STOCHASTIC AVERAGING OF THE EINSTEIN VACUUM EQUATIONS ON A
TOROIDAL RANDOM GEOMETRY: STABILITY CRITERIA AND INDUCED
COSMOLOGICAL CONSTANT TERMS

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Abstract. The Einstein vacuum equations on an n-dimensional toroidal geometry $\mathbb{M}^{n+1} = T^n \times \mathbb{R}^+$ reduce to a system of n-dimensional nonlinear ordinary differential equations in terms of the set of toroidal radii $(a_i(t))_{i=1}^n$ or the radial moduli fields $(\psi_i(t))_{i=1}^n = (\ln(a_i(t)))_{i=1}^n$ of the n-torus $T^n$. This geometry is also the basis of Kasner-Bianchi-type cosmologies. The equations are trivially satisfied for static or equilibrium solutions $\psi_i^E = \psi_i$ or radii $a_i^E = a_i$, describing an initially static micro-universe or toroidal 'vacuum bubble'. It is Lyapunov stable to short-pulse deterministic perturbations, which have a sharp Gaussian profile; the perturbed radii rapidly converge to new 'attractors' and therefore to new stable equilibria. These perturbations therefore induce transitions between stable states of the system. Introducing intrinsic classical Gaussian random fluctuations, with a regulated covariance, the radial moduli become Gaussian random fields parametrising a 'toroidal random geometry'. The randomly perturbed Einstein vacuum equations are now interpreted as an n-dimensional nonlinear randomly perturbed dynamical system. Non-vanishing 'cosmological constant' terms are retained within the stochastically averaged Einstein equations since they are nonlinear. This is analogous to averaging the Navier-Stokes equations in statistical turbulence theory, which yields an additional non-vanishing Reynolds term since like the Einstein equations they also are of nonlinear hyperbolic type. The expectations of the randomly perturbed toric radii can be estimated from a cluster-integral-type cumulant expansion method. The initially static micro-universe or bubble undergoes eternal noise-induced stochastic exponential growth or inflation, for regulated covariance functions. Random moduli fields within this scenario therefore act like a 'dark energy'. Finally, a class of random perturbations is considered for which the Einstein system is stable.

1. Introduction and motivation

This paper promotes the potential applications of stochastic and probabilistic methods within mathematical general relativity, as well as concepts from the theory of nonlinear deterministic and random dynamical systems. Stability within a general-relativistic cosmological context is approached by reducing the vacuum Einstein equations on a n-torus to a multi-dimensional nonlinear dynamical system of ordinary differential equations, with initial Cauchy data, which can then be perturbed by deterministic or random perturbations. A key concern with nonlinear systems is the determination of the steady state or stationary motions and their corresponding stability. The stability of equilibrium points is generally ascertained by linear stability analysis when the system is perfectly deterministic. This is usually the case for macroscopic systems in classical dynamics or dynamical systems theory, celestial mechanics, and for scenarios within gravitation and astrophysics [1,2,3,4,5,6,7,8,9]. Major technical results in stability analysis within pure general relativity have been the Christodoulou-Klainerman proof of the nonlinear stability of Minkowski space [10,11] and more recently proofs for black holes [12].

However, mesoscopic and microscopic systems in physics, chemistry and biology are not perfectly deterministic and are subject to intrinsic or extrinsic noise, fluctuations or random perturbations; indeed all known physical systems will invariably possess noise on some critical length scale, of either thermal or quantum origin. This requires the utilization of stochastic tools [13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31]. A challenge has been to extend existing techniques for linear systems to deal with nonlinear systems coupled to stochastic noise. Calculating properties of noisy stochastic nonlinear systems is generally fraught with difficulties however; in particular, what were established as stable fixed points via a deterministic stability analysis of a system may actually be unstable when subject to intrinsic stochastic perturbations or an external noise bath, where the coupling of the noise/fluctuations to the nonlinearity becomes a crucial issue. Conversely, an unstable deterministic system, described by either ODEs or PDEs, may actually become
interprets the cosmological problem as a random n-dimensional non-linear dynamical system. The outline of perturbations or fluctuations of the radial moduli fields which parametrize the metric of the hypertorus. This on a 'random toroidal geometry' can be derived. The randomness of the geometry arise from intrinsic random systems to this problem. In particular, the stochastically perturbed and averaged Einstein vacuum equations is the basis of Kasner-Bianchi type cosmological models—and then develop, and tentatively apply, methods (but rigorously) apply methods from classical stochastic functional analysis, and incorporate classical noise, as there is no complete or applicable theory of quantum gravity, it may still be possible to tentatively crucial. Cosmological density fluctuations are also taken to be Gaussian random fields \[39\]. Furthermore, \[38\]. Early universe cosmology is also a regime where gravitation and general relativity are applied on in relation to stochastic or chaotic inflation where a 'bubble' of vacuum will initially grow exponentially remove blowups and singularities which exist in the purely deterministic dynamical problem \[19,22\]. In deterministic situation. The theory of stochastic PDEs is also a growing area of research \[37\].

Another interesting property of random perturbations or noises is that they can sometimes dissipate or (3) In Section 4, the methods of Section 2 are applied such that the static or equilibrium solutions (1) In Section 2, we consider a specific class of n-dimensional nonlinear ODEs and consider methods to ascertain the effects of both 'short-pulse' deterministic perturbations and also random perturbations or noise on static or equilibrium solutions of these ODEs.

In Section 3, the Einstein vacuum equations \(\text{Ric}_{AB} = 0\) and \(\text{Ric}_{AB} = g_{AB} \Lambda\) for a cosmological constant \(\Lambda\) are formulated on the globally hyperbolic spacetime \(\mathbb{M}^{n+1} = \mathbb{T}^n \times \mathbb{R}^+\), where \(\mathbb{T}^n\) is an isotropic n-torus with metric \(g_{ij} = \delta_{ij} |2\pi a_i(t)|^2 dX^i \otimes dX^j \equiv 2\pi \delta_{ij} \exp(2\psi_i(t))dX^i \otimes dX^j\), parametrized by a set of real modulus functions \((\psi_i(t))_{i=1}^n\), which span \(\mathbb{R}^n\). The toroidal radii are then \(a_i(t) = \exp(\psi_i(t))\) for \(i = 1 \ldots n\). The Einstein vacuum equations reduce to n-dimensional sets of nonlinear autonomous ODEs for \(\psi_i(t)\) and \(a_i(t)\), and are essentially of the form discussed in Section 2. Static solutions \(a^E_i = \exp(\psi^E_i)\) and dynamic solutions \(a_i(t) = \exp(\psi_i(t))\) can be found. These essentially describe static and expanding 'Kasner universes' or 'rolling radii'.

In Section 4, the methods of Section 2 are applied such that the static or equilibrium solutions \(a^E_i = a^E\) are subjected to deterministic 'short-pulse' Gaussian perturbations and also to a continuous perturbation of constant amplitude. The perturbed \((L^2)\) norms \(\|\overline{a(t)} - a^E\|\) can be estimated and the asymptotic stability studied for \(\lim_{t \rightarrow \infty} \|\overline{a(t)} - a^E\|\), where \(a(t) = (a_1(t), \ldots , a_n(t))\)

In Section 5, the methods of Section 2 are applied such that the static or equilibrium solutions are subjected to random perturbations or noise which are taken to arise from intrinsic random fluctuations of the moduli fields \((\psi_i(t))\). The stochastically averaged Einstein equations then lead to extra non-vanishing terms that can be identified as a 'cosmological constant', which arises solely from the nonlinearity of the equations. This is analogous to a Reynolds number arising within stochastically averaged nonlinear Navier-Stokes equations. The randomly perturbed static and dynamical solutions are also solutions of the stochastically averaged Einstein equations.

In Section 6, the stochastic expectation or average \(M\{\|\overline{a(t)} - a^E\|\}\) of the stochastically perturbed norm \(\|\overline{a(t)} - a^E\|\) is estimated from a cluster expansion method with truncation at second order for a Gaussian dominance approximation. Using a regulated 2-point function ansatz the norm is estimated. The averaged norm \(M\{\|\overline{a(t)} - a^E\|\}\) then grows exponentially or "inflates" for eternity so that \(\lim_{t \rightarrow \infty} M\{\|\overline{a(t)} - a^E\|\} = \infty\) and with probability \(P(\|\overline{a(t)}\| = \infty) = 1\).

In the final section, a class of random perturbations are considered for which the Einstein system is stable.
2. 'Short-pulse' deterministic perturbations and random perturbations of an n-dimensional nonlinear system of autonomous ODEs

First the following systems of n-dimensional nonlinear ordinary differential equations are considered. Systems of nonlinear ODEs with this structure or form arise in cosmology when applying the Einstein vacuum equations to a n-dimensional toroidal spacetime, which will be derived in detail in Section 3. However, as a prerequisite the generic properties of these forms of nonlinear ODEs are discussed in relation to both 'short-pulse' deterministic and random perturbations or noise of their solutions. In particular, we are interested in the stochastically averaged differential equations.

**Proposition 2.1.** For all \( t \in \mathbb{R}^+ = [0, \infty) \) let \((\psi_i(t))_{i=1}^n \equiv \psi(t) = (\psi_1(t), ..., \psi_n(t))\) be a set of smooth real scalar functions spanning \( \mathbb{R}^n \), that describe the evolution of a nonlinear n-dimensional dynamical system from some initial data set \((\psi_i(0))_{i=1}^n \). Then let \((u_i(t))_{i=1}^n \equiv u(t) = (u_1(t), ..., u_n(t))\) be a set spanning \( \mathbb{R}^n \) such that for all \( t \in \mathbb{R}^+ \) and \( i = 1...n \), \( u_i(t) = \exp(\psi_i(t)) \) (2.1) or \( \psi_i(t) = \ln(u_i(t)) \). Then the process described by \( u_i(t) \) is essentially parametrized by the underlying functions \( \psi_i(t) \). It will be convenient to use the notations \( \partial_t \equiv d/dt \) and \( \partial_i \equiv d^2/dt^2 \) and retain summations throughout. Consider n-dimensional nonlinear ordinary differential equations of the very general form:

\[
\mathbf{H}_n \psi_i(t) = \sum_{i=1}^n \partial_t \psi_i(t) + \beta \sum_{i=1}^n \partial_i \psi_i(t) \partial_t \psi_i(t) = 0 \quad (2.2)
\]

with \( \beta > 0 \). If \( u_i(t) = \exp(\psi_i(t)) \), then \( \partial_t u_i(t) = \partial_t u_i(t)/u_i(t) \) and the second derivative is \( \partial_t \psi_i(t) = ((\partial_t u_i(t))/u_i(t)) - (\partial_t u_i(t)/u_i(t)u_i(t)) \), so that an equivalent set of differential equations is

\[
\mathbf{D}_n u_i(t) = \sum_{i=1}^n \partial_t u_i(t)/u_i(t) + (\beta - 1) \sum_{i=1}^n \partial_t u_i(t)/u_i(t)u_i(t) = 0 \quad (2.3)
\]

with some initial data \((u_i(0))_{i=1}^n \). For some \( C > 0 \), the inhomogeneous equations are

\[
\mathbf{H}_n \psi_i(t) \equiv \sum_{i=1}^n \partial_t \psi_i(t) + \beta \sum_{i=1}^n \partial_i \psi_i(t) \partial_t \psi_i(t) = \sum_{i=1}^n \psi_i(t) = C \quad (2.4)
\]

\[
\mathbf{D}_n u_i(t) = \sum_{i=1}^n \partial_t u_i(t)/u_i(t) + (\beta - 1) \sum_{i=1}^n \partial_t u_i(t)/u_i(t)u_i(t) = \sum_{i=1}^n C_i = C \quad (2.5)
\]

Here, \( \mathbf{H}_n \) and \( \mathbf{D}_n \) are nonlinear differential operators on \( \mathbb{R}^n \) such that

\[
\mathbf{H}_n(...) \equiv \sum_{i=1}^n \partial_t(...) + \beta \sum_{i=1}^n \partial_i(...) \partial_t(...) \quad (2.6)
\]

\[
\mathbf{D}_n(...) \equiv \sum_{i=1}^n \partial_t(...) + (\beta - 1) \sum_{i=1}^n \partial_i(...) \partial_t(...) \quad (2.7)
\]

The n-dimensional nonlinear differential equations can also be expressed in terms of \( L_2 \) norms.

**Corollary 2.2.** In terms of \( L_2 \) norms, (2.2) and (2.3) are

\[
\mathbf{H}_n \psi_i(t) = \left\| \sqrt{\partial_t \psi_i(t)} \right\|^2 + \beta \left\| \partial_t \psi_i(t) \right\|^2 \quad (2.8)
\]

\[
\mathbf{D}_n u_i(t) = \left\| \partial_t u_i(t)/u_i(t) \right\|^2 + (\beta - 1) \left\| \partial_t u_i(t)/u_i(t)u_i(t) \right\|^2 \quad (2.9)
\]

**Corollary 2.3.** The solution sets \((\psi_i(t))_{i=1}^n \equiv \psi(t) = (\psi_1(t), ..., \psi_n(t))\) and \((u_i(t))_{i=1}^n \equiv u(t) = (u_1(t), ..., u_n(t))\) for all \( t \in \mathbb{R}^+ \) describe a time-dependent vector in \( \mathbb{R}^n \). A trivial set of static or equilibrium solutions are \( \psi_i(t) = \psi_i(0) = \psi_i^E \) for all \( t \geq 0 \) and \( u_i^E = u_i^E(t) = \exp(\psi_i^E) \).

However, the inhomogeneous equations \( \mathbf{H}_n \psi_i(t) = C \) and \( \mathbf{D}_n u_i(t) = C \) can have no static or equilibrium solutions when \( C > 0 \).

**Lemma 2.4.** The equations \( \mathbf{H}_n \psi_i(t) = 0 \) and \( \mathbf{D}_n u_i(t) = 0 \) have the solutions

\[
\psi_i(t) = \psi_i(0) + q_i \ln |t| \quad (2.10)
\]

\[
u_i(t) = \psi_i(0)|t|^{q_i} \quad (2.11)\]
where $q_i \in \mathbb{R}$, provided that $\sum_{i=1}^{n} q_i = \beta \sum_{i=1}^{n} q_i^2$. If $q = q_i$ for $i = 1...n$ then $\psi_i(t) = \psi_i(0) + q \ln |t|$ and $u_i(t) = u_i(0) |t|^q$ are solutions if $\beta = 1/q$. The static or equilibrium solutions are $\psi_i(t) = \psi_i E$ and $u_i(t) = u_i^E$. The inhomogeneous equations $H_n \psi_i(t) = C$ and $D_n u_i(t) = C$ have the solutions

$$\psi_i(t) = \psi_i(0) + q_i t$$

$$u_i(t) = u_i(0) \exp(q_i t)$$

provided that the $q_i$ satisfy the constraints $\sum_{i=1}^{n} q_i = \beta \sum_{i=1}^{n} q_i^2$. If $q = q_i$ for $i = 1...n$, then $q = \pm (C/\beta n)^{1/2}$ and

$$\psi_i^{(\pm)}(t) = \psi_i(0) \pm (C/\beta n)^{1/2} t$$

$$u_i(t)^{\pm} = u_i(0) \exp(\pm (C/\beta n)^{1/2} t)$$

**Proof.** Since $\partial_t \psi_i(t) = q_i / t$ and $\partial_t u_i(t) = -q_i / t^2$

$$H_n \psi_i(t) = - \sum_{i=1}^{n} \frac{q_i}{t^2} + \beta \sum_{i=1}^{n} \frac{q_i^2}{t^2} = 0$$

$$\Rightarrow - \sum_{i=1}^{n} q_i + \beta \sum_{i=1}^{n} q_i^2 = 0$$

If $q_i = q$ for $i = 1...n$ then $- \sum_{i=1}^{n} q_i + \beta \sum_{i=1}^{n} q_i^2 = -nq + \beta nq^2$ so that $\beta = 1/q$. Similarly

$$D_n u_i(t) = \frac{q_i(q_i - 1)|t|^{q_i - 2}}{a_i(0)|t|^{q_i}} + (\beta - 1) \sum_{i=1}^{n} \frac{q_i(q_i - 1)|t|^{q_i - 2}}{a_i(0)}$$

$$= \sum_{i=1}^{n} (q_i^2 - q_i) + (\beta - 1) \sum_{i=1}^{n} q_i q_i$$

$$\Rightarrow - \sum_{i=1}^{n} q_i + \beta \sum_{i=1}^{n} q_i^2 = 0$$

which again gives $\beta = 1/q$ if $q_i = q$. Static or equilibrium solutions are simply any $\psi_i(t) = \psi_i E = \text{const.}$ and $u_i(t) = u_i^E = \exp(q_i E)$. Since $\partial_t \psi_i(t) = q_i$ and $\partial_t u_i(t) = 0$, then the inhomogeneous equation $H_n \psi_i(t) = C$ becomes $\beta \sum_{i=1}^{n} q_i q_i = C$ so that $\beta nq^2 = C$ if $q = q_i$ and $\beta = \pm (C/n\beta)^{1/2}$. Since $\partial_t u_i(t) = u_i(0) q_i \exp(q_i t)$ and $\partial_t u_i(t) = u_i(0) q_i \exp(q_i t)$ then for the inhomogeneous equation $D_n u_i(t) = C$

$$\sum_{i=1}^{n} \frac{u_i(0) q_i q_i \exp(q_i t)}{u_i(0) \exp(q_i t)} + (\beta - 1) \sum_{i=1}^{n} \frac{u_i(0) u_i(0) q_i q_i \exp(2q_i t)}{u_i(0) \exp(2q_i t)}$$

$$= \sum_{i=1}^{n} q_i^2 + (\beta - 1) \sum_{i=1}^{n} q_i q_i = C$$

If $q_i = q$ then $nq^2 + n(\beta - 1)q^2 = C$ so that again $q = \pm (C/n\beta)^{1/2}$. \hfill \Box

**Remark 2.5.** The nonlinear system described by (2.2) and (2.3) can be considered as special cases of the following general system of $n$-dimensional ODEs such that

$$H_n \psi_i(t) = \alpha \sum_{i=1}^{n} \partial_{tt} \psi_i(t) + \beta \sum_{i=1}^{n} (\partial_t \psi_i(t))^b + \gamma \sum_{i=1}^{n} (\psi_i(t))^a = 0$$

with $(\alpha, \beta, \gamma) \in \mathbb{R}^+$ and integers $(a, b) \in \mathbb{Z}$. In terms of $u_i(t) = \exp(\psi_i(t))$ the equation is equivalently

$$D_n a_i(t) = \alpha \sum_{i=1}^{n} \partial_{tt} u_i(t) - \alpha \sum_{i=1}^{n} \partial_t u_i(t) \partial_t(t) u_i(t) u_i(t)$$

$$+ \beta \sum_{i=1}^{n} (\partial_t u_i(t))^b + \gamma \sum_{i=1}^{n} (\ln(u_i(t)))^a = 0$$

These are essentially $n$-dimensional nonlinear polynomial autonomous systems. Equation (2.2) is the case for $\gamma = 0, b = 2$. For $\alpha = 0, \gamma = 1, \beta = 0, a = 2$, we have a system of Riccati equations.

$$\sum_{i=1}^{n} \partial_t \psi_i(t) + \gamma \sum_{i=1}^{n} (\psi_i(t))^2 = 0$$
with (non-global) solutions $\psi_i(t) = |t - \psi_i^{-1}(0)|^{-1}$. For $\alpha = 1, \beta = 0, \gamma = 1, \alpha = 1$, equation (2.17) reduces to a linear system of coupled simple harmonic oscillators
\begin{equation}
\sum_{i=1}^{n} \partial_t \psi_i(t) + \gamma \sum_{i=1}^{n} \psi_i(t) = 0
\end{equation}
with the basic solutions $\psi_i(t) = \psi_i(0) \cos(\sqrt{\gamma}t - \varphi)$, where $\varphi$ is a phase angle.

**Definition 2.6.** The following standard definitions are given

1. Let $u(t) = (u_1(t), \ldots, u_n(t))$. The $L_2$ norm of a vector $u_i(t)$ is $\|u_i(t)\| = (\sum_{i=1}^{n} |u_i(t)|^2)^{1/2}$ and the $L_p$-norm is $\|u_i(t)\| = (\sum_{i=1}^{n} |u_i(t)|^p)^{1/p}$.

2. A set of $n$ equilibrium stable fixed points is denoted $(u_i^E)_{i=1}^{n} = (u_1^E, \ldots, u_n^E)$ with $L_2$ norm $\|u^E\|$. For an isotropic equilibrium configuration, one can set $u_i^E = u^E$ for $i = 1, \ldots, n$. For initial data $\psi^E = \psi(0)$ or $u^E = u(0)$, the Cauchy developments for $t > 0$ are $\psi(t)$ and $u(t)$.

3. Given an Euclidean ball $B(L)$ of radius $L$, a set of stable fixed points $u_i^E$ spanning $\mathbb{R}^n$ is Lyapunov stable for all $B_1 \subset \mathbb{R}^n$ where $\|u_i^E\| \in B_1$ if $\exists B_2 \subset \mathbb{R}^n$ with $\|u_i^E\| \in B_2$, if for all $t > 0, \exists \delta > 0$ and $\delta < \epsilon$ such that $\|u(t) - u^E\| < \delta$ implies $\|u(t) - u^E\| < \epsilon$. One can choose any Euclidean ball $B(e)$ with $u_i^E \in B(e)$ for any small $\epsilon > 0$ such that all future states $\|\psi^E(t)\| \in B_2$ are trapped within $B(e)$ provided that they start out in a smaller ball $B(\delta) \subset B(e)$. So a finite ball $B(e)$ such that for all $t > 0, u(t) \in B(e)$.

4. Lyapunov stability requires a convergent norm such that $\exists K > 0$ whereby $0 < \lim_{t \to \infty} \|u(t) - u_i\| \leq K$. The norm is always taken to the $L_2$-norm although the definitions will still hold for $L_p$-norms.

5. The Lyapunov stability criterion is weaker than that of asymptotic stability. Suppose Lyapunov stability holds, then if $\exists B(\delta) \subset \mathbb{R}^n$ of radius $\delta$ such that $\|u(0) - u^E\| < \delta$ then $\lim_{t \to \infty} \|u(t) - u^E\| = 0$. If $u_i^E = 0$ for $i = 1, \ldots, n$ or $u^E = 0$, then the equilibrium points are just the origin of $\mathbb{R}^n$ and $\lim_{t \to 0} \|u(t)\| = 0$. For asymptotic stability, the norm converges to zero such that $\lim_{t \to \infty} \|u(t) - u^E\| = 0$.

6. The system is essentially unstable if $\lim_{t \to \infty} \|u(t) - u^E\| = \infty$.

For example, for the dynamic solution (2.10) with $u_i(0) = u_i^E$ we have the estimate
\begin{equation}
\lim_{t \to \infty} \|u(t) - u(0)\| \leq \lim_{t \to \infty} \|u(t)\| - \|u(0)\| = \lim_{t \to \infty} \left( \sum_{i=1}^{n} |u_i(0)|^2 |t|^{2q_i} \right)^{1/2} - \|u(0)\| = \lim_{t \to \infty} n^{1/2} u(0) |t|^{q_i} - \|u(0)\| = \infty
\end{equation}
where $u_i(0) = u(0)$ for $i = 1, \ldots, n$, so there is no convergence to equilibrium and the system is unstable. Similarly for $u_i(t) = u_i(0) \exp((C/n\beta)^{1/2}t)$ we have
\begin{equation}
\lim_{t \to \infty} \|u^{(+)}(t) - u(0)\| \leq \lim_{t \to \infty} \|u^{(+)}(t)\| - \|u(0)\| = \lim_{t \to \infty} \left( \sum_{i=1}^{n} |u_i(0)|^2 \exp(2(C/n\beta)^{1/2}t) \right)^{1/2} - \|u(0)\| = \lim_{t \to \infty} (n^{1/2} u(0) \exp((C/n\beta)^{1/2}t) - \|u(0)\|) = \infty
\end{equation}
with $u_i(0) = u(0)$ for $i = 1, \ldots, n$, giving an unbounded exponential expansion. However, the solution $u_i^{(-)}(t) = u_i(0) \exp(-(C/n\beta)^{1/2}t)$ is asymptotically stable in that
\begin{equation}
\lim_{t \to \infty} \|u^{(-)}(t) - u(0)\| \leq \lim_{t \to \infty} \|u^{(-)}(t)\| - \|u(0)\| \leq \lim_{t \to \infty} \left( \sum_{i=1}^{n} |u_i(0)|^2 \exp(-2(C/n\beta)^{1/2}t) \right)^{1/2} = \lim_{t \to \infty} (n^{1/2} u(0) \exp(-(C/n\beta)^{1/2}t)) = 0.
\end{equation}
The standard and practical method for evaluating the stability of fixed points in nonlinear dynamical systems and classical mechanics is linear stability analysis [1]. Consider a first-order NLDE. For small perturbations $\xi(t)$ around the equilibrium points so that $u_i(t) = u^E_i + \xi_i(t)$. NLDEs can be linearised by dropping higher-order terms and performing a ’normal-mode analysis’. However, in this paper, the aim is to study specific types of ’short-pulse’ and random perturbations while retaining the full nonlinearity of the equations; in particular, if the Einstein equations are reduced to an n-dimensional nonlinear system of ODEs within a dynamical systems interpretation, it is the nonlinearity that is of prime interest.

Nonlinear systems can also become chaotic, whereby the future evolution no longer becomes predictable from initial Cauchy data [9,47,48]. A useful ’acid test’ for chaos is the Lyapunov characteristic exponent (LCE) which gives the rate of exponential divergence from perturbed initial conditions.

**Definition 2.7.** The LCE of a dynamical system quantifies the rate of change or divergence or separation of initially infinitesimally close trajectories in phase space. For an n-dimensional system, if $u_i(t)$ and $\delta u_i(t)$ span $\mathbb{R}^n$ for $i = 1...n$, with $t \in [0, \infty)$ then let $\delta u_i(t) = u_i(t) - u_i(t)$ and $\|\delta u_i(t)\| = ||u(0) - u(0)||$. If

$$\frac{\|\delta u_i(t)\|}{\|\delta u_i(0)\|} = \frac{\|u(t) - u(t)\|}{\|u(0) - u(0)\|} = \left(\sum_{i=1}^{n} |u_i(t) - u_i(t)|^2\right)^{1/2} \sim \exp(Ly t)$$

then $Ly$ is a LCE of the system and $\delta u_i(t) \sim \delta u_i(0) \exp(\lambda t)$. The maximal LCE is the average deviation from the unperturbed state or orbit at time $t > 0$ and is established by the Oseledec Theorem [6] as

$$Ly = \lim_{t \to \infty} \lim_{\|\delta u(0)\| \to 0} \frac{1}{t} \ln \left(\frac{\|\delta u(t)\|}{\|\delta u(0)\|}\right)$$

Then:

1. In a chaotic region, future evolution is independent of initial conditions. When $\lambda < 0$, the orbits attract to a stable fixed point or ’attractors’ and the system exhibits asymptotic stability. If $\lambda = -\infty$, the system is superstable.
2. If $\lambda = 0$, the system is Lyapunov stable.
3. For all $\lambda > 0$, the system is unstable and nearby points or orbits diverge exponentially to arbitrary large separations.

In this paper, ”short-pulse” deterministic perturbations with respect to fully nonlinear ODEs of the form (2.2) or (2.3) will initially be considered, as a prerequisite to studying the effect of random perturbations or noise.

**Proposition 2.8.** Let $(U_\ell(t, \vartheta_i))_{1 \leq i \leq n}$ be a set of functions $U_\ell : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

1. $U_\ell(t, \vartheta_i) \to 0$ for $t \gg ||\vartheta||$, with $||\vartheta|| \ll 1$. For example, a set of highly peaked Gaussian functions with widths $\vartheta_i$ or ’smeared-out’ delta functions. Define a vector $V(t, \vartheta) = U_1(t, \epsilon_1), ..., U_n(t, \epsilon_n)$ and $
\vartheta = (\vartheta_1, ..., \vartheta_n)$ spanning $\mathbb{R}^n$. The $L_2$ norm is $||U_\ell(t, \vartheta_i)|| = U_\ell(t, \vartheta).
2. The functions $U_\ell(t, \vartheta_i)$ are sufficiently smooth such that the derivatives $\partial U_\ell(t, \vartheta_i)$ and $\partial U_\ell(t, \vartheta_i)$ exist and also rapidly decay for $t \gg ||\vartheta||$ so that $||\partial U(t, \vartheta)|| \to 0$ and $||\partial U(t, \vartheta)|| \to 0$
3. The integrals $\int_0^t U_\ell(t, \vartheta_i) d\tau > 0$ exist and are well defined. The integrals

$$\lim_{t \to \infty} \int_0^t U_\ell(t, \vartheta_i) d\tau, \lim_{t \to \infty} \int_0^t U_\ell(t, \vartheta) d\tau$$

may or may not converge, although we primarily are concerned with convergent integrals such that

$$\lim_{t \to \infty} \int_0^t U_\ell(t, \vartheta_i) d\tau = Q < \infty$$

Then if $\psi_i^E$ and $u_i^E$ are static equilibrium fixed points or solutions of (2.2) and (2.3) such that $H_n \psi_i^E = 0$ and $D_n a_i^E = 0$ then the perturbed static solutions are

$$\psi_i(t) = \psi_i^E + \int_0^t U_\ell(t, \vartheta) d\tau$$
\[
\overline{u_i(t)} = u_i^E \exp \left( \int_0^t U_i(\tau, \vartheta_i) \, d\tau \right) \equiv u_i^E B(t)
\]

Then
\[
\overline{u_i(t)} - u_i^E = u_i^E \exp \left( \int_0^t U_i(\tau, \vartheta) \, d\tau \right) - u_i^E < u_i^E \exp \left( \int_0^t U_i(\tau, \vartheta_i) \, d\tau \right)
\]
since \(u_i^E > 0\). The perturbed norm \(\|u(t) - u^E\|\) is then estimated as
\[
\|u(t) - u^E\| \leq \|u(t)\| - \|u^E\| = \|u_i^E \exp \left( \int_0^t U_i(\tau, \vartheta_i) \, d\tau \right)\| - \|u^E\|
\]
\[
= \left( \sum_{i=1}^n |u_i^E \exp \left( \int_0^t U_i(\tau, \vartheta_i) \, d\tau \right)|^2 \right)^{1/2} - \|u^E\|
\]
\[
\leq \left( \sum_{i=1}^n |u_i^E \exp \left( \int_0^t U_i(\tau, \vartheta_i) \, d\tau \right)|^2 \right)^{1/2} \equiv n^{1/2} u_i^E \exp \left( \int_0^t U_i(\tau, \vartheta) \, d\tau \right)
\]

if \(U_i(\tau, \vartheta_i) = U(\tau, \vartheta)\) and \(\vartheta_i = \vartheta\) for \(i = 1...n\), representing an isotropic set of perturbations. Evaluating the norm estimate then enables the asymptotic behavior and stability to be deduced for \(t \to \infty\). Stability and Lyapunov stability requires \(3K\) such that
\[
\lim_{t \to \infty} \|u(t) - u^E\| \leq \lim_{t \to \infty} n^{1/2} u_i^E \exp \left( \int_0^t U(\tau, \vartheta) \, d\tau \right) < K < \infty
\]
which will be the case if the integral \(\lim_{t \to \infty} \int_0^t U(\tau, \vartheta) \, d\tau\) converges.

**Lemma 2.9.** Equations (2.31) and (2.32) are solutions of the perturbed differential equations
\[
H_n \psi_i(t) = \|\sqrt{\beta} \partial U_i(t, \vartheta_i)\|_{L_2}^2 + \beta \|U_i(t, \vartheta_i)\|_{L_2}^2
\]
\[
D_n u_i(t) = \|\sqrt{\beta} \partial U_i(t, \vartheta_i)\|_{L_2}^2 + \beta \|U_i(t, \vartheta_i)\|_{L_2}^2
\]

If \(U_i(t, \vartheta_i) = U(t, \vartheta)\) for \(i = 1...n\), then equations (2.31) and (2.32) are solutions of the equivalent perturbed ODEs
\[
H_n \psi_i(t) = n \partial U_i(t, \vartheta) + n \beta |U_i(t, \vartheta)|^2 = S(t, \vartheta)
\]
\[
D_n u_i(t) = n \partial U_i(t, \vartheta) + n \beta |U_i(t, \vartheta)|^2 = S(t, \vartheta)
\]

**Proof.** The perturbed equation is
\[
H \psi_i(t) = \sum_{i=1}^n \partial U_i(t, \vartheta_i) + \beta \sum_{i=1}^n \partial U_i(t, \vartheta_i) \partial U_i(t, \vartheta_i)
\]
\[
= \sum_{i=1}^n \partial U_i(t, \vartheta_i) + \beta \sum_{i=1}^n U_i(t, \vartheta_i) U_i(t, \vartheta_i) = S_i(t, \vartheta_i)
\]

If \(U_i(t, \vartheta_i) = U(t, \vartheta)\) for \(i = 1...n\) then (2.36) follows. Using (2.3), the perturbed nonlinear system of ODEs is equivalently
\[
D_n u_i(t) = \sum_{i=1}^n \frac{\partial U_i(t, \vartheta_i)}{u_i(t)} + (\beta - 1) \sum_{i=1}^n \frac{\partial u_i(t)}{u_i(t)} \frac{\partial u_i(t)}{u_i(t)} = \sum_{i=1}^n \frac{u_i^E \partial U_i(t, \vartheta_i) B(t)}{u_i^E B(t)}
\]
\[
+ \sum_{i=1}^n \frac{u_i^E U_i(t, \vartheta_i) U_i(t, \vartheta_i) B(t)}{u_i^E B(t)} + (\beta - 1) \sum_{i=1}^n \frac{u_i^E U_i(t, \vartheta_i) U_i(t, \vartheta_i) B(t) B_i(t)}{u_i^E B_i(t) u_i^E B(t)}
\]
\[
= \sum_{i=1}^n \partial U_i(t, \vartheta_i) + \beta \sum_{i=1}^n U_i(t, \vartheta_i) U_i(t, \vartheta_i)
\]
\[
= \sum_{i=1}^n \partial U_i(t, \vartheta_i) + \beta \sum_{i=1}^n \delta_i U_i(t, \vartheta_i) U_i(t, \vartheta_i)
\]
\[ \| \sqrt{\partial_1 U_i(t, \vartheta_1)} \|_L^2 + \beta \| U_i(t, \vartheta_1) \|_L^2 = \sum_{i=1}^n S_i(t, \vartheta) \]  

(2.41)

where \( B_i(t) = \exp(\int_0^t U_i(t, \vartheta_1) \, d\vartheta) \). If \( U_i(t, \vartheta_1) = U(t, \vartheta) \) and \( \vartheta_1 = \vartheta \) for \( i = 1 \ldots n \) then

\[ D_n u_i(t) = n \partial_1 U(t, \vartheta) + n \beta U(t, \vartheta) \|_L^2 = S(t, \vartheta) \]  

(2.42)

But the perturbed equations converge back rapidly for \( t \gg \| \vartheta \| \) so that \( D_n u_i(t) \to 0 \) very rapidly for \( t \gg \| \vartheta \| \).

**Corollary 2.10.** Since \( \| U(t, \vartheta_1) \| \to 0 \) and \( \| \partial_1 U(t, \vartheta_1) \| \to 0 \) for \( t \gg \| \vartheta_1 \| \) then the perturbed equations decay rapidly to the unperturbed equations for \( t \gg \| \vartheta_1 \| \) so that

\[ \lim_{t \to \infty} H_n \psi_i(t) = \lim_{t \to \infty} \| \sqrt{\partial_1 U_i(t, \vartheta_1)} \|_L^2 + \lim_{t \to \infty} \| U(t, \vartheta) \|_L^2 = H_n \psi_i(t) \]  

(2.43)

\[ \lim_{t \to \infty} D_n u_i(t) = \lim_{t \to \infty} \| \sqrt{\partial_1 U_i(t, \vartheta_1)} \|_L^2 + \lim_{t \to \infty} \| U(t, \vartheta) \|_L^2 = D_n u_i(t) \]  

(2.44)

**Proposition 2.11.** Suppose \( u_i^E \) are a set of stable solutions of equilibrium fixed points of the system of ODEs \( D_n \psi_i(t) = D_n u_i^E = 0 \). Let \( \bar{u}_i(t) \) denote perturbations of the stable fixed points then \( \| \bar{u}(t) \| = \| u(t) - u^E \| \). If

\[ \| \bar{u}(t) - u^E \| = \| u(t) - u^E \| \sim \exp(Lyt) \]  

(2.45)

then \( Ly \) is essentially a LCE with stability if \( Ly < 0 \) and instability if \( Ly > 0 \). For example, if \( U_i(t, \vartheta) = A_i \) is a constant perturbation or amplitude then

\[ \bar{u}_i(t) = u_i^E \exp(\int_0^t A_i \, d\tau) = u_i^E \exp(A_i t) \]  

(2.46)

Using (2.33), the norm is estimated as \( \| u(t) - u^E \| < n^{1/2} u^E \exp(A t) \) if \( A_i = A \) so that

\[ \| \bar{u}(t) \| = \| u(t) - u^E \| \sim \exp(A t) = \exp(Lyt) \]  

(2.47)

so that \( A \) is essentially a Lyapunov exponent, with stability for \( A < 0 \) and instability for \( A > 0 \).

**Corollary 2.12.** Equation (2.46) is also a solution of the ODEs

\[ D_n \bar{u}_i(t) = \beta \| A_i \|_L^2 = n \beta \lambda^2 \]  

(2.48)

if \( A_i = A \) for all \( i = 1 \ldots n \), and \( \lambda^2 = \lambda^2 / \beta \). Since \( \partial_1 \bar{u}_i(t) = u_i^E A_i \exp(A_i t) \) and \( \partial_2 \bar{u}_i(t) = u_i^E A_i \exp(A_i t) \) then if \( A_i = A \)

\[ D_n \bar{u}_i(t) = \sum_{i=1}^n \frac{\partial_1 \bar{u}_i(t)}{u_i(t)} + (\beta - 1) \sum_{i=1}^n \frac{\partial_2 \bar{u}_i(t)}{u_i(t) u_j(t)} \]  

(2.49)

\[ = \sum_{i=1}^n \frac{u_i^E A_i A_i \exp(A_i t)}{u_i^E \exp(A_i t)} + (\beta - 1) \sum_{i=1}^n \frac{u_i^E u_i^E A_i A_i \exp(2A_i t)}{u_i^E u_i^E \exp(2A_i t)} \]  

(2.50)

\[ = \sum_{i=1}^n A_i A_i + (\beta - 1) \sum_{i=1}^n A_i A_i \]  

(2.51)

\[ = \beta \sum_{i=1}^n A_i A_i \equiv \beta \| A_i \|^2 = n \beta \lambda^2 \equiv n Ly^2 \]

if \( \lambda^2 = \lambda^2 / \beta \).
2.1. Random perturbations. Suppose instead, the nonlinear system described by (2.2) or (2.3) is subject to random perturbations, fluctuations or noise, either externally via an external coupled noise or thermal bath, or intrinsically. A random field can be defined as follows. (Appendix A.)

**Definition 2.13.** If \((\Omega, \mathcal{F}, \mu, T)\) is a probability space then for all \(t \in \mathbb{R}^+\) and \(\omega \in \Omega\) there is a map \(\mathcal{M}: \omega \times \mathbb{R}^+ \to \widehat{\mathcal{U}}(t, \omega) = \widehat{\mathcal{U}}(t)\). The stochastic expectation or average of any stochastic quantity or field \(\widehat{\mathcal{F}}(t, \omega)\) is obtained by integration over the measure \(\mu(\omega)\) so that \(\mathcal{M}\{\widehat{\mathcal{F}}(t, \omega)\} = \int_{\Omega} \widehat{\mathcal{F}}(t, \omega) d\mu(\omega)\). Note that all stochastic quantities will have an overhead tilde. For Gaussian free vector fields \(\mathcal{U}(t)\) with \(Q \geq 0\)

\[
\mathcal{M}\{\widehat{\mathcal{U}}(t, \omega)\} = \int_{\Omega} \widehat{\mathcal{U}}(t, \omega) d\mu(\omega) \leq Q
\]  

(2.52)

The map \(T: \Omega \to \Omega\) is a measure-preserving transformation such that \(\mu(T^{-1}B) = \mu(B)\) for all \(B \in \mathcal{F}\). For random dynamical systems \((\Omega, \mathcal{F}, \mu, T)\) one introduces the concepts of mixing and ergodicity such that for all \(A, B \in \mathcal{F}\) one has strong 2-mixing \(\mu(T^{-1}(A \cap B) \to \mu(A) \mu(B)\). The covariance for the Gaussian field \(\widehat{\mathcal{U}}(t)\) is formally

\[
\text{COV}_{ij}(t, s) = \int_{\Omega} \int_{\Omega} \mathcal{U}(t, \omega) \mathcal{U}(s, \xi) d\mu(\omega) d\mu(\xi)
\]

\[
\leq \mathcal{M}\{\mathcal{U}(t) \mathcal{U}(s)\} - Q^2 = \delta_{ij} J(\Delta; \varsigma) - Q^2
\]  

(2.53)

For a set of \(n\) correlated Gaussian noises \(\mathcal{U}(t)\), one has \(\mathcal{M}\{\mathcal{U}(t)\} = 0\) and a regulated 2-point function:

\[
\mathcal{M}\{\mathcal{U}(t) \mathcal{U}(s)\} = \delta_{ij} J(\Delta; \varsigma)
\]  

(2.54)

with \(\Delta = |t - s|\) and converges to white noise in the limit as \(\varsigma \to 0\), so that \(\mathcal{M}\{\mathcal{U}(t) \mathcal{U}(s)\} = \delta_{ij}\alpha\delta(t - s)\) for constant \(\alpha\). The standard Brownian motion is \(d\mathcal{B}(t) = \mathcal{W}(t)dt\). For a thermal bath of Gaussian white noise \(\mathcal{M}\{\mathcal{U}(t) \mathcal{U}(s)\} = \delta_{ij}\alpha k_B T\delta(t - s)\) where \(k_B\) is Boltzmann’s constant and \(T\) is the temperature. We assume non-white Gaussian random fields with regulated covariance throughout.

