Bosonic Hofstadter butterflies in synthetic antiferromagnetic patterns

Yury S Krivosenko, Ivan V Iorsh and Ivan A Shelykh

1 Department of Physics and Engineering, ITMO University, St. Petersburg 197101, Russia
2 Science Institute, University of Iceland, Dunhagi 3, IS-107, Reykjavik, Iceland

E-mail: y.krivosenko@gmail.com

Received 5 November 2020, revised 4 December 2020
Accepted for publication 7 January 2021
Published 25 January 2021

Abstract

The emergence of Hofstadter butterflies for bosons in synthetic-gauge-field antiferromagnetic (AFM) patterns is theoretically studied. We report on a specific tight-binding model of artificial AFM structures incorporating both nearest and next-to-nearest neighbour tunnelings and allowing for the formation of the fractal spectra even with the vanishing gauge field flux through the lattice. The model is applied to square and honeycomb lattices. Possible experimental realization is suggested for the lattices of microring resonators connected by waveguides. Finally, the structure of the butterflies is analyzed for different points in the magnetic Brillouin zone for both the ferromagnetic and AFM patterns.

Keywords: artificial gauge fields, tight-binding approximation, antiferromagnetic patterns

(Some figures may appear in colour only in the online journal)

1. Introduction

The fractal electron spectrum originating in a two-dimensional gas of electrons on a lattice subject to homogeneous magnetic field was first described in [1] by Hofstadter. Due to the characteristic shape of the spectrum, the effect was later called Hofstadter butterfly (HB). Since then HBs have been revealed in a variety of systems, ranging from electrons in 2D lattices [2–5] to the systems of trapped cold atoms [6] and exciton polaritons [7]. The latter two systems consist of electrically neutral particles, and thus to show the fractal spectra instead of the real magnetic field require synthetic gauge fields [7, 8].

There is a plethora of the physical effects related to the fractal nature of the spectrum. Padavić et al [9] reported on occurrence of HB in topological phase diagram of Su–Schrieffer–Heeger ladder. Du et al [10] applied Floquet theory and thus examined the influence of monochromatic field on HB in kagome and triangular lattices. Duncan et al [11] researched topological modes in quasicrystals and observed HBs as well. The authors of [12] gave an account of HB in square lattices with a synthetic magnetic field modified by external pump. Hafezi et al designed the artificial gauge field in the square lattice of microring resonators connected by the waveguides [13, 14]. Also, HB-like spectra were achieved in the system of microring resonators arranged in circle [15]. Otaki and Fukui [16] examined generalized two-dimensional Su–Schrieffer–Heeger model as an example of high-order topological insulators and described the appearance of HB-type spectra. Jaksh and Zoller [17] suggested the Raman-laser-assisted tunneling as the tool for creating gauge fields and hence the butterfly for neutral atoms in optical lattices, and Aidelsburger et al [18] followed the scheme.

The general hallmark of the above-mentioned investigations consists in the presence of non-zero flux of either real magnetic or artificial gauge field (hereafter, we refer to the flux as to the magnetic flux) through the lattice unit cell. In the present paper, we perform an attempt to reach the HB-type spectra in synthetic antiferromagnetic (AFM) structures realized on square and honeycomb lattices with both nearest-neighbour (NN) and next-to-nearest-neighbour (NNN) interactions. In this case, HBs can still be observed even for the case of the vanishing net flux through elementary cell. In this regard, our investigation can be compared to that performed by Haldane [19], where the author researched the quantum Hall effect without an external magnetic field, and to those by Kane and Mele [20] and Bernevig and Zhang [21] with the studies...
of quantum spin Hall effect, where the electrons with opposite spins experience opposite effective magnetic fields. Here, the role of pseudospin is played by sublattices, see below. Our result suggests, that the HB phenomena can be observed in a wider class of the systems, i.e. with the zero magnetic flux, which has been deficiently examined so far.

The paper is organized as follows. In section 2, the generic theoretical model is introduced and a scheme of experimental implementation is proposed. Then in section 3, the main results are presented and analyzed. The main findings are summarized in section 4.

