Aeppli-Bott-Chern cohomology and Deligne cohomology from a viewpoint of Harvey-Lawson’s spark complex

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Abstract

A cohomology theory for complex manifolds is constructed by using Harvey-Lawson’s spark complex. It is an extension of Deligne cohomology, possessing ring structure and refined Chern classes. The total refined Chern class in this cohomology satisfies the Whitney product formula.

1 Introduction

The theory of differential characters was founded by Cheeger and Simons (2) around 1970. It obtains intensive development in the last 20 years. Physicists realize that differential characters can be used in the mathematical formulation of generalized abelian gauge theories (3), and mathematicians found that they appear naturally in many mathematical problems (7, 8). The interaction between physics and mathematics stimulates lot of development in both disciplines and the theory of differential characters is extended to various generalized differential cohomologies. A particular nice construction of differential cohomologies was given by Harvey and Lawson through their theory of spark complexes which unifies many known results. Furthermore, by applying their theory, they constructed a ∂-analogue (6, 4) of differential characters for complex manifolds. The Harvey-Lawson spark group \( \hat{H}^k(X, p) \) of level \( p \) of a complex manifold \( X \) contains the analytic Deligne cohomology \( H^{k+1}_D(X, \mathbb{Z}(p)) \) as a subgroup and fits in a short exact sequence

\[
0 \to H^{k+1}_D(X, \mathbb{Z}(p)) \to \hat{H}^k(X, p) \to \mathbb{Z}^{k+1}(X, p) \to 0
\]

where \( \mathbb{Z}^{k+1}(X, p) \) is some subgroup of complex differential forms with integral periods. Deligne cohomology group \( H^{k+1}_D(X, \mathbb{Z}(p)) \) is usually defined by the hypercohomology group \( H^{k+1}(X, \mathbb{Z}(p)) \) of the Deligne complex of sheaves

\[
\mathbb{Z}(p) : 0 \to \mathbb{Z} \hookrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{p-1} \to 0
\]

where \( \Omega^k \) is the sheaf of holomorphic \( k \)-forms. Recall that the Aeppli and Bott-Chern cohomology of a complex manifold \( X \) can be defined by the hypercohomology of the complex of sheaves:

\[
B_{p,q}^* : 0 \to \mathbb{C} \to \mathcal{O} \oplus \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega^1} \to \cdots \to \Omega^{p-1} \oplus \overline{\Omega^{p-1}} \to \overline{\Omega}^p \to \cdots \to \overline{\Omega}^{q-1} \to 0
\]

where \( \overline{\Omega}^k \) be the sheaves of anti-holomorphic \( k \)-forms. We have

\[
H_{A,q}^p(X; \mathbb{C}) \cong \mathbb{H}^{p+q+1}(X, B_{p+1,q+1}^*) \text{ and } H_{BC}^{p,q}(X; \mathbb{C}) \cong \mathbb{H}^{p+q}(X, B_{p,q}^*)
\]

By this similarity to the definition of Deligne cohomology, we wish to have a differential cohomology \( \hat{H}(X, p, q) \) such that the role of Deligne cohomology is played by the hypercohomology group of the complex of sheaves

\[
0 \to \mathbb{Z} \to \mathcal{O} \oplus \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega^1} \to \cdots \to \Omega^{p-1} \oplus \overline{\Omega^{p-1}} \to \overline{\Omega}^p \to \cdots \to \overline{\Omega}^{q-1} \to 0
\]

This is the motivation of this paper and such theory is constructed in section 2.

The paper is organized as follow. In section 2, we review Harvey and Lawson’s theory of spark complexes and use it to construct a differential cohomology for complex manifolds. Such cohomology \( \hat{H}^{p,q}(X, p, q) \) is called the Harvey-Lawson spark group of level \( (p, q) \) of \( X \). A result mentioned above is proved, a \( 3 \times 3 \)-grid that relates Griffiths intermediate Jacobian and Hodge group is given and a Lefschetz
property is shown. In section 3, we establish a ring structure on the differential cohomology. In section 4, we construct refined Chern classes for hermitian vector bundles in our group and prove a Whitney product formula.

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2 Harvey-Lawson spark groups

We recall the construction of spark group given by Harvey and Lawson in [6].

Definition 2.1. (Spark complexes) Suppose that \( F^* = \oplus_{i\geq 0} F^i, E^* = \oplus_{i\geq 0} E^i, I^* = \oplus_{i\geq 0} I^i \) are cochain complexes and \( \Psi : I \to F \) is a morphism of cochain complexes, \( E^* \hookrightarrow F^* \) is an embedding with the following properties:

1. \( \Psi(I^k) \cap E^k = \{0\} \) for all \( k > 0 \),
2. \( \Psi : I^0 \to F^0 \) is injective,
3. the embedding induces an isomorphism \( H^k(E^*) \to H^k(F^*) \).

Then \( S = (F^*, E^*, I^*) \) is called a spark complex.

Definition 2.2. (Spark groups) Given a spark complex \( S = (F^*, E^*, I^*) \), a spark of degree \( k \) is a pair \( (a, r) \in F^k \oplus I^{k+1} \) which satisfies the spark equation

\[
\begin{align*}
da &= e - \Psi(r) \text{ where } e \in E^{k+1}, \\
\end{align*}
\]

Let \( H^k(F^*, E^*, I^*) \) be the collection of all sparks of degree \( k \) in \( (F^*, E^*, I^*) \). Two sparks \( (a, r), (a', r') \) of degree \( k \) are equivalent if there exists a pair \( (b, s) \in F^{k-1} \oplus I^k \) such that

\[
\begin{align*}
a - a' &= db + \Psi(s), \\
r - r' &= -ds.
\end{align*}
\]

We write \( \hat{H}^k(S) = S^k(F^*, E^*, I^*)/\sim \) for the group of equivalence spark classes of degree \( k \).

