Character sums with Beatty sequences on Burgess-type intervals

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Abstract
We estimate multiplicative character sums taken on the values of a non-homogeneous Beatty sequence \( \{ \lfloor \alpha n + \beta \rfloor : n = 1, 2, \ldots \} \), where \( \alpha, \beta \in \mathbb{R} \), and \( \alpha \) is irrational. Our bounds are nontrivial over the same short intervals for which the classical character sum estimates of Burgess have been established.

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1 Introduction

For fixed $\alpha, \beta \in \mathbb{R}$, the corresponding non-homogeneous Beatty sequence is the sequence of integers defined by

$$B_{\alpha, \beta} = ([\alpha n + \beta])_{n=1}^\infty,$$

where $[x]$ denotes the greatest integer $\leq x$ for every $x \in \mathbb{R}$. Beatty sequences arise in a variety of apparently unrelated mathematical settings, and because of their versatility, the arithmetic properties of these sequences have been extensively explored in the literature; see, for example, [1, 6, 15, 16, 19, 22] and the references contained therein.

In this paper, we study character sums of the form

$$S_k(\alpha, \beta, \chi; N) = \sum_{n \leq N} \chi([\alpha n + \beta]),$$

where $\alpha$ is irrational, and $\chi$ is a non-principal character modulo $k$. In the special case that $k = p$ is a prime number, the sums $S_p(\alpha, \beta, \chi; N)$ have been previously studied and estimated nontrivially for $N \geq p^{1/3+\varepsilon}$, where $\varepsilon > 0$; see [2, 3].

Here, we show that the approach of [1] (see also [6, 18]), combined with a bound on sums of the form

$$U_k(t, \chi; M_0, M) = \sum_{M_0 < m \leq M} \chi(m) e(tm) \quad (t \in \mathbb{R}), \quad (1)$$

where $e(x) = \exp(2\pi ix)$ for all $x \in \mathbb{R}$, yields a nontrivial bound on the sums $S_k(\alpha, \beta, \chi; N)$ for all sufficiently large $N$ (see Theorem 4.1 below for a precise statement). In particular, in the case that $k = p$ is prime, we obtain a nontrivial bound for all $N \geq p^{1/4+\varepsilon}$, which extends the results found in [2, 3].

It has recently been shown in [5] that for a prime $p$ the least positive quadratic non-residue modulo $p$ among the terms of a Beatty sequence is of size at most $p^{1/(4\sqrt{e}+1/2)+o(1)}$, a result which is complementary to ours. However, the underlying approach of [5] is very different and cannot be used to bound the sums $S_k(\alpha, \beta, \chi; N)$.

We remark that one can obtain similar results to ours by using bounds for double character sums, such as those given in [13]. The approach of this paper, however, which dates back to [1], seems to be more general and can...
be used to estimate similar sums with many other arithmetic functions $f(m)$ provided that appropriate upper bounds for the sums

$$V(t, f; M_0, M) = \sum_{M_0 < m \leq M} f(m) e(tm) \quad (t \in \mathbb{R})$$

are known. Such estimates have been obtained for the characteristic functions of primes and of smooth numbers (see [11] and [12], respectively), as well as for many other functions. Thus, in principle one can obtain asymptotic formulas for the number of primes or smooth numbers in a segment of a Beatty sequence (in the case of smooth numbers, this has been done in [4] by a different method).

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### 2 Notation

Throughout the paper, the implied constants in the symbols $O$ and $\ll$ may depend on $\alpha$ and $\varepsilon$ but are absolute otherwise. We recall that the notations $U = O(V)$ and $U \ll V$ are equivalent to the assertion that the inequality $|U| \leq cV$ holds for some constant $c > 0$.

We also use the symbol $o(1)$ to denote a function that tends to 0 and depends only on $\alpha$ and $\varepsilon$. It is important to note that our bounds are uniform with respect all of the involved parameters other than $\alpha$ and $\varepsilon$; in particular, our bounds are uniform with respect to $\beta$. In particular, the latter means that our bounds also apply to the shifted sums of the form

$$\sum_{M+1 \leq n \leq M+N} \chi([\alpha n + \beta]) = \sum_{n \leq N} \chi([\alpha n + \alpha M + \beta]),$$

and these bounds are uniform for all integers $M$.

