Reconstruction of a complex electromagnetic coefficient from partial measurements

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Abstract

We consider an inverse boundary value problem for the Maxwell equations with boundary data assumed known only in accessible part Γ of the boundary. We aim to prove a uniqueness result using the Dirichlet to Neumann data with measurements limited to an open part of the boundary and we seek to reconstruct the complex refractive index $n$ in the interior of a body. Further, using the impedance map restricted to Γ, we may identify locations of small volume fraction perturbations of the refractive index.

Key words: Inverse problem, Maxwell equations, electromagnetic coefficients, partial data, reconstruction

Mathematics Subject Classifications: (MSC2010) 35Q61, 78M35, 35R30

1 Introduction

In this paper, we consider the nonlinear inverse boundary value problems for the time-harmonic Maxwell’s equations in a bounded domain, that is, to reconstruct key and specific electromagnetic parameter: complex refractive index $n(x)$, as function of the spatial variable, from a specified set of electromagnetic field measurements taken on the boundary.

For a closely related problem to the one considered here, we refer the readers to the original work of Colton and Päivärinta in [11]. The authors showed that the refractive index $n(x)$ (corresponding to e.g., known constant $\mu$ but unknown $\varepsilon(x)$ and $\sigma(x)$) can be uniquely determined by the far-field patterns of scattered electric fields satisfying time-harmonic Maxwell’s equations. Their approach is based on the ideas, developed by Sylvester and Uhlmann in [33], of constructing CGO (Complex Geometric Optic) type of solutions. In this context, the unique recovery of electromagnetic parameters from the scattering amplitude was first proven in [11] under the assumption that the magnetic permeability $\mu$ is a constant. But, the unique recovery of general $C^2$-electromagnetic parameters $\mu$ and $\gamma$ from full boundary data was later proved in [27], and simplified in [29] by introducing the so-called generalized Sommerfeld potentials. Concerning boundary determination results, we may refer to [8,14,24,22,15,16] and [34].

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For a slightly more general approach and more background information, see also the review article [28].

Inverse problems with partial data for scalar elliptic equations have attracted considerable attention recently. In [6, 13], the authors use Carleman estimates in their approaches. In [2], the authors use the local Dirichlet-to-Newmann map to recover the unknown coefficient by measuring on part of the boundary, but in [13] the author uses reflection arguments.

However, in electromagnetic problems concerned with the partial data problem, namely, to determine the parameters from the impedance map only made on part of the boundary, there are not as many results as in the scalar case. It is shown in [8] that if the measurements \( \Lambda(f) \) is taken only on a nonempty open subset \( \Gamma \) of \( f = \nu \times E|_{\partial \Omega} \) supported in \( \Gamma \), where the inaccessible part \( \partial \Omega \setminus \Gamma \) is part of a plane or a sphere, the electromagnetic parameters can still be uniquely determined. Combined with the augmenting argument in [29], the proof in [8] generalized the reflection technique used in [13]. As for another well-known method in dealing with partial data problems based on the Carleman estimates [6, 18], there are however significant difficulties in generalizing the method to the full system of Maxwell’s equations, e.g., the CGO solutions constructed using Carleman estimates.

The novelty of this paper lies in the use of partial electromagnetic field measurements taken on accessible part \( \Gamma \) to recover a complex refractive index. These partial measurements are traduced by an exhibition of appropriate partial boundary measurements in the form of a restricted boundary mapping \( Z \), specifically the mapping from the tangential components of the electric field \( E \) to the tangential components of curl \( E \) on the nonempty part \( \Gamma \). In this article, we consider the mapping \( Z \) which is closer to a natural generalization of the resistive map considered in impedance imaging applications. Our ideas and our methods differ from the approaches developed by Brown et al. in [4] or by Caro et al. in [8, 9]. Other inverse problems in electromagnetism in settings different to the ones in this paper have been considered in [3, 17, 20, 21, 23, 35, 36, 30, 12, 15, 16].

The outline of the paper is as follows. In the second section we recall some useful notation and function space, and we formulate the underlined problem. In Section 3, We eliminate the magnetic field and we reduce previous Maxwell’s equations to a system of equations for electric field \( E \). The global uniqueness result is provided. In Section 4, we derive a formula for calculating the unknown refractive index \( n \) from the local impedance map \( Z_n \) and by the Fourier integral theorem. We conclude our paper in Section 5 by applying the reconstruction procedure described before for identifying the locations of small volume fraction perturbations of the refractive index.

2 Notation and problem formulation

2.1 Notation

In the present paper, the following notation is used. If \( \mathcal{F} \) is a function space, \( \mathcal{F}^p, p \in \mathbb{N} \), denotes the space of vector-valued functions with each of the \( p \) components in \( \mathcal{F} \). The usual \( L^2 \)-based Sobolev spaces are denoted by \( H^s, s \in \mathbb{R} \). On the boundary of \( \partial \Omega \), the Sobolev spaces of tangential fields are defined as

\[
TH^s(\partial \Omega) = \{ f \in (H^s(\partial \Omega))^3; \ f \cdot \nu = 0 \}.
\]

Here, \( \nu = \nu(x) \) is the exterior unit normal vector of the boundary at \( x \in \partial \Omega \). We also need spaces of tangential fields having extra regularity. Let "\( \text{div} \)" denote the surface divergence on \( \partial \Omega \) (see, e.g., [10] or [26] for the definition). We define

\[
TH^s_{\text{div}}(\partial \Omega) = \{ f \in TH^s(\partial \Omega); \ \text{div} \ f \in H^s(\partial \Omega) \}.
\]
Finally, we remind that $TH^{s}_{\text{div}}(\partial\Omega)$-spaces arise naturally through the tangential trace mapping acting on functions in the spaces of the type

$$H^{s}_{\text{div}}(\Omega) = \{ f \in (H^{s}(\Omega))^{3}; \text{ div} (\nu \times f|_{\partial\Omega}) \in H^{s-1/2}(\partial\Omega) \}.$$ 

Here $\times$ stands for the vector product. The div-spaces are discussed to some extent in the references [5], [10] and [31]. Throughout this paper, we use "$\cdot$" to denote the standard scalar product in $\mathbb{R}^{3}$.

### 2.2 Statement of the problem

Let $\Omega \subset \mathbb{R}^{3}$ be a nonempty, open, and bounded set having a $C^{2}$-smooth boundary $\partial\Omega$. The unit normal vector to $\partial\Omega$, which is directed into the exterior of $\Omega$, is denoted by $\nu$. Moreover, we assume that the exterior domain $\Omega_{e} := \mathbb{R}^{3} \setminus \overline{\Omega}$ is connected. Let $\Gamma$ be a smooth open subset of the boundary $\partial\Omega$ and $\Gamma_{c}$ denotes $\partial\Omega \setminus \overline{\Gamma}$.