For the convenience of the reader, we cite the following result from [6, Prop 1.8].

Proposition 2.3. There is a \( 3 \times 3 \) commutative grid of exact sequences associated to any spark complex \( S = (F^*, E^*, I^*) \)

\[
\begin{array}{ccc}
& 0 & 0 & 0 \\
0 & H^k(F^*) & H^k(E^*) & 0 \downarrow \\
0 & H^k(G) & \hat{H}^k(S) & \delta_1 \downarrow \\
0 & Ker^{k+1}(I) & H^{k+1}(I^*) & \Psi \downarrow \\
& 0 & 0 & 0
\end{array}
\]

where \( \Psi : I^* \to F^* \) is the morphism of cochain complexes which induces a group homomorphism \( \Psi_* : H^k(I^*) \to H^k(F^*) \), and \( H^k(F^*) := \text{Image} \Psi, Ker^{k+1}(I) := \text{Ker} \Psi, Z^{k+1}_k(E) = \{ e \in Z^{k+1}(E) | [e] = \Psi_*(\rho) \text{ for some } \rho \in H^{k+1}(I^*) \}, \hat{H}^k(S) = \text{kernel of } \delta_2, \text{ and } G \text{ is the cone complex formed by } \Psi : I^* \to F^*. \)
Let \( X \) be a complex manifold of complex dimension \( m \). We write \( \mathcal{E}_{cpt}^{p,q}(X) \) for the space of \((p,q)\)-forms with compact support on \( X \). The space of currents of degree \((p,q)\) on \( X \) is the topological dual space \( D_{\text{cpt}}^{p,q}(X) := \{ \mathcal{E}_{cpt}^{m-p,m-q} \}^\prime \). We write
\[
D^k(X,p,q) = \bigoplus_{i_1 + j_1 = k \atop i_2 + j_2 = k} D^{i_1,j_1}(X) \bigoplus D^{i_2,j_2}(X)
\]
and the counterpart of forms \( \mathcal{E}_{cpt}^k(X,p,q) \) is defined similarly.

Define \( d_{p,q} : D^{k-1}(X,p,q) \to D^k(X,p,q) \) by
\[
d_{p,q}(a, b) = (\pi_p a, \pi_q b)
\]
where
\[
\pi_p : D^k(X) \to \bigoplus_{i_1 + j_1 = k \atop i_2 + j_2 = k} D^{i_1,j_1}(X), \quad \pi_q : D^k(X) \to \bigoplus_{i_1 + j_1 = k \atop i_2 + j_2 = k} D^{i_2,j_2}(X)
\]
are the natural projections. It is easy to see that \( d_{p,q}^2 = 0 \) and hence \( (D^*(X,p,q), d_{p,q}) \) is a cochain complex. Let \( I^k(X) \) be the space of locally integral currents of degree \( k \) on \( X \). Define \( \Psi_{p,q} : I^k(X) \to D^k(X,p,q) \) by
\[
\Psi_{p,q}(r) = (\pi_p(r), \pi_q(r))
\]

**Proposition 2.4.** Let \( X \) be a complex manifolds of dimension \( m \). For \( \alpha \in \mathcal{E}^{p,q}(X), \beta \in \mathcal{E}^{m-p,m-q}(X) \), define \( \alpha(\beta) := \int_X \alpha \wedge \beta \). Then \( \mathcal{E}^{p,q}(X) \) may be considered as a subspace of \( D^{p,q}(X) \). With maps and differentials defined above, the triple \((D^*(X,p,q), \mathcal{E}^*(X,p,q), I^*(X))\) forms a spark complex.

**Proof.** It is well known that the inclusion map \( \mathcal{E}^*(X,p,q) \hookrightarrow D^*(X,p,q) \) is a quasi-isomorphism and \( \Psi : I^*(X) \to D^a(X,p,q) \) is injective. The fact that \( \mathcal{E}^*(X,p,q) \cap \Psi(I^*(X)) = \{ 0 \} \) follows from Appendix B. \( \square \)

**Definition 2.5.** For a complex manifolds \( X \), the \( k \)-th Harvey-Lawson spark group of level \((p,q)\) is the spark group
\[
\mathcal{H}^k(X,p,q) := \mathcal{H}^k(D^*(X,p,q), \mathcal{E}^*(X,p,q), I^*(X))
\]

**Proposition 2.6.** On a complex manifolds \( X \), the complex conjugation on currents induced by the complex structure of \( X \) induces a map \( \mathcal{H}^k(X,p,q) \to \mathcal{H}^k(X,q,p) \) defined by
\[
(a, b, r) \mapsto (\overline{a}, \overline{b}, \overline{r})
\]
which is an isomorphism.

**Proof.** Note that \( \overline{\pi_p(a)} = \pi_p(\overline{a}) \) and \( \overline{\pi_q(b)} = \pi_q(\overline{b}) \). If \( d_{p,q}(a, b) = (e_1, e_2) - (\pi_p(r), \pi_q(\overline{r})) \), then \( d_{p,q}(a, b) = (\pi_p(\overline{e_1}), \pi_q(\overline{e_2})) = (\overline{e_1}, \overline{e_2}) - (\pi_p(\overline{r}), \pi_q(\overline{\overline{r}})) \), so \( d_{q,p}(\overline{a}, \overline{b}) = (\pi_q(\overline{a}), \pi_p(\overline{b})) = (\overline{e_1}, \overline{e_2}) - (\pi_q(\overline{r}), \pi_p(\overline{\overline{r}})) \) which implies that \( (\overline{a}, \overline{b}, \overline{r}) \in \mathcal{H}^k(X,q,p) \). This map is well defined and applies it twice we get the minus identity which shows that it is an isomorphism. \( \square \)

