THE EIGENVALUE PROBLEM FOR A CLASS OF DEGENERATE OPERATORS RELATED TO THE NORMALIZED $p$-LAPLACIAN

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Abstract. In this paper, we investigate a weighted Dirichlet eigenvalue problem for a class of degenerate operators related to the $h$ degree homogeneous $p$-Laplacian

$$
\begin{aligned}
|Du|^{h-1} \Delta^N_p u + \lambda a(x)|u|^{h-1} u &= 0, \quad \text{in } \Omega, \\
 u &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
$$

Here $a(x)$ is a positive continuous bounded function in the closure of $\Omega \subset \mathbb{R}^n (n \geq 2)$, $h > 1$, $2 < p < \infty$, and $\Delta^N_p u = \frac{1}{p} |Du|^{2-p} \text{div} (|Du|^{p-2} Du)$ is the normalized version of the $p$-Laplacian arising from a stochastic game named Tug-of-War with noise. We prove the existence of the principal eigenvalue $\lambda_\Omega$, which is positive and has a corresponding positive eigenfunction for $p > n$. The method is based on the maximum principle and approach analysis to the weighted eigenvalue problem. When a parameter $\lambda < \lambda_\Omega$, we establish some existence and uniqueness results related to this problem. During this procedure, we also prove some regularity estimates including Hölder continuity and Harnack inequality.

1. Introduction. In this paper, we are interested in the weighted Dirichlet eigenvalue problem corresponding to the $h$ degree homogeneous $p$-Laplacian,

$$
|Du|^{h-1} \Delta^N_p u + \lambda a(x)|u|^{h-1} u = 0.
$$

To be more precise, consider the following problem:

$$
\begin{aligned}
|Du|^{h-1} \Delta^N_p u + \lambda a(x)|u|^{h-1} u &= 0, \quad \text{in } \Omega, \\
 u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $a(x) \in C(\Omega) \cap L^\infty(\Omega)$, $h > 1$, $2 < p < \infty$ and the normalized $p$-Laplacian is given by

$$
\Delta^N_p u := \frac{1}{p} |Du|^{2-p} \text{div} (|Du|^{p-2} Du) = \frac{1}{p} \text{Trace} \left( \frac{(p-2)Du \otimes Du}{|Du|^2} \right).
$$

Note that for $p = 2$ we get the equation $\frac{1}{2} |Du|^{h-1} \Delta u + \lambda a(x)|u|^{h-1} u = 0$. When $h = p - 1$, $|Du|^{h-1} \Delta^N_p u = \frac{1}{p} \Delta_p u$ is the variational $p$-Laplace operator (up to

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a constant) which is in divergence form. When \( h = 1 \), it is the normalized \( p \)-Laplacian \( \Delta_N^p u \) which comes from the stochastic game named Tug-of-War with noise [32]. Hence, the normalized \( p \)-Laplacian is also called the game \( p \)-Laplacian.

A Tug-of-War is a two-person, zero-sum game, i.e. two players compete and the gain of Player I equals the loss of Player II. Let \( \mu \) be a noise measure, that is, \( \mu \) is a mean zero, compactly supported probability measure on \( \mathbb{R}^n \) which is preserved by orthogonal transformations of \( \mathbb{R}^n \) which fix the first basis vector \( e_1 \). For each \( v \in \mathbb{R}^n \) and Borel-measurable set \( S \subset \mathbb{R}^n \), define \( \mu_v(S) = \mu(\Psi^{-1}(S)) \), where \( \Psi \) is a constant \( c \) times some orthonormal transformation of \( \mathbb{R}^n \), chosen so that \( \Psi(e_1) = v \).

Let \( \alpha = 1 + \inf \{ R : \mu(B(0, R)) = 1 \} \) and \( g \) be a final payoff function which is defined on the boundary \( \partial \Omega \) and continuous. The Tug-of-War with noise is played as follows: Fix an initial game state \( x_0 \in \Omega \). At the \( k \)th turn, a fair coin is tossed, and the player who wins the coin toss is allowed to make a move. If \( \text{dist}(x_{k-1}, \partial \Omega) > \alpha \varepsilon \), then the moving player chooses \( v_k \in \mathbb{R}^n \) with \( |v_k| \leq \varepsilon \) and sets \( x_k = x_{k-1} + v_k + z_k \), where \( z_k \) is a random noise vector sampled from \( \mu_{v_k} \). If \( \text{dist}(x_{k-1}, \partial \Omega) \leq \alpha \varepsilon \), then the moving player chooses an \( x_k \in \partial \Omega \) with \( |x_k - x_{k-1}| \leq \alpha \varepsilon \) and the game ends, with player I receiving a payoff of \( g(x_k) \) from player II. In [32], Peres and Sheffield showed that the Tug-of-War game with noise defined as above has a value \( u_\varepsilon \) which converges uniformly in \( \Omega \) to a function \( u \in C(\Omega) \) and \( u \) is the unique viscosity solution to

\[
\begin{align*}
\Delta_N^p u &= 0, \quad \text{in } \Omega, \\
u &= g, \quad \text{on } \partial \Omega.
\end{align*}
\]

The motivation to study the normalized \( p \)-Laplacian operator \( \Delta_N^p u \) stems partially from their connections to stochastic games and their applications to image processing. Note that \( \Delta_N^p u \) is homogeneous of degree 1, in contrast to the variational \( p \)-Laplacian, which is homogeneous of degree \( (p - 1) \). This leads to the fact that parabolic PDEs involving the game \( p \)-Laplacian are scaling invariant, which is especially useful for many applications in mathematical image processing. In [10], Does studied the corresponding parabolic operator \( u_t - \Delta_N^p u \) and applied it to the image processing. He showed that the brightness of an initial image does not affect the evolution process; but the choice of the parameter \( p \) affects in which direction the brightness evolves. For more applications in image processing, one can see [11, 12] etc. For more results about the normalized \( p \)-Laplacian and related topics about tug-of-war, one can see [19, 20, 21, 27, 28, 29] and the references therein.

Notice that

\[
\Delta_N^p u = \frac{p-2}{p} \Delta_N u + \frac{1}{p} \Delta u,
\]

where the normalized infinity Laplacian

\[
\Delta_N^\infty u := \text{Trace} \left( \frac{(p-2)Du \otimes Du}{|Du|^2} D^2 u \right) = \left( \frac{D^2 u Du}{|Du|} \right) \cdot \frac{Du}{|Du|}
\]

is closely related to \( L^\infty \)-variational problem and tug-of-war game, see for instance [1, 9, 22, 23, 24, 25, 30, 31] etc.

Here we analyze the case for general \( h > 1 \). It should be pointed out that, for the case \( h > 1 \), the operator \( |Du|^{h-1} \Delta_N^p u \) is highly degenerate when the gradient vanishes. Our main goal is to obtain the existence of the first eigenvalue and the corresponding positive eigenfunction for the weighted eigenvalue problem (1.2). The \( h \) degree homogeneous operator \( |Du|^{h-1} \Delta_N^p u \) is of intrinsic interest, because it is not only degenerate, but also has no variational structure and it is not in divergence form. Since the operator \( |Du|^{h-1} \Delta_N^p u \) is in non-divergence form and not even
defined at the critical points of $u$, the solutions throughout this paper are understood in the viscosity sense.

In this work, there are three main difficulties in the equation (1.1). One is the degeneracy of the generalized $p$-Laplace operator $|Du|^{h-1} \Delta_p^N u$ itself. The high degeneracy causes the difficulty to study the comparison principle for viscosity solutions of the equation. In order to overcome this difficulty, we take $U = \log u$ ($u > 0$) to transform the equation (1.1) to the equation

$$|DU(x)|^{h-1} \Delta_p^N U(x) + \frac{p-1}{p}|DU(x)|^{h+1} + \lambda a(x) = 0, \quad (1.3)$$

so that Jensen’s method [16] of the comparison principle can be carried out in the usual way. Another is to establish the uniform estimates which will allow us to use compactness arguments. To obtain the uniform estimate, we construct “suitable” barriers and establish the key estimate (3.1) in Section 3. The other difficulty is due to the non-divergence form of the operator $|Du|^{h-1} \Delta_p^N u$ which causes the argument based on variational methods does not work any more. In order to deal with this difficulty, we adopt the idea proposed by Berestycki, Nirenberg and Varadhan to characterize the principal eigenvalue based on the maximum principle to deal with this difficulty, we adopt the idea proposed by Berestycki, Nirenberg and Varadhan to characterize the principal eigenvalue based on the maximum principle.

