$\mathcal{N}=2$ supersymmetric extension of the Tremblay–Turbiner–Winternitz Hamiltonians on a plane

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Abstract

The family of Tremblay–Turbiner–Winternitz Hamiltonians $H_k$ on a plane, corresponding to any positive real value of $k$, is shown to admit an $\mathcal{N}=2$ supersymmetric extension of the same kind as that introduced by Freedman and Mende for the Calogero problem and based on an $osp(2/2, \mathbb{R}) \sim su(1, 1/1)$ superalgebra. The irreducible representations of the latter are characterized by the quantum number specifying the eigenvalues of the first integral of motion $X_k$ of $H_k$. Bases for them are explicitly constructed. The ground state of each supersymmetrized Hamiltonian is shown to belong to an atypical lowest-weight state irreducible representation.

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1. Introduction

Recently, an infinite family of exactly solvable quantum Hamiltonians on a plane

$$H_k = -\partial_r^2 - \frac{1}{r} \partial_r - \frac{1}{r^2} \partial_\varphi^2 + \omega^2 r^2 + \frac{k^2}{r^2} \left[ a(a-1) \sec^2 k \varphi + b(b-1) \csc^2 k \varphi \right],$$

$$0 \leq r < \infty, \quad 0 \leq \varphi < \frac{\pi}{2k}, \quad (1.1)$$

introduced by Tremblay, Turbiner and Winternitz (TTW) [1], has aroused a lot of interest because of the conjectured superintegrability of the Hamiltonians for all positive integer values of $k$. For the corresponding classical systems, it has been shown that all bounded trajectories are closed and that the motion is periodic for all integer and rational values of $k$ [2]. Such classical systems have been proved to be superintegrable [3] and generalizable to higher dimensions [4]. For the quantum Hamiltonians, the validity of the superintegrability conjecture has been demonstrated for any odd integer $k$ [5] by using a $D_{2k}$ extension of $H_k$ [6] and a Dunkl operator formalism previously employed in the $k=3$ case [7]. Furthermore, a
canonical operator method has been applied to provide a constructive proof that all quantum Hamiltonians with rational \( k \) are superintegrable [8].

Another attractive property of the TTW family (1.1) is that it includes, as special cases, several well-known Hamiltonians, which have been studied a lot in the literature both for their intrinsic mathematical properties and for their possible physical applications. They correspond to the Smorodinsky–Winternitz (SW) system \((k = 1)\) [9, 10], the rational \( BC_2 \) model \((k = 2)\) [11, 12] and the three-particle Calogero model [13] with some extra three-body interaction \((k = 3)\), initially introduced by Wolfes and also considered by Calogero and Marchioro (CMW model) [14, 15]. The fact that the \( k = 2 \) and \( k = 3 \) cases belong to the family of Calogero-type Hamiltonians associated with root systems of classical Lie algebras [11, 12] has surely contributed to the interest aroused by the TTW Hamiltonians.

Some years ago, an \( \mathcal{N} = 2 \) supersymmetric extension of the \( V \)-particle Calogero model, based on a dynamical \( osp(2/2, \mathbb{R}) \sim su(1, 1/1) \) superalgebra [16–20], has been introduced by Freedman and Mende [21]. Several aspects of this extension and of its generalization to other root systems than that of the \( A_{n-1} \) Lie algebra have been reviewed in the literature (see, e.g., [22–24]). These supermodels are exactly solvable and play an important role in many areas of physics, such as the study of superstrings, black holes, superconformal quantum mechanics, spin chains, etc.

Since then, there have been great theoretical advances in that field too. For instance, the nonuniqueness of the construction proposed by Freedman and Mende has been discussed [25, 26]. Calogero-like models have also been analyzed in terms of hidden nonlinear supersymmetries [27–29]. Furthermore, some \( \mathcal{N} = 4 \) supersymmetric extensions have been recently considered in connection with the \( su(1, 1/2) \) superalgebra [30–33] or more generally with \( D(2, 1; \alpha) \) [34].

The purpose of this paper is to show that the family of TTW Hamiltonians corresponding to any positive real value of \( k \) admits an \( \mathcal{N} = 2 \) supersymmetric extension including the standard one of Calogero-type Hamiltonians [21–24] as a special case for \( k = 2 \) and 3.

In section 2, we review some known realizations of the \( osp(2/2, \mathbb{R}) \) superalgebra in Cartesian coordinates. In section 3, we make a transformation to polar coordinates on a plane to deal with the case of the TTW Hamiltonians. Bases for the corresponding irreducible representations (irreps) are constructed in section 4. The \( k = 1, 2 \) and 3 examples are treated in detail in section 5. Finally, section 6 contains the conclusion.

2. Realizations of the \( osp(2/2, \mathbb{R}) \) superalgebra

The \( osp(2/2, \mathbb{R}) \) superalgebra is generated by eight operators, four even ones closing the \( sp(2, \mathbb{R}) \times so(2) \) Lie algebra and four odd ones, which separate into two \( k \)-fold \( sl_2 \) representations (irreps) are constructed in section 4. The \( k = 1, 2 \) and 3 examples are treated in detail in section 5. Finally, section 6 contains the conclusion.

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2. Realizations of the \( osp(2/2, \mathbb{R}) \) superalgebra

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From (2.1), it is clear that $K_0$ and $Y$ are the weight generators, while $K_-, V_-, W_-$ (resp. $K_+, V_+, W_+$) are the lowering (resp. raising) generators.

