Offline Dynamic Higher Connectivity

Richard Peng
GaTech
rpeng@cc.gatech.edu

Bryce Sandlund
UW-Madison
sandlund@cs.wisc.edu

Daniel D. Sleator
CMU
sleator@cs.cmu.edu

August 15, 2017

Abstract

We give the first $O(t \log t)$ time algorithm for processing a given sequence of $t$ edge updates and 3-vertex/edge connectivity queries in an undirected unweighted graph. Our approach builds upon a method by Eppstein [Epp94] that reduces graphs to smaller equivalents on a set of key vertices. It also leads to algorithms for offline 2-vertex/edge connectivity whose performances match those from Kaczmarz and Lacki [KL15].

1 Introduction

Dynamic graph data structures seek to maintain some property of a graph that’s modified by edge insertions/deletions. Connectivity queries ask for the existence of a path connecting two vertices $u$ and $v$ in the current graph. Aside from being a natural problem with a variety of applications, connectivity is representative of global and non-decomposable graph properties. That is, the insertion/deletion of a single edge may have consequences across the entire graph, and it’s difficult to combine solutions of arbitrary partitions of the graph to solve the problem. As a result, the study of maintenance of connectivity under graph updates is a central topic in graph data structures [Fre83, HK99, HdLT98, Tho00, AHLT05, KKM13, NS17].

Dynamic graph data structures are typically studied under the online setting, where each query needs to be processed before the next one is given. A less demanding variant is the offline setting, where the entire sequence of insertions/deletions/queries is provided as input to the algorithm. For many well-studied problems in dynamic graphs, one can obtain faster and simpler algorithms in the offline setting. For example, one of the first results in this direction is processing a sequence of $t$ inserts and deletes to a minimum spanning tree in $O(t \log n)$ time [Epp94], while in the online setting the current best bound in the pointer machine model is about $\log^4 n$ per update [HdLT98]. On the other hand, many lower bounds for data structures such as connectivity [PD04] and dynamic maintenance of flows [AWY15] are for processing a given sequence of queries. As a result, the offline model appears to be closer to the limitations of efficiency of answering multiple graph queries.

In this note we give better offline algorithms for maintain higher connectivity in undirected graphs. This approach is based on performing divide-and-conquer on the sequence of updates and building equivalent graphs on a small set of active vertices. Such ideas are present in previous works on graph timelines [LS13, KL15], as well as recent works on integrating graph sparsifiers into algorithms [ADK16, DKP17]. Our main result is:
Theorem 1.1. Given a sequence of \( t \) updates/queries on a graph of the form:

- Insert edge \((u,v)\).
- Delete edge \((u,v)\).
- Query if a pair of vertices \( u \) and \( v \) are 2-edge connected/3-edge connected/bi-connected/tri-connected in the current graph.

There exists an algorithm that answers all queries in \( O(t \log t) \) time.

For 2-edge connectivity and bi-connectivity, these bounds match the bounds from Kaczmarz-Lacki \cite{KL15} and are faster by polylog factors than online data-structures \cite{HdLT98}. For tri-connectivity and 3-edge connectivity, previous works only gave polylog time per update in the incremental setting \cite{DBT90} and about \( n^{2/3} \) per update in the fully dynamic setting \cite{GI91}. The paper is organized as follows:

- We describe our offline framework for reducing graphs to smaller equivalents in Section 2.
- We construct such equivalents for 2-edge connectivity and bi-connectivity in Section 3.
- In Section 4 we show that such ideas naturally extend to 3-edge connectivity.
- In Section 5 we develop a similar contraction process using the SPQR tree characterization of 2-vertex cuts (known as split pairs) to solve tri-connectivity.
- In Section 6 we conclude and give ideas for future work.

2 Processing Updates/Queries Offline

In this section we state our method for performing divide-and-conquer on the update sequence. As we will handle updates and queries in similar ways, we will denote insertions, deletions, and queries all as events. Specifically, we will label them in chronological order as

\[ x_1 \ldots x_t. \]

By duplicating edges, and adding extra insertions/deletions at the start/end, we may assume that each edge is inserted and deleted exactly once during these updates. For example, if edge \((1, 2)\) is deleted then inserted again later, the two occurrences of that edge are given different identities in our framework.

Because we’re given the entire sequence of modifications, for each edge we may associate an interval \([i_e, d_e]\), indicating that edge \( e \) was inserted at time \( i_e \) and removed at time \( d_e \). Plotting time along the \( x \) axis and edges on the \( y \) axis as in Figure 1 gives a convenient way to view the sequence of events.

A key idea from \cite{Epp94, LS13} is that if we process the events in a smaller time interval, we can work with a graph that goes through a smaller number of changes. For example, for the sequence of events corresponding to the timeline diagram shown in Figure 1 if we process the time interval \([6, 9]\), we will always have \( e_1 \) in our graph while never encountering \( e_2 \). There are only two edges, \( e_2 \) and \( e_3 \), that do get modified in this time interval. To summarize, if we try to answer queries from a range \([l, r]\), we will only deal with graphs containing two types of edges:

1. Edges affected by an event in this range (one or both of \( i_e, d_e \) is in \([l, r]\)), which we denote as non-permanent edges.
2. Edges present throughout the duration \((i_e < l \leq r < d_e)\), which we denote as permanent edges.
While there can be a large number of permanent edges, the number of non-permanent edges is limited by the number of time steps, $r - l + 1$. Therefore, the graph can be viewed as a large static graph on which a smaller number of events take place.

One issue with narrowing down to subintervals is that a deletion in this range might have its corresponding insertion outside ($i_e < l \leq d_e \leq r$). We handle this by treating this edge as a non-permanent edge that’s present at the start of this time interval. In this way, the more difficult edge deletion operation is replaced by insertion, when the edge becomes permanent in a particular time interval. Therefore, our reduction procedures that we will describe below need only preserve the structure of permanent edges as non-permanent edges are inserted into the graph.

Our goal will be to reduce this graph of permanent edges to one whose size is proportional to the number of events, or non-permanent edges.

**Definition 2.1.** An equivalent of a graph $G$ and $k$ events $x_1 \ldots x_k$ is a graph $H$ along with $k$ events $y_1 \ldots y_k$ such that the types of $x_i$ and $y_i$ are the same, and if $x_i$ is a query, the answer to $y_i$ is the same as the answer to $x_i$.

Both the offline minimum spanning tree algorithm [Epp94] and the result of [LS13] when applied to connectivity can be viewed as constructing such equivalents. We formalize the black-box implication of an efficient algorithm for finding small-sized equivalents below.

