SPECIAL EMBEDDINGS OF WEIGHTED SOBOLEV SPACES WITH NONTRIVIAL POWER WEIGHTS

PATRICK J. RABIER

ABSTRACT. In prior work, the author has characterized the real numbers $a, b, c$ and $1 \leq p, q, r < \infty$ such that the weighted Sobolev space $W^{(q, p)}_{(a, b)}(\mathbb{R}^N \setminus \{0\}) := \{u \in L^1_{loc}(\mathbb{R}^N \setminus \{0\}) : |x|^a u \in L^r(\mathbb{R}^N), |x|^b \nabla u \in (L^p(\mathbb{R}^N))^N\}$ is continuously embedded into $L^q(\mathbb{R}^N; |x|^{s}dx) := \{u \in L^1_{loc}(\mathbb{R}^N \setminus \{0\}) : |x|^s u \in L^q(\mathbb{R}^N)\}$.

This paper discusses the embedding question for $W^{(q, p)}_{(a, b)}(\mathbb{R}^N \setminus \{0\}) := \{u \in L^1_{loc}(\mathbb{R}^N \setminus \{0\}) : |x|^a u \in L^\infty(\mathbb{R}^N), |x|^b \nabla u \in (L^p(\mathbb{R}^N))^N\}$, which is not the space obtained by the formal substitution $q = \infty$ in the previous definition of $W^{(q, p)}_{(a, b)}(\mathbb{R}^N \setminus \{0\})$, unless $a = 0$.

The corresponding embedding theorem identifies all the real numbers $a, b, c$ and $1 \leq p, r < \infty$ such that $W^{(\infty, p)}_{(a, b)}(\mathbb{R}^N \setminus \{0\})$ is continuously embedded in $L^r(\mathbb{R}^N; |x|^{s}dx)$. A notable feature is that such embeddings exist only when $a \neq 0$ and, in particular, have no analog in the unweighted setting.

It is also shown that the embeddings are always accounted for by multiplicative rather than just additive norm inequalities. These inequalities are natural extensions of the Caffarelli-Kohn-Nirenberg inequalities which, in their known form, are restricted to functions of $C^0_0(\mathbb{R}^N)$ and do not incorporate supremum norms.

1. Introduction

Throughout this paper, $\mathbb{R}^N_+ := \mathbb{R}^N \setminus \{0\}$. Given $d \in \mathbb{R}$, the measure $|x|^d dx$ on $\mathbb{R}^N_+$ can be extended to a measure on $\mathbb{R}^N$ provided that the $|x|^d dx$-measure of $\{0\}$ is defined to be 0 (this must be specified if $d \leq -N$). If so, the space $L^s(\mathbb{R}^N; |x|^{d}dx), 0 < s < \infty$, coincides with the space of Lebesgue measurable functions $u$ on $\mathbb{R}^N$ such that $|x|^d u \in L^s(\mathbb{R}^N)$. The norm (quasi-norm when $0 < s < 1$) of $u \in L^s(\mathbb{R}^N; |x|^{d}dx)$ will be denoted by $||u||_{d, s} := |||x|^d u||_s$, where $||.||_s$ is the (quasi) norm of $L^s(\mathbb{R}^N)$.

If $a, b \in \mathbb{R}$ and $1 \leq p < \infty$ and $0 < q < \infty$, set

\begin{equation}
W^{1, (q, p)}_{(a, b)}(\mathbb{R}^N_+) := \{u \in L^1_{loc}(\mathbb{R}^N_+) : u \in L^q(\mathbb{R}^N; |x|^a dx), \quad \nabla u \in (L^p(\mathbb{R}^N; |x|^b dx))^N\},
\end{equation}

with (quasi) norm $||u||_{a, q} + ||\nabla u||_{b, p}$. Note that $W^{1, (q, p)}_{(a, b)}(\mathbb{R}^N)$ is not defined as the -usually smaller and unknown- closure of some subspace of smooth enough functions.

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Recently, the author has characterized all the real numbers \(a, b, c\) and \(1 \leq p, q, r < \infty\) (for \(1 \leq p < \infty\) and \(0 < q, r < \infty\) if \(N = 1\)) such that \(W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c \, dx)\) where, as usual, \(\hookrightarrow\) refers to continuous embedding, and showed that, with a single exception, this embedding is accounted for by a multiplicative inequality (10). This generalizes both the Sobolev embedding theorem in the unweighted case \(a = b = c = 0\) and the Caffarelli-Kohn-Nirenberg (CKN) inequalities [2] when \(a, b, c > -N\) and \(u \in C^\infty_0(\mathbb{R}^N)\).

If \(W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c \, dx)\), then \(W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c \, dx)\) since a smaller space is obtained when \(\mathbb{R}^N_a\) is replaced by \(\mathbb{R}^N\) in (1.1). However, it is only with \(\mathbb{R}^N\) that the admissible values of the parameters \(a, b, c, p, q, r\) have been exactly identified. If \(b \leq 0\), these admissible values are the same whether \(\mathbb{R}^N\) or \(\mathbb{R}^N_{\{a,b\}}\) is used. This is essentially trivial if \(N = 1\) and due to \(W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N) = W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N)\) if \(b \leq 0\) and \(N \geq 2\) (Remark 5.1).

The goal of the present paper is to show that, in a suitable form, the results of (10) can be extended when \(p\) and \(r\) are still finite but, roughly speaking, \(q = \infty\), although the problem is trivial if this statement is taken literally. Indeed, since \(L^\infty(\mathbb{R}^N; |x|^a \, dx) = L^\infty(\mathbb{R}^N)\) is independent of \(a\), the constant function \(u = 1\) gives an example when \(u \in L^\infty(\mathbb{R}^N; |x|^a \, dx)\) and \(\nabla u \in \{L^p(\mathbb{R}^N; |x|^c \, dx)\}^N\) irrespective of \(a, b, p, q, r\); yet \(u\) does not belong to \(L^r(\mathbb{R}^N; |x|^c \, dx)\) for any \(c \in \mathbb{R}\) and \(0 < r < \infty\). Thus, no embedding is true when \(q = \infty\) in (1.1).

As we state, the “correct” definition is given by

\[
\text{(1.2)} \quad W^{1,(\infty,p)}_{\{a,b\}}(\mathbb{R}^N) := \\
\{ u \in L^\infty_{\text{loc}}(\mathbb{R}^N) : |x|^a u \in L^\infty(\mathbb{R}^N), \quad \nabla u \in \{L^p(\mathbb{R}^N; |x|^b \, dx)\}^N \},
\]

equipped with the natural norm

\[
\text{(1.3)} \quad ||| |x|^a u|||_\infty + ||\nabla u||_{b,p}.
\]

As in the case of (1.1) when \(q < \infty\), the space \(W^{1,(\infty,p)}_{\{a,b\}}(\mathbb{R}^N)\) contains the space \(W^{1,(\infty,p)}_{\{a,b\}}(\mathbb{R}^N)\) obtained by replacing \(\mathbb{R}^N_{\{a,b\}}\) by \(\mathbb{R}^N\) in (1.2). Once again, both spaces coincide when \(N \geq 2\) and \(b \leq 0\); see Remark 5.1.

Unlike \(|x|^a u \in L^q(\mathbb{R}^N)\) with \(q < \infty\), which is just \(u \in L^q(\mathbb{R}^N; |x|^a \, dx)\), when \(a \neq 0\) the condition \(|x|^a u \in L^\infty(\mathbb{R}^N)\) is not of the type \(u \in L^s(\mu)\) for some measure \(\mu\) on \(\mathbb{R}^N\) or \(\mathbb{R}^N_{\{a,b\}}\) and some \(0 < s \leq \infty\). On the other hand, the space obtained by setting \(q = \infty\) in (1.1) is recovered if and only if \(a = 0\) in (1.2). Accordingly, \(a \neq 0\) is necessary for any embedding. In particular, the results of this paper do not generalize a property already familiar in classical, unweighted, Sobolev spaces. They are special to weighted spaces with nontrivial (power) weights, although \(b = 0\) is not ruled out.

Some notation must be introduced for the statement of the embedding Theorem 1.1 below, whose proof is the single purpose of this paper. As is customary, if \(1 \leq p < \infty\), then

\[
p^* = \infty \text{ if } p \geq N \text{ and } p^* = \frac{Np}{N-p} \text{ if } 1 \leq p < N.
\]
Next, we denote by \(c^0\) and \(c^1\) the two points

\[
eq c^0 := a r - N \quad \text{and} \quad c^1 := \frac{r(b - p + N)}{p} - N,
\]

where it is understood that \(a, b, p\) and \(r\) are given. The points \(c^0\) and \(c^1\) are distinct if and only if \(ap - N \neq b - p\). If so and if \(c\) is in the closed interval with endpoints \(c^0\) and \(c^1\), we set

\[
\theta_c := \frac{c - c^0}{c^1 - c^0},
\]

so that \(\theta_c \in [0, 1]\) and that

\[
c = \theta_c c^1 + (1 - \theta_c)c^0.
\]

In particular, \(\theta_c = 0\) and \(\theta_c = 1\) and, by \((1.4), (1.5)\) and \((1.6)\),

\[
\frac{c + N}{r} = \theta_c \frac{b - p + N}{p} + (1 - \theta_c)a.
\]

**Theorem 1.1.** Let \(a, b, c \in \mathbb{R}\) and \(1 \leq p, r < \infty\) be given (\(1 \leq p < \infty\) and \(0 < r < \infty\) if \(N = 1\)). Then, \(W^{1,(\infty,p)}_{\{a,b\}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)\) (and hence \(W^{1,2}_{\{a,b\}}(\mathbb{R}^N) \hookrightarrow W^{1,2}_{\{c,d\}}(\mathbb{R}^N)\)) if and only if \(a \neq 0\) and one of the following four conditions holds:

(i) \(ap - N\) and \(b - p\) are on the same side of \(-N\) (including \(-N\)), \(ap - N \neq b - p\), \(c\) is in the open interval with endpoints \(c^0\) and \(c^1\) and \(\theta_c \leq \frac{N}{r}\).

(ii) \(ap - N\) and \(b - p\) are strictly on opposite sides of \(-N\) (hence \(ap - N \neq b - p\)), \(c\) is in the open interval with endpoints \(c^0\) and \(-N\) and \(\theta_c \leq \frac{N}{r}\).

(iii) \(p \leq r \leq p^*\), \(a(b - p + N) > 0\) (i.e., \(ap - N\) and \(b - p\) are strictly on the same side of \(-N\)) and \(c = c^1\).

(iv) \(p < N, r > p^*, ap - N = b - p\) and \(c = c^1 = c^0\).

