On Binary de Bruijn Sequences from LFSRs with Arbitrary Characteristic Polynomials

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Abstract—We propose a construction of de Bruijn sequences by the cycle joining method from linear feedback shift registers (LFSRs) with arbitrary characteristic polynomial \( f(x) \). We study in detail the cycle structure of the set \( \Omega(f(x)) \) that contains all sequences produced by a specific LFSR on distinct inputs and provide an efficient way to find a state of each cycle. Our structural results lead to an efficient algorithm to find all conjugate pairs between any two cycles, yielding the adjacency graph. The approach provides a practical method to generate a large class of de Bruijn sequences. Many recently-proposed constructions of de Bruijn sequences are shown to be special cases of our construction.

Index Terms—Binary periodic sequence, LFSR, de Bruijn sequence, cycle structure, adjacency graph, cyclotomic number.

I. INTRODUCTION

A binary de Bruijn sequence of order \( n \) is a binary sequence with period \( N = 2^n \) in which each \( n \)-tuple occurs exactly once in one period of the sequence. There are \( 2^{2^n-n} \) such sequences \([1]\).

Some of the earliest applications of de Bruijn sequences are in communication systems. These sequences are generated in a deterministic way, yet satisfy the randomness criteria in \([2]\) Ch. 5 and are balanced, containing the same number of 1s and 0s. Many of the sequences achieve the upper bound on the linear complexity, making them hard to guess. In cryptography, they have been used as a source of pseudo-random numbers and in key-sequence generators of stream ciphers \([3]\) Sect. 6.3]. In computational molecular biology, one of the three assembly paradigms in DNA sequencing is called the de Bruijn graph. The approach provides a practical method to generate a large class of de Bruijn sequences. Many recently-proposed constructions of de Bruijn sequences are shown to be special cases of our construction.

A work by Bruckstein et al.\([5]\) highlighted some important roles that de Bruijn sequences play in robust positioning patterns. Such objects have numerous useful applications, e.g., in robotics, smart pens, and camera localization. There has also been an increased interest in deploying de Bruijn sequences in various spread spectrum applications \([6]\). Given comparable parameters, the number of de Bruijn sequences is much larger than that of the other sequences that had been previously considered. Some of the de Bruijn sequences may have just the right cross- and auto-correlation properties to fit perfectly in.

On the construction of binary de Bruijn sequences, two yardsticks are often used to measure the goodness of a method or algorithm, namely, the number of constructed sequences and the efficiency of the construction in terms of both time and memory requirements. In numerous applications of de Bruijn sequences that we are aware of, it is crucial to have a lot of sequences to choose from. Methods to generate all binary de Bruijn sequences have been available in the literature (see, e.g., \([7]\) and \([8]\)). These methods, however, require a large memory space or long running time or, worse, both.

Many consider Fredricksen’s survey \([7]\) a good starting point to recall various properties and construction methods up to the early 1980s. A well-known construction approach called the cycle joining (CJ) method begins with a given Feedback Shift Register (FSR) and joins all cycles produced by the FSR into a single cycle by identifying the conjugate pairs shared by any pair of cycles.

The cycle structure of a Linear FSR (LFSR) can be studied using tools from the algebra of polynomial rings. It is then natural to construct de Bruijn sequences by applying the cycle joining method to LFSRs. Some LFSRs with simple cycle structure, such as the maximal length LFSRs, pure cycling registers, and pure summing registers, have been used to generate de Bruijn sequences using the said method in \([7]\), \([9]\), \([10]\). Hauge and Helleseth established a connection between the cycles generated by LFSRs and irreducible cyclic codes in \([11]\). The number of de Bruijn sequences obtained from these LFSRs is related to cyclotomic numbers, which in general are hard to determine precisely.

Recently, some classes of de Bruijn sequences are studied by C. Li et al.. In \([12]\) and \([13]\) respectively the sequences are derived from LFSRs with characteristic polynomials \((1+x)^3 p(x)\) and \((1 + x^3) p(x)\), where \( p(x) \) is a primitive polynomial of degree \( n > 2 \). Further generalized results are given in \([14]\) to include products of primitive polynomials whose degrees are pairwise coprime, leading to coprime periods of the sequences that form the cycle structure. This generalization yields a relatively small number of de Bruijn sequences when compared to the one we are proposing.

In a recent preprint \([15]\) Li et al. discussed the cycle structure of LFSRs with \( f(x) \) the product of distinct irreducible polynomials and presented some results about the conjugate pairs between any two cycles in \( \Omega(f(x)) \).

We proposed an efficient construction from LFSRs whose
characteristic polynomials are products of two distinct irreducible polynomials and showed that the construction generates a large number of de Bruijn sequences in \[16\]. In another work \[17\], whose preliminary results were presented at SETA 2016, we discussed in detail how to determine the cycle structure and find a state for each cycle for an arbitrary polynomial \(f(x) \in \mathbb{F}_q[x]\). Drawing insights from them, this present work generalizes the construction to de Bruijn sequences from LFSRs with arbitrary polynomials as their characteristic polynomials.

The main contributions are as follows.

1) We propose a construction of de Bruijn sequences by the cycle joining method from LFSRs with an arbitrary characteristic polynomial \(f(x) \in \mathbb{F}_q[x]\). The cycle structure and adjacency graph are studied in sufficient details. This allows us to propose a fast method to find a state belonging to a particular cycle and to design practical algorithms to find all conjugate pairs shared by any two cycles. Our construction covers all previously studied constructions of de Bruijn sequences by the self method as special cases.

2) We exhibit how our method can be implemented to build a practical de Bruijn sequence generator, which is the first to the best of our knowledge. Worked-out examples highlight the crucial steps.

3) We support our theoretical results with actual implementation, confirming the practicality of our method for reasonable values of \(n\), even without several possible optimization tricks.

After this introduction come preliminary notions and known results in Section \[II\] Section \[III\] presents the cycle structure and a method to find a state belonging to a given cycle. Section \[IV\] determines the adjacency graph and is divided into two main parts. The first part establishes important properties of the conjugate pairs. These are then used to design an efficient algorithm to find all conjugate pairs between any pair of cycles in the second part.

Section \[V\] discusses the complexity that arises from using a polynomial with repeated roots as a characteristic polynomial and highlights some parts of our method that can still be useful to analyze the situation. Combining the algorithms, Section \[VI\] shows how the tools fit together nicely. It uses the examples presented in the preceding sections to derive a large number of de Bruijn sequences of order 7. We implemented the algorithms and present a summary of our implementation in Section \[VII\]. The last section contains a brief conclusion and some possible future directions.

II. PRELIMINARIES

For convenience, we recall needed definitions and results, mostly from \[2\ Ch. 4\].

An \(n\)-stage shift register is a circuit consisting of \(n\) consecutive storage units, each containing a bit, regulated by a clock. When it pulses, the bit in each storage unit is shifted to the next stage in line. A shift register turns into a binary code generator when one adds a feedback loop which outputs a new bit \(s_n\) based on the \(n\) bits \(s_0 = (s_0, \ldots, s_{n-1})\) called an initial state of the register. The corresponding feedback function \(f(x_0, \ldots, x_{n-1})\) is the Boolean function that outputs \(s_n\) on input \(s_0\).

A feedback shift register (FSR) outputs a binary sequence \(s = s_0, s_1, \ldots, s_n, \ldots\) satisfying the recursive relation

\[
s_{n+\ell} = f(s_\ell, s_{\ell+1}, \ldots, s_{\ell+n-1}) \quad \text{for } \ell = 0, 1, 2, \ldots.
\]

For \(N \in \mathbb{N}\), if \(s_{i+N} = s_i\) for all \(i \geq 0\), then \(s\) is \(N\)-periodic or with period \(N\) and one writes \(s = (s_0, s_1, s_2, \ldots, s_{N-1})\). The period of the all zero sequence \(0\) is 1. When the context is clear, 0 also denotes a string of zeroes or a zero vector. We call \(s_i, s_{i+1}, \ldots, s_{i+N-1}\) the \(i\)-th state of \(s\) and states \(s_{i-1}\) and \(s_{i+1}\) the predecessor and successor of \(s_i\), respectively.

Let \(\land\) and \(\lor\) denote the logical AND and OR respectively. Given a constant \(c \in \mathbb{F}_2\) and binary sequences (or vectors) \(a = (a_0, a_1, \ldots, a_{N-1})\) and \(b = (b_0, b_1, \ldots, b_{N-1})\), let

\[
a \land b := (a_0 \land b_0, a_1 \land b_1, \ldots, a_{N-1} \land b_{N-1})
\]

\[
a \lor b := (a_0 \lor b_0, a_1 \lor b_1, \ldots, a_{N-1} \lor b_{N-1})
\]

An FSR, distinct initial states generate distinct sequences, forming the set \(\Omega(f)\) of cardinality \(2^n\). All sequences in \(\Omega(f)\) are periodic if and only if the feedback function is non-singular, i.e., \(f\) can be written as \(f(x_0, x_1, \ldots, x_n) = x_0 + g(x_1, \ldots, x_n)\), where \(g(x_1, \ldots, x_n)\) is some Boolean function with domain \(\mathbb{F}_2^r\) \[18\ p. 116\]. In this paper, the feedback functions are all non-singular. An FSR is called linear or an LFSR if its feedback function is linear, and nonlinear or an NLFSR otherwise.

The characteristic polynomial of an \(n\)-stage LFSR with feedback function \(f(x_0, x_1, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} c_i x^i\) is the polynomial \(f(x) = x^n + \sum_{i=0}^{n-1} c_i x^i \in \mathbb{F}_2[x]\). A sequence \(s\) may have many characteristic polynomials. We call the characteristic polynomial with the lowest degree the minimal polynomial of \(s\). It represents the LFSR of shortest length that generates \(s\). Given an LFSR with characteristic polynomial \(f(x)\), the set \(\Omega(f)\) is also denoted by \(\Omega(f(x))\).

A sequence \(v\) is said to be a \(d\)-decimation sequence of \(s\), denoted by \(v = s^{(d)}\), if \(v_j = s_{jd}\) for all \(j \geq 0\).

For a sequence \(s\), the (left) shift operator \(L\) is given by \(Ls = L(s_0, s_1, \ldots, s_{N-1}) = (s_1, s_2, \ldots, s_{N-1}, s_0)\) with the convention that \(L^0s = s\). The set

\[
[s] := \{s, Ls, L^2s, \ldots, L^{N-1}s\}
\]

is a shift equivalent class or a cycle in \(\Omega(f)\).

