A Row-Wise Update Algorithm for Sparse Stochastic Matrix Factorization

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Abstract

Nonnegative matrix factorization arises widely in machine learning and data analysis. In this paper, for a given factorization of rank \( r \), we consider the sparse stochastic matrix factorization (SSMF) of decomposing a prescribed \( m \)-by-\( n \) stochastic matrix \( V \) into a product of an \( m \)-by-\( r \) stochastic matrix \( W \) and a sparse \( r \)-by-\( n \) stochastic matrix \( H \). With the prescribed sparsity level, we reformulate the SSMF as an unconstrained nonconvex-nonsmooth minimization problem and introduce a row-wise update algorithm for solving the minimization problem. We show that the proposed algorithm converges globally and the generated sequence converges to a special critical point of the cost function, which is a global minimizer over the \( W \)-factor as a whole and is nearly a global minimizer over each row vector of the \( H \)-factor. Numerical experiments on both synthetic and real data sets are given to demonstrate the effectiveness of our proposed algorithm.

Keywords. Nonnegative matrix factorization, stochastic matrix factorization, sparsity, alternating minimization, proximal gradient method

AMS subject classifications. 65K05, 90C06, 90C26

1 Introduction

Since the introduction of simple and efficient algorithms in the seminal work of Lee and Seung [24], nonnegative matrix factorization (NMF) has been widely adopted in various fields such as document clustering [3, 27], computer vision [8], recommendation systems [44], bioinformatics

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face recognition [21, 43, 45], acoustic signal processing [42], source separation [12, 16], and modeling default data via interactive hidden Markov model (IHMM) [10, 11].

In this paper, we consider the following sparse stochastic matrix factorization (SSMF) problem. Let \( V = (v_{ij}) \in \mathbb{R}^{m \times n} \) be a (row) stochastic data matrix, i.e., all its entries are nonnegative with each row summing to 1 (\( \sum_{j=1}^{n} v_{ij} = 1 \) for \( i = 1, \ldots, n \)). For a predetermined factor rank \( r < \min\{m, n\} \), the SSMF attempts to find two stochastic matrices \( W \in \mathbb{R}^{m \times r} \) and \( H \in \mathbb{R}^{r \times n} \) such that \( V \approx WH \), where the sparseness constraints are imposed on the \( H \)-factor.

In fact, the SSMF is a probabilistic model, i.e., the probabilistic latent semantic analysis (PLSA). Like NMF [13], the PLSA has been used in document clustering [17, 18]. There exist different methods for the sparse NMF (SNMF) since sparseness can achieve a sparse representation or data clustering. In particular, in [19], Hoyer presented the projected gradient based multiplicative algorithm for solving the following minimization problem:

\[
\min_{W,H \geq 0} C(W, H) = \frac{1}{2} \| V - WH \|_F^2 + \lambda \| H \|_1,
\]  

(1.1)

where \( V \geq 0 \) is the input data matrix, \( \lambda > 0 \) is a regularization parameter, and \( \| H \|_1 := \sum_{i,j} |h_{ij}| \). However, for the SSMF, the \( \ell_1 \)-norm of \( H \) is constant as \( H \) is a stochastic matrix. There exist other cost functions for measuring the factorization residual such as the generalized Kullback-Leibler (KL) divergence [13, 25] and Minkowski family of metrics, see for instance [35] and the references therein. Other regularizers include the entropic prior [33], the pseudo-Dirichlet prior [22], and the \( \ell_1/2 \)-regularization [31].

To our best knowledge, there exist only a few works on the SNMF with \( \ell_0 \) constraints. In particular, in [37], Xiu et al. gave a structured joint sparse NMF model:

\[
\min_{W,H} C(W, H) = \frac{1}{2} \| V - WH \|_F^2 + \lambda \text{tr}(H^T LH)
\]

subject to (s.t.) \( W \geq 0, \ H \geq 0, \ \| H \|_{2,0} \leq s \),

(1.2)

where \( L \) is the graph Laplacian matrix learned from the input data matrix \( V \geq 0 \), \( s \) is the prescribed sparsity level, and \( \| H \|_{2,0} \) denotes the \( \ell_{2,0} \) norm of \( H \), i.e., \( \| H \|_{2,0} := \text{card}(\{ t : \| H(t,:) \| \neq 0 \}) \). Also, an optimization algorithm based on the alternating direction method of multipliers was presented for solving the above model. However, it is not easy to choose an appropriate regularization parameter such that a good tradeoff between the residual term and the regularization term can be obtained [26, 36].

Recently, there have been some development in cardinality/\( \ell_0 \)-constrained optimization problems. In particular, several optimization methods were introduced for cardinality-constrained problems appeared in portfolio optimization and statistical learning [5, 7, 14]. In [38, 39], Xu et al. presented some projected gradient methods for cardinality constrained optimization appeared in compressed sensing, financial optimization and image processing. In [20], Kanzow et al. gave an augmented Lagrangian method for cardinality-constrained optimization problems.

In this paper, we directly apply \( \ell_0 \) constraints to measure the sparseness of \( H \)-factor in the SSMF. It is natural to add the sparsity to each row of the \( H \)-factor (i.e., \( \| H(t,:) \|_0 \leq s \) for all \( t = 1, \ldots, r \)) since the \( H \)-factor is stochastic. We find a solution to the SSMF by minimizing the distance between the input stochastic data matrix \( V \) and the product \( WH \) in Frobenius norm, where the required matrices \( W \) and \( H \) are both stochastic matrices with the sparse
the complement of $H$-factor in the sense that $\|H(t, :))\|_0 \leq s$ for all $t = 1, \ldots, r$. To our best knowledge, most optimization algorithms for solving the SNMF treat each of the two factors as a whole, which may lead to slow convergence rate. To develop an effective algorithm with simple update and global convergence, we propose a row-wise update algorithm for solving the SSMF. This is motivated by the alternating minimization (AM) method, see for instance the book [2], and the proximal alternating linearized minimization (PALM) [6] and the block coordinate update [40, 41] for nonconvex and nonsmooth optimization. We update the $W$-factor row by row via the projected gradient method and update the $H$-factor row by row by using the AM method or the cyclic projected gradient method, where the involved subproblems can be solved efficiently. In [2, Theorem 14.3], it has been shown that every limit point of the sequence generated by the AM method (i.e., the block coordinate descent method as in [4, Section 2.7]) is a stationary point, which is a coordinate-wise minimum. A PALM algorithm in [6] and the block coordinate update algorithms in [40, 41] were proposed for solving nonconvex-nonsmooth optimization. In particular, based on the Kurdyka-Lojasiewicz (KL) property, the proposed PALM algorithm and the block coordinate update algorithms were shown to converge globally to a critical point. For the SNMF, the $\ell_0$-norm $\| \cdot \|_0$ is a semi-algebraic function, which satisfies the KL property [6, Theorem 3 and Examples 2–3]. Then we reformulated the SSMF as an unconstrained nonconvex-nonsmooth minimization problem, where the objective function measures the difference between $V$ and $WH$ and the entry-wise nonnegativity and sparseness of $W$ and $H$. We establish the global convergence of the proposed method in the sense that the sequence generated by our method converges globally to a critical point, at which the objective function is globally minimized over the $W$-factor as a whole and is nearly globally minimized over each row vector of the $H$-factor. Numerical experiments on both synthetic and real data sets show that the proposed algorithm is more effective than the PALM method for solving the SSMF in terms of the reconstruction error.

Throughout this paper, we use the following notations. Let $\mathbb{R}^{m \times n}$ be the set of all $m \times n$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n \times 1}$. Let $\mathbb{R}^n$ be equipped with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Let $\mathbb{R}^{m \times n}$ be equipped with the Frobenius inner product $\langle \cdot, \cdot \rangle_F$ and its induced Frobenius norm $\| \cdot \|_F$. The superscript $^T$ stands for the transpose of a matrix or vector. $I_n$ denotes the identity matrix of order $n$. Let $| \cdot |$ be the absolute value of a real number or the components of a real vector and $\| \cdot \|_p$ be the matrix $p$-norm (especially $p = 1, 2, \infty$) and $\| \cdot \|_0$ denotes the number of nonzero entries of a vector or a matrix. We denote by $\lambda_{\max}(\cdot)$ the largest eigenvalue of a symmetric matrix. Let $\Pi_D(\cdot)$ denote the metric projection onto subset $D$ in $\mathbb{R}^n$ or $\mathbb{R}^{m \times n}$. For any matrix $A$, $A \geq 0$ means that $A$ is entrywise nonnegative and $A(i, :)$ and $A(:, j)$ denote respectively the $i$-th row and the $j$-th column of $A$. For any given point $(X_1, X_2) \in \mathbb{R}^{m_1 \times n_1} \times \subset \mathbb{R}^{m_2 \times n_2}$ and any subset $D_1 \times D_2 \subset \mathbb{R}^{m_1 \times n_1} \times \mathbb{R}^{m_2 \times n_2}$, the distance from $(X_1, X_2)$ to $D_1 \times D_2$ is determined by

$$\text{dist}((X_1, X_2), D_1 \times D_2) = \inf \{ \|(X_1, X_2) - (Y_1, Y_2)\|_F \mid (Y_1, Y_2) \in D_1 \times D_2 \}.$$

Let $[n] = \{1, 2, \ldots, n\}$ and for any set $S \subset [n]$, let $\text{card}(S)$ and $\mathcal{S}$ be the cardinality of $S$ and the complement of $S$ in $[n]$, respectively. For any set $S \subset [n]$, $x_S$ is the subvector of a vector $x$ with components indexed by $S$.

The rest of this paper is organized as follows. In Section 2 we reformulate the SSNMF as a nonconvex-nonsmooth minimization problem and then propose a new row-wise update algorithm.
for solving it. In Section 3, we establish the global convergence of the proposed algorithm. In Section 4, we report some numerical experiments to illustrate the effectiveness of our method. Finally, concluding remarks are given in Section 5.

2 A row-wise update algorithm

In this section, we reformulate the SSMF as an unconstrained nonconvex-nonsmooth minimization problem. Then we propose a row-wise update algorithm for solving the minimization problem.