Furthermore, when the embedding \(W^{1,(\infty,p)}_{\{a,b\}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx)\) holds, it is always characterized by a multiplicative inequality. Specifically:

(v) If \(a \neq 0\) and one of the conditions (i) to (iii) holds with \(ap - N \neq b - p\) (which is already assumed in (i) or (ii)), there is a constant \(C > 0\) such that

\[
\|u\|_{c,r} \leq C\|\nabla u\|_{b,p}^{\theta_{c,b,p}} \|x\|^{|\alpha|} u\|_{\infty}^{1 - \theta_{c,b,p}}, \quad \forall u \in W^{1,(\infty,p)}_{\{a,b\}}(\mathbb{R}^N),
\]

where \(\theta_{c,b,p}\) is given by \((1.4)\) and \((1.5)\).

(vi) If \(a \neq 0, ap - N = b - p\) and if \(p \leq r \leq p^*\), \(c = c^1 = c^0\), there is a constant \(C > 0\) such that

\[
\|u\|_{c^1,r} \leq C\|\nabla u\|_{b,p}, \quad \forall u \in W^{1,(\infty,p)}_{\{a,b\}}(\mathbb{R}^N).
\]

(vii) If \(a \neq 0, p < N, ap - N = b - p\) and if \(r > p^*\), \(c = c^1 = c^0\), there is a constant \(C > 0\) such that

\[
\|u\|_{c^1,r} \leq C\|\nabla u\|_{b,p}^{\theta_{c,b,p}} \|x\|^{|\alpha|} u\|_{\infty}^{1 - \theta_{c,b,p}}, \quad \forall u \in W^{1,(\infty,p)}_{\{a,b\}}(\mathbb{R}^N).
\]

The value \(r = \infty\) is not included in Theorem 1.1. This case requires a different treatment and will be discussed elsewhere in a more general framework. It is plain that the restriction \(\theta_c \leq \frac{N}{r}\) in (i) and (ii) is only relevant when \(p < N\) and \(r > p^*\).
To avoid misunderstandings, it should be stressed that if \( u \) is a distribution on \( \mathbb{R}^N \) whose restriction to \( \mathbb{R}_*^N \) is in \( W^{1, (\infty, p)}(\mathbb{R}_*^N) \), Theorem 1.1 addresses only the integrability of \( |x|^c |u|^r \) when \( u \) is viewed as a distribution on \( \mathbb{R}_*^N \). For instance, if \( u = \delta \) (Dirac delta), then \( \delta \) is the 0 function on \( \mathbb{R}_*^N \) and Theorem 1.1 with \( c = 0 \) and \( r = 1 \) implies the trivial \( 0 \in L^1(\mathbb{R}^N) \) but not the absurd \( \delta \in L^1(\mathbb{R}^N) \). It is only when \( u \in L^1_{\text{loc}}(\mathbb{R}^N) \) that the integrability of \( |x|^c |u|^r \) is independent of whether \( u \) is viewed as a distribution on \( \mathbb{R}^N \) or \( \mathbb{R}_*^N \).

The necessity of the conditions given in Theorem 1.1 is proved in the next section, where it is also shown that the inequality (1.8) in part (v) holds if the sufficiency of \( a \neq 0 \) together with (i), (ii) or (iii) is assumed (Corollary 2.2).

Since the norm (1.3) incorporates a supremum norm, a classical two-step approach to the sufficiency, first for some subclass of functions with bounded supports, followed by a denseness argument, is clearly hopeless in general. For instance, if \( b - p < ap - N \) and \( u \) is a smooth function on \( \mathbb{R}^N \) vanishing on a neighborhood of 0 and equal to \( |x|^{-a} \) for large \( |x| \), then \( u \in W^{1, (\infty, p)}(\mathbb{R}_*^N) \), but \( u \) cannot be approximated by functions of \( W^{1, (\infty, p)}(\mathbb{R}_*^N) \) with bounded support since \( \| |x|^a (u - v) \|_{\infty} \geq 1 \) for every such function \( v \).

The sufficiency of \( a \neq 0 \) plus one of the conditions (i) to (iv) is proved in four steps, after the short Section 3 of background material. The case \( r = p \) is resolved first (Section 4, Theorem 4.2) and used to handle \( 1 \leq r < p \) (Section 5, Theorem 5.2) and \( p < r \leq p^* \) (Section 6, Theorem 6.3). All three proofs also rely upon various special cases of the main embedding theorem in 110. Lastly, when \( 1 \leq p < N \) and \( r > p^* \), the embedding is deduced from the case \( r = p^* \) through a nonlinear “change of variable”. The (necessary) restriction \( \theta_{\cdot r} \leq \frac{N}{r} \) is crucial to the success of this procedure (Section 7, Theorem 7.3). The multiplicative inequalities (1.9) and (1.10) are proved in Theorems 4.2, 6.3 and 7.3.

When \( 1 \leq r \leq p \) (Sections 4 and 5), the embedding theorem is actually stronger than Theorem 1.1 since it proves the embedding into \( L^r(\mathbb{R}^N; |x|^c \, dx) \) of the larger space

\[
(1.11) \quad \widetilde{W}^{1, (\infty, p)}_{\{a, b\}} := \{ u \in L^1_{\text{loc}}(\mathbb{R}_*^N) : |x|^a u \in L^\infty(\mathbb{R}_*^N), \quad \partial_p u \in L^p(\mathbb{R}_*^N; |x|^b \, dx) \},
\]

with the weaker norm

\[
(1.12) \quad ||u||_{\{a, b\}, (\infty, p)} := |||x|^a u||_{\infty} + ||\partial_p u||_{b, p},
\]

where, in (1.11) and (1.12), \( \partial_p u = \nabla u \cdot \frac{1}{|x|} \) denotes the radial derivative of \( u \) (well defined for every distribution on \( \mathbb{R}_*^N \)). Thus, the embedding requires no integrability assumption about the first derivatives, except for the radial one. This is no longer true when \( r > p \), when the embedding is only proved for the space \( W^{1, (\infty, p)}_{\{a, b\}}(\mathbb{R}_*^N) \).

The multiplicative inequalities (1.8), (1.9) and (1.10) are extensions of the CKN inequalities 2 since, in addition to requiring \( u \in C^\infty_0(\mathbb{R}^N) \), the latter do not incorporate supremum norms. The same thing can be said of related inequalities of Maz’ya [6, Theorem 9], [7, p.127] (see also the expanded text [8]), more general but less explicit than the CKN inequalities.

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1. It also fails when \( q < \infty \), though for less obvious reasons; see 110.

2. We shall not need a notation for the norm of \( W^{1, (\infty, p)}_{\{a, b\}}(\mathbb{R}_*^N) \) in 113.
As a more concrete example, the real numbers $a, b$ and $1 \leq p, r < \infty$ such that $W_{(a,b)}^{1,(\infty,p)}(\mathbb{R}^N)$ is continuously embedded in the unweighted space $L'(\mathbb{R}^N)$ (so that $c = 0$) are characterized in Section 8 (Theorem 8.1). When $N = 1, r \geq p$ and $b = p \left(1 + \frac{1}{r} - \frac{1}{p}\right)$ (and $a > 0, c = 0$) we show that Theorem 8.1 can be deduced from a weighted Hardy-type inequality of Bradley [1], or even by a simple integration by parts if $a = b = p = r = 1$. We also show that if $b = c = 0$ (Corollary 8.2), another proof can be derived from Sobolev's inequality irrespective of $N$.

Even though the examples of Section 8 show that there is no doubt that the inequalities of this paper must be known for some values of the parameters, no systematic investigation seems to be on record, even for $C_0^\infty(\mathbb{R}^N)$ or $C_0^\infty(\mathbb{R}^N)$ and/or $N = 1$. On the other hand, for functions of $C_0^\infty(\mathbb{R}^N)$, the multiplicative inequalities, including those of [10], hold for a wider range of parameters than stipulated in Theorem 1.1. No proof of this claim will be given here, but when $p = q = r = 2$ and $c = \frac{a+b}{2} - 1$, a result of this type was recently obtained by Catrina and Costa [3].

The last section of [10] explains, in broad terms, how the embedding theorem of that paper can be used to prove more general ones when the weights have power-like singularities at a finite number of points and at infinity. The interested reader should have no difficulty to see how Theorem 1.1 above fits into that discussion.

**Remark 1.1.** Up to and including Section 8, the following will be used repeatedly: The Kelvin transform $x \in \mathbb{R}^N \mapsto x|x|^{-2} \in \mathbb{R}^N \setminus \{0\}$ induces an isometry from $W_{(a,b)}^{1,(\infty,p)}(\mathbb{R}^N)$ onto $\tilde{W}_{(-a,2p-2N-b)}^{1,(\infty,p)}(\mathbb{R}^N)$ and from $L'(\mathbb{R}^N; |x|^c dx)$ onto $L'(\mathbb{R}^N; |x|^{-2N-c} dx)$. In practice, this will be helpful to shorten proofs when two sets of assumptions about $a$ and $b$ are exchanged into one another by Kelvin transform.

Everywhere in the paper, $C > 0$ denotes a constant whose value may not be the same in different places. Also, $\zeta \in C_0^\infty(\mathbb{R}^N)$ is chosen once and for all such that $0 \leq \zeta \leq 1$, $\zeta(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\zeta(x) = 0$ if $|x| \geq 1$.

2. Necessity

In this section, we prove that the hypotheses of Theorem 1.1 are necessary in the more general case when $r > 0$ and $N$ is arbitrary; recall $r \geq 1$ is assumed in Theorem 1.1 when $N > 1$.

**Theorem 2.1.** Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty, 0 < r < \infty$ be given. Then, $W_{(a,b)}^{1,(\infty,p)}(\mathbb{R}^N)$ (hence a fortiori $\tilde{W}_{(a,b)}^{1,(\infty,p)}$) is not contained $L'(\mathbb{R}^N; |x|^c dx)$ if one of the following conditions holds:

(i) $a = 0$.
(ii) $c$ does not belong to the closed interval with endpoints $c^0$ and $c^1$.
(iii) $ap - N \neq b - p, c = c^0$.
(iv) $b - p \leq -N, a > 0$ or $b - p \geq -N, a < 0$ and $c$ does not belong to the open interval with endpoints $c^0$ and $-N$.

Furthermore, $W_{(a,b)}^{1,(\infty,p)}(\mathbb{R}^N)$ (hence a fortiori $\tilde{W}_{(a,b)}^{1,(\infty,p)}$) is not continuously embedded into $L'(\mathbb{R}^N; |x|^c dx)$ if:

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3In principle, this does not rule out $W_{(a,b)}^{1,(\infty,p)}(\mathbb{R}^N) \subset L'(\mathbb{R}^N; |x|^c dx)$. 

(v) \( c = c^1 \) and \( r < p \).

