LINEAR DIFFERENTIAL EQUATIONS FOR A FRACTIONAL SPIN FIELD

José L. Cortés\(^a\)\(^1\) and Mikhail S. Plyushchay\(^a,b\)\(^2\)

\(^a\)Departamento de Física Teórica, Facultad de Ciencias
Universidad de Zaragoza, 50009 Zaragoza, Spain
\(^b\)Institute for High Energy Physics, Protvino, Russia

Abstract

The vector system of linear differential equations for a field with arbitrary fractional spin is proposed using infinite-dimensional half-bounded unitary representations of the $SL(2, \mathbb{R})$ group. In the case of $(2j + 1)$-dimensional nonunitary representations of that group, $0 < 2j \in \mathbb{Z}$, they are transformed into equations for spin-$j$ fields. A local gauge symmetry associated to the vector system of equations is identified and the simplest gauge invariant field action, leading to these equations, is constructed.

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\(^2\) e-mail: cortes@cc.unizar.es
\(^3\) e-mail: mikhail@cc.unizar.es
1 Introduction

A (2+1)-dimensional space-time offers new possibilities which are not present in any higher dimensional case: due to the Abelian nature of the spatial rotation group, $SO(2)$, and the topology of many-particle configuration space, the spin of a relativistic particle can be an arbitrary real number, and a generalized statistics, intermediate between Bose and Fermi statistics, is also possible [1] (see also reviews [2] and references therein). The considerable interest to the field models of such particles, called anyons, is due to their application to different planar physical phenomena: the fractional quantum Hall effect, high-$T_c$ superconductivity and description of the physical processes in the presence of cosmic strings [3].

There are several field models realizing anyonic states. They appear as topological solitons in the O(3) nonlinear $\sigma$ model or the $CP^1$ model with the topological Hopf term [4], which turns in the low-energy limit into the $CP^1$ model with the Chern-Simons term [5]. In the Higgs models with the topological Chern-Simons term [6, 7], anyons are the electrically charged vortices, whereas in the models with the topologically massive vector gauge field, the anyons are the particles directly associated to the matter field [8]. But, since all these models contain other states, they do not give a minimal theory of anyons. In the best known approach, point particles, described by scalar or spinor fields, are coupled to a $U(1)$ gauge field, the so-called statistical gauge field, whose dynamics is governed by the Chern-Simons action [9, 2]. This statistical gauge field changes the spin and statistics of particles, but here it is not clear whether the only effect of the gauge field is to endow the particle with arbitrary spin or whether residual interactions are also present.

Therefore, we arrive at the natural question: is it possible to describe the anyons in a minimal way, without using the statistical Chern-Simons gauge field? For the purpose, one can turn over the group theoretical approach, generalizing ordinary approach to the description of bosonic and fermionic fields. Within such an approach, one can work with the multi-valued representations of the (2+1)-dimensional Lorentz group $SO(2,1)$ [10]–[12], or with the definite infinite-dimensional representations of its universal covering group $SO(2,1)$ (or $SL(2,\mathbb{R})$, isomorphic to it) [12]–[18]. Though, there is a close connection between these two possibilities [12], the problem of constructing the field actions in the case of using the multi-valued representations of the Lorentz group is open. At the same time, different variants of the
free field equations and corresponding actions were proposed for fractional spin fields within the framework of the approach dealing with the infinite-dimensional representations of $SL(2, \mathbb{R})$ [12], [14]–[16]. But here mutually connected problems of second quantization and spin-statistics relation are still unsolved. Therefore, strictly speaking, we cannot use the term ‘anyons’ for such fractional spin fields before establishing the spin-statistics relation, and it seems important to continue a search of new equations and corresponding actions for a fractional spin field in the framework of the group-theoretical approach.

