A UNIQUE DECOMPOSITION THEOREM FOR
TIGHT CONTACT 3-MANIFOLDS

FAN DING AND HANSJÖRG GEIGES

Abstract. It has been shown by V. Colin that every tight contact 3-manifold can be written as a connected sum of prime manifolds. Here we prove that the summands in this decomposition are unique up to order and contactomorphism.

1. Introduction

Unless stated otherwise, all 3-manifolds in this note are assumed to be closed, connected, and oriented. A 3-manifold is called non-trivial if it is not diffeomorphic to $S^3$. A non-trivial 3-manifold $P$ is said to be prime if in every connected sum decomposition $P = P_0 \# P_1$ one of the summands $P_0, P_1$ is $S^3$. It is known that every non-trivial 3-manifold $M$ admits a prime decomposition, i.e. $M$ can be written as a connected sum of finitely many prime manifolds. The main step in the proof of this fact is due to H. Kneser [10], cf. [7]. Moreover, as shown by J. Milnor [11], the summands in this prime decomposition are unique up to order and diffeomorphism.

The purpose of the present note is to prove the analogous result for tight contact 3-manifolds. The basis for the argument is a connected sum construction for such manifolds, due to V. Colin [1] and reproved by K. Honda [8]. Given a fixed connected sum decomposition $M = M_0 \# M_1$ of a 3-manifold $M$, Colin’s result says that tight contact structures $\xi_i$ on $M_i$, $i = 0, 1$, give rise to a tight contact structure $\xi_0 \# \xi_1$ on $M$, uniquely defined up to isotopy. Conversely, for any tight contact structure $\xi$ on $M$ there are — up to isotopy — unique tight contact structures $\xi_i$ on $M_i$, $i = 0, 1$, such that $\xi_0 \# \xi_1$ is the given contact structure $\xi$. The prime decomposition theorem for tight contact 3-manifolds is an immediate consequence.

Although Colin’s result goes a long way, it is not quite strong enough to prove the unique decomposition theorem for tight contact 3-manifolds. This is due to the fact that the system of 2-spheres in a given manifold $M$ defining the prime decomposition of $M$ is not, in general, unique up to isotopy. The argument for the unique decomposition of tight contact 3-manifolds given here closely follows the variant of Milnor’s argument given in J. Hempel’s book [7].

2. Colin’s results

In this section we collect the results from [1] that we shall need. We assume that the reader is familiar with the basics of contact topology at the level of [4], [9] and [5].

Lemma 1 ([1] Lemme 5). Let $(M, \xi)$ be a (not necessarily connected) tight contact 3-manifold. Given embeddings $f_0, f_1 : S^2 \to M$, there is a contact structure $\eta$ on
$S^2 \times [0, 1]$ such that the characteristic foliation $(S^2 \times \{i\})_\eta$ coincides with $S^2_{f_i} \xi_i$, $i = 0, 1$.\footnote{Here $S^2_{f_i} \xi_i$ denotes the characteristic foliation induced by the embedding $f_i$, that is, the pull-back to $S^2$ via $f_i$ of the characteristic foliation $(f_i(S^2))_\xi$.} This contact structure $\eta$ is unique up to isotopy rel boundary. \qed

We can now define surgery along a 0-sphere inside a given (not necessarily connected) tight contact 3-manifold $(M, \xi)$ as follows; this includes the formation of a connected sum.

Equip the 3-disc $D^3$ with its standard orientation. Let $\phi_0, \phi_1: D^3 \to M$ be embeddings such that $\phi_0$ reverses and $\phi_1$ preserves orientation, and whose images $B_i := \phi_i(D^3) \subset M$ are disjoint. Let $\eta$ be the contact structure on $S^2 \times [0, 1]$, constructed in the preceding lemma, with the property that $(S^2 \times \{i\})_\eta = (\partial D^3)_{\phi_i^* \xi}$. Then set

$$(M', \xi') = (M \setminus \text{Int}(B_0 \cup B_1), \xi) \cup_{\partial} (S^2 \times [0, 1], \eta),$$

where $\cup_{\partial}$ denotes the obvious gluing along the boundary.

