Green correspondence on centric Mackey functors over fusion systems.

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Abstract

In this paper we give a definition of (centric) Mackey functor over a fusion system which generalizes the notion of Mackey functor over a group. In this scenario and given some conditions on a related ring $\mathcal{R}$ we prove that the centric Burnside ring over a fusion system acts on any centric Mackey functor (Proposition 2.42) and that the Green correspondence can be extended to centric Mackey functors (Theorem 4.16). As a means to prove this generalization of the Green correspondence we also provide a decomposition of a particular product in $\mathcal{O}(\mathcal{F}^c)_{\mathcal{F}}$ in terms of the product in $\mathcal{O}((N_F^c)_{\mathcal{F}})$ (Theorem 4.14).

Contents

1 Introduction. 2

2 Background and first results. 3
  2.1 Fusion systems. .................................................... 3
  2.2 Mackey functors over fusion systems. .......................... 6
  2.3 The centric Burnside ring over a fusion system. ............. 11

3 Relative projectivity and the Higman’s criterion. 13

4 Green correspondence. 20
  4.1 Correspondence of endomorphisms. .............................. 20
  4.2 Conditional proof of the Green correspondence. ................ 24
  4.3 Proof of Proposition 4.12. ....................................... 31
  4.4 Proof of Proposition 4.13. ....................................... 35
  4.5 Proof of Theorem 4.14. ......................................... 38
  4.6 Proof of Proposition 4.15. ....................................... 44
1 Introduction.

Let \( \mathcal{R} \) be a complete DVR with residue field \( k \) of characteristic \( p \), let \( G \) be a finite group \( G \) and let \( H \) be a \( p \)-subgroup of \( G \). In [4] Green proves that there is a one-to-one correspondence between finitely generated indecomposable \( \mathcal{R}G \)-modules with vertex \( H \) and finitely generated indecomposable \( NG(H) \)-modules with vertex \( H \). Green later generalizes this same result in [5] to Green functors over \( G \) and Sasaki uses this in [11] to further extend the result to Mackey functors over \( G \). In this paper we will apply and adapt some of the techniques and arguments employed in [5] and [11] in order to obtain a similar one-to-one correspondence in the context of centric Mackey functors over a fusion system. To this end we organize the paper as follows.

In Section 2 we briefly recall the definitions of (saturated) fusion system (Definitions 2.1 and 2.6), of (centric) Mackey functor over a fusion system (Definitions 2.24 and 2.29) and of centric Burnside ring of a fusion systems (Definition 2.38). In this section we will also recall some well known properties regarding these concepts and prove 2 further results. The first one (Proposition 2.42) states that, under certain conditions regarding a related ring \( \mathcal{R} \), there is a unit preserving embedding of the centric Burnside ring over a fusion system \( \mathcal{F} \) into the center of the centric Mackey algebra over \( \mathcal{F} \) (Definition 2.28). The second result (Remark 2.36) states that, under certain conditions, the induction and restriction functors associated to centric Mackey functors over fusion systems (see Definition 2.27 and Notation 2.33) are left and right adjoint to one another.

In Section 3 we will introduce the concept of relative projectivity of a Mackey functor over a fusion system (Definition 3.1) and prove a generalization of Higman’s criterion to centric Mackey functors (Theorem 3.11). To do this we will need to define the transfer and restriction maps (Definition 3.4) and list some of the properties they satisfy (Proposition 3.5). These properties will later be needed during Subsections 4.4 and 4.5.

We will conclude with Section 4 where we will prove our 2 main results (Theorems 4.14 and 4.16). We will start this final section by stating and proving Proposition 4.7 which will play in the proof of Theorem 4.16 the same role that [5, Proposition 4.34] plays in the proof of the Green correspondence for Green functors. We will then state without proof Propositions 4.12, 4.13, and 4.15 and Theorem 4.14 and use them to prove Theorem 4.16. The reminder of the paper will then be dedicated to proving the above mentioned Propositions 4.12, 4.13 and 4.15 and Theorem 4.14.

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2 Background and first results.

In this section we will review the concepts of fusion systems, of Mackey functor over a fusion systems and of centric Burnside ring of a fusion system. The reader already familiar with such concepts can safely skip most of this section’s content only paying attention to the explicitly introduced notation, Definitions 2.18, 2.28, 2.31 , 2.37 and 2.40, Proposition 2.42 and Remark 2.43. These results are used to describe an action of the centric Burnside ring on any centric Mackey functor and interpret such action in terms of the maps $\theta_H$ and $\theta^H$ (see Definition 2.37). This action, as well as it’s interpretation, is analogous to the well known action of the Burnside ring of a group on any Mackey functor over a group. The reader already familiar with such action might skip those results as well. However be aware that Proposition 2.42 and Remark 2.43 will play a crucial role in Subsection 4.4.

2.1 Fusion systems.

What follows is a brief introduction to fusion systems which mostly aims to establish some notation. For a more thorough introduction please refer to [6]. In this subsection we also report the main result of [8, Section 4]. Namely, given a saturated fusion system $\mathcal{F}$ then the category $\mathcal{O}(\mathcal{F}^c)_{\cup}$ (see Definition 2.14) admits a product. We will also describe such product.

**Definition 2.1.** Let $p$ be a prime and let $S$ be a finite $p$-group. A **fusion system** over $S$ is a category $\mathcal{F}$ having as objects subgroups of $S$ and satisfying the following properties for every $H, K \leq S$:

1. Every morphism $\varphi \in \text{Hom}_\mathcal{F}(H, K)$ is an injective group homomorphism and the composition of morphisms in $\mathcal{F}$ is the same as the composition of morphisms in the category of groups.

2. $\text{Hom}_S(H, K) \subseteq \text{Hom}_\mathcal{F}(H, K)$ that is every group homomorphism from $H$ to $K$ that can be described as conjugation by an element of $S$ followed by inclusion is a morphism in $\mathcal{F}$.

3. If $\varphi \in \text{Hom}_\mathcal{F}(H, K)$ then, denoting by $\tilde{\varphi}$ the homomorphism obtained by looking at $\varphi$ as an isomorphism onto its image, we have that $\tilde{\varphi} \in \text{Hom}_\mathcal{F}(\varphi(H), H)$ and that $\tilde{\varphi}^{-1} \in \text{Hom}_\mathcal{F}(\varphi(H), H)$.

**Example 2.2.** The most common example of fusion system is obtained by taking a finite group $G$ containing $S$ and defining $\mathcal{F}_S(G)$ as the fusion system on $S$ whose morphisms are given by conjugation by elements in $G$ followed by inclusion. When $S = G$ we will often simply write $\mathcal{F}_S := \mathcal{F}_S(S)$ in order to keep notation simple.

The previous definition and example motivates the introduction of the following notation which will be useful throughout the paper.
Notation 2.3. From now on, unless otherwise specified, all groups will be understood to be finite, \( p \) will denote a prime integer, \( S \) will denote a finite \( p \)-group and \( \mathcal{F} \) will denote a fusion system over \( S \). Moreover, given subgroups \( H, K \leq S \) we will write \( H =_{\mathcal{F}} K \) if \( H \) and \( K \) are isomorphic in \( \mathcal{F} \), \( H \leq_{\mathcal{F}} K \) if exists \( J \leq K \) such that \( H =_{\mathcal{F}} J \) and \( H \prec_{\mathcal{F}} K \) is \( H \leq_{\mathcal{F}} K \) but \( H \neq_{\mathcal{F}} K \).

In the literature, when the term fusion system appears it is usually to refer to a particular type of fusion system called saturated fusion system.

Definition 2.4. Let \( H \leq S \). We say that \( H \) is fully \( \mathcal{F} \)-normalized if for every \( K =_{\mathcal{F}} H \) we have that \(|N_S(K)| \leq |N_S(H)|\).

Definition 2.5. Let \( H, K \leq S \) and let \( \varphi \) be a morphism from \( H \) to \( K \) in \( \mathcal{F} \). We define the \( \varphi \)-normalizer as the following subgroup of \( N_S(H) \)

\[
N_\varphi := \{ x \in N_S(H) : \exists z \in N_S(\varphi(H)) \text{ satisfying } \varphi(z\cdot h) = z\cdot \varphi(h) \quad \forall h \in H \}. 
\]

Definition 2.6. A fusion system \( \mathcal{F} \) is said to be saturated if the following 2 conditions are satisfied:

1. \( \text{Aut}_S(S) \) is a Sylow \( p \)-subgroup of \( \text{Aut}_\mathcal{F}(S) \).
2. For every \( H \leq S \) and every \( \varphi \in \text{Hom}_\mathcal{F}(H,S) \) such that \( \varphi(H) \) is fully \( \mathcal{F} \)-normalized exists \( \hat{\varphi} \in \text{Hom}_\mathcal{F}(N_\varphi,S) \) such that \( \hat{\varphi}\iota_{N_\varphi}^H = \varphi \). Here \( \iota_{N_\varphi}^H \) denotes the natural inclusion of \( H \) into \( N_\varphi \).

Notation 2.7. Throughout the paper we will use the same notation as in Definition 2.6 (or even just \( \iota \)) in order to denote the natural inclusion map.

Example 2.8. The fusion system \( \mathcal{F}_S(G) \) of Example 2.2 is saturated if \( S \) is a Sylow \( p \) subgroup of \( G \).

Example 2.9. Given a saturated fusion system \( \mathcal{F} \) and a fully \( \mathcal{F} \)-normalized subgroup \( H \leq S \) we can define the saturated fusion system \( N_\mathcal{F}(H) \) on \( N_S(H) \) by setting

\[
\text{Hom}_{N_\mathcal{F}(H)}(A,B) := \{ \varphi \in \text{Hom}_\mathcal{F}(A,B) | \exists \hat{\varphi} \in \text{Hom}_\mathcal{F}(AH,BH) \text{ s.t. } \iota_{BH}^{AH} \varphi = \hat{\varphi}\iota_{AH} \}. 
\]

If the subgroup \( H \) is understood we will often just denote \( N_\mathcal{F} := N_\mathcal{F}(H) \).

The previous definition and examples motivate the introduction of the following notation.

Notation 2.10. From now on every time we say fusion system we will be meaning saturated fusion system. In particular, whenever we write \( \mathcal{F}_S(G) \) it will be understood that \( G \) is a finite group having \( S \) as a Sylow \( p \)-subgroup.

There are two result regarding fusion systems that we will be using throughout this paper. We state them below without proof. The interested reader can refer to [6, Proposition 2.5 and Lemma 2.6] for proofs.
Lemma 2.11. Let $H \leq S$ be fully $\mathcal{F}$-normalized. Then $\text{Aut}_S(H)$ is a Sylow $p$-subgroup of $\text{Aut}_F(H)$.

Lemma 2.12. For any $H \leq S$ there exists a morphism $\varphi \in \text{Hom}_F(N_S(H), S)$ such that $\varphi(H)$ is fully $\mathcal{F}$-normalized.

For most part of this document we will not be dealing directly with fusion systems but rather with their orbit category or their centric full subcategory.

Definition 2.13. We define the orbit category of a fusion system $\mathcal{F}$ as the category $\mathcal{O}(\mathcal{F})$ having as objects the same objects as $\mathcal{F}$ and as morphisms

$$\text{Hom}_{\mathcal{O}(\mathcal{F})}(H, K) := \text{Aut}_K(K) \setminus \text{Hom}_\mathcal{F}(H, K),$$

where $H, K \leq S$ and $\text{Aut}_K(K)$ is acting on $\text{Hom}_\mathcal{F}(H, K)$ by post conjugation.

Definition 2.14. Let $H \leq S$. We say that $H$ is $\mathcal{F}$-centric if $C_S(K) \leq K$ for every $K =_\mathcal{F} H$. The centric subcategory of $\mathcal{F}$ (denoted by $\mathcal{F}^c$) is defined as the full subcategory of $\mathcal{F}$ having as objects $\mathcal{F}$-centric subgroups of $S$. Likewise we denote by $\mathcal{O}(\mathcal{F}^c)$ the full subcategory of $\mathcal{O}(\mathcal{F})$ having as objects the $\mathcal{F}$-centric subgroups of $S$. Finally we denote by $\mathcal{O}(\mathcal{F}^c)_{\bot}$ the additive completion of $\mathcal{O}(\mathcal{F}^c)$.

Notation 2.15. We will often abuse notation and, whenever there is no confusion, we will write $\mathcal{O}(\mathcal{F}^c)$ to refer to it’s additive completion $\mathcal{O}(\mathcal{F}^c)_{\bot}$.

There are two results regarding the category $\mathcal{O}(\mathcal{F}^c)_{\bot}$ that we will be using throughout this document. The first one will be directly used only in proof of Lemma 4.26 and we state it below without proof. The interested reader can find a proof in [6, Theorem 4.9].

Lemma 2.16. Every morphism in $\mathcal{O}(\mathcal{F}^c)$ (not $\mathcal{O}(\mathcal{F}^c)_{\bot}$) is an epimorphism in the categorical sense.

The second result will be used multiple times through the document and it’s the main reason why we will be mostly concerned with $\mathcal{F}^c$ instead of $\mathcal{F}$.

Proposition 2.17. ([8, Proposition 4.7]) The category $\mathcal{O}(\mathcal{F}^c)_{\bot}$ admits a product which is distributive with respect to its co-product.

In his paper Puig explicitly describes the product mentioned in Proposition 2.17. More precisely, given elements $H, K \in \mathcal{F}^c$ (seen as elements in $\mathcal{O}(\mathcal{F}^c)_{\bot}$) Puig defines their product $H \times_\mathcal{F} K$ (or simply $H \times K$ if $\mathcal{F}$ is clear) as follows: First take all pairs $(A, \varphi)$ with $A \leq H$ $\mathcal{F}$-centric and $\varphi \in \text{Hom}_{\mathcal{O}(\mathcal{F}^c)}(A, K)$. Then define the preorder $\preceq_H$ on the set of all such pairs by setting $(A, \varphi) \preceq_H (B, \psi)$ if and only if exists $h \in H$ such that $A^h \leq B$ and $\varphi c_h = \psi t_H^h A^h$. Then take all pairs that are maximal under such preorder and define the equivalence relation

$$(A, \varphi) \sim (B, \psi) \iff (A, \varphi) \preceq_H (B, \psi) \text{ and } (B, \psi) \preceq_H (A, \varphi)$$

5
Finally take only one representative for each equivalence class and define

\[ H \times K := \bigsqcup_{(A, \varphi)} A, \quad \pi^H_K := \bigsqcup_{(A, \varphi)} \iota^H_A, \quad \pi^K_H := \bigsqcup_{(A, \varphi)} \varphi, \]

where \((A, \varphi)\) run over the described set of maximal representatives and \(\pi^H_K\) and \(\pi^K_H\) denote the natural projections. To summarize the previous construction we will be introducing the following notation with which we conclude this subsection.

**Definition 2.18.** For every \(H, K \in \mathcal{F}^c\) we denote by \([H \times \mathcal{F} K]\) (or simply \([H \times K]\) if \(\mathcal{F}\) is clear) any set of representatives of the maximal pairs \((A, \varphi)\) described above.

### 2.2 Mackey functors over fusion systems.

This subsection is dedicated to giving a definition of (centric) Mackey functor over a fusion system and define the induction and restriction functors between different categories of Mackey functors. Moreover we will prove that, even though induction is not always right adjoint to restriction as in the case of Mackey functors over groups, it becomes right adjoint when restricting to particular sub-categories (Remark 2.36).

Let us start by giving a definition of Mackey functor over a fusion system. To do so we will use the same idea Bouc uses in order to define fused Mackey functor in [1].

**Definition 2.19.** Let \(G\) be a group and let \(f : G \to f(G)\) be a group isomorphism. We define the \(f\) **twisted diagonal of** \(G\) as the subgroup of \(f(G) \times G\) given by

\[ \Delta(G, f) := \{(f(x), x) \in f(G) \times G : x \in G\}. \]

**Definition 2.20.** Let \(H \leq K\) be subgroups of \(S\) and let \(\varphi : H \to \varphi(H)\) be an isomorphism in \(\mathcal{F}\). Denote by \(\overline{\mathbf{X}}\) the isomorphism class of a given biset \(X\) (over any pair of groups) and view \(\Delta(H, \text{Id}_H)\) as a subgroup of \(H \times K\). Then we can define the **restriction from** \(K\) **to** \(H\) as the isomorphism class of \((H, K)\)-bisets given by

\[ R^K_H := (H \times K) / \Delta(H, \text{Id}_H). \]

Likewise, viewing \(\Delta(H, \text{Id}_H)\) as a subgroup of \(K \times H\), we can define the **induction from** \(H\) **to** \(K\) as the isomorphism class of \((K, H)\)-bisets given by

\[ I^K_H := (K \times H) / \Delta(H, \text{Id}_H). \]

Finally we define the **conjugation by** \(\varphi\) as the isomorphism class of \((\varphi(H), H)\)-bisets given by

\[ c_\varphi := c_{\varphi, H} := (\varphi(H) \times H) / \Delta(H, \varphi). \]

**Remark 2.21.** With the notation above we have that \(c_{ch, \varphi} = c_\varphi\) for every \(h \in \varphi(H)\). With this in mind in future sections we will often abuse notation and, given an isomorphism \(\overline{\varphi}\) in \(\mathcal{O}(\mathcal{F}^c)\) with representative \(\varphi \in \mathcal{F}\), we will write \(c_{\overline{\varphi}}\) instead of \(c_\varphi\).
Our goal is that of defining Mackey functors as modules over a ring generated by elements of the form \( R^K_H, I^K_H \) and \( c_\varphi \) with \( H \leq K \leq S \) and \( \varphi \) an isomorphism in \( \mathcal{F} \). To do this we will start by defining a product between bisets. Given groups \( H, J', J \) and \( K \), an \((H, J)\)-biset \( X \) and a \((J', K)\)-biset \( Y \) we define the product of \( X \) and \( Y \) as the \((H, K)\)-biset given by

\[
XY := X \cdot Y := \begin{cases} 
X \times_J Y & \text{if } J = J' \\
\emptyset & \text{else}
\end{cases},
\]

where \( X \times_J Y \) is the \((H, K)\)-biset obtained as a quotient of the \((H, K)\)-biset \( X \times Y \) modulo the equivalence relation

\[
(x \cdot j) \times y \sim x \times (j \cdot y)
\]

for every \( x \in X \), \( y \in Y \) and \( j \in J \). It is straightforward to prove that, whenever the bisets \( X \) and \( Y \) are isomorphic to bisets \( X' \) and \( Y' \) respectively, then the \((H, K)\)-biset \( XY \) is isomorphic to the \((H, K)\)-biset \( X' \times Y' \). We can therefore define the product of two isomorphism classes of bisets to be the isomorphism class of the product of any two of their representatives. That is, using the same notation as in Definition 2.20, we define the product of two isomorphism classes of bisets as

\[
\overline{X} \overline{Y} := \overline{X} \cdot \overline{Y} := \overline{XY}.
\]

It is straightforward to prove that this product is associative.

