Collective light emission of a finite-size atomic chain

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Abstract – The radiative properties of collective electronic states for a finite one-dimensional atomic chain are investigated. The influence of the chain size on the collective damping rates is examined through the dependence on both the chain length and the lattice constant. The symmetric state damping rate is shown to grow up linearly with the length for short chains, in relative to the atomic transition wavelength, and to saturate for long ones. For large lattice constants the saturation is achieved for small number of chain atoms, and vice versa for small ones. The results can be adopted for any chain of active materials, e.g., an array of quantum dots or organic molecules on a linear matrix.

Linear chains of optically active materials are of big importance, mainly for their applications in quantum information processing [1], and their fundamental rule in the implementation of condensed matter collective effects. They can be realized in different set-ups, from a chain of organic molecules up to semiconductor nanostructures, which span a wide range of microscopic and macroscopic dimensions. In the recent years, optical lattice ultracold atoms have become of interest for different branches of physics [2]. Big attention is given for the realization of different condensed matter models that provide a test system for achieving a deep understanding of fundamental physics and answering open questions in the subject [3]. In general, the main objective is to consider optical lattice ultracold atoms as artificial crystals with a wide range of controllable parameters.

Optical lattices provide an ordered array of microtraps for ultracold atoms with lattice constant of half-wavelength of the laser [4,5]. Low-dimensional lattices can be achieved with different geometric structures and symmetries [6]. In conventional solid crystals the lattice constant and the symmetry of the lattice are fixed through the different chemical bonds that are responsible for the crystal formation. The advantage of optical lattices is due to the controllability of the lattice constant and symmetry through controlling the external laser field [2].

Collective states of electronic excitations play a central role in solid crystals and molecular clusters and they usually termed excitons [7,8]. They are induced by electrostatic interactions among the lattice atoms or molecules, where an electronic excitation can be delocalized in the crystal through energy transfer. Collective states can dominate the electrical and optical properties of the material, and especially they strongly affect the excitation lifetimes and give rise to dark and superradiant states. In such material the lattice constant is few angstroms which is much smaller than the electronic transition wavelength, and hence one can use electrostatic interactions, e.g., resonance dipole-dipole interactions, and to neglect radiative corrections altogether.

Electronic excitations in optical lattice ultracold atoms are of big importance, e.g., for optical lattice clocks [9], and for optical lattice Rydberg atoms [10,11]. We introduced excitons for optical lattice ultracold atoms [12,13], and concentrated mainly in the Mott insulator phase. We treated both large and finite atomic chains [14–16], and calculated the damping rate of excitons into free space [17–19], in exploiting the electrostatic interactions. But for typical optical lattices the lattice constant is few thousands of angstroms, which can be of the order of the electronic transition wavelength, and hence radiative corrections can be significant. Moreover, the presently realized optical lattices are of finite size, which includes tenths up to hundreds of sites.

In the present letter we investigate a linear finite chain of atoms where the lattice constant can take any value relative to the atomic transition wavelength. Finite atomic chains have been realized recently in a number of optical
lattice experiments [20,21]. We emphasize the influence of radiative corrections on the formation of collective sates and their damping rates, where we exploit general collective states with emphasis on the most symmetric one. Moreover, we derive the condition for the validity of applying electrostatic interactions. Few studies treated the collective effect on the optical properties of a finite atomic chain of several atoms [22], but an extensive study done for two atoms in the radiative regime [23]. We extract how the damping rate depends on the chainsize, namely on the number of atoms in the lattice for different lattice constants.

We consider a finite one-dimensional atomic lattice, where the number of atoms is $N$ with lattice constant $a$, as seen in fig. 1. The atoms are considered to be two-level systems with electronic transition energy $E_A = \hbar \omega A$. An electronic excitation can delocalize in the lattice by transferring among the atoms. The electronic excitation Hamiltonian is given by

$$H_{ex} = \sum_n \hbar \omega A \, B_n^\dagger B_n + \sum_{nm} \hbar J_{nm} B_n^\dagger B_m,$$

where $B_n^\dagger$ and $B_n$ are the creation and annihilation operators of an electronic excitation at atom $n$. For a single excitation the operators can be assumed to obey boson commutation relations.

