Generalized Conformal Transformation and Inflationary Attractors

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We investigate the inflationary attractors in models of inflation constructed from general scalar-tensor theories which the action can be completely transformed to the Einstein-Hilbert form through the general conformal transformation. The coefficient of the conformal transformation in our study depends on both the scalar field and its kinetic term. Therefore the relevant scalar-tensor theories display the subset of the class I of the degenerate higher order scalar-tensor theories in which both the scalar field and its kinetic term can non-minimally couple to gravity. We find that if the conformal coefficient $\Omega$ takes a multiplicative form such that $\Omega \equiv w(\phi)W(X)$ where $X$ is the kinetic term of the field $\phi$, the theoretical predictions from the theories can have usual universal attractor independent of any functions of $W(X)$. For the case where $\Omega$ takes an additive form, such that $\Omega \equiv w(\phi) + k(\phi)\Xi(X)$, we find that there are new $\xi$ attractors in addition to the universal ones. We analyze the inflationary observables of these models and compare them to the latest constraints from the Planck collaboration. We find that the observable quantities associated to these new $\xi$ attractors do not satisfy the constraints from Planck data at strong coupling limit.

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I. INTRODUCTION

The mechanism of cosmic inflation is a conceivable framework when one wants to describe our universe at the very early time. It can nicely address a number of issues of Big Bang cosmology.
More concretely, it paves the treatment of primordial fluctuations resulting in the large scale structures and the anisotropy in the temperature of the cosmic microwave background (CMB) observed today. In the simple version of the models, we require the presence of a scalar degree of freedom (inflaton), either as a fundamental scalar field, e.g. a Higgs field \([1-6]\) or a composite field \([7-12]\) (or even incorporated into gravity itself), in general as an effective scalar degree of freedom. More recently, a broad class of inflationary models, dubbed cosmological attractors \([13-16]\) has attracted a lot of attention. Cosmological attractor scenario for the inflationary models are being further developed in the past few years \([17-21]\).

Interestingly, the cosmological \(\alpha\)-attractors constitute most of the existing inflationary models with plateau-like potentials. These include the Starobinsky model and some generalized versions of the Higgs inflation. Regarding the \(\alpha\)-attractors, the flatness of the inflaton potential is achieved and protected by the existence of a pole in the kinetic term of the scalar field. Moreover, at large-field values, any non-singular inflaton potential acquires a universal plateau-like form when performing the (conformal) transformation. Regarding the hyperbolic geometry and the flatness of the Kahler potential in the supergravity context, the universal behaviors of these theories make very similar cosmological predictions preserving good agreement with the current observational data \([22]\). This class of models has certain universal predictions for the important cosmological observables, i.e. scalar spectral index \((n_s)\) and tensor-to-scalar ratio \((r)\). It has been shown that the non-minimal coupling between inflaton and gravity in the strong coupling limit can lead to attractor which the observational quantities are the same as the universal \(\alpha\) attractors \([14, 15]\). The general consideration for the relations between the inflationary attractor due to the non-minimal coupling, namely \(\xi\) attractors, and the \(\alpha\) attractors is presented in \([17]\).

In the present work, we extend analysis in the existing literature by considering the cases where the non-minimal coupling is also in the form of non-minimal kinetic coupling such that the term \(k(\phi)f(X)R\) appears in the action. Here, \(k(\phi)\) is an arbitrary function of the inflaton \(\phi\), \(f(X)\) is an arbitrary function of \(X\), \(X = -\partial_\mu \phi \partial^\mu \phi / 2\) is the kinetic term of the inflaton field and \(R\) is the Ricci scalar. In general such non-minimal coupling arises by applying the general conformal transformation, in which the conformal coefficient depends on both the scalar field and its kinetic term, to the Einstein-Hilbert action. In Sec.\([II]\), we set up the scalar-tensor theories with general non-minimal coupling based on the general conformal transformation of the Einstein-Hilbert action. We then consider the two-case scenarios. In Sec.\([III]\), we concentrate on the multiplicative form of the generalized conformal factor, i.e. \(\Omega = \Omega(X, \phi) = w(\phi)W(X)\). In this section, we derive both T-model potential and E-model potential and compute cosmological observables, i.e. \(n_s\), and \(r\). In Sec.\([IV]\), we choose the additive form of the generalized conformal
factor, i.e. $\Omega = \Omega(X, \phi) = w(\phi) + k(\phi)\Xi(X)$. We also compute the cosmological observables for T-model potential. Moreover, we consider theoretical predictions for the additive form of the conformal factor in the weak and strong coupling limits which are equivalent to large and small $\alpha$ limits respectively in our setup. In Sec. IV.C, we compare the obtained results of the cosmological observables with recent Planck 2015 data. Finally, we present our conclusion in the last section.

II. GENERAL SCALAR TENSOR THEORIES

Let us first consider general conformal transformation,

$$\tilde{g}_{\mu\nu} = \Omega(X, \phi) g_{\mu\nu}. \tag{1}$$

According to this transformation, a relation between the determinant of metric is

$$J_g \equiv \sqrt{-\tilde{g}} = \Omega^2. \tag{2}$$

and a relation between kinetic term in different frame is

$$\tilde{X} \equiv -\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \frac{X}{\Omega}. \tag{3}$$

Applying the transformation in Eq. (1) to the Einstein Hilbert action,

$$S_E = \int d^4x \sqrt{-\tilde{g}} \frac{1}{2} \tilde{R}, \tag{4}$$

we get [24]

$$S_J = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \Omega R + \frac{3}{4} \Omega^2 \partial_\alpha \phi \partial^\alpha \phi + \frac{3}{2\Omega} \Omega \partial_\alpha \phi \partial_\beta \phi \partial^\alpha X + \frac{3}{2\Omega} \Omega \partial_\alpha X \partial_\beta \phi \partial^\alpha \right]. \tag{5}$$

Here, we set the reduced Planck mass $M_P = (8\pi G)^{-1/2} = 1$. We now add kinetic term $-\sqrt{-\tilde{g}} h(\phi, X) \tilde{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi/2$ to the action in Eq. (4). Under the transformation given in Eq. (1), this kinetic term gives $-\sqrt{-g} h(\phi, X) g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi/2$ in the action in the Jordan Frame. Let us define the kinetic term of scalar field in the Jordan Frame as $-\sqrt{-g} G(\phi, X) g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi/2$. Hence, we have

$$-G(\phi, X) = \frac{3 \Omega^2}{2} - \Omega h(\phi, X), \tag{6}$$

and therefore

$$h = \frac{G + 3 \Omega^2/(2\Omega)}{\Omega}. \tag{7}$$
Based on the above analysis, we conclude that under the transformation given in Eq. (1) the action

\[ S_E = \int d^4 x \sqrt{-\tilde{g}} \left[ \frac{1}{2} \tilde{R} - \frac{G \Omega + 3 \Omega^2}{2 \tilde{g}} \partial_\alpha \phi \partial_\beta \phi \right] , \]  