Take now \( \mathcal{A} \) to be the abelian semigroup having as generators an artificial zero \((0)\) element and all the isomorphism classes of non-empty bisets over any pair of groups. The relations in \( \mathcal{A} \) are the following

\[
\overline{X} + \overline{Y} = \overline{X \sqcup Y}, \quad \overline{X} + \overline{Z} = \overline{Z} + \overline{X}, \quad 0 + 0 = 0, \quad 0 + \overline{Z} = \overline{Z} + 0 = \overline{Z},
\]

where \( X \) and \( Y \) are bisets over the same pair of groups, \( Z \) is a biset over any pair of groups and we are using the notation of Definition 2.20 to denote isomorphism classes of bisets. By sending \( \emptyset \) (seen as a biset over any 2 groups) to the element \( 0 \) in \( \mathcal{A} \) the previously defined product between isomorphism classes of bisets can be uniquely extended to \( \mathcal{A} \) in a way that satisfies the distributive property with \( + \). With this setup we have that \((\mathcal{A}, +, \cdot)\) is a semiring. We can now take the sub-semiring of \( \mathcal{A} \) generated by isomorphism classes of bisets of the form \( I^K_H, R^K_H \) and \( c_\varphi \) with \( H \leq K \leq S \) and \( \varphi \) an isomorphism in \( \mathcal{F} \). This sub-semiring can be used in order to define the Mackey algebra.

**Definition 2.22.** The **Mackey algebra of the fusion system \( \mathcal{F} \) on the ring \( \mathbb{Z} \)** (denoted by \( \mu_{\mathbb{Z}}(\mathcal{F}) \)) is the Grothendieck group of the previously described sub-semiring. Moreover, if \( \mathcal{R} \) is any commutative ring with unit, we define the **Mackey algebra over the fusion system \( \mathcal{F} \) on the ring \( \mathcal{R} \)** (or simply **Mackey algebra** if \( \mathcal{F} \) and \( \mathcal{R} \) are clear) as the ring

\[
\mu_{\mathcal{R}}(\mathcal{F}) := \mathcal{R} \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}(\mathcal{F}).
\]
Remark 2.23. Notice that the elements \( I^K_H, R^K_H \) and \( c_\varphi \) of the Mackey algebra \( \mu_R(\mathcal{F}) \) satisfy relations analogous to the similarly denoted elements in the Mackey algebra of a group (see [13, Section 3]). More precisely, the following are satisfied:

- For every \( h \in H \leq S \) then \( I^h_H = R^h_H = c_{c_h,H} \) is an idempotent in \( \mu_R(\mathcal{F}) \).
- For every \( H \leq K \leq J \leq S \) and every two composable isomorphisms \( \psi \) and \( \varphi \) in \( \mathcal{F} \) we have that
  \[
  R^K_H R^K_J = R^K_J, \quad I^K_H I^K_J = I^K_J, \quad c_\psi c_\varphi = c_{\psi \varphi}.
  \]
- For every \( H \leq K \leq S \) and every isomorphism \( \theta : K \to \theta(K) \) we have that
  \[
  c_{\theta,K} I^K_H = I^\theta(K)_{\theta(H)} c_{\theta,H}, \quad c_{\theta,H} R^K_H = R^\theta(K)_{\theta(H)} c_{\theta,K}.
  \]
- For every \( H, K \leq J \leq S \) we have that
  \[
  R^K_H R^K_J = \sum_{x \in [K \setminus J/H]} I^K_{(K \cap xH)} c_{c_x,(K^x \cap H)} R^K_{(K^x \cap H)}.
  \]
- All other combinations of induction restriction and conjugation are 0.

Once the Mackey algebra is defined it is immediate to define the Mackey functor.

**Definition 2.24.** A **Mackey functor over a fusion system** \( \mathcal{F} \) **on a commutative ring** \( \mathcal{R} \) (or simply **Mackey functor** if \( \mathcal{F} \) and \( \mathcal{R} \) are clear) is a finitely generated \( \mu_R(\mathcal{F}) \)-module. The **category of Mackey functors over** \( \mathcal{F} \) **on** \( \mathcal{R} \) **is the category** \( \mu_R(\mathcal{F})\text{-mod} \).

**Example 2.25.** Any globally defined Mackey functor (see [14]) naturally inherits a structure of Mackey functor over any fusion system \( \mathcal{F} \). Any conjugation invariant Mackey functor over a group \( G \) leads naturally to a Mackey functor over \( \mathcal{F}_S(G) \). The Mackey algebra \( \mu_R(\mathcal{F}) \) is itself a Mackey functor over \( \mathcal{F} \).

The previous definitions motivate the introduction of the following notation.

**Notation 2.26.** From now and until the end of this document \( \mathcal{R} \) will be denoting a commutative ring with unit.

Using this definition of Mackey functor over a fusion system we can borrow some results from ring theory to immediately define some induction and restriction functors.

**Definition 2.27.** Let \( H \in \mathcal{F} \) and let \( \mathcal{H} \subseteq \mathcal{F} \) be a fusion system on \( H \). It is straightforward to prove that \( \mu_R(\mathcal{H}) \subseteq \mu_R(\mathcal{F}) \). This allows us to define the **restriction functor from** \( \mathcal{F} \) **to** \( \mathcal{H} \) **as the functor**

\[
\downarrow^\mathcal{F}_{\mathcal{H}} : \mu_R(\mathcal{F})\text{-mod} \to \mu_R(\mathcal{H})\text{-mod},
\]
that sends any $\mu_\mathcal{R}(\mathcal{F})$-module $M$ to the $\mu_\mathcal{R}(\mathcal{H})$-module 

$$M \downarrow_{\mathcal{H}}^\mathcal{F} := \mu_\mathcal{R}(\mathcal{H}) M.$$ 

Analogously we can define the induction functor from $\mathcal{H}$ to $\mathcal{F}$ as the functor 

$$\uparrow_{\mathcal{H}}^\mathcal{F}: \mu_\mathcal{R}(\mathcal{H}) \text{-mod} \to \mu_\mathcal{R}(\mathcal{F}) \text{-mod},$$

that sends any $\mu_\mathcal{R}(\mathcal{H})$-module $N$ to the $\mu_\mathcal{R}(\mathcal{F})$-module 

$$N \uparrow_{\mathcal{H}}^\mathcal{F} := \mu_\mathcal{R}(\mathcal{F}) (N) := \mu_\mathcal{R}(\mathcal{H}) \otimes_{\mu_\mathcal{R}(\mathcal{H})} N.$$ 

These definitions of induction and restriction functors however lack a property that was present in the case of Mackey functors over groups and which plays a key role in extending the Green correspondence to Mackey functors over groups. Namely, given $M$ a Mackey functor over a group $G$ and given a subgroup $H \leq G$, it is a known result that the Mackey functor $M \downarrow_{H}^{G \times H}$ over $G$ satisfies for every $K \leq G$ the following equivalence of $\mathcal{R}$-modules 

$$M \downarrow_{H}^{G \times H}(K) \cong_{\mathcal{R}} \bigoplus_{x \in [K \setminus G/H]} M(Kx \cap H).$$

Trying to obtain a similar result in the case of Mackey functors over fusion systems however leads to many complications that can be traced back to the fact that the category $\mathcal{O}(\mathcal{F})_{\sqcup}$ does not always admit products. In order to avoid such complications Proposition 2.17 suggests that we should introduce the following

**Definition 2.28.** The centric Mackey algebra over a fusion system $\mathcal{F}$ on a ring $\mathcal{R}$ (or simply centric Mackey algebra if $\mathcal{F}$ and $\mathcal{R}$ are clear) is the $\mathcal{R}$-algebra defined as 

$$\mu_\mathcal{R}(\mathcal{F}^{c}) := \mu_\mathcal{R}(\mathcal{F}) / \mathcal{I}$$

where $\mathcal{I}$ is the 2 sided ideal of $\mu_\mathcal{R}(\mathcal{F})$ defined as 

$$\mathcal{I} := \sum_{H \leq S, H \notin \mathcal{F}^{c}} \mu_\mathcal{R}(\mathcal{F}) I_{H}^{H} \mu_\mathcal{R}(\mathcal{F}).$$

As before this leads immediately to a definition of centric Mackey functor.

**Definition 2.29.** A centric Mackey functor over a fusion system $\mathcal{F}$ on a ring $\mathcal{R}$ (or simply centric Mackey functor if $\mathcal{F}$ and $\mathcal{R}$ are clear) is a finitely generated $\mu_\mathcal{R}(\mathcal{F}^{c})$-module. The category of centric Mackey functors over $\mathcal{F}$ on $\mathcal{R}$ is the category $\mu_\mathcal{R}(\mathcal{F}^{c})$-mod.

**Remark 2.30.** Notice how there is a one-to-one correspondence between centric Mackey functors and Mackey functors $M$ such that $I_{H}^{H} \cdot M = 0$ for every $H \notin \mathcal{F}^{c}$. It is easily proven that this correspondence actually induces an embedding of the category of centric Mackey functors into the category of Mackey functors. Because of this we will often treat centric Mackey functors just as a particular type of Mackey functors.
This definition presents a problem towards defining induction and restriction. Namely, given $H \in \mathcal{F}^c$ and a fusion system $\mathcal{H}$ on $H$ such that $\mathcal{H} \subseteq \mathcal{F}$ we don’t necessarily have that $\mu_{\mathcal{R}}(\mathcal{H}^c) \subseteq \mu_{\mathcal{R}}(\mathcal{F}^c)$ since subgroups of $H$ that are $\mathcal{H}$-centric might not also be $\mathcal{F}$-centric. To fix this we need to introduce yet another definition.

**Definition 2.31.** Let $H$ and $\mathcal{H}$ be as above and define the double ideal $\mathcal{J}$ of $\mu_{\mathcal{R}}(\mathcal{H})$ as

$$\mathcal{J} := \langle I_K^\mathcal{H} \in \mu_{\mathcal{R}}(\mathcal{H}) \mid K \not\in \mathcal{F}^c \rangle.$$  

Then the $\mathcal{F}$-centric Mackey algebra over $\mathcal{H}$ on $\mathcal{R}$ (or simply $\mathcal{F}$-centric Mackey algebra if $\mathcal{H}$ and $\mathcal{R}$ are clear) is the $\mathcal{R}$-algebra defined as

$$\mu_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{H}) := \mu_{\mathcal{R}}(\mathcal{H}) / \mathcal{J}.$$  

Moreover we call $\mathcal{F}$-centric Mackey functor over $\mathcal{H}$ on $\mathcal{R}$ (or simply $\mathcal{F}$-centric Mackey functor if $\mathcal{H}$ and $\mathcal{R}$ are clear) any finitely generated $\mu_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{H})$-module. The category of $\mathcal{F}$-centric Mackey functors over $\mathcal{H}$ on $\mathcal{R}$ is the $\mathcal{R}$-category $\mu_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{H})$-mod.

**Remark 2.32.** The same arguments showed in Remark 2.30 can be used in order to prove that there is an embedding of the category $\mu_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{H})$-mod into the category $\mu_{\mathcal{R}}(\mathcal{H})$-mod and we can therefore see $\mathcal{F}$-centric Mackey functors over $\mathcal{H}$ as just a particular type of Mackey functors over $\mathcal{H}$.

With notation as above and keeping in mind Remarks 2.30 and 2.32 it is now easily proven that the induction functor $\uparrow_{\mathcal{H}}^{\mathcal{F}^c}$ sends $\mathcal{F}$-centric Mackey functor over $\mathcal{H}$ to centric Mackey functors over $\mathcal{F}$ and that the restriction functor $\downarrow_{\mathcal{H}}^{\mathcal{F}^c}$ sends centric Mackey functors over $\mathcal{F}$ to $\mathcal{F}$-centric Mackey functors over $\mathcal{H}$. Because of this we will be introducing the following notation.

**Notation 2.33.** With notation as above, whenever there is no possible confusion we will call restriction functor from $\mathcal{F}$ to $\mathcal{H}$ the functor from $\mu_{\mathcal{R}}(\mathcal{F}^c)$-mod to $\mu_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{H})$-mod that is induced by the functor $\downarrow_{\mathcal{H}}^{\mathcal{F}^c}$ of Definition 2.27 via the embeddings of Remarks 2.30 and 2.32. For simplicity’s sake we will also denote such functor simply by $\downarrow_{\mathcal{H}}^{\mathcal{F}^c}$ whenever there is no possible confusion. Likewise we will call induction functor from $\mathcal{H}$ to $\mathcal{F}$ the functor from $\mu_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{H})$-mod to $\mu_{\mathcal{R}}(\mathcal{F}^c)$-mod that is induced by the functor $\uparrow_{\mathcal{H}}^{\mathcal{F}^c}$ of Definition 2.27 via the embeddings of Remarks 2.30 and 2.32. Whenever there is no possible confusion we will also denote such functor simply by $\uparrow_{\mathcal{H}}^{\mathcal{F}^c}$.

**Remark 2.34.** It is easily proven that the functors $\downarrow_{\mathcal{H}}^{\mathcal{F}^c}$ and $\uparrow_{\mathcal{H}}^{\mathcal{F}^c}$ of Notation 2.33 are actually equivalent to the natural universal induction and restriction functors that arise from the inclusions $\mu_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{H}) \subseteq \mu_{\mathcal{R}}(\mathcal{F}^c)$.

With this setup it is now just a matter of straightforward calculation and application of the universal properties of products to prove the following result.

**Lemma 2.35.** Let $H \in \mathcal{F}^c$ and let $\mathcal{F}_H$ be as in Example 2.2. Then we have the following equivalence of centric Mackey functors over $\mathcal{F}$

$$\mu_{\mathcal{R}}(\mathcal{F}^c) \cdot 1_{\mu_{\mathcal{R}}(\mathcal{F}_H)} \cong \bigoplus_{K \in \mathcal{F}^c(A,\varphi) \in [H \times K]} I^K_{\varphi(A)\varphi} \mu_{\mathcal{R}}^{\mathcal{F}^c}(\mathcal{F}_H).$$
Where \( 1_{\mu_F^c(F_H)} \) is seen as an idempotent of \( \mu_R(F^c) \supseteq \mu_F^c(F_H) \).

**Remark 2.36.** This result is in fact telling us that \( \mu_R(F^c) \cdot 1_{\mu_R(F_H)} \) is a projective right \( \mu_F^c(F_H) \)-module. Using this together with Remark 2.34 it can be proven exactly as in the case of Mackey functors over groups that the functors \( \uparrow_{F_H}^F \) and \( \downarrow_{F_H}^F \) (seen as in Notation 2.33) are left and right adjoint to each other. This however is not true in general if we either replace \( F_H \) by a more general fusion subsystem on \( H \) or if we use the restriction and induction functors of Definition 2.27 without restricting to the appropriate sub-categories.

The decomposition given by Lemma 2.35 allows us to define the following maps between centric Mackey functors over \( F \) with which we conclude this subsection.

**Definition 2.37.** Let \( M \) be a centric Mackey functor over \( F \) and let \( H \in F^c \). We define the following centric Mackey functor over \( F \)

\[
M_H := M \downarrow_{F_H}^F \uparrow_{F_H}^F.
\]

Moreover we define the Mackey functor morphisms

\[
\theta^H_M : M_H \rightarrow M, \quad \theta^M_H : M \rightarrow M_H,
\]

by setting for every \( y \otimes x \in M_H \), every \( K \in F^c \) and every \( a \in I^K_K M \)

\[
\theta^H_M (y \otimes x) := y \cdot x, \quad \theta^M_H (a) := \sum_{(A, \varphi) \in [H \times K]} I^K_{\varphi(A)} c_{\varphi} \otimes c_{\varphi^{-1}} R^K_{\varphi(A)} a.
\]

If there is no possible confusion regarding the centric Mackey functor \( M \) we will write \( \theta_H := \theta^H_M \) and \( \theta^M := \theta^M_H \).

**2.3 The centric Burnside ring over a fusion system.**

In this subsection we will recall the definition of the centric Burnside ring over a fusion system due to Diaz and Libman [3] and describe how it can be embedded in the center of the centric Mackey algebra. This result is analogous to the known embedding of the Burnside ring of a group \( G \) on the center of the Mackey algebra of \( G \).

**Definition 2.38.** ([3, Definition 2.11]) The centric Burnside ring of a fusion system \( F \) (denoted by \( B^{F^c}_F \)) is the Grothendieck group of the isomorphism classes of \( O(F^c) \), where addition is given by taking the disjoint union and multiplication is given by taking the isomorphism class of the product of two representatives. Given a commutative ring \( R \) we also define the centric Burnside ring of \( F \) on \( R \) as

\[
B^{F^c}_R := R \otimes_{\mathbb{Z}} B^{F^c}.
\]

An important distinction between the ring \( B^{F^c}_R \) and the Burnside ring of a group is that, in general, the isomorphism class \( \overline{S} \) of \( S \) is not the identity in \( B^{F^c}_R \). However, under some conditions on \( R \) we have the following
Proposition 2.39. ([9, Proposition 4.8]) If every integer prime other than $p$ is invertible in $\mathcal{R}$ then the isomorphism class $S$ of $S$ is invertible in $B^F_{\mathcal{R}}$.

This result motivates the following definition.

**Definition 2.40.** We say that a ring $\mathcal{R}$ is $p$ local if all integer primes other than $p$ are invertible in $\mathcal{R}$.

**Remark 2.41.** Notice how the definition of $p$ local ring does not specify if $p$ is invertible or not. This distinction will not be relevant towards the results shown in this paper. It is however worth noting that, if $\mathcal{R}$ is $p$ local, a field and $p$ is invertible (i.e. $\mathcal{R}$ is a field of characteristic 0), then arguments analogous to those shown in proof of [12, Theorem 9.1] can be used in order to prove that $\mu_{\mathcal{R}}(\mathcal{F})$ and $\mu_{\mathcal{R}}(\mathcal{F}^c)$ are semisimple $\mathcal{R}$-algebras. This does not necessarily happen when $\mathcal{R}$ is a field of characteristic $p$.

Finally we have the following result which generalizes from the case of Mackey functors over a group.

**Proposition 2.42.** There exists an embedding of the centric Burnside ring $B^F_{\mathcal{R}}$ onto the center of the centric Mackey algebra $\mu_{\mathcal{R}}(\mathcal{F}^c)$. Moreover, if $\mathcal{R}$ is $p$ local then this embedding can be taken to preserve the identity. In particular, if $\mathcal{R}$ is $p$ local, $B^F_{\mathcal{R}}$ acts on any centric Mackey functor.

