Convergence of the Time Discrete Metamorphosis Model on Hadamard Manifolds

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Continuous image morphing is a classical task in image processing. The metamorphosis model proposed by Trouvé, Younes and coworkers [34, 45] casts this problem in the frame of Riemannian geometry and geodesic paths between images. The associated metric in the space of images incorporates dissipation caused by a viscous flow transporting image intensities and its variations along motion paths. In this paper, we propose a generalized metamorphosis model for manifold-valued images, where the range space is a finite dimensional Hadamard manifold. A corresponding time discrete version was presented in [36] based on the general variational time discretization proposed in [11]. Here, we prove for Hadamard valued images convergence of time discrete geodesic paths to a geodesic path in the continuous metamorphosis model and thereby in particular establish the existence of geodesic paths.

1 Introduction

Image morphing amounts to computing a visually appealing transition of two images such that image features in the reference image are mapped to corresponding image features in the target image whenever possible.

A particular model for image morphing known as image metamorphosis was proposed by Miller, Trouvé, and Younes [34, 45, 44]. It is based on the flow of diffeomorphism model and the large deformation diffeomorphic metric mapping (LDDMM), which dates back to the work of Arnold, Dupuis, Grenander and others [3, 4, 21, 8, 27, 32, 48, 47].

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From the perspective of the flow of diffeomorphism model, each point of the reference image is transported to the target image in an energetically optimal way such that the image intensity is preserved along the trajectories of the pixels. The metamorphosis model additionally allows for image intensity modulations along the trajectories by incorporating the magnitude of these modulations, which is reflected by the integrated squared material derivative of the image trajectories as a penalization term in the energy functional. Recently, the metamorphosis model has been extended to images in reproducing kernel Hilbert spaces [41], to functional shapes [14] and discrete measures [40]. For a more detailed exposition of these models we refer the reader to [50, 33] and the references therein.

A variational time discretization of the metamorphosis model for square-integrable images $L^2(\Omega, \mathbb{R}^m)$ was proposed in [11]. Furthermore, existence of discrete geodesic paths and the Mosco-convergence of the time discrete to the time continuous metamorphosis model was proven. In [36], the time discrete model was extended to the set of image $L^2(\Omega, \mathcal{H})$, where $\mathcal{H}$ denotes a finite dimensional Hadamard manifold. Recall that Hadamard manifolds are Hadamard spaces with a special Riemannian structure having non-positive sectional curvature (for details see below). In [6], it is revealed that many concepts of Banach spaces can be generalized to Hadamard spaces, which are therefore a proper choice for the analytical treatment of algorithms for manifold-valued images. They are at the same time very relevant for the processing of manifold-valued images in different applications.

Throughout the past years, manifold-valued images have received increased attention (see e.g. [7, 16, 30, 49, 9]). Some prominent applications for Hadamard manifold-valued images are the following:

- Diffusion tensor magnetic resonance imaging is an image acquisition method that incorporates in vivo magnetic resonance images of biological tissues driven by local molecular diffusion. The range space of the resulting images is frequently the space of symmetric and positive semidefinite matrices [5, 15, 23, 46].

- Retina data is commonly modeled as images with values in the manifold of univariate non-degenerate Gaussian probability distributions endowed with the Fisher metric [2, 10]. This space is isometric to the hyperbolic space, which can be exploited numerically.

In this paper, we prove convergence of the manifold-valued time discrete geodesic paths to geodesic paths in the proposed manifold-valued metamorphosis model, which coincides with the original metamorphosis energy functional in the Euclidean space. The proof of convergence in [11] imports as an essential ingredient a representation formula for images via integration of the weak material derivative along motion paths for the time continuous metamorphosis model in the Euclidean setting. Here, we no longer make use of such a representation formula. Indeed, our convergence result can thus be considered as a stronger result even in the case of images as pointwise maps into a Euclidean space.
Throughout this paper, we assume that the image domain $\Omega \subset \mathbb{R}^n$ for $n \in \{2, 3\}$ has Lipschitz boundary. Henceforth, we denote time continuous operators by calligraphic letters and time discrete operators by normal letters. We use standard notation for Lebesgue and Sobolev spaces on the image domain $\Omega$, i.e. $L^p(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$. The associated norms are denoted by $\| \cdot \|_{L^p(\Omega)}$ and $\| \cdot \|_{H^m(\Omega)}$, respectively, and the seminorm in $H^m(\Omega)$ is given by $| \cdot |_{H^m(\Omega)}$. For any $f, g \in H^m(\Omega)$, $m \geq 1$, we set

$$D^m f \cdot D^m g = \sum_{i_1, \ldots, i_m = 1}^n \frac{\partial^m f}{\partial i_1 \cdots \partial i_m} \cdot \frac{\partial^m g}{\partial i_1 \cdots \partial i_m}, \quad |D^m f| = (D^m f \cdot D^m f)^{1/2}.$$ 

Then, the Sobolev (semi-)norm is defined as

$$|f|_{H^m(\Omega)} = \|D^m f\|_{L^2(\Omega)}, \quad \|f\|_{H^m(\Omega)} = \left( \sum_{j=0}^m |f|_{H^j(\Omega)}^2 \right)^{1/2}.$$ 

The symmetric part of a matrix $A \in \mathbb{R}^{l,l}$ is denoted by $A^{\text{sym}}$, i.e. $A^{\text{sym}} = \frac{1}{2}(A + A^\top)$. We denote by $GL^+(n)$ the elements of $GL(n)$ with positive determinant, by $\mathbf{I}$ the identity matrix, and by $\text{Id}$ the identity map.

**Mosco–convergence** We conclude this section with a brief review of Mosco–convergence, which can be seen as a generalization of $\Gamma$–convergence. For further details we refer the reader to [20, 35].

**Definition 1** (Mosco–convergence). Let $(X, d)$ be a Hadamard space and let $\{J_k\}_{k \in \mathbb{N}}$ and $J$ be functionals mapping from $X$ to $\mathbb{R}$. Then the sequence $J_k$ is said to converge to $J$ in the sense of Mosco w.r.t. the topology induced by $d$ if the following holds:

1. For every sequence $\{x_k\}_{k \in \mathbb{N}} \subset X$ with $x_k \rightharpoonup x \in X$ it holds
   \[ J(x) \leq \liminf_{k \to \infty} J_k(x_k). \]  \hspace{1cm} (liminf-inequality)

2. For every $x \in X$ there exists a recovery sequence $\{x_k\}_{k \in \mathbb{N}} \subset X$ such that $x_k \to x \in X$ and
   \[ J(x) \geq \limsup_{k \to \infty} J_k(x_k). \]  \hspace{1cm} (limsup-inequality)

If in 1 the strong convergence of $x_k$ to $x$ in the topology induced by $d$ is required, then $J_k$ is said to $\Gamma$–converge to $J$ w.r.t. the topology induced by $d$.

This paper is organized as follows: In section 2 we briefly recall some preliminaries of Hadamard manifolds as well as the metamorphosis model in the Euclidean case and its time discretization on Hadamard manifolds. Then, in section 3 the novel manifold-valued metamorphosis model is introduced and the equivalence to the original metamorphosis model in the case of Euclidean spaces is proven. Section 4 is devoted to the temporal extension of all relevant quantities as required for the convergence proof. Finally, section 5 contains the precise statement of the convergence result in the manifold-valued case.
2 Review and preliminaries

In this section, we briefly present some preliminaries of Hadamard manifolds, a short introduction to the metamorphosis model in the Euclidean setting, and the manifold-valued time discrete metamorphosis model [11, 36].

2.1 Hadamard manifolds

In what follows, a short introduction of Hadamard manifolds is provided and the space of Hölder continuous functions on Hadamard manifolds is analyzed. For further details we refer the reader to the books [6, 13, 28].

![Comparison Triangle](image)

**Figure 1**: Comparison triangle in the Euclidean space $\mathbb{R}^2$ and geodesic triangle on a Hadamard manifold (Figure adapted from [6, Figure 1.1]).

**Hadamard manifolds** A metric space $(X, d)$ is *geodesic* if every two points $x, y \in X$ are connected by a shortest geodesic curve $\gamma_{x,y} : [0, 1] \to X$, which can be arclength parametrized, i.e.

$$d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = |s - t|d(\gamma_{x,y}(0), \gamma_{x,y}(1))$$  \hspace{1cm} (1)

for every $s, t \in [0, 1]$ such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$. A *geodesic triangle* $\triangle(p, q, r)$ in a geodesic space $(X, d)$ is composed of the vertices $p, q, r \in X$ and three geodesics joining these points. The corresponding comparison triangle $\triangle(\bar{p}, \bar{q}, \bar{r})$ (which is unique up to isometries) is a triangle in the Euclidean space $\mathbb{R}^2$ with vertices $\bar{p}, \bar{q}, \bar{r} \in \mathbb{R}^2$ such that the three line segments have the same side lengths as the corresponding geodesics of $\triangle(p, q, r)$, i.e.

$$d(p, q) = \|\bar{p} - \bar{q}\|, \quad d(p, r) = \|\bar{p} - \bar{r}\|, \quad d(r, q) = \|\bar{r} - \bar{q}\|.$$
A complete geodesic space \((\mathcal{H}, d)\) is called a Hadamard space if for every geodesic triangle \(\triangle(p, q, r) \in \mathcal{H}\) and \(x \in \gamma_{p,r}, y \in \gamma_{q,r}\) we have \(d(x, y) \leq \|\bar{x} - \bar{y}\|\), where \(\bar{x}, \bar{y}\) are the corresponding points in the comparison triangle \(\triangle(\bar{p}, \bar{q}, \bar{r}) \in \mathbb{R}^2\) (see section 2.1). Geodesic spaces satisfying the latter property are also called CAT(0) spaces. By [6, Proposition 1.1.3 and Corollary 1.2.5] the geometric CAT(0) condition is equivalent to \((\mathcal{H}, d)\) being a complete geodesic space with

\[
d^2(x, v) + d^2(y, w) \leq d^2(x, w) + d^2(y, v) + 2d(x, y)d(v, w)
\]

for every \(x, y, v, w \in \mathcal{H}\). The most prominent examples of Hadamard spaces are Hilbert spaces and Hadamard manifolds, which are defined as complete simply connected Riemannian manifolds with non-positive sectional curvature. Hyperbolic spaces and the manifold of positive definite matrices with the affine invariant metric are examples of Hadamard manifolds. Throughout this paper, we exclusively consider finite dimensional Hadamard manifolds, for which the existence of geodesic curves joining two arbitrary points is always valid. Recall that the Hopf–Rinow Theorem ceases to be true for general infinite dimensional manifolds [29].

A function \(f: \mathcal{H} \to \mathbb{R}\) is convex if for every \(x, y \in \mathcal{H}\) the function \(f \circ \gamma_{x,y}\) is convex, i.e.

\[
f(\gamma_{x,y}(t)) \leq (1 - t)f(\gamma_{x,y}(0)) + tf(\gamma_{x,y}(1))
\]

for all \(t \in [0, 1]\). In Hadamard spaces the distance is jointly convex [6, Proposition 1.1.5], i.e. for two geodesics \(\gamma_{x_1,x_2}, \gamma_{y_1,y_2}\) and \(t \in [0, 1]\) the relation

\[
d(\gamma_{x_1,x_2}(t), \gamma_{y_1,y_2}(t)) \leq (1 - t)d(x_1, y_1) + td(x_2, y_2)
\]

holds true. Thus, geodesics are in particular uniquely determined by their endpoints. For a bounded sequence \(\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{H}\), the function \(w: \mathcal{H} \to [0, +\infty)\) defined by

\[
w(x; \{x_n\}_{n \in \mathbb{N}}) := \limsup_{n \to \infty} d^2(x, x_n)
\]

has a unique minimizer, which is called the asymptotic center of \(\{x_n\}_{n \in \mathbb{N}}\) [6, p. 58]. A sequence \(\{x_n\}_{n \in \mathbb{N}}\) is said to converge weakly to a point \(x \in \mathcal{H}\) if it is bounded and \(x\) is the asymptotic center of each subsequence of \(\{x_n\}_{n \in \mathbb{N}}\) [6, p. 103]. Then, the notion of proper and (weakly) lower semicontinuous functions is analogous to Hilbert spaces.

