Improved Bounds Concerning the Maximum Degree of Intersecting Hypergraphs

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Abstract

For positive integers \( n > k > t \) let \( \binom{n}{k} \) denote the collection of all \( k \)-subsets of the standard \( n \)-element set \( \{1, \ldots, n\} \). Subsets of \( \binom{n}{k} \) are called \( k \)-graphs. A \( k \)-graph \( F \) is called \( t \)-intersecting if \( |F \cap F'| \geq t \) for all \( F, F' \in F \). One of the central results of extremal set theory is the Erdős-Ko-Rado Theorem which states that for \( n \geq (k - t + 1)(t + 1) \) no \( t \)-intersecting \( k \)-graph has more than \( \binom{n-t}{k-t} \) edges. For \( n \) greater than this threshold the \( t \)-star (all \( k \)-sets containing a fixed \( t \)-set) is the only family attaining this bound. Define \( F(i) = \{F \setminus \{i\} \mid i \in F \in F\} \). The quantity \( \varrho(F) = \max_{1 \leq i \leq n} |F(i)|/|F| \) measures how close a \( k \)-graph is to a star. The main result (Theorem 1.3) shows that \( \varrho(F) > 1/d \) holds if \( F \) is 1-intersecting, \( |F| > 2^d d^{2d+1} \binom{n-d-1}{k-d-1} \) and \( n \geq 4(d-1)dk \). Such a statement can be deduced from earlier results, however only for much larger values of \( n/k \) and/or \( n \). The proof is purely combinatorial, it is based on a new method: shifting ad extremis. The same method is applied to obtain a nearly optimal bound in the case of \( t \geq 2 \) (Theorem 1.4).

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1 Introduction

For positive integers \( n \geq k \), let \( [n] = \{1, \ldots, n\} \) be the standard \( n \)-element set and \( \binom{[n]}{k} \) the collection of its \( k \)-subsets. A family \( F \subset \binom{[n]}{k} \) is called \( t \)-intersecting if \( |F \cap F'| \geq t \) for all \( F, F' \in F \) and \( t \) a positive integer. In the case \( t = 1 \) we usually omit \( t \) and speak of intersecting families. Let us recall one of the fundamental results of extremal set theory.

Theorem 1 (Exact Erdős-Ko-Rado Theorem ([2], [4], [18])). Let \( k \geq t > 0 \), \( n \geq \)

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\(n_0(k, t) = (k - t + 1)(t + 1)\). Suppose that \(\mathcal{F} \subset \binom{[n]}{k}\) is \(t\)-intersecting. Then
\[
|\mathcal{F}| \leq \binom{n - t}{k - t}.
\] (1)

Let us note that \(|\mathcal{S}(n, k, t)| = \binom{n - t}{k - t}\) holds for the full star
\[
\mathcal{S}(n, k, t) = \left\{ S \in \binom{[n]}{k} : [t] \subset S \right\}
\]
and for \(n > n_0(k, t)\) up to isomorphism \(\mathcal{S}(n, k, t)\) is the only family to achieve equality in (1). The exact bound \(n_0(k, t) = (k - t + 1)(t + 1)\) is due to Erdős, Ko and Rado in the case \(t = 1\). For \(t \geq 15\) it was established in [4]. Wilson [18] closed the gap \(2 \leq t \leq 14\) by a proof valid for all \(t \geq 1\).

Let us recall some standard notation. Set \(\cap \mathcal{F} = \cap \{F : F \in \mathcal{F}\}\). If \(|\cap \mathcal{F}| \geq t\) then \(\mathcal{F}\) is called a \(t\)-star, for \(t = 1\) we usually omit the 1. If \(\cap \mathcal{F} = \emptyset\) then we call \(\mathcal{F}\) a non-trivial family.

For a subset \(E \subset [n]\) and a family \(\mathcal{F} \subset \binom{[n]}{k}\), define
\[
\mathcal{F}(E) = \{ F \setminus E : E \subset F \in \mathcal{F} \}, \quad \mathcal{F}(\overline{E}) = \{ F \in \mathcal{F} : F \cap E = \emptyset \}.
\]
In the case \(E = \{i\}\) we simply use \(\mathcal{F}(i)\) and \(\mathcal{F}(\overline{i})\) to denote \(\mathcal{F}(\{i\})\) and \(\mathcal{F}(\{\overline{i}\})\), respectively. In analogy,
\[
\mathcal{F}(u, v, \overline{w}) := \{ F \setminus \{u, v\} : F \in \mathcal{F}, F \cap \{u, v, w\} = \{u, v\} \}.
\]

Let us define the quantity
\[
\varrho(\mathcal{F}) = \max \left\{ \frac{|\mathcal{F}(i)|}{|\mathcal{F}|} : 1 \leq i \leq n \right\}.
\]
Since \(\varrho(\mathcal{F}) = 1\) if and only if \(\mathcal{F}\) is a star, in a way \(\varrho(\mathcal{F})\) measures how far a family is from a star.

A set \(T\) is called a \(t\)-transversal of \(\mathcal{F}\) if \(|T \cap F| \geq t\) for all \(F \in \mathcal{F}\). If \(\mathcal{F}\) is \(t\)-intersecting then each \(F \in \mathcal{F}\) is a \(t\)-transversal. Define
\[
\tau_t(\mathcal{F}) = \min\{|T| : T \text{ is a } t\text{-transversal of } \mathcal{F}\}.
\]
For \(t = 1\) we usually omit the 1.

**Proposition 2.** If \(\mathcal{F}\) is \(t\)-intersecting, then
\[
\varrho(\mathcal{F}) \geq \frac{t}{\tau_t(\mathcal{F})}.
\] (2)

**Proof.** Fix a \(t\)-transversal \(T\) of \(\mathcal{F}\) with \(|T| = \tau_t(\mathcal{F})\). Then
\[
t|\mathcal{F}| \leq \sum_{i \in T} |\mathcal{F}(i)| \leq |T| \cdot \max\{|\mathcal{F}(i)| : i \in T\},
\]
implying (2). \(\square\)
Obviously, $\tau_t(\mathcal{F}) = t$ if and only if $\mathcal{F}$ is a $t$-star.

**Example 3.** For $n > k > t > 0$ define

$$\mathcal{A}(n,k,t) = \left\{ A \in \binom{[n]}{k} : |A \cap [t+2]| \geq t+1 \right\}.$$ 

Clearly, $\mathcal{A} = \mathcal{A}(n,k,t)$ is $t$-intersecting, $\varrho(\mathcal{A}) = \frac{t+1+\varrho(1)}{t+2}$, $\tau_t(\mathcal{F}) = t+1$. We should note that for $2k-t < n < (k-t+1)(t+1)$, $|\mathcal{A}| > \binom{n-t}{k-t}$ with equality for $n = (k-t+1)(t+1)$.

In [3] it was shown that for any positive $\varepsilon$ and $n > n_1(k,t,\varepsilon)$, $\varrho(\mathcal{F}) < 1 - \varepsilon$ implies $|\mathcal{F}| \leq |\mathcal{A}|$ for any $t$-intersecting family $\mathcal{F} \subset \binom{[n]}{k}$. The value of $n_1(k,t,\varepsilon)$ is implicit in [3]. With careful calculation (cf. e.g. [10]) for fixed $\varepsilon > 0$ one can prove a bound quadratic in $k$. Dinur and Friedgut [1] introduced the so-called junta-method that leads to strong results for $n > ck$, however the value of the constant is large and it is further dependent on the particular problem (the same is true for the recent advances of Keller and Lifschitz [16]).

The aim of the present paper is to prove some similar results concerning $\varrho(\mathcal{F})$ for $t$-intersecting families for $n > ck$ with relatively small constants $c$. Let us state here our main result for the case $t = 1$.

**Theorem 4.** Let $n, k, d$ be integers, $k > d \geq 2$, $n \geq 4(d-1)dk$. If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting and $|\mathcal{F}| \geq 2^{d}2^{d+1}\binom{n-d-1}{k-d-1}$, then $\varrho(\mathcal{F}) > \frac{1}{d}$.

Let us stress once more that $\varrho(\mathcal{F}) > \frac{1}{d}$ follows from the results of [3] and [1] however only for much larger value of $n$.

For $t \geq 2$, we obtain the following result.

**Theorem 5.** Let $\mathcal{F} \subset \binom{[n]}{k}$ be a $t$-intersecting family with $t \geq 2$. If $|\mathcal{F}| > (t+1)\binom{n-1}{k-t-1}$ and $n \geq 2t(t+2)k$, then $\varrho(\mathcal{F}) > \frac{t}{t+1}$.