Consider first the boundary value problem of finding the electromagnetic fields $E$ and $H$ in a non-magnetic medium of bounded support $\Omega$:

\begin{align*}
\text{curl } E - i\omega \mu H &= 0, \quad \text{curl } H + i\omega \varepsilon n(x)E = 0, \quad \text{in } \Omega, \\
\nu \times E|_{\partial\Omega} &= f \in TH^{1/2}_{\text{div}}(\partial\Omega).
\end{align*}

Physically, $(E, H)$ is the time-harmonic electromagnetic field, $\omega > 0$ is its (fixed) frequency,

\begin{align*}
n(x) &= \frac{\varepsilon(x)}{\varepsilon_{0}} + i\frac{\sigma(x)}{\omega \varepsilon_{0}}
\end{align*}

is the refractive index of the medium, $\varepsilon$ denotes the electric permittivity of the conductor $\Omega$, $\mu$ denotes the magnetic permeability of $\Omega$, and $\sigma$ denotes its conductivity. Moreover, we recall that $\varepsilon_{0} > 0$ and $\mu_{0} > 0$ are respectively the permittivity and the permeability in the vacuum.

We assume the following conditions on the material parameters.

**Hypothesis 1** The permittivity $\varepsilon$, permeability $\mu$, and conductivity $\sigma$ are $C^{2}$ functions verifying the following properties.

- The magnetic permeability $\mu(=\mu_{0})$ is constant in this non-magnetic medium.
- For some positive constants $\varepsilon^{-}, \varepsilon^{+}$ and $\sigma^{+}$,

$$0 < \varepsilon^{-} \leq \varepsilon(x) \leq \varepsilon^{+}, \quad 0 \leq \sigma(x) \leq \sigma^{+}; \quad \text{for } x \in \overline{\Omega}.$$

- The function $\sigma$ and the difference $\varepsilon - \varepsilon_{0}$ are in $C^{2}_{0}(\Omega)$.

Now, under above properties we can claim that the refractive index $n$ is a complex function satisfying $n \in C^{2}(\Omega)$ for some $0 < \alpha < 1$, $Re(n) > 0$, $Im(n) \geq 0$. Moreover, if we denote by $n_{0}$ the refractive index in $\Omega_{e} := \mathbb{R}^{3} \setminus \overline{\Omega}$, then we have $n - n_{0} \in C^{2}_{0}(\Omega)$.

It is known [29] that the above boundary value problem \((1), (2)\) has a unique solution $(E, H) \in H^{1}_{\text{div}}(\Omega) \times H^{1}_{\text{div}}(\Omega)$ except for a discrete set of electric resonance frequencies $\omega_{n}$ when $\sigma = 0$. Assuming that $\omega$ is not a resonance frequency. Then according to [29, 27, 31, 32], and by considering electric field $E$ instead of the magnetic $H$ which considered in the previous references (see for example [29]), we can state that the following map

\begin{align*}
Z : TH^{1/2}_{\text{div}}(\partial\Omega) \to TH^{1/2}_{\text{div}}(\partial\Omega), \quad f \mapsto \nu \times H|_{\partial\Omega}
\end{align*}
called the impedance map is well defined. Recall that,
\[ H = \frac{1}{i\omega \mu_0} \text{curl} \ E, \]
and scaling in (4) by the complex constant \( i\omega \mu_0 \). Then, the following map
\[ Z : TH^{1/2}_d(\partial \Omega) \to TH^{1/2}_d(\partial \Omega), \quad f \mapsto \nu \times \text{curl} \ E|_{\partial \Omega}, \]
which denoted also by \( Z \) and still called the impedance map, is well defined. It is clear, that if \( \Omega \) is magnetic medium (\( \mu = \mu(x) \)), the mapping defined in (6) can not be comparable to those in the literature. This requires more attention and analysis, but we remove this consideration in the present paper.

## 3 Uniqueness result

We, now, eliminate the magnetic field from the equations (1)-(2) by dividing the first equation in (1) by \( i\omega \mu \) and taking the curl to obtain the following system of equations for electric field \( E \):
\[
\begin{align*}
\text{curl curl} \ E - k^2 n E &= 0 \quad \text{in } \Omega \\
\nu \times E &= f \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( k = \omega \sqrt{\mu_0 \varepsilon_0} \) is the wave number corresponding to the background, \( n \) the refractive index defined by (3), and \( f \in TH^{1/2}_d(\partial \Omega) \) is a given data on \( \partial \Omega \).

Introduce the trace space
\[
\tilde{TH}^{1/2}_d(\Gamma) := \left\{ f \in TH^{1/2}_d(\partial \Omega), f \equiv 0 \text{ on } \Gamma_c \right\}.
\]

Here and in the sequel we identify \( f \) defined only on \( \Gamma \) with its extension by 0 to all \( \partial \Omega \)

\text{(supp}(f)) \subset \Gamma).

**Remark 1** Let \( n \in C^2(\bar{\Omega}) \), and suppose that \( k^2 \) is not an eigenfrequency of the following problem:
\[
\begin{align*}
\left\{ \begin{array}{l}
(\text{curl curl} - k^2 n) E = 0, \quad \text{in } \Omega \\
\nu \times E|_{\partial \Omega} = f \in \tilde{TH}^{1/2}_d(\Gamma),
\end{array} \right.
\]
where \( \tilde{TH}^{1/2}_d(\Gamma) \) was defined by (8). Then, \( Z_n(f) = \nu \times \text{curl} \ E|_{\Gamma} \) is the local impedance map in this case.

Based on definition (8) and on Remark 1 the inverse boundary value problem is to recover \( n \) from the partial boundary measurements encoded as the well-defined local impedance map:
\[
(10) \quad Z_n : \tilde{TH}^{1/2}_d(\Gamma) \to TH^{1/2}_d(\partial \Omega), \quad f \mapsto \nu \times \text{curl} \ E|_{\Gamma}.
\]

We will prove the following main result, showing that partially boundary measurements for the Maxwell’s equations uniquely determine the material parameters in a bounded domain.

**Theorem 1** Let \( n_0 \) be the refractive index in \( \mathbb{R}^3 \setminus \bar{\Omega} \) such that \( n_i - n_0 \in C^0(\Omega) \), for \( i = 1, 2 \). Assume \( n_1 = n_2 \) almost everywhere in a neighborhood of the boundary \( \partial \Omega \). Suppose that \( k^2 \)
is not a resonant frequency for (7)-(8) associated to \( n_i, i = 1, 2 \). If the local impedance maps coincide,

\[
Z n_i(f) = Z n_j(f) \quad \text{for all } f \in \bar{TH}_{\text{div}}(\Gamma),
\]

then there exists \( \kappa = \kappa(\Omega) > 0 \) such that

\[
n_1 = n_2
\]

almost everywhere in \( \Omega \) whenever

\[
\|n_j - n_0\|_{W^{2, \infty}(\Omega)} < \kappa.
\]

Before proceeding with the proof of Theorem 1, we remind some well-known and original results.

