MODULI OF WEIGHTED STABLE CURVES AND LOG CANONICAL MODELS OF $\overline{M}_{g,n}$

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Abstract. We prove that Hassett’s spaces $\overline{M}_{g,A}$ are log canonical models of $\overline{M}_{g,n}$.

1. Introduction

In [9], Hassett constructed spaces $\overline{M}_{g,A}$ of weighted pointed stable curves, each of which is a compactification of $\overline{M}_{g,n}$. We establish nefness (and in some cases ampleness) of a number of divisors on $\overline{M}_{g,A}$ by exploiting the polarizing line bundle on the universal family over $\overline{M}_{g,A}$, which is known to be nef by Kollár’s semipositivity results [13]. As an application, we prove that the coarse moduli space of $\overline{M}_{g,A}$ is a log canonical model of $\overline{M}_{g,n}$ (Theorem 2) for an arbitrary weight vector $A$. In the case of $g = 0$, this solves Problem 7.1 of [9] (see also “Concluding Remarks” of [2]) and extends the results of [2], [5] and [14].

We now recall the definition of $\overline{M}_{g,A}$ from [9]: Given a weight vector $A = (a_1, \ldots, a_n) \in (0, 1]^n \cap \mathbb{Q}^n$, we say that an $n$-pointed connected at-worst nodal curve $(C; p_1, \ldots, p_n)$ of genus $g$ is $A$-stable if

1. $p_1, \ldots, p_n$ are nonsingular points of $C$.
2. $\omega_C(\sum_{i=1}^n a_i p_i)$ is ample.
3. $\text{mult}_x (\sum_{i=1}^n a_i p_i) \leq 1$, $\forall x \in C$.

The objects of the moduli stack $\overline{M}_{g,A}$ are flat families whose geometric fibers are $A$-stable curves of genus $g$.

Consider the universal family $(\pi: C \to \overline{M}_{g,A}, \{\sigma_i\}_{i=1}^n)$ of $A$-stable curves; here, $\{\sigma_i\}_{i=1}^n$ are the universal sections. By the definition above, the line bundle $\omega_{\pi}(\sum_{i=1}^n a_i \sigma_i)$ is $\pi$-nef. It turns out that it is also nef on $C$. We use this fact to produce nef divisors on $\overline{M}_{g,A}$. That $\omega_{\pi}(\sum_{i=1}^n a_i \sigma_i)$ is nef on $C$ was proved by Kollár in [13, Corollary 4.6 and Proposition 4.7]. When $a_i = 1$, $i = 1, \ldots, n$, i.e. $\overline{M}_{g,A} = \overline{M}_{g,n}$, the nefness of $\omega_{\pi}(\sum_{i=1}^n \sigma_i)$ was also established by Keel [10, Theorem 0.4]. We give yet another proof that $\omega_{\pi}(\sum_{i=1}^n a_i \sigma_i)$ is nef in Proposition 2.1; the argument is elementary and is included because it gives a finer information of curves on which $\omega_{\pi}(\sum_{i=1}^n a_i \sigma_i)$ has degree 0. The latter is used to establish ampleness of certain log canonical divisors on $\overline{M}_{g,A}$ (Theorem 1).

Even though we phrase all the results for an arbitrary genus, the positive genus case follows from that of genus 0 using a well-known positivity result on $\overline{M}_g$. For this reason, we isolate nef divisors on $\overline{M}_{0,n}$ produced by our construction in Section 5. These include all divisors contained in the GIT cone of $\overline{M}_{0,n}$ [2], as shown in Proposition 5.1. We leave open the question whether two cones are in fact the same.

We work over an algebraically closed base field of an arbitrary characteristic.
2. Main positivity result

Let \( B \) be a smooth curve. Consider a flat proper family \( \pi : C \to B \) of connected at-worst nodal curves of arithmetic genus \( g \). Suppose that \( \pi \) has \( n \), not necessarily distinct, sections \( \sigma_1, \ldots, \sigma_n : B \to C \).

Denote \( \sigma_i(B) \) by \( \Sigma_i \), and \( \sum_{i=1}^n \Sigma_i \) by \( \Sigma \). From now on, we assume that every \( \Sigma_i \) lies in \( C \setminus \text{Sing}(C/B) \). We denote both the relative dualizing line bundle of \( \pi \) and its numerical class by \( \omega \). We denote the numerical class of a fiber of \( \pi \) by \( F \).

**Proposition 2.1.** Let \( C \to B \) be a generically smooth family of nodal curves of arithmetic genus \( g \). Consider a weight vector \( A = (a_1, \ldots, a_n) \subset (0, 1]^n \cap \mathbb{Q}^n \) such that

\[
L := \omega + \sum_{i=1}^n a_i \Sigma_i
\]

is \( \pi \)-nef. Suppose further that sections \( \Sigma_{i_1}, \ldots, \Sigma_{i_k} \) can coincide only if \( \sum_{i=1}^k a_i \leq 1 \). Then \( L \) is a nef divisor on \( C \).

**Remark 1.** The assumptions of the above proposition are weaker than the condition of \( A \)-stability (see Section 1) in that they allow a collection of sections to intersect at finitely many points even when the sum of the associated weights is greater than 1.

