Adiabatic non-equilibrium steady states in the partition free approach

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Abstract

Consider a small sample coupled to a finite number of leads, and assume that the total (continuous) system is at thermal equilibrium in the remote past. We construct a non-equilibrium steady state (NESS) by adiabatically turning on an electrical bias between the leads. The main mathematical challenge is to show that certain adiabatic wave operators exist, and to identify their strong limit when the adiabatic parameter tends to zero. Our NESS is different from, though closely related with the NESS provided by the Jakšić-Pillet-Ruelle approach. Thus we partly settle a question asked by Caroli et al in 1971 regarding the (non)equivalence between the partitioned and partition-free approaches.

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1 Introduction

1.1 Generalities

This paper deals with the rigorous construction of adiabatic non-equilibrium steady states for mesoscopic systems which initially are fully coupled (or 'partition free') and at thermal equilibrium \[9, 14\]. The initial equilibrium state is broken down by slowly turning on an electrical bias between leads (i.e. inserting a d.c. battery), which in a certain way can be seen as slowly changing the chemical potentials of the leads coupled with the small sample.

In contrast with the above described partition-free setting, the 'partitioned procedure' is the one in which one starts with several decoupled reservoirs, each of them being at different equilibrium states. Let us assume for simplicity that they are in grand canonical Gibbs states having the same temperature but different chemical potentials. Then at \( t = 0 \) they are suddenly joined together with a sample, and the newly composed system is allowed to freely evolve until it reaches a steady state at \( t = \infty \). From a mathematical point of view this approach is by now very well understood, see for example \[1, 19, 33, 28, 5\] and references therein. One can allow the carriers to interact in the sample \[18\], and the theory still works. Note that even if we choose to turn on the coupling between the reservoirs in a time dependent way, the result will be the same \[15\].

One can ask which approach is more physical; here is a quote from a paper by Caroli et al \[8\] from 1971 -maybe the first very influential paper on the subject- who came with the following observation about the partitioned procedure:

One might raise a major objection to the above procedure; it amounts to establishing first the dc bias, and only later the coupling between the barrier and the electrode. Physically, it is the reverse that is true; the transfer matrix elements are always there, and the dc bias is established afterwards; it is not obvious that the corresponding limits can be interchanged.

The major achievement of our current paper is that we can now construct an adiabatic NESS in the partition free setting; let us explain how. The leads are already coupled with the sample, and at \( t = -\infty \) the full system is in a Gibbs equilibrium state at a given temperature and chemical potential. Then we adiabatically turn on a potential bias \( V \chi(\eta t) \) between the leads, modeling in this way a gradual appearance of a difference in the chemical potentials (here \( \chi(-\infty) = 0, \chi(0) = 1 \) and \( \eta > 0 \) is the adiabatic parameter). The final bias \( V \) does not need to be small; our results are beyond the linear response theory. The statistical density matrix \( \rho(\eta)(t) \) is found as the solution of a quantum Liouville equation, with the initial condition at \( t = -\infty \) given by the global Gibbs state.

In Theorem \[15\] we show the existence and compute the strong limit \( \rho_{ad} := \lim_{\eta \to 0} \rho(\eta)(t) \). The limit is \( t \)-independent, and contains - as in the partitioned procedure- two contributions: one from the discrete, and one from the continuous subspaces. Note that we do not have to take the Cesàro limit in order to insure convergence for the discrete part. The adiabatic limit takes care of the oscillations. The price we pay is that we need to demand that the point spectrum of certain Hamiltonians only consists from finitely many discrete eigenvalues. Most probably this condition is too strong, and getting rid of it remains an interesting open problem.

Even though the stationary density matrix of the partitioned procedure has a similar structure, it is different from the one we construct here. A careful comparison will be given elsewhere.

A future problem is to investigate the charge current and establish Landauer-Büttiker type formulas \[6, 7, 2, 3, 14\] in the partition free setting with a continuous model, and without the linear response approximation. In fact this was the starting point of a number of remarkable physical papers, see for example \[17, 23, 4\]. A first mathematically sound derivation of the L-B formula on a discrete model and under the linear response approximation was obtained in \[12\] and further investigated in \[13\]. In \[11\] we significantly improved the method of proof of \[12\], which also allowed us to extend the results to the continuous case.

Another challenging open problem is to extend the formalism in order to accommodate transient regimes (see \[26, 20, 24, 14\] and references therein), and locally interacting fermions \[34, 35\].

Finally, we want to stress that some of the technical conditions which we impose for our model (like smoothness of boundaries and potentials, working with only two parallel leads) can
be relaxed. We chose though to work under stronger conditions in order to give shorter proofs for
certain spectral and asymptotic completeness results, thus making the paper rather self-consistent.
In this way, the number of generic assumptions is kept to a minimum.

1.2 The model

Take two identical semi-infinite cylinders and couple them smoothly through a finite domain. The
cylinders will model the leads, while the connecting domain will represent the region where the
interesting physics takes place. The total configuration space \( \mathcal{L} \) is a subset of \( \mathbb{R}^{d+1} \) with \( d \geq 0 \). In
order to simplify presentation, we will assume that \( \mathcal{L} \) is cylinder-like, which means that for each
value of the longitudinal coordinate \( x_{||} \in \mathbb{R} \) the transverse coordinate \( x_{\perp} \) belongs to a bounded
cross-section \( \mathcal{D}(x_{||}) \subset \mathbb{R}^{d} \). Again for the sake of simplicity, we assume that the boundary
\( \Sigma := \partial \mathcal{L} \) (1.1)
defines a regular \( C^\infty \)-surface embedded in \( \mathbb{R}^{d+1} \).

Let us start with the description of the configuration space associated to one of our \( d + 1 \)
dimensional leads, namely the left one. Let \( \tilde{a} > 0 \). We let \( \tilde{I}_{-} := (-\infty, -\tilde{a}) \) model its longitudinal
dimension. Then we assume that:
\[
\mathcal{L} \cap \{\tilde{I}_{-} \times \mathbb{R}^{d}\} =: \tilde{I}_{-} \times \mathcal{D},
\]
where the transverse section \( \mathcal{D} \subset \mathbb{R}^{d} \) is supposed to be a bounded and simply connected open set
with a regular \( C^\infty \)-boundary \( \partial \mathcal{D} \). Thus the configuration space of the left cylinder is modeled in
a natural way by the set \( \tilde{I}_{-} \times \mathcal{D} \). Similarly, if \( \tilde{I}_{+} := (\tilde{a}, \infty) \), the configuration space of the right
cylinder is modeled by \( \tilde{I}_{+} \times \mathcal{D} \).

Now define:
\[
\tilde{C} := \mathcal{L} \cap \{[-\tilde{a}, \tilde{a}] \times \mathbb{R}^{d}\}. \quad (1.2)
\]
Thus the small sample is contained by a bounded and simply connected set \( \tilde{C} \subset \mathbb{R}^{d+1} \) which is
smoothly glued to the two leads. With these notations, the one particle configuration space can be
decomposed as:
\[
\mathcal{L} = (\tilde{I}_{-} \times \mathcal{D}) \cup \tilde{C} \cup (\tilde{I}_{+} \times \mathcal{D}). \quad (1.3)
\]
When we refer to the "coupled system", we mean that there are no internal walls between the
sample and leads. A particle will be free to flow inside the system, and to pass from one lead to
another via the sample. But it is not allowed to get out of \( \mathcal{L} \).

Now let us introduce the one particle Hamiltonian of the coupled system. In the sample \( \tilde{C} \) we
assume the existence of a potential \( w \in C^\infty_0 (\tilde{C}) \), which will be considered positive without loss of
generality. The kinetic energy of a particle living in \( \mathcal{L} \) will be modeled by the Laplace operator
\( -\Delta_{\mathcal{D}} \) with Dirichlet boundary conditions on \( \partial \mathcal{L} \) and having the domain \( \mathbb{H}^{1}_D (\mathcal{L}) := \mathbb{H}^{1}\_0 (\mathcal{L}) \cap \mathbb{H}^{2}_D (\mathcal{L}) \).
Thus the one-particle Hamiltonian is of the form:
\[
H := -\Delta_{\mathcal{D}} + w, \quad (1.4)
\]
with the same domain.

Regarding the spectral properties of \( H \), we will prove in Lemma 3.14 that its singular continuous
spectrum is absent. We will assume that the pure point spectrum consists of discrete and finitely
many eigenvalues:
\[
\sigma_{pp}(H) = \sigma_{disc}(H), \quad \# \sigma_{pp}(H) < \infty. \quad (1.5)
\]

Remark 1.1. This assumption means in particular that we do not allow embedded eigenvalues in
the continuous spectrum. To our knowledge, sufficient conditions to guarantee this property are
not known in dimension \( d + 1 \geq 2 \).
Let $\mathcal{H} := L^2(L)$, and let $a > \tilde{a}$. Define
\[
L_- := L \cap \{(-\infty, -a) \times \mathcal{D}\}, \quad L_+ := L \cap \{(a, \infty) \times \mathcal{D}\}, \quad \mathcal{C} := L \cap \{(-a, a) \times \mathbb{R}^d\}.
\]
(1.6)

We introduce three orthogonal projections:
\[
\Pi_- : \mathcal{H} \rightarrow \Pi_- := L^2(L_-), \quad \Pi_+ : \mathcal{H} \rightarrow \Pi_+ := L^2(L_+),
\]
\[
\Pi_0 : \mathcal{H} \rightarrow \Pi_0 := L^2(\mathcal{C}).
\]
(1.7)

Note that $\bar{\mathcal{C}}$ is completely included in the open set $\mathcal{C}$, and $L_{\pm}$ are “shorter” than the corresponding leads.