**Lemma 2.2.** If there is a linear-time algorithm $\text{Reduce}$ that takes a graph with $m$ edges along with $k$ events and outputs an equivalent with size at most $ck$ for some constant $c$ in $O(m)$ time, we can answer all queries in a list of events $x_1 \ldots x_t$ in $O(t \log t)$ time.

In order to give some intuition of this algorithm, first consider a simpler, two-level use of this routine for generating equivalents. We partition the event sequence into $\sqrt{t}$ subintervals of length $\sqrt{t}$. For all the subintervals, we can find $O(\sqrt{t})$ sized equivalents at a total cost of $O(t^{3/2})$. Then each subinterval can be processed by running linear-time static algorithms [HT73] for each query, giving an $O(t^{3/2})$ time algorithm. Our proof of Lemma 2.2 relies on partitioning the interval more gradually; instead, by a factor of 2 at each step.

---

**Figure 1:** A timeline diagram of 4 edge insertions (I)/deletions (D) and 3 queries (Q) plotted as described above.
Procedure \textbf{AnswerQueries}(G, x_1 \ldots x_k)

\begin{algorithmic}
\State \textbf{Input:} Graph $G = (V, E)$ of permanent edges for events $x_1, \ldots, x_k$.
\State Graph $G = (V, E)$ that’s present throughout the events and events $x_1 \ldots x_k$, where $k$ is a power of 2.
\State \textbf{Output:} Answer to all query events
\EndIf
\State if $k = 1$ then
\State Answer $x_1$ using static algorithm
\State else
\State $(G', y_1 \ldots y_k) \leftarrow \text{Reduce}(G, y_1 \ldots y_k)$
\State Run \textbf{AnswerQueries}($G', y_1 \ldots y_k/2$) to answer queries from $x_1 \ldots x_{k/2}$
\State Run \textbf{AnswerQueries}($G', y_{k/2+1} \ldots y_k$) to answer queries from $x_{k/2+1} \ldots x_k$
\State end if
\end{algorithmic}

Figure 2: Offline Algorithm for Processing a Sequence of $k$ Queries

\textbf{Proof.} Consider performing divide-and-conquer on a sequence of events using the pseudocode \textbf{AnswerQueries} shown in Figure 2. Note that $G$ starts as the empty graph, and that, on each call to \textbf{AnswerQueries}, Line 4 takes a graph $G$ of size at most $c \cdot 2^k$, where $c$ is the constant from Lemma 2.2, and reduces it to a graph $G'$ of size $ck$. It does this in $O(c \cdot 2^k) = O(k)$ time. The base case is argued similarly; $G$ is a graph of size at most 2$c$, thus the static algorithm charges constant time.

Therefore, the cost of each call to \textbf{AnswerQueries} is linear in input length; that is, $O(k)$. Since the interval sizes decrease geometrically, each event’s mapped versions are involved in at most $O(\log t)$ of these calls, giving a total cost of $O(t \log t)$.

Thus, being able to construct small-sized equivalents efficiently is a property that implies $O(t \log t)$ time offline algorithms. In the rest of this paper we will exhibit such routines for 2-edge connectivity, 3-edge connectivity, bi-connectivity, and tri-connectivity. An alternate approach of combining incremental data structures with persistence was also explored in [LS13]. Such incremental data structures exist for the properties that we maintain [WT92, DBT90], but the bounds for these data structures are amortized. While we believe it’s likely that some of these structures can be modified to have worst-case bounds, such bounds will likely have an extra log factor.

Our construction of equivalents is based on the dual view of connectivity through small cuts. Two vertices $u$ and $v$ are 2-edge connected, 3-edge connected, bi-connected, or tri-connected only if there does not exist a set of cut edges/vertices $C (u, v \notin C)$ whose removal separates them. The separation created by removing these edges/vertices can be described using $S$ and $T$ to denote the resulting partition of vertices (one way to create such partitions is to place connected components in the resulting graph on one of the two sides). This allows us to define a edge/vertex cut using $(C, S, T)$, where there are no edges between $S$ and $T$. The properties of 2-edge connected, 3-edge connected, bi-connected, or tri-connected are equivalent to the existence of a cut $(C, S, T)$ with $u \in S, v \in T$ and $C$ containing one edge, two edges, one vertex, or two vertices respectively.

This vertex-based characterization suggests that we only need to preserve the structure between the vertices involved in queries. As this number can be as high as the number of events, and both are bounded by $O(k)$, we instead preserve the structure between all vertices involved in events. We call such vertices \textbf{active} and denote them using $V^{\text{active}}$. Note that additional edges affected by updates will only be between two active vertices, so two cuts that give the same separation of active vertices will also cut these edges the same way. This motivates the following definition of equivalence of cuts w.r.t. active vertices.
Definition 2.3. Given graphs $G, H$ with active vertices $V_G^{\text{active}}, V_H^{\text{active}}$ (not necessarily distinct), as well as a surjection between them $f : V_G^{\text{active}} \rightarrow V_H^{\text{active}}$, a cut $(C_G, S_G, T_G)$ in $G$ is equivalent to a cut $(C_H, S_H, T_H)$ in $H$ if:

- $f(S_G \cap V_G^{\text{active}}) = S_H \cap V_H^{\text{active}}$  
- $f(T_G \cap V_G^{\text{active}}) = T_H \cap V_H^{\text{active}}$

Note that if $(C_G, S_G, T_G)$ is a vertex cut, these two conditions imply that $f(C_G \cap V_G^{\text{active}}) = C_H \cap V_H^{\text{active}}$.

Now, we say a cut $(C, S, T)$ separates a vertex subset $X \subset V$ if $X \cap S \neq \emptyset$ and $X \cap T \neq \emptyset$. For brevity, we will denote the size of a cut $(C, S, T)$ as the number of edges (edge-connectivity)/vertices (vertex-connectivity) in $C$ and sometimes write this as $|C|$. We call a separating cut $(C, S, T)$ of $X$ minimal if, among all separating cuts $(C', S', T')$ of $X$ with $X \cap S' = X \cap S$ and $X \cap T' = X \cap T$, we have $|C| \leq |C'|$. With this we define active-cut equivalence.

Definition 2.4. Two graphs $G, H$ with active vertices $V_G^{\text{active}}, V_H^{\text{active}}$ and a surjection $f : V_G^{\text{active}} \rightarrow V_H^{\text{active}}$ are active-cut equivalent for $l$-edge connectivity/l-vertex connectivity if:

1. For any minimal separating cut of $V_G^{\text{active}}$ in $G$ of size $< l$, an equivalent minimal separating cut of $V_H^{\text{active}}$ of the same size exists in $H$.

2. Any minimal separating cut of $V_G^{\text{active}}$ in $G$ of size $\geq l$ has no equivalent minimal separating cut of $V_H^{\text{active}}$ in $H$ of size $< l$.

