LARGE DEVIATION PRINCIPLE FOR REFLECTED SPDE ON INFINITE SPATIAL DOMAIN

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Abstract We study a large deviation principle for a reflected stochastic partial differential equation on infinite spatial domain. A new sufficient condition for the weak convergence criterion proposed by Matoussi, Sabbagh and Zhang (Appl. Math. Optim. 83: 849-879, 2021) plays an important role in the proof.

Keywords Stochastic partial differential equations, Reflection, Large deviation principle, Weak convergence approach

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1. Introduction

Consider the following stochastic partial differential equation (SPDE for short) with reflection:

$$
\begin{align*}
\frac{\partial u^\varepsilon}{\partial t} (t, x) &= \frac{\partial^2 u^\varepsilon}{\partial x^2} (t, x) + f(x, u^\varepsilon (t, x)) + \sqrt{\varepsilon} \sigma (x, u^\varepsilon (t, x)) \hat{W} (t, x) + \eta^\varepsilon (dx, dt); \\
u^\varepsilon (0, \cdot) &= u_0; \\
u^\varepsilon (t, 0) &= 0,
\end{align*}
$$

(1.1)

where \( \varepsilon > 0 \), \((t, x) \in \mathbb{R}_+^2; \hat{W} (t, x) \) denotes the space-time white noise defined on a complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), with \( \mathcal{F}_t = \sigma \{W(s,x) : 0 \leq s \leq t, x \geq 0\}; u_0 \) is a non-negative continuous function on \( \mathbb{R}_+ \); \( \eta^\varepsilon \) is a random measure on \( \mathbb{R}_+^2 \) to keep \( u^\varepsilon \) nonnegative and it is a part of the solution pair \((u^\varepsilon, \eta^\varepsilon)\).

When the spatial domain is a finite interval \([0, 1]\), the SPDE with reflection was first studied by Nualart and Pardoux [8], and they proved the existence and uniqueness of the solution in the case of \( \sigma(\cdot) = 1 \). When the volatility coefficients \( \sigma \) is Lipschitz with linear growth, Donati-Martin and Pardoux [3] proved the existence of the solution to the equation, Xu and Zhang [13] proved its uniqueness. Many interesting and important properties have been investigated, for example, the reversible probability measure in [4, 16]; the hitting properties of solutions in [1]; the Hölder continuity in [2]; the invariant measures in [6] and [14]; the large deviation principle (LDP for short) in [13] and [15]; the dimension-free Harnack inequalities and log-Harnack inequalities in [18] and [12]; the hypercontractivity in [11], and so on. The reflected SPDEs with fractional noises were studied in [10, 15]. We would like to refer the reader to [17] and references therein for more information on reflected SPDEs.

When the spatial domain is an infinite interval \( \mathbb{R} \) or \( \mathbb{R}_+ \), Otobe [9] proved the existence and uniqueness in the case of \( \sigma(\cdot) = 1 \), and proved the existence when \( \sigma \) is Lipschitz with linear growth. Uniqueness has also been shown by Hambly and Kalsi [5] for the equation on an unbounded domain provided that \( \sigma \) satisfies a Lipschitz condition, with a Lipschitz coefficient which decays exponentially fast in the spatial variable.

The purpose of this paper is to obtain the LDP for the solution of the reflected SPDE (1.1) on an unbounded domain.
**Condition 1.1.** Assume that there are some constants $C_{1,i} > 0$, $i = 1, 2, 3$, $R > 0$, $\delta \geq 0$ and $r \in \mathbb{R}$, $f$ and $\sigma$ are some measurable mappings from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$ satisfying the following conditions:

(I) For every $x \in [0, \infty)$, $u, v \in \mathbb{R}$, 
\[ |f(x, u) - f(x, v)| \leq C_{1,1}|u - v|. \]

(II) For every $x \in [0, \infty)$, $u \in \mathbb{R}$, 
\[ |f(x, u)| \leq C_{1,2}(e^{rx} + |u|). \]

(III) For every $x \in [0, \infty)$, $u, v \in \mathbb{R}$, 
\[ |\sigma(x, u) - \sigma(x, v)| \leq C_{1,3}e^{-\delta x}|u - v|. \]

(IV) For every $x \in [0, \infty)$, $u \in \mathbb{R}$, 
\[ |\sigma(x, u)| \leq Re^{-\delta x}(e^{rx} + |u|). \]

Fix $r$ in $\mathbb{R}$. We say that $u : \mathbb{R}_+ \to \mathbb{R}$ is in the space $\mathcal{L}_r$, if 
\[ \|u\|_{\mathcal{L}_r} := \sup_{x \geq 0} e^{-rx}|u(x)| < \infty. \tag{1.2} \]

We say that $u \in \mathcal{C}_r$ if $u \in \mathcal{L}_r$ and $u$ is continuous; $u : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ is in the space $\mathcal{C}_r^T$, if $u$ is continuous and 
\[ \|u\|_{\mathcal{C}_r^T} := \sup_{t \in [0, T]} \sup_{x \geq 0} e^{-rx}|u(t, x)| < \infty. \tag{1.3} \]

For any $r \in \mathbb{R}$, let 
\[ \|u\|_{t, \mathcal{C}_r} := \sup_{x \geq 0} e^{-rx}|u(t, x)|. \tag{1.4} \]

The following definition is taken from [5].

**Definition 1.1.** [5, Definition 4.1] A pair $(u^\varepsilon, \eta^\varepsilon)$ is called a solution of Eq. (1.1), if 

(i) $u^\varepsilon$ is a continuous $\mathcal{F}_t$-adapted process taking values in $\mathcal{C}_r^T$ and $u^\varepsilon(t, x) \geq 0$, a.s.

(ii) $\eta^\varepsilon$ is a random measure on $\mathbb{R}_+^2$ such that 
(a) for every measurable mapping $\psi : [0, \infty) \times [0, \infty) \to \mathbb{R}$, 
\[ \int_0^t \int_0^\infty \psi(s, x)\eta^\varepsilon(dx, ds) \text{ is } \mathcal{F}_t\text{-measurable.} \]

(b) $\int_0^\infty \int_0^\infty u^\varepsilon(s, x)\eta^\varepsilon(dx, ds) = 0$, a.s.

(iii) $(u^\varepsilon, \eta^\varepsilon)$ solves the parabolic SPDE in the following sense: for every $\varphi \in C_c^2(\mathbb{R}_+)$ with $\varphi(0) = 0$,
\begin{align*}
\int_0^\infty u^\varepsilon(t, x)\varphi(x)dx &= \int_0^\infty u_0(x)\varphi(x)dx + \int_0^t \int_0^\infty u^\varepsilon(s, x)\frac{\partial^2 \varphi}{\partial x^2}(x)dxds \\
&\quad + \int_0^t \int_0^\infty f(x, u^\varepsilon(s, x))\varphi(x)dxds \\
&\quad + \sqrt{\varepsilon} \int_0^t \int_0^\infty \sigma(x, u^\varepsilon(s, x))\varphi(x)W(dx, ds) \\
&\quad + \int_0^t \int_0^\infty \varphi(x)\eta^\varepsilon(dx, ds), \text{ a.s.} \tag{1.5}
\end{align*}
According to Hambly and Kalsi [5], we know the following result about the existence and uniqueness of the solution to Eq. (1.1).

**Proposition 1.1.** [5, Theorem 4.3 and Remark 4.10] Assume that Condition 1.1 holds with some $\delta > 0$. Then Eq. (1.1) admits a unique solution $u^\varepsilon$ in $C^T_r$. Moreover, $\mathbb{E}\left[\|u^\varepsilon\|_{C^T_r}^p\right] < \infty$ for any $p \geq 1$.