1.3 The state and the Liouville equation

We only work at the level of density matrices. In the remote past $t \rightarrow -\infty$ the electron gas is at equilibrium at a temperature $T > 0$ and a chemical potential $\mu$, moving in all the volume $L$. The gas is described by a quasi-free state, having as two-point function the usual Fermi-Dirac equilibrium density matrix operator:
\[
\rho_{eq}(H) := \frac{1}{1 + e^{(H - \mu)/kT}}.
\]
(1.8)

The system is driven out of equilibrium by slowly turning on an electric bias
\[
V = v_- \Pi_- + v_+ \Pi_+,
\]
where $v_{\pm}$ are real constants. We want to introduce the bias adiabatically with an adiabatic parameter $\eta > 0$, as a time-dependent potential $V_\eta(t) := \chi(\eta t) V$. One should have in mind $\chi(t) = e^t$, but only a few abstract properties of this function are really needed, namely:
\[
0 < \chi(t) < 1 \quad \text{and} \quad \chi'(t) > 0 \quad \text{if} \quad t < 0; \quad \chi(0) = 1; \quad \chi', |\chi''| \in L^1(\mathbb{R}_-).
\]
(1.10)

We will also need to consider the ‘bias’ with a fixed coupling constant $\kappa \in [0, 1]$. We introduce a family of operators:
\[
H(\kappa) := H + \kappa V.
\]
(1.11)

$E_{pp}(A)$ and $E_{ac}(A)$ will denote respectively the projector on the pure point and absolutely continuous spectral subspace of the self-adjoint operator $A$. In Lemma 3.14 we will prove that the singular continuous spectrum of $K(\kappa)$ is empty. We now make the following assumptions concerning the point spectrum:

**Hypothesis 1.2.**

1. $\forall \kappa \in [0, 1]$ the Hamiltonian $K(\kappa)$ has no eigenvalues embedded in the continuous spectrum;
2. $\dim E_{pp}(K(\kappa)) = N < \infty$, $\forall \kappa \in [0, 1]$, $\sigma_{pp}(K(\kappa)) = \{\varepsilon_j(\kappa)\}_{j=1}^N$;
3. $\min_{\kappa \in [0, 1]} \{\operatorname{dist}(\sigma_{pp}(K(\kappa)), \sigma_{ac}(K(\kappa)))\} \geq d > 0$.

In order to simplify the presentation, we will only work with $N = 2$ and adopt an extra assumption:

**Hypothesis 1.3.**

The eigenvalues $\{\varepsilon_j(\kappa)\}_{j \in \{1, 2\}}$ (which are real analytic functions of $\kappa \in [0, 1]$) can cross at most at one point $\kappa_0 \in (0, 1)$. This $\kappa_0$ corresponds to some unique $t_0 < 0$ where $\chi(t_0) = \kappa_0$ and $\chi'(t_0) > 0$. 


The time dependent Hamiltonian will be
\[ K(\chi(\eta t)) := H + \chi(\eta t)V, \quad (1.12) \]
having the constant domain equal to the domain of \( H \), i.e. \( \mathcal{H}_D(L) \). The evolution defined by the time-dependent Hamiltonian \( K(\chi(\eta t)) \) is described by a unitary propagator \( W_\eta(t) \), solution of the following Cauchy problem:
\[ \begin{cases} i\partial_t W_\eta(t) = K(\chi(\eta t))W_\eta(t) \\ W_\eta(0) = 1, \end{cases} \quad (1.13) \]
for \( t \in \mathbb{R} \). For any \( \eta > 0 \), the family \( \{K(\chi(\eta t))\}_{t \in \mathbb{R}} \) consists of self-adjoint operators in \( \mathcal{H} \) having a common domain equal to \( \mathcal{H}_D(L) \) and strongly differentiable with respect to \( t \in \mathbb{R} \) with a bounded self-adjoint norm derivative \( \partial_t K(\chi(\eta t)) = \eta \chi'(\eta t)V \).

Now using well known results quoted in [31, Th. X.70] we easily obtain that the problem (1.13) has a unique solution which is unitary and leaves the domain \( \mathcal{H}_D(L) \) invariant for any \( t \in \mathbb{R} \). Moreover, its adjoint satisfies the equation:
\[ i\partial_t W^*_\eta(t) = -W^*_\eta(t)K(\chi(\eta t)). \quad (1.14) \]
The object we are interested in is the time evolved density matrix \( \rho_\eta(t) \) which must be a solution of the Liouville equation, starting from the initial value \( \rho_{\eta_0}(H) \) at \( t \to -\infty \):
\[ i\partial_t \rho_\eta(t) = [K(\chi(\eta t)), \rho_\eta(t)], \quad n - \lim_{t \to -\infty} \rho_\eta(t) = \rho_{\eta_0}(H). \quad (1.15) \]
In the remaining part of our paper we will show that the unique solution \( \rho_\eta(t) \) of the Liouville equation has a strong limit when \( \eta \searrow 0 \), and compute it. In particular, we will see that the adiabatic limit is \( t \) independent.

### 1.4 The main result

In order to formulate our main result we need to define some new objects. First, we introduce the decoupled Hamiltonian obtained from \( H \) by introducing Dirichlet walls where the bias is discontinuous \( (x_i = \pm a) \). Remember that the decomposition (1.6) depends on \( a \), and the walls are inside the leads. Let \( \tilde{\Delta}_D \) be the self-adjoint Laplace operator defined in \( L^2(L) \) with Dirichlet conditions on \( \partial L_\pm \cup \partial C \); we have \( \Delta_D = \tilde{\Delta}_D, - \oplus \tilde{\Delta}_D, 0 \oplus \tilde{\Delta}_D, +, \) where their domains are denoted as follows:
\[ \mathcal{H}_D^0(L) := H^1_0(L) \cap H^2(L), \quad \mathcal{H}_D(L) := H^1_0(L) \cap H^2(L), \quad \mathcal{H}_D^0(L) := \mathcal{H}_D^0(L) \oplus \mathcal{H}_D^0(C) \ominus \mathcal{H}_D(L). \quad (1.16) \]

Let us note that due to the cylindrical symmetry of the regions \( L_\pm \) where the bias is piecewise constant, we can write
\[ \tilde{\Delta}_{D, \pm} = \mathcal{I}_\pm \otimes 1 + 1 \otimes \mathcal{L}_D \quad (1.17) \]
with \( \mathcal{L}_D \) the Laplacean on the bounded domain \( D \subset \mathbb{R}^d \) with Dirichlet conditions on the boundary \( \partial D \), and \( \mathcal{I}_\pm \) the operator of second derivative on \( \mathcal{I}_\pm \) with Dirichlet condition at \( \pm a \). The decoupled one particle Hamiltonian will be:
\[ \tilde{H} := -\tilde{\Delta}_D + w, \quad (1.18) \]
which is self-adjoint on the domain \( \mathcal{H}_D^0(L) \), having Dirichlet conditions on \( \partial L_\pm \cup \partial C \). As in the coupled case, we need to consider the bias with a fixed coupling constant \( \kappa \in [0, 1] \) and define \( \tilde{K}(\kappa) := \tilde{H} + \kappa V \). In order to formulate our main theorem we need the following lemma:

**Lemma 1.4.** The stationary wave operator \( \Xi_0 \) associated to the pair \( \{\tilde{K}(1), K(1)\} \):
\[ \Xi_0 := s - \lim_{s \searrow -\infty} e^{is\tilde{K}(1)}e^{-isK(1)}E_{\text{ac}}(\tilde{K}(1)). \]
exists and is a unitary operator from \( E_{\text{ac}}(\hat{K}(1))\mathcal{H} \) to \( E_{\text{ac}}(K(1))\mathcal{H} \). Moreover, the singular continuous spectrum of \( K(\kappa) \) is empty for all \( \kappa \in [0, 1] \).

And here is the main result:

**Theorem 1.5.** The adiabatic limit of the density matrix exists in the strong operator topology on \( \mathcal{B}(\mathcal{H}) \), is independent of \( t \) and given by:

\[
\rho_{ad} := s - \lim_{\eta \searrow 0} \rho_\eta(t) = \Xi_0 \rho_{eq}(\hat{H}) \Xi_0^* + \sum_{j=1}^N \rho_{eq}(\varepsilon_j(0)) E_j(K(1)),
\]

where \( \{\varepsilon_j(0)\}_{j=1}^N \) are the eigenvalues of \( H = K(0) \) in ascending order, while \( \{E_j(K(1))\}_{j=1}^N \) are the eigenprojections of \( H + V = K(1) \) obtained by analytically continuing \( \{E_j(K(\kappa))\}_{j=1}^N \) from \( \kappa = 0 \) to \( \kappa = 1 \).

**Remark 1.6.** Even though Lemma [1.4] is not surprising, its proof is not straightforward.

We also note that the adiabatic limit \( \rho_{ad} \) commutes with \( K(1) = H + V \), but it is not a function of \( K(1) \). Even though \( \rho_\eta(t) \) is a solution of a Liouville equation involving operators with no internal Dirichlet boundaries at \( \pm a \), the limit \( \rho_{ad} \) is expressed with the help of a comparison operator \( H + V \), depending on \( a \), and which appears naturally in the proof.

We will assume \( N = 2 \), but the result holds true for any finite \( N \). An interesting open problem is to study the case \( N = \infty \) and when the eigenvalues can enter the continuous spectrum while \( \kappa \) grows from 0 to 1. Another interesting situation is the one in which we have a degeneracy at \( \kappa = 0 \); this situation is related to the Gell-Mann and Low theorem for degenerate unperturbed states [30].

### 1.5 A useful expression of the density matrix

Before actually starting the study of the adiabatic limit, let us very quickly show that [1.15] has a solution, which can be put into a form which is particularly convenient for taking the adiabatic limit.

Define the unitary adiabatic wave operators

\[
\omega_\eta := n - \lim_{t \to -\infty} W_\eta^*(t)e^{-itH}, \quad \omega_\eta^* = n - \lim_{t \to -\infty} e^{itH}W_\eta(t).
\]

They converge in norm due to the following estimate \( (s < t) \):

\[
\|W_\eta^*(t)e^{-itH} - W_\eta^*(s)e^{-isH}\| \leq \int_s^t \left\| \frac{d}{d\tau} \{W_\eta^*(\tau)e^{-i\tau H}\} \right\| d\tau \leq \|V\| \int_s^t \chi(\eta\tau),
\]

where we use that \( \chi \in L^1(\mathbb{R}_-) \). Then by direct computation we can prove that the operator

\[
\rho_\eta(t) := W_\eta(t)\omega_\eta\rho_{eq}(H)\omega_\eta^*W_\eta^*(t)
\]

solves the Liouville equation. It also obeys the initial condition because we can write:

\[
0 = \lim_{t \to -\infty} \|\rho_\eta(t) - e^{-itH} \{\omega_\eta \rho_{eq}(H)\omega_\eta^* W_\eta^*(t)e^{-itH}\} e^{itH}\|
\]

\[
= \lim_{t \to -\infty} \|\rho_\eta(t) - e^{-itH} \rho_{eq}(H)e^{itH}\| = \lim_{t \to -\infty} \|\rho_\eta(t) - \rho_{eq}(H)\|.
\]

The above solution can be rewritten as:

\[
\rho_\eta(t) = W_\eta(t)\rho_\eta(0)W_\eta^*(t),
\]

where

\[
\rho_\eta(0) = \omega_\eta \rho_{eq}(H)\omega_\eta^*.
\]
Now let us show that it is enough to prove (1.19) for \( t = 0 \). Indeed, once this formula is proved for \( t = 0 \) it shows that the strong limit of \( \rho_\eta(0) \) when \( \eta \searrow 0 \) is commuting with \( K(1) = H + V \).