### 2.1 Aeppli-Bott-Chern cohomology as a hypercohomology

Fix a complex manifold \( X \). Let \( \Omega^k, \overline{\Omega}^k \) be the sheaves of holomorphic \( k \)-forms and anti-holomorphic \( k \)-forms on \( X \) respectively. Recall that the Aeppli and Bott-Chern cohomology for a complex manifold \( X \) can be defined by the hypercohomology of the complex of sheaves: if \( q \geq p \),

\[
B_{p,q}^k : 0 \to \mathcal{O} \oplus \overline{\mathcal{O}} \to \Omega^1 \oplus \overline{\Omega}^1 \to \cdots \to \Omega^{p-1} \oplus \overline{\Omega}^{p-1} \rightarrow \overline{\Omega}^p \rightarrow \cdots \to \overline{\Omega}^{q-1} \rightarrow 0,
\]

we have
\[
H^{p,q}_{\text{BC}}(X; \mathcal{O}) \cong H^{p+q+1}(X, B_{p+1,q+1}) \quad \text{and} \quad H^{p,q}_{\text{BC}}(X; \mathcal{O}) \cong H^{p+q}(X, B_{p,q}).
\]

Modifying accordingly we have the case for \( p \geq q \).
Definition 2.7. Let $\Omega^{\cdot \cdot \cdot \cdot \cdot \cdot < p < q}$ be the complex of sheaves
\[ O \oplus O \rightarrow \Omega^1 \oplus \Omega^1 \rightarrow \cdots \rightarrow \Omega^{p-1} \oplus \Omega^{p-1} \rightarrow \Omega^p \rightarrow \cdots \rightarrow \Omega^{q-1} \rightarrow 0 \]
if $p < q$, and the complex of sheaves:
\[ O \oplus O \rightarrow \Omega^1 \oplus \Omega^1 \rightarrow \cdots \rightarrow \Omega^{p-1} \oplus \Omega^{p-1} \rightarrow \Omega^p \rightarrow \cdots \rightarrow \Omega^{q-1} \rightarrow 0 \]
if $p \geq q$

Similar to the definition of Deligne cohomology, we define Aeppli-Bott-Chern cohomology as following.

Definition 2.8. The Aeppli-Bott-Chern cohomology $H_{ABC}^k(X; \mathbb{Z}(p,q))$ is defined to be the hypercohomology group $\mathbb{H}^k(X, \mathbb{Z} \rightarrow \Omega^{<p \cdot \cdot q})$. If without confusion, we will just call this cohomology the ABC cohomology.

Proposition 2.9. There is an isomorphism
\[ H_{ABC}^k(X; \mathbb{Z}(p,q)) \cong H^{k-1}(\text{Cone}(I^\bullet(X) \oplus \mathbb{Z})(X, p, q))) \]
where $\text{Cone}(I^\bullet(X) \oplus \mathbb{Z})(X, p, q)$ is the cone complex associated to the cochain morphism $\Psi_{p,q} : I^\bullet(X) \rightarrow D^\bullet(X, p, q)$.

Proof. We prove only the case $q \geq p$. There are acyclic resolutions
\[ Z \rightarrow I^\bullet \text{ and } \Omega^\bullet \oplus \bar{\Omega} \rightarrow D^\bullet \oplus D^\bullet \]
Define $\eta_k : I^k \rightarrow D^{k,0} \oplus D^{0,k}$ by
\[ \eta_k(r) = (\Pi_{k,0}(r), \Pi_{0,k}(r)) \]
where $\Pi_{i,j} : I^k \rightarrow D^{i,j}$ is the natural projection induced from the decomposition
\[ I^k \rightarrow D^k = \bigoplus_{i+j=k} D^{i,j} \]
Then we have a commutating diagram of sheaves:
\[
\begin{array}{ccc}
I^\bullet & \bigoplus_{i+j=k} & D^{0,0} \oplus D^{0,0} \bullet \\
\uparrow & & \uparrow \\
\uparrow Z & \rightarrow \Omega^\bullet \oplus \bar{\Omega} & \rightarrow \Omega^1 \oplus \bar{\Omega} \rightarrow \cdots
\end{array}
\]

Let $D^{i,j} = 0$ if $i$ or $j$ equals to -1. Then we have a more uniform expression of the resolution of sheaves
\[ \Omega^\bullet \oplus \bar{\Omega} \rightarrow D^{n_0,0} \oplus D^{0,n_0} \bullet \rightarrow D^{n_1,1} \oplus D^{1,n_1} \rightarrow \cdots \rightarrow D^{n_0,0} \oplus D^{0,n_0} \bullet \]
where
\[ n_i = \begin{cases} i, & \text{if } i < p \\ -1, & \text{if } i \geq p \end{cases} \quad n_i = \begin{cases} i, & \text{if } i < q \\ -1, & \text{if } i \geq p \end{cases} \]

Let $F^{i,j} = D^{n_0,0} \oplus D^{0,n_0}$, then $F^k := \bigoplus_{i+j=k} F^{i,j}$ and $F^k(X) = D^k(X, p, q)$. By Proposition A.3, the hypercohomology
\[ \mathbb{H}^k(X, \mathbb{Z} \rightarrow \Omega^{<p \cdot \cdot q}) \cong H^{k-1}(\text{Cone}(\Psi_{p,q} : I^\bullet(X) \rightarrow F^\bullet(X))) = H^{k-1}(\text{Cone}(\Psi_{p,q} : I^\bullet(X) \rightarrow D^\bullet(X, p, q))) \]
\[ \square \]

Corollary 2.10. There is a short exact sequence
\[ 0 \rightarrow H_{ABC}^{k+1}(X; \mathbb{Z}(p,q)) \rightarrow \tilde{H}^k(X, p, q) \delta^k \rightarrow Z^k_{p,q}(X, p, q) \rightarrow 0 \]

Proof. Consider the $3 \times 3$-grid in Proposition 2.3 associated to the spark complex $\mathcal{S} = (D^\bullet(X, p, q), \mathcal{E}^\bullet(X, p, q), I^\bullet(X))$. By result above, we may replace the cohomology of the cone complex in the middle row of the $3 \times 3$-grid by the ABC cohomology. \[ \square \]
Corollary 2.11. On a complex manifold $X$, the complex conjugation on currents induces an isomorphism between $H^k_{ABC}(X, p, q)$ and $H^k_{ABC}(X, q, p)$.

