Methods for computing $b$-functions associated with $\mu$-constant deformations
– Case of inner modality 2 –

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Abstract. New methods for computing parametric local \( b \)-functions are introduced for $\mu$-constant deformations of semi-weighted homogeneous singularities. The keys of the methods are comprehensive Gröbner systems in Poincaré-Birkhoff-Witt algebra and holonomic $\mathcal{D}$-modules. It is shown that the use of semi-weighted homogeneity reduces the computational complexity of $b$-functions associated with $\mu$-constant deformations. In the case of inner modality 2, local $b$-functions associated with $\mu$-constant deformations are obtained by the resulting method and given the list of parametric local $b$-functions.

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1. Introduction

We introduce a new algorithm for computing local $b$-functions of semi-weighted homogeneous polynomials, and study local $b$-functions of $\mu$-constant deformation of inner modality 2 singularities.

A $b$-function is an analytic invariant of a hypersurface singularity. In 1997, T. Oaku gave an algorithm for computing $b$-functions [20]. After that many researcher have improved the algorithm in the context of symbolic computation. However, as the computational complexity of the existing algorithms is still quite high, it is difficult to obtain $b$-functions in realistic time in many cases. The bottleneck lies in computing non-commutative Gröbner bases.

In a pioneering paper [22] published in 1978, T. Yano investigated $b$-functions and already considered $b$-functions associated with a $\mu$-constant deformations of singularities. He noticed that some roots of $b$-functions are not stable and change these values discontinuously under $\mu$-constant deformations. Later, in 1980’s, M. Kato and P. Cassou Nogués explicitly computed $b$-functions associated with a $\mu$-constant deformation for some cases in their seminal papers [3, 4, 8, 9].

In this paper, we consider $b$-functions of semi-weighted homogeneous singularities in the context of computational algebraic analysis. Upon using properties of semi-weighted homogeneous singularities, we construct an effective method for computing $b$-functions. Notably, the resulting algorithm explicitly compute parameter dependency of $b$-functions associated with $\mu$-constant deformations. The keys of our approach are comprehensive Gröbner systems in Poincaré-Birkhoff-Witt algebra and holonomic $\mathcal{D}$-modules. We show that the use of semi-weighted homogeneity allows us

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to design an effective algorithm for computing $b$-functions associated with $\mu$-constant deformations, which avoid the use of Gröbner bases computation in an elimination step. As an application, we study inner modality 2 singularities.

This paper is organized as follows. In section 2, we recall the notion of semi-weighted homogeneity and give a list of semi-weighted homogeneous polynomials whose inner modality is equal to two, the target singularities of the present paper. In section 3, we briefly recall a comprehensive Gröbner system, a method of computing parametric $b$-functions and review our previous results reported in \cite{17}. In section 4, we present an algorithm for computing algebraic local cohomology -constant deformations. Here we briefly recall a comprehensive Gröbner system approach to compute parametric $b$-functions and study inner modality 2 singularities. As an application, we give a list of semi-weighted homogeneous polynomials whose inner modality is equal to two, the target singularities of the present paper. In section 3, we briefly recall a comprehensive Gröbner system, a method of computing parametric $b$-functions and review our previous results reported in \cite{17} on the computation of $b$-functions associated with $\mu$-constant deformations.

Throughout this paper, we use the notation $x$ as the abbreviation of $n$ variables $x_1, \ldots, x_n$, $\mathbb{Q}$ as the field of rational numbers and $\mathbb{C}$ as the field of complex numbers. The set of natural numbers $\mathbb{N}$ includes zero. For elements $p_1, p_2, \ldots, p_r$ in a ring $R$, let $Id(p_1, p_2, \ldots, p_r)$ denote the ideal in $R$ generated by $p_1, p_2, \ldots, p_r$.

### 2. The list of inner modality 2 singularities

Let $w = (w_1, w_2, \ldots, w_n) \in \mathbb{N}^n$, $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ and $x^\alpha = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \in \mathbb{C}[x]$.

Let $|x^\alpha|_w$ denote the weighted degree $\sum_{i=1}^{n} w_i a_i$ of the monomial $x^\alpha$.

**Definition 2.1.** (1) A non-zero polynomial $f \in \mathbb{C}[x]$ is called weighted homogeneous of type $(d; w)$ if all monomials of $f$ have the same weighted degree $d$ w.r.t. $w$ where $d \in \mathbb{N}$.

(2) The polynomial $f$ is called semi-weighted homogeneous of type $(d; w)$ if $f$ is of the form $f = f_0 + g$ where $f_0$ is a weighted homogeneous polynomial of type $(d; w)$ with an isolated singularity at the origin $O$ in $\mathbb{C}^n$, and $f = f_0$ or $\text{ord}_w(f - f_0) > d$ where $\text{ord}_w(f) = \min \{|x^\alpha|_w : x^\alpha$ is a monomial of $f\}$. A monomial of $g$ is called an upper of $f_0$.

Note that the Milnor number at the origin of $f = f_0 + g$ and that of $f_0$ are same, and the embedded topological type of $f = f_0 + g$ singularity and that of $f_0$ are same $\cite{10, 24}$. Accordingly, $f = f_0 + g$ is called a $\mu$-constant deformation of $f_0$.

In $\cite{23}$, Yoshinaga and Suzuki gave lists of normal forms of quasihomogeneous (weighted homogeneous) functions with inner modality $\leq 4$. Table $\cite{1}$ quoted from $\cite{23}$ is the list of inner modality 2 singularities.

For example, $E_{18}$ in the table means the following.

(i) $f_0(x, y) = x^3 + y^{10}$ is a weighted homogeneous polynomial (of type $(30, 10, 3)$).

(ii) The Milnor number of $f_0(x, y)$ at the origin is equal to 18.

(iii) $f(x, y) = f_0(x, y) + c_{(1, 7)}xy^7 + c_{(1, 8)}xy^8$ is a semi-weighted homogeneous polynomial (of type $(30, 10, 3)$) where $c_{(1, 7)}, c_{(1, 8)} \in \mathbb{C}$.

### 3. Comprehensive Gröbner system approach

Here we briefly recall a comprehensive Gröbner system approach to compute parametric $b$-functions and review our previous results reported in $\cite{17}$ on the computation of $b$-functions associated with $\mu$-constant deformations.
Throughout the paper we assume that a linear partial differential operator is always represented in $D$ operators with coefficients in a polynomial ring $\mathbb{C}[x_1, \ldots, x_r]$.

For $g_1, \ldots, g_r \in \mathbb{C}[u]$, $V(g_1, \ldots, g_r) \subset \mathbb{C}^m$ denotes the affine variety of $g_1, \ldots, g_r$, i.e., $V(g_1, \ldots, g_r) = \{ \bar{u} \in \mathbb{C}^m | g_i(\bar{u}) = \cdots = g_r(\bar{u}) = 0 \}$. For $g_1, \ldots, g_r, g'_1, \ldots, g'_r \in \mathbb{C}[t]$, we call an algebraic constructible set $V(g_1, \ldots, g_r) \setminus V(g'_1, \ldots, g'_r) \subset \mathbb{C}^m$ a stratum. Notations $U_1, U_2, \ldots, U_\ell$ are frequently used to represent strata.

For every $\bar{u} \in \mathbb{C}^m$, the canonical specialization homomorphism $\sigma_\bar{u} : D[u] \langle s, \partial \rangle \to D \langle s, \partial \rangle$ (or $D[u] \to D$) is defined as the map that substitutes $u$ by $\bar{u}$ in $p(u, x, \partial, s, \partial) \in D[u] \langle s, \partial \rangle$. The

| type | weighted homo. $f_0$ | upper monomials | note |
|------|---------------------|-----------------|------|
| $E_{18}$ | $x^3 + y^3$ | $xy', xy^3$ | |
| $E_{19}$ | $x^3 + xy'$ | $y^3, y'z^3$ | |
| $E_{20}$ | $x^3 + y^3$ | $y^3, xy'$ | |
| $W_{17}$ | $x^3 + y^3$ | $y^3, y'$ | |
| $W_{18}$ | $x^3 + y^3$ | $y^3, y'$ | |
| $Z_{17}$ | $x^3y + y^3$ | $xy^3, y^3$ | |
| $Z_{18}$ | $x^3y + y^3$ | $y^3, y^3$ | |
| $Z_{19}$ | $x^3y + y^3$ | $y^3, xy^3$ | |
| $Q_{16}$ | $x^3 + yz^2 + y'$ | $xy^3, yz^2$ | |
| $Q_{17}$ | $x^3 + yz^2 + xy'$ | $y^3, yz^2$ | |
| $Q_{18}$ | $x^3 + y^3z^2 + y^3$ | $xy^3, y^3z^2$ | |
| $S_{16}$ | $x^3 + y^3z^2 + y^3$ | $y^3, y^3z^2$ | |
| $S_{17}$ | $x^3 + y^3z^2 + y^3$ | $y^3z, z^3$ | |
| $U_{16}$ | $x^3 + y^3z^2 + y^3$ | $y^3z^2, y^3z^2$ | |
| $J_{16}$ | $x^3 + y^3 + u_1x^2y^3$ | $y^3, 4u_1^3 + 27 \neq 0$ | |
| $W_{15}$ | $x^3 + y^3 + u_1x^2y^3$ | $y', u_1^3 - 4 \neq 0$ | |
| $Z_{15}$ | $x^3 + y^3 + u_1x^2y^3$ | $y^3, 4u_1^3 + 27 \neq 0$ | |
| $Q_{14}$ | $x^3 + y^3 + u_1x^2y^3 + xy^3$ | $y', u_1^3 - 4 \neq 0$ | |
| $S_{14}$ | $x^3 + y^3 + u_1x^2y^3 + xy^3$ | $z^3, u_1^3 - 4 \neq 0$ | |
| $U_{14}$ | $x^3 + x^2z^2 + u_1xy^3 + y^3z$ | $yz^3, u_1^3 + 1 \neq 0$ | |

Let $D = \mathbb{C}[x, \partial]$ denote the Weyl algebra, the ring of linear partial differential operators with coefficients in $\mathbb{C}[x]$, where $\partial = \{ \partial_1, \ldots, \partial_n \}$, $\partial_i = \frac{\partial}{\partial x_i}$, i.e.,

$$ D = \left\{ \sum_{\beta \in \mathbb{N}^n} h_\beta(x) \partial^\beta \mid h_\beta(x) \in \mathbb{C}[x] \right\}. $$

Throughout the paper we assume that a linear partial differential operator is always represented in the canonical form: each power product of a partial differential operator is written as $x^\alpha \partial^\beta$ where $\alpha, \beta \in \mathbb{N}^n$.