It is elementary to check that \( W_\eta(t) \) and \( W^{*}_\eta(t) \) converge in norm to \( e^{-itK(1)} \) and respectively \( e^{itK(1)} \) when \( \eta \searrow 0 \) (with \( t \) fixed). Since \( e^{\pm itK(1)} \) commutes with \( s - \lim \rho_\eta(0) \) it follows that the adiabatic strong limit of \( \rho_\eta(t) \) must also exist and equal the r.h.s of (1.19).

Moreover, due to the fact that the limits in (1.20) are in operator norm, it is easy to show that we have the identity:

\[
\rho_\eta(0) = n - \lim_{s \to -\infty} W^{*}_\eta(s)\rho_{eq}(H)W_\eta(s). 
\tag{1.26}
\]

It is important to note that the above norm limit is not uniform in \( \eta \), and this is the reason why the adiabatic limit is not straightforward. Formula (1.26) will be the starting point in what follows, and we will be interested in computing the double limit:

\[
\rho_{ad} = s - \lim_{\eta \searrow 0} \rho_\eta(0) = s - \lim_{\eta \searrow 0} \left\{ n - \lim_{s \to -\infty} W^{*}_\eta(s)\rho_{eq}(H)W_\eta(s) \right\}. 
\tag{1.27}
\]

### 2 A road map of the proof of the adiabatic limit

Since our proof of the adiabatic limit is quite long, in this section we will give a list of technical results leading to it and postpone their proofs for the next sections.

The two terms of (1.19) are coming from different spectral subspaces of \( H + V \): the first one from the absolutely continuous spectrum, and the second one from the discrete spectrum.

In Lemma 3.14 we will prove the absence of singular continuous spectrum for \( K(\kappa) \), thus we can consider the orthogonal decompositions

\[
\mathcal{H} = E_{ac}(\kappa)\mathcal{H} \oplus \left\{ \bigoplus_{1 \leq j \leq N} E_j(\kappa)\mathcal{H} \right\}, \tag{2.1}
\]

where \( E_{ac}(\kappa) := E_{ac}(K(\kappa)) \) and \( E_j(\kappa) := E_j(K(\kappa)) \). Let us remark here the important fact that due to the Rellich Theorem (Theorem II.61 in [20]) we can choose the eigenprojections of \( K(\kappa) \) to be real analytic functions of \( \kappa \) on the interval \([0, 1]\). Then we can write

\[
W^{*}_\eta(s)\rho_{eq}(H)W_\eta(s) = W^{*}_\eta(s)\rho_{eq}(H)E_{ac}(0)W_\eta(s) + \left\{ \sum_{1 \leq j \leq N} \rho_{eq}(\varepsilon_j(0)) W^{*}_\eta(s)E_j(0)W_\eta(s) \right\}.
\]

We will separately take the double limit as in (1.27) for both above terms.

### 2.1 The contribution of the discrete spectrum

Let us start our analysis with the pure-point part and compute

\[
s - \lim_{\eta \searrow 0} \left[ n - \lim_{s \to -\infty} W^{*}_\eta(s)E_j(0)W_\eta(s) \right].
\]

As \( V \) is a bounded analytic perturbation of \( H \), the map \([0, 1] \ni \kappa \mapsto E_j(\kappa) \) is - in particular - Lipschitz continuous in the uniform topology. Thus there exists a constant \( C > 0 \) such that:

\[
\|W^{*}_\eta(s)E_j(0)W_\eta(s) - W^{*}_\eta(s)E_j(\chi(\eta s))W_\eta(s)\| \leq C\chi(\eta s), \quad s \leq 0.
\tag{2.2}
\]

Thus we can replace \( E_j(0) \) with the analytically continued projection \( E_j(\chi(\eta s)) \) and the limit does not change. We will prove the following result (a weaker version of the gap-less adiabatic theorem, see [30] and references therein):
Proposition 2.7. Under our Hypothesis 1.2 the following limit exists in the uniform topology and we have the equality:

\[ n - \lim_{\eta \to 0} \left[ n - \lim_{s \to -\infty} W^*_\eta(s)E_j(\chi(\eta s))W_\eta(s) \right] = E_j(1), \]

which combined with immediately gives:

Corollary 2.8.

\[ n - \lim_{\eta \to 0} \left[ n - \lim_{s \to -\infty} W^*_\eta(s)E_j(0)W_\eta(s) \right] = E_j(1) \tag{2.3} \]

and

\[ n - \lim_{\eta \to 0} \left[ n - \lim_{s \to -\infty} e^{isH}E_{ac}(H)W_\eta(s)E_{pp}(K(1)) \right] = 0. \tag{2.4} \]

While concludes the proof of the adiabatic limit for the discrete part of the spectrum (even in the uniform topology), the limit in is a technical result which will play a role in the contribution of the continuous spectrum.

2.2 The contribution of the continuous spectrum

We will now focus our attention on the term coming from the absolutely continuous part of the spectrum:

\[ s - \lim_{\eta \to 0} \left[ s - \lim_{s \to -\infty} W^*_\eta(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s) \right]. \tag{2.5} \]

Due to we may conclude that

\[
\begin{align*}
& s - \lim_{\eta \to 0} \left[ s - \lim_{s \to -\infty} W^*_\eta(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s) \right] \\
& = s - \lim_{\eta \to 0} \left[ s - \lim_{s \to -\infty} W^*_\eta(s)E_{ac}(H)e^{-isH}\rho_{eq}(H)e^{isH}E_{ac}(H)W_\eta(s) \right] \\
& = s - \lim_{\eta \to 0} \left[ s - \lim_{s \to -\infty} E_{ac}(K(1))W^*_\eta(s)\rho_{eq}(H)E_{ac}(H)W_\eta(s)E_{ac}(K(1)) \right], \tag{2.6}
\end{align*}
\]

provided that the last double strong limit exists. Note that all errors go to zero in the uniform norm.

The next step in the proof is to replace \( \rho_{eq}(H) \) with \( \rho_{eq}(\dot{H})E_{ac}(\dot{H}) \) in (2.6). In order to show that we can do that replacement, let us write the identity:

\[ \{\rho_{eq}(\ddot{H})E_{ac}(\ddot{H}) - \rho_{eq}(H)\} E_{ac}(H)W_\eta(s) \tag{2.7} \]

\[ = -\rho_{eq}(H)E_{pp}(H)e^{-isH}E_{ac}(H) \{e^{isH}W_\eta(s)\} + \{\rho_{eq}(\dot{H}) - \rho_{eq}(H)\} e^{-isH}E_{ac}(H) \{e^{isH}W_\eta(s)\}. \]

When \( s \to -\infty \) both terms on the right hand side converge to zero due to the fact that \( e^{isH}W_\eta(s) \) is convergent in the operator norm, \( Ce^{-istA}P_{ac}(A) \) converges strongly to 0 for any A selfadjoint and \( C \) compact and using the fact that \( \rho_{eq}(\dot{H})E_{pp}(\dot{H}) \) is compact and the following result (see §5.1 for the proof):

Proposition 2.9. For any continuous function \( \Phi \in C(\mathbb{R}) \) which tends to zero to infinity, we have that \( \Phi(\dot{H}) - \Phi(\ddot{H}) \) is a compact operator.
Up to now we have shown that the limit in (2.10) must equal:

\[
\begin{align*}
s - \lim \eta \rightarrow 0 \left\{ s - \lim_{s \rightarrow -\infty} E_{ac}(K(1)) W_\eta^*(s) E_{ac}(H) \rho_{eq}(H) E_{ac}(H) W_\eta(s) E_{ac}(K(1)) \right\}. 
\end{align*}
\]  

(2.8)

For the next step we will need a comparison dynamics for \( W_\eta(t) \), generated by the operator with internal Dirichlet walls. To the decoupled Hamiltonian we can associate:

\[
\hat{K}(\chi(\eta t)) := \hat{H} + \chi(\eta t)V.
\]  

(2.9)

The associated evolution \( \hat{W}_\eta(t) \) is defined as the solution of the following Cauchy problem:

\[
\begin{align*}
\begin{cases}
    i\partial_t \hat{W}_\eta(t) = \hat{K}(\eta t) \hat{W}_\eta(t) \\
    \hat{W}_\eta(0) = 1
\end{cases}
\end{align*}
\]  

(its existence results by arguments similar to those concerning the existence of \( W_\eta(t) \)).

An important observation is the fact that \( \hat{\Delta}_D \) commutes with \( V \) so that we have

\[
\hat{W}_\eta(t) = e^{-i t \hat{H}} \left[ 1 + \Pi_- \left( e^{-i v_- \int_0^1 \chi(\eta u) du} - 1 \right) \right] + \Pi_+ \left( e^{-i v_+ \int_0^1 \chi(\eta u) du} - 1 \right)
\]  

(2.10)

with the exponentials in the second factor being just complex numbers. All terms commute which each other. Therefore the limit in (2.8) must equal:

\[
\begin{align*}
s - \lim \eta \rightarrow 0 \left\{ s - \lim_{s \rightarrow -\infty} E_{ac}(K(1)) W_\eta^*(s) E_{ac}(H) \hat{W}_\eta(s) E_{ac}(H) W_\eta(s) E_{ac}(K(1)) \right\}. 
\end{align*}
\]  

(2.11)

We state a result which will be proved later (see § 5.1):

**Proposition 2.10.** The following limits exist in the strong operator topology:

\[
\Xi_\eta := \lim_{s \rightarrow -\infty} E_{ac}(K(1)) W_\eta^*(s) E_{ac}(H) \hat{W}_\eta(s) E_{ac}(H).
\]  

(2.12)

One can see that the product of operators in the limit (2.12) coincides with the product of operators placed at the left of \( \rho_{eq}(H) \) in (2.11). At the same time, at the right of \( \rho_{eq}(H) \) is the adjoint of the same product.

