A mixed precision preconditioned Jacobi method for the symmetric eigenvalue problem

Zhiyuan Zhang∗ Zheng-Jian Bai†
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Abstract
The eigenvalue problem is a fundamental problem in scientific computing. In this paper, we first give the error analysis for a single step or sweep of Jacobi’s method in floating point arithmetic. Then we propose a mixed precision preconditioned Jacobi method for the symmetric eigenvalue problem: We first compute the eigenvalue decomposition of a real symmetric matrix by an eigensolver at low precision and we obtain a low-precision matrix of eigenvectors; Then by using the high-precision modified Gram-Schmidt orthogonalization process, a high-precision orthogonal matrix is obtained, which is used as an initial guess for Jacobi’s method. The rounding error analysis of the proposed method is established under some conditions. We also present a mixed precision preconditioned one-sided Jacobi method for the singular value problem and the corresponding rounding error analysis is discussed. Numerical experiments on CPUs and GPUs are reported to illustrate the efficiency of the proposed method over the original Jacobi method.

Keywords. Symmetric eigenvalue problem, singular value problem, Jacobi’s method, floating point arithmetic, mixed precision, rounding error analysis

AMS subject classifications. 65F15, 65G50

1 Introduction
The symmetric eigenvalue problem has widespread applications in scientific computing such as engineering computing [12, 51], numerical partial differential equations [36], and computing chemistry [13], etc.

The solution strategy of the symmetric eigenvalue problem depends on the structure of the given symmetric matrix and the desirable eigenvalues with or without associated eigenvectors. For example, when only a few extreme eigenvalues of a large, sparse, and symmetric matrix

∗School of Mathematical Sciences, Xiamen University, Xiamen 361005, People’s Republic of China (zyzhang510zg@stu.xmu.edu.cn).
†Corresponding author. School of Mathematical Sciences, Xiamen University, Xiamen 361005, People’s Republic of China (zjbai@xmu.edu.cn). The research of this author was partially supported by the National Natural Science Foundation of China grant 12371382.
are desired, Lanczos method and Jacobi-Davidson method [25, 45] are recommended. In general, a symmetric matrix can be reduced to tridiagonal form by finite Householder reflections or Givens rotations and there are some popular tridiagonalization based strategies for the symmetric eigenvalue problem [25, 40], e.g., the symmetric QR algorithm [9], the divide-and-conquer method [26], the bisection method [7], Sturm sequence method [27] and the method of multiple relatively robust representations (MR$^3$) [17]. Compared with these tridiagonalization based algorithm, there is another method directly applied to the original real symmetric matrix, i.e., Jacobi’s method. The Jacobi method is a very old method for diagonalizing a real symmetric matrix [34]. Recently, the Jacobi method has received much attention due to its natural suitability for parallel computation [8] and high accuracy in finite precision [16]. Another interesting aspect of Jacobi’s method with proper procedure ordering shows sweep-quadratic convergence rate after sufficient iterations [42, 47]. However, the Jacobi method is slower than a method based on tridiagonalization since it is conjectured that $O(n^3 \log n)$ operations are required for its standard implementation [44]. For an overview of the symmetric eigenvalue problem, one may refer to [40, 49] and [25, Chap.7, Chap.8, Chap.11].

To improve the efficiency of a numerical solver, in many engineering applications, one of the emerging strategies is to combine different precision arithmetics [5]. In the past decades, the most common IEEE 754 floating-point arithmetic in scientific computing has mainly been carried out in double precision (64 bit) and single precision (32 bit) [1]. Theoretically, single precision runs twice as fast as double precision in both communication and computation cost. And these two formats are supported by most of hardware architectures [3]. Recently, half precision (16-bit) floating point arithmetic has gradually been popular in the machine learning community. Half precision arithmetic is already available in some hardware (e.g., the NVIDIA V100 GPU), which runs faster in machine learning applications and also reduces memory storage and energy consumption. For higher precisions, there exists quadruple precision (128 bit) in some softwares [33] such as Advanpix Multiprecision Computing Toolbox for MATLAB [2].

Recently, based on different floating point precisions, many mixed precision algorithms have been proposed [10, 50, 52]. For mixed precision algorithms in numerical linear algebra, there are two survey papers [3, 33]. For the general eigenvalue problem, an earlier work given by Dongarra et al. [18, 19] was the mixed precision iterative refinement based on Newton’s method for computing eigenpairs of a matrix, which was extended to solving the symmetric eigenvalue problem by using the Sherman-Morrison formula [46]. For the symmetric eigenvalue problem, Petschow et al. [41] proposed a mixed precision MR$^3$-based eigensolver with improved accuracy and negligible performance penalty. Ogita and Aishima [38, 39] developed another novel iterative refinement for the symmetric eigenvalue decomposition. Very most recently, Gao et al. [28] proposed an elaborate mixed precision Jacobi singular value decomposition (SVD) algorithm which can achieve about 2x speedup comparable to LAPACK in x86-64 architecture.

The rounding error analysis has undisputed importance in numerical analysis, especially with the rise of mixed-precision computations in scientific computing. There exist some theoretical results on the error analysis of Jacobi’s method. In [49, p.279], it was showed that the computed diagonal entries of the updated matrix after some sweeps are closed to the eigenvalues of the original symmetric matrix $A$ with an error bound proportional to the product of machine precision and the norm of $A$. In [6, 16], it was established that the Jacobi method can compute the eigenvalues of a real symmetric positive definite diagonal scaling matrix with a uniformly
relative accuracy bound (see [37] for extended error analysis results). In [14], a backward error analysis was provided for the Cholesky–Jacobi method for the symmetric definite generalized eigenproblem. In [20], an high relative accuracy bound was provided for an orthogonal algorithm for the symmetric eigenproblem. In [21], it was showed that the eigenvalues of a symmetric matrix $A$ via the implicit Jacobi algorithm are computed with an error bound proportional to the product of machine precision and the spectral condition number of the eigenvector matrix of $A$.

In this paper, we first give the error analysis for a single step or sweep of the Jacobi method in floating point arithmetic. We derive the error bounds of the iterative matrix and its off-diagonal entries updated after one Jacobi rotation, and the computed diagonal entries of the updated matrix are closed to the eigenvalues of the original symmetric matrix with an error bound proportional to the product of machine precision and the norm of the original matrix. The error bounds of the off-diagonal entries of the iterative matrix updated after one sweep are established for the general and row cyclic order with distinct eigenvalues and the row-cyclic order with one multiple eigenvalue. Then we propose a mixed precision preconditioned Jacobi method for the symmetric eigenvalue problem. That is, by using an eigensolver to computing the eigenvalue decomposition of a real symmetric matrix at low precision, we can obtain a low-precision matrix of eigenvectors; Then, by using the high-precision modified Gram-Schmidt (MGS) orthogonalization process, a high-precision orthogonal matrix is obtained, which is employed as an initial guess for the Jacobi method. We give the rounding error analysis of the proposed mixed precision preconditioned Jacobi method with the cyclic ordering under some conditions. We also present a mixed precision preconditioned one-sided Jacobi method for the singular value problem and the corresponding rounding error analysis is studied. Finally, we report some numerical experiments to illustrate the efficiency of the proposed method over the original Jacobi method.

Throughout this paper, we use the following notation. Let $\mathbb{R}^{m \times n}$ be the set of all $m$-by-$n$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n \times 1}$. $I_n$ is an identity matrix of order $n$ and $e_s$ is the $s$th column of $I_n$. $1_n$ is an $n$-vector of all ones. Let $| \cdot |$ be the absolute value of a real number. Let $\| \cdot \|$ and $\| \cdot \|_F$ be the Euclidean vector norm or its induced matrix norm and the Frobenius matrix norm, respectively. The symbol “⊗” means the Kronecker product. The superscript “$^T$” stands for the transpose of a matrix or vector. For a symmetric matrix $G \in \mathbb{R}^{n \times n}$, we denote by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ its eigenvalues, arranged in decreasing order. For a matrix $G \in \mathbb{R}^{m \times n}$, we denote by $\sigma_1(G) \geq \sigma_2(G) \geq \cdots \geq \sigma_{\min\{m,n\}}(G) \equiv \sigma_{\min}(G) \geq 0$ its singular values, arranged in decreasing order. For a matrix $G = (g_{ij}) \in \mathbb{R}^{n \times n}$, let off$(G) := G - \text{diag}(g_{11}, \ldots, g_{nn})$ and $g_j$ be the $j$th column vector of $G$ for $j = 1, \ldots, n$.

The rest of the paper is organized as follows. In Section 2 we review some error analysis results on classical numerical methods for the eigenvalue problem and the singular value problem. In Section 3 we give the error analysis for a single step or sweep of the Jacobi method in floating point arithmetic. In Section 4 we propose a mixed precision preconditioned Jacobi method for the symmetric eigenvalue problem. The rounding error analysis is also discussed. In Section 5 we present a mixed precision preconditioned one-sided Jacobi method for the singular value problem. In Section 6 we present some numerical tests to demonstrate the efficiency of the proposed methods. Some concluding remarks are given in Section 7.
2 Preliminaries

In this section, we review some error analysis results on some numerical methods for symmetric eigenvalue problems and singular value problems. We first recall the following error bounds for the MGS method [31, Theorem 19.13].

Lemma 2.1 Let $A \in \mathbb{R}^{m \times n}$ with rank($A$) = $n$. Suppose the MGS method computes the approximate QR factorization $A \approx \hat{Q}\hat{R}$ in precision $\nu$, where $\hat{R} \in \mathbb{R}^{n \times n}$ is upper triangular and $\hat{Q} \in \mathbb{R}^{m \times n}$. Then there exist constants $\eta_i \equiv \eta_i(m, n)$ ($i = 1, 2, 3$) such that $\|A - \hat{Q}\hat{R}\| \leq \eta_1\|A\|\nu$, $\|\hat{Q}^T\hat{Q} - I_n\| \leq \eta_2\kappa(A)\nu$, and $\hat{Q} + \delta\hat{Q}$ is orthogonal with $\|\delta\hat{Q}\| \leq \eta_3\kappa(A)\nu$, where $\kappa(A) = \sigma_1(A)/\sigma_{\min}(A)$ is the condition number of $A$.

On the error analysis for symmetric eigenvalue problems and singular value problems, we have the following results (see [4, pp.104–105] and [4, pp.112–113]).

Lemma 2.2 Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. The computed symmetric eigenvalue decomposition $A \approx \hat{P}\hat{\Lambda}\hat{P}^T$ with $\hat{P} \in \mathbb{R}^{n \times n}$ and $\hat{\Lambda} = \text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_n) \in \mathbb{R}^{n \times n}$ via any eigensolver in LAPACK or EISPACK in precision $\nu$ is nearly the exact symmetric Schur decomposition of $A + E$, i.e., $A + E = (\hat{P} + \delta\hat{P})\hat{\Lambda}(\hat{P} + \delta\hat{P})^T$, where $\|E\| \leq p(n)\|A\|\nu$ and $\hat{P} + \delta\hat{P}$ is orthogonal with $\|\delta\hat{P}\| \leq p(n)\nu$. Here, $p(n)$ is a modestly growing function of $n$.

Lemma 2.3 Let $A \in \mathbb{R}^{m \times n}$ be a real matrix ($m \geq n$). The computed SVD $A \approx \hat{U}\hat{\Sigma}\hat{V}^T$ with $\hat{U} \in \mathbb{R}^{m \times m}$, $\hat{V} \in \mathbb{R}^{n \times n}$, and $\hat{\Sigma} = \text{diag}(\hat{\sigma}_1, \ldots, \hat{\sigma}_n) \in \mathbb{R}^{n \times n}$ via any SVD solver in LAPACK, LINPACK or EISPACK in precision $\nu$ is nearly the exact SVD of $A + E$, i.e., $A + E = (\hat{U} + \delta\hat{U})\hat{\Sigma}(\hat{V} + \delta\hat{V})^T$, where $\|E\| \leq p(m, n)\|A\|\nu$ and $\hat{U} + \delta\hat{U}$ and $\hat{V} + \delta\hat{V}$ are both orthogonal with $\|\delta\hat{U}\| \leq p(m, n)\nu$ and $\|\delta\hat{V}\| \leq p(m, n)\nu$. Here, $p(m, n)$ is a modestly growing function of $m$ and $n$.

Finally, we recall the perturbation bounds for eigenvalues and singular values [25, p.442 and p.487].

Lemma 2.4 If $G$ and $G + E$ are $n \times n$ real symmetric matrices, then $|\lambda_j(G + E) - \lambda_j(G)| \leq \|E\|$ for $j = 1, \ldots, n$ and $\sum_{j=1}^n (\lambda_j(G + E) - \lambda_j(G))^2 \leq \|E\|^2_{F}$.

Lemma 2.5 If $G$ and $G + E$ are $m \times n$ real matrices with $m \geq n$, then $|\sigma_j(G + E) - \sigma_j(G)| \leq \|E\|$ for $j = 1, \ldots, n$.

3 Jacobi’s method in floating point arithmetic

In this section, we first review Jacobi’s method for the symmetric eigenvalue problem. Then we rework the error analysis for one step/weep of Jacobi’s method in floating point arithmetic.
3.1 Jacobi’s method

Let $A$ be an $n \times n$ real symmetric matrix. The Jacobi method aims to construct a sequence of orthogonal updates $A^{(k+1)} = J_k^T A^{(k)} J_k$ such that the off-diagonal entries of $A^{(k+1)}$ are closer to zeros than $A^{(k)}$, where $A^{(0)} = A$ and $J_k$ is a Jacobi rotation. When $\text{off}(A^{(k)})$ is close to the zero matrix sufficiently, a computed eigenvalue decomposition of the original matrix $A$ is available.

Define a Jacobi rotation $J(i, j; c, s)$ by

$$J(i, j; c, s) = I_n + [e_i, e_j] \begin{bmatrix} c - 1 & s \\ -s & c - 1 \end{bmatrix} [e_i^T, e_j^T], \quad (3.1)$$

where $c, s \in \mathbb{R}$ is such that $c^2 + s^2 = 1$. Then we have the following result [25, §8.5].

**Lemma 3.1** Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then, for any index pair $(i, j)$ with $1 \leq i < j \leq n$, there exists a Jacobi rotation $J = J(i, j; c, s)$ defined by (3.1) such that, for the updated matrix $B = J^T A J$,

$$b_{ij} = b_{ji} = 0, \quad b_{ii}^2 + b_{jj}^2 = a_{ii}^2 + a_{jj}^2 + 2a_{ij}^2,$$

where $c = (1 + t^2)^{-1/2}$ and $s = tc$ with $t = 1/(\mu + \sqrt{1 + \mu^2})$ if $\mu \geq 0$ and $t = 1/(\mu - \sqrt{1 + \mu^2})$ if $\mu < 0$ for $\mu = (a_{jj} - a_{ii})/(2a_{ij})$. If $a_{ij} = 0$, then we set $(c, s) = (1, 0)$.

From Lemma 3.1, we observe that the updated matrix $B = J^T A J$ agrees with $A$ except in rows and columns $i$ and $j$ and $\|\text{off}(B)\|_F^2 = \|B\|_F^2 - \sum_{i=1}^n b_{ii}^2 = \|A\|_F^2 - \sum_{i=1}^n a_{ii}^2 + (a_{ii}^2 + a_{jj}^2 - b_{ii}^2 - b_{jj}^2) = \|\text{off}(A)\|_F^2 - 2a_{ij}^2$.

To minimize $\|\text{off}(B)\|_F$, a classical strategy is to choose the index $(i, j)$ such that the off-diagonal element $a_{ij}$ has the largest absolute value, i.e., $|a_{ij}| = \max_{p \neq q} |a_{pq}|$. This leads to the classical Jacobi algorithm, which is stated as Algorithm 3.1.

**Algorithm 3.1** Classical Jacobi’s method for the symmetric eigenvalue problem.

**Require:** A symmetric matrix $A \in \mathbb{R}^{n \times n}$ and a tolerance $\epsilon > 0$. Let $P = I_n$.

1. while $\|\text{off}(A)\|_F \geq \epsilon \|A\|_F$ do
2. Choose $(i, j)$ such that $|a_{ij}| = \max_{p \neq q} |a_{pq}|$.
3. Compute a cosine-sine group $(c, s)$ as in Lemma 3.1.
4. Set $A = J(i, j; c, s)^T A J(i, j; c, s)$ and $P = PJ(i, j; c, s)$.
5. end while

Let $A_k$ be the matrix $A_0 = A$ after $k$ Jacobi updates. Then Algorithm 3.1 converges linearly in the sense that $\|\text{off}(A_k)\|_F^2 \leq (1 - 1/N^k) \|\text{off}(A_0)\|_F^2$ [25, §8.5], where $N = n(n-1)/2$. Here, we refer to $N$ Jacobi updates as a sweep. The quadratic convergence of Algorithm 3.1 was established in [43] in the sense that for some constant $\alpha > 0$, $\|\text{off}(A_{k+N})\|_F \leq \alpha \|\text{off}(A_k)\|_F^2$ for $k$ sufficiently large.

We note that it is expensive to find the optimal index $(i, j)$ in each Jacobi update. A feasible alternative is to update $A$ by rows or columns. This is the so-called cyclic Jacobi algorithm, which is described as Algorithm 3.2 [24].

On the quadratic convergence of Algorithm 3.2, one may refer to [29, 47, 48].
Algorithm 3.2 Cyclic Jacobi's method for the symmetric eigenvalue problem.

Require: A symmetric matrix $A \in \mathbb{R}^{n \times n}$ and a tolerance $\epsilon > 0$. Let $P = I_n$.
1: while $\|\text{off}(A)\|_F > \epsilon\|A\|_F$ do
2: Choose $(i,j)$ in a general cyclic order or in the row cyclic order.
3: Compute a cosine-sine group $(c,s)$ as in Lemma 3.1.
4: Set $A = J(i,j;c,s)\hat{A}(i,j;c,s)$ and $P = PJ(i,j;c,s)$.
5: end while

3.2 Error analysis for one step of Jacobi’s method in floating point arithmetic

In this subsection, we consider the error analysis for one step of Jacobi’s method in floating point arithmetic. We use the standard model for floating point arithmetic [31, pp.40]

$$\text{fl}(x \text{ op } y) = (x \text{ op } y)(1 + \delta_1) = (x \text{ op } y)/(1 + \delta_2), \quad |\delta_1|, |\delta_2| \leq u, \quad \text{op} = +, -, *, /,$$

$$\text{fl}(\sqrt{x}) = \sqrt{x}(1 + \delta), \quad |\delta| \leq u,$$

where $u$ is the unit roundoff. Here, fl($x$) means floating-point operation of a real number $x$ at precision $u$.

We also recall the following lemma (see for instance [31, pp.63]).

Lemma 3.2 If $|\delta_i| \leq u$ and $\xi_i = \pm 1$ for $i = 1, 2, \ldots, n$, then $\prod_{i=1}^{n}(1 + \delta_i)^{\xi_i} = 1 + \theta_n$, where $|\theta_n| \leq \gamma_n := (1 - u)^{-n} - 1$.

Let $\tilde{\gamma}_j := (1 - u)^{-wj} - 1$ for a small integer constant $w > 0$ whose exact value is unimportant. In what follows, we denote by $\tilde{\theta}_j$ a quantity with $|\tilde{\theta}_j| \leq \tilde{\gamma}_j$.

For the rounding error analysis for the computed Jacobi rotation $\hat{J}(i,j;\hat{c},\hat{s})$, we have the following result [14, Lemma 3.2]. Here, the computed value in floating point arithmetic is denoted by $\hat{c}$.

Lemma 3.3 Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Suppose the exact Jacobi rotation $\hat{J} = J(i,j;c,s)$ constructed by Lemma 3.1 is such that $J^T A J$ has zeros in the $(i,j)$ and $(j,i)$ positions. Then the computed Jacobi rotation $\hat{J} = \hat{J}(i,j;\hat{c},\hat{s})$ is such that $\hat{c} = c(1 + \tilde{\theta}_1), \hat{s} = s(1 + \tilde{\theta}_1), \tilde{\theta}_1 = t(1 + \tilde{\theta}_1')$, where $c, s$ and $t$ are defined by Lemma 3.1.

We have the following result after one step of Jacobi’s method.

Lemma 3.4 If one step of Jacobi’s method is performed in the $(p,q)$ plane on the matrix $A_k = (\hat{a}_{ij}^{(k)})$ with the computed Jacobi rotation $\hat{J}_k = J(p,q;\hat{c}_k,\hat{s}_k)$ and the exact Jacobi rotation $J_k = J(p,q;c_k,s_k)$, then the computed $A_{k+1} = (\hat{a}_{ij}^{(k+1)})$ satisfies

(i) element invariance:

$$\hat{a}_{ij}^{(k+1)} = a_{ij}^{(k)} \quad \forall i, j \neq p, q,$$

(ii) proximity to zero:

$$|\hat{a}_{pq}^{(k+1)}| \leq (|\hat{a}_{pq}^{(k)}| + |s_kc_k|(|a_{pp}^{(k)}| + |a_{qq}^{(k)}|))\bar{\gamma}_4 \leq \sqrt{2}|\hat{a}_{pq}^{(k)}|^2 + |\hat{a}_{pp}^{(k)}|^2 + |\hat{a}_{qq}^{(k)}|^2 \cdot \bar{\gamma}_4,$$
and (iii) sum of squares controllability:

\[
|\hat{a}_{pq}^{(k+1)}|^2 + |\hat{a}_{qj}^{(k+1)}|^2 \leq (|\hat{a}_{pq}^{(k)}|^2 + |\hat{a}_{qj}^{(k)}|^2)(1 + 2\gamma_4) \quad \forall j \neq p, q.
\]

(3.4)

**Proof.** We first show (3.2). Observe that \(A_{k+1}\) agrees with \(A_k\) except in rows and columns \(p\) and \(q\). This implies that (3.2) holds.

Next, we show the inequality (3.3). It follows from Lemma 3.3 that

\[
\hat{a}_{pq}^{(k+1)} = \mathfrak{I} \left( \begin{bmatrix} \hat{c}_k & \hat{s}_k \end{bmatrix}^T \begin{bmatrix} \hat{a}_{pq}^{(k)} & \hat{a}_{qj}^{(k)} \\ \hat{a}_{qj}^{(k)} & \hat{a}_{qq}^{(k)} \end{bmatrix} \begin{bmatrix} \hat{c}_k \\ \hat{s}_k \end{bmatrix} \right)
\]

\[
= \mathfrak{I}(\hat{s}_k(\hat{c}_k\hat{a}_{pq}^{(k)} - \hat{s}_k\hat{a}_{qj}^{(k)}) + \hat{c}_k(\hat{c}_k\hat{a}_{qj}^{(k)} - \hat{s}_k\hat{a}_{qq}^{(k)}))
\]

\[
= \mathfrak{I}(\hat{s}_k(\hat{c}_k\hat{a}_{pq}^{(k)} - \hat{s}_k\hat{a}_{qj}^{(k)}))(1 + \theta_1) + \mathfrak{I}(\hat{c}_k(\hat{c}_k\hat{a}_{qj}^{(k)} - \hat{s}_k\hat{a}_{qq}^{(k)}))(1 + \theta_1)
\]

\[
= \hat{s}_k \cdot \mathfrak{I}(\hat{c}_k\hat{a}_{pq}^{(k)}(1 + \theta_2) - \hat{s}_k\hat{a}_{qj}^{(k)}(1 + \theta_2))(1 + \theta_2)
\]

\[
+ \hat{c}_k \cdot \mathfrak{I}(\hat{c}_k\hat{a}_{pq}^{(k)}(1 + \theta_2') - \hat{s}_k\hat{a}_{qj}^{(k)}(1 + \theta_2'))(1 + \theta_2')
\]

\[
= \hat{s}_k\hat{c}_k\hat{a}_{pq}^{(k)}(1 + \theta_4) - \hat{s}_k\hat{a}_{qj}^{(k)}(1 + \theta_4') + \hat{c}_k\hat{a}_{pq}^{(k)}(1 + \theta_4') - \hat{s}_k\hat{c}_k\hat{a}_{qj}^{(k)}(1 + \theta_4')
\]

\[
= \hat{s}_k\hat{c}_k\hat{a}_{pq}^{(k)}(1 + \theta_4) - \hat{s}_k\hat{a}_{qj}^{(k)}(1 + \theta_4') + \hat{c}_k\hat{a}_{pq}^{(k)}(1 + \theta_4') - \hat{s}_k\hat{c}_k\hat{a}_{qj}^{(k)}(1 + \theta_4')
\]

\[
= \hat{s}_k\hat{c}_k\hat{a}_{pq}^{(k)}(1 + \theta_4) - s_k^2\hat{a}_{qj}^{(k)}(1 + \theta_4') + \hat{c}_k\hat{a}_{pq}^{(k)}(1 + \theta_4') - \hat{s}_k\hat{c}_k\hat{a}_{qj}^{(k)}(1 + \theta_4')
\]

\[
= \hat{s}_k\hat{c}_k\hat{a}_{pq}^{(k)}\hat{\theta}_4 - s_k^2\hat{a}_{qj}^{(k)}\hat{\theta}_4' + \hat{c}_k\hat{a}_{pq}^{(k)}\hat{\theta}_4' - \hat{s}_k\hat{c}_k\hat{a}_{qj}^{(k)}\hat{\theta}_4',
\]

where the last equality uses the fact that \(\hat{a}_{pq}^{(k)}(c_k^2 - s_k^2) + (\hat{a}_{pq}^{(k)} - \hat{a}_{qj}^{(k)})c_k s_k = 0\). We note that \(c_k^2 + s_k^2 = 1\) and \(|s_k c_k| \leq 1/2(c_k^2 + s_k^2) = 1/2\). Then the inequality (3.3) follows from the Cauchy-Schwarz inequality.

Finally, we show the inequality (3.4). We have by Lemma 3.3, for any \(j \neq p, q\),

\[
|\hat{a}_{pq}^{(k+1)}|^2 + |\hat{a}_{qj}^{(k+1)}|^2 \leq (|\hat{a}_{pq}^{(k)}|^2 + |\hat{a}_{qj}^{(k)}|^2)(1 + 2\gamma_4)
\]

\[
\leq |(\hat{a}_{pq}^{(k)})^2 + (\hat{a}_{qj}^{(k)})^2| + 4|s_k c_k| |\hat{a}_{pq}^{(k)}\hat{a}_{qj}^{(k)}| \gamma_4
\]

\[
\leq |(\hat{a}_{pq}^{(k)})^2 + (\hat{a}_{qj}^{(k)})^2| + 2(|\hat{a}_{pq}^{(k)}| + |\hat{a}_{qj}^{(k)}|) \gamma_4
\]

\[
\leq |(\hat{a}_{pq}^{(k)})^2 + (\hat{a}_{qj}^{(k)})^2| + 2(|\hat{a}_{pq}^{(k)}| + |\hat{a}_{qj}^{(k)}|) \gamma_4 = (|(\hat{a}_{pq}^{(k)})^2 + (\hat{a}_{qj}^{(k)})^2|)(1 + 2\gamma_4).
\]

This completes the proof. \(\Box\)

On the error bound after one Jacobi rotation, we have the following result.
Lemma 3.5 If one step of Jacobi’s method is performed in the \((p, q)\) plane on the matrix \(A_k = (\hat{a}^{(k)}_{ij})\) with the computed Jacobi rotation \(\hat{J}_k = \hat{J}(p, q; \hat{c}_k, \hat{s}_k)\) and the exact Jacobi rotation \(J_k = J(p, q; c_k, s_k)\), then the computed \(A_{k+1}\) satisfies

\[
A_{k+1} = J_k^T A_k J_k + Y_k,
\]

where the symmetric matrix \(G_k \in \mathbb{R}^{n \times n}\) has zero entries except that the entries at the intersections of rows and columns \(p\) and \(q\) are the same as those of \(A_k\), and the symmetric matrix \(H_k \in \mathbb{R}^{n \times n}\) has zero entries except that \((H_k)_{pq} = \hat{a}^{(k)}_{pq}\) and \((H_k)_{ij} = \hat{a}^{(k)}_{ij}\) for all \(j \neq p, q\) and \((H_k)_{ip} = \hat{a}^{(k)}_{ip}\) and \((H_k)_{iq} = \hat{a}^{(k)}_{iq}\) for all \(i \neq p, q\).

