AN ESTIMATE FOR $F$-JUMPING NUMBERS VIA THE ROOTS OF THE BERNSTEIN-SATO POLYNOMIAL

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Abstract. Given a smooth, irreducible complex algebraic variety $X$ and a nonzero regular function $f$ on $X$, we give an effective estimate for the difference between the jumping numbers of $f$ and the $F$-jumping numbers of a reduction $f_p$ of $f$ to characteristic $p \gg 0$, in terms of the roots of the Bernstein-Sato polynomial $b_f$ of $f$. In particular, we get uniform estimates only depending on the dimension of $X$. As an application, we show that if $b_f$ has no roots of the form $-\lct(f) - n$, with $n$ a positive integer, then the $F$-pure threshold of $f_p$ is equal to the log canonical threshold of $f$ for $p \gg 0$ with $(p - 1) \lct(f) \in \mathbb{Z}$.

1. Introduction

Let $X$ be a smooth, irreducible complex algebraic variety and let $f \in \mathcal{O}_X(X)$ be a nonzero regular function defining the hypersurface $H \subset X$. One associates to $f$ a sequence of coherent ideals $\mathcal{J}(f^\lambda) \subseteq \mathcal{O}_X$, the multiplier ideals of $f$, depending on the parameter $\lambda \in \mathbb{R}_{\geq 0}$. They can be defined either in terms of a log resolution of the pair $(X, H)$, or in terms of integrability conditions. These ideals give interesting invariants of the singularities of the hypersurface $H$ and they play an important role in vanishing theorems. They satisfy $\mathcal{J}(f^\lambda) \subseteq \mathcal{J}(f^\mu)$ if $\mu \leq \lambda$ and there is an increasing sequence of positive rational numbers $(\lambda_m)_{m \geq 1}$, with $\lim_{m \to \infty} \lambda_m = \infty$, such that $\mathcal{J}(f^\lambda)$ is constant for $\lambda \in [\lambda_{m-1}, \lambda_m)$ for all $m \geq 1$ (with the convention that $\lambda_0 = 0$). These are the $F$-jumping numbers of $f$ and the smallest jumping number $\lambda_1$ is the log canonical threshold $\lct(f)$. For an introduction to multiplier ideals and jumping numbers, see [Laz04, Chapter 9].

It has long been understood that many classes and invariants of singularities that appear in birational geometry in characteristic 0 have analogues in positive characteristic, defined via the Frobenius morphism. Suppose now that $Y$ is a regular scheme of characteristic $p > 0$, which we assume to be $F$-finite (this means that the Frobenius morphism $F: Y \to Y$ is a finite morphism). If $g \in \mathcal{O}_Y(Y)$ is everywhere nonzero, then Hara and Yoshida [HY03] defined a sequence of ideals $\tau(g^\lambda) \subseteq \mathcal{O}_Y$, the (generalized) test ideals of $g$, depending on the parameter $\lambda \in \mathbb{R}_{\geq 0}$. These ideals turn out to satisfy many of the formal properties that multiplier ideals satisfy in characteristic 0. In particular, $\tau(g^\lambda) \subseteq \tau(g^\mu)$ if $\mu \leq \lambda$ and it was shown in [BMS09] that there is an increasing sequence of positive rational numbers $(\alpha_m)_{m \geq 1}$, with $\lim_{m \to \infty} \alpha_m = \infty$, such that $\tau(g^\lambda)$ is constant for $\lambda \in [\alpha_{m-1}, \alpha_m)$ for all $m \geq 1$ (with the convention that $\alpha_0 = 0$). These are the $F$-jumping numbers of $g$ and the smallest such invariant $\alpha_1$ is the $F$-pure threshold $\fpt(g)$, introduced by Takagi and Watanabe in [TW04].

The most interesting results and open problems in this area are related to the comparison between multiplier ideals and test ideals via reduction mod $p$. Suppose that $X$ is a smooth, irreducible complex algebraic variety and $f \in \mathcal{O}_X(X)$ is nonzero. After choosing a model
(\(X_A, f_A\)) of \((X, f)\) over a finitely generated \(\mathbb{Z}\)-subalgebra \(A \subseteq \mathbb{C}\), for every closed point \(t \in \text{Spec}(A)\) we obtain a pair \((X_t, f_t)\) in positive characteristic (note that the field \(k(t)\) is a finite field). As it is typical in this setting, we always allow replacing \(A\) by a localization \(A_a\), in order to preserve certain properties from characteristic 0 (for example, in this way we may assume that \(X_A\) is smooth over \(\text{Spec}(A)\) and that \(f_A\) defines a relative Cartier divisor over \(\text{Spec}(A)\)). The following is a key result from [IHY03]:

**Theorem 1.1.** With the above notation, after possibly replacing \(A\) by a localization \(A_a\), the following hold:

i) For every closed point \(t \in \text{Spec}(A)\), we have \(\tau(f^\lambda_t) \subseteq \mathcal{J}(f^\lambda)_t\) for all \(\lambda \in \mathbb{R}_{\geq 0}\).

ii) For every \(\lambda \in \mathbb{R}_{\geq 0}\), if \(t \in \text{Spec}(A)\) is a closed point such that \(\text{char } k(t) \gg 0\), then \(\tau(f^\lambda_t) = \mathcal{J}(f^\lambda)_t\).

