Semiclassical states in homogeneous loop quantum cosmology

Huahai Tan$^1$ and Yongge Ma$^2$

$^1$ Department of Physics, Tsinghua University, Beijing 100084, People’s Republic of China
$^2$ Department of Physics, Beijing Normal University, Beijing 100875, People’s Republic of China

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Abstract
Semiclassical states in homogeneous loop quantum cosmology (LQC) are constructed in two different ways. In the first approach, we firstly construct an exponentiated annihilation operator. Then a kind of semiclassical (coherent) state is obtained by solving the eigenequation of that operator. Moreover, we use these coherent states to analyse the semiclassical limit of the quantum dynamics. It turns out that the Hamiltonian constraint operator employed currently in homogeneous LQC has a correct classical limit with respect to the coherent states. In the second approach, the other kind of semiclassical state is derived from the mathematical construction of coherent states for compact Lie groups due to Hall.

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1. Introduction
As a candidate of the quantum gravity theory, loop quantum gravity is noticeable with its background independence [1–4]. However, the semiclassical analysis of the theory is still a crucial and open issue [5]. As minisuperspace models, loop quantum cosmology (LQC) carries out quantization by mimicking the construction of loop quantum gravity [6]. The symmetry-reduced models provide a mathematically simple arena in which to test the ideas and constructions in the full theory. In the spatially homogeneous and isotropic model, semiclassical states have been proposed to test the quantum-dynamical property of the theory [7]. Moreover, in the light of the quantum resolution of classical big bang singularity [8], semiclassical states are currently used to understand the quantum evolution of the universe across the deep Planck regime [9]. However, whether the above results are still robust in more complicated cases is a crucial question. It is thus desirable to do a similar semiclassical analysis in models with fewer symmetries, such as the homogeneous (non-isotropic) LQC formulated by Bojowald [10]. To achieve this aim, one still needs suitable semiclassical states in the homogeneous sector. On the other hand, the semiclassical analysis of the dynamics
in homogeneous LQC has not been carried out. This is a crucial theoretical criterion for the correctness of the quantum dynamics.

In this paper, we will construct the desired semiclassical states in the diagonal homogeneous model with vanishing intrinsic curvature in two independent ways. Recall that, the coherent states in the homogeneous and isotropic model arise from the construction in the polymer-like representation of a single-particle mimicking the construction in loop quantum gravity [11], since the former is also a quantum mechanical system with one degree of freedom. It is demonstrated by Ashtekar et al that the coherent states in the polymer representation are consistent with those in the traditional Schrödinger representation in the low-energy regime. Thus a first attempt is to generalize the construction to the homogeneous cosmology. We then use the coherent states constructed by this approach to test the classical limit of the Hamiltonian constraint operator proposed by Bojowald in homogeneous LQC. The result is positive. In addition, a mathematical approach is developed by Hall to construct coherent states on compact Lie groups [12, 13]. The construction can be directly applied to the diagonal homogeneous model whose configuration space is a submanifold of $SU(2)^3$, which is just a compact Lie group.

The rest of this paper is organized as follows. For readers’ convenience, a few basic elements in the diagonal homogeneous LQC are collected in section 2. The first kind of coherent state for the homogeneous model is constructed through Ashtekar’s approach in section 3, where a semiclassical analysis of the quantum dynamics is also carried out. In section 4, the other kind of coherent state is obtained by Hall’s mathematical approach. In section 5, the results are summarized and some possible applications of our semiclassical states are discussed.

2. Diagonal homogeneous model

We work with the diagonal homogeneous model of LQC with vanishing spatial curvature [10, 14], i.e. the homogeneous spin connection $\Gamma^i_a = 0$, and an Abelian symmetry group. At the classical level, the invariant connections can be reduced so that they depend only on a diagonal matrix,

$$A^i_a = c(I) A^i L^a, \quad \text{(1)}$$

where $\Lambda \in SO(3)$ is a rotation matrix, and $\omega^I_a$ ($I = 1, 2, 3$) are left-invariant 1-forms under homogeneous symmetry group, which act as the background structure. Similarly, the densitized triads can be reduced as

$$E^a_i = p^{(I)} A^i L^a X^a_I, \quad \text{(2)}$$

where $X^a_I$ are the left-invariant densitized vector fields dual to $\omega^I_a$. So the classical theory of homogeneous cosmology is reduced to a system with finite (three) degrees of freedom. A configuration is represented by $(c_1, c_2, c_3)$. The Gaussian and diffeomorphism constraints are naturally resolved by the symmetric reduction\(^3\). The Poisson bracket between the fundamental configuration and momentum variables in the phase space reads $\{c_I, p^J\} = \gamma \kappa \delta^I_J$, where $\gamma$ is the Barbero–Immirzi parameter and $\kappa$ is the gravitational constant. After the symmetry reduction, the simplified Hamiltonian reads

$$H = -2\gamma^{-1} \gamma^{-2} V \left( \frac{c_1 c_2}{p^1} + \frac{c_2 c_3}{p^2} + \frac{c_3 c_1}{p^3} \right). \quad \text{(3)}$$