**Proof.** (i) \( u = 1 \) provides a counterexample.

(ii) If \( c < \min \{ c^0, c^1 \} \), let \( u(x) := |x|^{-\frac{c+N}{r}} \zeta(x) \). Then, \( u \notin L^r(\mathbb{R}^N; |x|^r dx) \) since \( |x|^r u(x)|^r = |x|^{-N} \) on a neighborhood of 0, but \( u \in W^{1,(\infty,p)}(\mathbb{R}^N) \) since \( a - \frac{c+N}{r} > 0, b - p - \frac{p(c+N)}{r} > -N \), \( \zeta \) has compact support and \( \nabla \zeta \) has compact support and vanishes on a neighborhood of 0.

If \( c > \max \{ c^0, c^1 \} \), let \( u(x) := |x|^{-\frac{c+N}{r}}(1 - \zeta(x)) \) and argue as above, with obvious modifications.

(iii) If \( a - p - N \neq b - p \), then \( c^1 \neq c^0 \) and the argument of (ii) continues to work when \( c = c^0 \).

(iv) By Kelvin transform (Remark 1.1), it suffices to consider \( b - p \leq -N \) and \( a > 0 \). If so, \( a - p - N > b - p \) and \( c^1 \geq -N < c^0 \), so that \( W^{1,(\infty,p)}(\mathbb{R}^N) \nsubseteq L^r(\mathbb{R}^N; |x|^r dx) \) if \( c \geq c^0 \) by (ii) and (iii). If now \( c \leq -N \), then \( \zeta \notin L^r(\mathbb{R}^N; |x|^r dx) \) since \( \zeta = 1 \) on a neighborhood of 0 but \( \zeta \in W^{1,(\infty,p)}(\mathbb{R}^N) \) since \( a > 0 \), \( \zeta \) has compact support and \( \nabla \zeta \) has compact support and vanishes on a neighborhood of 0.

(v) The argument is different when \( a - p - N \neq b - p \) and when \( a - p - N = b - p \).

Case (v-1): \( a - p - N \neq b - p \).

By Kelvin transform and part (i), it suffices to consider the case when \( a < 0 \). By contradiction, if \( W^{1,(\infty,p)}(\mathbb{R}^N) \nrightarrow L^r(\mathbb{R}^N; |x|^r dx) \), then \( ||u||_{c,r} \leq C(||x|^a u||_{\infty} + ||\nabla u||_{b,p}) \) for every \( u \in W^{1,(\infty,p)}(\mathbb{R}^N) \). Upon replacing \( u(x) \) by \( u(\lambda x) \) with \( \lambda > 0 \), this yields \( ||u||_{c,r} \leq C(\lambda^k ||x|^a u||_{\infty} + ||\nabla u||_{b,p}) \) where \( k := \frac{b-p-N}{r} - a \neq 0 \) (this uses \( c^1 = \frac{r(b-p-N)}{p} - N \)). Thus, \( ||u||_{c,r} \leq C||\nabla u||_{b,p} \) by letting \( \lambda \) tend to 0 or \( \infty \). In particular, this holds when \( u(x) = f(|x|) \) with \( f \in W^{1,p}(0, \infty), f \geq 0, f = 0 \) on a neighborhood of 0 and \( f = M \) (constant) on a neighborhood of \( \infty \) (if so, \( u(x) = f(|x|) \) is in \( W^{1,(\infty,p)}(\mathbb{R}^N) \) irrespective of \( a < 0, b \in \mathbb{R} \) and \( p \geq 1 \) and so \( ||f||_{c^1+1,N-1,r} \leq C||f||_{b^1+1,N-1,p} \) for every such \( f \). That it is not so when \( 0 < r < p \) is shown in the proof of [10] Theorem 2.1 (iv).

Case (v-2): \( a - p - N = b - p \).

If so, \( c^1 = ar - N = c^0 \) and a direct rescaling as above is inoperative. By contradiction, if \( ||u||_{c^1,r} \leq C(||x|^a u||_{\infty} + ||\nabla u||_{b,p}) \) for every \( u \in W^{1,(\infty,p)}(\mathbb{R}^N) \), this inequality holds when \( u(x) = f(|x|) \) with \( f \in C^0(0, \infty) \) and then \( ||f||_{c^1+1,r} \leq C(||x|^a f||_{\infty} + ||f'||_{b^1+1,p}) \) where \( b = ap + p - N \) was used. Every such \( f \) has the form \( f(t) = t^{-a}g(\ln t) \) with \( g \in C^0(\mathbb{R}) \), whence \( ||g||_{r} \leq C(||g||_{\infty} + ||g'||_{p} + ||g'''||_{p}) \) (unweighted inequality) by the change of variable \( t = s \).

By choosing \( g \neq 0 \) and replacing \( g(s) \) by \( g(\lambda s) \) with \( \lambda > 0 \), it follows that \( \lambda^{-\frac{1}{r}}I_1 \leq C(I_2 + \lambda^{-\frac{1}{r}}I_3 + \lambda^{-\frac{1}{r}}I_4) \), where \( I_1, ..., I_4 \geq 0 \) are independent of \( \lambda > 0 \). This requires \( \frac{1}{r} \leq \frac{1}{p} \) and so \( r \geq p \). In other words, the embedding cannot be continuous if \( r < p \).

As a corollary, we find that when \( a - p - N \neq b - p \), the embedding is characterized by a multiplicative, rather than just additive, norm inequality:

**Corollary 2.2.** Let \( a, b, c \in \mathbb{R} \) and \( 1 \leq p < \infty, 0 < r < \infty \), be such that \( a - p - N \neq b - p \). Then, \( W^{1,(\infty,p)}(\mathbb{R}^N) \nrightarrow L^r(\mathbb{R}^N; |x|^r dx) \) if and only if \( a \neq 0, c \) is in the closed
interval with endpoints \(c^0\) and \(c^1\) and there is \(C > 0\) such that

\[
\|u\|_{c,r} \leq C\|\nabla u\|_{\infty}^{\theta_c} + \|x\|_\infty^{1-\theta_c}, \quad \forall u \in W^{1,\infty}(\mathbb{R}^N), \tag{2.1}
\]

where \(\theta_c\) is given by (1.4) and (1.3). The same property is true upon replacing \(W^{1,\infty}(\mathbb{R}^N)\) by \(W^{1,\infty}_{\{a,b\}}(\mathbb{R}^N)\) by (2.1) by

\[
\|u\|_{c,r} \leq C\|\partial_p u\|_{b,p}^{\theta_c} + \|x\|_\infty^{1-\theta_c}, \quad \forall u \in \tilde{W}^{1,\infty}(\mathbb{R}^N). \tag{2.2}
\]

Proof. In both cases, the sufficiency follows from the generalized arithmetic-geometric inequality. We prove the necessity for \(\tilde{W}^{1,\infty}(\mathbb{R}^N)\). Similar arguments work in the case of \(W^{1,\infty}(\mathbb{R}^N)\).

That \(a \neq 0\) is necessary was shown in part (i) of Theorem 2.1. Suppose then \(a \neq 0\) and \(W^{1,\infty}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^r dx)\). By part (ii) of Theorem 1.1, \(c^0\) is in the closed interval with (distinct) endpoints \(c^0\) and \(c^1\). Furthermore, \(\|u\|_{c,r} \leq C(\|\nabla a u\|_{\infty} + \|\partial_p u\|_{b,p})\) for every \(u \in \tilde{W}^{1,\infty}(\mathbb{R}^N)\). In this inequality, replace \(u(x)\) by \(u(\lambda x)\) with \(\lambda > 0\) to get

\[
\|u\|_{c,r} \leq C\lambda^{\frac{b-p}{p}} \|\nabla a u\|_{\infty} + C\lambda^{\frac{1-p}{p}} \|\partial_p u\|_{b,p} = C\lambda^{\theta_c} \|\nabla a u\|_{\infty} + C\lambda^{(1-\theta_c)\frac{b-p}{p}} \|\partial_p u\|_{b,p}.
\]

If \(c = c^0\) (\(c = c^1\)), then \(\theta_c = 0\) (\(\theta_c = 1\)), so that \(\|u\|_{c,r} \leq C\|\nabla a u\|_{\infty} \|\partial_p u\|_{b,p}\), i.e., (2.2) holds, by letting \(\lambda\) tend to \(0\) or to \(\infty\). Otherwise, (2.2) follows by minimizing the right-hand side of (2.3) for \(\lambda > 0\). This changes \(C\), which however remains independent of \(u\) even though the minimizer is of course \(u\)-dependent. In that regard, observe that if \(\theta_c > 0\), it follows from (2.3) that \(u = 0\) if \(\partial_p u = 0\), once again by letting \(\lambda\) tend to \(0\) or to \(\infty\). Thus, it is not restrictive to assume \(\|\nabla a u\|_{\infty} > 0\) and \(\|\partial_p u\|_{b,p} > 0\) in the minimization step.

The next corollary gives an additional necessary condition for the continuity of the embedding when \(r > p^*\).

**Corollary 2.3.** Let \(a, b, c \in \mathbb{R}\) and \(1 \leq p < N, p^* < r < \infty\) be given. If \(ap - N \neq \frac{p}{p^*} - b - p\) and \(W^{1,\infty}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^r dx)\), then \(\theta_c \leq \frac{p}{p^*} - 1 < 1\).

Proof. First, \(\theta_c \in [0,1]\) (even \((0,1]\)) by Theorem 2.1 irrespective of \(p\) and \(r\). Next, let \(\varphi \in C_c^\infty(\mathbb{R}^N), \varphi \neq 0\), be chosen once and for all. If \(x_0 \in \mathbb{R}^N\) and \(R := |x_0|\) is large enough, then \(\varphi(\cdot + x_0) \in C_c^\infty(\mathbb{R}^N) \subset W^{1,\infty}_{\{a,b\}}(\mathbb{R}^N)\) irrespective of \(a, b\) and \(p\). By using (2.1) with \(u = \varphi(\cdot + x_0)\) and by letting \(R \to \infty\), we get (because \(\sup \varphi\) is compact) \(R^\frac{p}{p^*} \|\varphi\|_r \leq \tilde{C} R^{\frac{p}{p^*} + a(1-\theta_c)} \|\nabla \varphi\|_{b,p}^{\theta_c} \|\varphi\|_{1-\theta_c}^{1-\theta_c}\) with \(C > 0\) independent of \(R > 0\) large enough. This implies \(\frac{p}{p^*} \leq \frac{b-p}{p} + a(1-\theta_c)\). By adding \(\frac{a}{r}\) to both sides and using (1.7), it follows that \(\theta_c \left(\frac{1}{p} - \frac{1}{N}\right) \leq \frac{1}{r}\). This is always true if \(p \geq N\) or if \(p < N\) and \(r \leq p^*\), but is equivalent to \(\theta_c \leq \frac{p}{p^*} < 1\) if \(p < N\) and \(r > p^*\). □

It is a simple matter to check that, together, Theorem 2.1 and Corollaries 2.2 and 2.3 imply that the hypotheses of Theorem 1.1 are necessary.
3. Background

In this section, we collect a few preliminary results needed at various stages of the proof of Theorem 1.1. The material in the first subsection is mostly taken from [10] Section 3. A proof is given only for Lemma 3.2, not used in that reference.