Proof. This follows from Proposition 2.6 by considering $H^k_{ABC}(X, p, q)$ as a subgroup of $\hat{H}^{k-1}(X, p, q)$. □

Definition 2.12. On a compact Kähler manifold $X$, we define the total Griffith’s $p$-th intermediate Jacobian to be the group

$$
\mathcal{T}_p(X) := (F^p H^{2p-1}(X; \mathbb{C})/H^{2p-1}(X; \mathbb{Z})) \bigoplus (\overline{F^p H^{2p-1}(X; \mathbb{C})/H^{2p-1}(X; \mathbb{Z})})
$$

where $F^p H^{2p-1}(X; \mathbb{C}) = \bigoplus_{i+j=2p-1} H^{i,j}(X)$ is the Hodge filtration and $\overline{F^p H^{2p-1}(X; \mathbb{C})}$ is the complex conjugation of $F^p H^{2p-1}(X; \mathbb{C})$.

Corollary 2.13. When $p = q, k = 2p - 1$, on a compact Kähler manifold $X$, the $3 \times 3$-grid has the form

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
TJ_p(X) & \tilde{H}^k_{ABC}(X, p, p) & d_{p,p} & \mathcal{E}^{2p-1}(X, p, p) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^k_{ABC}(X, \mathbb{Z}(p, p)) & \delta_1 & \delta_2 & Z^{2p}(X, p, p) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^{dp}(X) & (\Psi_{p-q})^* & H^{2p}(X, p, p) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 \\
\end{array}
\]

where $H^{dp}(X)$ is the group of Hodge classes.

Let $X$ be a complex manifold $X$. Recall that (see [6, 4]) the Harvey-Lawson spark group of level $p$ is the spark groups of the spark complex

\[
(D^\ast(X, p), E^\ast(X, p), I^\ast(X))
\]

where $D^k(X, p) = \bigoplus_{i+j=k} D^{i,j}(X)$, $E^k(X, p) = \bigoplus_{i+j=k} E^{i,j}(X)$, and $I^\ast(X) \to D^\ast(X, p)$ is the projection map. The Deligne cohomology group $H^{k+1}_{\mathcal{D}}(X, \mathbb{Z}(p))$ sits in the short exact sequence

\[
0 \to H^{k+1}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \tilde{H}^k(X, p) \to Z^{k+1}(X, p) \to 0
\]

Proposition 2.14. 1. We have a morphism between spark complexes

\[
I^\ast \quad D^\ast(X, p, q) \supseteq E^\ast(X, p, q)
\]

where the middle map is given by the natural projection. This morphism induces a morphism between short exact sequences:

\[
\begin{array}{cccc}
0 & H_A^k(X, \mathbb{Z}(p, q)) & \tilde{H}^{k-1}(X, p, q) & Z^k(X, p, q) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & H_{\mathcal{D}}^k(X, \mathbb{Z}(p)) & \tilde{H}^{k-1}(X, p) & Z^k(X, p) & 0 \\
\end{array}
\]
2. For X a complex manifold, there is a commutative diagram

\[ \begin{array}{ccc}
\tilde{H}^k(X, p, q) & \longrightarrow & \tilde{H}^k(X, q) \\
\downarrow & & \downarrow \\
\hat{H}^k(X, p) & \longrightarrow & H^k(X; \mathbb{Z})
\end{array} \]

given by natural projections which induces a commutative diagram

\[ \begin{array}{ccc}
H^k_{ABC}(X; \mathbb{Z}(p, q)) & \longrightarrow & H^k_{ABC}(X, q) \\
\downarrow & & \downarrow \\
H^k_{g}(X, Z(p)) & \longrightarrow & H^k(X; \mathbb{Z})
\end{array} \]

3. For X a compact Kähler manifold, k = p + q − 1, if \( H^{k+1}(X; \mathbb{Z}) \) is a free abelian group, then

\[ \tilde{H}^k(X, p, q) \cong (\mathbb{C}/\mathbb{Z})^t \oplus H^{k+1}(X; \mathbb{Z}) \oplus d_{p, q} \mathcal{E}^k(X, p, q) \]

where \( t = \dim C H^k(X; \mathbb{C}) \).

4. For X a complex manifold, there is a commutative diagram:

\[ \begin{array}{ccc}
\tilde{H}^k(X, p + 1, q + 1) & \longrightarrow & \mathbb{Z}^{k+1} \times (1 + 1, q + 1) \\
\downarrow & & \downarrow \\
H^{k+1}(X; \mathbb{Z}) & \longrightarrow & \bigoplus_{i+j=k} H^i_{A, j} (X) \bigoplus \bigoplus_{j+k=q} H^j_{A, j} (X)
\end{array} \]

where the right vertical arrow is given by \((e_1, e_2) \mapsto ([e_1], [e_2])\), the bottom horizontal arrow is induced by the projection \( \Pi_{i,j} : \mathbb{K}^{k+1} (X) \to D^{i,j} (X) \), and \( H^i_{A, j} (X) \) is the image of the homomorphism \( \Pi_{i,j} : H^{k+1} (X; \mathbb{Z}) \to H^i_{A, j} (X) \).

\[ \begin{array}{rcl}
Proof. & 1. & This follows directly from definition. \\
& 2. & The morphisms are \\
& & \[ [a, b, r] \longrightarrow [b, r] \] \\
& & \[ [a, r] \longrightarrow [r] \]
\]

3. In a compact Kähler manifold, \( k = p + q - 1 \), \( H^k(X; \Omega^{p+q-p, \bullet} < p, p, q) = \bigoplus_r H^r(X; \oplus_{r+k} H^{r,k} (X)) \).

