EQUIVARIANT $K$-CLASSES OF MATRIX ORBIT CLOSURES

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Abstract. We show that the equivariant $K$-theory class of an orbit closure
$\text{GL}_r v (\mathbb{K}^r)^n$ of $r \times n$ matrices depends only on the matroid of the matrix $v$. As consequences, the same is true of the equivariant cohomology class of this orbit, as well as of the isomorphism types of the cyclic $\text{GL}_r$ or $\mathfrak{S}_n$-module generated by the tensor product of the columns of $v$. We give formulae for certain coefficients of these matroid invariants, and for the entire invariants in some cases, as well as formulae for the behaviour of these invariants under certain matroid operations. We also give generators for the ideal of the orbit closure up to radical.

1. Introduction and statement of results

Let $v \in \mathbb{A}^{r \times n}$ be a matrix with entries in a field $\mathbb{K}$ of characteristic zero. Let $G$ be the product of $\text{GL}_r(\mathbb{K})$ with the algebraic $n$-torus $T = (\mathbb{K}^\times)^n$ which we view as the diagonal torus in $\text{GL}_n(\mathbb{K})$. In this way, $\mathbb{A}^{r \times n}$ carries an action of $G$ via $(g, t) \cdot v = gvt^{-1}$. Under this action $v$ has an orbit
$X^o_v = \text{GL}_r v T = \{ gvt^{-1} : g \in \text{GL}_r(\mathbb{K}), t \in T \}$.

The columns of $v$ may be thought of as a configuration $(v_1, \ldots, v_n)$ of vectors in $\mathbb{K}^r$. The configurations in $X^o_v$ are those that are projectively equivalent to $v$, in that their projectivizations, labelled point configurations in $\mathbb{P}^{r-1}$, are equal up to automorphisms of the ambient space.

The Zariski closure of this orbit is an irreducible subvariety of $\mathbb{A}^{r \times n}$, which we denote by $X_v$ and refer to as a matrix orbit closure. We denote the ideal of $X_v$ by $I_v$, and the coordinate ring of $\mathbb{A}^{r \times n}$ to which it belongs by
$R = \mathbb{K}[x_{i,j} : 1 \leq i \leq r, 1 \leq j \leq n]$.

The goal of our work is to thoroughly understand matrix orbit closures and their relations to the matroid of $v$. Recall that the matroid of a matrix $v \in \mathbb{A}^{r \times n}$, denoted $M(v)$, is the simplicial complex on $[n]$ whose faces index independent subsets of columns of $v$.

Our work was motivated by observed similarities between the cohomology class of a torus orbit closure in a Grassmannian, and the cyclic $\text{GL}_r$-module generated by a decomposable tensor in $(\mathbb{K}^r)^{\otimes n}$. These objects were observed to take strikingly similar forms, and while the former was known to be a matroid invariant, the latter was not. We will view both of these objects as being a shadow of the class of a matrix orbit closure in equivariant $K$-theory. Our main result is stated here and proved as Theorem 5.1.

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Theorem. The class of the matrix orbit closure of \( v \in A^{r \times n} \) in the G-equivariant K-theory of \( A^{r \times n} \) is a function of the matroid of \( v \).

The Zariski open subset of full rank matrices in \( A^{r \times n} \) is a principal GL\(_r\)-bundle over the Grassmannian \( G_r(n) \) of \( r \)-planes in \( k^n \) with projection \( \pi \). Accordingly, G-equivariant K-theory of \( A^{r \times n} \) is intimately related to T-equivariant K-theory of the Grassmannian, with the differences accounted for by phenomena within the locus of matrices of less than full rank.

The image of a matrix orbit closure in the Grassmannian is a torus orbit closure, and Speyer has shown that the K-classes of such varieties are invariants of the associated matroid [38]. Our main theorem builds on this result: we will show that the class in K-theory of the closure of the pullback of a T-invariant subvariety of the Grassmannian does not see the low rank matrices picked up in the closure. This is made precise in Theorem 5.2, which we paraphrase here.

Theorem. Let \( \pi^\dagger : K^G_0(A^{r \times n}) \to K^T_0(G_r(n)) \) denote the natural map (2.2). There exists a unique \( \mathbb{Z} \lbrack \text{Char}(T) \rbrack \)-module homomorphism

\[
s : K^T_0(G_r(n)) \to K^G_0(A^{r \times n}),
\]

providing a section of \( \pi^\dagger \) such that:

(a) For every G-invariant subvariety \( X \) of \( A^{r \times n} \) that contains a full rank matrix, we have

\[
s(K(\pi(X))) = K(X).
\]

(b) Every class in the image of \( s \) is a \( \mathbb{Z} \lbrack \text{Char}(T) \rbrack \)-linear combination of Schur polynomials \( s_{\lambda}(u) \) whose partition \( \lambda \) satisfies \( \lambda_1 \leq n - r \).

Here \( K(X) \) denotes the class of the structure sheaf of a subvariety \( X \) in its respective K-theory.

Thus, the difference between the torus orbit closures considered by Speyer and our matrix orbit closures might be compared with the difference between Schubert varieties and the matrix Schubert varieties of [30].

Theorems 5.1 and 5.2 should not be taken for granted, since the low rank matrices in a matrix orbit closure could still make up a complicated variety. A classification of the singularities of matrix orbit closures is not known and a priori could be quite complicated [11]. As one illustration of the complexity, smaller G orbit closures contained within a given matrix orbit closure do not provide a stratification. Among the smaller matrix orbit closures, there may be a continuum of unequal ones all with the same matroid. This stands in contrast to the case of the Schubert stratification of a Grassmannian, as well as the stratification of a toric variety by finitely many torus orbits.

Theorem 5.2 follows ideas for equivariant cohomology classes used by Fehér and Rimányi [19], and Fehér, Némethi and Rimányi [20]. The principal technology of this approach is an understanding of the stabilization of K-classes of orbit closures as one passes from equivariant K-theory of \( A^{(r-1) \times n} \) to that of \( A^{r \times n} \), by including the former as a coordinate subspace of the latter and taking GL\(_r\) orbit closures.

We give formulae in the basis of Schur functions for how this operation transforms classes in both equivariant K-theory, Proposition 4.3, and equivariant cohomology, Proposition 8.3.

From Theorem 5.1 we obtain the answer to a question that motivated our work, appearing as Theorems 7.2 and 7.9 below.
Theorem. The isomorphism type of the cyclic $GL_r$-module generated by a decomposable tensor $v_1 \otimes \cdots \otimes v_n \in (k^r)^\otimes n$ is a function of the unlabelled matroid of $v$. The same result holds with the symmetric group $\mathfrak{S}_n$ in place of $GL_r$.

Certain questions motivating our work still remain. We summarize them here along with references to our progress so far.

1. What are generators for the prime ideal of $X_v$? In Theorem 3.7, we give set-theoretical generators for this variety. In every computed case these generators actually yield the prime ideal of $X_v$. In Propositions 3.10 and 3.11, we describe two situations when our ideal is provably prime.

2. What can be said of the singularities of $X_v$? Are these varieties Cohen–Macaulay? Rational? In Propositions 3.10 and 3.11, we show that generic elements of $k^{2 \times n}$ and $k^{(n-2) \times n}$ have Cohen–Macaulay orbit closures.

3. What is the class of $X_v$ in the $G$-equivariant $K$-theory of $k^{r \times n}$? We show in Theorem 5.1 that this class is a matroid invariant taking the form

$$\sum_{\lambda, b \in \mathbb{N}^n} d_{\lambda, b} s_{\lambda}(u) t^b \in \mathbb{Z}[u_1, \ldots, u_r, t_1, \ldots, t_n],$$

where $d_{\lambda, b}$ are integers depending on $v$ and $s_{\lambda}(u)$ is a Schur polynomial. This does not address the matter of which matroid invariant it is. Ideally, we would like an explicit formula for $d_{\lambda, b}(v)$. In Theorem 7.17, we give a formula for $d_{\lambda, b}$ when $\lambda$ is a hook and $b \in \{0, 1\}^n$. In Proposition 6.5, we give a complete formula when $r = 2$.

4. What is the irreducible decomposition of the cyclic $GL_r$-module generated by a decomposable tensor? The answer is determined by the equivariant $K$-class of $X_v$. This is related to the problem of when symmetrizations of decomposable tensors are zero, a problem studied by Dias da Silva’s school. Here we address the same special cases as in (3). In Theorem 7.12, we give a formula for the multiplicity of a hook shape in this module. Proposition 7.11 gives a complete answer when $r = 2$.

5. What is the cohomology class, $T$-equivariant or ordinary, of a $T$-orbit closure in the Grassmannian $G_r(n)$? This question was studied by Kapranov [27], Klyachko [28], Liu [32] and Speyer [21, 38]. The answer, once again, is determined by the equivariant $K$-class of $X_v$. In Theorem 8.4, we give a positive formula for the equivariant cohomology class of the $T$-orbit closure of a sufficiently generic point in the Grassmannian. The proof relies on the equivariant localization theory of Goresky–Kottwitz–MacPherson. This theorem also yields a formula for the $G$-equivariant cohomology class of a sufficiently generic point in $k^{r \times n}$.

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2. Background material

2.1. Matroid terminology and background. White’s Theory of Matroids [42] serves as an excellent reference for the matroid theory needed here. For the convenience of the reader, we gather the required notions in this section.

A matroid is a simplicial complex $M$ on a finite ground set $E$ whose faces satisfy the following exchange axiom: for faces $I$ and $I'$ of $M$, if $|I| < |I'|$ then there is some $e \in I' \setminus I$ such that $I \cup \{e\}$ is a face of $M$. Two matroids are isomorphic if they are isomorphic as simplicial complexes: that is, if there is a bijection between their ground sets inducing a bijection between their faces. We will refer to the isomorphism type of a matroid as an unlabelled matroid.

For any matrix $v \in \mathbb{A}^{r \times n}$ the matroid of $v$, denoted $M(v)$, is the simplicial complex whose faces are those $I \subset [n]$ such that the columns of $v$ indexed by $I$ are linearly independent. Any matrix in the orbit $G \cdot v$ has the same matroid as $v$. The set of matrices in $\mathbb{A}^{r \times n}$ with a prescribed matroid is a subscheme of $\mathbb{A}^{r \times n}$ called a matroid stratum or a matroid realization space. It is a result of Sturmfels [39] that this is not a stratification in any nice sense (particularly that of Whitney). Worse, a matroid stratum can contain arbitrarily complicated singularities, a result referred to as Mnëv–Sturmfels universality [40].

Matroids that can be written as $M(v)$ for some $v \in \mathbb{A}^{r \times n}(k)$ are said to be realizable over $k$. The faces and non-faces of $M$ are called independent and dependent.
sets, respectively. The minimal dependent sets are called \textit{circuits} and the maximal independent sets are called \textit{bases}.

The \textbf{uniform matroid} of rank \( r \) on \( n \) elements, \( U_{r,n} \), is the matroid with ground set \([n]\) whose bases are all \( r \) element subsets of \([n]\). It is the matroid of a generic element of \( \mathbb{A}^{r \times n} \).

We denote the rank of a matrix \( v \) by \( \text{rk}(v) \). The \textbf{rank} \( \text{rk}(M) \) of a matroid \( M \) is the cardinality of a maximal independent set. In particular, \( \text{rk}(M(v)) = \text{rk}(v) \). On many occasions we will assume that the rank of matroids we deal with is full, i.e. equals \( r \). In particular, when we state the hypothesis “\( v \) has a uniform matroid”, we mean uniform of rank \( r \).

For any \( v \in \mathbb{A}^{r \times n} \), its \textbf{Gale dual} is any \( v^\perp \in \mathbb{A}^{(n-\text{rk}(v)) \times n} \) whose rows form a basis for the (right) kernel of \( v \). Thus, the Gale dual is determined up to the action of \( \text{GL}_{n-\text{rk}(v)} \) on \( \mathbb{A}^{(n-\text{rk}(v)) \times n} \). If \( v \) has full rank then Gale duality really is a duality, \( \text{GL}_r(v^\perp)^* = \text{GL}_r v \). To a matroid \( M \) we associate a \textbf{dual matroid} \( M^* \) whose bases are complements of bases of \( M \). If \( M(v) \) is the matroid of a matrix \( v \), then \( M(v^*) \) is the matroid \( M(v^\perp) \) of the Gale dual of \( v \).

The \textbf{direct sum} of two matroids on disjoint sets is the join of the two simplicial complexes. A matroid is said to be \textbf{connected} if it is indecomposable with respect to this operation. Any matroid \( M \) can be written uniquely as a direct sum of connected matroids, the constituents of which are called the \textbf{connected components} of \( M \). A \textbf{coloop} of \( M \) is cone-point, and a \textbf{loop} of \( M \) is cone point of \( M^* \). Thus, a coloop is in every base of \( M \) and a loop is in no base of \( M \).

The \textbf{rank partition} of \( M \) is the sequence of numbers \( \lambda(M) = (\lambda_1, \lambda_2, \ldots) \) determined by the condition that for all \( k \geq 1 \), its partial sums are sizes of the largest union of \( k \) independent sets of \( M \). It is a theorem of Dias da Silva \([16]\) that \( \lambda(M) \) is a partition (i.e., it is weakly decreasing). If \( M \) is loop-free then \( \lambda(M) \) is the maximum partition \( \lambda \) in dominance order such that \( M \) can be \( E \) can be partitioned into independent sets of sizes \( \lambda_1, \lambda_2, \ldots \).

The \textbf{restriction} of \( M \) to a subset \( J \subset E \), denoted \( M|J \), consists of those independent sets belonging to \( J \). The \textbf{contraction} of \( M \) by \( J \) is \( (M^*|J^*)^* \), where \( J^* = E \setminus J \), and is denoted \( M/J \). If \( M = M(v) \) is realizable then \( M/J \) is obtained as follows. Let \( A \in \text{End}(k^n) \) be a matrix whose kernel is spanned by \( \{v_j : j \in J\} \) and is generic with respect to this property. Then \( M/J \) is the matroid of \( Av \), with columns \( J \) deleted.

If there is a matroid \( M' \) with ground set \( E' \supset E \) such that \( M = M'|E \) then \( M'/(E' \setminus E) \) is a said to be a \textbf{quotient} of \( M \). It follows that every quotient of a realizable matroid is again realizable.

Let the indicator vector of a subset \( B \) of \([n]\) be \( e_B = \sum_{i \in B} e_i \). The \textbf{matroid base polytope} \( P(M) \) of a matroid \( M \) with ground set \([n]\) is the convex hull of the indicator vectors of the bases of \( M \) in \( \mathbb{R}^n \). It is a theorem of Edmonds \([14]\) that, among the polytopes \( P \) with vertices chosen from the set \( \{e_B : B \subseteq [n]\} \), matroid polytopes are exactly those that lay in a plane where the coordinate sum a positive integer and every edge of \( P \) has the form \( \text{conv}\{e_B, e_{B \cup \{j\}}\} \) for some \( B \subseteq [n] \) and \( i \in B, j \notin B \).

\subsection*{2.2. Equivariant K-theory.} In this section we define the basic elements we need from equivariant \( K \)-theory.

A \textbf{variety} is an integral separated scheme of finite type over \( k \). It will be important at various points in Section 5 that varieties are reduced.
2.2.1. **K-theory of coherent modules.** Let $X$ denote a variety carrying the action of an algebraic group $H$ and let $K^H_0(X)$ be the Grothendieck group of equivariant $H$-modules, that is, coherent $\mathcal{O}_X$-modules which carry an $H$ action that is compatible with the action on $X$. The cases that interest us are when $X = \mathbb{A}^{r \times n}$ and $G = \text{GL}_r \times T$, or when $X \subset \mathbb{A}^{r \times n}$ is a $G$-invariant subvariety. We refer to [34] for generalities on this $K$-theory.

If $X \subseteq Y$ is an inclusion of $H$-varieties, there is an exact sequence
\begin{equation}
K^H_0(X) \to K^H_0(Y) \to K^H_0(Y \setminus X) \to 0
\end{equation}
where the first two maps are a pushforward along the closed immersion followed by the restriction [34, §2.2.3]. We will repeatedly apply this in the case when $Y = \mathbb{A}^{r \times n}$. A particular case we will be interested in is when $X$ is the classical determinantal variety. In this case the sequence gives rise to the surjection
\[ K^G_0(\mathbb{A}^{r \times n}) \to K^G_0((\mathbb{A}^{r \times n})_{\text{fr}}), \]
where $(\mathbb{A}^{r \times n})_{\text{fr}}$ is the open subvariety of full rank matrices.

If $\phi : X \to Y$ is a principal $H$-bundle which is equivariant with respect to another group $H'$, there is an isomorphism $\phi^0 : K^H_0(Y) \xrightarrow{\sim} K^{H \times H'}_0(X)$ of rings [34, Proposition 3] even though these are naturally modules over the different base rings, $K^H_0(Y)$ and $K^{H \times H'}_0(X)$ respectively. In particular,
\[ \pi^0 : K^T_0(\mathbb{G}_r(n)) \xrightarrow{\sim} K^G_0((\mathbb{A}^{r \times n})_{\text{fr}}) \]
where $\pi : (\mathbb{A}^{r \times n})_{\text{fr}} \to \mathbb{G}_r(n)$ is the projection mapping, which is a principal $\text{GL}_r$-bundle. We conclude that there is a surjective map
\begin{equation}
\pi^1 : K^G_0(\mathbb{A}^{r \times n}) \to K^G_0((\mathbb{A}^{r \times n})_{\text{fr}}) \xrightarrow{\sim} K^T_0(\mathbb{G}_r(n)).
\end{equation}
Its kernel is an ideal in $K^G_0(\mathbb{A}^{r \times n})$ denoted by $I_r$.

A particular feature of torus-equivariant $K$-theory is that the classes of structure sheaves suffice to generate the entire $K$-theory.

**Proposition 2.1** ([1, Lemma 2.3]). Suppose that $H$ is a torus and $X$ is a $H$-variety. Then, $K^H_0(X)$ is generated by the classes of structure sheaves of $H$-invariant subvarieties of $X$, as a module over $K^H_0(\text{pt})$.

2.2.2. **K-polynomials of graded modules.** Let $R$ denote the polynomial ring $k[x_{i,j} : i \in [r], j \in [n]]$ and regard $\mathbb{A}^{r \times n}$ as $\text{Spec} R$. $R$ is graded by $\mathbb{Z}^r \times \mathbb{Z}^n$, the degree of $x_{i,j}$ being $a_i + b_j$, where $a_1, \ldots, a_r, b_1, \ldots, b_n$ are the standard basis vectors of $\mathbb{Z}^r \times \mathbb{Z}^n$. The grading group should be thought of as the weight lattice of the maximal torus in $G = \text{GL}_r \times T$ given by (the diagonal torus of $\text{GL}_r) \times T$.

Any finitely generated graded $R$-module $M = \bigoplus_{(a,b) \in \mathbb{Z}^r \times \mathbb{Z}^n} M_{(a,b)}$ has Hilbert series
\[ \text{Hilb}(M) = \sum_{(a,b) \in \mathbb{Z}^r \times \mathbb{Z}^n} \dim_k(M_{(a,b)})u^a t^b \in \mathbb{Z}[[u_1^{\pm 1}, \ldots, u_r^{\pm 1}, t_1^{\pm 1}, \ldots, t_n^{\pm 1}]] \]
By [35, Theorem 8.20], there is a Laurent polynomial $\mathcal{K}(M; u, t)$ such that
\[ \text{Hilb}(M) = \frac{\mathcal{K}(M; u, t)}{\prod_{i=1}^r \prod_{j=1}^n (1 - u_i t_j)}, \]
and we refer to this polynomial as the **$K$-polynomial** of $M$. In particular, if $X \subset \mathbb{A}^{r \times n}$ is a closed subscheme with defining ideal $I$, we write $\mathcal{K}(X; u, t)$ for the $K$-polynomial of $R/I$. Likewise, if $E$ denotes a coherent sheaf of $R$-modules we
denote by $K(\mathcal{E}; u, t)$ the $K$-polynomial of its module of global sections. We will often write the $K$-polynomials as $K(M)$, $K(X)$, etc.