2 Preliminaries

In this section, we recall some useful results that are needed in our proofs.

Define the lexicographic order $A <_L B$ for $A, B \in \binom{[n]}{k}$ by $A <_L B$ if and only if $\min \{ i : i \in A \setminus B \} < \min \{ i : i \in B \setminus A \}$. E.g., $\binom{1,2,9}{1,3,4} <_L \binom{1,2,9}{1,3,4}$. For $n > k > 0$ and $\binom{n}{k}$, $m > 0$ let $\mathcal{L}(n,k,m)$ denote the first $m$ sets $A \in \binom{[n]}{k}$ in the lexicographic order. For $X \subset [n]$ with $|X| > k > 0$ and $\binom{|X|}{k} \geq m > 0$, we also use $\mathcal{L}(X,k,m)$ to denote the first $m$ sets $A \in \binom{|X|}{k}$ in the lexicographic order.

For $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$, we say that $\mathcal{A}, \mathcal{B}$ are cross $t$-intersecting if $|A \cap B| \geq t$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A powerful tool is the Kruskal-Katona Theorem ([17, 15]), especially its reformulation due to Hilton [12].

**Hilton’s Lemma ([12]).** Let $n, a, b$ be positive integers, $n \geq a + b$. Suppose that $\mathcal{A} \subset \binom{[n]}{a}$ and $\mathcal{B} \subset \binom{[n]}{b}$ are cross-intersecting. Then $\mathcal{L}(n,a,|\mathcal{A}|)$ and $\mathcal{L}(n,b,|\mathcal{B}|)$ are cross-intersecting as well.
For $\mathcal{F} \subset \binom{[n]}{k}$ define the $\ell$th shadow of $\mathcal{F}$, 

$$\partial^\ell \mathcal{F} = \{ G : |G| = k - \ell, \exists F \in \mathcal{F} \text{ such that } G \subset F \}.$$ 

For $\ell = 1$ we often omit the superscript.

The following statement goes back to Katona [15]. Let us include the very short proof.

**Proposition 6.** Let $\mathcal{F} \subset \binom{[n]}{k}$ be an initial family. Then 

$$\partial \mathcal{F}(\bar{1}) \subset \mathcal{F}(1).$$ 

**Proof.** Indeed, if $E \subset F \in \mathcal{F}(\bar{1})$ and $E = F \setminus \{j\}$. Then by initiality $E \cup \{1\} \in \mathcal{F}$, i.e., $E \in \mathcal{F}(1)$. $\square$

The Katona Intersecting Shadow Theorem gives an inequality concerning the sizes of a $t$-intersecting family and its shadow.

**Katona Intersecting Shadow Theorem ([14]).** Suppose that $n \geq 2k - t$, $t \geq \ell \geq 1$. Let $\emptyset \neq \mathcal{A} \subset \binom{[n]}{k}$ be a $t$-intersecting family. Then 

$$|\partial^\ell \mathcal{A}| \geq |\mathcal{A}| \binom{2k-t}{k-\ell} \binom{2k-t}{k-\ell}$$

with equality holding if and only if $\mathcal{F}$ is isomorphic to $\binom{[2k-t]}{k}$.

Let us recall an important operation called shifting introduced by Erdős, Ko and Rado [2]. For $\mathcal{F} \subset \binom{[n]}{k}$ and $1 \leq i < j \leq n$, define 

$$S_{ij}(\mathcal{F}) = \{ S_{ij}(F) : F \in \mathcal{F} \},$$

where 

$$S_{ij}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\}, & j \in F, i \notin F \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}; \\ F, & \text{otherwise}. \end{cases}$$

It is well known (cf. [5]) that shifting preserves the $t$-intersecting property.

Let $(x_1, \ldots, x_k)$ denote the set $\{x_1, \ldots, x_k\}$ where we know or want to stress that $x_1 < \ldots < x_k$. Let us define the *shifting partial order* $\prec$ where $P \prec Q$ for $P = (x_1, \ldots, x_k)$ and $Q = (y_1, \ldots, y_k)$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq k$. This partial order can be traced back to [2]. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called initial if $F \prec G$ and $G \in \mathcal{F}$ always imply $F \in \mathcal{F}$. Note that an initial family $\mathcal{F}$ satisfies $S_{ij}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$. By repeated shifting one can transform an arbitrary $k$-graph into a shifted $k$-graph with the same number of edges. Note also that $|\mathcal{F}(1)| \geq |\mathcal{F}(2)| \geq \ldots \geq |\mathcal{F}(n)|$ for an initial family.

We need the following property of initial families.

**Proposition 7.** Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is initial and $t$-intersecting. Let $r \leq s < k - t$ and let $R \subset [s]$ with $|R| = r$. Then $\mathcal{F}(\bar{R}), \mathcal{F}(R, [s])$ are cross $(t + s - r)$-intersecting.
Proof. Suppose for contradiction that there exist $F \in \mathcal{F}(\binom{s}{k})$, $F' \in \mathcal{F}(R, \binom{s}{k})$ such that $|F \cap F'| = t + j \leq t - 1 + s - r$. Let $E \subset F \cap F'$ and $T \subset [s] \setminus R$ with $|E| = |T| = j + 1$. Then $F'' := F \cup T \setminus E$ satisfies $F'' \prec F$ whence $F'' \in \mathcal{F}$. However $|F' \cap F''| = |F \cap F'| - |E| = t - 1$, the desired contradiction.

We need a notion called pseudo $t$-intersecting, which was introduced in [7]. A family $\mathcal{F} \subset \binom{[n]}{k}$ is said to be pseudo $t$-intersecting if for every $F \in \mathcal{F}$ there exists $0 \leq i \leq k - t$ such that $|F \cap [2i + t]| \geq i + t$.

**Fact 8.** Let $\mathcal{F} \subset \binom{[n]}{k}$ be an initial family. If $[t - 1] \cup \{t + 1, t + 3, \ldots, 2k - t + 1\} \notin \mathcal{F}$, then $\mathcal{F}$ is pseudo $t$-intersecting.

*Proof.* Indeed, otherwise if $\mathcal{F}$ is pseudo $t$-intersecting then there exists $F \in \mathcal{F}$ such that $|F \cap [2i + t]| < i + t$ holds for all $i = 0, 1, \ldots, k - t$. By initiality it follows that

$$[t - 1] \cup \{t + 1, t + 3, \ldots, 2k - t + 1\} \in \mathcal{A},$$

a contradiction. □

**Theorem 9** ([4]). Let $\mathcal{F} \subset \binom{[n]}{k}$ be an initial family with $0 \leq t < k$. If $\mathcal{F}$ is pseudo $t$-intersecting, then

$$|\mathcal{F}| \leq \binom{n}{k - t}. \quad (5)$$

The following property is proved in [7]. Let us include a proof as well.

**Proposition 10** ([7]). Let $n > \max\{2a - t, 2b - t\}$. If $\mathcal{A} \subset \binom{[n]}{a}$, $\mathcal{B} \subset \binom{[n]}{b}$ are cross $t$-intersecting and both initial, then either both $\mathcal{A}$ and $\mathcal{B}$ are pseudo $t$-intersecting, or one of them is pseudo $(t + 1)$-intersecting.

*Proof.* If $\mathcal{A}$ is not pseudo $t$-intersecting, then there exists $A \in \mathcal{A}$ such that $|A \cap [2i + t]| < i + t$ holds for all $i = 0, 1, \ldots, a - t$. By initiality it follows that

$$A_0 := [t - 1] \cup \{t + 1, t + 3, \ldots, t + (a - t + 1) - 1\} \in \mathcal{A}. $$

Similarly, if $\mathcal{B}$ is not pseudo $(t + 1)$-intersecting then

$$B_0 := [t] \cup \{t + 2, t + 4, \ldots, 2b - t\} \in \mathcal{B}. $$

Note that $|A_0 \cap B_0| = t - 1$. By the cross $t$-intersecting property, we infer that if $\mathcal{B}$ is not pseudo $(t + 1)$-intersecting then $\mathcal{A}$ is pseudo $t$-intersecting. Similarly, if $\mathcal{A}$ is not pseudo $(t + 1)$-intersecting then $\mathcal{B}$ is pseudo $t$-intersecting. Thus the proposition follows. □

The following inequalities for cross $t$-intersecting families can be deduced from Proposition 10.
Corollary 11 ([4]). Suppose that \( A, B \subset \binom{[n]}{k} \) are cross \( t \)-intersecting, \( |A| \leq |B| \). Then either
\[
|B| \leq \binom{n}{k - t}
\]
(6) or
\[
|A| \leq \binom{n}{k - t - 1}.
\]
(7)

We need the following inequalities concerning binomial coefficients.

