ON HOLOMORPHIC CURVES IN ALGEBRAIC TORUS

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ABSTRACT. We study entire holomorphic curves in the algebraic torus, and show that they can be characterized by the “growth rate” of their derivatives.

1. INTRODUCTION

Let \( z = x + y\sqrt{-1} \) be the natural coordinate in the complex plane \( \mathbb{C} \), and let \( f(z) \) be an entire holomorphic function in the complex plane. Suppose that there are a non-negative integer \( m \) and a positive constant \( C \) such that
\[
|f(z)| \leq C|z|^m, \quad (|z| \geq 1).
\]
Then \( f(z) \) becomes a polynomial with \( \deg f(z) \leq m \). This is a well-known fact in the complex analysis in one variable. In this paper, we prove an analogous result for entire holomorphic curves in the algebraic torus \( (\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n \).

Let \([z_0 : z_1 : \cdots : z_n]\) be the homogeneous coordinate in the complex projective space \( \mathbb{C}P^n \). We define the complex manifold \( X \subset \mathbb{C}P^n \) by
\[
X := \{[1 : z_1 : \cdots : z_n] \in \mathbb{C}P^n | z_i \neq 0, \ (1 \leq i \leq n)\} \cong (\mathbb{C}^*)^n.
\]
\( X \) is a natural projective embedding of \( (\mathbb{C}^*)^n \). We use the restriction of the Fubini-Study metric as the metric on \( X \).

For a holomorphic map \( f : \mathbb{C} \to X \), we define its norm \( |df|(z) \) by setting
\[
|df|(z) := \sqrt{2}|df(\partial/\partial z)| \quad \text{for all } z \in \mathbb{C}.
\]
Here \( \partial/\partial z = \frac{1}{2}(\partial/\partial x - \sqrt{-1}\partial/\partial y) \), and the normalization factor \( \sqrt{2} \) comes from \( |\partial/\partial z| = 1/\sqrt{2} \).

The main result of this paper is the following.

**Theorem 1.1.** Let \( f : \mathbb{C} \to X \) be a holomorphic map. Suppose there are a non-negative integer \( m \) and a positive constant \( C \) such that
\[
|df|(z) \leq C|z|^m, \quad (|z| \geq 1).
\]
Then there are polynomials \( g_1(z), g_2(z), \cdots, g_n(z) \) with \( \deg g_i(z) \leq m + 1, \ (1 \leq i \leq n) \), such that
\[
f(z) = [1 : e^{g_1(z)} : e^{g_2(z)} : \cdots : e^{g_n(z)}].
\]

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Conversely, if a holomorphic map $f(z)$ is expressed by (3) with polynomials $g_i(z)$ of degree at most $m+1$, $f(z)$ satisfies the “polynomial growth condition” (2).

The direction (3) $\Rightarrow$ (2) is easier, and the substantial part of the argument is the direction (2) $\Rightarrow$ (3).

If we set $m = 0$ in the above, we get the following corollary.

**Corollary 1.2.** Let $f : \mathbb{C} \to X$ be a holomorphic map with bounded derivative, i.e., $|df|(z) \leq C$ for some positive constant $C$. Then there are complex numbers $a_i$ and $b_i$, $(1 \leq i \leq n)$, such that

$$f(z) = [1 : e^{a_1z+b_1} : e^{a_2z+b_2} : \cdots : e^{a_nz+b_n}].$$

This is the theorem of [BD, Appendice]. The author also proves this in [T, Section 6].

**Remark 1.3.** The essential point of Theorem 1.1 is the statement that the degrees of the polynomials $g_i(z)$ are at most $m+1$. Actually, it is easy to prove that if $f(z)$ satisfies the condition (2) then $f(z)$ can be expressed by (3) with polynomials $g_i(z)$ of degree at most $2m+2$. (See Section 4.)

