Robust strategies for lossy quantum interferometry

Lorenzo Maccone and Giovanni De Cillis
QUIT - Quantum Information Theory Group, Dipartimento di Fisica
“A. Volta” Università di Pavia, via A. Bassi 6, I-27100 Pavia, Italy.

We give a simple multiround strategy that permits to beat the shot noise limit when performing interferometric measurements even in the presence of loss. In terms of the average photon number employed, our procedure can achieve twice the sensitivity of conventional interferometric ones in the noiseless case. In addition, it is more precise than the (recently proposed) optimal two-mode strategy even in the presence of loss.

PACS numbers: 03.65.Ta,06.20.Dk,42.50.St

The shot noise limit is the minimum noise level that the Heisenberg uncertainty relations permit to achieve when classical states are employed in the apparatuses. Many quantum strategies have been proposed to beat the shot noise [1,2] and to achieve the ultimate Heisenberg limit [12], but virtually all of them are very sensitive to noise and loss of photons [3]. Only very recently some interferometric strategies were presented that can beat the shot noise even in the presence of relevant losses of photons [4,5]. These are all instances of parallel strategies [2], where both arms of the interferometer are sampled at the same time using a mode-entangled quantum state of the light (see Fig. 1). In addition to the parallel strategies, in quantum metrology it is also possible to achieve the Heisenberg limit using multiround (or sequential) strategies [6,7], which, in the noiseless case, are equivalent in terms of resources and of achievable precision [2].

FIG. 1: a) Parallel strategies for interferometry. Both arms of the interferometer are sampled at the same time using a two-mode entangled state. The phase factor $\varphi$ is imprinted in the state as a phase difference between the two modes. b) Sequential strategy proposed here. A loss-resistant single mode state is sent through the first interferometer arm, sampling the phase $\varphi$. Then a unitary transformation is applied and the state is sent back through the other interferometer arm. This is needed so that the final phase shift experienced by the state is only the relative phase in the interferometer. The state is measured after one (or more) round trips.

Here we detail how multiround strategies can be used to perform interferometry — see Fig. 1b. An appropriate input state is prepared (we will analyze two examples below). This state is fed into the first interferometer arm. It picks up a phase $\varphi + \vartheta$, where $\varphi$ is the interferometric phase we want to estimate, and $\vartheta$ is the absolute phase picked up by the free evolution in the arm (which is equal to the phase which would be picked up also in the reference arm). The main trick of multiround interferometry is the use of the unitary

$$U = \sum_{n=0}^{M} |M-n\rangle \langle n| + \sum_{n=M+1}^{\infty} |n\rangle \langle n|,$$

where $|n\rangle$ is the Fock basis and $M$ is the largest nonzero component of the initial input state. The purpose of this unitary is to permute the first $M$ components of the Fock state expansion of a state in such a way that, when the state is sent back through the reference arm, the absolute phase $\vartheta$ is removed from the state (only an irrelevant global phase factor $e^{iM\vartheta}$ persists). Thus, at the end of the round trip of Fig. 1b, (multiple round trips are also possible), only the relative phase $\varphi$ is imprinted on the state. A measurement is finally performed to estimate this phase.

