An explicit $P_1$ finite-element scheme for Maxwell’s equations with constant permittivity in a boundary neighborhood

L. Beilina *  V. Ruas †

December 25, 2019

Abstract

This paper is devoted to the complete convergence study of the approximation of Maxwell’s equations in terms of the sole electric field, by means of standard linear finite elements for the space discretization, combined with a well-known explicit finite-difference scheme for the time discretization. The analysis applies to the particular case where the dielectric permittivity has a constant value outside a sub-domain, whose closure does not intersect the boundary of the domain where the problem is defined. Optimal convergence results in natural norms are demonstrated under reasonable assumptions, provided a classical CFL condition holds.

Keywords CFL condition, explicit scheme, mass-lumping, Maxwell’s equations, $P_1$ finite elements

1 Motivation

The aim of this article is to present the convergence analysis of a $P_1$ lumped-mass FE scheme to solve hyperbolic Maxwell’s equations for the electric field in a domain of $\mathbb{R}^n$, $n = 2, 3$, with constant dielectric permittivity in a neighborhood of its boundary. The technique of analysis follows the main lines of [38, 40]. However the problem at hand is considerably more complex than those addressed in both books, in particular owing to its vector nature and the variable coefficient. The latter is the main reason that led the authors to carry out in detail the analysis of the $P_1$ lumped-mass approximation. Indeed, in spite of studies on this topic for other FEM (see e.g. [19]), to the best of the authors’ knowledge, this case had not yet undergone a rigorous analysis.

The standard continuous $P_1$ FEM is a tempting possibility to solve Maxwell’s equations due to its simplicity. However this method is not always well suited for this purpose. The first reason for this is the fact that in general the natural function space for the electric field is not the subspace of Sobolev space $[H^1]^n$ incorporating boundary conditions on tangential components, but rather the corresponding subspace of $H(\text{curl}) \cap H(\text{div})$. Indeed, if the domain in which an electric field with zero tangential components on its boundary is searched for has re-entrant corners, the subspace of

*Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-42196 Gothenburg, Sweden, e-mail: larisa@chalmers.se
†Institut Jean Le Rond d’Alembert, UMR 7190 CNRS - Sorbonne Université, F-75005, Paris, France, e-mail: vitoriano.ruas@upmc.fr
\([H^1]_\Omega \cap (H(\text{curl}) \cap H(\text{div}))\) incorporating such boundary conditions is a proper subspace of the corresponding subspace of \(H(\text{curl}) \cap H(\text{div})\). This is due to the so-called corner paradox (see e.g. [20], [21]). Another issue difficult to overcome with continuous Lagrange FE is precisely the prescription of zero tangential-components of the electric field on the boundary of a polytopic domain, which occur in many important applications. All this motivated the proposal by Nédélec about four decades ago of a family of \(H(\text{curl})\)-conforming methods to solve these equations (cf. [35]) also known as edge elements. These methods are still widely in use, as much as similar approaches well adapted to the above conditions (see e.g. [1], [36], [16] and [26]). New tools allowing for a finer reliability analysis of such methods are given in [25]. A comprehensive description of FEM for Maxwell’s equations can be found in [32]. Another approach to handle the static case was proposed in [39].

More recently specialists studied formulations of the static or the time-harmonic Maxwell’s equations suitable for a numerical solution with nodal elements rather than edge elements. In this respect we refer to [22], [27], [18], [2]. Such an evolution seems to be motivated by the fact that the \(P_1\) finite elements provide an inexpensive and reliable way to solve the time-dependent Maxwell’s equations, at least in some relevant practical situations. In this work we consider one such a case, characterized by the fact that the dielectric permittivity is constant in a neighborhood of the whole boundary of the domain of interest. This is because, at least in theory, whenever the dielectric permittivity is constant, the Maxwell’s equations simplify into as many wave equations as the space dimension under consideration. On the other hand we should emphasize that the study of nodal elements as applied to the time-dependent case is rather incipient. Actually this paper attempts to give a valid contribution to its field, by considering at a time all the aforementioned situations. More precisely a rigorous analysis is carried out, thereby establishing that a space discretization by means of conforming linear elements combined with an explicit mass lumping finite-difference scheme for the time discretization, gives rise to optimal approximations of the electric field, as long as a classical CFL condition is fulfilled. This numerical recipe is applied to a very simple augmented formulation of the Maxwell’s equations, as compared to those quoted above.

As a matter of fact this work can be viewed as both a continuation and the completion of studies presented in [3, 5] for a combination a the finite difference method in a sub-domain with constant permittivity with the finite element method in the complementary sub-domain. As pointed out above, the Maxwell’s equations reduces to the wave equation in the former case. Since the analysis of finite-difference methods for this type of equation is well established, only an explicit \(P_1\) finite element scheme for Maxwell’s equations is analyzed in this paper.

In [3, 5] a stabilized domain-decomposition finite-element/finite-difference approach for the solution of the time-dependent Maxwell’s system for the electric field was proposed and numerically verified. In these works [3, 5] different manners to handle a divergence-free condition in the finite-element scheme were considered. The main idea behind the domain decomposition methods in [3, 5] is that a rectangular computational domain is decomposed into two sub-domains, in which two different types of discretizations are employed, namely, the finite-element domain in which a classical \(P_1\) finite element discretization is used, and the finite-difference domain, in which the standard five- or seven-point finite difference scheme is applied, according to the space dimension. The finite element domain lies strictly inside the finite difference domain, in such a way that both domains overlap in two layers of structured nodes. First order absorbing boundary conditions [24]
are enforced on the boundary of the computational domain, i.e. on the outer boundary of the finite-difference domain. In [3, 5] it was assumed that the dielectric permittivity function is strictly positive and has a constant value in the overlapping nodes as well as in a neighborhood of the boundary of the domain. An explicit scheme was used both in the finite-element and finite-difference domains.

We recall that for a stable finite-element solution of Maxwell’s equation divergence-free edge elements are the most satisfactory from a theoretical point of view [32, 35]. However, the edge elements are less attractive for solving time-dependent problems, since a linear system of equations should be solved at every time iteration. In contrast, $P_1$ elements can be efficiently used in a fully explicit scheme with lumped-mass $P_1$ finite elements [23, 31]. On the other hand it is also well known that the numerical solution of Maxwell’s equations with nodal finite elements can result in unstable spurious solutions [33, 37]. Nevertheless a number of techniques are available to remove them, and in this respect we refer for example to [28, 29, 30, 34, 37]. In this work, similarly to [3, 5], this is achieved by adding a divergence-free condition term to the model equation for the electric field. Numerical tests given in [5] demonstrate that spurious solutions are removable indeed, in case an explicit $P_1$ finite-element solution scheme is employed.

Efficient usage of an explicit scheme with $P_1$ finite elements for the solution of coefficient inverse problems (CIPs), in the particular context described above was made evident in [6]. In many algorithms aimed at solving electromagnetic CIPs, a qualitative collection of experimental measurements is necessary on the boundary of a computational domain, in order to determine the dielectric permittivity function therein. In this case, in principle the numerical solution of the time-dependent Maxwell’s equations is required in the entire space $\mathbb{R}^3$ (see e.g. [6, 7, 8, 10, 11, 12], but instead it can be more efficient to consider Maxwell’s equations with a constant dielectric permittivity in a neighborhood of the boundary of a computational domain. The explicit scheme with a $P_1$ finite element space discretization considered in this work was numerically tested in the solution of the time-dependent Maxwell’s system in both two- and three-dimensional geometry (cf. [5]). It was also combined with a few algorithms to solve different CIPs for determining the dielectric permittivity function in connection with the time-dependent Maxwell’s equations, using both simulated and experimentally generated data (see [7, 8, 10, 11, 12]). The formal reliability analysis of such a method conducted here, corroborates the previously observed adequacy of this numerical approach.

An outline of this paper is as follows: In Section 2 we describe in detail the model problem being solved and give its equivalent variational form. In Section 3 we set up the discretizations of the model problem in both space and time. In Section 4 the stability analysis of the explicit scheme considered in the previous section is carried out while the corresponding consistency study is conducted in Section 5. Next we combine the results of the two previous sections to prove error estimates in Section 6. Underlying convergence results under the realistic assumption that the time step varies linearly with the mesh size as the meshes are refined are thus established. Finally in Section 7 some aspects of the reliability analysis are discussed and conclusions are drawn.

2 The model problem

In the case of a constant magnetic permeability the Maxwell’s equations of electromagnetism can be expressed in terms of the sole electric field $\mathbf{e} = (e_1, e_2, e_3)$. This will be the case of the problem studied in this work, defined in a bounded domain $\Omega$ of $\mathbb{R}^3$ with boundary $\partial\Omega$. 

3
First we consider that $\Omega = \Omega_{in} \cup \Omega_{out}$, where $\Omega_{in}$ is an interior open set whose boundary does not intersect $\partial \Omega$ and $\Omega_{out}$ is the complementary set of $\Omega_{in}$ with respect to $\Omega$. Being the unit outer normal vector on $\partial \Omega$, we denote by $\partial_{n}(\cdot)$ the outer normal derivative of a field on $\partial \Omega$.

We address here the case where $e$ satisfies absorbing boundary conditions, by assuming that $\Omega$ has no re-entrant corners, in which case corner singularities do not occur. The same hypothesis would be needed for Neumann boundary conditions (but not for Dirichlet). As far as the authors can see, the latter are of academic interest only. On the other hand, as pointed out in Section 1, absorbing boundary conditions correspond to situations addressed in [7, 8, 10, 11, 12]. Anyway, it is rather easy to see that the reliability analysis carried out in the sequel can be simplified to encompass other "uncoupled" boundary conditions, among which lie Dirichlet and Neumann conditions, irrespective of their practical meaning.

Now in case $e$ satisfies absorbing boundary conditions, let a pair of continuous functions $(e_0; e_1)$ be given in $[H^1(\Omega)]^3$, satisfying $\nabla \cdot (\varepsilon e_0) = \nabla \cdot (\varepsilon e_1) = 0$, where $\varepsilon$ is the dielectric permittivity. $\varepsilon$ is assumed to belong to $C^2(\Omega)$ and to fulfill $\varepsilon \equiv 1$ in $\Omega_{out}$ and $\varepsilon \geq 1$. Given $f \in \{L^2([0, T); L^2(\Omega))\}^3$ such that $\nabla \cdot f = 0 \\forall t \in (0, T)$, we wish to find an electric field $e \in \mathcal{V} := \{H^2([0, T); L^2(\Omega)) \cap L^2([0, T); H^1(\Omega))\}^3$ satisfying,

\[
\begin{align*}
\varepsilon \partial_{tt} e + \nabla \times \nabla \times e &= 0 & \text{in } \Omega \times (0, T), \\
\nabla \cdot (\varepsilon e) &= 0 & \text{in } \Omega \times (0, T), \\
e(\cdot, 0) &= e_0(\cdot), \text{ and } \partial_t e(\cdot, 0) &= e_1(\cdot) & \text{in } \Omega, \\
\partial_n e &= -\partial_{t} e & \text{on } \partial \Omega \times (0, T).
\end{align*}
\]

**Remark 2.1.** The assumption that $\varepsilon$ attains a minimum in an outer layer is not essential for our numerical method to work. However, as far as we can see, it is a condition that guarantees optimal convergence results. In the Section 7 a more elaborated discussion on this issue can be found. ■

We next consider a variational formulation for (2.1).

Let us denote the standard inner product of $[L^2(\Omega)]^M$ by $(\cdot, \cdot)$ for $M \in \{1, 2, 3\}$ and the corresponding norm by $\| \cdot \|$. Similarly we denote by $(\cdot, \cdot)_{\partial \Omega}$ the standard inner product of $[L^2(\partial \Omega)]^M$ and the associated norm by $\| \cdot \|_{\partial \Omega}$. Further, for a given non negative function $\omega \in L^\infty(\Omega)$ we introduce the weighted $L^2(\Omega)$-semi-norm $\| \cdot \|_{\omega} := \sqrt{\int_{\Omega} \omega \| \cdot \|^2 dx}$, which is actually a norm if $\omega \neq 0$ everywhere in $\tilde{\Omega}$. We also introduce, the notation $(A, B)_{\omega} := \int_{\Omega} \omega A \cdot B dx$ for two fields $A, B$ which are square integrable in $\Omega$. Notice that if $\omega$ is strictly positive this expression defines an inner product associated with the norm $\| \cdot \|_{\omega}$.

Now requiring that $\tilde{e}_{|t=0} = e_0$ and $\{\partial_t \tilde{e}\}_{|t=0} = e_1$, we wish to possibly find $\tilde{e}$ in the space $\mathcal{V}$ defined in (2.1), such that $\forall \nu \in [H^1(\Omega)]^3$ it holds,

\[
(\partial_{tt} \tilde{e}, \nu)_{\varepsilon} + (\nabla \tilde{e}, \nabla \nu) + (\nabla \cdot (\varepsilon \tilde{e}), \nabla \cdot \nu) - (\nabla \cdot \tilde{e}, \nabla \cdot \nu) + (\partial_t \tilde{e}, \nu)_{\partial \Omega} = (f, \nu) \\forall t \in (0, T). \tag{2.2}
\]

**Proposition 2.1.** Problem (2.2) has a unique solution, namely, the solution $e$ to Maxwell’s equations (2.1).

**Proof:** First we use the well known operator identity $\nabla \times \nabla \equiv -\nabla^2 + \nabla \cdot$ to rewrite the first equation of (2.1) as,

\[
\varepsilon \partial_{tt} e - \nabla^2 e + \nabla \cdot e = f \text{ in } \Omega \times (0, T). \tag{2.3}
\]
Now we subtract from (2.3) the second equation of (2.1) and take the $L^2$ inner product of each side of the resulting equation with an arbitrary $v \in [H^1(\Omega)]^3$. Then using integration by parts and taking into account that $\nabla \cdot e - \nabla \cdot \{\varepsilon e\} = 0$ and $-\partial_t e = \partial_h e$ on $\partial \Omega$, we readily obtain:

$$(\partial_t e, v) + (\nabla e, \nabla v) - (\nabla \cdot e, \nabla \cdot v) + (\nabla \cdot \{\varepsilon e\}, \nabla \cdot v) + (\partial_t e, v)_{\partial \Omega} = (f, v) \forall t \in (0, T).$$

(2.4)

This establishes that $\tilde{e} = e$ solves (2.2).

In order to prove the uniqueness of this solution, we resort to the following energy estimate derived in [5] for the solution of (2.2):

$$||\partial_t \tilde{e}||_2^2 + ||\nabla \tilde{e}||^2 + ||\nabla \cdot \tilde{e}||_{L^2}^2 \leq \mathcal{E} \left\{ \int_0^T ||f(\cdot, t)||^2 dt + ||e_0||^2 + ||e_1||^2 \right\},$$

(2.5)

where $\mathcal{E}$ is a constant independent of $f, e_0$, and $e_1$.

The uniqueness follows from (2.5) owing to the linearity of problem (2.2). □

3 Space-time discretization

Henceforth, for the sake of simplicity, we assume that $\Omega$ is a polyhedron. Moreover, throughout this article we denote the standard semi-norm of both $W^{m,\infty}(\Omega)$ and $C^m(\Omega)$ by $| \cdot |_{m,\infty}$ for $m > 0$ and the standard norm of either $L^\infty(\Omega)$ or $C^0(\Omega)$ by $\| \cdot \|_\infty$.

3.1 Space semi-discretization

Let $V_h$ be the usual $P_1$ FE-space of continuous functions related to a mesh $T_h$ fitting $\Omega$, consisting of tetrahedrons with maximum edge length $h$, belonging to a quasi-uniform family of meshes (cf. [15]). Each element $K \in T_h$ is to be understood as a closed set.