A well-known realization of this superalgebra uses $v$ commuting pairs of bosonic and fermionic creation and annihilation operators $a_i^\dagger$, $a_i$ and $b_i^\dagger$, $b_i$, $i = 1, 2, \ldots, v$, where $[a_i, a_j^\dagger] = \delta_{i,j}$ and $\{b_i, b_j^\dagger\} = \delta_{i,j}$, respectively. Such a realization is given by

$$K_0 = \frac{1}{2} \left( \sum_i a_i a_i^\dagger + \frac{v}{2} \right), \quad K_+ = \frac{1}{2} \sum_i a_i^\dagger, \quad K_- = \frac{1}{2} \sum_i a_i^2,$$

$$Y = \frac{1}{2} \left( \sum_i b_i b_i^\dagger - \frac{v}{2} \right), \quad V_+ = \frac{1}{\sqrt{2}} \sum_i a_i b_i^\dagger, \quad V_- = \frac{1}{\sqrt{2}} \sum_i a_i^\dagger b_i,$$  

$$W_+ = \frac{1}{\sqrt{2}} \sum_i a_i^\dagger b_i, \quad W_- = \frac{1}{\sqrt{2}} \sum_i a_i b_i,$$  

where all summations run over $1, 2, \ldots, v$. It is related to the superoscillator (see, e.g., [35]) for which

$$\mathcal{H}^e = 4\omega K_0 + Y, \quad Q = 2\sqrt{\omega} W_+, \quad Q^\dagger = 2\sqrt{\omega} V_-$$

provide a realization of the $sl(1/1)$ superalgebra of standard supersymmetric quantum mechanics

$$[\mathcal{H}^e, Q] = [\mathcal{H}^e, Q^\dagger] = 0, \quad (Q, Q^\dagger) = \mathcal{H}^e,$$

and the operators (2.3) generate a dynamical superalgebra.

The boson–fermion realization (2.3) can be generalized by including an additional contribution in the bosonic operators $a_i^\dagger = (-\partial_i + \omega x_i)/\sqrt{2\omega}$, $a_i = (\partial_i + \omega x_i)/\sqrt{2\omega}$ (with $\partial_i \equiv \partial/\partial x_i$) appearing in the odd generators $V_\pm, W_\pm$. The latter become

$$V_\pm = \frac{1}{2\sqrt{\omega}} \sum_i (\mp \partial_i + \omega x_i \mp \partial_i W) b_i^\dagger, \quad W_\pm = \frac{1}{2\sqrt{\omega}} \sum_i (\pm \partial_i + \omega x_i \pm \partial_i W) b_i,$$  

where $W$ denotes some function of the $x_i$’s. Then the even operators

$$K_0 = K_{0, B} + \Gamma, \quad K_\pm = K_{\pm, B} - \Gamma, \quad Y = \frac{1}{4} \sum_i \left( 2 x_i \partial_i W + [b_i^\dagger, b_i] \right),$$

$$K_{0, B} = D + \frac{1}{4\omega} \sum_i x_i^2, \quad K_{\pm, B} = -D + \frac{1}{4\omega} \sum_i x_i^2 \mp \frac{1}{4} \sum_i (2 x_i \partial_i + 1),$$  

$$D = \frac{1}{4\omega} \sum_i [-\partial_i^2 - \partial_i^2 W + (\partial_i W)^2], \quad \Gamma = \frac{1}{4\omega} \sum_{i,j} \partial_i^2 W b_i^\dagger b_j,$$

resulting from the anticommutation relations in (2.1), also satisfy the remaining defining relations of $osp(2/2, \mathbb{R})$ provided

$$\left[ \sum_i x_i \partial_i, D \right] = -2D, \quad \left[ \sum_i x_i \partial_i, \Gamma \right] = -2\Gamma, \quad \sum_i x_i \partial_i W = C,$$  

where $C$ is some constant. This is the kind of realization that leads to the dynamical superalgebra of Calogero-type Hamiltonians [21–24] if $x_i, i = 1, 2, \ldots, v$, denote the coordinates of $v$ particles on a line and $W$ is an appropriate solution of equation (2.7).
3. Realization of $osp(2/2, \mathbb{R})$ in polar coordinates for the TTW Hamiltonians

Let us consider the case where there are only two variables $x_1$, $x_2$, which are the Cartesian coordinates $x$, $y$ of a particle on a plane, and consequently two pairs of fermionic creation and annihilation operators, denoted by $(b^+_1, b_1)$ and $(b^+_2, b_2)$. On setting $x = r \cos \phi$, $y = r \sin \phi$, $W$ becomes a function of $r$, $\phi$, and equations (2.5), (2.6) and (2.7) are transformed into

$$V_\pm = \frac{1}{2\sqrt{\omega}} \left[ \left( \mp \sin \phi \partial_r \pm \frac{1}{r} \sin \phi \partial_\phi + \omega r \cos \phi \mp \cos \phi \partial_r W \pm \frac{1}{r} \cos \phi \partial_\phi W \right) b^+_1 \right. $$
$$\left. \left. + \left( \mp \sin \phi \partial_\phi \pm \frac{1}{r} \sin \phi \partial_r + \omega r \sin \phi \mp \sin \phi \partial_\phi W \pm \frac{1}{r} \sin \phi \partial_r W \right) b^+_2 \right] \right],$$

$W_\pm = \frac{1}{2\sqrt{\omega}} \left[ \left( \mp \sin \phi \partial_r \pm \frac{1}{r} \sin \phi \partial_\phi + \omega r \cos \phi \mp \cos \phi \partial_r W \pm \frac{1}{r} \cos \phi \partial_\phi W \right) b_1 \right. $$
$$\left. \left. \left. + \left( \mp \sin \phi \partial_\phi \pm \frac{1}{r} \sin \phi \partial_r + \omega r \sin \phi \mp \sin \phi \partial_\phi W \pm \frac{1}{r} \sin \phi \partial_r W \right) b_2 \right] \right].$$