(8)

becomes

\[ S_J = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} \Omega R - \frac{1}{2} G(\phi, X) \partial_\mu \phi \partial^\mu \phi + \frac{3}{2 \Omega^2} \Omega \Omega_X \partial_\alpha \phi \partial^\alpha X + \frac{3 \Omega^2}{4} \partial_\alpha X \partial_\beta X \right] . \]  

(9)

The potential term for the scalar field in the Einstein frame can be obtained by adding the term \(-\Omega^2 V_E(\phi)\) in the action in the Jordan frame. Thus under the general conformal transformation the action

\[ S_J = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} \Omega R + G(\phi, X) X - \Omega^2 V_E(\phi) + \frac{3}{2 \Omega^2} \Omega \Omega_X \partial_\alpha \phi \partial^\alpha X + \frac{3 \Omega^2}{4} \partial_\alpha X \partial_\beta X \right] . \]  

(10)

is equivalent to

\[ S_E = \int d^4 x \sqrt{-\tilde{g}} \left[ \frac{1}{2} \tilde{R} + \frac{G \Omega + 3 \Omega^2}{2 \tilde{g}} \tilde{X} - V_E(\phi) \right] . \]  

(11)

We note that the coefficients \(G(\phi, X)\) and \(\Omega(\phi, X)\) in the above Einstein action depend on kinetic terms \(X\) in general. We will consider in the subsequent sections the cases where the \(X\)-dependent terms in the Einstein action can cancel each other or can be transformed to \(\tilde{X}\).

The combination of the first and second terms in the action (10) is the Lagrangian of K-inflation, which can be defined as \(L_2 \equiv GX - \Omega^2 V_E(\phi)\). Using the definition \(3 \Omega \partial_\phi \Omega_X / (2 \Omega) \equiv F(\phi, X) + F_X(\phi, X) X\), the third term in the action can be integrated by parts yielding the cubic galileon term \(-F \Box \phi\). The fourth term in the action is a subset of the the degenerate higher order scalar-tensor theories (DHOST), so that it does not lead to Ostrogradski instability [25-29, 31]. Due to the existence of this term, the theory described by the action (10) belongs to the class I of DHOST theory in which the Laplacian instabilities arise from negative sound speed of the cosmological perturbations disappear [31]. Moreover, this theory satisfies the conditions for which propagation speed of gravitational waves equals to speed of light [31, 32].

In the following consideration, we will investigate the attractor of the observational predictions from the inflationary model described by the action (10) using its equivalent action given in (11).

### III. MULTIPLICATIVE FORM

We first consider the case where \(\Omega\) has an multiplicative form, such that

\[ \Omega(\phi, X) = w(\phi) W(X) . \]  

(12)
To make our consideration independent from the form of \( W(X) \), we set \( G(\phi, X) = g(\phi) W(X) \), so that Eq. (11) becomes

\[
S_E = \int d^4 x \sqrt{-\tilde{g}} \left[ \frac{1}{2} \tilde{R} - \frac{gw + 3w^2/2}{2w^2} \tilde{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V_E(\phi) \right].
\]

(13)

For suitable choices of field-redefinition, inflationary models described by the above action should have usual inflationary attractor. In terms of the canonical normalized field \( \psi \), the above action takes the form

\[
S_E = \int d^4 x \sqrt{-\tilde{g}} \left[ \frac{1}{2} \tilde{R} - \frac{1}{2} \tilde{g}^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi - V(\psi) \right],
\]

(14)

where

\[
d\psi^2 = \left( \frac{gw + 3w^2/2}{w^2} \right) d\phi^2.
\]

(15)

Since the action (13) is similar to the action in the Einstein frame for scalar-tensor theories with non-minimal coupling term \( w(\phi) \), we set \( w(\phi) = 1 + \xi f(\phi) \) with dimensionless coupling constant \( \xi \) and arbitrary function \( f(\phi) \). To obtain exact relation between \( \psi \) and \( w(\phi) \), the relation between \( w(\phi) \) and the kinetic coupling \( g(\phi) \) is supposed to satisfy the following condition [17],

\[
g(\phi) = \frac{1}{4\xi} \left( \frac{w^2}{\alpha} \right),
\]

(16)

then Eq. (15) gives \( \psi = \sqrt{3\alpha/2} \ln w(\phi) \). This yields

\[
w(\phi) = \exp(\sqrt{2/3\alpha} \psi),
\]

(17)

where \( \alpha = 1 + (6\xi)^{-1} \). Based on the above exact relation between \( \psi \) and \( w(\phi) \), the action (14) will be independent from \( w(\phi) \) if \( V_E(\phi) \) is a function of \( w(\phi) \). The slow roll parameters, \( \epsilon, \eta \), and the number of e-folding, \( N \), have the same forms as the standard slow-roll paradigm, and they read

\[
\epsilon = \frac{1}{2} \left( \frac{1}{V} \frac{dV}{d\psi} \right)^2, \quad \eta = \left( \frac{1}{V} \frac{d^2V}{d\psi^2} \right), \quad N = \int_{\psi_{\text{end}}}^{\psi_N} \frac{d\psi}{d\phi} \frac{V}{dV/d\psi},
\]

(18)

where \( \psi_{\text{end}} \) is the value of \( \psi \) at the end of inflation, and \( \psi_N \) is the value of psi at given \( N \).

### A. T-model potential

Regarding the relation in Eq. (17), if we take

\[
V_E(\phi) = V_0 \left[ \frac{w(\phi) - 1}{w(\phi) + 1} \right]^n,
\]

(19)
therefore, we obtain

\[ V(\psi) = V_0 \tanh^n \left( \frac{\psi}{\sqrt{6} \alpha} \right). \]  

Having used the effective potential in Eq. (20), the slow-roll parameters given in Eq. (18) become

\[ \epsilon = \frac{n^2}{12\alpha} \csc h^2 \left( \frac{\psi}{\sqrt{6} \alpha} \right) \sech^2 \left( \frac{\psi}{\sqrt{6} \alpha} \right), \quad \eta = \frac{n}{6\alpha} \left[ (n-1) \csc h^2 \left( \frac{\psi}{\sqrt{6} \alpha} \right) - 2 \right] \sech^2 \left( \frac{\psi}{\sqrt{6} \alpha} \right). \]  

(21)

Slow-roll inflation terminates when \( \epsilon = 1 \), so the field value at the end of inflation reads

\[ \epsilon(\psi_{\text{end}}) = 1 \rightarrow \cosh^2 \left( \frac{\psi_{\text{end}}}{\sqrt{6} \alpha} \right) = \frac{3\alpha + G(\alpha)}{6\alpha}, \]  

