The Free Energy of Hot QED at Fifth Order

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Abstract

The order $e^5$ contribution to the pressure of massless quantum electrodynamics at nonzero temperature is determined explicitly. An identity is also obtained relating a gauge-invariant piece of the pressure at order $e^{2n+3}$ ($n \geq 1$) (from diagrams with only one fermion loop) to the pressure at order $e^{2n}$. Prospects for higher order calculations are discussed and potential applications are mentioned.

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The three-loop contribution to the equation-of-state of massless quantum electrodynamics (QED) at temperature $T$ was obtained recently \[1\], thus extending the well known two-loop result \[2\]. While three-loop calculations in cold plasmas ($T = 0$, but nonzero chemical potential $\mu$) were done some time ago \[3\], same order calculations for hot plasmas have been delayed by technical complications due to the presence of statistical distribution factors in loop integrals. The three-loop calculation in hot QED \[1\] was stimulated by in part by a ground breaking three-loop calculation in hot $\phi^4$ theory \[4\].

The purpose of this letter is to further the results of \[1\] by determining explicitly the next order ($e^5$) contribution to the pressure of QED with $N$ massless electrons at nonzero $T$ (but $\mu = 0$). Higher orders will also be discussed qualitatively. The conventions here are similar to \[1\] and are reiterated for clarity: The imaginary time formalism is used, whereby the energies are discrete and imaginary, $q_0 = in\pi T$, $n$ being an odd (even) integer for fermions (bosons). Ultraviolet singularities are regularised by dimensional continuation ($4 \to D$) and renormalisation is through minimal subtraction. The wave-vector, $Q_\mu = (q_0, \vec{q})$ lives in a space with Minkowski metric, $g_{\mu\nu} = \text{diag}(1, -1, -1,..., -1)$ and the measure for loop integrals is denoted by

$$
\int \{dq\} \equiv T \sum_{q_0,\text{odd}} \int (dq),
$$

$$
\int [dq] \equiv T \sum_{q_0,\text{even}} \int (dq),
$$

where

$$
\int (dq) = \int \frac{d^{D-1}q}{(2\pi)^{D-1}}.
$$

Spinor traces are normalised according to $Tr(\gamma_\mu \gamma_\nu) = 4g_{\mu\nu}$, and the photon is in the Feynman gauge, $D_{\mu\nu}(Q) = g_{\mu\nu}/Q^2$.

Recall that in a QED plasma static electric fields are screened with an inverse screening length given to lowest order by $eT/\sqrt{3}$. Consequently a naive perturbative expansion using bare (unscreened) propagators produces severe infrared (IR) divergences in diagrams such as in Figs.1,2. When these diagrams are resummed one obtains a finite result of order $e^3$ \[3, 4\] which is nonanalytic in the coupling $e^2$. Just as the $e^3$ term is the “plasmon” correction to the two-loop $e^2$ contribution obtained
by dressing the photons, the order $e^5$ term to be presently calculated is the plasmon correction to the three-loop $e^4$ result. The object that is required in the analysis below is the renormalised, static, one-loop photon polarisation tensor $\Pi_{\mu\nu}(q_0 = 0, q)$. Gauge invariance, $Q_{\mu}\Pi^\mu\nu(Q) = 0$, implies $\Pi_{i0}(0, q) = 0$ (this is of course true to all orders) while explicit calculations yield $\Pi_{ij}(0, q \to 0) = O(e^2q^2)$ and

$$\Pi_{00}(0, q) \equiv m^2 + e^2q^2h(q) , \quad (2)$$

where $e$ is the renormalised coupling. The following limits will be required :

$$m^2(D \to 4) = e^2T^2N/3$$

The second term in (2) is the UV counterterm and $\mu$ is the mass scale of dimensional regularisation. The integral $f_2$ in (3) is defined through

