It takes two to tango: estimation of the zero-risk premium strike of a call option via joint physical and pricing density modeling

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Abstract

It is generally said that out-of-the-money call options are expensive and one can ask the question from which moneyness level this is the case. Expensive actually means that the price one pays for the option is more than the discounted average payoff one receives. If so, the option bears a negative risk premium. The objective of this paper is to investigate the zero-risk premium moneyness level of a European call option, i.e. the strike where expectations on the option’s payoff in both the $\mathcal{P}$- and $\mathcal{Q}$-world are equal. To fully exploit the insights of the option market we deploy the Tilted Bilateral Gamma pricing model to jointly estimate the physical and pricing measure from option prices. We illustrate the proposed pricing strategy on the option surface of stock indices, assessing the stability over time of the zero-risk premium strike of a European call option.

Keywords: Pricing density, Physical density, Bilateral Gamma, Tilted Bilateral Gamma, Call option, Risk premium

1. Introduction

Each event in the financial market is characterized by both its likelihood and its price, which is why financial engineers make a distinction between the so-called $\mathcal{P}$-world and $\mathcal{Q}$-world. The $\mathcal{P}$-world is the physical world in which payoffs are realized. A probability measure in this world estimates the real probability on the occurrence of a particular event. Differently, the $\mathcal{Q}$-world is an artificial setting under which one determines the price. Probabilities under the pricing measure $\mathcal{Q}$ do not describe real-world probabilities but they reflect prices, the price a representative market player is willing to pay for getting a dollar in a particular state of the market.

For a contingent claim, the (discounted) expected realized payoff is the (discounted) expectation of the payoff in the $\mathcal{P}$-world, whereas the arbitrage-free price is the discounted expectation of the payoff in the $\mathcal{Q}$-world. A contingent claim is considered expensive when expectations in the $\mathcal{Q}$-world exceed those in the $\mathcal{P}$-world. To capture the difference in expectation under the $\mathcal{P}$- and

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\( \mathcal{Q} \)-probability measures, the concept of a risk premium is introduced and here modeled as

\[
\text{risk premium} = \frac{\text{expected}_\mathcal{P} \text{ payoff} - \text{expected}_\mathcal{Q} \text{ payoff}}{\text{expected}_\mathcal{Q} \text{ payoff}}, \tag{1}
\]

i.e. the ratio of the difference between the expected payoff in the \( \mathcal{P} \)-world and the expected payoff in the \( \mathcal{Q} \)-world to the expected payoff in the \( \mathcal{Q} \)-world. An expensive claim then bears a negative risk premium whereas an inexpensive claim bears a positive risk premium.

This article contributes to the literature by studying risk premia in European call options. We especially focus on the option with a zero-risk premium, which is, for a fixed maturity, completely determined by the so-called zero-risk premium strike. Under specific modeling assumptions, we prove the existence of this zero-risk premium strike for European call options and the nonexistence for European put options, which immediately justifies our focus on call options. We additionally show that the zero-risk premium strike is unique, i.e. it indicates the transition point from inexpensive to expensive call options. Based on an empirical study, we uncover a general stability of the zero-risk premium strike over time, with small fluctuations around a slightly in-the-money level, on average.

In order to calculate the risk premium, we need information on the physical and pricing probability measure of the asset underlying the call option at the given maturity. Today, the rich variety of traded vanilla options provide us with valuable information on the \( \mathcal{Q} \)-measure and so the pricing distribution of an asset’s return. The estimation of a pricing density from option data is often preceded by the allocation of an option pricing model. In 1973, Black, Scholes and Merton made a significant breakthrough in asset modeling when publishing what has come to be known as the Black-Scholes market model (Black and Scholes, 1973; Merton, 1973). Later on, alternative pricing models such as the Variance Gamma model (Madan and Seneta, 1990; Madan et al., 1998) are successfully introduced to improve on the ideas of Black, Scholes and Merton. More recently, Küchler and Tappe (2008) suggested the four-parameter class of Bilateral Gamma processes to model the fluctuations of the financial market.

Instead of using option data, a physical return density is often inferred from historical time series data on the return of an asset, see for instance the digital moment estimation method as presented in Madan (2015). However, historical data is backward looking and only extended with one new observation each day. Leveraging the distributional wealth of the option market, we therefore elaborate on the methodology of Madan et al. (2020) to extract physical distributional information from option data. To this purpose, we deploy the Tilted Bilateral Gamma option pricing model which allows for the simultaneous extraction of model parameters according to both the physical and pricing probability measure.

The Tilted Bilateral Gamma model proceeds from the Bilateral Gamma model by imposing a U-shaped pricing kernel, or measure change, on the physical probability measure. Using this type of kernel, we distinguish from prior research on risk premia in options, such as that of Coval and Shumway (2001). The authors show that, under the assumption of a monotonically declining pricing kernel, risk premia in call options are always positive and increasing with the strike price. However, the existence of a declining pricing kernel is empirically rejected by many studies and evidence is mounting for a locally increasing pricing kernel. Cuesdeanu and Jackwerth (2018b) review the literature on and confirm the existence of this so-called pricing kernel puzzle. Among others, Bakshi et al. (2010), Christoffersen et al. (2013), Madan (2016) and Cuesdeanu and Jackwerth (2018a) report on a broad empirical support for the U-shape of the kernel.
The outline of the rest of the paper is as follows. Section 2 formalizes the definition of a risk premium and in particular a zero-risk premium strike and confirms its existence under certain modeling assumptions. The theory behind the Tilted Bilateral Gamma model is presented in Section 3, as well as the calibration methodology, which results in a joint estimation of physical and pricing distributional information from option prices. Section 4 elaborates on a numerical example, based on option surfaces of the S&P500 and DAX index. It reports on the empirical evolution and position of the zero-risk premium strike over time. Finally, Section 5 concludes.