Theorem 1.1 states that holomorphic curves in $X$ can be characterized by the growth rate of their derivatives. We can formulate this fact more clearly as follows:

Let $g_1(z), g_2(z), \cdots, g_n(z)$ be polynomials, and define $f : \mathbb{C} \to X$ by (3). We define the integer $m \geq -1$ by setting

$$m + 1 := \max(\deg g_1(z), \deg g_2(z), \cdots, \deg g_n(z)).$$

We have $m = -1$ if and only if $f$ is a constant map. $m$ can be obtained as the growth rate of $|df|$: 

**Theorem 1.4.** If $m \geq 0$, we have

$$\limsup_{r \to \infty} \frac{\max_{|z|=r} |df|(z)}{\log r} = m.$$ 

**Corollary 1.5.** Let $\lambda$ be a non-negative real number, and let $[\lambda]$ be the maximum integer not greater than $\lambda$. Let $f : \mathbb{C} \to X$ be a holomorphic map, and suppose that there is a positive constant $C$ such that

$$|df|(z) \leq C|z|^\lambda, \quad (|z| \geq 1).$$

Then we have a positive constant $C'$ such that

$$|df|(z) \leq C'|z|^{[\lambda]}, \quad (|z| \geq 1).$$
Proof. If \( f \) is a constant map, the statement is trivial. Hence we can suppose \( f \) is not constant. From Theorem 1.1, \( f \) can be expressed by (3) with polynomials \( g_i(z) \) of degree at most \([\lambda]+2\). Since \( f \) satisfies (3), we have

\[
\limsup_{r \to \infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} \leq \lambda.
\]

From Theorem 1.4 this shows \( \deg g_i(z) \leq [\lambda]+1 \) for all \( g_i(z) \). Then, Theorem 1.1 gives the conclusion.

\[\square\]

2. Proof of (3) \(\Rightarrow\) (2)

Let \( f : \mathbb{C} \to X \) be a holomorphic map. From the definition of \( X \), we have holomorphic maps \( f_i : \mathbb{C} \to \mathbb{C}^* \), \( (1 \leq i \leq n) \), such that \( f(z) = [1 : f_1(z) : \cdots : f_n(z)] \). The norm \( |df|(z) \) in (1) is given by

\[
|df|^2(z) = \frac{1}{4\pi} \Delta \log \left(1 + \sum_{i=1}^n |f_i(z)|^2\right), \quad (\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}).
\]

Suppose that \( f \) is expressed by (3), i.e., \( f_i(z) = \exp(g_i(z)) \) with a polynomial \( g_i(z) \) of degree \( \leq m+1 \). We will repeatedly use the following calculation in this paper.

\[
|df|^2 = \frac{1}{\pi} \left[ \sum_i \frac{|f_i'|^2}{(1+|f_i|^2)^2} + \frac{\sum_{i \neq j} |g_i' - g_j'|^2 |f_i|^2 |f_j|^2}{(1+\sum_i |f_i|^2)^2} \right],
\]

\[
\leq \frac{1}{\pi} \left[ \sum_i \frac{|f_i'|^2}{(1+|f_i|^2)^2} + \frac{\sum_{i \neq j} |g_i' - g_j'|^2 |f_i|^2 |f_j|^2}{(1+|f_i|^2)^2} \right],
\]

\[
= \frac{1}{\pi} \left[ \sum_i \frac{|f_i'|^2}{(1+|f_i|^2)^2} + \frac{\sum_{i \neq j} |(f_i/f_j)'|^2}{(1+|f_i/f_j|^2)^2} \right],
\]

\[
= \sum_i |df_i|^2 + \sum_{i \neq j} |d(f_i/f_j)|^2.
\]

Here we set

\[
|df_i| := \frac{|f_i'|}{\sqrt{\pi} 1 + |f_i|^2} \quad \text{and} \quad |d(f_i/f_j)| := \frac{|(f_i/f_j)'|}{\sqrt{\pi} 1 + |f_i/f_j|^2}.
\]

These are the norms of the differentials of the maps \( f_i, f_i/f_j : \mathbb{C} \to \mathbb{C}P^1 \).