3.1. The space $\widetilde{W}^{1,1}_{\text{loc}}$. If $u \in L^1_{\text{loc}}(\mathbb{R}^N)$, define the spherical mean of $u$

\[(3.1) \quad f_u(t) := (N\omega_N)^{-1} \int_{S^{N-1}} u(t\sigma)d\sigma,\]

where $\omega_N$ is the volume of the unit ball of $\mathbb{R}^N$. By Fubini’s theorem in spherical coordinates, $f_u(t)$ is well defined for a.e. $t > 0$ and $f_u \in L^1_{\text{loc}}(0, \infty)$. Note that $u$ is radially symmetric if and only if $u(x) = f_u(|x|)$. More generally, $u_S(x) := f_u(|x|)$ is the radial symmetrization of $u$.

Set

\[\widetilde{W}^{1,1}_{\text{loc}} := \{ u \in L^1_{\text{loc}}(\mathbb{R}^N) : \partial_\rho u \in L^1_{\text{loc}}(\mathbb{R}^N) \}.\]

If $u \in \widetilde{W}^{1,1}_{\text{loc}}$, then $f_u \in W^{1,1}_{\text{loc}}(0, \infty)$ and

\[(3.2) \quad \widetilde{W}^{1,1}_{\text{loc},-} := \{ u \in \widetilde{W}^{1,1}_{\text{loc}} : \lim_{t \to 0} f_u(t) = 0 \},\]

\[(3.3) \quad \widetilde{W}^{1,1}_{\text{loc},+} := \{ u \in \widetilde{W}^{1,1}_{\text{loc}} : \lim_{t \to \infty} f_u(t) = 0 \},\]

are well defined. A few properties needed later are spelled out in the next two lemmas.

**Lemma 3.1.** ([10] Lemma 3.5) If $f \in W^{1,1}_{\text{loc}}(0, \infty)$, $f \geq 0$ and $\lim_{t \to a+} f(t) = 0$ ($\lim_{t \to a} f(t) = 0$), then $f(t) \leq \int_0^t |f'(\tau)|d\tau$ ($f(t) \leq \int_t^\infty |f'(\tau)|d\tau$) for every $t > 0$.

**Lemma 3.2.** Let $a, b \in \mathbb{R}$ and $1 \leq p < \infty$ be given. If $u \in \widetilde{W}^{1,(\infty,p)}_{(a,b)}$, then

(i) $u \in \widetilde{W}^{1,1}_{\text{loc},-}$ if $a > 0$ and $u \in \widetilde{W}^{1,1}_{\text{loc},+}$ if $a < 0$.

(ii) $|u| \in \widetilde{W}^{1,(\infty,p)}_{(a,b)}$ and $||x|^av||_\infty = ||x|^av||_\infty$, $||\partial_\rho u||_{b,p} = ||\partial_\rho u||_{a,p}$.

(iii) $v := ([|u|^p]_s)^{1/p} \in \widetilde{W}^{1,(\infty,p)}_{(a,b)}$ and $||x|^av||_\infty \leq ||(|x|^av||_\infty$, $||\partial_\rho v||_{b,p} \leq ||\partial_\rho u||_{b,p}$.

**Proof.** Obviously, $\widetilde{W}^{1,(\infty,p)}_{(a,b)} \subset \widetilde{W}^{1,1}_{\text{loc}}$ irrespective of $a, b$ and $p$.

(i) By (3.1), $|x|^af_u$ is bounded on $(0, \infty)$. Thus, $\lim_{t \to 0} f_u(t) = 0$ if $a > 0$ and $\lim_{t \to 0^+} f_u(t) = 0$ if $a < 0$.

(ii) Since $u \in \widetilde{W}^{1,1}_{\text{loc}}$, then $\partial_\rho u = (\text{sgn} u)\partial_\rho u$ (as mentioned earlier). With this, the proof is trivial.

(iii) That $|x|^av \in L^\infty(\mathbb{R}^N)$ and $||x|^av||_\infty \leq ||x|^av||_\infty$ follows from $||x|^av||_\infty = ||x|^av||_\infty$ with $u$ replaced by $|u|^p$ and $(|u|^p)_s(x) := f_{u_p}(|x|)$ and from $||x|^av||_\infty = ||x|^av||_\infty$. The proof that $\partial_\rho v \in L^p(\mathbb{R}^N; \Sigma^b dx)$ with $||\partial_\rho v||_{b,p} \leq ||\partial_\rho u||_{b,p}$ is more delicate, but identical to the proof given in [10] Lemma 5.1 when, with the notation of that paper, $r = p \leq q < \infty$ and $u \in \widetilde{W}^{1,(q,p)}_{(a,b)}$. \qed
3.2. A Hardy-type inequality. If $\alpha < -1$ and $1 \leq p < \infty$, the inequality
\begin{equation}
\left( \int_0^\infty t^\alpha \left( \int_0^t g(\tau) d\tau \right)^p \right)^{\frac{1}{p}} \leq C \left( \int_0^\infty t^{\alpha+p} g(t)^p \right)^{\frac{1}{p}},
\end{equation}
holds for some constant $C > 0$ and every measurable function $g \geq 0$ on $(0, \infty)$. This is a special case of an inequality of Muckenhoupt [9] for general (compatible) weights. If $\alpha = -p$ with $p > 1$, Hardy’s inequality is recovered.

4. The embedding theorem when $r = p$

Let $d, b \in \mathbb{R}$ and $1 \leq p < \infty$. In analogy with (1.11), we define the space
\[ \tilde{W}^{1,(p,p)}_{\{d,b\}} := \{ u \in L^1_{\text{loc}}(\mathbb{R}^N) : u \in L^p(\mathbb{R}^N ; |x|^d dx) \}, \]
with norm $\|u\|_{\{d,b\},(p,p)} := \|u\|_{d,p} + \|\partial_p u\|_{b,p}$.

The next lemma is a special case of [10, Theorem 5.2].

**Lemma 4.1.** Let $b, c, d \in \mathbb{R}$ and $1 \leq p < \infty$ be given. Then, $\tilde{W}^{1,(p,p)}_{\{d,b\}} \hookrightarrow L^p(\mathbb{R}^N ; |x|^d |x|^p dx)$ (and hence $\tilde{W}^{1,(p,p)}_{\{d,b\}} \hookrightarrow \tilde{W}^{1,(p,p)}_{\{c,b\}}$) if one of the following conditions holds:

(i) $d$ and $b - p$ are on the same side of $-N$ (including $-N$), $d \neq b - p$ and $c$ is in the semi-open interval with endpoints $d$ (included) and $b - p$ (not included).

(ii) $d$ and $b - p$ are strictly on opposite sides of $-N$ and $c$ is in the semi-open interval with endpoints $d$ (included) and $-N$ (not included).

The conditions (i) and (ii) of the lemma are not necessary: There is a third option with no relevance to the issue of interest here.

**Theorem 4.2.** Let $a, b, c \in \mathbb{R}$ and $1 \leq p < \infty$ be given. Then, $\tilde{W}^{1,(\infty,p)}_{\{a,b\}} \hookrightarrow L^p(\mathbb{R}^N ; |x|^d |x|^p dx)$ if and only if $a \neq 0$ and one of the following three conditions holds:

(i) $ap - N \neq b - p$ are on the same side of $-N$ (including $b - p = -N$) and $c$ is in the open interval with endpoints $ap - N$ and $b - p$.

(ii) $ap - N$ and $b - p$ are strictly on opposite sides of $-N$ and $c$ is in the open interval with endpoints $ap - N$ and $-N$.

(iii) $a(b - p + N) > 0$ and $c = b - p$. If so, there is a constant $C > 0$ such that
\begin{equation}
\|u\|_{b-p,p} \leq C \|\partial_p u\|_{b,p}, \quad \forall u \in \tilde{W}^{1,(\infty,p)}_{\{a,b\}}.
\end{equation}

**Proof.** The necessity follows from Theorem 2.1 with $r = p$ (hence $c^0 = ap - N$ and $c^1 = b - p$). To prove the sufficiency, we first choose $\zeta \in C_0^\infty(\mathbb{R}^N)$ as in the end of the Introduction. It is readily checked that the multiplication by $\zeta$ or $1 - \zeta$ is continuous on $\tilde{W}^{1,(\infty,p)}_{\{a,b\}}$ (just notice that $|x|^{b-\alpha p}$ is integrable on Supp $\nabla \zeta$ irrespective of $a, b$ and $p$). Thus, the problem is reduced to showing that $\|\xi u\|_{c,p} \leq C \|\zeta u\|_{\{a,b\},(\infty,p)}$ and $\|(1 - \xi)u\|_{c,p} \leq C \|(1 - \zeta)u\|_{\{a,b\},(\infty,p)}$ where $C > 0$ is independent of $u$.

(i) By Kelvin transform, a proof is needed only when $ap - N > -N$ (i.e., $a > 0$) and $b - p \geq -N$. Since $ap - N \neq b - p$, this splits into the two cases $-N < ap - N < b - p$ and $-N \leq b - p < ap - N$.

Case (i-1): $-N < ap - N < b - p$.

Let $c \in (ap - N, b - p)$. That $\|\xi u\|_{c,p} \leq C \|\zeta u\|_{c,p} \leq C \|\xi u\|_{\{a,b\},(\infty,p)}$ is simply due to $|x|^{-\alpha p}$ being integrable on Supp $\zeta \subset B(0,1)$ since $c - ap > -N$.

\footnote{Note that $b - p = c^1$ in (1.2) when $r = p$ and that it is not assumed that $ap - N \neq b - p$.}
Next, pick \( d \in (-N, ap - N) \). Then, \( \| (1 - \zeta)u \|_{d,p} \leq C \| x^a (1 - \zeta)u \|_\infty \) because \( d - ap < -N \), so that \( x^{d-ap} \) is integrable on \( \text{Supp}(1 - \zeta) \subset \mathbb{R}^N \setminus B(0, \frac{1}{2}) \). Thus, \( (1 - \zeta)u \in \tilde{W}^{1,(p,p)}_{d,b} \) and \( \| (1 - \zeta)u \|_{d,b,(p,p)} \leq C \| (1 - \zeta)u \|_{(a,b),(\infty,p)} \). Since \( c \in (d, b - p) \) and \( d, b - p > -N \), part (i) of Lemma 3.2 yields \( \tilde{W}^{1,(p,p)}_{d,b} \hookrightarrow L^p(\mathbb{R}^N; |x|^d dx) \) and so \( \| (1 - \zeta)u \|_{c,p} \leq C \| (1 - \zeta)u \|_{(a,b),(\infty,p)} \) for another constant \( C \) by compounding inequalities.