Now if we can prove that \( \Xi_\eta^* \) can be written in the following way:

\[
\Xi_\eta^* = s - \lim_{s \rightarrow -\infty} E_{ac}(H) W_\eta^*(s) E_{ac}(H) W_\eta(s) E_{ac}(K(1)),
\]  

(2.13)

then the limit \( s \rightarrow -\infty \) in (2.11) would give:

\[
\Xi_\eta \rho(\hat{H}) \Xi_\eta^*.
\]  

(2.14)

Indeed, since Proposition 2.10 implies the existence of the weak limit:

\[
\Xi_\eta^* = w - \lim_{s \rightarrow -\infty} E_{ac}(H) W_\eta^*(s) E_{ac}(H) W_\eta(s) E_{ac}(K(1)),
\]  

then (2.13) holds if we can prove the existence of a strong limit. Now in order to prove that a strong limit exists, let us insert some operators in the following way:

\[
\begin{align*}
E_{ac}(H) W_\eta(s) e^{-i s H} E_{ac}(K(1)) \\
eq E_{ac}(H) W_\eta(s) e^{-i s H} \left\{ e^{i s H} W_\eta(s) \right\} E_{ac}(K(1)) \\
= E_{ac}(H) \left\{ W_\eta(s) e^{-i s H} \left\{ e^{i s H} e^{-i s H} E_{ac}(H) \right\} \right\} E_{ac}(K(1)).
\end{align*}
\]  

(2.15)
Let us investigate each curly bracket. The couple \( e^{is\hat{H}}W_\eta(s) \) converges in norm to \( \omega^*_\eta \) when \( s \to -\infty \). The factor \( \hat{W}_\eta^*(s)e^{-is\hat{H}} \) converges in norm too, see (2.11). Finally, the factor \( e^{is\hat{H}}e^{-isH}E_{ac}(H) \) converges strongly to the wave operator associated to the pair of Hamiltonians \( \{ \hat{H}, H \} \) as stated by the following proposition which we will prove later:

**Proposition 2.11.** The wave operator \( \omega^* := s - \lim_{s \to -\infty} e^{is\hat{H}}e^{-isH}E_{ac}(H) \) exists as a unitary map from \( E_{ac}(H)\mathcal{H} \) onto \( E_{ac}(\hat{H})\mathcal{H} \) and one has:

\[
s - \lim_{s \to -\infty} e^{is\hat{H}}e^{-isH}E_{ac}(H) = \omega^* = E_{ac}(H)\omega^*_\eta.
\]

Now we can introduce (2.12) and (2.13) in (2.11), and see that the contribution coming from the continuous part of the spectrum will be:

\[
s - \lim_{\eta \to 0} \Xi_\eta \rho(H)\Xi_\eta^*.
\]

(2.16)

The next step in our strategy is to prove that the strong limits of \( \Xi_\eta \) and \( \Xi_\eta^* \) exist when \( \eta \to 0 \), and they will equal the wave operators associated to the pair of Hamiltonians \( \{ \hat{K}(1), K(1) \} \). First, we need to be sure that these limiting operators exist and are complete, and this is stated by the following proposition:

**Proposition 2.12.**

1. For any \( \kappa \in [0, 1] \) we have \( E_{ac}(\hat{K}(\kappa)) = E_{ac}(H) \).
2. The following limits exist:

\[
s - \lim_{s \to -\infty} e^{s\hat{K}(1)}e^{-isK(1)}E_{ac}(H) =: \Xi_0 = E_{ac}(K(1))\Xi_0 E_{ac}(H);
\]

\[
s - \lim_{s \to -\infty} e^{s\hat{K}(1)}e^{-isK(1)}E_{ac}(K(1)) = \Xi_0^* = E_{ac}(H)\Xi_0^* E_{ac}(K(1)).
\]

(2.17)

Thus the wave operators associated to the pair \( \{ \hat{K}(1), K(1) \} \) exist and are complete.

The next technical result establishes the adiabatic limit for the wave operators \( \Xi_\eta \); note that Dollard [10] investigated a related problem in the case of short range and relatively bounded perturbations.

**Proposition 2.13.** \( \Xi_\eta \) has a strong limit when \( \eta \to 0 \) and moreover \( s - \lim_{\eta \to 0} \Xi_\eta = \Xi_0 \), where \( \Xi_0 \)

is the stationary wave operator associated to the pair \( \{ \hat{K}(1), K(1) \} \) and is unitary as a map from \( E_{ac}(\hat{H}) \) onto \( E_{ac}(K(1)) \).

We see that the very last thing to be shown in order to finish the computation of the adiabatic limit in (2.11), is the strong convergence of \( \Xi_\eta^* \) to \( \Xi_0^* \) when \( \eta \to 0 \). Due to the completeness of the wave operator \( \Xi_0 \) (point (2) in Proposition 2.12), we have that \( \Xi_0^* : E_{ac}(K(1))\mathcal{H} \to E_{ac}(H)\mathcal{H} \) is a unitary operator. Then:

\[
\| (\Xi_\eta^* - \Xi_0^*) f \|_{\mathcal{H}}^2 \leq 2\| f \|_{\mathcal{H}}^2 - 2\Re( \langle \Xi_\eta \Xi_0^* f, f \rangle ) \to_{\eta \to 0} 2\| f \|_{\mathcal{H}}^2 - 2\Re( \langle \Xi_0 \Xi_0^* f, f \rangle ) = 0
\]

for any \( f \in E_{ac}(K(1))\mathcal{H} \) and thus we have strong convergence of \( \Xi_\eta^* \) to \( \Xi_0^* \) when \( \eta \to 0 \) on \( E_{ac}(K(1))\mathcal{H} \).
With this, the proof of the adiabatic limit in (1.19) is concluded.

The next sections of the paper are devoted to the proofs of the above stated Propositions 2.7 and Corollary 2.8.

3 Absence of singular continuous spectrum

We give here the proof of the absence of the singular continuous spectrum for $K(\kappa)$ by establishing a limiting absorption principle. The main technical result of this section is the following lemma:

**Lemma 3.14.** Let $\kappa \in [0, 1]$. There exists a discrete set $\mathcal{N} \subset \mathbb{R}$ such that for any closed interval $I \subset \mathbb{R} \setminus \mathcal{N}$ we have the estimate (here $\langle x \rangle := \sqrt{x^2 + 1}$):

$$\sup_{z \in \{x + iy : x \in I, 0 < y < \delta\}} \left\| e^{-\langle Q_1 \rangle} R_\kappa(z) e^{-\langle Q_1 \rangle} \right\| \leq C(I, \delta, \kappa) < \infty. \quad (3.1)$$

In particular, $K(\kappa)$ has no singular continuous spectrum.

**Proof.** We use geometric perturbation theory. Let us define a quadratic partition of unity in the following way:

$$\chi^2 + \chi_0^2 + \chi_+^2 = 1, \quad \chi_\pm \in C^\infty(\mathbb{R}), \quad \chi_\pm(x) = 1 \text{ for } \pm x > 2a, \quad \chi_\pm(x) = 0 \text{ for } |x| < a$$

$$\chi_0 \in C^\infty(\mathbb{R}), \quad \chi_0(x) = 0 \text{ for } |x| > 2a, \quad \chi_0(x) = 1 \text{ for } |x| < a.$$  

Fix some $L > 2a$. Introduce the operator $K_{\kappa,L}$ obtained from $K(\kappa)$ on the region $L \cap (-L, L)$ by imposing Dirichlet boundary conditions at $x = \pm L$. The operator $K_{\kappa,L}$ has compact resolvent, and let us denote it with $R_{\kappa,L}(z)$. Here $z \in \mathbb{C} \setminus \sigma(K_{\kappa,L})$.

Now let us define an approximation for $R_\kappa(z)$ by the following formula:

$$\tilde{R}_\kappa(z) := \chi_-(Q_1) R_{\kappa,L}(z) \chi_-(Q_1) + \chi_0(Q_1) R_{\kappa,L}(z) \chi_0(Q_1) + \chi_+(Q_1) R_{\kappa,L}(z) \chi_+(Q_1).$$

Note that on the support of $\chi_\pm(Q_1)$ the differential operators $K(\kappa)$ and $\tilde{K}(\kappa)$ coincide, while on the support of $\chi_0(Q_1)$ the operators $K(\kappa)$ and $K_{\kappa,L}$ coincide, so that we can write

$$(K(\kappa) - z) \tilde{R}_\kappa(z) = \text{Id} + \left[ \tilde{H}, \chi_-(Q_1) \right] \tilde{R}_\kappa(z) \chi_-(Q_1) + [H, \chi_0(Q_1)] R_{\kappa,L}(z) \chi_0(Q_1)$$

$$+ \left[ \tilde{H}, \chi_+(Q_1) \right] \tilde{R}_\kappa(z) \chi_+(Q_1). \quad (3.2)$$

The above commutators are first order differential operators:

$$\left[ \tilde{H}, \chi_\pm(Q_1) \right] = -2i \chi_\pm'(Q_1) P_1 - \chi_\pm''(Q_1),$$

$$[H, \chi_0(Q_1)] = -2i \chi_0'(Q_1) P_1 - \chi_0''(Q_1). \quad (3.3)$$

Thus (3.2) can be put in the following form:

$$(K(\kappa) - z) \tilde{R}_\kappa(z) = \text{Id} + X(z), \quad e^{\langle Q_1 \rangle} X(z) \in \mathcal{B}(\mathcal{H}), \quad (3.4)$$

where the boundedness of $e^{\langle Q_1 \rangle} X(z)$ is due to the compact support of the functions appearing on the left-hand side of the operator $X(z)$. Thus we can write the identity:

$$e^{-\langle Q_1 \rangle} R_\kappa(z) e^{-\langle Q_1 \rangle} = e^{-\langle Q_1 \rangle} \tilde{R}_\kappa(z) e^{-\langle Q_1 \rangle} - e^{-\langle Q_1 \rangle} R_\kappa(z) X(z) e^{-\langle Q_1 \rangle}.$$
In what follows we will investigate how the norm of $e^{(Q_\eta)X(z)}$ tends to 0, we can write at least for those values of $z$ that:

$$e^{-\langle Q_\eta \rangle} R_{\eta}(z) e^{-\langle Q_\eta \rangle} = e^{-\langle Q_\eta \rangle} \overline{R_{\eta}(z)} e^{-\langle Q_\eta \rangle} \left[ 1 + e^{\langle Q_\eta \rangle} X(z) e^{-\langle Q_\eta \rangle} \right]^{-1}.$$ 

Now $e^{\langle Q_\eta \rangle} X(z) e^{-\langle Q_\eta \rangle}$ is compact and analytic in the upper complex plane, and has a bounded limit from above on any interval $I$ which avoids the discrete set of thresholds in the leads and the discrete spectrum of $K_{\kappa,L}$. Moreover, due to the exponential decaying weight on the right and the compactly supported cut-offs on the left, $e^{\langle Q_\eta \rangle} X(z) e^{-\langle Q_\eta \rangle}$ can be analytically continued to the set $\{x + iy | x \in I, -\delta < y < \delta\}$ for $\delta$ small enough. Thus we can then apply the analytic Fredholm alternative on this set and conclude that $\left[ 1 + e^{\langle Q_\eta \rangle} X(z) e^{-\langle Q_\eta \rangle} \right]^{-1}$ exists on $I$ outside a discrete set of points.

