Canonical systems and non-commutative geometry

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ABSTRACT: Inspired by ideas from non-commutative geometry, unions of moduli spaces of linear control systems are identified as open subsets of infinite Grassmannians.
1. Introduction

A linear control system $\Sigma$ of type $(m, n, p) \in \mathbb{N}^3$ is determined by the system of linear differential equations

$$\begin{cases}
\frac{dx}{dt} = Ax + Bu \\
y = Cx,
\end{cases}$$

where $u(t) \in \mathbb{C}^n$ is the input or control at time $t$, $x(t) \in \mathbb{C}^n$ is the state of the system and $y(t) \in \mathbb{C}^p$ is its output. That is, $\Sigma$ is described by a triple of matrices

$$\Sigma = (A, B, C) \in M_n(\mathbb{C}) \times M_{n \times m}(\mathbb{C}) \times M_{p \times n}(\mathbb{C}) = V_{m,n,p}$$

and is said to be equivalent to a system $\Sigma' = (A', B', C') \in V_{m,n,p}$ if and only if there is a basechange matrix $g \in GL_n = GL_n(\mathbb{C})$ in the state-space such that

$$\Sigma \sim \Sigma' \iff A' = gAg^{-1}, \quad B' = gB \quad \text{and} \quad C' = Cg^{-1}.$$ 

A system $\Sigma = (A, B, C) \in V_{m,n,p}$ is said to be completely controllable (resp. completely observable) if and only if the matrix

$$c(\Sigma) = \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} \quad (\text{resp.} \quad o(\Sigma) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix})$$

is of maximal rank. These conditions define $GL_n$-open subsets $V^{cc}_{m,n,p}$, resp. $V^{co}_{m,n,p}$, consisting of systems with trivial $GL_n$-stabilizer, whence we have corresponding orbit spaces

$$\text{sys}^{cc}_{m,n,p} = V^{cc}_{m,n,p}/GL_n \quad \text{and} \quad \text{sys}^{co}_{m,n,p} = V^{co}_{m,n,p}/GL_n.$$
which are known to be smooth quasi-projective varieties of dimension \((m + p)n\), see for example [13, Part IV]. A system \(\Sigma = (A, B, C) \in V_{m,n,p}\) is said to be canonical if it is both completely controllable and completely observable. The corresponding moduli space

\[
\text{sys}_{m,n,p}^c = (V_{m,n,p}^{cc} \cap V_{m,n,p}^{co})/GL_n
\]

classifies canonical systems having the same input-output behavior, that is, such that all the \(p \times m\) matrices \(CA^iB\) for \(i \in \mathbb{N}\) are equal [13, Part VI - VII]. Conversely, if \(F = \{F_j : j \in \mathbb{N}_+\}\) is a sequence of \(p \times m\) matrices such that the corresponding Hankel matrices

\[
H_{ij}(F) = \begin{bmatrix}
F_1 & F_2 & \ldots & F_j \\
F_2 & F_3 & \ldots & F_{j+1} \\
\vdots & \vdots & & \vdots \\
F_i & F_{i+1} & \ldots & F_{i+j-1}
\end{bmatrix}
\]

are such that there exist integers \(r\) and \(s\) such that \(rk \ H_{rs}(F) = rk \ H_{r+1,s+j}(F)\) for all \(j \in \mathbb{N}_+\), then \(F\) is realizable by a canonical system \(\Sigma = (A, B, C) \in V_{m,n,p}^c\) (for some \(n\) which is equal to \(rk \ H_{rs}(F)\)), that is,

\[
F_j = CA^{j-1}B \quad \text{for all } j \in \mathbb{N}_+,
\]

see for example [13, Part VI - VII] for connections between this realization problem and classical problems in analysis. These problems would be facilitated if there was an infinite dimensional manifold \(X\) together with a natural stratification

\[
X = \bigsqcup_n \text{sys}_{m,n,p}^c
\]

by the moduli spaces of canonical systems (for fixed \(m\) and \(p\) and varying \(n\)).