By referring to the works of Colton and Päivärinta [11], Sun and Uhlmann [32] and to the Calderón problem of electrostatics (see e.g., [7, 25] and [33]), we consider the inverse problem of determining the key electromagnetic parameters from the boundary measurement. Particularly, in this paper we assume that we can measure the values of \( Z_n(f) \) only on a nonempty open subset \( \Gamma \subset \partial \Omega \), and only for tangential boundary fields \( f \) supported in \( \Gamma \).

The common outline of the proof of Theorem 1 follows approximately the same lines as the proof of the global uniqueness theorem for the inverse conductivity problem given in [33], for an inverse boundary value problem for Maxwell’s equations in [32], for the uniqueness of a solution to an inverse scattering problem for electromagnetic waves in [11] or for proof of the global uniqueness theorem for the Schrödinger equation in [2]. Explicitly, one first proves an identity involving products of solutions of the equation under consideration as will be done in Lemma 1. Next, one proves a density result as in Lemma 2. Then one constructs vector CGO type solutions for the underlined problem (7)-(8) to obtain information, via this identity, of the Fourier transform of the unknown function. There are two main difficulties in carrying out this approach for the problem under consideration here. First, we cannot reduce Maxwell’s equations to a Schrödinger equation to proceed exactly as in [2] (for example). The best we can do is to reduce Maxwell’s equations to a system whose principal part is the Laplacian times the identity operator as done in [11, 32, 29, 27, 28]. We can then construct CGO solutions under appropriate smallness assumptions. Also, in our case we have to construct global solutions in order to guarantee that the solutions constructed satisfy the condition that the electric and magnetic field be divergence-free. In order to determine the unknown \( n \), one has to study the asymptotic expansion of these solutions in a free parameter. The second difficulty is that such CGO solutions for Maxwell’s equations do not have the property that \( R_\xi \) decays like \( O(1/|\xi|) \) (see e.g., [11, 29, 27, 28]), which was a key ingredient in the proof of the uniqueness in the scalar case. But, this is tackled in [11] by constructing appropriate \( R_\xi \) that decays to zero in certain distinguished directions as \( |\xi| \) tends to infinity. By carefully choice of several directions for \( \xi \), as will be defined in relation (28), such special set of solutions are enough to determine the refractive index.

To prove Theorem 1, we begin by the following lemma. This result generalizes the Alessandrini’s identity [1] for the conductivity equation to Maxwell system.

**Lemma 1** Let \( \mathcal{O} \subset \subset \Omega \) containing \( \text{supp}(n_j - n_0) \) (for \( j = 1, 2 \)), such that \( \mathcal{O} \) is a bounded domain with \( C^2 \) boundary. Let \( E_i \in H^1_{\text{div}}(\Omega) \) satisfying:

\[
(\frac{\nu}{n_i} \text{curl curl } n_i) E_i = 0 \quad \text{in } \Omega
\]

\[
\nu \times E_i|_{\Gamma} = 0, \quad i = 1, 2
\]

\[
\nu \times E_i|_{\partial\mathcal{O}} \neq 0, \quad i = 1, 2.
\]
Assume \( n_1 = n_2 \) almost everywhere in a neighborhood of the boundary \( \partial \Omega \) and \( Z_{n_1} = Z_{n_2} \). Then, \( \text{supp}(n_1 - n_2) \subset \mathcal{O} \) and

\[
(12) \quad \int_{\mathcal{O}} E_1 \cdot (n_1 - n_2) E_2 \, dx = 0.
\]

**Proof.** Firstly, to get \( \text{supp}(n_1 - n_2) \subset \mathcal{O} \), one may expand

\[
(n_1 - n_2)(x) = (n_1 - n_0)(x) - (n_2 - n_0)(x),
\]

and the result follows immediately by recalling that \( \text{supp}(n_j - n_0) \subset \mathcal{O} \) for \( j = 1, 2 \).

Now, let \( E_i, i = 1, 2 \) be solutions of (11). Then, by using Green’s theorem we have that

\[
\int_{\Omega} \nabla \cdot \mathbf{V} \cdot E_i \, dx = \int_{\Omega} \nabla \cdot E_i \cdot \mathbf{V} \, dx + \int_{\partial \Omega} (\nu \times \mathbf{V}) \cdot E_i \, ds(x),
\]

where \( ds(x) \) denotes surface measure.

If \( \mathbf{V}|_{\Gamma_\varepsilon} \equiv 0 \), the above relation becomes

\[
\int_{\Omega} \nabla \cdot \mathbf{V} \cdot E_i \, dx = \int_{\Omega} \nabla \cdot E_i \cdot \mathbf{V} \, dx + \int_{\Gamma} (\nu \times \mathbf{V}) \cdot E_i \, ds(x).
\]

Therefore, by replacing \( \mathbf{V} := \nabla \cdot E_i \) and using Green’s theorem for \( i = 1 \) and \( i = 2 \) respectively, relation (11) gives

\[
(13) \quad \int_{\Omega} E_1 \cdot (n_1 - n_2) E_2 \, dx = \frac{1}{k^2} \int_{\Gamma} [(\nu \times \nabla \cdot E_1) \cdot E_2 - E_1 \cdot (\nu \times \nabla \cdot E_2)] \, ds,
\]

Recall that \( \text{supp}(n_1 - n_2) \subset \mathcal{O} \subset \Omega \), then from (13) we immediately get

\[
(14) \quad \int_{\mathcal{O}} E_1 \cdot (n_1 - n_2) E_2 \, dx = \frac{1}{k^2} \int_{\Gamma} [(\nu \times \nabla \cdot E_1) \cdot E_2 - E_1 \cdot (\nu \times \nabla \cdot E_2)] \, ds.
\]

On the other hand, let \( \mathbf{V}_1 \in H^1(\Omega) \) be solution of \( (\frac{1}{k^2} \nabla \cdot \nabla - n_1) \mathbf{V}_1 \) in \( \Omega \) such that \( \nu \times \mathbf{V}_1|_{\Gamma_\varepsilon} = 0 \) and \( \nu \times \mathbf{V}_1|_{\Gamma} = \nu \times \mathbf{E}_2|_{\Gamma} \).