Next, we consider the Borel \(\sigma\)-algebra \(\mathcal{B}\) on \(\mathcal{H}\) on the open and bounded set \(\Omega \subset \mathbb{R}^n\). A measurable map \(f: \Omega \to \mathcal{H}\) belongs to \(L^p(\Omega, \mathcal{H}),\ p \in [1, \infty]\), if

\[
d_p(f, f_a) < \infty
\]

for any constant mapping \(f_a(\omega) = a\) with \(a \in \mathcal{H}\), where \(d_p\) is defined for two measurable maps \(f\) and \(g\) by

\[
d_p(f, g) := \begin{cases} \left( \int_{\Omega} d^p(f(\omega), g(\omega))\, d\omega \right)^{\frac{1}{p}}, & p \in [1, \infty), \\ \text{ess sup}_{\omega \in \Omega} d(f(\omega), g(\omega)), & p = \infty. \end{cases}
\]
Using the equivalence relation $f \sim g$ if $d_p(f, g) = 0$, the space $L^p(\Omega, \mathcal{H}) := L^p(\Omega, \mathcal{H})/\sim$ equipped with $d_p$ becomes a complete metric space. In the case $p = 2$ this space is a Hadamard space \cite[Proposition 1.2.18]{6}. Finally, for $f, g, h \in L^2$, which is slightly sharpened, noting that the functions \cite[Theorem 4]{6}, the set $L >$ and $L^2(\Omega, \mathcal{H}, w)$ with weight $w \in C^0([0, 1] \times \Omega, [c_1, c_2])$, $0 < c_1 < c_2$, the metric is given by

$$d_2^2(f, g) = \int_0^1 \int_\Omega d(f(t, x), g(t, x))^2 w(t, x) \, dx \, dt.$$ 

In the context of paths with finite path energy in the space of manifold-valued images, we observe Hölder continuity in time, which enables pointwise evaluations in time, in particular for $t = 0$ and $t = 1$. Next, we restate a density result given in \cite[Theorem 2]{30}, which is slightly sharpened, noting that the functions $h_k$ in the proof are actually Lipschitz continuous.

**Theorem 2.** Let $(\mathcal{H}, d)$ be a locally compact Hadamard space. Then the set of Lipschitz continuous functions mapping from $\Omega$ to $\mathcal{H}$ is dense in $L^p(\Omega, \mathcal{H})$ for $p \in [1, \infty)$.

Another classical property of Lebesgue spaces also transfers to the Hadamard setting:

**Lemma 3.** Let $f_k \in L^2((0, 1), L^2(\Omega, \mathcal{H}, w)$ be a convergent sequence with limit $f$. Then there exists a subsequence which converges a.e. as $k \to \infty$.

**Proof.** The Chebyshev inequality implies the convergence in measure, then we can apply \cite[Theorem 5.2.7 (i)]{31}.

Next, we define subsets of Hölder continuous functions with fixed parameters $\alpha \in (0, 1]$ and $L > 0$ by

$$A_{\alpha, L, w} := \left\{ f \in L^2((0, 1), L^2(\Omega, \mathcal{H}, w) : d_2(f(s), f(t)) \leq L |t - s|^{\alpha} \forall t, s \in [0, 1] \right\}.$$

**Theorem 4.** The set $A_{\alpha, L, w}$ is closed and convex. In particular, $A_{\alpha, L, w}$ is weakly closed.

**Proof.** First, we show closedness. By Lemma 3 we get an a.e. convergent subsequence. Assume there exists a point $t \in [0, 1]$, where this sequence does not converge. Then, we can choose $s \in [0, 1]$ arbitrarily close with $d_2(f_k(s), f(s)) \to 0$ as $k \to \infty$. This implies

$$d_2(f_k(t), f_l(t)) \leq 2L(t - s)^{\alpha} + d_2(f_k(s), f_l(s))$$

for all $k, l$ sufficiently large, which proves that the sequence $f_k(t)$ is a Cauchy sequence. The Hölder continuity of $f$ follows directly by approximation arguments.

Second, we show convexity. For $f_1, f_2 \in A_{\alpha, L, w}$ and the connecting geodesic $[0, 1] \ni r \mapsto \gamma_{f_1, f_2}^r \in L^p([0, 1], L^2(\Omega, \mathcal{H}, w)$ we obtain by the convexity of the Hadamard metric

$$d_2 \left( \gamma_{f_1, f_2}^r(s), \gamma_{f_1, f_2}^r(t) \right) = d_2 \left( \gamma_{f_1(s), f_2(s)}^r, \gamma_{f_1(t), f_2(t)}^r \right) \leq (1 - r)d_2(f_1(s), f_1(t)) + rd_2(f_2(s), f_2(t)) \leq L |t - s|^{\alpha}.$$

Finally, the weak closedness in the Bochner space follows by \cite[Lemma 3.2.1]{6}. 




The assumptions from lemma 5 hold true and let Corollary 6.

\[ \limsup_{j \to \infty} d_p(f \circ Y_j, f \circ Y) = 0. \]

If in addition \( Y_j \) converges to \( Y \) in \((C^{1,\alpha}(\Omega)) \), then \( \limsup_{j \to \infty} d_p(f \circ (Y_j)^{-1}, f \circ Y^{-1}) = 0. \)

The generalization of this result to the space \( L^2((0,1), L^2(\Omega, \mathcal{H})) \) is straightforward. Using the triangle inequality, we get the following corollary, which generalizes to \( L^2((0,1), L^2(\Omega, \mathcal{H})) \).

**Corollary 6.** Let the assumptions from lemma hold true and let \( \{f_j\}_{j \in \mathbb{N}} \subset L^p(\Omega, \mathcal{H}) \), \( p \in [1, \infty) \), be a sequence which converges to \( f \) in \( L^p(\Omega, \mathcal{H}) \). Then,

\[ \lim_{j \to \infty} d_p(f_j \circ Y_j, f \circ Y) = 0 \quad \text{and} \quad \lim_{j \to \infty} d_p(f_j \circ (Y_j)^{-1}, f \circ Y^{-1}) = 0. \]

**2.2 Metamorphosis model in Euclidean case**

In this subsection, we briefly introduce the space of images \( I: \Omega \to \mathbb{R} \) with a Riemannian structure from the perspective of the flow of diffeomorphisms model and the metamorphosis model. For further details we refer the reader to the literature mentioned in section 1.

**Flow of diffeomorphisms** In the flow of diffeomorphisms model, the temporal evolution of each pixel of the reference image along a trajectory is determined by a family of diffeomorphisms \( (Y(t))_{t \in [0,1]}: \Omega \to \mathbb{R}^n \) such that the brightness is preserved. The brightness constancy assumption is mathematically reflected by a vanishing material derivative \( \frac{D}{dt}I = \dot{I} + \nabla I \) along a motion path \( (I(t))_{t \in [0,1]} \) in the space of images, where \( \dot{Y}(t) = \dot{Y}(t) \circ Y^{-1}(t) \) denotes the time-dependent Eulerian velocity. Then, one defines the metric and the right-invariant path energy associated with this family of diffeomorphisms as follows

\[ g_{Y(t)}(\dot{Y}(t), \dot{Y}(t)) = \int_{\Omega} L[v(t), v(t)] \, dx, \quad \mathcal{E}((Y(t))_{t \in [0,1]}) = \int_0^1 g_{Y(t)}(\dot{Y}(t), \dot{Y}(t)) \, dt. \]

Throughout this paper, we consider the higher order operator

\[ L[v(t), v(t)] = \frac{\lambda}{2}(\text{tr} \varepsilon[v])^2 + \mu \text{tr}(\varepsilon[v]^2) + \gamma |D^m v|^2, \quad (5) \]

where \( \varepsilon[v] = (\nabla v)^{\text{sym}} \) refers to the symmetric part of the gradient and \( m > 1 + \frac{n}{2} \) as well as \( \lambda, \mu, \gamma > 0 \) are fixed constants. This particular choice of the operator \( L \) originates from
fluid mechanics, where the metric $g_{Y(t)}$ refers to a viscous dissipation in a multipolar fluid model as described in \[33\] \[26\] \[25\].

If $Y_A$ and $Y_B$ are diffeomorphisms and the energy $\mathcal{E}$ is finite for a general path $(Y(t))_{t \in [0,1]}$ with $Y(0) = Y_A$ and $Y(1) = Y_B$, then using the $H^m(\Omega)$-coerciveness of the metric $g_{Y(t)}$ the path is already a family of diffeomorphisms. In addition, an energy minimizing velocity field $v$ exists such that $\frac{d}{dt} Y(t, \cdot) = v(t, Y(t, \cdot))$ for every $t \in [0,1]$, see \[21\]. Furthermore, the corresponding path $I$ for two input images $I_A, I_B \in L^2(\Omega)$ has the particular form $I(t, \cdot) = I_A \circ Y^{-1}(t, \cdot)$.

In what follows, we investigate diffeomorphisms induced by velocity fields in the space $\mathcal{V} := H^m(\Omega, \mathbb{R}^n) \cap H^1_0(\Omega, \mathbb{R}^n)$.

The following theorem relates the norm of the induced flow to the integrated norm of the associated velocity field.

**Theorem 7.** Let $v \in L^2((0,1), \mathcal{V})$ be a velocity field. Then there exists a global flow $Y \in H^1((0,1], H^m(\Omega)^n)$ such that

\[
\begin{align*}
\frac{d}{dt} Y(t, x) &= v(t, Y(t, x)), \\
Y(0, x) &= x,
\end{align*}
\]

for a.e. $x \in \Omega$ and a.e. $t \in [0,1]$. In particular, $Y(t, \cdot)$ is a homeomorphism for all $t \in [0,1]$. Further, for $\alpha \in [0, m - 1 - \frac{n}{2})$ the following estimates hold

\[
\|Y\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))} + \|Y^{-1}\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))} \leq C \exp \left( C \int_0^1 \|v(s, \cdot)\|_{C^{1,\alpha}(\overline{\Omega})} \, ds \right),
\]

If $\alpha > 0$, then the constant $C$ depends on $\int_0^1 \|\nabla v(s, \cdot)\|_{C^{0,\alpha}(\overline{\Omega})} \, ds$. Finally, the solution operator $L^2((0,1), \mathcal{V}) \to C^0([0,1], C^1(\overline{\Omega}, \mathbb{R}^n))$ assigning a flow $Y$ to every velocity field $v$ is continuous w.r.t. to the weak topology in $L^2((0,1), \mathcal{V})$ and the $L^\infty([0,1], C^0(\overline{\Omega}))$-topology for $Y$.

**Proof.** Aside from eq. \[7\] the theorem corresponds to \[44\] Theorem 1 and Theorem 9]. The estimate for the first term in eq. \[7\] follows from \[44\] Lemma 7 and relies on Gronwall’s inequality when considering the $C^0([0,1], C^1(\overline{\Omega}))$-norm on the left-hand side and the $H^m(\Omega)$-norm in the exponent on the right-hand side. The generalization to the $C^{1,\alpha}(\overline{\Omega})$-norm in the exponent is straightforward. In what follows, we sketch the proof of \[7\] when employing the $C^0([0,1], C^{1,\alpha}(\overline{\Omega}))$-norm on the left-hand side.

Let $i \in \{1, \ldots, n\}$, $t \in (0,1)$ and $x, y \in \Omega$. Taking into account the aforementioned result we can assume $\|Y\|_{C^0([0,1],C^{1,\alpha}(\overline{\Omega}))} \leq C(v) := C \exp \left( C \int_0^1 \|v(s, \cdot)\|_{C^{1,\alpha}(\overline{\Omega})} \, ds \right)$. Then,
a nonrigorous comptutation yields

\[
\left| \frac{d}{dt} [\partial_t Y(t, x) - \partial_t Y(t, y)] \right| \leq |\nabla v(t, Y(t, x)) \cdot \partial_t Y(t, x) - \nabla v(t, Y(t, y)) \cdot \partial_t Y(t, y)|
\]

\[
\leq |\nabla v(t, Y(t, x)) - \nabla v(t, Y(t, y))| |\partial_t Y(t, x) - \partial_t Y(t, y)|
\]

\[
\leq C \|\nabla v(t, \cdot)\|_{C^{0, \alpha}(\Omega)} Y \|^{1+\alpha}_{C^{0, [0, 1], C^1(\Omega)}} |x - y|^\alpha + \|\nabla v(t, \cdot)\|_{C^{0, \alpha}(\Omega)} |\partial_t Y(t, x) - \partial_t Y(t, y)|
\]

\[
\leq C C(v)^{1+\alpha} \|\nabla v(t, \cdot)\|_{C^{0, \alpha}(\Omega)} |x - y|^\alpha + \|\nabla v(t, \cdot)\|_{C^{0, \alpha}(\Omega)} |\partial_t Y(t, x) - \partial_t Y(t, y)|.
\]

Thus,

\[
\frac{d}{dt} \left( \exp \left( - \int_0^t \|\nabla v(s, \cdot)\|_{C^{0, \alpha}(\Omega)} ds \right) |\partial_t Y(t, x) - \partial_t Y(t, y)| \right)
\]

\[
\leq C C(v)^{1+\alpha} \|\nabla v(t, \cdot)\|_{C^{0, \alpha}(\Omega)} \exp \left( - \int_0^t \|\nabla v(s, \cdot)\|_{C^{0, \alpha}(\Omega)} ds \right) |x - y|^\alpha
\]

\[
\leq C C(v)^{1+\alpha} \|\nabla v(t, \cdot)\|_{C^{0, \alpha}(\Omega)} |x - y|^\alpha.
\]

The integration of both sides w.r.t. \( t \) yields

\[
\exp \left( - \int_0^t \|\nabla v(s, \cdot)\|_{C^{0, \alpha}(\Omega)} ds \right) |\partial_t Y(t, x) - \partial_t Y(t, y)|
\]

\[
\leq C C(v)^{1+\alpha} \int_0^1 \|\nabla v(t, \cdot)\|_{C^{0, \alpha}(\Omega)} dt |x - y|^\alpha,
\]

which bounds the first term in (7) and the second term is estimated similarly by noting that \( Y^{-1}(t, \cdot) \) is the flow associated with the (backward) motion field \(-v(1 - t, \cdot)\).