(3.1)

$$K_0 = K_{0,B} + \Gamma, \quad K_\pm = K_{\pm,B} - \Gamma, \quad Y = \frac{1}{2} (r \partial_r W + b^+_1 b_1 + b^+_2 b_2 - 1),$$

$$K_{0,B} = D + \frac{1}{4} \omega^2, \quad K_{\pm,B} = -D + \frac{1}{4} \omega^2 \mp \frac{1}{2} (r \partial_r + 1),$$

$$D = \frac{1}{4\omega} \left[ -\partial_r^2 - \frac{1}{r} \partial_\phi - \frac{1}{r^2} \partial_\phi^2 - \partial_\phi^2 W - \frac{1}{r} \partial_r W - \frac{1}{r^2} \partial_\phi^2 W + (\partial_\phi W)^2 + \frac{1}{r^2} (\partial_\phi W)^2 \right],$$

$$\Gamma = \frac{1}{2\omega} \left[ \left( \cos^2 \phi \partial_\phi^2 W - \frac{2}{r} \sin \phi \cos \phi \partial_\phi^2 W + \frac{1}{r^2} \sin^2 \phi \partial_\phi^2 W + \frac{1}{r^2} \sin^2 \phi \partial_r W \right) + \frac{2}{r^2} \sin \phi \cos \phi \partial_\phi W \right] b^+_1 b_1 + \left[ \sin \phi \cos \phi \partial_\phi W + \frac{1}{r^2} (\cos^2 \phi - \sin^2 \phi) \partial_\phi^2 W \right. $$
$$\left. - \frac{1}{r^2} \sin \phi \cos \phi \partial_r W - \frac{1}{r} \sin \phi \cos \phi \partial_\phi W - \frac{1}{r^2} (\cos^2 \phi - \sin^2 \phi) \partial_\phi W \right] \left. \times \left( b^+_1 b_1 + b^+_2 b_2 \right) + \left[ \sin^2 \phi \partial_\phi^2 W + \frac{2}{r} \sin \phi \cos \phi \partial_\phi^2 W + \frac{1}{r^2} \cos^2 \phi \partial_\phi^2 W \right. $$
$$\left. + \frac{1}{r^2} \cos^2 \phi \partial_r W - \frac{2}{r^2} \sin \phi \cos \phi \partial_\phi W \right] b^+_2 b_2 \right] \right] \right].$$

(3.2)

and

$$[r \partial_r, D] = -2D, \quad [r \partial_r, \Gamma] = -2\Gamma, \quad r \partial_r W = C.$$  

(3.3)

Conditions (3.3) are readily satisfied if we assume that $W$ takes the form

$$W = C \ln r + F(\phi),$$

where $F(\phi)$ may be any (physically acceptable) function of $\phi$.

In order to be relevant to the TTW Hamiltonians $H'$, the realization (3.1), (3.2) should be such that the bosonic part $4\omega K_{0,B}$ of $H'$, defined in (2.4), reduces to (1.1), which means that $D$ should be given by

$$D = \frac{1}{4\omega} \left[ -\partial_r^2 - \frac{1}{r} \partial_\phi - \frac{1}{r^2} \partial_\phi^2 - \partial_\phi^2 W - \frac{1}{r} \partial_r W - \frac{1}{r^2} \partial_\phi^2 W + (\partial_\phi W)^2 + \frac{1}{r^2} (\partial_\phi W)^2 \right].$$

This will be so provided the function $F(\phi)$ and the constant $C$ satisfy the Riccati equation

$$-F'' + F^2 + C^2 = k^2 [a(a - 1) \sec^2 \phi + b(b - 1) \csc^2 \phi].$$

(3.4)
A solution is easily found to be given by

$$F(\varphi) = -a \ln \cos k\varphi - b \ln \sin k\varphi, \quad C = -k(a + b).$$  \hfill (3.5)

It is worth observing here that to choose (3.4) among all the solutions of (3.4), we have been guided by the known results for Calogero-like Hamiltonians to be reviewed in section 5.

On inserting (3.5) into (3.2), we obtain that for the even generators the operators \( \Gamma \) and \( Y \) become

$$\Gamma = \frac{k}{2\omega r^2} \left\{ a \sec^2 k\varphi \left[ \left( \cos(k - 2)\varphi \cos k\varphi + \frac{k}{2} (1 - \cos 2\varphi) \right) b^+_1 b_1 + \left( \sin(k - 2)\varphi \cos k\varphi + \frac{k}{2} \sin 2\varphi \right) b^+_2 b_2 + \left( \cos(k - 2)\varphi \sin k\varphi + \frac{k}{2} (1 - \cos 2\varphi) \right) b^+_3 b_3 + \left( \sin(k - 2)\varphi \sin k\varphi + \frac{k}{2} \sin 2\varphi \right) b^+_4 b_4 \right] \right\}$$

and

$$Y = \frac{1}{\sqrt{2}} \left[ b^+_1 b_1 + b^+_2 b_2 + b^+_3 b_3 + b^+_4 b_4 - k(a + b) - 1 \right],$$

respectively. The supersymmetrized TTW Hamiltonian then assumes the form

$$\mathcal{H}^s = \mathcal{H}_{k,B} + \mathcal{H}_{k,F}, \quad \mathcal{H}_{k,B} = \mathcal{H}_k, \quad \mathcal{H}_{k,F} = 4\omega (\Gamma + Y).$$

