ON A CLASS OF DIFFERENTIAL QUASI-VARIATIONAL-HEMIVARIATIONAL INEQUALITIES IN INFINITE-DIMENSIONAL BANACH SPACES

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(Communicated by Giuseppe Buttazzo)

ABSTRACT. A class of differential quasi-variational-hemivariational inequalities (DQVHI, for short) is studied in this paper. First, based on the Browder’s result, KKM theorem and monotonicity arguments, we prove the superpositionally measurability, convexity and strongly-weakly upper semicontinuity for the solution set of a general quasi-variational-hemivariational inequality. Further, by using optimal control theory, measurability of set-valued mappings and the theory of semigroups, we establish that the solution set of (DQVHI) is nonempty and compact. This kind of evolutionary problems incorporates various classes of problems and models.

1. Introduction. In this paper, for real infinite-dimensional Banach spaces $V$ and $V_1$, $K \subset V_1$ is a nonempty closed and convex subset, $E : D(E) \subset V \to V$ is the infinitesimal generator of a $C_0$-semigroup $e^{Et}$ in $V$, $\Psi : V_1 \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous functional, $F : V_1 \to V$ is a bounded linear operator, $H : [0,L] \times V \to 2^V$ is a set-valued mapping, $J^0(c; \tilde{c})$ denotes the generalized directional derivative of locally Lipschitz function $J : V_1 \to \mathbb{R}$ at $c$ in the direction $\tilde{c}$, $g : [0,L] \times V \to V_1^*$ and $Q : K \to V_1^*$ are given, which will be specified in Section 2.

Considering the above mentioned mathematical tools, we introduce a class of differential quasi-variational-hemivariational inequalities (DQVHI, for short) composed of an evolution equation and a quasi-variational-hemivariational inequality in infinite-dimensional Banach spaces:

\begin{align}
    s'(t) & \in Es(t) + Fc(t) + H(t, s(t)), \quad t \in [0,L], \\
    c(t) & \in S(K, g(t, s(t)) + Q(\cdot), J, \Psi), \quad a.e. \ t \in [0,L], \\
    s(0) & = s_0,
\end{align}

where $S(K, g(t, s(t)) + Q(\cdot), J, \Psi)$ stands for the solution set of the following quasi-variational-hemivariational inequality: find $c : [0,L] \to K$ such that

\begin{equation}
    \langle g(t, s(t)) + Q(c(t)), \tilde{c} - c(t) \rangle + J^0(c(t); \tilde{c} - c(t)) + \Psi(\tilde{c}) - \Psi(c(t)) \geq 0, \quad \forall \tilde{c} \in K.
\end{equation}

2020 Mathematics Subject Classification. Primary: 49J40; Secondary: 47J20.

Key words and phrases. Evolutionary computations, differential quasi-variational-hemivariational inequality, existence of solutions, bounded operator, evolutionary problem.

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According to Pang and Stewart [20], Pazy [22] and Liu et al. [12], the solutions of evolutionary problem (DQVHI) are understood in the following mild sense:

**Definition 1.1.** A pair of functions \((s, c)\), with \(s \in C([0, L]; V)\) and \(c : [0, L] \to K\) measurable, is said to be a mild solution of evolutionary problem (DQVHI) if

\[
s(t) = e^{Et}s_0 + \int_0^t e^{E(t-\tau)}[Fu(\tau) + h(\tau)]d\tau, \quad t \in [0, L],
\]

where \(c(t) \in S(K, g(t, s(t)) + Q(\cdot, J, \Psi))\) and \(h(t) \in H(t, s(t)), \ a.e. \ t \in [0, L]\).

Differential variational inequalities represent an important mathematical tool of variational and nonlinear analysis for studying real-life problems coming from natural sciences, operations research, engineering, and physical sciences. In finite-dimensional spaces, several contributions related to differential variational inequalities have been established so far (see, for instance, [3]-[9], [15], [17], [21], [25] and references therein). The case \(V = \mathbb{R}^n, V_1 = \mathbb{R}^m\) and \(E = 0\) is analyzed in Gwinner [5, 6], Liu et al. [15], Pang and Stewart [20], Li et al. [11]. Further, the theory of differential variational inequalities was extended to the more general level of infinite-dimensional Banach or Hilbert spaces. In this regard, the paper Liu et al. [14] studied the well-posedness and the generalized well-posedness associated with a class of differential mixed quasi-variational inequalities. To investigate some problems or phenomena governed by locally Lipschitz nonconvex superpotential functions, Cen et al. [2] extended the results included in Liu et al. [14]. Moreover, Migórski and Bai [18] introduced and studied a class of evolution subdifferential inclusions involving history-dependent operators. Also, some evolutionary problems governed by variational inequalities were analyzed in Liu et al. [12, 13] and Treantă [24].