Non-commutative geometry, as outlined by M. Kontsevich in [9], offers a possibility to glue together closely related moduli spaces into an infinite dimensional variety controlled by a non-commutative algebra. The individual moduli spaces are then recovered as moduli spaces of simple representations (of specific dimension vectors) of the non-commutative algebra. An illustrative example is contained in the recent work by G. Wilson and Yu. Berest [15] relating Calogero-Moser spaces to the adelic Grassmannian (see also [3] and [6] for the connection with non-commutative geometry). The main aim of the present paper is to offer another (and more elementary) example:

**Theorem 1** The equivalence classes of canonical systems with fixed input- and output-dimensions \(m\) and \(p\) form a specific open submanifold

\[
\bigsqcup_n \text{sys}_{m,n,p}^c \hookrightarrow \text{Gras}_{m+p}(\infty)
\]

of the infinite Grassmannian of \(m + p\)-dimensional subspaces.

This paper is organized as follows. In section two we show that Kontsevich’s approach is applicable to moduli spaces of canonical systems by proving that there is a natural one-to-one correspondence between equivalence classes of canonical systems with \(n\)-dimensional state space and
isomorphism classes of simple representations of dimension vector \((1, n)\) of the formally smooth path algebra of the quiver

\[
\begin{array}{cccccc}
\updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow \\
\updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow \\
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\updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow & \updownarrow \updownarrow \updownarrow \\
\end{array}
\]

with \(m\) arrows pointing right and \(p\) arrows pointing left. This observation gives a short proof of the following result, due to M. Hazewinkel ([13, thm. VI.2.5] or [8, (2.5.7)]):

**Theorem 2 (Hazewinkel)** The moduli space \(\text{sys}_{m,n,p}^c\) of canonical systems is a quasi-affine variety.

In section 3 we prove that the moduli spaces \(\text{sys}_{m,n,p}^{cc}\) (resp. \(\text{sys}_{m,n,p}^{co}\)) of completely controllable (resp. completely observable) systems are isomorphic to moduli spaces (in the sense of A. King [10]) of \(\theta\)-stable representations of dimension vector \((1, n)\) for this quiver, where \(\theta = (-n, 1)\) (resp. \(\theta = (n, -1)\)). By computing the cohomology of these moduli spaces, as in [14], we were then led to

**Theorem 3** The moduli space \(\text{sys}_{m,n,p}^{cc}\) of completely controllable systems has a cell decomposition identical to the natural cell decomposition of a vectorbundle of rank \((p + 1)n\) on the Grassmannian \(\text{Gras}_n(m + n - 1)\) with respect to the Schubert cells on the Grassmannian.

In an earlier version of this note we claimed that the moduli space itself is a vectorbundle over the Grassmannian. However, this cannot be the case when \(m = n\) as the referee kindly pointed out.

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2. Proof of theorem 2

Consider the quiver setting \((Q, \alpha)\) where the dimension vector is \(\alpha = (1, n)\) and the quiver \(Q\)

\[
\begin{array}{cccccc}
\uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow \\
\uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow \\
\uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow \\
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\uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow & \uparrow \uparrow \uparrow \uparrow \\
\end{array}
\]

has \(m\) arrows \(\{b_1, \ldots, b_m\}\) from left to right and \(p\) arrows \(\{c_1, \ldots, c_p\}\) from right to left. We can identify \(V_{m,n,p}\) with \(\text{rep}_\alpha Q\), where we associate to a system \(\Sigma = (A, B, C)\) the representation \(V_\Sigma\) which assigns to the arrow \(b_i\) (resp. \(c_j\)) the \(i\)-th column \(B_i\) of \(B\) (resp. the \(j\)-th row \(C_j\) of \(C\)) and the matrix \(A\) to the loop. The basechange action of \((\lambda, g) \in GL(\alpha) = \mathbb{C}^* \times GL_n\) acts on the representation \(V_\Sigma = (A, B_1, \ldots, B_m, C^1, \ldots, C^p)\) as follows:

\[(\lambda, g).V_\Sigma = (gA\lambda^{-1}, gB_1\lambda^{-1}, \ldots, gB_m\lambda^{-1}, \lambda C^1 g^{-1}, \ldots, \lambda C^p g^{-1}),\]
and as the central subgroup $C^*(1,1_n)$ acts trivially on $\text{rep}_\alpha Q$, there is a natural one-to-one correspondence between equivalence classes of systems in $V_{m,n,p}$ and isomorphism classes of $\alpha$-dimensional representations in $\text{rep}_\alpha Q$. If $CQ$ denotes the path algebra of the quiver $Q$, then it is well known that $CQ$ is a formally smooth algebra in the sense of [4], and that there is an equivalence of categories between finite dimensional right $CQ$-modules and representations of $Q$. It is perhaps surprising that the system theoretic notion of canonical system corresponds under these identifications to the algebraic notion of simple module.

