Soft robust solutions to possibilistic optimization problems

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Abstract

This paper discusses a class of uncertain optimization problems, in which unknown parameters are modeled by fuzzy intervals. The membership functions of the fuzzy intervals are interpreted as possibility distributions for the values of the uncertain parameters. It is shown how the known concepts of robustness and light robustness, for the interval uncertainty representation of the parameters, can be generalized to choose solutions under the assumed model of uncertainty in the possibilistic setting. Furthermore, these solutions can be computed efficiently for a wide class of problems, in particular for linear programming problems with fuzzy parameters in constraints and objective function. In this paper a theoretical framework is presented and results of some computational tests are shown.

Keywords: Robust optimization, possibility theory, fuzzy intervals

1 Introduction

In this paper we wish to investigate the following optimization problem with uncertain parameters:

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad \tilde{A} x \leq b \\
& \quad x \in \tilde{X} \subset \mathbb{R}_+^n
\end{align*}
\]

In formulation (1), \(x\) is an \(n\)-vector of decision variables, \(\tilde{A} = (\tilde{a}_{ij})\) is an \((m \times n)\)-matrix of imprecise constraint coefficients and \(c\) is an \(n\)-vector of objective function coefficients. To simplify presentation, we first assume that the vector \(c\) is precisely known. We will show later that the approach proposed in this paper can be easily extended to the case of uncertain objective function coefficients. An \(m\)-vector \(b\) of right hand sides is also assumed to be precisely known. It does not cause loss of generality, as we can always add artificial variables and include uncertain right hand sides in matrix \(\tilde{A}\). We will denote by \(\tilde{a}_{i}^T x \leq b_i\) the \(i\)th imprecise constraint in (1), where \(\tilde{a}_{i}, \ i \in [m]\), is the \(i\)th row of \(\tilde{A}\) (throughout the paper we will use the notation \([m] = \{1, \ldots, m\}\)). We assume that \(\tilde{X}\) is a bounded subset of \(\mathbb{R}_+^n\), where \(\mathbb{R}_+\) is the set of nonnegative reals. For example, if \(X\) is a bounded
polyhedron, then we get an uncertain linear programming problem. If \( X = \{0,1\}^n \), then (1) becomes an uncertain combinatorial optimization problem. For a particular realization of the constraint coefficients \( A \) (called scenario), we get a deterministic counterpart \( P \) of \( \widetilde{P} \), which is a traditional optimization problem.

A typical method of solving (1) consists in replacing the imprecise constraints with some crisp equivalents and solving the resulting mathematical programming problem (see, e.g., [1, 2]). The method of constructing such a problem depends on the interpretation of the imprecise parameters, which in turn, depends on the information available. In many cases the resulting model is harder to solve than the deterministic counterpart \( P \). If \( \tilde{a}_i, i \in [m] \), are vectors of random variables with known probability distributions, then stochastic optimization framework can be used (see, e.g., [1]). Namely, we can replace the imprecise constraints with chance constraints of the form

\[
\Pr(\tilde{a}_i^T x \leq b_i) \geq 1 - \epsilon_i,
\]

where \( \epsilon_i \in (0, 1) \) is a given risk (significance) level. In practice, however, it is often difficult or even impossible to provide the parameter distributions. Furthermore, the resulting problem with chance constraints can be computationally hard to solve [1].

If the probabilistic information about the parameters is not available, then robust optimization framework can be applied (see, e.g., [2], [3]). Suppose we only know that \( A \in U \subseteq \mathbb{R}^{m \times n} \), where \( U \) is a given uncertainty (scenario) set, containing all possible realizations (scenarios) of the uncertain constraint coefficients. Using the robust framework, problem (1) is then expressed as:

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b \quad \forall A \in U \\
& \quad x \in X
\end{align*}
\]

The solutions to (2), called strictly robust, can be very conservative, as we require that the constraints are satisfied for all possible realizations of the parameters (see [1]). Several methods of relaxing the strict robustness have been proposed in the existing literature. One of the most popular was introduced in [5], where it is assumed for each constraint, that only a subset of the imprecise parameters can take their worst values. Then, each constraint is satisfied with a reasonable probability. We will described this idea in more detail in Section 2.

Another method of soften (2) is to relax the right hand sides of the constraints, which leads to the concept of light robustness, originally proposed in [6] and further discussed in [7]. In typical situations, where everything goes smoothly without any disturbances, the constraint coefficients will take some nominal values \( \hat{A} \in U, \hat{A} = (\hat{a}_{ij}) \), where \( \hat{a}_{ij} \) is the nominal value of uncertain matrix coefficient \( \tilde{a}_{ij} \). A robust solution should be feasible in the nominal scenario and also not too far from optimality under this scenario. This can be modeled by adding the crisp constraints \( \hat{A}x \leq \hat{b} \) and \( c^T x \leq \hat{c} + \rho_0 \), where \( \hat{c} \) is the optimal objective value of (1) under the nominal scenario \( \hat{A} \) and \( \rho_0 \) is a fixed tolerance. Finally, the constraints should be satisfied for all scenarios with some possible tolerances (deviations). The goal is now to minimize a distance of the deviations to the zero-vector. The light-robustness counterpart of (1) takes
then the following form $[6,7]$:

$$
\begin{align*}
\min & \quad \|\gamma\| \\
\text{s.t.} & \quad Ax \leq b + \gamma \quad \forall A \in \mathcal{U} \\
& \quad Ax \leq b \\
& \quad c^T x \leq \hat{c} + \rho_0 \\
& \quad \gamma \geq 0 \\
& \quad x \in X
\end{align*}
$$

(3)

where $\|\cdot\|$ denotes a given norm.

In the classic stochastic approach a full probabilistic information about the problem parameters is available, while in the traditional robust approach we may only know the supports of the distributions of the random parameters. Many problems arising in practice are located between these two boundary cases. Namely, a partial information about parameter distributions, such as their mean (nominal) values and variances, is available. We can then seek solutions that hedges against the worst probability distributions which may appear. This leads to various robust distributionally models discussed, see for instance $[8,9]$.

Another method of modeling incomplete probabilistic information involves fuzzy sets with their possibilistic interpretation. We assume that $\tilde{a}_i, i \in [m]$, are vectors of fuzzy quantities with specified possibility distributions. Possibility distribution can be seen as an estimation (upper bound) on the unknown probability distribution and some methods of constructing it from the available data can be found in $[10,11]$. We can now utilize this additional possibilistic information to improve the solution robustness by using possibility and necessity measures. For example, we can replace the imprecise constraints with fuzzy chance constraints of the form $\Pi(\tilde{a}_i^T x \leq b_i) \geq 1-\epsilon_i$ or $N(\tilde{a}_i^T x \leq b_i) \geq 1-\epsilon_i$, where $\Pi$ and $N$ and possibility and necessity measures, respectively (see, e.g., $[12,14]$). For a deeper discussion of fuzzy set theory and possibility theory in optimization we refer the reader to $[15]$.

The aim of this paper is to extend the robust concepts proposed in $[5,7]$ to the fuzzy case in the possibilistic setting. As in $[5]$, we will assume that for each uncertain parameter an interval of possible values is provided, which is symmetric around its nominal (expected) value. However, in our approach a possibility distribution within this interval can also be prescribed. This possibility distribution can be seen as an upper bound on the unknown probability distribution (see, e.g., $[10]$). Now, some parameter values within this interval are more plausible than others, which extends and refines the traditional interval uncertainty representation. Following $[5]$, we make a reasonable assumption that in practical situations it is unlikely that all parameters will deviate from their nominal values at the same time. Accordingly, we specify at most how many coefficients in each constraint can deviate from their nominal values. Then, following $[6,7]$, we provide an acceptable increase in the cost of a solution found. In order to choose a robust solution, we propose two necessity measure based criteria. Using the first criterion we seek a solution, called a best necessarily feasible, for which we are sure with the highest degree that it is protected against the worst parameter realizations. The second criterion, called a best necessary soft feasibility, is a relaxation of the previous one and is similar in spirit to the idea of light robustness (see model (3)). It is worth pointing out that both criteria will lead to computationally tractable problems for some important special cases of $\tilde{P}$.