The set of sequences in \(\Omega(f)\) can be partitioned into cycles. If \(\Omega(f)\) consists of exactly \(r\) cycles \([s_1], [s_2], \ldots, [s_r]\) for some \(r \in \mathbb{N}\), then the cycle structure of \(\Omega(f)\) is

\[
\Omega(f) = [s_1] \cup [s_2] \cup \ldots \cup [s_r].
\]

A conjugate pair consists of a state \(v = (v_0, v_1, \ldots, v_{n-1})\) and its conjugate \(\bar{v} = (v_0 + 1, v_1, \ldots, v_{n-1})\). Cycles \(C_1\) and \(C_2\) are adjacent if they are disjoint and there exists \(v\) in \(C_1\) whose conjugate \(\bar{v}\) is in \(C_2\). Adjacent cycles \(C_1\) and \(C_2\) with the same feedback function \(g(x_0, x_1, \ldots, x_n)\) can be joined into a single cycle by interchanging the successors of \(v\) and \(\bar{v}\). The feedback function of the resulting cycle is

\[
h(x_0, \ldots, x_{n-1}) = g(x_0, \ldots, x_{n-1}) + \prod_{i=1}^{n-1} (x_i + v_i + 1).
\]
The feedback functions of the new de Bruijn sequences are completely determined once the corresponding conjugate pairs are found. It is therefore crucial to find all pairs and to place them in the adjacency graph.

**Definition 1.** ([79]) For an FSR with feedback function \( f \), its adjacency graph \( G \) is an undirected multigraph whose vertices correspond to the cycles in \( \Omega(f) \). There exists an edge between two vertices if and only if they are adjacent. A conjugate pair labels every edge. The number of edges between any pair of cycles is the number of conjugate pairs that they share.

By definition, \( G \) contains no loops. There is a one-to-one correspondence between the spanning trees of the adjacency graph \( G \) and the de Bruijn sequences constructed by the cycle matching method. The details can be found in [19] and [11]. The following result, a variant of the BEST (de Bruijn, Ehrenfest, Smith, and Tutte) Theorem adapted from [20 Sect. 7], provides the counting formula.

**Theorem 1.** (BEST) Let \( G \) be the adjacency graph of an FSR with vertex set \( \{V_1, V_2, \ldots, V_l\} \). Let \( M = (m_{i,j}) \) be the \( \ell \times \ell \) matrix derived from \( G \) in which \( m_{i,j} \) is the number of edges incident to vertex \( V_i \) and \( m_{i,j} \) is the negative of the number of edges between vertices \( V_i \) and \( V_j \) for \( i \neq j \). Then the number of the spanning trees of \( G \) is the cofactor of any entry of \( M \).

The cofactor of entry \( m_{i,j} \) in \( M \) is \((-1)^{i+j} \) times the determinant of the matrix obtained by deleting the \( i \)-th row and \( j \)-th column of \( M \).

An important tool we will need later is the generalized Chinese Remainder Theorem (CRT).

**Theorem 2.** (Generalized CRT) ([21] Thm. 2.4.2) Let \( 2 \leq k \in \mathbb{N} \). Given integers \( a_1, \ldots, a_k \) and positive integers \( m_1, \ldots, m_k \), there exists \( \ell \in \mathbb{N} \) such that

\[
\ell \equiv a_i \pmod{m_i} \quad \text{for all } i \in \{1, \ldots, k\}
\]

if and only if for arbitrary distinct integers \( 1 \leq i \neq j \leq k \), we have \( a_i \equiv a_j \pmod{\gcd(m_i, m_j)} \).

If \( \ell \) is a solution of this system of congruences, then \( \ell' \equiv \ell \pmod{\text{lcm}(m_1, \ldots, m_k)} \).

Let \( \{p_1(x), p_2(x), \ldots, p_s(x)\} \) be a set of \( s \) pairwise distinct irreducible polynomials over \( \mathbb{F}_2 \) and \( n_i := \deg(p_i(x)) \). From hereon, let

\[
f(x) := \sum_{i=1}^{s} p_i(x) \quad \text{and } n := \sum_{i=1}^{s} n_i.
\]

**III. THE CYCLE STRUCTURE AND A STATE IN EACH CYCLE**

This section is divided into two subsections. The first one recalls established results on the cycle structure of \( \Omega(f(x)) \). The second one provides a method to determine a state belonging to each of the cycles in \( \Omega(f(x)) \).

A. The Cycle Structure of \( \Omega(f(x)) \)

Let \( g(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_0 \in \mathbb{F}_2[x] \) be an irreducible polynomial of degree \( n \) with a root \( \beta \in \mathbb{F}_{2^n} \). Then there exists a primitive element \( \alpha \in \mathbb{F}_{2^n} \) such that \( \beta = \alpha^t \) for some \( t \in \mathbb{N} \) and \( e = 2^t - 1 \) is the order of \( \beta \).

Using the Zech logarithmic representation (see, e.g., [2 p. 39]), write \( 1 + \alpha^t = \alpha^{\tau_n(t)} \) where \( \tau_n(t) \) is the Zech logarithm relative to \( \alpha \). It induces a permutation on \( \{1, 2, \ldots, 2^n - 2\} \). Note that \( \tau_n(t) := \infty \) for \( t \equiv 0 \pmod{2^n - 1} \) and \( \alpha^\infty \equiv 0 \).

The cyclotomic classes \( C_i \subseteq \mathbb{F}_{2^n}, \) for \( 0 \leq i < t \), are

\[
C_i = \{\alpha^{i+s} \mid 0 \leq s < e\} = \{\alpha^i \beta^s \mid 0 \leq s < e\} = \alpha^i C_0.
\]

(1)

The cyclotomic numbers \((i,j)_i\), for \( 0 \leq i, j < t \), are

\[
(i,j)_i = |\{x \mid x \in C_i, x+1 \in C_j\}|.
\]

(2)

The theory of LFSRs in [2 Ch. 4] tells us that \( \Omega(g(x)) = \{0\} \cup \{n\} \) where \( \mathbf{n} \) is the \( m \)-sequence (also known as maximal sequence length) with period \( 2^n - 1 \) having the following shift-and-add property.

**Lemma 1.** ([2] Thm. 5.3) Let \( \mathbf{n} \) be an \( m \)-sequence with period \( 2^n - 1 \). Then, for \( 0 < i < 2^n - 1 \), there exists \( 0 < j < 2^n - 1 \) such that \( \mathbf{n} + L^i \mathbf{n} = L^j \mathbf{n} + \mathbf{j} = \tau_n(i) \).

If \( g(x) \) is primitive, then \( t = 1, e = 2^n - 1 \), and there exists only one cyclotomic class. Hence, \( \Omega(g(x)) = \{0\} \cup \{n\} \), where \( \mathbf{n} \) is the \( m \)-sequence and, hence, the cycle \( \{\mathbf{n}\} \) for \( 1 \leq i < t \) by (3).

The following result was proved in [15] by using the properties of cyclotomic numbers.

**Lemma 2.** Let \( g(x) \in \mathbb{F}_2[x] \) be an irreducible polynomial of degree \( n \) and order \( e \) (making \( t = (2^n - 1)/e \) with \( \Omega(g(x)) \) as presented in (3)). Then, for each triple \((i,j,k)\) with \( 0 \leq i, j, k < t \), we have

\[
(j-i,k-i) = \{|a \mid u_i + L^a u_j = L^b u_k; 0 \leq a, b < e\}.
\]

(4)

For a given \( f(x) \), since \( p_i(x) \) is irreducible of degree \( n_i \) and order \( e_i \), we have \( t_i := (2^{n_i} - 1)/e_i \) and write

\[
\Omega(p_i(x)) = \{0\} \cup \{s_{i_0}^1 \cup \{s_{i_1}^1 \cup \ldots \cup s_{i_{t_i-1}}^1\}.
\]

(5)

We can then infer the cycle structure of \( \Omega(f(x)) \) by some properties of LFSRs established in [15]. Here we provide another proof.

**Lemma 3.** The cycle structure of \( \Omega(f(x)) \) is given by \( \{\mathbf{n}\} \) with \( \gamma := \gcd(e, \text{lcm}(n_1, \ldots, n_s)) \). Letting

\[
f_i := \begin{cases} e_i & \text{if } a_i = 1 \\ 1 & \text{if } a_i = 0 \end{cases}
\]

and \( \delta := \gcd(f_s, \text{lcm}(f_1, \ldots, f_{s-1})) \),

\( \Omega(f(x)) \) is more succinctly expressed in \( \{\mathbf{n}\} \).

**Remark 1.** There are some noticeable facts regarding the representation of the cycles.

1. Equation (5) highlight the types of nonzero cycles in \( \Omega(f(x)) \). Cycles \( \{s_{i_0}^1 \} \) are in \( \Omega(p_i(x)) \). For \( S_i := \{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, s\} \) with \( i_j < i_k \) for
\[ \Omega(f(x)) = [0] \cup \left( \bigcup_{t_i \leq s} \bigcup_{j=0}^{t_i-1} [s_{j+1}^s] \right) \cup \left( \bigcup_{1 \leq t_1 < t_2 \leq s} \bigcup_{j_1=0}^{t_1-1} \bigcup_{j_2=0}^{t_2-1} \bigcup_{\ell_2=0}^{\gcd(e_2, e_1)-1} \left[ s_{j_1}^{i_1} + L^{\ell_2} s_{j_2}^{i_2} \right] \right) \cup \ldots \]

\[ \Omega(f(x)) = \bigcup_{a_i \in \mathbb{F}_2} \bigcup_{1 \leq \ell_i \leq s} \bigcup_{j_0=0}^{t_1-1} \bigcup_{j_0=0}^{t_2-1} \bigcup_{\ell_0=0}^{\gcd(f_2, f_1)-1} \bigcup_{\ell_2=0}^{\ell_1-1} \bigcup_{\ell_0=0}^{\delta-1} \left[ a_1 s_{j_1}^{i_1} + a_2 L^{\ell_2} s_{j_2}^{i_2} + \ldots + a_s L^{\ell_s} s_{j_s}^{i_s} \right]. \] (6)

Hence, sequences of the form (8) can be partitioned into at most \( \gcd(\eta, e_s) \) cycles. Thus, for given \( s \) and \( s_{j_s} \), they can be partitioned into exactly \( \gcd(\eta, e_s) \) cycles, each written as

\[ \left[ s_{j_1}^{i_1} + L^{\ell_2} s_{j_2}^{i_2} + \ldots + L^{\ell_s} s_{j_s}^{i_s} \right] \]

where \( 0 \leq \ell_s < \gcd(\eta, e_s) = \gcd(e_s, \lcm(e_1, \ldots, e_{s-1})) \).

Equation (9) follows directly from Equation (7) by the fact that if \( a_i = 0 \), then \( a_i s_{j_i} = 0 \), which can be safely ignored in the representation of the cycles.

Consider the number of cycles in \( \Omega(f(x)) \) that contain sequences with a given period. The \( 2^s \) possible periods can be expressed as \( \prod_{i=1}^{\ell_s} t^i \) with \( a_i \in \mathbb{F}_2 \).

1) There exists one all-zero cycle \([0]\).

2) There are \( t_1 \cdot t_2 \cdot \gcd(e_1, e_2) \) cycles containing sequences with period \( \lcm(e_1, e_2) \) for \( 1 \leq t_1 < t_2 \leq s \). This generalizes (16) Proof of Lem. 4 where \( s = 2 \).