Lemma 2.5. If $G$ and $H$ are active-cut equivalent via the mapping $f$, then $G$ with events $x_1, \ldots, x_k$ is equivalent to $H$ with events $y_1, \ldots, y_k$, where $y_i$ has the same type as $x_i$, but whose vertex names are given by the function $f$.

Proof. Consider $G'$ and $H'$ after a number of events. These graphs contains additional edges between vertices in $V_G^{\text{active}}$ and $V_H^{\text{active}}$, respectively. Suppose there is a query $x_i$ that asks for the connectivity of vertices $s$ and $t$. We will prove the minimum $st$-cut in $G'$ has the same size as the minimum $f(s)f(t)$-cut in $H'$ or that both cuts have size $\geq l$. We first show there is an $f(s)f(t)$-cut in $H'$ of the same size as the minimum $st$-cut in $G'$ if the latter cut has size $< l$. Let $(C_{G'}, S_{G'}, T_{G'})$ be a minimum $st$-cut in $G'$ and let $|C_{G'}| < l$. Let $X = S_{G'} \cap V_G^{\text{active}}$ and $Y = T_{G'} \cap V_G^{\text{active}}$. Note that although $f$ need not be injective, the rules of Definition 2.3 require any active vertices that map to the same vertex in $H$ be $\geq l$-connected, therefore $f(X) \cap f(Y) = \emptyset$.

For edge connectivity, note that the edges in $C_{G'}$ either exist in $G$ or were added in $G'$. If we remove the edges added in $G'$, we only reduce the size of $|C_{G'}|$, therefore by active-cut equivalence there is an equivalent minimal separating cut to $(C_{G'}, S_{G'}, T_{G'})$ in $H$. Since the added edges that cross from $X$ to $Y$ also cross $f(X)$ to $f(Y)$ in $H$, this gives an $f(s)f(t)$-cut in $H'$ of the same size as the minimum $st$-cut in $G'$.

For vertex connectivity, there must be no edges from $X$ to $Y$ in $G'$ since $(C_{G'}, S_{G'}, T_{G'})$ separates $s$ and $t$. Therefore, the addition of edges in $G'$ does not affect the size of this cut. By active-cut equivalence, there is an equivalent minimal separating cut to $(C_{G'}, S_{G'}, T_{G'})$ in $H$. Since there are no edges between $f(X)$ and $f(Y)$, this gives an $f(s)f(t)$-cut in $H'$ of the same size as the minimum $st$-cut in $G'$.

We may construct a minimal separating cut in $G'$ with the same size as a minimum $f(s)f(t)$-cut in $H'$ in the same way as above, so we omit the argument.

Note that there is a slight discrepancy between the requirements for edge and vertex connectivities. For edge connectivity, it is necessary to require that minimal separating cuts $< l$ have exactly the same size in $H$ as in $G$ due to the addition of edges in $G'$. For vertex connectivity, we only need to preserve the existence (or lack thereof) of cuts with size less than $l$: this is because additional edges can only be removed by removing their corresponding vertices, which leads to a different set of active vertices.
to remove. Another way to view this is that the choice of which active vertices to remove in the cut re-introduces this freedom in sizes of cutsets.

Note that these constructions do not take the events into account, but instead just the active vertices. For this type of construction, active-cut equivalence is in some sense also necessary as the updates can lead to a situation where the only edges missing are the ones from $S_G \cap V_G^{\text{active}}$ to $T_G \cap V_G^{\text{active}}$. The offline connectivity algorithm given in [LS13] can be viewed as a way to construct such active-cut equivalents. In their construction, the graph $H$ contains all active vertices, with $f$ being the identity mapping. The connectivity structure among these vertices is preserved by finding the connected components in $G$, and connecting together vertices in $H$ that are in the same component of $G$ using stars. Our constructions are also based on similar size reductions in components, but as we need to preserve higher connectivity structures, our reduction/replacement rules are more intricate.

3 Equivalent Graphs for 2-Edge Connectivity and Bi-connectivity

We now show offline algorithms for dynamic 2-edge connectivity and bi-connectivity by constructing the active-cut equivalents needed by Lemma 2.2 and Lemma 2.5. These two properties ask for the existence of a single edge/vertex whose removal separates query vertices $u$ and $v$. Since these cuts can affect at most one connected component, it suffices to handle each component separately.

The underlying structure for 2-edge connectivity and bi-connectivity are tree-like. This is perhaps more evident for 2-edge connectivity, where vertices on the same cycle belong to the same 2-edge connected component. We will first describe the reductions that we will make to this tree in Section 3.1, and adapt them to bi-connectivity in Section 3.2.

3.1 2-Edge Connectivity

Using depth-first search [HT73], we can identify all cut-edges in the graph and the 2-edge connected components that they partition the graph into. The case of edge cuts are slightly simpler conceptually, since we can combine vertices without introducing new cuts. Specifically, we show that each 2-edge connected component can be shrunk to single vertex.

**Lemma 3.1.** Let $S$ be a 2-edge connected component in $G$. Then mapping all vertices in $S$ to a single vertex $s$ in $H$, and endpoints of edges correspondingly, creates an active-cut equivalent graph.

**Proof.** The only cuts that we need to consider are ones that remove cut edges in $G$ or $H$. Since we only contracted vertices in a component, there is a one-to-one mapping of these edges from $G$ to $H$. Since $S$ is 2-edge connected, all vertices in it will be on the same side of one of these cuts. Furthermore, removing the same edge in $H$ leads to a cut with $s$ instead. Therefore, all active vertices in $S$ are mapped to $s$, and are therefore on the same side of the cut.

This allows us to reduce $G$ to a tree $H$, but the size of this tree can be much larger than $k$. Therefore we need to prune the tree by removing inactive leaves and length 2 paths whose middle vertex is inactive.

**Lemma 3.2.** If $G$ is a tree, the following two operations lead to active-cut equivalent graphs $H$.

- Removing an inactive leaf.
- Removing an inactive vertex with degree 2 and adding an edge between its two neighbors.

**Proof.** In the first case, the only cut in $G$ that no longer exist in $H$ is the one that removes the cut edge connecting the leaf with its unique neighbor. However, this places all active vertices in one component and thus does not separate $V_G^{\text{active}}$ and need not be represented in $H$.

In the second case, if a cut removes either of the edges incident to the degree 2 vertex, removing the new edge creates an equivalent cut since the middle vertex is inactive. Also, for a cut that removes the new edge in $H$, removing either of the two original edges in $G$ leads to an equivalent cut.
This allows us to bound the size of the tree by the number of active vertices, and therefore finish the construction.