There are some literature about the LDPs for SPDEs with reflections, e.g., Xu and Zhang [13], Yang and Zhou [15], Wang et al. [10], etc. Those works are forced on the situation when the spatial domain is a finite interval with the sup-norm. The purpose of this paper is to obtain the LDP result when the spatial domain is an infinite interval with the weighted sup-norm $\sup_{C^T_r}$ defined by (1.3). There is some extra complexity introduced in the case of an infinite spatial domain. Here, we adopt a new sufficient condition for the LDP (see Condition 2.1 below) proposed by Matoussi et al. [7], and we use some powerful estimates obtained in Hambly and Kasli [5] (see Appendix of this paper) in the proof.

The rest of this paper is organized as follows. In Section 2, we first recall a sufficient condition of the weak convergence criterion for the LDP given in [7], then we formulate the main result of the present paper. In Section 3, the skeleton equation is studied. Sections 4 and 5 are devoted to verifying two conditions for the weak convergence criterion. Finally, we give some useful estimates borrowed from [5] in Appendix.

2. A criterion for large deviations and main result

In this section, we first recall the definition of the LDP and the weak convergence criterion in [7]. Then we state the main result of this paper.

### 2.1. A criterion for LDP

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be a complete probability space, and the increasing family $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfies the usual conditions. Let $\mathcal{E}$ be a Polish space with the Borel $\sigma$-field $\mathcal{B}(\mathcal{E})$.

**Definition 2.1.** A function $I : \mathcal{E} \to [0, \infty]$ is called a rate function on $\mathcal{E}$, if for each $M < \infty$ the level set $\{y \in \mathcal{E} : I(y) \leq M\}$ is a compact subset of $\mathcal{E}$.

**Definition 2.2.** Let $I$ be a rate function on $\mathcal{E}$. The sequence $\{u^\varepsilon\}_{\varepsilon > 0}$ is said to satisfy a large deviation principle on $\mathcal{E}$ with the rate function $I$, if the following two conditions hold:

(a) For each closed subset $F$ of $\mathcal{E}$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(u^\varepsilon \in F) \leq \inf_{y \in F} I(y).$$

(b) For each open subset $G$ of $\mathcal{E}$,

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(u^\varepsilon \in G) \geq - \inf_{y \in G} I(y).$$

The Cameron-Martin space associated with the Brownian sheet $\{W(t, x), t \in [0, T], x \in [0, \infty)\}$ is given by

$$\mathcal{H} = \left\{ g = \int_0^T \int_0^\infty \dot{g}(s, x)dxds; \int_0^T \int_0^\infty \dot{g}^2(s, x)dxds < \infty \right\}. \quad (2.1)$$

Let

$$\|g\|_{\mathcal{H}} := \left( \int_0^T \int_0^\infty \dot{g}^2(s, x)dxds \right)^{\frac{1}{2}},$$
and

\[ S_N := \{ g \in \mathcal{H}; \| g \|_\mathcal{H} \leq N \}. \]  

(2.2)

The set \( S_N \) is endowed with the weak convergence topology. Let \( \mathcal{A} \) denote the class of all \( \mathcal{H} \)-valued \( \mathcal{F}_t \)-predictable processes, and let

\[ \mathcal{A}_N := \{ \phi \in \mathcal{A}; \phi(\omega) \in S_N, \text{P-a.s.} \}. \]

**Condition 2.1.** For any \( \varepsilon > 0 \), let \( \Gamma^\varepsilon \) be a measurable mapping from \( C([0,T] \times \mathbb{R}_+) \) into \( \mathcal{E} \). Suppose that there exists a measurable mapping \( \Gamma^0 : C([0,T] \times \mathbb{R}_+) \rightarrow \mathcal{E} \) such that

(a) For every \( N < +\infty \), let \( g_n, g \in S_N \) be such that \( g_n \rightarrow g \) weakly as \( n \rightarrow \infty \). Then, \( \Gamma^0 \left( \int_0^T \int_0^T \hat{g}(s,x)dxds \right) \) converges to \( \Gamma^0 \left( \int_0^T \int_0^T \hat{g}(s,x)dxds \right) \) in the space \( \mathcal{E} \).

(b) For every \( N < +\infty \), \( \{ g^\varepsilon \}_{\varepsilon > 0} \subset \mathcal{A}_N \) and \( \theta > 0 \),

\[
\lim_{\varepsilon \to 0} \mathbb{P}(\rho(Y^\varepsilon, Z^\varepsilon) > \theta) = 0,
\]

where \( Y^\varepsilon = \Gamma^\varepsilon \left( W + \frac{1}{\sqrt{\varepsilon}} \int_0^T \int_0^T \hat{g}(s,x)dxds \right) \), \( Z^\varepsilon = \Gamma^0 \left( \int_0^T \int_0^T \hat{g}(s,x)dxds \right) \) and \( \rho(\cdot, \cdot) \) stands for the metric in the space \( \mathcal{E} \).

Let \( I \) be defined by

\[
I(h) := \inf \left\{ \frac{1}{2} \int_0^T \int_{\mathbb{R}_+} \left( \frac{1}{2} \| g \|_\mathcal{H}^2 \right) \right\}, \quad h \in \mathcal{E},
\]

(2.3)

with the convention \( \inf \emptyset = \infty \).

Recall the following result from Matoussi et al. [7].

**Theorem 2.1.** ([7, Theorem 3.2]) Suppose that Condition 2.1 holds. Then, the family \( \{ \Gamma^\varepsilon(W) \}_{\varepsilon > 0} \) satisfies a large deviation principle with the rate function \( I \) defined by (2.3).

### 2.2. Main result

To state our main result, we need to introduce a map \( \Gamma^0 \). Given \( g \in \mathcal{H} \), consider the following deterministic integral equation (the skeleton equation):

\[
\begin{align*}
\frac{\partial u^g}{\partial t}(t,x) &= \frac{\partial^2 u^g}{\partial x^2}(t,x) + f(x, u^g(t,x)) + \sigma(x, u^g(t,x))\hat{g}(t,x) + \eta^\varepsilon(dx,dt); \\
u^g(0, \cdot) &= u_0; \\
u^g(\cdot, 0) &= 0.
\end{align*}
\]

(2.4)

Analogously to Definition 1.1, a pair of \( (u^g, \eta^\varepsilon) \) is called a solution of Eq. (2.4), if it satisfies the following conditions:

(i) \( u^g \) is in \( C^T_r \) and \( u^g(t,x) \geq 0 \).

(ii) \( \eta^\varepsilon \) is a measure on \( \mathbb{R}_+^2 \) such that \( \int_0^\infty \int_0^\infty u(t,x)\eta^\varepsilon(dx,dt) = 0 \).

(iii) \( (u^g, \eta^\varepsilon) \) solves the parabolic PDE in the following sense: for every \( \varphi \in C^2_r(\mathbb{R}_+) \) with \( \varphi(0) = 0 \), and for every \( t \geq 0 \),

\[
\int_0^\infty u^g(t,x)\varphi(x)dx = \int_0^\infty u_0(x)\varphi(x)dx + \int_0^t \int_0^\infty u^g(s,x)\frac{\partial^2 \varphi}{\partial x^2}(x)dxds \\
+ \int_0^t \int_0^\infty f(x, u^g(s,x))\varphi(x)dxds \\
+ \int_0^t \int_0^\infty \sigma(x, u^g(s,x))\varphi(x)\hat{g}(s,x)dxds \\
+ \int_0^t \int_0^\infty \varphi(x)\eta^\varepsilon(dx,ds). \quad (2.5)
\]
By Proposition 3.1 below, Eq. (2.4) admits a unique solution \( u^g \in C^T_r \) under Condition 1.1. For any \( g \in \mathcal{H} \), define
\[
\Gamma^0 \left( \int_0^T \int_0^T g(s, x) dx ds \right) := u^g. \tag{2.6}
\]

The following theorem is the main result of this paper.