3) Let \( 0 \leq t_1 < t_2 < \ldots < t_k \leq s \). Sequences with period \( \lcm(e_1, \ldots, e_k) \) are partitioned into

\[ \prod_{j=1}^{k} t_{j_1} \cdot \gcd(e_{j_1}, e_{j_2}) \prod_{r=3}^{k} \gcd(e_{j_r}, \lcm(\{e_{j_r}\}_{j=1}^{r-1})) = \prod_{j=1}^{k} \frac{(2^{n_j} - 1)}{\lcm(e_{j_1}, \ldots, e_{j_k})} \]

cycles. Sequences with period \( \lcm(e_1, \ldots, e_s) \), in particular, are partitioned into \( \frac{\prod_{j=1}^{s} (2^{n_j} - 1)}{\lcm(e_1, \ldots, e_s)} \) cycles.

**B. Finding a State belonging to Each Cycle**

Once the number of cycles in \( \Omega(f(x)) \) is determined, we want to efficiently store them. Since the periods of some of the cycles can be fairly large, one should avoid taking the naive approach of storing the whole cycle. In fact, if the feedback function is known, it suffices to store only one state belonging to each cycle. This section provides a method to determine a state of every cycle in \( \Omega(f(x)) \) and proposes a new state representation.

Recall the state \( s_i \) and its successor \( s_{i+1} \) of an \( n \)-stage FSR sequence \( s \) with feedback function \( f(x_0, \ldots, x_{n-1}) \) from Section II. A state operator \( T_n \) turns \( s_i \) into \( s_{i+1} \) with \( s_{i+n} = f(s_i, \ldots, s_{i+n-1}) \). The subscript \( n \) of \( T_n \) indicates that \( s_i \) is an \( n \)-stage state and is often omitted when no
confusion arises. If \( s_i \in [s] \) and \( e \) is the period of \( s \), then the \( e \) distinct states of \([s]\) are

\[
s_i, Ts_i = s_{i+1}, \ldots, T^{e-1}s_i = s_{i+e-1}.
\]

It suffices to identify just one state in a given cycle since applying \( T \) a suitable number of times generates all \( e \) distinct states. To reduce clutter, we use \( T \) to denote the state operator for distinct cycles with distinct stages.

Let \( g(x) \in \mathbb{F}_2[x] \) be an irreducible polynomial of degree \( n \), order \( e \), and \( \beta \) as a root. Hence, \( t = (2^n - 1)/e \) and \( \Omega(g(x)) \) is as given in \( \text{[3]} \). Exhaustive search is obviously an option to find one of the \( e \) distinct states belonging to each cycle \([u_i] \in \Omega(g(x))\). We provide a much better method that uses the notion of decimation.

Recall, e.g., from \([2] \) Sect. 4.6], that a \( d \)-decimation \( m^{(d)} \) of an \( m \)-sequence \( m \) is also an \( m \)-sequence if and only if \( \text{gcd}(d, 2^n - 1) = 1 \). It is well-known that the number of distinct \( m \)-sequences of period \( 2^n - 1 \), up to cyclic shifts, is \( \lambda := \phi(2^n - 1)/n \) where \( \phi(.) \) is the Euler totient function. There is a bijection between the set of all such sequences and the set of all primitive polynomials of degree \( n \) in \( \mathbb{F}_2[x] \).

The trace function from \( \mathbb{F}_{2^n} \) to \( \mathbb{F}_2 \) is given by \( \text{Tr}(x) = x + x^2 + \ldots + x^{2^n - 1} \). The sequence \( m = (m_0, m_1, \ldots, m_{2^n-2}) \) whose characteristic polynomial \( q(x) \) is primitive of degree \( n \) with a root \( \alpha \) satisfying \( \beta = \alpha^t \) can be described using

\[
m_i = \text{Tr}(\gamma \alpha^t) \neq 0 \quad \forall \gamma \in \mathbb{F}_{2^n}, \quad i = 0, 1, 2, \ldots, 2^n - 2.
\]

Without loss of generality, let \( \gamma = 1 \), i.e., \( m_i = \text{Tr}(\alpha^t) \). From \( m \), construct the \( t \) distinct \( t \)-decimation sequences, each of period \( e \):

\[
u_0 = m^{(t)}, \quad u_1 = (Lm^{(t)})^{(t)}, \quad u_{t-1} = (L^{t-1}m^{(t)}).
\]

Observe that the entries in the resulting sequences satisfy

\[
(u_k)_j = \text{Tr}(\alpha^{k+tj}) \quad 0 \leq k < t \quad 0 \leq j < e.
\]

Since \( \beta = \alpha^t \), each cycle \([u_i]\) is a cycle in \( \Omega(g(x)) \).

Starting from an arbitrary \( n \times t \) consecutive elements of \( m \), one can derive \( t \) distinct \( n \)-stage states by the above \( t \)-decimating process. It is then straightforward to verify that each of the derived states corresponds to one nonzero cycle.

In general, the order of the cycles in \( \Omega(g(x)) \) can be done arbitrarily. Since we want to use the correspondence between the cycles and the cyclotomic classes established in \([11]\) using Lemma \([2]\) we must ensure that \((1, 0, \ldots, 0)\) is the initial state of the cycle that we label \([u_0]\), corresponding to \( C_0 \).

**Proposition 1.** Let a non-primitive irreducible polynomial \( g(x) \) and its associated primitive polynomial \( q(x) \) be given. Then there exists an initial state \( s_0 \) such that \( q(x) \) generates an \( m \)-sequence \( m \) with \((1, 0, \ldots, 0) \in \mathbb{F}_2^n \) as the first \( n \) entries in \( m^{(t)} \).

**Proof.** Using the definition of characteristic polynomial, \( s_0 \) can be computed by solving a system of \( n \) linear equations. Write \( g(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + 1 \) and let \( A \) be its companion matrix: the \( n \times n \) matrix whose first row and last column are respectively \((0, \ldots, 0, 1)\) and \((1, a_1, a_2, \ldots, a_{n-1})^T \). The remaining entries form the identity \( I_{n-1} \) matrix. Then the respective first entry of the state vectors \( s_0, s_0A^t, s_0A^{2t}, \ldots, s_0A^{(n-1)t} \) must be \( 1, 0, 0, \ldots, 0 \). Solving the system gives us \( s_0 \).

Algorithm \([1] \) gives the steps to generate the required states.

**Algorithm 1 Finding a State in a Nonzero Cycle in \( \Omega(g(x)) \)**

**Input:** An irreducible polynomial \( g(x) \in \mathbb{F}_2[x] \).

**Output:** A state \( v_j \) of each nonzero cycle \([u_j]\) \( \in \Omega(g(x)) \).

1. \( e \leftarrow \text{order of } g(x) \); \( t \leftarrow (2^n - 1)/e \).
2. if \( t = 1 \) then
3. \( v_0 \leftarrow (1, 0, \ldots, 0) \in \mathbb{F}_2^n \) and break.
4. else
5. \( q(x) \leftarrow \text{an associated primitive polynomial of } g(x) \).
6. \( m \leftarrow \text{the sequence generated by } q(x) \) on input \( s_0 \in \mathbb{F}_2^n \).
7. for \( j \) from 0 to \( t - 1 \) do
8. \( u_j \leftarrow (L^j m)^{\ell} \).
9. \( v_j \leftarrow \text{first } n \text{ entries of } u_j \).
10. end for
11. end if

Given the set of all non-primitive irreducible polynomials and the set of all primitive polynomials, both of degree \( n \), we can perform a systematic check to match each element of order \( e \) in the former set to a subset of the latter set that contains all associated primitive polynomials. One of them can then be used in Line \( 5 \) of Algorithm \([1] \) This task can become computationally expensive for large values of \( n \) and \( t \) (small values of \( e \)). Decimation helps us keep the time complexity low by avoiding costly arithmetics.

**Lemma 4.** Let \( g(x) \) be a non-primitive irreducible polynomial of degree \( n \) with a root \( \beta \) and order \( e \). There are \( \phi(2^n - 1)/\phi(e) \) primitive polynomials that can be associated with \( g(x) \). Such a polynomial \( q(x) \) has degree \( n \) with a root \( \alpha \) that satisfies \( \beta = \alpha^e \).

**Proof.** Each of the roots of \( q(x) \), say \( \varphi \), satisfies \( \varphi^{2^n - 1} = 0 \). It is then clear that \( g(x) \) can be an associated primitive polynomial of only one irreducible polynomial of degree \( n \) and order \( e \). Two distinct non-primitive irreducible polynomials, both with degree \( n \) and order \( e \), have the same number of associated primitive polynomials. There are \( \phi(2^n - 1)/n \) primitive polynomials with degree \( n \) and there are \( \phi(e)/n \) non-primitive irreducible polynomials with degree \( n \) and order \( e \). Taking their ratio completes the proof.

Let \( q_j(x) \) be a primitive polynomial and let \( m_j \) be its corresponding \( m \)-sequence on input \((1, 0, \ldots, 0) \in \mathbb{F}_2^n \). The set of all shift inequivalent \( m \)-sequences is \( \{m_1, m_2, \ldots, m_j\} \). Note that one can also build this set from an arbitrary \( m \)-sequence \( m \) by simply collecting all the \( d_i \)-decimations \( \{m^{(d_i)}\} \) with \( d_i \) satisfying \( \text{gcd}(d_i, 2^n - 1) = 1 \). To avoid duplication, for \( i \neq k \), we add the requirement that \( d_i \) and \( d_k \) do not belong to the same conjugate coset, i.e., \( d_i \neq 2^k d_k \pmod{2^n - 1} \).

Using the first option, for each \( j \) derive \( m_j^{(t)} \) with period \( e \) and let \( v_j \) denote the first \( n \) entries in \( m_j^{(t)} \). Feed \( v_j \) as the initial state of the LFSR with characteristic polynomial \( g(x) \)
and call the resulting sequence $s_j$. If the first $2n$ entries of $s_j$ are equal to the first $2n$ entries of $m^{(i)}$, then $q_j(x)$ can be associated with $g(x)$. As $n$ grows and for smaller values of $e$, our saving becomes more prominent.

Here we supply an abbreviated list for $4 \leq n \leq 11$ by omitting the reciprocals. The $\text{reciprocal}$ of a polynomial $h(x) \in F_2[x]$ is $h^*(x) := x^{\deg(h(x))} h(1/x)$. The roots of $h^*$ are the inverses of the roots of $h(x)$. The reciprocal of an irreducible polynomial is again irreducible. The order of $h^*$ is equal to that of $h(x)$. The relationship between the roots of $h^*$ and those of $h(x)$ clearly implies that if $g(x)$ is an associated primitive polynomial of a non-primitive irreducible polynomial $g(x)$, then $q^*(x)$ is an associated primitive polynomial of $g^*(x)$. More details can be found in, e.g., [22, Ch. 40].

Only one associated primitive polynomial for each non-primitive irreducible polynomial is listed in Table I and only the coefficients are given, in descending degree of the monomials. In Entry 2 of $n = 6$, for example, 10101111 stands for $g(x) = x^9 + x^4 + x^2 + x + 1$. One of its three associated primitive polynomials is $p(x) = x^6 + x + 1$, written as 1000011. The self-reciprocal irreducible polynomials are shaded.

