Coordination for a Group of Autonomous Mobile Agents with Multiple Leaders

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Abstract: In this paper, we consider the coordination control of a group of autonomous mobile agents with multiple leaders. Different interconnection topologies are investigated. At first, a necessary and sufficient condition is proved in the case of fixed interconnection topology. Then a sufficient condition is proposed when the interconnection topology is switched. With a simple first-order dynamics model by using the neighborhood rule, both results show that the group behavior of the agents will converge to the polytope formed by the leaders.

Key Words: multi-agent systems, multiple leaders, convex set, polytope.

1 INTRODUCTION

Recent years have seen a large and growing literature concerned with the coordination of a group of autonomous agents, partly due to a broad application of multi-agent systems including flocking/swarming (e.g., [5, 6]), formation control (e.g., [1, 2]), and sensor networks (e.g., [8]). Leader-following problem is one of the important coordination problems in the studies of multiple mobile agents. Even in some leaderless cases, concepts like “virtual leader” are proposed to study cooperative behaviors [9, 13]. Usually, there is only one leader in the leader-following formulation, though sometimes, the leader may be active with unmeasurable states [4]. However, in some practical situations of formation or foraging, the formulation of multiple leaders may be needed. In [2], Lin et al. discussed an interesting model for a group of agents with straight-line formation containing two ”edge leaders”, where all the agents converge to a uniform distribution on the line segment specified by the two edge leaders. In [3], a simple model was given to simulate foraging and demonstrate that, the larger the group is, the smaller the proportion of “leaders” is needed to guide the group.

Inspired by [2, 3], in this paper we consider the coordination behavior of mobile agents with multiple leaders. By using a neighborhood rule, we show that a group of agents will converge to the polytope formed by the leaders (that is, the leaders forms the vertex set of the polytope) in two different cases of interconnection topologies associated with the agents and the leaders, and demonstrate that the collective behavior changes as the connectivity of the interconnection between agents and some leader increases through some simulations.

This paper is organized as follows. Section 2 presents the problem formulation for multiple leaders. Then, both fixed interconnection topology and switched topology are considered, and the corresponding coordination behaviors are analyzed in Sections 3 and Section 4. In Section 5, two numerical examples are shown. Finally, concluding remarks are given in Section 6.

2 PROBLEM FORMULATION

Before formulating our problem, we first introduce some basic concepts and notations in graph theory that will be used [10].

Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, A) \) be a weighted undirected graph of order \( n \) with a set of nodes \( \mathcal{V} = \{1, 2, ..., n\} \), set of edges \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \), and a weighted adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) with nonnegative elements. The node indexes belong to a finite index set \( \mathcal{I} = \{1, 2, ..., n\} \). An edge is denoted by \( (i, j) \), which means node \( i \) and node \( j \) are adjacent. A path from \( i \) to \( j \) in \( \mathcal{G} \) is a sequence of distinct nodes starting with \( i \) and ending with \( j \) such that the consecutive nodes are adjacent. If
there is a path between any two nodes of a graph $G$, then $G$
 is connected, otherwise disconnected. An induced subgraph
 $\mathcal{X}$ of $G$ that is maximal, subject to being connected, is called
 a component of $G$. The element $a_{ij}$ associated with the edge
 of the graph is positive, i.e. $a_{ij} > 0 \iff (i, j) \in \mathcal{E}$. Moreover,
 we assume $a_{ii} = 0$ for all $i \in I$. The set of neighbors of
 node $i$ is denoted by $N_i = \{ j \in V : (i, j) \in \mathcal{E}\}$.
 A diagonal matrix $D = diag(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n}$ is a degree
 matrix of $G$, whose diagonal elements $d_i = \sum_{j \in N_i} a_{ij}$ for
 $i = 1, \ldots, n$. Then the Laplacian of the weighted graph $G$
 is defined as $L = D - A \in \mathbb{R}^{n \times n}$. The following result is well-
 known in algebraic graph theory (e.g. [10]) and establishes
 a direct relationship between the graph connectivity and its
 Laplacian.

**Lemma 1** Let $G$ be a graph on $n$ vertices with Laplacian $L$. Denote the eigenvalues of $L$ by $\lambda_1(L), \ldots, \lambda_n(L)$
 satisfying $\lambda_1(L) \leq \cdots \leq \lambda_n(L)$. Then $\lambda_1(L) = 0$
 and $1 = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$ is its eigenvector. Moreover, if $G$
 is connected, $\lambda_2 > 0$.

