HODGE LEVEL FOR WEIGHTED COMPLETE INTERSECTIONS

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Abstract. We give lower bounds for Hodge numbers of smooth well formed Fano weighted complete intersections. In particular, we compute their Hodge level, that is, the maximal distance between non-trivial Hodge numbers in the same row of the Hodge diamond. This allows us to classify varieties whose Hodge numbers are like that of a projective space, of a curve, or of a Calabi–Yau variety of low dimension.

1. Introduction

Let $X$ be a smooth projective $n$-dimensional variety defined over the field of complex numbers. Hodge numbers of $X$ are among its most basic invariants. Thus it is interesting to understand the situation when they satisfy some minimality conditions.

Definition 1.1. We say that $X$ is $\mathbb{Q}$-homologically minimal, if its Hodge numbers are as small as possible, that is, the same as those of the $n$-dimensional projective space.

An old problem is to characterize varieties that satisfy a stronger property (for which being $\mathbb{Q}$-homologically minimal is a necessary condition): namely, varieties whose cohomology ring over $\mathbb{Z}$ is isomorphic to that of $\mathbb{P}^n$. Fujita [Fu80] showed that a smooth Fano variety of dimension at most five with the latter property is actually isomorphic to $\mathbb{P}^n$. Hirzebruch and Kodaira [HK57] proved the same result under different additional conditions; see also the work [KO73] of Kobayashi and Ochiai for another result of this kind.

As for varieties that are just $\mathbb{Q}$-homologically minimal, a smooth odd-dimensional quadric gives an example of such a variety that is not isomorphic to a projective space. It was checked by Ewing and Moolgavkar [EM76] that there are no other $\mathbb{Q}$-homologically minimal varieties among smooth complete intersections (in usual projective spaces). One can see from the classification of smooth del Pezzo surfaces and Fano threefolds (see [IP99, §12.2]) that in dimension 2 the only variety like this is $\mathbb{P}^2$, while in dimension 3 there are four (families of) examples: the projective space $\mathbb{P}^3$, the quadric, the del Pezzo threefold $V_5$ of anticanonical degree 40, and a Fano variety $V_{22}$ of Fano index 1 and anticanonical degree 22; while the first three of these threefolds are unique in their deformation classes, the fourth family is 6-dimensional. Wilson [Wil80] proved that in dimension 4 every smooth $\mathbb{Q}$-homologically minimal Fano variety is isomorphic to $\mathbb{P}^4$, with a possible (unlikely) exception of varieties with certain explicitly described properties. There are further examples of $\mathbb{Q}$-homologically minimal fivefolds (see below).

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Another property that implies $\mathbb{Q}$-homological minimality of $X$ is the existence in the derived category $D^b(X)$ of coherent sheaves on $X$ of a full exceptional collection of the minimal possible length, that is, of length $n + 1$; the fact that $n + 1$ is indeed the minimal possible length follows from additivity of Hochschild homology under semi-orthogonal decompositions, see [Kuz09, Corollary 7.5], and Hochschild–Kostant–Rosenberg theorem. It is known that such exceptional collection exists for $\mathbb{P}^n$ itself (see [Be78]); for odd dimensional-quadrics (see [Ka88]); for the varieties $V_5$ (see [Or91]) and $V_{22}$ (see [Kuz96]). It was proved in [GKMS13] that the only Fano fourfold $X$ with a full exceptional collection of length 5 in $D^b(X)$ is $\mathbb{P}^4$. There are three more examples of Fano fivefolds that have full exceptional collections of length 6 in derived categories of coherent sheaves, see [Kuz06, §6] (cf. also [Kuz18a, §1.4]); in particular, these varieties are $\mathbb{Q}$-homologically minimal. A conjecture of Bondal and Orlov predicts that a Fano variety $X$ of even dimension $n$ with a full exceptional collection in $D^b(X)$ of length $n + 1$ is isomorphic to a projective space.

It is known that if $D^b(X)$ contains a full exceptional collection, then $X$ is diagonal, due to additivity of Hochschild homology as above. Examples of diagonal varieties include even-dimensional quadrics and even-dimensional intersections of two quadrics. For varieties of the latter two classes, there is indeed a full exceptional collection in the derived category of coherent sheaves, see [Ka88] and [Kuz08, Corollary 5.7]. The following natural question is attributed to Bondal and Orlov: is it true that for every diagonal Fano variety $X$ there is a full exceptional collection in $D^b(X)$? It appeared that the answer to this question is negative: a counterexample was recently constructed by Kuznetsov [Kuz18b]. However, there is Lunts’ refinement of this question requiring, in addition, that $X$ is Tate, see [EL16, Conjecture 1.2] (note that the requirement for $X$ to be Fano is not necessary in this case). An alternative refinement is a requirement for the Grothendieck group $K_0(X)$ of the variety $X$ to be a free group. Besides that, there is another conjecture of Orlov predicting that every variety with a full exceptional collection is rational.

There are further classes of varieties that may be considered relatively simple from the point of view of derived categories of coherent sheaves. One of the interesting situations is when $D^b(X)$ contains an exceptional collection such that its semi-orthogonal complement in $D^b(X)$ is a category $\mathcal{A}_X$ with certain nice properties. For instance, $\mathcal{A}_X$ may itself be equivalent to the derived category of coherent sheaves on a variety of small dimension, or on a Calabi–Yau variety. One can easily see that if $\mathcal{A}_X$ is equivalent to the derived category of coherent sheaves on a curve, then again by additivity of Hochschild homology and Hochschild–Kostant–Rosenberg theorem for $X$ we obtain strong restrictions on the Hodge numbers of $X$. Namely, under an additional assumption that $h^{p,q}(X) = 0$ unless $p = q$ or $p + q = n$, the variety $X$ is of curve type in the following sense.

**Definition 1.4.** We say that $X$ is of curve type, if $n$ is odd, and one has $h^{p,q}(X) = 0$ unless $p = q$ or $\{p, q\} = \{\frac{n-1}{2}, \frac{n+1}{2}\}$.

Similarly ,one can easily see that if $\mathcal{A}_X$ is equivalent to the derived category of coherent sheaves on a Calabi–Yau variety of dimension $m$, then $X$ is of $m$-Calabi–Yau type in the following sense.
Definition 1.5. Given a positive integer $m$, we say that $X$ is of $m$-Calabi–Yau type, if $n$ has the same parity as $m$, one has $h^{n/2, n/2}(X) = 1$, and $h^{p,q}(X) = 0$ for $p < \frac{n-m}{2}$ or $p = \frac{n-m}{2}$, $p + q \neq n$. We say that $X$ is of K3 type, if it is of 2-Calabi–Yau type.

In other words, $X$ is of $m$-Calabi–Yau type if for the Hochschild homology of $D^b(X)$ one has $HH_s(D^b(X)) = 0$ for $s < -m$ and $s > m$, and
\[
\dim HH_{-m}(D^b(X)) = \dim HH_m(D^b(X)) = 1.
\]

There is a notion of $m$-Calabi–Yau category defined in terms of the Serre functor, see, for example, [Kuz15b, Definition 1.1]. Derived categories of $m$-dimensional non-commutative Calabi–Yau varieties (see [Kuz15b, §4.4]) are examples of categories of this type. Note that if $m$-Calabi–Yau category $\mathcal{T}$ is a derived category of an $m$-dimensional variety, one has $HH_s(\mathcal{T}) = 0$ for $s < -m$ and $s > m$. However such vanishing is not expected to hold for an arbitrary $m$-Calabi–Yau category.

Question 1.6. Does the vanishing hold in the geometric case, that is, when $\mathcal{T} = \mathcal{A}_X$ (cf. Example 1.10 below)?

Examples of varieties of curve type are given by Fano threefolds; for a Fano threefold $X$ a category $\mathcal{A}_X$ is indeed of curve type (that is, its only possibly non-vanishing Hochschild homology groups for $\mathcal{A}_X$ are of degrees $-1$, 0, and 1), although for some Fano threefolds this category is not equivalent to derived category of coherent sheaves on an actual curve. An example of a variety of curve type whose derived category of coherent sheaves contains the derived category of coherent sheaves on a curve as a semi-orthogonal complement to an exceptional collection is given by an odd-dimensional intersection of two quadrics, see [Kuz08, Corollary 5.7]. A cubic fourfold in $\mathbb{P}^5$ is a variety of K3 type; its derived category of coherent sheaves contains a derived category of a non-commutative K3 surface as a semi-orthogonal complement to an exceptional collection by [Kuz10]. Examples of varieties of 3-Calabi–Yau type are given by a smooth five-dimensional quartic hypersurface in the weighted projective space $\mathbb{P}(1^6,2)$, see [3] below for definitions and references concerning weighted projective spaces and subvarieties therein; a smooth five-dimensional complete intersection of a quadric and a cubic in $\mathbb{P}^7$; and a smooth seven-dimensional cubic in $\mathbb{P}^8$ (see [IM15, §3.1]). The derived categories of coherent sheaves on these varieties contain categories of 3-Calabi–Yau type as semi-orthogonal complements to exceptional collections, see [IM15, Proposition 4.6].

In this paper we classify $\mathbb{Q}$-homologically minimal varieties, diagonal varieties, varieties of curve type, varieties of K3 type, and varieties of 3-Calabi–Yau type among smooth Fano weighted complete intersections, see [2] for precise definitions. The reason to consider this class of varieties is as follows. One of the main ways to construct examples of Fano varieties is to describe them as complete intersections in already known ones, for instance, in Grassmannians, or in toric varieties whose properties are somewhat close to those of the usual projective space. Computing Hodge numbers for varieties of the former type is an interesting problem; some partial results in this direction were obtained in [FM18]. On the other hand, one may argue that toric varieties whose properties are most close to those of a projective space are weighted projective spaces. Fortunately, in this case we also have a nice and efficient way to compute Hodge numbers of a complete intersection.

Let $X$ be a smooth complete intersection of dimension $n$ in a $\mathbb{Q}$-factorial toric variety $Y$. From the Lefschetz-type theorem (see [Ma99, Proposition 1.4]) it follows
that $h^{p,q}(X) = h^{p,q}(Y)$ for $p + q \neq n$ and $h^{p,q}(X) \geq h^{p,q}(Y)$ for $p + q = n$. Moreover, if $Y$ is a weighted projective space, then $h^{p,p}(X) = 1$ for $p \neq \frac{n}{2}$. This means that the only "non-trivial" Hodge numbers of $X$ are the middle ones $h^{p,n-p}(X)$. Thus to check that the variety $X$ is $\mathbb{Q}$-homologically minimal, diagonal, of curve type, or of $m$-Calabi–Yau type, it is enough to check the relevant conditions for the middle row of the Hodge diamond.