2. Theory

2.1. Basic concepts, real space

We start with the bare Hamiltonians of the honeycomb (\( \hat{H}_h \)) and square (\( \hat{H}_s \)) lattices with two sites per unit cell. Both nearest neighbour and next-to-nearest neighbour hoppings are included, the corresponding amplitudes are \( t_1 \) and \( t_2 \), respectively. Neither magnetic nor other gauge field is yet present, polarization degree of freedom is not taken into account. Firstly, consider the following auxiliary operator:

\[
\hat{H}_{aux} = t_1 \sum_{n_1,n_2} \left[ |n_1,n_2\rangle \langle n_1,n_2| \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] \\
+ |n_1+1,n_2\rangle \langle n_1+1,n_2| \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
+ |n_1,n_2+1\rangle \langle n_1,n_2+1| \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
+ t_2 \sum_{n_1,n_2} \left[ |n_1+1,n_2\rangle \langle n_1,n_2| \otimes \sigma_0 \right] \\
+ |n_1,n_2+1\rangle \langle n_1,n_2| \otimes \sigma_0 + \text{h.c.},
\]

(1)

The operator is written in the basis of Wannier states \(|n_1,n_2\rangle \otimes |\alpha\rangle\) localized at the lattice sites. The variables \( n_1 \) and \( n_2 \) signify the coordinates of a unit cell along the crystallographic directions \( a_1 \) and \( a_2 \), see figure 1 (\( n_1 = 0, 1, \ldots, (N_1 - 1) \), \( n_2 = 0, 1, \ldots, (N_2 - 1) \)), \(|\alpha\rangle\) designates the inner-cell state \((\alpha = A, B)\). Tensor product serves to separate the external (cell-position, \(|n_1,n_2\rangle\)) and internal (inner-cell, \(|\alpha\rangle\)) states. The 2 \( \times \) 2 matrices and \( \sigma_0 \) (the identity matrix) act in the inner-cell states subspace and are thus two-dimensional. For the periodic boundary conditions, the terms \( n_1,n_2+1 \) are taken modulo \( N_1,N_2 \).

Then, \( \hat{H}_h = \hat{H}_{aux} + t_2 \sum_{n_1,n_2} \left[ |n_1,n_2+1\rangle \langle n_1+1,n_2| \otimes \sigma_0 + \text{h.c.} \right] \)

(2a)

and

\( \hat{H}_s = \hat{H}_{aux} + t_1 \sum_{n_1,n_2} \left[ |n_1+1,n_2+1\rangle \langle n_1,n_2| \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \text{h.c.} \right] \)

(2b)

Despite the fact that square lattice Hamiltonian can be constructed even with a single site per unit cell, we introduce here the two-sites unit cell for the sake of uniformity.

In this paper, we propose to design the FM and AFM patterns in compliance with the following scheme. First, within the tight-binding approximation, the influence of the magnetic field on the behaviour of spinless charged particles is manifested in the occurrence of the phase factor \( \exp(-i\theta(n_1,n_2)) \) at the hopping amplitudes, the so-called Peierls substitution. For hopping from site \( n_1 \) to \( n_2 \), the phase \( \theta(n_1,n_2) \) equals the linear integral [6]

\[
\theta(n_1,n_2) = \frac{2\pi}{\phi_0} \int_{n_1}^{n_2} \mathbf{A} \cdot d\mathbf{l} = 2\pi \cdot \frac{\Phi}{\phi_0} - \frac{\int_{n_1}^{n_2} \mathbf{A} \cdot d\mathbf{l}}{\phi_0},
\]

(3)

where \( \nabla \times \mathbf{A} = \mathbf{B} \) is the effective magnetic field corresponding to the artificial gauge field, \( \Phi \) is its flux through the unit cell, and \( \phi_0 \) is the quantum of magnetic field flux. We then utilize the fact that the honeycomb (square) lattice can be presented as two triangular (square) sublattices, \( A \) and \( B \), shifted with respect to each other (the red and blue sublattices in figure 1). Naturally, NNN hoppings leave the particle within the same sublattice, whereas NN hoppings correspond to the inter-sublattice process. Our main concept in constructing and distinguishing the FM and AFM phases consists in the assumption that the sublattices can be subject to the different gauge fields: the magnetic field applied to sublattice \( A \) is homogeneous, directed along \( z \)-axis and equals \( B_0 \), whereas it is \( \pm B_0 \) for the sublattice \( B \) in the FM (+) and AFM (−) phases. Moreover, the NN hopping matrix elements are assumed to be purely real and positive. The proposal for realization and justification of such a gauge field is presented in subsection 2.3.