Proposition 12 ([11]). Let \( n, k, i \) be positive integers. Then
\[
\binom{n - i}{k} \geq \frac{n - ik}{n} \binom{n}{k}, \text{ for } n > ik.
\]
(8)

Corollary 13. Let \( n, k, t \) be positive integers. If \( n \geq 2(t - 1)(k - t) \) and \( k > t \geq 2 \), then
\[
\binom{n - t - 2}{k - t - 2} \geq \frac{1}{2} \binom{n - 3}{k - t - 2}.
\]
(9)

Proof. Note that
\[
n \geq 2(t - 1)(k - t) = 2(t - 1)(k - t - 2) + 4(t - 1) > 2(t - 1)(k - t - 2) + 3.
\]
By (8) we have
\[
\binom{n - t - 2}{k - t - 2} \geq \frac{(n - 3) - (t - 1)(k - t - 2)}{n - 3} \binom{n - 3}{k - t - 2} \geq \frac{1}{2} \binom{n - 3}{k - t - 2}.
\]
\(\square\)

3 Shifting ad extremis and the proof of Theorem 4

Note that for initial families one can deduce Theorem 4 under much milder constraints (cf. [8]). The problem is that one cannot transform a general family into an initial family without increasing \( g(\mathcal{F}) \). To circumvent this difficulty we are going to apply the recently developed method of shifting ad extremis.

Let us define formally the notion of shifting ad extremis developed recently (cf. [6]). It can be applied to one, two or several families. For notational convenience we explain it for the case of two families in detail.

Let \( \mathcal{F} \subset \binom{[n]}{k} \), \( \mathcal{G} \subset \binom{[n]}{\ell} \) be two families and suppose that we are concerned, as usual in extremal set theory, to obtain upper bounds for \( |\mathcal{F}| + |\mathcal{G}| \), \( |\mathcal{F}||\mathcal{G}| \) or some other function \( f \) of \( |\mathcal{F}| \) and \( |\mathcal{G}| \). For this we suppose that \( \mathcal{F} \) and \( \mathcal{G} \) have certain properties (e.g., cross-intersecting and non-trivial). Since \( |S_{ij}(\mathcal{H})| = |\mathcal{H}| \) for all families \( \mathcal{H} \), it is convenient to apply \( S_{ij} \) simultaneously to \( \mathcal{F} \) and \( \mathcal{G} \). Certain properties, e.g., \( t \)-intersecting, cross-intersecting or \( \nu(\mathcal{F}) \leq r \) are known to be maintained by \( S_{ij} \). However, some other properties may be destroyed, e.g., non-triviality, \( g(\mathcal{G}) \leq c \), etc. Let \( \mathcal{P} \) be the collection of the latter properties that we want to maintain.
For any family $\mathcal{H}$, define the quantity

$$w(\mathcal{H}) = \sum_{H \in \mathcal{H}} \sum_{i \in H} i.$$ 

Obviously $w(S_{ij}(\mathcal{H})) \leq w(\mathcal{H})$ for $1 \leq i < j \leq n$ with strict inequality unless $S_{ij}(\mathcal{H}) = \mathcal{H}$.

**Definition 14.** Suppose that $\mathcal{F} \subseteq \binom{[n]}{k}$, $\mathcal{G} \subseteq \binom{[n]}{r}$ are families having property $\mathcal{P}$. We say that $\mathcal{F}$ and $\mathcal{G}$ have been *shifted ad extremis* with respect to $\mathcal{P}$ if $S_{ij}(\mathcal{F}) = \mathcal{F}$ and $S_{ij}(\mathcal{G}) = \mathcal{G}$ for every pair $1 \leq i < j \leq n$ whenever $S_{ij}(\mathcal{F})$ and $S_{ij}(\mathcal{G})$ also have property $\mathcal{P}$.

Let us show that we can obtain shifted ad extremis families by the following shifting ad extremis process. Let $\mathcal{F}$, $\mathcal{G}$ be cross-intersecting families with property $\mathcal{P}$. Apply the shifting operation $S_{ij}$, $1 \leq i < j \leq n$, to $\mathcal{F}, \mathcal{G}$ simultaneously and continue as long as the property $\mathcal{P}$ is maintained. By abuse of notation, we keep denoting the current families by $\mathcal{F}$ and $\mathcal{G}$ during the shifting process. If $S_{ij}(\mathcal{F})$ or $S_{ij}(\mathcal{G})$ does not have property $\mathcal{P}$, then we do not apply $S_{ij}$ and choose a different pair $(i', j')$. However we keep returning to previously failed pairs $(i, j)$, because it might happen that at a later stage in the process $S_{ij}$ does not destroy property $\mathcal{P}$ any longer. Note that the quantity $w(\mathcal{F}) + w(\mathcal{G})$ is a positive integer and it decreases strictly in each step. This guarantees that eventually we shall arrive at families that are shifted ad extremis with respect to $\mathcal{P}$.

Let $\mathcal{F}$, $\mathcal{G}$ be shifted ad extremis families. A pair $(i, j)$ is called *shift-resistant* if either $S_{ij}(\mathcal{F}) \neq \mathcal{F}$ or $S_{ij}(\mathcal{G}) \neq \mathcal{G}$.

In the case of several families, $\mathcal{F}_i \subseteq \binom{[n]}{k_i}$, $1 \leq i \leq r$. It is essentially the same. One important property that is maintained by simultaneous shifting is *overlapping*, namely the non-existence of pairwise disjoint edges $E_1 \in \mathcal{F}_1, \ldots, E_r \in \mathcal{F}_r$ (cf. [13]).

**Proof of Theorem 4.** Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be intersecting, $|\mathcal{F}| \geq 2^d d^{2d+1} \binom{n-d-1}{k-d-1}$ and $\varrho(\mathcal{F}) \leq \frac{1}{4}$. Without loss of generality, we may assume that $\mathcal{F}$ is shifted ad extremis for $\varrho(\mathcal{F}) \leq \frac{1}{4}$.

Then $S_{ij}(\mathcal{F}) \neq \mathcal{F}$ implies $\varrho(S_{ij}(\mathcal{F})) > \frac{1}{4}$. Thus, if a pair $(i, j)$ is shift-resistant then $|\mathcal{F}(i)| + |\mathcal{F}(j)| > |\mathcal{F}|/d$.

Let $P_1, \ldots, P_s$ be a maximal collection of pairwise disjoint shift-resistant pairs, $P_i = (x_i, y_i)$, $1 \leq i \leq s$. Clearly,

$$\sum_{1 \leq i \leq s} (|\mathcal{F}(x_i)| + |\mathcal{F}(y_i)|) \geq \frac{s}{d}|\mathcal{F}|. \quad (10)$$

For a pair of subsets $E_0 \subseteq E$, let us use the notation

$$\mathcal{F}(E_0, E) = \{ F \setminus E : F \in \mathcal{F}, F \cap E = E_0 \}.$$

Note that $\mathcal{F}(E, E) = \mathcal{F}(E)$ and $\mathcal{F}(\emptyset, E) = \mathcal{F}(E)$.

**Claim 15.** For all $D \in \binom{[n]}{d}$,

$$|\mathcal{F}(D)| \geq (d - 1)|\mathcal{F}(D)|. \quad (11)$$
Proof. For any subset $E \subset [n]$ note the identity
\[
\sum_{x \in E} |\mathcal{F}(x)| = \sum_{1 \leq j \leq |E|} \sum_{E_j \in \binom{E}{j}} j|\mathcal{F}(E_j, E)| \geq \sum_{1 \leq j \leq |E| - 1} \sum_{E_j \in \binom{E}{j}} |\mathcal{F}(E_j, E)| + |E||\mathcal{F}(E, E)|
\]
\[
\geq \sum_{E' \subset E, |E'| \geq 1} |\mathcal{F}(E', E)| + (|E| - 1)|\mathcal{F}(E)|.
\]
By $\sum_{E' \subset E} |\mathcal{F}(E', E)| = |\mathcal{F}|$, we infer that
\[
\sum_{x \in E} |\mathcal{F}(x)| \geq |\mathcal{F}| - |\mathcal{F}(E)| + (|E| - 1)|\mathcal{F}(E)|.
\]
(12)

If $|E| = d$, then $g(\mathcal{F}) \leq d$ implies that the left hand side of (12) is less than $|\mathcal{F}|$. Comparing with the right hand side yields (11).