In this paper we provide lower bounds for some middle Hodge numbers of smooth Fano weighted complete intersections. For this we use a well known method to compute Hodge numbers of weighted complete intersections as dimensions of graded components of some bigraded ring, see Theorem 2.8 below. As a corollary we describe all $\mathbb{Q}$-homologically minimal and diagonal smooth Fano weighted complete intersections, as well as ones of curve type and of 2- and 3-Calabi–Yau types. To make a long story short, there are no new varieties of these five types except for examples mentioned in the above discussion.

Our first result is a complete classification of smooth Fano weighted complete intersections that are $\mathbb{Q}$-homologically minimal, diagonal, or of curve type. We refer the reader to [Do82] and [IF00], or to §2 below, for relevant definitions. We will exclude the case of the projective space (that can be thought of as a complete intersection of codimension 0 in itself) from our considerations to simplify notation.

**Theorem 1.7.** Let $X$ be a smooth well formed Fano weighted complete intersection of dimension $n$ which is not an intersection with a linear cone. The following assertions hold.

(i) The variety $X$ is $\mathbb{Q}$-homologically minimal if and only if $X$ is an odd-dimensional quadric in $\mathbb{P}^{n+1}$.

(ii) The variety $X$ is diagonal if and only if either $n = 2$ (cf. Table 4), or $X$ is a quadric in $\mathbb{P}^{n+1}$, or $X$ is an even-dimensional intersection of two quadrics in $\mathbb{P}^{n+2}$.

(iii) The variety $X$ is of curve type if and only if either $n = 1$, or $n = 3$ (cf. Table 2), or $X$ is an odd-dimensional complete intersection of $k \leq 3$ quadrics in $\mathbb{P}^{n+k}$, or $X$ is a five-dimensional cubic in $\mathbb{P}^6$.

Our second result concerns smooth Fano weighted complete intersections that are of $m$-Calabi–Yau type for some $m$. For a weighted complete intersection $X$ of multidegree $(d_1, \ldots, d_k)$ in $\mathbb{P}(a_0, \ldots, a_N)$ we put

$$i_X = \sum a_i - \sum d_j.$$

**Theorem 1.8.** Let $X$ be a smooth well formed Fano weighted complete intersection of multidegree $(d_1, \ldots, d_k)$, $d_1 \leq \ldots \leq d_k$, which is not an intersection with a linear cone. Put $n = \dim X$, and let $m$ be a positive integer. Then $X$ is of $m$-Calabi–Yau type if and only if the following conditions hold:

- one has either $k = 1$ or $d_{k-1} < d_k$;
- the number $i_X$ is divisible by $d_k$;
- one has $m = n - \frac{2i_X}{d_k}$.

**Proposition 1.9.** Let $X$ be a smooth well formed Fano weighted complete intersection which is not an intersection with a linear cone. The following assertions hold.

(i) The variety $X$ is of K3 type if and only if $X$ is a four-dimensional cubic in $\mathbb{P}^5$.

(ii) The variety $X$ is of 3-Calabi–Yau type if and only if $X$ is either a five-dimensional quartic hypersurface in $\mathbb{P}(1^6, 2)$, or a five-dimensional complete intersection of a quadric and a cubic in $\mathbb{P}^7$, or a seven-dimensional cubic in $\mathbb{P}^8$. 

Using the method of the proof of Theorem \ref{thm:calabi-yau} one can also classify smooth Fano complete intersections of $m$-Calabi–Yau type for any given $m$.

The following example related to Theorem \ref{thm:calabi-yau} is well known to experts.

**Example 1.10** (cf. Question \ref{quest:calabi-yau}). Let $X$ be a smooth cubic in $\mathbb{P}^{n+1}$. Then $X$ is of $m$-Calabi–Yau type if and only if $n = 3m - 2$. By \cite[Corollary 4.2]{Kuz04} (or by \cite[Corollary 4.3]{Kuz04}, see also \cite[Corollary 4.2]{Kuz15b}) the derived category of coherent sheaves on a cubic of dimension $3m - 2$ contains an $m$-Calabi–Yau category as a semi-orthogonal complement to an exceptional collection. By \cite[Corollary 4.2]{Kuz04} the same assertion holds for other smooth hypersurfaces in weighted projective spaces satisfying the conditions of Theorem \ref{thm:calabi-yau} e.g., for $(2m-1)$-dimensional quartic hypersurfaces in $\mathbb{P}(1^{2m}, 2)$.

It would be interesting to know if this is also the case for complete intersections of larger codimension that satisfy the restrictions provided by Theorem \ref{thm:calabi-yau} e.g., for $(3m-4)$-dimensional complete intersections of a quadric and a cubic in $\mathbb{P}^{3m-2}$.

**Remark 1.11.** Fano varieties of 3-Calabi–Yau type were considered (and were called just varieties of Calabi–Yau type) in \cite{IM15} under a certain additional conditions. Namely, the authors of \cite{IM15} required that for an $n$-dimensional Fano threefold $X$ of 3-Calabi–Yau type one has $h^{p,0}(X) = 0$ for all $p$ and for any generator $\omega \in H^{n+2, n-1}(X)$ the contraction map

$$H^1(X, TX) \xrightarrow{\omega} H^n(X, \Omega_X^{n+1})$$

is an isomorphism, see condition (2) in \cite[Definition 2.1]{IM15} (note that there is a misprint in the index of the target cohomology group in \cite[Definition 2.1]{IM15}). Proposition \ref{prop:calabi-yau}(ii) shows that the examples found in \cite[§3.1]{IM15} give all possible varieties of this type among smooth Fano weighted complete intersections.

Keeping in mind the known results on the derived categories of coherent sheaves on varieties and exceptional collections therein (see \cite{KO05, Ka88, Kuz08, Kuz15b}, Corollary 4.2), we conclude that Theorem \ref{thm:calabi-yau} and Proposition \ref{prop:calabi-yau} imply the following assertion.

**Corollary 1.12.** Let $X$ be a smooth well formed Fano weighted complete intersection of dimension $n$ which is not an intersection with a linear cone. Then the following assertions hold.

(i) There is a full exceptional collection in the derived category $D^b(X)$ if and only if either $n = 2$, or $X$ is a quadric in $\mathbb{P}^{n+1}$, or $X$ is an even-dimensional intersection of two quadrics in $\mathbb{P}^{n+2}$.

(ii) The derived category $D^b(X)$ is of curve type if and only if $X$ is either a threefold, or an odd-dimensional complete intersection of $k \leq 3$ quadrics in $\mathbb{P}^{n+k}$, or a five-dimensional cubic in $\mathbb{P}^6$.

(iii) The derived category $D^b(X)$ contains a derived category of a non-commutative K3 surface as a semi-orthogonal complement to an exceptional collection if and only if $X$ is a four-dimensional cubic in $\mathbb{P}^5$.

**Remark 1.13.** If $X$ is either a five-dimensional quartic hypersurface in $\mathbb{P}(1^5, 2)$, or a five-dimensional complete intersection of a quadric and a cubic in $\mathbb{P}^7$, or a seven-dimensional cubic in $\mathbb{P}^8$, then the derived category $D^b(X)$ contains a subcategory of 3-Calabi–Yau type as a semi-orthogonal complement to an exceptional collection,
see [IM15, Proposition 4.6(3)]. It would be interesting to find out if the converse is also true.

The notions we have introduced above are related to the following classical notion.

**Definition 1.14** (cf. [R72, §1], [C80, §2a]). Let $X$ be a smooth projective variety of dimension $n$. Put

$$h(X) = \max\{q - p \mid h^{p,q}(X) \neq 0\}.$$  

The number $h(X)$ will be called the Hodge level of $X$.

Note that $h(X)$ is always non-negative. In the terminology of [R72], the number $h(X)$ is the maximal Hodge level of the Hodge structures on the cohomology groups $H^r(X,\mathbb{Z})$ for $0 \leq r \leq 2\dim X$. If $X$ is an $n$-dimensional smooth weighted complete intersection, then $h(X)$ equals the Hodge level of the Hodge structure on $H^n(X,\mathbb{Z})$. It is obvious that $h(X) = 0$ if and only if $X$ is diagonal; in particular, this holds for any smooth quadric. One has $h(X) \leq 1$ if $X$ is of curve type. If $X$ is of $m$-Calabi–Yau type (for instance, if $X$ is a Calabi–Yau variety of dimension $m$), then $h(X) = m$. Smooth complete intersections $X$ in a usual projective space such that $h(X) \leq 1$ were classified in [R72, §2].

Our next result describes some general properties of Hodge level for weighted complete intersections. For a weighted complete intersection $X$ of multidegree $(d_1,\ldots,d_k)$ in $\mathbb{P}(a_0,\ldots,a_N)$, where $d_1 \leq \ldots \leq d_k$, we put

$$p_X = \left\lceil \frac{i_X}{d_k} \right\rceil.$$  

**Proposition 1.15.** Let $X$ be a smooth well formed Fano weighted complete intersection of dimension $n$ which is not an intersection with a linear cone. If $X$ is an odd-dimensional quadric, then $h(X) = 0$. Otherwise $h(X) = n - 2p_X$.

Note that if $X$ is a Fano variety of dimension $n$, then $h^{0,n}(X) = 0$ by Kodaira vanishing, so that $h(X) \leq n - 2$. Proposition [1.15] implies the following.

**Corollary 1.16.** Let $X$ be a smooth well formed Fano weighted complete intersection of dimension $n \geq 2$ which is not an intersection with a linear cone. The following assertions hold.

(i) Suppose that either $i_X \leq 2$, or $i_X \leq 3$ and $X$ is not a complete intersection of quadrics in a projective space, or $i_X \leq 4$ and $X$ is not a complete intersection of quadrics and cubics in a projective space. Then $h(X) = n - 2$.

(ii) Suppose that $X$ is not a complete intersection of quadrics in a projective space. Then $h(X) \geq \frac{n-4}{3}$.

