Mental Age Compatibility: Quantification through the Convolution of Probability Distributions

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Abstract
We build on the empirical finding that a human being’s mental age is normally distributed around the chronological age. This opposes the frequent societal assumption “mental = chronological” which is known to be false in general but entertained for simplicity due to lack of methodology; hence disregarding that, f.e., people of different chronological ages can be much closer in their mental ages. As a quantitative approach on a scientific basis, we set up a general formula for the probability that two individuals of given ages are mentally within a certain range of years and investigate its implications i.a. by critically analyzing popular assumptions on age and computing statistical expectations within populations.
1 Introduction

It is very well known that human beings do not behave, think, and feel across the board according to the stereotypes assigned to their chronological ages. Statements like “he does not act like someone his age” or “she is far beyond her age” are heard often throughout different societies. The reality behind this and related phenomena is usually described within the context of mental age.
The fact that people can be mentally older or younger than what is usually referred to as their “age”, raises the question how far two individuals are actually apart in age on a mental level, which necessarily leads to questioning a lot of profound (legal, moral, ethical) views regarding “age disparities” anchored in common thinking and legislation.

In many societal issues, it is assumed that mental age is equal to chronological age. Of course, people are aware that in general these two ages cannot be equal; however, in the absence of a reliable calibration, such an assumption is nonetheless made for simplicity.

In this work, we propose a statistical method that promises to shed light on the issue and might provide guidance for practical applications. Although a statistical estimation cannot provide a conclusive answer to what is right or wrong in a particular case, it can furnish an enlightening – and quantitative – measure for good reasoning to reconsider accustomed ways of thinking and judging and so raise awareness of reality being in many cases and regards likely significantly different than commonly presupposed through socially conditioned mindsets.

The general concept of mental age is a measure of (emotional) intelligence and hence a rather multifaceted quantity across a spectrum of various cognitive abilities; e.g. the five components of emotional intelligence by Goleman [1]: self-awareness, self-regulation, social skills, empathy, motivation; the four factors of the Emotional Intelligence Inventory (EII) by Mayer and Salovey [2][3] (analyzed for its correlation to the Emotional Intelligence Scales (EIS) [4] by Tapia and Marsh II [5]): empathy, utilization of feelings, handling relationships, and self-control. That is why the mental age of an individual is not unique but dependent on the considered modality of intelligence and its components. It has also been subject to debate among groups of scholars, since there has been dissent about whether intelligence is permanently ingrained into someone’s DNA or variable through external factors of influence.

A number of studies and tests have been conducted on the topic, which obtained prolific results by focussing on specific or various aspects of interest. In particular, special point schemes have been designed for that purpose, on the basis of which subjects were tested. According to the Binet-Simon Intelligence Scales (IQ test) [6], later revised by Terman et Al. [7][8], scores were separately averaged by age groups and those averages used as benchmarks for the corresponding mental age. Regardless of the details (gender, culture, the modality and its components in question etc.), a broad finding was that the scores from manifold tests aimed at measuring intelligence are normally (Gaussian) distributed around the found average values of various chronological age groups. This makes intuitive and logical sense: people are more likely to be mentally closer to the average of their chronological age cohort; yet, even if they experience similar social structures and cultural habits and evolve through connatural forms of mental development within the same amount of time, there is a multiplicity of factors that give rise to individual divergences, that is, statistical deviations from the average. Given a mean value and a standard deviation, the principle of maximum entropy states a Gaussian form for the corresponding probability distribution.
It is important to note that the minutiae on the concept of intelligence and its in-depth measurement process are not relevant for the purpose of the current work. We centralize the empirical observation based on numerous studies that (emotional) intelligence is quantifiable, particularly assignable to age, and that the underlying mathematical structure for its distribution – and hence the distribution of mental age around a given chronological age – is a Gaussian probability density function (pdf).

Mental age is predominantly regarded to emotional intelligence here. From studies elaborating on its validation and estimation (e.g. [4, 5, 9–12]), with age groups ranging from older children (10-13 years) to young adults (18-29 years), we can distill a quite compact scope of guide values for the standard deviation.

In section 2, we set up the pdf for mental age and, in virtue of the convolution formalism, derive a formula for the probability that two people of given chronological ages are within a certain mental age range. We also set up probability formulae for further specific situations.

In section 3, we derive formulae for statistical expectations based on the aforementioned probability.

In section 4, we discuss ranges for the parameters, like the standard deviation, by reasoning natural bounds, applying empirical data from studies, and analyzing the mathematical structure.

In section 5, we elaborate in depth on the implications of the presented formalism, i.a. by computing explicit numerical results for given groups of people.

In the appendix, we provide an error analysis, accounting for the negligible impact of necessary approximations and the propagation of errors of parameters for the probability value.

2 Probability

2.1 General case

For a randomly chosen individual’s chronological age, $\mu$, we assume a normal distribution for the spectrum of possible mental ages, $x$,

$$g(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, \quad (1)$$

where $\sigma$ is the standard deviation, and ages are given in years throughout this work.

