ON THE SCATTERING PROBLEM FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH A POTENTIAL IN 2D

VLADIMIR GEORGIEV AND CHUNHUA LI

Abstract. We consider the scattering problem for the nonlinear Schrödinger equation with a potential in two space dimensions. Appropriate resolvent estimates are proved and applied to estimate the operator $A(s)$ appearing in commutator relations. The equivalence between the operators $(-\Delta)^s V$ and $(-\Delta)^s$ in $L^2$ norm sense for $0 \leq s < 1$ is investigated by using free resolvent estimates and Gaussian estimates for the heat kernel of the Schrödinger operator $-\Delta_V$. Our main result guarantees the global existence of solutions and time decay of the solutions assuming initial data have small weighted Sobolev norms. Moreover, the global solutions obtained in the main result scatter.

1. Introduction and main results

We consider the following nonlinear Schrödinger equation

\begin{equation}
\begin{cases}
    i\partial_t u + \frac{1}{2}(\Delta - V)u = \lambda|u|^{p-1}u, \\
    u(1,x) = u_0(x)
\end{cases}
\end{equation}

in $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, where $\Delta$ is the 2-dimensional Laplacian, $u = u(t,x)$ is a complex-valued unknown function, $t \geq 1$, $p > 2$, $\lambda \in \mathbb{C}\{0\}$, $\text{Im}\lambda \leq 0$, $V(x)$ is a real valued measurable function defined in $\mathbb{R}^2$.

In this paper we assume the time-independent potential $V(x)$ satisfying the following three hypotheses.

(H1) The real valued potential $V(x)$ is of the $C^1$ class on $\mathbb{R}^2$ and satisfies the decay estimate $|V(x)| + |x \cdot \nabla V(x)| \leq \frac{c}{<x>^\beta}$, where $c > 0$ and $\beta > 3$;

(H2) The potential $V(x)$ is non-negative;

(H3) Zero is a regular point.

We notice that the operator $\Delta_V = \Delta - V(x)$ is self-adjoint one by the assumption (H1). The assumption (H2) and the spectral theorem guarantee that the spectrum of $-\Delta_V \subset [0, \infty)$. The short range decay assumption (H1) implies that $-\Delta_V$ has no positive eigenvalues due to Agmon’s result in [2]. Combining this fact, the assumption (H3) and Theorem 6.1 in [11], we see that the spectrum of $-\Delta_V \subset [0, \infty)$ is absolutely continuous (as it was deduced also in [31]).

The assumption (H3) is not always necessary. We can see in Appendix II that stronger decay of the potential $V(x)$ with $\beta > 10$ in (H1) can guarantee that zero is a regular point provided $V \geq 0$ by Theorem 6.2 in [20]. Another situation, (H3) and appropriate resolvent estimates are obtained by Theorem 8.2 and Remark 9.2 in [28] under the additional assumption $\partial_r(rV(x)) \leq 0$.

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The importance of self-adjointness of quantum Hamiltonians has been shown, since the work of Von Neumann about 1930 (see [33]). After the Gross-Pitaevskii equation was presented in 1960s, many crucial problems in quantum mechanics can be reduced to the study of (1.1). However, there is few research about the asymptotic behavior of solutions for the nonlinear Schrödinger equation (1.1) (see [11], [13], [25], [27]).

In the case of V(x) ≡ 0, it is well known that p = 1 + \frac{2}{n} can be regarded as a borderline of the short range and long range interactions to the equation (1.1) (see [26], [36], [35] and [6]). The existence of modified wave operators of the cubic Schrödinger operators was first studied by Yajima in [37]. In [27] Mizumachi studied the asymptotic stability of a small solitary wave to the nonlinear Schrödinger equation (1.1) with V(x) ≡ 0 and λ ∈ \mathbb{R} \setminus \{0\} in \mathbb{R}. The existence of modified wave operators to the equation (1.1) with V(x) ≡ 0 and λ ∈ \mathbb{R} \setminus \{0\} in space dimensions n ≥ 4 was considered by Hayashi, Li and Naumkin in [17]. They obtained the two side sharp time decay estimates of solutions in the uniform norm. There have been some research about decay estimates of solutions to the subcritical nonlinear Schrödinger equation (1.1) with V(x) ≡ 0 and Imλ < 0 for arbitrarily large initial data (see e.g. [21] and [23]). Segawa, Sunagawa and Yasuda considered a sharp lower bound for the lifespan of small solutions to the subcritical Schrödinger equation (1.1) with V(x) ≡ 0 and Imλ > 0 in the space dimension n = 1, 2, 3 in [30]. For the systems of nonlinear Schrödinger equations, the existence of modified wave operators to a quadratic system in \mathbb{R}^2 was studied in [16], and initial value problem for a cubic derivative system in \mathbb{R} was investigated in [24].

When V(x) ≠ 0, the existence of wave operators for three dimensional Schrödinger operators with singular potentials was proved by Georgiev and Ivanov in [12]. Georgiev and Velichkov studied decay estimates for the nonlinear Schrödinger equation (1.1) with p > \frac{5}{3} in \mathbb{R}^3 in [13]. In [14], Cuccagna, Georgiev and Visciglia considered decay and scattering of small solutions to the nonlinear Schrödinger equation (1.1) with p > 3 in \mathbb{R}. Li and Zhao proved decay and scattering of solutions for the nonlinear Schrödinger equation (1.1) with 1 + \frac{2}{n} < p ≤ 1 + \frac{2}{n-2} in \mathbb{R}^3, when the space dimension n ≥ 3. Lp-boundedness of wave operators for two dimensional Schrödinger operators was first studied by Yajima in [37]. In [27] Mizumachi studied the asymptotic stability of a small solitary wave to the nonlinear Schrödinger equation (1.1) with p ≥ 3 in \mathbb{R}^2. As far as we know, the time decay and scattering problem for the supercritical nonlinear Schrödinger equations (1.1) with p > 2 in \mathbb{R}^2 has not been shown. In this paper, our aim is to study the time decay and scattering problem for (1.1) with V(x) under the assumptions (H1)−(H3) for p > 2.

We now introduce some notations. \mathcal{L}(\mathbb{R}^n) denotes usual Lebesgue space on \mathbb{R}^n for 1 ≤ p ≤ ∞. For m, s ∈ \mathbb{R}, weighted Sobolev space \mathcal{H}^{m,s}(\mathbb{R}^n) is defined by

\mathcal{H}^{m,s}(\mathbb{R}^n) = \left\{ f ∈ \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\mathcal{H}^{m,s}(\mathbb{R}^n)} = \| (1 + |x|^2)^{\frac{s}{2}} (I - \Delta)^{\frac{m}{2}} f \|_{L^2(\mathbb{R}^n)} < \infty \right\}.

We write \mathcal{H}^{m,0}(\mathbb{R}^n) = \mathcal{H}^{m}(\mathbb{R}^n) for simplicity. For s ≥ 0, the homogeneous Sobolev spaces are denoted by

\dot{\mathcal{H}}^{s,0}(\mathbb{R}^n) = \left\{ f ∈ \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{\mathcal{H}}^{s,0}(\mathbb{R}^n)} = \| (-\Delta)^{\frac{s}{2}} f \|_{L^2(\mathbb{R}^n)} < \infty \right\}.
and
\[ \mathcal{H}^{0,s}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\mathcal{H}^{0,s}(\mathbb{R}^n)} = \|x^s f\|_{L^2(\mathbb{R}^n)} < \infty \right\}. \]

For \( 1 \leq p \leq \infty \) and \( s > 0 \), we denote the space \( L^{p,s} \) with the norm
\[ \|f\|_{L^{p,s}} = \|x >^s f\|_{L^p(\mathbb{R}^n)}. \]

We define the dilation operator by
\[ (D_t \phi)(x) = \frac{1}{(it)^n} \phi \left( \frac{x}{t} \right) \]
for \( t \neq 0 \) and define \( M(t) = e^{-\frac{t}{2}|x|^2} \) for \( t \neq 0 \). Evolution operator \( U(t) \) is written as
\[ U(t) = M(-t)D_t FM(-t), \]
where \( F \) and \( F^{-1} \) denote the Fourier transform and its inverse respectively. The standard generator of Galilei transformations is given as
\[ J(t) = U(t)xU(-t) = x + it\nabla, \]
which is also represented as
\[ J(t) = M(-t)it\nabla M(t) \]
for \( t \neq 0 \). Fractional power of \( J(t) \) is defined as
\[ |J|^a(t) = U(t)|x|^aU(-t), \]
which is also represented as (see [19])
\[ |J|^a(t) = M(-t)(-it^2\Delta)^{\frac{a}{2}} M(t) \]
for \( t \neq 0 \). Moreover we have commutation relations with \( |J|^a \) and \( L = it\partial_t + \frac{1}{2}\Delta \) such that \( [L, |J|^a] = 0 \). In what follows, we denote several positive constants by the same letter \( C \), which may vary from one line to another. If there exists some constant \( C > 0 \) such that \( A \leq CF \), we denote this fact by “\( A \lesssim F \)”. Similarly, “\( A \sim F \)” means “\( A \lessgtr F \)” and “\( F \lesssim A \)”. Let \( A \) be a linear operator from Banach space \( X \) to Banach space \( Y \). We denote the operator norm of \( A \) by \( \|A\|_X \rightarrow Y \).