An immediate consequence of Corollary [1.16](ii) is the following.

**Corollary 1.17.** For every number $h$ the dimension of smooth well formed Fano weighted complete intersections $Y$ such that $Y$ is not an intersection with a linear cone, not a complete intersections of quadrics in a projective space, and $h(Y) < h$, is bounded.

**Remark 1.18.** Our results are motivated by applications to the derived categories of coherent sheaves on smooth varieties. However, if we allow singularities, then the cohomological dimension of the derived category of coherent sheaves on a variety becomes infinite, and it’s hard to study its semiorthogonal components of K3 type. On the other hand, if a
weighted complete intersection \(X\) is quasi-smooth (see Definition 5.1), then one can consider the derived category of a smooth stack \(\mathcal{X}\) with support at \(X\). Another approach is to consider the subcategory \(\text{Perf}(X) \subset D^b(X)\) of perfect complexes. Anyway, cohomology groups of quasi-smooth complete intersections admit pure Hodge structures (see [BC94, §11]), so one can ask which of them are of curve, K3, or \(m\)-Calabi–Yau type. Moreover, the approach to check this used in our paper can be applied to quasi-smooth weighted complete intersections.

However it turns out that the quasi-smoothness restriction is too weak to get a reasonable classification. It is easy to see that any (well formed) weighted projective space is a quasi-smooth \(\mathbb{Q}\)-homologically minimal variety (which can be considered as a weighted complete intersection of codimension 0 in itself). Furthermore, calculations of V. Alexeev show that there are 124 families of quasi-smooth well formed Fano hypersurfaces of K3 type in five-dimensional weighted projective spaces with weights up to 50; there are 122 families of quasi-smooth well formed Fano hypersurfaces of K3 type in seven-dimensional weighted projective spaces with weights up to 30; there are 105 families of quasi-smooth well formed Fano hypersurfaces of K3 type in nine-dimensional weighted projective spaces with weights up to 20, etc.

A nice observation due to A. Kuznetsov is that most of hypersurfaces from Alexeev’s list have birational transformations to varieties related to K3 surfaces, so categories of K3 surfaces naturally appear in their derived categories.

The paper is organized as follows. In §2 we give the relevant definitions, recall the method to compute Hodge numbers of weighted complete intersections and prove some auxiliary results. In §3 we obtain the bounds for Hodge numbers of Fano weighted complete intersections. In §4 we prove the main results of the paper, namely, Theorems 1.7 and 1.8, Propositions 1.9 and 1.15, and Corollary 1.16. In §5 we briefly discuss the quasi-smooth case. In §6 we discuss some open questions. In the appendix we provide the well known lists of two- and three-dimensional smooth well formed Fano weighted complete intersections.

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2. Preliminaries

We recall here some basic properties of weighted complete intersections. We refer the reader to [Do82] and [IF00] for more details. Put

\[
\mathbb{P} = \mathbb{P}(a_0, \ldots, a_N) = \text{Proj} \mathbb{C}[x_0, \ldots, x_N],
\]

where the weight of \(x_i\) equals \(a_i\). Without loss of generality we assume that \(a_0 \leq \ldots \leq a_N\). We will use the abbreviation

\[
(a^0_{r_0}, \ldots, a^m_{r_m}) = (a_0, \ldots, a_0, \ldots, a_m, \ldots, a_m),
\]

where \(r_0, \ldots, r_m\) will be allowed to be any positive integers. If some of \(r_i\) is equal to 1 we drop it for simplicity.

The weighted projective space \(\mathbb{P}\) is said to be well formed if the greatest common divisor of any \(N-1\) of the weights \(a_i\) is 1. Every weighted projective space is isomorphic to a
well formed one, see [Do82, 1.3.1]. If $\mathbb{P}$ is well formed, then the singular locus of $\mathbb{P}$ is a union of strata
\[
\{(x_0 : \ldots : x_N) \mid x_i = 0 \text{ for all } i \notin J\}
\]
for all subsets $J \subset \{0, \ldots, n\}$ such that the greatest common divisor of the weights $a_i$ for $i \in J$ is greater than 1, see [IF00, 5.15]. A subvariety $X \subset \mathbb{P}$ is said to be well formed if $\mathbb{P}$ is well formed and
\[
\text{codim}_X (X \cap \text{Sing} \mathbb{P}) \geq 2.
\]

We say that a subvariety $X \subset \mathbb{P}$ of codimension $k \geq 1$ is a weighted complete intersection of multidegree $(d_1, \ldots, d_k)$ if its weighted homogeneous ideal in $\mathbb{C}[x_0, \ldots, x_N]$ is generated by a regular sequence of $k$ homogeneous elements of degrees $d_1, \ldots, d_k$. The above condition is equivalent to the requirement that the codimension of (every irreducible component of) the variety $X$ equals $k$. This follows from a similar equivalence for the variety $C_X \subset \mathbb{A}^{N+1}$ defined as the closure of the preimage of $X$ under the natural projection $\mathbb{A}^{N+1} \setminus \{0\} \to \mathbb{P}$ and considered as an intersection of $k$ hypersurfaces in $\mathbb{A}^{N+1}$. The latter equivalence can be deduced, for instance, from [Ha77, Theorem II.8.21A(c)] or from [Ha77, Exercise II.8.4].

Note that $\mathbb{P}$ can be thought of as a complete intersection of codimension 0 in itself (which gives us a nice smooth Fano variety if $\mathbb{P} \cong \mathbb{P}^N$), but we do not consider this case for simplicity. The weighted complete intersection $X$ is said to be an intersection with a linear cone if one has $d_j = a_i$ for some $i$ and $j$. In this case one can exclude the $i$-th weighted homogeneous coordinate and think about $X$ as a weighted complete intersection in a weighted projective space of lower dimension, provided that $X$ is general enough, cf. [PSh16, Remark 5.2].

We will be interested in smooth well formed weighted complete intersections. Note that by [PSh16, Corollary 2.14] such varieties are automatically quasi-smooth (cf. Definition 5.1). This allows us to use many auxiliary results that were proved for quasi-smooth weighted complete intersections.

For a smooth well formed weighted complete intersection, one has the following relation between the weights $a_i$ and the degrees $d_j$.

**Lemma 2.1** (see [PSh16, Lemma 2.15], cf. [IF00, 6.12], [PST17, Proposition 3.1]). Let $X \subset \mathbb{P}$ be a smooth well formed weighted complete intersection of multidegree $(d_1, \ldots, d_k)$. Then for every $1 \leq t \leq k$ and every choice of $t$ weights $a_{i_1}, \ldots, a_{i_t}, i_1 < \ldots < i_t$, such that their greatest common divisor $\delta$ is greater than 1 there exist $t$ degrees $d_{s_1}, \ldots, d_{s_t}, s_1 < \ldots < s_t$, such that their greatest common divisor is divisible by $\delta$.

Consider the sheaf $\mathcal{O}_{\mathbb{P}}(1)$, see [Do82, 1.4.1]. Recall that $\mathcal{O}_{\mathbb{P}}(1)$ is usually not invertible. However, if $\mathbb{P}$ is well formed, the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to $\mathbb{P} \setminus \text{Sing} \mathbb{P}$ is an invertible sheaf, see [Do82, 1.5.5]. Let $X$ be a weighted complete intersection in $\mathbb{P}$. Denote by $\mathcal{O}_X(1)$ the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to $X$. If $X$ is smooth and well formed, then $X$ is contained in the smooth locus of $\mathbb{P}$, see for instance [PSh16, Proposition 2.11]. Hence the sheaf $\mathcal{O}_X(1)$ is a line bundle on $X$ in this case.

**Lemma 2.2** ([Ok16, Remark 4.2], [PST17, Proposition 2.3]). Let $X$ be a smooth well formed weighted complete intersection of dimension at least three in $\mathbb{P}$. Then the class of the line bundle $\mathcal{O}_{\mathbb{P}}(1)|_X$ is not divisible in $\text{Pic}(X)$. 

For a weighted complete intersection $X$ of multidegree $(d_1, \ldots, d_k)$ in $\mathbb{P}$ we denote

$$i_X = \sum a_i - \sum d_j.$$ 

It is easy to describe the canonical class of a weighted complete intersection.

**Theorem 2.3** (see [Do82, Theorem 3.3.4], [IF00, 6.14]). Let $X$ be a smooth well formed weighted complete intersection of multidegree $(d_1, \ldots, d_k)$ in $\mathbb{P}$. Then

$$\omega_X \simeq \mathcal{O}_X(-i_X).$$

In this paper we will be mostly concerned with Fano weighted complete intersections. There are various results bounding the relevant parameters. Recall from [PST17, Corollary 5.3(i)] that if there exists a smooth well formed Fano weighted complete intersection in $\mathbb{P}$, then $a_0 = 1$. In this case we will define the number $0 \leq l_p \leq N$ by the conditions

$$(2.1) \quad 1 = a_0 = \ldots = a_{l_p} < a_{l_p+1}$$

if $a_N > 1$, and put $l_p = N$ otherwise; thus $l_p + 1$ is the number of weights among $a_i$’s that are equal to 1.

**Theorem 2.4.** Let $X$ be a smooth well formed Fano weighted complete intersection of codimension $k$ and dimension $n = N - k$ in $\mathbb{P}$ which is not an intersection with a linear cone. Then the following inequalities hold:

(i) $a_N \leq N$;
(ii) $k \leq n$;
(iii) $l_p \geq k$.

**Proof.** Assertion (i) is proved in [PSh16, Theorem 1.1]. Assertion (ii) is [CCC11, Theorem 1.3]. Assertion (iii) follows from [PST17, Corollary 5.3(i)]. \qed

Using the bounds provided by Theorem 2.4 one can easily obtain the well known lists of all smooth Fano weighted complete intersections of small dimensions.

**Lemma 2.5.** Let $X$ be a smooth well formed Fano weighted complete intersection of dimension $n$ in $\mathbb{P}$ which is not an intersection with a linear cone. If $n = 1$, then $X$ is a conic in $\mathbb{P}^2$. If $n = 2$, then $X$ is one of the four types of del Pezzo surfaces listed in Table 1. If $n = 3$, then $X$ is one of the nine types of Fano threefolds listed in Table 2.