## 4 Adiabatic limit of the discrete subspace

In order to simplify our presentation, we adopt the conditions of Hypothesis [13] which means that we have $N = 2$ discrete eigenvalues which might cross at only one point when $\kappa$ varies. Moreover, they remain well isolated from the continuous spectrum. Under these conditions, Rellich’s Theorem (Theorem II.61 in [20]) states that the two eigenvalues are given by two real analytic functions $\{\epsilon_j(\kappa)\}_{j \in \{1,2\}}$ defined for $\kappa \in [0,1]$. If they cross at $\kappa_0 \in (0,1)$ and only there, then there must exist two constants $C > 0, M \in \mathbb{N}^*$ such that:

$$|\epsilon_1(\kappa) - \epsilon_2(\kappa)| \geq C|\kappa - \kappa_0|^M, \quad \kappa \in [0,1].$$ \hspace{1cm} (4.1)

Moreover, their corresponding orthogonal projections $E_j(\kappa)$ can also be chosen to be real analytic on $[0,1]$.

### 4.1 Proof of Proposition [2.7]

Let us focus on $j = 1$. We will have to show the equality:

$$E_1(1) = n - \lim_{\eta \searrow 0} \left[ n - \lim_{s \searrow -\infty} B_\eta(s) \right], \quad \text{with} \quad B_\eta(s) := W_\eta(s)^* E_1(\chi(\eta s)) W_\eta(s).$$ \hspace{1cm} (4.2)

This follows clearly from the next result.

**Lemma 4.15.** We have:

$$B_\eta(0) = E_1(\chi(0)) = E_1(1) \quad \text{and} \quad \lim_{\eta \searrow 0} \left\{ \sup_{s \leq 0} \| B_\eta(s) - B_\eta(0) \| \right\} = 0.$$ \hspace{1cm} (4.3)

**Proof.** The first two equalities are obvious. For the limit let us remember that there exists a unique critical time $t_0 < 0$ when $\chi(t_0) = \kappa_0$ which corresponds to the intersection of the two eigenvalues. Fix some $0 < \delta < 1$ (to be chosen later in a more precise way). We split the negative semi-axis $\mathbb{R}_-$ in three parts:

$$\mathbb{R}_- = \left( -\infty, \frac{t_0 - \eta^2}{\eta} \right] \cup \left[ \frac{t_0 - \eta^2}{\eta}, \frac{t_0 + \eta^2}{\eta} \right] \cup \left[ \frac{t_0 + \eta^2}{\eta}, 0 \right].$$ \hspace{1cm} (4.4)

In what follows we will investigate how $B_\eta(\cdot)$ changes when $s$ goes through each sub-interval.

**Near the crossing:** Let us first consider the interval in the middle $\left[ \frac{t_0 - \eta^2}{\eta}, \frac{t_0 + \eta^2}{\eta} \right]$. This is the "gapless region", but nevertheless, it is easiest to deal with. From the definition of $B_\eta(s)$ in (4.2), and since $K(\kappa)$ commutes with $E_1(\kappa)$, we have the important identity:

$$\partial_s B_\eta(s) = \eta \chi'(\eta s) W_\eta^*(s) E_1(\chi(\eta s)) W_\eta(s),$$ \hspace{1cm} (4.5)
where \( E'_1(\kappa) \) is uniformly bounded in \( \kappa \in [0, 1] \) due to the real analyticity of the projector. We write:

\[
B_\eta \left( \frac{t_0 + \eta \delta}{\eta} \right) - B_\eta \left( \frac{t_0 - \eta \delta}{\eta} \right) = \int_{t_0 - \eta \delta}^{t_0 + \eta \delta} \partial_s B_\eta(s) ds. \tag{4.6}
\]

This implies:

\[
\left\| B_\eta \left( \frac{t_0 + \eta \delta}{\eta} \right) - B_\eta \left( \frac{t_0 - \eta \delta}{\eta} \right) \right\| \leq C \eta \delta. \tag{4.7}
\]

**Outside the crossing:** In the other two intervals the eigenvalue \( \varepsilon_1(\chi(\eta s)) \) is isolated from the rest of the spectrum, as can be inferred from our Hypothesis 1.2 and 1.3. More precisely, let us show that it is situated at a distance larger than \( C \eta^{M_\delta} \) than the rest of the spectrum. Indeed, using the splitting from (4.1) we may write

\[
|\varepsilon_1(\chi(\eta s)) - \varepsilon_2(\chi(\eta s))| \geq C|\chi(\eta s) - \chi(t_0)|^M \geq \tilde{C} \eta^{M_\delta}
\]

for every \( s \) situated at a distance larger than \( \eta^{-1+\delta} \) from \( t_0/\eta \). Here \( \tilde{C} > 0 \) can be chosen uniformly in \( \eta \) because we assumed that \( \chi'(t) > 0 \) and \( |\chi''| \) is integrable.

It means that we can find a positively oriented simple contour \( \Gamma_\eta \) which only contains \( \varepsilon_1(\chi(\eta s)) \) and the following estimate holds true:

\[
D_\eta := \sup_{s \in \mathbb{R}_-} \sup_{\left[ \frac{t_0 - \eta \delta}{\eta}, \frac{t_0 + \eta \delta}{\eta} \right]} \sup_{z \in \Gamma_\eta} \left| (K(\chi(\eta s)) - z)^{-1} \right| \leq C \eta^{-M_\delta}. \tag{4.8}
\]

We can choose the length of the contour \( \Gamma_\eta \) to be of order \( 1/D_\eta \). We will treat this region by using a second order adiabatic development for the quasi-eigenprojector given by the adiabatic theory (see [29] and references therein). If

\[
X(\kappa) := \left[ E_1^+(\kappa) E_1^-(\kappa) E_1(\kappa) - E_1(\kappa) E_1^-(\kappa) E_1^+(\kappa) \right],
\]

we define:

\[
F_\eta(s) := B_\eta(s) + \eta \chi'(\eta s) W^*_\eta(s) Y(\chi(\eta s)) W_\eta(s) \tag{4.9}
\]

\[
Y(\chi(\eta s)) := -\frac{1}{2\pi} \oint_{\Gamma_{\eta}} dz \left( K(\chi(\eta s)) - z \right)^{-1} X(\chi(\eta s)) (K(\chi(\eta s)) - z)^{-1},
\]

where the operator \( Y(\kappa) \) is a solution to the commutator equation \( i[K(\kappa), Y(\kappa)] = -E'_1(\kappa) \). The operator \( F_\eta(s) \) is constructed in such a way that when we compute \( \partial_s F_\eta(s) \), the term \( \partial_s B_\eta(s) \) gets canceled and we have the identity:

\[
\partial_s F_\eta(s) = \eta^2 W^*_\eta(s) \left\{ \partial_x \frac{\chi'(x)}{2\pi} \oint_{\Gamma_{\eta}} dz \left( K(\chi(x)) - z \right)^{-1} X(\chi(x)) (K(\chi(x)) - z)^{-1} \right\} W_\eta(s). \tag{4.10}
\]

Note that \( Y(\chi(\eta s)) \) is a bounded operator obeying

\[
\| Y(\chi(\eta s)) \| \leq C \eta^{-M_\delta}, \quad s \in \mathbb{R}_- \setminus \left[ \frac{t_0 - \eta \delta}{\eta}, \frac{t_0 + \eta \delta}{\eta} \right], \tag{4.11}
\]

which is a consequence of (4.8) and because our choice of the contour \( \Gamma_\eta \). It follows that we can write a rough bound of the type

\[
\| \partial_s F_\eta(s) \| \leq C \eta^{2-2M_\delta} |\chi'(\eta s)|^2 + |\chi''(\eta s)|, \quad s \in \mathbb{R}_- \setminus \left[ \frac{t_0 - \eta \delta}{\eta}, \frac{t_0 + \eta \delta}{\eta} \right]. \tag{4.12}
\]
Thus on any sub-interval \([s_1, s_2]\) of the negative real axis where the above estimate holds true we can write:

\[
||F_\eta(s_1) - F_\eta(s_2)|| \leq C\eta^{1-2M\delta},
\]

(4.13)
due to the integrability properties of \(\chi\) (see (1.10)). From (4.10) and (4.11) we can derive the estimate:

\[
||B_\eta(s) - F_\eta(s)|| \leq C\eta^{1-M\delta}, \quad s \in \mathbb{R}_- \ \setminus \ \left[\frac{t_0 - \eta^\delta}{\eta}, \frac{t_0 + \eta^\delta}{\eta}\right].
\]

(4.14)

Up to a use of the triangle inequality, on any sub-interval \([s_1, s_2]\) of \(\mathbb{R}_-\) \([s_1, s_2]\) of \(\mathbb{R}_-\) \([s_1, s_2]\) of \(\mathbb{R}_-\) \([s_1, s_2]\) of \(\mathbb{R}_-\) \([s_1, s_2]\) of \(\mathbb{R}_-\) \([s_1, s_2]\) of \(\mathbb{R}_-\) we can write:

\[
||B_\eta(s_1) - B_\eta(s_2)|| \leq C\eta^{1-2M\delta}.
\]

(4.15)

This estimate together with (4.7) imply:

\[
||B_\eta(s) - B_\eta(0)|| = ||W_\eta^*(s)E_1(\chi(\eta s))W_\eta(s) - E_1(1)|| \leq C(\eta^\delta + \eta^{1-2M\delta}), \quad s \leq 0.
\]

Choose now any \(\delta \in (0, 1/(2M))\); then (4.3) is proved, which concludes the lemma.

\[\Box\]

4.2 Proof of Corollary 2.8

The limit in (2.3) is a trivial consequence of (2.2) and the result of Proposition 2.7. The proof of (of Lemmas 4.15 and 4.16) is a bit longer. We start with a lemma:

**Lemma 4.16.** At fixed \(\eta > 0\), we have the limit \(n - \lim_{s \to -\infty} B_\eta(s) = \omega_\eta E_1(0)\omega_\eta^*\).