Next, we introduce some notations in convex analysis [12]
 and linear algebra [11]. Let $S \subset \mathbb{R}^m$. $S$ is said to be convex
 if $(1-\gamma)x + \gamma y \in S$ whenever $x \in S$, $y \in S$ and $0 < \gamma < 1$. A vector sum $\gamma_1 x_1 + \cdots + \gamma_n x_n$ is called a convex
 combination of $x_1, \ldots, x_n$ if the coefficients $\gamma_i$ are all non-
 negative and $\gamma_1 + \cdots + \gamma_n = 1$. Here, $\gamma_i$ can be interpreted
 as proportions. The intersection of all convex sets containing
 $S$ is the convex hull of $S$, denoted by $co(S)$. The convex hull of
 a finite set of points $x_1, \ldots, x_n \in \mathbb{R}^m$ is a polytope,
 denoted by $co\{x_1, \ldots, x_n\}$. If $x_2 - x_1, \ldots, x_n - x_1$ are
 linearly independent, the set of points $x_1, \ldots, x_n$ is said to be
 affinely independent. Then the polytope is called an $n-1$
 dimensional simplex and $x_1, \ldots, x_n$ are called the vertices
 of the simplex.

In this paper, we consider a system consisting of $n$ agents
 and $k$ leaders, and the interconnection topology among them
can easily be described by a simple graph. The purpose of
 the leaders is to guide the multi-agent behavior. Denote the
 positions of these static leaders by $x_0^i \in \mathbb{R}^m$, $i = 1, \ldots, k$.
 The interconnection topology $G$ associated with $n$ agents can
 be fixed or variable but may not be connected. If we regard
 the polytope formed by leaders as a virtual node and if one
 agent can see a vertex (i.e. some leader) of the polytope, we
 say that the agent is connected to the virtual node. By "the
 graph, $\tilde{G}$, of this system is connected", we mean that at least
 one agent in each component of $\tilde{G}$ is connected to the virtual
 node. Then we define a diagonal matrix $B$ to be a leader
 adjacency matrix associated with $\tilde{G}$ with diagonal elements
 $b_i$ $(i \in I)$ such that each $b_i$ is some positive number if agent
 $i$ is connected to the virtual node and 0 otherwise.

A continuous-time dynamics of $n$ agents is described as follows:

$$\dot{x}_i = u_i,$$  \hspace{1cm} (1)

where $x_i \in \mathbb{R}^m$ can be the position of agent $i$ and $u_i \in \mathbb{R}^m$
 its interconnection control inputs for $i = 1, \ldots, n$. As usual,
 we propose a neighbor-based feedback control as follows:

$$u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) + \sum_{q=1}^k b_i^q(\sigma)(x_0^q - x_i);$$  \hspace{1cm} (2)

$$i = 1, \ldots, n,$$

where nonnegative functions $b_i^q(\sigma(t)) > 0$ if and only if
 agent $i$ is connected to leader $q$ ($q = 1, \ldots, k$) when $t \in
 [t_l, t_{l+1})$ for $l = 0, 1, \ldots, \text{switching signal } \sigma : [0, \infty) \rightarrow
 \mathcal{P} = \{1, \ldots, N\}$ (N \in \mathbb{Z}^+,$ denotes the total number of all
 possible digraphs) is a switching signal that determines the
 interconnection topology $\tilde{G}$. If $\sigma$ is a constant function, then
 the corresponding interconnection topology is fixed.

Denote

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^{mn}, x_0 = \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^n \end{pmatrix} \in \mathbb{R}^{km}.$$

Let $B_\sigma = diag\{b_1^0(\sigma(t)), \ldots, b_n^0(\sigma(t))\} \in \mathbb{R}^{n \times n}$ be a
 diagonal matrix with non-negative diagonal entry $b_i^q(\sigma(t))$ for
 $q = 1, \ldots, k$ and $B_\sigma = [B_1^0, \ldots, B_n^0] \in \mathbb{R}^{n \times nk}$. Let

$$\Xi = \{\xi = (\xi_1, \ldots, \xi_n)^T | \xi_i \in co\{x_0^1, \ldots, x_0^n\}\}.$$  \hspace{1cm} (3)

Then we rewrite the closed-loop system in a compact form:

$$\dot{x} \in F(x),$$

with a set-valued function

$$F(x) = \{-(H_p \otimes I_m)x + [B_p(I_k \otimes I_n)] \otimes I_m x_0 | p \in \mathcal{P}\}$$

and

$$H_p = L_p + B_p(I_k \otimes I_n),$$

where $\otimes$ is the Kronecker product.