Using Landau gauge, i.e. \( \mathbf{A} = (0, xB, 0) \), one can derive the linear integral in (3) and hence the phases \( \theta(n_1,n_2) \) [2–4]. The former is

\[
\int_{n_1}^{n_2} \mathbf{A} \cdot d\mathbf{l} = B \sin \varphi \left[ x(n_1,n_2,\tau) \lambda' + \cos \frac{\lambda'^2}{2} \right],
\]

(4)

The integral is evaluated for hopping from the site specified by the coordinates \( n_1 \) and \( n_2 \), and \( \tau \) to its next-to-nearest neighbour.
The sublattices $A$ and $B$ are marked by $\tau = 0$ and 1, respectively, $x(n_1, n_2, \tau)$ is the $x$-coordinate of the initial site, $\varphi$ is the angle between the hopping direction and $x$-axis, $\lambda'$ is hopping distance. For the honeycomb lattice $\lambda' = \lambda_0 \sqrt{3}$, and for the square one $\lambda_0$, see figure 1. The magnetic field $B$ equals $B_0$ for the both sublattices in the FM phase, and $\pm B_0$ for sublattice $A(\tau)$ and $B(\tau)$ in the AFM phase.

We arrange the coordinate axes so as the $y$-axis direction coincides with that of translation unit vector $a_2$. Hence, we obtain the initial positions as

$$x_0(n_1, n_2, \tau) = \frac{\lambda_0 (3n_1 + \tau)}{2},$$

$$x_4(n_1, n_2, \tau) = \frac{\lambda_0 (n_1 + \tau)}{2},$$

for the honeycomb and square lattices, respectively.

As soon as the NN hopping amplitudes are supposed not to vary, we split each of the Hamiltonians (2) into two parts responsible for NN and NNN hoppings and come to the Hamiltonians modified by the presence of the gauge field, $\hat{H}_{4(6)}$:

$$\hat{H}_{4(6)} = \hat{H}_{6(8)} + t_2 \sum_{n_1,n_2} \left| n_1 + 1, n_2 \right\rangle \left\langle n_1, n_2 \mid \hat{C}_n^\pm \right. + |n_1, n_2 + 1\rangle \langle n_1, n_2 | \hat{D}_n^\pm + \left. |n_1, n_2 + 1\rangle \langle n_1, n_2 | \hat{E}_n^\pm \right. + h.c. \right\rangle,$$ (6a)

and

$$\hat{H}_{4(6)} = \hat{H}_{4(6)} + t_2 \sum_{n_1,n_2} \left| n_1 + 1, n_2 \right\rangle \left\langle n_1, n_2 \mid \hat{F}_n^\pm + \left. |n_1, n_2 + 1\rangle \langle n_1, n_2 | \hat{G}_n^\pm \right. + h.c. \right\rangle,$$ (6b)

where $\hat{C}$, $\hat{D}$, and $\hat{F}$ are the $2 \times 2$ diagonal operators dependent on the magnetic field and pattern:

$$\hat{C}_n^\pm = \begin{pmatrix} e^{-i\pi(n+1/2)} & 0 \\ 0 & e^{i\pi(n+5/6)} \end{pmatrix},$$

$$\hat{D}_n^\pm = \begin{pmatrix} e^{-i\pi n} & 0 \\ 0 & e^{2i\pi(n+1/3)} \end{pmatrix},$$

$$\hat{F}_n^\pm = \begin{pmatrix} e^{-i\pi n} & 0 \\ 0 & e^{2i\pi(n+1/2)} \end{pmatrix}.$$ (7a - 7c)

The (+) superscript index denotes the FM (AFM) pattern revealed in the sign of the phase of the second diagonal terms, $\xi$ is the ratio $\Phi/\Phi_0$. Below, the $\pm$ superscripts are placed at the Hamiltonians and the $2 \times 2$ operators, if specifying the magnetic phase is crucial, and are omitted otherwise.