Furthermore, the odd generators in (3.1) turn out to be

$$V_\pm = \frac{1}{2\sqrt{\omega}} \left[ \left( \mp \cos \varphi \partial_\varphi \pm \frac{1}{r} \sin \varphi \partial_\varphi \mp \frac{r}{\cos k\varphi} \right) \left( \mp k \cos k\varphi \right) \right] \left[ b^+_1 \right] \pm \left[ \mp \cos \varphi \partial_\varphi \pm \frac{1}{r} \sin \varphi \partial_\varphi \mp \frac{r}{\cos k\varphi} \right] \left[ b^+_2 \right],$$

and

$$W_\pm = \frac{1}{2\sqrt{\omega}} \left[ \left( \mp \cos \varphi \partial_\varphi \pm \frac{1}{r} \sin \varphi \partial_\varphi \mp \frac{r}{\cos k\varphi} \right) \left( \mp k \cos k\varphi \right) \right] \left[ b^+_3 \right] \pm \left[ \mp \cos \varphi \partial_\varphi \pm \frac{1}{r} \sin \varphi \partial_\varphi \mp \frac{r}{\cos k\varphi} \right] \left[ b^+_4 \right],$$

with \( W_+ \) and \( V_- \) providing the two supercharge operators for the supersymmetrized Hamiltonian (3.8) through equation (2.4).

By introducing two ‘rotated’ pairs of fermionic creation and annihilation operators \((\tilde{b}^+_1, \tilde{b}_1)\) and \((\tilde{b}^+_2, \tilde{b}_2)\), defined by

$$\tilde{b}^+_1 = b^+_1 \cos \varphi + b^+_2 \sin \varphi, \quad \tilde{b}^+_2 = -b^+_3 \sin \varphi + b^+_4 \cos \varphi.$$
and similarly for $b_x, b_y$, equations (3.6), (3.7) and (3.9) can be recast in a somewhat simpler form:

$$\Gamma = \frac{k}{2\omega r} \left[ a \left( b_x^\dagger b_x - \tan k\varphi (b_y^\dagger b_y + b_y b_x^\dagger) + (k \sec^2 k\varphi - 1) b_y^\dagger b_y \right) 
+ b \left( b_x^\dagger b_x + \cot k\varphi (b_y^\dagger b_y + b_y b_x^\dagger) + (k \csc^2 k\varphi - 1) b_y^\dagger b_y \right) \right],$$

$$Y = \frac{1}{2} \left[ b_x^\dagger b_x + b_y^\dagger b_y - k(a + b) - 1 \right]$$
and

$$V_\pm = \frac{1}{2\sqrt{\omega}} \left[ b_x^\dagger \left( \mp \partial_r + \omega r \pm \frac{k(a + b)}{r} \right) \mp b_x \frac{1}{r} (\partial_r + ka \tan k\varphi - kb \cot k\varphi) \right],$$
$$W_\pm = \frac{1}{2\sqrt{\omega}} \left[ b_x^\dagger \left( \mp \partial_r + \omega r \mp \frac{k(a + b)}{r} \right) \mp b_x \frac{1}{r} (\partial_r - ka \tan k\varphi + kb \cot k\varphi) \right].$$

In the next section, we will proceed to determine the action of the osp(2/2, $\mathbb{R}$) generators on the TTW Hamiltonian eigenstates after extending the latter with fermionic degrees of freedom.

4. Irreducible representations of osp(2/2, $\mathbb{R}$) for the supersymmetrized TTW Hamiltonians

In [1], it has been shown that $H_k$ is exactly solvable and satisfies the eigenvalue equation

$$H_k \Psi_{N,n}(r, \varphi) = E_{N,n} \Psi_{N,n}(r, \varphi), \quad E_{N,n} = 2\omega \left[ 2N(2N + 2n + a + b)k + 1 \right],$$

where $N, n = 0, 1, 2, \ldots$. The wavefunctions can be written as

$$\Psi_{N,n}(r, \varphi) = N_{N,n} Z_N^2(2n+a+b)(z) \Phi_{n}^{(a,b)}(\varphi),$$

$$Z_N^2(2n+a+b)(z) = \left( \frac{z}{\alpha_0} \right)^{-(n+a+b)} L_N^{(2n+a+b)}(z) e^{-\frac{1}{2}z}, \quad z = \omega r^2,$$

$$\Phi_{n}^{(a,b)}(\varphi) = \cos^a k\varphi \sin^b k\varphi P_n^{(a-\frac{1}{2}, b-\frac{1}{2})}(\xi), \quad \xi = -\cos 2k\varphi,$$
in terms of Laguerre and Jacobi polynomials. In (4.1), $N_{N,n}$ denotes a normalization constant, which can be easily calculated from some known properties of these polynomials [36] and is given by

$$N_{N,n} = \tilde{N}_{N,n} N_{n}^{(a,b)}$$
with

$$\tilde{N}_{N,n} = (-1)^N \left( \frac{2\omega(2n+a+b)k+1 N!}{\Gamma(N+2n+a+b)k+1} \right)^{1/2},$$
$$N_{n}^{(a,b)} = \left( \frac{2kn!(2n+a+b)\Gamma(a+b+n)}{\Gamma(a+n+\frac{1}{2}) \Gamma(b+n+\frac{1}{2})} \right)^{1/2}.$$  

Observe that the optional phase factor $(-1)^N$ in $\tilde{N}_{N,n}$ has been introduced to get positive matrix elements for the $sp(2, \mathbb{R})$ generators $K_{\pm}$ in conformity with the conventional choice.