**Proof.** The proof becomes a long but straightforward computation once one realizes that, analogously to the case of Mackey functors over groups, the mentioned embedding

$$f : B^F_{\mathcal{R}} \hookrightarrow Z(\mu_{\mathcal{R}}(\mathcal{F}^c)),$$

is given by setting for every isomorphism class $\underline{H} \in B^F_{\mathcal{R}}$ of a group $H \in \mathcal{F}^c$

$$f(\underline{H}) := \sum_{K \in \mathcal{F}^c} \sum_{(A, \varphi) \in [H \times K]} I^K_{\varphi(A)} R^K_{\varphi(A)},$$

and extending by $\mathcal{R}$-linearity. Here $\varphi \in \mathcal{F}$ is a representative of $\varphi$ seen as an isomorphism onto its image. \hfill $\square$

**Remark 2.43.** From the explicit definition of the embedding given in proof of Proposition 2.42 and from Definition 2.37 we can deduce that, for any $H \in O(\mathcal{F}^c)$ and any centric Mackey functor $M$ on $\mathcal{F}$, the action of the isomorphism class $\underline{H}$ of $H$ on $M$ can be written as

$$\underline{H} \cdot = \theta_H \theta^H.$$

**Notation 2.44.** From now on and unless other wise specified, given $H \in \mathcal{F}^c$ the symbol $\underline{H}$ will be denoting the element in $B^F_{\mathcal{R}}$ corresponding to the isomorphism class of $H$ (just like in Propositions 2.39 and 2.42).
3 Relative projectivity and the Higman’s criterion.

In this section we define the concept of relative projectivity for a centric Mackey functor over a fusion system. This concept is analogous to the one appearing in the case of Mackey functors over groups. Following this analogy we also prove that, if \( R \) is \( p \)-local, any centric Mackey functor admits a defect set (see Definition 3.3).

With this definition in mind we then focus our attention on the endomorphism ring of a centric Mackey functor \( M \). In this scenario we define the transfer and restriction maps between the endomorphisms rings \( \text{End}(M) \) and \( \text{End}(M \downarrow^F_H) \) for any \( H \in \mathcal{F}^c \). These definitions are once again analogous to the ones appearing in the case of Mackey functors over groups. Following once again this analogy we conclude this section by proving that the Higman’s criterion can be extended to centric Mackey functors over fusion systems (Theorem 3.11). As in the case of Mackey functors over groups the Higman’s criterion allows us to study relative projectivity of \( M \) via the ring \( \text{End}(M) \).

If \( R \) is a complete local and \( p \)-local PID this relation allows us to decompose any centric Mackey functor into a direct sum of centric Mackey functors whose defect set is generated by a single element (called vertex). Once again this result is analogous to the one we have in the case of Mackey functors over groups.

Let us start giving a definition of relative projectivity.

**Definition 3.1.** Let \( M \) be a centric Mackey functor and let \( \mathcal{X} \) be a family of \( \mathcal{F} \)-centric subgroups of \( S \). Using the notation of Definition 2.37 we define

\[
M_{\mathcal{X}} := \bigoplus_{H \in \mathcal{X}} M_H, \quad \theta_{\mathcal{X}} := \bigoplus_{H \in \mathcal{X}} \theta_H, \quad \theta^{\mathcal{X}} := \sum_{H \in \mathcal{X}} \theta^H.
\]

Moreover, we say that \( M \) is **projective relative to** \( \mathcal{X} \) (or **\( \mathcal{X} \)-projective**) if \( \theta^{\mathcal{X}} \) is split surjective. If \( \mathcal{X} = \{ H \} \) for some \( H \in \mathcal{F}^c \) we say that \( M \) is **projective relative to** \( H \) (or **\( H \)-projective**).

There is a key difference between this definition of relative projectivity and the one given in the case of Mackey functors over groups. In the later case we have that any Mackey functor \( M \) over a group \( G \) is projective relative to \( G \). This is because, in this scenario, \( M_G = M \) and \( \theta_G = \text{Id}_M \). However, if \( N \) is a centric Mackey functor over \( \mathcal{F} \), it is not always true that \( N_S = N \) and that \( \theta_S = \text{Id}_S \). In fact we might not even have that \( N \) is \( S \)-projective. To solve this we must restrict ourselves to the case where \( R \) is \( p \)-local.

**Lemma 3.2.** Let \( R \) be \( p \)-local and let \( M \) be a centric Mackey functor over \( \mathcal{F} \) on \( R \). Then \( M \) is **\( S \)-projective**.

**Proof.** Since \( R \) is \( p \)-local then we have, from Proposition 2.39, that the centric Burnside ring \( B^F_R \) contains an inverse of \( S \). Then, using the notation of Remark 2.43, we can define

\[
u := \theta^S \left( S^{-1} \right) : M \to M_S.
\]

13
From Proposition 2.42 and again Remark 2.43 we can now deduce that
\[ \theta_S u = (\theta_S \theta_S) (S^{-1} \cdot) = (S \cdot) (S^{-1} \cdot) = \text{Id}_M. \]
This proves that \( \theta_S \) is split surjective or, equivalently, that \( M \) is \( S \)-projective thus concluding the proof.

This last result assures us that, if \( R \) is \( p \) local, then, for any centric Mackey functor \( M \) there exists a family of \( \mathcal{F} \)-centric subgroups of \( S \) relative to which \( M \) is projective. Following arguments analogous to those described in [15, Section 3] we can now prove that for all families \( \mathcal{X} \) and \( \mathcal{Y} \) of \( \mathcal{F} \)-centric subgroups of \( S \):

1. If \( M \) is \( \mathcal{X} \)-projective then it is \( \mathcal{Y} \)-projective whenever \( \mathcal{X} \subseteq \mathcal{Y} \)
2. If \( \mathcal{X} \) and \( \mathcal{Y} \) are closed under \( \mathcal{F} \)-subconjugacy and \( M \) is both \( \mathcal{X} \)-projective and \( \mathcal{Y} \)-projective then it is \( \mathcal{X} \cap \mathcal{Y} \)-projective
3. If \( M \) is \( \mathcal{X} \)-projective then \( M \) is \( \mathcal{X}^{\text{max}} \)-projective where \( \mathcal{X}^{\text{max}} \) is a sub-set of maximal elements of \( \mathcal{X} \) (under the preorder \( \leq_{F} \) of Notation 2.3) taken up to \( \mathcal{F} \)-isomorphism.

From these results it is straightforward to prove the existence and uniqueness of a minimal family of \( \mathcal{F} \)-centric subgroups of \( S \) that is closed under \( \mathcal{F} \)-subconjugacy and relative to which \( M \) is projective. This leads us to the following

**Definition 3.3.** Let \( R \) be \( p \) local and let \( M \) be a centric Mackey functor over \( \mathcal{F} \) on \( R \). We define the **defect set of** \( M \) (and denote it by \( \mathcal{X}_M \)) as the minimal family of \( \mathcal{F} \)-centric subgroups of \( S \) closed under \( \mathcal{F} \)-subconjugacy and relative to which \( M \) is projective. We call **defect group** of \( M \) any maximal element of \( \mathcal{X}_M \) under the preorder \( \leq_{F} \) (see Notation 2.3). If all defect groups are \( \mathcal{F} \)-conjugate we say that \( M \) admits a **vertex** and we call **vertex of** \( M \) (and denote it by \( V_M \)) any fully \( \mathcal{F} \)-normalized defect group of \( M \).

We would now like to be able to use the defect set of a centric Mackey functor in order to obtain a decomposition of such Mackey functor. More precisely, given a centric Mackey functor \( M \), we would like to obtain a decomposition of \( M \) into a direct sum of centric Mackey functors each of which admits a vertex that lies in the defect set of \( M \). It is known that, whenever \( R \) is a complete local \( PID \), such a result arises in the case of Mackey functors over a group. The rest of this section will be dedicated to proving that, in the case of centric Mackey functors over a fusion system we also have such a result as long as \( R \) is a complete local and \( p \) local \( PID \).

Let’s start by simplifying the problem. Let \( R \) be a complete local and \( p \) local \( PID \) and let \( M \) be a centric Mackey functor over \( \mathcal{F} \). From Remark 2.23 we can deduce that \( \mu_R (\mathcal{F}) \) (and therefore \( \mu_R (\mathcal{F}^c) \)) are finitely generated as \( R \)-modules. Since \( R \) is either
a field or a complete DVR (because it’s a complete local PID) then we can apply the Krull-Schmidt-Azumaya theorem (see [2, Theorem 6.12 (ii)]) to conclude that exists a unique (up to isomorphism) decomposition of $M$ as a direct sum of indecomposable centric Mackey functors

$$
\bigoplus_{i=1}^{n} N_i \cong M.
$$

This decomposition leads in turn to a decomposition of the morphism $\theta_{\mathcal{X}_M}^M$ as

$$
\theta_{\mathcal{X}_M}^M = \bigoplus_{i=1}^{n} \theta_{\mathcal{X}_M}^{N_i} : \bigoplus_{i=1}^{n} (N_i)_{\mathcal{X}_M} \to \bigoplus_{i=1}^{n} N_i.
$$

Here we are using the fact that induction and restriction commute with direct sum which follows from Remark 2.36 and the RAPL theorem. Moreover the morphisms $\theta_{\mathcal{X}_M}^{N_i}$ in the previous decomposition satisfy, by construction, that

$$
\theta_{\mathcal{X}_M}^{N_i} ((N_i)_{\mathcal{X}_M}) \subseteq N_i,
$$

for every $i = 1, \ldots, n$. Since $\theta_{\mathcal{X}_M}^M$ is split surjective by definition of $\mathcal{X}_M$ then we can conclude that each $\theta_{\mathcal{X}_M}^{N_i}$ is also split surjective. In other words each $N_i$ is projective relative to $\mathcal{X}_M$. By definition of defect set we can conclude that $\mathcal{X}_{N_i} \subseteq \mathcal{X}_M$. Therefore all we need to do is proving that every indecomposable Mackey functor admits a vertex since the previous statement ensures us that such vertex will be in $\mathcal{X}_M$.

To prove that we will start by defining what it means for an endomorphism of a Mackey functor $M$ to be projective relative to a family $\mathcal{X}$ of $F$-centric subgroups of $S$. We will then prove that if $\text{Id}_M$ is a local idempotent and it is projective relative to $\mathcal{X}$ then it is projective relative to a single subgroup of $\mathcal{X}$ (Remark 3.10). From [2, Proposition 6.10 (ii)] we know that $\text{Id}_M$ is local if and only if $M$ is indecomposable. Thus we can reinterpret the previous statement by saying that if a Mackey functor $M$ is indecomposable then $\text{Id}_M$ is projective relative to a single $F$-centric subgroup $W_M$ of $S$. Finally we will prove a generalization of Higman’s criterion (Theorem 3.11) which will tell us that $\text{Id}_M$ is $W_M$-projective if and only if $M$ is $W_M$-projective. This will prove that $W_M = V_M$ is in fact the vertex of $M$.

To sum up we only need to define relative projectivity for endomorphisms, and prove Remark 3.10 and Theorem 3.11. Once we did this we will have obtained a decomposition of a Mackey functor via its defect set.

To define relative projectivity of an endomorphism first we need to define the transfer map. The definitions below are analogous to those given for Mackey functors over groups.

**Definition 3.4.** Let $M$ be a centric Mackey functor over $\mathcal{F}$ and let $H \in \mathcal{F}^c$. We define the **transfer map from** $\mathcal{F}_H$ **to** $\mathcal{F}$ **on** $M$ as the map

$$
M_{\text{tr}}^{\mathcal{F}}_{\mathcal{F}_H} : \text{End} \left( M \downarrow_{\mathcal{F}_H}^{\mathcal{F}} \right) \to \text{End} (M),
$$

15
obtained by setting for every $f \in \text{End} \left( M \downarrow_{\mathcal{F}_H}^F \right)$, every $K \in \mathcal{F}^c$ and every $x \in I^K_K M$

$$M \text{tr}_{\mathcal{F}_H}^F (f)(x) := \sum_{(A, \varphi) \in [H \times K]} (I^K_K c_{\phi} f c_{\phi^{-1}} R^K_{\phi(A)}) (x).$$

Dually, given any fusion system $\mathcal{G} \subseteq \mathcal{F}$ we define the restriction map from $\mathcal{F}$ to $\mathcal{G}$ on $M$ as the $\mathcal{R}$-algebra map

$$M \text{r}_{\mathcal{G}}^\mathcal{F} : \text{End} \left( M \right) \to \text{End} \left( M \downarrow_{\mathcal{G}}^\mathcal{F} \right),$$

that sends every $g \in \text{End} \left( M \right)$ to

$$M \text{r}_{\mathcal{G}}^\mathcal{F} (g) := g \downarrow_{\mathcal{G}}^\mathcal{F} : = \downarrow_{\mathcal{G}}^\mathcal{F} (g).$$

Finally, given $\varphi \in \text{Hom}_\mathcal{F} (H, \varphi (H))$ we define the conjugation map from $\mathcal{F}_H$ to $\mathcal{F}_{\varphi (H)}$ on $M$ as the map

$$M c_{\varphi} : \text{End} \left( M \downarrow_{\mathcal{F}_H}^F \right) \to \text{End} \left( M \downarrow_{\mathcal{F}_{\varphi (H)}}^F \right),$$

obtained by setting for every $f \in \text{End} \left( M \downarrow_{\mathcal{F}_H}^F \right)$, every $K \in \mathcal{F}^c$ with $K \leq \varphi (H)$ and every $x \in I^K_K M \downarrow_{\mathcal{F}_{\varphi (H)}}^F$

$$(M c_{\varphi} (f))(x) := \varphi f (x) := (c_{\varphi} f c_{\varphi^{-1}}) x.$$ 

If the Mackey functor $M$ is clear we simply write

$$\text{tr}_{\mathcal{F}_H}^\mathcal{F} := M \text{tr}_{\mathcal{F}_H}^\mathcal{F}, \quad \text{r}_{\mathcal{G}}^\mathcal{F} := M \text{r}_{\mathcal{G}}^\mathcal{F}, \quad c_{\varphi} := M c_{\varphi}.$$ 

The following properties of the transfer, restriction and conjugation maps follow immediately from definition and the computations are analogous to those given for the case of Mackey functors over groups and will therefore be omitted.

**Proposition 3.5.** Let $M$ be a centric Mackey functor over $\mathcal{F}$. The following are satisfied.

1. For every $H \in \mathcal{F}^c$ and $h \in H$ we have that

$$\text{tr}_{\mathcal{F}_H}^\mathcal{F} = \text{r}_{\mathcal{F}_H}^\mathcal{F} = c_h = \text{Id}_{\text{End} \left( M \downarrow_{\mathcal{F}_H}^F \right)}.$$ 

2. For every $H \leq K \in \mathcal{F}^c$ we have

$$\text{r}_{\mathcal{F}_H}^\mathcal{F} \text{r}_{\mathcal{K}_H}^\mathcal{F} = \text{r}_{\mathcal{F}_K}^\mathcal{F}.$$ 

3. For every $H \leq K \in \mathcal{F}^c$ we have

$$\text{tr}_{\mathcal{F}_K}^\mathcal{F} \text{tr}_{\mathcal{F}_H}^\mathcal{F} = \text{tr}_{\mathcal{F}_H}^\mathcal{F}.$$
4. For all isomorphisms $\varphi, \psi \in \mathcal{F}^c$ that can be composed we have
\[ c_\psi c_\varphi = c_{\psi \varphi}. \]

5. For every $H \leq K \in \mathcal{F}^c$ and isomorphism $\varphi \in \text{Hom}_\mathcal{F}(K, \varphi(K))$ we have
\[ c_\varphi \text{tr}_{\mathcal{F}^c}^F(K) = \text{tr}_{\mathcal{F}^c}^F(\varphi(K))c_\varphi. \]

6. For every $H \leq K \in \mathcal{F}^c$ and isomorphism $\varphi \in \text{Hom}_\mathcal{F}(K, \varphi(K))$ we have
\[ c_\varphi \text{tr}_{\mathcal{F}^c}^F(K) = \text{tr}_{\mathcal{F}^c}^F(\varphi(K))c_\varphi. \]

7. For every $K, H \in \mathcal{F}^c$ we have
\[ r_{\mathcal{F}^c}^F(\text{tr}_{\mathcal{F}^c}^F(K)) = \sum_{(A, \varphi) \in [H \times K]} \text{tr}_{\mathcal{F}^c}^F(A) c_\varphi. \]

8. For every $H \in \mathcal{F}^c$, every $f \in \text{End}(M)$ and every $g \in \text{End}(M \downarrow_{\mathcal{F}^c}^F H)$ we have
\[ f \text{tr}_{\mathcal{F}^c}^F(g) = \text{tr}_{\mathcal{F}^c}^F(f \varphi g), \quad \text{tr}_{\mathcal{F}^c}^F(g) f = \text{tr}_{\mathcal{F}^c}^F(g \varphi f). \]

9. For every $H \in \mathcal{F}^c$ we have
\[ \text{tr}_{\mathcal{F}^c}^F(\text{tr}_{\mathcal{F}^c}^F(H)) = \text{Hom}(M, -(\overline{H})). \]

Remark 3.6. Property 3 is not in general satisfied if we replace $\mathcal{F}_K$ with another fusion system $\mathcal{F}_H \subseteq K \subseteq \mathcal{F}$. We have however something similar when $\mathcal{K} = N_\mathcal{F}(H)$ (see Proposition 4.15).

Definition 3.7. Let $M$ be a centric Mackey functor $M$ and let $H \in \mathcal{F}^c$. We define
\[ ^M \text{tr}^\mathcal{F}_H := \text{tr}^\mathcal{F}_H(M \downarrow_{\mathcal{F}^c}^F H). \]

Moreover, given a family $\mathcal{X}$ of $\mathcal{F}$-centric subgroups of $S$, we define
\[ ^M \text{tr}^\mathcal{F}_\mathcal{X} := \sum_{H \in \mathcal{X}} ^M \text{tr}^\mathcal{F}_H. \]

Whenever there is no possible confusion regarding the centric Mackey functor $M$ we will just write
\[ \text{tr}^\mathcal{F}_H := ^M \text{tr}^\mathcal{F}_H, \quad \text{tr}^\mathcal{F}_\mathcal{X} := ^M \text{tr}^\mathcal{F}_\mathcal{X}. \]

Remark 3.8. Notice how Point 8 of Proposition 3.5 tells us that both $\text{tr}^\mathcal{F}_H$ and $\text{tr}^\mathcal{F}_\mathcal{X}$ are two sided ideals of $\text{End}(M)$. 

17
With this setup we can finally define what it means for an endomorphism to be projective relative to a family of \( \mathcal{F} \)-centric subgroups of \( S \).

**Definition 3.9.** Let \( M \) be a centric Mackey functor over \( \mathcal{F} \), let \( f \) be an endomorphism of \( M \) and let \( \mathcal{X} \) be a family of \( \mathcal{F} \)-centric subgroups of \( S \). We say that \( f \) is **projective relative to** \( \mathcal{X} \) (or **\( \mathcal{X} \)-projective**) if \( f \in \text{tr}_F^{\mathcal{X}} \).

**Remark 3.10.** Assume that \( \text{Id}_M \) is projective relative to \( \mathcal{X} \). Since \( \text{tr}_F^{\mathcal{X}} \) is a two sided ideal of \( \text{End} (M) \) then we will have for every endomorphism \( f \in \text{End} (M) \) that
\[
f = f \text{Id}_M \in \text{tr}_F^{\mathcal{X}}.
\]
Therefore, \( f \) will also be \( \mathcal{X} \)-projective. Take now any local idempotent \( f \in \text{End} (M) \) and assume it is projective relative to \( \mathcal{X} \). Applying Point 8 of Proposition 3.5 we can conclude that
\[
f \text{End} (M) f = f \text{tr}_F^{\mathcal{X}} f = \sum_{H \in \mathcal{X}} f \text{tr}_H^{\mathcal{X}} f,
\]
where the last identity comes from definition. Since \( f \text{End} (M) f \) is a local ring by construction of \( f \) then we can deduce that exists at least one \( H \in \mathcal{X} \) such that
\[
f \text{End} (M) f = f \text{tr}_H^{\mathcal{X}} f,
\]
in particular \( f \) is \( H \)-projective.