Proof. Let \(J_k^T A_k J_k = (\hat{a}^{(k+1)}_{ij})\). Then we have

\[
\hat{a}^{(k+1)}_{pp} = \text{fl} \left( \begin{bmatrix} \hat{c}_k^2 & \hat{s}_k^2 & \hat{c}_k \hat{s}_k & - \hat{s}_k \hat{c}_k \end{bmatrix}^T \begin{bmatrix} \hat{a}^{(k)}_{pp} \\ \hat{a}^{(k)}_{pq} \\ \hat{a}^{(k)}_{qp} \\ \hat{a}^{(k)}_{qq} \end{bmatrix} \right),
\]

Thus, \(|\hat{a}^{(k+1)}_{pp} - \hat{a}^{(k)}_{pp}| \leq \left( c_k^2 |\hat{a}^{(k)}_{pp}| + s_k^2 |\hat{a}^{(k)}_{pq}| \right) + 2|s_k c_k||\hat{a}^{(k)}_{pq}| \gamma_4 \).

Analogously, we can show that \(|\hat{a}^{(k+1)}_{qq} - \hat{a}^{(k)}_{qq}| \leq \left( c_k^2 |\hat{a}^{(k)}_{pp}| + s_k^2 |\hat{a}^{(k)}_{pq}| \right) + 2|s_k c_k||\hat{a}^{(k)}_{pq}| \gamma_4 \).

For any \(j \neq p, q\), we have

\[
\hat{a}^{(k+1)}_{pq} = \text{fl} \left( \begin{bmatrix} \hat{c}_k \\ \hat{s}_k \end{bmatrix}^T \begin{bmatrix} \hat{a}^{(k)}_{pq} \\ \hat{a}^{(k)}_{qp} \end{bmatrix} \right) = \hat{c}_k \hat{a}^{(k)}_{pq} (1 + \theta_2) - \hat{s}_k \hat{a}^{(k)}_{qp} (1 + \theta_2'),
\]

and thus \(|\hat{a}^{(k+1)}_{pq} - \hat{a}^{(k)}_{pq}| \leq (|c_k||\hat{a}^{(k)}_{pq}| + |s_k||\hat{a}^{(k)}_{qp}|) \gamma_2 \).

In a similar way, we have \(|\hat{a}^{(k+1)}_{qj} - \hat{a}^{(k)}_{qj}| \leq (|s_k||\hat{a}^{(k)}_{pq}| + |c_k||\hat{a}^{(k)}_{qp}|) \gamma_2 \). Using (3.3) and the fact \(\hat{a}^{(k+1)}_{pq} = 0\), we obtain \(|\hat{a}^{(k+1)}_{qj} - \hat{a}^{(k)}_{qj}| \leq (|\hat{a}^{(k)}_{pq}| + |s_k c_k||\hat{a}^{(k)}_{pq}| + |\hat{a}^{(k)}_{qp}|) \gamma_4 \).
Therefore,

\[ \|Y_k\|^2_F = \|A_{k+1} - J_k^T A_k J_k\|^2_F \]
\[ = (\hat{a}_{pp}^{(k+1)} - \hat{a}_{pp}^{(k+1)})^2 + (\hat{a}_{qq}^{(k+1)} - \hat{a}_{qq}^{(k+1)})^2 + 2(\hat{a}_{pq}^{(k+1)} - \hat{a}_{pq}^{(k+1)}) \]
\[ + 2 \sum_{j \neq p,q} ((\hat{a}_{pj}^{(k+1)} - \hat{a}_{pj}^{(k+1)})^2 + (\hat{a}_{qj}^{(k+1)} - \hat{a}_{qj}^{(k+1)})^2) \]
\[ \leq (c_k^2 |\hat{a}_{pp}^{(k)}| + s_k^2 |\hat{a}_{qq}^{(k)}| + 2|s_k c_k| |\hat{a}_{pp}^{(k)}|)^2 \gamma_4^2 + (s_k^2 |\hat{a}_{qq}^{(k)}| + c_k^2 |\hat{a}_{pq}^{(k)}| + 2|s_k c_k| |\hat{a}_{pq}^{(k)}|^2) \gamma_4^2 \]
\[ + 2 \sum_{j \neq p,q} ((c_k |\hat{a}_{pj}^{(k)}| + |s_k| |\hat{a}_{qj}^{(k)}|)^2 + (|s_k| |\hat{a}_{pq}^{(k)}| + |c_k| |\hat{a}_{pq}^{(k)}|)^2) \gamma_4^2 \]
\[ \leq (1 + \sqrt{2} |s_k c_k|) |\hat{a}_{pp}^{(k)}| + (1 + \sqrt{2} |s_k c_k|) |\hat{a}_{qq}^{(k)}| + (\sqrt{2} + 4|s_k c_k|) |\hat{a}_{pq}^{(k)}|^2 \gamma_4^2 \]
\[ + 4 \sum_{j \neq p,q} (|\delta_{pj}^{(k)}|^2 + |\delta_{qj}^{(k)}|^2) \gamma_4^2 \]
\[ \leq (2(1 + \sqrt{2} |s_k c_k|)^2 + (1 + 2\sqrt{2} |s_k c_k|)^2) ((\hat{a}_{pp}^{(k)})^2 + (\hat{a}_{qq}^{(k)})^2 + 2(\hat{a}_{pq}^{(k)})^2) \gamma_4^2 \]
\[ + 4 \sum_{j \neq p,q} (|\delta_{pj}^{(k)}|^2 + |\delta_{qj}^{(k)}|^2) \gamma_4^2 \]
\[ \leq 2(1 + \sqrt{2}^2 ((\hat{a}_{pp}^{(k)})^2 + (\hat{a}_{qq}^{(k)})^2 + 2(\hat{a}_{pq}^{(k)})^2) \gamma_4^2 + 4 \sum_{j \neq p,q} (|\delta_{pj}^{(k)}|^2 + |\delta_{qj}^{(k)}|^2) \gamma_4^2 \]
\[ \equiv 2(3 + 2\sqrt{2}) \|G_k\|_F^2 \gamma_4^2 + 2 \|H_k\|_F^2 \gamma_4^2, \]

where the third inequality uses the fact that \(a_1^2 + a_2^2 + a_3^2 \leq (a_1 + a_2 + a_3)^2\) for all \(a_1, a_2, a_3 \geq 0\) and the Cauchy-Schwarz inequality, the fourth inequality uses the Cauchy-Schwarz inequality, and the fifth inequality uses the fact that \(|s_k c_k| \leq 1/2\). The lemma follows by taking the square root of the above inequality.

On the off-diagonal entries of the updated matrix after one step of Jacobi’s method, we have the following result.

**Lemma 3.6** If one step of Jacobi’s method is performed in the \((p, q)\) plane on the matrix \(A_k = (\hat{a}_{ij}^{(k)})\) with the computed Jacobi rotation \(J_k = J(p, q; \hat{c}_k, \hat{s}_k)\), then the computed \(A_{k+1}\) satisfies

\[ \|\text{off}(A_{k+1})\|_F^2 - \|\text{off}(A_k)\|_F^2 \leq -2|\hat{a}_{pq}^{(k)}|^2 + 2\|H_k\|_F^2 \gamma_4 + 2\|G_k\|_F^2 \gamma_4^2, \]

where \(G_k\) and \(H_k\) are defined as in Lemma 3.5. Moreover, for arbitrarily chosen index pairs \(\{(p_k, q_k)\}_{k=0}^{N-1}\), we have

\[ 2 \sum_{k=0}^{N-1} |\hat{a}_{pq_k}^{(k)}|^2 \leq \|\text{off}(A_0)\|_F^2 + 2 \sum_{k=0}^{N-1} \|H_k\|_F^2 \gamma_4 + 2 \sum_{k=0}^{N-1} \|G_k\|_F^2 \gamma_4^2. \]
Proof. It directly follows from Lemma 3.4 that, for the chosen index pair \((p, q)\),
\[
\|\text{off}(A_{k+1})\|_F^2 - \|\text{off}(A_k)\|_F^2 = \sum_{i \neq j} (|\hat{a}_{ij}^{(k+1)}|^2 - |\hat{a}_{ij}^{(k)}|^2)
\]
\[
= 2(\hat{a}_{pq}^{(k+1)})^2 - |\hat{a}_{pq}^{(k)}|^2 + 2 \sum_{j \neq p, q} (|\hat{a}_{pj}^{(k+1)}|^2 + |\hat{a}_{qj}^{(k+1)}|^2 - |\hat{a}_{pj}^{(k)}|^2 - |\hat{a}_{qj}^{(k)}|^2)
\]
\[
\leq 2(\hat{a}_{pp}^{(k)})^2 + |\hat{a}_{pq}^{(k)}|^2 + 2|\hat{a}_{pq}^{(k)}|^2 \gamma_4 - 2|\hat{a}_{pq}^{(k)}|^2 + 4 \sum_{j \neq p, q} (|\hat{a}_{pj}^{(k)}|^2 + |\hat{a}_{qj}^{(k)}|^2) \gamma_4
\]
\[
= 2\|G_k\|_F^2 \gamma_4^2 - 2|\hat{a}_{pq}^{(k)}|^2 + 2\|H_k\|_F^2 \gamma_4,
\]
which is (3.5). For arbitrarily chosen index pairs \(\{(p_k, q_k)\}_{k=0}^{N-1}\), we have by (3.5),
\[
\|\text{off}(A_N)\|_F^2 - \|\text{off}(A_0)\|_F^2 \leq 2 \sum_{k=0}^{N-1} \|G_k\|_F^2 \gamma_4^2 - 2 \sum_{k=0}^{N-1} |\hat{a}_{pq}^{(k)}|^2 + 2 \sum_{k=0}^{N-1} \|H_k\|_F^2 \gamma_4,
\]
which yields (3.6).
\[
\square
\]

Remark 3.7 We observe from Lemma 3.6 that \(A_k = (\hat{a}_{ij}^{(k)})\) moves closer to diagonal form with each Jacobi step if \(\gamma_4\) is such that \(\|H_k\|_F^2 \gamma_4 + \|G_k\|_F^2 \gamma_4^2 < |\hat{a}_{pq}^{(k)}|^2\). If \(|\hat{a}_{pq}^{(k)}| \leq u \min\{\|\hat{a}_{pp}^{(k)}\|, \|\hat{a}_{pq}^{(k)}\|\}\) or \(|\hat{a}_{pq}^{(k)}| \leq u(|\hat{a}_{pp}^{(k)}\hat{a}_{pq}^{(k)}|)^{1/2}\), then one may set \(\hat{a}_{pq}^{(k)} = \hat{a}_{pp}^{(k)} = 0\) [16, 14].

The following theorem states the error bound for one step of Jacobi’s method.

Theorem 3.8 Let \(A_k\) be the matrix \(A_0 = A\) after \(k\) Jacobi updates with the computed Jacobi rotations \(\{\hat{J}_j = \tilde{J}(p_j, q_j; \hat{c}_j, \hat{s}_j)\}_{j=0}^{k-1}\) in Algorithm 3.1. If \(2N \gamma_4 < 1\), then we have, for any \(k \geq 0\),
\[
\|\text{off}(A_{k+1})\|_F^2 \leq c\|\text{off}(A_k)\|_F^2 + 2\|G_k\|_F^2 \gamma_4^2,
\]
which is (3.7) where \(c = 1 - 1/N + 2 \gamma_4 \in (0, 1)\). Moreover, for any \(k \geq 1\), there exists some ordering \(\{\lambda_{\pi(w)}(A_0)\}_{w=1}^n\) of \(\{\lambda_1(A_0)\}_{w=1}^n\) such that
\[
\left(\sum_{w=1}^n (\hat{a}_{w}^{(k)} - \lambda_{\pi(w)}(A_0))\right)^{1/2} \leq \|\text{off}(A_k)\|_F + \gamma_4 \sum_{j=0}^{k-1} \varphi_j,
\]
which is (3.8) with \(\varphi_j = ((6 + 4\sqrt{2})\|G_j\|_F^2 + 2\|H_j\|_F^2)^{1/2}\) and \(G_j\) and \(H_j\) being defined as in Lemma 3.5 for \(j = 0, 1, \ldots, k - 1\).

Proof. From the procedure of Algorithm 3.1, it follows that, in the \(k\)th iteration, the index pair \((p_k, q_k)\) is chosen such that \(|\hat{a}_{pq}^{(k)}| = \max_{i \neq j} |\hat{a}_{ij}^{(k)}|\). Then, by (3.5) and using the definition of \(G_k\) and the fact that \(\|H_k\|_F^2 \leq \|\text{off}(A_k)\|_F^2\) and \(\text{off}(A_k)\|_F^2 \leq 2N |a_{pq}^{(k)}|^2\), we obtain (3.7), where \(c = 1 - 1/N + 2 \gamma_4 \in (0, 1)\) due to \(2N \gamma_4 < 1\).

Let \(J_j = \hat{J}(p_j, q_j; c_j, s_j)\) be the exact Jacobi rotation corresponding to \(\hat{J}_j\) for \(j = 0, 1, \ldots, k - 1\). Note that \(A_j\) and \(J_j^T A_j J_j\) have the same eigenvalues for \(j = 0, 1, \ldots, k - 1\). Then, for
any } k \geq 1 \text{, there exist some ordering } \{ \lambda_{(j)(i)}(A_j) \}_{i=1}^{n} \text{ of } \{ \lambda_{i}(A_j) \}_{i=1}^{n} \text{ with } D(A_j; \pi^{(j)}) = \text{diag}(\lambda_{(j)(1)}(A_j), \ldots, \lambda_{(j)(n)}(A_j)) \text{ for } j = 0, 1, \ldots, k - 1 \text{ such that }

\left( \sum_{i=1}^{n} (a_{ii}^{(k)} - \lambda_{(j)(i)}(A_0))^2 \right)^{1/2} = \| A_k - \text{off}(A_k) - D(A_0; \pi^{(0)}) \|_F 

\leq \| A_k - \text{off}(A_k) - D(J_{k-1}^T A_{k-1} J_{k-1}; \pi^{(k-1)}) \|_F + \| D(A_{k-1}; \pi^{(k-1)}) - D(A_0; \pi^{(0)}) \|_F 

\leq \| A_k - \text{off}(A_k) - J_{k-1}^T A_{k-1} J_{k-1} \|_F + \| D(A_{k-1}; \pi^{(k-1)}) - D(A_0; \pi^{(0)}) \|_F 

\leq \| \text{off}(A_k) \|_F + \| Y_{k-1} \|_F + \| D(A_{k-1}; \pi^{(k-1)}) - D(A_{j-2}; \pi^{(k-2)}) \|_F + \| D(A_{j-2}; \pi^{(k-2)}) - D(A_0; \pi^{(0)}) \|_F 

\leq \cdots \leq \| \text{off}(A_k) \|_F + \sum_{j=0}^{k-1} \| Y_j \|_F \leq \| \text{off}(A_k) \|_F + \tilde{\gamma}_4 \sum_{j=0}^{k-1} \varphi_j,

\text{where the second, third, fifth inequalities and so on use Lemma 2.4 and/or } Y_j = A_{j+1} - J_{j}^T A_j J_j \text{ for } j = k - 1, k - 2, \ldots, 0 \text{ as defined in Lemma 3.5, and the last inequality uses Lemma 3.5.} \]
3.3.1 The general cyclic order with distinct eigenvalues

We consider the error analysis for one sweep of the general cyclic Jacobi method in floating point arithmetic for a symmetric matrix \( A \in \mathbb{R}^{n \times n} \) with \( n \) distinct eigenvalues.

We first discuss the relationship between the Frobenius matrix norm of the off-diagonal entries of \( A_k \) and the minimal gap between the eigenvalues of \( A_k \) in the general cyclic Jacobi method.

**Lemma 3.11** If one step of Jacobi’s method is performed in the \((p, q)\) plane on the matrix \( A_k = (\tilde{a}_{ij}^{(k)}) \) with the computed Jacobi rotation \( \tilde{J}_k = J(p, q; \hat{c}_k, \hat{s}_k) \) and \( A \) has \( n \) distinct eigenvalues with \( d(A_k) := \min_{i \neq j} |\lambda_i(A_k) - \lambda_j(A_k)| > 0 \), then the computed \( A_{k+1} \) also has \( n \) distinct eigenvalues with \( d(A_{k+1}) \geq d(A_k) - 2\varphi_k\tilde{\gamma}_4 > 0 \), provided that \( 2\varphi_k\tilde{\gamma}_4 < d_k \), where \( \varphi_k = \left((6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^2\right)^{1/2} \) with \( G_k \) and \( H_k \) being defined as in Lemma 3.5.

**Proof.** Let \( J_k = J(p, q; c_k, s_k) \) be the exact Jacobi rotation. Then, by Lemma 2.4 we have, for any \( i \neq j \),

\[
|\lambda_i(A_{k+1}) - \lambda_j(A_{k+1})| \\
= |\lambda_i(A_k) - \lambda_j(A_k) + \lambda_i(A_k) - \lambda_j(A_k)| \\
\geq |\lambda_i(A_k) - \lambda_j(A_k) - |\lambda_i(A_k) - \lambda_j(A_k)|| \\
= |\lambda_i(A_k) - \lambda_j(A_k)| - |\lambda_i(A_{k+1}) - \lambda_j(A_{k+1})| \\
\geq d(A_k) - 2\|A_{k+1} - J_k^T A_k J_k\|,
\]

completing the proof of the lemma by invoking Lemma 3.5.

**Lemma 3.12** Suppose one step of Jacobi’s method is performed in the \((p, q)\) plane on the matrix \( A_k = (\tilde{a}_{ij}^{(k)}) \) with the computed Jacobi rotation \( \tilde{J}_k = J(p, q; \hat{c}_k, \hat{s}_k) \). Let \( \varphi_k = \left((6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^2\right)^{1/2} \) with \( G_k \) and \( H_k \) being defined as in Lemma 3.5. If \( A_k \) has \( n \) distinct eigenvalues with \( 4\|\text{off}(A_k)\|_F < d(A_k) \) and

\[
\frac{1}{8}\varphi_k^2\tilde{\gamma}_4^2 + \frac{1}{2}\|\text{off}(A_k)\|_F^2 \varphi_k\tilde{\gamma}_4 + \|H_k\|_F^2 \varphi_k\tilde{\gamma}_4 + \|G_k\|_F^2 \varphi_k\tilde{\gamma}_4 < |\tilde{a}_{pq}^{(k)}|^2,
\]

(3.9)

then we have \( 4\|\text{off}(A_{k+1})\|_F < d(A_{k+1}) \).

**Proof.** Let \( b_k = \|H_k\|_F^2 \varphi_k\tilde{\gamma}_4 + \|G_k\|_F^2 \varphi_k\tilde{\gamma}_4 \). From (3.9) we have \( b_k < |\tilde{a}_{pq}^{(k)}|^2 \). By Lemma 3.6, we have \( \|\text{off}(A_{k+1})\|_F < \|\text{off}(A_k)\|_F \) and \( \|\text{off}(A_{k+1})\|_F^2 + 2|\tilde{a}_{pq}^{(k)}|^2 - 2b_k > 0 \). This, together with (3.9) again, yields

\[
\left(\|\text{off}(A_{k+1})\|_F^2 + \frac{1}{4}\varphi_k^2\tilde{\gamma}_4^2\right) - \left(\|\text{off}(A_{k+1})\|_F^2 + 2|\tilde{a}_{pq}^{(k)}|^2 - 2b_k\right) \\
\leq \frac{1}{4}\varphi_k^2\tilde{\gamma}_4^2 + \|\text{off}(A_{k+1})\|_F^2 \varphi_k\tilde{\gamma}_4 + 2b_k - 2|\tilde{a}_{pq}^{(k)}|^2 \\
\leq \frac{1}{4}\varphi_k^2\tilde{\gamma}_4^2 + \|\text{off}(A_{k+1})\|_F^2 \varphi_k\tilde{\gamma}_4 + 2b_k - 2|\tilde{a}_{pq}^{(k)}|^2 < 0.
\]

(3.10)
By hypothesis, we have \(4\|\text{off}(A_k)\|_F < d(A_k)\). Then, by following the arguments similar to the proof of Lemma 3.11 we have

\[
\begin{align*}
d(A_{k+1}) & \geq d(A_k) - 2\varphi_k \tilde{\gamma}_4 > 4\|\text{off}(A_k)\|_F - 2\varphi_k \tilde{\gamma}_4. \\
& \geq 4\left(\|\text{off}(A_{k+1})\|_F^2 + 2(\delta_{pq}^{(k)})^2 - 2b_k\right)^{1/2} - 2\varphi_k \tilde{\gamma}_4 > 4\|\text{off}(A_{k+1})\|_F,
\end{align*}
\]

where the third inequality uses (3.5) and the last inequality uses (3.10).

\[\square\]

**Lemma 3.13** Let \(G = (g_{ij})\) be an \(n \times n\) real symmetric matrix with \(n\) distinct eigenvalues. If \(d(G) = \min_{i \neq j} |\lambda_i(G) - \lambda_j(G)| > 0\) and \(\|\text{off}(G)\|_F \leq d(G)/4\), then we have, for some ordering of \(\{\lambda_i(G)\}_{i=1}^n\), \(|g_{ii} - g_{jj}| \geq d(G)/2\) for all \(i \neq j\), and for \((p, q)\) plane on \(G\), the angle \(\theta_{pq}\) of the exact Jacobi rotation satisfies \(|\sin \theta_{pq}| \leq 2g_{pq}/d(G)\).

In the following, we study the error analysis for one sweep of the general cyclic Jacobi method in floating point arithmetic. Suppose that \(A \in \mathbb{R}^{n \times n}\) is symmetric with \(n\) distinct eigenvalues. In the general cyclic Jacobi method, all off-diagonal entries are annihilated (in the sense of floating point arithmetic) successively in some order. For convenience, we assume that \(\|\text{off}(A_0)\|_F < d(A_0)/4\). We use \((p_k, q_k)\) as the chosen annihilation position at the \(k\)th iteration of the whole general cyclic Jacobi method. During a fixed cycle of \(N\) consecutive rotations of a general cyclic ordering starting from \(k = 0\), the entries before annihilation are denoted by \(\{z_k\}_{k=0}^{N-1}\) and the computed and exact Jacobi rotations by \(\{\hat{J}(p_k, q_k; \hat{c}_k, \hat{s}_k)\}_{k=0}^{N-1}\) and \(\{J(p_k, q_k; c_k, s_k)\}_{k=0}^{N-1}\).

We now assume that, after the annihilation of the entry \(z_k\) at the \(k\)th iteration, its value will be affected only by a subset of the later Jacobi rotations with subscripts \(k_1, k_2, \ldots, k_r\) with \(r \leq 2(n - 2)\) being a function of \(k\). Let \(\hat{z}_{k,j}\) be the computed value of \(z_k\) after the rotation \(\hat{J}(p_k, q_k; \hat{c}_k, \hat{s}_k)\). Then we have by Lemma 3.4,

\[
\hat{z}_{k,1} = \text{fl}(\hat{z}_{k,0} \hat{c}_{k_1} \pm \hat{a}_k \hat{s}_{k_1}) = \hat{z}_{k,0} \hat{c}_{k_1}(1 + \delta_{k_1})(1 + \delta_{k_1}')(1 + \delta_{k_1}''), \\
\hat{z}_{k,2} = \text{fl}(\hat{z}_{i,1} \hat{c}_{k_2} \pm \hat{a}_k \hat{s}_{k_2}) = \hat{z}_{i,1} \hat{c}_{k_2}(1 + \delta_{k_2})(1 + \delta_{k_2}')(1 + \delta_{k_2}''), \\
\ldots \ldots \\
\hat{z}_{k,r} = \text{fl}(\hat{z}_{i,r-1} \hat{c}_{k_r} \pm \hat{a}_k \hat{s}_{k_r}) = \hat{z}_{i,r-1} \hat{c}_{k_r}(1 + \delta_{k_r})(1 + \delta_{k_r}')(1 + \delta_{k_r}''),
\]

where \(\hat{a}_{k_j}\) stands for an entry of \(A_{k_j}\) at either the \(p_{k_j}\) th or \(q_{k_j}\) th row or the \(p_{k_j}\) th or \(q_{k_j}\) th column except the intersections of rows and columns \(p_{k_j}\) and \(q_{k_j}\).

By Lemma 3.3, we have \(|\hat{c}| \leq 1 + \tilde{\gamma}_1\) and \(|\hat{s}| \leq |s|(1 + \tilde{\gamma}_1)\). Thus,

\[
|\hat{z}_{k,j}| \leq \left(|\hat{z}_{k,j-1}| + |\hat{a}_k||s_{k_j}|\right)(1 + \tilde{\gamma}_1)(1 + \gamma_2) \leq \left(|\hat{z}_{k,j-1}| + |\hat{a}_k||s_{k_j}|\right)(1 + \tilde{\gamma}_2),
\]

(3.11)
for all \(j = 0, 1, \ldots, r\). This implies that

\[
|\hat{z}_{k,r}| \leq (|\hat{z}_{k,r-1}| + |\hat{a}_{k_r}|s_{k_r}|)(1 + \gamma_2) \\
\leq ((|\hat{z}_{k,r-2}| + |\hat{a}_{k_{r-1}}||s_{k_{r-1}}|)(1 + \gamma_2) + |\hat{a}_{k_r}|s_{k_r}|)(1 + \gamma_2) \\
\leq \ldots \\
\leq |\hat{z}_{k,0}|(1 + \gamma_2)^r + |\hat{a}_{k_1}|s_{k_1}|(1 + \gamma_2)^{r-1} + \cdots \\
+ |\hat{a}_{k_{r-1}}||s_{k_{r-1}}|(1 + \gamma_2)^2 + |\hat{a}_{k_r}|s_{k_r}|(1 + \gamma_2)^r.
\]

Without loss of generality, we assume that row and column effects are divided into two segments, \([1, r_1]\) and \([r_1 + 1, r_1 + r_2]\) with \(r = r_1 + r_2\) (see Figure 3.1), i.e., \(\hat{a}_{kj} = e^T_{pk_j}A_{kj}e_{qk_j} = e^T_{pk_k}H_{kj}e_{k_qj}\) for \(j = 1, 2, \ldots, r_1\), and \(\hat{a}_{kj} = e^T_{pk_j}A_{kj}e_{qk} = e^T_{pk_j}H_{kj}e_{qk}\) for \(j = r_1 + 1, \ldots, r_1 + r_2\), where \(H_{kj}\)'s are defined as in Lemma 3.5. Hence,

\[
|\hat{z}_{k,r}| \leq (|\hat{z}_{k,0}| + \sum_{j=1}^{r_1} e^T_{pk_j}H_{kj}e_{k_qj}s_{kj} j + \sum_{j=r_1+1}^{r_1+r_2} e^T_{pk_j}H_{kj}e_{k_qj}s_{kj})(1 + \gamma_2)^r.
\]  

Figure 3.1: The diagram of the row-effected indices (in blue) and the column-effected indices (in red).