Let $u = \{ u_1, \ldots, u_m \}$ be variables such that $u \cap x = \emptyset$, $D[u]$ a ring of partial differential operators with coefficients in a polynomial ring $(\mathbb{C}[u])[x]$, i.e.,

$$ D[u] = \left\{ \sum_{\beta \in \mathbb{N}^n} h_\beta(u, x) \partial^\beta \mid h_\beta(u, x) \in (\mathbb{C}[u])[x] \right\}. $$

Let $D(s, \partial)$ (or $D[u] \langle s, \partial \rangle$) denote the Poincaré-Birkhoff-Witt (PBW) algebra $D \otimes_\mathbb{C} \mathbb{C}[s, \partial]$ (or $D[u] \otimes_\mathbb{C} \mathbb{C}[s, \partial]$) with a non-commutative relation $\partial s = s \partial - \partial$ and commutative relations $\partial_i x_i = x_i \partial_i$, $\partial_i s = s \partial_i$, $s x_i = x_i s$, $\partial_i \partial_i = \partial_i \partial_i$ ($1 \leq i \leq n$). There exist algorithms and implementations to compute Gröbner bases of given ideals in the non-commutative rings $D$ and $D(s, \partial)$ [5][11].
image \( \sigma_a \) of a set \( F \) is denoted by \( \sigma_a(F) = \{ \sigma_a(p) | p \in F \} \subset D(s, \partial_t) \). A symbol \( \text{Mono}(x \cup \partial \cup \{ s, \partial_t \}) \) is the set of monomials of \( x \cup \partial \cup \{ s, \partial_t \} \).

The main tool to compute \( b \)-functions associated with \( \mu \)-constant deformations, is a comprehensive Gröbner system in the PBW algebra. We adopt the following as a definition of comprehensive Gröbner systems.

**Definition 3.1 (CGS).** Let \( \succ \) be a monomial order on \( \text{Mono}(x \cup \partial \cup \{ s, \partial_t \}) \). Let \( F \) be a subset of \( D[u](s, \partial_t), U_1, U_2, \ldots, U_\ell \) strata in \( \mathbb{C}^m \) and \( G_1, \ldots, G_\ell \) subsets in \( D[u](s, \partial_t) \). If a finite set \( \mathcal{G} = \{(U_1, G_1), \ldots, (U_\ell, G_\ell)\} \) of pairs satisfies properties such that

1. \( U_i \neq \emptyset \) and \( U_i \cap U_j = \emptyset \) for \( 1 \leq i \neq j \leq \ell \),
2. for all \( u \in U_i, \sigma_a(G_i) \) is a minimal Gröbner basis of \( \text{Id}(\sigma_a(F)) \) w.r.t. \( \succ \) in \( D(s, \partial_t) \), and
3. for all \( u \in U_i \) and \( p \in G_i, \sigma_a(\text{hc}(p)) \neq 0 \) where \( \text{hc}(p) \) is the head coefficient of \( p \) in \( \mathbb{C}[u] \),

\( \mathcal{G} \) is called a comprehensive Gröbner system (CGS) on \( U_1 \cup \cdots \cup U_\ell \) for \( Id(F) \) w.r.t. \( \succ \). We simply say that \( \mathcal{G} \) is a comprehensive Gröbner system for \( Id(F) \) if \( U_1 \cup \cdots \cup U_\ell = \mathbb{C}^m \).

In our previous papers [14] [15], algorithms and implementations for computing comprehensive Gröbner systems in PBW algebras are introduced.

### 3.1. Global \( b \)-functions

Let \( f \) be a non-constant polynomial in \( \mathbb{C}[x] \). Then, the annihilating ideal of \( f^s \) is

\[
\text{Ann}(f^s) = \{ p \in D[s] | pf^s = 0 \}
\]

where \( s \) is an indeterminate, and

\[
D[s] = \left\{ \sum_{k \in \mathbb{N}, \beta \in \mathbb{N}^n} h_{k,\beta}(x)s^k \partial^\beta \middle| h_{k,\beta}(x) \in \mathbb{C}[x] \right\}.
\]

Consider the following left ideal \( I \) in the PBW algebra \( D(s, \partial_t) \).

\[
I = \text{Id} \left( f \cdot \partial_t + s, \partial_1 + \partial_t \cdot \frac{\partial f}{\partial x_1}, \partial_2 + \partial_t \cdot \frac{\partial f}{\partial x_2}, \ldots, \partial_n + \partial_t \cdot \frac{\partial f}{\partial x_n} \right).
\]

Briançon and Maisonobe show in [2] that \( \text{Ann}(f^s) = I \cap D[s] \). Thus, a basis of the ideal \( \text{Ann}(f^s) \) can be obtained by the Gröbner basis computation of \( I \) w.r.t. an elimination order for \( \partial_t \).

The global \( b \)-function or the global Bernstein-Sato polynomial of \( f \) is defined as the monic generator \( b_f(s) \) of \( (\text{Ann}(f^s) + \text{Id}(f)) \cap \mathbb{C}[s] \) where \( \text{Id}(f) \) is the ideal generated by \( f \). It is known that the \( b \)-function of \( f \) always has \( s + 1 \) as a factor and has a form \((s + 1)b_f(s) \), where \( b_f(s) \in \mathbb{C}[s] \). The polynomial \( b_f(s) \) is called the (global) reduced \( b \)-function of \( f \). The reduced \( b \)-function \( b_f(s) \) can be obtained by computing a Gröbner basis of \( \text{Ann}(f^s) + \text{Id}(f) \) and \( \text{Ann}(f^s) + \text{Id}(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \) w.r.t. a block order \( \{ x, \partial \} \gg s \).

Let \( f \) be a parametric polynomial in \( (\mathbb{C}[u])[x] \) where \( u \) are regarded as parameters. As mentioned previously, a CGS of the \( \text{Ann}(f^s) \) is computable by the algorithm for computing CGS’s in PBW algebras. Accordingly, global \( b \)-functions with parameters are computable by using CGS’s of the ideals \( \text{Ann}(f^s) + \text{Id}(f) \) and \( \text{Ann}(f^s) + \text{Id}(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \). We refer the reader to [14] and [15] for details.

**Algorithm 1. (Global \( b \)-functions)**

**Input:** \( U \subseteq \mathbb{C}^m, f \in (\mathbb{C}[u])[x] \): \( f \) is a non-constant polynomial with parameter \( u \),

**Output:** \( \mathcal{G} = \{ (U_1, \tilde{b}_1(s)), \ldots, (U_\ell, \tilde{b}_\ell(s)) \} \): for each \( i \in \{ 1, \ldots, \ell \} \) and \( \forall \bar{u} \in U_i, \tilde{b}_i(s) \) is the reduced \( b \)-function of \( \sigma_{\bar{u}}(f) \) and \( U = \bigcup_{i=1}^{\ell} U_i \).

**BEGIN**

\( \mathcal{G} \leftarrow \emptyset; J \leftarrow \{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \}; \)
\[ I \leftarrow \text{Id}(f \cdot \partial_t + s, \partial_1 + \partial_t \cdot \frac{\partial f}{\partial x_1}, \partial_2 + \partial_t \cdot \frac{\partial f}{\partial x_2}, \ldots, \partial_n + \partial_t \cdot \frac{\partial f}{\partial x_n}); \]

\[ \mathcal{P} \leftarrow \text{Compute a CGS of } I \text{ w.r.t. an elimination order for } \partial_t \text{ on } \mathcal{U}; \]

\[ \text{while } \mathcal{P} \neq \emptyset \text{ do} \]

\[ \text{Select } (\mathcal{U}', \mathcal{P}) \text{ from } \mathcal{P}; \]

\[ \mathcal{P} \leftarrow \mathcal{P} \setminus \{(\mathcal{U}', \mathcal{P})\}; \]

\[ \mathcal{B} \leftarrow \text{Compute a CGS of } \text{Id}(J \cup (P \cap D[s])) \text{ on } \mathcal{U}' \text{ w.r.t. a block order } \{x, \partial\} \gg s; \]

\[ \text{while } \mathcal{B} \neq \emptyset \text{ do} \]

\[ \text{Select } (\mathcal{U}'', \mathcal{B}) \text{ from } \mathcal{B}; \]

\[ \mathcal{G} \leftarrow \mathcal{G} \cup \{(\mathcal{U}'', \mathcal{B} \cap \mathbb{C}[s])\}; \]

\[ \text{end-while} \]

\[ \text{end-while} \]

\[ \text{return } \mathcal{G}; \]

\[ \text{END} \]

Note that the rationality of the roots of \( b \)-functions has been shown by [7]. Hence, the \( b \)-functions can be factorized into linear factors over \( \mathbb{Q} \).

**Example 1.** Let us consider \( U_{16} \) singularity \( f = x^3 + xz^2 + y^5 + u_1y^2z^2 + u_2y^3z^2 \) where \( u_1, u_2 \) are parameters. As \( f \) is a semi-weighted homogeneous, the Milnor number of the singularity at the origin is 16. Set \( b_u(s) = (s + \frac{14}{13})(s + \frac{16}{13})(s + \frac{18}{13})(s + \frac{20}{13})(s + \frac{22}{13})(s + \frac{24}{13})(s + \frac{26}{13})(s + \frac{28}{13}) \). Then the reduced \( b \)-functions of \( f \) is the following.