Case (i-\( 2 \)): \( -N \leq b - p < ap - N \).

If \( c \in (b - p, ap - N) \), what is now obvious is that \( \| (1 - \zeta)u \|_{c,p} \leq C \| x^a (1 - \zeta)u \|_\infty \leq C \| (1 - \zeta)u \|_{(a,b),(\infty,p)} \). To prove \( \| \zeta u \|_{c,p} \leq C \| \zeta u \|_{(a,b),(\infty,p)} \), choose \( d > ap - N \) and argue as in Case (i-\( 1 \)), with minor modifications. Specifically, \( \| \zeta u \|_{d,p} \leq C \| x^a \zeta u \|_\infty \) because \( x^{d-ap} \) is integrable on \( \text{Supp} \zeta \), so that \( \zeta u \in \tilde{W}^{1,(p,p)}_{d,b} \) with \( \| \zeta u \|_{d,b,(p,p)} \leq C \| \zeta u \|_{(a,b),(\infty,p)} \), while \( \tilde{W}^{1,(p,p)}_{d,b} \hookrightarrow L^p(\mathbb{R}^N; |x|^d dx) \) by part (i) of Lemma 3.1 since \( c \in (b - p, ap - N) \subset (b - p, d) \).

(ii) By Kelvin transform, it suffices to discuss the case when \( b - p < -N < ap - N \). Let \( c \in (-N, ap - N) \) be given. As in Case (i-\( 2 \)) above, it is plain that \( \| (1 - \zeta)u \|_{c,p} \leq C \| x^a (1 - \zeta)u \|_\infty \leq C \| (1 - \zeta)u \|_{(a,b),(\infty,p)} \). The proof that \( \| \zeta u \|_{c,p} \leq C \| \zeta u \|_{(a,b),(\infty,p)} \) proceeds as in Case (i-\( 2 \)), by first choosing \( d > ap - N \) to get \( \| \zeta u \|_{d,p} \leq C \| x^a \zeta u \|_\infty \), but next using part (ii) of Lemma 3.1 since \( c \in (-N, ap - N) \subset (-N, d) \).

(iii) It suffices to prove (4.1). By Kelvin transform, suppose \( a < 0, c = b - p < -N \) with no loss of generality. By part (ii) of Lemma 3.2 it is also not restrictive to assume \( u \geq 0 \) and, by part (iii) of that lemma, that \( u \) is radially symmetric since, when \( u \) is changed into \( [(u^p)_\sigma]^\frac{1}{p} \), the left-hand side of (4.1) is unchanged and its right-hand side is not increased.

Now, if \( u \geq 0 \) is radially symmetric, then \( u(x) = u_a(|x|) \) with \( u \in W^{1,1}_{loc}(0, \infty), f_u \geq 0 \) and (4.1) becomes

\[
|| f_u ||_{b-p+N-1,p} \leq C || f'_u ||_{b+N-1,p}.
\]

By part (i) of Lemma 3.2 \( u \in \tilde{W}^{1,1}_{loc,+} \) since \( a < 0 \), so that \( f_u(t) \leq \int_0^t f'_u(\tau) d\tau \) by (3.3) and Lemma 3.1. Thus, (4.2) follows from the Hardy-type inequality (3.3) with \( \alpha = b - p + N - 1 \) \((-1)\).

5. The Embedding Theorem when \( 1 \leq r < p \)

The embedding theorem when \( 1 \leq r < p \) (Theorem 5.2 below) will now be proved by combining Theorem 4.4 with the following special case of 10 Theorem 5.2.

Lemma 5.1. Let \( b, c, d \in \mathbb{R} \) and \( 1 \leq r < p < \infty \) be given \((1 \leq p < \infty \) and \( 0 < r < \infty \) if \( N = 1 \)). Then, \( \tilde{W}^{1,(p,p)}_{d,b} \hookrightarrow L^r(\mathbb{R}^N; |x|^c dx) \) if one of the following two conditions holds:

(i) \( d \) and \( b - p \) are on the same side of \( -N \) (including \(-N\)), \( d \neq b - p \), \( c \) is in the open interval with endpoints \( \frac{r(d+N)}{p} - N \) and \( \frac{r(b-p+N)}{p} - N \).

(ii) \( d \) and \( b - p \) are strictly on opposite sides of \( -N \) and \( c \) is in the open interval with endpoints \( \frac{r(d+N)}{p} - N \) and \(-N\).
Lemma 5.1 is also true if \( r = p \), when it coincides with Lemma 4.1, but the exposition is clearer by keeping the two statements separate.

**Theorem 5.2.** Let \( a, b, c \in \mathbb{R} \) and \( 1 \leq r < \infty \) be given. Then, \( W^{1,(\infty,p)}_{(a,b)} \hookrightarrow L^r(\mathbb{R}^N; |x|^\alpha dx) \) if and only if \( a \neq 0 \) and one of the following two conditions holds:

(i) \( ap - N \neq b - p \) are on the same side of \(-N\) (including \( b - p = -N \)) and \( c \) is in the open interval with endpoints \( c^0 \) and \( c^1 \).

(ii) \( ap - N \) and \( b - p \) are on the same side of \(-N\) and \( c \) is in the open interval with endpoints \( c^0 \) and \(-N\).

**Proof.** The necessity follows from Theorem 2.1.  
(i) By part (i) of Theorem 4.2 with \( c \) replaced by \( d \), it follows that \( W^{1,(\infty,p)}_{(a,b)} \hookrightarrow L^p(\mathbb{R}^N; |x|^\alpha dx) \) for every \( d \) in the open interval \( J \) with endpoints \( ap - N \) and \( b - p \). Of course, this implies \( W^{1,(\infty,p)}_{(a,b)} \hookrightarrow W^{1,(p,p)}_{(d,b)} \) for \( d \in J \). Since \( J \) is open and its endpoints are on the same side of \(-N\), it is plain that if \( d \in J \), then \( d \) and \( b - p \) are on the same side of \(-N\) and \( d \neq b - p \). Therefore, by Lemma 5.1, \( W^{1,(p,p)}_{(d,b)} \hookrightarrow L^r(\mathbb{R}^N; |x|^\alpha dx) \) for every \( d \in J \) and every \( c \) in the open interval \( I_d \) with endpoints \( \frac{r(d+N)}{p} - N \) and \( \frac{r(b-p+N)}{p} - N \).

Altogether, this yields \( W^{1,(\infty,p)}_{(a,b)} \hookrightarrow L^r(\mathbb{R}^N; |x|^\alpha dx) \) for every \( c \in \bigcup_{d \in J} I_d \) and it is obvious that this union is the open interval with endpoints \( ar - N = c^0 \) and \( \frac{r(b-p+N)}{p} - N = c^1 \).

(ii) Proceed as in (i), but now using parts (ii) of Theorem 4.2 and Lemma 5.1.

**Remark 5.1.** For the subspace of \( W^{1,(\infty,p)}_{(a,b)} \) of radially symmetric functions, Theorem 5.2 remains true if \( 0 < r < 1 \): Just use the theorem with \( N = 1 \) after replacing \( b \) and \( c \) by \( b + N - 1 \) and \( c + N - 1 \), respectively.

6. **THE EMBEDDING THEOREM WHEN \( p < r \leq p^* \)**

When \( r < p \), the proof of Theorem 5.2 shows that \( W^{1,(\infty,p)}_{(a,b)} \hookrightarrow L^r(\mathbb{R}^N; |x|^\alpha dx) \) if and only if \( W^{1,(\infty,p)}_{(a,b)} \hookrightarrow L^p(\mathbb{R}^N; |x|^\alpha dx) \) for some suitable \( d \in \mathbb{R} \). This feature is no longer true when \( r > p \), even when \( W^{1,(\infty,p)}_{(a,b)} \) is replaced by the smaller space \( W^{1,1}(\mathbb{R}^N) \). Accordingly, the strategy of proof will be different. We shall need two other special cases of the embedding theorem in [10]. For clarity, they are given in two separate statements. Lemma 6.1 is a rephrasing of parts (i) and (ii) of [10] Theorem 7.1 when \( p < r = q \leq p^* \) and the inequality in Lemma 6.2 below is proved in [10] Theorem 10.2.

**Lemma 6.1.** Let \( a, b, c, d \in \mathbb{R} \) and \( 1 \leq p < r < \infty \), \( r \leq p^* \) be given. Then, \( W^{1,(r,p)}_{(d,b)}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^\alpha dx) \) if one of the following conditions holds:

(i) \( d \) and \( b - p \) are on the same side of \(-N\) (including \(-N\)), \( \frac{d+N}{r} \neq \frac{b-p+N}{p} \), \( c \) is in the semi-open interval with endpoints \( d \) (included) and \( \frac{r(b-p+N)}{p} - N \) (not included).

(ii) \( d \) and \( b - p \) are strictly on opposite sides of \(-N\) and \( c \) is in the semi-open interval with endpoints \( c^0 \) and \( c^1 \).
Lemma 6.1 remains true if \( r = p \), when it is a special case of Lemma 4.1 since \( W^{1,(p,p)}_{(d,b)}(\mathbb{R}_b^+) \hookrightarrow W^{1,(p,p)}_{(d,b)} \).

**Lemma 6.2.** Let \( b \in \mathbb{R} \) and \( 1 \leq p < r < \infty, r \leq p^* \) be given. If \( b - p \neq -N \), then \( W^{1,(p,p)}_{(b-p,b)}(\mathbb{R}_b^+) \hookrightarrow L^r(\mathbb{R}^N; |x|^c \, dx) \), where (as in (1.4)), \( c^1 = \frac{r(b-p+N)}{p} - N \). Furthermore, there is a constant \( C > 0 \) such that

\[
||u||_{c^1,r} \leq C||\nabla u||_{b,p}, \quad \forall u \in W^{1,(p,p)}_{(b-p,b)}(\mathbb{R}_b^+).
\]

The conditions given in Lemmas 6.1 and 6.2 are not necessary, but they will suffice for our purposes.