The energy transfer among two atoms, $n$ and $m$, is a function of the interatomic distance and given by [24]

$$J(q A R_{nm}) = \frac{3 \Gamma_A}{4} \left\{ \frac{\sin(q A R_{nm})}{(q A R_{nm})^2} + \frac{\cos(q A R_{nm})}{(q A R_{nm})^3} \right\} \times \left( 1 - 3 \cos^2 \varphi \right) - \frac{\cos(q A R_{nm})}{q A R_{nm}} \left( 1 - \cos^2 \varphi \right),$$

where the distance between the two atoms is $R_{nm} = |n - m|a$, and $\mu$ is the magnitude of the electronic excitation transition dipole, which makes an angle $\varphi$ with the lattice direction, see fig. 1. $q_A$ is the atomic transition wavelength defined by $E_A = \hbar c q_A$. Here $\Gamma_A$ is the single excited atom damping rate [25]

$$\Gamma_A = \frac{\omega_A^3 \mu^2}{3 \pi \epsilon_0 \hbar c^3}.$$  

In the limit of $\lambda A > a$, where $\lambda A$ is the atomic transition wavelength, we can consider only energy transfer among nearest-neighbor atoms with $J(q AA)$ where we take $R_{nm} = a$.

For optical lattices we have the typical numbers $E_A = 1$ eV, with $\lambda A \approx 12405$ Å, and $q A \approx 4 \times 10^{-4}$ Å$^{-1}$. For $a = 1000$ Å we get $q A a \approx 0.5$, and $a/\lambda A \approx 0.08$. For $\varphi = 0^\circ$ we obtain $J(0.5)/\Gamma_A \approx -13.4$, and for $\varphi = 90^\circ$ we get $J(0.5)/\Gamma_A \approx 5.4$. For large $q AA$ the coupling tend to zero with oscillations, and the atoms are almost independent. In the limit $\lambda AA \gg a$, or $q AA \ll 1$, we can neglect the radiative terms, to get the electrostatic resonance dipole-dipole interaction

$$J \approx \frac{3 \Gamma_A}{4 (q AA)^3} \left( 1 - 3 \cos^2 \varphi \right).$$  

Using the previous numbers, $\varphi = 0^\circ$ yields $J(0.5)/\Gamma_A \approx -12$, and $\varphi = 90^\circ$ yields $J(0.5)/\Gamma_A \approx 6$, which are slightly different from the above exact results. For smaller $q AA$ we get much better agreement. In fig. 2 we plot eqs. (2) and (4) for $\varphi = 0^\circ$, and in fig. 3 for $\varphi = 90^\circ$. The results justify the use of electrostatic dipole-dipole interactions for optical lattice ultracold atoms when $q AA < 1$.  

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**Fig. 1:** A finite lattice of $N$ atoms. The lattice constant is $a$, and the transition dipole $\mu$ makes an angle $\varphi$ with the lattice direction.
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We motivate our study by two recent experiments. In [20] the strong evanescent field surrounding a tapered optical nanofiber is efficiently used for trapping, manipulating and detecting the atoms. Interference of two color evanescent fields are designed to form a periodic array of optical microtraps of lattice constant 5320 nm. The system was realized for cesium atoms with the transition lines D1 of 894 nm and D2 of 852 nm, which give \( q_{\lambda}\alpha \approx 4.2 \). In another approach the control of atomic positions in optical lattices has been demonstrated on the level of a single atom at a specific lattice site [21]. Several segments of one-dimensional finite arrays with different lengths and shapes were realized for \(^{77}\)Rb atoms with the transition lines D1 of 7948.8 nm and D2 of 780.2 nm and lattice constant of 5320 nm, and which yield \( q_{\lambda}\alpha \approx 3.7 \). Both cases go beyond the electrostatic regime and imply the radiative corrections. On the other hand, Rydberg atoms have recently become of big importance for the physical implementation of quantum information processing [26]. Rydberg excitations in Bose-Einstein condensates of rubidium atoms loaded into quasi–one-dimensional traps and in optical lattices experimentally realized by [10,11]. The huge transition dipole and the relatively small transition frequencies among nearest Rydberg states, which are in the range from few MHz up to few GHz, make the electrostatic interactions to survive up to large interatomic distances. Namely, the limit of \( q_{\lambda}\alpha \ll 1 \) can be held up to lattice constants of few microns, and in which the resonance dipole-dipole interactions dominate the higher order interactions [26].