We have \( f_i(z) = \exp(g_i(z)) \) and \( f_i(z)/f_j(z) = \exp(g_i(z) - g_j(z)) \), and the degrees of the polynomials \( g_i(z) \) and \( g_i(z) - g_j(z) \) are at most \( m+1 \). Then, the next Lemma gives the desired conclusion:

\[
|df|(z) \leq C|z|^m, \quad (|z| \geq 1),
\]

for some positive constant \( C \).
Lemma 2.1. Let \( g(z) \) be a polynomial of degree \( \leq m + 1 \), and set \( h(z) := e^{g(z)} \). Then we have a positive constant \( C \) such that

\[
|dh|(z) = \frac{1}{\sqrt{\pi}} \frac{|h'(z)|}{1 + |h(z)|^2} \leq C|z|^m, \quad (|z| \geq 1).
\]

Proof. We have

\[
\sqrt{\pi} |dh| = \frac{|g'|}{|h| + |h|^{-1}} \leq |g'| \min(|h|, |h|^{-1}) \leq |g'|.
\]

Since the degree of \( g'(z) \) is at most \( m \), we easily get the conclusion. \( \square \)

3. Preliminary estimates

In this section, \( k \) is a fixed positive integer.

The following is a standard fact in the Nevanlinna theory.

Lemma 3.1. Let \( g(z) \) be a polynomial of degree \( k \), and set \( h(z) = e^{g(z)} \). Then we have a positive constant \( C \) such that

\[
\int_1^r \frac{dt}{t} \int_{|z| \leq t} |dh|^2(z) \, dx \, dy \leq Cr^k, \quad (r \geq 1).
\]

Proof. Since \( |dh|^2 = \frac{1}{4\pi} \Delta \log(1 + |h|^2) \), Jensen’s formula gives

\[
\int_1^r \frac{dt}{t} \int_{|z| \leq t} |dh|^2 \, dx \, dy = \frac{1}{4\pi} \int_{|z|=r} \log(1 + |h|^2) \, d\theta - \frac{1}{4\pi} \int_{|z|=1} \log(1 + |h|^2) \, d\theta.
\]

Here \((r, \theta)\) is the polar coordinate in the complex plane. We have

\[
\log(1 + |h|^2) \leq 2 |\text{Re} \, g(z)| + \log 2 \leq Cr^k, \quad (r := |z| \geq 1).
\]

Thus we get the conclusion. \( \square \)

Let \( I \) be a closed interval in \( \mathbb{R} \) and let \( u(x) \) be a real valued function defined on \( I \). We define its \( C^1 \)-norm \( \|u\|_{C^1(I)} \) by setting

\[
\|u\|_{C^1(I)} := \sup_{x \in I} |u(x)| + \sup_{x \in I} |u'(x)|.
\]

For a Lebesgue measurable set \( E \) in \( \mathbb{R} \), we denote its Lebesgue measure by \( |E| \).

Lemma 3.2. There is a positive number \( \varepsilon \) satisfying the following: If a real valued function \( u(x) \in C^1[0, \pi] \) satisfies

\[
\|u(x) - \cos x\|_{C^1[0,\pi]} \leq \varepsilon,
\]

then we have

\[
|u^{-1}([-t, t])| \leq 4t \quad \text{for any } t \in [0, \varepsilon].
\]
Proof. The proof is just an elementary calculus. For any small number \( \delta > 0 \), if we choose \( \varepsilon \) sufficiently small, we have

\[ u^{-1}([-t, t]) \subset [\pi/2 - \delta, \pi/2 + \delta]. \]

Let \( x_1 \) and \( x_2 \) be any two elements in \( u^{-1}([-t, t]) \). From the mean value theorem, we have \( y \in [\pi/2 - \delta, \pi/2 + \delta] \) such that

\[ u(x_1) - u(x_2) = u'(y) (x_1 - x_2). \]

From \( \sin(\pi/2) = 1 \), we can suppose that

\[ |u'(y)| \geq 1/2. \]

Hence

\[ |x_1 - x_2| \leq 2 |u(x_1) - u(x_2)| \leq 4t. \]

Thus we get

\[ |u^{-1}([-t, t])| \leq 4t. \]

Using a scale change of the coordinate, we get the following.