\(^3\) Actually, there exists a residual gauge of the signatures of $p^I$, caused by fixing the background. $\text{sgn}(p^I)$ and $|p^I|$ are gauge invariant variables. But we may firstly consider $p^I$ as the gauge invariant variables, then eliminate the residual gauge at last.
There are no connection operators $\hat{c}^I$ in the quantum theory. Mimicking the full theory, quantum configuration variables are holonomies. So, at the quantum level, the configuration space $C^3$ consists of
\[
\{ \exp(c_1\Lambda^1 \tau_i) , \exp(c_2\Lambda^2 \tau_i) , \exp(c_3\Lambda^3 \tau_i) : c_I \in \mathbb{R}, \Lambda \in SO(3) \} \tag{4}
\]
instead of where $\tau_i$ represents the three generators of $SU(2)$.

It is easy to verify that the configuration space is a submanifold of $SU(2)^3$, but not a subgroup. Note that the quantum configuration space $C^3$ is regarded as a product of the three copies of the submanifold $C$ of $SU(2)$. The gauge-invariant measure on the quantum configuration space is the product Haar measure of three copies of $SU(2)$ as
\[
d\mu^H_{i_1,i_2,i_3} = (2\pi)^{-3} \sin^2 \left( \frac{c_1}{2} \right) \sin^2 \left( \frac{c_2}{2} \right) \sin^2 \left( \frac{c_3}{2} \right) dc_1 dc_2 dc_3. \tag{5}
\]
Thus, the Hilbert space is constructed as the space of square-integrable functions,
\[
H_S = L^2(C^3, d\mu^H) = \left[ L^2(C, d\mu_H) \right]^3 = \mathbb{C} \gamma_1 .
\]
Now consider a subspace $L^2(C, d\mu_H)$. The orthonormal bases of this space consist of holonomies up to a certain factor,
\[
\langle c_I | m_I \rangle = e^{\frac{i m_I c_I}{\sqrt{2} \sin \frac{c_I}{2}}}, \tag{6}
\]
where the factor in the denominator is due to integrability under the Haar measure, and $m_I$ take values in the collection $\mathbb{N}$ of non-negative numbers. The orthonormal basis in $H_S$ reads
\[
|m_{1,2,3}\rangle = |m_1\rangle \otimes |m_2\rangle \otimes |m_3\rangle. \tag{7}
\]
The symmetric states in the symmetric Hilbert space $H_S$ can be expanded as a finite linear combination of the basis,
\[
|\psi\rangle = \sum_{m_1,m_2,m_3} \psi_{m_1,m_2,m_3} |m_1,m_2,m_3\rangle. \tag{8}
\]
Since densitized triads are commutable, i.e. $\{p^I, p^J\} = 0$, there exists a momentum representation. The actions of fundamental operators—the configuration operator (holonomy) and the densitized triad operator—on the space of cylindrical functions are defined by multiplication and derivative, respectively. The action of a configuration operator on the basis of $L^2(C, d\mu_H)$ reads
\[
e^{c_1\Lambda_1 |m_I\rangle} = \frac{1}{2}(1 - 2i\Lambda_I)|m_I + 1\rangle + \frac{1}{2}(1 + 2i\Lambda_I)|m_I - 1\rangle, \tag{9}
\]
and
\[
\cos \frac{c_I}{2} |m_I\rangle = \frac{1}{2}((m_I + 1) |m_I + 1\rangle + |m_I - 1\rangle), \tag{10}
\]
\[
\sin \frac{c_I}{2} |m_I\rangle = \frac{1}{2i}((m_I + 1) |m_I + 1\rangle - |m_I - 1\rangle). \tag{11}
\]
Moreover, the densitized triad operator and its action on the basis read
\[
\hat{p}^I = -i\gamma_\ell_\pi^2 \left( \frac{\partial}{\partial c_I} + \frac{1}{2} \cot \frac{c_I}{2} \right), \tag{12}
\]
\[
\hat{p}^I |m_I\rangle = \frac{1}{2} \gamma \epsilon_\ell_\pi m_I |m_I\rangle. \tag{13}
\]
The eigenequation (13) shows that the bases (6) are eigenstates of the densitized triad operator. The volume operator in $\mathcal{H}_S$ is constructed by the densitized triad operators as
\[
\hat{V} = \sqrt{[\hat{p}^I \hat{p}^I \hat{p}^I]], \tag{14}
\]
and its eigenvalue reads
\[ \hat{V}|m_1, m_2, m_3\rangle = \left(\frac{1}{2} \gamma \ell_\text{Pl}^3\right)^{3/2} \sqrt{|m_1 m_2 m_3|}|m_1, m_2, m_3\rangle. \] (15)
The eigenvalues of the volume operator are measured by the cubic of Planck length. The
Hamiltonian constraint operator (with vanishing spin connection) is
\[ \hat{H} = 32i \gamma^{-3} \kappa^{-1} \ell_\text{Pl}^{-2} \left[ \sin \frac{c_1}{2} \cos \frac{c_1}{2} \sin \frac{c_2}{2} \cos \frac{c_2}{2} \left( \sin \frac{c_3}{2} \hat{V} \cos \frac{c_3}{2} - \cos \frac{c_3}{2} \hat{V} \sin \frac{c_3}{2} \right) \right. \]
\[ + \sin \frac{c_3}{2} \cos \frac{c_3}{2} \sin \frac{c_1}{2} \cos \frac{c_1}{2} \left( \sin \frac{c_2}{2} \hat{V} \cos \frac{c_2}{2} - \cos \frac{c_2}{2} \hat{V} \sin \frac{c_2}{2} \right) \]
\[ + \sin \frac{c_2}{2} \cos \frac{c_2}{2} \sin \frac{c_3}{2} \cos \frac{c_3}{2} \left( \sin \frac{c_1}{2} \hat{V} \cos \frac{c_1}{2} - \cos \frac{c_1}{2} \hat{V} \sin \frac{c_1}{2} \right) \]. (16)