2. The zero-risk premium strike of a European call option

The payoff of a contingent claim depends on the market performance of the underlying asset. The primary types of contingent claims are options. A European call option gives the right, not the obligation, to the buyer of the option to buy the underlying asset from the writer of the option at a fixed, future point \( T \) in time, called the maturity, for a predetermined price, called the strike \( K \).

Consider an asset \( S \), with level \( S_t \) at time \( t \). Let \( R_T = \ln(S_{t+T}) - \ln(S_t) \) be the \( T \)-period rate of return on this same asset. The payoff from buying a European call (\( EC \)) option on asset \( S \), at time \( t \), is then generally given by

\[
\text{payoff } EC(K, T) = (S_{t+T} - K)^+ = (S_t e^{R_T} - K)^+,
\]

\[
= \begin{cases} 
S_t e^{R_T} - K & \text{if } S_t e^{R_T} \geq K \\
0 & \text{if } S_t e^{R_T} < K.
\end{cases}
\]

2.1. Definition of a zero-risk premium strike

In the physical world, or \( \mathcal{P} \)-world, the market performance of an asset is modeled according to a physical probability density function. Using Equation (2), the expected payoff of the European call option under the physical return density \( f_{R_T} \) of asset \( S \) is determined as

\[
\text{expected}_\mathcal{P} \text{ payoff } EC(K, T) = \mathbb{E}_\mathcal{P}[(S_t e^{R_T} - K)^+],
\]

\[
= \int_{-\infty}^{+\infty} (S_t e^{x} - K)^+ f_{R_T}(x)dx.
\]

The discounted value of the expectation in Equation (3) results in the expected realized payoff at the time of buying the option.

In the pricing world, or \( \mathcal{Q} \)-world, the performance of the asset \( S \) is modeled using the corresponding pricing probability density function. As such, the expected payoff under the pricing measure \( \mathcal{Q} \) is determined as

\[
\text{expected}_\mathcal{Q} \text{ payoff } EC(K, T) = \mathbb{E}_\mathcal{Q}[(S_t e^{R_T} - K)^+],
\]

\[
= \int_{-\infty}^{+\infty} (S_t e^{x} - K)^+ g_{R_T}(x)dx,
\]

where \( g_{R_T} \) is the pricing return density of asset \( S \). Note that the arbitrage-free price of the option is given by the discounted value of the expectation in Equation (4).
Connecting the option’s expected payoff under the physical measure $\mathcal{P}$ to the corresponding expected payoff under the pricing measure $\mathcal{Q}$ naturally leads to the concept of a risk premium, defined as

$$
\text{risk premium } EC(K, T) = \frac{\mathbb{E}_P[(S_t e^{R_T} - K)^+]}{\mathbb{E}_Q[(S_t e^{R_T} - K)^+]} - \frac{\mathbb{E}_Q[(S_t e^{R_T} - K)^+]}{\mathbb{E}_Q[(S_t e^{R_T} - K)^+]} = 1.
$$

From Equation (5) we see that the risk premium is determined by the gap between the $\mathcal{P}$ and $\mathcal{Q}$ probability measures and so decided upon the shape and location of the pricing density with respect to the physical density. The risk premium is the return one can expect from buying an held-to-maturity European call option at time $t$ and it can be seen as a compensation directly related to the uncertainty on the future asset level. As options are often used for hedging purposes or to speculate on the perceived market uncertainty, one can expect negative risk premia in some cases. This will occur when the option’s price exceeds the discounted average payoff. The option is then also referred to as expensive.

For a fixed maturity $T$, we are interested in identifying the strike $K_{t,T}$ such that

$$
\mathbb{E}_P[(S_t e^{R_T} - K_{t,T})^+] = \mathbb{E}_Q[(S_t e^{R_T} - K_{t,T})^+],
$$

i.e. the strike where expectations on the payoff of a European call option are equal under both the $\mathcal{P}$ and $\mathcal{Q}$ probability measures. This strike thus determines the European call option with a zero-risk premium. We also accept this as the definition of the zero-risk premium strike and refer to it as $K_{t,T}$, recognizing the dependency upon the fixed maturity $T$ on the one hand and the moment of buying, time $t$, on the other hand. The zero-risk premium option with maturity $T$ is equivalently defined by the moneyness level $k_{t,T} = K_{t,T}/S_t$.

2.2. Conditions on the existence of a zero-risk premium strike

In what follows, we discuss the conditions that guarantee a solution to Equation (6) and so the existence of a call option’s zero-risk premium strike. We also briefly touch upon the European put option case to further substantiate our focus on call options.

The zero-risk premium strike of a European call option is defined by Equation (6), which we can rewrite in terms of the asset $S$ instead of the return $R_T$ as

$$
\int_{K_{t,T}}^{+\infty} (x - K_{t,T}) f_S(x) dx = \int_{K_{t,T}}^{+\infty} (x - K_{t,T}) g_S(x) dx,
$$

using the $T$-period physical and pricing density, respectively $f_S$ and $g_S$, of asset $S$ and the expressions in Equation (3) and Equation (4).

