde{T}_\ell(\theta)) H_{2n-2m}(\partial_2 \tilde{T}_\ell(\theta)).
\]

We are therefore left with the task of showing that the integrals

\[
\alpha_{2n,2n-2m} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} H_{2n}(y) H_{2n-2m}(z) e^{-\frac{y^2 + z^2}{2}} dydz,
\] (7.2.20)

where \( n \geq 0 \) and \( m = 0, \ldots, n \), can be evaluated according to (7.2.14). One elegant way for dealing with this task is to use the following Hermite polynomial expansion

\[
e^{\lambda y - \frac{x^2}{2}} = \sum_{a=0}^{+\infty} H_a(y) \frac{\lambda^a}{a!}, \quad \lambda \in \mathbb{R}.
\] (7.2.21)

We start by considering the integral

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} e^{\lambda y - \frac{x^2}{2}} e^{\mu z - \frac{y^2}{2}} e^{-\frac{y^2 + z^2}{2}} dydz = \frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} e^{-\frac{(y-\lambda)^2 + (z-\mu)^2}{2}} dydz.
\]
This integral coincides with the expected value of the random variable $W := \sqrt{Y^2 + Z^2}$ where $(Y, Z)$ is a vector of independent Gaussian random variables with variance one and mean $\lambda$ and $\mu$, respectively. Since $W^2 = Y^2 + Z^2$ has a non-central $\chi^2$ distribution (more precisely, $Y^2 + Z^2 \sim \chi^2(2, \lambda^2 + \mu^2)$) it is easily checked that the density of $W$ is given by

$$f_W(t) = \sum_{j=0}^{+\infty} e^{-((\lambda^2 + \mu^2)/2)(\lambda^2 + \mu^2)/2j}f_{2+2j}(t^2) 2t \mathbb{1}_{\{t > 0\}},$$

where $f_{2+2j}$ is the density function of a $\chi^2_{2+2j}$ random variable. The expected value of $W$ is therefore

$$2 \sum_{j=0}^{+\infty} e^{-((\lambda^2 + \mu^2)/2)(\lambda^2 + \mu^2)/2j} \int_0^{+\infty} f_{2+2j}(t^2) t^2 dt.\quad (7.2.23)$$

From the definition of $f_{2+2j}$ we have

$$\int_0^{+\infty} f_{2+2j}(t^2) t^2 dt = \frac{1}{2^{1+j} \Gamma(1+j)} \int_0^{+\infty} t^{2j+2}e^{-t^2/2} dt = \frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\pi}}{2^{1+j} \Gamma(1+j)}.$$

As a consequence,

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} e^{-\frac{(y-\lambda)^2 + (z-\mu)^2}{2}} dydz = 2e^{-((\lambda^2 + \mu^2)/2)\sum_{j=0}^{+\infty} ((\lambda^2 + \mu^2)/2j) \frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\pi}}{2^{1+j} \Gamma(1+j)}} =: F(\lambda, \mu).$$

We can develop the function $F$ as follows:

$$F(\lambda, \mu) = 2 \sum_{a=0}^{+\infty} \frac{(-1)^a \lambda^{2a}}{2^a a!} \sum_{b=0}^{+\infty} \frac{(-1)^b \mu^{2b}}{2^b b!} \sum_{j=0}^{+\infty} \frac{1}{j!} \sum_{l=0}^{j} \left(\begin{array}{c} j \\ l \end{array}\right) \lambda^{2l} \mu^{2j-2l} \frac{\prod_{i=1}^{1+j}(2i-1)\sqrt{\pi}}{2^{1+j} \Gamma(1+j)} =$$

$$= \sum_{a,b=0}^{+\infty} \frac{(-1)^a (-1)^b \prod_{i=1}^{1+j}(2i-1)\sqrt{\pi}}{2^a a! 2^b b! j!2^{2j} \Gamma(1+j)} \sum_{l=0}^{j} \left(\begin{array}{c} j \\ l \end{array}\right) \lambda^{2l+2a} \mu^{2j+2b-2l}.\quad (7.2.24)$$
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On the other hand

\[ F(\lambda, \mu) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} e^{\lambda y - \frac{\lambda^2}{2}} e^{\mu z - \frac{\mu^2}{2}} e^{-\frac{y^2 + z^2}{2}} \, dy \, dz \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} \sum_{a=0}^{+\infty} H_a(y) \left( \frac{\lambda}{a!} \right) \sum_{b=0}^{+\infty} H_b(z) \left( \frac{\mu}{b!} \right) e^{-\frac{y^2 + z^2}{2}} \, dy \, dz \]

\[ = \sum_{a,b=0}^{+\infty} \left( \frac{1}{a! b! 2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} H_a(y) H_b(z) e^{-\frac{y^2 + z^2}{2}} \, dy \, dz \right) \lambda^a \mu^b. \]

By the same reasoning as above, if \( a \) or \( b \) is odd, then the integral coefficient in the previous expression must be zero. Setting \( n := l + a \) and \( m := j + b - l \), we also have that

\[ \sum_{a,b=0}^{+\infty} \frac{(-1)^a}{2^a a!} \frac{(-1)^b}{2^b b!} \sum_{j=0}^{+\infty} \frac{\prod_{i=1}^{1+j} (2i - 1)}{j!^{2j} \Gamma(1 + j)} \sum_{l=0}^{j} \left( \frac{j}{l} \right) \lambda^{2l+2a} \mu^{2j+2b-2l} \]

\[ = \sum_{n,m} \sum_{j} \frac{\prod_{i=1}^{1+j} (2i - 1)}{j!^{2j} \Gamma(1 + j)} \sum_{l=0}^{j} \frac{(-1)^{(n-l)}}{2^{n-l} (n-l)!} \frac{(-1)^{m-l-j}}{2^{m-l-j} (m+l-j)!} \left( \frac{j}{l} \right) \lambda^{2n} \mu^{2m}. \]

Thus we obtain

\[ \alpha_{2n,2m} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} H_{2n}(y) H_{2m}(z) e^{-\frac{y^2 + z^2}{2}} \, dy \, dz \]

\[ = \frac{(2n)!(2m)!}{2^{n+m}} \sum_{j} (-1)^j \frac{\prod_{i=1}^{1+j} (2i - 1)}{2^j j! \Gamma(1 + j)} \sum_{l=0}^{j} \frac{(j)}{(n-l)! (m+l-j)!}. \]
Sec. 7.2 - Chaotic expansions

Representation (7.2.14) now follows from the computations:

\[
\alpha_{2n,2m} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} H_{2n}(y) H_{2m}(z) e^{-\frac{y^2 + z^2}{4}} \, dy \, dz
\]

\[
= (2n)!(2m)!! \sum_j (-1)^j \frac{\prod_{i=0}^{j} (2i) \sqrt{\pi}}{2^j j! \Gamma(1 + j)} \sum_{l=0}^{j} \frac{(j)}{(n - l)!(m + l - j)!}
\]

\[
= (2n)!(2m)!! \sum_j (-1)^j \frac{\prod_{i=0}^{j} (2i + 1) \sqrt{\pi}}{2^j j!} \sum_{l=0}^{j} \frac{(j)}{(n - l)!(m + l - j)!}
\]

\[
= \sqrt{\frac{\pi}{2}} \frac{(2n)!(2m)!! (n + m)!}{n!m!} \sum_{j=0}^{n+m} (-1)^j \frac{(n + m)!}{2^j j!} \binom{n + m}{j}
\]

\[
= \sqrt{\frac{\pi}{2}} \frac{(2n)!(2m)!! (n + m)!}{n!m!} \sum_{j=0}^{n+m} (-1)^j \frac{(n + m)!}{2^j j! (j + 1)!} \binom{n + m}{j}
\]

\[
= \sqrt{\frac{\pi}{2}} \frac{(2n)!(2m)!! (n + m)!}{n!m!} \sum_{j=0}^{n+m} (-1)^j \frac{(n + m)!}{2^j j! (j + 1)!} \binom{n + m}{j}.
\]

**Step 3: letting \( \varepsilon \to 0 \).** In view of Definition 7.1.2, the computations at Step 1 and Step 2 (together with the fact that the three random variables \( T_\ell(\theta) \), \( \partial_1 \tilde{T}_\ell(\theta) \) and \( \partial_2 \tilde{T}_\ell(\theta) \) are stochastically independent) show that, for fixed \( \theta \in S^2 \), the projection of the random variable

\[
\frac{1}{2\varepsilon} \mathbb{1}_{|z - \varepsilon, z + \varepsilon|}(T_\ell(\theta)) \sqrt{\partial_1 \tilde{T}_\ell(\theta)^2 + \partial_2 \tilde{T}_\ell(\theta)^2}
\]

on the chaos \( C_q \) equals

\[
\sum_{u=0}^{q} \sum_{m=0}^{u} \frac{\alpha_{m,u-m} \beta_{q-u}(z)}{(m)!(u - m)!(q - u)!} H_{q-u}(T_\ell(\theta)) H_{u-m}(\partial_1 \tilde{T}_\ell(\theta)) H_{u-m}(\partial_2 \tilde{T}_\ell(\theta))
\]

Since \( \int_{S^2} dx < \infty \), standard arguments based on Jensen inequality and dominated
convergence yield that, for every \( q \geq 1 \),

\[
\text{proj}(\mathcal{L}_\ell(z) \mid C_q) = \sqrt{\frac{\ell(\ell + 1)}{2}} \sum_{u=0}^{q} \sum_{m=0}^{u} \frac{\alpha_{m,u-m} \beta_{q-u}^{2q-u}(z)}{(m)!(u-m)!(q-u)!} \times \\
\times \int_{\mathbb{S}^2} H_{q-u}(T_\ell(\theta)) H_m(\partial_1 \tilde{T}_\ell(\theta)) H_{u-m}(\partial_2 \tilde{T}_\ell(\theta)) d\theta.
\]

One has that, as \( \varepsilon \to 0 \), \( \text{proj}(\mathcal{L}_\ell(z) \mid C_q) \) converges necessarily to \( \text{proj}(\mathcal{L}_\ell(z) \mid C_q) \) in probability. Using (7.2.18), we deduce from this fact that representation (7.2.16) is valid for every \( q \geq 1 \).

\[\square\]

**Remark 7.2.3.** The coefficients \( \alpha_{2n,2m} \) can be found also first using polar coordinates and then the explicit expression for Hermite polynomials [64]. Briefly,

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} \sqrt{y^2 + z^2} H_{2n}(y) H_{2m}(z) e^{-\frac{x^2 + y^2}{2}} dxdy = \\
= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{+\infty} \rho^2 H_{2n}(\rho \cos \vartheta) H_{2m}(\rho \sin \vartheta) e^{-\frac{\rho^2}{2}} d\rho d\vartheta = \\
= \frac{(2n)!(2m)!}{2\pi} \sum_{a=0}^{n} \frac{(-1)^a}{2^a a!(2n - 2a)!} \sum_{b=0}^{m} \frac{(-1)^b}{2^b b!(2m - 2b)!} \times \\
\times \int_0^{2\pi} \cos \vartheta^{2n-2a} \sin \vartheta^{2m-2b} d\vartheta \int_0^{+\infty} \rho^{2+2n-2a+2m-2b} e^{-\frac{\rho^2}{2}} d\rho.
\]

It remains to solve the previous integrals (which are well-known). \[\square\]

### 7.3 Asymptotic study of proj(\(\mathcal{L}_\ell(z)\mid C_2\))

In this section we find an explicit expression for the second order chaotic projection of the length of level curves.

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Proposition 7.3.1. We have

\[
\text{proj}(\mathcal{L}_\ell(z)|C_2) = \sqrt{\frac{\ell(\ell + 1)}{2}} \sqrt{\frac{\pi}{8}} \phi(z)z^2 \int_{S^2} H_2(T_\ell(x)) \, dx = \\
= \sqrt{\frac{\ell(\ell + 1)}{2}} \sqrt{\frac{\pi}{8}} \phi(z)z^2 \sum_{m=1}^{2\ell+1} \left( a^2_{\ell,m} - \frac{4\pi}{2\ell + 1} \right).
\]

Proof. The second chaotic projection is, omitting the factor \( \sqrt{\frac{\ell(\ell + 1)}{2}} \),

\[
\frac{\alpha_{0,0}\beta_2(z)}{2} \int_{S^2} H_2(T_\ell(x)) \, dx + \frac{\alpha_{0,2}\beta_0(z)}{2} \int_{S^2} H_2(\tilde{T}_\ell(x)) \, dx + \\
+ \frac{\alpha_{2,0}\beta_0(z)}{2} \int_{S^2} H_2(\tilde{\partial}_1 T_\ell(x)) \, dx = \frac{1}{2} \left( \alpha_{0,0}\beta_2(z) \right) \int_{S^2} (T_\ell(x)^2 - 1) \, dx + \\
+ \alpha_{2,0}\beta_0(z) \int_{S^2} ((\tilde{T}_\ell(x))^2 - 1) \, dx + \alpha_{2,0}\beta_0(z) \int_{S^2} ((\tilde{\partial}_1 T_\ell(x))^2 - 1) \, dx = \\
= \frac{1}{2} \left( \alpha_{0,0}\beta_2(z) \right) \int_{S^2} T_\ell(x)^2 \, dx + \frac{2}{\ell(\ell + 1)} \left( \partial_2 T_\ell(x) \right)^2 \, dx + \\
+ \frac{2}{\ell(\ell + 1)} \int_{S^2} (\partial_1 T_\ell(x))^2 \, dx - 4\pi \left( \alpha_{0,0}\beta_2(z) + \frac{4}{\ell(\ell + 1)} \alpha_{0,2}\beta_0(z) \right).
\]

Now, by Green’s formula, we have for \( j = 1, 2 \)

\[
\int_{S^2} (\partial_j T_\ell(x))^2 \, dx = - \int_{S^2} T_\ell(x) \partial_j^2 T_\ell(x) \, dx
\]

and putting things together

\[
\frac{1}{2} \left( \alpha_{0,0}\beta_2(z) \right) \int_{S^2} T_\ell(x)^2 \, dx + \frac{2}{\ell(\ell + 1)} \left( \partial_2 T_\ell(x) \right)^2 \, dx + \\
+ \alpha_{2,0}\beta_0(z) \frac{2}{\ell(\ell + 1)} \int_{S^2} (\partial_1 T_\ell(x))^2 \, dx - 4\pi \left( \alpha_{0,0}\beta_2(z) + \frac{4}{\ell(\ell + 1)} \alpha_{0,2}\beta_0(z) \right) = \\
= \frac{1}{2} \left( \alpha_{0,0}\beta_2(z) \right) \int_{S^2} T_\ell(x)^2 \, dx - \alpha_{0,2}\beta_0(z) \frac{2}{\ell(\ell + 1)} \int_{S^2} T_\ell(x) \partial_2^2 T_\ell(x) \, dx + \\
- \alpha_{2,0}\beta_0(z) \frac{2}{\ell(\ell + 1)} \int_{S^2} T_\ell(x) \partial_1^2 T_\ell(x) \, dx - 4\pi \left( \alpha_{0,0}\beta_2(z) + \frac{4}{\ell(\ell + 1)} \alpha_{0,2}\beta_0(z) \right).
\]
Sec. 7.3 - Asymptotic study of \( \text{proj}(L_{\ell}(z)|C_2) \)

\[
\begin{align*}
= \frac{1}{2} \left( \alpha_{0,0}\beta_{2}(z) \int_{S^2} T_{\ell}(x)^2 \, dx - \alpha_{0,2}\beta_{0}(z) \frac{2}{\ell(\ell + 1)} \int_{S^2} T_{\ell}(x)(\partial_1^2 T_{\ell}(x) + \partial_2^2 T_{\ell}(x)) \, dx + 
- 4\pi (\alpha_{0,0}\beta_{2}(z) + \frac{4}{\ell(\ell + 1)} \alpha_{0,2}\beta_{0}(z)) \right) = \\
= \frac{1}{2} \left( \alpha_{0,0}\beta_{2}(z) \int_{S^2} T_{\ell}(x)^2 \, dx - \alpha_{0,2}\beta_{0}(z) \frac{2}{\ell(\ell + 1)} \int_{S^2} T_{\ell}(x) \Delta T_{\ell}(x) \, dx + 
- 4\pi (\alpha_{0,0}\beta_{2}(z) + \frac{4}{\ell(\ell + 1)} \alpha_{0,2}\beta_{0}(z)) \right) = \\
= \frac{1}{2} \left( \alpha_{0,0}\beta_{2}(z) \int_{S^2} T_{\ell}(x)^2 \, dx + \alpha_{0,2}\beta_{0}(z) \frac{2}{\ell(\ell + 1)} \ell(\ell + 1) \int_{S^2} T_{\ell}(x)^2 \, dx + 
- 4\pi (\alpha_{0,0}\beta_{2}(z) + \frac{4}{\ell(\ell + 1)} \alpha_{0,2}\beta_{0}(z)) \right) = \\
= \frac{1}{2} \left( \alpha_{0,0}\beta_{2}(z) + 2\alpha_{0,2}\beta_{0}(z) \right) \int_{S^2} T_{\ell}(x)^2 \, dx - 4\pi (\alpha_{0,0}\beta_{2}(z) + 2\alpha_{0,2}\beta_{0}(z)) = \\
= \frac{1}{2} \left( \alpha_{0,0}\beta_{2}(z) + 2\alpha_{0,2}\beta_{0}(z) \right) \int_{S^2} (T_{\ell}(x)^2 - 1) \, dx = \\
= \frac{1}{2} \sqrt{\frac{\pi}{2}} \phi(z)z^2 \int_{S^2} H_2(T_{\ell}(x)) \, dx.
\end{align*}
\]

Moreover

\[
\int_{S^2} H_2(T_{\ell}(x)) \, dx = \int_{S^2} \sum_{m,m'} (a_{\ell,m}a_{\ell,m'}Y_{\ell,m}(x)Y_{\ell,m'}(x) - 1) \, dx = \\
= \sum_{m=1}^{2\ell+1} \left( a_{\ell,m}^2 - \frac{4\pi}{2\ell + 1} \right),
\]

since \( Y_{\ell,m} \) are an orthonormal family.

Now it immediately follows that

**Corollary 7.3.2.** *The second chaotic projection of the length \( L_{\ell}(z) \) vanishes if and only if \( z = 0 \).*

**Remark 7.3.3.** Previous computations in the proof of Proposition 7.3.1 indeed holds on every two dimensional compact Riemannian manifold, actually we can always use Green’s formula.
Sec. 7.4 - The CLT

7.4 The CLT

In this section we prove the main result of this chapter, that is a CLT for the length of $z$-level curve for $z \neq 0$. Let us first show the following.

**Lemma 7.4.1.** For $z \neq 0$, we have

$$\frac{\text{proj}(L_\ell(z)|C_2)}{\sqrt{\text{Var(\text{proj}(L_\ell(z)|C_2))}}} \xrightarrow{\mathcal{L}} Z,$$

where $Z \sim \mathcal{N}(0, 1)$.

