On Some Explicit Semi-stable Degenerations of Toric Varieties

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Abstract
We study semi-stable degenerations of toric varieties determined by certain partitions of their moment polytopes. Analyzing their defining equations we prove a property of uniqueness.

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1 Background
1.1 Polytopes and semi-stable partitions
In his paper [5], Hu provides a toric construction for semi-stable degenerations of toric varieties. We study the uniqueness of this construction for a toric variety $X$ in the particular case of a semi-stable partition of its moment polytope in two subpolytopes. Adapting a theorem by Strumfels on toric ideals (Lemma 4.1 in [9] and Section 2 in [8]) to particular open polytopes, we investigate the equations of the degeneration of $X$ as embedded variety.

Let $M \simeq \mathbb{Z}^n$ be a lattice and $N$ its dual. We consider polytopes $\Delta \subset M$ which describe smooth algebraic varieties $X_\Delta$; $\Delta$ determines the normal fan $\Sigma_{X_\Delta} \subset N$. Recall that convex polytopes $\Delta$ determine a toric manifold $X_\Delta$ together with an ample line bundle $L_\Delta$: $(X_\Delta, L_\Delta)$. If the polytope is non singular of dimension $n$, then $L_\Delta$ is very ample, we then have an embedding $X_\Delta \hookrightarrow \mathbb{P}^\ell$, for some $\ell$ [7].

Now fix a (compact) polytope $\Delta$ and suppose $\Delta \cap M = \{m_0, \ldots, m_\ell\}$. Take $x_0, \ldots, x_\ell$ as homogeneous coordinates in $\mathbb{P}^\ell$. We can define $X = X_\Delta$ as the closure in $\mathbb{P}^\ell$ of the image of the map

$$\varphi : (\mathbb{C}^*)^n \rightarrow \mathbb{P}^\ell$$

$$t \mapsto [t^{m_0}, \ldots, t^{m_\ell}], \quad (1)$$
where \( t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n \) and given \( u = (u_1, \ldots, u_n) \in \mathbb{Z}^n \) we use the notation \( t^u = t_1^{u_1} \cdots t_n^{u_n} \). Taking homogeneous coordinates in \( X_\Delta \), this map extends to a map \( X_\Delta \to \mathbb{P}^\ell \), which is an embedding under the assumption \( X_\Delta \) smooth (see [2]).

We assume that there exists a suitable finite partition \( \Gamma \) of \( \Delta \) in subpolytopes \( \{ \Delta_j\}_{j=1}^{k} \). We will assume that the toric varieties \( X_{\Delta_j} \) corresponding to each \( \Delta_j \) are also smooth. We call an open \( l \)-face \( \sigma \) of \( \Delta \) an \( l \)-face of \( \Gamma \) and we declare that the 0-faces of \( \Delta \) are not 0-faces of \( \Gamma \). Following [1, 5] we ask \( \Gamma \) to be semi-stable:

**Definition 1.1** \( \Gamma \) is semi-stable if for any \( l \)-face \( \sigma \) of \( \Gamma \), if \( \theta \) is a \( k \)-face of \( \Delta \) such that \( \sigma \subset \theta \), then there are exactly \( k-l+1 \) \( \Delta_j \)'s such that \( \theta \) is a face of each of them.

In fact:

**Theorem 1.2** [1, 5] If \( \{ \Delta_j\}_{j=1}^{k} \) is a semi-stable partition of \( \Delta \), then there exists a semi-stable degeneration of \( X_{\Delta} \), \( f : \tilde{X} \to \mathbb{C} \) with central fiber \( f^{-1}(0) = \bigcup_{j=1}^{k} X_{\Delta_j} \); the central fiber is completely described by the polytope partition \( \{ \Delta_j\}_{j=1}^{k} \).