H^k(X; \mathbb{C}) \). Note that \( H^k(E^i(X; p, q)) \cong \mathbb{H}^k(X; \Omega^{< p, p, q} \cup \mathcal{E}^i(X, p, q)) \). Now consider the \( 3 \times 3 \)-grid associated to the spark complex \( \mathcal{S} = (D^p (X, p, q), E^i (X, p, q), F^j (X)) \). Since \( H^{k+1}(X; \mathbb{Z}) \) is a free abelian group, the middle column of the \( 3 \times 3 \)-grid splits. Since \( d_{p, q} \mathcal{E}^k(X, p, q) \) is a vector space, the top row of the \( 3 \times 3 \)-grid also splits. Thus we have \( \tilde{H}^k(X, p, q) \cong \frac{H^k(E^i (X; p, q))}{H^k(E^i (X, p, q))} \oplus H^{k+1} (X; \mathbb{Z}) \oplus d_{p, q} \mathcal{E}^k(X, p, q) \) and the result follows.

4. Recall that the Aeppli cohomology is defined as \( H^{i,j}_\Lambda(X) = \frac{K_{\Lambda}}{\text{imag}(\Lambda)} \). For \((e_1, e_2) \in Z^k(X, p + 1, q + 1), \pi_{p+1} d e_1 = 0 \). By comparing the types of both sides, we get \( \partial d_e^{k+1, q+1} = 0 \). This implies that \( \partial \pi_{k+1} d_e^{k+1, q+1} = 0 \) for \( i = 1, 2, \ldots, p \). Similarly, \( \partial d_e^{k+1, q+1} = 0 \) for \( j = 1, 2, \ldots, q \).

So \((e_1, e_2) \in \bigoplus_{i+j=k} H^{i,j}_\Lambda(X) \bigoplus \bigoplus_{j+k=q} H^{j,j}_\Lambda(X) \). Note that if \( d \alpha = 0 \), then \( d \partial \alpha^{i,j} = 0 \) where \( \alpha = \sum_{i+j+k+1} \alpha^{i,j} \), and \( \Pi_{i,j} (d \beta) = \partial \beta^{i,j} + \partial \beta^{i,j} \) for \( \beta = \sum_{i+j+k} \beta^{i,j} \). This implies that \( d_{p, q} \) is well defined. The commutativity of this diagram is quite clear. Since \( \delta_1 \) is surjective, the right vertical homomorphism has image as indicated.
2.2 Lefschetz property

Let us recall that when $X$ is compact Kähler, the Lefschetz decomposition of forms induces a decomposition on currents. We summarize several properties that we need in the following: suppose that the dimension of $X$ is $n$.

1. $D^k(X) = \sum_{i \geq k} L^i P^{k-2i}(X)$ where $P^k(X) = \{ \alpha \in D^k(X) | [L_{n-k-1}] \alpha = 0 \}$ is the primitive part, $i_0 = \max \{ i-\alpha, 0 \}$, the Lefschetz operator $L^{n-k} : D^k(X) \to D^{2n-k}(X)$ is an isomorphism, and $L^i : D^k(X) \to D^{i+k}(X)$ is injective if $j \leq n-i$.

2. If $a = \sum_{i \geq k} L^i a_i \in D^k(X)$ is the Lefschetz decomposition of $a$ where $i_0 = \max \{ i-\alpha, 0 \}$, $a_i \in P^{k-2i}(X)$, define $Ta = \sum_{i \geq i_1} L^{-1-i_1} a_i$ where $i_1 = \max \{ i-\alpha, 1 \}$, then $T^{n-k}$ is the inverse of $L^{n-k} : D^k(X) \to D^{2n-k}(X)$ and $T^{n-k} \circ L^{n-k} = \text{id}_{n-1} : D^k(X) \to D^k(X)$ if $k \leq n$.

Proposition 2.15. Suppose that $p + q = k + 1$ and $k \leq n$, then the map $L^{n-k}$ induces monomorphisms

$L^{n-k} : \tilde{D}^{k-1}(X,p,q;\mathbb{Q}) \to \tilde{D}^{2n-k+1}(X,n-q,n-p;\mathbb{Q})$

and

$L^{n-k} : H^k_{ABC}(X;\mathbb{Z}(p,q)) \to H^{2n-k}_{ABC}(X;\mathbb{Z}(n-q,n-p))$

where $\mathbb{Q}$ indicates original groups tensored with $\mathbb{Q}$. These monomorphisms are isomorphisms if the primitive cohomology $PH^{k-1}(X;\mathbb{Q}) = 0$.

Proof. Note that $Ld = dL$, $Td = dT$ and $L^i \pi_p = \pi_{p+i} L^i$, $L^i \pi_q = \pi_{q+i} L^i$. The maps are well defined and injective by the properties of Lefschetz decomposition mentioned above. Note that for $[(a',b',r')] \in \tilde{H}^{n-k+1}(X,n-q,n-p;\mathbb{Q})$, we have $[(L^{n-k} a', T^{n-k} b', T^{n-k} r')] \in \tilde{H}^{k-1}(X,p,q;\mathbb{Q})$, and $L^{n-k}([T^{n-k} a', T^{n-k} b', T^{n-k} r']) = [(a' - a_{n-k-1}, b' - b_{n-k-1}, r')]$ where $a' = \sum_{i \geq n-k-1} L^i a_i$, $b' = \sum_{i \geq n-k-1} L^i b_i$ are the Lefschetz decomposition of $a'$ and $b'$. Thus if $PH^{k-1}(X;\mathbb{Q}) = 0$, then $a_{n-k-1} = dc$, $b_{n-k-1} = dc$, and $[(a_{n-k-1}, b_{n-k-1}, 0)] = 0 \in \tilde{H}^{k-1}(X,p,q;\mathbb{Q})$. By restriction, the same holds for Apell-Bott-Chern cohomology with $\mathbb{Q}$-coefficients. \hfill \Box

3 Ring structure on Harvey-Lawson spark groups of level $(p, q)$

Let $X$ be a complex manifold of compact dimension $m$.