**Proof.** The variance of the second chaotic projection (Proposition 7.3.1) is

$$\text{Var(\text{proj}(L_\ell(z)|C_2))} = \ell(\ell + 1) \frac{\pi}{16} \phi(z)^2 z^4 2 \cdot 4\pi \cdot 2\pi \int_0^\pi P_\ell(\cos \vartheta)^2 d\vartheta =$$

$$= \ell(\ell + 1) \frac{\pi}{16} \phi(z)^2 z^4 2 \cdot 4\pi \cdot 2\pi \frac{2}{2\ell + 1} = \ell(\ell + 1) \frac{\pi}{16} 2\pi e^{-z^2} z^4 2 \cdot 4\pi \cdot 2\pi \frac{2}{2\ell + 1} =$$

$$= \ell(\ell + 1) \frac{2}{2\ell + 1} \cdot \frac{\pi^2}{2} e^{-z^2} z^4 \sim \ell \cdot \frac{\pi^2}{2} e^{-z^2} z^4, \quad \ell \to +\infty,$$

where we used the identity (5.1.9)

$$\int_0^\pi P_\ell(\cos \vartheta)^2 d\vartheta = \frac{2}{2\ell + 1}.$$

Moreover we can rewrite the second chaotic projection as

$$\text{proj}(L_\ell(z)|C_2) = \sqrt{\frac{\ell(\ell + 1)}{2}} \sqrt{\frac{\pi}{8}} \phi(z) z^2 \sum_m \left( a_{\ell,m}^2 - \frac{4\pi}{2\ell + 1} \right) =$$

$$= \sqrt{\frac{\ell(\ell + 1)}{2}} \sqrt{\frac{\pi}{4}} \phi(z) z^2 \frac{4\pi}{2\ell + 1} \sqrt{2(2\ell + 1)} \sum_m \left( \left( \sqrt{\frac{2\ell + 1}{4\pi}} a_{\ell,m} \right)^2 - 1 \right).$$

Now we can apply the standard CLT to the sequence of normalized sums

$$\frac{1}{\sqrt{2(2\ell + 1)}} \sum_m \left( \left( \sqrt{\frac{2\ell + 1}{4\pi}} a_{\ell,m} \right)^2 - 1 \right) \xrightarrow{\mathcal{L}} Z,$$

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where $Z \sim \mathcal{N}(0,1)$. Finally this implies the CLT for the second chaotic projection

$$
\frac{\text{proj}(\mathcal{L}_\ell(z)|C_2)}{\sqrt{\text{Var} (\text{proj}(\mathcal{L}_\ell(z)|C_2))}} \xrightarrow{\ell} Z ,
$$

which conclude the proof. \hfill \Box

Now we can easily prove Theorem 7.1.1.

*Proof of Theorem 7.1.1.* We have, for $z \neq 0$,

$$
\lim_{\ell} \frac{\text{Var}(\text{proj}(\mathcal{L}_\ell(z)|C_2))}{\text{Var}(\mathcal{L}_\ell(z))} = 1 . \tag{7.4.24}
$$

It follows from the chaotic decomposition, that as $\ell \to \infty$

$$
\frac{\mathcal{L}_\ell(z)}{\sqrt{\text{Var}(\mathcal{L}_\ell(z))}} = \frac{\text{proj}(\mathcal{L}_\ell(z)|C_2)}{\sqrt{\text{Var}(\mathcal{L}_\ell(z))}} + o_p(1) ,
$$

therefore $\frac{\mathcal{L}_\ell(z)}{\sqrt{\text{Var}(\mathcal{L}_\ell(z))}}$ and $\frac{\text{proj}(\mathcal{L}_\ell(z)|C_2)}{\sqrt{\text{Var}(\mathcal{L}_\ell(z))}}$ have the same asymptotic distribution. Previous lemma allows to conclude the proof, recalling moreover that if the limit distribution is absolutely continuous, than the convergence in distribution is equivalent to the convergence in Kolmogorov distance. \hfill \Box
Chapter 8

Nodal lengths for arithmetic random waves

8.1 Introduction and main results

In this chapter we investigate the asymptotic behavior of nodal lengths for arithmetic random waves.

8.1.1 Arithmetic random waves

Let $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ be the standard 2-torus and $\Delta$ the Laplace operator on $\mathbb{T}$. We are interested in the (totally discrete) spectrum of $\Delta$ i.e. eigenvalues $E > 0$ of the Schrödinger equation

$$\Delta f + Ef = 0. \quad (8.1.1)$$

Let

$$S = \{ n \in \mathbb{Z} : n = a^2 + b^2 \text{ for some } a, b \in \mathbb{Z} \}$$

be the collection of all numbers expressible as a sum of two squares. Then the eigenvalues of (8.1.1) (also called “energy levels” of the torus) are all numbers of the form $E_n = 4\pi^2 n$ with $n \in S$. 

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In order to describe a Laplace eigenspace corresponding to $E_n$, denote $\Lambda_n$ to be the set of “frequencies”:

$$\Lambda_n := \{ \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \lambda_1^2 + \lambda_2^2 = n \}$$

of cardinality $|\Lambda_n|$. (Geometrically $\Lambda_n$ are all the standard lattice points lying on the centered radius-$\sqrt{n}$ circle.) For $\lambda \in \Lambda_n$ denote the complex exponential associated to the frequency $\lambda$

$$e_\lambda(\theta) = \exp(2\pi i \langle \lambda, \theta \rangle)$$

with $\theta = (\theta_1, \theta_2) \in \mathbb{T}$. The collection

$$\{e_\lambda(\theta)\}_{\lambda \in \Lambda_n}$$

of complex exponentials corresponding to frequencies $\lambda \in \Lambda_n$ is an $L^2$-orthonormal basis of the eigenspace of $\Delta$ corresponding to eigenvalue $E_n$. In particular, the dimension of $E_n$ equals the number of ways to express $n$ as a sum of two squares

$$\mathcal{N}_n := \dim E_n = |\Lambda_n|$$

(also denoted in the number theoretic literature $r_2(n) = |\Lambda_n|$). The number $\mathcal{N}_n$ is subject to large and erratic fluctuation; it grows \([?]\) on average as $\sqrt{\log n}$, but could be as small as 8 for (an infinite sequence of) prime numbers $p \equiv 1 \mod 4$, or as large as a power of $\log n$.

Following [61] and [34] we define the “arithmetic random waves” (random Gaussian toral Laplace eigenfunctions) to be the random fields

$$T_n(\theta) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda(\theta), \quad (8.1.2)$$

$\theta \in \mathbb{T}$, where the coefficients $a_\lambda$ are standard Gaussian i.i.d. save to the relations

$$a_{-\lambda} = \overline{a_\lambda}$$

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Sec. 8.1 - Introduction and main results

(ensuring that $T_n$ are real-valued). By the definition (8.1.2), $T_n$ is a centered Gaussian random field with covariance function

$$r_n(\theta, \zeta) = r_n(\theta - \zeta) := \mathbb{E}[T_n(\theta)\overline{T_n(\zeta)}] = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} e_\lambda(\theta - \zeta) = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \cos (2\pi \langle \theta - \zeta, \lambda \rangle),$$

$\theta, \zeta \in \mathbb{T}$ (by the standard abuse of notation). Note that $r_n(0) = 1$, i.e. $T_n$ is unit variance.

8.1.2 Nodal length: mean and variance

Consider the total nodal length of random eigenfunctions, i.e. the sequence the random variables $\{L_n\}_{n \in S}$ given by

$$L_n := \text{length}(T_n^{-1}(0)).$$

The expected value of $L_n$ was computed [61] to be

$$\mathbb{E}[L_n] = \frac{1}{2\sqrt{2}} \sqrt{E_n},$$

consistent to Yau’s conjecture [71, 26]. The more subtle question of asymptotic behaviour of the variance $\text{Var}(L_n)$ of $L_n$ was addressed [61], and fully resolved [34] as follows.

Given $n \in S$ define a probability measure $\mu_n$ on the unit circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$ supported on angles corresponding to lattice points in $\Lambda_n$:

$$\mu_n := \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda \theta}. $$

It is known that for a density 1 sequence of numbers $\{n_j\} \subseteq S$ the angles of lattice points in $\Lambda_n$ tend to be equidistributed in the sense that

$$\mu_{n_j} \Rightarrow \frac{d\theta}{2\pi}$$

(where $\Rightarrow$ is weak-* convergence of probability measures). However the sequence $\{\mu_n\}_{n \in S}$ has other weak-* partial limits [24, 34] (“attainable measures”), partially classified in [35].

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It was proved [34] that one has
\[ \text{Var}(\mathcal{L}_n) = c_n \frac{E_n}{N_n^2} (1 + o_{N_n \to \infty}(1)), \]  
(8.1.5)
where
\[ c_n = 1 + \frac{\hat{\mu}_n(4)^2}{512}, \]  
(8.1.6)
and for a measure \( \mu \) on \( \mathbb{S}^1 \),
\[ \hat{\mu}_n(k) = \int_{\mathbb{S}^1} z^{-k} \, d\mu_n(z) \]  
(8.1.7)
are the usual Fourier coefficients of \( \mu \) on the unit circle. As
\[ |\hat{\mu}_n(4)| \leq 1 \]
by the triangle inequality, the result (8.1.5) shows that the order of magnitude of \( \text{Var}(\mathcal{L}_n) \) is \( \frac{E_n}{N_n^2} \), that is, of smaller order than what would be a natural guess \( \frac{E_n}{N_n} \); this situation (‘arithmetic Berry’s cancellation’ – see [34]) is similar to the cancellation phenomenon observed by Berry in a different setting [11].

In addition, (8.1.5) shows that for \( \text{Var}(\mathcal{L}_n) \) to exhibit an asymptotic law (equivalent to \( \{c_n\} \) in (8.1.6) convergent along a subsequence) we need to pass to a subsequence \( \{n_j\} \subset S \) such that the limit
\[ \lim_{j \to \infty} |\hat{\mu}_{n_j}(4)| \]
exists. For example, if \( \{n_j\} \subset S \) is a subsequence such that \( \mu_{n_j} \Rightarrow \mu \) for some probability measure \( \mu \) on \( \mathbb{S}^1 \), then (8.1.5) reads (under the usual extra-assumption \( N_{n_j} \to \infty \))
\[ \text{Var}(\mathcal{L}_{n_j}) \sim c(\mu) \frac{E_{n_j}}{N_{n_j}^2} \]  
(8.1.8)
with
\[ c(\mu) = 1 + \frac{\hat{\mu}(4)^2}{512}, \]
where, here and for the rest of the chapter, we write \( a_n \sim b_n \) to indicate that the two positive sequences \( \{a_n\} \) and \( \{b_n\} \) are such that \( a_n/b_n \to 1 \), as \( n \to \infty \). Note that the
set of the possible values for the 4th Fourier coefficient \( \hat{\mu}(4) \) covers the whole interval \([-1, 1]\) (see [34, 35]). This implies in particular that the possible values of the constant \( c(\mu) \) cover the whole interval

\[
\left[ \frac{1}{512}, \frac{1}{256} \right];
\]

the above discussion provides a complete classification of the asymptotic behaviour of \( \text{Var}(\mathcal{L}_n) \).

8.1.3 Main results

Let \( \{n_j : j \geq 1\} \subset S \) be a sequence within \( S \), and assume that \( \lim_{j \to \infty} N_{n_j} = \infty \). As it is customary, generic subsequences of \( \{n_j\} \) will be denoted by \( \{n_j'\} \), \( \{n_j''\} \), and so on. Our principal aim in this chapter is to study the asymptotic behaviour, as \( j \to \infty \), of the distribution of the sequence of normalized random variables

\[
\tilde{\mathcal{L}}_{n_j} := \frac{\mathcal{L}_{n_j} - \mathbb{E}[\mathcal{L}_{n_j}]}{\sqrt{\text{Var}[\mathcal{L}_{n_j}]}}; \quad j \geq 1.
\]

(8.1.9)

Since, in this setting, the variance (8.1.5) diverges to infinity, it seems reasonable to expect a central limit result, that is, that the sequence \( \tilde{\mathcal{L}}_{n_j}, j \geq 1 \), converges in distribution to a standard Gaussian random variable. Our main findings not only contradict this somewhat naive prediction, but also show the following non-trivial facts:

(i) the sequence \( \{\tilde{\mathcal{L}}_{n_j}\} \) does not necessarily converge in distribution, and

(ii) the adherent points of the sequence \( \{\tilde{\mathcal{L}}_{n_j} : j \geq 1\} \) (in the sense of the topology induced by the convergence in distribution of random variables) coincide with the distributions spanned by a class of linear combinations of independent squared Gaussian random variables; such linear combinations are moreover parameterized by the adherent points of the numerical sequence

\[
j \mapsto |\hat{\mu}_{n_j}(4)|, \quad j \geq 1.
\]
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One should note that the phenomenon described at Point (ii) is consistent with the fact that the variance $\text{Var}(\mathcal{L}_n)$ explicitly depends on the constant $\hat{\mu}_n(4)^2$ (see (8.3.31)). In order to formally state our main findings, we introduce some further notation: for every $\eta \in [0, 1]$, we write $\mathcal{M}_\eta$ to indicate the random variable

$$\mathcal{M}_\eta := \frac{1}{2\sqrt{1 + \eta^2}}(2 - (1 + \eta)X_1^2 - (1 - \eta)X_2^2),$$

where $X = (X_1, X_2)$ is a two-dimensional centered Gaussian vector with identity covariance matrix (more information on the distributions of the random variables $\mathcal{M}_\eta$ is provided in Proposition 8.1.3). For every $n \in S$, we write

$$\mathcal{M}^n := \mathcal{M}_{|\hat{\mu}_n(4)|},$$

where the quantity $\hat{\mu}_n(4)$ is defined according to formula (8.1.7).

The following statement is the main result of the chapter.

**Theorem 8.1.1.** Let the above notation and assumptions prevail. Then, the sequence $\{D(\tilde{\mathcal{L}}_n^j) : j \geq 1\}$ is relatively compact with respect to the topology of weak convergence, and a subsequence $\{\tilde{\mathcal{L}}_{n'_j}\}$ admits a limit in distribution if and only if the corresponding numerical subsequence $\{|\hat{\mu}_{n'_j}(4)| : j \geq 1\}$ converges to some $\eta \in [0, 1]$, and in this case

$$\tilde{\mathcal{L}}_{n'_j} \xrightarrow{d} \mathcal{M}_\eta.$$

In particular, letting $d$ denote either the Kolmogorov distance (4.1.11), or an arbitrary distance metrizing weak convergence on $\mathcal{P}$ (the space of all probability measures on $\mathbb{R}$ - see Chapter 4) this implies that

$$\lim_{j \to \infty} d(\tilde{\mathcal{L}}_{n_j}, \mathcal{M}^n_j) = 0.$$  

(8.1.12)

The next result is a direct consequence of Theorem 8.1.1, of [27, Theorem 11.7.1] and of the fact that $\{\tilde{\mathcal{L}}_{n_j}\}$ is a bounded sequence in $L^2$: it shows that one can actually couple the elements of the sequences $\{\tilde{\mathcal{L}}_{n_j}\}$ and $\{\mathcal{M}^n_j\}$ on the same probability space, in such a way that their difference converges to zero almost surely and in $L^p$, for every $p < 2$. 

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Corollary 8.1.2. There exists a probability space \((\Omega^*, \mathcal{F}^*, \mathbb{P}^*)\) as well as random variables \(\{A_j, B_j : j \geq 1\}\) defined on it such that, for every \(j \geq 1\), \(A_j \xrightarrow{d} \tilde{L}_n\), \(B_j \xrightarrow{d} \mathcal{M}^n\), and, as \(j \to \infty\),

\[ A_j - B_j \to 0, \quad \text{a.s.} \quad \mathbb{P}^*. \]

Also, for every \(p \in (0, 2)\), \(\mathbb{E}^*[|A_j - B_j|^p] \to 0\).

We conclude this section by stating some elementary properties of the random variables \(\mathcal{M}_\eta, \eta \in [0, 1]\), whose proof (left to the reader) can be easily deduced from the representation

\[ \mathcal{M}_\eta = a(\eta)H_2(X_1) + b(\eta)H_2(X_2), \tag{8.1.13} \]

where \(H_2(x) = x^2 - 1\) is the second Hermite polynomial, \(a(\eta) := -(1 + \eta)/\sqrt{4(1 + \eta^2)}\) and \(b(\eta) := -(1 - \eta)/\sqrt{4(1 + \eta^2)}\), as well as from the (classical) results presented in [53, Section 2.7.4].

In what follows, we will use the elementary fact that, if \(\mathcal{M}_\eta\) is the random variable defined in (8.1.10) and if \(\eta \to \eta_0 \in [0, 1]\), then \(\mathcal{M}_\eta \xrightarrow{d} \mathcal{M}_{\eta_0}\).

Proposition 8.1.3 (About \(\mathcal{M}_\eta\)). Let the above notation prevail.

(i) For every \(\eta \in [0, 1]\), the distribution of \(\mathcal{M}_\eta\) is absolutely continuous with respect to the Lebesgue measure, with support equal to \((-\infty, (1 + \eta^2)^{-1/2})\).

(ii) For every \(\eta \in [0, 1]\), the characteristic function of \(\mathcal{M}_\eta\) is given by

\[ \varphi_\eta(\mu) := \mathbb{E}[\exp(i\mu \mathcal{M}_\eta)] = \frac{e^{-i\mu(a(\eta) + b(\eta))}}{\sqrt{(1 - 2i\mu a(\eta))(1 - 2i\mu b(\eta))}}, \quad \mu \in \mathbb{R}. \]

(iii) For every \(\eta \in [0, 1]\), the distribution of \(\mathcal{M}_\eta\) is determined by its moments (or, equivalently, by its cumulants). Moreover, the sequence of the cumulants of \(\mathcal{M}_\eta\), denoted by \(\{\kappa_p(\mathcal{M}_\eta) : p \geq 1\}\), admits the representation: \(\kappa_p(\mathcal{M}_\eta) = 2^{p-1}(p - 1)!(a(\eta)^p + b(\eta)^p)\), for every \(p \geq 1\) (in particular, \(\mathcal{M}_\eta\) has unit variance).

(iv) Let \(\eta_0, \eta_1 \in [0, 1]\) be such that \(\eta_0 \neq \eta_1\). Then, \(\mathbb{D}(\mathcal{M}_{\eta_0}) \neq \mathbb{D}(\mathcal{M}_{\eta_1})\).
We observe that Point 4 in the previous statement is an immediate consequence of Point 1 and of the fact that the mapping $\eta \mapsto (1 + \eta^2)^{-1/2}$ is injective on $[0, 1]$. In the next section, we will discuss the role of chaotic expansions in the proofs of our main findings.

8.1.4 Chaos and the Berry cancellation phenomenon

As in the previous chapter, the proofs of our results rely on a pervasive use of Wiener-Itô chaotic expansions for non-linear functionals of Gaussian fields (the reader is referred to the two monographs [53, 56] for an exhaustive discussion).

According to (8.1.2), the arithmetic random waves considered in this work are built starting from a family of complex-valued Gaussian random variables $\{a_\lambda : \lambda \in \mathbb{Z}^2\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and verifying the following properties: (a) each $a_\lambda$ has the form $x_\lambda + iy_\lambda$, where $x_\lambda$ and $y_\lambda$ are two independent real-valued Gaussian random variables with mean zero and variance $1/2$; (b) $a_\lambda$ and $a_\tau$ are stochastically independent whenever $\lambda \notin \{\tau, -\tau\}$, and (c) $a_\lambda = \overline{a_{-\lambda}}$. We define $A$ to be the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables $\xi$ having the form $\xi = za_\lambda + \overline{z}a_{-\lambda}$, where $\lambda \in \mathbb{Z}^2$ and $z \in \mathbb{C}$. It is easily verified that $A$ is a real centered Gaussian space (that is, a linear space of jointly Gaussian centered real-valued random variables, that is stable under convergence in $L^2(\mathbb{P})$).