\( \tilde{X} \) is constructed by a lift of \( \Delta \) (see Definition [1.3]). From Theorem 2.8 in [5], \( \tilde{X} \) is unique: we study the uniqueness of \( \tilde{X} \) for semi-stable partitions of \( \Delta \) in two subpolytopes \( \Delta_1, \Delta_2 \), and we describe its defining equations. In particular, in Section 2 of [5], Hu shows that the ordering (arbitrarily fixed) \( \{ \Delta_1, \ldots, \Delta_k \} \) of the polytopes in \( \Gamma \) determines a piecewise affine function on the partition \( F : \Delta \to \mathbb{R} \), which takes rational values on the points in the lattice \( M \). \( F \) can be chosen to be concave and it is called lifting function.

**Definition 1.3**

\( \tilde{\Delta}_F = \{(m, \tilde{m}) \in M \times \mathbb{Z} \text{ such that } m \in \Delta \text{ and } \tilde{m} \geq F(m)\} \)

is an open lifting (here simply lift) of \( \Delta \) with respect to \( \Gamma \).

There are many possible lifts of \( \Delta \) with respect to \( \Gamma \); if \( \Gamma \) consists of two subpolytopes, then two lifts exist. By construction there exists a morphism \( f : \tilde{X}_F := X_{\tilde{\Delta}_F} \to \mathbb{C} \) which realizes a semi-stable degeneration of \( X \). As
before we have embeddings $X \hookrightarrow \mathbb{P}^\ell$ and $\tilde{X}_F \hookrightarrow \mathbb{P}^\ell \times \mathbb{C}$. In particular we can define $\tilde{X}_F$ as the closure in $\mathbb{P}^\ell \times \mathbb{C}$ of the image of the map:

$$\psi_F : ((\mathbb{C}^*)^n \times \mathbb{C}) \to \mathbb{P}^\ell \times \mathbb{C}$$

$$(t, \lambda) \mapsto ([\lambda^{F(m_0)} t^{m_0}, \lambda^{F(m_1)} t^{m_1}, \ldots, \lambda^{F(m_\ell)} t^{m_\ell}], \lambda).$$

Theorem 2.8 in [5] claims that the image of $\psi := \psi_F$, and hence $\tilde{X}_F$, is independent of the lifting function $F$.

We explicitly study this statement for semi-stable partitions of $\Delta$ in two subpolytopes. If $\Gamma$ consists of two subpolytopes $\Delta_1, \Delta_2$, then we can construct two possible lifting functions $F, G$ and then $\Delta$ has two lifts, say $\Delta_F$ and $\Delta_G$. In particular let $y_1, \ldots, y_n$ be coordinates in $\mathbb{R}^n \supset \Delta$ and let

$$a_1y_1 + \ldots + a_ny_n + a_{n+1} = 0$$

be an equation of the cut $\Delta_1 \cap \Delta_2$ in the lattice, where we take $a_1, \ldots, a_{n+1} \in \mathbb{Z}$ such that for all $m_j = (m_{1j}, \ldots, m_{nj}) \in \Delta_2 \cap M$ we have

$$a_1m_{1j} + \ldots + a_nm_{nj} + a_{n+1} \geq 0.$$

Following the construction in [5], the functions $F, G$ we obtain look like:

$$F(m_j) = \begin{cases} 0 & \text{if } m_j \in \Delta_1 \\ L_F(m_j) := a_1m_{1j} + \ldots + a_nm_{nj} + a_{n+1} & \text{if } m_j \in \Delta_2, \end{cases}$$

$$G(m_j) = \begin{cases} L_G(m_j) := -a_1m_{1j} - \ldots - a_nm_{nj} - a_{n+1} & \text{if } m_j \in \Delta_1 \\ 0 & \text{if } m_j \in \Delta_2. \end{cases}$$

We prove that the two non-compact toric varieties defined by the open polytopes $\tilde{\Delta}_F$ and $\tilde{\Delta}_G$ have the same toric ideals. To do this we adapt a Strumfels’s theorem on toric ideals (Lemma 4.1 in [9] and Section 2 in [8]) to this non-compact context.