Setting $V_h := [V_h]^3$ we define $e_{0h}$ (resp. $e_{1h}$) to be the usual $V_h$-interpolate of $e_0$ (resp. $e_1$). Then the space semi-discretized problem to solve is

Find $e_h \in V_h$ such that $\forall v \in V_h$

$$(\partial_t e_h, v) + (\nabla e_h, \nabla v) - (\nabla \cdot e_h, \nabla \cdot v) - (\partial_t e_h, v)_{\partial \Omega} = (f, v),$$

(3.6)

$$e_h(\cdot, 0) = e_{0h}(\cdot) \text{ and } \partial_t e_h(\cdot, 0) = e_{1h}(\cdot) \text{ in } \Omega.$$

Remark 3.1. Although we do not need to handle problem (3.6) in the sequel, we can assert that it is well-posed. Indeed it is nothing but the system of second order ordinary differential equations $\tilde{e}_{tt} + B(\varepsilon)\tilde{e} = \tilde{f}$ with initial conditions $\tilde{e}_{t=0} = \tilde{e}_0$ and $\tilde{e}_{tt=0} = \tilde{e}_1$, where $\tilde{e}(t)$ is the vector of nodal values of $e_h(x, t)$ for $t \in [0, T]$, $B(\varepsilon)$ is a time-independent positive definite matrix, $\tilde{e}_0$ and $\tilde{e}_1$ are the vectors of nodal values of $e_0$ and $e_1$ respectively, and $\tilde{f}$ is a time-dependent vector derived from $f$. □
3.2 Full discretization

To begin with we consider a natural centered time-discretization scheme to solve (3.6), namely:

Given a number \( N \) of time steps we define the time increment \( \tau := T/N \). Then defining \( f^k \) to be \( f \) at time \( t = k\tau \), we approximate \( e_h(k\tau) \) by \( e_h^k \) for \( k = 1, 2, ..., N \) according to the following FE scheme for \( k = 1, 2, ..., N - 1 \):

\[
\begin{align*}
&\left( \frac{e_h^{k+1} - 2e_h^k + e_h^{k-1}}{\tau^2}, v \right) + \left( \nabla e_h^k, \nabla v \right) + \left( \nabla \cdot e_h^k, \nabla \cdot v \right) - \left( \nabla \cdot e_h^k, \nabla \cdot v \right) \\
&+ \left( \frac{e_h^{k+1} - e_h^{k-1}}{2\tau}, v \right)_{\partial \Omega} = (f, v) \quad \forall v \in V_h, \\
&\quad e_h^0 = e_{0h} \text{ and } e_h^1 = e_h^0 + \tau e_{1h} \text{ in } \Omega.
\end{align*}
\]  

(3.7)

Owing to its coupling with \( e_h^k \) and \( e_h^{k-1} \) on the left hand side of (3.7), \( e_h^{k+1} \) cannot be determined explicitly by (3.7) at every time step. In order to enable an explicit solution we resort to the classical mass-lumping technique. We recall that for a constant \( \varepsilon \) this consists of replacing on the left hand side the inner product \((u, v)_\varepsilon \) (resp. \((u, v)_{\varepsilon, h} \)) by an inner product \((u, v)_{\varepsilon, h} \) (resp. \((u, v)_{\partial \Omega, h} \)) using the trapezoidal rule to compute the integral of \( \int_K \varepsilon u_K \cdot v_K \, dx \) (resp. \( \int_{K \cap \partial \Omega} u_K \cdot v_K \, dS \)), for every element \( K \) in \( \mathcal{T}_h \), where \( u \) stands for \( e_h^{k+1} - 2e_h^k + e_h^{k-1} \) (resp. \( e_h^{k+1} - e_h^{k-1} \)). It is well-known that in this case the matrix associated with \((\varepsilon e_h^{k+1}, v)_h \) (resp. \((e_h^{k+1}, v)_{\partial \Omega, h} \)) for \( v \in V_h \), is a diagonal matrix. In our case \( \varepsilon \) is not constant, but the same property will hold if we replace in each element \( K \) the integral of \( \varepsilon u_K \cdot v_K \) in a tetrahedron \( K \in \mathcal{T}_h \) or of \( u_F \cdot v_F \) in a face \( F \subset \partial \Omega \) of a certain tetrahedron \( K \in \mathcal{T}_h \) as follows:

\[
\begin{align*}
\int_K \varepsilon u_K \cdot v_K \, dx & \approx \varepsilon(G_K) \text{ volume}(K) \sum_{i=1}^{4} \frac{u(S_{K,i}) \cdot v(S_{K,i})}{4}, \\
\int_{F \subset \partial \Omega} u_F \cdot v_F \, dS & \approx \text{area}(F \subset \partial \Omega) \sum_{i=1}^{3} \frac{u(R_{F,i}) \cdot v(R_{F,i})}{3},
\end{align*}
\]

where \( S_{K,i} \) are the vertexes of \( K, i = 1, 2, 3, 4, G_K \) is the centroid of \( K \) and \( R_{F,i} \) are the vertexes of a face \( F \subset \partial \Omega \) of certain tetrahedrons \( K \in \mathcal{T}_h, i = 1, 2, 3 \).

Before pursuing we define the auxiliary function \( \varepsilon_h \) whose value in each \( K \in \mathcal{T}_h \) is constant equal to \( \varepsilon(G_K) \). Furthermore we introduce the norms \( \| \cdot \|_{\varepsilon,h} \) and \( \| \cdot \|_h \) of \( V_h \), given by \( (\cdot, \cdot)_{\varepsilon,h}^{1/2} \) and \( (\cdot, \cdot)_h^{1/2} \), respectively. Similarly we denoted by \( \| \cdot \|_{\partial \Omega,h} \) the norm defined by \( (\cdot, \cdot)_{\partial \Omega,h}^{1/2} \). Then still denoting the approximation of \( e_h(k\tau) \) by \( e_h^k \), for \( k = 1, 2, ..., N \) we
determine \( e_h^{k+1} \) by,
\[
\begin{align*}
&\left( e_h^{k+1} - 2e_h^k + e_h^{k-1}, \nu \right)_{\varepsilon,h} + (\nabla e_h^k, \nabla \nu) + (\nabla \cdot \varepsilon e_h^k, \nabla \cdot \nu) - (\nabla \cdot e_h^k, \nabla \cdot \nu) \\
&+ \left( e_h^{k+1} - e_h^{k-1}, \nu \right)_{\partial \Omega,h} = (f^k, \nu) \quad \forall \nu \in V_h,
\end{align*}
\]
(3.8)

\( e_0^h = e_{0h} \) and \( e_1^h = e_{0h}^0 + \tau e_{1h} \) in \( \Omega \).

Now we recall a result given in Lemma 3 of [13], which allows us to assert that the norm \( \| v \|_{\varepsilon,h} \) is bounded above by \( \| v \|_{\varepsilon,h,h} \) \( \forall v \in V_h \). In order to prove such a result we use the barycentric coordinates \( \lambda_{K,i} \) of tetrahedron \( K \in T_h, i = 1, 2, 3, 4 \). We have \( u|_K = \sum_{i=1}^4 u(S_{K,i})\lambda_{K,i} \). Since
\[
\int_K \varepsilon u_k^2 dx \leq \varepsilon(G_K)\text{volume}(K)\left(1 + \delta_{i,j}\right)/20,
\]
after straightforward manipulations we obtain,
\[
\int_K \varepsilon u_k^2 dx \leq \varepsilon(G_K)\text{volume}(K)\sum_{i=1}^4 u^2(S_{K,i}).
\]
This immediately implies that
\[
\| v \|_{\varepsilon,h} \leq \| v \|_{\varepsilon,h,h} \quad \forall v \in V_h.
\]
(3.9)

For the same reason we have,
\[
\| v \|_{\partial \Omega} \leq \| v \|_{\partial \Omega,h} \quad \forall v \in V_h.
\]
(3.10)

It is noteworthy that problem (3.8) has a unique solution for every \( k \).

4 Stability analysis

In order to conveniently prepare the subsequent steps of the reliability study of scheme (3.8), following a technique thoroughly exploited in [38], we carry out the stability analysis of a more general form thereof, namely:
\[
\begin{align*}
&\left( e_h^{k+1} - 2e_h^k + e_h^{k-1}, \nu \right)_{\varepsilon,h} + (\nabla e_h^k, \nabla \nu) + (\nabla \cdot \varepsilon e_h^k, \nabla \cdot \nu) - (\nabla \cdot e_h^k, \nabla \cdot \nu) \\
&+ \left( e_h^{k+1} - e_h^{k-1}, \nu \right)_{\partial \Omega,h} = F^k(\nu) + (d^k, \nabla \cdot \nu) + G^k(\nu) \quad \forall \nu \in V_h,
\end{align*}
\]
(4.11)

\( e_0^h = e_{0h} \) and \( e_1^h = e_{0h}^0 + \tau e_{1h} \) in \( \Omega \).

where for every \( k \in \{1, 2, \ldots, N - 1\} \), \( F^k \) and \( G^k \) are given bounded linear functionals over \( V_h \) and the space of traces over \( \partial \Omega \) of fields in \( V_h \) equipped with the norms \( \| \cdot \|_h \) and \( \| \cdot \|_{\partial \Omega,h} \) respectively. We denote by \( | F^k |_h \) and \( | G^k |_{\partial \Omega,h} \) the underlying norms of both functionals. \( d^k \) in turn is a
given function in $L^2(\Omega)$ for $k \in \{1, 2, \ldots, N - 1\}$.

Rigorous reliability analyses of the approximation of time-dependent problems by lumpingmass finite element schemes have been carried out by several authors since long. A celebrated work on the subject for linear elements is [14]. However the case of variable coefficients in the time-derivative term seems to have been overlooked. That is why in this work we endeavor to address it in detail, in the framework of Maxwell’s equation. Of course the technique employed below can be applied as such to other problems with variable coefficients. Now taking $v = e_h^{k+1} - e_h^k$ in (4.11) we get for $k = 1, 2, \ldots, N - 1$,

$$
\left( \frac{e_h^{k+1} - 2e_h^k + e_h^{k-1}}{\tau}, e_h^{k+1} - e_h^k \right) + (\nabla e_h^k, \nabla e_h^{k+1} - \nabla e_h^k) \\
+ \left( \nabla \cdot (\varepsilon - 1) e_h^k, \nabla \cdot e_h^{k+1} - \nabla \cdot e_h^k \right) + \left( \frac{e_h^{k+1} - e_h^k}{2\varepsilon}, e_h^{k+1} - e_h^k \right)_{\partial \Omega, h}
$$

(4.12)

Noting that $e_h^{k+1} - 2e_h^k + e_h^{k-1} = (e_h^{k+1} - e_h^k) - (e_h^k - e_h^{k-1})$ and that $e_h^{k+1} - e_h^k = (e_h^{k+1} - e_h^{k-1}) + (e_h^k - e_h^{k-1})$, the following estimate trivially holds for equation (4.11):

$$
\left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\|^2_{\varepsilon, h} - \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|^2_{\varepsilon, h} + I_1 + I_2 + 2\varepsilon \left\| \frac{e_h^{k+1} - e_h^k}{2\varepsilon} \right\|^2_{\partial \Omega, h}

\leq |F|^k_h \left\| e_h^{k+1} - e_h^k \right\|_{h} + \|d^k\| \|\nabla \cdot (e_h^{k+1} - e_h^k)\|

+ |G|^k_{\partial \Omega, h} \left\| e_h^{k+1} - e_h^k \right\|_{\partial \Omega, h}

(4.13)

where

$I_1 := (\nabla e_h^{k+1}, \nabla \{e_h^{k+1} - e_h^k\})$; and

$I_2 := (\nabla \cdot (\varepsilon - 1) e_h^k, \nabla \cdot \{e_h^{k+1} - e_h^k\})$.

Next we estimate the terms $I_1$ and $I_2$ given by (4.13).

First of all it is easy to see that

$$
I_1 = \frac{1}{2} (\|\nabla e_h^{k+1}\|^2 + \|\nabla e_h^k\|^2 - \|\nabla (e_h^{k+1} - e_h^k)\|^2) - \frac{1}{2} (\|\nabla e_h^k\|^2 + \|\nabla e_h^{k-1}\|^2 - \|\nabla (e_h^k - e_h^{k-1})\|^2).

(4.14)

Next we note that $I_2 = J_1 + J_2$ where,

$$
J_1 := (\nabla \cdot e_h^k, \nabla \cdot \{e_h^{k+1} - e_h^k\})_{\varepsilon - 1}
$$

and

$$
J_2 := (\nabla \varepsilon \cdot e_h^k, \nabla \cdot \{e_h^{k+1} - e_h^k\})

(4.15)
Similarly to (4.14) we can write,
\[
J_1 = \frac{1}{2}\left(\|\nabla \cdot e_k^{k+1}\|_{L^2\cap \partial\Omega}^2 + \|\nabla \cdot e_k^k\|_{L^2\cap \partial\Omega}^2 - \|D \cdot (e_h^{k+1} - e_h^k)\|_{L^2\cap \partial\Omega}^2\right) \quad (4.16)
\]

Now observing that \(\nabla \varepsilon \equiv 0\) on \(\partial \Omega\), we integrate by parts \(J_2\) given by (4.15), to get
\[
J_2 = - (\nabla \{\nabla \cdot e_h^k, e_h^{k+1} - e_h^k\}). \quad (4.17)
\]

Let us rewrite \(J_2\) as \(J_2 = M_1 + M_2\) where,
\[
M_1 := - (\nabla \nabla \varepsilon e_h^k, e_h^{k+1} - e_h^k)
\]
and
\[
M_2 := - (\{\nabla e_h^k\}^T \nabla \varepsilon, e_h^{k+1} - e_h^k) \quad (4.18)
\]

\(M_1\) in turn can be rewritten as \(M_1 = N_1 + N_2\) where,
\[
N_1 := - \tau \left(\nabla \nabla \varepsilon e_h^k, \frac{e_h^{k+1} - e_h^k}{\tau}\right)
\]
and
\[
N_2 := - \tau \left(\nabla \nabla \varepsilon e_h^k, \frac{e_h^k - e_h^{k-1}}{\tau}\right). \quad (4.19)
\]

Then we further observe that
\[
N_1 = - \tau^2 \sum_{i=1}^k \left(\nabla \nabla \varepsilon e_h^i, \frac{e_h^{i+1} - e_h^{i-1}}{\tau}\right) - \tau \left(\nabla \nabla \varepsilon e_h^0, \frac{e_h^{k+1} - e_h^k}{\tau}\right). \quad (4.20)
\]

and hence,
\[
N_1 \geq - \tau^2 \epsilon_{2,\infty} \left\|\frac{e_h^{k+1} - e_h^k}{\tau}\right\| \left\|\sum_{i=1}^k \frac{e_h^i - e_h^{i-1}}{\tau}\right\| - \tau \epsilon_{2,\infty} \left\|e_h^0\right\| \left\|\frac{e_h^{k+1} - e_h^k}{\tau}\right\|
\]
or yet,
\[
N_1 \geq - \tau \epsilon_{2,\infty} \left\|\frac{e_h^{k+1} - e_h^k}{\tau}\right\| \left\{\tau \sqrt{k} \left(\sum_{i=1}^k \left\|\frac{e_h^i - e_h^{i-1}}{\tau}\right\|^2\right)^{1/2} + \left\|e_h^0\right\|\right\},
\]
and noting that \(k \leq T/\tau\) we get
\[
N_1 \geq - \tau \epsilon_{2,\infty} \left\|\frac{e_h^{k+1} - e_h^k}{\tau}\right\| \left\{\frac{T}{\tau} \left(\sum_{i=1}^k \left\|\frac{e_h^i - e_h^{i-1}}{\tau}\right\|^2\right)^{1/2} + \left\|e_h^0\right\|\right\}. \quad (4.21)
\]
Applying to (4.21) Young’s inequality \( ab \leq \delta a^2 / 2 + b^2 / (2\delta) \) \( \forall a, b \in \mathbb{R} \) and \( \delta > 0 \) with \( \delta = 1 \), we easily conclude that

\[
N_1 \geq -\frac{\tau}{2}\varepsilon_2,\infty^2 \left( 2\left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\|^2 + \tau T \sum_{i=1}^{k} \left\| \frac{e_h^i - e_h^{i-1}}{\tau} \right\|^2 + \|e_h^0\|^2 \right). \tag{4.22}
\]