We start in presenting the free space radiation field and its coupling to a finite atomic chain. The free space radiation field Hamiltonian is

\[
H_{\text{rad}} = \sum_{q\lambda} E_{pb}(q) a_{q\lambda}^\dagger a_{q\lambda},
\]  

where \( a_{q\lambda}^\dagger \) and \( a_{q\lambda} \) are the creation and annihilation operators of a photon with wave vector \( q \) and polarization \( \lambda \), respectively. The photon energy is \( E_{pb}(q) = \hbar c q \). The electric field operator is

\[
\hat{E}(r) = i \sum_{q\lambda} \frac{\hbar c q}{2\epsilon_0 V} \left\{ a_{q\lambda} e_{q\lambda} e^{iq \cdot r} - a_{q\lambda}^\dagger e_{q\lambda}^* e^{-iq \cdot r} \right\},
\]

where \( e_{q\lambda} \) is the photon polarization unit vector, and \( V \) is the normalization volume.

The atomic transition dipole operator is \( \hat{\mu} = \mu \sum_{n=1}^N (B_n + B_n^\dagger) \). The matter-field coupling is given formally by the electric dipole interaction \( H_I = -\hat{\mu} \cdot \hat{E} \). In the rotating wave approximation and for linear polarization, we get

\[
H_I = -i \sum_{q\lambda,n} \frac{\hbar c q}{2\epsilon_0 V} (\mu \cdot e_{q\lambda}) \\
\times \left\{ a_{q\lambda} B_n^\dagger e^{i q \cdot n} a_n - a_{q\lambda}^\dagger B_n e^{-i q \cdot n} a_n \right\}.
\]

In the following light we treat a single electronic excitation in the atomic chain. We start in treating the most symmetric collective state and then the general collective state.

We consider a single excitation in the symmetric collective state \( |i\rangle_s = \frac{1}{\sqrt{N}} \sum_n |g_1, \ldots, g_N\rangle \), which is an eigenstate of the Hamiltonian in the limit of \( q_{\lambda}\alpha > 1 \) with \( J/\Gamma_s < 1 \), where the atoms are almost independent. The other limit of \( q_{\lambda}\alpha < 1 \) treated by us before [12–19].

We calculate the damping rate of such collective state through the emission of a photon into free space and the damping into the final ground state \( |f\rangle = |g_1, \ldots, g_N\rangle \). We apply the Fermi golden rule to calculate the collective symmetric state damping rate,

\[
\Gamma_s = \frac{2\pi}{N} \sum_{q\lambda} \langle f | H_I | i \rangle^2 \delta(E_A - E_{pb}),
\]

which reads

\[
\Gamma_s = \sum_{q\lambda} \frac{\pi q c_{q\lambda}}{e_0VN} (\mu \cdot e_{q\lambda})^2 \sum_{n=1}^N e^{-iq \cdot n} a_n \delta(E_A - E_{pb}).
\]

The summation over the photon polarization yields

\[
\sum_{q\lambda} (\mu \cdot e_{q\lambda})^2 = \mu^2 - \frac{q^2 |\mu|^2}{\hbar c q}.
\]

The summation over \( q \) can be converted into the integral \( \int_{-\pi}^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \int_0^\infty q^2 dq \). We use \( q = q_0 \sin \theta \cos \phi, q_0 \sin \theta \sin \phi, q_0 \cos \theta \), and the transition dipole is taken to be \( \mu = \mu_0 \sin \theta \cos \phi \). The integration over \( \phi \), and the change of the variable \( y = q_{\lambda}\alpha \sin \theta \), gives

\[
\Gamma_s = \frac{\mu^2 q_{\lambda}^2}{8\pi e_0 h a N} \int_{-q_{\lambda}\alpha}^{+q_{\lambda}\alpha} dy \sum_{n=1}^N e^{-inY} \left| \int_0^\infty F(q_{\lambda}\alpha y) (3 \cos^2 \theta - 1) \right|^2.
\]

Using the relation \( \sum_{n=1}^N e^{-inY} = N + \sum_{n=1}^N e^{-inY} \), we reach, after the integration over \( y \), the result

\[
\Gamma_s = \Gamma_A \left\{ 1 + \frac{2}{N} \sum_{n=1}^N F(q_{\lambda}\alpha(n - m)) \right\},
\]

where

\[
F(x) = \frac{3}{2} \left( \frac{\sin x}{x} - (1 - \cos^2 \phi) \right) \left( \frac{\sin x}{x^2} - \frac{\cos x}{x^3} \right) (1 - 3 \cos^2 \phi).
\]

Using the previous numbers, for \( \phi = 0^\circ \) we get \( F(0.5) = 0.9752 \), and for \( \phi = 90^\circ \) we get \( F(0.5) = 0.9507 \), which justifies the use of \( F = 1 \) for optical lattice ultracold atoms in the limit of \( q_{\lambda}\alpha < 1 \).