If $M = M_0 + M_1$ is the topological sum of two connected tight contact 3-manifolds $(M_0, \xi_0)$, $(M_1, \xi_1)$, and $B_i \subset M_i$, $i = 0, 1$, then $M'$ is the connected sum $M_0 \# M_1$ of $M_0$ and $M_1$, and we write $\xi_0 \# \xi_1$ for the contact structure $\xi'$ in this specific case. We also use the notation $(M_0, \xi_0) \# (M_1, \xi_1)$ for this connected sum of tight contact 3-manifolds. As in the topological case, this connected sum operation is commutative and associative; these are consequences of the discussion that follows. From Theorem 5 below it follows that $(S^3, \xi_{st})$, the 3-sphere with its unique tight contact structure, serves as the neutral element.

**Lemma 2 (\cite{F-K} Corollaire 8).** Let $(M', \xi')$ be a contact 3-manifold and $f_t: S^2 \to M'$, $t \in [0, 1]$ an isotopy of embeddings. If the spheres $S_i := f_i(S^2), i = 0, 1$, are convex with respect to $\xi'$, and $(M' \setminus S_i, \xi')$ is tight, then so is $(M' \setminus S_i, \xi')$. \qed

**Lemma 3 (\cite{F-K} Proposition 9).** The manifold $(M', \xi')$ obtained, in the way described above, via 0-surgery on a tight contact 3-manifold $(M, \xi)$, is tight and only depends, up to contactomorphism, on the isotopy class of the embeddings $\phi_0$, $\phi_1$. \qed

\section{3. The unique decomposition theorem}

We can now formulate the unique decomposition theorem for tight contact 3-manifolds.

**Theorem 4.** Every non-trivial tight contact 3-manifold $(M, \xi)$ is contactomorphic to a connected sum

$$(M_1, \xi_1) \# \cdots \# (M_k, \xi_k)$$

of finitely many prime tight contact 3-manifolds. The summands $(M_i, \xi_i), i = 1, \ldots, k$, are unique up to order and contactomorphism.

The proof of this theorem requires a few preparations. Besides Colin’s results, the most important ingredient is the following theorem of Ya. Eliashberg.

**Theorem 5 (\cite{E} Theorem 2.1.3).** Two tight contact structures on the 3-disc $D^3$ which induce the same characteristic foliation on the boundary $\partial D^3$ are isotopic rel boundary. \qed
First of all, we observe that there is a well-defined procedure for capping off a compact tight contact 3-manifold with boundary consisting of a collection of 2-spheres. Indeed, suppose that \((M, \xi)\) is a tight contact 3-manifold with boundary \(\partial M = S_1 + \cdots + S_k\), where each \(S_i\) is diffeomorphic to \(S^2\). Choose orientation-reversing diffeomorphisms \(f_i: \partial D^3 \to S_i\). By a reasoning as in Colin’s proof of Lemma 1, one finds an orientation-preserving embedding \(g: \partial D^3 \to \hat{S}^3\) such that \(S_i^2 = g^{-1}_i\hat{S}^3\); here \(\hat{S}^3\) denotes the standard tight contact structure on \(\mathbb{R}^3\) (which is the restriction of \(\xi_{\text{st}}\) on \(S^3\) to the complement of a point). The tight contact structure \(\eta_i := g^*_i\xi_{\text{st}}\) on \(D^3\) — which by Theorem 3 is uniquely determined by the characteristic foliation it induces on the boundary — can then be used to form the closed contact manifold

\[(\hat{M}, \hat{\xi}) = (M, \xi) \cup (D^3, \eta_1) \cup \cdots \cup (D^3, \eta_k),\]

where the gluing is defined by the embeddings \(f_i\).