In this paper, motivated by the above research works and based on the Browder’s theorem, KKM theorem, optimal control theory, measurability of set-valued mappings and the theory of semigroups, we focus on the existence of solutions for (DQVHI) in infinite-dimensional separable reflexive Banach spaces. More precisely, we consider several properties associated with the solution set of (DQVHI) as the superpositionally measurability, strongly-weakly upper semicontinuity, compactness and convexity. Next, we formulate and prove the main result and establish that the solution set of problem (DQVHI) is nonempty and compact. The main point is that (DQVHI) is described by superpotential functions, which are locally Lipschitz and may be nonconvex. This kind of evolutionary problems involving quasi-variational-hemivariational inequalities incorporates various classes of problems and models and here we extend the previous works dealing with such type of problems.

The present paper is structured as follows. The first part of Section 2 contains definitions and preliminary results to be used in the sequel. Further, we set an existence theorem for quasi-variational-hemivariational inequalities closely related to evolutionary problem (DQVHI). Also, the strongly-weakly upper semicontinuity, compactness and superpositionally measurability are investigated for the solution set of the considered quasi-variational-hemivariational inequality. The final part of Section 2 focuses on the existence and qualitative properties of solution set for evolutionary problem (DQVHI).

2. **Main results.** In this section, we will investigate the existence of solutions and properties of solution set for (DQVHI) in infinite-dimensional Banach spaces. First, we recall some notations, notions and results which will be useful in the sequel.
For any nonempty set $S$, we denote by $2^S$ the collection of its nonempty subsets. Also, we introduce $\mathcal{K}(S) := \{D \in 2^S : D \text{ is compact} \}$ and $\mathcal{K}_c(S) := \{D \in 2^S : D \text{ is compact and convex} \}$. The strong convergence and the weak convergence in a Banach space $V$ are specified by “$\to$” and “$\rightharpoonup$”, respectively. The symbol $w - X$ is used for the space $X$ endowed with the weak topology. For a normed space $X$, we denote by $X^*$ its topological dual.

**Definition 2.1.** (see [8]) Let $f : [0, L] \to 2^V$ be a set-valued mapping. $f$ is said to be *measurable* if, for every open subset $O \subset V$, the set \( \{t \in [0, L] : f(t) \subset O \} \) is measurable on $\mathbb{R}$.

**Definition 2.2.** Let $V$ and $V_1$ be Banach spaces and let an interval $I \subset R$. We say that $U : I \times V \to 2^{V_1}$ is *superpositionally measurable* if, for every measurable set-valued mapping $s : I \to 2^V$, the composition $\Phi : I \to 2^{I_1}$, $\Phi(t) = U(t, s(t))$ is measurable.

**Lemma 2.3.** (see [4]) Let $K$ be a nonempty subset of a Hausdorff topological vector space $V_1$ and let $\mathcal{G} : K \to 2^{V_1}$ be a set-valued mapping with the properties:

(i) for any $\{\tilde{c}_1, \cdots, \tilde{c}_n\} \subset K$, one has that its convex hull $co\{\{\tilde{c}_1, \cdots, \tilde{c}_n\}$ is included in $\bigcup_{i=1}^n \mathcal{G}(\tilde{c}_i)$ (i.e., $\mathcal{G}$ is a KKM mapping);

(ii) $\mathcal{G}(\tilde{c})$ is closed in $V_1$ for every $\tilde{c} \in K$;

(iii) $\mathcal{G}(\tilde{c}_0)$ is compact in $V_1$ for some $\tilde{c}_0 \in K$.

Then it holds $\bigcap_{\tilde{c} \in K} \mathcal{G}(\tilde{c}) \neq \emptyset$.

**Lemma 2.4.** (see [26]) If $U : I \times V \to \mathcal{K}(V_1)$ satisfies the Carathéodory condition or $U$ is upper or lower semicontinuous, then $U$ is superpositionally measurable.

**Definition 2.5.** Let $J : V_1 \to \mathbb{R}$ be a locally Lipschitz function. The *generalized (Clarke) directional derivative* of $J$ at $c \in V_1$ in the direction $\tilde{c} \in V_1$, denoted by $J^0(c; \tilde{c})$, is defined by

$$J^0(c; \tilde{c}) = \limsup_{\lambda \to 0^+, z \to c} \frac{J(z + \lambda \tilde{c}) - J(z)}{\lambda}.$$ 

**Definition 2.6.** The *generalized gradient* (subdifferential) of $J$ at $c \in V_1$, denoted by $\partial J(c)$, is a subset of the dual space $V_1^*$ given by

$$\partial J(c) = \{c^* \in V_1^* : J^0(c; \tilde{c}) \geq \langle c^*, \tilde{c} \rangle_{V_1^* \times V_1}, \forall \tilde{c} \in V_1 \}.$$ 

The next lemma provides basic properties of the generalized directional derivative and the generalized gradient (see, for example, [19]).