Definition 3.1. Let $(D^k(X))^2 = D^k(X) \oplus D^k(X), (E^k(X))^2 = E^k(X) \oplus E^k(X)$, and $\Psi : I^k(X) \to (D^k(X))^2$ be defined by $r \mapsto (r, r)$. Then $((D^* X)^2, (E^* X)^2, I^*(X))$ is a spark complex. Let

$\tilde{H}_{D^2}(X) := \tilde{H}^k((D^* X)^2, (E^* X)^2, I^*(X))$

To define the ring structure on $\tilde{H}_{D^2}(X)$, we need a modified version of [5] Thm D.1. If $(a, r)$ is a spark and $da = e - r$, we write $d_1 a = e, d_2 a = r$.

Lemma 3.2. For given $\alpha \in \tilde{H}^k(X), \beta \in \tilde{H}^l(X)$ with $k + l \leq 2m$ and $(a_1, a_2, r) \in \alpha$, there is representative $(b_1', b_2', s') \in \beta$ such that if $d(a_1, a_2) = (e_1, e_2) - (r, r)$, $d(b_1', b_2') = (\bar{e}_1, \bar{e}_2) - (s, s)$, then $a_1 \wedge b_1', a_1 \wedge s', r \wedge b_1', r \wedge s, a_2 \wedge b_2', a_2 \wedge s', r \wedge b_2'$ are well defined and $r \wedge s'$ is rectifiable.

Proof. Let us recall the construction in [5] Thm D.1. For $[(a, R)] \in \tilde{H}^k(X), [(b, S)] \in \tilde{H}^l(X)$ with $k + l \leq 2m$, $db = \psi - S$, there is a current $b' := f_{e_\chi} b + \chi + \eta$ where $\chi$ is a smooth $\ell$-form, $\eta$ is a smooth $d$-closed $\ell$-form, for which $a \wedge b', a \wedge d b', R \wedge b'$ and $R \wedge d b'$ are well defined, the last one is rectifiable and $(b', f_{e_{\chi}} S)$ is equivalent to $(b, S)$. The functions $f_{e_{\chi}} : X \to X$ are diffeomorphisms close to identity parametrized by points $\xi \in \mathbb{R}^N$ for some $N$. Note that $db' = \psi - f_{e_{\chi}} S$ and $d b' = f_{e_{\chi}} S$. Now we fix two representatives $(a_1, a_2, r) \in \alpha, (b_1, b_2, s) \in \beta$. Since $[(a_1, r)] \in \tilde{H}^k(X), [(b_1, s)] \in \tilde{H}^l(X)$, by the construction above, we may choose $s \in \mathbb{R}^N$ such that $a_1 \wedge f_{e_{\chi}} b_1, a_1 \wedge f_{e_{\chi}} s, r \wedge f_{e_{\chi}} b_1, r \wedge f_{e_{\chi}} s, a_2 \wedge f_{e_{\chi}} b_2, a_2 \wedge f_{e_{\chi}} s, r \wedge f_{e_{\chi}} s$ are all simultaneously well defined and $r \wedge f_{e_{\chi}} s$ is rectifiable.

As in the Harvey-Lawson-Zweck’s construction, there exist some smooth forms $\chi, \eta_1, \chi_2, \eta_2$ and

$b_1' := f_{e_{\chi}} b_1 + \chi + \eta_1, \quad b_2' := f_{e_{\chi}} b_2 + \chi_2 + \eta_2$
such that \((b'_1, f'_{\ell}, s)\) and \((b'_2, f'_{\ell}, s)\) are equivalent to \((b_1, s)\) and \((b_2, s)\) respectively in \(\tilde{H}_\ell(X)\). So by definition, \((b'_1, b'_2, f'_{\ell}, s)\) ∈ \(\beta\) and the products mentioned in the statement of the Lemma are well defined and \(r \land f_{s_{\ell}} s\) is rectifiable.

If \(da = \phi - R, db = \psi - S\) and the product is well defined for these two sparks, we write
\[
a \ast b := a \land \psi + (-1)^{k+1} R \land b
\]

**Definition 3.3.** Suppose that \(\alpha \in \tilde{H}^k_{D_2}(X), \beta \in \tilde{H}^\ell_{D_2}(X)\) with \(k + \ell \leq 2m\). For any representative \((a_1, a_2, r) \in \alpha\), choose representative \((b'_1, b'_2, s') \in \beta\) according to the Lemma above, we define
\[
\alpha \ast \beta := [(a_1 \ast b'_1, a_2 \ast b'_2, r \land s')] \in \tilde{H}^{k+\ell+1}_{D_2}(X)
\]

As in the proof of [5] Thm 3.5, we verify that the product is well defined and graded-commutative. This gives us the following result.

**Proposition 3.4.** \(\tilde{H}^*_D(X)\) is a graded-commutative ring.

The following result is proved as in [10] Proposition 10.2.

**Theorem 3.5.** The map \((\pi_p, \pi'_q, id)_k : \tilde{H}^k_{D_2}(X) \to \tilde{H}^k(X, p, q)\) defined by
\[
[(a, b, r)] \mapsto [(\pi_p(a), \pi'_q(b), r)]
\]
is a surjective group homomorphism and the kernel of the map \((\pi_p, \pi'_q, id)_k\) is an ideal of \(\tilde{H}^*_{D_2}(X)\).