Similarly to (4.22),

\[
N_2 \geq -\frac{\tau}{2}\varepsilon_2,\infty^2 \left( 2\left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|^2 + \tau T \sum_{i=1}^{k} \left\| \frac{e_h^i - e_h^{i-1}}{\tau} \right\|^2 + \|e_h^0\|^2 \right). \tag{4.23}
\]

Combining (4.22) and (4.23) we come up with

\[
M_1 \geq -\tau\varepsilon_2,\infty^2 \left( \left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\|^2 + \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|^2 + L_k \right), \tag{4.24}
\]

where

\[
L_k := \tau T \sum_{i=1}^{k} \left\| \frac{e_h^i - e_h^{i-1}}{\tau} \right\|^2 + \|e_h^0\|^2.
\]

As for \( M_2 \) given by (4.18) we have:

\[
M_2 \geq -\varepsilon_1,\infty \left\| \nabla e_h^k \right\| \left\| e_h^{k+1} - e_h^{k-1} \right\| \geq -\tau\varepsilon_1,\infty \left\| \nabla e_h^k \right\| \left( \left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\| + \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\| \right)
\]

or yet

\[
M_2 \geq -\frac{\tau}{2}\varepsilon_1,\infty^2 \left( 2\left\| \nabla e_h^k \right\|^2 + \left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\|^2 + \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|^2 \right). \tag{4.25}
\]

Now we recall (4.13) together with (3.10) and notice that for every square-integrable field \( \mathbf{A} \) in \( \Omega \) we have \( \|\mathbf{A}\| \leq \|\mathbf{A}\|_{\varepsilon_A} \). Then taking into account that \( \|\nabla \cdot \mathbf{v}\| \leq \sqrt{3}\|\nabla \mathbf{v}\| \) \( \forall \mathbf{v} \in [H^1(\Omega)]^3 \), and using Young’s inequality with \( \delta = \tau \), \( \delta = 1/\tau \) and \( \delta = \tau \), respectively, we easily obtain the following estimates:

\[
|F|^h \|e_h^{k+1} - e_h^{k-1}\| \leq \frac{\tau}{2}|F|^2_h + \tau \left( \left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\| + \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|^2 \right),
\]

\[
|d^k| \|\nabla \cdot (e_h^{k+1} - e_h^{k-1})\| \leq \frac{3}{2\tau}|d|^2 + \tau(\|\nabla e_h^{k+1}\|^2 + \|\nabla e_h^{k-1}\|^2), \tag{4.26}
\]

\[
|G|^h_{\partial\Omega,h} \|e_h^{k+1} - e_h^{k-1}\|_{\partial\Omega,h} \leq \frac{\tau}{2}|G|^2_{\partial\Omega,h} + 2\tau \left\| \frac{e_h^{k+1} - e_h^{k-1}}{\tau} \right\|_{\partial\Omega,h}^2.
\]
where in the first and the second inequality we also used the fact that \( \|A \pm B\|^2 \leq 2(\|A\|^2 + \|B\|^2) \) for all square-integrable fields \( A \) and \( B \).

Now we collect (4.14), (4.15), (4.16), (4.17), (4.18) and (4.24), (4.25), (4.26) to plug them into (4.13). Using the fact that \( \| \cdot \| \leq \| \cdot \|_{\varepsilon h} \leq \| \cdot \|_{\varepsilon h,h} \) we obtain for \( 1 \leq k \leq N - 1 \):

\[
(A_k - B_k) + (C_k - D_k) \leq \frac{\tau}{2} \left| F^k \right|_{h}^2 + \frac{3}{2 \tau} \left\| d^k \right\|_{\partial\Omega}^2 + \frac{\tau}{2} \left| G^k \right|_{\partial\Omega}^2,
\]

where

\[
A_k := \left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\|_{\varepsilon h,h}^2 - \frac{\tau}{2} (2 + |\varepsilon|_{1,\infty} + 2|\varepsilon|_{2,\infty}) \left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\|_{\varepsilon h,h}^2 + \frac{1}{2} \left\{ (1 - 2\tau) \left\| \nabla e_h^{k+1} \right\|^2 + (1 - \tau|\varepsilon|_{1,\infty}) \left\| \nabla e_h^k \right\|^2 \right\}
\]

\[
B_k := \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|_{\varepsilon h,h}^2 + \frac{\tau}{2} (2 + |\varepsilon|_{1,\infty} + 2|\varepsilon|_{2,\infty}) \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|_{\varepsilon h,h}^2 + \frac{1}{2} \left\{ (1 + 2\tau) \left\| \nabla e_h^{k-1} \right\|^2 + (1 + \tau|\varepsilon|_{1,\infty}) \left\| \nabla e_h^k \right\|^2 \right\} + \tau|\varepsilon|_{2,\infty} L_k
\]

\[
C_k := -\frac{1}{2} \left\{ \left\| \nabla (e_h^{k+1} - e_h^k) \right\|^2 + \left\| \nabla \cdot (e_h^{k+1} - e_h^k) \right\|^2 \right\} + \frac{1}{2} \left( \left\| \nabla \cdot e_h^{k+1} \right\|^2_{\varepsilon_{-1}} + \left\| \nabla \cdot e_h^k \right\|^2_{\varepsilon_{-1}} \right)
\]

\[
D_k := -\frac{1}{2} \left\{ \left\| \nabla (e_h^k - e_h^{k-1}) \right\|^2 + \left\| \nabla \cdot (e_h^k - e_h^{k-1}) \right\|^2 \right\} + \frac{1}{2} \left( \left\| \nabla \cdot e_h^k \right\|^2_{\varepsilon_{-1}} + \left\| \nabla \cdot e_h^{k-1} \right\|^2_{\varepsilon_{-1}} \right).
\]

Setting

\[
\eta := 2 + |\varepsilon|_{1,\infty} + 2|\varepsilon|_{2,\infty};
\]

\[
\theta := |\varepsilon|_{1,\infty};
\]

\[
\rho := T^2|\varepsilon|_{2,\infty},
\]

we can rewrite \( A_k \) and \( B_k \) as follows:

\[
A_k := \left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\|_{\varepsilon h,h}^2 - \frac{\tau}{2} \left\| \frac{e_h^{k+1} - e_h^k}{\tau} \right\|_{\varepsilon h,h}^2 + \frac{1}{2} \left( \left\| \nabla e_h^{k+1} \right\|^2 + \left\| \nabla e_h^k \right\|^2 \right) - \tau \left( \left\| \nabla e_h^{k+1} \right\|^2_{\varepsilon_{-1}} + \frac{\theta}{2} \left\| \nabla e_h^k \right\|^2_{\varepsilon_{-1}} \right);
\]

\[
B_k := \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|_{\varepsilon h,h}^2 + \frac{\tau}{2} \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|_{\varepsilon h,h}^2 + \frac{1}{2} \left( \left\| \nabla e_h^{k-1} \right\|^2 + \left\| \nabla e_h^k \right\|^2 \right) + \tau \left( \left\| \nabla e_h^{k-1} \right\|^2_{\varepsilon_{-1}} + \frac{\theta}{2} \left\| \nabla e_h^k \right\|^2_{\varepsilon_{-1}} \right) + \frac{\tau \rho}{T^2} L_k
\]

Then we note that for \( 1 \leq k \leq m - 1 \) with \( 2 \leq m \leq N - 1 \),

\[
A_{k-1} - B_k = -\tau \eta \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|_{\varepsilon h,h}^2 - \tau \left( 1 + \frac{\theta}{2} \right) \left( \left\| \nabla e_h^k \right\|^2 + \left\| \nabla e_h^{k-1} \right\|^2 \right) - \frac{\tau \rho}{T^2} L_k.
\]
It follows that

\[
\sum_{k=1}^{m} (A_k - B_k) = A_m - B_1 + \sum_{k=2}^{m} (A_{k-1} - B_k)
\]

\[
= \frac{1}{2} \left(2 - \tau \eta\right) \left\| \frac{e^{m+1}_h - e^m_h}{\tau} \right\|_{e,h}^2 + \frac{1}{2} \left(1 - 2\tau\right) \left\| \nabla e^{m+1}_h \right\|^2 + \frac{1}{2} \left(1 - \tau \theta\right) \left\| \nabla e^m_h \right\|^2
\]

\[
- \frac{1}{2} \left(2 + \tau \eta\right) \left\| \frac{e^1_h - e^0_h}{\tau} \right\|_{e,h}^2 - \frac{1}{2} \left(1 + 2\tau\right) \left\| \nabla e^0_h \right\|^2 - \frac{1}{2} \left(1 + \tau \theta\right) \left\| \nabla e^1_h \right\|^2 - \frac{\tau \rho}{T^2} L_1
\]

\[
- \sum_{k=2}^{m} \left\{ \tau \eta \left\| \frac{e^k_h - e^{k-1}_h}{\tau} \right\|_{e,h}^2 + \frac{\tau \rho}{T^2} \left( \left\| \nabla e^k_h \right\|^2 + \left\| \nabla e^{k-1}_h \right\|^2 \right) \right\}
\]

(4.31)

On the other hand, since \(C_{k-1} = D_k\) for all \(m \geq k \geq 2\), recalling (4.27) we easily derive,

\[
\sum_{k=1}^{m} (C_k - D_k) = C_m - D_1
\]

\[
= \frac{1}{2} \left\{ \left\| \nabla (e^{m+1}_h - e^m_h) \right\|^2 + \left\| \nabla \cdot (e^{m+1}_h - e^m_h) \right\|^2 \right\} + \frac{1}{2} \left( \left\| \nabla \cdot e^{m+1}_h \right\|_{e-1}^2 + \left\| \nabla \cdot e^m_h \right\|_{e-1}^2 \right)
\]

\[
+ \frac{1}{2} \left( \left\| \nabla (e^1_h - e^0_h) \right\|^2 + \left\| \nabla \cdot (e^1_h - e^0_h) \right\|^2 \right) - \frac{1}{2} \left( \left\| \nabla \cdot e^1_h \right\|_{e-1}^2 + \left\| \nabla \cdot e^0_h \right\|_{e-1}^2 \right).
\]

(4.32)

Extending to \(k = 1\) the summation range on the right hand side of (4.31) by adjusting the coefficients of the terms involving \(e^j_h\) for \(j = 0, 1\) and combining the resulting relation with (4.32), we obtain for \(2 \leq m \leq N - 1\):

\[
\sum_{k=1}^{m} (A_k + C_k) - \sum_{k=1}^{m} (B_k + D_k)
\]

\[
= \frac{1}{2} \left(2 - \tau \eta\right) \left\| \frac{e^{m+1}_h - e^m_h}{\tau} \right\|_{e,h}^2 + \frac{1}{2} \left(1 - 2\tau\right) \left\| \nabla e^{m+1}_h \right\|^2 + \frac{1}{2} \left(1 - \tau \theta\right) \left\| \nabla e^m_h \right\|^2
\]

\[
- \frac{1}{2} \left(2 - \tau \eta\right) \left\| \frac{e^1_h - e^0_h}{\tau} \right\|_{e,h}^2 - \frac{1}{2} \left(1 - 2\tau\right) \left\| \nabla e^0_h \right\|^2 - \frac{1}{2} \left(1 - \tau \theta\right) \left\| \nabla e^1_h \right\|^2
\]

\[
- \sum_{k=1}^{m} \left\{ \tau \eta \left\| \frac{e^k_h - e^{k-1}_h}{\tau} \right\|_{e,h}^2 + \frac{\tau \rho}{T^2} \left( \left\| \nabla e^k_h \right\|^2 + \left\| \nabla e^{k-1}_h \right\|^2 \right) \right\}
\]

\[
- \frac{1}{2} \left\{ \left\| \nabla (e^{m+1}_h - e^m_h) \right\|^2 + \left\| \nabla \cdot (e^{m+1}_h - e^m_h) \right\|^2 \right\} + \frac{1}{2} \left( \left\| \nabla \cdot e^{m+1}_h \right\|_{e-1}^2 + \left\| \nabla \cdot e^m_h \right\|_{e-1}^2 \right)
\]

\[
+ \frac{1}{2} \left\{ \left\| \nabla (e^1_h - e^0_h) \right\|^2 + \left\| \nabla \cdot (e^1_h - e^0_h) \right\|^2 \right\} - \frac{1}{2} \left( \left\| \nabla \cdot e^1_h \right\|_{e-1}^2 + \left\| \nabla \cdot e^0_h \right\|_{e-1}^2 \right).
\]

(4.33)

On the other hand, recalling (4.24) we note that for \(2 \leq m \leq N - 1\)

\[
\sum_{k=1}^{m} L_k \leq \sum_{k=1}^{m} \left( \tau T \sum_{i=1}^{m} \left\| \frac{e^i_h - e^{i-1}_h}{\tau} \right\|_{e,h}^2 + \left\| e^0_h \right\|^2 \right) = T m \tau \sum_{k=1}^{m} \left\| \frac{e^k_h - e^{k-1}_h}{\tau} \right\|_{e,h}^2 + m \left\| e^0_h \right\|^2
\]

(4.34)
In short since \( m \tau \leq T \), from (4.34) we easily derive for \( 2 \leq m \leq N - 1 \):

\[
\frac{\tau \rho}{T^2} \sum_{k=1}^{m} L_k \leq \tau \rho \sum_{k=1}^{m} \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\| + \frac{\rho}{T} \| e_0^0 \|^2. 
\]

(4.35)

Plugging (4.35) into (4.33) and summing up both sides of (4.27) from \( k = 1 \) through \( k = m \) for \( 2 \leq m \leq N - 1 \) by using (4.29), (4.33) yields:

\[
\frac{1}{2} (2 - \tau \eta) \left\| \frac{e_h^{m+1} - e_h^m}{\tau} \right\|^2 \leq \frac{1}{2} (2 - \tau) \| \nabla e_h^{m+1} \|^2 + \frac{1}{2} (1 - \tau \theta) \| \nabla e_h^m \|^2 \\
- \frac{1}{2} (2 - \tau \eta) \left\| \frac{e_h^1 - e_h^0}{\tau} \right\|^2 \leq \frac{1}{2} (1 - \tau \theta) \| \nabla e_h^0 \|^2 - \frac{1}{2} (1 - \tau) \| \nabla e_h^1 \|^2 \\
- \sum_{k=1}^{m} \left\{ \tau (\eta + \rho) \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|^2 \leq \frac{1}{2} \frac{\tau m}{2} \left( \| \nabla e_h^1 \|^2 + \| \nabla e_h^0 \|^2 \right) \right\} \\
- \sum_{k=1}^{m} \left\{ \frac{1}{2} \left\{ \| \nabla (e_h^m - e_h^0) \|^2 + \| \nabla \cdot (e_h^m - e_h^0) \|^2 \right\} \leq \frac{1}{2} \left( \| \nabla \cdot e_h^m \|^2 + \| \nabla \cdot e_h^0 \|^2 \right) \right\} \leq \sum_{k=1}^{m} \left( \frac{\tau}{2} |k| f_h^2 + \frac{3}{2} \sum_{k=1}^{m} \| d \|^2 + \frac{\tau^2}{2} \| G_h^k \|_{\Omega_h}^2 \right) + \| e_h^1 \|^2_{\epsilon, h} + \frac{\tau}{2} \left( \| \nabla e_h^1 \|^2 + \| \nabla \cdot e_h^1 \|^2 \right) + \frac{1}{2} \left( \| \nabla e_h^0 \|^2 + \| \nabla \cdot e_h^0 \|^2 \right). 
\]

(4.36)