This multiround interferometry employs the same average energy and the same modes as the conventional (parallel) strategies, but it can achieve twice the sensitivity and it is more robust against noise. In fact, we show that, with an appropriate choice of inputs, our protocol permits to estimate the phase with an error which is smaller than what is achieved by the strategy detailed in Ref. [4], which is claimed to be the optimal two-mode strategy. In the presence of loss, this is not unexpected, as parallel protocols rely on entanglement which is notoriously fragile to noise. Instead, in the noiseless case, this can be seen easily with a simple example. Recall that the Heisenberg limit [12] is essentially an application of the time-energy uncertainty, $\Delta \varphi \Delta h \geq 1/2$, where $h$ is the generator of the unitary that inserts the phase $\varphi$ into the system [13], namely $h = a^\dagger a$ ($a$ being the annihilation operator of the first arm of the interferometer). Optimal two-mode states, such as the NOON state $|(N0) + (0N)\rangle/\sqrt{2}$, have $\Delta h = N/2$. However, the corresponding single-mode “NO” state of same average number of photons, namely the state $|(2N) + |0\rangle\rangle/\sqrt{2}$, has $\Delta h = N$. Both states achieve the Heisenberg-limited sensitivity of $\Delta \varphi = 1/(2\Delta h)$, but in terms of the average number of photons $N$, the NOON state can achieve...
\( \Delta \varphi_{\text{NOON}} = 1/N \), whereas the NO state can achieve \( \Delta \varphi_{\text{NO}} = 1/(2N) \), i.e. twice the sensitivity. The NO state is, however, just as sensitive to noise as the NOON state: the loss of a single photon renders both states useless to phase estimation.

The rest of the paper is devoted to presenting two examples of loss-resistant multiround interferometry based on two different input states, the optimal phase state and the single-mode M&M state. Both are able to beat the optimal parallel strategy for some values of parameters.

We start by considering the single-mode optimal phase state \( |\psi_{\text{opt}}\rangle \) introduced in \([3,4]\) (and later extended to the multi-mode case in \([10]\), building on \([11]\)), i.e. the state

\[
|\psi_{\text{opt}}\rangle \equiv \sqrt{\frac{2}{M+1}} \sum_{n=0}^{M} \sin \left(\frac{\pi}{M+1} \left( n + 1/2 \right) \right) |n\rangle ,
\]

where \( M \) is a parameter identifying the average number of photons \( N = M/2 \). In the noiseless case, this state can be used to achieve the ultimate precision \([12]\) in the absolute phase estimation, i.e. the Heisenberg scaling \( \sim 1/N \). Moreover, this state is highly robust to loss, since the loss tends to deplete primarily the components with large Fock numbers \( n \) which are not very populated in this state. Following the scheme of Fig. 1, this state is sent through the first arm of the interferometer, where it is subject to a phase shift of \( \varphi + \vartheta \), where \( \varphi \) is the interferometric phase we want to estimate, and \( \vartheta \) is the absolute phase picked up by the free evolution in the arm. During this transit, the state is also evolved by the loss map, described by the Kraus operators \( K_i \equiv (\eta^{-1} - 1)^{i/2} a^i a^i / \sqrt{M} \), where \( a \) is the annihilation operator of the optical mode and \( \eta \) is its transmissivity or quantum efficiency. (Note that the loss and the phase accumulation commute, so that the order in which we apply these two transformations is irrelevant.) Then, the state is subject to the unitary evolution \( U \) of Eq. 11 and it is sent back along the reference arm. Thanks to the permutation of the Fock components that \( U \) applies to the state, the free evolution is effectively reversed, so that the absolute phase \( \vartheta \) that was picked up in the first arm is removed while the radiation travels back through the reference arm. Also in the reference arm the state is typically subject to the loss (although there are interesting cases where the reference may be considered noiseless). Finally, the state is subject to the measurement. For the optimal phase state, a good measurement \([4]\) can be obtained by considering the orthogonal POVM composed by the projectors on the Pegg-Barnett states

\[
|\Phi_l\rangle \equiv \frac{1}{\sqrt{M+1}} \sum_{n=0}^{M} e^{i\varphi_l} |n\rangle \quad \text{with} \quad \Phi_l = \frac{2\pi l}{M+1},
\]

with \( l = 0, \ldots, M \). The RMS of the probability distribution obtained from this POVM is a function of the interferometer phase \( \varphi \). Its minimum value gives the minimum error that our interferometer can achieve, which is plotted as the continuous lines in Fig. 2. In addition, we have also directly estimated the error through the Holevo variance \([13]\)