**Lemma 2.14.** The random field \(\mathcal{U}(t)\) also has the properties [18]:

1. Stochastic continuity such that for any \((t, s)\) and \(\epsilon > 0\) one has the probabilities

\[
P(\|\mathcal{U}_t - \mathcal{U}_s\| > \epsilon) = 0
\]  

(2.55)

For any pair \((t_1, t_2)\) and \((\alpha, \beta, K) > 0\)

\[
\mathcal{M}\{\|\mathcal{U}_{t_2} - \mathcal{U}_{t_1}\|^\alpha\} \leq K|t_2 - t_1|^\beta + 1
\]  

(2.56)

2. \(\exists T\) such that for any \(t > T\) and some \((\epsilon, \delta) > 0\)

\[
P\left(\left\|\frac{1}{T}\int_{t_0}^{t_0 + t} \mathcal{U}(s)ds - \frac{1}{T}\int_{t_0}^{t_0 + t} \mathcal{M}\{\mathcal{U}(s)\} ds\right\| > \delta\right) < \epsilon
\]  

(2.57)

The effect of such noise or random perturbations on nonlinear classical systems has become a subject of considerable interest. Coupling noise to classical nonlinear ODEs and PDEs is a powerful and useful methodology with applications to turbulence, chaos and pattern formation [14,15,25,26,27,28]. Noise can destabilize a stable system, or a system considered stable to deterministic perturbations, but can also stabilize an unstable system [31,32,33,34,35,36]. It can also smooth out or dissipate blowups or singularities which exist for the purely deterministic problem. For multiplicative noise or random perturbations \(\mathcal{U}(t)\), equation (2.19) can become a nonlinear stochastic differential equation

\[
\alpha \sum_{i=1}^{n} \partial_i \mathcal{U}(t) + \beta \sum_{i=1}^{n} (\partial_i \mathcal{U}(t)) + \sum_{i=1}^{n} (\gamma + \mathcal{U}(t)) (\mathcal{U}(t))^n = 0
\]  

(2.58)

Setting \(\alpha = a = 1\) and \(\beta = 0\) for example, gives

\[
\sum_{i=1}^{n} \partial_i \mathcal{U}(t) + \sum_{i=1}^{n} (\gamma + \mathcal{U}(t)) (\mathcal{U}(t))^n \equiv \sum_{i=1}^{n} \partial_i \mathcal{U}(t) + \sum_{i=1}^{n} \gamma(t) \mathcal{U}(t)
\]  

(2.59)
which describes a linear set of \( n \) noisy harmonic oscillators with random frequencies \( \gamma(t) \). One can also consider static or equilibrium solutions \( \psi_i^E \) and subject these to stochastic perturbations so that \( \psi_i(t) = \psi_i^E + \mathcal{W}_i(t) \) and with averaged norms \( M\{ \| \psi(t) - \psi_i^E \| \} \).

Extending the classical Oseledec Theorem [6] to noisy or random systems presents technical challenges, but the following proposition for Lyapunov characteristic exponents can be considered for the randomly perturbed deterministic or equilibrium solutions [18].

**Proposition 2.15.** Suppose \( u_i^E \) are a set of stable solutions of equilibrium fixed points of the system of ODEs \( D_n u_i(t) = 0 \) or \( D_n u_i^E = 0 \). Let \( \tilde{u}_i(t) \) be the stochastic perturbations of the stable points with \( \tilde{u}_i(t) = u_i^E \exp(\int_0^t \mathcal{W}(\tau) d\tau) \) then \( \| \delta \tilde{u}(t) \| = \| \tilde{u}(t) - u_i^E \| \). If

\[
\| u^E \|^{-1} M\left\{ \left| \delta \tilde{u}(t) \right| \right\} \equiv \| u^E \|^{-1} M\left\{ \left| \tilde{u}(t) - u^E \right| \right\} \sim \exp(Ly) \tag{2.60}
\]

then \( Ly \) is a LCE with stability if \( Ly < 0 \) and instability if \( Ly > 0 \) and so by analogy with (2.28)

\[
Ly = \lim_{t \uparrow \infty} \frac{1}{t} \| u^E \|^{-1} \log M\left\{ \left| \tilde{u}(t) \right| \right\} = \lim_{t \uparrow \infty} \frac{1}{t} \| u^E \|^{-1} \log M\left\{ \left| \tilde{u}(t) - u^E \right| \right\} \tag{2.61}
\]

The general \( \ell \)-th moment Lyapunov characteristic exponent (LCE) can be defined as follows

**Definition 2.16.** If \( \tilde{u}_i(t) \) is a solution of a SDE or a randomly perturbed deterministic solution with initial data \( u_0 = u^E \), then the \( \ell \)th moment is

\[
Ly(\ell) = \lim_{t \uparrow \infty} \frac{1}{t} \log M\left\{ \left| \tilde{u}(t) - u^E \right|^\ell \right\} \tag{2.62}
\]

The stability criteria are then

1. If \( Ly(\ell) > 0 \) then \( M\{ \| \tilde{u}(t) - u^E \|^{\ell} \} \to \infty \) as \( t \to \infty \) and the randomly perturbed system cannot reach a new stable state. The system is then unstable.
2. If \( Ly(\ell) = -\infty \) then the randomly perturbed system is superstable.
3. If \( Ly(\ell) < 0 \) then \( M\{ \| \tilde{u}(t) - u^E \|^{\ell} \} \to 0 \) then the randomly perturbed system is stable.
4. If \( Ly(\ell) = 0 \) then \( \exists(B, C) > 0 \) such for \( t > 0 \) one has \( B \leq M\{ \| \tilde{u}(t) - u^E \|^{\ell} \} \leq C \).

Using (2.62) then

\[
Ly(\ell) = \lim_{t \uparrow \infty} \frac{1}{t} \log M\left\{ \left| \tilde{u}(t) - u^E \right|^\ell \right\}
\]

\[
\leq \lim_{t \uparrow \infty} \frac{1}{t} \log \left( u^E |^\ell /2 M\left\{ \exp \left( \zeta \ell \int_0^t \mathcal{W}(s) ds \right) \right\} \right)
\]

\[
= \lim_{t \uparrow \infty} \frac{1}{t} \log(u^E |^\ell /2) + \lim_{t \uparrow \infty} \frac{1}{t} \log M\left\{ \exp \left( \zeta \ell \int_0^t \mathcal{W}(s) ds \right) \right\}
\]

\[
= \lim_{t \uparrow \infty} \frac{1}{t} \log M\left\{ \exp \left( \zeta \ell \int_0^t \mathcal{W}(s) ds \right) \right\} \tag{2.63}
\]

which again depends on the convergence properties of the stochastic integral.

**Definition 2.17.** The \( \ell \)-th order mean \( M\{ \| u(t) - u^E \|^{\ell} \} \) is also associated with the characteristic function \( \Psi(t)(z) \) where \( z \) is complex with \( z \in \mathbb{C} \) and \( L = \log \| \tilde{u}(t) - u^E \| \). Then

\[
\Psi(z) = M\left\{ \exp(izLy(t)) \right\} = M\left\{ iz \log \| \tilde{u}(t) - u^E \| \right\} \tag{2.64}
\]

If \( z = \ell \in \mathbb{Z} \) then \( \Psi(\ell) = M\{ \exp(iz \ln \| \tilde{u}(t) - u^E \|) \} \). The Lyapunov functional in the complex plane then has the representation

\[
Ly(iz) = \lim_{t \uparrow \infty} \frac{1}{t} \log \Psi(z; t) = \lim_{t \uparrow \infty} \frac{1}{t} M\left\{ \exp(iz \ln \| \tilde{u}(t) - u^E \|) \right\} \tag{2.65}
\]
Proposition 2.18. Consider an $n$-dimensional 1st-order linear system of the general form

$$
\sum_{i=1}^{n} \frac{\partial u_i(t)}{u_i(t)} = \zeta \sum_{i=1}^{n} \frac{f_i(t)}{\beta_i} \equiv n\zeta \frac{f(t)}{\beta}
$$

(2.66)

if $f_i(t) = f(t)$ and $\beta_i = \beta$ for $i = 1...n$, and where $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta > 0$ and $\beta_i$ are constants. If $f(t) = 0$ then $\sum_{i=1}^{n} \frac{\partial u_i(t)}{u_i(t)} = 0$, which is also equivalent to $\sum_{i=1}^{n} \partial u_i(t) = 0$. There is then a trivial set of stable or equilibrium solutions $u_i(t) = \alpha_i^\circ$ for $i = 1...n$ for the homogeneous equations. For the inhomogeneous equations the solution is

$$
u_i(t) = u_i(0) \exp \left( \frac{\zeta}{\beta_i} \int_{0}^{t} f_i(\tau) d\tau \right)
$$

(2.67)

and we can set $\beta_i = 1$. Let $\mathcal{W}_i(t)$ be a set of $n$ independent Gaussian white noises and let $\mathcal{W}_i(t)$ be a non-white Gaussian noise with correlation $\zeta$ so that for any $t, s \in \mathbb{R}^+$ $\mathbb{M}\{\mathcal{W}_i(t)\} = 0$ and with a regulated 2-point function $\mathbb{M}\{\mathcal{W}_i(t)\mathcal{W}_i(s)\} = \delta_{ij} J(\Delta; \zeta)$, where $J(0; \zeta) < \infty$ such that

$$
\lim_{\tau \to 0} \mathbb{M}\{\mathcal{W}_i(t)\mathcal{W}_i(s)\} = \mathbb{M}\{\mathcal{W}_i(t)\mathcal{W}_i(s)\} = \alpha \delta_{ij} \delta(t - s)
$$

(2.68)

Also, unlike for white noise, the derivative $\partial_t \mathcal{W}_i(t)$ exists. (Appendix A.) The following tentative SDEs are then possible:

$$
\sum_{i=1}^{n} \frac{\partial \hat{u}_i(t)}{\hat{u}_i(t)} = \zeta \sum_{i=1}^{n} f_i(t) + \zeta \sum_{i=1}^{n} \mathcal{W}_i(t)
$$

(2.69)

$$
\sum_{i=1}^{n} \frac{\partial \hat{u}_i(t)}{\hat{u}_i(t)} = \zeta \sum_{i=1}^{n} f_i(t) + \zeta \sum_{i=1}^{n} \mathcal{W}_i(t)
$$

(2.70)

where $\zeta > 0$. Using the Stratanovitch interpretation (Appendix A) the rules of ordinary calculus apply so that the solutions of (2.69) and (2.70) are

$$
\hat{u}_i(t) = u_i(0) \exp \left( \zeta \int_{0}^{t} f_i(\tau) d\tau + \zeta \int_{0}^{t} \mathcal{W}_i(\tau) d\tau \right)
$$

(2.71)

$$
\hat{u}_i(t) = u_i(0) \exp \left( \zeta \int_{0}^{t} f_i(\tau) d\tau + \zeta \int_{0}^{t} \mathcal{W}_i(\tau) d\tau \right)
$$

(2.72)

Then for any $t > 0$, $u(t) = \{\hat{u}_1(t), ..., \hat{u}_n(t)\}$ is essentially a random matrix. Concentrating on (2.72), the expected value or stochastic average $\mathbb{M}\{\hat{u}_i(t)\}$ is

$$
\mathbb{M}\{\hat{u}_i(t)\} = u_i(0) \exp \left( \zeta \int_{0}^{t} f_i(\tau) d\tau \right) \mathbb{M}\left\{ \exp \left( \zeta \int_{0}^{t} \mathcal{W}_i(\tau) d\tau \right) \right\}
$$

(2.73)

and if $f_i(t) = 0$ for $i = 1...n$ then $\mathbb{M}\{\hat{u}_i(t)\} = u_i(0)\mathbb{M}\{\exp(\zeta \int_{0}^{t} \mathcal{W}_i(\tau) d\tau)\}$

The stochastic integral (2.73) exists and can be shown to be well defined. (Appendix A.)

As an example of stability or instability induced by noise or random perturbations, consider again equation (2.66) which describes a (linear) $n$-dimensional system subject to white noise. This SDE can be solved exactly and one can then apply the Lyapunov exponent (2.62) to the solution to test stability.

Lemma 2.19. (Noise-induced destabilisation and stabilisation). Let $u_i(0)$ be initial data for an $n$-dimensional linear system and let $\alpha, \beta \in \mathbb{R}$. Let $\mathcal{W}(t)$ be a white noise and $d\mathcal{B}(t) = \mathcal{W}(t)dt$, the standard Brownian motion with $\mathcal{B}(0) = 0$. Then:
(1) The $n$-dimensional stable system

$$\sum_{i=1}^{n} \partial_t u_i(t) = -\alpha \sum_{i=1}^{n} u_i(t)$$  \hspace{1cm} (2.74)

with solution $u_i(t) = u_i(0) \exp(-\alpha t)$ which is randomly perturbed as

$$\sum_{i=1}^{n} \partial_t u_i(t) = -\alpha \sum_{i=1}^{n} u_i(t) + \zeta \sum_{i=1}^{n} u_i(t) \mathcal{W}(t)$$  \hspace{1cm} (2.75)

and which is equivalent to the $n$-dimensional Brownian motion

$$\sum_{i=1}^{n} d\tilde{u}_i(t) = -\alpha \sum_{i=1}^{n} u_i(t) dt + \zeta \sum_{i=1}^{n} u_i(t) d\mathcal{B}(t)$$  \hspace{1cm} (2.76)

is destabilised by the noise or random perturbation if $(-\alpha - \frac{1}{2} \zeta^2 > 0)$ but remains stable if $(-\alpha - \frac{1}{2} \zeta^2 < 0$.

(2) The $n$-dimensional unstable system

$$\sum_{i=1}^{n} \partial_t u_i(t) = \alpha \sum_{i=1}^{n} u_i(t)$$  \hspace{1cm} (2.77)

with solution $\hat{u}(t) = u(0) \exp(\alpha t)$ is stabilized by random perturbations of the form

$$\sum_{i=1}^{n} d\hat{u}_i(t) = \alpha \sum_{i=1}^{n} u_i(t) dt + \zeta \sum_{i=1}^{n} u_i(t) d\mathcal{B}(t)$$  \hspace{1cm} (2.78)

if $\alpha - \frac{1}{2} \zeta^2 < 0$.

Proof. If $\mathcal{F}(u_i(t))$ is a $C^2$-differentiable functional of $u_i(t)$ and $D = d/du_i(t)$ then the Ito Lemma gives

$$d\mathcal{F}(\hat{u}_i(t)) = D\mathcal{F}(u_i(t)) d\hat{u}_i(t) + \frac{1}{2} D^2 \mathcal{F}(u_i(t)) (\zeta^2 u_i(t))^2 dt$$

$$= D\mathcal{F}(u_i(t)) (-\alpha u_i(t) dt + \zeta u(t) \mathcal{B}(t)) + \frac{1}{2} D^2 \mathcal{F}(u_i(t)) (\zeta^2 u_i(t))^2 dt$$  \hspace{1cm} (2.79)

so that for $\mathcal{F}(\hat{u}_i(t)) = \log u_i(t)$

$$d\mathcal{F}(\hat{u}_i(t)) = \frac{1}{u_i(t)} d\hat{u}_i(t) - \frac{\zeta^2}{2} \frac{1}{u_i(t)^2} |u_i(t)|^2 dt$$

$$= (-\alpha - \frac{1}{2} \zeta^2) dt + \zeta d\mathcal{B}(t)$$  \hspace{1cm} (2.80)

The solution is

$$\log |u_i(t)| = \log |u_i(0)| + \int_0^t (-\alpha - \frac{1}{2} \zeta^2) ds + \zeta \int_0^t d\mathcal{B}(s)$$  \hspace{1cm} (2.81)

so that

$$\hat{u}_i(t) = u_i(0) \exp \left(- (\alpha - \frac{1}{2} \zeta^2) t + \zeta \mathcal{B}(t) \right)$$  \hspace{1cm} (2.82)

The LCE is then

$$\hat{L} = \lim_{t \to \infty} \frac{1}{t} \log(M\{\hat{u}_i(t)\}) = -\alpha - \frac{1}{2} \zeta^2$$  \hspace{1cm} (2.83)

If $\hat{L} = -\alpha - \frac{1}{2} \zeta^2 < 0$ then the system it remains stable but if $\hat{L} = -\alpha - \frac{1}{2} \zeta^2 < 0$ then it is unstable to the random perturbations. Repeating with $\alpha$ replacing $-\alpha$, shows that noise will stabilise the unstable system (2.77) for $\alpha - \frac{1}{2} \zeta^2 < 0$. \hfill $\square$

We consider now only the non-white random perturbations.

Lemma 2.20. Let $f_i(t) = 0$. If $\tilde{\mathcal{F}}(\hat{u}(t)) = \hat{\mathcal{F}}(\hat{u}(t))$ and $u_i^E = u_i^E$ for $i = 1...n$ then the estimates for the expectations of the norms $M\{||\hat{u}(t) - u_i^E||\}$ and moments $M\{||\hat{u}(t) - u_i^E||^*\}$ for integers $t \in \mathbb{Z}$ are

$$M\{||\hat{u}(t) - u_i^E||\} \leq u_i^E n^{1/2} M \left\{ \exp \left( \zeta \int_0^t \hat{\mathcal{F}}(\tau) d\tau \right) \right\} = u_i^E n^{1/2} I(t)$$  \hspace{1cm} (2.84)
solution can very often be found for nonlinear equations. Such FP equations are often impossible to solve although in the infinite-time relaxation limit, the equilibrium equation or Fokker-Planck (FP) equation. However, this is only possible for first-order equations. Again, more appropriate to consider the maxima of a probability density distribution function.

\[
\text{Proof.} \quad \text{The system is asymptotically stable if } \lim_{t \to \infty} N\{\|\hat{u}(t) - u^E\|\} = 0 \text{ and Lyapunov stable if } \exists K > 0 \text{ such that } \lim_{t \to \infty} M\{\|\hat{u}(t) - u^E\|\} < K
\]

(2) Random perturbations then destabilize the system if \(\lim_{t \to \infty} M\{\|\hat{u}(t) - u^E\|\} = \infty\)

The evaluation of stability then requires an estimate of the stochastic integrals \(I(t)\) or \(I(t, \ell)\).

As before, a key question of interest is often to determine the extent to which such non-white random perturbations or noise can induce transitions between stable states of a system, especially a nonlinear system, or whether noise will actually destabilize the system: what was established as a stable point via a deterministic linear stability analysis may actually be unstable, or at best ‘quasi-stable’, in the presence of stochastic noise. However, in general, most SNLDEs will be impossible to solve. Due to the presence of noise terms it is usually more appropriate to consider the maxima of a probability density distribution function \(\mathcal{P}(u(t), t)\) rather than fixed points of the dynamics \([21, 22]\). The \(\mathcal{P}(u(t), t)\) would be stationary solutions of a Kolmogorov forward equation or Fokker-Planck (FP) equation. However, this is only possible for first-order equations. Again, such FP equations are often impossible to solve although in the infinite-time relaxation limit, the equilibrium solution can very often be found for nonlinear equations.

One could consider the following candidates for 2nd-order n-dimensional nonlinear SDES

\[
D_n\dot{u}_i(t) = \sum_{i=1}^{n} \frac{\partial u_i(t)}{u_i(t)} + (\beta - 1) \sum_{i=1}^{n} \frac{\partial u_i(t) \partial u_i(t)}{u_i(t)u_j(t)} = \zeta \sum_{i=1}^{n} \mathcal{W}_i(t) \quad (2.88)
\]
\[ D_n \hat{u}_i(t) = \sum_{i=1}^{n} \frac{\partial u_i(t)}{u_i(t)} + (\beta - 1) \sum_{i=1}^{n} \frac{\partial u_i(t) \partial u_i(t)}{u_i(t) u_j(t)} = \zeta \sum_{i=1}^{n} \mathcal{V}_i(t) \]  

(2.89)

or

\[ H_n \hat{v}_i(t) = \sum_{i=1}^{n} \delta_{i \ell} \psi_i(t) + \beta \sum_{i=1}^{n} \delta_{i \ell} \psi_i(t) \partial_t \psi_i(t) = \zeta \sum_{i=1}^{n} \mathcal{V}_i(t) \]  

(2.90)

However, these are impossible to solve. Instead one could substitute randomly perturbed solutions of the original deterministic equations and substitute back into the original deterministic equations, and then take the stochastic expectation or average. Because of the nonlinearity, additional terms can be induced within the stochastically averaged equations.

**Proposition 2.21.** Let \( \psi(t) \) and \( u_i(t) \) be deterministic solutions of (2.2) and (2.3) and let \( \{ \mathcal{W}_i(t) \} \) be a Gaussian white noise regulated so that \( M\{ \mathcal{W}_i(t) \} = 0 \), derivative \( \partial_t \mathcal{W}_i(t) \) and \( M\{ \partial_t \mathcal{W}_i(t) \partial_t \mathcal{W}_i(t) \} = \delta_{ij} J(0; \zeta) < \infty \). Let the randomly perturbed solution be

\[ \hat{\psi}_i(t) = \psi_i(t) + \zeta \int_0^t \mathcal{W}_i(\tau)d\tau \equiv \psi_i(t) + \zeta \int_0^t d\mathcal{W}_i(\tau) \]  

(2.91)

then since \( u_i(t) = \exp(\psi_i(t)) \)

\[ \hat{u}_i(t) = u_i(t) \exp \left( \zeta \int_0^t \mathcal{W}(\tau)d\tau \right) \equiv u_i(t) \mathcal{R}_i(t) \]  

(2.92)

Equation (2.91) is equivalent to the stochastic differential equation

\[ d\hat{\psi}(t) = d\psi(t) + \zeta d\mathcal{W}(t) \]  

(2.93)

For white noise \( \mathcal{W}_i(t) = \mathcal{W}_i(t) \), this is a simple linear Brownian motion \( d\hat{\psi}(t) = d\psi(t) + \zeta d\mathcal{W}(t) \equiv d\psi(t) + \zeta \mathcal{W}(t)dt \). Equations (2.91) and (2.92) are then solutions of the stochastically averaged systems of differential equations

\[ M\{ H_n \hat{\psi}_i(t) \} = H_n \hat{\psi}_i(t) + \zeta^2 \beta M\left\{ \mathcal{W}_i(t) \right\}^2 = \zeta^2 \beta M\left\{ \mathcal{W}_i(t) \right\}^2 \]  

(2.94)

\[ M\{ D_n \hat{u}_i(t) \} = D_n u_i(t) + \zeta^2 \beta M\left\{ \mathcal{W}_i(t) \right\}^2 = \zeta^2 \beta M\left\{ \mathcal{W}_i(t) \right\}^2 \]  

(2.95)

or

\[ M\{ H_n \hat{\psi}_i(t) \} = H_n \hat{\psi}_i(t) + C = C \]  

(2.96)

\[ \tilde{M}\{ D_n \hat{u}_i(t) \} = D_n u_i(t) + C = C \]  

(2.97)

when \( H_n \hat{\psi}_i(t) = D_n u_i(t) = 0 \) and where \( C = \zeta^2 \beta n J(0; \zeta) \) when \( \mathcal{W}_i(t) = \mathcal{W}(t) \).

**Proof.** If \( \psi_i(t) \) is a solution of the deterministic equations \( H_n \psi_i(t) = 0 \) then the randomly perturbed equations are

\[ H_n \hat{\psi}_i(t) = \sum_{i=1}^{n} \partial_t \psi_i(t) + \beta \sum_{i=1}^{n} \partial_t \psi_i(t) \partial_t \psi_i(t) \]  

(2.98)

The derivatives of (2.91) are \( \partial_t \hat{\psi}_i(t) = \partial_t \psi_i(t) + \zeta \mathcal{W}_i(t) \) and \( \partial_t \hat{\psi}_i(t) = \partial_t \psi_i(t) + \zeta \partial_t \mathcal{W}_i(t) \) so that (2.98) becomes

\[ H_n \hat{\psi}_i(t) = \sum_{i=1}^{n} \partial_t \psi_i(t) + \beta \sum_{i=1}^{n} \partial_t \psi_i(t) \partial_t \psi_i(t) + \zeta \sum_{i=1}^{n} \partial_t \mathcal{W}_i(t) + 2 \zeta \beta \sum_{i=1}^{n} \mathcal{W}_i(t) \partial_t \psi_i(t) + \zeta^2 \beta \sum_{i=1}^{n} \mathcal{W}_i(t) \hat{\psi}_i(t) \]  

(2.99)

Taking the expectation and using \( M\{ \mathcal{W}_i(t) \} = 0 \) and \( M\{ \partial_t \mathcal{W}_i(t) \} = 0 \) gives

\[ M\{ H_n \hat{\psi}_i(t) \} = \sum_{i=1}^{n} \partial_t \psi_i(t) \]
\[ + \beta \sum_{i=1}^{n} \partial_{i}\psi_{i}(t)\partial_{i}\psi_{i}(t) + \zeta^{2} \beta \sum_{i=1}^{n} \mathcal{M}\left\{ \mathcal{W}_{i}(t)\mathcal{W}_{i}(t) \right\} \]

\[ = \mathbf{H}_{n}\psi_{i}(t) + \zeta^{2} \beta \sum_{i=1}^{n} \mathcal{M}\left\{ \mathcal{W}_{i}(t)\mathcal{W}_{i}(t) \right\} \]

\[ = \mathbf{H}_{n}\psi_{i}(t) + \zeta^{2} \beta \mathcal{M}\left\{ \left\| \mathbf{W}_{i}(t) \right\|^{2}_{L_{2}} \right\} \]

\[ = \mathbf{H}_{n}\psi_{i}(t) + \zeta^{2} \beta \sum_{i=1}^{n} \delta_{i}J(0, \zeta) \]

\[ = \mathbf{H}_{n}\psi_{i}(t) + \zeta^{2} \beta nJ(0; \zeta) \equiv \mathbf{H}_{n}\psi_{i}(t) + C = C \]  

(2.100)

The randomly perturbed ODE for \( \mathbf{u}_{i}(t) \) is

\[ \mathbf{D}_{n}\mathbf{u}_{i}(t) = \sum_{i=1}^{n} \frac{\partial_{i}u_{i}(t)}{u_{i}(t)} + (\beta - 1) \sum_{i=1}^{n} \frac{\partial_{i}u_{i}(t)\partial_{i}u_{i}(t)}{u_{i}(t)u_{j}(t)} \]  

(2.101)

Next, the derivatives of \( \mathbf{u}_{i}(t) \) are \( \partial_{i}\mathbf{u}_{i}(t) = \zeta u_{i}(t)\mathbf{W}_{i}(t)\mathbf{B}_{i}(t) = \mathbf{B}_{i}(t)\partial_{i}u_{i}(t) \) and \( \partial_{i}\mathbf{u}_{i}(t) = \zeta^{2} u_{i}(t)\mathbf{W}_{i}(t)\mathbf{B}_{i}(t) + \zeta u_{i}(t)\mathbf{W}_{i}(t)\mathbf{B}_{i}(t) + (\partial_{i}u_{i}(t))\mathbf{B}_{i}(t) + \zeta (\partial_{i}u_{i}(t))\mathbf{B}_{i}(t) + \zeta u_{i}(t)(\partial_{i}u_{i}(t))\mathbf{B}_{i}(t) + (\partial_{i}u_{i}(t))(\partial_{i}u_{i}(t))\mathbf{B}_{i}(t) \). Then (2.95) becomes

\[ \mathbf{D}_{n}\mathbf{u}_{i}(t) = \zeta^{2} \sum_{i=1}^{n} \frac{u_{i}(t)\mathbf{W}_{i}(t)\mathbf{B}_{i}(t)(\partial_{i}\mathbf{B}_{i}(t))}{u_{i}(t)\mathbf{B}_{i}(t)} + \zeta \sum_{i=1}^{n} \frac{u_{i}(t)(\partial_{i}\mathbf{W}_{i}(t))(\partial_{i}\mathbf{B}_{i}(t))}{u_{i}(t)\mathbf{B}_{i}(t)} \]

\[ + \zeta^{2} \sum_{i=1}^{n} \frac{(\partial_{i}u_{i}(t))(\partial_{i}\mathbf{W}_{i}(t))(\partial_{i}\mathbf{B}_{i}(t))}{u_{i}(t)\mathbf{B}_{i}(t)} + \zeta^{2} (\beta - 1) \sum_{i=1}^{n} \frac{u_{i}(t)\mathbf{W}_{i}(t)\mathbf{B}_{i}(t)u_{i}(t)\mathbf{W}_{i}(t)\mathbf{B}_{i}(t)}{u_{i}(t)u_{i}(t)\mathbf{B}_{i}(t)\mathbf{B}_{i}(t)} \]

\[ + 2\zeta(\beta - 1) \sum_{i=1}^{n} \frac{u_{i}(t)\mathbf{W}_{i}(t)(\partial_{i}u_{i}(t))(\partial_{i}\mathbf{B}_{i}(t))}{u_{i}(t)u_{i}(t)\mathbf{B}_{i}(t)\mathbf{B}_{i}(t)} \]

\[ + (\beta - 1) \sum_{i=1}^{n} \frac{(\partial_{i}u_{i}(t))(\partial_{i}\mathbf{B}_{i}(t))(\partial_{i}u_{i}(t))(\partial_{i}\mathbf{B}_{i}(t))}{u_{i}(t)u_{i}(t)\mathbf{B}_{i}(t)\mathbf{B}_{i}(t)} \]  

(2.102)

Cancelling the \( \mathbf{B}_{i}(t) \) terms

\[ \mathbf{D}_{n}\mathbf{u}_{i}(t) = \zeta^{2} \sum_{i=1}^{n} \frac{(\partial_{i}u_{i}(t))\mathbf{W}_{i}(t)\mathbf{W}_{i}(t)}{u_{i}(t)} + \zeta \sum_{i=1}^{n} (\partial_{i}(\mathbf{W}_{i}(t))) \]

\[ = \zeta \sum_{i=1}^{n} \frac{(\partial_{i}u_{i}(t))\mathbf{W}_{i}(t)}{u_{i}(t)} + \zeta^{2} \sum_{i=1}^{n} \frac{(\partial_{i}u_{i}(t))\mathbf{W}_{i}(t)}{u_{i}(t)} + \zeta^{2} (\beta - 1) \sum_{i=1}^{n} \frac{\mathbf{W}_{i}(t)(\partial_{i}u_{i}(t))}{u_{i}(t)} \]

\[ + (\beta - 1) \sum_{i=1}^{n} \frac{(\partial_{i}u_{i}(t))(\partial_{i}u_{i}(t))}{u_{i}(t)u_{i}(t)} \]  

(2.103)

and taking the stochastic average

\[ \mathbf{M}\left\{ \mathbf{D}_{n}\mathbf{u}_{i}(t) \right\} = \sum_{i=1}^{n} \frac{\partial_{i}u_{i}(t)}{u_{i}(t)} + (\beta - 1) \sum_{i=1}^{n} \frac{\partial_{i}u_{i}(t)(\partial_{i}u_{i}(t))}{u_{i}(t)u_{i}(t)} + \zeta^{2} \beta \sum_{i=1}^{n} \mathbf{M}\left\{ \mathbf{W}_{i}(t)\mathbf{W}_{i}(t) \right\} \]

(2.104)

which is

\[ \mathbf{M}\left\{ \mathbf{D}_{n}\mathbf{u}_{i}(t) \right\} = \mathbf{D}_{n}u_{i}(t) + \zeta^{2} \beta \sum_{i=1}^{n} \mathbf{M}\left\{ \mathbf{W}_{i}(t)\mathbf{W}_{i}(t) \right\} \]
Proposition 2.23. for stability in general probabilistic terms can be tentatively defined as follows:

\[ u(t) \text{ with initial conditions } u(0) \]

Remark 2.22. The non-vanishing terms which arise in the stochastically averaged system of equations are due to the nonlinearity of the equations. For a linear system, the stochastically averaged equations will reduce back to the original deterministic equations. For example, in (2.46), the stochastically perturbed equations are:

\[
\sum_{i=1}^{n} \frac{\partial \hat{u}_i(t)}{\hat{u}_i(t)} = \sum_{i=1}^{n} \frac{\partial u_i(t) \hat{B}_i(t)}{u_i(t) \hat{B}_i(t)} + \sum_{i=1}^{n} \frac{u_i(t) \hat{W}_i(t) \hat{B}_i(t)}{u_i(t) \hat{B}_i(t)}
\]

Taking the stochastic average gives back the original ODE so that

\[
\mathbb{M} \left\{ \sum_{i=1}^{n} \frac{\partial \hat{u}_i(t)}{\hat{u}_i(t)} \right\} = \sum_{i=1}^{n} \frac{\partial u_i(t)}{u_i(t)} + \zeta \sum_{i=1}^{n} \hat{W}_i(t) = \zeta \sum_{i=1}^{n} f_i(t)
\]

since \( \mathbb{M}\{\hat{W}_i(t)\} = 0 \). If \( \mathbb{M}\{\hat{W}_i(t)\} > 0 \) then an extra term can arise also for averaged linear equations.

2.2. Stability criteria. Given the random perturbations and the random norm \( \|\hat{u}(t) - u^E\| \) the conditions for stability in general probabilistic terms can be tentatively defined as follows:

Proposition 2.23. Given the random perturbations \( \hat{u}_i(t) = \psi^E_i + \int_0^t \hat{W}(\tau) d\tau \) which gives \( \hat{u}_i(t) = u^E_i \exp(\int_0^t \hat{W}(\tau) d\tau) \), with initial conditions \( u_i(0) = u^E_i \) for a set of stable fixed points \( u^E_i \) then:

1. The system is stable in probability for all \( t > 0 \) if for any \( L > 0 \) there is a \( t > 0 \) such that \( \mathbb{P}\{\|\hat{u}(t) - u^E\| \leq |L|\} = 1 \) or \( \mathbb{P}\{\|\hat{u}(t) - u^E\| > |L|\} = 0 \). So there is a ball \( \mathbb{B}(L) \) of radius \( L \) containing \( \|\hat{u}(t) - u^E\| \) for any \( t > 0 \).
2. There is no noise-induced blowup or singularity for any finite \( t > 0 \) if \( \mathbb{P}\{\|\hat{u}(t) - u^E\| = \infty\} = 0 \).
3. The system is unstable if for any \( L > 0 \) and any \( t > 0 \) if \( \mathbb{P}\{\|\hat{u}(t) - u^E\| > |L|\} = 1 \). Instability can also be defined asymptotically as \( \lim_{t \to \infty} \mathbb{P}\{\|\hat{u}(t) - u^E\| = \infty\} = 1 \).
4. If \( \mathbb{B}(L) \subset \mathbb{R}^n \) is an Euclidean ball of radius \( L \) then if the norm is contained within \( \mathbb{B}(L) \) at any \( t > 0 \) then \( \|\hat{u}(t) - u^E\| \in \mathbb{B}(L) \). If this holds asymptotically for \( t > 0 \) then the randomly perturbed system is stable so that for some \( L > 0 \):

\[
\lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| \in \mathbb{B}(L)) = \lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| \leq |L|) = 1
\]

or

\[
\lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| \in \mathbb{B}(L)) = \lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| \geq |L|) = 0
\]

or

\[
\lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| \in \mathbb{B}(\infty)) = \lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| = \infty) = 0
\]

5. The randomly perturbed system is unstable if for any ball \( \mathbb{B}(L) \subset \mathbb{R}^n \):

\[
\lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| \in \mathbb{B}(L)) = \lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| \leq |L|) = 0
\]

or

\[
\lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| \in \mathbb{B}(L)) = \lim_{t \to \infty} \mathbb{P}(\|\hat{u}(t) - u^E\| > |L|) = 1
\]
The set is sub-Gaussian if
\[
\lim_{t \uparrow \infty} P(\|\tilde{u}(t) - u^E\| \in \mathcal{B}(\infty)) \equiv \lim_{t \uparrow \infty} P(\|\tilde{u}(t) - u^E\| = \infty) = 1
\] (2.113)

Equivalently for all \( p \geq 1 \)
\[
\lim_{t \uparrow \infty} P(\|\tilde{u}(t) - u^E\|^p \in \mathcal{B}(L)) \equiv \lim_{t \uparrow \infty} P(\|\tilde{u}(t) - u^E\|^p \leq |L|) = 0
\] (2.114)
\[
\lim_{t \uparrow \infty} P(\|\tilde{u}(t) - u^E\|^p \notin \mathcal{B}(L)) \equiv \lim_{t \uparrow \infty} P(\|\tilde{u}(t) - u^E\|^p > |L|) = 1
\] (2.115)

(6) The system is \( p \)-stable if for all \( p \geq 1 \) and some \( |L| > 0 \)
\[
E\left\{\left\|\tilde{u}(t) - u^E_i\right\|^p\right\} \leq |L|
\] (2.116)
or \( M\|\tilde{u}(t) - u^E_i\|^p \in \mathcal{B}(L) \). It is asymptotically \( p \)-stable if
\[
\lim_{t \uparrow \infty} M\left\{\left\|\tilde{u}(t) - u^E_i\right\|^p\right\} = 0
\] (2.117)

(7) The system is exponentially \( p \)-stable if \( \exists \) constants \( \{A, Q\} > 0 \) such that
\[
M\left\{\left\|\tilde{u}(t) - u^E_i\right\|^p\right\} \leq A\|u^E\| \exp(-Q|t - t_0|)
\] (2.118)
and exponentially \( p \)-unstable if
\[
M\left\{\left\|\tilde{u}(t) - u^E_i\right\|^p\right\} \leq A\|u^E\| \exp(+Q|t - t_0|)
\] (2.119)

Then \( \lim_{t \uparrow \infty} M\{\tilde{u}(t) - u^E\|^p \} \to 0 \) or \( \lim_{t \uparrow \infty} M\{\tilde{u}(t) - u^E\|^p \} \to \infty \)

**Definition 2.24.** Given \( \gamma \in (0, 1) \), and \( L > 0 \), the \( \gamma \)-basins of attraction (\( \gamma \)-BOA) are the sets
\[
\{ u^E \in \mathbb{R}^n : P(||u(t) - u^E\|| = 0) \geq \gamma \}
\] (2.120)
\[
\{ u^E \in \mathbb{R}^n : P(||u(t) - u^E\|| \leq |L|) \geq \gamma \}
\] (2.121)

Given a set of random variables, it is possible to establish expressions, bounds and estimates for these probabilistic stability criteria.

**Definition 2.25.** If set of random variables \( (\tilde{u}_i(t))_{i=1}^n = (\tilde{u}_1, \ldots, \tilde{u}_n(t)) \), representing random perturbations of an initially static or equilibrium set \( u^E_i \) are Gaussian, then for any \( L > 0 \) and for some \( C > 0 \).
\[
P(\tilde{S}(t) - M\tilde{S}(t) \geq L) \equiv P\left(\frac{1}{n} \sum_{i=1}^n \tilde{u}_i(t) - \frac{1}{n} \sum_{i=1}^n M\{\tilde{u}_i(t)\} \geq L\right)
\]
\[
\equiv P\left(\frac{1}{n} \sqrt{\frac{1}{n}} \|\tilde{u}_i(t)\|^2 - \frac{1}{n} \|M\{\tilde{u}_i(t)\}\|^2 \geq L\right)
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \frac{1}{C} \exp \left( - \frac{2n^2|L|^2}{C} \right)
\] (2.122)

The set is sub-Gaussian if
\[
P(\tilde{S}(t) - M\tilde{S}(t) \geq L) \equiv P\left(\frac{1}{n} \sum_{i=1}^n \tilde{u}_i(t) - \frac{1}{n} \sum_{i=1}^n M\{\tilde{u}_i(t)\} \geq L\right)
\]
\[
\equiv P\left(\frac{1}{n} \|\sqrt{\tilde{u}_i(t)}\|^2 - \frac{1}{n} \|M\{\tilde{u}_i(t)\}\|^2 \geq L\right)
\]
\[
\leq \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{2n^2|L|^2}{C} \right)
\] (2.123)
Lemma 2.26. If the set of random variables \( \tilde{u}_i(t) \) is Gaussian or sub-Gaussian then the stability of condition of (2.113) also holds so that

\[
P(\tilde{S}(t) - M\{\tilde{S}(t)\} = \infty) = P\left( \frac{1}{n} \sum_{i=1}^{n} \tilde{u}_i(t) - \frac{1}{n} \sum_{i=1}^{n} M\{\tilde{u}_i(t)\} = \infty \right)
\]

\[
= \exp\left( -\frac{2n^2|L|^2}{\sum_{i=1}^{n} |u_i^{E*} - u_i^{E}|^2} \right)
\]

so that there is zero probability that the perturbed system will blow up in any finite time or be asymptotically unstable as \( t \to \infty \).

In particular, if the set is sub-Gaussian then it is bounded and the Hoeffding inequality and the Chernoff bound inequality apply. This suggests that if a randomly perturbed set \( \tilde{u}_i(t) \) is sub-Gaussian then it is bounded and therefore the system is stable to the random perturbations and vice versa.

Proposition 2.27. Let \( u_i^{E} \) be a set of static equilibrium solutions of a nonlinear ODE of the form \( D_\nu u_i^{E} = 0 \). Let the randomly perturbed set of solutions be \( \tilde{u}_i(t) \). Let \( \tilde{u}_i(t) \) be 'attractors' or new stable equilibrium fixed points such that the perturbed system converges as \( \tilde{u}_i(t) \to u_i^{E*} \) for some finite \( t \gg 0 \) or as \( t \to \infty \). Then for all finite \( t > 0 \) the set is bounded in that

\[ u_i^{E} \leq \tilde{u}_i(t) \leq u_i^{E*} \]

1. The Hoeffding inequality applies and is then

\[
P(\tilde{S}(t) - M\{\tilde{S}(t)\} \geq L) = \exp(\beta L)M\{\exp(-\beta(\tilde{S}(t) - M\{\tilde{S}(t)\}))\}
\]

Then Lemma 2.21 holds and the randomly perturbed system is stable in probability, otherwise it is unstable

\[
P(\tilde{S}(t) - M\{\tilde{S}(t)\} = \infty) = \exp(\beta L)M\{\exp\left( -\beta \left( \tilde{u}_i(t) - u_i^{E} \right) \right) \}
\]

The Chernoff bound can also be expressed as

\[
P(\|\tilde{u}_i(t) - u_i^{E}\| \leq |L|) \leq \exp(\beta |L|)M\{\exp\left( -\beta \left( \|\tilde{u}_i(t) - u_i^{E}\| \right) \right) \}
\]

or asymptotically as

\[
\lim_{t \to \infty} P(\|\tilde{u}_i(t) - u_i^{E}\| \leq |L|) \leq \lim_{t \to \infty} \exp(\beta |L|)M\{\exp\left( -\beta \left( \|\tilde{u}_i(t) - u_i^{E}\| \right) \right) \}
\]

then

\[
\lim_{t \to \infty} P(\|\tilde{u}_i(t) - u_i^{E}\| \leq |L|) = 0
\]

if \( \exp(-\beta(\|\tilde{u}_i(t) - u_i^{E}\|)) \to 0 \) as \( t \to \infty \) and the randomly perturbed variables are not bounded.
These exponential inequalities are valid for linear combinations of bounded independent random variables, and in particular for the average. But one is often more interested in controlling the maximum or supremum of the set in terms of the maximal estimates.