Thus taking into account that \( e_h^1 - e_h^0 = \tau e_{1h} \), leaving on the left hand side only the terms with superscripts \( m + 1 \) and \( m \), and increasing the coefficients of \( \| \nabla e_j^l \|^2 \) for \( j = 0, 1 \) and \( \| e_{1h} \|^2_{\epsilon, h} \), we derive for \( 2 \leq m \leq N - 1 \):

\[
\frac{1}{2} (2 - \tau \eta) \left\| \frac{e_h^{m+1} - e_h^m}{\tau} \right\|^2 \leq \frac{1}{2} (2 - \tau) \| \nabla e_h^{m+1} \|^2 + \frac{1}{2} (1 - \tau \theta) \| \nabla e_h^m \|^2 \\
- \frac{1}{2} \left\{ \| \nabla (e_h^{m+1} - e_h^m) \|^2 + \| \nabla \cdot (e_h^{m+1} - e_h^m) \|^2 \right\} \leq \frac{1}{2} \left| \| \nabla \cdot e_h^m \|^2 + \| \nabla \cdot e_h^0 \|^2 \right| \leq \sum_{k=1}^{m} \left\{ \frac{\tau}{2} |k| f_h^2 + \frac{3}{2} \sum_{k=1}^{m} \| d \|^2 + \frac{\tau^2}{2} \| G_h^k \|_{\Omega_h}^2 \right\} + \| e_h^1 \|^2_{\epsilon, h} + \frac{\tau}{2} \left( \| \nabla e_h^1 \|^2 + \| \nabla \cdot e_h^1 \|^2 \right) + \frac{1}{2} \left( \| \nabla e_h^0 \|^2 + \| \nabla \cdot e_h^0 \|^2 \right) + \frac{\rho}{T} \| e_0^0 \|^2. 
\]

(4.37)

Now we recall a classical inverse inequality (cf. [15]), according to which,

\[
\| \nabla v \| \leq C h^{-1} \| v \| \leq C h^{-1} \| v \|_{\epsilon, h} \text{ for all } v \in V_h, \]

(4.38)

where \( C \) is a mesh-independent constant, and we apply the trivial upper bound \( \| \nabla \cdot v \|_{\epsilon, h} \leq \sqrt{3} \| v \|_{\infty} \| \nabla v \| \text{ for all } v \in V_h. \)
Let us assume that $\tau$ satisfies the following CFL-condition:

$$\tau \leq h/\nu \text{ with } \nu = C(1 + 3\|\varepsilon - 1\|_\infty)^{1/2}. \quad (4.39)$$

Then we have, $\forall \mathbf{v} \in V_h$:

$$\|\nabla \mathbf{v}\|^2 + \|\nabla \cdot \mathbf{v}\|_{-1}^2 \leq \nu^2 \frac{\tau^2}{h^2} \|\mathbf{v}\|^2 \leq \frac{\|\mathbf{v}\|^2}{\tau^2} \quad (4.40)$$

This means that

$$\|\nabla (e^{m+1}_h - e^m_h)\|^2 + \|\nabla \cdot (e^{m+1}_h - e^m_h)\|_{\varepsilon-1}^2 \leq \left\|\frac{e^{m+1}_h - e^m_h}{\tau}\right\|^2 \quad (4.41)$$

Plugging (4.41) into (4.37) we come up with,

$$\frac{1}{2} (1 - \tau\eta) \left\|\frac{e^{m+1}_h - e^m_h}{\tau}\right\|^2_{\varepsilon, h} + \frac{1}{2} (1 - 2\tau) \|\nabla e^{m+1}_h\|^2 + \frac{1}{2} (1 - \tau\theta) \|\nabla e^m_h\|^2$$

$$\leq \sum_{k=1}^{m} \left\{ \tau (\eta + \rho) \left\|\frac{e^k_h - e^{k-1}_h}{\tau}\right\|^2_{\varepsilon, h} + \frac{\tau}{2} (2 + \theta) \left( \|\nabla e^k_h\|^2 + \|\nabla e^{k-1}_h\|^2 \right) \right\}$$

$$+ \frac{\tau}{2} \left| F^k_h \right|^2 + \frac{3}{2\tau} \|d^k\|^2 + \frac{\tau}{2} \left| G^k \right|^2 \|_{\partial \Omega, h} + \frac{\tau}{2} \left( \|\nabla e_1h\|^2 + \|\nabla \cdot e_1h\|_{\varepsilon-1}^2 \right)$$

$$+ \frac{1}{2} \left( \|\nabla e^k_h\|^2 + \|\nabla e^0_h\|^2 \right) + \frac{\tau}{2} \left( \|\nabla \cdot e^1_h\|_{\varepsilon-1}^2 + \|\nabla \cdot e^0_h\|_{\varepsilon-1}^2 \right) + \frac{\rho}{\tau} \|e^0_h\|^2. \quad (4.42)$$

Next we note that both $1 - 2\tau$ and $1 - \tau\theta$ are bounded below by $1 - \tau\eta$ and moreover $\frac{\tau}{2} (2 + \theta)$ is obviously bounded above by $\eta + \rho$. Therefore it is easy to see that (4.42) can be transformed into:

$$\frac{1}{2} (1 - \tau\eta) \left\|\frac{e^{m+1}_h - e^m_h}{\tau}\right\|^2_{\varepsilon, h} + \|\nabla e^{m+1}_h\|^2 + \|\nabla e^m_h\|^2$$

$$\leq S_N + E_0 + \sum_{k=1}^{m} \tau (\eta + \rho) \left\|\frac{e^k_h - e^{k-1}_h}{\tau}\right\|^2_{\varepsilon, h} + \|\nabla e^k_h\|^2 + \|\nabla e^{k-1}_h\|^2, \quad (4.43)$$

where

$$S_N := \sum_{k=1}^{N-1} \left( \frac{\tau}{2} \left| F^k_h \right|^2 + \frac{3}{2\tau} \|d^k\|^2 + \frac{\tau}{2} \left| G^k \right|^2 \|_{\partial \Omega, h} \right)$$

$$E_0 := \|e_1h\|^2_{\varepsilon, h} + \frac{\tau^2}{2} \left( \|\nabla e_1h\|^2 + \|\nabla \cdot e_1h\|_{\varepsilon-1}^2 \right)$$

$$+ \frac{1}{2} \left( \|\nabla e^1_h\|^2 + \|\nabla e^0_h\|^2 \right) + \frac{1}{2} \left( \|\nabla \cdot e^1_h\|_{\varepsilon-1}^2 + \|\nabla \cdot e^0_h\|_{\varepsilon-1}^2 \right) + \frac{\rho}{\tau} \|e^0_h\|^2. \quad (4.44)$$
Let us assume that \( \tau \leq \frac{1}{2\eta} \). Then from (4.44) we have

\[
\left\| \frac{e_h^{m+1} - e_h^{m}}{\tau} \right\|_{\varepsilon_h,h}^2 + \left\| \nabla e_h^{m+1} \right\|^2 + \left\| \nabla e_h^m \right\|^2 \\
\leq \left\{ S_N + E_0 + \sum_{k=1}^{m} \tau (\eta + \rho) \left( \left\| \frac{e_h^k - e_h^{k-1}}{\tau} \right\|_{\varepsilon_h,h}^2 + \left\| \nabla e_h^k \right\|^2 + \left\| \nabla e_h^{k-1} \right\|^2 \right) \right\},
\]

(4.45)

Now setting

\[
\beta = 4N \tau (\eta + \rho) = 4T \left\{ 2 + |\varepsilon|_{1,\infty} + (2 + T^2) |\varepsilon|_{2,\infty} \right\},
\]

(4.46)

from the discrete Grönwall’s Lemma and (3.9) from (4.45) we derive for all \( m \leq N - 1 \):

\[
\left\| \frac{e_h^{m+1} - e_h^{m}}{\tau} \right\|_{\varepsilon_h,h}^2 + \left\| \nabla e_h^{m+1} \right\|^2 + \left\| \nabla e_h^m \right\|^2 \leq 4(S_N + E_0)e^\beta,
\]

(4.47)

as long as \( \tau \leq \min \{ \frac{h}{\nu}, 1/(2\eta) \} \), where \( \nu, \eta - \rho \) and \( \beta \) are defined in (4.39), (4.28) and (4.46), respectively, and in the expression of \( E_0 \), \( e_1h \) is to be replaced by \( e_0h + \tau e_1h \).

5 Scheme consistency

Before pursuing the reliability study of our scheme we need some approximation results related to the Maxwell’s equations. The arguments employed in this section found their inspiration in Thomée [40] and in Ruas [38].

5.1 Preliminaries

Akin to [40] and [38], optimal orders of consistency are attained by assuming that \( \Omega \) is convex. Thus henceforth we assume that \( \Omega \) is a convex polyhedron. In this case one may reasonably assume that for every \( g \in [L_0^2(\Omega)]^3 \), where \( L_0^2(\Omega) \) is the subspace of \( L^2(\Omega) \) consisting of functions whose integral in \( \Omega \) equals zero, the solution \( v_g \in [H^1(\Omega) \cap L_0^2(\Omega)]^3 \) of the equation

\[
-\nabla^2 v_g - \nabla [(\varepsilon - 1) \nabla \cdot v_g] = g \quad \text{in} \quad \Omega
\]

\[
\partial_n v_g = 0 \quad \text{on} \quad \partial \Omega.
\]

(5.48)

belongs to \( [H^2(\Omega)]^3 \).

Another result that we take for granted in this section is the existence of a constant \( C \) such that,

\[
\forall g \in [L_0^2(\Omega)]^3 \text{ it holds } \| \mathcal{H}(v_g) \| \leq C \| g \|,
\]

(5.49)

where \( \mathcal{H}(\cdot) \) is the Hessian of a function or field. (5.49) is a result whose grounds can be found in analogous inequalities applying to the scalar Poisson problem and to the Stokes system. In fact (5.48) can be viewed as a problem half way between the (vector) Poisson problem and a sort of generalized Stokes system, both with homogeneous Neumann boundary conditions. In order to
create such a system we replace in (5.48) \(-(\varepsilon - 1)\nabla \cdot \mathbf{v}_g\) by a fictitious pressure \(p_g\). Then the resulting equation is supplemented by the relation \((\varepsilon - 1)\nabla \cdot \mathbf{v}_g + p_g = 0\) in \(\Omega\). Akin to the classical Stokes system, the operator associated with this system is weakly coercive over \([H^1(\Omega) \cap L_0^2(\Omega)]^3 \times L_0^2(\Omega)\) equipped with the natural norm. This can be verified by choosing in the underlying variational form a test field \(\mathbf{w} = ((\varepsilon - 1/2)\mathbf{v}_g\) in the first equation, and a test-function \(q = p_g\) in the second equation. Thus the convexity of \(\Omega\) strongly supports (5.49).

Now in order to establish the consistency of the explicit scheme (3.8) we first introduce an auxiliary field \(\hat{\mathbf{e}}_h(\cdot, t)\) belonging to \(\mathbf{V}_h\) for every \(t \in [0, T]\), uniquely defined up to an additive field depending only on \(t\) as follows, for every \(t \in [0, T]\):

\[
(\nabla \hat{\mathbf{e}}_h(\cdot, t), \nabla \mathbf{v}) + (\nabla \cdot \hat{\mathbf{e}}_h(\cdot, t), \nabla \cdot \mathbf{v})_{e-1} = (\nabla \mathbf{e}(\cdot, t), \nabla \mathbf{v}) + (\nabla \cdot \mathbf{e}(\cdot, t), \nabla \cdot \mathbf{v})_{e-1} \forall \mathbf{v} \in \mathbf{V}_h.
\]

(5.50)

The time-dependent additive field up to which \(\hat{\mathbf{e}}_h(\cdot, t)\) is defined can be determined by requiring that \(\int_{\Omega}(\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t))d\mathbf{x} = 0\ \forall t \in [0, T]\).

Let us further assume that for every \(t \in [0, T]\) \(\mathbf{e}(\cdot, t) \in [H^2(\Omega)]^3\). In this case, from classical approximation results based on the interpolation error, we can assert that,

\[
\|\nabla \{\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)\}\| + \|\nabla \cdot \{\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)\}\|_{e-1} \leq \hat{C}_1 h\|\mathbf{H}(\mathbf{e})(\cdot, t)\| \forall t \in [0, T],
\]

(5.51)

where \(\hat{C}_1\) is a mesh-independent constant.

Let us show that there exists another mesh-independent constant \(\hat{C}_0\) such that for every \(t \in [0, T]\) it holds,

\[
\|\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)\| \leq \hat{C}_0 h^2\|\mathbf{H}(\mathbf{e})(\cdot, t)\| \forall t \in [0, T],
\]

(5.52)

With this aim we use the assumed convexity of \(\Omega\) and apply the classical Aubin-Nitsche trick (cf. [15]). Since \(\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t) \in [L_0^2(\Omega)]^3\) for every \(t\), in the case under consideration this consists of writing:

\[
\|\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)\| = \sup_{\mathbf{g} \in [L_0^2(\Omega)]^3 \setminus \{0\}} \frac{\{\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t), \mathbf{g}\}}{\|\mathbf{g}\|} \forall t \in [0, T].
\]

(5.53)

Defining \(\mathbf{W} := \{\mathbf{w} \mid \mathbf{w} \in [H^2(\Omega) \cap L_0^2(\Omega)]^3, \partial_n \mathbf{w} = \mathbf{0} \text{ on } \partial \Omega\}\), owing to (5.49) we have \(\forall t \in [0, T]\):

\[
\|\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)\| \leq C \sup_{\mathbf{w} \in \mathbf{W} \setminus \{0\}} \frac{\{\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t), -\nabla^2 \mathbf{w} - \nabla\{\varepsilon - 1\} \nabla \cdot \mathbf{w}\}}{\|\mathbf{H}(\mathbf{w})\|}.
\]

(5.54)

Since \(\partial_n \mathbf{w} = \mathbf{0}\) and \(\varepsilon = 1\) on \(\partial \Omega\) we may integrate by parts the numerator in (5.54) to obtain for every \(t \in [0, T]\),

\[
\|\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)\| \leq C \sup_{\mathbf{w} \in \mathbf{W} \setminus \{0\}} \frac{(\nabla [\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)], \nabla \mathbf{w}) + (\nabla \cdot [\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)], \nabla \cdot \mathbf{w})_{e-1}}{\|\mathbf{H}(\mathbf{w})\|}.
\]

(5.55)

Taking into account (5.50) the numerator of (5.55) can be rewritten as

\[
(\nabla [\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)], \nabla \mathbf{w} - \mathbf{v}) + (\nabla \cdot [\hat{\mathbf{e}}_h(\cdot, t) - \mathbf{e}(\cdot, t)], \nabla \cdot [\mathbf{w} - \mathbf{v}])_{e-1} \forall \mathbf{v} \in \mathbf{V}_h.
\]
Then choosing \( v \) to be the \( \mathbf{V}_h \)-interpolate of \( w \), taking into account (5.51) we easily establish (5.52) with \( \hat{C}_0 = C \hat{C}^2_1 \).

To conclude these preliminary considerations, we refer to Chapter 5 of [38], to infer that the second order time-derivative \( \partial_t \hat{e}_h(\cdot, t) \) is well defined in \([H^1(\Omega)]^3\) for every \( t \in [0, T] \), as long as \( \partial_t e(\cdot, t) \) lies in \([H^1(\Omega)]^3\) for every \( t \in [0, T] \). Moreover, provided \( \partial_t e \in [H^2(\Omega)]^3 \) for every \( t \in [0, T] \), the following estimate holds:

\[
\| \partial_t \{ \hat{e}_h(\cdot, t) - e(\cdot, t) \} \| \leq \hat{C}_0 h^2 \| \mathcal{H}(\partial_t e)(\cdot, t) \| \quad \forall t \in [0, T].
\] (5.56)

In addition to the results given in this sub-section, we recall that, according to the Sobolev Embedding Theorem, there exists a constant \( C_T \) depending only on \( T \) such that it holds:

\[
\| u \|_{L^\infty(0,T)} \leq C_T \| u \|_{L^2(0,T)}^2 + \| u_t \|_{L^2(0,T)}^{1/2} \quad \forall u \in H^1(0,T).
\] (5.57)

In the remainder of this work we assume a certain regularity of \( e \), namely,

**Assumption**: The solution \( e \) to equation (2.1) belongs to \([H^4(\{0, 0\})]^3\).

Notice that this assumption requires that \( f \in [H^2(\{0, 0\})]^3 \) and \( e_j \in [H^{7/2-j}(\Omega)]^3 \) for \( j = 0, 1 \).