Eliashberg’s theorem entails that we arrive at a contactomorphic manifold if instead of gluing discs along the \(S_i\) we first perturb the boundary spheres into convex spheres \(S'_i\) in the interior of \((M, \xi)\), cut off the spherical shell between \(S_i\) and \(S'_i\), and then glue discs along the \(S'_i\). In the same way that Lemma 2 enters the proof of Lemma 3 in Colin’s argument, one can use it here to conclude that \((\hat{M}, \hat{\xi})\) is tight.

Given an embedded 2-sphere \(S \subset \text{Int}(M)\), we can find a product neighbourhood \(S \times [-1, 1] \subset M\) of \(S \equiv S \times \{0\}\). Set \(M_S = M \setminus (S \times (-1, 1))\). Again by Theorem 5, the contactomorphism type of \((\hat{M}_S, \hat{\xi})\) is independent of the choice of this product neighbourhood; this follows by comparing the resulting manifolds using two given product neighbourhoods with a third manifold constructed from a product neighbourhood contained in the first two. In particular, this justifies our notation \((\hat{M}_S, \hat{\xi})\).

**Lemma 6.** If \(S_0\) and \(S_1\) are isotopic 2-spheres in \(\text{Int}(M)\), then \((\hat{M}_{S_0}, \hat{\xi})\) and \((\hat{M}_{S_1}, \hat{\xi})\) are contactomorphic.

**Proof.** This is clear if \(S_1\) is isotopic to \(S_0\) inside a product neighbourhood \(S_0 \times (-1, 1)\). The general case follows by an argument very similar to Colin’s proof of Lemma 2.

Given a connected sum decomposition \(M = M_0 \# M_1\) of a closed, connected 3-manifold with a tight contact structure \(\xi\), let \(S \subset M\) be an embedded sphere defining this connected sum, i.e. \(\hat{M}_S = M_0 + M_1\). The described constructions imply that

\[(M, \xi) = (M_0, \hat{\xi}|_{M_0}) \# (M_1, \hat{\xi}|_{M_1}).\]

So the topological prime decomposition of \(M\) also gives us a decomposition of \((M, \xi)\) into prime tight contact 3-manifolds. The only remaining issue is the uniqueness of this decomposition up to contactomorphism of the summands.

A 3-manifold \(M\) is said to be **irreducible** if every embedded 2-sphere bounds a 3-disc in \(M\). Clearly, irreducible 3-manifolds (except \(S^3\)) are prime. There is but one orientable prime 3-manifold that is not irreducible, namely, \(S^2 \times S^1\) [7]. Lemma
In a connected sum $M = M_0 \# S^2 \times S^1$ we obviously find an embedded non-separating 2-sphere $S$ such that $\hat{M}_S = M_0$; simply take $S$ to be a fibre of $S^2 \times S^1$ not affected by the connected sum construction.

In the argument proving that the number of summands $S^2 \times S^1$ in a prime decomposition of $M$ is uniquely determined by $M$, the crucial lemma is that for any two non-separating 2-spheres $S_0, S_1 \subset \hat{M}$ there is a diffeomorphism of $M$ sending $S_0$ to $S_1$. In the presence of a contact structure, this statement needs to be weakened slightly; the following is sufficient for our purposes.

**Lemma 7.** Let $(M, \xi)$ be a (connected) tight contact 3-manifold and $S_0, S_1 \subset M$ two non-separating 2-spheres. Then $(\hat{M}_{S_0}, \xi)$ and $(\hat{M}_{S_1}, \xi)$ are contactomorphic.

**Proof.** By the preceding lemma we may assume that $S_0$ and $S_1$ are in general position with respect to each other, so that $S_0 \cap S_1$ consists of a finite number of embedded circles. We use induction on the number $n$ of components of $S_0 \cap S_1$.