2.2. Reciprocal space and Harper Hamiltonians

As soon as the gauge phases in equations (6) and (7) do not depend on $n_2$, the Fourier transform of the Hamiltonians along this direction can be straightforwardly performed. At this stage, the localized state $|n_1, n_2\rangle$ is presented as

$$|n_1, n_2\rangle = \frac{1}{2\pi} \sum_{k_2} |k_2\rangle \otimes |n_1\rangle e^{-i\pi k_2},$$ (8)

where $k_2$ designates the corresponding wave number: it takes on the values of $2\pi m_2/N_2$ with $m_2$ listing the integers within the $[-N_2/2, N_2/2)$ range. Thus, collecting equations (1), (2), (6), and (8), we arrive at

$$\hat{H}_{6d} = \sum_{k_2} |k_2\rangle \langle k_2 | \otimes \hat{H}_{6d2}$$

with

$$\hat{H}_{6d2} = \sum_{n_1} \left\{ t_1 \left| n_1\right\rangle \langle n_1 | \otimes \begin{pmatrix} 0 & e^{-i\pi k_2} \\ 1 & 0 \end{pmatrix} \right. + |n_1 + 1\rangle \langle n_1 | \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} + t_2 \left[ |n_1 + 1\rangle \langle n_1 | \otimes \hat{C}_n^+ + e^{-i\pi k_2} |n_1\rangle \langle n_1 | \otimes \hat{D}_n - e^{-i\pi k_2} |n_1\rangle \langle n_1 | \otimes \hat{F}_n \right\} + h.c.,$$ (9a)

and

$$\hat{H}_{4d} = \sum_{k_2} |k_2\rangle \langle k_2 | \otimes \hat{H}_{4d2}$$

with

$$\hat{H}_{4d2} = \sum_{n_1} \left\{ t_1 \left| n_1\right\rangle \langle n_1 | \otimes \begin{pmatrix} 0 & e^{-i\pi k_2} \\ 1 & 0 \end{pmatrix} \right. + |n_1 + 1\rangle \langle n_1 | \otimes \begin{pmatrix} 0 & 1 + e^{-i\pi k_2} \\ 0 & 0 \end{pmatrix} \right\} + t_2 \left[ |n_1 + 1\rangle \langle n_1 | \otimes \hat{C}_n^+ + e^{-i\pi k_2} |n_1\rangle \langle n_1 | \otimes \hat{F}_n \right\} + h.c.$$ (9b)

Henceforth, we constrain ourselves to rational magnetic fluxes: $\xi = \Phi/\Phi_0 = p/q \in \mathbb{Q} \cap (0, 1)$, where $p$ and $q$ are coprime integers. Within the assumption, all the $2 \times 2$ operators in sums (9) (the right-side terminal operators in each summand) become translationally invariant:

$$\hat{G}_{n+q} = \hat{G}_n,$$ (10)

whereafter $\hat{G}$ stands for $\hat{C}$, $\hat{D}$, or $\hat{F}$, and the translation period $Q$ is $2q$ and $q$ for the honeycomb and square lattices, respectively. The Hamiltonians (9a) and (9b) thus acquire translational symmetry with respect to the shift by $N_1Q$ along $a_1$. Then, we extend the system along $a_1$ so as it got $N_1Q$ unit cells in this dimension. Hereupon, $n_1$ changes its variation range to $0, 1, \ldots, (QN_1 - 1)$ and can be rewritten as $n_1 = n'_1Q + s_1$ where $n'_1 = 0, 1, \ldots, (N_1 - 1)$ and $s_1 = 0, 1, \ldots, (Q - 1)$. This eventually leads to $\hat{G}_{n_1} = \hat{G}_{n_1}$. The system begins to effectively possess of $N_1$ supercells each of which contains $Q$ original unit cells.
The intermediate Hamiltonians expressed in terms of $n'_1$ and $k_2$, as well as the final explicit forms of the bulk Hamiltonians defined as

$$\hat{H}_{6(4)} = \sum_{k'_1, k_2} |k'_1, k_2\rangle \langle k'_1, k_2| \otimes \hat{H}_{6(4)\text{bulk}}$$  \hspace{1cm} (11)

are omitted here due to their cumbersomeness and are presented in appendix. What is to be mentioned about them, is that, initially, both bulk Hamiltonians can be represented as the sum

$$\hat{H}_{\text{bulk}} = \sum_m \alpha_m \hat{U}_m \otimes \hat{J}_m,$$  \hspace{1cm} (12)

where each $\alpha_m$ is a constant proportional to either $t_1$ or $t_2$, $\hat{U}_m$ and $\hat{J}_m$ are $Q \times Q$ and $2 \times 2$ matrices, respectively. Operators $\hat{U}_m$ and $\hat{J}_m$ generally inherit the information on the intra-supercell hoppings and the internal degree of freedom of the initial unit cell, respectively. The inter-supercell tunneling is conventionally incorporated in $e^{\pm ik'_1}$ and $e^{\pm ik_2}$ factors.