Now taking \( u = \| \mathcal{H}(e)(\cdot, t) \| \) we have \( u_t(t) = \int_{\Omega} \{ \mathcal{H}(e) : \mathcal{H}(\partial_t e) \} (x, t) dx \), where \( : \) denotes the inner product of two constant tensors of order greater than or equal to three. Then by the Cauchy-Schwarz inequality and taking into account Assumption *, it trivially follows from (5.57) that the following upper bound holds:

\[
\| \mathcal{H}(e)(\cdot, t) \| \leq C_T \| \mathcal{H}(e) \|_{L^2(0,T)}^2 + \| \mathcal{H}(\partial_t e) \|_{L^2(0,T)}^{1/2} \quad \forall t \in (0, T). \] (5.58)

As a complement to the above ingredients we extend the inner products \( (\cdot, \cdot)_{e,h} \) and \( (\cdot, \cdot)_{\mathcal{D}h,h} \), and associated norms \( \| \cdot \|_{e,h} \) and \( \| \cdot \|_{\mathcal{D}h,h} \) in a semi-definite manner, to fields in \([L^2(\Omega)]^N, N \leq 3\) as follows:

First of all, \( \forall K \in T_h \) let \( \Pi_K : L^2(K) \to P_1(K) \) be the standard orthogonal projection operator onto the space \( P_1(K) \) of linear functions in \( K \). We set

\[
\forall u, v \in [L^2(\Omega)]^N, (u, v)_{e,h} := \sum_{K \in T_h} \varepsilon(G_K/volume(K)) \sum_{i=1}^4 \Pi_K u(S_{K,i}) \cdot \Pi_K v(S_{K,i}).
\]

Let us generically denote by \( F \subset \partial \Omega \) a face of tetrahedron \( K \) such that \( area(K \cap \partial \Omega) > 0 \). Moreover we denote by \( \Pi_F(v) \) the standard orthogonal projection of a function \( v \in L^2(F) \) onto the space of linear functions on \( F \). Similarly we define:

\[
\forall u, v \in [L^2(\partial \Omega)]^N, (u, v)_{\mathcal{D}h,h} := \sum_{F \subset \partial \Omega} \frac{area(F)}{3} \sum_{i=1}^3 \Pi_F u(R_{F,i}) \cdot \Pi_F v(R_{F,i}).
\]

17
It is noteworthy that whenever \( u \) and \( v \) belong to \( \mathbf{V}_h \), both semi-definite inner products coincide with the inner products previously defined for such fields.

The following results hold in connection with the above inner products:

**Lemma 5.1.** Let \( \Delta_{\varepsilon_h}(u, v) := (u, v)_{\varepsilon_h,h} - (u, v)_{\varepsilon_h} \) for \( u, v \in [L^2(\Omega)]^N \). There exist mesh independent constants \( c_{\Omega} \) such that \( \forall u \in [H^1(\Omega)]^3 \) and \( \forall v \in \mathbf{V}_h \),

\[
|\Delta_{\varepsilon_h}(u, v)| \leq c_{\Omega} \| \varepsilon \|_\infty \ h \| \nabla u \| \| v \|_h. \tag{5.59}
\]

**Lemma 5.2.** Let \( \Gamma_h(\gamma(u), \gamma(v)) := (\gamma(u), \gamma(v))_{\partial\Omega,h} - (\gamma(u), \gamma(v))_{\partial\Omega} \) for \( u, v \in [H^1(\Omega)]^N \), where \( \gamma(w) \) represents the trace on \( \partial\Omega \) of a function \( w \in H^1(\Omega) \). Let also \( \nabla_{\partial\Omega} \) be the piecewise tangential gradient operator defined face by face of \( \partial\Omega \). There exist a mesh independent constant \( c_{\partial\Omega} \) such that \( \forall u \in [H^2(\Omega)]^3 \) and \( \forall v \in \mathbf{V}_h \),

\[
|\Gamma_h(\gamma(u), \gamma(v))| \leq c_{\partial\Omega} h \| \nabla_{\partial\Omega} \gamma(u) \|_{\partial\Omega} \| \gamma(v) \|_{\partial\Omega,h}. \tag{5.60}
\]

The proof of Lemma 5.1 is based on the Bramble-Hilbert Lemma, and we refer to [13] for more details. Lemma 5.2 in turn follows from the same arguments combined with the Trace Theorem, which ensures that \( \nabla_{\partial\Omega} \gamma(u) \) is well defined in \([L^2(\partial\Omega)]^3\) if \( u \in [H^2(\Omega)]^3 \). Incidentally the Trace Theorem allows us to bound above the right hand side of (5.60) in such a way that the following estimate also holds for another mesh independent constant \( c_{\partial\Omega} \):

\[
|\Gamma_h(\gamma(u), \gamma(v))| \leq c_{\partial\Omega} h \| \nabla_{\partial\Omega} \gamma(u) \|_{\partial\Omega}^2 + \| \nabla u \|_{\partial\Omega}^2 \| \gamma(v) \|_{\partial\Omega,h}. \tag{5.61}
\]

To conclude we prove the validity of the following upper bounds:

**Lemma 5.3.** \( \forall v \in [L^2(\Omega)]^3 \) it holds

\[
\|v\|_{\varepsilon_h,h} \leq \sqrt{5} \|v\|_{\varepsilon_h,h}.
\]

**Proof.** Denoting by \( \Pi_h(v) \) the function defined in \( \Omega \) whose restriction to every \( K \in T_h \) is \( \Pi_K(v) \) for a given \( v \in [L^2(\Omega)]^3 \), from an elementary property of the orthogonal projection we have

\[
\| \Pi_h(v) \|_{\varepsilon_h} \leq \| v \|_{\varepsilon_h}, \ \forall v \in [L^2(\Omega)]^3. \tag{5.62}
\]

Now taking \( v \) such that \( v|_K \in P_1(K) \) \( \forall K \in T_h \), by a straightforward calculation using the expression of \( v|_K \) in terms of barycentric coordinates we have:

\[
\int_K \varepsilon(G_K)v|_K^2 dx = \frac{\varepsilon(G_K)\text{volume}(K)}{10} \left\{ \sum_{i=1}^{4} |v(S_{K,i})|^2 + \sum_{i=1}^{3} \sum_{j=i+1}^{4} v(S_{K,i})v(S_{K,j}) \right\}.
\]

It trivially follows that

\[
\int_K \varepsilon(G_K)v|_K^2 dx = \frac{\varepsilon(G_K)\text{volume}(K)}{20} \left\{ \sum_{i=1}^{4} |v(S_{K,i})|^2 + \left[ \sum_{i=1}^{4} v(S_{K,i}) \right]^2 \right\}.
\]
and finally
\[ 5 \int_{K} \varepsilon(G_K)v_{K}^{2}dx \geq \frac{\varepsilon(G_K)\text{volume}(K)}{4} \sum_{i=1}^{4} |v(S_{K,i})|^2. \]
This immediately yields Lemma 5.3, taking into account (5.62).

**Lemma 5.4.** \( \forall v \in [L^2(\partial\Omega)]^3 \) it holds \( \|v\|_{\partial\Omega,h} \leq 2 \|v\|_{\partial\Omega}. \)

The proof of this Lemma is based on arguments entirely analogous to Lemma 5.3.

### 5.2 Residual estimation

To begin with we define for \( k = 0, 1, \ldots, N \) functions \( \hat{e}_h^k \in V_h \) by \( \hat{e}_h^k(\cdot) := e_h(\cdot, k\tau) \). In the sequel for any function or field \( A \) defined in \( \Omega \times (0, T) \), \( A^k(\cdot) \) denotes \( A(\cdot, k\tau) \), except for other quantities carrying the subscript \( h \) such as \( e_h^k \).

Let us substitute \( e_h^k \) by \( \hat{e}_h^k \) for \( k = 2, 3, \ldots, N \) on the left hand side of the first equation of (3.8) and take also as initial conditions \( \hat{e}_h^j \) instead of \( e_h^j \), \( j = 0, 1 \).

The case of the initial conditions will be dealt with in the next section in the framework of the convergence analysis. The variational residual \( E_h^k(v) \) in turn, resulting from the above substitution, \( E_h^k \) being a linear functional acting on \( V_h \), can be expressed in the following manner:

\[
E_h^k(v) = (\{\partial_t e\}^k, v)_{e} + (\nabla e^k, \nabla v) + (\nabla \cdot e^k, \nabla \cdot v)_{e} + (\nabla \varepsilon \cdot e^k, \nabla \varepsilon \cdot v) + (\{\partial_t e\}^k, v)_{\partial\Omega} - (f^k, v) + F^k(v) + (\bar{d}^k, v) + G^k(v) \forall v \in V_h, 
\]

where
\[
\bar{d}^k = \nabla \varepsilon \cdot \{\hat{e}_h^k - e^k\},
\]

and \( F^k(v) \) and \( G^k(v) \) can be written as follows:

\[
F^k(v) = F_1^k(v) + F_2^k(v) + F_3^k(v) + F_4^k(v),
\]

with
\[
F_1^k(v) = (T^k(\hat{e}_h - e), v)_{e, \Omega},
F_2^k(v) = \Delta e_h(T^k(e), v),
F_3^k(v) = ((\varepsilon_h - \varepsilon)T^k(e), v),
F_4^k(v) = (T^k(e) - \{\partial_t e\}^k, v)_{e},
\]

\( T^k \) being the finite-difference operator defined by,

\[
T^k(\cdot) := \{\cdot\}^{k+1} - 2\{\cdot\}^{k} + \{\cdot\}^{k-1},
\]

and

\[
G^k(v) = G_1^k(v) + G_2^k(v) + G_3^k(v),
\]

with
\[
G_1^k(v) = (Q^k(\hat{e}_h - e), v)_{\partial\Omega},
G_2^k(v) = \Gamma_h(Q^k(e), v),
G_3^k(v) = (Q^k(e) - \{\partial_t e\}^k, v)_{\partial\Omega},
\]

19
\( Q^k \) being the finite-difference operator defined by,

\[
Q^k(\cdot) := \frac{\{\cdot\}^k + \{\cdot\}^{k-1}}{2\tau}.
\]  

(5.68)

Detailed explanations about the above decomposition of \( E_h^k(v) \) are as follows:

First of all we use (5.50) to write \( \forall v \in V_h \)

\[
\{(\partial_{tt} e)^k, v\} + (\nabla \cdot \hat{e}^k_h, \nabla v) + (\nabla \cdot \hat{e}^k_h, \nabla \cdot v)_{\varepsilon - 1} + (\nabla \varepsilon \cdot e^k, \nabla \cdot v) - (f^k, v) = 0.
\]

From this equality we may write \( \forall v \in V_h \):

\[
E_h^k(v) := (T^k(\hat{e}^k_h), v)_{\varepsilon, h} + (\nabla \hat{e}^k_h, \nabla v) + (\nabla \cdot \hat{e}^k_h, \nabla \cdot v)_{\varepsilon - 1} + (\nabla \varepsilon \cdot e^k, \nabla \cdot v) - (f^k, v)
\]

\[
= \mathbf{F}^k(v) + (\hat{d}^k, \nabla \cdot v) + \mathbf{G}^k(v)
\]

where, \( \mathbf{F}^k(v) := (T^k(\hat{e}^k_h), v)_{\varepsilon, h} - \{(\partial_{tt} e)^k, v\}_e \) \( \forall v \in V_h, \) \( \hat{d}^k := \nabla \varepsilon \cdot \{\hat{e}^k_h - e^k\} \) and \( \mathbf{G}^k(v) := (Q^k(\hat{e}^k_h), v)_{\partial \Omega, h} - \{(\partial_{tt} e)^k, v\}_{\partial \Omega} \).

Next we proceed to an obvious splitting of \( \mathbf{F}^k(v) \) into the sum of four terms, namely,

\[
\mathbf{F}^k(v) = \{(T^k(\hat{e}^k_h), v)_{\varepsilon, h} - (T^k(e), v)_{\varepsilon, h}\} + \{(T^k(e), v)_{\varepsilon, h} - (T^k(e), v)_{\varepsilon}\} + \{(T^k(e), v)_{\varepsilon} - (\partial_{tt} e)^k, v\}_e.
\]

We can set up a similar splitting of \( \mathbf{G}^k(v) \) into the sum of the three terms \( \mathbf{G}^k_i(v), i = 1, 2, 3. \)

Finally, recalling the definitions of \( \Delta \varepsilon \) and \( \Gamma \) given by (5.1) and in Lemma 5.2, respectively, we readily come up with (5.65) and (5.67).

Notice that, under Assumption*, both \( \partial_t e(\cdot, t) \) and \( \partial_{tt} e(\cdot, t) \) belong to \( [H^1(\Omega)]^3 \) for every \( t \in [0, T] \). Hence we can define \( \partial_t \hat{e}_h \) from \( \partial_t e \) and \( \partial_{tt} \hat{e}_h \) from \( \partial_{tt} e \) in the same way as \( \hat{e}_h \) is defined from \( e \). Moreover straightforward calculations lead to,

\[
T^k(\cdot) = \frac{1}{\tau^2} \left\{ \int_{(k-1)\tau}^{k\tau} \left( \int_t^{k\tau} \partial_{tt} \{\cdot\} \, ds \right) \, dt + \int_{k\tau}^{(k+1)\tau} \left( \int_t^{(k+1)\tau} \partial_{tt} \{\cdot\} \, ds \right) \, dt \right\}.
\]

(5.69)

Furthermore another straightforward calculation allows us writing:

\[
T^k(e) = \{\partial_{tt} e\}^k + R^k(e)
\]

where

\[
R^k(e) := \frac{1}{2\tau^2} \left[ -\int_{(k-1)\tau}^{k\tau} \{t - (k - 1)\tau\}^2 \partial_{tt} e \, dt + \int_{k\tau}^{(k+1)\tau} \{(k + 1)\tau - t\}^2 \partial_{tt} e \, dt \right].
\]

(5.70)

Similarly,

\[
Q^k(\cdot) = \frac{1}{2\tau} \int_{(k-1)\tau}^{(k+1)\tau} \partial_t \{\cdot\} \, dt,
\]

(5.71)
and

\[ Q^k(e) = \{\partial_t e\}^k + S^k(e), \]

where

\[ S^k(e) = \frac{1}{2\tau} \left[ \int_{(k-1)\tau}^{k\tau} \{t - (k - 1)\tau\} \partial_t e dt + \int_{k\tau}^{(k+1)\tau} \{(k + 1)\tau - t\} \partial_t e dt \right]. \tag{5.72} \]

Now we note that the sum of the terms on the first line of the expression (5.63) of \( E_h^k(v) \) equals zero because of (2.2) at time \( t = k\tau \). Therefore the functions \( \hat{e}_h^k \in V_h \) are the solution of the following problem, for \( k = 1, 2, \ldots, N - 1 \):

\[
\begin{align*}
(\varepsilon_h \frac{\hat{e}_h^{k+1} - 2\hat{e}_h^k + \hat{e}_h^{k-1}}{\tau^2}, v) + (\nabla \hat{e}_h^k, \nabla v) + (\nabla \cdot \{\varepsilon \hat{e}_h^k\}, \nabla \cdot v) - (\nabla \cdot \hat{e}_h^k, \nabla \cdot v) \\
+ \left( \frac{\hat{e}_h^{k+1} - \hat{e}_h^{k-1}}{2\tau}, v \right)_{\partial\Omega} = \bar{F}^k(v) + (\hat{d}^k, \nabla \cdot v) + G^k(v) \quad \forall v \in V_h,
\end{align*}
\]

where \( \hat{e}_h^0(\cdot) = \hat{e}_h(\cdot, 0) \) and \( \hat{e}_h^k(\cdot) = \hat{e}_h(\cdot, \tau) \) in \( \Omega \), \( \bar{d}^k, \bar{F}^k \) and \( G^k \) being given by (5.64), (5.65)-(5.66) and (5.67)-(5.68).