As opposed to a European call option, a European put option gives the right, not the obligation, to the buyer of the option to sell the underlying asset to the writer of the option at a fixed, future point $T$ in time, called the maturity, for a predetermined price, called the strike $K$. The payoff from buying a European put ($EP$) option on asset $S$, at time $t$, is then generally given by

$$
\text{payoff } EP(K, T) = (K - S_{t+T})^+ = (K - S_t e^{R_T})^+,
$$

$$
\begin{cases} 
K - S_t e^{R_T} & \text{if } S_t e^{R_T} \leq K \\
0 & \text{if } S_t e^{R_T} \geq K.
\end{cases}
$$

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The equivalent of Equation (7) for a European put option with the same features then becomes
\[
\int_0^{K_{t,T}} (K_{t,T} - x) f_S(x) \, dx = \int_0^{K_{t,T}} (K_{t,T} - x) g_S(x) \, dx.
\] (9)

Representing the corresponding \( T \)-period cumulative distribution functions of \( f_S \) and \( g_S \) as \( F_S \) and \( G_S \), integration by parts of Equation (7) and Equation (9) respectively leads to
\[
\int_{K_{t,T}}^{+\infty} (1 - F_S(x)) \, dx = \int_{K_{t,T}}^{+\infty} (1 - G_S(x)) \, dx,
\] (10)

for the European call option and
\[
\int_0^{K_{t,T}} F_S(x) \, dx = \int_0^{K_{t,T}} G_S(x) \, dx,
\] (11)

for the European put option. Based on Equation (10) and Equation (11), we now define
\[
c(K) = \int_K^{+\infty} (1 - F_S(x)) \, dx - \int_K^{+\infty} (1 - G_S(x)) \, dx,
\] (12)
\[
p(K) = \int_0^K F_S(x) \, dx - \int_0^K G_S(x) \, dx.
\] (13)

**Proposition 1.** If for all \( x \in (0, +\infty) \) it holds that
\[
F_S(x) \leq G_S(x),
\]

there will not exist a zero-risk premium strike for the European call option on asset \( S \), neither for the European put option on this same asset.

**Proof.** The strike \( K \) is a zero-risk premium strike for the European call option on asset \( S \) if \( c(K) = 0 \). Likewise, \( K \) is a zero-risk premium strike for the European put option on asset \( S \) if \( p(K) = 0 \). It is therefore sufficient to show that there exists no such strike for both functions \( c \) and \( p \).

The expressions in Equation (12) and Equation (13) result in respectively \( c(\infty) = 0 \) and \( p(0) = 0 \). Besides, it is easy to see that for each \( K \in [0, +\infty) \) :
\[
c'(K) = F_S(K) - G_S(K) = p'(K),
\] (14)

and so
\[
c'(0) = p'(0) = c'(\infty) = p'(\infty) = 0.
\] (15)

Since \( c \) and \( p \) have the same derivative, it holds that
\[
c(K) = - \int_K^{+\infty} c'(u) \, du = - \int_K^{+\infty} p'(u) \, du = p(K) - p(\infty),
\] (16)
\[
p(K) = \int_0^K p'(u) \, du = \int_0^K c'(u) \, du = c(K) - c(0).
\] (17)

In general, the fundamental drift of the asset will exceed the risk-free rate of return to reflect risk compensation. Under the conditions of arbitrage-free pricing we then have
\[
c(0) > 0,
\] (18)
such that Equation (17) results in
\[ p(\infty) = -c(0) < 0. \] (19)

The condition that \( \forall x \in (0, +\infty) : F_S(x) \leq G_S(x) \) now easily translates in both \( c'(K) \leq 0 \) and \( p'(K) \leq 0 \) for each value of \( K \in (0, +\infty) \), using the equality in Equation (14). \( c' \leq 0 \) together with \( c(0) > 0 \) and \( c(\infty) = 0 \) leads to the conclusion that \( c \) can never be zero, meaning that there exists no solution to Equation (7) and no zero-risk premium strike for the call option.

\( p' \leq 0 \) together with \( p(0) = 0 \) and \( p(\infty) < 0 \) leads to the conclusion that \( p \) is always negative and so no zero-risk premium strike for the put option exists either, which ends the proof. A graphical clarification can be found in Figure 1a and 1b.

First, note that the condition in Proposition 1 can be translated into \( F_S \) first-order stochastically dominating \( G_S \). Second, in Proposition A1, we show that the positioning of the density functions as in Figure 1c, i.e. exactly one point of intersection, results in first order stochastic dominance of the respective cumulative density functions. No zero-risk premium for both the European call and European put option will exist in that situation.

Next, a sufficient condition on the existence of a zero-risk premium strike for call options is derived in Proposition 2.

**Proposition 2.** If the cumulative distribution functions \( F_S \) and \( G_S \) of asset \( S \) cross exactly once, meaning that there is a unique \( x \in (0, +\infty) \) such that
\[ 0 < F_S(x) = G_S(x) < 1, \]
there exists a zero-risk premium strike for the European call option on this asset. Moreover, the zero-risk premium strike is unique. Under the same condition, there will not exist a zero-risk premium strike for the European put option on asset \( S \).