This proof can be further generalized to \( C^0([0, 1], C^{k, \alpha}(\overline{\Omega})) \)-norms provided that \( m \) is sufficiently large.

**Remark 8.** Analogous results hold when replacing \( V \) by \( C^{1, \alpha}(\overline{\Omega}) \) with zero boundary condition [74, Chapter 8]. Furthermore, the mapping \( v \to Y^v \) is Lipschitz continuous in \( v \), i.e.

\[
\|Y_v(t, \cdot) - Y_{\tilde{v}}(t, \cdot)\|_{C^0(\Omega)} \leq \left( 1 + C \exp(C) \right) \int_0^1 \|v(s, \cdot) - \tilde{v}(s, \cdot)\|_{C^0(\Omega)} ds,
\]

where \( C = \int_0^1 \|v(s, \cdot)\|_{C^{1, \alpha}(\Omega)} ds \) [74, (8.16)].

**Metamorphosis** The metamorphosis model can be regarded as a generalization of the flow of diffeomorphisms model, in which the brightness constancy assumption is replaced by a quadratic penalization of the material derivative, which in particular allows for intensity modulations along the trajectories. Thus, as a first attempt one could define the metric and the path energy in the metamorphosis model associated with the family of images \((I(t))_{t \in [0, 1]} : \overline{\Omega} \to \mathbb{R}^n\) and a penalization parameter \( \delta > 0 \) as follows

\[
g(\hat{I}, \hat{I}) = \min_{v: \Omega \to \mathbb{R}^n} \int_{\Omega} L[v, v] + \frac{1}{\delta} \left( \frac{D}{\partial t} I \right)^2 \, dx , \quad E(I) = \int_0^1 g(\hat{I}(t), \hat{I}(t)) \, dt . \quad (8)
\]
This shows that the flow of diffeomorphisms model can be seen as the limit case of the metamorphosis model for \( \delta \to 0 \).

However, there are two major problems related with \([8]\). Clearly, in general paths in the space of images do not exhibit any smoothness properties—neither in space nor in time. Thus, the evaluation of the material derivative \( \frac{D}{d\tau}I^2 \) is not well-defined. Moreover, since different pairs of velocity fields \( v \) and material derivatives \( \frac{D}{d\tau}I \) can imply the same time derivative of the image path \( \dot{I} \), the restriction to equivalence classes of pairs \((v, \frac{D}{d\tau}I)\) is required, where two pairs are equivalent if and only if they induce the same temporal change of the image path \( \dot{I} \).

To tackle both problems, Trouvé and Younes \([44]\) proposed a nonlinear geometric structure in the space of images \( L^2(\Omega) := L^2(\Omega, \mathbb{R}) \). In detail, for a given velocity field \( v \in L^2((0,1), \mathcal{V}) \) and an image path \( I \in L^2((0,1), L^2(\Omega)) \) the material derivative is replaced by the function \( z \in L^2((0,1), L^2(\Omega)) \) known as the weak material derivative, which is uniquely determined by

\[
\int_0^1 \int_{\Omega} \eta z \, dx \, dt = - \int_0^1 \int_{\Omega} (\partial_t \eta + \text{div}(v \eta)) I \, dx \, dt
\]

for \( \eta \in C^\infty_c((0,1) \times \Omega) \). Moreover, for all \( I \in L^2(\Omega) \) the associated tangent space \( T_I L^2(\Omega) \) is defined as \( T_I L^2(\Omega) = \{I\} \times W/N_I \), where \( W = \mathcal{V} \times L^2(\Omega) \) and

\[
N_I = \left\{ w = (v, z) \in W : \int_{\Omega} z\eta + I \text{div}(\eta v) \, dx = 0 \ \forall \eta \in C^\infty_c(\Omega) \right\}.
\]

As usual, the associated tangent bundle is given by \( TL^2(\Omega) = \bigcup_{I \in L^2(\Omega)} T_I L^2(\Omega) \).

Then, following Trouvé and Younes, a regular path in the space of images (denoted by \( I \in H^1([0,1], L^2(\Omega)) \)) is a curve \( I \in C^0([0,1], L^2(\Omega)) \) such that there exists a measurable path \( \gamma : [0,1] \to TL^2(\Omega) \) with bounded \( L^2 \)-norm in space and time and \( \pi(\gamma) = I \), where \( \pi(I,(v,z)) = I \) refers to the projection onto the image manifold and \((I,(v,z))\) denotes the equivalence class, such that

\[
- \int_0^1 \int_{\Omega} I \partial_t \eta \, dx \, dt = \int_0^1 \int_{\Omega} z\eta + I \text{div}(\eta v) \, dx \, dt
\]

for all \( \eta \in C^\infty_c((0,1) \times \Omega) \). In this paper, we use the alternative definition of the weak material derivative

\[
I(t,Y(t,\cdot)) - I(s,Y(s,\cdot)) = \int_s^t z(r,Y(r,\cdot)) \, dr \quad \text{for all} \ s > t \in [0,1]
\]

for a given flow \( Y \) \([44]\). Finally, if we assume the \( \mathcal{V} \)-coercivity of the operator \( L \), then the path energy in the metamorphosis model for a regular path \( I \in H^1([0,1], L^2(\Omega)) \) is defined as

\[
\mathcal{E}(I) = \int_0^1 \inf_{(v,z) \in T_{I(t)} L^2(\Omega)} \int_{\Omega} L[v,v] + \frac{1}{\delta} z^2 \, dx \, dt. \tag{9}
\]

The existence of energy minimizing paths in the space of images (known as geodesic curves), i.e. solutions of the boundary value problem

\[
\min \{\mathcal{E}(\tilde{I}) : \tilde{I} \in H^1([0,1], L^2(\Omega)), \ \tilde{I}(0) = I_A, \ \tilde{I}(1) = I_B\}
\]
for fixed images $I_A, I_B \in L^2(\Omega)$, is proven in [44]. In addition, one can prove the existence of minimizing $(v, z) \in T_{I(t)}L^2(\Omega)$.

We remark that all results of this paper can be easily generalized to the space of multichannel or color images $L^2(\Omega, \mathbb{R}^C)$ for $C \geq 2$ color channels with minor modifications.

### 2.3 Manifold-valued time discrete metamorphosis model

Now, we pick up the time discrete metamorphosis model for manifold-valued images, for which convergence is studied in this paper. The model itself was thoroughly analyzed in [36] and extends the variational time discretization of the classical metamorphosis model proposed in [11].

Fix $\gamma, \delta, \epsilon > 0$ and $m > 1 + \frac{n}{2}$, and let $H$ be any finite dimensional Hadamard manifold. For two manifold-valued images $I, \tilde{I} \in L^2(\Omega, H)$ and an admissible deformation $\varphi \in A_{\epsilon} = \{\varphi \in H^m(\Omega, \Omega) : \det D\varphi > \epsilon \text{ in } \Omega, \varphi = \text{Id} \text{ on } \partial\Omega\}$ the time discrete energy for pairs of images is defined as

$$R(I, \tilde{I}) = \inf_{\varphi \in A_{\epsilon}} R(I, \tilde{I}, \varphi),$$

where

$$R(I, \tilde{I}, \varphi) = \int_{\Omega} W(D\varphi(x)) + \gamma |D^m\varphi(x)|^2 \, dx + \frac{1}{\delta} d_2^2(I, \tilde{I} \circ \varphi)$$

for an elastic energy density $W$. Here, $d_2^2(\cdot, \cdot)$ replaces the squared $L^2$-norm in the time discrete metamorphosis model. The energy $R$ can be considered as a numerically feasible approximation of the squared Riemannian distance in the underlying image space [42]. Throughout this paper, we assume that $W$ satisfies the following conditions:

(W1) $W \in C^4(\text{GL}^+(n), \mathbb{R}_0^+)$ is polyconvex.

(W2) There exist constants $C_{W,1}, C_{W,2}, r_W > 0$ such that for all $A \in \text{GL}^+(n)$ the following growth estimates hold true:

$$W(A) \geq C_{W,1}|A^{\text{sym}} - \text{Id}|^2, \quad \text{if } |A - \text{Id}| < r_W, \quad (10)$$

$$W(A) \geq C_{W,2}, \quad \text{if } |A - \text{Id}| \geq r_W. \quad (11)$$

(W3) The energy density admits the following representation at $\text{Id}$:

$$W(\text{Id}) = 0, \quad DW(\text{Id}) = 0, \quad (12)$$

$$\frac{1}{2} D^2W(\text{Id})(A, A) = \lambda (\text{tr} A)^2 + \mu \text{ tr} \left((A^{\text{sym}})^2\right). \quad (13)$$

The assumption [W1] is required for the lower semicontinuity of the energy functional. Furthermore, [W2] enforces the convergence of the optimal deformations to the identity in the limit $K \to \infty$. Finally, [W3] ensures the compatibility of $W$ with the elliptic
operator \( L \) (cf. \([5]\)). Note that \([\text{W1}]\) and \([\text{W3}]\) are identical to \([\text{W1}]\) and \([\text{W3}]\).

We recall that in \([\text{W2}]\) a growth estimate of the form

\[
W(A) \geq C(\det A)^{-s} - C
\]  

for \( s > n - 1 \) and a positive constant \( C \) instead of \([\text{W2}]\) is assumed. This modification additionally requires essentially bounded images in order to ensure that the deformations are homeomorphic. However, in order to use the Hadamard space of square-integrable images, we have to use \([\text{W2}]\) instead, which in particular results in diffeomorphic deformations.

The time discrete path energy for \( K + 1 \) images \( I = (I_0, \ldots, I_K) \in (L^2(\Omega, \mathcal{H}))^{K+1} \), \( K \geq 2 \), is defined as the weighted sum of the discrete energies \( R \) evaluated at consecutive images, i.e.

\[
J_K(I) := \inf_{\varphi = (\varphi_1, \ldots, \varphi_K) \in \mathcal{A}^K} \left\{ J_K(I, \varphi) := K \sum_{k=1}^K R(I_{k-1}, I_k, \varphi_k) \right\}.
\]  

(15)

For two fixed images \( I_A = I_0, I_B = I_K \in L^2(\Omega, \mathcal{H}) \) a \((K + 1)\)-tuple \( I = (I_0, \ldots, I_K) \in (L^2(\Omega, \mathcal{H}))^{K+1} \) is called a discrete geodesic curve if

\[
J_K(I) \leq J_K((I_0, \tilde{I}_1, \ldots, \tilde{I}_{K-1}, I_K))
\]

for all \((\tilde{I}_1, \ldots, \tilde{I}_{K-1}) \in (L^2(\Omega, \mathcal{H}))^{K-1}\). The existence of discrete geodesic curves has been shown in \([36]\) Section 3] under different assumptions with respect to the energy density. However, by slightly altering these proofs the existence under the assumptions \([\text{W1}]\) \([\text{W3}]\) (cf. also \([22]\)) can be checked. Note that in general neither the discrete geodesic curve nor the associated set of deformations is uniquely determined. The Mosco–convergence of a temporal extension of \( J_K \) to \( E \) in the Euclidean case was proven in \([11]\).

Figure 3 shows different discrete geodesic paths for \( K = 4, 8, 16 \) connecting two slices of DTI images given in Figure 2. In particular, one experimentally observes an indication of convergence for increasing \( K \).
3 Manifold-valued metamorphosis model

In this section, we propose a (time continuous) metamorphosis energy functional $\mathcal{J}$ for manifold-valued images in $L^2(\Omega, \mathcal{H})$, where $\mathcal{H}$ is a finite dimensional Hadamard manifold. This energy is identified in section 5 as the Mosco-type limit of the above time discrete path energy. A straightforward generalization of the weak notion of the material derivative $z$ is unfeasible. Hence, this is replaced by an inequality relating distances between images along the motion path and the associated scalar material derivative $z$.

We prove the equivalence of this novel energy functional with the classical metamorphosis model for $\mathbb{R}^C$-valued images, where the scalar material derivative coincides with the norm of the classical material derivative.

