Gauge transformations in lattice chiral theories

Werner Kerler

_Institut für Physik, Humboldt-Universität, D-12489 Berlin, Germany_

Abstract

We show that gauge-transformation properties of correlation functions in chiral gauge theories on the finite lattice are determined in a general way.

1 Introduction

Gauge invariance of the chiral determinant on the lattice has been considered a major problem [1]. A particular construction aiming at gauge invariance has been presented in Ref. [2]. Using finite transformations we here show that gauge-transformation properties of correlation functions in chiral gauge theories on the finite lattice are determined in a general way, so that there is no freedom for adjustments by constructions. A careful consideration of equivalence classes of pairs of bases is important in this context. Our results hold also in the presence of zero modes and for any value of the index.

We also add a corresponding analysis of the subject in terms of gauge variations. Within this we find that fully exploiting the covariance requirement for the current of Ref. [2] leads to the same result as follows from the indicated consideration of equivalence classes of pairs of bases. On the other hand, the behavior of the effective action anticipated in Ref. [2] in its special case and the view of the anomaly cancelation there turn out not to conform with the actual results.

In Section 2 we collect some relations needed. In Section 3 we use finite transformations to analyze the general cases with both and with only one of the chiral projections depending on the gauge field. In Section 4 we consider variations of the effective action and the special case of Ref. [2]. Section 5 contains conclusions and discussions.
2 General relations

2.1 Chiral projections

The chiral projections $\bar{P}_+$ and $P_-$ obey $\bar{P}_+^\dagger = \bar{P}_+ = \bar{P}_+^2$, $P_-^\dagger = P_- = P_-^2$ and
\[ \bar{P}_+ D = DP_-, \]
where $D$ is the Dirac operator. For the numbers of anti-Weyl and Weyl degrees of freedom $\bar{N} = \text{Tr} \bar{P}_+$ and $N = \text{Tr} P_-$ we have the two possibilities $\bar{N} = d$, $N = d - I$ or $\bar{N} = d + I$, $N = d$, where $d = \frac{1}{2} \text{Tr} \mathbb{1}$. Requiring $D$ to be $\gamma_5$-Hermitian and normal, $I$ is its index.

The chiral projections may also be expressed as
\[ P_- = \frac{1}{2}(1 + \gamma_5 G), \quad \bar{P}_+ = \frac{1}{2}(1 - \gamma_5 \bar{G}), \]
with $\gamma_5$-Hermitian and unitary operators $G$ and $\bar{G}$, which according to (2.1) satisfy
\[ D + \bar{G}D^\dagger G = 0. \] (2.3)

2.2 Basic fermionic functions

In terms of Grassmann variables basic fermionic correlation functions – which do not vanish identically and of which linear combinations make up general functions – for the Weyl degrees of freedom are of form
\[ \langle \chi_{i_{r+1}} \cdots \chi_{i_N} \bar{\chi}_{j_{r+1}} \cdots \bar{\chi}_{j_N} \rangle_f = s_r \int d\bar{\chi}_N \cdots d\bar{\chi}_1 d\chi_N \cdots d\chi_1 e^{-\bar{\chi}M\chi} \chi_{i_{r+1}} \cdots \chi_{i_N} \bar{\chi}_{j_{r+1}} \cdots \bar{\chi}_{j_N}, \]
where we put $s_r = (-1)^{r N - r(r+1)/2}$. The fermion fields $\bar{\psi}_{\sigma'}$ and $\psi_\sigma$ are given by $\bar{\psi} = \bar{\chi} \bar{u}^\dagger$ and $\psi = u \chi$ with bases $\bar{u}_{\sigma' j}$ and $u_{\sigma i}$ which satisfy
\[ P_- = uu^\dagger, \quad u^\dagger u = 1_w, \quad \bar{P}_+ = \bar{u}\bar{u}^\dagger, \quad \bar{u}^\dagger \bar{u} = 1_\bar{w}. \] (2.5)