**Proof.** Since price dominates probability in the left tail, it is expected for all \( x \) close to zero that
\[ f_S(x) - g_S(x) < 0. \] (20)

Suppose that \( F_S \) and \( G_S \) cross exactly once at strike \( K_c \), i.e. \( F_S(K_c) = G_S(K_c) \). Combining the results in Equation (14) and Equation (20), it then holds that
\[ \forall 0 < K < K_c : c'(K) = p'(K) = F_S(K) - G_S(K) \leq 0, \] (21)

and \( c \) and \( p \) are decreasing functions for all \( K \) larger than \( K_c \). Besides, Equation (14) results in
\[ \forall K > K_c : c'(K) = p'(K) = F_S(K) - G_S(K) \geq 0, \] (22)

and both \( c \) and \( p \) are increasing functions for all \( K \) larger than \( K_c \).

From Equation (12) we see that
\[ c(K_c) = \int_{K_c}^{+\infty} (1 - F_S(x))dx - \int_{K_c}^{+\infty} (1 - G_S(x))dx, \]
\[ = \int_{K_c}^{+\infty} (G_S(x) - F_S(x))dx < 0, \]
and $c$ is also negative for all $K \geq K_c$. As $c(\infty) = 0$ and $c$ only increases for $K > K_c$, there will not exist a $K \in [K_c, +\infty)$ such that $c(K) = 0$. However, as $c$ decreases over all $0 < K < K_c$, $c(0) > 0$ and $c(K_c) < 0$, there exists a unique $K_T \in (0, K_c)$ such that $c(K_T) = 0$. This $K_T$ is called the zero-risk premium strike for the European call option on asset $S$.

As $p(0) = 0$ and $p$ only decreases for all $K \in (0, K_c)$, there will not exist a $K$ in this region such that $p(K) = 0$. Also, as $p$ only increases over all $K > K_c$ and $p(\infty) < 0$, there will not exist a $K \in [K_c, +\infty)$ such that $p(K) = 0$ and thus no zero-risk premium strike for the European put option on asset $S$. A graphical clarification can again be found in Figure 1d and 1e.

The density functions, resulting from the cumulative distribution functions in Figure 1d, are added in Figure 1f. In Proposition A2, we show that the situation as presented in Figure 1f, i.e. two density functions crossing exactly twice, results in cumulative distribution functions that meet the conditions in Proposition 2. In that case, a unique zero-risk premium strike for the European call option exists.

3. Joint density estimation methodology

An accurate estimation of both the physical density and the pricing density of the underlying asset are crucial in determining the risk premium of a European call option. In what follows, we advocate
the U-shape of the measure change between the $\mathcal{P}$ and $\mathcal{Q}$ probability measures, which gives rise to the pricing strategy of Madan et al. (2020).

3.1. The pricing density as U-shaped perturbation of the physical density

The pricing density of an asset’s return arises naturally from the corresponding physical density, acknowledging a U-shaped pricing kernel, also often called change-of-measure function. Following Cochrane (2005), we accept the existence of a pricing kernel $m(R)$ such that the price $p_t$ at time $t$ of a security paying out a cash-flow $cf(R)$ after a period of length $T$ equals

$$p_t = \exp(-rT)\mathbb{E}_\mathcal{P}[m(R)cf(R)],$$ (23)

with $r$ the $T$-period risk-free rate of return. That way, the price of a European call option at time $t$ is represented as

$$\text{price } EC(K, T) = \exp(-rT) \int_{-\infty}^{+\infty} (S_t e^x - K)^+ m(x)f_{R_t}(x)dx.$$ (24)

The pricing kernel thus relates the price of a security to its expected payoff under measure $\mathcal{P}$, i.e. it informs us on how to transform subjective probabilities into pricing ones. As such, the kernel reflects a representative market player’s assessment on different states of the market: it is more valuable to earn a dollar in a state of the market where the own wealth is low (Cuesdeanu and Jackwerth, 2018b).

According to this preference-based perspective, Bakshi and Madan (2008) present a theoretical framework that gives rise to a U-shaped pricing kernel. The theory relies on the aggregation of the assessment of two classes of heterogeneous, risk averse investors, either holding a long position in the market or a short position. Allowing for short positions is a crucial assumption that leads to a convex, U-shaped pricing kernel.

On the one hand, long investors lose wealth in negative return states of the market. Being risk averse and seeking loss protection, these investors are willing to pay a premium for receiving a dollar in highly negative market situations. On the other hand, short investors are losing wealth in a positive return state of the market. The same reasoning leads to the willingness to pay a premium in highly positive return states. To summarize, the declining part of the U-shaped kernel thus reflects the risk aversion of investors having a long position in the market, while the inclining part represents the risk aversion of investors having a short position in the market.

Following Madan et al. (2020), we construct the U-shaped pricing kernel, connecting the pricing density $g$ to the physical density $f$, as the weighted sum of two exponential functions. We define

$$g(x) = C \cdot \left[(1 - p) \cdot e^{-\eta x} + p \cdot e^{\zeta x}\right] \cdot f(x).$$ (25)

The constant $C$ is needed to ensure that $g$ is a proper density function and so

$$C^{-1} = \int_{-\infty}^{\infty} \left[(1 - p) \cdot e^{-\eta x} + p \cdot e^{\zeta x}\right] \cdot f(x)dx.$$ (26)

Investors’ preferences are thus characterized introducing the parameters $\eta$, $\zeta$ and $p$. The first parameter $\eta$ represents the risk aversion coefficient for being in a long position and likewise, $\zeta$ represents the risk aversion coefficient for being in a short position. The last parameter $p$ takes on
values between 0 and 1 and weighs the importance of the declining part of the U-shape against the
importance of the inclining part.