If $n = 0$, we may find disjoint product neighbourhoods $S_i \times [-1, 1] \subset M$, $i = 0, 1$. In case $M \setminus (S_0 \cup S_1)$ is not connected, we may assume that the identifications of these neighbourhoods with a product have been chosen in such a way that $S_0 \times \{1\}$ and $S_1 \times \{1\}$ lie in the same component of $M \setminus (S_0 \cup S_1)$. As described above, we then obtain a well-defined tight contact manifold $(\hat{M}, \hat{\xi})$ by capping off the boundary components $S_i \times \{\pm 1\}$ of

$$M \setminus (S_0 \times (-1, 1) \cup S_1 \times (-1, 1))$$

with 3-discs $D_0^+ \cup D_0^-$. Our assumptions imply that $D_0^+ + D_0^- \subset \hat{M}$, by performing 0-surgery with respect to these embeddings of $S_0 \times D^3$, we obtain $(\hat{M}_{S_0}, \hat{\xi})$ and $(\hat{M}_{S_1}, \hat{\xi})$, respectively, so the result follows from Lemma 7.

If $n > 0$, then some component $J$ of $S_0 \cap S_1$ bounds a 2-disc $D \subset S_1$ with $\text{Int}(D) \cap S_0 = \emptyset$. Let $E'$ and $E''$ be the 2-discs in $S_0$ bounded by $J$, and set $S_0' = D \cup E'$ and $S_0'' = D \cup E''$. At least one of $S_0'$ and $S_0''$, say $S_0''$, is non-separating.² Move $S_0''$ slightly so that it becomes a smoothly embedded sphere disjoint from $S_0$ and intersecting $S_1$ in fewer than $n$ circles. Then two applications of the inductive assumption prove the inductive step. □

**Proof of Theorem 3** As indicated above, it only remains to prove the uniqueness statement. Thus, let

$$(M_1, \xi_1) \# \cdots \# (M_k, \xi_k)$$

and

$$(M_1^*, \xi_1^*) \# \cdots \# (M_l^*, \xi_l^*)$$

be two prime decompositions of a given tight contact 3-manifold $(M, \xi)$. Without loss of generality we assume³ $k \leq l$ and use induction on $k$. For $k = 1$ there is nothing to prove. Now assume $k > 1$ and the assumption to be proved for prime decompositions with fewer than $k$ summands.

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²Since $S_0$ is non-separating, there is a loop $\gamma$ in $M$ (in general position with respect to all spheres in question) that intersects $S_0$ in a single point, say one contained in the interior of $E'$. If $S_0''$ is separating, then $\gamma$ intersects it in an even number of points. Since $\gamma$ does not intersect $E''$, these points all lie in $D$. So $\gamma$ intersects $S_0''$ in an odd number of points, which means that $S_0''$ is non-separating.

³Of course, from the topological prime decomposition theorem, one already knows that $k = l$, but this does not help to simplify the present proof.
(i) Suppose some $M_i$ (say $M_k$) is diffeomorphic to $S^2 \times S^1$. Then $M$ contains a non-separating 2-sphere. By applying the argument in the footnote to the preceding proof to this non-separating 2-sphere and the 2-spheres defining the splitting of $M$ into the connected sum of the $M_j^*$, one finds a non-separating 2-sphere in at least one of these summands, say $M_i^*$, which therefore must be a copy of $S^2 \times S^1$. By a folklore theorem of Eliashberg, there is a unique tight contact structure on $S^2 \times S^1$, cf. [1] for an outline proof and [6] for a complete proof. Thus, $(M_k, \xi_k)$ is contactomorphic to $(M_i^*, \xi_i^*)$. Let $S_0, S_1$ be a fibre in $M_k, M_i^*$, respectively. From Theorem [5] it follows that

$$(\widehat{M}_{S_0}, \widehat{\xi}) = (M_1, \xi_1) \# \cdots \# (M_{k-1}, \xi_{k-1})$$

and

$$(\widehat{M}_S, \widehat{\xi}) = (M_i^*, \xi_i^*) \# \cdots \# (M_{l-1}^*, \xi_{l-1}^*),$$

and by the preceding lemma these two manifolds are contactomorphic. So the conclusion of the theorem follows from the inductive assumption.