This paper is organized as follows. In Section $2$ we recall the concepts of robustness and light robustness proposed in $[5,7]$. In Section $3$ we apply possibility theory to model the
uncertain problem parameters. We introduce a possibilistic model of uncertainty and provide its interpretation. In Section 4 we propose a concept of choosing a solution, which extends the traditional robust approach to the fuzzy (possibilistic) case. In Section 5 we further generalize the concept from Section 4 by using the idea similar to light robustness. In Section 6 we show how the uncertain objective function can be considered in our model. In Section 7 we provide an algorithm for solving the problem and identify the special cases which can be solved in polynomial time. Finally, in Section 8 we show the results of some experiments, which suggest that taking additional information about the uncertain parameters into account may lead to better solutions.

2 Robust and light robust solutions under interval uncertainty

In this section we briefly recall the robust and light robust approaches proposed in [5–7]. Consider the $i$th imprecise constraint $\tilde{a}_i^T x \leq b_i$. Suppose that $\tilde{a}_{ij}$ is a random variable, symmetrically distributed around its nominal (expected) value $\hat{a}_{ij}$. Hence, the value of $\tilde{a}_{ij}$ is only known to belong to the support $[\hat{a}_{ij} - \overline{a}_{ij}, \hat{a}_{ij} + \overline{a}_{ij}]$ of $\tilde{a}_{ij}$. Let $U_i$ be the Cartesian product of the supports and $\Gamma_i$ be an integer parameter in $[0, n]$, called protection level, which specifies the maximal number of coefficients in the constraint, whose values can be different from their nominal ones. Accordingly, define

$$S_i = \{(a_{ij})_{j \in [n]} \in \mathbb{R}^n : |\{j : a_{ij} \neq \hat{a}_{ij}\}| \leq \Gamma_i\}.$$  

(4)

We will consider all scenarios $a_i$, which are in $S_i \cap U_i$. Using the robust approach (2), we can rewrite the imprecise constraint as

$$\max_{a_i \in S_i \cap U_i} a_i^T x \leq b_i,$$  

(5)

which can be equivalently expressed as

$$a_i^T x + \max_{\{N \subseteq [n] : |N| \leq \Gamma_i\}} \sum_{j \in N} \overline{a}_{ij} x_j \leq b_i,$$  

(6)

where $\hat{a}_i$ is the vector of nominal constraint coefficient values, $\hat{a}_i = (\hat{a}_{ij})_{j \in [n]}$. Using the linear programming duality, the inequality (6) can be represented as the following system of linear constraints (see [5] for details):

$$\begin{align*}
\hat{a}_i^T x + \Gamma_i w_i + \sum_{j \in [n]} p_{ij} & \leq b_i \\
w_i + p_{ij} & \geq \overline{a}_{ij} x_j & j \in [n] \\
w_i & \geq 0, p_{ij} & \geq 0 & j \in [n]
\end{align*}$$  

(7)

It was shown in [5], that if $\tilde{a}_{i1}, \ldots, \tilde{a}_{in}$ are independent random variables, then the probability that the $i$th constraint will be violated is at most $\exp(-\Gamma_i^2/(2n))$. For example, in order to ensure that the probability of the constraint violation is not greater than 0.01, one should fix $\Gamma_i \geq \min\{n, 3.11\sqrt{n}\}$.

Applying (7) to all constraints we get the following robust counterpart of $\tilde{P}$:

$$\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad \hat{a}_i^T x + \Gamma_i w_i + \sum_{j \in [n]} p_{ij} \leq b_i \quad i \in [m] \\
& \quad w_i + p_{ij} \geq \overline{a}_{ij} x_j & i \in [m], j \in [n] \\
& \quad w_i \geq 0, p_{ij} \geq 0 & i \in [m], j \in [n] \\
& \quad x \in \mathbb{X}
\end{align*}$$  

(8)
The protection levels $\Gamma_i$ allow the decision maker to control the conservatism of the model. However, it is still assumed that a subset of the parameters will take the largest values of the corresponding supports. The probability of occurrence of the extreme values can be much less than other values within the supports. Model (8) does not take any additional information about the parameter distributions into account. In the next sections we will extend (8) to the case, in which possibility distributions for the parameters are specified.

Under the model of uncertainty assumed in this section, the light robust counterpart of (1) (see also (3)) takes the following form \cite{6,7}:

$$
\begin{align*}
\min & \|\gamma\| \\
\text{s.t.} & \hat{A}_i^T x + \Gamma_i w_i + \sum_{j \in [n]} p_{ij} x_j \leq b_i + \gamma_i, \quad i \in [m] \\
& w_i + p_{ij} \geq \bar{\alpha}_{ij} x_j, \quad i \in [m], j \in [n] \\
& \hat{A}x \leq \tilde{b} \\
& c^T x \leq \hat{c} + \rho_0 \\
& \gamma_i \geq 0, w_i \geq 0, p_{ij} \geq 0, \quad i \in [m], j \in [n] \\
& x \in \mathbb{X}
\end{align*}
$$

where $\rho_0 \geq 0$ is a fixed tolerance controlling the price of robustness, $\hat{c}$ is the optimal objective value of the deterministic counterpart under the nominal scenario $\hat{A}$, and $\|\cdot\|$ is a given norm. Model (9) is more flexible than (8). It allows us to fix a tradeoff between the robustness of a solution and its price (modeled by the parameter $\rho_0$). However, similarly to model (8), only the information contained in the supports of the uncertain parameters is exploited.

### 3 Possibilistic model of uncertainty

Possibility theory provides a framework of dealing with incomplete information. Its key feature is using two dual set functions, called possibility and necessity measures. A detailed description of possibility theory can be found in book \cite{11}. We now briefly describe (following \cite{10,16}) its main components, together with the interpretation assumed in this paper. The primitive object of possibility theory is a possibility distribution, which assigns to each element $u$ in universal set $\Omega$ a degree of possibility $\pi_\tilde{u}(u) \in [0,1]$. Function $\pi_\tilde{u}$ reflects the more or less plausible values of the unknown quantity $\tilde{u}$ taking values in $\Omega$. The possibility degree of an event $A \subseteq U$ is then

$$
\Pi(A) = \sup_{u \in A} \pi_\tilde{u}(u).
$$

Accordingly, the degree of necessity of an event $A \subseteq \Omega$ is

$$
N(A) = 1 - \pi_\tilde{u}(\overline{A}) = \inf_{u \notin A} (1 - \pi_\tilde{u}(u)),
$$

where $\overline{A}$ is the complement of $A$. The necessity measure satisfies the minitivity axiom, i.e. for any two events $A, B \subseteq \Omega$

$$
N(A \cap B) = \min \{N(A), N(B)\}.
$$

There are several interpretations of the possibility and necessity measures. In this paper we assume that possibility measure $\Pi$ is equivalent to the family $P(\Pi)$ of probability measures.
such that $\mathbf{P}(\Pi) = \{\Pr : \forall A \text{ measurable}, \Pr(A) \leq \Pi(A)\}$ or, equivalently, $\mathbf{P}(\Pi) = \{\Pr : \forall A \text{ measurable}, \Pr(A) \geq \Pi(A)\}$. Hence possibility distribution can be seen as an estimation (upper bound) on the unknown probability distribution, and for each event $A \subseteq \Omega$, $\Pi(A) \leq \Pr(A) \leq \Pi(A)$.

Consider uncertain parameter $\tilde{a}_{ij}$ in matrix $\tilde{A}$. In the approach described in Section 2 we only know the support $[\hat{a}_{ij} - \underline{a}_{ij}, \hat{a}_{ij} + \overline{a}_{ij}]$ of $\tilde{a}_{ij}$. However, in real applications more information about $\tilde{a}_{ij}$ can be provided, which can be utilized to improve the quality of the computed solution. In our model we assume that $\tilde{a}_{ij}$ is a fuzzy interval, whose membership function is continuous, symmetrically distributed around the nominal value $\hat{a}_{ij}$ and the support equal to $[\hat{a}_{ij} - \underline{a}_{ij}, \hat{a}_{ij} + \overline{a}_{ij}]$ (see Fig. 1). The membership function $\pi_{\tilde{a}_{ij}}$ of fuzzy interval $\tilde{a}_{ij}$ is interpreted as a possibility distribution for $\tilde{a}_{ij}$.