**Theorem 2.2.** Assume that Condition 1.1 holds with some \( \delta > 0 \). Then the family \( \{u^g\}_{g > 0} \) in Eq. (1.1) satisfies an LDP in the space \( (C^T_r, \| \cdot \|_{C^T_r}) \) with the rate function \( I \) given by
\[
I(h) := \inf_{\{g \in \mathcal{H} : h = \Gamma^0(\int_0^T g(s, x) dx ds)\}} \left\{ \frac{1}{2} \int_0^T \int_0^\infty g^2(s, x) dx ds \right\}, \quad h \in \Gamma^0(\mathcal{H}),
\tag{2.7}
\]
with the convention \( \inf \emptyset = \infty \).

**Proof.** According to Theorem 2.1, it suffices to show that the conditions (a) and (b) in Condition 2.1 are satisfied. Condition (a) will be proved in Proposition 4.1, and Condition (b) will be verified in Proposition 5.1. The proof is complete. \( \square \)

3. The skeleton equation

3.1. The deterministic obstacle problem on \( \mathbb{R}_+ \). Recall the following result for the deterministic obstacle problem on \( \mathbb{R}_+ \) from Hambly and Kalsi [5].

**Definition 3.1.** [5, Definition 2.5] Fix \( r \in \mathbb{R} \). Let \( v \in C^T_r \) be such that \( v(t, 0) = 0 \) for every \( t \in [0, T] \) and \( v(0, \cdot) \leq 0 \). We say that the pair \( (z, \eta) \) satisfies the heat equation with the obstacle \( v \) and exponential growth \( r \) on \( \mathbb{R}_+ \), if

(i) \( z \in C^T_r, z(t, 0) = 0, z(0, x) = 0 \) and \( z \geq v \).
(ii) \( \eta \) is a measure on \( (0, \infty) \times [0, T] \) such that \( \int_0^T \int_0^\infty (z(s, x) - v(s, x)) \eta(dx, ds) = 0 \).
(iii) \( z \) weakly solves the PDE
\[
\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + \eta. \tag{3.1}
\]

That is, for every \( t \in [0, T] \) and every \( \varphi \in C_c^\infty(\mathbb{R}_+) \) with \( \varphi(0) = 0 \),
\[
\int_0^\infty z(t, x) \varphi(x) dx = \int_0^t \int_0^\infty z(s, x) \varphi'(x) dx ds + \int_0^t \int_0^\infty \varphi(x) \eta(dx, ds).
\]

**Theorem 3.1.** [5, Theorem 2.6] For every \( r \in \mathbb{R} \) and every \( v \in C^T_r \) such that \( v(t, 0) = 0 \) for every \( t \in [0, T] \) and \( v(0, \cdot) \leq 0 \), there exists a unique solution \( (z, \eta) \) to the heat equation on \( \mathbb{R}_+ \) with Dirichlet boundary condition and the obstacle \( v \). Furthermore, if \( v_1, v_2 \in C^T_r \), then
\[
\|z_1 - z_2\|_{C^T_r} \leq C_{3.1} \|v_1 - v_2\|_{C^T_r}, \tag{3.2}
\]
where \( C_{3.1} = C_{3.1}(r, T) \in (0, \infty) \) and \( z_i \) is the solution to the obstacle problem corresponding to \( v_i, i = 1, 2 \).

3.2. The skeleton equation. In this part, we prove the existence and uniqueness of the solution to the skeleton equation (2.4). The following notations will be used throughout the rest of the paper.
and define $p$ is the solution of the following reflected PDE:

$$G(t, x, y) := \frac{1}{\sqrt{4\pi t}} \left[ \exp \left( -\frac{(x-y)^2}{4t} \right) - \exp \left( -\frac{(x+y)^2}{4t} \right) \right]. \quad (3.3)$$

For any $r \in \mathbb{R}$, let

$$G_r(t, x, y) := e^{-r(x-y)}G(t, x, y), \ t > 0, \ x, y \in \mathbb{R}._{+}. \quad (3.4)$$

**Proposition 3.1.** Assume that $g \in \mathcal{H}$ and Condition 1.1 holds with some $\delta = 0$. Then Eq. (2.4) admits a unique solution in the space $(C^T_r, \|\cdot\|_{C^T_r})$.

**Proof.** Existence. Assume that $g \in S_N$ for some $N \geq 1$. We use a Picard argument to prove the existence of the solution. Let

$$v^0_1(t, x) = \int_0^\infty G(t, x, y)u_0(y)dy + \int_0^t \int_0^\infty G(t-s, x, y)f(y, u_0(y))dyds$$

$$+ \int_0^t \int_0^\infty G(t-s, x, y)\sigma(y, u_0(y))\dot{g}(s, y)dyds, \quad (3.5)$$

with $v^0_1(0, 0) = 0$. Denote the solution of Eq. (3.1) with the obstacle $v = -v^0_1$ by $(z^0_1, \eta^0_1)$ and set $u^0_1 = z^0_1 + v^0_1$. Then $(u^0_1, \eta^0_1)$ is a solution of the following reflected PDE:

$$\frac{\partial u^0_1}{\partial t}(t, x) = \frac{\partial^2 u^0_1}{\partial x^2}(t, x) + f(x, u_0(x)) + \sigma(x, u_0(x))\dot{g}(t, x) + \eta^0_1(dx, dt), \quad (3.6)$$

with $u^0_1(0, x) = u_0(x)$ and $u^0_1(t, 0) = 0$ for any $t \in [0, T], x \in \mathbb{R}._{+}$.

Inductively, put

$$v^0_n(t, x) = \int_0^\infty G(t, x, y)u_0(y)dy + \int_0^t \int_0^\infty G(t-s, x, y)f(y, u^0_{n-1}(s, y))dyds$$

$$+ \int_0^t \int_0^\infty G(t-s, x, y)\sigma(y, u^0_{n-1}(s, y))\dot{g}(s, y)dyds, \quad (3.7)$$

and define $(z^0_n, \eta^0_n)$ as the solution of Eq. (3.1) with $v = -v^0_n$. Let $u^0_n = z^0_n + v^0_n$. Then $(u^0_n, \eta^0_n)$ is the solution of the following reflected PDE:

$$\frac{\partial u^0_n}{\partial t}(t, x) = \frac{\partial^2 u^0_n}{\partial x^2}(t, x) + f(x, u^0_{n-1}(t, x)) + \sigma(x, u^0_{n-1}(t, x))\dot{g}(t, x) + \eta^0_n(dx, dt), \quad (3.8)$$

with $u^0_n(0, x) = u_0(x)$ and $u^0_n(t, 0) = 0$ for any $t \in [0, T], x \in \mathbb{R}._{+}$.