Following the definition as in [3, 6, 33], we let

$$\lambda_\Omega := \sup \{ \lambda : \exists v > 0 \in \overline{\Omega} \text{ such that } |Dv|^{h-1} \Delta_p^N v + \lambda a(x)|v|^{h-1}v \leq 0 \}. \quad (1.4)$$

Because $v(x) \equiv \|a(x)\|_{L^\infty(\Omega)}^{1/h} > 0$ is a supersolution with $\lambda = -1/\|a(x)\|_{L^\infty(\Omega)}$, $\lambda_\Omega$ is well defined. Our main aim is to prove that $\lambda_\Omega$ is really the smallest Dirichlet eigenvalue of the operator $-|Du|^{h-1} \Delta_p^N u$ in $\Omega$. To do so, now we first give the following maximum principle when the parameter $\lambda$ is less than $\lambda_\Omega$.

**Theorem 1.1.** Let $h > 1$, $2 \leq p < \infty$, $\Omega$ be a bounded domain, a positive function $a(x) \in C(\Omega) \cap L^\infty(\Omega)$ and $\lambda < \lambda_\Omega$. Suppose that $u \in C(\overline{\Omega})$ satisfies

$$|Du(x)|^{h-1} \Delta_p^N u(x) + \lambda a(x)|u(x)|^{h-1}u(x) \geq 0, \forall x \in \Omega$$

in the viscosity sense. If $u \leq 0$ on $\partial \Omega$, then there holds $u \leq 0$ in $\Omega$.

One should notice that Theorem 1.1 gives $\lambda_\Omega$ a characterization, that is $\lambda_\Omega$ is the supremum of the real numbers $\lambda$ such that the operator $|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1}u$ satisfies the maximum principle in $\Omega$. Due to the high degeneracy of the equation, we first use the logarithmic transformation $U = \log u$ ($u > 0$) to turn the equation (1.2) into (1.3). And then we can use the perturbation argument of viscosity solutions to compare super and subsolutions. Based on the perturbation idea, we establish a series of comparison results related to the weighted eigenvalue problem in Section 2. Then using the characterization of the maximum principle of the operator $|Du|^{h-1} \Delta_p^N u$ and the Arzelà-Ascoli compactness criteria, we get the following existence result of the positive eigenfunction corresponding to the principal eigenvalue $\lambda_\Omega$.

**Theorem 1.2.** Let $h > 1$, $2 \leq n < p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded domain and a positive function $a(x) \in C(\Omega) \cap L^\infty(\Omega)$. Then there exists a positive viscosity
solution \( u \in C(\overline{\Omega}) \) satisfying

\[
\begin{aligned}
&Du^{h-1} N_p u + \lambda_\Omega a(x) |u|^{h-1} u = 0, \quad \text{in } \Omega, \\
&u = 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

Moreover, \( u \) is locally Hölder continuous with exponent \( \frac{p-n}{p-1} \).

We remark that the above theorem shows that \( \lambda_\Omega \) defined as in (1.4) is the first Dirichlet eigenvalue of the weighted eigenvalue problem. In order to obtain the existence result we need to establish the Hölder estimate (see the Theorem 3.3 below) which is a direct result of the Lemma 3.2. It should be pointed out that the key estimate (3.1) in Section 3 depends heavily on the construction of the barrier function. Therefore, we obtain the existence result for \( p > n \) here. Note that it would be interesting to consider the case for \( 2 < p \leq n \).

The method we use in the proof of Theorem 1.2 is based on the Perron’s method and the compactness analysis as in [3] for the uniformly elliptic linear operators. See also [4, 5, 6, 7, 23, 24, 26, 33, 34] etc. It should be pointed out that the “good” structure of the operator makes it possible that we can find an appropriate viscosity supersolution. We remark that due to the high degeneracy of \( |Du|^{h-1} N_p u \) (non-degenerate only in the direction of the gradient of \( u \)), we do not know whether it admits sign changing solutions corresponding to the eigenvalue \( \lambda_\Omega \). In [5, 6], Birindelli and Demengel obtain the existence of the first eigenvalue \( \lambda_{1,p} \) of the following eigenvalue problem

\[
\begin{aligned}
&\Delta_p^N u + \lambda u = 0, \quad \text{in } \Omega, \\
&u = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

which is the case \( h = 1 \) and \( a(x) \equiv 1 \) in (1.2). In [26], the authors prove that \( \lambda_{1,p} \) converges to the first eigenvalue corresponding to the limiting operator \( \Delta_\infty^N u \). In [13], the authors prove the interior Hölder estimates for the spatial gradients of the viscosity solutions to the parabolic equation

\[
u_t = |Du|^{\gamma} \Delta_p^N u,
\]

where \( \gamma > -1 \).

Note that the operator \( |Du|^{h-1} N_p u \) tends to \( |Du|^{h-1} \Delta_\infty^N u \) as \( p \to \infty \), and the Dirichlet eigenvalue problem for the limit operator was studied in [23]. It would be interesting to consider the relationship of the first eigenvalue between these two operators.

The organization of this paper is as follows. In Section 2, we give some necessary definitions and prove the maximum principle and comparison principles of the viscosity solutions by the perturbation method. In Section 3, for the case \( p > n \), we will establish some necessary regularity estimates including Harnack inequality and Hölder continuity of a nonnegative viscosity supersolution to the inhomogeneous equation \( |Du|^{h-1} N_p u + \lambda a(x) |u|^{h-1} u = f \), where \( f \) is a nonpositive continuous function. The idea is to construct a fine barrier function and then the key estimate (3.1) follows. In Section 4, for any \( 2 \leq n < p < \infty \), we first establish the existence of positive viscosity solution of the approximate problem by Perron’s method. And then the main existence result, Theorem 1.2, is proved by the approximating procedure.
2. Maximum Principle and Comparison Results. In this section, we give the definition of viscosity solutions of equation (1.1). A transformation of equation (1.1) is introduced together with its viscosity solutions. Finally, we establish the comparison results which are needed to verify the validity of $\lambda$. Noting that for $h > 1$ the operator $|Du|^{h-1}\Delta_p^N u$ is continuous, we can adopt the standard definition of viscosity solutions [8, 14, 15] etc. First, we rewrite the equation (1.1) as

$$F_h(D^2u, Du) + \lambda a(x)|u|^{h-1}u = 0, \quad x \in \Omega,$$

where $F_h : \mathbb{S} \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ and

$$F_h(M, q) := \frac{p-2}{p}|q|^{h-3}(Mq) \cdot q + \frac{1}{p}|q|^{h-1}\text{Trace}(M).$$

Here $\mathbb{S}$ denotes the set of $n \times n$ real symmetric matrices. Since $h > 1$, we have $\lim_{h \to 0} F_h(M, q) = 0$ for arbitrary $M \in \mathbb{S}$. Therefore, we introduce the continuous extension of $F_h$ as follows,

$$F_h(M, q) := \begin{cases} F_h(M, q), & \text{if } q \neq 0, \\ 0, & \text{if } q = 0. \end{cases}$$

This means that the singularity can be removed for $h > 1$. Now we can give the definition of viscosity solutions by the usual theory.

**Definition 2.1.** Let $h > 1, 2 \leq p < \infty, \Omega \subset \mathbb{R}^n$ be a bounded domain and a positive function $a(x) \in C(\Omega) \cap L^\infty(\Omega)$. An upper semi-continuous function $u : \Omega \to \mathbb{R}$ is called a viscosity subsolution of equation (2.1) in $\Omega$ if, whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u(x_0) = \varphi(x_0)$ and $u(x) < \varphi(x)$ for all $x \in \Omega$ with $x \neq x_0$, then

$$F_h(D^2\varphi(x_0), D\varphi(x_0)) + \lambda a(x_0)|\varphi(x_0)|^{h-1}\varphi(x_0) \geq 0. \quad (2.3)$$

Similarly, a lower semi-continuous function $u : \Omega \to \mathbb{R}$ is called a viscosity supersolution of equation (2.1) in $\Omega$ if, whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u(x_0) = \varphi(x_0)$ and $u(x) > \varphi(x)$ for all $x \in \Omega$ with $x \neq x_0$, then

$$F_h(D^2\varphi(x_0), D\varphi(x_0)) + \lambda a(x_0)|\varphi(x_0)|^{h-1}\varphi(x_0) \leq 0. \quad (2.4)$$

A continuous function $u$ is called a viscosity solution of equation (2.1) if it is at the same time a viscosity subsolution and a viscosity supersolution.