**Proof.** The surface \( C \) has at-worst \( A_k \) singularities, which are Du Val (the discrepancies of the canonical divisor are zero). It follows that the statement holds for \( C \) if and only if it holds for the minimal desingularization of \( C \). From now on, we assume that \( C \) is smooth.

**Proof in the case \( g \geq 2 \).** Assume that \( C \subset C \) is an irreducible curve not contained in the fiber of \( \pi \). Denote by \( f : C \to C_{\text{min}} \) the relative minimal model of \( C \) over \( B \). We have \( \omega_{C/B} = f^*\omega_{C_{\text{min}}/B} + E \), where \( E \) is an effective vertical divisor. By [8, Theorem 6.33], \( \omega_{C_{\text{min}}/B} \) is nef on \( C_{\text{min}} \). It follows that \( \omega_{C/B} \cdot C \geq 0 \) and \( -\Sigma_i^2 = \omega_{C/B} \cdot \Sigma_i \geq 0 \). We conclude that \( L \cdot C \geq 0 \). Together with \( \pi \)-nefness assumption this proves that \( L \) is nef. \( \Box \)

**Proof in the case \( g = 1 \).** In this case, \( \omega_{C/B} \) is a linear combination of effective divisors supported in fibers (see [6, Theorem 15, p. 176]). Therefore, \( -\Sigma_i^2 = \omega_{C/B} \cdot \Sigma_i \geq 0 \). By the virtue of effectivity of \( \omega_{C/B} \) and \( \pi \)-nefness assumption, to prove that \( L = \omega_{C/B} + \sum_{i=1}^n a_i \Sigma_i \) is nef, we need to show that \( L \cdot \Sigma_i \geq 0 \), for every \( i = 1, \ldots, n \). Let \( \Sigma_{j_1}, \ldots, \Sigma_{j_k} \) be all the sections coinciding with \( \Sigma_i \). Then

\[
L \cdot \Sigma_i \geq (\omega_{C/B} + \sum_{\ell=1}^k a_{j_\ell} \Sigma_{j_\ell}) \cdot \Sigma_i
= -\Sigma_i^2 (1 - \sum_{\ell=1}^k a_{j_\ell}) \geq 0.
\]

**Proof in the case \( g = 0 \).** We need to show that \( C \cdot L \geq 0 \) for every irreducible curve \( C \) on \( C \). When \( C \) lies in the fiber, \( C \cdot L \geq 0 \) by assumption. Next, the proof breaks into two parts: in the first, we deal with the case when \( C \) is a section; in the second, we show that a general case reduces to the former.

First, suppose that \( C \) is a section. By successively blowing-down \((-1\)-curves not meeting \( C \), we reduce to the case when \( C \to B \) is a \( \mathbb{P}^1 \)-bundle. We consider two cases: \( C^2 < 0 \) and \( C^2 \geq 0 \).
Case I: \( C \) is a negative section of \( C \to B \). Without loss of generality, we can assume that sections \( \Sigma_1, \ldots, \Sigma_k \) coincide with \( C \). Then compute

\[
C \cdot L = C \cdot (\omega + \sum_{i=1}^{n} a_i \Sigma_i)
\]

\[
= -C^2 + \sum_{i=1}^{k} a_i C^2 + \sum_{i=k+1}^{n} C \cdot a_i \Sigma_i
\]

\[
= -C^2 (1 - \sum_{i=1}^{k} a_i) + \sum_{i=k+1}^{n} a_i (C \cdot \Sigma_i) \geq 0,
\]

since \( \sum_{i=1}^{k} a_i \leq 1 \) by assumption.

Case II: \( C \) is a non-negative sections of \( C \to B \). To make the computation more transparent, let \( \{ \Sigma_i \}_{i=1}^{k} \), \( \{ \Sigma_i \}_{i=k+1}^{\ell} \) and \( \{ \Sigma_i \}_{i=\ell+1}^{n} \) be the sections that, respectively, coincide with \( C \), have negative self-intersection and are neither of the first two. A \( \mathbb{P}^1 \)-bundle \( C \to B \) has a unique section of negative self-intersection, say, \( E^2 = -r \). Therefore, \( \Sigma_i = E \) for all \( i = k+1, \ldots, \ell \). By our assumption, we have

\[
\sum_{i=1}^{k} a_i \leq 1,
\]

\[
\sum_{i=k+1}^{\ell} a_i \leq 1,
\]

\[
\sum_{i=\ell+1}^{n} a_i \geq 2.
\]

A non-negative section \( \Sigma \) on \( C \) satisfies \( \Sigma^2 \geq r \). Also, since \( (C - \Sigma_i)^2 = 0 \), we have

\[
C \cdot \Sigma_i = \frac{1}{2} (C^2 + \Sigma_i^2), \quad i = 1, \ldots, n.
\]

Now compute

\[
C \cdot L = C \cdot (\omega + \sum_{i=1}^{n} a_i \Sigma_i)
\]

\[
= -C^2 + \sum_{i=1}^{k} (C \cdot a_i \Sigma_i)
\]

\[
= C^2 (\sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} \frac{a_i}{2} - 1) + \frac{1}{2} (-r \sum_{i=k+1}^{\ell} a_i + \sum_{i=\ell+1}^{n} a_i \Sigma_i^2)
\]

\[
\geq r (\sum_{i=1}^{k} a_i + \sum_{i=\ell+1}^{n} a_i - 1) > 0.
\]

Finally, we consider a general case when \( C \) is an arbitrary irreducible curve on \( C \) that does not lie in the fiber of \( \pi \).