**Lemma 3.3.** Given a graph $G$ with $m$ edges and $k$ active vertices $V_G^\text{active}$, a active-cut equivalent of size $O(k)$ for 2-edge connectivity can be constructed in $O(m)$ time.

*Proof.* We can find all the cut edges and 2-edge connected components in $O(m)$ time using depth-first search [HT73], and reduce the resulting structure to a tree $H$ using Lemma 3.1. On $H$, we repeatedly apply Lemma 3.2 to obtain $H'$.

In $H'$, all leaves are active, and any inactive internal vertex has degree at least 3. Therefore the number of such vertices can be bounded by $O(k)$, giving a total size of $O(k)$.

### 3.2 Bi-connectivity

All cut-vertices (articulation points) can also be identified using DFS, leading to a structure known as the block-tree. However, several modifications are needed to adapt the ideas from Section 3.1. The main difference is that we can no longer replace each bi-connected component with a single vertex in $H$, since cutting such vertices corresponding to cutting a much larger set in $G$. Instead, we will need to replace the bi-connected components with simpler bi-connected graphs such as cycles.

**Lemma 3.4.** Replacing a bi-connected component with a cycle containing all its cut-vertices and active vertices, and mapping the active vertices accordingly gives an active-cut equivalent graph.

*Proof.* As this mapping maintains the bi-connectivity of the component, it does not introduce any new cut-vertices. Therefore, $G$ and $H$ have the same set of cut vertices and the same block-tree structure. Note that the actual order the active vertices appear in does not matter, since they will never be separated. The claim follows similarly to Lemma 3.1.

The block-tree also needs to be shrunk in a similar manner. Note that the fact that blocks are connected by shared vertices along with Lemma 3.4 implies the removal of inactive leaves. Any leaf component with no active vertices aside from its cut vertex can be reduced to the cut vertex, and therefore be removed. The following is an equivalent of the degree two removal part of Lemma 3.2.

**Lemma 3.5.** Two bi-connected components $C_1$ and $C_2$ with no active vertices that share cut vertex $w$ and are only incident to one other cut vertex each, $u$ and $v$ respectively, can be replaced by an edge connecting $u$ and $v$ to create an active-cut equivalent graph.

*Proof.* As we have removed only $w$, any cut vertex in $H$ is also a cut vertex in $G$. As $C_1$ and $C_2$ contain no active vertices, this cut would induce the same partition of active vertices.

For the cut given by removing $w$ in $G$, removing $u$ in $H$ gives the same cut since $C_1$ has no active vertices (which in turn implies that $u$ is not active). Note that the removal of $u$ may break the graph into more pieces, but our definition of cuts allows us to place these pieces on two sides of the cut arbitrarily.

Note that Lemma 3.4 may need to be applied iteratively with Lemma 3.5 since some of the cut vertices may no longer be cut vertices due to the removal of components attached to them.

**Lemma 3.6.** Given a graph $G$ with $m$ edges and $k$ active vertices $V_G^\text{active}$, an active-cut equivalent of size $O(k)$ for bi-connectivity can be constructed in $O(m)$ time.

*Proof.* We can find all the initial block-trees using depth-first search [HT73]. Then we can apply Lemmas 3.4 and 3.5 repeatedly until no more reductions are possible. Several additional observations are needed to run these reduction steps in $O(m)$ time. As each cut vertex is removed at most once, we can keep a counter in each component about the number of cut vertices on it. Also, the second time we run Lemma 3.4 on a component, it’s already a cycle, so the reductions can be done without examining the entire cycle by tracking it in a doubly linked list and removing vertices from it.
It remains to bound the size of the final block-tree. Each leaf in the block-tree has at least one active vertex that’s not its cut vertex. Therefore, the block-tree contains at most $O(k)$ leaves and therefore at most $O(k)$ internal components with 3 or more cut vertices, as well as $O(k)$ components containing active vertices. If these components are connected by paths with 4 or more blocks in the block tree, then the two middle blocks on this path meet the condition of Lemma 3.6 and should have been removed by Lemma 3.5. This gives a bound of $O(k)$ on the number of blocks, which in turn implies an $O(k)$ bound on the number of cut vertices. The edge count then follows from the fact that Lemma 3.4 replaces each component with a cycle, whose number of edges is linear in the number of vertices.

4 3-Edge Connectivity

We now extend our algorithms to 3-edge connectivity. Our starting point is a statement similar to Lemma 3.1, namely that we can contract all 3-edge connected components.

Lemma 4.1. Let $S$ be a 3-edge connected component in $G$. Then mapping all vertices in $S$ to a single vertex $s$ in $H$, and endpoints of edges correspondingly, creates an active-cut equivalent graph.

Proof. A two-edge cut will not separate a 3-edge connected component. Therefore all active vertices in $S$ fall on one side of the cut, to which vertex $s$ may also fall. The proof follows analogously to Lemma 3.1.

Such components can also be identified in $O(m)$ time using depth-first search [Tsi09]. So the preprocessing part of this algorithm is the same as with the 2-connectivity cases. However, the graph after this shrinking step is no longer a tree. Instead, it is a cactus, which in its simplest terms can be defined as:

Definition 4.2. A cactus is an undirected graph where each edge belongs to at most one cycle.

On the other hand, cactuses can also be viewed as a tree with some of the vertices turned into cycles. Such a structure essentially allows us to repeat the same operations as in Section 3 after applying the initial contractions.

Lemma 4.3. A connected undirected graph with no nontrivial 3-edge connected component is a cactus.

Proof. We prove by contradiction. Let $G$ be a graph with no nontrivial 3-edge connected component. Suppose there exists two simple cycles $a$ and $b$ in $G$ with more than one vertex, and thus at least one edge, in common.

Call the vertices in the first simple cycle $a_1, \ldots, a_n$ and the second simple cycle $b_1, \ldots, b_m$, in order along the cycle.

Since these cycles are not the same, there must be some vertex not common to both cycles. Without loss of generality, assume (by flipping $a$ and $b$) that $b$ is not a subset of $a$, and (by shifting $b$ cyclically) that $b_1$ is only in $b$ and not $a$.

Now let $b_{first}$ be the first vertex after $b_1$ in $b$ that is common to both cycles, aka.

$\text{first} \overset{\text{def}}{=} \min_i b_i \in a$. \hspace{1cm} (1)

and let $b_{last}$ be the last vertex in $b$ common to both cycles

$\text{last} \overset{\text{def}}{=} \max_i b_i \in a$. \hspace{1cm} (2)

The assumption that these two cycles have more than 1 vertex in common means that

$\text{first} < \text{last}$. \hspace{1cm} (3)

\footnote{Some ‘virtual’ edges are needed in this construction because a vertex can still belong to multiple cycles.}
We claim \( b_{\text{first}} \) and \( b_{\text{last}} \) are 3-edge connected.