Our main theorem is stated as follows:

**Theorem 1.1.** Assume that \( V(x) \) satisfies (H1) – (H3). Let \( p > 2 \). Then there exist constants \( c_0 > 0 \) and \( C_0 > 0 \) such that for any \( \epsilon \in (0, c_0) \) and \( \|u_0\|_{H^\alpha(\mathbb{R}^2) \cap H^{0,s}(\mathbb{R}^2)} \leq \epsilon \), where \( 1 < \alpha < 2 \), the solution \( u \) to (1.1) satisfies the time decay estimates
\[ \|u\|_{L^\infty(\mathbb{R}^2)} \leq C\alpha^{-1} \]
for \( t \geq 1 \). Moreover there exists \( u_+ \in L^2(\mathbb{R}^2) \) such that
\[ \lim_{t \to \infty} \|u(t) - e^{it^2\Delta} u_+\|_{L^2(\mathbb{R}^2)} = 0. \]

To prove Theorem 1.1, we introduce the operators \( |J|^a \) and \( A(s) \) derived from some commutation relations. The properties of operators \( |J|^a \) and \( A(s) \) are shown in Section 2. We present Strichartz estimates by Proposition 3.2 and Proposition 3.3 in Section 3. We have
\[ \|A(s)u\|_{L^4(\mathbb{R}^2)} \lesssim \|u\|_{L^{4'}(\mathbb{R}^2)} \]
Proposition 2.1. Given in the following two propositions.

4, where \( \frac{1}{q} + \frac{1}{s} = 1 \). Then we show

\[
(1.5) \quad \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)}
\]

for all \( 0 \leq s < 1 \) (see Lemma 5.4) and

\[
(1.6) \quad \left\| (-\Delta)^{\frac{s}{2}} f - (-\Delta)^{\frac{s}{2}} \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \| f \|_{L^{1+\frac{s}{2}}(\mathbb{R}^2)}
\]

for all \( 1 \leq s < 2 \) and \( 0 < \sigma < 1 \) (see (5.13) in Lemma 5.3) in Section 5. We prove our main theorem by using Strichartz estimates, (1.4) and (1.6) in Section 6. In Section 7, we give the proofs of properties of operators \(|J_V|^s \) and \(A(s)\). We show that zero is not resonance in Section 8.

2. Operators \(|J_V|^s \) and \(A(s)\)

We will introduce the operators \(|J_V|^s \) and \(A(s)\) to consider appropriate Sobolev norms and to study the asymptotic behavior of solutions to the equation (1.1).

Setting \( M(t) = e^{-\frac{\tau}{t} |x|^2} \), we may define \(|J_V|^s(t) := M(-t) (-t^2 \Delta)^{\frac{s}{2}} M(t)\). We shall use the standard notation \([B, D] = BD - DB\) for the commutator of two operators \(B\) and \(D\). The key commutator properties of the operator \(|J_V|^s(t)\) are given in the following two propositions.

**Proposition 2.1.** Let \( A(s) := s (-\Delta)^{\frac{s}{2}} + \left[ x \cdot \nabla, (-\Delta)^{\frac{s}{2}} \right] \). For \( s > 0 \), we have

\[
(2.1) \quad \left[ i\partial_t + \frac{1}{2} \Delta V, |J_V|^s(t) \right] = it^{s-1}M(-t)A(s)M(t)
\]

in two space dimensions.

We also have

**Proposition 2.2.** Let \( W := 2V + x \cdot \nabla V \). For \( 0 < s < 2 \), we obtain

\[
(2.2) \quad A(s) = c(s) \int_{0}^{\infty} \tau^{\frac{s}{2}} (\tau - \Delta V)^{-1} W (\tau - \Delta V)^{-1} d\tau
\]

in two space dimensions, where \( c(s)^{-1} = \int_{0}^{\infty} \tau^{\frac{s}{2}-1}(\tau + 1)^{-1}d\tau \).

Proposition 2.1 and Proposition 2.2 are well-known from [11] for the case of one-dimensional Schrödinger equation (1.1) with a potential. For the convenience of readers, we give proofs of these propositions in the appendix I of this paper.

3. Strichartz Estimates

Strichartz estimates are important tools to investigate asymptotic behavior of solutions to some evolution equations, such as Schrödinger equations and wave equations. The well known homogeneous Strichartz estimate

\[
\left\| e^{\frac{\tau}{t} \Delta} f \right\|_{L^{p_2}(\mathbb{R}; L^{q_2}(\mathbb{R}^n))} \lesssim \| f \|_{L^2(\mathbb{R}^n)}
\]

and inhomogeneous Strichartz estimate

\[
\left\| \int_{s < t} e^{\frac{\tau}{t} (t-s) \Delta} F(s, \cdot) ds \right\|_{L^{p_2}(\mathbb{R}; L^{q_2}(\mathbb{R}^n))} \lesssim \| F \|_{L^{p_1}(\mathbb{R}; L^{q_1}(\mathbb{R}^n))}
\]
hold for $n \geq 2$, $f \in L^2(\mathbb{R}^n)$, and $F \in L^{p_j'}(\mathbb{R}; L^{q_j'}(\mathbb{R}^n))$ if $\frac{2}{p_j} + \frac{n}{q_j} = \frac{2}{q_j} \leq 2 \leq p_j \leq \infty$, $2 \leq q_j \leq \frac{2n}{n-2}$, $q_j \not= \infty$, $p_j', q_j'$ are the dual exponents of $p_j$ and $q_j$, $j = 1, 2$ (see e.g. [22]). We note that both endpoints $(p_j, q_j) = (\infty, 2)$ and $(p_j, q_j) = (2, \frac{2n}{n-2})$ are included in the situation of $n \geq 3$, and only the endpoint $(p_j, q_j) = (\infty, 2)$ is included in the case of $n = 2$ for $j = 1, 2$.

In recent years, a large number of works on Strichartz estimates for Schrödinger equations with potentials $V(x)$ have been investigated (see e.g. [10], [9], [13], [3], [4], [27], [28], [31]). However, the study of Strichartz estimates for 2d Schrödinger equations is essentially restricted to the cases of smallness of the magnetic potential and electric potential (see [34]), smallness of the magnetic potential while the electric potential can be large (see [3]), very fast decay of the potential and assumption that zero is a regular point (see [27]), or $V \geq 0$ and $\partial_t(rV) \leq 0$ (see [28]). In [9], Strichartz estimates for Schrödinger equations with the inverse-square potential $\frac{1}{r^2}$ in two space dimensions were considered by Burq, Planchon, Stalker and Tahvildar-Zadeh, where $a$ is a real number. In [27], Mizumachi presented Strichartz estimates by the $L^\infty - L^1$ estimates in [31]. To state the dispersive estimate in [31], we recall the notion zero is a regular point as follow:

**Definition 3.1.** (see [31]) Let $V \not= 0$ and set $U = \text{sign} V, v = |V|^{\frac{2}{n}}$. Let $P_v$ be the orthogonal projection onto $v$ and set $Q = I - P_v$. And let

$$
(G_0 f)(x) := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| f(y) dy.
$$

We say that zero is a regular point of the spectrum of $-\Delta V$, provided $Q(U + vG_0 v)Q$ is invertible on $QL^2(\mathbb{R}^2)$.

We have

**Proposition 3.1.** (Dispersive Estimate in [31]) Let $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a measurable function such that $|V(x)| \leq C(1 + |x|)^{-\beta}, \beta > 3$. Assume in addition that zero is a regular point of the spectrum of $-\Delta V$. Then we have

$$
\|e^{-\frac{\pi}{4}t\Delta V} P_{ac}(H)f\|_{L^\infty(\mathbb{R}^2)} \lesssim |t|^{-1}\|f\|_{L^1(\mathbb{R}^2)}
$$

for all $f \in L^1(\mathbb{R}^2)$.

The requirement that zero is a regular point is the analogue of the usual condition that zero is neither an eigenvalue nor a resonance (generalized eigenvalue) of $-\Delta V$. Under the assumptions of Proposition[31], the spectrum of $-\Delta V$ on $[0, \infty)$ is purely absolutely continuous, and that the spectrum is pure point on $(-\infty, 0)$ with at most finitely many eigenvalues of finite multiplicities (See [31]). Moreover, any point on the real line different from zero is not a resonance due to the results in [14]. Therefore, unique candidate for resonant point is the origin and the assumption zero is regular means that zero is not resonance too.

Next, we need the definition of admissible couples appearing in the Schr"odinger estimates. The couple $(p, q)$ of positive numbers $p \geq 2, q \geq 2$ is called Schr"odinger admissible if it satisfies

$$
\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, (p, q) \not= (2, \infty).
$$

We have the following homogeneous Strichartz estimate by Proposition[31], Theorem 6.1 in [1], and the methods in [22]. We omit the proof.
Proposition 3.2. (Homogeneous Strichartz Estimate) Let \((p, q)\) be a Schrödinger admissible pair. If (H1) – (H3) are satisfied, then we obtain

\[
\left\| e^\frac{\tau}{2} \Delta \varphi \right\|_{L^p(\mathbb{R}; L^q(\mathbb{R}^2))} \lesssim \| \varphi \|_{L^2(\mathbb{R}^2)}
\]

holds for all \(\varphi \in L^2(\mathbb{R}^2)\).