Estimating \( \|\bar{d}^k\| \) is a trivial matter. Indeed, since \( e^k \in [H^2(\Omega)]^3 \), from (5.52) we immediately obtain,

\[ \|\bar{d}^k\| \leq C_0 h^2 \varepsilon |1, \infty\|H(e^k)\|. \tag{5.74} \]

Therefore consistency of the scheme will result from suitable estimations of \( |\bar{F}^k|_h \) and \( |G^k|_{\partial\Omega, h} \) in terms of \( e, \varepsilon, \tau \) and \( h \), which we next carry out.

First of all we derive some upper bounds for the operators \( T^k(\cdot), Q^k(\cdot), R^k(\cdot) \) and \( S^k(\cdot) \). With this aim we denote by \( | \cdot | \) the euclidean norm of \( \mathbb{R}^M \), for \( M \geq 1 \).

From (5.69) and the Cauchy-Schwarz inequality, we easily derive for every \( x \in \Omega \) and \( u \) such that \( \{ \partial_t u \}(\cdot, t) \in [H^2(\Omega)]^3 \forall t \in (0, T) \),

\[
|T_h^k(u(x))| \leq \frac{1}{\tau^2} \left\{ \int_{(k-1)\tau}^{k\tau} \left[ \int_{t}^{k\tau} |\{\partial_t u\}(x)| dt \right] ds \right\} + \int_{k\tau}^{(k+1)\tau} \left[ \int_{k\tau}^{t} |\{\partial_t u\}(x)| ds \right] dt \\
\leq \frac{1}{\tau^2} \left\{ \int_{(k-1)\tau}^{k\tau} \left[ \int_{(k-1)\tau}^{k\tau} |\{\partial_t u\}(x)| ds \right] dt \right\} + \int_{k\tau}^{(k+1)\tau} \left[ \int_{k\tau}^{(k+1)\tau} |\{\partial_t u\}(x)| ds \right] dt \\
= \frac{1}{\tau^2} \left\{ \int_{(k-1)\tau}^{k\tau} |\{\partial_t u\}(x)| dt \right\}.
\]

It follows that, for every \( u \) such that \( \{ \partial_t u \}(\cdot, t) \in [H^2(\Omega)]^3 \forall t \in (0, T) \) we have,

\[
|T_h^k(u(x))| \leq \sqrt{\frac{\tau}{2}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} |\{\partial_t u\}(x)|^2 dt \right]^{1/2}. \tag{5.75}
\]

21
Furthermore, from (5.70) and the inequality \( a + b \leq [2(a^2 + b^2)]^{1/2} \ \forall a, b \in \mathbb{R} \), for every \( x \in \Omega \) we obtain:

\[
|R^k \{ e(x) \}| \leq \frac{1}{2} \left[ \int_{(k-1)\tau}^{k\tau} |\{\partial_{tt} e\}(x)| \, dt + \int_{k\tau}^{(k+1)\tau} |\{\partial_{tt} e\}(x)| \, dt \right] \\
\leq \frac{\sqrt{\tau}}{2} \left[ \left\{ \int_{(k-1)\tau}^{k\tau} |\{\partial_{tt} e\}(x)|^2 \, dt \right\}^{1/2} + \left\{ \int_{k\tau}^{(k+1)\tau} |\{\partial_{tt} e\}(x)|^2 \, dt \right\}^{1/2} \right].
\]

It follows that,

\[
|R^k \{ e(x) \}| \leq \frac{\sqrt{\tau}}{2} \left[ \int_{(k-1)\tau}^{(k+1)\tau} |\{\partial_{tt} e\}(x)|^2 \, dt \right]^{1/2}.
\] (5.76)

On the other hand from (5.71) and the Cauchy-Schwarz inequality we trivially have for every \( x \in \partial \Omega \) and \( u \) such that \( \gamma \{\partial \} \{\cdot, t\} \in [H^{3/2}(\partial \Omega)]^3 \ \forall t \in (0, T) \):

\[
|Q^k \{ u(x) \}| \leq \frac{1}{\sqrt{2\tau}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} |\{\partial_t u\}(x)|^2 \, dt \right]^{1/2}.
\] (5.77)

Finally, similarly to (5.76), from (5.72) for every \( x \in \partial \Omega \) we successively derive,

\[
|S^k \{ e(x) \}| \leq \frac{1}{2} \left[ \int_{(k-1)\tau}^{k\tau} |\{\partial_t e\}(x)| \, dt + \int_{k\tau}^{(k+1)\tau} |\{\partial_t e\}(x)| \, dt \right] \\
\leq \frac{\sqrt{\tau}}{2} \left[ \left\{ \int_{(k-1)\tau}^{k\tau} |\{\partial_t e\}(x)|^2 \, dt \right\}^{1/2} + \left\{ \int_{k\tau}^{(k+1)\tau} |\{\partial_t e\}(x)|^2 \, dt \right\}^{1/2} \right].
\]

Therefore it holds,

\[
|S^k \{ e(x) \}| \leq \frac{\sqrt{\tau}}{2} \left[ \int_{(k-1)\tau}^{(k+1)\tau} |\{\partial_t e\}(x)|^2 \, dt \right]^{1/2}.
\] (5.78)

Notice that bounds entirely analogous to (5.75) and (5.77) hold for \( \nabla T^k (\cdot) = T^k \{\nabla \cdot \} \) and \( \nabla_{\partial \Omega} Q^k (\cdot) = Q^k \{\nabla_{\partial \Omega} \cdot \} \), that is, \( \forall x \in \Omega \),

\[
|\nabla T^k_h (e)(x)| \leq \frac{\sqrt{\tau}}{2} \left[ \int_{(k-1)\tau}^{(k+1)\tau} |\{\partial_t \nabla e\}(x)|^2 \, dt \right]^{1/2},
\] (5.79)

and \( \forall x \in \partial \Omega \),

\[
|\nabla_{\partial \Omega} Q^k (e)(x)| \leq \frac{1}{\sqrt{2\tau}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} |\{\partial_t \nabla_{\partial \Omega} e\}(x)|^2 \, dt \right]^{1/2}.
\] (5.80)
Next we estimate the four terms in the expression (5.65) of $\mathbf{F}^k(v)$. With the use of (5.75) and of Lemma 5.3 followed by a trivial manipulation, we successively have:

$$|\mathbf{F}_1^k(v)| \leq \varepsilon \parallel \mathbf{v} \parallel \sqrt{\frac{2}{\tau}} \left[ \int_{(k-1)^r}^{(k+1)^r} \| \{ \partial_t (\hat{e}_h - \mathbf{e}) \}(\cdot, t) \|^2 dt \right]^{1/2} \parallel \mathbf{v} \parallel_h$$

$$\leq \sqrt{5} \varepsilon \parallel \mathbf{v} \parallel \sqrt{\frac{2}{\tau}} \left[ \int_{(k-1)^r}^{(k+1)^r} \| \{ \partial_t (\hat{e}_h - \mathbf{e}) \}(\cdot, t) \|^2 dt \right]^{1/2} \parallel \mathbf{v} \parallel_h$$

Recalling (5.65) and applying (5.66) to (5.81), we come up with,

$$|\mathbf{F}_1^k(v)| \leq \hat{C}_t h^2 \sqrt{\frac{10}{\tau}} \varepsilon \parallel \mathbf{v} \parallel \sqrt{\frac{2}{\tau}} \left[ \int_{(k-1)^r}^{(k+1)^r} \| \{ \mathcal{H}(\partial_t \mathbf{e}) \}(\cdot, t) \|^2 dt \right]^{1/2} \parallel \mathbf{v} \parallel_h$$

Next, combining (5.59) and (5.79) we immediately obtain.

$$|\mathbf{F}_2^k(v)| \leq c_\Omega h \varepsilon \parallel \mathbf{v} \parallel \sqrt{\frac{2}{\tau}} \left[ \int_{(k-1)^r}^{(k+1)^r} \| \{ \partial_t \nabla \mathbf{e} \}(\cdot, t) \|^2 dt \right]^{1/2} \parallel \mathbf{v} \parallel_h$$

Further, from (5.75), the fact that $\parallel \mathbf{v} \parallel \leq \parallel \mathbf{v} \parallel_h$ [13] and the standard estimate $\parallel \varepsilon_h - \varepsilon \parallel_\infty \leq C_\infty h \varepsilon_1,1,\infty$ where $C_\infty$ is a mesh-independent constant, we easily derive,

$$|\mathbf{F}_3^k(v)| \leq C_\infty h \varepsilon_1,1,\infty \sqrt{\frac{2}{\tau}} \left[ \int_{(k-1)^r}^{(k+1)^r} \| \{ \partial_t \mathbf{e} \}(\cdot, t) \|^2 dt \right]^{1/2} \parallel \mathbf{v} \parallel_h$$

Finally by (3.9), (5.70) and (5.76), we have

$$|\mathbf{F}_4^k(v)| \leq \varepsilon \parallel \mathbf{v} \parallel \sqrt{\frac{2}{\tau}} \left[ \int_{(k-1)^r}^{(k+1)^r} \| \{ \partial_{ttt} \mathbf{e} \}(\cdot, t) \|^2 dt \right]^{1/2} \parallel \mathbf{v} \parallel_h$$

Now we turn our attention to the three terms in the expression (5.67) of $\mathbf{G}^k(v)$. First of all, owing to Assumption* and standard error estimates, we can write for a suitable mesh-independent constant $\hat{C}_1$:

$$\parallel \{ \nabla \partial_t (\hat{e}_h - \mathbf{e}) \}(\cdot, t) \parallel \leq \hat{C}_1 h \parallel \{ \mathcal{H}(\partial_t \mathbf{e}) \}(\cdot, t) \parallel, \forall t \in [0, T].$$

On the other hand, by the Trace Theorem there exists a constant $C_{Tr}$ depending only on $\Omega$ such that,

$$\parallel \{ \partial_t (\hat{e}_h - \mathbf{e}) \}(\cdot, t) \parallel_{\partial \Omega} \leq C_{Tr} \parallel \{ \partial_t (\hat{e}_h - \mathbf{e}) \}(\cdot, t) \parallel^2 + \parallel \{ \nabla \partial_t (\hat{e}_h - \mathbf{e}) \}(\cdot, t) \parallel^2 \parallel \mathbf{G}^k(v) \parallel^{1/2} \forall t \in [0, T].$$

(5.87)
Hence by \((5.56), (5.86)\) and \((5.87)\), we have, for a suitable mesh-independent constant \(C_B\):
\[
\| \{ \partial_t(\hat{e}_h - e) \} \|_{\partial \Omega} \leq C_B h \| \{ \mathcal{H}(\partial_t e) \} \|_{\partial \Omega} \forall t \in [0, T].
\] (5.88)

Now recalling \((5.65)\) and taking into account \((5.80)\) and Lemma 5.4, similarly to \((5.81)\) we first obtain:
\[
|G_1^k (v)| \leq \frac{2}{\sqrt{2} \tau} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \| \{ \mathcal{H}(\partial_t e) \} \|_{\partial \Omega}^2 dt \right]^{1/2} \| v \|_{\partial \Omega, h}. \] (5.89)

Then using \((5.88)\) we immediately establish,
\[
|G_1^k (v)| \leq C_B h \sqrt{2} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \| \{ \mathcal{H}(\partial_t e) \} \|_{\partial \Omega}^2 dt \right]^{1/2} \| v \|_{\partial \Omega, h}. \] (5.90)

Next we switch to \(G_2^k\). Using \((5.61)\) and \((5.80)\), similarly to \((5.83)\) we derive,
\[
|G_2^k (v)| \leq C_B h \sqrt{2} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \| \{ \mathcal{H}(\partial_t e) \} \|_{\partial \Omega}^2 dt \right]^{1/2} \| v \|_{\partial \Omega, h}. \] (5.91)

As for \(G_3^k\), taking into account \((5.72)\) together with \((5.78)\) and \((3.10)\), we obtain:
\[
|G_3^k (v)| \leq \sqrt{2} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \| \{ \partial_t e \} \|_{\partial \Omega}^2 dt \right]^{1/2} \| v \|_{\partial \Omega, h}. \] (5.92)

Then using the Trace Theorem (cf. \((5.87)\)), we finally establish,
\[
|G_3^k (v)| \leq C_T \sqrt{\frac{\tau}{2}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \| \{ \partial_t e \} \|_{\partial \Omega}^2 dt \right]^{1/2} \| v \|_{\partial \Omega, h}. \] (5.93)

Now collecting \((5.82), (5.83), (5.84)\) and \((5.85)\) we can write,
\[
|\bar{B}^k|_h \leq C_0 h^2 \sqrt{\frac{10}{\tau}} \| \varepsilon \|_{\infty} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \| \{ \mathcal{H}(\partial_t e) \} \|_{\partial \Omega}^2 dt \right]^{1/2} + \varepsilon \| \varepsilon \|_{1, \infty} \sqrt{\frac{2}{\tau}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \| \{ \partial_t \nabla e \} \|_{\partial \Omega}^2 dt \right]^{1/2} + C_0 h \| \varepsilon \|_{1, \infty} \sqrt{\frac{2}{\tau}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \| \{ \partial_t \nabla e \} \|_{\partial \Omega}^2 dt \right]^{1/2} + \| \varepsilon \|_{\infty} \sqrt{\frac{2}{\tau}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \| \{ \partial_t e \} \|_{\partial \Omega}^2 dt \right]^{1/2}. \] (5.94)
On the other hand, (5.90), (5.91) and (5.93) yield,

\[
\begin{align*}
|G^k|\Omega_h & \leq C_B h^{\frac{3}{2}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \left( \| \mathcal{H}(\partial_t e) \| \right) (\cdot,t) \| \right\|^2 dt \right]^{1/2} \\
+ C_{\Omega h} h^{\frac{3}{2}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \left( \| \{ \nabla \partial_t e \} \right) (\cdot,t) \| \right\|^2 dt \right]^{1/2} \\
+ C_{\Omega h} h^{\frac{3}{2}} \left[ \int_{(k-1)\tau}^{(k+1)\tau} \left( \| \{ \nabla \partial_t e \} \right) (\cdot,t) \| \right\|^2 dt \right]^{1/2}.
\end{align*}
\]

(5.95)

Then, taking into account (5.73) and the stability condition (4.39), by inspection we can assert that the consistency of scheme (3.8) is an immediate consequence of (5.74), (5.94) and (5.95).

6 Convergence results

In order to establish the convergence of scheme (3.8) we combine the stability and consistency results obtained in the previous sections. With this aim we define \(e^k_h := \hat{e}^k_h - e^k_h\) for \(k = 0, 1, 2, \ldots, N\). By linearity we can assert that the variational residual on the left hand side of the first equation of (3.8) for \(k = 2, 3, \ldots, N\), when the \(e^k_h\)'s are replaced with the \(e^k_h\)'s and \(e^j_h\) is replaced with \(\hat{e}^j_h\) for \(j = 0, 1\), is exactly \(E^k_h(v)\), since the residual corresponding to the \(e^k_h\)'s vanishes by definition. The initial conditions \(\hat{e}^j_h\) for \(j = 0\) and \(j = 1\) corresponding to the thus modified problem have to be estimated. This is the purpose of the next subsection.

6.1 Initial-condition deviations

Here we turn our attention to the estimate of \(E_0\), which accounts for the deviation in the initial conditions appearing in the stability inequality (4.47) that applies to the modification of (3.8) when \(e^k_h\) is replaced by \(\hat{e}^k_h\).

