Independence of the Yoshikawa eighth move and a minimal generating set of band moves

by

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Abstract. Yoshikawa moves were introduced a quarter-century ago and are still actively used by researchers. For any marked graph diagram we will define its twisted diagram and its mirror cut surface. By using a surface-link group of a mirror cut surface of a twisted diagram we will prove the independence of the Yoshikawa eighth move. As a consequence we will establish a minimal generating set of band moves for links with bands.

1. Introduction. A surface-link is a closed 2-manifold smoothly (or piecewise linearly and locally flatly) embedded in the Euclidean 4-space. A marked graph diagram is a link diagram possibly with 4-valent vertices equipped with markers that can represent a surface-link.

It is known that the set \{Ω₁, ..., Ω₈, Ω₄', Ω₆'\} of ten types of moves (presented in Fig. 3), called Yoshikawa moves, is a generating set of moves that relate two marked graph diagrams presenting equivalent surface-links. In [3], [4] it is shown that any Yoshikawa move from the set \{Ω₁, Ω₂, Ω₃, Ω₆, Ω₆', Ω₇\} is independent of the other nine types. It has been an open problem (see [4]) whether Ω₈ is independent of the other Yoshikawa moves.

We can translate the marked graph diagrams in \(\mathbb{R}^2\) to links with bands in \(\mathbb{R}^3\); then instead of ten types of moves we have the set \{M₁, M₂, M₃, M₄\} of four generating types of moves (presented in Fig. 13). Therefore, independence of some Yoshikawa moves yields a minimal generating set of moves on links with bands.

We will prove the following two main theorems.

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Theorem 1. The Yoshikawa move $\Omega_8$ cannot be realized by a finite sequence of Yoshikawa moves of the other nine types.

Theorem 2. The set $\{M_1, M_2, M_3, M_4\}$ is a minimal generating set of moves on links with bands.

This paper is organized as follows. In Section 2, we review marked graphs corresponding to surface-links in the hyperbolic splitting position and their diagrams. In Section 3, for any marked graph diagram we define its twisted diagram and its mirror cut surface, and by using a surface-link group of a mirror cut surface of a twisted diagram we prove Theorem 1. In Section 4, we prove Theorem 2.

2. Preliminaries. Throughout this paper, we work in the standard smooth category. An embedding (or its image) of a closed (i.e. compact, without boundary) surface $F$ into $\mathbb{R}^4$ is called a surface-link (or surface-knot if it is connected). Two surface-links are equivalent (or have the same type, which is denoted also by $\cong$) if there exists an orientation preserving homeomorphism of the four-space $\mathbb{R}^4$ to itself (or equivalently an auto-homeomorphism of the four-sphere $S^4$) mapping one of those surfaces onto the other.

We will use the word classical when thinking about the theory of embeddings of circles $S^1 \sqcup \cdots \sqcup S^1 \hookrightarrow \mathbb{R}^3$ modulo an ambient isotopy of $\mathbb{R}^3$ with their planar or spherical generic projections.

To describe a knotted surface in $\mathbb{R}^4$, we will use transverse cross-sections $\mathbb{R}^3 \times \{t\} \subset \mathbb{R}^4$ for $t \in \mathbb{R}$, denoted by $\mathbb{R}^3_t$. This method introduced by Fox and Milnor was presented in [1].

It is well-known ([5], [6], [8]) that any surface-link $F$ admits a hyperbolic splitting, i.e. there exists a surface-link $F'$ satisfying the following: $F'$ is equivalent to $F$ and has only finitely many Morse critical points, all maximal points of $F'$ lie in $\mathbb{R}^3_1$, all minimal points of $F'$ lie in $\mathbb{R}^3_{-1}$, and all saddle points of $F'$ lie in $\mathbb{R}^3_0$.