The symmetric damping rate is written in the form

\[
\Gamma_s(N) = \Gamma_A \left\{ 1 + \frac{2}{N} \sum_{n=1}^{N-1} \frac{N - n}{N} F(q_{\lambda}\alpha n) \right\}.
\]
Let us consider $F(q_A an)$ to represent a bond between two atoms that are separated by a distance $(an)$, then in the above summation the function $F(q_A an)$ is multiplied by the number of bonds of this length which is $(N - n)$. In the limit of $q_A a \ll 1$ we get $F \approx 1$, and then $A_s(N) \approx N A$. In the limit of $q_A a \gg 1$ we get $F \approx 0$, and then $A_s(N) \approx A$ with oscillations.

Now we emphasize the dependence of the symmetric state damping rate on the size of the chain. Namely, on the number of atoms, $N$, for different lattice constants, $a$. We plot the scaled damping rate $A_s/A$ as a function of $N$ for different values of $q_A a$. In figs. 4–6 we plot for $q_A a = 0.001$, $q_A a = 0.1$, and $q_A a = 1$, in the two cases of $\varphi = 0^\circ$ and $\varphi = 90^\circ$. The damping rate of the symmetric state grows linearly with the number of atoms for small $N$, where $A_s \sim N A$. The damping rate approaches a finite value for large $N$ with $A_s$ that can be much smaller than $N A$. For $q_A a \gg 1$ the damping rate approaches the finite value faster than for $q_A a \ll 1$. Significant difference appears between the damping rates for $\varphi = 0^\circ$ and $\varphi = 90^\circ$ for large $N$.

Similar conclusions can be presented in terms of the chain length, which is $L = a(N - 1)$. The system has two length scales, the lattice constant $a$ and the chain length $L$, that need to be compared to the atomic transition wavelength $\lambda_A$. The limit of small lattice constant $a \ll \lambda_A$ will not be considered here. It has been treated by us in other works for very short chains [19], that is in the limit of $L \ll \lambda_A$, and which implies the inclusion of the energy transfer among the atoms. The linear behavior of the collective damping rate as a function of the chain length is obtained for short chains with $\lambda_A > L$, and the deviation from the linear dependence started around $\lambda_A \sim L$. The saturation is achieved for long chains with $\lambda_A < L$. The limit of $a \gg \lambda_A$ yields the expected result of $A_s \sim A$, for completely independent atoms.

Here we consider the case of a single excitation but for a general collective state, which is given by $|i\rangle = \frac{1}{N} \sum_i C_i |g_1,\ldots,e_i,\ldots,g_N\rangle$, where $C_i = \pm 1$, and $\sum_i C_i^2 = N$. Equation (9) reads

$$
\Gamma = \frac{\mu^2 q_A^2}{8\pi\epsilon_0 haN} \int_{-q_A a}^{q_A a} dy \left| \sum_{n=1}^{N} C_n e^{-iny} \right|^2 \times \left[ (1 + \cos^2 \varphi) - \frac{y^2}{(q_A a)^2} (3 \cos^2 \varphi - 1) \right]. \quad (13)
$$

We use

$$
\left| \sum_{n=1}^{N} C_n e^{-iny} \right|^2 = N + \sum_{n \neq m=1}^{N} C_n C_m e^{-(n-m)y}, \quad (14)
$$

where $C_n C_m = \pm 1$. After integration over $y$, we get

$$
\Gamma(N) = A \left\{ 1 + \frac{2}{N} \sum_{n \neq m=1}^{N} C_n C_m F(q_A a(n-m)) \right\}. \quad (15)
$$

Here we present the results for two examples. For $N = 2$ we have the symmetric state $|i\rangle_s = \frac{|e_1, g_2\rangle + |g_1, e_2\rangle}{\sqrt{2}}$, with...
the damping rate \( \Gamma_{s} = \Gamma_{A} \{1 + F(q_{AA})\} \), and the antisymmetric state \( |i\rangle_{a} = \frac{|e_{1}, g_{2}, g_{3}\rangle - |g_{1}, e_{2}, g_{3}\rangle}{\sqrt{3}} \), with the damping rate \( \Gamma_{a} = \Gamma_{A} \{1 - F(q_{AA})\} \). The results for \( N = 2 \) agree with the known ones [23]. In the limit of \( q_{AA} \ll 1 \), for the symmetric state we get \( \Gamma_{s} \approx 2\Gamma_{A} \), and for the antisymmetric one we get \( \Gamma_{a} \approx 0 \). In the limit of \( q_{AA} \gg 1 \) for the symmetric and antisymmetric states we get \( \Gamma \approx \Gamma_{A} \).