Recall that the set $\tilde{a}_{ij}^{\lambda} = \{v \in \mathbb{R} : \pi_{\tilde{a}_{ij}}(v) \geq \lambda\}$, $\lambda \in (0, 1]$, is called a $\lambda$-cut of $\tilde{a}_{ij}$ and contains all values of $\tilde{a}_{ij}$ whose possibility of occurrence is at least $\lambda$. We will assume that $\tilde{a}_{ij}^{0}$ is the support of $\tilde{a}_{ij}$. The sets $\tilde{a}_{ij}^{\lambda} = [\hat{a}_{ij} - \alpha_{ij}(\lambda), \hat{a}_{ij} + \alpha_{ij}(\lambda)]$, $\lambda \in [0, 1]$, form a nested family of closed intervals with centers equal to the nominal value $\hat{a}_{ij}$. The bound $\alpha_{ij}(\lambda)$ is a continuous, strictly decreasing function in $[0, 1]$, such that $\alpha_{ij}(0) = \overline{a}_{ij}$. For example, if $\tilde{a}_{ij}$ is a symmetric triangular fuzzy interval, then $\alpha_{ij}(\lambda) = \overline{a}_{ij} \cdot (1 - \lambda)$. One can, however, use also generalized functions $\alpha_{ij}(\lambda) = \overline{a}_{ij} \cdot (1 - \lambda^{2})$, $z > 0$, to better reflect the uncertainty (see Fig. 1). Namely, the smaller the value of $z$ the less uncertainty is associated with $\tilde{a}_{ij}$. For large $z$, $\tilde{a}_{ij}$ tends to the traditional closed interval. Applying (10) and the continuity of $\pi_{\tilde{a}_{ij}}$ yield

$$N(\tilde{a}_{ij}^{\lambda}) = 1 - \lambda.$$ 

Hence $\Pr(\tilde{a}_{ij}^{\lambda}) \geq 1 - \lambda$ and the probability that the value of $\tilde{a}_{ij}$ falls within $\tilde{a}_{ij}^{\lambda}$ is at least $1 - \lambda$.

Let $a_{i} = (a_{i1}, \ldots, a_{im}) \in \mathbb{R}^{n}$ be a scenario describing a realization of $\tilde{a}_{i}$ (a state of the world) in the $i$th imprecise constraint $a_{i}^{T}x \leq b_{i}$. The degree of possibility that scenario $a_{i} = (a_{i1}, \ldots, a_{im})$ will occur is provided by the following joint possibility distribution $\pi_{\tilde{a}_{i}}$ on the set of all possible scenarios, induced by possibility distributions $\pi_{\tilde{a}_{ij}}$, (see, e.g., [17]):

$$\pi_{\tilde{a}_{i}}(a_{i}) = \min_{j \in [n]} \pi_{\tilde{a}_{ij}}(a_{ij}). \quad (12)$$
We can now compute the set of all scenarios whose possibility of occurrence is at least \( \lambda \in [0,1] \) in the following way:

\[
U_i^\lambda = \{ \mathbf{a}_i \in \mathbb{R}^n : \pi_{\mathbf{\tilde{a}}_i}(\mathbf{a}_i) \geq \lambda \}
\]
\[
= \mathbf{\tilde{a}}_i^\lambda \times \mathbf{\tilde{a}}_2^\lambda \times \cdots \times \mathbf{\tilde{a}}_n^\lambda
\]  

(13)

and \( U_i^0 = \mathbf{\tilde{a}}_i^0 \times \cdots \times \mathbf{\tilde{a}}_n^0 \). Now \( N(U_i^\lambda) = 1 - \lambda \), \( \lambda \in [0,1] \), so the probability that \( \mathbf{a}_i \) will fall within \( U_i^\lambda \) is at least \( 1 - \lambda \).

### 4 A robust approach to possibilistic optimization problems

In this section we generalize the approach proposed in [5] (see Section 2) to the fuzzy case. We will use the possibilistic interpretation of the uncertain parameters, described in Section 3. Consider an imprecise constraint \( \mathbf{\tilde{a}}_i^T \mathbf{x} \leq b_i \), in which vector \( \mathbf{\tilde{a}}_i \) has a possibility distribution described as (12). As in Section 2 we provide a protection level \( \Gamma_i \), which is an integer in \([0,n]\) and bounds the number of components in \( \mathbf{\tilde{a}}_i \) whose realization values are different than their nominal ones. We can now compute the possibility of the event that the constraint will be \( \Gamma_i \)-protected for a given solution \( \mathbf{x} \in \mathbb{X} \) (\( \mathbf{x} \) is called \( \Gamma_i \)-feasible):

\[
\Pi(\mathbf{x} \text{ is } \Gamma_i \text{-FEAS}) = \sup_{\{\mathbf{a}_i \in S_i : \mathbf{\tilde{a}}_i^T \mathbf{x} \leq b_i\}} \pi_{\mathbf{\tilde{a}}_i}(\mathbf{a}_i),
\]

(14)

where \( S_i \) is defined as (4). Applying the duality between the possibility and necessity measures gives the degree of necessity that a solution \( \mathbf{x} \) is \( \Gamma_i \)-feasible:

\[
N(\mathbf{x} \text{ is } \Gamma_i \text{-FEAS}) = 1 - \Pi(\mathbf{x} \text{ is not } \Gamma_i \text{-FEAS})
\]
\[
= 1 - \sup_{\{\mathbf{a}_i \in S_i : \mathbf{\tilde{a}}_i^T \mathbf{x} > b_i\}} \pi_{\mathbf{\tilde{a}}_i}(\mathbf{a}_i).
\]

(15)

Observe that the quantity

\[
\sup_{\{\mathbf{a}_i \in S_i : \mathbf{\tilde{a}}_i^T \mathbf{x} > b_i\}} \pi_{\mathbf{\tilde{a}}_i}(\mathbf{a}_i)
\]

is the possibility of the event that the constraint is not protected, i.e. it can be violated under the assumption that at most \( \Gamma_i \) components of \( \mathbf{\tilde{a}}_i \) are different from their nominal values. Hence \( N(\mathbf{x} \text{ is } \Gamma_i \text{-FEAS}) \geq 1 - \lambda \), \( \lambda \in (0,1) \), if and only if for all coefficient scenarios \( \mathbf{a}_i \) such that \( \mathbf{a}_i \in S_i \) and \( \pi_{\mathbf{\tilde{a}}_i}(\mathbf{a}_i) \geq \lambda \), the inequality \( \mathbf{a}_i^T \mathbf{x} \leq b_i \) holds. Taking (13), we get the following proposition:

**Proposition 1** For each \( \lambda \in [0,1] \), \( N(\mathbf{x} \text{ is } \Gamma_i \text{-FEAS}) \geq 1 - \lambda \) if and only if

\[
\max_{\mathbf{a}_i \in S_i \cap U_i^\lambda} \mathbf{a}_i^T \mathbf{x} \leq b_i.
\]

(16)

We can now provide the following probabilistic interpretation of our model. If \( N(\mathbf{x} \text{ is } \Gamma_i \text{-FEAS}) \geq 1 - \lambda \), then the constraint is \( \Gamma_i \)-protected with probability at least \( 1 - \lambda \). Observe that (16) is a parametrized version of (5). Hence, it can be replaced with the system of constraints (7) in which \( \pi_{ij} \) is replaced with \( \alpha_{ij}(\lambda) = \pi_{ij} \cdot (1 - \lambda^2) \).

Let \( \hat{c} \) be the optimal objective value of the deterministic counterpart of \( \mathbf{\tilde{P}} \) under the nominal scenario \( \mathbf{\tilde{A}} \) and \( \rho_0 \geq 0 \) be a given tolerance parameter. Consider the crisp constraint

\[
\mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0,
\]

(17)

7
which ensures that the cost of solution $\mathbf{x}$ must be of some predefined distance from the optimal cost $\hat{c}$. The parameter $\rho_0$ controls the price of robustness of our model (see [5]). Namely, the greater is the value of $\rho_0$ the more relaxed is the optimality of the solution.