where $1_w$ and $1_\bar{w}$ are the identity operators in the spaces of the Weyl and anti-Weyl degrees of freedom, respectively. Specifying the fermion action as $\bar{\chi}M\chi = \bar{\psi}D\psi$ we then have for basic correlation functions of the fermion fields
\[ \langle \psi_{\sigma_{r+1}} \cdots \psi_{\sigma_N} \bar{\psi}_{\sigma_{r+1}} \cdots \bar{\psi}_{\sigma_N} \rangle_f = \frac{1}{r!} \sum_{\sigma_1 \cdots \sigma_r} \sum_{\bar{\sigma}_1 \cdots \bar{\sigma}_N} \bar{\Upsilon}_{\bar{\sigma}_1 \cdots \bar{\sigma}_N} Y_{\sigma_1 \cdots \sigma_N} D_{\sigma_1 \sigma_1} \cdots D_{\sigma_r \sigma_r} \] (2.6)

with the alternating multilinear forms
\[ Y_{\sigma_1 \cdots \sigma_N} = \sum_{i_1, \ldots, i_N = 1}^N \epsilon_{i_1 \ldots i_N} u_{\sigma_1 i_1} \cdots u_{\sigma_N i_N}, \]
\[ \bar{\Upsilon}_{\bar{\sigma}_1 \cdots \bar{\sigma}_N} = \sum_{j_1, \ldots, j_N = 1}^N \epsilon_{j_1 \ldots j_N} \bar{u}_{\bar{\sigma}_1 j_1} \cdots \bar{u}_{\bar{\sigma}_N j_N}. \] (2.7)

(2.8)
2.3 Subsets of bases

By (2.5) the bases are only fixed up to unitary transformations, \( u^{[S]} = uS, \bar{u}^{[\bar{S}]} = \bar{u}\bar{S} \). While the chiral projections are invariant under such transformations, the forms \( \Upsilon_{\sigma_1...\sigma_N} \) and \( \bar{\Upsilon}_{\bar{\sigma}_1...\bar{\sigma}_N} \) get multiplied by factors \( \det_w S \) and \( \det_{\bar{w}} \bar{S} \), respectively. Therefore in order that general correlation functions remain invariant, we have to impose

\[
\det_w S \cdot \det_{\bar{w}} \bar{S}^\dagger = 1. \tag{2.9}
\]

This is so since firstly in full correlation functions only a phase factor independent of the gauge field can be tolerated. Secondly this factor must be 1 in order that in general functions, which involve linear combinations of basic functions, individual basis transformations in its parts leave the interference terms in the moduli of the amplitudes invariant.

Condition (2.9) has important consequences. Without it all bases related to a chiral projection are connected by unitary transformations. With it the total set of pairs of bases \( u \) and \( \bar{u} \) decomposes into inequivalent subsets, beyond which legitimate transformations do not connect. These subsets of pairs of bases obviously are equivalence classes. Clearly the formulation of the theory must be restricted to one of such classes (which raises the question which choice is appropriate for describing physics).

Different ones of the indicated equivalence classes are obviously related by pairs of basis transformations \( S, \bar{S} \) for which

\[
\det_w S \cdot \det_{\bar{w}} \bar{S}^\dagger = e^{i\Theta} \quad \text{with} \quad \Theta \neq 0 \tag{2.10}
\]

holds. The phase factor \( e^{i\Theta} \) then determines how the results of the formulation of the theory with one class differ from the results of the formulation with the other class.

3 Gauge transformations

A gauge transformation \( D' = \mathcal{T} D \mathcal{T}^\dagger \) of the Dirac operator by (2.1) implies the corresponding transformations

\[
P'_- = \mathcal{T} P_- \mathcal{T}^\dagger, \quad \bar{P}'_+ = \mathcal{T} \bar{P}_+ \mathcal{T}^\dagger \tag{3.1}
\]

of the chiral projections. In view of (2.3) (and since on the lattice \( D^\dagger \neq -D \)) at least one of them should depend on the gauge field. Thus the cases are of interest where none or where only one of the chiral projections commutes with \( \mathcal{T} \).

3.1 Non-constant chiral projections

We first consider the case where \( [\mathcal{T}, P_-] \neq 0 \) and \( [\mathcal{T}, \bar{P}_+] \neq 0 \). To get the behavior of the bases we start from the fact that conditions (2.5) must be satisfied such that relations (3.1) hold. It is obvious that given a solution \( u \) of the conditions (2.5), then \( Tu \) is a
solution of the transformed conditions (2.5). Analogous considerations apply to \(\bar{u}\). All solutions are then obtained by performing basis transformations.