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1 It is worth observing here that these new fermionic operators should only be seen as a convenient tool to write the generators and their action on wavefunctions in a concise way since their dependence on $\varphi$ breaks the commutativity of bosonic and fermionic degrees of freedom.
After multiplication by the fermionic vacuum state $|0\rangle$ (i.e., $b_i|0\rangle = b_i|0\rangle = \bar{b}_i|0\rangle = 0$), the wavefunctions (4.1) yield eigenstates of the supersymmetrized TTW Hamiltonian (3.8) with eigenvalues

$$E_{N,n} = E_{N,n} - E_{0,0} = 4\omega(N + nk).$$

Such extended wavefunctions also turn out to be eigenstates of the $osp(2/2, \mathbb{R})$ weight generators $K_0$ and $Y$ corresponding to the eigenvalues $\tau + N$ and $q$, where

$$\tau = \left(\frac{n + a + b}{2}\right) k + \frac{1}{2}, \quad q = -\frac{1}{2}[(a + b)k + 1].$$

All the states $\Psi_{N,n,0}(0)$ with a definite value of $n$ (hence of $\tau$) and $N = 0, 1, 2, \ldots$ belong to a $sp(2, \mathbb{R})$ lowest-weight state (LWS) irrep characterized by $\tau$ and will be denoted by

$$|\tau, \tau + N, q\rangle = \Psi_{N,n,0}(0).$$

(4.2)

They indeed satisfy the relations

$$K_0|\tau, \tau + N, q\rangle = (\tau + N)|\tau, \tau + N, q\rangle,$$

$$K_+|\tau, \tau + N, q\rangle = [(N + 1)(2\tau + N)]^{1/2}|\tau, \tau + N + 1, q\rangle,$$

$$K_-|\tau, \tau + N, q\rangle = [N(2\tau + N - 1)]^{1/2}|\tau, \tau + N - 1, q\rangle,$$

which can be easily checked by rewriting $K_\pm$ as

$$K_\pm = -\frac{1}{4\omega} H_k + \frac{1}{2} z \mp \left(z \theta_2 + \frac{1}{2}\right) - \Gamma$$

and using some well-known properties of Laguerre polynomials [36].

The odd generators $W_\pm$ annihilate the zero-fermion states (4.2), whereas the remaining odd generators $V_\pm$ may lead to one- and two-fermion states. After some straightforward calculations, equations (3.10) and (4.1) yield

$$V_+|\tau, \tau + N, q\rangle = \frac{N\omega}{\sqrt{2}} \left[[N + (n + a + b)k + 1]Z_N^{(2n+a+b)}(z) - (N + 1)Z_N^{(2n+a+b)}(z)\right]
\times \Phi_n^{(a,b)}(\psi)\bar{b}^1 - (n + a + b)k Z_N^{(2n+a+b)}(z)\Phi_{n-1}^{(a+1,b+1)}(\psi)\bar{b}^1|0\rangle,$$

$$V_-|\tau, \tau + N, q\rangle = \frac{N\omega}{\sqrt{2}} \left[[N + nk]Z_N^{(2n+a+b)}(z) - [N + (2n + a + b)k]Z_N^{(2n+a+b)}(z)\right]
\times \Phi_n^{(a,b)}(\psi)\bar{b}^1 + (n + a + b)k Z_N^{(2n+a+b)}(z)\Phi_{n+1}^{(a+1,b+1)}(\psi)\bar{b}^1|0\rangle,$$

$$V_0V_-|\tau, \tau + N, q\rangle = -V_-V_+|\tau, \tau + N, q\rangle = \frac{N\omega}{\sqrt{2}} \left[(n + a + b)k Z_N^{(2n+a+b)}(z)\Phi_{n+1}^{(a+1,b+1)}(\psi)\bar{b}^1\right]|0\rangle.$$

(4.3)

From (2.1), it is obvious that the three states in (4.3) are eigenstates of $K_0$ and $Y$ with eigenvalues $(\tau + N + \frac{1}{2}, q + \frac{1}{2}), (\tau + N - \frac{1}{2}, q + \frac{1}{2})$ and $(\tau + N, q + 1)$, respectively.

These one- and two-fermion states can be easily normalized by using the Hermiticity properties and anticommutation relations of the $osp(2/2, \mathbb{R})$ generators. The results read

$$|+, \tau + N + \frac{1}{2}, q + \frac{1}{2}\rangle = [N + (n + a + b)k + 1]^{-1/2}V_+|\tau, \tau + N, q\rangle,$$

$$|-, \tau + N - \frac{1}{2}, q + \frac{1}{2}\rangle = (N + nk)^{-1/2}V_-|\tau, \tau + N, q\rangle,$$

$$|\pm, \tau + N, q + 1\rangle = [n(n + a + b)k^3]^{-1/2}V_0V_-|\tau, \tau + N, q\rangle.$$

(4.4)

It can also be shown that the one-fermion states with the same eigenvalue of $K_0$ are not orthogonal and that their overlap is given by

$$\left<+, \tau + N - \frac{1}{2}, q + \frac{1}{2}|-, \tau + N - \frac{1}{2}, q + \frac{1}{2}\right> = \left(\frac{N[N + (2n + a + b)k]}{[N + (n + a + b)k](N + nk)}\right)^{1/2}$$

for $N = 1, 2, \ldots$. 

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The results obtained so far for one- and two-fermion states correspond to generic values of \( N \) and \( n \). To construct from them basis states for \( sp(2, \mathbb{R}) \) irreps we have to distinguish between vanishing and nonvanishing values of \( n \).