**Lemma 2.7.** Let $V_1$ be a Banach space. If $J : U \to \mathbb{R}$ is a locally Lipschitz function on a subset $U \subset V_1$, then:

(i) for every $c \in U$, the set $\partial J(c)$ is a nonempty, convex, and weakly compact subset of $V_1^*$; more precisely, $\partial J(c)$ is bounded by the Lipschitz constant $L_c > 0$ of $J$ near $c$;

(ii) the graph of $\partial J$ is closed in $V_1 \times (w - V_1^*)$ topology, namely, if $\{c_k\} \subset U$ and $\{\zeta_k\} \subset V_1^*$ are sequences such that $\zeta_k \in \partial J(c_k)$ and $c_k \to c$ in $V_1$, $\zeta_k \to \zeta$ weakly in $V_1^*$, then we have $\zeta \in \partial J(c)$ ($w - V_1^*$ denotes the space $V_1^*$ equipped with weak topology);

(iii) the set-valued operator $U \ni c \to \partial J(c) \subseteq V_1^*$ is upper semicontinuous from $U$ into $w - V_1^*$;

(iv) for each $\tilde{c} \in V_1$, there exists $z_{\tilde{c}} \in \partial J(c)$ such that

$$J^0(c; \tilde{c}) = \max\{\langle z, \tilde{c} \rangle : z \in \partial J(c)\} = \langle z_{\tilde{c}}, \tilde{c} \rangle.$$
(v) the function $U \ni \tilde{c} \to J^0(c; \tilde{c}) \in \mathbb{R}$ is finite, positively homogeneous, subadditive on $U$ and satisfies $|J^0(c; \tilde{c})| \leq L_c \| \tilde{c} \|_{V_1}$;
(vi) $J^0(c; \tilde{c})$ as a function of $(c, \tilde{c})$ is upper semicontinuous, and as a function of $\tilde{c}$ alone is Lipschitz of rank $\lambda$ on $U$;
(vii) $J^0(c; \tilde{c}) = (-J^0(c; \tilde{c}))$, for all $c, \tilde{c} \in U$.

The first main result of this section is as follows.

**Theorem 2.8.** Let $\mathcal{K}$ be a nonempty closed and convex subset of a reflexive Banach space $V_1$ and $J : V_1 \to \mathbb{R}$ be a locally Lipschitz function. Assume that:

(i) $Q : \mathcal{K} \to V_1^*$ is monotone on $\mathcal{K}$, that is

$$\langle Q(\tilde{c}) - Q(c), \tilde{c} - c \rangle \geq 0, \quad \forall c, \tilde{c} \in \mathcal{K},$$

and satisfy

$$\lim_{\lambda \to 0^+} \inf_{c \in \mathcal{K}} \frac{\langle Q(\lambda c + (1 - \lambda)\tilde{c}), \tilde{c} - c \rangle}{\| \tilde{c} \|_{V_1}} \leq \langle Q(\tilde{c}), \tilde{c} - c \rangle, \quad \forall c, \tilde{c} \in \mathcal{K};$$

(ii) $\Psi : V_1 \to (-\infty, +\infty]$ is a proper, convex, and lower semicontinuous functional;

(iii) if the set $\mathcal{K}$ is unbounded in $V_1$, there exist $c_0 \in \mathcal{K}$ and an $r > 0$ such that

$$\langle Q(\tilde{c}), \tilde{c} - c_0 \rangle + J^0(c_0; \tilde{c} - c_0) + \Psi(\tilde{c}) - \Psi(c_0) > 0, \quad \forall \tilde{c} \in \mathcal{K}, \| \tilde{c} \|_{V_1} > r$$

and satisfy

$$\lim_{\| \tilde{c} \|_{V_1} \to \infty} \inf_{c \in \mathcal{K}} \frac{\langle Q(\tilde{c}), \tilde{c} - c \rangle + J^0(c; \tilde{c} - c) + J^0(c_0; \tilde{c} - c_0) + \Psi(\tilde{c}) - \Psi(c_0)}{\| \tilde{c} \|_{V_1}} = +\infty;$$

(iv) $J$ satisfies $J^0(c; \tilde{c} - c) + J^0(c; c - \tilde{c}) = 0$.