2.1 Reformulation

As in [24], for the SSMF, we measure the approximation quantity by the residual $\|V - WH\|_F$ in the Frobenius norm. Then we can rewrite the SSMF as the following minimization problem:

$$\min_{W \in \mathbb{R}^{m \times r}, H \in \mathbb{R}^{r \times n}} f(W, H) = \frac{1}{2}\|V - WH\|_F^2$$

subject to

$$W \in G_1, \quad H \in G_2 \cap G_3,$$

where $G_1 := \{W = (w_{ij}) \in \mathbb{R}^{m \times r} \mid W \geq 0, \sum_{t=1}^{r} w_{it} = 1, i = 1, \ldots, m\}$, $G_2 := \{H = (h_{ij}) \in \mathbb{R}^{r \times n} \mid H \geq 0, \sum_{j=1}^{n} h_{tj} = 1, t = 1, \ldots, r\}$, and $G_3 := \{H \in \mathbb{R}^{r \times n} \mid \|H(t,:)\|_0 \leq s, t = 1, \ldots, r\}$.

We point out that, if $s = n$, then the SSMF (2.1) is reduced to the stochastic matrix factorization (SMF).

It is easy to see that the SSMF (2.1) can be reduced to the following unconstrained minimization problem:

$$\min_{W \in \mathbb{R}^{m \times r}, H \in \mathbb{R}^{r \times n}} F(W, H) := f(W, H) + \delta_{G_1}(W) + \delta_{G_2 \cap G_3}(H),$$

where $\delta_K$ is the indicator function of a set $K$, i.e., $\delta_K(x) = 0$ if $x \in K$ and $\delta_K(x) = +\infty$ if $x \notin K$.

In problem (2.2), $f$ is nonconvex and smooth and both $\delta_{G_1}$ and $\delta_{G_2 \cap G_3}$ are nonsmooth. One may solve problem (2.2) by the PALM method in [6], which is described in Algorithm 2.1 below.

In Algorithm 2.1, both the $W$-factor and the $H$-factor are updated as a whole, which may cause slow convergence. To overcome this drawback, we develop an efficient algorithm with simpler update for solving the SSMF (2.1).

In the following, we present a row-wise update algorithm for solving the SSMF (2.1), where both the $W$-factor and the $H$-factor are updated row-by-row. To do so, we let

$$P_n := \{x \in \mathbb{R}^n \mid x \geq 0, \sum_{j=1}^{n} x_j = 1\}, \quad Q^s_n := \{x \in \mathbb{R}^n \mid \|x\|_0 \leq s\}.$$

For convenience, we let

$$f(W, H) = \frac{1}{2}\|V - [w_1, \ldots, w_m]^T [h_1, \ldots, h_r]^T\|_F^2 \equiv f(w_1, \ldots, w_m, h_1, \ldots, h_r),$$

where $W = [w_1, \ldots, w_m]$ and $H = [h_1, \ldots, h_r]$.

4
Algorithm 2.1 A PALM method for problem (2.2)

Step 0. Choose \( W^0 \in G_1, H^0 \in G_2 \cap G_3, \delta_1 > 0, \delta_2 > 0 \). Let \( k := 0 \).

Step 1. Take \( \mu_k = 1/(\|H^k\|^2_F + \delta_1) \) and compute
\[
W^{k+1} = \Pi_{G_1}(W^k - \mu_k \nabla_W f(W^k, H^k)).
\]

Step 2. Take \( \nu_k = 1/(\|W^{k+1}\|^2_F + \delta_2) \) and compute
\[
H^{k+1} = \Pi_{G_2 \cap G_3}(H^k - \nu_k \nabla_H f(W^{k+1}, H^k)).
\]

Step 3. Replace \( k \) by \( k + 1 \) and go to Step 1.

for all \( W := [w_1, \ldots, w_m]^T \in \mathbb{R}^{m \times r} \) and \( H := [h_1, \ldots, h_r]^T \in \mathbb{R}^{r \times n} \). Also, we define the function \( F \) by
\[
F(W, H) = f(w_1, \ldots, w_m, h_1, \ldots, h_r) + \sum_{i=1}^m \delta_{P_i}(w_i) + \sum_{t=1}^r \delta_{P_n}(h_t) + \sum_{t=1}^r \delta_{Q_n}(h_t)
\]
(2.4)

for all \( W := [w_1, \ldots, w_m]^T \in \mathbb{R}^{m \times r} \) and \( H := [h_1, \ldots, h_r]^T \in \mathbb{R}^{r \times n} \). Then the SSMF (2.1) can be rewritten equivalently as the following unconstrained minimization problem:
\[
\min_{w_1, \ldots, w_m \in \mathbb{R}^r, h_1, \ldots, h_r \in \mathbb{R}^n} F(w_1, \ldots, w_m, h_1, \ldots, h_r).
\]
(2.5)

To avoid confusion, we refer to the minimization problem (2.5) as the SSMF.

Let \( V := [v_1, \ldots, v_m]^T \). Then the partial gradient \( \nabla_{w_i} f \) of \( f \) defined in (2.3) at \( (w_1, \ldots, w_m, h_1, \ldots, h_r) \in \mathbb{R}^r \times \cdots \times \mathbb{R}^r \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \) is given by
\[
\nabla_{w_i} f(w_1, \ldots, w_m, h_1, \ldots, h_r) = H(H^T w_i - v_i)
\]
(2.6)

with the Lipschitz constant \( \|HH^T\|_2 \) for \( i = 1, \ldots, m \). Moreover, the partial gradient \( \nabla_{h_t} f \) of \( f \) at \( (w_1, \ldots, w_m, h_1, \ldots, h_r) \in \mathbb{R}^r \times \cdots \times \mathbb{R}^r \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \) is given by
\[
\nabla_{h_t} f(w_1, \ldots, w_m, h_1, \ldots, h_r) = -(U_t^T - h_t W(:, t) W(:, t)),
\]
(2.7)

with the Lipschitz constant \( \|W(:, t)\|^2 \) for \( t = 1, \ldots, r \), where \( U_t := V - \sum_{j=1}^{t-1} W(:, j) h_j^T - \sum_{j=t+1}^r W(:, j) h_j^T, \ 1 \leq t \leq r \).

We must point out that the function \( F \) defined by (2.4) is nonconvex-nonsmooth as \( f \) is nonconvex and smooth while \( \delta_{P_i}, \delta_{P_n}, \) and \( \delta_{Q_n} \) are all nonsmooth. However, we note that \( \delta_{P_i}, \delta_{P_n}, \) and \( \delta_{Q_n} \) are all lower semicontinuous (lsc). As noted in [1], the \( l_0 \)-norm \( \| \cdot \|_0 \) is
semialgebraic and thus satisfies the KL property (see for instance [16, 40, 41]). Hence, $F$ is
semialgebraic and thus is a KL function. It is natural to solve the SSMF (2.5) by the PALM
method in [6] and the generated sequence converges to a critical point of $F$. From [2, Chapter
14] and [11, Section 2.7]), one can see that, when we apply the AM method to the SSMF (2.5),
every limit point of the sequence generated by the AM method is a stationary point and it is
also a row-wise minimum under some assumptions. This means that the AM method may find
a better critical point. By combining the AM method with the PALM method, we provide a
row-wise update algorithm for solving the SSMF (2.5), which is stated in Algorithm 2.2 below.

Algorithm 2.2 A row-wise update algorithm for solving the SSMF (2.5)

**Step 0.** Choose $W^0 := [w_0^0, \ldots, w_m^0]^T \in G_1$, $H^0 := [h_0^0, \ldots, h_r^0]^T \in G_2 \cap G_3$, $\delta_1 > 0$, $\delta_2 > 0$, $c > 0$. Let $k := 0$.

**Step 1.** For $i = 1, \ldots, m$, take $\mu_{ik} = \min \{c, \frac{\|\nabla \phi_k(w_i^k)\|^2}{\|H_i^k\|^2} \}$ and compute

$$w_i^k = \Pi_{P_1}(w_i^k - \mu_{ik} \nabla \phi_k(w_i^k)) \quad (2.8)$$

such that if $\Phi_k(w_i^k) - \Phi_k(\bar{w}_i^k) \geq \frac{\delta_1}{2} \|w_i^k - \bar{w}_i^k\|^2$, then set $w_i^{k+1} = \bar{w}_i^k$; else set

$$w_i^{k+1} = \Pi_{P_1}(w_i^k - \mu_{ik} \nabla \phi_k(w_i^k)), \quad (2.9)$$

where $\phi_k(w_i) := f(w_i^k, \ldots, w_{i-1}^k, w_i, w_{i+1}^k, \ldots, w_m^k, H_k)$, $\Phi_k(w_i) := \phi_k(w_i) + \delta_{P_n}(w_i)$, and $\mu_{ik} = 1/\|H_k^k\|^2 + \delta_1$. Set

$$W^{k+1} := [w_1^{k+1}, \ldots, w_m^{k+1}]^T.$$

**Step 2.** For $t = 1, \ldots, r$, if $\|W^{k+1}(t)\| = 0$, then set $h_i^{k+1} = h_i^k$; else compute

$$\tilde{h}_i^k = \arg\min_{h_t \in P_n \cap Q_n} \psi_k(h_t) \quad (2.10)$$

such that if $\Psi_k(h_i^k) - \Psi_k(\tilde{h}_i^k) \geq \frac{\delta_2}{2} \|h_i^k - \tilde{h}_i^k\|^2$, then set $h_i^{k+1} = \tilde{h}_i^k$; else set

$$h_i^{k+1} = \Pi_{P_n \cap Q_n}(h_i^k - \nu_{tk} \nabla \psi_k(h_i^k)), \quad (2.11)$$

where $\psi_k(h_t) := f(W^{k+1}, h_1^{k+1}, \ldots, h_{i-1}^{k+1}, h_t, h_{i+1}^{k+1}, \ldots, h_r^{k+1})$, $\Psi_k(h_t) := \psi_k(h_t) + \delta_{P_n}(h_t) + \delta_{Q_n}(h_t)$, and $\nu_{tk} = 1/\|W^{k+1}(t)\|^2 + \delta_2$. Set

$$H^{k+1} := [h_1^{k+1}, \ldots, h_r^{k+1}]^T.$$

**Step 3.** Replace $k$ by $k + 1$ and go to Step 1.

Regarding Algorithm 2.2, we have the following comments.