By using Proposition 3.2 and a result of Christ-Kiselev lemma (Lemma A.1 in [8]), we have the following result. We skip the proof here.

Proposition 3.3. (Inhomogeneous Strichartz Estimate) Let \(a, b \in \mathbb{R}\) and let \((p_j, q_j)\) be Schrödinger admissible pairs for \(j = 1, 2\). Assume \(V(x)\) satisfy the hypotheses (H1) – (H3). Then we have

\[
\left\| \int_a^t e^\frac{\tau}{2} (t-s) \Delta \varphi \, d\tau \right\|_{L^p(\mathbb{R}; L^q(\mathbb{R}^2))} \lesssim \| \varphi \|_{L^p(\mathbb{R}; L^q(\mathbb{R}^2))},
\]

\[
\forall \varphi \in L^p(\mathbb{R}; L^q(\mathbb{R}^2)) \cap L^{p'_j}(\mathbb{R}^2),
\]

where \(p'_j, q'_j\) are the dual exponents of \(p_j, q_j, j = 1, 2\).

4. The estimates of \(A(s)\)

To derive estimates of \(A(s)\), we use free resolvent estimates, following the approach of [26].

Lemma 4.1. (Free Resolvent Estimates)

i): For any \(1 < q < \infty, \ 0 < s_0 \leq 1\), one can find \(C = C(q, s_0) > 0\) so that for any \(\tau > 0\) we have

\[
\left\| (\tau - \Delta)^{-1} f \right\|_{L^q(\mathbb{R}^2)} \leq C \tau^{-s_0} \| f \|_{L^q(\mathbb{R}^2)}, \quad \frac{1}{k} = \frac{1}{q} + 1 - s_0;
\]

ii): For any

\[
1 < q < \infty, \ 0 < s_0 \leq 1, \ a > 2(1 - s_0),
\]

one can find \(C = C(q, s_0, a) > 0\) so that for any \(\tau > 0\) we have

\[
\left\| (\tau - \Delta)^{-1} < x >^{-a} f \right\|_{L^q(\mathbb{R}^2)} \leq C \tau^{-s_0} \| f \|_{L^q(\mathbb{R}^2)};
\]

iii): For any

\[
1 < q < \infty, \ 0 < s_0 \leq 1, \ a > 2(1 - s_0),
\]

one can find \(C = C(q, s_0, a) > 0\) so that for any \(\tau > 0\) we have

\[
\left\| < x >^{-a} (\tau - \Delta)^{-1} f \right\|_{L^q(\mathbb{R}^2)} \leq C \tau^{-s_0} \| f \|_{L^q(\mathbb{R}^2)}.
\]

Proof. To prove (4.1), we take advantage of the fact that the Green function

\[
G(x - y; \tau) = (\tau - \Delta)^{-1} (x - y)
\]

of the operator \((\tau - \Delta)^{-1}\) can be computed explicitly, indeed we have

\[
G(x; \tau) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{-ix\xi} \frac{d\xi}{\tau + |\xi|^2} = (2\pi)^{-1} K_0(\sqrt{\tau} |x|),
\]
where $K_0(r)$ is the modified Bessel function of order 0. We have the following estimates of $K_0(r)$,

\begin{equation}
|K_0(r)| \lesssim \begin{cases} 
|\ln r|, & \text{if } 0 \leq r \leq 1; \\
\frac{1}{r}, & \text{if } r > 1.
\end{cases}
\end{equation}

This estimate implies $K_0(|x|) \in L^m(\mathbb{R}^2)$ for any $m \in [1, \infty)$. In this way we deduce

\[
\int_{\mathbb{R}^2} |G(x; \tau)|^m \, dx = (2\pi)^{-m} \int_{\mathbb{R}^2} |K_0(\sqrt{\tau}|x|)|^m \, dx = \frac{(2\pi)^{-m}}{\tau} \int_{\mathbb{R}^2} |K_0(|y|)|^m \, dy = \frac{c_1}{\tau},
\]

where $y = \sqrt{\tau}|x|$ and we can write

\begin{equation}
\|K_0(\sqrt{\tau})\|_{L^m(\mathbb{R}^2)} = \frac{c_2}{\tau^{1/m}}, \quad \forall m \in [1, \infty).
\end{equation}

Applying the Young inequality

\[
\left\| (\tau - \Delta)^{-1} f \right\|_{L^q(\mathbb{R}^2)} = \left\| K_0(\sqrt{\tau}) \ast f \right\|_{L^q(\mathbb{R}^2)} \leq \left\| K_0(\sqrt{\tau}) \right\|_{L^m(\mathbb{R}^2)} \|f\|_{L^k(\mathbb{R}^2)},
\]

where $1 + 1/q = 1/m + 1/k$, combining with \eqref{eq:4.5} and choosing $s_0 = 1/m$, we deduce \eqref{eq:4.1}. So the assertion i) is verified.

To get \eqref{eq:4.2}, we apply the estimate \eqref{eq:4.1}

\[
\left\| (\tau - \Delta)^{-1} < x >^{-a} f \right\|_{L^q(\mathbb{R}^2)} \lesssim \tau^{-s_0} \left\| < x >^{-a} f \right\|_{L^k(\mathbb{R}^2)},
\]

and via the Hölder inequality

\[
\left\| < x >^{-a} f \right\|_{L^k(\mathbb{R}^2)} \lesssim \| f \|_{L^{s_0}(\mathbb{R}^2)}, a > 2 \left( \frac{1}{k} - \frac{1}{q} \right)
\]

and we arrive at \eqref{eq:4.2}.

Finally \eqref{eq:4.3} follows from \eqref{eq:4.2} by a duality argument.

This completes the proof. \hfill \square

\begin{remark}
The estimates \eqref{eq:4.1}, \eqref{eq:4.2} are not valid for $q = \infty$, $s_0 = 0$, $k = 1$, but they are valid for

\[1 \leq q \leq \infty, \quad 0 < s_0 \leq 1, \quad \frac{1}{k} = \frac{1}{q} + 1 - s_0.\]

In particular, they are true for $q = k = 1, s_0 = 1$.

Further, \eqref{eq:4.3} holds when $1 < q \leq \infty$ as in Lemma \ref{lem:4.1}, but also in the following case

\[q = 1, \quad 0 < s_0 \leq 1, \quad a > 2(1 - s_0).\]

In Proposition \ref{prop:2.2}, for $0 < s < 2$ we have

\begin{equation}
A(s) = c(s) \int_0^\infty \tau^{\frac{3}{2}} (\tau - \Delta_V)^{-1} W (\tau - \Delta_V)^{-1} \, d\tau,
\end{equation}

where $W = 2V + x \cdot \nabla V$.

Let $G(t, x, y) = e^{t\Delta_V}(x, y)$ be the heat kernel of the Schrödinger operator $-\Delta_V$, i.e. it solves

\[
\begin{aligned}
\partial_t G &= (\Delta - V)G, \\
G(0, x, y) &= \delta(x - y),
\end{aligned}
\]
where \( y \in \mathbb{R}^2 \). Similarly,
\[
e^{t(\Delta)}(x, y) = c_4 t^{-1} \exp \left\{ -\frac{|x - y|^2}{4t} \right\}
\]
is the heat kernel of \(-\Delta\), so that
\[
e^{\alpha t(\Delta)}(x, y) = c_4 t^{-1} \exp \left\{ -\frac{|x - y|^2}{4\alpha t} \right\}, \quad \forall \alpha > 0.
\]

Since we consider the case \( V(x) \geq 0 \), one can use Feynman-Kac formula and the results in [32] deduce the heat kernel estimate
\[
0 \leq e^{t(\Delta)V}(x, y) \lesssim t^{-1} \exp \left\{ -\frac{|x - y|^2}{4\beta t} \right\},
\]
(4.7)

where \( \beta > 0 \). Using (4.7), we get the following estimate

**Lemma 4.2.** Assume that the hypotheses (H1) and (H2) are satisfied. Then there exists positive \( \beta \) such that
\[
0 \leq e^{t(\Delta)V}(x, y) \lesssim e^{\beta t(\Delta)}(x, y).
\]
(4.8)

Without assumption \( V \geq 0 \) one can use the estimates from [5] and deduce only the estimate
\[
|e^{t(\Delta)V}(x, y)| \lesssim e^{\gamma t(\Delta)}(x, y)e^{\omega t},
\]
where \( \gamma > 0 \) and \( \omega \) depends on \( V(x) \). This is not sufficient for our goal to control the solution to (1.1) with a potential \( V(x) \).