Then, for each element $w \in V_1^*$, there exists $c \in \mathcal{K}$ such that

$$\langle w + Q(c), \tilde{c} - c \rangle + J^0(c; \tilde{c} - c) + \Psi(\tilde{c}) - \Psi(c) \geq 0, \quad \forall \tilde{c} \in \mathcal{K}, \quad (3)$$

if and only if, for each element $w \in V_1^*$, there exists $c \in \mathcal{K}$ such that

$$\langle w + Q(\tilde{c}), \tilde{c} - c \rangle + J^0(c; \tilde{c} - c) + \Psi(\tilde{c}) - \Psi(c) \geq 0, \quad \forall \tilde{c} \in \mathcal{K}. \quad (4)$$

Moreover, the solution set of $3$ is nonempty, convex and closed in $V_1$.

**Proof.** If $c \in \mathcal{K}$ is a solution of $3$, by using the monotonicity of $Q$ in assumption (i), it follows that $c \in \mathcal{K}$ is also a solution of $4$. Conversely, assume that $c \in \mathcal{K}$ is a solution of $4$. Taking into account the convexity of the set $\mathcal{K}$, for all $\lambda \in (0, 1)$ and all $\tilde{c} \in \mathcal{K}$, it results that $c_\lambda := (1 - \lambda)c + \lambda \tilde{c} \in \mathcal{K}$. In consequence, we have

$$\langle w + Q(c_\lambda), c_\lambda - c \rangle + J^0(c; c_\lambda - c) + \Psi(c_\lambda) - \Psi(c) \geq 0,$$

and, by assumption (ii) and Lemma 2.7, we get

$$\langle w + Q(c_\lambda), \tilde{c} - c \rangle + J^0(c; \tilde{c} - c) + \Psi(\tilde{c}) - \Psi(c) \geq 0.$$

Passing to the limit as $\lambda \to 0^+$ in the above inequality (see the second part of hypothesis (i)), we obtain that $c \in \mathcal{K}$ is also a solution of $3$.

Further, in order to prove the other assertions of our theorem, we consider the following two cases:

**Case 1.** $\mathcal{K}$ is bounded in $V_1$. Consider the set-valued mapping $\mathcal{G} : \mathcal{K} \to 2^\mathcal{K}$ defined as

$$\mathcal{G}(\tilde{c}) := \{ c \in \mathcal{K} : \langle w + Q(\tilde{c}), \tilde{c} - c \rangle + J^0(c; \tilde{c} - c) + \Psi(\tilde{c}) - \Psi(c) \geq 0, \quad \forall \tilde{c} \in \mathcal{K} \}.$$

It is of course immediate (from above equivalent conclusion) that $\mathcal{G}(\tilde{c})$ is a convex set and $\tilde{c} \in \mathcal{G}(\tilde{c})$, whenever $\tilde{c} \in \mathcal{K}$.
Now, let us show that $\mathcal{G}(\bar{c})$ is weakly closed in $V_1$, for all $\bar{c} \in \mathcal{K}$. Let $c_n \subset \mathcal{G}(\bar{c})$ be a sequence with $c_n \rightharpoonup c$ in $V_1$, Therefore, we have
\[ \langle w + Q(\bar{c}), \bar{c} - c_n \rangle + J^0(c_n; \bar{c} - c) + \Psi(\bar{c}) - \Psi(c_n) \geq 0, \quad \forall \bar{c} \in \mathcal{K}. \]
Passing to the limit as $n \to +\infty$ in the above inequality, by assumption (ii) and Lemma 2.7, it results
\[ \langle w + Q(\bar{c}), \bar{c} - c \rangle + J^0(c; \bar{c} - c) + \Psi(\bar{c}) - \Psi(c) \geq 0, \quad \forall \bar{c} \in \mathcal{K}, \]
which means that $c \in \mathcal{G}(\bar{c})$.

Further, we prove that the set-valued mapping $\mathcal{G}$ is a KKM mapping (see Lemma 2.3). Suppose, by contradiction, that there exists $\{\bar{c}_1, \cdots, \bar{c}_n\} \subset \mathcal{K}$ and $c_0 = \sum_{i=1}^n \lambda_i \bar{c}_i$, with $\lambda_i \in [0,1]$ and $\sum_{i=1}^n \lambda_i = 1$, satisfying $c_0 \notin \bigcup_{i=1}^n \mathcal{G}(\bar{c}_i)$, that is
\[ \langle w + Q(\bar{c}_i), \bar{c}_i - c_0 \rangle + J^0(c_0; \bar{c}_i - c_0) + \Psi(\bar{c}_i) - \Psi(c_0) < 0, \quad \forall i \in \{1,2,\cdots, n\}. \]
By using the monotonicity of $Q$, it follows
\[ \langle w + Q(c_0), \bar{c}_i - c_0 \rangle + J^0(c_0; \bar{c}_i - c_0) + \Psi(\bar{c}_i) - \Psi(c_0) < 0, \quad \forall i \in \{1,2,\cdots, n\}, \]
from which it arises the contradiction
\[ 0 = \langle w + Q(c_0), c_0 - c_0 \rangle + J^0(c_0; c_0 - c_0) + \Psi(c_0) - \Psi(c_0) \leq \sum_{i=1}^n \lambda_i \left[ \langle w + Q(c_0), \bar{c}_i - c_0 \rangle + J^0(c_0; \bar{c}_i - c_0) + \Psi(\bar{c}_i) - \Psi(c_0) \right] < 0. \]