Remark 3.10 proves that any local idempotent projective relative to a family \( \mathcal{X} \) is also projective relative to a single element in \( \mathcal{X} \). From our previous discussion we are now only left with proving that an indecomposable centric Mackey functor \( M \) is projective relative to \( H \) if and only if \( \text{Id}_M \) is projective relative to \( H \). As in the case of Mackey functors over groups this result is provided by a generalization of the Higman’s criterion which we report below.

**Theorem 3.11.** (Higman’s criterion) Let \( M \) be an indecomposable centric Mackey functor over \( \mathcal{F} \) and let \( H \in \mathcal{F}^c \). Then, the following are equivalent:

1. There is an \( \mathcal{F} \)-centric Mackey functor \( N \) over \( \mathcal{F}_H \) such that \( M \) is a direct summand of \( N \uparrow_{\mathcal{F}_H} \).
2. \( \text{Id}_M \) is \( H \)-projective.
3. \( \text{End} (M) = \text{tr}_H^{\mathcal{F}} \).
4. \( \theta_H \) is an epimorphism. Moreover, given centric Mackey functors \( N, L \) over \( \mathcal{F} \) and \( \mu_{\mathcal{F}} (\mathcal{F}^c) \)-module morphisms \( \varphi : N \rightarrow L \) and \( \psi : M \rightarrow L \) with \( \varphi \) surjective, if exists \( f : M \downarrow_{\mathcal{F}_H} \rightarrow N \downarrow_{\mathcal{F}_H} \) making the following diagram commute

\[
\begin{array}{c}
M \downarrow_{\mathcal{F}_H} \downarrow f \downarrow & \psi \downarrow_{\mathcal{F}_H} \downarrow \downarrow \downarrow L \downarrow_{\mathcal{F}_H} \downarrow 0 \\
N \downarrow_{\mathcal{F}_H} \downarrow \downarrow \downarrow & \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow 0
\end{array}
\]
then exists \( \hat{f} : M \to N \) such that \( \varphi \hat{f} = \psi \).

5. \( \theta^H \) is a monomorphism. Moreover, given centric Mackey functors \( N, L \) over \( \mathcal{F} \) and \( \mu_H(\mathcal{F}^c) \)-module morphisms \( \varphi : L \hookrightarrow N \) and \( \psi : L \to M \) with \( \varphi \) injective, if exists \( f : N \downarrow_{\mathcal{F}^H} \to M \downarrow_{\mathcal{F}^H} \) making the following diagram commute

\[
\begin{array}{ccc}
0 & \rightarrow & \psi \downarrow_{\mathcal{F}^H} \\
\downarrow & & \downarrow \psi \\
L \downarrow_{\mathcal{F}^H} & \rightarrow & N \downarrow_{\mathcal{F}^H} \\
\downarrow f & & \downarrow f \\
M \downarrow_{\mathcal{F}^H} & \rightarrow & \end{array}
\]

then there exists \( \hat{f} : N \to M \) such that \( \hat{f} \varphi = \psi \).

6. \( \theta^H \) is an epimorphism. Moreover, given a centric Mackey functor \( N \) over \( \mathcal{F} \) and an epimorphism \( \varphi : N \to M \), if \( \varphi \downarrow_{\mathcal{F}^H} \) splits then \( \varphi \) splits.

7. \( \theta^H \) is an monomorphism. Moreover, given a centric Mackey functor \( N \) over \( \mathcal{F} \) and a monomorphism \( \varphi : M \hookrightarrow N \), if \( \varphi \downarrow_{\mathcal{F}^H} \) splits then \( \varphi \) splits.

8. \( \theta^H \) is split surjective (equivalently \( M \) is \( H \)-projective).

9. \( \theta^H \) is split injective.

10. \( M \) is a direct summand of \( M^H \).

**Proof.** The proof of this generalization of the Higman’s criterion is analogous to that of [7, Theorem 2.2]. There is however one detail that makes the generalization harder. In [7] Hiroshi uses the fact that, given a Mackey functor \( N \) over a finite group \( H \) and given another finite group \( G \geq H \) then there is a decomposition of the form

\[
N \uparrow_{\mathcal{F}^H}^G = \bigoplus_{x \in [H \backslash G / H]} \left( \mathcal{F} \left( N \downarrow_{H \cap G \cap H}^H \right) \right) \uparrow_{H \cap G \cap H}^H .
\]

More precisely, denoting by \( \pi_N \) the endomorphism of \( N \uparrow_{\mathcal{F}^H}^G \) corresponding to the projection onto

\[
\left( \mathcal{F} \left( N \downarrow_{H \cap G \cap H}^H \right) \right) \uparrow_{H \cap G \cap H}^H \cong N
\]

Hiroshi uses the identity

\[
tr_H^G (\pi_N) = \text{Id}_{N^H}.
\]

In the case of Mackey functor over fusion systems these results are replaced by Lemma 4.20 and Corollary 4.21 respectively. With these analogous results the same arguments used in [7, Theorem 2.2] work here.

\[\square\]
4 Green correspondence.

In this section we will prove the main result of this paper. Namely we will prove Theorem 4.16 which is a generalization of the Green correspondence to centric Mackey functors over fusion systems.

We will start on Subsection 4.1 by proving Proposition 4.7 which is a ring theoretical version of [5, Proposition 4.34]. We will then continue in Subsection 4.2 where we will state without proof 4 different results which we will use together with Proposition 4.7 in order to prove Theorem 4.16. The remaining 4 subsections will then be dedicated to proving the 4 results that are only stated in Subsection 4.2.

4.1 Correspondence of endomorphisms.

The goal of this subsection is that of stating and proving Proposition 4.7 which is a ring theoretical version of [5, Proposition 4.34]. This result will become in Subsection 4.2 one of the key-stones for proving Theorem 4.16.

Let’s start with some notation.

**Definition 4.1.** Let $A$ and $B$ be rings (non necessarily having a unit) and let $f : A \to B$ be a surjective ring morphism. We say that $f$ is a near isomorphism if and only if

$$A \cdot \ker(f) = \ker(f) \cdot A = 0.$$

Let us give some examples of near isomorphisms that will be of use later on.

**Example 4.2.** Any isomorphism of rings $f : A \to B$ is a near isomorphism since $\ker(f) = 0$. Conversely if $A$ has a unit and $f$ is a near isomorphism then $f$ is an isomorphism.

**Example 4.3.** Let $A$ be a ring and let $I$ and $J$ be two sided ideals of $A$ satisfying

$$JA, AJ \subseteq I \subseteq J.$$

Then the natural projection map $\pi : A/I \to A/J$ is a near isomorphism. This is immediate from definition since $\pi$ is surjective (because $I \subseteq J$) and $\ker(\pi) = J$. Where $J$ is the image of $J$ via the projection map from $A$ to $A/I$.

**Example 4.4.** Let $A, B$ and $C$ be rings and let $f : A \to B$ and $g : B \to C$ be ring homomorphisms. If $gf$ is a near isomorphism and $f$ is surjective then both $f$ and $g$ are near isomorphisms. To prove this start by noticing that, since $\ker(f) \subseteq \ker(gf)$, then

$$A \ker(f) \subseteq A \ker(gf) = 0, \quad \ker(f) A \subseteq \ker(gf) A = 0.$$

Since $f$ is surjective by hypothesis this proves that $f$ is a near isomorphism.
On the other hand, since $f$ is surjective, then, for every $c \in \ker (g) \subseteq B$ we can take $a_c \in A$ such that $f(a_c) = c$ and we will have $a_c \in \ker (gf)$. Take now any $b \in B$ and define $a_b$ as before. Since $gf$ is a near isomorphism the we have that

$$bc = f(a_b)f(a_c) = f(abac) = f(0) = 0.$$  

This proves that $B \cdot \ker (g) = 0$. Analogously we can prove that $\ker (g) \cdot B = 0$ thus proving that $g$ is a near isomorphism and concluding the proof.

**Example 4.5.** Let $A, B$ and $C$ be rings, let $f : A \to B$ be a near isomorphism and let $g : B \hookrightarrow C$ be an isomorphism then $gf$ is a near isomorphism. To prove this start by noticing that, since both $f$ and $g$ are surjective, then $gf$ is also surjective. Moreover, since $g$ is an isomorphism we have that $\ker (gf) = \ker (f)$. The result now follows from the fact that $f$ is a near isomorphism.

The importance of near isomorphisms comes from the following well known lemma which we state without proving.

**Lemma 4.6.** ([5, Lemma 4.22]) Let $A$ and $B$ be $\mathcal{R}$-algebras and let $f : A \to B$ be a near isomorphism. Denote by $E(A)$ and $E(B)$ the sets of idempotents of $A$ and $B$ respectively. Then the following are satisfied

1. $f$ induces an isomorphism from $E(A)$ to $E(B)$.
2. Given a local idempotent $x \in E(A)$ then $f(x) \in E(B)$ is also a local idempotent.
3. Two idempotents $x$ and $y$ in $E(A)$ are conjugate to one another if and only if $f(x)$ and $f(y)$ are also conjugate.

With this in mind we can now prove a ring theoretical version of [5, Proposition 4.34].

**Proposition 4.7.** Let $A$ and $B$ be $\mathcal{R}$-algebras, let $C$ be a two sided ideal of $A$ inheriting it’s $\mathcal{R}$-algebra structure, let $I$, $J$ and $K$ be two sided ideals of $C$, $A$ and $B$ respectively and let $f : C \to B$ and $g : B \to C + J$ be $\mathcal{R}$-linear maps. Assume that the following are satisfied:

1. $(C \cap J)C, C(C \cap J) \subseteq I \subseteq C \cap J$,
2. $g(K) \subseteq J$,
3. $f(I) \subseteq K$,
4. $f$ is surjective.
5. $g$ sends idempotents to idempotents.
6. The $\mathcal{R}$-linear maps $\overline{f} : C/I \to B/K$ and $\overline{g} : B/K \to (C + J)/J$ are actually $\mathcal{R}$-algebra morphisms.
7. The natural isomorphism \( s : C/(C \cap J) \to (C + J)/J \) and the natural projection \( q : C/I \to C/(C \cap J) \) satisfy \( sq = \overline{f} \).

8. For every idempotent \( x \in A \) there exists a unique (up to conjugation) decomposition of \( x \) as a finite sum of orthogonal local idempotents.

Then for every local idempotent \( b \in B - K \) (i.e. \( b \in B \) but \( b \notin K \)) there is exactly one local idempotent \( a \in A \) in the decomposition of \( g(b) \in A \) as finite sum of orthogonal local idempotents such that \( a \in C - J \). Moreover we have the following equivalences

\[
    f(a) \equiv b \mod K, \quad g(b) \equiv a \mod J.
\]

**Proof.** Take \( b \) as in the statement. Since \( g \) preserves idempotents we can conclude that \( g(b) \) is an idempotent of \( A \). From Condition 8 we obtain a decomposition of \( g(b) \) as a sum of orthogonal local idempotents

\[
    g(b) = \sum_{i=0}^{n} a_i.
\]

Notice that, since both \( C \) and \( J \) are two sided ideals of \( A \) then \( C + J \) is also a two sided ideal of \( A \). Since all the \( a_i \) are pairwise orthogonal then, for every \( i = 0, \ldots, n \) we have that

\[
    a_i = a_i g(b) \in C + J.
\]

Since \( C + J \) is an ideal of \( A \) we can use the previous equation to deduce that

\[
    a_i (C + J) a_i \subseteq a_i A a_i = a_i (a_i A a_i) a_i \subseteq a_i (C + J) a_i,
\]

and, therefore

\[
    a_i (C + J) a_i = a_i A a_i.
\]

By definition of local idempotent we can then conclude that all the \( a_i \) are local idempotents of \( C + J \) and not just of \( A \).

Notice now that the projection of a local idempotent onto a quotient ring is either a local idempotent of the projection ring or 0. Since \( b \notin K \) by construction then we can conclude that its projection \( \overline{b} \) onto \( B/K \) is a local idempotent. Likewise, for every \( i = 0, \ldots, n \), we have that the projection \( \overline{a_i} \) of \( a_i \) onto \( (C + J)/J \) is either 0 or a local idempotent of \( (C + J)/J \). From Examples 4.2, 4.3, 4.4 and 4.5 we can now deduce that \( \overline{f}, \overline{g}, q, s \) and their compositions are all near isomorphisms. In particular, from Lemma 4.6 we have that \( \overline{g} \) preserves local idempotents. Therefore, since \( \overline{b} \) is a local idempotent, we have that \( \overline{g(b)} = \sum_{i=0}^{n} \overline{a_i} \) is a local idempotent. Since local idempotents are primitive we can conclude that exists exactly one \( i \in \{0, \ldots, n\} \) such that \( \overline{a_i} \neq 0 \). We can assume without loss of generality that such \( i \) is 0 and we will have

\[
    \overline{a_i} = \begin{cases} 
    \overline{g(b)} & \text{if } i = 0 \\
    0 & \text{else}
    \end{cases}
\]
In other words for every \( i = 1, \ldots, n \) we have that \( a_i \in J \) while

\[
g(b) \equiv a_0 \mod J.
\]

Since both \( C \) and \( J \) are two sided ideals of \( A \) then we can also conclude that \( a_0Ca_0 \) and \( a_0Ja_0 \) are two sided ideals of \( a_0(C + J)a_0 \). Moreover, since \( a_0 \) is a local idempotent of \( C + J \) then we have that \( a_0(C + J)a_0 \) is a local ring by definition. Since

\[
a_0Ca_0 + a_0Ja_0 = a_0(C + J)a_0,
\]

then we can conclude that either

\[
a_0(C + J)a_0 = a_0Ca_0 \subseteq C,
\]

or

\[
a_0(C + J)a_0 = a_0Ja_0 \subseteq J,
\]

or both. Since \( a_0 \notin J \) by construction and \( a_0 \in a_0(C + J)a_0 \) then we can deduce that

\[
a_0(C + J)a_0 \neq a_0Ja_0,
\]

and, therefore

\[
a_0(C + J)a_0 = a_0Ca_0.
\]

In particular we have that \( a_0 \in C \). By taking \( a := a_0 \) we have proved the first half of the statement and the second equivalence. We are therefore only left with proving the first equivalence.

Since \( C \) is an ideal of \( A \) we have once again that \( a \) is a local idempotent of \( C \). Denote now by \( \overline{a} \) the projection of \( a \) into \( C/I \). If we now join the equation \( sq = \overline{g} \overline{f} \) with the previous discussion we deduce that

\[
\overline{g} \overline{f} (\overline{a}) = sq (\overline{a}) = \overline{a} = \overline{g} (\overline{b}).
\]

Since \( \overline{a} \) is a local idempotent of \( (C + J)/J \) as previously stated then, from Lemma 4.6, and the previous identity we can conclude that

\[
\overline{f} (\overline{a}) = \overline{b}.
\]

Equivalently, since \( f(I) \subseteq K \), we have that

\[
f(a) \equiv b \mod K.
\]

Thus concluding the proof.

\[\square\]

Remark 4.8. Notice how, as mentioned in the proof, the second equivalence is telling us that all idempotents of the decomposition of \( g(b) \) other than \( a \) lie in \( J \).

Let’s conclude this subsection by giving an example where Proposition 4.7 can be applied in order to prove [5, Proposition 4.34].

23
Example 4.9. Let $\mathcal{R}$ be a complete local PID, let $G$ be a finite group and let $M$ be a Green functor over $G$ on $\mathcal{R}$ (see first definition of [5, Subsection 1.3]). Let also $D$ and $H$ be subgroups of $G$ such that $N_G(D) \leq H$. Then we can apply Proposition 4.7 by setting

$$A := \text{End} \left( M \downarrow_{D}^{G} \right),$$
$$B := \text{tr}_{D}^{G} \left( \text{End} \left( M \downarrow_{D}^{G} \right) \right),$$
$$C := \text{tr}_{D}^{H} \left( \text{End} \left( M \downarrow_{D}^{G} \right) \right),$$
$$K := \sum_{x \in G - H} \text{tr}_{D \cap D}^{G} \left( \text{End} \left( M \downarrow_{D \cap D}^{G} \right) \right),$$
$$I := \sum_{x \in G - H} \text{tr}_{D \cap D}^{H} \left( \text{End} \left( M \downarrow_{D \cap D}^{G} \right) \right),$$
$$J := \sum_{x \in G - H} \text{tr}_{D \cap D}^{H} \left( \text{End} \left( M \downarrow_{D \cap D}^{G} \right) \right),$$
$$f := \text{tr}_{G}^{H},$$
$$g := \text{tr}_{G}^{H}.$$

With this setup then [5, Proposition 4.34] follows from Proposition 4.7 and the first remark after [5, Hypothesis 4.31].

4.2 Conditional proof of the Green correspondence.

In [11, Proposition 3.1] Sasaki uses [5, Proposition 4.34] in order to extend the Green correspondence to Mackey functors over groups. In this subsection we will follow similar ideas and use Proposition 4.7 in order to extend the Green correspondence to centric Mackey functors over fusion systems (Theorem 4.16). Unfortunately the same arguments used by Sasaki do not translate easily to centric Mackey functors over fusion systems and in order to prove Theorem 4.16 we will need to get 4 further results which we will state now and prove in the following subsections. Hence the “conditional” part in this subsection’s title.