Note that it will be meaningful to calculate the risk premium and especially the zero-risk premium
strike of a call option under the assumption of a U-shaped pricing kernel. Indeed, this kernel lifts
both tails of the physical density, which results in exactly two points of intersection of the physical
and corresponding pricing density. An example is given later on, in Figure 3a and 3b. Combining
the results in Proposition A2 and 2, the existence and uniqueness of the zero-risk premium strike
is confirmed.

3.2. The simultaneous calibration procedure

The pricing of European options now enables the extraction of information on both the physical
density $f$ and the pricing density $g$, using the relation as given in Equation (25). Classically, one
determines an optimal set of parameters for a pricing model with pricing density $g$, by minimizing
the distance between available market prices on European options and the respective model prices.
We evaluate this distance in terms of the root mean squared error (RMSE), which is defined as

$$\text{RMSE} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\text{market price}_i - \text{model price}_i)^2}, \quad (27)$$

with $N$ the total number of available market prices.

To calculate the model price, $MoEC(K, T)$, of a European call option, we use the pricing formula
of Carr and Madan (1999):

$$MoEC(K, T) = \exp\left(-\frac{\alpha \log(K)}{\pi}\right) \int_{-\infty}^{\infty} \exp\left(-i\nu \log(K)\right) \varrho(\nu) d\nu, \quad (28)$$

where

$$\varrho(\nu) = \frac{\exp(-rT)\mathbb{E}_Q[\exp(i(\nu - (\alpha + 1)i)\log(S_T))]}{\alpha^2 + \alpha - \nu^2 + i(2\alpha + 1)\nu}, \quad (29)$$

and $\alpha$ a positive constant equal to 1.5. This formula makes use of the characteristic function $\phi_g$ of
the log-price process under the pricing measure $Q$. The numerical technique used to approximate
the integral in Equation (28) is based on Fast Fourier Transforms (FFT) and Simpson’s integral
weighting scheme. An extensive discussion on this technique can be found in Madan and Schoutens
(2016). Combining the model price of a European call option with the put-call parity of Stoll
(1969), the price of a European put option with the same features is calculated.

The characteristic function $\phi_g$, defined as

$$\phi_g(u) = \int_{-\infty}^{+\infty} \exp(iux)g(x)dx,$$ associated to the pricing density $g$, results from the physical characteristic function $\phi_f$, combining
Equation (25) and Equation (26) into

$$\phi_g(u) = \frac{(1-p) \cdot \phi_f(u + i\eta) + p \cdot \phi_f(u - i\zeta)}{(1-p) \cdot \phi_f(i\eta) + p \cdot \phi_f(-i\zeta)}. \quad (30)$$

Equation (25) and Equation (30) directly relate the pricing density $g$ to the physical density $f$. We
see that an optimal set of parameters for the pricing density $g$ can be split into a set of parameters
characterizing the U-shaped measure change on the one hand, and a set of parameters characterizing
the physical density $f$ on the other hand. The calibration procedure thus results in a simultaneous
extraction of information on both densities.
3.3. From Bilateral Gamma to Tilted Bilateral Gamma

The theory in Section 3.1 and 3.2 only enables the extraction of physical distributional information from option data if preceded by the allocation of a distributional family to the characteristic function $\phi_f$. Madan and Seneta (1990) and Madan et al. (1998) suggested the (general) Variance Gamma (VG) process to model stock market logarithmic returns as an alternative for the Black-Scholes (BS) model. Following Madan and Schoutens (2016), a Variance Gamma process $X^{VG} = \{X^{VG}_t \mid t \geq 0\}$ is defined in terms of two independent Gamma processes $X^{G}_1 = \{X^{G}_{1,t} \mid t \geq 0\}$ and $X^{G}_2 = \{X^{G}_{2,t} \mid t \geq 0\}$ with scale parameters respectively equal to $b_p$ and $b_n$ and a common shape parameter $c$ by

$$X^{VG}_t = X^{G}_{1,t} - X^{G}_{2,t} \sim VG(b_p, b_n, ct). \quad (31)$$

The characteristic function of the Variance Gamma process at time $t$ can be expressed as

$$\phi_{X^{VG}_t}(u) = \left(\frac{1}{1 - iub_p}\right)^{ct} \left(\frac{1}{1 + iub_n}\right)^{ct}. \quad (32)$$

The Variance Gamma model does not distinguish between the speed of the upward and downward movement of the stock as the almost surely non-decreasing Gamma processes have a common speed parameter $c$. However, among others, Madan and Wang (2017) convince that, in reality, the market is going up more frequently and with smaller steps compared to the downward movement.

The Bilateral Gamma (BG) model of Küchler and Tappe (2008) arises from the Variance Gamma model but encounters different speed as well as scale parameters for the Gamma distributed components. Comparable to the work of Madan et al. (2020), we introduce the parameters $c_p$ and $c_n$ such that the characteristic function of the Bilateral Gamma process at time $t$ follows from Equation (32) as

$$\phi_{X^{BG}_t}(u) = \left(\frac{1}{1 - iub_p}\right)^{c_p t} \left(\frac{1}{1 + iub_n}\right)^{c_n t}. \quad (33)$$

For stability reasons in the calibration of the model parameters, we shift attention from the $\{b, c\}$-parameterization of a Gamma process towards a mean($\mu$)-variance($\sigma^2$)-parameterization. If, in general, $b = \frac{\sigma^2}{\mu}$ and $c = \frac{\mu^2}{\sigma^2}$, Equation (33) is equivalent to

$$\phi_{X^{BG}_t}(u) = \left(\frac{1}{1 - iu \sigma_p^2}\right)^{\frac{\sigma_p^2 t}{\mu_p}} \left(\frac{1}{1 + iu \sigma_n^2}\right)^{\frac{\sigma_n^2 t}{\mu_n}}. \quad (34)$$

The Bilateral Gamma model does not explicitly incorporate a drift term. The process will however inherit an implicit drift, related to the movement of the up and down Gamma processes and equal to $\mu_p - \mu_n$.