In addition (2.9) is to be satisfied, i.e. these considerations are to be restricted to an equivalence class of pairs of bases. Accordingly the original class \(uS, \bar{u}\bar{S}\) and the transformed one \(u'S, \bar{u}'\bar{S}'\) are related by

\[
u'S' = \mathcal{T}uS\mathcal{S}, \quad \bar{u}'\bar{S}' = \mathcal{T}\bar{u}\bar{S}\mathcal{S},
\] (3.2)

where \(u, \bar{u}, S, \bar{S}\) satisfy (2.5) and (2.9), respectively, and \(u', \bar{u}', S', \bar{S}'\) their transformed versions. For full generality of (3.2) we have included the unitary transformations \(S(\mathcal{T}, \mathcal{U})\) and \(\bar{S}(\mathcal{T}, \mathcal{U})\) obeying

\[
det_{w}S(1, \mathcal{U})(det_{w}\bar{S}(1, \mathcal{U}))^* = 1,
\] (3.3)

with \(S\) combining as \(S(\mathcal{T}_a, \mathcal{U})S(\mathcal{T}_b, \mathcal{T}_a\mathcal{U}\mathcal{T}_a^\dagger) = S(\mathcal{T}_b\mathcal{T}_a, \mathcal{T}_b\mathcal{T}_a\mathcal{U}\mathcal{T}_a^\dagger\mathcal{T}_b^\dagger)\) and \(\bar{S}\) analogously. Inserting (3.2) into (2.6) we get for the transformation of correlation functions

\[
\langle \psi_{\sigma_1'}^* \ldots \psi_{\sigma_R'}^* \tilde{\psi}_{\sigma_1} \ldots \tilde{\psi}_{\sigma_R} \rangle = e^{i\bar{\vartheta}} \sum_{\sigma_1, \ldots, \sigma_R} \sum_{\bar{\sigma}_1, \ldots, \bar{\sigma}_R} \mathcal{T}_{\sigma_1'} \ldots \mathcal{T}_{\sigma_R'} \langle \psi_{\sigma_1} \ldots \psi_{\sigma_R} \tilde{\psi}_{\bar{\sigma}_1} \ldots \tilde{\psi}_{\bar{\sigma}_R} \rangle \mathcal{T}_{\bar{\sigma}_1}^\dagger \ldots \mathcal{T}_{\bar{\sigma}_R}^\dagger,
\] (3.4)

where

\[
e^{i\bar{\vartheta}} = det_{w}S \cdot det_{w}\bar{S}\mathcal{S}
\] (3.5)

In (3.5) so far \(\bar{\vartheta} \neq 0\) for \(\mathcal{T} \neq 1\) is admitted with the ambiguity of the many possible choices of \(S\) and \(\bar{S}\). However, (3.3) with \(\bar{\vartheta} \neq 0\) obviously is just of form (2.10) related to transformations to arbitrary inequivalent subsets of pairs of bases, which ultimately cannot be tolerated. According to (2.9) then fixing to \(\bar{\vartheta} = 0\) is appropriate and the correlation functions turn out to transform gauge-covariantly.

### 3.2 One constant chiral projection

In the special case where \([\mathcal{T}, P_-] \neq 0\) and \([\mathcal{T}, \bar{P}_+] = 0\), the equivalence class of pairs of bases always contains members where \(\bar{P}_+\) is represented as \(\bar{P}_+ = \bar{u}_c\bar{u}_c^\dagger\) with constant \(\bar{u}_c\). Indeed, given a pair \(u, \bar{u}\) we note that \(\bar{u}\) is generally related to \(\bar{u}_c\) by a basis transformation \(\bar{u} = \bar{u}_c\bar{S}_c\). Thus transforming \(u\) as \(u = u_eS_e\), where \(S_e\) is subject to \(det_{w}S_e \cdot det_{w}\bar{S}_e^\dagger = 1\), according to (2.9) the pair \(u_e, \bar{u}_c\) is in the same equivalence class as the pair \(u, \bar{u}\). For a transformed pair \(u', \bar{u}'\) we analogously get the equivalent pair \(u'_e, \bar{u}_c\). Then instead of (3.2) we have

\[
u'_eS'_e = \mathcal{T}u_eS\mathcal{S}, \quad \bar{u}_c\bar{S}_c = \text{const},
\] (3.6)

where \(S\) and \(\bar{S}_c\) as well as \(S'\) and \(\bar{S}_c\) satisfy (2.9), so that

\[
det_{w}S'_e = det_{w}S,
\] (3.7)