Notation 4.10. From now on and unless otherwise specified $H$ will denote a fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric subgroup of $S$ and we will denote

$$\mathcal{X} := \{ K \trianglelefteq H : K \in \mathcal{F}^{e} \},$$
$$\mathcal{Y} := \{ K \leq_{\mathcal{F}} H : K \leq N_{S}(H), K \in \mathcal{F}^{e} \text{ and } K \neq H \}.$$

Remark 4.11. Notice that

$$\mathcal{X} = \{ K \in \mathcal{Y} : K \leq H \}.$$

Proposition 4.12. Let $A, B \in \mathcal{F}^{e}$ and let $N$ be an $\mathcal{F}$-centric Mackey functor over $\mathcal{F}_{A}$. Then, for every $(C, \theta) \in [A \times B]$ there exists an $\mathcal{F}$-centric Mackey functor $N_{(C, \theta)}$ over $\mathcal{F}_{\theta(C)}$ such that

$$N \uparrow_{\mathcal{F}_{A}}^{\mathcal{F}_{B}} \downarrow_{\mathcal{F}_{\theta(C)}}^{\mathcal{F}_{B}} = \bigoplus_{(C, \theta) \in [A \times B]} N_{(C, \theta)} \uparrow_{\mathcal{F}_{\theta(C)}}^{\mathcal{F}_{B}}.$$

Moreover let $M$ be an $H$-projective, $\mathcal{F}$-centric Mackey functor over $N_{\mathcal{F}}$ on $\mathcal{R}$. If $\mathcal{R}$ is a complete local and $p$-local PID then we also have that

$$M \uparrow_{N_{\mathcal{F}}}^{\mathcal{F}} \downarrow_{N_{\mathcal{F}}}^{\mathcal{F}} = M \oplus M'$$

for some $\mathcal{Y}$-projective, $\mathcal{F}$-centric Mackey functor $M'$ over $N_{\mathcal{F}}$. 24
Proposition 4.13. Let \( M \) be a centric Mackey functor over \( \mathcal{F} \) on \( \mathcal{R} \) and let \( f \) be an endomorphism of \( M \downarrow_{\mathcal{F}_H} \). If \( \mathcal{R} \) is \( p \)-local then, for every \( K \in \mathcal{Y} \) exists an endomorphism \( f_K \in \text{End}(M \downarrow_{\mathcal{F}_K}) \) such that

\[
\tau_{M}^F \tau_{N_S}^F (f) = \tau_{N_S}^F (f) + \sum_{K \in \mathcal{Y}} \tau_{N_S}^F (f_K).
\]

More precisely, for every \( K \in \mathcal{Y} \) we can take

\[
f_K := \sum_{(A, \varphi) \in [H \times N_S(H)]} \tau_{N_S}^F \left(N_S(H)^{-1}\right) \left(c_{\varphi} \tau_{N_S}^F \right) (f),
\]

where \( \varphi \in \mathcal{F} \) is any representative of \( \varphi \) seen as an isomorphism onto its image, we are using the notation of Remark 2.43 when referring to the endomorphism \( \tau_{N_S}^F \) and we are regarding \( N_S(H) \) as an element of \( B^{\mathcal{F}_K} \).

Theorem 4.14. Let \( K \in \mathcal{F}_c \). There exists a subset \([N_S \times \mathcal{F}_K] \subseteq [N_S(H) \times \mathcal{F}_K] \) such that for every \( (A, \varphi) \in [N_S \times \mathcal{F}_K] \) there is a subgroup \( N_S^N \leq N_S(H) \) such that \( A \leq N_S^N \) and exists a (necessarily unique) morphism \( \varphi \in \text{Hom}_{O(\mathcal{F}_c)}(N_S^N, K) \) satisfying that \( \varphi|_A = \varphi \).

Moreover we have the following isomorphism of sets

\[
\bigsqcup_{(A, \varphi) \in [N_S \times \mathcal{F}_K]} [H \times N_S^N] \cong [H \times \mathcal{F}_K]
\]

which is realized by a bijective map \( \Upsilon \) obtained by sending any \( (B, \psi) \in [H \times N_S^N] \) to \( (B^h, \psi \psi c_h) \in [H \times \mathcal{F}_K] \) for some appropriate \( h \in H \). With an appropriate choice of the elements in \([H \times \mathcal{F}_K]\) we can take each such \( h \) to be the identity element.

Proposition 4.15. Let \( \mathcal{R} \) be \( p \)-local and let \( M \) be a centric Mackey functor over \( \mathcal{F} \) on \( \mathcal{R} \). Then the map

\[
\tau_{N_S}^F : \tau_{N_S}^F \rightarrow \tau_{N_S}^F
\]

defined by setting \( \tau_{N_S}^F (\tau_{N_S}^F (f)) := \tau_{N_S}^F (f) \).

for every \( f \in \text{End}(M \downarrow_{\mathcal{F}_H}) \) is a well defined \( \mathcal{R} \)-module morphism. Moreover the quotient map

\[
\tau_{N_S}^F : \tau_{N_S}^F / \tau_{N_S}^F \rightarrow \tau_{N_S}^F / \tau_{N_S}^F
\]

is a well defined surjective morphism of \( \mathcal{R} \)-algebras.

With these results in mind we can now finally prove the existence of a Green correspondence for centric Mackey functors over fusion systems.
Theorem 4.16. (Green correspondence) Let $\mathcal{R}$ be a complete local and $p$ local PID, let $M$ be an indecomposable centric Mackey functor over $\mathcal{F}$ on $\mathcal{R}$ with vertex $H$ and let $N$ be an indecomposable $\mathcal{F}$-centric Mackey functor over $N_\mathcal{F}$ (see example 2.9 and notation 2.10) on $\mathcal{R}$ with vertex $H$. For any decomposition

$$M \downarrow_{N_\mathcal{F}}^\mathcal{F} := \bigoplus_{i=0}^n M_i, \quad N \uparrow_{N_\mathcal{F}}^\mathcal{F} := \bigoplus_{j=0}^m N_j,$$

of $M \downarrow_{N_\mathcal{F}}^\mathcal{F}$ and $N \uparrow_{N_\mathcal{F}}^\mathcal{F}$ into direct sums of indecomposable Mackey functors there exist a unique $i \in \{0, \ldots, n\}$ such that $M_i$ has vertex $H$ and a unique $j \in \{0, \ldots, m\}$ such that $N_j$ has vertex $H$. Without loss of generality we can assume that $i = j = 0$. Moreover we have that

$$V_{M_i} \in \mathcal{Y} \text{ for every } i = 1, \ldots, n,$$

and that

$$V_{N_j} \in \mathcal{X} \text{ for every } j = 1, \ldots, m,$$

where $\mathcal{X}$ and $\mathcal{Y}$ come from Notation 4.10.

Finally, if we define the Green correspondents

$$M_{N_\mathcal{F}} := M_0, \quad N^\mathcal{F} := N_0,$$

we have that

$$(M_{N_\mathcal{F}})^{\mathcal{F}} \cong M, \quad (N^\mathcal{F})_{N_\mathcal{F}} \cong N.$$

Proof. Let us start by proving that we can apply Proposition 4.7 in this context. More precisely we want to check that the conditions 1-8 of Proposition 4.7 are met when we set

$$A := \text{End } (M \downarrow_{N_\mathcal{F}}^\mathcal{F}), \quad B := \text{tr}_{\mathcal{H}}^N = \text{End } (M),$$

$$C := \text{tr}_{\mathcal{H}}^N, \quad K := \text{tr}_{\mathcal{X}}^N,$$

$$I := \text{tr}_{\mathcal{X}}^N, \quad J := \text{tr}_{\mathcal{Y}}^N,$$

$$f := \text{tr}_{\mathcal{X}}^\mathcal{F}, \quad g := \text{r}_{N_\mathcal{F}}^\mathcal{F}.$$

Condition 1 follows from the fact that $\mathcal{X} \subseteq \mathcal{Y}$ and Points 7 and 8 of Proposition 3.5.

Condition 2 follows from Proposition 4.13 and again the fact that $\mathcal{X} \subseteq \mathcal{Y}$.

Conditions 3 and 4 follow from definition of $\text{tr}_{N_\mathcal{F}}^\mathcal{F}$ and Proposition 4.15.

Condition 5 follows from definition of $\text{r}_{N_\mathcal{F}}^\mathcal{F}$.

First half of Condition 6 follows from Proposition 4.15 while the second half follows from Proposition 4.13 and the fact that $\text{r}_{N_\mathcal{F}}^\mathcal{F}$ is an $\mathcal{R}$-algebra morphism.

Condition 7 follows from definition of $\text{r}_{N_\mathcal{F}}^\mathcal{F}$, $\text{tr}_{N_\mathcal{F}}^\mathcal{F}$, $s$ and $q$ and Proposition 4.13.
To prove that Condition 8 is satisfied notice that, because of Remark 2.23, the \( \mathcal{R} \)-algebra \( \mu_{\mathcal{R}}^{F^c}(N_F) \) is finitely generated as an \( \mathcal{R} \)-module. Therefore we can apply the Krull-Schmidt-Azumaya theorem (see [2, Theorem 6.12 (ii)]) together with [2, Proposition 6.10 (ii)] to conclude that for any idempotent \( f \in M \downarrow_{N_F}^F \) (which is associated to the summand \( f(M \downarrow_{N_F}^F) \) of \( M \downarrow_{N_F}^F \)) there is a unique decomposition of \( f \) (up to conjugation) as a sum of orthogonal local idempotents (which is associated to the decomposition of \( f(M \downarrow_{N_F}^F) \) as direct sum of indecomposable summands).

Since \( \mathcal{R} \) is a complete local \( \text{PID} \) then, from [2, Proposition 6.10 (ii)], we have that \( \text{End}(M) \) is a local ring and, in particular, \( \text{Id}_M \) is a local idempotent of \( \text{End}(M) \). Notice also that \( r_{N_F}^F(\text{Id}_M) = \text{Id}_{M \downarrow_{N_F}^F} \). Write now \( \text{Id}_{M \downarrow_{N_F}^F} \) as a sum of orthogonal local idempotents in \( \text{End}(M \downarrow_{N_F}^F) \)

\[
\text{Id}_{M \downarrow_{N_F}^F} = \sum_{i=0}^{n} \varepsilon_i.
\]

Applying Proposition 4.7 with \( b = \text{Id}_M \) we can deduce that exists a unique \( i \in \{0, \ldots, n\} \) such that \( \varepsilon_i \in \text{tr}^{N_F}_{H} - \text{tr}^{N_F}_{Y} \) (i.e. \( \varepsilon_i \) has vertex \( H \)) while for every \( j \neq i \) we have that \( \varepsilon_j \in \text{tr}^{N_F}_{Y} \) (i.e. \( \varepsilon_j \) has vertex in \( Y \)). Without loss of generality we can take \( i = 0 \). From Higman’s criterion we can deduce that \( \varepsilon_0(M \downarrow_{N_F}^F) \) has vertex \( H \) while \( \varepsilon_j(M \downarrow_{N_F}^F) \) has vertex in \( Y \) for every \( j \geq 1 \). By taking

\[
M_{N_F} := \varepsilon_0(M \downarrow_{N_F}^F)
\]

we obtain the first part of the statement.

Let’s now move to the second part. Let \( N \) be as in the statement. From Higman’s criterion there exists an \( \mathcal{F} \)-centric Mackey functor \( P \) over \( \mathcal{F}_H \) and an \( \mathcal{F} \)-centric Mackey functor \( Q \) over \( N_F \) such that

\[
N \oplus Q \cong P \uparrow_{\mathcal{F}_H}^{N_F}.
\]

Fixing this Mackey functor \( P \) we can take the following direct sum decomposition into indecomposable summands

\[
P \uparrow_{\mathcal{F}_H}^{F} = \bigoplus_{k=0}^{l} P_k
\]

Since \( N \) is indecomposable then we can deduce from Proposition 4.12 that exists some \( k \in \{0, \ldots, l\} \) such that \( N \) is isomorphic to a summand of \( P_k \downarrow_{N_F}^F \). Assume without loss of generality that \( k = 0 \). From the previous equation we have that \( P_0 \) is \( H \)-projective. Moreover, if it was \( \mathcal{X} \)-projective then, from Higman’s criterion, Proposition 4.13 and the fact that \( \mathcal{X} \subseteq \mathcal{Y} \), we could deduce that \( N \) is \( \mathcal{Y} \)-projective. This contradicts the hypothesis that \( N \) has vertex \( H \). Thus we can conclude that \( P_0 \) has vertex \( H \). From what we have already proven we can then conclude that

\[
N = (P_0)_{N_F}.
\]
In other words every $\mathcal{F}$-centric Mackey functor over $N_F$ on $\mathcal{R}$ is of the form $L_{N_F}$ for some centric Mackey functor $L$ over $\mathcal{F}$ on $\mathcal{R}$. If we now prove that exists some $\mathcal{V}$-projective centric Mackey functor $V$ satisfying

$$M_{N_F} \uparrow_{N_F}^F \cong M \oplus V,$$

then we could define

$$(M_{N_F})^\mathcal{F} := M,$$

and since for every $\mathcal{F}$-centric Mackey functor $N$ over $N_F$ with vertex $H$ exists a centric Mackey functor $L$ over $\mathcal{F}$ such that $N = L_{N_F}$ then we would have

$$(N^\mathcal{F})_{N_F} = (L_{N_F})_{N_F} = L_{N_F} = N.$$

Thus we only need to prove that Equation (1) is satisfied.

Let us start by introducing some notation. From the first part of the proof and from Higman’s criterion we know that exist an $\mathcal{Y}$-projective $\mathcal{F}$-centric Mackey functor $U$ over $N_F$ and an $H$-projective $\mathcal{F}$-centric Mackey functor $N$ over $N_F$ such that

$$(M_{N_F})_H = M_{N_F} \oplus N, \quad M \downarrow_{N_F}^\mathcal{F} = M_{N_F} \oplus U.$$

Here we are using the notation of Definition 2.37. Moreover, since the induction functor $\uparrow_{N_F}^\mathcal{F}$ is additive (because of Remark 2.36) and $\theta^H_{M_{N_F}}$ maps $M_{N_F}$ onto the corresponding component in $(M_{N_F})_H$ then we can conclude that the image of $M \downarrow_{N_F}^\mathcal{F} \uparrow_{N_F}^\mathcal{F}$ via $\theta^H_{M_{N_F}} \uparrow_{N_F}^\mathcal{F}$ lies in the summand $M_{N_F} \uparrow_{N_F}^\mathcal{F} \oplus U_H \uparrow_{N_F}^\mathcal{F}$ of $M_H$. On the other hand, because of Theorem 4.14, we have that for every $K \leq S$ and every $x \in I_K^M$ then

$$\theta^H_{M} (x) = \sum_{(A, \varphi) \in [N_F \times S \times K]} I_K^{H \varphi \psi (N_F) \varphi (N_F)} (c_{\varphi^{-1} R_K^N \varphi}) (x),$$

$$= \sum_{(A, \varphi) \in [N_F \times S \times K]} \left( \sum_{(B, \psi) \in [H \times N_F \times K_{K, \psi}]} I_K^{H \varphi \psi (N_F) \varphi (N_F) \varphi} (c_{\varphi^{-1} R_K^N \varphi}) (x) \right),$$

$$= \theta^H_{M_{N_F}} \uparrow_{N_F}^\mathcal{F} \left( \sum_{(A, \varphi) \in [N_F \times S \times K]} I_K^{H \varphi \psi (N_F) \varphi (N_F)} (c_{\varphi^{-1} R_K^N \varphi}) (x) \right).$$

We can therefore conclude that

$$\theta^H_{M} (M) \subseteq \theta^H_{M_{N_F}} \uparrow_{N_F}^\mathcal{F} \left( M \downarrow_{N_F}^\mathcal{F} \uparrow_{N_F}^\mathcal{F} \right) \subseteq M_{N_F} \uparrow_{N_F}^\mathcal{F} \oplus U_H \uparrow_{N_F}^\mathcal{F}.$$

Since $\theta^H_{M}$ is split injective and $M$ is indecomposable then, from Krull-Schmidt-Azumaya (see [2, Theorem 6.12 (ii)]) we can conclude that $M$ is either a summand of $M_{N_F} \uparrow_{N_F}^\mathcal{F}$ or
a summand of $U_H \uparrow_{N_F}^F$. Since $U$ is $\mathcal{Y}$-projective then, from the first part of Proposition 4.12 we can conclude that $U_H$ is also $\mathcal{Y}$-projective. Moreover, from Higman’s criterion, we have that $U_H$ is, by construction, $H$-projective. Since $U_H$ is both $H$-projective and $\mathcal{Y}$-projective then, from Remark 4.11 and the arguments exposed right before Definition 3.3, we can conclude that $U_H$ is $\mathcal{X}$-projective. From Higman’s criterion we can now conclude that $U_H \uparrow_{N_F}^F$ is $\mathcal{X}$-projective. Since $M$ has vertex $H$ by hypothesis we can conclude that $M$ is not a summand of $U_H \uparrow_{N_F}^F$ and, therefore, it must be a summand of $M_{N_F} \uparrow_{N_F}^F$. If we now prove that $M$ is the unique summand of $M_{N_F} \uparrow_{N_F}^F$ having vertex $H$ then, since $M_{N_F} \uparrow_{N_F}^F$ is $H$-projective, we will be able to conclude that all other summands of $M_{N_F} \uparrow_{N_F}^F$ are $\mathcal{X}$-projective. From the first part of the proof it suffices to prove that there is a unique summand of $M_{N_F} \uparrow_{N_F}^F$ having vertex $H$. This is however an immediate consequence of the second part of Proposition 4.12 and the fact that $M_{N_F}$ has vertex $H$. We have therefore proven that $M$ is the unique summand of $M_{N_F} \uparrow_{N_F}^F$ having vertex $H$ while all other summands have vertex in $\mathcal{X}$ thus concluding the proof.

Before concluding this subsection let us see some examples where Theorem 4.16 is satisfied or can be applied.

**Example 4.17.** Using techniques similar to those shown in [14] it can be proven that the simple centric Mackey functors over $F$ on $R$ are in one-to-one correspondence with pairs of the form $(H,V)$ where $H$ runs over $F$-centric subgroups of $S$ taken up to isomorphism and $V$ runs over simple $R\text{Out}_F(H)$-modules taken up to isomorphism. Let’s denote by $S_{H,V}^F$ the simple centric Mackey functor over $F$ associated to the pair $(H,V)$. With this notation it is straightforward to check (see Example 4.18) that $S_{H,V}^F$ is a summand of $S_{H,V}^F \downarrow_{N_F}^F$. Thus, from Theorem 4.16, we can conclude that $S_{H,V}^F$ is a summand of $S_{H,V}^F \uparrow_{N_F}^F$ and that

$$(S_{H,V}^F)_{N_F} = S_{H,V}^F,$$

$$(S_{H,V}^F)^F = S_{H,V}^F.$$

Let’s see now a more concrete example.

**Example 4.18.** Let $F$ to be a fusion system. For example we can take the fusion system

$$F_1 := F_{D_8}(GL_2(3))$$

or the Ruiz-Viruel exotic fusion system $F_2$ on $\mathbb{T}^{1+2}_+$ having two $F$-orbits of elementary abelian subgroups of rank 2 the first of which has 6 elements and the second had 2 elements (see [10, Theorem 1.1]).

Choose now $H \in F$ minimal under the preorder $\leq_F$. For $F_1$ we can take $H_1$ to be any one of the two characteristic elementary abelian subgroups of rank 2. For $F_2$ we can take $H_2$ to be one of the two elementary abelian subgroups of rank two whose $F_2$-orbit contains only 2 elements.
In order to visualize this example it might help to notice that
\[ N_{\mathcal{F}_1}(H_1) = \mathcal{F}_{D_8}(S_4), \]
Which follows after a straightforward calculation. Likewise, from [10, Theorem 1.1], we can deduce that
\[ N_{\mathcal{F}_2}(H_2) = \mathcal{F}_{T_1+2}(L_3(7)\cdot3). \]
Take now a decomposition of the idempotent \( I_H^H \in \mu_R(\mathcal{F}_c) \) into a sum of orthogonal local idempotents
\[ \sum x = I_H^H. \]
We can take this decomposition because of the Krull-Schmidt-Azumaya theorem and the same arguments employed at the beginning of proof of Theorem 4.16. Notice that, for each \( i \) we have that \( I_H^H x_i = x_i I_H^H = x_i \neq 0 \). Define now \( x = x_0 \). For example, in the case of \( \mathcal{F}_1 \) we can take
\[ x := \frac{2}{3} I_{H_1} - \frac{1}{3} c_\varphi - \frac{1}{3} c_\varphi^2 \]
where \( \varphi \) is one of the two \( \mathcal{F}_1 \) automorphisms on \( H_1 \) with order 3. Recall that
\[ \text{Aut}_{\mathcal{F}_1}(H_1) \cong S_3 \]
and, therefore, \( \varphi \) can be defined this way.