Suppose first that \( C^2 < 0 \). The intersection number \( \omega \cdot C \) equals to \( -C^2 \) plus the number of branch points of \( \tilde{C} \to B \), where \( \tilde{C} \) is the normalization of \( C \). Hence \( \omega \cdot C > 0 \). If \( C \neq \Sigma_i \), then clearly \( C \cdot L > 0 \). The case \( C = \Sigma_i \) is already considered.
Suppose now that \( C^2 \geq 0 \). We will proceed to prove by contradiction that \( C \cdot L \geq 0 \). If \( C \cdot L < 0 \), then for an arbitrary ample divisor \( H \) on \( C \), we can find \( 0 < \epsilon \ll 1 \) such that \( (C + \epsilon H) \cdot L < 0 \). Since \( C \) is a nef divisor, \( C + \epsilon H \) is ample by the Kleiman’s criterion [12]. In particular, we can find a smooth curve \( D \subset C \) whose numerical class is a multiple of \( C + \epsilon H \) and such that \( D \) avoids (finitely many points) \( \text{Sing}(C/B) \). Consider the induced map \( D \to B \) and the fiber product \( C' := C \times_B D \).

\[
\begin{array}{cccc}
C' & \xrightarrow{\tau} & D & \xrightarrow{\pi} & C \\
\downarrow & & \downarrow & & \downarrow \\
\uparrow & & \uparrow & & \uparrow \\
B & \xrightarrow{\pi} & C \\
\end{array}
\]

The projection \( C' \to D \) has a section \( \tau : D \to C' \), whose image in \( C' \) maps onto \( D \) in \( C \). Let \( \Sigma_i' \) be the preimage of \( \Sigma_i \) on \( C \). Introduce the line bundle

\[
L' = \omega_{C'/D} + \sum_{i=1}^{n} a_i \Sigma_i'.
\]

Then \( L' \) is the pullback of \( L \) from \( C \). Moreover, \( \tau(D) \cdot L' = D \cdot L \), by the projection formula. Noting that the surface \( C' \) and the line bundle \( L' \) satisfy the assumptions of the proposition, we must have \( \tau(D) \cdot L' \geq 0 \), by the first part of the argument. A contradiction.

\[\square\]

3. Nef divisors on \( \overline{M}_{g,A} \)

3.1. Tautological divisors. Let \( A = (a_1, \ldots, a_n) \) be a weight vector with \( a_i \in (0,1] \cap \mathbb{Q} \) and \( 2g - 2 + \sum_{i=1}^{n} a_i > 2 \). Consider the Hassett’s moduli space \( \overline{M}_{g,A} \) parameterizing \( A \)-stable pointed genus \( g \) curves [9]. It is a smooth Deligne-Mumford stack and carries a universal family \((\pi : C \to \overline{M}_{g,A}, \sigma_1, \ldots, \sigma_n)\). We consider the following tautological divisor classes on the stack \( \overline{M}_{g,A} \):

1. The Hodge class \( \lambda = c_1(\pi_{*}\omega_{C/B}) \).
2. The kappa class \( \kappa = \pi_{*}(c_1^2(\omega_{C/B})) \) (this is different from \( \kappa_1 \) of [1]).
3. The psi-class \( \psi_i = \pi_{*}(\omega_{C/B}^{g,n}) \); the total psi-class is \( \psi := \sum_{i=1}^{n} \psi_i \).
4. The boundary divisors \( \Delta_{i,j} := \pi_{*}(\sigma_i \cdot \sigma_j) \). These are denoted \( D_{I,J}(A) \), where \( I = \{i, j\} \) and \( J = \{1, \ldots, n\} \setminus I \), in [9]. The sum of all \( \Delta_{i,j} \) is denoted by \( \Delta_{s} \).
5. The sum, denoted \( \Delta_{\text{nodal}} \), of the boundary divisors parameterizing nodal curves. This is denoted by \( \nu \) in loc. cit.

The Mumford’s relation gives \( \kappa = 12\lambda - \Delta_{\text{nodal}} \) (cf. [1]). In the case of \( \overline{M}_{g,n} \), we use the usual notation \( \Delta \) for the total boundary. Note for the future that under the natural reduction morphism \( \overline{M}_{g,n} \to \overline{M}_{g,A} \), the divisor \( \Delta \) in \( \overline{M}_{g,n} \) pushes forward to \( \Delta_{s} + \Delta_{\text{nodal}} \).

In the case of \( \overline{M}_{g,n} \), we also consider boundary divisors \( \Delta_{S} \) for every subset \( S \subset \{1, \ldots, n\} \) satisfying \( |S| \geq 2 \) (and in the case of \( g = 0 \), \( n - |S| \geq 2 \)). The generic point of \( \Delta_{S} \) is a reducible curve with a single node and an irreducible component of genus \( 0 \) marked by sections \( \{\sigma_i\}_{i \in S} \).
3.2. Main result.