\(\square\)

Claim 16. For all $D \in {\binom{[n]}{d}}$,
\[
|\mathcal{F}(D)| < d \binom{n - d - 1}{k - d - 1}.
\]
(13)

Proof. For convenience assume that $D = [n - d + 1, n]$. Then $\mathcal{F}(D) \subset \binom{[n-d]}{k-d}$, $\mathcal{F}(\overline{D}) \subset \binom{[n-d]}{k-d}$ and $\mathcal{F}(D), \mathcal{F}(\overline{D})$ are cross-intersecting. If
\[
|\mathcal{F}(D)| \geq d \binom{n - d - 1}{k - d - 1} > \sum_{1 \leq j \leq d} \binom{n - d - j}{k - d - 1} + \binom{n - 2d - 2}{k - d - 2},
\]
then $L(n - d, k - d, |\mathcal{F}(D)|)$ contains
\[
\left\{ A \in \binom{[n-d]}{k-d} : A \cap [d] \neq \emptyset \right\} \cup \left\{ A \in \binom{[d+1,n-d]}{k-d} : \{d+1,d+2\} \subset A \right\}.
\]
By Hilton’s Lemma, we have
\[
L(n - d, k - d, |\mathcal{F}(D)|) \subset \left\{ B \in \binom{[n-d]}{k} : [d] \subset B \text{ and } B \cap \{d+1,d+2\} \neq \emptyset \right\}.
\]
It follows that
\[
|\mathcal{F}(\overline{D})| \leq \binom{n - 2d - 1}{k - d - 1} + \binom{n - 2d - 2}{k - d - 1} < |\mathcal{F}(D)|,
\]
contradicting (11).

\(\square\)

Claim 17.
\[
s \leq d^2 - d.
\]
(14)
Proof. Assume that \( s \geq d^2 - d + 1 \). Define \( E = P_1 \cup \ldots \cup P_{d^2 - d + 1} \) and
\[
\mathcal{F}_j = \{ F \in \mathcal{F} : |F \cap E| = j \}.
\]
Clearly \( |E| = 2(d^2 - d + 1) \) and
\[
|\mathcal{F}_j| = \sum_{E_j \in \binom{E}{j}} |\mathcal{F}(E_j, E)|. \quad (15)
\]
By (13) we have
\[
\sum_{D \in \binom{E}{d}} |\mathcal{F}(D)| < \binom{2(d^2 - d + 1)}{d} d^{n - d - 1} \binom{k - d - 1}{d}.
\]
Note that for any set \( F \in \mathcal{F} \) with \( F \cap E = E_j \) and \( d \leq j \leq |E| \), \( F \) is counted \( \binom{j}{d} \) times in \( \sum_{D \in \binom{E}{d}} |\mathcal{F}(D)| \). By (15) and \( \binom{j}{d} \geq j \) for \( j > d \), it follows that
\[
\sum_{D \in \binom{E}{d}} |\mathcal{F}(D)| = \sum_{d \leq j \leq |E|} \sum_{E_j \in \binom{E}{j}} \binom{j}{d} |\mathcal{F}(E_j, E)| \geq |\mathcal{F}_d| + \sum_{d < j \leq |E|} j|\mathcal{F}_j|.
\]
By (13) we obtain that
\[
|\mathcal{F}_d| + \sum_{d < j \leq |E|} j|\mathcal{F}_j| \leq \sum_{D \in \binom{E}{d}} |\mathcal{F}(D)| < \binom{2(d^2 - d + 1)}{d} d^{n - d - 1} \binom{k - d - 1}{d}. \quad (16)
\]
Applying (10) with \( s = d^2 - d + 1 \),
\[
\frac{d^2 - d + 1}{d} |\mathcal{F}| \leq \sum_{x \in E} |\mathcal{F}(x)| = \sum_{1 \leq j \leq |E|} j|\mathcal{F}_j|
< (d - 1) \sum_{1 \leq j \leq d} |\mathcal{F}_j| + |\mathcal{F}_d| + \sum_{d < j \leq |E|} j|\mathcal{F}_j|
\overset{(16)}{<} (d - 1)|\mathcal{F}| + d \binom{2(d^2 - d + 1)}{d} d^{n - d - 1} \binom{k - d - 1}{d}.
\]
It follows that
\[
|\mathcal{F}| < d^2 \binom{2(d^2 - d + 1)}{d} d^{n - d - 1} \binom{k - d - 1}{d}.
\]
Let \( c(d) = d^2(2(d^2 - d + 1)) \). For \( d \geq 4 \), since \( e^d < 4^d - 1 \leq d^d - 1 \), using \( \binom{n}{k} < \left( \frac{en}{k} \right)^k \) we have
\[
c(d) < 2^d d^d d^{d+2} < 2^d d^{2d+1},
\]
contradicting our assumption \( |\mathcal{F}| \geq 2^d d^{2d+1} \left( \frac{n-d-1}{k-d-1} \right) \). For \( d = 2, 3 \), it can be checked directly that \( c(d) < 2^d d^{2d+1} \), contradicting our assumption as well. \( \square \)
Fix $X \subset [n]$ with $|X| = 2d^2 - 2d$ and $P_1 \cup \cdots \cup P_s \subset X$. Define
\[ \mathcal{T} = \{ T \subset [n] : |T| \leq d, \ |\mathcal{F}(T)| > (2d^2)^{-|T|} |\mathcal{F}| \} . \]
By (10), there exists $x \in X$ such that
\[ |\mathcal{F}(x)| \geq \frac{1}{2d} |\mathcal{F}| > \frac{1}{2d^2} |\mathcal{F}| , \]
implying $\mathcal{T} \neq \emptyset$. By (13) and $|\mathcal{F}| \geq 2d^{d+1} \binom{n-d-1}{k-d-1}$, we know that for every $D \in \binom{[n]}{d}$,
\[ |\mathcal{F}(D)| < d \left( \frac{n-d-1}{k-d-1} \right) \leq (2d^2)^{-d} |\mathcal{F}| . \]
Thus $|T| \leq d - 1$ for each $T \in \mathcal{T}$.
Now choose $T \in \mathcal{T}$ such that $|T| = t$ is maximum. Clearly $t \geq 1$. Note that the maximality of $t$ implies that for every $Z \subset [n]$ with $t < |Z| \leq d$
\[ |\mathcal{F}(Z)| \leq (2d^2)^{-|Z|} |\mathcal{F}| , \quad (17) \]
Set $\mathcal{A} = \mathcal{F}(T, X \cup T)$ and $U = [n] \setminus (X \cup T)$. Assume that
\[ U = \{ u_1, u_2, \ldots, u_m \} \text{ with } u_1 < u_2 < \cdots < u_m . \]
Let $Q = \{ u_1, u_2, \ldots, u_{2d-t} \}$. Note that $\mathcal{A}(Q) = \mathcal{F}(T, X \cup T \cup Q)$. By (17) we have
\[ |\mathcal{A}(Q)| > |\mathcal{F}(T)| - \sum_{x \in (X \cup T) - Q} |\mathcal{F}(T \cup \{ x \})| > (2d^2)^{-t} |\mathcal{F}| - (2d^2 - 2d + 2d - t)(2d^2)^{-(t+1)} |\mathcal{F}| = \frac{t}{(2d^2)^{t+1}} |\mathcal{F}| . \]
Then by $|\mathcal{F}| \geq 2d^{d+1} \binom{n-d-1}{k-d-1}$ we infer that
\[ |\mathcal{A}(Q)| > \binom{n-d-1}{k-d-1} = \frac{n-d-1}{(k-t) - (d+1-t)} . \quad (18) \]
\[ \textbf{Claim 18.} \text{ For every } S \subset X \setminus T , \]
\[ |\mathcal{F}(S, X \cup T)| \leq 2d^{d-1} \binom{n-d-1 - |S|}{k-d-1 - |S|} . \quad (19) \]
\[ \textbf{Proof.} \text{ Let } \mathcal{B} = \mathcal{F}(S, X \cup T) . \text{ Recall that } P_1, P_2, \ldots, P_s \text{ is a maximal collection of pairwise disjoint shift-resistant pairs and } P_1 \cup P_2 \cup \cdots \cup P_s \subset X . \text{ Then } \mathcal{F} \text{ is initial on } [n] \setminus X . \text{ It follows that } \mathcal{A} \subset \binom{[n] \setminus (X \cup T)}{k-t} , \mathcal{B} \subset \binom{[n] \setminus (X \cup T)}{k-|S|} \text{ are initial and cross-intersecting. For any } R \subset Q \text{ with } |R| = r \leq d , \text{ we have } \mathcal{A}(Q) \subset \binom{[n] \setminus (X \cup T \cup Q)}{k-t} \text{ and } \mathcal{B}(R, Q) \subset \binom{[n] \setminus (X \cup T \cup Q)}{k-|S|} - r . \]