(22)

where \( G(\alpha) = \sqrt{3\alpha \sqrt{3\alpha + n^2}} \). The number of e-foldings for the change of the field \( \psi \) from \( \psi_N \) to \( \psi_{\text{end}} \) is given by

\[ N = \int_{\psi_{\text{end}}}^{\psi_N} \frac{d\psi}{dV/d\psi} = \frac{3\alpha}{n} \cosh^2 \left( \frac{\psi_N}{\sqrt{6} \alpha} \right) - \frac{3\alpha}{n} \cosh^2 \left( \frac{\psi_{\text{end}}}{\sqrt{6} \alpha} \right), \]  

(23)

where \( \psi_N \) represents the field value corresponding to the horizon crossing at e-folding \( N \) of the observed CMB modes. From the above expression, we write instead

\[ N + N_e = \frac{3\alpha}{n} \cosh^2 \left( \frac{\psi_N}{\sqrt{6} \alpha} \right) \quad \text{with} \quad N_e = \frac{1}{2n} \left( 3\alpha + G(\alpha) \right). \]  

(24)

In terms of \( N \), we have

\[ \epsilon_N = \frac{n^2}{12\alpha} \left( \frac{1}{\frac{3\alpha}{n}} (N + N_e) \right) \left( \frac{1}{\frac{3\alpha}{n}} (N + N_e) - 1 \right) \]  

\[ = \frac{4nN^2 + 3n\alpha + 4N\sqrt{3\alpha (n^2 + 3\alpha)}}{3n\alpha}, \]  

(25)

\[ \eta_N = \frac{n}{6\alpha} \left[ (n-1) \left( \frac{1}{\frac{3\alpha}{n}} (N + N_e) - 1 \right) - 2 \right] \left( \frac{1}{\frac{3\alpha}{n}} (N + N_e) \right) \]  

\[ = -\frac{2(nN - 3\alpha + \sqrt{3\alpha (n^2 + 3\alpha})}{4nN^2 + 3n\alpha + 4\sqrt{3\alpha (n^2 + 3\alpha)}}. \]  

(26)

(27)

(28)

We can also test our predictions with the experimental results by using the relative strength of the tensor perturbation, i.e. the tensor-to-scalar ratio \( r \) and the spectral index of curvature perturbation \( n_s \). In terms of the slow-roll parameters, these observables are written as

\[ r = 16 \epsilon_N, \quad n_s = 1 - 6 \epsilon_N + 2 \eta_N. \]  

(29)

In terms of \( N \), we write

\[ r = \frac{12n\alpha}{nN^2 + G(\alpha)N + 3n\alpha/4}, \]  

(30)

\[ n_s = \frac{n(4(-2 + N)N - 3\alpha) + 4(-1 + N)\sqrt{3\alpha (n^2 + 3\alpha)}}{4nN^2 + 3n\alpha + 4N\sqrt{3\alpha (n^2 + 3\alpha)}} \]  

\[ = \frac{1 - \frac{2}{N} - \frac{3\alpha}{4N^2} + \frac{1}{2N} \left( 1 - \frac{1}{N} \right) G(\alpha)}{1 + \frac{1}{2N} G(\alpha) + \frac{3\alpha}{4N^2}}. \]  

(31)

(32)
To the lowest order in the slow-roll approximation, the inflationary predictions in terms of the number of e-foldings in the Einstein frame parameters for this model read:

\[ n_s = 1 - \frac{2n + 4}{4N + n}, \quad r = \frac{16n}{4N + n} \text{ for } \alpha \gg 1 & \alpha \gg n, \]  
\[ n_s = 1 - \frac{2}{N}, \quad r = \frac{12\alpha}{N^2} \text{ for } \alpha \ll 1. \]  

The above expressions for \( n_s \) and \( r \) in the large and small \( \alpha \) limits are computed by treating \( \alpha \) as a free parameter which controls the slope of \( V(\psi) \). From the definition of \( \alpha \) in terms of the coupling constant \( \xi \), we have \( \alpha \to \infty \) in the weak coupling \( \xi \to 0 \) limit and \( \alpha \to 1 \) in the strong \( \xi \gg 1 \) limit. We will see in the numerical investigation plotting in figure (1) that in the strong coupling limit \( (\alpha = 1) \), the observable quantities also converge to the universal attractor regime in Eq. (34) [17, 33, 34]. This regime corresponds to the part of the \( n_s - r \) plane favored by the Planck data [30]. For small coupling limit, the predictions converge to Eq. (33) which are in agreement with [33] if \( n \) is replaced by \( 2n \).

**B. E-model potential**

Regarding the relation in Eq. (17), we take \( V_E(\phi) = V_0 \left[ 1 - w^{-1}(\phi) \right]^n \). Therefore, we obtain

\[ V(\psi) = V_0 \left[ 1 - \exp(-\sqrt{2/3\alpha \psi}) \right]^n. \]  

For this form of the potential, it is difficult to write the time-varying parts of the inflationary predictions \( r \) and \( n_s \) solely in terms of the number of e-folding as in Eqs. (30) and (32). Hence, we consider the inflationary predictions for this case in the large and small \( \alpha \) limits. In the large \( \alpha \) limit, the above potential coincides with the simplest chaotic inflation model with \( \psi^n \)-potential. In the limit \( \alpha \gg 1 \), i.e. \( \sqrt{2/3\alpha} \ll 1 \), we write

\[ V(\psi) = V_0 \left[ 1 - \exp(-\sqrt{2/3\alpha \psi}) \right]^n = V_0 \left[ 1 - \exp(-\sqrt{2/3\alpha \psi}) \right]^n \approx \frac{2}{3\alpha} V_0 \psi^n = \tilde{V}_0 \psi^n. \]  

The slow-roll parameters take the form

\[ \epsilon = \frac{n^2}{2\psi^2}, \quad \eta = \frac{n(n - 1)}{\psi^2}. \]  

Slow-roll inflation terminates when \( \epsilon = 1 \), so the field value at the end of inflation reads

\[ \epsilon = \frac{n^2}{2\psi^2} = 1 \implies \psi_{\text{end}} = \frac{n}{\sqrt{2}}. \]  

The number of e-foldings for the change of the field \( \psi \) from \( \psi_N \) to \( \psi_{\text{end}} \) is given by

\[ N + N_e = \frac{\psi_N^2}{2n} \quad \text{with} \quad N_e = \frac{\psi_{\text{end}}^2}{2n}. \]
Therefore, in terms of $N$, the values of $n_s$ and $r$ for the large $\alpha$ limit are given by

$$n_s = 1 - \frac{2n + 4}{4N + n}, \quad r = \frac{16n}{4N + n}, \quad \text{for} \quad \alpha \gg 1. \quad (40)$$

However, in the small $\alpha$ limit, i.e. $\alpha \ll 1$, the potential in Eq. (35) becomes

$$V(\psi) \simeq V_0 \left[ 1 - n \exp(-\sqrt{2/3\alpha \psi}) \right]. \quad (41)$$