One should notice that the contact condition in the above definition can be relaxed to hold only in a neighborhood of $x_0$. One can also give the equivalent definition using semi-jets. See [8] for more details about general theory of viscosity solutions.

If $u$ is the viscosity solution (supersolution, or subsolution) in Definition 2.1, we say that it satisfies $F_h(D^2u, Du) + \lambda a(x)|u|^{h-1}u = (\leq, \geq)0$ in the viscosity sense.

Before establishing the maximum principle, we introduce a transformation of the original equation (1.1), namely, if $u > 0$ solves the differential equation (1.1) and $U = \log u$, then a simple calculation yields

$$|DU(x)|^{h-1}\Delta_p^N U(x) + \frac{p-1}{p}|DU(x)|^{h+1} + \lambda a(x) = 0.$$
and
\[
F_h \left( D^2 \varphi(x_0), D\varphi(x_0) \right) + \frac{p - 1}{p} |D\varphi(x_0)|^{p+1} + \lambda a(x_0) \leq 0,
\]
respectively, we get the definition of viscosity solutions of equation (1.3).

**Remark 2.1.** If \( u \) is positive in \( \Omega \), one can easily check that \( u \) is a viscosity solution of equation (1.1) in \( \Omega \) if and only if \( U = \log u \) is a viscosity solution of equation (1.3) in \( \Omega \).

Next, we will give the following fundamental comparison principle by the perturbation argument in [8] for a slightly more general equation allowing for the first order term
\[
|Du|^{h-1} \Delta_p^N u + G(Du) + H(x, u) = 0, \quad x \in \Omega,
\]
where \( H : \Omega \times \mathbb{R} \to \mathbb{R} \) is continuous and \( G(q) : \mathbb{R}^n \to \mathbb{R} \) is a continuous function satisfying \( G(0) = 0 \). Note that one can define the viscosity solutions of equation (2.5) similar to Definition 2.1.

**Theorem 2.1.** Let \( h > 1, 2 \leq p < \infty, G(q) \in C(\mathbb{R}^n), G(0) = 0, \Omega \subset \mathbb{R}^n \) be a bounded domain, \( H \in C(\Omega \times \mathbb{R}) \) and \( K \in C(\Omega \times \mathbb{R}) \). Suppose that \( u \in C(\overline{\Omega}) \) and \( v \in C(\overline{\Omega}) \) satisfy
\[
|Du(x)|^{h-1} \Delta_p^N u(x) + G(Du(x)) + H(x, u(x)) \geq 0, \quad \forall x \in \Omega
\]
and
\[
|Dv(x)|^{h-1} \Delta_p^N v(x) + G(Dv(x)) + H(x, v(x)) \leq 0, \quad \forall x \in \Omega
\]
in the viscosity sense. If \( \sup_{\Omega} (u - v) > \sup_{\partial\Omega} (u - v) \), then there is a point \( x_0 \in \Omega \) such that
\[
(u - v)(x_0) = \sup_{\Omega} (u - v) \quad \text{and} \quad H(x_0, u(x_0)) \geq K(x_0, v(x_0)).
\]

**Proof.** Suppose that there exists an interior point \( x_0 \in \Omega \) such that
\[
M := u(x_0) - v(x_0) = \sup_{\Omega} (u - v) > \sup_{\partial\Omega} (u - v).
\]
We consider
\[
w_j(x, y) = u(x) - v(y) - \frac{j}{4} |x - y|^4, \quad j \in \mathbb{N}, \quad (x, y) \in \Omega \times \Omega.
\]
Denote by \((x_j, y_j)\) the maximum point of \( w_j \) over \( \overline{\Omega} \times \overline{\Omega} \) and \( M_j = w_j(x_j, y_j) \).

Noticing (2.6) and Proposition 3.7 in [8], we get
\[
\lim_{j \to \infty} M_j = \lim_{j \to \infty} \left( u(x_j) - v(y_j) - \frac{j}{4} |x_j - y_j|^4 \right) = M
\]
and
\[
\lim_{j \to \infty} \frac{j}{4} |x_j - y_j|^4 = 0.
\]
Then we have
\[
x_j \to x_0 \quad \text{and} \quad y_j \to x_0, \quad \text{as} \quad j \to \infty,
\]
and \( x_j, y_j, x_0 \) are interior points of \( \Omega \) for \( j \) large enough.

Denote
\[
\varphi(x) = \frac{j}{4} |x - y_j|^4, \quad \phi(y) = -\frac{j}{4} |x_j - y|^4.
\]
Then one can check that the functions \( u - \varphi \) and \( v - \phi \) have a local maximum at \( x_j \) and a local minimum at \( y_j \) respectively. From now on, we will consider only \( j \) large enough and divide the proof into two cases: either \( x_j \neq y_j \) or \( x_j = y_j \).
Case 1: If \( x_j = y_j \), it is obvious that \( D \varphi(x_j) = 0 \), \( D^2 \varphi(x_j) = 0 \), \( D \phi(y_j) = 0 \) and \( D^2 \phi(y_j) = 0 \). Since \( u \) and \( v \) are viscosity subsolution and supersolution respectively, there holds
\[
H(x_j, u(x_j)) \geq 0 \geq K(y_j, v(y_j)).
\]
Passing to the limit as \( j \to \infty \), we have \( H(x_0, u(x_0)) \geq K(x_0, v(x_0)) \).

Case 2: If \( x_j \neq y_j \), we use jets and maximum principle for semi-continuous functions of sums for \( w \) which means that there exist \( n \times n \) symmetric matrices \( X_j \) and \( Y_j \) such that \( Y_j - X_j \) positive semi-definite, that is \( Y_j - X_j \geq 0 \), and
\[
\begin{align*}
(j|x_j - y_j|^2(x_j - y_j), X_j) & \in J^{2,+}u(x_j), \\
(j|x_j - y_j|^2(x_j - y_j), Y_j) & \in J^{2,-}v(y_j).
\end{align*}
\]
One can see more for the notation and relevant definitions [8]. Since \( u \) and \( v \) are a viscosity subsolution and a supersolution respectively, we can conclude that
\[
0 \leq j^{h-1}|x_j - y_j|^{3h-3} \left( \frac{p-2}{p} \left\langle X_j \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \right\rangle + \frac{1}{p} \text{Trace}(X_j) \right) + G \left( j|x_j - y_j|^2(x_j - y_j) \right) + H(x_j, u(x_j)) \]
\[
\leq j^{h-1}|x_j - y_j|^{3h-3} \left( \frac{p-2}{p} \left\langle Y_j \frac{x_j - y_j}{|x_j - y_j|}, \frac{x_j - y_j}{|x_j - y_j|} \right\rangle + \frac{1}{p} \text{Trace}(Y_j) \right) + G \left( j|x_j - y_j|^2(x_j - y_j) \right) + K(y_j, v(y_j)) + H(x_j, u(x_j)) - K(y_j, v(y_j)),
\]
where we have used \( Y_j - X_j \geq 0 \). We let \( j \to \infty \) to obtain \( H(x_0, u(x_0)) \geq K(x_0, v(x_0)) \). \( \square \)

Now we will utilize Theorem 2.1 and logarithmic transformation to establish the comparison results related to the eigenvalue problems (1.2).

**Theorem 2.2.** Let \( h > 1 \), \( 2 \leq p < \infty \), \( \Omega \subset \mathbb{R}^n \) be a bounded domain and a positive function \( \alpha(x) \in C(\Omega) \cap L^\infty(\Omega) \) and \( \lambda < \mu \) be real numbers. Suppose that \( u \in C(\overline{\Omega}) \), \( v \in C(\overline{\Omega}) \) and \( v > 0 \) satisfy
\[
|Du(x)|^{h-1} \Delta_p^N u(x) + \lambda \alpha(x)|u(x)|^{h-1}u(x) \geq 0, \quad \forall x \in \Omega
\]
and
\[
|Dv(x)|^{h-1} \Delta_p^N v(x) + \mu \alpha(x)|v(x)|^{h-1}v(x) \leq 0, \quad \forall x \in \Omega
\]
in the viscosity sense. Then \( u \leq 0 \) on \( \partial \Omega \) implies \( u \leq 0 \) in \( \Omega \).