We show this by constructing three edge-disjoint paths connecting \( b_{\text{first}} \) and \( b_{\text{last}} \). Since both \( b_{\text{first}} \) and \( b_{\text{last}} \) occur in \( a \), we may take the two paths formed by cycle \( a \) connecting \( b_{\text{first}} \) and \( b_{\text{last}} \), which are clearly edge-disjoint.

By construction, vertices

\[
\begin{align*}
  b_{\text{last}+1}, \ldots, b_m, b_1, \ldots, b_{\text{first}-1}
\end{align*}
\]  

(4)

are not shared with \( a \). Thus they form a third edge-disjoint path connecting \( b_{\text{first}} \) and \( b_{\text{last}} \), and so the claim follows. Therefore, a graph with no 3-edge connected vertices, and thus no nontrivial 3-edge connected component has the property that two simple cycles have at most one vertex in common. \( \square \)

With this structural statement, we can then repeat the reductions from the 2-edge active-cut equivalent algorithm from Section 3.1 to produce the 3-edge active-cut equivalent graph.

**Lemma 4.4.** Given a graph \( G \) with \( m \) edges and \( k \) active vertices \( V_G^{\text{active}} \), a active-cut equivalent of size \( O(k) \) for 3-edge connectivity can be constructed in \( O(m) \) time.

**Proof.** Lemma 3.3 means that we can reduce the graph to a cactus after \( O(m) \) time preprocessing.

First consider the tree where the cycles are viewed as vertices. Note that in this view, a vertex that’s not on any cycle is also viewed as a cycle of size 1. This can be pruned in a manner analogous to Lemma 3.2:

1. Cycles containing no active vertices and incident to 1 or 2 other cycles can be contracted to a single vertex.
2. Inactive single-vertex cycles incident to 1 other cycle can be removed.

This procedure takes \( O(m) \) time and produces a graph with at most \( O(k) \) leaves. Correctness of the first rule follows by replacing a cut of the two edges within an inactive cycle by a cut of the single contracted vertex with one of its neighbors. The second rule does not affect minimal separating cuts. So it remains to reduce the length of degree 2 paths and the sizes of the cycles themselves.

As in Lemma 3.2, all vertices of degree 2 can be replaced by an edge between its two neighbors. This bounds the length of degree 2 paths and reduces the size of each cycle to at most twice its number of incidences with other cycles. This latter number is in turn bounded by the number of leaves of the tree of cycles. Hence, this contraction procedure reduces the total size to \( O(k) \).

We remark that this is not identical to iteratively removing inactive vertices of degrees at most 2. With that rule, a cycle can lead to a duplicate edge between pairs of vertices, and a chain of such cycles needs to be reduced in length.

## 5 Tri-connectivity

Tri-connectivity queries involving \( s \) and \( t \) ask for the existence of a separation pair \( \{u, v\} \) whose removal disconnects \( s \) from \( t \). In this section we will extend our techniques to offline tri-connectivity. To do so, we rely on a tree-like structure for the set of separation pairs in a bi-connected graph, the SPQR tree [HT73, DBT90]. We review these structures in Section 5.1 and show how to trim them in Section 5.2. As these trees require that the graphs are bi-connected, we extend this subroutine to our full construction in Section 5.3.
5.1 SPQR Trees

We now review the definition of SPQR Trees. We will follow the model given in Chapter 2 of [Wei02]. For a more thorough description of SPQR Trees, please refer to [Wei02].

SPQR trees are based on the definition of a split pair, which generalizes separation pairs by allowing the extra case of \( \{u, v\} \) being an edge of \( G \). A split component of the split pair \( \{u, v\} \) is either an edge connecting them, or a maximal connected subgraph \( G' \) of \( G \) such that removing \( \{u, v\} \) does not disconnect \( G' \).

The SPQR tree is defined recursively on a graph \( G \) with a special split pair \( \{s, t\} \). This process can be started by picking an arbitrary edge as the root. Each node \( \mu \) in the tree \( T \) has an associated graph denoted as its skeleton, \( \text{skeleton}(\mu) \), and is associated with an edge in the skeleton of its parent \( \nu \), called the virtual edge of \( \mu \) in \( \text{skeleton}(\nu) \). In this way, each virtual edge of a node \( \nu \) corresponds to a child of \( \nu \). For contrast, we will also use real edges to denote edges that are present in \( G \).

- Trivial Case: if \( G \) is a single edge from \( s \) to \( t \), then \( T \) contains a single Q-node whose skeleton is \( G \) itself.
- Series Case: If the removal of the (virtual) edge \( st \) creates cut-vertices, then these cut vertices partition \( G \) into blocks \( G_1 \ldots G_k \) and the block-tree has a cycle-like structure. The root of \( T \) is then an S-node, and \( \text{skeleton}(\mu) \) is the cycle containing these cut vertices with virtual edges corresponding to the blocks.
- Parallel Case: If the split pair \( \{s, t\} \) creates split components \( G_1 \ldots G_k \) with \( k \geq 2 \), the root of \( T \) is a P-node and the skeleton contains \( k \) parallel virtual edges from \( s \) to \( t \) corresponding to the split components.
- Rigid Case: If none of the above cases apply, then the root of \( T \) is an R-node \( \mu \) and \( \text{skeleton}(\mu) \) is a tri-connected graph where each edge corresponds to a split pair \( \{s_i, t_i\} \), and the corresponding child contains the union of all split components generated by this pair.

Note that by this construction, each edge in the SPQR tree corresponds to a split-pair. An additional detail that we omitted is that the construction of R-nodes picks only the split pairs that are maximal w.r.t. the edge \( st \). This detail is unimportant to our trimming routine, but is discussed in [Wei02]. Our algorithm acts directly upon the SPQR tree, and we make use of several important properties of this tree in our reduction routines.

When viewed as an unrooted tree, the SPQR tree is unique with all leaves as Q-nodes. For simplicity, we will refer to this tree with all Q-nodes removed as the simplified SPQR tree, or \( T_{\text{simple}} \). Also, it suffices to work on the skeletons of nodes, and the only candidates for separation pairs that we need to consider are:

1. Two cut vertices in an S-node.
2. The split pair corresponding to a P-node.
3. Endpoints of an edge in an R-node.

5.2 Trimming SPQR Trees

We now show how to convert a bi-connected graph with \( k \) active vertices to an active-cut equivalent with \( O(k) \) vertices and edges. Our algorithm makes a sequence of modifications on the SPQR trees similar to the trimming from Section 3. We will call a vertex \( u \) internal to a node \( \mu \) of the SPQR tree if \( u \) is contained in the split graph of \( \mu \), and \( u \) is not part of the split pair associated with \( \mu \). We call a vertex \( u \) exact to a node \( \mu \) if it is internal to \( \mu \) but not to any descendants of \( \mu \). Exact vertices provide a way to count nodes according to active vertices without reusing active vertices for multiple nodes.
We first give a way to remove split components with no internal active vertices, which we will refer to as inactive split components.