Now, given tolerance $\rho_0 \geq 0$, we wish to compute a solution, which satisfies all the constraints with the highest necessity degree. Namely, we focus on the following optimization problem:

$$\text{Nec} \tilde{P} : \max_{\{\mathbf{x} \in \mathbb{X} : \mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0\}} N(\mathbf{\Lambda}_{i=1}^m \mathbf{x} \text{ is } \Gamma_i\text{-Feas}).$$

An optimal solution $\mathbf{x}^*$ to $\text{Nec} \tilde{P}$ is called a best necessarily feasible solution. Using the minitivity axiom (see (11)), we can rewrite (18) as follows:

$$\max_{\{\mathbf{x} \in \mathbb{X} : \mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0\}} \min_{i \in [m]} n(\mathbf{x} \text{ is } \Gamma_i\text{-Feas}),$$

which in turn can be expressed as follows:

$$\max (1 - \lambda) \text{ s.t. } N(\mathbf{x} \text{ is } \Gamma_i\text{-Feas}) \geq 1 - \lambda \quad i \in [m]$$
$$\mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0$$
$$\mathbf{x} \in \mathbb{X}$$

(19)

By Proposition 1, we can rewrite (19) as

$$\max (1 - \lambda) \text{ s.t. } \max_{\mathbf{a}_i \in \mathcal{S}(\mathcal{U}(\hat{a}_i))} \mathbf{a}_i^T \mathbf{x} \leq b_i \quad i \in [m]$$
$$\mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0$$
$$\mathbf{x} \in \mathbb{X}$$

(20)

Finally, applying (7), we can represent $\text{Nec} \tilde{P}$ as the following mathematical programming problem:

$$\max (1 - \lambda) \text{ s.t. } \mathbf{\tilde{a}_i}^T \mathbf{x} + \Gamma_i \mathbf{w}_i + \sum_{j \in [n]} p_{ij} \leq b_i \quad i \in [m]$$
$$\mathbf{w}_i + p_{ij} \geq \alpha_{ij}(\lambda) x_j \quad i \in [m], j \in [n]$$
$$\mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0$$
$$\mathbf{w}_i \geq 0, p_{ij} \geq 0 \quad i \in [m], j \in [n]$$
$$0 \leq \lambda \leq 1$$
$$\mathbf{x} \in \mathbb{X}$$

(21)

where $\alpha_{ij}(\lambda) = \tilde{a}_{ij} \cdot (1 - \lambda^z)$. Model (21) is nonlinear due to the terms $\alpha_{ij}(\lambda) x_j$. A method of solving it will be shown in Section 7.

5 A soft robust approach to possibilistic optimization problems

In this section we propose a more general and flexible concept for choosing a robust solution to (11). Consider again the uncertain constraint $\mathbf{\tilde{a}_i}^T \mathbf{x} \leq b_i$, where $\mathbf{\tilde{a}_i}$ has a possibility distribution being as in (12). Solution $\mathbf{x}$ is feasible for scenario $\mathbf{a}_i$ if the crisp constraint $\mathbf{a}_i^T \mathbf{x} \leq b_i$
is satisfied. Following the idea of light robustness \textsuperscript{[6,7]} (see also \textsuperscript{[3]}), we relax the concept of feasibility by allowing some violation of the constraint. We assume that \( \mathbf{x} \) should now satisfy a flexible constraint under scenario \( \mathbf{a}_i \), which is of the form \( \mathbf{a}_i^T \mathbf{x} \leq \overline{B}_i \), where \( \overline{B}_i \) is a fuzzy set in \( \mathbb{R} \) with membership function \( \mu_{\overline{B}_i} \). The value of \( \mu_{\overline{B}_i}(\mathbf{a}_i^T \mathbf{x}) \) is the extent to which \( \mathbf{a}_i^T \mathbf{x} \) satisfies the flexible constraint. If \( \mu_{\overline{B}_i}(v) = 1 \) for \( v \leq b_i \) and \( \mu_{\overline{B}_i}(v) = 0 \) for \( v > b_i \), then the flexible constraint reduces to the crisp one. In order to model the right hand side of the flexible constraint, we will use fuzzy set \( \overline{B}_i \), shown in Fig. 2. Namely, \( \mu_{\overline{B}_i} \) is nonincreasing, \( \mu_{\overline{B}_i}(v) = 1 \) for \( v \leq b_i \) and \( \mu_{\overline{B}_i}(v) = 0 \) for \( v \geq b_i + \overline{b}_i \), where \( \overline{b}_i \geq 0 \) is a parameter denoting the maximal allowed constraint violation. Let

\[
\mu_{\overline{B}_i}^{-1}(\lambda) = \sup\{v : \mu_{\overline{B}_i}(v) \geq \lambda\}, \lambda \in (0, 1]
\]

be the pseudoinverse of \( \mu_{\overline{B}_i} \). We get \( \mu_{\overline{B}_i}^{-1}(\lambda) = b_i + \gamma_i(\lambda) \), where \( \gamma_i(\lambda) \) is nonincreasing function of \( \lambda \in [0, 1] \) such that \( \gamma_i(1) = 0 \). We will fix \( \mu_{\overline{B}_i}^{-1}(0) = b_i + \gamma_i(0) = b_i + \overline{b}_i \). One can choose, for example, \( \gamma_i(\lambda) = \overline{b}_i \cdot (1 - \lambda^2) \) for some \( z \geq 0 \) (see Fig. 2). Notice that the larger is the value of \( z \) the larger tolerance for the constraint violation is allowed.

![Figure 2: Fuzzy set \( \overline{B}_i \) with \( \mu_{\overline{B}_i}^{-1}(\lambda) = b_i + \gamma_i(\lambda) = b_i + \overline{b}_i \cdot (1 - \lambda^2) \), \( z \geq 0 \), representing the right hand side of the \( i \)th flexible constraint.](image)

We can now compute the possibility of the event that the soft constraint will be \( \Gamma_i \)-protected for a given solution \( \mathbf{x} \in \mathbb{X} \), i.e. the degree of possibility that \( \mathbf{x} \) is \( \Gamma_i \)-soft feasible:

\[
\Pi(\mathbf{x} \text{ is } \Gamma_i\text{-Feas}) = \sup_{\mathbf{a}_i \in S_i} \min\{\pi_{\overline{a}_i}(\mathbf{a}_i), \mu_{\overline{B}_i}(\mathbf{a}_i^T \mathbf{x})\}. \tag{22}
\]

Note that in \( \text{(22)} \) we jointly consider the uncertainty (induced by the uncertain coefficients in \( \overline{a}_i \)) and flexibility of the \( i \)th constraint (see \textsuperscript{[1,7]})). Accordingly, the degree of necessity that a solution \( \mathbf{x} \) is \( \Gamma_i \)-soft feasible is defined as follows:

\[
N(\mathbf{x} \text{ is } \Gamma_i\text{-Feas}) = 1 - \Pi(\mathbf{x} \text{ is not } \Gamma_i\text{-Feas}) \tag{23}
\]

\[
= 1 - \sup_{\mathbf{a}_i \in S_i} \min\{\pi_{\overline{a}_i}(\mathbf{a}_i), 1 - \mu_{\overline{B}_i}(\mathbf{a}_i^T \mathbf{x})\}.
\]

Thus \( N(\mathbf{x} \text{ is } \Gamma_i\text{-Feas}) \geq 1 - \lambda, \lambda \in (0, 1], \) if and only if for all scenarios \( \mathbf{a}_i \) such that \( \mathbf{a}_i \in S_i \) and \( \pi_{\overline{a}_i}(\mathbf{a}_i) \geq \lambda \), the inequality \( \mu_{\overline{B}_i}(\mathbf{a}_i^T \mathbf{x}) \geq 1 - \lambda \) holds. This inequality is equivalent to \( \mathbf{a}_i^T \mathbf{x} \leq \mu_{\overline{B}_i}^{-1}(1 - \lambda) = b_i + \gamma_i(1 - \lambda) \). Hence, \textsuperscript{(13)} leads to the following proposition:
Proposition 2 For each $\lambda \in [0,1]$, $N(\mathbf{x} \text{ is } \Gamma_i, \text{Feas}) \geq 1 - \lambda$ if and only if

$$\max_{\mathbf{a}_i \in S(U^i)} \mathbf{a}_i^T \mathbf{x} \leq b_i + \gamma_i(1 - \lambda),$$

(24)

where $\gamma_i(1 - \lambda) = \overline{b}_i \cdot (1 - (1 - \lambda)^z)$. 