By Theorem 3.1, we obtain that for $n \geq 2$,

$$\|u^0_n - u^0_{n-1}\|_{C^T}$$

$$\leq 2C_{3,1}\|v^0_n - v^0_{n-1}\|_{C^T}$$

$$\leq 2C_{3,1} \sup_{t \in [0, T]} \sup_{x \geq 0} e^{-r x} \left| \int_0^\infty \int_0^\infty G(t-s, x, y) \left( f(y, u^0_{n-1}(s, y)) - f(y, u^0_{n-2}(s, y)) \right) dyds \right|$$

$$+ 2C_{3,1} \sup_{t \in [0, T]} \sup_{x \geq 0} e^{-r x} \left| \int_0^\infty \int_0^\infty G(t-s, x, y) \left( \sigma(y, u^0_{n-1}(s, y)) - \sigma(y, u^0_{n-2}(s, y)) \right) \dot{g}(s, y) dyds \right|$$

$$= 2C_{3,1} \left[ I_1(T) + I_2(T) \right]. \quad (3.9)$$
By Lemma 6.1, Condition 1.1 (I) and Hölder’s inequality, we have that for any \( p > 4 \),

\[
I_1(T)^p \leq C_{6,1}^p \left[ \int_0^T \| f(\cdot, u_{n-1}^g) - f(\cdot, u_{n-2}^g) \|_{t, \mathcal{L}_r} dt \right]^p \\
\leq C_{6,1}^p \cdot C_{1,1}^p \left[ \int_0^T \| u_{n-1}^g - u_{n-2}^g \|_{t, \mathcal{L}_r} dt \right]^p \\
\leq C_{6,1}^p \cdot C_{1,1}^p \cdot T^{p-1} \int_0^T \| u_{n-1}^g - u_{n-2}^g \|_{t, \mathcal{L}_r}^p dt. \quad (3.10)
\]

Since \( g \in S_N \), by Hölder’s inequality and Condition 1.1 (III), we have that for any \( p > 4 \), \( t \in [0, T] \),

\[
| e^{-r_x} \int_0^t \int_0^\infty G(t-s, x, y) \left( \sigma(y, u_{n-1}^g(s, y)) - \sigma(y, u_{n-2}^g(s, y)) \right) \dot{g}(s, y) dy ds |^p \\
= | \int_0^t \int_0^\infty G_r(t-s, x, y) e^{-r_y} \left( \sigma(y, u_{n-1}^g(s, y)) - \sigma(y, u_{n-2}^g(s, y)) \right) \dot{g}(s, y) dy ds |^p \\
\leq C_{1,3}^p \left( \int_0^t \int_0^\infty G_r(t-s, x, y)^2 e^{-2r_y} |u_{n-1}^g(s, y) - u_{n-2}^g(s, y)|^2 dy ds \right)^{\frac{p}{2}} \\
\cdot \left( \int_0^t \int_0^\infty \dot{g}^2(s, y) dy ds \right)^{\frac{p}{2}} \\
\leq C_{1,3}^p \cdot N^p \left( \int_0^t \left( \int_0^\infty G_r(t-s, x, y)^2 dy \right) \| u_{n-1}^g - u_{n-2}^g \|_{s, \mathcal{L}_r}^2 ds \right)^{\frac{p}{2}} \\
\leq C_{1,3}^p \cdot N^p \cdot \left( \int_0^T \left( \int_0^\infty G_r(t-s, x, y)^2 dy \right)^{\frac{p-2}{2}} ds \right)^{\frac{p}{p-2}} \cdot \int_0^T \| u_{n-1}^g - u_{n-2}^g \|_{s, \mathcal{L}_r}^p ds, \quad (3.11)
\]

where \( G_r(t-s, x, y) \) is defined in (3.4). By Lemma 6.3 (i), we have

\[
I_2(T)^p \leq C_{1,3}^p \cdot C_{6,3} \cdot N^p \cdot T^{\frac{p+4}{4}} \cdot \int_0^T \| u_{n-1}^g - u_{n-2}^g \|_{t, \mathcal{L}_r}^p dt. \quad (3.12)
\]

By (3.9), (3.10) and (3.12), there exists a positive constant \( C_{3,2} = C_{3,2}(r, p, N, T) \) such that

\[
\| u_n^g - u_{n-1}^g \|^p_{C_T^r} \leq C_{3,2} \int_0^T \| u_{n-1}^g - u_{n-2}^g \|^p_{t, \mathcal{L}_r} dt \\
\leq C_{3,2}^{n-1} \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-2}} \| u_1^g - u_0^g \|^p_{t_{n-1}, \mathcal{L}_r} dt_{n-1} \cdots dt_1 dt_2 dt_1 \\
\leq C_{3,2}^{n-1} \| u_1^g - u_0^g \|^p_{C_T^r} \cdot \frac{T^{n-1}}{(n-1)!}. \quad (3.13)
\]

In particular, by using the same procedure, we have

\[
\| u_1^g \|^p_{C_T^r} < +\infty.
\]

Therefore, for any \( m \geq n \geq 1 \),

\[
\| u_m^g - u_n^g \|^p_{C_T^r} \leq \| u_1^g - u_0^g \|^p_{C_T^r} \cdot \sum_{k=n}^{m-1} \left( \frac{C_{3,2}^k T^k}{k!} \right)^p \\
\longrightarrow 0, \text{ as } n, m \to \infty. \quad (3.14)
\]
Hence, \( \{u^q_n\}_{n \geq 1} \) is Cauchy in \( \mathcal{C}_r^T \), and its limit is denoted by
\[
u^q(t, x) = \lim_{n \to \infty} u^q_n(t, x).
\]

Next, we prove that \( u^q \) is a solution to Eq. (2.4).

Let
\[
\tilde{v}^q(t, x) = \int_0^\infty G(t, x, y)u_0(y)dy + \int_0^t \int_0^\infty G(t - s, x, y)f(y, u^q(s, y))dyds
\]
\[+ \int_0^t \int_0^\infty G(t - s, x, y)\sigma(s, u^q(s, y))\dot{g}(s, y)dyds.
\]

Define \( \tilde{w}^q = \tilde{v}^q + \tilde{z}^q \), where \( \tilde{z}^q \), together with a measure \( \tilde{\eta}^q \) on \( (0, \infty) \times [0, T] \), solves the obstacle problem Eq. (3.1) with obstacle \( -\tilde{v}^q \). Thus, by Definition 3.1, we have the following results:

(i) \( z \in \mathcal{C}_r^T, \tilde{z}^q(t, 0) = 0, \tilde{z}^q(0, x) = 0 \) and \( \tilde{z}^q \geq -\tilde{v}^q \), which implies that \( \tilde{w}^q \geq 0 \).

(ii) For any \( T > 0 \), \( \int_0^T \int_0^\infty \tilde{g}^q(s, x, \tilde{w}^q(s, x))\tilde{\eta}^q(dx, ds) = 0 \), which is equivalent to
\[
\int_0^T \int_0^\infty \tilde{w}^q(s, x)\tilde{\eta}^q(dx, ds) = 0.
\]

(iii) \( \tilde{z}^q \) weakly solves the PDE
\[
\frac{\partial \tilde{z}^q}{\partial t} = \frac{\partial^2 \tilde{z}^q}{\partial x^2} + \tilde{\eta}^q.
\]

Putting (3.16) and (3.17) together, we obtain that for every \( \varphi \in C_c^2(\mathbb{R}_+) \) with \( \varphi(0) = 0 \), and for every \( t \geq 0 \),
\[
\int_0^\infty \tilde{w}^q(t, x)\varphi(x)dx = \int_0^\infty \tilde{w}^q(0, x)\varphi(x)dx + \int_0^t \int_0^\infty \tilde{w}^q(s, x)\frac{\partial^2 \varphi}{\partial x^2}(x)dxds
\]
\[+ \int_0^t \int_0^\infty f(x, \tilde{w}^q(s, x))\varphi(x)dxds
\]
\[+ \int_0^t \int_0^\infty \sigma(x, \tilde{w}^q(s, x))\varphi(x)\dot{g}(s, x)dxds
\]
\[+ \int_0^t \int_0^\infty \varphi(x)\tilde{\eta}^q(dx, ds).
\]

As in the proof of (3.13), by (3.15), we obtain that
\[
\|u^q_n - \tilde{w}^q\|_{\mathcal{C}_r^T}^p \leq C_{3.2} \int_0^T \|u^q_{n-1} - u^q\|_{\mathcal{L}_r^p}^p dt
\]
\[\leq C_{3.2} T \|u^q_{n-1} - u^q\|_{\mathcal{C}_r^T}^p \to 0,
\]

which implies that \( \tilde{w}^q = u^q \). Thus, \( (\tilde{w}^q, \tilde{\eta}^q) \) is a solution to Eq. (2.4).