- If \((u_1, u_2) \) belongs to \( \mathbb{C}^2 \setminus \{V(u_1(27u_1^4 + 256u_2^2)\), then \( \tilde{b}_f(s) = b_u(s)(s + \frac{14}{13})(s + \frac{16}{13}) \).
- If \((u_1, u_2) \) belongs to \( \{V(27u_1^4 + 256u_2^2)\} \setminus \{V(u_1, u_2)\} \), then \( \tilde{b}_f(s) = b_u(s)(s + \frac{2}{3})(s + \frac{14}{13})(s + \frac{16}{13}) \).
- If \((u_1, u_2) \) belongs to \( \{V(u_1)\} \setminus \{V(u_1, u_2)\} \), then \( \tilde{b}_f(s) = b_u(s)(s + \frac{17}{15})(s + \frac{19}{15}) \).
- If \((u_1, u_2) \) belongs to \( \{V(u_1, u_2)\} \setminus \{V(u_1)\} \), then \( \tilde{b}_f(s) = b_u(s)(s + \frac{21}{15})(s + \frac{23}{15}) \).

Note that if \((u_1, u_2) \) belongs to \( \{V(27u_1^4 + 256u_2^2)\} \setminus \{V(u_1, u_2)\} \), then \( f \) has three isolated singularities \((0, 0, 0) \) and \( \left( \frac{3u_1^3}{64u_2^2}, \frac{u_1}{4u_2}, \pm \sqrt{-16}{27u_1^4u_2^2} \right) \) in \( \mathbb{C}^3 \). In other cases, \( f \) has one isolated singularity \((0, 0, 0) \).

### 3.2. Local \( b \)-functions

Let

\[ \mathcal{D}[s] = \left\{ \sum_{k \in \mathbb{N}, \beta \in \mathbb{N}^n} h_{k,\beta}(x) s^k \partial^\beta \middle| h_{k,\beta}(x) \in \mathbb{C}[x]_q \right\} \]

where \( \mathbb{C}[x]_q = \{g_1(x)/g_2(x) \mid g_1(x), g_2(x) \in \mathbb{C}[x], g_2(q) \neq 0\} \) the localization of \( \mathbb{C}[x] \) at \( q \in \mathbb{C}^n \).

The local \( b \)-function of a non-constant polynomial \( f \in \mathbb{C}[x] \) at \( q \) is defined as the monic polynomial \( b_{f,q}(s) \) of the minimal degree for \( p \in \mathcal{D}[s] \) and \( b_{f,q}(s) \in \mathbb{C}[s] \) satisfying \( p \cdot f^{s+1} = a(x)b_{f,q}(s) \cdot f^s \)

where \( a(x) \in \mathbb{C}[x]_q \). The reduced \( b \)-function of \( f \) at \( q \), written as \( \tilde{b}_{f,q}(s) \), is \( b_{f,q}(s)/(s + 1) \).

In [13], Mebkhout and Narváez-Macarro show the fact

\[ \tilde{b}_f(s) = \text{LCM}(\tilde{b}_{f,q}(s)|q \in \text{Sing}(f)) \]

where \( \text{Sing}(f) \) is the singular locus of \( \mathbb{V}(f) \), i.e., \( \text{Sing}(f) = \{f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n\} \).

We borrow from [22] the following theorem.

**Theorem 3.2.** Let \( f \in \mathbb{C}[x], \gamma \in \mathbb{Q} \) and set

\[ M_{(\gamma, f)} = \mathcal{D}[s]/(\text{Ann}(f^s) + \text{Id}(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n) + \text{Id}(s - \gamma)). \]

Then, if \( \tilde{b}_f(\gamma) \neq 0 \), then \( M_{(\gamma, f)} = \{0\} \), and if \( \tilde{b}_f(\gamma) = 0 \), then \( M_\gamma \) is a holonomic \( D \)-module and \( \text{supp}(M_{(\gamma, f)}) \subseteq \text{Sing}(f) \) where \( \text{supp}(M_{(\gamma, f)}) \) is the support of \( M_{(\gamma, f)} \).
Lemma 3.3. Using the same notation as in Theorem 3.2 let $\gamma$ be a root of $\tilde{b}_{f,q}(s) = 0$. Then, $q \in \text{supp}(M_{\gamma,f})$.

Algorithms and implementations for computing $M_{\gamma,f}$ and $\text{supp}(M_{\gamma,f})$, have been already introduced in [11] and in [14, 15, 20].

After here, we consider local $b$-functions of semi-weighted homogeneous polynomials at $O \in \mathbb{C}^n$. For a semi-weighted homogeneous polynomial, the following property is known.

Theorem 3.4 (C.2.1.6. [1]). For a semi-weighted homogeneous polynomial $f \in \mathbb{C}[x]$, the local $b$-function $\tilde{b}_{f,0}(s)$ is square-free, i.e., $\tilde{b}_{f,0}(s) = 0$ has no multiple roots.

Note that, there is a possibility that the global $b$-function of a semi-weighted homogeneous polynomial has multiple roots, whereas as Theorem 3.4 says, the local $b$-function of a semi-weighted homogeneous polynomial has no multiple roots.

We turn to parametric cases. Let $f$ be a semi-weighted homogeneous polynomial in $(\mathbb{C}[u])[x]$. Then, by utilizing Lemma 3.3 and Theorem 3.4 we are able to construct an algorithm for computing local $b$-functions of $f$ at $O \in \mathbb{C}^n$.

Algorithm 2. (Local $b$-functions at $O$)

**Input:** $U \subseteq \mathbb{C}^n$, $f = f_0 + g \in (\mathbb{C}[u])[x]$: $\forall \tilde{u} \in U$, $\sigma_{\tilde{u}}(f_0)$ is a weighted homogeneous polynomial with an isolated singularity at $O$ and $\sigma_{\tilde{u}}(f)$ is a semi-weighted homogeneous polynomial.

**Output:** $G_0 = \{(U_1, \tilde{b}_1(s)), \ldots, (U_\ell, \tilde{b}_\ell(s))\}$: for each $i \in \{1, \ldots, \ell\}$ and $\forall \tilde{u} \in U_i$, $\tilde{b}_i(s)$ is the local reduced $b$-function of $\sigma_{\tilde{u}}(f)$ at $O$ (i.e., $\tilde{b}_i(s) = \tilde{b}_{\sigma_{\tilde{u}}(f),0}(s)$) and $U = \bigcup_{i=1}^{\ell} U_i$.

**BEGIN**

$G_0 \leftarrow \emptyset$; $G \leftarrow \text{Execute Algorithm 1 on } U$;

while $G \neq \emptyset$ do

Select $(U', b(s))$ from $G$; $G \leftarrow G \setminus \{(U', b(s))\}$;

$E \leftarrow \text{Compute all roots of } b(s) = 0$;

$\tilde{b} \leftarrow 1$;

while $E \neq \emptyset$ do

Select $\gamma$ from $E$; $E \leftarrow E \setminus \{\gamma\}$;

if $O \subseteq \text{supp}(M_{\gamma,f})$ then

$\tilde{b} \leftarrow \tilde{b} \cdot (s - \gamma)$;

end-if

end-while

$G_0 \leftarrow G_0 \cup \{(U', \tilde{b})\}$

end-while

return $G_0$;

END

Example 2. Let us consider Example 1 again. The Milnor number $\mu$ of the singularity $x_3^3 + x_2 z^2 + y^5 = 0$, at the origin $O$, is 16, and the $\mu$-constant deformation is given by $f = x^3 + x_2 z^2 + y^5 + u_1 y^2 z^2 + u_2 y z^2$ where $u_1, u_2$ are parameters.

According to Algorithm 2, we compute the local $b$-function $\tilde{b}_{f,0}(s)$. We consider the case where $(u_1, u_2) \in \mathbb{V}(27u_1^4 + 256u_2) \setminus \mathbb{V}(u_1, u_2)$ and examine the holonomic $D$-module $M_{-\frac{1}{2},f}$ associated with the factor $s + \frac{1}{2}$ of $\tilde{b}_{f}(s)$. Direct computation shows

$$\text{supp}(M_{-\frac{1}{2},f}) = \mathbb{V}(64u_2^2 x - 3u_1^3, 4u_2 y - u_1, 16u_2 z^2 - u_1^2).$$
Since \( O \notin \text{supp}(M_{(-\frac{s}{2},f)}) \), \( s + \frac{3}{2} \) is not a factor of \( \bar{b}_{f,0}(s) \). The supports associated with other roots, contain the origin \( O \). Therefore, we obtain Table 2 as \( \bar{b}_{f,0}(s) \) where \( b_{\mu}(s) \) is from Example 1.

### Table 2. List of local \( b \)-functions

| stratum                  | (Local) \( \bar{b}_{f,0}(s) \) |
|--------------------------|---------------------------------|
| \( \mathbb{C}^2 \setminus \mathcal{V}(u_1) \) | \( b_{\mu}(s)(s + \frac{14}{15}) \) |
| \( \mathcal{V}(u_1) \setminus \mathcal{V}(u_2) \) | \( b_{\mu}(s)(s + \frac{17}{15}) \) |
| \( \mathcal{V}(u_1, u_2) \) | \( b_{\mu}(s)(s + \frac{20}{15}) \) |

As the example above shows that Algorithm 2 gives a computation method of \( b \)-functions associated with \( \mu \)-constant deformations. We implemented Algorithm 2 in the computer algebra system Risa/Asir [19], and we tried to compute all problems of Table 1 for three months by using 3 computers: PC1 [OS: Linux, CPU: Xeon E3-1225, 3.2 GHz, Memory: 126 GB], PC2 [OS: Linux, CPU: Xeon E3-1230, 3.3 GHz, Memory: 504 GB] and PC3 [OS: Windows 10, CPU: Core i7-5930k, 3.5 GHz, Memory: 64 GB].

All bases of the annihilating ideals of \( f^* \) in Table 1 were successfully obtained. It turned out by this computer experiment that the cost of computation of the part \((*1)\) of Algorithm 1 (i.e., computing a CGS of \( \text{Ann}(f^*) + Id(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \)) w.r.t. an elimination order) is quite high. Among 20 cases, our implementation could not return 12 \( b \)-functions within three months and output only 8 \( b \)-functions: \( b \)-functions of \( E_{18}, E_{20}, W_{18}, Z_{17}, Q_{16}, Q_{17}, S_{17}, U_{16} \). We see that the direct use of Algorithm 2 is not adequate for computing \( b \)-functions associated with \( \mu \)-constant deformations. In order to overcome difficulties, we improve the method presented in this section by specializing Algorithm 2 to handle semi-weighted homogeneous cases.