![Fig. 1. Rules for smoothing a marker](image)

The zero section $\mathbb{R}^3_0 \cap F'$ of the surface $F'$ in the hyperbolic splitting position described above gives us then a 4-regular graph. We assign to each vertex a marker that informs us about one of the two possible types of saddle points (see Fig. 1) depending on the shape of the section $\mathbb{R}^3_{-\epsilon} \cap F'$ or $\mathbb{R}^3_\epsilon \cap F'$ for a small real number $\epsilon > 0$. The resulting (rigid-vertex) graph is called the marked graph presenting $F$. 
Making a projection in general position of this graph to \( \mathbb{R}^2 \times \{0\} \times \{0\} \subset \mathbb{R}^4 \) and assigning types to classical crossings between regular arcs, we obtain a marked graph diagram. For a marked graph diagram \( D \), we denote by \( L_+(D) \) and \( L_-(D) \) the classical link diagrams obtained from \( D \) by smoothing every vertex as presented in Fig. 1 for the \( +\epsilon \) and \( -\epsilon \) case respectively. We call \( L_+(D) \) and \( L_-(D) \) the positive resolution and the negative resolution of \( D \), respectively.

An abstractly created marked graph diagram is a ch-diagram (or is admissible) if both its resolutions are trivial classical link diagrams.

A band on a link \( L \) is an image of an embedding \( b : I \times I \to \mathbb{R}^3 \) intersecting the link \( L \) precisely in the subset \( b(\partial I \times I) \). A link with bands, \( LB \), in \( \mathbb{R}^3 \) is a pair \((L,B)\) consisting of a link \( L \) in \( \mathbb{R}^3 \) and a finite set \( B = \{b_1, \ldots, b_n\} \) of pairwise disjoint \( n \) bands spanning \( L \).

![Fig. 2. A band corresponding to a marked vertex](image)

By an ambient isotopy of \( \mathbb{R}^3 \), we shorten the bands of a link with bands \( LB \) so that each band is contained in a small 2-disk. Replacing the neighborhood of each band by the neighborhood of a marked vertex as in Fig. 2, we obtain a marked graph, called the marked graph associated with \( LB \). Conversely, when a marked graph \( G \) in \( \mathbb{R}^3 \) is given, by replacing each marked vertex with a band as in Fig. 2, we obtain a link with bands \( LB(G) \), called the link with bands associated with \( G \).

Let \( D \) be a ch-diagram with associated link with bands \( LB(D) = (L,B) \), \( L = L_-(D) \), \( B = \{b_1, \ldots, b_n\} \), let \( \Delta_1, \ldots, \Delta_a \subset \mathbb{R}^3 \) be mutually disjoint 2-disks with \( \partial(\bigcup_{j=1}^a \Delta_j) = L_+(D) \), and let \( \Delta'_1, \ldots, \Delta'_b \subset \mathbb{R}^3 \) be mutually disjoint 2-disks with \( \partial(\bigcup_{k=1}^b \Delta'_k) = L_-(D) \). We define \( S(D) \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4 \), the surface-link corresponding to the diagram \( D \), by the following cross-sections:

\[
(\mathbb{R}^3, S(D) \cap \mathbb{R}^3_1) = \begin{cases}
(\mathbb{R}^3, \emptyset) & \text{for } t > 1, \\
(\mathbb{R}^3, L_+(D) \cup \bigcup_{j=1}^a \Delta_j) & \text{for } t = 1, \\
(\mathbb{R}^3, L_+(D)) & \text{for } 0 < t < 1, \\
(\mathbb{R}^3, L_-(D) \cup \bigcup_{i=1}^n b_i) & \text{for } t = 0, \\
(\mathbb{R}^3, L_-(D)) & \text{for } -1 < t < 0, \\
(\mathbb{R}^3, L_-(D) \cup \bigcup_{k=1}^b \Delta'_k) & \text{for } t = -1, \\
(\mathbb{R}^3, \emptyset) & \text{for } t < -1.
\end{cases}
\]
It is known that the surface-knot type of \( S(D) \) does not depend on the choices of trivial disks (cf. [6]). It is straightforward from the construction of \( S(D) \) that \( D \) is a marked graph diagram presenting \( S(D) \).

In [10] Yoshikawa introduced local moves on admissible marked graph diagrams that do not change the corresponding surface-link types and conjectured that the converse is also true. This was resolved ([9], [7], [4]) as follows. Any two marked graph diagrams representing the same type of surface-link are related by a finite sequence of Yoshikawa local moves presented in Fig. 3 and an isotopy of the diagram in \( \mathbb{R}^2 \).

An orientable surface-link in \( \mathbb{R}^4 \) is \textit{unknotted} if it is equivalent to a surface embedded in \( \mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4 \). A marked graph diagram for an unknotted \textit{standard sphere} \( S^2 \) is shown in Fig. 4(a), and an unknotted \textit{standard torus} \( T^2 \) is in Fig. 4(d).