We can now provide the following probabilistic interpretation of our model. If $N(\mathbf{x} \text{ is } \Gamma_i, \text{Feas}) \geq 1 - \lambda$, then the $i$th constraint is $\Gamma_i$-protected with the tolerance $\gamma_i(1 - \lambda)$, with probability at least $1 - \lambda$.

In the approach described in Section 4 we required that $\mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0$, where $\hat{c}$ is the optimal objective value of the deterministic counterpart under the nominal scenario $\hat{A}$ and $\rho_0 \geq 0$ is the assumed tolerance. We can replace this crisp constraint with a flexible constraint of the form $\mathbf{c}^T \mathbf{x} \lessgtr \tilde{C}$, where $\tilde{C}$ is a fuzzy set shown in Fig. 2 with the pseudoinverse $\mu_C^{-1}(\lambda) = \hat{c} + \zeta(\lambda) = \hat{c} + \rho_0 \cdot (1 - \lambda^z)$, where the interpretation of $\hat{c}$ and $\rho_0$ is the same as in Section 4. Now, $\mu_C(\mathbf{c}^T \mathbf{x})$ expresses a preference (satisfaction) about the deviation of $\mathbf{c}^T \mathbf{x}$ from $\hat{c}$ (less deviations are more preferred). We can define the necessity degree that the flexible constraint $\mathbf{c}^T \mathbf{x} \lessgtr \tilde{C}$ is satisfied as follows:

$$N(\mathbf{c}^T \mathbf{x} \lessgtr \tilde{C}) = 1 - \sup_{\{c : \mathbf{c}^T \mathbf{x} > c\}} \mu_C(c),$$

(25)

The following proposition is analogous to Proposition 2.

Proposition 3 For each $\lambda \in [0,1]$, $N(\mathbf{c}^T \mathbf{x} \lessgtr \tilde{C}) \geq 1 - \lambda$ if and only if

$$\mathbf{c}^T \mathbf{x} \leq \hat{c} + \zeta(1 - \lambda),$$

(26)

where $\zeta(1 - \lambda) = \rho_0 \cdot (1 - (1 - \lambda)^z)$.

Notice that we can control the flexibility of the constraint $\mathbf{c}^T \mathbf{x} \lessgtr \tilde{C}$ by changing the parameter $z$. If $z = 0$, then the computed solution must be optimal under the nominal scenario. On the other hand, if $z > 0$ is large, then the constraint tends to the crisp constraint $\mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0$, which was used in the model discussed in Section 4.

We can now extend model (18) by considering the following optimization problem:

$$\text{SOFT-NEC } \mathcal{P} : \max_{\mathbf{x} \in \mathbb{X}} \max_{i=1}^m (\mathbf{x} \text{ is } \Gamma_i, \text{Feas}) \land \mathbf{c}^T \mathbf{x} \lessgtr \tilde{C}).$$

(27)

An optimal solution to (27) is called a best necessary soft feasible. Using the minitivity axiom, Proposition 2 and 3, and applying the same reasoning as in Section 4, we can represent SOFT-NEC $\mathcal{P}$ as follows:

$$\max (1 - \lambda)$$

s.t.  

$$\mathbf{a}_i^T \mathbf{x} + \Gamma_i w_i + \sum_{j \in [n]} p_{ij} \leq b_i + \gamma_i(1 - \lambda) \quad i \in [m]$$

$$w_i + p_{ij} \geq \alpha_{ij}(\lambda)x_j \quad i \in [m], j \in [n]$$

$$\mathbf{c}^T \mathbf{x} \leq \hat{c} + \zeta(1 - \lambda)$$

$$w_i, p_{ij} \geq 0 \quad i \in [m], j \in [n]$$

$$\mathbf{x} \in \mathbb{X}$$

$$\lambda \in [0,1]$$

(28)
Assume that the maximum accepted magnitude of the constraint violation equals \( \rho \). Some intermediate value of \( \rho \) is the same as the optimal robust solution to (8). It can be reasonable to choose \( \rho = 1 \) (the ratio \( \rho \)-feasibility equals 1. In fact, for \( \rho = 6.29 \) (the ratio \( \rho \)-feasibility equals 0.64), which results in a large deterioration in the objective value, then the degree of necessary \( \Gamma_1 \)-feasibility equals 1. In fact, for \( \rho = 6.29 \), a best necessarily feasible solution is the same as the optimal robust solution to (8). It can be reasonable to choose some intermediate value of \( \rho \). For example, if we fix \( \rho = 3 \) (the ratio \( \rho \)-feasibility equals 0.3), then we get solution \( \mathbf{x}^* = (1, 0.6, 0.6, 0) \) with the degree of necessary \( \Gamma_1 \)-feasibility equal to 0.44.

Let us now compute a best necessarily soft feasible solution to (30) by solving the model (28). Assume that the maximum accepted magnitude of the constraint violation equals \( \overline{b}_1 = 2 \), i.e. it is at most 33\% of its nominal value equal to 6. The crisp right hand side in (30) is thus replaced with fuzzy set \( \overline{B}_1 \) with \( \mu_{\overline{B}_1}^{-1}(\lambda) = 6 + 2(1 - \lambda) \). We also replace the crisp constraint \( \mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0 \) with the flexible constraint \( \mathbf{c}^T \mathbf{z} \leq \overline{C} \), where \( \overline{C} \) is a fuzzy set with the pseudoinverse \( \hat{c} + \rho_0 \cdot (1 - \lambda) \). As in the previous model, \( \hat{c} = -10 \) and \( \rho_0 \) is a parameter denoting the maximum accepted tolerance.

Let us first investigate the deterioration of the objective function for various \( \rho_0 \) (see Fig. 4). Let \( \mathbf{x}^* \) be an optimal solution to (21) or (28) for a fixed \( \rho_0 \) and consider the ratio \( d(\rho_0) = \frac{\|\mathbf{c}^T \mathbf{x}^* - \hat{c}\|}{\hat{c}} \). Observe that for (21) the ratio \( d(\rho_0) \) grows linearly with \( \|\rho_0 / \hat{c}\| \). This is due to the constraint \( \mathbf{c}^T \mathbf{x} \leq \hat{c} + \rho_0 \), which is tight at \( \mathbf{x}^* \). Different behavior can be observed
Figure 3: The optimal objective values of (21) and (28), depending on the ratio $|\gamma_0/\hat{c}|$.

Figure 4: The ratio $d(\rho_0) = |(c^T x^* - \hat{c})/\hat{c}|$, where $x^*$ is an optimal solution to (21) or (28), depending on the ratio $|\rho_0/\hat{c}|$.

if $x^*$ is an optimal solution to (28). In general, the ratio $d(\rho_0)$ can be smaller, due to the constraint $c^T x \leq \hat{c} + \zeta(1 - \lambda) = \hat{c} + \rho_0 \cdot \lambda$, which is tight at $x^*$ and $\lambda^* \in [0, 1]$. Hence, model (28) returns solutions with smaller price of robustness.

In Fig. 3 the optimal objective functions of (21) and (28) are compared. For smaller ratios $|\rho_0/\hat{c}|$ the objective value of (28) is greater. This is the effect of relaxation of the constraint which dominates the preference imposed on the objective value. The situation reverses for larger ratios $|\rho_0/\hat{c}|$, where the preference about the objective value is relaxed. Then a solution computed has smaller price of robustness but also is less protected against the constraint violation.