For this potential, the slow-roll parameters are

$$\epsilon \simeq \frac{n^2}{3\alpha \left( e^{\sqrt{2/3\alpha \psi}} \right)^2}, \quad \eta = -\frac{2n e^{-\sqrt{2/3\alpha \psi}}}{3\alpha}. \quad (42)$$

Slow-roll inflation terminates when $\epsilon = 1$, so the field value at the end of inflation reads

$$\epsilon(\psi_{\text{end}}) = 1 = \frac{n^2 e^{-2\sqrt{2/3\alpha \psi}}}{3\alpha} \rightarrow \psi_{\text{end}} = \sqrt{\frac{3\alpha}{8}} \ln \left( \frac{n^2}{3\alpha} \right). \quad (43)$$

The number of e-foldings for the change of the field $\psi$ from $\psi_N$ to $\psi_{\text{end}}$ is given by

$$N = \int_{\psi_{\text{end}}}^{\psi_N} d\psi \frac{V}{dV/d\psi} \simeq \frac{3\alpha e^{\sqrt{2/3\alpha \psi_N}}}{2n} - N_e \quad \text{with} \quad N_e = \frac{\sqrt{3\alpha}}{2}. \quad (44)$$

In terms of $N$, we can write slow-roll parameters as

$$\epsilon_N = \frac{n^2}{3\alpha \left( \frac{2n(N + N_e)}{3\alpha} \right)^2}, \quad \eta_N = -\frac{1}{N + N_e}. \quad (45)$$

Therefore, in terms of $N$, the values of $n_s$ and $r$ for the small $\alpha$ limit are given by

$$n_s = 1 - \frac{2}{N}, \quad r = \frac{12\alpha}{N^2} \quad \text{for} \quad \alpha \ll 1. \quad (46)$$

It follows from Eqs. (40) and (46) that when $\alpha$ is sufficiently large or small, the predictions for the E-model also converge to the attractor given in Eq. (33) or the universal attractor given in (34) respectively. Both T model and E model have the same $\alpha$ attractors because the potentials for the T model and E model have the same asymptotic behavior when $\alpha \ll 1$ and $\alpha \gg 1$. We conclude that the $\alpha$ attractors can be achieved from models by using the generalized conformal transformation. Moreover, in our multiplicative form models, where the conformal factor can be separated in to two parts as in Eq. (12) and $G = g(\phi)W(X)$, the attractors do not depend on the function $W(X)$.

**IV. ADDITIVE FORM**

Let us now consider the case where $\Omega$ has additive form, i.e.,

$$\Omega = w(\phi) + k(\phi)\Xi(X), \quad (47)$$
where \( k(\phi) \) and \( \Xi(X) \) are dimensionless. For this case, Eq. (11) becomes

\[
S_E = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \tilde{R} + \left( \frac{G}{\Omega} + \frac{3}{2} \left( \frac{w_\phi + k_\phi \Xi}{w + k \Xi} \right)^2 \right) \tilde{X} - V_E(\phi) \right].
\] (48)

This action will be reduced to Eq. (13) if \( k(\phi) = k_1 w(\phi) \) and \( G = g(\phi)(1 + k_1 \Xi) \), where \( k_1 \) is constant. Hence, the above action is a possible generalization of the action in Eq. (13). When \( \Omega \) is separated as in Eq. (47), \( w(\phi) \) will represent non-minimal coupling and \( k(\phi) \Xi(X) \) will represent the non-minimal kinetic coupling between \( \phi \) and gravity. In analogy to the consideration in section (III), we set \( w(\phi) \equiv 1 + \xi f(\phi) \) and \( k(\phi) \equiv \xi_k f_k(\phi) \), where \( \xi \) and \( \xi_k \) are dimensionless constants while \( f(\phi) \) and \( f_k(\phi) \) are arbitrary functions. In the weak non-minimal kinetic coupling limit, i.e., \( |k(\phi) \Xi(X)| \ll |w(\phi)| \), the kinetic terms of \( \phi \) in the action (48) becomes

\[
\left( \frac{G}{\Omega} + \frac{3}{2} \left( \frac{w_\phi + k_\phi \Xi}{w + k \Xi} \right)^2 \right) \tilde{X} = \left( \frac{G}{w} + \frac{3}{2} \left( \frac{w_\phi}{w} \right)^2 \right) \tilde{X}.
\] (49)

The above kinetic term is similar to that in Eq. (13), so that when the non-minimal kinetic coupling is weak the usual attractor can exist. In the limit where the non-minimal kinetic coupling is strong but the non-minimal coupling is weak, i.e., \( |k(\phi) \Xi(X)| \gg |w(\phi)| \) and \( w(\phi) \simeq 1 \), the kinetic terms of \( \phi \) in the action (48) becomes

\[
\left( \frac{G}{\Omega} + \frac{3}{2} \left( \frac{w_\phi + k_\phi \Xi}{w + k \Xi} \right)^2 \right) \tilde{X} = \left( \frac{G}{k \Xi} + \frac{3}{2} \left( \frac{k_\phi}{k} \right)^2 \right) \tilde{X}.
\] (50)

Thus the usual \( \alpha \) attractor can exist if \( G \equiv \xi k \left( \frac{k^2}{k} \right) \) and

\[
g(\phi) = \frac{1}{\xi_k} \left( \frac{k^2}{k} \right).
\] (51)

In general when both the non-minimal and non-minimal kinetic couplings are not weak, the action in Eq. (48) depends on the kinetic term \( X \) in Jordan frame. The kinetic term \( X \) can be eliminated from this action using Eq. (3) to convert \( X \) to \( \tilde{X} \) as

\[
\tilde{X} = \frac{X}{\phi + k(\phi) \Xi(X)}, \quad \Rightarrow \quad k(\phi) \Xi(X) \tilde{X} + w(\phi) \tilde{X} - X = 0.
\] (52)

For the simplest case where \( \Xi(X) \equiv X/\Lambda \) and \( \Lambda \) is constant with dimension of mass\(^4\), the above equation yields

\[
X = \frac{\tilde{X} w}{1 - k \tilde{X}/\Lambda}.
\] (53)

Therefore

\[
\Omega = \frac{w}{1 - k \tilde{X}/\Lambda}.
\] (54)
Inserting Eqs. (53) and (54) into Eq. (48), we get
\[ S_E = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \tilde{R} + \frac{G}{w} \left( 1 - k \frac{\tilde{X}}{\Lambda} \right) + \frac{3}{2w^2} \left( w_\phi + \frac{1}{\Lambda} (k_\phi w - w_\phi k) \tilde{X} \right)^2 \tilde{X} - V_E(\phi) \right\}. \] (55)

In principle, the function $G$ can be chosen such that the Lagrangian in the above action is a linear function of $\tilde{X}$, consequently we will obtain exactly the same inflationary attractor as discussed in the previous section. For such choices of $G$, the term $1 + kX/\Lambda$ will appear in the denominator of $G$ in the Jordan frame, and therefore the Lagrangian of scalar field $L_2$ does not take usual form for $k$-inflation. In the following consideration, we will see that if $G$ is a polynomial function of $X$, the action can contain non-linear $\tilde{X}$-term, and consequently the inflationary predictions have different attractor from Eqs. (33) and (34).