The manifold-valued metamorphosis energy functional $\mathcal{J}: L^2((0,1) \times \Omega, \mathcal{H}) \rightarrow [0, \infty]$ is defined as follows

$$\mathcal{J}(I) := \inf_{(v,z) \in C(I)} \int_0^1 \int_{\Omega} L[v, v] + \frac{1}{\delta} z^2 \, dx \, dt.$$  

(16)
Here, \( C(I) \) is the set of pairs \((v, z) \in L^2((0, 1), \mathcal{V}) \times L^2((0, 1), L^2(\Omega))\) such that the flow \( Y \) defined by
\[
\frac{d}{dt} Y(t, x) = v(t, Y(t, x)) \quad \text{for} \ (t, x) \in [0, 1] \times \Omega, \quad Y(0, x) = x \quad \text{for} \ x \in \Omega
\]
satisfies for all \( t < s \in [0, 1] \) the inequality
\[
d(I(t, Y(t, \cdot)), I(s, Y(s, \cdot))) \leq \int_t^s z(r, Y(r, \cdot)) \, dr. \tag{18}
\]

For a given image curve \( t \mapsto I(t, Y(t, \cdot)) \) the associated tangential vectors at different times are in general contained in different tangent spaces. In particular, the norm of the material derivative \( z \) at time \( t \) depends on the image path. The definition of the material derivative via the variational inequality \((18)\) avoids this technical difficulty. Let us now verify the equivalence of both versions of the metamorphosis model for \( \mathbb{R}C \)-valued images. In the classical model, the \((C\text{-dimensional})\) material derivative \( \hat{z} \) is defined via the equation
\[
I(t, Y(t, \cdot)) - I(s, Y(s, \cdot)) = \int_t^s \hat{z}(r, Y(r, \cdot)) \, dr \tag{19}
\]
for all \( t < s \in [0, 1] \), whereas the scalar material derivative \( z \) obeys the inequality
\[
|I(t, Y(t, \cdot)) - I(s, Y(s, \cdot))| \leq \int_t^s z(r, Y(r, \cdot)) \, dr. \tag{20}
\]

In fact, the equivalence is already implied by the following proposition.

**Proposition 9.** For every \( z \) fulfilling \((20)\) there exists a \( \hat{z} \) fulfilling \((19)\) with \( z \geq |\hat{z}| \). Vice versa, for every \( \hat{z} \) fulfilling \((19)\) there exists a \( z \) fulfilling \((20)\) with \( z = |\hat{z}| \).

**Proof.** For given \( \hat{z} \) the result follows from the triangle inequality by choosing \( z = |\hat{z}| \). To prove the converse, let \( z \) solve \((20)\). Taking the \( L^2 \)-norm on both sides implies
\[
\|I(t, Y(t, \cdot)) - I(s, Y(s, \cdot))\|_{L^2(\Omega)} \leq \int_t^s \|z(r, Y(r, \cdot))\|_{L^2(\Omega)} \, dr,
\]
i.e. the function \( t \mapsto I(t, Y(t, x)) \) is \( AC^2([0, 1], L^2(\Omega)) \) in the sense of [1, Definition 1.1.1]. Using [1, Remark 1.1.3] one can additionally infer the a.e. differentiability with derivative \( Z \in L^2((0, 1), L^2(\Omega)) \) such that
\[
I(t, Y(t, x)) - I(0, Y(0, x)) = \int_0^t Z(r, x) \, dr = \int_0^t \hat{z}(r, Y(r, x)) \, dr
\]
with \( \hat{z}(r, x) := Z(r, X(r, x)) \). Here, \( X(r, \cdot) \) is the spatial inverse of \( Y(r, \cdot) \). Now set
\[
B = \{(r, x) \in [0, 1] \times \Omega : z(r, Y(r, x)) < |\hat{z}(r, Y(r, x))|\}
\]
and assume that the Lebesgue measure of \( B \) is strictly positive. Note that \( B \) can be approximated with finite unions of disjoint semi-open cuboids [13, Theorem 1.4]. By
taking into account [11, Theorem 1.1.2/Remark 1.1.3] one gets for every such cuboid $[t_1, t_2] \times D \subset [0, 1] \times \Omega$ that
\[
\int_{t_1}^{t_2} \int_D |\tilde{z}(t, Y(t, x))|^2 \, dx \, dt \leq \int_{t_1}^{t_2} \int_D z(t, Y(t, x))^2 \, dx \, dt.
\]
Combining this estimate with the dominated convergence theorem we conclude
\[
\int_B |\tilde{z}(t, Y(t, x))|^2 \, dx \, dt \leq \int_B z(t, Y(t, x))^2 \, dx \, dt.
\]
This yields a contradiction to the definition of the set $B$. Hence, $z \geq |\tilde{z}|$ a.e. in $t$ and $x$.

4 Temporal extension operators

In this section, temporal extensions of all relevant quantities required for the convergence proof of the time discrete metamorphosis are proposed, which in particular allows an explicit solution to the optimality conditions [17] and [15]. We remark that the subsequent construction is similar to [11] with two major modifications, namely the definition of the interpolated image sequence (22) and the weak material derivative [25], which are related to the manifold structure.

For fixed $K \in \mathbb{N}$, let a discrete image path $I_K = (I_{K,0}, \ldots, I_{K,K}) \in L^2(\Omega, H)^{K+1}$ be given. The existence of the corresponding optimal deformations $\varphi_K = (\varphi_{K,1}, \ldots, \varphi_{K,K}) \in A^H_K$ satisfying [15] is proven in [36, Section 3]. We refer to $\tau = K^{-1}$ as the time step size and the image $I_{K,k}$ is associated with the time step $t_{K,k} = k\tau$. For $k = 1, \ldots, K$, we define the discrete transport map $y_{K,k} : [t_{K,k-1}, t_{K,k}] \times \overline{\Omega} \to \overline{\Omega}$ as
\[
y_{K,k}(t, x) := x + (t - t_{K,k-1}) K(\varphi_{K,k}(x) - x). \tag{21}
\]
If
\[
\max_{k=1,\ldots,K} \|\varphi_{K,k} - \text{Id}\|_{C^{1,\alpha}(\overline{\Omega})} < 1,
\]
we can use [17, Theorem 5.5-1/Theorem 5.5-2] to infer that $\det(Dy_{K,k}(t, \cdot)) > 0$ holds and that $y_{K,k}(t, \cdot)$ is invertible with inverse $x_{K,k}(t, \cdot)$. The validity of this assumption is proven below and is tacitly assumed for all further considerations.

Next, we define the extension operator $I^\text{int}_K : L^2(\Omega, H)^{K+1} \times A^H_K \to L^2((0, 1), L^2(\Omega, H))$, which is given for $t \in [t_{K,k-1}, t_{K,k})$ and a.e. $x \in \Omega$ by
\[
I^\text{int}_K(I_K, \varphi_K)(t, x) := \gamma(t_{K,k-1}, t, \varphi_{K,k})(K(t - t_{K,k-1}))(x_{K,k}(t, x)). \tag{22}
\]
Thus, $I^\text{int}_K$ uniquely describes for given $I_K$ and $\varphi_K$ a blending on the manifold along the transport path governed by $y_{K,k}$.

In what follows, we set $w_{K,k} = K(\varphi_{K,k} - \text{Id})$ and define the piecewise constant (in time) velocity $w_K = w_K(\varphi_K) \in L^2((0, 1), \mathcal{V})$ as
\[
w_K(\varphi_K)|_{[t_{K,k-1}, t_{K,k})} := w_{K,k}.
\]
Furthermore, we define the discrete velocity field \( v_K : \mathcal{V}^K \rightarrow L^2((0,1), C^{1,\alpha}(\Omega)) \),
\[
v_K(\varphi_K)(t, x) := K(\varphi_{K,k} - \text{Id})(x_{K,k}(t, x))
\]
for \( t \in [t_{K,k-1}, t_{K,k}) \) and a.e. \( x \in \Omega \), which is constant along time discrete paths.
Note that the extension operator \( v_K \) merely admits a \( C^{1,\alpha} \)-regularity. To see this, we note that the concatenation of two Hölder continuous functions \( f \in C^{1,\alpha}(\Omega) \) and \( g \in C^{1,\alpha}(\Omega, \Omega) \) is again Hölder continuous [12, Lemma 2.2] and the estimate
\[
\| f \circ g \|_{C^{1,\alpha}(\Omega)} \leq C \| f \|_{C^{1,\alpha}(\Omega)} (1 + \| g \|_{C^{1,\alpha}(\Omega)})^2
\]
easily follows. Taking into account [12] Theorem 2.1 we infer that \( x_{K,k}(t, \cdot) \in C^{1,\alpha}(\Omega) \) and
\[
D(x_{K,k}(t, \cdot)) = K^{-1} \text{Inv} \left( K^{-1} \mathbf{f} + (t - t_{K,k-1})(D\varphi_{K,k} - \mathbf{1})(x_{K,k}(t, \cdot)) \right),
\]
where \( \text{Inv} : GL(n) \rightarrow GL(n) \) denotes the smooth inversion operator. Since \( \Omega \) is bounded and \( x_{K,k}(t, \cdot) \) is a diffeomorphism, we get
\[
\| x_{K,k}(t, \cdot) \|_{C^{1,\alpha}(\Omega)} \leq C \left( 1 + K^{-1} \max_{k=1 \ldots K} \| \varphi_{K,k} - \text{Id} \|_{C^{1,\alpha}(\Omega)} \right). \tag{23}
\]
This implies that \( v_K(t, \cdot) \in C^{1,\alpha}(\Omega) \) and
\[
\| v_K(t, \cdot) \|_{C^{1,\alpha}(\Omega)} \leq C \| w_{K,k}(t, \cdot) \|_{C^{1,\alpha}(\Omega)} (1 + K^{-1} \| w_{K,k}(t, \cdot) \|_{C^{1,\alpha}(\Omega)})^2. \tag{24}
\]
As a last preparatory step, we define the discrete path \( Y_K : [0,1] \times \overline{\Omega} \rightarrow \Omega \), which is the concatenation of all small diffeomorphisms \( y_{K,k} \) along the motion path. In detail, the mapping is defined for \( t \in [0, t_{K,1}] \) by \( Y_K(t,x) := y_{K,1}(t,x) \) and then recursively for \( k = 2, \ldots, K \) and \( t \in (t_{K,k-1}, t_{K,k}] \) by
\[
Y_K(t, x) := y_{K,k}(t, Y_K(t_{K,k-1}, x))
\]
for all \( x \in \Omega \). The spatial inverse of \( Y_K \) is denoted by \( X_K \). Finally, we define the material derivative \( z_K \in L^2((0,1), L^2(\Omega)) \) for \( t \in [t_{K,k-1}, t_{K,k}) \) as
\[
z_K(t, x) := Kd(I_{K,k-1}(x_{K,k}(t,x)), I_{K,k} \circ \varphi_{K,k}(x_{K,k}(t,x))). \tag{25}
\]
In the following proposition it is shown that the temporal extensions of the images, the velocities, the material derivatives and the discrete paths indeed constitute a time continuous solution to [17] and [18].

**Proposition 10 (Admissible extension).** For \( I_K \in L^2(\Omega, \mathcal{H})^{K+1} \) and corresponding optimal deformations \( \varphi_K \in \mathcal{A}^K \), the tuple \( (I_{K}^{\text{ext}}(I_K, \varphi_K), v_K(\varphi_K), Y_K, z_K) \) is a solution to [17] and [18].
Proof. By definition it holds that \( Y_K(0, x) = x \) for all \( x \in \Omega \). For \( t \in (t_{k-1}, t_k) \) and \( x \in \Omega \) we get
\[
\frac{d}{dt} Y_K(t, x) = \frac{d}{dt} y_k(t, Y_K(t_{k-1}, x)) = K(\varphi_k - \text{Id})(Y_K(t_{k-1}, x)) = v_K(\varphi_K)(t, Y_K(t, x)).
\]
Therefore, \( Y_K \) is a solution of (17) in a weak sense according to remark 8. A short computation shows for \( s \leq t \in [t_{K,k-1}, t_{K,k}] \) that
\[
\begin{aligned}
d(I_{K,k-1}^\text{int}(I_K, \varphi_K)(t, Y_K(t, x)), I_{K,k}^\text{int}(I_K, \varphi_K)(s, Y_K(s, x))) & = d\left(\gamma_{I_{K,k-1}I_K, \varphi_K,k}(K(t - t_{K,k-1}))(Y_K(t_{K,k-1}, x)), \gamma_{I_{K,k-1}I_K, \varphi_K,k}(K(s - t_{K,k-1}))(Y_K(t_{K,k-1}, x))\right) \\
& = K(t - s)d\left(\gamma_{I_{K,k-1}I_K, \varphi_K,k}(K(t_{K,k-1} - 1))(Y_K(t_{K,k-1} - 1, x)), \gamma_{I_{K,k-1}I_K, \varphi_K,k}(K(t_{K,k-1} - 1))(Y_K(t_{K,k-1} - 1, x))\right) \\
& \leq \int_s^t z_K(r, Y_K(r, x)) \, dr.
\end{aligned}
\]
If \( s \) and \( t \) are not in the same time interval, we can use the triangle inequality multiple times. This concludes the proof. \qed

The next lemma allows to bound the \( H^m(\Omega) \)-norm of the displacements by a function solely depending on the energy \( R \).