The Hamiltonians at the first magnetic Brillouin zone (FMBZ) $\Gamma$-points ($k'_1 = k_2 = 0$) pairwise coincide for FM and AFM patterns:

$$\hat{H}^+_{\text{bulk}}(\Gamma) = \hat{H}^-_{\text{bulk}}(\Gamma),$$  \hspace{1cm} (13a)

$$\hat{H}^-_{\text{bulk}}(\Gamma) = \hat{H}^+_{\text{bulk}}(\Gamma),$$  \hspace{1cm} (13b)

which follows from the equality $\hat{G}^+ + (\hat{G}^+)^\dagger = \hat{G}^- + (\hat{G}^-)^\dagger$ and the diagonal (complementary) manner of their inclusion into the Hamiltonians for the square (honeycomb) structures.

Considering the FM and AFM square lattice bulk Hamiltonians as functions of quasi momentum $\mathbf{k} = (k'_1, k_2)$, one obtains that they produce the same Hofstadter butterflies in the following cases:

$$\hat{H}^+_{\text{bulk}}(k'_1, k_2) \sim \hat{H}^-_{\text{bulk}}(\pm k'_1, \pm k_2),$$  \hspace{1cm} (14a)

$$\hat{H}^-_{\text{bulk}}(k'_1, k_2) \sim \hat{H}^+_{\text{bulk}}(\pm k'_1, \pm k_2)$$  \hspace{1cm} (14b)

for all the combinations of $+$ and $-$ signs. Contrarily, the honeycomb lattice Hamiltonians generally differ.

### 2.3. Experimental realization

The realization we propose, see figure 2, is based on the papers by Hafezi et al [13, 14]. Each site here is a microring resonator. The resonators (red and blue rounds) are connected by the waveguides (red, blue, and grey solid elliptic lines). The phase acquirement can be achieved here, e.g., by tuning the relative lengths of the waveguides connecting the same pair of resonators, see references [13, 14]. On the other hand, no additional phase gain for the NN hoppings can be guaranteed by the equal lengths of grey waveguides. The microring resonators generally host photons of two polarizations, clockwise and counter-clockwise, thus introducing the pseudospin into the system. In the absence of specific scatterers, the pseudospin components can be considered as uncoupled thus separating the system into two independent subsystems. Here, we choose the counter-clockwise polarization (explicitly shown in the figure) and demonstrate only the square AFM pattern.

To arrange the waveguides, we utilize three nominal layers: two layers to internally connect the sites within sublattice $A$ and sublattice $B$ (the red and blue waveguides), and another one to link the nearest neighbours (the grey ones). The lengths of $A(B)$-waveguides are designed to result in $-(+\pi\xi)$ flux through the square plaquette presented by the green area. The effective phase shift equals 0 for the grey waveguides (nearest neighbour hoppings).

### 3. Results and discussion

Figures 3(A) and (B) demonstrate the eigenenergies of the bulk Hamiltonians, (A.3a) and (A.3b), respectively, computed at the $\Gamma$-points, $k'_1 = k_2 = 0$. The spectra are evaluated for the conventional values of the magnetic flux, $\xi \in (0, 1)$, namely $\xi = p/q$ with $q = 197$ and $p = 1, 2, \ldots , (q - 1)$. The pair of NN and NNN transitions amplitudes, $(t_1, t_2)$, gradually changes from (0, 1) to (1, 0). The eigenenergy is measured in the units same to those of $t_1$ and $t_2$.