From \( \Lambda n_1 = \Lambda n_2 \) it follows that

\[
(15) \quad \nu \times \mathbf{V}_1|_{\Gamma_\varepsilon} = 0, \nu \times \mathbf{V}_1|_{\Gamma} = \nu \times \mathbf{E}_2|_{\Gamma}
\]

gives

\[
\nu \times \nabla \cdot \mathbf{V}_1|_{\Gamma} = \nu \times \nabla \cdot \mathbf{E}_2|_{\Gamma}.
\]

Then by Green’s theorem again

\[
(16) \quad 0 = \int_{\Omega} E_1 \cdot (n_1 - n_1) \mathbf{V}_1 \, dx = \int_{\Gamma} [(\nu \times \nabla \cdot E_1) \cdot \mathbf{V}_1 - E_1 \cdot (\nu \times \nabla \cdot \mathbf{V}_1)] \, ds
\]

\[
= - \int_{\Gamma} [(\nu \times \nabla \cdot E_1) \cdot (\nu \times \mathbf{V}_1) - (\nu \times \nabla \cdot \mathbf{V}_1) \cdot (\nu \times E_1)] \, ds.
\]

The last relation may be deduced by a triple product (e.g., by using the Levi-Civita symbol we write \( (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \varepsilon_{ijk} a^i b^j c^k \)).

Then from (14), (13), and (16) we deduce the desired identity (12). \( \square \)

The second Lemma states that the set of solutions of the Maxwell’s equations with boundary data 0 on \( \Gamma_\varepsilon \) is dense in \( L^2(\mathcal{O}) \) in the set of all solutions.
Lemma 2 Let $n \in L^\infty(\Omega)$. Let $\mathcal{O}$ be as in Lemma 1 such that $\Omega \setminus \overline{\mathcal{O}}$ is connected. Let us define

$$\overline{S}(\Omega) = \left\{ V \in H^2(\Omega) \mid \frac{1}{k^2} \text{curl curl } n V = 0 \text{ in } \Omega, \nu \times V = 0 \text{ on } \Gamma_c \right\}$$

and

$$S(\Omega) = \left\{ V \in H^2(\Omega) \mid \frac{1}{k^2} \text{curl curl } n V = 0 \text{ in } \Omega \right\}.$$

Then $\overline{S}(\Omega)$ is dense in $S(\Omega)$ according to $L^2(\mathcal{O})$ norm.

Proof. We first define Green's function $G(x, y)$ for (7) as a $3 \times 3$ matrix valued function solution of:

$$\begin{cases}
    (\text{curl curl } - k^2 n) G(x, y) = -I_3 \delta(x - y) \text{ in } \Omega, \\
    \nu \times G(x, y) = 0 \text{ on } x \in \partial \Omega
\end{cases}$$

where $I_3$ is the $3 \times 3$ identity matrix. In the above notation the curl operator acts on matrices column by column. The Green function $G$ is given by

$$G(x, y) = \Phi(x, y) \left[ I_3 + \frac{\nabla_x \nabla_x}{k^2} \right],$$

where the scalar function $\Phi$ means the outgoing fundamental solution for the Helmholtz operator "$\Delta + k^2 n$" and given by

$$\Phi(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}.$$  

As an example, the first column of $G(x, y)$ equals

$$\begin{pmatrix}
    \Phi(x, y) \\
    0 \\
    0
\end{pmatrix} + \frac{1}{k^2} \nabla_x \nabla_x \begin{pmatrix}
    \Phi(x, y) \\
    0 \\
    0
\end{pmatrix}.$$

Multiplying equation (7) by $G(x, y) \cdot V$ ($V \in \mathbb{R}^3$), integrating by parts in the domain $\Omega$, and using the relation (5) we immediately get a convenient integral representation formula for the electric field called Stratton-Chu formula. For more detail about this representation, one can see Theorem 6.1 in [10].

Subsequently, suppose there exists $V \in S(\Omega)$ such that

$$\int_{\mathcal{O}} V \cdot \nabla' dx = 0, \text{ for all } V' \in \overline{S}(\Omega).$$

Define the vector valued function

$$W(x) := - \int_{\mathcal{O}} G(x, y) \nabla(y) dy \in H^2(\Omega).$$

Then, by referring to (17) we find that

$$(\text{curl curl } - k^2 n) W(x) = \begin{cases}
    \nabla \text{ in } \mathcal{O}, \\
    0 \text{ in } \Omega \setminus \overline{\mathcal{O}}.
\end{cases}$$

Moreover, by (19) we may write:

$$\nu(x) \times W(x) = - \int_{\mathcal{O}} \nu(x) \times G(x, y) \nabla(y) dy.$$  

Since for any $x \in \partial \Omega$, $\nu(x) \times G(x, y) = 0$ ($\forall y \in \mathcal{O}$), we have $\nu \times W|_{\partial \Omega} = 0$.

On the other hand, for all $V' \in \overline{S}(\Omega)$ integration by parts yields

$$V'(x) = \int_{\partial \Omega} \nu(y) \times \text{curl } G(x, y) V'(y) \, ds(y).$$
On the other hand, by Green’s formula we get
\[HV \equiv \text{geometric-optics solution (CGO) for the Maxwell’s equations (7). More precisely, they con-}\]
\[\text{tinue principle to} \quad \nu \times \text{curl} W = 0, \quad \text{in} \Gamma.\]

Now, by the unique continuation principle, it follows that \(W(x) = 0\) for all \(x \in \Omega \setminus \overline{\Omega}\), and \(\nu \times \text{curl} W = \nu \times W = 0\), on \(\partial \Omega\).

On the other hand, by Green’s formula we get
\[
\int_{\Omega} |V|^2 dx = \int_{\Omega} V \cdot \nabla dx = \int_{\Omega} (\text{curl curl} - k^2 n)W \cdot \nabla dx
\]
\[= \int_{\Omega} W \cdot (\text{curl curl} - k^2 n)\nabla dx = 0.\]

Hence, \(V \equiv 0\) in \(\Omega\). To achieve the proof of our density result, we can apply again the unique continuation principle to \(V\) to find that \(V \equiv 0\) in \(\Omega\).

**Proof of Theorem** Thanks to Sylvester and Uhlmann [33], we can construct complex geometric-optics solution (CGO) for the Maxwell’s equations (7). More precisely, they constructed their CGO solution to Schrödinger’s equation by looking for a solution in the form
\[u(x) = e^{i\xi \cdot x} (1 + R_\xi(x))\] where \(\xi \in \mathbb{R}^3\) satisfying \(\xi, \xi = 0\) and \(R_\xi\) decays like \(O(1/|\xi|)\).

After that, Sun and Uhlman [32] and Colton and Paviranta [11] proved that the CGO solution of the Maxwell’s equations may be of the form:
\[V = e^{x \cdot \xi} [\eta + \Psi(x, \xi)]; \quad \xi, \eta \in \mathbb{C}^3,\]
with \(\Psi \in H^1_\delta (\mathbb{R}^3)\) and \(\Psi = O(1)\) as \(|z| \to +\infty\) (\(z\) is a distinguished direction from \(\xi\)). Here, \(L^2_3(\mathbb{R}^3)\) denotes the the Hilbert space
\[L^2_3(\mathbb{R}^3) = \{ \mathbf{g} \in L^2_{\text{loc}}(\mathbb{R}^3); \int_{\mathbb{R}^3} (1 + |x|^2)^{\delta/2} |\mathbf{f}(x)|^2 \, dx < +\infty \}, \quad \text{for} -1 < \delta < 0,\]
and \(H^1_3(\mathbb{R}^3)\) denotes the corresponding Sobolev space. Moreover, \(\xi\) and \(\eta\) are complex constant vectors satisfying \(\xi \cdot \xi = k^2\), and \(\xi \cdot \eta = 0\).