Since \( H \) is minimal \( \mathcal{F} \)-centric then, by construction of \( x \), we have that \( x \in \mu_{R_c}(N_x) \). Moreover, since \( \mu_{R_c}(N_x) \) is contained in \( \mu_R(\mathcal{F}_c) \) we have that \( x \) is also a primitive idempotent in \( \mu_{R_c}(N_x) \) (remember that every local idempotent is primitive). With this setup we can defined the following 2 Mackey functors that are necessarily indecomposable
\[ M := \mu_R(\mathcal{F}_c) x, \quad N := \mu_{R_c}(N_x) x. \]
We can now define the maps
\[ u_H^M : M \rightarrow M_H, \quad u_H^N : N \rightarrow N_H, \]
by setting
\[ u_H^M(x) = x \otimes x, \quad u_H^N(x) = x \otimes x. \]
With this setup we have that
\[ \theta_H^M u_H^M = \text{Id}_M, \quad \theta_H^N u_H^N = \text{Id}_N, \]
thus proving that both \( M \) and \( N \) are \( H \)-projective. Since \( H \) is minimal \( \mathcal{F} \)-centric we can conclude that \( H \) is in fact the vertex of both \( M \) and \( N \). We now have by construction that
\[ M = N \uparrow_{N_x}^\mathcal{F}. \]
which satisfies one part of the Green correspondence. From minimality of $H$ it is now straightforward to prove that
\[ M \downarrow_{N_F}^F = \bigoplus_{K=F} \bigoplus_{\psi \in \text{Aut}_{N_F}(K) \setminus \text{Hom}_F(H,K)} \mu^F_R(N_F) c_\psi x, \quad (2) \]
\[ = N \bigoplus_{K=F} \bigoplus_{\psi \in \text{Aut}_{N_F}(K) \setminus \text{Hom}_F(H,K)} \mu^F_R(N_F) c_\psi x. \quad (3) \]

For example, since $H_1$ is characteristic then, for $F=F_1$ we would have that $M \downarrow_{N_F}^F = N$. In the case of $F_2$ the outer sum on equation (3) would iterate over the only subgroup other than $H_2$ that is $F_2$-isomorphic to $H_2$.

Finally we can use similar methods as before in order to prove that the summands in equation (2) are $K$-projective (for the corresponding $K=F_H$). Thus, since every $K=F_H$ other than $H$ belongs to $\mathcal{Y}$ we can conclude from equation (3) that the remaining part of the Green correspondence is satisfied.

### 4.3 Proof of Proposition 4.12.

In the case of Mackey functors over groups there exists a Mackey formula for the composition of the induction and restriction functors ([13, Proposition 5.3]). This result is however not easily translated to the case of Mackey functors over fusion systems. For instance, given fusion systems $\mathcal{F}, \mathcal{H} \subseteq \mathcal{G}$ it is unclear what the Mackey formula decomposition of $\downarrow^\mathcal{G}_{\mathcal{H}} \circ \uparrow^\mathcal{F}_{\mathcal{H}}$ should be. In this subsection we will study the behavior of the composition of some particular functors of this forms. In some cases this will lead us to a decomposition via a Mackey formula. The main results of this subsection are Lemmas 4.20 and 4.23 each of which proves one half of Proposition 4.12 stated in Subsection 4.2.

**Definition 4.19.** Let $H$ and $K$ be isomorphic $p$-groups and let $\varphi : H \hookrightarrow K$ be a group isomorphism. Notice that the isomorphism $\varphi$ induces an isomorphism of $\mathcal{R}$-algebras $\hat{\varphi} : \mu_R(\mathcal{F}_H) \hookrightarrow \mu_R(\mathcal{F}_K)$. In particular we have an inverse morphism $\hat{\varphi}^{-1}$ from $\mu_R(\mathcal{F}_K)$ to $\mu_R(\mathcal{F}_H)$. This in turn induces the exact restriction of scalars functor $(\hat{\varphi}^{-1})^*$ from $\mu_R(\mathcal{F}_H)$-mod to $\mu_R(\mathcal{F}_K)$-mod. For every Mackey functor $M$ over $\mathcal{F}_H$ we define the conjugation of $M$ by $\varphi$ as the Mackey functor on $\mathcal{F}_K$ given by
\[ \varphi M := (\hat{\varphi}^{-1})^*(M). \]

With this notation we can prove the first part of Proposition 4.12.

**Lemma 4.20.** Let $H, K \in \mathcal{F}^c$ and let $M$ be an $\mathcal{F}$-centric Mackey functor over $\mathcal{F}_H$ then
\[ M \uparrow_{\mathcal{F}_H}^\mathcal{F}_K \cong \bigoplus_{(A,\varphi) \in [H \times K]} (\hat{\varphi}(M \downarrow_{\mathcal{F}_A}^\mathcal{F}_H)) \uparrow_{\mathcal{F}_{K\varphi(A)}}. \]
Proof. From Lemma 2.35 and Remark 2.34 we have that

\[ M \uparrow_{F_H} F = \mu_R(\mathcal{F}) \cdot I \mu_R(\mathcal{F}_H) \otimes \mu_R(\mathcal{F}_H) M \downarrow_{F_H}, \]

\[ \cong \bigoplus_{J \in F \mathcal{F} (B, \psi) \in [H \times J]} I_{\mathcal{F}_H}^J \mathcal{F}_\psi \otimes \mu_R(\mathcal{F}_H) I_B M. \]

Therefore, again from Remark 2.34 we have that

\[ M \uparrow_{F_H} F \downarrow_{F_K} \cong \bigoplus_{J \in F \mathcal{F} (B, \psi) \in [H \times J]} I_{\mathcal{F}_H}^J \mathcal{F}_\psi \otimes \mu_R(\mathcal{F}_H) I_B M. \]

From [8, Proposition 4.9] we know that the category \( \mathcal{O}(\mathcal{F}_c) \) admits pullbacks. Denote by \( \times_K \) the natural pullback onto \( K \). From the universal properties of products and pullbacks we know that for any \( J \in F \mathcal{F} \) such that \( J \leq K \) then

\[ H \times J \cong (H \times K) \times_K J. \]

Notice now that \( K \) is a final object in the category \( \mathcal{O}(\mathcal{F}_c) \) and, therefore, pullback onto \( K \) in this category are equivalent to products in this category. From [8, Proposition 4.9] we know that such product coincides (except for non \( \mathcal{F} \)-centric elements) with the pullback into \( K \) in the category \( \mathcal{O}(\mathcal{F}_c) \). Joining this with Lemma 2.35 we obtain the following equivalences of \( \mathcal{R} \)-modules

\[ M \uparrow_{F_H} F \downarrow_{F_K} \cong \bigoplus_{J \in F \mathcal{F} (A, \varphi) \in [H \times K]} I_{\mathcal{F}_H}^J \mathcal{F}_\varphi \otimes I_{\mathcal{F}_A}^J \mathcal{F}_\varphi (M \downarrow_{F_A}), \]

\[ \cong \bigoplus_{J \in F \mathcal{F} (A, \varphi) \in [H \times K]} I_{\mathcal{F}_H}^J \mathcal{F}_\varphi \otimes I_{\mathcal{F}_A}^J \mathcal{F}_\varphi (M \downarrow_{F_A}), \]

\[ = \bigoplus_{(A, \varphi) \in [H \times K]} (\mathcal{F} (M \downarrow_{F_A})) \uparrow_{F_K} F_{\varphi(A)}. \]

Proving that this equivalence extends to an equivalence of \( \mu_R^c(\mathcal{F}_K) \)-modules is just a straightforward calculation that the interested reader can easily perform.

Using Lemma 4.20 it is just a matter of long but straightforward computations to prove the following

32
Corollary 4.21. Let $H \in \mathcal{F}^c$ and let $M$ be an $\mathcal{F}$-centric Mackey functor over $\mathcal{F}_H$. Denote by $\pi_M$ the endomorphism of $M \uparrow_{\mathcal{F}_H}^{\mathcal{F}}$ given by the projection onto the component

$$\left(\text{Id}_H \downarrow_{\mathcal{F}_H}^{\mathcal{F}} M\right) \uparrow_{\mathcal{F}_H}^{\mathcal{F}} \cong M$$

of the decomposition in Lemma 4.20 then

$$\text{tr}_{\mathcal{F}_H}^\mathcal{F} (\pi_M) = \text{Id}_{M\uparrow_{\mathcal{F}_H}^{\mathcal{F}}}.$$ 

Corollary 4.21 completes the proof of the Higman’s criterion (Theorem 3.11) while Lemma 4.20 proves the first part of Proposition 4.12. In order to prove the second part we first need the following.

Lemma 4.22. Let $\mathcal{R}$ be complete local and $p$-local PID and let $M$ be an $\mathcal{F}$-centric Mackey functor over $\mathcal{F}_H$ on $\mathcal{R}$. Then we have that

$$M \uparrow_{\mathcal{F}_H}^{\mathcal{F}} \downarrow_{\mathcal{F}_N}^{\mathcal{F}} \cong M \uparrow_{\mathcal{F}_H}^{\mathcal{F}} \bigoplus_{K \in \mathcal{Y}} M_K$$

where each $M_K$ is an $\mathcal{F}$-centric $K$-projective Mackey functor over $N_\mathcal{F}$.

Proof. Since $\mathcal{R}$ is $p$-local then we can use Lemma 3.2 to deduce that the map

$$\theta_{N_\mathcal{S}(\mathcal{H})} : M \uparrow_{\mathcal{F}_H}^{\mathcal{F}} \downarrow_{N_\mathcal{S}(\mathcal{H})}^{N_\mathcal{F}} \rightarrow M \uparrow_{\mathcal{F}_H}^{\mathcal{F}} \downarrow_{N_\mathcal{F}}^{N_\mathcal{S}(\mathcal{H})},$$

is split surjective. From Lemma 4.20 we can also deduce that

$$M \uparrow_{\mathcal{F}_H}^{\mathcal{F}} \downarrow_{N_\mathcal{S}(\mathcal{H})}^{N_\mathcal{F}} \cong \bigoplus_{(A, \varphi) \in [H \times N_\mathcal{S}(\mathcal{H})]} \left(\varphi \downarrow_{\mathcal{F}_A}^{\mathcal{F}} \left( M \downarrow_{\mathcal{F}_A}^{\mathcal{F}} \right) \right) \uparrow_{N_\mathcal{F}}^{N_\mathcal{S}(\mathcal{H})}. \quad (4)$$

If we now define for every $K \in \mathcal{Y}$

$$M'_K := \bigoplus_{(A, \varphi) \in [H \times N_\mathcal{S}(\mathcal{H})], \varphi(A) = K} \varphi \downarrow_{\mathcal{F}_A}^{\mathcal{F}} \left( M \downarrow_{\mathcal{F}_A}^{\mathcal{F}} \right),$$

Then we can re-write equation (4) as

$$M \uparrow_{\mathcal{F}_H}^{\mathcal{F}} \downarrow_{N_\mathcal{S}(\mathcal{H})}^{N_\mathcal{F}} \cong \left( \bigoplus_{(H, \varphi) \in [H \times N_\mathcal{S}(\mathcal{H})]} \left( \varphi \downarrow_{\mathcal{F}_H}^{\mathcal{F}} \left( M \downarrow_{\mathcal{F}_H}^{\mathcal{F}} \right) \right) \uparrow_{N_\mathcal{F}}^{N_\mathcal{S}(\mathcal{H})} \right) \oplus \left( \bigoplus_{K \in \mathcal{Y}} M'_K \uparrow_{\mathcal{F}_K}^{N_\mathcal{F}} \right). \quad (5)$$

Notice how, from Higman’s criterion, each $M_K \uparrow_{\mathcal{F}_K}^{N_\mathcal{F}}$ is $K$-projective while each each $(\varphi M) \uparrow_{\mathcal{F}_H}^{N_\mathcal{F}}$ is $H$-projective. Since $\theta_{N_\mathcal{S}(\mathcal{H})}$ is split surjective then we know that $M \uparrow_{\mathcal{F}_H}^{\mathcal{F}} \downarrow_{N_\mathcal{F}}^{N_\mathcal{S}(\mathcal{H})}$ is a summand of $M \uparrow_{\mathcal{F}_H}^{\mathcal{F}} \downarrow_{N_\mathcal{S}(\mathcal{H})}^{N_\mathcal{F}}$. On the other hand, since $\mathcal{R}$ is a complete local
We can apply the Krull-Schmidt-Azumaya theorem (see [2, Theorem 6.12 (ii)]) in order to obtain a decomposition of $M \uparrow_{F \downarrow N_F}^F$ from the decomposition shown in equation (5). In other words we can write

$$M \uparrow_{F \downarrow N_F}^F = M' \oplus \bigoplus_{K \in \mathcal{Y}} M_K$$

Where each $M_K$ is a (possibly zero) summand of the elements $M'_K \uparrow_{F \downarrow N_F}^F$ in equation (5) and are therefore $K$-projective while $M'$ is a summand of

$$N := \bigoplus_{(H, \varphi) \in [H \times N_S(H)]} (\varphi(M)) \uparrow_{F \downarrow N_F}^F$$

which is necessarily isomorphic to the image of $N$ via $\theta_{N_S(H)}$ (because $\theta_{N_S(H)}$ is split surjective). Thus we are only left with proving that the image of $N$ via $\theta_{N_S(H)}$ is isomorphic to $M \uparrow_{F \downarrow N_F}^F$.

Given $x \in \varphi(M)$ and $I_{\psi(J)}^{K} \psi(x) \in \mu_{N_F}$ we have that

$$\theta_{N_S(H)}(y \otimes x) = y \otimes x,$$

from which we deduce that

$$\bigoplus_{(H, \varphi) \in [H \times N_S(H)]} (\varphi(M)) \uparrow_{F \downarrow N_F}^F = \mu_{N_F}(N_F) \otimes M = M \uparrow_{F \downarrow N_F}^F \subseteq M \uparrow_{F \downarrow N_F}^F.$$

Thus concluding the proof.

We can finally prove the second part of Proposition 4.12.

**Lemma 4.23.** Let $\mathcal{R}$ be a complete local and $p$ local PID and let $M$ be an $H$-projective $\mathcal{F}$-centric Mackey functor over $N_F$ on $\mathcal{R}$. Then we have that

$$M \uparrow_{N_F \downarrow N_F}^F = M \oplus M'$$

for some $\mathcal{Y}$-projective, $\mathcal{F}$-centric Mackey functor $M'$ over $N_F$.

**Proof.** From Higman’s criterion we know that exists an $\mathcal{F}$-centric Mackey functor $N$ over $\mathcal{F}_H$ and an $\mathcal{F}$-centric Mackey functor $U$ over $N_F$ such that

$$M \oplus U \cong N \uparrow_{\mathcal{F}_H}^F.$$

Since induction and restriction preserve direct sum decomposition then, from proof of Lemma 33 we obtain the isomorphism

$$f : M \uparrow_{N_F \downarrow N_F}^F \oplus U \uparrow_{N_F \downarrow N_F}^F \hookrightarrow N \uparrow_{\mathcal{F}_H}^F \oplus \bigoplus_{K \in \mathcal{Y}} N_K.$$
Moreover, from the proof of Lemma 33 we know that the isomorphism $f$ sends the sub-module $M \oplus U$ of $M \uparrow_{N_F} \downarrow_{N_F} \oplus U \uparrow_{N_F} \downarrow_{N_F}$ isomorphically onto the summand $N \uparrow_{N_F}$ of the right hand side. Therefore we obtain the inclusion $$M \uparrow_{N_F} \downarrow_{N_F} \hookrightarrow M \oplus \bigoplus_{K \in \mathcal{Y}} N_K.$$ Since $\mathcal{R}$ is complete local and $p$ local and the image of $M \uparrow_{N_F} \downarrow_{N_F}$ by $f$ includes the sub-module $M$ on the right hand side then we then apply the Krull-Schmidt-Azumaya theorem in order to conclude that exists a summand $M'$ of $\bigoplus_{K \in \mathcal{Y}} N_K$ (which is necessarily $\mathcal{Y}$-projective) such that $$M \uparrow_{N_F} \downarrow_{N_F} \simeq M \oplus M'. $$ Thus concluding the proof. 

Proposition 4.12 follows immediately from Lemmas 4.20 and 4.23.

4.4 Proof of Proposition 4.13.

Let $G$ be a group and let $H$ and $K$ be subgroups of $G$ such that $K \leq N_G (H)$. Then we have the following decomposition of double coset representatives

$$[K \backslash G/H] = \bigsqcup_{x \in [N_G (H)/G/H]} [K \backslash N_G (H) / (N_G (H) \cap xH)] x. \quad (6)$$

This can be used, in the case of Mackey functors over groups, in order to prove that for any Mackey functor $M$ over $G$

$$r_{N_G(H)}^G \mathsf{tr}_H^G = \sum_{x \in [N_G (H)/G/H]} \mathsf{tr}_{N_G (H) \cap r^H x}^H c_{c_x} r_{N_G (H) \cap r^H x}. \quad (7)$$

Where $\mathsf{tr}_{N_G(H)}^G$ and $r_{H}^G$ are the transfer and restriction maps of the Endomorphism Mackey functor $\mathsf{End} (M)$ (see [11, Definition 2.7]). Even though evidence suggests that a result analogous to Equation (6) might be possible for fusion systems (replacing the double cosets by products in $\mathcal{O} (F^c)_{\mathcal{U}}$ we only succeeded on proving it for particular cases. We did however manage to work around this inconvenient and still obtain a result analogous to Equation (7) (Proposition 4.13).

Let us start by introducing some notation.

Notation 4.24. Let $\mathcal{C}$ be a category, let $X, Y, X'$ and $Y'$ be objects in $\mathcal{C}$ and let $\varphi$ be a morphism from $X$ to $Y$ in $\mathcal{C}$. Through this paper we will use the following standard notation

$$\varphi_\ast := \mathsf{Hom} (X', -) (\varphi) : \mathsf{Hom} (X', X) \to \mathsf{Hom} (X', Y),$$

$$\varphi^* := \mathsf{Hom} (-, Y') (\varphi) : \mathsf{Hom} (Y, Y') \to \mathsf{Hom} (X, Y').$$
With this notation and Point 9 of Proposition 3.5 taking $H = S$ we can obtain the following result via straightforward calculations.

**Lemma 4.25.** Let $R$ be $p$ local. Then $\overline{N}_S(H) \in B_R^{(N_F)_c}$ has an inverse (see Proposition 2.39 and Notation 2.44) and the following identity holds for every centric Mackey functor

$$r_{NF}^F tr_{FH}^F = \sum_{(A, \varphi) \in [H \times N_S(H)]} \left( N_S(H)^{-1} \right)_{tr_{FH}(A)} c_{\varphi} r_{FA}^F.$$

We also need the following lemma which wouldn’t hold if we weren’t restricting to $F$-centric Mackey functors.