Leveraging the theory in Section 3.2, we assume the characteristic function $\phi_f$ equal to the one in Equation (34) of the four-parameter Bilateral Gamma family: $\phi_f^{BG}(u) = \phi_{X^{BG}_t}(u)$. This assumption gives rise to the seven-parameter Tilted Bilateral Gamma (TBG) pricing model. According to Equation (30), the Tilted Bilateral Gamma characteristic function for the log-return $x$ at time $t$,
under the $Q$-measure is given by

\[
\phi_{TBG}^Q(x; t) = (1 - p) \left( \frac{1}{1 - (ix - \eta) \sigma_p^2 \mu_p} \right) \left( \frac{1}{1 + (ix - \eta) \sigma_n^2 \mu_n} \right) + p \left( \frac{1}{1 - (ix + \zeta) \sigma_p^2 \mu_p} \right) \left( \frac{1}{1 + (ix + \zeta) \sigma_n^2 \mu_n} \right). \tag{35}
\]

A calibration of the Tilted Bilateral Gamma model on option data results in a set of 7 optimal values for the parameters in Equation (35), from which the corresponding subset of Bilateral Gamma parameters completely determines the characteristic function in Equation (34). The translation of this information from option data into risk premia for call options, as given by Equation (5), requires an expression for the asset return density under the $P$ and $Q$ probability measures. Although a closed-form expression exists for the characteristic function under both the Bilateral Gamma and Tilted Bilateral Gamma model, we cannot find a simple expression for the physical and pricing density respectively. However, if the characteristic function $\phi_X$ of a univariate random variable $X$ is integrable, the relationship between this function and the probability density function $f_X$ is given by the inverse Fourier transform

\[
f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixu} \phi_X(u) du. \tag{36}
\]

This continuous Fourier transform is approximated numerically using again Fast Fourier Transforms (FFT) and Simpson’s integral weighting scheme, which finally enables the conversion of the characteristic function into distributional information.

4. Numerical results

The S&P500 index is used to illustrate the proposed pricing strategy and opportunities of the Tilted Bilateral Gamma model, as well as to assess the zero-risk premium strike of a one-month European call option. In comparison with this American stock index, we also use the DAX index, consisting of German constituents only. The data features of both indices are summarized in Table 1.

The option surfaces are cleaned by only using European-style out-of-the-money call and put options with a strike to spot distance smaller or equal to 30% of the spot price and a maturity between 30 and 60 days. As such, both in-the-money options and far away out-of-the-money options are eliminated to mitigate possible illiquidity concerns. Not taking into account the direction of the transaction, the mid-prices of the options are used as the available market prices.

4.1. The pricing performance and the quality of physical extraction

We first examine how well the Tilted Bilateral Gamma model is able to fit the option surface of the S&P500 index. To this purpose, we recalibrate the model on every business day for which we have option data available. Details on the resulting RMSE time series are given in Figure 2a. To compare, we add the evolution of the RMSE under the Black-Scholes, Variance Gamma and Bilateral Gamma option pricing model. It is clear that the one-parameter Black-Scholes model
Table 1. Summary of the data features of both the S&P500 and DAX index.

|                | S&P500                          | DAX                            |
|----------------|---------------------------------|---------------------------------|
| Data collection| January 2, 2018 - August 29, 2018 | January 4, 2013 - April 3, 2020 |
| Frequency      | daily                           | weekly (every Friday)           |
| Available option surfaces | 167                             | 380                            |
| Currency       | USD                             | Euro                           |

performs poorly. The Variance Gamma model already leads to a significant improvement on the pricing performance. Extending this model to the Bilateral Gamma model results in an even better fit. In general, the Tilted Bilateral Gamma model outperforms the previous models with an average RMSE of 0.7559 over 167 calibration points, compared to an average RMSE for the other models of 5.2984, 1.4340 and 0.9625 respectively.

For a quality check of the implied Bilateral Gamma physical density we follow Madan et al. (2020), measuring the performance of the model using the probability integral transform. Let $x_t$ be the true one-month ahead return at time $t$ of the S&P500 index. Denote by $F_{BG}$ the Bilateral Gamma cumulative distribution function for a monthly maturity. Define

$$u_t = F_{BG}(x_t, \mu_{p,t}, \sigma_{p,t}, \mu_{n,t}, \sigma_{n,t}),$$

with $t$ across the 167 estimation dates, using the optimal calibrated parameter values at each day. For a successful extraction of physical information, these data points $u_t$ should be uniformly distributed. In order to evaluate this, we graph the sorted values of $u_t$ against the cumulative distribution function of a uniformly distributed random variable. Though not perfect, a fairly promising plot is shown in Figure 2b.