For \( N = 3 \), for the symmetric state

\[
|i\rangle_{s} = \frac{|e_{1}, g_{2}, g_{3}\rangle + |g_{1}, e_{2}, g_{3}\rangle + |g_{1}, g_{2}, e_{3}\rangle}{\sqrt{3}}, \quad (16)
\]

we get

\[
\Gamma_{s} = \Gamma_{A} \left\{1 + \frac{2}{3} \left[2F(q_{AA}) + F(2q_{AA})\right]\right\}. \quad (17)
\]

For the antisymmetric state

\[
|i\rangle_{a} = \frac{|e_{1}, g_{2}, g_{3}\rangle - |g_{1}, e_{2}, g_{3}\rangle + |g_{1}, g_{2}, e_{3}\rangle}{\sqrt{3}}, \quad (18)
\]

we get

\[
\Gamma_{a} = \Gamma_{A} \left\{1 - \frac{2}{3} \left[2F(q_{AA}) - F(2q_{AA})\right]\right\}. \quad (19)
\]

In the limit of \( q_{AA} \ll 1 \) we have \( F \approx 1 \), then for the symmetric state we get \( \Gamma_{s} \approx 3\Gamma_{A} \), and for the antisymmetric one we get \( \Gamma_{a} \approx \Gamma_{A}/3 \). In the limit of \( q_{AA} \gg 1 \) we have \( F \approx 0 \), then for the symmetric and antisymmetric states we get \( \Gamma \approx \Gamma_{A} \) with oscillations. In fig. 7 we plot \( \Gamma/\Gamma_{A} \) for the symmetric state of \( N = 3 \) as a function of \( q_{AA} \), for the polarizations \( \varphi = 0^\circ \) and \( \varphi = 90^\circ \). In fig. 8 the plot is for the antisymmetric state.

The damping rates of the symmetric states, in a chain of \( N \) atoms, tend to \( N \) times that of a single excited atom, that is \( \Gamma_{s} \sim N\Gamma_{A} \), for small lattice constants. But, the antisymmetric states become metastable and even dark in the same limit. The damping rates of both the symmetric and antisymmetric states for large lattice constant tend to that of a single excited atom as expected, that is \( \Gamma_{A} \), with oscillations.

As a summary, in the present letter we investigated optical properties of a one-dimensional atomic chain, in which the lattice constant can range from a few angstroms to thousands of angstroms. Namely, the lattice constant can change from being smaller than the atomic transition wavelength up to a much larger one. In the limit of lattice constant smaller than the atomic transition wavelength the electrostatic interactions are applicable. For small lattice constants, the electrostatic interactions are responsible for the formation of excitons. For large lattice constants, the inclusion of radiative corrections are necessary, which is the main issue of the letter.

We conclude that for large lattice constants, the radiative corrections are included, and in this regime we found that the coupling parameter for the energy transfer among even the nearest-neighbor atom sites is smaller than a single excited atom damping rate. Hence, the energy transfer is not favorable, and we treated the atoms as independently setting on the lattice sites. Then, we calculated the damping rates of different collective electronic excitations including the radiative corrections by considering the effect of the existence of all the other atom sites, despite their large distances from the excited atom. Big attention we paid to the most symmetric state, where we emphasized the influence of the chain size via the dependence of its damping rate on the number of atoms for different lattice constants. We found the symmetric damping rate to behave linearly at small atom numbers and saturate at large numbers. Namely, to behave linearly for short chains and saturate for long ones. The damping rate of the symmetric and antisymmetric collective states for large lattice constants tend to that of a single excited atom with oscillations due to the radiative effect through the exchange of virtual photons. The differences between damping rates of collective states appear for small lattice constants.
constants, in which the symmetric states have superradiant damping rate, that is \( N \) times the single excited atom rate. Here, part of the antisymmetric states become dark with zero damping rate, and the other part is metastable with a fraction of the single excited atom damping rate.

The results of the present letter are illustrated in terms of optical lattice ultracold atoms, but they are general and can be adopted for any chain of optically active material. For example, chains of semiconductor quantum dots fit exactly in the regime of large lattice constant, where radiative corrections are unavoidable, and the lifetimes of their collective states can be treated according to the present letter. Another system that exploits the regime of the present letter is a lattice of large organic molecules sitting on a matrix with a given large lattice constant, the collective damping rate is expected to behave according to our present results. The properties of the collective states presented in this letter make them promising qubits for the implementation of quantum information processing.

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