Lemma 11. Under the assumptions (W1) and (W2) there exists a continuous and monotonously increasing function \( \theta : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) with \( \theta(0) = 0 \) such that
\[
\| \varphi - \text{Id} \|_{H^m(\Omega)} \leq \theta \left( R(I, \tilde{I}, \varphi) \right)
\]
for all \( I, \tilde{I} \in L^2(\Omega, \mathcal{H}) \) and all \( \varphi \in \mathcal{A}_\epsilon \). Furthermore, \( \theta(x) \leq C(x + x^2)^{\frac{1}{2}} \) for a constant \( C > 0 \).

Proof. Set \( \overline{R} = R(I, \tilde{I}, \varphi) \). The Gagliardo-Nirenberg interpolation inequality [39] implies
\[
\| \varphi - \text{Id} \|_{H^m(\Omega)} \leq C \left( \| \varphi - \text{Id} \|_{L^2(\Omega)} + |\varphi - \text{Id}|_{H^m(\Omega)} \right). \tag{26}
\]
The \( H^m(\Omega) \)-seminorm of the displacement can be controlled as follows
\[
|\varphi - \text{Id}|_{H^m(\Omega)} = |\varphi|_{H^m(\Omega)} \leq \sqrt{\overline{R}}, \tag{27}
\]
which is implied by the definition of \( \overline{R} \). Since \( \varphi \in H^m(\Omega, \Omega) \), this already shows for \( \alpha \in \left(0, m - 1 - \frac{n}{2}\right) \) that
\[
\| \varphi - \text{Id} \|_{C^{1,\alpha}(\overline{\Omega})} \leq C \| \varphi - \text{Id} \|_{H^m(\Omega)} \leq C + C\sqrt{\overline{R}}. \tag{28}
\]
To control the lower order term appearing on the right-hand side of (26), we first define the set \( \Omega' = \{ x \in \Omega : |D\varphi(x) - \text{Id}| < r_W \} \). Then, by using (10) and (11) we obtain
\[
|\Omega \setminus \Omega'|_{C_{W,2}} \leq \int_{\Omega} W(D\varphi) \, dx \leq \overline{R},
\]
which implies \( |\Omega \setminus \Omega'| \leq \frac{\overline{R}}{C_{W,2}} \). Hence, by taking into account the embedding \( H^m(\Omega) \hookrightarrow C^1(\overline{\Omega}) \), Korn’s inequality as well as (28) we deduce
\[
\int_{\Omega} |(D\varphi)^{\text{sym}} - 1|^2 \, dx = \int_{\Omega'} |(D\varphi)^{\text{sym}} - 1|^2 \, dx + \int_{\Omega \setminus \Omega'} |(D\varphi)^{\text{sym}} - 1|^2 \, dx \\
\leq \int_{\Omega} \frac{W(D\varphi)}{C_{W,1}} \, dx + |\Omega \setminus \Omega'| \left( C + C\sqrt{\overline{R}} \right)^2 \\
\leq \frac{\overline{R}}{C_{W,1}} + \frac{\overline{R}}{C_{W,2}} \left( C + C\overline{R} \right).
\]
Thus, the lemma follows by combining (26), (27) and (29).

5 Convergence of time discrete geodesic paths

In this section, we prove the convergence of time discrete geodesic paths to a time continuous minimizer of (16). Indeed, we prove the full \( \liminf \)-inequality of the definition of Mosco–convergence, but only construct recovery sequences in the context of the \( \limsup \)-inequality for specific paths in the space of Hadamard–valued images (36), which suffices to establish the convergence of time discrete minimizers. We give here a comprehensive proof of the convergence result, even though the general procedure follows the Mosco–convergence proof in the Euclidean setting \[11\] with major differences due to the manifold setting. These differences are highlighted throughout the proof. In what follows, we pass to subsequences several times and to increase readability, we frequently avoid relabeling of subsequences if obvious.

As a first step, we extend the functional \( J_K : L^2(\Omega, H)^K \rightarrow [0, \infty] \) to an operator \( J_K : L^2([0, 1], L^2(\Omega, H)) \rightarrow [0, \infty] \) by
\[
J_K(I) = \begin{cases} 
J_K(I_K, \varphi_K), & \text{if } I = I^m_K(I_K, \varphi_K), \text{ where } \varphi_K \text{ is a minimizer of (15)}, \\
\infty, & \text{else}.
\end{cases}
\]
The ingredients for the convergence proof are the \( \liminf \)-inequality (Theorem 12) and the existence of a recovery sequence for specific paths (Theorem 13).

**Theorem 12** (\( \liminf \)-inequality). Under the assumptions \((W1), (W2)\) and \((W3)\) the time discrete path energy \( J_K \) satisfies the \( \liminf \)-inequality for \( J \) w.r.t. the \( L^2(\Omega, H) \)-topology.

**Proof.** Let \( I_K \in L^2([0, 1], L^2(\Omega, H)) \) be a sequence which weakly converges to an image path \( I \in L^2([0, 1], L^2(\Omega, H)) \). If we exclude the trivial case \( \liminf_{K \to \infty} J_K(I_K) = \infty \)
and eventually pass to a subsequence (without relabeling), we may assume

\[ \mathcal{J}_K(I_K) \leq \mathcal{J} < \infty \]

for all \( K \in \mathbb{N} \). By definition of \( \mathcal{J}_K \) this directly implies \( I_K = I_K^{\text{int}}(I_K, \varphi_K) \) with \( I_K = (I_{K,0}, \ldots, I_{K,K}) \in L^2(\Omega, \mathcal{H})^{K+1} \) and \( \varphi_K = (\varphi_{K,1}, \ldots, \varphi_{K,K}) \in \mathcal{A}_K^\varepsilon \). In particular, by incorporating lemma \( \text{(11)} \) we deduce

\[
\max_{k \in \{1, \ldots, K\}} \| \varphi_{K,k} - \text{Id} \|_{C^{1,\alpha}(\overline{\Omega})} \leq C \max_{k \in \{1, \ldots, K\}} \| \varphi_{K,k} - \text{Id} \|_{H^m(\Omega)} \leq C_\theta (\mathcal{J} K^{-1}) \leq CK^{-\frac{1}{2}}.
\]

Note that for \( K \) sufficiently large \( Y_K, X_K, v_K \) and \( z_K \) exist due to section \( 4 \).

1. Lower semicontinuity of the weak material derivative

Let us remark that this step resembles the first step of the proof in the Euclidean setting replacing the squared \( L^2 \)-norm by the squared distance in the Hadamard manifold.

A straightforward computation shows

\[
\begin{align*}
\int_0^1 \int_\Omega z_K^2 \, dx \, dt &= \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \int_\Omega K^2 d(I_{K,k-1}(x_{K,k}(t,x)), I_{K,k} \circ \varphi_{K,k}(x_{K,k}(t,x)))^2 \, dx \, dt \\
&= \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \int_\Omega K^2 d(I_{K,k-1}(x), I_{K,k} \circ \varphi_{K,k}(x))^2 \det(Dy_{K,k}(t,x)) \, dx \, dt.
\end{align*}
\]

Next, we want to bound the difference of \( \det(Dy_{K,k}) \) and 1 in the \( L^\infty \)-norm. Thus, we have

\[ Dy_{K,k}(t,x) = 1 + K(t - t_{k-1})(D \varphi_{K,k}(x) - 1), \]

which together with the local Lipschitz continuity of the determinant implies

\[ \| \det(Dy_{K,k}(t,x)) - 1 \|_{L^\infty([t_{k-1}, t_k] \times \Omega)} \leq C \| \varphi_{K,k} - \text{Id} \|_{C^{1,\alpha}(\overline{\Omega})}. \]

Hence, we can deduce

\[
\begin{align*}
\left| \sum_{k=1}^K K^2 \int_{t_{k-1}}^{t_k} \int_\Omega d(I_{K,k-1}(x), I_{K,k} \circ \varphi_{K,k}(x))^2(Dy_{K,k}(t,x)) - 1 \right| \, dx \, dt \\
\leq \frac{1}{\delta} \mathcal{J} C \max_{k=1, \ldots, K} \| \varphi_{K,k} - \text{Id} \|_{C^{1,\alpha}(\overline{\Omega})} \leq \frac{1}{\delta} \mathcal{J} C \max_{k=1, \ldots, K} \| \varphi_{K,k} - \text{Id} \|_{H^m(\Omega)}.
\end{align*}
\]

Taking into account \( (30) \) this ultimately leads to

\[
\lim_{K \to \infty} \int_0^1 \int_\Omega z_K^2 \, dx \, dt = \lim_{K \to \infty} K \sum_{k=1}^K \int_\Omega d(I_{K,k-1}(x), I_{K,k} \circ \varphi_{K,k}(x))^2 \, dx.
\]
This also shows the uniform boundedness of $z_K \in L^2((0,1), L^2(\Omega))$, which implies the existence of a weakly convergent subsequence with limit $z \in L^2((0,1), L^2(\Omega))$. Hence, using the weak lower semicontinuity of the norm one arrives at

$$
\int_0^1 \int_\Omega z^2 \, dx \, dt \leq \liminf_{K \to \infty} \int_0^1 \int_\Omega z^2_K \, dx \, dt
= \liminf_{K \to \infty} K \sum_{k=1}^K \int_\Omega d(I_{K,k-1}(x), I_{K,k} \circ \varphi_{K,k}(x))^2 \, dx.
$$

2. **Lower semicontinuity of the viscous dissipation** We highlight that this step differs from the corresponding step appearing in [11] due to the modification of the assumption (W2), where the overall structure persists.

Note that the velocity fields $v_K = v_K(\varphi_K)$ are not necessarily in $L^2((0,1), V)$. The sequence $w_K = w_K(\varphi_K) \in L^2((0,1), V)$ is uniformly bounded in $L^2((0,1), V)$. To see this, we first assume that $K$ is sufficiently large such that $\max_{k=1,\ldots,K} \|D\varphi_{K,k-1}\|_{C^0(\Omega)} < r_W$ (see [W2]), which is possible due to (30). Then, using Korn’s inequality, the Poincaré inequality as well as (W2) we obtain

$$
\int_0^1 \int_\Omega |w_K|^2 \, dx \, dt \leq C \sum_{k=1}^K \int_{t_{k-1}}^{t_{k}} \int_\Omega K^2 |(D\varphi_{K,k})^{\text{sym}} - I|^2 \, dx \, dt
\leq CK \sum_{k=1}^K \int_\Omega \frac{W(D\varphi_{K,k})}{C_{W,1}} \, dx \leq \frac{CJ}{C_{W,1}},
$$

$$
\int_0^1 \int_\Omega |D^m w_K|^2 \, dx \, dt = \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \int_\Omega K^2 |D^m(\varphi_{K,k} - \text{Id})|^2 \, dx \, dt
= \sum_{k=1}^K K \int_\Omega |D^m \varphi_{K,k}|^2 \, dx \leq \frac{J}{\gamma}.
$$

The Gagliardo–Nirenberg inequality implies the uniform boundedness of the sequence $w_K$ in $L^2((0,1), V)$. By passing to a subsequence (again labeled in the same way) we can deduce $w_K \rightharpoonup v \in L^2((0,1), V)$ for $K \to \infty$.