To explain the distinguished direction, we may refer to [32] to write:
\[\xi = sp + \frac{l}{2} + g(s)\omega_1, \quad \text{and} \quad \eta = l - \frac{|l|^2}{2s}\omega_1 + \frac{g(s)}{s} l \quad (i^2 = -1),\]
where \(s > 0\), \(l \in \mathbb{R}^3\), \(\rho = w_1 + iw_2\) for \(w_i \in \mathbb{S}^2\) such that \(w_1l = w_2l = w_1w_2 = 0\), and
\[g(s) := \frac{|l|^2 + 4k^2}{4s + 2\sqrt{4s^2 + |l|^2} + 4k^2} - 1.\]

From previous results (e.g., [32]) and from (23), we can construct CGO solution of \(\Omega\) in \(\mathbb{R}^3\) as follows.

**Proposition 1** Let \(\mathbf{n} \in C^2(\Omega)\) be as in \([17]\). Extend \(\mathbf{n} = \mathbf{n}_0\) in \(\Omega_c = \mathbb{R}^3 \setminus \overline{\Omega}\). Let \(\xi\) and \(\eta\) be as in (24), and let \(-1 < \delta < 0\). Then there exist \(\kappa_1 = \kappa_1(\Omega) > 0\) and \(r > 0\) such that if \(s > r\) and
\[\|\mathbf{n} - \mathbf{n}_0\|_{W^{2,\infty}(\Omega)} \leq \kappa_1,\]
then there is a unique solution of (7) in \(\mathbb{R}^3\) of the form
\[V = e^{x \cdot \xi} [\eta + \Psi_\mathbf{n}(x, \xi)],\]
for \(|\xi|\) sufficiently large, with \(\Psi_\mathbf{n} \in H^1_3(\mathbb{R}^3)\) and \(\Psi_\mathbf{n} = O(1)\) as \(s \to +\infty\).
Concerning the proof of Proposition 1, one can follow the proof of Theorem 1.6 in [32] by making the necessary changes that needed in our problem here. Now, from Proposition 1 we can remark the following.

**Remark 2** From [24] and Proposition 1, the vector valued function \( \Psi_{\Omega} \) decays to zero in certain distinguished directions as \( \xi \to +\infty \). In particular \( \Psi_{\Omega} = O(1) \) as \( s \to +\infty \), and this suffices for our purpose to prove our main theorem.

To proceed with the proof, we define

\[
\tilde{n}_j = \begin{cases} 
  n_j & \text{in } \Omega, \\
  n_0 & \text{in } \mathbb{R}^3 \setminus \Omega.
\end{cases}
\]

Then by Proposition 1 for \( j = 1, 2 \) and for \(-1 < \delta < 0\), there exist \( \kappa_1^{(j)} > 0 \) and \( r_j > 0 \) such that if

\[
s > \tilde{r} = \max(r_1, r_2), \quad \text{and } ||\tilde{n}_j - n_0||_{W^{2,\infty}(\Omega)} \leq \tilde{\kappa}_1.
\]

where \( \tilde{\kappa}_1 = \min(\kappa_1^{(1)}, \kappa_1^{(2)}) \) we can construct solutions of the problem (curl curl \(-k^2 n_j\) \( V \)) \( j = 0, 1 \) \( k^2 \) in \( \mathbb{R}^3 \) of the form

\[
V_j = e^{x \cdot \xi_j}[\eta_j + \Psi_{\tilde{n}_j}(x, \xi_j)], \quad j = 1, 2
\]

with \( \Psi_{\tilde{n}_j}(x, \xi_j) \in L^2(\mathbb{R}^3) \). Moreover, by Remark 2 \( \Psi_{\tilde{n}_j}(x, \xi_j) \) decays to zero in certain distinguished directions as \( \xi \to +\infty \). Precisely, \( \Psi_{\tilde{n}_j}(x, \xi_j) = O(1) \) as \( s \to +\infty \).

On the other hand, from [24] we can expand that \( \xi = sw_1 + g(s)w_1 + i(sw_2 + \frac{1}{2}) \). Then, we may define

\[
\xi_j = -(1)^j[s + g(s)]w_1 + i[\frac{1}{2} - (-1)^jsw_2], \quad \text{for } j = 1, 2 \quad (i^2 = -1);
\]

where \( s, g(s), w_1, w_2 \) and \( l \) given as in [24]. Consequently, we have \( \xi_1 + \xi_2 = il \).

To complete the proof, we write down,

\[
\int_{\Omega} V_1|_{\partial \Omega} \cdot (\tilde{n}_1 - \tilde{n}_2)V_2|_{\partial \Omega} \, dx = \int_{\Omega} V_1|_{\partial \Omega} \cdot (n_1 - n_2)V_2|_{\partial \Omega} \, dx.
\]

Having \( \text{supp}(n_1 - n_2) \subset \partial \Omega \), we get

\[
\int_{\Omega} V_1|_{\partial \Omega} \cdot (n_1 - n_2)V_2|_{\partial \Omega} \, dx = \int_{\partial \Omega} V_1|_{\partial \Omega} \cdot (n_1 - n_2)V_2|_{\partial \Omega} \, dx.
\]

Since \( V_i|_{\partial \Omega} \in S(\Omega) \), we can apply Lemma 2 to state that for \( i = 1, 2 \), \( V_i|_{\partial \Omega} \) can be approximated by elements of \( S(\Omega) \) in in \( L^2(\partial \Omega) \) norm. Therefore,

\[
\int_{\partial \Omega} V_1|_{\partial \Omega} \cdot (n_1 - n_2)V_2|_{\partial \Omega} \, dx
\]

may be approximated by

\[
\int_{\partial \Omega} \tilde{V}_1 \cdot (n_1 - n_2)\tilde{V}_2 \, dx,
\]

where \( \tilde{V}_i \in H^1(\Omega) \) solution of

\[
(\text{curl curl } - k^2 n_i)\tilde{V}_i = 0 \quad \text{in } \Omega \\
\nu \times \tilde{V}_i|_{\Gamma_c} = 0, \quad i = 1, 2.
\]

But, by Lemma 3 we have

\[
\int_{\partial \Omega} \tilde{V}_1 \cdot (n_1 - n_2)\tilde{V}_2 \, dx = 0.
\]
Thus,
\begin{equation}
\int_{\Omega} V_1|\Omega \cdot (n_1 - n_2)\mathbf{V}_2|\Omega \, dx = \int_{\Omega} V_1|\Omega \cdot (n_1 - n_2)\mathbf{V}_2|\Omega \, dx = 0.
\end{equation}