Note that, at the end of the complete set of \(N\) rotations, the off-diagonal entries of \(\text{off}(A_N)\) is composed of \(\hat{z}_{k,r}\)'s. By (3.12) we have

\[
|\text{off}(A_N)| = \sum_{k=0}^{N-1} |\hat{z}_{k,r}| \cdot E_k \leq \left( \sum_{k=0}^{N-1} |\hat{z}_{k,0}| \cdot E_k + \sum_{k=0}^{N-1} |s_k||H_k|| \right)(1 + \gamma_2)^{2n-4},
\]

where \(E_k \in \mathbb{R}^{n \times n}\) has zero entries except \(E_k(p_k, q_k) = 1 = E_k(q_k, p_k)\). Therefore,

\[
\|\text{off}(A_N)\|_F = \|\text{off}(A_N)\|_F \\
\leq \left\| \sum_{k=0}^{N-1} |\hat{z}_{k,0}| \cdot E_k \right\|_F(1 + \gamma_2)^{2n-4} + \left\| \sum_{k=0}^{N-1} |s_k||H_k|| \right\|_F(1 + \gamma_2)^{2n-4}.
\]  

(3.13)
In the following, we estimate the two items on the right-hand side of (3.13). We note that \( \hat{z}_{k,0} \) denotes the value of \( z_k \) after its annihilation. By Lemma 3.4 we have

\[
\left\| \sum_{k=0}^{N-1} |\hat{z}_{k,0}| \cdot E_k \right\|_F \leq \left( 2 \sum_{k=0}^{N-1} \|G_k\|_F^2 \right)^{1/2} \gamma_4. \tag{3.14}
\]

Let \( \tilde{\gamma}_4 \) be such that

\[
\frac{1}{8} \varphi_k^2 \tilde{\gamma}_4^2 + \frac{1}{2} \|\text{off}(A_k)\|_F \varphi_k \tilde{\gamma}_4 + \|H_k\|_F^2 \tilde{\gamma}_4^2 + \|G_k\|_F^2 \tilde{\gamma}_4^2 < |\hat{a}_{pkqk}|^2, \tag{3.15}
\]

for \( k = 0, 1, \ldots, N-1 \), and

\[
d(A_0) \geq d + \sum_{k=0}^{N-1} 2 \varphi_k \tilde{\gamma}_4 \tag{3.16}
\]

for some \( d > 0 \). Then by Lemmas 3.12–3.13 we have

\[
\|\text{off}(A_k)\|_F < d(A_k)/4 \quad \text{and} \quad |s_k| \leq 2|z_k|/d(A_k), \tag{3.17}
\]

for \( k = 0, 1, \ldots, N-1 \). Hence,

\[
\left\| \sum_{k=0}^{N-1} |s_k| \|H_k\|_F (1 + \tilde{\gamma}_2)^{2n-4} \right\|_F \leq \max_{0 \leq k \leq N-1} \|H_k\|_F \sum_{k=0}^{N-1} |s_k| \|H_k(1 + \tilde{\gamma}_2)^{2n-4} \|
\]

\[
\leq (1 + \tilde{\gamma}_2)^{2n-4} \max_{0 \leq k \leq N-1} \|H_k\|_F \sum_{k=0}^{N-1} |s_k| \left( \sum_{k=0}^{N-1} |s_k|^2 \right)^{1/2}
\]

\[
\leq (1 + \tilde{\gamma}_2)^{2n-4} \max_{0 \leq k \leq N-1} \|H_k\|_F \left( \sum_{k=0}^{N-1} \frac{|s_k|^2}{(d(A_k))^2} \right)^{1/2}
\]

\[
\leq (1 + \tilde{\gamma}_2)^{2n-4} \sqrt{2N} \cdot \max_{0 \leq k \leq N-1} \|H_k\|_F \left( \sum_{k=0}^{N-1} \frac{2|z_k|^2}{d(A_k)^2} \right)^{1/2}
\]

\[
\leq \alpha_1 \|\text{off}(A_0)\|_F \left( \|\text{off}(A_0)\|_F^2 + 2 \sum_{k=0}^{N-1} \|H_k\|_F^2 \tilde{\gamma}_4 + 2 \sum_{k=0}^{N-1} \|G_k\|_F^2 \tilde{\gamma}_4^2 \right)^{1/2}
\]

\[
\leq \alpha_1 \|\text{off}(A_0)\|_F^2 + \alpha_1 \sum_{k=0}^{N-1} \|H_k\|_F^2 \tilde{\gamma}_4 + \alpha_1 \sum_{k=0}^{N-1} \|G_k\|_F^2 \tilde{\gamma}_4^2, \tag{3.18}
\]

where the third inequality uses the Cauchy-Schwarz inequality, the fourth inequality uses (3.17), the sixth inequality uses (3.6), the last inequality uses the fact that \( a_1 \sqrt{a_1^2 + a_2^2} \leq a_1^2 + \frac{1}{2} a_2^2 \) for all \( a_1, a_2 \geq 0 \), and

\[
\alpha_1 = (1 + \tilde{\gamma}_2)^{2n-4} \sqrt{2N} \cdot \max_{0 \leq k \leq N-1} \|H_k\|_F \min_{0 \leq k \leq N-1} d(A_k) \|\text{off}(A_0)\|_F \tag{3.19}
\]
Substituting (3.14) and (3.18) into (3.13) yields

\[
\|\text{off}(A_N)\|_F \leq \alpha_1\|\text{off}(A_0)\|_F^2 + \alpha_1 \sum_{k=0}^{N-1} \|H_k\|_F^2 \tilde{\gamma}_4 + \alpha_1 \sum_{k=0}^{N-1} \|G_k\|_F^2 \tilde{\gamma}_4^2
\]

\[
+ \left( 2 \sum_{k=0}^{N-1} \|G_k\|_F^2 \right)^{1/2} \tilde{\gamma}_4 (1 + \tilde{\gamma}_2)^{2n-4} \leq \alpha_1\|\text{off}(A_0)\|_F^2 + \beta_1 \tilde{\gamma}_4,
\]

where

\[
\beta_1 = \alpha_1 \left( \sum_{k=0}^{N-1} \|H_k\|_F^2 + \sum_{k=0}^{N-1} \|G_k\|_F^2 \tilde{\gamma}_4 \right) + (1 + \tilde{\gamma}_2)^{2n-4} \left( 2 \sum_{k=0}^{N-1} \|G_k\|_F^2 \right)^{1/2}.
\]

Finally, we show that \(\alpha_1 < \infty\). It follows from (3.16) and Lemma 3.11 that

\[
d(A_{k+1}) \geq d(A_k) - 2\varphi_k \tilde{\gamma}_4 \geq \cdots \geq d(A_0) - \sum_{j=0}^{k} 2\varphi_j \tilde{\gamma}_4 \geq d(A_0) - \sum_{j=0}^{N-1} 2\varphi_j \tilde{\gamma}_4 \geq d > 0,
\]

for \(k = 0, 1, \ldots, N-1\). This implies that \(\min_{0 \leq k \leq N-1} d(A_k) \geq d > 0\). It follows from (3.15) and Lemma 3.6 that \(\|\text{off}(A_N)\|_F \leq \|\text{off}(A_{N-1})\|_F \leq \cdots \leq \|\text{off}(A_0)\|_F\) and thus \(\max_{0 \leq k \leq N-1} \|\text{off}(A_k)\|_F \leq \|\text{off}(A_0)\|_F\). Thus, \(\alpha_1 \leq (1 + \tilde{\gamma}_2)^{2n-4}\sqrt{2N/d} < \infty\).

Based on the above analysis, we have the following result on the error bound for one sweep of the general cyclic Jacobi method in floating point arithmetic.

**Theorem 3.14** Let \(A_k\) be the matrix \(A_0 = A\) after \(k\) Jacobi updates, which is generated by Algorithm 3.2 in floating point arithmetic, where the index pairs \((p_k, q_k)\) are chosen in a general cyclic order. Let \(\varphi_k = ((6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^2)^{1/2}\) with \(G_k\) and \(H_k\) being defined as in Lemma 3.5 with \((p, q) = (p_k, q_k)\) for all \(k \geq 0\). Suppose \(A_0\) has \(n\) distinct eigenvalues and \(\|\text{off}(A_0)\|_F < d(A_0)/4\). If \(\tilde{\gamma}_4\) satisfies the conditions in (3.15)-(3.16), then we have \(\|\text{off}(A_N)\|_F \leq \alpha_1\|\text{off}(A_0)\|_F^2 + \beta_1 \tilde{\gamma}_4\), where the constants \(\alpha_1 > 0\) and \(\beta_1 > 0\) are defined by (3.19) and (3.20), respectively.

**Remark 3.15** In Theorem 3.14, it is easy to see that the condition (3.15) indicates the descent condition in Remark 3.7. The condition in (3.16) is sufficient to guarantee that \(2\varphi_k \tilde{\gamma}_4 < d(A_k)\) (which is not easy to check numerically) for \(k = 0, 1, \ldots, N-1\) and \(\alpha_1\) is a finite constant. We also assume that \(\|\text{off}(A_0)\|_F < d(A_0)/4\) for convenience. In fact, one may assume that \(\|\text{off}(A_t)\|_F < d(A_0)/4\) for some \(t > 0\) and the corresponding error bound can be established similarly by replacing the condition in (3.16) by \(d(A_0) \geq d + \sum_{k=0}^{t+N-1} 2\varphi_k \tilde{\gamma}_4\). This is reasonable since the condition (3.15) is such that \(\|\text{off}(A_t)\|_F\) moves closer to zero and it follows from (3.16) and Lemma 3.11 that \(d(A_t) \geq d(A_0) - \sum_{k=0}^{t} 2\varphi_k \tilde{\gamma}_4 \geq d > 0\).

**Remark 3.16** We see from Theorem 3.14 that if \(\tilde{\gamma}_4 = 0\), then \(A_k\) has the same eigenvalues as \(A_0\), \(d(A_k) = d(A_0)\), and \(\|H_k\|_F \leq \|A_k\|_F \leq \|A_0\|_F\) for all \(k \geq 0\). Thus \(\|\text{off}(A_N)\|_F \leq \sqrt{2N\|\text{off}(A_0)\|_F^2/d(A_0)}\). This coincides with Wilkinson’s result [48].
3.3.2 The row-cyclic order with distinct eigenvalues

We study the error analysis for one sweep of Algorithm 3.2 in floating point arithmetic for a symmetric matrix $A \in \mathbb{R}^{n \times n}$ with $n$ distinct eigenvalues, where all off-diagonal entries are annihilated successively in the row-cyclic order. For simplicity, as in section 3.3.1, we also assume that $\|\text{off}(A_0)\|_F < d(A_0)/4$ and a fixed cycle of $N$ consecutive rotations of a row-cyclic ordering. We will use $(p_k, q_k)$ as the chosen annihilation position at the $k$th iteration of the whole row-cyclic Jacobi method. During a fixed cycle of $N$ consecutive rotations of a row-cyclic ordering starting from $k = 0$, the entries before annihilation are still denoted by $\{z_k\}_{k=0}^{N-1}$ and the computed and exact Jacobi rotations by $\{\hat{J}(p_k, q_k; c_k, s_k)\}_{k=0}^{N-1}$ and $\{J(p_k, q_k; c_k, s_k)\}_{k=0}^{N-1}$.

We first discuss, after the annihilation of the strictly upper diagonal entries in the first row, how their values are affected by the later Jacobi rotations. To illustrate the effects of the annihilation, we list the affected indices in Table 3.1.

Table 3.1: Subsequent effects of annihilation on entries in positions $\{(1, q)\}_{q=2}^n$.

| position | row-indexed annihilation | column-indexed annihilation |
|----------|-------------------------|-----------------------------|
| (1, 2)   | (1, 3), . . . , (1, n)  | (2, 3), . . . , (2, n)      |
| (1, 3)   | (1, 4), . . . , (1, n)  | (2, 3), (3, 4), . . . , (3, n) |
| .        | .                       | .                           |
| (1, p)   | (1, p + 1), . . . , (1, n) | (2, p), . . . , (p − 1, p), (p, p + 1), . . . , (p, n) |
| .        | .                       | .                           |
| (1, n − 2)| (1, n − 1), (1, n) | (2, n − 2), . . . , (n − 3, n − 2), (n − 2, n − 1), (n − 2, n) |
| (1, n − 1)| (1, n) | (2, n − 1), . . . , (n − 2, n − 1), (n − 2, n − 1), (n − 1, n) |
| (1, n)   | /                       | (2, n), . . . , (n − 1, n)  |

We see from Table 3.1 that the entries in the positions $\{(1, q)\}_{q=2}^n$ are affected exactly $n − 2$ times after completing their annihilation.

As (3.11), for any $2 \leq q \leq n − 1$, we have $|\hat{a}_{1q}^{(k+1)}| \leq (|\hat{a}_{1q}^{(k)}| + |\hat{a}_{q,k+2}^{(k)}||s_k|(1 + \tilde{\gamma}_2))$ for $k = q − 1, . . . , n − 2$. This indicates that

$$
|\hat{a}_{1q}^{(n−1)}| \leq (|\hat{a}_{1q}^{(n−2)}| + |\hat{a}_{q,n−2}^{(n−3)}||s_{n−2}|)(1 + \tilde{\gamma}_2) \\
\leq (|\hat{a}_{1q}^{(n−3)}| + |\hat{a}_{q,n−1}^{(n−3)}||s_{n−3}|)(1 + \tilde{\gamma}_2)^2 + |\hat{a}_{q,n}^{(n−2)}||s_{n−2}|(1 + \tilde{\gamma}_2) \\
\leq \cdots \leq |\hat{a}_{1q}^{(q−1)}|(1 + \tilde{\gamma}_2)^{n−q} + \sum_{k=q−1}^{n−2} |s_k||\hat{a}_{q,k+2}^{(k)}|(1 + \tilde{\gamma}_2)^{n−1−k}.
$$

Then, using the inequality $(|a_1| + |a_2|)^2 \leq 2(a_1^2 + a_2^2)$ for all $a_1, a_2 \in \mathbb{R}$ and the Cauchy-Schwarz inequality, we have for any $2 \leq q \leq n − 1$,

$$
|\hat{a}_{1q}^{(n−1)}|^2 \leq 2|\hat{a}_{1q}^{(q−1)}|^2(1 + \tilde{\gamma}_2)^{2(n−q)} + 2\left(\sum_{k=q−1}^{n−2} |s_k||\hat{a}_{q,k+2}^{(k)}|(1 + \tilde{\gamma}_2)^{n−1−k}\right)^2 \\
\leq 2|\hat{a}_{1q}^{(q−1)}|^2(1 + \tilde{\gamma}_2)^{2(n−q)} + 2\left(\sum_{k=q−1}^{n−2} |s_k|^2\right)^2 \left(\sum_{k=q−1}^{n−2} |\hat{a}_{q,k+2}^{(k)}|^2(1 + \tilde{\gamma}_2)^{2(n−1−k)}\right)^2.
$$

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Thus,
\[
\sum_{q=2}^{n} |a_{i,q}^{(n-1)}|^2 \leq 2 \sum_{q=2}^{n} |a_{1,q}^{(q-1)}|^2 (1 + \bar{\gamma}_2)^{2(n-q)}
\]
\[
+ 2 \left( \sum_{k=0}^{n-2} |s_k|^2 \right) \cdot \left( \sum_{q=2}^{n-1} \sum_{k=q-1}^{n-2} |a_{q,k+2}^{(k)}|^2 (1 + \bar{\gamma}_2)^{2(n-1-k)} \right)
\]
\[
\leq 2 \sum_{q=2}^{n} |a_{1,q}^{(q-1)}|^2 (1 + \bar{\gamma}_2)^{2(n-q)}
\]
\[
+ 2 \left( \sum_{k=0}^{n-2} |s_k|^2 \right) \cdot \left( \sum_{q=3}^{n} \sum_{p=2}^{n-2} |a_{p,q}^{(q-2)}|^2 (1 + \bar{\gamma}_2)^{2(n+1-q)} \right). \quad (3.22)
\]

By Lemma 3.4, we have
\[
\begin{cases}
|\hat{a}_{q-1,q}^{(q-2)}|^2 + |\hat{a}_{1,q}^{(q-2)}|^2 \leq (\hat{a}_{q-1,q}^{(q-3)})^2 + |\hat{a}_{1,q}^{(q-3)}|^2)(1 + 2\bar{\gamma}_4), \\
(\hat{a}_{q-2}^{(q-2)}) = \hat{a}_{pq}^{(q-3)} = \cdots = \hat{a}_{pq}^{(0)}, q \leq p \leq n
\end{cases}
\]
and \( a_{ij}^{(q-2)} = \hat{a}_{ij}^{(q-3)}, 2 \leq i \leq q - 2, q \leq j \leq n, q \geq 4 \). Then for any \( q \geq 4 \), we have
\[
\sum_{p=1}^{q-1} |\hat{a}_{p,q}^{(q-2)}|^2 = \sum_{p=2}^{q-2} |\hat{a}_{p,q}^{(q-2)}|^2 + (|\hat{a}_{q-1,q}^{(q-2)}|^2 + |\hat{a}_{1,q}^{(q-2)}|^2)
\]
\[
\leq \sum_{p=2}^{q-2} |\hat{a}_{p,q}^{(q-3)}|^2 + (|\hat{a}_{q-1,q}^{(q-3)}|^2 + |\hat{a}_{1,q}^{(q-3)}|^2)(1 + 2\bar{\gamma}_4)
\]
\[
= \sum_{p=2}^{q-3} |\hat{a}_{p,q}^{(q-4)}|^2 + (|\hat{a}_{q-2,q}^{(q-3)}|^2 + |\hat{a}_{1,q}^{(q-3)}|^2) + |\hat{a}_{q-1,q}^{(q-3)}|^2(1 + 2\bar{\gamma}_4) + 2|\hat{a}_{1,q}^{(q-3)}|^2\bar{\gamma}_4
\]
\[
\leq \sum_{p=2}^{q-4} |\hat{a}_{p,q}^{(q-5)}|^2 + (|\hat{a}_{q-3,q}^{(q-4)}|^2 + |\hat{a}_{1,q}^{(q-4)}|^2) + \sum_{p=q-2}^{q-1} |\hat{a}_{p,q}^{(p-2)}|^2(1 + 2\bar{\gamma}_4) + 2 \sum_{k=q-4}^{q-3} |\hat{a}_{1,q}^{(k)}|^2\bar{\gamma}_4
\]
\[
\leq \cdots
\]
\[
\leq \left( |\hat{a}_{2,q}^{(1)}|^2 + |\hat{a}_{1,q}^{(1)}|^2 \right) + \sum_{p=3}^{q-1} |\hat{a}_{p,q}^{(p-2)}|^2(1 + 2\bar{\gamma}_4) + 2 \sum_{k=1}^{q-3} |\hat{a}_{1,q}^{(k)}|^2\bar{\gamma}_4
\]
\[
\leq \sum_{p=1}^{q-1} |\hat{a}_{p,q}^{(0)}|^2(1 + 2\bar{\gamma}_4) + 2 \sum_{k=0}^{q-3} |\hat{a}_{1,q}^{(k)}|^2\bar{\gamma}_4.
\]

In addition, for \( q = 3 \), we have \( \sum_{p=1}^{q-1} |\hat{a}_{p,q}^{(q-2)}|^2 = |\hat{a}_{1,3}^{(1)}|^2 + |\hat{a}_{2,3}^{(1)}|^2 \leq (|\hat{a}_{1,3}^{(0)}|^2 + |\hat{a}_{2,3}^{(0)}|^2)(1 + 2\bar{\gamma}_4). \)
Therefore,

$$\sum_{q=3}^{n} \sum_{p=2}^{q-1} \sum_{k=0}^{q-3} |b_{p,q}^{(k)}|^{2} (1 + \tilde{\gamma}_{2})^{2(n + 1 - q)} \leq \sum_{q=3}^{n} \left( \sum_{p=1}^{q-1} |b_{p,q}^{(0)}|^{2} (1 + 2\tilde{\gamma}_{4}) + 2 \sum_{k=0}^{q-3} |b_{1,q}^{(k)}|^{2} \tilde{\gamma}_{4} \right) (1 + \tilde{\gamma}_{2})^{2(n + 1 - q)}. \quad (3.23)$$

We now study, after the annihilation of the entries in the positions \{(1, q)\}_{q=2}^{n}, how their values are affected by the later rotations. We first consider how their values are affected after the annihilation of the entries in the positions \{(2, q)\}_{q=3}^{n}. We can easily check that

$$\sum_{q=2}^{n} |a_{1,q}^{(2n-3)}|^{2} \leq (|a_{12}^{(2n-4)}|^{2} + |a_{1n}^{(2n-4)}|^{2}) (1 + 2\tilde{\gamma}_{4}) + \sum_{q=3}^{n-1} |a_{1,q}^{(2n-3)}|^{2}$$

$$= (|a_{12}^{(2n-4)}|^{2} + |a_{1n}^{(2n-4)}|^{2}) (1 + 2\tilde{\gamma}_{4}) + \sum_{q=3}^{n-1} |a_{1,q}^{(2n-4)}|^{2}$$

$$= (|a_{12}^{(2n-4)}|^{2} + |a_{1}^{(n-1)}|^{2}) (1 + 2\tilde{\gamma}_{4}) + \sum_{q=3}^{n-1} |a_{1,q}^{(2n-4)}|^{2}$$

$$= (|a_{12}^{(2n-4)}|^{2} + |a_{1}^{(n-1)}|^{2}) (1 + 2\tilde{\gamma}_{4}) + \sum_{q=3}^{n-1} |a_{1,q}^{(2n-4)}|^{2}$$

$$\leq (|a_{12}^{(2n-5)}|^{2} + |a_{1}^{(n-1)}|^{2}) (1 + 2\tilde{\gamma}_{4}) + \sum_{q=3}^{n-1} |a_{1,q}^{(2n-4)}|^{2}$$

$$= (|a_{12}^{(2n-5)}|^{2} + |a_{1}^{(n-1)}|^{2}) (1 + 2\tilde{\gamma}_{4}) + \sum_{q=3}^{n-1} |a_{1,q}^{(2n-4)}|^{2}$$

$$\leq \ldots$$

$$\leq (|a_{12}^{(n-1)}|^{2} + |a_{13}^{(n-1)}|^{2}) (1 + 2\tilde{\gamma}_{4}) + \sum_{q=2}^{n-2} |a_{1,q}^{(n-2+q)}|^{2} \tilde{\gamma}_{4} + \sum_{q=2}^{n-2} |a_{1,q+2}^{(n-2+q)}|^{2} (1 + 2\tilde{\gamma}_{4})$$
\[
\begin{align*}
\sum_{q=1}^{n} |\hat{a}_{1,q}^{(n-1)}|^2 & = |\hat{a}_{12}^{(n-1)}|^2 + 2 \sum_{q=1}^{n-2} |\hat{a}_{12}^{(n-2+q)}|^2 \gamma_4 + \sum_{q=1}^{n-2} |\hat{a}_{1,q+2}^{(n-1)}|^2 (1 + 2\gamma_4) \\
\leq & \sum_{q=1}^{n-2} |\hat{a}_{12}^{(n-2+q)}|^2 \gamma_4 + \sum_{q=0}^{n-2} |\hat{a}_{1,q+2}^{(n-1)}|^2 (1 + 2\gamma_4) \\
= & \sum_{q=1}^{n-2} |\hat{a}_{12}^{(n-2+q)}|^2 \gamma_4 + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(n-1)}|^2 (1 + 2\gamma_4).
\end{align*}
\]

To summarize, we have, for \( r = 2, \ldots, n - 1 \),
\[
\begin{align*}
\sum_{q=r}^{n} |\hat{a}_{1,q}^{(n(r+1)/2)}|^2 & \leq \sum_{q=1}^{n-r} |\hat{a}_{1,q}^{(n(1-r)/2)}|^2 \gamma_4 + \sum_{q=r}^{n} |\hat{a}_{1,q}^{(n(r-1)-(r-1)/2)}|^2 \gamma_4 + \sum_{q=r}^{n} |\hat{a}_{1,q}^{(n(r-1)-(r-1)/2)}|^2 (1 + 2\gamma_4).
\end{align*}
\]

We note that
\[
\begin{align*}
\hat{a}_{12}^{(3n-6)} & = \hat{a}_{12}^{(2n-3)}, \\
\hat{a}_{12}^{(4n-10)} & = \hat{a}_{12}^{(3n-6)}, \hat{a}_{13}^{(4n-10)} = \hat{a}_{13}^{(3n-6)}, \\
\vdots & \\
\hat{a}_{1,j}^{(n(r+1)/2)} & = \hat{a}_{1,j}^{(n(r-1)-(r-1)/2)}, 2 \leq j \leq r - 1, \\
\vdots & \\
\hat{a}_{1,j}^{(n-1)/2} & = \hat{a}_{1,j}^{(n-1)/2}, 2 \leq j \leq n - 2.
\end{align*}
\]

Let \( \nu(r) = nr - r(r + 1)/2, r = 1, 2, \ldots, n - 1 \). Then
\[
\begin{align*}
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(\nu(r))}|^2 & \leq \sum_{q=2}^{n-r} |\hat{a}_{1,q}^{(\nu(r-1)|+q-1)|2}\gamma_4 + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(\nu(r-1)|2}\gamma_4 + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(\nu(r-1)|2}(1 + 2\gamma_4),
\end{align*}
\]

and thus
\[
\begin{align*}
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r))}|^2 & \leq \sum_{q=2}^{n-r} |\hat{a}_{1,q}^{(nu(r-1)|+q-1)|2}(1 + 2\gamma_4) + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r-1)|2}\gamma_4 + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r-1)|2}(1 + 2\gamma_4),
\end{align*}
\]

which reads as follows:
\[
\begin{align*}
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(2n-3)|2} & \leq \sum_{q=2}^{n} |\hat{a}_{1,q}^{(n-1)|2}(1 + 2\gamma_4) + \sum_{q=2}^{n-2} |\hat{a}_{1,q}^{(n-2+q)|2}\gamma_4, \\
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(3n-6)|2} & \leq \sum_{q=2}^{n} |\hat{a}_{1,q}^{(2n-3)|2}(1 + 2\gamma_4) + \sum_{q=2}^{n-3} |\hat{a}_{1,q}^{(2n-3+q-1)|2} - \sum_{q=2}^{n} |\hat{a}_{1,q}^{(3n-6)|2}\gamma_4, \\
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(4n-10)|2} & \leq \sum_{q=2}^{n} |\hat{a}_{1,q}^{(3n-6)|2}(1 + 2\gamma_4) + \sum_{q=2}^{n-4} |\hat{a}_{1,q}^{(3n-6+q-1)|2} - \sum_{q=2}^{n} |\hat{a}_{1,q}^{(3n-6)|2}\gamma_4, \\
\ldots & \\
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r)|2} & \leq \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r-1)|+q-1)|2}(1 + 2\gamma_4) + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r-1)|2}\gamma_4 + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r-1)|2}(1 + 2\gamma_4),
\end{align*}
\]

and
\[
\begin{align*}
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r)|2} & \leq \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r-1)|+q-1)|2}(1 + 2\gamma_4) + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r-1)|2}\gamma_4 + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(r-1)|2}(1 + 2\gamma_4), \\
\ldots & \\
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(n-1)/2-1)|2} & \leq \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(n-1)|+q-1)|2}(1 + 2\gamma_4) + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(n-1)|2}\gamma_4 + \sum_{q=2}^{n} |\hat{a}_{1,q}^{(nu(n-1)|2}(1 + 2\gamma_4).
\end{align*}
\]
This implies that

\[ \sum_{q=2}^{n} |\hat{a}_{1,q}^{(N)}|^2 \leq \sum_{q=2}^{n} |\hat{a}_{1,q}^{(n-1)}|^2 (1 + 2\tilde{\gamma}_4)^{n-2} + 2 \sum_{q=1}^{n-2} |\hat{a}_{1,2}^{(n-2+q)}|^2 (1 + 2\tilde{\gamma}_4)^{n-3} \cdot \tilde{\gamma}_4 + 2 \sum_{r=3}^{n-1} \left( \sum_{q=1}^{n-r} |\hat{a}_{1,r}^{((r-1)+q-1)}|^2 - \sum_{q=2}^{r-1} |\hat{a}_{1,q}^{(r-1)}|^2 \right) (1 + 2\tilde{\gamma}_4)^{n-1-r} \cdot \tilde{\gamma}_4 \]

\[ \leq \sum_{q=2}^{n} |\hat{a}_{1,q}^{(n-1)}|^2 (1 + 2\tilde{\gamma}_4)^{n-2} (1 + 2\zeta_1 \tilde{\gamma}_4), \quad (3.24) \]

where \( \zeta_1 = |\zeta_{11} + \zeta_{12}| / \sum_{q=2}^{n} |\hat{a}_{1,q}^{(n-1)}|^2 (1 + 2\tilde{\gamma}_4)^{n-2} \) with \( \zeta_{11} = \sum_{q=1}^{n-2} |\hat{a}_{1,2}^{(n-2+q)}|^2 (1 + 2\tilde{\gamma}_4)^{n-3} \) and

\[ \zeta_{12} = \sum_{r=3}^{n-1} \left( \sum_{q=1}^{n-r} |\hat{a}_{1,r}^{((r-1)+q-1)}|^2 - \sum_{q=2}^{r-1} |\hat{a}_{1,q}^{(r-1)}|^2 \right) (1 + 2\tilde{\gamma}_4)^{n-1-r}. \]

Using (3.22), (3.23), and (3.24) we have

\[ \sum_{q=2}^{n} |\hat{a}_{1,q}^{(N)}|^2 \leq \sum_{q=2}^{n} |\hat{a}_{1,q}^{(n-1)}|^2 (1 + 2\tilde{\gamma}_4)^{n-2} (1 + 2\zeta_1 \tilde{\gamma}_4) \]

\[ \leq 2 \sum_{q=2}^{n} |\hat{a}_{1,q}^{(q-1)}|^2 (1 + \tilde{\gamma}_2)^{2(n-q)} (1 + 2\tilde{\gamma}_4)^{n-2} (1 + 2\zeta_1 \tilde{\gamma}_4) + 2(1 + 2\tilde{\gamma}_4)^{n-2} (1 + 2\zeta_1 \tilde{\gamma}_4) \left( \sum_{k=0}^{n-2} |s_k|^2 \right) \]

\[ \left( \sum_{q=3}^{n} \sum_{p=1}^{q-1} |\hat{a}_{p,q}^{(0)}|^2 (1 + 2\tilde{\gamma}_4) (1 + \tilde{\gamma}_2)^{2(n+1-q)} + 2 \sum_{q=3}^{n} \sum_{k=0}^{q-3} |\hat{a}_{1,q}^{(k)}|^2 \tilde{\gamma}_4 (1 + \tilde{\gamma}_2)^{2(n+1-q)} \right) \]

\[ \leq 2 \sum_{q=2}^{n} ||G_{q-2}||_{F}^2 \tilde{\gamma}_4^2 (1 + \tilde{\gamma}_2)^{2(n-q)} (1 + 2\tilde{\gamma}_4)^{n-2} (1 + 2\zeta_1 \tilde{\gamma}_4) + (1 + 2\tilde{\gamma}_4)^{n-2} (1 + 2\zeta_1 \tilde{\gamma}_4) \left( \sum_{k=0}^{n-2} |s_k|^2 \right) \]

\[ \quad \left( \| \text{off}(A_0) \|^2_{F} (1 + 2\tilde{\gamma}_4) (1 + \tilde{\gamma}_2)^{2(n-2)} + 4 \sum_{q=3}^{n} \sum_{k=0}^{q-3} |\hat{a}_{1,q}^{(k)}|^2 \tilde{\gamma}_4 (1 + \tilde{\gamma}_2)^{2(n+1-q)} \right), \]

where the third inequality uses Lemma 3.4 with \( G_k \) being defined as in Lemma 3.5.