Throughout the paper, $R^+ = [0, \infty)$, $\| \cdot \|$ denotes the Eu-
clidean norm and $\langle \cdot, \cdot \rangle$ denotes inner product on $\mathbb{R}^{mn}$.

For non-empty $\Xi \subset \mathbb{R}^{mn}$, $d_\Xi : \mathbb{R}^{mn} \rightarrow R^+$ denotes its Eu-
clidean distance function given by $d_\Xi(x) = \frac{1}{\xi \in \Xi} ||x - \xi||^2$.

The objective of this work is to lead all the agents to enter the
region formed by the leaders; namely, $x_i$ $(i = 1, \ldots, n)$, for
each agent $i$, will be contained in a convex hull of $x^j_0$ ($j = 1, \ldots, k$) as $t \to \infty$. In other words,
\[
\lim_{t \to \infty} d_2(x(t)) = 0. \tag{4}
\]
In the following sections, the convergence of the system (3) will be studied based on $d_2(x)$, with either fixed interconnection topology or switched interconnection topology.

3 FIXED INTERCONNECTION TOPOLOGY

In this section, we will focus on the convergence analysis of a group of dynamic agents with fixed interconnection topology. In this case, the subscript $\sigma$ can be dropped.

Then the differential inclusion (3) becomes
\[
\dot{x} = -(H \otimes I_m)x + [B(I_k \otimes 1_n)] \otimes I_mx_0. \tag{5}
\]
The next lemma shows the relationship between the positive definiteness of a matrix $H = L + B(1_k \otimes I_n)$ and the connectivity of $\mathcal{G}$, which was proved in [4].

Lemma 2 If graph $\mathcal{G}$ is connected, then the symmetric matrix $H$ associated with $\mathcal{G}$ is positive definite.

Then a main result associated with the fixed interconnection topology is given as follows:

Theorem 1 For the system (3), (4) holds if and only if $\mathcal{G}$ is connected.

Proof: (Sufficiency) Let $\bar{x} = x - [H^{-1}B(I_k \otimes 1_n)] \otimes I_mx_0$. Then, from equation (5), we have
\[
\dot{\bar{x}} = -(H \otimes I_m)\bar{x}. \tag{6}
\]
For any initial value $\bar{x}(t_0)$ at initial time $t_0$, the solution of the system (6) is
\[
\bar{x}(t) = e^{-(H \otimes I_m)(t-t_0)} \bar{x}(t_0), \ t \geq t_0.
\]
From Lemma 2 the eigenvalues of $H \otimes I_m$ are positive, and therefore, $\bar{x}(t) \to 0$; namely, $x \to x^* = [H^{-1}B(I_k \otimes 1_n)] \otimes I_mx_0$, as $t \to \infty$.

Then we only need to prove that each vector $x^*_i \in R^m$ ($i = 1, \ldots, n$) can be represented by a convex combination of $x^p_0 \in R^m$ for $p = 1, \ldots, k$. It is equivalent to prove that $[H^{-1}B(I_k \otimes 1_n)] \otimes I_m$ is a row stochastic matrix which is a non-negative matrix and the sum of the entries in every row equals 1.

For $H$, there exists a positive number $\alpha$ such that $H = \alpha I - M$, where $M$ is a non-negative matrix and $\alpha > \rho(M)$ with $\rho(M) = \max\{\sqrt{\lambda}, \lambda \text{ is an eigenvalue of } M^T M\}$. In fact, the eigenvalue $\lambda_i(H) = \alpha - \lambda_i(M) > 0, i = 1, \ldots, n$.

Thus,
\[
(\alpha I - M)^{-1} = \frac{1}{\alpha} (I + \frac{1}{\alpha} M + \frac{1}{\alpha^2} M^2 + \cdots) \geq 0_{n \times n}
\]
or equivalently, $(H^{-1} \otimes I_m)$ is a non-negative matrix, and so is $[H^{-1}B(I_k \otimes 1_n)] \otimes I_m$.