- We note that, for any fixed $H = [h_1, \ldots, h_r]^T \in \mathbb{R}^{r \times n}$, the function $F$ defined in (2.4) is
separable with respect to the variables $w_i$’s, i.e.,

$$F(W, H) = \sum_{i=1}^{m} \frac{1}{2} \|H^T w_i - v_i\|^2 + \sum_{i=1}^{m} \delta_{\mathcal{P}_i}(w_i) + \sum_{t=1}^{r} \delta_{\mathcal{P}_n}(h_t) + \sum_{t=1}^{r} \delta_{\mathcal{Q}_s}(h_t).$$

Hence, in Step 1 of Algorithm 2.2 we update $w_k$’s separately via the proximal gradient method. Also, we have by the definition of $\Phi_k$, for $i = 1, \ldots, m$,

$$\Phi_k(w_k^i) - \Phi_k(\bar{w}_k^i) = \frac{1}{2} \left( \|H^k w_k^i - v_i\|^2 - \|H^k \bar{w}_k^i - v_i\|^2 \right),$$

which is easy to estimate.

• We note that, for any fixed $W = [w_1, \ldots, w_m]^T \in \mathbb{R}^{m \times r}$, $F$ is non-separable with respect to the variables $h_t$’s. Thus, in Step 2 of Algorithm 2.2 it is preferred to using the AM method to update $h_t$’s. That is, for each $1 \leq t \leq r$, we update $h_t^k$ by (2.10) if the minimum in (2.10) is uniquely attained, which satisfies the sufficient decrease condition $\Psi_k(h_t^k) - \Psi_k(\bar{h}_t^k) \geq \frac{\delta_1}{2} \|h_t^k - h_t^k\|^2$; Otherwise, we update $h_t^k$ via the proximal gradient scheme (2.11).

• For the scalar $\mu_{ik}$ defined in Step 1 of Algorithm 2.2 we have the following bounds. Since $H_k = [h_k^1, \ldots, h_k^r]$ with $h_t^k \in \mathcal{P}_n \cap \mathcal{Q}_s$ for $t = 1, \ldots, r$, we have $\|H_k\|_1 = \max_{1 \leq j \leq n} \sum_{t=1}^r h_{tj} \leq r$ and $\|H_k\|_\infty = 1$. Thus,

$$\|H^k(H^k)^T\|_2 \leq \|H^k\|_2 \leq \|H^k\|_1 \|H^k\|_\infty = \|H^k\|_1 \leq r. \quad (2.12)$$

If $\mu_{ik}$ is determined by (2.8), then from (2.12) we have

$$1/r \leq 1/\|H^k(H^k)^T\|_2 \leq \mu_{ik} \leq c. \quad (2.13)$$

If $\mu_{ik}$ is determined by (2.9), then from (2.12) we have

$$1/(r + \delta_1) \leq \mu_{ik} \leq 1/\delta_1. \quad (2.14)$$

• All subproblems in Algorithm 2.2 can be solved efficiently (see Section 3.1).

3 Convergence analysis

In this section, we first derive the projection onto $\mathcal{P}_n \cap \mathcal{Q}_s$. Then we establish the global convergence of Algorithm 2.2.

3.1 Projection onto $\mathcal{P}_n \cap \mathcal{Q}_s$

We note that, for any given $y \in \mathbb{R}^n$, $z = \Pi_{\mathcal{P}_n \cap \mathcal{Q}_s}(y)$ is the unique solution to the following minimization problem:

$$\min_{S \subseteq [n], \text{card}(S) = s} \frac{1}{2} \|z - y\|^2 \quad \text{s.t.} \quad z \in M^*_S := \{z \in \mathbb{R}^n \mid \sum_{i \in S} z_i = 1, z_S \geq 0, z_{\overline{S}} = 0\}. \quad (3.1)$$

For problem (3.1), we have the following result.
Proposition 3.1. Let $y \in \mathbb{R}^n$ and $1 \leq s \leq n$. Suppose $\pi = \{\pi(1), \ldots, \pi(n)\}$ is a permutation such that $y_{\pi(1)} \geq y_{\pi(2)} \geq \cdots \geq y_{\pi(n)}$. Let $S_\pi = \{\pi(1), \ldots, \pi(s)\} \subset [n]$. Then a solution $z^* \in \mathbb{R}^n$ to problem \ref{eq:3.1} is given by

$$z^* := \arg\min_{z \in M_{S_\pi}^n} \frac{1}{2} \|z - y\|^2.$$

\textit{Proof.} Without loss of generality, we assume that $y_1 \geq y_2 \geq \cdots \geq y_n$. In this case, $S_\pi = \{1, \ldots, s\}$. For any $S = \{j_1, \ldots, j_s\} \subset [n]$ with $j_1 < j_2 < \cdots < j_s$, there exists a permutation matrix $P(S) \in \mathbb{R}^{n \times n}$ such that $z \in M_{S_\pi}^n$ if and only if $z = P(S)w$ for some $w \in M_{S_\pi}^n$. Let $\hat{z} := \arg\min_{z \in M_{S_\pi}^n} \frac{1}{2} \|z - y\|^2$ and we have $\hat{z} = P(S)\hat{w} \in M_{S_\pi}^n$, where

$$\hat{w} := \arg\min_{w \in M_{S_\pi}^n} \frac{1}{2} \|w - (P(S))^T y\|^2.$$

By the definitions of $z^*$ and $\hat{w}$ we have

$$\frac{1}{2} \|z^* - y\|^2 \leq \frac{1}{2} \|\hat{w} - y\|^2 = \frac{1}{2} \sum_{i=1}^{s} (\hat{w}_i - y_i)^2 + \frac{1}{2} \sum_{i=s+1}^{n} y_i^2 = \frac{1}{2} \sum_{i=1}^{s} \hat{w}_i^2 + \frac{1}{2} \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{s} \hat{z}_j y_i = \frac{1}{2} \sum_{i=1}^{s} \hat{z}_j^2 + \frac{1}{2} \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{s} \hat{z}_j y_i,$$

(3.3)

where the last equality uses $\hat{z} = P(S)\hat{w}$.

By hypothesis, $y_1 \geq y_2 \geq \cdots \geq y_n$ and $j_1 < j_2 < \cdots < j_s$. Thus, $y_i \geq y_{j_i}$ for $i = 1, \ldots, s$. We note that $\hat{z}_{j_i} \geq 0$ for $i = 1, \ldots, s$ since $\hat{z} \in M_{S_\pi}^n$. Hence,

$$-\hat{z}_{j_i} y_i \leq -\hat{z}_{j_i} y_{j_i} \quad \text{for} \quad i = 1, \ldots, s.$$

This, together with (3.3), implies that

$$\frac{1}{2} \|z^* - y\|^2 \leq \frac{1}{2} \sum_{i=1}^{s} \hat{z}_j^2 + \frac{1}{2} \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{s} \hat{z}_j y_{j_i} = \frac{1}{2} \sum_{i=1}^{s} (\hat{z}_j - y_{j_i})^2 + \frac{1}{2} \sum_{i \notin S} y_i^2 = \frac{1}{2} \|\hat{z} - y\|^2.$$ 

By the arbitrariness of $S$, the proof is complete. \hfill \Box

Remark 3.2. In \cite[Theorem 2.4]{39}, Xu et al. gave another way for finding a closed-form projection onto $\mathcal{P}_n \cap \mathcal{Q}_n^s$.

Based on Proposition 3.1 and the projection of a vector onto the probability simplex \cite[59], we can find $z = \Pi_{\mathcal{P}_n \cap \mathcal{Q}_n^s}(y)$ of a given vector $y \in \mathbb{R}^n$, which is described in Algorithm 3.3.

We point out that the complexity of Algorithm 3.3 is $O(n \log n)$, which is dominated by sorting the elements of $y$. 

8
Algorithm 3.3 Projection onto \( P_n \cap Q_n^r \)

**Step 0.** Given a vector \( y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n \) and a permutation \( \pi = \{\pi(1), \ldots, \pi(n)\} \) such that \( y_{\pi(1)} \geq y_{\pi(2)} \geq \cdots \geq y_{\pi(n)} \).

**Step 1.** Find \( \rho = \max_j \{ j \mid y_{\pi(j)} - \frac{1}{j} \left( \sum_{r=1}^{j} y_{\pi(r)} - 1 \right) > 0 \} \) and define \( \beta = \frac{1}{\rho} \left( \sum_{r=1}^{\rho} y_{\pi(r)} - 1 \right) \).

**Step 2.** Set \( z := (z_1, \ldots, z_n)^T \in \mathbb{R}^n \) with \( z_{\pi(j)} = \max\{y_{\pi(j)} - \beta, 0\} \) for \( j = 1, \ldots, s \) and \( z_{\pi(j)} = 0 \) for \( j = s + 1, \ldots, n \).

3.2 Global convergence of Algorithm [2.2]

In this subsection, we establish the global convergence of Algorithm [2.2]. In the following, we present some necessary lemma. We first recall the lemma for a continuously differentiable function ([4, 30]).

**Lemma 3.3** Let \( g : \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function, where the gradient \( \nabla g \) is Lipschitz-continuous with a Lipschitz constant \( L_g \). Then one has

\[
g(y_2) \leq g(y_1) + \langle y_2 - y_1, \nabla g(y_1) \rangle + \frac{L_g}{2} \|y_2 - y_1\|^2, \quad \forall y_1, y_2 \in \mathbb{R}^n.
\]

For the sequence \( \{h_t^k\} \) generated by Algorithm [2.2] we have the following result.

**Lemma 3.4** Let \( (W^k, H^k) \) be the current iterate generated by Algorithm [2.2] and \( W^{k+1} \) be defined in (2.8). For \( t = 1, \ldots, r \), if \( \|W^{k+1}(; t)\| > 0 \), then

\[
\tilde{h}_t^k = \Pi_{P_n \cap Q_n^r} \left( \frac{1}{\|W^{k+1}(; t)\|^2} (U_t^k)^T W^{k+1}(; t) - \Pi_{P_n \cap Q_n^r} \left( h_t^k - \frac{1}{\|W^{k+1}(; t)\|^2} \nabla_{h_t} \psi_k(h_t^k) \right) \right),
\]

where \( U_t^k := V - \sum_{j=1}^{t-1} W^{k+1}(; j)(h_j^{k+1})^T - \sum_{j=t+1}^{r} W^{k+1}(; j)(h_j^k)^T \).