\[
\Delta \Phi = (S_{\Phi}^2 - 1)^{1/2}, \quad \text{where} \quad S_{\Phi} = |\langle e^{i\Phi} \rangle| \quad \text{is the average value of the function} \quad e^{i\Phi} \quad \text{weighted with the probability obtained from the continuous POVM} \quad |\Phi\rangle \langle \Phi| \quad \text{d} \Phi,
\]

obtained using the phase states of Eq. 3 for arbitrary \( \Phi \). The Holevo variance is more appropriate than the RMS for the estimation of the phase error, since the phase is a periodic quantity \([13]\). However, as is clear from our plots, the Holevo variance is well approximated by the RMS when these two quantities are small enough (compared to \( 2\pi \)).

It is tedious but straightforward to calculate that, after a round trip characterized by the same quantum efficiency \( \eta \) in both arms, the state \( |\psi_{\text{opt}}\rangle \) is evolved into

\[
\rho = \frac{2}{M+1} \sum_{i,j=0}^{M} (1 - \eta)^{i+j} \eta^{M-j} \sum_{n,m} \omega_n \omega_m e^{i\omega (m-n)} |n\rangle \langle m|,
\]

where the second sum runs between \( \max(0, i-j) \) and \( M-j \). From this state, one can estimate the probability distribution of the POVM of Eq. 3, namely

\[
p(l) = \frac{2}{(M+1)^2} \sum_{i,j} (1 - \eta)^{i+j} \eta^{M-j} \sum_n \omega_n e^{-i\eta (\varphi + \varphi_l)} |n\rangle \langle n|.\]

The error in the estimation of the phase from a measurement on \( \rho \) is given by the RMS of \( n \), plotted in Fig. 2, as a function of the average photon number \( N \) and quantum efficiency \( \eta \).

The second input state we consider is a single-mode analogous of the M&M state introduced in \([3]\), i.e. the state

\[
(|M\rangle + |M'\rangle)/\sqrt{2},
\]

where \( M > M' \) and whose average photon number is \( N = (M + M')/2 \). Again, it is straightforward to obtain the output state, after the round trip of Fig. 1:

\[
\sigma = \sum_{j=-\delta}^{M'} \alpha_j |j+\delta\rangle \langle j+\delta| + \sum_{j=0}^{M} \beta_j |j\rangle \langle j| + \frac{1}{2} \sum_{j=0}^{M'} \gamma_j [e^{-i\delta \varphi} |j+\delta\rangle \langle j| + e^{i\delta \varphi} |j+\delta\rangle \langle j|] ,
\]

\[
\alpha_j = \sum_{i=\max(0,j)}^{M'} f_{ij} (M' \ i) \ i+\delta \ i-j \ /2, \quad (5)
\]

\[
\beta_j = \sum_{i=j}^{M} f_{ij} (M \ i) \ i \ /2, \quad (6)
\]

\[
\gamma_j = \sum_{i=j}^{M'} f_{ij} [ (M' \ i) (M \ i) (i+\delta) (i) ] \ /2, \quad (7)
\]

with \( \delta = M - M' \) and \( f_{ij} \equiv (1 - \eta)^{2i-j} \eta^{M-i+j} \). To extract the phase from the state \( \sigma \), in analogy to the
two-mode case of [3], we can measure the observable
\[ A = \sum_{k=0}^{M'} (M - k)\langle M' - k\rangle + |M' - k\rangle\langle M - k\rangle. \]  
Then, the error on the phase can be obtained from the RMS of \( A \) using error propagation, namely
\[ \Delta \varphi = \Delta A |\frac{\partial}{\partial \varphi} \langle A \rangle| = \frac{\sqrt{\Theta - \cos^2(\delta \varphi) \Gamma^2}}{\delta |\sin(\delta \varphi) \Gamma|}, \]  
where \( \Theta = \sum_{k=0}^{M'} \alpha_k + \alpha_k \beta_{k-\delta} + \beta_{k+\delta} + \beta_k \) and \( \Gamma = \sum_{k=0}^{M'} \gamma_k \). This quantity is plotted in Fig. 3 from which it is evident that, also in this case, the multiround protocol can achieve a better sensitivity than the optimal two-mode one. The single-mode M&M state can take advantage of most of the implementation ideas presented in [2] for the two-mode case. It may appear surprising that the state \( \sigma \) permits to achieve a greater precision than the NOON state \((|N0\rangle + |0N\rangle)/\sqrt{2}\) even for high values of \( N \). As discussed above, this is essentially due to the fact that a single mode state performs better than a two-mode state in terms of the average number of photons \( N \).