Further, we get the following Lemma

**Lemma 4.3.** Let \( V(x) \) satisfy (H1) and (H2). We have
\[
\|A(s)f\|_{L^q(\mathbb{R}^2)} \lesssim \|f\|_{L^q(\mathbb{R}^2)}
\]
for \( 1 \leq q \leq 2 \) and \( 1 < s < 2 \), where \( \frac{1}{q} + \frac{1}{q'} = 1 \).
(4.10)

**Proof.** By \((\tau + a)^{-1} = f_0^\infty e^{-(\alpha + \tau)t}dt\), we obtain
\[
(\tau - \Delta V)^{-1} f_\pm = \int_{0}^{\infty} e^{-(\tau - \Delta V)t} f_\pm dt
\]
\[
= \int_{0}^{\infty} e^{-\tau t} e^{t\Delta V} f_\pm dt.
\]
(4.11)

Since \( e^{t(\Delta)V}(x, y) \) is the heat kernel of \(-\Delta V\), then we have
\[
e^{t(\Delta)V} f_\pm(x) = \int_{\mathbb{R}^2} e^{t(\Delta)V}(x, y) f_\pm(y) dy.
\]
(4.12)

By the estimate (4.8) in Lemma 4.2 and (4.12), from (4.11) there exist positive \( \beta \) such that
\[
\left| (\tau - \Delta V)^{-1} f_\pm \right| \lesssim \int_{0}^{\infty} e^{-\tau t} e^{\beta t} f_\pm dt
\]
\[
\lesssim \left( \frac{\tau}{\beta - \Delta} \right)^{-1} f_\pm.
\]
(4.13)
Given any \( q \in [1, 2] \) we can apply Proposition 2.2 and via the Hölder inequality to get

\[
\| A(s)f \|_{L^q(\mathbb{R}^2)} \lesssim \int_0^\infty \tau^{\frac{q}{2}} \left\| (\tau - \Delta)^{-1} W(\tau - \Delta)^{-1} f \right\|_{L^q(\mathbb{R}^2)} d\tau
\]

\[
\lesssim \left( \| f_+ \|_{L^{q'}(\mathbb{R}^2)} + \| f_- \|_{L^{q'}(\mathbb{R}^2)} \right) \| W \|_{L^{\omega(q)}(\mathbb{R}^2)}
\]

\[
\times \int_1^\infty \tau^{\frac{q}{2}} \left\| \left( \frac{\tau}{\beta} - \Delta \right)^{-1} \right\|_{L^q(\mathbb{R}^2) \to L^q(\mathbb{R}^2)} d\tau
\]

\[
\times \left\| \left( \frac{\tau}{\beta} - \Delta \right)^{-1} \right\|_{L^{q'}(\mathbb{R}^2) \to L^{q'}(\mathbb{R}^2)} d\tau
\]

\[
+ \left( \| f_+ \|_{L^{q'}(\mathbb{R}^2)} + \| f_- \|_{L^{q'}(\mathbb{R}^2)} \right) \| f \|_{L^{\omega(q)}(\mathbb{R}^2)} \| W \|_{L^{\omega(q)}(\mathbb{R}^2)}
\]

\[
\times \int_0^1 \tau^{\frac{q}{2}} \left\| \left( \frac{\tau}{\beta} - \Delta \right)^{-1} \right\|_{L^q(\mathbb{R}^2) \to L^q(\mathbb{R}^2)} d\tau
\]

\[
\times \left\| \left( \frac{\tau}{\beta} - \Delta \right)^{-1} \right\|_{L^{q'}(\mathbb{R}^2) \to L^{q'}(\mathbb{R}^2)} d\tau,
\]

where \( \omega = \omega(q) \) is determined by \( \frac{1}{\omega} = \frac{2}{q} - 1 \) and \( a = a(q) \) is an appropriate parameter to be chosen so that we can apply Lemma 4.1 (with \( s_0 = 1 \) in (4.11), \( s_1 = 3/4 \) in (4.12) and (4.13)), i.e. we have to require

\[
\frac{1}{2} < a(q) < 1 + \frac{\beta}{2} - \frac{2}{q},
\]

where \( \beta > 3 \) is from our assumption (H1). Then we can write

\[
\| A(s)f \|_{L^q(\mathbb{R}^2)} \lesssim \| f \|_{L^{q'}(\mathbb{R}^2)} \| W \|_{L^{\omega(q)}(\mathbb{R}^2)} \int_1^\infty \tau^{\frac{q}{2}}^{-2} d\tau
\]

\[
+ \| f \|_{L^{q'}(\mathbb{R}^2)} \| f \|_{L^{\omega(q)}(\mathbb{R}^2)} \int_0^1 \tau^{\frac{q}{2}}^{-2} d\tau
\]

\[
\lesssim \| f \|_{L^{q'}(\mathbb{R}^2)} \| W \|_{L^{\omega(q)}(\mathbb{R}^2)} + \| f \|_{L^{q'}(\mathbb{R}^2)} \| f \|_{L^{\omega(q)}(\mathbb{R}^2)} \| W \|_{L^{\omega(q)}(\mathbb{R}^2)}.
\]

Note that our choice guarantees that we have

\[
\| x > 2a \|_{L^\infty(\mathbb{R}^2)} \lesssim \| x \|_{L^{2a}(\mathbb{R}^2)} + \sum_{j=1}^2 \| x_j \|_{L^{2a}(\mathbb{R}^2)} \leq C.
\]

So the assertion is proved.

\[ \square \]

**Remark 4.2.** By using the similar method as Lemma 4.3, we also have

\[
\| A(s)f \|_{L^q(\mathbb{R}^2)} \lesssim \| f \|_{L^q(\mathbb{R}^2)}
\]

for \( 1 \leq q \leq \infty \) and \( 1 < s < 2 \).
5. Equivalence of \((-\Delta V)^{\frac{s}{2}}\) and \((-\Delta)^{\frac{s}{2}}\) in \(L^2(\mathbb{R}^2)\) norm sense

To estimate \(\|J_V^{\gamma}(|u|^p-u)|\|_{L^2(\mathbb{R}^2)}\) which will be mentioned below, we study the operator \((-\Delta V)^{\frac{s}{2}}\) via heat kernels of some Schrödinger operators \(-\Delta_V\) and \(-\beta\Delta\) on \(\mathbb{R}^2\), where \(\beta > 0\). By Lemma 4.2, we obtain the following lemma.

**Lemma 5.1.** Assume that the hypotheses (H1) and (H2) are satisfied. For \(s \geq 0\), we have

\[
\left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| (-\Delta V)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)}.
\]

**Proof.** Obviously, (5.1) holds in the case of \(s = 0\). We show our attention to the situation of \(s > 0\). We show that

\[
\left\| (-\Delta V)^{\frac{s}{2}} (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}
\]

for \(s > 0\).

Using

\[
a^{-\frac{s}{2}} = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty e^{-at} t^{\frac{s}{2}-1} dt
\]

for \(s > 0\), we have

\[
\left( (-\Delta V)^{-\frac{s}{2}} g \right)(x) = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty e^{\Delta_V t} g(x) t^{\frac{s}{2}-1} dt.
\]

Since (H1) and (H2) say \(-\Delta_V\) is a positive self-adjoint operator on \(L^2(\mathbb{R}^2)\), then we have for every \(t > 0\), \(e^{\Delta_V t}\) has a jointly continuous integral kernel \(e^{\Delta_V t}(x, y)\). Thus we have

\[
e^{\Delta_V t}g(x) = \int_{\mathbb{R}^2} e^{\Delta_V t}(x, y)g(y)dy.
\]

By the estimate 4.5 in Lemma 4.2, we have

\[
0 \leq e^{\Delta_V t}g(x) = \int_{\mathbb{R}^2} e^{t\beta \Delta}(x, y)g(y)dy \lesssim e^{\beta t\Delta}g(x)
\]

for \(g \geq 0\), where \(e^{\beta t \Delta}(x, y)\) is the heat kernel of the Schrödinger operator \(-\beta \Delta, \beta > 0\). Then we have from (5.3) and (5.4)

\[
(-\Delta V)^{-\frac{s}{2}} g(x) \lesssim (-\Delta)^{-\frac{s}{2}} g(x)
\]

for \(s > 0\) and \(g \geq 0\). Thus we have

\[
(-\Delta V)^{-\frac{s}{2}} (-\Delta)^{\frac{s}{2}} f(x) \lesssim f(x)
\]

for \(s > 0\) and \(f \geq 0\), where \(f = (-\Delta)^{-\frac{s}{2}} g\). Let \(B_s = (-\Delta V)^{-\frac{s}{2}} (-\Delta)^{\frac{s}{2}}\). Decomposing \(f = f_+ - f_-\), \(g = g_+ - g_-\) we can deduce

\[
< B_s f, g > \leq \|f\|_{L^1(\mathbb{R}^2)} \|g\|_{L^2(\mathbb{R}^2)}
\]
for $s > 0$, without requiring $f \geq 0, g \geq 0$. The inequality
\[
\|B_s f\|_{L^2} \lesssim \|f\|_{L^2(\mathbb{R}^2)}
\]
holds for $s > 0$. Therefore we have the estimate (5.2).

\[\square\]

**Remark 5.1.** It is difficult to obtain Gaussian estimates for heat kernel of the Schödinger operator $-\Delta + V(x)$, especially for $V \leq 0$. There are some sharp Gaussian estimates for heat kernel of the Schödinger operator $-\Delta + V(x)$ with $V \geq 0$ (see e.g. [5], [11] and [38]). Especially, the sharp Gaussian Estimates for heat kernel of the Schödinger operator $-\Delta + V(x)$ with nontrivial $V \geq 0$ in $\mathbb{R}^2$ or $\mathbb{R}^1$ fail (see [7]).