We first make an observation.

**Lemma 3.6.** \(\text{Ker}(\pi_p, \pi'_q, id)_k = \{a \in \tilde{H}^k(X)|\exists (a, b, 0) \in \alpha, a, b\text{ smooth }, \pi_p(a) = 0, \pi'_q(b) = 0\}\).

**Proof.** Suppose \((a, b, r) \in \alpha \in \text{Ker}(\pi_p, \pi'_q, id)_k\). Then there is \((a', b', s) \in D^{k-1}(X, p, q) \oplus I^k(X)\) such that \((\pi_p(a), \pi'_q(b)) = d_{p,q}(a', b') + (\pi_p(s), \pi'_q(s))\) and \(r = -ds\). Let \(\tilde{a} = a - da' - \Psi(s), \tilde{b} = b - db' - \Psi(s)\) and \(\tilde{r} = r + ds = 0\), then \((a, b) - (\tilde{a}, \tilde{b}) = d(a', b') + \Psi(s)\). So \((\tilde{a}, \tilde{b}, 0) \in \alpha\). The other direction is clear. □

**Proof.** Suppose \(\alpha \in \text{Ker}(\pi_p, \pi'_q, id)_k, \beta \in \tilde{H}^k_{D_2}(X)\), choose representatives \((a, b, 0) \in \alpha\) such that \(\pi_p(a) = 0, \pi'_q(b) = 0\), and \((a', b', r') \in \beta\) such that the product is well defined. Suppose \(D(a', b') = (e_1, e_2) - (r', r')\), then
\[
\alpha \ast \beta = [(a \land e_1 + (-1)^{k+1}0 \land r', b \ast e_2 + (-1)^{k+1}0 \land r', 0 \land r')] = [(a \land e_1, b \ast e_2, 0)] \in \text{Ker}(\pi_p, \pi'_q, id)_k
\]

So the kernel is an ideal of \(\tilde{H}^*_D(X)\).

To show the surjectivity, we pick \([(a, b, r)] \in \tilde{H}^k(X, p, q)\). Then by definition, \(d_{p,q}(a, b) = (e_1, e_2) - \Psi_{p,q}(r)\) and \(dr = 0\). From the isomorphism \(H^{k+1}(\mathcal{E}^k(X)^2) \cong H^{k+1}(\mathcal{E}^k(X)^2)\), there is \((a_0, b_0) \in D^{k+1}(X)^2, (e_0, f_0) \in D^{k+1}(X)^2\) such that \(d(a_0, b_0) = (e_0, f_0) - (r, r)\). So \(d_{p,q}(a_0, b_0) = (\pi_p(a_0), \pi'_q(b_0)) = (e_1, e_2) - (\pi_p, \pi'_q)(e_0, f_0)\). By [2] Lemma 1.5., \((a, b) - (a_0, b_0) = (g_1, g_2) + d_{p,q}(h_1, h_2)\) where \((g_1, g_2) \in \mathcal{E}^k(X, p, q), (h_1, h_2) \in D^{k-1}(X, p, q)\). Let \((\tilde{a}, \tilde{b}) = (a_0, b_0) + (g_1, g_2) + d(h_1, h_2)\). Then \(d(\tilde{a}, \tilde{b}) = (e_0, f_0) + d(g_1, g_2) - \Psi(r)\). This implies that \([(\tilde{a}, \tilde{b}, r)] \in \tilde{H}^{k+1}_{D_2}(X)\). Note that \((\pi_p, \pi'_q)(\tilde{a}, \tilde{b}) = (\pi_p, \pi'_q)(a_0, b_0) + (g_1, g_2) + (\pi_p, \pi'_q)d(h_1, h_2) = (\pi_p, \pi'_q)(a, b) = (a, b)\). This proves the surjectivity. □

**Definition 3.7.** Fix \(p, q\). Let \(\Pi = \bigoplus_{k=0}^{2n}(\pi_p, \pi'_q, id)_k\). Then by Theorem above, the kernel of \(\Pi\) is an ideal of \(\tilde{H}^*_D(X)\) and \(\Pi\) is surjective. So we have a group isomorphism
\[
\tilde{H}^*(X, p, q) \cong \tilde{H}^*_{D_2}(X)/\text{Ker}\Pi
\]
The right hand side has a natural ring structure and we define the ring structure of \(\tilde{H}^*(X, p, q)\) by this isomorphism.

The ring structure on the Harvey-Lawson spargk group \(\tilde{H}^*(X, p, q)\) induces a ring structure on the ABC cohomology \(H^*_A^{BC}(X, \mathbb{Z}(*, *))\).
Definition 3.8. We define a product \( H^*_{ABC}(X; \mathbb{Z}(p,q)) \times H^*_{ABC}(X; \mathbb{Z}(p',q')) \rightarrow H^{*+\ell}_{ABC}(X; \mathbb{Z}(p+p', q+q')) \) by
\[
(\alpha, \beta) \mapsto (\pi_{p+p'}, \pi'_{q+q'}, \text{id})(\tilde{\alpha} \ast \tilde{\beta})
\]
where \( \tilde{\alpha} \in \tilde{H}^{k-1}_F(X), \tilde{\beta} \in \tilde{H}^t_{D^2}(X) \) are lifts of \( \alpha \) and \( \beta \) respectively.

To verify that this product is well defined, we refer the reader to the proof of \([4, \text{Theorem 6.6}]\) where a similar verification for Deligne cohomology was done. The following result is also clear from the definition.

Corollary 3.9. The natural map \( T : \bigoplus_{k,p,q} H^k_{ABC}(X; \mathbb{Z}(p,q)) \rightarrow \bigoplus_{k,p} H^k_{ABC}(X; \mathbb{Z}(p)) \) induced from the projection \([a_1,a_2,r]) \mapsto [a_1,r] \) is a ring homomorphism.