1.2 Toric ideals

In [8] Sottile describes the ideal $I$ of the compact toric variety $X$ (toric ideal) defined as the closure of the image of a map [1], following Strumfels’s book [9].
Take \( x_0, \ldots, x_\ell \) as homogeneous coordinates in \( \mathbb{P}^\ell \). With the notation of the previous section, suppose \( m_j = (m_{1j}, \ldots, m_{nj}) \), \( j = 0, \ldots, \ell \) and consider the \((n+1) \times (\ell+1)\) matrix
\[
\mathcal{A}^+ = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
m_{10} & m_{11} & \ldots & m_{1\ell} \\
\vdots & \vdots & \ddots & \vdots \\
m_{n0} & m_{n1} & \ldots & m_{n\ell}
\end{pmatrix}.
\]

Observe that if \( u \in \mathbb{Z}^{\ell+1} \), then we may write \( u \) uniquely as \( u = u^+ - u^- \), where \( u^+, u^- \in \mathbb{N}^{\ell+1} \), but \( u^+ \) and \( u^- \) have no non-zero components in common. For instance, if \( u = (1, -2, 1, 0) \), then \( u^+ = (1, 0, 1, 0) \) and \( u^- = (0, 2, 0, 0) \) (Sottile’s notation).

We therefore have:

**Theorem 1.4** ([8], Corollary 2.3)

\[ I = \langle x u^+ - x u^- \mid u \in \ker(\mathcal{A}^+) \text{ and } u \in \mathbb{Z}^{\ell+1} \rangle. \]

There are no simple formulas for a finite set of generators of a general toric ideal. An effective method for computing a finite set of equations defining \( X_\Delta \) in \( \mathbb{P}^\ell \) is applying elimination theory algorithms to its parametrization in homogeneous coordinates. These algorithms are implemented in the well known computer algebra system Maplesoft [4].

## 2 First examples

To illustrate the previous section, we describe the semi-stable degenerations of a curve and a surface determined by a subdivision of their moment polytopes in two subpolytopes.

### 2.1 The twisted cubic

The twisted cubic \( X \subset \mathbb{P}^3 \) can be defined as \( \mathbb{P}^1 \) embedded in \( \mathbb{P}^3 \) by cubics, that is, as the toric curve \( (X_\Delta, \mathcal{L}_\Delta) = (\mathbb{P}^1, \mathcal{O}(3)) \), where \( \Delta \) is the polytope below.

Here \( M = \mathbb{Z} \), \( \Delta \cap M = \{m_j = j, j = 0, \ldots, 3\} \), \( X \) is the closure of the image of

\[
\varphi : \mathbb{C}^* \to \mathbb{P}^3 \\
t \mapsto [1, t, t^2, t^3],
\]
Figure 1: The moment polytope $\Delta$ of the twisted cubic $X \subset \mathbb{P}^3$.

which extends to the embedding

$$X_\Delta \hookrightarrow \mathbb{P}^3 \quad (v_0, v_1) \mapsto [v_1^3, v_0 v_1^2, v_0^2 v_1, v_0^3],$$

where $v_0, v_1$ are homogeneous coordinates in $X_\Delta$.

The toric ideal of $X$ is of course computed to be

$$I = \langle x_0 x_2 - x_1^2, x_1 x_3 - x_2^2, x_0 x_3 - x_1 x_2 \rangle.$$

Now consider the semi-stable partition $\{\Delta_1, \Delta_2\}$ of $\Delta$, where $\Delta_1 = [0, 1] \subset \mathbb{R}$ and $\Delta_2 = [1, 3] \subset \mathbb{R}$. This partition gives the semi-stable degeneration of $X$ to the union of two curves $X_1 \cup X_2$, where $X_1 = (\mathbb{P}^1, \mathcal{O}(1))$ and $X_2 = (\mathbb{P}^1, \mathcal{O}(2))$.