**Lemma 4.26.** Let $A, H \in F^c$ such that $A \leq H$. Then we have that

$$[A \times_{NF} N_S(H)] = \{ (A, \varphi) \mid \varphi \in Hom_{O(N_F)}(A, N_S(H)) \}.$$

Moreover we have that

$$Hom_{O(N_F)}(A, N_S(H)) \cong Hom_{O(N_F)}(H, N_S(H)).$$

In particular

$$[A \times_{NF} N_S(H)] \cong [H \times_{NF} N_S(H)] \cong Hom_{O(N_F)}(A, N_S(H)),$$

and

$$|[A \times_{NF} N_S(H)]| = |[H \times_{NF} N_S(H)]| = |Hom_{O(N_F)}(A, N_S(H))|.$$

**Proof.** Since $A \leq H$ then, for any subgroup $B \leq A$, we have that $HB = H$. Analogously we also have that $HN_S(H) = N_S(H)$. From definition of $N_F$ these identities imply that all morphisms on $N_F$ from subgroups of $A$ to $N_S(H)$ can be lifted to morphisms from $H$ to $N_S(H)$. In particular, they can be lifted to morphisms from $A$ to $N_S(H)$. Because of the definition of product in $N_F^c$ this implies that

$$[A \times_{NF} N_S(H)] = \{ (A, \varphi) \mid \varphi \in Hom_{O(N_F)}(A, N_S(H)) \},$$

$$\cong Hom_{O(N_F)}(A, N_S(H)).$$

This proves the first part of the statement.

In order to prove the second part it suffices to prove that the following map is bijective

$$(i_A^H)^* : Hom_{O(N_F)}(H, N_S(H)) \to Hom_{O(N_F)}(A, N_S(H)).$$

Where $i_A^H$ is as in Notation 2.7. From Lemma 2.16 we know that $i_A^H$ is surjective. Since the contravariant functor $Hom_{O(N_F)}(\_ \_, N_S(H))$ is right adjoint we know from the RAPL theorem that it is left exact. Joining both these facts we can conclude that $(i_A^H)^*$ is injective.

On the other hand, from definition of $N_F$, we have that for every morphism $\varphi : A \to N_S(H)$ in $O(N_F)$ exists a morphism $\hat{\varphi} : H \to N_S(H)$ in $O(N_F)$ such that $\varphi = \hat{\varphi} i_A^H$. This proves that $(i_A^H)^*$ is also surjective thus concluding the proof. \qed
We can now finally obtain the last ingredient we need to prove Proposition 4.13.

**Lemma 4.27.** Let \( R \) be \( p \)-local and let \( M \) be a centric Mackey functor over \( \mathcal{F} \) on \( R \). Then \( N_S(H) \in B^{N_F}_R \) has an inverse (see Proposition 2.39 and Notation 2.44) and the following identity holds on \( M \)

\[
\sum_{\varphi \in \text{Hom}_{O}(H,N_S(H))} \left( N_S(H)^{-1} \right)_* \text{tr}^{N_F}_{\mathcal{F}_H} c_\varphi = \text{tr}^{N_F}_{\mathcal{F}_H}.
\]

Where \( \varphi \in \mathcal{F} \) is any representative of \( \varphi \) seen as an isomorphism onto its image. Equivalently

\[
\sum_{\varphi \in \text{Hom}_{O}(H,N_S(H))} \text{tr}^{N_F}_{\mathcal{F}_H} c_\varphi = \left( N_S(H) \right)_* \text{tr}^{N_F}_{\mathcal{F}_H}.
\]

**Proof.** A straightforward computation from Definition 3.4 shows that

\[
\text{tr}^{N_F}_{\mathcal{F}_H} c_\varphi = \text{tr}^{N_F}_{\mathcal{F}_H},
\]

for every automorphism \( \varphi \in \text{Hom}_{O}(N_F)(H,N_S(H)) \) and \( \varphi \) as in the statement. This implies together with Lemma 4.26 that

\[
\sum_{\varphi \in \text{Hom}_{O}(N_F)(H,N_S(H))} \text{tr}^{N_F}_{\mathcal{F}_H} c_\varphi = \left| H \times_{N_F} N_S(H) \right| \text{tr}^{N_F}_{\mathcal{F}_H}.
\]

Let’s now look at the right hand side of the equation we want to prove.

Let \( f \in \text{End} \left( M \right)^{N_F}_{\mathcal{F}_H} \). From Point 8 of Proposition 3.5 we have that

\[
N_S(H) \cdot \text{tr}^{N_F}_{\mathcal{F}_H} (f) = \text{tr}^{N_F}_{\mathcal{F}_H} \left( r^{N_F}_{\mathcal{F}_H} \left( N_S(H) \right) \cdot f \right).
\]  

(8)

Let \( A \in \mathcal{F}^c \) such that \( A \leq H \) and let \( a \in I^A_{M} \left( M \right)^{N_F}_{\mathcal{F}_H} \). From Proposition 2.42 and using the same notation we have that

\[
r^{N_F}_{\mathcal{F}_H} \left( N_S(H) \right)(a) = \sum_{(B,\psi) \in [N_S(H) \times_{N_F} A]} I^A_{\psi(B)} R^A_{\psi(B)}(a).
\]

Applying Lemma 4.26 we can now deduce that

\[
r^{N_F}_{\mathcal{F}_H} \left( N_S(H) \right)(a) = \left| H \times_{N_F} N_S(H) \right| (a).
\]

In other words \( r^{N_F}_{\mathcal{F}_H} \left( N_S(H) \right) \) acts as multiplication by \( \left| H \times_{N_F} N_S(H) \right| \). Replacing this into Equation (8) and using the first half of the proof we can deduce that

\[
N_S(H) \cdot \text{tr}^{N_F}_{\mathcal{F}_H} (f) = \left| H \times_{N_F} N_S(H) \right| \text{tr}^{N_F}_{\mathcal{F}_H} = \sum_{\varphi \in \text{Hom}_{O}(N_F)(H,N_S(H))} \text{tr}^{N_F}_{\mathcal{F}_H} c_\varphi.
\]

Thus concluding the proof. \( \Box \)
We are now finally able to prove Proposition 4.13 and conclude this subsection.

**Proof.** (of Proposition 4.13) From definition of $N_F$ we know that
\[ \{(A, \varphi) \in [H \times N_S(H)] \mid \varphi(A) = H\} = \{(H, \varphi) \mid \varphi \in \text{Hom}_{\mathcal{O}(N_F)}(H, N_S(H))\} . \]

Using this together with Lemma 4.27 and with its same notation we can deduce that
\[ (N_S(H)^{-1}) \sum_{(A, \varphi) \in [H \times N_S(H)]} \text{tr}^N_{F_{\varphi(A)}} c_{\varphi} \text{tr}^F_{\varphi(A)} = \text{tr}^N_{F_H} . \]

Define now
\[ \mathcal{G} := \{(A, \overline{\varphi}) \in [H \times N_S(H)] : \varphi(A) \neq H\} . \]

For every $(A, \overline{\varphi}) \in \mathcal{G}$ we have that $\varphi(A) \in \mathcal{Y}$ by definition. Taking $f$ and $f_K$ as in the statement for every $K \in \mathcal{Y}$ we can deduce from Point 8 of Proposition 3.5 and definition of $f_K$ that
\[ (N_S(H)^{-1}) \sum_{(A, \overline{\varphi}) \in \mathcal{G}} \text{tr}^N_{F_{\varphi(A)}} ((c_{\varphi} \text{tr}^F_{\varphi(A)}) (f)) = \sum_{K \in \mathcal{Y}} \text{tr}^N_{F_K} (f_K) . \]

Joining both results with Lemma 4.25 we can conclude that
\[ \left( r^F_{N_S} \text{tr}^F_{N_S} \right) (f) = \text{tr}^N_{F_H} (f) + \sum_{K \in \mathcal{Y}} \text{tr}^N_{F_K} (f_K) . \]

just as we wanted to prove. \(\square\)

### 4.5 Proof of Theorem 4.14.

Let $G$ be a group and let $K$, $J$ and $H$ be subgroups of $G$ such that $H \leq J$. Then we have the following decomposition of double coset representatives
\[ [K\backslash G/H] = \bigsqcup_{x \in [K\backslash G/J]} x [(K^x \cap J) \backslash J/H] . \quad (9) \]

This can be used, in the case of Mackey functors over groups, in order to prove that for any Mackey functor $M$ on $G$ the transfer maps of $\text{End}(M)$ (see [11, Definition 2.7]) compose correctly. That is, using the notation of Subsection 4.4, Equation (9) can be used to prove that $\text{tr}^G_H \circ \text{tr}^J_H = \text{tr}^G_H$. We would like to obtain a similar result for Mackey functors over fusion systems. That is we want to prove Proposition 4.15. In order to do this we will first need to prove Theorem 4.14 which will be replacing Equation (9) in the case of fusion systems. Notice how Theorem 4.14 has a scope beyond that of
proving Proposition 4.15 since it is also a key element of the proof of Theorem 4.16 and we believe it could be of its own interest.

Since Theorem 4.14 is essentially a result analogous to Equation (9) in the case of fusion system we should start by finding elements analogous to the elements \([K \setminus G/J] \) and \(K^x \cap J\) of Equation (9) but in the context of fusion systems.

**Lemma 4.28.** Let \(A,K \in \mathcal{F}^c\) with \(A \leq N_S(H)\) and let \(\varphi \in \text{Hom}_\mathcal{F}(A,K)\). Then there is a unique maximal subgroup

\[
N_{\varphi}^N \leq N_K(\varphi(A)),
\]

such that

\[
\text{Aut}_{N_{\varphi}^N}(\varphi(A)) \leq \varphi \text{Aut}_{N_{\varphi}}(A).
\]

We call this subgroup the \textbf{normalizer after} \(\varphi\) \textbf{in} \(N_{\varphi}\).

Moreover there exist a fully \(N_{\varphi}\)-normalized subgroup \(A' \leq N_S(H)\), an isomorphism \(\theta \in \text{Aut}_{N_{\varphi}}(A',A)\) and a subgroup \(N_{\varphi \theta}^N\) of \(N_{NS(H)}(A')\) containing \(A'\) such that

\[
\text{Aut}_{N_{\varphi \theta}^N}(\varphi(A)) = \varphi \theta \text{Aut}_{N_{\varphi \theta}}(A').
\]

We call any morphism of the form \(\varphi \theta\) with \(\theta\) as before \(N_{\varphi}^-\text{-top of} \ \varphi\) and denote it by \(\varphi_{\varphi\theta}^N\). We also call \textbf{normalizer before} \(\varphi\) \textbf{in} \(N_{\varphi}\) any group of the form \(N_{\varphi}^N\) with the properties of \(N_{\varphi \theta}^N\). We will often simply write

\[
N_{\varphi}^N := N_{\varphi \theta}^N
\]

and call it \textbf{normalizer before} \(\varphi\) \textbf{in} \(N_{\varphi}\).

**Proof.** Let us start by defining

\[
N_{\varphi}^N := \{x \in N_K(\varphi(A)) : \varphi^{-1}c_x \varphi \in \text{Aut}_{N_{\varphi}}(A)\}.
\]

With this definition it is straightforward to check that \(N_{\varphi}^N\) is indeed a subgroup of \(N_K(\varphi(A))\) and satisfies the desired properties. It is also straightforward to check that any other subgroup of \(N_K(\varphi(A))\) with the same properties is contained in \(N_{\varphi}^N\).

This proves that \(N_{\varphi}^N\) is the unique maximal subgroup of \(N_K(\varphi(A))\) with the desired properties.

Let’s now prove the second half of the statement. Let \(A' = \mathcal{F}\) \(A\) be fully \(N_{\varphi}\)-normalized and let \(\alpha\) be an isomorphism in \(N_{\varphi}\) from \(A'\) to \(A\). Since \(N_{\varphi}^N \leq S\) then it is a \(p\)-group. Therefore \(\text{Aut}_{N_{\varphi}^N}(\varphi(A))^{\varphi \alpha}\) is also a \(p\)-group. By construction of \(A'\) and \(\alpha\) we also have that

\[
\text{Aut}_{N_{\varphi}^N}(\varphi(A))^{\varphi \alpha} \leq \text{Aut}_{N_{\varphi}}(A').
\]

Applying Lemma 2.11 we can deduce that exists \(\beta \in \text{Aut}_{N_{\varphi}}(A')\) such that

\[
\text{Aut}_{N_{\varphi}^N}(\varphi(A))^{\varphi \alpha \beta} \leq \text{Aut}_{N_{NS(H)(A')}}(A').
\]

39
We can now set \( \theta := \alpha \beta \) and define
\[
N^{N_F}_{\varphi} := \left\{ x \in N_{N_S(H)}(A') : c_x \in \text{Aut}_{N^{N_F}_{\varphi}}(\varphi(A))^{\varphi \theta} \right\}.
\]

With this definition it is straightforward to check that \( N^{N_F}_{\varphi} \) is indeed a subgroup of \( N_{N_S(H)}(A') \) containing \( A' \) and that
\[
\text{Aut}_{N^{N_F}_{\varphi}}(\varphi(A)) = \varphi^{\theta} \text{Aut}_{N^{N_F}_{\varphi}}(A').
\]

thus concluding the proof. \( \square \)

**Remark 4.29.** Notice that whenever \( \varphi^{N_F} \) can be lifted to \( N^{N_F}_{\varphi} \) then we necessarily have that
\[
\varphi^{N_F}(N^{N_F}_{\varphi}) = N^{N_F}_{\varphi},
\]

Moreover we can always take \((\varphi^{N_F})^{N_F} = \varphi^{N_F}\).

**Notation 4.30.** Given a morphism \( \overline{\varphi} \) in \( O(F) \) with representative \( \varphi \in F \) we will write \( \overline{\varphi}^{N_F} \) to indicate any morphism in \( O(F) \) with representative \( \varphi^{N_F} \). Likewise we will denote by \( N^{N_F}_{\overline{\varphi}} \) and \( N^{N_F}_{\overline{\varphi}}N \) the subgroups \( N^{N_F}_{\varphi} \) and \( N^{N_F}_{\varphi}N \) respectively.

The subgroups of the form \( N^{N_F}_{\varphi} \) with \((A, \varphi) \in [H \times K]\) will play in Theorem 4.14 the role that the elements of the form \( K^x \cap J \) play in Equation (9).

Notice now that, given subgroups \( H, K \in F^c \) a pair \((A, \varphi) \in [H \times K]\) and an isomorphism \( \theta \in \text{Hom}_{O(F)}(A', A) \), then \((A', \varphi \theta)\) is also one of the maximal pairs of Definition 2.18. In particular, if we take \( \theta \) so that \( \varphi \theta = \varphi^{N_F} \), then, pre-composing with conjugation by an element of \( H \) if necessary, we have that \((A', \varphi^{N_F}) \in [H \times K]\). This allows us to give the following definition which gives us the elements that will play in Theorem 4.14 the role that the element \([K \setminus G/J]\) plays in Equation (9).

**Definition 4.31.** Let \( K \in F^c \). We define the subset \([N_F \times K]\) of \([H \times K]\) as follows.
First we define an equivalence relation \( \sim \) in \([H \times K]\) by setting \((A, \varphi) \sim (A', \varphi')\) if and only if exists an isomorphism \( \psi \in \text{Hom}_{O(F)}(A', A) \) such that \( \varphi = \varphi' \psi \). From the arguments above and proof of Lemma 4.28 we can now take for each equivalence class in \([H \times K]/ \sim \) a representative \((A, \varphi)\) such that \( A \) is fully \( N_F \)-normalized and \( \varphi = \varphi^{N_F} \). We define \([N_F \times K]\) to be the set of those representatives.

As we said before the elements \([N_F \times K]\) and \( N^{N_F}_{\varphi} \) play in Theorem 4.14 the role that the elements \([K \setminus G/J]\) and \( K^x \cap J \) play in Equation (9). Notice however that, in Equation (9) the elements \( x \) in \([K \setminus G/J]\) are multiplying on the left the sets of double coset representatives \([K^x \cap J]/J/H]\). Therefore, in order for the analogy to hold, for every \((A, \varphi) \in [N_F \times K]\) and every \((B, \psi) \in [H \times N_F N^{N_F}_{\varphi}]\), the morphism \( \varphi^{N_F} \psi \) needs to be well define. In other words we need to be able to lift \( \varphi^{N_F} \) to a unique morphism from \( N^{N_F}_{\varphi} \) to \( K \). Such lifting is given by the following.
Proposition 4.32. Let $K \in \mathcal{F}^c$ and let $(A, \varphi) \in [N_{\mathcal{F}} \times K]$. Take now a representative $\varphi \in \mathcal{F}^c$ of $\overline{\varphi}$. Then there exists a morphism $\hat{\varphi} \in \text{Hom}_{\mathcal{F}} \left( N^{N_{\mathcal{F}}} \varphi, K \right)$ such that $\varphi = \hat{\varphi} i_{A}^{N_{\mathcal{F}}}$ where $i_{A}^{N_{\mathcal{F}}}$ denotes the natural inclusion. In particular, from Remark 4.29, we have that

$$\hat{\varphi} \left( N^{N_{\mathcal{F}}} \varphi \right) = N_{\varphi} \leq N_{K} \left( \varphi (A) \right).$$

In particular, from Lemma 2.16, there is a unique morphism in $\mathcal{O}(\mathcal{F}^c)$ from $N^{N_{\mathcal{F}}} \varphi$ to $K$ lifting $\overline{\varphi}$. We will denote such morphism by $\overline{\varphi}$.

Proof. From Lemma 4.28 we have that for every $x \in N^{N_{\mathcal{F}}} \varphi$ exists $k \in N_{K} \left( \varphi (A) \right)$ such that the following identity of morphisms holds

$$\varphi c_{x} = c_{k} \varphi : A \to \varphi (A).$$

Take now a fully $\mathcal{F}$-normalized subgroup $B =_{\mathcal{F}} \varphi (A)$. Since $B$ is fully $\mathcal{F}$-normalized then from Lemma 2.12 we can take a morphism $\psi \in \text{Hom}_{\mathcal{F}} \left( N_{S} \left( \varphi (A) \right), N_{S} (B) \right)$ such that $\psi \varphi (A) = B$. Then, from the previous identity and since $\psi$ can be applied on $k$ by construction, we obtain

$$\psi \varphi c_{x} = \psi c_{k} \varphi = c_{\psi (k)} \psi \varphi : A \to B. \quad (10)$$

We can therefore conclude that $x \in N_{\psi \varphi}$ (see Definition 2.5). Since this works for every $x \in N^{N_{\mathcal{F}}} \varphi$ we can conclude that $N^{N_{\mathcal{F}}} \varphi \leq N_{\psi \varphi}$. Thus, since $\mathcal{F}$ is saturated, we can lift $\psi \varphi$ to a morphism $\theta$ from $N^{N_{\mathcal{F}}} \varphi$ to $N_{S} (B)$. Joining this lifting with Equation (10) we obtain the identity

$$c_{\theta (x)} \psi \varphi = c_{\psi (k)} \psi \varphi.$$

Since $A$ (and therefore $B$) are $\mathcal{F}$-centric by construction we can now deduce that exists $\psi (c) \in B$ such that

$$\theta (x) = \psi (k c) \in \psi \left( N_{K} \left( \varphi (A) \right) \right).$$

We can therefore conclude that

$$\theta \left( N^{N_{\mathcal{F}}} \varphi \right) \leq \psi \left( N_{K} \left( \varphi (A) \right) \right).$$

This allows us to define the map $\hat{\varphi} : N^{N_{\mathcal{F}}} \varphi \to N_{K} \left( \varphi (A) \right)$ by setting for every $x \in N^{N_{\mathcal{F}}} \varphi$

$$\hat{\varphi} (x) := \psi^{-1} \theta (x).$$

thus concluding the proof.