Note: Very strictly spoken, we would need to consider a truncated Gaussian pdf for the mental age distribution, with a lower bound within the period $x_0 \in [-0.75, 0]$, since that is the point where a human being starts to develop ($x = 0$ set at birth), and an upper bound at some higher age; however, the contributions of outer Gaussian sectors, especially in case of the product of two Gaussians, are negligibly small, as we show in the error discussion in the appendix.
If we randomly choose two people of ages \( \mu_1 \) and \( \mu_2 \), respectively, what is the probability that their mental ages are lying at most \( d \) years apart? Going with (1), the probability of person 1 having their mental age within \([x_1, x_1 + dx]\) is \( g_1 (x_1; \mu_1, \sigma_1) \) and person 2 being within the range \([x_1 - d, x_1 + d]\) is \( \int_{x_1-d}^{x_1+d} g_2 (x_2; \mu_2, \sigma_2^2) \) \( dx_2 \), so that the sought result can be very well approximated by taking the integral of the product of these expressions:

\[
p (d; \mu_1, \sigma_1, \mu_2, \sigma_2) = \int_{-\infty}^{\infty} dx_1 \int_{x_1-d}^{x_1+d} dx_2 g_1 (x_1; \mu_1, \sigma_1^2) g_2 (x_2; \mu_2, \sigma_2^2).
\]

(2)

This can also be expressed through the convolution of \( g_1 \) and \( g_2 \), where \( \mu_2 \) needs to be given the opposite sign\(^1\), integrated over the mental age span:

\[
p (d; \mu_1, \sigma_1, \mu_2, \sigma_2) = \int_{-d}^{d} dy (g_1 * g_2) (y; \mu_1, \sigma_1^2, -\mu_2, \sigma_2^2).
\]

(3)

Applying

\[
\mathcal{N} (\mu_1, \sigma_1^2) * \mathcal{N} (-\mu_2, \sigma_2^2) = \mathcal{N} (\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)
\]

(4)
to (1), and using the cumulative distribution function (cdf),

\[
\Phi \left( \frac{x-\mu}{\sigma} \right) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{x} dy e^{-\frac{1}{2} \left( \frac{y-\mu}{\sigma} \right)^2}
\]

(5)

we can solve (3) to

\[
p (d; \mu_1, \sigma_1, \mu_2, \sigma_2) = \Phi \left( \frac{\mu_1 - \mu_2 + d}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) - \Phi \left( \frac{\mu_1 - \mu_2 - d}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right).
\]

(6)

2.2 Special case

Assuming that the mental age of one person is known to be \( x_1 \), the probability of them to be mentally \( d \) or less years away from another randomly chosen person with pdf \( g (x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \) is computed by

\[
p (d, x_1; \mu, \sigma) = \int_{x_1-d}^{x_1+d} dx g (x_1; \mu, \sigma^2) = \Phi \left( \frac{x_1 - \mu + d}{\sigma} \right) - \Phi \left( \frac{x_1 - \mu - d}{\sigma} \right),
\]

(7)

which can be easily obtained by using (2) and setting the pdf of the person of known mental age equal to a Dirac delta function with peak at \( x = x_1 \).

\(^1\)In fact, it does not matter which \( \mu \) gets the negative sign; the idea is to imply a pdf for \( \Delta \mu \) by the convolution.
2.3 Normalized case

Given two randomly chosen individuals of ages $\mu_2 \leq \mu_1$, one can normalize the probability as per

$$p_0 (d; \mu_1, \sigma_1, \mu_2, \sigma_2) = \frac{p(d; \mu_1, \sigma_1, \mu_2, \sigma_2)}{p(d; \mu_2, \sigma_2, \mu_2, \sigma_2)},$$

where

$$p(d; \mu_2, \sigma_2, \mu_2, \sigma_2) \equiv p(d; \sigma_2) = \Phi \left( \frac{d}{\sqrt{2\sigma_2}} \right) - \Phi \left( -\frac{d}{\sqrt{2\sigma_2}} \right) = \text{erf} \left( \frac{d}{\sqrt{2\sigma_2}} \right),$$

so that $p_0 = 1$ if both individuals are of the same chronological age.

Setting the probability for people of different ages in relation to the probability for people of the same age, as the normalized expression does, provides a measure for a reasonable and proportionate evaluation of the former. We will revisit it in section 4.1.2 within a more detailed discussion of the parameters.

2.4 Individual case

In order to compute the probability that one individual of chronological age $\mu$ is within a certain mental age range $[x_1, x_2]$ is

$$p(x_1 \leq x \leq x_2; \mu, \sigma) = \int_{x_1}^{x_2} \text{d}x \cdot g(x; \mu, \sigma^2) = \Phi \left( \frac{x_2 - \mu}{\sigma} \right) - \Phi \left( \frac{x_1 - \mu}{\sigma} \right).$$

In particular, this means for a symmetric span around $\mu$ and an upper and lower bound, respectively,

$$p(\mu - d \leq x \leq \mu + d; \mu, \sigma) = \Phi \left( \frac{d}{\sigma} \right) - \Phi \left( -\frac{d}{\sigma} \right) = \text{erf} \left( \frac{d}{\sqrt{2}\sigma} \right)$$

$$p(x \geq x_0; \mu, \sigma) = 1 - \Phi \left( \frac{x_0 - \mu}{\sigma} \right)$$

$$p(x \leq x_0; \mu, \sigma) = \Phi \left( \frac{x_0 - \mu}{\sigma} \right),$$

where we used that $\lim_{x \to -\infty} \Phi(x) = 0$ and $\lim_{x \to \infty} \Phi(x) = 1$.