The ring $R$ has the action of $G$ given by $((g, t) \cdot f)(v) = f(g^{-1}vt)$. The decomposition of $R$ into its various graded pieces $R_{(a,b)}$ is a refinement of the irreducible decomposition of $R$ as a $G$-module; it is precisely the refinement into weight spaces. It is important to note that the gradation and weight space decompositions have the (unfortunate) property that if $f \in R_{(a,b)}$, then $f$ has $GL_r$-weight $-a$. The arguably more natural convention of setting $\deg(x_{i,j}) = b_j - a_i$ results in ugly formulas and does not agree with the standard grading of $R$. Thus, given a $G$-equivariant graded module $M$ we pass back and forth between its character, as a $G$-module, and its Hilbert series by inverting all the $u$ variables.

It follows that the category of finitely generated graded $G$-equivariant $R$-modules equals the category of equivariant coherent $R$-modules, and we have

$$K^0_G(A^{r \times n}) = \mathbb{Z}[u_1, \ldots, u_r, t_1, \ldots, t_n][u_1^{-1}, \ldots, u_r^{-1}, t_1^{-1}, \ldots, t_n^{-1}]_{\mathcal{S}_r},$$

where $\mathcal{S}_r$ acts on the $u$ variables. Here we have written $e_r(-)$ for the elementary symmetric polynomial in its arguments. Thus, the $K$-polynomials $K(X)$ and $K(\mathcal{E})$ are the equivariant $K$-classes of the structure sheaf of $X$ and the global sections of $\mathcal{E}$.

More generally, we will employ the following standard notation.

- $\text{Par}_r$ is the set of partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ of length at most $r$.
- $s_\lambda(x)$ is a Schur polynomial in the list of variables $x = (x_1, \ldots, x_k)$.
- $e_k(x) = s_{\lambda_k}(x)$ is an elementary symmetric polynomial.
- $h_k(x) = s_k(x)$ is a complete homogeneous symmetric polynomial.

It will also be useful to give meaning to $s_\lambda(x_1, \ldots, x_k)$ when $\lambda$ is a $k$-tuple of integers (of any signs) that need not be a partition. We do this using the determinantal formula

$$s_\lambda(x_1, \ldots, x_k) = \frac{\det((x_{i}^{\lambda_i+j})_{i,j})}{\det((x_{i}^{k-j})_{i,j})}.$$  

In particular, the set

$$\{s_\lambda(x_1, \ldots, x_k) : \lambda \in \mathbb{Z}^k \text{ is nonincreasing}\}$$

is a basis for the symmetric Laurent polynomials in $x_1, \ldots, x_k$.

2.3. EQUIVARIANT COHOMOLOGY. Though we will speak of the $H$-equivariant cohomology of a variety with the action of a linear algebraic group $H$, we are really thinking of the Chow cohomology ring of its $H$-invariant algebraic cocycles modulo rational equivalence. This equivocation is forgiven since all the varieties in question have sufficiently nice stratifications that the two notions coincide [22, Example 19.1.11].

By a form of the Grothendieck–Riemann–Roch theorem, the $G$-equivariant cohomology class of an invariant algebraic subset $X \subset \mathbb{A}^{r \times n}$ is obtained from its $K$-polynomial as the term of $K(X; 1 - u, 1 - t)$ whose total degree is $\text{codim}(X)$. That is, replace each $u_i$ and $t_j$ with $1 - u_i$ and $1 - t_j$ and retain from the resulting power series only those summands $c(u)^a t^b$ where $a + b = \text{codim}(X)$. We denote this polynomial by $\mathcal{C}(X) = \mathcal{C}(X; u, t)$. It is also called the multidegree [35] of $X$. The
The points of \( X_v \). In this section we discuss the geometry of the matrix orbit closures \( X_v \), with respect to the \( G \) orbits they comprise.

**Proposition 3.1.** The closure of a \( G \)-orbit in \( \mathbb{A}^{r \times n} \) is an irreducible affine variety. If \( v \) has a matroid of rank \( r \) with \( c \) connected components, then

\[
\dim(X_v) = r^2 + n - c.
\]

**Proof.** Since \( G \) is a connected linear algebraic group the first claim follows. The second follows since the stabilizer of \( v \) is seen to be a \( c \)-dimensional torus inside the diagonal torus of \( G \).

Recall that \( \pi \) is the projection of the \( \text{GL}_r \)-bundle \((\mathbb{A}^{r \times n})^{fr} \to \mathbb{G}_r(n)\). Consider the case that \( v \in (\mathbb{A}^{r \times n})^{fr} \). It is well known that \( \pi(v)T \) is the toric variety associated to the matroid polytope of \( M(v) \). The \( T \)-orbits in \( \pi(v)T \) are in bijection with the faces of the matroid base polytope \( P(M(v)) \). One can give a combinatorial description of the faces of the matroid polytope as follows [2, Proposition 2]. Let \( S_\bullet \) be a flag of subsets

\[
\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} = [n].
\]

Every face of \( P(M(v)) \) is of the form \( P(M(v)_{S_\bullet}) \) where

\[
M(v)_{S_\bullet} = \bigoplus_{i=1}^{k+1} (M(v)|_{S_i})/S_{i-1}.
\]

Two different flags can produce the same matroid, but there is only one \( T \)-orbit in \( \pi(v)T \) with a given matroid. A realization of this result in terms of torus orbit closures is obtained as follows. Rescale column \( i \in S_j \setminus S_{j-1} \) of \( v \) by \( s^j \). Projecting this matrix into \( \mathbb{G}_r(n) \) we obtain a subspace \( \pi(v)\lambda(s) \), where \( \lambda(s) \) is a one-parameter subgroup of \( T \), i.e., an element of \( T(\mathbb{k}(s)) \). Here \( \mathbb{k}(s) \) is the field of Laurent series in \( s \) over \( \mathbb{k} \). Taking the limit \( \lim_{s \to 0} \pi(v)\lambda(s) \) yields a point of \( \pi(v)T \) with matroid \( M(v)_{S_\bullet} \). Every \( T \)-orbit in \( \pi(v)T \) is reached in this way and so our argument is complete.

The pullback \( \pi^{-1}(\lim_{s \to 0} \pi(v)\lambda(s)) \) is the \( G \)-orbit of a full rank matrix in \( X_v \) whose matroid is \( M(v)_{S_\bullet} \). We call any such matrix a **projection of \( v \) along the flag \( S_\bullet \)**. As before, there is only one \( G \)-orbit in \( X_v \) whose points have a prescribed matroid of the form \( \bigoplus_{i=1}^{k+1} M(v)|_{S_i}/S_{i-1} \).

The next result shows that all elements of \( X_v \) are obtained by projecting \( v \) along some flag and applying some element \( g \in \text{End}(\mathbb{k}^r) \) on the left.
Proposition 3.2. Suppose that $v$ has rank $r$ and $w \in X_v$ is a matrix of rank less than $r$. Then there is a matrix $w' \in X_v$ whose rank is that of $v$, and $w = gw'$ for some singular $g \in \text{End}(k^r)$.

Proof. Let $V = k^{r \times n}$ and suppose that $w$ has rank $\ell$. After applying an element of $\text{GL}_r$, and relabeling the columns of our matrices, we may assume that $w$ has a row equal to zero and its first $\ell$ columns are the first $\ell$ standard basis vectors. By the valuative criterion for properness, there is an element $(g(s), t(s))$ of $G(k((s))) = \text{GL}_r(k((s))) \times T(k((s)))$ such that $g(s)v t(s) \in k[[s]] \otimes_k V$ and

$$g(s)v t(s) \equiv w \mod s.$$  

Applying an element of $\text{GL}_r(k[[a]])$ we may assume that the first $\ell$ columns of $g(s)v t(s)$ are the first $\ell$ standard basis vectors. Let $\nu_i$ be the least of the non-negative integers that appears as an exponent in row $i$ of $g(s)v t(s)$. The limit

$$\text{diag}(s^{-\nu_1}, \ldots, s^{-\nu_r})(g(s)v t(s))$$

as $s \to 0$ gives an element $w'$ that has rank strictly larger than $w$. If the rank of $w'$ is not the rank of $v$ then, by induction, there is some $w'' \in X_v$ of rank $r$ and $g \in \text{End}(k^r)$ such that $gw'' = w'$. Applying an element of $\text{End}(k^r)$ that zeros out the appropriate rows, we bring $w'$ to $w$. □

Corollary 3.3. If $w \in X_v$ then there is a flag of sets $S_\bullet$ such that the matroid of $w$ is a quotient of

$$\bigoplus_{i=1}^{k+1}(M(v)|S_i)/S_{i-1}.$$  

Conversely, every quotient of such a matroid occurs as the matroid of some $w \in X_v$.

Proof. Combining the remarks above about faces of the matroid polytope $P(M(v))$ with Proposition 3.2, we obtain the first claim. The converse follows since the contraction of a matroid by a set of elements is realized by applying an element of $\text{End}(k^r)$ to a vector configuration in $k^r$. This means that every quotient of $\bigoplus_{i=1}^{k+1}(M(v)|S_i)/S_{i-1}$ is the matroid of a point in the $\text{End}(k^r)$-orbit of $\pi^{-1}(\pi(v)T)$. □

Remark 3.4. The correspondence between matroids and orbits in $X_v$ is not in general bijective as the following example shows. If

$$v = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Then for every $\mu \in k$,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & \mu - 1 & 1 & \mu \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \in X_v \implies \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & \mu \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in X_v.$$  

The matrices of the latter form are all projectively inequivalent. This stands in contrast to the situation with $\pi(v)T$, where orbits are in bijection with the matroids of the points in the orbit closure.
3.2. The ideal of an orbit closure. Since $X_v$ is an irreducible affine variety it is the vanishing locus of a prime ideal $I_v \subset R$. In this section we discuss this ideal. Our main result is Theorem 3.7, which gives a finite generating set for an ideal $I'_v$ given by minors of certain matrices, for which $\sqrt{I'_v} = I_v$.

3.2.1. The ideal $I'_v$. We now give the polynomial conditions for a matrix to lie in $X_v$. Recall from Section 2.1 the notion of Gale duality. For $v \in \mathbb{A}^{r \times n}$, its Gale dual is any $v^\perp \in \mathbb{A}^{(n-rk(v)) \times n}$ whose rows form a basis for the kernel of $v$. For any $w = (w_1, \ldots, w_n) \in X_v^\circ$, the vectors
\[
  w_1 \otimes v_1^1, \quad w_2 \otimes v_2^1, \quad \ldots, \quad w_n \otimes v_n^1
\]
are linearly dependent. This can be seen by expanding a linear combination in the standard basis of $\mathbb{k}^r \otimes \mathbb{k}^{n-r}$. By continuity this holds for any $u \in X_v$. More is true:

**Proposition 3.5** (Kapranov [27]). Suppose that $w \in \mathbb{A}^{r \times n}$ has a connected matroid of full rank. If the collection of tensors
\[
  w_1 \otimes v_1^1, \quad w_2 \otimes v_2^1, \quad \ldots, \quad w_n \otimes v_n^1
\]
forms a circuit in $\mathbb{k}^r \otimes \mathbb{k}^{n-r}$ then $w \in X_v^\circ$.

For a subset $J$ of $[n]$, let $v_J$ be the submatrix of $v$ on the columns indexed by $J$, so that the rank $rk(M|J)$ in the matroid of $v$ is the dimension of the span of these columns in $\mathbb{k}^r$. The Gale dual of $v_J$ is not $(v^\perp)_J$, but it is a projection of this configuration. This fact is matroidally manifested by the equality $(M|J)^c = M^*/J^c$ where $J^c$ is the complement of $J$ in the ground set of $M$.

**Theorem 3.6.** For any $v \in \mathbb{A}^{r \times n}$, a matrix $w$ is in $X_v$ if and only if for every $J = \{j_1, \ldots, j_\ell\} \subset [n]$, the tensors
\[
  \{w_{j_i} \otimes (v^\perp_J)_i : i = 1, \ldots, \ell\},
\]
are linearly dependent.

The proof of the theorem can be found in Section 3.3 below.

An immediate consequence of Theorem 3.6 is the next theorem, giving set-theoretic equations for $X_v$. Let $x$ denote the matrix of variables $x_{i,j}$, and let $x_j$ denote the $j$-th column $(x_{1,j}, \ldots, x_{r,j})^t$ of $x$. For each subset $J = \{j_1, \ldots, j_\ell\} \subset [n]$ we form the matrix $x_J \otimes v_J^\perp$, whose columns are the tensors $x_{j_i} \otimes (v^\perp_J)_i \in R^r \otimes \mathbb{k}^{n-rk(v_J)}$. There exists a linear dependence among the columns of $x_J \otimes v_J^\perp$ if and only if all its size $|J|$ minors vanish. As such:

**Theorem 3.7.** Let the size $|J|$ minors of the matrices $x_J \otimes v_J^\perp$, $J \subset [n]$, generate the ideal $I'_v \subset R$. Then $\sqrt{I'_v} = I_v$.

**Remark 3.8.** There are two special cases that occur when applying this result. The first occurs when a subconfiguration $v_J$ consists of linearly independent vectors. In this case $v_J^\perp$ is a configuration of $n$ null vectors. We interpret $x_J \otimes v_J^\perp$ to be the zero matrix in this case. The second special case is when $v_J$ has $U_{|J|-1,|J|}$ as its matroid. In this case the dimensions of the matrix $x_J \otimes v_J^\perp$ are $(|J|-1)$-by-$|J|$, and hence all its size $|J|$ minors vanish.
3.2.2. On the primality of $I'_v$. We first make a conjecture.

**Conjecture 3.9.** The ideal $I'_v$ is equal to $I_v$.

We can prove this conjecture in two special cases.

**Proposition 3.10.** Suppose $v \in \mathbb{A}^{(n-2) \times n}$ and that $v$ has a uniform matroid of rank $n - 2$. Then $I'_v = I_v$ and $R/I'_v$ is a Cohen–Macaulay ring.

*Proof.* The hypothesis on $v$ ensures that it has a full dimensional orbit. The codimension of the orbit closure is thus $n - 3$. It follows that the codimension of $I'_v$ is $n - 3$.

If $v \in \mathbb{A}^{(n-2) \times n}$ then Remark 3.8 implies that $I'_v$ is generated by the size $(n - 2)$ minors of $x \circ v^\perp$ --- it is a determinantal ideal. Since $x \circ v^\perp$ has dimension $2(n - 2)$-by-$n$, $I'_v$ has the expected codimension. We apply [17, Corollary 4] to conclude that $I'_v$ is prime and hence $I'_v = I_v$. The cited result also implies that $R/I'_v$ is a Cohen–Macaulay ring. □

The second case of primality is Gale dual to the first.

**Proposition 3.11.** Suppose that $v \in \mathbb{A}^{2 \times n}$ and that $v$ has a uniform matroid of rank 2. Then $I'_v = I_v$ and $R/I'_v$ is a Cohen–Macaulay ring.

The proof of this result follows by constructing a third ideal from $v$ with the desired properties. This ideal is contained in $I'_v$ and we will show that the former ideal cuts out $X_v$. Specifically, let $I''_v$ denote the ideal generated by the size 4 minors of the 4-by-$n$ matrix

$$x \circ v = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{bmatrix} \otimes v_1 \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \otimes v_2 \cdots \begin{bmatrix} x_{1n} \\ x_{2n} \end{bmatrix} \otimes v_n.$$

Given two integers $a < b \in [n]$, we let $p_{ab}(v)$ denote the determinant of the 2-by-2 submatrix of $v$ with columns $a$ and $b$. Similarly define $p_{ab}(x)$. It is an immediate calculation that the minors of $x \circ v$ are all of the form

$$p_{ab}(v)p_{cd}(v)p_{ac}(x)p_{bd}(x) - p_{ac}(v)p_{bd}(v)p_{ab}(x)p_{cd}(x).$$

This polynomial is obtained from the equality of the cross ratio

$$\frac{p_{ab}(v)p_{cd}(v)}{p_{ac}(v)p_{bd}(v)} = \frac{p_{ab}(gvt)p_{cd}(gvt)}{p_{ac}(gvt)p_{bd}(gvt)},$$

which holds on the orbit $X^0_v$, which is open in its closure.

**Proposition 3.12.** The vanishing locus of $I''_v$ is $X_v$.

*Proof.* The result follows by induction on $n$. If $n = 4$ then $\text{codim } X_v = 1$, and so $I_v$ must be principal. Since $I''_v$ is principal and $I_v$ cannot be generated by a linear form or a constant, we must have equality.

Suppose that $n > 4$ and let $w = (w_1, \ldots, w_n)$ be a 2-by-$n$ matrix in the vanishing locus of $I''_v$. If $w$ has a zero column then $w \in X_v$ by induction on $n$. Hence, we assume that no column of $w$ is zero. Assume that $w$ has a pair of parallel columns. The fact that $w$ vanishes at all the generators of $I''_v$ implies that at least $n - 1$ of the columns of $w$ are parallel. A simple calculation proves that $w \in X_v$.

Finally, assume that $w$ has no parallel columns. By induction we can bring the first $n - 1$ columns of $w$ to those of $v$ by a projective transformation. Since the first, second, third and last column of $w$ have a presribed cross ratio, we see that $w_n$ must be a non-zero scalar multiple of $v_n$. □
Proof of Proposition 3.11. By [17, Corollary 4] we know that $I''_v$ is prime. A short calculation gives $I''_v \subset I'_v$ and we conclude that $I''_v = I'_v = I_v$. The cited result also yields the fact that $R/I''_v$ is a Cohen–Macaulay ring. □

Proposition 3.11 allows us to explicitly determine the $K$-polynomial $K(X_v)$ when $M(v) = U_{2,n}$.

**Proposition 3.13.** Let $v \in K^{2 \times n}$ have a uniform matroid. The $K$-polynomial of $X_v$ is

$$K(X_v) = 1 - \sum_{\lambda=(\lambda_1 \geq \lambda_2), 2 \leq \lambda_2, \lambda_1 + \lambda_2 \leq n} (-1)^{\lambda_1} s_{\lambda}(1,1)s_{\lambda}(u)e_{\lambda}(t).$$

**Proof.** The degeneracy locus of the map $\psi_v$ defined by the matrix $x \otimes v$ is the variety of $I_v$. Therefore, $I_v$ is resolved $G$-equivariantly by the Eagon-Northcott complex $C_*(\psi_v) \to I_v \to 0$, wherein

$$C_m(\psi_v) = \text{Sym}^{m-4}((\text{End}_R(R^2)) \otimes \bigwedge^m R^n), \quad m = 4, 5, \ldots, n.$$ 

This is a minimal resolution since $\text{depth}(I_v) = \text{codim}(I_v) = n - 3$ [18, Theorem A2.10], as we have shown above.

When $R^n$ and $R^2$ are graded by characters of the diagonal torus in $G$ acting on $R$, all the maps in $(\bigwedge^2 R^2) \otimes_R C_* \to I_v \to 0$ are linear maps. To compute the $K$-polynomial of the terms of the resolution it suffices to compute the character of the $G$-module

$$\left(\bigwedge^2 k^2 \otimes \text{Sym}^m \text{End}(k^2)\right) \otimes \bigwedge^m k^n.$$ 

The character of the $\text{GL}_2(k)$-module $\text{Sym}^m \text{End}(k^2)$ has been computed by Désarménien, Kung and Rota [15] as

$$\sum_{\lambda=(\lambda_1 \geq \lambda_2), t=m} s_{\lambda}(1,1)s_{\lambda}(u_1, u_2).$$

The proposition follows. □

3.3. **Proof of Theorem 3.6.** The “only if” direction of the theorem is true by the discussion proceeding Proposition 3.5. Our proof of the “if” direction is by induction.

**Lemma 3.14.** Suppose that for every $r' < r$ Theorem 3.6 is true for $K^{r' \times n}$. Then to prove the theorem for $K^{r \times n}$, we may assume that $v \in (K^{r \times n})^{fr}$.