Its action on the basis reads
\[ \hat{H}|n_1, n_2, n_3\rangle = \gamma^{-3} \kappa^{-1} \ell_\text{Pl}^{-2} [(V_{n_1, n_2, n_3 + 1} - V_{n_1, n_2, n_3 - 1}) (|n_1 + 2, n_2 + 2, n_3\rangle \]
\[ - |n_1 - 2, n_2 + 2, n_3\rangle - |n_1 + 2, n_2 - 2, n_3\rangle + |n_1 - 2, n_2 - 2, n_3\rangle) \]
\[ + (V_{n_1, n_2 + 1, n_3} - V_{n_1, n_2 - 1, n_3}) (|n_1 + 2, n_2, n_3 + 2\rangle \]
\[ - |n_1 - 2, n_2, n_3 + 2\rangle - |n_1 + 2, n_2, n_3 - 2\rangle + |n_1 - 2, n_2, n_3 - 2\rangle) \]
\[ + (V_{n_1 + 1, n_2, n_3} - V_{n_1 - 1, n_2, n_3}) (|n_1, n_2 + 2, n_3 + 2\rangle \]
\[ - |n_1, n_2 - 2, n_3 + 2\rangle - |n_1, n_2 + 2, n_2 - 2\rangle + |n_1, n_2 - 2, n_3 - 2\rangle)]. \] (17)

3. Coherent states and semiclassical analysis

Now we begin to construct coherent states in the kinematical Hilbert space of the homogeneous
LQC by the annihilation operator approach following the idea of Ashtekar et al. We then check
whether the semiclassical limit of the Hamiltonian constraint operator with respect to these
states is consistent with the classical Hamiltonian constraint (3).

The coherent states for quantum-mechanical systems are defined as those states satisfying
following conditions.

1. They belong to the Hilbert space, but every coherent state can be labelled with a point in
classical phase space. In this sense, classical mechanics can be considered as a quantum
system that only includes all the coherent states.
2. Ehrenfest theorem, i.e. the expectation values of the elementary (configuration and
momentum) operators (and of their commutators divided by \(i\hat{\hbar}\), respectively) under a
coherent state labelled by a classical phase agree with, up to leading order in \(\hbar\), the values
of corresponding elementary functions (and of their Poisson brackets, respectively) at that
phase.
3. The uncertainty relation of the elementary observables under coherent states is minimum,
e.g., \(\Delta x \Delta p = \hbar/2\) in one-dimensional single-particle quantum mechanics.

For a quantum theory with some classical theory as its classical limit, one would expect that
there exist enough semiclassical (coherent) states which can represent all the classical solutions.
As we have seen, the framework of loop quantum cosmology (or loop quantum gravity) looks
disparate from that of conventional quantum mechanics (or quantum field theory). Hence it
becomes a crucial task to check its classical limit by constructing semiclassical states. In
general case, the eigenstates of annihilation operator \(\hat{a}\) in quantum mechanics satisfy the
above conditions and are usually also used as a definition of coherent states by physicists.
Following the idea of Ashtekar et al in the construction of coherent states for a single-particle
system [11], we will construct coherent states in the light of an exponentiated annihilation operator in our diagonal homogeneous model.