**Proof.** This can be seen by writing

\[
B_\eta(s) = \{W_\eta^*(s)e^{-isH}\} \{e^{isH} E_1(\chi(\eta s))e^{-isH}\} \{e^{isH}W_\eta(s)\} \to_{s \to -\infty} \omega_\eta E_1(0)\omega_\eta^*
\]

where we used the fact that each parenthesis converges in norm (even though not uniformly in \(\eta\)).

\[\Box\]

**Corollary 4.17.** (of Lemmas 4.15 and 4.16)

\[
\lim_{\eta \to 0} \||\omega^*_\eta (E_1(0)\omega_\eta^* - E_1(1))|| = 0.
\]

(4.16)

We can now prove (2.4):

\[
\begin{align*}
& n - \lim_{\eta \to 0} \left[ n - \lim_{s \to -\infty} \frac{n - \lim_{s \to -\infty} e^{isH}E_{ac}(H)W_\eta(s)E_{pp}(K(1))}{\omega_\eta^* n - \lim_{s \to -\infty} W_\eta^*(s)\{1 - E_{pp}(H)\}W_\eta(s)E_{pp}(K(1))} \right] \\
& = n - \lim_{\eta \to 0} \left[ n - \lim_{s \to -\infty} \frac{\omega_\eta^* \{E_{pp}(K(1)) - \omega_\eta E_{pp}(K(0))\omega_\eta^*\}}{\omega_\eta^*} \right] E_{pp}(K(1)) = 0,
\end{align*}
\]

(4.17)

where we used Corollary 4.17 and the fact that \(\omega^*_\eta = n - \lim_{s \to -\infty} e^{isH}W_\eta(s)\) has norm one.

5 Existence and completeness of stationary wave operators

In this Section we analyze the pair of Hamiltonians \(\tilde{\hat{H}} + \kappa V, \hat{H} + \kappa V\) and prove Propositions 2.9, 2.11 and 2.12.
5.1 Proof of Proposition 2.9

We can approximate the function $\Phi$ in the uniform norm with a sequence of $C^\infty_0(\mathbb{R})$ functions $\Phi_n$. Thus if we can prove the proposition for smooth and compactly supported functions, then we are done. For such a $\Phi$, we can apply for example the Helffer-Sjöstrand formula (or any other norm convergent functional calculus involving the resolvent) and argue that we can approximate in norm the difference $\Phi_n(H) - \Phi_0(H)$ with a linear combination of differences of resolvents of the type

$$\sum_{j=1}^N C_j \left\{ (H - z_j)^{-1} - (\hat{H} - z_j)^{-1} \right\},$$

where $C_j$ are complex coefficients and $z_j$ are complex numbers with nonzero imaginary part. Thus one can reduce the problem to showing that

$$(H - z)^{-1} - (\hat{H} - z)^{-1} =: R(z) - \hat{R}(z)$$

is compact for some $z$ with $\text{Im}(z) > 0$.

Our decoupled Hamiltonian $\hat{H}$ (see (1.10)–(1.15)) is a direct sum of three commuting operators, and we have $\sigma_{ac}(\hat{H}) = \emptyset$ and

$$\sigma_{pp}(\hat{H}) = \sigma_{pp}(\Pi_0 \hat{H} \Pi_0) = \sigma(\Pi_0 \hat{H} \Pi_0) \subset \mathbb{R}_+.$$

Let us denote by $\{w_n\}_{n \in \mathbb{N}}$ the complete orthonormal set of eigenvectors of $\Sigma_{\mathcal{D}}$ in $L^2(\mathcal{D})$ (see (1.17)), having eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ so that $\sigma_{pp}(\Sigma_{\mathcal{D}}) = \{\lambda_n\}_{n \in \mathbb{N}}$; let $P_n$ be the 1-dimensional orthogonal projection on $w_n$ in $L^2(\mathcal{D})$. In particular,

$$\sigma_{ac}(\hat{H}) = \sigma_{ac}(\Pi_- \hat{H} \Pi_- \oplus \Pi_+ \hat{H} \Pi_+) = \sigma(\Pi_- \hat{H} \Pi_- \oplus \Pi_+ \hat{H} \Pi_+) = [\lambda_1, \infty).$$

Then for $z \in \mathbb{C} \setminus [0, \infty)$ we have

$$\hat{R}(z) = \oplus_{n \in \mathbb{N}} \left[ (1_+ - (z - \lambda_n))^{-1} \pi_+ \oplus (1_- - (z - \lambda_n))^{-1} \pi_- \right] P_n \oplus (-\Delta_D + w - z)^{-1}$$

with $\pi_{\pm}: L^2(\mathbb{R}) \to L^2(\mathcal{I}_{\pm})$ the orthogonal projections and $(1 - (z - \lambda_n))^{-1}$ the resolvent of the longitudinal kinetic energy on $\mathcal{I}_{\pm}$ with Dirichlet conditions at $\pm a$.

In order to study the Hamiltonian $H$ and its relation with $\hat{H}$, let us first observe that they are two self-adjoint extensions of the same symmetric operator

$$D_0 := -\Delta_D + w : C^\infty_0(L_- \cup \mathcal{C} \cup L_+) \to \mathcal{H}.$$

Let $D^\ast_0$ be the adjoint of this symmetric operator. In order to compare the two resolvents, $R(z)$ and $\hat{R}(z)$ for $z \in \mathbb{C} \setminus [0, \infty)$, we note that $D^\ast_0$ extends both self-adjoint operators $H$ and $\hat{H}$ so that:

$$\left( R(z) - \hat{R}(z) \right) \mathcal{H} \subset \ker(D^\ast_0 - z).$$

Notice that for $u \in \ker(D^\ast_0 - z)$ the distribution $D_0^* u - z u$ has support in the part of the boundary $\mathcal{D}_- \cup \mathcal{D}_+$, where $\mathcal{D}_{\pm} := L_{\pm} \cap \{\pm a\} \times \mathbb{R}^d$; thus on $L_- \cup L_+$ they satisfy the equation:

$$-\Delta_D u_{\pm} = z u_{\pm}$$

with the boundary condition $u_{\pm}|_{\mathcal{I}_{\pm} \times \partial \mathcal{D}} = 0$, for $u_{\pm} := u|_{L_{\pm}}$. Then standard arguments show that our vectors $u_{\pm} \in \mathcal{H}_{\pm}$ must be of the form $u_{\pm} = \oplus_{n \in \mathbb{N}} (u_{\pm,n} \otimes w_n)$, where the functions
$u_{\pm,n} \in L^2(\mathcal{I}_{\pm})$ satisfy the equation $L_{\pm} u_{\pm,n} = (z - \lambda_n)u_{\pm,n}$. Thus $u_{\pm,n} = \beta_{\pm,n}e^{\xi_{\pm,n}x}$ with $\xi_{\pm,n}$ the unique complex square root of $z - \lambda_n$ having $\pm \text{Re}\xi_{\pm,n} < 0$. Let us observe that due to the fact that $z \in \mathbb{C} \setminus [0,\infty)$ and $\lambda_n > 0$, our sequence $\{|\text{Re}\xi_{\pm,n}|\}$ contains strictly positive numbers, and moreover, diverges with $n$. Thus the infimum below is positive:

$$
\gamma_0(z) := \inf_{n \in \mathbb{N}} |\text{Re}\xi_{\pm,n}| > 0. \quad (5.2)
$$

We have thus proved the following statement (here $Q_1$ is the multiplication operator with the longitudinal coordinate):

**Lemma 5.18.** Let $z \in \mathbb{C} \setminus [0,\infty)$ and $\gamma_{\pm} \in \mathbb{R}_+ \setminus \{0\}$ be such that $0 < \gamma_{\pm} < \gamma_0(z)$ (defined in (5.2)), then we have the following estimations of exponential decay:

$$
\left\| e^{\pm \gamma_{\pm} Q_1 \Pi_{\pm} (R(z) - \hat{\circ} R(z)) e^{\pm \gamma_{\pm} Q_1 \Pi_{\pm}}} \right\| \leq c, \quad (5.3)
$$

and for $\Psi_\alpha(x) := e^{\alpha \sqrt{x^2 + 1}}$ (with $0 \leq \alpha < \gamma_0(z)$) we have:

$$
\left\| \Psi_\alpha(Q_1) (R(z) - \hat{\circ} R(z)) \Psi_\alpha(Q_1) \right\| \leq c. \quad (5.4)
$$

Taking into account that $R(z)\mathcal{H}$ and $\hat{\circ} R(z)\mathcal{H}$ are both contained in $H^1(L_-) \oplus H^1(\mathcal{C}) \oplus H^1(L_+)$, the above estimate (5.4) together with the compactness of Sobolev embeddings for compact domains, imply that $R(z) - \hat{\circ} R(z)$ are compact operators for any $z \in \mathbb{C} \setminus \mathbb{R}$.

### 5.2 Proof of Proposition 2.11

The absence of singular continuous spectrum will be proved later on in Lemma 5.19. Here we only show completeness of wave operators by using the Birman-Kuroda method [37]. In other words, we want to show that the difference between some large enough powers of the resolvents is a trace class operator.

We need to elaborate on the previous definition of $\Psi_\alpha \in C^\infty(\mathbb{R})$ introduced in Lemma 5.18. We now allow any $\alpha \in \mathbb{R}$ and further more:

$$
\Psi_\alpha(x) = e^{\alpha \sqrt{x^2 + 1}}, \quad \text{so that} \quad \left| (\partial^n \ln \Psi_\alpha)(x) \right| \leq C_s|\alpha|, \quad s \geq 1, \quad x \in \mathbb{R}. \quad (5.5)
$$

We observe that $\Psi_\alpha(x)$ is invertible everywhere on $\mathbb{R}$, and if $\alpha > 0$ then $\Psi_\alpha^{-1} \in L^k(\mathbb{R})$ for any $k \geq 1$. Another fact we shall use here is that the following multiple commutator is bounded:

$$
\left[ P_1, [P_1, \ldots [P_1, w] \ldots] \right] \quad (5.6)
$$

where $P_1 := -i \partial_x$ and $x$ denotes the first (longitudinal) variable of $\mathbb{R}^{d+1}$.

**Lemma 5.19.** Fix $z \in \mathbb{C} \setminus [0,\infty)$. Then there exist $k_d \in \mathbb{N}$ large enough and $\alpha(z) > 0$ small enough such that for any $k \geq k_d$ the operators $\hat{\circ} R(z)^k$ and $\Psi_\alpha(Q_1)\hat{\circ} R(z)^k \Psi_\alpha(Q_1)^{-1}$ with $|\alpha| < \alpha(z)$ are bounded operators from $L^2(\mathcal{L})$ into $BC(\mathcal{L})$ (the bounded continuous functions on $\mathcal{L}$).

**Proof.** The result for $\hat{\circ} R(z)^k$ is based on the fact that

$$
\hat{\circ} R(z)^k \mathcal{H} \subset \left( H^{2k}(\mathcal{L}_-) \cap H^1_0(\mathcal{L}_-) \right) \oplus \left( H^{2k}(\mathcal{C}) \cap H^1_0(\mathcal{C}) \right) \oplus \left( H^{2k}(\mathcal{L}_+) \cap H^1_0(\mathcal{L}_+) \right) \quad (5.7)
$$

which is based on the commutator estimate in (5.6). Now if $k$ is large enough (depending only on the dimension $d$), the right hand side becomes a subset of $BC(\mathcal{L})$. 