To perform further analysis, it is necessary to specify form of $w(\phi), k(\phi)$ and $G(\phi, X)$. In practice, one may write these functions in concrete forms, or keeps one of them generic and then write the other two functions in terms of it. Here, we consider the second possibility by writing $G(\phi, X) = g(\phi) \gamma(\phi, X)$, where
\[ \gamma(\phi, X) \equiv f_0 + f_1(\phi) \frac{X}{\Lambda} + f_2(\phi) \frac{X^2}{\Lambda^2} \ldots, \] (56)
where all coefficients $f_0, f_1, f_2, \ldots$ are dimensionless and $f_0$ is constant. Similarly to Eq. (16), $g(\phi)$ is written in terms of $w(\phi)$ as
\[ g(\phi) = \alpha^2 \frac{w_\phi^2}{w}, \] (57)
where $\alpha \equiv 1/(2\sqrt{\xi})$. For the case where the non-minimal kinetic coupling disappears, the action in Eq. (55) will not depend on the form of $w(\phi)$ if we can write the action in terms of a new field variable $\psi$ similar to that in Eq. (17). When both the non-minimal and non-minimal kinetic couplings appear in the action, is also possible to write the action in Eq. (55) in the form that is independent from the form of $w(\phi)$ by introducing suitable relation between $k(\phi)$ and $w(\phi)$. Let us define
\[ k(\phi) \equiv \kappa \alpha^2 \left( \frac{w_\phi}{w} \right)^2, \] (58)
where $\kappa$ is constant, so that the action (55) can be written as
\[ S_E = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \tilde{R} + \gamma(\phi, X) \left( 1 - \kappa \frac{X_\psi}{\Lambda} \right) + \frac{3}{2\alpha^2} \left( 1 - 3 \kappa \frac{X_\psi}{\Lambda} + 2 \kappa \frac{w_\phi w}{w_\phi^2} \frac{X_\psi}{\Lambda} \right)^2 X_\psi - V_E(\psi) \right\}, \] (59)
where $X_\psi \equiv -\tilde{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi/2$ and $\psi$ is defined from
\[ w(\phi) = \exp \left( \frac{\psi}{\alpha} \right). \] (60)
It can be seen that the action in Eq. (59) still depend on the form of \( w(\phi) \) unless \( w_{\phi\phi}/w_\phi^2 \) is constant. The constancy of the ratio \( w_{\phi\phi}/w_\phi^2 \) is possible for various forms of \( w(\phi) \), for examples, \( w \sim e^{\xi \phi}, w \sim \cosh(\xi \phi) \), etc, and also \( w = (1 + \xi \phi^b) \) with large coupling constant \( \xi \).

Moreover, this ratio is expected to be nearly constant for arbitrary form of \( w(\phi) \) when \( \phi \) slowly vary with time. Hence, it is reasonable to suppose that the ratio \( w_{\phi\phi}/w_\phi^2 \) is constant and can be quantified by

\[
\frac{w_{\phi\phi}}{w_\phi^2} = \lambda,
\]

where \( \lambda \) is a constant parameter. Inserting the above relation in to Eq. (59), we get

\[
S_E = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \tilde{R} + \left[ \gamma(\phi, X) \left( 1 - \kappa \frac{X_\psi}{\Lambda} \right) + \frac{3}{2\alpha^2} \left( 1 - 3\kappa \frac{X_\psi}{\Lambda} + 2\kappa\lambda \frac{X_\psi}{\Lambda} \right)^2 \right] X_\psi - V_E(\psi) \right\}, \tag{62}
\]

Firstly, we consider the case where \( \gamma(\phi, X) \) is constant, but not equal to \(-3/(2\alpha^2)\). For this case, the slow-roll evolution of \( \psi \) during inflation suggests that the \( \tilde{X}^2 \)-term and \( \tilde{X}^3 \)-term in the action (62) can be neglected. Consequently, the theoretical predictions from inflationary model described by the action (62) obeys the attractor in Eqs. (33) and (34) under suitable redefinition of parameter \( \alpha \).

For the case of \( \gamma(\phi, X) = -3/(2\alpha^2) \), the linear \( X_\psi \)-term in the action (62) disappears, so that under the slow roll approximation the kinetic term of \( \psi \) is proportional to \( X_\psi^2 \), such that the action becomes

\[
S_E \simeq \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \tilde{R} + \frac{X_\psi^2}{\Lambda_2^2} - V_E(\psi) \right\}, \tag{63}
\]

where

\[
\Lambda_2 \equiv \left[ \frac{3\kappa}{2\alpha^2} (4\lambda - 5) \right]^{-1} \Lambda, \tag{64}
\]

The observable quantities for this case will be discussed in the subsequent studies.

The other interesting form of \( \gamma(X) \) is the form where \( \gamma(X) \) is a linear function of \( X \) as

\[
\gamma(\phi, X) = \frac{3}{2\alpha^2} \left[ (5 - 4\lambda) \frac{X k(\phi)}{\Lambda w(\phi)} - 1 \right]. \tag{65}
\]

The above equation can be written in terms of \( X_\psi \) as

\[
\gamma = \frac{3}{2\alpha^2} \left[ (5 - 4\lambda) \kappa \frac{X_\psi}{\Lambda - \kappa X_\psi} - 1 \right]. \tag{66}
\]

Inserting this relation in to Eq. (62), we get

\[
S_E = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} \tilde{R} + \frac{X_\psi^3}{\Lambda_2^2} - V_E(\psi) \right\}, \tag{67}
\]
where

$$\Lambda_3 \equiv \left[ \frac{3\kappa^2}{2\alpha^2} (2\lambda - 3)^2 \right]^{-1} \Lambda,$$  \hspace{1cm} (68)

The observable predictions from inflationary model described by the actions in Eqs. (63) and (67) have different attractor from that in Eqs. (33) and (34). We will study this attractor in the following considerations. In general, it is also possible to set \( \gamma(\phi, X) \propto X^m \) where \( m \geq 2 \). Nevertheless, it leads to the term that is proportional to \( X^{m+2} \) which is negligible in the slow roll limit. To compute the observable quantities, we use the slow-roll approximation in which the evolution equations derived from the actions (63) and (67) can be written as

$$H^2 \simeq \frac{1}{3} V_E(\psi), \quad \text{and} \quad (\psi')^{2q-1} = -A_q \Lambda_q^{-1} \frac{1}{V_E} \frac{dV_E}{d\psi},$$  \hspace{1cm} (69)

where \( H \) is a Hubble parameter, \( a \) is a cosmic scale factor, a prime denotes derivative with respect to \( \ln a \), \( A_q \equiv 6^{q-1}/q \), and \( q = 2, 3 \) for \( X_2^q \) and \( X_3^q \) models respectively. Since the form of the equation of motion for scalar field \( \psi \) is different from that for the usual canonical normalized field, we have to compute the slow roll parameter \( \epsilon \) and number of e-folding \( N \) from their definitions:

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \quad \text{and} \quad N \equiv \int H \, dt.$$  \hspace{1cm} (70)

Using Eq. (69), the relations in Eq. (70) can be written as

$$\epsilon = \frac{A_q^{1/(2q-1)}}{2} \Lambda_q^{(q-1)/(2q-1)} \frac{1}{V_E^{1/(3q-1)/(2q-1)}} \left( \frac{dV_E}{d\psi} \right)^{2q/(2q-1)},$$  \hspace{1cm} (71)

and

$$N = A_q^{1/(1-2q)} \Lambda_q^{(1-q)/(2q-1)} \int_{\psi_{\text{end}}}^{\psi_N} d\psi \frac{V_E^{1/(2q-1)}}{(dV_E/d\psi)^{1/(2q-1)}}.$$  \hspace{1cm} (72)

### A. T-model

To obtain the potential for the T-model, we set \( V_E(\phi) = V_0 \Lambda_\alpha \left[ \left( w - 1 \right)/\left( w + 1 \right) \right]^n \), so that we obtain

$$V_E(\psi) = V_0 \Lambda_q \left[ \tanh \left( \psi/2\alpha \right) \right]^n,$$  \hspace{1cm} (73)

where \( V_0 \) is dimensionless constant which is supposed to be order of unity. In the following analytical analysis, we will restrict ourselves to the case \( n = 2 \) in which the analytical expressions for \( n_s \) and \( r \) in terms of the number of e-folding can be straightforwardly obtained. This
restriction will be relaxed when numerical analysis is performed in section \[ IVB \]. Substituting this potential in to Eqs. (71) and (72) and then setting \( n = 2 \), we get

\[
\epsilon = \frac{1}{2} \left( \frac{A_q}{V_0^{(q-1)} \alpha^{2q}} \right)^{1/(2q-1)} \left[ \sinh^2 \left( \frac{\psi}{2\alpha} \right) \cosh^{2/(2q-1)} \left( \frac{\psi}{2\alpha} \right) \right]^{-1},
\]

(74)

\[
N = (2q - 1) \left[ \frac{V_0^{q-1} \alpha^{2q}}{A_q} \cosh^2 \left( \frac{\psi}{2\alpha} \right) \right]^{1/(2q-1)} \left| \frac{\psi}{\psi_{\text{end}}} \right|^N.
\]

(75)

In this case, value of the field \( \psi \) at the end of inflation cannot be computed analytically from the relation \( \epsilon = 1 \). Hence, we define

\[
N_e \equiv (2q - 1) \left[ \frac{V_0^{q-1} \alpha^{2q}}{A_q} \cosh^2 \left( \frac{\psi_{\text{end}}}{2\alpha} \right) \right]^{1/(2q-1)},
\]

(76)

so that Eq. (75) can be written as

\[
N + N_e = A \cosh^{2/(2q-1)} \left( \frac{\psi_N}{2\alpha} \right),
\]

(77)

where

\[
A \equiv (2q - 1) \left[ \frac{V_0^{q-1} \alpha^{2q}}{A_q} \right]^{1/(2q-1)}.
\]

(78)

For inflaton with non-canonical kinetic terms, the spectral index and the tensor-to-scalar ratio are given by [35, 36]

\[
n_s = 1 - 2\epsilon - \frac{d}{d\psi} \left( \ln \epsilon \right) - \frac{d}{d\psi} \left( \ln c_s^2 \right),
\]

(79)

\[
r = 16c_s \epsilon,
\]

(80)

where \( c_s \equiv \sqrt{\left( \partial P/\partial X^\psi \right)/\left( \partial \rho/\partial X^\psi \right)} = 1/\sqrt{2q-1} \) is the propagation speed of the scalar perturbations, \( P = X^\psi \Lambda^{q-1} - V(\psi) \) is the pressure of \( \psi \), and \( \rho \equiv (2q - 1)X^\psi \Lambda^{q-1} + V(\psi) \) is the energy density of \( \psi \). Eq. (74) can be written in terms of the number of e-folding using Eq. (77) as

\[
\epsilon_N = \frac{(2q - 1)A^2(2q - 1)}{2(N + N_e) \left( (N + N_e)^{2q-1} - A^{2q-1} \right)}.
\]

(81)

Using Eqs. (74) and (77), Eqs. (79) and (80) can be written as

\[
n_s = 1 - \frac{2q}{N + N_e} - \frac{4q - 2A^2(2q - 1)}{(N + N_e) \left( (N + N_e)^{2q-1} - A^{2q-1} \right)}.
\]

(82)

\[
r = 16c_s \frac{(2q - 1)A^2(2q - 1)}{2(N + N_e) \left( (N + N_e)^{2q-1} - A^{2q-1} \right)}.
\]

(83)
For \( \alpha \gg 1 \) or equivalently in the weak coupling limit \( \xi \ll 1 \), the condition \( \epsilon = 1 \) at the end of inflation yields \( \psi_{\text{end}} \simeq \alpha \sqrt{(4q - 2)/A} \), and therefore

\[
N_e \simeq \left( 1 + \frac{1}{2A} \right) A \simeq A + \frac{1}{2}, \tag{84}
\]

Substituting the above relation in to Eqs. (83) and (82), we get

\[
n_s \simeq 1 - \frac{8q - 4}{(4q - 2)N + (2q - 1)}, \quad r \simeq 16\frac{\sqrt{2q - 1}}{(4q - 2)N + (2q - 1)}. \tag{85}
\]

Interestingly, the theoretical predictions in this limit do not depend on \( \kappa \) which controls relative strength between the non-minimal and non-minimal kinetic couplings. In contrast, for \( \alpha \ll 1 \) or equivalently strong coupling \( \xi \gg 1 \) limit, \( \epsilon = 1 \) gives \( e^{2q\psi_{\text{end}}/((2q-1)\alpha)} \simeq 2^{4q/(2q-1)}(2q - 1)/(2A) \), so that

\[
N_e \simeq \left( \frac{5}{2} \right)^{1/2q} A^{(2q-1)/2q} \propto \alpha, \tag{86}
\]

and consequently

\[
n_s \simeq 1 - \frac{2q}{N} + O \left( \frac{\alpha}{N^2} \right), \quad r \simeq \frac{8}{3}(2q - 1)^{2q-1/2} \frac{V_0^{(q-1)/2} \alpha^{2q} N^{-2q}}{V_0^{2q+1}} + O \left( \frac{\alpha^{2q+1}}{N^{2q+1}} \right). \tag{87}
\]

Again, the inflationary predictions do not depend on \( \kappa \). Note that \( r \) is independent from \( \kappa \) because the coefficient of the potential defined in Eq. (73) is in the form of \( V_0 \Lambda q \) with constant \( V_0 \). Instead of setting \( V_0 \) to be constant, if we set \( \tilde{V}_0 \equiv V_0 \Lambda q \) to be constant independent of \( \kappa \) and \( \alpha \), the expression for \( r \) will depend on \( \kappa \) when \( V_0 \) is replaced by \( \tilde{V}_0 \).