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By Proposition 7, we infer that $A(\overline{Q})$, $B(R, Q)$ are cross $(2d - t - r + 1)$-intersecting. Since $r \leq d$ implies $2d - t - r + 1 \geq d + 1 - t$, by (18) and (5) we see that $A(\overline{Q})$ is not pseudo $(2d - t - r + 1)$-intersecting. By Proposition 10, it follows that $B(R, Q)$ is pseudo $(2d - t - r + 2)$-intersecting. Thus by (5) we have

$$|B(R, Q)| \leq \left( \frac{n - |X \cup T \cup Q|}{k - |S| - r - (2d - t - r + 2)} \right) = \left( \frac{n - |X \cup T \cup Q|}{k - 2d - 2 + t - |S|} \right).$$

Since $t \leq d - 1$, $|X \cup T \cup Q| \geq |S| + 2d - t \geq |S| + d + 1$ and

$$\frac{n - d - 1 - |S|}{2} \geq k - d - 1 - |S| > k - 2d - 2 + t - |S|,$$

we infer that

$$|B(R, Q)| \leq \left( \frac{n - d - 1 - |S|}{k - 2d - 2 + t - |S|} \right) < \left( \frac{n - d - 1 - |S|}{k - d - 1 - |S|} \right).$$

Moreover, $|B(R)| \leq \left( \frac{n - d - 1 - |S|}{k - d - 1 - |S|} \right)$ for $|R| = d + 1$. Thus,

$$|B| = \sum_{R \subset Q} |B(R, Q)| = \sum_{R \subset Q, |R| \leq d} |B(R, Q)| + \sum_{R \subset Q, |R| = d + 1} |B(R, Q)|
\leq \sum_{R \subset Q, |R| \leq d} |B(R, Q)| + \sum_{R \subset Q, |R| = d + 1} |B(R)|
\leq \sum_{0 \leq i \leq d} \binom{2d - t}{i} \left( \frac{n - d - 1 - |S|}{k - d - 1 - |S|} \right) + \binom{2d - t}{d + 1} \left( \frac{n - d - 1 - |S|}{k - d - 1 - |S|} \right)
\leq \left( \frac{n - d - 1 - |S|}{k - d - 1 - |S|} \right) \sum_{0 \leq i \leq d + 1} \binom{2d - 1}{i}
\leq 2^{2d-1} \left( \frac{n - d - 1 - |S|}{k - d - 1 - |S|} \right). \quad \square$$

By (19),

$$|F(T)| = \sum_{S \subset X \setminus T} |F(S, X \cup T)| < \sum_{0 \leq j \leq |X \setminus T|} \binom{|X \setminus T|}{j} 2^{2d-1} \left( \frac{n - d - 1 - j}{k - d - 1 - j} \right)
\leq \sum_{0 \leq j \leq |X \setminus T|} \left( \frac{2d^2 - 2d}{j} \right) 2^{2d-1} \left( \frac{n - d - 1 - j}{k - d - 1 - j} \right).$$

Note that $n \geq 4d(d - 1)k$ implies

$$\frac{\binom{2d^2 - 2d}{j + 1} \binom{n - d - 2 - j}{k - d - 2 - j}}{\binom{2d^2 - 2d}{j} \binom{n - d - 1 - j}{k - d - 1 - j}} = \frac{(2d^2 - 2d - j)(k - d - 1 - j)}{(j + 1)(n - d - 1 - j)} \leq \frac{(2d^2 - 2d)k}{n} \leq \frac{1}{2}.$$
It follows that
\[
\sum_{0 \leq j \leq |X \setminus T|} \binom{2d^2 - 2d}{j} \binom{n - d - 1 - j}{k - d - 1 - j} < \binom{n - d - 1}{k - d - 1} \sum_{0 \leq i \leq \infty} 2^{-i} = 2^{n - d - 1}.
\]
Thus,
\[
|F(\bar{T})| < 2^2 \binom{n - d - 1}{k - d - 1} < \frac{1}{d} |F|
\]
and therefore
\[
\sum_{x \in T} |F(x)| \geq |F| - |F(\bar{T})| > \frac{d - 1}{d} |F|.
\]
Since \(|T| = t \leq d - 1\), there exists some \(x \in T\) with \(|F(x)| > \frac{1}{d} |F|\), contradicting \(\varrho(F) \leq \frac{1}{d}\).
Thus the theorem holds.

4 Proof of Theorem 5

In this section we consider the maximum degree ratio problem for \(t\)-intersecting families.

Let us recall the \(t\)-covering number \(\tau_t(F)\):
\[
\tau_t(F) = \min \{|T| : |T \cap F| \geq t \text{ for all } F \in F\}.
\]
It should be clear that \(\tau_t(F) = t\) if and only if \(F\) is a \(t\)-star. Proposition 2 yields
\[
\varrho(F) \geq \frac{t}{\tau_t(F)} \tag{20}
\]
for any \(t\)-intersecting family \(F \subset \binom{[n]}{k}\).

We say that a \(t\)-intersecting family \(F\) is saturated if any addition of an extra \(k\)-set to \(F\) would destroy the \(t\)-intersecting property.

In the case \(\tau_t(F) = t + 1\) one can improve on (20).

**Proposition 19.** Suppose that \(F \subset \binom{[n]}{k}\) is \(t\)-intersecting, \(n \geq 2k\), \(\tau_t(F) \leq t + 1\) and \(F\) is saturated. Then \(\varrho(F) > \frac{t + 1}{t + 2}\).

**Proof.** Without loss of generality let \([t + 1]\) be a \(t\)-transversal of \(F\), i.e., \(|F \cap [t + 1]| \geq t\) for all \(F \in F\). Define
\[
F_i = \{ F \setminus [t + 1] : F \in F, F \cap [t + 1] = [t + 1] \setminus \{i\} \}
\]
and \(F_0 = F([t + 1])\). By saturatedness \(F_0 = \binom{[t+2,n]}{k-t-1}\). Obviously, \(F_i, F_j\) are cross-intersecting for \(1 \leq i < j \leq t + 1\). By Hilton’s Lemma, \(\min\{|F_1|, |F_j|\} \leq \binom{n-t-2}{k-t-1}\). Assume by symmetry \(|F_1| \leq |F_2| \leq \ldots \leq |F_{t+1}|\). Then
\[
|F_1| \leq \binom{n-t-2}{k-t-1} < \binom{n-t-1}{k-t-1} = |F_0|.
\]
Note that
\[ |\mathcal{F}(1)| = |\mathcal{F}_2| + \cdots + |\mathcal{F}_{t+1}| + |\mathcal{F}_0| > (t + 1)|\mathcal{F}_1|, \quad |\mathcal{F}(1)| = |\mathcal{F}_1|. \]
Thus
\[ g(\mathcal{F}) \geq \frac{|\mathcal{F}(1)|}{|\mathcal{F}(1)| + |\mathcal{F}(1)|} > \frac{t + 1}{t + 2}. \]

**Remark 20.** Considering all \( k \)-subsets of \([2k - t]\) shows that without some conditions on \(|\mathcal{F}|\) one cannot hope to prove better than \( g(\mathcal{F}) \geq \frac{k}{2k - t} \).

In the case of cross \( t \)-intersecting families, \( t \geq 2 \), we cannot apply Hilton’s Lemma. To circumvent this difficulty we prove a similar albeit somewhat weaker inequality.