**Lemma 1** The following are equivalent:

1. $\Sigma = (A, B, C) \in V_{m,n,p}$ is a canonical system,
2. $V_\Sigma = (A, B_1, \ldots, B_m, C^1, \ldots, C^p) \in \text{rep}_\alpha Q$ is a simple representation.

**Proof.** 1 $\Rightarrow$ 2 : If $V_\Sigma$ has a proper subrepresentation of dimension vector $\beta = (1, l)$ for some $l < n$, then the rank of the control-matrix $c(\Sigma)$ is at most $l$, contradicting complete controllability. If $V_\Sigma$ has a proper subrepresentation of dimension vector $\beta' = (0, l)$ with $l \neq 0$, then the observation-matrix $o(\Sigma)$ has rank at most $n - l$, contradicting complete observability. 2 $\Rightarrow$ 1 : If $rk\ c(\Sigma) = l < n$ then there is a proper subrepresentation of dimension vector $(1, l)$ of $V_\Sigma$. If $rk\ o(\Sigma) = n - l$ with $l > 0$, then there is a proper subrepresentation of dimension vector $(0, l)$ of $V_\Sigma$.

From [12] we recall that for a general quiver setting $(Q, \alpha)$ the isomorphism classes of $\alpha$-dimensional semi-simple representations are classified by the affine algebraic quotient variety

$$\text{rep}_\alpha Q//GL(\alpha) = \text{iss}_\alpha Q$$

whose coordinate ring is generated by all traces along oriented cycles in the quiver $Q$. If $\alpha$ is the dimension vector of a simple representation, this affine quotient has dimension $1 - \chi_Q(\alpha, \alpha)$ where $\chi_Q$ is the Euler form of $Q$. Moreover, the isomorphism classes of simple representations form a Zariski open smooth subvariety of $\text{iss}_\alpha Q$. Specializing these general results from [12] to the case of interest, we recover Hazewikels theorem.

**Theorem 4 (Hazewinkel)** The moduli space $\text{sys}_{m,n,p}^C$ of canonical systems is a smooth quasi-affine variety of dimension $(m + p)n$.

In fact, combining the theory of local quivers (see for example [11]) with the classification of all quiver settings having a smooth quotient variety due to Raf Bocklandt [2], it follows that (unless $m = p = 1$) $\text{sys}_{m,n,p}^C$ is precisely the smooth locus of the affine quotient variety $\text{iss}_\alpha Q$.

**3. Proof of theorem 3**

For $(Q, \alpha)$ a quiver setting on $k$ vertices and if $\theta \in \mathbb{Z}^k$, a representation $V \in \text{rep}_\alpha Q$ is said to be $\theta$-semistable (resp. $\theta$-stable) if and only if for every proper non-zero subrepresentation $W$ of $V$ we have that $\theta.\beta \geq 0$ (resp. $\theta.\beta > 0$), where $\beta$ is the dimension vector of $W$. In the special case
when \( \alpha = (1, n) \) and \( Q \) is the quiver introduced before, there are essentially two different stability structures on \( \text{rep}_\alpha Q \) determined by the integral vectors
\[
\theta_+ = (-n, 1) \quad \text{and} \quad \theta_- = (n, -1)
\]
By the identification of \( \text{rep}_\alpha Q \) with \( V_{m,n,p} \) and the proof of lemma 1 we have

**Lemma 2** For \( \theta_+ = (-n, 1) \) the following are equivalent:
1. \( \Sigma \in V_{m,n,p} \) is completely controllable,
2. \( V_\Sigma \in \text{rep}_\alpha Q \) is \( \theta_+ \)-stable.

For \( \theta_- = (n, -1) \) the following are equivalent:
1. \( \Sigma \in V_{m,n,p} \) is completely controllable,
2. \( V_\Sigma \in \text{rep}_\alpha Q \) is \( \theta_- \)-stable.