**Example 1.** Consider $g(x) = x^4 + x^3 + x^2 + x + 1$, an irreducible polynomial of order 5, with $p(x) = x^4 + x + 1$ as the associated primitive polynomial. Let $m$ be the $m$-sequence that $p(x)$ generates on input $(1, 0, 0, 0, 0)$:

$$(1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0).$$

Computing $(L^j m)^{(3)}$ with $j \in \{0, 1, 2\}$ gives us $[u_j]$. We take as the respective initial states the first 4 entries of $(L^j m)^{(3)}$: $(1, 0, 0, 0), (0, 1, 1, 1), \text{and} (0, 0, 1, 0).$

Based on known states of the cycles in $\Omega(p_1(x))$, we can efficiently determine a state in each of the cycles in $\Omega(f(x))$. For each $1 \leq i \leq s$, construct the $n \times n$ matrix $P_i$ in the following manner. The $j$-th row of $P_i$ is the first $n$ bits of the sequence generated by the LFSR with characteristic polynomial $p_i(x)$ whose $n_j$-stage initial state has 1 in the $j$-th position and 0 elsewhere. Combine the resulting matrices into $P_{n \times n} = \begin{pmatrix} P_1 & P_2 & \cdots & P_s \end{pmatrix}$.

**Proposition 2.** $\mathcal{P}$ is of full rank, i.e., $\text{rank}(\mathcal{P}) = n$.

**Proof.** Let $\alpha_{i,j}$ denote the $j$-th row of $P_i$. We show that the rows $\alpha_{i,j}$ of $\mathcal{P}$ are linearly independent for all $(i, j)$ with $1 \leq i \leq s$ and $1 \leq j \leq n_i$.

Note that $\alpha_{i,j}$ is the first $n_i$ bits of the sequence from LFSR with characteristic polynomial $p_i(x)$ and initial $n_i$-stage state $(0, \ldots, 0, 1, 0, \ldots, 0) \in F_2^{n_i}$, where the unique 1 is in the $j$-th position. For a fixed $i$, it is clear that $\text{rank}(P_i) = n_i$.

For a contradiction, suppose that there exists a linear combination of the $n$ rows of $\mathcal{P}$

$$\sum_{i=1}^{n} \sum_{j=1}^{n_i} a_{i,j} \alpha_{i,j} = 0$$

with not all $a_{i,j} = 0$. Without loss of generality, write

$$\sum_{j=1}^{n_1} a_{1,j} \alpha_{1,j} = \sum_{i=2}^{n} \sum_{j=1}^{n_i} a_{i,j} \alpha_{i,j}$$

(10) with not all $a_{1,j} = 0$.

The left hand side of (10) is the first $n$ bits of a sequence in $\Omega(p_1(x))$ with a nonzero initial state of length $n_1$ while the right hand side is the first $n$ bits of a sequence in $\Omega(p_2(x) \cdots p_s(x))$ with a nonzero initial state of length $n-n_1$.

Since $\deg(p_1(x))$ and $\deg(p_2(x) \cdots p_s(x))$ are $< n$, if the first $n$ bits of these two sequences are equal, then they must be the same sequence. Hence, there exists a sequence that simultaneously belongs to both $\Omega(p_1(x))$ and $\Omega(p_2(x) \cdots p_s(x))$. Since the irreducible polynomials $p_1(x), \ldots, p_s(x)$ are pairwise distinct, this sequence must be 0, a contradiction.

Let $\mathcal{P}$ be already constructed. Let $v \in F_2^n$ and $a_i \in F_2^{n_i}$, with $1 \leq i \leq s$, be respectively the $n$-stage and $n_i$-stage states of the sequences in $\Omega(f(x))$ and $\Omega(p_i(x))$. Two facts are immediately clear.
1. $\mathcal{P}$ provides a one-to-one correspondence between $v$ and $(a_1, a_2, \ldots, a_s)$ via $v = (a_1, a_2, \ldots, a_s)\mathcal{P}$.

2. $\mathcal{P}$ and $T$ commute since $Tv = T[(a_1, a_2, \ldots, a_s)\mathcal{P}] = (Ta_1, Ta_2, \ldots, Ta_s)\mathcal{P}$.

Hence, any sequence $s \in \Omega(f(x))$ with initial state $v$ can be written as the sum of sequences $s_i$ from $\Omega(p_i(x))$ with corresponding initial states $a_{i}$ for all $i$. We can conveniently use $(a_1, a_2, \ldots, a_s)$ to represent the state $v$.

A sequence $s$ generated by $f(x)$ has a longer period and is more complex to study than the *component sequences* $s_i$. Representing the state $v$ of $s$ in terms of states $a_i$ of corresponding sequences $s_i$ helps us study the global properties while relying on local properties only. The new state representation offers a significant gain in storage efficiency. In addition, a state belonging to each cycle in $\Omega(f(x))$ can be quickly computed using the representation.

Suppose that we have obtained the set $A_i$ of states corresponding to the $t_i + 1$ distinct cycles in $\Omega(p_i(x))$ given in (5). Letting $0$ be the state of $[0]$ and $a_i^0$ a nonzero state of $[s_i^0]$, we get

$$A_i := \{a_0^i, a_1^i, \ldots, a_{t_i - 1}^i, a_i^0 = 0\}.$$

For convenience, let each of the states be the initial state of its corresponding sequence. Then

$$v = (a^1_{j_1}, a^2_{j_2}, \ldots, a^s_{j_s})\mathcal{P} \text{ with } a^i_{j_i} \in A_i \quad (11)$$

can be taken as an initial state of a sequence $s \in \Omega(f(x))$. Note that $s$ has the form

$$a_1s^1_{j_1} + a_2s^2_{j_2} + \ldots + a_ss^s_{j_s},$$

where $a_i s^i_{j_i} = 0$ if $a_i = 0$. For all other cases, $a_i = 1$.

For $1 \leq i \leq s$, let $\ell_i$ be a nonnegative integer and let $v$ be as given in (11). By the properties of $\mathcal{P}$ and $T$,

$$w = (T^{\ell_1}a^1_{j_1}, T^{\ell_2}a^2_{j_2}, \ldots, T^{\ell_s}a^s_{j_s})\mathcal{P};$$

is a state of cycle

$$[a_1L^{\ell_1}s^1_{j_1} + a_2L^{\ell_2}s^2_{j_2} + \ldots + a_sL^{\ell_s}s^s_{j_s}] = [a_1s^1_{j_1} + a_2L^{\ell_2}s^2_{j_2} + \ldots + a_sL^{\ell_s}s^s_{j_s}].$$

This approach enables us to quickly find a state belonging to any cycle in $\Omega(f(x))$.

**Example 2.** Let

$$f(x) = (x + 1)(x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1).$$

Note that $p_3(x)$ is not primitive. One gets $\mathcal{P}_1 = [1]$,

$$\mathcal{P}_2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \text{ and}$$

$$\mathcal{P}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$
The relevant cycles and sets of states are
\[ \Omega(p_1(x)) = \{0 \} \cup \{ s_0^3 = 1 \}, A_1 = \{ a_0^1 = (1), a_1^1 = (0) \}, \]
\[ \Omega(p_2(x)) = \{0 \} \cup \{ s_0^2 = (1, 0, 1) \}, \]
\[ A_2 = \{ a_0^2 = (1, 0), a_2^2 = (0, 0) \}, \]
\[ \Omega(p_3(x)) = \{0 \} \cup \{ s_0^3 = (0, 1, 0, 0, 1) \} \cup \{ s_1^3 = (0, 1, 1, 1, 1) \} \]
\[ \cup \{ s_2^3 = (0, 0, 1, 0, 1) \}, \]
\[ A_3 = \{ a_0^3 = (1, 0, 0, 0), a_1^3 = (0, 1, 1, 1), \]
\[ a_2^3 = (0, 0, 0, 0) \}. \]

The periods of the nonzero sequences in \(\Omega(p_i(x))\) are 1, 3, and 5. We write each cycle in \(\Omega(f(x))\) as \(\{a_1 1 + a_2 s_0^2 + a_3 s_0^3\}\) with \(a_i \in F_2\) and \(j \in \{0, 1, 2\}\).

Choosing \(a = (a_0, a_1, a_2)^T = (1, 0, 1, 0, 0, 0)\) implies that \(v = a P = (1, 1, 0, 0, 0, 0)\) is a state of \([1 + s_0^2 + s_0^3]\). A state of each of the cycles in \(\Omega(f(x))\) can be similarly derived. Table I lists down the states.

Example 2 demonstrates how our approach quickly finds a state belonging to any cycle in \(\Omega(f(x))\). If \(f(x)\) is the product of many distinct irreducible polynomials whose corresponding periods are not coprime, then there are less computational tasks to perform and our method works even better.

IV. THE ADJACENCY GRAPH OF \(\Omega(f(x))\)

We now focus on finding all conjugate pairs between any two cycles in \(\Omega(f(x))\) to rapidly construct the adjacency graph of LFSR with characteristic polynomial \(f(x)\). The first subsection presents the properties of conjugate pairs between any two cycles in \(\Omega(f(x))\). The second proposes a generic algorithm to find all conjugate pairs between two cycles.

We assume that each nonzero cycle \([s_i^j] \in \Omega(p_i(x))\) in \(\mathbb{F}_2\) corresponds to the suitable cyclotomic class. This enables us to use Lemma 2. This assumption does not lead to a loss of generality since if the do not correspond, then the resulting adjacency graph will be permutation equivalent to the one that we obtain.

A. Properties of the Conjugate Pairs

Let \(C_1 = [s_1]\) and \(C_2 = [s_2]\) be two not necessarily distinct cycles in \(\Omega(f(x))\) and let the special state \(S := (1, 0) \in \mathbb{F}_2^2\) belong to cycle \([s_0] \in \Omega(f(x))\). If \((v, \tilde{v})\) is a conjugate pair between \(C_1\) and \(C_2\), then \(\tilde{v} = v\). Thus, \(C_1\) and \(C_2\) share at least a conjugate pair if and only if some shift of \(s_1\) plus some shift of \(s_2\) is equal to \(s_0\), i.e., there exist two integers \(\ell\) and \(\ell'\) satisfying

\[ L^{\ell} s_1 + L'^{\ell'} s_2 = s_0. \]

Since \(S \in \mathbb{F}_2^2\) has \(n - 1\) consecutive 0s, \(f(x)\) is the minimal polynomial of \([s_0]\). Hence, \([s_0]\) can be determined with the help of \(P^{-1}\) and (11). Without loss of generality, let

\[ [s_0] = \{ L^{c_1} s_1^d + L^{c_2} s_2^d + \ldots + L^{c_s} s_d^d \}. \]  

(12)

where \(c_1, c_2, \ldots, c_s\) and \(d_1, d_2, \ldots, d_s\) are some suitable integers. In fact, if the initial states are appropriately chosen, one can make \(c_k = d_k = 0\) for \(1 \leq k \leq s\). For our purposes, doing so is not required.