Recall that the kernel of the Hamiltonian constraint operator does not belong to the kinematic Hilbert space $\mathcal{H}_S$, because they are not normalizable states in general. So the physical states should be in their natural home $Cyl^*_S$, the algebraic dual of the symmetric cylindrical function space $Cyl_S$. One has a natural inclusion, the Gel’fand triplet,

$$Cyl_S \subset \mathcal{H}_S \subset Cyl^*_S.$$  

From the viewpoint of the polymer representation of a single particle, the Schrödinger representation depicts the low-energy physics of a quantum mechanics system, while the intersection of the polymer Hilbert space $\mathcal{H}_S$ and the Schrödinger Hilbert space contains only the zero element. So, to study the classical limit of our LQC, one should consider $Cyl^*_S$ which contains both polymer states and Schrödinger states. It is necessary to ask the question whether $Cyl^*_S$ is big enough to contain desired coherent states. To answer this question, one should firstly try to construct the creation operator $\hat{a}^\dagger$ on $Cyl^*_S$ and treat its eigenvectors as coherent states. However, the flaw of $Cyl^*_S$ is that there is no natural inner product on it. Thus we are not able to calculate the expectation values of quantum operators. We note that an arbitrary element in $Cyl^*_S$ can formally be written as

$$\langle \Psi \rangle = \sum_{m_1, m_2, m_3} \Psi_{m_1, m_2, m_3} (m_1, m_2, m_3).$$

Thus one may extract a combination of certain finite terms in the expression of $\langle \Psi \rangle$ as an element projected from $Cyl^*_S$ into $Cyl_S$. The projected states are called the ‘shadows’ of $\langle \Psi \rangle$ in $Cyl_S$ in [11]. The action of a coherent state on a cylindrical function, which is a linear combination of a finite collection of $|m_1, m_2, m_3\rangle$, in $Cyl_S$ is the same as the action of its shadow $\langle \Psi_{\text{shad}} \rangle$ on the same collection of basis. Each shadow state only captures a part of the complete properties of the full state.

Now we consider the Gel’fand triplet for one copy of the symmetric Hilbert space, $Cyl_{S,1} \subset \mathcal{H}_{S,1} = L^2(C, dq_H) \subset Cyl^*_{S,1}$. Formally, we may consider a dimensionless ‘annihilation operator’

$$\hat{a}^\dagger = \frac{\sqrt{\gamma}}{\gamma \ell^2_{Pl} d} \left( \hat{p}^\dagger - \frac{\gamma \ell^2_{Pl} d^2}{4} \hat{c}_1 \right),$$

where $p^\dagger$ has the dimension of $[L]^2$, $c_1$ is dimensionless, and $d$ is a dimensionless quantity usually representing the tolerance scaled by $\ell^2_{Pl}$. Note that in Schrödinger quantum mechanics $\hat{a}^\dagger$ has the same eigenstates as those of the conventional annihilation operator in the configuration representation. Coherent states are supposed to be the eigenvectors of the creation operator $\hat{a}^\dagger$ in $Cyl^*_S$. However, the problem is that there is no connection operator $\hat{c}_1$ in homogeneous LQC. The key idea of Ashtekar et al is to use the exponentiated creation operator to solve the problem. For all real number $\alpha$, using the Baker–Hausdorff–Campbell identity, we have

$$e^{\sqrt{2} \alpha (\hat{a}^\dagger)^\dagger} = e^{\frac{-\sqrt{2}}{\gamma \ell^2_{Pl} d} \frac{\gamma}{\gamma \ell^2_{Pl} d} V(\alpha d) e^{-\frac{d^2}{4}}}.$$  \hspace{1cm} (18)

where $V(\mu) = \exp \left( \frac{d}{2\ell^2_{Pl}} \right)$ belongs to the Weyl algebra on $\mathcal{H}_{S,1}$, thus also on $Cyl^*_{S,1}$. Since $V(\mu)$ is a well-defined operator in $\mathcal{H}_{S,1}$, it turns out that equation (18) is well defined on $Cyl^*_{S,1}$. The coherent states in $Cyl^*_{S,1}$ are obtained by solving the eigenequation