For a general stability structure \( \theta \) and quiver setting \( (Q, \alpha) \), A. King \([10]\) introduced and studied the moduli space \( \text{moduli}_\theta^\alpha Q \) of \( \theta \)-semistable representations, the points of which classify isomorphism classes of direct sums of \( \theta \)-stable representations. In the case of interest to us we have
\[
\text{sys}_m^{cc} = \text{moduli}_\theta^\alpha Q \quad \text{and} \quad \text{sys}_m^{co} = \text{moduli}_\theta^\alpha Q.
\]
In \([14]\) the Harder-Narasimham filtration associated to a stability structure was used to compute the cohomology of the moduli spaces \( \text{moduli}_\theta^\alpha Q \) (at least if the quiver \( Q \) has no oriented cycles). For general quivers the same methods can be applied to compute the number of \( \mathbb{F}_q \)-points of these moduli spaces, where \( \mathbb{F}_q \) is the finite field of \( q = p^l \) elements. In the case of interest to us, we get the rational functions
\[
\begin{align*}
\# \text{moduli}_\theta^\alpha Q (\mathbb{F}_q) &= q^{n(p+1)} \prod_{i=1}^{n} \frac{q^{m+i-1} - 1}{q^i - 1}, \\
\# \text{moduli}_\theta^\alpha Q (\mathbb{F}_q) &= q^{n(m+1)} \prod_{i=1}^{n} \frac{q^{p+i-1} - 1}{q^i - 1},
\end{align*}
\]
which suggests that the moduli space \( \text{sys}_m^{cc} \) is a vectorbundle of rank \( n(p+1) \) over the Grassmannian \( \text{Gras}_n(m+n-1) \), and that the moduli space \( \text{sys}_m^{co} \) is a vectorbundle of rank \( n(m+1) \) over \( \text{Gras}_n(p+n-1) \).

To a completely controllable \( \Sigma = (A, B, C) \) one associates its Kalman code \( K_\Sigma \), which is an array of \( n \times m \) boxes \( \{(i,j) \mid 0 \leq i < n, 1 \leq j \leq \} \), ordered lexicographically, with exactly \( n \) boxes painted black. If the column \( A^iB_j \) is linearly independent of all column vectors \( A^kB_l \) with \( (k,l) < (i,j) \) we paint box \( (i,j) \) black. From this rule it is clear that if \( (i,j) \) is a black box so are \( (i',j) \) for all \( i' \leq i \). That is, the Kalman code \( K_\Sigma \) (which only depends on the \( GL_n \)-orbit of \( \Sigma \)) looks like
Assume $\kappa = K_\Sigma$ has $k$ black boxes on its first row at places $(0, i_1), \ldots, (0, i_k)$. Then we assign to $\kappa$ the strictly increasing sequence

$$1 \leq j_{\kappa}(1) = i_1 < j_{\kappa}(2) = i_2 < \ldots < j_{\kappa}(k) = i_k \leq m$$

and another sequence $p_{\kappa}(1), \ldots, p_{\kappa}(k)$, where $p_{\kappa}(j)$ is the total number of black boxes in the $i_j$-th column of $\kappa$, that is,

$$p_{\kappa}(1) + p_{\kappa}(2) + \ldots + p_{\kappa}(k) = n.$$ 

It is clear that there is a one-to-one correspondence between Kalman codes and pairs of functions satisfying these conditions. Further, define the strictly increasing sequence

$$h_{\kappa}(0) = 0 < h_{\kappa}(1) = p_{\kappa}(1) < \ldots < h_{\kappa}(j) = \sum_{i=1}^{j} p_{\kappa}(i) < \ldots < h_{\kappa}(k) = n.$$ 

With these notations we have the following canonical form for $\Sigma = (A, B, C) \in V_{cc}^{\kappa}$, which is essentially [5, lemma 3.2]:

**Lemma 3** For a completely reachable system $\Sigma = (A, B, C)$ with Kalman code $\kappa = K_\Sigma$, there is a unique $g \in GL_n$ such that $g.(A, B, C) = (A', B', C')$ with

- $B'_{j_{\kappa}(i)} = 1_{h_{\kappa}(i-1)+1}$ for all $1 \leq i \leq k$.
- $A'_{i+1} = 1_{i+1}$ for all $i \notin \{h_{\kappa}(1), h_{\kappa}(2), \ldots, h_{\kappa}(k)\}$.
- All entries in the remaining columns of $A'$ and $B'$ are determined as the quotient of two specific $n \times n$ minors of $c(\Sigma)$.
- $C' = Cg^{-1}$.

This allows us to prove theorem 3:

**Theorem 5** The moduli space $\text{sys}^{cc}_{m,n,p}$ of completely controllable systems has a cell decomposition identical to the natural cell decomposition of a vectorbundle of rank $n(p + 1)$ over the Grassmann manifold $\text{Gras}_n(m + n - 1)$.