It is clear that \([0]\) and \([s_0]\) share a unique conjugate pair.

To compute the exact number of shared conjugate pairs between \(C_1\) and \(C_2\), knowing important properties of the cycles in \(\Omega(p_i(x))\) is crucial. Suppose that we have

\[ \Gamma_i(j_1, j_2) := \{(u, v) \mid L^{u + \ell} s_1^j + L'^{v + \ell'} s_2^j = s_0^u, \ 0 \leq u, v < e_i \}. \]  

(13)

If \(s_1^j = 0\), then \(u = 0\). If \(s_2^j = 0\), then \(v = 0\). If only one of \(s_1^j\) and \(s_2^j\) is \(0\), then the other must be \(s_d^d\) and \(\Gamma_i(j_1, j_2) = \{(0, 0)\}\). If both \(s_1^j\) and \(s_2^j\) are \(\neq 0\), then \(\Gamma_i(j_1, j_2) = (j_1 - d_1, j_2 - d_2)\) is \((j_2 - d_2, j_1 - d_1)\) defined in Lemma 2.

Following Equation (7), we let

\[ C_1 = [s_1] = [a_1 L^{\ell_1} s_1^j + a_2 L'^{\ell'} s_2^j + \ldots + a_s L^{\ell_s} s_s^j], \]

\[ C_2 = [s_2] = [a'_1 L^{\ell_1} s_1^j + a'_2 L'^{\ell'} s_2^j + \ldots + a'_s L^{\ell_s} s_s^j]. \]  

(14)

The period of both \(s_1\) and \(s_2\) is \(\text{lcm}(f_1, \ldots, f_s)\). Let \(E_1 := \{ i \mid a_i = 1, 1 \leq i \leq s \}\) and \(E_2 := \{ j \mid a'_j = 1, 1 \leq j \leq s \}\).

Lemma 5. Take \(C_1\) and \(C_2\) from (12) and \(\Gamma_i(j_1, j_2)\) in (13).

1) The following conditions are necessary for \(C_1\) and \(C_2\) to share at least a conjugate pair.

a) \(E_1 \cup E_2 = \{1, 2, \ldots, s\}\).

b) If \(i \in E_1 \cap E_2\), then \(s_1^{j_i} = s_2^{j_i}\). Similarly, if \(i \in E_1^{\ell} \cap E_2\), then \(s_1^{j_i} = s_2^{j_i}\).

c) If \(i \in E_1 \cap E_2\), then \(j_i - d_i, j'_i - d_i > 0\).

2) The number of conjugate pairs between \(C_1\) and \(C_2\) is equal to the number of tuples \((u_1, v_1), \ldots, (u_s, v_s)\) that satisfy two requirements.

a) \((u_i, v_i) \in \Gamma_i(j_1, j_2)\) for \(1 \leq i \leq s\).

b) The systems of congruences

\[ c_1 + u_i - \ell_i \equiv c_2 + u_i - \ell_j \]  

for all \(i \neq j \in E_1\),

\[ c_1 + v_i - \ell_i \equiv c_2 + v_j - \ell'_j \]  

for all \(i \neq j \in E_2\),

hold modulo \(g\text{cd}(e_i, e_j)\).

3) The sum of the numbers of conjugate pairs between any two cycles over all possible values for \(\ell_1, \ldots, \ell_s\) and \(\ell'_1, \ldots, \ell'_s\) is equal to \(\sum_{i \in E_1 \cap E_2} (j_i - d_i, j'_i - d_i)\).

As the \(\ell\) and \(\ell'\) range through all of their respective values, it may happen that \(C_1 = C_2\). When this is the case, we count the conjugate pairs \((v, \tilde{v})\) and \((\tilde{v}, v)\) separately even though they are the same.

Proof. If \(C_1\) and \(C_2\) share a conjugate pair, then there exist integers \(\ell\) and \(\ell'\) satisfying \(L^{\ell} s_1 + L'^{\ell'} s_2 = s_0\). By definitions,

\[ L^{\ell} s_1 + L'^{\ell'} s_2 = \sum_{i=1}^{s} a_i L^{\ell+\ell_i} s_1^j + a'_i L^{\ell' + \ell'_i} s_2^j = \sum_{i=1}^{s} L^{c_i} \left( a_i L^{\ell+\ell_i-c_i} s_1^j + a'_i L^{\ell' + \ell'_i-c'_i} s_2^j \right) = \sum_{i=1}^{s} L^{c_i} s^i_d. \]

Since sequences \(s^i_d\) correspond to distinct irreducible characteristic polynomials, we must have

\[ a_i L^{\ell+\ell_i-c_i} s_1^j + a'_i L^{\ell' + \ell'_i-c'_i} s_2^j = s^i_d. \]

Therefore, \(a_i \lor a'_i = 1\), \((\ell + \ell_i - c_i, \ell' + \ell'_i - c_i) \in \Gamma_i(j_1, j'_1)\), and \(|\Gamma_i(j_1, j'_1)| > 0\), proving Statement 1.
Let \((u_i, v_i)\) be one of the \((j_i - d_i, j'_i - d_i)_{i_1}\) tuples in \(\Gamma_i(j_i, j'_i)\). To ensure the existence of \(\ell\) and \(\ell'\), the following systems of congruences must have solutions modulo \(f_i\) for all \(i\):

\[
\begin{align*}
  u_i &\equiv \ell + \ell_i - c_i \pmod{e_i} & \forall i \\
  v_i &\equiv \ell' + \ell'_i - c_i \pmod{e_i} & \forall i
\end{align*}
\]

By the definition of \(f_i\), the systems reduce to

\[
\begin{align*}
  \ell &\equiv c_i + u_i - \ell_i \pmod{e_i} & \forall i \\
  \ell' &\equiv c_i + v_i - \ell'_i \pmod{e_i} & \forall i
\end{align*}
\]

with \(u_i = v_i = 0\) whenever \(i \notin E_1 \cap E_2\).

Theorem 2 says that solutions exist if and only if the following systems of congruences hold modulo \(\gcd(e_i, e_j)\):

\[
\begin{align*}
  c_i + u_i - \ell_i &\equiv c_j + u_j - \ell_j \pmod{e_i} & \forall i \neq j \\
  c_i + v_i - \ell'_i &\equiv c_j + v_j - \ell'_j \pmod{e_i} & \forall i \neq j
\end{align*}
\]

This completes the proof of Statement 2.

As \(\ell_i\) and \(\ell'_i\) range through their possible values, for a given \((u_i, v_i)\) there exists a corresponding \((\ell, \ell')\) satisfying the above systems simultaneously. If \((j_i - d_i, j'_i - d_i)_{i_1}\) possible choices for \((u_i, v_i)\) if both corresponding sequences are \(\neq 0\) but only one choice if one of them is \(0\). Thus, the sum of the numbers of conjugate pairs between \(C_1\) and \(C_2\) over all possible \(\ell_i\) and \(\ell'_i\) is

\[
\prod_{i \in E_1 \cap E_2} (j_i - d_i, j'_i - d_i)_{i_1},
\]

When \(C_1 = C_2\), we double count each of their conjugate pairs since both \((v, \bar{v})\) and \((\bar{v}, v)\) appear and are counted separately, despite being exactly the same. Statement 3 is established.

Remark 2. A similar result to Lemma 5 was given in [15] Thm. 4. We present Lemma 5 with a proof using our notations for this work to be self-contained.

To determine the number of conjugate pairs between two chosen cycles, for each possible \((\ell_i, \ell'_i)\) tuple, we check how many \((u_1, v_1), \ldots, (u_s, v_s)\) tuples make the systems solvable. We will provide an efficient procedure to do so.

Lemma 5 says that in general it is hard to determine the exact number of conjugate pairs between any two cycles in \(\Omega(f(x))\). In some special cases the problem becomes easier. If, for all \(1 \leq i \leq s\), the periods of the nonzero sequences in \(\Omega(p_i(x))\) are pairwise coprime, then, by Lemma 5,

\[
\begin{align*}
  [s_0] &\equiv [s_{d_1}^1 + s_{d_2}^2 + \ldots + s_{d_s}^s] \\
  C_1 &\equiv [a_1s_{d_1}^1 + a_2s_{d_2}^2 + \ldots + a_s s_{d_s}^s] \\
  C_2 &\equiv [a_1's_{d_1}^1 + a_2's_{d_2}^2 + \ldots + a_s' s_{d_s}^s]
\end{align*}
\]

We obtain the following corollary to Lemma 5

**Corollary 1.** For all \(1 \leq i \leq s\) let the periods of the nonzero sequences in \(\Omega(p_i(x))\) be pairwise coprime. Let \(C_1\) and \(C_2\) be as given in (15). They share at least a conjugate pair if and only if the following requirements are satisfied for all \(i\):

1) \(E_1 \cup E_2 = \{1, 2, \ldots, s\}\).

2) If \(a_i = 1\) and \(a'_i = 0\), then \(s_{j_i}^i = s_{d_i}^i\).

3) If \(a_i = 0\) and \(a'_i = 1\), then \(s_{j_i}^i = s_{d_i}^i\).

4) If \(a_i = a'_i = 1\), then \((j_i - d_i, j'_i - d_i)_{i_1} > 0\).

Distinct \(C_1\) and \(C_2\) share \(\prod_{i \in E_1 \cap E_2} (j_i - d_i, j'_i - d_i)_{i_1}\) conjugate pairs. We halve the number when \(C_1 = C_2\).

If \(p_1(x)\) in Corollary 1 are all primitive, then \(t_i = 1\) for all \(i\) and there is a conjugate pair between \(C_1\) and \(C_2\) in (13) if and only if \(E_1 \cup E_2 = \{1, 2, \ldots, s\}\). When \(C_1 \neq C_2\), the exact number of conjugate pairs is \(\prod_{i \in E_1 \cap E_2} (2n^i - 2)\). When \(C_1 = C_2\), we halve the number.

We add the assumption that \(p_1(x) = x + 1\) and keep \(p_1(x)\) primitive for \(i \geq 2\). If \(1 \in E_1 \cap E_2\), i.e., \(a_1 = a'_1 = 1\), then there is no conjugate pair between the corresponding pairs of cycles since \(n_1 = 1\), making \(2n^i - 2 = 0\).

Keeping \(p_1(x) = x + 1\), we prove a result that generalizes [16] Prop. 10. It covers LFSRs with arbitrary characteristic polynomial \(q(x)\) having \((x + 1)\) as a factor.

**Proposition 3.** Consider the LFSR with an arbitrary characteristic polynomial \(q(x) \in \mathbb{F}_2[x]\). If \((x + 1) | q(x)\), then \(v\) and \(\bar{v}\) never belong to the same cycle.

**Proof.** Let \(p_1(x) := (x + 1)\) and

\[
q(x) := \prod_{i=1}^s p_i^a(x) \quad \text{with} \quad a_i \in \mathbb{N}.
\]

If \([s] \in \Omega(q(x))\) contains a conjugate pair, then \(q(x)\) must be its minimal polynomial.