**Proof.** Suppose that $v \in K^{r \times n}$ has rank $r' < r$ and that $u \in K^{r' \times n}$ has rank larger than $r'$. Replacing $v$ with a matrix in $\text{GL}_n v$ we may assume that the last $r - r'$ rows of $v$ are zero. Let $J = \{j_1, \ldots, j_{r'+1}\} \subset [n]$ denote a set of indices of size $r' + 1$ such that $u_J$ has rank $r' + 1$. Not every column of $v_J$ can be a coloop of the matroid $M(v)|J$, so by throwing away elements of $J$, assume that $v_J$ is coloop free. Consider the tensors

$$u_{j_1} \otimes (v_J^1)_1, \quad u_{j_2} \otimes (v_J^1)_2, \ldots, \quad u_{j_{r'+1}} \otimes (v_J^1)_{r'+1},$$

These are linearly independent, since the columns of $u_J$ are linearly independent, except in the case that one of the columns of $v_J^1$ is zero. However, every column of $v_J^1$ is non-zero since $M(v)|J$ is coloop free.

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Suppose that for every $J \subseteq [n]$, the tensors in (3.1), with $u$ taking the place of $w$, are linearly dependent. Then $u$ has rank at most $r'$. Replacing $u$ with a $\text{GL}_r$-translate, we may assume that the latter $r - r'$ rows of $u$ and $v$ are zero. Ignoring the latter $r - r'$ rows of $u$ and $v$, we can appeal to the truth of Theorem 3.6 for $\mathbb{A}^{r' \times n}$, thus proving the lemma.

**Lemma 3.15.** Suppose that for every $n' < n$ Theorem 3.6 is true for matrices in $\mathbb{A}^{r \times n'}$. Then, to prove the theorem for $v \in \mathbb{A}^{r \times n}$, we may assume that $v$ has a connected matroid.

Proof. Suppose that $v$ has a disconnected matroid and, after permuting columns of $v$, write $v = [v' \ w']$ where $M(v) = M(v') \oplus M(w')$. After applying an element of $\text{GL}_r$, we may assume that the first $r'$ rows of $v'$ are zero, and the latter $r'' = r - r'$ rows of $v'$ are zero, so that $v$ is a direct sum of matrices. From this we see that $v'$ is also a direct sum of matrices.

Pick $w \in \mathbb{A}^{r \times n}$ satisfying conditions (3.1) and write $w = [w' \ w'']$, where $w'$ and $w''$ have the same numbers of columns as $v'$ and $v''$. It follows that $w' \in X_{v'}$ and $w'' \in X_{v''}$ by our induction hypothesis.

By Proposition 3.2 there are configurations $\tilde{w}' \in X_{v'}$, $\tilde{w}'' \in X_{v''}$, of rank $r'$ and $r''$ respectively, and matrices $g, h \in \text{End}(k'^r)$ such that $g\tilde{w}' = w'$ and $h\tilde{w}'' = w''$. We may assume that $\tilde{w}''$ has its latter $r''$ rows equal to zero, and that the first $r''$ rows of $\tilde{w}'$ are zero. Thus, $[\tilde{w}' \ \tilde{w}'''] \in X_v$. Taking the first $r'$ columns of $g$ and the latter $r''$ columns of $h$ and forming a new matrix $A$ from these, we have that $A[\tilde{w}' \ \tilde{w}'''] = w$, which is thus in $X_v$. □

We are now in a good place to prove the theorem.

Proof of Theorem 3.6. We have reduced to the case that $v$ has full rank and a connected matroid. This implies that the orbit $X^\circ_v$ has dimension $r^2 + n - 1$. If $n = r + 1$ then $r^2 + n - 1 = r(r + 1)$ and so $X_v = \mathbb{A}^{r \times (r + 1)}$. The tensors in (3.1) are dependent by a dimension count, so the theorem is true in this case. We will assume that $n > r + 1$ below.

Assume that we have $u \in \mathbb{A}^{r \times n}$ such that for all $J \subseteq [n]$, the tensors in (3.1) are linearly dependent. We prove that $u \in X_v$ by induction on $n$.

We start with the case when the rank of $u$ is less than $r$. Assume that the last column of $u$ is non-zero, since we are done if it is. Applying elements of $\text{GL}_r$ we may assume that the $r$th row of $u$ is all zeros and the last entry of $v_n$ is non-zero.

Let $u' = (u_1, \ldots, u_{n-1})$ and likewise for $v$. By induction on $n$, we know that there is some element $(g(s), t(s)) \in \text{GL}_r(k((s))) \times (k((s))^\times)^{n-1}$ such that $g(s) \ v' \ t(s)$ has coordinates in $k[[s]]$ and

$$g(s) \ v' \ t(s) \equiv u' \mod s.$$ 

Since the bottom row of $u'$ is all zeros we can replace the bottom row of $g(s)$ with $(0, \ldots, 0, s^m)$, $m \gg 0$, and obtain the same reduction modulo $s$. Now let $t'(s) = (t(s), s^m) \in (k((s))^\times)^n$ and consider the matrix

$$g(s) \ v' \ t'(s).$$

Setting $s = 0$ yields a matrix whose first $n - 1$ columns agree with those of $w$ and whose last column is the $r$th standard basis vector of $k^r$. Applying the element of $\text{End}(k^r)$ that fixes the first $r - 1$ basis vectors and sends the last to $u_n$, we bring this matrix to $u$. We conclude that $u \in X_v$. □
Suppose that $u$ has rank $r$. If $u$ has a connected matroid then Proposition 3.5 shows that $u \in X^\circ_n$. We thus reduce to the case that $u$ has a disconnected matroid. Our goal is to show that $M(u)$ has a connected component $K$ such that the orbit of $u_K$ equals the orbit of $v_K$.

For any $J \subset [n]$, the rank of $(v_J)^\perp$ is \( \dim \ker(v_J) = |J| - \rk_{M(u)}(J) \). If $K \subset J$, then the restriction of $(v_J)^\perp$ to the columns indexed by $K$ has rank \( \dim \ker(v_J) - \dim \ker(v_{J\setminus K}) \).

Since the tensors in $(3.1)$ are dependent, it follows that for any $J \subset [n]$ there is a connected component $K$ of $M(u)$ with $J \cap K$ non-empty and \( \dim \ker(v_J) - \dim \ker(v_{J\setminus K}) \) linearly independent dependences among $u_{J\cap K}$. That is,

\[
\dim \ker(u_{J\cap K}) \geq \dim \ker(v_J) - \dim \ker(v_{J\setminus K}),
\]

and hence

\[
(3.2) \quad \rk u_{J\cap K} \leq \rk v_J - \rk v_{J\setminus K}.
\]

for some connected component $K$ of $M(u)$. Applying (3.2) with $J = [n]$ we obtain a component $K_1$ of $M(u)$. Apply (3.2) again with $J = [n] \setminus K_1$ and obtain a connected component $K_2$ of $M(u)$. Continue in this way to obtain $K_1, \ldots, K_t$, an ordering of the components of $M(u)$. Summing the inequalities obtained from (3.2) yields,

\[
\rk(u_{K_1}) + \rk(u_{K_2}) + \cdots + \rk(u_{K_t}) \leq (\rk v - \rk v_{[n] \setminus K_1}) + (\rk v_{[n] \setminus K_1 \cup K_2}) + \cdots + (\rk v_{K_t} - \rk v_0)
\]

The left and right sides of this are both $r$ and hence all the inequalities above are all equalities. It follows that \( \rk(u_{K_t}) = \rk(v_{K_t}) \). We know that $u_{K_t} \in X_{v_{K_t}}$ by the induction hypothesis, and thus $v_{K_t}$ is connected because $u_{K_t}$ is. We conclude from Proposition 3.5 that the orbit of $v_{K_t}$ equals the orbit of $u_{K_t}$. We thus take $u_{K_t} = v_{K_t}$.

Setting $K'_t = [n] \setminus K_t$, there is some $g(s), t(s)$ such that $g(s)v_{K'_t} t(s) \equiv u_{J_G} \mod s$. Since the first \( \rk(u_{K_t}) \) rows of $u_{K_{t'}}$ can be taken to be zero, we replace the first \( \rk(u_{K_t}) \) rows of $g(s)$ with the corresponding rows of $s^m \Id_r$ for $m \gg 0$, and apply $g(s), (s^{-m}, \ldots, s^{-m}, t(s))$ to $v$. The result is $u$. \[ \square \]

4. Stabilization

The basic operation we consider in this section is embedding a $G$-invariant subvariety $X \subset \mathbb{A}^{r \times n}$ in a matrix space with more rows, and stabilizing it under the larger general linear group action. We extend this notion to equivariant coherent modules, and describe what it does at the level of $K$-polynomials.

In this section we emphasize the size of the general linear group appearing as a factor in $G$, denoting $G = \GL_r(\mathbb{k}) \times T$ by $G_r$. Let $R_r$ denote the polynomial ring $\mathbb{k}[x_{i,j} : i \in [r], j \in [n]]$, and $J_r$ the ideal in $R_r$ generated by the size $r$ minors of the coordinate matrix $[x_{i,j}]$.

4.1. Stabilization of modules. Let $V$ be a finite dimensional representation of $G_r$ whose dual is a polynomial representation. According to our convention on the grading of $R$, this ensures that $K$-polynomial of $R_r \otimes V$ is a polynomial in $u_1, \ldots, u_r$.

Let $V'$ be a representation of $G_{r+1}$ whose character is obtained from that of $V$ by replacing $s_\lambda(1/u)$ with $s_\lambda(1/u, 1/u_{r+1})$. Since we are only interested in the
Stabilization of $4.2$. Lemma 4.1. at the level of Hilbert series. (K will now show its class in stabilization replaced by $r$ which we view as a subset of $1$. By \[ \text{Proof.} \]

Here $G$ Lemma 4.2. Suppose that $M$ is thus obtained from the difference between that of $R$ is $\text{Hilb}(\lambda) = 0 \to \to R + V \to M \to 0$. $\text{Hilb}(M') = \sum_{\lambda \in \text{Par}_r, a \in \mathbb{N}^n} d_{\lambda, a} s_{\lambda}(u_1, \ldots, u_r) t^n$. \[ \text{Hilb}(M') = \sum_{\lambda \in \text{Par}_r, a \in \mathbb{N}^n} d_{\lambda, a} s_{\lambda}(u_1, \ldots, u_r, u_{r+1}) t^n. \]

Proof: This follows since $\text{Hilb}$ can be thought of as computing the character of its argument. Every such character is a Schur-positive symmetric polynomial in the $1/u$ variables with coefficients in the $t$'s. The character of $R \otimes V'$ is obtained from the character of $R \otimes V$ by replacing $s_{\lambda}(1/u)$ with $s_{\lambda}(1/u, 1/u_{r+1})$ plus some $\mathbb{Z}[t^{\pm 1}]$-linear combination of Schur polynomials in $(1/u, 1/u_{r+1})$ all of whose length is $r + 1$. These latter terms are precisely the character of $J_{r+1} \otimes V'$. The character of $GL_{r+1} N$ consists of the character of $N$, where each Schur polynomial $s_{\lambda}(1/u)$ is replaced by $s_{\lambda}(1/u, 1/u_{r+1})$, along with a sum of Schur polynomials whose length is $r + 1$.

The difference between the characters of $R \otimes V'$ and $GL_{r+1} N + J_{r+1} \otimes V'$ is thus obtained from the difference between that of $R \otimes V$ and $N$ by replacing $s_{\lambda}(1/u)$ with $s_{\lambda}(1/u, 1/u_{r+1})$. The lemma follows.

Given a $G_r$-stable reduced subscheme $X \subset \mathbb{A}^{r \times n}$, we let $X'$ denote the smallest $G_{r+1}$-stable subscheme of $\mathbb{A}^{(r+1) \times n}$ that contains $X$, namely $G_{r+1} X$. This is consistent with the above notation in that if $R/I$ is the ring of functions on $X$, then $(R/I)'$ is the ring of functions on $X'$.

Lemma 4.2. Suppose that $M$ has a presentation as in (4.1), and the support of $M$ is $X \subset \mathbb{A}^{r \times n}$. Then, the support of $M'$ is $X'$.

Proof: By \[18, \text{Corollary 2.7}], it is sufficient to prove the following fact: $\text{ann}(M') = G_{r+1} \text{ann}(M) + J_{r+1}$. Here $\text{ann}(-)$ is the annihilator of its argument. Since the ideal $\text{ann}(M')$ is $G_{r+1}$-invariant and contains $\text{ann}(M) + J_{r+1}$, we argue at the level of highest weight vectors. Every highest weight vector whose $GL_{r+1}$-weight is a partition of length $r + 1$ is in $\text{ann}(M')$, and in $J_{r+1}$. A highest weight vector in
\( R_{r+1} \) whose \( \text{GL}_{r+1} \)-weight has length \( r \) or less is in \( R_r \subset R_{r+1} \). Thus, such a highest weight vector is contained in \( \text{ann}(M') \) if and only if it is in \( \text{ann}(M) \). \( \square \)

We now define a collection of linear operators

\[ \rho_k : \mathbb{Z}[u_1, \ldots, u_r]^{e_r} \rightarrow \mathbb{Z}[u_1, \ldots, u_r, u_{r+1}]^{e_{r+1}}, \]

which we extend to a map \( \mathbb{Z}[u, t^{\pm 1}]^{e_r} \rightarrow \mathbb{Z}[u, u_{r+1}, t^{\pm 1}]^{e_{r+1}} \) by letting them act linearly on the \( t \) variables. In view of our extension of the notation \( s_\lambda \) in Section 2.2, \( \rho_k \) can be concisely defined by

\[ \rho_k s_\lambda(u_1, \ldots, u_r) := s_{\lambda+1,k}(u_1, \ldots, u_r, u_{r+1}) \]

when \( \lambda \) is a partition with \( r \) parts (possibly zero), and \( \lambda+1, k \) is the sequence \( \lambda_1 + 1, \ldots, \lambda_r + 1, k \). Recall that, using the determinantal formula, this means

\[ (\rho_k s_\lambda)(u_1, \ldots, u_{r+1}) = \frac{\det(u^\lambda_j r+1-j)}{\det(u^r+1-j)}_{i,j=1,\ldots,r+1} \]

where \( \lambda_{r+1} = k \). Alternatively, \( \rho_k s_\lambda(u) \) equals \( \pm s_\mu(u, u_{r+1}) \) if there is a partition \( \mu \) containing \( \lambda \) such that the skew shape \( \mu \setminus \lambda \) is a ribbon\(^1\) containing the first box in row \( r+1 \), where the sign is +, respectively −, if \( \mu \setminus \lambda \) has an odd, respectively even, number of rows; and \( \rho_k s_\lambda(u) = 0 \) if there is no such \( \mu \).

We also collect these operators \( \rho_k \) into a sum

\[ \rho = \sum_{k=0}^n e_k(-t) \rho_k : \mathbb{Z}[u, t^{\pm 1}]^{e_r} \rightarrow \mathbb{Z}[u, u_{r+1}, t^{\pm 1}]^{e_{r+1}}. \]

We will sometimes abuse notation and allow \( \rho \) to denote the analogous operator on the ring of symmetric polynomials in \( r-1 \) variables. The argument of \( \rho \) makes clear which operator is being referred to.

**Proposition 4.3.** If \( M \) has a presentation as in (4.1), then,

\[ K(M') = \rho K(M). \]

In particular, if \( X \) is a reduced subscheme of \( \mathbb{A}^{r \times n} \), then \( K(X') = \rho K(X) \).

**Proof.** Consider the following sum, corresponding to a single Schur function within the right side of (4.3):

\[ \sum_{k=0}^n e_k(-t) \rho_k s_\lambda(u_1, \ldots, u_r). \]

We expand along the last row the numerator in our determinantal definition of \( \rho_k \), corresponding to the introduced \( \lambda_{r+1} = k \). This turns the displayed sum above into

\[ \sum_{k=0}^n e_k(-t) \rho_k s_\lambda(u_1, \ldots, u_r). \]

\[ \sum_{k=0}^n e_k(-t) \frac{\det(u^\lambda_j r+1-j)}{\det(u^r+1-j)}_{i,j=1,\ldots,r+1} \]

\[ = \sum_{k=0}^n e_k(-t) \frac{\det(u^\lambda_j r+1-j)}{\det(u^r+1-j)}_{i,j=1,\ldots,r+1}. \]

\(^1\)The letter \( \rho \) can be taken to stand for “ribbon”, or for “raising” following [20].
Moving the inner summation outside, we may rewrite the sum over \( k \) as a product, yielding

\[
\sum_{\ell=1}^{r+1} \frac{\prod_{j=1}^{n} (1-u_{\ell}t_j) (-1)^{\ell-1} \det(u_i^{\lambda_i+r+1-j})_{i=1,\ldots,\ell,j=1,\ldots,r} \det(u_i^{r+1-j})_{i,j=1}^{r+1}}{\det(u_i^{r+1-j})_{i,j=1}^{r+1}} = \sum_{\ell=1}^{r+1} \prod_{j=1}^{n} (1-u_{\ell}t_j) s_{\lambda+1}(u_1, \ldots, \hat{u}_{\ell}, \ldots, u_{r+1}),
\]

where \( \lambda + 1 \) is an ad hoc notation for the partition on \( r \) parts such that \( (\lambda + 1)_i = \lambda_i + 1 \), and we’ve used the Vandermonde identity. Since \( \lambda \) only appears in this expression in a single \( s_{\lambda} \), in which the rest of the expression is linear, and since the \( \rho \) are linear in the \( t \) variables as well, the right side of (4.3) equals

\[
\sum_{\ell=1}^{r+1} \prod_{j=1}^{n} (1-u_{\ell}t_j) \frac{(\prod_{i \neq \ell} u_i) \Hilb(M; u_1, \ldots, \hat{u}_{\ell}, \ldots, u_{r+1}, t) \prod_{j \neq \ell} \prod_{j}(1-u_{\ell}t_j)}{\prod_{j \neq \ell} (u_{\ell} - u_{\ell})},
\]

where we have also used the definition of the \( K \)-polynomial.

Now, the coefficient of a monomial \( t^b \) in the double product \( \prod_{i \neq \ell} \prod_{j}(1-u_{\ell}t_j) \) is \( \prod_{j} h_{b_j}(u_1, \ldots, \hat{u}_{\ell}, \ldots, u_{r+1}) \). So, using the expansion of \( \Hilb(M) \) in (4.2), the coefficient of \( t^a \) in the right side of (4.3) is

\[
\sum_{b,\lambda} \sum_{\ell=1}^{r+1} \prod_{j=1}^{n} \left( h_{b_j}(u_1, \ldots, u_{r+1}) - u_{\ell} h_{b_j-1}(u_1, \ldots, u_{r+1}) \right) \frac{(\prod_{i \neq \ell} u_i) d_{\lambda,a-b} s_{\lambda}(u_1, \ldots, \hat{u}_{\ell}, \ldots, u_{r+1})}{\prod_{j \neq \ell} (u_{\ell} - u_{\ell})},
\]

The monomials of \( h_{b_j}(u_1, \ldots, u_{r+1}) - u_{\ell} h_{b_j-1}(u_1, \ldots, u_{r+1}) \) are just the degree \( b_j \) monomials in the variables \( u_1, \ldots, u_{r+1} \), except for those that use \( u_{\ell} \); that is, this function equals \( h_{b_j}(u_1, \ldots, \hat{u}_{\ell}, \ldots, u_{r+1}) \). So the above coefficient is

\[
\sum_{b,\lambda} \sum_{\ell=1}^{r+1} \frac{(\prod_{i \neq \ell} u_i) d_{\lambda,a-b} s_{\lambda}(u_1, \ldots, \hat{u}_{\ell}, \ldots, u_{r+1})}{\prod_{j \neq \ell} (u_{\ell} - u_{\ell})}.
\]

By an argument exactly parallel to the one we used initially, expanding the determinantal definition of the Schur functions along their last row, this is equal to

\[
\sum_{b,\lambda} d_{\lambda,a-b}(s_{\lambda} \prod_{j} h_{b_j})(u_1, \ldots, u_{r+1}),
\]

where the product \( s_{\lambda} \prod_{j} h_{b_j} \) is still computed in the ring of symmetric functions in only \( r \) variables. Therefore, by Lemma 4.1, the above expression is the coefficient of \( t^a \) in

\[
\Hilb(M'; u_1, \ldots, u_{r+1}, t) \prod_{i=1}^{r+1} \prod_{j=1}^{n} (1-u_{i}t_j),
\]

which equals \( K(M') \). This proves (4.3). \( \square \)
5. Dependence of the $K$-polynomial on the matroid

This section is dedicated to the proof of Theorem 5.1 below.