Consider the extreme situations when one of the amplitudes equals zero. The honeycomb lattice with $t_1 = 0$ and $t_2 = 1$ is equivalent to the pair of independent ferromagnetic (FM) triangular lattices (cf [4]). Similarly, the square lattice can be treated as a pair of independent FM square ones (cf [1]). In these cases, the eigenstates are at least doubly degenerate. On the other hand, the case when $t_1 = 1$ and $t_2 = 0$ corresponds to the absence of magnetic field effect and no dependence of the eigenenergies on $\xi$, as it can be clearly seen from the right
Figure 3. Hofstadter-type spectra for the (A) honeycomb and (B) square lattices, at the FMBZ $\Gamma$-points. The values of amplitudes $t_1$ and $t_2$ are indicated in the insets. The green solid lines demonstrate the eigenenergies of the corresponding $\hat{H}_{\text{nnn}}$. The blue solid lines represent the eigenenergies calculated within the stationary perturbation theory (see the text for details). (C) The butterflies calculated for (i)–(iv) honeycomb and (v) and (vi) square lattices for different magnetic patterns. The tunneling amplitudes are $t_1 = t_2 = 0.5$, the magnetic phases and points of the FMBZ are specified in the insets. (A)–(C) Everywhere, the horizontal axis is responsible for the magnetic flux, $\xi = p/q$ with $q = 197$ and $q = 1, 2, \ldots, q - 1$, the vertical axis is the eigenenergy. The units of measurement are arbitrary but the same for $t_1$, $t_2$, and $E$.

panels of figures 3(A) and (B). The lattices are then equivalent to honeycomb and square ones with the NN hopping only.

In addition, second from the left panels in figures 3(A) and (B) contain green and blue solid lines representing the highest energetic states of NNN-hopping parts of the bulk Hamiltonians (A.3) with $t_2 = 3/4$ (green lines) and their splitting by the influence of corresponding NN-hopping parts (with $t_1 = 1/4$) treated within the first order of stationary perturbation theory (the sets of blue lines). As can be seen, such approach describes the effect for small $t_1$ and weak gauge fields sufficiently well.

To illustrate the difference between the AFM and FM phases, the Hofstadter-type spectra were calculated for other points in the FMBZ. Figure 3(C) shows the butterflies for the (i)–(iv) honeycomb and (v) and (vi) square structures computed for $t_1 = t_2 = 1/2$. K and K’ points of the hexagonal FMBZ are located at $(2\pi/3, -2\pi/3)$ and vice versa. M point of the square FMBZ is positioned at $(\pi/2, \pi/2)$. One can easily see that AFM and FM structures generally result in the different butterfly spectra. The interesting aspect of figures 3(C)(i) and (C)(iii) is that the spectra are very similar but still are not completely equal (the absence of the exact match can be detected by the direct comparison of the raw output data of the calculations and is poorly visible in the butterfly charts).

Finally, figures 3(C)(v) and (C)(vi) exhibit the HBs for FM and AFM square lattices at $M$-point, which was chosen instead of the conventional $M$-point ($k_1' = k_2 = \pi$) due to the following peculiarity of the square lattice bulk Hamiltonian (A.3b): $k_1$ equal to $\pm\pi$ eliminates the NN hopping part of the
Hamiltonian (A.3b) as the matrix \((0, 1 + e^{-ik_2}), (0, 0)\) can be, after a certain algebra, factorized out in the tensor product proportional to \(t_1\).

Our results can be compared to those of Otaki and Fukui [16], where the authors consider gradual variation of Hofstadter butterflies arising in the 2D generalization of Su–Schrieffer–Heeger model. In contrast with figure 3, their HB-spectra retain reflection symmetry with respect to \(E = 0\). Another point of connection is the symmetry enclosed in equation (14), the analogy of which is reported in the work [16]. As well, the results resembling those presented in our paper can be found in the work by Hasegawa and Kohmoto [22], where Hofstadter butterflies distortions are examined in twisted bilayer graphene, where such features as splitting of the highest energetic states highlighted by the blue and green lines in figure 3 were reported.

4. Conclusion

We have developed the theory describing both the FM and AFM patterns with nearest and next-to-nearest neighbours hoppings in the honeycomb and square lattices. The gauge field has been assumed to alter only the next-to-nearest neighbour tunnelings, the additional phase has not been acquired during transitions to nearest neighbours. The major finding of the study was that a HB can arise in the AFM structures with zero total gauge field flux through the lattice.