3 Statistical expectation

The above calculated probability is a mutual statement between two age groups, that is, it states the likelihood of a selected pair of people of given ages to be mentally compatible. Therefore, the computation of the statistical expectation of compatible pairs is rather straightforward. However, the statistical expectations within the groups themselves, like the number of people in one group that have at least one or more compatible counterparts in the other group, are interesting to consider as well; especially since there might be overlaps, that is, multiple people could be compatible.
with the same person or people.

Let us assume that one age group consists of $N_1$ and the other of $N_2$ individuals. The statistical expectation of the number of compatible pairs $n_{12}$ is simply the total number of possible pairs multiplied by the obtained probability:

$$n_{12} = N_1N_2p.$$  

(14)

The question is now, how many people of group 1 and group 2 compose these pairs.

It is quickly shown that the mean number of people in group 1 that are mentally compatible with one randomly chosen individual in group 2 and vice versa amount to

$$\bar{n}_{1\rightarrow 2} = N_1p \text{ and } \bar{n}_{2\rightarrow 1} = N_2p.$$  

(15)

The probability that at least one member of group 1 is mentally compatible with a randomly chosen member of group 2, amounts to $1 - (1 - p)^{N_1}$; so the expected value of members of group 2 that have at least one mentally compatible counterpart in group 1 would be obtained by multiplying the last result with $N_2$. For group 2, it works analogously. Hence the expected values are

$$n_1 = N_1 \left[ 1 - (1 - p)^{N_2} \right] \text{ and } n_2 = N_2 \left[ 1 - (1 - p)^{N_1} \right].$$  

(16)

One could go even further and ask for the number of people in one group that are mentally compatible with at least $k$ people in the other group. Since the Binomials distribution,

$$Pr (k; N, p) = \binom{N}{k}p^k (1 - p)^{N-k},$$  

(17)

states the probability of exactly $k$ successes out of $N$ independent trials, each subject to probability $p$, the probability of at least $k$ successes amounts to

$$Pr (m \geq k; N, p) = 1 - \sum_{m=0}^{k-1} Pr (m; N, p) = 1 - \sum_{m=0}^{k-1} \binom{N}{m}p^m (1 - p)^{N-m} \approx 1 - \frac{1}{\sqrt{2\pi Np(1-p)}} \int_0^k dx e^{-\frac{1}{2} \frac{(x-Np)^2}{Np(1-p)}}$$  

$$= 1 - \Phi \left( \frac{k-Np}{\sqrt{Np(1-p)}} \right) + \Phi \left( -\sqrt{\frac{Np}{1-p}} \right),$$  

(18)

where the approximation by a normal distribution is justified for sufficiently big $N$ fulfilling the condition

$$N > 9 \max \left( \frac{p}{1-p}, \frac{1-p}{p} \right).$$  

(19)

Note: (16) can be inferred from setting $k = 1$ in the first line of (18).
4 Parameter discussion

In this section, we infer ranges for the standard deviation \( \sigma \) and the mental age difference \( d \) from empirical data and a detailed analysis of the pdf, followed by a discussion of some interesting implications.

4.1 Ranges of parameters

4.1.1 Standard deviation

From the findings in [4, 5, 9–12], some of which are distinguishing between multiple sub-scales regarding different components of emotional intelligence, we obtain a reliable span of values for the statistical dispersions of the most significant domains of mental age, ranging over various age groups. One observation from the studies is that the average scores from the applied tests are about proportional to the chronological ages of the participating groups. This suggests that results scattered around the mean of one age group likely evolve accordingly as age progresses – and so the dispersion of those results. Hence one may write

\[
s \equiv \frac{\sigma}{\mu} = \text{const.} \tag{20}
\]

Regardless of the considered component of (emotional) intelligence, standard deviations obtained from said works are mostly scattered in between 10\% and slightly above 20\% of the mean value\(^2\). The full range of the scatter presumably results in part from focusing on different groups of people and factors of interest throughout the studies, like culture, gender etc.

In the following calculations, we will account for the bounds:

\[
0.1\mu \leq \sigma \leq 0.2\mu. \tag{21}
\]

We will see that the outcome could depend quite sensitively on slight shifts in that percentage, which is why a separate computation for the interval bounds is definitely sensible.