4.2. The risk premium of a European call option under the Tilted Bilateral Gamma model

Figure 3a shows the implied physical Bilateral Gamma return density and estimated pricing Tilted Bilateral Gamma return density of the S&P500 index on March 15, 2018 (chosen arbitrarily) with a maturity equal to one month. Around the zero return, the physical density lies above the pricing density with almost no shift in the center. For large returns in absolute value, the physical density lies below the pricing density, meaning that more probability mass is carried in the tails of this pricing density.

The illustrated relative position and shape of the pricing density with respect to the physical density result from the assumption of a U-shaped pricing kernel. The calibrated, non-normalized U-shape on March 15, 2018, as set in Equation (25), is given in Figure 3b. The parameter values determining this U-shape are given in Table 2. To compare, we also report on the average parameter values across all calibration dates. The higher $\eta$ and $\zeta$ realizations result in a more pronounced U-shape on March 15, compared to the average shape, also shown in Figure 3b. Moreover, investors generally assign higher prices to payoffs in negative return states, which implies more risk-aversion to large negative returns compared to large positive returns.

Next, we calculate the risk premium of European call options, with a maturity equal to one month, using again the calibration results on March 15, 2018. Moneyness is ranging from 10% in-the-money to 10% out-of-the-money. Calculating the risk premium for each moneyness level results in
Figure 2. (a) Evolution of the RMSE over time between optimal Black-Scholes, Variance Gamma, Bilateral Gamma and Tilted Bilateral Gamma model prices and market prices of plain-vanilla options on the S&P500 index. A calibration is conducted on each business day between January 2, 2018 and August 29, 2018. (b) Empirical cumulative distribution function under the Bilateral Gamma model, evaluated at the true monthly return of the S&P500 index, in comparison with the identity function as being the cumulative distribution function of a uniform random variable.

Table 2. Calibrated values on March 15, 2018, for the parameters $\eta$, $\zeta$ and $p$ defining the U-shaped measure change as set in Equation (25). The time series average values are determined as the mean parameter values across all calibration dates.

|              | $\eta$        | $\zeta$        | $p$      |
|--------------|---------------|----------------|---------|
| March 15, 2018 | 22.4030       | 7.0985         | 0.6472  |
| Time series average | 19.5495       | 6.0611         | 0.5689  |

The descending graph of Figure 4. The zero-risk premium moneyness level $k_T$ amounts around 98% of the spot price, as determined by the intersection point of the curve and the zero-axis. The same descending shape can be found across all different calibration dates.

The average realized returns for held-to-maturity options on the S&P500 index during the sample period are presented in Figure B1. We observe a similar behavior than the one in Figure 4, for different maturities encountered.

4.3. Evolution of the zero-risk premium strike

Repeating the work of Section 4.2, we assess on the evolution of the smoothed zero-risk premium strike over time, as displayed in Figure 5. We observe an average zero-risk premium moneyness level around 98.46%. The smoother highlights the fluctuations of the zero-risk premium strike around this mean level. These fluctuations are rather small in absolute value, which reveals some general stability over time.

Across the different calibration dates, we find the fairly consistent result that the zero-risk premium strike of a one-month held-to-maturity European call option on the S&P500 index is located slightly
in-the-money, though close to the at-the-money level. This means that only further away in-the-money call options are priced at a positive risk premium, while out-of-the-money options are expensive as resulting in a negative risk premium.

Further, the calibration procedure of Section 3.2 is also executed based on option prices with underlying the DAX index. Performing the same analysis results in the evolution of the zero-risk premium moneyness level of a one-month held-to-maturity European call option on the DAX index, as is shown in Figure 6. The moneyness levels are calculated week-to-week (see Table 1). The average zero-risk premium level amounts around 88.94%, which is lower than the average level on the S&P500 index.

5. Conclusion

The risk premium of a European call option, defined as the relative difference in expected payoff under the $P$ and $Q$ probability measures, is presented as arising naturally from pricing and physical distributional information on the return of the underlying asset. While historical time series are classically used to estimate a physical distribution, we use evidence from the option market to extract information on both the physical and corresponding pricing distribution. Heterogeneity in beliefs about return outcomes and allowing risk averse investors to take short positions in the equity market give rise to a U-shaped pricing kernel connecting the pricing density to the physical density. These assumptions result in the so-called Tilted Bilateral Gamma pricing model. Leveraging the distributional wealth of the option market, a calibration of the model on an option surface allows us to simultaneously extract information on both physical and pricing densities.

Under the assumption of a U-shaped pricing kernel, risk premia in European call options are proven to be negative for strikes beyond a certain threshold. For a fixed maturity, we empirically show that this threshold, the zero-risk premium strike, is typically located slightly in-the-money. With small fluctuations around a mean level, the zero-risk premium strike appears to follow a rather stable pattern over time.
Figure 4. Risk premium of the European call option with maturity equal to one month and varying moneyness levels. The underlying asset is the S&P500 index on March 15, 2018. The zero-risk premium moneyness level amounts around 97.53% of the spot price.

Figure 5. Evolution over time of the zero-risk premium strike of a European call option on the S&P500 index, with a fixed maturity of one month. The average moneyness level amounts around 98.46%.

Appendix A.

Proposition A1. \( f \) and \( g \) are two density functions, with respective support \((a_f, b_f)\) and \((a_g, b_g)\). Given is that
\[
a_g \leq a_f < b_g \leq b_f,
\]
and there exists a unique \( c \in (-\infty, +\infty) \) such that
\[
\begin{align*}
  & f(x) < g(x) \quad x \in (a_g, c) \\
  & f(x) = g(x) \neq 0 \quad x = c \\
  & f(x) > g(x) \quad x \in (c, b_f). 
\end{align*}
\]

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Figure 6. Evolution over time of the zero-risk premium strike of a European call option on the DAX index, with a fixed maturity of one month. The average moneyness level amounts around 88.94%.