We have shown that the AFM and FM Hamiltonians and butterflies coincide in the \(\Gamma\)-points and confirmed the differences for other points in the first magnetic Brillouin zone. Accidental similarities between the butterflies have been also disclosed. For several cases, first order perturbation theory has been applied and demonstrated good agreement with the exact calculations for small magnetic fields.

In the end, this study enriches the class of the systems where the HB may be observed, by the materials with the AFM order.

Acknowledgments

The authors acknowledge support from Russian Science Foundation (project No. 18-72-10110). The work of IAS was also supported by Icelandic Science Foundation (Project “Hybrid Polaritons’). YSK is grateful to Dr A Nalitov for the useful discussion.

Appendix. Intermediate and final Hamiltonians

In the real space along \(a_1\), the Hamiltonians expressed in terms of \(k_2\) and \(n_1\) are

\[
\hat{H}_{a_2} = \sum_{\sigma_1} \frac{1}{t_1} \left[ |n'| \langle n'| \otimes \left( \hat{I}_0 \otimes \begin{pmatrix} 0 & e^{-ik_2} \\ 1 & 0 \end{pmatrix} \right) \right] + \frac{1}{t_2} \left[ |n' \rangle \langle n'| \otimes \left( \hat{T}_0 \otimes \begin{pmatrix} 0 & 1 + e^{-ik_2} \\ 0 & 0 \end{pmatrix} \right) \right] + \frac{1}{t_2} \left[ |n' \rangle \langle n'| \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right]
\]

Here, all the empty positions in the matrices indicate zeros. \(\hat{O}\) is the \(2 \times 2\) zero matrix, the indices \(Q\) and \(2Q\) explicitly show the dimensions of the corresponding matrix. \(\hat{I}_0\) is the \(Q\)-dimensional identity matrix, \(\hat{T}_0\) and \(\hat{T}_s\) are auxiliary \(S\)-dimensional matrices:

\[
\hat{T}_0 = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \quad \hat{T}_s = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}
\]
with $S$ being equal to either $Q$ or $2Q$. The superscript $T$ implies matrix transposition. Performing the Fourier transform over $n'_1$ coordinate responsible for the supercell position, we finally get the explicit expressions for the Hamiltonians in the reciprocal space:

$$
\hat{H}_{6\text{bulk}} = t_1 \left[ \hat{T}_Q \otimes \begin{pmatrix} 0 & e^{-ik_2} \\ 1 & 0 \end{pmatrix} + \hat{T}_0 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + e^{-ik_2} \hat{T}_Q \right]
$$

$$
+ t_2 \left[ \begin{pmatrix} \hat{\mathcal{C}}_0 \\ \hat{\mathcal{C}}_1 \\ \vdots \\ \hat{\mathcal{C}}_{Q-2} \end{pmatrix} \right] \begin{pmatrix} \hat{D}_0 \\ \hat{D}_1 \\ \vdots \\ \hat{D}_{Q-1} \end{pmatrix}_{2Q}
$$

$$
+ e^{-ik_2} \hat{T}_Q \otimes \hat{\mathcal{C}}_{Q-1} + e^{ik_1-ik_2} \hat{T}_Q \otimes \hat{\mathcal{C}}_{Q-1} + \text{h.c.} \right]
$$

(A.3a)

and

$$
\hat{H}_{4\text{bulk}} = t_1 \left[ \hat{T}_Q \otimes \begin{pmatrix} 0 & e^{-ik_2} \\ 1 & 0 \end{pmatrix} + \hat{T}_0 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + e^{-ik_2} \hat{T}_Q \right]
$$

$$
+ e^{ik_2} \hat{T}_0 \otimes \begin{pmatrix} 0 & 1 + e^{-ik_2} \\ 0 & 0 \end{pmatrix} + e^{-ik_1} \hat{T}_{Q-1} \otimes \hat{\mathcal{F}}_{Q-1}
$$

$$
+ t_2 \left[ \begin{pmatrix} \hat{\mathcal{F}}_0 \\ \hat{\mathcal{F}}_1 \\ \vdots \\ \hat{\mathcal{F}}_{Q-1} \end{pmatrix} \right] \begin{pmatrix} \hat{\mathcal{F}}_0 \\ \hat{\mathcal{F}}_1 \\ \vdots \\ \hat{\mathcal{F}}_{Q-1} \end{pmatrix}_{2Q}
$$