$$f_n \equiv \int \frac{dQ}{(Q^2)^n} = (2^{2n+1-D} - 1)b_n ,$$

$$b_n \equiv \int \frac{dQ}{(Q^2)^n}$$

Consider now the three-loop, order $e^4N$, diagram $G_1$ shown in Fig.3a. It is given by

$$\frac{G_1}{\mu^{8-2D}} = -e^4N \left\{ \frac{dK}{dQ} dP \right\} \frac{Tr(\gamma_\mu K\gamma_\alpha D^{\alpha\sigma}(P)(K-P)\gamma_\sigma K\gamma_\nu D^{\nu\mu}(Q)(K-Q))}{K^4(K-Q)^2(K-P)^2} . \quad (4)$$

From the behaviour of the polarisation tensor discussed above, it is deduced that when one or more self-energy subdiagrams are inserted along the photon lines of $G_1$, IR divergences occur only for the $\Pi_{00}(0, 0) = m^2$ insertions . These may be resummed into an effective propagator

$$D_{\mu\nu}(Q) \to \left( \frac{g_{\mu\nu}}{Q^2} - \frac{g_{\mu0} g_{\nu0} \delta_{q_0,0}}{Q^2} \right) + \frac{g_{\mu0} g_{\nu0} \delta_{q_0,0}}{Q^2 - m^2}$$

$$= \frac{g_{\mu\nu}}{Q^2} + \frac{m^2 g_{\mu0} g_{\nu0} \delta_{q_0,0}}{q^2(q^2 + m^2)}$$

$$\equiv D_{\mu\nu}(Q) + D^\ast_{\mu\nu}(Q) . \quad (5)$$
When the effective propagator (5) is used for the $Q$-photon in $G_1$ the correction $D^*(Q)$ makes the $q$-integral of order $e$, $\int \frac{d^3q \, m^2}{q^2(q^2 + m^2)} \sim m \sim e$, and therefore its contribution to $G_1$ is $O(e^5)$. Similarly if both photons are dressed, the extra correction to $G_1$ will be of order $e^6$. Thus the order $e^5$ contribution from dressing $G_1$ is obtained by keeping one photon bare, using the correction $D^*$ for the other, and multiplying the result by two:

$$
\frac{\delta G_1}{\mu^{8-2D}} = -e^4 N T \int (dq) \frac{m^2 \delta_{q0,0}}{q^2(q^2 + m^2)} \int \frac{dK [dP] \text{Tr}(\gamma_0 K \gamma_0(K - P)\gamma^\alpha K \gamma_0(K - Q))}{K^4P^2(K - Q)^2(K - P)^2}.
$$

(6)

Now scale $q = m \bar{x}$ in the above equation. This results in an overall external factor $e^4m \sim e^5$ and since the $K, P$ integrals are infrared safe in the $m \to 0$ limit, one obtains the exact $e^5$ contribution from $\delta G_1$ as

$$
\frac{G_{15}}{\mu^{8-2D}} = -e^4 N T m^{D-3} \int \frac{(dx)}{x^2(x^2 + 1)} \int \frac{dK [dP] \text{Tr}(\gamma_0 K \gamma_0(K - P)\gamma^\alpha K \gamma_0 K)}{K^6P^2(K - P)^2}.
$$

(7)

Notice that for the $e^5$ calculation, the original complicated three-loop integral has factorised completely into the product of a finite integral and a relatively simple two-loop integral. A similar analysis for diagram $G_2$ (Fig.3b) gives for the $e^5$ contribution,

$$
\frac{G_{25}}{\mu^{8-2D}} = \frac{-e^4 N T m^{D-3}}{2} \int \frac{(dx)}{x^2(x^2 + 1)} \int \frac{dK [dP] \text{Tr}(\gamma_0 K \gamma_0(K - P)\gamma^\alpha K \gamma_0 K)}{K^4P^2(K - P)^4}.
$$

(8)