The two possible lifting functions are

$$F(j) = \begin{cases} 0 & j = 0, 1 \\ j - 1 & j = 2, 3 \end{cases}, \quad G(j) = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}.$$

Using the notation of [2], in local coordinates the embeddings of $\tilde{X}_F$ and $\tilde{X}_G$ in $\mathbb{P}^3 \times \mathbb{C}$ are $([1, t, \lambda t^2, \lambda^2 t^3], \lambda)$ and $([\lambda, t, t^2, t^3], \lambda)$, while in homogeneous coordinates these are

$$([v_1^3, v_0 v_1^2, \lambda v_0^2 v_1, \lambda^2 v_0^3], \lambda)$$
and
\[ (\lambda v_1^3, v_0 v_1^2, v_0^2 v_1, v_0^3, \lambda). \]
We therefore observe that $\tilde{X}_F$ and $\tilde{X}_G$ have different parametric equations, nevertheless it is easy to see that both of them are defined in $\mathbb{P}^3 \times \mathbb{C}$ by the equations
\[ x_0 x_2 - \eta x_1^2 = 0, x_1 x_3 - x_2^2 = 0, x_0 x_3 - \eta x_1 x_2 = 0, \]
where $\eta$ is the non-homogeneous coordinate in $\mathbb{C}$. These equations can also be found applying elimination theory algorithms to the two parametrizations in homogeneous coordinates, computations can be performed by hand or using computer algebra systems.

2.2 $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in a point

Consider the polytope $\Delta$ in figure 3 with its associated normal fan. The toric surface $X$ determined by $\Delta$ is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in a point and embedded in $\mathbb{P}^7$. In local coordinates it is the closure of the image of
\[ \varphi : (\mathbb{C}^*)^2 \rightarrow \mathbb{P}^7 \]
\[ (t_1, t_2) \mapsto [1, t_1, t_1^2 t_2, t_2, t_1 t_2, t_1^2 t_2, t_2, t_1^2], \]
while taking homogeneous coordinates $v_0, \ldots, v_4$ for $X_\Delta$ (one for each facet of $\Delta$), the embedding is
\[ X_\Delta \hookrightarrow \mathbb{P}^7 \]
\[ (v_0, \ldots, v_4) \mapsto [v_2^3 v_4^2, v_0 v_2 v_3^2 v_4^2, v_0^2 v_3^2 v_4^2, v_1 v_2^2 v_3 v_4^2, v_0 v_1 v_2 v_3 v_4, \]
\[ v_0^2 v_1 v_4, v_1^2 v_2 v_3, v_0 v_1^2 v_2 v_4]. \]
Consider the semi-stable partition \( \{ \Delta_1, \Delta_2 \} \) of \( \Delta \):

![Diagram](image)

Figure 4: A semistable partition of \( X \).

This partition gives the semi-stable degeneration of \( X \) to the union of two surfaces \( X_1 \cup X_2 \), where \( X_1 = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( X_2 = \mathbb{F}^1 \).

The two possible lifting functions are

\[
F(m_j) = \begin{cases} 
0 & j = 0, \ldots, 5 \\
1 & j = 6, 7 
\end{cases},
\]

\[
G(m_j) = \begin{cases} 
1 & j = 0, 1, 2 \\
0 & j = 3, \ldots, 7 
\end{cases}.
\]

In local coordinates the embeddings of \( \tilde{X}_F \) and \( \tilde{X}_G \) in \( \mathbb{P}^7 \times \mathbb{C} \) are

\[
([1, t_1, t_1^2, t_2, t_1^2 t_2, t_1^2 t_2 x_1, \lambda t_2 t_1^2], \lambda)
\]

and

\[
([\lambda, \lambda t_1, \lambda t_1^2, t_2, t_1^2 t_2, t_1^2 t_2 x_1, \lambda], \lambda).
\]

We have embeddings

\[
\iota_F : \tilde{X}_F \hookrightarrow \mathbb{P}^7 \times \mathbb{C}
\]

\[
(v_0, \ldots, v_4, \lambda) \mapsto (v_0^2 v_1^3 v_4^2, v_0 v_2 v_3^2 v_4, v_0 v_3^2 v_4, v_1 v_2^2 v_4, v_0 v_1 v_2 v_3^2, v_0 v_1^2 v_2 v_3^2, v_0 v_1 v_2 v_3 v_4, v_0 v_1 v_2 v_3^2, v_0 v_1^2 v_2 v_3^2, v_0 v_1^2 v_2 v_3^2, v_0 v_1^2 v_2 v_4, \lambda),
\]

and

\[
\iota_G : \tilde{X}_G \hookrightarrow \mathbb{P}^7 \times \mathbb{C}
\]

\[
(v_0, \ldots, v_4, \lambda) \mapsto (v_0^2 v_1^3 v_4^2, v_0 v_2 v_3^2 v_4, v_0^2 v_3 v_4^2, v_1 v_2^2 v_4, v_0 v_1 v_2 v_3^2, v_0 v_1^2 v_2 v_3^2, v_0 v_1 v_2 v_3 v_4, v_0 v_1 v_2 v_3^2, v_0 v_1^2 v_2 v_3^2, v_0 v_1^2 v_2 v_4, \lambda).
\]