**Uniqueness.** Given two solutions \( (u^q_1, \eta^q_1) \) and \( (u^q_2, \eta^q_2) \) with the same initial value, by the similar method used in the proof of (3.13), we obtain that
\[
\|u^q_1 - u^q_2\|_{\mathcal{C}_r^T} \leq C_{3.2} \int_0^T \|u^q_1 - u^q_2\|_{\mathcal{C}_r^T}^p dt.
\]

By Gronwall’s inequality, we have \( u^q_1 = u^q_2 \).

The proof is complete.
4. Verification of Condition 2.1 (a)

In this section, we will show the continuity of the skeleton equation. Recall that $S_N$ is defined by (2.2).

**Proposition 4.1.** Assume that Condition 1.1 holds with some $\delta = 0$. Then the mapping $u^g : g \in S_N \mapsto u^g \in C^T_r$ is continuous.

**Proof.** Let $g, g_n, n \geq 1$, be in $S_N$ such that $g_n \to g$ weakly as $n \to \infty$. Let $u^{g_n}$ and $u^g$ be the solutions to Eq. (2.4) associated with $g_n$ and $g$, respectively. It is sufficient to prove

$$\lim_{n \to \infty} \|u^{g_n} - u^g\|_{C^T_r} = 0. \quad (4.1)$$

Let

$$v^{g_n}(t, x) = \int_0^t \int_0^\infty G(t, x, y)u_0(y)dy + \int_0^t \int_0^\infty G(t - s, x, y)f(y, u^{g_n}(s, y))dyds$$

$$+ \int_0^t \int_0^\infty G(t - s, x, y)\sigma(y, u^{g_n}(s, y))\dot{g}_n(s,y)dyds, \quad (4.2)$$

and

$$v^g(t, x) = \int_0^t \int_0^\infty G(t, x, y)u_0(y)dy + \int_0^t \int_0^\infty G(t - s, x, y)f(y, u^g(s, y))dyds$$

$$+ \int_0^t \int_0^\infty G(t - s, x, y)\sigma(y, u^g(s, y))\dot{g}(s,y)dyds. \quad (4.3)$$

By Theorem 3.1, we obtain that

$$\|u^{g_n} - u^g\|_{C^T_r} \leq 2C_{3,1} \sup_{t \in [0,T]} \sup_{x \geq 0} e^{-rx} \left| \int_0^t \int_0^\infty G(t - s, x, y)(f(y, u^{g_n}(s, y)) - f(y, u^g(s, y)))dyds \right|$$

$$+ 2C_{3,1} \sup_{t \in [0,T]} \sup_{x \geq 0} e^{-rx} \left| \int_0^t \int_0^\infty G(t - s, x, y)\sigma(y, u^{g_n}(s, y))\dot{g}_n(s,y)dyds \right|$$

$$+ 2C_{3,1} \sup_{t \in [0,T]} \sup_{x \geq 0} e^{-rx} \left| \int_0^t \int_0^\infty G(t - s, x, y)\sigma(y, u^g(s, y))\dot{g}(s,y)dyds \right|$$

$$=: 2C_{3,1} \left[ K_1(T) + K_2(T) + K_3(T) \right]. \quad (4.4)$$

By using the same method as that in the proofs of (3.10) and (3.12), we have that for any $p > 4$,

$$K_1(T)^p \leq C_{6,1}^p \cdot C_{1,1}^p \cdot T^{p-1} \int_0^T \|u^{g_n} - u^g\|_{C^T_r}^p dt, \quad (4.5)$$

$$K_2(T)^p \leq C_{1,3}^p \cdot C_{6,3}^p \cdot N^p \cdot T^{p-1} \int_0^T \|u^{g_n} - u^g\|_{C^T_r}^p dt. \quad (4.6)$$

Set

$$F_n(t, x) := \int_0^t \int_0^\infty G(t - s, x, y)\sigma(y, u^g(s, y))\dot{g}_n(s,y)dyds.$$
Putting (4.4), (4.5) and (4.6) together, we have that there exists a positive $C_{4,1} = C_{4,1}(r, p, N, T)$ such that
\[\|u^n - u^g\|_{C^T}^p \leq C_{4,1} \left\| u^{g_0} - u^g \right\|_{C^T}^p + 2^p C_{3,1}^p \cdot \|F_n\|_{C^T}^p.\]

By the Gronwall inequality, we have
\[\|u^n - u^g\|_{C^T}^p \leq 2^p C_{3,1}^p \cdot \|F_n\|_{C^T}^p \cdot \left(1 + C_{4,1}T \cdot e^{C_{4,1}T^2}\right). \tag{4.7}\]

Consequently, if
\[\lim_{n \to \infty} \|F_n\|_{C^T} = 0, \tag{4.8}\]
then (4.1) holds. Now, it remains to prove (4.8).

By Condition 1.1 (IV), Hölder’s inequality and Lemma 6.3 (i), there exists $C_{4,2} = C_{6,3}^2 \cdot R \cdot T^{\frac{1}{2}}$ such that for any fixed $x \in \mathbb{R}_+, t \in [0, T],$
\[\int_0^t \int_0^\infty G(t - s, x, y)^2 \left|\sigma(y, u^g(s, y))\right|^2 dy ds \]
\[\leq 2R^2 \cdot e^{2ry} \cdot \int_0^t \int_0^\infty G_r(t - s, x, y)^2 \left(1 + e^{-2ry} |u^g(s, y)|^2\right) dy ds \]
\[\leq 2R^2 \cdot e^{2ry} \cdot \int_0^t \left(\int_0^\infty G_r(t - s, x, y)^2 dy\right)^\frac{p}{p-2} \left[T^\frac{2}{p} + \left(\int_0^t \|u^g\|_{L^p}^p ds\right)^\frac{2}{p}\right] ds \]
\[\leq 2C_{6,3}^2 \cdot R^2 \cdot T^{\frac{1}{2}} \cdot e^{2ry} \cdot \left(1 + \|u^g\|_{C^T}^2\right)^\frac{2}{p} < \infty. \tag{4.9}\]

Consequently, since $g_n$ converges weakly to $g$ in $\mathcal{H}$, we have that for any fixed $x \in \mathbb{R}_+, t \in [0, T],$
\[\lim_{n \to \infty} F_n(t, x) = 0. \tag{4.10}\]

We claim that $\{F_n(t, x); t \geq 0, x \geq 0\}_{n \geq 1}$ is relatively compact in the space $(C^T, \| \cdot \|_{C^T})$, or equivalently, $\{e^{-rx} F_n(t, x); t \geq 0, x \geq 0\}_{n \geq 1}$ is relatively compact in $C([0, T] \times \mathbb{R}_+)$ with respect to the sup-norm $\|f\|_\infty = \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}_+} |f(t, x)|$. Combining this fact with (4.10), we obtain (4.8).