**Remark 2.6.** By Lemma 2.2 and Theorem 2.3 the number $i_X$ equals the Fano index of $X$ provided that $X$ is a smooth Fano variety of dimension $n \geq 3$. By Lemma 2.5 this is also the case if $X$ is a del Pezzo surface, i.e. a smooth Fano variety of dimension $n = 2$. However, if $X$ is a conic in $\mathbb{P}^2$, then $i_X = 1$, while the Fano index of $X$ equals 2.

It is possible to bound $i_X$ in terms of dimension of a smooth Fano weighted complete intersection $X$.

**Theorem 2.7.** Let $X$ be a smooth well formed Fano weighted complete intersection of multidegree $(d_1, \ldots, d_k)$ and dimension $n = N - k \geq 2$ in $\mathbb{P}$ which is not an intersection with a linear cone. Then

(i) $X$ is not isomorphic to $\mathbb{P}^n$;
(ii) one has $i_X \leq n$. 

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Proof. By Lemma 2.5 we may assume that \( n \geq 3 \), so that by Remark 2.6 the Fano index of \( X \) equals \( i_X \). Suppose that \( X \cong \mathbb{P}^n \), so that \( i_X = n + 1 \).

Recall that \( a_0 \leq \ldots \leq a_N \). We may also assume that \( d_1 \leq \ldots \leq d_k \). Then one has \( d_{k-i} > a_{N-i} \) for all \( 0 \leq i \leq k-1 \), see [PSh16, Lemma 3.1(i),(ii)]. On the other hand, for the number \( l_p \) defined by (2.1) one has
\[
l_p \geq i_X = n + 1 = N - k + 1
\]
by [PST17, Corollary 5.11]. This implies
\[
\prod_{j=1}^{k} d_j > \prod_{i=N-k+1}^{N} a_i = \prod_{i=0}^{N} a_i.
\]

Now let \( H \) be the class of the line bundle \( \mathcal{O}_X(1) \) in \( \text{Pic}(X) \). Then \( H \) is not divisible in \( \text{Pic}(X) \) by Lemma 2.2, which means that \( H \) is the class of a hyperplane on \( \mathbb{P}^n \). Hence, one has
\[
1 = H^n = \frac{\prod_{j=1}^{k} d_j}{\prod_{i=0}^{N} a_i} > 1,
\]
which gives a contradiction. This proves assertion (i).

To deduce assertion (ii) from (i), recall that the Fano index of an arbitrary smooth \( n \)-dimensional Fano variety \( Y \) is bounded by \( n \) provided that \( Y \) is not isomorphic to \( \mathbb{P}^n \), see for instance [IP99, Corollary 3.1.15]. □

Put
\[
S = \mathbb{C}[x_0, \ldots, x_N, w_1, \ldots, w_k].
\]
Let \( f_1, \ldots, f_k \) be polynomials in \( \mathbb{C}[x_0, \ldots, x_N] \) of weighted degrees \( d_1, \ldots, d_k \) that generate the weighted homogeneous ideal of \( X \). Let
\[
F = F(f_1, \ldots, f_k) = w_1 f_1 + \ldots + w_k f_k \in S.
\]
Denote by \( J = J(F) \) the ideal in \( S \) generated by
\[
\frac{\partial F}{\partial w_1}, \ldots, \frac{\partial F}{\partial w_k}, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_N}.
\]
Put
\[
R = R(f_1, \ldots, f_k) = S/J.
\]

The algebra \( S \) is bigraded by \( \text{deg}(x_s) = (0, a_s) \) and \( \text{deg}(w_j) = (1, -d_j) \), so that \( F \) is a bihomogeneous polynomial of bidegree \((1,0)\). Thus, the bigrading descends to the ring \( R \).

Suppose that \( X \) is smooth, and put \( n = N - k = \dim X \). Let \( h^{n-q,q}_{pr}(X) \) be primitive middle Hodge numbers of \( X \), that is,
\[
h_{pr}^{p,q}(X) = h_{pr}^{p,q}(X)
\]
for \( p \neq q \) and
\[
h_{pr}^{p,q}(X) = h^{p,q}(X) - 1
\]
otherwise. The following theorem will be our main tool to compute and estimate Hodge numbers of weighted complete intersections.

**Theorem 2.8** (see [Di95], [Gr69], [Na97, Proposition 2.16], [Ma99, Theorem 3.6]). One has
\[
h_{pr}^{q,n-q}(X) = \dim R_{q,-i_X}.
\]
Recall that the number \( l_\mathbb{P} \) is defined by (2.1). In §3 we will need the following notion.

**Definition 2.9.** We say that a weighted complete intersection \( X \subset \mathbb{P} \) of codimension \( k \) is of Fermat type if \( X \) is given by equations of the form

\[
\begin{align*}
\alpha_{1,0}x_0^{d_1} + \ldots + \alpha_{1,l_p}x_{l_p}^{d_1} + \hat{f}_1 &= 0, \\
\quad \ldots \quad \ldots \\
\alpha_{k,0}x_0^{d_k} + \ldots + \alpha_{k,l_p}x_{l_p}^{d_k} + \hat{f}_k &= 0,
\end{align*}
\]

(2.3)

where \( \hat{f}_j, 1 \leq j \leq k \), are weighted homogeneous polynomials of degree \( d_j \) that depend only on variables \( x_{l_p+1}, \ldots, x_N \).

Note that in the notation of Definition 2.9 one has \( l_\mathbb{P} \geq k \) by Theorem 2.4(iii).

**Lemma 2.10.** Suppose that there exists a smooth well formed weighted complete intersection \( X \subset \mathbb{P} \). Then there exists a smooth well formed weighted complete intersection of Fermat type of the same multidegree in \( \mathbb{P} \).

**Proof.** Let \( X \) be a smooth well formed weighted complete intersection in \( \mathbb{P} \) given by equations \( f_1 = \ldots = f_k = 0 \), where the weighted degree of \( f_j \) equals \( d_j \). Write \( f_j = g_j + \hat{f}_j \), where \( \hat{f}_j \) is a weighted homogeneous polynomial of degree \( d_j \) that depends only on variables \( x_{l_p+1}, \ldots, x_N \), while every monomial of \( g_j \) is divisible by some of the variables \( x_0, \ldots, x_{l_p} \). For every \( 1 \leq j \leq k \) define a hypersurface \( D_j \) in \( \mathbb{P} \) by equation

\[
\alpha_{j,0}x_0^{d_j} + \ldots + \alpha_{j,l_p}x_{l_p}^{d_j} + \hat{f}_j = 0,
\]

(2.4)

where \( \alpha_{j,i} \) are general coefficients.

Let \( X' \) be the intersection of \( D_1, \ldots, D_k \), so that \( X' \) is given by equations (2.3). We claim that \( X' \) is a weighted complete intersection (of multidegree \( (d_1, \ldots, d_k) \)), so by construction it of Fermat type.

Let \( \Pi \subset \mathbb{P} \) be the stratum given by equations \( x_{l_p+1} = \ldots = x_N = 0 \). Put

\[
X'_j = D_1 \cap \ldots \cap D_j
\]

for \( 0 \leq j \leq k \), so that \( X'_0 = \mathbb{P} \) and \( X'_k = X' \). For every \( 0 \leq t \leq k \) define the variety \( X'_t \) as the intersection of \( X'_j \) with the stratum \( \Pi \). One can easily see that every Weil divisor on \( \mathbb{P} \) and hence on \( X'_j \) is \( \mathbb{Q} \)-Cartier. Hence the codimension of \( X'_j \) in \( X'_{j-1} \) equals the codimension of \( X''_j \) in \( X''_{j-1} \). Since \( \Pi \cong \mathbb{P}^{l_\mathbb{P}} \) and the coefficients \( \alpha_{j,i} \) in (2.4) are general, the latter codimension equals 1. This means that the codimension of \( X' = X'_k \) in \( \mathbb{P} \) equals \( k \), so that \( X' \) is a weighted complete intersection.

To deduce the assertion of the lemma, check that \( X' \) is well formed and smooth. This follows from the claim that \( X' \) is disjoint from the singular locus \( \Sigma \) of \( \mathbb{P} \). Indeed, let \( \Lambda \subset \mathbb{P} \) be the stratum given by equations \( x_0 = \ldots = x_{l_p} = 0 \). Since every monomial of \( g_j \) vanishes at \( \Lambda \), we see that \( X \cap \Lambda = X' \cap \Lambda \). On the other hand, \( \Sigma \) is contained in \( \Lambda \), see [DF00, 5.15]. Since \( X \) is well formed and smooth, it must be disjoint from \( \Sigma \) by [Di86, Proposition 8], which means that \( X' \) is disjoint from \( \Sigma \) as well. In particular, this implies that \( X' \) is well formed. The rest thing we need to do is to prove that \( X' \) does not have singularities outside of \( \Sigma \).

Let \( D_j \) be the linear system of all hypersurfaces given by equations of the form (2.4), and let \( D'_j \) be its restriction to \( X'_{j-1} \). We claim that if \( D'_j \) has a base point on \( X'_{j-1} \), then it has a base point in \( \Sigma \cap X'_{j-1} \). Indeed, suppose that \( P \in X'_{j-1} \) is a base point of \( D'_j \).
Then it is also a base point of $D_j$. Obviously, the base locus $Bs D_j$ is contained in $\Lambda$. The stratum $\Lambda$ is itself a (possibly not well formed) weighted projective space

$$\hat{P} \cong \mathbb{P}(a_{l+1}, \ldots, a_N)$$

with weighted homogeneous coordinates $x_{l+1}, \ldots, x_N$. The restriction of $D_j$ to $\Lambda$ is the complete linear system $\hat{D}_j$ of hypersurfaces of weighted degree $d_j$ in $\hat{P}$, so $Bs D_j = Bs \hat{D}_j$. Using the action of the automorphism group of $\hat{P}$, we see that $Bs \hat{D}_j$ is a (possibly empty) union $\hat{\Lambda}$ of strata given by the vanishing of some coordinates among $x_{l+1}, \ldots, x_N$. Thus $\hat{\Lambda}$, considered as a subset of $\mathbb{P}$, contains a singular point of $\mathbb{P}$, since $\hat{\Lambda} \ni P$ is nonempty.

Applying Bertini’s theorem we see that singularities of a general member of $D_j'$ are contained in $Bs D_j'$. The claim above shows that if the general member is singular, it has singularity on $\Sigma$. Now the assertion of the lemma follows from the fact that $X' = X_k'$ is disjoint from $\Sigma$. □

The following simple computation concerning complete intersections in usual projective spaces will be used in the proof of Lemma 3.2.