6 Treating the uncertain objective function

In this section we show how the model discussed in Section 5 can be extended to take the uncertainty in the objective function into account. Consider the following generalization of $\bar{P}$:

$$\begin{align*}
\min_{\bar{c}^T x} & \\
\text{s.t.} & \bar{A}x \leq \bar{b} \\
x & \in \bar{X} \subset \mathbb{R}_+^n
\end{align*}$$
which is similar to (1) with the exception that \( \hat{c} \) is now \( n \)-vector of uncertain objective function coefficients. We will again use the possibilistic model of uncertainty, described in Section 3. Namely, \( \hat{c}_j, j \in [n] \), are modeled as fuzzy intervals with membership functions \( \pi_{\hat{c}_j} \), symmetrically distributed around the nominal values \( \hat{c}_j \) and supports \( [\hat{c}_j - \overline{\hat{c}}_j, \hat{c}_j + \overline{\hat{c}}_j] \). We will use \( \hat{c}_j^\lambda = [\hat{c}_j - \beta_j(\lambda), \hat{c}_j + \beta_j(\lambda)] \) to denote the \( \lambda \)-cut of \( \hat{c}_j \), where \( \beta_j(\lambda) = \pi_j(1 - \lambda^2) \) for a fixed \( \lambda > 0 \) (see Fig. 1). If \( \hat{c} \) is scenario describing a realization of the uncertain objective function coefficients, then applying the same reasoning as previously (see (12)), we can compute

\[
\pi_{\hat{c}}(c) = \min_{j \in [n]} \pi_{\hat{c}_j}(c_j).
\]

Then \( U_0^\lambda = \{ c \in \mathbb{R}^n : \pi_{\hat{c}}(c) \geq \lambda \} = \lambda_1^\lambda \times \lambda_2^\lambda \times \cdots \times \lambda_n^\lambda \) and \( U_0^0 = \lambda_1^0 \times \cdots \times \lambda_n^0 \). Now \( N(U_0^\lambda) = 1 - \lambda \), \( \lambda \in [0, 1] \), so the probability that \( c \) will fall within \( U_0^\lambda \) is at least \( 1 - \lambda \).

There are many approaches to deal with fuzzy objective function in uncertain problems of type (31). Often, \( \hat{c}^T x \) is replaced with minimizing \( r(\hat{c}^T x) \), where \( r \) is a real-valued ranking function \( \| \cdot \|_\infty \) \( \| \cdot \|_1 \). In another approach, a fuzzy goal \( \hat{g} \) is associated with the imprecise objective function and one can replace the fuzzy objective with maximizing \( N(\hat{c}^T x \leq \hat{g}) \), which is interpreted as the necessity degree of achieving the goal \( \hat{g} \). This concept can be soften \( \| \cdot \|_\infty \) \( \| \cdot \|_1 \) by maximizing \( N(\hat{z}(x) \leq \hat{g}) \), where \( \hat{z}(x) \) is a fuzzy set whose membership function describes a possibility distribution of the maximum regret of \( x \) (the maximum distance to the optimality of \( x \)). A drawback of this approach is that it may lead to problems computationally intractable (see, e.g., [21]).

In order to extend the approach presented in Section 5 to uncertain objective function, let us introduce additional variable \( x_0 \) and rewrite (31) as follows:

\[
\begin{align*}
\min & \quad x_0 \\
\text{s.t.} & \quad \hat{c}^T x - x_0 \leq 0 \\
& \quad \bar{A} x \leq b \\
& \quad x \in \mathcal{X}
\end{align*}
\]

(32)

Observe that (32) has crisp objective function and one additional imprecise constraint of the form \( \hat{c}^T x - x_0 \leq 0 \). Hence, it is of the form (1). We can treat this new constraint just in the same way as the original constraints (see Section 5). Namely, let us define a protection level \( \Gamma_0 \), being an integer in \([0, n]\). Then

\[
\mathcal{S}_0 = \{ (c_j)_{j \in [n]} \in \mathbb{R}^n : |\{ j : c_j \neq \hat{c}_j \}| \leq \Gamma_0 \}.
\]

Let us introduce fuzzy set \( \hat{B}_0 \) (see Fig. 2) with the pseudoinverse \( \mu^{-1}_{\hat{B}_0}(\lambda) = \gamma_0(\lambda) = \overline{b}_0 \cdot (1 - \lambda^2) \). Accordingly, we can define

\[
N((x_0, x) \text{ is } \Gamma_0-\text{FEAS}) = 1 - \Pi((x_0, x) \text{ is not } \Gamma_0-\text{FEAS}) = 1 - \sup_{c \in \mathcal{S}_0} \min\{ \pi_{\hat{c}}(c), 1 - \mu_{\hat{B}_0}(c^T x - x_0) \}.
\]

(33)

The following proposition is analogous to Proposition 2

**Proposition 4** For each \( \lambda \in [0, 1] \), \( N((x_0, x) \text{ is } \Gamma_0-\text{FEAS}) \geq 1 - \lambda \) if and only if

\[
\max_{c \in \mathcal{S}_0 \cap U_0^\lambda} c^T x - x_0 \leq \gamma_0(1 - \lambda),
\]

(34)

where \( \gamma_0(1 - \lambda) = \overline{b}_0 \cdot (1 - (1 - \lambda)^2) \).
Let \( \hat{c} \) be the optimal objective value of the deterministic counterpart of (32) under the nominal scenario \((\hat{A}, \hat{c})\). The flexible constraint \( c^T x \leq \hat{C} \), considered in Section 5, becomes then \( x_0 \leq \hat{C} \), where \( \hat{C} \) is defined in the same way as in Section 5. We can now extend SOFT-NEC \( \hat{P} \) (see (27)) by using the necessity degree of conjunction of the following events:

\[
\land_{i=1}^{m} (x) \leq \Gamma_{i} - \hat{\text{FEAS}}) \land ((x, x_0) \leq \Gamma_{0} - \hat{\text{FEAS}}) \land x_0 \leq \hat{C}.
\]

Taking Proposition 4 into account and applying the same reasoning as in Section 4, we can represent SOFT-NEC \( \hat{P} \) as the following mathematical programming problem:

\[
\begin{align*}
\text{max} \quad & (1 - \lambda) \\
\text{s.t.} \quad & \hat{c}^T x + \Gamma_0 w_0 + \sum_{j \in [n]} q_j - x_0 \leq \gamma_0 (1 - \lambda) \\
& w_0 + q_j \geq \beta_j(\lambda)x_j \quad j \in [n] \\
& \hat{a}_i^T x + \Gamma_i w_i + \sum_{j \in [n]} p_{ij} \leq b_i + \gamma_i(1 - \lambda) \quad i \in [m] \\
& x_0 \leq \hat{c} + \zeta(1 - \lambda) \\
& w_i \geq 0 \quad i \in [m] \cup \{0\} \\
& q_j \geq 0 \quad i \in [m], j \in [n] \\
& p_{ij} \geq 0 \quad i \in [m], j \in [n] \\
& x \in \mathbb{X} \\
& \lambda \in [0, 1]
\end{align*}
\]

Observe that the variable \( x_0 \) can be eliminated from (35), which yields:

\[
\begin{align*}
\text{max} \quad & (1 - \lambda) \\
\text{s.t.} \quad & \hat{c}^T x + \Gamma_0 w_0 + \sum_{j \in [n]} q_j \leq \hat{c} + \zeta(1 - \lambda) + \gamma_0 (1 - \lambda) \\
& w_0 + q_j \geq \beta_j(\lambda)x_j \quad j \in [n] \\
& \hat{a}_i^T x + \Gamma_i w_i + \sum_{j \in [n]} p_{ij} \leq b_i + \gamma_i(1 - \lambda) \quad i \in [m] \\
& w_i \geq 0 \quad i \in [m] \cup \{0\} \\
& q_j \geq 0 \quad i \in [m], j \in [n] \\
& p_{ij} \geq 0 \quad i \in [m], j \in [n] \\
& x \in \mathbb{X} \\
& \lambda \in [0, 1]
\end{align*}
\]

where \( \alpha_{ij}(\lambda) = \pi_{ij} \cdot (1 - \lambda^z) \), \( \beta_j(\lambda) = \pi_j \cdot (1 - \lambda^z) \), \( \gamma_i(1 - \lambda) = \bar{b}_i \cdot (1 - (1 - \lambda)^z) \), and \( \zeta(1 - \lambda) = \rho_0 \cdot (1 - (1 - \lambda)^z) \). Model (36) generalizes (21) and (28). Indeed, if there is no uncertainty in the objective, then \( e = \hat{c}, b_0 = 0 \), and \( \pi_j = 0 \) for each \( j \in [n] \). Then the first two constraints of (36) reduce to \( e^T x \leq \hat{c} + \zeta(1 - \lambda) \), which yields (28). Fixing further large \( z \) in \( \zeta(1 - \lambda) = \rho_0 \cdot (1 - (1 - \lambda)^z) \) and \( \bar{b}_i = 0 \) for all \( i \in [m] \) leads to (21).