### 4. Computing local cohomology solutions to a holonomic \( D \)-module

Here we introduce an algorithm for computing local cohomology solutions of the holonomic \( D \)-module \( M_{(\gamma,f)} \). The algorithm will be utilized as a key tool in the new computation method of \( b \)-functions.

All local cohomology classes, in this paper, are algebraic local cohomology classes that belong to the set defined by

\[
H^n_{[O]}(\mathbb{C}[x]) = \lim_{k \to \infty} \text{Ext}^n_{\mathbb{C}[x]}(\mathbb{C}[x]/\langle x_1, x_2, \ldots, x_n \rangle^k, \mathbb{C}[x])
\]

where \( \langle x_1, x_2, \ldots, x_n \rangle \) is the maximal ideal generated by \( x_1, \ldots, x_n \). We adopt notations used in [16] to represent algebraic local cohomology classes, namely, we represent an algebraic local cohomology class as a polynomial \( \sum c_\lambda \xi^\lambda \) where \( \xi \) is the abbreviation of \( n \) variables \( \xi_1, \ldots, \xi_n \), \( c_\lambda \in \mathbb{C} \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \). The multiplication is defined as

\[
x^\alpha \ast \xi^\lambda = \begin{cases} 
\xi^{\lambda - \alpha}, & \lambda_i \geq \alpha_i, i = 1, \ldots, n, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n \) and \( \lambda - \alpha = (\lambda_1 - \alpha_1, \ldots, \lambda_n - \alpha_n) \in \mathbb{N}^n \). The partial derivative by \( \frac{\partial}{\partial x_i} \) is defined as

\[
\frac{\partial}{\partial x_i} (\xi_1^{\lambda_1} \xi_2^{\lambda_2} \cdots \xi_i^{\lambda_i} \cdots \xi_n^{\lambda_n}) = -(\lambda_i + 1)\xi_1^{\lambda_1} \xi_2^{\lambda_2} \cdots \xi_i^{\lambda_i + 1} \cdots \xi_n^{\lambda_n}.
\]
Let fix a monomial order $\succ$. For a given algebraic local cohomology class of the form

$$\psi = c_\lambda \xi^\lambda + \sum_{\xi^\lambda \succ \xi^\nu} c_\nu \xi^\nu, \quad c_\lambda \not= 0$$

we call $\xi^\lambda$ the head monomial, $c_\lambda$ the head coefficient and $\xi^\nu$ the lower monomials. We write the head monomial as $\text{hm}(\psi)$.

Let $f$ be a holomorphic function defined on an open neighborhood $X$ of the origin $O$ of the $n$-dimensional complex space $\mathbb{C}^n$, with an isolated singularity at the origin. Let $\gamma$ be a root of the local reduced $b$-function $\tilde{b}_{f,0}(s)$ at $O$. Let $G'$ be a minimal Gröbner basis of $\text{Ann}(f^*) + \text{Id}(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) + \text{Id}(s - \gamma)$ w.r.t. a monomial order satisfying $\{x, \partial\} \gg s$ in $D[s]$. Set $G_{(\gamma,f)} = G' \setminus \{s - \gamma\}$, then $M_{(\gamma,f)} = D/\text{Id}(G_{(\gamma,f)})$. We define a set $H_{M_{(\gamma,f)}}$ to be the set of algebraic local cohomology classes in $H^n_O(\mathbb{C}[x])$ that are annihilated by $G_{(\gamma,f)}$:

$$H_{M_{(\gamma,f)}} = \left\{ \psi \in H^n_O(\mathbb{C}[x]) \mid h \ast \psi = 0, \forall h \in G_{(\gamma,f)} \right\}.$$

Since $H_{M_{(\gamma,f)}}$ is the algebraic local cohomology solution space of the holonomic $D$-module $M_{(\gamma,f)}$, we have the following.

**Theorem 4.1.** The set $H_{M_{(\gamma,f)}}$ is a finite dimensional vector space.

Here we introduce an algorithm for computing a basis of the vector space $H_{M_{(\gamma,f)}}$.

**Lemma 4.2.** Using the same notation as in above, let $P_0 = G_{(\gamma,f)} \cap \mathbb{C}[x]$, $F_0 = P_0 \cup \{f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\} \subset \mathbb{C}[x]$. Set

$$H_{F_0} = \left\{ \psi \in H^n_O(\mathbb{C}[x]) \mid h \ast \psi = 0, \forall h \in F_0 \right\}.$$

Then, $H_{M_{(\gamma,f)}} \subseteq H_{F_0}$.

Since $P_0 \subseteq G_{(\gamma,f)}$, Lemma 4.2 holds. Note that $F_0 \subseteq \mathbb{C}[x]$, thus, a basis of the vector space $H_{F_0}$ can be obtained by the algorithm [16, 21]. An algorithm for computing a basis of $H_{M_{(\gamma,f)}}$ is the following.

**Algorithm 3.** (A basis of $H_{M_{(\gamma,f)}}$)

**Input:** $f \in \mathbb{C}[x]$: a polynomial with an isolated singularity at $O$. $\gamma \in \mathbb{Q}$: a root of $\tilde{b}_{f,0}(s)$. Fix a monomial order $\succ$ on $\mathbb{C}[\xi]$.

**Output:** $\Psi$: a basis of the vector space $H_{M_{(\gamma,f)}}$.

**BEGIN**

$G_{(\gamma,f)} \leftarrow \text{Compute } G_{(\gamma,f)}; \quad \{f_1, \ldots, f_r\} \leftarrow G_{(\gamma,f)} \cap \mathbb{C}[x]$;

$F_0 \leftarrow \{f_1, \ldots, f_r\} \cup \{f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\}; \quad P_1 \leftarrow G_{(\gamma,f)} \setminus \{f_1, \ldots, f_r\}$;

$G_0 \leftarrow \text{Compute a basis of the vector space } H_{F_0}; \quad /* G_0 \text{ is echelon form. */}$

$\Psi \leftarrow \emptyset; \quad L \leftarrow \emptyset$;

**while** $G_0 \not= \emptyset$ **do**

Select $\psi$ whose head monomial is the smallest in $\text{hm}(G_0); G_0 \leftarrow G_0 \setminus \{\psi\}$;

$\varphi \leftarrow \psi + \sum_{\xi^\lambda \in L} c_i \xi^\lambda; \quad /* \text{The symbol } c_i \text{ is an indeterminate. */}$

$E \leftarrow \text{Make a system of linear equations with } c_i \text{ from } \{p \ast \psi = 0 | p \in P_1\};$

**if** $E$ has a solution **then**

$\varphi' \leftarrow \text{Substitute the solution into } c_i \text{ of } \varphi;$

$\Psi \leftarrow \Psi \cup \{\varphi'\}$;

**else**

*/

**END**
Theorem 4.3. Algorithm 3 returns a basis of the vector space $H_{M(\gamma,f)}$ and terminates.

**Proof.** As $H_{F_0}$ is the finite dimensional vector space, the set $G_0$ is finite. Thus, this algorithm terminates. Since each element in the output $\Psi$ satisfies linear partial differential equation $G_{(\gamma,f)}$, we have $\Psi \subseteq H_{M(\gamma,f)}$. Since $H_{M(\gamma,f)} \subseteq H_{F_0}$, we have $\text{Span}(\Psi) = H_{M(\gamma,f)}$. Furthermore, each element in $\Psi$ has a form $\psi + \sum c_i \zeta_i$, they are linearly independent. Therefore, the algorithm returns a basis of the vector space $H_{M(\gamma,f)}$.

We illustrate Algorithm 3 with the following example

**Example 3.** Let us consider $f = x^3 + y^2 + y^7 + xy^5 + xz^2 \in \mathbb{C}[x,y,z]$ that defines an isolated singularity at the origin. By computing $b_{f,0}$, we have rational numbers $\gamma = -\frac{10}{21}$ and $\bar{\gamma} = -\frac{1}{3}$ as roots of $b_{f,0}(\gamma) = 0$. Let us execute Algorithm 3 to get bases of the vector spaces $H_{M(-\frac{10}{21},f)}$ and $H_{M(-\frac{1}{3},f)}$.

Let $\xi, \eta, \zeta$ denote symbols correspond to the variables $x, y, z$, and $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$, $\partial_z = \frac{\partial}{\partial z}$. The monomial order $\succ$ is the degree lexicographic with $\xi \succ \eta \succ \zeta$.

- In case $\gamma = -\frac{10}{21}$, then $G_{(-\frac{10}{21},f)} = \{x, y, z\}$. Thus, it is obvious that $H_{M(-\frac{10}{21},f)} = \text{Span}(1)$.
- In case $\gamma = -\frac{1}{3}$, then $G_{(-\frac{1}{3},f)} = \{x^2, xy, xz, yx, x^2, 12x - y^3, 4x^2\partial_y + x + 2y^2, 81xz\partial_z - 1367x + 32y^3 - 1152y^2\partial_y - 48y^2 - 768yz\partial_z - 5376y, 41472x\partial_x + 61503x + 1064y^2 + 13824y\partial_y + 576y + 41472z\partial_z + 138240, 35378x + 124416y^2\partial_z + 184509y^2 - 47664y\partial_y^2 - 41742y\partial_y + 75744y - 1492992z\partial_z - 62208z\partial_z - 3981312z\partial_y - 165888\}$. Set

$$P_0 = G_{(-\frac{1}{3},f)} \cap \mathbb{C}[x,y,z]$$

A basis $G_0$ of the vector space of $H_{F_0}$ is $G_0 = \{1, \xi, \eta, \zeta, \eta^3 + \frac{1}{12} \xi\}$. Set

$$P_1 = G_{(-\frac{1}{3},f)} \setminus P_0 = \{24x\partial_y + x + 2y^2, 81xz\partial_z - 1367x + 32y^3 - 1152y^2\partial_y - 48y^2 - 768yz\partial_z - 5376y, 41472x\partial_x + 61503x + 1064y^2 + 13824y\partial_y + 576y + 41472z\partial_z + 138240, 35378x + 124416y^2\partial_z + 184509y^2 - 47664y\partial_y^2 - 41742y\partial_y + 75744y - 1492992z\partial_z - 62208z\partial_z - 3981312z\partial_y - 165888\}, \Psi = \emptyset$$

Thus, renew $L$ as $L \cup \{1\}$.