Lemma 3.3. There is a positive number \( \varepsilon \) satisfying the following: If a real valued function \( u(x) \in C^1[0, 2\pi] \) satisfies

\[ \|u(x) - \cos kx\|_{C^1[0, 2\pi]} \leq \varepsilon, \]

then we have

\[ |u^{-1}([-t, t])| \leq 8t \quad \text{for any } t \in [0, \varepsilon]. \]

Proof.

\[ u^{-1}([-t, t]) = \bigcup_{j=0}^{2k-1} u^{-1}([-t, t]) \cap [j\pi/k, (j + 1)\pi/k]. \]

Applying Lemma 3.2 to \( u(x/k) \), we have

\[ |u^{-1}([-t, t]) \cap [0, \pi/k]| \leq 4t/k. \]

In a similar way,

\[ |u^{-1}([-t, t]) \cap [j\pi/k, (j + 1)\pi/k]| \leq 4t/k, \quad (j = 0, 1, \cdots, 2k - 1). \]

Thus we get the conclusion.

Let \( E \) be a subset of \( \mathbb{C} \). For a positive number \( r \), we set

\[ E(r) := \{ \theta \in \mathbb{R}/2\pi \mathbb{Z} | re^{i\theta} \in E \}. \]

In the rest of the section, we always assume \( k \geq 2 \).
Lemma 3.4. Let $C$ be a positive constant, and let $g(z) = z^k + a_1 z^{k-1} + \cdots + a_k$ be a monic polynomial of degree $k$. Set

$$E := \{ z \in \mathbb{C} : |\text{Re} g(z)| \leq C|z| \}.$$ 

Then we have a positive number $r_0$ such that

$$|E(r)| \leq 8C/r^{k-1}, \quad (r \geq r_0).$$

Proof. Set $v(z) := \text{Re}(a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k)$. Then we have

$$|\text{Re} g(re^{i\theta})| \leq C r \iff |\cos k\theta + v(re^{i\theta})/r^k| \leq C/r^{k-1}.$$ 

Set $u(\theta) := \cos k\theta + v(re^{i\theta})/r^k$. It is easy to see that

$$\|u(\theta) - \cos k\theta\|_{C^1[0,2\pi]} \leq \text{const}/r, \quad (r \geq 1).$$

Then we can apply Lemma 3.3 to this $u(\theta)$, and we get

$$|E(r)| = |u^{-1}([-C/r^{k-1}, C/r^{k-1}])| \leq 8C/r^{k-1}, \quad (r \gg 1).$$

□

The following is the key lemma.

Lemma 3.5. Let $g(z) = a_0 z^k + a_1 z^{k-1} + \cdots + a_k$ be a polynomial of degree $k$, $(a_0 \neq 0)$. Set

$$E := \{ z \in \mathbb{C} : |\text{Re} g(z)| \leq |z| \}.$$ 

Then we have a positive number $r_0$ such that

$$|E(r)| \leq \frac{8}{|a_0|r^{k-1}}, \quad (r \geq r_0).$$

Proof. Let $\text{arg} a_0$ be the argument of $a_0$, and set $\alpha := \text{arg} a_0/k$. We define the monic polynomial $g_1(z)$ by

$$g_1(z) := \frac{1}{|a_0|} g(e^{-i\alpha}z) = z^k + \cdots.$$ 

Then we have

$$|\text{Re} g(re^{i\theta})| \leq r \iff |\text{Re} g_1(re^{i(\theta+\alpha)})| \leq r/|a_0|.$$ 

Hence the conclusion follows from Lemma 3.4. □

Lemma 3.6. Let $g(z)$ be a polynomial of degree $k$, and we define $E$ as in Lemma 3.5. Set $h(z) := e^{g(z)}$. Then we have

$$\int_{\mathbb{C} \setminus E} |dh|^2(z) \, dx \, dy < \infty.$$
PROOF. Since $|h| = e^{\text{Re}g}$, the argument in the proof of Lemma 2.1 gives

$$\sqrt{\pi} |dh| \leq |g'| \min(|h|, |h|^{-1}) \leq |g'| e^{-\text{Re}g}.$$ 

$g'(z)$ is a polynomial of degree $k - 1$, and we have $|\text{Re}g| > |z|$ for $z \in \mathbb{C} \setminus E$. Hence we have a positive constant $C$ such that

$$|dh|(z) \leq C|z|^{k-1} e^{-|z|}, \quad \text{if } z \in \mathbb{C} \setminus E \text{ and } |z| \geq 1.$$ 