For the sum of (7) and (8) I obtain

$$
G_{15} + G_{25} = -e^4 m^{D-3} T \mu^{8-2D} N S 2(2 - D)(b_1 - f_1)(4 - D)f_2
$$

$$
= -\frac{e^5 T^4 N^{3/2}}{64\pi^3\sqrt{3}} + O(D - 4).
$$

(9)

(10)

Pleasantly, while the individual expressions $G_{15}$ and $G_{25}$ require the evaluation of some two-loop integrals, their sum (9) depends only on the trivial one-loop integrals $b_n$ and $f_n$, defined previously, and $S = \int \frac{(dx)}{x^2(x^2 + 1)}$. The final result as $D \to 4$ in (10) is finite as required since the contribution of $e^5$ diagrams coming from electron-wavefunction and vertex renormalsations cancels as in the $e^4$ calculation [1] because of the Ward identity $Z_1 = Z_2$, and the restriction to massless electrons.
The order $e^4N^2$ diagram is shown in Fig.1. Inserting self-energy subdiagrams along the photon lines produces the equivalent set of diagrams shown in Fig. 2. Power counting now indicates that only if the self-energies in Fig.2 are $\Pi_{00}(0, q)$ can one obtain contributions of lower order than $e^6$. These insertions sum to

$$-\frac{T}{2} \int (dq) \left[ \ln \left( 1 + \frac{\Pi_{00}(0, q)}{q^2} \right) - \frac{\Pi_{00}(0, q)}{q^2} + \frac{1}{2} \left( \frac{\Pi_{00}(0, q)}{q^2} \right)^2 \right]. \quad (11)$$

However (11) contains as a leading contribution part of the lower order $e^3$ piece due to $\Pi_{00}(00) = m^2$. Subtracting from (11) the same expression but with $\Pi_{00}(0, q)$ replaced by $m^2$ gives, upon using (2),

$$-\frac{T}{2} \int (dq) \left[ \ln \left( 1 + e^{2q^2} h(q) \right) - e^2 h(q) + \frac{e^2 h(q)}{2q^2} \left( 2m^2 + e^2 q^2 h(q) \right) \right]. \quad (12)$$

This contains all terms of order $e^5N^{5/2}$ but also many subleading terms. Again the scaling $\vec{q} = m\vec{x}$ helps to identify the exact $e^5$ pieces. Dropping subleading terms one finds

$$G_{35} = -\frac{T e^2 m^{D-1}}{2} \int (dx) x^2 h(0) \left( \frac{1}{x^2 + 1} - \frac{1}{x^2} + \frac{1}{x^4} \right) \quad (13)$$

$$\rightarrow -\frac{e^5 T^4 N^{5/2}}{8\pi \sqrt{27}} \gamma - 1 + \ln(4/\pi) + \frac{e^5 T^4 N^{5/2}}{8\pi \sqrt{27}} \frac{\ln(T/\mu)}{6\pi^2}. \quad (14)$$

The expression (13) has a simple interpretation and could have been written down by inspecting the diagrams in Fig. 2: It is the subleading contribution from the sum of infinite diagrams when, for each diagram, the subleading $e^2q^2$ piece is kept of exactly one self-energy while the leading $e^2T^2$ pieces are taken from the remaining self-energies. The $e^5N^{5/2} \ln(T/\mu)$ term in the answer (14) is a remnant of wavefunction renormalisation. This logarithm cancels against a similar term that is generated from the lower order $e^3N^{3/2}$ plasmon term when the pressure is written in terms of the temperature dependent coupling

$$e^2(T) = e^2 \left( 1 + \frac{e^2 N}{6\pi^2} \ln \frac{T}{\mu} \right) + O(e^6). \quad (15)$$

Alternatively one may also eliminate such logarithms by simply choosing $\mu = T$.