The assertion in ii) is the deepest one. We note that the condition on \(\text{char } k(t)\) here can’t be made independent of \(\lambda\), in general. However, we have the following open problem concerning the relation between multiplier ideals and test ideals:

**Conjecture 1.2.** With the above notation, there is a dense subset \(T\) of closed points in \(\text{Spec}(A)\) such that \(\tau(f^\lambda_t) = \mathcal{J}(f^\lambda)_t\) for all \(t \in T, \lambda \in \mathbb{R}_{\geq 0}\).\n
It is known that this conjecture is equivalent to a deep conjecture in arithmetic geometry, the Weak Ordinarity conjecture (see [Mus12] and [MS11], as well as [BST17] for the case of a singular ambient variety). It is instructive to see what the above theorem and conjecture say about the relation between \(\text{lct}(f)\) and \(\text{fpt}(f_t)\). First, the assertion in Theorem 1.1i) says that after possibly replacing \(A\) by a localization \(A_a\), we may assume that \(\text{fpt}(f_t) \leq \text{lct}(f)\) for all closed points in \(t \in \text{Spec}(A)\). On the other hand, the result in Theorem 1.1ii) implies that

\[
\lim_{\text{char } k(t) \to \infty} \text{fpt}(f_t) = \text{lct}(f).
\]

Finally, Conjecture 1.2 predicts that there is a dense subset \(T\) of closed points in \(\text{Spec}(A)\) such that \(\text{fpt}(f_t) = \text{lct}(f)\) for all \(t \in T\).

Our main goal in this note is to give an effective estimate of the limit in (1) and a generalization of this to arbitrary jumping numbers, in terms of the roots of the *Bernstein-Sato polynomial* \(b_f(s)\) of \(f\). Recall that \(b_f(s)\) is the monic polynomial of smallest degree such that

\[
b_f(s)f^s \in \mathcal{D}_X[s] \cdot f^{s+1},
\]

where \(\mathcal{D}_X\) is the sheaf of differential operators on \(X\). Here \(f^s\) has to be interpreted as a formal symbol on which differential operators act in the expected way. The existence of a polynomial \(b_f(s)\) as in (2) was proved by Bernstein in [Ber72] when \(X = \mathbb{A}^n\) and by Kashiwara [Kas76] in general (in fact, for arbitrary holomorphic functions on complex manifolds). Furthermore, it was shown in [Kas76] that all roots of \(b_f\) are negative rational numbers.

The following is our estimate for the \(F\)-pure thresholds of the reductions of \(f\).

**Theorem 1.3.** Let \(X\) be a smooth complex algebraic variety and \(f \in \mathcal{O}_X(X)\) nonzero. If \((X_A, f_A)\) gives a model of \((X, f)\) over \(A\), then after possibly replacing \(A\) by a localization \(A_a\), for every closed point \(t \in \text{Spec}(A)\) with \(\text{char } k(t) = p_t\), the following holds: if \(i\) is a positive integer such that \((p_t - i) \cdot \text{lct}(f) \in \mathbb{Z}\), then

\[
\text{fpt}(f_t) > \text{lct}(f) - \frac{m_i}{p_t},
\]
where \( m_i = \max \{ \beta \mid b_f(-\beta) = 0, \beta - (i \cdot \lct(f) + 1 - [i \cdot \lct(f)]) \in \mathbb{Z}_{\geq 0} \}. \)

Note that if we write \( \lct(f) = \frac{a}{b} \), with \( a \) and \( b \) relatively prime, then it is enough to consider in the theorem only those \( i \) with \( 1 \leq i \leq b \) and relatively prime to \( b \). In particular, we get an explicit lower bound for \( \fpt(f_i) \), in terms of the roots of \( b_f \), that works for all \( t \). Using a lower bound for the roots of the Bernstein-Sato polynomial from [Sai94], we will also see that we may replace \( m_i \) in the theorem by \( \dim(X) \) (see Remark 3.2).

In fact, we prove a more general statement (see Theorem 3.1), in which \( \lct(f) \) is replaced by any jumping number of \( f \). The main ingredient in the proof of Theorem 1.3 is a result from [MZ13] which says that there is such a lower bound, but for which \( m_i \) is only determined, in a complicated way, by a log resolution (this, in turn, relied on the techniques in [HY03] that give (1)). The other ingredient is the elementary observation, which goes back to [MTW05], that for every closed point \( t \in \text{Spec}(A) \), the integer \( [\fpt(f_i)_{p_t}] - 1 \) gives a root of \( b_f \) mod \( p_t \).

We note that the assertion in Theorem 1.3 when \( \lct(f) = 1 \) has been recently obtained by Dodd in [Dod22], using completely different methods. In fact, his result provided us with the motivation for revisiting this circle of ideas.

It is well-known that \( -\lct(f) \) is a root of \( b_f \) (in fact, this is the largest root of the Bernstein-Sato polynomial, see [Ko97, Section 10]). When \( b_f \) has no roots of the form \( -\lct(f) - n \), with \( n \in \mathbb{Z}_{>0} \), we obtain the following result in the direction of the conjectural existence of prime reductions with \( F \)-pure threshold equal to \( \lct(f) \).