**Proof.** By the definition of \( \tilde{h}_t^k \), we have

\[
\tilde{h}_t^k = \arg\min_{h_t \in P_n \cap Q_n^r} f(W^{k+1}, h_1^{k+1}, \ldots, h_{t-1}^{k+1}, h_t, h_{t+1}^{k+1}, \ldots, h_r^k) = \arg\min_{h_t \in P_n \cap Q_n^r} \frac{1}{2} \| (U_t^k)^T - h_t(W^{k+1}(; t))^T \|^2_2.
\]

Let \( 1 \leq t \leq r \) be fixed. By hypothesis, we have \( \|W^{k+1}(; t)\| > 0 \). Then one can construct an orthogonal matrix

\[
Q = \left[ \frac{W^{k+1}(; t)}{\|W^{k+1}(; t)\|}, q_2, \ldots, q_m \right] \in \mathbb{R}^{m \times m}.
\]

By (2.7) we have

\[
\nabla_{h_t} \psi_k(h_t^k) = -((U_t^k)^T - h_t^k(W^{k+1}(; t))^T) W^{k+1}(; t) = -(U_t^k)^T W^{k+1}(; t) + h_t^k \|W^{k+1}(; t)\|^2.
\]
It follows from the orthogonal invariance of the Frobenius matrix norm that

\[
\begin{align*}
\tilde{h}_t^k &= \arg\min_{h_t \in \mathcal{P}_n \cap \mathcal{Q}_n^\delta} \frac{1}{2} \left\| (U_t^k)^T Q - h_t(W^{k+1}(; t))^T Q \right\|^2_F \\
&= \arg\min_{h_t \in \mathcal{P}_n \cap \mathcal{Q}_n^\delta} \frac{1}{2} \frac{1}{\| W^{k+1}(; t) \|^2} \left\| (U_t^k)^T W^{k+1}(; t) - \| W^{k+1}(; t) \| h_t \right\|^2 \\
&= \arg\min_{h_t \in \mathcal{P}_n \cap \mathcal{Q}_n^\delta} \frac{\| W^{k+1}(; t) \|^2}{2} \left\| h_t - \frac{1}{\| W^{k+1}(; t) \|^2} (U_t^k)^T W^{k+1}(; t) \right\|^2 \\
&= \arg\min_{h_t \in \mathcal{P}_n \cap \mathcal{Q}_n^\delta} \frac{\| W^{k+1}(; t) \|^2}{2} \left\| h_t - \left( h_t^k - \frac{1}{\| W^{k+1}(; t) \|^2} \nabla_{h_t} \psi_k(h_t^k) \right) \right\|^2.
\end{align*}
\]

Lemma 3.4 shows the minimization in (2.10) is in fact a projection onto \( \mathcal{P}_n \cap \mathcal{Q}_n^\delta \), which can be computed via Algorithm 3.3.

We also have the following useful properties for Algorithm 2.2.

**Lemma 3.5** Let \( \{ (W^k, H^k) \} \) be the sequence generated by Algorithm 2.2. Then for any \( k \geq 0 \),

\[
\Phi_k(w_i^k) - \Phi_k(w_i^{k+1}) \geq \frac{\delta_1}{2} \| w_i^k - w_i^{k+1} \|^2,
\]

for \( i = 1, 2, \ldots, m \) and

\[
\Psi_k(h_t^k) - \Psi_k(h_t^{k+1}) \geq \frac{\delta_2}{2} \| h_t^k - h_t^{k+1} \|^2,
\]

for \( t = 1, 2, \ldots, r \).

**Proof.** We first prove that (3.4) holds for all \( k \geq 0 \). Let \( k \geq 0 \) and \( 1 \leq i \leq m \) be fixed. If \( \Phi_k(w_i^k) - \Phi_k(w_i^{k+1}) \geq \frac{\delta_1}{2} \| w_i^k - w_i^{k+1} \|^2 \), then we have \( w^{k+1} = w_i^k \). In this case, it is natural that (3.4) holds. Otherwise, \( w_i^{k+1} \) is given by (2.9), i.e.,

\[
w_i^{k+1} = \arg\min_{w_i \in \mathbb{R}^r} \{ \langle w_i - w_i^k, \nabla \phi_k(w_i^k) \rangle + \frac{1}{2\mu_ik} \| w_i - w_i^k \|^2 + \delta_{P_r}(w_i) \},
\]

which gives rise to

\[
\langle w_i^{k+1} - w_i^k, \nabla \phi_k(w_i^k) \rangle + \frac{1}{2\mu_ik} \| w_i^{k+1} - w_i^k \|^2 + \delta_{P_r}(w_i^{k+1}) \leq \delta_{P_r}(w_i^k).
\]

By Lemma 3.3 with \( g = \phi_k \), we have

\[
\phi_k(w_i^{k+1}) + \delta_{P_r}(w_i^{k+1}) \leq \phi_k(w_i^k) + \langle w_i^{k+1} - w_i^k, \nabla \phi_k(w_i^k) \rangle + \frac{\| H^k(H^k)^T \|_2}{2} \| w_i^{k+1} - w_i^k \|^2 + \delta_{P_r}(w_i^{k+1}).
\]
Thus, together with (3.6), yields
\[
\phi_k(w_i^{k+1}) + \delta P_i(w_i^{k+1}) \leq \phi_k(w_i^k) + \delta P_i(w_i^k) - \frac{1}{\mu_k} \left( \frac{1}{2} \mu_k - \|H^k(H^k)^T\|_2 \right) \|w_i^{k+1} - w_i^k\|^2.
\]
By hypothesis, \(\mu_k = 1/(\|H^k(H^k)^T\|_2 + \delta_1)\). Thus, we have
\[
0 < \delta_1 = \frac{1}{\mu_k} - \|H^k(H^k)^T\|_2.
\]
Therefore,
\[
\Phi_k(w_i^k) - \Phi_k(w_i^{k+1}) \geq \frac{\delta_1}{2} \|w_i^k - w_i^{k+1}\|^2.
\]
Next, we show that (3.5) holds for \(t = 1, 2, \ldots, r\) and for all \(k \geq 0\). Let \(1 \leq l \leq r\) and \(k \geq 0\) be fixed. We establish (3.5) in different cases.

(i) If \(\|W^{k+1}(:,t)\| = 0\), then \(h_t^{k+1} = h_t^k\). In this case, it is obvious that (3.5) holds.

(ii) Suppose \(\|W^{k+1}(:,t)\| \neq 0\). If \(\psi_k(h_t^k) - \psi_k(h_t^k) \geq \frac{\delta_2}{2} \|h_t^k - h_t^k\|^2\), then \(h_t^{k+1} = h_t^k\) and thus (3.5) also holds. Otherwise, \(h_t^{k+1}\) is determined by (2.11), i.e.,
\[
h_t^{k+1} = \arg\min_{h \in \mathbb{R}^n} \{h_t^k - h_t^k, \nabla \psi_k(h_t^k) + \frac{1}{2\nu_{tk}} \|h_t^k - h_t^k\|^2 + \delta P_n(h_t^k) + \delta Q_n(h_t^k)\}.
\]

Thus,
\[
\langle h_t^{k+1} - h_t^k, \nabla \psi_k(h_t^k) \rangle + \frac{1}{2\nu_{tk}} \|h_t^{k+1} - h_t^k\|^2 + \delta P_n(h_t^{k+1}) + \delta Q_n(h_t^{k+1}) \leq \delta P_n(h_t^k) + \delta Q_n(h_t^k).
\]

By Lemma 3.3 for \(g = \psi_k\) we have
\[
\psi_k(h_t^{k+1}) + \delta P_n(h_t^{k+1}) + \delta Q_n(h_t^{k+1}) \\
\leq \psi_k(h_t^k) + \langle h_t^{k+1} - h_t^k, \nabla \psi_k(h_t^k) \rangle + \frac{\|W^{k+1}(:,t)\|^2}{2} \|h_t^{k+1} - h_t^k\|^2 + \delta P_n(h_t^{k+1}) + \delta Q_n(h_t^{k+1}).
\]
This, together with (3.8), yields
\[
\psi_k(h_t^{k+1}) + \delta P_n(h_t^{k+1}) + \delta Q_n(h_t^{k+1}) \\
\leq \psi_k(h_t^k) + \delta P_n(h_t^k) + \delta Q_n(h_t^k) + \frac{1}{2\nu_{tk}} \left( \frac{1}{2} - \|W^{k+1}(:,t)\|^2 \right) \|h_t^{k+1} - h_t^k\|^2.
\]
By hypothesis, \(\nu_{tk} = 1/(\|W^{k+1}(:,t)\|^2 + \delta_2)\). Then we obtain (3.5).

The following lemma shows the monotone decreasing of the sequence \(\{F(W^k, H^k)\}\) generated by Algorithm 2.2

Lemma 3.6 The sequence \(\{F(W^k, H^k)\}\) generated by Algorithm 2.2 is monotonic decreasing and for all \(k \geq 0\),
\[
F(W^k, H^k) - F(W^{k+1}, H^{k+1}) \geq \frac{\delta_1}{2} \|W^{k+1} - W^k\|^2 + \frac{\delta_2}{2} \|H^{k+1} - H^k\|^2.
\]
Proof. If follows from Lemma 3.5 that, for any $k \geq 0$,
\[
F(W^k, H^k) - F(W^{k+1}, H^{k+1}) = F(W^k, H^k) - F(W^{k+1}, H^k) + F(W^{k+1}, H^k) - F(W^{k+1}, H^{k+1})
\]
\[
= \sum_{i=1}^{m} (\Phi_k(w_i^k) - \Phi_k(w_i^{k+1})) + \sum_{t=1}^{r} (\Psi_k(h_t^k) - \Psi_k(h_t^{k+1}))
\]
\[
\geq \frac{\delta_1}{2} \sum_{i=1}^{m} \|w_i^{k+1} - w_i^k\|^2 + \frac{\delta_2}{2} \sum_{t=1}^{r} \|h_t^{k+1} - h_t^k\|^2 = \frac{\delta_1}{2} \|W^{k+1} - W^k\|^2_F + \frac{\delta_2}{2} \|H^{k+1} - H^k\|^2_F.
\]
This shows that $\{F(W^k, H^k)\}$ is monotonic decreasing and bounded below. \hfill \Box

Remark 3.7 From Lemma 3.6 it follows that for any integer $q > 0$,
\[
\sum_{k=0}^{q} \|(W^{k+1}, H^{k+1}) - (W^k, H^k)\|^2_F \leq \frac{2}{\delta_3} (F(W^0, H^0) - F(W^{q+1}, H^{q+1})) \leq \frac{2}{\delta_3} F(W^0, H^0),
\]
where $\delta_3 := \min\{\delta_1, \delta_2\}$, which implies that
\[
\sum_{k=0}^{\infty} \|(W^{k+1}, H^{k+1}) - (W^k, H^k)\|^2_F < \infty.
\]

The following lemma gives a subgradient lower bound for the successive iterate gap. We refer to Definition A.1 and Lemma A.2 for the subgradient of $F$.