**Theorem 6.3.** Let \( a,b,c \in \mathbb{R} \) and \( 1 \leq p < r < \infty, r \leq p^* \), be given. Then, \( W^{1,(\infty,\infty)}_{(a,b)}(\mathbb{R}^N_a) \hookrightarrow L^r(\mathbb{R}^N; |x|^c \, dx) \) if and only if \( a \neq 0 \) and one of the following three conditions holds:

(i) \( a \) \( \neq -N \) \( b - p \) are on the same side of \( -N \) (including \( b - p = -N \)) and \( c \) is in the open interval with endpoints \( c^0 \) and \( c^1 \).

(ii) \( a \neq -N \) and \( b - p \) are strictly on opposite sides of \( -N \) and \( c \) is in the open interval with endpoints \( c^0 \) and \( c^1 \).

(iii) \( a(b-p+N) > 0 \) and \( c = c^1 \). If so, there is a constant \( C > 0 \) such that

\[
||u||_{c^1,r} \leq C||\nabla u||_{b,p}, \quad \forall u \in W^{1,(\infty,\infty)}_{(a,b)}(\mathbb{R}_a^+).
\]

**Proof.** Once again, the necessity follows from Theorem 2.1 and we only address the sufficiency.

(i) When \( d \in \mathbb{R} \) runs over the open interval with endpoints \( c^0 = ar - N \) and \( b - p + a(r-p) = \frac{b}{c} c^1 + (1-\frac{b}{c}) c^0 \), the point \( c_d := d - a(r-p) \) runs over the open interval with endpoints \( ap - N \) and \( b - p \).

Note that since \( 0 < \frac{b}{c} < 1 \), the point \( d \) above lies in the open interval with endpoints \( c^0 \) and \( c^1 \). Since both \( ap - N \) and \( b - p \) are on the left (or right) of \( -N \), then both \( c^0 \) and \( c^1 \) are on the left (or right) of \( -N \), so that \( d \) and \( b - p \) are always on the same side of \( -N \).

Furthermore, if \( d \) is close enough to \( c^0 \), then \( \frac{d+N}{r} \neq \frac{b-p+N}{p} \) since this holds when \( d = c^0 \) (recall \( ap - N \neq b - p \)). This assumption is retained in the subsequent considerations.

By part (i) of Theorem 4.2 with \( c \) replaced by \( c_d := d - a(r-p) \), it follows that \( ||u||_{c_d,p} \leq C||u||_{(a,b),(\infty,\infty)} \) when \( u \in \overline{W}^{1,(\infty,\infty)}_{(a,b)} \). If so, \( |x|^c u \in L^\infty(\mathbb{R}^N) \) and \( r > p \) yield \( |x|^{a(r-p)}|u|^{r-p} \in L^\infty(\mathbb{R}^N) \) with \( \||x|^{a(r-p)}|u|^{r-p}\|_\infty = |||x|^a u||_\infty^{r-p} \). As a result, \( |x|^c u = (|x|^{a(r-p)}|u|^{r-p})(|x|^c u)p \in L^1(\mathbb{R}^N) \) and \( ||u||_{\infty,\infty} = |||x|^c u||_{1} \leq \|||x|^c u||_{\infty,\infty}||x|^c u||_{\infty,\infty} \leq C^p ||u||_{(a,b),(\infty,\infty)}^p \). This shows that \( \overline{W}^{1,(\infty,\infty)}_{(a,b)} \hookrightarrow L^r(\mathbb{R}^N; |x|^d \, dx) \). In particular, \( W^{1,(\infty,\infty)}_{(a,b)}(\mathbb{R}^N_a) \hookrightarrow L^r(\mathbb{R}^N; |x|^c \, dx) \) and so \( W^{1,(\infty,\infty)}_{(a,b)}(\mathbb{R}_a^+) \hookrightarrow L^r(\mathbb{R}^N; |x|^d \, dx) \) and \( W^{1,(\infty,\infty)}_{(a,b)}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N; |x|^c \, dx) \).

As shown earlier, \( d \) and \( b - p \) are on the same side of \( -N \) and \( \frac{d+N}{r} \neq \frac{b-p+N}{p} \). Therefore, by part (i) of Lemma 6.1, if \( c \) is in the semi-open interval with endpoints \( d \) (included) and \( \frac{b-p+N}{p} - N = c^3 \) (not included), then \( W^{1,(r,p)}_{(d,b)}(\mathbb{R}^N_a) \hookrightarrow L^r(\mathbb{R}^N; |x|^c \, dx) \) and so, from the above, \( W^{1,(\infty,\infty)}_{(a,b)}(\mathbb{R}_a^+) \hookrightarrow L^r(\mathbb{R}^N; |x|^c \, dx) \). Since \( d \)
can be chosen arbitrarily close to $c^0$, this embedding does hold for every $c$ in the open interval with endpoints $c^0$ and $c^1$, as claimed.

(ii) Proceed as above, but now using the parts (ii) of Theorem 4.2 and of Lemma 6.1 (note that $ap - N$ and $c^0 = ar - N$ are always on the same side of $-N$, so that $d$ and $b - p$ as well as $c_d$ and $b - p$ are on opposite sides of $-N$ if $d$ is close to $c^0$).

(iii) First, $W^{1,\infty,p}(\mathbb{R}_a^+) \hookrightarrow W^{1,\infty,p}(\mathbb{R}_a^+)$ is locally Lipschitz continuous, the latter by part

(ii) of Theorem 4.2 so that $W^{1,\infty,p}(\mathbb{R}_a^+) \hookrightarrow W^{1,\infty,p}(\mathbb{R}_a^+)$ Next, by Lemma

6.2 $W^{1,\infty,p}(b-p,b)(\mathbb{R}_a^+) \hookrightarrow L^r(\mathbb{R}_a^+;|x|^c dx)$ and (6.1) holds since $b - p \neq -N$ and $c^1 = \frac{r(b-p+N)}{p} - N$.

\[ \square \]

Remark 6.1. Set $a_r := c^1 = \frac{r(b-p+N)}{p} - N$ to make explicit the $r$-dependence.

By part (iii) of Theorem 6.3, $W^{1,\infty,p}(\mathbb{R}_a^+) \hookrightarrow W^{1,\infty,p}(\mathbb{R}_a^+)$ if $a(b - p + N) > 0$. As pointed out in the Introduction of [10], the space $W^{1,\infty,p}(\mathbb{R}_a^+)$ is actually independent of $p \leq r < p^*$, $r < \infty$. Although this will be proved elsewhere ([11]), it seems of interest to report that if $N = 1$ or $p > N > 1$ (hence $p^* = \infty$), this space also coincides with $W^{1,\infty,p}(\mathbb{R}_a^+)$ when $a = \frac{b-p+N}{p}$. In other words, if $N = 1$ or $p > N > 1$ and if $b - p + N \neq 0$, then $W^{1,\infty,p}(\mathbb{R}_a^+) = W^{1,\infty,p}(\mathbb{R}_a^+)$ for every $p \leq r < \infty$, with equivalent norms.

7. The embedding theorem when $p < N$ and $r > p^*$

When $p < N$ and $r > p^*$, the embedding theorem will be deduced from the case $r = p^*$ in Theorem 6.3 after changing $u \in W^{1,\infty,p}(\mathbb{R}_a^+)$ into $|u|^{\frac{r}{p^*}}$. The details of the procedure follow.

Lemma 7.1. Let $a, b \in \mathbb{R}$ and $1 \leq p < N, p^* < r < \infty$, be given. If $u \in W^{1,\infty,p}(\mathbb{R}_a^+)$, then $|u|^{\frac{r}{p^*}} \in W^{1,\infty,p}(\mathbb{R}_a^+)$ where

\[ a^* := \frac{ar}{p^*} \quad \text{and} \quad b^* := b + ap\left(\frac{r}{p^*} - 1\right) \]

and $\| |x|^a |u|^{\frac{r}{p^*}} \| \| = \| |x|^a u \|^{\frac{r}{p^*}}$, \( \| \nabla |u|^{\frac{r}{p^*}} \|_{b^*,p} \leq \frac{r}{p^*} \| \nabla u \|_{b^*,p} \| |x|^a u \|^{\frac{r}{p^*} - 1} \|

Proof. Since $u \in W^{1,\infty,p}(\mathbb{R}_a^+) \subset L^\infty(\mathbb{R}_a^+)$, it is clear that $|u|^{\frac{r}{p^*}} \in L^\infty(\mathbb{R}_a^+)$. Thus, everything is a routine verification if it is shown that $\nabla \left(|u|^{\frac{r}{p^*}}\right)$ (as a distribution on $\mathbb{R}_a^+$) is ACK \( \frac{r}{p^*} [u]^{\frac{r}{p^*} - 1} \nabla u \in L^p(W_9^*) \subset \text{L}^1(\mathbb{R}_a^+)\.)

The next Lemma is just a special case of Theorem 6.3.

Lemma 7.2. Let $a, b, c \in \mathbb{R}$ and $1 \leq p < N, p^* < r < \infty$ be given. If $a^*$ and $b^*$ are defined by (7.1), then $W^{1,\infty,p}(\mathbb{R}_a^+) \hookrightarrow L^p(\mathbb{R}_a^+; |x|^c dx)$ if and only if $a \neq 0$ and one of the following three conditions holds:

(i) $a^* p - N \neq b^* - p$ are on the same side of $-N$ (including $b^* - p = -N$) and $c$ is in the open interval with endpoints $c^0 = ar - N$ and $c^{1*} := \frac{p(b^* - p + N)}{p} - N = \ldots$
\( \frac{\varepsilon}{r} c^1 + \left( 1 - \frac{\varepsilon}{r} \right) c^0. \)

(ii) \( a^* p - N \) and \( b^* - p \) are strictly on opposite sides of \( -N \) and \( c \) is in the open interval with endpoints \( c^0 \) and \( -N. \)

(iii) \( a(b^* + p + N) > 0 \) and \( c = c^1. \) Furthermore, there is a constant \( C > 0 \) such that

\[
\|u\|_{c^1, p^*} \leq C \|\nabla u\|_{b^*, p^*} \quad \forall u \in W^{1, (\infty, p)}(\mathbb{R}^n).
\]

**Proof.** Observe that \( c^0 := ar - N = a^* p^* - N \) and use Theorem 6.3 with \( a, b, r \) replaced by \( a^*, b^*, p^* \), respectively.

Recall the definition of \( \theta_c \) in (1.4) and (1.5) when \( c \) is in the closed interval with endpoints \( c^0 \) and \( c^1 \neq 0. \)

**Theorem 7.3.** Let \( a, b, c \in \mathbb{R} \) and \( 1 \leq p < N, p^* < r < \infty \) be given. Then, \( W^{1, (\infty, p)}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n; |x|^d dx) \) if and only if \( a \neq 0 \) and one of the following three conditions holds:

(i) \( ap - N \neq b - p \) are on the same side of \( -N \) (including \( b - p = -N \)), \( c \) is in the open interval with endpoints \( c^0 \) and \( c^1 \) and \( \theta_c \leq \frac{\varepsilon}{r} \) (i.e., \( c \) is in the semi-open interval with endpoints \( c^0 \) (not included) and \( c^1 = \frac{\varepsilon}{r} c^1 + \left( 1 - \frac{\varepsilon}{r} \right) c^0 \) (included)).