(ii) It remains to deal with the case where all the $M_i$ are irreducible. Arguing as before (with the roles of the two connected sum decompositions reversed), we see that each $M_j^*$ must be irreducible. Choose a separating 2-sphere $S \subset M$ such that the closures $U, V$ of the components of $M \setminus S$ satisfy

$$(\widehat{U}, \widehat{\xi}_{|U}) = (M_1, \xi_1) \# \cdots \# (M_{k-1}, \xi_{k-1})$$

and $$(\widehat{V}, \widehat{\xi}_{|V}) = (M_k, \xi_k).$$

Similarly, there exist pairwise disjoint 2-spheres $T_1, \ldots, T_{l-1}$ in $M$ such that — with $W_1, \ldots, W_l$ denoting the closures of the components of $M \setminus (T_1 \cup \ldots \cup T_{l-1})$, and $\xi_j$ the restriction of $\xi$ to $W_j$ — we have $(\widehat{W}_j, \widehat{\xi}_j) = (M_j^*, \xi_j^*)$, $j = 1, \ldots, l$.

Suppose that the system $T_1, \ldots, T_{l-1}$ of embedded spheres has been chosen in general position with respect to $S$ and with $S \cap (T_1 \cup \ldots \cup T_{l-1})$ having the minimal number of components among all such systems.

Here we have to enter a caveat. The notation suggests that $W_1$ has boundary $T_1$, the $W_j$ with $j \in \{2, \ldots, l-1\}$ have boundary $T_{j-1} \cup T_j$, and $W_l$ has boundary $T_{l-1}$. In fact, some of the reasoning in the proof given in [7] seems to rely on such an assumption. However, under the minimality condition we have just described, it is perfectly feasible that some of the $W_j$ have several boundary components (i.e. the connected sum looks like a tree rather than a chain). In particular, the numbering of the $W_j$ is not meant to suggest any kind of order in which they are glued together.

We claim that the minimality condition implies $S \cap (T_1 \cup \ldots \cup T_{l-1}) = \emptyset$.

Assuming this claim, we have $S \subset W_j$ for some $j \in \{1, \ldots, l\}$. Since $W_j = M_j^*$ is irreducible, $S$ bounds a 3-cell $B$ in $M_j$. Thus, $S$ cuts $W_j$ into two pieces $X$ and $Y$, where say $Y = S^3$. By the uniqueness of the tight contact structure on $S^3$ we have in fact $(\widehat{Y}, \widehat{\xi}_{|Y}) = (S^3, \xi_{st})$. Moreover, $(\widehat{X}, \widehat{\xi}_{|X}) = (M_j^*, \xi_j^*)$ by Theorem [5].

Of the 3-discs in $M_j^*$ used for forming the connected sum with one or several of the other prime manifolds, at least one has to be contained in $B$, otherwise $S$ would bound a disc in $M$. This means that of the closures $U, V$ of the two components of $M \setminus S$, the one containing $Y$ must contain at least one of $S^3$.

\footnote{The contact structure $\widehat{\xi}_{|U}$ is the same as the restriction of the contact structure $\widehat{\xi}$ (defined on $\widehat{M}_S = \hat{U} + V$) to $\hat{U}$.}
Thus, in the case $Y \subset V$, the numbering (including that of $W_j$) can be chosen in such a way that $W_1, \ldots, W_{j-1}, W_j, \ldots, W_l \subset V$, with $j \leq l - 1$. (The case with $X \subset V$ and $Y \subset U$ is analogous; here $j \geq 2$.) With Theorem 5, and in particular the fact that $(S^3, \xi_{st})$ is the neutral element for the connected sum operation, we conclude that

$$(M_1, \xi_1) \# \cdots \# (M_{k-1}, \xi_{k-1}) = (\hat{U}, \xi_{U})$$

$$(\hat{W}_1, \hat{\xi}_1) \# \cdots \# (\hat{W}_{j-1}, \hat{\xi}_{j-1}) \# (\hat{X}, \hat{\xi}_{X}) = (M_1^*, \xi_1^*) \# \cdots \# (M_j^*, \xi_j^*)$$

and

$$(M_k, \xi_k) = (\hat{V}, \hat{\xi}_{V})$$

$$(\hat{W}_{j+1}, \hat{\xi}_{j+1}) \# \cdots \# (\hat{W}_1, \hat{\xi}_1) = (M_{j+1}^*, \xi_{j+1}^*) \# \cdots \# (M_1^*, \xi_1^*).$$

Since $M_k$ is prime, we must have $j = l - 1$, hence $(M_k, \xi_k) = (M_1^*, \xi_1^*)$. Once again, the theorem follows from the inductive assumption.