**Lemma 2.11.** Suppose that $X \subset \mathbb{P}^l = \text{Proj} \mathbb{C}[x_0, \ldots, x_l]$ is a complete intersection of hypersurfaces given by homogeneous polynomials $f_1, \ldots, f_s$. Let $V$ be the graded component of degree $r$ of the quotient algebra $\mathbb{C}[x_0, \ldots, x_l]/(f_1, \ldots, f_s)$. Then

$$\dim V \geq \binom{r + l - s}{r}.$$  

Moreover, if $r > 0$, then one has

$$\dim V \geq \binom{r + l - s}{r} + s.$$  

**Proof.** Let $d_1, \ldots, d_s$ be the degrees of the polynomials $f_1, \ldots, f_s$, respectively. The Poincaré series of the complete intersection $X$ is given by the well known formula

$$P_X(z) = \frac{(1 - z^{d_1}) \cdots (1 - z^{d_s})}{(1 - z)^{l+1}} = \frac{(1 + z + \ldots + z^{d_1-1}) \cdots (1 + z + \ldots + z^{d_s-1})}{(1 - z)^{l+s+1}},$$

see for instance [St78, Corollary 3.3]. This immediately implies the equality

(2.5) $$\dim V = \sum_{t_1=0}^{d_1-1} \cdots \sum_{t_s=0}^{d_s-1} \binom{r - t_1 - \ldots - t_s + l - s}{l - s}.$$  

To obtain the first lower bound for $\dim V$ consider the summand corresponding to $t_1 = \ldots = t_s = 0$ on the right hand side of (2.5). To obtain the second lower bound in the case of positive $r$ consider also the summands corresponding to

$$t_j = 1, \quad t_1 = \ldots = t_{j-1} = t_{j+1} = \ldots = t_s = 0$$

for all $1 \leq j \leq s$. □
3. Bounds for Hodge numbers

Throughout this section $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_N)$, $a_0 \leq \ldots \leq a_N$, denotes a well formed weighted projective space. As in §2 define the number $0 \leq l_{\mathbb{P}} \leq N$ by conditions (2.1) if $a_N > 1$, and put $l_{\mathbb{P}} = N$ otherwise; in other words, $l_{\mathbb{P}} + 1$ is the number of $x_i$’s of weight $a_i = 1$. Besides this, we use the following notation.

Notation. Let $X$ be a smooth well formed Fano weighted complete intersection of multi-degree $(d_1, \ldots, d_k)$ in $\mathbb{P}$. Suppose that $d_1 \leq \ldots \leq d_k$. Define $s_X$ to be the maximal index $s$ such that $d_s < d_k$; if $d_1 = \ldots = d_k$, we put $s_X = 0$. For convenience we denote $d_X = d_k$.

Lemma 3.1. Let $X$ be a smooth well formed Fano weighted complete intersection in $\mathbb{P}$ which is not an intersection with a linear cone. The following assertions hold.

(i) If $d_X = 2$, then $X$ is a complete intersection of $k$ quadrics in $\mathbb{P} \cong \mathbb{P}^N$.

(ii) If $d_X = 3$, then $X$ is a complete intersection of $s_X$ quadrics and $k - s_X$ cubics in $\mathbb{P} \cong \mathbb{P}^N$.

Proof. One has $a_i \leq d_X$ for all $0 \leq i \leq N$ by Lemma 2.1. Since $X$ is not an intersection with a linear cone, we see that $a_i \neq d_X$ for all $0 \leq i \leq N$. In particular, if $d_X = 2$, then we have $a_i = 1$ for all $0 \leq i \leq N$, which proves assertion (i).

Now suppose that $d_X = 3$. Let $d$ be the minimum of the degrees of the defining equations of $X$. If for some $i$ one has $a_i = 2$, then one must have $d = 2$ by Lemma 2.1. This again means that $X$ is an intersection with a linear cone, which is not the case by assumption. Thus $a_i = 1$ for all $0 \leq i \leq N$, which proves assertion (ii). \qed

Given a smooth well formed Fano weighted complete intersection $X$ in $\mathbb{P}$, we put

$$p_X = \left[ \frac{i_X}{d_X} \right] \quad \text{and} \quad r_X = p_X d_X - i_X,$$

so that $i_X = p_X d_X$ if and only if $r_X = 0$. The strategy of our proof of Theorems 1.7 and 1.8 and Proposition 1.15 is to find, following the notation introduced before Theorem 2.8, a certain more or less explicit subspace in $R_{p_X, -i_X}$ to show that the corresponding Hodge number does not vanish.

Lemma 3.2. Let $X$ be a smooth well formed Fano weighted complete intersection in $\mathbb{P}$. One has

$$h_{pr}^{p_X, n-p_X}(X) \geq \left( \frac{p_X + k - s_X - 1}{p_X} \right) \cdot \left( \frac{r_X + l_{\mathbb{P}} - s_X}{r_X} \right)$$

if $r_X < d_X - 1$, and

$$h_{pr}^{p_X, n-p_X}(X) > \left( \frac{p_X + k - s_X - 1}{p_X} \right) \cdot \left( \frac{(d_X - 1 + l_{\mathbb{P}} - s_X)}{d_X - 1} \right) + s_X - l_{\mathbb{P}} - 1$$

if $r_X = d_X - 1$.

Proof. Recall that the Hodge numbers are constant in smooth families of projective varieties since they are upper semicontinuous, and their sums are dimensions of cohomology, which are locally constant by Ehresmann theorem. Hence by Lemma 2.10 one can assume that $X$ is of Fermat type. Furthermore, we can assume that the coefficients $\alpha_{i,j}$ in the expansion of the polynomials $f_j$ as in (2.24) are general.
Define the ideal $J'$ in $S = \mathbb{C}[x_0, \ldots, x_N, w_1, \ldots, w_k]$ as one generated by
\[ x_{l_0+1}, \ldots, x_N, w_1, \ldots, w_{s_X}, (x_0, \ldots, x_{l_0})^{dx}, f_1, \ldots, f_{s_X}, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_{l_0}}, \]
where $(x_0, \ldots, x_{l_0})^{dx}$ denotes the ideal generated by all monomials of degree $d_X$ in the variables $x_0, \ldots, x_{l_0}$. Since $X$ is of Fermat type, it is straightforward to check that $J \subseteq J'$.

Let $\bar{J}$ be the image of $J'$ in $R$, so that $\bar{J}$ is the ideal in $R$ generated by
\[ x_{l_0+1}, \ldots, x_N, w_1, \ldots, w_{s_X}, (x_0, \ldots, x_{l_0})^{dx}, \frac{\partial F_R}{\partial x_0}, \ldots, \frac{\partial F_R}{\partial x_{l_0}}, \]
where $F_R$ is the image of $F$ in $R$. Set $\bar{R} = R/\bar{J} \cong S/J'$.

Denote by $\tilde{f}_j$ the polynomial obtained from $f_j$ by substituting $x_{l_0+1} = \ldots = x_N = 0$; in other words, $\tilde{f}_j$ is the “Fermat part” of $f_j$, and $\tilde{f}_j = f_j - \hat{f}_j$ in the notation of Definition 2.9. Note that since the coefficients $\alpha_{j,i}$ are general, the polynomials $\tilde{f}_j$ define a complete intersection in the usual projective space $\mathbb{P}^p$. Let $\bar{F}$ be the polynomial obtained from $F$ by substituting
\[ x_{l_0+1} = \ldots = x_N = w_1 = \ldots = w_{s_X} = 0, \]
so that
\[ \bar{F} = \sum_{j=s_X+1}^k w_j \tilde{f}_j. \]

Let $\bar{J}$ be the ideal in $\mathbb{C}[x_0, \ldots, x_{l_0}, w_{s_X+1}, \ldots, w_k]$ generated by
\[ (x_0, \ldots, x_{l_0})^{dx}, \frac{\partial \bar{F}}{\partial x_0}, \ldots, \frac{\partial \bar{F}}{\partial x_{l_0}}, \tilde{f}_1, \ldots, \tilde{f}_{s_X}, \]
so that
\[ S/J' \cong \mathbb{C}[x_0, \ldots, x_{l_0}, w_{s_X+1}, \ldots, w_k]/\bar{J}. \]

Also, define $\bar{R}^0$ as the quotient of $\mathbb{C}[x_0, \ldots, x_{l_0}]$ by the ideal generated by
\[ (x_0, \ldots, x_{l_0})^{dx}, \tilde{f}_1, \ldots, \tilde{f}_{s_X}, \]
and let $\bar{J}^0$ be the ideal in $\bar{R}^0[w_{s_X+1}, \ldots, w_k]$ generated by
\[ \frac{\partial \bar{F}}{\partial x_0}, \ldots, \frac{\partial \bar{F}}{\partial x_{l_0}}. \]

We have isomorphisms of bigraded algebras
\[ \bar{R} \cong \mathbb{C}[x_0, \ldots, x_{l_0}, w_{s_X+1}, \ldots, w_k]/\bar{J} \cong \bar{R}^0[w_{s_X+1}, \ldots, w_k]/\bar{J}^0. \]

Note that the ideal $\bar{J}^0$ is generated by $l_0 + 1$ polynomials, and each of them is linear in the variables $w_i$ of degree $d_X - 1$ in variables $x_0, \ldots, x_{l_0}$. Consider the grading on the algebra $\mathbb{C}[w_{s_X+1}, \ldots, w_k]$ such that the variables $w_j$ have degree 1. For all non-negative integers $p$ and $r$, we have
\[ \dim \bar{R}_{p,r-pd_X} = \dim \bar{R}^0_r \cdot \dim \mathbb{C}[w_{s_X+1}, \ldots, w_k]_p = \dim \bar{R}^0_r \cdot \left( \frac{p + k - s_X - 1}{p} \right). \]
if \( r \leq d_X - 2 \), and

\[
\dim \bar{R}_{p,d_X-1-pd_X} = \\
= \dim \bar{R}^0_{d_X-1} \cdot \dim \mathbb{C}[w_{s_X+1}, \ldots, w_k, p - (l_{p} + 1)] \cdot \dim \mathbb{C}[w_{s_X+1}, \ldots, w_k, p-1] = \\
= \dim \bar{R}^0_{d_X-1} \cdot \left( \frac{p + k - s_X - 1}{p} \right) \cdot \left( l_{p} + 1 \right) \cdot \left( \frac{p + k - s_X - 2}{p-1} \right).
\]