In what follows, the letters $m$ and $n$ always denote non-negative integers unless indicated otherwise.

We use $\lfloor x \rfloor$ and $\{ x \}$ to denote the greatest integer $\leq x$ and the fractional part of $x$, respectively.
Finally, recall that the discrepancy $D(M)$ of a sequence of (not necessarily distinct) real numbers $a_1, \ldots, a_M \in [0,1)$ is defined by

$$D(M) = \sup_{I \subseteq [0,1)} \left| \frac{V(I, M)}{M} - |I| \right|,$$

where the supremum is taken all subintervals $I = (c, d)$ of the interval $[0,1)$, $V(I, M)$ is the number of positive integers $m \leq M$ such that $a_m \in I$, and $|I| = d - c$ is the length of $I$.

### 3 Preliminaries

It is well known that for every irrational number $\alpha$, the sequence of fractional parts $\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \ldots$, is uniformly distributed modulo 1 (for instance, see [17, Example 2.1, Chapter 1]). More precisely, let $D_{\alpha, \beta}(M)$ denote the discrepancy of the sequence $(a_m)_{m=1}^M$, where

$$a_m = \{\alpha m + \beta\} \quad (m = 1, 2, \ldots, M).$$

Then, we have:

**Lemma 3.1.** Let $\alpha$ be a fixed irrational number. Then, for all $\beta \in \mathbb{R}$ we have

$$D_{\alpha, \beta}(M) \leq 2D_{\alpha, 0}(M) = o(1) \quad (M \to \infty),$$

where the function implied by $o(1)$ depends only on $\alpha$.

When more information about $\alpha$ is available, the bound of Lemma 3.1 can be made more explicit. For this, we need to recall some familiar notions from the theory of Diophantine approximations.

For an irrational number $\alpha$, we define its type $\tau$ by the relation

$$\tau = \sup \left\{ \vartheta \in \mathbb{R} : \liminf_{q \to \infty, \ q \in \mathbb{Z}^+} \ q^\vartheta \|\alpha q\| = 0 \right\}.$$  

Using Dirichlet’s approximation theorem, it is easy to see that $\tau \geq 1$ for every irrational number $\alpha$. The celebrated theorems of Khinchin [14] and of Roth [20] assert that $\tau = 1$ for almost all real numbers $\alpha$ (with respect to Lebesgue measure) and all algebraic irrational numbers $\alpha$, respectively; see also [7, 21].

The following result is taken from [17, Theorem 3.2, Chapter 2]:

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**Lemma 3.2.** Let $\alpha$ be a fixed irrational number of type $\tau < \infty$. Then, for all $\beta \in \mathbb{R}$ we have

$$D_{\alpha, \beta}(M) \leq M^{-1/\tau + o(1)} \quad (M \to \infty),$$

where the function implied by $o(1)$ depends only on $\alpha$.

Next, we record the following property of type:

**Lemma 3.3.** If $\alpha$ is an irrational number of type $\tau < \infty$ then so are $\alpha^{-1}$ and $a\alpha$ for any integer $a \geq 1$.

Finally, we need the following elementary result, which describes the set of values taken by the Beatty sequence $B_{\alpha, \beta}$ in the case that $\alpha > 1$:

**Lemma 3.4.** Let $\alpha > 1$. An integer $m$ has the form $m = \lfloor \alpha n + \beta \rfloor$ for some integer $n$ if and only if

$$0 < \{ \alpha^{-1}(m - \beta + 1) \} \leq \alpha^{-1}.$$ 

The value of $n$ is determined uniquely by $m$.