The conclusion follows from this estimate. \qed

4. PROOF OF $(2) \Rightarrow (3)$

Let $f = [1 : f_1 : f_2 : \cdots : f_n] : \mathbb{C} \to X$ be a holomorphic map with $|df|(z) \leq C|z|^m$, $(|z| \geq 1)$. Since $\exp : \mathbb{C} \to \mathbb{C}^*$ is the universal covering, we have entire holomorphic functions $g_i(z)$ such that $f_i(z) = e^{g_i(z)}$. We will prove that all $g_i(z)$ are polynomials of degree $\leq m + 1$. The proof falls into two steps. In the first step, we prove all $g_i(z)$ are polynomials. In the second step, we show $\text{deg} g_i(z) \leq m + 1$. The second step is the harder part of the proof.

Schwarz’s formula gives

$$\pi r^k g_i^{(k)}(0) = k! \int_{|z|=r} \text{Re} (g_i(z)) e^{-k\sqrt{-1}\theta} d\theta = k! \int_{|z|=r} |f_i(z)| e^{-k\sqrt{-1}\theta} d\theta, \quad (k \geq 1).$$

We have

$$|\log |f_i|| \leq \log(|f_i| + |f_i|^{-1}) = \log(1 + |f_i|^2) - \log |f_i| \leq \log(1 + \sum |f_j|^2) - \log |f_i|.$$ 

Hence

$$\pi r^k |g_i^{(k)}(0)| \leq k! \int_{|z|=r} \log(1 + \sum |f_j|^2) d\theta - k! \int_{|z|=r} \log |f_i| d\theta.$$ 

Since $\log |f_i| = \text{Re} g_i(z)$ is a harmonic function, the second term in the above is equal to the constant $-2\pi k! \text{Re} g_i(0)$. Since $|df|^2 = \frac{1}{4\pi} \Delta \log(1 + \sum |f_j|^2)$, Jensen’s formula gives

$$\frac{1}{4\pi} \int_{|z|=r} \log(1 + \sum |f_j|^2) d\theta - \frac{1}{4\pi} \int_{|z|=1} \log(1 + \sum |f_j|^2) d\theta = \int_1^r \frac{dt}{t} \int_{|z|\leq t} |df|^2(z) dxdy.$$ 

Thus we get

$$(8) \quad \frac{r^k}{4k!} |g_i^{(k)}(0)| \leq \int_1^r \frac{dt}{t} \int_{|z|\leq t} |df|^2(z) dxdy + \text{const}.$$ 

Since $|df|(z) \leq C|z|^m$, $(|z| \geq 1)$, this shows $g_i^{(k)}(0) = 0$ for $k \geq 2m + 3$. Hence $g_i(z)$ are polynomials.

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The idea of using Schwarz’s formula is due to [BD, Appendice]. The author gives a different approach in [T, Section 6].
Next we will prove \( \deg g_i(z) \leq m+1 \). We define \( E_i, E_{ij} \subset \mathbb{C} \), \((1 \leq i \leq n, 1 \leq i < j \leq n)\), by setting

\[
\begin{align*}
\deg g_i(z) &\leq m + 1 \implies E_i := \emptyset, \\
\deg g_i(z) &\geq m + 2 \implies E_i := \{ z \in \mathbb{C} \mid |\Re g_i(z)| \leq |z| \}, \\
\deg(g_i(z) - g_j(z)) &\leq m + 1 \implies E_{ij} := \emptyset, \\
\deg(g_i(z) - g_j(z)) &\geq m + 2 \implies E_{ij} := \{ z \in \mathbb{C} \mid |\Re(g_i(z) - g_j(z))| \leq |z| \}.
\end{align*}
\]

We set \( E := \bigcup_i E_i \cup \bigcup_{i<j} E_{ij} \). Then we have \( E(r) = \bigcup_i E_i(r) \cup \bigcup_{i<j} E_{ij}(r) \) for \( r > 0 \).