4 K-theory and refined Chern classes

By a result of Cheeger and Simons \([2]\), each smooth complex vector bundle with unitary connection \( \nabla \) over a smooth manifold \( X \) has refined Chern classes \( \hat{c}_k(E, \nabla) \in \tilde{H}^{2k-1}(X) \) with \( \delta_1(\hat{c}_k(E, \nabla)) = c_k(\Omega^\nabla) \) the Chern-Weil form, and \( \delta_2(\hat{c}_k(E, \nabla)) = c_k(E) \) the Chern class of \( E \). They also proved a Whitney product formula
\[
\hat{c}(E \oplus E', \nabla \oplus \nabla') = \hat{c}(E, \nabla) \ast \hat{c}(E', \nabla')
\]
where \( \hat{c} \) is the total refined Chern class. In this section, we are going to define refined Chern classes in ABC cohomology and prove some results analogous to the classical counterparts.

Definition 4.1. Let \( X \) be a complex manifold and \( E \) be a complex vector bundles with unitary connections \( \nabla \) over \( X \). Suppose that \( \hat{c}_k(E, \nabla) = [(a, r)] \in \tilde{H}^{2k-1}(X) \). Then \([(a, r)] \in \tilde{H}^{2k-1}(X, k, k) \) and we define
\[
\hat{f}_k(E, \nabla) := (\pi_k, \pi'_{k}, \text{id})[(a, r)] \in \tilde{H}^{2k-1}(X, k, k)
\]

Remark 4.2. If \( E \) is a hermitian bundle and \( \nabla \) is the hermitian connection associated to the hermitian metric of \( E \), then the Chern-Weil form \( \hat{c}_k(\Omega^\nabla) \) is of type \( (k, k) \) and hence \( \delta_1(\hat{f}(E, \nabla)) = 0 \). This implies that \( \hat{f}_k(E, \nabla) \in H^*_{ABC}(X, k(k)) \).

Proposition 4.3. Let \( E \) be a hermitian vector bundle over a complex manifold \( X \) and \( \nabla \) be the canonical connection associated to the hermitian metric of \( E \).

1. The class \( \hat{f}(E, \nabla) \in H^*_{ABC}(X, k(k)) \) is independent of the choice of hermitian metric on \( E \).

2. Under the canonical map from \( H^{2k}_{ABC}(X, k(k)) \rightarrow H^k_{ABC}(X, k(k)) \), the class \( \hat{f}_k \) is sent to \( \hat{c}_k \) where \( \hat{c}_k \) is the Harvey-Lawson’s refined Chern class.

Proof. Suppose that \( \hat{c}_k(E, \nabla_1) = [(a_1, r_1)] \in \tilde{H}^{2k-1}(X), \hat{c}_k(E, \nabla_2) = [(a_2, r_2)] \in \tilde{H}^{2k-1}(X) \). By \([6, \text{Proposition 12.1}]\), Harvey and Lawson showed that their refined Chern classes in Deligne cohomology are independent of the choice of hermitian metric on \( E \), hence \([(\pi_a a_1, r)] = [(\pi_a a_2, r)] \in H^k_{ABC}(X, k(k)) \).

This means that there exist \( b \in D^k(X, k), s \in I^k(X) \) such that
\[
\begin{aligned}
\pi_a a_1 - \pi_a a_2 &= \pi_k d + \pi_k(s), \\
r_1 - r_2 &= -ds,
\end{aligned}
\]
Note that \( \pi_a a = \pi'_k a \) and \( d \) is a real operator. By taking the complex conjugation of the first equation, we get
\[
\pi'_k \pi_a a = \pi'_k d + \pi'_k(s)
\]
Together with equations above, this means that \( \hat{f}_k(E, \nabla_1) = [(\pi'_k a_1, \pi'_k a_1, r_1)] = [(\pi'_k a_2, \pi'_k a_2, r_2)] = \hat{f}_k(E, \nabla_2) \). The class \( \hat{f}_k \) is sent to \( \hat{c}_k \) follows directly from the definition.

Definition 4.4. If \( E \) is a hermitian vector bundle of rank \( k \) on a complex manifold \( X \), we write the total refined Chern class to be
\[
\hat{f}(E) := 1 + \hat{f}_1(E) + \cdots + \hat{f}_k(E) \in \bigoplus_{i=0}^k H^*_{ABC}(X, \mathbb{Z}(i, i))
\]
We first observe that the product in $\hat{H}^\bullet(X)$ commutes with the complex conjugation.

**Lemma 4.5.** For $[(a_1, r_1)] \in \hat{H}^k(X), [(a_2, r_2)] \in \hat{H}^\ell(X)$, we have

$$[(a_1, r_1)] \ast [(a_2, r_2)] = [(a_1, r_1)] \ast [(a_2, r_2)]$$

**Theorem 4.6.** Let $E$ and $F$ be two hermitian vector bundles on a complex manifold $X$. There is a Whitney product formula

$$\hat{f}(E \oplus F) = \hat{f}(E) \ast \hat{f}(F)$$

**Proof.** By the Whitney product formula of Harvey-Lawson’s refined Chern classes [6, Theorem 12.2], $\hat{d}(E \oplus F) = \hat{d}(E) \ast \hat{d}(F)$. This provides the desire formula for the first component of $\hat{f}(E \oplus F)$. By Lemma above, we have $\hat{d}(E \oplus F) = \hat{d}(E) \ast \hat{d}(F)$. This provides the desire formula for the second component of $\hat{f}(E \oplus F)$. 

By [4, Theorem 8.4] and argument similar to the proof above, we have the following result.

**Theorem 4.7.** For any short exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

of holomorphic vector bundles over $X$, we have

$$\hat{f}(E_2) = \hat{f}(E_1) \ast \hat{f}(E_2)$$

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