It follows from Eqs. (85) and (87) that at \( \alpha \gg 1 \) and \( \alpha \ll 1 \) limits, the expressions for observable quantities, \( n_s \) and \( r \), converge to the forms that are similar to Eqs. (33) and (34) up to some constant factor. The existence of these convergences do not depend on the form of \( w(\phi) \) but of course depend on the relation among \( w(\phi) \), \( k(\phi) \) and \( G(\phi, X) \). Moreover, Eqs. (85) and (87) are computed in the large and small \( \alpha \) limits, in which various potentials take similar forms, especially the potentials for the T model and E model described in the previous sections. Hence, the convergent of the observable quantities to Eqs. (85) or (87) at asymptotic value of \( \alpha \) can imply the inflationary attractor.

In more general cases where \( n \) is not restricted to be two, the number of e-folding will depend on Hypergeometric functions so that it is not possible to write \( \epsilon \) in terms of the number of e-folding. In this situation, it is difficult to write analytic expressions for \( n_s \) and \( r \) in terms of the number of e-folding.

### B. Theoretical predictions for large and small \( \alpha \) limits

As mentioned in section III B, the potentials of the T model and E model have the same asymptotic behavior in the large and small \( \alpha \) limits. Since the inflationary attractors are char-
acterized by these asymptotic behaviors, we investigate in this section inflationary predictions for the models described by Eq. (69) in the large and small $\alpha$ limits instate of replting the calculations in the previous section for the E model.

In the limit $\alpha \gg 1$, the potential in Eq. (73) becomes

$$V_E(\psi) \simeq V_0 \Lambda q \left( \frac{\psi}{2\alpha} \right)^n.$$  \hspace{1cm} (88)

Replacing the potential in Eq. (73) by this approximated potential, it can be shown that $\epsilon$ and $N$ are given by

$$\epsilon = \frac{1}{2} \left( \frac{A_q n^2 q^3 \psi^{n-2q-nq}}{V_0^{(q-1)} (2\alpha)^{n-q}} \right)^{1/(2q-1)},$$  \hspace{1cm} (89)

$$N = \frac{2q - 1}{2q + nq - n} \left[ \frac{V_0^{q-1} (2\alpha)^{n-q}}{n A_q^{n-2q-nq}} \right]^{1/(2q-1)} - \frac{n}{2} \frac{2q - 1}{2q + nq - n}.$$  \hspace{1cm} (90)

Combining the above two equations, we can write $\epsilon$ in terms of the number of e-folding as

$$\epsilon_N = \frac{n}{2} \frac{2q - 1}{2q + nq - n} \left[ N + \frac{2q - 1}{2q + nq - n} \right]^{-1}.$$  \hspace{1cm} (91)

Inserting the above results into Eqs. (79) and (80), we get

$$n_s = 1 - 2 \frac{2q + 3nq - 2n}{2(2q + nq - n) N + n(2q - 1)}, \quad r = \frac{16}{\sqrt{2q - 1}} \frac{2q - n}{2(2q + nq - n) N + n(2q - 1)}.$$  \hspace{1cm} (92)

In the limit $\alpha \ll 1$, the potential in Eq. (73) becomes

$$V_E(\psi) \simeq V_0 \Lambda q (1 - 2ne^{-\psi/\alpha}),$$  \hspace{1cm} (93)

and therefore we have

$$\epsilon = \frac{1}{2} B \left[ \frac{e^{-2q\psi/\alpha}}{(1 - 2ne^{-\psi/\alpha})^{3q-1}} \right]^{1/(2q-1)} \simeq \frac{1}{2} B e^{-2q\psi/(2q-1)\alpha},$$  \hspace{1cm} (94)

$$N = \frac{2n}{B} \left( (2q - 1) - \frac{nq}{1 - q} e^{-\psi/\alpha} \right) e^{\psi/(2q-1)\alpha} \psi_N^{-\psi/\psi_{end}} \frac{V_0^{q-1}}{2A_q} \left( \frac{V_0^{q-1}}{2A_q} \right)^{1/2q}.$$  \hspace{1cm} (95)

where

$$B \equiv \frac{A_q (2n)^{2q}}{V_0^{q-1} \alpha^{2q}}.$$  \hspace{1cm} (96)

In terms of the number of e-folding, $\epsilon$ can be written as

$$\epsilon_N \simeq \frac{1}{2} (2q - 1)^{2q} \alpha^{2q} \frac{V_0^{q-1}}{A_q} \left[ N + (2q - 1) \alpha \left( \frac{V_0^{q-1}}{2A_q} \right)^{1/2q} \right]^{-2q}.$$  \hspace{1cm} (97)
Hence, for this case, Eqs. (79) and (80) yield

\[ n_s \simeq 1 - \frac{2q}{N}, \quad r \simeq (2q - 1)^2q \cdot \frac{8}{\sqrt{2q - 1}} \cdot \frac{V_0^{q-1}}{A_q} \frac{\alpha^{2q}}{N^{2q}}. \]  

(98)

Eqs. (92) and (98) are the generalization of the attractor in Eqs. (85) and (87). These equations will become the attractors in Eqs. (33) and (34) if \( q = 1 \) for suitable redefinition of \( \alpha \). We will see in the numerical investigation that due to the factor \( 2q \) in the expression for \( n_s \) in Eq. (98), the value of \( n_s \) in small \( \alpha \) limit is less than the observational bound and value from universal attractor at large number of e-folding, e.g., at \( N = 60 \). This is put a tight constraint on the inflationary attractor in the strong coupling \( \xi \gg 1 \) limit for the models where \( q > 1 \).

FIG. 1: The plots show how \( n_s \) and \( r \) evolve with the changing of \( \alpha \). In all plots, \( n_s \) and \( r \) are evaluated at \( N = 60 \).

In order to compute the observable quantities from the inflationary models whose dynamics are governed by evolution equations in Eq. (69) and potential is given by Eq. (73), we integrate Eq. (69) and compute the observable quantities numerically for various values of \( q, n \) and \( \alpha \).

In Fig. 1, we plot the predictions of the model in the \( n_s - \log(\alpha) \) and \( r - \log(\alpha) \) plane for various values of the parameters \( n \) and \( q \). From Fig. 1, we discover that our results for \( \alpha < O(10) \) with any values of \( n \) and \( q \) show an attractor behavior, but with only \( q = 1 \) display an universal attractor given in Eq. (34). From our definition of \( \alpha = 1/(2\sqrt{\xi}) \), we see that the attractor can be achieved when \( \xi > O(10^{-3}) \) which is in agreement with Ref. [14]. In addition, from Fig. 1, in case of large \( \alpha \) the attractor can be achieved when \( \xi < O(10^{-4}) \).