**Corollary 1.4.** Let \( X \) be a smooth complex algebraic variety and \( f \in \mathcal{O}_X(X) \) nonzero such that \( b_f(-\lct(f) - n) \neq 0 \) for every \( n \in \mathbb{Z}_{>0} \). If \( (X_A, f_A) \) gives a model of \( (X, f) \), then after possibly replacing \( A \) by a localization \( A_t \), for every closed point \( t \in \text{Spec}(A) \) with \( \text{char}(k(t)) = p_t \), we have \( \fpt(f_t) = \lct(f) \) as long as \( (p_t - 1) \cdot \lct(f) \in \mathbb{Z} \).

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### 2. A BRIEF REVIEW OF \( F \)-JUMPING NUMBERS

Let \( Y \) be a regular \( F \)-finite scheme of characteristic \( p > 0 \) and let \( g \in \mathcal{O}_Y(Y) \) be everywhere nonzero (in other words, \( g \) is nonzero on every connected component of \( Y \)). Note that since \( Y \) is regular, the Frobenius morphism \( F : X \to X \) is also flat by a famous result of Kunz [Kun69]. We do not recall the definition of the test ideals \( \tau(g^t) \) since we will not need it. We refer the reader to [HY03] for the original definition and to [BMS08] for a simpler one in our setting (when the ambient scheme is assumed to be regular and \( F \)-finite).

We only review here the description of the \( F \)-jumping numbers of \( g \) as \( F \)-thresholds, following [MTW05] and [BMS08]. We note that if \( Y = U_1 \cup \ldots \cup U_N \) is an open cover, then the set of \( F \)-jumping numbers of \( g \) is the union of the sets of the jumping numbers of \( g|_{U_i} \), for \( 1 \leq i \leq N \). By taking a suitable affine cover of \( Y \), we may thus reduce the study of \( F \)-jumping numbers to the case of an affine scheme. In order to conform to the presentation in [MTW05] and [BMS08], we assume that \( Y \) is affine and let \( R = \mathcal{O}_Y(Y) \).
Given a proper ideal $J \subseteq R$ such that $g \in \text{rad}(J)$, for every $e \geq 1$ we put

$$\nu_g^J(p^e) = \max\{r \in \mathbb{Z}_{\geq 0} \mid g^r \notin J[p^e]\}.$$ 

Recall that $J[p^e] = (h^p^e \mid h \in J)$. The flatness of the Frobenius morphism gives

$$(J[p^{e+1}] : g^{pr}) = (J[p^e] : g^r)^[p],$$

hence $g^r \notin J[p^e]$ implies $g^{pr} \notin J[p^{e+1}]$. We thus have

$$\frac{\nu_g^J(p^e)}{p^e} \leq \frac{\nu_g^J(p^{e+1})}{p^{e+1}} \quad \text{for all} \quad e \geq 1$$

and the $F$-threshold of $g$ with respect to $J$ is

$$c^J(g) := \sup_{e \geq 1} \frac{\nu_g^J(p^e)}{p^e} = \lim_{e \to \infty} \frac{\nu_g^J(p^e)}{p^e}.$$

One can show that $c^J(g) < \infty$ and we have

$$\frac{\nu_g^J(p^e)}{p^e} < c^J(g) \quad \text{for all} \quad e \geq 1$$

(see [MTW05, Remark 1.2] and [MTW05, Proposition 1.7(5)]). What’s special when working with a principal ideal such as $(g)$, is that if $g^{r+1} \in J[p^e]$, then $g^{(r+1)p} \in J[p^{e+1}]$. We thus have

$$\frac{\nu_g^J(p^e) + 1}{p^e} \leq \frac{\nu_g^J(p^{e+1})}{p^{e+1}} \quad \text{for all} \quad e \geq 1,$$

which immediately implies

$$c^J(g) \leq \frac{\nu_g^J(p^e) + 1}{p^e} \quad \text{for all} \quad e \geq 1.$$  

By combining the inequalities (3) and (4), we obtain

$$\nu_g^J(p^e) + 1 = [c^J(g) \cdot p^e] \quad \text{for all} \quad e \geq 1$$

(see [MTW05, Proposition 1.9]).

The $F$-thresholds are relevant for us since the set of $F$-jumping numbers of $g$ coincides with the set of all $F$-thresholds of $g$, when the ideal $J$ varies (see [BMS08, Corollary 2.30]). We note that unlike in loc. cit. we do not consider 0 as an $F$-jumping number, which corresponds to the fact that we require $J$ to be a proper ideal of $\mathcal{O}_X$.

As we have mentioned in the Introduction, a basic result about the $F$-jumping numbers of $g$ is that they form a discrete set of rational numbers (see [BMS08, Theorem 3.1] for the case when $R$ is essentially of finite type over a field and [BMS09, Theorem 1.1] for the general case). We note that the corresponding result about jumping numbers in characteristic 0 is an immediate consequence of their description in terms of a log resolution, see [Laz04, Lemma 9.3.21].