**Proof.** We argue by contradiction. Suppose \( u \) is positive somewhere in \( \Omega \). Then there exists \( x_0 \in \Omega \) such that \( u(x_0) > 0 \) and
\[
\frac{u(x_0)}{v(x_0)} = \sup \frac{u}{v} > 0 \geq \sup \frac{u}{v} \partial \Omega.
\]
It is clear that there exists some neighborhood \( \Omega_0 \subset \subset \Omega \) of \( x_0 \) such that \( u(x) > 0 \) in \( \Omega_0 \).

Set
\[
U(x) = \log u(x) \quad \text{and} \quad V(x) = \log v(x), \quad x \in \Omega_0.
\]
Obviously, \( U - V \) attains its local maximum at \( x_0 \in \Omega_0 \). That is
\[
U(x_0) - V(x_0) = \sup_{x \in \Omega_0} (U - V).
\]
Since $u$ and $v$ are a viscosity subsolution and supersolution respectively, Remark 2.1 implies that
\[
|DU(x)|^{h-1} \Delta_p^N U(x) + \frac{p-1}{p} |DU(x)|^{h+1} + \lambda a(x) \geq 0, \text{ in } \Omega_0
\]
and
\[
|DV(x)|^{h-1} \Delta_p^N V(x) + \frac{p-1}{p} |DV(x)|^{h+1} + \mu a(x) \leq 0, \text{ in } \Omega_0
\]
in the viscosity sense. Theorem 2.1 implies $\lambda \geq \mu$ which contradicts to the assumption $\lambda < \mu$.

\textbf{Proof of Theorem 1.1.} Taking $\mu = \lambda_\Omega$ in Theorem 2.2, we get Theorem 1.1 immediately.

As a result of Theorem 2.2 we have

\textbf{Corollary 2.1.} Let $h > 1$, $2 \leq p < \infty$, a positive function $a(x) \in C(\Omega) \cap L^\infty(\Omega)$ and $\lambda < \lambda_\Omega$. Suppose that $u \in C(\overline{\Omega})$ is a viscosity solution of
\[
\begin{cases}
|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u = 0, & \text{ in } \Omega, \\
u = 0, & \text{ on } \partial \Omega.
\end{cases}
\]
Then we have $u \equiv 0$ in $\Omega$.

\textbf{Proof.} Using Theorem 2.2, one has $u \leq 0$ in $\Omega$. Since the operator $|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u$ is odd, we have $-u$ is also a solution. Then the conclusion follows immediately.

\textbf{Remark 2.2.} For any $\lambda < \lambda_\Omega$, Corollary 2.1 implies that the maximum principle holds for the operator $|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u$. This means that such $\lambda$ is not an eigenvalue. In fact, if we show that $\lambda_\Omega$ is indeed an eigenvalue, then Corollary 2.1 will give us a characterization for the smallest eigenvalue. That is $\lambda_\Omega$ is the supremum of the real numbers $\lambda$ such that the maximum principle holds for the operator $|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u$ in $\Omega$.

Next we shall use Theorem 2.1 to establish the following comparison result of the inhomogeneous problem. The idea comes from [17] which deals with a problem related to the infinity Laplacian.

\textbf{Theorem 2.3.} Let $h > 1$, $2 \leq p < \infty$, a positive function $a(x) \in C(\Omega) \cap L^\infty(\Omega)$ and $f \in C(\Omega)$. Suppose that $u$ and $v$ are a viscosity subsolution and supersolution, respectively, of the equation
\[
|Dw|^{h-1} \Delta_p^N w + \lambda a(x)|w|^{h-1} w = f.
\]
Suppose that either
\[
\lambda < \lambda_\Omega \quad \text{and} \quad f(x) < 0, \quad \text{for all } x \in \Omega
\]
or
\[
0 < \lambda < \lambda_\Omega \quad \text{and} \quad f(x) \leq 0, \quad \text{for all } x \in \Omega.
\]
Suppose also $v > 0$ on $\partial \Omega$ and $u \leq v$ on $\partial \Omega$. Then we have $u \leq v$ in $\Omega$.

\textbf{Proof.} First we claim that $v > 0$ in $\Omega$. Let $w := -v$. Noting $v$ is a viscosity supersolution, we have
\[
|Dw|^{h-1} \Delta_p^N w + \lambda a(x)|w|^{h-1} w \geq -f(x) \geq 0, \quad x \in \Omega
\]
and \(w \leq 0\) on \(\partial \Omega\). Then one has \(w \leq 0\) in \(\Omega\) by Theorem 1.1. If \(\lambda \leq 0\) and there exists \(x_0 \in \Omega\) such that \(v(x_0) = 0\), since \(\varphi(x) = 0\) is a test function, then one has \(f(x_0) \geq 0\) which is a contradiction. Hence \(v > 0\) in \(\Omega\). If \(\lambda > 0\), we have the bounded viscosity supersolutions \(v\) of \([Dv]^{h-1} \Delta \phi \leq 0\) is indeed weak supersolutions of the standard \(p\)-Laplacian [18]. Therefore, the strong minimum principle also implies \(v > 0\) in \(\Omega\).

Now for the case \(f < 0\), we will prove \(u \leq v\) in \(\Omega\). Arguing by contradiction, we assume that there exists \(x_0 \in \Omega\) such that

\[
1 < \frac{u(x_0)}{v(x_0)} = \sup_{x \in \Omega} \frac{u(x)}{v(x)}. \tag{2.7}
\]

Then there exists some neighborhood \(\Omega_0\) of \(x_0\) such that \(u > v > 1\) in \(\overline{\Omega}_0\) (otherwise we can rescale \(f\)).

Set \(U(x) = \log u(x)\) and \(V(x) = \log v(x)\) for \(x \in \Omega_0\). Due to (2.7), we have

\[
0 < U(x_0) - V(x_0) = \sup_{x \in \overline{\Omega}_0} (U - V).
\]

Again Remark 2.1 implies

\[
|DU(x)|^{h-1} \Delta \phi U(x) + \frac{p-1}{p} |DU(x)|^{h+1} + \lambda a(x) - f(x)e^{-hU(x)} \geq 0, \quad \text{in} \quad \Omega_0
\]

and

\[
|DV(x)|^{h-1} \Delta \phi V(x) + \frac{p-1}{p} |DV(x)|^{h+1} + \lambda a(x) - f(x)e^{-hV(x)} \leq 0, \quad \text{in} \quad \Omega_0
\]

in the viscosity sense. Applying Theorem 2.1 with \(G(q) = \frac{p-1}{p} |q|^{h+1}, H(x, U(x)) = \lambda a(x) - f(x)e^{-hU(x)}\) and \(K(x, V(x)) = \lambda a(x) - f(x)e^{-hV(x)}\), one can get that

\[
f(x_0)e^{-hU(x_0)} \leq f(x_0)e^{-hV(x_0)}.
\]

Since \(f < 0\) in \(\Omega\), we have \(U(x_0) \leq V(x_0)\) which is also a contradiction.

For the case \(f \leq 0\), we first perturb \(v\) so that it becomes a strict viscosity supersolution. Set

\[
w = v - m \alpha, \quad \text{in} \quad \Omega_0,
\]

where \(0 < m = \inf_{\Omega_0} v\) and \(0 < \alpha < 1\) to be determined.

It is obvious \(w \geq m(1 - \alpha) > 0\) in \(\Omega_0\). Due to (2.7), there exists an interior point \(x_0\) such that

\[
1 < \frac{u(x_0)}{w(x_0)} = \sup_{x \in \Omega_0} \frac{u(x)}{w(x)} \tag{2.8}
\]

for \(\alpha\) sufficiently close to 0. Denoting \(M = \sup_{\Omega_0} v\), then for \(\mu > 0\), by direct calculation we have

\[
|Dw(x)|^{h-1} \Delta \phi w(x) + \mu a(x)w(x)h^{-1}w(x) = |Dv(x)|^{h-1} \Delta \phi v(x) + \mu a(x)(v - m\alpha)^h
\]

\[
\leq f(x) - \lambda a(x)e^h(x) + \mu a(x)(v - m\alpha)^h
\]

\[
\leq f(x) + a(x)e^h(x)\left(\mu(1 - m\alpha/M)^h - \lambda\right)
\]

where we have chosen \(\lambda < \mu < \lambda(1 - m\alpha/M)^{-h}\).