4.1.2 Mental age difference

The allowed mental age difference is, as already mentioned, subject to individual and cultural morality; however, since it would not really make sense to choose the allowed mental age difference stricter than the natural mental dispersion within one age group, it is reasonable to take the first

\(^2\)Bar-On and others used to normalize their score histograms to the common IQ-distribution with \( \mu = 100 \) and \( \sigma = 15 \).
standard deviation of the younger of two considered age groups as its lower bound:

\[ d \geq \min (\sigma_1, \sigma_2). \] (22)

There is also another interesting and meaningful way of deriving a measure for \( d \). Instead of looking at the pdf for mental age itself, as done so far, one may ask for the pdf of mental age differences within the same chronological age group. This can be obtained in an analogous fashion to above from the convolution of the pdf for mental age with itself:

\[
h(d) = \int_{-\infty}^{\infty} dx g(x; \mu, \sigma^2) g(x - d; \mu, \sigma^2)
\]

\[
= \int_{-\infty}^{\infty} dx g(x; \mu, \sigma^2) g(d - x; -\mu, \sigma^2)
\]

\[
= (g \ast g)(d; 0, 2\sigma^2),
\] (23)

from which we read off

\[ \sigma_d = \sqrt{2}\sigma. \] (24)

Note that, because \( d \geq 0 \), the true pdf of mental age differences within the same chronological age group is a half-normal distribution:

\[ \hat{h}(d) = 2h(d) \text{ for } d \geq 0. \] (25)

The sought measure is the mean mental age difference, so the expectation of \( d \) under this pdf:

\[ \langle d \rangle_{\hat{h}} = \sqrt{\frac{2}{\pi}} \sigma_d = \frac{2}{\sqrt{\pi}} \sigma \approx 1.128 \sigma, \] (26)

which is going to be the main reference value for \( d \) henceforth.

The dispersion of values around this mean amounts to

\[ \hat{\sigma}_d = \sqrt{\langle (d - \langle d \rangle_{\hat{h}})^2 \rangle_{\hat{h}}} = \sigma_d \sqrt{1 - \frac{2}{\pi}} = \sigma \sqrt{2 \left(1 - \frac{2}{\pi}\right)} \approx 0.853\sigma. \] (27)

The scope for the mental age differences can now be inferred from the intersection of the interval given by \( \langle d \rangle_{\hat{h}} \pm \hat{\sigma}_d \) and the lower bound \( d \geq \sigma \) in virtue of (22); hence, with (26) – (27) and the fact that \( \langle d \rangle_{\hat{h}} - \hat{\sigma}_d \approx 0.276\sigma < \sigma \), we have

\[ \sigma \leq d \leq 1.981\sigma. \] (28)
For different chronological ages, $\mu_1 \neq \mu_2$, it is thereby, in accordance with (22),

$$\sigma = \min (\sigma_1, \sigma_2).$$  \hspace{1cm} (29)

Another interesting consequence are the actual probabilities for two randomly chosen people from the same chronological age group. Using (9), we find

$$p(d = t\sigma, \mu, \sigma, \mu, \sigma) \equiv p(t) = \Phi\left(\frac{\mu}{\sqrt{\sigma}}\right) - \Phi\left(-\frac{\mu}{\sqrt{\sigma}}\right) = \operatorname{erf}\left(\frac{\mu}{2}\right).$$ \hspace{1cm} (30)

Note that this result is independent of both chronological age and standard deviation, since substituting $d$ with any constant multiple of $\sigma$ cancels the contribution of $\sigma$ completely. Tab. 1 lists the same-age probabilities of a few special cases for $d$:

| $d$ | $\sigma$ | $\frac{2}{\sqrt{\pi}}$ | $\sqrt{2}\sigma$ | $(d)_h + \hat{\sigma}d$ |
|-----|-----------|-------------------------|------------------|------------------------|
| $p$ | 0.52      | 0.58                    | 0.68             | 0.84                   |

Tab 1: $p$ vs. $d$ for some special cases

So, the probability that two randomly chosen people from the same chronological age group are mentally compatible is less than 60%, if the benchmark (20) is applied. This is an essential finding, as it provides a frame of reference for interpreting probabilities regarding different chronological age groups in the right (moral/ethical/legal) proportion.

### 4.2 Age ratio

Assuming $\mu_1 \geq \mu_2$ and writing, as in the last subsection, $\sigma_{1,2} = s_{1,2}\mu_{1,2}$ and $d = t\sigma_2$, we find

$$p = \Phi\left(\frac{\mu_1 - 1 + s_2 t}{\sqrt{s^2_1\left(\frac{\mu_1}{\mu_2}\right)^2 + s_2^2}}\right) - \Phi\left(\frac{\mu_2 - 1 - s_2 t}{\sqrt{s^2_1\left(\frac{\mu_2}{\mu_2}\right)^2 + s_2^2}}\right).$$ \hspace{1cm} (31)

So, besides the coefficients $s_1$, $s_2$, and $t$, the probability only depends on the ratio of the chronological ages – not their absolute values. If the relationship (20) was not applied, then the absolute values of $\mu_1$ and $\mu_2$ might be implicit in the choices of $s_1$ and $s_2$; unless we assume $s_1 = s_2$, in which case the latter, including their potential age-dependences, cancel.

One interesting implication is that, for example, the probability of mental compatibility between two people aged 16 and 24 is the same as for two people aged 24 and 36.