Notice that
\[\|F_n\|_{C^T} \leq \sup_{t \in [0, T]} \sup_{x \geq 0} \int_0^t \int_0^\infty G_r(t - s, x, y)e^{-ry} \left|\sigma(y, u^g(s, y))\right| |\hat{g}_n(s, y) - \hat{g}(s, y)| dy ds \]
\[\leq \left(\sup_{t \in [0, T]} \sup_{x \geq 0} \int_0^t \int_0^\infty G_r(t - s, x, y)^2 e^{-2ry} \left|\sigma(y, u^g(s, y))\right|^2 dy ds\right)^\frac{1}{2} \]
\[\cdot \left(\int_0^T \int_0^\infty |\hat{g}_n(s, y) - \hat{g}(s, y)|^2 dy ds\right)^\frac{1}{2}. \tag{4.11}\]

By using the same method as that in the proof of (4.9), we have that for any $t \in [0, T],$
\[\sup_{t \in [0, T]} \sup_{x \geq 0} \int_0^t \int_0^\infty G_r(t - s, x, y)^2 e^{-2ry} \left|\sigma(y, u^g(s, y))\right|^2 dy ds \]
Since $g_n, g \in S_N, N \geq 1$, by (4.11) and (4.12), we have

$$\sup_{n \geq 1} \|F_n\|_{c_T^p} \leq C_{6.3}^p \cdot R^2 \cdot T^{\frac{1}{2}} \cdot \left(1 + \|u^g\|_{c_T^p}^2\right)^{\frac{1}{2}} < +\infty.$$ (4.12)

On the other hand, we claim that $\{e^{-rt}F_n(t,x); t \geq 0, x \geq 0\}_{n \geq 1}$ is also equi-continuous. Then, according to the Arzelà-Ascoli theorem, we obtain that $\{F_n\}_{n \geq 1}$ is relatively compact in the space $\left(C_t^p, \cdot \|c_T^p\right)$.

Next, we prove that $\{e^{-rt}F_n(t,x); t \geq 0, x \geq 0\}_{n \geq 1}$ is equi-continuous. Notice that

$$\left| e^{-rt}F_n(s,y) - e^{-rt}F_n(t,x) \right|^{p} \leq \left| e^{-rt}F_n(s,y) - e^{-rt}F_n(s,x) \right| + \left| e^{-rt}F_n(s,x) - e^{-rt}F_n(t,x) \right|.$$ (4.13)

Since $g_n, g \in S_N, N \geq 1$, by Hölder’s inequality, Condition 1.1 (IV) and Lemma 6.3 (iii), we have that for any $p \geq 4$ and for any $s \in [0,T], x, y \in \mathbb{R}_+$,

$$\left| e^{-rt}F_n(s,y) - e^{-rt}F_n(s,x) \right|^{p} \leq R^p \cdot N^p \cdot T \cdot \left(1 + \|u^g\|_{c_T^p} \right)^{\frac{p}{2}} \left| s - t \right|^{\frac{p}{4}}.$$ (4.14)

Similarly, by Hölder’s inequality, Condition 1.1 (IV) and Lemma 6.3 (ii), we have that for any $p \geq 4$ and for any $s, t \in [0,T], x \in \mathbb{R}_+$,

$$\left| e^{-rt}F_n(s,x) - e^{-rt}F_n(t,x) \right|^{p} \leq C_{6.4} \cdot R^p \cdot N^p \cdot T \cdot \left(1 + \|u^g\|_{c_T^p} \right)^{\frac{p}{2}} \left| s - t \right|^{\frac{p}{4}}.$$ (4.15)

Putting (4.13), (4.14) and (4.15) together, we have that for any $p \geq 4$ and for any $s, t \in [0,T], x, y \in \mathbb{R}_+$,

$$\left| e^{-rt}F_n(s,y) - e^{-rt}F_n(t,x) \right|^{p} \leq C_{4.3} \left(1 + \|u^g\|_{c_T^p} \right) \cdot \left(\left| s - t \right|^{\frac{p}{4}} + |x - y|^{\frac{p}{4}}\right),$$

where $C_{4.3} = C_{4.3}(r, p, N, R, T) \in (0, \infty)$.

The proof is complete.
5. Verification of Condition 2.1 (b)

For any $\varepsilon > 0$, let $\Gamma^\varepsilon : C([0, T] \times \mathbb{R}_+) \to C^T$ be the map defined by
\begin{equation}
\Gamma^\varepsilon (W(\cdot, \cdot)) := u^\varepsilon,
\end{equation}
where $u^\varepsilon$ stands for the solution to Eq. (1.1).

Let $\{g^\varepsilon\}_{\varepsilon > 0} \subset \mathcal{A}_N$, $N \geq 1$, be a given family of stochastic processes. Denote
\[ \tilde{u}^\varepsilon := \Gamma^0 \left( \int_0^t \int_0^\infty \hat{g}^\varepsilon(s, x)dxds \right), \]
where $\Gamma^0$ is defined by (2.6). Then, $\tilde{u}^\varepsilon$, together with a random measure $\tilde{\eta}^\varepsilon$, solves the following equation:
\begin{equation}
\begin{aligned}
\frac{\partial \tilde{u}^\varepsilon}{\partial t}(t, x) &= \frac{\partial^2 \tilde{u}^\varepsilon}{\partial x^2}(t, x) + f(x, \tilde{u}^\varepsilon(t, x)) + \sigma(x, \tilde{u}^\varepsilon(t, x))\hat{g}^\varepsilon(t, x) + \tilde{\eta}^\varepsilon(dx, dt); \\
\tilde{u}^\varepsilon(0, \cdot) &= u_0; \\
\tilde{u}^\varepsilon(\cdot, 0) &= 0.
\end{aligned}
\end{equation}

Denote
\[ \tilde{u}^\varepsilon := \Gamma^\varepsilon \left( W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^t \int_0^\infty \hat{g}^\varepsilon(s, x)dxds \right), \]
where $\Gamma^\varepsilon$ is defined by (5.1). By the Girsanov theorem, it is easily to see that $\tilde{u}^\varepsilon$, together with a random measure $\tilde{\eta}^\varepsilon$, solves the following stochastic obstacle problem:
\begin{equation}
\begin{aligned}
\frac{\partial \tilde{u}^\varepsilon}{\partial t}(t, x) &= \frac{\partial^2 \tilde{u}^\varepsilon}{\partial x^2}(t, x) + f(x, \tilde{u}^\varepsilon(t, x)) + \sqrt{\varepsilon} \sigma(x, \tilde{u}^\varepsilon(t, x))\tilde{W}(t, x) \\
&\quad + \sigma(x, \tilde{u}^\varepsilon(t, x))\hat{g}^\varepsilon(t, x) + \tilde{\eta}^\varepsilon(dx, dt); \\
\tilde{u}^\varepsilon(0, \cdot) &= u_0; \\
\tilde{u}^\varepsilon(\cdot, 0) &= 0.
\end{aligned}
\end{equation}

**Proposition 5.1.** Assume that Condition 1.1 holds with some $\delta > 0$. Then, for every $N \geq 1$, $\{g^\varepsilon\}_{\varepsilon > 0} \subset \mathcal{A}_N$ and for any $p \geq 1$, we have
\[ \lim_{\varepsilon \to 0} \mathbb{E} \left( \| \tilde{u}^\varepsilon - \tilde{u}^\varepsilon \|_{C^T}^p \right) = 0. \]

Before proving Proposition 5.1, we give the following lemma in preparation.