In the present paper we propose new equations for fractional spin fields, which, in our opinion, have definite advantages with respect to those from refs. [12], [14]–[16]. They are linear differential equations, and the corresponding fields here, unlike those from equations proposed in refs. [14], [16], carry irreducible representations of $SL(2, \mathbb{R})$. In this sense the proposed equations are similar to equations for ‘semions’ (i.e. fields with spin $\pm(1/4 + n)$, $n = 0, \pm 1, ...$), which have been proposed in [13], [17], and generalized to the case of arbitrary spin fields with the help of the deformed Heisenberg algebra in a recent paper [18]. The equations which we shall construct, have the following property of ‘universality’: if we choose in them $(2j + 1)$-dimensional nonunitary representation of $SL(2, \mathbb{R})$, we will get the equations for a massive field with integer or half-integer spin $j$. In particular, at $j = 1/2$ and $j = 1$ these equations are reduced to the Dirac equation and to the equation for a topologically massive vector gauge field, respectively. On the other hand, the choice of infinite-dimensional unitary representation of the discrete type series, restricted from below or from above, which is the only additional possibility allowing nontrivial solutions, gives the equations for a field with fractional (arbitrary) spin. Therefore, the proposed equations give some link between the ordinary description of bosonic integer and fermionic half-integer spin fields, and the fields with arbitrary spin. Moreover, as we shall see, they represent by themselves the first example of equations which fix the choice of infinite-dimensional unitary representations of $SL(2, \mathbb{R})$ group for the description of fractional spin fields. Also, we shall see that the vector system of linear equations satisfy some identity which, being a consequence of the choice of irreducible representation of $SL(2, \mathbb{R})$, can be used as a dynamical principle for the construction of the corresponding gauge invariant field action.

The paper is organized as follows. In sect. 2 we investigate the equa-
tion, which, in general case, establishes a mass-spin relation for \((2j + 1)\)-
or infinite-component fields, depending on the choice of the corresponding
representation of \(SL(2, R)\). Except for the case \(j = 1/2\) and \(j = 1\), it does
not describe irreducible representations of \((2+1)\)-dimensional quantum me-
chanical Poincaré group \(ISO(2,1)\). Then, in sect. 3, proceeding from this
equation, we find in a simple way the system of equations which describe a
relativistic field with arbitrary (fixed) spin and fixed mass, i.e., an irreducible
representation of \(ISO(2,1)\). A first attempt in the direction of identifying
the field action is pointed out. Sect. 4 is devoted to the discussion of the re-
results and to the concluding remarks. Here, in particular, we demonstrate that
the proposed equations unambiguously fix the choice of the representations of
the discrete series \(D^\pm_\alpha\) of \(SL(2, R)\) for the description of fractional spin fields,
and that they are the only possible linear vector equations for such fields.

\section{Mass-spin equation}

Let us consider the \((2+1)\)-dimensional field equation \[14, 15\]
\[
(P J - \varepsilon \alpha m) \Psi = 0,
\]
\[2.1\]
where \(\varepsilon = +1\) or \(-1\), and \(J^\mu\) are the generators of the \(SL(2, R)\) group, which
satisfy the algebra:
\[
[J^\mu, J^\nu] = -i\epsilon^{\mu\nu\lambda} J^\lambda.
\]
\[2.2\]
Here, a real parameter \(\alpha \neq 0\) defines the value of the \(SL(2, R)\) Casimir
operator:
\[
J^2 = -\alpha(\alpha - 1),
\]
\[2.3\]
i.e. in the case \(\alpha = -j, j > 0\) being integer or half-integer, we suppose the
choice of a \((2j + 1)\)-dimensional irreducible nonunitary representation \(\tilde{D}_j\) of
\(SL(2, R)\), whereas in the case \(\alpha > 0\) we mean the choice of an irreducible
unitary infinite-dimensional representation of the discrete type series \(D^\pm_\alpha\) of
that group \[19\]. These two types of representations are characterized by the
following property: they have a lowest or a highest state annihilated by the
corresponding operator \(J_+\) or \(J_-\) (see below) in the case of representations
\(D^\pm_\alpha\), or both such states in the case of finite-dimensional representations \(\tilde{D}_j\).
Let us turn over the case of finite-dimensional representations and first consider the simplest nontrivial case of the spinor representation

$$J^\mu = -\frac{1}{2}\gamma^\mu,$$  \hspace{1cm} (2.4)

$$\gamma^\mu \gamma^\nu = -g^{\mu\nu} + i\epsilon^{\mu\nu\lambda}\gamma^\lambda, \quad \gamma^0 = \sigma^3, \quad \gamma^i = i\sigma^i, \quad i = 1, 2,$$  \hspace{1cm} (2.5)

where $\sigma^a, a = 1, 2, 3,$ are the Pauli matrices. It is this simplest case that will help us to find the equations we are looking for. Generators (2.4) correspond to the 2-dimensional irreducible nonunitary representation $D_{1/2}$ with $-\alpha = j = 1/2,$ and reduce equation (2.1) to the (2+1)-dimensional Dirac equation:

$$(P\gamma - \varepsilon m)\Psi = 0.$$  \hspace{1cm} (2.6)