Set \(W(x) = \log w(x)\) for all \(x \in \Omega_0\). We have

\[
|DW(x)|^{h-1} \Delta \phi W(x) + \frac{p-1}{p} |DW(x)|^{h+1} + \mu a(x) - f(x)e^{-hW(x)} \leq 0, \quad \text{in} \quad \Omega_0
\]
in the viscosity sense and
\[ 0 < U(\hat{x}_0) - W(\hat{x}_0) = \sup_{x \in \Omega_0} (U - W) \]
by (2.8). Using Theorem 2.1 once again, we get
\[ \lambda a(\hat{x}_0) - f(\hat{x}_0) e^{-hU(\hat{x}_0)} \geq \mu a(\hat{x}_0) - f(\hat{x}_0) e^{-hW(\hat{x}_0)}. \]
Therefore,\[
\lambda a(\hat{x}_0) - f(\hat{x}_0) e^{-hW(\hat{x}_0)} - e^{-hU(\hat{x}_0)} \geq (\mu - \lambda)a(\hat{x}_0) > 0,
\]
which is a contradiction to the assumption \( f \leq 0 \) in \( \Omega \).

**Remark 2.3.** When \( \lambda = 0 \), the restriction condition of \( v > 0 \) on \( \partial \Omega \) can be dropped since a constant can be added to the viscosity supersolution \( v \).

The comparison principle yields the following uniqueness result immediately.

**Theorem 2.4.** Let \( h > 1, 2 \leq p < \infty, \) a positive function \( a(x) \in C(\Omega) \cap L^{\infty}(\Omega) \), \( f \in C(\Omega) \), \( g \in C(\partial \Omega) \) and \( g \) is positive. Suppose either
\[ f < 0 \quad \text{and} \quad \lambda < \lambda_{\Omega}\]
or
\[ f \leq 0 \quad \text{and} \quad 0 < \lambda < \lambda_{\Omega}. \]
Then the Dirichlet problem
\[
\begin{cases}
|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} = f, & \text{in } \Omega, \\
u = g, & \text{on } \partial \Omega
\end{cases}
\] (2.9)
has at most one viscosity solution.

**Remark 2.4.** Since the operator \( |Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} \) is odd, similar to Theorem 2.3, one can also prove the comparison principle for viscosity solutions, and get the uniqueness of the viscosity solutions of (2.9) when \( g \) is negative and either
\[ f > 0 \quad \text{and} \quad \lambda < \lambda_{\Omega}\]
or
\[ f \geq 0 \quad \text{and} \quad 0 < \lambda < \lambda_{\Omega}. \]

Arguing as in the proof of Theorem 2.3, we can also establish the following comparison result related to the eigenvalue problem.

**Theorem 2.5.** Let \( h > 1, 2 \leq p < \infty, \lambda > 0 \) be a real number and a positive function \( a(x) \in C(\Omega) \cap L^{\infty}(\Omega) \). Let \( u \in C(\Omega) \) and \( v \in C(\partial \Omega) \) be a viscosity subsolution and supersolution, respectively, of the equation
\[ |Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} = 0. \]
Suppose also \( v > 0 \) on \( \partial \Omega \). If \( u \) is positive somewhere in \( \Omega \), then one must have
\[ \sup_{\Omega} u = \sup_{\partial \Omega} v. \]
where we have used $\gamma > \text{dist}({\mathcal{O}})$, which will allow us to have a compactness criteria that will be useful in next section.

We also establish several regularity results including Hölder estimate and barrier functions based on the special structure of the operator. Besides the Harnack inequality, we also establish several regularity results including Hölder estimate which will allow us to have a compactness criteria that will be useful in next section.

Now we denote $\beta := \frac{p-n}{p-1}$ $(p > n \geq 2)$ and consider

$$
\Phi(x) := \left(r^\beta - |x - x_0|^\beta \right)^\gamma, \quad x \in B_r(x_0),
$$

where $\gamma > 1$ is constant. It is obvious that $\Phi(x)$ is positive in $B_r(x_0)$ and vanishes on $\partial B_r(x_0)$. Letting $s = |x - x_0|$ and $\Phi(s) := (r^\beta - s^\beta)^\gamma$, then for $x \neq x_0$, a direct calculation leads to

$$
D\Phi(x) = \Phi'(s) \frac{x-x_0}{|x-x_0|}.
$$

and

$$
D^2\Phi(x) = \Phi''(s) \frac{(x-x_0) \otimes (x-x_0)}{|x-x_0|^2} + \Phi'(s) \left( \frac{1}{|x-x_0|} - \frac{(x-x_0) \otimes (x-x_0)}{|x-x_0|^3} \right),
$$

where $\otimes$ denotes the tensor product. Hence, we have

$$
|D\Phi(x)|^{h-1} \Delta_p N \Phi(x) = |D\Phi(x)|^{h-1} \left( \frac{p-2}{p} \Delta_p U \Phi(x) + \frac{1}{p} \Delta \Phi(x) \right)
$$

$$
= |\Phi'(s)|^{h-1} \left( \frac{p-2}{p} \Phi''(s) + \frac{1}{p} \left( \Phi''(s) + \frac{n-1}{s} \Phi'(s) \right) \right)
$$

$$
= |\Phi'(s)|^{h-1} \left( \frac{p-1}{p} \Phi''(s) + \frac{n-1}{p} \cdot \frac{1}{s} \Phi'(s) \right)
$$

$$
> 0
$$

where we have used $\gamma > 1$.

In particular, we have proven the following lemma:

**Lemma 3.1.** Let $h > 1$, $p > n \geq 2$, $r > 0$, $x_0 \in \mathbb{R}^n$, and $\gamma > 1$. Then for any $x \in B_r(x_0)$ and $x \neq x_0$, $\Phi(x) = (r^\beta - |x - x_0|^\beta)^\gamma$ is a strict viscosity subsolution of $|Du|^{h-1} \Delta_p N u = 0$.

*Proof.* The fact that a classical subsolution is a viscosity subsolution follows easily from the definition of a viscosity subsolution. \qed

Now we use the perturbation argument to establish the following key estimate.

**Lemma 3.2.** Let $h > 1$, $p > n \geq 2$, $\lambda > 0$, $a(x) \geq 0$ and $f \leq 0$ be continuous in a domain $\Omega$. Suppose that $u \in C(\Omega)$ is a nonnegative viscosity supersolution of $|Du|^{h-1} \Delta_p N u + \lambda a(x)|u|^{h-1} u = f$ in $\Omega$. Letting $x_0 \in \Omega$, $u(x_0) > 0$ and $0 < r \leq \text{dist}(x_0, \partial \Omega)$, we have

$$
u(x) \geq \varphi(x) := \frac{u(x_0)}{r^\beta} \left(r^\beta - |x - x_0|^\beta \right), \quad x \in B_r(x_0).
$$

Furthermore, we have

$$
u(x) \geq u(x_0)(1 - 1/2^\beta)
$$

for all $x \in B_{r/2}(x_0)$. \qed
There exists a point $B_\Omega x$ such that $u(x) = \varphi(x) = \inf_{B_\Omega(x)} (u - \varphi) < 0$. 

Since $u(x_0) = \varphi(x_0)$ and $u(x) \geq \varphi(x)$ on $\partial B_\Omega(x_0)$, we have $x_s \neq x_0$.

Denote 
$$\theta(x) := \varphi(x)^\gamma = \left(\frac{u(x_0)}{r^\beta}\right)^\gamma (r^\beta - |x - x_0|^\beta)^\gamma = \left(\frac{u(x_0)}{r^\beta}\right)^\gamma \Phi(x), \quad x \in B_\Omega(x_0).$$

Noting (3.2), we can choose $\gamma > 1$ sufficiently close to 1 such that $u - \theta$ attains its negative minimum at the interior point $x_s^\gamma$ and $x_s^\gamma \neq x_0$. Since $\theta$ is smooth in a neighborhood of $x_s^\gamma$, it can be viewed as a test function of $u$ at $x_s^\gamma$. By the definition of viscosity supersolution of $u$, we obtain 

$$|D\theta(x_s^\gamma)|^{h-1} \Delta_p^N \theta(x_s^\gamma) \leq |D\theta(x_s^\gamma)|^{h-1} \Delta_p^N \theta(x_s^\gamma) + \lambda a(x_s^\gamma) |\theta(x_s^\gamma)|^{h-1} \theta(x_s^\gamma)$$

$$= \left(\frac{u(x_0)}{r^\beta}\right)^{\gamma h} [D\Phi(x_s^\gamma)]^{h-1} \Delta_p^N \Phi(x_s^\gamma) + \lambda a(x_s^\gamma) |\Phi(x_s^\gamma)|^{h-1} \Phi(x_s^\gamma)]$$

$$\leq f(x_s^\gamma) \leq 0.$$

Lemma 3.1 implies $|D\Phi(x_s^\gamma)|^{h-1} \Delta_p^N \Phi(x_s^\gamma) > 0$, which leads to a contradiction. $\square$

An immediate consequence of Lemma 3.2 is the following strict minimum principle. Other strong minimum(maximum) principles have been established in [2, 6] for fully nonlinear operators.