**Definition 8.1.4.** For every $q = 0, 1, 2, ...$ the $q$th Wiener chaos associated with $A$, written $C_q$, is the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables with the form

$$H_{p_1}(\xi_1)H_{p_2}(\xi_2) \cdots H_{p_k}(\xi_k),$$

where the integers $p_1, ..., p_k \geq 0$ verify $p_1 + \cdots + p_k = q$, and $(\xi_1, ..., \xi_k)$ is a real centered Gaussian vector with identity covariance matrix extracted from $A$ (note that, in particular, $C_0 = \mathbb{R}$).

Again $C_q \perp C_m$ (where the orthogonality holds in the sense of $L^2(\mathbb{P})$) for every $q \neq m$. 

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and moreover

\[ L^2(\Omega, \sigma(A), \mathbb{P}) = \bigoplus_{q=0}^{\infty} C_q, \quad (8.1.14) \]

that is: each real-valued functional \( F \) of \( A \) can be (uniquely) represented in the form

\[ F = \sum_{q=0}^{\infty} \text{proj}(F | C_q), \quad (8.1.15) \]

where \( \text{proj}(\bullet | C_q) \) stands for the projection operator onto \( C_q \), and the series converges in \( L^2(\mathbb{P}) \). Plainly, \( \text{proj}(F | C_0) = \mathbb{E}F \). Now recall the definition of \( T_n \) given in (8.1.2): the following elementary statement shows that the Gaussian field

\[ \left\{ T_n(\theta), \frac{\partial}{\partial \theta_1} T_n(\theta), \frac{\partial}{\partial \theta_2} T_n(\theta) : \theta = (\theta_1, \theta_2) \in \mathbb{T} \right\} \]

is a subset of \( A \), for every \( n \in S \).

**Proposition 8.1.5.** Fix \( n \in S \), let the above notation and conventions prevail. Then, for every \( j = 1, 2 \) one has that

\[ \partial_j T_n(\theta) := \frac{\partial}{\partial \theta_j} T_n(\theta) = \frac{2\pi i}{\sqrt{N_n}} \sum_{(\lambda_1, \lambda_2) \in \Lambda_n} \lambda_j a_{\lambda} e_{\lambda}(\theta), \quad (8.1.16) \]

and therefore \( T_n(\theta), \partial_1 T_n(\theta), \partial_2 T_n(\theta) \in A \), for every \( \theta \in \mathbb{T} \). Moreover, for every fixed \( \theta \in \mathbb{T} \), one has that \( T_n(\theta), \partial_1 T_n(\theta), \partial_2 T_n(\theta) \) are stochastically independent.

We shall often use the fact that

\[ \text{Var}[\partial_j T_n(\theta)] = \frac{4\pi^2}{N_n} \sum_{\lambda \in \Lambda_n} \lambda_j^2 = 4\pi^2 \frac{n}{2}, \]

and, accordingly, for \( \theta = (\theta_1, \theta_2) \in \mathbb{T} \) and \( j = 1, 2 \), we will denote by \( \partial_j \tilde{T}_n(\theta) \) the normalized derivative

\[ \partial_j \tilde{T}_n(\theta) := \frac{1}{2\pi} \sqrt{\frac{2}{n}} \frac{\partial}{\partial \theta_j} T_n(\theta) = \sqrt{\frac{2}{n}} \frac{i}{\sqrt{N_n}} \sum_{\lambda \in \Lambda_n} \lambda_j a_{\lambda} e_{\lambda}(\theta). \quad (8.1.17) \]
The next statement gathers together some of the main technical achievements of the present chapter. It shows in particular that the already evoked ‘arithmetic Berry cancellation phenomenon’ (see [34], as well as [11]) – according to which the variance of the nodal length $L_n$ (as defined in (8.1.3)) has asymptotically the same order as $E_n N_n^2$ (rather than the expected order $E_n N_n$) – is a consequence of the following two facts:

(i) The projection of $L_n$ on the second Wiener chaos $C_2$ is exactly equal to zero for every $n \in S$ (and so is the projection of $L_n$ onto any chaos of odd order $q \geq 3$).

(ii) The variance of $\text{proj}(L_n | C_4)$ has the order $\frac{E_n}{N_n^2}$, as $N_n \to \infty$, and one has moreover that

$$\text{Var}(L_n) = \text{Var}(\text{proj}(L_n | C_4)) + o\left(\frac{E_n}{N_n^2}\right).$$

Note that, in principle, if $\text{proj}(L_n | C_2)$ did not vanish, then the sequence $n \mapsto \text{Var}(\text{proj}(L_n | C_2))$ would have provided the leading term (of the order $\frac{E_n}{N_n}$) in the asymptotic development of $\text{Var}(L_n)$.

**Proposition 8.1.6 (Berry cancellation phenomenon).** For every fixed $n \in S$, one has that

$$\text{proj}(L_n | C_2) = \text{proj}(L_n | C_{2k+1}) = 0, \quad k = 0, 1, \ldots,$$

Moreover, if $\{n_j : j \geq 1\} \subset S$ is a sequence contained in $S$ such that $\lim_{j \to \infty} N_{n_j} = \infty$, then (as $j \to \infty$)

$$\text{Var}(L_{n_j}) \sim c(\mu_{n_j}) \frac{E_{n_j}}{N_{n_j}^2} \sim \text{Var}(\text{proj}(L_{n_j} | C_4)),$$

and therefore,

$$\sum_{k=3}^{\infty} \text{Var}(\text{proj}(L_{n_j} | C_{2k})) = o\left(\frac{E_{n_j}}{N_{n_j}^2}\right).$$

**Remark 8.1.7.** Nodal lengths of Gaussian Laplace eigenfunctions $T_\ell$, $\ell \in \mathbb{N}$, on the two-dimensional sphere have the same qualitative behavior as their toral counterpart. Indeed, in Proposition 7.3.1 it is shown that the second chaotic term in the Wiener-Itô
expansion of the length of level curves $T^{-1}_\ell(u)$, $u \in \mathbb{R}$ disappears if and only if $u = 0$. These findings shed some light on the Berry’s cancellation phenomenon, indeed they explain why the asymptotic variance of the length of level curves respects the natural scaling – except for the nodal case [68, 69].

**Conjecture 8.1.8.** Consider Gaussian eigenfunctions $T$ on some manifold $\mathbb{M}$ and define as usual the $u$-excursion set as

$$A_u(T, \mathbb{M}) := \{x \in \mathbb{M} : T(x) > u\}, \quad u \in \mathbb{R}.$$ 

Toral (resp. spherical) nodal lengths can be viewed as the length of the boundary of $A_u$ for $u = 0$ for $\mathbb{M} = \mathbb{T}$ the 2-torus (resp. $\mathbb{M} = S^2$ the 2-sphere). In this sense, as stated in the Introduction of this thesis, they represent a special case of the second Lipschitz-Killing curvature of $A_u$, $u \in \mathbb{R}$ (see [1] for the definition and a comprehensive treatment of Lipschitz-Killing curvatures on Gaussian excursion sets). We conjecture that for excursion sets $A_u$ of Gaussian eigenfunctions on compact manifolds $\mathbb{M}$ the projection of each Lipschitz-Killing curvature on the second-order Wiener chaos vanishes if and only if $u = 0$; clearly the proof of this conjecture would represent a major step towards a global understanding of the Berry’s cancellation phenomenon. In the two-dimensional case, there are three Lipschitz-Killing curvatures, which correspond to the area, half the boundary length and the Euler-Poincaré characteristic of the excursion sets; for the 2-sphere, we refer to [47, 44, 16] for results supporting our conjecture in the case of the area and the Euler-Poincaré characteristic, and to Remark 8.1.7 and Chapter 7 for the boundary lengths.

### 8.1.5 Plan

The rest of the chapter is organized as follows: §8.2 contains a study of the chaotic representation of nodal lengths, §8.3 focuses on the projection of nodal lengths on the fourth Wiener chaos, whereas §8.4 contains a proof of our main result.
8.2 Chaotic expansions

The aim of this section is to derive an explicit expression for each projection of the type $\text{proj}(\mathcal{L}_n \mid C_q)$, $q \geq 1$. In order to accomplish this task, as in Chapter 7, we first focus on a sequence of auxiliary random variables $\{\mathcal{L}^\varepsilon_n : \varepsilon > 0\}$ that approximate $\mathcal{L}_n$ in the sense of the $L^2(\mathbb{P})$ norm.

8.2.1 Preliminary results

Fix $n \geq 1$, and let $T_n$ be defined according to (8.1.2). Define, for $\varepsilon > 0$, the approximating random variables

$$\mathcal{L}^\varepsilon_n := \frac{1}{2\varepsilon} \int_{\mathbb{T}} 1_{[-\varepsilon,\varepsilon]}(T_n(\theta)) \| \nabla T_n(\theta) \| d\theta .$$

(8.2.19)

Lemma 8.2.1. We have

$$\lim_{\varepsilon \to 0} \mathbb{E}[|\mathcal{L}^\varepsilon_n - \mathcal{L}_n|^2] = 0 .$$

Proof. We have that

$$\lim_{\varepsilon \to 0} \mathcal{L}^\varepsilon_n = \mathcal{L}_n , \quad \text{a.s.}$$

Moreover, for every $\varepsilon$

$$|\mathcal{L}^\varepsilon_n - \mathcal{L}_n|^2 \leq 2(\|\mathcal{L}^\varepsilon_n\|^2 + (\mathcal{L}_n)^2) \leq 2((12\sqrt{4\pi^2n})^2 + (\mathcal{L}_n)^2) ,$$

where the last equality follows from Lemma 3.2 in [61]. We can hence apply dominated convergence theorem to conclude. 

We observe that the previous result suggests that the random variable $\mathcal{L}_n$ can be formally written as

$$\mathcal{L}_n = \int_{\mathbb{T}} \delta_0(T_n(\theta)) \| \nabla T_n(\theta) \| d\theta ,$$

(8.2.20)

where $\delta_0$ denotes the Dirac mass in $0$.
8.2.2 Chaotic expansion of nodal length $L_n$

In view of the convention (8.1.17), we will rewrite (8.2.19) as

$$L_n^\varepsilon = \frac{1}{2\varepsilon} \sqrt{4\pi^2} \sqrt{\frac{n}{2}} \int_{T} 1_{[-\varepsilon,\varepsilon]}(T_n(\theta)) \sqrt{\partial_1 \tilde{T}_n(\theta)^2 + \partial_2 \tilde{T}_n(\theta)^2} \, d\theta.$$  \hspace{1cm} (8.2.21)

Recall from Chapter 7 the two collection of coefficients $\{\alpha_{2n,2m} : n, m \geq 1\}$ and $\{\beta_{2l} : l \geq 0\}$, that are connected to the (formal) Hermite expansions of the norm $\| \cdot \|$ in $\mathbb{R}^2$ and the Dirac mass $\delta_0(\cdot)$ respectively. These are given by (7.2.13) and (7.2.14)

$$\beta_{2l} := \frac{1}{\sqrt{2\pi}} H_{2l}(0),$$

where $H_{2l}$ denotes the $2l$-th Hermite polynomial and

$$\alpha_{2n,2m} = \sqrt{\frac{\pi}{2}} \frac{(2n)!(2m)!}{n!m!} \frac{1}{2^{n+m}} p_{n+m} \left( \frac{1}{4} \right),$$

where for $N = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$ (as in (7.2.15))

$$p_N(x) := \sum_{j=0}^{N} (-1)^j \left( -1 \right)^N \binom{N}{j} \frac{(2j+1)!}{(j!)^2} x^j,$$

being the so-called swinging factorial restricted to odd indices. The following result provides the key tool in order to prove Proposition 8.1.6.

**Proposition 8.2.2 (Chaotic expansion of $L_n$).** Relation (8.1.18) holds for every $n \in S$ and also, for every $q \geq 2$,

$$\text{proj}(L_n \mid C_{2q}) = \sqrt{\frac{4\pi^2n}{2}} \sum_{u=0}^{q} \sum_{k=0}^{u} \alpha_{2k,2u-2k} \beta_{2q-2u} \frac{1}{(2k)!(2u-2k)!(2q-2u)!} \int_{T} H_{2q-2u}(T_n(\theta)) H_{2k}(\partial_1 \tilde{T}_n(\theta)) H_{2u-2k}(\partial_2 \tilde{T}_n(\theta)) \, d\theta. \hspace{1cm} (8.2.22)$$

As a consequence, one has the representation

$$L_n = \mathbb{E}L_n \sqrt{4\pi^2n} \sum_{q=2}^{+\infty} \sum_{u=0}^{q} \sum_{k=0}^{u} \alpha_{2k,2u-2k} \beta_{2q-2u} \frac{1}{(2k)!(2u-2k)!(2q-2u)!} \times \int_{T} H_{2q-2u}(T_n(\theta)) H_{2k}(\partial_1 \tilde{T}_n(\theta)) H_{2u-2k}(\partial_2 \tilde{T}_n(\theta)) \, d\theta, \hspace{1cm} (8.2.23)$$

where the series converges in $L^2(\mathbb{P})$. 

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Proof of Proposition 8.2.2. The proof of the chaotic projection formula is based on the same arguments as the proof of Proposition 7.2.2, therefore we can skip details.

Let us show that the proj(\(L_n \mid C_2\)) vanishes. It equals the quantity

\[
\sqrt{4\pi^2} \sqrt{n} \left( \frac{\alpha_0 \beta_2}{2} \int_\mathbb{T} H_2(T_n(\theta)) \, d\theta + \frac{\alpha_2 \beta_0}{2} \int_\mathbb{T} H_2(\partial_2 T_n(\theta)) \, d\theta + \right.
\]

\[
\left. \frac{\alpha_2 \beta_0}{2} \int_\mathbb{T} H_2(\partial_1 T_n(\theta)) \, d\theta \right).
\]

Using the explicit expression \(H_2(x) = x^2 - 1\), we deduce that

\[
\int_\mathbb{T} H_2(T_n(\theta)) \, d\theta = \int_\mathbb{T} (T_n(\theta)^2 - 1) \, d\theta = \int_\mathbb{T} \left( \frac{1}{N_n} \sum_{\lambda, \lambda' \in \Lambda_n} a_\lambda \overline{a}_{\lambda'} e_{\lambda, \lambda'}(\theta) - 1 \right) \, d\theta
\]

\[
= \frac{1}{N_n} \sum_{\lambda, \lambda' \in \Lambda_n} a_\lambda \overline{a}_{\lambda'} \underbrace{\int_\mathbb{T} e_{\lambda, \lambda'}(\theta) \, d\theta}_{\delta_{\lambda}^{\lambda'}} - 1 = \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} (|a_\lambda|^2 - 1),
\]

where \(\delta_{\lambda}^{\lambda'}\) is the Kronecker symbol (observe that \(\mathbb{E}[|a_\lambda|^2] = 1\), hence the expected value of the integral \(\int_\mathbb{T} H_2(T_n(\theta)) \, d\theta\) is 0, as expected). Analogously, for \(j = 1, 2\) we have that

\[
\int_\mathbb{T} H_2(\partial_j T_n(\theta)) \, d\theta = \int_\mathbb{T} \left( \frac{2}{n N_n} \sum_{\lambda, \lambda' \in \Lambda_n} \lambda_j \lambda'_j a_\lambda \overline{a}_{\lambda'} e_{\lambda, \lambda'}(\theta) - 1 \right) \, d\theta
\]

\[
= \frac{2}{n N_n} \sum_{\lambda \in \Lambda_n} \lambda_j^2 |a_\lambda|^2 - 1 = \frac{1}{N_n n} \sum_{\lambda \in \Lambda_n} 2 \lambda_j^2 (|a_\lambda|^2 - 1),
\]

where the last equality follows from the elementary identity

\[
\sum_{\lambda \in \Lambda_n} \lambda_j^2 = \frac{n N_n}{2}.
\]
Since \( \alpha_{2n,2m} = \alpha_{2m,2n} \) we can rewrite \( \text{proj}(\mathcal{L}_n \mid C_2) \) as

\[
\sqrt{4\pi^2} \sqrt{\frac{n}{2}} \left( \frac{\alpha_{0,0}\beta_2}{2} \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} (|a_{\lambda}|^2 - 1) + \frac{\alpha_{0,2}\beta_0}{2} \frac{1}{\mathcal{N}_n} \frac{n}{2} \sum_{\lambda \in \Lambda_n} (\lambda_1^2 + \lambda_2^2)(|a_{\lambda}|^2 - 1) \right)
\]

\[
= \sqrt{4\pi^2} \sqrt{\frac{n}{2}} \frac{1}{2\mathcal{N}_n} \left( \alpha_{0,0}\beta_2 \sum_{\lambda \in \Lambda_n} (|a_{\lambda}|^2 - 1) + 2\alpha_{0,2}\beta_0 \sum_{\lambda \in \Lambda_n} (|a_{\lambda}|^2 - 1) \right)
\]

\[
= \sqrt{4\pi^2} \sqrt{\frac{n}{2}} \frac{1}{2\mathcal{N}_n} \left( \alpha_{0,0}\beta_2 + 2\alpha_{0,2}\beta_0 \right) \sum_{\lambda \in \Lambda_n} (|a_{\lambda}|^2 - 1).
\]

Easy computations show that

\[
\alpha_{0,0} = \sqrt{\frac{\pi}{2}}, \quad \alpha_{0,2} = \alpha_{2,0} = \frac{1}{2} \sqrt{\frac{\pi}{2}}, \quad \beta_0 = \frac{1}{\sqrt{2\pi}}, \quad \beta_2 = -\frac{1}{\sqrt{2\pi}},
\]

and therefore

\[
\text{proj}(\mathcal{L}_n \mid C_2) = \sqrt{4\pi^2} \sqrt{\frac{n}{2}} \frac{1}{2\mathcal{N}_n} \left( -\frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{2\pi}} + 2\frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{2\pi}} \right) \sum_{\lambda \in \Lambda_n} (|a_{\lambda}|^2 - 1)
\]

\[
= \sqrt{4\pi^2} \sqrt{\frac{n}{2}} \frac{1}{2\mathcal{N}_n} \left( -\frac{1}{2} + \frac{1}{2} \right) \sum_{\lambda \in \Lambda_n} (|a_{\lambda}|^2 - 1) = 0,
\]

thus concluding the proof. \( \square \)

### 8.3 Asymptotic study of \( \text{proj}(\mathcal{L}_n \mid C_4) \)

#### 8.3.1 Preliminary considerations

As anticipated in the Introduction, one of the main findings of the present chapter is that, whenever \( \mathcal{N}_{n_j} \to \infty \), the asymptotic behaviour of the normalized sequence \( \{\widetilde{\mathcal{L}}_{n_j}\} \) in (8.1.9) is completely determined by that of the fourth order chaotic projections

\[
\text{proj}(\widetilde{\mathcal{L}}_{n_j} \mid C_4) = \frac{\text{proj}(\mathcal{L}_{n_j} \mid C_4)}{\sqrt{\text{Var}[\mathcal{L}_{n_j}]}} \quad j \geq 1.
\]
The aim of this section is to provide a detailed asymptotic study of the sequence appearing in (8.3.24), by using in particular the explicit formula (8.2.22). For the rest of the chapter, we use the notation

\[ \psi(\eta) := \frac{3 + \eta}{8}, \quad \eta \in [-1, 1], \]  

and will exploit the following elementary relations, valid as \( N_{n_j} \to \infty \):

(i) if \( \hat{\mu}_{n_j}(4) \to \eta \in [-1, 1] \), then, for \( \ell = 1, 2 \),

\[ \frac{2}{n_j^2 N_{n_j}} \sum_{\lambda = \langle \lambda_1, \lambda_2 \rangle \in \Lambda_{n_j}} \lambda_4^\ell \to \psi(\eta); \]  

(ii) if \( \hat{\mu}_{n_j}(4) \to \eta \in [-1, 1] \), then

\[ \frac{2}{n_j^2 N_{n_j}} \sum_{\lambda = \langle \lambda_1, \lambda_2 \rangle \in \Lambda_{n_j}} \lambda_1^2 \lambda_2^2 \to \frac{1}{2} - \psi(\eta). \]  

Note that (8.3.26)–(8.3.27) follow immediately from the fact that, for every \( n \),

\[ \hat{\mu}_n(4) = \frac{1}{n^2 N_n} \sum_{\lambda \in \Lambda_n} (\lambda_1^4 + \lambda_2^4 - 6\lambda_1^2 \lambda_2^2), \]

as well as from elementary symmetry considerations. We will also use the identity:

\[ 64 \psi(\eta)^2 - 48 \psi(\eta) + 10 = \eta^2 + 1. \]  

One of our principal tools will be the following multivariate central limit theorem (CLT).