Next, suppose that we have (26). Then taking into account Proposition 1, substituting (27) into (29), using (28), considering Remark 2, and letting \( s \to +\infty \). We conclude by the Fourier integral theorem that:
\[(n_1 - n_2)(-l) = 0, \quad \forall l \in \mathbb{R}^3.\]

The hats denoting the Fourier transforms of the corresponding functions. The theorem is now proved. \( \square \)

4 Reconstruction of \( n \)

Let \( \hat{n} \in C^2(\bar{\Omega}) \) be a known function. Assume that \( n = \hat{n} \) almost everywhere in a neighborhood of \( \partial \Omega \). Denote \( \mathcal{O} \subset \subset \Omega, \mathcal{O} \) bounded open with \( C^2 \)-boundary containing \( supp(n - \hat{n}) \). In this section we derive a formula for calculating \( n \) from the local impedance map \( Z_n : \tilde{TH}_d^2(\Gamma) \to TH_\mathcal{O}^2(\partial \Omega) \).

Assume that \( Z_n \) is known, then for any \( \mathbf{E}, \mathbf{V} \in H^1_{\partial \Omega}(\Omega) \) satisfying
\[
\begin{align*}
& (\text{curl}\, \text{curl} - k^2n)\mathbf{E} = 0 \quad \text{in} \ \Omega \\
& (\text{curl}\, \text{curl} - k^2\hat{n})\mathbf{V} = 0 \quad \text{in} \ \Omega \\
& \nu \times \mathbf{E}|_{\Gamma^c} = \nu \times \mathbf{V}|_{\Gamma^c} = 0,
\end{align*}
\]
we have
\begin{equation}
\int_{\Omega} \mathbf{E} \cdot (n - \hat{n})\mathbf{V} \, dx = \int_{\Gamma} \nu \times \mathbf{E} \cdot (Z_n - Z_{\hat{n}})(\nu \times \mathbf{V}) \, ds,
\end{equation}
where \( Z_{\hat{n}} \) denotes the local impedance map associated to the refractive index \( \hat{n} \).

Extend \( n \) and \( \hat{n} \) by \( n_0 \) in \( \mathbb{R}^3 \). Let \( \xi \in C^2 \setminus \{0\} \) with \( \xi \cdot \xi = 0 \). Define \( \mathbf{E}_\xi \) to be the solution of
\[
\begin{align*}
& (\text{curl}\, \text{curl} - k^2n_0)\mathbf{E}_\xi = 0 \quad \text{in} \ \mathbb{R}^3 \setminus \overline{\Omega}, \\
& (\text{curl}\, \text{curl} - k^2\hat{n})\mathbf{E}_\xi = 0 \quad \text{in} \ \mathbb{R}^3.
\end{align*}
\]
subject to the radiation condition
\begin{equation}
(32) 
-x^2 \mathbf{E}_\xi - \eta \in \mathbf{L}^2_{\delta} = \{ f : \int_{\mathbb{R}^3} (1 + |x|^2)^3 |f(x)|^2 \, dx < +\infty \}, \quad \text{for} \ -1 < \delta < 0, \ \eta \in \mathbb{R}^3.
\end{equation}

According to Proposition 11 and to Proposition 2.11 in [32], one can easily expand:
\begin{equation}
(33) \quad \mathbf{E}_\xi(x) = e^{x \xi}[\eta + (d_1 + \tilde{d}_1)\rho + d_2 s + D/s + R],
\end{equation}
where the scalar functions \( d_1, \tilde{d}_1, d_2 \), and the vector functions \( D \) and \( R \) satisfy respectively:
\[
\begin{align*}
& d_1 = d_1(x, \rho, \tilde{l})|l|; \quad \|d_1\| < C, \\
& \tilde{d}_1 = \tilde{d}_1(x, \rho, s, l); \quad \lim_{s \to \infty} \|\tilde{d}_1\| = 0, \\
& d_2 = \sqrt{\frac{D}{\mu_\infty}} - 1 = 0, \\
& D = D_0(x, \rho, \tilde{l}) + D_1(x, \rho, \tilde{l})|l| + D_2(x, \rho, \tilde{l})|l|^2, \quad \|D_i\| < C; \ i = 0, 1, 2, \\
& R = R(x, \rho, s, l); \quad \lim_{s \to \infty} s\|R\| = 0,
\end{align*}
\]
where \( \tilde{l} = \frac{l}{|l|}, C \) is a positive constant independent of \( \delta \) and \( n \).
Remark 3  To simplify our method, we shall set

\begin{equation}
A + G = \eta + (d_1 + d_1)\rho + d_2l + D/s + R,
\end{equation}

where \( A := \eta + d_1\rho + d_2l \) satisfies a transport equation type, and the remainder \( G \) satisfies \( \lim_{s \to \infty} ||G||_{L^1} = 0. \)

Therefore, from (33)-(34) we get:

\[ \nu \times \text{curl} \ E_\xi|_\Gamma = -((\xi \times E_\xi) \times \nu|_\Gamma) + \nu \times \text{curl} \ (A + G - \eta)e^{x \cdot \xi}|_\Gamma \quad \forall \ x \in \Gamma, \]

and the Jacobi identity immediately gives

\[ \nu \times \text{curl} \ E_\xi|_\Gamma = -(\nu \times E_\xi) \times \xi|_\Gamma - (\xi \times \nu) \times E_\xi|_\Gamma + \nu \times \text{curl} \ (A + G - \eta)e^{x \cdot \xi}|_\Gamma \quad \forall \ x \in \Gamma. \]

Since

\[ \nu \times \text{curl} \ E_\xi|_\Gamma(x) = Z_n(E_\xi|_\Gamma) \]

we obtain that \( E_\xi \) solves the following equation on the open surface \( \Gamma \):

\begin{equation}
Z_n(\nu \times E_\xi|_\Gamma) + \mathcal{N}_\xi(\nu \times E_\xi|_\Gamma)(x) = (A + G)e^{x \cdot \xi}, \quad \forall \ x \in \Gamma,
\end{equation}

where \( \mathcal{N}_\xi \) is the operator defined by \( \mathcal{N}_\xi : \overline{T \mathcal{H}^{\frac{3}{2}}_{\text{div}}(\Gamma)} \to \overline{T \mathcal{H}^{\frac{3}{2}}_{\text{div}}(\partial\Omega)}, \nu \times f \mapsto (\nu \times f) \times \xi + C_\xi(\nu \times f) \)

with \( C_\xi(\nu \times f) = (\xi \times \nu) \times f \) is a bounded map on \( \overline{T \mathcal{H}^{\frac{3}{2}}_{\text{div}}(\Gamma)} \).

Then, the following holds.