(2) Take $\zeta$ whose head monomial is the smallest in $\text{hm}(G_0)$ w.r.t. $\succ$. Renew $G_0$ as $G_0 \setminus \{\zeta\}$. Set $\varphi = \zeta + c$ where $c$ is an indeterminate. Then $\{p \ast \varphi | p \in P_1\} = \{0, 0, 41472e, -62208c + 1492992e\}$. Thus, when $c = 0$, then $\varphi \in H_{M(-\frac{1}{3},f)}$. Renew $\Psi$ as $\Psi \cup \{\zeta\}$.

(3) Take $\eta$ whose head monomial is the smallest in $\text{hm}(G_0)$. Renew $G_0$ as $G_0 \setminus \{\eta\}$. Set $\varphi = \eta + c$ where $c$ is an indeterminate. Then, $\{p \ast \varphi | p \in P_1\} = \{0, -2304, (41472c + 27648\eta, (75744 - 62208c) + (1492992e - 20736)e)\}$. Obviously, the second element $-2304$ is not zero. Thus, renew $L$ as $\{\eta, 1\}$.

(4) Take $\eta^2$ whose head monomial is the smallest in $\text{hm}(G_0)$. Renew $G_0$ as $G_0 \setminus \{\eta\}$. Set $\varphi = \eta^2 + c_1 \eta + c_2$ where $c_1, c_2$ are indeterminates. Then, $(24x\partial_y + x + 2y^2) \ast \varphi = 2 \neq 0$ where $24x\partial_y + x + 2y^2 \in P_1$. Hence, $\varphi \notin H_{M(-\frac{1}{3},f)}$ and renew $L$ as $\{\eta^2, \eta, 1\}$. 


(5) Take $\eta^3 + \frac{1}{12}\xi$ from $G_0$. Renew $G_0$ as $G_0 \cap \{\eta^3 + \frac{1}{12}\xi\}$ where $c_1, c_2, c_3$ are indeterminates. Set $\varphi = \eta^3 + \frac{1}{12}\xi + c_1\eta^2 + c_2\eta + c_3$. Then, 

$\{p \circ \varphi | p \in P_1\} = \{2c_1 + \frac{1}{12}, 48c_1 + 2304c_2 + \frac{266}{3}, (1152c_1 + 48)\eta, 1064c_1 + 576c_2 + 41472c_3 + \frac{20501}{4} + (576c_1 + 27648c_2 + 1064)\eta + (13824c_1 + 576)\eta^2, 184509c_1 + 75744c_2 + 62208c_3 + \frac{17689}{6} + (75744c_1 - 20736c_2 + 1492992c_3 + 184509)\eta - (124416c_1 + 5184)\eta^2 + (20736c_1 + 1990656c_2 + 75744)\eta^3 + (1492992c_1 + 62208)\eta^3\}$.

Solve the following system of linear equations that are from $\{p \circ \varphi | p \in P_1\}$.

\[
\begin{align*}
2c_1 + \frac{1}{12} &= 0, \\
48c_1 + 2304c_2 + \frac{266}{3} &= 0, \\
1152c_1 + 48 &= 0, \\
1064c_1 + 576c_2 + 41472c_3 + \frac{20501}{4} &= 0, \\
576c_1 + 27648c_2 + 1064 &= 0, \\
13824c_1 + 576 &= 0, \\
184509c_1 + 75744c_2 + 62208c_3 + \frac{17689}{6} &= 0, \\
75744c_1 - 20736c_2 + 1492992c_3 + 184509 &= 0, \\
124416c_1 + 5184 &= 0, \\
20736c_1 + 1990656c_2 + 75744 &= 0, \\
1492992c_1 + 62208 &= 0.
\end{align*}
\]

Then, $c_1 = -\frac{1}{248}, c_2 = -\frac{65}{1728}, c_3 = 0$. Renew $\Psi$ as $\{\eta, \eta^3 + \frac{1}{12}\xi - \frac{1}{248}\eta^2 - \frac{65}{1728}\eta\}$.

(6) Since $G_0 = \emptyset$, Algorithm 3 stops. Therefore, $\Psi$ is a basis of the vector space $H_{\gamma,n}(\frac{1}{4}, f)$.

By using a framework presented in [16], we have extended Algorithm 3 to handle parametric cases. The resulting algorithm that compute the parametric local cohomology solution space of parametric holonomic $D$-module $M(\gamma, f)$ is implemented in the computer algebra system Risa/Asir.

5. New algorithm

Here we introduce a new algorithm for computing reduced $b$-functions $\tilde{b}_{f,0}$ of a semi-weighted homogeneous polynomial. As we described in subsection 3.2, the computational complexity of computing a Gröbner basis of $\text{Ann}(f^*) + \text{Id}(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is quite high. In order to overcome the difficulty, we adopt the idea introduced by Levandovskyy and Martin-Morales [12] and address to the computation of $b$-functions associated with $\mu$-constant deformations.

Let $f = f_0 + g \in \mathbb{C}[x]$ where $f_0$ is the weighted homogeneous part, $d \in \mathbb{N}$, $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ and $g$ is a linear combination of upper monomials.

5.1. Properties of semi-weighted homogeneous singularities

First we review some properties of semi-weighted homogeneous singularities that are needed for constructing a new algorithm.

Definition 5.1. Let $f_0$ be a weighted homogeneous polynomial of type $(d; w)$ with an isolated singularity at the origin $O$. The Poincaré polynomial of $f_0$ is the univariate polynomial defined to be

\[ P_{(d, w)}(t) = \frac{t^{d-w_1} - 1}{t^{w_1} - 1} \cdot \frac{t^{d-w_2} - 1}{t^{w_2} - 1} \cdots \frac{t^{d-w_n} - 1}{t^{w_n} - 1}. \]

It is well-known that all roots of $\tilde{b}_{f_0,0}$ can be computed by the Poincaré polynomial.

Theorem 5.2. Let $f_0$ be a weighted homogeneous polynomial of type $(d; w)$ with an isolated singularity at the origin $O$, and $w_0 = \sum_{i=1}^n w_i$. Let $P_{(d, w)}(t) = \sum_{i=1}^r c_i t^{c_i} = (c_i \neq 0)$ be the Poincaré polynomial of type $(d; w)$. Then, the set of roots of $\tilde{b}_{f_0,0}(s) = 0$ is equal to $\{-\frac{w_i + mw}{d} | 1 \leq i \leq r\}$.

The following properties are from [6, 22].

Theorem 5.3. Let $\gamma_1, \ldots, \gamma_r$ be the set of roots of $\tilde{b}_{f,0}(s) = 0$ and let $\mu$ be the Milnor number of the singularity $f = 0$ at the origin $O$. Then, $\mu = \sum_{i=1}^r \dim_{\mathbb{C}}(H_{\gamma_i, f})$. 

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5.2. New algorithm

The new algorithm mainly consists of the following three steps,

Step 1: to compute candidates $\gamma$s of the roots of $\tilde{b}_{f,0}(s) = 0$,
Step 2: to check whether $\tilde{b}_{f,0}(\gamma) = 0$ or $\tilde{b}_{f,0}(\gamma) \neq 0$,
Step 3: to check whether $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ is equal to $\{\gamma | \tilde{b}_{f,0}(\gamma) = 0\}$ or not, where $\gamma_1, \ldots, \gamma_r$ are roots of $\tilde{b}_{f,0}(s) = 0$.

The following lemma quoted from [7], tells us how to compute the candidates. 

**Lemma 5.4.** Let $E_0 = \{\gamma \in \mathbb{Q} | \tilde{b}_{f,0}(\gamma) = 0\}$ and $E = \{\gamma \in \mathbb{Q} | \tilde{b}_{f,0}(\gamma) = 0\}$. Then, $E$ is a subset of $E' = \{\gamma + k | \gamma \in E_0, k \in \mathbb{Z}, -n < \gamma + k < 0\}$ where $\mathbb{Z}$ is the set of integers.

Since $E_0$ is determined by Theorem 5.2 it is easy to obtain $E'$. Empirically, it is sufficient to check $k = 0, 1, 2$. Hence, in Step 1, we use

$E' = \{\gamma + k | \gamma \in E_0, k \in \{0, 1, 2\}, -n < \gamma + k < 0\}$

as $s$ set of candidates of the roots.

Next, in Step 2, we have to check whether $\gamma \in E'$ is a root of $\tilde{b}_{f,0}(s) = 0$ or not. We borrow the idea of Levandovskyy and Martin-Morales [12].

**Lemma 5.5.** Let $H$ be a basis of $\text{Ann}(f^s)$ in $D[s]$ and $\gamma \in \mathbb{Q}$. Let $G$ be a minimal Gröbner basis of $\text{Id}(H \cup \{f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\} \cup \{s - \gamma\})$ w.r.t. a block order with $x \cup \partial \gg s$. Then, if $s - \gamma \in G$, $s - \gamma$ is a factor of the global $b$-function of $f$.

**Remark:** The computational speed of computing a minimal Gröbner basis of $\text{Id}(H \cup \{f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\} \cup \{s - \gamma\})$ is much faster than that of $\text{Id}(H \cup \{f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\})$, because the degree of $s - \gamma$ is 1. Moreover, for a candidate $\gamma \in E'$, it is sufficient to check the single root because of Theorem 3.4. However, as we are considering “local” $b$-functions at the origin $O$, we need to check the support of $\gamma$ (i.e., $\text{supp}(M_{(\gamma, f)})$) if $s - \gamma \in G$.

As we know how to compute a CGS in $D[s]$, we can naturally extend the idea to parametric cases. We have implemented the parametric version of Lemma 5.5 in Risa/Asir.