Theorem 1. The following divisors

\[ A = A(a_1, \ldots, a_n) = \kappa + \psi + \sum_{i<j}(a_i + a_j)\Delta_{ij}, \]
\[ B = B(a_1, \ldots, a_n) = \kappa + \sum_{i=1}^{n}(2a_i - a_i^2)\psi_i + \sum_{i<j}(2a_i a_j)\Delta_{ij}, \]
\[ C_i = C_i(a_1, \ldots, a_n) = (1 - a_i)\psi_i + \sum_{j \neq i} a_j\Delta_{ij}, \quad \text{for each } i = 1, \ldots, n, \]

are nef on \( \overline{M}_{g,n} \).

Proof. Adopt the convention that for a generically singular family \( C \rightarrow B \), the sections \( B \rightarrow C \) associated to the conductor have weight 1. Then the divisors under consideration are functorial with respect to the boundary stratification (cf. [5]). Therefore, it suffices to check that they have non-negative degree on every curve \( \pi : (C; \sigma_1, \ldots, \sigma_n) \rightarrow B \) with an irreducible total space \( C \).

First, by Proposition 2.1, the divisor

\[ L := \omega + \sum_{i=1}^{n} a_i \sigma_i \]

is nef on \( C \). In particular, it is pseudoeffective and has a non-negative self-intersection. We will show that the intersection numbers \( A, B \) and \( C_i \) are non-negative by expressing each of them as an intersection of \( L \) with a pseudoeffective class on \( C \).

For \( A \), we note that \( \omega + \sum_{i=1}^{n} \sigma_i \) is an effective combination of \( L \) and \( \sigma_i, 1 \leq i \leq n \). Therefore,

\[ 0 \leq (\omega + \sum_{i=1}^{n} \sigma_i) \cdot L = \kappa + \psi + \sum_{i<j}(a_i + a_j)\Delta_{ij} = A. \]

For \( B \), we have

\[ 0 \leq L^2 = (\omega + \sum_{i=1}^{n} a_i \sigma_i)^2 = \kappa + \sum_{i=1}^{n}(2a_i - a_i^2)\psi_i + \sum_{i<j}(2a_i a_j)\Delta_{ij} = B. \]

For \( C_i \), we have

\[ 0 \leq L \cdot \sigma_i = (1 - a_i)\psi_i + \sum_{j \neq i} a_j\Delta_{ij} = C_i. \]

\[ \square \]

3.3. Ampleness result. With a bit more work, we can prove that the divisor \( A \) is ample.

We begin with a few preliminaries.

Definition 1. Let \( A = (a_1, \ldots, a_n) \) be a weight vector. Suppose that \( a_1 = \sum_{k=1}^{k} b_k \). Set \( B = (b_1, \ldots, b_k, b_{k+1}, \ldots, b_{n+k-1}) \), where \( b_{k+i} = a_{i+1} \) for \( i \geq 1 \).

A replacement morphism \( \chi : \overline{M}_{g,A} \rightarrow \overline{M}_{g,B} \) is a morphism which sends an \( A \)-stable curve \( (C \rightarrow B; \sigma_1, \ldots, \sigma_n) \) to a \( B \)-stable curve \( (C \rightarrow B; \tau_1, \ldots, \tau_k, \sigma_2, \ldots, \sigma_n) \), where the sections \( \{\tau_i\}_{1 \leq i \leq k} \) are equal to the section \( \sigma_1 \).
Lemma 3.1 (Pull-back formulae for $\chi$). Keep the notation of Definition 1. Under the replacement morphism $\chi: \overline{\mathcal{M}}_{g,A} \to \overline{\mathcal{M}}_{g,B}$ the divisor $A(b_1, \ldots, b_{n+k-1})$ pulls back according to the following formula
\[
\chi^* (A(b_1, \ldots, b_{n+k-1})) = A(a_1, \ldots, a_n) + (1 - a_1)\psi_1.
\]

Proof. This follows from a straightforward generalization of [5, Lemma 2.9]. The proof is analogous and so is omitted.

Lemma 3.2. The divisor $\kappa + \psi$ is a positive linear combination of $\Delta_*$ and all boundary divisors on the space of weighted pointed rational curves $\overline{\mathcal{M}}_{0,A}$.

Proof. On $\overline{\mathcal{M}}_{0,n}$, we have $\kappa = -\Delta$ and a well-known relation
\[
\kappa + \psi = \sum_{r=2}^{\lfloor n/2 \rfloor} \frac{r(n-r) - n + 1}{n-1} \Delta_r.
\]
This is a positive combination of $\Delta_2$ and all other boundary divisors for $n \geq 4$. The general case follows by pushing forward the relation (2) to $\overline{\mathcal{M}}_{0,A}$ via the reduction morphism.

Proposition 3.1. The divisor $A = A(a_1, \ldots, a_n) = \kappa + \psi + \sum_{i<j} (a_i + a_j)\Delta_{ij}$ on $\overline{\mathcal{M}}_{g,A}$ is a pullback of an ample divisor on the coarse moduli space.

