Supersymmetry on the Lattice

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Context: Why lattice and supersymmetry?

Lattice discretization provides non-perturbative, gauge-invariant regularization of gauge theories.

It can improve our understanding of strongly coupled field theories!

Supersymmetry is extremely interesting, especially non-perturbatively.

- More generally, symmetries simplify systems
  \[\rightarrow\] Insight into confinement, symmetry breaking, conformal field theories, etc.

- Dualities: gauge–gravity (AdS / CFT)
  \[\rightarrow\] potential non-perturbative definition of string theory

Lattice studies can also tell us something about the black holes (if gauge-gravity duality is correct)!
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Only few have been explored. Much to do!
There is a problem with supersymmetry in discrete space-time

Supersymmetry generalizes Poincaré symmetry, adding spinorial generators \( Q \) and \( \bar{Q} \) to translations, rotations, boosts.

The algebra includes \( \bar{Q}Q + \bar{Q}Q = 2\sigma^\mu P_\mu \),

\( P_\mu \) generates infinitesimal translations, which don’t exist on the lattice

\( \implies \) supersymmetry explicitly broken at classical level

Explicitly broken supersymmetry \( \implies \) relevant susy-violating operators (typically many)

Fortunately, there are certain theories where we can exactly preserve a subset of SUSY algebra on the lattice.
$\mathcal{N} = 4$ SYM is a particularly interesting theory

—The only known 4d system with a supersymmetric lattice formulation
—Context for development of AdS/CFT correspondence
—Important for studies of Quark-Gluon Plasma (QGP) at strong couplings
—Arguably simplest non-trivial field theory in four dimensions

Basic features:

- SU($N$) gauge theory with four fermions $\psi^I$ and six scalars $\phi^{IJ}$, all massless and in adjoint rep.

- Supersymmetric: 16 supercharges $Q^I_{\alpha}$ and $\overline{Q}^I_{\dot{\alpha}}$ with $I = 1, \cdots, 4$
  Fields and Q’s transform under global $SU(4) \simeq SO(6)$ R symmetry

- Conformal: $\beta$ function is zero for any ’t Hooft coupling $\lambda = g^2_{YM}N$
so that the full improved action becomes

$$S_{imp} = S'_{exact} + S_{closed} + S'_{soft}$$

$$S'_{exact} = \frac{N}{2\lambda_{lat}} \sum_n \text{Tr} \left[ -\mathcal{F}_{ab}(n)\mathcal{F}_{ab}(n) - \chi_{ab}(n)D^{(+)}_{a}\psi_{b}(n) - \eta(n)D^{(-)}_{a}\psi_{a}(n) + \frac{1}{2} \left( D^{(-)}_{a}\mathcal{U}_{a}(n) + G\sum_{a\neq b} (\det \mathcal{P}_{ab}(n) - 1)1_{N} \right)^{2} \right] - S_{det}$$

$$S_{det} = \frac{N}{2\lambda_{lat}} G \sum_{n} \text{Tr}[\eta(n)] \sum_{a\neq b} [\det \mathcal{P}_{ab}(n)] \text{Tr} \left[ \mathcal{U}_{b}^{-1}(n)\psi_{b}(n) + \mathcal{U}_{a}^{-1}(n + \hat{b})\psi_{a}(n + \hat{b}) \right]$$

$$S_{closed} = -\frac{N}{8\lambda_{lat}} \sum_{n} \text{Tr} \left[ \epsilon_{abcde} \chi_{dc}(n + \hat{\mu}_{a} + \hat{\mu}_{b} + \hat{\mu}_{c})D^{(-)}_{c}\chi_{ab}(n) \right]$$

$$S'_{soft} = \frac{N}{2\lambda_{lat}} \mu^{2} \sum_{n} \sum_{a} \left( \frac{1}{N} \text{Tr} [\mathcal{U}_{a}(n)\overline{\mathcal{U}_{a}(n)}] - 1 \right)^{2}$$

The lattice action is obviously very complicated!