Let us first define,

\[
\begin{align*}
\hat{e}_{1h} & := (\hat{e}_{1h} - \hat{e}^0_{1h})/\tau \\
e_{1h} & := e_{1h} - e_{1h},
\end{align*}
\]

(6.96)

\[
\begin{align*}
\hat{E}_0 & := \| e_{1h} \|^2_{\varepsilon_h,h} + \frac{\tau^2}{2} \left( \| \nabla e_{1h} \|^2 + \| \nabla \cdot e_{1h} \|^2_{\varepsilon_h,h-1} \right) \\
& \quad + \frac{1}{2} \left( \| \nabla e_{1h} \|^2 + \| \nabla e_{1h} \|^2_{\varepsilon_h,h-1} \right) + \| \nabla \cdot e_{1h} \|^2_{\varepsilon_h,h-1} + \| \nabla \cdot e_{1h} \|^2_{\varepsilon_h,h-1} + T|\varepsilon|_{2,\infty} \| e_{1h} \|^2.
\end{align*}
\]

Recalling that \(e_{1h} = e_{1h} + \tau \hat{e}_{1h}\) we have \(\hat{e}_{1h} = \hat{e}_{1h} + \tau \hat{e}_{1h}\). Thus, taking either \(A = \nabla e_{1h}\) or \(A = \nabla \cdot e_{1h}\) and either \(B = \tau \nabla e_{1h}\) or \(B = \tau \nabla \cdot e_{1h}\), we apply twice the inequality \(\|A + B\|^2/2 \leq \|A\|^2 + \|B\|^2\) to (6.96) together with Lemma 5.3, to obtain,

\[
\begin{align*}
\hat{E}_0 & \leq \frac{3}{2} \left\{ \| \nabla e_{1h} \|^2 + \| \nabla \cdot e_{1h} \|^2_{\varepsilon_h,h-1} + \tau^2 \left( \| \nabla e_{1h} \|^2 + \| \nabla e_{1h} \|^2_{\varepsilon_h,h-1} \right) \right\} \\
& \quad + T|\varepsilon|_{2,\infty} \| e_{1h} \|^2 + 5|\varepsilon|_{\infty} \| e_{1h} \|^2.
\end{align*}
\]

(6.97)
Finally using the inequality \( \| \nabla \cdot v \|^{2}_{-1} \leq 3 \| \varepsilon - 1 \|_{\infty} \| \nabla v \|^{2} \) \( \forall v \in [H^{1}(\Omega)]^{3} \), after straightforward manipulations we easily derive from (6.97):

\[
E_{0} \leq \frac{3(1 + 3 \| \varepsilon - 1 \|_{\infty})}{2} (\| \nabla e_{0}^{h} \|^{2} + \tau^{2} \| \nabla e_{1h}^{h} \|^{2}) + T \| \varepsilon \|_{2,\infty} \| e_{0}^{h} \|^{2} + 5 \| \varepsilon \|_{\infty} \| e_{1h} \|^{2}. \tag{6.98}
\]

We next use the obvious splitting \( \bar{e}_{h}^{0} = (e_{h}^{0} - e_{0}) + (e_{0} - e_{h}^{0}) \), together with the trivial one,

\[
\bar{e}_{1h} = \frac{e_{h}^{1} - e_{h}^{0}}{\tau} - e_{1h} = (\{ \partial_{t} \bar{e}_{h} \}_{t=0} - e_{1}) + \frac{1}{\tau} \int_{0}^{T} (\tau - t) \partial_{tt} \bar{e}_{h} \, dt + (e_{1} - e_{1h}). \tag{6.99}
\]

Then plugging (6.99) into (6.98), since \( \sum_{i=1}^{p} \| A_{i} \|^{2} \leq p \sum_{i=1}^{p} \| A_{i} \|^{2} \) for any set of \( p \) functions or fields \( A_{i} \), we obtain:

\[
\frac{1}{\tau^{2}} \left| \int_{0}^{T} (\tau - t) \nabla \partial_{tt} \bar{e}_{h} \, dt \right|^{2} \leq \left| \int_{0}^{T} \nabla \partial_{tt} \bar{e}_{h} \, dt \right|^{2} \leq \tau \left[ \int_{0}^{T} |\nabla \partial_{tt} \bar{e}_{h}|^{2} \, dt \right]^{1/2}. \tag{6.100}
\]

Owing to a trivial upper bound and to the Cauchy-Schwarz inequality we easily derive

\[
\frac{1}{\tau^{2}} \left| \int_{0}^{T} (\tau - t) \nabla \partial_{tt} \bar{e}_{h} \, dt \right|^{2} \leq \tau \int_{0}^{T} |\nabla \partial_{tt} \bar{e}_{h}|^{2} \, dt \leq (1 + 3 \| \varepsilon - 1 \|_{\infty}) \tau \int_{0}^{T} |\nabla \partial_{tt} e|^{2} \, dt. \tag{6.101}
\]

The last inequality in (6.101) follows from the definition of \( \bar{e}_{h} \). Indeed we know that,

\[
(\nabla \partial_{tt} \bar{e}_{h}, \nabla v) + (\nabla \cdot \partial_{tt} \bar{e}_{h}, \nabla v)_{\varepsilon - 1} = (\nabla \partial_{tt} e, \nabla v) + (\nabla \cdot \partial_{tt} e, \nabla v)_{\varepsilon - 1} \forall v \in V_{h}. \tag{6.102}
\]

Taking \( v = \bar{e}_{h} \) by the Cauchy-Schwarz inequality, we easily obtain:

\[
\| \nabla \partial_{tt} \bar{e}_{h} \|^{2} + \| \nabla \cdot \partial_{tt} \bar{e}_{h} \|^{2}_{\varepsilon - 1}^{1/2} \leq \| \nabla \partial_{tt} e \|^{2} + \| \nabla \cdot \partial_{tt} e \|^{2}_{\varepsilon - 1}^{1/2}, \tag{6.103}
\]

or yet

\[
\| \nabla \partial_{tt} \bar{e}_{h} \| \leq \sqrt{1 + 3 \| \varepsilon - 1 \|_{\infty}} \| \nabla \partial_{tt} e \|. \tag{6.104}
\]

which validates (6.101).

On the other hand according to (5.56) we have \( \| \partial_{tt} \bar{e}_{h} \| \leq \| \partial_{tt} e \| + \hat{C}_T h^{2} \| H(\partial_{tt} e) \| \). This yields,

\[
\left| \int_{0}^{T} (\tau - t) \partial_{tt} \bar{e}_{h} \, dt \right|^{2} \leq \tau \int_{0}^{T} |\nabla \partial_{tt} \bar{e}_{h}|^{2} \, dt \leq 2 \tau \left[ \int_{0}^{T} |\nabla \partial_{tt} e|^{2} \, dt + \hat{C}_T h^{4} \int_{0}^{T} \| H(\partial_{tt} e) \|^{2} \, dt \right]. \tag{6.105}
\]

Incidentally we point out that Assumption * and (5.57) allow us to assert that
\[ \partial_t \nabla e \in [L^2(\Omega \times (0, T))]^3; \]
\[ \| \partial_t e \| \in [L^\infty(0, T)]^3; \]
\[ \{ \partial_t e \}_{t=0} = e_1 \in [H^2(\Omega)]^3; \]
\[ \{ e \}_{t=0} = e_0 \in [H^2(\Omega)]^3. \]

Clearly enough, besides (5.52) and (5.51), we will apply to (6.100) standard estimates based on the interpolation error in Sobolev norms (cf. [15]), together with the following obvious variants of (5.51) and (5.56), namely,
\[ \| \nabla \partial_t (\hat{\epsilon}_e - e) (\cdot, 0) \| \leq \bar{C}_1 h \| \mathcal{H}(e_1) \|, \quad (6.106) \]

Then taking into account that \( \tau \leq 1/(2\eta) \), from (6.100)-(6.101)-(6.105)-(6.106) and Assumption*, we conclude that there exists a constant \( \bar{C}_0 \) depending on \( \Omega, T, \) and \( \varepsilon \), but neither on \( h \) nor on \( \tau \), such that,
\[ \bar{E}_0 \leq \bar{C}_0^2 \left[ (h^2 + \tau h^2 + h^4) \| \mathcal{H}(e_0) \|^2 + (h^2 + \tau^2 h^2) \| \mathcal{H}(e_1) \|^2 + \tau^3 \| \| \nabla \partial_t e \| \|^2_{L^2(0, T)} \ight. \\
\left. + \tau^2 \| \| \partial_t e \| \|^2_{L^\infty(0, T)} + \tau h^4 \int_0^\tau \| \mathcal{H}(\partial_t e) \|^2 dt \right]. \]
(6.107)

Notice that, starting from (5.57) with \( u = \| \nabla \partial_t e \| \), similarly to (5.58), we obtain
\[ \| \| \partial_t e \| \|_{L^\infty(0, T)} \leq C_T \| \| \partial_t e \| \|^2_{L^2(0, T)} + \| \| \partial_t e \| \|^2_{L^2(0, T)} \right]^{1/2}. \]

Thus noting that \( h \leq \text{diam}(\Omega) \), using again the upper bound \( \tau \leq 1/(2\eta) \) and extending the integral to the whole interval \((0, T)\) in (6.107), from the latter inequality we infer the existence of another constant \( \bar{C}_0 \) independent of \( h \) and \( \tau \), such that,
\[ \bar{E}_0 \leq \bar{C}_0^2 \left[ h^2 \{ \| \mathcal{H}(e_0) \|^2 + \| \mathcal{H}(e_1) \|^2 + \| \mathcal{H}(\partial_t e) \|^2_{L^2(0, T)} \} \right. \\
\left. + \tau^2 \| \partial_t e \|^2_{L^2(0, T)} + \| \| \nabla \partial_t e \| \|^2_{L^2(0, T)} \right]. \]
(6.108)

### 6.2 Error estimates

In order to fully exploit the stability inequality (4.47) we further define,
\[ \bar{S}_N := \sum_{k=1}^{N-1} \left( \frac{\tau}{2} | \hat{F}^k |^2 h + \frac{3}{2} \| d^k \|^2 + \frac{\tau}{2} | \hat{G}^k |^2 \right), \]
(6.109)

According to (5.73), in order to estimate \( \bar{S}_N \) under the regularity Assumption* on \( e \), we resort to the estimates (5.74), (5.94) and (5.95). Using the inequality \( |a \cdot b| \leq |a||b| \) for \( a, b \in \mathbb{R}^M \), it is easy to see that there exists two constants \( \bar{C}_F \) and \( \bar{C}_G \) independent of \( h \) and \( \tau \) such that
\[ \sum_{k=1}^{N-1} \frac{\tau}{2} | \hat{F}^k |^2 h \leq \bar{C}_F (h^2 + \tau^2) \| e \|^2_{H^4(\Omega \times (0, T))}, \]
(6.110)
On the other hand, recalling (5.58) we have \( k = 1, 2, \ldots, N - 1 \):

\[
\| \mathcal{H}(e^k) \|^2 \leq C_T^2 \int_0^T \left[ \| \mathcal{H}(e) \|^2 + \| \mathcal{H}(\partial_t e) \|^2 \right] dt.
\]

Therefore, since \( N = T/\tau \) we have:

\[
\sum_{k=1}^{N-1} \| \mathcal{H}(e^k) \|^2 \leq C_T^2 T \int_0^T \left[ \| \mathcal{H}(e) \|^2 + \| \mathcal{H}(\partial_t e) \|^2 \right] dt
\]

It follows from (6.113) and (5.74) that,

\[
\sum_{k=1}^{N-1} \frac{3}{2\tau} \| \mathcal{H} \|^2 \leq \tilde{C}_0^2 C_T^2 T \| \mathcal{H} \|^2 \leq 3h^4 \frac{3h^4}{2\tau^2} \left( \| |e| \|^2_{H^2(\Omega \times (0,T))} + \| |e| \|^2_{H^3(\Omega \times (0,T))} \right)
\]

Plugging (6.110), (6.114) and (6.111) into (6.109) we can assert that there exists a constant \( \tilde{C} \) depending on \( \Omega, T \) and \( \varepsilon \), but neither on \( h \) nor on \( \tau \), such that,

\[
\tilde{S}_N \leq \tilde{C}^2 \left( h^2 + \tau^2 + \frac{h^4}{\tau^2} \right) \left( \| e \|^2_{H^4(\Omega \times (0,T))} \right)
\]

Now recalling (4.47)-(4.46), provided (4.39) holds, together with \( \tau \leq 1/(2\eta) \), we have:

\[
\left[ \frac{\| e_{h,m+1} - e_{h,m} \|}{\tau} + \| \nabla e_{h,m+1} \|^2 + \| \nabla e_{h,m} \|^2 \right]^{1/2} \leq 2\sqrt{S_N + E_0 e^{\beta/2}}.
\]

This implies that, for \( m = 1, 2, \ldots, N - 1 \), it holds:

\[
\left[ \frac{\| e_{h,m+1} - e_{h,m} \|}{\tau} \right]^2 + \| \nabla (e_{h,m+1} - e_{h,m}) \|^2 \leq 2\sqrt{S_N + E_0 e^{\beta/2}}.
\]

Let us define a function \( e_h \) in \( \bar{\Omega} \times [0, T] \) whose value at \( t = k\tau \) equals \( e_h^k \) for \( k = 1, 2, \ldots, N \) and that varies linearly with \( t \) in each time interval \([k-1]\tau, k\tau\), in such a way that \( \partial_t e_h(x, t) = \frac{e_h^k(x) - e_h^{k-1}(x)}{\tau} \) for every \( x \in \bar{\Omega} \) and \( t \in ([k-1]\tau, k\tau) \).
Now we define \( \tilde{A}^{m+1/2}(\cdot) \) for any function or field \( A(\cdot, t) \) to be the mean value of \( A(\cdot, t) \) in \((m\tau, [m+1]\tau)\), that is \( \tilde{A}^{m+1/2} = \tau^{-1} \int_{m\tau}^{(m+1)\tau} A(\cdot, t) \, dt \). Clearly enough we have
\[
\left\| \frac{e^{m+1}_h - e^m_h}{\tau} - \frac{\tilde{e}^{m+1}_h - \tilde{e}^m_h}{\tau} \right\| \leq \left\| \{ \partial_t \tilde{e}^m_h \}^{m+1/2} - \{ \partial_t e^m \}^{m+1/2} \right\|^2. \tag{6.118}
\]
and also recalling (6.113) and (5.58)
\[
\left\| \frac{e^{m+1}_h - e^m_h}{\tau} - \frac{\tilde{e}^{m+1}_h - \tilde{e}^m_h}{\tau} \right\| \leq \left\| \int_{m\tau}^{(m+1)\tau} \frac{\partial_t \tilde{e}^m_h - \partial_t e^m}{\tau} \, dt \right\|^2 \\
\leq \left\| \left[ \int_{m\tau}^{(m+1)\tau} (\partial_t \tilde{e}^m_h - \partial_t e^m)^2 \, dt \right]^{1/2} \right\|^2 \\
\leq \frac{1}{\tau} \int_{m\tau}^{(m+1)\tau} \| \partial_t \tilde{e}^m_h - \partial_t e^m \|^2 \, dt
\]
which implies that
\[
\left\| \frac{e^{m+1}_h - e^m_h}{\tau} - \frac{\tilde{e}^{m+1}_h - \tilde{e}^m_h}{\tau} \right\| \leq C_h \tilde{C}_h h^2 \| \text{diam}(\Omega) \| \| e^m \|^2_{H^2(\Omega \times (0, T))}. \tag{6.119}
\]
On the other hand from (5.51) and (5.58) we have for \( j = m \) or \( j = m+1 \):
\[
\| \nabla (\tilde{e}^j_h - e^j) \|^2 \leq \tilde{C}_h^2 h^2 \| \mathcal{H}(e^j) \|^2 \leq \tilde{C}_h^2 \tilde{C}_h^2 h^2 \int_0^T \| \mathcal{H}(e^j) \|^2 + \| \mathcal{H}(\partial_t e^j) \|^2 \, dt,
\]
which yields,
\[
\| \nabla (\tilde{e}^j_h - e^j) \|^2 \leq C_h \tilde{C}_h h^2 \| e^m \|^2_{H^2(\Omega \times (0, T))}. \tag{6.120}
\]
Now using Taylor expansions about \( t = (m+1/2)\tau \) together with some arguments already exploited in this work, it is easy to establish that for a certain constant \( C \) independent of \( h \) and \( \tau \) it holds,
\[
\left\| \{ \tilde{\partial}_t e^m_h \}^{m+1/2} - [\partial_t e^m_h]^{m+1/2} \right\| \leq C \tau^2 \| e^m \|^2_{H^2(\Omega \times (0, T))}, \tag{6.121}
\]
where for every function \( g \) defined in \( \bar{\Omega} \times [0, T] \) and \( s \in \mathbb{R}^+ \), \( g^s(\cdot) = g(\cdot, s\tau) \).
Noticing that \( \{ \tilde{\partial}_t e^m_h \}^{m+1/2}(\cdot) \) is nothing but \( [\partial_t e^m_h]^{m+1/2}(\cdot) \), collecting (6.117), (6.118), (6.119), (6.120), (6.121), together with (6.108) and (6.115), we have thus proved the following a priori error estimate for scheme (3.8):

**Theorem 6.1.** Provided the CFL condition (4.39) is satisfied and \( \tau \) also satisfies \( \tau \leq 1/[2\eta] \), under Assumption * on \( e \), there exists a constant \( C \) depending only on \( \Omega, \varepsilon \) and \( T \) such that
\[
\max_{1 \leq m \leq N-1} \left\| [\partial_t (e^m_h - e^m)]^{m+1/2} \right\| + \max_{2 \leq m \leq N} \| \nabla (e^m_h - e^m) \| \\
\leq C(\tau + h + h^2/\tau) \left\{ \| e^m \|^2_{H^2(\Omega \times (0, T))} + \| \mathcal{H}(e^0) \| + \| \mathcal{H}(e_1) \| \right\}. \tag{6.122}
\]
In short, as long as τ varies linearly with h, first order convergence of scheme (3.8) in terms of either τ or h is thus established in the sense of the norms on the left hand side of (6.122).