Proof. We claim that a sufficiently small neighborhood (in the Euclidean topology with the usual norm $\| \cdot \|$ of $A$ lies inside the nef cone of the vector space $\text{NS}(\overline{\mathcal{M}}_{g,A}) \otimes \mathbb{Q}$). The statement then follows from Kleiman’s criterion [12] applied to the coarse moduli space of $\overline{\mathcal{M}}_{g,A}$. The proof is by induction on dimension. When dimension is 1 (i.e., when $g = 0$ and $|A| = 4$, or $g = 1$ and $|A| = 1$) the proof is by direct computation.

Suppose $\dim \overline{\mathcal{M}}_{g,A} > 1$. By the functoriality of $A$ and by the induction assumption, a sufficiently small perturbation of $A$ is still ample when restricted to any boundary divisor in $\Delta_*$. It remains to show that for any $P \in \text{NS}(\overline{\mathcal{M}}_{g,A}) \otimes \mathbb{Q}$ satisfying $0 \leq \|P\| \ll 1$, the perturbed divisor $A + P$ has non-negative degree on any family $C \to B$ with a generically smooth fiber of genus $g$.

The proof falls into two parts: $g = 0$ and $g \geq 1$.

Case of $g = 0$: First, suppose that no sections of negative self-intersection are coincident. Then for $0 < \epsilon \ll 1$ the divisor $(1 - \epsilon)\omega + \sum a_i \sigma_i$ is nef on $C$ by Proposition 2.1 (see also Remark 1). It follows that the divisor
\[
A(\epsilon) = \left( (1 - \epsilon)\omega + \sum a_i \sigma_i \right) (\omega + \sum \sigma_i) = A - \epsilon(\kappa + \psi).
\]
is non-negative on $B$. By Lemma 3.2, the divisor $\kappa + \psi$ is a positive linear combination of all boundary divisors and all divisors in $\Delta_*$. Since irreducible components of these divisors generate $\text{NS}(\overline{\mathcal{M}}_{0,A})$, we conclude that $\epsilon(\kappa + \psi) + P$ is an effective linear combination of these generators. Therefore, $A + P = A(\epsilon) + \epsilon(\kappa + \psi) + P$ intersects $B$ non-negatively.

Suppose that $k \geq 2$ sections $\{\sigma_i\}_{\ell=1}^k$ satisfying $\sum_{\ell=1}^k a_{i\ell} \leq 1$ and $\sigma^2_{i\ell} < 0$ are coincident. Let $\chi: \overline{\mathcal{M}}_{0,A^i} \to \overline{\mathcal{M}}_{0,A}$ be the (closed immersion) replacement morphism replacing a section $\tau$ of weight $\sum_{\ell=1}^k a_{i\ell}$ by $k$ sections $\{\sigma_{i\ell}\}_{\ell=1}^k$ of weights $\{a_{i\ell}\}_{\ell=1}^k$. Then by Lemma 3.1, the divisor $A$ pulls back to the sum of $\psi_\tau$ and the divisor $A(\sum_{\ell=1}^k a_{i\ell}, a_{k+1}, \ldots, a_n)$ on $\overline{\mathcal{M}}_{0,A'}$. The latter is ample by the induction assumption $(\dim(\overline{\mathcal{M}}_{0,A^i}) = \dim(\overline{\mathcal{M}}_{0,A}) - k + 1)$. In particular, the divisor $A + P$ pulls back to the sum of ample divisor and $\psi_\tau$. By the assumption $\sigma^2_{i\ell} < 0$, hence $\psi_\tau$ has positive degree on the curve $B$. It follows from the projection formula that $B \cdot (A + P) \geq 0$. 
Case of \( g \geq 1 \): Denote by \( f : \overline{M}_{g,n} \to \overline{M}_{g,A} \) the contraction morphism. Suppose that no sections of \((C \to B; \sigma_1, \ldots, \sigma_n)\) are coincident. Then \( B \) is not entirely contained in \( f(\text{Exc}(f)) \). As in the case of \( g = 0 \), for \( 0 < \epsilon \ll 1 \), the divisor \((1 - \epsilon)\omega + \sum a_i \sigma_i\) is nef on \( C \). It follows that the divisor
\[
A(\epsilon) = \left((1 - \epsilon)\omega + \sum a_i \sigma_i\right) \left(\omega + \sum \sigma_i\right) = A - \epsilon(\kappa + \psi)
\]
is non-negative on \( B \). It is well-known that the divisor \( \kappa + \psi = 12\lambda - \Delta + \psi \) is ample on \( \overline{M}_{g,n} \) \cite{7, 10}. Then for any \( P \in \text{NS}(\overline{M}_{g,A}) \otimes \mathbb{Q} \) satisfying \( |P| \ll 1 \), the divisor \( f^*(P + \epsilon(\kappa + \psi)) \) is an effective combination of an ample divisor and an \( f \)-exceptional divisor. Hence, \( B \cdot (P + \epsilon(\kappa + \psi)) \geq 0 \).