For \( n = 0 \), as a result of (4.3) and of the properties \( Z_{-1}^{(a+b)}(z) = \Phi_{-1}^{(a+1,b+1)}(\varphi) = 0 \), it turns out that the ground state \(|\tau, \tau, q\rangle = \Psi_{0,0}(0)\rangle \) of \( \mathcal{H}' \) is annihilated not only by \( K_- \) and \( W_- \) but also by \( V_- \). Hence it is an \( osp(2/2, \mathbb{R}) \) LWS and the states obtained from it by means of the raising generators form a basis for a so-called atypical LWS irrep with \( \tau = -q \) [19]. The latter is known to decompose into two \( sp(2, \mathbb{R}) \times so(2) \) irreps characterized by \( (\tau)(q) \) and \( (\tau + \frac{1}{2})(q + \frac{1}{2}) \), respectively. This is confirmed by setting \( n = 0 \) in the generic results (4.3) and (4.4). We indeed obtain

\[
|\tau + N + \frac{1}{2}, q + \frac{1}{2}\rangle = +, \tau + N + \frac{1}{2}, q + \frac{1}{2}, \quad |\pm, \tau + N, q + 1\rangle = 0
\]

for \( N = 0, 1, 2, \ldots \). In the \( n = 0 \) case, we may therefore set

\[
|\tau + \frac{1}{2}, \tau + N + \frac{1}{2}, q + \frac{1}{2}\rangle = +, \tau + N + \frac{1}{2}, q + \frac{1}{2}, \quad N = 0, 1, 2, \ldots
\]

It is worth observing that since \( Q \) and \( Q^\dagger \) in (2.4) annihilate the ground state of \( \mathcal{H}' \), supersymmetry is unbroken.

For nonvanishing values of \( n \), the situation is more complicated since the three states in (4.3) and (4.4) are nonzero. It is, however, straightforward to show that the one-fermion states can be combined into basis states of two \( sp(2, \mathbb{R}) \) irreps, characterized by \( \tau = -\frac{1}{2} \) and \( \tau + \frac{1}{2} \), respectively. The latter can be written as

\[
|\tau - \frac{1}{2}, \tau + N - \frac{1}{2}, q + \frac{1}{2}\rangle = \alpha_N|\tau + N - \frac{1}{2}, q + \frac{1}{2} + \beta_N|\tau + N - \frac{1}{2}, q + \frac{1}{2}, |	au + \frac{1}{2}, \tau + N + \frac{1}{2}, q + \frac{1}{2}\rangle = \gamma_N|\tau - \frac{1}{2}, \tau + N + \frac{1}{2}, q + \frac{1}{2} + \delta_N|\tau + N + \frac{1}{2}, q + \frac{1}{2},
\]

with

\[
\alpha_N = \left(\frac{[N + (2n + a + b)k](N + nk)}{n(2n + a + b)k^2}\right)^{1/2},
\]

\[
\beta_N = \left(-\frac{N[N + (n + a + b)k]}{n(2n + a + b)k^2}\right)^{1/2},
\]

\[
\gamma_N = \left(\frac{(N + 1)(N + nk + 1)}{(n + a + b)(2n + a + b)k^2}\right)^{1/2},
\]

\[
\delta_N = \left(-\frac{[N + (n + a + b)k + 1][N + (2n + a + b)k + 1]}{(n + a + b)(2n + a + b)k^2}\right)^{1/2}
\]

On the other hand, the two-fermion states belong to a single \( sp(2, \mathbb{R}) \) LWS irrep specified by \( \tau \):

\[
|\tau, \tau + N, q + 1\rangle = |\pm, \tau + N, q + 1\rangle.
\]

Hence, for any \( n \neq 0 \) there exists an \( osp(2/2, \mathbb{R}) \) irrep, which is not a LWS one and decomposes into four \( sp(2, \mathbb{R}) \times so(2) \) irreps \( (\tau)(q) \), \( (\tau - \frac{1}{2})(q + \frac{1}{2}) \), \( (\tau + \frac{1}{2})(q + \frac{1}{2}) \) and \( (\tau)(q + 1) \).

We conclude that the eigenstates of any supersymmetrized TTW Hamiltonian may be separated into basis states of an infinite collection of \( osp(2/2, \mathbb{R}) \) irreps, each member of the set being characterized by a given value of the quantum number \( n \) associated with the angular part of \( H_k \) and determining the eigenvalue \( (2n + a + b)^2k^2 \) of the first integral of motion \( X_1 \) [1, 5]. The corresponding eigenvalues of the second- and third-order Casimir operators [20]

\[
C_2 = K_0(K_0 - 1) - Y(Y + 1) - K_sK_- + V_-W_+ - V_+W_-,
\]

\[
C_3 = (K_0 + Y)(K_0 - Y - 1)(Y + \frac{1}{2}) - (Y + \frac{1}{2})K_sK_- + \frac{3}{2}[K_-V_+ - (K_0 - 3Y)V_-]W_+ + \frac{3}{2}[K_-V_- - (K_0 + 3Y)V_+]W_-
\]
The Hamiltonian of the $BC$ is usually written in Cartesian coordinates, wherein it is also separable. For simplicity’s sake, we will restrict ourselves here to the explicit relations obtained for $\mathcal{H}'$ and $Q$, defined in (2.4), as the remaining $osp(2/2, \mathbb{R})$ generators can be easily dealt with in the same way.