Since $\mathcal{K}$ is a bounded, closed and convex set in the reflexive Banach space $V_1$, it follows that $\mathcal{K}$ is weakly compact in $V_1$. Consequently, $\mathcal{G}(\bar{c})$ is weakly compact in $V_1$ for each $\bar{c} \in \mathcal{K}$. Now, we are in a position to apply Lemma 2.3 ensuring that $\bigcap_{\bar{c} \in \mathcal{K}} \mathcal{G}(\bar{c}) \neq \emptyset$. Hence the solution set of problem 4 is nonempty, so the same is true for the solution set of problem 3.

Case 2. $\mathcal{K}$ is unbounded in $V_1$. For every integer $n \geq 1$, consider the bounded, closed and convex subset of $V_1$
\[ \mathcal{K}_n := \{ s \in \mathcal{K} : \| s - c_0 \|_{V_1} \leq n \}, \]
where $c_0 \in \mathcal{K}$ is given in assumption (iii). In accordance with the previous case, we can find $c_n \in \mathcal{K}_n$ such that
\[ \langle w + Q(c_n), \bar{c} - c_n \rangle + J^0(c_n; \bar{c} - c_n) + \Psi(c_n) \geq 0, \quad \forall \bar{c} \in \mathcal{K}_n. \]
In the following, let us show that there exists an integer $k \geq 1$ such that
\[ \| c_k - c_0 \|_{V_1} < k. \]
By contradiction, assume that $\| c_n - c_0 \|_{V_1} = n$, for every integer $n \geq 1$. Putting $\bar{c} = c_0$ in 5, we get (see assumption (iv))
\[ \langle w + Q(c_n), c_n - c_0 \rangle + J^0(c_n; c_n - c_0) + \Psi(c_n) - \Psi(c_0) \leq 0, \]
which contradicts assumption (iii) provided $n$ is sufficiently large. Hence the claim in 6 is true. By using 6, for $y \in \mathcal{K}$ and sufficiently small $t > 0$, we have
\[ \| c_k + t(y - c_k) - c_0 \|_{V_1} < k. \]
Next, set $\bar{c} = c_k + t(y - c_k)$ and $n = k$ in 5. By hypothesis (ii) and Lemma 2.7, it follows
\[ \langle w + Q(c_k), y - c_k \rangle + J^0(c_k; y - c_k) + \Psi(y) - \Psi(c_k) \geq 0, \]
that is, $c = c_k$ is a solution of quasi-variational-hemivariational problem 3.
Further, by using the equivalence between quasi-variational-hemivariational problems 3 and 4 and assumption (ii), we conclude that the solution set for 3 is closed and convex in $V_1$. The proof is now complete.

**Theorem 2.9.** (see [1]) Let $\mathcal{K}$ be a nonempty closed and convex subset of a reflexive Banach space $V_1$ and consider $Q : \mathcal{K} \rightarrow V_1^*$ is monotone on $\mathcal{K}$ and hemicontinuous (that is, the real function $t \mapsto \langle Q(c_1 + tc_2), c_3 \rangle$, $\forall c_1, c_2, c_3 \in \mathcal{K}$ is continuous on $[0,1]$). Also, we suppose there exists $c_0 \in \mathcal{K}$ such that

$$\liminf_{c \in \mathcal{K}, \|c\|_{V_1} \rightarrow +\infty} \frac{\langle Q(c), c - c_0 \rangle}{\|c\|_{V_1}} = +\infty.$$  

Then, for each element $w \in V_1^*$, there exists $c \in \mathcal{K}$ such that

$$\langle Q(c) + w, \tilde{c} - c \rangle \geq 0, \quad \forall \tilde{c} \in \mathcal{K}.$$  

**Remark 1.** Theorem 2.8 extends some results derived in Liu and Zeng [16], Liu et al. [12] and it is based on the Browder’s theorem (see Theorem 2.9). Assertion 4 is an extension of Minty’s technique and assumption (iii) expresses a generalized coercivity condition.

The first part of the proof of Theorem 2.8 implies:

**Corollary 1.** Let $\mathcal{K}$ be a nonempty compact and convex subset of a real reflexive Banach space $V_1$ and $J : V_1 \rightarrow \mathbb{R}$ be a locally Lipschitz function satisfying (iv) in Theorem 2.8. Assume that $Q : \mathcal{K} \rightarrow V_1^*$ and $\Psi : V_1 \rightarrow (-\infty, +\infty]$ verify conditions (i), (ii) in Theorem 2.8. Then the conclusion of Theorem 2.8 is valid.