Additionally, since
\[
(H \otimes I_m)(1_n \otimes 1_m) = (\sum_{q=1}^{k} B^q 1_n) \otimes 1_m,
\]
we have
\[
(H \otimes I_m)^{-1}(\sum_{q=1}^{k} B^q 1_n) \otimes 1_m
\]
\[
= ((H \otimes I_m)^{-1}[B(I_k \otimes 1_n)] \otimes I_m)(1_k \otimes 1_m)
\]
\[
= ([H^{-1}B(I_k \otimes 1_n)] \otimes I_m)(1_k \otimes 1_m) = 1_n \otimes 1_m.
\]
Note that $[H^{-1}B(I_k \otimes 1_n)] \otimes I_m$ is a matrix with every row sum equal to 1, which leads to the conclusion.

(Necessity) Suppose that $\mathcal{G}$ is not connected. Without loss of generality, we also suppose that there are $\kappa$ components in $\mathcal{G}$, where some components are connected with the virtual node. We can renumber the nodes of $\mathcal{G}$ such that Laplacian matrix associated with $\mathcal{G}$ takes the following form
\[
L = \begin{pmatrix}
L_{11} & 0 \\
0 & L_{22}
\end{pmatrix}
\]
and, correspondingly,
\[
B^q = \begin{pmatrix}
B_{11}^q & 0 \\
0 & 0
\end{pmatrix}
\]
where $L_{11} \in R^{s \times s} (0 \leq s < n)$ is the Laplacian associated with those components connected with the virtual node, $B_{11}^q \in R^{s \times s}$ is a nonzero diagonal matrix and 0’s denote some appropriate zero matrices.

Let $x = (y_1^T, y_2^T)^T$ with $y_1 \in R^m; y_2 \in R^{(n-s)m}$. Then (5) become:
\[
\dot{y}_1 = -(H_1 \otimes I_m)y_1 + [B_1(I_k \otimes 1_s)] \otimes I_mx_0,
\]
\[
\dot{y}_2 = -(L_{22} \otimes I_m)y_2,
\]
where $B_1 = [B_{11}^1, \ldots, B_{11}^k], H_1 = L_{11}+B_1(I_k \otimes 1_s)$. From the proof for the sufficient condition, it follows that $s$ agents corresponding to $y_1$ will approach to the leaders for an arbitrary initial conditions. However, the other agents corresponding to $y_2$ will stay static or diverge to some distinct locations which can arbitrarily exist and may not belong to the polytope $co\{x^1_0, \ldots, x^k_0\}$. This leads to a contradiction.

Remark 1 If $k = 1$, then Theorem 4 will be consistent with Theorem 4 in (4).
4 SWITCHED INTERCONNECTION TOPOLOGY

In this section, we consider the system (3) associated with switched interconnection topology. A result is given as follows.

Theorem 2 For the system (3), if $\bar{G}_p$ ($p \in P$) is connected, then (4) holds.

Proof: Since the topology $\bar{G}_p$ ($p \in P$) is connected, from Lemma 2 $H_p$ ($p \in P$) keeps positive definite, and moreover, $[H_p^{-1} B_p(I_k \otimes 1_n)] \otimes I_m x_0 \in \Xi$, from the differential inclusion (5), we have

$$\dot{x} \in \tilde{F}(x) = \left\{ -(H_p \otimes I_m)(x - \xi) | \xi \in \Xi, p \in P \right\}. \quad (7)$$

Consider the Dini derivative of $d_{\Xi}(x(t))$,

$$D^+ d_{\Xi}(x(t)) = \inf_{\xi \in \Xi} \frac{x - \xi, \dot{x}}{\xi \in \Xi} \leq -\lambda_1 d_{\Xi}(x(t)), \quad (8)$$

where $\lambda_1$ is the smallest (positive) eigenvalue of all possible $H_p$ for $p \in P$. Hence, $d_{\Xi}(x(t))$ is a decreasing nonnegative function and $d_{\Xi}(x(t)) = 0$ if and only if $x(t) \in \Xi$, then we have $d_{\Xi}(x(t)) \to 0$ as $t \to \infty$.