To summarize, we have, for \( r = 1, 2, \ldots, n - 1, \)

\[ \sum_{q=r+1}^{n} |\hat{a}_{r,q}^{(N)}|^2 = \sum_{q=2}^{n-r+1} |\hat{a}_{r-1+1,r-1+q}^{(r-1)+(n-r+1)(n-r)/2}|^2 \]
Thus, \( A_k^{(r-1)} = A_k^{(r-1)+k} \) for \( n, r : n \) and

\[
\zeta_r = |\zeta_{r1} + \zeta_{r2}|/ \sum_{q=2}^{n-r+1} |\tilde{a}_{r,r-1+q}^{(r-1)+n-r}|^2 (1 + 2\tilde{\gamma}_4)^{n-r-1}
\]

with \( \zeta_{r1} = \sum_{q=1}^{n-r-1} |\tilde{a}_{r,r+1}^{(r-1)+n-r+q}|^2 (1 + 2\tilde{\gamma}_4)^{n-r-2} \) and

\[
\zeta_{r2} = \sum_{j=2}^{n-r} \sum_{q=1}^{n-r-1-j} |\tilde{a}_{r,r-1+q}^{(r-1)+(n-r+1)(j-1)-(j-1)j/2+q-1}|^2 (1 + 2\tilde{\gamma}_4)^{n-r-j}
\]

Thus,

\[
\sum_{r=1}^{n-1} \sum_{q=r+1}^{n} |\tilde{a}_{r,q}^{(N)}|^2
\]

\[
\leq 2 \sum_{r=1}^{n-1} \sum_{q=2}^{n-r} \|G_{t(r-1)+q-2}\|^2 P_4^2 (1 + \tilde{\gamma}_2)^2 (1 + 2\tilde{\gamma}_4)^{n-r-1} (1 + 2\zeta_r\tilde{\gamma}_4)
\]

\[
+ \sum_{r=1}^{n-1} \sum_{k=0}^{n-r+1-q-3} |s_{t(r-1)+k}|^2 \cdot \left( \|\text{off}(A_0^{(r-1)})\|^2 P_4 \right) (1 + 2\tilde{\gamma}_4)^2 (1 + 2\tilde{\gamma}_2)^{2(n-r-2)} (1 + 2\tilde{\gamma}_4)^{n-r-1} (1 + 2\zeta_r\tilde{\gamma}_4)
\]

\[
\leq 2(1 + 2\zeta\tilde{\gamma}_4)(1 + \tilde{\gamma}_2)^2 (1 + 2\tilde{\gamma}_4)^{n-2} \sum_{k=0}^{N-1} \|G_k\|^2 P_4^2 \tilde{\gamma}_4^2
\]

\[
+(1 + 2\zeta\tilde{\gamma}_4)(1 + \tilde{\gamma}_2)^2 (1 + 2\tilde{\gamma}_4)^{n-1} \xi^2 \sum_{k=0}^{N-1} |s_k|^2,
\]

where \( \zeta, \xi \geq 0 \) are defined by

\[
|\tilde{a}_{r,r-1+q}^{(r-1)+n-r}^2| = 1, \quad \xi = \max_{1 \leq r \leq n-1} \zeta_r, \quad \zeta_1 = \max_{1 \leq r \leq n-1} \|\text{off}(A_0^{(r-1)})\|^2 P_4^2 + 4 \sum_{q=3}^{n-r+1} \sum_{k=0}^{q-3} |\tilde{a}_{r,r-1+q}^{(r-1)+n-r}^2| (1 + 2\tilde{\gamma}_4)^{n-r-1}.
\]
Suppose that the conditions (3.15) and (3.16) are satisfied. This implies that (3.17) holds. Hence,

\[
\|\text{off}(A_N)\|_F^2 = 2 \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} |\hat{a}_{r,q}^{(N)}|^2
\]

\[
\leq 4 \sum_{k=0}^{N-1} \|G_k\|_F^2 \gamma_4^2 (1 + \gamma_6)^2 (1 + 2\zeta_4) + 2 \sum_{k=0}^{N-1} |s_k|^2 (1 + \gamma_6)^2 (1 + 2\zeta_4) \xi^2
\]

\[
\leq 4 \sum_{k=0}^{N-1} \|G_k\|_F^2 \gamma_4^2 (1 + \gamma_6)^2 (1 + 2\zeta_4) + 2 \sum_{k=0}^{N-1} 4|\tilde{z}_k|^2 (1 + \gamma_6)^2 (1 + 2\zeta_4) \xi^2
\]

\[
\leq 4 \sum_{k=0}^{N-1} \|G_k\|_F^2 \gamma_4^2 (1 + \gamma_6)^2 (1 + 2\zeta_4) + 4(1 + \gamma_6)^2 (1 + 2\zeta_4) \|\text{off}(A_0)\|_F^2.
\]

\[
\xi^2 \left( \min_{0 \leq k \leq N-1} (d(A_k))^2 \|\text{off}(A_0)\|_F^2 \right)^{-1} (\|\text{off}(A_0)\|_F^2 + 2 \sum_{k=0}^{N-1} \|H_k\|_F^2 \gamma_4 + 2 \sum_{k=0}^{N-1} \|G_k\|_F^2 \gamma_4^2)
\]

\[
\leq \left( \alpha_2 \|\text{off}(A_0)\|_F^2 + \beta_2 \gamma_4 \right)^2,
\]

where the first inequality follows from \((1 + 2\zeta_4) \leq (1 + \gamma_4)^2 \) and \((1 + \gamma_2)(1 + \gamma_4) \leq (1 + \gamma_6)\), the second inequality uses (3.17), the third inequality uses Lemma 3.6, and the fourth inequality is obtained by applying \(a_1^2 + a_2^2 \leq (a_1^2 + a_2^2/2)^2\) for all \(a_1, a_2 \in \mathbb{R}\), and

\[
\alpha_2 = 2(1 + \gamma_6)^{n-1} (1 + 2\zeta_4)^{1/2} \xi \left( \min_{0 \leq k \leq N-1} d(A_k) \|\text{off}(A_0)\|_F \right)^{-1}
\]

and

\[
\beta_2 = \alpha_2 \left( \sum_{k=0}^{N-1} \|H_k\|_F^2 + \sum_{k=0}^{N-1} \|G_k\|_F^2 \gamma_4 \right)
\]

\[
+ 2 \left( \sum_{k=0}^{N-1} \|G_k\|_F^2 \right)^{1/2} (1 + \gamma_6)^{n-2} (1 + 2\zeta_4)^{1/2}. \quad (3.28)
\]

Finally, we show that \(\xi \left( \min_{0 \leq k \leq N-1} d(A_k) \|\text{off}(A_0)\|_F \right)^{-1} < \infty\). Under the conditions (3.15) and (3.16), it follows from Lemmas 3.11 and 3.6 that \(d(A_k) \geq \tilde{d} > 0\) for all \(0 \leq k \leq N - 1\) and \(\|\text{off}(A_k^{(r-1)})\|_F \leq \|\text{off}(A_{r,(r-1)+k})\|_F \leq \|\text{off}(A_0)\|_F\) for all \(1 \leq r \leq n - 1\) and \(k \geq 0\). Interchanging
the sum of $\sum_{q=3}^{n-r+1} \sum_{k=0}^{q-3} |\hat{a}_{r,r-1+q}^{(\nu(r-1)+k)}|^2$ yields

$$\xi^2 = \max_{1 \leq r \leq n-1} \left( \|\text{off}(A_0^{(r-1)})\|_F^2 + 4 \sum_{k=0}^{n-r-2} \sum_{q=3+k}^{n-r+1} |\hat{a}_{r,r-1+q}^{(\nu(r-1)+k)}|^2 \gamma_4 \right)$$

$$\leq \max_{1 \leq r \leq n-1} \left( \|\text{off}(A_0^{(r-1)})\|_F^2 + 4 \sum_{k=0}^{n-r-2} \|\text{off}(A_k^{(r-1)})\|_F^2 \gamma_4 \right)$$

$$\leq (1 + 4(n - 2)\gamma_4)\|\text{off}(A_0)\|_F^2.$$  

Therefore, $\xi(\min_{0 \leq k \leq N-1} d(A_k))\|\text{off}(A_0)\|_F^{-1} \leq \sqrt{1 + 4n\gamma_4/d} \leq (1 + 2n\gamma_4)/d < \infty.$

Based on the above analysis, we have the following error bound for one sweep of the row cyclic Jacobi method in floating point arithmetic.

**Theorem 3.17** Let $A_k$ be the matrix $A_0 = A$ after $k$ Jacobi updates, which is generated by Algorithm 3.2 in floating point arithmetic, where the index pairs $\{(p_k, q_k)\}$ are chosen in the row cyclic order. Let $\varphi_k = ((6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^2)^{1/2}$ with $G_k$ and $H_k$ being defined as in Lemma 3.5 with $(p, q) = (p_k, q_k)$ for all $k \geq 0$. Suppose $A_0$ has $n$ distinct eigenvalues and $\|\text{off}(A_0)\|_F < d(A_0)/4$. If $\gamma_4$ satisfies the conditions in (3.15)–(3.16), then we have $\|\text{off}(A_N)\|_F \leq \alpha_2\|\text{off}(A_0)\|_F^2 + \beta_2\gamma_4$, where the constants $\alpha_2 > 0$ and $\beta_2 > 0$ are defined by (3.27) and (3.28), respectively.

**Remark 3.18** In Theorem 3.17, we note that $\alpha_2 < 2(1 + \gamma_4)^{n-1}(1 + 2\zeta 4)^{1/2}(1 + 2n\gamma_4)/d$ and $\alpha_1 \leq (1 + \gamma_4)^{2n-4}\sqrt{2}/d$. One may expect that $\alpha_2$ is significantly smaller than $\alpha_1$ defined by (3.19) if $1 + 2\zeta 4$ is not too large for a sufficient small $\gamma_4$. This is observed from the later numerical results (see Section 6.3).

### 3.3.3 The row-cyclic order with one multiple eigenvalue

We study the error analysis for one sweep of Algorithm 3.2 in floating point arithmetic for a symmetric matrix $A \in \mathbb{R}^{n \times n}$ with only one multiple eigenvalue, where all off-diagonal entries are annihilated successively in the row-cyclic order. Under the same assumptions as in section 3.3.2, without loss of generality, we assume that only the eigenvalue $\lambda_1(A)$ is multiple with multiplicity $m$ and that the diagonal elements $\hat{a}_{11}^{(k)}, \ldots, \hat{a}_{mm}^{(k)}$ are sufficiently close to $\lambda_1(A)$ for all $k \geq 1$.

For any $k \geq 1$, we define the intra cluster distance

$$d_i(A_k) := \max_{1 \leq i, j \leq m, i \neq j} |\lambda_i(A_k) - \lambda_j(A_k)|$$

and the inter cluster distance

$$d_r(A_k) := \min_{1 \leq i \leq n, m+1 \leq j \leq n, i \neq j} |\lambda_i(A_k) - \lambda_j(A_k)|.$$  

We first extend Lemma 3.11 to the case of partially distinct eigenvalues.
Lemma 3.19 If one step of Jacobi’s method is performed in the \((p,q)\) plane on the matrix \(A_k = (\hat{a}_{ij}^{(k)})\) with the computed Jacobi rotation \(J_k = J(p; \hat{c}_k, \hat{s}_k)\) and \(A_k\) has \(n-m\) distinct eigenvalues \(\{\lambda_j(A_k)\}_{j=m+1}^n\) with \(d_r(A_k) > 0\), then we have \(d_r(A_k+1) \geq d_r(A_k) - 2\varphi_k\tilde{\gamma}_4 > 0\), provided that \(2\varphi_k\tilde{\gamma}_4 < d_r(A_k)\), where \(\varphi_k = (6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^1/2\) with \(G_k\) and \(H_k\) being defined as in Lemma 3.5.

**Proof.** Let \(J_k = J(p; q; c_k, s_k)\) be the exact Jacobi rotation. Then, by Lemma 2.4 we have, for any \(1 \leq i \leq n\) and \(m + 1 \leq j \leq n\) with \(i \neq j\),

\[
|\lambda_i(A_{k+1}) - \lambda_j(A_{k+1})| \\
\geq |\lambda_i(A_k) - \lambda_j(A_k)| - |\lambda_i(A_{k+1}) - \lambda_j(A_k)| \\
= |\lambda_i(A_k) - \lambda_j(A_k)| - |\lambda_i(A_{k+1}) - \lambda_j(A_k)| \\
\geq d_r(A_k) - 2\|A_{k+1} - J_k^{\top}A_kJ_k\|, 
\]

completing the proof of the lemma by invoking Lemma 3.5. \(\Box\)

**Remark 3.20** By Lemma 3.19, we have, for any \(k \geq 0\),

\[
d_r(A_{k+1}) \geq d_r(A_k) - 2\varphi_k\tilde{\gamma}_4 \geq d_r(A_0) - 2\sum_{t=0}^k \varphi_t\tilde{\gamma}_4. \tag{3.29}
\]

This shows that if \(2\sum_{t=0}^k \varphi_t\tilde{\gamma}_4 < d_r(A_0)\), then \(d_r(A_{k+1}) > 0\), i.e., \(A_{k+1}\) has \(n-m\) distinct eigenvalues \(\{\lambda_j(A_k)\}_{j=m+1}^n\) for all \(k \geq 0\). Moreover, by Lemmas 2.4 and 3.5 with the fact that \(d_a(A_0) = 0\) we have

\[
d_a(A_{k+1}) = \max_{1 \leq i, j \leq m, i \neq j} |\lambda_i(A_{k+1}) - \lambda_j(A_{k+1})| \\
\leq \max_{1 \leq i, j \leq m, i \neq j} |\lambda_i(A_k) - \lambda_j(A_k)| \\
+ \max_{1 \leq i, j \leq m, i \neq j} \{|\lambda_i(A_{k+1}) - \lambda_i(A_k)| + |\lambda_j(A_{k+1}) - \lambda_j(A_k)|\} \\
\leq d_a(A_k) + 2\|Y_k\| \leq \cdots \leq 2\sum_{t=0}^k \|Y_t\| \leq 2\sum_{t=0}^k \varphi_t\tilde{\gamma}_4. \tag{3.30}
\]

This shows that the eigenvalues \(\{\lambda_j(A_k)\}_{j=1}^m\) of \(A_{k+1}\) are close to each other in floating point arithmetic.

We now extend Lemma 3.13 to the case of partially distinct eigenvalues.

**Lemma 3.21** Let \(A_k = (\hat{a}_{ij}^{(k)})\) be the matrix \(A_0 = A\) after \(k\) Jacobi updates in the row-cyclic order. Suppose \(2\sum_{t=0}^k \varphi_t\tilde{\gamma}_4 < d_r(A_0)\) and the diagonal entries \(\{\hat{a}_{tt}^{(k)}\}_{t=1}^m\) are sufficiently close to \(\lambda_1(A)\) and the diagonal entries \(\{\hat{a}_{tt}^{(k)}\}_{t=m+1}^n\) are sufficiently close to the \(n-m\) distinct eigenvalues \(\{\lambda_t(A)\}_{t=m+1}^n\), respectively. If \(||\text{off}(A_k)||_F \leq d_r(A_k)/4\), then we have, for some ordering of \(\{\lambda_t(A_k)\}_{t=m+1}^n\) and some ordering of \(\{\lambda_t(A_k)\}_{t=1}^m\), \(|\hat{a}_{ij}^{(k)} - \hat{a}_{ij}^{(k)}| \geq d_r(A_k)/2\) for all \(1 \leq i \leq
n, m + 1 \leq j \leq n, i \neq j$, and for $(p_k, q_k)$ plane on $A_k$ with $1 \leq p_k \leq n$ and $\max\{m + 1, p_k + 1\} \leq q_k \leq n$, the angle $\theta_{p_kq_k}$ of the corresponding exact Jacobi rotation $J_k = J(p_k, q_k; c_k, s_k)$ satisfies $|\sin \theta_{p_kq_k}| \leq 2|\hat{a}_{p_kq_k}^{(k)}|/d_r(A_k)$.

**Proof.** It follows from Remark 3.20 that the $n - m$ eigenvalues $\lambda_j(A_k)_{j=m+1}^n$ are distinct and the eigenvalues $\{\lambda_j(A_k)\}_{j=1}^m$ are close to each other in floating point arithmetic. By hypothesis, $\hat{a}_{tt}^{(k)}_{t=1}$ are sufficiently close to $\lambda_i(A)$ and $\hat{a}_{tt}^{(k)}_{t=m+1}$ are sufficiently close to the $n - m$ distinct eigenvalues $\lambda_r(A)_{t=m+1}$, respectively. Hence, $\hat{a}_{tt}^{(k)}_{t=1}$ are sufficiently close to each other and $\hat{a}_{tt}^{(k)}_{t=m+1}$ are all distinct. Therefore, from Lemma 2.4 we have, for some ordering of $\{\lambda_t(A_k)\}_{t=1}^m$ and some ordering of $\{\lambda_t(A)\}_{t=m+1}$,

$$
|\hat{a}_{ii}^{(k)} - \hat{a}_{jj}^{(k)}| \geq |\lambda_i(A_k) - \lambda_j(A_k)| - |\hat{a}_{ii}^{(k)} - \lambda_i(A_k)| - |\hat{a}_{jj}^{(k)} - \lambda_j(A_k)| \\
\geq d_r(A_k) - 2\|\text{off}(A_k)\|_F \geq d_r(A_k)/2,
$$

for all $1 \leq i \leq n, m + 1 \leq j \leq n, i \neq j$. Then the second conclusion of this lemma follows from Lemma 3.13. \qed

By using the arguments similar to those of Lemmas 3.12 and 3.19, we have the following result.

**Lemma 3.22** Suppose one step of Jacobi’s method is performed in the $(p, q)$ plane on the matrix $A_k = (\hat{a}_{ij}^{(k)})$ with the computed Jacobi rotation $J_k = J(p, q; \hat{c}_k, \hat{s}_k)$. Let $\varphi_k = ((6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^2)^{1/2}$ with $G_k$ and $H_k$ being defined as in Lemma 3.5. If $A_k$ has $n - m$ distinct eigenvalues $\lambda_j(A_k)_{j=m+1}^n$ with $4\|\text{off}(A_k)\|_F < d_r(A_k)$ and

$$
\frac{1}{8} \varphi_k^2 \gamma_4 + \frac{1}{2} \|\text{off}(A_k)\|_F \varphi_k \gamma_4 + \|H_k\|_F^2 \gamma_4 + \|G_k\|_F^2 \gamma_4^2 < |\hat{a}_{pq}^{(k)}|^2,
$$

(3.31)

then we have $4\|\text{off}(A_k)\|_F < d_r(A_k+1)$.

**Proof.** Let $b_k = \|H_k\|_F \varphi_k \gamma_4 + \|G_k\|_F^2 \gamma_4^2$. From (3.31) we have $b_k < |\hat{a}_{pq}^{(k)}|^2$. By Lemma 3.6, we have $\|\text{off}(A_{k+1})\|_F < \|\text{off}(A_k)\|_F$ and $\|\text{off}(A_{k+1})\|_F^2 + 2|\hat{a}_{pq}^{(k)}|^2 - 2b_k > 0$. This, together with (3.31) again, yields

$$
(\|\text{off}(A_{k+1})\|_F^2 + \frac{1}{2} \varphi_k \gamma_4)^2 - (\|\text{off}(A_{k+1})\|_F^2 + 2|\hat{a}_{pq}^{(k)}|^2 - 2b_k) \\
\leq \frac{1}{4} \varphi_k^2 \gamma_4^2 + \|\text{off}(A_{k+1})\|_F \varphi_k \gamma_4 + 2b_k - 2|\hat{a}_{pq}^{(k)}|^2 \\
\leq \frac{1}{4} \varphi_k^2 \gamma_4^2 + \|\text{off}(A_k)\|_F \varphi_k \gamma_4 + 2b_k - 2|\hat{a}_{pq}^{(k)}|^2 < 0.
$$

(3.32)

By hypothesis, we have $4\|\text{off}(A_k)\|_F < d_r(A_k)$. Then, by following the arguments similar to the proof of Lemma 3.19 we have

$$
d_r(A_{k+1}) \geq d_r(A_k) - 2\varphi_k \gamma_4 > 4\|\text{off}(A_k)\|_F - 2\varphi_k \gamma_4 \\
\geq 4(\|\text{off}(A_{k+1})\|_F^2 + 2|\hat{a}_{pq}^{(k)}|^2 - 2b_k)^{1/2} - 2\varphi_k \gamma_4 > 4\|\text{off}(A_{k+1})\|_F,
$$

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where the third inequality uses (3.5) and the last inequality uses (3.32).