(ii) \( ap - N \) and \( b - p \) are strictly on opposite sides of \( -N, c \) is in the open interval with endpoints \( c^0 \) and \( -N \) and \( \theta_c \leq \frac{\varepsilon}{r} \) (i.e., \( \theta_c \in (0, 1) \) is defined and \( c \) is in the open interval with endpoints \( c^0 \) and \( -N \) if \( \theta_c \leq \frac{\varepsilon}{r} \), or in the semi-open interval with endpoints \( c^0 \) (not included) and \( c^1 \) (included) if \( \theta_c > \frac{\varepsilon}{r} \)).

(iii) \( ap - N = b - p \) and \( c = c^1 \) (i.e. \( c^0 \)). If so, there is a constant \( C > 0 \) such that

\[
\|u\|_{c, r} \leq C \left|\frac{\nabla u}{b, p}\right| |x|^n |u|^{1 - \frac{r}{p^*}}, \quad \forall u \in W^{1, (\infty, p)}(\mathbb{R}^n).
\]

**Proof.** The necessity follows from Theorem 2.1 and Corollary 2.3, and the sufficiency is proved below.

(i) By (7.1), the hypothesis \( ap - N \neq b - p \) shows that \( a^* p - N \neq b^* - p \). Furthermore, since \( ap - N \) and \( b - p \) are both on the left (or right) \( -N \), it is readily checked that the same thing is true of \( a^* p - N \) and \( b^* - p \). Another important point is that \( b^* - p \neq -N \). Otherwise, \( b - p + N = -ap \left( \frac{p^*}{p} - 1 \right) \). Since \( r > p^* \), this implies \( b - p + N < 0 \) if \( a > 0 \) and \( b - p + N > 0 \) if \( a < 0 \) so that, in either case, \( ap - N \) and \( b - p \) are strictly on opposite sides of \( -N \), in contradiction with the standing hypotheses.

By part (i) of Lemma 7.2, if \( c \) is in the open interval with endpoints \( c^0 \) and \( c^1 \), there is a constant \( C > 0 \) such that \( ||v||_{c^0, r^*} \leq C (|| |x|^{a} v||_{\infty} + ||\nabla v||_{b^*, p^*}) \) for every \( v \in W^{1, (\infty, p)}(\mathbb{R}^n) \). By Lemma 7.1, this holds with \( v = |u|^{p^*} \) and \( u \in W^{1, (\infty, p)}(\mathbb{R}^n) \).

Since (also by Lemma 7.1)

\[
|| |x|^a |u|^{p^*}||_{\infty} = || |x|^a u||^{p^*}_{\infty} \quad \text{and} \quad ||\nabla |u|^{p^*}||_{b^*, p^*} \leq \frac{r}{p^*} ||\nabla u||_{b, p} || |x|^a u||^{1 - \frac{r}{p^*}}_{\infty},
\]

it follows that \( ||u||_{c, r} \leq C (|| |x|^a u||^{1 - \frac{r}{p^*}}_{\infty} (|| |x|^a u||_{\infty} + ||\nabla u||_{b, p} ||^{p^*}) \leq C (|| |x|^a u||_{\infty} + ||\nabla u||_{b, p} ||^{p^*}) \), whence \( W^{1, (\infty, p)}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n; |x|^d dx) \).

Since \( b - p \neq -N \), the same argument, but with part (iii) of Lemma 7.2 instead of part (i), shows that \( W^{1, (\infty, p)}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n; |x|^d dx) \) also when \( c = c^1. \)
(ii) The main difference with (i) is that \( ap - N \) and \( b - p \) being strictly on opposite sides of \( -N \) does not imply the same thing for \( a^* p - N \) and \( b^* - p \). The argument is split into three cases, depending upon the relative values of \( \theta_{-N} \) and \( \frac{c^0}{c^1} \). Observe that since \( ap - N \) and \( b - p \) are strictly on opposite sides of \( -N \), the same thing is true of \( c^0 \) and \( c^1 \), so that \( -N \) is in the open interval with endpoints \( c^0 \) and \( c^1 \) and \( \theta_{-N} \in (0, 1) \) is defined.

Case (ii-1): \( \theta_{-N} > \frac{c^0}{c^1} \).

Since \( c^1 = \frac{c^0}{r} c^1 + \left(1 - \frac{c^0}{r}\right) c^0 \) with \( c^0 \neq c^1 \) amounts to \( \theta_{c^1} = \frac{c^0}{r} \), it follows that \( \theta_{-N} > \theta_{c^1} \). In particular, \( c^1 \neq -N \) and so \( b^* - p \neq -N \) since \( c^1 = \frac{p}{p}(b^* - p + N) - N \), a formula which also shows that \( c^1 \) and \( b^* - p \) are on the same side of \( -N \). On the other hand, \( \theta_{-N} > \theta_{c^1} \) also means that \( c^1 \) and \( c^0 = ar - N \) are on the same side of \( -N \). Since \( a \neq 0 \) and \( a^* \neq 0 \) have the same sign, \( ar - N \) and \( a^* p - N \) are also on the same side of \( -N \). As a result, \( a^* p - N \) and \( b^* - p \neq -N \) are on the same side of \( -N \). Therefore, the proof of (i) can be repeated verbatim, to the effect that \( W_{(a,b)}^{r,(\infty,p)}(\mathbb{R}^N) \rightarrow L^r(\mathbb{R}^N, |x|^c dx) \) for every \( c \) in the semi-open interval with endpoints \( c^0 \) (not included) and \( c^1 \) (included).

Case (ii-2): \( \theta_{-N} = \frac{c^0}{c^1} \).

Then, \( -N = c^1 \) and so \( b^* - p = -N \). Evidently, this ensures that \( a^* p - N \) and \( b^* - p \) are still on the same side of \( -N \) (though of course not strictly). The proof of (i) still shows that \( W_{(a,b)}^{r,(\infty,p)}(\mathbb{R}^N) \rightarrow L^r(\mathbb{R}^N, |x|^c dx) \) for \( c \) in the open interval with endpoints \( c^0 \) and \( c^1 = -N \) (but not when \( c = -N \)).

Case (ii-3): \( \theta_{-N} < \frac{c^0}{c^1} \).

By arguing as in Case (ii-1), \( a^* p - N \) and \( b^* - p \) are now strictly on opposite sides of \( -N \). By combining part (ii) of Lemma \[7.2\] and Lemma \[7.1\] in the same way as in the proof of (i), it follows that \( W_{(a,b)}^{r,(\infty,p)}(\mathbb{R}^N) \rightarrow L^r(\mathbb{R}^N, |x|^c dx) \) for every \( c \) in the open interval with endpoints \( c^0 \) and \( -N \).

(iii) If \( ap - N = b - p \), then \( c^0 = c^1 = c^*^1 \), the latter since \( c^1 = \frac{c^0}{r} c^1 + \left(1 - \frac{c^0}{r}\right) c^0 \) and \( c^0 = c^1 \). If \( a > 0 \) (\( a < 0 \)), then \( b - p > -N \) and \( b^* - p > -N \) (\( b - p < -N \) and \( b^* - p < -N \)). Thus, \( a(b^* - p + N) > 0 \), so that \[7.3\] follows from Lemma \[7.1\] from \[7.2\] with \( u \) replaced by \(|u|^{\frac{1}{r^*}} \) and from \[7.3\].

8. An example

As an application of Theorem \[1.1\] we characterize all the real numbers \( a, b \) and \( 1 \leq p, r < \infty \) such that \( W_{(a,b)}^{r,(\infty,p)}(\mathbb{R}^N) \rightarrow L^r(\mathbb{R}^N) \), i.e., such that \( c = 0 \) is admissible in Theorem \[1.1\]. We merely give the result and leave the routine (though somewhat tedious) verification to the reader. This verification is easier by using Theorems \[4.2\] \[5.2\] \[6.3\] and \[7.3\] rather than Theorem \[1.1\] (equivalent to all four theorems together, but not phrased in terms of the relative values of \( r, p \) and \( p^* \)). It is also helpful to notice that \( ap - N \) and \( b - p \) are on the same side of \( -N \) (including \( -N \)) if and only if \( a(b - p + N) \geq 0 \) and strictly on opposite sides of \( -N \) if and only if \( a(b - p + N) < 0 \).

**Theorem 8.1.** Let \( a, b \in \mathbb{R} \) and \( 1 \leq p < \infty \) be given. Then, \( W_{(a,b)}^{r,(\infty,p)}(\mathbb{R}^N) \rightarrow L^r(\mathbb{R}^N) \) if and only if \( a > 0 \) and one of the following conditions holds:

(i) \( 1 \leq r < p \) and
(i-1) $a < \frac{N}{r}$ and $b > Np \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right)$ or 

(i-2) $a > \frac{N}{r}$ and $b < Np \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right)$.

(ii) $0 < r \leq p^*$ and

(ii-1) $a < \frac{N}{r}$ and $b \geq Np \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right)$ or

(ii-2) $a > \frac{N}{r}$ and $b \leq Np \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right)$ or

(ii-3) $a = \frac{N}{r}$ and $b = Np \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right)$.

(iii) $p < N, r > p^*$ and

(iii-1) $a < \frac{N}{r}$ and $b \geq arp \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right)$ or

(iii-2) $a > \frac{N}{r}$ and $b \leq arp \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right)$ or

(iii-3) $a = \frac{N}{r}, b = arp \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right) = Np \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right)$.

For the subspace of radially symmetric functions of $W^{1, (\infty, p)}(\mathbb{R}_+^N)$, part (i) remains true when $0 < r < p$. (Of course, the embedding properties of radially symmetric functions into $L^r(\mathbb{R}^N)$ have nothing to do with the case $N = 1$ of Theorem 3.1 because the reduction to $N = 1$ not only requires changing $b$ into $b + N - 1$ but also $c$ into $c + N - 1$, which does not preserve the value $c = 0$).

Whenever $u \in W^{1, (\infty, p)}(\mathbb{R}_+^N)$ and $a < \frac{N}{r}$ ($a > \frac{N}{r}$) is assumed, the integrability of $|u|^r$ near the origin (at infinity) is obvious. What is not obvious is that integrability at infinity (near the origin) is also true when the complementary condition about $b$ holds. In (ii-3) and (iii-3), $a = \frac{N}{r}$ alone does not suffice for $|u|^r$ to be integrable near the origin or at infinity, so that both properties also depend upon the complementary condition $b = pN \left( \frac{1}{N} + \frac{1}{r} - \frac{1}{p} \right)$.