### 7 Solving the problem

Let us focus on solving SOFT-NEC \( \hat{P} \). We will study the most general model (36), in which an uncertain objective function is taken into account. For a fixed value of \( \lambda \in [0, 1] \), all the
constraints in (36) (possibly, except for the ones describing \( x \in X \)) become linear. Let \( X^\lambda \subseteq X \) be the set of feasible solutions to (36) for a fixed value of \( \lambda \in [0, 1] \). Since all the functions \( \alpha_{ij}(\lambda), \beta_j(\lambda), \gamma_i(\lambda), \zeta(\lambda) \) are nonincreasing, we get \( X^\lambda_1 \subseteq X^\lambda_2 \) if \( \lambda_1 \leq \lambda_2 \). Consequently, (36) can be solved by computing the smallest value \( \lambda_{\text{min}} \in [0, 1] \) for which \( X^{\lambda_{\text{min}}} \) is nonempty. This can be done by applying a binary search in the interval \([0, 1]\) (see Algorithm 1).

**Algorithm 1: Solving Soft Nec-\( \tilde{\mathcal{P}} \) with accuracy \( \varepsilon > 0 \)**

1. \( \lambda \leftarrow 1, \lambda \leftarrow 0; \)
2. \( \hat{c} \leftarrow \hat{c}^T x^* = \min \{ e^T x : \hat{A} x \leq b, x \in X \} ; \)
3. while \( |\lambda - \lambda| > \varepsilon \) do
   4. \( \lambda \leftarrow \lambda + (\lambda - \lambda)/2; \)
   5. if there exists \( x \) feasible to (36) for \( \lambda \) then
      6. \( x^* \leftarrow x, \lambda \leftarrow \lambda \)
   7. else \( \lambda \leftarrow \lambda; \)
8. return \( x^*, 1 - \lambda; \)

The running time of Algorithm 1 depends on the complexity of the problem which must be solved in Steps 2 and 5, i.e. checking the feasibility of (36) for a fixed \( \lambda \in [0, 1] \). In Step 2 the feasibility of (36) is implicitly checked for \( \lambda = 1 \). Indeed, it is easily seen that this task can be reduced to solving the deterministic counterpart of \( \tilde{\mathcal{P}} \) under the nominal scenario \((\hat{A}, \hat{c})\), since such solution \( x^* \) computed is always feasible to (36) for \( \lambda = 1 \). Thus the computational complexity of Steps 2 and 5 depends on the structure of the set \( X \). If the feasibility can be checked in \( T(|I|) \) time, where \( |I| \) is the size of (36), then Algorithm 1 runs in \( O(\lceil \log \varepsilon^{-1} \rceil T(|I|)) \) time, because the feasibility must be tested at most \( \lceil \log \varepsilon^{-1} \rceil + 1 \) times. If \( T(|I|) \) is polynomial in size \( |I| \), then Algorithm 1 runs in polynomial time and Soft-Nec-\( \tilde{\mathcal{P}} \) can be solved in polynomial time with a fixed accuracy \( \varepsilon > 0 \). In the next section we will identify some important special cases of \( \tilde{\mathcal{P}} \) for which this is the case.

### 7.1 Tractable problems

If \( X \) is a polyhedron in \( \mathbb{R}_+^n \), then \( \tilde{\mathcal{P}} \) is an uncertain linear programming problem. In this case (36), for a fixed \( \lambda \in [0, 1] \), is a system of linear constraints over \( \mathbb{R}_+^n \), whose feasibility can be tested in polynomial time. In consequence, Soft-Nec-\( \tilde{\mathcal{P}} \) can be then solved in polynomial time with a fixed accuracy \( \varepsilon > 0 \). If the integrality assumptions on some variables are imposed or \( X \subseteq \{0, 1\}^n \), then checking the feasibility of (21), for a fixed \( \lambda \in [0, 1] \), is NP-hard in general. We now describe a special case of such a problem, which can be solved efficiently.

Consider the following combinatorial optimization problem with uncertain costs:

\[
\begin{align*}
\min & \quad \hat{c}^T x \\
\text{s.t.} & \quad x \in X \subseteq \{0, 1\}^n
\end{align*}
\]

(37)

Notice that (37) is a special case of (31), in which the uncertainty affects only the objective
function. From (36), it may be concluded that (37) can be expressed as follows:

$$\max (1 - \lambda)$$

s.t. $$\hat{c}^T \mathbf{x} + \Gamma_0 w_0 + \sum_{j \in [n]} q_j \leq \hat{c} + \zeta(1 - \lambda) + \gamma_0(1 - \lambda)$$

$$w_0 + q_j \geq \beta_j(\lambda)x_j \quad j \in [n]$$

$$w_0 \geq 0$$

$$\mathbf{x} \in \mathbb{X} \subseteq \{0, 1\}^n$$

$$\lambda \in [0, 1]$$

(38)

where $$\hat{c} = \min_{\mathbf{x} \in \mathbb{X}} \hat{c}^T \mathbf{x}.$$ Using similar relation as the one between (5) and (7), we can equivalently express (38) as

$$\max 1 - \lambda$$

s.t. $$\max_{\mathbf{c} \in \mathbb{S}_0 \cap \mathbb{U}_0^\lambda} \mathbf{c}^T \mathbf{x} \leq \hat{c} + \zeta(1 - \lambda) + \gamma_0(1 - \lambda)$$

$$\mathbf{x} \in \mathbb{X} \subseteq \{0, 1\}^n$$

(39)

where $$\mathbb{S}_0$$ and $$\mathbb{U}_0^\lambda$$ were defined in Section 6. In order to test the feasibility of (39) of a fixed $$\lambda$$, we can first solve the problem

$$\min_{\mathbf{x} \in \mathbb{X}} \max_{\mathbf{c} \in \mathbb{S}_0 \cap \mathbb{U}_0^\lambda} \mathbf{c}^T \mathbf{x}$$

(40)

and check then if the optimal objective value of (40) is not greater than $$\hat{c} + \zeta(1 - \lambda) + \gamma_0(1 - \lambda)$$. To solve (40) we can use the algorithm proposed in [22, Theorem 3]. It consists of solving $$n + 1$$ deterministic counterparts of problem (40) in $$(n + 1)T(n)$$ time, where $$T(n)$$ is the time required to solve one deterministic problem. The algorithm for solving (39) is an adaptation of Algorithm 1 (it is enough to solve deterministic problem under the nominal costs $$\mathbf{c}$$ in Step 2 and apply the algorithm proposed in [22, Theorem 3] in Step 5). Its overall running time is now $$O(nT(n)[\log \epsilon^{-1}])$$, where $$\epsilon > 0$$ is a given accuracy. The algorithm is polynomial under the assumption that solving the deterministic counterpart of $$\tilde{P}$$ can be done in polynomial time. This is true for such problems as: shortest path, minimum spanning tree, minimum assignment, etc.