As it is the case for jumping numbers in characteristic 0, some $\lambda > 1$ is an $F$-jumping number of $g$ if and only if $\lambda - 1$ has the same property. This follows, for example, from the fact that $\tau(g^\alpha) = g \cdot \tau(g^{\alpha - 1})$ for every $\alpha > 1$ (see for example [BMS08, Proposition 2.25]). A more interesting fact, that is peculiar to positive characteristic, is that if $\lambda$ is an $F$-jumping number of $g$, then so is $p\lambda$. This follows from the fact that $p \cdot c^J(g) = c^{f_p^J}(g)$, see [BMS08, Proposition 3.4(1)]. By combining these two facts one can show that the $F$-pure
threshold \( \text{fpt}(g) \) can’t lie in certain intervals: more precisely, for every \( e \geq 1 \) and every integer \( a, \) with \( 1 \leq a \leq p^e - 1, \) we have

\[
\text{fpt}(g) \not\in \left( \frac{a}{p^e}, \frac{a}{p^e - 1} \right).
\]

This is due to the fact that if \( \text{fpt}(g) \in \left( \frac{a}{p^e}, \frac{a}{p^e - 1} \right), \) then \( p^e \cdot \text{fpt}(g) - a \) is an \( F \)-jumping number of \( g \) that is \( < \text{fpt}(g), \) a contradiction; see \([BMS09, \text{Proposition 4.3}]\) for details.

We end this discussion of \( F \)-jumping numbers with the following result that we will need:

**Proposition 2.1.** Suppose that \( Y = \text{Spec}(R) \) is a smooth affine scheme of finite type over the finite field \( k \) and \( k \subseteq k' \) is a finite field extension. If \( g \in R \) is nowhere zero and \( g' = g \otimes 1 \in R' = R \otimes_k k', \) then

\[
\tau(g^\lambda) = \tau(g^\lambda) \cdot R' \quad \text{for all } \lambda \in \mathbb{R}_{\geq 0}.
\]

In particular, \( g \) and \( g' \) have the same \( F \)-jumping numbers.

The first assertion is a very special case of \([HT04, \text{Theorem 3.3}]\), which applies since the homomorphism \( R \to R' \) is finite and étale. The second assertion is an immediate consequence.

## 3. The proof of the main result

Our goal is to prove the following more general version of Theorem 1.3.

**Theorem 3.1.** Let \( X \) be a smooth, irreducible complex algebraic variety, \( f \in \mathcal{O}_X(X) \) nonzero, \( \lambda > 0 \) a jumping number of \( f, \) and \( \lambda' < \lambda \) such that \( \mathcal{J}(f^\alpha) \) takes the same value for all \( \alpha \in [\lambda', \lambda]. \) If \((X_A, f_A)\) gives a model of \((X, f)\) over \( A, \) then after possibly replacing \( A \) by a localization \( A_\alpha, \) for every closed point \( t \in \text{Spec}(A) \) with \( \text{char} k(t) = p_t, \) there is an \( F \)-jumping number \( \mu \) for \( f_t \) in the interval \( (\lambda', \lambda], \) and for every such \( \mu, \) if \( (p_t - i)\lambda \in \mathbb{Z}, \) then

\[
\mu > \lambda - \frac{m_i}{p_t},
\]

where \( m_i = \max \{ \beta \mid b_f(-\beta) = 0, \beta - (\lambda i + 1 - \lfloor \lambda i \rfloor) \in \mathbb{Z}_{\geq 0} \}. \)

**Remark 3.2.** In the setting of Theorem 3.1, if \( \lambda = \frac{a}{p}, \) with \( a, b \) relatively prime, and

\[
I = \{ i \mid 1 \leq i \leq b, \gcd(i, b) = 1 \},
\]

then for every \( t \) there is \( i \in I \) such that \( (p_t - i)\lambda \in \mathbb{Z}. \) The assertion in the theorem thus implies that if \( m = \max_{i \in I} m_i, \) then \( \mu > \lambda - \frac{m}{p_t}. \)

In fact, we can get a uniform bound just in terms of \( n = \dim(X). \) More precisely, it follows from \([Sai94] \) that all roots of \( b_f(s) \) are \( > -n; \) therefore, in the situation in the theorem, we have \( \mu > \lambda - \frac{n}{p_t} \) for all \( t. \)

Note that by taking \( \lambda = \text{let}(f) \) in Theorem 3.1, we deduce the assertion in Theorem 1.3. Hence from now on we focus on Theorem 3.1.

**Remark 3.3.** If \( X, f, \) and \( A \) are as in the theorem, and \( \lambda \in \mathbb{R}_{>0} \) is not a jumping number of \( f, \) then there is \( \lambda' < \lambda \) such that \( \mathcal{J}(f^\lambda) = \mathcal{J}(f^{\lambda'}). \) Applying Theorem 1.1 for both \( \lambda \) and \( \lambda', \) we see that if \( t \in \text{Spec}(A) \) is a closed point with \( \text{char} k(t) \gg 0, \) then there is no \( F \)-jumping number of \( f_t \) in the interval \( (\lambda', \lambda]. \)
Before giving the proof of Theorem 3.1, we make a few preliminary remarks concerning models for \((X, f)\) over \(A\). This is standard material, for more details we refer to [MS11, Section 2.2].

The assumption in Theorem 3.1 is that \(A \subseteq C\) is a finite type algebra over \(Z\) and \(X_A\) is a scheme of finite type over \(A\) and \(f_A \in \mathcal{O}_{X_A}(X_A)\) are such that we have an isomorphism \(X_A \times_A C \cong X\) such that the pull-back of \(f_A\) is mapped to \(f\). By generic smoothness, after possibly replacing \(A\) by a localization \(A_t\), we may and always assume that \(X_A\) is smooth over \(\text{Spec}(A)\) and \(f_A\) defines a relative Cartier divisor in \(X_A\) over \(\text{Spec}(A)\).