**Proposition 2**  Assume that \( k^2 \) is not an eigenfrequency of \( (\text{curl} \ \text{curl} - k^2 \mathbf{n}) \) in \( \Omega \). Suppose that \( E_\xi \) is a solution of (31)-(32), then \( \nu \times E_\xi|_\Gamma \) solves (35) uniquely.

Now, let \( \xi \in \mathbb{C}^3 \setminus \{0\} \) with \( \xi \cdot \xi = 0 \). Let \( \nu \times E_\xi|_\Gamma \in \overline{T \mathcal{H}^{\frac{3}{2}}_{\text{div}}(\Gamma)} \) be the solution of (35). Then, according to Section 2, we may have the following representation:

\begin{equation}
- \int_\Omega \nu \times (G(x, y) \ E_\xi) \ dy = \nu \times e^{x \cdot \xi}(\eta + \Psi_\mathbf{n}(x, \xi)),
\end{equation}

where

\begin{equation}
\Psi_\mathbf{n}(x, \xi) = O(1) \quad \text{as} \ s \to +\infty.
\end{equation}

Moreover, a carefully analysis on properties of operators \( Z_\mathbf{n} \) and \( \mathcal{N}_\xi \) immediately gives, by relation (35):

\begin{equation}
\nu \times E_\xi|_\Gamma = (Z_\mathbf{n} + \mathcal{N}_\xi)^{-1}((A + G)e^{x \cdot \xi}), \quad \forall \ x \in \Gamma.
\end{equation}

Hence, we have the following reconstruction formula.

**Theorem 2**  Let \( \mathbf{n} \in C^2(\bar{\Omega}) \) be a given function. Assume that \( k^2 \) is not an eigenfrequency of \( (\text{curl} \ \text{curl} - k^2 \mathbf{n}) \) in \( \Omega \), and \( \mathbf{n} = \mathbf{n} \) almost everywhere in a neighborhood of \( \partial\Omega \). Then

\begin{equation}
(\mathbf{n} - \mathbf{n})(-l) = \lim_{s \to +\infty} \int_\Gamma (Z_\mathbf{n} + \mathcal{N}_\xi)^{-1}((A + G)e^{x \cdot \xi}|_\Gamma) \cdot (Z_\mathbf{n} - Z_\mathbf{n})(Z_\mathbf{n} + \mathcal{N}_\xi)^{-1}((A + G)e^{x \cdot \xi}|_\Gamma) \ ds(x).
\end{equation}

**Proof.**  An major step of the proof was given in the previous approaches. Now, from (28) we may write

\[ \xi_1 + \xi_2 = i l \quad (\text{for} \ i^2 = -1). \]

By applying relation (36), we can pose

\[ E = E_\xi = e^{x \cdot \xi_1}(\eta_1 + \Psi_\mathbf{n}(x, \xi_1)), \]
and  
\[ V = E_\xi = e^{x \cdot \xi}(\eta_2 + \Psi_\tilde{n}(x, \xi_2)), \]

with \( \eta_1 \cdot \eta_2 = 1 \), to obtain from (30) and (38) that

\[
\int_{\Omega} e^{x \cdot (\xi_1 + \xi_2)} \left[ \eta_1 \cdot \eta_2 + \eta_2 \cdot \Psi_\tilde{n}(x, \xi_1) + \eta_1 \cdot \Psi_\tilde{n}(x, \xi_2) \right] (n - \tilde{n})(x) \, dx \\
\quad + \Psi_\tilde{n}(x, \xi_1) \cdot \Psi_\tilde{n}(x, \xi_2) \right] (n - \tilde{n})(x) \, dx \\
= \int_{\Gamma} (Z_n + N_{\xi_1})^{-1}((A + G)e^{x \cdot \xi_1}|_{\Gamma}) \cdot (Z_n \\
- Z_{\tilde{n}})((Z_n + N_{\xi_2})^{-1}((A + G)e^{x \cdot \xi_2}|_{\Gamma})) \, ds(x).
\]

(39)

Now, by using (28), we immediately get \( |\xi_i| \leq s \), for \( i = 1, 2 \) if \( s > \hat{r} \) (where \( \hat{r} \) given by Proposition 1).

Thus, by (37), Remark 3, the left hand side of relation (39) may be written as:

\[
\lim_{s \to +\infty} \int_{\Omega} e^{ix \cdot l} \left[ \eta_1 \cdot \eta_2 + \eta_2 \cdot \Psi_\tilde{n}(x, \xi_1) + \eta_1 \cdot \Psi_\tilde{n}(x, \xi_2) + \Psi_\tilde{n}(x, \xi_1) \cdot \Psi_\tilde{n}(x, \xi_2) \right] (n - \tilde{n})(x) \, dx \\
= \lim_{s \to +\infty} \int_{\Omega} e^{ix \cdot l} \left[ 1 + o\left(\frac{1}{s}\right) + o\left(\frac{1}{s}\right) + o\left(\frac{1}{s}\right) \right] (n - \tilde{n})(x) \, dx \\
= (n - \tilde{n})(-l).
\]

The theorem now follows by considering the limit of expression (39) as \( s \to +\infty \). \( \square \)

5 Application: reconstruction of the locations of small volume fraction perturbations of the refractive index

The aim of this section is to apply the reconstruction procedure described in Section 3 for identifying the locations of small volume fraction perturbations of the refractive index. Assume that \( \Omega \subset \mathbb{R}^3 \) contains a finite number of inhomogeneities, each of the form \( z_j + \alpha B_j \), where \( B_j \subset \mathbb{R}^3 \) is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is \( B_\alpha = \bigcup_{j=1}^m (z_j + \alpha B_j) \) with

\[(z_i + \alpha B_i) \cap (z_j + \alpha B_j) = \emptyset, \text{ for } i \neq j.\]

The points \( z_j \in \Omega, j = 1, \ldots, m \), which determine the location of the inhomogeneities, are assumed to satisfy the following inequalities:

\[(40) \quad |z_j - z_i| \geq c_0 > 0, \forall j \neq l \quad \text{and} \quad \text{dist}(z_j, \partial \Omega) \geq c > 0, \forall j,\]

where \( c \) is a positive constant. Assume that \( \alpha > 0 \), the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to \( \mathbb{R}^3 \setminus \overline{\Omega} \) is larger than \( c \). Let \( \Gamma \subset \partial \Omega \) be a given open subset of \( \partial \Omega \). Let \( n(x) \in C^2(\Omega) \) denote the unperturbed refractive index. We assume that \( n(x) \) is known on a neighborhood of the boundary \( \partial \Omega \). Let \( n_j(x) \in C^2(z_j + \alpha B_j) \) denote the refractive index of the \( j \)-th inhomogeneity, \( z_j + \alpha B_j \). Introduce the perturbed refractive index

\[(41) \quad n_\alpha(x) = \begin{cases} n(x), & x \in \Omega \setminus \overline{B}_\alpha, \\ n_j(x), & x \in z_j + \alpha B_j, \ j = 1 \ldots m. \end{cases}\]
Let us introduce the (perturbed) Maxwell equations in the presence of the inhomogeneities $\mathcal{B}_\alpha$

\begin{align}
\text{(curl curl} - k^2 n_\alpha) E_\alpha = 0 \quad \text{in } \Omega \\
\nu \times E_\alpha = f \in \overline{TH}^\perp_{\text{div}}(\Gamma), \quad \text{on } \partial \Omega
\end{align}

and define the local impedance map associated to $n_\alpha$ by $Z_{n_\alpha}(f) = \text{curl } E_\alpha \times \nu |_{\Gamma}$ for all $f \in \overline{TH}^\perp_{\text{div}}(\Gamma)$. Let $E$ denote the solution to the Maxwell equations with the boundary condition $E \times \nu = f$ on $\partial \Omega$ in the absence of any inhomogeneities and $Z_n$ be the local impedance map associated to $n$.