The angular momentum operator

$$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \epsilon_{\mu\nu\lambda}J^\lambda$$  \hspace{1cm} (2.7)

is not hermitian and therefore it is necessary to use the indefinite ‘internal’ Dirac scalar product

$$(\Psi_1, \Psi_2) = \overline{\Psi}_1^a \Psi_2^a, \quad \overline{\Psi} = \Psi^+ \gamma^0,$$

and the Klein-Gordon equation

$$(P^2 + m^2)\Psi = 0,$$  \hspace{1cm} (2.8)

following from (2.6), we conclude that in the case $-\alpha = j = 1/2$ initial equation (2.1) describes a particle with mass $M = m$ and spin $s = -\varepsilon/2,$ where $s$ is the eigenvalue of the relativistic spin operator

$$S = -\frac{1}{2\sqrt{-P^2}}\epsilon_{\mu\nu\lambda}P^\mu M^\nu\lambda.$$  \hspace{1cm} (2.9)

In the case of the vector representation ($\alpha = -1$),

$$(J_\mu)^\alpha_\beta = -i\epsilon^\alpha_{\mu\beta},$$  \hspace{1cm} (2.10)
we have $J^2 = -2$, and eq. (2.1) becomes the equation for the topologically massive vector field [20]:

$$(-i \epsilon^{\alpha \beta} P_{\mu} + \epsilon m g^\alpha_\beta) \Psi^\beta = 0. \quad (2.11)$$

From (2.11) it follows that $P_{\mu} \Psi^\mu = 0$, and that the field $\Psi^\mu$ satisfies the Klein-Gordon equation (2.8). Then, using definition (2.9), we conclude that the field $\Psi^\mu$ has spin $s = -\epsilon$. $J^\mu$ and the angular momentum operator (2.7) are hermitian with respect to the obvious indefinite ‘internal’ scalar product

$$(\Psi_1, \Psi_2) = \Psi_1^* g_{\alpha \beta} \Psi_2^\beta.$$  

Putting $\Psi_\alpha = \frac{1}{2} \epsilon_{\alpha \beta \gamma} F^{\beta \gamma}$, $F^{\alpha \beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, we can rewrite eq. (2.11) in the form of equations of motion for the field strength tensor [20]:

$$(g_{\mu \lambda} \partial_\nu + \frac{1}{2} m \epsilon_{\mu \nu \lambda}) F^{\nu \lambda} = 0. \quad (2.12)$$

An arbitrary $(2j + 1)$-dimensional nonunitary representation $\tilde{D}_j$ of $SL(2, R)$ can be obtained from the corresponding $(2j + 1)$-dimensional representation $D_j$, $j = 1/2, 1, 3/2, ...$, of the $SU(2)$ group. Indeed, let the hermitian operators $J^a$, $a = 1, 2, 3$, be the generators of $SU(2)$ group in the representation $D_j$, i.e.

$$[J^a, J^b] = i \epsilon^{abc} J^c, \quad (2.13)$$

and

$$J^a J^a = j(j + 1). \quad (2.14)$$

Then the substitution

$$J_0 = J^3, \quad J^i = -i J^i, \quad i = 1, 2, \quad (2.15)$$

gives us the operators $J^\mu$ satisfying commutation relations (2.2), and condition (2.3) with $\alpha = -j$. Then to have an angular momentum operator (2.7) as a hermitian one, it is necessary to use the corresponding indefinite scalar product, which we do not write here for the general case. Note only that representation (2.4) is exactly representation (2.15) for $j = 1/2$ if we put $J^a = \sigma^a/2$, whereas representation (2.10) is connected with the corresponding representation (2.15) with $(J^b)^{ac} = i \epsilon^{abc}$ at $j = 1$ via the unitary transformation:

$$U \tilde{J}^{\mu} U^{-1} = J^\mu,$$
where we denoted the operators (2.15) as \( \tilde{J}^\mu \), and \( U \) is the unitary diagonal 3 \( \times \) 3-matrix with nonzero elements: \( U^0_0 = -i, \ U^1_1 = U^2_2 = 1 \).