5. The $k = 1, 2$ and 3 cases

The purpose of this section is to establish some connections between the outcomes of section 3 and some known results corresponding to $k = 1, 2$ and 3. For simplicity’s sake, we directly get $\mathcal{H}' = -\partial_x^2 - \partial_y^2 + \omega^2(x^2 + y^2) + \frac{a(a - 1)}{x^2} + \frac{b(b - 1)}{y^2}$

is usually written in Cartesian coordinates, wherein it is also separable. For $k = 1$ in (3.6), (3.7), (3.8) and (3.9) and going back from $r, \varphi$ to $x, y$, we directly get

$\mathcal{H}' = -\partial_x^2 - \partial_y^2 + \omega^2(x^2 + y^2) + \frac{a^2}{x^2} + \frac{b^2}{y^2} + 2\omega(b^2 b_x + b^1 b_y) + \frac{a}{x^2}[b^1_x, b_x]$  

+ $\frac{b}{y^2}[b^1_y, b_y] - 2\omega(a + b + 1)$,

$Q = \left(-\partial_x + \omega x - \frac{a}{x}\right)b_x + \left(-\partial_y + \omega y - \frac{b}{y}\right)b_y$.

Similar results would be obtained either by supersymmetrizing separately the two one-dimensional Cartesian Hamiltonians or by setting $W = -a \ln|x| - b \ln|y|$ in equations (2.5) and (2.6).

5.1. SW model ($k = 1$)

The SW Hamiltonian $[9, 10]$  

$H_1 = -\partial_x^2 - \partial_y^2 + \omega^2(x^2 + y^2) + \frac{a(a - 1)}{x^2} + \frac{b(b - 1)}{y^2}$

is usually written in Cartesian coordinates, wherein it is also separable. For setting $k = 1$ in (3.6), (3.7), (3.8) and (3.9) and going back from $r, \varphi$ to $x, y$, we directly get

$\mathcal{H}' = -\partial_x^2 - \partial_y^2 + \omega^2(x^2 + y^2) + \frac{a^2}{x^2} + \frac{b^2}{y^2} + 2\omega(b^2 b_x + b^1 b_y) + \frac{a}{x^2}[b^1_x, b_x]$  

+ $\frac{b}{y^2}[b^1_y, b_y] - 2\omega(a + b + 1)$,

$Q = \left(-\partial_x + \omega x - \frac{a}{x}\right)b_x + \left(-\partial_y + \omega y - \frac{b}{y}\right)b_y$.

Similar results would be obtained either by supersymmetrizing separately the two one-dimensional Cartesian Hamiltonians or by setting $W = -a \ln|x| - b \ln|y|$ in equations (2.5) and (2.6).

5.2. BC2 model ($k = 2$)

The Hamiltonian of the $BC_2$ model is given by $[12]$  

$H_2 = -\partial_x^2 - \partial_y^2 + \omega^2(x^2 + y^2) + 2a(a - 1) \left(\frac{1}{(x - y)^2} + \frac{1}{(x + y)^2}\right) + b(b - 1) \left(\frac{1}{x^2} + \frac{1}{y^2}\right)$.

For $k = 2$, equations (3.6), (3.7), (3.8) and (3.9) easily yield

$\mathcal{H}' = -\partial_x^2 - \partial_y^2 + \omega^2(x^2 + y^2) + 2a^2 \left(\frac{1}{(x - y)^2} + \frac{1}{(x + y)^2}\right) + b^2 \left(\frac{1}{x^2} + \frac{1}{y^2}\right)$  

+ $2\omega(b^1_x b_x + b^1_y b_y) + a \left(\frac{1}{(x - y)^2}[b^1_x - b^1_y, b_x - b_y] + \frac{1}{(x + y)^2}[b^1_x + b^1_y, b_x + b_y]\right)$  

+ $b \left(\frac{1}{x^2}[b^1_x, b_x] + \frac{1}{y^2}[b^1_y, b_y]\right) - 2\omega(2a + b + 1)$,

$Q = \left[-\partial_x + \omega x - a \left(\frac{1}{x - y} + \frac{1}{x + y}\right) - \frac{b}{x}\right]b_x$  

+ $\left[-\partial_y + \omega y + a \left(\frac{1}{x - y} - \frac{1}{x + y}\right) - \frac{b}{y}\right]b_y$.  

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which would also result from section 2 by using

\[ W = -a(\ln|x-y|+\ln|x+y|) - b(\ln|x|+\ln|y|) \]

in agreement with [23].

5.3. CMW model \((k = 3)\)

In contrast with the two previous cases, the comparison of the results obtained in section 3 for \(k = 3\) with those of the supersymmetrized CMW model is more involved because the latter is a three-particle system, which can only be interpreted as a planar problem after eliminating the centre-of-mass motion. On starting from [14]

\[ H_{\text{CMW}} = \sum_i \left( -\partial_i^2 + \omega^2 x_i^2 \right) + a(a - 1) \sum_{i,j \neq i,j} \frac{1}{x_{ij}} + 3b(b - 1) \sum_{i,j \neq i,j} \frac{1}{y_{ij}} \]

where \(x_{ij} = x_i - x_j (i \neq j)\), \(y_{ij} = x_i + x_j - 2x_k (i \neq j \neq k \neq i)\) and all indices run over 1, 2, 3, we indeed obtain

\[ H_{\text{CMW}} = H_{\text{rel}} + H_{\text{cm}}, \quad H_{\text{rel}} = H_3, \quad H_{\text{cm}} = -\partial_X^2 + \omega^2 X^2, \]

by setting \(r \cos \varphi = x_{12}/\sqrt{2}\), \(r \sin \varphi = y_{12}/\sqrt{6}\) and \(X = (x_1 + x_2 + x_3)/\sqrt{3}\).