**Proof.** Define a map $\phi : V_{cc}^{cc}_{m,n,p} \rightarrow \text{Gras}_n(m + n - 1)$ by sending a completely reachable system $\Sigma = (A, B, C)$ to the point in $\text{Gras}_n(m + n - 1)$ determined by the $n \times (m + n - 1)$ matrix

$$M_\Sigma = \begin{bmatrix} B'_1 & \ldots & B'_m & A'_1 & \ldots & A'_{n-1} \end{bmatrix},$$

where $(A', B', C')$ is the canonical form of $\Sigma$ given by the previous lemma. By construction, $M_\Sigma$ has rank $n$ with invertible $n \times n$ matrix determined by the columns

$$I_{\kappa} = \{j_{\kappa}(1) < \ldots < j_{\kappa}(k) < m + c_1 < \ldots < m + c_{n-k} \} \subset \{1, \ldots, m + n - 1\},$$

where $\{c_1, \ldots, c_{n-k}\} = \{1, \ldots, n\} - \{h_{\kappa}(1), \ldots, h_{\kappa}(k)\}$. As all remaining entries of $(A', B')$ are determined by $c(\Sigma)$ it follows that $\phi(\Sigma)$ depends only on the $GL_n$-orbit of $\Sigma$, whence the map factorizes through

$$\psi : \text{sys}^{cc}_{m,n,p} \rightarrow \text{Gras}_n(m + n - 1),$$

- 6 –
and we claim that $\psi$ is surjective. To begin, all multi-indices $I = \{1 \leq d_1 < d_2 < \ldots < d_n \leq n + m - 1\}$ are of the form $I_\kappa$ for some Kalman code $\kappa$. Define

$$\{d_1, \ldots, d_n\} = \{i_1, \ldots, i_k\} \cup \{m + c_1, \ldots, m + c_{n-k}\}$$

with $i_j \leq m$ and $1 \leq c_j < n$, and let $\{e_1 < \ldots < e_k\} = \{1, \ldots, n\} - \{c_1, \ldots, c_{n-k}\}$, and set $e_0 = 0$. Construct the Kalman code $\kappa$ having $e_j - e_{j-1}$ black boxes in the $i_j$-th column and verify that $I$ is indeed $I_\kappa$.

Gras$_n(m + n - 1)$ is covered by modified Schubert cells $S_I$ (isomorphic to some affine space) consisting of points such that the $I$-minor is invertible, where $I$ is a multi-index $\{d_1, \ldots, d_n\}$, and the dimension of the subspace spanned by the first $k$ columns is $i$ iff $k < d_i + 1$. A point in $S_I$ can be taken such that the $d_i$-th column is equal to

$$1_{I_\kappa(I-1)+1} \quad \text{for } d_i \leq m$$

$$1_{j+1} \quad \text{for } d_i = m + i,$$

where $I = I_\kappa$. This determines a $n \times (n+m-1)$ matrix $[B_1 \ldots B_m A_1 \ldots A_{n-1}]$, and choosing any last column $A_n$ and any $p \times n$ matrix $C$ we obtain a system $\Sigma = (A, B, C)$ which is completely controllable, and which is mapped to the given point under $\psi$. This finishes the proof.

Because the map $(A, B, C) \longrightarrow (A^t, C^t, B^t)$ defines a duality between $V^{co}_{m,n,p}$ and $V^{cc}_{p,n,m}$, we have a similar result for the moduli spaces of completely observable systems.

**Theorem 6** The moduli space of completely observable systems $\text{sys}^{co}_{m,n,p}$ has a cell decomposition identical to that of a vectorbundle of rank $n(p+1)$ over the Grassmann manifold $\text{Gras}_n(p+n-1)$.

### 4. Proof of theorem 1

The counting argument of the previous section gives us also a conjectural description of the infinite dimensional variety admitting a stratification by the moduli spaces $\text{sys}^{cc}_{m,n,p}$. It follows from the explicit rational form of $\# \text{sys}^{cc}_{m,n,p}(\mathbb{F}_q)$ and the $q$-binomial theorem that

$$\sum_{n=0}^{\infty} \# \text{sys}^{cc}_{m,n,p}(\mathbb{F}_q) t^n = \prod_{i=1}^{m} \frac{1}{1 - q^{p+i}t}$$

In the special case when $p = 0$ we recover the cohomology of the infinite Grassmannian $\text{Gras}_m(\infty)$ of $m$-dimensional subspaces of a countably infinite dimensional vectorspace. For $p \geq 1$ we only get a factor of the cohomology of $\text{Gras}_{m+p}(\infty)$, which led to the following result.