**Hypothesis 2** Throughout this section we suppose that: the constant $k^2 = \omega^2 \varepsilon_0 \mu_0$ is such that the natural weak formulation of the problem (42), in the absence of any inhomogeneities, has a unique solution.

The goal in this section is to identify efficiently, by using Theorem 2, the locations $\{z_j\}_{j=1}^m$ of the small inhomogeneities $\mathcal{B}_\alpha$ from the knowledge of the difference between the local impedance maps $Z_{n_\alpha} - Z_n$ on $\Gamma$.

Let $V$ be any function in $\tilde{S}(\Omega)$, where $\tilde{S}(\Omega)$ is given by Lemma 2. Then by referring to [3], the following asymptotic formula (we shall not detail the proof, but we refer to the reference so quoted for closely techniques concerning a magnetic field $H_\alpha$) can be derived:

**Theorem 3** Suppose (40), (41) and Hypothesis 2 are satisfied. There exists $0 < \alpha_0$ such that, given an arbitrary $f \in \overline{TH}^\perp_{\text{div}}(\Gamma)$, and any $0 < \alpha < \alpha_0$, the boundary value problem (42) has a unique (weak) solution $E_\alpha$. The constant $\alpha_0$ depends on the domains $B_j, \Omega$, the constants and the number $m$, but is otherwise independent of the points $z_j; j = 1, \cdots, m$. Let $E$ denote the unique (weak) solution to the boundary value problem (42), in the absence of any inhomogeneities. Then, for $V \in \tilde{S}(\Omega)$ we have:

\begin{align}
\int_{\Gamma} (Z_{n_\alpha}(E_\alpha \times \nu) \cdot V - E_\alpha \cdot Z_n(V \times \nu)) ds(x) = \alpha^3 \sum_{j=1}^m |n(z_j)| - n_j(z_j)(M_j E(z_j)) V(z_j) + o(\alpha^4),
\end{align}

where $M_j = (m_{p,q}^j)_{1 \leq p,q \leq 3}$ is a $3 \times 3$ positive, symmetric, definite matrix (called the (rescaled) polarization tensor of the inhomogeneity set $B_j$) and the remainder $o(\alpha^4)$ is independent of the set of points $\{z_j\}_{j=1}^m$.

**Proof.** The existence and uniqueness of solution to problem (42) is completely fixed in [3], when the solution is a magnetic field $H_\alpha$. Concerning our work here, one can use the well-known relation (50) to justify also the existence and uniqueness (weakly) of solution to problem (42) for $E_\alpha$.

We focus our attention, now, to justify (40). Regarding Theorem 1 in [3], the authors developed an asymptotic formula concerning the perturbation, $(H_\alpha - H_0) \times \nu |_{\partial \Omega}$, in the (tangential) boundary magnetic field, caused by the presence of the inhomogeneities ($\alpha \to 0$). Based on (50), we may write

\begin{align}
H \times \nu = \frac{1}{ij\omega} \text{curl } E \times \nu, \quad \text{on } \partial \Omega.
\end{align}

As we said before that we don’t give a detail to the proof of this theorem. But, we may insert relation (44) into the formula provided by Theorem 1 in [3] p.774, and use $\nu \times (E_\alpha \times \nu)$ as the projection of $E_\alpha$ onto the tangent plane of $\partial \Omega$. Thus, by using a vector triple product and by assumption in this paper that the permeability $\mu = \mu_0$ (fixed), we can rescale the polarization tensor and we may simplify the formula in the reference by using the definitions of both $Z_{n_\alpha}$. 

and $Z_n$ to get precisely (43).

In order to get simple equations for the unknown parameters, namely, for the points \{$z_j\}_{j=1}^m$ and the values \{$n_j(z_j)\}_{j=1}^m$, we may make suitable choices for the test functions $V$ in $\tilde{S}(\Omega)$. Similar idea was used in the literature, and the associated numerical experiments have been successfully conducted in the case of the (piecewise constant) conductivity problem with boundary measurements on all of $\partial \Omega$. According to Section 3, we may define

$$\Lambda(\xi_1, \xi_2) = \int_{\Gamma} (Z_n + N_{\xi_1})^{-1}((A + G)e^{x\xi_1}|\Gamma) \cdot (Z_n - Z_n)((Z_n$$

$$+ N_{\xi_2})^{-1}((A + G)e^{x\xi_2}|\Gamma)) \, ds(x).$$

From Theorem 2 and Sobolev’s embedding theorem we can take

$$E(z_j) = e^{z_j \xi_1}(\eta_1 + \Psi_n(x, \xi_1))$$

$$V(z_j) = e^{z_j \xi_2}(\eta_2 + \Psi_n(x, \xi_2))$$

with $\eta_1 \cdot \eta_2 = 1$ and $\xi_1 + \xi_2 = il$ (for $i^2 = -1$) to obtain from (43) that

$$\Lambda(\xi_1, \xi_2) = \int_{\Gamma} (Z_n, (E \times \nu) \cdot V - E \cdot Z_n(V \times \nu)) \, ds = \alpha^3 \sum_{j=1}^m (n(z_j)$$

$$- n_j(z_j)) M^j e^{ilz_j} + o(\alpha^4).$$

Then, by neglecting the remainders $o(\alpha^4)$ in (46) we may achieve the proof of the following result.

**Corollary 1** Suppose that we have all hypothesis of Theorem 3. Let $\Lambda(\xi_1, \xi_2)$ be defined by (45). Then, the locations \{$z_j\}_{j=1}^m$ are obtained as supports of the inverse Fourier transform of $\Lambda(\xi_1, \xi_2)$.

Finally, it follows from Corollary 1 that the centers \{$z_j: j = 1, \ldots, m\}$ can be recovered easily, and therefore the values $n_j(z_j)$ (for $j = 1, \ldots, m$) could be obtained by solving a linear system arising from (46). The extension to general geometries, and/or to anisotropic domain, would allow us to deal with real-life applications. This may be considered in further paper.

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