**Proposition 8.3.1.** Assume that the subsequence \( \{n_j\} \subset S \) is such that \( N_{n_j} \to \infty \) and \( \hat{\mu}_{n_j}(4) \to \eta \in [-1, 1] \). Define

\[ H(n_j) = \left( \begin{array}{c} H_1(n_j) \\ H_2(n_j) \\ H_3(n_j) \\ H_4(n_j) \end{array} \right) := \frac{1}{n_j \sqrt{N_{n_j}/2}} \sum_{\lambda = \langle \lambda_1, \lambda_2 \rangle \in \Lambda_{n_j}} (|a_{\lambda}|^2 - 1) \left( \begin{array}{c} n_j \\ \lambda_1^2 \\ \lambda_2^2 \\ \lambda_1 \lambda_2 \end{array} \right). \]
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Then, as \( n_j \to \infty \), the following CLT holds:

\[
H(n_j) \overset{d}{\to} Z(\eta) = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix},
\]

(8.3.29)

where \( Z(\eta) \) is a centered Gaussian vector with covariance

\[
\Sigma = \begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2} - \psi & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} - \psi & 0 \\
\frac{1}{2} & \frac{1}{2} - \psi & \psi & 0 \\
0 & 0 & 0 & \frac{1}{2} - \psi
\end{pmatrix}.
\]

(8.3.30)

with \( \psi = \psi(\eta) \), as defined in (8.3.25).

**Proof.** Each component of the vector \( H(n_j) \) is an element of the second Wiener chaos associated with \( A \) (see Section 8.1.4). As a consequence, according e.g. to [53, Theorem 6.2.3], in order to prove the desired result it is enough to establish the following relations (as \( n_j \to \infty \)): (a) the covariance matrix of \( H(n_j) \) converges to \( \Sigma \), and (b) for every \( k = 1, 2, 3, 4 \), \( H_k(n_j) \) converges in distribution to a one-dimensional centered Gaussian random variable. Point (a) follows by a direct computation based on formulae (8.3.26)–(8.3.27), as well as on the fact that the random variables in the set

\[
\{ |a_\lambda|^2 - 1 : \lambda \in \Lambda(n_j), \lambda_2 \geq 0 \}
\]

are centered, independent, identically distributed and with unit variance. To prove Point (b), write \( \Lambda_{n_j}^+ := \{ \lambda \in \Lambda(n_j), \lambda_2 \geq 0 \} \) and observe that, for every \( k \) and every \( n_j \), the random variable \( H_k(n_j) \) has the form

\[
H_k(n_j) = \sum_{\lambda \in \Lambda_{n_j}^+} c_k(n_j, \lambda) \times (|a_\lambda|^2 - 1)
\]

where \( \{c_k(n_j, \lambda)\} \) is a collection of positive deterministic coefficients such that

\[
\max_{\lambda \in \Lambda_{n_j}^+} c_k(n_j, \lambda) \to 0,
\]
as \( n_j \to \infty \). An application of the Lindeberg criterion, e.g. in the quantitative form stated in [53, Proposition 11.1.3], consequently yields that \( H_k(n_j) \) converges in distribution to a Gaussian random variable, thus concluding the proof.

\( \square \)

**Remark 8.3.2.** The eigenvalues of the matrix \( \Sigma \) are given by: \( \{0, \frac{3}{2}, \frac{1}{2} - \psi, 2\psi - \frac{1}{2}\} \).

Following [34], we will abundantly use the structure of the *length-4 correlation set of frequencies*:

\[
S_n(4) := \{(\lambda, \lambda', \lambda'', \lambda''') \in (\Lambda_n)^4 : \lambda + \cdots + \lambda''' = 0\}.
\]

It is easily seen that an element of \( S_n(4) \) necessarily verifies one of the following properties (A)–(C):

- **(A)**: \[
\begin{align*}
\lambda &= -\lambda' \\
\lambda'' &= -\lambda'''.
\end{align*}
\]
- **(B)**: \[
\begin{align*}
\lambda &= -\lambda'' \\
\lambda' &= -\lambda'''.
\end{align*}
\]
- **(C)**: \[
\begin{align*}
\lambda &= -\lambda''' \\
\lambda' &= -\lambda''.
\end{align*}
\]

We also have the following identities between sets:

\[
\{(\lambda, \lambda', \lambda'', \lambda''') : (A) \text{ and } (B) \text{ are verified}\} = \{(\lambda, \lambda', \lambda'', \lambda''') : \lambda = -\lambda' = -\lambda'' = \lambda'''\},
\]

\[
\{(\lambda, \lambda', \lambda'', \lambda''') : (A) \text{ and } (C) \text{ are verified}\} = \{(\lambda, \lambda', \lambda'', \lambda''') : \lambda = -\lambda' = \lambda'' = -\lambda'''\},
\]

\[
\{(\lambda, \lambda', \lambda'', \lambda''') : (B) \text{ and } (C) \text{ are verified}\} = \{(\lambda, \lambda', \lambda'', \lambda''') : \lambda = \lambda' = -\lambda'' = -\lambda'''\},
\]

whereas, for \( n \neq 0 \), there is no element of \( S_n(4) \) simultaneously verifying (A), (B) and (C). In view of these remarks, we can apply the inclusion-exclusion principle to deduce that, for \( n \neq 0 \),

\[
|S_n(4)| = 3\mathcal{N}_n(\mathcal{N}_n - 1).
\]

In the next subsections, we will establish a non-central limit theorem for the sequence defined in (8.3.24).
8.3.2 Non-central convergence of the fourth chaotic projection: statement

One of the main achievements of the present chapter is the following statement.

**Proposition 8.3.3.** Let \( \{n_j\} \subset S \) be such that \( N_{n_j} \to \infty \) and \( \hat{\mu}_{n_j} \to \eta \in [-1, 1] \); set
\[
v(n_j) := \sqrt{\frac{4\pi n_j}{512}} \frac{1}{N_{n_j}}, \quad j \geq 1. \tag{8.3.31}
\]
Then, as \( n_j \to \infty \),
\[
Q(n_j) := \frac{\text{proj}(\mathcal{L}_{n_j} | C_4)}{v(n_j)} \overset{d}{\to} 1 + Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2, \tag{8.3.32}
\]
where the four-dimensional vector \( Z^\top = Z^\top(\eta) = (Z_1, Z_2, Z_3, Z_4) \) is defined in \( (8.3.29) \).
Moreover, one has that
\[
\text{Var}(1 + Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2) = 1 + \eta^2. \tag{8.3.33}
\]

Since the multidimensional CLT stated in \( (8.3.29) \) implies that
\[
(H_1(n_j)^2, H_2(n_j)^2, H_3(n_j)^2, H_4(n_j)^2) \overset{d}{\to} (Z_1^2, Z_2^2, Z_3^2, Z_4^2),
\]
in order to prove Proposition 8.3.3 it is sufficient to show that
\[
Q(n_j) = H_1(n_j)^2 - 2H_2(n_j) - 2H_3(n_j)^2 - 4H_4(n_j)^2 + R(n_j), \tag{8.3.34}
\]
where, as \( n_j \to \infty \), the sequence of random variables \( \{R_{n_j}\} \) converges in probability to some numerical constant \( \alpha \in \mathbb{R} \). To see this, observe that, in view of \( (8.1.5) \) and of the orthogonal chaotic decomposition \( (8.1.14) \), one has that \( \{Q(n_j)\} \) is a centered sequence of random variables such that \( \sup_j \mathbb{E}[Q(n_j)^2] < \infty \). By uniform integrability, this fact implies that, as \( n_j \to \infty \),
\[
\mathbb{E}[Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2] + \alpha = \lim_{n_j \to \infty} \mathbb{E}[Q(n_j)] = \lim_{n_j \to \infty} 0 = 0,
\]
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and therefore, since $E[Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2] = 1 - 2\psi - 2\psi - 4(2^{-1} - \psi) = -1$, one has necessarily that $\alpha = 1$. (Note that our results yield indeed a rather explicit representation of the term $R(n_j)$, so that the fact that $\alpha = 1$ can alternatively be verified by a careful bookkeeping of the constants appearing in the computations to follow).

Our proof of (8.3.34) is based on a number of technical results that are gathered together in the next subsection. These results will be combined with the following representation of $\text{proj}(\mathcal{L}_{n_j} | C_4)$, that is a direct consequence of (8.2.22) in the case $q = 2$:

$$\begin{align*}
\text{proj}(\mathcal{L}_{n_j} | C_4) &= \sqrt{4\pi^2} \sqrt{\frac{n}{2}} \left( \frac{\alpha_0\beta_4}{4!} \int_T H_4(T_n(\theta)) \, d\theta \right) \\
&\quad + \frac{\alpha_0\beta_0}{4!} \int_T H_4(\partial_2 \tilde{T}_n(\theta)) \, d\theta + \frac{\alpha_4\beta_0}{4!} \int_T H_4(\partial_1 \tilde{T}_n(\theta)) \, d\theta + \\
&\quad + \frac{\alpha_0\beta_2}{2!2!} \int_T H_2(T_n(\theta)) H_2(\partial_2 \tilde{T}_n(\theta)) \, d\theta + \\
&\quad + \frac{\alpha_2\beta_2}{2!2!} \int_T H_2(T_n(\theta)) H_2(\partial_1 \tilde{T}_n(\theta)) \, d\theta + \\
&\quad + \frac{\alpha_2\beta_0}{2!2!} \int_T H_2(\partial_1 \tilde{T}_n(\theta)) H_2(\partial_2 \tilde{T}_n(\theta)) \, d\theta,
\end{align*}$$

(8.3.35)

where the coefficients $\alpha_\cdot$, $\beta_\cdot$ are defined according to equation (7.2.14) and equation (7.2.13), respectively.

### 8.3.3 Some ancillary lemmas

The next four lemmas provide some useful representations for the six summands appearing on the right-hand side of (8.3.35). In what follows, $n$ always indicates a positive integer different from zero and, moreover, in order to simplify the discussion we sometimes use the shorthand notation:

$$\sum_\lambda = \sum_{\lambda=(\lambda_1, \lambda_2) \in \Lambda_n}, \quad \sum_{\lambda, \lambda'} = \sum_{\lambda, \lambda' \in \Lambda_n} \quad \text{and} \quad \sum_{\lambda, \lambda_2 \geq 0} = \sum_{\lambda=(\lambda_1, \lambda_2) \geq 0 \in \Lambda_n}.$$
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in such a way that the exact value of the integer \( n \) will always be clear from the context. Also, the symbol \( \{ n_j \} \) will systematically indicate a subsequence of integers contained in \( S \) such that \( N_{n_j} \to \infty \) and \( \hat{\mu}_{n_j}(4) \to \eta \in [-1, 1] \), as \( n_j \to \infty \). As it is customary, we write ‘\( P \to \)’ to denote convergence in probability, and we use the symbol \( o_P(1) \) to indicate a sequence of random variables converging to zero in probability, as \( N_n \to \infty \).

Lemma 8.3.4. One has the following representation:

\[
\int_T H_4(T_n(\theta)) \, d\theta = \frac{6}{N_n} \left( \frac{1}{\sqrt{N_n/2}} \sum_{\lambda, \lambda' \geq 0} (|a_\lambda|^2 - 1) + o_P(1) \right)^2 - \frac{3}{N_n^2} \sum_\lambda |a_\lambda|^4.
\]

Also, as \( n_j \to \infty \),

\[
\frac{3}{N_{n_j}} \sum_\lambda |a_\lambda|^4 \xrightarrow{P} 6.
\]

Proof. Using the explicit expression \( H_4(x) = x^4 - 6x^2 + 3 \), we deduce that

\[
\int_T H_4(T_n(\theta)) \, d\theta = \int_T (T_n(\theta)^4 - 6T_n(\theta)^2 + 3) \, d\theta
\]

\[
= \frac{1}{N_n^2} \sum_{\lambda, \lambda', \lambda'' \in \Lambda_n} a_\lambda a_\lambda' a_\lambda'' \int_T \exp(2\pi i \langle \lambda - \lambda' + \lambda'' - \lambda''' \rangle, \theta) \, d\theta +
\]

\[
- 6 \cdot \frac{1}{N_n} \sum_{\lambda, \lambda' \in \Lambda_n} a_\lambda a_\lambda' \int_T \exp(2\pi i \langle \lambda - \lambda' \rangle, \theta) \, d\theta + 3
\]

\[
= \frac{1}{N_n^2} \sum_{\lambda, \lambda' + \lambda'' - \lambda''' = 0} a_\lambda a_\lambda' a_\lambda'' a_\lambda'' - 6 \cdot \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} |a_\lambda|^2 + 3,
\]

where the subscript \( \lambda - \lambda' + \lambda'' - \lambda''' = 0 \) indicates that \( (\lambda, -\lambda', \lambda'', -\lambda''') \in S_n(4) \). Owing to the structure of \( S_n(4) \) discussed above, the previous expression simplifies to

\[
3 \cdot \frac{1}{N_n^2} \left( \sum_{\lambda, \lambda' \in \Lambda_n} |a_\lambda|^2 |a_{\lambda'}|^2 - \sum_\lambda |a_\lambda|^4 \right) - 6 \cdot \frac{1}{N_n} \sum_{\lambda \in \Lambda_n} |a_\lambda|^2 + 3
\]

\[
= 3 \cdot \frac{1}{N_n} \left( \frac{1}{\sqrt{N_n}} \sum_{\lambda \in \Lambda_n} (|a_\lambda|^2 - 1) \right)^2 - 3 \cdot \frac{1}{N_n^2} \sum_{\lambda \in \Lambda_n} |a_\lambda|^4
\]

\[
= \frac{6}{N_n} \left( \frac{1}{\sqrt{N_n/2}} \sum_{\lambda, \lambda' \geq 0} (|a_\lambda|^2 - 1) + o_P(1) \right)^2 - \frac{3}{N_n^2} \sum_\lambda |a_\lambda|^4,
\]

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where $o_P(1) = -(2N_{nj})^{-1/2}( |a_{(n-1/2,0)}|^2 + |a_{(-n-1/2,0)}|^2 - 2)$, thus immediately yielding the first part of the statement (after developing the square). The second part of the statement follows from a standard application of the law of large numbers to the sum,

$$
\frac{3}{N_{nj}} \sum_{\lambda} |a_{\lambda}|^4 = \frac{3}{N_{nj}^{1/2}} \sum_{\lambda, \lambda_2 \geq 0} |a_{\lambda}|^4 + o_P(1),
$$

as well as from the observation that the set $\{|a_{\lambda}|^4 : \lambda \in \Lambda_{nj}, \lambda_2 \geq 0\}$ is composed of i.i.d. random variables such that $\mathbb{E}|a_{\lambda}|^4 = 2$.

\[\square\]

**Lemma 8.3.5.** For $\ell = 1, 2$,

$$
\int_{\mathbb{T}} H_4(\partial \tilde{T}_n(\theta)) d\theta = \frac{24}{N_n} \left[ \frac{1}{\sqrt{N_n^{1/2}}} \sum_{\lambda, \lambda_2 \geq 0} \left( \frac{\lambda_2^2}{n} \left( |a_{\lambda}|^2 - 1 \right) \right) + o_P(1) \right]^2 +
$$

$$
- \left( \frac{2}{n} \right)^2 \frac{3}{N_n^2} \sum_{\lambda} \lambda_1^4 |a_{\lambda}|^4.
$$

Moreover, as $n_j \to \infty$,

$$
\left( \frac{2}{n_j} \right)^2 \frac{3}{N_{nj}} \sum_{\lambda} \lambda_1^4 |a_{\lambda}|^4 \xrightarrow{p} 24 \psi(\eta).
$$
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\textbf{Proof.} The proof is similar to that of Lemma 8.3.4. We have that

\[
\int_\mathcal{T} H_4(\partial_\ell \widetilde{T}_n(\theta)) \, d\theta = \int_\mathcal{T} (\partial_\ell \widetilde{T}_n(\theta)^4 - 6 \partial_\ell \widetilde{T}_n(\theta)^2 + 3) \, d\theta
\]

\[
= \frac{1}{\mathcal{N}_n^2} \frac{2}{n^2} \sum_{\lambda, \lambda' \in \Lambda_n} \lambda T_{\ell} \lambda' T_{\ell} \bar{\lambda} \bar{\lambda}' a_{\lambda \lambda'} \int_\mathcal{T} \exp(2\pi i (\lambda - \lambda' + \lambda'' - \lambda''' \theta)) \, d\theta + \frac{6}{\mathcal{N}_n} \frac{1}{n} \sum_{\lambda \in \Lambda_n} \lambda^2 |a_\lambda|^2 + 3
\]

\[
= \frac{3}{\mathcal{N}_n^2} \frac{4}{n^2} \left( \sum_{\lambda, \lambda'} \lambda^2 |a_\lambda|^2 |a_{\lambda'}|^2 - \sum_{\lambda} \lambda^4 |a_\lambda|^4 \right) - \frac{6}{\mathcal{N}_n} \frac{3}{n^2} \sum_{\lambda \in \Lambda_n} \lambda^2 |a_\lambda|^2 + 3
\]

\[
= \frac{24}{\mathcal{N}_n} \left[ \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda, \lambda' \geq 0} \left( \frac{\lambda^2}{n} (|a_{\lambda'}|^2 - 1) \right) + o_F(1) \right]^2 - \left( \frac{2}{n} \right)^2 \frac{3}{\mathcal{N}_n^2} \sum_{\lambda} \lambda^4 |a_\lambda|^4.
\]

To conclude the proof, we observe that

\[
\left( \frac{2}{n_j} \right)^2 \frac{3}{\mathcal{N}_{n_j}} \sum_{\lambda} \lambda^4 |a_\lambda|^4 = o_F(1) + \left( \frac{2}{n_j} \right)^2 \frac{3}{\mathcal{N}_{n_j}/2} \sum_{\lambda, \lambda' \geq 0} \lambda^2 (|a_{\lambda'}|^4 - 2) + \frac{24}{n^2 \mathcal{N}_{n_j}} \sum_{\lambda, \lambda' \geq 0} \lambda^4 := K_1(n_j) + K_2(n_j),
\]

so that the conclusion follows from (8.3.26), as well as from the fact that, since the random variables \{|a_\lambda|^4 - 2 : \lambda \in \Lambda_{n_j}, \, \lambda_2 \geq 0\} are i.i.d., square-integrable and centered and \(\lambda^4/n^2 \leq 1\), \(\mathbb{E} K_1(n_j)^2 = O(\mathcal{N}_{n_j}^{-1}) \rightarrow 0\).

\[\square\]