As an immediate corollary we have the following.

Corollary 4.33. Let $K \in \mathcal{F}^c$ and $(A, \varphi) \in [N_{\mathcal{F}} \times K]$. Then we have that $A = N^{N_{\mathcal{F}}} \varphi \cap H$. 

41
Proof. By construction we know that $A \leq N^N_{\psi}$, and $A \leq H$. Therefore $A \leq N^N_{\psi} \cap H$. On the other hand Proposition 4.32 tells us that we can lift $\varphi$ to $N^N_{\psi} \cap H$. From maximality of the pair $(A, \varphi)$ we can conclude that $A = N^N_{\psi} \cap H$ as we wanted to prove. \hfill $\square$

Take now any $\mathcal{F}$-centric $K \leq N_S (H)$, any $(A, \overline{\varphi}) \in [H \times N_{\mathcal{F}} K]$ and any representative $\varphi \in \mathcal{F}$ of $\overline{\varphi}$. Since every morphism on $N_{\mathcal{F}}$ restricts to an automorphism of $H$ we have that $\varphi (A) \leq H \cap K$. From definition of $N_{\mathcal{F}}$ this implies that $\varphi$ can be lifted to an automorphism $\psi$ of $H$. We can now take $B \leq H$ such that $\psi (B) = H \cap K$. From what we said before we necessarily have that $A \leq B$. From maximality of the pair $(A, \overline{\varphi})$ this necessarily implies that $A = B$ and, therefore, that $\varphi (A) = H \cap K$. Joining this with Corollary 4.33 we now obtain the following.

**Corollary 4.34.** Let $K$ be an $\mathcal{F}$-centric subgroup of $N_S (H)$ and let $(A, \overline{\varphi}) \in [N_{\mathcal{F}} \times K]$. For every $(B, \overline{\psi}) \in [H \times N_{\mathcal{F}} N^N_{\psi}]$ and every representative $\psi \in N_{\mathcal{F}}$ of $\psi$ we have that

$$\psi (B) = N^N_{\psi} \cap H = A.$$  

We have now gathered all ingredients needed to prove Theorem 4.14 and conclude this subsection.

**Proof.** (of Theorem 4.14) Let us start by proving that the map $\Upsilon$ is well defined. Take $(A, \overline{\varphi})$ and $(B, \overline{\psi})$ as in the statement. From the universal property of products we know that there exists a unique $(C, \overline{\theta}) \in [H \times K]$ and an element $h \in H$ such that $C \geq B^h$ and $\theta_{C^h} = \overline{\varphi} \overline{\psi} c_h$. Here we are using Notation 2.7 and the notation of Proposition 4.32. Then, all we need to do to prove that $\Upsilon$ is well defined is showing that $C = B^h$. We can now take morphisms $\psi$, $\varphi$ and $\theta$ in $\mathcal{F}$ that are representatives of $\overline{\psi}$, $\overline{\varphi}$ and $\overline{\theta}$ respectively and such that the identity $\theta_{C^h} = \varphi \psi c_h$ is satisfied. From definition of $N_{\mathcal{F}}$ we can now take an automorphism of $H$ lifting $\psi$. We will abuse notation denoting such automorphism also by $\psi$. Since $\psi$ is an automorphism of $H$ then it maps $h C$ to a subgroup of $H$. Joining this with Corollary 4.34 we can re-write the previous identity of morphisms as

$$\theta_{C^h} \psi^{-1} \varphi = \varphi : A \rightarrow K.$$  

Where $\varphi \in \mathcal{F}$ is a representative of $\overline{\varphi}$ obtained by restricting $\hat{\varphi}$. From maximality of the pair $(A, \overline{\varphi}) \in [H \times K]$ and again Corollary 4.34 we can now deduce that

$$\psi (h C) = A = \psi (B).$$

In particular we have that $C = B^h$ thus proving that $\Upsilon$ is well defined.

Let us prove that $\Upsilon$ is surjective. By construction of $[N_{\mathcal{F}} \times K]$ we know that for every $(C, \theta) \in [H \times N_{\mathcal{F}} K]$ exists a unique $(A, \varphi) \in [N_{\mathcal{F}} \times K]$ and an isomorphism $\psi$ from $C$ to
A in $O(N_F)$ such that $\theta = \varphi \psi$. Thus using again Notation 2.7 we just need to prove that the following holds up to pre-conjugation by an element of $h$

$$\left( C, \iota_A^{N_F \varphi} \psi \right) \in \left[ H \times_{N_F} N_{\varphi}^N \right].$$

This however is immediate from maximality of the elements in $\left[ H \times_{N_F} N_{\varphi}^N \right]$, the fact that $\psi$ maps onto $A$ and Corollary 4.34.

Finally let us prove that $\Upsilon$ is injective. Let $\left( A, \varphi \right), \left( A', \varphi' \right) \in [N_F \times N]$ and assume that exist $\left( B, \psi \right) \in \left[ H \times_{N_F} N_{\varphi}^N \right]$ and $\left( B', \psi' \right) \in \left[ H \times_{N_F} N_{\varphi'}^N \right]$ such that

$$T \left( \left( B', \psi' \right) \right) = T \left( \left( B, \psi \right) \right).$$

By construction exists $h \in H$ such that

$$\left( B', \varphi' \psi' c_h \right) = \left( B^h, \varphi' \psi c_h \right).$$

Take now morphisms $\psi, \psi', \varphi$ and $\varphi'$ in $F$ that are representatives of $\overline{\psi}, \overline{\psi}', \overline{\varphi}$ and $\overline{\varphi}'$. From Corollary 4.34 we can rewrite the previous identity (post conjugating by some element in $k$ if necessary) as the following identity of morphisms

$$\varphi' \psi' c_{h^{-1}} \psi^{-1} = \varphi : A \to K.$$

Notice that the map $\psi' c_{h^{-1}} \psi^{-1}$ in the previous identity is an isomorphism of $N_F$. From construction of $[N_F \times K]$ this implies that $\left( A, \varphi \right) = \left( A', \varphi' \right)$. In particular there exists $k \in K'$ satisfying

$$c_{\varphi^{-1} (k)} \psi' = \psi c_h : B^h \to N_{\varphi}^N.$$ 

From Corollary 4.34 and the previous identity we can then deduce that

$$c_k \in \text{Aut}_K \left( \varphi \left( A \right) \right) \cap \varphi \text{Aut}_{N_F} \left( A \right),$$

which proves that

$$k \in N_{\varphi}^N.$$

Using Remark 4.29 we obtain that

$$\varphi^{-1} (k) \in N_{\varphi}^N.$$ 

From definition of the orbit category we can then conclude that $\left( B, \overline{\psi} \right) = \left( B', \overline{\psi'} \right)$ thus proving that $\Upsilon$ is also injective and therefore concluding the proof. □
4.6 Proof of Proposition 4.15.

As we explained at the beginning of Subsection 4.5, Equation (9) can be used in the case of Mackey functors over groups in order to prove that transfer maps compose nicely. During this subsection we will show how a similar result can be obtain in the case of centric Mackey functor by using Theorem 4.14 instead of Equation (9). More precisely during this subsection we will be using Theorem 4.14 in order to prove that we can define an $R$-linear map

$$tr^{F}_{N,F}: tr^{N,F}_{H} \to tr^{F}_{H},$$

satisfying

$$tr^{F}_{N,F} tr^{N,F}_{H} = tr^{F}_{H},$$

and such that it can be restricted to an $R$-algebra homomorphism

$$tr^{F}_{N,F}: tr^{N,F}_{H}/tr^{N,F}_{X} \to tr^{F}_{H}/tr^{F}_{X}.$$

Let us start by defining $tr^{F}_{N,F}$ and proving that it is $R$-linear.

**Lemma 4.35.** Let $R$ be $p$ local and let $M$ be a centric Mackey functor over $F$ on $R$. Then the map $tr^{F}_{N,F}$ from $tr^{N,F}_{H}$ to $tr^{F}_{H}$ obtained by setting

$$tr^{F}_{N,F} (tr^{N,F}_{H} (f)) = tr^{F}_{H} (f)$$

for every $f \in \text{End}(M \downarrow^{F}_{F,H})$ is surjective and $R$-linear.

**Proof.** If $tr^{F}_{N,F}$ is well defined then it is straightforward to check (using $R$-linearity of $tr^{N,F}_{H}$ and $tr^{F}_{H}$) that it is $R$-linear. Surjectiveness on the other hand follows immediately from definition. Thus we only need to prove that $tr^{F}_{N,F}$ is well defined or, equivalently, that for every $f$ as in the statement the image $tr^{F}_{H} (f)$ depends only of the image $tr^{N,F}_{H} (f)$. From Proposition 2.39 and Point 9 of Proposition 3.5 we have that

$$tr^{F}_{H} = \left( S^{-1} \right) \cdot tr^{F}_{S,F} tr^{F}_{F,H}.$$

Therefore it suffices to prove that we can write $r^{F}_{F_S} tr^{F}_{F_H}$ in terms of $tr^{N,F}_{F_H}$ From Proposition 3.5 and Theorem 4.14 we now have that

$$r^{F}_{F_S} tr^{F}_{F_H} = \sum_{(C,B) \in [H \times S]} tr^{F}_{F_S} \cdot tr^{F}_{F_C} \cdot c_{B} tr^{F}_{F_C},$$

$$= \sum_{(A,B) \in [N_{F} \times S]} \sum_{(B,\overline{B}) \in [H \times N_{F}]} tr^{F}_{F_{\phi(B)}} c_{\phi} c_{\psi} tr^{F}_{F_B}.$$
Where \( \theta, \varphi, \psi \in \mathcal{F} \) are representatives of \( \overline{\theta}, \overline{\varphi} \) and \( \overline{\psi} \) seen as isomorphisms onto their images. For every \((A, \varphi) \in [N_{\mathcal{F}} \times S]\) let \( \hat{\varphi} \) be a representative of \( \overline{\varphi} \) seen as an isomorphism and such that it restricts to \( \varphi \). Applying now Propositions 3.5 and 4.32 with this same notation we obtain that

\[
\sum_{(B, \psi) \in [H \times N_{\mathcal{F}} \times S]} \text{tr}^F_{\mathcal{F}_S} c_{\varphi} c_{\psi} \text{tr}^F_{\mathcal{F}_H} = \text{tr}^F_{\mathcal{F}_S} c_{\hat{\varphi}} \text{tr}^N_{\mathcal{F}_N} \text{tr}^N_{\mathcal{F}_H}.
\]

Combining both results we can conclude that

\[
\text{tr}^F_{\mathcal{F}_S} \text{tr}^F_{\mathcal{F}_H} = \sum_{(A, \varphi) \in [N_{\mathcal{F}} \times S]} \text{tr}^F_{\mathcal{F}_S} c_{\hat{\varphi}} \text{tr}^N_{\mathcal{F}_N} \text{tr}^N_{\mathcal{F}_H}.
\]

Thus concluding the proof.  

\textit{Remark 4.36.} In the previous proof the choices of representatives are not important because of Point 1 of Proposition 3.5.

From the previous lemma we can also deduce that the map \( \text{tr}^F_{N_{\mathcal{F}}} \) sends \( \text{tr}^N_{\mathcal{F}} \) to \( \text{tr}^F_{\mathcal{F}} \). This allows us to define the surjective map of \( \mathcal{R} \)-modules \( \text{tr}^F_{N_{\mathcal{F}}} \) as the quotient map of \( \text{tr}^F_{N_{\mathcal{F}}} \). This quotient map moreover has the following additional property.

\textbf{Lemma 4.37.} With the assumptions of Lemma 4.35 and notation above we have that the map \( \text{tr}^F_{N_{\mathcal{F}}} \) is multiplicative. In particular, from Lemma 4.35, \( \text{tr}^F_{N_{\mathcal{F}}} \) is a surjective map of \( \mathcal{R} \)-algebras.

\textit{Proof.} During this proof we will use the symbol \( \equiv \) in order to represent the equivalence class of endomorphisms with representative the endomorphism \( \cdot \).

Let \( f \) and \( g \) be any two endomorphisms of \( M \downarrow_{\mathcal{F}} \). Since \( \text{tr}^F_{\mathcal{F}} \) is an ideal of \( \text{tr}^N_{\mathcal{F}} \), then, from Point 8 of Proposition 3.5, we have that

\[
\text{tr}^F_{N_{\mathcal{F}}} \left( \text{tr}^N_{\mathcal{F}} (f) \right) \text{tr}^F_{N_{\mathcal{F}}} (g) = \text{tr}^F_{\mathcal{F}_H} (f) \text{tr}^F_{\mathcal{F}_H} (g),
\]

\[
= \text{tr}^F_{\mathcal{F}_H} (f) \text{tr}^F_{\mathcal{F}_H} (g),
\]

\[
= \text{tr}^F_{\mathcal{F}_H} \left( \text{tr}^F_{\mathcal{F}_H} (f) \text{tr}^F_{\mathcal{F}_H} (g) \right),
\]

\[
= \text{tr}^F_{N_{\mathcal{F}}} \left( \text{tr}^F_{\mathcal{F}_H} \left( \text{tr}^F_{\mathcal{F}_H} (f) \text{tr}^F_{\mathcal{F}_H} (g) \right) \right).
\]

Thus all we need to do is proving that

\[
\text{tr}^F_{N_{\mathcal{F}}} \left( \text{tr}^F_{\mathcal{F}_H} (f) \text{tr}^F_{\mathcal{F}_H} (g) \right) = \text{tr}^F_{N_{\mathcal{F}}} (f) \text{tr}^F_{N_{\mathcal{F}}} (g).
\]

Notice that for every \((A, \overline{\varphi}) \in [H \times H]\) such that \( \overline{\varphi} (A) \neq H \) for some representative \( \varphi \in \mathcal{F} \) of \( \overline{\varphi} \), we necessarily have \( \varphi (A) \in \mathcal{X} \). Therefore, if we define

\[
g_K := \sum_{(A, \overline{\varphi}) \in [H \times H]} c_{\varphi} \text{tr}^F_{\mathcal{F}_A} (g),
\]

\[
(A, \overline{\varphi}) \in [H \times H], \quad \varphi (A) = K
\]

45
for every $K \in \mathcal{X}$ then, applying Proposition 3.5, we will have that
\[ \text{tr}^N_{F_H} (f \text{tr}^F_{F_K} (g_K)) = \text{tr}^N_{F_K} (r^F_{F_K} (f) \text{tr}^F_{F_K} (g_K)) \in \text{tr}^N_{X}. \]

It now follows that
\[ \sum_{(A, \varphi) \in [H \times H]} \text{tr}^N_{F_H} \left( f \text{tr}^F_{F_{\varphi(A)}} c_{\varphi} \left( r^F_{F_A} (g) \right) \right) = \sum_{K \in \mathcal{X}} \text{tr}^N_{F_H} \left( f \text{tr}^F_{F_K} (g_K) \right) = 0. \]

On the other hand, by construction of $N_F$, we have that
\[ [H \times N_F H] = \{ (A, \varphi) \in [H \times H] | A = \varphi (A) = H \}. \]

Notice that if $\varphi (A) = H$ for a representative of $\varphi$ then it will be true for all of them. Therefore, from Proposition 3.5, and the previous result we obtain that
\[ \text{tr}^N_{F_H} \left( f \text{tr}^F_{F_H} (g) \right) = \sum_{(A, \varphi) \in [H \times H]} \text{tr}^N_{F_H} \left( f \text{tr}^F_{F_{\varphi(A)}} c_{\varphi} \left( r^F_{F_A} (g) \right) \right), \]
\[ = \sum_{(A, \varphi) \in [H \times N_F H]} \text{tr}^N_{F_H} \left( f \text{tr}^F_{F_{\varphi(A)}} c_{\varphi} \left( r^F_{F_A} (g) \right) \right), \]
\[ = \text{tr}^N_{F_H} (f) \text{tr}^N_{F_H} (g). \]

Thus concluding the proof. \[ \square \]

Proposition 4.15 is now an immediate consequence of Lemmas 4.35 and 4.37.

References

[1] Serge Bouc. Fused Mackey functors. Geom. Dedicata, 176:225–240, 2015.

[2] C.W. Curtis and I. Reiner. Methods of Representation Theory: Vol.: 1. : With Applications to Finite Groups and Orders. Pure and Applied Mathematics - Wiley. John Wiley & Sons, 1981.

[3] Antonio Díaz and Assaf Libman. The Burnside ring of fusion systems. Advances in Mathematics, 222(6):1943–1963, December 2009.

[4] J. A. Green. A transfer theorem for modular representations. Journal of Algebra, 1(1):73–84, 1964.
[5] J. A. Green. Axiomatic representation theory for finite groups. *Journal of Pure and Applied Algebra*, 1(1):41–77, 1971.

[6] Markus Linckelmann. Introduction to fusion systems. In *In Group representation theory, EPFL*, pages 79–113. Press, 2007.

[7] Hirosi Nagao and Yukio Tsushima. *Representations of finite groups*. Academic Press, London, 1989.

[8] Lluís Puig. Frobenius categories. *Journal of Algebra*, 303(1):309–357, 2006.

[9] Sune P. Reeh. *PhD thesis: Burnside rings of fusion systems*. PhD thesis, University of Copenhagen, 2014.

[10] Albert Ruiz and Antonio Viruel. The classification of $p$-local finite groups over the extraspecial group of order $p^3$ and exponent $p$. *Mathematische Zeitschrift*, 248:45–65, 09 2004.

[11] Hiroki Sasaki. Green correspondence and transfer theorems of Wielandt type for $G$-functors. *Journal of Algebra*, 79(1):98–120, 1982.

[12] Jacques Thévenaz and Peter Webb. Simple Mackey functors. In *roc. of 2nd International Group Theory Conference, Bressanone*, 1990.

[13] Jacques Thévenaz and Peter Webb. The structure of Mackey functors. *Transactions of the American Mathematical Society*, 347(6):1865–1961, 1995.

[14] Peter Webb. Two classifications of simple Mackey functors with applications to group cohomology and the decomposition of classifying spaces. *Journal of Pure and Applied Algebra*, 88(1):265–304, 1993.

[15] Peter Webb. A guide to Mackey functors. *Handbook of Algebra*, 2, 12 2000.