Important note: This result is to be interpreted under the consideration that the same factor $t$ is chosen for the allowed mental age difference; in particular, a randomly chosen 16 year-old is as likely to be mentally at most 1.6 years apart from a randomly chosen 24 year-old as a randomly
chosen 24 year-old is to be mentally at most 2.4 years apart from a randomly chosen 36 year-old, if one chooses the lowest bound with \( s_1 = s_2 = 0.1 \) and \( t = 1 \).

## 5 Applications

We apply the findings derived above to real-life examples. The implications of the math derived above are illustrated by computing statistical expectations for the mental compatibility of particular groups of individuals and analyzing popular age rules.

### 5.1 High school and college students

The usual age range in high school is 14-18 years and in college 18-22 years. If we take the youngest age of 14 years as a reference, then the probability that an individual in high school or college is mentally compatible with someone of chronological age 14 is illustrated in fig. 1. It is very interesting to see that about one ninth of people aged 14 and 18 are mentally compatible under minimum conditions; especially if contrasted with the fact that the probability for two 14 year-olds under the same conditions is only a bit more than one half.

![Fig. 1: Mental compatibility between age cohorts in high school/college and 14-y.o.’s; thin lines represent the upper and lower bounds and thicker lines the benchmark for \( s = 0.15 \) and \( t = \frac{2}{\sqrt{\pi}} \).](image)

Left: regular probabilities; right: normalized probabilities.

Another example: Considering the scopes of \( \sigma \) and \( d \), as explained in subsection 4.1, two people of chronological ages 16 and 20, respectively, are mentally compatible within a probability range \( 0.16 \leq p \leq 0.65 \); the benchmark at \( s = 0.15 \) and \( t = \frac{2}{\sqrt{\pi}} \) is \( p = 0.33 \). So at least about one sixth
of pairs formed by 16 and 20 year-olds can be considered mentally compatible within the present framework.

These results are significant, since they strongly suggest the potential of collaboration between students of different ages and merged classes of different grades for special projects – possibly even between high school and college students.

According to the National Center for Education Statistics (NCES) [17], the number of students attending grades 9 through 12 at public high schools in the United States was about 15.1 Million in Fall 2017. Assuming an approximate equipartition of this number across the four grades, every age group encompassed about 3.78 Million students. In the following, we will compute a few statistical expected values, as outlined in section 3, taking the minimum case calculated above with \( p \approx \frac{1}{9} \):

- With (14) at least \((3.78 \times 10^6)^2 \cdot \frac{1}{9} \approx 1.59 \times 10^{12} = 1.59\) trillion (out of \((3.78 \times 10^6)^2 \approx 14.3\) trillion possible) pairs of American public high school students aged 14 and 18, respectively, were mentally compatible in Fall 2017. (Note: The vast size of the last result is due to a huge number of intersections among the pairs)

- The expected number of students aged 18 who are mentally compatible with a randomly chosen 14 year-old – and vice versa – amounts with (15) to at least \(3.78 \times 10^6 \cdot \frac{1}{9} = 420,000\).

- According to (16), the expected values within both age groups, counting how many students have one or more mentally compatible counterparts in the other group, are infinitesimally close to 3.78 Million; that is, nearly every 14 year-old and nearly every 18 year-old has one or more – based on the average of 420 thousand, rather a great many of – persons in the other group whom they are mentally compatible with.

### 5.2 Age limits

If we were to set up an age limit in a way that is consistent with the present mathematical framework, we would first of all define a mental – not chronological – age limit \( x_{\text{min}} \), as most subjects pertaining to age laws, like adulthood, really mean a state of mind rather than cell aging (albeit by correlation – not by causation). Then we would need to agree on a limit (minimum or maximum) probability/expected relative frequency \( p_{\text{lim}} \) to decide on the threshold share of people within a “qualifying” chronological age group that are above or below the stipulated mental age, where \( \mu_{\text{lim}} \), with associated standard deviation \( \sigma_{\text{lim}} = s\mu_{\text{lim}} \), is the limit chronological age fulfilling that condition.

As a consequence, \( p_{\text{lim}} = 0.5 \) generally implies \( x_{\text{lim}} = \mu_{\text{lim}} \), which also follows immediately from the axial symmetry of a Gaussian around the vertical axis at its mean value, in case of negligible skewness. This means in particular for \( p_{\text{lim}} \geq 0.5 \): \( \mu_{\text{min}} \geq x_{\text{min}} \) and \( \mu_{\text{max}} \leq x_{\text{max}} \).
For lower age limits \((\text{lim} = \text{min})\), it is, together with (12),

\[
p_{\text{min}} = p \left( x \geq x_{\text{min}}; \mu_{\text{min}}, \sigma_{\text{min}} \right) = 1 - \Phi \left( \frac{x_{\text{min}} - \mu_{\text{min}}}{\sigma_{\text{min}}} \right), \tag{32}
\]

which has to be solved for \(\mu_{\text{min}}\) to get the minimum chronological age for the sought age limit,

\[
\mu_{\text{min}} = \frac{x_{\text{min}}}{1 + s \Phi^{-1}(1-p_{\text{min}})}, \tag{33}
\]

where \(\Phi^{-1}\) is the inverse cdf or the quantile function.