$$\left( \Psi_{\alpha} \right) \left[ e^{\frac{-\sqrt{2}}{\gamma \ell^2_{Pl} d} \frac{\gamma}{\gamma \ell^2_{Pl} d} V(\alpha d) e^{-\frac{d^2}{4}}} \right] = e^{\sqrt{2} \alpha \lambda_0} \langle \Psi_{\alpha} \rangle.$$  \hspace{1cm} (19)
The distribution state $|\Psi_{a_I}| \in \text{Cyl}_S^*$ can be formally expanded as an infinite summation $|\Psi_{a_I}| = \sum_{m_I} \hat{\Psi}_{a_I}(m_I)|m_I\rangle$. Its shadow in $\text{Cyl}_S$ is $|\Psi_{a_I}^{\text{shad}}| = \sum_{m_I} \hat{\Psi}_{a_I}(m_I)|m_I\rangle$, where $j$ is finite. The coefficients $\Psi_{a_I}(m_I)$ in the eigen equation of the exponentiated creation operator are constrained by

$$\Psi_{a_I}(m_I + \alpha d) = \exp \left[ \sqrt{2} \alpha \hat{a}_d^I - \frac{am_I}{d} - \frac{\alpha^2}{2} \right] \Psi_{a_I}(m_I),$$

for all $\alpha$. It is easy to verify that the shadow state of $|\Psi_{a_I}|$ which peaks at $(P^I, C_I)$ in the classical phase space is, up to a normalization constant, written as

$$|\Psi_{a_I}^{\text{shad}}| = \sum_{m_I} \left[ \exp \left( -\frac{1}{2d^2} (m_I - M_I)^2 \right) \exp \left( -i \frac{C_I}{2} (m_I - M_I) \right) \right] |m_I\rangle,$$

where $P^I = \frac{1}{2} p^I \ell_\text{Pl}^2$ and $\frac{1}{2} \gamma^2 \ell_\text{Pl}^2 d$ is the ‘tolerance’ for the quantum fluctuation of $P^I$. For the semiclassical case, one has $d \gg 1$. To simulate the classical behaviour, we should specify $M_I \gg d$, which means a large volume compared to the Planck volume, and $C_I \ll 1$, which means small external curvature, thus late time. Finally, we arrive at a Gaussian-type shadow state of the desired coherent state $|\Psi_{a_0}|$ in $\text{Cyl}_S^*$ by directly composing the three copies in $\text{Cyl}_S$,

$$|\Psi_{a_0}^{\text{shad}}| = \sum_{m_1, m_2, m_3} \left[ \exp \left( -\frac{1}{2d^2} \sum_{I=1}^{3} (m_I - M_I)^2 \right) \exp \left( -i \frac{C_I}{2} \sum_{I=1}^{3} (m_I - M_I) \right) \right] |m_1, m_2, m_3\rangle.$$

(22)

Note that, \textit{a priori}, there is no guarantee that any meaningful calculation in the framework of homogeneous LQC can isolate semiclassical states corresponding to the standard coherent states in Schrödinger quantum mechanics. So, the existence of the Gaussian-type coherent states in $\text{Cyl}_S^*$ for homogeneous LQC is a nontrivial result. Since the coefficients in expression (22) coincide with the coherent states in Schrödinger quantum mechanics, in the semiclassical regime the relevant physics extracted from the conventional quantum cosmology could also be obtained from homogeneous LQC. One may take the viewpoint that in the homogenous models LQC is more fundamental as it incorporates the underlying discrete nature of quantum geometry, while the conventional quantum cosmology is a kind of ‘coarse-grained’ description of the fundamental theory. On the other hand, it is expected that the singularity problem, which still exists in conventional quantum cosmology, can also be resolved in homogeneous LQC. As in the single-particle case, our toy model also provides enlightening implications on full loop quantum gravity how low-energy physics can arise from a suitable semiclassical treatment.

Now we come to the semiclassical analysis of the homogeneous LQC. This is a crucial step to check the correctness of the quantum setting. So our next task is to apply the shadow states (22) to compute the ‘expectation value’ of Hamiltonian constraint operator (16), which is constructed to understand the semiclassical information as

$$\frac{\langle \Psi_{a_0} | \hat{H} | \Psi_{a_0}^{\text{shad}} \rangle}{\langle \Psi_{a_0}^{\text{shad}} | \Psi_{a_0}^{\text{shad}} \rangle},$$

(23)

for any shadow state. Operated by the Hamiltonian constraint operator, the three copies in (22) will intercross each other. So the computation process is not so simple as in the homogeneous isotropic case [7], whose configuration is $\mathbb{R}^4$. We begin with the norm of the shadow state:

$$\langle \Psi_{a_0}^{\text{shad}} | \Psi_{a_0}^{\text{shad}} \rangle = (\sqrt{\pi d})^3 \left[ 1 + \mathcal{O}(e^{-\pi^2 d^2}) + \mathcal{O}((M_I/d)^{-2}) \right].$$