Let $1 \leq k \leq N$ be fixed. As in [47], we partition $A_k \in \mathbb{R}^{n \times n}$ into the following form

$$
A_k = \begin{bmatrix}
B_1^{(k)} & B_2^{(k)} \\
(B_2^{(k)})^T & B_3^{(k)}
\end{bmatrix}, \quad B_1^{(k)} \in \mathbb{R}^{m \times m},
$$

(3.33)

where the diagonal entries of $B_1^{(k)}$ are sufficiently close to $\lambda_1(A)$. We estimate the quantity $\|\text{off}(B_1^{(k)})\|_F$. The matrix $A_k$ admit the following spectral decomposition:

$$
A_k = Q_k \text{diag}(\lambda_1(A_k), \ldots, \lambda_n(A_k))Q_k^T,
$$

where $Q_k \in \mathbb{R}^{n \times n}$ is orthogonal matrix. Then we have

$$
\begin{bmatrix}
B_1^{(k)} - \lambda_1(A_k)I_m & B_2^{(k)} \\
(B_2^{(k)})^T & B_3^{(k)} - \lambda_1(A_k)I_{n-m}
\end{bmatrix} := A_k - \lambda_1(A_k)I_n
$$

$$
= Q_k \text{diag}(\lambda_1(A_k) - \lambda_1(A_k), \ldots, \lambda_m(A_k) - \lambda_1(A_k), \ldots, \lambda_n(A_k) - \lambda_1(A_k))Q_k^T.
$$

Let

$$
Q_k \text{diag}(\lambda_1(A_k) - \lambda_1(A_k), \ldots, \lambda_m(A_k) - \lambda_1(A_k), 0, \ldots, 0)Q_k^T := \begin{bmatrix}
W_1^{(k)} & W_2^{(k)} \\
(W_2^{(k)})^T & W_3
\end{bmatrix} = W_k, \quad W_k \in \mathbb{R}^{m \times m}.
$$

Then

$$
A_k - \lambda_1(A_k)I_n - W_k = \begin{bmatrix}
B_1^{(k)} - \lambda_1(A_k)I_m - W_1^{(k)} & B_2^{(k)} - W_2^{(k)} \\
(B_2^{(k)})^T - (W_2^{(k)})^T & B_3^{(k)} - \lambda_1(A_k)I_{n-m} - W_3^{(k)}
\end{bmatrix}
$$

$$
= Q_k \text{diag}(0, \ldots, 0, \lambda_m+1(A_k) - \lambda_1(A_k), \ldots, \lambda_n(A_k) - \lambda_1(A_k))Q_k^T.
$$

We note that, for any $1 \leq i \leq 3$, $\|W_i^{(k)}\| \leq \|W_i^{(k)}\|_F \leq \sqrt{m} \cdot d_a(A_k)$. If

$$
\|\text{off}(A_k)\|_F + \sqrt{m}d_a(A_k) < d_r(A_k),
$$

(3.34)

then we have

$$
|\lambda_j(B_3^{(k)}) - \lambda_1(A_k)| \geq |\lambda_{j+m}(A_k) - \lambda_1(A_k)| - |\lambda_j(B_3^{(k)})|
$$

$$
\geq d_r(A_k) - \|\text{off}(A_k)\|_F > \sqrt{m}d_a(A_k)
$$

for all $1 \leq j \leq n - m$. This means $\sigma_{\min}(B_3^{(k)} - \lambda_1(A_k)I_{n-m}) - \|W_3^{(k)}\| > 0$ by Lemma 2.5 and thus $B_3^{(k)} - \lambda_1(A_k)I_{n-m} - W_3^{(k)}$ is nonsingular. By using congruent transformation we obtain

$$
C_k^T(A_k - \lambda_1(A_k)I_n - W_k)C_k = \begin{bmatrix}
\Delta A^{(k)} & 0 \\
0 & B_3^{(k)} - \lambda_1(A_k)I_{n-m} - W_3^{(k)}
\end{bmatrix},
$$

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This implies that
\[ \Delta A^{(k)} = B_1^{(k)} - \lambda_1(A_k)I_m - W_1^{(k)} \]
\[ - (B_2^{(k)} - W_2^{(k)})(B_3^{(k)} - \lambda_1(A_k)I_{n-m} - W_3^{(k)})^{-1}(B_2^{(k)} - W_2^{(k)})^T, \]

\[ C_k = \begin{bmatrix} I_m & 0 \\ -(B_3^{(k)} - \lambda_1(A_k)I_{n-m} - W_3^{(k)})^{-1}(B_2^{(k)} - W_2^{(k)})^T & I_{n-m} \end{bmatrix}. \]

We note that, under the condition (3.34), we have \( d(A_k) > 0 \) and thus the rank of \( A_k - \lambda_1(A_k)I_n - W_k \) is \( n - m \). Also, \( C_k \) is nonsingular. Hence, \( \Delta A^{(k)} = 0 \), i.e.,

\[ B_1^{(k)} - \lambda_1(A_k)I_m = W_1^{(k)} + (B_2^{(k)} - W_2^{(k)})(B_3^{(k)} - \lambda_1(A_k)I_{n-m} - W_3^{(k)})^{-1}(B_2^{(k)} - W_2^{(k)})^T. \]

This implies that
\[
\| \text{off}(B_1^{(k)}) \|_F^2 \leq 2\|W_1^{(k)}\|_F^2 + \frac{2(2\|B_2^{(k)}\|_F^2 + 2\|W_2^{(k)}\|_F^2)^2}{(\min_{1 \leq j \leq n-m} |\lambda_j(B_3^{(k)}) - \lambda_1(A_k)| - \|W_3^{(k)}\|)}
\leq 2md_a^2(A_k) + \frac{2(2\|B_2^{(k)}\|_F^2 + 2md_a^2(A_k))^2}{(d_r(A_k) - \|\text{off}(A_k)\|_F - \sqrt{md_a(A_k)})^2}.
\] (3.35)

Furthermore, if
\[
\frac{4}{4\sqrt{m} + 3} + a + 2\sum_{j=0}^{N-1} \phi_j \varphi_4 < \frac{3}{4\sqrt{m} + 3} d_r(A_0),
\] (3.36)
for some \( a > 0 \), then using (3.29) and (3.30), we have

\[ 4\sqrt{md_a(A_k)} \leq 4\sqrt{m}\left(2\sum_{j=0}^{k-1} \varphi_j \varphi_4\right) \leq 4\sqrt{m}\left(2\sum_{j=0}^{N-1} \varphi_j \varphi_4\right)
\leq 3\left(d_r(A_0) - 2\sum_{j=0}^{N-1} \phi_j \varphi_4\right) \leq 3\left(d_r(A_0) - 2\sum_{j=0}^{N-1} \phi_j \varphi_4\right) \leq 3d_r(A_k).
\]

Thus (3.34) holds if \( 4\|\text{off}(A_k)\|_F < d_r(A_k) \).

Under conditions (3.15), (3.36), and

\[ d_r(A_0) \geq d + \sum_{k=0}^{N-1} 2\phi_k \varphi_4 \] (3.37)

for some \( d > 0 \), it follows from Lemma 3.22 that

\[ \|\text{off}(A_k)\|_F \leq d_r(A_k)/4 \quad \text{and} \quad \|\text{off}(A_k)\|_F \leq \|\text{off}(A_0)\|_F. \] (3.38)
Using (3.21) we have

\[
\sum_{q=m+1}^{n} |a_{1,q}^{(n-1)}|^2 \leq 2 \sum_{q=m+1}^{n} |a_{1,q}^{(q-1)}|^2 (1 + \gamma_2)^{2(n-q)} + 2 \left( \sum_{k=m}^{n-2} s_k^2 \right) \cdot \left( \sum_{q=m+1}^{n} \sum_{k=q-1}^{n-2} |a_{q,k+2}^{(k)}|^2 (1 + \gamma_2)^{2(n-1-k)} \right)
\]

\[
\leq 2 \sum_{q=m+1}^{n} |a_{1,q}^{(q-1)}|^2 (1 + \gamma_2)^{2(n-q)} + 2 \left( \sum_{k=m}^{n-2} s_k^2 \right) \cdot \left( \sum_{q=m+2}^{n} \sum_{p=m}^{q-1} |a_{p,q}^{(q-2)}|^2 (1 + \gamma_2)^{2(n+1-q)} \right)
\]

\[
\leq 2 \sum_{q=2}^{n} |a_{1,q}^{(q-1)}|^2 (1 + \gamma_2)^{2(n-q)} + 2 \left( \sum_{k=m}^{n-2} s_k^2 \right) \cdot \left( \sum_{q=3}^{n} \sum_{p=2}^{q-1} |a_{p,q}^{(q-2)}|^2 (1 + \gamma_2)^{2(n+1-q)} \right).
\]

(3.39)

Using (3.39), (3.23), and (3.24) we have

\[
\sum_{q=2}^{n} |a_{1,q}^{(N)}|^2 \leq \sum_{q=2}^{n} |a_{1,q}^{(n-1)}|^2 (1 + 2\gamma_4)^{n-2}(1 + 2\zeta_1 \gamma_4)
\]

\[
\leq \sum_{q=2}^{m} |a_{1,q}^{(n-1)}|^2 (1 + 2\gamma_4)^{n-2}(1 + 2\zeta_1 \gamma_4)
\]

\[
+ \sum_{q=m+1}^{n} |a_{1,q}^{(n-1)}|^2 (1 + 2\gamma_4)^{n-2}(1 + 2\zeta_1 \gamma_4)
\]

\[
\leq \sum_{q=2}^{m} |a_{1,q}^{(n-1)}|^2 (1 + 2\gamma_4)^{n-2}(1 + 2\zeta_1 \gamma_4)
\]

\[
+ 2 \sum_{q=2}^{n} |a_{1,q}^{(q-1)}|^2 (1 + \gamma_2)^{2(n-q)}(1 + 2\gamma_4)^{n-2}(1 + 2\zeta_1 \gamma_4)
\]

\[
+ 2(1 + 2\gamma_4)^{n-2}(1 + 2\zeta_1 \gamma_4) \left( \sum_{k=m}^{n-2} s_k^2 \right)
\]

\[
\left( \sum_{q=3}^{n} \sum_{p=1}^{q-1} |a_{p,q}^{(0)}|^2 (1 + 2\gamma_4) + 2 \sum_{k=0}^{n-3} |a_{1,q}^{(k)}|^2 (1 + \gamma_2)^{2(n+1-q)} \right)
\]

29
where the last inequality uses the definitions of \( \zeta \) and \( \xi^2 \) as in (3.26) and the fact that \((1 + 2\tilde{\gamma}_4)^2 \leq (1 + \tilde{\gamma}_4)^2 \) and \((1 + \tilde{\gamma}_2)(1 + \tilde{\gamma}_4) \leq (1 + \tilde{\gamma}_6) \).

It follows from that (3.36) that

\[
\frac{3}{4} (d_r(A_0) - 2 \sum_{t=0}^{N-1} \varphi_t \tilde{\gamma}_4) - 2\sqrt{m} \sum_{t=0}^{N-1} \varphi_t \tilde{\gamma}_4 \geq a > 0. \tag{3.41}
\]

From (3.29), (3.30), (3.33), (3.35), (3.38), and (3.41) we have

\[
\sum_{q=2}^{m} |\tilde{a}_{1,q}^{(n-1)}|^2 \leq \frac{1}{2} \|\text{off}(B_1^{(n-1)})\|_F^2 \\
\leq m d_a^2 (A_{n-1}) + \frac{(2 \|\text{off}(A_{n-1})\|_F^2 + 2md_a^2 (A_{n-1}))^2}{(d_r(A_{n-1}) - \|\text{off}(A_{n-1})\|_F - \sqrt{md_a (A_{n-1}))^2} \\
\leq m d_a^2 (A_{n-1}) + \frac{(2 \|\text{off}(A_0)\|_F^2 + 2md_a^2 (A_{n-1}))^2}{(\frac{3}{4} d_r(A_{n-1}) - 2 \sum_{t=0}^{N-1} \varphi_t \tilde{\gamma}_4) - 2\sqrt{m} \sum_{t=0}^{N-1} \varphi_t \tilde{\gamma}_4) \tag{3.42}
\]

\[
\leq 4m \left( \sum_{t=0}^{N-1} \varphi_t \right)^2 \tilde{\gamma}_4^2 + 4m \left( \frac{2 \|\text{off}(A_0)\|_F^2 + 8m(\sum_{t=0}^{N-1} \varphi_t \tilde{\gamma}_4)^2}{(\frac{3}{4} d_r(A_0) - 2 \sum_{t=0}^{N-1} \varphi_t \tilde{\gamma}_4) - 2\sqrt{m} \sum_{t=0}^{N-1} \varphi_t \tilde{\gamma}_4) \right)^2 \\
\leq 4m \left( \sum_{t=0}^{N-1} \varphi_t \right)^2 \tilde{\gamma}_4^2 + a^{-2} (2 \|\text{off}(A_0)\|_F^2 + 8m(\sum_{t=0}^{N-1} \varphi_t \tilde{\gamma}_4)^2). \tag{3.43}
\]
This, together with (3.40) and (3.26), yields

\[
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(N)}|^{2} \leq 2 \sum_{q=2}^{n} \|G_{q-2}\|_{F}^{2}(1 + \tilde{\gamma}_{4})^{2(n-2)}(1 + 2\tilde{\gamma}_{4}) + \left( \sum_{k=m}^{n-2} |s_{k}|^{2} \right)(1 + \tilde{\gamma}_{6})^{2(n-1)}(1 + 2\tilde{\gamma}_{4})\xi^{2}
\]

\[
+ \left( 4m \left( \sum_{t=0}^{N-1} \phi_{t} \right)^{2} + a^{-2} \left( 2 \|\text{off}(A_{0})\|_{F}^{2} + 8m \left( \sum_{t=0}^{N-1} \phi_{t} \right)^{2} \right) \right) \xi^{2}
\]

\[
+ \left( 4m \left( \sum_{t=0}^{N-1} \phi_{t} \right)^{2} + a^{-2} \left( 2 \|\text{off}(A_{0})\|_{F}^{2} + 8m \left( \sum_{t=0}^{N-1} \phi_{t} \right)^{2} \right) \right) \xi^{2}
\]

(3.42)

where \(A_{k}^{(r-1)} = A_{i(r-1)+k}(r : n, r : n)\). Using (3.25), (3.26) and the fact that \((1 + 2\tilde{\gamma}_{4}) \leq (1 + \tilde{\gamma}_{4})^{2}\) and \((1 + \tilde{\gamma}_{2})(1 + \tilde{\gamma}_{4}) \leq (1 + \tilde{\gamma}_{6})\), we obtain, for \(r = m, \ldots, n-1\),

\[
\sum_{q=2}^{n} |\hat{a}_{1,q}^{(N)}|^{2} \leq 2 \sum_{q=2}^{n-1} \|G_{q-1}+q-2\|_{F}^{2}(1 + \tilde{\gamma}_{6})^{2(n-2)}(1 + 2\tilde{\gamma}_{4})
\]

\[
+ \sum_{k=m+1-r}^{n-r-1} |s_{i(r-1)+k}|^{2}(1 + \tilde{\gamma}_{6})^{2(n-1)}(1 + 2\tilde{\gamma}_{4})\xi^{2}
\]

(3.43)

Let \(\Omega = \bigcup_{1 \leq p < q \leq m} \{(p, q)\}\). By Lemma 3.21 we have

\[
\sum_{(p_{k}, q_{k}) \in \Omega} |s_{k}|^{2} \leq \frac{4 \sum_{(p_{k}, q_{k}) \in \Omega} |a_{p_{k}, q_{k}}^{(k)}|^{2}}{\min_{(p_{k}, q_{k}) \in \Omega} \sum_{p_{k}, q_{k}} d_{p_{k}, q_{k}}^{2} \sum_{p_{k}, q_{k}} d_{p_{k}, q_{k}}^{2} (A_{k})} \leq \frac{4 \sum_{k=0}^{N-1} |a_{p_{k}, q_{k}}^{(k)}|^{2}}{\min_{(p_{k}, q_{k}) \in \Omega} \sum_{p_{k}, q_{k}} d_{p_{k}, q_{k}}^{2} \sum_{p_{k}, q_{k}} d_{p_{k}, q_{k}}^{2} (A_{k})}
\]

\[
\leq \frac{2 \|\text{off}(A_{0})\|_{F}^{2} + 2 \sum_{k=0}^{N-1} \|H_{k}\|_{F}^{2}(1 + \tilde{\gamma}_{4}) + 2 \sum_{k=0}^{N-1} \|G_{k}\|_{F}^{2}(1 + \tilde{\gamma}_{4})}{\min_{(p_{k}, q_{k}) \in \Omega} \sum_{p_{k}, q_{k}} d_{p_{k}, q_{k}}^{2} \sum_{p_{k}, q_{k}} d_{p_{k}, q_{k}}^{2} (A_{k})}
\]

(3.44)

where the last inequality follows from (3.6).
From (3.42), (3.43), and (3.44) we have

\[ \| \operatorname{off}(A_N) \|_F^2 = 2 \sum_{r=1}^{n-1} \sum_{q=r+1}^{n} |\hat{a}_{r,q}^{(N)}|^2 = 2 \sum_{r=1}^{m-1} \sum_{q=r+1}^{m} |\hat{a}_{r,q}^{(N)}|^2 + 2 \sum_{r=m+1}^{n} \sum_{q=r+1}^{n} |\hat{a}_{r,q}^{(N)}|^2 \]

\[ \leq 4 \sum_{k=0}^{n-1} \sum_{q=k+1}^{n} \| G_{k+1} \|_F^2 \| \gamma_4 \|^2 (1 + 2 \gamma_4)(1 + \gamma_6)^{2(n-2)} \]

\[ + 2 \sum_{r=1}^{m-1} \sum_{q=r+1}^{m} |s_{r,q}|^2 (1 + 2 \gamma_4)(1 + \gamma_6)^{2(n-1)} \xi^2 \]

\[ + 2(m - 1) \left( 4m \left( \sum_{t=0}^{N-1} \phi_t \right)^2 \gamma_4^2 + \alpha^{-2} \left( 2 \| \operatorname{off}(A_0) \|_F^2 + 8m \left( \sum_{t=0}^{N-1} \phi_t \right)^2 \right) \right) (1 + \gamma_4)^{2(n-2)} (1 + 2 \gamma_4) \]

\[ = 4 \sum_{k=0}^{N-1} \| G_k \|_F^2 \| \gamma_4 \|^2 (1 + 2 \gamma_4)(1 + \gamma_6)^{2(n-2)} + 2 \sum_{(p_k,q_k) \notin \Omega} |s_{k,q}|^2 (1 + 2 \gamma_4)(1 + \gamma_6)^{2(n-1)} \xi^2 \]

\[ + 2(m - 1) \left( 4m \left( \sum_{t=0}^{N-1} \phi_t \right)^2 \gamma_4^2 + \alpha^{-2} \left( 2 \| \operatorname{off}(A_0) \|_F^2 + 8m \left( \sum_{t=0}^{N-1} \phi_t \right)^2 \right) \right) (1 + \gamma_4)^{2(n-2)} (1 + 2 \gamma_4) \]

\[ \leq 4 \sum_{k=0}^{N-1} \| G_k \|_F^2 \| \gamma_4 \|^2 (1 + 2 \gamma_4)(1 + \gamma_6)^{2(n-2)} + 4 \left( \min_{(p_k,q_k) \notin \Omega} d^2(A_k) \| \operatorname{off}(A_0) \|_F^2 \right)^{-1} \| \operatorname{off}(A_0) \|_F^2 \cdot \]

\[ \left( \| \operatorname{off}(A_0) \|_F^2 + 2 \sum_{k=0}^{N-1} \| H_k \|_F^2 \gamma_4 + 2 \sum_{k=0}^{N-1} \| G_k \|_F^2 \gamma_4^2 \right) (1 + 2 \gamma_4)(1 + \gamma_6)^{2(n-1)} \xi^2 \]

\[ + 2(m - 1) \left( 4m \left( \sum_{t=0}^{N-1} \phi_t \right)^2 \gamma_4^2 + \alpha^{-2} \left( 2 \| \operatorname{off}(A_0) \|_F^2 + 8m \left( \sum_{t=0}^{N-1} \phi_t \right)^2 \right) \right) (1 + \gamma_4)^{2(n-2)} (1 + 2 \gamma_4) \]

\[ \leq (\alpha_2 \| \operatorname{off}(A_0) \|_F^2 + \beta_2 \gamma_4)^2 \]

\[ + 2(m - 1) \left( 4m \left( \sum_{t=0}^{N-1} \phi_t \right)^2 \gamma_4^2 + \alpha^{-2} \left( 2 \| \operatorname{off}(A_0) \|_F^2 + 8m \left( \sum_{t=0}^{N-1} \phi_t \right)^2 \right) \right) (1 + \gamma_4)^{2(n-2)} (1 + 2 \gamma_4) \]

\[ \leq \left( \alpha_2 \| \operatorname{off}(A_0) \|_F^2 + \beta_2 \gamma_4 + (1 + \gamma_4)^{n-2} (1 + 2 \gamma_4)^{1/2} \sqrt{8(m - 1)m} \sum_{t=0}^{N-1} \phi_t \right) \gamma_4 \]

\[ + \sqrt{2(m - 1)}(1 + \gamma_4)^{n-2} (1 + 2 \gamma_4)^{1/2} \alpha^{-1} \left( 2 \| \operatorname{off}(A_0) \|_F^2 + 8m \left( \sum_{t=0}^{N-1} \phi_t \right)^2 \right) \]

\[ = (\alpha_3 \| \operatorname{off}(A_0) \|_F^2 + \beta_3 \gamma_4)^2, \]
Thus, \( \xi \) and \( \| \) interchanging the sum of \( P \) uses the fact that \( \alpha_2^2(a_1^2 + a_2^2) \leq (a_1^2 + a_2/2)^2 \) for all \( a_1, a_2 \in \mathbb{R} \), the fourth inequality uses the fact that \( a_1 + a_2 \leq (\sqrt{a_1^2 + a_2^2})^2 \) for all \( a_1, a_2 \geq 0 \), the last inequality uses the fact that \( \alpha_1^2 + a_2^2 + a_3^2 \leq (a_1 + a_2 + a_3)^2 \) for all \( a_1, a_2, a_3 \geq 0 \), and

\[
\alpha_2' = 2(1 + \tilde{\gamma}_6)^{n-1}(1 + 2\tilde{\gamma}_4)^{1/2} \xi \left( \min_{(p_k, q_k) \notin \Omega} d_r(A_k) \right) \| \text{off}(A_0) \|_F^{-1},
\]

\[
\beta_2' = \alpha_2' \left( \sum_{k=0}^{N-1} \| H_k \|_F^2 + \sum_{k=0}^{N-1} \| G_k \|_F^2 \tilde{\gamma}_4^2 \right) + 2 \left( \sum_{k=0}^{N-1} \| G_k \|_F^2 \right)^{1/2} (1 + \tilde{\gamma}_6)^{n-2}(1 + 2\tilde{\gamma}_4)^{1/2},
\]

\[
\alpha_3 = \alpha_2' + 2\alpha_1^{-1} \sqrt{2(m - 1)}(1 + \tilde{\gamma}_4)^{n-2}(1 + 2\tilde{\gamma}_4)^{1/2}, \tag{3.45}
\]

and

\[
\beta_3 = \beta_2' + \sqrt{8(m - 1)m(1 + \tilde{\gamma}_4)^{n-2}(1 + 2\tilde{\gamma}_4)^{1/2}} \sum_{i=0}^{N-1} \varphi_i \\
+ \alpha_1^{-1} \left( 8m \sqrt{2(m - 1)}(1 + \tilde{\gamma}_4)^{n-2}(1 + 2\tilde{\gamma}_4)^{1/2} \left( \sum_{i=0}^{N-1} \varphi_i \right)^2 \right) \tilde{\gamma}_4. \tag{3.46}
\]

Finally, we show that \( \xi \left( \min_{0 \leq k \leq N-1} d(A_k) \right) \| \text{off}(A_0) \|_F^{-1} < \infty \). Under the conditions (3.15), (3.36), and (3.37), it follows from Lemmas 3.19 and 3.6 that \( d(A_k) \geq d > 0 \) for all \( 0 \leq k \leq N - 1 \) and \( \| \text{off}(A_k^{(r-1)}) \|_F \leq \| \text{off}(A_0) \|_F \) for all \( 1 \leq r \leq n - 1 \) and \( k \geq 0 \). Then, interchanging the sum of \( \sum_{q=3}^{n-r+1} \sum_{k=0}^{q-3} \| d_{r+1-q} \|_F^2 \) yields

\[
\xi^2 = \max_{1 \leq r \leq n-1} \left( \| \text{off}(A_0^{(r-1)}) \|_F^2 + 4 \sum_{k=0}^{n-r-2} \sum_{q=3+k}^{n-r+1} | d_{r+1-q} |^2 \tilde{\gamma}_4^2 \right) \\
\leq \max_{1 \leq r \leq n-1} \left( \| \text{off}(A_0^{(r-1)}) \|_F^2 + 4 \sum_{k=0}^{n-r-2} \| \text{off}(A_k^{(r-1)}) \|_F^2 \tilde{\gamma}_4 \right) \\
\leq (1 + 4(n - 2)\tilde{\gamma}_4) \| \text{off}(A_0) \|_F^2.
\]

Thus, \( \xi \left( \min_{0 \leq k \leq N-1} d(A_k) \right) \| \text{off}(A_0) \|_F^{-1} \leq \sqrt{1 + 4n\tilde{\gamma}_4/d} \leq (1 + 2n\tilde{\gamma}_4)/d < \infty \).

Based on the above analysis, we have the following error bound for one sweep of the row cyclic Jacobi method in floating point arithmetic.

**Theorem 3.23** Let \( A_k \) be the matrix \( A_0 = A \) after \( k \) Jacobi updates, which is generated by Algorithm 3.2 in floating point arithmetic, where the index pairs \( \{(p_k, q_k)\} \) are chosen in the row cyclic order. Let \( \varphi_k = \left( (6 + 4\sqrt{2})\| G_k \|_F^2 + 2\| H_k \|_F^2 \right)^{1/2} \) with \( G_k \) and \( H_k \) being defined as in Lemma 3.5 with \( (p, q) = (p_k, q_k) \) for all \( k \geq 0 \). Suppose \( A_0 \) has only one multiple eigenvalue \( \lambda_1(A) \) with multiplicity \( m \) and the remaining \( n - m \) eigenvalues \( \{\lambda_i(A)\}_{i=m+1}^n \) are distinct, and \( \| \text{off}(A_0) \|_F < d_r(A_0)/4 \). If \( \tilde{\gamma}_4 \) satisfies the conditions (3.15), (3.36), and (3.37), then we have \( \| \text{off}(A_N) \|_F \leq \alpha_3 \| \text{off}(A_0) \|_F \beta_3 \tilde{\gamma}_4 \), where the constants \( \alpha_3 > 0 \) and \( \beta_3 > 0 \) are defined by (3.45) and (3.46), respectively.
4 A mixed precision preconditioned Jacobi method for the symmetric eigenvalue problem

In this section, we propose a mixed precision preconditioned Jacobi algorithm for computing the eigenvalue decomposition of an $n$-by-$n$ real symmetric matrix $A$. We first compute an approximate eigenvalue decomposition $A \approx ZDZ^T$ in low precision $\nu$, where $Z \in \mathbb{R}^{n \times n}$ and $D = \text{diag}(d_1, \ldots, d_n)$. To improve the orthogonality of $Z$, we use the MGS method to $Z$ in high precision $\omega \ll \nu$ and we obtain a high accuracy orthogonal matrix $Q \in \mathbb{R}^{n \times n}$. One may also employ Householder orthogonalization to orthogonalize $Z$, which is less computationally efficient than MGS [25, §5.2.9]. Then we use the matrix $Q$ as an initial guess for the Jacobi method for computing the eigenvalue decomposition of $A$.

Based on the above analysis, we present our mixed precision preconditioned Jacobi algorithm in Algorithm 4.1, where “In precision $\omega$” means computed and stored in precision $\omega$.

Algorithm 4.1 A mixed precision preconditioned cyclic Jacobi method for the symmetric eigenvalue decomposition.

Require: A symmetric matrix $A \in \mathbb{R}^{n \times n}$ and a tolerance $\epsilon > 0$.
1: $[Z, \Phi] = \text{eig} (A)$ \Comment{Symmetric QR factorization in precision $\nu$ and store $Z$ at precision $\omega$}
2: $Q = \text{MGS} (Z)$ \Comment{MGS orthogonalization in precision $\omega$}
3: Set $T = Q^T AQ$ and $P = Q$ \Comment{In precision $\omega$}
4: while $\|\text{off}(T)\|_F > \epsilon \|A\|_F$ do
5: for $i = 1, \ldots, n - 1$ do
6: for $j = i + 1, \ldots, n$ do
7: Compute a cosine-sine group $(c, s)$ as in Lemma 3.1 with $A = T$. \Comment{In precision $\omega$}
8: Set $T = J(i, j, \theta)^T T J(i, j, \theta)$ and $P = P J(i, j, \theta)$. \Comment{In precision $\omega$}
9: end if
10: end for
11: end for
12: end while

We observe from Theorems 3.17 and 3.23 that, to take advantage of the Jacobi method as much as possible, it is desirable that the Frobenius norm of the off-diagonal entries of the matrix $Q^T AQ$ in Step 3 of Algorithm 4.1 is small enough.

In the following, we give the error analysis of Algorithm 4.1. We first give an estimate of the distance between $Z$ and $Q$ generated by Algorithm 4.1.

Lemma 4.1 Suppose that $Z \in \mathbb{R}^{n \times n}$ in Algorithm 4.1 is computed by any eigensolver in LAPACK or EISPACK in precision $\nu$ and $Q \in \mathbb{R}^{n \times n}$ in Algorithm 4.1 is computed by using the MGS method to $Z$ in precision $\omega$ ($\omega \ll \nu$). Then there exist constants $h_t \equiv h_t(n)$ for $t = 1, 2$ such that $\|Z - Q\|_F \leq h_1 \nu + h_2 \omega$.