**Lemma 5.1.** Consider an inactive non-Q split component $G'$ produced by the split pair $\{u, v\}$. We may create an active-cut equivalent graph $H$ by the following replacement rule: if $u$ and $v$ are $\geq 3$-vertex connected in $G$, we may replace $G'$ with the edge $uv$; otherwise, we may replace $G'$ with a vertex $x$ and edges $ux, vx$.

**Proof.** Consider a cut in $G$. If $u$ and $v$ are on the same side of the cut, an equivalent cut exists in $H$ by placing $u, v$, and possibly $x$ on the same side of the cut. Similarly, a cut in $G$ that removes either $u$ or $v$ can be made in $H$ by removing the same vertex. If a cut in $G$ separates $u$ and $v$, $u$ and $v$ are not $\geq 3$-vertex connected in $G$ and a vertex $w$ in $G'$ must have been removed since $G'$ connects $u$ and $v$. Removing $x$ instead of $w$ creates an equivalent cut in $H$.

In the other direction, if $u$ and $v$ were $\geq 3$-vertex connected in $G$, also no cut in $H$ separates $u$ and $v$. If $u$ and $v$ were not $\geq 3$-vertex connected in $G$, there exists a vertex $w$ in $G'$ whose removal separates $u$ and $v$ in $G'$. Therefore the cut in $H$ produced by vertices $\{x, z\}$ for some $z \notin G'$ that separates $u$ and $v$ can be made in $G$ by removing $\{w, z\}$ instead. All other cuts in $H$ can be formed in $G$ by placing $G'$ on the same side as any remaining vertex in $\{u, v\}$. \qed

An important consequence of Lemma 5.1 is that an inactive split component is a leaf in $T_{\text{simple'}}$. Because of this, it will be easier to think of inactive split components in the same way we think of Q-nodes. We define the tree $T_{\text{simple'}}$ as the tree $T_{\text{simple}}$ with inactive split component leaves removed. Every leaf $\mu$ in $T_{\text{simple'}}$ must have an internal active vertex $u$. Furthermore, any vertex internal to a node $\eta$ is only internal to ancestors and possibly descendants of $\eta$. Since $\mu$ is a leaf in $T_{\text{simple'}}$, it has no descendants. It follows that $u$ is exact for $\mu$ and the tree $T_{\text{simple'}}$ has $O(k)$ leaves.

We will use this to bound the number of leaves in $T$. However, if $G$ contains an R-node whose skeleton contains a complete graph between active vertices, the number of leaves in $T$ can still be $\Omega(k^2)$. Therefore, we need another rule to process each skeleton of $T_{\text{simple'}}$ so that we can bound its size by the number of its exact active vertices and split components.

**Lemma 5.2.** There exists a constant $c_0$ such that any node $\mu$ in $T_{\text{simple'}}$ whose skeleton contains a total of $k$ exact active vertices and virtual edges corresponding to active split components can be replaced by a node whose skeleton contains $c_0 \cdot k$ vertices/edges to give an active-cut equivalent graph.

In other words, the number of children of $\mu$ in $T$ is at most $c_0 \cdot k$.

**Proof.** We consider the cases where the node is of type S, P, R separately.

If the node is a P-node, it suffices to consider the case where there are 3 or more inactive split components incident to the cut pair. Removing at most 2 vertices from these components will leave at least one of them intact, and therefore not change the connectivity between the separation pair, and therefore the other components. Therefore, all except 3 of these components can be discarded.

If the node is an S-node, any virtual edge in the cycle corresponding to an inactive split component that’s not incident to two active vertices is replaced by actual edges via Lemma 5.1. Furthermore, two consecutive real edges in the cycle connecting three inactive vertices can be reduced to a single edge by removing the middle vertex in a manner analogous to Lemma 5.1. Therefore the size of the cycle is proportional to the number of exact active vertices and active split components.

For an R-node, we can identify all vertices that are either exact and active, or are incident to virtual edges corresponding to active split components. Then we can simply connect these vertices together using a tri-connected graph of linear size (e.g. a wheel graph) and add the active split components back between their respective separation pairs. Every cut in the new skeleton with a child in $T_{\text{simple}}$ can be produced by the original graph, and the only cuts in the original graph that can’t be produced in the new one are cuts isolating an inactive component. Therefore the two graphs are active-cut equivalent. \qed

Before we can bound the number of nodes in $T_{\text{simple'}}$, we must eliminate long paths where a node has only one active child, which in turn has only one active child, etc. This is only possible if there exists
Lemma 5.4. Let \( \mu_1, \mu_2, \) and \( \mu_3 \) be a parent-child sequence of SPQR nodes where \( \mu_2 \) and \( \mu_3 \) are associated with split pairs \{u, v\} and \{x, y\} respectively, \( \mu_3 \) is the single active child of \( \mu_2 \), \( \mu_2 \) is the single active child of \( \mu_1 \), and either none of \( u, v, x, \) and \( y \) are active or \( u = x \) and is active. We may replace \( \mu_2 \) with \( \mu_3 \), effectively replacing vertex \( u \) with \( x \) and \( v \) with \( y \).

Proof. In \( G \), cuts induced by the removal of \{u, v\} and \{x, y\} both give the same partition of active vertices. Therefore, we only need one in \( H \), which is given by the cut \{x, y\}. All other cuts in \( G \) not present in \( H \) only separate inactive split components, which need not be represented in \( H \).

In the other direction, every cut in \( H \) is still present in \( G \). Thus the new graph \( H \) preserves active-cut equivalence.

We can now bound the number of nodes in \( T_{\text{simple}} \) after no reductions via Lemmas 5.1, 5.2, and 5.3 are possible.

Lemma 5.4. Consider a graph \( G \) where no reductions via Lemmas 5.1, 5.2, and 5.3 are possible. There is a constant \( c_1 \) such that the number of nodes in \( G \)’s SPQR tree \( T \) is no more than \( c_1 \cdot k \), where \( k \) is the total number of active vertices in \( G \).