(24)
Note that, in order to get equation (24) we used the Poisson resummation formula
\[ \sum_n e^{-\epsilon(n-N)^2} f(n) = \sum_n \int e^{-\epsilon(y-N)^2} f(y) e^{2\pi isn} \, dy, \]  
(25)
and the method of steepest descent (see the appendix of [7]). They lead to a very useful approximation formula:
\[ \sum_n e^{-\epsilon(n-N)^2} f(n) = \sqrt{\pi} e^{-1} f(N) (1 + O(e^{-\pi^2/\epsilon^2}) + O((N\epsilon)^{-2})). \]  
(26)

Set \( \epsilon = 1/d \), the calculation of \( \langle \Psi_{\text{shad}} | \hat{H} | \Psi_{\text{shad}} \rangle \) can be divided into three parts. The difference among them is only an index permutation, since
\[
\langle \Psi_{\text{shad}} | \hat{H} | \Psi_{\text{shad}} \rangle = \gamma^{-3} \kappa^{-1} \frac{1}{d^2} \sum_{m_1, m_2, m_3, n_1, n_2, n_3} \exp\left( -\frac{\epsilon^2}{2} \sum_{I=1}^{3} ((n_I - M_I)^2 + (m_I - M_I)^2) \right) \exp\left( i \sum_{I=1}^{3} C_I (n_I - m_I) \right) 
\times \left[ \left( V_{n_1, n_2, n_3+1} - V_{n_1, n_2, n_3-1} \right) \left( \delta_{m_1, n_1+2} \delta_{m_2, n_2+2} - \delta_{m_1, n_1+2} \delta_{m_2, n_2+2} - \delta_{m_1, n_1+2} \delta_{m_2, n_2+2} - \delta_{m_1, n_1+2} \delta_{m_2, n_2+2} \right) \delta_{m_3, n_3} 
+ \left( V_{n_1, n_2+1, n_3} - V_{n_1, n_2-1, n_3} \right) \left( \delta_{m_1, n_1+2} \delta_{m_2, n_2+2} - \delta_{m_1, n_1+2} \delta_{m_2, n_2+2} - \delta_{m_1, n_1+2} \delta_{m_2, n_2+2} - \delta_{m_1, n_1+2} \delta_{m_2, n_2+2} \right) \delta_{m_3, n_3} 
- \delta_{m_1, n_1-2} \delta_{m_2, n_2+2} + \delta_{m_1, n_1-2} \delta_{m_2, n_2+2} + \delta_{m_1, n_1-2} \delta_{m_2, n_2+2} + \delta_{m_1, n_1-2} \delta_{m_2, n_2+2} \right] 
\equiv \gamma^{-3} \kappa^{-1} \frac{1}{d^2} \sum_{m_1, m_2, m_3, n_1, n_2, n_3} (A + B + C) 
\times \left[ \exp\left( -\frac{\epsilon^2}{2} \sum_{I=1}^{3} ((n_I - M_I)^2 + (m_I - M_I)^2) \right) \exp\left( i \sum_{I=1}^{3} C_I (n_I - m_I) \right) \right].
\]

We show the calculation of the first part as follows. The other two are similar. It is easy to see that

Part 1 = \( \gamma^{-3} \kappa^{-1} \frac{1}{d^2} \sum_{m_1, m_2, m_3, n_1, n_2, n_3} A \)
\times \left[ \exp\left( -\frac{\epsilon^2}{2} \sum_{I=1}^{3} ((n_I - M_I)^2 + (m_I - M_I)^2) \right) \exp\left( i \sum_{I=1}^{3} C_I (n_I - m_I) \right) \right] \]
\equiv \gamma^{-3} \kappa^{-1} \frac{1}{d^2} \sum_{m_1, m_2, m_3, n_1, n_2, n_3} e^{-\epsilon^2 \sum_{I=1}^{3} (n_I - M_I)^2}
\times \left[ e^{-i(C_1 + C_2)} \left( V_{n_1-1, n_2-1, n_3+1} - V_{n_1-1, n_2-1, n_3-1} \right) - e^{i(C_1 + C_2)} \left( V_{n_1+1, n_2+1, n_3+1} - V_{n_1+1, n_2+1, n_3-1} \right) 
- e^{-i(C_1 - C_2)} \left( V_{n_1-1, n_2+1, n_3+1} - V_{n_1-1, n_2+1, n_3-1} \right) 
+ e^{i(C_1 - C_2)} \left( V_{n_1+1, n_2-1, n_3+1} - V_{n_1+1, n_2-1, n_3-1} \right) \right].
Using equation (26), we get

\[
\text{Part I} = g^{-3} \kappa^{-1} \ell_p^2 (\sqrt{\pi} d)^2 V_{M_1, M_2, M_3} \left( \sqrt{1 + \frac{1}{M_3}} - \sqrt{1 - \frac{1}{M_3}} \right) \times \left[ \mathrm{e}^{-i(C_1 C_2)} \sqrt{1 - \frac{1}{M_1}} \left( \frac{1}{M_2} - \mathrm{e}^{i(C_1 - C_2)} \right) \sqrt{1 + \frac{1}{M_2}} \left( \frac{1}{M_1} - \mathrm{e}^{i(C_1 + C_2)} \right) \sqrt{1 + \frac{1}{M_1}} \right]
\]