**Theorem 5.1.** The class $K(X_v)$ is a function of the matroid $M(v)$.

By Proposition 4.3, it is sufficient in proving Theorem 5.1 to restrict our attention to those $v$ of full rank. We adopt this assumption throughout the present section.

Recall from Section 2.2 the surjection
$$
\pi^\dagger : K^G_0(A^{r \times n}) \to K^T_0(\mathbb{G}_r(n)),
$$
whose kernel is $I_{fr}$. The image of the class $K(X_v)$ under $\pi^\dagger$ is the class $K(\pi(v)^T)$ in the $T$-equivariant $K$-theory of $\mathbb{G}_r(n)$. It is proved in [38] (see also [21]) that this latter class is a function of the matroid $M(v)$. In view of this, Theorem 5.1 will follow directly from the construction of a suitable section of $\pi^\dagger$, which is done in part (a) of Theorem 5.2 below.

Let $L_r = A^{r \times n} \setminus (A^{r \times n})_{fr}$ be the locus of matrices with rank at most $r - 1$. Recall from the end of Section 2.2 that we have extended the definition of the Schur symmetric functions to define $s_\lambda$ for any sequence $\lambda$ of integers, and that if $\lambda$ ranges over nonincreasing length $r$ sequences we get a basis for symmetric Laurent polynomials in $r$ variables.

**Theorem 5.2.** There exists a unique $\mathbb{Z}[t^\pm]$-module homomorphism
$$
s : K^T_0(\mathbb{G}_r(n)) \to K^G_0(A^{r \times n}),
$$
providing a section of $\pi^\dagger$, such that:

(a) For every $G$-invariant subvariety $X$ of $A^{r \times n}$ not contained in $L_r$, we have
$$
s(K(\pi(X))) = K(X).
$$

(b) Every class in the image of $s$ has an expansion in the Schur basis of Laurent polynomials for $K^G_0(A^{r \times n})$ of the form
$$
\sum_{\lambda=(\lambda_1 \geq \cdots \geq \lambda_r) \in \mathbb{Z}^r} \sum_{a \in \mathbb{Z}^n} c_{\lambda,a} s_\lambda(u) t^a
$$
in which $\lambda_r \geq 0$, $a_j \geq 0$ for all $j$ and $\lambda_1 \leq n - r$.

Our approach to the proof of Theorem 5.2 is guided by the cohomological arguments in [19, 20]. However, where these arguments proceed via degree considerations, we must make a more involved argument, mindful of the supports of the particular $K$-classes obtained.

5.1. Avoiding ideals. The crucial property of the symmetric Laurent polynomials in Theorem 5.2(b) is that no nonzero equivariant $K$-class supported on $L_r$ can satisfy the bounds $0 \leq \lambda_i \leq n - r$ on the parts of the sequences $\lambda$ appearing in its expansion. This may be proved by the machinery of this subsection (Example 5.6). In fact we discuss here the shape of classes supported on general invariant subvarieties; the more important application will be to subvarieties of our orbit closures $X_v$.

**Definition 5.3.** Given an invariant subvariety $X$ of a variety $Y$ with an action of a linear algebraic group $H$, define its **avoiding ideal** $A_H(X)$ in $K^H_0(Y)$ to be the kernel of the restriction from $Y$ to $Y \setminus X$. 

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We will generally omit the subscript $H$ in the notation. For example, $I_t$ is by definition $A(L_t)$.

We begin with some general facts on avoiding ideals. Recall the exact sequence of a pair (2.1),

\[(5.1) \quad K^H_0(X) \to K^H_0(Y) \to K^H_0(Y \setminus X) \to 0.\]

This allows us to identify the avoiding ideal of $X \subset \mathbb{A}^{r \times n}$ as being generated by those coherent equivariant $R$-modules whose support is contained in $X$.

**Lemma 5.4.** Let the one-dimensional torus $H = k^\times$ act linearly on the affine space $A$, and $X$ be a $H$-invariant subvariety. Then the ideal $\mathbb{Q}A_H(X)$ of $K^H_0(A) \otimes \mathbb{Q}$ is generated by $\mathcal{K}(\mathcal{O}_X)$.

**Proof.** Denote the generators of the character ring of $H$ by $t^{\pm 1}$. The ring $K^H_0(A) \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}[t^{\pm 1}]$, and its filtration by codimension of structure sheaves of subvarieties is carried by the isomorphism to the filtration of $\mathbb{Q}[t^{\pm 1}]$ by powers of $t - 1$. Suppose $\mathbb{Q}A(X)$ is not generated by $\mathcal{K}(\mathcal{O}_X)$. There must, then, be a subvariety $Y \subseteq X$ such that $\mathcal{K}(\mathcal{O}_Y) \notin \langle \mathcal{K}(\mathcal{O}_X) \rangle$, since $K^H_0(A)$ is generated by classes of subvarieties of $X$ (by Proposition 2.1) and by sequence (2.1) the classes of these subvarieties within $K^H_0(A)$ generate $A(X)$. Choose such a $Y$ of maximal dimension, so that neither $\mathcal{K}(\mathcal{O}_Y)$ nor $\mathcal{K}(\mathcal{O}_X)$ lie in $(t - 1)A(X)$. Since $X$ is itself a variety, $Y$ cannot equal $X$ and therefore has strictly smaller dimension than $X$. It follows that $\mathcal{K}(\mathcal{O}_Y)$ and $\mathcal{K}(\mathcal{O}_X)$ are $\mathbb{Q}$-linearly independent elements of $A(X)/(t - 1)A(X)$. But this quotient cannot have $\mathbb{Q}$-dimension $\geq 2$, as $\mathbb{Q}[t^{\pm 1}]$ is a PID. \(\square\)

In general, if $H$ is a torus acting linearly on the affine space $A$, the equivariant $K$-theory ring $K^H_0(A)$ is the Laurent polynomial ring $\mathbb{Z}[\text{Char}(H)] \cong \mathbb{Z}[t^{\pm 1}, \ldots, t^{\pm 1}]$. Accordingly, given a class $c \in K^H_0(A)$, we can construct its support $\text{supp}(c)$ as a subset of the character lattice $\text{Char}(H)$. This should not be confused with the support of a module which may realize the class $c \in K^H_0(A) \otimes \mathbb{Q}$.

**Proposition 5.5.** Given a functional $\nu : \text{Char}(H) \to \mathbb{R}$, the minimal value of the “width”

\[(5.2) \quad \max\{\nu(\chi) : \chi \in \text{supp}(c)\} - \min\{\nu(\chi) : \chi \in \text{supp}(c)\}\]

as $c$ ranges over nonzero elements of $A(X)$ is attained by $c = \mathcal{K}(\mathcal{O}_X)$.

The context in which we will use this lemma will, of course, be where $A = \mathbb{A}^{r \times n}$ with our $G = GL_r \times T$ action, inside which $H$ is the maximal torus. By choosing $\nu$ to take value 1 on $u_r$ and 0 on the other variables, the maximal value attained by $\nu(\chi)$ among terms $\chi$ appearing in $s_{\lambda}(u)t^a$ is $\lambda_1$. Moreover, the $K$-class of any subvariety is polynomial in the $t$ and $u$ variables and contains 1 as a monomial, so the global minimum of $\nu(\chi)$ for $\chi$ in the support of the class equals 0. Therefore, if we expand a class $\mathcal{K}(X)$ of a subvariety as

\[\mathcal{K}(X) = \sum_{\lambda, a} c_{\lambda, a} s_{\lambda}(u)t^a,
\]

the difference in equation (5.2) is the length of the longest row of any $\lambda$ such that $c_{\lambda, a} \neq 0$. 

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Example 5.6. The $K$-class of $L_r$ is
\[
\rho(1) = 1 - \sum_{i=1}^{n-r+1} s_{(i,1^{r-1})}(u)e_{r+i-1}(-t),
\]
which is polynomial in the $u$ variables, and whose last term is $s_{(n-r+1,1^{r-1})}(u)e_n(-t)$. It follows that any polynomial element of $I_{nt} = A(L_r)$ must have degree at least $n - r + 1$ in $u_r$. This fact could have been used to replace the invocation of the Schubert basis in the proof of Theorem 5.2, below.

**Proof of Proposition 5.5.** Suppose that there exists a nonzero $d \in A(X)$, such that the width (5.2) is lesser for $c = d$ than for $c = K(O_X)$. We may preserve this inequality of widths after perturbing $\nu$ to another functional $\nu'$ whose image is in $\mathbb{Q}$; moreover we may insist that $\nu'$ is injective on $\supp(d) \cup \supp(K(O_X))$.

Let $H'$ be a one-dimensional subtorus of $H$ so that, identifying $\text{Char}(H')$ with the free abelian group generated by a suitable element of $\mathbb{Q}$, the restriction map of characters $|H': \text{Char}(H) \to \text{Char}(H')$ is identified with $\nu'$. If $c$ is a class in $K_0^H(A)$, let us write $\supp'(c)$ for the support of $c|_{H'}$, taken as a subset of $\mathbb{Q}$. Because of the injectivity in the choice of $\nu'$, we have that $\supp'(c) = \nu'(\supp(c))$ when $c$ equals either $d$ or $K(O_X)$, and accordingly,

\[
\max \supp'(d) - \min \supp'(d) < \max \supp'(K(O_X)) - \min \supp'(K(O_X)).
\]

But this contradicts Lemma 5.4, since (after base change to $\mathbb{Q}$) the class $d|_{H'}$ must be a nonzero multiple of $K(O_X)|_{H'}$, and multiplication cannot decrease the width. \qed

Now let $j$ denote the coordinate embedding $\mathbb{A}^{r\times n} \to \mathbb{A}^{(r+1)\times n}$. Let $X$ be a $G_r = GL_r \times T$-invariant subvariety of $\mathbb{A}^{r\times n}$, and $X^i$ a $G_{r+1} = GL_{r+1} \times T$-invariant subvariety of $\mathbb{A}^{(r+1)\times n}$ such that $X = j^{-1}(X^i)$ — for example, $X^i$ may be $X' = GL_{r+1}j(X)$. The next proposition is an exact $K$-theoretic analogue of [19, Theorem 2.1], and the proof given there also applies here, *mutatis mutandis*.

**Proposition 5.7.** If any class in equivariant $K$-theory contained in $A(X)$ is expanded as
\[
\sum_{i \geq 0} p_i(u_1, \ldots, u_r, t_1, \ldots, t_n) u_r^i \in K_0^{G_{r+1}}(\mathbb{A}^{r\times n}),
\]
each $p_i$ lies in $A(X)$.

5.2. **Degree bounds for invariant subvarieties.** Compare the following result to [20, Theorem 7.4].

**Proposition 5.8.** Let $X \subset \mathbb{A}^{r\times n}$ be a $G_r$-invariant subvariety not contained in the locus $L_r$ of matrices with rank at most $r - 1$. When expanding
\[
K(X; u, t) = \sum_{\lambda, a} c_{\lambda, a} s_{\lambda}(u) t^a \in K_0^{G_r}(\mathbb{A}^{r\times n}),
\]
we have that $\lambda_r \geq 0$ and $\lambda_1 \leq n - r$ if $c_{\lambda, a} \neq 0$.

**Proof.** The bound $\lambda_r \geq 0$ holds because the ideal of $X$ is a quotient of $R_r = k[x_{i,j} : i \in [r], j \in [n]]$, which is nonnegatively graded in the $u$ variables.

Let $k$ be the maximum value of $\lambda_1$ occurring among all $\lambda$ with $c_{\lambda, a} \neq 0$, and suppose $k > n - r$. For brevity, define $K_r$ to be $K_0^{GL_r \times T}(\mathbb{A}^{r\times n})$ for each $r$. For
example, the function $\rho$ introduced in Section 4 is in fact, for each $r$, a function $\rho : K_r \to K_{r+1}$.

Let $K_{r,k}$ be the $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ submodule of $K_r$ spanned by the functions $s_{\lambda}(u)$ with $\lambda_1 \leq k$. Define functions $\tau : K_{r,k} \to K_{r-1,k}$ for each $r$ by $\mathbb{Z}[t_1, \ldots, t_n]$-linear extension from $\tau(s_{(k,\mu)}(u)) = s_{\mu}(u)$ and $\tau(s_{\lambda}(u)) = 0$ if $\lambda_1 < k$. That is, $\tau$ extracts the coefficient $p_k$ of $u_k^{-k}$ when a class is expanded in the manner of Proposition 5.7.

For brevity, write $c$ for $\mathcal{K}(X)$, so that $\rho(c)$ equals $\mathcal{K}(X')$. By Propositions 5.7 and 5.9, $\tau(\rho(c)) = \rho(\tau(c))$ is an element of $\mathcal{A}(X)$.

Applying Proposition 5.7 again, we find that $\tau(c)$ is an element of $\mathcal{A}(X \cap \mathbb{A}^{(r-1)\times n})$, where $\mathbb{A}^{(r-1)\times n}$ is the coordinate subvariety. By the sequence (2.1), the class $\tau(c)$ can be representated as a linear combination of sheaves supported on $X \cap \mathbb{A}^{(r-1)\times n}$. Since $\mathcal{A}(X \cap \mathbb{A}^{(r-1)\times n})$ is an ideal in $K_{r-1}$, $s_1(u)^m \cdot \tau(c)$ is realized as a linear combination of sheaves supported on $X \cap \mathbb{A}^{(r-1)\times n}$. It follows from Lemma 4.2 that for $m \geq 0$, $\rho(s_1(u)^m \cdot \tau(c))$ is a class with support in $(X \cap \mathbb{A}^{(r-1)\times n})' \subseteq X$.

We conclude that the following class corresponds to a linear combination of sheaves supported on $(X \cap \mathbb{A}^{(r-1)\times n})' \subseteq X$:

$$
\sum_{i=0}^{k-n+r} (-1)^i \binom{k-n+r}{i} s_1(u)^i \cdot \rho(s_1(u)^{k-n+r-i} \cdot \tau(c)).
$$

This class is divisible by $e_r(u) = (u_1 \cdots u_r)$, since setting $u_r = 0$ reduces the sum to

$$
\sum_{i=0}^{k-n+r} (-1)^i \binom{k-n+r}{i} s_1(u)^{k-n+r-i} \cdot \tau(c) = 0.
$$

Division by $e_r(u)$ can be realized as twisting by a line bundle on the sheaf level, and so does not change the support of a sheaf realizing a class. Hence,

$$
d := \frac{1}{e_r(u)e_n(-\ell)} \sum_{i=0}^{k-n+r} (-1)^i \binom{k-n+r}{i} s_1(u)^{i} \cdot \rho(s_1(u)^{k-n+r-i} \cdot \tau(c)),
$$

lies in $\mathcal{A}(X \cap \mathbb{A}^{(r-1)\times n})' \subseteq \mathcal{A}(X)$. It follows from Lemma 5.11 that $\tau(d) = \tau(c)$.

We claim that $c \neq d$. For this we use the fact that $X$ is not contained in $L_r$ to know that $c = \mathcal{K}(X) \notin \mathcal{A}(X \cap \mathbb{A}^{(r-1)\times n})'$ since $(X \cap \mathbb{A}^{(r-1)\times n})' \subseteq X$.

As such, the class $c-d \in \mathcal{A}(X)$ is a non-zero polynomial in the $u$ variables, and contains only terms $s_{\lambda}(u)t^a$ with $\lambda_1 < k$. This is a contradiction to Proposition 5.3, applied to $\mathcal{A}(X)$ (as explained after the statement of the proposition).

We can now prove Theorem 5.2, and thereby Theorem 5.1, the main result of this section.

Proof of Theorem 5.2. As a $\mathbb{Z}[t^\pm]$-module, the equivariant $K$-ring $K_r^T(\mathcal{G}_r(n))$ has the basis $\{s_{\lambda}(u) : \lambda \in \text{Par}_r, \lambda_1 \leq n-r\}$, which is a diagonal change of coordinates of the basis given by the classes of Schubert varieties. Our notation is chosen so that $\pi^+ s_{\lambda}(u)) = s_{\lambda}(u)$ for these classes. Therefore the unique section $s$ of $\pi^+$ satisfying (b) sends $s_{\lambda}(u)$ to $s_{\lambda}(u)$ as well.

It follows from Proposition 5.8 that, for any invariant subvariety $X$ in question, $\mathcal{K}(X)$ is supported on Schur functions $s_{\lambda}(u)$ of the shape appearing in (b). Since $\mathcal{K}(X) \in (\pi^+)^{-1}(\mathcal{K}(\pi(X)))$ and the class $\mathcal{K}(\pi(X))$ has exactly one preimage under $\pi^+$ of the needful shape, this preimage must be $\mathcal{K}(X)$, proving (a).
We conclude with a few combinatorial results used in the proof of Proposition 5.8.

**Proposition 5.9.** Maintaining the notation from the proof of Proposition 5.8, we have $\tau \rho = \rho \tau$.

**Proof.** We show that for $1 \leq j \leq n$, $\rho_j$ and $\tau$ commute. Whatever Schur function $\pm \rho_j s_\lambda(u) = s_\mu$ is has largest part at most $\max(\lambda_1, j - r) \leq \max(\lambda_1, n - r)$. Since $k > n - r$, the longest part of $s_\mu$ is less than $k$ if and only if $\lambda_1 < k$, and this longest part is equal to $k$ if and only if $\lambda_1 = k$. It follows that $\tau \rho_j s_\lambda(u) = \rho_j \tau s_\lambda(u)$. □

**Lemma 5.10.** We have the equality of operators

$$e_k(u)\rho_j - \rho_j e_k(u) = (\rho_{j+1} - \rho_j) e_{k-1}(u),$$

where the symmetric functions of $u$ act by multiplication, and are in $r - 1$ variables to the right of $\rho$ and $\tau$ to the left.

**Proof.** Let $\alpha$ be the sequence $\lambda_1 + r - 1, \lambda_2 + r - 2, \ldots, \lambda_{r-1} + 1, j$. Applying both sides of the identity to be proved to $s_\lambda(u)$ and multiplying by the Vandermonde determinant gives

$$\sum_{\beta \in \{0, 1\}^r \atop \sum \beta = k} \det(u_i^{\alpha_j + \beta_j}) - \sum_{\beta \in \{0, 1\}^{r-1} \times \{0\} \atop \sum \beta = k} \det(u_i^{\alpha_j + \beta_j}) = \sum_{\beta \in \{0, 1\}^{r-1} \times \{1\} \atop \sum \beta = k} \det(u_i^{\alpha_j + \beta_j}),$$

which is true. □

**Lemma 5.11.** Maintain the notation of the proof of Proposition 5.8. Let $s_\lambda$ be a Schur polynomial in variables $u_1, \ldots, u_{r-1}$ whose first part is at most $k$. The image of the sum

$$(5.3) \quad \frac{1}{e_r(u)} \sum_{i=0}^{k-n+r} (-1)^i \binom{k-n+r}{i} s_1(u)^{k-n+r} \rho_i(s_1(u_1, \ldots, u_{r-1})^i s_\lambda)$$

under $\tau$ is $e_n(-t)s_\lambda$.

**Proof.** Let $\phi_j : K_{r-1} \to K_r$ be the $\mathbb{Z}[t^\pm]$-linear operator which acts on Schur polynomials $s_\lambda$ by the rule

$$s_\lambda \mapsto \sum_{i=0}^n \rho_{i+j}(s_\lambda) e_i(-t).$$

By Lemma 5.10, $\phi_{j+1} = s_1 \phi_j - \phi_j s_1$ for every $j$. The operator $\phi_0$ is our $\rho$, whereas the operator occurring in (5.3) is $1/e_r(u)$ times

$$\phi_{k-n+r} = s_\lambda \mapsto \sum_{i=0}^n \rho_{i+k-n+r}(s_\lambda) e_i(-t).$$

The longest row of any shape appearing in this sum is $k + 1$, which only occurs for the $n$th summand. Further, $\rho_{k+r}s_\lambda$ is at once computed to be $s_{(k+1, \lambda+1)}$, where $\lambda + 1$ means add one to every part of $\lambda$. Dividing the sum by $e_r(u)$ and applying $\tau$ yields the lemma. □

6. **Matroid constructions and the $K$-polynomial**

In this section we consider some common matroid operations as operations on vector configurations, and see how these manifest themselves at the level of $K$-classes.
6.1. **Direct sum.** In order to understand how direct sums interact with $K$-classes we first consider a concatenation operation.