Proof. As explained earlier, the tree \( T_{\text{simple}} \) has \( O(k) \) leaves. To bound the total number of nodes, we must consider the length of a path of degree 2 nodes in \( T_{\text{simple}} \). If a node \( \mu \) on this path has a split pair \{u, v\} with an active vertex, say \( u \), internal to its parent \( \eta \) (implying \( \eta \) does not have \( u \) in its split pair), then \( u \) is exact for \( \eta \) and we may associate \( u \) with \( \eta \). Otherwise, consider two adjacent degree 2 nodes to which this does not apply. The parent may have an active vertex in its split pair not shared with its child, however then its child cannot have an active vertex in its split pair or else it fits the above situation (the active vertex in the child’s split pair is internal to the parent). Therefore this can only happen once until the parent and child have at most one active vertex shared in their split pairs, and the procedure of Lemma 5.3 may be used to replace the parent node with the child. It follows that we may associate each active vertex along this path to a constant number of SPQR nodes.

Since the number of nodes of degree \( \geq 3 \) in a tree is bounded by the number of leaves, the above shows \( T_{\text{simple}} \) has \( O(k) \) nodes. We now consider the full tree \( T \). This tree adds inactive split components with a constant number of Q-node leaves as well as Q-node leaves themselves to active split components. By Lemma 5.2 for any node \( \mu \) in \( T_{\text{simple}} \), we add at most \( c_0 \cdot k \) extra children to \( \mu \) in \( T \), where \( k \) is the sum of active vertices exact to \( \mu \) and the number of active split components, which are children of \( \mu \) in \( T_{\text{simple}} \). The sum of the degrees of all the vertices in \( T_{\text{simple}} \) is \( O(k) \), therefore the number of extra nodes in \( T \) is also \( O(k) \). Therefore \( T \) has \( O(k) \) nodes.

We now consider applying Lemmas 5.1, 5.2, and 5.3 on \( G \) to produce an equivalent graph \( H \) in linear time.

Lemma 5.5. Given a bi-connected graph \( G \) and \( k \) active vertices, we may find the SPQR tree associated with \( G \) and apply the reduction rules given by Lemmas 5.1, 5.2, and 5.3 until exhaustion in linear time. From this can be constructed a graph \( H \) active-cut equivalent to \( G \) with \( O(k) \) vertices and edges.

Proof. The SPQR tree can be found in linear time [BM88]. Lemma 5.4 shows that continued application of Lemmas 5.1, 5.2, and 5.3 produces an SPQR tree \( T \) with \( O(k) \) nodes. As each leaf of \( T \) is an edge of the graph it represents, this shows the resulting graph represented has \( O(k) \) vertices and edges.

The rules given by Lemmas 5.1, 5.2, and 5.3 can be applied until exhaustion in \( O(m) \) time. First, we may apply Lemma 5.1 by traversing each split component of \( T \) and applying the lemma when possible.
Next, we may apply Lemma 5.2 by a similar traversal. Finally, the rule of Lemma 5.3 can be applied by keeping a stack of the SPQR nodes of a depth-first traversal of the SPQR tree \( T \). All three rules require a single traversal of \( T \) each and thus can be done in \( O(m) \) total time.

5.3 Constructing the Full Equivalent

We now extend this trimming routine for bi-connected graphs to arbitrary graphs. By an argument similar to that at the start of Section 3, we may assume that \( G \) is connected. Therefore we need to work on the block-tree in a manner similar to Section 3.2. We first show that we may invoke the trimming routine from Section 5.2 on each bi-connected component.

**Lemma 5.6.** Let \( B \) be a bi-connected component in \( G \) and \( B' \) be a cut-equivalent graph for \( B \) where the active vertices consist of all cut vertices and active vertices in \( B \). Replacing \( B \) with \( B' \) gives \( H \) that's active-cut equivalent to \( G \).

**Proof.** By symmetry of the setup it suffices to show that any cut in \( G \) has an equivalent cut in \( H \). Since \( B \) is two-connected, any cut in \( B \) that does not remove at least 2 vertices in \( B \) results in all vertices of \( B \) on the same side of the cut. This means any vertex in \( B \) that’s not a cut-vertex in \( G \) can be added in without changing the cut. This also means that removing these vertices has the same effect in \( G \) and \( H \).

Therefore it suffices to consider cuts that remove 2 vertices in \( B \). By assumption, there exists a cut in \( H \) that results in the same set of active vertices in \( C \), and cut vertices incident to \( B \) on one side. The structure of the block tree gives that the rest of the graph is connected to \( B \) via one of the cut vertices, and the side that this cut vertex is on dictates the side of the cut that part is on. Therefore, the rest of the graph, and therefore the active vertices not in \( B \) will be partitioned the same way by this cut.

It remains to reduce the block tree, which we do in a way analogous to Section 3.2. However, the proofs need to be modified for separation pairs instead of a single cut edge/vertex.

**Lemma 5.7.** A bi-connected component containing no active vertices and incident to only one cut vertex can be removed (while keeping the cut-vertex) to create an equivalent graph.

**Proof.** Any cut in \( G \) can be mapped over to \( H \) by removing the same set of vertices (minus the ones in this component). Given a cut in \( H \), removing the same set of vertices in \( G \) result in a cut with this component added to one side. As there are no active vertices in this cut, the result separates the active vertices in the same manner.

Long paths of bi-connected components can be handled in a way similar to Lemma 3.5.

**Lemma 5.8.** Two bi-connected components \( C_1 \) and \( C_2 \) with no active vertices that share cut vertex \( w \) and only incident to one other cut vertex each, \( u \) and \( v \) respectively, can be replaced by an edge connecting \( u \) and \( v \).

**Proof.** As there is a path from \( u \) to \( v \) through these components and \( w \), a minimal separating cut in \( G \) either removes one of \( u \), \( v \), or \( w \), or has \( uv \) on the same side. In the first case, removing \( u \) or \( v \) leads to the same cut as none of them are active, while in the second case the edge \( uv \) does not cross the cut. Since \( uv \) is connected by an edge in \( H \), in a cut in \( H \) they’re either on the same side, or one of \( u \) or \( v \) is removed. In either case, an equivalent cut in \( G \) can be formed by assigning what remains of \( C_1 \) and \( C_2 \) to the same side as any remaining vertex.

Our final construction can be obtained by applying these routines to the block tree first, then Lemma 5.6.

**Lemma 5.9.** Given a graph \( G \) with \( m \) edges and \( k \) active vertices \( V_G^{\text{active}} \), an active-cut equivalent of size \( O(k) \) for tri-connectivity can be constructed in \( O(m) \) time.
Proof. By a proof similar to Lemma 3.6 applying Lemmas 5.7 and 5.8 repeatedly leads to an equivalent graph $H$ whose block-tree contains $O(k)$ components and cut vertices.

The total number of cut vertices and active vertices summed over all bi-connected components is $O(k)$. Therefore, applying Lemma 5.6 on each component gives the size bound.