For full generality in (3.6) the unitary transformation \(\bar{S}(\mathcal{T}, \mathcal{U})\) is included which obeys

\[
det_{w}\bar{S}(1, \mathcal{U}) = 1.
\] (3.8)
We next note that because of $[\mathcal{T}, \hat{P}_+]=0$ we can rewrite $\bar{u}_c$ as

$$\bar{u}_c = \mathcal{T} \bar{u}_c S_{\mathcal{T}}$$

(3.9)

where $S_{\mathcal{T}}$ is unitary. With this and (3.6) we get for the transformation of the correlation functions (2.6) the form

$$\langle \psi_{\sigma_1}' \ldots \psi_{\sigma_R}' \bar{\psi}_{\bar{\sigma}_1}' \ldots \bar{\psi}_{\bar{\sigma}_R}' \rangle_{\mathcal{T}} = e^{i\vartheta} \det_{\vartheta} S_{\mathcal{T}}^\dagger \sum_{\sigma_1, \ldots, \sigma_R, \bar{\sigma}_1, \ldots, \bar{\sigma}_R} \mathcal{T}_{\sigma_1}' \ldots \mathcal{T}_{\sigma_R}' \langle \psi_{\sigma_1} \ldots \psi_{\sigma_R} \bar{\psi}_{\bar{\sigma}_1} \ldots \bar{\psi}_{\bar{\sigma}_R} \rangle_{\mathcal{T}} \mathcal{T}_{\sigma_1}^\dagger \ldots \mathcal{T}_{\sigma_R}^\dagger,$$

(3.10)

where $e^{i\vartheta} = \det_{\vartheta} \bar{S}$. The ambiguity of the many possible choices of $\bar{S}$ here is fixed by noting that $\vartheta \neq 0$ is again related to transformations to arbitrary inequivalent subsets of pairs of bases, which are prevented by choosing $\vartheta = 0$.

For the calculation of the factor $\det_{\vartheta} S_{\mathcal{T}}^\dagger = \det_{\vartheta} (\bar{u}_c^\dagger \mathcal{T} \bar{u}_c)$ in (3.11) we note that with $[\mathcal{T}, \hat{P}_+]=0$ and $\mathcal{T} = e^B$ we get $\bar{u}_c^\dagger \mathcal{T} \bar{u}_c = \bar{u}_c^\dagger e^{B+} \bar{u}_c$ and the simultaneous eigen-equations $B \mathcal{T}_+ \bar{u}_c^\dagger = \omega_j \bar{u}_c^\dagger$ and $\mathcal{T}_+ \bar{u}_c^\dagger = \bar{u}_c^\dagger$. Since $\bar{u}_c^\dagger = \bar{u}_c \bar{S}$ with unitary $\bar{S}$, we obtain

$$\det_{\vartheta} (\bar{u}_c^\dagger e^{B+} \bar{u}_c) = \prod_j e^{\omega_j} = \exp(\text{Tr}(B P_+)),$$

so that using $P_+ = \frac{1}{2}(1 + \sigma_5)\mathbb{I}$ we have

$$\det_{\vartheta} S_{\mathcal{T}}^\dagger = \exp(\frac{1}{2} \text{Tr} B),$$

(3.11)

where in detail $\text{Tr} B = 4i \sum_{n, \ell} b_n^\ell \text{tr}_{\ell} T^\ell$ with constants $b_n^\ell$ and group generators $T^\ell$.