To reduce barriers to entry our parallel code is publicly developed at

github.com/daschaich/susy

Evolved from MILC lattice QCD code. The public code is presented in

arXiv:1410.6971
Physics result: Static potential is Coulombic at all $\lambda$

Static potential $V(r)$ from $r \times T$ Wilson loops:

$$W(r, T) \propto e^{-V(r)} T$$

Fit $V(r)$ to Coulombic or confining form

$$V(r) = A - \frac{C}{r}$$

$$V(r) = A - \frac{C}{r} + \sigma r$$

$C$ is Coulomb coefficient

$\sigma$ is string tension

Fits to confining form always produce vanishing string tension $\sigma = 0$
Coupling dependence of Coulomb coefficient

Perturbation theory predicts \( C(\lambda) = \lambda/(4\pi) + O(\lambda^2) \)

AdS/CFT predicts \( C(\lambda) \propto \sqrt{\lambda} \) for \( N \to \infty, \lambda \to \infty, \lambda \ll N \)

**Left:** Agreement with perturbation theory for \( N = 2, \lambda \lesssim 2 \)

**Right:** Tantalizing \( \sqrt{\lambda} \)-like discrepancy for \( N = 3, \lambda \gtrsim 1 \)
Holographic applications

**AdS/CFT correspondence:**
\[ \mathcal{N} = 4 \text{ SYM} \text{ has dual description as strings in low energy string theory on } AdS_5 \times S_5. \]
Supergravity (SUGRA, low-energy) limit requires \( \lambda, N \to \infty \) with \( \lambda \ll N \)

**Applications: General holographic dualities**
Maximally superymmetric YM in \( p + 1 \) dim dual to Dp-branes
At finite temperature \( T \) and in decoupling limit described by black holes in type II SUGRA
Decoupling limit: \( N \to \infty \) and \( t = T/\lambda^{\frac{1}{3-p}} \ll 1 \)
SYM in one dimension

SYM consisting of sixteen supercharges in 1d at low temperatures with large N is conjectured to be dual to a black hole with N units of charge at same temperature. Energy of the black hole has been computed in SUGRA (Klebanov and Tseytlin [hep-th/9604089]):

$$\epsilon \sim 7.41 N^2 t^{14/5}$$

with $\epsilon = E/\lambda^{1/3}$ and $t = T/\lambda^{1/3}$.

This has been checked on the SYM side with great success upto first order corrections in $\alpha'$. In fact, leading order in $\alpha'$ is a prediction of lattice simulations.
Polyakov line in 1d case is non-vanishing even at low T, no phase transition in 1d, as predicted by the gauge/gravity correspondence. There is only single deconfined phase.
Revisiting p=0 with improved action and public code

Figure: Initial results with $4 \leq N \leq 7$ without $N_T \to \infty$ continuum extrapolations. Lower temperatures can only be accessed with larger N. At high temperatures, we see the expected asymptotic value of 6. [hep-th/0710.2188]
Unlike 1-d case discussed before, the maximally supersymmetric theory in two dimensions has a deconfinement/confinement phase transition. It is conjectured to be related to the phase transition between black hole/black string in the dual supergravity theory.

We construct dimensionless coupling given by $\hat{\lambda} = \lambda \beta^2$, where $\beta = aN_t$. Other dimensionless quantities related to size of spatial and temporal directions can be defined as:

$$r_x = \sqrt{\lambda} R \quad \text{and} \quad r_\tau = \sqrt{\lambda \beta}$$

The energy power law from supergravity calculations is predicted:

$$\epsilon \sim N^2 t^3 \sqrt{\lambda} R \quad \forall \quad t \ll 1$$

with, $\epsilon = E/\sqrt{\lambda}$, $t = T/\lambda^{1/2}$ defined as the dimensionless energy and temperature respectively.
Figure: Expected phase diagram for maximally supersymmetric \( \mathcal{N} = (8, 8) \) in two dimensions. Previous work: arXiv 1008.4964
Interesting open problem - Free energy of $\mathcal{N} = 4$ SYM

The free energy was calculated at strong coupling using AdS/CFT correspondence in [Gubser et. al, Phys.Rev. D54 (1996) 3915, hep-th/9602135]. It was suggested that the leading term in expansion of $F$ has the form:

$$F = -f(g_{YM}^2 N) \frac{\pi^2}{6} N^2 V T^4$$

where, $f(g_{YM}^2 N)$ is (possibly!) a smooth function which interpolates between a weak coupling limit of 1 and a strong coupling limit of 3/4.