The fine-structure constant at temperature $T$ is $\alpha(T) = e^2(T)/4\pi$. Defining $g^2 = \alpha(T) N/\pi$, the pressure of QED with $N$ massless Dirac fermions at nonzero temperature, $T$, then follows from Refs. [2, 1] and eqns.(10), (14) and (15),
\[
\frac{P}{T^4} = a_0 + g^2 a_2 + g^3 a_3 + g^4 (a_4 + b_4/N) + g^5 (a_5 + b_5/N) + O(g^6),
\]  
(16)

with

\[
a_0 = \frac{\pi^2}{45} (1 + \frac{7}{4}N),
\]
(17)
\[
a_2 = -\frac{5\pi^2}{72},
\]
(18)
\[
a_3 = \frac{2\pi^2}{9\sqrt{3}},
\]
(19)
\[
a_4 = -0.8216 \pm 10^{-4},
\]
(20)
\[
b_4 = 1.2456 \pm 1.5 \times 10^{-4},
\]
(21)
\[
a_5 = \frac{\pi^2 [1 - \gamma - \ln(4/\pi)]}{9\sqrt{3}} = 0.11473...,
\]
(22)
\[
b_5 = -\frac{\pi^2}{2\sqrt{3}}.
\]
(23)

Some observations are made on the short perturbation series above. For \(N = 1\) the fourth order \((g^4)\) coefficient \((a_4 + b_4) = 0.4240 \pm 0.00025\) is positive\(^\dagger\) while the fifth order coefficient \((a_5 + b_5) = -2.734...\) is negative. Thus there is a sign alternation separately within the even orders \((g^0, g^2, g^4)\) and the odd orders \((g^3, g^5)\). On the other hand if one considers a double expansion in \(g\) and \(1/N\) then the \((1/N)^0\) series \(\{a_n\}\) shows sign alternation and so does the \((1/N)\) series \(\{b_n\}\), at least for the available terms. I do not know if this empirical regularity is suggestive of a general result.

In \(T = 0\) QED the large order behaviour of perturbation theory with respect to the expansion parameter \(e^2\) was studied many years ago \([3, 7]\). By contrast, at \(T \neq 0\), resumming the IR divergences creates an expansion in \(\sqrt{e^2}\) and to my knowledge an asymptotic analysis in this case is lacking. Some information about the perturbation series can be obtained in the large \(N\) expansion. To leading order in \(1/N\) (the \(\{a_n\}\) series), the diagrams which contribute are precisely those in Figs.1,2 (plus the order \(e^2\) diagram not shown) with full one-loop self-energy insertions. Consider now only the odd terms \(\{a_{2n+1}, n = 1, 2, \ldots\}\), these are the plasmon \((e^{2n+1})\) contributions in the large \(N\) limit. From power counting as before one deduces that all the \(a_{2n+1}\) terms are contained in the expression

\[
-\frac{T}{2} \int \frac{d^3q}{(2\pi)^3} \left[ \ln \left( 1 + \frac{\Pi_{00}(0, q)}{q^2} \right) - \frac{\Pi_{00}(0, q)}{q^2} \right].
\]
(24)

\(^\dagger\)This value may be very closely approximated by \(3\sqrt{2}/10 = 0.42426...\); also note that \(a_4 \approx -\pi^2/12 = -0.8224...\)
Here $\Pi_{00}(0,q)$ is the finite one-loop self-energy of eq.(2) at $D=4$ and renormalisation scale $\mu = T$. A systematic expansion of (24) in $e$ determines the series $\{a_{2n+1}\}$ (parts of $a_{2n}$ are also contained in the expansion) and it might be possible to determine $a_{2n+1}$ in closed form for any $n$.