On the other hand, by Lemma 2.11 one has

\[
\dim R^0_r \geq \left( \frac{r + l_{p} - s_X}{r} \right),
\]

and moreover

\[
\dim \bar{R}^0_r \geq \left( \frac{r + l_{p} - s_X}{r} \right) + s_X
\]

if \( r > 0 \). Since \( d_X \geq 2 \), we have in particular

\[
\dim \bar{R}^0_{d_X-1} \geq \left( \frac{d_X - 1 + l_{p} - s_X}{d_X - 1} \right) + s_X.
\]

Note that

\[
\dim R_{p_X,-i_X} = \dim R_{p_X,r_X-pX d_X} \geq \dim \bar{R}_{p_X,r_X-pX d_X}.
\]

Therefore, we have

\[
\dim R_{p_X,-i_X} \geq \left( \frac{r_X + l_{p} - s_X}{r_X} \right) \cdot \left( \frac{p_X + k - s_X - 1}{p_X} \right)
\]

if \( r_X < d_X - 1 \), so that the first assertion of the lemma is implied by Theorem 2.8. In the case if \( r_X = d_X - 1 \), we get

\[
\dim R_{p_X,-i_X} \geq \left( \frac{p_X + k - s_X - 1}{p_X} \right) \cdot \left( \frac{d_X - 1 + l_{p} - s_X}{d_X - 1} \right) + s_X - l_{p} - 1
\]

when \( r_X = d_X - 1 \). Now Theorem 2.8 implies the second assertion of the lemma as well. \( \square \)

Lemma 3.3. Let \( X \) be a smooth well formed Fano weighted complete intersection in \( \mathbb{P} \). Suppose that \( k = s_X + 1 \) and \( r_X = 0 \). Then \( h_{p_X,-s_X}^{p_X,p_X}(X) = 1 \).

Proof. Bidegree estimates show that the bigraded component of bidegree \( (p_X, -i_X) \) in the algebra \( S = \mathbb{C}[x_0, \ldots, x_N, w_1, \ldots, w_k] \) is generated by the monomial \( w_k^{p_X} \). On the other hand, one obviously has \( w_k^{p_X} \not\in J \), and hence the bigraded component \( R_{p_X,-i_X} \) is one-dimensional. Thus the assertion follows from Theorem 2.8. \( \square \)

Now we are able to deduce the following positivity result.
Corollary 3.4. Let $X$ be a smooth well formed Fano weighted complete intersection in $\mathbb{P}$ which is not an intersection with a linear cone. Suppose that $X$ is not a complete intersection of quadrics in $\mathbb{P} = \mathbb{P}^N$. Then $h^{p_X,n-p_X}_{pr}(X)$ is positive. Moreover, one has $h^{p_X,n-p_X}_{pr}(X) = 1$ if and only if $k = s_X + 1$ and $r_X = 0$.

Proof. Note that $d_X > 2$ by Lemma 3.1(i), and $s_X < k$ by definition. Also, one has $l_\varphi \geq k$ by Theorem 2.4(iii). In particular, we have

\begin{equation}
\left( \frac{r_X + l_\varphi - s_X}{r_X} \right) \geq 1,
\end{equation}

and the equality holds if and only if $r_X = 0$. Since $p_X > 0$, we also have

\begin{equation}
\left( \frac{p_X + k - s_X - 1}{p_X} \right) \geq 1,
\end{equation}

and the equality holds if and only if $k = s_X + 1$.

Suppose that $r_X < d_X - 1$. Then

\begin{equation}
h^{p_X,n-p_X}_{pr}(X) \geq \left( \frac{p_X + k - s_X - 1}{p_X} \right) \cdot \left( \frac{r_X + l_\varphi - s_X}{r_X} \right)
\end{equation}

by Lemma 3.2. By (3.1) and (3.2) this gives $h^{p_X,n-p_X}_{pr}(X) \geq 1$, and the equality can hold only if $k = s_X + 1$ and $r_X = 0$. On the other hand, if $k = s_X + 1$ and $r_X = 0$, then $h^{p_X,n-p_X}_{pr}(X) = 1$ by Lemma 3.3.

Now suppose that $r_X = d_X - 1$. Then

\begin{equation}
h^{p_X,n-p_X}_{pr}(X) > \left( \frac{p_X + k - s_X - 1}{p_X} \right) \cdot \left( \frac{d_X - 1 + l_\varphi - s_X}{d_X - 1} + s_X - l_\varphi - 1 \right)
\end{equation}

Furthermore, since $d_X \geq 3$, we have

\begin{equation}
\left( \frac{d_X - 1 + l_\varphi - s_X}{d_X - 1} \right) + s_X - l_\varphi - 1 \geq (d_X - 1 + l_\varphi - s_X) + s_X - l_\varphi - 1 = d_X - 2 \geq 1.
\end{equation}

Therefore, keeping in mind inequality (3.2), we obtain $h^{p_X,n-p_X}_{pr}(X) > 1$ in this case. \hfill \Box

Lemma 3.5. Let $X$ be a smooth well formed Fano weighted complete intersection in $\mathbb{P}$. Then $h^{q,n-q}_{pr}(X) = 0$ for every $q < p_X$.

Proof. Suppose that $h^{q,n-q}_{pr}(X) > 0$ for some $q < p_X$. Then Theorem 2.8 implies that there is a monomial

\begin{equation}
m_w \cdot m_x \in S = \mathbb{C}[x_0, \ldots, x_N, w_1, \ldots, w_k]
\end{equation}

of bidegree $(q, -i_X)$, where $m_w$ is a monomial in $w_i$ of degree $q$, and $m_x$ is a monomial in $x_j$. Let $(q, -t)$ be the bidegree of $m_w$. Then $t \leq qd_X$, and the bidegree of $m_x$ is $(0, t - i_X)$. But since $q < p_X - 1$, one has

\begin{equation}
t - i_X \leq qd_X - i_X \leq (p_X - 1)d_X - i_X = r_X - d_X < 0,
\end{equation}

which gives a contradiction. \hfill \Box

Lemma 3.6. Let $X \subset \mathbb{P}^N$ be a smooth Fano complete intersection of $k$ quadrics of dimension $n = N - k$ in $\mathbb{P}^N$. The following assertions hold.

(i) Suppose that $k = 1$, so that $X$ is a quadric hypersurface. Then $h^{q,n-q}_{pr}(X) = 0$ for all $q$, with the only exception $h^{n,n}_{pr}(X) = 1$ for even dimension $n$. 

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(ii) Suppose that $k > 1$. Then $h_{pr}^{q,n-q}(X) = 0$ for $q < p_X$. Moreover, one has
\[
h_{pr}^{p_X,n-p_X}(X) = \left(\frac{N-1}{2} \right) > 1
\]
for even $i_X = N - 2k + 1$, and
\[
h_{pr}^{p_X,n-p_X}(X) \geq k \cdot \left(\frac{N}{2} k + 1 \right) - \left(\frac{N}{2} - k\right) > 1.
\]
for odd $i_X$.

Proof. Assertion (i) is obvious and well known.

Let us prove assertion (ii). The vanishing of $h_{pr}^{q,n-q}(X)$ for all $q < p_X$ follows from Lemma 3.5. For the estimates from below we will use Theorem 2.8. Similarly to the proof of Lemma 3.2, one can assume that $X$ is of Fermat type. We have $i_X = n - k + 1$ and $p_X = \left\lceil \frac{n-k+1}{2} \right\rceil$.

Suppose that $i_X$ is even, so that $p_X = \frac{n-k+1}{2}$. Let $\tilde{J}$ be the ideal in $R$ generated by the variables $x_0, \ldots, x_N$. Then there is a natural surjective map of bigraded algebras
\[
R \to R/\tilde{J} \cong \mathbb{C}[w_1, \ldots, w_k].
\]
Note that the kernel of this map has zero intersection with $R_{p_X,i_X}$, and consider the new grading on $\mathbb{C}[w_1, \ldots, w_k]$ such that the variables $w_1, \ldots, w_k$ have degree 1. One has
\[
h_{pr}^{p_X,n-p_X}(X) = \dim R_{p_X,-i_X} = \dim \mathbb{C}[w_1, \ldots, w_k]_{p_X} = \left(\frac{n+k-1}{2} \right) = \left(\frac{N-1}{2} k - 1\right) > 1.
\]

Now suppose that $i_X$ is odd, so that $p_X = \frac{n-k}{2} + 1$. Starting from the complete intersection $X$ written in the Fermat form (2.3) and making linear changes of coordinates if necessary, we can assume that the quadric polynomials generating the homogeneous ideal of $X$ have the form
\[
f_j = x_{j-1}^2 + \sum_{i \geq k} \alpha_{j,i} x_i^2, \quad 1 \leq j \leq k.
\]
Let $\tilde{J}$ be the ideal in $R$ generated by
\[
(x_0, \ldots, x_{k-1})^2, x_k, \ldots, x_N,
\]
where $(x_0, \ldots, x_{k-1})^2$ is the ideal generated by all monomials of degree 2 in $x_0, \ldots, x_{k-1}$. Let $\tilde{J}$ be the ideal in $\mathbb{C}[x_0, \ldots, x_{k-1}, w_1, \ldots, w_k]$ generated by $(x_0, \ldots, x_{k-1})^2$ and $k$ monomials
\[
w_1 x_0, \ldots, w_k x_{k-1}.
\]
Then there is a natural surjective map of bigraded algebras
\[
R \to R/\tilde{J} \cong \mathbb{C}[x_0, \ldots, x_{k-1}, w_1, \ldots, w_k]/\tilde{J}.
\]
Thus the dimension $\dim R_{p_X,-i_X} = \dim R_{p_X,1-2p_X}$ is bounded from below by the dimension of the bigraded component of bidegree $(p_X, 1 - 2p_X)$ of the algebra $\mathbb{C}[x_0, \ldots, x_{k-1}, w_1, \ldots, w_k]/\tilde{J}$. Define the grading on $\mathbb{C}[w_1, \ldots, w_k]$ so that the degree
of the variables \( w_1, \ldots, w_k \) equals 1. One has

\[
\dim R_{p_X,-i_X} \geq \dim \mathbb{C}[x_0,\ldots,x_{k-1},w_1,\ldots,w_k]_{p_X,1-2p_X} - k \cdot \dim \mathbb{C}[w_1,\ldots,w_k]_{p_X-1} = \\
= \dim \mathbb{C}[x_0,\ldots,x_{k-1}] \cdot \dim \mathbb{C}[w_1,\ldots,w_k]_{p_X} - k \cdot \dim \mathbb{C}[w_1,\ldots,w_k]_{p_X-1} = \\
= k(\dim \mathbb{C}[w_1,\ldots,w_k]_{p_X} - \dim \mathbb{C}[w_1,\ldots,w_k]_{p_X-1}).
\]

Therefore, we have

\[
h^{p_X,n-p_X}(X) = \dim R_{p_X,-i_X} \geq k \cdot \left( \binom{k+p_X-1}{p_X} - \binom{k+p_X-2}{p_X-1} \right) = \\
k \cdot \left( \left( \frac{N}{2} - k + 1 \right) - \left( \frac{N}{2} - 1 \right) \right) \geq k > 1. \quad \square
\]

Recall that if \( X \) is a smooth Fano complete intersection of quadrics and at least one cubic in \( \mathbb{P}^N \), then

\[
r_X = 3 \left\lceil \frac{i_X}{3} \right\rceil - i_X \in \{0,1,2\}.
\]

We summarize the results of this section in terms of the Hodge level \( h(X) \), see Definition 1.14.