The supersymmetrized CMW Hamiltonian and the corresponding supercharge are obtained by inserting

\[ W = -\frac{a}{2} \sum_{i,j \neq i,j} \ln|x_{ij}| - \frac{b}{2} \sum_{i,j \neq i,j} \ln|y_{ij}| \]

in (2.5) and (2.6) and they are given by

\[ H_{\text{CMW}}^{\epsilon} = \sum_i \left( -\partial_i^2 + \omega^2 x_i^2 \right) + a^2 \sum_{i,j \neq i,j} \frac{1}{x_{ij}} + 3b^2 \sum_{i,j \neq i,j} \frac{1}{y_{ij}} + 2\omega \sum_i b_i^\dagger b_i \]

\[ + a \sum_{i,j \neq i,j} \frac{1}{x_{ij}} [b_i^\dagger b_i - b_j^\dagger b_j] + b \sum_{i,j,k \neq i,j,k} \frac{1}{y_{ij}} ([b_i^\dagger b_i + b_j + b_j^\dagger + 2b_k - 2b_k^\dagger] - [b_k^\dagger + b_j + b_j^\dagger - 2b_k]) \]

\[ (5.1) \]

\[ Q_{\text{CMW}}^{\epsilon} = \sum_i \left[ -\partial_i + \omega x_i - a \sum_{j \neq i} \frac{1}{x_{ij}} - b \left( \sum_{j \neq i} \frac{1}{y_{ij}} - b_{ij}^\dagger \sum_{j \neq i} \frac{1}{y_{ij}} \right) \right] b_i, \]

in terms of three pairs of fermionic creation and annihilation operators \(b_i^\dagger, b_i, i = 1, 2, 3\).

Let us now make for the latter the same kind of orthogonal transformation as that performed for the coordinates

\[ b_1^\dagger = \frac{1}{\sqrt{2}} (b_1^\dagger - b_2^\dagger), \quad b_2^\dagger = \frac{1}{\sqrt{6}} (b_1^\dagger + b_2^\dagger - 2b_3^\dagger), \quad b_3^\dagger = \frac{1}{\sqrt{3}} (b_1^\dagger + b_2^\dagger) \]

and similarly for the annihilation operators. Then, after some calculations, equation (5.1) can be rewritten as

\[ H_{\text{CMW}}^{\epsilon} = H_{\text{rel}}^{\epsilon} + H_{\text{cm}}^{\epsilon}, \quad Q_{\text{CMW}}^{\epsilon} = Q_{\text{rel}}^{\epsilon} + Q_{\text{cm}}^{\epsilon}. \]
where

\[ H_{\text{rel}} = -\frac{\partial^2}{\partial \varphi^2} + \omega^2 r^2 + \frac{1}{r^2 \cos^2 \varphi} \left( a + b_1^1, b_x \right) \]

\[ + \frac{1}{r^2 \cos^2 \left( \varphi - \frac{2 \pi}{3} \right)} a \left( a + b_1^1, b_y \right) \]

\[ + \frac{1}{r^2 \cos^2 \left( \varphi - \frac{4 \pi}{3} \right)} b \left( b + b_x^1 + b_y^1, b_x \right) \]

\[ + \frac{1}{r^2 \sin^2 \left( \varphi - \frac{2 \pi}{3} \right)} b \left( b + b_x^1 - b_y^1, b_y \right) \]

\[ + 2 \omega \left( b_1^1 b_x + b_1^1 b_y \right) - 2 \omega (3a + 3b + 1), \]  

\[ (5.2) \]

\[ Q_{\text{rel}} = \left( -\cos \varphi \partial_x + \frac{1}{r} \sin \varphi \partial_x + \omega r \cos \varphi = \frac{3a}{r} \cos 2\varphi + \frac{3b}{r} \sin 2\varphi \right) \]

\[ + \left( -\sin \varphi \partial_y - \frac{1}{r} \cos \varphi \partial_y + \omega r \sin \varphi = \frac{3a}{r} \sin 2\varphi + \frac{3b}{r} \cos 2\varphi \right) \]

\[ b_x \]

\[ \text{and} \]

\[ H_{\text{cm}} = -\partial_x^2 + \omega^2 X^2 + 2\omega \left( b_1^x b_X - \frac{1}{2} \right), \]

\[ Q_{\text{cm}} = \left( -\partial_x + \omega X \right) b_X. \]

It is immediately clear that \( Q_{\text{rel}} \) in (5.3) coincides with \( 2\sqrt{\omega} W \) obtained by setting \( k = 3 \) in (3.9). To show that \( H_{\text{rel}} \) in (5.2) also reduces to the supersymmetrized TTW Hamiltonian with \( k = 3 \), given in (3.6), (3.7) and (3.8), requires some work, but this can be easily done by employing well-known trigonometric identities similar to those used in [5, 6], thereby completing the comparison.

6. Conclusion

In this paper, we have obtained, for any positive real \( k \), an \( \mathcal{N} = 2 \) supersymmetric extension \( \mathcal{H} \) of the TTW Hamiltonians \( H_k \) on a plane, which generalizes the known ones for \( k = 1, 2 \) and 3. Such a supersymmetrized TTW Hamiltonian has an \( osp(2/2, \mathbb{R}) \) dynamical superalgebra, whose irreps have been shown to be characterized by the quantum number \( n \) specifying either the angular wavefunctions of \( H_k \) or the eigenvalues of its first integral of motion \( X_k \). Bases for these irreps have been explicitly built and the irrep containing the ground state of \( \mathcal{H} \) has been identified as an atypical LWS one.

Several interesting questions are raised by the results obtained in this work, such as the feasibility of constructing higher \( \mathcal{N} \) extensions of \( H_k \), the possible existence of hidden nonlinear supersymmetries and the relation between the present \( \mathcal{N} = 2 \) supersymmetric extension and the previously considered one based on the dihedral group \( D_{2k} \). We hope to come back to some of these issues in forthcoming publications.
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