The full $e^5$ calculation was simple because of the factorisation of the diagrams $G_1$ and $G_2$. In general, let the gauge-invariant contribution to the pressure at order $e^{2n+2}$ ($n \geq 1$), from diagrams with one-fermion loop, be $P_{2n+2}^{1F}$. Then the order $P_{2n+3}^{1F}$ contribution is obtained by dressing and using the factorisation property:

$$P_{2n+3}^{1F} = \int \frac{(dq) \ m^2}{q^2(q^2 + m^2)} \frac{\delta P_{2n+2}^{1F}}{\delta D_{00}(0,q \to 0)} V ,$$

$$= \frac{eTN^{1/2}}{4\pi\sqrt{3}} \frac{\delta P_{2n+2}^{1F}}{\delta D_{00}(0)} V ,$$

$$= \frac{eT^2N^{1/2}}{8\pi\sqrt{3}} \hat{\Pi}_{00}^{1F,2n+2}(0,0) ,$$

$$= \frac{e^3T^2N^{1/2}}{8\pi\sqrt{3}} \frac{\partial^2 P_{2n+3}^{1F}}{\partial \mu^2} |_{\mu=0} , n \geq 1 .$$

Equation (27) follows from the earlier equation upon using the relation $\hat{\Pi}_{II}^{\mu\nu} = T \hat{\Pi}^{\mu\nu}/2V$, where $V =$ volume, $\hat{PI}$ refers to one-particle-irreducible and $\hat{\Pi}$ is the full self-energy. The final equation (28) then follows from the identity $\hat{\Pi}_{00}(00) = e^2\partial P^2/\partial \mu^2$. The result (28) is that mentioned in the abstract. It is valid for the case of $N$ massless electrons at zero chemical potential $\mu$. For the case $n = 1$ one may verify that (28) is satisfied by computing the right-hand-side from known two-loop results (2, 8) and comparing it with the left-hand-side given by (10).

Eq.(28) suggests that the $e^7$ contribution should be calculable because knowledge of three-loop diagrams is at hand (4, 8) (diagrams with more than one fermion loop are not covered by (28) but they are easier and are handled as for the case Fig.1 discussed above). If extra cancellations occur, as in the $e^5$ computation, then the $e^7$ calculation might reduce to a two-loop calculation. Note that the factorisation is successful because fermions in imaginary time have IR safe propagators. If one attempted to calculate the $\lambda^5$ contribution to the pressure of $\lambda^2\phi^4$ theory, the diagram in Fig. 4 is involved. Now the exact factorisation fails because the two-loop

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$^3$An alternative derivation using the ring-summation formula (8), and the extension to massive electrons at nonzero chemical potential will be presented elsewhere.
subdiagram has a logarithmic singularity coming from the zero modes of the bosonic propagators. Nevertheless the calculation in this case is still feasible because only the leading $T^2(\ln k + \text{constant})$ piece of the two-loop self-energy subdiagram (with inflowing momentum $(0, k)$) is needed to determine the order $T^4 \lambda^5 (\ln \lambda + \text{constant})$ contribution.

Calculations similar to those discussed in [1] and here have been performed to obtain directly the screening masses in hot QED to high-order [4], and future applications to quantum chromodynamics [10] are envisaged. One speculative application for QED will be mentioned: there has been much discussion in the literature (see, e.g., [11] and references therein) concerning a possible strong-coupling phase (at $T = 0$) of QED with an ultraviolet-stable fixed point. Usual perturbative arguments indicate that at exponentially high temperatures the QED coupling will be strong. If a temperature driven transition to a stable nonperturbative phase is possible, then a resummed high-order perturbation series [7] for hot QED might be useful for some studies.

In conclusion, some of the questions raised in the last three paragraphs are left for future investigations [12].

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Figure Captions

Fig.1:
The order $e^4N^2$ diagram. It has an infrared singularity which contributes to the lower ($e^3$) order. In Figs.1-3 the wavy line is the photon.

Fig.2:
Diagrams generated from Fig.1 by photon-polarisation insertions.

Fig.3:
The order $e^4N$ diagrams which produce $e^5N^{3/2}$ terms when the photons are dressed.

Fig.4:
The nontrivial three-loop diagram of $\lambda^2\phi^4$ theory studied in [4]. It will contribute a $\lambda^5\ln\lambda$ piece to the free energy when dressed.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9406318v1