Proof. By Lemma 2.2, there exists a matrix $\delta Z \in \mathbb{R}^{n \times n}$ such that $Z + \delta Z$ is orthogonal, where $\|\delta Z\| \leq p(n) \nu$. Thus,

$$\|Z\| \leq \|Z + \delta Z\| + \|\delta Z\| = 1 + \|\delta Z\| \leq 1 + p(n) \nu \equiv c_1.$$ (4.1)
This, together with the orthogonality of $Z + \delta Z$, yields

\[
\begin{align*}
\lambda_{\max}(Z^T Z) &\leq 1 + (2\|\delta Z\|\|Z\| + \|\delta Z\|^2) \leq 1 + \chi_1, \\
\lambda_{\min}(Z^T Z) &\geq 1 - (2\|\delta Z\|\|Z\| + \|\delta Z\|^2) \geq 1 - \chi_1,
\end{align*}
\]

provided that $\chi_1 = (2c_1 + p(n)\nu)p(n)\nu < 1$. Hence, $Z$ is nonsingular and $\text{rank}(Z) = n$. In addition,

\[
\kappa(Z) = \frac{\sigma_{\max}(Z)}{\sigma_{\min}(Z)} = \sqrt{\frac{\lambda_{\max}(Z^T Z)}{\lambda_{\min}(Z^T Z)}} \leq \sqrt{\frac{1 + \chi_1}{1 - \chi_1}}.
\tag{4.2}
\]

It follows from Lemma 2.1 and (4.1) that there exists an upper triangular matrix $R \in \mathbb{R}^{n \times n}$ such that

\[
Z = QR + F_1, \quad \|F_1\| \leq \eta_1\|Z\|\omega \leq \eta_1 c_1 \omega \equiv c_2 \omega,
\tag{4.3}
\]

where $Q$ is such that

\[
Q^T Q = I + F_2, \quad \|F_2\| \leq \eta_2 \kappa(Z) \omega \leq \sqrt{\frac{1 + \chi_1}{1 - \chi_1}} \cdot \eta_2 \omega \equiv c_3 \omega.
\tag{4.4}
\]

We now give an upper bound and a lower bound to each diagonal entry of $R$. From (4.4) we have

\[
\|Q\| \leq \sqrt{1 + \|F_2\|} \leq 1 + \|F_2\|.
\tag{4.5}
\]

Using (4.3) and (4.4) we have $Q^T Z = R + F_2 R + Q^T F_1$, which implies that

\[
\|R\| \leq \|Q\|\|Z\| + \|F_2\|\|R\| + \|Q\|\|F_1\|.
\]

Thus,

\[
\|R\| \leq \frac{\|Q\|}{1 - \|F_2\|} (\|Z\| + \|F_1\|) \leq \frac{1 + \|F_2\|}{1 - \|F_2\|} (\|Z\| + \|F_1\|) \\
\leq \left(1 + \frac{2\|F_2\|}{1 - \|F_2\|}\right) (\|Z\| + \|F_1\|) \\
\leq \left(1 + \frac{2\chi_2}{1 - \chi_2}\right)(1 + \eta_1\omega)(1 + p(n)\nu) \equiv 1 + g_1 \nu,
\tag{4.6}
\]

provided that $\chi_2 = c_3 \omega < 1$, where

\[
g_1 = \eta_1 \omega \nu^{-1} + (1 + \eta_1\omega)p(n) + \frac{2c_3 \omega \nu^{-1} + 2\eta_1 \chi_2 \omega \nu^{-1} + 2\chi_2 p(n) + 2\eta_1 \chi_2 p(n)\omega}{1 - \chi_2}.
\]

Similarly, we have by (4.3),

\[
(Z + \delta Z)R^{-1} = Q + (F_1 + \delta Z)R^{-1}.
\]

This, together with the orthogonality of $Z + \delta Z$, yields

\[
\|R^{-1}\| = \|(Z + \delta Z)R^{-1}\| \leq \|Q\| + (\|F_1\| + \|\delta Z\|)\|R^{-1}\|.
\]

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Using (4.3)–(4.5) we have
\[ \| R^{-1} \| \leq \frac{\| Q \|}{1 - (\| F_1 \| + \| \delta Z \|)} \leq \frac{1 + \| F_2 \|}{1 - (\| F_1 \| + \| \delta Z \|)} \leq \frac{1 + c_3 \omega}{1 - \chi_3} \equiv 1 + g_2 \nu, \]  
provided that \( \chi_3 = c_2 \omega + p(n) \nu < 1 \), where \( g_2 = ((c_2 + c_3) \omega ^{-1} + p(n))/(1 - \chi_3) \).

If \( g_2 \nu < 1 \), then
\[ 1 - g_2 \nu \leq \frac{1}{1 + g_2 \nu} \leq \frac{1}{\| R^{-1} \|} \leq \tau_s \leq \| R \| \leq 1 + g_1 \nu, \ s = 1, 2, \ldots, n. \tag{4.8} \]

Next, we bound the spectral norm of the strictly upper triangular part of \( R \). Substituting (4.3) and (4.4) into \((Z + \delta Z)^T(Z + \delta Z) = I \) yields
\[ I - R^T R = R^T F_2 R + (F_1 + \delta Z)^T Q R + R^T Q^T (F_1 + \delta Z) + (F_1 + \delta Z)^T (F_1 + \delta Z). \]

Then, by (4.3), (4.4), (4.5), and (4.6) we have
\[
\| I - R^T R \| \leq \| R \|^2 \| F_2 R \| + 2 (\| F_1 \| + \| \delta Z \|) \| Q \| \| R \| + (\| F_1 \| + \| \delta Z \|)^2 \\
\leq (1 + g_1 \nu)^2 c_3 \omega + 2(c_2 \omega + p(n) \nu) \sqrt{1 + \chi_2(1 + g_1 \nu) + (c_2 \omega + p(n) \nu)^2} \equiv c_4 \nu,
\]
where \( c_4 = c_3(1 + g_1 \nu)^2 \omega ^{-1} + 2(c_2 \omega ^{-1} + p(n)) \sqrt{1 + \chi_2(1 + g_1 \nu) + \chi_3(c_2 \omega ^{-1} + p(n))}. \) This, together with (4.7), yields
\[
\| R^{-1} - R^T \| \leq \| I - R^T R \| \| R^{-1} \| \leq c_4(1 + g_2 \nu) \nu \equiv c_5 \nu,
\]
which implies that
\[
| \hat{r}_{st} | \leq \| R^{-1} - R^T \| \leq c_5 \nu \quad \forall s < t. \tag{4.9}
\]

It follows from (4.8) and (4.9) that
\[
\| R - I \|_F = \left( \sum_{i=1}^{n} (\hat{r}_{ii} - 1)^2 + \sum_{i<j} \hat{r}_{ij}^2 \right)^{1/2} \leq \nu \sqrt{n \max \{ g_1, g_2 \}^2 + \frac{n(n-1)}{2} c_5^2} \equiv c_6 \nu.
\]

Therefore, by (4.3), (4.4), and (4.5) we have
\[
\| Z - Q \|_F \leq \| Q \| \| R - I \|_F + \| F_1 \|_F \leq \sqrt{1 + \| F_2 \|} \| R - I \|_F + \sqrt{n} \| F_1 \| \\
\leq \sqrt{1 + c_3 \omega} \cdot c_6 \nu + \sqrt{n} \cdot c_2 \omega \equiv h_1 \nu + h_2 \omega.
\]

On the Frobenius norm of the off-diagonal entries of the matrix \( Q^T A Q \) generated in Step 3 of Algorithm 4.1, we have the following result.

**Theorem 4.2** Suppose that \( Z \in \mathbb{R}^{n \times n} \) in Algorithm 4.1 is computed by any eigensolver in LAPACK or EISPACK in precision \( \nu \) and \( Q \in \mathbb{R}^{n \times n} \) in Algorithm 4.1 is computed by using the MGS method to \( Z \) in precision \( \omega \) (\( \omega \ll \nu \)). Then there exists a constant \( \tilde{\psi}_1 \equiv \psi_1(n) \) such that \( \| \text{off}(Q^T A Q) \|_F \leq \tilde{\psi}_1 \| A \| \nu. \)

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Proof. By Lemma 2.2 we know there exists a symmetric matrix $E \in \mathbb{R}^{n \times n}$ such that $A + E$ admits the following exact eigenvalue decomposition:

\begin{equation}
A + E = (Z + \delta Z)D(Z + \delta Z)^T, \quad D = \text{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n},
\end{equation}

where $\|E\| \leq p(n)\|A\|v$ and $Z + \delta Z$ is orthogonal with $\|\delta Z\| \leq p(n)v$. We note that $Q$ is computed by using the MGS method to $Z$ with precision $\omega$. It follows that

$$Q^T AQ - D = (Q - Z + Z)^T A(Q - Z + Z) - D = (Q - Z)^T A(Q - Z) + (Q - Z)^T AZ + Z^T A(Q - Z) + (Z^T AZ - D).$$

On the other hand, using (4.10) we have $D = (Z + \delta Z)^T(A + E)(Z + \delta Z)$, i.e.,

$$D - Z^T AZ = Z^T A\delta Z + \delta Z^T AZ + \delta Z^T A\delta Z + (Z + \delta Z)^T E(Z + \delta Z).$$

Thus,

$$\|Z^T AZ - D\|_F \leq 2\|A\|_F \|Z\|\|\delta Z\| + \|A\|_F \|\delta Z\|^2 + \|E\|_F \leq \|A\|_F(2\|Z\| + \|\delta Z\|)\|\delta Z\| + \|E\|_F.$$  

By Lemma 4.1 and using (4.1) and (4.11), we have

$$\|\text{off}(Q^T AQ)\|_F \leq \|Q^T AQ - D\|_F \leq \|(Q - Z)^T A(Q - Z)\|_F + 2\|(Q - Z)^T AZ\|_F + \|Z^T AZ - D\|_F \leq \|A\|(Q - Z)\|_F + 2\|A\|\|Z\|\|\delta Z\| + \|A\|_F(2\|Z\| + \|\delta Z\|)\|\delta Z\| + \|E\|_F \leq \|A\|(Q - Z)\|_F + \sqrt{n}\|\|\delta Z\|)\|\delta Z\| + \sqrt{n}\|E\|_F \leq (2c_1 + h_1v + h_2\omega)\|A\|(h_1v + h_2\omega) + \sqrt{n}\|A\|(2c_1 + p(n)v)p(n)v + \sqrt{n}\|A\|p(n)v \leq \tilde{\psi}_1\|A\|v,$$

where $\tilde{\psi}_1 = (2c_1 + h_1v + h_2\omega)(h_1 + h_2\omega v^{-1}) + \sqrt{n}p(n)(1 + 2c_1 + p(n)v)$. \hfill \square

The following theorem gives an error bound for one sweep of Jacobi’s method in Algorithm 4.1. Here, $\text{fl}(\cdot)$ is floating-point operation at precision $\omega$.

**Theorem 4.3** Let $T_k = (i_{ij}^{(k)})$ be the matrix $T_0 = \text{fl}(Q^T AQ)$ after $k$ Jacobi updates in Algorithm 4.1, where the $k$th computed Jacobi rotation is $J_k = J(p_k, q_k, \hat{c}_k, \hat{\delta}_k)$. Let $\gamma_n := (1 - \omega)^{-n} - 1$, $\tilde{\gamma}_j := (1 - \omega)^{-w_j} - 1$ for a small integer constant $w > 0$, and $\varphi_k = ((6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^2)^{1/2}$, where $G_k$ and $H_k$ are defined as in Lemma 3.5 with $A_k = T_k$ and $(p, q) = (p_k, q_k)$ for all $k \geq 0$. Suppose $T_0$ has $n$ distinct eigenvalues with $d(T_0) > 0$. If the precisions $\omega$ and $v$ satisfy $\omega \ll v$, $4\psi_1\|A\|v < d(T_0)$ for some constant $\psi_1 = \tilde{\psi}_1(n) + v^{-1}\gamma_n(2 + \gamma_n)\|Q\|T\|A\|Q\|F/\|A\|$, $d(T_0) \geq d + \sum_{k=0}^{N-1} \varphi_k \tilde{\gamma}_k$ for some $d > 0$, and

$$\frac{1}{8}\varphi_k\tilde{\gamma}_k^2 + \frac{1}{2}\|\text{off}(T_k)\|_F\varphi_k\tilde{\gamma}_k + \|H_k\|_F^2\tilde{\gamma}_k^2 + \|G_k\|_F^2\tilde{\gamma}_k^2 \leq |i_{ij}^{(k)}|_2^2,$$

for all $k = 0, 1, \ldots, N - 1$, then there exist two constants $\alpha, \beta > 0$ such that $\|\text{off}(T_N)\|_F \leq \alpha\|\text{off}(T_0)\|_F^2 + \beta\tilde{\gamma}_k$, where $N = (n - 1)/2$ and $\tilde{\psi}_1(n)$ is defined as Theorem 4.2.
Proof. The computed symmetric matrix congruence $T_0 = \text{fl}(Q^T AQ)$ satisfies $|T_0 - Q^T AQ| \leq \gamma_n(2 + \gamma_n)|Q|^T|A||Q|$ [31, Sect. 3], and thus $\|T_0 - Q^T AQ\|_F \leq \gamma_n(2 + \gamma_n)||Q|^T|A||Q||_F$. By Theorem 4.2 we have

$$\|\text{off}(T_0)\|_F \leq \|T_0 - Q^T AQ\|_F + \|\text{off}(Q^T AQ)\|_F \leq \psi_1 \|A\|_v$$

for some constant $\psi_1 \equiv \tilde{\psi}_1(n) + v^{-1}\gamma_n(2 + \gamma_n)||Q|^T|A||Q||_F/\|A\|$ Therefore, we have $4\|\text{off}(T_0)\|_F < d(T_0)$ provided that $4\psi_1 \|A\|_v < d(T_0)$. The theorem is established by using Theorem 3.17.

In Theorem 4.3, it is worth pointing out that

$$(||Q|^T|A||Q||_F/\|A\|) \leq \|Q\|_F^2 ||A||/\|A\| \leq n^{3/2}(1 + c_3\omega).$$

by using (4.4)–(4.5). Hence, $\psi_1$ is not too large. We also note that the error bound of Algorithm 4.1 is established under the assumption that $\psi_1 \|A\|_v < d(T_0)/4$ for some constant $\psi_1 = \psi_1(n)$ and there exists a clear gap between the eigenvalues of $T_0 = \text{fl}(Q^T AQ)$. In the following theorem, we establish the error analysis of Algorithm 4.1 under the assumption that there exists a clear gap between the eigenvalues of the original matrix $A$.

**Theorem 4.4** Let $T_k = (t^{(k)}_{ij})$ be the matrix $T_0 = \text{fl}(Q^T AQ)$ after $k$ Jacobi updates in Algorithm 4.1, where the $k$th computed Jacobi rotation is $J_k = J(p_k, q_k; \delta_k, \tilde{\delta}_k)$. Let $\gamma_n := (1 - \omega)^{-n} - 1, \tilde{\gamma}_j := (1 - \omega)^{-w_j} - 1$ for a small integer constant $w > 0$, and $\varphi_k = ((6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^2)^{1/2}$, where $G_k$ and $H_k$ are defined as in Lemma 3.5 with $A_k = T_k$ and $(p, q) = (p_k, q_k)$ for all $k \geq 0$. Suppose $A$ has $n$ distinct eigenvalues with $\|A\| > 0$. If the precisions $\omega$ and $v$ satisfy $\omega \ll v, 4\psi_1 \|A\|_v < (1 - \rho_1)d(A)$ for two constants $\psi_1 = \tilde{\psi}_1(n) + v^{-1}\gamma_n(2 + \gamma_n)||Q|^T|A||Q||_F/\|A\|$ and $0 < \rho_1 < 1, 2\psi_2 \|A\|_v \leq \rho_1d(A)$ for some constant $\psi_2 = \tilde{\psi}_2(n), (1 - \rho_1)d(A) \geq \tilde{d} + \sum_{k=0}^{N-1} \varphi_k \tilde{\gamma}_4$ for some $\tilde{d} > 0$, and

$$\frac{1}{\varphi_k^2} \tilde{\gamma}_4^2 + \frac{1}{2} \|\text{off}(T_k)\|_F \varphi_k \tilde{\gamma}_4 + \|H_k\|_F^2 \tilde{\gamma}_4 + \|G_k\|_F^2 \tilde{\gamma}_4^2 \leq |l^{(k)}_{ij}|^2,$$

for all $k = 0, 1, \ldots, N - 1$, then there exist two constants $\alpha, \beta > 0$ such that $\|\text{off}(T_N)\|_F \leq \alpha \|\text{off}(T_0)\|_F + \beta \tilde{\gamma}_4$, where $\tilde{\psi}_1(n)$ is defined as in Theorem 4.2.

Proof. By Lemma 2.1, there exists a matrix $\delta Q \in \mathbb{R}^{n \times n}$ such that $Q + \delta Q$ is orthogonal with

$$\|\delta Q\| \leq \eta_3 \kappa(Z)\omega \leq \sqrt{\frac{1 + \chi_1}{1 - \chi_1}} \cdot \eta_3 \omega \equiv \eta_4 \omega,$$

where the second inequality follows from (4.2). We note that

$$(Q + \delta Q)^T A (Q + \delta Q) = Q^T AQ + Q^T A\delta Q + \delta Q^T AQ + \delta Q^T A\delta Q.$$

By hypothesis, $T_0 = \text{fl}(Q^T AQ) = Q^T AQ + \Delta_1$ with $\|\Delta_1\| \leq \|\Delta_1\| \leq \gamma_n(2 + \gamma_n)||Q|^T|A||Q||$. It follows from Lemma 2.4, (4.4) and (4.5) that, for any $1 \leq k \leq n$,

$$|\lambda_k(A) - \lambda_k(T_0)| \leq 2\|A\||Q|^T||\delta Q|| + \|A\||\delta Q\| + \gamma_n(2 + \gamma_n)||Q|^T|A||Q|| \leq 2\|A\||Q|^T + \|\delta Q\| + \gamma_n(2 + \gamma_n)||Q|^T|A||Q|| \leq \psi_2 \|A\|_v.$$
where \( \psi_2 = (2\sqrt{1 + c_0^2} + \eta_1 \omega)\eta_4 + \omega^{-1}\gamma_n(2 + \gamma_n)||Q^T A||Q||/||A||. \)

For any \( i \neq j \), we have

\[
|\lambda_i(A) - \lambda_j(A)| \leq |\lambda_i(A) - \lambda_i(T_0)| + |\lambda_i(T_0) - \lambda_j(T_0)| + |\lambda_j(T_0) - \lambda_j(A)| \\
\leq |\lambda_i(T_0) - \lambda_j(T_0)| + 2\psi_2\|A\|\omega, \\
|\lambda_i(A) - \lambda_j(A)| \geq ||\lambda_i(T^{(0)}) - \lambda_j(T_0)\| - |\lambda_i(A) - \lambda_i(T_0)| - |\lambda_j(T_0) - \lambda_j(A)| \\
\geq |\lambda_i(T_0) - \lambda_j(T_0)| - 2\psi_2\|A\|\omega.
\]

If \( 2\psi_2\|A\|\omega \leq \rho_1 d(A) \) for some \( 0 < \rho_1 < 1 \), then we have

\[
\frac{1}{1 + \rho_1} |\lambda_i(T_0) - \lambda_j(T_0)| \leq |\lambda_i(A) - \lambda_j(A)| \leq \frac{1}{1 - \rho_1} |\lambda_i(T_0) - \lambda_j(T_0)|.
\]

By hypothesis, \( d(A) > 0 \). Thus, \( d(T_0) \geq (1 - \rho_1)d(A) \). The theorem follows from Theorem 4.3 provided that \( \psi_1\|A\|\upsilon < (1 - \rho_1)d(A)/4 \).

In Theorem 4.4, we need an additional condition that \( 2\psi_2\|A\|\omega \leq \rho_1 d(A) \) for some constant \( \psi_2 = \psi_2(n) \) and \( 0 < \rho_1 < 1 \). In fact, it is much weaker than the condition that \( \psi_1\|A\|\upsilon < (1 - \rho_1)d(A)/4 \) for some constant \( \psi_1 = \psi_1(n) \) since \( \omega \ll \upsilon \). Therefore, the later is an essential condition for the error analysis of Algorithm 4.1 under the assumption that there exists a clear gap between the eigenvalues of the original matrix \( A \).

## 5 A mixed precision preconditioned one-sided Jacobi method for the SVD

In this section, we propose a mixed precision preconditioned one-sided Jacobi method for computing the SVD of a real matrix. As we know, the one-sided Jacobi method for the singular value problem was originally mentioned in [30].

We first describe the one-sided Jacobi algorithm. Let \( A \) be an \( m \times n \) real matrix \((m \geq n)\). The one-sided Jacobi algorithm aims to construct a sequence of orthogonal updates \( A^{(k+1)} = A^{(k)}J_k \) such that the columns of \( A^{(k+1)} \) are mutually orthogonal sufficiently, where \( A^{(0)} = A \) and \( J_k \) is a Jacobi rotation defined by (3.1). Then, the computed SVD of \( A \) is available by columns scaling of the updated \( A^{(k+1)} \).

On how to orthogonalize two columns of \( A \), we have the following result [15].

**Lemma 5.1** Let \( A \) be an \( m \times n \) real matrix \((m \geq n)\). For any index pair \((i, j)\) with \( 1 \leq i < j \leq n \), if \( a_i^T a_j \neq 0 \), then there exists a Jacobi rotation \( J = J(i, j; \theta) \) defined by (3.1) such that, for the updated matrix \( B = AJ \),

\[
b_i^T b_j = 0, \quad b_i^2 + b_j^2 = a_i^2 + a_j^2 + 2a_i a_j, 
\]

where \( c = \cos \theta = (1 + t^2)^{-1/2} \) and \( s = tc with |\theta| \leq \pi/4 \). Here, \( t = 1/(\mu + \sqrt{1 + \mu^2}) \) if \( \mu \geq 0 \) and \( t = 1/(\mu - \sqrt{1 + \mu^2}) \) if \( \mu < 0 \), where \( \mu = (a_j^T a_i - a_i^T a_j)/(2a_i^T a_j) \).
Algorithm 5.1 Cyclic one-sided Jacobi’s method for the SVD.

Require: A matrix $A \in \mathbb{R}^{m \times n}$ ($m \geq n$) with rank$(A) = n$ and a tolerance $\epsilon > 0$. Let $V = I_n$.

1: while $\|\text{off}(A^T A)\|_F > \epsilon \|A^T A\|_F$ do
2:     for $i = 1, \ldots, n - 1$ do
3:       for $j = i + 1, \ldots, n$ do
4:         Compute a cosine-sine group $(c, s)$ as in Lemma 5.1.
5:        Set $A = A J(i, j, \theta)$ and $V = V J(i, j, \theta)$.
6:     end for
7:  end for
8: end while

In fact, the Jacobi rotation $J = J(i, j; \theta)$ defined by Lemma 5.1 is such that the off-diagonal entries $(i, j)$ and $(j, i)$ in the symmetric matrix $B^T B = J^T A^T A J$ are zeros. A row-cyclic one-sided Jacobi algorithm for the SVD is stated as Algorithm 5.1.

For a reliable implementation of Algorithm 5.1, one may refer to [22].

As in Section 4, we propose a mixed precision preconditioned one-sided Jacobi algorithm for computing the SVD of a real matrix with full column rank, which is stated as Algorithm 5.2.

Here, $\omega \leq u \leq \upsilon$.

Algorithm 5.2 A mixed precision preconditioned one-sided Jacobi method for the SVD.

Require: A matrix $A \in \mathbb{R}^{m \times n}$ ($m \geq n$) with rank$(A) = n$ and a tolerance $\epsilon > 0$.

1: $[\sim, \sim, Z] = \text{svd}(A)$ \hspace{1cm} \triangleright \text{SVD in precision } \upsilon \text{ and store } Z \text{ in precision } \omega
2: \text{Q} = \text{MGS}(Z) \hspace{1cm} \triangleright \text{Modified Gram-Schmidt orthogonalization in precision } \omega
3: \text{Set } C = A \text{Q} \text{ and } V = \text{Q} \hspace{1cm} \triangleright \text{In precision } \omega
4: \text{while } \|\text{off}(C^T C)\|_F > \epsilon \|C^T C\|_F \text{ do}
5:     for $i = 1, \ldots, n - 1$ do
6:       for $j = i + 1, \ldots, n$ do
7:         Compute a cosine-sine group $(c, s)$ as in Lemma 5.1 with $A = C$.
8:        Set $C = C J(i, j, \theta)$ and $V = V J(i, j, \theta)$. \triangleright \text{In precision } \omega
9:     end for
10: end while
11: Reorder the columns of $[C^T, V^T]^T$ with $\|c_1\| \geq \cdots \geq \|c_n\| > 0$.
12: Compute $\sigma_j = \|c_j\|$ and $u_j = c_j / \sigma_j$ for $j = 1, \ldots, n$. \triangleright \text{In precision } \omega
13: Set $U = [u_1, \ldots, u_n], V = [v_1, \ldots, v_n]$, and $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$. \triangleright \text{In precision } \omega

Remark 5.2 In Algorithms 5.1–5.2, the given matrix $A$ is assumed to be of full column rank. In fact, these algorithms can be used to compute the SVD of a general matrix [30]. When $m \gg n$, one may use the rank-revealing QR factorization as a preconditioner for these algorithms [23].

In the following, we give the error analysis of Algorithm 5.2. We first give an estimate of the distance between $Z$ and $Q$ generated by 5.2.
Lemma 5.3 Suppose that \( Z \in \mathbb{R}^{n \times n} \) in Algorithm 5.2 is computed by any svd solver in LAPACK, LINPACK or EISPACK in precision \( \nu \) and \( Q \in \mathbb{R}^{n \times n} \) in Algorithm 5.2 is computed by using the MGS method to \( Z \) in precision \( \omega \) (\( \omega \ll \nu \)). Then there exist constants \( f_i \equiv f_i(m, n) \) for \( i = 1, 2 \) such that \( \| Z - Q \|_F \leq f_1 \nu + f_2 \omega \).

Proof. The lemma follows from the arguments similar to that of Lemma 4.1.

On the orthogonalization of the columns of \( AQ \) generated in Step 3 of Algorithm 5.2, we have the following result.

Theorem 5.4 Suppose that \( Z \in \mathbb{R}^{n \times n} \) in Algorithm 5.2 is computed by any svd solver in LAPACK, LINPACK or EISPACK in precision \( \nu \) and \( Q \in \mathbb{R}^{n \times n} \) in Algorithm 5.2 is computed by using the MGS method to \( Z \) in precision \( \omega \) (\( \omega \ll \nu \)). Then there exists a constant \( \tilde{\zeta}_1 \equiv \tilde{\zeta}_1(m, n) \) such that \( \| \text{off}((AQ)^T(AQ)) \|_F \leq \tilde{\zeta}_1 \| A \|^2 \nu \).