**Lemma 5.2.** Let $V(x)$ satisfy (H1) and (H2). For any $0 < s < 2$ and for any $q > 2$, we have
\[
\left\| (-\Delta^s)^{\frac{1}{2}} f - (-\Delta)^{\frac{1}{2}} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^q(\mathbb{R}^2)}.
\]

**Proof.** Since $(\tau + a)^{-1} = \int_0^\infty e^{-(a+\tau)t}dt$, we have
\[
(\tau - \Delta)^{-1} g(x) = \int_0^\infty e^{-\tau t} e^{\Delta t} g(x) dt.
\]
Let $e^{t\Delta V}(x, y)$ be the heat kernel of the Schödinger operator $-\Delta V$. Then we have
\[
e^{\Delta V t} g(x) = \int_{\mathbb{R}^2} e^{t\Delta V}(x, y) g(y) dy.
\]
By Lemma 4.2, we have positive $\beta$ such that
\[
(\tau - \Delta)^{-1} g(x) \lesssim \int_0^\infty e^{-\tau t} e^{\beta t} g(x) dt
\]
\[
\lesssim \left( \frac{\tau}{\beta} - \Delta \right)^{-1} g(x)
\]
for $g \geq 0$. Then we have
\[
\| (\tau - \Delta)^{-1} g \|_{L^2(\mathbb{R}^2)} \lesssim \left\| \left( \frac{\tau}{\beta} - \Delta \right)^{-1} \right\|_{L^2(\mathbb{R}^2)}
\]
without requiring $g \geq 0$.

Now we can use the relation
\[
(-\Delta)^{\frac{1}{2}} f = c(s)(-\Delta V) \int_0^\infty \tau^{\frac{1}{2}} (\tau - \Delta V)^{-1} f d\tau
\]
with
\[
c(s)^{-1} = \int_0^\infty \tau^{\frac{1}{2}} (\tau + 1)^{-1} d\tau
\]
for $0 < s < 2$. Therefore, we have

\[
(-\Delta_V)^{\frac{s}{2}} f = c(s)(-\Delta_V) \int_0^\infty \tau^{\frac{s}{2} - 1} (\tau - \Delta_V)^{-1} f d\tau \\
= c(s)(-\Delta_V) \int_0^\infty \tau^{\frac{s}{2} - 1} [(\tau - \Delta_V)^{-1} - (\tau - \Delta)^{-1}] f d\tau \\
+ c(s)(-\Delta_V) \int_0^\infty \tau^{\frac{s}{2} - 1} (\tau - \Delta)^{-1} f d\tau.
\]

(5.8)

Using the relations

\[
(-\Delta_V) \left[ (\tau - \Delta_V)^{-1} - (\tau - \Delta)^{-1} \right] \\
= -(-\Delta_V) (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} \\
= -(\tau - \Delta_V) (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} + \tau (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} \\
= -(\tau - \Delta)^{-1} + \tau (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1},
\]

we find

\[
(-\Delta_V)^{\frac{s}{2}} f = -c(s) \int_0^\infty \tau^{\frac{s}{2} - 1} V (\tau - \Delta)^{-1} f d\tau + c(s) \int_0^\infty \tau^{\frac{s}{2} - 1} (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} f d\tau \\
- c(s) \Delta \int_0^\infty \tau^{\frac{s}{2} - 1} (\tau - \Delta)^{-1} f d\tau + c(s) \int_0^\infty \tau^{\frac{s}{2} - 1} V (\tau - \Delta)^{-1} f d\tau \\
= (-\Delta)^{\frac{s}{2}} f + c(s) \int_0^\infty \tau^{\frac{s}{2}} (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} f d\tau.
\]

Therefore we obtain

\[
(-\Delta_V)^{\frac{s}{2}} f = (-\Delta)^{\frac{s}{2}} f + c(s) \int_0^\infty \tau^{\frac{s}{2}} (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} f d\tau
\]

(5.9) for $0 < s < 2$.

By (5.7) and the Hölder inequality with $1/q + 1/r = 1/2$, we have

\[
\int_0^\infty \tau^{\frac{s}{2}} (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} f \left\|_{L^2(\mathbb{R}^2)} \right. \right. \\
\lesssim \|f\|_{L^q(\mathbb{R}^2)}\|V\|_{L^r(\mathbb{R}^2)} \int_0^\infty \tau^{\frac{s}{2}} \left\| \left( \frac{\tau}{\beta} - \Delta \right)^{-1} \right\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \left\| (\tau - \Delta)^{-1} \right\|_{L^q(\mathbb{R}^2) \rightarrow L^q(\mathbb{R}^2)} d\tau \\
\times \int_0^1 \tau^{\frac{s}{2}} \left\| \left( \frac{\tau}{\beta} - \Delta \right)^{-1} \left< x >^{-a} \right\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \left\| < x >^{-a} (\tau - \Delta)^{-1} \right\|_{L^q(\mathbb{R}^2) \rightarrow L^q(\mathbb{R}^2)} d\tau,
\]

here $a$ is chosen so that $a > 1 - s/2$. Now taking

\[
a = 1 - \frac{s}{2} + \varepsilon, \quad s_0 = \frac{1}{2} + \frac{s}{4} - \frac{\varepsilon}{4},
\]

where $0 < \varepsilon < \frac{s}{2}$, we can apply Lemma 4.1 since

\[
\frac{a}{2} + s_0 = 1 + \varepsilon \left( \frac{1}{2} - \frac{1}{4} \right) > 1
\]
and get
\[
(5.11) \int_0^{\infty} \| \tau^{\frac{1}{2}} (\tau - \Delta_V)^{-1} V (\tau - \Delta)^{-1} f \|_{L^2(\mathbb{R}^2)} d\tau \leq \| f \|_{L^q(\mathbb{R}^2)} \| V \|_{L^r(\mathbb{R}^2)} \int_1^{\infty} \tau^{\frac{1}{2} - 2s} d\tau + \| f \|_{L^q(\mathbb{R}^2)} \int_0^1 \tau^{\frac{1}{2} - 2s_0} d\tau
\]
\[
\leq \| f \|_{L^q(\mathbb{R}^2)},
\]

since
\[
\frac{s}{2} - 2s_0 = -1 + \frac{\varepsilon}{2} > -1
\]
and
\[
r(3 - 2a) = r(1 + s - 2\varepsilon) > 2.
\]

\[\square\]

**Lemma 5.3.** Let \( V(x) \) satisfy (H1) and (H2). Then we have the estimates

a): for any \( 0 \leq s < 1 \) we have
\[
(5.12) \quad \left\| (-\Delta_V)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)};
\]

b): for any \( 1 \leq s < 2 \), and \( 0 < \sigma < 1 \), we have
\[
(5.13) \quad \left\| (-\Delta_V)^{\frac{s}{2}} f - (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| (-\Delta)^{\frac{\sigma}{2}} f \right\|_{L^2(\mathbb{R}^2)} \left\| f \right\|_{L^2(\mathbb{R}^2)}^{1 - \frac{\sigma}{2}}.
\]

**Proof.** We have (5.12) by Lemma 5.2 and the Sobolev embedding
\[
(5.14) \quad \| f \|_{L^q(\mathbb{R}^2)} \lesssim \| (-\Delta)^{\frac{s}{2}} f \|_{L^2(\mathbb{R}^2)},
\]
where \( \frac{s}{2} = \frac{1}{2} - \frac{1}{q} \) and \( q > 2 \).

Applying the Sobolev embedding (5.14) with \( \sigma = s \), where \( \frac{s}{2} = \frac{1}{2} - \frac{1}{q} \) and \( q > 2 \), and the interpolation inequality
\[
\left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| (-\Delta)^{\frac{\sigma}{2}} f \right\|_{L^2(\mathbb{R}^2)} \left\| f \right\|_{L^2(\mathbb{R}^2)}^{1 - \frac{\sigma}{\sigma}}
\]
with \( \theta = \frac{\sigma}{s} \), we get (5.13) from Lemma 5.2.

\[\square\]

**Remark 5.2.** For any \( 1 < s < 2 \), we have
\[
(5.15) \quad \left\| (-\Delta_V)^{\frac{s}{2}} f - (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \lesssim \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \left\| f \right\|_{L^2(\mathbb{R}^2)}^{1 - \frac{s}{2}},
\]
due to (5.13) and Lemma 5.2.

By Lemma 5.1 and 5.12 in Lemma 5.3, we have the following equivalence property directly.

**Lemma 5.4.** Suppose that (H1) and (H2) are satisfied. For any \( 0 \leq s < 1 \), we have
\[
(5.16) \quad \left\| (-\Delta_V)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} \sim \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)}.
\]

To estimate \( \left\| (-\Delta_V)^{\frac{s}{2}} M(t) (|u|^{p-1} u) \right\|_{L^2(\mathbb{R}^2)} \) with \( 1 < s < 2 \), we need the estimate about \( \| u \|_{L^\infty(\mathbb{R}^2)} \). We consider the following lemma.
Lemma 5.5. Suppose that (H1) and (H2) are satisfied. Then for any $1 < s < 2$, we have

$$
\|f\|_{L^\infty(R^2)} \lesssim \|(-\Delta V)^{\frac{s}{2}} f\|_{L^2(R^2)} \|f\|_{L^2(R^2)}^{1-s} \tag{5.17}
$$

Proof. By the Hölder inequality, we have

$$
\|F\|_{L^1(R^2)} \lesssim \|F\|_{L^2(|\xi| \leq \tau)}
$$

$$
+ \|\xi^s F\|_{L^2(|\xi| \geq \tau)} \|\xi|^{-s} \|F\|_{L^2(|\xi| \geq \tau)}
$$

$$
\lesssim \tau \|F\|_{L^2(R^2)} + \frac{1}{2(s-1)} \tau^{1-s} \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(R^2)} \tag{5.18}
$$

for any $s > 1$ and $\tau > 0$. Let $\tau = \left(\frac{1}{2(s-1)} \right) \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(R^2)}^{\frac{1}{s}} \|f\|_{L^2(R^2)}^{1-s}$. Then we have

$$
\tau \|f\|_{L^2(R^2)} = \left(\frac{1}{2(s-1)} \right)^{\frac{1}{s}} \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(R^2)}^{\frac{1}{s}} \|f\|_{L^2(R^2)}^{1-s}. \tag{5.19}
$$

By (5.18), (5.19) and Lemma 5.1, we have our desired result. \qed

Remark 5.3. By Lemma 5.3 and $|J_V|^a(t) = M(-t)(-t^2 \Delta V)^{\frac{a}{2}} M(t)$, we have

$$
\|f(t, \cdot)\|_{L^\infty(R^2)} \lesssim \|(-\Delta)^{\frac{a}{2}} M(t) f(t, \cdot)\|_{L^2(R^2)} \|M(t) f(t, \cdot)\|_{L^2(R^2)}^{1-\frac{a}{2}} \tag{5.20}
$$

for $1 < s < 2$.