An embedded projective plane \( P^2 \) in \( \mathbb{R}^4 \) is \textit{unknotted} if it is equivalent to a surface whose marked graph diagram is of an unknotted \textit{standard projective plane}; the unknotted depicted projective plane in Fig. 4(b) (resp. Fig. 4(c)) is a \textit{positive} \( P^2_+ \) (resp. a \textit{negative} \( P^2_- \)). A nonorientable surface-knot is \textit{unknotted} if it is equivalent to some finite connected sum of unknotted projective planes.

**Proposition 3 ([10]).** The relations \( H1, H2, H3, H4 \) presented in Fig. 5 applied to \( ch \)-diagrams preserve the type of the associated surface-link.
3. Independence of the Yoshikawa eighth move. A standard stabilization/destabilization is defined by adding/deleting an unknotted handle attached locally. The surface-link group of a surface-link $F$, denoted by $\text{Gr}(F)$, is the fundamental group $\pi_1(\mathbb{R}^4\setminus \text{int}(N(F)))$ where $N(F)$ is a tubular neighborhood of $F$.

**Definition 4.** Let $D$ be a marked graph diagram such that $L_-(D)$ is a trivial link and let $\Delta_1', \ldots, \Delta_b' \subset \mathbb{R}^3$ be mutually disjoint 2-disks with $\partial(\bigcup_{k=1}^b \Delta_k') = L_-(D)$. Denote by $LB(D)$ the associated link with bounds $(L, B)$ with $L = L_-(D)$ and $B = \{b_1, \ldots, b_n\}$, and let $B^* = \{b_1^*, \ldots, b_n^*\}$ be the dual bands of $B$ (associated to changing the corresponding markers to the other type). We define the mirror cut surface of $D$, denoted by $MC(D) \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$, as a surface-link defined by the following cross-sections:

$$
\left(\mathbb{R}_t^3, MC(D) \cap \mathbb{R}_t^3\right) = \begin{cases} 
(\mathbb{R}_t^3, \emptyset) & \text{for } t > 2, \\
(\mathbb{R}_t^3, L_-(D) \cup \bigcup_{k=1}^b \Delta_k') & \text{for } t = 2, \\
(\mathbb{R}_t^3, L_-(D)) & \text{for } 1 < t < 2, \\
(\mathbb{R}_t^3, L_+(D) \cup \bigcup_{i=1}^n b_i^*) & \text{for } t = 1, \\
(\mathbb{R}_t^3, L_+(D)) & \text{for } 0 < t < 1, \\
(\mathbb{R}_t^3, L_-(D) \cup \bigcup_{i=1}^n b_i) & \text{for } t = 0, \\
(\mathbb{R}_t^3, L_-(D)) & \text{for } -1 < t < 0, \\
(\mathbb{R}_t^3, L_+(D) \cup \bigcup_{k=1}^b \Delta_k') & \text{for } t = -1, \\
(\mathbb{R}_t^3, \emptyset) & \text{for } t < -1.
\end{cases}
$$

The mirror cut surface of a given diagram $D$ can be intuitively seen as the result of making a cut of the surface-link $S(D)$ corresponding to $D$ on
the level of \( t = 1/2 \), making a copy of its lower part and gluing it to the mirror image of its copy along their common boundary link \( L_+(D) \). In other words, put a mirror in the cut level and see the whole surface-link from below. Saddles in the middle cross-sections are depicted in Fig. 6.

![Fig. 6. Saddles change by the mirror cut transformation](image)

**Definition 5.** For a marked graph diagram \( D \), let \( \mathcal{O}(D) \) denote the set of all possible orientations of \( L_-(D) \). For a fixed \( o \in \mathcal{O}(D) \), define the **twisted graph diagram** of \( D \), denoted by \( D^t \), as follows. If in a neighborhood of a marked vertex of \( D \), the orientations of strands are coherent (see top of Fig. 7), call this vertex **type d** and do not change the diagram \( D^t \) in that neighborhood. If the orientations near a marked vertex in \( D \) are not coherent (see bottom left of Fig. 7), call this vertex **type e** and change it by adding a classical crossing as shown at the bottom right of the figure. Note that the marked graph diagram \( D^t \) may not be admissible.