It remains to verify that

$$
\int_0^1 \int_\Omega L[v,v] \leq \liminf_{K \to \infty} K \sum_{k=1}^K \int_\Omega W(D\varphi_{K,k}) + \gamma |D^m \varphi_{K,k}|^2 \, dx.
$$

The second order Taylor expansion around $t_{K,k-1}$ of the function $t \mapsto W(I + (t - t_{K,k-1}) D w_{K,k})$ evaluated at $t = t_{K,k}$ yields

$$
W(D\varphi_{K,k}) = W(I) + K^{-1} DW(I)(Dw_{K,k}) + \frac{1}{2K^2} D^2 W(I)(Dw_{K,k}, Dw_{K,k}) + r_{K,k}
= K^{-2} \left( \frac{\lambda}{2} (\text{tr}(\varepsilon[w_{K,k}]))^2 + \mu \text{tr}(\varepsilon[w_{K,k}]^2) \right) + r_{K,k}.
$$

(33)
Here, the lower order terms vanish due to (12) and the last equality follows from (13). The remainder satisfies $|r_{K,k}| \leq CK^{-3} |Dw_{K,k}|^3$ if $K$ is large enough due to eq. (30). Then,
\[
K \sum_{k=1}^{K} \int_{\Omega} W(D\varphi_{K,k}) + \gamma |D^m \varphi_{K,k}|^2 \, dx
\]
\[
= K^{-1} \sum_{k=1}^{K} \int_{\Omega} \frac{\lambda}{2} \left( \text{tr}(\varepsilon[w_{K,k}]) \right)^2 + \mu \text{tr}(\varepsilon[w_{K,k}])^2 + \gamma |D^m w_{K,k}|^2 \, dx + K \sum_{k=1}^{K} \int_{\Omega} r_{K,k} \, dx ,
\]
and the remainder is of order $K^{-\frac{1}{2}}$. To see this, we apply (30), lemma 11 and the uniform bound on the energy to deduce
\[
K \sum_{k=1}^{K} \int_{\Omega} |r_{K,k}| \, dx \leq CK \sum_{k=1}^{K} \int_{\Omega} K^{-3} |Dw_{K,k}|^3 \, dx
\]
\[
\leq CK \max_{k=1,\ldots,K} \|\varphi_{K,k} - \text{Id}\|_{C^1(\overline{\Omega})} \sum_{k=1}^{K} \|\varphi_{K,k} - \text{Id}\|_{H^m(\Omega)}^2
\]
\[
\leq CK \theta(\mathcal{J} K^{-1}) \sum_{k=1}^{K} \theta(R(I_{K,k-1}, I_{K,k}, \varphi_{K,k})) \leq CK^{\frac{1}{2}} \sum_{k=1}^{K} R(I_{K,k-1}, I_{K,k}, \varphi_{K,k}) \leq C \mathcal{J} K^{-\frac{1}{2}}.
\]
Finally, a standard weak lower semicontinuity argument [19] shows
\[
\liminf_{K \to \infty} K \sum_{k=1}^{K} \int_{\Omega} W(D\varphi_{K,k}) + \gamma |D^m \varphi_{K,k}|^2 \, dx
\]
\[
= \liminf_{K \to \infty} \int_{0}^{1} \int_{\Omega} \frac{\lambda}{2} \left( \text{tr}(\varepsilon[w_{K}]) \right)^2 + \mu \text{tr}(\varepsilon[w_{K}])^2 + \gamma |D^m w_{K}|^2 \, dx \, dt
\]
\[
\geq \int_{0}^{1} \int_{\Omega} \frac{\lambda}{2} \left( \text{tr}(\varepsilon[v]) \right)^2 + \mu \text{tr}(\varepsilon[v])^2 + \gamma |D^m v|^2 \, dx \, dt ,
\]
which implies the weak lower semicontinuity of the path energy along the sequence $\{I_K\}_{K \in \mathbb{N}}$.

3. Verification of the admissibility of the limit
Finally, it remains to verify that $(I,v,Y,z)$ for a suitable $Y$ is a solution of (17) and (18). We have already pointed out that the manifold-valued metamorphosis energy functional necessitates a variational inequality, which results in significant modifications of this step compared to [11].

Let $\tilde{Y}$ denote the solution of
\[
\frac{d}{dt} \tilde{Y}(t,x) = v(t,\tilde{Y}(t,x)) \quad \text{for } (t,x) \in [0,1] \times \Omega ,
\]
\[
\tilde{Y}(0,x) = x \quad \text{for } x \in \Omega ,
\]
which exists due to theorem [7]. Furthermore, (24) and the uniform boundedness of $w_K \in L^2((0,1), \mathcal{V})$ imply that the sequence $v_K$ is uniformly bounded in $L^2((0,1), C^{1,\alpha}(\overline{\Omega}))$. 

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Then, incorporating remark 8, one can infer that \( Y_K \) is uniformly bounded in \( C^0([0,1], C^{1,\alpha}((0,1), \Omega)) \), and by exploiting Hölder’s inequality we can even show that the sequence is uniformly bounded in \( C^{0,\frac{1}{2}}([0,1], C^{1,\alpha}((0,1), \Omega)) \). Hence, by using the compact embedding of Hölder spaces, the sequence \( Y_K \) converges strongly to some \( Y \) in \( C^{0,\beta}([0,1], C^{1,\beta}((0,1), \Omega)) \) for some \( \beta \in (0, \min(\frac{1}{2}, \alpha)) \).

It is left to verify that \( \tilde{Y} = Y \). To this end, we denote the solutions associated with \( w_K \) by \( \tilde{Y}_K \). Then,

\[
\|Y - \tilde{Y}\|_{C^0([0,1] \times \Omega)} \leq \|Y - Y_K\|_{C^0([0,1] \times \Omega)} + \|Y_K - \tilde{Y}_K\|_{C^0([0,1] \times \Omega)} + \|\tilde{Y}_K - \tilde{Y}\|_{C^0([0,1] \times \Omega)}.
\]

Here, the first term converges to zero as shown above and the last term converges to zero by theorem 7. Then, we can estimate as follows

\[
\|Y_K - \tilde{Y}_K\|_{C^0([0,1] \times \Omega)} \leq C \sum_{k=1}^K \int_{t_{K,k-1}}^{t_{K,k}} \|w_{K,k}(s, x_{K,k}(s, \cdot)) - w_{K,k}(s, \cdot)\|_{C^0(\Omega)} \, ds \tag{34}
\]

\[
\leq C \sum_{k=1}^K \int_{t_{K,k-1}}^{t_{K,k}} \|w_{K,k}(s, \cdot)\|_{H^m(\Omega)} \|y_{K,k}(s, \cdot) - Id\|_{C^0(\Omega)} \, ds
\]

\[
\leq C \|w_K\|_{L^2([0,1], H^m(\Omega))} \max_{k=1, \ldots, K} \|\varphi_{K,k} - Id\|_{C^0(\Omega)}.
\]

Here, the first inequality can be deduced from remark 8. Furthermore, the second inequality follows from the Lipschitz control of \( x \mapsto w_{K,k}(s, x_{K,k}(s, x)) - w_{K,k}(s, x) \), where the Lipschitz constant is bounded by \( C \|w_{K,k}(s, \cdot)\|_{H^m(\Omega)} \), and the third results from the Cauchy–Schwarz estimate. The uniform control of \( w_K \) and (30) imply \( Y = \tilde{Y} \) and by Hölder’s inequality \( Y \in C^{0,\frac{1}{2}}([0,1], C^{1,\alpha}((0,1), \Omega)) \). Finally, \( X_K \) is uniformly bounded in \( C^{0,\frac{1}{2}}([0,1], C^{1,\alpha}((0,1), \Omega)) \) due to remark 8. Thus, (17) is fulfilled.

Next, note that

\[
\int_{\Omega} d(I_K(t, Y_K(t, x)), I_K(s, Y_K(s, x)))^2 \, dx \leq \int_{\Omega} \left(\int_t^s z_K(r, Y_K(r, x)) \, dr\right)^2 \, dx
\]

\[
\leq |s - t| \int_t^s z_K(r, Y_K(r, x))^2 \, dr \, dx.
\]

By the uniform boundedness of \( z_K \) in \( L^2((0,1), L^2(\Omega)) \) we achieve that \( I_K \circ Y_K \in A_{\frac{1}{2}, L, |\det D Y|} \) for some appropriate \( L \). In what follows, we prove the weak convergence of a subsequence of \( I_K \circ Y_K \) to \( I \circ Y \in A_{\frac{1}{2}, L, |\det D Y|} \). To this end, we observe

\[
\limsup_{K \to \infty} \|I_K\|_{L^2}^2 = \limsup_{K \to \infty} \int_{\Omega} d(I_K, I)^2 \, dx
\]

\[
= \limsup_{K \to \infty} \int_0^1 \int_{\Omega} d(I_K(t, Y_K(t, x)), I(t, Y_K(t, x)))^2 \, |\det D Y_K| \, dx \, dt
\]

\[
= \limsup_{K \to \infty} \int_0^1 \int_{\Omega} d(I_K(t, Y_K(t, x)), I(t, Y(t, x)))^2 \, |\det D Y| \, dx \, dt,
\]

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where we used the transformation formula, the uniform convergence of $DY_K$, the metric triangle inequality and the convergence of $I(t, Y_K(t, x))$ to $I(t, Y(t, x))$ (see lemma 5). To sum up, this proves the weak convergence of $I_K \circ Y_K$ according to eq. (4) and by theorem 4 the limit is also contained in $A_{\frac{1}{2}, L, \mathrm{det} \, DY}$. Finally, it remains to verify (18). Assume there exist $s < t \in [0, 1]$ such that the set

$$B := \left\{ x \in \Omega : d(I(s, Y(s, x)), I(t, Y(t, x))) > \int_s^t z(r, Y(r, x)) \, dr \right\}$$

has positive Lebesgue measure. From the joint convexity of the metric $d(\cdot, \cdot)$ and the distance on $L^2(B, \mathcal{H})$ one observes that the mapping $(I, \tilde{I}) \mapsto \left( \int_B d(I(s, x), \tilde{I}(t, x))^2 \, dx \right)^{\frac{1}{2}}$ is continuous and convex on $A_{\frac{1}{2}, L, \mathrm{det} \, DY}$. Now, this implies weak lower semicontinuity [6 Lemma 3.2.3] and we obtain

$$\int_B d(I(s, Y(s, x)), I(t, Y(t, x))) \, dx \leq \liminf_{K \to \infty} \int_B d(I_K(s, Y_K(s, x)), I_K(t, Y_K(t, x))) \, dx$$

$$\leq \liminf_{K \to \infty} \int_B \int_s^t z_K(r, Y_K(r, x)) \, dr \, dx = \int_B \int_s^t z(r, Y(r, x)) \, dr \, dx ,$$

where the last equality follows from the weak convergence of $z_K$ combined with the strong convergence of $Y_K$, which also implies the weak convergence of $z_K \circ Y_K$. This yields a contradiction and concludes the proof of the liminf-inequality. \hfill \Box

**Theorem 13** (Recovery sequence). Let $I_A, I_B \in L^2(\Omega, \mathcal{H})$ be fixed input images and $z \in L^2([0, 1], L^2(\Omega, \mathbb{R}_+) )$. Furthermore, let $v \in L^2((0, 1), \mathcal{V})$ be a velocity field with corresponding global flow $Y \in H^1([0, 1], H^m(\Omega, \Omega))$ related by [6]. Assume that

$$d(I_A(x), I_B \circ Y(1, x)) \leq \int_0^1 z(s, Y(s, x)) \, ds \tag{35}$$

for a.e. $x \in \Omega$ and (W1), (W2) and (W3) hold true. If an image path $I \in L^2([0, 1], L^2(\Omega, \mathcal{H}))$ is given by

$$I(t, Y(t, x)) = \gamma_{I_A(x), I_B \circ Y(1, x)}(\alpha(t, x)) , \tag{36}$$

where $\alpha(t, x) := \int_0^t z(s, Y(s, x)) \, ds \left( \int_0^1 z(s, Y(s, x)) \, ds \right)^{-1}$, which is set to zero if the second term vanishes. Then there exists a recovery sequence such that the limsup-inequality in Definition 7 is valid.

**Remark 14.** It is easy to verify that $I(t, Y(t, \cdot)) \in L^2(\Omega, \mathcal{H})$ for a.e. $t \in [0, 1]$, which immediately implies $t \mapsto I(t, Y(t, \cdot)) \in L^2([0, 1], L^2(\Omega, \mathcal{H}))$ using continuity arguments. The parametrization of the geodesic $\gamma_{I_A(x), I_B \circ Y(1, x)}$ via $\alpha(t, x)$ can be regarded as the generalization of a linear blending of $\mathbb{R}^C$-valued images along motion paths considered in the classical metamorphosis model.

**Proof.** We proceed in several steps.
1. Construction of the recovery sequence

Note that the definitions of the piecewise constant velocity fields and the deformations coincide with the corresponding construction in the proof of \cite{11}, whereas the constructions of the image sequence based on the weak material derivative and the reconstructed flow field significantly differ.