**Corollary 3.7 (Proposition 1.15).** Let \( X \) be a smooth well formed Fano weighted complete intersection which is not an intersection with a linear cone. If \( X \) is an odd-dimensional quadric, then \( h(X) = 0 \). Otherwise \( h(X) = n - 2p_X \).

**Proof.** If \( X \) is a complete intersection of quadrics in \( \mathbb{P}^N \), then the assertion follows from Lemma 3.6. Otherwise it is given by Corollary 3.4 and Lemma 3.5. \( \square \)

4. PROOFS OF THE MAIN RESULTS

In this section we prove Theorems 1.7 and 1.8, Propositions 1.9 and 1.15, and Corollary 1.16. We use the notation introduced in the beginning of §3.

**Remark 4.1.** Let \( X \) be a smooth \( n \)-dimensional Fano variety. We always have \( h^{0,n}(X) = 0 \) by Kodaira vanishing. This implies that \( X \) is always diagonal provided that \( n \leq 2 \), and it is always of curve type provided that \( n = 3 \).

We start with the case of complete intersections of quadrics and cubics in the usual projective space.

**Lemma 4.2.** Let \( X \subset \mathbb{P}^N \) be a smooth \( n \)-dimensional Fano complete intersection of quadrics and cubics (including the case when there are either no quadrics or no cubics). Then

(i) \( X \) is \( \mathbb{Q} \)-homologically minimal if and only if \( X \) is an odd-dimensional quadric;

(ii) \( X \) is diagonal if and only if \( X \) is a quadric, or an even-dimensional intersection of two quadrics, or a cubic surface;

(iii) \( X \) is of curve type if and only if \( X \) is either an odd-dimensional intersection of at most three quadrics, or a threefold, or a cubic fivefold;

(iv) \( X \) is of 2-Calabi–Yau type if and only if \( X \) is a cubic fourfold;
(v) \( X \) is of 3-Calabi–Yau type if and only if \( X \) is either a seven-dimensional cubic or a five-dimensional intersection of a quadric and a cubic.

**Proof.** Assertion (i) immediately follows from Corollary 3.7. In what follows we will assume that \( X \) is not an odd-dimensional quadric. Let \( X \) be a complete intersection of \( k_1 \) quadrics and \( k_2 \) cubics. Note that in every case the sufficiency of the provided conditions is straightforward to check.

One has

\[
p_{X} = \left\lceil \frac{n - k_1 + 1}{2} \right\rceil
\]

if \( k_2 = 0 \) and

\[
p_{X} = \left\lceil \frac{n - k_1 - 2k_2 + 1}{3} \right\rceil
\]

otherwise. This is less or equal to \( \frac{n}{2} \), and the equality holds if and only if \( X \) is a quadric, an even-dimensional intersection of two quadrics, or a cubic surface. Thus, using Corollary 3.7, we obtain assertion (ii).

Suppose that \( X \) is of curve type. Then \( n \) is odd, and it follows from Corollary 3.7 that \( p_{X} = \frac{n - 1}{2} \). If \( k_2 = 0 \), we conclude that \( X \) is a complete intersection of at most three quadrics. If \( k_2 > 0 \), then \( X \) is either a threefold, or a cubic fivefold. This gives assertion (iii).

Now suppose that \( X \) is of 2-Calabi–Yau type or of 3-Calabi–Yau type. We know from Lemma 3.6 that \( k_2 > 0 \). Hence it follows from Corollary 3.7 that

\[ r_{X} = 0 \text{ and } k_2 = 1. \]

If \( n \) is even (so that \( X \) is of 2-Calabi–Yau type), then

\[
p_{X} = \frac{n}{2} - 1
\]

by Corollary 3.7. This implies

\[
\frac{n - k_1 - 2k_2 + 1}{3} = \frac{n}{2} - 1,
\]

which means that \( X \) is a cubic fourfold. If \( n \) is odd (so that \( X \) is of 3-Calabi–Yau type), then

\[
p_{X} = \frac{n - 3}{2}
\]

by Corollary 3.7. This implies

\[
\frac{n - k_1 - 2k_2 + 1}{3} = \frac{n - 3}{2},
\]

which means that \( X \) is either a seven-dimensional cubic or a five-dimensional complete intersection of a quadric and a cubic. Therefore, we obtain assertions (iv) and (v). \( \square \)

**Lemma 4.3.** Let \( X \) be a smooth well formed Fano weighted complete intersection of dimension \( n \geq 3 \) in \( P \) which is not an intersection with a linear cone. Suppose that \( d_{X} \geq 4 \). Then

(i) \( X \) is not diagonal (and in particular not \( \mathbb{Q} \)-homologically minimal);

(ii) \( X \) is of curve type if and only if \( n = 3 \);

(iii) \( X \) is not of 2-Calabi–Yau type;

(iv) \( X \) is of 3-Calabi–Yau type if and only if \( X \) is a five-dimensional quartic hypersurface in \( P(1^6, 2) \).
Proof. Recall that $i_X \leq n$ by Theorem 2.7(ii). Therefore, by Corollary 3.7 one has

\[
(4.1) \quad h(X) = n - 2 \left\lceil \frac{i_X}{d_X} \right\rceil \geq n - 2 \left\lceil \frac{n}{4} \right\rceil.
\]

Since $n \geq 3$, we see that $h(X) > 0$, which implies assertion (i).

To prove the remaining assertions, we can restrict to the case $n > 5$. Indeed, for $n = 4$ and $n = 5$ it follows from Remark 4.1 and the classification of such varieties (see [PSh16, §5], cf. [Kü97, Proposition 2.2.1]) that $X$ is not of curve type and not of 2-Calabi–Yau type; moreover, $X$ is of 3-Calabi–Yau type if and only if $X$ is a five-dimensional quartic hypersurface in $\mathbb{P}(1^6,2)$. We deduce from (4.1) that

\[
h(X) \geq \frac{n}{2} - 2,
\]

which means that for $n > 5$ the variety $X$ is not of curve type. This proves assertion (ii).

Now suppose that $X$ is of $m$-Calabi–Yau type for some $m$. Then we know from Corollary 3.4 that $i_X$ is divisible by $d_X$, so that (4.1) reads

\[
m = h(X) = n - 2 \frac{i_X}{d_X} \geq n - 2 \frac{n}{4} = \frac{n}{2}.
\]

This implies assertion (iii), and shows that one can have $h(X) = 3$ only for $n = 6$. However, if $X$ is of 3-Calabi–Yau type, its dimension must be odd by definition, and thus we obtain assertion (iv) as well. □

Now we are ready to prove our main results.

Proof of Theorem 1.7 and Proposition 1.9. By Lemma 2.5 we can assume that $\dim X \geq 3$. Therefore, by Lemma 4.3 we can assume that $d_X \leq 3$. According to Lemma 3.1 this means that $X$ is a complete intersection of quadrics and cubics in $\mathbb{P} = \mathbb{P}^N$. Now the required assertions follow from Lemma 4.2. □

Proof of Theorem 1.8. The Hodge level $h(X)$ is computed by Corollary 3.7. Thus, the assertion follows from Corollary 3.4 and Lemma 3.6. □

Note also that one can obtain an alternative proof of Proposition 1.9 using Theorem 1.8.

The assertion of Proposition 1.15 is given by Corollary 3.7. Finally, we prove Corollary 1.16.

Proof of Corollary 1.16. Suppose that the assumptions of assertion (i) hold. Then $X$ is not an odd-dimensional quadric. Hence

\[
h(X) = n - 2 \left\lceil \frac{i_X}{d_X} \right\rceil
\]

by Corollary 3.7. In particular, one has $h(X) = n - 2$ if $i_X \leq 2$. If $i_X \geq 3$ and $X$ is not a complete intersection of quadrics, then $d_X \geq 3$ by Lemma 3.1(i), and one also has $h(X) = n - 2$. If $i_X \geq 4$ and $X$ is not a complete intersection of quadrics and cubics, then $d_X \geq 4$ by Lemma 3.1, and again $h(X) = n - 2$.

Now suppose that $X$ is not a complete intersection of quadrics in a projective space. Then $d_X \geq 3$ by Lemma 3.1(i). Therefore, Corollary 3.7 and Theorem 2.7(ii) give

\[
h(X) = n - 2 \left\lceil \frac{i_X}{d_X} \right\rceil \geq n - 2 \left\lceil \frac{n}{3} \right\rceil.
\]
Since $\left\lceil \frac{n}{3} \right\rceil \leq \frac{n+2}{3}$, we obtain

$$h(X) \geq n - 2 \left( \frac{n + 2}{3} \right) = \frac{n - 4}{3},$$

which gives assertion (ii). □

5. QUASI-SMooth COMPLETE INTERSECTIONS

In this section we briefly discuss quasi-smooth weighted complete intersections.