Next, taking account of the following approximations:

\[
\sqrt{1 + \frac{1}{M_I}} = 1 + \frac{1}{2M_I} + \mathcal{O}(M_I^{-2}),
\]

and

\[
\mathrm{e}^{i(C_1 + C_2)} = 1 + i(C_1 + C_2) - \frac{1}{2}(C_1 + C_2)^2 + \mathcal{O}((C_1 + C_2)^3),
\]

we get

\[
\text{Part I} = -2(\sqrt{\pi} d)^2 \kappa^{-1} g^{-2} V_{M_1, M_2, M_3} \frac{C_1 C_2}{p^3} \left( 1 + \mathcal{O}((M_I/d)^{-2}) + \mathcal{O}(M_I^{-2}) + \mathcal{O}(e^{-\pi^2d^2}) + \mathcal{O}((C_I)^3) \right).
\]

Finally, we arrive at the semiclassical limit of homogeneous LQC by composing the three copies together,

\[
\langle \Psi_{\text{shad}} | \hat{A} | \Psi_{\text{shad}} \rangle = -2 \kappa^{-1} g^{-2} V_{M_1, M_2, M_3} \left( \frac{C_1 C_2}{p^3} + \frac{C_2 C_3}{p^1} + \frac{C_3 C_1}{p^2} \right) \left( 1 + \mathcal{O}((M_I/d)^{-2}) + \mathcal{O}(M_I^{-2}) + \mathcal{O}(e^{-\pi^2d^2}) + \mathcal{O}(C_I) \right). (28)
\]

Thus, under the conditions of large volume and late time of the universe one has \( M_I \gg d \) and \( C_I \ll 1 \), which further imply for our diagonal homogeneous model \( M_I C_I \sim M_I^{1/4} \gg 1 \).

For simplicity, let us illustrate the validity of the analogue of the last formula in homogeneous and isotropic cosmology. From equation (4.1.37) in [15] it is easy to find that the multiplication of the corresponding momentum and configuration variables can be estimated as \( p_c \sim a^2 \dot{a} \sim a^{3/2} \sim p_b^3 \), where \( a \) denotes the scale factor. It turns out that a similar estimation is still valid in the diagonal homogeneous cosmology. Therefore the ‘expectation value’ (28) agrees with the classical constraint (3) up to some small corrections. Thus we have proved that the Hamiltonian constraint operator \( \hat{A} \) in homogeneous LQC has a correct classical limit with respect to the coherent state \( \langle \Psi \rangle \).

### 4. On Hall’s coherent states

It is shown by Hall in [12] that, on a compact Lie group \( G \), the coherent transformation is an isomorphism from the space of square-integrable functions \( L^2(G, d\mu_H) \) to the holomorphic square-integrable function space \( \mathcal{H}(G^C) \cap L^2(G^C, d\rho) \), that is

\[
\phi_{\rho} : L^2(G, d\mu_H) \longrightarrow \mathcal{H}(G^C) \cap L^2(G^C, d\rho), \quad (29)
\]
where $G^C$ is the complexification of $G$, $d\mu_H$ and $d\rho$ are, respectively, the Haar measure on $G$ and the heat kernel measure on $G^C$. The complexification of $G$ can be carried out by a suitable complexifier $C$ [16],

$$g = \sum_{n=0}^{\infty} \frac{i^n}{n!} [g, C]_{(n)} ,$$

which is a suitable function on the cotangent bundle of $G$, where the Poisson bracket is naturally defined. Then $\phi_g(g)$ is a coherent state in $L^2(G, d\mu_H)$, whose subscript $g$ presents its peak in the classical phase space. In the appendix of [12], the concrete form of a coherent state in $L^2(G, d\mu_H)$ is constructed as

$$\phi_g(g) = \frac{\rho_t(g^{-1}g)}{\sqrt{\rho_t(g)}} ,$$

where the heat kernel of the compact Lie group $G$ reads

$$\rho_t(g) = \sum_{\pi \in \hat{G}} \dim V_\pi e^{-\lambda_\pi t/2} \chi_\pi(g).$$

Note that, there is a natural complex analytic continuation $\rho_t(g)$ for $g \in G^C$. $\hat{G}$ is the set of irreducible representation equivalence classes, $\dim V_j$ is the dimension of the representation space $V$, $\lambda_\pi$ is the factor of the Casimir operator, i.e. $\pi(\Delta) = -\lambda_\pi I$, and $\chi_\pi(g)$ is the character of $g$ in representation $\pi$. The convergence of (33) is also proved in [12].