From Lemma 3.5 we have positive constants \( r_0 \) and \( C' \) such that

\[
(9) \quad |E(r)| \leq C'/r^{m+1}, \quad (r \geq r_0).
\]

We have

\[
(10) \quad \int_1^r \frac{dt}{t} \int_{|z| \leq t} |df|^2(z) \, dx \, dy = \int_1^r \frac{dt}{t} \int_{E \cap \{ |z| \leq t \}} |df|^2(z) \, dx \, dy + \int_1^r \frac{dt}{t} \int_{E \cap \{ |z| \leq t \}} |df|^2(z) \, dx \, dy.
\]

Using (9) and \( |df|(z) \leq C|z|^m \), \((|z| \geq 1)\), we can estimate the first term in (10) as follows:

\[
\int_{E \cap \{ |z| \leq t \}} |df|^2(z) \, dx \, dy \leq C^2 \int_{E \cap \{ |z| \leq t \}} r^{2m+1} \, dr \, d\theta = C^2 \int_1^t r^{2m+1} |E(r)| \, dr.
\]

If \( t \geq r_0 \), we have

\[
\int_{r_0}^t r^{2m+1} |E(r)| \, dr \leq C' \int_{r_0}^t r^m \, dr = \frac{C'}{m+1} t^{m+1} - \frac{C'}{m+1} r_0^{m+1}.
\]

Thus

\[
(11) \quad \int_1^r \frac{dt}{t} \int_{E \cap \{ |z| \leq t \}} |df|^2(z) \, dx \, dy \leq \text{const} \cdot r^{m+1}, \quad (r \geq 1).
\]

Next we will estimate the second term in (10) by using the inequality (7) given in Section 2:

\[
|df|^2 \leq \sum_i |df_i|^2 + \sum_{i<j} |d(f_i/f_j)|^2.
\]

If \( \deg g_i(z) \leq m + 1 \), Lemma 3.1 gives

\[
\int_1^r \frac{dt}{t} \int_{E \cap \{ |z| \leq t \}} |df_i|^2(z) \, dx \, dy \leq \int_1^r \frac{dt}{t} \int_{|z| \leq t} |df_i|^2(z) \, dx \, dy \leq \text{const} \cdot r^{m+1}.
\]

If \( \deg g_i(z) \geq m + 2 \), Lemma 3.6 gives

\[
\int_{E \cap \{ |z| \leq t \}} |df_i|^2(z) \, dx \, dy \leq \int_{E_i \cap \{ |z| \leq t \}} |df_i|^2(z) \, dx \, dy \leq \text{const}.
\]
The terms for \(|d(f_i/f_j)|\) can be also estimated in the same way, and we get

\[
(12) \quad \int_1^r \frac{dt}{t} \int_{E \cap \{|z| \leq t\}} |df|^2(z) \, dxdy \leq \text{const} \cdot r^{m+1}, \quad (r \geq 1).
\]

From (10), (11), (12), we get

\[
\int_1^r \frac{dt}{t} \int_{|z| \leq t} |df|^2(z) \, dxdy \leq \text{const} \cdot r^m + 1, \quad (r \geq 1).
\]

From (8), this shows \(g_i^{(k)}(0) = 0\) for \(k \geq m+2\). Thus \(g_i(z)\) are polynomials with \(\text{deg} \, g_i(z) \leq m+1\). This concludes the proof of Theorem 1.1.

5. Proof of Theorem 1.4 and a corollary

5.1. Proof of Theorem 1.4. The proof of Theorem 1.4 needs the following lemma.

**Lemma 5.1.** Let \(k \geq 1\) be an integer, and let \(\delta\) be a real number satisfying \(0 < \delta < 1\). Let \(g(z) = a_0z^k + a_1z^{k-1} + \cdots + a_k\) be a polynomial of degree \(k\), \((a_0 \neq 0)\). We set \(h(z) := e^{g(z)}\) and define \(E \subset \mathbb{C}\) by

\[
E := \{z \in \mathbb{C} \mid |\text{Re} g(z)| \leq |z|^\delta\}.
\]

Then we have

\[
\int_{\mathbb{C}\setminus E} |dh|^2 < \infty,
\]

and there is a positive number \(r_0\) such that

\[
|E(r)| \leq \frac{8}{|a_0|^r r^{-\delta}}, \quad (r \geq r_0).
\]

**Proof.** This can be proven by the methods in Section 3. We omit the detail. \(\square\)

Let \(g_1(z), g_2(z), \cdots, g_n(z)\) be polynomials, and define the holomorphic map \(f : \mathbb{C} \to X\) and the integer \(m \geq -1\) by (3) and (4). Here we suppose \(m \geq 0\), i.e., \(f\) is not a constant map. We will prove Theorem 1.4.