In the case of \( j > 1 \), the corresponding \((2j + 1)\)-component field \( \Psi \) satisfying the equation (2.1) with \( \alpha = -j \), does not describe an irreducible representation of the \((2+1)\)-dimensional quantum mechanical Poincaré group \( \text{ISO}(2, 1) \). Indeed, first of all we note that the equation has no nontrivial solutions in the cases \( p^2 > 0 \) and \( p^2 = 0 \) since according to (2.15), operators \( J^i \) and \( J^0 \pm J^i \) have no real nonzero eigenvalues. Therefore, the nontrivial solutions may exist only for \( p^2 < 0 \). Then, passing over to the rest frame \( p = 0 \) via the corresponding Lorentz transformation, and using the representation where the operator \( J^0 \) is diagonal, we find the solutions of the equation (2.1):

\[
\Psi_r \propto \delta(p^0 - \varepsilon^0 M|\!|r|\!) \delta(p) .
\]

(2.16)

Here \( r = -j, -j + 1, ..., j - 1, j \), except for the value \( r = 0 \) for integer \( j \), \( \varepsilon^0 = \varepsilon \cdot \text{sign} \ r \),

\[
M|\!|r|\!| = m \frac{j}{|\!|r|\!|}
\]

(2.17)
is the mass of the corresponding state, whereas, according to (2.9), its spin is \( s = -\varepsilon |\!|r|\!| \). Therefore, we conclude that eq. (2.1) describes two states with fixed mass \( M = m \) and spin \( s = -\varepsilon j \) only in the cases when \( -\alpha = j = 1/2 \) and \( 1 \). These two states differ in their energy signs. In all other cases eq. (2.1) describes a set of \( 2N \) states, where \( N = j \) and \( N = (2j + 1)/2 \) for the cases of integer and half-integer \( j \)'s, respectively.

Now let us turn to the case of irreducible unitary infinite-dimensional representations of the discrete series \( D^+_{\alpha} \) or \( D^-_{\alpha} \) of \( \text{SL}(2, \mathbb{R}) \). These representations are characterized by the value of the Casimir operator (2.3) with \( \alpha > 0 \), and by the eigenvalues of the operator \( J^0 \): \( J^0_0 = \alpha + n \) and \( J^0_0 = - (\alpha + n) \) in these two series, respectively, where \( n = 0, 1, 2, ... [19] \). In the representation \( D^+_{\alpha} \) with diagonal operator \( J^0 \), the matrix elements of \( J^\mu \) are [15]:

\[
J^0_{kn} = -(\alpha + n) \delta_{k,n},
\]

(2.18)

\[
J^+_k = -\sqrt{(2\alpha + n - 1)n \cdot \delta_{k+1,n}}, \quad J^-_k = -\sqrt{(2\alpha + n)(n + 1) \cdot \delta_{k-1,n}},
\]

(2.19)

where \( J^\pm = J^1 \mp iJ^2 \) and \( k, n = 0, 1, 2, ... \). The representation \( D^-_{\alpha} \) can be obtained from (2.18), (2.19) through the substitution [12]:

\[
J^i_0 \rightarrow -J^i_0, \quad J^i_1 \rightarrow -J^i_1, \quad J^i_2 \rightarrow J^i_2.
\]

(2.20)
Here generators $J^\mu$ are hermitian with respect to the positive definite scalar product $(\Psi_1, \Psi_2) = \Psi_1^* \Psi_2$.

In the cases of the infinite-dimensional representations $D_\alpha^\pm$, eq. (2.1) is a $(2 + 1)$-dimensional analog of the Majorana equation [21], which appears as the equation for the physical subspace in the model of the relativistic particle with torsion [13]. Moreover, the mass spectrum (2.17) appears in that model too: it is the spectrum of the model in the euclidean space-time. Note also that the action for the model of the relativistic particle with torsion, in turn, appeared as the effective action for a charged particle interacting with a $U(1)$ statistical gauge field [22].