\textbf{Lemma 8.3.6.} One has that

\[
\int_\mathcal{T} H_2(T_n(\theta)) \left( H_2(\partial_1 \widetilde{T}_n(\theta)) + H_2(\partial_2 \widetilde{T}_n(\theta)) \right) \, d\theta 
\]

\[
= \frac{4}{\mathcal{N}_n} \left\{ \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda, \lambda' \geq 0} (|a_\lambda|^2 - 1) + o_F(1) \right\}^2 - \frac{2}{\mathcal{N}_n^2} \sum_{\lambda} |a_{\lambda'}|^4.
\]

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Proof. For $\ell = 1, 2,$
\[
\int_{\mathcal{T}} H_2(T_n(\theta)) H_2(\partial_{\ell} \overline{T}_n(\theta)) \, d\theta = \int_{\mathcal{T}} (T_n(\theta)^2 - 1)(\partial_{\ell} \overline{T}_n(\theta)^2 - 1) \, d\theta
\]
\[
= \int_{\mathcal{T}} \left( \frac{1}{N_n} \sum_{\lambda, \lambda'} a_{\lambda} \overline{a}_{\lambda'} e_{\lambda}(\theta) e_{-\lambda'}(\theta) - 1 \right) \left( \frac{2}{n N_n} \sum_{\lambda, \lambda''} \lambda_{\ell}'' \lambda_{\ell}' a_{\lambda} \overline{a}_{\lambda'} e_{\lambda''}(\theta) e_{-\lambda''}(\theta) - 1 \right) \, d\theta
\]
\[
= \frac{2}{n N_n^2} \sum_{\lambda, \lambda''} \lambda_{\ell}'' \lambda_{\ell}' a_{\lambda} \overline{a}_{\lambda'} e_{\lambda''}(\theta) e_{-\lambda''}(\theta) - \frac{1}{N_n} \sum_{\lambda} |a_{\lambda}|^2 - \frac{2}{n N_n} \sum_{\lambda} \lambda_{\ell}^2 |a_{\lambda}|^2 + 1.
\]
An application of the inclusion-exclusion principle yields that the first summand in the previous expression equals
\[
\frac{2}{n N_n^2} \left( \sum_{\lambda, \lambda'} \lambda_{\ell}^j |a_{\lambda}|^2 |a_{\lambda'}|^2 + 2 \sum_{\lambda, \lambda'} \lambda_{\ell} j |a_{\lambda}|^2 |a_{\lambda'}|^2 - 2 \sum_{\lambda} \lambda_{\ell}^j |a_{\lambda}|^4 + \sum_{\lambda} \lambda_{\ell}^2 |a_{\lambda}|^4 \right)
\]
Using the relation $a_{-\lambda} = \overline{a}_{\lambda},$ we also infer that
\[
\sum_{\lambda, \lambda'} \lambda_{\ell} j |a_{\lambda}|^2 |a_{\lambda'}|^2 = \left( \sum_{\lambda} \lambda_{\ell} |a_{\lambda}|^2 \right)^2 = 0.
\]
Summing the terms corresponding to $\partial_1$ and $\partial_2$ we deduce that the left-hand side of (8.3.36) equals obtain
\[
= \frac{2}{n N_n^2} \left\{ \sum_{\lambda, \lambda'} (\lambda_{\ell}^2 + \lambda_{\ell}'^2) |a_{\lambda}|^2 |a_{\lambda'}|^2 - \sum_{\lambda} (\lambda_{\ell}^2 + \lambda_{\ell}'^2) |a_{\lambda'}|^4 \right\}
\]
\[
- \frac{2}{N_n} \sum_{\lambda} |a_{\lambda}|^2 - \frac{2}{n N_n} \sum_{\lambda} (\lambda_{\ell}^2 + \lambda_{\ell}'^2) |a_{\lambda}|^2 + 2 =
\]
\[
= \frac{2}{n N_n^2} \left\{ \sum_{\lambda, \lambda'} n |a_{\lambda}|^2 |a_{\lambda'}|^2 - \sum_{\lambda} n |a_{\lambda'}|^4 \right\} - \frac{2}{N_n} \sum_{\lambda} |a_{\lambda}|^2 - \frac{2}{n N_n} \sum_{\lambda} n |a_{\lambda}|^2 + 2
\]
\[
= \frac{2}{N_n^2} \left\{ \sum_{\lambda, \lambda'} |a_{\lambda}|^2 |a_{\lambda'}|^2 - \sum_{\lambda} |a_{\lambda'}|^4 \right\} - \frac{2}{N_n} \sum_{\lambda} |a_{\lambda}|^2 - \frac{2}{N_n} \sum_{\lambda} |a_{\lambda}|^2 + 2
\]
\[
= \frac{2}{N_n} \left\{ \sqrt{\frac{2}{N_n}} \sum_{\lambda, \lambda' \geq 0} \left( |a_{\lambda}|^2 - 1 \right) + o_p(1) \right\}^2 - \frac{2}{N_n^2} \sum_{\lambda} |a_{\lambda'}|^4,
\]
which corresponds to the desired conclusion. \qed
Our last lemma allows one to deal with the most challenging term appearing in formula (8.3.35).

**Lemma 8.3.7.** We have that

\[
\int H_2(\partial_1 \tilde{T}_n) H_2(\partial_2 \tilde{T}_n) \, d\theta
\]

\[
= -4 \left[ \frac{1}{\sqrt{N_n/2}} \frac{1}{n} \sum_{\lambda, \lambda_2 \geq 0} \lambda_2^2 (|a_\lambda|^2 - 1) \right]^2 - 4 \left[ \frac{1}{\sqrt{N_n/2}} \frac{1}{n} \sum_{\lambda, \lambda_2 \geq 0} \lambda_1^2 (|a_\lambda|^2 - 1) + o_\mathbb{P}(1) \right]^2 
\]

\[
+ 4 \left[ \frac{1}{\sqrt{N_n/2}} \sum_{\lambda, \lambda_2 \geq 0} (|a_\lambda|^2 - 1) + o_\mathbb{P}(1) \right]^2 
\]

\[
+ 16 \left[ \frac{1}{\sqrt{N_n/2}} \frac{1}{n} \sum_{\lambda, \lambda_2 \geq 0} \lambda_1 \lambda_2 (|a_\lambda|^2 - 1) + o_\mathbb{P}(1) \right]^2 - \frac{12}{n^2 N_n^2} \sum_{\lambda} \lambda_1^2 \lambda_2^2 |a_\lambda|^4.
\]

And the following convergence takes place as \( n_j \to \infty \):

\[
\frac{12}{n_j^2 N_n^2} \sum_{\lambda} \lambda_1^2 \lambda_2^2 |a_\lambda|^4 \xrightarrow{\mathbb{P}} 12 - 24 \psi(\eta).
\]

**Proof.** One has that

\[
\int H_2(\partial_1 \tilde{T}_n) H_2(\partial_2 \tilde{T}_n) \, d\theta
\]

\[
= \frac{4}{n^2 N_n^2} \sum_{\lambda - \lambda' + \lambda'' - \lambda''' = 0} \lambda_1 \lambda_1' \lambda_2 \lambda_2' a_\lambda a_{\lambda'} \overline{a}_\lambda a_{\lambda''} \overline{a}_{\lambda'''} \tag{8.3.37}
\]

\[
- \frac{2}{n N_n} \sum_{\lambda} \lambda_1^2 |a_\lambda|^2 - \frac{2}{n N_n} \sum_{\lambda} \lambda_2^2 |a_\lambda|^2 + 1. \tag{8.3.38}
\]

First of all, we note that

\[
\mathbb{E} \left[ \frac{2}{n N_n} \sum_{\lambda} (\lambda_1^2 + \lambda_2^2) |a_\lambda|^2 \right] = \mathbb{E} \left[ \frac{2}{N_n} \sum_{\lambda} |a_\lambda|^2 \right] = 2.
\]

Let us now focus on (8.3.37). Using the structure of \( S_4(n) \) recalled above, we obtain

\[
\frac{4}{n^2 N_n^2} \sum_{\lambda - \lambda' + \lambda'' - \lambda''' = 0} \lambda_1 \lambda_1' \lambda_2 \lambda_2' a_\lambda a_{\lambda'} \overline{a}_\lambda a_{\lambda''} \overline{a}_{\lambda'''}
\]

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We have that
\[ A = \frac{1}{n^2 N_n^2} \left\{ \sum_{\lambda, \lambda'} \lambda_1^2 (\lambda_2')^2 |a_\lambda|^2 |a_{\lambda'}|^2 + 2 \sum_{\lambda, \lambda'} \lambda_1 \lambda_2 \lambda_1' \lambda_2' |a_\lambda|^2 |a_{\lambda'}|^2 - 3 \sum_{\lambda} \lambda_1^2 \lambda_2^2 |a_\lambda|^4 \right\}. \]

Let us now write
\[ \frac{4}{n^2 N_n^2} \sum_{\lambda, \lambda'} \lambda_1^2 (\lambda_2')^2 |a_\lambda|^2 |a_{\lambda'}|^2 := A, \]
\[ \frac{4}{n^2 N_n^2} \sum_{\lambda, \lambda'} \lambda_1 \lambda_2 \lambda_1' \lambda_2' |a_\lambda|^2 |a_{\lambda'}|^2 := B, \]
\[ -3 \frac{4}{n^2 N_n^2} \sum_{\lambda} \lambda_1^2 \lambda_2^2 |a_\lambda|^4 := C, \]
\[ \frac{4}{n^2 N_n^2} \left\{ -N_n \frac{n}{2} \sum_{\lambda} |a_\lambda|^2 + \frac{N_n^2 n^2}{4} \right\} := D. \]

We have that \( A \) equals
\[ \frac{4}{n^2 N_n^2} \sum_{\lambda, \lambda'} \lambda_1^2 (\lambda_2')^2 |a_\lambda|^2 |a_{\lambda'}|^2 \]
\[ = \frac{4}{n^2 N_n^2} \sum_{\lambda, \lambda'} \lambda_1^2 (\lambda_2')^2 |a_\lambda|^2 |a_{\lambda'}|^2 + \sum_{\lambda, \lambda'} \lambda_1^2 (\lambda_2')^2 |a_\lambda|^2 |a_{\lambda'}|^2 \]
\[ = \frac{4}{n^2 N_n^2} \sum_{\lambda, \lambda'} \left( n - \frac{3}{2} \right) |a_\lambda|^2 |a_{\lambda'}|^2 + \sum_{\lambda, \lambda'} \lambda_1^2 (n - (\lambda_1')^2) |a_\lambda|^2 |a_{\lambda'}|^2 \]

an expression that can be rewritten as
\[ \frac{4}{n^2 N_n^2} \sum_{\lambda, \lambda'} \lambda_1^2 (\lambda_2')^2 |a_\lambda|^2 |a_{\lambda'}|^2 - \sum_{\lambda, \lambda'} \lambda_1^2 (\lambda_1')^2 |a_\lambda|^2 |a_{\lambda'}|^2 \]
\[ + \frac{4}{n^2 N_n^2} \sum_{\lambda, \lambda'} (\lambda_2')^2 |a_\lambda|^2 |a_{\lambda'}|^2 + n \sum_{\lambda, \lambda'} \lambda_1^2 |a_\lambda|^2 |a_{\lambda'}|^2 \]
\[ = \frac{4}{n^2 N_n^2} \sum_{\lambda, \lambda'} \lambda_1^2 (\lambda_2')^2 |a_\lambda|^2 |a_{\lambda'}|^2 - \sum_{\lambda, \lambda'} \lambda_1^2 (\lambda_1')^2 |a_\lambda|^2 |a_{\lambda'}|^2 \]
\[ + \frac{4}{n^2 N_n^2} \sum_{\lambda, \lambda'} (\lambda_2')^2 |a_\lambda|^2 |a_{\lambda'}|^2 + n \sum_{\lambda, \lambda'} \lambda_1^2 |a_\lambda|^2 |a_{\lambda'}|^2 \].
As a consequence,

\[
A + D = -4 \frac{1}{n^2 N_n^2} \frac{1}{2} \left[ \sum_\lambda \lambda_2^2 (|a_\lambda|^2 - 1) \right]^2
\]

\[
-4 \frac{1}{n^2 N_n^2} \frac{1}{2} \left[ \sum_\lambda \lambda_1^2 (|a_\lambda|^2 - 1) \right]^2
\]

\[
+4 \frac{1}{N_n^2} \frac{1}{2} \left[ \sum_\lambda (|a_\lambda|^2 - 1) \right]^2.
\]

On the other hand,

\[
B = 4 \frac{1}{n^2 N_n^2} \frac{1}{2} \sum_{\lambda, \lambda'} \lambda_1 \lambda_2 \lambda_1' \lambda_2' |a_\lambda|^2 |a_{\lambda'}|^2
\]

\[
= 4 \frac{1}{n^2 N_n^2} \frac{1}{2} \left[ \sum_\lambda \lambda_1 \lambda_2 (|a_\lambda|^2 - 1) \right]^2.
\]

The last assertion in the statement, which concerns the term \(C\) defined above, is a direct consequence of (8.3.27) and of an argument similar to the one that concluded the proof of Lemma 8.3.5.

\(\square\)

### 8.3.4 End of the proof of Proposition 8.3.3

Plugging the explicit expressions appearing in Lemma 8.3.4, Lemma 8.3.5, Lemma 8.3.6 and Lemma 8.3.7 into (8.3.35) (and exploiting the fact that \(p_2(1/4) = -1/8\)), one deduces after some standard simplification that representation (8.3.34) is indeed valid, so that the conclusion of Proposition 8.3.3 follows immediately. In order to prove relation (8.3.33), introduce the centered Gaussian vector \(\tilde{Z}^\top := (\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3, \tilde{Z}_4)\), with
covariance matrix given by

\[ \tilde{\Sigma} := \begin{pmatrix}
1 & \frac{1}{2\sqrt{\psi}} & \frac{1}{2\sqrt{\psi}} & 0 \\
\frac{1}{2\sqrt{\psi}} & 1 & \frac{1}{2\sqrt{\psi}} & -1 \\
\frac{1}{2\sqrt{\psi}} & \frac{1}{2\sqrt{\psi}} & -1 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}. \]

Then,

\[
\text{Var} \left[ Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2 \right] = \text{Var} \left[ \tilde{Z}_1^2 - 2\psi \tilde{Z}_2^2 - 2\psi \tilde{Z}_3^2 - 4\left(\frac{1}{2} - \psi\right) \tilde{Z}_4^2 \right] \\
= \text{Var} \left[ H_2(\tilde{Z}_1) - 2\psi H_2(\tilde{Z}_2) - 2\psi H_2(\tilde{Z}_3) - 4\left(\frac{1}{2} - \psi\right) H_2(\tilde{Z}_4) \right] \\
= 2 + 8\psi^2 + 8\psi^2 + 32\left(\frac{1}{2} - \psi\right)^2 - 4\psi \text{Cov} \left[ H_2(\tilde{Z}_1), H_2(\tilde{Z}_2) \right] \\
- 4\psi \text{Cov} \left[ H_2(\tilde{Z}_1), H_2(\tilde{Z}_3) \right] + 8\psi^2 \text{Cov} \left[ H_2(\tilde{Z}_2), H_2(\tilde{Z}_3) \right] \\
= 2 + 8\psi^2 + 8\psi^2 + 32\left(\frac{1}{2} - \psi\right)^2 \\
- 8\psi\left(\frac{1}{2\sqrt{\psi}}\right)^2 - 8\psi\left(\frac{1}{2\sqrt{\psi}}\right)^2 + 16\psi^2\left(\frac{1}{2\psi} - 1\right)^2 \\
= 2 + 8\psi^2 + 8\psi^2 + 32\left(\frac{1}{4} + \psi^2 - \psi\right) - 2 - 2 + 16\psi^2\left(\frac{1}{4\psi^2} + 1 - \frac{1}{\psi}\right) \\
= 64\psi^2 - 48\psi + 10,
\]

and the conclusion follows from (8.3.28).

### 8.4 End of the proof of Theorem 8.1.1

A direct computation (obtained e.g. by diagonalising the covariance matrix \( \Sigma \) appearing in (8.3.30)) reveals that, for every \( \eta \in [-1, 1] \), the random variable

\[
\frac{1}{\sqrt{1 + \eta^2}} \left( 1 + Z_1^2 - 2Z_2^2 - 2Z_3^2 - 4Z_4^2 \right)
\]
has the same law as $M_{|\eta|}$, as defined in (8.1.10). This implies, in particular, that such a random variable has unit variance, and has a distribution that does not depend on the sign of $\eta$. Now let the assumptions and notations of Theorem 8.1.1 prevail (in particular, $N_{n_j} \to \infty$). Since the sequence $\{ |\hat{\mu}_{n_j}(4) | : j \geq 1 \}$ is nonnegative and bounded by 1, there exists a subsequence $\{n'_j\}$ such that $|\hat{\mu}_{n'_j}(4)|$ converges to some $\eta \in [0,1]$. It follows that $\{n'_j\}$ necessarily contains a subsequence $\{n''_j\} \subset \{n'_j\}$ such that one of the following properties holds: either (i) $\hat{\mu}_{n''_j}(4)$ converges to $\eta$, or (ii) $\hat{\mu}_{n''_j}(4)$ converges to $-\eta$. Now, if $\{n''_j\}$ is of type (i), then our initial remarks together with (8.1.5) and (8.3.32) imply that

$$
\lim_{n''_j \to \infty} \frac{\text{Var}[\mathcal{L}_{n''_j}]}{\text{Var}[\text{proj}(\mathcal{L}_{n''_j} | C_4)]} = 1.
$$

In view of the chaotic decomposition (8.1.15), this result implies that, as $n''_j \to \infty$,

$$
\tilde{\mathcal{L}}_{n''_j} = \frac{\text{proj}(\mathcal{L}_{n''_j} | C_4)}{\sqrt{\text{Var}[\mathcal{L}_{n''_j}]}} + o_P(1),
$$

and consequently that $\tilde{\mathcal{L}}_{n''_j}$ converges in distribution to $\mathcal{M}_\eta$, by virtue of Proposition 8.3.3. An analogous argument shows that, if $\{n''_j\}$ is of type (ii), then necessarily $\tilde{\mathcal{L}}_{n''_j}$ converges in distribution to $\mathcal{M}_{|\eta|} = \mathcal{M}_\eta$. The results described above readily imply the following three facts: (a) if the subsequence $\{n'_j\} \subset \{n_j\}$ is such that $|\hat{\mu}_{n'_j}(4)| \to \eta \in [0,1]$, then $\tilde{\mathcal{L}}_{n'_j} \overset{d}{\to} \mathcal{M}_\eta$, (b) any subsequence $\{n'_j\} \subset \{n_j\}$ contains a further subsequence $\{n''_j\} \subset \{n'_j\}$ such that $|\hat{\mu}_{n''_j}(4)|$ converges to some $\eta \in [0,1]$, and therefore $\tilde{\mathcal{L}}_{n''_j} \overset{d}{\to} \mathcal{M}_\eta$, and (c) if the subsequence $\{n'_j\}$ is such that $|\hat{\mu}_{n'_j}(4)|$ is not converging, then $\tilde{\mathcal{L}}_{n'_j}$ is not converging in distribution, since in this case the set $\{D(\tilde{\mathcal{L}}_{n'_j})\}$ has necessarily two distinct adherent points (thanks to Point 4 in Proposition 8.1.3). The first part of the statement is therefore proved. To prove (8.1.12), use fact (b) above to deduce that, for every subsequence $\{n'_j\}$, there exists a further subsequence $\{n''_j\}$ such that $\tilde{\mathcal{L}}_{n''_j} \overset{d}{\to} \mathcal{M}_\eta$ and $\mathcal{M}^{n''_j} \overset{d}{\to} \mathcal{M}_\eta$ (where we have used the notation (8.1.11)), and consequently

$$
d(\tilde{\mathcal{L}}_{n''_j}, \mathcal{M}^{n''_j}) \leq d(\tilde{\mathcal{L}}_{n''_j}, \mathcal{M}_\eta) + d(\mathcal{M}_\eta, \mathcal{M}^{n''_j}) \longrightarrow 0, \quad n''_j \to \infty.
$$
The previous asymptotic relation is obvious whenever $d$ is a distance metrizing weak convergence on $\mathcal{P}$. To deal with the case where $d = d_K$ equals the Kolmogorov distance, one has to use the standard fact that, since $\mathcal{M}_\eta$ has an absolutely continuous distribution, then $X_n \xrightarrow{d} \mathcal{M}_\eta$ if and only if $d_K(X_n, \mathcal{M}_\eta) \to 0$. The proof of Theorem 8.1.1 is complete.
Part 3
Spin random fields
Chapter 9

Representation of Gaussian isotropic spin random fields

This chapter is based on the second part of [8]: we investigate spin random fields on the sphere, extending the representation formula for Gaussian isotropic random fields on homogeneous spaces of a compact group in Chapter 2 to the spin case. Moreover we introduce a powerful tool for studying spin random fields and more generally random sections of homogeneous vector bundles, that is, the “pullback” random field.