In the sequel, we denote all the solutions of the quasi-variational-hemivariational inequality 3 by $\mathcal{S}(\mathcal{K}, w + Q(\cdot), J, \Psi)$. By using of Theorem 2.8, we also establish the following two results, closely related to evolutionary problem (DQVHI).

**Lemma 2.10.** Under the same hypotheses of Theorem 2.8, for each integer $n > 0$ there exists a constant $M_n > 0$ such that

$$\|c\|_{V_1} \leq M_n, \quad c \in \mathcal{S}(\mathcal{K}, w + Q(\cdot), J, \Psi), \quad \forall w \in \mathcal{B}(n, V_1^*) := \{w \in V_1^* : \|w\|_{V_1^*} \leq n\}.$$  

**Proof.** Arguing by contradiction, we assume that there exists $N_0 > 0$ such that

$$\sup_{w \in \mathcal{B}(N_0, V_1^*)} \{\|c\|_{V_1} : c \in \mathcal{S}(\mathcal{K}, w + Q(\cdot), J, \Psi)\} = +\infty.$$  

In consequence, there exist $w_k \in \mathcal{B}(N_0, V_1^*)$ and $c_k \in \mathcal{S}(\mathcal{K}, w_k + Q(\cdot), J, \Psi)$ such that $\|c_k\|_{V_1} > k$ ($k = 1, 2, \ldots$). By assumption (iii) of Theorem 2.8, it follows that there are a function $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with $p(k) \rightarrow +\infty$ as $k \rightarrow +\infty$, and a constant $M > 0$ such that for each $\|c\|_{V_1} > M$, we have

$$\langle Q(c), c - \tilde{c}_0 \rangle + J^0(\tilde{c}_0; c - \tilde{c}_0) + \Psi(c) - \Psi(\tilde{c}_0) \geq p(\|c\|_{V_1}) \|c\|_{V_1}. \quad \tilde{c}_0 \in \mathcal{K}.$$  

Therefore, for $k > M$, one has $\|c_k\|_{V_1} > M$ and (see assumption (iv) of Theorem 2.8)

$$\langle w_k + Q(c_k), \tilde{c}_0 - c_k \rangle + J^0(c_k; \tilde{c}_0 - c_k) + \Psi(\tilde{c}_0) - \Psi(c_k) \leq [\|w_k\|_{V_1^*} - p(\|c_k\|_{V_1})] \|c_k\|_{V_1}$$

$$+ \|w_k\|_{V_1^*} \|\tilde{c}_0\|_{V_1} \leq [N_0 - p(\|c_k\|_{V_1})] \|c_k\|_{V_1} + N_0 \|\tilde{c}_0\|_{V_1} < 0,$$

as $k$ large enough. This is a contradiction, which completes the proof.
Further, for $g : [0, L] \times V \to V_1^*$, introduce a set-valued mapping $Z : [0, L] \times V \to 2^{V_1}$ as follows

$$Z(t, s) := \{ c \in K : \langle g(t, s) + Q(c), \hat{c} - c \rangle + J^0(c; \hat{c} - c) + \Psi(\hat{c}) - \Psi(c) \geq 0, \forall \hat{c} \in K \}.$$ 

**Theorem 2.11.** Under the same hypotheses of Theorem 2.8, if $g : [0, L] \times V \to V_1^*$ is a continuous and uniformly bounded function, then:

(i) the set-valued mapping $Z$ is strongly-weakly upper semicontinuous;

(ii) there exists a positive constant $\psi > 0$ such that, for all $s \in C([0, L]; V)$, we have

$$\| Z(t, s(t)) \| := \sup_{c \in Z(t, s(t))} \| c \|_{V_1} \leq \psi, \quad \forall t \in [0, L];$$

(iii) the set-valued mapping $Z$ is superpositionally measurable.

**Proof.** (i) In accordance with Kamenskoo et al. [8], we have to prove that $Z^-(C) := \{(t, s) \in [0, L] \times V : Z(t, s) \cap C \neq \emptyset \}$ is strongly closed in $[0, L] \times V$, for each weakly closed subset $C$ of $V_1$. Therefore, we need to prove that, if the sequence $(t_n, s_n) \in Z^-(C)$ and $(t_n, s_n) \to (t, s)$, then $(t, s) \in Z^-(C)$. By $(t_n, s_n) \in Z^-(C)$ it results that there exists $c_n \in \mathcal{S}(K, g(t_n, s_n) + Q(\cdot), J, \Psi)$, for $n \in \mathbb{N}$. By using of the uniform boundedness of $g$, we may assume that $\| g(t_n, s_n) \|_{V_2} \leq k_0$, where $k_0$ is a positive constant. According to Lemma 2.10, there exists a constant $M_{k_0} > 0$ such that $\| c_n \|_{V_1} \leq M_{k_0}$, for $n \in \mathbb{N}$. Consequently, the sequence $\{c_n\}$ is relatively weakly compact in the reflexive Banach space $V_1$ and, without loss of generality, we may assume $c_n \to c$. Since $c_n \in \mathcal{S}(K, g(t_n, s_n) + Q(\cdot), J, \Psi)$, for $n \in \mathbb{N}$, we get