It remains to prove the claim. Arguing by contradiction, we assume that $S \cap (T_1 \cup \ldots \cup T_{l-1}) \neq \emptyset$. Then we find a 2-disc $D \subset S$ with $\partial D \subset T_i$ for some $i \in \{1, \ldots, l-1\}$, and $\text{Int}(D) \cap (T_1 \cup \ldots \cup T_{l-1}) = \emptyset$. This disc is contained in $W_j$ for some $j \in \{1, \ldots, l\}$. For ease of notation we assume that $i = j = 1$, and that $W_2$ is the other component adjacent to $T_1$.

Let $E', E''$ be the 2-discs in $T_1$ bounded by $\partial D$. Since $\hat{W}_1$ is irreducible, the sets $D \cup E'$ and $D \cup E''$ (which are homeomorphic copies of $S^2$) bound 3-cells $B', B''$ in $\hat{W}_1$. One of these must contain the other, otherwise it would follow that $\hat{W}_1$ can be obtained by capping off the 3-cell $B' \cup D \cup B''$, and thus would be a 3-sphere.

So suppose that $B'' \subset B'$. Then $D \cup E'$ can be deformed into a smooth 2-sphere $T_1'$ that meets $S$ in fewer components than $T_1$, see Figure 1. In the complement $M \setminus (T_1' \cup T_2 \cup \ldots \cup T_{l-1})$ we still find $W_3, \ldots, W_l$, but $W_1, W_2$ have been changed.
to new components $W'_1, W'_2$. Write $\xi'_1, \xi'_2$, respectively, for the restriction of $\xi$ to these components. We are done if we can show that

$$(\hat{W}'_i, \hat{\xi}'_i) = (\hat{W}_i, \hat{\xi}_i), \quad i = 1, 2,$$

because then the new system of spheres $T'_1, T_2, \ldots, T_{l-1}$ contradicts the minimality assumption on $T_1, T_2, \ldots, T_{l-1}$.

The 2-sphere $T'_1$ is isotopic to $T_1$ in $\hat{W}_1$: simply move $D \subset T'_1$ across the ball $B''$ to $E''$. But beware that $T'_1$ need not be isotopic to $T_1$ in $W_1$ or $M$. However, $B''$ lies on the same side of $T_1$ as $W_1$, so $T'_1$ is isotopic to $T_1$ in $\hat{W}_1 \cup W_2 = \hat{W}'_1 \cup W'_2$.

Cutting this latter manifold open along $T_1$ and then capping off with discs gives the disjoint union of $(\hat{W}_1, \hat{\xi}_1)$ and $(\hat{W}_2, \hat{\xi}_2)$; cutting it open along $T'_1$ and capping off yields the disjoint union of $(\hat{W}'_1, \hat{\xi}'_1)$ and $(\hat{W}'_2, \hat{\xi}'_2)$. From Lemma it follows that the results of either procedure are contactomorphic.

This was the last point we had to show in order to conclude the proof of the unique decomposition theorem. □

**Remark.** There is obviously no unique decomposition theorem for overtwisted contact 3-manifolds. For instance, start with a connected sum of two distinct prime tight contact 3-manifolds. Now perform a Lutz twist in one or the other summand, preserving the topology of the manifold and the homotopy class of the contact structure as a 2-plane field. By Eliashberg’s classification of overtwisted contact structures, the resulting manifolds are contactomorphic, and we obviously have two distinct connected sum decompositions.

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