From Theorem 1.1, we have

\[
|df|(z) \leq \text{const} \cdot |z|^m, \quad (|z| \geq 1).
\]

It follows

\[
\limsup_{r \to \infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} \leq m.
\]

We want to prove that this is actually an equality. Suppose

\[
\limsup_{r \to \infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} \leq m.
\]

Then, if we take \(\varepsilon > 0\) sufficiently small, we have a positive number \(r_0\) such that

\[
|df|(z) \leq |z|^{m-\varepsilon}, \quad (|z| \geq r_0).
\]
Schwarz’s formula gives the inequality (8):

\[ \frac{r^k}{4k!}\left|g_i^{(k)}(0)\right| \leq \int_1^r \frac{dt}{t} \int_{|z|\leq t} |df(z)|^2 \, dx \, dy + \text{const}, \quad (k \geq 0). \]

Let \( \delta \) be a positive number such that \( 0 < \delta < 2\varepsilon \). We define \( E_i \) and \( E_{ij} \), \((1 \leq i \leq n, 1 \leq i < j \leq n)\), by setting

\[
\deg g_i(z) \leq m \Rightarrow E_i := \emptyset,
\]
\[
\deg g_i(z) = m + 1 \Rightarrow E_i := \{ z \in \mathbb{C} \mid |\text{Re } g_i(z)| \leq |z|^\delta \},
\]
\[
\deg (g_i(z) - g_j(z)) \leq m \Rightarrow E_{ij} := \emptyset,
\]
\[
\deg (g_i(z) - g_j(z)) = m + 1 \Rightarrow E_{ij} := \{ z \in \mathbb{C} \mid |\text{Re } (g_i(z) - g_j(z))| \leq |z|^\delta \}.
\]

We set \( E := \bigcup_i E_i \cup \bigcup_{i<j} E_{ij} \). Then, if we take \( r_0 \) sufficiently large, we have

\[ |E(r)| \leq \text{const}/r^{m+1-\delta}, \quad (r \geq r_0). \]

We have

\[
\int_1^r \frac{dt}{t} \int_{|z|\leq t} |df(z)|^2 \, dx \, dy
= \int_1^r \frac{dt}{t} \int_{E \cap \{ |z| \leq t \}} |df(z)|^2 \, dx \, dy + \int_1^r \frac{dt}{t} \int_{E^c \cap \{ |z| \leq t \}} |df(z)|^2 \, dx \, dy.
\]

From (13) and (15), the first term can be estimated as in Section 4

\[
\int_1^r \frac{dt}{t} \int_{E \cap \{ |z| \leq t \}} |df(z)|^2 \, dx \, dy \leq \text{const} \cdot r^{m+1-(2\varepsilon-\delta)}, \quad (r \geq 1).
\]

Using Lemma 5.1 and the inequality \( |df|^2 \leq \sum_i |df_i|^2 + \sum_{i<j} |d(f_i/f_j)|^2 \), we can estimate the second term:

\[
\int_1^r \frac{dt}{t} \int_{E^c \cap \{ |z| \leq t \}} |df(z)|^2 \, dx \, dy \leq \text{const} \cdot \log r + \text{const} \cdot r^m, \quad (r \geq 1).
\]

Thus we get

\[
\int_1^r \frac{dt}{t} \int_{E \cap \{ |z| \leq t \}} |df(z)|^2 \, dx \, dy \leq \text{const} \cdot r^{m+1-(2\varepsilon-\delta)}, \quad (r \geq 1).
\]

Note that \( 2\varepsilon - \delta \) is a positive number. Using this estimate in (14), we get

\[ g_i^{(k)}(0) = 0, \quad (k \geq m + 1). \]

This shows \( \deg g_i(z) \leq m \). This contradicts the definition of \( m \).