**Lemma 5.1.** Assume that Condition 1.1 holds with some $\delta > 0$. Then, for every $N \geq 1$, $\{g^\varepsilon\}_{\varepsilon > 0} \subset \mathcal{A}_N$ and for any $p \geq 1$, we have
\begin{equation}
\mathbb{E} \left[ \| \tilde{u}^\varepsilon \|_{C^T}^p \right] < +\infty \quad \text{and} \quad \mathbb{E} \left[ \| \tilde{u}^\varepsilon \|_{C^T}^p \right] < +\infty.
\end{equation}

**Proof.** Let
\begin{equation}
\tilde{u}^\varepsilon(t, x) = \int_0^\infty G(t, x, y)u_0(y)dy + \int_0^t \int_0^\infty G(t - s, x, y)f(y, \tilde{u}^\varepsilon(s, y))dyds \\
+ \sqrt{\varepsilon} \int_0^t \int_0^\infty G(t - s, x, y)\sigma(y, \tilde{u}^\varepsilon(s, y))\tilde{W}(dy, ds) \\
+ \int_0^t \int_0^\infty G(t - s, x, y)\sigma(y, \tilde{u}^\varepsilon(s, y))\hat{g}^\varepsilon(s, y)dyds
= : J_{1, \varepsilon}(t, x) + J_{2, \varepsilon}(t, x) + J_{3, \varepsilon}(t, x) + J_{4, \varepsilon}(t, x).
\end{equation}
Set $\tilde{z}^\varepsilon = \tilde{u}^\varepsilon - \tilde{v}^\varepsilon$. Then $(\tilde{z}^\varepsilon, \tilde{\eta}^\varepsilon)$ be the solution of Eq. (3.1) with the obstacle $-\tilde{v}^\varepsilon$. For any $n \geq 1$, let

$$
\tau_n := \inf \left\{ t \geq 0 : \sup_{x \geq 0} e^{-rx} |\tilde{u}^\varepsilon(t, x)| \geq n \right\},
$$

and let $J_{i, \varepsilon}(t, x) := J_{i, \varepsilon}(t \wedge \tau_n, x)$, $i = 1, \cdots, 4$. Since $u_0 \in C_r$, there exists a positive constant $C_{5,1} = C_{5,1}(r, p, T)$ such that

$$
\mathbb{E}\left[ \left\| J_{1, \varepsilon}^p \right\|_{C_T^p}^p \right] = \mathbb{E} \left[ \sup_{t \in [0, T]} \sup_{x \geq 0} \int_0^\infty G_r(t \wedge \tau_n, x, y) e^{-rt} u_0(y) dy \right]^p \leq \mathbb{E} \left[ \sup_{t \in [0, T]} \sup_{x \geq 0} \int_0^\infty G_r(t \wedge \tau_n, x, y) dy \cdot \left\| u_0 \right\|_{L_r} \right]^p \leq C_{5,1} \mathbb{E} \left[ \left\| u_0 \right\|_{C_r}^p \right],
$$

where in the above last inequality we used $\int_0^\infty G_r(t, x, y) dy \leq C_{5,2}(r, t) \leq C_{5,2}(r, T)$ for any $t \in [0, T]$ from (4.8) in [5]. By Lemma 6.1, Condition 1.1 (II) and Hölder's inequality, we have that for any $p > 4$,

$$
\mathbb{E}\left[ \left\| J_{2, \varepsilon}^p \right\|_{C_T^p}^p \right] \leq C_{6,1}^p \cdot \mathbb{E} \left[ \int_0^T \left\| f(\cdot, \tilde{u}^\varepsilon) \right\|_{t \wedge \tau_n, L_r} dt \right]^p \leq C_{6,1}^p \cdot C_{1,2}^p \cdot \mathbb{E} \left[ \int_0^T \left(1 + \left\| \tilde{u}^\varepsilon \right\|_{t \wedge \tau_n, L_r} \right) dt \right]^p \leq C_{6,1}^p \cdot C_{1,2}^p \cdot T^{p-1} \cdot \mathbb{E} \left[ \int_0^T \left(1 + \left\| \tilde{u}^\varepsilon \right\|_{t \wedge \tau_n, L_r} \right)^p dt \right].
$$

Since $\tilde{u}^\varepsilon \in L^p(\Omega; L^\infty([0, T \wedge \tau_n]; L_r))$, by Condition 1.1 (IV), we have

$$
\sigma(\cdot, \tilde{u}^\varepsilon(\cdot \wedge \tau_n, \cdot)) \in L^p(\Omega; L^\infty([0, T \wedge \tau_n]; L_{r-\delta})).
$$

Thus, by Lemma 6.2 Condition 1.1 (IV), we have that for any $p > 12$,

$$
\mathbb{E}\left[ \left\| J_{3, \varepsilon}^p \right\|_{C_T^p}^p \right] \leq \varepsilon^\frac{p}{2} \cdot C_{6,2}^p \cdot \mathbb{E} \left[ \int_0^T \left\| \sigma(\cdot, \tilde{u}^\varepsilon) \right\|_{t \wedge \tau_n, L_{r-\delta}}^p dt \right] \leq \varepsilon^\frac{p}{2} \cdot C_{6,2}^p \cdot R^p \cdot \mathbb{E} \left[ \int_0^T \left(1 + \left\| \tilde{u}^\varepsilon \right\|_{t \wedge \tau_n, L_r} \right)^p dt \right] \leq \varepsilon^\frac{p}{2} \cdot C_{6,2}^p \cdot R^p \cdot \mathbb{E} \left[ \int_0^T \left(1 + \left\| \tilde{u}^\varepsilon \right\|_{t \wedge \tau_n, L_r} \right)^p dt \right].
$$

As in (3.11), by Condition 1.1 (IV), we have that for any $p > 4$,

$$
\left| e^{-rx} \int_0^{t \wedge \tau_n} \int_0^\infty G(t \wedge \tau_n - s, x, y) \sigma(y, \tilde{u}^\varepsilon(s, y)) \tilde{g}^\varepsilon(s, y) dy ds \right|^p \leq R^p \cdot \left( \int_0^{t \wedge \tau_n} \left( \int_0^\infty G_r(t \wedge \tau_n - s, x, y)^2 dy \right)^{\frac{p}{2}} ds \right)^{\frac{p}{2}} \cdot \int_0^{t \wedge \tau_n} \left(1 + \left\| \tilde{u}^\varepsilon \right\|_{s, L_r} \right)^p ds \cdot \left| \int_0^T \int_0^\infty \tilde{g}^\varepsilon(s, y)^2 dy ds \right|^\frac{p}{2}.
$$
where \( C_{5.3} = C_{6.3} \cdot R_p \cdot T^{\frac{p-3}{4}} \).

Putting (5.5), (5.6), (5.7), (5.8) and (5.9) together, by Gronwall’s inequality, we obtain that there exists a constant \( C_{5.4} = C_{5.4}(\varepsilon, r, p, N, T, \| u_0 \|_{C_r}) \) such that

\[
E \left[ \| \tilde{\nu}^\varepsilon \|_{C_{r,T}^p}^p \right] \leq C_{5.4},
\]

where \( C_{5.4} \) is independent on \( n \). Letting \( n \to \infty \), we have

\[
E \left[ \| \tilde{\nu}^\varepsilon \|_{C_T^p}^p \right] \leq C_{5.4}.
\]

By Theorem 3.1, we have

\[
E \left[ \| \tilde{\nu}^\varepsilon \|_{C_T^p}^p \right] \leq 2^p \cdot C_{3.1}^p \cdot E \left[ \| \tilde{\nu}^\varepsilon \|_{C_T^p}^p \right] < \infty.
\]

The proof for \( \tilde{\nu}^\varepsilon \) is similar and is omitted here. The proof is complete. \( \square \)

**Proof of Proposition 5.1.** Recall \( \tilde{\nu}^\varepsilon(t, x) \) defined by (5.5), and let

\[
\tilde{\nu}^\varepsilon(t, x) = \int_0^\infty G(t, x, y)u_0(y)dy + \int_0^t \int_0^\infty G(t-s, x, y)f(y, \tilde{\nu}^\varepsilon(s, y))dyds + \int_0^t \int_0^\infty G(t-s, x, y)\sigma(y, \tilde{\nu}^\varepsilon(s, y))\dot{\tilde{\nu}}^\varepsilon(s, y)dyds.
\]

Then \((\tilde{\nu}^\varepsilon - \tilde{\nu})\) and \((\tilde{\nu} - \tilde{\nu}^\varepsilon, \tilde{\eta}^\varepsilon)\) are the solutions of Eq. (3.1) with the obstacles \(-\tilde{\nu}^\varepsilon\) and \(-\tilde{\nu}^\varepsilon\), respectively.