![Fig. 7. Rules for twisting a marked graph diagram](image)

By an ambient isotopy of \( \mathbb{R}^4 \) we can obtain the following ch-diagram of the hyperbolic splitting of the mirror cut surface for the twisted ch-diagram.

**Proposition 6.** Let \( D \) be a ch-diagram with a fixed \( o \in \mathcal{O}(D) \). A marked graph diagram presenting \( MC(D^t) \), denoted by \( \mathcal{W}(D) \), can be obtained by
replacing each marked vertex of $D$ by the rule presented in Fig. 8, with the orientation in the neighborhood of the new vertex induced from $o$.

![Diagram](image)

Fig. 8. A marked graph diagram presentation of $W(D)$

**Remark 7.** Note that the moves from Proposition 3 can be obtained by a combination of Yoshikawa moves $\Omega_1, \ldots, \Omega_7, \Omega'_4, \Omega'_6$ and an ambient isotopy of $\mathbb{R}^2$. If we change the order of marked crossings and classical crossings appearing at the bottom right of Fig. 8 the local diagram can be transformed into the shape shown at the bottom right of Fig. 8 by using $H3, H4, \Omega_2, \Omega_5$.

**Lemma 8.** Let $V$ be a set of marked vertices of a marked graph diagram $D_1$ presenting a surface-link $F_1$. Define $D_2$ as the graph diagram obtained by switching markers on each vertex from $V$ to the other type (i.e. rotating them by $90^\circ$ around the vertex). If $D_2$ is a ch-diagram (i.e. is admissible) presenting a surface-link $F_2$ then $\text{Gr}(F_2) \cong \text{Gr}(F_1)$.

**Proof.** By the group calculation method presented in [1] we see that the presentation relations remain the same when markers type are switched and the corresponding surface is still closed. So the surface-link group (obtained from a marked graph diagram) cannot detect types of markers, as long as the diagram remains admissible.

**Remark 9.** As a consequence of Lemma 8 and Proposition 3 and by looking at Yoshikawa’s table of surface-link marked graph diagrams [10], we can easily obtain (without calculations) the isomorphisms $\text{Gr}(2_{1}^{1}) \cong \text{Gr}(0_{1})$ and $\text{Gr}(10_{1}^{-2,-2}) \cong \text{Gr}(8_{1}^{-1,-1})$.

Considering a marked graph diagram of the unknotted standard torus presented in Fig. 4(d), the above together with Lemma 8 and Proposition 3 yields the following.

**Corollary 10.** A standard stabilization/destabilization of a surface-link does not change its group.
For a ch-diagram $D$ let us define $\mathcal{M}(D)$ as an (unordered) set of groups over all possible orientations:

$$\mathcal{M}(D) = \{ \text{Gr}(MC(D^o)) | o \in \mathcal{O}(D) \}.$$  

**Proposition 11.** For a marked graph diagram $D$ the set $\mathcal{M}(D)$ is an invariant of Yoshikawa moves $\Omega_1, \ldots, \Omega_7, \Omega_4', \Omega_6'$.

**Proof.** For a fixed $o \in \mathcal{O}(D)$ we investigate how these Yoshikawa moves performed on the diagram $D$ change the marked graph diagram $W(D)$ obtained as in Proposition 6. It is clear that $\Omega_1, \Omega_2, \Omega_3$ do not change the type of the surface-link $S(W(D)) \cong MC(D^o)$ because those moves do not involve marked vertices. Performing $\Omega_4, \Omega_4'$ on $D$ produces changes on $W(D)$ that can be easily obtained using $\Omega_1, \ldots, \Omega_4, \Omega_4'$, no matter whether the marked vertex involved was type $d$ or $e$.

We see that in the case of an $\Omega_5$ move the diagram $W(D)$ is changed by moves that can be generated by $\Omega_1, \ldots, \Omega_5, \Omega_4'$, because in this case the orientation type ($d$ or $e$) of the marked vertex involved does not change, and this transformation can be done by an ambient isotopy of $\mathbb{R}^3$.