Analogously, for upper age limits \((\text{lim} = \text{max})\), we find with (13),

\[
p_{\text{max}} = p \left( x \leq x_{\text{max}}; \mu_{\text{max}}, \sigma_{\text{max}} \right) = \Phi \left( \frac{x_{\text{max}} - \mu_{\text{max}}}{\sigma_{\text{max}}} \right), \tag{34}
\]

and hence

\[
\mu_{\text{max}} = \frac{x_{\text{max}}}{1 + s \Phi^{-1}(p_{\text{max}})}, \tag{35}
\]

### 5.2.1 Example 1: age of majority

The age of majority is the age at which human beings enter adulthood. Although the complexity behind the complete definition is not relevant for our purposes here, it shall be remarked that it has varied from culture to culture throughout history and so given rise to a multiplicity of age laws in the world [18]. In the vast majority of countries, that number was set at 18, in some it is as small as 14, and in some as big as 21.

In reality, a limit for the chronological – not mental – age is taken as a measure; set mostly to \(\mu_{\text{min}} = 18\), the expression (33) implies the corresponding mental age limits, depending on the choices of \(s\) and \(p_{\text{min}}\), as depicted in fig. 2 below. It is obvious that those choices sensitively affect the mental age bar, which raises the question of the underlying reasoning.

**Upshot:** The fact that the legislators behind such age laws only refer to the chronological age either means that they don’t ascribe any relevance to the concept of mental age per se or simply equate both ages. The latter case would imply the choice \(p_{\text{min}} = 0.5\), for which \(x_{\text{min}} = \mu_{\text{min}}\) for all values of \(s\), if one presumed reasoning only remotely akin to the present work. Since that is rather unlikely, however, it seems that at best there is a mere focus on the statistical average disregarding the significant implications of the standard deviation, as detailed in this work; or they are well aware of it but dodge the question for lack of adequate theoretical support.

### 5.2.2 Example 2: senior age

The age from which a person is defined as “elder” ranges between 60 and 65 years. If one defines “senior” as a mostly mental state, one may analyze it as an upper age limit in an analogous fashion
Since those age numbers are in reality also solely of chronological nature, we would use (35) to find the corresponding mental age limits within the range of parameters. Fig. 2 depicts them for $\mu_{\text{max}} = 60$.

The upshot regarding that social system of chronological age limits would be similar to the one stated above for the age of majority.

Fig. 2: Mental age limits vs. limit probabilities for given chronological age limits: 18 (l.), 60 (r.).

5.3 The “half-your-age-plus-seven” rule

A fairly common rule of thumb in determining a lower (chronological) age bound to a “socially acceptable” gamut with regard to dating relationships is the so-called “half-your-age-plus-seven” rule [13], which is supposed to be taken literally:

$$\mu_{\text{min}} = \frac{1}{2}\mu + 7,$$

(36)

where $\mu$ is the chronological age of the inquiring person.

This rule can also be inverted to infer an upper age bound:

$$\mu_{\text{max}} = 2\mu - 14.$$

(37)

Albeit of unclear origin, this rule is reported as once not being meant as a boundary for appropriateness but rather a measure for the ideal (younger) age of a man’s bride [14][16]. In any
In view of mental age compatibility, finding a bound for chronological ages would most sensibly connect to defining a bound for the probability \( \Phi \). For the remainder of this subsection, \( \mu \) is the younger age and \( \Delta \) the maximum difference upward in terms of “social acceptibility”; then we would write, after some algebra,

\[
p_{\min} = \Phi \left( \frac{\Delta + t s_1}{\sqrt{s_1^2 + s_2^2 (1 + \frac{\Delta}{\mu})^2}} \right) - \Phi \left( \frac{\Delta - t s_1}{\sqrt{s_1^2 + s_2^2 (1 + \frac{\Delta}{\mu})^2}} \right),
\]

where we used \( \sigma_1 = s_1 \mu, \sigma_2 = s_2 (\mu + \Delta) \), and \( d = t \sigma_1 \) with \( t \geq 1 \) in accordance with (28).

The last expression is equivalent to the relation

\[
\frac{\Delta}{\mu} = f(p_{\min}, s_1, s_2, t),
\]

where the RHS is a constant, unless the parameters \( s_1 \) and \( s_2 \) are assumed to harbor a dependence on \( \mu \) and \( \Delta \). This is congruent with subsection 4.2.

Assuming \( s_1, s_2 = \text{const.} \) in accordance with (20), (38) requires a proportional relationship,

\[
\Delta = m \mu \text{ or } \mu_{\text{max}} = (m + 1) \mu,
\]

in order for the result to be a constant; otherwise, the relation would have a more complicated form, although it really appears that the empirically determined values of \( s \) are rather dense within the interval \([0.1, 0.2]\) and thus have not much of an impact within their scope of variation.