In our implementation, the command $\text{para\_ann1}$ (or $\text{para\_ann}$) returns a CGS of $\text{Ann}(f^s)$, and the command $\text{root\_check}$ (or $\text{root\_check11}$) returns a CGS of $\text{Id}(\text{Ann}(f^s) \cup \{f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\} \cup \{s - \gamma\})$ where $\gamma \in \mathbb{Q}$. In Example 4, $\text{ANN}$ is a CGS of $\text{Ann}(f^s)$. The form $[[S1], [S2]]$ means a stratum $\forall(S1) \setminus \forall(S2)$.

**Example 4.** Let us consider $S_{16}$ singularity. The $\mu$-constant deformation is given by $f = x^2z + yz^2 + xy^4 + u_1y^6 + u_2z^3$ where $u_1, u_2$ are parameters. Let us check whether $17s + 19$ is a factor of $\tilde{b}_f(s)$ or not.

```
[2727] F=x^2*z+y*z^2+x*y^4+u1*y^6+u2*z^3$s
[2728] ANN=para_ann1(F, [u1, u2], [x, y, z])$
[2729] roots_check(ANN, F, 17*s+19, [u1, u2], [x, y, z]);
[10], (486+u2*u1^13-1143+u2^2*u1^9-639+u2^3*u1^5-68+u2^4+u1])
[17*s+19, z, x, y^2, (4030+u1^4+4913+dy*u1-9826+u2)*y+11560+u2^2+z+9826+u1]
[[27*u1^4+4+u2], [u1, u2]]
[17*s+19, z, x, y^2, (140711*u1^3+9826+dy)*y+19652]
[[u1, u2], [1]]
[1]
```
\[
[[6+u_1^4-17*u_2], [u_1, u_2]]
\]
\[
(17*s+19, z, x, y^2, (-562*u_1^3-4913*dy)*y-9826)
\]

\[
[[3+u_1^4+u_2], [u_1, u_2]]
\]
\[
(17*s+19, z, x, y^2, (33508*u_1^3+4913*dy)*y+9826)
\]

\[
[[u_1], [u_1, u_2]]
\]
\[
(17*s+19, z, y, x)
\]

\[
[[u_2], [u_1, u_2]]
\]
\[
(17*s+19, z, x, y^2, (4030*u_1^3+4913*dy)*y+9826)
\]

The monomial order \( \succ \) used in the computation above is a block order \( \{ \partial_x, \partial_y, \partial_z \} \succ \{ x, y, z \} \succ s \) which is specified on Mono(\( \{ \partial_x, \partial_y, \partial_z \} \)) as the total degree lexicographic monomial order with \( \partial_x \succ \partial_y \succ \partial_z \), and on Mono(\( \{ x, y, z \} \)) as the total degree lexicographic monomial order with \( z \succ y \succ x \).

Let \( I \) denote the ideal generated by \( \text{Ann}(f^s) \) and \( \{ f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \}, 17s+19 \} \) in the ring \( (D[u_1, u_2])[s] \), i.e., \( \text{Id}(\text{Ann}(f^s) \cup \{ f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \}, 17s+19 \}) \).

The meanings of the output above are the following.\\
1. If \( (u_1, u_2) \) belongs to \( U_1 = \left( C^2 \setminus \mathbb{V}(486u_2u_1^3 - 1143u_2^2u_1^9 - 639u_2^3u_1^8 - 68u_2^4u_1^7) \right) = \left( C^2 \setminus \mathbb{V}((27u_1^4 + 4u_2)(6u_1^7 - 17u_2)(3u_1^4 + u_1^2)u_2^2) \right) \), then
   \[
   G_1 = \{ 17s + 19, z, x, y^2, (4030u_1^4 + 4913\partial_y u_1 - 9826u_2)y + 11560u_2^2z + 9826u_2 \}
   \]
   is a minimal Gröbner basis of \( I \) w.r.t. \( \succ \). As \( 17s + 19 \in G_1 \) and \( G_1 \cap C[x, y, z] = \{ z, x, y^2 \} \), \( 17s + 19 \) is a factor of \( \tilde{b}_f(s) \) and \( \text{supp}(M(-1/2, f)) = \{ O \} \), namely, \( 17s + 19 \) is a factor of \( \tilde{b}_{f,0}(s) \).

2. If \( (u_1, u_2) \) belongs to \( U_2 = \mathbb{V}(27u_1^4 + 4u_2) \setminus \mathbb{V}(u_1^2, u_2) \), then \( G_2 = \{ 17s + 19, z, x, y^2, (140711u_1^3 + 9826\partial_y u_1) + 19652 \} \) is a minimal Gröbner basis of \( I \) w.r.t. \( \succ \). As \( 17s + 19 \in G_2 \) and \( C[x, y, z] = \{ z, x, y^2 \} \), \( 17s + 19 \) is a factor of \( \tilde{b}_f(s) \).

3. If \( (u_1, u_2) \) belongs to \( U_3 = \mathbb{V}(u_1, u_2) \), then \( G_3 = \{ 1 \} \) is a minimal Gröbner basis of \( I \) w.r.t. \( \succ \). As \( 17s + 19 \notin G_3 \), \( 17s + 19 \) is not a factor of \( \tilde{b}_f(s) \).

4. If \( (u_1, u_2) \) belongs to \( U_4 = \mathbb{V}(6u_1^4 - 17u_2) \setminus \mathbb{V}(u_1^2, u_2) \), then \( G_4 = \{ 17s + 19, z, x, y^2, (-562u_1^3 - 4913\partial_y u_1) + 9826 \} \) is a minimal Gröbner basis of \( I \) w.r.t. \( \succ \). As \( 17s + 19 \in G_4 \) and \( G_4 \cap C[x, y, z] = \{ z, x, y^2 \} \), \( 17s + 19 \) is a factor of \( \tilde{b}_f(s) \).

5. If \( (u_1, u_2) \) belongs to \( U_5 = \mathbb{V}(3u_1^4 + u_2) \setminus \mathbb{V}(u_1, u_2) \), then \( G_5 = \{ 17s + 19, z, x, y^2, (33508u_1^3 + 4913\partial_y u_1) + 9826 \} \) is a minimal Gröbner basis of \( I \) w.r.t. \( \succ \). As \( 17s + 19 \in G_5 \) and \( G_5 \cap C[x, y, z] = \{ z, x, y^2 \} \), \( 17s + 19 \) is a factor of \( \tilde{b}_f(s) \).

6. If \( (u_1, u_2) \) belongs to \( U_6 = \mathbb{V}(u_1) \setminus \mathbb{V}(u_1, u_2) \), then \( G_6 = \{ 17s + 19, z, y, x \} \) is a minimal Gröbner basis of \( I \) w.r.t. \( \succ \). As \( 17s + 19 \in G_6 \) and \( G_6 \cap C[x, y, z] = \{ z, y, x \} \), \( 17s + 19 \) is a factor of \( \tilde{b}_{f,0}(s) \).

7. If \( (u_1, u_2) \) belongs to \( U_7 = \mathbb{V}(u_2) \setminus \mathbb{V}(u_1, u_2) \), then \( G_7 = \{ 17s + 19, z, x, y^2, (4030u_1^3 + 4913\partial_y u_1) + 9826 \} \) is a minimal Gröbner basis of \( I \) w.r.t. \( \succ \). As \( 17s + 19 \in G_7 \) and \( G_7 \cap C[x, y, z] = \{ z, x, y^2 \} \), \( 17s + 19 \) is a factor of \( \tilde{b}_{f,0}(s) \).

Note that \( 17s + 19 \in G_i \) \( (i = 1, 2, 4, 5, 6, 7) \) and \( 17s + 19 \notin G_3 \).

Since
\[
U_1 \cup U_2 \cup U_4 \cup U_5 \cup U_6 \cup U_7 = C^2 \setminus U_3,
\]
we have
i) if \( (u_1, u_2) \neq (0, 0) \), then \( 17s + 19 \) is a factor of \( \tilde{b}_f(s) \) and \( \text{supp}(M(-1/2, f)) = \{ O \} \), namely,
Methods for computing \( b \)-functions

17\( s \) + 19 is a factor of (local) \( \tilde{b}_{f,0}(s) \).

ii) if \( u_1 = u_2 = 0 \), then 17\( s \) + 19 is not a factor of (global) \( \tilde{b}_f(s) \).

In Step 3, we apply Theorem \[5.3\] and we use Algorithm 3 for computing \( \dim_{\mathbb{C}}(H_{M(\gamma,f)}) \).

---

**Algorithm 4.** (Local \( b \)-function at \( O \))

**Input:** \( f = f_0 + g \in \mathbb{C}[x] \): a semi-weighted homogeneous polynomial with an isolated singularity at \( O \).

\( H \): a basis of \( \text{Ann}(f^*) \). \( \mu(f_0) \in \mathbb{N} \): the Milnor number of \( f_0 \) at the origin \( O \).

\( \succ \): a monomial order satisfying \( \{x, \partial\} \succ s \).

**Output:** \( b \): the reduced \( b \)-function \( \tilde{b}_{f,0}(s) \).

**BEGIN**

\[
\begin{align*}
\quad & b \leftarrow 1; \quad \mu \leftarrow 0; \quad k \leftarrow 0; \quad J \leftarrow \{ f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \}; \\
& E_0 \leftarrow \text{Compute all roots of } \tilde{b}_{f_0,0}(s) \text{ by the Poincaré polynomial}; \\
& \text{while } \mu \neq \mu(f_0) \text{ do} \\
& \quad E' \leftarrow \{ \gamma + k | \gamma \in E_0, -n < \gamma + k < 0 \}; \\
& \quad \text{while } E' \neq \emptyset \text{ do} \\
& \quad \quad \text{Select } \gamma \text{ from } E'; \quad E' \leftarrow E' \setminus \{ \gamma \}; \\
& \quad \quad G \leftarrow \text{Compute a minimal Gröbner basis of } Id(H \cup J \cup \{ s - \gamma \}) \text{ w.r.t. } \succ; \\
& \quad \text{if } (s - \gamma \in G) \text{ and } (O \in \text{supp}(M_{(\gamma,f)})) \text{ then} \\
& \quad \quad b \leftarrow b \cdot (s - \gamma); \\
& \quad \quad \mu \leftarrow \mu + \dim_{\mathbb{C}}(H_{M_{(\gamma,f)}}); \\
& \quad \quad \text{end-if} \\
& \quad k \leftarrow k + 1; \\
& \text{end-while} \\
& \text{return } b; \\
\end{align*}
\]

**END**

As we described in section 3 and 4, a CGS of \( \text{Ann}(f^*) \) and a basis of the vector space \( H_{M(\gamma,f)} \) with parameters, are computable. Thus, Algorithm 4 can be generalized to parametric cases, too. We have computed all \( b \)-functions of Table \[4\] by utilizing the generalized algorithm. We have obtained all \( b \)-functions of Table \[4\] successfully. The \( b \)-functions of Table \[4\] are given in Section 6.