Lemma 2.28. Let \( u_i^E \) be a set of \( n \) equilibrium solutions of a nonlinear ODE \( D_n u_i^E = 0 \) and let \( \hat{u}_i(t) \) be the set of \( n \) randomly perturbed solutions. Let \( \sup_{1 \leq i \leq n} \hat{u}_i(t) \) be the supremum or maximum of the set. If the set is bounded it is sub-Gaussian and vice-versa so that for some \( \theta > 0 \)

\[
\mathbb{P}(\sup_{1 \leq i \leq n} \hat{u}_i(t) \geq |L|) \leq \exp\left(-\frac{L^2}{2C^2}\right) \tag{2.131}
\]

Then \( \exists(C, B, D) > 0 \) such that the maximal inequalities hold and the system is stable so that for all \( t \in \mathbb{R}^+ \cup \mathbb{R}^+ \cup \mathbb{R}^+ \)

\[
\mathcal{M}\left\{ \sup_{1 \leq i \leq n} \hat{u}_i(t) \right\} \leq C \sqrt{2 \log(n)} \leq B < \infty \tag{2.132}
\]

\[
\lim_{t \to \infty} \mathcal{M}\left\{ \sup_{1 \leq i \leq n} \hat{u}_i(t) \right\} \leq C \sqrt{2 \log(n)} \leq B < \infty \tag{2.133}
\]

\[
\mathbb{P}(\sup_{1 \leq i \leq n} \hat{u}_i(t) \geq |L|) \leq n \exp\left(-\frac{L^2}{2C^2}\right) \leq D < \infty \tag{2.134}
\]

\[
\lim_{t \to \infty} \mathbb{P}(\sup_{1 \leq i \leq n} \hat{u}_i(t) \geq |L|) \leq n \exp\left(-\frac{L^2}{2C^2}\right) \leq D < \infty \tag{2.135}
\]

and

\[
\mathbb{P}(\sup_{1 \leq i \leq n} \hat{u}_i(t) = \infty) = 0 \tag{2.136}
\]

\[
\lim_{t \to \infty} \mathbb{P}(\sup_{1 \leq i \leq n} \hat{u}_i(t) = \infty) = 0 \tag{2.137}
\]

Proof. For any \( \xi > 0 \)

\[
\mathcal{M}\left\{ \sup_{1 \leq i \leq n} u_i(t) \right\} = \frac{1}{\xi} \mathcal{M}\left\{ \log\left(\exp(\xi \sup_{1 \leq i \leq n} \hat{u}_i(t))\right) \right\}
\]

\[
= \frac{1}{\xi} \log\mathcal{M}\left\{ \exp(\xi \sup_{1 \leq i \leq n} \hat{u}_i(t)) \right\}
\]

\[
= \frac{1}{\xi} \log\mathcal{M}\left\{ \sup_{1 \leq i \leq n} \exp(\xi \hat{u}_i(t)) \right\}
\]

\[
= \frac{1}{\xi} \log\left(\sum_{i=1}^{n} \mathcal{M}\left\{ \exp(\xi \hat{u}_i(t)) \right\}\right) \leq \frac{1}{\xi} \log\left(\sum_{i=1}^{n} \exp(\frac{1}{2}C^2\xi^2)\right)
\]

\[
= \frac{1}{\xi} \log\left(n \exp\left(\frac{1}{2}C^2\xi^2\right)\right)
\]

\[
= \frac{1}{\xi} \log(n) + \frac{1}{2}C^2\xi
\]

choosing \( \xi = \sqrt{2 \log(n)/C^2} \) then gives the maximal inequalities (2.132) or (2.133). Next

\[
\mathbb{P}\left(\sup_{1 \leq i \leq n} \hat{u}_i(t) \geq |L|\right) = \mathbb{P}\left(\bigcup_{i=1}^{n} u_i(t) \geq |L|\right)
\]

\[
\leq \sum_{i=1}^{n} \mathbb{P}(\hat{u}_i(t) \geq |L|)
\]

\[
\leq n \exp\left(-\frac{L^2}{2C^2}\right) \leq D < \infty \tag{2.139}
\]

so that (2.134) and (2.135) follow \[\square\]
Lemma 2.29. Using the basic Markov inequality \( P[|X| ≥ |L|] ≤ |L|^{-1} M \{ |X| \} \) for a random variable \( \hat{X} \) and any \( |L| > 0 \), the following estimate can be made for the probability that the stochastic norm \( \| \hat{u}(t) - u^E \| \) is outside a ball \( B(L) \) of any radius \( |L| \) at any time \( t > 0 \). Using the estimate (2.86)

\[
P(\| \hat{u}(t) - u^E \| ≥ |L|) ≤ |L|^{-1} M \left\{ \left\| \hat{u}(t) - u^E \right\| \right\}
\]

\[
= |L|^{-1} n^{1/2} |u^E| M \left\{ \exp \left( \zeta \int_0^t \hat{\mathcal{W}}(\tau) d\tau \right) \right\}
\]

\[
= |L|^{-1} n^{1/2} |u^E| I(t) ≤ 1 \tag{2.140}
\]

with \( \hat{\mathcal{W}}(t) = \hat{\mathcal{W}}(t) \) for \( i = 1 \ldots n \). If \( \lim_{t \to \infty} P(\| \hat{u}(t) - u^E \| ≥ |L|) = 0 \) for all finite \( R \) then the system is stable. There is no noise-induced blowup for all finite \( t > 0 \) if

\[
P(\| \hat{u}(t) - u^E \| = ∞) = \lim_{L \to \infty} u^E n^{1/2} |L|^{-1} M \left\{ \exp \left( \zeta \int_0^t \hat{\mathcal{W}}(\tau) d\tau \right) \right\} = 0 \tag{2.141}
\]

If \( \exists L > 0 \) such that \( 0 ≤ \| \hat{u}(t) - u^E \| ≤ L \) for all \( t > 0 \) then the system is stable to random perturbations. Given the stochastic norm \( \| \hat{u}(t) - u^E \| \), probabilistic stability criteria can also be established using a stronger Chernoff bound estimate.

Theorem 2.30. Let \( u^E \) be equilibrium solutions such that \( D_n u^E = 0 \), then for a non-white noise perturbation \( \hat{\mathcal{W}}(t) \), the randomly perturbed \( L_2 \) norms are given by \( \| \hat{u}(t) - u^E \| = u^E n^{1/2} \exp(\int_0^t \hat{\mathcal{W}}(\tau) d\tau) \). Now let \( B(L) \subset \mathbb{R}^n \) be an Euclidean ball of radius \( |L| \) and \( \ell \in \mathbb{Z} \). Then we can make the estimate

\[
\lim_{t \to \infty} P(\| \hat{u}(t) - \hat{u}^E \| \in B(L)) = \lim_{t \to \infty} P(\| \hat{u}(t) - \hat{u}^E \| \leq |L|)
\]

\[
≤ \lim_{t \to \infty} \exp(+β|L|) \exp \left( -β |u^E| n^{1/2} \right) M \left\{ \exp \left( \zeta \int_0^t \hat{\mathcal{W}}(\tau) d\tau \right) \right\} \left( -\frac{1}{\ell} \right) \tag{2.142}
\]

The asymptotic probabilistic stability criteria can then be stated as follows:

1. The system is asymptotically unstable to the random perturbations for \( t \to \infty \) if the stochastic norm can never be contained within a ball \( B(L) \) of any finite radius \( |L| \). The probability is zero such that

\[
\lim_{t \to \infty} P(\| \hat{u}(t) - \hat{u}^E \| \in B(L)) = \lim_{t \to \infty} P(\| \hat{u}(t) - \hat{u}^E \| \leq |L|)
\]

\[
≤ \lim_{t \to \infty} \exp(+β|L|) \exp \left( -β |u^E| n^{1/2} \right) M \left\{ \exp \left( \zeta \int_0^t \hat{\mathcal{W}}(\tau) d\tau \right) \right\} \left( -\frac{1}{\ell} \right) \tag{2.143}
\]

which is the case if the stochastic integral diverges

\[
\lim_{t \to \infty} I(t) = \lim_{t \to \infty} M \left\{ \exp \left( \zeta \int_0^t \hat{\mathcal{W}}(\tau) d\tau \right) \right\} = \infty \tag{2.144}
\]

2. The system is asymptotically stable to the random perturbations if the stochastic norm is always contained within a ball of any finite radius \( |L| \), with finite or unit probability.

\[
\lim_{t \to \infty} P(\| \hat{u}(t) - \hat{u}^E \| \in B(L)) = \lim_{t \to \infty} P(\| \hat{u}(t) - \hat{u}^E \| \leq |L|)
\]

\[
≤ \lim_{t \to \infty} \exp(+β|L|) \exp \left( -β |u^E| n^{1/2} \right) M \left\{ \exp \left( \zeta \int_0^t \hat{\mathcal{W}}(\tau) d\tau \right) \right\} \left( -\frac{1}{\ell} \right) \tag{2.145}
\]

which is the case if the stochastic integral converges such that

\[
\lim_{t \to \infty} I(t, \ell) = \lim_{t \to \infty} M \left\{ \exp \left( \zeta \int_0^t \hat{\mathcal{W}}(\tau) d\tau \right) \right\} = Q < \infty \tag{2.146}
\]
For a set of independent random variables \( X = \{X_1, \ldots, X_n\} \) with \( L_2 \) norm \( ||X|| \)

\[
P\left( \sum_{i=1}^{n} |X_i| \leq L \right) \leq \inf_{\beta > 0} \exp(\beta L) \prod_{i=1}^{n} \exp(-\beta X_i)
\]

and using the estimate (2.86) the Chernoff estimate (2.150) is

\[
\lim_{t \to \infty} P[||\hat{u}(t) - u^E|| \leq |L|] = \lim_{t \to \infty} P[||\hat{u}(t) - u^E|| \in B(L)]
\]

The stability criteria can then be determined if one can explicitly estimate the stochastic integral

\[
I(t) = \mathcal{M} \left\{ \exp \left( \zeta \int_{0}^{t} \tilde{W}(\tau) d\tau \right) \right\}
\]

\[
\lim_{t \to \infty} I(t) = \lim_{t \to \infty} \mathcal{M} \left\{ \exp \left( \zeta \int_{0}^{t} \tilde{W}(\tau) d\tau \right) \right\}
\]

This will be done in Section 6. A stability criterion can also be derived from a Hoeffding inequality which provides an upper bound on the probability that the sum of a set of bounded independent (sub-Gaussian) random variables deviates from its expected value by more than a specified amount.
Proposition 2.31. As before, let $(\tilde{u}_i(t))_{i=1}^n = (\tilde{u}_1(t) \ldots \tilde{u}_n(t))$ be the set of random variables due to random perturbations of the initially static equilibria $u_i^E$ such that $\tilde{u}_i(t) = u_i^E \exp(\zeta \int_0^t \mathcal{U}(s)ds)$. Then $u_i(t)$ solves an averaged equation of the form (-) such that $D_\lambda u_i(t) = \lambda$. Suppose the perturbed solutions converge to 'attractors' or new equilibrium points within a finite time such that $\hat{u}_i(t) \to u_i^{E\ast}$. Then $\hat{u}_i^2 \leq \hat{u}_i(t) \leq u_i^{E\ast}$ for all finite $t > 0$.

$$\hat{S}(t) = \frac{1}{n} \sum_{i=1}^n \tilde{u}_i(t) = \frac{1}{n} (\tilde{u}_1(t) + \ldots + \tilde{u}_n(t))$$

(2.154)

$$M\{\hat{S}(t)\} = \frac{1}{n} \sum_{i=1}^n M\{\tilde{u}_i(t)\}$$

(2.155)

Then

$$P(\hat{S}(t) - M\{\hat{S}(t)\}) = P\left(\frac{1}{n} \sum_{i=1}^n |\tilde{u}_i(t)| - \frac{1}{n} \sum_{i=1}^n M\{\tilde{u}_i(t)\} \geq |L|\right)$$

$$= P\left(\frac{1}{n} \sum_{i=1}^n u_i^n \exp\left(\zeta \int_0^t \mathcal{U}(s)ds\right) - \frac{1}{n} \sum_{i=1}^n u_i^n M\left\{\exp\left(\zeta \int_0^t \mathcal{U}(s)ds\right)\right\} \geq |L|\right)$$

$$= P\left(\frac{1}{n} \sum_{i=1}^n \sqrt{u_i^n} \exp\left(\frac{\zeta}{2} \int_0^t \mathcal{U}(s)ds\right) - \frac{1}{n} \sum_{i=1}^n \sqrt{u_i^n} M\left\{\exp\left(\zeta \int_0^t \mathcal{U}(s)ds\right)\right\} \geq |L|\right)$$

$$\leq \frac{\exp(-2n^2|L|^2)}{\sum_{i=1}^n |u_i^{E\ast} - u_i^n|^2} \leq \frac{\exp(-2n^2L^2)}{\|u_i^{E\ast} - u_i^n\|^2}$$

(2.156)

Hence if $\|u_i^{E\ast} - u_i^n\|^2 < \infty$ for all finite $t \in \mathbb{R}^+$ then there is zero probability of blowup or asymptotic instability for any finite $t > 0$, so that

$$P\left(\frac{1}{n} \sqrt{u_i^n} \exp\left(\frac{\zeta}{2} \int_0^t \mathcal{U}(s)ds\right) \|L\|_2^2 - \frac{1}{n} \sqrt{u_i^n} M\left\{\exp\left(\zeta \int_0^t \mathcal{U}(s)ds\right)\right\} \|L\|_2^2 = \infty\right) = 0$$

(2.157)

$$\lim_{t \to \infty} P\left(\frac{1}{n} \sqrt{u_i^n} \exp\left(\frac{\zeta}{2} \int_0^t \mathcal{U}(s)ds\right) \|L\|_2^2 - \frac{1}{n} \sqrt{u_i^n} M\left\{\exp\left(\zeta \int_0^t \mathcal{U}(s)ds\right)\right\} \|L\|_2^2 = \infty\right) = 0$$

(2.158)

If however, $u_i^{E\ast} \to \infty$

$$P(\hat{S}(t) - M\{\hat{S}(t)\} \geq |L|)$$

$$= P\left(\frac{1}{n} \sqrt{u_i^n} \exp\left(\frac{\zeta}{2} \int_0^t \mathcal{U}(s)ds\right) \|L\|_2^2 - \frac{1}{n} \sqrt{u_i^n} M\left\{\exp\left(\zeta \int_0^t \mathcal{U}(s)ds\right)\right\} \|L\|_2^2 \geq |L|\right)$$

$$= \lim_{t \to \infty} \frac{\exp(-2n^2L^2)}{\|u_i^{E\ast} - u_i^n\|^2} = 1$$

(2.159)

and there is then unit probability that the growth of the norm of perturbed solutions cannot be contained within any finite $L > 0$. Hence, the system is asymptotically unstable to the random perturbations.

Note that bounded random variables are always sub-Gaussian. Sub-Gaussianity is then a necessary criteria for stability or convergence of the randomly perturbed system, described by $\tilde{u}_i(t)$, to new attractors or equilibrium fixed points.

3. The Einstein vacuum equations on $\mathbb{T}^n \times \mathbb{R}^+$ as an n-dimensional autonomous system of nonlinear ODEs: static and dynamical solutions

Sets of multi-dimensional nonlinear autonomous ODEs with the structure of the form (2.2) and (2.3) arise within general relativity when one reduces the Einstein vacuum equations on a n-dimensional toroidal
A spacetime manifold by N coupled equations of N classical particles of mass \( m \) system evolves against a fixed background reference geometry, essentially (and non-gravitational, time and space are absolute and the notion of mechanical phase space is clear: the variation equation is \( g_{AB} \) and \( R_{AB} \) for example, exhibit uniform curvature blowup along a pair of spacelike hypersurfaces, (\( \Sigma \) Friedmann-Lemaître-Robertson-Walker (FLRW) model [48,49,50,51,52,53]. Closed-universe FLRW solutions of the Kasner-Bianchi.

If \( \Lambda = 0 \) then the vacuum equations are incomplete to the past and future. In the absence of matter \( T_{AB} \) and where \( \Phi(\mathbf{r}_i, t) \) be considered from initial data, although the problem even for \( N=3 \) can become chaotic and the future relativistic systems therefore do not appear to be dynamical systems in the usual sense in that they do not provide an obvious set of parameters ‘evolving in time’.

\[ \text{Ric}_{AB} - \frac{1}{2} g_{AB} R + \Lambda g_{AB} = T_{AB} \] (3.1)

and \( \nabla_A T^{AB} = 0 \) is the energy conservation condition, where \( \nabla_A \) is the covariant derivative and \( \Lambda \) is a fixed cosmological constant [48,49,50,51,52,53]. In this paper, the spacetime is cosmological with metric \( ds^2 = -dt^2 + \delta_{ij}(t)dX^i dX^j \) and with scale factors \( a_i(t) \). It is globally hyperbolic, that is, foliated with compact spacelike Cauchy hypersurfaces \( \Sigma_t \). In particular, the nonlinear ODEs that we will consider arise from reduction of the Einstein vacuum equations on \( M^{n+1} = T^n \times \mathbb{R}^+ \), where \( T^n \) is an n-torus and \( \mathbb{R}^+ = [0, \infty) \), and containing no matter so that \( T_{AB} = 0 \). The dominant cosmological model with matter and/or radiation is the Friedmann-Lemaître-Robertson-Walker (FLRW) model [48,49,50,51,52,53]. Closed-universe FLRW solutions for example, exhibit uniform curvature blowup along a pair of spacelike hypersurfaces, (\( \Sigma_{\text{Tang}}, \Sigma_{\text{Crunch}} \)) signifying the Big Bang and the Big Crunch. In particular, this is exemplified by the blowup of the invariant \( K = \text{Riem}_{ABCD} \text{Riem}^{ABCD} \) along the hypersurfaces (\( \Sigma_{\text{Tang}}, \Sigma_{\text{Crunch}} \)); so the spacetime is geodesically incomplete to the past and future. In the absence of matter \( T_{AB} = 0 \) the Einstein equations reduce to \( \text{Ric}_{AB} - \frac{1}{2} g_{AB} R + \Lambda g_{AB} = 0 \) or \( \text{Ric}_{AB} = g_{AB} \Lambda \), describing deSitter or anti-deSitter space if \( \Lambda > 0 \) or \( \Lambda < 0 \). If \( \Lambda = 0 \) then the vacuum equations are \( \text{Ric}_{AB} = 0 \) which still admit dynamical anisotropic cosmological solutions of the Kasner-Bianchi.

**Remark 3.1.** General relativity admits a well-posed Cauchy initial-value formulation [11,53,54]. The initial data is given by the set \( \mathcal{D} = \{ \Sigma_0, g_{ij}(0), k_{ij}(0) \} \), where \( g_{ij}(0) \) is the Riemannian 3-metric on \( \Sigma_0 \), with \( g_{00} = -1 \) and \( g_{0i} = 0 \). \( k_{ij}(0) \) is the covariant symmetric tensor, and the constraints on the initial data are \( \text{Ric}_{00} = 0 \) and \( R_{0i} = 0 \), equivalent to the Codazzi and Gauss constraint conditions. For the Cauchy evolution, the 1st variation equation is

\[ \partial_t g_{ij}(t) \equiv -2k_{ij}(t) \] (3.2)

A spacetime manifold \( M^{n+1} \) is then a development of the initial data \( \mathcal{D} \) and there is an imbedding \( \mathcal{I} : \Sigma \rightarrow M^{3+1} \)

The equivalence class of all maximal Cauchy Einstein developments of \( \mathcal{D} \) are related by diffeomorphisms.

In theories of ordinary linear and nonlinear dynamical systems at the Newtonian level, both gravitational and non-gravitational, time and space are absolute and the notion of mechanical phase space is clear: the system evolves against a fixed background reference geometry, essentially (\( \mathbb{R}^4, \eta_{\alpha\beta} \)). For example, a ‘cloud’ of N classical particles of mass m interacting gravitationally—for example, a globular star cluster— with coordinates \( r_1(t), ..., r_N(t) \) and velocities \( v_1(t), ..., v_N(t) \) is an N-body Newtonian dynamical system described by N coupled equations

\[ \frac{dv_i(t)}{dt} = -Gm \sum_{i \neq j} \frac{r_i(t) - r_j(t)}{|r_i(t) - r_j(t)|^3} \equiv -m \nabla U(r_1...r_n) \] (3.3)

\[ U(r_1...r_n(t)) = \sum_{i < j} \Phi(r_i(t) - r_j(t)) \] (3.4)

and where \( \Phi(r_i(t) - r_j(t)) = -G/|r_i(t) - r_j(t)| \) is the Newtonian potential between pairs of particles. The Hamiltonian is \( H = \sum_{i=1}^N \frac{1}{2} m v_i(t)^2 + m^2 U(r_1...r_n) \). The dynamical evolution of the system can be considered from initial data, although the problem even for \( N=3 \) can become chaotic and the future evolution cannot be predicted from the initial data. But in general relativistic space and time they assume a dynamical role with a space-time (\( M^{3+1}, g \)) that is a solution of the Einstein equations. General relativistic systems therefore do not appear to be dynamical systems in the usual sense in that they do not provide an obvious set of parameters ‘evolving in time’.
One way to deal with general relativity on $\mathbb{M}^{n+1}$ and on a higher-dimensional manifold $\mathbb{M}^{n+1}$ is to break the space-time covariance of the formulation and use an ADM split [49,50] exploiting the foliation of the manifold $\mathbb{M}^{n+1}$ with the space-like hypersurfaces $\Sigma$. The metric $g_{AB}$ on $\mathbb{M}^{n+1}$ induces an n-metric $g_{ij}$ on the hypersurface $\Sigma$ so that

$$ds^2 = -\phi^2 dt^2 + g_{ij}(dX^i + \phi^j) \otimes (dX^j + \phi^i dt)$$ (3.5)

where $\phi$ and $\phi^i$ are the lapse function and shift vectors. Setting $\phi^i = 0$ and $\phi = 1$ gives the typical cosmological metric form

$$ds^2 = -dt^2 + g_{ij}dX^i \otimes dX^j = -dt^2 + \delta_{ij}a^2(t)dX^i \otimes dX^j$$ (3.6)

Then $\mathbb{M}^{n+1} = \Sigma_t \times \mathbb{R}^+$. Using the ADM split, and the well-defined Cauchy formulation, the evolution of the n-metric $g_{ij}$ as $\partial_t g_{ij}(t)$ via the 1st variation equation and can then be interpreted as a 'nonlinear dynamical system', on somewhat equal terms as conventional dynamical systems which possess degrees of freedom evolving in time.

The Einstein vacuum equations $\text{Ric}_{AB} = 0$ are applied to a hypertoroidal cosmological-type metric (3.6) and the static and dynamical solutions are considered. The Einstein vacuum equations will then reduce to a system of nonlinear ODEs in terms of the first and second derivatives of the scale factors $\partial_\alpha(t)$ and $\partial_\beta a(t)$ which can be interpreted as a nonlinear dynamical system, structurally of the from (2.2) and (2.3).

**Definition 3.2.** An $(n+1)$-dimensional toroidal space-time has the following properties

1. $\mathbb{M}^{n+1}$ is the product $\mathbb{M}^{n+1} = \mathbb{T}^n \times \mathbb{R}^+$, where $\mathbb{T}^n$ is an isotropic n-torus.
2. The Einstein vacuum equations are $\text{Ric}_{AB} = 0$ and $(\mathbb{M}^{n+1}, g_{AB})$ is a solution.
3. Topologically, $\mathbb{T}^n$ is an isotropic or anisotropic n-torus, for which the constant time slices $\Sigma_t$ are n-dimensional tori with 'rolling radii' $a_i(t)$; that is, radii that depend only the time parameter $t \in \mathbb{R}^+$.
4. Topologically, $\mathbb{T}^n$ is a Cartesian product of $n$ circles so that $\mathbb{T}^n = S^1 \times S^2 \times \ldots \times S^n$.
5. Retaining summations, the metric is a solution of the Einstein vacuum equations of the form

$$ds^2 = \sum_{A=0}^{n+1} \sum_{B=0}^{n+1} g_{AB}dX^A \otimes dX^B = -dt^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}dX^i \otimes dX^j$$ (3.7)

with $g_{00} = -1$ and $g_{ij} = (2\pi a_i(t))^2$, with all other off-diagonal components vanishing.
6. Each $X^i = X^i + 2\pi a_i(t)$ takes values in a circle of radius $a_i(t)$, where $i = 1$ to $n$. The constant time slices or Cauchy spacelike surfaces $\Sigma_t$ are n-dimensional tori with 'rolling radii' $a_i(t)$.
7. The radii $a_i(t)$ can be parametrized by a set of $n$ scalar modulus functions (moduli) $(\psi_i(t))_{i=1}^{n}$ so that

$$\psi_i: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \quad \text{for} \quad i = 1...n, \quad \text{and} \quad a_i(t) = \exp(\psi_i(t))$$

for $i = 1...n$. The metric (3.7) then becomes

$$ds^2 = -dt^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}dX^i \otimes dX^j = -dt^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} (2\pi)^2 \delta_{ij} \exp(2\psi_i(t))dX^i \otimes dX^j$$ (3.8)

8. If $\psi_i(t) = \psi_i^E = \psi^E$, then $a_i(t) = a_i^E = \exp(\psi_i^E)$ and the dimensions are stable or constant as $t \rightarrow \infty$. This describes a 'static hypertoroidal universe'.

In general, a metric on an n-torus $\mathbb{T}^n$ has $\frac{1}{2}n(n + 1)$ moduli, namely $n$ radii and $\frac{1}{2}(n - 2)(n - 3)$ angles. However, one usually chooses $g_{ij}(t) = 0$ thus freezing the "rolling angles" for $i \neq j$ and considering only the 'rolling radii'[52,53].

**Definition 3.3.** The spatial volume $V_\mathbf{g}(t)$ of the toroidal n-metric (3.7) or (3.8) is defined as

$$V_\mathbf{g}(t) = \prod_{i=1}^{n} \exp(\psi_i(t)) \equiv \exp \left( \sum_{i=1}^{n} \psi_i(t) \right) \equiv \prod_{i=1}^{n} a_i(t)$$ (3.9)

and the $L_{(2,1)}$ norm of the diagonal n-metric is

$$\|\mathbf{g}(t)\|_{(2,1)} = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} |g_{ij}(t)|^2 \right)^{1/2} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |g_{ij}(t)|^2 \right)^{1/2}$$ (3.10)
The Frobenius norm can also be used such that
\[ \|g(t)\|_F = \left( \sum_{j=1}^{n} \sum_{i=1}^{n} |g_{ij}(t)|^2 \right)^{1/2} = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |g_{ij}(t)|^2 \right)^{1/2} \] (3.11)

Remark 3.4. Such metrics also arise in toroidal compactification of Kaluza-Klein and superstring theories [54,55,56,57,58,59,60], whereby a theory on a manifold \( \mathbb{R}^{m+n+1} \) where can be compactified as \( \mathbb{R}^{m+n+1} \to M^{m+1} \times T^{n} \) such that
\[ ds^2 = -dt^2 + \sum_{a,b} g_{ab} dX^a dX^b + \sum_{i,j} \delta_{ij} (2\pi a_i(t))^2 dX^i \otimes dX^j \]
\[ = -dt^2 + \sum_{a,b} g_{ab} dX^a dX^b + \sum_{i,j} \delta_{ij} (2\pi)^2 \exp(2\psi(t)) dX^i \otimes dX^j \] (3.12)

with \( a, b = 1...m \) and \( i, j = 1...n \). For example, \( n = 6 \) and \( m = 3 \) for a toroidal compactification of a superstring theory or \( n = 1 \) and \( m = 3 \) for a basic 5-dimensional Kaluza-Klein compactification on a circle. For M-theory one has \( n = 10 \)

Remark 3.5. The metric (3.7) or (3.8) also represents the higher-dimensional generalization of the Kasner solutions found in 4-dimensional Bianchi-Type I cosmological models [47,48]
\[ ds^2 = -dt^2 + \sum_{i=1}^{n} t^{2p_i} (dX_i)^2 \] (3.13)

with the Kasner constraints \( \sum_{i=1}^{n} p_i = \sum_{n=1}^{n} p_i = 1 \). For \( n=4 \), this is the Bianchi Type-I Universe
\[ ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 \] (3.14)

where \( p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \).

The space is anisotropic if at least two of the three \( p_i \) are different. The Kasner constraints only hold for a pure vacuum and are lost in the presence of a fluid matter source. Kasner-like solutions are also the building blocks of string cosmology [57], and for \( n+1 = 11 \) they represent vacuum cosmological solutions of the low-energy effective limit of M-theory or 11-dimensional supergravity [58].

Theorem 3.6. For empty ‘toroidal universes’ with no matter and with \( \Lambda = 0 \) The Einstein vacuum field equations are
\[ G_{AB} = \text{Ric}_{AB} - \frac{1}{2} g_{AB} R = 0 \] (3.15)

or \( \text{Ric}_{AB} = 0 \). Using the metric ansatz (3.8) with \( g_{00} + 1 \) the Einstein equations can be reduced to a system of ordinary nonlinear differential equations.
\[ H \psi_i(t) = \sum_{i=1}^{n} \partial_t \psi_i(t) + \frac{1}{2} \sum_{i=1}^{n} \partial_t \psi_i(t) \partial_t \psi_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_t \psi_i(t) \partial_t \psi_j(t) = 0 \] (3.16)

Since \( a_i(t) = \exp(\psi(t)) \), an equivalent set of differential equations is
\[ D_n a_i(t) = \sum_{i=1}^{n} \partial_t a_i(t) - \frac{1}{2} \sum_{i=1}^{n} \partial_t a_i(t) \partial_t a_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_t a_i(t) \partial_t a_i(t) = 0 \] (3.17)

where \( H_n \) and \( D_n \) are now the nonlinear differential operators on \( \mathbb{R}^n \) such that
\[ H_n(... \equiv \sum_{i=1}^{n} \partial_t (...) + \frac{1}{2} \sum_{i=1}^{n} \partial_t (...) \partial_t (...) + \frac{1}{2} \sum_{i=1}^{n} \partial_t (...) \partial_t (...) \] (3.18)
\[ D_n(... \equiv \sum_{i=1}^{n} \partial_t (...) - \frac{1}{2} \sum_{i=1}^{n} \partial_t (...) \partial_t (...) + \frac{1}{2} \sum_{i=1}^{n} \partial_t (...) \partial_t (...) = 0 \] (3.19)
Proof. Following [56], we introduce an orthonormal basis of one-forms \( e^0 = dt \) and \( e^i = 2\pi a_i(t)dX^i \). The Cartan structure equations

\[
R^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j \tag{3.20}
\]

are then solved for the spin connection 1-forms so that \( \omega^i_j = (a_i(t)/a_j(t))e^i \) and \( \omega^i_i = 0 \), and the structure equations for curvature give the curvature 2-forms

\[
R_{ij} = d\omega^i_j + \omega^i_k \wedge \omega^k_j \tag{3.21}
\]

The only nonvanishing components of the curvature tensor are

\[
\text{Riem}^0_{0\alpha} = \sum_{i=1}^{n} \frac{a_i(t)}{a_i(t)} \tag{3.22}
\]

\[
\text{Riem}^i_{j\alpha} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial a_i(t)\partial a_j(t)}{a_i(t)a_j(t)} \tag{3.23}
\]

The Ricci tensor components and curvature scalar are then

\[
\text{Ric}_{00} = -\sum_{i=1}^{n} \frac{\partial a_i(t)}{a_i(t)}; \text{Ric}_{ij} = \sum_{i=1}^{n} \frac{\partial a_i(t)}{a_i(t)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{a_i(t)\partial a_j(t)}{a_i(t)a_j(t)} \tag{3.24}
\]

\[
R = 2\sum_{i=1}^{n} \frac{\partial a_i(t)}{a_i(t)} - \sum_{i=1}^{n} \frac{\partial a_i(t)\partial a_i(t)}{a_i(t)a_i(t)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial a_i(t)\partial a_j(t)}{a_i(t)a_j(t)} \tag{3.25}
\]

In terms of the moduli \( \psi_i(t) \) where \( a_i(t) = \exp(\psi_i(t)) \), the Ricci curvature scalar for the toroidal metric (3.8) is

\[
R = -2g^{00} \sum_{i=1}^{n} \frac{\partial \psi_i(t)}{\partial x^i} + \sum_{i=1}^{n} \frac{\partial \psi_i(t)\partial \psi_i(t)}{\partial x^i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \psi_i(t)\partial \psi_j(t)}{\partial x^i} \tag{3.26}
\]

with \( R_{00} = 0 \) and \( R_{ij} = 0 \) for \( i \neq j \). We set \( g_{00} = -1 \) as a 'gauge choice' then (3.26) becomes

\[
R = 2\sum_{i=1}^{n} \frac{\partial \psi_i(t)}{\partial x^i} + \sum_{i=1}^{n} \frac{\partial \psi_i(t)\partial \psi_i(t)}{\partial x^i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \psi_i(t)\partial \psi_j(t)}{\partial x^i} \tag{3.27}
\]

The vacuum Einstein equations are \( \text{Ric}_{AB} = g_{AB}R = 0 \) or simply \( R = 0 \), giving a set of nonlinear differential equations in terms of the radial moduli functions

\[
H_i \psi_i(t) \equiv \sum_{i=1}^{n} \frac{\partial \psi_i(t)}{\partial x^i} + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial \psi_i(t)\partial \psi_i(t)}{\partial x^i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \psi_i(t)\partial \psi_j(t)}{\partial x^i} = 0 \tag{3.28}
\]

Since \( a_i(t) = \exp(\psi(t)) \), an equivalent set of differential equations in terms of the toroidal radii is

\[
D_i a_i(t) \equiv \sum_{i=1}^{n} \frac{\partial a_i(t)}{a_i(t)} - \frac{1}{2} \sum_{i=1}^{n} \frac{\partial a_i(t)\partial a_i(t)}{a_i(t)a_i(t)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial a_i(t)\partial a_j(t)}{a_i(t)a_j(t)} = 0 \tag{3.29}
\]

\[\square\]

Remark 3.7. In keeping with a dynamical systems interpretation, the action for pure Einstein gravity in \( n \) dimensions is

\[
S = \int_{M^{n+1}} d^{n+1}x \sqrt{-\det g_{n+1}} R \tag{3.30}
\]

and \( \delta S = 0 \) gives the vacuum field equations. Using (3.27) this can be written as

\[
S = \int d\tau \int d^n x \left( \prod_{k=1}^{n} \exp(\psi_k(t)) \right) \times \left( \sum_{i=1}^{n} 2\partial \psi_i(t) + \sum_{i=1}^{n} \partial \psi_i(t)\partial \psi_i(t) + \sum_{i=1}^{n} \partial \psi_i(t)\partial \psi_i(t) \right)
\]
Lemma 3.9. The Einstein vacuum equations $\mathbf{Ric}_{AB} = 0$ on $\mathbb{T}^n \times \mathbb{R}^+$ in the form $\mathbf{H}_n \psi_i(t) = 0$ and $\mathbf{D}_n a_i(t) = 0$, and for some initial data $\mathcal{D} = \{ t = 0, \Sigma_0, \psi_i(0) = \psi_i^E, a_i(0) = a_i^E \}$ have the power-law solutions for $t > 0$.

$$\psi_i(t) = \psi_i^E + \ln |t|^{p_i} \equiv \psi_i^E + p_i \ln |t|$$  \hspace{1cm} (3.42)
\[ a_i(t) = a^E_i|t|^{p_i} \quad (3.43) \]

provided the Kasner constraints are satisfied

\[ \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} p_i \quad (3.44) \]

Proof. The derivatives of (3.42) are \( \partial_t \psi_i(t) = p_i t^{-2} \) and \( \partial_p \psi_i(t) = -p_j t^{-2} \) so that

\[ H_n \psi_i(t) = -\sum_{i=1}^{n} p_i t^{-2} + \frac{1}{2} \sum_{i=1}^{n} p_i p_i t^{-2} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j t^{-2} = 0 \quad (3.45) \]

which holds if \( \sum_{i=1}^{n} p_i^2 = \sum_{i=1}^{n} p_i \). Given the set of modulus functions \( \psi_i(t) \) then the radii are \( a_i(t) = \exp(\psi_i(t)) \) so that

\[ a_i(t) = \exp(\psi^E_i)|t|^{p_i} \equiv a^E_i|t|^{p_i} \quad (3.46) \]

The derivatives are \( \partial_t a_i(t) = a^E_i p_i |t|^{p_i-1} \) and \( \partial_p a_i(t) = a^E_i p_i (p_i - 1) |t|^{p_i-2} \) and (3.29) becomes

\[
D_n a_i(t) = \sum_{i=1}^{n} a^E_i |p_i| (p_i - 1)|t|^{p_i-2} - \frac{1}{2} \sum_{i=1}^{n} a^E_i a^E_j p_i (p_i - 1) |t|^{p_i-1} |t|^{p_j-1} \frac{1}{a_i|t|^{p_i} a_j|t|^{p_j}} = 0 \quad (3.47)
\]

which is

\[ \sum_{i=1}^{n} \frac{p_i (p_i - 1)}{t^2} - \frac{1}{2} \sum_{i=1}^{n} p_i p_i + \frac{1}{2} \sum_{i=1}^{n} p_i p_j = 0 \quad (3.48) \]

Canceling the \( |t|^{p_i-2} \) term gives

\[ - \sum_{i=1}^{n} p_i + \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i=1}^{n} p_i p_j = 0 \quad (3.49) \]

If \( m_{ij} = p_i p_j \) is a diagonal matrix with \( p_i = p_j \) for \( i, j = 1 \ldots n \) then the Kasner constraints follow so that

\[ - \sum_{i=1}^{n} p_i + \frac{1}{2} \sum_{i=1}^{n} m_{ii} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \]

\[ = - \sum_{i=1}^{n} p_i + \frac{1}{2} \sum_{i=1}^{n} m_{ii} + \frac{1}{2} \sum_{i=1}^{n} m_{ii} = - \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} m_{ii} \quad (3.50) \]

which gives \( \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} p_i^2 \).

When \( p_i \neq 0 \) the vacuum Einstein equations on the n-torus \( \mathbb{T}^n \) are a solvable system provided the \( p_i \) obey the Kasner constraints as in the Bianchi model cosmologies. For the Bianchi-I model, for example

\[ p_1^2 + p_2^2 + p_3^2 = p_1 + p_2 + p_3 = 1 \quad (3.51) \]

Choosing the ordering \( p_1 < p_2 < p_3 \), these can also be parametrized by the Khalatnikov-Lifshitz parameter \( u \) which was introduced for cosmological applications studying oscillatory and chaotic behavior within Kasner epochs [61,62,63,64,65].

\[ p_1 = -\frac{u}{1 + u + u^2}; p_2 = \frac{1 + u}{1 + u + u^2}; \]

\[ p_3 = \frac{u(1 + u)}{1 + u + u^2} \quad (3.52) \]

As \( u \) varies over \( u \geq 1 \) then \( p_1, p_2, p_3 \) can take on all permissible values such that \( \frac{1}{3} \leq p_1 \leq 0 \) with \( 0 \leq p_2 \leq \frac{2}{3} \) and \( \frac{2}{3} \leq p_3 \leq 1 \). In general, the \( p_i \) can be chosen in any arbitrary way provided that the Kasner constraints are always satisfied. The cases where two of the \( p_i \) are equal are given by the Bianchi-I triplets \( \mathfrak{B} = (0, 0, 1) \) and \( \mathfrak{B} = (-1/3, 2/3, 2/3) \). If the \( p_i \) are considered coordinates on \( \mathbb{T}^n \) then these constraints can be regarded as spatially flat and homogenous solutions of the Einstein equations that reside on the intersection \( I = \mathbb{H}^n \bigcap \mathbb{H}^n \) of a hypersphere \( \mathbb{H}^n \) with a fixed hyper plane \( \mathbb{H}^n \). The Ricci curvature scalar \( R \) then vanishes at all times along \( I \) so that \( R = 0 \) or \( R_{AB} = 0 \), as expected for vacuum solutions of the Einstein equations.
For $p_i > 0$ for some $i$ then $a_i(t) \to \infty$ as $t \to \infty$ so the solution grows or 'rolls out'. For $p_i < 0$ then $a_i(t) \to 0$ and the solution collapses or becomes singular in a finite time. These are the rolling radii solutions and it is possible for regions of this universe to expand while other regions collapse. For example, for the Bianchi-I triplet $\mathfrak{B} = (-1/3, 2/3, 2/3)$, the rolling radii are

$$a_1(t) = a_1^E |t|^{-1/3} \equiv a_1(0)|t|^{-1/3} \quad (3.53)$$

$$a_2(t) = a_2^E |t|^{2/3} \equiv a_2(0)|t|^{2/3} \quad (3.54)$$

$$a_3(t) = a_3^E |t|^{2/3} \equiv a_3(0)|t|^{2/3} \quad (3.55)$$

**Definition 3.10.** The Kretchmann scalar invariant, and the expansion and shear associated with the Cauchy hypersurfaces $\Sigma_t = \text{const.}$ can be defined as follows [58]:

1. The Kretchmann scalar for this cosmology is $K = \text{Riem}_{ABCD}\text{Riem}^{ABCD} \sim t^{-4}$ so that the dynamic Kasner solutions have Big-Bang singularities along the past boundary $\Sigma_t = 0$. In terms of the moduli

$$K(t) = 4 \sum_{i=1}^{n} \partial_t \psi_i(t) + 4 \sum_{i=1}^{n} \partial_t \psi_i(t) \partial_t \psi_i + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (\partial_t \psi_i(t) \partial_t \psi_j(t))^2$$

$$= \sum_{i=1}^{n} -p_i^2 t^{-2} + \sum_{i=1}^{n} p_i^2 t^{-2} + \sum_{i} \sum_{j} p_i p_j t^{-4} \sim t^{-4} \quad (3.56)$$

so that $K(0) = \text{Riem}_{ABCD}\text{Riem}^{ABCD} = \infty$.

2. The expansion $\chi(t)$ associated with the Cauchy surface $\Sigma_t$ is

$$\chi(t) = \sum_{i=1}^{n} \partial_t \psi_i(t) $$

$$= \sum_{i=1}^{n} \sum_{j} \left| \partial_t \psi_i(t) - \partial_t \psi_j(t) \right|^2 \quad (3.57)$$

3. The shear is

$$\mathcal{S}^2(t) = \sum_{i} \sum_{j} \left| \partial_t \psi_i(t) - \partial_t \psi_j(t) \right|^2$$

$$= \sum_{i} \sum_{j} \left| \partial_t \psi_i(t) - \partial_t \psi_j(t) - 2 \partial_\psi_i(t) \partial_t \psi_j(t) + \partial_t \psi_j(t) \partial_t \psi_j(t) \right| \quad (3.58)$$

The following defines an 'eternal' static Kasner torus universe for all $t > 0$. This is essentially the (trivial) equilibrium or static solution defined by a set of fixed points $a_i(0) = a_i^E$ for some initial time $T > 0$ since $K = \infty$.