Proof. By Lemma 2.3, there exists a matrix \( E \in \mathbb{R}^{m \times n} \) such that \( A + E \) admits the following exact SVD:

\[
A + E = (Y + \delta Y)S(Z + \delta Z)^T, \quad S = \text{diag}(s_1, \ldots, s_n) \in \mathbb{R}^{m \times n},
\]

where \( \| E \| \leq p(m, n)\| A \| \nu \) and \( Y + \delta Y \) and \( Z + \delta Z \) are both orthogonal with \( \| \delta Y \| \leq p(m, n)\nu \) and \( \| \delta Z \| \leq p(m, n)\nu \). Using the orthogonality of \( Z + \delta Z \) we have

\[
\| Z \| \leq \| Z + \delta Z \| + \| \delta Z \| = 1 + \| \delta Z \| \leq 1 + p(m, n)\nu \equiv \tau_1.
\]

We note that \( Q \) is computed by using the MGS method to \( Z \) in precision \( \omega \). Then we have

\[
Q^T A^T AQ - S^T S = (Q - Z + Z)^T A^T A(Q - Z + Z) - S^T S
\]

\[
= (Q - Z)^T A^T A(Q - Z) + (Q - Z)^T A^T AZ
\]

\[
+ Z^T A^T A(Q - Z) + (Z^T A^T AZ - S^T S).
\]

On the other hand, it follows from (5.1) that \( S^T S = (Z + \delta Z)^T(A + E)^T(A + E)(Z + \delta Z) \), i.e.,

\[
S^T S - Z^T A^T AZ = Z^T A^T \delta Z + \delta Z^T A^T AZ + \delta Z^T A^T \delta Z
\]

\[
+ (Z + \delta Z)^T(A^T E + E^T A + E^T E)(Z + \delta Z).
\]

Thus,

\[
\| Z^T A^T AZ - S^T S \| \leq 2\| A \|^2\| Z \|\| \delta Z \| + \| A \|^2\| \delta Z \|^2 + 2\| A \|\| E \| + \| E \|^2.
\]

By Lemma 5.3 and using (5.2) and (5.3) we obtain

\[
\| \text{off}(Q^T A^T AQ) \|_F \leq \| Q^T A^T AQ - S^T S \|_F
\]

\[
\leq \| A \|^2\| Q - Z \|_F^2 + 2\| A \|^2\| Z \|\| Q - Z \|_F + \| Z^T A^T AZ - S^T S \|_F
\]

\[
\leq \| A \|^2\| Q - Z \|_F\| Q - Z \|_F + 2\| Z \| + \sqrt{n}\| Z^T A^T AZ - S^T S \|_F
\]

\[
\leq \| A \|^2\| Q - Z \|_F\| Q - Z \|_F + 2\| Z \| + \sqrt{n}\| Z \|^2 + \| \delta Z \|^2\| \delta Z \| + \sqrt{n}\| A \|^2\| \delta Z \| + \| A \|\| E \| + \| E \|^2
\]

\[
\equiv \tilde{\zeta}_1 \| A \|^2 \nu,
\]

41
where \( \tilde{\zeta}_1 = (2\tau_1 + f_1v + f_2\omega)(f_1 + f_2\omega v^{-1}) + 2\sqrt{n}(\tau_1 + 1 + p(m,n)v)p(m,n). \)

In the following theorem, we give an error bound for one sweep of Algorithm 5.2. Here, \( \text{fl}(\cdot) \) is floating-point operation at precision \( \omega \).

**Theorem 5.5** Let \( C_k \) be the matrix \( C_0 = \text{fl}(AQ) \) after \( k \) Jacobi updates in Algorithm 5.2, where the \( k \)th computed Jacobi rotation is \( \hat{J}_k = J(p_k, q_k; \hat{e}_k, \hat{s}_k) \). Let \( \gamma_n := (1 - \omega)^{-n} - 1 \), \( \tilde{\gamma}_j := (1 - \omega)^{-w_j} - 1 \) for a small integer constant \( w > 0 \) and \( \varphi_k = ((6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^2)^{1/2} \), where \( G_k \) and \( H_k \) are defined as in Lemma 3.5 with \( A_k = C_k^T C_k \) and \( (p,q) = (p_k, q_k) \) for all \( k \geq 0 \). Suppose \( C_0 \) has \( n \) distinct singular values with \( d(C_0^T C_0) = \min_{s \neq t} |\sigma_s^2(C_0) - \sigma_t^2(C_0)| > 0 \). If the precisions \( \omega \) and \( v \) satisfy \( \omega \ll v \), \( q_1\|A\|_2^2v < d(C_0^T C_0) \) for some constant \( q_1 \), then

\[
\tilde{\zeta}_1 = \tilde{\zeta}_1 + 2\sqrt{n}\gamma_n v^{-1}\|AQ\||A||Q||/\|A\|^2 + \sqrt{n}\gamma_n^2 v^{-1}\|A||Q||/\|A\|^2,
\]

where \( \tilde{\zeta}_1 = \text{fl}(AQ) \) satisfies \( \|C_0 - AQ\| = |\Delta_2| \leq \gamma_n\|A||Q| \) [31, Sect. 3]. Thus,

\[
\|\text{off}(C_0^T C_0)\|_F = \|\text{off}((AQ + \Delta_2)^T(AQ + \Delta_2))\|_F \leq \|AQ^T A\|^2_F + 2\|AQ\|\|\Delta_2\|^2_F + \|\Delta_2^T \Delta_2\|^2_F \leq \|AQ^T A\|^2_F + 2\sqrt{n}\|A||Q||/\|A\|^2 + \sqrt{n}\gamma_n^2\|A||Q||/\|A\|^2.
\]

By Theorem 5.4 we have \( \|\text{off}(C_0^T C_0)\|_F \leq \zeta_1\|A\|^2v \) for some constant \( \zeta_1 \equiv \zeta_1(m,n) \) and there exists a clear gap between the singular values of \( C_0 = \text{fl}(AQ) \), which is not easy to estimate in practice. In the following theorem, we establish the error analysis of Algorithm 5.2 under the assumption that there exists a clear gap between the singular values of the original matrix \( A \).

**Theorem 5.6** Let \( C_k \) be the matrix \( C_0 = \text{fl}(AQ) \) after \( k \) Jacobi updates in Algorithm 5.2, where the \( k \)th computed Jacobi rotation is \( \hat{J}_k = J(p_k, q_k; \hat{e}_k, \hat{s}_k) \). Let \( \gamma_n := (1 - \omega)^{-n} - 1 \), \( \tilde{\gamma}_j := (1 - \omega)^{-w_j} - 1 \) for a small integer constant \( w > 0 \) and \( \varphi_k = ((6 + 4\sqrt{2})\|G_k\|_F^2 + 2\|H_k\|_F^2)^{1/2} \),
where $G_k$ and $H_k$ are defined as in Lemma 3.5 with $A_k = C_k^T C_k$ and $(p,q) = (p_k,q_k)$ for all $k \geq 0$. Suppose $A$ has $n$ distinct singular values with $d(A^T A) > 0$. If the precisions $\omega$ and $\nu$ satisfy $\omega \ll \nu$ and $4\zeta||A||^2 \nu < (1-\rho_2)d(A^T A)$ for two constants

$$\zeta_1 = \tilde{\zeta}_1 + 2\sqrt{n}\nu^{-1}||AQ||/||A||^2 + \sqrt{n}\nu^{-1}||AQ||/||A||^2$$

and $0 < \rho_2 < 1$, $2\zeta_2||A||^2 \omega \leq \rho_2d(A^T A)$ for some constant $\zeta_2 = \zeta_2(m,n)$, $(1-\rho_2)d(A^T A) \geq d + \sum_{k=0}^{N-1} \varphi_k \gamma_k$, and

$$\frac{1}{8}\frac{2}{\gamma_k \gamma_4} + \frac{1}{2}\|\text{off}(C_k^T C_k)\|_F |\varphi_\gamma| + \|H_k\|_F^2 \gamma_4 + \|G_k\|^2 \gamma_4 \leq \|(C_k^T C_k)_{p_kq_k}\|^2,$$

for all $k = 0,1,\ldots,N-1$, then there exist two constants $\alpha, \beta > 0$ such that $\|\text{off}(C_N^T C_N)\|_F \leq \alpha\|\text{off}(C_0^T C_0)\|_F^2 + \beta \gamma_4$, where $\tilde{\zeta}_1$ is defined as Theorem 5.4.

Proof. By Lemma 2.1, there exists a matrix $\delta Q \in \mathbb{R}^{n \times n}$ such that $Q + \delta Q$ is orthogonal, where $||\delta Q|| \leq \eta_4 \omega$ with $\eta_4 = \eta_4(n)$ being defined by (4.13). Then

$$||Q|| \leq ||Q + \delta Q|| + ||\delta Q|| = 1 + ||\delta Q|| \leq 1 + \eta_4 \omega \equiv \eta_5.$$

(5.4)

We note that

$$(Q + \delta Q)^T A^T A(Q + \delta Q) = Q^T A^T A Q + Q^T A^T A \delta Q + \delta Q^T A^T A Q + \delta Q^T A^T A \delta Q.$$  

By hypothesis, $C_0 = \text{fl}(AQ) = AQ + \Delta_2$ with $\|\Delta_2\| \leq \|\Delta_2\| \leq \gamma_n \|A||Q||$. From Lemma 2.4 and (5.4) we obtain, for any $1 \leq k \leq n$,

$$|\sigma_k^2(A) - \sigma_k^2(C_0)| = |\lambda_k(A^T A) - \lambda_k(C_0^T C_0)|$$

$$\leq 2||A||^2 ||Q||^2 ||\delta Q|| + ||A||^2 ||\delta Q||^2 + 2\|\Delta_2\||||AQ|| + \|\Delta_2\||^2$$

$$\leq ||A||^2 (2\|Q|| + ||\delta Q||)||\delta Q|| + 2\gamma_n |||A||Q||||AQ|| + \gamma_n^2 |||A||Q||^2$$

$$\leq ||A||^2 (2\eta_5 + \eta_4 \omega) \eta_4 \omega + 2\gamma_n |||A||Q||||AQ|| + \gamma_n^2 |||A||Q||^2 \equiv \zeta_2 ||A||^2 \omega,$$

where $\zeta_2 = (2\eta_5 + \eta_4 \omega) \eta_4 \omega + 2\gamma_n \omega^{-1}(|||A||Q||||AQ||)/||A||^2 + \gamma_n^2 \omega^{-1}(|||A||Q||^2)/||A||^2$.

For any $i \neq j$, we have

$$|\sigma_i^2(A) - \sigma_j^2(A)| = |\lambda_i(A^T A) - \lambda_j(A^T A)|$$

$$\leq |\lambda_i(A^T A) - \lambda_i(C_0^T C_0) + \lambda_i(C_0^T C_0) - \lambda_j(C_0^T C_0) + |\lambda_j(C_0^T C_0) - \lambda_j(A^T A)|$$

$$\leq |\lambda_i(C_0^T C_0) - \lambda_j(C_0^T C_0) + 2\zeta_2 ||A||^2 \omega = |\sigma_i^2(C_0) - \sigma_j^2(C_0)| + 2\zeta_2 ||A||^2 \omega$$

and

$$|\sigma_i^2(A) - \sigma_j^2(A)| = |\lambda_i(A^T A) - \lambda_j(A^T A)|$$

$$\geq ||\lambda_i(C_0^T C_0) - \lambda_j(C_0^T C_0) - |\lambda_i(A^T A) - \lambda_i(C_0^T C_0) + |\lambda_j(C_0^T C_0) - \lambda_j(A^T A)|$$

$$\geq |\lambda_i(C_0^T C_0) - \lambda_j(C_0^T C_0) - 2\zeta_2 ||A||^2 \omega = |\sigma_i^2(C_0) - \sigma_j^2(C_0)| - 2\zeta_2 ||A||^2 \omega.$$  

If $2\zeta_2 ||A||^2 \omega \leq \rho_2d(A^T A)$ for some $0 < \rho_2 < 1$, then we have

$$\frac{1}{1 + \rho_2} |\sigma_i^2(C_0) - \sigma_j^2(C_0)| \leq |\sigma_i^2(A) - \sigma_j^2(A)| \leq \frac{1}{1 - \rho_2} |\sigma_i^2(C_0) - \sigma_j^2(C_0)|.$$  

By hypothesis, $d(A^T A) > 0$. Hence, $d(C_0^T C_0) \geq (1-\rho_2)d(A^T A) > 0$. The theorem follows from Theorem 3.17 if $\zeta_1||A||^2 \nu < (1-\rho_2)d(A^T A)/4$. 

\[\square\]
6 Numerical Experiments

In this section, we present some numerical experiments to illustrate the effectiveness of Algorithms 4.1 and 5.2 for computing the symmetric eigenvalue decomposition and the SVD. Numerical experiments were implemented in C++ and linked with Intel oneAPI Math Kernel Library (oneMKL) running on a workstation of CentOS equipped with an Intel(R) Xeon(R) Gold 6348 CPU at 2.60 GHz, 250GB of RAM and NVIDIA A30 Tensor Core GPU. In our numerical tests, we set $\nu = 2^{-24}$ (single precision) and $\omega = u = 2^{-53}$ (double precision) for Algorithms 4.1 and 5.2.

In Algorithm 4.1, we employ oneMKL routine ‘LAPACKE_ssyev’ to compute the eigenvalues and associated eigenvectors of a given matrix in precision $\nu$. In Algorithm 5.2, we use ‘LAPACKE_sgesvd’ and the associated left and right singular vectors of a given matrix as an approximate SVD in precision $\nu$. Besides, various precisions were simulated by software Advanpix [2] in MATLAB R2022a.

In our numerical tests, “Res.”, “OR-P.”, “CT.”, “JU.”, and “SP.” denote the computed relative residual $\|A_p - \tilde{P}\text{diag}(t_{11}, \ldots, t_{nn})\|_F/\|A\|_F$ (or $\|AV - U\Sigma\|_F/\|A\|_F$), the measure of orthonormality $\|P^T P - I_n\|_F/\sqrt{n}$ of $P$, the running time in seconds, the Jacobi updates, and the number of sweeps at the final iterates of the corresponding algorithms, respectively. Also, for Algorithms 3.2 and 4.1, we set $\epsilon = 20.0 \times \omega$. For Algorithms 5.1–5.2, we set $\epsilon = \omega$ and the algorithms are stopped if the ratio of Jacobi updates to $N := n(n-1)/2$ is less than $2 \times 10^{-4}$.

For comparison, Algorithms 3.2 and 5.1 are implemented at the machine precision $u$.

6.1 The symmetric eigenvalue problem

In this subsection, we consider following two examples.

Example 6.1 Let $A$ be an $n \times n$ random symmetric and positive definite matrix with pre-assigned singular value generated by MATLAB 2022a’s gallery (‘randsvd’, n, -kappa, mode) with $\kappa(A) = \text{kappa}$. We report our numerical results for (a) mode = 1: the large eigenvalue is 1 and the rest of the eigenvalues are $1/kappa$, (b) mode = 2: the small eigenvalue is $1/kappa$ and the rest of the eigenvalues are 1, (c) mode = 3: geometrically distributed eigenvalues, (d) mode = 4: arithmetically distributed eigenvalues, and (e) mode = 5: random eigenvalues with uniformly distributed logarithm.

Example 6.2 We consider the case of multiple eigenvalues. Let $A = P_*\text{diag}(\text{lam} \otimes 1_r)P_*^T$ be an $n \times n$ random symmetric and positive definite matrix with $n = rs$, where the orthogonal matrix $P_* \in \mathbb{R}^{n\times n}$ is randomly generated by the built-in functions randn and orth in MATLAB R2022a and the vector $\text{lam} \in \mathbb{R}^s$ of exact eigenvalues is generated as follows: (a) mode = 1: $\text{lam} = [1; \text{ones}(s-1,1)*1/kappa]$, (b) mode = 2: $\text{lam} = [\text{ones}(s-1,1); 1/kappa]$, (c) mode = 3: $\text{lam} = \text{lam} \ast \text{linspace}(-1,0,s)$, (d) mode = 4: $\text{lam} = \text{linspace}(0,1,s)$, (e) mode = 5: $\text{lam} = \text{lam} \ast (-\text{rand}(s,1))$.

In Table 6.1, we report the numerical results for Example 6.1. We observe from Table 6.1 that Algorithm 4.1 preserves the high accuracy of Algorithm 3.2 for all test matrices and even produces a slightly better orthonormality of $P$ than Algorithm 3.2 (especially for the cases
that \( \text{mode} = 3, 4, 5 \). Moreover, Algorithm 4.1 works much better than \( v \) for the cases that \( \text{mode} = 3, 4, 5 \) in terms of the computing time, the number of rotations, and sweeps.

To show the effectiveness of the initial guess \( Q \) generated by Algorithm 4.1, in Figure 6.1, we plot the quantities \( \| \text{off} (A) \|_F \), \( \| \text{off} (T_0) \|_F \), \( \text{bd}/10 \), \( \text{bd} \), \( 10 \text{bd} \), and \( d(A)/4 \) versus the dimension \( n \) for Example 6.1 with \( \text{mode} = 4 \) (left) and \( \text{mode} = 5 \) (right), respectively. Here, \( T_0 = \Phi(Q^T A Q) \), \( d(A) = \min_{\lambda_i(A) \neq \lambda_j(A)} |\lambda_i(A) - \lambda_j(A)| \) and \( \text{bd} = n \| A \|_V \) is an approximation estimate of the theoretical bound \( \psi_1 \| A \|_V \), which is obtained in Theorem 4.4. We also see from Table 6.1 and Figure 6.1 that Algorithm 4.1 is much efficient over Algorithm 3.2, where the preprocessed matrix \( Q \) is such that \( \| \text{off} (T_0) \|_F \) is much less than \( \| \text{off} (A) \|_F \), despite the quantity \( \psi_1 \| A \|_V \) is not necessarily less than \( d(A)/4 \). Moreover, in Table 6.2, we compare the performance of Algorithm 4.1, where the starting guess \( Q \) was computed in two ways: both eig and MGS in double precision and eig in single precision and MGS in double precision. Here, “init-Res.,” “init-OR-Q.” mean the computed relative residual \( \| A Z - Z \Phi \|_F / \| A \|_F \) and the measure of orthonormality \( \| Q^T Q - I_n \|_F / \sqrt{n} \) of \( Q \), respectively. We see from Tables 6.1–6.2 that the starting guess via the mixed precision may accelerate the Jacobi iteration. Even both eig and MGS are implemented in double precision, the Jacobi iteration can further improve the accuracy.

To investigate the sensitivity of the lower precision in the proposed algorithm, we utilize Advanpix to simulate the performance of Algorithm 4.1 for Example 6.2 with different precisions. Figures 6.2–6.3 describe \( \| \text{off} (T_j N) \|_F \) versus \( j \) (the number of sweeps) for various choices of \( \nu \). We observe from Figures 6.2–6.3 that Algorithm 4.1 is much more efficient than Algorithm 3.2 for different precisions, especially for the original matrices with uniform distribution of eigenvalues and large condition numbers.

![Figure 6.1: Different quantities versus n for Example 6.1.](image)

To further illustrate the effectiveness of Algorithm 4.1, we also implement Algorithm 3.2 and Algorithm 4.1 in parallel. Here, we utilize NVIDIA cuSOLVER library 1 and NVIDIA cuBLAS library 2 and the parallel ordering of the rotation set \( \{(i, j) \mid 1 \leq i < j \leq n\} \) can be taken as some non-overlapping order, e.g., the merry-go-round ordering \( (i, j) = (1, 2), (3, 4), \ldots, (n - 1, n), (1, 4), \ldots, (n - 3, n - 1), (1, 6), \ldots, (n, n - 2) \). The pre-processing stage in Steps 1–3 of Algo-

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1https://docs.nvidia.com/cuda/cusolver
2https://docs.nvidia.com/cuda/cublas
| Algorithm | Res. | OR-P. | CT. | SP. | Res. | OR-P. | JU. | CT. |
|-----------|-----|------|-----|-----|-----|------|-----|-----|
| LAPACKE | 1.62e-15 | 3.11e-15 | 0.05 | | 1.60e-15 | 8.11e-16 | 0.178N | 0.08 |
| dgesvd | (1, 10) | | | | (2, 10) | | | |
| Alg. 3.2 | | | | | (3, 10) | | | |
| Alg. 4.1 | | | | | (4, 10) | | | |
| | | | | | (5, 10) | | | |
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Table 6.2: Numerical results for Example 6.1 with $n = 2048$.

| (mode, kappa) | double-double solver | single-double solver |
|---------------|----------------------|----------------------|
|               | init-Res. | init-OR-Q. | Res. | OR-P. | JU. | CT. | SP. | init-Res. | init-OR-Q. | Res. | OR-P. | JU. | CT. | SP. |
| (1, $10^3$)   | 2.58e-15  | 1.09e-15  | 1.58e-15  | 1.14e-15  | 0.013N | 2.94 | 1 | 1.11e-06 | 1.39e-15  | 2.99e-15  | 1.48e-15  | 0.025N | 3.36 | 1   |
| (2, $10^3$)   | 1.79e-15  | 1.19e-15  | 1.42e-15  | 1.19e-15  | 0.001N | 2.44 | 1 | 1.46e-06 | 1.27e-15  | 2.49e-15  | 1.28e-15  | 0.001N | 2.11 | 1   |
| (3, $10^3$)   | 4.51e-15  | 1.23e-15  | 3.70e-15  | 2.00e-15  | 0.414N | 20.88 | 1 | 2.73e-06 | 1.42e-15  | 4.14e-15  | 4.49e-15  | 2.002N | 76.21 | 3   |
| (4, $10^3$)   | 6.17e-15  | 1.28e-15  | 4.03e-15  | 1.47e-15  | 0.075N | 6.44 | 1 | 3.18e-06 | 1.42e-15  | 3.24e-15  | 4.49e-15  | 1.994N | 76.14 | 2   |
| (5, $10^3$)   | 4.71e-15  | 1.24e-15  | 3.82e-15  | 2.01e-15  | 0.421N | 21.40 | 1 | 2.60e-06 | 1.43e-15  | 4.17e-15  | 4.50e-15  | 2.002N | 77.63 | 3   |
| (1, $10^4$)   | 3.13e-15  | 1.07e-15  | 2.52e-15  | 1.39e-15  | 0.491N | 22.41 | 1 | 1.40e-06 | 1.37e-15  | 3.34e-15  | 2.71e-15  | 0.587N | 27.52 | 1   |
| (2, $10^4$)   | 1.78e-15  | 1.19e-15  | 1.41e-15  | 1.19e-15  | 0.001N | 2.89 | 1 | 1.45e-06 | 1.28e-15  | 2.50e-15  | 1.28e-15  | 0.001N | 2.75 | 1   |
| (3, $10^4$)   | 4.48e-15  | 1.22e-15  | 3.47e-15  | 2.32e-15  | 0.640N | 32.79 | 1 | 2.56e-06 | 1.42e-15  | 4.33e-15  | 4.52e-15  | 2.035N | 86.76 | 3   |
| (4, $10^4$)   | 5.49e-15  | 1.32e-15  | 5.07e-15  | 1.51e-15  | 0.075N | 6.67 | 1 | 3.89e-06 | 1.42e-15  | 2.98e-15  | 4.49e-15  | 1.996N | 81.18 | 2   |
| (5, $10^4$)   | 4.26e-15  | 1.22e-15  | 3.51e-15  | 2.33e-15  | 0.638N | 30.82 | 1 | 2.56e-06 | 1.45e-15  | 4.31e-15  | 4.51e-15  | 2.023N | 87.41 | 3   |
| (1, $10^5$)   | 3.60e-15  | 1.08e-15  | 2.78e-15  | 3.12e-15  | 0.910N | 37.86 | 1 | 1.50e-06 | 1.37e-15  | 3.44e-15  | 4.32e-15  | 0.937N | 47.80 | 1   |
| (2, $10^5$)   | 1.76e-15  | 1.19e-15  | 1.41e-15  | 1.19e-15  | 0.001N | 2.67 | 1 | 1.39e-06 | 1.27e-15  | 2.50e-15  | 1.26e-15  | 0.001N | 2.35 | 1   |
| (3, $10^5$)   | 4.63e-15  | 1.21e-15  | 3.29e-15  | 2.51e-15  | 0.761N | 40.33 | 1 | 2.36e-06 | 1.42e-15  | 4.45e-15  | 4.59e-15  | 2.097N | 91.02 | 4   |
| (4, $10^5$)   | 5.50e-15  | 1.32e-15  | 5.09e-15  | 1.51e-15  | 0.075N | 6.86 | 1 | 2.86e-06 | 1.42e-15  | 3.02e-15  | 4.49e-15  | 1.995N | 76.60 | 2   |
| (5, $10^5$)   | 3.95e-15  | 1.21e-15  | 3.34e-15  | 2.51e-15  | 0.762N | 35.58 | 1 | 2.08e-06 | 1.42e-15  | 4.43e-15  | 4.59e-15  | 2.099N | 89.77 | 4   |
| (1, $10^6$)   | 3.47e-15  | 1.08e-15  | 2.87e-15  | 3.22e-15  | 0.992N | 40.33 | 1 | 2.07e-06 | 1.36e-15  | 3.50e-15  | 3.33e-15  | 0.993N | 41.66 | 1   |
| (2, $10^6$)   | 1.78e-15  | 1.18e-15  | 1.39e-15  | 1.18e-15  | 0.001N | 2.69 | 1 | 1.68e-06 | 1.28e-15  | 2.50e-15  | 1.28e-15  | 0.001N | 2.29 | 1   |
| (3, $10^6$)   | 4.45e-15  | 1.20e-15  | 3.17e-15  | 2.61e-15  | 0.827N | 42.89 | 1 | 2.07e-06 | 1.42e-15  | 4.54e-15  | 4.60e-15  | 2.179N | 85.75 | 4   |
| (4, $10^6$)   | 5.47e-15  | 1.32e-15  | 5.05e-15  | 1.51e-15  | 0.076N | 6.64 | 1 | 2.88e-06 | 1.42e-15  | 3.00e-15  | 4.48e-15  | 1.995N | 89.05 | 2   |
| (5, $10^6$)   | 3.67e-15  | 1.20e-15  | 3.23e-15  | 2.64e-15  | 0.840N | 36.07 | 1 | 1.80e-06 | 1.42e-15  | 4.55e-15  | 4.70e-15  | 2.228N | 87.31 | 4   |
Algorithm 4.1 was implemented on the NVIDIA CUDA routine, where the function ‘cusolverDnSyevd’ was employed as the eigensolver in precision \( \nu \) and the MGS method was replaced by the Householder QR factorization, which can theoretically guarantee higher orthogonality [25, §5.2].

The numerical results for Example 6.1 with different \( n \) are displayed in Tables 6.3–6.4. Here, “CT-Pre.” means the running time for the pre-processing stage in Steps 1–3 of (parallel) Algorithm 4.1 and “CT-J.” means the running time for the stage of the Jacobi procedure of (parallel) Algorithm 4.1 or (parallel) Algorithm 3.2. We see from Tables 6.3–6.4 that, as parallel Algorithm 3.2, parallel Algorithm 4.1 can significantly improve the efficiency of Algorithm 4.1. As expected, Algorithm 4.1 (parallel version, respectively) works much better than Algorithm 3.2 (parallel version, respectively) in terms of the total computing time.

The numerical results for Example 6.2 with different \( n \) are reported in Tables 6.5–6.6, where the values of \( \|\text{off}(Q^T AQ)\|_F \) and \( d(A)/4 \) were calculated under CPU environment. We observe from Tables 6.5–6.6 that parallel Algorithm 4.1 is much more efficient than Algorithm 4.1 in
Table 6.3: Numerical results for Example 6.1 with \( n = 2048 \).

| (mode, kappa) | parallel Alg. 3.2 | parallel Alg. 4.1 |
|---------------|-------------------|-------------------|
|               | Res. OR-P. CT-J. SP. | Res. OR-P. CT-Pre. CT-J. SP. |
| (3, 10^3)    | 1.61e-12 1.52e-12 0.88 15 | 2.20e-13 2.39e-13 0.14 0.12 3 |
| (4, 10^3)    | 1.36e-12 1.37e-12 0.66 12 | 2.24e-13 2.24e-13 0.14 0.12 2 |
| (5, 10^3)    | 1.65e-12 1.58e-12 0.85 15 | 2.25e-13 2.51e-13 0.14 0.13 3 |
| (3, 10^4)    | 1.75e-12 1.65e-12 0.94 17 | 2.29e-13 2.88e-13 0.14 0.14 3 |
| (4, 10^4)    | 1.37e-12 1.37e-12 0.66 12 | 2.24e-13 2.24e-13 0.14 0.12 2 |
| (5, 10^4)    | 1.76e-12 1.67e-12 0.95 17 | 2.33e-13 2.96e-13 0.14 0.15 4 |
| (3, 10^5)    | 1.04e-14 9.65e-15 379.91 14 | 4.50e-15 4.50e-15 2.22 76.42 3 |
| (4, 10^5)    | 1.08e-14 9.34e-15 358.10 11 | 3.70e-15 4.48e-15 2.17 76.09 2 |
| (5, 10^5)    | 1.03e-14 9.75e-15 388.71 15 | 4.39e-15 4.50e-15 2.36 76.22 3 |
| (3, 10^6)    | 1.03e-14 9.81e-15 391.40 15 | 4.55e-15 4.52e-15 2.52 76.78 3 |
| (4, 10^6)    | 1.07e-14 9.33e-15 355.92 12 | 3.76e-15 4.49e-15 2.29 74.76 2 |
| (5, 10^6)    | 1.02e-14 9.86e-15 390.06 15 | 4.61e-15 4.52e-15 2.43 76.44 3 |
| (3, 10^7)    | 9.79e-15 9.90e-15 394.39 16 | 4.67e-15 4.58e-15 2.43 78.35 4 |
| (4, 10^7)    | 1.04e-14 9.82e-15 348.81 11 | 3.72e-15 4.49e-15 2.26 74.79 2 |
| (5, 10^7)    | 1.00e-14 9.97e-15 402.72 17 | 4.60e-15 4.61e-15 2.30 79.21 4 |
| (3, 10^8)    | 9.66e-15 1.01e-14 409.20 17 | 4.78e-15 4.66e-15 2.40 81.36 3 |
| (4, 10^8)    | 1.05e-14 9.30e-15 351.52 11 | 3.40e-15 4.49e-15 2.37 75.61 2 |
| (5, 10^8)    | 1.00e-14 1.01e-14 417.28 18 | 4.84e-15 4.66e-15 2.20 82.55 4 |

terms of the total computing time.