Passing over to the rest frame $p = 0$ in the case when $p^2 < 0$, we find the solutions of this equation:

$$\Psi_n \propto \delta(p^0 - \varepsilon \varepsilon' M_n) \delta(p),$$

where $\varepsilon' = +1$ and $-1$ for representations $D_\alpha^+$ and $D_\alpha^-$, respectively. Their masses and spins are

$$M_n = m \frac{\alpha}{\alpha + n}, \quad s_n = \varepsilon(\alpha + n).$$

If we take the direct sum of representations, $D_\alpha^+ \oplus D_\alpha^-$, we will have the states with both energy signs in the massive sector [12]. Majorana equation (2.1) as well as its $(3 + 1)$-dimensional analog [21], besides massive solutions also has massless and tachyonic solutions (see ref. [13]).

To single out the state with highest mass, $M_0 = m$, and lowest spin, $s_0 = \varepsilon \alpha$, and to get rid of massless and tachyonic solutions, one can supplement equation (2.1), linear in $P^\mu$, with the Klein-Gordon equation (2.8) [12, 13, 16]. Obviously, in the case of finite-dimensional representations $\tilde{D}_j$ and for the corresponding choice $\alpha = -j$, these two equations single out two states with mass $M = m$ and spin $s = -\varepsilon j$, differing in their energy sign.

3 Linear differential equations for fractional spin

Since eqs. (2.4) and (2.8) are completely independent in the case of representations $D_\alpha^\pm$ (as well as in the case of representations $\tilde{D}_j$, $j \neq 1/2, 1$),
they are not very suitable equations as a basis for constructing the action and quantum theory of the fractional spin field. In this section we shall construct the set of linear differential equations for the field with arbitrary fractional spin in such a way that both equations (2.1) and (2.8) will appear as a consequence of them.

To find such equations, let us multiply eq. (2.6) by an invertible operator $\frac{1}{2}\gamma^\mu$. Then we obtain the vector system of three equations:

$$L_\mu \Psi = 0,$$

with

$$L_\mu \equiv (\alpha P_\mu - i \epsilon_{\mu\nu\lambda} P^\nu J^\lambda + \epsilon m J_\mu),$$

where $J_\mu = -\frac{1}{2}\gamma_\mu$ and $\alpha = -\frac{1}{2}$. These equations are equivalent to eq. (2.6). Let us show now that in the general case, i.e. for the choice of any representation $D_j$ or $D_{\pm\alpha}$, these equations are equivalent to eqs. (2.1) and (2.8).

Indeed, multiplying eq. (3.1) by $J_\mu$, $P_\mu$ and $i \epsilon_{\mu\nu\lambda} P^\nu J^\lambda$, we correspondingly get:

$$\left(\alpha - 1\right) \left(P J - \epsilon \alpha m\right) \Psi = 0,$$

$$\left(\alpha (P^2 + m^2) + \epsilon m(P J - \epsilon \alpha m)\right) \Psi = 0,$$

$$\left(\alpha (\alpha - 1)(P^2 + m^2) + \left(P J + \epsilon (\alpha - 1)m\right)(P J - \epsilon \alpha m)\right) \Psi = 0.$$

Whence we immediately arrive at the desired conclusion for $\alpha > 0$, $\alpha \neq 1$, and $\alpha = -j$. As for the case $\alpha = 1$, in which eq. (3.3) disappears, we note that eqs. (3.4) and (3.5) have no nontrivial solutions in the massless case $P^2 = 0$, and as a result, these two equations also are equivalent to equations (2.1) and (2.8). Therefore, the vector set of three equations (3.1) describes a relativistic field with spin $s = \epsilon \alpha$ and mass $M = m$ for any corresponding choice of irreducible representations $D_j$ or $D_{\pm\alpha}$.

Moreover, by direct verification one can get convinced that in the general case any two equations from eqs. (3.1) are equivalent to the complete set of three equations. For example, in the case of representation $D_{\pm\alpha}$ it can be easily done with the help of the explicit form of the generators (2.18), (2.19). Therefore, the presence of three equations (3.1) gives us a covariant set of linear differential equations for the description of an arbitrary spin field. As a consequence, we have the following relation

$$R^\mu L_\mu \equiv 0$$

9
with
\[ R_\mu = \left( (\alpha - 1)^2 g_{\mu\nu} - i(\alpha - 1)\epsilon_{\mu\nu\lambda} J^\lambda + J_\nu J_\mu \right) P^\nu, \] (3.7)
which reflects the dependence of eqs. (3.1) in a covariant way.