Lemma 3.8 Let $\{W^k, H^k\}$ be the sequence generated by Algorithm 2.2. For any $k \geq 0$, define $\mathbf{A}^{k+1} = [\mathbf{a}_1^{k+1}, \ldots, \mathbf{a}_m^{k+1}]^T$ with
\[
\mathbf{a}_i^{k+1} := \nabla_{w_i} f(W^{k+1}, H^{k+1}) - \nabla \phi_k(w_i^k) - \frac{1}{\mu_{ik}} (w_i^{k+1} - w_i^k),
\]
for $i = 1, \ldots, m$ and $\mathbf{B}^{k+1} = [\mathbf{b}_1^{k+1}, \ldots, \mathbf{b}_r^{k+1}]^T$ with
\[
\mathbf{b}_t^{k+1} := \nabla_{h_t} f(W^{k+1}, H^{k+1}) - \nabla \psi_k(h_t^k) - \frac{1}{\alpha_{tk}} (h_t^{k+1} - h_t^k),
\]
for $t = 1, \ldots, r$, where $\alpha_{tk} = 1$ if $\|W^{k+1}(; t)\| = 0$; else $\alpha_{tk} = 1/\|W^{k+1}(; t)\|^2$ if $\Psi_k(h_t^k) - \Psi_k(h_t^k) \geq \delta_2 \|h_t^k - h_t^k\|^2$ and $\alpha_{tk} = \nu_{tk}$ if $\Psi_k(h_t^k) - \Psi_k(h_t^k) < \frac{\delta_2}{2} \|h_t^k - h_t^k\|^2$. Then $(\mathbf{A}^{k+1}, \mathbf{B}^{k+1}) \in \partial F(W^{k+1}, H^{k+1})$ and
\[
\|(\mathbf{A}^{k+1}, \mathbf{B}^{k+1})\|_F \leq \sqrt{\delta_4} \|(W^{k+1}, H^{k+1}) - (W^k, H^k)\|_F,
\]
for $\delta_4 := \max\{2(\delta_1 + 2r)^2, 2(\|V\|_2 + 2\sqrt{mr})^2 + (2m + \delta_2)^2 r (r + 1)/2\}$. 

12
Proof. Let $k \geq 0$ and $1 \leq i \leq m$ be fixed. We first show that $(A^{k+1}, B^{k+1}) \in \partial F(W^{k+1}, H^{k+1})$. It is easy to see that the iterate $w_i^{k+1}$ defined in Algorithm 2.2 is a solution to the following minimization problem:

$$\min_{w_i \in \mathbb{R}^r} \langle w_i - w_i^k, \nabla \phi_k(w_i^k) \rangle + \frac{1}{2\mu_{ik}} \|w_i - w_i^k\|_F^2 + \delta_{P_i}(w_i).$$

Then there exists an element $\theta_i^{k+1} \in \partial \delta_{P_i}(w_i^{k+1})$ such that

$$\nabla \phi_k(w_i^k) + \frac{1}{\mu_{ik}}(w_i^{k+1} - w_i^k) + \theta_i^{k+1} = 0.$$

By definition, $\nabla_{w_i} f(W^{k+1}, H^{k+1}) + \theta_i^{k+1} \in \partial w_i F(W^{k+1}, H^{k+1})$. Therefore,

$$a_i^{k+1} := \nabla_{w_i} f(W^{k+1}, H^{k+1}) - \nabla \phi_k(w_i^k) - \frac{1}{\mu_{ik}}(w_i^{k+1} - w_i^k) \in \partial w_i F(W^{k+1}, H^{k+1}).$$

Hence, we have $A^{k+1} \in \partial_W F(W^{k+1}, H^{k+1})$ since $a_i^{k+1} \in \partial w_i F(w_i^{k+1}, H^{k+1})$ for $i = 1, \ldots, m$.

Next, we show that $B^{k+1} \in \partial_H F(W^{k+1}, H^{k+1})$ by proving that $b_t^{k+1} \in \partial_{h_t} F(W^{k+1}, H^{k+1})$ for $t = 1, \ldots, r$. Let $1 \leq t \leq r$ be fixed. We show that $b_t^{k+1} \in \partial_{h_t} F(W^{k+1}, H^{k+1})$ in different cases.

(i) Suppose $\|W^{k+1}(; t)\| = 0$. In this case, we have

$$\nabla \psi_k(h_t^k) = -((U_t^k)^T h_t^k(W^{k+1}(; t))^T) W^{k+1}(; t) = 0,$$

where $U_t^k$ is defined as in Lemma 3.4. By Algorithm 2.2, we have $h_t^{k+1} = h_t^k$. Since $0 \in \partial \delta_{P_n}(h_t^{k+1})$ and $0 \in \partial \delta_{Q_n}(h_t^{k+1})$, we know that $\nabla_{h_t} f(W^{k+1}, H^{k+1}) \in \partial_{h_t} F(W^{k+1}, H^{k+1})$. Thus, $b_t^{k+1} := \nabla_{h_t} f(W^{k+1}, H^{k+1}) - \nabla \psi_k(h_t^k) - (h_t^{k+1} - h_t^k) = \nabla_{h_t} f(W^{k+1}, H^{k+1}) \in \partial_{h_t} F(W^{k+1}, H^{k+1}).$

(ii) Suppose $\|W^{k+1}(; t)\| \neq 0$. If $\Psi_k(h_t^k) - \psi_k(h_t^k) \geq \frac{\delta}{2}\|h_t^k - h_t^k\|^2$, then $h_t^{k+1} = h_t^k$. By Lemma 3.4 we have

$$h_t^k = \Pi_{P_n \cap Q_n}(h_t^k - \frac{1}{\|W^{k+1}(; t)\|^2} \nabla \psi_k(h_t^k)),
$$

which is a solution to the following minimization problem:

$$h_t^{k+1} = \arg\min_{h_t \in \mathbb{R}^r} \left\{\langle h_t - h_t^k, \nabla \psi_k(h_t^k) \rangle + \frac{1}{2\alpha_{tk}} \|h_t - h_t^k\|^2 + \delta_{P_n}(h) + \delta_{Q_n}(h)\right\},$$

where $\alpha_{tk} = 1/\|W^{k+1}(; t)\|^2$. Then there exists an element $\zeta_t^{k+1} \in \partial \delta_{P_n}(h_t^{k+1})$ and $\omega_t^{k+1} \in \partial \delta_{Q_n}(h_t^{k+1})$ such that

$$\nabla \psi_k(h_t^k) + \frac{1}{\alpha_{tk}} (h_t^{k+1} - h_t^k) + \zeta_t^{k+1} + \omega_t^{k+1} = 0.$$

By definition, $\nabla_{h_t} f(W^{k+1}, H^{k+1}) + \zeta_t^{k+1} + \omega_t^{k+1} \in \partial_{h_t} F(W^{k+1}, H^{k+1})$. Therefore,

$$b_t^{k+1} := \nabla_{h_t} f(W^{k+1}, H^{k+1}) - \nabla \psi_k(h_t^k) - \frac{1}{\alpha_{tk}} (h_t^{k+1} - h_t^k) \in \partial_{h_t} F(W^{k+1}, H^{k+1}).$$
On the other hand, if $\Psi_k(h^k_t) - \Psi_k(h^k_t) < \frac{\delta_1}{2} \|h^k_t - \bar{h}^k\|^2$, then we have (2.11),

$$h^{k+1}_t = \Pi_{\mathcal{P}_n \cap Q_n} \left( h^k_t - \nu_k \nabla \psi_k(h^k_t) \right),$$

which is a solution to the following minimization problem:

$$h^{k+1}_t = \arg \min_{h_t \in \mathbb{R}^r} \left\{ \langle h_t - h^k_t, \nabla \psi_k(h^k_t) \rangle + \frac{1}{2\alpha_{tk}} \| h_t - h^k_t \|^2 + \delta_{\mathcal{P}_n}(h) + \delta_{\mathcal{Q}_n}(h) \right\},$$

where $\alpha_{tk} = \nu_k$. Then there exists an element $\zeta^{k+1}_t \in \partial \delta_{\mathcal{P}_n}(h^{k+1}_t)$ and $\varsigma^{k+1}_t \in \partial \delta_{\mathcal{Q}_n}(h^{k+1}_t)$ such that

$$\nabla \psi_k(h^k_t) + \frac{1}{\alpha_{tk}} (h^{k+1}_t - h^k_t) + \zeta^{k+1}_t + \varsigma^{k+1}_t = 0.$$

By definition, $\nabla h_t f(W^{k+1}, H^{k+1}) + \zeta^{k+1}_t + \varsigma^{k+1}_t \in \partial h_t F(W^{k+1}, H^{k+1})$. Therefore,

$$b^{k+1}_t := \nabla h_t f(W^{k+1}, H^{k+1}) - \nabla \psi_k(h^k_t) - \frac{1}{\alpha_{tk}} (h^{k+1}_t - h^k_t) \in \partial h_t F(W^{k+1}, H^{k+1}).$$

Next, we show that (3.9) holds. By the definition of $A^{k+1}$ and $\phi_k$ we have

$$A^{k+1} := \nabla_W f(W^{k+1}, H^{k+1}) - \nabla_W f(W^k, H^k) - D^k(W^{k+1} - W^k),$$

where $D^k = \text{diag}(1/\mu_{1k}, \ldots, 1/\mu_{mk})$. Using (2.13) and (2.14) we have

$$\|D^k\|_2 \leq r + \delta_1. \tag{3.10}$$

In addition, since $W^k \in \mathcal{G}_1$ we have

$$\|W^k\|_2^2 \leq \|W^k\|_1 \|W^k\|_\infty = \|W^k\|_1 \leq m. \tag{3.11}$$

From (2.12), (3.10), and (3.11) we have

$$\|A^{k+1}\|_F \leq \|\nabla_W f(W^{k+1}, H^{k+1}) - \nabla_W f(W^k, H^k)\|_F + \|D^k(W^{k+1} - W^k)\|_F$$

$$= \|((W^{k+1}H^{k+1} - V)(H^{k+1})^T - (W^kH^k - V)(H^k)^T\|_F + \|D^k(W^{k+1} - W^k)\|_F$$

$$\leq \|V\|_2 \|H^{k+1} - H^k\|_F + \|H^k(H^k)^T\|_2 W^{k+1} - W^k\|_F + \|W^{k+1}\|_2 H^k\|_2 H^{k+1} - H^k\|_F$$

$$+ \|W^{k+1}\|_2 H^{k+1} - H^k\|_F + \|D^k\|_2 W^{k+1} - W^k\|_F$$

$$\leq \|V\|_2 + 2\sqrt{m} \|H^{k+1} - H^k\|_F + (\delta_1 + 2r) \|W^{k+1} - W^k\|_F. \tag{3.12}$$