Finally, suppose that two sections are coincident. Then \( B \) lies in the image of some replacement morphism \( \chi \). By Lemma 3.1 and the induction assumption, the divisor \( A + P \) pulls back to a sum of an ample divisor and a \( \psi \) class. Since the self-intersection of any section in a family of stable curves of positive genus is non-positive, we are done. \( \square \)

4. Certain log-canonical models of \( \overline{M}_{g,n} \)

By \cite[Section 3.1.1]{9}, the canonical class of \( \overline{M}_{g,A} \) is
\[
K_{\overline{M}_{g,A}} = 13\lambda - 2\Delta_{\text{nodal}} + \psi = \lambda + \kappa - \Delta_{\text{nodal}} + \psi.
\]
The ample divisor \( A \) of Proposition 3.1 can be written as
\[
A = K_{\overline{M}_{g,A}} + \sum_{i < j} (a_i + a_j)\Delta_{i,j} + \Delta_{\text{nodal}} - \lambda.
\]
Since \( \lambda \) is always nef, we see that \( A + \lambda = K_{\overline{M}_{g,A}} + \sum_{i < j} (a_i + a_j)\Delta_{i,j} + \Delta_{\text{nodal}} \) is a natural ample log canonical divisor on \( \overline{M}_{g,A} \). Using this, we can give an affirmative answer to \cite[Problem 7.1]{9}. The statement is cleanest for \( g = 0 \), when all the spaces under consideration are smooth schemes. For \( g > 0 \), \( \overline{M}_{g,A} \) is only a smooth proper Deligne-Mumford stack.

We denote its coarse moduli space by \( \overline{M}_{g,A} \).

**Theorem 2.** The coarse moduli space \( \overline{M}_{g,A} \) is a log canonical model of \( \overline{M}_{g,n} \). Namely,
\[
\overline{M}_{g,A} = \text{Proj} \bigoplus_{m \geq 0} H^0(\overline{M}_{g,n}, m(K_{\overline{M}_{g,n}} + \sum_{i < j; a_i + a_j \leq 1} (a_i + a_j)\Delta_{i,j} + \Delta_{\text{rest}})),
\]
where \( \Delta_{\text{rest}} = \Delta - \sum_{i < j; a_i + a_j \leq 1} \Delta_{i,j} \).

**Proof.** Consider the reduction morphism \( f : \overline{M}_{g,n} \to \overline{M}_{g,A} \). Note that
\[
K_{\overline{M}_{g,n}} + \sum_{i < j} (a_i + a_j)\Delta_{i,j} + \Delta_{n} = A + \lambda
\]
By Proposition 3.1, the divisor \( A \) is ample on \( \overline{M}_{g,A} \). Since \( \lambda \) is nef, the sum \( A + \lambda \) is ample.

It remains to observe that \( f_*(K_{\overline{M}_{g,n}} + \sum_{i < j; a_i + a_j \leq 1} (a_i + a_j)\Delta_{i,j} + \Delta_{\text{rest}}) = A + \lambda \) and the discrepancy divisor
\[
\left(K_{\overline{M}_{g,n}} + \sum_{i < j; a_i + a_j \leq 1} (a_i + a_j)\Delta_{i,j} + \Delta_{\text{rest}}\right) - f^*(A + \lambda)
\]
is an effective combination of exceptional divisors with all discrepancies in the interval \([0, \infty)\). Indeed,

\[
f^*(A + \lambda) = \left( K_{\overline{M}_{0,n}} + \sum_{i<j; a_i + a_j \leq 1} (a_i + a_j)\Delta_{i,j} + \Delta_{\text{rest}} \right) + \sum_{S: |S| \geq 3} (|S| - 1) \left( 1 - \sum_{i \in S} a_i \right) \Delta_S \]

\[+ \sum_{\sum_{i \in S} a_i \leq 1} \left( a_i + a_j \right) \Delta_{i,j} \]

\[+ \sum_{r \geq 3} \Delta_r \]

\[\square\]

5. Nef divisors on \(\overline{M}_{0,n}\)

5.1. The case of \(g = 0\). We keep the conventions of Sections 2 and 3.1. On \(\overline{M}_{0,n}\), we let \(\Delta_r\) to be the union of all boundary divisors \(\Delta_S\) with \(|S| = r\). Recall that the canonical divisor of \(\overline{M}_{0,n}\) has class \(K_{\overline{M}_{0,n}} = \psi - 2\Delta_{\text{nodal}}\). We now restate Theorems 1 and 2 for \(\overline{M}_{0,n}\).

**Theorem 3.** The Hassett’s space \(\overline{M}_{0,A}\) is a log canonical model of \(\overline{M}_{0,n}\). Namely,

\[
\overline{M}_{0,A} = \text{Proj} \bigoplus_{m \geq 0} H^0(\overline{M}_{0,n}, m(K_{\overline{M}_{0,n}} + \sum_{i<j} (a_i + a_j)\Delta_{i,j} + \sum_{r \geq 3} \Delta_r)).
\]

**Theorem 4.** The divisor

\[
A(a_1, \ldots, a_n) = \psi + \sum_{i<j} (a_i + a_j)\Delta_{i,j} - \Delta_{\text{nodal}},
\]

is ample on \(\overline{M}_{0,A}\) and the divisors

\[
B(a_1, \ldots, a_n) = \sum_{i=1}^n (2a_i - a_i^2)\psi_i + \sum_{i<j} (2a_i a_j)\Delta_{i,j} - \Delta_{\text{nodal}},
\]

\[
C(a_1, \ldots, a_n) = \sum_{i=1}^n (1 - a_i)\psi_i + \sum_{j \neq i} (a_i + a_j)\Delta_{i,j}
\]

are nef on \(\overline{M}_{0,A}\).