## 4 Variational approach

### 4.1 General relations

We define general gauge-field variations of a function $\phi(U)$ by

$$\delta \phi(U) = \frac{d \phi(U(t))}{d t} \bigg|_{t=0}, \quad U_\mu(t) = e^{iB^{\mu}_{\text{left}}} U_\mu e^{-iB^{\mu}_{\text{right}}},$$

(4.1)

where $(U_\mu)_{n'n} = U_{\mu n'} \delta^4_{n'n} \cdot \delta^{4}_{n'n}$ and $(B^{\mu}_{\text{left/right}})_{n'n} = B^{\mu}_{\text{left/right}} \delta^4_{n'n}$. The special case of gauge transformations is then described by

$$B^{\mu}_{\text{left}} = B^{\mu}_{\text{right}} = B.$$

(4.2)

Varying the logarithm of the general condition (2.9) gives

$$\text{Tr}_\vartheta (S^\dagger \delta S) - \text{Tr}_\vartheta (\bar{S}^\dagger \delta \bar{S}) = 0.$$

(4.3)

Instead of $\det_{\vartheta} S \cdot \det_{\vartheta} \bar{S}^\dagger = 1$, as needed in general functions involving linear combinations of basic functions, this obviously reflects the weaker condition $\det_{\vartheta} S \cdot \det_{\vartheta} \bar{S}^\dagger = \text{const}$. Relation (4.3) can also be expressed in terms of bases as

$$\text{Tr}(\delta(uS)(uS)^\dagger) - \text{Tr}(\delta(\bar{u}S)(\bar{u}S)^\dagger) = \text{Tr}(\delta u u^\dagger) - \text{Tr}(\delta \bar{u} \bar{u}^\dagger),$$

(4.4)
which indicates that $\text{Tr}(\delta u u^\dagger) - \text{Tr}(\delta \bar{u} \bar{u}^\dagger)$ remains invariant within the extended subset of bases specified by $\det w S \cdot \det \bar{w} \bar{S}^\dagger = \text{const.}$

Requiring absence of zero modes of $D$ (and thus also restricting to the vacuum sector) the effective action can be considered, for the variation of which one gets

$$\delta \ln \det \bar{w} w M = \text{Tr}(P_- D^{-1} \delta D) + \text{Tr}(\delta u u^\dagger) - \text{Tr}(\delta \bar{u} \bar{u}^\dagger).$$ (4.5)

### 4.2 Gauge transformations

In the special case of gauge transformations we can use the definition (4.1) and the finite transformation relations to get the related variations. For operators with $O(U(t)) = T(t) O(U(0)) T^\dagger(t)$ and $T(t) = e^{t B}$ this gives

$$\delta G O = [B, O].$$ (4.6)

In the case $[T, P_-] \neq 0$, $[T, \bar{P}_+] \neq 0$ according to (3.2) we have for the bases $u(t) = T(t) u(0) S_p(t)$, $\bar{u}(t) = T(t) \bar{u}(0) \bar{S}_p(t)$ where $S_p = SS^\dagger$, $\bar{S}_p = \bar{S} \bar{S}^\dagger$ and obtain

$$\delta^G u = B u + u S_p^\dagger \delta^G S_p, \quad \delta^G \bar{u} = B \bar{u} + \bar{u} \bar{S}_p^\dagger \delta^G \bar{S}_p.$$ (4.7)

Using these relations the terms in the variation of the effective action become

$$\text{Tr}(P_- D^{-1} \delta^G D) = \text{Tr}(B \bar{P}_+) - \text{Tr}(B P_-),$$ (4.8)

$$\text{Tr}(\delta^G u u^\dagger) = \text{Tr}(B \bar{P}_-) + \text{Tr}_w(S_p^\dagger \delta^G S_p), \quad \text{Tr}(\delta^G \bar{u} \bar{u}^\dagger) = \text{Tr}(B \bar{P}_+) + \text{Tr}_w(\bar{S}_p^\dagger \delta^G \bar{S}_p),$$ (4.9)

so that with (4.3) we get $\delta^G \ln \det \bar{w} w M = \text{Tr}_w(S^\dagger \delta^G S) - \text{Tr}_w(\bar{S}^\dagger \delta^G \bar{S})$. Because the exclusion of transformations to inequivalent subsets of pairs of bases leads to the relation $\text{Tr}_w(S^\dagger \delta^G S) - \text{Tr}_w(\bar{S}^\dagger \delta^G \bar{S}) = 0$, we thus obtain

$$\delta^G \ln \det \bar{w} w M = 0,$$ (4.10)

as expected according to the results for finite transformations.