On the other hand, by the definition of $b^{k+1}_t$ and $\psi_k$ and using (2.7) we have, for $t = 1, \ldots, r$,

$$b^{k+1}_t = \nabla h_t f(W^{k+1}, H^{k+1}) - \nabla h_t f(W^{k+1}, h^{k+1}_1, \ldots, h^{k+1}_{t-1}, h^k_t, \ldots, h^k_r) - \frac{1}{\alpha_{tk}} (h^{k+1}_t - h^k_t)$$

$$= \left( \sum_{j=t}^r (h^{k+1}_j - h^k_j) (W^{k+1}(::, j))^T \right) W^{k+1}(::, t) - \frac{1}{\alpha_{tk}} (h^{k+1}_t - h^k_t)$$
and thus
\[ \| \mathbf{b}^{k+1} \| \leq \left( \sum_{j=t}^{r} \| \mathbf{b}^{k+1} - \mathbf{b}^{k} \| \| (W^{k+1}(\cdot, j)) \| \right) \| W^{k+1}(\cdot, t) \| + (\| W^{k+1}(\cdot, t) \| + \delta_{2}) \| \mathbf{h}^{k+1} - \mathbf{h}^{k} \| \]
\leq m \sum_{j=t}^{r} \| \mathbf{b}^{k+1} - \mathbf{b}^{k} \| + (m + \delta_{2}) \| \mathbf{h}^{k+1} - \mathbf{h}^{k} \| \leq (2m + \delta_{2}) \sum_{j=t}^{r} \| \mathbf{h}^{k+1} - \mathbf{h}^{k} \|,

where the second inequality uses the fact that \( \| W^{k+1}(\cdot, j) \|^{2} \leq m \) and
\[
1/\alpha_{tk} \leq \max(\| W^{k+1}(\cdot, t) \|^{2} + \delta_{2}, 1) \leq m + \delta_{2}.
\]

Hence,
\[
\| B^{k+1} \|^{2}_{F} = \sum_{t=1}^{r} \| \mathbf{b}^{k+1} \|^{2} \leq \sum_{t=1}^{r} (2m + \delta_{2}) \left( \sum_{j=t}^{r} \| \mathbf{h}^{k+1} - \mathbf{h}^{k} \| \right)^{2}
\leq (2m + \delta_{2}) \sum_{t=1}^{r} (r - t + 1) \sum_{j=t}^{r} \| \mathbf{h}^{k+1} - \mathbf{h}^{k} \|^{2}
\leq (2m + \delta_{2})^{2} \frac{r(r + 1)}{2} \| H^{k+1} - H^{k} \|^{2}_{F}.
\]

From (3.12) and (3.13) we obtain
\[
\| (A^{k+1}, B^{k+1}) \|^{2}_{F} = \| A^{k+1} \|^{2}_{F} + \| B^{k+1} \|^{2}_{F}
\leq 2(\| V \|_{2} + 2\sqrt{m}r)^{2} \| H^{k+1} - H^{k} \|^{2}_{F} + 2(\delta_{1} + 2r)^{2} \| W^{k+1} - W^{k} \|_{F}^{2}
+ (2m + \delta_{2})^{2} \frac{r(r + 1)}{2} \| H^{k+1} - H^{k} \|^{2}_{F} \leq \delta_{4} \| (W^{k+1}, H^{k+1}) - (W^{k}, H^{k}) \|^{2}_{F},
\]
which implies that (3.9) holds.

In the following, we set \( \mathcal{L}(W^{0}, H^{0}) \) to be the set of all accumulation points of the sequence \( \{(W^{k}, H^{k})\} \) generated by Algorithm 2.2. Regarding the set \( \mathcal{L}(W^{0}, H^{0}) \), we have the following result.

**Lemma 3.9** Let \( \{(W^{k}, H^{k})\} \) be the sequence generated by Algorithm 2.2. Then \( \mathcal{L}(W^{0}, H^{0}) \) is a nonempty and compact set and the function \( F(W, H) \) is finite and constant on \( \mathcal{L}(W^{0}, H^{0}) \).

**Proof.** It is obvious that \( \mathcal{L}(W^{0}, H^{0}) \) is nonempty and compact since the sequence \( \{(W^{k}, H^{k})\} \) is bounded. By Lemma 3.6, we know that the sequence \( \{F(W^{k}, H^{k})\} \) converges to a finite limit \( F^{*} \). For any \( (W^{*}, H^{*}) \in \mathcal{L}(W^{0}, H^{0}) \), there exists a subsequence \( \{(W^{k_{q}}, H^{k_{q}})\} \) such that \( \lim_{q \to \infty} (W^{k_{q}}, H^{k_{q}}) = (W^{*}, H^{*}) \). Thus, \( F(W^{*}, H^{*}) = \lim_{q \to \infty} F(W^{k_{q}}, H^{k_{q}}) = F^{*} \). Hence, \( F \) is finite and constant on \( \mathcal{L}(W^{0}, H^{0}) \).

On the global convergence of Algorithm 2.2, we have the following result.

15
Theorem 3.10 Let \( \{(W^k, H^k)\} \) be the sequence generated by Algorithm 2.2. Then every accumulation point of \( \{(W^k, H^k)\} \) is a critical point of \( F \).

**Proof.** The proof is similar to that of [6, Lemma 5(i)] by using Remark 3.7 and Lemmas 3.8–3.9.

Finally, on the convergence of the sequence \( \{(W^k, H^k)\} \) generated by Algorithm 2.2, we have the following result. The proof follows from the similar arguments of [6, Theorem 1] by using Lemmas 3.5–3.6, Lemmas 3.8–3.9, and Theorem 3.10. Hence, we omit it here.

Theorem 3.11 Let \( \{(W^k, H^k)\} \) be the sequence generated by Algorithm 2.2. Then the sequence \( \{(W^k, H^k)\} \) converges to a critical point of \( F \).

The following theorem shows that the sequence \( \{(W^k, H^k)\} \) generated by Algorithm 2.2 converges to a special critical point of \( F \).

Theorem 3.12 Let \( \{(W^k, H^k)\} \) be the sequence generated by Algorithm 2.2. Suppose \( \{(W^k, H^k)\} \) converges to \( (W^*, H^*) \). Then we have

\[
F(W^*, H^*) \leq F(W, H^*), \quad \forall W \in G_1 \tag{3.14}
\]

and for \( t = 1, \ldots, r \),

\[
F(W^*, H^*) \leq F(W^*, h_1^*, \ldots, h_{t-1}^*, h_t, h_{t+1}^*, \ldots, h_r^*) + \delta_2, \tag{3.15}
\]

for all \( h_t \in P \cap Q_n^s \).

**Proof.** We first show that (3.14). Let \( 1 \leq i \leq m \) be fixed. From Step 1 of Algorithm 2.2 we see that \( w_i^{k+1} \in P \) solves the following minimization problem:

\[
\min_{w_i \in \mathbb{R}^r} \frac{1}{2\mu_{ik}} \|w_i - (w_i^k - \mu_{ik} \nabla \phi_k(w_i^k))\|^2 \\
\text{s.t. } e^T w_i = 1, \quad w_i \geq 0. \tag{3.16}
\]

Then there exists two Lagrange multipliers \( \gamma_{k+1} \) and \( p^{k+1} \) such that the following first-order optimization conditions hold:

\[
\begin{cases}
\frac{1}{\mu_{ik}} (w_i^{k+1} - w_i^k) + \nabla \phi_k(w_i^k) + \gamma_{k+1} e - p^{k+1} = 0, \\
e^T w_i^{k+1} = 1, \quad w_i^{k+1} \geq 0,
\end{cases} \tag{3.17}
\]

We can obtain that

\[
\begin{cases}
\gamma_{k+1} = -\frac{1}{\mu_{ik}} (w_i^{k+1} - w_i^k) + \frac{1}{\mu_{ik}} \nabla \phi_k(w_i^k) + \gamma_{k+1} e, \\
p^{k+1} = \frac{1}{\mu_{ik}} (w_i^{k+1} - w_i^k) + \nabla \phi_k(w_i^k) + \gamma_{k+1} e.
\end{cases}
\]
We note that $\nabla \phi_k(w^k_i) = H^k((H^k)^T w^k_i - v_i)$. By hypothesis, $\{(W^k, H^k)\}$ converges to $(W^*, H^*)$. From Remark 3.7 we know that $\|W^{k+1} - W^k\|_F \to 0$ as $k \to \infty$. Also, $\{\mu_k\}$ is bounded. Taking $k \to \infty$ yields

$$
\begin{align*}
\lim_{k \to \infty} \gamma_{k+1} &= -\langle w^*_i, H^*((H^*)^T w^*_i - v_i) \rangle \equiv \gamma_s,
\lim_{k \to \infty} p^{k+1} &= H^*((H^*)^T w^*_i - v_i) + \gamma_s e \equiv p^*.
\end{align*}
$$

This, together with (3.17), implies that

$$
\begin{align}
H^*((H^*)^T w^*_i - v_i) + \gamma_s e - p^* &= 0,
\langle e^T w^*_i, 1, w^*_i \rangle &\geq 0,
\langle p^*, w^*_i \rangle &= 0, 
\end{align}
$$

(3.18)

It is obvious that $w^*_i$ satisfies conditions (3.18) for some Lagrange multipliers $\gamma_s$ and $p^*$ and $H^*(H^*)^T$ is positive semi-definite. Thus, $w^*_i$ is a global solution of the following minimization problem:

$$
\min_{w_i \in \mathbb{R}^r} \frac{1}{2} \|(H^*)^T w_i - v_i\|^2 \\
\text{s.t.} \quad e^T w_i = 1, \quad w_i \geq 0.
$$

(3.19)

We note that

$$
f(W, H^*) = \frac{1}{2} \|V - WH^*\|_F^2 = \sum_{i=1}^{m} \frac{1}{2} \|(H^*)^T w_i - v_i\|^2
$$

and $W \in G_1$ if and only if $e^T w_i = 1, w_i \geq 0 (i = 1, \ldots, m)$ for all $W := [w_1, \ldots, w_m]^T \in \mathbb{R}^{m \times r}$. It is easy to see that $W^* := [w^*_1, \ldots, w^*_m]^T$ is a global solution of the following minimization problem:

$$
\min_{W \in \mathbb{R}^{m \times r}} \min_{W \in \mathbb{R}^{m \times r}} f(W, H^*) = \frac{1}{2} \|V - WH^*\|_F^2 \\
\text{s.t.} \quad W \in G_1.
$$

(3.20)

Therefore,

$$
F(W^*, H^*) \leq F(W, H^*), \quad \forall W \in G_1.
$$

Next, we establish (3.14) for $t = 1, \ldots, r$. Let $1 \leq t \leq r$ be fixed. We show (3.14) in different cases.