If \(A \subseteq A' \subseteq C\), where \(A'\) is another finite type algebra over \(Z\), then we have \(X_{A'} = X \times_A A'\) and the image \(f_{A'}\) of \(f_A\) in \(\mathcal{O}_{X_{A'}}(X_{A'})\), which give a model for \((X, f)\) over \(A'\). By general properties of finite type morphisms, there is a nonzero \(a \in A\) such that \(\text{Spec}(A_a)\) is contained in the image of \(\text{Spec}(A')\). Moreover, if \(t \in \text{Spec}(A_a)\) is a closed point, then there is a closed point \(t' \in \text{Spec}(A')\) whose image is \(t\). In particular, the field extension \(k(t) \subset k(t')\) is an extension of finite fields, hence we may apply Proposition 2.1 to the elements of an affine open cover of \(X_t\) to conclude that \(f_t\) and \(f_{t'}\) have the same \(F\)-jumping numbers. Therefore it is enough to prove the theorem for \(X_{A'}\) and \(f_{A'}\), hence we are free to replace \(A\) by a larger algebra with the same properties.

We can now prove our main result.

Proof of Theorem 3.1. If \(X = U_1 \cup \ldots \cup U_N\) is an affine open cover, then \(b_f = \text{lcm}(b_f|_{U_i})\). Since the set of jumping numbers of \(f\) is the union of the sets of jumping numbers of \(f|_{U_i}\), for \(1 \leq i \leq N\), it is straightforward to see that it is enough to prove the assertion in the theorem for each \(U_i\). Hence from now on we may and will assume that \(X = \text{Spec}(R)\) is affine.

Suppose that \(X_A = \text{Spec}(R_A)\). For every closed point \(t \in \text{Spec}(A)\), let \(R_t = R_A \otimes_A k(t)\) be the \(k(t)\)-algebra corresponding to \(X_t\). By definition of the Bernstein-Sato polynomial, there is \(P(s) \in D_R[s]\) such that
\[
b_f(s)f^s = P(s) \cdot f^{s+1},
\]
where \(D_R\) is the ring of differential operators of \(R\). After possibly replacing \(A\) by a larger finitely generated \(Z\)-subalgebra of \(C\), we may assume that the denominators of the roots of \(b_f\) are invertible in \(A\) and that we have \(P_A(s) \in D_{R_{A/A}}^{(0)}[s]\) such that
\[
b_f(s)f_A^s = P_A(s) \cdot f_A^{s+1},
\]
where \(D_{R_{A/A}}^{(0)} \subseteq \text{End}_A(R_A)\) is the subring generated by \(R_A\) and \(\text{Der}_A(R_A)\). In this case, for every closed point \(t \in \text{Spec}(A)\) with \(\text{char} k(t) = p_t\), we get a corresponding relation
\[
\overline{b_f}(s)f_t^s = P_t(s) \cdot f_t^{s+1},
\]
where \(\overline{b_f} \in \mathbb{F}_{p_t}[s]\) is the image of \(b_f\) and \(P_t(s)\) has coefficients in the subring \(D_{R_t/k(t)}^{(0)} \subseteq \text{End}_{k(t)}(R_t)\) generated by \(R_t\) and \(\text{Der}_{k(t)}(R_t)\). In particular, for every \(m \in \mathbb{Z}_{\geq 0}\), we have
\[
\overline{b_f}(m)f_t^m \in D_{R_t/k(t)}^{(0)} \cdot f_t^{m+1} \subseteq R_t.
\]
A key observation, going back to [MTW05, Proposition 3.11] is that for every proper ideal \(J\) in \(R_t\) and every \(e \geq 1\), we have
\[
\overline{b_f}((p_t^e)^J) = 0 \text{ in } \mathbb{F}_{p_t}.
\]
Indeed, if we take \( m = \nu_{f_t}^J(p_t^e) \), then by definition we have \( f^m \not\in J[p_t^e] \), while \( f^{m+1} \in J[p_t^e] \). Since the ideal \( J[p_t^e] \) is preserved by the action of \( D_{R_t/k(t)}^{(0)} \), the assertion in (8) is a consequence of the formula in (7).

The other main ingredient in our proof is [MZ13, Theorem B(i)], which says that there is \( C > 0 \) such that after possibly replacing \( A \) by some localization \( A_o \), we may assume that for every closed point \( t \in \text{Spec}(A) \) with \( \text{char } k(t) = p_t \), there is always an \( F \)-jumping number \( \mu \in (\lambda', \lambda] \) for \( f_t \) and every such \( F \)-jumping number satisfies

\[
\lambda = \frac{C}{p_t} \leq \mu \leq \lambda.
\]

Without any loss of generality, we may and will assume that \( C \in \mathbb{Z} \), in which case it follows from (9) that

\[
0 \leq \lfloor \lambda p_t \rfloor - \lfloor \mu p_t \rfloor \leq C. \tag{10}
\]

Let us pick one such \( t \in \text{Spec}(A) \) and \( \mu \) as above. Using the description of \( F \)-jumping numbers as \( F \)-thresholds discussed in the previous section, we see that there is a proper ideal \( J \) in \( R_t \) such that \( \mu = c^J(f_t) \). In this case, it follows from (5) that \( \lfloor \mu p_t \rfloor = \nu_{f_t}^J(p_t) + 1 \).