9.1 Random sections of vector bundles

We now investigate the case of Gaussian isotropic spin random fields on $S^2$, with the aim of extending the representation result of Theorem 2.2.3. As stated in the Introduction of this thesis, these models have received recently much attention (see [36], [38] or [40]), being motivated by the modeling of CMB data. Actually our point of view begins from [38].

We consider first the case of a general vector bundle. Let $\xi = (E, p, B)$ be a finite-dimensional complex vector bundle on the topological space $B$, which is called the base...
space. The surjective map
\[ p : E \longrightarrow B \]  
(9.1.1)
is the bundle projection, \( p^{-1}(x) \), \( x \in B \) is the fiber above \( x \). Let us denote \( \mathcal{B}(B) \) the Borel \( \sigma \)-field of \( B \). A section of \( \xi \) is a map \( u : B \rightarrow E \) such that \( p \circ u = id_B \). As \( E \) is itself a topological space, we can speak of continuous sections.

We suppose from now on that every fibre \( p^{-1}(x) \) carries an inner product and a measure \( \mu \) is given on the base space. Hence we can consider square integrable sections, as those such that
\[ \int_B \langle u(x), u(x) \rangle_{p^{-1}(x)} \, d\mu(x) < +\infty \]
and define the corresponding \( L^2 \) space accordingly.

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space.

**Definition 9.1.1.** A random section \( T \) of the vector bundle \( \xi \) is a collection of \( E \)-valued random variables \( (T_x)_{x \in B} \) indexed by the elements of the base space \( B \) such that the map \( \Omega \times B \ni (\omega, x) \mapsto T_x(\omega) \) is \( \mathcal{F} \otimes \mathcal{B}(B) \)-measurable and, for every \( \omega \), the path
\[ B \ni x \mapsto T_x(\omega) \in E \]
is a section of \( \xi \), i.e. \( p \circ T_\cdot(\omega) = id_B \).

Continuity of a random section \( T \) is easily defined by requiring that for every \( \omega \in \Omega \) the map \( x \mapsto T_x \) is a continuous section of \( \xi \). Similarly a.s. continuity is defined. A random section \( T \) of \( \xi \) is a.s. square integrable if the map \( x \mapsto \|T_x(\omega)\|_{p^{-1}(x)}^2 \) is a.s. integrable, it is second order if \( E[\|T_x\|_{p^{-1}(x)}^2] < +\infty \) for every \( x \in B \) and mean square integrable if
\[ E\left[ \int_B \|T_x\|_{p^{-1}(x)}^2 \, d\mu(x) \right] < +\infty . \]
As already remarked in [38], in defining the notion of mean square continuity for a random section, the naive approach
\[ \lim_{y \to x} E[\|T_x - T_y\|^2] = 0 \]
is not immediately meaningful as $T_x$ and $T_y$ belong to different (even if possibly isomorphic) spaces (i.e. the fibers). A similar difficulty arises for the definition of strict sense invariance w.r.t. the action of a topological group on the bundle. We shall investigate these points below.

A case of particular interest to us are the homogeneous (or twisted) vector bundles. Let $G$ be a compact group, $K$ a closed subgroup and $\mathcal{X} = G/K$. Given an irreducible unitary representation $\tau$ of $K$ on the complex (finite-dimensional) Hilbert space $H$, $K$ acts on the Cartesian product $G \times H$ by the action

$$ k(g, z) := (g k, \tau(k^{-1})z) . $$

Let $G \times_\tau H = \{\theta(g, z) : (g, z) \in G \times H\}$ denote the quotient space of the orbits $\theta(g, z) = \{(g k, \tau(k^{-1})z) : k \in K\}$ under the above action. $G$ acts on $G \times_\tau H$ by

$$ h \theta(g, z) := \theta(h g, z) . \quad (9.1.2) $$

The map $G \times H \to \mathcal{X} : (g, z) \to gK$ is constant on the orbits $\theta(g, z)$ and induces the projection

$$ G \times_\tau H \owns \theta(g, z) \xrightarrow{\pi^-_\tau} gK \in \mathcal{X} $$

which is a continuous $G$-equivariant map. $\xi_\tau = (G \times_\tau H, \pi^-_\tau, \mathcal{X})$ is a $G$-vector bundle: it is the homogeneous vector bundle associated to the representation $\tau$. The fiber $\pi^-_\tau^{-1}(x)$ is isomorphic to $H$ for every $x \in \mathcal{X}$ (see [66]). More precisely, for $x \in \mathcal{X}$ the fiber $\pi^-_\tau^{-1}(x)$ is the set of elements of the form $\theta(g, z)$ such that $gK = x$. We define the scalar product of two such elements as

$$ \langle \theta(g, z), \theta(g, w) \rangle_{\pi^-_\tau^{-1}(x)} = \langle z, w \rangle_H \quad (9.1.3) $$

for some fixed $g \in G$ such that $gK = x$, as it is immediate that this definition does not depend on the choice of such a $g$. Given a function $f : G \to H$ satisfying

$$ f(gk) = \tau(k^{-1})f(g) , \quad (9.1.4) $$

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then to it we can associate the section of $\xi$ 

$$u(x) = u(gK) = \theta(g, f(g))$$

(9.1.5)

as again this is a good definition, not depending of the choice of $g$ in the coset $gK$. The functions $f$ satisfying to (9.1.4) are called right $K$-covariant functions of type $\tau$ (functions of type $\tau$ from now on).

More interestingly, also the converse is true.

**Proposition 9.1.2.** Given a section $u$ of $\xi$, there exists a unique function $f$ of type $\tau$ on $G$ such that $u(x) = \theta(g, f(g))$ where $gK = x$. Moreover $u$ is continuous if and only if $f : G \to H$ is continuous.

**Proof.** Let $(g_x)_{x \in \mathcal{X}}$ be a measurable selection such that $g_xK = x$ for every $x \in \mathcal{X}$. If $u(x) = \theta(g_x, z)$, then define $f(g_x) := z$ and extend the definition to the elements of the coset $g_xK$ by $f(g_xk) := \tau(k^{-1})z$; it is easy to check that such a $f$ is of type $\tau$, satisfies (9.1.5) and is the unique function of type $\tau$ with this property.

Note that the continuity of $f$ is equivalent to the continuity of the map

$$F : g \in G \to (g, f(g)) \in G \times H.$$  (9.1.6)

Denote $pr_1 : G \to \mathcal{X}$ the canonical projection onto the quotient space $\mathcal{X}$ and $pr_2 : G \times H \to G \times_\tau H$ the canonical projection onto the quotient space $G \times_\tau H$. It is immediate that

$$pr_2 \circ F = u \circ pr_1.$$  

Therefore $F$ is continuous if and only if $u$ is continuous, the projections $pr_1$ and $pr_2$ being continuous open mappings.

We shall again call $f$ the *pullback* of $u$. Remark that, given two sections $u_1, u_2$ of $\xi$ and their respective pullbacks $f_1, f_2$, we have

$$\langle u_1, u_2 \rangle := \int_\mathcal{X} \langle u_1(x), u_2(x) \rangle_{\pi_\tau^{-1}(x)} dx = \int_G \langle f_1(g), f_2(g) \rangle_H dg$$  (9.1.7)

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so that \( u \leftrightarrow f \) is an isometry between the space \( L^2(\xi_\tau) \) of the square integrable sections of \( \xi_\tau \) and the space \( L^2(G) \) of the square integrable functions of type \( \tau \).

The left regular action of \( G \) on \( L^2(G) \) (also called the representation of \( G \) induced by \( \tau \))

\[
L_h f(g) := f(h^{-1}g)
\]

can be equivalently realized on \( L^2(\xi_\tau) \) by

\[
U_h u(x) = hu(h^{-1}x)
\]  

(9.1.8)

We have

\[
U_h u(gK) = hu(h^{-1}gK) = h\theta(h^{-1}g, f(h^{-1}g)) = \theta(g, f(h^{-1}g)) = \theta(g, L_h f(g))
\]

so that, thanks to the uniqueness of the pullback function:

**Proposition 9.1.3.** If \( f \) is the pullback function of the section \( u \) then \( L_h f \) is the pullback of the section \( U_h u \).

Let \( T = (T_x)_{x \in X} \) be a random section of the homogeneous vector bundle \( \xi_\tau \). As, for fixed \( \omega \), \( x \mapsto T_x(\omega) \) is a section of \( \xi_\tau \), by Proposition 9.1.2 there exists a unique function \( g \mapsto X_g(\omega) \) of type \( \tau \) such that \( T_{gK}(\omega) = \theta(g, X_g(\omega)) \). We refer to the random field \( X = (X_g)_{g \in G} \) as the pullback random field of \( T \). It is a random field on \( G \) of type \( \tau \), i.e. \( X_{gk}(\omega) = \tau(k^{-1})X_g(\omega) \) for each \( \omega \). Conversely every random field \( X \) on \( G \) of type \( \tau \) uniquely defines a random section of \( \xi_\tau \) whose pullback random field is \( X \). It is immediate that

**Proposition 9.1.4.** Let \( T \) be a random section of \( \xi_\tau \).

- a) \( T \) is a.s. square integrable if and only if its pullback random field \( X \) is a.s. square integrable.
- b) \( T \) is second order if and only if its pullback random field \( X \) is second order.
- c) \( T \) is a.s. continuous if and only if its pullback random field \( X \) is a.s. continuous.
Proposition 9.1.4 introduces the fact that many properties of random sections of the homogeneous bundle can be stated or investigated through corresponding properties of their pullbacks, which are just ordinary random fields to whom the results of previous sections can be applied. A first instance is the following definition.

**Definition 9.1.5.** The random section $T$ of the homogeneous vector bundle $\xi_{\tau}$ is said to be mean square continuous if its pullback random field $X$ is mean square continuous, i.e.,

$$\lim_{h \to g} \mathbb{E}[\|X_h - X_g\|_H^2] = 0 .$$

(9.1.9)

Recalling Definition 1.3.6, we state now the definition of strict-sense invariance. Let $T$ be an a.s. square integrable random section of $\xi_{\tau}$. For every $g \in G$, the “rotated” random section $T^g$ is defined as

$$T^g_x(\cdot) := g^{-1}T_{gx}(\cdot)$$

(9.1.10)

which is still an a.s. square integrable random section of $\xi_{\tau}$. For any square integrable section $u$ of $\xi_{\tau}$, let

$$T(u) := \int_X \langle T_x, u(x) \rangle_{\pi^{-1}(x)} \, dx .$$

(9.1.11)

**Definition 9.1.6.** Let $T$ be an a.s. square integrable random section of the homogeneous vector bundle $\xi_{\tau}$. It is said to be (strict-sense) $G$-invariant or isotropic if and only if for every choice of square integrable sections $u_1, u_2, \ldots, u_m$ of $\xi_{\tau}$, the random vectors

$$(T(u_1), \ldots, T(u_m)) \quad \text{and} \quad (T^g(u_1), \ldots, T^g(u_m)) = (T(U_g u_1), \ldots, T(U_g u_m))$$

(9.1.12)

have the same law for every $g \in G$.

**Proposition 9.1.7.** Let $T$ be an a.s. square integrable random section of $\xi_{\tau}$ and let $X$ be its pullback random field on $G$. Then $X$ is isotropic if and only if $T$ is an isotropic random section.
Proof. Let us denote $X(f) := \int_G \langle X_g, f(g) \rangle_H \, dg$. Thanks to Proposition 9.1.3 the equality in law (9.1.12) is equivalent to the requirement that for every choice of square integrable functions $f_1, f_2, \ldots, f_m$ of type $\tau$ (i.e. the pullbacks of corresponding sections of $\xi_\tau$), the random vectors

$$(X(f_1), \ldots, X(f_m)) \quad \text{and} \quad (X(L_g f_1), \ldots, X(L_g f_m)) \quad (9.1.13)$$

have the same law for every $g \in G$. As $L^2_\tau(G)$ is a closed subspace of $L^2(G)$ and is invariant under the left regular representation of $G$, every square integrable function $f : G \to H$ can be written as the sum $f^{(1)} + f^{(2)}$ with $f^{(1)} \in L^2_\tau(G)$, $f^{(2)} \in L^2(G) \perp$. As the paths of the random field $X$ are of type $\tau$ we have $X(f^{(2)}) = X(L_h f^{(2)}) = 0$ for every $h \in G$ so that

$$X(f) = X(f^{(1)}) \quad \text{and} \quad X(L_h f) = X(L_h f^{(1)}). \quad (9.1.14)$$

Therefore (9.1.13) implies that, for every choice $f_1, f_2, \ldots, f_m$ of square integrable $H$-valued functions on $G$, the random vectors

$$(X(L_g f_1), \ldots, X(L_g f_m)) \quad \text{and} \quad (X(f_1), \ldots, X(f_m)) \quad (9.1.15)$$

have the same law for every $g \in G$ so that the pullback random field $X$ is a strict-sense isotropic random field on $G$.

\[\square\]

As a consequence of Proposition 1.3.9 we have

**Corollary 9.1.8.** Every a.s. square integrable, second order and isotropic random section $T$ of the homogeneous vector bundle $\xi_\tau$ is mean square continuous.

In order to make a comparison with the pullback approach developed above, we briefly recall the approach to the theory of random fields in vector bundles introduced by Malyarenko in [38]. The main tool is the scalar random field associated to the random section $T$ of $\xi = (E, p, B)$. More precisely, it is the complex-valued random field $T^{sc}$ indexed by the elements $\eta \in E$ given by

$$T^{sc}_\eta := \langle \eta, T_b \rangle_{p^{-1}(b)}, \quad b \in B, \eta \in p^{-1}(b). \quad (9.1.16)$$
$T^{sc}$ is a scalar random field on $E$ and we can give the definition that $T$ is mean square continuous if and only if $T^{sc}$ is mean square continuous, i.e., if the map

$$E \ni \eta \mapsto T^{sc}_\eta \in L^2_C(\mathbb{P})$$

(9.1.17)
is continuous. Given a topological group $G$ acting with a continuous action $(g, x) \mapsto gx, g \in G$ on the base space $B$, an action of $G$ on $E$ is called associated if its restriction to any fiber $p^{-1}(x)$ is a linear isometry between $p^{-1}(x)$ and $p^{-1}(gx)$. In our case of interest, i.e. the homogeneous vector bundles $\xi_\tau = (G \times_\tau H, \pi_\tau, \mathcal{F})$, we can consider the action defined in (9.1.2) which is obviously associated. We can now define that $T$ is strict sense $G$-invariant w.r.t. the action of $G$ on $B$ if the finite-dimensional distributions of $T^{sc}$ are invariant under the associated action (9.1.2). In the next statement we prove the equivalence of the two approaches.

**Proposition 9.1.9.** The square integrable random section $T$ of the homogeneous bundle $\xi_\tau$ is mean square continuous (i.e. its pullback random field on $G$ is mean square continuous) if and only if the associated scalar random field $T^{sc}$ is mean square continuous. Moreover if $T$ is a.s. continuous then it is isotropic if and only if the associated scalar random field $T^{sc}$ is $G$-invariant.

**Proof.** Let $X$ be the pullback random field of $T$. Consider the scalar random field on $G \times H$ defined as $X^{sc}_{(g,z)} := \langle z, X_g \rangle_H$. Let us denote $pr$ the projection $G \times H \to G \times_\tau H$: keeping in mind (9.1.3) we have

$$T^{sc} \circ pr = X^{sc},$$

(9.1.18)
i.e.

$$T^{sc}_{\theta(g,z)}(\omega) = X^{sc}_{(g,z)}(\omega)$$

for every $(g, z) \in G \times H, \omega \in \Omega$. Therefore the map $G \times H \ni (g, z) \mapsto T^{sc}_{\theta(g,z)} \in L^2_C(\mathbb{P})$ is continuous if and only if the map $G \times H \ni (g, z) \mapsto X^{sc}_{(g,z)} \in L^2_C(\mathbb{P})$ is continuous, the projection $pr$ being open and continuous. Let us show that the continuity of the latter map is equivalent to the mean square continuity of the pullback random field $X$,
which will allow to conclude. The proof of this equivalence is inspired by the one of a similar statement in [38], §2.2.

Actually consider an orthonormal basis \( \{ v_1, v_2, \ldots, v_{\dim \tau} \} \) of \( H \), and denote \( X^i = \langle X, v_i \rangle \) the \( i \)-th component of \( X \) w.r.t. the above basis. Assume that the map \( G \times H \ni (g, z) \mapsto X^\text{sc}_{(g, z)} \in L^2_\mathbb{C}(P) \) is continuous, then the random field on \( g \mapsto X^\text{sc}_{(g, v_i)} = X^i_g \) is continuous for every \( i = 1, \ldots, \dim \tau \). As \( \mathbb{E}[|X^i_g - X^i_h|^2] = \mathbb{E}[|X^i_g - X^i_h|^2] \),

\[
\lim_{h \to g} \mathbb{E}[\|X_g - X_h\|^2_{H}] = \lim_{h \to g} \sum_{i=1}^{\dim \tau} \mathbb{E}[|X^i_g - X^i_h|^2] = 0 .
\]

Suppose that the pullback random field \( X \) is mean square continuous. Then for each \( i = 1, \ldots, \dim \tau \)

\[
0 \leq \limsup_{h \to g} \mathbb{E}[|X^i_g - X^i_h|^2] \leq \lim_{h \to g} \mathbb{E}[\|X_g - X_h\|^2_{H}] = 0
\]

so that the maps \( G \ni g \mapsto X^i_g \in L^2_\mathbb{C}(\mathbb{P}) \) are continuous. Therefore

\[
\lim_{(h, w) \to (g, z)} \mathbb{E}[|X^\text{sc}_{(h, w)} - X^\text{sc}_{(g, z)}|^2] \leq 2 \sum_{i=1}^{\dim \tau} \lim_{(h, w) \to (g, z)} \mathbb{E}[|w^i X^i_h - z^i X^i_g|^2] = 0 ,
\]

\( a_i \) denoting the \( i \)-th component of \( a \in H \).