\( \tilde{X}_F \) and \( \tilde{X}_G \) have different parametric equations. We find that \( \tilde{X}_F, \tilde{X}_G \) are both defined in \( \mathbb{P}^7 \times \mathbb{C} \) by the following nine quadratic equations:

\[
x_3 x_5 - x_4^2 = 0, x_2 x_6 - \lambda x_4^2 = 0, x_1 x_6 - \lambda x_3 x_4 = 0,
\]

\[
x_1 x_5 - x_2 x_4 = 0, x_1 x_4 - x_2 x_3 = 0, x_0 x_6 - \lambda x_3^2 = 0,
\]

\[
x_0 x_5 - x_2 x_3 = 0, x_0 x_4 - x_1 x_3 = 0, x_0 x_2 - x_1^2 = 0.
\]
Omitting $\lambda$ in these equations we obtain a set of equation for $X_\Delta$ embedded in $\mathbb{P}^7$: these are the same equations one can compute from (3) through elimination.

3 Main results

We use the notation of the previous sections.

Let $I_F$ be the ideal of all polynomials in the coordinates $x_0, \ldots, x_\ell, \eta$ homogeneous in $x_0, \ldots, x_\ell$ and vanishing on $\tilde{X}_F$, where $\eta$ is the non-homogeneous coordinate in $\mathbb{C}$. In analogy with the compact case we use the notation $z^u = x_0^{u_0} \ldots x_\ell^{u_\ell} \eta^{u_{\ell+1}}$, with $u = (u_0, \ldots, u_\ell, u_{\ell+1}) \in \mathbb{Z}^{\ell+2}$.

Consider the $(n+2) \times (\ell+2)$ matrix

$$B^+ = B_F^+ = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ m_{10} & m_{11} & \cdots & m_{1\ell} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{n\ell} & 0 \\ F(m_0) & F(m_1) & \cdots & F(m_\ell) & 1 \end{pmatrix}.$$ 

Lemma 3.1 $I_F$ is the linear span of all binomials $z^u - z^v$ with vectors $u, v \in \mathbb{N}^{\ell+2}$ such that $B^+ u = B^+ v$.

Proof. We follow Theorems 2.1 and 2.2 [8].

A binomial $z^u - z^v$, with $u, v \in \mathbb{N}^{\ell+2}$, vanishing on $\psi((\mathbb{C}^*)^n \times \mathbb{C})$ needs to be homogeneous in the coordinates $x_0, \ldots, x_\ell$, i.e.

$$\sum_{i=0}^{\ell} u_i = \sum_{i=0}^{\ell} v_i. \quad (4)$$

Therefore we prove that $I_F$ is the linear span of all binomials $z^u - z^v$ with vectors $u, v$ such that (4) holds and $Bu = Bv$, where

$$B = B_F = \begin{pmatrix} m_{10} & m_{11} & \cdots & m_{1\ell} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ m_{n0} & m_{n1} & \cdots & m_{n\ell} & 0 \\ F(m_0) & F(m_1) & \cdots & F(m_\ell) & 1 \end{pmatrix}.$$
Consider a monomial $z^u$ and restrict it to $\psi((\C^*)^n \times C)$:

$$z^u|_{\psi((\C^*)^n \times C)} = (x_0^{u_0} \cdots x_\ell^{u_\ell} \eta^{u_{\ell+1}})|_{\psi((\C^*)^n \times C)} = (t_1^{m_1 u_0} \cdots t_n^{m_n u_0} \lambda^{F(m_0)})^{u_0} \cdots (t_1^{m_1 u_\ell} \cdots t_n^{m_n u_\ell} \lambda^{F(m_\ell)})^{u_\ell} \cdot \lambda^{u_{\ell+1}} = (t_1^{m_1 u_0 + \ldots + m_\ell u_\ell} \cdots t_n^{m_0 u_0 + \ldots + m_n u_\ell} \lambda^{F(m_0) u_0 + \ldots + F(m_\ell) u_\ell + u_{\ell+1}} = T^{Bu},$$

with $T = (t_1, \ldots, t_n, \lambda)$.