Rewriting (37) to \( \frac{\Delta}{\mu} = 1 - \frac{14}{\mu} \), one sees directly that \( \Delta \) and \( \mu \) deviate significantly from being proportional for realistic values of \( \mu \), since the approximation \( \frac{\Delta}{\mu} \approx 1 \) would require \( \mu \geq 280 \) to have an error \( \frac{14}{\mu} \leq 0.05 \) (or \( \mu \geq 140 \) if one goes with (40b)). Even allowing \( s_1 \) and \( s_2 \) to vary within \([0.1, 0.2]\), (38) cannot nearly be fulfilled with the substitution \( \frac{\Delta}{\mu} = 1 - \frac{14}{\mu} \) (see fig. 3).
**Fig. 3:** The half-your-age-plus-seven rule applied to the probability formalism with various combinations of $s_1, s_2 \in [0.1, 0.2]$

*Upshot:* If one sets the “half-your-age-plus-seven” rule as a basis, one does not arrive at a consistent lower-bound statement for the probability, as far as the current mathematical framework is concerned. Briefly, the “half-your-age-plus-seven” rule is not compatible with the present findings. Given that the formalism presented here is a consistently established mathematical model based on empirical data from scientific studies and the “half-your-age-plus-seven” rule a loose rule of thumb without scientific foundation nor clear origin reflecting the purpose it is claimed to have these days, the former merits preferential consideration by any rational means.

We conclude this subsection with a table displaying the values for $m$ from the proportional relationship between $\Delta$ and $\mu$, according to [40], for a few special cases of the $s$-values (we assume $s_1 = s_2$ independent of age) and $p_{\text{min}}$:

| $s$   | $p_{\text{min}} = 0.05$ | $p_{\text{min}} = 0.1$ | $p_{\text{min}} = 0.15$ |
|-------|-------------------------|------------------------|-------------------------|
| $s = 0.1$ | 0.39                    | 0.32                   | 0.28                    |
| $s = 0.15$ | 0.64                    | 0.51                   | 0.43                    |
| $s = 0.2$ | 0.92                    | 0.71                   | 0.59                    |

Tab. 2: $m$-values for $p_{\text{min}}$ vs. $s$
6 Conclusion

We have set up a formula for computing the probability that two individuals of given chronological ages are within a certain mental age difference. From there, we derived expressions for statistical expectations and condensed ranges and benchmarks for the parameters through empirical data from studies and in-depth analysis of the presented mathematical structure. With these findings, we could investigate the validity of popular assumptions about age differences and existing rules regarding age limits through detailed qualitative and quantitative elaboration.

For the entirety of high school and college students in the US from a given year, we computed the probabilities between cohorts that are further apart in age and derived the statistical expectations of both pairs and individuals that are mentally compatible. The results showed clearly that even between 14-year-olds and 18-year-olds there is a notable number of mentally compatible people that is far from any negligence.

We critically examined the sensibility of socially imposed chronological age limits by depicting the notable span of associated mental ages following from the presented formalism within the scope of parameters and concluded a lack of the concept of mental age in such legislations.

Finally, we questioned the “half-your-age-plus-seven” rule and found that it is not compatible with the presented statistical formalism. Given that a both empirically supported and mathematically consistent fundament is in clear contrast to a loose rule of thumb of unclear origin, said rule could be shown to stand on not too solid ground.

Altogether, we derived a formal basis on which we could bring forward substantial reasoning to question the largely prevailing construct of ideas around age.

More in-depth directed research in the field, i.a. in form of more specifically targeted studies, will help refine the methodology, particularly in terms of parameter accuracy, and increase the precision – and hence expressiveness – of results.

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A Appendix: Error discussion

A.1 Outer sectors

As mentioned earlier, the outer age sectors \((x < 0)\), which have been included in the integral to arrive at the closed-form solution \((\ref{eq:closed_form})\), have negligible contribution. This can be seen by secting \((\ref{eq:inner})\), which is the product of two Gaussians integrated over the area spanned by
\{(x_1, x_2) \in \mathbb{R}^2 | x_1 - d \leq x_2 \leq x_1 + d\}. In particular, one decomposes this domain into lines, defined by \(x_2 = x_1 + \Delta \ (\Delta \leq -d \leq d)\), and looks at the corresponding one-dimensional slices of the product function,

\[
\frac{1}{2\pi \sigma_1 \sigma_2} e^{-\frac{1}{2} \left( \frac{(x_1 - \mu_1)^2}{\sigma_1^2} + \frac{(x_2 - \mu_2 + \Delta)^2}{\sigma_2^2} \right)} = \frac{1}{2\pi \sigma_1 \sigma_2} e^{-\frac{1}{2} \left( \frac{(\mu_1 - \mu_2 + \Delta)^2}{\sigma_2^2} \right)} e^{-\frac{1}{2} \left( \frac{(x_1 - \mu_1)^2}{\sigma_1^2} \right)},
\]

(41)

where

\[
\mu_{12} = \frac{(\mu_2 - \Delta)\sigma_1^2 + \mu_1 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad \text{and} \quad \sigma_{12} = \frac{\sigma_1 \sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}.
\]

(42)

The trick is now to compare the integral of (41) over the outer sector, \([-\infty, 0]\), with the entire one used above, \([-\infty, \infty]\), for which the rewritten form on the RHS is very helpful:

\[
\int_{-\infty}^{0} dx_1 e^{-\frac{1}{2} \left( \frac{x_1 - \mu_{12}}{\sigma_{12}} \right)^2} = \frac{1}{\sqrt{2\pi \sigma_{12}}} \int_{-\infty}^{0} e^{-\frac{1}{2} \left( \frac{x - \mu_{12}}{\sigma_{12}} \right)^2} = \Phi \left( \frac{-\mu_{12}}{\sigma_{12}} \right).
\]