**Proposition 6.1.** Suppose that $v^1 \in \mathbb{A}^{r \times n_1}$ has its last $r'$ rows equal to zero and $v^2 \in \mathbb{A}^{r \times n_2}$ has its first $r - r'$ rows equal to zero. Let $v = (v^1, v^2)$ be the concatenation of $v^1$ and $v^2$. Then, the $K$-polynomial of $v$ is the product of the $K$-polynomials of $v^1$ and $v^2$.

Combining this result with Proposition 4.3 allows us to see how the direct sum of vector configurations manifests itself at the level of $K$-polynomials.

**Corollary 6.2.** Let $v^1$ and $v^2$ be vector configurations in $\mathbb{A}^{r_1 \times n_1}$ and $\mathbb{A}^{r_2 \times n_2}$, and $v^1 \oplus v^2$ their direct sum in $\mathbb{A}^{r \times n}$. Then,

$$K(X_{v^1 \oplus v^2}) = \rho^{r_2}K(X_{v^1}) \cdot \rho^{r_1}K(X_{v^2})$$

**Proof of Proposition 6.1.** There is a projection of $\mathbb{A}^{r \times n}$ onto the first $n_1$ columns and the last $n_2$ columns. The orbit of $v$ is the intersection of the pullbacks of the orbits of $v^1$ and $v^2$ under these projections. It follows that the ideal of the orbit of $v^1 \oplus v^2$ is the sum of the inclusions of the ideals of $v^1$ and $v^2$ into $R$. Denote these ideals and their inclusion by $I_{v^1}$ and $I_{v^2}$. Since these ideals are in different variables we conclude that

$$\text{Hilb}(R/I_{v^1 \oplus v^2}) = \text{Hilb}(R/(I_{v^1} + I_{v^2})) = \text{Hilb}(R/I_{v^1} \otimes R/I_{v^2})$$

and hence that the $K$-polynomial of $R/I_v$ is

$$\text{Hilb}(R/I_{v^1}) \text{Hilb}(R/I_{v^2}) \prod_{i=1}^{r} \prod_{j=1}^{n} (1 - u_i t_j)$$

The $K$-polynomial of $R/I_{v^1}$ is

$$K(R_1/I_{v^1}) \prod_{i=1}^{r} \prod_{j=n_1+1}^{n} (1 - u_i t_j),$$

where $R_1$ is the coordinate ring of $\mathbb{A}^{r \times n_1}$. Similarly, the $K$-polynomial of $R/I_{v^2}$ is

$$K(R_2/I_{v^2}; u, t) \prod_{i=1}^{r} \prod_{j=1}^{n_1} (1 - u_i t_j),$$

and from this the result follows. \hfill $\square$

6.2. **Parallel extension.** Here we are concerned with the effect on the $K$-polynomial of duplicating a column of $v \in \mathbb{A}^{r \times (n-1)}$, which corresponds to a parallel extension of the underlying matroid. In view of Section 6.1, it is just as informative to compare the matrix with duplicated column to a matrix of the same size with one of the duplicated columns replaced by zero. This gives the next theorem a particularly nice form.

Let $\delta_{n-1}$ be the $(n - 1)$th Demazure operator on the $t$ variables, given by

$$\delta_{n-1}(f) = \frac{t_{n-1} f - t_n \sigma_{n-1} f}{t_{n-1} - t_n}$$

for $f \in \mathbb{Z}[t_1, \ldots, t_n, u_1, \ldots, u_r]$, where $\sigma_{n-1} \in \mathbb{S}_n$ is the transposition $(n-1 \ n)$, and $\mathbb{S}_n$ acts by permuting the $t$ variables.
Theorem 6.3. Suppose that the last two columns of \( v^\parallel \in \mathbb{A}^{r \times n} \) are nonzero and equal, and \( v \in \mathbb{A}^{r \times n} \) is obtained from \( v^\parallel \) by changing the last column to zero. Then

\[
\mathcal{K}(X_{v^\parallel}) = \delta_{n-1}\mathcal{K}(X_v)
\]

This theorem comes quickly from a \( \mathbb{P}^1 \)-bundle construction like the one used for Schubert varieties [8, 12]. To extend the parallelism of these two situations, Schubert varieties \( \Omega_{\lambda} \in \mathbb{G}_r(n) \) are in bijection with Schubert matroids \( M_{\lambda} \), in such a way that a generic point of \( \Omega_{\lambda} \) has matroid \( M_{\lambda} \). If the indexing partition \( \lambda \) of one Schubert matroid satisfies \( \lambda_1 = n - r - 1 \), and \( \lambda' = (n - r, \lambda_2, \ldots, \lambda_r) \) is obtained from it by adding a box, then \( n \) is in \( M_{\lambda} \) and \( M_{\lambda'} \) is a parallel extension of \( M_{\lambda} \backslash \{n\} \), while \( \mathcal{K}(X_{\lambda'}) = \delta_{n-1}\mathcal{K}(X_{\lambda}) \in K^0(\mathbb{G}_r(n)) \).

The precise statement we will use is the following “sweeping lemma”. It is a \( K \)-theoretic analogue of the cohomological lemma [29, Lemma 2.2.1], whose statement we have mimicked closely.

Lemma 6.4. Let \( P \subseteq B \subseteq T \) be a triple of Lie groups such that \( P/B \cong \mathbb{P}^1 \) and \( T \) is a torus acting with weight \( \mu \) on \( P/B \). Let \( r \in N_n(T) \) be an element of the normalizer, inducing an automorphism \( r \) of \( T^* \) such that \( r \cdot \mu = -\mu \).

Let \( V \) be a \( P \)-representation and \( X \subseteq V \) a \( B \)-invariant subvariety. Then in \( K^0_T(V) \) we have the equality

\[
d[P \cdot X] = \frac{[X] - \chi(r \cdot [X])}{1 - \chi}
\]

where \( d \) is the degree of the map \( P \times^B X \to P \cdot X \) (or 0 if \( Y \) is \( P \)-invariant).

The proof goes through as in [29], except that for a torus \( T \) acting on \( \mathbb{P}^1 \) via the weight \( \mu \), the relation that attains in \( K^0_T(\mathbb{P}^1) \) is

\[
\{(0)\} - \mu(\{\infty\}) = 1 - \mu,
\]

as can still be checked by equivariant localization.

Proof of Theorem 6.3. In any matrix \( w \in X_{v^\parallel} \), the last two columns \( w_{n-1} \) and \( w_n \) are parallel. Let \( a \in k^p \) be one of \( w_{n-1} \) and \( w_n \) which is nonzero, if either is, or 0 if \( w_{n-1} = w_n = 0 \). Then \( (w_1, \ldots, w_{n-2}, a, 0) \) is in \( X_v \), and we can write \( w = (w_1, \ldots, w_{n-2}, y_1a, y_2a) \) for some generically unique choice of \( (y_1 : y_2) \in \mathbb{P}^1(k) \).

Let the variety \( X \subseteq \mathbb{A}^{r \times n} \) be \( X_v \). Since \( v_n = 0 \), this orbit is in fact \( B \)-equivariant where

\[
B = \text{GL}_r \times (k^*)^{n-2} \times \left\{ \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \right\} \cong \text{GL}_r \times T = G
\]

and the \( \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \) factor acts on the last two columns. Let \( P = \text{GL}_r \times (k^*)^{n-2} \times \text{GL}_2 \), so that as above \( P \cdot X = X_{v^\parallel} \), and the map \( P \times^B X \to P \cdot X \) is degree 1. Take the \( T \) of the lemma to be the maximal torus in \( G \). Then \( \mu = t_n/t_{n-1} \), and we will take \( r \) to be \( (1, 1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) \), whose action is the same as that of \( \sigma_{n-1} \). Then the theorem is immediate from Lemma 6.4.

We can now combine Proposition 3.13 and Theorem 6.3 to obtain an explicit formula for the \( K \)-polynomial of an arbitrary \( v \in \mathbb{A}^{2 \times n} \). We will only formulate our result for those \( v \) of full rank with no columns equal to zero. If the 7th column of \( t \) is zero, then \( \mathcal{K}(X_v) \) is simply the \( K \)-class where this column is deleted multiplied by \( (1 - u_1t_1)(1 - u_2t_2) \).
If $v \in \mathbb{A}^{2 \times n}$ has no zero columns then we define its **parallelism partition** to be the decreasing sequence of sizes of its rank one flats.

**Proposition 6.5.** Suppose that $v \in \mathbb{A}^{2 \times n}$ has rank two and no zero columns. Write

$$K(X_v; u, t) = \sum_{\substack{b \in \{0, 1\}^n \\atop 0 \leq k \leq |b|/2}} d_{k, b}(v)s_{(|b| - k, k)}(u)t^b,$$

as we may. Then,

1. $d_{0, 0}(v) = 1$, and $d_{0, b}(v) = 0$ for all other $b \in \{0, 1\}^n$.
2. If $k = 1$ and $v_b$ has rank one then $d_{k, b}(v) = (-1)^{|b| + 1}$.
3. If $k = 1$ and the rank of $v_b$ is two then $d_{k, b}(v) = 0$.
4. If $k \geq 2$ and $v_b$ has parallelism partition $\mu = (\mu_1 \geq \cdots \geq \mu_\ell)$, $\ell \geq 4$ and $\mu_1 + \cdots + \mu_{k-1} \geq 2k - 1$ then $d_{k, b}(v) = (-1)^{|b| + 1}(\mu_1' + \cdots + \mu_{k-1}' - 2k + 1)$.

Otherwise, $d_{k, b}(v) = 0$.

**Proof.** We express Theorem 6.3 in the following way: If $v' \in \mathbb{A}^{2 \times (n+1)}$ is obtained by duplicating the last column of $v \in \mathbb{A}^{2 \times n}$ then

$$K(\text{GL}_2v'T^{n+1}) = \delta_n(K(\text{GL}_2vT^n)(1 - u_1t_{n+1})(1 - u_2t_{n+1})).$$

Temporarily write $K(v)$ and $K(v')$ for the $K$-polynomials of the orbit closures of $v$ and $v'$.

Since, by induction, $K(v)$ is square free in the $t$-variables, we may uniquely write $K(v) = K(v)_0 + K(v)_1t_n$, where $t_n$ does not appear in $K(v)_0$. A simple computation yields

$$\delta_n(1 - u_1t_{n+1})(1 - u_2t_{n+1}) = 1 - u_1u_2t_{n+1},$$

$$\delta_n t_n(1 - u_1t_{n+1})(1 - u_2t_{n+1}) = t_n + t_{n+1} - (u_1 + u_2)t_{n+1},$$

and it follows that

$$(6.1) \quad K(v') = K(v)_0 - s_{(1, 1)}(v)t_n t_{n+1} K(v)_0$$

$$+ K(v)_1 t_n + K(v)_1 t_{n+1} - s_{(1, 1)}(v)t_{n+1} K(v)_1.$$

We conclude that $K(v')$ is square free in the $t$-variables, does not contain any Schur polynomials of partitions of length 1, and the coefficient of any $t^b$ in $K(v')$ that does not contain both $t_n$ and $t_{n+1}$ is as described in the proposition.

Suppose that $t^b = t^n t_{n+1}$, $k \geq 1$, and write $(a, 1)$ for the exponent vector of $t^a t_n$. Then (6.1) implies that

$$d_{k, b}(v') = -\left(d_{k-1, a}(v) + d_{k-1, (a, 1)}(v) + d_{k, (a, 1)}(v)\right).$$

Since $d_{k, (a, 1)}(v) = d_{k, (a, 1)}(v')$, by our computation above, this yields

$$(6.2) \quad d_{k, b}(v') + d_{k, (a, 1)}(v') = -\left(d_{k-1, a}(v) + d_{k-1, (a, 1)}(v)\right).$$

What follows from here is a tedious check that the coefficients described in the proposition obey (6.2). To preserve the reader’s patience, we will not provide the details of all possible cases, which are many. This will be forgiven since the case $k = 1$ is addressed in much greater generality by Theorem 7.17.
We will focus on the least degenerate case, when \( k \geq 3 \) and \( d_{k-1,a}(v), d_{k-1,(a,1)}(v) \) and \( d_{k,(a,1)}(v') = d_{k,(a,1)}(v') \) are all non-zero. In this case, induction yields

\[
\begin{align*}
d_{k-1,a}(v) &= (-1)^{|a|+1} (\mu'_1(a) + \cdots + \mu'_{k-2}(a) - 2k + 3), \\
d_{k-1,(a,1)}(v) &= (-1)^{|a|+2} (\mu'_1(a,1) + \cdots + \mu'_{k-2}(a,1) - 2k + 3), \\
d_{k,(a,1)}(v') &= (-1)^{|a|+2} (\mu'_1(a,1) + \cdots + \mu'_{k-2}(a,1) + \mu'_{k-1}(a,1) - 2k + 1).
\end{align*}
\]

Here, \( \mu(a) \) and \( \mu(a,1) \) are the parallelism partitions of \( v_a \) and \( v_{(a,1)} \). Now, \( \mu(a) \) and \( \mu(a,1) \) differ in exactly one position. Hence, the sum of the first two terms is \((-1)^{|a|+2}\), unless the number of vectors parallel to \( v_n \) in \( v_a \) is \( k - 1 \) or larger. If the latter happens then the \( d_{k-1,a}(v) + d_{k-1,(a,1)}(v) = 0 \). This implies that \( d_{k,b}(v') \) differs from \( d_{k,(a,1)}(v') \) by \( \pm 1 \) or 0, according to whether the number of vectors parallel to \( v_n \) in \( v_{(a,1)} \) is larger than \( k - 1 \), or not. Hence \( d_{k,b}(v') \) is given by the formula

\[
\mu'_1(b) + \cdots + \mu'_{k-1}(b) - 2k + 1,
\]

where \( \mu(b) \) is the parallelism partition of \( v_b \). The remaining cases are left to the reader.

\[ \square \]

6.3. Matroid base polytope subdivisions. Several matroid functions of interest, among them the Tutte and multivariate Tutte polynomials of Sections 6.3 and 7.5, are valuations: that is, they share a measure-like additivity property when viewed as functions of the corresponding matroid base polytopes. See the introduction of [13] for a short survey.

For a polytope \( P \) in \( \mathbb{R}^n \), let \([P] : \mathbb{R}^n \rightarrow \mathbb{R}\) denote its indicator function, taking value 1 at points of \( P \) and 0 at other points.

**Definition 6.6.** A function from the set of rank \( r \) matroids on \([n]\) to an abelian group is a valuation if it has the form \( M \mapsto \phi([P(M)]) \) for some abelian group homomorphism \( \phi \).

Suppose that the matroid base polytopes \( P(M_1), \ldots, P(M_k) \) form a subdivision of \( P(M) \), namely, that \( P(M_1), \ldots, P(M_k) \) are the maximal cells of a geometric polyhedral complex whose total space is \( P(M) \). There is then an inclusion-exclusion relation among the indicator functions of \( P(M) \) and the cells:

\[
[P(M)] = \sum_{i=1}^n [P(M_i)] - \sum_{i<j} [P(M_i) \cap P(M_j)] + \cdots \pm \left[ \bigcap_{i=1}^n P(M_i) \right].
\]

All of the nonempty intersections in the above relation are matroid base polytopes. Therefore, the values of any valuation must satisfy the analogous inclusion-exclusion relation for every matroid base polytope subdivision.

It is noted in [21, Example 3.5] that the equivariant \( K \)-class \( K_{\pi(v)T}(\mathbb{G}_r(n)) \) is a valuation as a function of the matroid \( M(v) \). Moreover, this function can be extended to a valuation taking values in \( K^*_0(\mathbb{G}_r(n)) \) which is defined on the set of all rank \( r \) matroids on \([n]\), not only the realizable ones \( M(v) \). Theorem 5.2 establishes that the class \( K(X_v) \) in \( K^*_{GL_r \times T}(\mathbb{A}^r \times n) \) is a \( \mathbb{Z} \)-linear function of the corresponding class \( K(\pi(v)T) \). The next proposition follows immediately.
Proposition 6.7. There is a valuation \( y \) on rank \( r \) matroids on \([n]\) taking values in

\[
\left\{ \sum_{\lambda \in \text{Par}_r, \lambda_1 \leq n-r; \ a} c_{\lambda,a} s_{\lambda}(u) t^a \right\}
\]

such that \( y(M(v)) = \mathcal{K}(X_v) \) for all \( v \in \mathbb{A}^{r \times n} \).

7. The Tensor Module

The tensor module of \( v \in \mathbb{A}^{r \times n} \) is the cyclic \( \text{GL}_r \)-module in \((k^r)^\otimes n\) generated by

\[ v_1 \otimes \cdots \otimes v_n. \]

We denote the tensor module of \( v \) by \( G(v) \). In this section we consider the connection between the tensor module, the \( K \)-class, and the matroid of \( v \).

7.1. \( O(1, \ldots, 1) \) and the Tensor Module. Let \((\mathbb{A}^{r \times n})^{nz}\) denote the space of matrices in \( \mathbb{A}^{r \times n} \) with no column equal to zero. This is a principal \( T \)-bundle over the product of projective spaces \((\mathbb{P}^{r-1})^n\). We let \( j : (\mathbb{A}^{r \times n})^{nz} \to \mathbb{A}^{r \times n} \) denote the inclusion, and \( p : (\mathbb{A}^{r \times n})^{nz} \to (\mathbb{P}^{r-1})^n \) the projection. The tensor module \( G(v) \) can be constructed from the line bundle \( O(1, \ldots, 1) \) on \((\mathbb{P}^{r-1})^n\), which is the external tensor product of the \( O(1) \)'s on each factor.

The inverse image \( j^{-1} X_v \) is the intersection of \( X_v \) with \((\mathbb{A}^{r \times n})^{nz}\) and the projection of this to \((\mathbb{P}^{r-1})^n\) is the \( \text{GL}_r \)-orbit closure of \( p(v) \).

Proposition 7.1. For \( v \in (\mathbb{A}^{r \times n})^{nz} \), \( G(v) \) is dual as a \( \text{GL}_r \)-module to the global sections of \( O(1, \ldots, 1)^{\mathbb{Z}[\text{GL}_r p(v)]} \). The character of \( G(v) \), as a \( \text{GL}_r \)-module, is the coefficient of \( t_1 \cdots t_n \) in \( \text{Hilb}(X_v) \).

Proof. The dual of \( G(v) \) consists of those multilinear polynomials defined on \( \text{GL}_r p(v) \).

This proves the first claim. The second follows since the character of \( O(1, \ldots, 1)^{\mathbb{Z}[\text{GL}_r p(v)]} \) is obtained from the coefficient in \( \mathbb{Z}[u_1, \ldots, u_r] \) of \( t_1 \cdots t_n \) in the multigraded Hilbert series of \( X_v \) by replacing \( u \) with \( 1/u \). Taking the dual representation at the level of characters corresponds to replacing each \( u_i \) with \( 1/u_i \). The second claim follows. \( \square \)

Theorem 7.2. The isomorphism type of \( G(v) \), as a \( \text{GL}_r \)-module, is a function of the unlabelled matroid of \( v \).

Proof. This follows immediately from Theorem 5.1 and Proposition 7.1. \( \square \)

We will occasionally need to use facts about the tensor module of \( v \) and all of its parallel extensions. As such, we set up the notation for this now. Given \( b \in \mathbb{N}^n \), we let \( v_b \) denote the vector configuration obtained from \( v \) by duplicating the \( i \)th column \( b_i \) times (or omitting it if \( b_i = 0 \)). Let \( v_b^\otimes \) be tensor product of the vectors in the configuration \( v_b \), in order, and define \( G(v_b) \) to be the cyclic \( \text{GL}_r \)-module in \((k^r)^\otimes |b|\) generated by \( v_b^\otimes \). The obvious generalization of Proposition 7.1 is true for \( v_b \).