Assume that \( T \) is a.s. continuous and let us show that it is isotropic if and only if the associated scalar random field \( T^\text{sc} \) is \( G \)-invariant. Note first that, by (9.1.18) and \( (T^\text{sc})^h = (X^\text{sc})^h \circ pr \) for any \( h \in G \), \( T^\text{sc} \) is \( G \)-invariant if and only if \( X^\text{sc} \) is \( G \)-invariant. Actually if the random fields \( X^\text{sc} \) and its rotated \( (X^\text{sc})^h \) have the same law, then \( T^\text{sc} = \text{law } X^\text{sc} \) and vice versa. Now recalling the definition of \( X^\text{sc} \), it is obvious that \( X^\text{sc} \) is \( G \)-invariant if and only if \( X \) is isotropic.

\( \square \)
9.2 Random sections of the homogeneous line bundles on $S^2$

We now concentrate on the case of the homogeneous line bundles on $X = S^2$ with $G = SO(3)$ and $K \cong SO(2)$. For every character $\chi_s$ of $K$, $s \in \mathbb{Z}$, let $\xi_s$ be the corresponding homogeneous vector bundle on $S^2$, as explained in the previous section. Given the action of $K$ on $SO(3) \times \mathbb{C}$: $k(g, z) = (gk, \chi_s(k^{-1})z)$, $k \in K$, let $E_s := SO(3) \times_s \mathbb{C}$ be the space of the orbits $E_s = \{\{\theta(g, z), (g, z) \in G \times \mathbb{C}\}$ where $\theta(g, z) = \{(gk, \chi_s(k^{-1})z); k \in K\}$. If $\pi_s : E_s \ni \theta(g, z) \rightarrow gK \in S^2$, $\xi_s = (E_s, \pi_s, S^2)$ is an homogeneous line bundle (each fiber $\pi_s^{-1}(x)$ is isomorphic to $\mathbb{C}$ as a vector space).

The space $L^2(\xi_s)$ of the square integrable sections of $\xi_s$ is therefore isomorphic to the space $L^2_s(SO(3))$ of the square integrable functions of type $s$, i.e. such that, for every $g \in G$ and $k \in K$,

$$f(gk) = \chi_s(k^{-1})f(g) = \overline{\chi_s(k)f(g)}.$$  \hspace{1cm} (9.2.19)

Let us investigate the Fourier expansion of a function of type $s$.

**Proposition 9.2.1.** Every function of type $s$ is an infinite linear combination of the functions appearing in the $(-s)$-columns of Wigner’s $D$ matrices $D^\ell$, $\ell \geq |s|$. In particular functions of type $s$ and type $s'$ are orthogonal if $s \neq s'$.

**Proof.** For every $\ell \geq |s|$, let $\hat{f}(\ell)$ be as in (1.2.16). We have, for every $k \in K$,

$$\hat{f}(\ell) = \sqrt{2\ell + 1} \int_{SO(3)} f(g)D^\ell(g^{-1}) \, dg = \sqrt{2\ell + 1} \chi_s(k) \int_{SO(3)} f(gk)D^\ell(g^{-1}) \, dg = \sqrt{2\ell + 1} \chi_s(k) \int_{SO(3)} f(g)D^\ell(k^{-1}g^{-1}) \, dg = \sqrt{2\ell + 1} \chi_s(k)D^\ell(k)f(\ell),$$  \hspace{1cm} (9.2.20)

i.e. the image of $\hat{f}(\ell)$ is contained in the subspace $H^{(-s)}_\ell \subset H_\ell$ of the vectors such that $D^\ell(k)v = \chi_{-s}(k)v$ for every $k \in K$. In particular $\hat{f}(\ell) \neq 0$ only if $\ell \geq |s|$, as for every $\ell$
the restriction to $K$ of the representation $D^\ell$ is unitarily equivalent to the direct sum of the representations $\chi_m$, $m = -\ell, \ldots, \ell$ as recalled at the end of §2.

Let $\ell \geq |s|$ and $v_{-\ell}, v_{-\ell+1}, \ldots, v_\ell$ be the orthonormal basis of $H_\ell$ as in (1.2.14), in other words $v_m$ spans $H^m_\ell$, the one-dimensional subspace of $H_\ell$ formed by the vectors that transform under the action of $K$ according to the representation $\chi_m$. It is immediate that

$$\hat{f}(\ell)_{i,j} = \langle \hat{f}(\ell)v_j, v_i \rangle = 0 , \quad (9.2.21)$$

unless $i = -s$. Thus the Fourier coefficients of $f$ vanish but those corresponding to the column $(-s)$ of the matrix representation $D^\ell$ and the Peter-Weyl expansion (1.2.17) of $f$ becomes, in $L^2(SO(3))$,

$$f = \sum_{\ell \geq |s|} \sqrt{2\ell + 1} \sum_{m = -\ell}^\ell \hat{f}(\ell)_{-s,m} D^\ell_{m,-s} , \quad (9.2.22)$$

We introduced the spherical harmonics in (1.2.18) from the entries $D^\ell_{m,0}$ of the central columns of Wigner’s $D$ matrices. Analogously to the case of $s = 0$, for any $s \in \mathbb{Z}$ we define for $\ell \geq |s|, m = -\ell, \ldots, \ell$

$$-s Y_{\ell,m}(x) := \theta(g, \sqrt{\frac{2\ell + 1}{4\pi}}D_{m,s}(g)) , \quad x = gK \in \mathbb{S}^2 . \quad (9.2.23)$$

$-s Y_{\ell,m}$ is a section of $\xi_s$ whose pullback function (up to a factor) is $g \mapsto D^\ell_{m,-s}(g)$ (recall the relation $D_{m,s}(g) = (-1)^{m-s}D^\ell_{m,-s}(g)$, see [40] p. 55 e.g.). Therefore thanks to Proposition 9.2.1 the sections $-s Y_{\ell,m}, \ell \geq |s|, m = -\ell, \ldots, \ell$, form an orthonormal basis of $L^2(\xi_s)$. Actually recalling (9.1.3) and (considering the total mass equal to $4\pi$ on the sphere and to 1 on $SO(3)$)

$$\int_{\mathbb{S}^2} -s Y_{\ell,m} -s Y_{\ell',m'} \, dx = 4\pi \int_{SO(3)} \frac{2\ell + 1}{4\pi} D^\ell_{m,s}(g) \frac{2\ell' + 1}{4\pi} D^\ell_{m',s}(g) \, dg = \delta_\ell \delta_{m} \delta_{\ell'} \delta_{m'} .$$

The sections $-s Y_{\ell,m}, \ell \geq |s|, m = -\ell, \ldots, \ell$ in (9.2.23) are called spin $-s$ spherical harmonics. Recall that the spaces $L^2_s(SO(3))$ and $L^2(\xi_s)$ are isometric through the
identification \( u \longleftrightarrow f \) between a section \( u \) and its pullback \( f \) and the definition of the scalar product on \( L^2(\xi_s) \) in (9.1.7). Proposition (9.2.1) can be otherwise stated as

Every square integrable section \( u \) of the homogeneous line bundle \( \xi_s = (\mathcal{E}_s, \pi_s, \mathbb{S}^2) \) admits a Fourier expansion in terms of spin \(-s\) spherical harmonics of the form

\[
    u(x) = \sum_{\ell \geq |s|} \sum_{m=-\ell}^{\ell} u_{\ell,m} Y_{\ell,m}(x) ,
\]

where \( u_{\ell,m} := \langle u, -s Y_{\ell,m} \rangle \), the above series converging in \( L^2(\xi_s) \).

In particular we have the relation

\[
    u_{\ell,m} = \int_{\mathbb{S}^2} u(x) -s Y_{\ell,m}(x) \, dx = 4\pi \int_{SO(3)} f(g) \sqrt{\frac{2\ell + 1}{4\pi} D^\ell_{m,s}(g)} \, dg = (-1)^{s-m} \sqrt{4\pi(2\ell + 1)} \int_{SO(3)} f(g) \frac{D^\ell_{-m,-s}(g)}{\sqrt{4\pi}} \, dg = (-1)^{s-m} \sqrt{4\pi} \hat{f}(\ell)_{-s,-m} .
\]

**Definition 9.2.2.** Let \( s \in \mathbb{Z} \). A square integrable function \( f \) on \( SO(3) \) is said to be \( bi-s \)-associated if for every \( g \in SO(3), k_1, k_2 \in K \),

\[
    f(k_1 g k_2) = \chi_s(k_1) f(g) \chi_s(k_2) .
\]

Of course for \( s = 0 \) \( bi-0 \)-associated is equivalent to \( bi-K \)-invariant. We are particularly interested in \( bi-s \)-associated functions as explained in the remark below.

**Remark 9.2.3.** Let \( X \) be an isotropic random field of type \( s \) on \( SO(3) \). Then its associate positive definite function \( \phi \) is \( bi-(s) \)-associated. Actually, assuming for simplicity that \( X \) is centered, as \( \phi(g) = \mathbb{E}[X_g X_\varepsilon] \), we have, using invariance on \( k_1 \) and type \( s \) property on \( k_2 \),

\[
    \phi(k_1 g k_2) = \mathbb{E}[X_{k_1 g_k_2} X_\varepsilon] = \mathbb{E}[X_{g_k_2} X_{k_1^{-1}}] = \chi_s(k_1^{-1}) \mathbb{E}[X_g X_\varepsilon] \chi_s(k_2^{-1}) = \chi_{-s}(k_1) \phi(g) \chi_{-s}(k_2) .
\]
Let us investigate the Fourier expansion of a bi-s-associated function $f$: note first that a bi-s-associated function is also a function of type $(-s)$, so that $\hat{f}(\ell) \neq 0$ only if $\ell \geq |s|$ as above and all its rows vanish but for the $s$-th. A repetition of the computation leading to (9.2.20) gives easily that

$$\hat{f}(\ell) = \chi_{-s}(k) \hat{f}(\ell) D^\ell(k)$$

so that the only non-vanishing entry of the matrix $\hat{f}(\ell)$ is the $(s,s)$-th.

Therefore the Fourier expansion of a bi-s-associated function $\phi$ is

$$f = \sum_{\ell \geq |s|} \sqrt{2\ell + 1} \alpha_\ell D^\ell_{s,s}, \quad (9.2.26)$$

where we have set $\alpha_\ell = \hat{f}(\ell)_{s,s}$.

Now let $T$ be an a.s. square integrable random section of the line bundle $\xi_s$ and $X$ its pullback random field. Recalling that $X$ is a random function of type $s$ and its sample paths are a.s. square integrable, we easily obtain the stochastic Fourier expansion of $X$ applying (9.2.22) to the functions $g \mapsto X_g(\omega)$. Actually define, for every $\ell \geq |s|$, the random operator

$$\tilde{X}(\ell) = \sqrt{2\ell + 1} \int_{SO(3)} X_g D^\ell(g^{-1}) \, dg. \quad (9.2.27)$$

The basis of $H_\ell$ being fixed as above and recalling (9.2.22), we obtain, a.s. in $L^2(SO(3))$,

$$X_g = \sum_{\ell \geq |s|} \sqrt{2\ell + 1} \sum_{m=-\ell}^\ell \tilde{X}(\ell)_{-s,m} D^\ell_{m,s}(g). \quad (9.2.28)$$

If $T$ is isotropic, then by Definition 9.1.6 its pullback random field $X$ is also isotropic in the sense of Definition 1.3.6. The following is a consequence of well known general properties of the random coefficients of invariant random fields (see [9] Theorem 3.2 or [38] Theorem 2).

**Proposition 9.2.4.** Let $s \in \mathbb{Z}$ and $\xi_s = (\mathcal{E}_s, \pi_s, S^2)$ be the homogeneous line bundle on $S^2$ induced by the $s$-th linear character $\chi_s$ of $SO(2)$. Let $T$ be a random section
of \( \xi_s \) and \( X \) its pullback random field. If \( T \) is second order and strict-sense isotropic, then the Fourier coefficients \( X(\ell)_{-s,m} \) of \( X \) in its stochastic expansion (9.2.28) are pairwise orthogonal and the variance, \( c_\ell \), of \( \hat{X}(\ell)_{-s,m} \) does not depend on \( m \). Moreover \( \mathbb{E}[\hat{X}(\ell)_{-s,m}] = 0 \) unless \( \ell = 0, s = 0 \).

For the random field \( X \) of Proposition 9.2.4 we have immediately

\[
\mathbb{E}[|X_g|^2] = \sum_{\ell \geq |s|} (2\ell + 1) c_\ell < +\infty . \tag{9.2.29}
\]

The convergence of the series above is also a consequence of Theorem 1.4.5, as the positive definite function \( \phi \) associated to \( X \) is given by

\[
\phi(g) = \mathbb{E}[X_g X_e] = \sum_{\ell \geq |s|} (2\ell + 1) c_\ell \sum_{m=-\ell}^\ell D_{m,-s}(g) D_{m,-s}(e) = \sum_{\ell \geq |s|} (2\ell + 1) c_\ell \sum_{m=-\ell}^\ell D_{m,-s}(g) D_{m,-s}(e) = \sum_{\ell \geq |s|} (2\ell + 1) c_\ell D_{\ell,-s,-s}(g) .
\]

**Remark 9.2.5.** Let \( X \) be a type \( s \) random field on \( SO(3) \) with \( s \neq 0 \). Then the relation \( X_{gk} = \chi_s(k^{-1})X_g \) implies that \( X \) cannot be real (unless it is vanishing). If in addition it was Gaussian, then, the identity in law between \( X_g \) and \( X_{gk} = \chi_s(k^{-1})X_g \) would imply that, for every \( g \in G \), \( X_g \) is a complex Gaussian r.v.

### 9.3 Construction of Gaussian isotropic spin random fields

We now give an extension of the construction of §2.1 and prove that every complex Gaussian random section of a homogeneous line bundle on \( S^2 \) can be obtained in this way, a result much similar to Theorem 2.2.3. Let \( s \in \mathbb{Z} \) and let \( \xi_s \) be the homogeneous line bundle associated to the representation \( \chi_s \).

Let \( (X_n)_n \) be a sequence of i.i.d. standard Gaussian r.v.’s on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\), and \( \mathcal{H} \subset L^2(\Omega, \mathcal{F}, \mathbb{P}) \) the complex Hilbert space generated by \( (X_n)_n \).
Let \((e_n)_n\) be an orthonormal basis of \(L^2(SO(3))\) and define an isometry \(S\) between \(L^2(SO(3))\) and \(H\) by

\[
L^2(SO(3)) \ni \sum_k \alpha_k e_k \mapsto \sum_k \alpha_k X_k \in H.
\]

Let \(f \in L^2(SO(3))\), we define a random field \(X^f\) on \(SO(3)\) by

\[
X^f_g = S(L_g f),
\] (9.3.30)

\(L\) denoting as usual the left regular representation.

**Proposition 9.3.1.** If \(f\) is a square integrable bi-\(s\)-associated function on \(SO(3)\), then \(X^f\) defined in (9.3.30) is a second order, square integrable Gaussian isotropic random field of type \(s\). Moreover it is complex Gaussian.

**Proof.** It is immediate that \(X^f\) is second order as

\[
E[|X^f_g|^2] = \|L_g f\|^2_2 = \|f\|^2_2.
\]

It is of type \(s\) as for every \(g, k \in K\),

\[
X^f_{gk} = S(L_{gk} f) = \chi_{s}(k^{-1}) S(L_g f) = \chi_{s}(k^{-1}) X^f_g.
\]

Let us prove strict-sense invariance. Actually, \(S\) being an isometry, for every \(h \in SO(3)\)

\[
E[X^f_{hg} \overline{X^f_{hg}'}] = E[S(L_{hg} f) \overline{S(L_{hg}' f)}] = \langle L_{hg} f, L_{hg}' f \rangle_2 = \langle L_g f, L_{g'} f \rangle_2 = E[X^f_{g} \overline{X^f_{g'}}].
\]

Therefore the random fields \(X^f\) and its rotated \((X^f)^h\) have the same covariance kernel. Let us prove that they also have the same relation function. Actually we have, for every \(g, g' \in SO(3)\),

\[
E[X^f_g X^f_{g'}] = E[S(L_{hg} f) S(L_{hg'} f)] = \langle L_{hg} f, L_{hg'} f \rangle_2 = \langle L_g f, L_{g'} f \rangle_2 = 0 \quad (9.3.31)
\]

as the function \(L_{g'} f\) is bi-(\(-s\))-associated and therefore of type \(s\) and orthogonal to \(L_g f\) which is of type \(-s\) (orthogonality of functions of type \(s\) and \(-s\) is a consequence of Proposition 9.2.1).

In order to prove that \(X^f\) is complex Gaussian we must show that for every \(\psi \in L^2(SO(3))\), the r.v.

\[
Z = \int_{SO(3)} X^f_g \psi(g) \, dg
\]

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is complex Gaussian. As $Z$ is Gaussian by construction we must just prove that $\mathbb{E}[Z^2] = 0$. But as, thanks to (9.3.31), $\mathbb{E}[X^f_g X^f_h] = 0$

\[
\mathbb{E}[Z^2] = \mathbb{E} \left[ \int_{SO(3)} \int_{SO(3)} X^f_g X^f_h \psi(g) \psi(h) \, dg \, dh \right] = \\
= \int_{SO(3)} \int_{SO(3)} \mathbb{E}[X^f_g X^f_h] \psi(g) \psi(h) \, dg \, dh = 0.
\]

\[\square\]

Let us investigate the stochastic Fourier expansion of $X^f$. Let us consider first the random field $X^\ell$ associated to $f = D^\ell_{s,s}$. Recall first that the r.v. $Z = S(D^\ell_{s,s})$ has variance $\mathbb{E}[|Z|^2] = \|D^\ell_{s,s}\|^2 = (2\ell + 1)^{-1}$ and that $D^\ell_{m,s} = (-1)^{m-s} D^\ell_{-m,-s}$. Therefore

\[
X^\ell_g = S(L_g D^\ell_{s,s}) = \sum_{m=-\ell}^\ell S(D^\ell_{m,s}) D^\ell_{s,m}(g^{-1}) = \\
= \sum_{m=-\ell}^\ell S(D^\ell_{m,s}) D^\ell_{m,s}(g) = \sum_{m=-\ell}^\ell S(D^\ell_{m,s})(-1)^{m-s} D^\ell_{m,-s}(g).
\]

Therefore the r.v.’s

\[
a_{\ell,m} = \sqrt{2\ell + 1} S(D^\ell_{m,s})(-1)^{m-s}
\]

are complex Gaussian, independent and with variance $\mathbb{E}[|a_{\ell,m}|^2] = 1$ and we have the expansion

\[
X^\ell_g = \frac{1}{\sqrt{2\ell + 1}} \sum_{m=-\ell}^\ell a_{\ell,m} D^\ell_{m,-s}(g).
\]

(9.3.32)

Note that the coefficients $a_{\ell,m}$ are independent complex Gaussian r.v.’s. This is a difference with respect to the case $s = 0$, where in the case of a real random field, the coefficients $a_{\ell,m}$ and $a_{\ell,-m}$ were not independent. Recall that random fields of type $s \neq 0$ on $SO(3)$ cannot be real.

In general, for a square integrable bi-$s$-associated function $f$

\[
f = \sum_{\ell \geq |s|} \sqrt{2\ell + 1} a_{\ell} D^\ell_{s,s}
\]

(9.3.33)
with
\[ \|f\|_2^2 = \sum_{\ell \geq |s|} |\alpha_{\ell}|^2 < +\infty , \]
the Gaussian random field \( X_f \) has the expansion
\[ X_f^g = \sum_{\ell \geq |s|} \alpha_{\ell} \sum_{m=\ell}^\ell \alpha_{\ell,m} D_{m,-s}^\ell(g) , \tag{9.3.34} \]
where \((a_{\ell,m})_{\ell,m}\) are independent complex Gaussian r.v.’s with zero mean and unit variance.