**Definition 3.11.** The toroidal spacetime $\mathbb{M}^{n+1} = \mathbb{T}^n \times \mathbb{R}^+$ is an eternal static 'Kasner universe' if the following hold:

1. The initial data $\mathcal{D} = \{t = 0, \Sigma_0 = 0, g_{ij}(0), k_{ij}(0), \psi_i(0) = \psi_i^E, a_i(t) = a_i^E \}$ with $g_{oo} = -1, g_{io} = 0$ with constraints $\mathcal{R}_{oo} = 0$ and $\mathcal{R}_{io} = 0$, and there is no development of the data for any $t > 0$.

2. The Einstein vacuum equations $\text{Ric}_{AB} = 0$ exist on $\mathbb{M}^{n+1}$ in the form $H_n \psi_i(t) = 0$ for all $t \in \mathbb{R}^+$, and $\mathbf{D}_n \psi_i^E = 0$.

3. For all $t \in \mathbb{R}^+$, the first variation equation vanishes so that $\partial_t g_{ij}(t) = 2k_{ij}(t) = 0$.

4. For all $t > T$, the toroidal radii are $a_i(t) = a_i^E = \exp(\psi_i^E)$ corresponding to the set or static moduli $\psi_i^E$ so that

$$H_n \psi_i^E = 0 \quad (3.59)$$

$$\mathbf{D}_n a_i^E = 0 \quad (3.60)$$

5. For all $t \geq T$ the initially (static) $n$-metric is

$$ds^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}dX^i \otimes dX^j = \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \exp(2\psi_i^E) dX^i \otimes dX^j$$
(6) The spatial volume of the static hyper-toroidal Kasner universe is

\[ V_\mathbf{g}(t) = \prod_{i=1}^{n} \exp(\psi_i^E) = \exp\left( \sum_{i=1}^{n} \psi_i^E \right) = \prod_{i=1}^{n} a_i^E \]

If \( \psi_i^E = \psi^E \) and \( a_i^E = a^E \) for \( i = 1, \ldots, n \) then the static spatial volume is

\[ V_\mathbf{g}(t) = \prod_{i=1}^{n} \exp(\psi_i^E) = n \exp(\psi^E) \]

(7) The \( L_{(2,1)} \) norm of the static \( n \)-metric is

\[ \|g^E\|_{(2,1)} = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} |g_{ij}^E|^2 \right)^{1/2} \equiv \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |g_{ii}^E|^2 \right)^{1/2} \] (3.61)

Note that the static radii \( a_i^E \) can be arbitrarily small. One could have \( a_i^E \sim L_p \), where \( L_p \) is the Planck length. In subsequent sections, rigorous stability criteria are developed for both deterministic and stochastic perturbations of this initially static 'micro-universe'.

Lemma 3.12. Suppose now, the cosmological constant \( \Lambda \) is not zero, then the classical Einstein-Hilbert action is

\[ S = \frac{1}{\kappa} \int_{M^{n+1}} d^{n+1}x (-g_{n+1})^{1/2}(R - 2\Lambda) \] (3.62)

where \( \kappa = 1/16\pi G_N \) and \( G_N \) is the Newton constant. On \( M^{n+1} = T^n \times \mathbb{R}^+ \), this gives the Einstein equations as the sets of \( n \)-dimensional inhomogeneous nonlinear ordinary differential equations for \( \psi_i(t) \) and \( a_i(t) \) as

\[ \frac{1}{2} R = H_n \psi_i(t) = \frac{1}{2} \sum_{i=1}^{n} \partial_t \psi_i(t) + \frac{1}{2} \sum_{i=1}^{n} \partial^t \psi_i(t) \partial^t \psi_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_t \psi_i(t) \partial_t \psi_i(t) = \frac{\Lambda(1+n)}{(1-n)} \equiv \lambda \] (3.63)

\[ \frac{1}{2} R = D_n a_i(t) = \frac{1}{2} \sum_{i=1}^{n} \partial_t a_i(t) a_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_t a_i(t) \partial_t a_j(t) = \frac{\Lambda(1+n)}{(1-n)} \equiv \lambda \] (3.64)

These are essentially of the form (2.4) and (2.5).

Proof. The variation of the action \( \delta S = 0 \) gives the Einstein field equations coupled to the cosmological constant

\[ \text{Ric}_{AB} - \frac{1}{2} g_{AB} R + g_{AB} \Lambda = 0 \] (3.65)

Multiplying by \( g^{AB} \) and using \( g^{AB} g_{AB} = (n+1) \) then

\[ g^{AB} \text{Ric}_{AB} - \frac{1}{2} g^{AB} g_{AB} R + g^{AB} g_{AB} \Lambda \]

\[ = R - \frac{1}{2}(n+1)R - \Lambda(n+1) = 0 \] (3.66)

so that

\[ \frac{1}{2} R = \Lambda \frac{1+n}{1-n} \] (3.67)

Using (3.28) and (3.29), then gives the equivalent sets of inhomogeneous \( n \)-dimensional ordinary nonlinear differential equations

\[ \frac{1}{2} R = H_n \psi_i(t) = \Lambda \frac{1+n}{1-n} \equiv \frac{1}{2} \lambda \] (3.68)

\[ \frac{1}{2} R = D_n a_i(t) = \Lambda \frac{1+n}{1-n} \equiv \frac{1}{2} \lambda \] (3.69)

For these equations, there are no equilibrium or static solutions of the form \( \psi_i(t) = \psi^E_i \) and \( a_i(t) = a_i^E = \exp(\psi_i^E) \) and so any solutions are necessarily dynamical. A cosmological constant term is then expected to drive an expansion or collapse of the toroidal Kasner universe.
Lemma 3.13. Let \( t \in \mathbb{R}^+ \) with initial data \( \mathcal{D} = (\psi_i(0) \equiv \psi_i^E, a_i(0) \equiv a_i^E) \) and the conditions of Definition 3.10. Then the solutions of the Einstein equations with a cosmological constant term are

\[
\begin{align*}
H_n \psi_i(t) &= \lambda \\
D_n a_i(t) &= \lambda
\end{align*}
\]  
are of the general form

\[
\begin{align*}
\psi_i^{(\pm)}(t) &= \psi_i^E + q_i(n, \lambda) t = \psi_i(0) + q_i(n, \lambda) t \\
a_i^{(\pm)}(t) &= a_i^E \exp(q_i(n, \lambda)t) = a_i^E \exp(q_i(n, \lambda)t)
\end{align*}
\]  
where \( q_i(n, \lambda) \) are some constant functions of \( n \) and \( \lambda \), where for each \( i = 1...n, q_i \in \mathbb{R}^+ \) provided that

\[
\frac{1}{2} \sum_{i=1}^{n} q_i(n, \lambda) q_j(n, \lambda) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_i(n, \lambda) q_j(n, \lambda) = \lambda
\]  

If \( q_i = q_j = q = \text{const.} \) for all \( i = 1...n \) then \( q = (\lambda/n)^{1/2} \). The full solutions are then

\[
\begin{align*}
\psi_i^{(+)}(t) &= \psi_i^E + (\lambda/n)^{1/2} t \\
a_i^{(+)}(t) &= a_i^E \exp((\lambda/n)^{1/2} t) = a_i(0) \exp((\lambda/n)^{1/2} t)
\end{align*}
\]  

for an expanding universe driven by a cosmological constant so that \( H_n \psi_i^{(+)}(t) = \lambda \) and \( D_n a_i^{(+)}(t) = \lambda \)

\[
\begin{align*}
\psi_i^{(-)}(t) &= \psi_i^E - (\lambda/n)^{1/2} t \\
a_i^{(-)}(t) &= a_i^E \exp(-(\lambda/n)^{1/2} t) = a_i(0) \exp(-(\lambda/n)^{1/2} t)
\end{align*}
\]  

for a collapsing universe, such that \( H_n \psi_i^{(-)}(t) = \lambda \) and \( D_n a_i^{(-)}(t) = \lambda \).

Proof. The derivatives of \( \psi_i(t) \) are simply \( \partial_t \psi_i(t) = q_i \) and \( \partial_{tt} \psi_i(t) = 0 \) so that (3.70) becomes

\[
H_n \psi_i(t) = \frac{1}{2} \sum_{i=1}^{n} q_i(n, \lambda) q_i(n, \lambda) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_i(n, \lambda) q_j(n, \lambda) = \lambda
\]  

If \( q_i = q_j = q \) for \( i, j = 1...n \) then one can choose \( M_{ij} = \delta_{ij} q^2 \) and \( M_{ii} = \delta_{ii} q^2 \), each having \( n \) nonzero terms giving

\[
H_n \psi_i(t) = \frac{1}{2} nq^2 + \frac{1}{2} nq^2 = nq^2 = \lambda
\]  

so that \( q = (\lambda/n)^{1/2} \). The same result also follows from the Einstein equations \( D_n a_i(t) = \lambda \) so that (3.71) becomes

\[
D_n a_i(t) = \frac{\sum_{i=1}^{n} |a_i(0)|^2 |q_i(n, \lambda)| |X_i(t)|^2}{|a_i(0)|^2 |X_i(t)|^2} - \frac{1}{2} \sum_{i=1}^{n} \frac{|a_i(0)|^2 |q_i(n, \lambda)|^2 |X_i(t)|^2}{|a_i(0)|^2 |X_i(t)|^2}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{|a_i(0) a_j(0) q_i(n, \lambda) q_j(n, \lambda) X_i(t) X_j(t)}{|a_i(0) a_j(0) |X_i(t)|X_j(t)|} = \lambda
\]  

where \( X_i(t) = \exp(q_i(n, \lambda)t) \). Cancelling terms

\[
D_n a_i(t) = \sum_{i=1}^{n} |q_i(n, \lambda)|^2 - \frac{1}{2} \sum_{i=1}^{n} |q_i(n, \lambda)|^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_i(n, \lambda) q_j(n, \lambda)
\]

\[
= \sum_{i=1}^{n} m_{ij}(n, \lambda) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}(n, \lambda) = \lambda
\]  

Again, if \( q_i = q_j = q \) for \( i = 1...n \) then the matrices \( m_{ij} \) and \( m_{ii} \) each have \( n \) terms and one can choose \( m_{ij} = \delta_{ij} q^2 \) and \( m_{ii} = \delta_{ii} q^2 \). Equation (3.82) reduces to \( n|q(n, \lambda)|^2 = \lambda \) so that \( q(n, \lambda) = \pm \left( \frac{\lambda}{n} \right)^{1/2} \) as before. \( \square \)
4. 'Short-pulse' and continuous perturbations of static solutions: stability criteria

In this section, stability criteria of the static Kasner-type hypertorus universe or 'vacuum bubble' are considered in relation to 'short-pulse' Gaussian deterministic perturbations and also continuous 'step' perturbations of the static or constant moduli fields $\psi_i^E$. In Section 5, stability criteria for random perturbations or 'noise' are developed and compared. But in each case, the Einstein equations will now be interpreted as a nonlinear multi-dimensional dynamical system of ordinary differential equations with initial data, which are then subject to such perturbations; indeed, from a purely mathematical perspective, the form of these nonlinear ODEs could be considered independently of any general relativistic considerations. On $M^{n+1} = \mathbb{T}^n \times \mathbb{R}^+$, the Einstein vacuum equations reduce to the general ODES are $H_i\psi_i^E = 0$ and $D_\alpha a_i(t) = 0$. If the static solutions are subject to small perturbation $f_i(t)$ for some $f_i : \mathbb{R}^+ \to \mathbb{R}^+$, where $|f_i(t)| \ll 1$ about these static equilibrium points then

$$\bar{\psi}_i(t) = \psi_E + f_i(t)$$  \hspace{1cm} (4.1)

In general one can proceed to study stability via a nonlinear stability analysis. Rather than linearize the equations, the effect of deterministic 'short-pulse' perturbations on the fully nonlinear equations will be considered. The nonlinearity should be retained since it is a crucial feature of general relativity and gravitational systems.

Consider first, a set of delta-function 'impulse' perturbations of the form

$$\bar{\psi}_i(t) = \psi_E^i + \zeta \int_0^t \delta_i(\tau)d\tau$$  \hspace{1cm} (4.2)

or $\bar{\psi}_i(t) = \psi_E^i + \zeta \delta_i(t)$, with $\zeta > 0$. This is convergent since $\lim_{t \to \infty} \| \int_0^t \delta_i(\tau)d\tau \| = 1$ but the derivative $\partial_i \delta_i(t)$ does not exist. However, the delta functions can be 'smeared out' into very narrow sharply peaked functions such as Gaussians or power-law distributions represented as $U_i(t, \vartheta_i)$ with finite widths $\vartheta_i$ such that $\lim_{\|\vartheta\| \to 0} U_i(t, \vartheta_i) = \delta_i(t)$. The following perturbations can be considered

$$\bar{\psi}_i(t) = \psi_E^i + \zeta \int_0^t U_i(\tau, \vartheta_i)d\tau = \psi_E^i + \zeta H(\tau, \vartheta_i)$$  \hspace{1cm} (4.3)

or

$$\bar{\psi}_i(t) = \psi_E^i + \zeta U_i(t, \vartheta_i)$$  \hspace{1cm} (4.4)

where $\zeta > 0$. In this paper, we will use the integral form (4.3).

4.1. General stability criteria. In this subsection it will be shown that initially static toroidal universes or 'vacuum bubbles' are stable to these types of narrow and sharply peaked deterministic perturbations.

**Proposition 4.1.** Let $M^{n+1} = \mathbb{T}^n \times \mathbb{R}^+$ be a globally hyperbolic spacetime where $\mathbb{T}^n$ is the n-torus. The following hold:

1. The initial data $D = \{ t = 0, \Sigma_0 = 0, g_{ij}(0), k_{ij}(0), \psi_i(0) = \psi_i^E, a_i(0) = a_i^E \}$ with $g_{oo} = -1, g_{oo} = 0$ with constraints $R_{oo} = 0$ and $R_{icoo} = 0$, and conditions (2), (3) and (4) of Definition 3.8.

2. A set of smooth functions $(U_i(t, \vartheta_i))_{i=1}^n$ span $\mathbb{R}^m$ where $\vartheta_i$ are the widths such that $U_i(t, \vartheta_i) = 0$ for $t \gg |\vartheta|$ and $\lim_{t \to \infty} \| U(t, \vartheta) \| = 0$. (E.g., a set of Gaussians with widths $\{ \vartheta_i \}$). These can be represented as a vector $U_i(t, \vartheta_i) = U_1(t, \vartheta_1, ... , U_n(t, \vartheta_n)$ with norms $\| U(t, \vartheta) \| \equiv \| U_i(\vartheta_i) \|$

3. The derivative $\partial_i U_i(t, \vartheta_i)$ exists and $\partial_i U_i(t, \vartheta_i) \to 0$ for $t \gg |\vartheta_i|$.

The initially static torus is isotropic if $\psi_1^E = \psi_2^E = ... = \psi_n^E$. General deterministic perturbations of the moduli functions $\psi_i^E = \psi_i^E$ can be of the form

$$\bar{\psi}_i(t) = \psi_i^E + \int_0^t U_i(\tau, \vartheta_i)d\tau = \psi_i^E + \mathcal{H}_i(t, \vartheta_i)$$  \hspace{1cm} (4.5)

If at least any two of the set $U_i(t, \vartheta_i)$ are different, then the perturbations are anisotropic so that $U_i(t, \vartheta_i) \neq U_j(t, \vartheta_j)$ for any $i \neq j$. For example, if the $U_i(t, \vartheta_i)$ are a set of Gaussians such that $U_i(t, \vartheta_i) = A_i \exp(-t^2/2\vartheta_i)$ then $U_i(t, \vartheta_i) \neq U_j(t, \vartheta_j)$ if $A_i \neq A_j$ and/or $\vartheta_i \neq \vartheta_j$. If the perturbations are isotropic then $U_i(t, \vartheta_i) = U(t, \vartheta)$ for $i = 1...n$. 

Proposition 4.2. Given the perturbations of the static solutions
\[ \psi_1(t) = \psi_2^E + \int_0^t U_1(\tau, \vartheta_1)d\tau \equiv \psi_1^E + \mathcal{H}_1(t, \vartheta_1) \]
(4.6)
\[ \psi_2(t) = \psi_1^E + \int_0^t U_2(\tau, \vartheta_2)d\tau \equiv \psi_2^E + \mathcal{H}_2(t, \vartheta_2) \]
(4.7)
\[ \psi_3(t) = \psi_1^E + \int_0^t U_3(\tau, \vartheta_3)d\tau \equiv \psi_3^E + \mathcal{H}_3(t, \vartheta_3) \]
(4.8)
The initially static radii \( a_i^E = a_E \) of the n-torus are then anisotropically perturbed as
\[ a_i(t) = a_i^E \exp(\int_0^t U_i(\tau, \vartheta_i)d\tau) = a_i^E \exp(\mathcal{H}_i(t, \vartheta_i)) = a_i^E X_i(t) \]
(4.9)
And for \( \mathbb{T}^3 \), the anisotropically perturbed radii are
\[ a_1(t) = a_1^E \exp(\int_0^t U_1(\tau, \vartheta_1)d\tau) = a_1^E \exp(\mathcal{H}_1(t, \vartheta_1)) = a_1^E X_1(t) \]
(4.10)
\[ a_2(t) = a_2^E \exp(\int_0^t U_2(\tau, \vartheta_2)d\tau) = a_2^E \exp(\mathcal{H}_2(t, \vartheta_2)) = a_2^E X_2(t) \]
(4.11)
\[ a_3(t) = a_3^E \exp(\int_0^t U_3(\tau, \vartheta_3)d\tau) = a_3^E \exp(\mathcal{H}_3(t, \vartheta_3)) = a_3^E X_3(t) \]
(4.12)
For isotropic perturbations with \( U_i(t, \vartheta_i) = U(t, \vartheta) \) for \( i = 1...n \) then
\[ \bar{\psi}_i(t) = \psi_i^E + \int_0^t U(\tau, \vartheta)d\tau \equiv \psi_i^E + \mathcal{H}(t, \vartheta) \]
(4.13)
so the isotropically perturbed modulus functions are \( \bar{\psi}_i(t) = \psi_i^E + \int_0^t U_1(\tau, \vartheta_1)d\tau \equiv \psi_i^E + \mathcal{H}(t, \vartheta) \) and so on.

**Proposition 4.2.** Given the perturbations of the static solutions
\[ \bar{\psi}_1(t) = \psi_i^E + \int_0^t U_i(\tau, \vartheta_i)d\tau \]
(4.14)
\[ \bar{a}_i(t) = a_i^E \exp(\int_0^t U_i(\tau, \vartheta_i)d\tau) \]
(4.15)
with \( ||\vartheta|| < 1 \) and the convergence
\[ \gamma_i = \lim_{t \to \infty} \int_0^t U_i(\tau, \vartheta_i)d\tau = \lim_{t \to \infty} \int_0^t U_i(\tau, \vartheta_i)d\tau < \infty \]
(4.16)
\[ \|\gamma\| = \lim_{t \to \infty} \left\| \int_0^t U(\tau, \vartheta)d\tau \right\| = \lim_{t \to \infty} \left\| \int_0^t U(\tau, \vartheta)d\tau \right\| < \infty \]
(4.17)
Then for \( t \gg ||\vartheta|| \)
\[ \bar{\psi}_1(t) \to \psi_1^E + \gamma_1 \equiv \psi_1^{E*} \]
(4.18)
\[ \bar{a}_1(t) \to a_1^E \exp(\gamma_1) \equiv a_1^{E*} \]
(4.19)
where the points \( \psi_1^{E*} \) and \( a_1^{E*} \) are 'attractors'. For the perturbed radii of the 3-torus \( \mathbb{T}^3 \) for example
\[ a_1(t) \to a_1^E \exp(\gamma_1) \equiv a_1^{E*} \]
\[ a_2(t) \to a_2^E \exp(\gamma_2) \equiv a_2^{E*} \]
\[ a_3(t) \to a_3^E \exp(\gamma_3) \equiv a_3^{E*} \]
Given the deterministic perturbations \( \bar{\psi}(t) = \psi_1^E + \int_0^t U_1(\tau; \vartheta_1)d\tau \) and \( \bar{a}(t) = a_1^E \exp(\int_0^t U_1(\tau)d\tau) \), for initial data \( \psi_i(0) = \psi_i^E \) the \( L_2 \) norms are estimated as
\[ \|\psi(t) - \psi_i^E\| = \left( \sum_{i=1}^n |\psi_i(t) - \psi_i^E|^2 \right)^{1/2} \]
Proposition 4.3.

If the integrals $|\mathcal{H}(t, \vartheta)|^m = \left| \int_0^t \mathcal{U}(\tau, \vartheta) d\tau \right|^m$ converge for all $m \in \mathbb{Z}$, then the perturbation series converges.

For very sharply peaked functions with $\|\vartheta\| \ll 1$ the perturbed radii should converge very quickly to the attractor points. The existence of attractors is also linked with the Lyapunov stability of the system.

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Definition 4.4. Given the equilibrium points $a^E_i$, let $\|a\|$ be contained within an Euclidean ball $B(L) \subset \mathbb{R}^n$ of radius $|L|$, then the system is asymptotically stable if $\|a(t) - a^E\| \in B(L)$ for all $t > 0$ or

$$\lim_{t \to \infty} \|a(t) - a^E\| = 0$$

and $a_i(t) \to a^E_i$ for $i = 1,...,n$.

The static system is unstable to the perturbations if

$$\lim_{t \to \infty} \|a(t) - a^E\| = \infty$$

(1)

The static system is perturbatively stable if

$$\lim_{t \to \infty} \|a(t) - a^E\| = 0$$

(2)

The static system is not stable to the perturbations if

$$\lim_{t \to \infty} \|a(t) - a^E\| = \infty$$

(3)

For stability, the attractors will satisfy the perturbed differential equations in order to be new equilibrium stable fixed points of the system so that $H_n \psi^* = 0$ and $D_n a^* = 0$.

Proposition 4.5. Let $\mathbb{M}^{n+1} = \mathbb{T}^n \times \mathbb{R}^+$ be a globally hyperbolic space-time where $\mathbb{T}^n$ is the n-torus. The following hold: the initial data set $\mathcal{D} = \{t = 0, \Sigma_0 = 0, g_{ij}(0), k_{ij}(0), \psi_i(0) = \psi_i^E, a_i(0) = R^E \}$ with $g_{oo} = -1, g_{io} = 0$ with constraints $\text{Ric}_{oo} = 0$ and $\text{Ric}_{io} = 0$. Then given the modulus perturbations $\psi_i(t) = \psi_i^E + \int_0^t U_i(\tau, \vartheta_i) d\tau$ or $a_i(t) = a_i^E \exp\left(\int_0^t U_i(\tau, \vartheta_i) d\tau\right)$, the perturbed Einstein equations are

$$H_n \psi_i(t) = S(t, \vartheta_i)$$

(4.27)

$$D_n a_i(t) = S(t, \vartheta_i)$$

(4.28)

where

$$S(t, \vartheta_i) = \sum_{i=1}^n \partial_i U_i(t, \vartheta) + \frac{1}{2} \sum_{i=1}^n U_i(t, \vartheta_i) U_i(t, \vartheta_j) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n U_i(t, \vartheta_i) U_i(t, \vartheta_j)$$

$$= \sum_{i=1}^n \partial_i U_i(t, \vartheta) + \frac{1}{2} \sum_{i=1}^n U_i(t, \vartheta_i) + \frac{1}{2} \sum_{i=1}^n \left|U_i(t, \vartheta_i)\right|^2 + \frac{1}{2} \sum_{j=1}^n \left|U_j(t, \vartheta_j)\right|^2$$

$$= \left\|\partial_i U_i(t, \vartheta)\right\|^2_{L_2} + \frac{1}{2} \left\|U_i(t, \vartheta_i)\right\|^2_{L_2} + \frac{1}{2} \left\|U_j(t, \vartheta_j)\right\|^2_{L_2}$$

(4.29)

If $U_i(t, \vartheta_i) = U(t, \vartheta)$ then the perturbed Einstein equations are

$$H_n \psi_i(t) = n \partial_i U(t, \vartheta) + \frac{1}{2} n U(t, \vartheta) = S(t, \vartheta)$$

(4.30)

$$D_n a_i(t) = n \partial_i U(t, \vartheta) + \frac{1}{2} n U(t, \vartheta) = S(t, \vartheta)$$

(4.31)

and $\lim_{t \to \infty} D_n a_i(t) = \lim_{t \to \infty} S(t, \vartheta) = 0$ and $\lim_{t \to \infty} D_n a_i(t) = \lim_{t \to \infty} S(t, \vartheta) = 0$. Also $[H_n \psi_i(t)]_{t>\vartheta} = D_n a_i(t)_{t>\vartheta} = 0$ for very sharply peaked `short-pulse’ perturbations. This is equivalent to

$$\sum_{i=1}^n \partial_i a^E_i \frac{a^E_i}{a^*_i} = \frac{1}{2} \sum_{i=1}^n \partial_i a^E_i \frac{a^E_i}{a^*_i} + \frac{1}{2} \sum_{i \neq j} \partial_i a^E_i \partial_j a^E_j = 0$$

(4.32)

for the attractors $a^E_i$.

$$a^E_i = a^E_i \lim_{t \to \infty} \left(\int_0^t U_i(t, \vartheta_i) d\tau\right) \equiv a^E_i \exp(A_i) = a^E_i \exp(H_i(t, \vartheta_i))$$

(4.33)
Proof. The perturbed Einstein vacuum equations are

\[ H_n \psi_i(t) = H_n \psi_i^E + \sum_{i=1}^n \partial_i H(t, \vartheta_i) + \sum_{i=1}^n \partial_t H(t, \vartheta_i) \partial_t H(t, \vartheta_i) \]

(4.34)

Since \( \partial_i H(t, \vartheta_i) = U_i(t, \vartheta_i) \) and \( \partial_t H(t, \vartheta_i) = \partial_t U_i(t, \vartheta_i) \),

\[ H_n \psi_i(t) = H_n \psi_i^E + \sum_{i=1}^n \partial_i U_i(t, \vartheta_i) \]

(4.35)

Similarly,

\[ D_n a_i(t) = \sum_{i=1}^n \frac{\partial_t a_i(t)}{a_i(t)} - \frac{1}{2} \sum_{i=1}^n \frac{\partial_t a_i(t) \partial_t a_i(t)}{a_i(t) a_j(t)} + \frac{1}{2} \sum_{i=1}^n \frac{\partial_t a_i(t) \partial_t a_i(t)}{a_i(t) a_j(t)} \]

(4.36)

should give the same result.

\[ D_n a_i(t) = D_n a_i^E + \sum_{i=1}^n \partial_t U_i(t, \vartheta_i) + \sum_{i=1}^n \sum_{j=1}^n U_i(t, \vartheta_i) U_j(t, \vartheta_j) \]

(4.37)

Cancelling terms gives

\[ D_n a_i(t) = D_n a_i^E + \sum_{i=1}^n \partial_t U_i(t, \vartheta_i) + \sum_{i=1}^n \sum_{j=1}^n U_i(t, \vartheta_i) U_j(t, \vartheta_j) \]

(4.38)

or

\[ D_n a_i(t) = \sum_{i=1}^n \partial_t U_i(t, \vartheta_i) + \frac{1}{2} \left\| U_i(t, \vartheta_i) \right\|^2 + \frac{1}{2} \left\| U_j(t, \vartheta_j) \right\|_{L^2} + \frac{1}{2} \left\| U_i(t, \vartheta_i) \right\|_{L^2} \]

(4.39)

Setting \( U_i = U_j = U \) then gives (4.30) and (4.31). \( \square \)

Lemma 4.6. Given the initial data \( \mathcal{D} \), the evolution of the corresponding perturbed n-metric \( g_{ij}(t) \) is

\[ g_{ij}(t) = 2\delta_{ij} \pi \exp(2\psi_i(t)) \exp \left( \int_0^t U_i(\tau, \vartheta_i) d\tau \right) \equiv g_{ij}(0) \exp \left( \int_0^t U_i(\tau, \vartheta_i) d\tau \right) \]

(4.40)

For a 3-torus \( T^3 \), for example, the perturbed 3-metric components are

\[ g_{11}(t) = g_{11}(0) \exp \left( \int_0^t U_1(\tau, \vartheta_1) d\tau \right) \equiv g_{11}(0) \exp(\mathcal{H}_1(t, \vartheta_1)) \]

(4.41)

\[ g_{22}(t) = g_{22}(0) \exp \left( \int_0^t U_2(\tau, \vartheta_2) d\tau \right) \equiv g_{22}(0) \exp(\mathcal{H}_2(t, \vartheta_2)) \]

(4.42)

\[ g_{33}(t) = g_{33}(0) \exp \left( \int_0^t U_3(\tau, \vartheta_3) d\tau \right) \equiv g_{33}(0) \exp(\mathcal{H}_3(t, \vartheta_3)) \]

(4.43)
Lemma 4.7. The evolution of the norms and volume of the perturbed n-metric can be defined:

1. The norm of the perturbed n-metric is

\[
\|\mathbf{g}(t)\| = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} |g_{ij}(t)|^2 \right)^{1/2} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |g_{ij}(t)|^2 \right)^{1/2}
\]

then

\[
\lim_{t \to \infty} \|\mathbf{g}(t)\| = \lim_{t \to \infty} \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |g_{ij}(t \exp(2\psi_i^E) \exp(2\mathcal{H}_i(t, \vartheta_i)|^2 \right)^{1/2}
\]

which converges if the integral converges.

2. Given the initially static spatial volume of the hyper-toroidal geometry \( \mathbb{T}^n \)

\[
|V_{g}^E| = \prod_{i=1}^{n} \exp(\psi_i^E) = \exp \left( \sum_{i=1}^{n} \psi_i^E \right) = \prod_{i=1}^{n} a_i^E
\]

the perturbed volume is

\[
\overline{V_{g}(t)} = \prod_{i=1}^{n} \exp \left( \psi_i^E + \int_{0}^{t} \mathcal{U}_i(\tau, \vartheta_i) d\tau \right)
\]

\[
= \exp \left( \sum_{i=1}^{n} \psi_i^E \right) \exp \left( \sum_{i=1}^{n} \int_{0}^{t} \mathcal{U}_i(\tau, \vartheta_i) d\tau \right)
\]

\[
= |V_{g}^E| \exp \left( \sum_{i=1}^{n} \int_{0}^{t} \mathcal{U}_i(\tau, \vartheta_i) d\tau \right)
\]

\[
= |V_{g}^E| \exp \left( n \int_{0}^{t} \mathcal{U}(\tau, \vartheta) d\tau \right)
\]

(4.47)

if \( \psi_1^E = \psi^E \) and \( a_1^E = a^E \) so that instability occurs if \( \lim_{t \to \infty} \overline{V_{g}(t)} = \infty \) and stability occurs for

\[
\lim_{t \to \infty} \overline{V_{g}(t)} = \lim_{t \to \infty} |V_{g}^E| \exp \left( n \int_{0}^{t} \mathcal{U}(\tau, \vartheta) d\tau \right) = V_{g}^{E*} < \infty
\]

(4.48)
4.2. Short-pulse Gaussian profile perturbations. The previous propositions for stability criteria are now demonstrated for very sharp Gaussian 'short-pulse' perturbations of the modulus functions. Various functions can satisfy the conditions of Proposition 4.1 but the Gaussian is the most convenient. It is shown that a Kasner-type static universe or 'toroidal vacuum bubble' is stable to this form of perturbation.

Proposition 4.8. Setting 

$$U_i(t, \vartheta_i) \equiv G_i(t, \vartheta) = A_i \exp(-t^2/2\vartheta_i^2)$$  \hspace{1cm} (4.49)

with amplitudes $A_i$ and widths $\vartheta_i$ with $||\vartheta|| \ll 1$, the perturbed modulus functions are

$$\psi_i(t) = \psi_i^E + \int_0^t G_i(\tau, \vartheta_i)d\tau = A_i \int_0^t \exp(-\tau^2/2\vartheta_i^2)d\tau < \infty$$  \hspace{1cm} (4.50)

Then the integral converges if

$$Y_i = \lim_{t \to \infty} \int_0^t G_i(\tau, \vartheta_i)d\tau = \lim_{t \to \infty} \int_0^t G_i(\tau, \vartheta_i)d\tau < \infty$$  \hspace{1cm} (4.51)

or

$$||Y|| = \lim_{t \to \infty} \left| \int_0^t G_i(\tau, \vartheta)d\tau \right| = \lim_{t \to \infty} \left| \int_0^t G_i(\tau, \vartheta)d\tau \right| < \infty$$  \hspace{1cm} (4.52)

The incomplete Gaussian integral is the error function 'erf' so that

$$\psi_i(t) = \psi_i^E + \int_0^t G_i(\tau, \vartheta_i)d\tau \equiv \psi_i^E + A_i(\pi/2)^{1/2}\vartheta_i^{-1} \text{erf}(t/\sqrt{2}\vartheta_i)$$  \hspace{1cm} (4.53)

The estimate is then

$$\left| \int_0^t G_i(\tau, \vartheta)d\tau \right| = \left( \sum_{i=1}^n \left| \int_0^t G_i(\tau, \vartheta_i)d\tau \right|^2 \right)^{1/2}$$

$$= \left( \sum_{i=1}^n A_i \int_0^t \exp(-\tau^2/2\vartheta_i^2)d\tau \right)^{1/2}$$

$$= \left( \sum_{i=1}^n |A_i|^2 \int_0^t \exp(-\tau^2/2\vartheta_i^2)d\tau \right)^{1/2}$$

$$= \left( \sum_{i=1}^n |A_i|^2 \vartheta_i^{-1} \text{erf}(t/\sqrt{2}\vartheta_i) \right)^{1/2}$$  \hspace{1cm} (4.54)

Since $\text{erf}(t/\sqrt{2}\vartheta) \to 1$ as $t \to \infty$ or for $t \gg ||\vartheta||$ if $||\vartheta|| \ll 1$ then

$$||Y|| = \lim_{t \to \infty} \left| \int_0^t G_i(\tau, \vartheta)d\tau \right| = \lim_{t \to \infty} \left| \int_0^t G_i(\tau, \vartheta)d\tau \right|$$

$$= \lim_{t \to \infty} \left( \sum_{i=1}^n |A_i|^2 \vartheta_i^{-1} \text{erf}(t/\sqrt{2}\vartheta_i) \right)^{1/2} = \left( \sum_{i=1}^n |A_i|^2 \vartheta_i^{-1} \right)^{1/2}$$  \hspace{1cm} (4.55)

If $A_i = A$, $\vartheta_i = \vartheta$ for $i = 1...n$ then

$$||Y|| = \left( \sum_{i=1}^n |A_i|^2 \vartheta_i^{-1} \right)^{1/2} = \left( \sum_{i=1}^n |A_i|^2 \vartheta_i^{-1} \right)^{1/2} = (\pi/2)^{1/4} (nA^2\vartheta^{-1})^{1/2}$$  \hspace{1cm} (4.56)

The perturbed toroidal radii are

$$a_i(t) = \exp(\psi_i(t)) = \exp(\psi_i^E) \exp \left( \int_0^t G_i(\tau, \vartheta_i)d\tau \right)$$

$$\equiv a_i^E \exp(A(\pi/2)^{1/2}\vartheta_i^{-1} \text{erf}(t/\sqrt{2}\vartheta_i))$$  \hspace{1cm} (4.57)

so that the attractors for $t \to \infty$ or $t \gg ||\vartheta||$ are

$$\overline{a_i(t)} \to a_i^E \exp(A(\pi/2)^{1/2}\vartheta_i^{-1}) = a_i^E \exp(Y_i) \equiv a_i^{E*}$$  \hspace{1cm} (4.58)

and so the perturbed radii converge to new stable values or attractors $a_i^*$. 

Lemma 4.9. The $L_2$-norms are
\[ \| \bar{\psi}(t) - \psi^E \| = \left\| \int_0^t G(t, \vartheta) d\tau \right\| = \left( \frac{\pi}{2} \right)^{1/2} \left( \sum_{i=1}^n A_i^2 \vartheta_i^{-1} \text{erf}(t / \sqrt{2} \vartheta_i) \right) \] (4.59)
which follows from (4.21). Then
\[
\lim_{t \to \infty} \| \bar{\psi}(t) - \psi^E \| = \lim_{t \to \infty} \left( \frac{\pi}{2} \right)^{1/2} \left( \sum_{i=1}^n A_i^2 \vartheta_i^{-1} \text{erf}(t / \sqrt{2} \vartheta_i) \right) \] (4.60)
If $A_i = A$ and $\vartheta_i = \vartheta$,
\[
\lim_{t \to \infty} \| \bar{\psi}(t) - \psi^E \| = \left( \frac{\pi}{2} \right)^{1/2} \left( \sum_{i=1}^n (A_i^2 \vartheta_i^{-1}) \right)^{1/2} = \left( \frac{\pi}{2} \right)^{1/4} (nA^2 \vartheta^{-1})^{1/2} \] (4.61)
The $L_2$-norm for the perturbed radii is
\[
\| a(t) - a^E \| = \| a(t) \| - \| a^E \| = \left\| a^E \exp \left( \int_0^t G(t, \vartheta) d\tau \right) \right\| - \| a^E \| \\
= \left( \sum_{i=1}^n |a_i^E| \exp \left( A_i \int_0^t \exp(-\tau^2 / \sqrt{2} \vartheta_i^2) d\tau \right) \right)^{1/2} - \left( \sum_{i=1}^n |a_i^E|^2 \right)^{1/2} \] (4.62)
Then asymptotically
\[
\lim_{t \to \infty} \| a(t) - a^E \| \leq \left( \sum_{i=1}^n |a_i|^2 \exp \left( 2A_i (\pi / 2)^{1/4} \vartheta_i^{-1} \right) \right)^{1/2} - \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2} \] (4.63)
If $A_i = A$, $\vartheta_i = \vartheta$, $a_i^E = a$, then
\[
\lim_{t \to \infty} \| a(t) - a^E \| \leq n^{1/2} a^E \exp \left( A (\pi / 2)^{1/4} \vartheta^{-1} \right) - n^{1/2} a^E \] (4.64)
Lemma 4.10. Given the initially static spatial volume of the hyper-toroidal geometry $\mathbb{T}^n$
\[ V_g^E = \prod_{i=1}^n \exp(\psi_i^E) = \exp \left( \sum_{i=1}^n \psi_i^E \right) = \prod_{i=1}^n a_i^E. \] (4.65)
The perturbed spatial volume is
\[
\bar{V}_g(t) = \prod_{i=1}^n \exp \left( \psi_i^E + \int_0^t G_i(\tau, \vartheta_i) d\tau \right) \\
= \exp \left( \sum_{i=1}^n \psi_i^E \right) \exp \left( \sum_{i=1}^n \int_0^t G_i(\tau, \vartheta) d\tau \right) = |V_g^E| \exp \left( \sum_{i=1}^n A_i \int_0^t \exp(-\tau^2 / 2 \vartheta_i^2) d\tau \right) \] (4.66)
evolves to a new stable but larger volume $\psi$ with $V$. The following theorem for short-pulse Gaussian perturbations of the nonlinear ODE system can now be established.

Theorem 4.11. Given the following:

1. The initial data $D = [t = 0, \Sigma_o = 0, g_{ij}(0), k_{ij}(0), \psi_i(0) = \psi_i^E, a_i(0) = a_i^E]$ with $g_{oo} = -1, g_{io} = 0$ with constraints $\text{Ric}_{oo} = 0$ and $\text{Ric}_{io} = 0$, and there is no development of the data for $t > 0$ in the absence of perturbations.

2. The Einstein vacuum equations $\text{Ric}_{AB} = 0$ exist on $\mathbb{M}^{n+1}$ in the form $H_n \psi_i(t) = 0$ for all $t \in \mathbb{R}^+$, and $D_n \psi_i^E = 0$ and the toroidal radii are $a_i(t) = a_i^E = \exp(\psi_i^E)$ corresponding to the set or static moduli $\psi_i^E$.

3. Short-pulse Gaussian functions $(G_i(t, \vartheta))_{i=1}^n$ spanning $\mathbb{R}^m$ where $\vartheta_i$ are the widths such that $G_i(t, \vartheta) = 0$ for $t \gg \|\vartheta\|$ and $\lim_{t \to \infty} \|G_i(t, \vartheta)\| = 0$. (E.g., a set of Gaussians with widths $\vartheta_i$). The derivatives $\partial_t G_i(t, \vartheta_i)$ exist and $\partial_t G_i(t, \vartheta_i) \to 0$ for $t \gg \|\vartheta\|$.

4. The perturbed moduli are

$$\overline{\psi}_i(t) = \psi_i^E + \int_0^t G_i(t, \vartheta_i) d\tau$$

The perturbed Einstein equations are

$$H_n \overline{\psi}_i(t) = S(t, \vartheta_i)$$

where

$$S(n, t, \vartheta_i) = \sum_{i=1}^n \partial_t G_i(t, \vartheta_i) + \sum_{i=1}^n \sum_{j=1}^n G_i(t, \vartheta_i) G_j(t, \vartheta_j) + \sum_{i=1}^n G_i(t, \vartheta_i) \vartheta_i$$

$$= -\frac{1}{\vartheta_i \sqrt{2}} \mathcal{H}_i(t/\vartheta_i \sqrt{2}) A_i(t) + \sum_{i=1}^n \sum_{j=1}^n A_i A_j E_i(t) E_j(t) + \sum_{i=1}^n A_i A_i(t) E_i(t)$$

where $E_i(t) \equiv \exp(-t^2/2\vartheta_i^2)$ and $\mathcal{H}_i$ is the first-order Hermite polynomial. Then as $t \uparrow \infty$ or for $t \gg \|\vartheta\|$,

$$\lim_{t \uparrow \infty} H_n \overline{\psi}_i(t) = \lim_{t \uparrow \infty} S(n, t, \vartheta_i) = 0$$

$$\lim_{t \uparrow \infty} D_n \overline{a}_i(t) = \lim_{t \uparrow \infty} S(n, t, \vartheta_i) = 0$$

or $[H_n \overline{\psi}_i(t)]_{t > \|\vartheta\|} = S(t, \vartheta_i)_{t > \|\vartheta\|}$ and $[D_n \overline{a}_i(t)]_{t > \|\vartheta\|} = J(t, \vartheta_i)_{t > \|\vartheta\|}$. 