5 NUMERICAL EXAMPLES

In this section, we give numerical simulations to illustrate the coordination of multi-agent systems with multiple leaders. The following two examples are considered (see Fig. 1 and Fig. 2).

In the first example, we take $n = 5, k = 2, m = 1, t_0 = 0$ and the initial positions of agents and leaders are given as follows:

$$x_1(0) = 5, x_2(0) = 5.5, x_3(0) = 6, x_4(0) = 7, x_5(0) = 6.5, x_0^1 = 1, x_0^2 = 2.$$ 

Then the simulation results show that the group of agents will approach to the segment connecting the two leaders in Figs 3 – 6. In Fig. 4, when the number of agents linked to leader 1 increases, the agent group will move to leader 1 more closely. Moreover, in Fig. 6, when agent 5 is sensed by other agents, then it also moves to leader 1. In Fig. 5, though agent 2 is not connected with other agents but linked to leader 1, then it will reach the locality of leader 1 finally.
In the second simulation example, we take $n = 5$, $k = 3$, $m = 2$, $t_0 = 0$ and the initial positions of agents and leaders are given as follows:

\[ x_1(0) = (0, 0)^T, \quad x_2(0) = (1, 0)^T, \quad x_3(0) = (2, 0)^T, \]
\[ x_4(0) = (3, 0)^T, \quad x_5(0) = (4, 0)^T, \quad x_0^1 = (1, 1)^T, \]
\[ x_0^2 = (2, 2)^T, \quad x_0^3 = (1, 2)^T. \]

Here, the agents take a straight-line formation at the beginning.

From Fig. 7, we observe that this group of agents will enter the triangle formed by three leaders. Moreover, during the evolution, agents 2, 3, 4, and 5 still remain straight-line formation.

6 CONCLUSIONS

This paper addressed a coordination problem of a multi-agent system with multiple leaders. This group of agents were shown to approach to the region "contained" by the leaders if the interconnection graph is connected. Two interconnection cases, fixed topology and switched topology, were discussed. Moreover, numerical simulations were given to illustrate the theoretical analysis.

REFERENCES

[1] A. Fax, and R. M. Murray, Information flow and cooperative control of vehicle formations, IEEE Trans. on Automatic Control, Vol. 49, No. 9, 1465-1476, 2004.
[2] Z. Lin, and B. Francis, M. Maggiore, Necessary and sufficient graphical conditions for formation control of unicycles, IEEE Trans. Automatic Control, Vol. 50, No. 1, 121-127, 2005.
[3] I. D. Couzin, J. Krause, N. R. Franks, S. A. Levin, Effective leadership and decisionmaking in animal groups on the move, Nature, Vol. 433, 513-516, 2005.
[4] Y. Hong, J. Hu and L. Gao, Tracking control for multi-agent consensus with an active leader and variable topology, to appear in Automatica, 2006.
[5] C. W. Reynolds, Flocks, herds, and schools: a distributed behavioral model, Computer Graphics, ACM SIGGRAPH ’87 Conference Proceedings, Vol. 21, No. 4, 25–34, 1987.
[6] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Schochet, Novel type of phase transitions in a system of self-driven particles, Physical Review Letters, Vol. 75, 1226–1229, 1995.
[7] A. Jadbabaie, J. Lin, and A. S. Morse, Coordination of groups of mobile agents using nearest neighbor rules, IEEE Trans. Automatic Control, Vol. 48, No. 6, 988–1001, 2003.
[8] J. Cortes and F. Bullo, Coordination and geometric optimization via distributed dynamical systems, SIAM J. Control Optimization, Vol. 44, No. 5, 2005.
[9] H. Shi, L. Wang, and T. Chu, Virtual leader approach to coordinated control of multiple mobile agents with asymmetric interactions, Physica D, Vol. 213, 51-65, 2006.
[10] C. Godsil and G. Royle, Algebraic Graph Theory, New York: Springer-Verlag, 2001.
[11] R. Horn and C. Johnson, Matrix Analysis, New York: Cambridge University Press, 1985.
[12] R. T. Rockafellar, Convex Analysis, New Jersey: Princeton University Press, 1972.
[13] R. Olfati-Saber, Flocking for multi-agent dynamic systems: algorithms and theory, IEEE Trans. Automatic Control, Vol. 51, No. 3, 401-420, 2006.