**Proposition 21.** Let \( n, k, \ell, t, s \) be integers, \( s > t \geq 2 \), \( k, \ell > s \). Suppose that \( \mathcal{F} \subseteq \binom{[n]}{k} \) and \( \mathcal{G} \subseteq \binom{[n]}{\ell} \) are cross \( t \)-intersecting. Assume that \( |\mathcal{G}| > \binom{n}{\ell - s} \). Then
\[ |\mathcal{F}| < \binom{s - 1}{t} \binom{n - s - 1}{k - t} + 2^{s} \binom{n - t - 1}{k - t - 1}. \]  \hspace{1cm} (21)

Moreover, if \( n \geq s(k - t) \), then
\[ |\mathcal{F}| < \binom{s - 1}{t} \binom{n - s}{k - t} + \frac{2}{s - 1} \binom{s}{t - 1} \binom{s - 1}{t - 1} + 2 \binom{s}{t + 1} \binom{n - s}{k - t - 1}. \]  \hspace{1cm} (22)

**Proof.** Assume the contrary. Without loss of generality, we can suppose that \( \mathcal{F} \) and \( \mathcal{G} \) are initial (\( S_{ij} \) does not change \(|\mathcal{F}|, |\mathcal{G}|\)). Since \( |\mathcal{G}| > \binom{n}{\ell - s} \), by Theorem 9 we infer that \( \mathcal{G} \) is not pseudo \( s \)-intersecting. That is, there exists \( G \in \mathcal{G} \) such that \( |G \cap [2i + s]| < i + s \) for all \( i = 0, 1, \ldots, \ell - s \). It follows that
\[ (1, 2, \ldots, s - 1, s + 1, s + 3, \ldots) =: G_0 \in \mathcal{G}. \]

Let \( T_0 \in \binom{[s - 1]}{t - 1} \). Then by the cross \( t \)-intersecting property
\[ T_0 \cup (s, s + 2, s + 4, \ldots) \notin \mathcal{F}. \]

Define \( T = T_0 \cup \{s\} \). Then \( \mathcal{F}(T, [s]) \subseteq \binom{[s + 1, n]}{k - t} \) and
\[ E_0 := (s + 2, s + 4, \ldots, s + 2(k - t)) \notin \mathcal{F}(T, [s]). \]

By Fact 8, \( \mathcal{F}(T, [s]) \) is pseudo intersecting. Thus,
\[ |\mathcal{F}(T, [s])| \leq \binom{n - s}{k - t - 1}. \]  \hspace{1cm} (23)

For \( R \subseteq [s], \)
\[ |\mathcal{F}(R, [s])| \leq \binom{n - s}{k - |R|}. \]  \hspace{1cm} (24)

We shall use (24) for \( R \) with \(|R| > t\) and \(|R| = t\) but \( s \notin R \).
Claim 22. For $R \subseteq [s]$ with $|R| = t - i$ and $i \geq 1$, $F(R, [s])$ is pseudo $(2i+1)$-intersecting.

Proof. Let

$$\tilde{R} := (s+1, s+2, \ldots, s+2i-1, s+2i, s+2i+2, \ldots, s+2(k-t)).$$

Set $Q = [t-1] \cup (s, s+2, \ldots, s+2(k-t))$ and note $|Q \cap G_0| = t-1$ whence $Q \notin F$. Since $Q \not\subset R \cup \tilde{R}, R \cup \tilde{R} \notin F$, i.e., $\tilde{R} \notin F(R, [s])$. By Fact 8, we infer that $F(R, [s])$ is pseudo $(2i+1)$-intersecting.

For $|R| = t - i$ with $1 \leq i \leq t$, by Claim 22

$$|F(R, [s])| \leq \binom{n-s}{k-(t-i)-2i-1} = \binom{n-s}{k-t-i-1}.$$

Now

$$|F| = \sum_{R \subseteq [s]} |F(R, [s])|$$

$$= \sum_{R \subseteq [s], |R| \leq t-1} |F(R, [s])| + \sum_{R \subseteq \binom{[t]}{i}} |F(R, [s])| + \sum_{R \subseteq [s], |R| \geq t+1} |F(R, [s])|$$

$$\leq \sum_{0 \leq i \leq t-1} \binom{s}{i} \binom{n-s}{k-2t+i-1} + \binom{s-1}{t} \binom{n-s}{k-t} + \binom{s-1}{t-1} \binom{n-s}{k-t-1}$$

$$+ \sum_{t+1 \leq i \leq s} \binom{s}{i} \binom{n-s}{k-i}.$$

(25)

Using $(\binom{n-s}{k-2t+i-1}) < (\binom{n-s}{k-t-1})$ for $i \leq t-1$ and $(\binom{n-s}{k-i}) \leq (\binom{n-s}{k-t-1})$ for $i \geq t+1$, we conclude that

$$|F| < \binom{s-1}{t} \binom{n-s}{k-t} + \sum_{0 \leq i \leq s} \binom{s}{i} \binom{n-s}{k-t-1} - \binom{s-1}{t} \binom{n-s}{k-t-1}$$

$$\leq \binom{s-1}{t} \binom{n-s-1}{k-t} + 2^s \binom{n-s}{k-t-1}.$$

This proves (21).

If $n \geq s(k-t)$ then for $1 \leq i \leq t - 1$

$$\frac{\binom{s}{i} \binom{n-s}{k-2t+i-1}}{\binom{s-1}{i-1} \binom{n-s}{k-2t+i-2}} = \frac{(s-i+1)(n-s-k+2t-i+2)}{i(k-2t+i-1)}$$

$$\geq \frac{(s-t+2)(n-s-k+t+3)}{(t-1)(k-t-2)}$$

$$> \frac{(s-t+2)(s-1)(k-t-1)}{(t-1)(k-t-2)}$$

$$> 2.$$
and
\[
\binom{n-s}{k-t-2} \leq \binom{k-t-1}{n-s-k+t+2} < \binom{k-t-1}{s-1(k-t-1)} \leq \frac{1}{s-1}.
\]

It follows that
\[
\sum_{0 \leq i \leq t-1} \binom{s}{i} \binom{n-s}{k-2t+i-1} < \binom{s}{t-1} \binom{n-s}{k-t-2} \sum_{i=0}^{\infty} 2^{-i} = 2 \binom{s}{t-1} \binom{n-s}{k-t-2} < \frac{2}{s-1} \binom{s}{t-1} \binom{n-s}{k-t-1}.
\]

For \( t+1 \leq i \leq s-1 \),
\[
\binom{s}{i+1} \binom{n-s}{k-i-1} = \frac{(s-i)(k-i)}{(i+1)(n-s-k+i+1)} \leq \frac{(s-t-1)(k-t-1)}{(t+2)(n-s-k+t+2)} < \frac{(s-t-1)(k-t-1)}{(t+2)(s-1)(k-t-1)} < \frac{1}{2}.
\]

It follows that
\[
\sum_{t+1 \leq i \leq s} \binom{s}{i} \binom{n-s}{k-i} < \binom{s}{t+1} \binom{n-s}{k-t-1} \sum_{i=0}^{\infty} 2^{-i} = 2 \binom{s}{t+1} \binom{n-s}{k-t-1}.
\]  \hspace{1cm} (27)

Combining (25), (26) and (27), we conclude that
\[
|F| < \frac{2}{s-1} \binom{s}{t-1} \binom{n-s}{k-t-1} + \binom{s-1}{t} \binom{n-s}{k-t} + \binom{s-1}{t-1} \binom{n-s}{k-t-1}
\]
\[
+ 2 \binom{s}{t+1} \binom{n-s}{k-t-1} = \binom{s-1}{t} \binom{n-s}{k-t} + \frac{2}{s-1} \binom{s}{t-1} + \binom{s-1}{t-1} + 2 \binom{s}{t+1} \binom{n-s}{k-t-1}.
\]

Consider the obvious construction:
\[
\mathcal{G} = \left\{ G \in \binom{[n]}{\ell} : [s] \subset G \right\}, \quad \mathcal{F} = \left\{ F \in \binom{[n]}{k} : |F \cap [s]| \geq t \right\}.
\]
Then $\mathcal{F}, \mathcal{G}$ are cross $t$-intersecting and

$$|\mathcal{G}| = \binom{n-s}{t-s}, \quad |\mathcal{F}| = \binom{s}{t} \binom{n-s}{k-t} + \sum_{t<j<s} \binom{s}{j} \binom{n-s}{k-j},$$

showing that (21) does not hold for $|\mathcal{G}| \leq \binom{n-s}{t-s}$.