The associated positive definite function of \( X_f \), \( \phi_f(g) := \mathbb{E}[X_f^g X_f^\ell] \) is bi-\((-s)-\)associated (Remark 9.2.3) and continuous (Theorem 1.4.5) and, by (2.1.2), is related to \( f \) by
\[ \phi_f = f * ̂f(g^{-1}) . \]

This allows to derive its Fourier expansion:
\[ \phi_f^g(g) = f * ̂f(g^{-1}) = \int_{SO(3)} f(h) ̂f(gh) dh = \]
\[ = \sum_{\ell,\ell' \geq |s|} \sqrt{2\ell + 1} \sqrt{2\ell' + 1} \alpha_{\ell,\ell'} \int_{SO(3)} D_{s,s}^\ell(h) D_{s,s}^{\ell'}(gh) dh = \]
\[ = \sum_{\ell,\ell' \geq |s|} \sqrt{2\ell + 1} \sqrt{2\ell' + 1} \alpha_{\ell,\ell'} \sum_{j=-\ell}^\ell \left( \int_{SO(3)} D_{s,s}^\ell(h) D_{j,s}^{\ell'}(h) dh \right) D_{s,j}^{\ell'}(g) = \]
\[ = \sum_{\ell \geq |s|} |\alpha_{\ell}|^2 D_{-s,-s}^{\ell}(g) . \]

Note that in accordance with Theorem 1.4.5, as \(|D_{-s,-s}^{\ell}(g)| \leq D_{-s,-s}(e) = 1\), the above series converges uniformly.

Conversely, it is immediate that, given a continuous positive definite bi-\((-s)-\)associated function \( \phi \), whose expansion is
\[ \phi_f(g) = \sum_{\ell \geq |s|} |\alpha_{\ell}|^2 D_{-s,-s}^{\ell}(g) , \]
by choosing
\[ f(g) = \sum_{\ell \geq |s|} \sqrt{2\ell + 1} \beta_{\ell} D_{-s,-s}(g) \]
with \(|\beta_{\ell}| = \sqrt{\alpha_{\ell}}\), there exist a square integrable bi-s-associated function \( f \) as in (9.3.33) such that \( \phi(g) = f * \tilde{f}(g^{-1}) \). Therefore, for every random field \( X \) of type \( s \) on \( SO(3) \) there exists a square integrable bi-s-associated function \( f \) such that \( X \) and \( X^{f} \) coincide in law. Such a function \( f \) is not unique.

From \( X^{f} \) we can define a random section \( T^{f} \) of the homogeneous line bundle \( \xi_{s} \) by
\[ T_{x}^{f} := \theta(g, X_{g}^{f}) , \quad (9.3.35) \]
where \( x = gK \in S^{2} \). Now, as for the case \( s = 0 \) that was treated in §2.1, it is natural to ask whether every Gaussian isotropic section of \( \xi_{s} \) can be obtained in this way.

**Theorem 9.3.2.** Let \( s \in \mathbb{Z} \setminus \{0\} \). For every square integrable, isotropic, (complex) Gaussian random section \( T \) of the homogeneous \( s \)-spin line bundle \( \xi_{s} \), there exists a square integrable and bi-s-associated function \( f \) on \( SO(3) \) such that
\[ T^{f} \overset{\text{law}}{=} T . \quad (9.3.36) \]
Such a function \( f \) is not unique.

**Proof.** Let \( X \) be the pullback random field (of type \( s \)) of \( T \). \( X \) is of course mean square continuous. Let \( R \) be its covariance kernel. The function \( \phi(g) := R(g, e) \) is continuous, positive definite and bi-(\(-s\))-associated, therefore has the expansion
\[ \phi = \sum_{\ell \geq |s|} \sqrt{2\ell + 1} \beta_{\ell} D_{-s,-s}^{\ell} , \quad (9.3.37) \]
where \( \beta_{\ell} = \sqrt{2\ell + 1} \int_{SO(3)} \phi(g) D_{-s,-s}^{\ell} dg \geq 0 \). Furthermore, by Theorem 1.4.5, the series in (9.3.37) converges uniformly, i.e.
\[ \sum_{\ell \geq |s|} \sqrt{2\ell + 1} \beta_{\ell} < +\infty . \]
Now set $f := \sum_{\ell \geq |s|}(2\ell + 1)\sqrt{\beta_\ell}D_{s,s}$. Actually, $f \in L^2_s(SO(3))$ as $\|f\|_{L^2(SO(3))}^2 = \sum_{\ell \geq |s|}(2\ell + 1)\beta_\ell < +\infty$ so that it is bi-s-associated.

Note that every function $f$ of the form $f = \sum_{\ell \geq |s|}(2\ell + 1)\alpha_\ell D_{s,s}$ where $\alpha_\ell$ is such that $\alpha_\ell^2 = \beta_\ell$ satisfies (9.3.36) (and clearly every function $f$ such that $\phi(g) = f \ast \tilde{f}(g^{-1})$ is of this form). □

9.4 The connection with classical spin theory

There are different approaches to the theory of random sections of homogeneous line bundles on $S^2$ (see [32], [36], [38], [51] e.g.). In this section we compare them, taking into account, besides the one outlined in §6, the classical Newman and Penrose spin theory ([51]) later formulated in a more mathematical framework by Geller and Marinucci ([32]).

Let us first recall some basic notions about vector bundles. From now on $s \in \mathbb{Z}$. We shall state them concerning the complex line bundle $\xi_s = (E_s, \pi_s, S^2)$ even if they can be immediately extended to more general situations. An atlas of $\xi_s$ (see [33] e.g.) can be defined as follows. Let $U \subset S^2$ be an open set and $\Psi$ a diffeomorphism between $U$ and an open set of $\mathbb{R}^2$. A chart $\Phi$ of $\xi_s$ over $U$ is an isomorphism

$$\Phi : \pi_s^{-1}(U) \rightarrow \Psi(U) \times \mathbb{C},$$

(9.4.38)

whose restriction to every fiber $\pi_s^{-1}(x)$ is a linear isomorphism $\leftrightarrow \mathbb{C}$. An atlas of $\xi_s$ is a family $(U_j, \Phi_j)_{j \in J}$ such that $\Phi_j$ is a chart of $\xi_s$ over $U_j$ and the family $(U_j)_{j \in J}$ covers $S^2$.

Given an atlas $(U_j, \Phi_j)_{j \in J}$, For each pair $i, j \in J$ there exists a unique map (see [33] Prop. 2.2) $\lambda_{i,j} : U_i \cap U_j \rightarrow \mathbb{C} \setminus 0$ such that for $x \in U_i \cap U_j, z \in \mathbb{C},$

$$\Phi_i^{-1}(\Psi_i(x), z) = \Phi_j^{-1}(\Psi_j(x), \lambda_{i,j}(x)z).$$

(9.4.39)

The map $\lambda_{i,j}$ is called the transition function from the chart $(U_j, \Phi_j)$ to the chart
Sec. 9.4 - The connection with classical spin theory

(U_i, \Phi_i). Transition functions satisfy the cocycle conditions, i.e. for every i, j, l ∈ J

\begin{align*}
\lambda_{j,i} &= 1 & & \text{on } U_j, \\
\lambda_{j,i} &= \lambda_{i,j}^{-1} & & \text{on } U_i \cap U_j, \\
\lambda_{l,i} \lambda_{i,j} &= \lambda_{l,j} & & \text{on } U_i \cap U_j \cap U_l.
\end{align*}

Recall that we denote K ∼= SO(2) the isotropy group of the north pole as in §6, §7, so that S^2 ∼= SO(3)/K. We show now that an atlas of the line bundle ξ_s is given as soon as we specify

a) an atlas (U_j, Ψ_j)_{j∈J} of the manifold S^2,

b) for every j ∈ J a family (g^j_x)_{x∈U_j} of representative elements g^j_x ∈ G with g^j_x K = x.

More precisely, let (g^j_x)_{x∈U_j} be as in b) such that x ↦ g^j_x is smooth for each j ∈ J.

Let η ∈ π_s^{-1}(U_j) ⊆ E_s and x := π_s(η) ∈ U_j, therefore η = θ(g^j_x, z), for a unique z ∈ C. Define the chart Φ_j of ξ_s over U_j as

\[ \Phi_j(\eta) = (Ψ_j(x), z) . \] (9.4.40)

Transition functions of this atlas are easily determined. If η ∈ ξ_s is such that x = π_s(η) ∈ U_i ∩ U_j, then \( \Phi_j(\eta) = (Ψ_j(x), z_j) \), \( \Phi_i(\eta) = (Ψ_i(x), z_i) \). As g^j_x K = g^i_x K, there exists a unique k = k_{i,j}(x) ∈ K such that g^j_x = g^i_x k, so that η = θ(g^j_x, z_j) = θ(g^i_x k, z_j) = \( e^{iω} u \). Therefore

\[ \lambda_{i,j}(x) = χ_s(k) . \] (9.4.41)

The spin s concept was introduced by Newman and Penrose in [51]: a quantity u defined on S^2 has spin weight s if, whenever a tangent vector ρ at any point x on the sphere transforms under coordinate change by ρ' = e^{iω} ρ, then the quantity at this point x transforms by u' = e^{iω} u. Recently, Geller and Marinucci in [32] have put this notion in a more mathematical framework modeling such a u as a section of a complex line bundle on S^2 and they describe this line bundle by giving charts and fixing transition functions to express the transformation laws under changes of coordinates.
More precisely, they define an atlas of $S^2$ as follows. They consider the open covering
$$(U_R)_{R \in SO(3)}$$
of $S^2$ given by
$$U_e := S^2 \setminus \{x_0, x_1\} \quad \text{and} \quad U_R := RU_e,$$
where $x_0$ is the north pole (as usual), $x_1$ is the south pole. On $U_e$ they consider the usual spherical coordinates $(\vartheta, \phi)$, $\vartheta =$ colatitude, $\phi =$ longitude and on any $U_R$ the “rotated” coordinates $(\vartheta_R, \phi_R)$ in such a way that $x$ in $U_e$ and $Rx$ in $U_R$ have the same coordinates.

The transition functions are defined as follows. For each $x \in U_R$, let $\rho_R(x)$ denote the unit tangent vector at $x$, tangent to the circle $\vartheta_R = \text{const}$ and pointing to the direction of increasing $\phi_R$. If $x \in U_{R_1} \cap U_{R_2}$, let $\psi_{R_2,R_1}(x)$ denote the (oriented) angle from $\rho_{R_1}(x)$ to $\rho_{R_2}(x)$. They prove that the quantity
$$e^{i\psi_{R_2,R_1}(x)}$$
satisfies the cocycle relations (9.4.40) so that this defines a unique (up to isomorphism) structure of complex line bundle on $S^2$ having (9.4.43) as transition functions at $x$ (see [33] Th. 3.2).

We shall prove that this spin line bundle is the same as the homogeneous line bundle $\xi_s = (E_s, \pi_s, S^2)$. To this aim we have just to check that, for a suitable choice of the atlas $(U_R, \Phi_R)_{R \in SO(3)}$ of $\xi_s$ of the type described in a), b) above, the transition functions (9.4.41) and (9.4.43) are the same. Essentially we have to determine the family $(g^R_x)_{R \in SO(3), x \in U_R}$ as in b).

Recall first that every rotation $R \in SO(3)$ can be realized as a composition of three rotations: (i) a rotation by an angle $\gamma_R$ around the z axis, (ii) a rotation by an angle $\beta_R$ around the y axis and (iii) a rotation by an angle $\alpha_R$ around the z axis (the so called z-y-z convention). $(\alpha_R, \beta_R, \gamma_R)$ are the Euler angles of $R$. Therefore the rotation $R$ acts on the north pole $x_0$ of $S^2$ as mapping $x_0$ to the new location on $S^2$ whose spherical coordinates are $(\beta_R, \alpha_R)$ after rotating the tangent plane at $x_0$ by an angle $\gamma_R$. In each coset $S^2 \ni x = gK$ let us choose the element $g_x \in SO(3)$ as the rotation such that
$g_x x_0 = x$ and having its third Euler angle $\gamma_{g_x}$ equal to 0. Of course if $x \neq x_0, x_1$, such $g_x$ is unique.

Consider the atlas $(U_R, \Psi_R)_{R \in SO(3)}$ of $S^2$ defined as follows. Set the charts as

$$\Psi_e(x) := (\beta_{g_x}, \alpha_{g_x}), \quad x \in U_e, \quad (9.4.44)$$

$$\Psi_R(x) := \Psi_e(R^{-1}x), \quad x \in U_R. \quad (9.4.45)$$

Note that for each $R$, $\Psi_R(x)$ coincides with the “rotated” coordinates $(\vartheta_R, \varphi_R)$ of $x$. Let us choose now the family $(g^R_x)_{x \in U_R, R \in SO(3)}$. For $x \in U_e$ choose $g^e_x := g_x$ and for $x \in U_R$

$$g^R_x := Rg^{-1}_R x. \quad (9.4.46)$$

Therefore the corresponding atlas $(U_R, \Phi_R)_{R \in SO(3)}$ of $\xi_s$ is given, for $\eta \in \pi^{-1}_s(U_R)$, by

$$\Phi_R(\eta) = (\Psi_R(x), z), \quad (9.4.47)$$

where $x := \pi_s(\eta) \in U_R$ and $z$ is such that $\eta = \theta(g^R_x, z)$. Moreover for $R_1, R_2 \in SO(3)$, $x \in U_{R_1} \cap U_{R_2}$ we have

$$k_{R_2, R_1}(x) = (g^{-1}_{R_2} x)^{-1} R_2^{-1} R_1 g^{-1}_{R_1} x \quad (9.4.48)$$

and the transition function from the chart $(U_{R_1}, \Phi_{R_1})$ to the chart $(U_{R_2}, \Phi_{R_2})$ at $x$ is given by (9.4.41)

$$\lambda_{R_2, R_1}^{(-s)}(x) := \chi_s(k). \quad (9.4.49)$$

From now on let us denote $\omega_{R_2, R_1}(x)$ the rotation angle of $k_{R_2, R_1}(x)$. Note that, with this choice of the family $(g^R_x)_{x \in U_R, R \in SO(3)}$, $\omega_{R_2, R_1}(x)$ is the third Euler angle of the rotation $R_2^{-1} R_1 g^{-1}_{R_1} x$.

**Remark 9.4.1.** Note that we have

$$R^{-1} g_x = g^{-1}_R x,$$

i.e. $g^R_x = g_x$, in any of the following two situations
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a) \( R \) is a rotation around the north-south axis (i.e. not changing the latitude of the points of \( S^2 \)).

b) The rotation axis of \( R \) is orthogonal to the plane \([x_0, x]\) (i.e. changes the colatitude of \( x \) leaving its longitude unchanged).

Note that if each of the rotations \( R_1, R_2 \) are of type a) or of type b), then

\[
k_{R_2, R_1}(x) = g_{R_2^{-1}x}^{-1}R_2^{-1}R_1g_{R_1^{-1}x}^{-1} = (R_2g_{R_2^{-1}x})^{-1}R_1g_{R_1^{-1}x}^{-1} = g_x^{-1}g_x = \text{the identity}
\]

and in this case the rotation angle of \( k_{R_2, R_1}(x) \) coincides with the angle \(-\psi_{R_2, R_1}(x)\), as neither \( R_1 \) nor \( R_2 \) change the orientation of the tangent plane at \( x \).

Another situation in which the rotation \( k \) can be easily computed appears when \( R_1 \) is the identity and \( R_2 \) is a rotation of an angle \( \gamma \) around an axis passing through \( x \). Actually

\[
k_{R_2, e}(x) = g_x^{-1}R_2^{-1}g_x
\]

which, by conjugation, turns out to be a rotation of the angle \(-\gamma\) around the north-south axis. In this case also it is immediate that the rotation angle \( \omega_{R_2, R_1}(x) \) coincides with \(-\psi_{R_2, R_1}(x)\).

We have already shown in Remark 9.4.1 that \( \omega_{R_2, R_1}(x) = -\psi_{R_2, R_1}(x) \) in two particular situations: rotations that move \( y_1 = R_1^{-1}x \) to \( y_2 = R_2^{-1}x \) without turning the tangent plane and rotations that turn the tangent plane without moving the point. In the next statement, by combining these two particular cases, we prove that actually they coincide always.

**Lemma 9.4.2.** Let \( x \in U_{R_1} \cap U_{R_2} \), then \( \omega_{R_2, R_1}(x) = -\psi_{R_2, R_1}(x) \).
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Proof. The matrix $R_2^{-1}R_1$ can be decomposed as $R_2^{-1}R_1 = EW$ where $W$ is the product of a rotation around an axis that is orthogonal to the plane $[x_0, y_1]$ bringing $y_1$ to a point having the same colatitude as $y_2$ and of a rotation around the north-south axis taking this point to $y_2$. By Remark 9.4.1 we have $Wg_{y_1} = g_{Wy_1} = g_{y_2}$. $E$ instead is a rotation around an axis passing by $y_2$ itself.

We have then, thanks to (9.4.50) and (9.4.51)

$$k_{R_2,R_1}(x) = k_{R_1^{-1}R_2,e}(R_1^{-1}x) = k_{W^{-1}E^{-1},e}(y_1) = g_{W^{-1}g_{y_1}}^{-1}EWg_{y_1} = g_{y_2}^{-1}Eg_{y_2} = k_{E^{-1},e}(y_2).$$

By the previous discussion, $\omega_{E^{-1},e}(y_2) = -\psi_{E^{-1},e}(y_2)$. To finish the proof it is enough to show that

$$\psi_{R_2,R_1}(x) = \psi_{E^{-1},e}(y_2). \tag{9.4.52}$$

Let us denote $\rho(x) = \rho_e(x)$ the tangent vector at $x$ which is parallel to the curve $\vartheta = \text{const}$ and pointing in the direction of increasing $\varphi$. Then in coordinates

$$\rho(x) = \frac{1}{\sqrt{x_1^2 + x_2^2}}(-x_2, x_1, 0)$$

and the action of $R$ is given by ([32], §3) $\rho_R(x) = R\rho(R^{-1}x)$. As $W\rho(y_1) = \rho(y_2)$ ($W$ does not change the orientation of the tangent plane),

$$\langle \rho_{R_2}(x), \rho_{R_1}(x) \rangle = \langle R_2\rho(R_2^{-1}x), R_1\rho(R_1^{-1}x) \rangle = \langle R_1^{-1}R_2\rho(R_2^{-1}x), \rho(R_1^{-1}x) \rangle =$$

$$= \langle W^{-1}E^{-1}\rho(EWR_1^{-1}x), \rho(W^{-1}E^{-1}R_2^{-1}x) \rangle = \langle E^{-1}\rho(Ey_2), W\rho(W^{-1}y_2) \rangle =$$

$$= \langle E^{-1}\rho(y_2), W\rho(y_1) \rangle = \langle E^{-1}\rho(y_2), \rho(y_2) \rangle ,$$

so that the oriented angle $\psi_{R_2,R_1}(x)$ between $\rho_{R_2}(x)$ and $\rho_{R_1}(x)$ is actually the rotation angle of $E^{-1}$. 

\[\square\]
This is the part of my thesis that maybe I like most. Indeed, here I can write whatever I want, with no definition, label, rule... For the same reasons it is the most difficult for me, as if I feel something, then I deeply feel it. With no rule, logic step and reason. And to write it down with no guideline and of course no usual word, I should work hard somehow.

If you think that what I will write for you is not enough, then yes, you are right. But I am quite sure that you know how much indebted I am with you for your words, help, contribution... whatever.

Part 1

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