**Theorem 3.1.** Let $h > 1$, $p > n \geq 2$, $\lambda \geq 0$, $a(x) \geq 0$ and $f \leq 0$ be continuous in $\Omega$. Suppose that $u \in C(\Omega)$ is a nonnegative viscosity supersolution of $|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u = f$ in $\Omega$. Then either $u \equiv 0$ or $u > 0$ in $\Omega$.

**Proof.** Denote $S = \{x \in \Omega : u(x) > 0\}$. It is clear that $S$ is open. Let $y$ be a limit point of $S$. Then either $y \in \partial \Omega$ or $y \in S$. If $y \in \Omega$, then there exists a sufficiently small ball $B_r(y)$ which is compactly contained in $\Omega$. Since $y$ is a limit point of $S$, there exists a point $z \in B_{r/4}(y) \cap S$. Due to $u(z) > 0$, $y \in B_{r/4}(z) \subset B_{r/2}(y)$ and $B_{r/2}(z) \subset B_r(y)$, Lemma 3.2 implies that $u(y) \geq u(z)(1 - 1/2^\beta) > 0$. That is $y \in S$. Hence $S$ is both open and closed. Since $\Omega$ is connected, we have $S = \Omega$. $\square$

Although the operator $|Du|^{h-1} \Delta_p^N u$ is degenerate elliptic, we will show its solutions enjoy the Harnack inequality as well.

**Theorem 3.2.** Let $h > 1$, $p > n \geq 2$, $\lambda \geq 0$, $a(x) \geq 0$ and $f \leq 0$ be continuous in $\Omega$. Suppose that $u$ is a nonnegative viscosity supersolution of $|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u = f$ in $\Omega$. Letting $x_0 \in \Omega$, $u(x_0) > 0$ and $0 < r \leq \text{dist}(x_0, \partial \Omega)$, we have

$$\inf_{B_{r/2}(x_0)} u(x) \geq \left(\sup_{B_{r/2}(x_0)} u(x)\right) \cdot \left(1 - 1/2^\beta\right)^3.$$

**Proof.** Denote $u(x_s) = \inf_{B_{r/2}(x_0)} u$. Lemma 3.2 implies

$$u(x_*) \geq u(x_0) \cdot (1 - 1/2^\beta).$$

For arbitrary $x \in B_{r/2}(x_0)$, letting $\tau = \frac{x_0 + x}{2}$, it is obvious that $x_0 \in B_{r/4}(\tau)$ and $\tau \in B_{r/4}(x)$. Using Lemma 3.2 once again, we have

$$u(x_0) \geq u(\tau) \cdot (1 - 1/2^\beta)$$
and
\[ u(\overline{y}) \geq u(x) \cdot (1 - 1/2^\beta) . \]
With these three inequalities in hand, we get
\[ u(x_*) \geq u(x) \cdot (1 - 1/2^\beta)^3 \]
for \( \forall x \in B_{r/2}(x_0) \). This clearly implies
\[ u(x_*) \geq \left( \sup_{B_{r/2}(x_0)} u(x) \right) \cdot (1 - 1/2^\beta)^3 . \]

Now we use Lemma 3.2 to establish the local Hölder estimate. The main interest
of these estimates is that they are stable in \( p \), and apply to the whole range \( n < p < \infty \) with all the parameters involved varying continuously.

**Theorem 3.3.** Let \( h > 1 \), \( p > n \geq 2 \), \( \lambda \geq 0 \), \( a(x) \geq 0 \) and \( f \leq 0 \) be continuous in \( \Omega \). Suppose that \( u \) is a nonnegative viscosity supersolution of \( |Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u = f \) in \( \Omega \). Letting \( y \in \Omega \) and \( r = \text{dist}(y, \partial\Omega) > 0 \), then for all \( x \in B_{r/4}(y) \), one has
\[ |u(x) - u(y)| \leq \frac{\sup_{\Omega} u}{1 - 1/2^\beta} \cdot \frac{|x - y|^\beta}{r^\beta} . \]

**Proof.** For arbitrary \( x \in B_{r/4}(y) \), by (3.1), we have
\[ u(x) - u(y) \geq \frac{u(y)}{r^\beta} \left( r^\beta - |x - y|^\beta \right) - u(y) = -\frac{u(y)}{r^\beta} |x - y|^\beta . \]
Since \( y \in B_{r/2}(x) \subset B_r(x) \), another application of (3.1) to \( B_{r/2}(x) \) implies that
\[ u(y) - u(x) \geq -\frac{u(x)}{r^\beta} |x - y|^\beta . \]
From the above two inequalities, we can conclude that
\[ -\frac{u(y)}{r^\beta} |x - y|^\beta \leq u(x) - u(y) \leq \frac{u(x)}{r^\beta} |x - y|^\beta . \]  
(3.3)

Noting \( y \in B_{r/2}(x) \), Lemma 3.2 implies that
\[ u(y) \geq u(x) \cdot (1 - 1/2^\beta) . \]  
(3.4)

Taking into account (3.3) and (3.4), we have
\[ -u(y) \cdot \frac{|x - y|^\beta}{r^\beta} \leq u(x) - u(y) \leq \frac{u(x)}{1 - 1/2^\beta} \cdot \frac{|x - y|^\beta}{r^\beta} . \]

Therefore, we have
\[ |u(x) - u(y)| \leq \frac{u(y)}{1 - 1/2^\beta} \cdot \frac{|x - y|^\beta}{r^\beta} \leq \frac{\sup_{\Omega} u}{1 - 1/2^\beta} \cdot \frac{|x - y|^\beta}{r^\beta} . \]

The local Hölder continuity is proven. \( \square \)

**Theorem 3.4** (Liouville Theorem). Let \( h > 1 \), \( p > n \geq 2 \), \( \lambda \geq 0 \), \( a(x) \geq 0 \) and \( f \leq 0 \) be continuous in the whole space \( \mathbb{R}^n \). Suppose that \( u \) is a nonnegative viscosity supersolution of
\[ |Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u = f \]
in \( \mathbb{R}^n \). Then \( u \) is a constant function.
4. Existence Results. In this section, we first introduce an approximate equation of the weighted eigenvalue problem (1.2). By establishing the existence of positive viscosity solutions of the approximate problems, we prove the existence of positive solutions of the eigenvalue problem (1.2) with $\lambda = \lambda_0$ based on the compact criterion. Then the proof of Theorem 1.2 is given. To ensure the approximate solutions satisfying the homogeneous boundary condition, one should find the suitable barrier functions. Based on the particular structure of the operator $|Du|^p - \Delta_p u$, we can construct a good barrier without assuming any regularity of $\partial \Omega$. In this procedure, we also establish the existence result for more general boundary condition. See more related results in [17] for the infinity Laplacian and [5, 6, 26] for the normalized $p$-Laplacian.

Now we use Perron’s method to prove the existence of the approximate problem.