7 Some important remarks and conclusions

I- As previously noted, the approach advocated in this work was extensively and successfully tested in the framework of the solution of CIPs governed by Maxwell’s equations. More specifically it was used with minor modifications to solve both the direct problem and the adjoint problem, as steps of an adaptive algorithm to determine the unknown dielectric permittivity. More details on this procedure can be found in [10, 12].

II- The method studied in this paper was designed to handle composite dielectrics structured in such a way that layers with higher permittivity are completely surrounded by layers with a (constant) lower permittivity, say with unit value. It should be noted however that the assumption that the minimum value of ε (equal to one) in the outer layer was made here only to simplify things. Actually under the same assumptions (6.122) also applies to the case where ε in inner layers is allowed to be smaller than in the outer layer, say ε < 1. For instance, if ε > 2/3 the upper bound (6.122) also holds for a certain mesh-independent constant C. This is because under such an assumption on ε it is possible to guarantee that the auxiliary problems (5.50) and (5.48) are coercive.

On the other hand in case ε can be less than or equal to 2/3, the convergence analysis of scheme (3.8) is a little more laborious. The key to the problem is a modification of the variational form (2.2) as follows. First of all we set ε_{min} = \min_{x \in \Omega} \epsilon(x). Then we recast (2.2) for every t ∈ (0, T) as : ∀v ∈ [H^{1}(\Omega)]^{3} it holds,

(\epsilon \partial \epsilon \mathbf{e}, \mathbf{v}) + (\nabla \mathbf{e}, \nabla \mathbf{v}) + \left(\nabla \cdot \left\{\frac{\epsilon \mathbf{e}}{\epsilon_{min}}\right\}, \nabla \cdot \mathbf{v}\right) − (\nabla \cdot \mathbf{e}, \nabla \cdot \mathbf{v}) + (\partial_{t} \mathbf{e}, \mathbf{v})_{\partial \Omega} = 0. \quad (7.123)

Akin to (2.2), problem (7.123) is equivalent to Maxwell’s equations (2.1). Indeed, integrating by parts in (7.123), for all v ∈ [H^{1}(\Omega)]^{3} we get,

(\epsilon \partial \epsilon \mathbf{e}, \mathbf{v}) + (\nabla \times \nabla \times \mathbf{e}, \mathbf{v}) − (\nabla \nabla \cdot \{\epsilon \mathbf{e}\}, \mathbf{v})
+ (\partial_{n} \mathbf{e} + \partial_{t} \mathbf{e}, \mathbf{v})_{\partial \Omega} + (\nabla \cdot \{\epsilon_{min}^{-1} \epsilon \mathbf{e}\} − \nabla \cdot \mathbf{e}, \mathbf{v} \cdot \mathbf{n})_{\partial \Omega} = 0 \quad (7.124)

Besides the Maxwell’s equations in \Omega × (0, T), this time the conditions on \partial \Omega × (0, T) are those resulting from (7.124), that is,

(\partial_{n} \mathbf{e} + \partial_{t} \mathbf{e}) \cdot \mathbf{n} + \nabla \cdot \{\epsilon_{min}^{-1} \epsilon \mathbf{e}\} − \nabla \cdot \mathbf{e} = 0,
(\partial_{n} \mathbf{e} + \partial_{t} \mathbf{e}) \times \mathbf{n} = 0. \quad (7.125)

Since \nabla \cdot \mathbf{e} = \nabla \cdot \{\epsilon \mathbf{e}\} = 0 on \partial \Omega, where \mathbf{e} is the solution of Maxwell’s equations in strong form (2.1), this field necessarily satisfies the boundary conditions (7.125) as well. Then in the same way as the solution of (2.2), this implies that the solution of (7.123) also fulfills \nabla \cdot \{\epsilon \mathbf{e}\} = 0.

However in this case, given g ∈ [L^{2}_{0}(\Omega)]^{3}, the auxiliary problem (5.48) must be modified into,

\begin{align*}
-\nabla^{2} \mathbf{v} − \nabla \{\epsilon_{min}^{-1} \epsilon − 1\} \nabla \cdot \mathbf{v} & = \mathbf{g} \quad \text{in } \Omega, \\
\partial_{n} \mathbf{v} \cdot \mathbf{n} + \nabla \cdot \{\epsilon_{min} \mathbf{e} \mathbf{v}\} − \nabla \cdot \mathbf{v} & = 0, \quad \text{on } \partial \Omega, \\
\partial_{n} \mathbf{v} \times \mathbf{n} & = 0. \quad (7.126)
\end{align*}
As a matter of fact this is the only real difference to be taken into account in order to extend to (7.123) the convergence analysis conducted in this paper. More precisely the final result that can be expected to hold for the fully discrete analog of (7.123), defined as (3.8) under the same assumptions, is an $O(h^\mu)$ error estimate, where $\mu \in (0, 1]$ is such that the solution of (7.126) belongs to $[H^{1+\mu}(\Omega)]^3$ for every $g \in [L^2_0(\Omega)]^3$.

III- Another issue that is worth a comment is the practical calculation of the term $(\nabla \cdot \varepsilon^k h, \nabla \cdot v)$ in (3.8). Unless $\varepsilon$ is a simple function such as a polynomial, it is not possible to compute this term exactly. That is why we advocate the use of the trapezoidal rule do carry out these computations. At the price of small adjustments in some terms involving norms of $\varepsilon$, the thus modified scheme remains stable in the sense of (4.47). Moreover the qualitative convergence result (6.122) also holds, provided we require a little more regularity from $\varepsilon$. We skip details for the sake of brevity.

As a conclusion, in this work we studied from the theoretical point of view a scheme to solve Maxwell’s equations of electromagnetism in terms of the sole electric field, combining an explicit finite difference time discretization with a lumped-mass $P_1$ finite element space discretization. After presenting the problem under consideration for an electric field satisfying absorbing boundary conditions, we supplied the scheme’s detailed description. A priori error estimates were next established under rather stringent regularity assumptions for both the dielectric permittivity and the electric field, provided a classical CFL condition is satisfied. However our methodology is likely to be effective for a dielectric permittivity having a much weaker regularity, and consequently the electric field as well. This assertion is supported by numerical evidence supplied in [4]. Although this study was carried out for three-dimensional problems it also applies to two-dimensional ones. Moreover, after performing pertinent simplifications, it can be extended to other boundary conditions such as Neumann conditions, as long as $\varepsilon$ is constant in a neighborhood of the boundary of the spacial domain. Additionally we indicated that the case where the dielectric permittivity does not attain a minimum at the boundary can be dealt with in a similar manner, eventually with some erosion of convergence rates. In short we undoubtedly indicated in both this work and [9] (see also [4]) that Maxwell’s equations can be efficiently solved with conforming linear finite elements in many relevant particular cases, among which lies the problem (2.1).

Acknowledgments: The research of the first author is supported by the Swedish Research Council grant VR 2018-03661. The second author gratefully acknowledges the financial support provided by CNPq/Brazil through grant 307996/2008-5.

References

[1] F. Assous, P. Degond, E. Heintze and P. Raviart, On a finite-element method for solving the three-dimensional Maxwell equations, J.Comput.Physics, 109, pp.222–237, 1993.

[2] S. Badia and R. Codina, A Nodal-based finite element approximation of the Maxwell problem suitable for singular solutions, SIAM J. Numer. Anal., 50-2 (2012), 398417.
[3] L. Beilina, M. Grote, Adaptive hybrid finite element/difference method for Maxwell’s equations, *TWMS J. of Pure and Applied Mathematics*, 1-2 (2010), 176-197.

[4] L. Beilina and V. Ruas, Numerical validation of an explicit $P_1$ finite-element scheme for Maxwell’s equations in a polygon with variable permittivity away from its boundary, *arXiv:1905.03619 [math.NA]*, 2019.

[5] L. Beilina, Energy estimates and numerical verification of the stabilized Domain Decomposition Finite Element/Finite Difference approach for time-dependent Maxwell’s system, *Cent. Eur. J. Math.*, 11-4 (2013), 702-733.

[6] L. Beilina, M. V. Klibanov, *Approximate global convergence and adaptivity for Coefficient Inverse Problems*, Springer, New York, 2012.

[7] L. Beilina, M. Cristofol and K. Niinimaki, Optimization approach for the simultaneous reconstruction of the dielectric permittivity and magnetic permeability functions from limited observations, *Inverse Problems and Imaging*, 9-1 (2015), 1-25.

[8] L. Beilina, N. T. Thanh, M.V. Klibanov and J. B. Malmberg, Globally convergent and adaptive finite element methods in imaging of buried objects from experimental backscattering radar measurements, *Journal of Computational and Applied Mathematics*, Elsevier

[9] L. Beilina and V. Ruas, Convergence of explicit $P_1$ finite-element solutions to Maxwell’s equations, to appear in *Mathematical and Numerical Approaches for Multi-wave Inverse Problems*, Springer Proceedings in Mathematics and Statistics, to appear.

[10] J. Bondestam-Malmberg and L. Beilina, An adaptive finite element method in quantitative reconstruction of small inclusions from limited observations, *Applied Mathematics and Information Sciences*, 12-1 (2018), 119.

[11] J. Bondestam-Malmberg, L. Beilina, Iterative regularization and adaptivity for an electromagnetic coefficient inverse problem, *AIP Conference Proceedings*, p.1863, art.nr.370002, 2017.

[12] J. Bondestam-Malmberg, *Efficient Adaptive Algorithms for an Electromagnetic Coefficient Inverse Problem*, doctoral thesis, University of Gothenburg, Sweden, 2017.

[13] J.H. Carneiro de Araujo, P.D. Gomes and V. Ruas, Study of a finite element method for the time-dependent generalized Stokes system associated with viscoelastic flow. *J. Computational Applied Mathematics*, 234-8 (2010), 2562-2577.

[14] C. M. Chen and V. Thomée, The lumped mass finite element method for a parabolic problem, *J. Austral. Math. Soc. Ser. B*, 26 (1985), 329–354.

[15] P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North Holland, 1978.

[16] P. Ciarlet Jr and J. Zou, Fully discrete finite element approaches for time-dependent Maxwell’s equations, *Numerische Mathematik*, 82-2 (1999), 193–219.
[17] P. Ciarlet Jr, Augmented formulations for solving Maxwell equations, *Computer Methods in Applied Mechanics and Engineering*, 194(25) (2005), 559–586.

[18] P. Ciarlet Jr and E. Jamelot, Continuous Galerkin methods for solving the time-dependent Maxwell equations in 3D geometries, *J. Comp. Physics*, 226(1) (2007), 1122-1135.

[19] G. Cohen and P. Monk, Gauss point mass lumping schemes for Maxwell’s equations, *Numerical Methods for Partial Differential Equations*, 14(1) (1998), 63–88.

[20] M. Costabel, A coercive bilinear form for Maxwell’s equations, *J. Math. Anal. Appl.*, 157(2) (1991), 527–541.

[21] M. Costabel and M. Dauge, Singularities of Maxwell’s equations on polyhedral domains, Analysis, Numerics and Applications of Differential and Integral Equations, *Pitman Research Notes in Mathematics Series*, Volume 379, p.69–76, 1998.

[22] M. Costabel and M. Dauge, Weighted regularization of Maxwell’s equations in polyhedral domains. A rehabilitation of nodal finite elements, *Num. Math.*, 93(2) (2002), 239–277.

[23] A. Elmkies and P. Joly, Finite elements and mass lumping for Maxwell’s equations: the 2D case. *Numerical Analysis, C. R. Acad. Sci. Paris*, 324 (1997), 1287–1293.

[24] B. Engquist and A. Majda, Absorbing boundary conditions for the numerical simulation of waves *Math. Comp.*, 31 (1977), 629–651.

[25] A. Ern and J.-L. Guermond, Finite element quasi-interpolation and best approximation, *ESAIM Math. Model. Numer. Anal.*, 21(4) (2017), 1367–1385.

[26] A. Ern and J.-L. Guermond, Analysis of the edge finite element approximation of the Maxwell equations with low regularity solutions, *Comp. Math with Appl.*, 75(3) (2018), 918–932.

[27] E. Jamelot, *Résolution des équations de Maxwell avec des éléments finis de Galerkin continu*, thèse doctorale, École Polytechnique, Palaiseau, France, 2005.

[28] B. Jiang, *The Least-Squares Finite Element Method. Theory and Applications in Computational Fluid Dynamics and Electromagnetics*, Springer-Verlag, Heidelberg, 1998.

[29] B. Jiang, J. Wu and L. A. Povinelli, The origin of spurious solutions in computational electromagnetics, *Journal of Computational Physics*, 125 (1996), 104–123.

[30] J. Jin, *The finite element method in electromagnetics*, Wiley, 1993.

[31] P. Joly, *Variational Methods for Time-dependent Wave Propagation Problems*, Lecture Notes in Computational Science and Engineering, Springer, 2003.

[32] P. Monk, *Finite Element Methods for Maxwell’s Equations*, Clarendon Press, 2003.

[33] P. B. Monk and A. K. Parrott, A dispersion analysis of finite element methods for Maxwell’s equations, *SIAM J. Sci. Comput.*, 15 (1994), 916–937.
[34] C.D. Munz, P. Omnes, R. Schneider, E. Sonnendrucker and U. Voss, Divergence correction techniques for Maxwell solvers based on a hyperbolic model, *J. Comp. Phys.*, 161(2000), 484-511.

[35] J.-C. Nédélec, Mixed finite elements in $\mathbb{R}^3$, *Numerische Mathematik*, 35 (1980), 315-341.

[36] S. Nicaise, Edge elements on anisotropic meshes and approximation of the Maxwell equations, *SIAM J. Numer. Anal.*, 39(3), 784-816.

[37] K. D. Paulsen, D. R. Lynch, Elimination of vector parasites in finite element Maxwell solutions, *IEEE Transactions on Microwave Theory Technologies*, 39 (1991), 395–404.

[38] V. Ruas, *Numerical Methods for Partial Differential Equations. An Introduction*, Wiley, 2016.

[39] V. Ruas and M.A. Silva Ramos, A Hermite Method for Maxwells Equations, *Applied Mathematics and Information Sciences*, 12-2 (2018), 271283.

[40] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer, 2nd ed., 1997.