**Remark:** We tried to compute local \( b \)-functions associated with \( \mu \)-constant deformations of inner modality 3 singularities by using a computer PC1 (see section 3). We could obtain none of CGSs of \( \text{Ann}(f^*) \) within three months and thus we could not use the method described in section 5. We expect however that local \( b \)-functions associated with \( \mu \)-constant deformations can be computed by using the proposed method provided that parametric bases of \( \text{Ann}(f^*) \) is given.

In this paper, we introduce a new algorithm for computing parametric \( \tilde{b}_{f,0} \) of a semi-weighted homogeneous polynomial by improving Algorithm 2. The resulting algorithm has better performance than Algorithm 2. The resulting algorithm and Algorithm 2 use the same CGS algorithm described in \[14, 15\]. The difference lies in the way of its using. The versatility of CGS algorithm allows the specialization and the improvement.
6. List of $b$-functions associated with $\mu$-constant deformations

Here all $b$-functions $\bar{b}_{f,0}(s)$ of $\mu$-constant deformation of inner modality 2 singularities, are presented.

Currently, our Risa/Asir implementation is in the following webpage

https://www.math.ias.tokushima-u.ac.jp/~nabesima/bfunction/bfunc2.html

Two computation times “CGS of $\text{Ann}(f^*)$” and “parametric version of Algorithm 4” are also presented in each $\mu$-constant deformation. The time is given in CPU seconds. The computer [OS: Windows 10, CPU: intel core i9-7900X, 3.30 GHz, Memory: 128 GB] was used.

- $E_{18}$: $f = x^3 + y^{10} + u_1xy^7 + u_2xy^8$

| stratum                      | $b_{f,0}(s)$ |
|------------------------------|--------------|
| $\mathbb{C}^2 \setminus \mathcal{V}(u_1)$ | $b_{u_2}(s)(s + \frac{14}{30})(s + \frac{17}{30})$ |
| $\mathcal{V}(u_1) \setminus \mathcal{V}(u_1, u_2)$ | $b_{u_2}(s)(s + \frac{23}{30})(s + \frac{23}{30})$ |
| $\mathcal{V}(u_1, u_2)$ | $b_{u_2}(s)(s + \frac{33}{30})(s + \frac{18}{30})$ |

$b_{u_2}(s) = (s + \frac{13}{30})(s + \frac{16}{30})(s + \frac{19}{30})(s + \frac{22}{30})(s + \frac{23}{30})(s + \frac{23}{30})(s + \frac{23}{30})$
\hspace{1cm} $\times(s + \frac{29}{30})(s + \frac{30}{30})(s + \frac{21}{30})(s + \frac{22}{30})(s + \frac{23}{30})(s + \frac{23}{30})(s + \frac{23}{30})(s + \frac{23}{30})$
\hspace{1cm} $\times(s + \frac{20}{30})(s + \frac{21}{30})(s + \frac{22}{30})(s + \frac{23}{30})(s + \frac{23}{30})(s + \frac{23}{30})(s + \frac{23}{30})(s + \frac{23}{30})$.

CGS of $\text{Ann}(f^*)$: 3141 parametric version of Algorithm 4: 5.156

- $E_{19}$: $f = x^3 + xy^7 + u_1y^{11} + u_2y^{12}$

| stratum                      | $b_{f,0}(s)$ |
|------------------------------|--------------|
| $\mathbb{C}^2 \setminus \mathcal{V}(u_1)$ | $b_{u_2}(s)(s + \frac{12}{30})(s + \frac{31}{30})$ |
| $\mathcal{V}(u_1) \setminus \mathcal{V}(u_1, u_2)$ | $b_{u_2}(s)(s + \frac{12}{30})(s + \frac{31}{30})$ |
| $\mathcal{V}(u_1, u_2)$ | $b_{u_2}(s)(s + \frac{12}{30})(s + \frac{31}{30})$ |

$b_{u_2}(s) = (s + \frac{9}{30})(s + \frac{11}{30})(s + \frac{13}{30})(s + \frac{15}{30})(s + \frac{17}{30})(s + \frac{18}{30})(s + \frac{19}{30})$
\hspace{1cm} $\times(s + \frac{29}{30})(s + \frac{21}{30})(s + \frac{22}{30})(s + \frac{23}{30})(s + \frac{24}{30})(s + \frac{25}{30})(s + \frac{26}{30})(s + \frac{27}{30})(s + \frac{28}{30})$.

CGS of $\text{Ann}(f^*)$: 10410 parametric version of Algorithm 4: 45.5

- $E_{20}$: $f = x^3 + y^{11} + u_1xy^8 + u_2xy^9$

| stratum                      | $b_{f,0}(s)$ |
|------------------------------|--------------|
| $\mathbb{C}^2 \setminus \mathcal{V}(u_1)$ | $b_{u_2}(s)(s + \frac{13}{30})(s + \frac{12}{30})$ |
| $\mathcal{V}(u_1) \setminus \mathcal{V}(u_1, u_2)$ | $b_{u_2}(s)(s + \frac{13}{30})(s + \frac{12}{30})$ |
| $\mathcal{V}(u_1, u_2)$ | $b_{u_2}(s)(s + \frac{13}{30})(s + \frac{12}{30})$ |

$b_{u_2}(s) = (s + \frac{14}{30})(s + \frac{17}{30})(s + \frac{20}{30})(s + \frac{23}{30})(s + \frac{25}{30})(s + \frac{26}{30})(s + \frac{28}{30})(s + \frac{31}{30})$
\hspace{1cm} $\times(s + \frac{33}{30})(s + \frac{34}{30})(s + \frac{35}{30})(s + \frac{37}{30})(s + \frac{38}{30})(s + \frac{41}{30})(s + \frac{43}{30})(s + \frac{46}{30})$.

CGS of $\text{Ann}(f^*)$: 2667 parametric version of Algorithm 4: 164

- $W_{17}$: $f = x^4 + xy^5 + u_1y^7 + u_2y^8$

| stratum                      | $b_{f,0}(s)$ |
|------------------------------|--------------|
| $\mathbb{C}^2 \setminus \mathcal{V}(u_1)$ | $b_{u_2}(s)(s + \frac{9}{30})(s + \frac{12}{30})$ |
| $\mathcal{V}(u_1) \setminus \mathcal{V}(u_1, u_2)$ | $b_{u_2}(s)(s + \frac{12}{30})(s + \frac{20}{30})$ |
| $\mathcal{V}(u_1, u_2)$ | $b_{u_2}(s)(s + \frac{13}{30})(s + \frac{12}{30})$ |

$b_{u_2}(s) = (s + \frac{8}{30})(s + \frac{11}{30})(s + \frac{13}{30})(s + \frac{14}{30})(s + \frac{15}{30})(s + \frac{17}{30})(s + \frac{18}{30})(s + \frac{19}{30})$
\hspace{1cm} $\times(s + \frac{20}{30})(s + \frac{21}{30})(s + \frac{22}{30})(s + \frac{23}{30})(s + \frac{24}{30})(s + \frac{25}{30})(s + \frac{27}{30})$.

CGS of $\text{Ann}(f^*)$: 354.1 parametric version of Algorithm 4: 7.547

- $W_{18}$: $f = x^4 + y^7 + u_1x^2y^4 + u_2x^2y^5$
\[
\begin{align*}
\text{CGS of Ann}(f^a): 149.8 & \quad \text{parametric version of Algorithm 4: 2.438} \\
\text{CGS of Ann}(f^a): 546.1 & \quad \text{parametric version of Algorithm 4: 5.344} \\
\text{CGS of Ann}(f^a): 984.2 & \quad \text{parametric version of Algorithm 4: 5.141} \\
\text{CGS of Ann}(f^a): 50730 & \quad \text{parametric version of Algorithm 4: 3.953} \\
\text{CGS of Ann}(f^a): 1073 & \quad \text{parametric version of Algorithm 4: 2.813}
\end{align*}
\]

\[
\begin{array}{|c|c|}
\hline
C^2 \setminus V(u_1) & b_a(s)(s + \frac{13}{35})(s + \frac{16}{35}) \\
\hline
V(u_1) \setminus V(u_1, u_2) & b_a(s)(s + \frac{17}{35})(s + \frac{19}{35}) \\
\hline
V(u_1, u_2) & b_a(s)(s + \frac{41}{35})(s + \frac{45}{35}) \\
\hline
\end{array}
\]
• $Q_{18}: f = x^3 + yz^2 + y^8 + u_1 xy^6 + u_2 xz^2$

| $\mathbb{C}^2 \setminus \mathbb{V}(u_1)$ | $b_u(s)(s + \frac{47}{15})(s + \frac{29}{17})$ |
| $\mathbb{V}(u_1) \setminus \mathbb{V}(u_1,u_2)$ | $b_u(s)(s + \frac{55}{17})(s + \frac{29}{17})$ |
| $\mathbb{V}(u_1,u_2)$ | $b_u(s)(s + \frac{65}{17})(s + \frac{29}{17})$ |

$b_u(s) = (s + \frac{43}{48})(s + \frac{49}{48})(s + \frac{55}{48})(s + \frac{59}{48})(s + \frac{61}{48})(s + \frac{64}{48})(s + \frac{65}{48})(s + \frac{67}{48})$

$\times(s + \frac{79}{47})(s + \frac{83}{47})(s + \frac{85}{47})(s + \frac{86}{47})(s + \frac{87}{47})(s + \frac{88}{47})$