**Theorem 4.1.** Let $h > 1$, $2 \leq n < p < \infty$, $0 < \lambda < \lambda_0$, a positive function $a(x) \in C(\overline{\Omega})$, $f \in C(\overline{\Omega})$ and $f < 0$ in $\Omega$. Then, there exists $u \in C(\overline{\Omega})$ a viscosity solution of

$$
\begin{align*}
|Du|^h - \Delta_p u + \lambda a(x)|u|^h &= f, & \text{in } \Omega, \\
0 &< u, & \text{in } \Omega, \\
0 &= u, & \text{on } \partial \Omega.
\end{align*}
$$

**Proof.** Noticing that Theorem 2.4 provides the uniqueness result for this problem, we need to construct a viscosity subsolution and supersolution based on Perron’s method. We start with constructing a supersolution. Recalling the definition of $\lambda_0$, we can take a positive function $v \in C(\overline{\Omega})$ such that

$$
|Du|^h - \Delta_p v + \lambda a(x)|v|^h \leq 0
$$

for $0 < \lambda < \lambda_0$. For any fixed $\eta$ with $0 < \eta < 1$, set

$$
w(x) = v(x) - m\eta, \quad \text{in } \Omega,
$$

where $m := \min_{\overline{\Omega}} v > 0$. Then, we get $w \geq m(1 - \eta) > 0$ in $\overline{\Omega}$ and

$$
|Dw|^h - \Delta_p w + \mu a(x)|w|^h w = |Dv|^h - \Delta_p v + \mu a(x)(v - m\eta)^h
\leq -\lambda a(x)v^h + \mu a(x)(v - m\eta)^h
\leq a(x)v^h (-\lambda + \mu \left(1 - \frac{mn}{v^h}\right)^h).
$$

Choosing $\mu$ such that $\lambda < \mu < \lambda (1 - mn/M)^{-h}$, where $M := \max_{\overline{\Omega}} v$, we obtain

$$
|Dw|^h - \Delta_p w + \lambda a(x)|w|^h w \leq (\lambda - \mu)a(x)|w|^h w \leq (\lambda - \mu)a_0 m^h < 0, \quad x \in \Omega,
$$

where $a_0 := \inf_{\overline{\Omega}} a(x) > 0$. Defining

$$
W(x) = \left(\frac{\inf_{\overline{\Omega}} f}{(\lambda - \mu)a_0 m^h}\right)^{1/h} w(x), \quad x \in \Omega,
$$

we need to construct a viscosity subsolution and supersolution based on Perron’s method. We can take a positive function $v \in C(\overline{\Omega})$ such that

$$
|Du|^h - \Delta_p u + \lambda a(x)|u|^h = f, \quad v(x) = \inf_{\overline{\Omega}} f, \quad x \in \Omega.
$$

By interchanging $x$ with $y$, we obtain $u(x) \geq v(y)$. □
Recalling $\beta$ now choosing $\alpha < 0$ as a test function in the definition of viscosity solutions, a contradiction.

$x^h$ from the Perron’s method. In particular, one has $Dv \equiv 0$ in $\Omega$ and it is clear that $Cv$ is the desired positive viscosity supersolution to

$$\text{data. For any given } W, \text{ now we modify } W \text{ to get a supersolution of this equation with the right boundary data. For any given } z \in \partial \Omega, \text{ set}$$

$$v_z(x) = \frac{1}{n} |x - z|^\alpha, \quad x \in \Omega,$$

where $0 < \alpha < 1$ to be determined. Direct calculations yield

$$|Dv_z|^{h-1} \Delta_p^N v_z = |x - z|^{(\alpha - 1)(h - 1)} \left( \frac{(p - 1)(\alpha - 1)}{p} |x - z|^{\alpha - 2} + \frac{n - 1}{p} |x - z|^{\alpha - 2} \right)$$

$$= \frac{(p - 1)(\alpha - 1) + n - 1}{p} |x - z|^{(\alpha - 1)(h - 1) + \alpha - 2}.$$

Recalling $\beta = \frac{p - n}{p - 1} > 0$, if we choose $0 < \alpha < \min\{\beta, 1\}$, we have $\frac{(p - 1)(\alpha - 1) + n - 1}{p} < 0$. Hence, it follows that

$$|Dv_z|^{h-1} \Delta_p^N v_z(x) = \frac{(p - 1)(\alpha - 1) + n - 1}{p} |x - z|^{(\alpha - 1)(h - 1) + \alpha - 2}$$

$$= \alpha^h [(p - 1)(\alpha - 1) + n - 1] |x - z|^{-(h + 1)} v_z^h$$

$$= \alpha^h [(p - 1)(\alpha - 1) + n - 1] |x - z|^{-(h + 1)} v_z^h + \alpha^h [(p - 1)(\alpha - 1) + n - 1] |x - z|^{-h - 1}.$$

Now choosing $0 < \rho < \frac{1}{2} \min \left\{ \left( 2p \lambda \alpha^{\frac{1}{p - 1}} \|a\|_{L^\infty(\Omega)} / \|(p - 1)(1 - \alpha) - n + 1\| \right)^{-\frac{1}{p - 1}}, \left( 2p \lambda \alpha^{\frac{1}{p - 1}} \sup_{\Omega} (-f) / \|(p - 1)(1 - \alpha) - n + 1\| \right)^{-\frac{1}{p - 1}} \right\},$

we obtain

$$|Dv_z|^{h-1} \Delta_p^N v_z + \lambda a(x)|v_z|^{h-1} v_z \leq f$$

in $B_{2\rho}(z) \cap \Omega$. Taking $C \geq \max \left\{ 1, \alpha \rho^\alpha \sup_{\Omega} W \right\}$, one has

$$Cv_z(x) \geq W(x) \text{ in } \Omega \setminus B_{\rho}(z)$$

and it is clear that $Cv_z$ is also a viscosity supersolution. Noting that the minimum of two supersolutions is also a supersolution, we get that the function

$$V(x) = \inf_{z \in \partial \Omega} \left\{ \min \{ Cv_z(x), W(x) \} \right\}$$

is the desired positive viscosity supersolution to $|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u = f$, vanishing on $\partial \Omega$.

It is obvious that $\hat{u}(x) \equiv 0$ is a nonnegative subsolution of the equation $|Du|^{h-1} \Delta_p^N u + \lambda a(x)|u|^{h-1} u = f$. At this point, the existence of a nonnegative solution $u$ follows from the Perron’s method. In particular, one has $u > 0$ in $\Omega$. Otherwise, there would exist a point $x_0 \in \Omega$ such that $u(x_0) = 0$ and, since $u$ is nonnegative, one could use $0$ as a test function in the definition of viscosity solutions, a contradiction. □
Remark 4.1. Let \( h > 1, 2 \leq n < p < \infty, \lambda > 0 \), a positive function \( a(x) \in C(\overline{\Omega}) \), \( f \in C(\overline{\Omega}) \) and \( f < 0 \) in \( \Omega \). If \( v \in C(\overline{\Omega}) \) is a positive viscosity supersolution to \( |Du|^{h-1} \Delta_p^N u + \lambda a(x) |u|^{h-1} u = f \) in \( \Omega \), then for all \( \mu \) satisfying \( \lambda < \mu < \lambda + \frac{A (B + \varepsilon)^h}{\inf_{\overline{\Omega}} (-f) - \lambda Ah^2 B^{h-1}} \), where \( A := \sup_{\Omega} a(x) > 0 \), \( B := \sup_{\Omega} f > 0 \),

\[
0 < \varepsilon < \min \left\{ 1, \frac{1}{\lambda Ah^2 B^{h-1}} \right\},
\]

there exists a strictly positive viscosity solution \( w \) satisfying \( |Dw|^{h-1} \Delta_p^N w + \mu a(x) |w|^{h-1} w \leq 0 \).

In fact, it is clear that \( v_\varepsilon := v + \varepsilon \) (\( 0 < \varepsilon < 1 \)) verifies

\[
|Dv_\varepsilon|^{h-1} \Delta_p^N v_\varepsilon + \lambda a(x) |v_\varepsilon|^{h-1} v_\varepsilon \leq \lambda \varepsilon h^2 B^{h-1} a(x) + f.
\]

One can easily check that

\[
0 \geq |Dv_\varepsilon|^{h-1} \Delta_p^N v_\varepsilon + \lambda a(x) |v_\varepsilon|^{h-1} v_\varepsilon - \lambda \varepsilon h^2 B^{h-1} a(x) - f
\]

\[
\geq |Dv_\varepsilon|^{h-1} \Delta_p^N v_\varepsilon + \lambda a(x) |v_\varepsilon|^{h-1} v_\varepsilon + \inf_{\overline{\Omega}} (-f) - \lambda Ah^2 B^{h-1} x
\]

\[
A (B + \varepsilon)^h a(x) |v_\varepsilon|^{h-1} v_\varepsilon.
\]

Therefore, we have \( |Dv_\varepsilon|^{h-1} \Delta_p^N v_\varepsilon + \mu a(x) |v_\varepsilon|^{h-1} v_\varepsilon \leq 0 \) for all \( \mu \) such that \( \lambda < \mu < \lambda + \frac{A (B + \varepsilon)^h}{\inf_{\overline{\Omega}} (-f) - \lambda Ah^2 B^{h-1}} \).

Following the strategy of the proof in Theorem 4.1, we can prove the existence result for more general boundary problem.