The norm of the perturbed metric converges or relaxed back to a new value or 'attractor'

$$\lim_{t \uparrow \infty} \|g(t)\|_{(2,1)} = \lim_{t \uparrow \infty} \left( \sum_{i=1}^n \left[ g_{ii}^E + 2\delta_{ii} A_i(\pi/2)^{1/2} \text{erf}(t/2\vartheta_i) \right] \right)^{1/2} = \|g\|$$

$\psi^E_i = \psi_i^E, a_i^E = a_i^E$ and $A_i = A, \vartheta_i = \vartheta$. Asymptotic stability then occurs since as the volume of the space evolves to a new stable but larger volume $\psi$ with $V$.
(5) If \( A_i = A_j = A \) and \( \partial_i = \partial \) for \( i, j = 1 \ldots n \) then

\[
\begin{align*}
H_n \psi(t) &= -\frac{n}{\sqrt{2\vartheta}} \mathcal{H}_i(t/\sqrt{2\vartheta}) A E(t) + n A^2 E(t) = S(n, t, \vartheta) \\
D_n a_i(t) &= -\frac{n}{\sqrt{2\vartheta}} \mathcal{H}_i(t/\sqrt{2\vartheta}) A E(t) + n A^2 E(t) = S(n, t, \vartheta)
\end{align*}
\]  

(4.75) 

(4.76)

Proof. For Gaussian perturbations

\[
\lim_{t \to \infty} \| \psi(t) - \psi^E \| \equiv \lim_{t \to \infty} \| \text{erf}(t/2\vartheta) \| < \infty
\]

since \( \lim_{t \to \infty} \text{erf}(t) \to 1 \). If \( G_i(t, \vartheta_i) = G_i(t, \vartheta) \) for all \( i = 1 \ldots n \). Similarly,

\[
\lim_{t \to \infty} \| \mathbf{a}(t) - \mathbf{a}^E \| \leq \lim_{t \to \infty} \| a_i \| \| \exp(2\text{erf}(t/2\vartheta)) \| - 1\| < \infty
\]

From (3.25)

\[
\begin{align*}
H_n \psi(t) &= H_n \psi^E + \sum_{i=1}^{n} \partial_i H_i(t) \\
&+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i H_i(t) \partial_j H_j(t) + \frac{1}{2} \sum_{i=1}^{n} \partial_i H_i(t) \partial_j H_j(t) \\
&= \sum_{i=1}^{n} \partial_i G_i(t, \vartheta_i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} G_i(t, \vartheta_i) G_j(t, \vartheta_j) + \frac{1}{2} \sum_{i=1}^{n} G_i(t, \vartheta_i) G_i(t, \vartheta_i)
\end{align*}
\]

(4.78)

so for the set of Gaussian functions (4.49)

\[
\begin{align*}
H_n \psi(t) &= \sum_{i=1}^{n} \partial_i H_i(t, \vartheta_i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j E_i(t) E_j(t) + \frac{1}{2} \sum_{i=1}^{n} A_i A_i E_i(t) E_i(t)
\end{align*}
\]

(4.79)

which is

\[
\begin{align*}
H_n \psi(t) &= -\sum_{i=1}^{n} \frac{1}{\vartheta \sqrt{2}} \mathcal{H}_i(t/\vartheta \sqrt{2}) A_i E_i(t) \\
&+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j E_i(t) E_j(t) + \frac{1}{2} \sum_{i=1}^{n} A_i A_i E_i(t) E_i(t)
\end{align*}
\]

(4.80)

where the derivative \( \partial_i G_i(t, \vartheta_i) \) is given by the Hermite polynomials \( \mathcal{H}_m(x) \) with \( \mathcal{H}_1(x) = 2x \) such that

\[
\partial_i G_i(t, \vartheta) = -\mathcal{H}_1(t/\sqrt{2\vartheta}) \exp(-t^2/2\vartheta^2) \equiv \mathcal{H}(t/\sqrt{2\vartheta}) E_i(t)
\]

(4.81)

Then \( H_n \psi(t) \to 0 \) very rapidly for \( t > |\vartheta| \). Indeed, the nonlinearity accelerates the rate of the convergence to zero. The perturbed equation \( D_n a_i(t) \) gives the same result. The perturbed radii are

\[
\tilde{a}_i(t) = a_i^E \exp \left( A_i \int_0^t \exp(\tau^2/2\vartheta^2) d\tau \right) = a_i^E \exp \left( A_i \int_0^t E_i(\tau) d\tau \right) = a_i^E X_i(t)
\]

(4.82)

so that

\[
\begin{align*}
D_n a_i(t) &= -\sum_{i=1}^{n} a_i^E A_i \mathcal{H}_i(t/\sqrt{2\vartheta}) E_i(t) X_i(t) + \sum_{i=1}^{n} a_i^E A_i A_i E_i(t) E_i(t) X_i(t)
\end{align*}
\]

(4.83)

Cancelling terms then gives (4.71). Equations (4.75) and (4.76) then follow from setting \( A_i = A \) and \( \vartheta_i = \vartheta \) for \( i = 1 \ldots n \).

\[ \Box \]

Lemma 4.12. Given a dynamic solution \( \psi = p_i \log |t| \), the perturbed quantities defined in Definition 3.9 rapidly converge back to their unperturbed forms.
(1) The perturbed Kretschmann scalar \( \mathbf{K}(t) \) for the dynamical solutions converges or 'relaxes' back to its original form for the short-pulse Gaussian perturbations so that

\[
\lim_{t \to \infty} \mathbf{K}(t) = \lim_{t \to |\theta|} \mathbf{K}(t) = \mathbf{K}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{i} p_{j} p_{j} t^{-4} \tag{4.84}
\]

(2) The perturbed expansion \( \chi(t) \) converges or relaxes back such that

\[
\overline{\chi(t)} \to \chi(t) \tag{4.85}
\]

(3) The perturbed shear converges as

\[
\overline{\mathcal{S}^{2}(t)} \to \mathcal{S}^{2}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{p_{i}}{t} \right) \left( \frac{p_{j}}{t} \right) - 2 \left( \frac{p_{i} p_{j}}{t^{2}} \right) \tag{4.86}
\]

Proof. The perturbed moduli are

\[
\overline{\psi_{i}(t)} = p_{i} \ln |t| + A_{i} \int_{0}^{t} \exp(-\tau^{2}/2\vartheta^{2})d\tau \tag{4.87}
\]

with derivatives \( \frac{\partial \psi_{i}(t)}{\partial t} = (p_{i}/t) + A_{i} \exp(-t^{2}/2\vartheta^{2}) \) so that for \( t \gg |\vartheta| \) or \( t \gg |\vartheta| \ll 0 \) so that

\[
\mathbf{K}(t) \to \mathbf{K}(t) = -\sum_{i=1}^{n} \frac{p_{i}^{2}}{t^{2}} + 4 \sum_{i=1}^{n} p_{i}^{2} t^{-2} + 2 \sum_{i=1}^{n} \left( \frac{p_{i}}{t} \right) \left( \frac{p_{i}}{t} \right) - 2 \left( \frac{p_{i} p_{i}}{t^{2}} \right) \tag{4.89}
\]

The perturbed expansion is

\[
\overline{\chi(t)} = \sum_{i=1}^{n} \left| \frac{\partial \psi_{i}(t)}{\partial t} \right| = \sum_{i=1}^{n} \left( \frac{p_{i}}{t} \right) + A_{i} \chi(t) \tag{4.90}
\]

which rapidly converges back to \( \chi(t) \) as \( t \to \infty \) or \( t \gg |\vartheta| \). Using (3.58), the perturbed shear is given as

\[
\overline{\mathcal{S}^{2}(t)} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{p_{i}}{t} + A_{i} E_{i}(t) \right) \left( \frac{p_{j}}{t} + A_{j} E_{j}(t) \right) - 2 \left( \frac{p_{i} p_{j}}{t^{2}} \right) \tag{4.91}
\]

so that for \( t \to \infty \) or \( t \gg |\vartheta| \)

\[
\overline{\mathcal{S}^{2}(t)} \to \mathcal{S}^{2}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{p_{i}}{t} \right) \left( \frac{p_{j}}{t} \right) - 2 \left( \frac{p_{i} p_{j}}{t^{2}} \right) \tag{4.92}
\]

\[
\square
\]

4.3. Continuous amplitude perturbation as a 'cosmological constant'. Given the static Kasner micro-universe, it is now shown that it is unstable to a continuous 'step' perturbation of the modulus functions which has constant amplitude and which enters the Einstein equations as a 'cosmological constant-like' term.

**Theorem 4.13.** Given \( M^{n+1} = T^{n} \times \mathbb{R}^{(+)} \) and the following:

1. The initial data \( \mathcal{D} = [t = 0, \Sigma_{o} = 0, g_{ij}(0), l_{ij}(0), \psi_{i}(0) = \psi^{E}_{i}, a_{i}(0) = a^{E}_{i}] \) with \( g_{oo} = -1, g_{io} = 0 \) with constraints \( \text{Ric}_{oo} = 0 \) and \( \text{Ric}_{io} = 0 \).
The Einstein vacuum equations $\mathbf{Ric}_{AB} = 0$ exist on $\mathbb{M}^{n+1}$ in the form $\mathbf{H}_n \psi^E_i = 0$.

The initially static toroidal radii are $a_i^E = \exp(\psi_i^E)$.

There is a set of functions $A_i(t)$ such that $A(0) = 0$ and $A_i(t) = A = \text{const.}$ for all $t > 0$.

The initially static modulus functions $\psi_i^E$ and radii $a_i^E$ are perturbed so that

$$\overline{\psi_i(t)} = \psi_i^E + \int_0^t A_i(\tau) d\tau \equiv \psi_i^E + A_i t$$

(4.93)

$$\overline{a_i(t)} = a_i^E \exp \left( \int_0^t A_i d\tau \right) \equiv a_i^E \exp(A_i t) = a_i^E \chi_i(t)$$

(4.94)

The derivatives are then $\partial_t \overline{\psi_i(t)} = A_i$ and $\partial_t \overline{\psi_i(t)} = 0$ and $\partial_t \overline{a_i(t)} = a_i^E A$ and $\partial_t \overline{a_i(t)} = a_i^E A^2 \chi_i(t)$. The norms are then

$$||\overline{\psi}(t) - \psi^E|| = \left| \int_0^t A d\tau \right| = \left( \sum_{i=1}^n \left| \int_0^t A_i d\tau \right|^2 \right)^{1/2} \equiv n^{1/2} \left| \int_0^t A d\tau \right| = n^{1/2} A t$$

(4.95)

$$||\overline{a}(t) - a^E|| = \left| a_i^E \exp \left( \int_0^t A_i d\tau \right) \right| = n^{1/2} |a_i^E| \exp(A t)$$

(4.96)

so that $\lim_{t \to \infty} ||\overline{\psi}(t) - \psi^E|| = \infty$ and $\lim_{t \to \infty} ||\overline{a}(t) - a^E|| = \infty$. Equations (4.93) and (4.94) are solutions of the perturbed Einstein equations so that

$$\mathbf{H}_n \overline{\psi}(t) = \mathbf{H}_n \psi^E + n A^2 \equiv \lambda$$

(4.97)

$$\mathbf{D}_n \overline{a_i(t)} = \mathbf{D}_n a_i^E + n A^2 \equiv \lambda$$

(4.98)

if $A_i = A$ for $i = 1 \ldots n$ so that $A = n A^2$ is an induced cosmological constant and we recover the Einstein equations (3.70) and (3.71). This is equivalent to the result of Lemma 5.12, which gives the solutions for the Einstein equations (3.70) and (3.71) with a cosmological constant term.

Proof. The perturbed Einstein equations are

$$\mathbf{H}_n \psi_i(t) = \sum_{i=1}^n \partial_i \psi_i(t) + \frac{1}{2} \sum_{i=1}^n \partial_i \psi_i(t) \partial_j \psi_j(t) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \partial_i \psi_i(t) \partial_j \psi_j(t)$$

(4.99)

if $A_i = A$ for $i = 1 \ldots n$. The perturbed equations for the toroidal radii are

$$\mathbf{D}_n \overline{a_i(t)} = \sum_{i=1}^n a_i^E A_i \chi_i^2 \chi_i - \frac{1}{2} \sum_{i=1}^n a_i^E a_i^E \chi_i \chi_i |A_i|^2 + \frac{1}{2} \sum_{i=1}^n a_i^E a_i^E \chi_i \chi_i |A_i|^2$$

= $\sum_{i=1}^n A_i A_i - \frac{1}{2} \sum_{i=1}^n A_i A_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n A_i A_j$

(4.100)

$$= n A^2 - \frac{1}{2} n A^2 + \frac{1}{2} n A^2 \equiv \lambda$$

Corollary 4.14. The spatial volume "inflates" or grows exponentially

$$\lim_{t \to \infty} \overline{V}(t) = \lim_{t \to \infty} |\mathbf{vol}^E| \exp(n A t) \equiv \exp(Lyt) = \infty$$

(4.101)

A second corollary is that $A$ is also a Lyupunov exponent.

Corollary 4.15. From Proposition (2.11), it follows that

$$\frac{||a(t) - a^E||}{a^E} \sim \exp(A t)$$

(4.102)

so that $A$ is essentially a LCE and the system is unstable for $A > 0$ and can never reach equilibrium or stability.
5. Random perturbations and stochastically averaged Einstein vacuum equations

The previous analysis established the perturbative stability to a deterministic Gaussian impulse perturbation of the radial modulus functions. But what were established as stable points or static equilibrium solutions, via the deterministic stability analysis, may be unstable in the presence of stochasticity, random fluctuations or 'noise'. The coupling of random fluctuations or noise to the inherent nonlinearity of the problem now becomes a crucial issue. Random perturbations of the radial modulus functions \( \{ \psi_i(t) \} \) are now considered, and the Einstein equations are then interpreted as an n-dimensional nonlinear system of differential equations coupled to Gaussian perturbations or random fields. The stochastic average of the systems of differential equations can also be computed and this leads to a non-vanishing extra terms due to nonlinearity—these then enter as induced 'cosmological constant' terms.

Applying the methods of Section 2, the most general random perturbations of the initially static moduli \( \psi_i^E \) are of the (Stratanovitch) stochastic integral form

\[
\hat{\psi}_i(t) = \psi_i^E + \zeta \int_0^t f(\tau) \hat{\mathcal{W}}_i(\tau) d\tau
\]

or

\[
\hat{\psi}_i(t) = \psi_i(t) + \zeta \int_0^t f(\tau) \hat{\mathcal{W}}_i(\tau) d\tau
\]

for initially dynamical solutions, where \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is some smooth continuous function and \( \hat{\mathcal{W}}_i(t) \) is an n-dimensional Gaussian random vector field, and \( \zeta > 0 \) is a constant. Then \( \hat{a}_i(t) = \exp(\hat{\psi}_i(t)) \) are the randomly perturbed radii. The randomly perturbed Einstein system of nonlinear ODEs is \( H_n \hat{\psi}_i(t) \) or \( H_n \hat{\psi}_i(t) \). The stochastically averaged Einstein system is then

\[
\mathbf{M} \left\{ H_n \hat{\psi}_i(t) \right\} = \mathbf{H}_n \psi_i^E + \text{terms} = \text{terms}
\]

\[
\mathbf{M} \left\{ D_n \hat{a}_i(t) \right\} = D_n a_i^E + \text{terms} = \text{terms}
\]

for initial static moduli or radii, and

\[
\mathbf{M} \left\{ H_n \hat{\psi}_i(t) \right\} = \mathbf{H}_n \psi_i(t) + \text{terms} = \text{terms}
\]

\[
\mathbf{M} \left\{ D_n \hat{a}_i(t) \right\} = D_n a_i(t) + \text{terms} = \text{terms}
\]

for randomly perturbed dynamical solutions. In each case, new terms are induced in the stochastically averaged equations. We first make the following preliminary definitions.

**Definition 5.1.** Given a set of Gaussian random fields \( \mathcal{W}_i(t) \equiv \mathcal{W}_i(t) \) for \( i = 1...n \) then \( \mathbf{M} \left\{ \mathcal{W}_i(t) \right\} = 0 \) and the regulated covariance is \( \text{COV}(t,s) = \mathbf{M} \left\{ \mathcal{W}_i(t)\mathcal{W}_j(s) \right\} = \delta_{ij} J(\Delta,\zeta) \) with \( \Delta = |t-s| \) and \( \zeta \) a correlation length with \( |\zeta| \ll 1 \). It is regulated so that \( \text{COV}(t,t) = \mathbf{M} \left\{ \mathcal{W}_i(t)\mathcal{W}_j(t) \right\} = N_{ij} J(0,\zeta), \infty \), where \( N_{ij} \) is an \( n \times n \) matrix which can the Kronecker delta \( \delta_{ij} \). Let \( \mathcal{S}_{ij}(t) = \mathcal{W}_i(t)\mathcal{W}_j(t) \) then \( \text{COV}(t,t) = \mathbf{M} \left\{ \mathcal{S}_{ij}(t) \right\} \). Then

\[
\sqrt{\text{COV}(t,t)} = \sqrt{\mathbf{M} \left\{ \mathcal{S}_{ij}(t) \right\}} = \delta_{ij} \sqrt{J(0,\zeta)} \equiv \delta_{ij} \sqrt{J(0,\zeta)} < \infty
\]

We can define the following \( L_2 \) and Frobenius norms

\[
\mathbf{M} \left\{ \sum_{i=1}^{n} \mathcal{W}_i(t)\mathcal{W}_i(t) \right\} = \sum_{i=1}^{n} \mathbf{M} \left\{ \mathcal{W}_i(t)\mathcal{W}_i(t) \right\} = \sum_{i=1}^{n} \mathbf{M} \left\{ |\mathcal{W}_i(t)|^2 \right\}_\text{via FubiniThm}
\]

\[
= \sum_{i=1}^{n} \left\| \mathbf{M} \left\{ |\mathcal{W}_i(t)|^2 \right\} \right\|^2_{L_2}
\]

and

\[
\mathbf{M} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{W}_i(t)\mathcal{W}_j(t) \right\} =
\]
Theorem 5.2. Let $\mathbb{M}^{n+1} = \mathbb{T}^n \times \mathbb{R}^+$ with the conditions for an initially static torus in equilibrium.

(1) The initial data $\mathfrak{D} = \{ t = 0, \Sigma_0 = 0, g_{ij}(0), \psi_i(0) = \psi_i^E, a_i(0) = a_i^E \}$ with $g_{oo} = -1, g_{io} = 0$ with constraints $R_{oo} = 0$ and $R_{oo} = 0$.

(2) The Einstein vacuum equations $R_{AB} = 0$ are defined on $\mathbb{M}^{n+1}$ in the form $H_n \psi_i^E = 0$ for all $t \in \mathbb{R}^+$, and $D_n a_i^E = 0$.

(3) The static radii are equal so the torus is static in that $a_i^E = a^E \equiv \exp(\psi_i^E)$.

(4) The Gaussian random fields $\hat{\mathcal{U}}_i(t, \xi)$ are colored or Gaussian-correlated such that $M\{ \hat{\mathcal{U}}_i(t) \} = 0$ and with a regulated 2-point function

$$
M \left\{ \hat{\mathcal{U}}_i(t, \xi) \hat{\mathcal{U}}_i(s, \xi) \right\} = N_{ij} \exp \left( -\frac{|\Delta|^q}{\xi^q} \right) = N_{ij} C \exp \left( -\frac{|\Delta|^q}{\xi^q} \right) = N_{ij} J(\Delta; \xi) = \delta_{ij} J(\Delta; \xi) \quad (5.10)
$$

where $\Delta = |t - s|$ with $N_{ij}$ an $n \times n$ matrix such as $\delta_{ij}$, $C > 0$, and $q = 1, 2$. The equal-time correlation is finite or regulated so that $M\{ \hat{\mathcal{U}}_i(t) \hat{\mathcal{U}}_i(t) \} = N_{ij} J(0; \xi) < \infty$ and the derivative $\delta_{ij} \hat{\mathcal{U}}_i(t)$ exists. (Appendix A.) Also $N_{ij} J(\Delta; \xi) \to 0$ for $t > |\xi|$.

(5) The stochastically perturbed modulus functions and radii are (with $f(t) = 1$)

$$
\hat{\psi}_i(t) = \psi_i^E + \zeta \int_0^t \hat{\mathcal{U}}_i(\tau) d\tau \equiv \psi_i^E + \zeta \hat{\mathcal{U}}_i(t) \quad (5.11)
$$

$$
\hat{a}_i(t) = a_i^E \exp \left( \zeta \int_0^t \hat{\mathcal{U}}_i(\tau) d\tau \right) \equiv a_i^E \hat{\mathcal{B}}_i(t) \quad (5.12)
$$

where $\zeta > 0$ is a real parameter. The "toroidal random geometry" $\mathbb{T}^n$ is then defined by the stochastic $(n+1)$-metric with the stochastic average

$$
M \left\{ d^2 \hat{s} \right\} = -dt^2 + \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} |a_i^E|^2 M \left\{ \hat{\mathcal{B}}_i(t) \hat{\mathcal{B}}_i(t) \right\} dX^i \otimes dX^j \quad (5.13)
$$

Equations (5.11) and (5.12) are then solutions of the stochastically averaged Einstein equations for this random geometry such that

$$
M \left\{ H_n \hat{\psi}_i(t) \right\} = M \left\{ \sum_{i=1}^n \partial_\tau \hat{\psi}_i(t) + \frac{1}{2} \sum_{i=1}^n \partial_\tau \hat{\psi}_i(t) \partial_\tau \hat{\psi}_i(t) + \frac{1}{2} \sum_{i=1}^n \partial_\tau \hat{\psi}_i(t) \partial_\tau \hat{\psi}_j(t) \right\} = \lambda \quad (5.14)
$$

$$
M \left\{ D_n \hat{a}_i(t) \right\} = M \left\{ \sum_{i=1}^n \partial_\tau \hat{a}_i(t) - \frac{1}{2} \sum_{i=1}^n \partial_\tau \hat{a}_i(t) \partial_\tau \hat{a}_i(t) + \frac{1}{2} \sum_{i=1}^n \partial_\tau \hat{a}_i(t) \partial_\tau \hat{a}_j(t) \right\} = \lambda \quad (5.15)
$$

or

$$
M \left\{ H_n \hat{\psi}_i(t) \right\} = \frac{1}{2} \left\| M \left\{ |\hat{\mathcal{U}}_i(t)|^2 \right\} \right\|_{L_2} + \frac{1}{2} \zeta^2 \left\| M \left\{ \dot{\mathcal{U}}_i(t) \right\} \right\|_{F} = \lambda_1 + \lambda_2 = \lambda \quad (5.16)
$$

$$
M \left\{ D_n \hat{a}_i(t) \right\} = \frac{1}{2} \left\| M \left\{ |\hat{\mathcal{U}}_i(t)|^2 \right\} \right\|_{L_2} + \frac{1}{2} \zeta^2 \left\| M \left\{ \dot{\mathcal{U}}_i(t) \right\} \right\|_{F} = \lambda_1 + \lambda_2 = \lambda \quad (5.17)
$$

where $\lambda$ is an induced positive 'cosmological constant' term given by

$$
\lambda = \frac{1}{2} \left\| M \left\{ |\hat{\mathcal{U}}_i(t)|^2 \right\} \right\|_{L_2} + \frac{1}{2} \zeta^2 \left\| M \left\{ \dot{\mathcal{U}}_i(t) \right\} \right\|_{F} = \frac{1}{2} \zeta^2 \sum_{i=1}^n \alpha_i J(0; \xi) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} J(0; \xi) = \lambda_1 + \lambda_2 = \lambda \quad (5.18)
$$
If \( \hat{U}_i(t) = \mathcal{U}(t) \) for all \( i = 1 \ldots n \) and \( \alpha_{ij} = \delta_{ij} \) then \( \lambda = \frac{1}{2} \zeta^2 n^2 J(0; \varsigma) \).

**Proof.** The derivatives are \( \partial_t \hat{\psi}_i(t) = \mathcal{U}_i(t) \) and \( \partial_t \hat{\psi}_i(t) = \partial_t \mathcal{U}_i(t) \) since \( \mathcal{U}(t) = \int_0^t \mathcal{U}(\tau) d\tau \). The stochastically perturbed Einstein equations are

\[
\mathbf{H}_n \hat{\psi}_i(t) = \sum_{i=1}^n \partial_{\tau} \hat{\psi}_i(t) + \frac{1}{2} \sum_{j=1}^n \partial_{\tau} \hat{\psi}_i(t) \partial_{\tau} \hat{\psi}_j(t) + \frac{1}{2} \sum_{i=1}^n \partial_{\tau} \hat{\psi}_i(t) \partial_{\tau} \hat{\psi}_i(t)
\]

\[
= \zeta \sum_{i=1}^n \partial_{\tau} \mathcal{U}_i(t) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \mathcal{U}_i(t) \mathcal{U}_i(t) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \mathcal{U}_i(t) \mathcal{U}_i(t) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \mathcal{U}_i(t) \mathcal{U}_i(t) \tag{5.19}
\]

Taking the stochastic average or mean \( M\{\ldots\} \) produces new non-vanishing finite terms such that

\[
M\left\{ \mathbf{H}_n \hat{\psi}_i(t) \right\} = M\left\{ \zeta \sum_{i=1}^n \partial_{\tau} \mathcal{U}_i(t) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \mathcal{U}_i(t) \mathcal{U}_i(t) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \mathcal{U}_i(t) \mathcal{U}_i(t) \right\}
\]

\[
\equiv \zeta \sum_{i=1}^n M\{ \partial_{\tau} \mathcal{U}_i(t) \} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n M\{ \mathcal{U}_i(t) \mathcal{U}_j(t) \} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n M\{ \mathcal{U}_i(t) \mathcal{U}_j(t) \} \tag{5.20}
\]

The linear term vanishes since \( M\{ \partial_{\tau} \mathcal{U}(t) \} = 0 \) where \( M\{ \partial_{\tau} \mathcal{U}_i(t) \} \equiv \partial_{\tau} M\{ \mathcal{U}_i(t) \} = 0 \) so that

\[
M\left\{ \mathbf{H}_n \hat{\psi}_i(t) \right\} = \frac{1}{2} \zeta^2 \sum_{i=1}^n M\{ \mathcal{U}_i(t) \mathcal{U}_i(t) \} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n M\{ \mathcal{U}_i(t) \mathcal{U}_j(t) \}
\]

\[
= \frac{1}{2} \zeta^2 \sum_{i=1}^n \delta_{ii} M\{ \mathcal{U}_i(t) \mathcal{U}_i(t) \} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} M\{ \mathcal{U}_i(t) \mathcal{U}_j(t) \}
\]

\[
= \frac{1}{2} \zeta^2 \sum_{i=1}^n \delta_{ii} J(0; \varsigma) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} J(0; \varsigma) \tag{5.21}
\]

\[
= \frac{1}{2} \zeta^2 n J(0; \varsigma) + \frac{n}{2} \zeta^2 J(0; \varsigma) = \zeta^2 n J(0; \varsigma) \equiv \lambda_1 + \lambda_2 = \lambda \tag{5.22}
\]

Or equivalently, to get (5.16)

\[
M\left\{ \mathbf{H}_n \hat{\psi}_i(t) \right\} = \frac{1}{2} \zeta^2 \sum_{i=1}^n M\{ \mathcal{U}_i(t) \mathcal{U}_i(t) \} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n M\{ \mathcal{U}_i(t) \mathcal{U}_j(t) \}
\]

\[
= \frac{1}{2} \zeta^2 \sum_{i=1}^n \delta_{ii} M\{ \mathcal{U}_i(t) \mathcal{U}_i(t) \} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} M\{ \mathcal{U}_i(t) \mathcal{U}_j(t) \}
\]

\[
= \frac{1}{2} \zeta^2 \sum_{i=1}^n M\{ |\mathcal{U}_i(t)|^2 \} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n M\{ A_{ij}(t) \}
\]

\[
= \frac{1}{2} \zeta^2 \sum_{i=1}^n \sqrt{M\{ |\mathcal{U}_i(t)|^2 \}}^2 + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \sqrt{M\{ A_{ij}(t) \}}^2 \tag{5.23}
\]

The same result must also follow from the nonlinear Einstein system of ODES for the radii. The derivatives of \( \tilde{a}_i(t) \) are

\[
\partial_{\tau} \tilde{a}_i(t) = a^E \zeta \hat{\mathcal{U}}_i(t) \exp \left( \zeta \int_0^t \hat{\mathcal{U}}(\tau) d\tau \right) \equiv a^E \zeta \hat{\mathcal{U}}_i(t) \tilde{B}_i(t) \tag{5.24}
\]
and $\partial_t \tilde{a}_i(t) = a_E \zeta \mathcal{U}_i(t) \mathcal{W}_i(t) \mathcal{B}_i(t) + a_E \zeta \partial_t \mathcal{U}_i(t) \mathcal{B}_i(t)$. The stochastically perturbed Einstein equations for the toroidal radii are

\[
D_n \tilde{a}_i(t) = \zeta \sum_{i=1}^n \frac{\partial_t \tilde{a}_i(t)}{\tilde{a}_i(t)} - \frac{1}{2} \zeta^2 \sum_{i=1}^n \frac{\partial_t \tilde{a}_i(t) \partial_t \tilde{a}_i(t)}{\tilde{a}_i(t) \tilde{a}_j(t)} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \frac{\partial_t \tilde{a}_i(t) \partial_t \tilde{a}_i(t)}{\tilde{a}_i(t) \tilde{a}_j(t)}
= \zeta \sum_{i=1}^n \frac{a_i^E \partial_t \mathcal{U}_i(t) \mathcal{B}_i(t)}{a_i^E \mathcal{B}_i(t)} + \zeta^2 \sum_{i=1}^n \frac{a_i^E \mathcal{U}_i(t) \mathcal{W}_i(t) \mathcal{B}_i(t)}{a_i^E \mathcal{B}_i(t)}
- \frac{1}{2} \zeta^2 \sum_{i=1}^n \frac{a_i^E \mathcal{U}_i(t) \mathcal{W}_i(t) \mathcal{B}_i(t)}{a_i^E \mathcal{B}_i(t) a_i \mathcal{B}_i(t)} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \frac{a_i^E a_j^E \mathcal{U}_i(t) \mathcal{B}_i(t) \mathcal{B}_j(t)}{a_i^E \mathcal{B}_i(t) a_i a_j \mathcal{B}_j(t)}
= \sum_{i=1}^n \zeta \partial_t \mathcal{W}_i(t) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \mathcal{W}_i(t) \mathcal{W}_i(t) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \mathcal{W}_i(t) \mathcal{W}_j(t) \tag{5.25}
\]

Taking the stochastic average $\mathcal{M}\{\ldots\}$ gives

\[
\mathcal{M}\left\{D_n \tilde{a}_i(t)\right\} = \frac{1}{2} \zeta^2 \sum_{i=1}^n \mathcal{M}\left\{\mathcal{W}_i(t) \mathcal{W}_i(t)\right\} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \mathcal{M}\left\{\mathcal{W}_i(t) \mathcal{W}_j(t)\right\}
= \frac{1}{2} \zeta^2 \sum_{i=1}^n \delta_i \mathcal{M}\left\{\mathcal{W}_i(t) \mathcal{W}_i(t)\right\} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} \mathcal{M}\left\{\mathcal{W}_i(t) \mathcal{W}_j(t)\right\}
= \zeta^2 \sum_{i=1}^n \delta_i J(0; \zeta) + \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} J(0; \zeta)
= \frac{1}{2} \zeta^2 n J(0; \zeta) + \frac{1}{2} \zeta^2 n J(0; \zeta) = \zeta^2 n J(0; \zeta) = \lambda \tag{5.26}
\]

as required. \hfill \Box

**Corollary 5.3.** If the form (5.1) is used for the random perturbations with $f(t) = 1$ then the stochastic averaging $\mathcal{M}\{\ldots\}$ gives

\[
\mathcal{M}\left\{H_n \dot{\mathcal{U}}_i(t)\right\} = \mathcal{M}\left\{\zeta \sum_{i=1}^n \partial_t \mathcal{W}_i(t) + \zeta \sum_{i=1}^n \partial_t \mathcal{W}_i(t)\right\}
+ \frac{1}{2} \mathcal{M}\left\{\zeta^2 \sum_{i=1}^n \sum_{j=1}^n f(t) f(t) \mathcal{W}_i(t) \mathcal{W}_j(t) + \frac{1}{2} \zeta^2 f(t) f(t) \sum_{i=1}^n \mathcal{W}_i(t) \mathcal{W}_i(t)\right\}
= \zeta \sum_{i=1}^n \mathcal{M}\left\{\partial_t \mathcal{W}_i(t)\right\} + \frac{1}{2} \zeta^2 f(t) f(t) \sum_{i=1}^n \sum_{j=1}^n \mathcal{M}\left\{\mathcal{W}_i(t) \mathcal{W}_j(t)\right\}
+ \frac{1}{2} \zeta^2 f(t) f(t) \sum_{i=1}^n \mathcal{M}\left\{\mathcal{W}_i(t) \mathcal{W}_i(t)\right\} \tag{5.27}
\]

The first term vanishes since $\mathcal{M}\{\partial_t \mathcal{W}_i(t)\} = 0$ where $\mathcal{M}\{\partial_t \mathcal{W}_i(t)\} \equiv \partial_t \mathcal{M}\{\mathcal{W}_i(t)\} = 0$ so that

\[
\mathcal{M}\left\{H_n \dot{\mathcal{U}}_i(t)\right\} = \frac{1}{2} \zeta^2 f(t) f(t) \sum_{i=1}^n \mathcal{M}\left\{\mathcal{W}_i(t) \mathcal{W}_j(t)\right\} + \frac{1}{2} \zeta^2 f(t) f(t) \sum_{i=1}^n \sum_{j=1}^n \mathcal{M}\left\{\mathcal{W}_j(t) \mathcal{W}_i(t)\right\}
= \frac{1}{2} \zeta^2 f(t) f(t) \sum_{i=1}^n \delta_i \mathcal{M}\left\{\mathcal{W}_i(t) \mathcal{W}_i(t)\right\} + \frac{1}{2} \zeta^2 f(t) f(t) \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} \mathcal{M}\left\{\mathcal{W}_i(t) \mathcal{W}_j(t)\right\}
= \frac{1}{2} \zeta^2 n f(t) f(t) J(0; \zeta) + \frac{1}{2} \zeta^2 f(t) f(t) \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} J(0; \zeta)
= \frac{1}{2} \zeta^2 n f(t) f(t) J(0; \zeta) + \frac{1}{2} \zeta^2 f(t) f(t) J(0; \zeta) = \zeta^2 n f(t) f(t) J(0; \zeta) \equiv \lambda \tag{5.28}
\]
Then
\[ M \left\{ H_n \tilde{\psi}_i(t) \right\} = \zeta^2 n f(t) f(t) J(0; \zeta) \equiv \lambda(t) \] (5.29)
and the induced cosmological constant term is now time dependent. Similarly,
\[ M \left\{ D_n \tilde{a}_i(t) \right\} = \zeta^2 n f(t) f(t) J(0; \zeta) \equiv \lambda(t) \] (5.30)

Remark 5.4. The induced cosmological constant term in the stochastically averaged vacuum equations arises purely from the nonlinearity of the Einstein equations. Intrinsic stochastic fluctuations of the moduli therefore act like a "dark energy". The cosmological constant \( \lambda \) can be made as small as required by fine tuning the parameter \( \zeta \) or reducing or "diluting" the intensity of the fluctuations so that \( J(0; \zeta) > 0 \) but with \( J(0; \zeta) \sim 0 \).

Remark 5.5. Rephrasing Remark 1.1, there is no analog of this for a purely linear theory. The non-vanishing terms which arise in the stochastically averaged system of equations are due to the nonlinearity of the equations. For example, given the linear ODEs
\[ L_n \psi_i(t) \equiv \sum_{i=1}^{n} \frac{\partial \tilde{a}_i(t)}{\tilde{a}_i(t)} \]
then there are trivial equilibrium solutions \( \psi_i(t) = E \) such that \( L_n \psi^E = 0 \). If \( \tilde{\psi}_i(t) = \psi^E + \zeta \int_0^t \tilde{\zeta}(\tau) \, d\tau \). For a linear system, the stochastically averaged equations will reduce back to the original deterministic equations. The randomly perturbed equations are
\[ L_n \tilde{\psi}_i(t) = \sum_{i=1}^{n} \frac{\partial \tilde{a}_i(t)}{\tilde{a}_i(t)} = \sum_{i=1}^{n} \tilde{a}_i(t) \tilde{\zeta}_i(t) \tilde{\zeta}_i(t) + \tilde{\zeta}_i(t) \tilde{\zeta}_i(t) \tilde{\zeta}_i(t) \] (5.32)
The stochastically averaged equations are then
\[ M \left\{ L_n \tilde{a}_i(t) \right\} = M \left\{ \sum_{i=1}^{n} \frac{\partial \tilde{a}_i(t)}{\tilde{a}_i(t)} \right\} \]
\[ = \sum_{i=1}^{n} \tilde{a}_i(t) M \left\{ \tilde{\zeta}_i(t) \right\} = \sum_{i=1}^{n} \tilde{a}_i(t) = 0 \] (5.33)
since \( M \{ \tilde{\zeta}_i(t) \} = 0 \) for a Gaussian random field, and so no new constant terms are induced for a stochastically averaged linear set of ODEs.

The random perturbations of the radial moduli must have a regulated covariance to induce cosmological constant terms in the stochastically averaged Einstein system, that are both finite and positive. White noise perturbations will induce delta-function singularities in the averaged system.

Lemma 5.6. let \( \psi^E \) be a set of equilibrium moduli solutions of the Einstein system \( H_n \psi^E = 0 \) and let \( \tilde{\psi}_i(t) \) be a dynamical set of solutions such that \( H_n \tilde{\psi}_i(t) = 0 \). Let \( \{ \tilde{\zeta}_i(t) \} \) be a set of Gaussian white noises with
\[ M \{ \tilde{\zeta}_i(t) = 0 \} \text{ and } M \{ \tilde{\zeta}_i(t), \tilde{\zeta}_j(t) \} = \alpha \delta_{ij} \delta(t - s) \] such that the randomly perturbed moduli are
\[ \tilde{\psi}_i(t) = \psi_i(t) + \zeta \int_0^t f(\psi(s)) d\tilde{\zeta}(s) \equiv \psi_i(t) + \zeta \int_0^t f(\psi(s)) \tilde{\zeta}(s) ds \] (5.34)
\[ \tilde{\psi}_i(t) = \psi^E_i(t) + \zeta \int_0^t f(\psi(s)) d\tilde{\zeta}(s) \equiv \psi^E_i(t) + \zeta \int_0^t f(\psi(s)) \tilde{\zeta}(s) ds \] (5.35)
where \( f(\psi(s)) \) is some smooth \( C^2 \)-differentiable functional and \( d\tilde{\zeta}(t) = (t)dt \) is the standard Brownian motion or Weiner process. This is equivalent to the stochastic DE or 'Langevin equation'
\[ d\tilde{\psi}_i(t) = d\psi_i(t) + \zeta f(\psi(t)) d\tilde{\zeta}(t) \] (5.36)
The stochastically averaged Einstein system then has a delta-function singularity such that
\[ M \left\{ H_n \tilde{\psi}_i(t) \right\} = H_n \psi_i(t) + n \alpha \delta(0) = \infty \] (5.37)
Lemma 5.7. Given white-noise perturbations of the static radial moduli set \((\psi_i(t))_{i=1}^n\) of the form

\[
\hat{\psi}(t) = \psi_i^E + \zeta \int_0^t f(\psi_i(s))dW(s)
\]

or

\[
\hat{\psi}(t) = \psi_i(t) + \zeta \int_0^t f(\psi_i(s))dW(s)
\]

with the conditions

\[
\|f(\psi_i(t))\|_t^t \leq K \|\psi_i(t)\|_t^2
\]

and

\[
\int_{t_0}^t \|f(\psi_i(s))\|^2 ds < \infty
\]

then the \(l^{th}\)-order moments are finite and bounded for all finite \(t > t_0\) and grow exponentially, with the estimates

\[
\mathcal{M}\left\{\|\sup_{t \leq T} \hat{\psi}_i(t)\|^l\right\} \leq \|\psi(t)\|_t^l \exp\left(\frac{1}{2}K(\ell - 1)|T - t_0|\right)
\]

\[
\mathcal{M}\left\{\|\sup_{t \leq T} \hat{\psi}_i(t)\|^l\right\} \leq \|\psi_i\|_t^l \exp\left(\frac{1}{2}K(\ell - 1)|T - t_0|\right)
\]

The proof is given in Appendix B.

As an example of specific regulated random modulus perturbations, one can apply an Ornstein-Uhlenbeck process, which is well-defined [21,22].

Lemma 5.8. Let the radial modulus perturbations be of the form

\[
\hat{\psi}_i(t) = \psi_i^E + \zeta \int_0^t \mathcal{H}_i(s)ds \equiv \psi_i^E + \zeta \int_0^t \mathcal{O}_i(s)ds
\]
where $\mathcal{O}_i(s)$ is the OU process. Then
\[
\partial_t \hat{\psi}_i(t) = \mathcal{O}_i(t) = \exp(-At)\mathcal{O}_i(0) + \sigma \exp(-At) \int_0^\infty \exp(As) d\mathcal{W}(s)
\]
\[
= \exp(-At)\mathcal{O}_i(0) + \sigma \int_0^t \exp(-A|t-s|) d\mathcal{W}(s)
\]
if $\mathcal{O}_i(0) = 0$. This is a solution of the linear stochastic DE
\[
d\mathcal{O}_i(t) = -A\mathcal{O}_i(t) dt + \sigma d\mathcal{W}(t)
\]  
Then the covariance is
\[
M\{\partial_t \hat{\psi}_i(t) \partial_t \hat{\psi}_j(t)\} = M\{\mathcal{O}_i(t) \mathcal{O}_j(s)\}
\]
\[
= M\{\sigma \int_0^t \exp(-A|t-s|) d\mathcal{W}(s)\}^2
\]
\[
= A\delta_{ij} \exp(-A|t-s|) \equiv \delta_{ij} J(\Delta; \sigma)
\]
which is regulated such that $M\{\partial_t \hat{\psi}_i(t) \partial_t \hat{\psi}_j(t)\} = A\delta_{ij}$. The averaged Einstein system is then
\[
M\{\mathbf{H}_n \hat{\psi}_i(t)\} = \mathbf{H}_n \psi_i(t) + nA\sigma^2 = nA\sigma^2 \equiv \lambda > 0
\]

**Proof.** From (5.48), the average of the 2nd derivative is zero, so that $M\{\partial_{tt} \hat{\psi}(t)\} = 0$. The averaged randomly perturbed Einstein system is then
\[
M\{\mathbf{H}_n \hat{\psi}_i(t)\} = \frac{1}{2} \sum_{i=1}^n M\{\sigma \int_0^t \exp(-A|t-s|) d\mathcal{W}_i(s)\}^2
\]
\[
+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n M\{\sigma \int_0^t \exp(-A|t-s|) d\mathcal{W}_i(s) \sigma \int_0^t \exp(-A|t-s|) d\mathcal{W}_j(s)\}
\]
\[
\equiv \frac{1}{2} M\{\sigma \int_0^t \exp(-A|t-s|) d\mathcal{W}_i(s)\}^2
\]
\[
+ M\{\sigma^2 \int_0^t \exp(-A|t-s|) d\mathcal{W}_i(s) \int_0^t \exp(-A|t-s|) d\mathcal{W}_j(s)\}
\]
\[
\equiv \frac{1}{2} \sigma^2 \sum_{i=1}^n \delta_{ii} A + \frac{1}{2} \sigma^2 \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} A
\]
\[
= \frac{1}{2} \sigma^2 nA + \frac{1}{2} \sigma^2 nA = \sigma^2 nA \equiv \lambda
\]

Similarly, one could average the equations in terms of the perturbed radii to obtain the same result.