**Corollary 23.** Let $\mathcal{F} \subset \binom{[n]}{k}$ be $t$-intersecting with $n \geq (t+2)(k-t)$ and $|\mathcal{F}| > (t+1)\binom{n-1}{k-t-1}$. If $\varrho(\mathcal{F}) < \frac{t}{t+1}$, then for every $P \in \binom{[n]}{2}$,

$$|\mathcal{F}(P)| \leq (t+1)\left(\frac{n-t-2}{k-t-2}\right) + \frac{5t^2 + 19t + 24}{6} \binom{n-t-3}{k-t-3}. \tag{28}$$

**Proof.** If there exists $\{x, y\} \subset [n]$ such that

$$|\mathcal{F}(x, y)| > (t+1)\left(\frac{n-t-2}{k-t-2}\right) + \frac{5t^2 + 19t + 24}{6} \binom{n-t-3}{k-t-3},$$

then for every $P \in \binom{[n]}{2}$,

$$|\mathcal{F}(x, y)| > (t+1)\left(\frac{n-t-4}{k-t-2}\right) + \left(\frac{2}{t+1}\left(\frac{t+2}{t-1}\right) + \left(\frac{t+1}{t-1}\right) + 2\left(\frac{t+2}{t+1}\right)\right) \binom{n-t-4}{k-t-3},$$

note that $\mathcal{F}(x, y) \subset \binom{[n]\setminus\{x,y\}}{k}$, $\mathcal{F}(\bar{x}, \bar{y}) \subset \binom{[n]\setminus\{x,y\}}{k}$ are cross $t$-intersecting, by applying Proposition 21 with $s = t+2$ we infer

$$|\mathcal{F}(\bar{x}, \bar{y})| \leq \binom{n-2}{k-t-2}.$$ 

Since $\varrho(\mathcal{F}) < \frac{t}{t+1}$ implies

$$|\mathcal{F}(\bar{x})|, |\mathcal{F}(\bar{y})| > \frac{1}{t+1}|\mathcal{F}| > \binom{n-1}{k-t-1},$$

it follows that

$$\mathcal{F}(x, y) \geq |\mathcal{F}(\bar{x})| - |\mathcal{F}(\bar{x}, \bar{y})| > \binom{n-2}{k-t-1}$$

and

$$\mathcal{F}(x, y) \geq |\mathcal{F}(\bar{y})| - |\mathcal{F}(\bar{x}, \bar{y})| > \binom{n-2}{k-t-1}.$$ 

But $\mathcal{F}(\bar{x}, \bar{y}), \mathcal{F}(x, y)$ are cross $t$-intersecting. This contradicts Corollary 11. \qed

**Lemma 24.** Let $\mathcal{F} \subset \binom{[n]}{k}$ be initial, $t$-intersecting, $n \geq 2(t+1)(k-t)$ and $|\mathcal{F}| \geq 2t(t+1)(t+2)\binom{n-t-4}{k-t-2}$ then

$$\varrho(\mathcal{F}) > \frac{t}{t+1}.$$

**Proof.** Consider a subset $P \subset [t+1], |P| \leq t-1$. 

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Claim 25. $\mathcal{F}(P, [t + 1])$ is $1 + 2(t - |P|)$-intersecting.

Proof. Suppose for contradiction that $\tilde{F}_1, \tilde{F}_2 \in \mathcal{F}(P, [t + 1])$ satisfy $\tilde{F}_1 \cap \tilde{F}_2 = D$ with $|D| \leq (2 - |P|)$. Since $\mathcal{F}$ is $t$-intersecting, we infer $|D| \geq t - |P|$. If $|D| = t - |P|$ then choose $y \in D$ and $x \in [t + 1] \setminus P$ and set $F_1 = \tilde{F}_1 \cup P$, $F_2 = (\tilde{F}_2 \cup P \cup \{x\}) \setminus \{y\}$. By initiality $F_2 \prec \tilde{F}_2 \cup P$ implies $F_2 \in \mathcal{F}$. But $|F_1 \cap F_2| = |P| + |D| - 1 = t - 1$, a contradiction.

If $|D| \geq t + 1 - |P|$, then choose $E \subset D$, $|E| = t + 1 - |P|$ and set $F_1 = \tilde{F}_1 \cup P$, $F_2 = (\tilde{F}_2 \cup [t + 1]) \setminus E$. Then $F_1 \cap F_2 = P \cup D \setminus E$ whence

$$|F_1 \cap F_2| = |P| + |D| - |E| \leq |P| + 2t - 2|P| - (t + 1 - |P|) = t - 1.$$ 

Now $F_1 \in \mathcal{F}$ and $\tilde{F}_2 \cup P \in \mathcal{F}$ by definition and $F_2 \prec \tilde{F}_2 \cup P$. Hence $F_2 \in \mathcal{F}$ contradicting the $t$-intersecting property.

Define $\mathcal{F}_i = \mathcal{F}([t + 1] \setminus \{i\}, [t + 1])$. By initiality

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_{t+1}. $$

Apply Claim 25 with $P = [t + 1] \setminus \{1\}$, $\mathcal{F}_1$ is intersecting. Thus by (4) $|\partial \mathcal{F}_1| \geq |\mathcal{F}_1|$. By initiality $\partial \mathcal{F}_1 \subset \mathcal{F}_0 := \mathcal{F}([t + 1])$. Then

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| + |\mathcal{F}_2| + \ldots + |\mathcal{F}_{t+1}| \geq (t + 2)|\mathcal{F}_1|. \quad (29)$$

For any $P \in \binom{[t + 1]}{t-j}$, by Claim 25 we know $\mathcal{F}(P, [t + 1])$ is $(2j + 1)$-intersecting. Note that $n \geq 2(t + 1)(k - t) > (2j + 2)(k - t - j)$ for $j = 1, \ldots, t$. By (1), we infer

$$|\mathcal{F}(P, [t + 1])| \leq \binom{n - t - 1 - 2j - 1}{k - (t - j) - 2j - 1} = \binom{n - t - 2 - 2j}{k - t - 1 - j}.$$ 

Note that $\mathcal{F}(P, [t + 1])$ is $(k - t + j)$-uniform and $j \geq k - t$ implies $2j + 1 > k - t + j$. It follows that $|\mathcal{F}(P, [t + 1])| = 0$ for $j \geq k - t$. Thus,

$$|\mathcal{F}(\bar{P})| = \sum_{P \subseteq [2, t+1]} |\mathcal{F}(P, [t + 1])|$$

$$= |\mathcal{F}_1| + \sum_{0 \leq i \leq t-1} \sum_{P \subseteq \binom{[2, t+1]}{t-j}} |\mathcal{F}(P, [t + 1])|$$

$$\leq |\mathcal{F}_1| + \sum_{1 \leq j \leq t-1} \sum_{t-j \leq k-1} \binom{t}{t-j} \binom{n - t - 2 - 2j}{k - t - 1 - j}.$$ 

For $k = t + 2$, we have

$$|\mathcal{F}(\bar{P})| \leq |\mathcal{F}_1| + t \binom{n - t - 4}{k - t - 2}. $$

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For $2 \leq j \leq \min\{t, k - t - 1\}$, 
\[
\binom{t}{j} \binom{n-t-2-j}{k-t-1-j} = \frac{(t-j+1)(k-t-j)(n-k-j)}{j(n-t-2j)(n-2t-1)} \leq \frac{(t-1)(k-t-2)(n-k-2)}{2(n-3t)(n-3t-1)}.
\]
Since $n \geq 2(t+1)(k-t)$ and $k \geq t + 3$ implies that
\[
\frac{n-k-2}{n-3t} < 2, \quad \frac{(t-1)(k-t-2)}{n-3t-1} < \frac{1}{2},
\]
it follows that $\binom{t}{j} \binom{n-t-2-j}{k-t-1-j} < \frac{1}{2} \binom{t}{j} \binom{n-t-2-j}{k-t-1-j}$. Thus,
\[
|\mathcal{F}(\overline{1})| \leq |\mathcal{F}| + \sum_{1 \leq j \leq t} \binom{t}{j} \binom{n-t-2-2j}{k-t-1-j} = |\mathcal{F}| + t \binom{n-t-4}{k-t-2} \sum_{i=0}^{\infty} 2^{-i} = |\mathcal{F}| + 2t \binom{n-t-4}{k-t-2}.
\]
By (29) and $|\mathcal{F}| \geq 2t(t+1)(t+2)\binom{n-t-4}{k-t-2}$, it follows that
\[
|\mathcal{F}(\overline{1})| \leq \frac{1}{t+2} |\mathcal{F}| + 2t \binom{n-t-4}{k-t-2} \leq \frac{1}{t+2} |\mathcal{F}| + \frac{1}{(t+1)(t+2)} |\mathcal{F}| = \frac{1}{t+1} |\mathcal{F}|.
\]
Thus the lemma follows. 

Proof of Theorem 5. Suppose to the contrary that $|\mathcal{F}| > (t+1)\binom{n-1}{k-t-1}$ and $\varrho(\mathcal{F}) \leq \frac{t}{t+1}$. Since $n \geq 2t(t+2)k \geq 4(t+2)k$, we infer
\[
|\mathcal{F}| > (t+1)\frac{n-1}{k-t-1} \binom{n-2}{k-t-2} > 4(t+1)(t+2) \binom{n-t-2}{k-t-2}.
\]
Shift $\mathcal{F}$ ad extremis for $\varrho(\mathcal{F}) \leq \frac{t}{t+1}$ and let $\mathcal{H}$ be the graph formed by the shift-resistant pairs. For every $P \in \binom{n}{2}$, by (28) and $n \geq 2t(t+2)k > \frac{5t^2+19t+24}{6}k$ we infer
\[
|\mathcal{F}(P)| < (t+1) \binom{n-t-2}{k-t-2} + \frac{5t^2+19t+24}{6} \binom{n-t-3}{k-t-3} < (t+2) \binom{n-t-2}{k-t-2}.
\]
Claim 26. $\mathcal{H}$ is intersecting.