In conclusion, we have given a strategy for determining the relative phase in an interferometer using single-mode states that are sent through the interferometer in a round trip, interleaved by the unitary \( U \) of Eq. 1. This entails that a) in the noiseless case a double sensitivity can be reached over the optimal two-mode states (such as the NOON state) in terms of the average number of photons, b) in the lossy case we can achieve a better phase sensitivity than what is claimed to be the optimal two-mode strategy [2], proving that multiround protocols are preferable in the presence of noise. The robustness in the face of loss stems from two main properties. On one side there is no entanglement between different modes, and it is well known that entanglement is very sensitive to noise [3]. On the other side, the fact that we are using a single mode permits to double the phase sensitivity over the two-mode entangled case, since all the photons travel through one mode only. One may object that the increased phase sensitivity arises because we are devoting more resources to the estimation. This objection is unfounded since the average number of photons employed in the two strategies is the same, \( N \). One cannot even say that in the two-mode strategies the phase \( \varphi \) is sampled by less photons, because the number of photons that travel through an arm of an interferometer is an undefined quantity (the “which path” information is complementary to the phase information). One can only bound the number of photons traveling through each arm with the total number of photons, \( N \), injected in the interferometer.

We thank R. Demkowicz-Dobrzenski for having kindly provided the data of the optimal two-mode state of [2].

[1] For a recent review, see V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).
[2] V. Giovannetti, S. Lloyd, L. Maccone, Phys. Rev. Lett. 96, 010401 (2006); S. L. Braunstein, Nature 440, 617 (2006).
[3] G. Gilbert, M. Hamrick, Y.S. Weinstein, J.Opt. Soc. Am. B, 25, 1336 (2008).
[4] U. Dorner, R. Demkowicz-Dobrzenski, B. J. Smith, J. S. Lundeen, W. Wasilewski, K. Banaszek, I. A. Walmsley, [arXiv:0807.3659] [quant-ph] (2008).
[5] S.D. Huver, C.F. Wildfeuer, J.P. Dowling, arXiv:0805.0296 [quant-ph] (2008).
[6] A. Luis, Phys. Rev. A 65, 025802 (2002).
[7] B.L. Higgins, D.W. Berry, S.D. Bartlett, H.M. Wiseman, and G.J. Pryde, Nature 450, 393 (2007).
[8] H.M. Wiseman and R.B. Killip, Phys. Rev. A 56, 944 (1997).
[9] V. Bužek, R. Derka, and S. Massar, Phys. Rev. Lett. 82, 2207 (1999).
[10] D.W. Berry and H.M. Wiseman, Phys. Rev. Lett. 85, 5098 (2000).
[11] B.C. Sanders and G.J. Milburn, Phys. Rev. Lett. 75, 2944 (1995); B.C. Sanders, G.J. Milburn, and Z. Zhang, J. Mod. Opt. 44, 1309 (1997).
[12] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A 54, R4649 (1996).
[13] A.S. Holevo, Probabilistic and statistical aspects of quantum theory (North Holland pub. co., Amsterdam, 1982).
[14] S. L. Braunstein, C. M. Caves and G. J. Milburn, Ann. Phys. 247, 135 (1996); S. L. Braunstein, C. M. Caves, Phys. Rev. Lett. 72, 3439 (1994).