Let us choose a positive integer \( d \) such that the coefficients of \( b_f \) lie in \( \frac{1}{d} \mathbb{Z} \) and \( d \lambda^i \in \mathbb{Z} \) for \( i \leq \deg(b_f) \). We may and will assume that \( p_t \) does not divide \( d \). We deduce from (8) that

\[
b_f(\lfloor \mu p_t \rfloor - 1) \equiv 0 \pmod{p_t} \text{ in } \frac{1}{d} \mathbb{Z}. \tag{11}
\]

Let \( k = \lfloor \lambda p_t \rfloor - \lfloor \mu p_t \rfloor \), so \( 0 \leq k \leq C \) by (10). Since we can write \( \lambda p_t = \lambda(p_t - i) + \lambda i \) and \( \lambda(p_t - i) \in \mathbb{Z} \), it follows that

\[
\lfloor \lambda p_t \rfloor = \lambda(p_t - i) + \lfloor \lambda i \rfloor.
\]

The condition (11) thus implies

\[
b_f(\lambda(p_t - i) + \lfloor \lambda i \rfloor - k - 1) \equiv 0 \pmod{p_t} \text{ in } \frac{1}{d} \mathbb{Z},
\]

and thus

\[
b_f(\lfloor \lambda i \rfloor - \lambda i - k - 1) \equiv 0 \pmod{p_t} \text{ in } \frac{1}{d} \mathbb{Z}. \tag{12}
\]

Since \( k \) is an integer bounded independently of \( t \) and \( \mu \), it can take only finitely many values. After possibly inverting finitely many prime integers, we may assume that every \( k \) that appears when we vary \( t \in \text{Spec}(A) \), appears for infinitely many primes \( p_t \). In this case, condition (12) implies that for every such \( k \), we have

\[
b_f(\lfloor \lambda i \rfloor - \lambda i - k - 1) = 0. \tag{13}
\]

By definition, we thus have

\[
k + 1 + \lambda i - \lfloor \lambda i \rfloor \leq m_i. \tag{14}
\]

Note now that we have

\[
\mu > \frac{\lfloor \mu p_t \rfloor - 1}{p_t} = \frac{\lfloor \lambda p_t \rfloor - k - 1}{p_t} = \frac{\lambda(p_t - i) + \lfloor \lambda i \rfloor - k - 1}{p_t} \geq \frac{\lambda - m_i}{p_t},
\]

where the last inequality follows from (14). This completes the proof of the theorem. \( \square \)

We next deduce our result on the equality between \( \text{let}(f) \) and \( \text{fpt}(f_t) \).
Proof of Corollary 1.4. It follows from Theorem 1.3 that under our assumption, we may assume that for every closed point \( t \in \text{Spec}(A) \) such that \( (p_t - 1) \cdot \text{lct}(f) \in \mathbb{Z} \), we have

\[
\text{fpt}(f_t) > \text{lct}(f) - \frac{\text{lct}(f)}{p_t},
\]

where \( p_t = \text{char} \ k(t) \). Moreover, we also have

\[
\text{fpt}(f_t) \leq \text{lct}(f)
\]

by Theorem 1.1i). In order to prove that \( \text{fpt}(f_t) = \text{lct}(f) \), it is enough to show that we can’t have \( \text{fpt}(f_t) \in \left( \text{lct}(f) - \frac{\text{lct}(f)}{p_t}, \text{lct}(f) \right) \). This follows from (6). Indeed, let us write \( \text{lct}(f) = \frac{a}{b} \), with \( \gcd(a, b) = 1 \). Since \( (p_t - 1) \cdot \text{lct}(f) \in \mathbb{Z} \), we can write \( p_t - 1 = bc \) for some positive integer \( c \). We then have

\[
\text{lct}(f) = \frac{ac}{pt - 1} \quad \text{and} \quad \text{lct}(f) - \frac{\text{lct}(f)}{pt} = \frac{ac}{pt},
\]

hence (6) gives the assertion we need.

\[\square\]

Example 3.4. Let \( f = x^2 + y^3 \in \mathbb{C}[x, y] \). It is well-known that \( \text{lct}(f) = \frac{5}{6} \) (see [Laz04, Example 9.2.15]) and in fact

\[
b_f(s) = \left( s + \frac{5}{6} \right) (s + 1) \left( s + \frac{7}{9} \right)
\]

(see for example [Kas03, Example 6.19]). In this case we can take \( A = \mathbb{Z} \) and the model given by \( (A_2^2, f) \), where we view \( f \) as an element of \( \mathbb{Z}[x, y] \). If we denote by \( f_p \) the image of \( f \) in \( \mathbb{F}_p[x, y] \), with \( p \gg 0 \), then it follows from Corollary 1.4 that if \( p \equiv 1 \pmod{6} \), then \( \text{fpt}(f_p) = \text{lct}(f) = \frac{5}{6} \), and it follows from Theorem 1.3 that if \( p \equiv 5 \pmod{6} \), then

\[
\text{fpt}(f_p) \geq \frac{5}{6} - \frac{7}{6p}.
\]

In fact, it is known that in this case \( \text{fpt}(f_p) = \frac{5}{6} - \frac{1}{6p} \), see [MTW05, Example 4.3].