**Proof.** It is easy to see that an integer $m$ has the form $m = \lfloor \alpha n + \beta \rfloor$ for some integer $n$ if and only if the inequalities

$$\frac{m - \beta}{\alpha} \leq n < \frac{m - \beta + 1}{\alpha}$$

hold, and since $\alpha > 1$ the value of $n$ is determined uniquely. \hfill \Box

## 4 Character Sums

For every real number $\varepsilon > 0$ and integer $k \geq 1$, we put

$$B_{\varepsilon}(k) = \begin{cases} k^{1/4+\varepsilon} & \text{if } k \text{ is prime;} \\ k^{1/3+\varepsilon} & \text{if } k \text{ is a prime power;} \\ k^{3/8+\varepsilon} & \text{otherwise.} \end{cases} \tag{3}$$

**Theorem 4.1.** Let $\alpha > 0$ be a fixed irrational number, and let $\varepsilon > 0$ be fixed. Then, uniformly for all $\beta \in \mathbb{R}$, all non-principal multiplicative characters $\chi$ modulo $k$, and all integers $N \geq B_{\varepsilon}(k)$, we have

$$S_k(\alpha, \beta; N) = o(N) \quad (k \to \infty),$$

where the function implied by $o(N)$ depends only on $\alpha$ and $\varepsilon$.\hfill \Box
Proof. We can assume that $\varepsilon < 1/10$, and this implies that $B_{\varepsilon}(k) \leq k^{2/5}$ in all cases. Observe that it suffices to prove the result in the case that $B_{\varepsilon}(k) \leq N \leq k^{1/2}$. Indeed, assuming this has been done, for any $N > k^{1/2}$ we put $N_0 = \lfloor k^{9/20} \rfloor$ and $t = \lfloor N/N_0 \rfloor$; then, since $B_{\varepsilon}(k) \leq N_0 \leq k^{1/2}$ we have

$$S_k(\alpha, \beta; N) = \sum_{j=0}^{t-1} \sum_{n \leq N_0} \chi(\lfloor \alpha n + jN_0 + \beta \rfloor) + \sum_{tN_0 < n \leq N} \chi(\lfloor \alpha n + \beta \rfloor)$$

$$= \sum_{j=0}^{t-1} S_k(\alpha + \alpha jN_0, \beta; N_0) + O(N_0)$$

$$= o(tN_0) + O(Nk^{-1/20}) = o(N) \quad (k \to \infty)$$

using the fact that our bounds are uniform with respect to $\beta$.

We first treat the case that $\alpha > 1$. Put $\gamma = \alpha^{-1}$, $\delta = \alpha^{-1}(1 - \beta)$, $M_0 = \lfloor \alpha + \beta - 1 \rfloor$, and $M = \lfloor \alpha N + \beta \rfloor$. From Lemma 3.3 we see that

$$S_k(\alpha, \beta, \chi; N) = \sum_{M_0 < m < M} \chi(m) = \sum_{M_0 < m < M} \chi(m) \psi(\gamma m + \delta), \quad (4)$$

where $\psi(x)$ is the periodic function with period one for which

$$\psi(x) = \begin{cases} 
1 & \text{if } 0 < x \leq \gamma; \\
0 & \text{if } \gamma < x \leq 1.
\end{cases}$$

By a classical result of Vinogradov (see [23, Chapter 2, Lemma 2]) it is known that for any $\Delta$ such that

$$0 < \Delta < \frac{1}{8}$$

and

$$\Delta \leq \frac{1}{2} \min\{\gamma, 1 - \gamma\},$$

there is a real-valued function $\psi_{\Delta}(x)$ with the following properties:

- $\psi_{\Delta}(x)$ is periodic with period one;
- $0 \leq \psi_{\Delta}(x) \leq 1$ for all $x \in \mathbb{R}$;
- $\psi_{\Delta}(x) = \psi(x)$ if $\Delta \leq x \leq \gamma - \Delta$ or $\gamma + \Delta \leq x \leq 1 - \Delta$;
• $\psi_\Delta(x)$ can be represented as a Fourier series

$$\psi_\Delta(x) = \gamma + \sum_{j=1}^{\infty} \left( g_j e(jx) + h_j e(-jx) \right),$$

where the coefficients $g_j, h_j$ satisfy the uniform bound

$$\max\{|g_j|, |h_j|\} \ll \min\{j^{-1}, j^{-2}\Delta^{-1}\} \quad (j \geq 1).$$

Therefore, from (4) we derive that

$$S_k(\alpha, \beta, \chi; N) = \sum_{M_0 < m \leq M} \chi(m) \psi_\Delta(\gamma m + \delta) + O(V(\mathcal{I}, M_0, M)), \quad (5)$$

where $V(\mathcal{I}, M_0, M)$ denotes the number of integers $M_0 < m \leq M$ such that

$$\{\gamma m + \delta\} \in \mathcal{I} = [0, \Delta) \cup (\gamma - \Delta, \gamma + \Delta) \cup (1 - \Delta, 1).$$

Since $|\mathcal{I}| \ll \Delta$, it follows from Lemma 3.1 and the definition (2) that

$$V(\mathcal{I}, M_0, M) \ll \Delta N + o(N), \quad (6)$$

where the implied function $o(N)$ depends only on $\alpha$.