For a contradiction, suppose that the minimal polynomial is \(p(x)\) with \(\deg(p(x)) < \deg(q(x))\). Then there is \(k \in \mathbb{Z}\) such that \(s + L^k s\) contains \(S\) as a state and has characteristic polynomial \(p(x)\). The sequence containing \(S\) as a state, however, must have \(q(x)\) as its minimal polynomial.

Any cycle containing a conjugate pair must then have the form \([s_1 + L^{a} s_2 + \ldots + L^{a} s_s]\) for \(i_1, i_2, \ldots, i_s \in \mathbb{Z}\) with \(p_i^{a_i}(x)\) being the minimal polynomial of \(s_i\) for all \(i\). Thus, for some \(\ell \in \mathbb{Z}\), \([s_1 + L^{i_1} s_2 + \ldots + L^{i_s} s_s]\) + \(L^{\ell}([s_1 + L^{i_1} s_2 + \ldots + L^{i_s} s_s])\) + \(L^{\ell}([s_1 + L^{i_1} s_2 + \ldots + L^{i_s} s_s])\) + \(L^{\ell}([s_1 + L^{i_1} s_2 + \ldots + L^{i_s} s_s])\) = \(L^{\ell} s_1 + L^{\ell} s_2 + \ldots + L^{\ell} s_s\) with \(p_i^{a_i}(x)\) being the characteristic polynomial of \(s_i\).

In particular, \(L^{\ell} s_1' = s_1 + L^{\ell} s_1\) must be the sum of two sequences having the same minimal polynomial \((x + 1)^{n_1}\). Its period is a power of 2. By [23] Lem. 4.1, the degree of the minimal polynomial of \(L^{\ell} s_1\) is less than \(a_1\). Hence, the degree of the minimal polynomial of the resulting sequence \(L^{\ell} s_1' + L^{\ell} s_2' + \ldots + L^{\ell} s_s'\) must be less than \(\deg(q(x))\). Thus, it cannot contain the special state \(S\).

In the construction of de Bruijn sequences by the cycle joining method, the conjugate pairs between any cycle and itself are never used. To take advantage of Proposition 3 we henceforth assume that \(p_1(x) = x + 1\). This implies \(s_1 = 1\). Lemma 5 can still be used to determine the conjugate pairs between any two distinct cycles.

Some additional properties can be easily inferred. Let \(s_0\) be as in (12), \(s_1\) and \(s_2\) in (14), and \(s_{j_1}^i = s_{j_1}^{i_1} = s_{j_1}^{i_1} = 1\). If \([s]\) is a cycle in \(\Omega(f(x))\), then clearly so is \([s + 1]\).
Corollary 2. Let \( C_1 = [s_1] \) and \( C_2 = [s_2] \) be two cycles in \( \Omega(f(x)) \) with \( p_1(x) = x + 1 \).

1) For \( C_1 \) and \( C_2 \) to be adjacent, \( a_1 + a'_1 = 1 \).
2) If \((v, \tilde{v})\) is a conjugate pair between \( C_1 = [s_1] \) and \( C_2 = [s_2] \), then \((v+1, \tilde{v}+1)\) is a conjugate pair between \([s_1 + 1]\) and \([s_2 + 1]\).

Utilizing these facts simplifies the process of finding the conjugate pairs. This was exhibited in (4) for \( f(x) = (x+1)\prod_{i=2}^{l} p_i(x) \) where the \( p_i(x) \)'s are primitive polynomials with pairwise coprime periods. Corollary 2 tells us that the same applies to a bigger class of polynomials.

B. Finding the Conjugate Pairs

Based on the proof of Lemma 4 this subsection develops some algorithms. Deploying them allows us to find all conjugate pairs between any two cycles in \( \Omega(f(x)) \) and, hence, to construct the adjacency graph \( G \). Note that any conjugate pair can be written as \((v, v + S)\) with \( S = (1, 0) \in \mathbb{F}_2^n \).

We start with a roadmap. Suppose that we have already constructed the matrix \( P \) and found a state of each cycle in \( \Omega(p_i(x)) \) for all \( 1 \leq i \leq s \) by Algorithm 1.

1) Use Algorithm 2 to find the representation

\[
\mathcal{S} = \left\{ T^{c_1} a_i^1, T^{c_2} a_i^2, \ldots, T^{c_r} a_i^r \right\} \mathcal{P}.
\]

(17)

2) Consider \( \Omega(p_i(x)) \) for a fixed \( i \) and choose any two cycles in it. Use Algorithm 3 to find all possible pairs \((v_1, v_2)\) satisfying \( v_1 + v_2 = T^{c_i} a_i^c \).

3) Algorithm 4 builds on the results of Algorithm 3 to output all conjugate pairs between a chosen pair of cycles in \( \Omega(f(x)) \). Running Algorithm 4 for all possible pairs of cycles yields the data needed to construct the adjacency graph.

Algorithm 2 Finding the Special State \( \mathcal{S} \)

Input: \( \mathcal{P}, \mathcal{S} = (1, 0, \ldots, 0) \in \mathbb{F}_2^n \).

Output: \( [[(c_1, d_1), \ldots, (c_s, d_s)] \) such that \( (17) \) holds.

1: \((v_1, \ldots, v_s) \leftarrow \mathcal{S} \mathcal{P}^{-1}

2: \text{for } i \text{ from 1 to } s \text{ do}

3: \text{for } j \text{ from 0 to } t_i - 1 \text{ do}

4: \text{if } v_i = a_i^c \text{ then}

5: \text{store } (c_i, d_i) \leftarrow (k, j)

6: \text{else}

7: \text{store } a_i^j \leftarrow T^j a_i^c

8: \text{end if}

9: \text{end for}

10: \text{end for}

11: \text{return } ((c_1, d_1), \ldots, (c_s, d_s))

Given \( \mathcal{S} \mathcal{P}^{-1} \), Algorithm 2 requires at most \( \sum_{i=1}^{s} (2^{n_i} - 1) \) searches. 

There are several notable facts about Algorithm 3

1) The input consists of \( t_i \) states, each corresponding to one of the \( t_i \) nonzero cycles in \( \Omega(p_i(x)) \). The output is the set of all tuples \((\ell, -m)\) that ensure \( T^\ell a_i^j + T^m a_i^k = T^\ell a_i^c \), i.e. the corresponding state pairs sum to \( T^\ell a_i^c \).

2) Given \( C_1 = \{0\} \) and \( C_2 \) any nonzero cycle, the required value follows directly from the representation of \( \mathcal{S} \) and there is no need to run Algorithm 3 in this case. When the pair is \((0, a_i^c)\), the counter is 1 if and only if \( a_i^j = a_i^c \). Hence, \( pair_{t_i,d_i} = (0, c_i) \) and \( counter_{t_i,d_i} = 1 \).

3) For a chosen pair \( C_1 \) and \( C_2 \), if Algorithm 3 yields a defining pair \((x_1, x_2)\), then the defining pair for a “conjugate pair” between \( C_2 \) and \( C_1 \) will be \((x_2, x_1)\). Hence, for each \( p_i(x) \), it suffices to take \( \left\{ x_{i,2} \right\} \), instead of \( t_{i,2} \), distinct \((j, k)\) tuples.

4) The total number of suitable \((\ell, -m)\) tuples is a cycloomatic number with specific parameters. Conversely, prior knowledge of relevant cycloomatic numbers helps optimize the algorithm. In each iteration, if the corresponding cycloomatic number is 0, then no “conjugate pair” exists and the process can be terminated. If, at some point in the run, the number of pairs that the algorithm has found is equal to the cycloomatic number, we break the run.
5) If p_i(x) is primitive, then the algorithm is unnecessary. The desired results can be computed using the Zech logarithm \tau_{n}^{(\ell)}. Recall that if a is an n-stage state of an m-sequence, then Lemma 4 says that a + T^a = T^{\tau_{n}^{(\ell)}}a when \ell \neq 0. If, in (v_1, \ldots, v_s) = SP^{-1}, v_i = T^v a, then an output (y, y') of Algorithm 3 for p_i(x) satisfies T^v a + T^{y'} a = T^{v} a. Hence, T^y a = T^v a + T^{y'} a = T^v (a + T^{y'-v}) a. Thus, the desired output must be \{ (y, c + \tau_{n}(y - c)) | y \in \{0, 1, \ldots, 2^n - 2\} \} and knowing \tau_{n}(y) is sufficient to find the tuples.

Algorithm 4 uses the output of Algorithm 3 which pinpoints the "conjugate pairs" shared by pairs of cycles in \Omega(p_i(x)) for all i. With C_1 = [s_1] and C_2 = [s_2] in \Omega(f(x)) given in (14), Algorithm 4 takes as input the states v_1 of C_1 and v_2 of C_2 written in the form

v_1 = (a_1T^v_{a_1}, a_2T^v_{a_2}, \ldots, a_sT^v_{a_s}) P,

v_2 = (a_1T^{v'}_{a_1}, a_2T^{v'}_{a_2}, \ldots, a_sT^{v'}_{a_s}) P. \quad (18)

Theorem 3. Algorithm 4 is correct.

Proof. If C_1 and C_2 are adjacent, then there are integers \ell and \ell' satisfying T^\ell v_1 + T^{\ell'} v_2 = S. In particular, for each i,
a_i T^{\ell+i} a_1, a_i T^{\ell'+i} a_1',

Hence, (\ell + \ell_i, \ell' + \ell'_i) must be a pair (u_k, v_k) obtained from Algorithm 3 i.e.,

\ell + \ell_i \equiv v_k, \text{ (mod } f_i) \text{ and } \ell' + \ell'_i \equiv v_k, \text{ (mod } f_i).

We know that \ell and \ell' exist if and only if the following systems of congruences can be simultaneously solved.

\ell \equiv u_{k_i}, \text{ (mod } f_i) \text{ for all } 1 \leq i \leq s,

\ell' \equiv u_{k_i'}, \text{ (mod } f_i) \text{ for all } 1 \leq i \leq s.

For this to happen, Theorem 2 requires that the systems

u_{k_i} - \ell_i \equiv u_{k_j}, \ell_j \text{ (mod } \gcd(f_i, f_j)),

v_{k_i} - \ell'_i \equiv v_{k_j}, \ell'_j \text{ (mod } \gcd(f_i, f_j))

hold for 1 \leq i \neq j \leq s.

Algorithm 4 checks whether this requirement is met. Given 2 \leq i \leq s, the verification is performed for all 1 \leq m < i. All relevant congruences are certified to hold simultaneously. The process is terminated at the first instance when one of the congruences fails to hold.

For a chosen set \{(u_{k_i}, v_{k_i})\}_{i=1}^s, once the systems of congruences are certified to hold, then \ell and \ell' exist, making (T^\ell v_1, T^{\ell'} v_2) a conjugate pair. Since \ell + \ell_i \equiv u_{k_i}, \text{ (mod } f_i), (v, v') is a conjugate pair with

v = (a_1T^{u_{k_i}} a_1, a_2T^{u_{k_i}} a_2, \ldots, a_sT^{u_{k_i}} a_s) P.