(i) Suppose $\|W^{k+1}(:, t)\| = 0$. In this case, by Step 2 of Algorithm 2.2 we have $h^k_{t+1} = h^k_t$ and

$$
F(W^{k+1}, h^{k+1}_1, \ldots, h^{k+1}_{t-1}, h^{k+1}_{t+1}, \ldots, h^{k}) = F(W^{k+1}, h^{k+1}_1, \ldots, h^{k+1}_{t-1}, h_t, h^{k}_{t+1}, \ldots, h^{k})
= \frac{1}{2} \|V - \sum_{j=1}^{t-1} W^{k+1}(:, j)(h^{k+1}_j)^T - \sum_{j=t+1}^{r} W^{k+1}(:, j)(h^k_j)^T\|_F^2,
$$

(3.21)

for all $h_t \in \mathcal{P}_n \cap \mathcal{Q}^*_n$. 

17
(ii) Suppose $\|W^{k+1}(; t)\| \neq 0$. If $\Psi_k(h_t^k) - \Psi_k(\bar{h}_t^k) \geq \frac{\delta_2}{2}\|h_t^k - \bar{h}_t^k\|^2$, then $h_t^{k+1} = \bar{h}_t^k$, i.e.,

$$h_t^{k+1} = \arg\min_{h_t \in \mathbb{R}^n} F(W^{k+1}, h_t^{k+1}, \ldots, h_{t-1}^{k+1}, h_t, h_{t+1}^k, \ldots, h_k^k).$$

Thus, we have

$$F(W^{k+1}, h_1^{k+1}, \ldots, h_t^{k+1}, h_{t-1}^{k+1}, h_t, h_{t+1}^k, \ldots, h_k^k) \leq F(W^{k+1}, h_1^{k+1}, \ldots, h_t^{k+1}, h_t, h_{t+1}^k, \ldots, h_k^k),$$

(3.22)

for all $h_t \in \mathcal{P}_n \cap Q_n^s$.

On the other hand, if $\Psi_k(h_t^k) - \Psi_k(\bar{h}_t^k) < \frac{\delta_2}{2}\|h_t^k - \bar{h}_t^k\|^2$, then $h_t^{k+1}$ is determined by (2.11), i.e.,

$$h_t^{k+1} = \arg\min_{h_t \in \mathcal{P}_n \cap Q_n^s} \{\langle h_t - h_t^k, \nabla \psi_k(h_t^k) \rangle + \frac{1}{2\nu_k}\|h_t - h_t^k\|^2\}.$$

We note that $\psi_k$ is a quadratic function. By simple calculation, we find that, for any $h_t \in \mathcal{P}_n \cap Q_n^s$,

$$\psi_k(h_t) = \psi_k(h_t^k) + \langle h_t - h_t^k, \nabla \psi_k(h_t^k) \rangle + \frac{\|W^{k+1}(; t)\|^2}{2}\|h_t - h_t^k\|^2.$$

Thus,

$$\psi_k(h_t^{k+1}) = \psi_k(h_t^k) + \langle h_t^{k+1} - h_t^k, \nabla \psi_k(h_t^k) \rangle + \frac{\|W^{k+1}(; t)\|^2}{2}\|h_t^{k+1} - h_t^k\|^2$$

$$= \psi_k(h_t^k) + \langle h_t^{k+1} - h_t^k, \nabla \psi_k(h_t^k) \rangle + \frac{1}{2\nu_k}\|h_t^{k+1} - h_t^k\|^2 - \frac{\delta_2}{2}\|h_t^{k+1} - h_t^k\|^2$$

$$\leq \psi_k(h_t^k) + \langle h_t - h_t^k, \nabla \psi_k(h_t^k) \rangle + \frac{1}{2\nu_k}\|h_t - h_t^k\|^2 - \frac{\delta_2}{2}\|h_t^{k+1} - h_t^k\|^2$$

$$= \psi_k(h_t) + \frac{\delta_2}{2}\|h_t - h_t^k\|^2 - \frac{\delta_2}{2}\|h_t^{k+1} - h_t^k\|^2,$$

for all $h_t \in \mathcal{P}_n \cap Q_n^s$. This means that

$$F(W^{k+1}, h_1^{k+1}, \ldots, h_t^{k+1}, h_{t-1}^{k+1}, h_t, h_{t+1}^k, \ldots, h_k^k)$$

$$\leq F(W^{k+1}, h_1^{k+1}, \ldots, h_t^{k+1}, h_t, h_{t+1}^k, \ldots, h_k^k) + \frac{\delta_2}{2}\|h_t - h_t^k\|^2 - \frac{\delta_2}{2}\|h_t^{k+1} - h_t^k\|^2,$$

(3.23)

for all $h_t \in \mathcal{P}_n \cap Q_n^s$. By hypothesis, $\{(W^K, H^K)\}$ converges to $(W^*, H^*)$. Then, by using the similar arguments as in Theorem 3.10 to the inequalities (3.21)-(3.23), we have

$$F(W^*, h_1^*, \ldots, h_{t-1}^*, h_t^*, h_{t+1}^*, \ldots, h_r^*)$$

$$\leq F(W^*, h_1^*, \ldots, h_{r-1}^*, h_r, h_{r+1}^*, \ldots, h_r^*) + \frac{\delta_2}{2}\|h_t - h_t^*\|^2,$$

for all $h_t \in \mathcal{P}_n \cap Q_n^s$. We note that $\|h_t - h_t^*\|^2 \leq \|h_t\|^2 + \|h_t^*\|^2 \leq 2$ since $h_t, h_t^* \in \mathcal{P}_n \cap Q_n^s$. Therefore, (3.15) follows. ∎
Remark 3.13 From Theorem 3.12, we observe that the point \((W^*, H^*)\) is a global minimum of \(F\) over the \(W\)-factor as a whole. Also, the point \((W^*, H^*)\) is nearly a global minimum of \(F\) over each row vector of the \(H\)-factor if the parameter \(\delta_2 > 0\) is chosen to be sufficiently small. In the upcoming numerical tests, one can see that the solution to the SSMF obtained by the proposed algorithm may have a smaller factorization residual than the PALM method.

4 Numerical experiments

In this section, we report the numerical performance of Algorithm 2.2 for solving the SSMF (2.5) over synthetic and real data. To illustrate the effectiveness of our method, we compare Algorithm 2.2 with the PALM algorithm (i.e., Algorithm 2.1) proposed in [6]. All the numerical tests were implemented in MATLAB R2020a on a linux server (20-core, Intel (R) Xeon (R) Gold 6230 @ 2.10 GHz, 32 GB RAM).

In our numerical tests, for both algorithms, we randomly choose the same initial point \((W_0, H_0)\), and the stopping criterion is set to be
\[
\frac{\|W^k H^k - W^{k-1} H^{k-1}\|}{\|W^{k-1} H^{k-1}\|} \leq \text{tol},
\]
where “\(\text{tol}\)” is a prescribed tolerance. For Algorithms 2.1 and 2.2, we also set \(\delta_1 = 10^{-5}\), \(\delta_2 = 10^{-6}\), and \(c = 10.0\). In addition, we use ‘strue’, ‘ITmax’, ‘ct.’, and ‘res.’ to denote the number of non-zeros in the true solution, the largest number of iterations, the total computational in seconds, and the relative residual \(\|V - W^k H^k\|_F/\|V\|_F\) at the final iterates of the corresponding algorithms, accordingly.

4.1 Synthetic data

We first consider the following numerical example with synthetic data [28].

Example 4.1 We consider the SSMF with fixed \((m, r, n)\) and sparsity. Here, \(V = \hat{W} \hat{H} \in \mathbb{R}^{400 \times 200}\) with \(\hat{W} = \Pi_{G_1}(W) \in \mathbb{R}^{400 \times 15}\) and \(\hat{H} = \Pi_{G_2}(H) \in \mathbb{R}^{15 \times 200}\) with the true sparsity of each row of \(\hat{H}\) being \(ts\), where \(W\) and \(H\) are randomly generated by using \texttt{rand}. We report our numerical results for the prescribed sparsity \(s = ts = 10, 20, 30, 40, 50\).

Table 4.1 displays the numerical results for Example 4.1 by running Algorithms 2.1 and 2.2 over 100 randomly generated \(V\) and initial points. Here, we choose \(\text{tol} = 10^{-5}\) and \(\text{ITmax} = 4000\), and the recovery is considered to be successful if the relative reconstruction error \(\|V - W^k H^k\|_F/\|V\|_F\) is less than 1% at the final iterate \((W^k, H^k)\). Here, ‘ct.’ means the averaged total computing time over the number of successful reconstructions.

We can observe from Table 4.1 that with the increase of prescribed sparsity, the successful probability of reconstruction of our algorithm is much higher than Algorithm 2.1. At the same time, our algorithm is much effective over Algorithm 2.1 in terms of the computational time.

Example 4.2 We consider the SSMF with fixed \((m, r, n)\) and different prescribed sparsity. Here, \(V = \hat{W} \hat{H} \in \mathbb{R}^{400 \times 200}\), where \(\hat{W} = \Pi_{G_1}(W) \in \mathbb{R}^{400 \times 15}\) and \(\hat{H} = \Pi_{G_2}(H) \in \mathbb{R}^{15 \times 200}\) with the true...
sparsity of each row of $\hat{H}$ being $ts = 30$, where $W$ and $H$ are randomly generated by using $\text{rand}$. We report our numerical results for fixed $V$ and the prescribed sparsity $s = 30, 31, 32, 33, 34, 35$.