This shows that in the hypothesis (4), $z^u - z^v$ vanishes on $\psi((\C^*)^n \times C)$ (and hence belongs to $I_F$) if and only if $Bu = Bv$.

Now we show that these binomials generate $I_F$ as a $C$-vector space: we follow Strumfels’s book [9]. Strumfels considers the (compact) toric variety defined as in (1) and doesn’t deal with the homogeneous vs. non-homogeneous question.

Fix a monomial ordering $>$ on $\C[x_0, \ldots, x_\ell, \eta]$, and remember that this is a well-ordering on the set of monomials $z^u$. Suppose the set $R$ of polynomials $f \in I_F$ which cannot be written as a $C$-linear combination of binomials as above is non-empty and take $f \in R$ such that

$$LM_>(f) = \min_{g \in R} LM_>(g),$$

where $LM_>(f)$ is the leading monomial of $f$ with respect to $>$. We can suppose $f$ to be monic, so that its leading term $LT_>(f)$ is its leading monomial, let this be the monomial $z^u$.

When we restrict $f$ to $\psi((\C^*)^n \times C)$ we get an expression containing $T^{Bu}$ as a term and which is equal to zero. Hence the term $T^{Bu}$ must cancel in this expression. This means that there is some other monomial $z^v$ appearing in $f$ such that $Bu = Bv$ and (4) holds.

Moreover $z^u > z^v$. The polynomial

$$f' := f - z^u + z^v$$

belongs to $I_F$ and to $R$ but since $LM_>(f) > LM_>(f')$, we get a contradiction. □

**Theorem 3.2** $I_F = \langle z^u^+ - z^u^- | u \in \ker(B^+) \rangle$ and $u \in \Z^{\ell+2}$.
Proof. On one hand, \( u \in \ker(B^+\!\!_+) \) if and only if \( B^+\!\!_+u = B^+\!\!_-u \). On the other hand, we show that if \( B^+\!\!_+v = B^+\!\!_-w \) (and \([4]\) holds), then \( z^v - z^w = h(z^u - z^w) \), for some polynomial \( h \) and vector \( u \in \ker(B^+\!\!_+) \cap \mathbb{Z}^{\ell+2} \); the statement will then follow from the theorem.

If \( B^+\!\!_+v = B^+\!\!_-w \), then \( v - w \in \ker(B^+\!\!_+) \).

\[
z^v - z^w = z^w(z^v - w - 1) = z^wz^{-(v-w)}(z^{(v-w)+} - z^{(v-w)-})
= z^w-(v-w)(z^{v-w} - z^{v-w})
\]

It is easy to show that \( w - (v - w) - \in \mathbb{N}^{\ell+2} \). □

Now let \( G \) be the second lift, then we can consider the matrix \( B^+_G \) and characterize the toric ideal \( I_G \) of \( \tilde{X}_G \) as above. In general \( \tilde{X}_G \) will have a different parametrization from the one of \( \tilde{X}_F \), moreover the normal fans are different.

Our main result is

**Theorem 3.3** \( \tilde{X}_F \) and \( \tilde{X}_G \) have the same equations in \( \mathbb{P}^{\ell} \times \mathbb{C} \), i.e. \( I_F = I_G \).