Proof. Suppose that there are disjoint pairs $(a_1, b_1), (a_2, b_2) \in \mathcal{H}$. Set $\mathcal{G}_i = \{F \in \mathcal{F}: F \cap \{a_i, b_i\} \neq \emptyset\}, i = 1, 2$. Since $\varrho(S_{a_i, b_i}(\mathcal{F})) > \frac{t}{t+1} |\mathcal{F}|$, we infer $|\mathcal{G}_i| > \frac{t}{t+1} |\mathcal{F}|$. By (31) we have
\[
|\mathcal{G}_1 \cap \mathcal{G}_2| \leq \sum_{i=1,2} \sum_{j=1,2} \mathcal{F} \{a_i, b_j\} < 4(t+2) \binom{n-t-2}{k-t-2}.
\]
It follows that
\[ |F| \geq |G_1| + |G_2| - |G_1 \cap G_2| > \frac{2t}{t+1} |F| - 4(t+2) \binom{n-t-2}{k-t-2} \]
\[ > \frac{2t}{t+1} |F| - \frac{1}{t+1} |F| \geq |F|, \]
(a contradiction. \)

Note that \( n \geq 2t(t+2)k \) implies
\[ |F| > (t+1) \binom{n-1}{k-1} > 2t(t+1)(t+2) \binom{n-t-4}{k-t-2}. \] (32)

By Lemma 24, we may assume that \( H \neq \emptyset \). For convenience assume that \( (n-1,n) \in H \).

Let
\[ A = \left\{ A \in \left[ \binom{n-2}{k-1} : A \cup \{x\} \in F \text{ with } x = n-1 \text{ or } x = n \right] \right\}, \quad B = \left[ \binom{n-2}{k} \right] \cap F. \]

Since \( g(S_{n-1,n}(F)) > \frac{t}{t+2} |F| \) implies
\[ |F(n-1,n)| + |F(n-1,n) \cup F(n-1,n)| > \frac{t}{t+1} |F|, \]
by (31) and \( t \geq 2 \) we infer
\[ A(\bar{1},\bar{2}) \geq \frac{t}{t+1} |F| - |F(n-1,n)| - \sum_{i \in \left\{ 1,2 \right\}, j \in \left\{ n-1,n \right\}} |F(i,j)| \]
\[ \geq \frac{t}{t+1} |F| - 5(t+2) \binom{n-t-2}{k-t-2} \]
\[ \geq 4t(t+2) \binom{n-t-2}{k-t-2} - 5(t+2) \binom{n-t-2}{k-t-2} \]
\[ \geq 3(t+2) \cdot \frac{1}{2} \binom{n-3}{k-t-2} \]
\[ \geq \binom{n-4}{k-t-2}. \] (33)

Fix \( R \subset [2] \) with \( |R| \leq 1 \). Since \( A, B \) are initial and cross \( t \)-intersecting, by Proposition 7 we infer that \( A(\bar{1},\bar{2}) \) and \( B(R, [2]) \) are cross \((t+2-|R|)\)-intersecting. By (33) we know that \( A(\bar{1},\bar{2}) \) is not pseudo \((t+2-|R|)\)-intersecting. By Proposition 10 we infer that \( B(R, [2]) \) is pseudo \((t+3-|R|)\)-intersecting. Therefore,
\[ |B(R, [2])| \leq \binom{n-4}{k-|R|-(t+3-|R|)} = \binom{n-4}{k-t-3}. \]
Note that (31) implies $\mathcal{B}([2]) < (t + 2)\binom{n-t-2}{k-t-2}$. Thus,

$$|\mathcal{B}| = \sum_{R \subseteq [2]} |\mathcal{B}(R, [2])|$$

$$< 3\left(\frac{n-4}{k-t-3}\right) + (t + 2)\binom{n-t-2}{k-t-2}$$

$$= \frac{3(k-t-2)}{n-3}\left(\frac{n-3}{k-t-2}\right) + (t + 2)\binom{n-t-2}{k-t-2}$$

$$\overset{(9)}{\leq} \frac{6(k-t-2)}{n-3}\left(\frac{n-t-2}{k-t-2}\right) + (t + 2)\binom{n-t-2}{k-t-2}$$

$$< (t + 3)\binom{n-t-2}{k-t-2}.$$ Then $\varrho(\mathcal{F}) \leq \frac{t}{t+1}$ implies

$$|\mathcal{F}(n-1, n)| = |\mathcal{F}(n-1)| - |\mathcal{B}| > \frac{1}{t+1}|\mathcal{F}| - (t + 3)\binom{n-t-2}{k-t-2} > (3t + 5)\binom{n-t-2}{k-t-2}$$

and

$$|\mathcal{F}(n, n)| = |\mathcal{F}(n)| - |\mathcal{B}| > \frac{1}{t+1}|\mathcal{F}| - (t + 3)\binom{n-t-2}{k-t-2} > (3t + 5)\binom{n-t-2}{k-t-2}.$$ Now by (31)

$$|\mathcal{F}(\{1, n-1\}, n)| \geq |\mathcal{F}(n-1, n)| - |\mathcal{F}(1, n)| > (2t + 3)\binom{n-t-2}{k-t-2}$$

and

$$|\mathcal{F}(\{1, n\}, n-1)| \geq |\mathcal{F}(n-1, n)| - |\mathcal{F}(1, n-1)| > (2t + 3)\binom{n-t-2}{k-t-2}.$$ By (9),

$$(2t + 3)\binom{n-t-2}{k-t-2} > \frac{2t + 3}{2}\binom{n-3}{k-t-2} > \binom{n-3}{k-t-2},$$

this contradicts the fact that $\mathcal{F}(\{1, n-1\}, n), \mathcal{F}(\{1, n\}, n-1) \subseteq \binom{[2, n-2]}{k-1}$ are cross $(t+1)$-intersecting. Thus the theorem holds. $\square$

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References

[1] I. Dinur and E. Friedgut. Intersecting families are essentially contained in juntas. *Comb. Probab. Comput.*, 18: 107–122, 2009.

[2] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets, Quart. *J. Math. Oxford Ser.*, 12: 313–320, 1961.

[3] P. Frankl. On intersecting families of finite sets. *J. Combinatorial Theory, Ser. A*, 24: 146–161, 1978.

[4] P. Frankl. The Erdős-Ko-Rado theorem is true for $n = ckt$. *Coll. Math. Soc. J. Bolyai*, 18: 365–375, 1978.

[5] P. Frankl. The shifting technique in extremal set theory. *Surveys in Combinatorics*, 123: 81–110, 1987.

[6] P. Frankl. On the maximum of the sum of the sizes of non-trivial cross-intersecting families. *Combinatorica*, 44: 15–35, 2024.

[7] P. Frankl and G.O.H. Katona. On strengthenings of the intersecting shadow theorem. *J. Combinatorial Theory, Ser. A*, 184: 105510, 2021.

[8] P. Frankl and A. Kupavskii. Simple juntas for shifted families. *Discrete Anal.*, 14: 18 pp, 2020.

[9] P. Frankl and N. Tokushige. On $r$-cross intersecting families of sets. *Comb. Probab. Comput.*, 20:749–752, 2011.

[10] P. Frankl and J. Wang. Intersections and distinct intersections in cross-intersecting families. *Europ. J. Combin.*, 110: 103665, 2022.

[11] P. Frankl and J. Wang. A product version of the Hilton-Milner-Frankl theorem. *Sci. China Math.*, 67: 455–474, 2024.

[12] A.J.W. Hilton. The Erdős-Ko-Rado Theorem with valency conditions. *unpublished manuscript*, 1976.

[13] H. Huang, P.-S. Loh, and B. Sudakov. The size of a hypergraph and its matching number. *Comb. Probab. Comput.*, 21(3) : 442–450, 2012.

[14] G.O.H. Katona. Intersection theorems for systems of finite sets. *Acta Math. Acad. Sci. Hung.*, 15: 329–337, 1964.

[15] G.O.H. Katona. A theorem of finite sets. *Theory of Graphs.Proc. Colloq. Tihany, Akad. Kiadó*, 187–207, 1966.

[16] N. Keller and N. Lifshitz. The junta method for hypergraphs and the Erdős-Chvátal simplex conjecture. *Adv. Math.*, 392:107991, 2021.

[17] J.B. Kruskal. The number of simplices in a complex. *Mathematical Optimization Techniques*, 251:251–278, 1963.

[18] R. M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica* 4: 247–257, 1984.