For completeness, let us write here the commutation relations of the operators \( L_\mu \):
\[ [L_\mu, L_\nu] = -i m \epsilon_{\mu\nu\lambda} \left( L^\lambda + \frac{P^\lambda}{m} (PJ - \varepsilon \alpha m) \right), \] (3.8)
and note, that in the case \( \alpha \neq 1 \) they can be rewritten in the following simple form:
\[ [L_\mu, L_\nu] = -i m \epsilon_{\mu\nu\lambda} \left( g^{\lambda\rho} + \frac{P^\lambda J^\rho}{m(\alpha - 1)} \right) L_\rho. \]

As the simplest action leading to the proposed equations (3.1), we can take
\[ A = \int \mathcal{L} d^3 x, \quad \mathcal{L} = \bar{\chi}^\mu L_\mu \Psi + \bar{\Psi} L_\mu^\dagger \chi^\mu + c \cdot \bar{\Psi} (PJ - \varepsilon \alpha m) \Psi, \] (3.9)
where \( c \) is an arbitrary real parameter and \( \chi_\mu = \chi_\mu^a \) and \( \bar{\chi}_\mu = \bar{\chi}_\mu^a \) are mutually conjugate fields with index \( a \) taking values in the chosen representation of \( SL(2, R) \) group. The variation of the action (3.9) with respect to \( \bar{\chi}^\mu \) gives equations (3.1), whereas the \( \bar{\Psi} \)-variation gives
\[ L_\mu^\dagger \chi^\mu + c \cdot (PJ - \varepsilon \alpha m) \Psi = 0, \] (3.10)
and, besides, we have corresponding equations for the conjugate fields \( \bar{\Psi} \) and \( \bar{\chi}^\mu \). Hence, the basic field satisfy the equations which we want to have, and from (3.10) we conclude that
\[ L_\mu^\dagger \chi^\mu = 0 \]
for any choice of \( c \). Now it is necessary to get convinced that the fields \( \chi^\mu \) and \( \bar{\chi}^\mu \)are pure auxiliary fields. In the simplest way this can be done within the Hamiltonian formalism which we hope to present in a future work.
4 Discussion and conclusions

We have proposed the system of linear differential equations (3.1) for a fractional spin field using infinite-dimensional representations $D^\pm_\alpha$. They have the form of a covariant (vector) set of matrix infinite-dimensional equations, from which only two equations are independent and the presence of the third one allows to have a covariant set of equations. One can show that eq. (3.1) is in fact the only possible linear vector set of equations for a fractional spin field. In other words, if we take an arbitrary linear combination of the operators $mJ_{\mu}$, $P_{\mu}$ and $\epsilon_{\mu\nu\lambda}P^{\nu}J^{\lambda}$ as the operator $L_{\mu}$ and then demand that equations of the form (3.1) would be equivalent to eqs. (2.1) and (2.8), we shall obtain for the operators $L_{\mu}$ the form (1.2).

Moreover, the following more general remarkable property of eqs. (3.1) is valid. Let us take a set of linear differential equations of the form (3.1) as the equations for a (2+1)-dimensional field, assuming that $L_{\mu} = \alpha P_{\mu} - i\beta \epsilon_{\mu\nu\lambda}P^{\nu}J^{\lambda} + \epsilon m J_{\mu}$, and that the generators $J_{\mu}$ are the most general translation-invariant Lorentz group generators satisfying the commutation relations (2.2) (i.e. not fixing the choice of a representation of $SO(2,1)$ from the very beginning). In this case the parameters $\alpha$ and $\beta$ are arbitrary dimensionless constants. Then, multiplying these linear equations by the operators $mJ_{\mu}$, $P_{\mu}$ and $-i\epsilon_{\mu\nu\lambda}P^{\nu}J^{\lambda}$, we find that there are only two possible cases in which eqs. (3.1) are consistent. The first case is trivial and corresponds to the choice of a trivial representation for generators: $J_{\mu} = 0$, and, therefore, to a trivial system with $p_{\mu} = 0$. In the nontrivial case there is an arbitrariness in the normalization of the operator $L_{\mu}$ which can be fixed by putting $\beta = 1$. Then the system of vector equations (3.1) will be equivalent to the equations (2.1), (2.8) and

\[(J^2 + \alpha(\alpha - 1))\Psi = 0. \quad (4.1)\]