**Theorem 7** The disjoint union $\bigsqcup_n \text{sys}^{cc}_{m,n,p}$ is the open subset of the infinite dimensional Grassmann manifold $\text{Gras}_{m+p}(\infty)$ which is the union of all standard affine open sets corresponding to a multi-index set $I = \{1 \leq d_1 < d_2 < \ldots < d_{m+p}\}$ such that

$$\{m + 1, m + 2, \ldots, m + p, m + p + n\} \subset I.$$
Proof. Let $\Sigma = (A, B, C)$ be a completely controllable system in canonical form represented by the point $p_\Sigma \in \text{sys}^{cc}_{m,n,p}$. Consider the $n \times (m + p + n)$ matrix

$$L_\Sigma = \begin{bmatrix} B & C \end{bmatrix} A.$$

The submatrix $M_\Sigma = \begin{bmatrix} B_1 & \ldots & B_m & A_1 & \ldots & A_{n-1} \end{bmatrix}$ has rank $n$, whence so has $L_\Sigma$, and $p_\Sigma$ determines a point in $\text{Gras}_n(n + m + p)$. Under the natural duality

$$\text{Gras}_n(m + p + n) \xrightarrow{\nabla} \text{Gras}_{m+p}(m + p + n),$$

the point $p_\Sigma$ is mapped to the point determined by the $(m + p) \times (m + p + n)$ matrix $N_\Sigma$ whose rows give a basis for the linear relations holding among the columns of $L_\Sigma$. Because $M_\Sigma$ has rank $n$ it follows that the columns of $C$ and the last column $A_n$ of $A$ are linearly dependent of those of $M_\Sigma$. As a consequence the matrix

$$N_\Sigma = \begin{bmatrix} U_1 & \ldots & U_m & V_1 & \ldots & V_p & W_1 & \ldots & W_n \end{bmatrix}$$

has the property that the submatrix $\begin{bmatrix} V_1 & \ldots & V_p & W_n \end{bmatrix}$ has rank $p + 1$. This procedure defines a morphism

$$\text{sys}^{cc}_{m,n,p} \xrightarrow{\gamma_n} \text{Gras}_{m+p}(m + p + n),$$

the image of which is the open union of all standard affine opens determined by a multi-index set $I = \{1 \leq d_1 < d_2 < \ldots < d_{m+p} \leq m + p + n\}$ satisfying

$$\{m+1, m+2, \ldots, m+p, m+p+n\} \subset I.$$

Therefore, the image of the morphism

$$\bigsqcup_n \text{sys}^{cc}_{m,n,p} \xrightarrow{\nabla \gamma_n} \text{Gras}_{m+p}(\infty)$$

is the one of the statement of the theorem. The dimension $n$ of the system corresponding to a point in this open set of $\text{Gras}_{m+p}(\infty)$ is determined by $d_{m+p} = m + p + n$.

By the duality between $\text{sys}^{cc}_{m,n,p}$ and $\text{sys}^{co}_{p,n,m}$ used in the previous section we deduce:

**Theorem 8** The disjoint union $\bigsqcup_n \text{sys}^{co}_{m,n,p}$ is the open subset of $\text{Gras}_{m+p}(\infty)$ which is the union of all standard affine opens corresponding to a multi-index set $I = \{1 \leq d_1 < d_2 < \ldots < d_{m+p} \}$ such that

$$\{1, 2, \ldots, m, m+p+n\} \subset I.$$

This, in turn, proves theorem 1:

**Theorem 9** The disjoint union $\bigsqcup_n \text{sys}^{cc}_{m,n,p}$ of all moduli spaces of canonical systems with fixed input- and output-dimension $m$ and $p$ is the open subset of the infinite Grassmannian $\text{Gras}_{m+p}(\infty)$ of $m + p$-dimensional subspaces of a countably infinite dimensional vectorspace which is the intersection of all possible standard open subsets $X_I$ and $X_J$, where $I$ and $J$ are multi-index sets satisfying the conditions

$$\{m+1, m+2, \ldots, m+p, m+p+n\} \subset I \quad \text{and} \quad \{1, 2, \ldots, m, m+p+n\} \subset J.$$
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