6.2 The singular value problem

In this subsection, we compare Algorithms 5.1–5.2 for computing the SVD of a real matrix. We consider the following example.

**Example 6.3** Let \( A \) be an \( m \times n \) random matrix with pre-assigned singular value generated by MATLAB 2022a’s gallery (’randsvd’, \([m,n], \kappa, \text{mode}\)) with \( \kappa(A) = \kappa \). We report our numerical results for (a) \( \text{mode} = 1 \): the large singular value is equal to 1 and the rest of the singular values are equal to \( 1/\kappa \), (b) \( \text{mode} = 2 \): the small singular value is equal to \( 1/\kappa \) and the rest of the singular values are equal to 1, (c) \( \text{mode} = 3 \): geometrically distributed singular values, (d) \( \text{mode} = 4 \): arithmetically distributed singular values, and (e) \( \text{mode} = 5 \): random singular values with uniformly distributed logarithm.

The numerical results for Example 6.3 are reported in Table 6.7. We see from Table 6.7 that Algorithm 5.2 works more efficient than Algorithm 5.1 for the cases \( \text{mode} = 3, 4, 5 \).
In this subsection, we will investigate the quantity 6.3 Numerical verification

Table 6.4: Numerical results for Example 6.1 with \( n = 4096 \).

| (mode, kappa) | Res. | OR-P. | CT-J. | SP. | Res. | OR-P. | CT-Pre. | CT-J. | SP. |
|---------------|------|-------|-------|-----|------|-------|---------|-------|-----|
| (3, 10^3)    | 5.74e-12 | 5.03e-12 | 4.86 | 7 | 4.59e-13 | 5.29e-13 | 0.76 | 0.70 | 1 |
| (4, 10^3)    | 4.44e-12 | 4.43e-12 | 4.02 | 6 | 4.60e-13 | 4.61e-13 | 0.75 | 0.69 | 1 |
| (5, 10^3)    | 5.41e-12 | 5.08e-12 | 5.10 | 7 | 4.70e-13 | 5.47e-13 | 0.76 | 0.80 | 2 |
| (3, 10^4)    | 5.77e-12 | 5.65e-12 | 5.36 | 7 | 4.87e-13 | 6.25e-13 | 0.77 | 0.77 | 2 |
| (4, 10^4)    | 4.32e-12 | 4.32e-12 | 4.00 | 6 | 4.60e-13 | 4.61e-13 | 0.75 | 0.69 | 1 |
| (5, 10^4)    | 6.62e-12 | 5.66e-12 | 5.39 | 7 | 4.97e-13 | 6.33e-13 | 0.77 | 0.91 | 2 |
| (3, 10^5)    | 6.49e-12 | 5.99e-12 | 5.86 | 8 | 5.18e-13 | 7.90e-13 | 0.78 | 0.86 | 2 |
| (4, 10^5)    | 4.35e-12 | 4.35e-12 | 4.00 | 6 | 4.60e-13 | 4.61e-13 | 0.76 | 0.69 | 1 |
| (5, 10^5)    | 5.30e-12 | 5.92e-12 | 6.16 | 8 | 5.24e-13 | 7.98e-13 | 0.77 | 0.97 | 2 |
| (3, 10^6)    | 6.62e-12 | 6.45e-12 | 6.35 | 9 | 5.45e-13 | 1.00e-12 | 0.78 | 1.06 | 2 |
| (4, 10^6)    | 4.40e-12 | 4.40e-12 | 4.06 | 6 | 4.60e-13 | 4.61e-13 | 0.76 | 0.70 | 1 |
| (5, 10^6)    | 7.13e-12 | 6.55e-12 | 6.56 | 9 | 5.51e-13 | 1.04e-12 | 0.77 | 1.15 | 2 |

6.3 Numerical verification

In this subsection, we will investigate the quantity \( \zeta \) in Theorem 3.17. We note that \( (1+2\tilde{\gamma}_4)^n \approx 1 \) for \( \tilde{\gamma}_4 = (1-u)^{-4w} - 1 \). One may expect that

\[
\zeta_r \approx \tilde{\zeta}_r := |\tilde{\zeta}_r| + \tilde{\zeta}_r^2 = \sum_{q=2}^{n-r} |a_{r,r-1+q}^{(u)(r-1)+n-r+1}|^2,
\]

where \( \tilde{\zeta}_r = \sum_{q=1}^{n-r-1} |a_{r,r+1}^{(u)(r-1)+n-r+1}|^2 \) and

\[
\tilde{\zeta}_r^2 = \sum_{j=3}^{n-r} \sum_{q=1}^{n-r-j} |\tilde{a}_{r,r-1+j}^{(u)(r-1)+(n-r)(j-1)}|^2 - \sum_{j=3}^{n-r} \sum_{q=2}^{n-r-j} |\tilde{a}_{r,r-1+q}^{(u)(r-1)+(n-r)(j-1)}|^2.
\]

Figure 6.4 describes the quantity \( \tilde{\zeta} = \max_{1 \leq r \leq n-1} \tilde{\zeta}_r \) versus \( j \) (the number of sweeps) in double precision for some test matrices of Example 6.1 with different choices of \( (\text{mode}, \text{kappa}) \). We see from Figure 6.4 that \( \tilde{\zeta} \) and \( \zeta \) are of order of \( n \). Thus one may expect that the quantities \( \tilde{\zeta}\tilde{\gamma}_4 \) and \( \zeta\tilde{\gamma}_4 \) are not too large since \( \tilde{\gamma}_4 \) is small enough.

7 Conclusions

In this paper, we give the error analysis for a single step or sweep of the Jacobi method in floating point arithmetic. Then we propose a mixed precision preconditioned Jacobi method
| (mode, kappa) | $||\text{off}(Q^T AQ)||_F$ | $d(A)/4$ | Res. | OR-P. | CT-Pre. | CT-J. | SP. | Res. | OR-P. | CT-Pre. | CT-J. | SP. |
|---------------|--------------------------|-----------------|-------|-------|--------|-------|-----|-------|-------|--------|-------|-----|
| (1, $10^3$)   | 1.97e-06                 | 2.50e-01        | 2.52e-14 | 4.82e-15 | 0.13   | 0.07  | 3   | 3.07e-15 | 1.93e-15 | 1.93   | 8.08  | 1   |
| (2, $10^3$)   | 7.14e-07                 | 2.50e-01        | 1.50e-14 | 1.50e-14 | 0.17   | 0.14  | 6   | 4.51e-15 | 1.33e-15 | 1.84   | 0.27  | 1   |
| (3, $10^3$)   | 2.75e-05                 | 3.40e-06        | 3.35e-13 | 3.12e-13 | 0.18   | 0.30  | 6   | 5.02e-15 | 4.48e-15 | 2.46   | 71.36 | 3   |
| (4, $10^3$)   | 7.19e-05                 | 4.89e-04        | 3.11e-13 | 3.04e-13 | 0.19   | 0.24  | 6   | 3.78e-15 | 4.48e-15 | 2.40   | 70.46 | 2   |
| (5, $10^3$)   | 2.30e-05                 | 1.02e-07        | 3.45e-13 | 3.21e-13 | 0.18   | 0.28  | 6   | 5.02e-15 | 4.48e-15 | 2.31   | 70.45 | 3   |
| (1, $10^4$)   | 2.87e-06                 | 2.50e-01        | 1.81e-14 | 5.11e-15 | 0.17   | 0.08  | 3   | 3.47e-15 | 3.17e-15 | 2.24   | 31.55 | 1   |
| (2, $10^4$)   | 6.15e-07                 | 2.50e-01        | 7.95e-15 | 7.64e-15 | 0.18   | 0.13  | 5   | 4.56e-15 | 1.32e-15 | 1.82   | 0.27  | 1   |
| (3, $10^4$)   | 2.23e-05                 | 4.55e-07        | 3.39e-13 | 3.39e-13 | 0.19   | 0.28  | 6   | 5.04e-15 | 4.50e-15 | 2.68   | 71.75 | 3   |
| (4, $10^4$)   | 7.21e-05                 | 4.89e-04        | 3.11e-13 | 3.05e-13 | 0.18   | 0.25  | 6   | 3.70e-15 | 4.48e-15 | 2.51   | 70.15 | 2   |
| (5, $10^4$)   | 2.16e-05                 | 1.64e-08        | 3.36e-13 | 3.45e-13 | 0.19   | 0.28  | 6   | 5.14e-15 | 4.51e-15 | 2.33   | 72.04 | 3   |
| (1, $10^5$)   | 3.40e-06                 | 2.50e-01        | 2.68e-14 | 5.67e-15 | 0.18   | 0.07  | 3   | 3.62e-15 | 3.33e-15 | 2.43   | 37.14 | 1   |
| (2, $10^5$)   | 5.19e-07                 | 2.50e-01        | 1.27e-14 | 1.23e-14 | 0.17   | 0.14  | 6   | 4.61e-15 | 1.30e-15 | 1.96   | 0.22  | 1   |
| (3, $10^5$)   | 1.87e-05                 | 5.70e-08        | 3.48e-13 | 4.11e-13 | 0.19   | 0.30  | 7   | 5.10e-15 | 4.56e-15 | 2.42   | 73.50 | 3   |
| (4, $10^5$)   | 7.36e-05                 | 4.89e-04        | 3.12e-13 | 3.04e-13 | 0.19   | 0.24  | 6   | 4.48e-15 | 4.48e-15 | 2.40   | 70.82 | 2   |
| (5, $10^5$)   | 1.77e-05                 | 2.00e-09        | 3.44e-13 | 4.15e-13 | 0.18   | 0.30  | 7   | 5.09e-15 | 4.55e-15 | 2.19   | 73.11 | 3   |
| (1, $10^6$)   | 3.48e-06                 | 2.50e-01        | 2.34e-14 | 5.06e-15 | 0.18   | 0.05  | 2   | 3.55e-15 | 3.35e-15 | 2.35   | 35.74 | 1   |
| (2, $10^6$)   | 3.96e-07                 | 2.50e-01        | 1.56e-14 | 1.56e-14 | 0.18   | 0.15  | 6   | 4.59e-15 | 1.30e-15 | 1.80   | 0.27  | 1   |
| (3, $10^6$)   | 1.60e-05                 | 6.85e-09        | 3.62e-13 | 5.13e-13 | 0.19   | 0.35  | 8   | 5.24e-15 | 4.63e-15 | 2.26   | 76.06 | 3   |
| (4, $10^6$)   | 7.33e-05                 | 4.89e-04        | 3.12e-13 | 3.05e-13 | 0.19   | 0.24  | 6   | 4.46e-15 | 4.48e-15 | 2.44   | 70.76 | 2   |
| (5, $10^6$)   | 1.40e-05                 | 1.51e-10        | 3.58e-13 | 5.13e-13 | 0.19   | 0.34  | 8   | 5.20e-15 | 4.63e-15 | 2.09   | 76.08 | 5   |
Table 6.6: Numerical results for Example 6.2 with n = 4096.

| m | k | l | (m, k, l) | ∥off(Q^T A Q)∥_F | d(A) | 4Res. | OR-P. | CT-Pre. | CT-J. | SP. | Res. | OR-P. | CT-Pre. | CT-J. | SP. |
|---|---|---|-----------|----------------|------|-------|-------|--------|--------|-----|-----|-------|-------|--------|--------|-----|
| 1 | 1 | 0 | (1, 10^3) | 2.31e-06 | 2.50e-01 | 5.41e-14 | 5.94e-15 | 0.79 | 0.33 | 1 | 4.04e-15 | 1.99e-15 | 35.72 | 23.04 | 1 |
| 2 | 1 | 0 | (2, 10^3) | 5.85e-07 | 2.50e-01 | 1.68e-14 | 1.72e-14 | 0.80 | 0.56 | 2 | 5.89e-15 | 1.71e-15 | 43.40 | 1.97 | 1 |
| 3 | 1 | 0 | (3, 10^3) | 5.65e-05 | 1.69e-06 | 6.93e-13 | 6.91e-13 | 0.81 | 1.82 | 4 | 6.50e-15 | 6.33e-15 | 35.89 | 1040.80 | 3 |
| 4 | 1 | 0 | (4, 10^3) | 1.44e-04 | 2.44e-04 | 6.20e-13 | 6.15e-13 | 0.82 | 1.74 | 3 | 4.83e-15 | 6.32e-15 | 36.15 | 1012.79 | 2 |
| 5 | 1 | 0 | (5, 10^3) | 5.56e-05 | 2.32e-08 | 6.95e-13 | 7.11e-13 | 0.79 | 1.78 | 3 | 6.52e-15 | 6.33e-15 | 42.69 | 1033.98 | 3 |
| 1 | 2 | 0 | (1, 10^4) | 3.30e-06 | 2.50e-01 | 5.70e-14 | 5.99e-15 | 0.80 | 0.28 | 1 | 4.67e-15 | 4.17e-15 | 34.12 | 403.45 | 1 |
| 2 | 2 | 0 | (2, 10^4) | 7.41e-07 | 2.50e-01 | 1.95e-14 | 2.02e-14 | 0.81 | 0.71 | 2 | 5.96e-15 | 1.71e-15 | 32.30 | 1.95 | 1 |
| 3 | 2 | 0 | (3, 10^4) | 4.53e-05 | 2.26e-07 | 7.14e-13 | 7.51e-13 | 0.79 | 1.68 | 3 | 6.74e-15 | 6.37e-15 | 35.62 | 1021.47 | 3 |
| 4 | 2 | 0 | (4, 10^4) | 1.44e-04 | 2.44e-04 | 6.15e-13 | 6.10e-13 | 0.80 | 1.75 | 3 | 4.86e-15 | 6.33e-15 | 61.18 | 950.80 | 2 |
| 5 | 2 | 0 | (5, 10^4) | 4.14e-05 | 1.93e-09 | 7.22e-13 | 7.72e-13 | 0.81 | 1.78 | 3 | 6.77e-15 | 6.37e-15 | 34.37 | 1037.00 | 3 |
| 1 | 3 | 0 | (1, 10^5) | 4.30e-06 | 2.50e-01 | 5.64e-14 | 5.88e-15 | 0.79 | 0.31 | 1 | 4.91e-15 | 4.62e-15 | 43.51 | 561.86 | 1 |
| 2 | 3 | 0 | (2, 10^5) | 5.11e-07 | 2.50e-01 | 1.58e-14 | 1.50e-14 | 0.77 | 0.68 | 2 | 5.92e-15 | 1.71e-15 | 58.46 | 1.96 | 1 |
| 3 | 3 | 0 | (3, 10^5) | 3.65e-05 | 3.40e-09 | 7.36e-13 | 1.08e-12 | 0.80 | 1.77 | 3 | 7.01e-15 | 6.59e-15 | 35.89 | 1112.71 | 5 |
| 4 | 3 | 0 | (4, 10^5) | 1.47e-04 | 2.44e-04 | 6.22e-13 | 6.17e-13 | 0.82 | 1.73 | 3 | 5.74e-15 | 6.32e-15 | 62.06 | 1014.28 | 2 |
| 5 | 3 | 0 | (5, 10^5) | 2.83e-05 | 1.06e-10 | 7.53e-13 | 1.17e-12 | 0.79 | 1.88 | 3 | 6.98e-15 | 6.62e-15 | 59.68 | 1024.97 | 6 |
| 1 | 4 | 0 | (1, 10^6) | 5.16e-06 | 2.50e-01 | 4.78e-14 | 5.97e-15 | 0.79 | 0.31 | 1 | 4.95e-15 | 4.67e-15 | 59.52 | 480.34 | 1 |
| 2 | 4 | 0 | (2, 10^6) | 6.22e-07 | 2.50e-01 | 1.90e-14 | 1.99e-14 | 0.77 | 0.68 | 2 | 5.89e-15 | 1.71e-15 | 55.73 | 1.93 | 1 |
| 3 | 4 | 0 | (3, 10^6) | 3.11e-05 | 3.40e-09 | 7.36e-13 | 1.08e-12 | 0.80 | 1.77 | 3 | 7.01e-15 | 6.59e-15 | 35.89 | 1112.71 | 5 |
| 4 | 4 | 0 | (4, 10^6) | 1.46e-04 | 2.44e-04 | 6.22e-13 | 6.17e-13 | 0.82 | 1.73 | 3 | 5.74e-15 | 6.32e-15 | 62.06 | 1014.28 | 2 |
| 5 | 4 | 0 | (5, 10^6) | 2.83e-05 | 1.06e-10 | 7.53e-13 | 1.17e-12 | 0.79 | 1.88 | 3 | 6.98e-15 | 6.62e-15 | 59.68 | 1024.97 | 6 |

Note: OR-P. = Ordinary Preconditioning, CT-Pre. = Complete Orthogonalization Preconditioning, CT-J. = Complete Orthogonalization Preconditioning + Jacobi Iteration, SP = Sparsity Preserving, Res. = Residual.
Table 6.7: Numerical results for Example 6.3 with $m \times n = 2048 \times 1024$.

| Rule (mode, kappa) | Alg. 5.1 | | | | Alg. 5.2 | | | |
|-------------------|---------|---------|---------|---------|---------|---------|---------|---------|
|                   | Res.    | OR-U.   | OR-V.   | JU.     | CT.     | SP.     | Res.    | OR-U.   | OR-V.   | JU.     | CT.     | SP.     |
| (1, $10^3$)       | 1.26e-15| 1.12e-14| 4.71e-16| 0.011N  | 4.25    | 4       | 1.82e-15| 1.78e-14| 1.19e-15| 0.006N  | 8.98    | 3       |
| (2, $10^3$)       | 1.34e-16| 2.35e-15| 1.06e-16| 0.004N  | 2.13    | 2       | 8.96e-16| 2.56e-15| 1.15e-15| 0.002N  | 6.81    | 1       |
| (3, $10^3$)       | 6.48e-15| 2.36e-14| 7.01e-15| 10.501N | 71.07   | 18      | 2.81e-15| 2.66e-14| 3.04e-15| 1.737N  | 19.72   | 5       |
| (4, $10^3$)       | 6.47e-15| 2.08e-14| 6.58e-15| 9.262N  | 61.67   | 15      | 3.08e-15| 1.87e-14| 3.19e-15| 1.905N  | 20.50   | 5       |
| (5, $10^3$)       | 6.51e-15| 2.33e-14| 7.03e-15| 10.642N | 72.66   | 19      | 2.81e-15| 2.68e-14| 3.04e-15| 1.719N  | 19.61   | 5       |
| (1, $10^4$)       | 1.45e-15| 3.58e-14| 1.03e-15| 0.140N  | 4.91    | 4       | 2.29e-15| 4.18e-14| 2.20e-15| 0.717N  | 17.84   | 8       |
| (2, $10^4$)       | 1.41e-16| 2.28e-15| 1.13e-16| 0.004N  | 2.06    | 2       | 8.99e-16| 2.55e-15| 1.15e-15| 0.003N  | 6.77    | 1       |
| (3, $10^4$)       | 6.56e-15| 2.57e-14| 7.24e-15| 11.291N | 78.41   | 21      | 2.75e-15| 2.57e-14| 3.02e-15| 1.729N  | 19.76   | 5       |
| (4, $10^4$)       | 6.49e-15| 2.08e-14| 6.59e-15| 9.316N  | 63.23   | 16      | 3.08e-15| 1.85e-14| 3.18e-15| 1.900N  | 20.62   | 5       |
| (5, $10^4$)       | 6.61e-15| 2.52e-14| 7.34e-15| 11.570N | 79.64   | 21      | 2.75e-15| 2.49e-14| 3.03e-15| 1.740N  | 19.54   | 5       |
| (1, $10^5$)       | 2.34e-15| 4.33e-14| 2.88e-15| 1.652N  | 15.99   | 8       | 3.13e-15| 4.44e-14| 3.85e-15| 2.879N  | 28.65   | 9       |
| (2, $10^5$)       | 1.33e-16| 2.27e-15| 1.10e-16| 0.004N  | 1.98    | 2       | 8.98e-16| 2.54e-15| 1.15e-15| 0.003N  | 6.56    | 1       |
| (3, $10^5$)       | 6.65e-15| 2.77e-14| 7.53e-15| 12.376N | 83.99   | 24      | 2.73e-15| 2.52e-14| 3.01e-15| 1.785N  | 20.54   | 6       |
| (4, $10^5$)       | 6.52e-15| 2.10e-14| 6.62e-15| 9.373N  | 62.79   | 16      | 3.08e-15| 1.85e-14| 3.18e-15| 1.900N  | 20.33   | 5       |
| (5, $10^5$)       | 6.44e-15| 2.69e-14| 7.51e-15| 12.239N | 84.24   | 24      | 2.73e-15| 2.47e-14| 3.02e-15| 1.787N  | 19.22   | 5       |
| (1, $10^6$)       | 3.12e-15| 4.32e-14| 4.25e-15| 3.760N  | 26.93   | 9       | 3.66e-15| 4.35e-14| 4.71e-15| 4.471N  | 35.79   | 9       |
| (2, $10^6$)       | 1.35e-16| 2.29e-15| 1.07e-16| 0.004N  | 1.96    | 2       | 8.98e-16| 2.56e-15| 1.16e-15| 0.003N  | 6.33    | 1       |
| (3, $10^6$)       | 6.68e-15| 2.95e-14| 7.70e-15| 13.002N | 88.17   | 26      | 2.71e-15| 2.31e-14| 3.03e-15| 1.862N  | 19.23   | 5       |
| (4, $10^6$)       | 6.50e-15| 2.04e-14| 6.61e-15| 9.364N  | 60.09   | 16      | 3.08e-15| 1.85e-14| 3.18e-15| 1.908N  | 18.24   | 4       |
| (5, $10^6$)       | 6.77e-15| 2.94e-14| 7.83e-15| 13.426N | 93.02   | 27      | 2.72e-15| 2.35e-14| 3.01e-15| 1.845N  | 18.65   | 4       |
Table 6.8: Numerical results for Example 6.3 with 
\( m \times n \) = 4096 \times 2048.

| Rule | Alg. 5.1 | Alg. 5.2 | (mode, \( \kappa \)) | Res. | OR-U. | OR-V. | JU. | CT. | SP. |
|------|----------|----------|------------------------|------|-------|-------|-----|-----|-----|
| (1, 10^3) | 1.66e-15 | 1.40e-14 | 4.66e-16 | 0.004N | 48.28 | 5 | 2.31e-15 | 1.42e-14 | 1.55e-15 | 0.002N | 82.61 | 4 |
| (2, 10^3) | 1.33e-16 | 3.36e-15 | 1.11e-16 | 0.002N | 22.83 | 2 | 9.21e-16 | 3.48e-15 | 1.49e-15 | 0.001N | 59.62 | 1 |
| (3, 10^3) | 9.33e-15 | 4.67e-14 | 1.00e-14 | 10.978N | 944.57 | 19 | 3.89e-15 | 5.57e-14 | 4.26e-15 | 1.742N | 213.95 | 5 |
| (4, 10^3) | 9.39e-15 | 4.26e-14 | 9.52e-15 | 9.831N | 792.14 | 16 | 4.25e-15 | 3.84e-14 | 4.43e-15 | 1.900N | 209.26 | 5 |
| (5, 10^3) | 9.41e-15 | 4.82e-14 | 1.01e-14 | 11.112N | 803.27 | 19 | 3.87e-15 | 5.56e-14 | 4.24e-15 | 1.714N | 188.50 | 5 |
| (1, 10^4) | 1.70e-15 | 4.22e-14 | 8.08e-16 | 0.038N | 41.00 | 4 | 2.48e-15 | 7.68e-14 | 1.99e-15 | 0.160N | 144.13 | 9 |
| (2, 10^4) | 1.27e-16 | 3.18e-15 | 1.11e-16 | 0.002N | 19.15 | 2 | 9.23e-16 | 3.49e-15 | 1.49e-15 | 0.002N | 58.02 | 1 |
| (3, 10^4) | 9.41e-15 | 5.14e-14 | 1.04e-14 | 11.794N | 862.50 | 22 | 3.80e-15 | 5.43e-14 | 4.22e-15 | 1.745N | 187.00 | 5 |
| (4, 10^4) | 9.37e-15 | 4.22e-14 | 9.51e-15 | 9.817N | 696.45 | 17 | 4.25e-15 | 3.81e-14 | 4.43e-15 | 1.903N | 214.96 | 5 |
| (5, 10^4) | 9.45e-15 | 5.02e-14 | 1.04e-14 | 11.890N | 877.83 | 23 | 3.81e-15 | 5.34e-14 | 4.23e-15 | 1.754N | 187.01 | 5 |
| (1, 10^5) | 2.56e-15 | 8.09e-14 | 2.75e-15 | 0.758N | 136.91 | 9 | 3.79e-15 | 8.64e-14 | 4.54e-15 | 1.976N | 240.94 | 8 |
| (2, 10^5) | 1.32e-16 | 3.16e-15 | 1.10e-16 | 0.002N | 19.27 | 2 | 9.24e-16 | 3.49e-15 | 1.49e-15 | 0.002N | 58.02 | 1 |
| (3, 10^5) | 9.55e-15 | 6.01e-14 | 1.08e-14 | 12.759N | 961.53 | 25 | 3.78e-15 | 4.76e-14 | 4.24e-15 | 1.833N | 182.91 | 4 |
| (4, 10^5) | 9.38e-15 | 4.22e-14 | 9.50e-15 | 9.827N | 690.15 | 16 | 4.25e-15 | 3.81e-14 | 4.43e-15 | 1.904N | 202.13 | 5 |
| (5, 10^5) | 9.75e-15 | 5.42e-14 | 1.07e-14 | 12.522N | 926.41 | 25 | 3.79e-15 | 4.82e-14 | 4.24e-15 | 1.831N | 189.47 | 5 |

Note: \( \text{mode} \), \( \text{OR-U} \), \( \text{OR-V} \), \( \text{JU} \), \( \text{CT} \), \( \text{SP} \).
for computing the eigenvalue decomposition of a real symmetric matrix and a mixed precision preconditioned one-sided Jacobi method for the singular value problem. The corresponding rounding error analysis is studied. Our numerical experiments show the efficiency of the proposed mixed precision Jacobi method over the classical Jacobi method. Moreover, our algorithms can achieve higher speedup on GPUs. An interesting question is how to develop a mixed precision method for the generalized eigenvalue problem. This needs further study.

**Conflict of Interest Statement** The authors declare that they have no conflict of interest.

**Data Availability Statement** All data generated or analysed during this study are included in this manuscript.

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