The argument on the very right can be estimated like

\[\frac{-\mu_{12}}{\sigma_{12}} = \frac{-(\mu_2 - \Delta)\sigma_1^2 + \mu_1 \sigma_2^2}{\sigma_1 \sigma_2 \sqrt{\sigma_1^2 + \sigma_2^2}} \leq -\frac{\mu_2 \sigma_1^2 + \mu_1 \sigma_2^2}{\sigma_1 \sigma_2 \sqrt{\sigma_1^2 + \sigma_2^2}} + \frac{\sigma_1}{\sigma_2 \sqrt{\sigma_1^2 + \sigma_2^2}} d \leq -\frac{\mu_2 - d}{\sigma_2},\]


where we assumed \(\frac{\mu_1}{\sigma_1} = \frac{\mu_2}{\sigma_2}\), as explained in section 4. Furthermore, assuming \(0.1 \leq \frac{\sigma}{\mu} \leq 0.2\) and \(d \sim \sigma\), we have

\[\Phi \left( \frac{-\mu_{12}}{\sigma_{12}} \right) < \Phi \left( \frac{-\mu_2}{\sigma_2} \right) \leq \Phi \left( -4 \right) \approx 3.17 \times 10^{-5}.
\]

Since this ratio bound holds for every slice of the integration domain, the contribution from the sector \(x < 0\) is negligible. The same applies to the sector \(x > \max(\mu_1, \mu_2)\) far beyond human age.

### A.2 Error propagation

The main uncertainty in this endeavor comes from the proper estimation of the standard deviations, \(\sigma_1\) and \(\sigma_2\), and the choice of the allowed mental age difference, \(d\). The error propagation resulting from these amounts to

\[
\Delta p (d; \mu_1, \sigma_1, \mu_2, \sigma_2) = \frac{\Delta d}{\sqrt{2\pi (\sigma_1^2 + \sigma_2^2)}} \left[ e^{-\frac{1}{2} \left( \frac{\mu_1 - \mu_2 + d}{\sigma_1^2 + \sigma_2^2} \right)^2} + e^{-\frac{1}{2} \left( \frac{\mu_1 - \mu_2 - d}{\sigma_1^2 + \sigma_2^2} \right)^2} \right] \left[ (\mu_1 - \mu_2 + d) e^{-\frac{1}{2} \left( \frac{\mu_1 - \mu_2 + d}{\sigma_1^2 + \sigma_2^2} \right)^2} - (\mu_1 - \mu_2 - d) e^{-\frac{1}{2} \left( \frac{\mu_1 - \mu_2 - d}{\sigma_1^2 + \sigma_2^2} \right)^2} \right].
\]

(43)

Let us compare the impacts of the single errors. The ratio of the standard deviation and the
mental age part is

\[
\frac{\Delta p|_{\Delta d=0}}{\Delta p|_{\Delta \sigma,1,2=0}} = \frac{1}{\sigma_1^2 + \sigma_2^2} \left( (\mu_1 - \mu_2) \tanh \left( \frac{2d(\mu_1 - \mu_2)}{\sigma_1^2 + \sigma_2^2} \right) + d \right) \frac{\sigma_1 \Delta \sigma_1 + \sigma_2 \Delta \sigma_2}{\Delta d}.
\]

(44)

If we take \(d = a\sigma_1, \mu_1 - \mu_2 = b\sigma_1, \) and \(\sigma_1 = c\sigma_2, \) with \(a, b, c > 1, \) this becomes

\[
\frac{\Delta p|_{\Delta d=0}}{\Delta p|_{\Delta \sigma,1,2=0}} = \frac{c}{1+c^2} \left[ b \tanh \left( \frac{2ab^2}{1+c^2} \right) + a \right] \frac{c\Delta \sigma_1 + \Delta \sigma_2}{\Delta d} > \frac{ac}{1+c^2} \left( \frac{2b^2c^2}{1+c^2+2abc} + 1 \right) \frac{\Delta \sigma_1 + \Delta \sigma_2}{\Delta d} > \frac{1}{2} \frac{a+b}{1+c} \frac{\Delta \sigma_1 + \Delta \sigma_2}{\Delta d},
\]

and

\[
\frac{\Delta p|_{\Delta d=0}}{\Delta p|_{\Delta \sigma,1,2=0}} < (a + b) \frac{\Delta \sigma_1 + \Delta \sigma_2}{\Delta d}
\]

(45)

where we used

\[
\frac{x}{x+1} < \tanh x < 1, \text{ for } x > 0.
\]

(47)

In case \(\Delta \sigma_1 \approx \Delta \sigma_2 \approx \Delta d, \) we get

\[
\frac{a+b}{1+c} < \frac{\Delta p|_{\Delta d=0}}{\Delta p|_{\Delta \sigma,1,2=0}} < 2 (a + b).
\]

(48)

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