+ e^{-ik_1} \hat{T}_{Q-1} \otimes \hat{\mathcal{F}}_{Q-1} + \text{h.c.} \right]
$$

(A.3b)

References

[1] Hofstadter D R 1976 Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields Phys. Rev. B 14 2239–49
[2] Godfrey G and Fekete P 1997 Hofstadter butterfly for the hexagonal lattice Phys. Rev. B 56 3787–91
[3] Oh G-Y 2000 Energy spectrum of a triangular lattice in a uniform magnetic field: effect of next-nearest-neighbor hopping J. Korean Phys. Soc. 37 534–9
[4] Oh G-Y Feb 2001 Comment on ‘Hofstadter butterfly for the hexagonal lattice’ Phys. Rev. B 63 087301
[5] Li J, Wang Y-F and Gong C-D 2011 Tight-binding electrons on triangular and kagomé lattices under staggered modulated magnetic fields: quantum Hall effects and Hofstadter butterflies J. Phys.: Condens. Matter. 23 156002
[6] Yilmaz F, Unal F N and Oktel M Ö 2015 Evolution of the Hofstadter butterfly in a tunable optical lattice Phys. Rev. A 91 063628
[7] Banerjee R, Liew T C H and Kyriienko O 2018 Realization of Hofstadter’s butterfly and a one-way edge mode in a polaritonic system Phys. Rev. B 98 075412
[8] Celi A, Massignan P, Ruseckas J, Goldman N, Spielman I B, Juzeliūnas G and Lewenstein M 2014 Synthetic gauge fields in synthetic dimensions Phys. Rev. Lett. 112 043001
[9] Padavić K, Hegde S S, DeGottardi W and Vishveshwara S 2018 Topological phases, edge modes, and the Hofstadter butterfly in coupled Su–Schrieffer–Heeger systems Phys. Rev. B 98 024205
[10] Du L, Chen Q, Barr A D, Barr A R and Fiete G A 2018 Floquet Hofstadter butterfly on the kagome and triangular lattices Phys. Rev. B 98 245145
[11] Duncan C W, Manna S and Nielsen A E B 2020 Topological models in rotationally symmetric quasicrystals Phys. Rev. B 101 115413
[12] Colella E, Mivehvar F, Piazza F and Ritsch H 2019 Hofstadter butterfly in a cavity-induced dynamic synthetic magnetic field Phys. Rev. B 100 224306
[13] Hafezi M, Demler E A, Lukin M D and Taylor J M 2011 Robust optical delay lines with topological protection Nat. Phys. 7 907–12
[14] Hafezi M, Mittal S, Fan J, Migdall A and Taylor J M 2013 Imaging topological edge states in silicon photonics Nat. Photon. 7 1001–5
[15] Zimmerling T J and Van V 2020 Generation of Hofstadter’s butterfly spectrum using circular arrays of microring resonators Opt. Lett. 45 714–7
[16] Onuki Y and Fukui T 2019 Higher-order topological insulators in a magnetic field Phys. Rev. B 100 245108
[17] Jaksh D and Zoller P 2003 Creation of effective magnetic fields in optical lattices: the Hofstadter butterfly for cold neutral atoms New J. Phys. 5 56
[18] Aidelsburger M, Atala M, Lohse M, Barreiro J T, Paredes B and Bloch I 2013 Realization of the Hofstadter Hamiltonian with ultracold atoms in optical lattices Phys. Rev. Lett. 111 185301
[19] Haldane F D M 1988 Model for a quantum Hall effect without Landau levels: condensed-matter realization of the ‘parity anomaly’ Phys. Rev. Lett. 61 2015–8
[20] Kane C L and Mele E J 2005 Quantum spin Hall effect in graphene Phys. Rev. Lett. 95 226801
[21] Bernevig B A and Zhang S-C 2006 Quantum spin Hall effect Phys. Rev. Lett. 96 106802
[22] Hasegawa Y and Kohmoto M 2013 Periodic Landau gauge and quantum Hall effect in twisted bilayer graphene Phys. Rev. B 88 125426

ORCID iDs

Yury S Krivosenko https://orcid.org/0000-0002-5562-4334