### 5.1. Analogy with statistical hydrodynamical turbulence and averaged Navier-Stokes equations.

**Remark 5.9.** The stochastically averaged Einstein equations giving induced cosmological constant terms are also strongly analogous to what occurs when random fields are coupled to the Navier-Stokes equations in order to incorporate or describe the randomness of turbulence. Because the Navier-Stokes equations are also nonlinear a non-vanishing term arises or is induced when the stochastic Navier-Stokes equations are stochastically averaged or ’Reynolds averaged’. This is essentially an induced Reynolds stress tensor or Reynolds number $[26,27,28,29,30]$. The Einstein and Navier-Stokes equations are also similar in that they are nonlinear PDEs of hyperbolic type and both describe the dynamical evolution of a continuum approximation.”
Suppose non-white random vector fields $\hat{u}(x, t)$ are coupled to a fluid flow $u_i(x, t) \subset \mathbb{D} \subset \mathbb{R}^n$ in a domain $\mathbb{D}$ and described by the Navier-Stokes equations with some boundary conditions and initial data. In order to incorporate or describe turbulence in the fluid, one can couple a random field $\hat{u}(x, t)$, usually Gaussian, that varies randomly in both space and time. The following theorem for stochastically averaged Navier-Stokes equations for a randomly perturbed fluid flow, is then an analog of Theorem (5.1).

**Theorem 5.10.** The random field has the following properties

1. The fluid flow satisfies the laminar nonlinear Navier-Stokes PDEs
   \[
   \mathbf{N}_n \hat{u}_i(x, t) = \sum_{i=1}^{n} \partial_i \hat{u}_i(x, t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \partial^j [\hat{u}_j(x, t) \hat{u}_i(x, t)] \
   - \nu \sum_{i=1}^{n} \Delta \hat{u}_i(x, t) + \sum_{i=1}^{n} \partial_i p(x, t) - \sum_{i=1}^{n} f_i(x, t)
   \] (5.53)
   with some suitable initial data and boundary conditions.
2. The smooth/laminar Navier-Stokes flow $u_i(x, t)$ is randomly perturbed as
   \[
   \hat{u}_i(x, t) = u_i(x, t) + \hat{u}_i(x, t)
   \] (5.54)
3. The spatio-temporal random field $\hat{u}(x, t)$ is Gaussian with expectation $\mathbb{E}(\hat{u}_i(x, t)) = 0$
4. The 2-point function is of the form
   \[
   \mathcal{R}_{ij}(x, y, s, t) = \mathbb{E}\left\{\hat{u}_i(x, t) \hat{u}_j(y, s)\right\} = \delta_{ij} \Xi(|x-y|; \xi) J(|t-s|; \varsigma)
   \] (5.55)
   where $\xi$ is a spatial correlation length and $\varsigma$ a temporal correlation. Then $\mathcal{R}_{ij}(x, y, t, s) \rightarrow 0$ for $|x-y| \gg \xi$ and/or $|t-s| \gg \varsigma$.
5. It is regulated such that
   \[
   \mathcal{R}_{ij}(x, y, t, s) = \lim_{x \rightarrow y} \lim_{t \uparrow s} \mathbb{E}\left\{\hat{u}_i(x, t) \hat{u}_j(y, s)\right\} = \lim_{x \rightarrow y} \lim_{t \uparrow s} \delta_{ij} \Xi(0; \xi) J(0; \varsigma) < \infty
   \] (5.56)
6. The derivative exists such that
   \[
   \partial^i \mathcal{R}_{ij}(x, y, t, s) = \partial^i \mathbb{E}\left\{\hat{u}_i(x, t) \hat{u}_j(y, s)\right\} = (\partial^i \Xi_j(|x-y|; \xi)) J(|t-s|; \varsigma) \equiv \delta_{ij} \partial^i \Xi(|x-y|; \xi) J(|t-s|; \varsigma)
   \] (5.57)
   and the limit of the derivative is regulated such that
   \[
   \partial^i \mathcal{R}_{ij}(x, y, t, s) = \lim_{x \rightarrow y} \lim_{t \uparrow s} \partial^i \mathbb{E}\left\{\hat{u}_i(x, t) \hat{u}_j(y, s)\right\} = \lim_{x \rightarrow y} \lim_{t \uparrow s} \delta_{ij} \partial^i \Xi(0; \xi) J(0; \varsigma) < \infty
   \] (5.58)

Then the stochastically averaged NS equations are

\[
\mathbb{E}\left\{\mathbf{N}_n \hat{u}_i(x, t)\right\} = \mathbf{N}_n u_i(x, t) + \partial^i \mathcal{R}_{ij}(x, x, t, t) = \mathbf{N}_n u_i(x, t) + \delta_{ij} \partial^i \Xi(0; \xi) J(0; \varsigma)
\] (5.59)

and the averaged incompressibility condition still holds such that $\mathbb{E}\{\partial_i \hat{u}_i(x, t)\} = 0$

**Proof.** The randomly perturbed or turbulent flow is $\hat{u}_i(x, t) = u_i(x, t) + \hat{u}_i(x, t)$. So that perturbed PDEs are

\[
\mathbf{N}_n \hat{u}_i(x, t) = \sum_{i=1}^{n} \partial_i \hat{u}_i(x, t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \partial^j [\hat{u}_j(x, t) \hat{u}_i(x, t)]
\]
- \nu \sum_{i=1}^{n} \Delta \tilde{u}_i(x, t) + \sum_{i=1}^{n} \partial_i p(x, t) - \sum_{i=1}^{n} f_i(x, t)

= \sum_{i=1}^{n} \partial_i u_i(x, t) + \sum_{i=1}^{n} \partial_i \mathcal{U}_i(x, t) + \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(x, t) u_j(x, t)

+ \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(x, t) \mathcal{U}_j(x, t) + \sum_{i=1}^{n} \sum_{j=1}^{n} u_j(x, t) \mathcal{U}_i(x, t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{U}_i(x, t) \mathcal{U}_j(x, t)

- \nu \sum_{i=1}^{n} \Delta \tilde{u}_i(x, t) - \nu \sum_{i=1}^{n} \Delta \mathcal{U}_i(x, t) + \sum_{i=1}^{n} \partial_i p(x, t) - \sum_{i=1}^{n} f_i(x, t) \quad (5.60)

Taking the stochastic expectation or average,

\mathbf{M}\{\mathbf{N}_n \tilde{u}_i(x, t)\} = \sum_{i=1}^{n} \partial_i u_i(x, t) + \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(x, t) u_j(x, t)

\equiv \sum_{i=1}^{n} \partial_i u_i(x, t) + \sum_{i=1}^{n} \mathbf{M}\{\partial_i \mathcal{U}_i(x, t)\} + \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(x, t) u_j(x, t)

+ \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(x, t) \mathbf{M}\{\mathcal{U}_j(x, t)\} + \sum_{i=1}^{n} \sum_{j=1}^{n} u_j(x, t) \mathbf{M}\{\mathcal{U}_i(x, t)\}

+ \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{M}\{\mathcal{U}_i(x, t) \mathcal{U}_j(x, t)\} - \nu \sum_{i=1}^{n} \Delta \tilde{u}_i(x, t) - \nu \sum_{i=1}^{n} \mathbf{M}\{\Delta \mathcal{U}_i(x, t)\}

+ \sum_{i=1}^{n} \partial_i p(x, t) - \sum_{i=1}^{n} f_i(x, t) \quad (5.61)

The linear terms containing \mathcal{U}_i(x, t) and \partial_i \mathcal{U}_i(x, t) vanish, however, averaging over the nonlinear convective term induces an additional stress tensor term

\mathbf{M}\{\mathbf{N}_n \tilde{u}_i(x, t)\} = \mathbf{N}_n u_i(x, t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \partial^j \mathbf{M}\{\tilde{\mathcal{U}}_i(x, t) \tilde{\mathcal{U}}_j(x, t)\}

= \mathbf{N}_n u_i(x, t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \partial^j \mathbf{M}\{\tilde{\mathcal{U}}_i(x, t) \tilde{\mathcal{U}}_j(x, t)\}

\equiv \partial^j \mathcal{R}_{ij}(x; x, t; t) = \delta_{ij} \partial^j \Xi(0; \xi J(0; \zeta)) \quad (5.62)

The averaged incompressibility condition is

\mathbf{M}\{\partial_i \tilde{u}_i(x, t)\} = \partial_i u_i(x, t) + \mathbf{M}\{\partial_i \mathcal{U}_i(x, t)\} = \partial_i \mathbf{M}\{\mathcal{U}_i(x, t)\} = 0 \quad (5.63)

5.2. Random perturbations of the dynamical solutions. Using Lemma (3.8) there is also a dynamical solution of the stochastic Einstein equations.

**Theorem 5.11.** Given the set of dynamical power-law solutions (3.44) and (3.45) whereby \psi_i(t) = \psi_i^E + p_i \ln |t| is a solution of \mathbf{H}_n \psi_i(t) = 0 and \alpha_i(t) = \alpha_i^E |t|^{p_i} is a solution of \mathbf{D}_n \alpha_i(t) = 0 where \pi_i satisfy the Kasner constraints, then the following stochastic solutions

\tilde{\psi}(t) = \psi_i^E + p_i \ln |t| + \int_0^t \tilde{\mathcal{U}}_i(\tau)d\tau \equiv \psi_i(t) + \int_0^t \tilde{\mathcal{U}}_i(\tau)d\tau \quad (5.64)
The derivatives are
are dynamical solutions of the stochastically averaged Einstein equations

\begin{align}
\hat{\alpha}_i(t) &= \alpha^E_i \exp \left( \int_{0}^{t} \hat{W}_i(r) \, dr \right) = a^E_i (t) \nonumber \\
\hat{\alpha}_i(t) &= \alpha^E_i \exp \left( \int_{0}^{t} \hat{W}_i(r) \, dr \right) = a^E_i (t) \\
\end{align}

(5.65)

or

\begin{align}
\mathbf{M} \{ \mathbf{H}_n \hat{\psi}_i(t) \} &= \mathbf{M} \left\{ \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) + \frac{1}{2} \sum_{i \neq j} \partial \hat{\psi}_i(t) \partial \hat{\psi}_j(t) + \frac{1}{2} \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) \partial_t \hat{\psi}_i(t) \right\} = \lambda \\
\mathbf{M} \{ \mathbf{D}_n \hat{\alpha}_i(t) \} &= \mathbf{M} \left\{ \sum_{i=1}^{n} \partial_t \hat{\alpha}_i(t) + \frac{1}{2} \sum_{i \neq j} \partial \hat{\alpha}_i(t) \partial \hat{\alpha}_i(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_t \hat{\alpha}_i(t) \partial_t \hat{\alpha}_j(t) \right\} = \lambda \\
\end{align}

(5.66)

\begin{align}
\mathbf{M} \{ \mathbf{H}_n \hat{\psi}_i(t) \} &= \mathbf{H}_n \hat{\psi}_i(t) + \frac{1}{2} \lambda^2 \left\| \mathbf{M} \{ \hat{W}_i(t) \} \right\|_{L^2}^2 + \frac{1}{2} \lambda^2 \left\| \mathbf{M} \{ \hat{W}_j(t) \} \right\|_{F}^2 \\
&= \mathbf{H}_n \hat{\psi}_i(t) + \lambda_1 + \lambda_2 = \lambda \\
\mathbf{M} \{ \mathbf{D}_n \hat{\alpha}_i(t) \} &= \mathbf{D}_n \hat{\alpha}_i(t) + \frac{1}{2} \lambda^2 \left\| \mathbf{M} \{ \hat{W}_i(t) \} \right\|_{L^2}^2 + \frac{1}{2} \lambda^2 \left\| \mathbf{M} \{ \hat{W}_j(t) \} \right\|_{F}^2 \\
&= \mathbf{D}_n \hat{\alpha}_i(t) + \lambda_1 + \lambda_2 = \lambda \\
\end{align}

(5.68)

\begin{align}
\mathbf{H}_n \hat{\psi}_i(t) &= \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) + \frac{1}{2} \sum_{i \neq j} \partial \hat{\psi}_i(t) \partial \hat{\psi}_j(t) + \frac{1}{2} \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) \partial_t \hat{\psi}_i(t) \\
&= \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) + \frac{1}{2} \sum_{i \neq j} \partial \hat{\psi}_i(t) \partial \hat{\psi}_j(t) + \frac{1}{2} \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) \partial \hat{\psi}_i(t) \\
&+ \sum_{i=1}^{n} \partial_t \hat{W}_i(t) + \frac{1}{2} \sum_{i \neq j} \partial \hat{\psi}_i(t) \hat{W}_j(t) + \frac{1}{2} \sum_{i=1}^{n} \hat{W}_i(t) \hat{W}_i(t) \\
&+ \frac{1}{2} \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) \hat{W}_j(t) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{W}_i(t) \hat{W}_j(t) \\
\end{align}

(5.70)

which is upon taking the stochastic expectation

\begin{align}
\mathbf{M} \{ \mathbf{H}_n \hat{\psi}_i(t) \} &= \mathbf{M} \left\{ \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) + \frac{1}{2} \sum_{i \neq j} \partial \hat{\psi}_i(t) \partial \hat{\psi}_j(t) + \frac{1}{2} \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) \partial_t \hat{\psi}_i(t) \right\} \\
&= \sum_{i=1}^{n} \left( -\frac{p_i}{\ell^2} \right) + \frac{1}{2} \sum_{i \neq j} \sum_{j=1}^{n} \frac{p_i p_j}{\ell^2} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{p_i p_j}{\ell^2} \\
&+ \sum_{i=1}^{n} \partial_t \mathbf{M} \{ \hat{W}_i(t) \} + \frac{1}{2} \sum_{i=1}^{n} \mathbf{M} \{ \partial_t \hat{W}_i(t) \hat{W}_j(t) \} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{M} \{ \hat{W}_i(t) \hat{W}_j(t) \} \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{M} \{ \partial_t \hat{W}_i(t) \hat{W}_j(t) \} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{M} \{ \hat{W}_i(t) \hat{W}_j(t) \} \\
\end{align}

(5.71)
Due to the nonlinearity, and since $M\{\hat{\mathcal{U}}(t)\} = 0$ terms involving the non-vanishing correlations $M\{\hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)\}$ are retained so that equation (5.71) reduces to

$$M\{H_n\hat{\psi}_i(t)\} = M\left\{\sum_{i=1}^{n} \partial_i\hat{\psi}_i(t) + \frac{1}{2} \sum_{i\neq j} \partial_i\hat{\psi}_i(t)\partial_j\hat{\psi}_j(t) + \frac{1}{2} \sum_{i=1}^{n} \partial_i\hat{\psi}_i(t)\partial_i\hat{\psi}_i(t)\right\}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} M\{\hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M\{\hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)\}$$

(5.72)

Taking $p_i = p_j = 1$ for all $i, j = 1...n$, then

$$\sum_{i=1}^{n} \left(-\frac{p_i}{t^2}\right) + \sum_{i=1}^{n} p_i p_j \frac{1}{t^2} + \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \frac{1}{t^2} = -n\left(\frac{1}{t^2}\right) + \frac{1}{2} n \frac{1}{t^2} + \frac{1}{2} n \frac{1}{t^2} = 0$$

(5.73)

so that

$$M\{H_n\hat{\psi}_i(t)\} = \frac{1}{2} \sum_{i=1}^{n} M\{\hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M\{\hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)\}$$

$$= \frac{1}{2} \sum_{i=1}^{n} \delta_{ii} M\{\hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} M\{\hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)\}$$

$$= \frac{1}{2} \zeta^2 \sum_{i=1}^{n} \sqrt{M\{|\mathcal{U}_i|^2\}}^2 + \frac{1}{2} \zeta^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{M\{|\mathcal{S}_{ij}(t)|^2\}}^2$$

$$= \frac{1}{2} \zeta^2 \left\|\sqrt{M\{|\mathcal{U}_i|^2\}}\right\|^2_{L^2} + \frac{1}{2} \zeta^2 \left\|\sqrt{M\{|\mathcal{S}_{ij}(t)|^2\}}\right\|^2_{F}$$

$$= \frac{1}{2} \sum_{i=1}^{n} \delta_{ii} J(0; \zeta) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} J(0; \zeta)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \delta_{ii} J(0; \zeta) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} J(0; \zeta) = \frac{1}{2} \sum_{i=1}^{n} \delta_{ii} J(0; \zeta) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} J(0; \zeta)$$

(5.74)

and the result is proved. To prove (5.69), first establish the derivatives $\partial_t\hat{\mathcal{S}}_i(t) = \hat{\mathcal{U}}(t)\hat{\mathcal{S}}_i(t)$ and $\partial_t\hat{\mathcal{S}}_i(t) = \hat{\mathcal{U}}(t)\partial_t\hat{\mathcal{S}}_i(t)$ and then $\partial_{\alpha_i^{(1)}}(t) = a_i E_1 |t|^\mu \partial_t\hat{\mathcal{S}}_i(t) + a_i E_1 |t|^\mu - 1 \beta_i$ and $\partial_t a_i^{(1)}(t) = a_i E_1 |t|^\mu \partial_t\hat{\mathcal{S}}_i(t) + a_i E_1 |t|^\mu t |t|^\mu - 1 \partial_t\hat{\mathcal{S}}_i(t)$. The stochastically perturbed Einstein system of nonlinear ODEs becomes

$$D_n a_i^{(1)}(t) = \sum_{i=1}^{n} a_i E_1 |t|^\mu \hat{\mathcal{U}}(t)\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t) + \sum_{i=1}^{n} \frac{a_i E_1 |t|^\mu \partial_t\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)}{\alpha_1 E_1 |t|^\mu \hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)}$$

$$+ \sum_{i=1}^{n} \frac{a_i E_1 |t|^\mu \partial_t\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)}{\alpha_1 E_1 |t|^\mu \hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)} + \sum_{i=1}^{n} \frac{a_i E_1 |t|^\mu |t|^\mu - 1 \partial_t\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)}{\alpha_1 E_1 |t|^\mu \hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)}$$

$$- \frac{1}{2} \sum_{i=1}^{n} a_i E_1 |t|^\mu |t|^\mu - 1 \partial_t\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)$$

$$- \frac{1}{2} \sum_{i=1}^{n} 2a_i E_1 |t|^\mu |t|^\mu - 1 \beta_i |t|^\mu - 1 \partial_t\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)$$

$$- \frac{1}{2} \sum_{i=1}^{n} \frac{a_i E_1 |t|^\mu |t|^\mu - 1 \partial_t\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)}{\alpha_1 E_1 |t|^\mu \hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)}$$

$$- \frac{1}{2} \sum_{i=1}^{n} \frac{a_i E_1 |t|^\mu |t|^\mu - 1 \partial_t\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)}{\alpha_1 E_1 |t|^\mu \hat{\mathcal{S}}_i(t)\hat{\mathcal{S}}_i(t)}$$
Now setting \( p_i = 1 \) for \( i = 1 \ldots n \) so that \( p_i = p_j \) for \( i \neq j \) and \( \tilde{\mathcal{W}}_i(t) = \tilde{\mathcal{W}}(t) \) for \( i = 1 \ldots n \) gives

\[
\mathcal{M}\{D_n a_i(t)\} = \frac{1}{2} \sum_{i=1}^{n} \delta_{ii} J(0; \varsigma) - \frac{1}{2} \sum_{i=1}^{n} |p_i|^2 t^{-2} \\
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} J(0; \varsigma) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_ip_j t^{-2} \\
= \frac{1}{2} n J(0; \varsigma) - \frac{1}{2} n^2 t^{-2} + \frac{1}{2} n J(0; \varsigma) + \frac{1}{2} n^2 t^{-2} = nJ(0, \varsigma) \equiv \lambda
\]

and the result is proved. \( \square \)

5.3. Random perturbations and averaged Einstein equations with a pre-existing cosmological constant. We can also consider random perturbations of the dynamical cosmological solutions. The Einstein equations on a toroidal spacetime geometry with a pre-existing cosmological constant \( \tilde{\lambda} \) are given by (3.70) and (3.71)

\[
\mathbf{H}_n \psi(t) = \tilde{\lambda}
\]
where $\lambda = \Lambda(1 + n)/(1 - n)$, and where $\text{Ric}_{AB} = g_{AB}\Lambda$ are the underlying Einstein vacuum equations with cosmological constant. These have expanding and collapsing solutions such that $a_i^{(\pm)}(t) = a_i(0) \exp(\pm(\lambda/n)^{1/2}t)$. Randomly perturbing the fields $\psi_i(t)$ and taking the stochastic average of the perturbed equations will then add a new contribution to $\lambda$. This additional contribution is small if the fluctuations are weak.

**Lemma 5.12.** Given the conditions of Thm (5.1) but with EVEs $\text{Ric}_{AB} = g_{AB}\Lambda$, then on $\mathbb{T}^n \times \mathbb{R}^+$ the Einstein equations are $H_n \hat{\psi}_i(t) = \lambda$ or $D_n \hat{a}_i(t) = \lambda$, where $\lambda = \Lambda(1 + n)/(1 - n)$ with 'inflating' solutions $a_i(t) = \exp((\lambda/n)^{1/2}t), a_n \hat{\psi}_i(t) = \psi_i(0) + (\lambda/n)^{1/2}$. If $M\{\hat{\Psi}_i(t)\} = 0$ and the 2-point function is regulated as $M\{\hat{\Psi}_i(t)\hat{\Psi}_j(t)\} = \delta_{ij}J(0, \theta) = \delta_{ij}C$ for Gaussian random perturbations or noise, then the randomly perturbed solution (with $\zeta = 1$) are $\hat{a}_i(t) = a_i(0) \exp((\lambda/n)^{1/2}t) \exp \left( \int_0^t \hat{\Psi}(\tau) d\tau \right)$ is a solution of the stochastically averaged Einstein systems of differential equations

\[
M\{H_n \hat{\psi}_i(t)\} = \lambda + \frac{1}{2} \zeta^2 \left\| M\{\hat{\Psi}_i(t)\hat{\Psi}_i(t)\} \right\|_F^2 + \frac{1}{2} \zeta^2 \left\| M\{\hat{\Psi}_i(t)\hat{\Psi}_j(t)\} \right\|_F^2
\]

or

\[
M\{D_n \hat{a}_i(t)\} = \lambda + (\lambda_1 + \lambda_2) = \lambda \quad \text{(5.84)}
\]

where $\lambda$ is an induced cosmological constant contribution arising from the nonlinearity, and if $\hat{\Psi}_i(t) = \hat{\Psi}(t)$ for all $i = 1...n$ then $\lambda = nC$. The averaged effect of the random perturbation is to boost the expansion rate.

**Proof.** Writing $a_i(t) = a_i(0) \exp((\lambda/n)^{1/2}t) \exp \left( \int_0^t \hat{\Psi}(\tau) d\tau \right) = a_i(t) \hat{B}_i(t)$

where $\hat{a}_i(t) = a_i(0) \exp((\lambda/n)^{1/2}t)$. The derivatives are $\partial_i \hat{B}_i(t) = \hat{\Psi}_i(t) \hat{B}_i(t)$ and $\partial_i \hat{a}_i(t) = a_i(t) \hat{B}_i(t) \hat{B}_i(t) + \hat{B}_i(t) \partial_i a_i(t)$. The second derivative is $\partial_i \hat{a}_i(t) = a_i(t) \hat{\Psi}_i(t) \hat{B}_i(t) + a_i(t) \hat{B}_i(t) \hat{B}_i(t) + \partial_i a_i(t) \hat{\Psi}_i(t) \hat{B}_i(t) \hat{B}_i(t) + \partial_i a_i(t) \hat{\Psi}_i(t) \hat{B}_i(t) \hat{B}_i(t)$.

The perturbed Einstein equations are then

\[
D_n \hat{a}_i(t) = \sum_{i=1}^n \frac{\partial_i a_i(t)}{a_i(t)} - \frac{1}{2} \sum_{i=1}^n \frac{\partial_i a_i(t) \partial_j a_i(t)}{a_i(t) a_j(t)} + \frac{1}{2} \sum_{i=1}^n \frac{\partial_i a_i(t) \partial_j a_i(t)}{a_i(t) a_j(t)}
\]

(5.87)
Cancelling terms and taking the stochastic average $\text{M}\{\ldots\}$ with $\text{M}\{\hat{\mathcal{U}}(t)\} = 0$ and $\text{M}\{\partial_t \hat{\mathcal{U}}(t)\} = 0$

\[
\text{M}\left\{D_n a_i(t)\right\} = \sum_{i=1}^{n} \frac{\partial_t \hat{a}_i(t)}{a_i(t)} - \frac{1}{2} \sum_{i=1}^{n} \frac{\partial_t a_i(t) \partial_t a_i(t)}{a_i(t) a_j(t)} + \frac{1}{2} \sum_{j=1}^{n} \sum_{j=1}^{n} \frac{\partial_t a_i(t) \partial_t a_i(t)}{a_i(t) a_j(t)}
\]

\[
+ \sum_{i=1}^{n} \text{M}\left\{\hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t)\right\} - \frac{1}{2} \sum_{i=1}^{n} \text{M}\left\{\hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t)\right\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{M}\left\{\hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t)\right\}
\]

\[
\equiv D_n a_i(t) + \sum_{i=1}^{n} \text{M}\left\{\hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t)\right\} - \frac{1}{2} \sum_{i=1}^{n} \text{M}\left\{\hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t)\right\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{M}\left\{\hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t)\right\}
\]

\[
\equiv \lambda + \frac{1}{2} \sum_{i=1}^{n} \delta_{ii} \text{M}\left\{\hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t)\right\} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \text{M}\left\{\hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t)\right\}
\]

\[
= \left(\frac{1}{2} \zeta^2 \sum_{i=1}^{n} \sqrt{\text{M}\{\mathcal{U}_i(t)^2\}}\right)^2 + \left(\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{\text{M}\{\mathcal{J}_{ij}(t)\}}\right)^2
\]

\[
= \lambda + \frac{1}{2} \sum_{i=1}^{n} \delta_{ii} \mathcal{J}(0; \zeta) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} \mathcal{J}(0; \zeta)
\]

\[
= \lambda + \frac{1}{2} n \mathcal{J}(0; \zeta) + \frac{1}{2} \mathcal{J}(0; \zeta) = \lambda + \lambda_1 + \lambda_2
\]

(5.88)

where we have taken $\hat{\mathcal{U}}(t) = \hat{\mathcal{U}}(t)$ for $i = 1\ldots n$ as before. This then reduces to

\[
\text{M}\left\{D_n \hat{a}_i(t)\right\} = \lambda + n \text{M}\left\{\hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t)\right\} = \lambda + n \mathcal{J}(t, t; \zeta) \equiv \lambda + \lambda
\]

(5.89)

\[
\square
\]

5.4. Kretschmann invariant, shear and expansion. The final lemma of this section considers the averaged Kretschmann invariant, the averaged expansion and the averaged shear for a dynamical solution $\psi_i(t)$ which is randomly perturbed.

**Lemma 5.13.** If $\hat{\mathcal{U}}(t) = \hat{\mathcal{U}}(t)$, the averaged Kretschmann invariant is shifted as

\[
\text{M}\left\{\mathcal{K}(t)\right\} = \mathcal{K}(t) + 6n \zeta^2 \mathcal{J}(0; \zeta)
\]

(5.90)

The averaged expansion remains invariant so that

\[
\text{M}\left\{\mathcal{X}(t)\right\} = \mathcal{X}(t)
\]

(5.91)

The averaged shear is shifted as

\[
\text{M}\left\{\mathcal{G}^2(t)\right\} = \mathcal{G}^2(t) + 4n \zeta^2 \mathcal{J}(0; \zeta)
\]

(5.92)

**Proof.** The randomly perturbed Kretschmann invariant is

\[
\hat{\mathcal{K}}(t) = 4 \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) + 4 \sum_{i=1}^{n} \partial_t \hat{\psi}_i(t) \partial_t \hat{\psi}_i(t) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_t \hat{\psi}_i(t) \partial_t \hat{\psi}_j(t)
\]

\[
= 4 \sum_{i=1}^{n} \partial_t \psi_i(t) + \zeta \hat{\mathcal{U}}(t) \right) + \sum_{i=1}^{n} \left( \partial_t \psi_i(t) \partial_t \psi_i(t) + 4 \partial_t \psi_i(t) \zeta \hat{\mathcal{U}}(t) + \zeta^2 \hat{\mathcal{U}}(t) \hat{\mathcal{U}}(t) \right)
\]
\[ + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \partial_{t} \psi_{i}(t) \partial_{t} \psi_{j}(t) + \zeta \partial_{t} \psi_{i}(t) \hat{\psi}_{i}(t) + \zeta \partial_{t} \psi_{j}(t) \hat{\psi}_{j}(t) + \zeta^{2} \hat{\psi}_{i}(t) \hat{\psi}_{j}(t) \right) \]  

(5.93)

Taking the stochastic expectation

\[
M \left\{ \hat{\mathbf{K}}(t) \right\} = 4 \sum_{i=1}^{n} \partial_{t} \psi_{i}(t) + 4 \sum_{i=1}^{n} \partial_{t} \psi_{i}(t) \partial_{t} \psi_{i}(t) + 2 \sum_{i=1}^{n} \partial_{t} \psi_{i}(t) \partial_{t} \psi_{j}(t) 
+ 4 \zeta^{2} \sum_{i=1}^{n} M \left\{ \hat{\psi}_{i}(t) \hat{\psi}_{i}(t) \right\} + 2 \zeta^{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M \left\{ \hat{\psi}_{i}(t) \hat{\psi}_{j}(t) \right\} 
= K(t) + 4 \zeta^{2} n J(0; \zeta) + 2 \zeta^{2} n J(0; \zeta) = K(t) + 6 \zeta^{2} J(0; \zeta) 
\]  

(5.94)

The randomly perturbed expansion is

\[ \hat{\chi}(t) = \sum_{i=1}^{n} \partial_{t} \psi_{i}(t) + \sum_{i=1}^{n} \zeta \hat{\psi}_{i}(t) \]  

(5.95)

so that

\[ M \left\{ \hat{\chi}(t) \right\} = \chi(t) + \sum_{i=1}^{n} M \left\{ \hat{\psi}_{i}(t) \right\} = \chi(t) \]  

(5.96)

Finally, the stochastically averaged shear is

\[
\hat{\mathbf{E}}^{2}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \partial_{t} \hat{\psi}_{i}(t) \partial_{t} \hat{\psi}_{j}(t) - 2 \partial_{t} \hat{\psi}_{i}(t) \partial_{t} \hat{\psi}_{j}(t)(t) + \partial_{t} \hat{\psi}_{j}(t) \partial_{t} \hat{\psi}_{j}(t) \right) 
= \partial_{t} \psi_{i}(t) \partial_{t} \psi_{j}(t) + \zeta^{2} \partial_{t} \psi_{i}(t) \hat{\psi}_{i}(t) + \zeta \partial_{t} \psi_{j}(t) \hat{\psi}_{j}(t) + \partial_{t} \psi_{j}(t) \partial_{t} \psi_{j}(t) 
+ 2 \zeta \partial_{t} \psi_{i}(t) \hat{\psi}_{j}(t) + \zeta \partial_{t} \psi_{j}(t) \hat{\psi}_{i}(t) - \zeta \partial_{t} \psi_{i}(t) \hat{\psi}_{j}(t) - \zeta \partial_{t} \psi_{j}(t) \hat{\psi}_{i}(t) \]  

(5.97)

Again, taking the stochastic average this reduces to

\[ M \left\{ \hat{\mathbf{E}}^{2}(t) \right\} = \mathbf{E}^{2}(t) + \zeta^{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M \left\{ \hat{\psi}_{i}(t) \hat{\psi}_{j}(t) \right\} 
- 2 \zeta^{2} M \left\{ \hat{\psi}_{i}(t) \hat{\psi}_{j}(t) \right\} + \zeta^{2} \sum_{i=1}^{n} M \left\{ \hat{\psi}_{i}(t) \hat{\psi}_{j}(t) \right\} 
= \mathbf{E}^{2}(t) + \zeta^{2} \sum_{i=1}^{n} \delta_{ii} J(0; \zeta) - \zeta^{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij} J(0; \zeta) = \mathbf{E}^{2}(t) \]  

(5.98)

\[ \square \]

6. Cumulant cluster integral expansion

We will use the Euclidean or \( L_{2} \)-norms of the stochastically perturbed fields \( \hat{\psi}_{i}(t) \) and the radii \( \hat{\alpha}_{i}(t) \) and we wish to evaluate the estimates \( M \left\| \hat{\psi}_{i}(t) - \psi_{i}^{E} \right\| \) and \( M \left\| \hat{\alpha}(t) - \alpha_{i}^{E} \right\| \), and then the asymptotic estimate \( \lim_{t \to \infty} M \left\| \hat{\alpha}(t) - \alpha^{E} \right\| \). The expectation of the norm for the moduli is zero

\[ M \left\{ \left\| \hat{\psi}_{i}(t) - \psi_{i} \right\| \right\} = M \left\{ \left( \sum_{i=1}^{n} \left| \hat{\psi}_{i}(t) - \psi_{i} \right|^{2} \right)^{1/2} \right\} 
= M \left\{ \left( \sum_{i=1}^{n} \left| \int_{0}^{t} \hat{\psi}_{i}(\tau) d\tau \right|^{2} \right)^{1/2} \right\} = n^{1/2} \zeta M \left\{ \left| \int_{0}^{t} \hat{\psi}(\tau) d\tau \right| \right\} = 0 \]  

(6.1)
if $\hat{\mathcal{V}}(t) = \hat{V}(t)$ for $i = 1...n$. The asymptotic behavior $\lim_{t \to \infty} \|\hat{a}(t) - a^E\|$ essentially determines whether the initially static Kasner universe with $\hat{a}_i(0) = a^E_i$ is stable or unstable to stochastic perturbations of the static moduli fields $\psi_i^E$.

**Lemma 6.1.** Given the initially static spatial volume of the hyper-toroidal geometry $\mathbb{T}^n$

\[
\nu^E = \prod_{i=1}^n \exp(\psi_i^E) = \exp \left( \sum_{i=1}^n \psi_i^E \right) = \prod_{i=1}^n a_i^E \quad (6.2)
\]

The randomly perturbed spatial volume is

\[
\hat{\nu}_g(t) = \prod_{i=1}^n \exp \left( \psi_i^E + \zeta \int_0^t \hat{\mathcal{V}}_i(\tau) d\tau \right)
\]

\[
= \exp \left( \sum_{i=1}^n \psi_i^E \right) \exp \left( \zeta \sum_{i=1}^n \int_0^t \hat{\mathcal{V}}_i(\tau) d\tau \right)
\]

\[
= |\nu^E| \exp \left( \zeta \sum_{i=1}^n \int_0^t \hat{\mathcal{V}}_i(\tau) d\tau \right)
\]

The stochastic expectation is then

\[
\nu(t) = M \left\{ \hat{\nu}_g(t) \right\} = |\nu^E| M \left\{ \exp \left( \zeta \sum_{i=1}^n \int_0^t \hat{\mathcal{V}}_i(\tau) d\tau \right) \right\}
\]

\[
= |\nu^E| M \left\{ \exp \left( n \zeta \int_0^t \hat{\mathcal{V}}(\tau) d\tau \right) \right\}
\]

(6.4)

if $\psi_i^E = \psi^E$. Asymptotic stability of the spatial volume then occurs if

\[
\lim_{t \to \infty} M \left\{ \hat{\nu}_g(t) \right\} = \lim_{t \to \infty} |\nu^E| M \left\{ \exp \left( n \zeta \int_0^t \hat{\mathcal{V}}(\tau) d\tau \right) \right\} < \infty
\]

while instability occurs if

\[
M \left\{ \hat{\nu}_g(t) \right\} = \lim_{t \to \infty} |\nu^E| M \left\{ \exp \left( n \zeta \int_0^t \hat{\mathcal{V}}(\tau) d\tau \right) \right\} = \infty
\]

(6.5)

The norm of the randomly perturbed metric is

\[
\|\hat{g}(t)\|_{(2,1)} = \sum_{j=1}^n \left( \sum_{i=1}^n |\hat{g}_{ij}(t)|^2 \right)^{1/2} = \sum_{i=1}^n \left( \sum_{i=1}^n |\hat{g}_{ii}(t)|^2 \right)^{1/2}
\]

\[
= \sum_{i=1}^n \left( \sum_{i=1}^n |\exp(2\psi_i(t))|^2 \right)^{1/2}
\]

\[
= \sum_{i=1}^n \left( \sum_{i=1}^n \delta^{ii} \exp(2\psi_i^E) \exp \left( 2 \zeta \int_0^t \hat{\mathcal{V}}_i(\tau) d\tau \right) \right)^{1/2}
\]

(6.7)

The stochastic average is then

\[
M \left\{ \|\hat{g}(t)\|_{(2,1)} \right\} = \sum_{i=1}^n M \left\{ \left( \sum_{i=1}^n \delta^{ii} \exp(2\psi_i^E) \exp \left( 2 \zeta \int_0^t \hat{\mathcal{V}}_i(\tau) d\tau \right) \right)^{1/2} \right\}
\]

\[
< \sum_{i=1}^n \sum_{i=1}^n \delta^{ii} \exp(2\psi_i^E) M \left\{ \exp \left( 2 \zeta \int_0^t \hat{\mathcal{V}}_i(\tau) d\tau \right) \right\}
\]
For the stochastically perturbed radii $\hat{a}_i(t)$, the $L_2$ norm is estimated as

$$\|\hat{a}(t) - a^E\| \leq \left(n|a^E|^2 \exp \left(\zeta \int_0^t \hat{U}(\tau) d\tau\right)\right)^{1/2}$$

$$= n^{1/2}a^E \exp \left(\zeta \int_0^t \hat{U}(\tau) d\tau\right)$$

(6.9)

where $\hat{U}(t) = \hat{U}(t)$ for $i = 1...n$ and $a_i(t)$ is a solution of $D_n a_i(t) = 0$. The expectation is then estimated as

$$M \left\{ \|\hat{a}(t) - a^E\| \right\} \leq n^{1/2}a^E M \left\{ \exp \left(\zeta \int_0^t \hat{U}(\tau) d\tau\right) \right\}$$

(6.10)

Then

1. If $\lim_{t \to \infty} M \left\{ \|\hat{a}(t) - a^E\| \right\} = 0$, then the initially static toroidal universe is asymptotically stable to the random perturbations.
2. If $\lim_{t \to \infty} M \left\{ \|\hat{a}(t) - a^E\| \right\} = 0$, then the initially static Kasner universe is ‘Lyapunov stable’ to the random perturbations.
3. If $\lim_{t \to \infty} M \left\{ \|\hat{a}(t) - a^E\| \right\} = 0$ then the initially static Kasner universe is unstable to the random perturbations and will undergo a stochastically induced expansion to infinity.

The asymptotic behavior of the norms then requires the estimation of the stochastic integral. In particular, it will be shown that the stochastically induced expansion is exponential in nature so that the expanding universe essentially inflates from a static (non-singular) Kasner state.