Table 4.1 lists the numerical results for Example 4.1 by running Algorithms 2.1 and 2.2 over 100 randomly generated initial points. Here we choose $\text{tol} = 10^{-5}$ and $\text{ITmax} = 4000$, and the recovery is considered to be successful if the relative reconstruction error $\|V - W^k H^k\|_F / \|V\|_F$ is less than 1%.

We see from Table 4.1 that our algorithm is much more effective than Algorithm 2.1 in terms of both the successful probability of reconstruction and the computing time.

| $s$ | Alg. 2.1 probability | ct. | Alg. 2.2 probability | ct. |
|-----|----------------------|-----|----------------------|-----|
| 10  | 0%                   | 0   | 64%                  | 3.4106 |
| 20  | 1%                   | 31.432 | 95%                  | 2.4371 |
| 30  | 23%                  | 22.868 | 100%                 | 3.4075 |
| 40  | 66%                  | 19.014 | 100%                 | 4.0772 |
| 50  | 62%                  | 15.580 | 100%                 | 4.9952 |

Table 4.2: Numerical results for Example 4.1

Example 4.2 We consider the SSMF with varied $(m, n)$ and fixed factorization rank and prescribed sparsity. Here, $V = \hat{W} \hat{H} \in \mathbb{R}^{200 \times 100}$, where $\hat{W} = \Pi_{G_1}(W) \in \mathbb{R}^{200 \times 15}$ and $\hat{H} = \Pi_{G_2}(H) \in \mathbb{R}^{15 \times 100}$ with the true sparsity of each row of $\hat{H}$ being $ts = 15$, where $W$ and $H$ are randomly generated by using $\text{rand}$. We report our numerical results for $j = 1, 2, 3, 4, 5$ and the prescribed sparsity $s = 20$.

Table 4.2 gives the numerical results for Example 4.2 by running Algorithms 2.1 and 2.2 over 100 randomly generated $V$ and initial points. Here, we choose $\text{tol} = 10^{-5}$ and $\text{ITmax} = 4000$, and the recovery is considered to be successful if the relative reconstruction error $\|V - W^k H^k\|_F / \|V\|_F$ is less than 1%.

We see from Table 4.2 that our algorithm is much more effective than Algorithm 2.1 in terms of both the successful probability of reconstruction and the computing time.

| $s$ | Alg. 2.1 probability | ct. | Alg. 2.2 probability | ct. |
|-----|----------------------|-----|----------------------|-----|
| 30  | 25%                  | 21.688 | 100%                 | 3.4756 |
| 31  | 22%                  | 20.401 | 99%                  | 3.8104 |
| 32  | 29%                  | 22.943 | 100%                 | 3.6208 |
| 33  | 31%                  | 21.267 | 100%                 | 3.6449 |
| 34  | 35%                  | 21.525 | 99%                  | 3.6948 |
| 35  | 45%                  | 21.068 | 100%                 | 3.7500 |

Table 4.2: Numerical results for Example 4.2

Example 4.3 We consider the SSMF with varied $(m, n)$ and fixed factorization rank and prescribed sparsity. Here, $V = \hat{W} \hat{H} \in \mathbb{R}^{200 \times 100}$, where $\hat{W} = \Pi_{G_1}(W) \in \mathbb{R}^{200 \times 15}$ and $\hat{H} = \Pi_{G_2}(H) \in \mathbb{R}^{15 \times 100}$ with the true sparsity of each row of $\hat{H}$ being $ts = 15$, where $W$ and $H$ are randomly generated by using $\text{rand}$. We report our numerical results for $j = 1, 2, 3, 4, 5$ and the prescribed sparsity $s = 20$.

Table 4.3 gives the numerical results for Example 4.3 by running Algorithms 2.1 and 2.2 over 100 randomly generated $V$ and initial points. Here, we choose $\text{tol} = 10^{-5}$ and $\text{ITmax} = 6000$, and the recovery is considered to be successful if the relative reconstruction error $\|V - W^k H^k\|_F / \|V\|_F$ is less than 1%.

We see from Table 4.3 that our algorithm is much more effective than Algorithm 2.1 in terms of both the successful probability of reconstruction and the computing time.

| $s$ | Alg. 2.1 probability | ct. | Alg. 2.2 probability | ct. |
|-----|----------------------|-----|----------------------|-----|
| 30  | 25%                  | 21.688 | 100%                 | 3.4756 |
| 31  | 22%                  | 20.401 | 99%                  | 3.8104 |
| 32  | 29%                  | 22.943 | 100%                 | 3.6208 |
| 33  | 31%                  | 21.267 | 100%                 | 3.6449 |
| 34  | 35%                  | 21.525 | 99%                  | 3.6948 |
| 35  | 45%                  | 21.068 | 100%                 | 3.7500 |

Table 4.3: Numerical results for Example 4.3

4.2 Real data
In this subsection, we consider a numerical example in document recognition [28].
Example 4.4 We apply Algorithm 2.2 to data converted from grayscale images from the MNIST Handwritten Digits data set. We arbitrarily choose 800 $20 \times 20$ images of handwritten digit 3, which are vectorized to column vectors and normalized to have total sum of one. Then we use the vectorized and normalized images to form the $800 \times 400$ target matrix $V$. We would like to decompose $V$ as the product of an $800 \times 196$ stochastic matrix $W$ and a $196 \times 400$ sparse stochastic matrix $H$.

Figures 4.1–4.2 give the decomposition results for Example 4.4 with $s = 50, 60$, where $tol = 10^{-3}$ and $IT_{\text{max}} = 5000$. Here, we only show 36 reshaped columns of the factors as images because of space limits. Table 4.4 displays the relative reconstruction error and the total computing time and Figure 4.3 shows the convergence and computing time curves.

We see from Figures 4.1–4.2 that the $W$-factor possesses complex structures for both algorithms while the $H$-factor generated by our algorithm contains more zeros than Algorithm 2.1. Moreover, our method is more effective than Algorithm 2.1 in document recognition. We also observe from Table 4.4 and Figure 4.3 that, compared with Algorithm 2.1, our algorithm can obtain smaller factorization residual though we need more computing cost in each iteration.

Table 4.4: Numerical results for Example 4.4

| $s$ | Alg. 2.1 | Alg. 2.2 |
|-----|---------|---------|
|     | res. ct. | res. ct. |
| 50  | 0.3512  | 0.1319  | 233.87 |
| 60  | 0.3525  | 0.1300  | 408.63 |

5 Concluding remarks

In this paper, we have considered the sparse stochastic matrix factorization (SSMF), which is rewritten as an unconstrained nonconvex-nonsmooth minimization problem. Then a row-wise update algorithm is proposed for solving the minimization problem. The global convergence of the proposed algorithm is established. In particular, the sequence generated by our algorithm converges to a special critical point of the cost function, which is shown to be a global minimizer over the $W$-factor as a whole and is nearly a global minimizer over each row of the $H$-factor. Numerical experiments on both synthetic and real data sets demonstrate the effectiveness of our algorithm.

http://yann.lecun.com/exdb/mnist/
Figure 4.1: Constructed images with $s = 50$ via Alg. 2.1 (top) and Alg. 2.2 (bottom).

Figure 4.2: Constructed images with $s = 60$ via Alg. 2.1 (top) and Alg. 2.2 (bottom).

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Figure 4.3: Convergence and computing time curves for one of the tests ($s = 50$).

Figure 4.4: Convergence and computing time curves for one of the tests ($s = 60$).

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Appendix A

In this appendix, we present some preliminary results on subgradients of nonsmooth functions. We first recall definitions of subdifferentials (subgradients) for nonsmooth functions in [29, 32].

**Definition A.1** Let $h : \mathbb{R}^n \to (-\infty, +\infty]$ be a proper lower semicontinuous (lsc) function. Then, the set

$$
\hat{\partial} h(\bar{a}) := \{ r \in \mathbb{R}^n | \liminf_{a \to \bar{a}} \frac{h(a) - h(\bar{a}) - \langle r, a - \bar{a} \rangle}{\|a - \bar{a}\|} \geq 0 \}.
$$

is the presubdifferential or Fréchet subdifferential of $h$ at $\bar{a} \in \text{dom } h$ and we set $\hat{\partial} h(\bar{a}) := \emptyset$ if $\bar{a} \notin \text{dom } h$. Moreover, the set

$$
\partial h(\bar{a}) := \{ r \in \mathbb{R}^n | \exists a^k \to \bar{a}, h(a^k) \to h(\bar{a}) \text{ and } r^k \in \hat{\partial} h(a^k) \to r \text{ as } k \to \infty \}
$$

(A.1)

is the limiting subdifferential of $h$ at $\bar{a} \in \mathbb{R}^n$.

From [32, Theorem 8.6], it follows that, for any $\bar{a} \in \text{dom } g$, $\hat{\partial} h(\bar{a}) \subset \partial h(\bar{a})$ and $\hat{\partial} h(\bar{a})$ is convex and closed while $\partial h(\bar{a})$ is closed. A point $\bar{a} \in \mathbb{R}^n$ is called a critical point of $h$ if $0 \in \partial h(\bar{a})$.

For the partial subdifferential of $F$ defined by (2.4), we have the following result from [1] [32].
Lemma A.2 Let $F$ be defined in [2.4]. Then for any

$$(W, H):= (w_1, \ldots, w_m, h_1, \ldots, h_r) \in \mathbb{R}^r \times \cdots \times \mathbb{R}^r \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n,$$

we have $\partial F(W, H) = (\partial_{w_1} F(W, H), \ldots, \partial_{w_m} F(W, H), \partial_{h_1} F(W, H), \ldots, \partial_{h_r} F(W, H)),$ where

$\partial_{w_i} F(w_1, \ldots, w_m, h_1, \ldots, h_r) = \{\nabla_{w_i} f(w_1, \ldots, w_m, h_1, \ldots, h_r) + \partial &\delta_{P_r}(w_i)\},$

for $i = 1, \ldots, m$ and

$\partial_{h_t} F(w_1, \ldots, w_m, h_1, \ldots, h_r) = \{\nabla_{h_t} f(w_1, \ldots, w_m, h_1, \ldots, h_r) + \partial &\delta_{P_n}(h_t) + \partial &\delta_{Q_n}(h_t)\},$

for $t = 1, \ldots, r.$