5.2. \(\text{SL}_2\) quotients of \((\mathbb{P}^1)^n\). In [2], Alexeev and Swinarski introduced a subcone, called the GIT cone, of the nef cone of \(\overline{M}_{0,n}\). The GIT cone is generated by pullbacks of natural polarizations on GIT quotients \((\mathbb{P}^1)^n / \mathbb{G} \text{-SL}_2\). Here, we show that the GIT cone is contained in the cone generated by divisors of Theorem 4. We do not preclude a possibility that two cones coincide. The idea of proof is simple: We observe that the GIT polarization on \((\mathbb{P}^1)^n / \mathbb{G} \text{-SL}_2\) is proportional to \(A(x_1, \ldots, x_n)\).

In what follows, we regard \((\mathbb{P}^1)^n / \mathbb{G} \text{-SL}_2\) as a good moduli space of an Artin moduli stack\(^1\) of weighted \(n\)-pointed rational curves. In particular, all tautological divisors introduced in Section 3.1 make sense on \((\mathbb{P}^1)^n / \mathbb{G} \text{-SL}_2\).

\(^1\)The stack is Deligne-Mumford when the linearization is typical, i.e. there are no strictly semistable points.
Proposition 5.1. Let $\vec{x} = (x_1, \ldots, x_n)$ be such that $x_1 + \cdots + x_n = 2$. Then the natural GIT polarization on $(\mathbb{P}^1)^n/\pi \mathcal{SL}_2$ is proportional to $A(x_1, \ldots, x_n) = \psi + \sum_{i<j}(x_i+x_j)\Delta_{ij} - \Delta_{nodal}$.

Proof. We treat $(\mathbb{P}^1)^n$ as the parameter space of $n$ ordered points on $\mathbb{P}^1$. Consider the universal family $\pi: (\mathbb{P}^1)^n \times \mathbb{P}^1 \to (\mathbb{P}^1)^n$ with sections $\tau_1, \ldots, \tau_n$: $(\mathbb{P}^1)^n \to (\mathbb{P}^1)^n \times \mathbb{P}^1$. For $i = 1, \ldots, n$, set $H_i = p_i^* O_{\mathbb{P}^1}(1)$. Then $\tau_i = H_i + H_{n+1}$, for $i = 1, \ldots, n$, and the relative dualizing sheaf of the universal family is $\omega = -2H_{n+1}$. We compute

\[
(\omega + \sum_{i=1}^n x_i \tau_i) (\omega + \sum_{i=1}^n \tau_i) = \left( \sum_{i=1}^n x_i H_i \right) \left( (n-2)H_{n+1} + \sum_{i=1}^n H_i \right).
\]

Pushing forward via $\pi$, we obtain that on $(\mathbb{P}^1)^n$

\[
\pi_* (\omega + \sum_{i=1}^n x_i \tau_i) (\omega + \sum_{i=1}^n \tau_i) = (n-2) \sum_{i=1}^n x_i H_i.
\]

By definition, an integer multiple of $O_{(\mathbb{P}^1)^n}(\sum_{i=1}^n x_i H_i)$ descends to the GIT polarization on $(\mathbb{P}^1)^n/\pi \mathcal{SL}_2$. Hence, the divisor class $\pi_*(\omega + \sum_{i=1}^n x_i \tau_i) (\omega + \sum_{i=1}^n \tau_i)$ descends to a multiple of the GIT polarization. On the other hand, since $\omega$ and $\tau_i$ are defined functorially, the divisor class $\pi_* (\omega + \sum_{i=1}^n x_i \tau_i) (\omega + \sum_{i=1}^n \tau_i)$ descends to $A(x_1, \ldots, x_n)$ on $(\mathbb{P}^1)^n/\pi \mathcal{SL}_2$. The statement follows.

For completeness, we record what divisor classes on $(\mathbb{P}^1)^n$ descend to tautological divisor classes $\psi$, $\Delta_{ij}$ and $\Delta_{nodal}$ on $(\mathbb{P}^1)^n/\pi \mathcal{SL}_2$. On $(\mathbb{P}^1)^n$,

\[
\pi_* (-\tau_i^2) = \pi_* (-H_i + H_{n+1})^2 = -2H_i.
\]

It follows that $O_{(\mathbb{P}^1)^n}(-2H_i)$ descends to $\psi_i$. Further, since $(\tau_i - \tau_j)^2 = 0$ on $(\mathbb{P}^1)^n \times \mathbb{P}^1$, we conclude that

\[
\Delta_{ij} = -\frac{1}{2}(\psi_i + \psi_j).
\]

Finally, $\Delta_{nodal} = 0$ because all curves parameterized by $(\mathbb{P}^1)^n/\pi \mathcal{SL}_2$ are $\mathbb{P}^1$s.