Proposition 7.3. The character of \( G(v_b) \) is the coefficient of \( t^b \) in \( \text{Hilb}(X_v) \).
7.2. **Support of the tensor module.** The **support** of the tensor module is the collection of partitions of \( n \) that index the irreducible representations appearing in the irreducible decomposition of \( G(v) \). The **rank partition** of a matroid \( M \) is the sequence \( \lambda(M(v)) = (\lambda_1, \lambda_2, \lambda_3, \ldots) \) determined by the condition that
\[
\lambda_1 + \lambda_2 + \cdots + \lambda_k
\]
is the size of the largest union of \( k \) independent sets in \( M \).

**Theorem 7.4** (Dias da Silva [16]). The rank partition of \( M \) is a partition. If \( M \) has no loops then there is a set partition of the ground set of \( M \) into independent sets of sizes \( \mu = (\mu_1 \geq \mu_2 \geq \ldots \mu_k) \vdash n \) if and only if \( \mu \leq \lambda(M) \) in dominance order.

The following result is related to a generalization of Gamas’s theorem on the vanishing of symmetrized tensors (see [4, 16]).

**Proposition 7.5.** The tensor module \( G(v) \) has an irreducible submodule of highest weight \( \mu \) if and only if \( \mu \geq \lambda(M(v))^t \). Further, \( \mu \geq \lambda(M(v))^t \) if and only if there is a standard Young tableau of shape \( \mu \) whose columns index independent sets of \( M(v) \).

As an immediate corollary we have the following result.

**Corollary 7.6.** Given \( b \in \mathbb{N}^n \), the coefficient of \( s_{\mu}(u)^{b} \) in the multigraded Hilbert series of \( X_r \) is positive if and only if \( \mu \geq \lambda(M(v)) \). The latter condition happens if and only if there is a semi-standard Young tableau of shape \( \mu \) whose columns index independent sets of \( M(v) \).

7.3. **Schur-Weyl duality.** One can study the irreducible decomposition of the tensor module using the representation theory of the symmetric group.

We will denote the cyclic \( \mathfrak{S}_n \)-module in \((k^r)^{\otimes n}\) generated by \( v_1 \otimes \cdots \otimes v_n \) by \( \mathfrak{S}(v) \).

**Proposition 7.7.** The tensor module \( G(v) \) is Schur–Weyl dual to \( \mathfrak{S}(v) \). That is, there are isomorphisms,
\[
\text{Hom}_{\text{GL}_r}(G(v), (k^r)^{\otimes n}) \cong \mathfrak{S}(v) \quad \text{Hom}_{\mathfrak{S}_n}(\mathfrak{S}(v), (k^r)^{\otimes n}) \cong G(v),
\]
of \( \mathfrak{S}_n \)-modules and \( \text{GL}_r \)-modules, respectively.

**Proof.** Either of the above isomorphisms is defined as
\[
\varphi \mapsto \varphi(v_1 \otimes \cdots \otimes v_n).
\]
Such a homomorphism, say \( \varphi : G(v) \rightarrow (k^r)^{\otimes n} \), extends to a map of \( \text{GL}_r \)-modules \( \tilde{\varphi} \in \text{End}_{\text{GL}_r}((k^r)^{\otimes n}) \). Schur–Weyl duality [23, Section 6.2] asserts that \( \tilde{\varphi} \in \mathbf{k}\mathfrak{S}_n \).

It follows that
\[
\varphi(v_1 \otimes \cdots \otimes v_n) \in \mathfrak{S}(v).
\]
The other isomorphism is proved similarly. \( \square \)

Combining Propositions 7.3 and 7.7 we obtain a second proof of Lemma 4.1. Indeed the isomorphism type of \( \mathfrak{S}(v) \) visibly does not change when we embed \( v \) in a matrix space with more rows.

The irreducible representations of \( \mathfrak{S}_n \) that can appear in \((k^r)^{\otimes n}\), and hence \( \mathfrak{S}(v) \), are indexed by partitions of \( n \) with at most \( r \) parts. The irreducible representations of \( \text{GL}_r \) that can appear in \( G(v) \) are indexed by the exact same set of partitions, as we have discussed.
Corollary 7.8. For any partition \( \lambda \) of \( n \), the multiplicity of \( \lambda \) in \( \mathcal{G}(v) \) is equal to the multiplicity of \( \lambda \) in \( G(v) \).

Proof. By another formulation of Schur–Weyl duality \([23, \text{Section 6.1}]\), the functors \( \text{Hom}_{\mathfrak{S}_n}(-, (k^r)^{\otimes n}) \) and \( \text{Hom}_{\text{GL}_r}(-, (k^r)^{\otimes n}) \) take an irreducible indexed by \( \lambda \) to an irreducible indexed by \( \lambda \). Since these functors commute with direct sums we are done.

From this one immediately deduces the last of the main results from the introduction.

Theorem 7.9. The isomorphism type of the \( \mathfrak{S}_n \)-module \( \mathcal{G}(v) \) is a function of the unlabelled matroid of \( v \).

As a first application of Proposition 7.1 and Corollary 7.8, we extract from Proposition 3.13 the character of the tensor module for the uniform matroid in rank 2.

Corollary 7.10. Let \( v \in \mathbb{A}_2^{2 \times n} \) have uniform matroid. The character of the tensor module \( G(v) \) is

\[
\chi_{\mathcal{G}(v)}(u) = s_{(n,0)}(u) + \sum_{\ell=1}^{n/2} (n-2\ell+1)s_{(n-\ell,\ell)}(u).
\]

Proof. To see this we take the coefficient of \( e_n(t) = t_1 \cdots t_n \) in the product of the \( K \)-polynomial \( K(X;u,t) \) from Proposition 3.13 with \( 1/\prod_{i=1}^n (1-u_1 t_i)(1-u_2 t_i) \). Writing

\[
K(X;u,t) = 1 + p_4(u)e_4(t) - p_5(u)e_5(t) - \cdots + (-1)^n p_n(u)e_n(t),
\]

we see that the coefficient of \( e_n(t) \) in the product in question is

\[
(u_1 + u_2)^n + \binom{n}{4} (u_1 + u_2)^{n-4} p_4(u) - \binom{n}{5} (u_1 + u_2)^{n-5} p_5(u) + \cdots + (-1)^n p_n(u).
\]

Setting \( u_1 = u_2 = 1 \) in this formula tells us the dimension of \( G(v) \). We use the fact that \( p_i(1,1) = (-1)^{i+1} \sum_{k=2}^{i/2} (i-2k+1)^2 = (-1)^{i+1} \binom{i-1}{3} \). From this we obtain

\[
\dim G(v) = 2^n + \sum_{i=4}^n (-1)^{i+1} \binom{n}{i} \binom{i-1}{3} 2^{n-i} = (n^3 + 5n + 6)/6.
\]

Since the multiplicity of the Specht module indexed by \( (n-k,k) \) in \( (k^2)^{\otimes n} \) is at most \( (n-2k+1) \), by Schur–Weyl duality, we see that the multiplicity of \( (n-k,k) \) in \( G(v) \) is likewise bounded. Also, the multiplicity of \( (n) \) in \( G(v) \) is at most one since \( \text{Sym}^n(k^2) \) is an irreducible \( \text{GL}_2 \) module. If any of these multiplicities were less than these trivial upper bounds we would have

\[
(n^3 + 5n + 6)/6 < (n+1) + \sum_{i=1}^{n/2} (n-2k+1)^2 = (n^3 + 5n + 6)/6.
\]

It follows that each multiplicity is as large as possible in \( G(v) \), thus proving the proposition.

Using Proposition 6.5, it is possible to extract the isomorphism type of the tensor module of any rank two configuration \( v \in \mathbb{A}_2^{2 \times n} \) with no zero columns. In practice, the computation becomes an endless checking of cases. We state the result here,
refering the reader to [7, Theorem 3.5.1] for a proof avoiding the technology of K-theory, and relying further on Schur–Weyl duality.

**Proposition 7.11.** Let \( v \in k^{2 \times n} \) have rank two and no columns equal to zero. Let \( \mu = (\mu_1 \geq \mu_2 \geq \ldots) \) denote the parallelism partition of \( v \). Then, the character of the tensor module of \( v \) is

\[
s_{(n,0)}(u) + \sum_{k=1}^{n/2} \max(\mu'_1 + \cdots + \mu'_k - 2k + 1, 0)s_{(n-k,k)}(u).
\]

### 7.4. Hook shapes and broken circuits.

A partition is called a **hook** if it has at most one part that is not equal to one. The multiplicity of a hook shape \( \lambda \) in \( \mathcal{S}(v) \) (equivalently, \( G(v) \)) is determined by the subcomplex of non-broken circuit sets of \( M(v) \). In this section we make this result explicit.

For a matroid \( M \) with ground set contained in a totally ordered set, a **broken circuit** of \( M \) is a containment minimal dependent set with its smallest ordered element removed. A set is said to be an **nbc set** if does not contain any of the broken circuits of \( M \). The collection of nbc sets of \( M \) forms a subcomplex of \( M \) whose structure is well studied. The nbc sets of \( M(v) \) are known to be intimately related to the cohomology ring of the complement in \( (k^r)^* \) of the hyperplanes given by the vanishing of the linear functionals \( v_1, \ldots, v_n \) on \( (k^r)^* \). In particular, enumerating the nbc sets by corank yields the coefficients of the Poincaré polynomial of this variety.

Here is our main result relating hook shapes and non-broken circuits.

**Theorem 7.12.** The multiplicity of \( (n-k+1,1^{k-1}) \) in \( \mathcal{S}(v) \) is the number of nbc bases of the truncation of \( M(v) \) to rank \( k \), if \( k \leq \text{rk}(M(v)) \). It is zero otherwise.

We will let \( \lambda_{n,k} \) denote the hook shape that is a partition of \( n \) with length \( k \), i.e., \( \lambda_{n,k} = (n-k+1,1^{k-1}) \).

**Proof.** The element

\[
\sum_{\sigma \in \mathcal{S}_{[k]}} \sum_{\tau \in \mathcal{S}_{[n]\setminus[k]}} (-1)^{\ell(\sigma)} \sigma \tau \in k\mathcal{S}_n
\]

acts as a projector from \( k\mathcal{S}_n \) to the sum of two irreducible Specht modules, one of shape \( \lambda_{n,k+1} \), the other of shape \( \lambda_{n,k} \). This follows from the Pieri rule and the fact that the above element is a product of a row symmetrizer and a column anti-symmetrizer.

Since \( \mathcal{S}(v) \) is a cyclic module, it follows that the sum of the multiplicities of \( \lambda_{n,k+1} \) and \( \lambda_{n,k} \) in it is the dimension of the vector space

\[
\mathcal{S}(v) \sum_{\sigma \in \mathcal{S}_{[k]}} \sum_{\tau \in \mathcal{S}_{[n]\setminus[k]}} (-1)^{\ell(\sigma)} \sigma \tau \subset \bigotimes_{i=1}^{k} (k^r) \otimes \text{Sym}^{n-k}(k^r) \subset (k^r)^{\otimes n}.
\]

The image of this space is spanned by the tensors

\[
\left\{ \bigotimes_{i \in I} v_i \otimes \prod_{j \notin I} v_j : I \in \binom{[n]}{k} \right\}
\]
If \( k = r \) then the wedges simply record whether \( I \) is a basis of \( M(v) \). In this case we can forget the wedges and simply look at the dimension of the vector space spanned by

\[
\left\{ \prod_{j \notin B} v_j : B \in \mathcal{B}(M(v)) \right\} \subset \text{Sym}^{n-r}(k^r),
\]

where \( \mathcal{B}(M(v)) \) denotes the bases of \( M(v) \). By a result of Orlik and Terao (reproved as [5, Corollary 2.3]) the dimension of this vector space is the number of nbc bases of \( M(v) \), which agrees with the statement of the theorem, since the hook \( \lambda_{n, r+1} \) does not appear in \( \mathcal{G}(v) \) (even in \( (k^r)^{\otimes n} \)).

In case \( k < r \) we project \( v \) onto a generic \( k \)-dimensional subspace through the origin, to obtain a new configuration \( v' \). The multiplicity of \( \lambda_{n, k} \) in \( \mathcal{G}(v') \) is the number of nbc bases of the truncation of \( M \) to rank \( k \). Since \( \mathcal{G}(v') \) is a homomorphic image of \( \mathcal{G}(v) \) this gives a lower bound for the multiplicity of the length \( k \) hook in \( \mathcal{G}(v) \). It follows from [6, Theorem 5.4] that this multiplicity in \( \mathcal{G}(v) \) is at most the number of nbc bases of the truncation of \( M(v) \) to rank \( k \) and from this the theorem follows. \( \square \)

The Tutte polynomial of a matroid \( M \) is the unique polynomial \( T_M = T_M(x, y) \in \mathbb{Z}[x, y] \) satisfying the conditions:

- T1. \( T_M(x, y) = x \) if \( M \) is rank zero on one element and \( T_M(x, y) = y \) if \( T \) is rank one on one element.
- T2. \( T_{M \oplus N} = T_M T_N \).
- T3. If \( e \) is neither a loop nor an isthmus of \( M \), then \( T_M = T_{M \setminus e} + T_{M / e} \).

It is well known that the Tutte evaluation \( q^{rk(M)}T_M(1 + 1/q, 0) \) is the generating function for the nbc sets of \( M \) by their rank, as is seen by appealing to the deletion contraction recurrence (T3).

**Corollary 7.13.** Let \( d_{\lambda, n, k} \) denote the multiplicity of \( \lambda_{n, k} \) in \( G(v) \). Then

\[
\sum_{k=0}^{r} d_{\lambda_{n, k}, 1} q^{k-1}(q + 1) = q^{rk(M(v))}T_{M(v)}(1 + 1/q, 0).
\]

**Proof.** First, \( q^{rk(M(v))}T_{M(v)}(1 + 1/q, 0) \) is the generating function for nbc sets of \( M(v) \) by their rank. Next, the number of nbc bases of the truncation of \( M(v) \) to rank \( k \) plus the number of nbc bases of the truncation of \( M(v) \) to rank \( k + 1 \) is the number of nbc sets of \( M(v) \) of size \( k \). To see this add the element 1 to each size \( k \) nbc set of \( M(v) \) that does not already contain it. The sets that already contained 1 were the nbc bases of the truncation to rank \( k \), the other sets correspond to the nbc bases of the truncation to rank \( k + 1 \). \( \square \)

Reformulating the above result in terms of the multigraded Hilbert series of \( X_v \) yields the following result.

**Corollary 7.14.** Write

\[
\text{Hilb}(X_v) = \sum_{\lambda \in \text{Par}_r, b \in \mathbb{N}^n} d_{\lambda, b} s_\lambda(u) t^b.
\]

Then,

\[
\sum_{k=0}^{r} d_{\lambda_{n, k}, b} q^{k-1}(q + 1) = q^{rk(M(v_b))}T_{M(v_b)}(1 + 1/q, 0).
\]

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Proof. The coefficient in question is the multiplicity of \( \lambda_{[b],k} \) in \( G(v_b) \). Since this is the multiplicity of \( \lambda_{[b],k} \) in \( \mathcal{S}(v) \), by Corollary 7.8, the result follows from Proposition 7.3 and Corollary 7.13.

7.5. Hooks and the multivariate Tutte polynomial. In this section we use the multivariate Tutte polynomial of Sokal to describe the hook shapes that appear in the \( K \)-polynomial of a matrix orbit closure.

We start with a definition. Given a matroid \( M \) with ground set \([n] \) we define

\[
\tilde{Z}_M(q; t_1, \ldots, t_n) = \sum_{b \in [0,1]^n} q^{-rk(M|b)} t^b.
\]

This is the multivariate Tutte polynomial of \( M \), due to Sokal [37], which is also known to statistical physicists as the \( q \)-state Potts model partition function.

Lemma 7.15 (Ardila–Postnikov [3, Lemma 6.6]). The generating function for the Tutte polynomials of \( M|b \), as \( b \) ranges over \( \mathbb{N}^n \), can be expressed as

\[
\sum_{b \in \mathbb{N}^n} (x - 1)^{-rk(M|b)} T_{M|b}(x, y)t^b = \frac{1}{\prod_{j=1}^n(1-t_j)} \tilde{Z}_M \left( (x-1)(y-1); \frac{(y-1)t_1}{1-yt_1}, \ldots, \frac{(y-1)t_n}{1-yt_n} \right).
\]

Using Lemma 4.1 we can unambiguously extend the Hilbert series of \( X_v \) to a symmetric function in the infinitely many variables \( u_1, u_2, \ldots \), with coefficients in \( \mathbb{Z}[t_1, \ldots, t_n] \). From this symmetric function we mod out those Schur functions in \( u \) that are not hook shapes, as well as monomials \( s_\lambda(u)^b \) where \( |\lambda| \neq |b| \).

We do the former, since we are interested in hook shapes here, and the latter since these shapes do not show up in any of the Hilbert series of matrix orbit closures. It is a consequence of the Littlewood–Richardson rule that the quotient of \( \mathbb{Z}[u_1, \ldots, u_r, \ldots]^{S_\infty} \) by these relations is isomorphic, as a ring, to \( \mathbb{Z}[t_1, \ldots, t_n] \). The image of \( s_{\lambda_{[b],k}}(u)^b \) under this isomorphism is \( q^{k-1}(q+1)t^b \).

We take the image of the Hilbert series \( \text{Hilb}(X_v) \) in \( \mathbb{Z}[t_1, \ldots, t_n] \), which has the form

\[
\sum_{b \in \mathbb{N}^n} \sum_{k=0}^\infty d_{\lambda_{[b],k}} q^{k-1}(q+1)t^b = \sum_{b \in \mathbb{N}^n} q^{rk(M|b)} T_{M|b}(1 + 1/q, 0)t^b,
\]

the second equality being Corollary 7.14. Lemma 7.15 then condenses the sum to

\[
\frac{1}{\prod_{j=1}^n(1-t_j)} \tilde{Z}_M(-q^{-1}; -t_1, \ldots, -t_n).
\]

We can now state our result on hook shapes in the \( K \)-polynomial of \( X_v \).

Proposition 7.16. The enumerator of hook shapes in the \( K \)-polynomial of \( X_v \) is

\[
\prod_{j=1}^n \frac{1 - (q+1)t_j + q(q+1)t_j^2 - q^2(q+1)t_j^3 + \cdots}{(1-t_j)} \tilde{Z}_M(-q^{-1}; -t_1, \ldots, -t_n).
\]

in the following sense: The coefficient of \( q^{k-1}(q+1)t^b \), \( k \leq r \), is equal to the coefficient of \( s_{\lambda_{[b],k}}(u_1, \ldots, u_r)^b \) in the \( K \)-polynomial.

Proof. This follows from the definition of \( K \)-polynomial, after we push the denominator of the Hilbert series \( \prod_{j=1}^n \prod_{i=1}^\infty (1-u_it_j) \) into \( \mathbb{Z}[q, t_1, \ldots, t_n] \).
Theorem 7.17. Take \( b \in \{0,1\}^n \). The coefficient of \( s_{\lambda_{|b|}}(u)^b \) in the \( K \)-polynomial of \( X_v \) is \((-1)^k\) if \( b \) indexes a rank \( k \) dependent set of \( M(v) \). It is zero otherwise.