6. Proof of Theorem 1.1

We define the function space $X_T$ as follows

$$
X_T = \left\{ f \in C([1, T]; S') : \|f\|_{X_T} = \|J_V|^\alpha f\|_{L^\infty((1, T], L^2(R^2))} + \sup_{t \in [1, T]} \|f\|_{L^2(R^2)} \right\},
$$

where $T > 1$ and $1 < \alpha < 2$. Since we can obtain the local existence of solutions to the equation (1.1) by the standard contraction mapping principle, we skip the proof in this section. Multiplying both sides of the equation (1.1) by $|J_V|^\alpha$ and using Proposition 2.1, we have

$$
\left( i \partial_t + \frac{1}{2} \Delta V \right) |J_V|^\alpha u = it^{\alpha-1} M(-t) A(\alpha) M(t) u + \lambda |J_V|^\alpha (|u|^{p-1} u). \tag{6.1}
$$

Let $|J_V|^\alpha u = u_\alpha^I + u_\alpha^T$. We consider

$$
\begin{cases}
  (i \partial_t + \frac{1}{2} \Delta V) u_\alpha^I = \lambda |J_V|^\alpha (|u|^{p-1} u), \\
  u_\alpha^I(1) = |J_V|^\alpha (u_0),
\end{cases} \tag{6.2}
$$

and

$$
\begin{cases}
  (i \partial_t + \frac{1}{2} \Delta V) u_\alpha^T = it^{\alpha-1} M(-t) A(\alpha) M(t) u, \\
  u_\alpha^T(1) = 0.
\end{cases} \tag{6.3}
$$
for $p > 2$, and $t \geq 1$, where $u = u(t, x)$ is a real valued unknown function, $x \in \mathbb{R}^2$, $\Delta_V = \Delta - V(x)$, $|J_V|^\alpha(t) = M(-t) \left(-t^2 \Delta_V\right)^{\frac{\alpha}{2}} M(t)$, $M(t) = e^{-\frac{t}{2}|x|^2}$ and $A(\alpha) = \alpha \left(-\Delta_V\right)^{\frac{\alpha}{2}} + \left[x \cdot \nabla, (-\Delta_V)^{\frac{\alpha}{2}}\right]$.

First we consider the integral equation

\[ (6.4) u^I_\alpha = e^{\frac{i}{2} t \Delta_V} e^{-\frac{i}{2} t \Delta_V} |J_V|^\alpha(1) u_0 - i \lambda \int_1^t e^{\frac{i}{2} (t-\tau) \Delta_V} |J_V|^\alpha \left(|u|^{p-1} u\right)(\tau) d\tau \]

associated with (6.2). For simplicity, we let $|J_V|^\alpha \left(|u|^{p-1} u\right) = F_\alpha$. Then from (6.4) we have

\[ (6.5) u^I_\alpha = e^{\frac{i}{2} t \Delta_V} e^{-\frac{i}{2} t \Delta_V} |J_V|^\alpha(1) u_0 - i \lambda \int_1^t e^{\frac{i}{2} (t-\tau) \Delta_V} F_\alpha(\tau) d\tau. \]

We also have

\[ (6.6) u^{II}_\alpha = \int_1^t e^{\frac{i}{2} (t-\tau) \Delta_V} \tau^{\alpha-1} M(-\tau) A(\alpha) M(\tau) u(\tau) d\tau \]

from (6.5). By Proposition 3.2 and Proposition 3.3 from (6.6) we have

\[ (6.7) \| u^I_\alpha \|_{L^\infty([1,T];L^2(\mathbb{R}^2))} \lesssim \| |J_V|^\alpha(1) u_0 \|_{L^2(\mathbb{R}^2)} + \| F_\alpha \|_{L^1([1,T];L^2(\mathbb{R}^2))}, \]

where $F_\alpha = |J_V|^\alpha \left(|u|^{p-1} u\right)$, $|J_V|^\alpha(t) = M(-t) \left(-t^2 \Delta_V\right)^{\frac{\alpha}{2}} M(t)$, and $M(t) = e^{-\frac{t}{2}|x|^2}$.

By (6.10) in Lemma 3.3 and Lemma 3.4 in [13], we obtain

\[ (6.8) \| F_\alpha \|_{L^2(\mathbb{R}^2)} = \| |J_V|^\alpha \left(|u|^{p-1} u\right) \|_{L^2(\mathbb{R}^2)} \]

\[ \lesssim \left( \left\| |J|^\alpha \left(|u|^{p-1} u\right) \|_{L^2(\mathbb{R}^2)} + t^{\alpha-\sigma} \left\| |J|^\alpha \left(|u|^{p-1} u\right) \|_{L^2(\mathbb{R}^2)} \right\|_{L^2(\mathbb{R}^2)} \right) \]

\[ \lesssim \| u \|_{L^{p-1}(\mathbb{R}^2)} \left( \left\| |J|^\alpha \|_{L^2(\mathbb{R}^2)} + t^{\alpha-\sigma} \left\| \left| J \right|^\alpha u \|_{L^2(\mathbb{R}^2)} \right\|_{L^2(\mathbb{R}^2)} \right) \]

for $p > 2$, $1 < \alpha < 2$ and $0 < \sigma < 1$. By Lemma 3.1 from (6.8) we obtain

\[ (6.9) \| F_\alpha \|_{L^2(\mathbb{R}^2)} \lesssim \| u \|_{L^{p-1}(\mathbb{R}^2)} \left( \left\| |J_V|^\alpha u \|_{L^2(\mathbb{R}^2)} + t^{\alpha-\sigma} \left\| |J_V|^\alpha u \|_{L^2(\mathbb{R}^2)} \right\|_{L^2(\mathbb{R}^2)} \right) \]

for $p > 2$, $1 < \alpha < 2$ and $0 < \sigma < 1$. By (5.29) in Remark 5.3 from (6.9) we get

\[ (6.10) \| F_\alpha \|_{L^2(\mathbb{R}^2)} \lesssim \left( \left\| t^{-1} \left| |J_V|^\alpha u \|_{L^2(\mathbb{R}^2)} \right\|_{L^2(\mathbb{R}^2)} \right\|_{L^2(\mathbb{R}^2)} \right)^{p-1} \]

\[ \times \left( \left\| |J_V|^\alpha u \|_{L^2(\mathbb{R}^2)} + t^{\alpha-\sigma} \left\| |J_V|^\alpha u \|_{L^2(\mathbb{R}^2)} \right\|_{L^2(\mathbb{R}^2)} \right) \],
Then we obtain

$$
\frac{\|F_\alpha\|_{L^1([1, T]; L^2(\mathbb{R}^2))}}{16}
\lesssim\|u_0\|_{L^2(\mathbb{R}^2)}\|J\|^{\alpha}u\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))}^{\frac{p-1}{\alpha}}
+\|u_0\|_{L^2(\mathbb{R}^2)}\|J\|^{\alpha}u\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))}^{\frac{p-1}{\alpha}}
$$

for 0 < p < 2 + \alpha - \sigma < 0 for p > 2, where 1 < \alpha < \frac{3}{2}
and \frac{2}{3} < \sigma < 1. By (6.13) in Lemma 3.3 and (6.11), then we have

$$
\|u_\alpha^H\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))} \lesssim \|J\|^{\alpha}(1)u_0\|_{L^2(\mathbb{R}^2)} + \|J\|^{\alpha}(1)u_0\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}
+\|u_0\|_{L^2(\mathbb{R}^2)}\|J\|^{\alpha}u\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))}^{\frac{p-1}{\alpha}}
+\|u_0\|_{L^2(\mathbb{R}^2)}\|J\|^{\alpha}u\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))}^{\frac{p-1}{\alpha}}
$$

for p > 2, where 1 < \alpha < \frac{2}{3} and \frac{2}{3} < \sigma < 1.