In the case $[T, P_-] \neq 0$, $[T, \bar{P}_+] = 0$ according to (3.6) we get instead of (4.7)

$$\delta^G u_e = B u_e + u_e \bar{S}_p^\dagger \delta^G \bar{S}_p, \quad \delta^G \bar{u}_e = 0,$$ (4.11)

where $\bar{S}_p = S \bar{S} S^\dagger$, which with (3.7) and $P_+ = \frac{1}{2} (1 + \gamma_5) \mathbb{1}$ gives for the effective action $\delta^G \ln \det \bar{w} w M = \text{Tr}_w(\bar{S}^\dagger \delta^G \bar{S}) + \frac{1}{2} \text{Tr} B$. Excluding transformations to inequivalent subsets of pairs of bases here means that $\text{Tr}_w(S^\dagger \delta^G S) = 0$, so that we remain with

$$\delta^G \ln \det \bar{w} w M = \frac{1}{2} \text{Tr} B,$$ (4.12)

again as expected according to the results for finite transformations.
4.3 Special case of Lüscher

Lüscher [2] considers the variation of the effective action, imposing the Ginsparg-Wilson relation \( \{ \gamma_5, D \} = D \gamma_5 D \) and using chiral projections which correspond to the choice \( \tilde{G} = \mathbb{1} \) and \( G = \mathbb{1} - D \) in (2.2). He assumes \( \bar{u} = \text{const} \), so that the last term in (4.3) is absent and condition (4.3) reduces to \( \text{Tr}(\delta \bar{S}) = 0 \). He defines a current \( j_{\mu n} \) by

\[
\text{Tr}(\delta u u^\dagger) = -i \sum_{\mu, n} \text{tr}_g(\eta_{\mu n} j_{\mu n}), \quad \delta U_{\mu n} = \eta_{\mu n} U_{\mu n}, \quad (4.13)
\]

and requires it to transform gauge-covariantly.

His generator is given by \( \eta_{\mu n} = B_{\mu, n+\hat{n}} - U_{\mu n} B_{\mu n} \) in terms of our left and right generators. We get explicitly

\[
j_{\mu n} = i(U_{\mu n} \rho_{\mu n} + \rho_{\mu n}^\dagger U_{\mu n}^\dagger), \quad \rho_{\mu n, \alpha' \alpha} = \sum_{j, \alpha} u_{j \alpha}^\dagger \frac{\partial u_{j \alpha}}{\partial U_{\mu n, \alpha' \alpha}}. \quad (4.14)
\]

The requirement of gauge-covariance \( j_{\mu n}' = e^{B_{\mu n} \hat{n}} j_{\mu n} e^{-B_{\mu n} \hat{n}} \) because of \( U_{\mu n}' = e^{B_{\mu n} \hat{n}} U_{\mu n} e^{-B_{\mu n} \hat{n}} \) implies that one must have

\[
\rho_{\mu n}' = e^{B_{\mu n} \hat{n}} \rho_{\mu n} e^{-B_{\mu n} \hat{n}}, \quad (4.15)
\]

which with \( u' = T u S \bar{S} S^\dagger \) and (3.7) leads to the condition

\[
\sum_{j, k} \bar{S}_{kj} \frac{\partial \bar{S}_{jk}}{\partial U_{\mu n, \alpha' \alpha}} = 0. \quad (4.16)
\]

From (4.16) and \( \bar{S}^{-1} = \bar{S}^\dagger \) it follows that

\[
\text{Tr}_w(\delta \bar{S}^\dagger \delta \bar{S}) = 0. \quad (4.17)
\]

With this we arrive at the interesting result that the covariance requirement for Lüscher’s current leads just to what we have found before to follow from the exclusion of transformations to inequivalent subsets of pairs of bases.