The following example was pointed out by the anonymous referee.

Example 3.5. Consider the polynomials \( f, g \in \mathbb{C}[x, y] \), where

\[
f = x^5 + y^4 \quad \text{and} \quad g = x^5 + x^3y^2 + y^4.
\]

The roots of \( b_f(s) \) are the negatives of

\[
\frac{9}{20}, \frac{11}{20}, \frac{13}{10}, \frac{17}{20}, \frac{19}{20}, \frac{1}{10}, \frac{21}{20}, \frac{23}{20}, \frac{27}{20}, \frac{31}{20}
\]

and the roots of \( b_g(s) \) are the negatives of

\[
\frac{9}{20}, \frac{11}{20}, \frac{13}{10}, \frac{7}{10}, \frac{9}{20}, \frac{19}{20}, 1, \frac{21}{20}, \frac{23}{20}, \frac{13}{20}, \frac{27}{20}, \frac{31}{20}
\]

(see [Yan78, Sections 11 and 18]). This is a well-known example in which \( f \) and \( g \) are part of a family of isolated singularities, with constant Milnor number, but such that the Bernstein-Sato polynomials are different. In both cases we may take \( A = \mathbb{Z} \) and the models given by \( (A_2^2, f) \) and \( (A_2^2, g) \), respectively, where we view \( f \) and \( g \) as elements of \( \mathbb{Z}[x, y] \). Note that we have \( \text{lct}(f) = \frac{30}{20} = \text{lct}(g) \) and

\[
\text{fpt}(f_p) = \frac{9}{20} \text{ if } p \equiv 1 \pmod{20} \quad \text{and} \quad \text{fpt}(f_p) = \frac{9p-11}{20p} \text{ if } p \equiv 19 \pmod{20}
\]

(this follows, for example, from [Her15, Theorem 3.1]), while

\[
\text{fpt}(g_p) = \frac{9}{20} \text{ if } p \equiv 1 \pmod{20} \quad \text{and} \quad \text{fpt}(g_p) = \frac{9p-11}{20(p-1)} \text{ if } p \equiv 19 \pmod{20}
\]
(see [MTW05, Example 4.5]). Note that this matches the predictions provided by Corollary 1.4, namely that
\[ \text{fpt}(f_p) = \text{lct}(f) \quad \text{and} \quad \text{fpt}(g_p) = \text{lct}(g) \quad \text{if} \quad p \equiv 1 \pmod{20} \]
and those provided by Theorem 1.3, namely that
\[ \text{fpt}(f_p) > \text{lct}(f) - \frac{31}{20p} \quad \text{and} \quad \text{fpt}(g_p) > \text{lct}(g) - \frac{11}{20p} \quad \text{if} \quad p \equiv 19 \pmod{20}. \]
Note that in this case the bound satisfied by \( g \) when \( p \equiv 19 \pmod{20} \) is not satisfied by \( f \) (in fact, for \( f \) this becomes equality), but this is allowed due to the presence of the root \(-\frac{31}{20}\) of \( b_f(s) \).

**Remark 3.6.** Suppose now that \( X \) is a smooth, irreducible, \( n \)-dimensional complex algebraic variety and \( a \) is an arbitrary nonzero coherent ideal sheaf on \( X \). Recall that we can associate multiplier ideals and test ideals to non-principal ideal as well (see [Laz04, Chapter 9] for the case of multiplier ideals and [BMS08] for the case of test ideals) Let \( \lambda > 0 \) be a jumping number of \( a \), and let \( \lambda' < \lambda \) be such that \( J(a^\alpha) \) takes the same value for all \( \alpha \in [\lambda', \lambda) \). We fix an integer \( d > \lambda \). We claim that if \((X_A, a_A)\) gives a model of \((X, a)\), then after possibly replacing \( A \) by a localization \( A_\alpha \), for every closed point \( t \in \text{Spec}(A) \) with \( \text{char } k(t) = p_t \), there is an \( F \)-jumping number \( \mu \) for \( a_t \) in the interval \([\lambda', \lambda]\), and for every such \( \mu \), we have
\[ \mu > \lambda - \frac{dn}{p_t}. \]
Indeed, in order to see this, we may again assume that \( X = \text{Spec}(R) \) is affine and let \( f_1, \ldots, f_r \in R \) be generators of \( a \). For \( 1 \leq i \leq d \), let \( h_i \) be a general linear combination of \( f_1, \ldots, f_r \), with \( \mathbb{C} \)-coefficients, and let \( h = \prod_{i=1}^d h_i \). In this case it follows from [Laz04, Proposition 9.2.28] that
\[ J(a^\alpha) = J(h^{\alpha/d}) \quad \text{for all} \quad \alpha \leq \lambda. \]
After possibly enlarging \( A \), we may assume that we have a model \( h_A \in R_A \) for \( h \), such that \( h_A \in a_A^d \).