Below, we give more direct proofs of the sufficiency part of Theorem 8.1 in three simpler special cases. In these proofs, the role played by the integrability properties of $\nabla u$ (i.e., by the complementary condition referred to above) becomes more apparent and connections with (semi) classical results are revealed. On the other hand, the arguments depend upon the problem at hand and do not suggest a general procedure to prove Theorem 8.1, let alone Theorem 1.1, in any generality.

**Example 1.** Let $N = 1$ and $a = b = 1, p = r = 1$. Clearly, the embedding properties are the same when $\mathbb{R}_+$ is replaced by $(0, \infty)$. Thus, by (ii-3), $u \in L^1(0, \infty)$ if $tu \in L^\infty(0, \infty)$ and $tu' \in L^1(0, \infty)$. Since this assumption is unaffected by changing $u$ into $|u|$, it is not restrictive to assume $u \geq 0$. Then, $u$ is locally absolutely continuous on $(0, \infty)$ and the formula $\int_0^\beta u = \beta u(\beta) - \alpha u(\alpha) - \int_0^\beta tu'$ for every $0 < \alpha < \beta < \infty$ shows that, indeed, $u \in L^1(0, \infty)$ since the right-hand side is bounded irrespective of $\alpha$ and $\beta$.

**Example 2.** More generally, still with $N = 1$, assume $r \geq p$ and $b = p \left( 1 + \frac{1}{r} - \frac{1}{p} \right)$.

By (ii) of Theorem 8.1 $u \in L^r(0, \infty)$ if $t^a u \in L^\infty(0, \infty)$ for some $a > 0$ and $t^{\frac{a}{p}} u' \in L^p(0, \infty)$. Since $a > 0$, it follows from part (i) of Lemma 3.2 and from Lemma 5.1 that $|u(t)| \leq \int_0^t |u'(\tau)| d\tau$. On the other hand, $\left( \int_0^\infty \left( \int_0^t |u'(\tau)| d\tau \right)^p dt \right)^{\frac{1}{p}} \leq C \left( \int_0^\infty t^b |u(t)| |dt|^{\frac{1}{p}} \right)^{\frac{1}{b}}$ by an inequality of Bradley [11, 21, p. 40] generalizing the Hardy-type inequality (3.3), which gives again $u \in L^r(0, \infty)$. 
Example 3. Other special cases of Theorem [8.1] can be given alternate proofs, including when \( N > 1 \). For example, if \( b = 0 \) in Theorem [8.1], then (i-1), (ii) and (iii-3) cannot occur and the necessary and sufficient conditions for \( W^{1,(\infty,p)}_{\{0\}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \) take the much simpler form:

**Corollary 8.2.** Let \( a \in \mathbb{R} \) and \( 1 \leq p < \infty \) be given. Then, \( W^{1,(\infty,p)}_{\{0\}}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N) \) if and only if \( a > 0 \) and one of the following conditions holds:

(i) \( p < N \) and \( a > \frac{N}{r} \).

(ii) \( p = N \) and \( r = p^* \).

(iii) \( p < N \) and \( r > p^* \).

With the help of the right trick, the sufficiency part of Corollary [8.2] can also be proved by classical arguments (necessity still relies on Section [2]). Since \( N = 1 \) is trivial, we assume \( N \geq 2 \) and, for brevity, we only show how \( W^{1,(\infty,p)}_{\{0\}}(\mathbb{R}^N) \subset L^r(\mathbb{R}^N) \) can be recovered without elaborating on the continuity issue.

We shall need the preliminary remark that \( W^{1,(\infty,p)}_{\{a\}}(\mathbb{R}^N) = W^{1,(\infty,p)}_{\{a,0\}}(\mathbb{R}^N) \) for every \( a \in \mathbb{R} \) when \( N \geq 2 \). To see this, note that \( \mathbb{R}^N \) is the union of finitely many open half-spaces \( H_j \). If \( u \in W^{1,(\infty,p)}_{\{a\}}(\mathbb{R}^N) \), then \( \nabla u \in L^p(H_j) \) and so \( u \in W^{1,p}(B \cap H_j) \) where \( B \) is the unit ball of \( \mathbb{R}^N \). Since \( j \) is arbitrary, it follows that \( u \in W^{1,p}(B \setminus \{0\}) \).

By [4] (Theorem 2.44) or by the so-called “absolutely continuous” characterization of Sobolev functions ([12], Theorem 2.1.4), \( W^{1,p}(B \setminus \{0\}) = W^{1,p}(B) \subset L^r(\mathbb{R}^N) \) when \( N \geq 2 \). In particular, \( \nabla u \) as a distribution on \( B \) coincides with \( \nabla u \) as a distribution on \( B \setminus \{0\} \). Thus, \( \nabla u \) is the same as a distribution on \( \mathbb{R}^N \) or \( \mathbb{R}^N \), so that \( u \in W^{1,(\infty,p)}_{\{a\}}(\mathbb{R}^N) \).

**Remark 8.1.** By a similar argument, \( W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N) = W^{1,(q,p)}_{\{a\}}(\mathbb{R}^N) \) when \( N \geq 2 \) and \( b \leq 0 \), irrespective of \( 1 \leq p < \infty, 1 \leq q \leq \infty \) and \( a \in \mathbb{R} \). However, equality need not hold when \( b \geq 0 \). For example, if \( u \in C^\infty(\mathbb{R}^N) \) has bounded support and coincides with \( |x|^{-N} \) on a neighborhood of \( 0 \), then \( u \) is not integrable near \( 0 \). Thus, \( u \) is not in \( W^{1,(q,p)}_{\{a\}}(\mathbb{R}^N) \) for any values of the parameters, but \( u \in W^{1,(q,p)}_{\{a,b\}}(\mathbb{R}^N) \) if \( a \) and \( b \) are large enough (depending on \( p \) and \( q \)).

By using \( W^{1,(\infty,p)}_{\{a\}}(\mathbb{R}^N) = W^{1,(\infty,p)}_{\{a,0\}}(\mathbb{R}^N) \), a “direct” proof that \( a > 0 \) plus any one of the conditions (i) to (iii) of Corollary [8.2] suffices for \( W^{1,(\infty,p)}_{\{a\}}(\mathbb{R}^N) \subset L^r(\mathbb{R}^N) \) goes as follows.

Suppose first that \( p < N \) and let \( u \in W^{1,(\infty,p)}_{\{a\}}(\mathbb{R}^N) \) be given. By Sobolev’s inequality, there is a constant \( C > 0 \) independent of \( u \) such that \( ||u - c||_p \leq C||\nabla u||_p \) for some \( c \in \mathbb{R} \). In addition, \( |x|^nu \in L^\infty(\mathbb{R}^N) \) with \( a > 0 \) implies that \( u \) tends (essentially) uniformly to 0 at infinity. Thus, \( c = 0 \), so that \( u \in L^p(\mathbb{R}^N) \) and \( ||u||_p \leq C||\nabla u||_p \). Not only does this shows that (ii) suffices, but it also corroborates the inequality (1.8) or (1.19) of Theorem 1.1 when \( a \neq \frac{N}{p} - 1 \) (or \( a = \frac{N}{p} - 1 \), \( b = 0 \) and \( r = p^* \) (so that \( c^1 = 0 \) in 1.4), whence \( \theta_0 = 1 \)).

Next, if \( r > p^* \), the same argument that \( u \) tends to 0 at infinity and \( u \in L^{p^*}(\mathbb{R}^N) \) shows that \( |u|^r \) is integrable at infinity. If \( a < \frac{N}{r} \), then \( |u|^r \) is also integrable near 0, which proves that (iii) suffices. On the other hand, if \( r < p^* \), then \( u \in L^{p^*}(\mathbb{R}^N) \) implies \( u \in L^{p^*}_{\text{loc}}(\mathbb{R}^N) \) while \( |u|^r \) is integrable at infinity if \( a > \frac{N}{r} \). This proves that (i) suffices when \( p < N \).
It only remains to prove the sufficiency of (i) when $p \geq N$ and $1 \leq r < \infty$. As before, $a > \frac{N}{r}$ implies that $|u|^r$ is integrable at infinity. Once again, let $B$ denote the unit ball of $\mathbb{R}^N$ and $H_j$ a finite collection of open hyperplanes such that $\bigcup H_j = \mathbb{R}^N$. The condition $\nabla u \in L^p(\mathbb{R}^N)$ implies $\nabla u \in L^p(B \cap H_j)$. Therefore, $u \in W^{1,p}(B \cap H_j)$ and so $u \in L^r(B \cap H_j)$ by the Sobolev embedding theorem. As a result, $u \in L^r(B \setminus \{0\}) = L^r(B)$ and the proof is complete.

REFERENCES

[1] Bradley, S. C., Hardy inequalities with mixed norms, *Canad. Math. Bull.* 21 (1978) 405-408.
[2] Caffarelli, L., Kohn, R. and Nirenberg, L., First order interpolation inequalities with weights, *Compos. Math.* 53 (1984) 259-275.
[3] Catrina, F. and Costa, D. G., Sharp weighted-norm inequalities for functions with compact support in $\mathbb{R}^N \setminus \{0\}$, *J. Differential Equations* 246 (2009) 164-182.
[4] Heinonen, J., Kilpeläinen, T. and Martio, O., *Nonlinear potential theory of degenerate elliptic equations*, Oxford Univ. Press, Oxford 1993.
[5] Marcus, M. and Mizel, V. J., Absolute continuity on tracks and mappings of Sobolev spaces, *Arch. Rational Mech. Anal.* 45 (1972) 294-320.
[6] Maz’ya, V. G., On certain integral inequalities for functions of many variables, *J. Math. Sciences* 1 (1973) 205-234.
[7] Maz’ya, V. G., *Sobolev spaces*, Springer-Verlag, Berlin, 1985.
[8] Maz’ya, V., *Sobolev spaces with applications to elliptic partial differential equations*, Grund. Math. Wiss. Vol. 342, Springer, Heidelberg 2011.
[9] Muckenhoupt, B., Hardy’s inequality with weights, *Studia Math.* 44 (1972) 31-38.
[10] Rabier, P. J., Embeddings of weighted Sobolev spaces and generalized Caffarelli-Kohn-Nirenberg inequalities, *J. Anal. Math.* (to appear) Arxiv preprint 1106.4874.
[11] Rabier, P. J., Boundedness properties of Sobolev functions in spaces with power weights (in preparation).
[12] Ziemer, W. P., *Weakly differentiable functions*, Springer-Verlag, Berlin, 1989.

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
E-mail address: rabier@imap.pitt.edu