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In order to prove the same inclusion for the other operator, let us note that we can use a
Combes-Thomas type rotation [10]: \( \Psi_\alpha(Q_1)^\circ H \Psi_\alpha(Q_1)^{-1} = \overset{\circ}{H} + T_\alpha \), where \( T_\alpha \) is a first order
differential operator which has the following mapping property:

\[
T_\alpha : H^k(L_-) \oplus H^k(C) \oplus H^k(L_+) \rightarrow H^{k-1}(L_-) \oplus H^{k-1}(C) \oplus H^{k-1}(L_+).
\] (5.8)

Now if \( |\alpha| \) is small enough, one can prove by induction with respect to \( k \) that

\[
\| \overset{\circ}{H}^k \Psi_\alpha(Q_1)^\circ R(z)^k \Psi_\alpha(Q_1)^{-1} \| < \infty,
\] (5.9)

which means:

\[
\Psi_\alpha(Q_1)^\circ R(z)^k \Psi_\alpha(Q_1)^{-1} \mathcal{H} \subset H^{2k}(L_-) \oplus H^{2k}(C) \oplus H^{2k}(L_+)
\] (5.10)

and we are done.

**Corollary 5.20.** Let \( F \in L^2(\mathbb{R}) \). Then there exists \( k_d \in \mathbb{N} \) depending on the dimension \( d \) such that
for any \( z \in \mathbb{C} \setminus [0, \infty) \) and any \( k \geq k_d \) we have that \( F(Q_1) \circ R(z)^k \) and \( F(Q_1) \Psi_\alpha(Q_1) \circ R(z)^k \Psi_\alpha(Q_1)^{-1} \)
with \( |\alpha| < \alpha(z) \), are Hilbert-Schmidt operators on \( \mathcal{H} \).

**Proof.** Let us denote by \( T \) either \( \overset{\circ}{R}(z)^k \) or \( \overset{\circ}{\Psi}_\alpha(Q_1) \circ R(z)^k \Psi_\alpha(Q_1)^{-1} \) appearing in the previous
lemma. For any \( f \in \mathcal{H} \) we have that \( T f \) is a bounded and continuous function. Then for any fixed \( x \in L \), the mapping

\[
\mathcal{H} \ni f \mapsto (T f)(x) \in \mathbb{C}
\] (5.11)

defines a bounded linear functional on \( \mathcal{H} \), uniformly bounded in \( x \). The Riesz representation
theorem allows us to conclude that \( T \) has an integral kernel obeying

\[
\sup_{x \in L} \int |T(x,y)|^2 \, dy < \infty.
\] (5.12)

Thus for any function \( F \in L^2(\mathbb{R}) \), the operators \( F(Q_1)T \) have integral kernels of class \( L^2(L \times L) \),
hence they are Hilbert-Schmidt.

**Lemma 5.21.** Fix \( z \in \mathbb{C} \setminus [0, \infty) \). Then there exist \( k_d \in \mathbb{N} \) large enough and \( \alpha(z) > 0 \) small enough,
such that for any \( k \geq k_d \) we have that \( F(Q_1) \overset{\circ}{\circ} R(z)^k \) and \( F(Q_1) \Psi_\alpha(Q_1) \circ R(z)^k \Psi_\alpha(Q_1)^{-1} \)
with \( |\alpha| < \alpha(z) \) are Hilbert-Schmidt operators on \( \mathcal{H} \) for any measurable function \( F \in L^2(\mathbb{R}) \).

**Proof.** We use similar arguments with those for \( \overset{\circ}{H} \) but this time repeated for \( H \). We do not give
further details.

The final technical result needed for the Birman-Kuroda theorem is the following:

**Lemma 5.22.** Let \( z \in \mathbb{C} \setminus [0, \infty) \). Then there exists \( n_d \in \mathbb{N} \) large enough such that for any \( n \geq n_d \)
we have that \( \left[ R(z)^n - \overset{\circ}{\circ} R(z)^n \right] \in \mathcal{B}_1(\mathcal{H}) \) (the set of trace class operators).

**Proof.** Let us fix \( z \in \mathbb{C} \setminus [0, \infty) \). We start with the formula (valid for any \( p \in \mathbb{N} \)):

\[
R(z)^p - \overset{\circ}{\circ} R(z)^p = \sum_{0 \leq j \leq p-1} R(z)^j (R(z) - \overset{\circ}{\circ} R(z))^j \overset{\circ}{\circ} R(z)^{p-1-j}.
\] (5.13)

Let us choose \( 0 < \alpha < \min\{\alpha(z), \gamma_0(z)\} \), where \( \gamma(z) \) is the same as in Lemma 5.18. Let us choose
\( p \geq 4k_d + 1 \) and observe that in (5.13) we either have \( j \geq 2k_d \) or \( p - j - 1 \geq 2k_d \). The idea is to prove that operators of the type

\[
R(z)^{2k_d} \Psi_\alpha(Q_1)^{-1} \text{ or } \Psi_\alpha(Q_1)^{-1} \overset{\circ}{\circ} R(z)^{2k_d}
\]
are trace class, which together with Lemma 5.18 would finish the proof. Indeed, we can write:

\[ R(z)^{2k} \Psi_{\alpha}(Q_1)^{-1} = \left\{ \Psi_{-\alpha/2}(Q_1)[\Psi_{\alpha/2}(Q_1)R(z)^{k} \Psi_{-\alpha/2}(Q_1)] \right\} \cdot \left\{ \Psi_{-\alpha/2}(Q_1)[\Psi_{\alpha}(Q_1)R(z)^{k} \Psi_{-\alpha}(Q_1)] \right\}, \quad (5.14) \]

where the right hand side is - according to Lemma 5.21 - a product of two Hilbert-Schmidt operators. The other operator can be treated in a similar way, up to taking the adjoint. The proof is over.

\[ \square \]

5.3 Proof of Proposition 2.12

We now want to study the pair of Hamiltonians \( K(\kappa) = H + \kappa V \) and \( \overset{\circ}{K}(\kappa) = H + \kappa V \), for any \( \kappa \in [0, 1] \) and prove Proposition 2.12. The only difficulty comes from the fact that the perturbation \( V \) has a singular commutator with \( H \) and, at the same time, it does not tend to zero at infinity.

But for \( \overset{\circ}{K}(\kappa) \) there is no difficulty due to the fact that \( \overset{\circ}{H} \) commutes with the bias \( V \), while the last one is just a multiple of the identity on each orthogonal subspace in the decomposition \( \mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+ \).

In fact, the only result which cannot be obtained just like in the previous section is the following lemma:

**Lemma 5.23.** Fix \( z \in \mathbb{C} \setminus [0, \infty) \). There exists \( k_d' \in \mathbb{N} \) large enough such that for any any \( k \geq k_d' \) we have that the operators \( F(Q_1)R_{\kappa}(z)^k \) and \( F(Q_1)\Psi_{\alpha}(Q_1)R_{\kappa}(z)^k\Psi_{\alpha}(Q_1)^{-1} \) with \( |\alpha| < \alpha(z) \) (here \( \Psi_{\alpha} \) is as in (5.13)) are Hilbert-Schmidt operators on \( \mathcal{H} \) for any function \( F \in L^2(\mathbb{R}) \).

**Proof.** The perturbation \( V \) is still relatively bounded with respect to \( H \) with 0 relative bound, but its commutator with \( H \) defined as a sesquilinear form on the domain of \( H \) is singular. We observe that the main difficulty comes from the fact that the range of the operator \( R_{\kappa}(z)^k \) is no longer contained in the Sobolev space \( H^{2k}(L) \) but only in \( H^2(L) \) for any \( k \in \mathbb{N} \), due to the singularity of the commutator of \( -\Delta \) with \( V \). In fact the situation is a bit better due to the fact that \( V \) (being constant in the \( \mathcal{D} \)-space) commutes with all derivatives with respect to directions from \( \mathcal{D} \). Thus, using the results in [24], we may conclude that:

\[ R_{\kappa}(z)^k \mathcal{H} \subset H^2(\mathbb{R}; H^{2(k-1)}(\mathbb{R}^d)) \cap L^2(L) \subset BC(\mathbb{R}; H^{2(k-1)}(\mathbb{R}^d)) \cap L^2(L) \subset \]

\[ \subset BC(\mathbb{R}; BC(\mathbb{R}^d)) \cap L^2(L) \subset BC(L) \]

for \( k \geq k_d' \) depending only on the dimension \( d \). Thus the proof goes on exactly as in Section 3 and we are done with \( F(Q_1)R_{\kappa}(z)^k \).

Regarding \( F(Q_1)\Psi_{\alpha}(Q_1)R_{\kappa}(z)^k\Psi_{\alpha}(Q_1)^{-1} \), we need to replace \( \Psi_{\alpha} \) with a function \( \tilde{\Psi}_{\alpha} \) which is constant in a small neighborhood of \( \pm \alpha \). In this case, when we write

\[ \tilde{\Psi}_{\alpha}K(\kappa)\tilde{\Psi}_{\alpha}^{-1} = K(\kappa) + \tilde{T}_{\alpha} \]

we see that \( \tilde{T}_{\alpha} \) equals zero around the points where \( V \) is discontinuous. Therefore, if \( |\alpha| \) is small enough we will have

\[ \|K(\kappa)^k\tilde{\Psi}_{\alpha}R_{\kappa}(z)^k\tilde{\Psi}_{\alpha}^{-1}\| < \infty, \quad (5.15) \]

and the proof goes in the same way as in the previous section.

\[ \square \]

6 Study of \( \Xi_\eta \) and its adiabatic limit

Let us recall a few facts about the decoupled system:

\[ \sigma_{sc}(\overset{\circ}{K}(\kappa)) = \emptyset, \quad \mathcal{H}_{ac}(\overset{\circ}{K}(\kappa)) = \mathcal{H}_- \oplus \mathcal{H}_+, \quad \mathcal{H}_{pp}(\overset{\circ}{K}(\kappa)) = \mathcal{H}_0, \quad \forall \kappa \in [0, 1], \]

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\[
\hat{H} = \Pi_- \left( \Delta_D^+ - \Delta_D^- \right) \Pi_+ + \Pi_+ \left( \Delta_D^+ - \Delta_D^- \right) \Pi_-
\]

\[
W\eta(s)E_{ac}\hat{H} = \left[ \Pi_- e^{-i\nu^- f_\eta(x,\nu)du} + \Pi_+ e^{-i\nu^+ f_\eta(x,\nu)du} \right] e^{-is\hat{H}}E_{ac}\hat{H}
\]

and using the notations defined earlier \((1.17)\), it is well known that:

\[
\Delta_D^\pm = \Pi^\pm 1 + 1 \otimes \Sigma_D, \quad \sigma(\Pi^\pm) = \sigma_{ac}(\Pi^\pm) = [0, \infty), \quad \sigma(\Sigma_D) = \sigma_{pp}(\Sigma_D) \subset \mathbb{R}_+.
\]

Thus \(\sigma_{ac}(\hat{H}) = [\inf \sigma(\Sigma_D), \infty)\) and has the set of thresholds \(T = \sigma_{pp}(\Sigma_D)\).