Through lattice, we can explore the behavior of the free energy at intermediate couplings which might be useful for determining the exact form of $f(g_{YM}^2 N)$. 
Recapitulation

- Lattice supersymmetry is both enticing and challenging.
- $\mathcal{N} = 4$ SYM is practical to study on the lattice thanks to exact preservation of susy subalgebra $Q^2 = 0$.
- The theory is simple; the lattice action is complicated.
  → Public code to reduce barriers to entry.
- The static potential is always Coulombic.
  For $N = 2$ $C(\lambda)$ is consistent with perturbation theory.
  For $N = 3$ we may be seeing behavior predicted by AdS/CFT.
- Lower dimensional theories with maximal number of supercharges are very interesting for studies related to duality.
Thank you!
Thank you!

Funding and computing resources
Backup(s)

Figure: AdS/CFT correspondence. [Ref: Holographic Duality in Condensed Matter Physics, Zaanen et. al CUP 2015]
Regime of valid SUGRA description

To have a valid supergravity description, we need:

- Radius of curvature should be large in units of $\alpha'$. This implies $g_{\text{eff}} \gg 1$.

- String coupling should be small: $g_s \approx \frac{g_{\text{eff}}^{(7-p)/2}}{N} \ll 1$

We can combine both requirements to get a constraint on the effective dimensionless coupling we can probe on the lattice.

$$1 \ll g_{\text{eff}}^2 \ll N^{\frac{4}{7-p}}$$
Supplement: The (absence of a) sign problem

In lattice gauge theory we compute operator expectation values

\[ \langle O \rangle = \frac{1}{Z} \int [dU][d\overline{U}] \ O \ e^{-S_B[U,\overline{U}]} \ \text{pf} \mathcal{D}[U,\overline{U}] \]

\[ \text{pf} \mathcal{D} = |\text{pf} \mathcal{D}| e^{i\alpha} \] can be complex for lattice \( \mathcal{N} = 4 \) SYM

\[ \rightarrow \text{Complicates interpretation of } [e^{-S_B} \ \text{pf} \mathcal{D}] \text{ as Boltzmann weight} \]

Have to **reweight** “phase-quenched” (pq) calculations

\[ \langle O \rangle_{pq} = \frac{1}{Z_{pq}} \int [dU][d\overline{U}] \ O \ e^{-S_B[U,\overline{U}]} \ |\text{pf} \mathcal{D}| \]

\[ \langle O \rangle = \frac{\langle O e^{i\alpha} \rangle_{pq}}{\langle e^{i\alpha} \rangle_{pq}} \]

**Sign problem:** This breaks down if \( \langle e^{i\alpha} \rangle_{pq} \) is consistent with zero
Illustration of sign problem and its absence

- With **periodic temporal fermion boundary conditions**, we have an obvious sign problem, \( \langle e^{i\alpha} \rangle_{pq} \) consistent with zero.

- With **anti-periodic BCs** and all else the same, \( \langle e^{i\alpha} \rangle_{pq} \approx 1 \) → phase reweighting not even necessary.

\[ \mathcal{N} = 4 \text{ SYM, } U(2) \]
\[ 3^3 \times 4 \]

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Pfaffian phase dependence on volume and $N$

No indication of a sign problem with anti-periodic BCs

- $1 - \langle \cos(\alpha) \rangle \ll 1$ means $\text{pf } \mathcal{D} = |\text{pf } \mathcal{D}| e^{i\alpha}$ nearly real and positive
- Fluctuations in pfaffian phase don’t grow with the lattice volume
- Insensitive to number of colors $N = 2, 3, 4$
- To be revisited with the improved action

Hard calculations

- Each $4^3 \times 6$ measurement required $\sim 8$ days, $\sim 10\text{GB}$ memory
- Parallel $\mathcal{O}(n^3)$ algorithm
Twisted $\mathcal{N} = 4$ SYM

Everything transforms with integer spin under $\text{SO}(4)_{tw} \rightarrow$ no spinors

- $Q^I_\alpha$ and $\overline{Q}^I_{\dot{\alpha}} \rightarrow Q$, $Q_a$ and $Q_{ab}$
- $\psi^I$ and $\overline{\psi}^I \rightarrow \eta$, $\psi_a$ and $\chi_{ab}$
- $A_\mu$ and $\Phi^{IJ} \rightarrow A_a = (A_\mu, \phi) + i(B_\mu, \phi)$ and $\overline{A}_a$