For homogeneous LQC, although the configuration space $C^3$ is not a group, we can still use the compact group method to construct coherent states, because all the treatment is carried out in the compact group $SU(2)^3$. Now, the configuration is $g = e^{i\gamma \kappa /\Lambda_1} \otimes e^{i\gamma \kappa /\Lambda_2} \otimes e^{i\gamma \kappa /\Lambda_3}$, and we choose a complexifier as $C = \frac{i}{2} (p^1)^2 \otimes (p^2)^2 \otimes (p^3)^2$. Taking account of the direct product structure of $C^3$, in the following we present the calculation only for one copy of them, since the other two are of the same type. Then the complexification is written as

$$g = \sum_{n=0}^{\infty} \frac{i^n}{n!} [g, C]_{(n)} ,$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} (\gamma \kappa \Lambda_1 p^1)^n e^{i\gamma \kappa \Lambda_1} ,$$

$$= e^{i(\gamma \kappa p^1 /\Lambda_1)}.$$

Explicitly, it is a kind of complexification of the quantum configuration space $C$ and corresponds to a point $(c_j, p^j)$ in the classical phase space. For the $SU(2)$ group, the irreducible representations are labelled by their dimensions $\dim V_j = (2j + 1)$. The character of a representation reads

$$\chi_j(g) = \frac{\sin \left(\frac{(2j+1)c_1}{2}\right)}{\sin \frac{c_1}{2}} = \frac{\exp(i(2j + 1)c_1/2) - \exp(-i(2j + 1)c_1/2)}{\exp(ic_1/2) - \exp(-ic_1/2)}$$

$$= \frac{1}{\sqrt{2}} \left((2j + 1) - | - 2j - 1|\right) .$$

Hence the heat kernel in (33) becomes

$$\rho_t(g) = \sum_{j} (2j + 1) \exp(-j(j+1)t/2) \frac{\sin \left(\frac{(2j+1)c_1}{2}\right)}{\sin \frac{c_1}{2}} .$$
Note that equation (37) is explicitly an eigenstate of the volume operator with eigenvalue 0, thus \(|p| = 0\). Substituting (34) and (36) into (32), we get the coherent states:

\[
\phi_g(g) = \sum_{n=1}^{\infty} \frac{n}{\sqrt{2}} e^{-(n^2 - 1)t/8} \exp\left( -\gamma \kappa P_I / 2 \right) \exp\left( -i (C_I - \gamma I)/2 \right) \sin(mc_I / 2) \sin(c_I / 2), \tag{38}
\]

where \((C_I, P_I)\) is a point in classical phase space, which is peaked by \(\phi_g(g)\). However, the structure of coherent states (38) is too complicated. Further work is needed for its applications.

5. Discussion

In previous sections, two kinds of coherent states for homogeneous LQC have been obtained. The coherent states constructed by Ashtekar’s approach lie in the dual space \(\text{Cyl}^*_S\), which does not carry a natural inner product. However, one may extract the information of a coherent state from its shadows in \(\text{Cyl}_S\). We further show that the Hamiltonian constraint operator in homogeneous LQC has a correct classical limit with respect to these shadow states under the large volume and late time limit. Our result confirms that the current construction of the Hamiltonian operator (16) is feasible at least in theoretical sense. On the other hand, the coherent states constructed by Hall’s compact Lie group method are rather complicated. For a concrete application of the last kind of coherent state one still needs to develop some manageable calculation method for it.

The possible applications of the coherent states constructed in this paper are fascinating. For instance, one may derive an effective Hamiltonian expression by specifying certain coherent state peaked at a particular classical solution for homogeneous cosmology. Recall that in isotropic LQC, a new repulsive force associated with the non-perturbative quantum geometry comes into play in Planck regime, which prevents the formation of the big bang singularity [9]. It is desirable to check in the anisotropic case if this qualitative picture would still emerge and if the quantum evolution of the universe across the deep Planck regime could be simulated numerically. We leave these open issues for further investigation [17]. As our model has less symmetries than the isotropic one, we may also try to generalize other remarkable achievements in isotropic LQC, such as the account of inflation [18, 19]. Besides its own meaning in quantum cosmology, the semiclassical analysis in the homogeneous model will certainly provide valuable hints for the investigation of full loop quantum gravity.

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