Let us proceed to $\Omega_6, \Omega_6'$ moves on the diagram $D$; the corresponding changes on $W(D)$ are shown in Fig. 9. We see that in the case of an $\Omega_6$ move the surface-link $S(W(D))$ is changed by a standard stabilization/destabilization, because the marked vertex involved has to be of type $d$, so by Corollary 10 the group $\text{Gr}(S(W(D)))$ does not change its isomorphism class. In the case of an $\Omega_6'$ move the diagram $W(D)$ is changed by moves that can be generated by $\Omega_1, \Omega_6, \Omega_6'$.
Finally, we investigate the change on $W(D)$ by performing an $\Omega_7$ move on the diagram $D$, which is depicted in Fig. 10 where (taking additionally their mirror images) all the possible cases are covered. We see there that transformations (two-sided arrows) follow from moves $H3, H4, \Omega_1, \ldots, \Omega_5, \Omega'_4, \Omega_7$. The case where all four marked vertices involved in an $\Omega_7$ move are of type $e$ cannot occur, as a consequence of the definition of type $e$ and investigation of all possible types of orientations of the classical link $L_-(D)$ involved in the move.

Proof of Theorem 1. Let $D_1$ and $D_2$ be two ch-diagrams presented in Fig. 11. They are diagrams of equivalent surface-links of $S^2 \sqcup P^2_-$ as they differ by just one Yoshikawa move (four classical crossings are changed). However, $\mathcal{M}(D_1) \neq \mathcal{M}(D_2)$, because the set of multivariable first elementary ideals for the corresponding elements in $\mathcal{M}(D_1)$ is \{\langle 0 \rangle \} and for $\mathcal{M}(D_2)$ it is \{\langle (x+1)(x-1), (x+1)(y-1) \rangle \}. Therefore, by Proposition 11 the diagram $D_1$ cannot be transformed into $D_2$ by using Yoshikawa moves $\Omega_1, \ldots, \Omega_7, \Omega'_4, \Omega'_6$ and an ambient isotopy of $\mathbb{R}^2$. ■
4. A minimal generating set of band moves. Examples of links with bands of unknotted surfaces are presented in Fig. 12 for examples of nontrivial links with bands refer to [2].

We have analogous (to the marked graph diagram case) moves for links with bands ([2], [7]), i.e. any two links with bands representing the same type of surface-link are related by a finite sequence of local moves presented in Fig. 13 (and an isotopy of $\mathbb{R}^3$). The moves $M_1, M_2, M_3, M_4$ are called a cup move, cap move, band-slide, band-pass respectively.
Proof of Theorem 2. Directly from the correspondence between markers and bands (see Fig. 2), we get a correspondence between $M_1, M_2, M_3, M_4$ and $\Omega'_6, \Omega_6, \Omega_7, \Omega_8$ respectively and the independence of any move in $\{\Omega_6, \Omega'_6, \Omega_7, \Omega_8\}$ from the other nine types of Yoshikawa moves. ■

References

[1] R. H. Fox, A quick trip through knot theory, in: Topology of 3-manifolds and Related Topics, Prentice-Hall, 1962, 120–167.
[2] M. Jabłonowski, On a banded link presentation of knotted surfaces, J. Knot Theory Ramif. 25 (2016), art. 1640004, 11 pp.
[3] Y. Joung, J. Kim and S. Y. Lee, Ideal coset invariants for surface-links in $\mathbb{R}^4$, J. Knot Theory Ramif. 22 (2013), art. 1350052, 25 pp.
[4] Y. Joung, J. Kim and S. Y. Lee, On generating sets of Yoshikawa moves for marked graph diagrams of surface-links, J. Knot Theory Ramif. 24 (2015), art. 1550018, 21 pp.
[5] S. Kamada, Non-orientable surfaces in 4-space, Osaka J. Math. 26 (1989), 367–385.
[6] A. Kawauchi, T. Shibuya and S. Suzuki, Descriptions on surfaces in four-space, I; Normal forms, Math. Sem. Notes Kobe Univ. 10 (1982), 72–125.
[7] C. Kearton and V. Kurlin, All 2-dimensional links in 4-space live inside a universal 3-dimensional polyhedron, Algebr. Geom. Topol. 8 (2008), 1223–1247.
[8] S. J. Lomonaco, Jr., The homotopy groups of knots I. How to compute the algebraic 2-type, Pacific J. Math. 95 (1981), 349–390.
[9] F. J. Swenton, On a calculus for 2-knots and surfaces in 4-space, J. Knot Theory Ramif. 10 (2001), 1133–1141.
[10] K. Yoshikawa, An enumeration of surfaces in four-space, Osaka J. Math. 31 (1994), 497–522.

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