Theorem 4.2. Let \( h > 1, 2 \leq n < p < \infty, 0 \leq \lambda < \lambda_\Omega, \) a positive function \( a(x) \in C(\overline{\Omega}) \), \( f \in C(\overline{\Omega}) \), \( f < 0 \) in \( \Omega \) and \( g \in C(\partial \Omega) \) nonnegative. Then there exists a positive viscosity solution \( u \in C(\overline{\Omega}) \) satisfying

\[
\begin{cases}
|Du|^{h-1} \Delta_p^N u + \lambda a(x) |u|^{h-1} u = f & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega.
\end{cases}
\]

Proof. Using Perron’s method again, we will find a viscosity subsolution and a supersolution of (4.3) satisfying the inhomogeneous boundary condition. In [32], the existence and uniqueness of a nonnegative solution to the problem

\[
\begin{cases}
\Delta_p^N u = 0 & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega
\end{cases}
\]

were obtained. Obviously, it is also a viscosity subsolution of (4.3).

Next we construct the desired viscosity supersolution. In the process of proving Theorem 4.1, we have established the fact that there exists a positive function \( W \) satisfying (4.2).

Denote \( \tilde{W} = CW \), where \( C \geq \max \left\{ \frac{1}{\lambda_\Omega} \sup_{\partial \Omega} g, 1 \right\} \). One can check that \( \tilde{W} \) is a viscosity solution of \( |D\tilde{W}|^{h-1} \Delta_p^N \tilde{W} + \lambda a(x) \tilde{W}|^{h-1} \tilde{W} \leq f \) in \( \Omega \) and \( \tilde{W} \geq g \) on \( \partial \Omega \).

In order to get a viscosity supersolution with the right boundary data, we define for any given \( z \in \partial \Omega \) the function

\[
v_z(x) = g(z) + \frac{C}{\alpha} |x - z|^{\alpha},
\]
where \( C \geq 1 \) to be determined. The choices \( p > n \geq 2 \) and \( 0 < \alpha < \min\{\frac{p-n}{p-1}, 1\} \)
ensure that
\[
|Dv_z|^{h-1} \Delta_p^N v_z = \frac{C^h[(p-1)(\alpha-1) + n - 1]}{p} |x-z|^{-(1-\alpha)h-1} \to -\infty, \text{ as } x \to z.
\]
Then, there exists some \( \rho > 0 \) depending on \( p, n, h, \lambda, \alpha, \|a\|_{L^\infty(\Omega)}, \sup g \) and
\( \sup(-f) \), but independent of \( z \) and \( C \geq 1 \), such that \( |Dv_z|^{h-1} \Delta_p^N v_z + \lambda a(x)|v_z|^{h-1}v_z \leq f \) in \( B_{2\rho}(z) \cap \Omega \). Taking \( C \) sufficiently large we obtain that \( v_z(x) \geq \bar{W}(x) \) in \( \Omega \backslash B_{\rho}(z) \), thus
\[
V(x) = \inf_{z \in \partial \Omega} \left( \min \left\{ v_z(x), \bar{W}(x) \right\} \right)
\]
is the desired positive viscosity supersolution and \( V = g \) on the boundary. Once
again, Perron’s method guarantees the existence of a nonnegative viscosity solution
\( u \) satisfying (4.3) and Theorem 3.2 implies the positivity of it in \( \Omega \).

With the uniform estimate in Theorem 3.3 and the existence of the approximate
problem in 4.1, we can now turn our attention to the proof of Theorem 1.2 which
is similar to that for the normalized \( p \)-Laplacian case [5, 6, 26] and the infinity
Laplacian in [17, 23].

**Proof of Theorem 1.2.** Consider an increasing sequence of numbers \( \mu_j \)
converging to \( \lambda_\Omega \). Theorem 4.1 ensures that there exists a positive continuous function \( u_j \)
satisfying
\[
\begin{cases}
|Du_j|^{h-1} \Delta_p^N u_j + \mu_ja(x)|u_j|^{h-1}u_j = -1, & \text{in } \Omega, \\
u_k = 0, & \text{on } \partial \Omega.
\end{cases}
\]
Denoting \( M_j = \sup_{\Omega} u_j \), we first show that the sequence \( \{M_j\} \) is unbounded. Indeed,
if we suppose that it is bounded, Theorem 3.3 implies that the sequence \( \{u_j\} \) is
locally equicontinuous and hence convergent (up to a subsequence) locally uniformly
to a positive viscosity solution \( u \) of
\[
\begin{cases}
|Du|^{h-1} \Delta_p^N u + \lambda_\Omega a(x)|u|^{h-1}u = -1, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]
Therefore, by Remark 4.1, there exists a strictly positive viscosity solution \( v \) of
\[
|Dv|^{h-1} \Delta_p^N v + \lambda a(x)|v|^{h-1}v \leq 0
\]
for any \( \lambda \) satisfying \( \lambda_\Omega < \lambda < \lambda_\Omega + \frac{1 - \lambda_\Omega A \varepsilon B^{h-1}}{A(B + \varepsilon)^h} \), where \( A := \sup_{\Omega} a(x) > 0 \),
\( B := \sup_{\Omega} v > 0 \), \( 0 < \varepsilon < \min \left\{ 1, \frac{1}{\lambda_\Omega A h^2 B^{h-1}} \right\} \). This contradicts to the definition
of \( \lambda_\Omega \). Hence, we have \( M_j \to +\infty \).

Now letting \( v_j = \frac{u_j}{M_j} \), one can easily check that \( v_j \) satisfies
\[
\begin{cases}
|Dv_j|^{h-1} \Delta_p^N v_j + \mu_j a(x)|v_j|^{h-1}v_j = -1, & \text{in } \Omega, \\
v_j = 0, & \text{on } \partial \Omega
\end{cases}
\]
in the viscosity sense and \( \sup_{\Omega} v_j(x) = 1 \). Since \( \{v_j\} \) are uniformly bounded, again
Theorem 3.3 implies they are locally equicontinuous. Therefore, \( \{v_j\} \) (up to a
subsequence) converges locally uniformly to a nonnegative function \( v \). Now we
claim that \( v \) is nontrivial. In fact, based on the barrier argument we have that the maximum of \( v_j \) is attained at a positive uniform distance from \( \partial \Omega \). Therefore, Theorem 3.3 provides a uniform \( C_\beta \) estimate near the points at which \( v_j \) attains its maximum, \( x_j \). This means that there exists a uniform radius \( r \) such that \( v_j \geq 1/2 \) in \( B_r(x_j) \). Since \( x_j \) converge to a point inside \( \Omega \) (extracting a subsequence if necessary), we have that the limit \( v \) is nontrivial.

By the stability result of viscosity solutions, we have the limit function \( v \) satisfies

\[
\begin{cases}
|Dv|^{h-1}\Delta_p^N v + \lambda_\Omega a(x)|v|^{h-1}v = 0, & \text{in } \Omega, \\
v = 0, & \text{on } \partial \Omega.
\end{cases}
\]

Since \( v \) is nonnegative and nontrivial and then strictly positive in \( \Omega \) by the same arguments used in the proof of Theorem 2.3. Therefore, \( v \) is the desired eigenfunction related to the eigenvalue \( \lambda_\Omega \). It is obvious that \( v \) is locally Hölder continuous with exponent \( \beta \) and we have completed the proof of Theorem 1.2.

Recall that, for \( \lambda < \lambda_\Omega \), solutions of (1.2) are the zero-solutions by Corollary 2.1. Thus \( \lambda_\Omega \) is the smallest value such that the problem (1.2) has a nontrivial solution. However, due to the high degeneracy of \( |Du|^{h-1}\Delta_p^N u \), we don’t know whether the principal eigenvalue is single. One should notice that Theorem 2.5 means the local uniqueness for the positive eigenfunction associated to the same eigenvalue. See also [17] for the infinity Laplacian case.

**Corollary 4.1.** Let \( h > 1, n < p < \infty \) and a positive function \( a(x) \in C(\Omega) \cap L^\infty(\Omega) \). If \( u \) and \( v \) are positive eigenfunctions corresponding to the same Dirichlet eigenvalue \( \lambda \geq \lambda_\Omega \) of the equation

\[
|Du|^{h-1}\Delta_p^N u + \lambda a(x)|u|^{h-1}u = 0,
\]

then there holds

\[
\sup_{x \in \Omega'} \frac{u(x)}{v(x)} = \sup_{x \in \partial \Omega'} \frac{u(x)}{v(x)}
\]

for any \( \Omega' \) compactly contained in \( \Omega \).

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