If \( F \) and \( G \) are the cumulative distribution functions of respectively \( f \) and \( g \), then \( F \) will first order stochastically dominate \( G \), i.e. \( F(x) \leq G(x) \), for all \( x \in (-\infty, +\infty) \).

Proof. First of all, it is clear that for all \( x \in (-\infty, a_g] \), it holds that \( F(x) = G(x) = 0 \), and for all \( x \in [b_f, +\infty) \), we have \( F(x) = G(x) = 1 \).

Second, for all \( x \in (a_g, c] \) we have

\[
F(x) = \int_{-\infty}^{x} f(y)dy < \int_{-\infty}^{x} g(y)dy = G(x). \quad \text{(A.1)}
\]

Now, suppose \( F \) and \( G \) intersect at least once, i.e. there exists a point \( x_c \in (a_g, b_f) \) such that \( F(x_c) = G(x_c) \). According to the result in Equation (A.1), \( x_c \) must be strictly larger than \( c \). Since \( f \) and \( g \) both integrate to 1, we have

\[
\int_{-\infty}^{x_c} f(x)dx + \int_{x_c}^{+\infty} f(x)dx = \int_{-\infty}^{x_c} g(x)dx + \int_{x_c}^{+\infty} g(x)dx,
\]

and so

\[
F(x_c) + \int_{x_c}^{+\infty} f(x)dx = G(x_c) + \int_{x_c}^{+\infty} g(x)dx,
\]

This results in

\[
\int_{x_c}^{b_f} [g(x) - f(x)] dx = \int_{b_f}^{b_g} f(x)dx. \quad \text{(A.2)}
\]

However, since \( f(x) > g(x) \) for all \( x \in (c, b_g) \), we have a strictly negative result in the left hand side of Equation (A.2) and a positive result in the right hand side. We conclude that Equation (A.2) cannot be valid and \( F(x) < G(x) \) must hold for all \( x \in (a_g, b_f) \). This means that \( F \) stochastically dominates \( G \).
Proposition A2. \( f \) and \( g \) are two density functions, with respective support \((a_f, b_f)\) and \((a_g, b_g)\). Given is that
\[
a_g \leq a_f < b_f \leq b_g,
\]
and there exist exactly two points, \(c_1\) and \(c_2\), such that \(c_1 < c_2\) and
\[
\begin{align*}
&f(x) < g(x) \quad x \in (a_g, c_1) \\
&f(x) = g(x) \neq 0 \quad x = c_1 \\
&f(x) > g(x) \quad x \in (c_1, c_2) \\
&f(x) = g(x) \neq 0 \quad x = c_2 \\
&f(x) < g(x) \quad x \in (c_2, b_g).
\end{align*}
\]
If \( F \) and \( G \) are the cumulative distribution functions of respectively \( f \) and \( g \), there exists a unique point \( x_c \) such that \( 0 < F(x_c) = G(x_c) < 1 \).

Proof. First, for all \( x \in (a_g, c_1] \) we have
\[
F(x) = \int_{-\infty}^{x} f(y) dy < \int_{-\infty}^{x} g(y) dy = G(x). \tag{A.3}
\]
Second, for \( x \) smaller than, but close to \( b_g \), we have that \( f(x) < g(x) \). Since \( g \) has a fatter right tail, \( G \) will reach the value of 1 slower than \( F \) and so \( G(x) < F(x) \), for these values of \( x \). Since \( G \) is above \( F \) when reaching values close to 0, but below \( F \) when reaching values close to 1, it is clear that \( F \) and \( G \) must intersect at least once in a point \( x_c \).

Since \( f(x) < g(x) \) for all \( x \in (a_g, c_1) \), we have that \( F \) is flatter than \( G \) until \( c_1 \), and so \( x_c \) must be larger than \( c_1 \). Also, \( F \) is steeper than \( G \) for all \( x \in (c_1, c_2) \), so \( F \) possibly intersects \( G \) in this region, but only once, at a unique point. For \( x \) larger than \( c_2 \), \( F \) is again flatter than \( G \), and will not cross it in this region. Since \( x_c \) must exist, this means that \( x_c \in (c_1, c_2) \) holds and this point is unique. \( \square \)

Appendix B.

In Figure B1, we report on the realized return of held-to-maturity call options on the S&P500 index, with varying moneyness level and maturity. The realized return is calculated as
\[
\text{Return} = \frac{\text{Payoff}_{t+T} - \text{Price}_t}{\text{Price}_t},
\]
with \( t \) varying over all business days between January 2, 2018 and August 29, 2018. The given return is the average return over all values of \( t \).

In general, we observe a declining behavior, for increasing moneyness, with some minor deviations for longer maturities. For the options with a one and two months maturity, the zero-risk premium strike lies around the at-the-money-level. This coincides with our findings of Section 4.3. Note that for the longer maturity of 6 months, the realized average returns are all negative for the considered moneyness range.
**Figure B1.** The average realized return of held-to-maturity call options on the S&P500 index, over the sample period from January 2, 2018 to August 29, 2018. 4 different maturity levels are encountered, and moneyness varies from 0.8 to 1.2.

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