$$\langle g(t_n, s_n) + Q(c_n), \hat{c} - c_n \rangle + J^0(c_n; \hat{c} - c_n) + \Psi(\hat{c}) - \Psi(c_n) \geq 0, \forall \hat{c} \in K.$$ 

In virtue of the monotonicity of $Q$, it follows

$$\langle g(t, s) + Q(\hat{c}), \hat{c} - c_n \rangle + J^0(c_n; \hat{c} - c_n) + \Psi(\hat{c}) - \Psi(c_n) \geq 0, \forall \hat{c} \in K.$$ 

Applying assumption (ii) of Theorem 2.8, the continuity of $g$ and $n \to \infty$, we have

$$\langle g(t, s) + Q(\hat{c}), \hat{c} - c \rangle + J^0(c; \hat{c} - c) + \Psi(\hat{c}) - \Psi(c) \geq 0, \forall \hat{c} \in K,$$

that is, $c \in \mathcal{S}(K, g(t, s) + Q(\cdot), J, \Psi)$ (since the problem 4 is equivalent with the problem 3). By the weak closedness of $C$, we get $c \in Z(t, s) \cap C$, that is, $Z$ is strongly-weakly upper semicontinuous.

(ii) By hypothesis, $g : [0, L] \times V \to V_1^*$ is a continuous and uniformly bounded function. Therefore, for any $s \in C([0, L]; V)$, $\| g(t, s(t)) \|_{V_1^*}$ is uniformly bounded for all $t \in [0, L]$. By Lemma 2.10, there exists a positive constant $\psi$ such that

$$\| Z(t, s(t)) \| := \sup_{c \in Z(t, s(t))} \| c \|_{V_1} \leq \psi, \quad \forall t \in [0, L].$$

(iii) Since $Z$ is strongly-weakly upper semicontinuous with weakly compact convex values, then assertion (iii) holds true (see Lemma 2.4). The proof is complete.

In the following, consider the set-valued mapping $H : [0, L] \times V \to K_\nu(V)$ such that:

(a) for each $s \in V$, the set-valued mapping $H(\cdot, s) : [0, L] \to K_\nu(V)$ is measurable;

(b) for a.e. $t \in [0, L]$, the set-valued mapping $H(t, \cdot) : V \to K_\nu(V)$ is upper semicontinuous;

(c) for a.e. $t \in [0, L]$ and for all $s \in V$, there exists $\rho \in L^2([0, L])$ such that

$$\| h(t) \|_{V} \leq \rho(t)(1 + \| s(t) \|_{V}), \quad \forall h(t) \in H(t, s(t)).$$
function of interval $iL$ given by

$$P_H(s) := \{ h \in L^2([0, L]; V) : h(t) \in H(t, s(t)), \text{ a.e. } t \in [0, L] \}$$

is well-defined.

The next theorem represents the central result of this paper. It investigates the existence of solutions for (DQVHI).

**Theorem 2.12.** Let $e^{Et}$ be a compact $C_0$-semigroup, $F : V_1 \to V$ a bounded linear operator and the assumptions (a)-(c) for $H$ be fulfilled. Under the hypotheses of Theorem 2.11, (DQVHI) has at least one mild solution $(s, c)$.

**Proof.** According to Pazy [22], for $c \in L^2([0, L]; V_1)$, the mild solutions of

$$s'(t) \in Es(t) + Fc(t) + H(t, s(t)), \quad t \in [0, L],$$

$$s(0) = s_0$$

may denoted by

$$s(t) = e^{Et}s_0 + \int_0^t e^{E(t-\tau)}[Fu(\tau) + h(\tau)]d\tau, \quad t \in [0, L], h \in P_H(s).$$

By hypothesis and Rykaczewski [23] it follows that 7 is solvable, that is, there exists $s$ satisfying 8.