Proof. Since we are only interested in the square-free monomials in the \( t \)'s, we work modulo \((t_1^2,\ldots,t_n^2)\). By Proposition 7.16, the image of the \( K \)-polynomial in \( \mathbb{Z}[[q,t_1,\ldots,t_n]]/(t_1^2,\ldots,t_n^2) \) is

\[
\prod_{j=1}^{n} \frac{1 - t_j(1 + q)}{(1 - t_j)} Z_{M(v)}(-q^{-1};-t_1,\ldots,-t_n) \equiv \prod_{j=1}^{n} (1 - t_j q) Z_{M(v)}(-q^{-1};-t_1,\ldots,-t_n).
\]

Give the right-hand side of this equality the name \( \text{FakeDep}_{M(v)}(q;t_1,\ldots,t_n) \). By [37, Eq. (4.18a)] we see that \( \text{FakeDep} \) satisfies the recurrence

\[
\text{FakeDep}_{M(v)} = (1 - t_j q) \text{FakeDep}_{M(v-v_i)} + t_j q \text{FakeDep}_{M(v/v_i)}, \quad (v_i \neq 0).
\]

We define the multivariate polynomial

\[
\text{Dep}_{M(v)}(q;t_1,\ldots,t_n) = 1 + \sum_{b \in \{0,1\}^n, b \neq (0,\ldots,0)} \sum_{\text{rk}(M(v_b)) < |b|} (-1)^{\text{rk} M(v_b)} q^{\text{rk} M(v_b) - 1} (q + 1)^b.
\]

The sum is over the dependent sets of \( M(v) \). It is straightforward that \( \text{Dep} \) satisfies the same recurrence as \( \text{FakeDep} \). Further, if \( v_i \) is the zero vector then

\[
\text{Dep}_{M(v)}(q;t_1,\ldots,t_n) = (1 - q t_i) \text{Dep}_{M(v-v_i)}(q;t_1,\ldots,t_n),
\]

and likewise for \( \text{FakeDep} \). Since both \( \text{FakeDep} \) and \( \text{Dep} \) evaluate to 1 when \( v = (v_1) \), \( v_1 \neq 0 \), and to \((1 - t_i q)\) when \( v = (0) \) it follows that they are equal in general. The theorem follows by taking the coefficient of \( q^{k(M(v_b))-1}(q + 1)^b \) in \( \text{Dep}_{M(v)} \). \( \square \)

8. EQUIVARIANT COHOMOLOGY

In this section we discuss equivariant cohomology classes of matrix orbit closures, which we introduced in Section 2.3. The culmination of this is an explicit, positive formulation for the equivariant cohomology class of both \( \pi(v)T \) and \( X_v \) when \( v \in \mathbb{A}^{r \times n} \) is generic.

8.1. Stabilization for cohomology. The part of this section used in the sequel, Theorem 8.1, is a complete formal analogue of Theorem 5.2 for cohomology, and follows by similar techniques. This is necessary to be able to transfer computations done on the Grassmannian itself to our primary varieties of interest, subvarieties of \( \mathbb{A}^{r \times n} \). For completeness, we also construct the cohomological raising operators which carry cohomology classes of subvarieties of \( \mathbb{A}^{r \times n} \) to their \( \text{GL}_{r+c} \)-orbit closures in \( \mathbb{A}^{(r+c)\times n} \).

For the purposes of this section, we write

\[
\pi^\dagger : H^*_G(\mathbb{A}^{r \times n}) \rightarrow H^*_G(\mathbb{A}^{r \times n})^\dagger \rightarrow H^*_G(\mathbb{G}_T(n)).
\]

Denote the kernel of this cohomological map \( \pi^\dagger \) by \( J_T \).

Theorem 8.1. There exists a unique \( \mathbb{Z}[t] \)-module homomorphism

\[
s : H^*_G(\mathbb{G}_T(n)) \rightarrow H^*_G(\mathbb{A}^{r \times n}),
\]

providing a section of \( \pi^\dagger \), such that:
For every \( G \)-invariant subvariety \( X \) of \( \mathbb{A}^{r \times n} \) not contained in \( L_r \), we have
\[
s(\mathcal{C}(\pi(X))) = \mathcal{C}(X).
\]

(b) In the Schur function expansion of any class in the image of \( s \),
\[
\sum_{\lambda=(\lambda_1 \geq \cdots \geq \lambda_r) \in \mathbb{N}^r} c_{\lambda,a} s_{\lambda}(u) t^a
\]
we have \( \lambda_1 \leq n - r \).

We first state a corollary of Proposition 5.8, which holds because the computation of \( \mathcal{C}(X) \) from \( K(\pi(X)) \) cannot increase the total degree in any variable.

**Corollary 8.2.** Let \( X \) be a \( G \)-invariant subvariety not contained in the locus \( L_r \) of matrices with rank at most \( r - 1 \). When expanding
\[
\mathcal{C}(X; u, t) = \sum_{\lambda, a} c_{\lambda,a} s_{\lambda}(u)t^a \in K_G^0(\mathbb{A}^{r \times n}),
\]
we have that \( \lambda_1 \leq n - r \) if \( c_{\lambda,a} \neq 0 \).

**Proof of Theorem 8.1.** As before, in view of the Schubert basis of
\[
H_T^*(\mathbb{G}_T(n)) = \mathbb{Z}[t,u]^{{\mathbb{G}_T}}/J_\mathbb{T},
\]
there is exactly one \( \mathbb{Z}[t] \)-linear map satisfying (b). The fact that this map also satisfies (a) follows from Corollary 8.2, which shows that the desired preimage \( \mathcal{C}(X) \) of \( \mathcal{C}(\pi(X)) \) is in fact the one in the image of the section \( s \) satisfying (b).

Next, we introduce the cohomological analogue of the operator \( \rho \) from Section 4. By contrast to that situation, if \( X \subseteq \mathbb{A}^{r \times n} \) and \( X' = \text{GL}_{r+1} X \), a formula for the operator which will transform \( \mathcal{C}(X) \) to \( \mathcal{C}(X') \) cannot be stated uniformly; the formula depends on the difference in codimensions. So for each \( c \geq 1 \), define
\[
\rho_{\mathcal{H}}^{(c)} : Z[u,t]^{{\mathbb{G}_T}} \to Z[u,u_{r+1}, \ldots, u_{r+c}, t]^{{\mathbb{G}_T}}
\]
by
\[
\rho_{\mathcal{H}}^{(c)} = (-1)^c e_{r+c}(u)^{-c} \sum_{k_1, \ldots, k_c} e_{k_1}(t) \cdots e_{k_c}(t) \rho_{n+c-k_1} \cdots \rho_{n+c-k_c}.
\]

**Proposition 8.3.** Let \( X \) be a \( G = G_r \)-stable subvariety of \( \mathbb{A}^{r \times n} \) not contained in the locus of rank \( r - 1 \) matrices, and let \( X' = \text{GL}_{r+c} X \) be the smallest \( G_{r+c} = \text{GL}_{r+c} \times T \)-stable subscheme of \( \mathbb{A}^{(r+c) \times n} \) containing it. Then
\[
\mathcal{C}(X') = \rho_{\mathcal{H}}^{(c)} \mathcal{C}(X).
\]

Note that if \( X \) is contained in the locus of rank \( r - 1 \) matrices, then \( X \) itself is a 
raising of some smaller subvariety to which the hypotheses of the proposition apply.

Another formula for \( \mathcal{C}(X') \), obtained by equivariant localization, appears as [20, Theorem 7.1], but it is not of as useful a form for working with expansions in the Schur basis. (It asserts that \( \mathcal{C}(X') \) is a sort of symmetrization of \( \mathcal{C}(X) \) after accounting for the cohomology analogue of the denominators in the Hilbert series.) In the same work, [20, Theorem 7.6] is the special case of Proposition 8.3 where the \( t \) variables are evaluated at 0.
Proof. The matrix multiplication map $K^{r+c} \times X \to X'$ is surjective, and its fibers are generically of dimension $r^2$: if $xv = v'$ for some $v' \in K^{r+c+n}$ whose rows span an $r$-space, then the rows of $v$ are some basis of this $r$-space and the choice of $v$ fixes $x$. Thus $\dim X' = \dim X + cr$, so that $\text{codim} X' = \text{codim} X + c(n-r)$. The degree of $\rho^{(c)}_H$ is also $c(n-r)$.

Proposition 4.3 implies that $K(X') = \rho^c K(X)$. Temporarily let

$$g : \mathbb{Z}[u, t^{\pm 1}]^{\mathcal{S}_r} \to \mathbb{Z}[u, t]^{\mathcal{S}_r},$$

(or the same with $r+c$ variables $u$) be the operator so that $C(X) = g K(X)$, that is, $g$ substitutes $1-u$ for $u$ and $1-t$ for $t$ and then takes the lowest-degree term. It is thus enough to show that $g\rho^c$ and $\rho^{(c)}_H g$ act identically on a basis for the symmetric polynomials in the $u$ variables which are homogeneous with respect to the grading in powers of $1-u$. This implies they act identically on every homogeneous $K$-class, except perhaps for those in the kernel of $\rho^{(c)}_H$, but our dimension assumption on $X$ ensures that $K(X)$ is not in this kernel. We will use the Schur evaluations $\{s_\lambda(1-u)\}$.

It will also be convenient to index these by the sequences of exponents appearing in the Vandermonde determinant. So if $\alpha$ is a sequence of $k$ integers and $v$ a sequence of $k$ indeterminates, let $S_\alpha(v)$ be the ratio

$$S_\alpha = \frac{\det(v^{\alpha_j}_{i,j})}{\det(v^{(e_i)\gamma}_{i,j})}. $$

If $\alpha$ is nonnegative and strictly decreasing, then $S_\alpha$ equals the Schur function $s_\lambda$ with $\lambda_i = \alpha_i - k+i$. Also write $|\alpha| = |\lambda| = (\sum_{i=1}^k \alpha_i) - \binom{k}{2}$.

By [33, Ex. 1.3.10],

$$g\rho^c S_\alpha(1-u) = g\rho^c \sum_\beta \left| \begin{pmatrix} \alpha_i \\ \beta_j \end{pmatrix} \right| (-1)^{|\beta|} S_\beta(u)$$

where for visual spareness we use vertical bars for determinant, and $\beta$ ranges over decreasing sequences of $r$ nonnegative integers. By definition of $\rho$,

$$g\rho^c S_\alpha(1-u) = \sum_{\beta,\ell} \left| \begin{pmatrix} \alpha_i \\ \beta_j \end{pmatrix} \right| (-1)^{|\beta|} e_\ell(-t) S_{\beta + e, \ell_1 + (c-1), \ldots, \ell_e}(u)$$

$$= g e_{r+c}(u)^c \sum_{\beta,\ell} \left| \begin{pmatrix} \alpha_i \\ \beta_j \end{pmatrix} \right| (-1)^{|\beta|} e_\ell(-t) S_{\beta, \ell_1, \ldots, \ell_e+c}(u)$$

where $\ell$ is a sequence of $c$ integers, and $e_\ell$ is the product $e_{\ell_1} \cdots e_{\ell_e}$. So by loc. cit. again

$$g\rho^c S_\alpha(1-u) = g e_{r+c}(u)^c \sum_{\beta,\ell,\gamma} \left| \begin{pmatrix} \alpha_i \\ \beta_j \end{pmatrix} \right| (-1)^{|\beta|} e_\ell(-t)$$

$$\cdot \left| \begin{pmatrix} \gamma, \ell_1 - 1, \ldots, \ell_e - c \end{pmatrix} \right| (-1)^{|\gamma|} S_{\gamma}(1-u)$$

where $\gamma$ ranges over length $r+1$ decreasing sequences of nonnegative integers, and the binomial coefficient $\binom{\gamma}{\ell}$ must be interpreted as the coefficient of $x^b$ in the power series $(1+x)^a$, taking nonzero values when $a < 0$. 

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The substitution of $1-u$ for $u$ is an involution, which we use after expanding the second determinant along all of the rows containing an $\ell_i - i$:

$$gp^c S_{\alpha}(1-u) = g e_{r+c}(u)^c \sum_{\beta,\ell,\gamma,m} \left( \sum_{\beta_j} \right) \left( -1 \right)^{1+|\gamma|} e_t(-t) \cdot$$

$$\cdot \left( \sum_{\beta_j} \right) \left( \ell_1 - 1 \right) \cdots \left( \ell_c - c \right) (-1)^{|\gamma|+\text{sgn}(m)} S_{\gamma}(1-u)$$

$$= g e_{r+c}(u)^c \sum_{\ell,\gamma,m} \delta_{\alpha,\gamma,m} e_t(-t).$$

$$\left( \ell_1 - 1 \right) \cdots \left( \ell_c - c \right) (-1)^{|\gamma|+\text{sgn}(m)} S_{\gamma}(1-u)$$

where $m$ is a sequence of $c$ indices, $\text{sgn}(m)$ is the number of transpositions needed to move columns $m_1, \ldots, m_c$ to the right side of the matrix in that order, and $\gamma_m$ is $\gamma$ with its $m_1$th, $\ldots$, and $m_c$th components removed.

If $\alpha = \gamma_m$, then $\gamma$ is obtained from $\alpha$ by inserting new parts in positions $m_1, \ldots, m_c$. Let $p_i = \gamma_m$ for $i = 1, \ldots, c$. Then, moving all of these parts to the end of the sequence, we have $S_{\gamma} = (-1)^{\text{sgn}(m)} S_{\alpha,p}$. Now, given a sequence $p$ of length $c$, there is at most one $m$ such that inserting $p_i$ into $\alpha$ to take position $m_i$ for each $i$ will yield a decreasing sequence; and if there is no such $m$, then some integer is repeated in the concatenated list $\alpha, p$, so $S_{\alpha,p} = 0$. So we may drop $m$ as an index of summation and use only $p_i$, which ranges over all length $c$ sequences.

$$gp^c S_{\alpha}(1-u) = g e_{r+c}(u)^c \sum_{\ell, p} e_t(-t) \left( \ell_1 - 1 \right) \cdots \left( \ell_c - c \right)$$

$$\cdot \left( -1 \right)^{1-\left( \ell_i - i \right) + \sum_i p_i} S_{\alpha,p}(1-u)$$

By formula (4.10.15) of [25],

$$gp S_{\alpha}(1-u) = g e_{r+c}(u)^c \sum_{\ell, k, p} e_k(1-t) \left( \prod_{i=1}^{c} (-1)^{l_i-k_i} \left( n-k_i \right) \left( \ell_i - i \right) \left( p_i \right) \right).$$

$$\cdot \left( -1 \right)^{1-\left( \ell_i - i \right) + \sum_i p_i} S_{\alpha,p}(1-u)$$

$$= g e_{r+c}(u)^c \sum_{k, p} e_k(1-t) \left( \prod_{i=1}^{c} (-1)^{n-k_i} \left( k_i - i \right) \left( p_i - n + k_i \right) \right).$$

(8.1)

In the grading by powers of $1-u$ and $1-t$, the function $e_k(1-t)$ is homogeneous of degree $\sum_i k_i$, and $S_{\alpha,p}(1-u)$ is homogeneous of degree $|S_{\alpha,p}| = |a| + \sum_i p_i + \left( \ell - p \right)$, while $e_{r+c}(u)^c$ is inhomogeneous but contains 1 as a constant term. Writing $a_i = p_i - n + k_i$, it follows that the least degree appearing in the $(k, p)$ term of sum (8.1) is

$$|a| + cn + \left( \frac{r}{2} \right) - \left( \frac{r+c}{2} \right) + \sum_i a_i.$$

If any $a_i$ is negative, then the binomial coefficient in which it is the lower argument vanishes. If $a_i$ and $a_j$ are equal for some $i \neq j$, then exchanging $k_i$ with $k_j$ and $p_i$ with $p_j$ contributes another term which is identical except in that a single
transposition has been applied to the index of \( S_{\alpha,p}(1-u) \), which negates the value. Thus all of these terms cancel from the summation, and the surviving terms with the least degree appearing in them are those in which the \( a_i \) are a permutation of \( c-1, c-2, \ldots, 1, 0 \). Note that this least degree is

\[
|\alpha| + cn + \left( \frac{r}{2} \right) - \left( \frac{r+c}{2} \right) + \left( \frac{c}{2} \right) = |\alpha| + cn - cr,
\]

concordant with our earlier computation of the degree of \( \rho_H^{(c)} \).

If we fix sequences \( k \) and \( p \) so that \( a = (c-1, \ldots, 0) \), then the terms of sum (8.1) contributing some integer multiple of \( e_k(1-t)S_{\alpha,p}(1-u) \) are those corresponding to \( (\sigma, k, \sigma p) \) for some permutation \( \sigma \in \mathfrak{S}_c \). The subsum of these terms is

\[
g e_{r+c}(u)^c(-1)^{\binom{c}{2}} - \sum_i p_i e_k(1-t)S_{\alpha, p}(1-u) .
\]

wherein the summation appearing here is a determinant,

\[
\det \left[ \left( k_i - j \right) \frac{c-j}{c-i} \right]_{i,j}.
\]

Using a factorization into triangular matrices, this determinant equals

\[
\det \left[ \left( c-i \right) \frac{c-j}{c-h} \right]_{i,h} \det \left[ \left( k_j - c \right) \frac{h-j}{h-i} \right]_{j,h} = 1 \cdot (-1)^{cn - \sum k_i}.
\]

We use this to eliminate \( p \) as an index of summation in (8.1), taking \( p_i = n + c - i - k_i \). In the exponent of -1, we use \( \sum_i p_i + \sum_i k_i = \binom{c}{2} + cn \).

\[
\rho_H^{(c)} g S_{\alpha}(1-u) = g e_{r+c}(u)^c \sum_k (-1)^{\binom{c}{2}} e_k(1-t) S_{\alpha+n+c-1-k_1, \ldots, n+k_c}(1-u) = \sum_k (-1)^{cr} e_k(t) S_{\alpha+n+c-1-k_1, \ldots, n+k_c}(u).
\]

This, at last, is seen to agree with

\[
\rho_H^{(c)} g S_{\alpha}(1-u) = \rho_H^{(c)} S_{\alpha}(u)
\]

\[
= (-1)^{cr} e_{r+c}(u)^{-cr} \sum_k e_k(t) \rho_{n+c-k_1} \cdots \rho_{n+c-k_c} S_{\alpha}(u) = (-1)^{cr} e_{r+c}(u)^{-cr} \sum_k e_k(t) S_{\alpha+n+c-2c-1-k_1, \ldots, n+c-k_c}(u),
\]

which completes the proof. \( \square \)

8.2. **Vector bundles on the Grassmannian.** We use the notations \( t_i \) and \( u_i \) for the classes in \( H^*_T(\mathbb{G}_r(n)) \) that are images of the similarly-named classes in \( H^*_G(k^{r \times n}) \), and we use the symbols \( C \) and \( K \) for cohomology class and \( K \)-classes on \( \mathbb{G}_r(n) \).

On \( \mathbb{G}_r(n) \) we have two equivariant universal vector bundles: \( S \), the tautological bundle or universal subbundle, whose fiber over \( x \in \mathbb{G}_r(n) \) is the subspace \( x \subset \mathbb{C}^n \); and \( Q \), the universal quotient bundle, with fiber \( \mathbb{C}^{n-x} \). It is possible to lift \( S \) to a coherent equivariant \( R \)-module as follows. Let \( k^n \) be the representation of \( T \) where \( t \) acts via the character \( t_i^{-1} \) on the \( i \)th factor. Then the \( K \)-polynomial of \( R \otimes k^n \) is
we can describe a set of vector bundles on \( K \) class. This is done by substituting \( 1 - u_i \) and \( 1 - t_j \) for \( u_i \) and \( t_j \) in \( K(S) \) and taking the degree one term, yielding \( c_1(S) = e_1(-u) \) and hence \( c_k(S) = e_k(-u) \). The exact sequence

\[
0 \to S \to k^n \to Q \to 0
\]

of vector bundles on \( G_r(n) \) is transformed to the Whitney sum formula in \( H^*_R(G_r(n)) \),

\[
\prod_{i=1}^n (1 + t_i) = c(k^n) = c(Q)c(S),
\]

from which one can solve for the \( c_k(Q) \) in terms of the \( c_k(S) \).

Using Theorem 8.1 and Proposition 8.3 we can describe a set of \( \mathbb{Z}[t_1, \ldots, t_n] \)-module generators for the ideal \( J_{fr} \). Namely, \( J_{fr} \) is generated by the images of the operators \( \rho_H^{(c)} \). But as an ideal a smaller generating set suffices, namely

\[
J_{fr} = \left( \sum_{a+b=k} e_a(t)h_b(u) : k > n-r \right).
\]

Letting \( J \) be the ideal on the right hand side, \( J \) is contained in \( J_{fr} \) in view of equation (8.2), because the Chern classes \( c_k(Q) \) vanish for \( k > n-r = \text{rank } Q \). On the other hand, \( J_{fr} \) is contained in \( J \) because the quotient \( \mathbb{Z}[u, t]^\Sigma_r / J \) is generated as a \( \mathbb{Z}[t] \)-module by the \( s_\lambda(u) \) with \( \lambda_1 \leq n-r \): indeed, any polynomial \( s_\lambda(u) \) with \( \lambda_1 > n-r \) is contained in the ideal \( (h_k(u) : k > n-r) \) by Giambelli’s formula, so such \( s_\lambda(u) \) can be nontrivially reduced modulo \( J \) in a lexicographic term order with \( u > t \). Therefore no proper quotient of \( \mathbb{Z}[u, t]^\Sigma_r / J \) can be \( H^*_R(G_r(n)) = \mathbb{Z}[u, t]^\Sigma_r / J_{fr} \), which is known to be a free \( \mathbb{Z}[t] \)-module with the same set of generators, \( \{ s_\lambda(u) : \lambda_1 \leq n-r \} \).