We now have seen in different ways that in the special case considered by Lüscher one obtains the definite result

\[
\delta \ln \det_{\bar{q}w} M = \frac{1}{2} \text{Tr } B \quad (4.18)
\]

which leaves no room for adjustments by particular constructions. Furthermore, it has also become obvious that aiming at \( \delta \ln \det_{\bar{q}w} M = 0 \) in Ref. [2] does not to conform with the actual result (4.18).
5 Conclusions and discussions

We have given an unambiguous derivation of the gauge-transformation properties of correlation functions in chiral gauge theories on the finite lattice using finite transformations. In the case where both of the chiral projections are gauge-field dependent the exclusion of switching to arbitrary inequivalent subsets of pairs of bases leads to gauge covariance. In the cases where one of the chiral projections is constant a factor depending on the particular gauge transformation remains. A careful consideration of equivalence classes of pairs of bases has been important in our analysis. Our results have been seen to hold also in the presence of zero modes and for any value of the index.

We have also considered the subject in terms of variations of the effective action (which implies restriction to absence of zero modes and the vacuum sector). In this context we have shown that satisfying the covariance requirement for Lüscher’s current quite remarkably leads to the same result as the exclusion of switching to inequivalent subsets of pairs of bases. It has furthermore turned out that the behavior anticipated in Ref. [2] for the effective action in the special case there does not conform with the actual result.

Altogether it has become obvious that (whatever the detailed gauge-field dependences might be) the gauge-transformation properties of correlation functions in chiral gauge theories on the finite lattice are determined in a general way and cannot be adjusted by particular constructions, as has been tried to do in literature.

In Ref. [2] a main argument was that without the anomaly cancelation condition one would be unable to cancel the anomaly term. However, as has been seen in Section 4.2 the anomaly term \( \text{Tr}(P_+ D^{-1} \delta \varepsilon D) = \text{Tr}(B \tilde{P}_+) - \text{Tr}(BP_-) \) in the case where both chiral projections are gauge-field dependent is compensated by the basis contribution \( \text{Tr}(BP_-) - \text{Tr}(B \tilde{P}_+) \), while with \( \tilde{P}_+ \) being constant by the basis contribution \( \text{Tr}(BP_-) \) there is compensation up to the quantity \( \frac{1}{2} \text{Tr} B \).

There is no contradiction to continuum perturbation theory since in the limit one arrives just at the usual situation where the anomaly cancelation condition is needed to get gauge invariance of the chiral determinant. This is seen from the consideration of perturbation theory in Ref. [4], of which we here briefly mention some main features. In the limit the chiral projections become constant and their products with propagators get the appropriate forms. Using the notation \( D = D_0 + D_1 \), \( u = u_0 + u_1 \) and \( \tilde{u} = \tilde{u}_0 + \tilde{u}_1 \), where the quantities with indices 0 refer to the free case, the vertices decompose as

\[
P_{+0} D_1 P_{-0} + \tilde{u}_0 u_1^\dagger D u_1 u_0^\dagger + \tilde{u}_0 u_1^\dagger D P_{-0} + \tilde{P}_{+0} D u_1 u_0^\dagger.
\]  

(5.1)

Since the chiral projections get constant in the limit and accordingly then are described by constant bases, the terms in \( (5.1) \) which rely on \( u_1 \) and \( \tilde{u}_1 \) vanish in the limit. For the surviving contributions thus agreement with continuum perturbation theory becomes obvious at lower order. With appropriate locality properties of the Dirac operator this extends to higher orders.

With respect to the chiral determinant thus what happens is that in the limit the contributions are no longer there which when performing gauge transformations on the
finite lattice produce the compensating terms. Furthermore, in the limit obviously also
the particular cases with one constant chiral projection are no longer distinct from the
other ones. In this way in all cases one arrives at the usual continuum result.

Acknowledgement

I wish to thank Michael Müller-Preussker and his group for their kind hospitality.

References

[1] For an overview see: M. Golterman, Nucl. Phys. B (Proc. Suppl.) 94 (2001) 189.
[2] M. Lüscher, Nucl. Phys. B549 (1999) 295; Nucl. Phys. B568 (2000) 162.
[3] P.H. Ginsparg, K.G. Wilson, Phys. Rev. D25 (1982) 2649.
[4] W. Kerler, Nucl. Phys. B680 (2004) 51.