C. Contact with recent Planck data

In this section, we compare our results in Eqs. (92) and (98) with Planck 2015 data. Note once that the potentials of the T model and E model have the same asymptotic behavior in the large and small \( \alpha \) limits. In the small \( \alpha \) limit, we compared our results with the Planck 2015
measurement by placing the predictions in the \((n_s - r)\) plane with different values of \(q\) while kept \(N = 60\), illustrated in Fig.2. We notice that with \(q = 1\) our results lie within 2\(\sigma\) C.L. of Planck 2015 contours. However, when \(q > 1\) the results are far outside 2\(\sigma\) C.L. of Planck 2015 contours. In addition, from the right panel of Fig.1, for various values of \(n\) and \(q\) at strong coupling limit our model provides \(r < 0.064\) in precise agreement with the improved value recently reported in [23].

**FIG. 2:** In case of small values of \(\alpha\), we compare the theoretical predictions in the \((n_s - r)\) plane for small \(\alpha\) with Planck 2015 results for TT, TE, EE, +lowP and assuming ΛCDM + r [22].

However, in the large \(\alpha\) limits, with \(n = 2\) our results lie within 1\(\sigma\) C.L. of Planck 2015 contours for \(q = 1\) & 2, while within 2\(\sigma\) C.L. of Planck 2015 contours for \(q = 3\), illustrated in the upper-left panel of Fig.3. Moreover, our results lie far outside 2\(\sigma\) C.L. of Planck 2015 when \(q = 1, n = 4\), but lie within 1\(\sigma\) C.L. of Planck 2015 when \(q = 3\) with \(n = 4\), displayed in the upper-right panel of Fig.3.
FIG. 3: In case of large values of $\alpha$, we compare the theoretical predictions in the $(n_s - r)$ plane for large $\alpha$ with Planck015 results for TT, TE, EE, +lowP and assuming $\Lambda$CDM + $r$ [22].

In the lower-panel of Fig. 3 with $n = 5$ we observe that when $q = 2$ the results lie outside $1\sigma$ C.L. of Planck 2015 contours, while $q = 3$ our results lie inside $1\sigma$ C.L. of Planck 2015. In addition, we can deduce that when $q > 3$ in this case the results lie well within $2\sigma$ C.L. of Planck 2015 contours. Interestingly we conclude that the greater values of $q$ take, the better results lie well within $2\sigma$ C.L. of Planck 2015 contours for any $n$.

V. CONCLUSION

Among the few viable inflationary models, the $\alpha$ attractors, in light of presently existing CMB data, has received particular attention. In the present work, we investigated the inflationary attractors in models of inflation constructed from general scalar-tensor theories in which the action is obtained by performing general conformal transformation to the Einstein-Hilbert action. Since the coefficient of the conformal transformation in our study depends on both the scalar field and its kinetic term, the non-minimal coupling due to both the field and its kinetic term can appear in the resulting action. This action presents a subset of the class I of the DHOST theories, so that the theories associated to this action free from the Ostrogradski instability.

We then considered the two-case scenarios. We first concentrated on the multiplicative form
of the generalized conformal factor, i.e. $\Omega = \Omega(X, \phi) = w(\phi)W(X)$. We derived both T-model potential and E-model potential and compute cosmological observables, i.e. $n_s$, and $r$. We demonstrated that the $\alpha$ attractors can be achieved from models by using the generalized conformal transformation. Moreover, in our multiplicative form models, the attractors do not depend on the function $W(X)$. From the definition of $\alpha$ in terms of the coupling constant $\xi$, we have $\alpha \to \infty$ in the weak coupling $\xi \to 0$ limit and $\alpha \to 1$ in the strong $\xi \gg 1$ limit. In the strong coupling limit, the predictions converge to the universal attractor regime in Eq. (34) [17, 33, 34] which corresponds to the part of the $n_s - r$ plane favored by the Planck data [30]. For small coupling limit, the predictions converge to Eq. (33) which are in agreement with [33] if $n$ is replaced by $2n$.

In addition, we have chosen the additive form of the generalized conformal factor, i.e. $\Omega = \Omega(X, \phi) = w(\phi) + k(\phi)X$. We also compute the cosmological observables for T-model potential. We have found that in our choice of the relation among the functions of the coefficients, the inflationary predictions do not depend on both $w(\phi)$ and relative strength between the non-minimal kinetic and usual non-minimal couplings. However, in some choices of the relation among the functions of the coefficients, the kinetic term of the redefined field that governs dynamics of inflation takes a non-linear form, e.g., $X^2$ and $X^3$. In these situations, the inflationary predictions converge to new attractors given by Eqs. (92) and (98) in the weak and strong coupling limits respectively. For the additive form of the conformal factor, the parameter $\alpha$ is defined such that the weak and strong coupling limits are equivalent to large and small $\alpha$ respectively. From our numerical calculation, we discovered that the attractor can be achieved for the strong coupling limit and the weak one when $\xi > O(10^{-3})$ and $\xi < O(10^{-4})$, respectively.

We confronted the obtained results of the cosmological observables with recent Planck 2015 data. More concretely, in the small $\alpha$ limit, we compared our results given in Eq. (98) with the Planck 2015 measurement by placing the predictions in the $(n_s - r)$ plane with different values of $q$ while kept $N = 60$, illustrated in Fig. 2. We notice that with $q = 1$ our results lie within $1\sigma$ C.L. of Planck 2015 contours. However, when $q > 1$ the results are not satisfied the observational bound of the Planck 2015 contours. However, in the large $\alpha$ limits given in Eq. (92), with $n = 2$ our results lie within $1\sigma$ C.L. of Planck 2015 contours for $q = 1$ & 2, while within $2\sigma$ C.L. of Planck 2015 contours for $q = 3$, illustrated in the upper-left panel of Fig. 3. Moreover, our results lie far outside $2\sigma$ C.L. of Planck 2015 when $q = 1, n = 4$, but lie within $1\sigma$ C.L. of Planck 2015 when $q = 3$ with $n = 4$, displayed in the upper-right panel of Fig. 3. Notice that the greater values of $q$ take, the better results lie well within $2\sigma$ C.L. of Planck 2015 contours for $n = 4$. In the lower-panel of Fig. 3 with $n = 5$ we observe that when $q = 2$ the results lie outside $1\sigma$ C.L.
of Planck 2015 contours, while \( q = 3 \) our results lie inside 1\( \sigma \) C.L. of Planck 2015. In addition, we can deduce that when \( q > 3 \) in this case the results lie well within 2\( \sigma \) C.L. of Planck 2015 contours.

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