When we write a symmetric function of a vector bundle \( \mathcal{E} \), we mean that symmetric function of its Chern roots, so that \( e_k(\mathcal{E}) = c_k(\mathcal{E}) \). For later reference, we give explicit expansions of these symmetric functions in the \( t \) and \( u \) variables for the vector bundles \( S^\vee \) and \( Q \), in the first instance using the additional fact that dualizing a vector bundle negates alternate Chern classes.

\[
s_\nu(S^\vee) = s_\nu(u)
\]

\[
s_\nu(Q) = \omega(s_\nu(u, t)) = \sum_{\lambda, \mu} c^\nu_{\lambda, \mu} s_\lambda(t)s_\mu(u)
\]

Here \( \omega \) is the usual operation on \( \mathbb{Z}[u_1, \ldots, u_r]^\Sigma_r \) that transposes Schur polynomials, extended \( \mathbb{Z}[t] \)-linearly [33, I.2.7]. On symmetric functions in infinitely many variables, \( \omega \) is an involution; in our setting, it is an involution as long as no part of a partition exceeds \( r \).

### 8.3. Equivariant localization for Grassmannians

To prove the main theorem of this section (Theorem 8.4), we will make use of the results of Chang–Skjelbred [10], commonly referred to as the Goresky–Kottwitz–MacPherson theory of equivariant localization [24], applied to \( G_r(n) \). We deploy it in the fashion of [21], ultimately converting the computation of equivariant cohomology classes for our
toric varieties to a problem of lattice point generating functions of rational cones. Below we state the background material specialized to the setting in which we use it. For a more general-purpose overview, see [31, Section 2].

The Grassmannian $G_r(n)$ has a finite set of $T$-fixed points: they are the $r$-dimensional coordinate subspaces of $k^n$. We denote by $x_B$ the fixed point in which the coordinates not fixed to zero are those indexed by the set $B \in \binom{[n]}{r}$. The inclusion $\iota$ of this discrete fixed set $G_r(n)^T$ into $G_r(n)$ induces a restriction map

$$\iota^*: H^*_T(G_r(n)) \to H^*_T(G_r(n)^T).$$

Its target $H^*_T(G_r(n)^T)$ is a direct sum of polynomial rings $H^*_T(\mathfrak{a}) = \mathbb{Z}[t_1, \ldots, t_n]$, one for each fixed point. The restriction of the class of a $T$-equivariant subvariety to a fixed point $x$ will equal the restriction of this class to an affine space $\mathfrak{a}$ containing $x$ on which $T$ acts linearly, under the natural isomorphism $H^*_T(\mathfrak{a}) = \mathbb{Z}[t_1, \ldots, t_n] = H^*_T(\mathfrak{a})$. We will let $c|x \in \mathbb{Z}[t_1, \ldots, t_n]$ denote the restriction of the class $c$ to $x$.

Since the Grassmannian is equivariantly formal, the results of [10] imply that $\iota^*$ is injective. The theory of Goresky–Kottwitz–MacPherson identifies the image of $\iota^*$, via consideration of the finite set of one-dimensional $T$-orbits on $G_r(n)$, as the tuples of polynomials $f = (f_B : B \in \binom{[n]}{r})$ such that

$$f_B - f_{B \cup \{i\}} \in (t_j - t_i)$$

for all $i \in B$ and $j \notin B$.

If $V$ is a vector bundle, a Schur polynomial $s_\lambda$ of the Chern roots of $V$ localizes at a fixed point to the sum of the characters by which $T$ acts on the tangent space of $S^\lambda(V^\vee)$, where $S^\lambda$ is a Schur functor. For the vector bundles $S^\vee$ and $Q$, the resulting localizations are

$$s_\lambda(S^\vee)|_{x_B} = s_\lambda(-t_i : i \in B)$$
$$s_\lambda(Q)|_{x_B} = s_\lambda(t_j : j \notin B)$$

The Plücker embedding embeds $G_r(n)$ equivariantly in $\mathbb{P}(\binom{[n]}{r})^{-1}$, on which $T$ acts diagonally. A fixed point $x_B$ of $G_r(n)$ is sent to a coordinate point in $\mathbb{P}(\binom{[n]}{r})^{-1}$, and the character by which $T$ acts on the corresponding coordinate is $t^B = \prod_{i \in B} t_i$.

### 8.4. The cohomology class for the uniform matroid.

In this section we compute the equivariant cohomology class of those $T$-orbit closures in $G_r(n)$ whose matroid is uniform, obtaining a new positive equivariant formula for this cohomology class. Our formula stands alongside that of Klyachko for the same class in [28, Theorem 6].

Let $v \in k^{r \times n}$ have full rank, so that $\pi(v) \in G_r(n)$ is the row span of $v$. The torus orbit closure $\pi(v)\mathcal{O} \subset G_r(n)$ is the projection of $X_v \cap (k^{r \times n})^{fr}$ to the Grassmannian.

**Theorem 8.4.** Given $v \in k^{r \times n}$ whose matroid is uniform, the class of $\pi(v)\mathcal{O}$ in $T$-equivariant cohomology is

$$C(\pi(v)\mathcal{O}) = \sum_{\lambda, \mu} c_\lambda^{(n-r-1)^r-1} s_\lambda(S^\vee) s_\mu(Q).$$
Therefore the class of $X_v$ in $H^*_T(\mathbb{A}^{r \times n})$ is
\[
\mathcal{C}(X_v) = \sum_{\lambda, \mu, v, \xi} c_{\lambda, \mu, v, \xi}^{(n-r-1)^r-1} c_{\mu, v}^s s_\lambda(u) s_{\mu'}(t) s_v(u)
= \omega(s_{(r-1)n-r-1}(u, u, t)).
\]

The partition $(n-r-1)^r-1$ is the $(r-1) \times (n-r-1)$ rectangle, and if the $(\lambda, \mu)$ term of sum (8.3) is nonzero, then $c_{\mu, v}^{(n-r-1)^r-1}$ equals 1, and $\mu$ is the complement of $\lambda$ within this rectangle.

As promised, we proceed by equivariant localization, so the first step is to understand the class of $\pi(v)T$ localized at a $T$-fixed point.

**Lemma 8.5.** The $T$-equivariant cohomology class of $\pi(v)T$ localized at $x_B$ is
\[
(8.4) \quad \mathcal{C}(\pi(v)T)|_{x_B} = \prod_{i \in B, j \notin B} (t_j - t_i) \sum_{(i_1, \ldots, i_n)} \frac{1}{(t_{i_2} - t_{i_1})(t_{i_3} - t_{i_1}) \cdots (t_{i_n} - t_{i_{n-1}})},
\]
where the sums range over permutations $(i_1, \ldots, i_n) \in S_n$ whose lex-first basis is $B$.

Note that the sum occurring in Lemma 8.5 is zero if $B$ is not a base of $M(v)$. When $M(v)$ is uniform, the sum ranges over those permutations that have the elements of $B$ in their first $r$ positions and are arbitrary thereafter.

**Proof.** Following the approach of [21], we first identify $\pi(v)T$ as a toric variety. Viewing toric varieties as images of monomial maps [35, Chapters 7,10], the normalization of $\pi(v)T$ is the toric variety of the polytope given as the convex hull of the characters corresponding to the $T$-fixed points it contains. By [41], the variety $\pi(v)T$ is already normal, and therefore is the toric variety just stated. The $T$-fixed points in $\pi(v)T$ are those $x_B$ such that $B$ is a basis of the matroid $M(v)$. The corresponding characters are $\{t^B : B$ is a basis$\}$, whose convex hull is the matroid base polytope $P(M(v))$ of $M(v)$, defined in Section 2.1.

If the toric variety $\pi(v)T$ contains the fixed point $x_B$, then its restriction to the $T$-invariant translate of the big Schubert cell around $x_B$ is the corresponding affine patch of $\pi(v)T$, in Fulton’s construction: that is, it is the affine toric subvariety consisting of the orbits whose closures contain $x_B$. Explicitly, this affine scheme is $\text{Spec } k[C]$, where $C$ is the semigroup of lattice points in the tangent cone to $P(M(v))$ at the vertex $e_B$. Parallel to Section 2.2.2, the $K$-class of $\pi(v)T|_{x_B}$ is then the product of $\text{Hilb}(k[C])$ with $\prod_{i \in B, j \notin B} (1 - t_j/t_i)$.

The Hilbert series $\text{Hilb}(k[C])$ is the finely-graded lattice point enumerator of $C$. We claim
\[
(8.5) \quad \text{Hilb}(k[C]) = \sum_{(i_1, \ldots, i_n)} \text{Hilb cone}(e_{i_2} - e_{i_1}, \ldots, e_{i_n} - e_{i_{n-1}})
= \sum_{(i_1, \ldots, i_n)} \frac{1}{(1 - t_{i_2}/t_{i_1})(1 - t_{i_3}/t_{i_2}) \cdots (1 - t_{i_n}/t_{i_{n-1}})}
\]
where the sums range over permutations $(i_1, \ldots, i_n) \in S_n$ whose lex-first basis is $B$. To see this, apply Brion’s theorem to the triangulation of the dual of this cone into type A Weyl chambers. The cones in the triangulation are unimodular, and their lattice point generators are those given in the second line.
Altogether,
\[ K(\pi(v)T)_{|x_B} = \prod_{i \in B, j \notin B} (1 - t_j/t_i) \sum_{(i_1, \ldots, i_n)} \left( \sum_{(i_1, \ldots, i_n)} \frac{1}{(1 - t_{i_2}/t_{i_1}) \cdots (1 - t_{i_n}/t_{i_{n-1}})} \right) \]

This becomes the equation to be proved upon replacing each \( t_i \) with \( 1 - t_i \), and then extracting the lowest degree term of the resulting power series. (Note that taking the lowest-degree term can be done one factor at a time.)

**Proof of Theorem 8.4.** The second equation of the theorem follows from the first, by Theorem 8.1.

By equivariant localization, it is enough to show the claimed equality after restriction to each \( x_B \), in \( H^*_T(x_B) \cong \mathbb{Z}[t_1, \ldots, t_n] \). On one hand, the restriction of the right side of (8.3) at \( x_B \) is
\[
\sum_{\lambda, \mu} c^{(n-r-1)r-1}_{\lambda \mu} s_\lambda(-t_i : i \in B) s_{\mu'}(t_j : j \notin B).
\]

We massage the formula for \( C(\pi(v)T)_{|x_B} \) in Lemma 8.5, and show that it equals the above polynomial.

Let us temporarily write \( f(i_1, \ldots, i_n) \) for \( 1/(t_{i_2} - t_{i_1}) \cdots (t_{i_n} - t_{i_{n-1}}) \). We have
\[
\frac{f(i_1, \ldots, \hat{i}_r, \ldots, i_n)}{f(i_1, \ldots, i_n)} = \frac{t_{i_{r+1}} - t_{i_r}}{(t_{i_r} - t_{i_{r-1}})(t_{i_{r+1}} - t_{i_r})} = \frac{1}{t_{i_{r+1}} - t_{i_r}} - \frac{1}{t_{i_{r-1}} - t_{i_r}}
\]
and corresponding identities when \( s = 1 \) or \( s = n \). Thus, if \( \ell \) is a list of indices, we have a telescoping sum
\[
\sum_{\ell' : \ell \text{ in } \ell' \text{ with } i \text{ dropped}} f(\ell') = \frac{f(\ell)}{t_j - t_i}.
\]

Grouping the terms of the sum in (8.4) by \( i_r \) and repeatedly applying the above identity and its order-reversed counterpart, we get
\[
(8.6) \quad C(\pi T)_{|x_B} = \prod_{i \in B, j \notin B} (t_j - t_i) \cdot \sum_{i_r \in B} \left( \prod_{i \in B \setminus i_r} \frac{1}{t_{i-r} - t_i} \right) \left( \prod_{j \notin B} \frac{1}{t_j - t_{i_r}} \right).
\]

We next invoke the following variant of the Cauchy identity:
\[
\prod_{t \in T, v \in V} (t - v) = \sum_{\nu, \mu} c^{(|V|/\nu)}_{\mu} s_\nu(t \in T) s_{\mu'(-v \in V)}
\]

In our localized cohomology class, we combine the first and last products in (8.6) and apply the Cauchy identity with \( (T, V) = \{ -t_i : i \in B \setminus i_r \}, \{ -t_j : j \notin B \} \)

and giving
\[
\sum_{i_r \in B} \left( \prod_{i \in B \setminus i_r} \frac{1}{t_{i_r} - t_i} \right) \left( \sum_{\nu, \mu} c^{(n-r-1)n-1}_{\nu \mu} \frac{\det(-t_j^{i_r})_{i \in B \setminus i_r}}{\det(-t_j^{i_r})_{i \in B \setminus i_r}} \cdot s_{\mu'}(t_j : j \notin B) \right)
\]
where the \( s_\nu \) is written as a ratio of determinants. Now, combine the remaining product term in the above display into the Vandermonde determinant in the denominator. The sum over \( i_r \in B \) can then be read as an expansion along the last row of the determinantal formula for \( s_\lambda(-t_i : i \in B) \), where \( \lambda \) is obtained from \( \nu \) by decrementing every part, and the term vanishes if \( \nu \) has fewer than \( r - 1 \).
parts. If this is the case and $c^{(n-r)^{r-1}}_{\lambda,\mu}$ equals 1, so that $\nu$ and $\mu$ are complements in a $(r-1) \times (n-r)$ rectangle, it follows that $\lambda$ and $\mu$ are complements in a $(r-1) \times (n-r-1)$ rectangle, so our localized class is
\[
\sum_{\lambda,\mu} c^{(n-r)^{r-1}}_{\lambda,\mu} s_{\lambda}(-t_i : i \in B)s_{\mu'}(t_j : j \not\in B).
\]
This agrees with the localization of (8.3) and the theorem follows.

□

Corollary 8.6. Let $v \in A^{r \times n}$ have a uniform rank $r$ matroid. The degree of $X_v$ is
\[
\sum_{\lambda,\mu} c^{(n-r-1)^{r-1}}_{\lambda,\mu} s_{\lambda}(1, \ldots, 1)s_{\mu'}(1, \ldots, 1)
\]
where each symmetric polynomial takes $r$ arguments.

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References

[1] D. Anderson, S. Payne. Operational K-theory. arXiv:1301.0425, 2013.
[2] F. Ardila, C. Klivans. The Bergman complex of a matroid and phylogenetic trees. J. Comb. Theory, Ser. B 96(1):38–49, 2006.
[3] F. Ardila, A. Postnikov. Combinatorics and geometry of power ideals. Trans. Amer. Math. Soc. 362 (2010), no. 8, 4357–4384, 2010.
[4] A. Berget. Equality of symmetrized tensors and the flag variety. Linear Algebra Appl., 438(2):658–656, 2013.
[5] A. Berget. Products of linear forms and Tutte polynomials. European J. of Combin., 31(7):1924–1935, 2010.
[6] A. Berget. Tableaux in the Whitney module of a matroid. Sémin. Lothar. Combin. 63, Art. B63f, 17 pp, 2010.
[7] A. Berget. Symmetries of tensors. PhD Thesis, University of Minnesota, 2009.
[8] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Schubert cells and cohomology of the spaces $G/P$. Russian Mathematical Surveys 28:1–26, 1973.
[9] M. Brion. Equivariant cohomology and equivariant intersection theory. Notes available from http://www-fourier.ujf-grenoble.fr/~mbion/notes.html.
[10] T. Chang, T. Skjelbred. The topological Schur lemma and related results. Ann. of Math. (2) 100 (1974), 307–321.
[11] C. Chindris. On the geometry of orbit closures for representation-infinite algebras. Gasly. Math. J., 54(3):629–636.
[12] M. Demazure. Désingularization des variétés de Schubert généralisées. Ann. Sc. ENS sér. 4, 7:53-88, 1974.
[13] H. Derksen. A. Fink. Valuative invariants for polymatroids. Adv. Math. 225(4):1840–1892, 2010.
[14] J. Edmonds. Submodular functions, matroids, and certain polyhedra. in Comb. Structures and their App., eds. Guy, Hanani, Sauer, Schonheim. Gordon and Breach, New York, 69–87, 1970.
[15] J. Désarménien and J. Kung and G.C. Rota. Invariant theory, Young bitableaux and combinatorics, Adv. Math. 27, 1978.
[16] J. A. Dias da Silva. On the $\mu$-colorings of a matroid. Linear and Multilinear Algebra, 27(1):25–32, 1990.
[17] J. Eagon, M. Hochster. Cohen-Macaulay Rings, Invariant Theory, and the Generic Perfection of Determinantal Loci. American J. Math., 93(4):1020–1058, 1971.
[18] D. Eisenbud. *Commutative algebra with a view toward algebraic geometry*. Springer-Verlag, 1995.

[19] L. Fehér, R. Rimányi. On the structure of Thom polynomials of singularities. *Bull. Lond. Math. Soc.* 39(4):541–549, 2007.

[20] L. Fehér, A. Némethi, R. Rimányi. Equivariant classes of matrix matroid varieties. *Comment. Math. Helv.* 87:861–889, 2012.

[21] W. Fulton. *Intersection theory*, second edition, volume 3 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge*. Springer-Verlag, 1998.

[22] W. Fulton, J Harris. *Representation theory. A first course*. Springer-Verlag, 1991.

[23] A. Fink and D. Speyer. K-classes of matroids and equivariant localization. *Duke Math. J.*, 161(14):2699–2723, 2012.

[24] A. Knutson. Introduction to geometric representation theory. Course notes, available from [www.math.cornell.edu/~allenk/courses/10fall/notes.pdf](http://www.math.cornell.edu/~allenk/courses/10fall/notes.pdf)

[25] G. M. MacLaury. Theory of matroids, volume 26 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1986.

[26] S.H. Lee, R. Vakil. Mnev–Sturmfels universality for schemes. *arXiv:1202.3934*, 2012.

[27] J. Oxley. *Matroid theory*. Oxford University Press, 1992.

[28] D. Speyer. A matroid invariant via the K-theory of the Grassmannian. *Adv. Math.*, 221(3):882–913, 2009.

[29] S. Vakil, M. Miller. Tutte-Thom polynomials for matroids. *J. Combin. Theory Ser. A* 110(2):221–260, 2004.

[30] D. Speyer. A matroid invariant via the K-theory of the Grassmannian. *Adv. Math.*, 221(3):882–913, 2009.

[31] A. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. *Surveys in combinatorics 2005*, London Math. Soc. Lecture Notes 327, 2005.

[32] R. Liu, Specht modules and Schubert varieties for general diagrams. Ph.D. Thesis, Massachusetts Institute of Technology, 2010.

[33] I. G. MacDonald. *Symmetric functions and Hall polynomials*. Oxford University Press, New York, 1995.

[34] A. Merkurjev. Equivariant K-theory. In *Handbook of K-theory, vol. 2*, pages 925–954. Springer-Verlag, Berlin, 2005.

[35] E. Miller and B. Sturmfels. *Combinatorial commutative algebra*. Springer-Verlag, 2005.

[36] D. Speyer. A matroid invariant via the K-theory of the Grassmannian. *Adv. Math.*, 221(3):882–913, 2009.

[37] A. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. *Surveys in combinatorics 2005*, London Math. Soc. Lecture Notes 327, 2005.

[38] B. Sturmfels. On the matroid stratification of Grassmann varieties, specialization of coordinates, and a problem of N. White. *Adv. Math.*, 75:202–211, 1989.

[39] S. H. Lee, R. Vakil. Mnev–Sturmfels universality for schemes. *arXiv:1202.3934*, 2012.

[40] N. White. The basis monomial ring of a matroid. *Adv. Math.*, 24(3):292–297, 1977.

[41] N. White. *Theory of matroids*, volume 26 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1986.