6 Conclusion

We believe that our approach has several natural extensions. Our framework from Section 2 is parallelizable and the reduction operations that we perform on block-trees and SPQR-trees are similar to the ones in parallel tree-contraction [MR91]. As a result, it should be possible to combine our results with parallel algorithms for finding static decompositions [TV85, MR87, EV12] to obtain polylog depth, nearly-linear work parallel algorithms.

Furthermore, it’s likely that our method can be adapted to maintain a variety of other graph-based properties in the offline setting. A closer look at our constructions of equivalent graphs for bi-connectivity and tri-connectivity shows that much of the complications are due to the fact that eventful vertices themselves can be removed. Therefore, we believe that edge-cuts might be more natural to study for higher connectivity. The existence of tree-like structures for edge-cuts [GH61] also make this direction more appealing.

References

[ADK+16] Ittai Abraham, David Durfee, Ioannis Koutis, Sebastian Krinninger, and Richard Peng. On fully dynamic graph sparsifiers. CoRR, abs/1604.02094, 2016. Available at: https://arxiv.org/abs/1604.02094.

[AHLT05] Stephen Alstrup, Jacob Holm, Kristian De Lichtenberg, and Mikkel Thorup. Maintaining information in fully dynamic trees with top trees. ACM Trans. Algorithms, 1(2):243–264, October 2005.

[AWY15] Amir Abboud, Virginia Vassilevska Williams, and Huacheng Yu. Matching triangles and basing hardness on an extremely popular conjecture. In Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC 2015, Portland, OR, USA, June 14-17, 2015, pages 41–50, 2015. Available at: https://people.csail.mit.edu/virgi/MatchTria.pdf.

[BM88] Daniel Bienstock and Clyde L. Monma. On the complexity of covering vertices by faces in a planar graph. SIAM J. Comput., 17(1):53–76, February 1988.

[DBT90] Giuseppe Di Battista and Roberto Tamassia. On-line graph algorithms with spqr-trees. In Proceedings of the seventeenth international colloquium on Automata, languages and programming, pages 598–611, New York, NY, USA, 1990. Springer-Verlag New York, Inc.

[DKP+17] David Durfee, Rasmus Kyng, John Peebles, Anup B. Rao, and Sushant Sachdeva. Sampling random spanning trees faster than matrix multiplication. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017, pages 730–742, 2017. Available at: https://arxiv.org/abs/1611.07451.

[Epp94] David Eppstein. Offline algorithms for dynamic minimum spanning tree problems. J. Algorithms, 17(2):237–250, September 1994.

[EV12] James Alexander Edwards and Uzi Vishkin. Brief announcement: speedups for parallel graph triconnectivity. In Proceedings of the 24th ACM symposium on Parallelism in algorithms and architectures, SPAA ’12, pages 190–192, New York, NY, USA, 2012. ACM.
[Fre83] Greg N. Frederickson. Data structures for on-line updating of minimum spanning trees. In Proceedings of the fifteenth annual ACM symposium on Theory of computing, STOC ’83, pages 252–257, New York, NY, USA, 1983. ACM.

[GH61] R. E. Gomory and T. C. Hu. Multi-terminal network flows. Journal of the Society for Industrial and Applied Mathematics, 9(4):pp. 551–570, 1961.

[GI91] Zvi Galil and Giuseppe F Italiano. Fully dynamic algorithms for edge connectivity problems. In Proceedings of the twenty-third annual ACM symposium on Theory of computing, pages 317–327. ACM, 1991.

[HdLT98] Jacob Holm, Kristian de Lichtenberg, and Mikkel Thorup. Poly-logarithmic deterministic fully-dynamic algorithms for connectivity, minimum spanning tree, 2-edge, and biconnectivity. In Proceedings of the thirtieth annual ACM symposium on Theory of computing, STOC ’98, pages 79–89, New York, NY, USA, 1998. ACM.

[HK99] Monika Rauch Henzinger and Valerie King. Randomized fully dynamic graph algorithms with polylogarithmic time per operation. JOURNAL OF THE ACM, 46:516, 1999.

[HT73] John E. Hopcroft and Robert Endre Tarjan. Dividing a graph into triconnected components. SIAM J. Comput., 2(3):135–158, 1973.

[KK13] Bruce Kapron, Valerie King, and Ben Mountjoy. Dynamic graph connectivity in polylogarithmic worst case time. SODA, 2013.

[KL15] Adam Karaczmarz and Jakub Lacki. Fast and simple connectivity in graph timelines. In Frank Dehne, Jorg-Rudiger Sack, and Ulrike Stege, editors, Algorithms and Data Structures: 14th International Symposium, WADS 2015, Victoria, BC, Canada, August 5-7, 2015. Proceedings, pages 458–469, 2015. Available at: https://arxiv.org/abs/1506.02973.

[LS13] Jackub Lacki and Piotr Sankowski. Reachability in graph timelines. ITCS, 2013.

[MR87] Gary L. Miller and Vijaya Ramachandran. A new graph triconnectivity algorithm and its parallelization. In Combinatorica, pages 254–263, 1987.

[MR91] Gary L. Miller and John H. Reif. Parallel tree contraction part 2: Further applications. SIAM JOURNAL ON COMPUTING, 20(6):1128–1147, 1991.

[NS17] Danupon Nanongkai and Thatchaphol Saranurak. Dynamic spanning forest with worst-case update time: adaptive, las vegas, and o(n^{1/2 - \epsilon})-time. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017, pages 1122–1129, 2017. Available at: https://arxiv.org/abs/1611.03745.

[PD04] Mihai Patrascu and Erik D. Demaine. Lower bounds for dynamic connectivity. In Proceedings of the thirty-sixth annual ACM symposium on Theory of computing, STOC ’04, pages 546–553, New York, NY, USA, 2004. ACM.

[Tho00] Mikkel Thorup. Near-optimal fully-dynamic graph connectivity. In Proceedings of the thirty-second annual ACM symposium on Theory of computing, STOC ’00, pages 343–350, New York, NY, USA, 2000. ACM.

[Tsi09] Yung H Tsin. Yet another optimal algorithm for 3-edge-connectivity. Journal of Discrete Algorithms, 7(1):130–146, 2009.

[TV85] Robert Endre Tarjan and Uzi Vishkin. An efficient parallel biconnectivity algorithm. SIAM J. Comput., 14(4):862–874, 1985.
[Wei02] Rene Weiskircher. *New Applications of SPQR-Trees in Graph Drawing*. PhD thesis, Universität des Saarlandes, 2002.

[WT92] Jeffery Westbrook and RobertE. Tarjan. Maintaining bridge-connected and biconnected components on-line. *Algorithmica*, 7:433–464, 1992.