By Proposition 3.3, we have from (6.13)

$$
\|u_\alpha^H\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))} \lesssim \|e^{\|\alpha\|\theta}M(-t)A(\alpha)M(t)u\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))}
$$

for 1 < \alpha < 2, where \( M(t) = e^{-\frac{t}{2}x^2}, A(\alpha) = \alpha(-\Delta V)^{\frac{\alpha}{2}} + [x \cdot \nabla, (-\Delta V)^{\frac{\alpha}{2}}] \),
\( \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}, \frac{1}{p_1} + \frac{1}{q_1} = 1, \frac{1}{p_1} + \frac{1}{q_1} = 1 \), and 1 < q_1 < 2. Let 1 < \alpha < \frac{3}{2}. By Lemma 4.3 and the Sobolev inequality

$$
\|u\|_{L^q(\mathbb{R}^2)} \lesssim \|(-\Delta)^{\frac{\alpha}{2}}u\|_{L^p(\mathbb{R}^2)}^{\frac{q}{2}} \|u\|_{L^2(\mathbb{R}^2)}^{\frac{1-q}{2}}
$$

for 0 < \theta < \frac{1}{\alpha}, where q_1 = \frac{2}{\theta - \alpha}, from (6.13) we have

$$
\|u_\alpha^H\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))} \lesssim \|e^{\|\alpha\|\theta}M(-t)A(\alpha)M(t)u\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))}
$$

for 0 < \theta < \frac{1}{\alpha}, where \( \frac{1}{p_1} + \frac{1}{q_1} = 1, \frac{1}{p_1} + \frac{1}{q_1} = 1, \frac{1}{p_1} + \frac{1}{q_1} = \frac{3}{2}, q_1 = \frac{2}{\theta - \alpha} \) and 1 < q_1 < 2. For 1 < \alpha < \frac{3}{2}, we choose \theta \in \left(\frac{2}{3}, \frac{1}{\alpha}\right). Then we have

$$
\|u_\alpha^H\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))} \lesssim \|e^{\|\alpha\|\theta}M(-t)A(\alpha)M(t)u\|_{L^\infty((1, T]; L^2(\mathbb{R}^2))}
$$
\[(\alpha - 1 - \alpha \theta) p' + 1 = -\frac{\alpha(3\theta - 2)}{2 - \alpha \theta} < 0 \text{ for } 1 < \alpha < \frac{3}{2}, \text{ where } \frac{3}{2} < \theta < \frac{1}{\alpha}, \text{ we have} \]

\[(6.15) \quad \|t^{\alpha - 1 - \alpha \theta}\|_{L^p((1, T])} \leq C. \]

By (6.14), (6.15) and Lemma 5.1 we have

\[(6.16) \quad \|u_\alpha\|_{L^\infty([1, T]; L^2(\mathbb{R}^2))} \leq \|u_0\|_{\dot{H}^\alpha(\mathbb{R}^2)} \|J_V|^{\alpha} u\|_{L^\infty([1, T]; L^2(\mathbb{R}^2))}^{\frac{3}{2}} \]

for \(1 < \alpha < \frac{3}{2}. \)

Using (6.12) and (6.16), we have

\[
\|J_V|^{\alpha} u\|_{L^\infty([1, T]; L^2(\mathbb{R}^2))} \leq \|u_\alpha\|_{L^\infty([1, T]; L^2(\mathbb{R}^2))} + \|u_\alpha\|_{L^\infty([1, T]; L^2(\mathbb{R}^2))} \]

\[
\leq \|u_0\|_{H^\alpha(\mathbb{R}^2) \cap \dot{H}^{1, \alpha}(\mathbb{R}^2)} + \|u_0\|_{H^\alpha(\mathbb{R}^2) \cap \dot{H}^{1, \alpha}(\mathbb{R}^2)} \|

\[
\leq \|\|J_V|^{\alpha} u\|\|_{L^\infty([1, T]; L^2(\mathbb{R}^2))}^{\frac{\alpha - 1 + \alpha - 1}{2}} \]

for \(p > 2, \text{ where } 1 < \alpha < \frac{3}{2} \text{ and } \frac{3}{2} < \sigma < 1. \) Then for a fixed \(C > 0 \) we have

\[(6.17) \quad \|J_V|^{\alpha} u\|_{L^\infty([1, T]; L^2(\mathbb{R}^2))} \leq C\|u_0\|_{H^\alpha(\mathbb{R}^2) \cap \dot{H}^{1, \alpha}(\mathbb{R}^2)}, \]

if \(\|u_0\|_{H^\alpha(\mathbb{R}^2) \cap \dot{H}^{1, \alpha}(\mathbb{R}^2)} \) is small enough. By a standard continuity argument and Remark 5.3, we have the time decay estimate (1.2) if \(\epsilon_0 \) is small enough. From (1.1), we have

\[
u(t) = e^{\frac{\pi}{2} |x|^2} \left( e^{-\frac{\pi}{2} |x|^2} u_0 - i \lambda \int_1^t e^{-\frac{\pi}{2} |x|^2} (|x|^{p-1} u)(\tau)d\tau \right). \]

Let \(u_+ = e^{-\frac{\pi}{2} |x|^2} u_0 - i \lambda \int_1^\infty e^{-\frac{\pi}{2} |x|^2} (|x|^{p-1} u)(\tau)d\tau. \) The we have

\[
u(t) = e^{\frac{\pi}{2} |x|^2} u_+ + i \lambda \int_1^\infty e^{-\frac{\pi}{2} |x|^2} (|x|^{p-1} u)(\tau)d\tau. \]

We omit the scattering (1.3) by a standard argument from the time decay estimate (1.2). We omit the proof here.

7. Appendix I

Let \(M(t) = e^{-\frac{\pi}{2} |x|^2} \) and \([B, D] = BD - DB. \) To prove Proposition 2.1 and Proposition 2.2 we consider the following lemmas (see [11]).

Lemma 7.1. We have the following identities:

\[(7.1) \quad [i \partial_t, M(-t)] = \frac{|x|^2}{2t^2} M(-t), \]

and

\[(7.2) \quad [i \partial_t, M(t)] = -\frac{|x|^2}{2t^2} M(t). \]
Proof. Since
\[ [i\partial_t, M(-t)]f = i\partial_t(M(-t)f) - M(-t)i\partial_t f \]
\[ = \frac{|x|^2}{2t^2} M(-t)f, \]
then we obtain the first identity (7.1).

By using the similar method, we get the second identity (7.2).

\( \square \)

Lemma 7.2. We have
\[
[\Delta, M(-t)] = M(-t) \left( \frac{in}{t} - \frac{|x|^2}{t^2} + \frac{2i}{t} x \cdot \nabla t \right),
\]
and
\[
[\Delta, M(t)] = M(t) \left( -\frac{in}{t} - \frac{|x|^2}{t^2} + \frac{2i}{t} x \cdot \nabla t \right),
\]
where \( n \) is the generic space dimension.

Proof. By some calculations, we have
\[
[\Delta, M(-t)]f
= \Delta(M(-t)f) - M(-t)\Delta f
= M(-t)\Delta f + \Delta(M(-t))f + 2\nabla M(-t) \cdot \nabla f - M(-t)\Delta f
= \Delta(M(-t))f + 2\nabla M(-t) \cdot \nabla f
= M(-t) \left( \frac{in}{t} - \frac{|x|^2}{t^2} + \frac{2i}{t} x \cdot \nabla t \right) f.
\]

Taking complex conjugates, we get the second identity (7.4).

\( \square \)

We have the following commutator relations

\( \text{Lemma 7.3.} \)

\[
[i\partial_t + \frac{1}{2} \Delta, M(-t)] = \frac{1}{2} M(-t) \left( \frac{in}{t} + \frac{2i}{t} x \cdot \nabla t \right),
\]
and
\[
[i\partial_t + \frac{1}{2} \Delta, M(t)] = M(t) \left( -\frac{in}{2t} - \frac{|x|^2}{t^2} - \frac{i}{t} x \cdot \nabla t \right),
\]
where \( n \) is the generic space dimension.

Proof. By Lemmas 7.1 and 7.2, we have
\[
[i\partial_t + \frac{1}{2} \Delta, M(-t)] f
= [i\partial_t, M(-t)]f + \frac{1}{2} [\Delta, M(-t)]f
= \frac{|x|^2}{2t^2} M(-t)f + \frac{1}{2} M(-t) \left( \frac{in}{t} - \frac{|x|^2}{t^2} + \frac{2i}{t} x \cdot \nabla t \right) f
= \frac{1}{2} M(-t) \left( \frac{in}{t} + \frac{2i}{t} x \cdot \nabla t \right) f.
\]

By Lemmas 7.1 and 7.2, we also get the commutator relation (7.6).