The velocity field \( v \) corresponding to \( Y \) is approximated by piecewise constant functions \( w_K \in L^2((0,1),\mathcal{V}) \) given by \( w_K(t,x) = w_{K,k}(x) \) for \( t \in [t_{K,k-1},t_{K,k}) \) and a.e. \( x \in \Omega \), where

\[
 w_{K,k}(x) = \int_{t_{K,k-1}}^{t_{K,k}} v(s, x) \, ds.
\]

A standard argument shows that the sequence \( w_K \) converges to \( v \) in \( L^2((0,1),\mathcal{V}) \) \cite{24}. Moreover, we define a sequence of diffeomorphisms \( \varphi_K = (\varphi_{K,1}, \ldots, \varphi_{K,K}) \in H^m(\Omega,\mathbb{R}^n)^K \) by

\[
 \varphi_{K,k} = \text{Id} + K^{-1}w_{K,k}.
\]

A straightforward computation leads to

\[
 \max_{k \in \{1,\ldots,K\}} \| \varphi_{K,k} - \text{Id} \|_{C^1(\overline{\Omega})} = \max_{k \in \{1,\ldots,K\}} K^{-1} \left\| \int_{t_{K,k-1}}^{t_{K,k}} v(s, \cdot) \, ds \right\|_{C^1(\overline{\Omega})} \leq \max_{k \in \{1,\ldots,K\}} CK^{-1} \int_{t_{K,k-1}}^{t_{K,k}} \| v(s, \cdot) \|_{H^m(\Omega)} \, ds \leq CK^{-\frac{1}{2}} \left( \int_0^1 \| v(s, \cdot) \|_{H^m(\Omega)}^2 \, ds \right)^{\frac{1}{2}}.
\]

As before, choosing \( K \) sufficiently large ensures \( \varphi_K \in \mathcal{A}_K^K \). Hence, we can use the construction from section \ref{construction} to obtain \( v_K, Y_K \) and \( X_K \). Additionally, the same arguments as in the third part of the proof of theorem \ref{regularity} establish the regularity and the convergence results of the flows associated with \( v_K \). Then, to define an approximation of \( I \) we use the geodesic connection between \( I_A \) and \( I_B \circ Y(1,\cdot) \) along the path \( Y_K \) with the parametrization \( t \to \min(1,\alpha_K(t,x)) \) as follows

\[
 G_K(t,Y_K(t,x)) = \gamma_{I_A(x),I_B \circ Y(1,x)} \left( \min \{ 1, \alpha_K(t,x) \} \right),
\]

where \( \beta_K = \max(\|z(\cdot,Y_K(\cdot,\cdot)) - z(\cdot,Y(\cdot,\cdot))\|_{L^2((0,1) \times \Omega)}, K^{-1}) \) the function

\[
 \alpha_K(t,x) = \begin{cases} 
 0 & \text{if } \int_0^t z(s,Y(s,x)) \, ds = 0, \\
 \frac{\int_0^t z(s,Y_K(s,x)) \, ds}{\sqrt{\beta_K}} & \text{if } 0 < \int_0^t z(s,Y(s,x)) \, ds \leq \sqrt{\beta_K}, \\
 \frac{\int_0^t z(s,Y_K(s,x)) \, ds}{\sqrt{\beta_K}} & \text{else},
\end{cases}
\]

lies in \( L^2(\Omega,\mathbb{R}) \). This specific form of \( \alpha_K \) is needed in the last step of the proof to identify the limit of the recovery sequence. It is easy to see that \( \beta_K \to 0 \) and \( \int_0^t z(s,Y_K(s,x)) \, ds \to \int_0^t z(s,Y(s,x)) \, ds \) for \( K \to \infty \). Hence, \( \alpha_K(t,\cdot) \to \alpha(t,\cdot) \) in \( L^2(\Omega) \) for a.e. \( t \in [0,1] \) as \( K \to \infty \), which is proven below. Finally, the recovery sequence is defined as \( I_K = I_K^{\text{re}}(I_K, \varphi_K) \), where

\[
 I_K = (I_{K,0}, I_{K,1}, \ldots, I_{K,K}) = (G_K(t_{K,0},\cdot), \ldots, G_K(t_{K,K},\cdot))
\]

and \( \varphi_K \) refers to a vector of corresponding optimal deformations.
2. Verification of the limsup-inequality  Note that this step is very similar to the corresponding step in [11] with modifications necessitated by the manifold structure. A straightforward computation reveals

\[
J^K(I_K) = J_K(I_K, \varphi_K) \leq J_K[I_K, \varphi_K]
\]

\[
= K \sum_{k=1}^{K} \int_{\Omega} W(D\varphi_K) + \gamma |D^m \varphi_K|^2 + \frac{1}{\delta}(I_{K,k} - 1, I_{K,k} \circ \varphi_{K,k})^2 \, dx.
\]

The estimate (37) for \( K \) sufficiently large as well as the uniform boundedness of \( Y_K \) in \( C^{0,1}(\Omega) \) imply via a Taylor expansion

\[
\max_{k \in \{1, \ldots, K\}} \sup_{t \in [t_{K,k-1}, t_{K,k}]} \|1 - \det(Dx_K(t, \cdot))\|_{C^0(\Omega)} \\
\leq \max_{k \in \{1, \ldots, K\}} \sup_{t \in [t_{K,k-1}, t_{K,k}]} \left\| \frac{\det(Dy_K(t, \cdot)) - \det(Dy_K(t_{K,k-1}, \cdot))}{\det(Dy_K(t, \cdot))} \right\|_{C^0(\Omega)} \leq CK^{-\frac{1}{2}}.
\]

(39)

Taking into account (38), (1) and (35), we deduce for a.e. \( x \in \Omega \) that

\[
d(I_{K,k-1} \circ Y_K(t_{K,k-1}, x), I_{K,k} \circ Y_K(t_{K,k}, x)) \\
\leq d(I_A(x), I_B \circ Y(1, x)) |\alpha_K(t_{K,k}, x) - \alpha_K(t_{K,k-1}, x)| \leq \int_{t_{K,k}}^{t_{K,k-1}} z(s, Y_K(s, x)) \, ds.
\]

(40)

Hence, for any \( k = 1, \ldots, K \) we infer from (39), (40) and Jensen’s inequality that

\[
\int_{\Omega} d(I_{K,k-1} \circ \varphi_{K,k})^2 \, dx \\
= \int_{\Omega} d(I_{K,k-1} \circ Y_K(t_{K,k-1}, x), I_{K,k} \circ Y_K(t_{K,k}, x))^2 \, \det(DY_K(t_{K,k-1}, x)) \, dx \\
\leq \int_{\Omega} \left( \int_{t_{K,k-1}}^{t_{K,k}} z(s, Y_K(s, x)) \, ds \right)^2 \, \det(DY_K(t_{K,k-1}, x)) \, dx \\
\leq \frac{1}{K} \int_{t_{K,k-1}}^{t_{K,k}} \int_{\Omega} z^2(s, x) \, \det(Dx_K(s, x)) \, dx \, ds \\
\leq \frac{1}{K} \left( 1 + CK^{-\frac{1}{2}} \right) \int_{t_{K,k-1}}^{t_{K,k}} \int_{\Omega} z^2(s, x) \, dx \, ds.
\]

(41)

Furthermore, the same Taylor argument as in eq. (33) implies

\[
\int_{\Omega} W(D\varphi_K) + \gamma |D^m \varphi_K|^2 \, dx \leq K^{-2} \int_{\Omega} L[w_{K,k}, w_{K,k}] \, dx + CK^{-3} \int_{\Omega} |Dw_{K,k}|^3 \, dx.
\]

(42)
A direct application of Jensen’s inequality shows
\[
\int_\Omega L[w_{K,k}, w_{K,k}] \, dx = \int_\Omega \frac{\lambda}{2} \left( \text{tr} \left( \varepsilon \int_{t_{K,k-1}}^{t_{K,k}} v(s, x) \, ds \right) \right)^2 \\
+ \mu \text{tr} \left( \varepsilon \int_{t_{K,k-1}}^{t_{K,k}} v(s, x) \, ds \right)^2 + \gamma \left| D^m \int_{t_{K,k-1}}^{t_{K,k}} v(s, x) \, ds \right|^2 \, dx \\
\leq K \int_\Omega \int_{t_{K,k-1}}^{t_{K,k}} L[v, v] \, dt \, dx.
\] (43)

The derivation of a bound for the remainder of the Taylor expansion in (42) incorporates a Sobolev embedding theorem, Jensen’s inequality and \( v \in L^2((0, 1), \mathcal{V}) \), and is similar to eq. (33):
\[
\| w_{K,k} \|_{C^1(\Omega)}^2 \leq C \sum_{l=1}^{K} \| w_{K,l} \|_{H^m(\Omega)}^2 \leq C \sum_{l=1}^{K} \int_{t_{K,l-1}}^{t_{K,l}} \| v(t, \cdot) \|_{H^m(\Omega)}^2 \, dt \\
\leq CK \int_0^1 \| v(t, \cdot) \|_{H^m(\Omega)}^2 \, dt \leq CK.
\]

Hence, \( \max_{k=1, \ldots, K} \| w_{K,k} \|_{C^1(\Omega)} \leq CK^{\frac{1}{2}} \), which yields, again in combination with Jensen’s inequality, the estimate
\[
\sum_{k=1}^{K} \int_\Omega |Dw_{K,k}|^3 \, dx \leq \max_{k=1, \ldots, K} \| w_{K,k} \|_{C^1(\Omega)} \sum_{l=1}^{K} \int_{t_{K,l-1}}^{t_{K,l}} |Dv(t, x)|^2 \, dt \, dx \\
\leq C K^{\frac{1}{2}} \sum_{l=1}^{K} \int_{t_{K,l-1}}^{t_{K,l}} |Dv(t, x)|^2 \, dt \, dx \leq CK^{\frac{3}{2}}.
\] (44)

Altogether, by taking into account the estimates (41), (42), (43) and (44) we conclude
\[
\mathcal{J}_K(I_K) \leq \mathcal{J}_K(I_K, \varphi_K) = K \sum_{k=1}^{K} \int_\Omega W(D\varphi_{K,k}) + \gamma |D^m \varphi_{K,k}|^2 + \frac{1}{\delta} d(I_{K,k-1}, I_{K,k} \circ \varphi_{K,k})^2 \, dx \\
\leq \sum_{k=1}^{K} \left( \int_{t_{K,k-1}}^{t_{K,k}} L[v, v] + CK^{-1} |Dw_{K,k}|^3 + \frac{1}{\delta} \left( 1 + CK^{-\frac{1}{2}} \right) z^2(t, x) \, dx \, dt \right) \\
\leq \int_0^1 \int_\Omega L[v, v] + \frac{1}{\delta} z^2(t, x) \, dx \, dt + CK^{-\frac{1}{2}} + C \frac{1}{\delta} K^{-\frac{1}{2}} = \mathcal{J}[I] + O(K^{-\frac{1}{2}}),
\]

which readily implies the \textit{limsup-inequality}.

3. Identification of the recovery sequence limit  It remains to verify the convergence 
\( I_K \to I \) in \( L^2([0, 1], L^2(\Omega, \mathcal{H})) \) as \( K \to \infty \). To this end, we prove that every subsequence has a convergent subsequence with limit \( I \). This step substantially differs from the corresponding step in the proof of [11]. Note that \( Y_K \) has a convergent subsequence in
With these results at hand, we are able to use lemma 5 in the Bochner space setting to conclude
\[ \|z(\cdot, Y_K(\cdot, \cdot)) - z(\cdot, Y(\cdot, \cdot))\|_{L^2((0,1) \times \Omega)} \to 0 \]
as \( K \to \infty \). Setting \( \Omega_K = \{x \in \Omega: 0 < \int_0^1 z(s, Y(s,x)) \, ds \leq \sqrt{\beta_K} \} \) we infer for a.e. \( t \in [0,1] \) that
\[
\begin{align*}
\int_\Omega (\alpha_K(t,x) - \alpha(t,x))^2 \, dx & \leq \int_\Omega (\alpha_K(t,x) - \alpha(t,x))^2 \, dx + \frac{\|z(\cdot, Y_K(\cdot, \cdot)) - z(\cdot, Y(\cdot, \cdot))\|^2_{L^2((0,1) \times \Omega)}}{\beta_K} \\
& \leq \int_\Omega 2 \left( \frac{\int_0^1 z(s, Y(s,x)) \, ds}{\sqrt{\beta_K}} - \frac{\int_0^1 z(s, Y(s,x)) \, ds}{\beta_K} \right)^2 \, dx + 3\beta_K \\
& \leq 2|\Omega_K| + 3\beta_K \to 0.
\end{align*}
\]
(45)

Here, the second inequality results from an insertion of \( \beta_K^{-\frac{1}{2}} \int_0^1 z(s, Y(s,x)) \, ds \) and the triangle inequality. Next, we prove the convergence of a subsequence of \( G_K(\cdot, Y_K(\cdot, \cdot)) \) to \( I(\cdot, Y(\cdot, \cdot)) \), which in combination with corollary 6 implies the convergence of this subsequence of \( G_K \) to \( I \) in \( L^2((0,1), L^2(\Omega, H)) \). To this end, using (1) we estimate
\[
\begin{align*}
d_2(G_K(\cdot, Y_K(\cdot, \cdot)), I(\cdot, Y(\cdot, \cdot)))^2 & \leq \int_0^1 \int_\Omega |d(I_A(x), I_B \circ Y(1, x))|^2 \min(1, \alpha_K(t,x)) - \alpha(t,x) \, dx \, dt,
\end{align*}
\]
for which a subsequence converges to zero by eq. (45), lemma 3 and the dominated convergence theorem. In what follows, we restrict to this subsequence. Using the convergence of \( G_K \), we can show that also \( I_K^{\text{nt}}(I_K, \varphi_K) \) converges to \( I \). This follows directly from the estimate
\[
\begin{align*}
d_2(G_K(\cdot, Y_K(\cdot, \cdot)), I_K^{\text{nt}}(I_K, \varphi_K)(\cdot, Y_K(\cdot, \cdot)))^2 & \leq C \sum_{k=1}^K \left( \int_{t_{K,k-1}}^{t_{K,k}} \int_\Omega K^{-2} z^2(s, Y_K(s,x)) \, dx \, ds \\
+ \int_{t_{K,k-1}}^{t_{K,k}} \int_\Omega d(I_{K,k}(Y_K(t_{K,k-1}, x)), I_K^{\text{nt}}(I_K, \varphi_K)(s, Y_K(s,x)))^2 \, dx \, ds \right) \\
& \leq CK^{-2}\|z(\cdot, Y_K(\cdot, \cdot))\|^2_{L^2((0,1) \times \Omega)}.
\end{align*}
\]
Here, for the second inequality we exploit the argument form (40) for the geodesic distance \( d(G_K(s, Y_K(s,x)), I_{K,k}(Y_K(t_{K,k-1}, x))) \) together with a Cauchy–Schwarz inequality and the last inequality follows from the same arguments together with the definition of \( I_K^{\text{nt}} \) (22).
It remains to show that $d_2(I_{K}^{int}(I_K, \varphi_K), I_{K}^{int}(I_K, \varphi_K)) \to 0$. We get
\[
d_2(I_{K}^{int}(I_K, \varphi_K), I_{K}^{int}(I_K, \varphi_K))^2 \leq C \sum_{k=1}^{K} \int_{t_{k-1}}^{t_k} \int_{\Omega} \left( d(I_{K,k-1} \circ x_{K,k}, I_{K,k-1} \circ \varphi_{K,k})^2 
+ d(I_{K,k} \circ \varphi_{K,k} \circ x_{K,k}, I_{K,k} \circ \varphi_{K,k} \circ \varphi_{K,k})^2 \right) dx \, dt,
\]
where we used the convexity of the metric $d$ (3) and the fact $K(t - t_{K,k-1}) \leq 1$ for all $t \in [t_{K,k-1}, t_{K,k})$. Here, $\varphi_{K,k}$ and $\varphi_{K,k}$ denote the discrete transport map associated with $\varphi_{K,k}$ and the inverse map, respectively.