In (14), the p_i(x)’s are primitive and e_1, \ldots, e_s are pairwise coprime. The conjugate pairs can be found without Algorithm 4. Let C_1 = [a_1s_1 + \ldots + a_s s_s] and C_2 = [b_1s_1 + \ldots + b_s s_s].

Algorithm 4 All Conjugate Pairs between 2 Cycles in \Omega(f(x))

Input: v_1, v_2 defined in (18).

Output: All conjugate pairs between distinct C_1 and C_2.

1: for i from 1 to s do
2: \text{c} \leftarrow \text{counter}_{j_i, j'_i}^1 \quad \triangleright \text{Obtained from Algorithm 3}
3: if c = 0 then
4: return: there is no conjugate pair; break
5: end if
6: end for
7: for k_i from 0 to \text{counter}_{j_i, j'_i}^1 do
8: Take (u_{k_i}, v_{k_i})
9: \ldots
10: for 0 \leq k_s < \text{counter}_{j_s, j'_s}^1 do
11: Take (u_{k_i}, v_{k_i})
12: if \left\{ \begin{array}{l}
\text{u}_{k_i} - \ell \equiv \text{u}_{k_m} - \ell_m \\
\text{v}_{k_i} - \ell' \equiv \text{v}_{k_m} - \ell'_m
\end{array} \right. \text{ holds in modulo } \gcd(f_i, f_m) \text{ for all } 1 \leq m < i then
13: continue to the next for-loop
14: else
15: \text{k}_i \leftarrow \text{k}_i + 1; \text{ continue in this for-loop}
16: end if
17: \ldots
18: for 0 \leq k_s < \text{counter}_{j_s, j'_s}^1 do
19: Take (u_{k_i}, v_{k_i})
20: if \left\{ \begin{array}{l}
\text{u}_{k_i} - \ell \equiv \text{u}_{k_m} - \ell_m \\
\text{v}_{k_i} - \ell' \equiv \text{v}_{k_m} - \ell'_m
\end{array} \right. \text{ holds in modulo } \gcd(f_i, f_m) \text{ for all } 1 \leq m < s then
21: \text{v} \leftarrow (a_1T^{u_{k_i}} a_1, \ldots, a_sT^{u_{k_i}} a_s) P
22: return the conjugate pair (v, v')
23: else
24: \text{k}_s \leftarrow \text{k}_s + 1; \text{ continue in this for-loop}
25: end if
26: \ldots
27: \ldots
28: \ldots
29: \ldots
30: end for

with a_i \lor b_i = 1. Given \text{SP}^{-1} = (a_1, \ldots, a_s), it was shown that \text{v} = (v_1, \ldots, v_s) P in the conjugate pair (v, \text{v'}) has

v_i = \left\{ \begin{array}{ll}
0 & \text{if } a_i = 0 \text{ and } b_i = 1, \\
\text{a}_i & \text{if } a_i = 1 \text{ and } b_i = 0, \\
\text{vi} & \in \mathbb{F}_2^n \setminus \{0, \text{a}_i\} \text{ if } a_i = b_i = 1.
\end{array} \right.

If the periods are not coprime, then the situation is more involved. One must ensure that the required systems of congruences hold simultaneously. We devise Algorithm 4 to cover this more general situation.

The overall running time can be further improved by using the properties in Subsection [V.X] to rule out pairs of cycles with no conjugate pairs prior to running Algorithm 4.

Since Algorithm 4 builds upon the results of Algorithm 3, sufficient storage must be allocated to have them ready at hand. The two algorithms can in fact be merged. In Algorithm 4, at the precise step when the "conjugate pairs" between the specified two cycles in \Omega(p_i(x)) are needed, a search for
them can be executed to cut down on the storage requirement. Anticipating the need to find all conjugate pairs between any two cycles in $\Omega(f(x))$ in various applications and noting that the data are reasonably small, storing them is the option we prefer.

Algorithm 4 transforms the problem of finding conjugate pairs between two cycles with a more complicated characteristic polynomial into one of finding “conjugate pairs” between two cycles with simpler characteristic polynomials. Once we have determined the “conjugate pairs” for each $p_i(x)$, solving systems of congruences leads us to the desired conjugate pairs for $f(x)$. For two cycles with minimal polynomial $f(x)$, exhaustive search requires around $(\text{lcm}(e_1, \ldots , e_s))^2$ computations. Our algorithm needs around $\sum_{i=1}^s (e_i)^2$ computations. The gain is most prominent when $e_1, \ldots , e_s$ are coprime and nearly equal.

V. THE CASE OF REPEATED ROOTS

This section briefly considers characteristic polynomials with repeated roots

$$q(x) := \prod_{i=1}^s p_i^{b_i}(x) \text{ with } b_i > 1 \text{ for some } i.$$  

We have recently determined the cycle structure of $\Omega(q(x))$ in [17]. Here we provide only a brief outline of the idea.

The number of cycles in $\Omega(p_i^{b_i}(x))$ can be derived from the counts provided in [24] Thm. 8.63.

1) The only cycle with period 1 is $[0]$.

2) There are $t_1$ cycles containing sequences with period $e_1$.

3) Let $\chi_i$ be the smallest positive integer such that $2^{\chi_i} \geq b_i$ and let $1 \leq j < \chi_i$. Sequences with period $e_1 \cdot 2^j$ are partitioned into $\rho$ cycles whenever $b_i > 2$ while those with period $e_1 \cdot 2^{\chi_i}$ whenever $b_i \geq 2$ are partitioned into $\psi$ cycles where

$$\rho := \frac{2n_1^{e_1} \cdot 2^{\chi_i} - 2n_i^{e_1} \cdot 2^{\chi_i-1}}{e_1 \cdot 2^j} \text{ and } \psi := \frac{2n_1^{e_1} \cdot 2^{\chi_i} - 2n_i^{e_1} \cdot 2^{\chi_i-1}}{e_1 \cdot 2^{\chi_i}}.$$  

Based on the states of the cycles in $\Omega(p_i(x))$ we give a detailed procedure to derive the states of the remaining cycles in $\Omega(p_i^{b_i}(x))$. For brevity, all states are considered to have the same length $b_i \cdot n_i$. Using the cycles in $\Omega(p_i^{b_i}(x))$ for all $i$, we can determine all cycles in $\Omega(q(x))$ and prove results similar to the ones in Lemma 3.

Once the cycle structure of $\Omega(q(x))$ is known, one can use the methods discussed above to study the properties of the conjugate pairs.

Copying the construction of $\mathcal{P}$, we build $\tilde{\mathcal{P}}$ by replacing the original polynomial $p_i(x)$ by $p_i^{b_i}(x)$. One can similarly prove that $\tilde{\mathcal{P}}$ is of full rank. Proceed to determine the new representation of $\mathcal{S}$ as $(v_1, \ldots , v_s) \tilde{\mathcal{P}}$. Here, $v_i \in \mathbb{F}_2^{b_i \cdot n_i}$ is a state of a cycle in $\Omega(p_i^{b_i}(x))$ with minimal polynomial $p_i^{b_i}(x)$. Deploy Algorithm 5 with inputs the states of all cycles in $\Omega(p_i^{b_i}(x))$. Given two cycles, the outputs would be pairs of states $(x, y)$ satisfying $x + y = v_i$. Now, this is where complication arises since we are no longer able to leverage on tools or results from relevant cyclotomic numbers, plus there are a lot of distinct cycles in $\Omega(p_i^{b_i}(x))$.

The proof of the following result on the pair $(x, y)$ is straightforward.

**Proposition 4.** Let $x$ and $y$ be two states belonging to respective cycles $C_1 = [s_1]$ and $C_2 = [s_2]$ in $\Omega(p_i^{b_i}(x))$. If there exist $a, b \in \mathbb{Z}$ satisfying $T^a x + T^b y = v_i$, then at least one of $s_1$ or $s_2$ must have $p_i^{b_i}(x)$ as minimal polynomial. If $p_i(x) = 1 + x$, then exactly one of $s_1$ and $s_2$ has minimal polynomial $(1 + x)^k$.

Using the cycle structure of $\Omega(p_i^{b_i}(x))$ and properties of LFSRs, it may be possible to derive more results on the pair $(x, y)$. We leave them as open problems.

Algorithm 4 can be modified to find all conjugate pairs between any two cycles in $\Omega(q(x))$ and, hence, construct the adjacency graph. One needs to exercise greater care here since the cycles in $\Omega(p_i^{b_i}(x))$ may have distinct periods. This affects the application of the generalized CRT (Theorem 2).

Algorithms 3 and 4 perform better for large $s$, i.e., when $f(x)$ has more distinct irreducible polynomials as its factors. Algorithm 3 is more efficient when there are less cycles and the periods are small. Since $q(x)$ has repeated roots one has to treat the cycles in $\Omega(p_i^{b_i}(x))$ separately according to their respective periods. A modified version of Algorithm 3 takes more time here since the overall degree is $b_i \cdot n_i$ and there are a lot of cycles to pair up. Thus, in practice, it is generally not advisable to use $q(x)$.

In the literature, studies on the case of characteristic polynomials with repeated roots have been quite limited, e.g., $q(x) = (x + 1)^n$ in [25] and $q(x) = (x + 1)^p(x)$ with $p(x)$ having no repeated roots or is primitive done in [12], [26]. Because $p_i(x) = (x + 1)$ is the simplest nontrivial polynomial, initial studies looked into cases with $(x + 1)^b \mid q(x)$ for $b \in \mathbb{N}$.

VI. GENERATING THE DE BRUIJN SEQUENCES

After implementing Algorithms 1 to 4 we obtain all of the conjugate pairs between any two adjacent cycles in $\Omega(f(x))$ and, hence, the adjacency graph $G$. The edges between a specific pair of vertices in $G$ correspond to the conjugate pairs shared by the represented cycles. Derive the graph $\tilde{G}$ from $G$ by bundling together multiple edges incident to the same pair of vertices into one edge. An edge in $\tilde{G}$ corresponds to a set of conjugate pairs. Next, we use Algorithm 5 to generate all spanning trees in $\tilde{G}$.

Our implementation is in python. Readers who prefer to code in C may opt to use D. Knuth’s implementation of [27] Alg. S. pp. 464 ff.] named grayspan [28]. Its running time is roughly estimated to be $O(\mu + \nu + \zeta_G)$ where $\mu$ is the total number of edges in $\tilde{G}$, $\nu$ is the number of cycles in $\Omega(f(x))$, and $\zeta_G$ is the number of the spanning trees in $\tilde{G}$. Notice that $\mu < 2^{n-1}$, $\nu \leq \frac{1}{n} \sum_{d|n} \phi(d)2^{n/d}$ (see, e.g., [29]), and $\zeta_G$ is much larger than both $\mu$ and $\nu$. Hence, $O(\mu + \nu + \zeta_G) = O(\zeta_G)$. Using Algorithm 5 instead of Algorithm S from [27] is reasonable in our context since we know a lot about $G$ already. Algorithm S covers the most general case and is more involved. Comparing the running time, our approach also takes $O(\zeta_G)$.
Algorithm 5 Finding all Spanning Trees in $\hat{G}$

Input: $\hat{G}$ with $V(\hat{G}) := \{V_1, \ldots, V_6\}$ and edge set $E(\hat{G})$.