Proof. Reorder the \( m_j \)'s such that \( m_0, \ldots, m_r \in \Delta_1 - \Delta_2, m_{r+1}, \ldots, m_s \in \Delta_1 \cap \Delta_2 \) and \( m_{s+1}, \ldots, m_\ell \in \Delta_2 - \Delta_1 \), then we have

\[
B^+_F = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 \\
m_{10} & \ldots & m_{1r} & m_{1,r+1} & \ldots & m_{1s} & m_{1,s+1} & \ldots & m_{1\ell} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
m_{n0} & \ldots & m_{nr} & m_{n,r+1} & \ldots & m_{ns} & m_{n,s+1} & \ldots & m_{n\ell} & 0 \\
0 & \ldots & 0 & \ldots & 0 & L_F(m_{s+1}) & \ldots & L_F(m_\ell) & 1
\end{pmatrix}
\]

and

\[
B^+_G = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 0 \\
m_{10} & \ldots & m_{1r} & m_{1,r+1} & \ldots & m_{1s} & m_{1,s+1} & \ldots & m_{1\ell} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
m_{n0} & \ldots & m_{nr} & m_{n,r+1} & \ldots & m_{ns} & m_{n,s+1} & \ldots & m_{n\ell} & 0 \\
L_G(m_0) & \ldots & L_G(m_r) & 0 & \ldots & 0 & \ldots & 0 & 1
\end{pmatrix}
\]

Let \( E \) be the \((n + 2) \times (n + 2)\) elementary matrix

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
a_{n+1} & a_1 & \ldots & a_n & 1
\end{pmatrix}
\in SL_{n+2}(\mathbb{Z})
\]

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we have
\[ E \cdot \mathcal{B}_G^+ = \mathcal{B}_F^+, \]
and hence
\[ \ker \mathcal{B}_F^+ = \ker \mathcal{B}_G^+. \]
The theorem follows from Theorem (3.2).

Going back to the examples above, if \( X \) is the twisted cubic, we have
\[
\mathcal{B}_F^+ = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 2 & 3 & 0 \\
1 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]
\[
\mathcal{B}_G^+ = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 2 & 3 & 0 \\
0 & 0 & 1 & 2 & 1 \\
\end{pmatrix},
\]
and \( E \) is the 3 \times 3 elementary matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & -1 & 1 \\
\end{pmatrix} \in SL_3(\mathbb{Z}).
\]

In the case of \( \mathbb{P}^1 \times \mathbb{P}^1 \) blown up in a point, we have
\[
\mathcal{B}_F^+ = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]
\[
\mathcal{B}_G^+ = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix},
\]
and
\[
E = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & -1 & 1 \\
\end{pmatrix} \in SL_4(\mathbb{Z}).
\]

It would be interesting to extend such results to semi-stable partitions of a polytope \( \Delta \) in an arbitrary number of subpolytopes.
References

[1] V. Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. Math., 155 (2002) 611–708.

[2] D. Cox, *What is a toric variety?*, Topics in algebraic geometry and geometric modeling, 203–223, Contemp. Math., 334, Amer. Math. Soc., Providence, RI, (2003); also available at http://www.amherst.edu/~dacox/

[3] W. Fulton, *Introduction to Toric Varieties*, Ann. of Math. Studies, 13, Princeton University Press, (1993).

[4] Maplesoft. Maple10. [http://maplesoft.com].

[5] S. Hu, *Semistable Degeneration of Toric Varieties and Their Hypersurfaces*, Communications in Analysis and Geometry, Volume 14, Number 1 (2006), 59–89; arXiv:math.AG/0110091, (2001) 1-26.

[6] M. Marchisio - V. Perduca, *On Some Properties of Explicit Toric Degenerations*, Bollettino U.M.I. (8) 9-B (2006), 779-784.

[7] T. Oda, *Convex Bodies and Algebraic Geometry*, 15, Springer-Verlag, Berlin Heidelberg (1988).

[8] F. Sottile, *Toric ideals, real toric varieties, and the moment map*, Topics in algebraic geometry and geometric modeling, 225–240, Contemp. Math., 334, Amer. Math. Soc., Providence, RI, (2003); arXiv:math.AG/0212044.

[9] B. Strumfels, *Gröbner Bases and Convex Polytopes*, American Mathematical Society, University Lecture Series, Volume 8, Providence, RI (1996). MR 97b:13034.
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