The twisted-scalar supersymmetry $Q$ acts as

1. $Q A_a = \psi_a$  
2. $Q \chi_{ab} = -\overline{F}_{ab}$
3. $Q \eta = d$

- $Q \psi_a = 0$
- $Q \overline{A}_a = 0$
- $Q d = 0$

- bosonic auxiliary field with e.o.m. $d = \overline{D}_A A_a$

1. $Q$ directly interchanges bosonic $\leftrightarrow$ fermionic d.o.f.
2. The susy subalgebra $Q^2 \cdot = 0$ is manifest
Lattice $\mathcal{N} = 4$ SYM

The lattice theory is very nearly a direct transcription

- Covariant derivatives $\rightarrow$ finite difference operators
- Gauge fields $A_a \rightarrow$ gauge links $U_a$

\[
Q A_a \rightarrow Q U_a = \psi_a \\
Q \chi_{ab} = -\overline{F}_{ab} \\
Q \eta = d
\]

- Naive lattice action retains same form as continuum action and remains supersymmetric, $QS = 0$

**Geometrical formulation facilitates discretization**

| $\eta$ live on lattice sites | $\psi_a$ live on links |
| $\chi_{ab}$ connect opposite corners of oriented plaquettes |

Orbifolding / dimensional deconstruction produces same lattice system
Five links in four dimensions $\rightarrow A_4^*$ lattice

—Can picture $A_4^*$ lattice as 4d analog of 2d triangular lattice

—Preserves $S_5$ point group symmetry

—Basis vectors are non-orthogonal and linearly dependent

$S_5$ irreps precisely match onto irreps of twisted $SO(4)_{tw}$

\[ 5 = 4 \oplus 1 : \quad \mathcal{U}_a \rightarrow A_\mu + iB_\mu, \quad \phi + i\bar{\phi} \]
\[ \psi_a \rightarrow \psi_\mu, \quad \bar{\eta} \]
\[ 10 = 6 \oplus 4 : \quad \chi_{ab} \rightarrow \chi_{\mu\nu}, \quad \bar{\psi}_\mu \]
Twisted $\mathcal{N} = 4$ SYM on the $A_4^*$ lattice
— We have exact gauge invariance
— We exactly preserve $Q$, one of 16 supersymmetries
— The $S_5$ point group symmetry provides twisted R & Lorentz symmetry in the continuum limit

The high degree of symmetry has important consequences

- Moduli space preserved to all orders of lattice perturbation theory → no scalar potential induced by radiative corrections
- $\beta$ function vanishes at one loop in lattice perturbation theory
- Real-space RG blocking transformations preserve $Q$ and $S_5$
- Only one marginal tuning to recover $Q_a$ and $Q_{ab}$ in the continuum

The theory is almost suitable for practical numerical calculations...
New development

Scalar potential softly breaks $\mathcal{Q}$ supersymmetry

Plaquette determinant can be made $\mathcal{Q}$-invariant

Basic idea: Modify the equations of motion $\rightarrow$ moduli space

$$d(n) = \bar{D}_a^{(-)} U_a(n) \rightarrow \bar{D}_a^{(-)} U_a(n) + G \sum_{a \neq b} [\det P_{ab}(n) - 1]$$

Produces much smaller violations of $\mathcal{Q}$ Ward identity $\langle s_B \rangle = 9N^2 / 2$
Numerical complications

1. Complex gauge field $\implies U(N) = SU(N) \otimes U(1)$ gauge invariance
   $U(1)$ sector decouples only in continuum limit

2. $Q U_a = \psi_a \implies$ gauge links must be elements of algebra
   Resulting flat directions required by supersymmetric construction
   but must be lifted to ensure $U_a = 1_N + A_a$ in continuum limit

We need to add two deformations to regulate flat directions

SU($N$) scalar potential

$$\propto \mu^2 \sum_a (\text{Tr} [U_a U_a^\dagger] - N)^2$$

U(1) plaquette determinant

$$\sim G \sum_{a \neq b} (\det P_{ab} - 1)$$

Scalar potential **softly** breaks $Q$ supersymmetry

susy-violating operators vanish as $\mu^2 \to 0$

Plaquette determinant can be made $Q$-invariant (new development)