Output: All spanning trees in $\hat{G}$.

1: $VL \leftarrow \{V_i\}$ \quad \triangleright \text{for convenience, set $V_1 = \emptyset$}
2: $EL \leftarrow \emptyset$ \quad \triangleright \text{initiate the edge list}
3: $i \leftarrow 1$
4: Perform $ST(VL, EL, i)$ through $V(\hat{G})$
5: procedure SPAN(TREES $(ST(VL, EL, i))$
6: \hspace{1cm} $\nabla \leftarrow VL_i$
7: \hspace{1cm} if $|EL| = \nu - 1$ then
8: \hspace{2cm} return $EL$
9: \hspace{1cm} else
10: \hspace{2cm} $B_{now} \leftarrow \{V_j : (\nabla, V_j) \in E(\hat{G}), V_j \notin VL\}$
11: \hspace{2cm} for each $X \subseteq B_{now}$ do
12: \hspace{3cm} $E_{new} \leftarrow \{(\nabla, V_k) : V_k \in X\}$
13: \hspace{3cm} $VL \leftarrow VL \cup B_{now}$
14: \hspace{3cm} $EL \leftarrow EL \cup E_{new}$
15: \hspace{2cm} $i \leftarrow i + 1$
16: \hspace{2cm} $ST(VL, EL, i)$
17: \hspace{1cm} end for
18: \hspace{1cm} end if
19: end procedure

Now comes the cycle joining procedure. Let $\nabla$ denote the last $n - 1$ bits of $v \in \mathbb{F}_2^n$. For each edge in a spanning tree $\Upsilon_G$ of $\hat{G}$, take a conjugate pair $(v, \bar{v})$ from the edge’s corresponding set of all conjugate pairs to derive a spanning tree $\Upsilon_G$ of $G$. Define a new set $E(\Upsilon_G) := \{\nabla_1, \ldots, \nabla_{\nu - 1}\}$. Starting from an arbitrary initial state, i.e., any vector in $\mathbb{F}_2^n$, use the LFSR with feedback function \( f(x_0, \ldots, x_{n-1}) \) to generate a sequence \( a_0, a_1, \ldots, a_{n-1} \). For each input \( (a_0, \ldots, a_{n-1}) \) and let

\[
\begin{align*}
    a_i &= f(a_i, \ldots, a_{i-1}) & \text{if NOT}, \\
    a_i &= f(a_i, \ldots, a_{i-1}) + 1 & \text{if YES}.
\end{align*}
\]

The resulting sequence is de Bruijn with feedback function

\[ h(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1}) + \sum_{w \in E} \sum_{i=1}^{n-1} (x_i + w_i + 1). \]

Performing the cycle joining procedure to all spanning trees in $G$ gives us all de Bruijn sequences in this class.

It is now time to tie up all examples pertaining to $f(x) = (x + 1)(x^2 + x + 1)(x^4 + x^3 + x^2 + x + 1)$ done above. Table III in Example 2 has listed all cycles in $\Omega(f(x))$ complete with the corresponding states. The vertices in the adjacency graph are labeled $V_1, V_2, \ldots, V_{16}$ according to the ordering of the cycles in that table.

Since $SP^{-1} = (1, 1, 1, 1, 0, 1, 0) = (1, L^2s_0^2, L^2s_3^2)$ is in $[1 + s_0^2 + s_2^2]$, $S$ is the initial state of $L^2 (1 + s_0^2 + s_2^2)$.

Using the appropriate results from Example 2 and running Algorithm 3 yield all the conjugate pairs between any two cycles. We summarize the count in Table III. The complete adjacency graph $G$ has edges labeled by the conjugate pairs.

| Edge | # | Edge | # | Edge | # |
|------|---|------|---|------|---|
| $V_1, V_{16}$ | 1 | $V_2, V_{13}$ | 1 | $V_2, V_{16}$ | 2 |
| $V_3, V_{14}$ | 2 | $V_4, V_{15}$ | 1 | $V_4, V_{16}$ | 2 |
| $V_5, V_{10}$ | 1 | $V_6, V_{12}$ | 1 | $V_6, V_{13}$ | 2 |
| $V_7, V_{11}$ | 2 | $V_6, V_{15}$ | 4 | $V_6, V_{16}$ | 4 |
| $V_7, V_{14}$ | 1 | $V_7, V_{12}$ | 2 | $V_7, V_{13}$ | 2 |
| $V_8, V_{12}$ | 2 | $V_8, V_{14}$ | 2 | $V_8, V_{15}$ | 2 |

Let $M_1$ be the diagonal $8 \times 8$ matrix with entries, in order, 1, 3, 5, 5, 15, 15, 15. Then the derived matrix $\hat{M}$ is given by

\[
\hat{M} = \begin{pmatrix}
    M_1 & M_2 \\
    M_2 & M_1
\end{pmatrix}
\]

with

\[
M_1 = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & 0 & -1 \\
    0 & 0 & 0 & 0 & -1 & 0 & -2 \\
    0 & 0 & 0 & 0 & 0 & -2 & -1 \\
    0 & 0 & 0 & -2 & -1 & 0 & 2 \\
    0 & 0 & 0 & 0 & 0 & 0 & -2 \\
    0 & 0 & -2 & -1 & -2 & 0 & 2 \\
    0 & 0 & -4 & -2 & 0 & 0 & 2 \\
    -1 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The cofactor of any entry in $\hat{M}$ gives 12, 485, 394, 432 ≈ 2^{33,54} as the number of constructed de Bruijn sequences of order 7.

The matrix representation $\hat{M}$ of $\hat{G}$ has main diagonal entries 1, 2, 3, 3, 3, 6, 6, 6, 1, 2, 3, 3, 6, 6, 6. For $i \neq j$, the $(i, j)$ entries is $-1$ whenever $V_i$ and $V_j$ are adjacent and 0 otherwise. Hence, $\zeta_G = 1, 4, 51, 520 \approx 2^{39,47}$, which is easier to process computationally than $\zeta_G \approx 2^{33,54}$.

The polynomial $f(x)$ studied in [14] is the product of $x + 1$ and some primitive polynomials with coprime periods. Such a choice greatly simplifies the procedure since the resulting cycle structure is simple and the conjugate pairs can be deduced directly. Their main motivation was to be able to generate the sequences quickly. The main drawback is the relatively low number of sequences generated.

The construction in [14] Sect. IV, for example, only produces sequences from a special spanning tree named maximal spanning tree. The number of de Bruijn sequences generated by [14] Alg. 1] is $O(2^{2^{n^2} - 1})$. From $f(x) = (x + 1)p_1(x)p_2(x)$, the number of sequences generated by [14] Alg. 2.] is $O(2^{3^n})$. To illustrate the point more concretely, the number of all de Bruijn sequences of order 8 that can be generated from $f(x) = (x + 1)(x^3 + x^2 + 1)(x^4 + x^3 + 1)$ is 926,016. Using the same $f(x)$, [14] Alg. 2] produces only 592,704 of them.

VII. Implementation

A software implementing Algorithms 1 to 5 and the cycle joining routine was written in python version 2.7. The platform was a laptop with Ubuntu 16.04 LTS operating system powered by an intel Hasswell processor i5-4300U CPU 1.90GHz. We used only one core and capped the available
memory to 3GB. Table IV presents some of the implementation results based on four basic scenarios.

1) Users want to determine the number $\nu$ of distinct cycles in $\Omega(f(x))$, the adjacency graph $G$ and its corresponding matrix $M$, the number $\zeta_G$ of de Bruijn sequences that can be generated, and the number $\zeta_{\tilde{G}}$ of all spanning trees in $\tilde{G}$. We implemented Algorithms 1 to 4 derived $\tilde{G}$ from $G$, and applied the BEST theorem to produce the desired results. The running time was $\text{Run}_1$.

2) Users need to build a library of de Bruijn sequences of order $n$. Our implementation covered the whole process. Once a tree in $\tilde{G}$ had been identified, the corresponding set of tree(s) in $G$ was generated. For each of these trees, we applied the cycle joining method to generate all de Bruijn sequences in this class. The running time to generate and write 1024 = $2^{10}$ de Bruijn sequences to a file was $\text{Run}_2$ with $0 \in \mathbb{F}_2$ as the initial state in the cycle joining process.

3) Users require a number of randomly generated de Bruijn sequences. To generate a de Bruijn sequence, we chose a random tree $\tilde{G}$ before choosing one among the corresponding trees in $G$ randomly, say $\Gamma_G$. We then used an arbitrary $v \in \mathbb{F}_2^n$ as the initial state in the cycle joining routine. To generate and write 1024 sequences to a file from the inputted $f(x)$ took $\text{Run}_3$.

4) Users require only one de Bruijn sequence chosen uniformly at random from among all of the constructible de Broujin sequences. After the adjacency graph $G$ has been determined, we applied Broder's elegant algorithm [30] Alg. Generate] on $G$ (not on $\tilde{G}$) to get a uniformly random tree $\Gamma_G$. Note that the algorithm's expected running time per generated tree is $O(\nu \log \nu)$ for most graphs. The worst case value is $O(\nu^2)$. We chose an arbitrary initial state $v$ from $\mathbb{F}_2^n$ and performed the cycle joining routine with $\tilde{G}$ and $v$ as ingredients. Once $G$ had been determined, the rest of the process was very fast.

Entry 3 is our example while Entry 6 is an example in [14]. Often, using only one non-primitive irreducible polynomial as $f(x)$, especially the one with the lowest possible order $e$ for a given $n$, is sufficient to generate a comparatively large number of sequences as shown, e.g., in Entries 16 and 18.

### VIII. Conclusion and Future Directions

In this work, we put forward a method to generate a large class of binary de Bruijn sequences from LFSRs with an arbitrary characteristic polynomial. We pay special attention to the case where $f(x)$ is the product of $s$ pairwise distinct irreducible polynomials. The related structures are studied in details. Our approach covers all previously proposed constructions utilizing the cycle joining method as special cases. We build a practical de Bruijn sequence generator based on our method and back our claims with solid implementation results. Our basic implementation scenarios can be extended to cover more application-dependent purposes.
Several open problems remain. Determining the exact structure of LFSRs whose characteristic polynomials have repeated roots is a worthy challenge. Deriving a reasonably tight lower bound on the number of de Bruijn sequences that can be constructed also merits a closer look. Two important generalization projects naturally emerge. One is to consider an arbitrary characteristic polynomial in the ring $\mathbb{F}_q[x]$. The other is to use the cycle joining method to generate de Bruijn sequences from NFSRs.

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