For every $t \in [0, L]$ and $M_E := \max_{t \in [0, L]} ||e^{Et}||$, from 8, we have the following estimates

$$\| s(t) \|_V \leq \| e^{Et}s_0 \|_V + \int_0^t \| e^{E(t-\tau)}[Fu(\tau) + h(\tau)] \|_V d\tau$$

$$\leq M_E \left( \| s_0 \|_V + \| F \| \| c \|_{L^2([0, L]; V_1)} L^{1/2} \right) + M_E \int_0^t \rho(\tau)(1 + \| s(\tau) \|_V) d\tau,$$

or, equivalently, by the Gronwall’s inequality,

$$\| s(t) \|_V \leq M_E \left( \| s_0 \|_V + \| F \| \| c \|_{L^2([0, L]; V_1)} L^{1/2} + \| \rho \|_{L^2([0, L])} \right)e^{M_E \| \rho \|_{L^2([0, L])}}.$$

Next, we prove the existence of solutions for (DQVHI). Set

$$c_k(t) = \sum_{i=0}^{k-1} c(i) \chi_{\left(\frac{iL}{k}, \frac{(i+1)L}{k}\right)}(t) \in Z(iL_k, s_k(iL_k)), \quad t \in \left[\frac{iL_k}{k}, \frac{(i+1)L_k}{k}\right), 0 \leq i \leq k - 1,$$

where $[0, L] = \bigcup_{i=1}^{k-1} [t_i, t_{i+1}] \cup \{L\}$, with $t_i := \frac{iL}{k}$, and $\chi_{\left(\frac{iL}{k}, \frac{(i+1)L}{k}\right)}$ is the character function of interval $\left[\frac{iL}{k}, \frac{(i+1)L}{k}\right)$. Then, there exists $s_k(\cdot)$ such that

$$s_k(t) = e^{Et}s_0 + \int_0^t e^{E(t-\tau)}[Fu_k(\tau) + h_k(\tau)]d\tau, \quad t \in [0, L], h_k \in P_H(s_k).$$

For the sequences $\{c_k\}$, it follows from Theorem 2.11 that there is $r_1 > 0$ such that $\| c_k \|_{L^2([0, L]; V_1)} \leq r_1$. Also, by the above computations, we get $\| s_k \|_{C([0, L]; V)} \leq r_0$, where $r_0 > 0$ is a constant. Thus, by assumption (c) of $H$, it follows that there is $r_2 > 0$ such that $\| h_k \|_{L^2([0, L]; V)} \leq r_2$. Without loss of generality, we may assume that $c_k \rightharpoonup c^* \in L^2([0, L]; V_1)$ and $h_k \rightharpoonup h^*$ in $L^2([0, L]; V)$. By using that $e^{Et}$ is
a compact $C_0$-semigroup and according to Li et al. [10], it results that $s_k \to s^*$ in $C([0, L]; V)$, where

$$s^*(t) := e^{E_t} s_0 + \int_0^t e^{E(t-\tau)}[Fu^*(\tau) + h^*(\tau)]d\tau.$$ 

Further, by applying Mazur theorem (see Li et al. [11]), we obtain that there are $a_{il} \geq 0$, $b_{il} \geq 0$ with $\sum_{i \geq 1} a_{il} = \sum_{i \geq 1} b_{il} = 1$ such that

$$c_l(t) := \sum_{i \geq 1} a_{il} c_{l+i}(t) \in C^2([0, L]; V_1), \text{ a.e. } t \in [0, L],$$

$$h_l(t) := \sum_{i \geq 1} b_{il} h_{l+i}(t) \in C^2([0, L]; V), \text{ a.e. } t \in [0, L].$$

Since $H(t, \cdot)$ is upper semicontinuous, $s_k \to s^*$ in $C([0, L]; V)$, then for $\epsilon > 0$ and $k > 0$ large enough, we get

$$H(t, s_k(t)) \subset H(t, s^*(t)) + B_{\epsilon},$$

where $B_{\epsilon}$ is a ball in $V$ centered in origin and radius $\epsilon$. Due to the convexity of $H(t, s^*(t)) + B_{\epsilon}$, it follows that $h_l(t) \in H(t, s^*(t)) + B_{\epsilon}$, a.e. $t \in [0, L]$. Since $h_l(t) \to h^*(t)$ as $l \to \infty$, we get $h^*(t) \in H(t, s^*(t)) + B_{\epsilon}$. For $\epsilon > 0$ arbitrarily, it results $h^*(t) \in \overline{H(t, s^*(t))} = H(t, s^*(t))$, a.e. $t \in [0, L]$. By the same arguments, we get $c^*(t) \in Z(t, s^*(t)), \text{ a.e. } t \in [0, L]$ such that $s^*(t) = e^{E_t} s_0 + \int_0^t e^{E(t-\tau)}[Fu^*(\tau) + h^*(\tau)]d\tau$, a.e. $t \in [0, L]$ and $h^* \in P_H(s^*)$. The proof is complete.

Acknowledgments. The author would like to thank anonymous referees for their careful reading and constructive suggestions that substantially improved the revision of the manuscript.

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Received November 2020; revised March 2021.

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