To prepare this, we first apply theorem 2 and approximate the input images $I_A$ and $I_B$ by Lipschitz continuous functions $I_{A,j}$ and $I_{B,j}$, where the Lipschitz bound is assumed to be smaller than $C_j$ and the maximum value is bounded by $C_j$. We define spatially Lipschitz continuous approximations of $\alpha_K$ by $\alpha_{K,j}(t, \cdot) = \alpha_K(t, \cdot) \cdot \kappa_j$, where $\{\kappa_j\}_{j \in \mathbb{N}} \subset C_0(\mathbb{R}^n)$ is a family of non-negative mollifiers with mass 1 such that $\alpha_{K,j}$ is Lipschitz continuous with Lipschitz constant bounded by $C_j$. Analogously, we approximate $\alpha$ by $\alpha_j$. The estimate eq. (45) implies
\[
\|\alpha_{K,j}(t, \cdot) - \alpha_j(t, \cdot)\|_{L^2(\Omega)} \leq C \|\alpha_K(t, \cdot) - \alpha(t, \cdot)\|_{L^2(\Omega)} \leq C(2|\Omega_K| + 3\beta_K).
\]

Next, $G_K$ is approximated by
\[
G_{K,j}(t, Y_K(t, x)) = \gamma_{I_{A,j}(x)}(t \cdot, Y(t, x)) \left( \min(1, \alpha_{K,j}(t, x)) \right),
\]
which is spatially Lipschitz continuous with Lipschitz constant $C_j$. To verify this, it suffices to prove the Lipschitz continuity of the function $G_{K,j}(t, Y_K(t, \cdot))$ for all $t \in [0, 1]$ since $X_K$ is uniformly bounded in $C^{\frac{1}{2}}([0, 1], C_1(\Omega))$. For $x_1, x_2 \in \Omega$ we get
\[
\begin{align*}
&d(G_{K,j}(t, Y_K(t, x_1)), G_{K,j}(t, Y_K(t, x_2))) \\
\leq &d(G_{K,j}(t, Y_K(t, x_1)), \hat{G}_{K,j}(t, x_1, x_2)) + d(\hat{G}_{K,j}(t, x_1, x_2), G_{K,j}(t, Y_K(t, x_2))) \\
\leq &d(I_{A,j}(x_1), I_{A,j}(x_2)) + d(I_{B,j} \circ Y_K(1, x_1), I_{B,j} \circ Y_K(1, x_2)) \\
&+ |\alpha_{K,j}(t, x_1) - \alpha_{K,j}(t, x_2)| d(I_{A,j}(x_1), I_{B,j} \circ Y_K(1, x_1)) \\
\leq &C_j|x_1 - x_2|.
\end{align*}
\]

Here, we use for comparison
\[
\hat{G}_{K,j}(t, x_1, x_2) = \gamma_{I_{A,j}(x_1)}(t \cdot, Y(t, x_1)) \left( \min(1, \alpha_{K,j}(t, x_2)) \right)
\]
on the discrete curve $t \mapsto Y_K(t, x_1)$. Next, we approximate $I_{K,k}$ by $I_{K,k,j}(x) := G_{K,j}(t_{K,k}, x)$ and show uniform convergence in $K, k$ of the associated approximation error to zero. To this end, we verify that
\[
\begin{align*}
d_2(I_{K,k,j}, I_{K,k}) \leq &C d_2 \left( \gamma_{I_A}(t \cdot, Y(t, x)) \left( \min(1, \alpha_{K,j}(t_{K,k}, \cdot)) \right), \gamma_{I_A}(t \cdot, Y(t, x)) \left( \min(1, \alpha_K(t_{K,k}, \cdot)) \right) \right) \\
\leq &C \left( \int_{\Omega} \left( d(I_{A}(x), I_{B} \circ Y(x, \cdot)) \right)^2 \min \left(1, |\alpha_{K,j}(t_{K,k}, x) - \alpha_K(t_{K,k}, x)| \right) dx \right)^{\frac{1}{2}} \\
\leq &C \left( \epsilon + C_\epsilon \text{ess sup}_{t \in [0, 1]} \|\alpha_{K,j}(t, \cdot) - \alpha_K(t, \cdot)\|_{L^2(\Omega)} \right),
\end{align*}
\]
where we first use the transformation formula together with the uniform boundedness of the determinants and then split the domain Ω into a set, where \(d(I_A(x), I_B \circ Y(1, x))^2 \leq C_\epsilon\), and the remainder. By the monotone convergence theorem, the remainder can be assumed to be smaller than a fixed \(\epsilon\) with \(C_\epsilon\) chosen sufficiently large. We observe that 
\[
\text{ess sup}_{t \in [0, 1]} \|\alpha_{K,i}(t, \cdot) - \alpha_{K}(t, \cdot)\|_{L^2(\Omega)} \text{ converges to zero for } j \to \infty \text{ uniformly in } K.
\]
Thus, the approximation error converges to zero uniformly in \(K\) and \(k\).

Now, we are able to prove the convergence of the first integral in (47). To this end, we fix \(j = \min\{l \in \mathbb{N}: l \geq K \} \). Then,
\[
\sum_{k=1}^K \int_{I_{K,k-1}}^{I_{K,k}} \int_{\Omega} d(I_{K,k-1} \circ x_{K,k}(t, x), I_{K,k-1} \circ \overline{\tau}_{K,k}(t, x))^2 \, dx \, dt 
\]
\[
\leq C \sum_{k=1}^K \int_{I_{K,k-1}}^{I_{K,k}} \int_{\Omega} d(I_{K,k-1, j} \circ x_{K,k}(t, x), I_{K,k-1, j} \circ \overline{\tau}_{K,k}(t, x))^2 + 
\]
\[
d(I_{K,k-1, j} \circ x_{K,k}(t, x), I_{K,k-1, j} \circ \tau_{K,k}(t, x))^2 + 
\]
\[
d(I_{K,k-1, j} \circ \tau_{K,k}(t, x), I_{K,k-1} \circ \overline{\tau}_{K,k}(t, x))^2 \, dx \, dt .
\]
Here, the first and the last term converge to zero as \(K \to \infty\) by using the transformation formula and the uniform boundedness of the determinants. For the second term we obtain
\[
\sum_{k=1}^K \int_{I_{K,k-1}}^{I_{K,k}} \int_{\Omega} d(I_{K,k-1, j} \circ x_{K,k}(t, x), I_{K,k-1, j} \circ \tau_{K,k}(t, x))^2 \, dx \, dt 
\]
\[
\leq C j^2 \max_{k=1, \ldots, K} \|\text{Id} - \tau_{K,k} \circ y_{K,k}\|_{C^2([t_{K,k-1}, t_{K,k}] \times \overline{\Omega})}^2 
\]
\[
\leq C K^\frac{1}{2} \max_{k=1, \ldots, K} \|\tau_{K,k} - y_{K,k}\|_{C^2([t_{K,k-1}, t_{K,k}] \times \overline{\Omega})}^2 
\]
\[
\leq C K^\frac{1}{2} \max_{k=1, \ldots, K} \left\{\|\tau_{K,k} - \text{Id}\|_{C^2(\overline{\Omega})}^2, \|\tau_{K,k} - \text{Id}\|_{C^1(\overline{\Omega})}^2\right\} \leq C K^{-\frac{1}{2}}.
\]
For the last inequality we incorporate (37) for \(\varphi_{K,k}\) and (30) for \(\overline{\tau}_{K,k}\). The convergence of the second integral in (47) follows by an analogous reasoning, which concludes the proof.

We conclude this section with the desired convergence statement for discrete geodesic paths.

**Theorem 15** (Convergence of discrete geodesic paths). Let \(I_A, I_B \in L^2(\Omega, \mathcal{H})\) and suppose that the assumptions (W1), (W2) and (W3) hold true. For every \(K \in \mathbb{N}\) let \(I_K\) be a minimizer of \(\mathcal{J}_K\) subject to \(I_K(0) = I_A\) and \(I_K(1) = I_B\). Then, a subsequence of \(\{I_K\}_{K \in \mathbb{N}}\) converges weakly in \(L^2([0, 1], L^2(\Omega, \mathcal{H}))\) to a minimizer of the continuous path energy \(\mathcal{J}\) as \(K \to \infty\), and the associated sequence of discrete energies converges to the minimal continuous path energy.
Proof. By a comparison argument we deduce that the path energy $\mathcal{J}_K$ is bounded by $\mathcal{J} = \frac{1}{2}d_2(I_A, I_B)^2$. For optimal vectors of images $I_K$ and deformations $\varphi_K$ we apply the construction of the extension in time from section 4. In particular, $\mathcal{J}_K(I_K, \varphi_K) \leq \mathcal{J}$ for all $K \in \mathbb{N}$. Using [23] and eq. (31) we conclude that $z_k$ is uniformly bounded in $L^2((0,1) \times \Omega)$. Then, remark 8 together with (24) and eq. (30) imply the uniform boundedness of $Y_K, X_K$ in $C^0([0,1], C^{1,\alpha}((\Omega)))$. By incorporating inequality (18), we obtain for a constant function $f_a(x) = a$ with $a \in H$ that

$$d_2(I_K(t, \cdot), f_a) \leq C(d_2(I_K(t, Y(t, \cdot)), I_A)) + d_2(I_A, f_a) \leq C(\|z_K\|_{L^2((0,1) \times \Omega)} + 1).$$

Therefore, $\{I_K\}_{K \in \mathbb{N}}$ is uniformly bounded in $L^\infty([0,1], L^2(\Omega, \mathcal{H}))$ and a subsequence converges weakly to some $I \in L^2([0,1], L^2(\Omega, \mathcal{H}))$.

Now, we assume that there exists an image path $\tilde{I} \in L^2([0,1], L^2(\Omega, \mathcal{H}))$ with corresponding optimal tuple $(\tilde{I}, \tilde{v}, \tilde{Y}, \tilde{z})$ satisfying (17) and (18) such that $\mathcal{J}[\tilde{I}] < \mathcal{J}[I]$. (49)

Note that such a tuple exists due to the weak lower semi-continuity and the weak closedness of (17) and (18), which follows by similar arguments like in step 3 of the proof of theorem 12. Without restriction we assume that $\tilde{I}$ has the form

$$\tilde{I}(t, \tilde{Y}(t, x)) = \gamma_{I_A(x), I_B \circ \tilde{Y}(1, x)} \left( \int_0^t \tilde{z}(s, \tilde{Y}(s, x)) \, ds \right).$$

Then, using the lim sup-estimate shown in theorem 13, there exists a sequence $\{\tilde{I}_K\}_{K \in \mathbb{N}} \subset L^2((0,1], L^2(\Omega, \mathcal{H}))$ satisfying $\limsup_{K \to \infty} \mathcal{J}_K[\tilde{I}_K] \leq \mathcal{J}[\tilde{I}]$. Thus, we obtain

$$\mathcal{J}[I] \leq \liminf_{K \to \infty} \mathcal{J}_K[I_K] \leq \limsup_{K \to \infty} \mathcal{J}_K[\tilde{I}_K] \leq \mathcal{J}[\tilde{I}],$$

which contradicts (49). Hence, $I$ minimizes the continuous path energy over all admissible image paths. Finally, the discrete path energies converge to the limiting path energy along a subsequence, i.e. $\lim_{K \to \infty} \mathcal{J}_K[I_K] = \mathcal{J}[I]$, which again follows from eq. (50) by using $\tilde{I} = I$. □

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