**Remark 5.2.** The following is also true:

\[ \limsup_{r \to \infty} \frac{\max_{|z| \leq r} \log |df(z)|}{\log r} = m. \]
Proof. We have
\[ m = \limsup_{r \to \infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} \leq \limsup_{r \to \infty} \frac{\max_{|z| \leq r} \log |df|(z)}{\log r}. \]
And we have \(|df|(z) \leq \text{const} \cdot |z|^m, (|z| \geq 1)\). Thus
\[ \limsup_{r \to \infty} \frac{\max_{|z| \leq r} \log |df|(z)}{\log r} \leq m. \]
\[ \square \]

5.2. Order of the Shimizu-Ahlfors characteristic function. For a holomorphic map \( f : \mathbb{C} \to X \), we define the Shimizu-Ahlfors characteristic function \( T(r, f) \) by
\[ T(r, f) := \int_1^r \frac{dt}{t} \int_{|z| \leq t} |df|^2(z) \, dx \, dy, \quad (r \geq 1). \]
The order \( \rho_f \) of \( T(r, f) \) is defined by
\[ \rho_f := \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}. \]
\( \rho_f \) can be obtained as the growth rate of \( |df| \):

Corollary 5.3. For a holomorphic map \( f : \mathbb{C} \to X \), we have
\[ \rho_f < \infty \iff \limsup_{r \to \infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} < \infty. \]
If these values are finite and \( f \) is not a constant map, then we have
\[ \rho_f = \limsup_{r \to \infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} + 1. \]

Proof. If \( \rho_f < \infty \), the estimate (14) shows that \( f \) can be expressed by (3) with polynomials \( g_1(z), \ldots, g_n(z) \). Then we have
\[ \limsup_{r \to \infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} < \infty. \]
The proof of the converse is trivial.

Suppose \( \rho_f < \infty \). Then we can express \( f \) by \( f(z) = [1 : e^{g_1(z)} : \ldots : e^{g_n(z)}] \) with polynomials \( g_1(z), \ldots, g_n(z) \). We set \( f_i(z) := e^{g_i(z)} \), and define the integer \( m \) by (14). Theorem 1.4 gives
\[ \limsup_{r \to \infty} \frac{\max_{|z|=r} \log |df|(z)}{\log r} + 1 = m + 1. \]
The estimate (14) gives
\[ m + 1 \leq \rho_f. \]
Since \( |df| = \frac{1}{4\pi} \Delta \log(1 + \sum |f_i|^2) \), Jensen’s formula gives
\[ T(r, f) = \frac{1}{4\pi} \int_{|z|=r} \log(1 + \sum_i |f_i|^2) \, d\theta - \frac{1}{4\pi} \int_{|z|=1} \log(1 + \sum_i |f_i|^2) \, d\theta. \]
Since $\deg g_i(z) \leq m + 1$, we have
\[ \log(1 + \sum_i |f_i|^2) \leq \text{const} \cdot r^{m+1}, \quad (r \geq 1). \]

Hence
\[ \rho_f \leq m + 1. \]

Thus we get
\[ \rho_f = m + 1 = \limsup_{r \to \infty} \max_{|z|=r} \frac{\log |df|(z)}{\log r} + 1. \]

**Remark 5.4.** Of course, the statement of Corollary 5.3 is not true for general entire holomorphic curves in the complex projective space $\mathbb{C}P^n$. For example, let $f : \mathbb{C} \to \mathbb{C}P^1$ be a non-constant elliptic function. Since $|df|$ is bounded all over the complex plane, we have
\[ \limsup_{r \to \infty} \max_{|z|=r} \frac{\log |df|(z)}{\log r} = 0. \]

And it is easy to see
\[ \rho_f = 2 \neq \limsup_{r \to \infty} \max_{|z|=r} \frac{\log |df|(z)}{\log r} + 1. \]

**References**

[BD] F. Berteloot, J. Duval, Sur l’hyperbolicité de certains complémentaires, Enseign. Math. 47 (2001) 253-267

[T] M. Tsukamoto, A packing problem for holomorphic curves, preprint, arXiv: math.CV/0605353

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