### 6.1 Proof of Proposition 2.10

Here we are interested in the strong limit when \(s \to -\infty\) of:

\[
E_{ac}(K_1)W^s_\eta(s)E_{ac}(\hat{H})W\eta(s)E_{ac}(\hat{H}).
\]

Let us start by noting that we can replace \(E_{ac}(\hat{H})\) with the identity in the above product, and still get the same strong limit (if it exists). The explanation is that we can write:

\[
E_{pp}(\hat{H})W\eta(s)E_{ac}(\hat{H}) = \left\{ E_{pp}(\hat{H}) e^{-is\hat{H}}E_{ac}(\hat{H}) \right\} e^{is\hat{H}}W\eta(s)
\]

and use the fact that \(e^{is\hat{H}}W\eta(s)\) converges in norm, while \(E_{pp}(\hat{H}) e^{-is\hat{H}}E_{ac}(\hat{H})\) converges strongly to zero when \(s \to -\infty\) because \(E_{pp}(\hat{H})\) is compact. Thus it is enough to study the existence of a strong limit when \(s \to -\infty\) of:

\[
\Xi_\eta(s) := E_{ac}(K_1)W^s_\eta(s)W\eta(s)E_{ac}(\hat{H}).
\]

For any \(\delta > 0\) let \(\mathcal{V}_\delta\) be the set of vectors \(f \in \mathcal{H}_{ac}(\hat{H})\) with compact spectral support with respect to \(\hat{H}\) at distance larger than \(\delta\) from all thresholds. Clearly, \(\{\mathcal{V}_\delta\}_{\delta > 0}\) is dense in \(\mathcal{H}_{ac}(\hat{K}_\delta) = \mathcal{H}_{ac}(\hat{H})\). It is thus enough to show the existence of the limit \(\lim_{s \to -\infty} \Xi_\eta(s)f\) for \(f \in \mathcal{V}_\delta\). As any vector \(f \in \mathcal{H}_{ac}(\hat{H})\) is of the form \((f_-, f_+) \in \mathcal{H}_- \oplus \mathcal{H}_+\) we will treat the two situations separately.

The idea is to use a variant of Cook’s method. We have the following identities:

\[
\Xi_\eta(s)f = \Xi_{\eta}(s) = \Phi_\eta(s) - \Psi_\eta(s).
\]

Without loss of generality, let us assume that \(f \in \mathcal{H}_+ \cap \mathcal{V}_\delta\). Since \(f\) is with compact support in the spectral measure of \(\hat{H}\), there exist a finite number \(N\) of transverse eigenvectors \(\{w_n\}\) of \(\Sigma_D\) in \(L^2(\mathcal{D})\) corresponding to the eigenvalues \(\{\lambda_n\}\) and so that

\[
f(x, x) = \sum_{n=1}^{N} w_n(x) \int_{\mathbb{R}} \sin[k(x - a)]f_n(k)dk.
\]
where \( f_n \) are smooth, compactly supported, with a support which does not contain the points \( k^2 < \delta \). Then we have

\[
\{e^{-is\hat{P}}f\}(x, x_\perp) = \sum_{n=1}^N w_n(x_\perp) \int_{\mathbb{R}} e^{-is(k^2 + \lambda_n)} \sin[k(x - a)] f_n(k) dk,
\]

\[
\{\hat{W}_\eta(s)f\}(x, x_\perp) = e^{-iv + \int_0^s \chi(\eta t)dt} \sum_{n=1}^N w_n(x_\perp) \int_{\mathbb{R}} e^{-is(k^2 + \lambda_n)} \sin[k(x - a)] f_n(k) dk. \tag{6.3}
\]

Moreover, for \( j \geq 1 \) we have:

\[
\{\hat{W}_\eta(s)(\hat{K}(\chi(\eta s)) + 1)^j f\}(x, x_\perp) = e^{-iv + \int_0^s \chi(\eta t)dt} \sum_{n=1}^N w_n(x_\perp) \int_{\mathbb{R}} e^{-is(k^2 + \lambda_n)} [k^2 + v + \chi(\eta s) + \lambda_n]^j \sin[k(x - a)] f_n(k) dk.
\]

By standard integration by parts arguments, due to the support condition of \( f_n \), we can prove the following estimate:

\[
\| e^{-\alpha(Q_t)} \{W_\eta(s)(\hat{K}(\chi(\eta s)) + 1)^j f\} \| \leq (1 + |x - a|)^N C_N(f, j), \quad |s| > 1, \forall N \in \mathbb{N} \tag{6.5}
\]

or (taking \( N = 2 \))

\[
\| e^{-\alpha Q} \{\hat{W}_\eta(s)(\hat{K}(\chi(\eta s)) + 1)^j f\} \| \leq \frac{C(f, j, \alpha)}{1 + s^2}. \tag{6.6}
\]

We need only one more ingredient. Using the same methods as in subsection 5.1 one can prove the following estimation similar to (5.3) for the given time dependent objects:

\[
\left\| e^{\alpha(Q_t)} \Pi_{\pm} \left[ (\hat{K}(\chi(\eta s)) + 1)^{-1} - (\hat{\chi}(\chi(\eta s)) + 1)^{-1} \right] e^{\alpha(Q_t)} \Pi_{\pm} \right\| \leq c, \tag{6.7}
\]

for \( \alpha \) small enough. The above constant can be chosen uniformly with respect to \( s \).

Now we can go back to (5.1) and investigate the structure of \( \Psi_\eta(s) \). Let us show that it will converge to zero when \( s \to -\infty \). Indeed, the difference of resolvents provides the exponential localization near the sample. But then we know that the adiabatic decoupled free evolution decays with \( s \), as in (6.6). We conclude:

\[
\lim_{s \to -\infty} \Xi_\eta(s) f = \lim_{s \to -\infty} \Phi_\eta(s), \tag{6.8}
\]

provided that the limit on the right hand side exists. We shall show that \( \Phi_\eta(s) \) has an absolutely integrable derivative with respect to \( s \). Let us differentiate \( \Phi_\eta(s) \) with respect to \( s \). We obtain the identity:

\[
- i \partial_s \Phi_\eta(s) = - E_{ac}(K_1) W^*_\eta(s) \left[ (\hat{K}(\chi(\eta s)) + 1)^{-1} - (\hat{\chi}(\chi(\eta s)) + 1)^{-1} \right] \hat{W}_\eta(s)(\hat{K}(\chi(\eta s)) + 1)^2 f
+ i \eta \chi(\eta s) E_{ac}(K_1) W^*_\eta(s)(\hat{K}(\chi(\eta s)) + 1)^{-1} V \left[ (\hat{K}(\chi(\eta s)) + 1)^{-1} - (\hat{\chi}(\chi(\eta s)) + 1)^{-1} \right] \hat{W}_\eta(s)(\hat{K}(\chi(\eta s)) + 1)f.
\]

We see that using (6.7) and (6.8) we can write:

\[
\| \partial_s \Phi_\eta(s) \| \leq \frac{C}{1 + s^2}. \tag{6.10}
\]
Thus \( \lim_{s \to -\infty} \Phi_\eta(s) \) exists and equals:

\[
\Xi_\eta f = \Phi_\eta(0) (6.11)
\]

\[
- i \int_{-\infty}^{0} E_{ac}(K_1) W_\eta^*(s) \left[ (K(\chi(\eta s)))^{-1} - (\hat{K}(\chi(\eta s)))^{-1} \right] W_\eta(s) (\hat{K}(\chi(\eta s)) + 1)^2 f ds
\]

\[
- \int_{0}^{\eta} \eta \chi'(\eta s) E_{ac}(K_1) W_\eta^*(s) (K(\chi(\eta s)))^{-1}
\]

\[
\cdot V \left[ (K(\chi(\eta s)))^{-1} - (\hat{K}(\chi(\eta s)))^{-1} \right] W_\eta(s) (\hat{K}(\chi(\eta s)) + 1)^2 f ds.
\]

The proof of Proposition 2.10 is over.

### 6.2 Proof of Proposition 2.13

First, let us compute the limit \( \eta \downarrow 0 \) in (6.11). We can apply the Lebesgue dominated convergence theorem in (6.11) and obtain:

\[
\lim_{\eta \downarrow 0} \Xi_\eta f = E_{ac}(K_1)(K(1) + 1)^{-1} \hat{\phi}(K(1) + 1) f (6.12)
\]

\[
- i \int_{-\infty}^{0} E_{ac}(K_1) e^{isK(1)} \left[ (K(1) + 1)^{-1} - (\hat{K}(\chi(\eta s))^{-1}) \right] e^{-is\hat{K}(1)} (\hat{K}(1) + 1)^2 f ds.
\]

Second, let us show that the above right hand side coincides with \( \Xi_0 f \). Indeed, let us look at the vector \( E_{ac}(K_1) e^{isK(1)} e^{-is\hat{K}(1)} f \), where \( f \in \mathcal{V}_3 \). As before, we can decompose the vector as :

\[
\Phi_0(s) - \Psi_0(s)
\]

\[
\Phi_0(s) := E_{ac}(K_1) e^{isK(1)} (K(1) + 1)^{-1} (\hat{K}(1) + 1)^{-1} e^{-is\hat{K}(1)} (\hat{K}(1) + 1)^2 f,
\]

\[
\Psi_0(s) := E_{ac}(K_1) e^{isK(1)} \left[ (K(1) + 1)^{-1} - (\hat{K}(\chi(\eta s)))^{-1} \right] e^{-is\hat{K}(1)} (\hat{K}(1) + 1)^2 f.
\]

Using the previous propagation estimates which were shown to be uniform in \( \eta \), we can repeat the same argument which led us to (6.11) but with \( \eta = 0 \) from the beginning. This will give a formula for \( \Xi_0 f \) which will coincide with the right hand side of (6.12). The proof is over.

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