**Theorem 6.3.** Setting $\hat{U}_i(t) = \hat{U}(t)$ for $i = 1...n$ for Gaussian random fields with $M\{\hat{U}(t)\} = 0$ and defined by the regulated 2-point function, the stochastic integral in (6.8) can be estimated as

$$Y(t) = M \left\{ \exp \left(\zeta \int_0^t \hat{U}(\tau) d\tau\right) \right\} \sim \exp \left(\frac{1}{2} \hat{a}^E \int_0^t \int_0^{r_1} d\tau_1 d\tau_2 \left\{ \hat{U}(\tau_1) \hat{U}(\tau_2) \right\} \right)$$

(6.11)

**Proof.** The proof depends on evaluating the stochastic integral

$$Y(t) = M \left\{ \exp \left(\zeta \int_0^t \hat{U}(\tau) d\tau\right) \right\} \equiv M \left\{ \exp \left(\zeta \int_0^t \hat{U}(\tau) d\tau\right) \right\}$$

(6.12)

by a cluster expansion method or Van Kampen expansion, similar to cluster integral techniques used in stochastic analysis and statistical mechanics [13,66,67]. First, the $m$-point correlations or moments $M\{t;m\}$ and the $m$-point cumulants $C(t;m)$ are

$$M\{t;m\} = M \left\{ \hat{U}(t_1) \times ... \hat{U}(t_m) \right\} = M \left\{ \prod_{\xi=1}^m \hat{U}(t_\xi) \right\}$$

(6.13)

$$C(t;m) = C \left\{ \hat{U}(t_1) \times ... \hat{U}(t_m) \right\} = C \left\{ \prod_{\xi=1}^m \hat{U}(t_\xi) \right\}$$

(6.14)

The second-order cumulants are for example

$$C \left\{ \hat{U}(t_1) \hat{U}(t_2) \right\} = M \left\{ \hat{U}(t_1) \hat{U}(t_2) \right\} + M \left\{ \hat{U}(t_1) \right\} M \left\{ \hat{U}(t_2) \right\}$$

(6.15)

so that $C\{\hat{U}(t_1) \hat{U}(t_2)\} = M\{\hat{U}(t_1) \hat{U}(t_2)\}$ if $\{\hat{U}(t_1)\} = 0$. The moment and cumulant $m$-point correlations can be related to the generating functions for the stochastic process $\hat{U}(t)$

$$\hat{\Psi}[\hat{U}(t)] = \sum_{m=0}^{\infty} \frac{\hat{m}^m}{m!} \int_0^t \int_0^{t-1} ... \int_0^{t-m} d\tau_1 ... d\tau_m C \left\{ \prod_{\xi=0}^m \hat{U}(\tau_\xi) \right\}$$

(6.15)
\[ \equiv \sum_{m=0}^{\infty} \frac{\xi^m}{m!} \prod_{\xi=1}^{m} \int d\tau_{\xi} \mathcal{C} \left\{ \prod_{\xi=0}^{m} \mathcal{U} \left( \tau_{\xi} \right) \right\} \] (6.16)

while the cumulant-generating functional is

\[ \Phi \left[ \mathcal{U} \left( t \right) \right] = \sum_{m=1}^{\infty} \frac{\xi^m}{m!} \prod_{\xi=1}^{m} \int_{0}^{t} \int_{0}^{t_{m-1}} \cdots \int_{0}^{t_{1}} d\tau_{1} \cdots d\tau_{m} \mathcal{C} \left\{ \prod_{\xi=1}^{m} \mathcal{U} \left( \tau_{\xi} \right) \right\} \]
\[ \equiv \sum_{m=1}^{\infty} \frac{\xi^m}{m!} \prod_{\xi=1}^{m} \int d\tau_{\xi} \mathcal{C} \left\{ \prod_{\xi=1}^{m} \mathcal{U} \left( \tau_{\xi} \right) \right\} \] (6.17)

Note that the first summation begins form \( m = 0 \) whereas the second begins from \( m = 1 \). These can be written more succintly as

\[ \Psi \left[ \hat{\mathcal{U}} \left( t \right) \right] = \sum_{m=0}^{\infty} \frac{\xi^m}{m!} \int \mathcal{D}_{m}[\tau] \mathcal{M} \left\{ \prod_{\xi=1}^{m} \mathcal{U} \left( \tau_{\xi} \right) \right\} \] (6.18)

\[ \Phi \left[ \hat{\mathcal{U}} \left( t \right) \right] = \sum_{m=1}^{\infty} \frac{\xi^m}{m!} \int \mathcal{D}_{m}[\tau] \mathcal{C} \left\{ \prod_{\xi=1}^{m} \mathcal{U} \left( \tau_{\xi} \right) \right\} \] (6.19)

where \( \int \mathcal{D}_{m}[\tau] \) is a 'path integral'. But the moment-generating functional is equal to the integral

\[ \Psi \left[ \mathcal{U} \left( t \right) \right] = \exp \left( \int_{0}^{t} \mathcal{U} \left( \tau \right) d\tau \right) \] (6.20)

The relation between the generating functionals is

\[ \Phi = \ln \Psi \] (6.21)

so that \( \Psi = \exp(\Phi) \). Hence

\[ \Psi \left[ \hat{\mathcal{U}} \left( t \right) \right] = \exp \left( \int_{0}^{t} \mathcal{U} \left( \tau \right) d\tau \right) \]
\[ \equiv \sum_{m=0}^{\infty} \frac{\xi^m}{m!} \int \mathcal{D}_{m}[\tau] \mathcal{M} \left\{ \prod_{\xi=1}^{m} \mathcal{U} \left( \tau_{\xi} \right) \right\} \] (6.22)

or equivalently using (6.18)

\[ \Psi \left[ \mathcal{U} \left( t \right) \right] = \exp \left( \int_{0}^{t} \mathcal{U} \left( \tau \right) d\tau \right) \]
\[ \equiv \sum_{m=0}^{\infty} \frac{\xi^m}{m!} \int \mathcal{D}_{m}[\tau] \mathcal{C} \left\{ \prod_{\xi=1}^{m} \mathcal{U} \left( \tau_{\xi} \right) \right\} \] (6.23)

Expanding (6.23)

\[ \mathcal{M} \left\{ \exp \left( \int_{0}^{t} \mathcal{U} \left( \tau \right) d\tau \right) \right\} = \exp \left( \int \mathcal{D}_{1}[\tau] \mathcal{C} \left\{ \hat{\mathcal{U}} \left( \tau \right) \right\} \right) \]
\[ + \frac{\xi^2}{2} \int \mathcal{D}_{2}[\tau] \mathcal{M} \left\{ \mathcal{U} \left( \tau_{1} \right) \mathcal{U} \left( \tau_{2} \right) \right\} \]
\[ + \cdots + \int \mathcal{D}_{m}[\tau] \mathcal{M} \left\{ \mathcal{U} \left( \tau_{1} \right) \times \cdots \times \mathcal{U} \left( \tau_{m} \right) \right\} \] (6.24)
Although there are some subtle technical issues regarding temporal ordering \cite{68,69} the series can be truncated at second order for a Gaussian process with \( M\{ \hat{W}(t) \} = 0 \) so that
\[
M \left\{ \exp \left( \zeta \int_0^t \hat{W}(\tau)d\tau \right) \right\} = \exp \left( \frac{1}{2} \zeta^2 \int_0^t \int_0^\tau C \left\{ \hat{W}(\tau_1)\hat{W}(\tau_2) \right\} d\tau_1 d\tau_2 \right) = \exp \left( \frac{1}{2} \zeta^2 \int_0^t \int_0^\tau C \left\{ \hat{W}(\tau_1)\hat{W}(\tau_2) \right\} d\tau_1 d\tau_2 \right) (6.25)
\]
which is
\[
M \left\{ \exp \left( \zeta \int_0^t \hat{W}(\tau)d\tau \right) \right\} = \exp \left( \frac{1}{2} \zeta^2 \int_0^t \int_0^\tau d\tau_1 d\tau_2 M \left\{ \hat{W}(\tau_1)\hat{W}(\tau_2) \right\} \right) (6.26)
\]
Choosing \( \zeta = 1 \) gives
\[
M \left\{ \exp \left( \int_0^t \hat{W}(\tau)d\tau \right) \right\} = \exp \left( \frac{1}{2} \int_0^t \int_0^\tau d\tau_1 d\tau_2 M \left\{ \hat{W}(\tau_1)\hat{W}(\tau_2) \right\} \right) (6.27)
\]
and so the result follows. \( \square \)

6.1. Results for some classical regulated 2-point functions. The norm \( M\{ \| \hat{a}(t) - a^E \| \} \) and its asymptotic behavior can be computed for viable regulated 2-point functions, such as that for 'colored noise' or for Gaussian-correlated noise.

\textbf{Theorem 6.4.} Let the conditions of Theorem (5.1) hold for an initially static or stationary and isotropic hypertoroidal spacetime such that \( a_0(t) = a^E = a^E \) and \( \psi(t) = \psi^E = \psi^E \) so that the Einstein equations are \( D_n a^E = H_n \psi^E 0 \). Introducing Gaussian stochastic perturbations \( \hat{W}(t) \) of the moduli then \( \hat{\psi}(t) = \psi^E + \int_0^t \hat{W}(\tau)d\tau \). The radii are \( \hat{a}_i(t) = a^E \exp(\int_0^t \hat{W}(\tau,\varsigma)d\tau) \).

(1) If the regulated 2-point function of the random perturbations is of the colored noise or Ornstein-Uhlenbeck form with correlation \( \varsigma \) then set \( \hat{W}(t) = \bar{W}(t) \) so that the regulated 2-point function is
\[
J(\Delta; \varsigma) = M \left\{ \hat{W}(t)\hat{W}(s) \right\} = \frac{C}{\varsigma} \exp \left( -\frac{|t - s|}{\varsigma} \right) \quad (6.28)
\]
\[
J(0; \varsigma) = M \left\{ \hat{W}(t)\bar{W}(t) \right\} = \frac{C}{\varsigma} \quad (6.29)
\]
where \( \varsigma \) is the correlation time and \( \bar{W}(t) \) is a solution of the (Ornstein-Uhlenbeck) linear stochastic DE
\[
\partial_t \bar{W}(t) = -\frac{1}{\varsigma} \bar{W}(t) + \frac{1}{\varsigma} \bar{W}(t) = \quad (6.30)
\]
and \( \bar{W}(t) \) is a white noise with \( M\{ \hat{W}(t)\bar{W}(t) \} = \alpha \delta(t - s) \). The expectation of the randomly perturbed norms then evolve as
\[
\delta M\left\{ \| \hat{a}_i(t) \| \right\} = M\left\{ \| \hat{a}_i(t) - a_i^E \| \right\} \leq a^E n^{1/2} M\left\{ \exp \left( \int_0^t \bar{W}(\tau)d\tau \right) \right\} \nonumber \\
\leq a^E n^{1/2} \exp (Ct - C \exp(-\beta t) + C) \sim a^E n^{1/2} \exp(Ct) \quad (6.31)
\]
for large \( t \).

(2) If the random perturbations or noise is Gaussian correlated then \( M\{ \hat{W}(t) \} = 0 \) with 2-point function
\[
J(\Delta; \varsigma) = M \left\{ \hat{W}(t)\hat{W}(s) \right\} = \frac{C}{\varsigma^2} \exp \left( -\frac{|t - s|^2}{\varsigma^2} \right) \quad (6.32)
\]
with \( J(0; \varsigma) = M\{ \hat{W}(t)\hat{W}(s) \} = \frac{C}{\varsigma^2} \). Then similarly for \( t \gg \varsigma \)
\[
\delta M\left\{ \| \hat{a}_i(t) \| \right\} = M\left\{ \| \hat{a}_i(t) - a_i^E \| \right\} \sim a^E n^{1/2} \exp(\lambda t) \quad (6.33)
\]
Then on average, in both cases, the randomly perturbed radii then evolve exponentially or 'inflate.'
Proof. The averaged perturbed Einstein equations follow from

\[
M \{ \mathbf{D}_n \tilde{\mathbf{a}}(t) \} = \frac{1}{2} \xi^2 n \Sigma(0; \varsigma) \equiv \frac{1}{2} \xi^2 n \frac{C}{\varsigma} \equiv \lambda
\]  

(6.34)

The expectation of the stochastically perturbed norms is estimated as

\[
M\left\{ \left\| \delta_n \tilde{\mathbf{a}}(t) \right\| \right\} = M\left\{ \left\| \mathbf{a}(t) - \mathbf{a}^E \right\| \right\}
\]

\[
\leq a^E n^{1/2} \mathbf{M} \left\{ \exp \left( \int_0^t \tilde{\mathcal{W}}(\tau) d\tau \right) \right\}
\]

\[
= a^E n^{1/2} \exp \left( \frac{1}{2} \xi^2 \int_0^t \int_0^{\tau_1} d\tau_1 d\tau_2 \mathbf{M} \left\{ \tilde{\mathcal{W}}(\tau_1) \tilde{\mathcal{W}}(\tau_2) \right\} \right)
\]

\[
a^E n^{1/2} \exp \left( \frac{C\varsigma^2}{2} \int_0^t \int_0^{\tau_1} d\tau_1 d\tau_2 \exp \left( -\frac{\tau_1 - \tau_2}{\varsigma} \right) \right)
\]

\[
a^E n^{1/2} \exp \left( \frac{C\varsigma^2}{2} \int_0^t \int_0^{\tau_1} d\tau_1 d\tau_2 \exp \left( -\frac{t}{\varsigma} \right) \right)
\]

\[
a^E n^{1/2} \exp \left( \frac{1}{2} \xi^2 C t + \frac{1}{2} \gamma^2 C (1 - \exp(-t/\varsigma)) \right)
\]

(6.35)

For large \( t \gg \varsigma \), the estimate is

\[
M\left\{ \delta \left\| \tilde{\mathbf{a}}(t) \right\| \right\} \sim M\left\{ \left\| \tilde{\mathbf{a}}(t) - \mathbf{a}^E \right\| \right\}
\]

\[
= a^E n^{1/2} \exp \left( \frac{1}{2} \xi^2 C t \right) \equiv a^E n^{1/2} \exp(Qt)
\]  

(6.36)

For the Gaussian-correlated 2-point function

\[
M\left\{ \left\| \delta_n \tilde{\mathbf{a}}(t) \right\| \right\} = M\left\{ \left\| \tilde{\mathbf{a}}(t) - \mathbf{a}^E \right\| \right\}
\]

\[
\leq a^E n^{1/2} \mathbf{M} \left\{ \exp \left( \int_0^t \tilde{\mathcal{W}}(\tau) d\tau \right) \right\}
\]

\[
= a^E n^{1/2} \exp \left( \frac{1}{2} \xi^2 \int_0^t \int_0^{\tau_1} d\tau_1 d\tau_2 \mathbf{M} \left\{ \tilde{\mathcal{W}}(\tau_1) \tilde{\mathcal{W}}(\tau_2) \right\} \right)
\]

\[
a^E n^{1/2} \exp \left( \frac{1}{2} \xi^2 \int_0^t \int_0^{\tau_1} d\tau_1 d\tau_2 \exp \left( -\frac{\tau_1 - \tau_2}{\varsigma^2} \right) \right)
\]

\[
a^E n^{1/2} \exp \left( \frac{1}{2} \xi^2 \int_0^t \int_0^{\tau_1} d\tau_1 d\tau_2 \exp \left( -\frac{t}{\varsigma} \right) \right)
\]

\[
a^E n^{1/2} \exp \left( \frac{1}{4} C \mu^2 \int_0^t \int_0^{\tau_1} \frac{\varsigma}{\pi^{1/2}} \exp(-t^2/\varsigma^2) d\tau_1 d\tau_2 \exp(f(\tau_1/\varsigma)) \right)
\]

\[
a^E n^{1/2} \exp(1/2 C \mu^2 f(t/\varsigma) + \frac{1}{2} C \mu^2 \exp(-t^2/\varsigma^2))
\]

(6.37)

and for large \( t \gg \varsigma \) one has \( \exp(f(t/\varsigma)) = 1 \) and \( \exp(-t^2/\varsigma^2) = 0 \) so that

\[
M\left\{ \left\| \delta_n \tilde{\mathbf{a}}(t) \right\| \right\} = M\left\{ \left\| \tilde{\mathbf{a}}(t) - \mathbf{a}^E \right\| \right\} \sim a^E n^{1/2} \exp(1/2 \mu^2 t) = a^E n^{1/2} \exp(Qt)
\]  

(6.38)

\[\square\]
Corollary 6.5. Asymptotically, the system is then unstable to the stochastic perturbations since

$$\lim_{t \to \infty} M\left\{ \| \delta \hat{a}(t) \| \right\} = \lim_{t \to \infty} M\left\{ \| \hat{a}(t) - a^E \| \right\} \sim \lim_{t \to \infty} a^E n^{1/2} \exp(\frac{1}{2} \zeta^2 Ct) = \infty$$

so the system grows exponentially or undergoes a noise-induced inflation for eternity. Also

$$M\left\{ \delta \| \hat{a}_i(t) \| \right\} \| a^E \|^{-1} \equiv M\left\{ \| \hat{a}_i(t) - a^E \| \right\} \| a^E \|^{-1} \sim \exp(Qt)$$

so that $Q$ plays the role of a positive Lyapunov exponent.

Since the stochastic integral has been estimated, one can reprise Theorem 2.18. for the probability of instability in terms of a Chernoff bound

**Corollary 6.6.** The probability that $\lim_{t \to \infty} \| \hat{a}(t) - a^E \|$ is bounded by any finite $|L|$ is zero-hence the randomly perturbed static Kasner universe is unstable and expands forever. From (2.79) $\exists \xi > 0$ such that

$$\lim_{t \to \infty} P[\| \hat{a}(t) - a^E \| \leq |L|]$$

$$\leq \liminf_{t \to \infty} \exp(\beta |L|) M\left\{ \exp\left( -\beta \| a(t) - a^E \| \right) \right\}$$

$$= \liminf_{t \to \infty} \exp(\beta |L|) M\left\{ a^E n^{1/2} \exp\left( \beta \int_0^t \varpi(\tau) d\tau \right) \right\}$$

$$= \liminf_{t \to \infty} \exp(\beta |L|) \exp\left( -\beta |a^E| n^{1/2} \left\| M\left\{ \exp\left( \gamma \int_0^t \varpi(\tau) d\tau \right) \right\} \right\|^{1/\gamma}$$

$$= \liminf_{t \to \infty} \exp(\beta |L|) \exp(-\beta |a^E| n^{1/2} \exp(Qt)) = 0$$

Since

$$\lim_{t \to \infty} P[\| \hat{a}_i(t) - a^E_i \| \leq |L|] + \lim_{t \to \infty} P[\| \hat{a}_i(t) - a^E_i \| > |L|] = 1$$

then

$$\lim_{t \to \infty} P[\| \hat{a}_i(t) - a^E_i \| > |L|] = 1$$

$$\lim_{t \to \infty} P[\| \hat{a}_i(t) - a^E_i \| \leq |L|] = 0$$

so the randomly perturbed system is asymptotically unstable in probability and never reaches equilibrium since the perturbed norm can never be contained with any ball of radius $L$.

The same conclusion follows from the Hoeffding Lemma of Proposition 2.22. **Lemma 6.7.** As before, let $(\hat{a}_i(t))_{i=1}^n = (\hat{a}_1(t)...\hat{a}_n(t))$ be the set of randomly perturbed radii due to random perturbations of the initially static equilibria $a^E_i$ such that $\hat{a}_i(t) = a^E_i \exp(\zeta \int_0^t \varpi_i(s) ds)$. Then $\hat{a}_i(t)$ solves the averaged Einstein equation such that $M\{ D_n \hat{a}(t) \} = \lambda$. If the the randomly perturbed actually radii converged to `attractors' or new equilibria within a finite time such that $\hat{a}_i(t) \to a_{E_i}^*$, then $a_{E_i}^* \leq \hat{a}_i(t) \leq a_{E_i}^*$ for all finite $t > 0$. If

$$\hat{S}(t) = \frac{1}{n} \sum_{i=1}^n \hat{a}_i(t) = \frac{1}{n} (\hat{a}_1(t) + ... + \hat{a}_n(t))$$

$$M\{ \hat{S}(t) \} = \frac{1}{n} \sum_{i=1}^n M\{ \hat{a}_i(t) \}$$

If however, $\nu_{E_i}^* \to \infty$ the the random variables $\hat{a}_i(t)$ are no longer bounded. The estimate is then

$$P(\hat{S}(t) - M\{ \hat{S}(t) \} \geq |L|)$$

$$= P\left( \frac{1}{n} \left\| \sqrt{a^E_i} \exp\left( \frac{\zeta}{2} \int_0^t \varpi_i(s) ds \right) \right\| \geq |L| \right)$$

$$= \lim_{a_{E_i}^* \to \infty} \exp(-2n^2 L^2 \left\| a_{E_i}^* - a_i^E \right\|^{-2}) = 1$$
Given the Einstein equations

Lemma 6.9. From (6.34) or (6.36),\( a_i \) and there is then unit probability that the growth of the norm of perturbed solutions cannot be contained within any finite \( L > 0 \). Hence, the system is asymptotically unstable in probability to the random perturbations.

The averaged solution also satisfies the Einstein equations with a cosmological constant.

**Lemma 6.8.** Let \( A_i(t) = M\{a_i^+(t)\} = a^E_i \exp(Q^{1/2}t) \), which is the stochastic average as computed from (6.34) and Thm. (6). Then \( A^+ \) is a solution of the system of deterministic nonlinear ODES

\[
D_n A_i(t) = \sum_{i=1}^{n} \frac{\partial_t a_i(t)}{a_i^+(t)} - \frac{1}{2} \sum_{i=1}^{n} \frac{\partial_t A_i(t) \partial_t A_i(t)}{A_i(t)A_i(t)} + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial_t A_i(t) \partial_t A_j(t)}{A_i(t)A_j(t)} = \lambda = nV^2
\]

where \( \lambda \) is a cosmological constant term and \( Q_i = Q \). This is equivalent to the system of ODES

**Proof.** From (6.34) or (6.36), \( A_i(t) = a^E_i \exp(Qit) \). The derivatives are: \( \partial_t A^+_i(t) = a^E_i Q_i \exp(Qit) \) and \( \partial_t A^+_i(t) = a^E_i Q_i \). \( \square \)

### 6.2. Lyapunov exponents and relation to induced cosmological constant terms

The final result establishes that the perturbed Einstein vacuum equations of (4.97) and (4.98) for a constant perturbation, namely \( D_n a_i(t) \) and the stochastically averaged Einstein equations \( M\{D_n \bar{a}_i(t)\} \) are equivalent in that an initially static toroidal universe expands exponentially or inflates under either of these perturbations.

**Lemma 6.9.** Given the Einstein equations \( D_n a^E(t) = H_n \psi(t) = 0 \) describing a static hypertoroidal micro-universe, the continuous deterministic and stochastic perturbations are

\[
\bar{a}_i(t) = a^E \exp \left( \int^t A d\tau \right)
\]

\[
\hat{a}_i(t) = a^E \exp \left( \int^t \hat{A}(\tau) d\tau \right)
\]

arising from \( \bar{a}_i(t) = \psi^E + \int^t A d\tau = \psi^E + A t \) and \( \hat{a}_i(t) = \psi^E + \int^t \hat{A}(\tau) d\tau \). Using previous results it follows that we have the equivalent estimates

\[
\frac{\|\bar{a}(t)\|}{a^E} = \frac{\|\bar{a}(t) - a^E\|}{a^E} \sim \exp(Qt) \equiv \exp(Ly_1t)
\]

\[
M\left\{ \frac{\|\hat{a}(t)\|}{a^E} \right\} \equiv M\left\{ \frac{\|\hat{a}(t) - a^E\|}{a^E} \right\} \sim \exp(Qt) \equiv \exp(Ly_2t)
\]

It follows that \( Ly_1, Ly_2 \) are essentially Lyapunov exponents so that

\[
Ly_1 \sim \lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{\|\bar{a}(t) - a^E\|}{\|a^E\|} \right)
\]

\[
Ly_2 \sim \lim_{t \rightarrow \infty} \frac{1}{t} \ln M\left( \|\hat{a}(t) - a^E\| \right) \sim \lim_{t \rightarrow \infty} \frac{1}{t} \ln M\left( \|\hat{a}(t) - a^E\| \right)
\]

The perturbed radii can be shown to be equivalent to the characteristic function, and the \( \ell^{th} \)-order Lyapunov exponent (LE) can be estimated as follows.
Lemma 6.10. let $a^E$ be a set of equilibrium solutions of the $n$-dimensional Einstein system $D_n a_i(t) = 0$, and as before let $\widehat{a}_i(t)$ be the randomly perturbed solutions which are a solution of the averaged Einstein system $D_n a_i(t) = \lambda$ as in (3.72). Then

$$
\operatorname{Ly}(iz) = \operatorname{Ly}(\ell) \sim n^{\ell/2}|u^E|\lambda > 0
$$

(6.56)

where $\Psi(-i\ell) = \Psi(z)$ is the characteristic function and $\operatorname{Ly}(iz)$ is the LE.

Proof.

$$
\Psi(z) = M\{\exp(i\bar{z}\ln(\bar{a}(t) - a^E))\}
$$

(6.57)

where $z = -i\ell$. The LE is estimated as

$$
\operatorname{Ly}(iz) = \lim_{t \to \infty} \frac{1}{t} \ln \left\{ \exp(i\bar{z}\ln(\bar{a}(t) - a^E)) \right\}
$$

$$
= \lim_{t \to \infty} \frac{1}{t} \ln M\{\exp(i\bar{z}\ln(\bar{a}^E) + i\bar{z}\varpi(s))\}
$$

$$
= \lim_{t \to \infty} \frac{1}{t} \ln M\{\exp(i\bar{z}\ln(\bar{a}^E) + i\bar{z}\varpi(s))\}
$$

(6.58)

where $z = -i\ell$, and $Q > 0$ implies instability.

6.3. Alternative Estimate. As a consistency check, it can also be shown that the estimate

$$
M\{\left\| \bar{a}(t) - a^E \right\| \sim n^{\ell/2}a^E \exp(Q|t - t_o|) \}
$$

(6.59)

also arises via a different method, when only certain conditions are imposed on the covariance of the field $\varpi_i(t)$.

Theorem 6.11. Let $\widehat{\varpi}_i(t) = \varpi_i(t)$ be a Gaussian random field such that $M\{\widehat{\varpi}_i(t)\} = 0$ and $M\{\widehat{\varpi}(t)\varpi(s)\} = J(\Delta; \varsigma) \equiv J(s - t; \varsigma)$. The following hold:
Then the following estimates can be made
\[
M\{\|\hat{\mathcal{U}}(t)\|_2^2\} \leq C_1 \int_{t_0}^{\infty} M\{\|\hat{\mathcal{U}}(t)\hat{\mathcal{U}}(s)\|\} = \int_{t_0}^{\infty} J(t-s;\zeta)ds \leq C_2 \tag{6.60}
\]
(2) The random field \(\hat{\mathcal{U}}(t)\) has a Fourier expansion
\[
\hat{\mathcal{U}}(t) = \sum_{\xi=1}^{\infty} (E_{\xi})^{1/2} \varphi(t)\hat{\mathcal{U}}(\xi) \tag{6.61}
\]
where \(M\{\hat{\mathcal{U}}(\xi)\hat{\mathcal{U}}(\xi)\} = 1\) and where \(E_{\xi}\) and \(\varphi(t)\) are the normalised eigenfunctions and eigenvalues such that \(|\int_{t_0}^{t} J(t-s;\zeta)|\varphi(s)ds = E \varphi(t)\) and where
\[
\int_{t_0}^{t} |\hat{\mathcal{U}}(s)|^2 ds = \sum_{\xi=1}^{\infty} E_{\xi}|\hat{\mathcal{U}}(\xi)|^2 \tag{6.62}
\]
is the Parsaval identity.

(3)
\[
M\left\{\exp\left(\beta E_{\xi}|\hat{\mathcal{U}}(\xi)|^2\right)\right\} = (1 - 2\beta E_{\xi})^{-1/2} \tag{6.63}
\]

for \(\beta < (2E_{\xi})^{-1}\)

Then the following estimates can be made
\[
M\left\{\exp\left(\beta \int_{t_0}^{t} |\hat{\mathcal{U}}(t)|^2 dt\right)\right\} = \prod_{\xi=1}^{\infty} (1 - 2\beta E_{\xi})^{-1} \tag{nonumber}
\]
\[
M\left\{\left\|a(t) - a^E\right\|\right\} = n^{1/2} a^E \exp\left(\zeta \int_{t_0}^{t} \hat{\mathcal{U}}(s)ds\right) \tag{6.64}
\]
\[
< n^{1/2} a^E \exp\left(\zeta [C_1^{1/2} + \frac{1}{2}\zeta C_2]|t - t_0|\right) \tag{6.65}
\]
\[
= n^{1/2} a^E \exp(Q|t - t_0|)
\]
which agrees with the estimate (6.38).

Proof. Using (6.60) and (6.61) it follows that
\[
\sum_{\xi=1}^{\infty} E_{\xi} = M\left\{\int_{t_0}^{t} \left|\hat{\mathcal{U}}(s)\right|^2\right\} \leq C_1|t - t_0| \tag{6.66}
\]
Now \(E_{\max} = \sup_{\xi}(E_{\xi})_{\xi=1}^{\infty}\) and we can choose \(E = E_1\). Using the Parsaval identity
\[
M\left\{\exp\left(\beta \int_{t_0}^{t} |\hat{\mathcal{U}}(t)|^2 dt\right)\right\} = M\left\{\exp\left(\sum_{\xi=1}^{\infty} \beta E_{\xi}|\hat{\mathcal{U}}(\xi)|^2\right)\right\} \tag{6.67}
\]
\[
\prod_{\xi=1}^{\infty} M\left\{\exp\left(\beta E_{\xi}|\hat{\mathcal{U}}(\xi)|^2\right)\right\} = \prod_{\xi=1}^{\infty} (1 - 2\beta E_{\xi})^{-1/2}
\]
Next
\[
E_1 \leq \int_{t_0}^{t} \int_{t_0}^{t} \|J(s, \tau; \zeta)||\varphi_1(t)\varphi_1(s)dsd\tau
\]
\[
\leq \frac{1}{2} \int_{t_0}^{t} \int_{t_0}^{t} \|J(s, \tau; \zeta)||\varphi_1(t)^2 + \varphi_2(t)^2|dsdt \leq C_2 \tag{6.68}
\]
so that $\mathcal{E}_1 \leq C_2$. Using the basic inequality $(1 + x) < \exp(x)$ for $x > 0$

\[
M\left\{ \exp \left( \beta \int_{t_o}^{t} |\widehat{\mathcal{W}}(t)|^2 dt \right) \right\} = \prod_{\xi=1}^{\infty} (1 - 2\beta\mathcal{E}_\xi)^{-1/2}
\]

\[
= \prod_{\xi=1}^{\infty} (1 + 2\beta\mathcal{E}_\xi + 4\beta^2\mathcal{E}_\xi^2(1 - 2\beta\mathcal{E}_\xi)^{-1})^{1/2}
\]

\[
= \exp \left( \beta(1 - 2\beta\mathcal{E}_1)^{-1} \sum_{\xi=1}^{\infty} \mathcal{E}_\xi \right)
\]

\[
\leq \exp(\beta(1 - 2\beta\mathcal{E}_1)^{-1})C_1|t - t_o|
\]

(6.69)

Finally, applying the basic inequality $x < \frac{1}{2}x^2 + (2\beta)^{-1}$ gives

\[
M\left\{ \left\| \hat{a}(t) - a^E \right\| \right\} = n^{1/2}a^{E}M\left\{ \exp \left( \zeta \int_{t_o}^{t} |\mathcal{W}(s)|^2 ds \right) \right\}
\]

\[
\leq n^{1/2}a^{E} \exp \left( \frac{\zeta}{2\beta} |t - t_o| \right)M\left\{ \exp \left( \frac{1}{2}E_1 \beta \int_{t_o}^{t} |\widehat{\mathcal{W}}(s)|^2 ds \right) \right\}
\]

\[
= n^{1/2}a^{E} \exp \left( \left[ \frac{E_1}{2\beta} + \frac{1}{2}E_1 \beta C_1 (1 - \beta E_1 C_2)^{-1} \right] |t - t_o| \right)
\]

(6.70)

Setting $\beta = \mathcal{E}_1 C_2 + C_1^{1/2})^{-1}$ then gives (-) so that

\[
M\left\{ \left\| \hat{a}(t) - a^E \right\| \right\} = n^{1/2}a^{E}M\left\{ \exp \left( \zeta \int_{t_o}^{t} |\mathcal{W}(s)|^2 ds \right) \right\} \sim \exp(Q|t - t_0|)
\]

(6.71)

which agrees with previous estimates. \(\square\)

6.4. Random perturbations of two Bianchi-I type cosmologies. For dynamic solutions describing a universe that has some dimensions expanding and some collapsing, the average effect of the random perturbations is to 'boost' both the expansion or collapse of those toroidal radii which are already expanding or collapsing. The following lemmas illustrate the averaged asymptotic expansion behavior of some Bianchi-I cosmologies subject to the random perturbations or noise.

**Lemma 6.12.** Given the Bianchi-I triplet $\mathcal{B} = (p_1, p_2, p_3) = (-1/3, 2/3, 2/3)$ for $n=3$ or $\mathbb{T}^3$ the radii collapse and expand as for the Bianchi-I triplet $\mathcal{B} = (-1/3, 2/3, 2/3)$, the rolling radii are

\[
a_1(t) = a_1^E |t|^{-1/3} \equiv a_1(0)|t|^{-1/3}
\]

(6.72)

\[
a_2(t) = a_2^E |t|^{2/3} \equiv a_2(0)|t|^{2/3}
\]

(6.73)

\[
a_3(t) = a_3^E |t|^{2/3} \equiv a_3(0)|t|^{2/3}
\]

(6.74)

where $a_1(0) = a_2(0) = a_3(0)$. The randomly perturbed radii are then

\[
\hat{a}_1(t) = a_1^E |t|^{-1/3} \exp \left( \zeta \int_{0}^{t} |\mathcal{W}(\tau)| d\tau \right)
\]

(6.75)

\[
\hat{a}_2(t) = a_2^E |t|^{-1/3} \exp \left( \zeta \int_{0}^{t} |\mathcal{W}(\tau)| d\tau \right)
\]

(6.76)

\[
\hat{a}_3(t) = a_3^E |t|^{-1/3} \exp \left( \zeta \int_{0}^{t} |\mathcal{W}(\tau)| d\tau \right)
\]

(6.77)

with stochastic expectations

\[
\mathbf{A}_1(t) = M\left\{ \hat{a}_1(t) \right\} = a_1^E |t|^{-1/3}M\left\{ \exp \left( \zeta \int_{0}^{t} |\mathcal{W}(\tau)| d\tau \right) \right\}
\]

(6.78)

\[
\sim a_1^E |t|^{-1/3} \exp(Qt)
\]

(6.79)

\[
\mathbf{A}_2(t) = M\left\{ \hat{a}_1(t) \right\} = a_1^E |t|^{-1/3}M\left\{ \exp \left( \zeta \int_{0}^{t} |\mathcal{W}(\tau)| d\tau \right) \right\}
\]

(6.80)
Hence, the collapse and expansions are boosted by an exponential factor \( \exp(Qt) \). Given the Bianchi-I triplet \( \mathcal{B} = (p_1, p_2, p_3) = (0, 0, 1) \) for \( n=3 \) or \( \mathbb{T}^3 \) one dimension expands and two remain static

\[
\begin{align*}
    a_1(t) &= a_1^E |t|^{-1/3} = a_1(0) = a_1^E \\
    a_2(t) &= a_2^E |t|^{2/3} = a_2(0) = a_2^E \\
    a_3(t) &= a_3^E |t|^{2/3} = a_3(0)|t| = a_3^E |t|
\end{align*}
\]

where \( a_1(0) = a_2(0) = a_3(0) \). The randomly perturbed radii are then

\[
\begin{align*}
    \tilde{a}_1(t) &= a_1^E \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right) \\
    \tilde{a}_2(t) &= a_2^E \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right) \\
    \tilde{a}_3(t) &= a_3^E |t| \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right)
\end{align*}
\]

with stochastic expectations

\[
\begin{align*}
    \mathbf{A}_1(t) &= \mathbf{M}\left\{ \tilde{a}_1(t) \right\} = a_1^E \mathbf{M} \left\{ \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right) \right\} \sim a_1^E \exp(Qt) \\
    \mathbf{A}_2(t) &= \mathbf{M}\left\{ \tilde{a}_2(t) \right\} = a_2^E \mathbf{M} \left\{ \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right) \right\} \sim a_2^E \exp(Qt) \\
    \mathbf{A}_3(t) &= \mathbf{M}\left\{ \tilde{a}_3(t) \right\} = a_3^E |t| \mathbf{M} \left\{ \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right) \right\} \sim a_3^E |t|^{2/3} \exp(Qt)
\end{align*}
\]

The static dimensions are then exponentially boosted and 'inflate', while the expanding dimension now expands faster.

**Lemma 6.13.** The randomly perturbed radii for the Bianchi-I triplet \( (p_1, p_2, p_3) = (-1/3, 2/3, 2/3) \)

\[
\begin{align*}
    \tilde{a}_1(t) &= a_1^E |t|^{-1/3} \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right) \\
    \tilde{a}_2(t) &= a_2^E |t|^{2/3} \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right) \\
    \tilde{a}_3(t) &= a_3^E |t|^{2/3} \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right)
\end{align*}
\]

and the Bianchi-I triplet \( (p_1, p_2, p_3) = (0, 0, 1) \),

\[
\begin{align*}
    \tilde{a}_1(t) &= a_1^E \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right) \\
    \tilde{a}_2(t) &= a_2^E \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right) \\
    \tilde{a}_3(t) &= a_3^E |t| \exp\left( \zeta \int_0^t \hat{\mathcal{H}}(\tau)\,d\tau \right)
\end{align*}
\]

are solutions of the stochastically averaged Einstein vacuum equations such that

\[
\mathbf{M}\left\{ \mathbf{D}_n \tilde{a}_i(t) \right\} = nJ(0; \zeta) \equiv \lambda
\]
Proof. Taking the stochastic expectation, only the nonlinear terms are nonvanishing so that

\[
\text{M} \{ D_n a_i(t) \} = \sum_{i=1}^{n} \text{M} \{ \tilde{\mathcal{W}}_i(t) \tilde{\mathcal{W}}_i(t) \}
\]

\[
- \frac{1}{2} \sum_{i=1}^{3} \text{M} \{ \tilde{\mathcal{W}}_i(t) \tilde{\mathcal{W}}_i(t) \} - \frac{1}{2} \sum_{i=1}^{3} |p_i|^2 t^{-2} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{3} \text{M} \{ \tilde{\mathcal{W}}_i(t) \tilde{\mathcal{W}}_i(t) \}
\]

\[
= \frac{1}{2} \sum_{i=1}^{3} \delta_{ii} \text{M} \{ \tilde{\mathcal{W}}_i(t) \tilde{\mathcal{W}}_i(t) \} - \frac{1}{2} \sum_{i=1}^{3} |p_i|^2 t^{-2} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{3} \delta_{ij} J(0; \varsigma) + \frac{1}{2} \sum_{i=2}^{3} \sum_{j=1}^{3} p_i p_j t^{-2}
\]

\[
= \frac{1}{2} \sum_{i=1}^{3} \delta_{ii} J(0; \varsigma) - \frac{1}{2} \sum_{i=1}^{3} |p_i|^2 t^{-2} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{3} \delta_{ij} J(0; \varsigma) + \frac{1}{2} \sum_{i=2}^{3} \sum_{j=1}^{3} p_i p_j t^{-2}
\]

(6.100)

If \( m_{ij} = p_i p_j \) set \( m_{ij} = p_i p_j = p_i^2 \) if \( i = j \) and \( m_{ij} = 0 \) if \( i \neq j \). Then \( M_{ij} \) is a diagonal matrix.

\[
\text{M} \{ D_n a_i(t) \} = \frac{1}{2} \sum_{i=1}^{3} \delta_{ii} J(0; \varsigma) - \frac{1}{2} \sum_{i=1}^{3} |p_i|^2 t^{-2}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{3} \delta_{ij} J(0; \varsigma) + \frac{1}{2} \sum_{i=2}^{3} \sum_{j=1}^{3} p_i p_j t^{-2}
\]

\[
= \frac{1}{2} \sum_{i=1}^{3} \delta_{ii} J(0; \varsigma) - \frac{1}{2} t^{-2} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{3} \delta_{ij} J(0; \varsigma) + \frac{1}{2} t^{-2} = n J(0; \varsigma) \equiv \lambda
\]

(6.101)

and since

\[
\sum_{i=1}^{n} |p_i|^2 = |\tilde{a}_1|^2 + |\tilde{a}_2|^2 + |\tilde{a}_3|^2 = 1
\]

(6.102)

for \( (p_1, p_2, p_3) = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \) and

\[
\sum_{i=1}^{n} |p_i|^2 = |1|^2 = 1
\]

(6.103)

for \( (p_1, p_2, p_3) = (0, 0, 1) \).  

\[
\begin{align*}
\text{7. Stability of the Einstein System to a Class of Random Perturbations} \\
\end{align*}
\]

In this final section, it is shown that a class of random perturbations of the radial moduli can lead to stability of the Einstein system. We can consider random perturbations of the radial moduli fields of the form \( \tilde{\psi}_i(t) = \psi_i^E + \tilde{\mathcal{W}}(t) \), rather than taking the integral over the field as in (5.2).

**Proposition 7.1.** Let the conditions of Theorem (5.1) hold and also the following:

1. The moduli fields are randomly perturbed as \( \tilde{\psi}(t) = \psi^E + \zeta \tilde{\mathcal{W}}(t) \) so that the perturbed toroidal radii are

\[
\tilde{a}_i(t) = a_i^E + \zeta \tilde{\mathcal{W}}_i(t)
\]

(7.1)

2. The first and second derivatives of the random field exist so that \( \tilde{\mathcal{W}}(t) = \partial_t \tilde{\mathcal{W}}(t) \) and \( \partial_t \tilde{\mathcal{W}}(t) = \partial_t \tilde{\mathcal{W}}(t) \) with \( \text{M} \{ \tilde{\mathcal{W}}(t) \} = 0 \)

3. The 2-point function of the fields is of the form

\[
\text{M} \{ \tilde{\mathcal{W}}_i(t) \tilde{\mathcal{W}}_j(s) \} = \delta_{ij} \Xi(\Delta; \varsigma)
\]

(7.2)

\[
\text{M} \{ \tilde{\mathcal{W}}_i(t) \tilde{\mathcal{W}}_j(s) \} = \delta_{ij} J(\Delta; \varsigma)
\]

(7.3)

where \( \Delta = |t - s| \) and is regulated so that \( \text{M} \{ \tilde{\mathcal{W}}(t) \tilde{\mathcal{W}}(s) \} = \delta_{ij} \Xi(0; \varsigma) < \infty. \)
Then the stochastically averaged Einstein vacuum equations are

\[ M \left\{ H_n \hat{\psi}(t) \right\} = n \zeta^2 \Xi(0; \zeta) \equiv \lambda \]  

(7.4)

\[ M \left\{ D_n \hat{\alpha}(t) \right\} = n \zeta^2 \Xi(0; \zeta) \equiv \lambda \]  

(7.5)

and the perturbed norm has the asymptotic behavior

\[ \lim_{t \to \infty} M \left\{ \left\| a(t) - a^E \right\| \right\} \leq \lim_{t \to \infty} a^E n^{1/2} M \left\{ \exp(\zeta \hat{\mathcal{W}}(t)) \right\} \]  

(7.6)

**Proof.** The derivatives are \( \partial_t \hat{\psi}(t) = \zeta \partial_t \mathcal{W}(t) = \zeta \mathcal{F}(t) \) and \( \partial_{tt} \hat{\psi}(t) = \zeta \partial_t \mathcal{W}(t) = \zeta \partial_{tt} \mathcal{F}(t) \). The randomly perturbed Einstein equations are

\[ H_n \hat{\psi}(t) = \sum_{i=1}^n \partial_{tt} \psi_i(t) + \frac{1}{2} \sum_{i=1}^n \partial_i \hat{\psi}_i(t) \partial_i \hat{\psi}_i(t) + \frac{1}{2} \sum_{i=1}^n \partial_i \hat{\psi}_i(t) \partial_t \hat{\psi}_i(t) \]

\[ = \sum_{i=1}^n \partial_t \mathcal{F}_i(t) + \frac{1}{2} \sum_{i=1}^n \mathcal{F}_i(t) \mathcal{F}_i(t) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \mathcal{F}_i(t) \mathcal{F}_j(t) \]  

(7.7)

Taking the expectation

\[ M \left\{ H_n \hat{\psi}(t) \right\} = \frac{1}{2} \zeta^2 \sum_{i=1}^n M \left\{ \mathcal{F}_i(t) \mathcal{F}_i(t) \right\} + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \left\{ \mathcal{F}_i(t) \mathcal{F}_j(t) \right\} \]

\[ = \frac{1}{2} \zeta^2 \sum_{i=1}^n \delta_{ii} \Xi(0; \zeta) + \frac{1}{2} \zeta^2 \sum_{i=1}^n \sum_{j=1}^n \delta_{ij} \Xi(0; \zeta) \]

\[ = \frac{1}{2} n \zeta^2 \Xi(0; \zeta) + \frac{1}{2} n \zeta^2 \Xi(0; \zeta) = n \zeta^2 \Xi(0; \zeta) \equiv \lambda \]  

(7.8)

since \( \{ \mathcal{F}(t) \} = 0 \) and assuming \( \mathcal{F}_i(t) = \mathcal{F}_j(t) = \mathcal{F}(t) \).