CGS of Ann($f^*$): 1516 parametric version of Algorithm 4: 6.844

• $S_{16}: f = x^2 z + y z^2 + x y^4 + u_1 y^6 + u_2 z^3$

| $\mathbb{C}^2 \setminus \mathbb{V}(u_1)$ | $b_u(s)(s + \frac{47}{15})(s + \frac{15}{17})$ |
| $\mathbb{V}(u_1) \setminus \mathbb{V}(u_1,u_2)$ | $b_u(s)(s + \frac{47}{15})(s + \frac{15}{17})$ |
| $\mathbb{V}(u_1,u_2)$ | $b_u(s)(s + \frac{47}{15})(s + \frac{36}{17})$ |

$b_u(s) = (s + \frac{19}{27})(s + \frac{19}{27})(s + \frac{23}{27})(s + \frac{24}{27})(s + \frac{25}{27})(s + \frac{26}{27})$

$\times(s + \frac{28}{27})(s + \frac{30}{27})(s + \frac{31}{27})(s + \frac{32}{27})(s + \frac{33}{27})(s + \frac{34}{27})$

CGS of Ann($f^*$): 7107 parametric version of Algorithm 4: 18.98

• $S_{17}: f = x^2 z + y z^2 + y^6 + u_1 y^4 z + u_2 z^3$

| $\mathbb{C}^2 \setminus \mathbb{V}(u_1)$ | $b_u(s)(s + \frac{47}{15})(s + \frac{29}{17})$ |
| $\mathbb{V}(u_1) \setminus \mathbb{V}(u_1,u_2)$ | $b_u(s)(s + \frac{47}{15})(s + \frac{29}{17})$ |
| $\mathbb{V}(u_1,u_2)$ | $b_u(s)(s + \frac{47}{15})(s + \frac{36}{17})$ |

$b_u(s) = (s + \frac{21}{27})(s + \frac{23}{27})(s + \frac{24}{27})(s + \frac{25}{27})(s + \frac{29}{27})(s + \frac{31}{27})(s + \frac{32}{27})(s + \frac{33}{27})$

$\times(s + \frac{34}{27})(s + \frac{35}{27})(s + \frac{36}{27})(s + \frac{37}{27})(s + \frac{38}{27})(s + \frac{39}{27})(s + \frac{40}{27})$

CGS of Ann($f^*$): 26220 parametric version of Algorithm 4: 3.063

• $U_{16}: x^3 + x z^2 + y^5 + u_1 y^2 z^2 + u_2 y^4 z^2$

| $\mathbb{C}^2 \setminus \mathbb{V}(u_1)$ | $b_u(s)(s + \frac{41}{15})(s + \frac{17}{15})$ |
| $\mathbb{V}(u_1) \setminus \mathbb{V}(u_1,u_2)$ | $b_u(s)(s + \frac{41}{15})(s + \frac{17}{15})$ |
| $\mathbb{V}(u_1,u_2)$ | $b_u(s)(s + \frac{41}{15})(s + \frac{36}{17})$ |

$b_u(s) = (s + \frac{14}{15})(s + \frac{16}{15})(s + \frac{18}{15})(s + \frac{20}{15})(s + \frac{22}{15})(s + \frac{24}{15})(s + \frac{26}{15})(s + \frac{28}{15})$

$\times(s + \frac{30}{15})(s + \frac{32}{15})(s + \frac{34}{15})(s + \frac{36}{15})(s + \frac{38}{15})(s + \frac{40}{15})$

CGS of Ann($f^*$): 3141 parametric version of Algorithm 4: 5.156

• $J_{16}: f = x^3 + y^9 + u_1 x^2 y^3 + u_2 y^{10} \quad (4u_1^2 + 27 \neq 0)$

| stratrum | $b_{f,0}(s)$ | note |
|----------|--------------|------|
| $\mathbb{C}^2 \setminus \mathbb{V}((4u_1^3 + 27)u_1u_2)$ | $b_u(s)$ | $\text{dim}_C(H_{M(\delta^2/0, f)}) = 2$ |
| $\mathbb{V}(u_1u_2) \setminus \mathbb{V}(4u_1^3 + 27)$ | $b_u(s)(s + \frac{14}{15})$ | $\text{dim}_C(H_{M(\delta^2/0, f)}) = 1,$ |
| | | $\text{dim}_C(H_{M(\delta^2/9, f)}) = 1$ |

$b_u(s) = (s + \frac{4}{15})(s + \frac{6}{15})(s + \frac{8}{15})(s + \frac{10}{15})(s + \frac{12}{15})(s + \frac{14}{15})(s + \frac{16}{15})(s + \frac{18}{15})$

CGS of Ann($f^*$): 55.83 parametric version of Algorithm 4: 304.6

• $W_{15}: f = x^4 + y^6 + u_1 x^2 y^3 + u_2 y^7 \quad (u_2^2 - 4 \neq 0)$

| $\mathbb{C}^2 \setminus \mathbb{V}((u_1^2 - 4)u_1u_2)$ | $b_u(s)$ | $\text{dim}_C(H_{M(-7/12, f)}) = 2$ |
| $\mathbb{V}(u_1u_2) \setminus \mathbb{V}(u_1^2 - 4)$ | $b_u(s)(s + \frac{19}{12})$ | $\text{dim}_C(H_{M(-7/12, f)}) = 1,$ |
| | | $\text{dim}_C(H_{M(-19/12, f)}) = 1$ |

$b_u(s) = (s + \frac{5}{12})(s + \frac{7}{12})(s + \frac{9}{12})(s + \frac{11}{12})(s + \frac{13}{12})(s + \frac{15}{12})$

$\times(s + \frac{17}{12})(s + \frac{19}{12})(s + \frac{21}{12})(s + \frac{23}{12})$.
• $Z_{15}$: $f = x^3y + y^7 + u_1 x^2 y^3 + u_2 y^8$ ($4u_1^3 + 27 \neq 0$)

| $\mathbb{C}^2 \setminus \mathcal{V}(4u_1^3 + 27)u_1u_2)$ | $b_4(s)$ | $\dim(\mathcal{H}_{M_{(-4/7,l)}})$ |
|----------------|---------|-----------------|
| $\mathcal{V}(u_1u_2) \setminus \mathcal{V}(4u_1^3 + 27)$ | $b_4(s)(s + \frac{11}{10})$ | $2$ |
| $\dim(\mathcal{H}_{M_{(-4/7,l)}})$ | $1$ | $1$ |
| $\dim(\mathcal{H}_{M_{(-11/7,l)}})$ | $1$ | $1$ |

$b_4(s) = (s + \frac{9}{10})(s + \frac{11}{10})(s + \frac{13}{10})(s + \frac{15}{10})(s + \frac{17}{10})(s + \frac{19}{10})(s + \frac{21}{10})(s + \frac{23}{10})$.

• $Q_{14}$: $f = x^3 + yz^2 + u_1 x^2 y^2 + xy^4 + u_2 y^7$ ($u_1^2 - 4 \neq 0$)

| $\mathbb{C}^2 \setminus \mathcal{V}(u_2^2 - 4)(4u_2^2 - 15)u_2)$ | $b_4(s)$ | $\dim(\mathcal{H}_{M_{(-13/12,l)}})$ |
|----------------|---------|-----------------|
| $\mathcal{V}(u_2(4u_2^2 - 15)) \setminus \mathcal{V}(u_2^2 - 4)$ | $b_4(s)(s + \frac{2}{5})$ | $2$ |
| $\dim(\mathcal{H}_{M_{(-13/12,l)}})$ | $1$ | $1$ |
| $\dim(\mathcal{H}_{M_{(-25/12,l)}})$ | $1$ | $1$ |

$b_4(s) = (s + \frac{13}{10})(s + \frac{15}{10})(s + \frac{17}{10})(s + \frac{19}{10})(s + \frac{21}{10})(s + \frac{23}{10})$.

• $S_{14}$: $f = x^2 z + y^5 + u_1 x^3 z + u_2 z^3$ ($u_1^2 - 4 \neq 0$)

| $\mathbb{C}^2 \setminus \mathcal{V}(u_2^2 - 4)(3u_2^2 - 10)u_2)$ | $b_4(s)$ | $\dim(\mathcal{H}_{M_{(-11/10,l)}})$ |
|----------------|---------|-----------------|
| $\mathcal{V}(u_2(3u_2^2 - 10)) \setminus \mathcal{V}(u_2^2 - 4)$ | $b_4(s)(s + \frac{2}{5})$ | $2$ |
| $\dim(\mathcal{H}_{M_{(-11/10,l)}})$ | $1$ | $1$ |
| $\dim(\mathcal{H}_{M_{(-21/10,l)}})$ | $1$ | $1$ |

$b_4(s) = (s + \frac{1}{10})(s + \frac{3}{10})(s + \frac{5}{10})(s + \frac{7}{10})(s + \frac{9}{10})(s + \frac{11}{10})(s + \frac{13}{10})(s + \frac{15}{10})(s + \frac{17}{10})(s + \frac{19}{10})$.

• $U_{14}$: $f = x^3 + xz^2 + u_1 xy^3 + y^3 z + u_2 y^5$ ($u_1^2 + 1 \neq 0$)

| $\mathbb{C}^2 \setminus \mathcal{V}(u_2^2 + 1)(2u_2^2 + 3)u_2)$ | $b_4(s)$ | $\dim(\mathcal{H}_{M_{(-10/9,l)}})$ |
|----------------|---------|-----------------|
| $\mathcal{V}(u_2(2u_2^2 + 3)) \setminus \mathcal{V}(u_2^2 + 1)$ | $b_4(s)(s + \frac{2}{9})$ | $2$ |
| $\dim(\mathcal{H}_{M_{(-10/9,l)}})$ | $1$ | $1$ |
| $\dim(\mathcal{H}_{M_{(-19/9,l)}})$ | $1$ | $1$ |

$b_4(s) = (s + \frac{8}{9})(s + \frac{10}{9})(s + \frac{12}{9})(s + \frac{14}{9})(s + \frac{16}{9})(s + \frac{17}{9})$.

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