To estimate the sum in (5), we insert the Fourier expansion for $\psi_\Delta(\gamma m + \delta)$ and change the order of summation, obtaining

$$\sum_{M_0 < m \leq M} \chi(m) \psi_\Delta(\gamma m + \delta) = \gamma U_k(0, \chi; M_0, M)$$

$$+ \sum_{j=1}^{\infty} g_j e(\delta j) U_k(\gamma j, \chi; M_0, M) + \sum_{j=1}^{\infty} h_j e(-\delta j) U_k(-\gamma j, \chi; M_0, M),$$

where the sums $U_k(t, \chi; M_0, M)$ are defined by (1).

Since $M - M_0 \ll N$, using the well known results of Burgess [8, 9, 10] on bounds for partial Gauss sums, it follows that for any fixed $\varepsilon > 0$ there exists $\eta > 0$ such that

$$U_k(a/k, \chi; M_0, M) \ll N^{1-\eta} \quad (7)$$

holds uniformly for all $N \geq B_\varepsilon(k)$ and all integers $a$; clearly, we can assume that $\eta \leq 1/10$.
Put \( r = \lfloor \gamma k \rfloor \). Then, for any integer \( n \), we have
\[
e(\gamma n) - e(\gamma n/k) \ll |\gamma n - rn/k| \leq |n|k^{-1},
\]
which implies that
\[
U_k(\gamma j, \chi; M_0, M) = U_k(rj/k, \chi; M_0, M) + O(N^2k^{-1}|j|).
\]
Using (7) in the case that \( |j| \leq kN^{-1-\eta} \) we derive that
\[
U_k(\gamma j, \chi; M_0, M) \ll N^{1-\eta},
\]
and for \( |j| > kN^{-1-\eta} \) we use the trivial bound
\[
|U_k(\gamma j, \chi; M_0, M)| \ll N.
\]
Consequently,
\[
\sum_{M_0 < m \leq M} \chi(m) \psi(\gamma m + \delta) \ll N^{1-\eta} \sum_{j \leq kN^{-1-\eta}} (|g_j| + |h_j|) + N \sum_{j > kN^{-1-\eta}} (|g_j| + |h_j|)
\]
\[
\ll N^{1-\eta} \sum_{j \leq kN^{-1-\eta}} j^{-1} + N\Delta^{-1} \sum_{j > kN^{-1-\eta}} j^{-2}
\]
\[
\ll N^{1-\eta} \log k + N^{2+\eta} \Delta^{-1} k^{-1}.
\]
Since \( N^2 \leq k \leq N^4 \), we see that
\[
\sum_{M_0 < m \leq M} \chi(m) \psi(\gamma m - \delta) \ll N^{1-\eta} \log N + N^{\eta} \Delta^{-1}.
\]
(8)

Inserting the bounds (6) and (8) into (5), choosing \( \Delta = N^{(\eta-1)/2} \), and taking into account that \( 0 < \eta \leq 1/10 \), we complete the proof in the case that \( \alpha > 1 \).

If \( \alpha < 1 \), put \( a = \lfloor \alpha^{-1} \rfloor \) and write
\[
S_k(\alpha, \beta, \chi; N) = \sum_{n \leq N} \chi(\lfloor \alpha n + \beta \rfloor)
\]
\[
= \sum_{j=0}^{a-1} \sum_{m \leq (N-j)/a} \chi(\lfloor a\alpha m + \alpha j + \beta \rfloor)
\]
\[
= \sum_{j=0}^{a-1} S_k(\alpha a, \alpha j + \beta, \chi; (N-j)/a).
\]

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Applying the preceding argument with the irrational number $\alpha > 1$, we conclude the proof.