The norm estimate is

\[ \lim_{t \to \infty} M \left\{ \left\| \hat{a}(t) - a^E \right\| \right\} \leq \lim_{t \to \infty} M \left\{ \left\| \hat{a}(t) \right\| \right\} \]

\[ < \lim_{t \to \infty} M \left\{ \left( \sum_{i=1}^n |a_i^E \exp(\zeta \hat{\mathcal{W}}(t))|^2 \right)^{1/2} \right\} \]

\[ = \lim_{t \to \infty} M \left\{ \left( n |a_i^E|^2 \exp(2 \zeta \hat{\mathcal{W}}(t)) \right)^{1/2} \right\} \]

\[ = \lim_{t \to \infty} a^E n^{1/2} M \left\{ \exp(\zeta \hat{\mathcal{W}}(t)) \right\} \]  

(7.9)

so that for stability

\[ \lim_{t \to \infty} M \left\{ \left\| \hat{a}(t) - a^E \right\| \right\} < \infty \]  

(7.10)

and for instability

\[ \lim_{t \to \infty} M \left\{ \left\| \hat{a}(t) - a^E \right\| \right\} = \infty \]  

(7.11)

This involves an estimation of \( M \left\{ \exp(\zeta \hat{\mathcal{W}}(t)) \right\} \) using a cluster expansion as before. In particular, if the set is sub-Gaussian then it is bounded and the Hoeffding inequality and the Chernoff bound inequality apply (ref). This suggests that if a randomly perturbed set \( \hat{\alpha}(t) \) is sub-Gaussian then it is bounded and therefore the system is stable to the random perturbations.
Lemma 7.2. If the random perturbations are of the form $\hat{a}_i(t) = a_i^E \exp(\zeta \mathcal{U}_i(t))$ for a Gaussian random field with $M\{\mathcal{U}_i(t)\} = 0$ and $M\{\mathcal{U}_i(t)\mathcal{U}_i(s)\} = J(\Delta; \sigma)$ and regulated $M\{\mathcal{U}_i(t)\mathcal{U}_i(t)\} = J(0; \sigma) < \infty$ then for all $t > 0$

$$M\left\{\hat{a}_i(t)\right\} \sim a_i^E \exp(\frac{1}{2}\zeta^2 J(0; \zeta)) < \infty \quad (7.12)$$

Proof. Interpreting $\Phi(t) = M\{\exp(\zeta \mathcal{U}_i(t))\}$ as a moment-generating function (MGF) then the corresponding cumulant generating function (CGF) is

$$\Phi(t) = \log \Psi(t) = \log M\left\{\exp(\zeta \mathcal{U}_i(t))\right\} \quad (7.13)$$

The CGF has the McLauren power-series representation which can be truncated at second order for Gaussian random fields so that

$$\Phi(t) = \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} C\{\mathcal{U}_i(t)^n\} = \zeta C\{\mathcal{U}_i(t)\} + \frac{1}{2} \zeta^2 C\{\mathcal{U}_i(t)\mathcal{U}_i(t)\} + ...$$

$$\equiv \zeta M\left\{\mathcal{U}_i(t)\right\} + \frac{1}{2} \zeta^2 M\left\{\mathcal{U}_i(t)\mathcal{U}_i(t)\right\} + ... = \frac{1}{2} \zeta^2 J(0; \zeta) \quad (7.14)$$

Hence

$$\Psi(t) = M\left\{\exp(\zeta \mathcal{U}_i(t))\right\} = \exp(\frac{1}{2} \zeta^2 J(0; \zeta)) \quad (7.15)$$

so that

$$M\{\hat{a}_i(t)\} \sim a_i^E \exp(\frac{1}{2} \zeta^2 J(0; \zeta)) < \infty \quad (7.16)$$

Corollary 7.3. The norm estimate is

$$\lim_{t \to \infty} M\left\{\left\|\hat{a}(t) - a^E\right\|\right\} \leq \lim_{t \to \infty} M\left\{\left\|\hat{a}(t)\right\|\right\}$$

$$= \lim_{t \to \infty} a_i^E n^{1/2} M\left\{\exp(\zeta \mathcal{U}(t))\right\} = n^{1/2} a_i^E \exp(\frac{1}{2} \zeta^2 J(0; \zeta)) < \infty \quad (7.17)$$

The Hoeffding and maximal estimates from Section 1 can now be applied.

Proposition 7.4. Let $a_i^E$ be a set of static equilibrium solutions of the Einstein system of nonlinear ODEs $D_n a_i^E = 0$. Let the randomly perturbed set of solutions be $\hat{a}_i(t) = a_i^E \exp(\zeta \mathcal{U}_i(t))$ with

$$M\left\{\hat{a}_i(t)\right\} = a_i^E \exp(\frac{1}{2} \zeta^2 J(0; \zeta)) < \infty \quad (7.18)$$

Let $\hat{a}_i^{E*}$ be ’attractors’ or new stable equilibrium fixed points such that the perturbed system converges as $\hat{a}_i(t) \to a_i^{E*}$ for some finite $t \geq 0$ or $t \to \infty$. Then for all finite $t > 0$ the set is bounded in that $\exists B > 0$ such that

$$a_i^E \leq \hat{a}_i(t) \leq a_i^{E*} < M\left\{\hat{a}_i(t)\right\} \leq B$$

(1) The Hoeffding inequality applies and for any $L > 0$ is

$$P\left\{\bar{S}(t) - M\{\bar{S}(t)\} \geq L\right\} = P\left\{\frac{1}{n} \sum_{i=1}^{n} \hat{a}_i(t) - \frac{1}{n} \sum_{i=1}^{n} M\{\hat{a}_i(t)\} \geq L\right\}$$

$$= P\left\{\frac{1}{n} \left\|\hat{a}_i(t)\right\| - \frac{1}{n} \left\|M\{\hat{a}_i(t)\}\right\| \geq L\right\}$$

$$\leq \exp\left(-\frac{2n^2|L|^2}{\sum_{i=1}^{n} \left\|B - a_i^E\right\|^2}\right)$$

$$= \exp\left(-\frac{2n^2|L|^2}{\left\|B - a_i^E\right\|^2}\right) \quad (7.19)$$
The Chernoff bound can also be expressed as

\[
P(\hat{S}(t) - M\{\hat{S}(t)\} = \infty) = P\left(\frac{1}{n} \sum_{i=1}^{n} a_i(t) - \frac{1}{n} \sum_{i=1}^{n} M\{\hat{a}_i(t)\} = \infty\right)
\]

\[
\equiv P\left(\frac{1}{n} \|\hat{a}_i(t)\| - \frac{1}{n} M\{\hat{a}_i(t)\} = \infty\right) = 1
\]

(7.20)

The Chernoff bound can also be expressed as

\[
P(\|\hat{a}_i(t) - a_i^E\| \leq |L|) \leq \exp(\beta|L|)M\left\{\exp\left(-\beta\left(\|\hat{a}_i(t) - a_i^E\|\right)\right)\right\}
\]

then

\[
P(\|\hat{a}_i(t) - a_i^E\| \leq |L|) \neq 0
\]

(7.21)

if \(M\left\{\exp(-\beta(\|\hat{a}_i(t) - a_i^E\|))\right\} < \infty\) and the randomly perturbed radii are bounded.

These exponential inequalities are valid for linear combinations of bounded independent random variables, and in particular for the average. But one is often more interested in controlling the maximum or supremum of the set in terms of the maximal estimates.

Lemma 7.5. Let \(a_i^E\) be a set of \(n\) equilibrium solutions of the Einstein system \(D_n a_i^E = 0\) and let \(\hat{a}_i(t)\) be the set of \(n\) randomly perturbed solutions. Let \(\sup_{1 \leq i \leq n} \hat{a}_i(t)\) be the supremum or maximum of the set. If the set if bounded it is sub-Gaussian and vice-versa so that

\[
P\left(\sup_{1 \leq i \leq n} \hat{a}_i(t) \geq |L|\right) \leq \exp\left(-\frac{L^2}{2C^2}\right)
\]

and

\[
\hat{a}_i^E \leq \sup_{1 \leq i \leq n} \hat{a}_i(t) \leq M\{\hat{a}_i(t)\} \leq B
\]

(7.24)

Then \(\exists (L, C, B, D) > 0\) such that the maximal inequalities hold and the system is stable so that for all \(t \in \mathbb{R}^+ \cup \infty\)

\[
M\left\{\sup_{1 \leq i \leq n} \hat{a}_i(t)\right\} \leq C \sqrt{2\log(n)} \leq B < \infty
\]

(7.25)

\[
\lim_{t \uparrow \infty} M\left\{\sup_{1 \leq i \leq n} \hat{a}_i(t)\right\} \leq C \sqrt{2\log(n)} \leq B < \infty
\]

(7.26)

\[
P\left(\sup_{1 \leq i \leq n} \hat{a}_i(t) \geq |L|\right) \leq n \exp\left(-\frac{L^2}{2C^2}\right) \leq D < \infty
\]

(7.27)

\[
\lim_{t \uparrow \infty} P\left(\sup_{1 \leq i \leq n} \hat{a}_i(t) \geq |L|\right) \leq n \exp\left(-\frac{L^2}{2C^2}\right) \leq D < \infty
\]

(7.28)

The probability of blowup or asymptotic instability is zero so that

\[
P\left(\sup_{1 \leq i \leq n} \hat{a}_i(t) = \infty\right) = 0
\]

(7.29)

\[
\lim_{t \uparrow \infty} P\left(\sup_{1 \leq i \leq n} \hat{a}_i(t) = \infty\right) = 0
\]

(7.30)

8. Conclusion

In this paper, it has been tentatively explored how one might incorporate classical randomness and stochasticity into general relativity within the context of specific solvable cosmological models, in order to incorporate the effects of fluctuations or ‘noise’. In particular, cosmological constant terms arise when one stochastically averages the nonlinear Einstein equations formulated on a random toroidal geometry, in analogy with induced Reynolds stresses and numbers within hydrodynamical turbulence theory when the Navier-Stokes PDEs are averaged.
APPENDIX A: APPENDIX A: GAUSSIAN RANDOM FIELDS

In this appendix, the definitions, existence, properties, correlations, statistics, derivatives and integrals are defined for random scalar vector fields (RVFS) on \( \mathbb{R}^n \). Details can be found in a number of texts [19,20,21,22,23,24,68].

A.1. Existence and statistical correlations of random fields.

Definition A.1. Let \((\Omega, \mathcal{F}, \mu)\) be a probability space. Within the probability triplet, \((\Omega, \mathcal{F})\) is a measurable space, where \(\mathcal{F}\) is the \(\sigma\)-algebra (or Borel field) that should be interpreted as being comprised of all reasonable subsets of the state space \(\Omega\). Then \(\mu\) is a function such that \(\mu : \mathcal{F} \rightarrow [0,1]\), so that for all \(A \in \mathcal{F}\), there is an associated probability \(\mu(A)\). The measure is a probability measure when \(\mu(\Omega) = 1\). The probability space obeys the Kolmogorov axioms:

- \(\mu(\Omega) = 1\).
- \(0 \leq \mu(A_i) \leq 1\) for all sets \(A_i \in \mathcal{F}\).
- If \(A_i \cap A_j = \emptyset\), then \(\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)\).

These are standard (and abstract) definitions within probability theory, stochastic functional analysis and ergodic theory.

Definition A.2. Let \(t \in \mathbb{R}^+\) and let \((\Omega, \mathcal{F}, \mu)\) be a probability space. Let \(\mathcal{U}(x;\omega)\) be a random scalar function that depends only on \(t \in \mathbb{R}\) and also \(\omega \in \Omega\). Given any pair \((t,\omega)\) there is a mapping \(M : \mathbb{R} \times \Omega \rightarrow \mathbb{R}\) such that

\[
M : (\omega, x) \rightarrow \mathcal{U}(t;\omega)
\]

so that \(\mathcal{U}(t,\omega)\) is a random Variable or field on \(\mathbb{R}^+\) with respect to the probability space \((\Omega, \mathcal{F}, \mu)\). The stochastic field is then essentially a family of random variables \(\{\mathcal{U}(t;\omega)\}\) defined with respect to \((\Omega, \mathcal{F}, \mu)\) and \(\mathbb{R}^+\).

The scalar random field can also include a spatial variable \(x \in \mathbb{R}^3\) so that given any triplet \((x,t,\omega)\) there is a mapping \(\mathfrak{M} : \mathbb{R}^+ \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^+\) such that

\[
\mathfrak{M} : (\omega, x, t) \rightarrow \mathcal{U}(x,t;\omega)
\]

However, it will be sufficient to consider fields that vary randomly in time only. The expected value of the random field with respect to \((\Omega, \mathcal{F}, \mu)\) is defined as follows.

Definition A.3. Given the random scalar field \(\mathcal{U}(t;\omega)\), then if \(\int_{\Omega} \|\mathcal{U}(t;\omega)\|d\mu(\omega) < \infty\), the expectation of \(\mathcal{U}(t;\omega)\) is

\[
M \left\{ \mathcal{U}(t;\omega) \right\} = \int_{\Omega} \mathcal{U}(t;\omega)d\mu(\omega) \tag{A.1}
\]

Definition A.4. Let \((\Omega, \mathcal{F}, \mu)\) be a probability space, then an \(L_p(\Omega, \mathcal{F}, \mu)\) space or an \(L_p\)-space for \(p \geq 1\) is a linear normed space of random scalar fields that satisfies the conditions

\[
M \left\{ \|\mathcal{U}(t;\omega)\|^p \right\} = \int_{\Omega} \|\mathcal{U}(t;\omega)\|^p d\mu(\omega) < \infty \tag{A.2}
\]

and the corresponding norm \(L_p\) norm is

\[
\|\mathcal{U}(x)\| = \left(M \left\{ \|\mathcal{U}(t;\omega)\|^p \right\} \right)^{1/p} \tag{A.3}
\]

with the usual \(L_2\) Euclidean norm for \(p = 2\). When \(p = 2\) the fields are second-order random fields. Note that an \(L_2\)-space equipped with the scalar product

\[
M \left\{ \mathcal{U}(x,\omega) \otimes \mathcal{U}'(t,\omega) \right\} = \int_{\Omega} \mathcal{U}(x,\omega) \otimes \mathcal{U}'(t,\omega)d\mu(\omega) \tag{A.4}
\]

is also a Hilbert space.

The second-order correlations, moments and covariances are of the most interest.
Definition A.5. Let \( t, s \in \mathbb{R}^+ \) and let \( \omega, \xi \in \Omega \). The expectations of mean values of the fields \( \hat{U}(t, \omega) \) and \( \hat{U}(y, \xi) \) are

\[
M(t) = M(\hat{U}(t)) = \int_{\Omega} \hat{U}(t, \omega) d\mu(\omega) \tag{A.5}
\]

\[
M(s) = M(\hat{U}(s)) = \int_{\Omega} \hat{U}(s, \xi) d\mu(\omega) \tag{A.6}
\]

then the 2nd-order moment or stochastic expectation is

\[
M\{(\hat{U}(t) \otimes \hat{U}(s))\} = \int_{\Omega} \int_{\Omega} \hat{U}(t, \omega) \otimes \hat{U}(s, \xi) d\mu(\omega) d\mu(\xi) \tag{A.7}
\]

The covariance is then

\[
COV(t, s) = M\{((\hat{U}(t) - M(t))(\hat{U}(s) - M(s)))\} \tag{A.8}
\]

or

\[
COV(t, s) = \int_{\Omega} \int_{\Omega} (\hat{U}(t; \omega) - M(t)) \otimes (\hat{U}(s; \omega) - M(s)) d\mu(\omega) d\mu(\xi) \tag{A.9}
\]

so that

\[
COV(t, s) = M\{(\hat{U}(t) \hat{U}(s)) - M(t)M(s)\} \tag{A.10}
\]

Definition A.6. Given a set of fields \( \hat{U}(t_1), ..., \hat{U}(t_n) \) at points \( t_1, ..., t_n \in \mathbb{R}^+ \) then the \( m \)-th order moments and cumulants are

\[
M\{\hat{U}(t_1)...\hat{U}(t_m)\} \tag{A.11}
\]

\[
COV\{\hat{U}(t_1)...\hat{U}(t_m)\} \tag{A.12}
\]

where at second order

\[
COV\{\hat{U}(t)\hat{U}(s)\} = COV(t, s) = M\{\hat{U}(t)\hat{U}(s)\} - M(t)M(s) \tag{A.13}
\]

The covariance must have the following important properties

Lemma A.7. A function \( COV(t, s) \) is formally a covariance if the following are satisfied:

1. Let \( t_\alpha, s_\beta \in \mathbb{R}^+ \) with \( \alpha, \beta \in \mathbb{Z} \). Then any covariance \( COV(t_\alpha, s_\beta) \) is always nonnegative semi-definite such that for any \( q_\alpha, q_\beta > 0 \)

\[
\sum_\alpha \sum_\beta q_\alpha q_\beta COV(t_\alpha, s_\beta) \geq 0
\]

2. Symmetry, \( COV(s, t_\beta) = COV(s, t) \)

3. \( \lim_{\|t-s\| \to \infty} COV(t, s) = 0 \). If \( COV\{\hat{U}(t)\hat{U}(s)\} \equiv COV(t, s) = 0 \) then \( \hat{U}(s) \) and \( \hat{\psi}(s) \) are uncorrelated.

Proof. To prove (1),

\[
M\left\{ \sum_{\alpha=1}^N q_\alpha [\hat{U}(t_\alpha) - M(t_\alpha)]^2 \right\} = \sum_\alpha \sum_\beta q_\alpha q_\beta M\left\{ [\hat{U}(t_\alpha) - M(t_\alpha)] \otimes [\hat{U}(t_\beta) - M(t_\beta)] \right\} \geq 0 \tag{A.14}
\]

A lognormal scalar random field is defined as follows

Definition A.8. Let \( \hat{U}(t) \) be a scalar random field, then there is a scalar random field \( \hat{B}(t) \) such that

\[
\hat{B}(t) = \exp(\hat{U}(t)) \tag{A.15}
\]

with inverse \( \hat{U}(x) = \ln(\hat{F}(x)) \)

These definitions now extend naturally to random vector fields \( \hat{U}_i(t) \) for all \( i = 1 \) to \( n \).
Definition A.9. Let \( x^i \subset \mathbb{D} \subset \mathbb{R}^3 \) be Euclidean coordinates and let \( (\Omega, \mathcal{F}, \mu) \) be a probability space. Let \( \widehat{\mathcal{U}}(x; \omega) \) be a random vector function that depends on \( t \subset \mathbb{R}^+ \) and also \( \omega \in \Omega \). Given any pair \((t, \omega)\) there is a mapping \( \mathcal{M} : \mathbb{R}^+ \times \Omega \to \mathbb{R}^n \) such that
\[
\mathcal{M} : (\omega, t) \to \widehat{\mathcal{U}}(t; \omega)
\]
so that \( \widehat{\mathcal{U}}(t, \omega) \) is a random vector field spanning \( \mathbb{R}^n \) with respect to the probability space \((\Omega, \mathcal{F}, \mu)\).

The expected value of the random vector field with respect to \((\Omega, \mathcal{F}, \mu)\) is defined as before

Definition A.10. Given the random vector field \( \widehat{\mathcal{U}}(t; \omega) \), then if \( \int_{\Omega} \| \widehat{\mathcal{U}}(x; \omega) \| d\mu(\omega) < \infty \), the expectation of \( \widehat{\mathcal{U}}(t; \omega) \) is
\[
\mathcal{M}\left\{ \widehat{\mathcal{U}}(t; \omega) \right\} = \int_{\Omega} \widehat{\mathcal{U}}(t, \omega) d\mu(\omega) \tag{A.16}
\]

Definition A.11. An \( L_p(\Omega, \mathcal{F}, \mu) \) space or an \( L_p \)-space for \( p \geq 1 \) is a linear normed space of random fields that satisfies the conditions
\[
\mathcal{M}\left\{ \| \widehat{\mathcal{U}}(t; \omega) \|^p \right\} = \int_{\Omega} \left( \sum_{i=1}^{n} |\widehat{\mathcal{U}}(t; \omega)|^p \right)^{1/p} d\mu(\omega) < \infty \tag{A.17}
\]
with the usual Euclidean or \( L_2 \) norm for \( p = 2 \). The second-order correlations, moments and covariances are now

Definition A.12. Let \( t, s \in \mathbb{R}^+ \) and let \( \omega, \xi \in \Omega \). The expectations of mean values of the fields \( \widehat{\mathcal{U}}(t, \omega) \) and \( \widehat{\mathcal{U}}(s, \xi) \) are
\[
\mathcal{M}_i(t) = \mathcal{M}\left\{ \widehat{\mathcal{U}}(t, \omega) \right\} = \int_{\Omega} \widehat{\mathcal{U}}(t, \omega) d\mu(\omega) \tag{A.18}
\]
\[
\mathcal{M}_j(s) = \mathcal{M}\left\{ \widehat{\mathcal{U}}(s, \xi) \right\} = \int_{\Omega} \widehat{\mathcal{U}}(s, \xi) d\mu(\xi) \tag{A.19}
\]
then the 2nd-order moment or expectation is
\[
\mathcal{M}\left\{ \widehat{\mathcal{U}}(t; \omega) \widehat{\mathcal{U}}(t; \omega) \right\} = \int_{\Omega} \int_{\Omega} \widehat{\mathcal{U}}(t, \omega) \widehat{\mathcal{U}}(s, \xi) d\mu(\omega)d\mu(\xi) \tag{A.20}
\]
The covariance is then
\[
\text{COV}_{ij}(t, s) = \mathcal{M}\left\{ (\widehat{\mathcal{U}}(t; \omega) - \mathcal{M}_i(t))(\widehat{\mathcal{U}}(s; \xi) - \mathcal{M}_j(s)) \right\} \tag{A.21}
\]
or
\[
\text{COV}_{ij}(t, s) = \int_{\Omega} \int_{\Omega} (\widehat{\mathcal{U}}(t; \omega) - \mathcal{M}_i(t)) \odot (\widehat{\mathcal{U}}(s; \xi) - \mathcal{M}_j(s)) d\mu(\omega)d\mu(\xi) \tag{A.22}
\]
so that
\[
\text{COV}_{ij}(t, s) = \mathcal{M}\left\{ (\widehat{\mathcal{U}}(t; \omega) - \mathcal{M}_i(t))(\widehat{\mathcal{U}}(s; \xi) - \mathcal{M}_j(s)) \right\} \tag{A.23}
\]

Definition A.13. Given a set of fields \( \widehat{\mathcal{U}}(t_1), \ldots, \widehat{\mathcal{U}}(t_m) \) at points \( t_1 \ldots t_m \in \mathbb{R}^+ \) then the \( m \)th-order moments and cumulants are
\[
\mathcal{M}\left\{ \widehat{\mathcal{U}}(t_1) \ldots \widehat{\mathcal{U}}(t_m) \right\} \tag{A.24}
\]
\[
\mathcal{C}\left\{ \widehat{\mathcal{U}}(t_1) \ldots \widehat{\mathcal{U}}(t_m) \right\} \tag{A.25}
\]
where at second order
\[
\mathcal{C}\left\{ \widehat{\mathcal{U}}(t) \widehat{\mathcal{U}}(s) \right\} \equiv \text{COV}_{ij}(t, s) = \mathcal{M}\left\{ (\widehat{\mathcal{U}}(t; \omega) - \mathcal{M}_i(t))(\widehat{\mathcal{U}}(s; \xi) - \mathcal{M}_j(s)) \right\} \tag{A.26}
\]
The covariance tensor is again nonnegative semi-definite so that
\[ \sum_{\alpha}^{N} \sum_{\beta}^{N} q_{\alpha} q_{\beta} \text{COV}_{i_\alpha j_\beta}(t_\alpha, s_\beta) \geq 0 \]
with symmetry \( \text{COV}_{ij}(t, s) = \text{COV}_{ij}(t, s) \) and \( \lim_{||t-s|| \to \infty} \text{COV}(t, s) = 0 \). If
\[ C \bigg\{ \mathcal{W}_i(t) \mathcal{W}_j(t) \bigg\} = \text{COV}_{ij}(t, s) = 0 \quad (A.27) \]
then \( \mathcal{W}_i(s) \) and \( \mathcal{W}_j(s) \) are uncorrelated.

A very important class of random fields are the Gaussian random vector fields (GRVFS) which are characterized only by their first and second moments. The GRVFS can also be isotropic, homogenous and stationary. The details will be made more precise but the advantages of GRVFS are briefly enumerated.

1. GRVFS have convenient mathematical properties which generally simplify calculations; indeed, many results can only be evaluated using Gaussian fields.
2. A GRVF can be classified purely by its first and second moments and high-order moments and cumulants can be ignored.
3. Gaussian fields accurately describe many natural stochastic processes including Brownian motion.
4. A large superposition of non-Gaussian fields can approach a Gaussian field.

For this paper, the following definitions are sufficient for isotropic GRVFS.

**Definition A.14.** Any GRVF has normal probability density functions. The following always hold:

1. The first moment vanishes so that
\[ M_i(x) = M \bigg\{ \mathcal{W}_i(t; \omega) \bigg\} = \int_{\Omega} \mathcal{W}_i(t; \omega) d\mu(\omega) = 0 \]
2. The covariance then reduces to
\[ \text{COV}_{ij}(t, s) \equiv C \bigg\{ \mathcal{W}_i(t) \mathcal{W}_j(s) \bigg\} = M \bigg\{ \mathcal{W}_i(t) \mathcal{W}_j(s) \bigg\} = J_{ij}(\Delta; \varsigma) \]

**Definition A.15.** The GRVF is isotropic if \( \text{COV}_{ij}(t, s) = J_{ij}(\Delta; \varsigma) \) depends only on the separation \( \Delta = |t-s| \) and is stationary if \( \text{COV}_{ij}(t + \delta t, s + \delta s) = J_{ij}(\Delta; \varsigma) \). Hence, the 2-point function or Greens function is translationary invariant.

**Definition A.16.** An important class of random fields are white noises which have the delta-function correlated 2-point function. But for white noises \( \text{COV}_{ij}(t, t) = \infty \) so the equal-time correlation diverges. The standard differential for Brownian motion is then \( d\mathcal{W}(t) = \mathcal{W} dt \). If the noise has a finite correlation time \( \varsigma \) then a regulated covariance or 2-point function is possible such that
\[ \text{COV}_{ij}(t, s) = M \bigg\{ \mathcal{W}_i(t) \mathcal{W}_j(s) \bigg\} = \alpha \delta_{ij} J(|\Delta|; \varsigma) \quad (A.28) \]
where now \( \text{COV}_{ij}(t, t) = J(0; \varsigma) < \infty \). Examples are colored noise or the Ornstein-Uhlenbeck process.

**Definition A.17.** A GRVF \( \mathcal{W}_i(t) \) is almost surely continuous at \( x \in \mathbb{R}^+ \) if
\[ \mathcal{W}_i(t + \varsigma) \rightarrow \mathcal{W}_i(t) \quad (A.29) \]
as \( \varsigma \to 0 \)

When this holds for all \( t \in \mathbb{R}^+ \) then this is known as ’sample function continuity’. The following result due to Adler [24,68], gives the sufficient condition for continuous sample paths.

**Lemma A.18.** Let \( \mathcal{W}_i(t) \) be a non-white GRVF. Then if for some \( C > 0 \) and \( \lambda > 0 \) with \( \eta > \lambda \)
\[ M \bigg\{ \left| \mathcal{W}_i(t + \varsigma) - \mathcal{W}_i(t) \right|^\lambda \bigg\} \leq \frac{C|\varsigma|^{2\eta}}{|\ln |\varsigma||^{1+\eta}} \quad (A.30) \]
If \( \mathcal{W}_i(t) \) is a Gaussian random field with continuous \( \text{COV}_{ij}(t, s) \) then given some \( C > 0 \) and some \( \epsilon > 0 \)
\[ M \bigg\{ \left| \mathcal{W}_i(t + \varsigma) - \mathcal{W}_i(t) \right|^2 \bigg\} \leq \frac{C}{|\ln |\varsigma||^{1+\epsilon}} \quad (A.31) \]
Theorem A.19. If \( \phi \) is a convex function and \( X \) is a random variable or field then Jensen’s inequality is that statement
\[
\phi(\mathbb{M}\{X\}) \leq \mathbb{M}\{\phi(X)\}
\]
(A.32)

The differentiability of a RGVF is defined as follows

Definition A.20. Let \( \hat{\mathcal{W}}_t(t) \) be a RVF and let \( t \in \mathbb{R} \). Then
\[
\partial_t \hat{\mathcal{W}}_t(t) \equiv \frac{d}{dt} \hat{\mathcal{W}}_t(t) = \lim_{\zeta \to 0} \left( \frac{\hat{\mathcal{W}}_t(t + \zeta) - \hat{\mathcal{W}}_t(t)}{\zeta} \right)
\]
(A.33)
for all \( t \in \mathbb{R}^+ \). It follows that
\[
\lim_{\zeta \to 0} \mathbb{M}\left\{ \left( \frac{\hat{\mathcal{W}}_t(t + \zeta) - \hat{\mathcal{W}}_t(t)}{\zeta} - \partial_t \hat{\mathcal{W}}_t(t) \right)^2 \right\} = 0
\]
(A.34)

The second-order derivative is
\[
\partial_{tt} \hat{\mathcal{W}}_t(t) = \lim_{\zeta \to 0} \lim_{\xi \to 0} \frac{1}{\zeta \xi} \left[ \hat{\mathcal{W}}_t(t + \zeta + \xi) - \hat{\mathcal{W}}_t(t + \zeta) - \hat{\mathcal{W}}_t(t + \xi) + \hat{\mathcal{W}}_t(t) \right]
\]
(A.35)

An alternative (and better) definition in the mean-square sense is as follows
\[
\lim_{h \to 0} \lim_{g \to 0} \mathbb{M}\left\{ \left( \frac{\hat{\mathcal{W}}_t(t + \zeta) - \hat{\mathcal{W}}_t(t)}{\zeta} - \frac{\hat{\mathcal{W}}_t(t + \xi) - \hat{\mathcal{W}}_t(t)}{\xi} \right)^2 \right\} = 0
\]
(A.36)

Clearly the existence of derivatives requires that the 2-point function or covariance is non-white and therefore finite or regulated at \( s = t \).

This leads to the following important lemma which establishes the correlations for a gradient of a RVF.

Lemma A.21. A SRF is differentiable iff:

1. The first moment \( \mathbb{M}_t(x) = \mathbb{M}\{\hat{\mathcal{W}}_t(t)\} \) is differentiable.
2. The covariance \( \text{COV}_{ij}(t, s) \) exists and is finite at all points \( t = s \in \mathbb{R}^+ \).

Then
\[
\partial_t \text{COV}_{ij}(t, s) \equiv \partial_t \text{COV}_{ij}(t, s) = \mathbb{M}\{\hat{\mathcal{W}}_t(t) \hat{\mathcal{W}}_s(s)\}
\]
(A.37)

A.2. Stochastic Integration. Having established existence of the derivative of a SRVF, stochastic integrals can be defined as a mean-square Riemann integration.

Definition A.22. Let \( \hat{\mathcal{W}}_i(x) \) be a RGVF spanning \( \mathbb{R}^n \). Let \( f(t) \) be a deterministic continuous and bounded function such that \( f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \). The stochastic integral is the mean-square Riemann integral
\[
\hat{\mathcal{I}}_i(t) = \int_{\mathbb{R}^+} f(t) \hat{\mathcal{W}}_i(t) dt \equiv \int_{\mathbb{R}^+} f(t) \hat{\mathcal{W}}_i(t, \zeta) dt
\]
(A.38)

The integral exists if the limit of the Riemann sum exists
\[
\hat{\mathcal{I}}_i^{(m)}(t) = \sum_{\eta=1}^{m} f(t_\eta) \otimes \hat{\mathcal{W}}_i(t_\eta) \Delta(t_\eta)
\]
(A.39)
where \( \Delta(t_\eta) \) is a line element. The integral then exists if
\[
\hat{\mathcal{I}}_i(t) = \lim_{m \to \infty} \hat{\mathcal{I}}_i^{(m)}(t)
\]
(A.40)

Since \( \mathbb{M}\{\hat{\mathcal{W}}(x)\} = 0 \) then \( \mathbb{M}\{\hat{\mathcal{W}}(y)\} = 0 \). When \( f(t) = 1 \) then
\[
\hat{\mathcal{I}}_i(t) = \int_{\mathbb{R}^+} \hat{\mathcal{W}}_i(t) dt
\]
(A.41)

This definition leads to the following corollary.
Corollary A.23. A SRVF $\widehat{\mathcal{H}}(x)$ is mean-square Riemann integrable iff

$$M\left\{\widehat{\mathcal{F}}(t)\widehat{\mathcal{F}}(s)\right\} = \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) f(s) \text{Cov}_{ij}(t, s) dt ds < \infty$$

$$\lim_{m \to \infty} \sum_{\xi=1}^{m} \sum_{\eta=1}^{m} f(t_\eta) f(s_\eta) M\left\{\widehat{\mathcal{H}}_{1}(t_\xi) \otimes \widehat{\mathcal{H}}_{2}(s_\eta)\right\} \Delta(t_\xi) \Delta(x_\eta)$$

For a Gaussian RVF this is equivalent to

$$M\left\{\widehat{\mathcal{F}}(s)\widehat{\mathcal{F}}(y')\right\} = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} f(t) f(s) M\left\{\widehat{\mathcal{H}}(t)\widehat{\mathcal{H}}(s')\right\} dt ds < \infty$$

Proposition A.24. For a GRVF, the 2-point correlation is

$$M\left\{\widehat{\psi}(t)\widehat{\psi}(s)\right\} = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} M\left\{\widehat{\mathcal{H}}_{1}(t) \otimes \widehat{\mathcal{H}}_{2}(s)\right\} dt ds$$

The n-point correlation is

$$M\{\widehat{\psi}(t_1) \times ... \times \widehat{\psi}(t_n)\} = ... \int dt_1...dt_m M\{\widehat{\mathcal{H}}_{1}(t_1) \times ... \times \widehat{\mathcal{H}}_{1}(t_1)\}$$

which can be expressed in a path integral form as

$$M\{\widehat{\psi}(t_1) \times ... \times \widehat{\psi}(t_n)\} = \int \mathcal{D}_n t M\{\widehat{\mathcal{H}}_{1}(t_1) \times ... \times \widehat{\mathcal{H}}_{1}(t_1)\}$$

Theorem A.25. Let $X$ be a random variable or field and let $X_i$ be set of $n$ random variables or fields, then Fubini’s theorem states that

$$M\left\{\int X\right\} = \int M\left\{X\right\}$$

or

$$M\left\{\sum_{i=1}^{n} X_i\right\} = \sum_{i=1}^{n} M\left\{X_i\right\}$$

A.3. Ito and stochastic integrals. The Ito and and Stratanovich interpretations of stochastic integrals are defined as follows

Definition A.26. Let $f(t)$ be a continuous function of $t$. If $\mathcal{W} = [0, T]$ is partitioned so that $t_{\xi=0} = t_0$ then $t_0 < t_1 < t_2 < ... < t_\xi$ and $t_\xi = t_y$. For $\mathcal{H} = \mathcal{H}(t)$, a Weiner process then the following Riemann-Steiljes sum over $\mathcal{W}$ defines an Ito integral

$$\widehat{\mathcal{F}}(t) = \int_{t_y}^{t_x} f(\tau) d\mathcal{H}(\tau) \equiv \int_{t_y}^{t_x} f(\tau) \mathcal{H}(\tau) d\tau$$

$$\sum_{\xi=0}^{N-1} f(t_{\xi}^N) \mathcal{H}(t_{\xi+1}^N) - \mathcal{H}(t_{\xi}^N)$$

in the limit that partitions $\{t_{\xi}^N\} \to 0$. For each $i = 1$ to $n - 1$. This interpretation essentially always takes the minimum value of the pair $[f((t_{\xi+1}^N), f(t_{\xi}^N))]$. The alternative Stratanovich interpretation always takes the averaged value $\frac{1}{2} [f((t_{\xi+1}^N), f(t_{\xi}^N))]$ so that the Stratanovich stochastic integral is

$$\widehat{\mathcal{F}} = \int_{t_y}^{t_x} f(\tau) d\mathcal{H}(\tau) = \int_{t_y}^{t_x} f(\tau) \mathcal{H}(\tau) d\tau$$

$$= \sum_{\xi=0}^{n-1} \frac{1}{2} [f(t_{\xi+1}^N) \mathcal{H}(t_{\xi+1}^N) - \mathcal{H}(t_{\xi}^N)]$$

For the Gaussian random field $\mathcal{W}(t)$ with regulated 2-point function

$$\widehat{\mathcal{F}} = \int_{t_y}^{t_x} f(\tau) d\mathcal{H}(\tau) = \int_{t_y}^{t_x} f(\tau) \mathcal{H}(\tau) d\tau$$
\[
\sum_{\xi=0}^{n-1} \frac{1}{2} [f(t^\xi_{\xi+1}) - f(t^\xi_\xi)] [\hat{\mathcal{W}}(t^\xi_{\xi+1}) - \hat{\mathcal{W}}(t^\xi_\xi)]
\] (A.51)

The Ito interpretation requires the Ito calculus and \( d\hat{\mathcal{W}}(t)d\hat{\mathcal{W}}(t) = dt \) and \( (dt^2) = 0 \). However, within the Stratanovitch interpretation the rules of ordinary calculus apply. Two important properties of Ito integrals are zero mean and an Ito isometry such that

\[
\mathbb{M} \left( \int_0^T f(t) d\hat{\mathcal{W}}(t) \right) = 0
\] (A.52)

and

\[
\mathbb{M} \left( \int_0^T f(t) d\hat{\mathcal{W}}(t) \right)^2 = \int_0^T \mathbb{M} f(t)^2 dt
\] (A.53)

\[
\mathbb{M} \left( \int_0^T f(t) d\hat{\mathcal{W}}(t) \int_0^T Y(t) d\hat{\mathcal{W}}(t) \right)^2 = \int_0^T \mathbb{M} (f(t) Y(t)) dt
\] (A.54)

**Proposition A.27.** Let \( \hat{\mathcal{B}}_i(t) \) be a random field which is the lognormal of a field \( \mathcal{B}(x) \) so that \( \hat{\mathcal{B}}(t) = \ln(\mathcal{B}(t)) \). Then

\[
\hat{\mathcal{B}}(t) = \exp(\hat{\mathcal{F}}(t)) \equiv \exp \left( \int_0^t \hat{\mathcal{F}}(s) ds \right)
\] (A.55)

The expectation is

\[
\mathbb{M} \{ \hat{\mathcal{B}}(x) \} = \mathbb{M} \{ \exp(\hat{\mathcal{F}}(t)) \} \equiv \mathbb{M} \left\{ \exp \left( \int_0^t \hat{\mathcal{F}}(s) ds \right) \right\}
\] (A.56)

**Appendix B. Appendix B: Proof of Lemma**

**Lemma B.1.** Given white-noise perturbations of the static radial moduli set \( \{\psi_i(t)\}_{i=1}^n \) of the form

\[
\hat{\psi}(t) = \psi_i(t) + \zeta \int_0^t f(\psi_i(s)) d\mathcal{W}(s)
\] (B.1)

with the conditions

\[
\|f_i(\psi_i(t))\| \leq K \|\psi_i(t)\|^2
\] (B.2)

and

\[
\int_0^t \left\| f(\psi_i(s)) \right\|^2 ds < \infty
\] (B.3)

then the \( \ell^\text{th} \)-order moments are finite and bounded for all finite \( t > t_o \) and grow exponentially, with the estimates

\[
\mathbb{M} \left\{ \left\| \sup_{t \leq \ell} \hat{\psi}_i(t) \right\| \right\|^\ell \leq \|\psi_i\|\| \exp(\frac{1}{2}K\ell(\ell - 1)(T - t_0))
\] (B.4)

\[
\mathbb{M} \left\{ \left\| \sup_{t \leq T} \hat{\psi}_i(t) \right\| \right\|^\ell \leq \|\psi_i\|\| \exp(\frac{1}{2}K\ell(\ell - 1)(T - t_0))
\] (B.5)

**Proof.** Let \( X(\hat{\psi}(t)) \) be a \( C^2 \)-differentiable functional of \( \hat{\psi}(t) \) then by Ito’s Lemma

\[
dX(\hat{\psi}(t)) = \nabla X(\psi_i(t)) d\hat{\psi}_i(t) + \frac{1}{2} \nabla^2 X(\psi_i(t)) d[\hat{\psi}, \hat{\psi}]_i(t)
\]

\[
\equiv (\nabla X(\psi_i(t))) d\hat{\psi}_i(t) + \zeta f(\psi_i(t)) d\mathcal{W}(t) + \frac{1}{2} (\nabla^2 X(\psi_i(t))) \| f(\psi_i(t)) \|^2 dt
\] (B.6)

where \( \nabla = d/d\psi_i(t) \) and \( [\hat{\psi}, \hat{\psi}]_i(t) \) is the quadratic variation. Integrating

\[
X(\hat{\psi}(t)) = X(\psi_i^T) + \int_0^t \nabla X(\psi_i(s)) d\psi_i(s)
\]

\[
+ \zeta \int_{t_0}^t \nabla X(\psi_i(t)) \| f(\psi_i(s)) \|^2 ds + \zeta^2 \frac{1}{2} \int_{t_0}^t (\nabla^2 X(\psi_i(t))) \| f(\psi_i(s)) \|^2 ds
\] (B.7)
Averaging then gives
\[
M\left\{X(\hat{\psi}_i(t))\right\} = X(\psi_i^E) + \int_0^t \nabla M\{X(\psi_i(s))\} d\psi_i(s)
\]
\[
+ \zeta^2 \frac{1}{2} \int_{t_0}^t (\nabla^2 M\{X(\psi_i(t))\}) \|f(\psi_i(s))\|^2 ds
\]  
(B.8)

Now letting \(X(\hat{\psi}_i(t)) = \|\psi_i(t)\|^\ell\)
\[
M\left\{\|\hat{\psi}_i(t)\|^\ell\right\} = \|\psi_i^E\|^\ell + \int_0^t \nabla M\{\|\psi_i(t)\|^\ell\} d\psi_i(s)
\]
\[
+ \zeta^2 \frac{1}{2} \int_{t_0}^t (\nabla^2 M\{\|\psi_i(t)\|^\ell\}) \|f(\psi_i(s))\|^2 ds
\]
\[
= \|\psi_i^E\|^\ell + \ell \int_{t_0}^t M\{\|\psi_i(s)\|^{\ell-1}\} d\psi_i(t) + \frac{1}{2} \zeta^2 \ell (\ell - 1) \int_{t_0}^t \|f(\psi_i(t))\|^2 M\{\|\psi_i(s)\|^{\ell-2}\} ds
\]
\[
\leq \|\psi_i^E\|^\ell + \ell \int_{t_0}^t M\{\|\psi_i(s)\|^{\ell-1}\} d\psi_i(t) + \frac{1}{2} \zeta^2 \ell (\ell - 1) \int_{t_0}^t K \|\psi_i(t)\|^2 M\{\|\psi_i(s)\|^{\ell-2}\} ds
\]
\[
\leq \|\psi_i^E\|^\ell + \ell \int_{t_0}^t M\{\|\psi_i(s)\|^{\ell-1}\} d\psi_i(t) + \frac{1}{2} \zeta^2 \ell (\ell - 1) \int_{t_0}^t K M\{\|\psi_i(s)\|^\ell\} ds
\]
\[
\leq \int_{t_0}^t K M\{\|\psi_i(s)\|^\ell\} ds
\]  
(B.9)

The Gronwall lemma then gives the estimates (B4) and (B5). \qed
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