Eq. (4.1) is simply the condition of irreducibility, and one can check that the system of eqs. (2.1), (2.8) and (4.1) is consistent only in the case of the choice of either finite-dimensional nonunitary representations $\tilde{D}_j$, or the infinite-dimensional unitary representations $D^\pm_\alpha$.

Therefore, eq. (3.1) is the most general vector set of linear differential equations for a fractional (arbitrary) spin field in $2+1$ dimensions, whose consistency fixes the choice of unitary representations of the universal covering group of (2+1)-dimensional Lorentz group. Let us notice here that refs.
have used representations $D^\pm_\alpha$ for the description of fractional spin fields proceeding, in fact, simply from the first quantized theory of the relativistic point particle with torsion, and have not excluded for the purpose the choice of other unitary infinite-dimensional representations of the principal and supplementary continuous series of the $SL(2,R)$ group (see refs. [19] and a discussion in ref. [24]). After fixing the representation we have property (3.6) as a simple consequence of the irreducibility condition $J^2 = -\alpha(\alpha - 1)$, and, as a result, the number of independent equations for the description of arbitrary spin fields here is the same as in the spinor-like system of equations from the recent paper [18].

To conclude, let us list some related problems to be solved.

1. We have constructed the simplest action (3.9) leading to the proposed equations (3.1). The action is invariant with respect to the local transformations: $\delta \chi_\mu = R^\dagger_\mu \lambda$, $\delta \Psi = 0$, $\lambda$ being an arbitrary field, due to the identity (3.6). Therefore, the following question arises: is it possible to derive the fractional spin field action from a gauge symmetry principle based on the local transformations $\delta \chi_\mu = R^\dagger_\mu \lambda$? It is not clear whether more complicated choices of the field action (including the possibility of having additional auxiliary fields) will be equivalent to the action (3.9) or whether some new basic ingredient should be identified in order to understand the formulation of a free fractional spin system. It would be interesting to investigate the possible relationship between the proposed approach to the description of a fractional spin field and the approach based on the use of a $U(1)$ statistical gauge field [9]. Revealing such possible relationship seems very important because there are some reasons to expect that anyons can occur only in gauge theories, or in theories with a hidden local gauge invariance [4].

2. One can verify that the prescription of a simple substitution $P_\mu \rightarrow P_\mu - eA_\mu$ in equations (3.1) to describe the interaction with the simplest $U(1)$ gauge (electromagnetic) field is consistent only in the case of the spinor representation (2.4). Therefore, the introduction of the interaction of the fractional spin field with gauge fields remains an open problem in the present approach.

3. The next interesting problem consists in the construction of a corresponding singular classical model of a relativistic particle, whose quantization would lead to equations (3.1) as the equations for the physical states of the system. Note, that corresponding classical models leading in an analogous way to equations (2.1) and (2.8), and to ‘semionic’ equations were constructed
in refs. [12, 16], and [17], respectively.

4. The most interesting and intriguing problem within the approach considered in this paper is the problem of second quantization of the fractional spin field. The solution of this problem would answer the question of spin-statistics relation for such fields. In connection with this problem, we would like to make two remarks. First, we note, that the infinite-component nature of the fractional spin field within the present approach can be considered as some indication of a hidden nonlocal nature of the theory, and, therefore, can be treated in favour of an existence of a spin-statistics relation [7, 25].

Second, let us point out, that when performing the second quantization of a fractional spin field within Hamiltonian approach, an infinite number of Hamiltonian constraints must appear, which are to single out only one physical component (like Ψ₀ from eq. (2.21)) from the infinite component basic field Ψₙ(x). This infinite set of constraints should appropriately be taken into account.

At last, let us point out here that it seems interesting to investigate the system of (four) linear vector differential equations for (3+1)-dimensional field in an analogous way, starting from the generalization of eqs. (3.1) to the (3+1)-dimensional case.

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