For an irrational number $\alpha$ of type $\tau < \infty$, we proceed as in the proof of Theorem 4.1 using Lemma 3.2 instead of Lemma 3.1 and also applying Lemma 3.3; this yields the following statement:

**Theorem 4.2.** Let $0 < \alpha$ be a fixed irrational number of type $\tau < \infty$. For every fixed $\varepsilon > 0$ there exists $\rho > 0$, which depends only on $\varepsilon$ and $\tau$, such that for all $\beta \in \mathbb{R}$, all non-principal multiplicative characters $\chi$ modulo $k$, and all integers $N \geq B_{\varepsilon}(k)$, we have

$$S_k(\alpha, \beta; \chi; N) \ll Nk^{-\rho}.$$ 

**References**

[1] A. G. Abercrombie, ‘Beatty sequences and multiplicative number theory’, *Acta Arith.* 70 (1995), 195–207.

[2] W. Banks and I. E. Shparlinski, ‘Non-residues and primitive roots in Beatty sequences’, *Bull. Aust. Math. Soc.* (to appear).

[3] W. Banks and I. E. Shparlinski, ‘Short character sums with Beatty sequences’, *Math. Res. Lett.* (to appear).

[4] W. Banks and I. E. Shparlinski, ‘Prime divisors in Beatty sequences’, *J. Number Theory* (to appear).

[5] W. Banks, M. Z. Garaev and I. E. Shparlinski, ‘Density of non-residues in short intervals’, *Preprint* 2006, 1-15.

[6] A. V. Begunts, ‘An analogue of the Dirichlet divisor problem’, *Moscow Univ. Math. Bull.* 59 (2004), no. 6, 37–41.

[7] Y. Bugeaud, *Approximation by algebraic numbers*, Cambridge Univ. Press, Cambridge, 2004.

[8] D. A. Burgess, ‘Partial Gaussian sums’, *Bull. London Math. Soc.* 20 (1988), no. 6, 589–592.
[9] D. A. Burgess, ‘Partial Gaussian sums, II’, Bull. London Math. Soc. 21 (1989), no. 2, 153–158.

[10] D. A. Burgess, ‘Partial Gaussian sums. III’, Glasgow Math. J. 34 (1992), no. 2, 253–261.

[11] H. Davenport, Multiplicative number theory, 2nd ed., Springer-Verlag, New York 1980.

[12] E. Fouvry and G. Tenenbaum, ‘Entiers sans grand facteur premier en progressions arithmétiques’, London Math. Soc. 63 (1991), 449–494.

[13] J. Friedlander and H. Iwaniec, ‘Estimates for character sums’, Proc. Amer. Math. Soc. 119 (1993), 365–372.

[14] A. Y. Khinchin, ‘Zur metrischen Theorie der diophantischen Approximationen’, Math. Z. 24 (1926), no. 4, 706–714.

[15] T. Komatsu, ‘A certain power series associated with a Beatty sequence’, Acta Arith. 76 (1996), 109–129.

[16] T. Komatsu, ‘The fractional part of \( n\vartheta + \varphi \) and Beatty sequences’, J. Théor. Nombres Bordeaux 7 (1995), 387–406.

[17] L. Kuipers and H. Niederreiter, Uniform distribution of sequences, Wiley-Interscience, New York-London-Sydney, 1974.

[18] G. S. Lü and W. G. Zhai, ‘The divisor problem for the Beatty sequences’, Acta Math. Sinica 47 (2004), 1213–1216 (in Chinese).

[19] K. O’Bryant, ‘A generating function technique for Beatty sequences and other step sequences’, J. Number Theory 94 (2002), 299–319.

[20] K. F. Roth, ‘Rational approximations to algebraic numbers’, Mathematika 2 (1955), 1–20 (see also ‘Corrigendum’, 2 (1955), 168).

[21] W. M. Schmidt, Diophantine Approximation, Springer-Verlag, Berlin, 1980.

[22] R. Tijdeman, ‘Exact covers of balanced sequences and Fraenkel’s conjecture’, Algebraic number theory and Diophantine analysis (Graz, 1998), 467–483, de Gruyter, Berlin, 2000.
[23] I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*, Dover Publ., NY, 2004.