By Theorem 3.1, (5.2), (5.3), (5.5) and (5.10), we obtain that

\[
\begin{align*}
\| \tilde{\nu}^\varepsilon - \tilde{\nu} \|_{C_T^p} & \leq 2C_{3.1} \sup_{t \in [0, T]} \sup_{x \geq 0} e^{-t\varepsilon} \left| \int_0^t \int_0^\infty G(t-s, x, y) \left( f(y, \tilde{\nu}^\varepsilon(s, y)) - f(y, \tilde{\nu}(s, y)) \right) dyds \right| \\
& \quad + 2C_{3.1} \sup_{t \in [0, T]} \sup_{x \geq 0} e^{-t\varepsilon} \left| \sqrt{\varepsilon} \int_0^t \int_0^\infty G(t-s, x, y)\sigma(y, \tilde{\nu}^\varepsilon(s, y))W(dy, ds) \right| \\
& \quad + 2C_{3.1} \sup_{t \in [0, T]} \sup_{x \geq 0} e^{-t\varepsilon} \left| \int_0^t \int_0^\infty G(t-s, x, y) \left( \sigma(y, \tilde{\nu}^\varepsilon(s, y)) - \sigma(y, \tilde{\nu}(s, y)) \right) \dot{\tilde{\nu}}^\varepsilon(s, y)dyds \right| \\
& =: 2C_{3.1} \left[ Q_{1,\varepsilon}(T) + Q_{2,\varepsilon}(T) + Q_{3,\varepsilon}(T) \right].
\end{align*}
\]

As in (3.10), we have that for any \( p > 4 \),

\[
E \left[ Q_{1,\varepsilon}(T)^p \right] \leq C_{6.1}^p \cdot C_{1,1}^p \cdot T^{p-1} \cdot E \left[ \int_0^T \| \tilde{\nu}^\varepsilon - \tilde{\nu} \|_{C_T^p}^p dt \right].
\]

Since \( \tilde{\nu}^\varepsilon \in L^p(\Omega; C_T^p) \), by Condition 1.1 (IV), we have

\[
\sigma(\cdot, \tilde{\nu}^\varepsilon(\cdot, \cdot)) \in L^p(\Omega; C_{r-\delta}^T).
\]
As in (5.8), by Lemma 5.1, we have that for any $p > 12$,
\[
\mathbb{E}[Q_{2,\varepsilon}(T)^p] \leq e^p \cdot C_{6,2} \cdot R^p \cdot \mathbb{E} \left[ \int_0^T (1 + \|\tilde{u}\|_{C_1^p})^p \, dt \right] \to 0, \text{ as } \varepsilon \to 0. \tag{5.13}
\]
As in (5.9), by Condition 1.1 (III), we have that for any $p > 4$,
\[
Q_{3,\varepsilon}(T)^p \leq C_{1,3}^p \cdot C_{6,3} \cdot T^{p-4} \cdot N^p \int_0^T \|\tilde{u}\|_{C_1^p}^p \, dt. \tag{5.14}
\]
Putting (5.11), (5.12), (5.13) and (5.14) together, by the Gronwall inequality, we obtain that for any $p > 12$,
\[
\mathbb{E}\left[\|\tilde{u}^\varepsilon - \tilde{u}\|_{C_1^p}^p\right] \leq \mathbb{E}[Q_{2,\varepsilon}(T)^p] \left(1 + C_{5,5} T \cdot e^{C_{5,5} T} \right) \to 0, \text{ as } \varepsilon \to 0,
\]
where $C_{5,5} = 2^p \cdot C_{3,1}^p \cdot C_{6,1}^p \cdot C_{1,1}^p \cdot T^{p-1} + C_{6,3}^p \cdot C_{6,3} \cdot T^{p-4} \cdot N^p$.
The proof is complete. \qed

6. APPENDIX

Recall the functions $G(t, x, y)$ and $G_r(t, x, y)$ given by (3.3) and (3.4), respectively.

**Lemma 6.1.** [5, Lemma 4.5] Let $r \in \mathbb{R}$. Suppose that $u \in L^1([0, T]; \mathbb{L}_r)$. Then we have that for any $t \in [0, T]$, there exists a positive constant $C_{6,1} = C_{6,1}(r, T)$ such that

\[
\sup_{\tau \in [0, t]} \sup_{x \geq 0} \left| \int_0^\tau \int_0^\infty G(\tau - s, x, y) u(s, y) dy ds \right| \leq C_{6,1} \int_0^t \|u\|_{s, \mathbb{L}_r} ds. 
\]

**Lemma 6.2.** [5, Proposition 4.8] Let $r \in \mathbb{R}$. Suppose that $u \in L^p(\Omega; L^\infty([0, T]; \mathbb{L}_{r-\varepsilon}))$. Then for any $t \in [0, T]$ and for any $p > 12$, there exists a positive constant $C_{6,2} = C_{6,2}(r, p, T, \varepsilon)$ such that

\[
\mathbb{E} \left[ \sup_{\tau \in [0, t]} \sup_{x \geq 0} \left| \int_0^\tau \int_0^\infty G(\tau - s, x, y) u(s, y) W(dy, ds) \right|^p \right] \leq C_{6,2} \mathbb{E} \left[ \int_0^t \|u\|_{s, \mathbb{L}_{r-\varepsilon}} ds \right].
\]

**Lemma 6.3.** [5, Proposition A.1] Fix $r \in \mathbb{R}$, $T > 0$. Then, for any $p > 4$, we have the following results:

(i) For every $t, s \in [0, T]$, there exists a positive constant $C_{6,3} = C_{6,3}(r, p, T)$ such that

\[
\sup_{x \geq 0} \left( \int_s^t \left[ \int_0^\infty G_r(t - u, x, z)^2 du \right]^{\frac{p}{p-2}} dz \right)^\frac{2}{p-2} \leq C_{6,3}|t - s|^{\frac{4}{p-2}}.
\]

(ii) For every $t, s \in [0, T]$, there exists a positive constant $C_{6,4} = C_{6,4}(r, p, T)$ such that

\[
\sup_{x \geq 0} \left( \int_0^s \left[ \int_0^\infty \left| G_r(t - u, x, z) - G_r(s - u, x, z) \right|^2 du \right]^{\frac{p}{p-2}} dz \right)^\frac{2}{p-2} \leq C_{6,4}|t - s|^{\frac{4}{p-2}}.
\]

(iii) For every $x, y \in [0, \infty)$, there exists a positive constant $C_{6,5} = C_{6,5}(r, p, T)$ such that

\[
\sup_{s \leq 0} \left( \int_0^s \left[ \int_0^\infty \left| G_r(s - u, x, z) - G_r(s - u, y, z) \right|^2 du \right]^{\frac{p}{p-2}} dz \right)^\frac{2}{p-2} \leq C_{6,5}|x - y|^{\frac{4}{p-2}}.
\]
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