**Definition 5.1** (see [IF00, Definition 6.3]). Let $\pi: \mathbb{A}^{N+1} \setminus \{0\} \rightarrow \mathbb{P}$ be the natural projection to a weighted projective space. A subvariety $X \subset \mathbb{P}$ is called quasi-smooth if $\pi^{-1}(X)$ is smooth.

The following result allows one to check quasi-smoothness of weighted hypersurfaces in terms of weights and degrees.

**Theorem 5.2** ([IF00, Theorem 8.1]). Let $X$ be a general hypersurface in $\mathbb{P}(a_0, \ldots, a_N)$ of degree $d$, which is not an intersection with a linear cone. Then $X$ is quasi-smooth if and only if for any $0 \leq s \leq N$ and for any non-empty subset $I = \{i_0, \ldots, i_s\} \subset \{0, \ldots, N\}$

there either exists a monomial $x_{i_0}^{m_{i_0}} \cdots x_{i_s}^{m_{i_s}}$ of degree $d$, or for any $\mu = 1, \ldots, k+1$ there exist monomials $x_{i_0}^{m_{i_0,e_\mu}} \cdots x_{i_s}^{m_{i_s,e_\mu}} x_{e_\mu}$ of degree $d$ for $s+1$ distinct variables $e_\mu$.

Note that there is a pure Hodge structure on the cohomology of quasi-smooth weighted complete intersections. Theorem 2.8 actually holds in the quasi-smooth case, see [Ma99, Theorem 3.6].

Let us consider some examples from the Alexeev’s list (see Remark 1.18).

**Example 5.3.** Let $X$ be a general hypersurface of degree 4 in $\mathbb{P}(1,1,1,2,2)$. Then $X$ is a well formed Fano hypersurface, and $X$ is quasi-smooth by Theorem 5.2. One has $h^{1,3}(X) = \dim (R_{1,-4}) = 1$ by Theorem 2.8. Thus, $X$ is of K3 type. Moreover, by [Kuz15b, Corollary 4.2], the derived category of coherent sheaves on $X$ has a semiorthogonal decomposition on a category of K3 type and three exceptional objects.

**Example 5.4.** Let $X$ be a general hypersurface of degree 16 in $\mathbb{P}(1,4,5,6,8,8,8,8)$. Then $X$ is a well formed Fano hypersurface, and $X$ is quasi-smooth by Theorem 5.2. One has $h^{2,4}(X) = \dim (R_{2,-32}) = 1$ by Theorem 2.8. Thus, $X$ is of K3 type. Moreover, in the same way as in Example 5.3, the derived category of coherent sheaves on $X$ contains a category of K3 type.

**Example 5.5.** Let $X$ be a hypersurface of degree 6 in $\mathbb{P}(1,1,1,3,3,3)$. Then $X$ is a well formed Fano hypersurface, and $X$ is quasi-smooth by Theorem 5.2. One has $h^{1,3}(X) = \dim (R_{1,-6}) = 1$ by Theorem 2.8. Thus, $X$ is of K3 type. Moreover, in the same way as in Example 5.3, the derived category of coherent sheaves on $X$ contains a category of K3 type.
In this section we discuss some remaining questions and problems.

Theorem 1.7 gives non-vanishing of some middle Hodge numbers for Fano weighted complete intersections. This non-vanishing holds for trivial reasons for Calabi–Yau varieties, since for $n$-dimensional Calabi–Yau variety $X$ one has $h^{0,n}(X) = 1$. Calculations in some examples shows that $h^{0,n}(X) > 0$ for many smooth $n$-dimensional weighted complete intersections $X$ of general type. On the other hand, if one drops the assumption that $X$ is a weighted complete intersection, there are plenty of examples of $\mathbb{Q}$-homologically minimal surfaces of general type known as fake projective planes, see, for example, [Mu79], [CS10]. Also, there are examples of $\mathbb{Q}$-homologically minimal fourfolds of general type, see [PG06].

**Question 6.1.** Do there exist smooth weighted complete intersections of general type that are $\mathbb{Q}$-homologically minimal, diagonal, of curve type, or of $m$-Calabi–Yau type for small $m$? Is it true that $h^{0,n}(X) > 0$ for all smooth $n$-dimensional weighted complete intersections $X$ of general type?

In all examples of smooth Fano weighted complete intersections we have considered middle Hodge numbers that are closer to the center of the Hodge diamond are larger than those that are further from the center: for an $n$-dimensional variety $X$ one has $h^{p,n-p}(X) \leq h^{q,n-q}(X)$ if $p < q \leq \frac{n}{2}$. This is not true for some Calabi–Yau varieties: there are rigid Calabi–Yau threefolds, that are ones with $h^{1,2}(X) = 0$, while $h^{3,0}(X) = 1$, see, for example, [Og96]. However we do not know rigid Calabi–Yau weighted complete intersections.

**Question 6.2.** Let $X$ be a smooth $n$-dimensional weighted complete intersection. Is it true that $h^{p,n-p}(X) \leq h^{q,n-q}(X)$ if $p < q \leq \frac{n}{2}$? If not, is this true if we also assume that $X$ is a Fano variety, or a Calabi–Yau variety?

There is a generalization of the construction of Fano varieties as complete intersections. Namely, one can consider zeros of sections of equivariant vector bundles on Grassmannians or partial flag manifolds, see, for instance, [Kü95], [Kuz15a], [Kuz16].

**Problem 6.3.** Bound Hodge numbers and compute Hodge level for such varieties.

From the point of view of classification of Fano varieties the most complicated and interesting case is the case of Fano index 1 and Picard rank 1. By Corollary 1.16(i), for an $n$-dimensional weighted complete intersection $X$ of this type, one has $h^{1,n-1}(X) > 0$. There is an approach to computing this number via Mirror Symmetry. That is, one can construct Calabi–Yau compactified (toric) Landau–Ginzburg models for a big class of Fano varieties (see [Prz13], [PShl4], [Prz16], [Prz17]), cf. also the notion of a tame compactified Landau–Ginzburg model in [KKP17]. Given such a Landau–Ginzburg model $LG_X$, one can define the number $k_{LG_X}$ as the number of irreducible components of reducible fibers of $LG_X$ minus the number of the reducible fibers. It is expected that one has $k_{LG_X} = h^{1,n-1}(X)$. This is proved for Picard rank 1 Fano threefolds in [Prz13] and for Fano complete intersections in usual projective spaces in [PShl5a].

This approach can be applied to any smooth Fano variety, not necessary to a weighted complete intersection, provided its weak Landau–Ginzburg model is known. Most of the known constructions of Picard rank 1 Fano varieties are via complete intersections in weighted projective spaces or Grassmannians. Fortunately, there are constructions of
weak Landau–Ginzburg models for some of them, for instance, for weighted complete intersections of Cartier divisors (see [Prz11]) or of two hypersurfaces (see [PSh17b]), and for complete intersections in Grassmannians (see [BCFKS98], [PSh14], [PSh15b], [PSh17a]); these constructions are based on Givental’s approach [G97].

**Problem 6.4.** Construct a Calabi–Yau compactification $LG_X$ of a weak Landau–Ginzburg model for a smooth Fano weighted complete intersection $X$ of dimension $n$ and compute $k_{LG_X}$. The fact that $k_{LG_X} > 0$ (that is, that the Landau–Ginzburg model has a reducible fiber) is conjecturally equivalent to the fact that $h(X) = n - 2$.

**Remark 6.5.** Except for $\mathbb{Q}$-homological minimality there are other notions of minimality for Fano varieties. For instance, in [Prz07b] (see also [Prz07a]) the notion of quantum minimality was introduced: a Fano variety is quantum minimal if the subring in cohomology generated by the anticanonical class is closed with respect to quantum multiplication; in other words, it contains the minimal possible subring of small quantum cohomology ring containing the anticanonical class. Note that by [Ga02, Lemma 5.5] (see also [Prz07a]) all smooth Fano weighted complete intersections are quantum minimal.

**Appendix A. Smooth Fano weighted complete intersections of small dimension**

In Tables 1 and 2 we list smooth well-formed Fano weighted complete intersections that are not intersections with linear cones and have dimensions 2 and 3, respectively; according to the convention of §2 we exclude projective spaces from these lists. The classification of del Pezzo surfaces is well known, as well as the classification of Picard rank 1 Fano threefolds (see [Is77]). On the other hand, by the Lefschetz-type theorem smooth well-formed weighted complete intersections of dimension at least 3 in weighted projective spaces have Picard rank 1, see [Ma99, Proposition 1.4], so the three-dimensional case indeed can be extracted from [Is77]. An alternative way to compile our lists is to use the bounds for discrete invariants of weighted projective spaces and Fano weighted complete intersections therein provided by Theorem 2.4. An advantage of this approach is that one does not need to check that certain Fano varieties cannot be re-embedded into weighted projective spaces as weighted complete intersections.

For every line in each of the two tables, in the first column we put the ambient weighted projective space, in the second one we list degrees of hypersurfaces that determine our weighted complete intersection, and in the third one we put the unique non-trivial Hodge number, that is, $h^{1,1}$ in the two-dimensional case and $h^{1,2}$ in the three-dimensional case.

| $\mathbb{P}$ | Degrees | $h^{1,1}$ |
|-------------|---------|-----------|
| $\mathbb{P}(1^2, 2, 3)$ | 6 | 9 |
| $\mathbb{P}(1^3, 2)$ | 4 | 8 |
| $\mathbb{P}^3$ | 3 | 7 |
| $\mathbb{P}^4$ | 2,2 | 6 |
| $\mathbb{P}^3$ | 2 | 2 |

**Table 1.** Del Pezzo surfaces
Table 2. Fano threefolds

|   | Degrees | $h^{1,2}$ |
|---|---------|-----------|
| $\mathbb{P}(1^4, 3)$ | 6        | 52        |
| $\mathbb{P}^4$       | 4        | 30        |
| $\mathbb{P}^5$       | 2, 3     | 20        |
| $\mathbb{P}^6$       | 2, 2, 2  | 14        |
| $\mathbb{P}(1^3, 2, 3)$ | 6    | 21        |
| $\mathbb{P}(1^4, 2)$ | 4        | 10        |
| $\mathbb{P}^4$       | 3        | 5         |
| $\mathbb{P}^5$       | 2, 2     | 2         |
| $\mathbb{P}^4$       | 2        | 0         |

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