We conclude that the GIT polarization on $(\mathbb{P}^1)^n/\pi \mathcal{SL}_2$ is written in terms of tautological divisor classes as

\[
A(x_1, \ldots, x_n) = \sum_{i=1}^n \psi_i + \sum_{i<j}(x_i+x_j)\Delta_{ij} - \Delta_{nodal} = \frac{(n-2)}{2} \sum_{i=1}^n x_i \psi_i.
\]

\[\square\]

5.3. Possible extensions. The main positivity result of Proposition 2.1 can be used to obtain other nef divisors on $\overline{\mathcal{M}}_{0,n}$. We present now an example of such application.

5.3.1. Case of $n = 6$. We consider a weight vector $A = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2)$. Clearly, $\overline{\mathcal{M}}_{0,A} \cong \overline{\mathcal{M}}_{0,6}$. Since the Fulton’s conjecture holds for $n = 6$ ([11, Theorem 1.2] or [4, Theorem 2]), the symmetric nef cone of $\overline{\mathcal{M}}_{0,A}$ has two extremal rays generated by divisor classes dual to $F$-curves. In terms of tautological divisor classes they are $2\Delta_{nodal} - \psi$ and $\psi + \Delta_s$. The latter is $\frac{3}{2} \cdot C(1/3, \ldots, 1/3)$, where $C(1/3, \ldots, 1/3)$ is the nef divisor of Theorem 4. We establish that the former is nef using Proposition 2.1 and a case-by-case analysis.

Proposition 5.2. The divisor $2\Delta_{nodal} - \psi$ is nef on $\overline{\mathcal{M}}_{0,A}$. 

Proof. First, note that the relation $5\psi + 2\Delta_s - 9\Delta_{\text{nodal}}$ holds in the Picard group of $\overline{M}_{0,4}$. It follows that

$$2\Delta_{\text{nodal}} - \psi = \frac{1}{9}(\psi + 4\Delta_s) = 2\psi + \frac{2}{3}\Delta_s - \Delta.$$  

Consider a family $C \rightarrow B$ of $\mathcal{A}$-stable curves over a smooth proper curve $B$. If the total space $C$ is not irreducible, then it is necessarily a union of two $(1,1/2,1/2,1/2)$-stable families over $B$. One sees readily that $\psi = -3\Delta_{\text{nodal}}$ and $\Delta_{\text{nodal}} > 0$ in this case, and so $2\Delta_{\text{nodal}} - \psi > 0$.

Suppose that $C$ is an irreducible surface. If all sections have positive self-intersection, we are done. If there are no coincident sections, then $\Delta_s \geq 0$ and by Proposition 2.1 the divisor $\omega + \Sigma$ is nef on $C$. It follows that $\psi + 2\Delta_s - \Delta_{\text{nodal}} = (\omega + \Sigma)^2 \geq 0$. In particular, $\psi + 4\Delta_s \geq 0$, and we are done.

Suppose now that there are two coincident sections $\Sigma_1 = \Sigma_2$, necessarily of negative self-intersection and disjoint from $\Sigma_{\text{rest}} = \sum_{i=3}^{6}\Sigma_i$. The fibered surface $\pi : C \rightarrow B$ is obtained by successive blow-ups from a $\mathbb{P}^1$-bundle over $B$ with a negative section $\Sigma_1$. It follows that the divisor $\omega + \frac{1}{3}\Sigma$ is an effective combination of the fiber class and $(-1)$-curves in the fibers of $\pi$. Moreover, these $(-1)$-curves are disjoint from $\Sigma_1$. By the stability assumption, $\omega + \frac{1}{3}\Sigma_{\text{rest}}$ has intersection 0 with the fiber class and positive intersection with $(-1)$-curves disjoint from $\Sigma_1$. We conclude that

$$0 \leq (\omega + \frac{1}{3}\Sigma) \cdot (\omega + \frac{1}{2}\Sigma_{\text{rest}}) = 2\psi + \frac{2}{3}\sum_{i=3}^{6}\psi_i + \frac{1}{3}\sum_{3 \leq i < j \leq 6}\Delta_{ij} - \Delta_{\text{nodal}}.$$  

If sections $\{\Sigma_i\}_{i=3}^{6}$ are distinct, then

$$2\psi + \frac{2}{3}\sum_{i=3}^{6}\psi_i + \frac{1}{3}\sum_{3 \leq i < j \leq 6}\Delta_{ij} - \Delta_{\text{nodal}}$$

$$\leq \frac{2}{3}\psi + \frac{2}{3}\sum_{i=3}^{6}\psi_i + \frac{2}{3}\sum_{3 \leq i < j \leq 6}\Delta_{ij} - \Delta_{\text{nodal}} = \frac{2}{3}\psi + \frac{2}{3}\Delta_s - \Delta_{\text{nodal}},$$

and we are done because of Equation (6).

Finally, if there are two pairs of coincident sections among $\{\Sigma_i\}_{i=1}^{6}$, then by replacing every pair of coincident sections by a section of weight 1, we reduce to proving that the divisor $\frac{2}{3}\psi + \frac{2}{3}\Delta_s - \Delta_{\text{nodal}}$ is nef on $\overline{M}_{0,(1,1,1)}$. The space under consideration is isomorphic $\mathbb{P}^1$. The degree of $\frac{2}{3}\psi + \frac{2}{3}\Delta_s - \Delta_{\text{nodal}}$ on the universal family is $4/3 + 2/3 - 2 = 0$.

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