We may assume the existence of an \( F \)-jumping number for \( a_t \) in the interval \([\lambda', \lambda]\) by Theorem 1.1 (more precisely, by its version for arbitrary ideals), hence we only need to prove that
\[ \tau(a_t^{\lambda'}) = \tau(a_t^{\lambda - \frac{dn}{p_t}}). \]
Note that we may assume that we have the equalities
\[ \tau(a_t^{\lambda'}) = J(a^{\lambda'})t = J(h^{\lambda'}_{-\lambda})t = \tau(h_{-\lambda}^{\lambda'}), \]
where the first and third equalities follow from (1.1), the second one follows from (16), and the fourth one follows from Remark 3.2. On the other hand, we have the inclusions
\[ \tau(h_{-\lambda}^{\lambda'} \frac{n}{p_t}) \subseteq \tau(a_t^{\lambda - \frac{dn}{p_t}}) \subseteq \tau(a_t^{\lambda'}), \]
where the first inclusion follows from \( h_t \in a_t^d \) and the second one follows from \( \lambda' \leq \lambda - \frac{dn}{p_t} \). By combining (18) and (19), we obtain, in particular, the equality (17). This completes the proof of our assertion.

We finally note that for an arbitrary ideal \( a \) there is a notion of Bernstein-Sato polynomial, whose roots are negative rational numbers, see [BMS06]. It is an interesting question whether
the bound in (15) can be refined by taking into account these roots as in Theorem 3.1. In fact, everything in the proof of the theorem carries through in this setting, with the exception of one point: it is not clear that having $\lambda p_t - C \leq \mu p_t$ implies that $\lambda p_t - \nu_J(p_t)$ stays bounded.

References

[Ber72] I. N.Bernštejn, Analytic continuation of generalized functions with respect to a parameter, Funkcional. Anal. i Priložen. 6 (1972), no. 4, 26–40. \[2

[BST17] B. Bhattacharya, K. Schwede, and S. Takagi, The weak ordinarity conjecture and F-singularities, Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata’s sixtieth birthday, Adv. Stud. Pure Math., vol. 74, Math. Soc. Japan, Tokyo, 2017, pp. 11–39. \[2

[BMS08] M. Blickle, M. Mustată, and K. E. Smith, Discreteness and rationality of F-thresholds, Michigan Math. J. 57 (2008), 43–61. Special volume in honor of Melvin Hochster. \[3, 4, 5, 9

[BMS09] M. Blickle, M. Mustată, and K. E. Smith, F-thresholds of hypersurfaces, Trans. Amer. Math. Soc. 361 (2009), no. 12, 6549–6565. \[1, 4, 5

[BMS06] N. Budur, M. Mustată, and M. Saito, Bernstein-Sato polynomials of arbitrary varieties, Compos. Math. 142 (2006), no. 3, 779–797. \[9

[Dod22] C. Dodd, Differential operators, gauges, and mixed Hodge modules, preprint arXiv:2210.12611 (2022). \[3

[HT04] N. Hara and S. Takagi, On a generalization of test ideals, Nagoya Math. J. 175 (2004), 59–74. \[5

[HY03] N. Hara and K.-I. Yoshida, A generalization of tight closure and multiplier ideals, Trans. Amer. Math. Soc. 355 (2003), no. 8, 3143–3174. \[1, 2, 3

[Her15] D. J. Hernández, F-invariants of diagonal hypersurfaces, Proc. Amer. Math. Soc. 143 (2015), no. 1, 87–104. \[8

[Kas76] M. Kashiwara, B-functions and holonomic systems. Rationality of roots of B-functions, Invent. Math. 38 (1976/77), no. 1, 33–53. \[12

[Kas03] , D-modules and microlocal calculus, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003. Translated from the 2000 Japanese original by Mutsumi Saito; Iwanami Series in Modern Mathematics. \[8

[Kol97] J. Kollár, Singularities of pairs, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. \[3

[Kun69] E. Kunz, Characterizations of regular local rings of characteristic p, Amer. J. Math. 91 (1969), 772–784. \[3

[Laz04] R. Lazarsfeld, Positivity in algebraic geometry II, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 49, Springer-Verlag, Berlin, 2004. \[1, 4, 8, 9

[Mus12] M. Mustată, Ordinary varieties and the comparison between multiplier ideals and test ideals II, Proc. Amer. Math. Soc. 140 (2012), no. 3, 805–810. \[12

[MS11] M. Mustată and V. Srinivas, Ordinary varieties and the comparison between multiplier ideals and test ideals, Nagoya Math. J. 204 (2011), 125–157. \[6

[MTW05] M. Mustată, S. Takagi, and K.-i. Watanabe, F-thresholds and Bernstein-Sato polynomials, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2005, pp. 341–364. \[3, 4, 6, 8, 9

[MZ13] M. Mustată and W. Zhang, Estimates for F-jumping numbers and bounds for Hartshorne-Speiser-Lyubeznik numbers, Nagoya Math. J. 210 (2013), 133–160. \[3, 7

[Sai94] M. Saito, On microlocal b-function, Bull. Soc. Math. France 122 (1994), no. 2, 163–184. \[3, 5

[TW04] S. Takagi and K.-i. Watanabe, On F-pure thresholds, J. Algebra 282 (2004), no. 1. \[1

[Yan78] T. Yano, On the theory of b-functions, Publ. Res. Inst. Math. Sci. 14 (1978), no. 1, 111–202. \[8

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