\( \square \)
Lemma 7.4. Let $\Delta_V = \Delta - V(x)$. For $s \geq 0$, we have

\begin{equation}
\left[i\partial_t + \frac{1}{2}\Delta_V, (-t^2\Delta_V)^\frac{s}{2}\right] = \frac{is}{t} (-t^2\Delta_V)^\frac{s}{2}.
\end{equation}

Proof. By the commutator relation $\left[(-\Delta_V)^\frac{s}{2}, \Delta_V\right] = 0$, we have

\begin{equation}
\left[i\partial_t + \frac{1}{2}\Delta_V, (-t^2\Delta_V)^\frac{s}{2}\right] f = \left[i\partial_t, (-t^2\Delta_V)^\frac{s}{2}\right] f + \frac{1}{2} \left[\Delta_V, (-t^2\Delta_V)^\frac{s}{2}\right] f
\end{equation}

By some simple calculations, we have

\begin{equation}
\left[i\partial_t, (-t^2\Delta_V)^\frac{s}{2}\right] f = i\partial_t \left[(-t^2\Delta_V)^\frac{s}{2} f\right] - i \left[(-t^2\Delta_V)^\frac{s}{2}\right] \partial_t f
\end{equation}

Combining (7.8) and (7.9), we have our desired result. \qed

7.1. Proof of Proposition 2.1. Since $[B, DE] = [B, D]E + D[B, E]$, then we have

\begin{equation}
\left[i\partial_t + \frac{1}{2}\Delta_V, |J\nu|^\sigma(t)\right] f
\end{equation}

\begin{align*}
&= \left[i\partial_t + \frac{1}{2}\Delta_V, M(-t) (-t^2\Delta_V)^\frac{s}{2} M(t)\right] f \\
&= \left[i\partial_t + \frac{1}{2}\Delta, M(-t) (-t^2\Delta_V)^\frac{s}{2} M(t)\right] f \\
&\quad + M(-t) \left[i\partial_t + \frac{1}{2}\Delta_V, (-t^2\Delta_V)^\frac{s}{2} M(t)\right] f.
\end{align*}
By Lemmas 7.3, 7.4 and \([B, DE] = [B, D]E + D[B, E]\), we have

\[
(7.11) \quad \begin{align*}
&\left[i\partial_t + \frac{1}{2} \Delta, M(-t)\right] (-t^2 \Delta_V)^\frac{s}{2} M(t)f \\
&\quad + M(-t) \left[i\partial_t + \frac{1}{2} \Delta_V, (-t^2 \Delta_V)^\frac{s}{2} M(t)\right] f \\
&= \frac{i}{t} |J_V|^s(t) f + \frac{i}{t} M(-t) x \cdot \nabla (-t^2 \Delta_V)^\frac{s}{2} M(t)f \\
&\quad + M(-t) \left[i\partial_t + \frac{1}{2} \Delta_V, (-t^2 \Delta_V)^\frac{s}{2} \right] M(t)f \\
&\quad + M(-t) \left(-t^2 \Delta_V\right)^\frac{s}{2} \left[i\partial_t + \frac{1}{2} \Delta_V, M(t)\right] f \\
&= \frac{i}{t} |J_V|^s(t) f + \frac{i}{t} M(-t) x \cdot \nabla (-t^2 \Delta_V)^\frac{s}{2} M(t)f \\
&\quad + M(-t) \frac{is}{t} (-t^2 \Delta_V)^\frac{s}{2} M(t)f \\
&\quad + M(-t) \left(-t^2 \Delta_V\right)^\frac{s}{2} M(t) \left(-\frac{i}{t} - \frac{|x|^2}{t^2} - i \frac{x \cdot \nabla}{t}\right) f \\
&= \frac{is}{t} |J_V|^s(t) f + \frac{i}{t} M(-t) \left[x \cdot \nabla, (-t^2 \Delta_V)^\frac{s}{2} M(t)\right] f \\
&\quad - M(-t) \left(-t^2 \Delta_V\right)^\frac{s}{2} \frac{|x|^2}{t^2} M(t)f.
\end{align*}
\]

Using \([B, DE] = [B, D]E + D[B, E]\) and \((-t^2 \Delta_V)^\frac{s}{2} [x \cdot \nabla, M(t)] = (-t^2 \Delta_V)^\frac{s}{2} x \cdot (\nabla M(t)) f\), we have

\[
(7.12) \quad \begin{align*}
&\frac{is}{t} |J_V|^s(t) f + \frac{i}{t} M(-t) \left[x \cdot \nabla, (-t^2 \Delta_V)^\frac{s}{2} M(t)\right] f \\
&\quad - M(-t) \left(-t^2 \Delta_V\right)^\frac{s}{2} \frac{|x|^2}{t^2} M(t)f \\
&= it^{s-1} M(-t) [s(-\Delta_V)^\frac{s}{2} M(t)f + \frac{i}{t} M(-t)[x \cdot \nabla, (-t^2 \Delta_V)^\frac{s}{2}] M(t)f \\
&\quad + \frac{i}{t} M(-t)(-t^2 \Delta_V)^\frac{s}{2} [x \cdot \nabla, M(t)] f - M(-t) (-t^2 \Delta_V)^\frac{s}{2} \frac{|x|^2}{t^2} M(t) f \\
&= it^{s-1} M(-t) A(s) M(t)f,
\end{align*}
\]

where \(A(s) = s(-\Delta_V)^\frac{s}{2} + \left[x \cdot \nabla, (-\Delta_V)^\frac{s}{2}\right]\).

Combining (7.10), (7.11) and (7.12), we complete the proof of (2.1).

### 7.2. Proof of Proposition 2.2

Let \(S = x \cdot \nabla\). By the formula

\[
(-\Delta_V)^\frac{s}{2} f = c(s)(-\Delta_V) \int_0^\infty \tau^{\frac{s}{2} - 1} (\tau - \Delta_V)^{-1} f \, d\tau
\]

for \(0 < s < 2\), where \(c(s)^{-1} = \int_0^\infty \tau^{\frac{s}{2} - 1}(\tau + 1)^{-1} d\tau\), we get

\[
(7.13) \quad A(s) = s(-\Delta_V)^\frac{s}{2} + c(s) \int_0^\infty \tau^{\frac{s}{2} - 1}[S, -\Delta_V(\tau - \Delta_V)^{-1}] d\tau.
\]
Using \([B, DE] = [B, D]E + D[B, E]\), we have

\[
\begin{align*}
[S, -\Delta_V (\tau - \Delta_V)^{-1}] &= [S, -\Delta_V (\tau - \Delta_V)^{-1} - \Delta_V [S, (\tau - \Delta_V)^{-1}]] \\
&= [S, -\Delta_V (\tau - \Delta_V)^{-1} + \Delta_V (\tau - \Delta_V)^{-1} [S, -\Delta_V (\tau - \Delta_V)^{-1}].
\end{align*}
\]

Since \(\Delta_V = \Delta - V(x)\), we obtain

\[
[S, -\Delta] + [S, V] = -(x \cdot \nabla) \Delta + \Delta (x \cdot \nabla) + x \cdot \nabla V - V x \cdot \nabla = 2\Delta + SV
\]

By (7.14) and (7.15), we have

\[
\begin{align*}
[S, -\Delta_V (\tau - \Delta_V)^{-1}] &= 2\Delta_V (\tau - \Delta_V)^{-1} + W(\tau - \Delta_V)^{-1} \\
&= 2\Delta_V (\tau - \Delta_V)^{-1} + W(\tau - \Delta_V)^{-1} (2\Delta_V + W)(\tau - \Delta_V)^{-1} \\
&= 2\tau \Delta_V (\tau - \Delta_V)^{-2} + \tau (\tau - \Delta_V)^{-1} W(\tau - \Delta_V)^{-1}.
\end{align*}
\]

By (7.13) and (7.16), we get

\[
\begin{align*}
A(s) &= s (-\Delta_V)^{\frac{3}{2}} + 2c(s) \int_0^\infty \tau^{\frac{3}{2}} \Delta_V (\tau - \Delta_V)^{-2} d\tau \\
&\quad + c(s) \int_0^\infty \tau^{\frac{3}{2}} (\tau - \Delta_V)^{-1} W(\tau - \Delta_V)^{-1} d\tau.
\end{align*}
\]

Since

\[
s (-\Delta_V)^{\frac{3}{2}} = -2c(s) \int_0^\infty \tau^{\frac{3}{2}} \Delta_V (\tau - \Delta_V)^{-2} d\tau
\]

by integrating by parts, we have our desired result from (7.17).

8. Appendix II: zero is not resonance

In this section we can prove the lack of resonance at the origine, i.e. we shall prove that the origin is not resonance point, recalling that the definition of resonance used in [20] and Theorem 6.2 guarantee that zero is a resonance point can be characterized by the existence of solution

\[
\Psi(x) = c_0 + \Psi_0(x), \quad c_0 \in \mathbb{C}, \quad \Psi_0 \in L^q(\mathbb{R}^2), \quad \exists q \in (2, \infty)
\]

to the equation

\[
-\Delta \Psi + V \Psi = 0.
\]

Using Lemma 6.4 and the relation (6.94) in [20], assuming \(\beta > 10\), we can deduce further

\[
\Psi_0(x) = O((\langle x \rangle)^{-1}), \quad \nabla \Psi_0(x) = O((\langle x \rangle)^{-2}).
\]
Rewriting (8.1) in the form

$$-\Delta \Psi_0 + V \Psi = 0,$$

multiplying by $\Psi$ and integrating over $|x| \leq R$, we get

$$\int_{|x| \leq R} |\nabla \Psi_0(x)|^2 dx - c_0 \int_{|x| = R} \partial_r \Psi_0(x) dS_x + \int_{|x| \leq R} V(x) |\Psi(x)|^2 dx = 0.$$

The asymptotics of $\Psi_0, \partial_r \Psi_0$ enables one to take the limit $R \to \infty$ and arrive at

$$\int_{\mathbb{R}^2} |\nabla \Psi_0(x)|^2 dx + \int_{\mathbb{R}^2} V(x) |\Psi(x)|^2 dx = 0$$

and the assumption $V \geq 0$ implies $\Psi = 0$.

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