8 Computational experiments

In this section we show the results of some computational tests. Our goal is to compare the soft robust approach in the possibilistic setting, proposed in Section 5, to the concept of light robustness presented in [6, 7]. We examine uncertain linear programming problem of the following form:

$$\min \mathbf{c}^T \mathbf{x}$$

s.t. $$\tilde{a}_i^T \mathbf{x} \leq b_i \quad i \in [m]$$

$$\mathbf{x} \in [0, 1]^n$$

(41)

We assume that the objective function is deterministic (only the constraints are uncertain). An instance I of the problem (41) is generated as follows:

1. the number of variables $$n = 100$$ and the number of constraints $$m = 5$$;

2. each cost $$c_j, j \in [n]$$, is a random integer, uniformly distributed in the interval $$[-100, -1];
3. the nominal value of the constraint coefficient \( \hat{a}_{ij} \) is a random integer, uniformly distributed in the interval \([1, 100]\) and the bound \( \sigma_{ij} \) is set to \( \sigma \cdot \hat{a}_{ij} \), where \( \sigma \) is a random number uniformly distributed in the interval \([0, 1]\);

4. we fix \( \rho_i = 0.3 \sum_{j \in [m]} \hat{a}_{ij} \) for each \( i \in [m] \).

We set the protection levels \( \Gamma_i = 30 \) for each \( i \in [m] \). In the light robustness concept (see model (22)) we use the \( ||\gamma||_\infty = \max\{\gamma_1, \ldots, \gamma_m\} \) norm. In the soft robust approach we assume the 10\% tolerance for the constraint violation, i.e. \( \tilde{b}_i = 0.1b_i \) for each \( i \in [m] \). For the membership functions of all fuzzy sets we fix \( z = 1 \), so their membership functions are piecewise linear. In particular, the uncertain coefficients \( \hat{a}_{ij} \) are triangular fuzzy intervals. Let \( \hat{c} \) be the optimal objective value of the deterministic counterpart of (41) under the nominal scenario \( \hat{A} \). We will choose \( \rho_0 = p \cdot \hat{c} \) for \( p \in \{0, 0.2\%, 0.4\%, \ldots, 10\%\} \).

Let \( \mathbf{x} \in \mathbb{X} \) be a solution to (41). We will compute the distance of \( \mathbf{x} \) to the optimum under the nominal scenario as follows:

\[
d(\mathbf{x}) = \frac{\mathbf{c}^T \mathbf{x} - \hat{c}}{|\hat{c}|}.
\]

The value of \( d(\mathbf{x}) \) is the price of robustness of \( \mathbf{x} \). In order to evaluate the a posteriori quality of \( \mathbf{x} \) we use the following Monte Carlo simulation. For each coefficient \( \hat{a}_{ij} \), independently, we generate its value (realization) as follows. First we choose uniformly at random \( \lambda \in [0, 1] \) and then uniformly at random the realization \( a_{ij} \in [\hat{a}_{ij} - \pi_{ij}(1 - \lambda), \hat{a}_{ij} + \pi_{ij}(1 - \lambda)] \). Observe that realizations closer to \( \hat{a}_{ij} \) are more probable. This gives us a scenario \( \mathbf{A} = (a_{ij}) \in \mathbb{R}^{n \times m} \), which provides a deterministic counterpart of (41). For this deterministic problem we compute the magnitude of the constraint violation of \( \mathbf{x} \), i.e. the value \( \text{viol}(\mathbf{x}, \mathbf{A}) = \max_{i \in [m]}[\sum_{j}[a_{ij} x_{j} - b_i]/b_i]^{+} \), where \( [y]^{+} = \max\{0, y\} \). After generating a set \( \mathcal{A} \) of 1000 random scenarios, we computed the fraction of the scenarios under which \( \mathbf{x} \) is infeasible, i.e.

\[
\#\text{infeas}(\mathbf{x}) = \frac{|\mathbf{A} \in \mathcal{A} : \text{viol}(\mathbf{x}, \mathbf{A}) > 0|}{1000}
\]

and the average magnitude of the constraint violation

\[
\text{aviol}(\mathbf{x}) = \frac{1}{1000} \sum_{\mathbf{A} \in \mathcal{A}} \text{viol}(\mathbf{x}, \mathbf{A}).
\]

The quantities \( d(\mathbf{x}) \), \#\text{infeas}(\mathbf{x}) \) and \( \text{aviol}(\mathbf{x}) \) can be seen as a posteriori evaluation of the quality of \( \mathbf{x} \).

The experiments were performed as follows. For each \( p \in \{0, 0.2\%, 0.4\%, \ldots, 10\%\} \) we generated 100 instances \( I_1, \ldots, I_{100} \) as shown in points (14) and (15). For each instance \( I_i \), we fixed \( \rho_0 = p \cdot \hat{c}_i \) and computed an optimal light robust solution \( \mathbf{x}^{L}_i \), by solving (9), and a best necessarily soft feasible solution \( \mathbf{x}^{S}_i \), by solving (28). We computed the average qualities of the solutions. Namely, the average qualities of optimal light robust solutions are

\[
d^{L}(p) = \frac{1}{100} \sum_{i \in [100]} d(\mathbf{x}^{L}_i),
\]

\[
\#\text{infeas}^{L}(p) = \frac{1}{100} \sum_{i \in [100]} \#\text{infeas}(\mathbf{x}^{L}_i),
\]

\[
\text{aviol}^{L}(p) = \frac{1}{100} \sum_{i \in [100]} \text{aviol}(\mathbf{x}^{L}_i).
\]
\[ \text{aviol}^L(p) = \frac{1}{100} \sum_{i \in [100]} \text{aviol}(x_i^L). \]

The value of \#infeas^L(p) can be interpreted as the fraction of 100 000 deterministic counterparts for which an optimal light robust solution was infeasible (at least one constraint was violated) for a fixed \( p \). Accordingly, the value of aviol^L(p) is the average magnitude of the infeasibility. The quantities \( d^S(p) \), \#infeas^S(p) and aviol^S(p) for the set of best necessarily soft feasible solutions are computed in the same way.

![Figure 5: Average prices of robustness for various \( p = \rho_0/|\hat{c}| \).](image1)

![Figure 6: Fractions of infeasible solutions for various \( p = \rho_0/|\hat{c}| \).](image2)

Fig. 5 shows the average prices of robustness of the computed solutions for various ratios \( p = \rho_0/|\hat{c}| \). One can observe that \( x^S \) have smaller prices of robustness than \( x^L \). Furthermore, the difference between the prices becomes greater for larger \( p \). This observation can be explained as follows. In model (9) we use the constraint \( c^T x \leq \hat{c} + \rho_0 \), which is tight at the optimum. So, the figure of \( d^L(p) \) is linear. In contrast, in the model (28) we use the flexible constraint, which yields \( c^T x \leq \hat{c} + \zeta(1 - \lambda) = \hat{c} + \lambda \rho_0 \). Because, \( \lambda \in [0,1] \), the cost of the solutions \( x^S \) can be closer to \( \hat{c} \).

Fig. 6 and 7 show the fractions of infeasible solutions and the average magnitude of constraints violations for both tested approaches. If \( p = 0 \), then both \( x^S \) and \( x^L \) must be optimal under \( \hat{c} \) (their prices of robustness equal 0). In this case they robustness is very weak, i.e. almost all deterministic counterparts are infeasible. Increasing \( p \) (equivalently, the tolerance \( \rho_0 \)), we can improve the robustness of both \( x^S \) and \( x^L \). For \( p \geq 10\% \) almost all
deterministic counterparts are feasible. However, the average price of robustness of $x^L$ is 0.1 whereas the average price of robustness of $x^H$ is about 0.06. For $p \in (0, 7.5\%)$ the solutions $x^S$ are more robust than $x^L$, have smaller average magnitude of the constraints violation and also have smaller price of robustness. We can thus conclude that taking the possibilistic information into account can improve the quality of the obtained solutions.

Figure 7: Average magnitudes of infeasibility for various $p = \rho_0/|\hat{c}|$.

9 Conclusions

In this paper we have proposed a new concept of choosing a solution in uncertain optimization problems, in which unknown parameters are modeled by fuzzy intervals whose membership functions are regarded as possibility distributions for their values. In the traditional robust approach the values of uncertain parameters are only known to belong to a given uncertainty set $U$. We then seek a solution which behaves reasonably under the worst parameter realizations in $U$. This traditional robust approach has some well-known drawbacks. It does not take any additional information connected with $U$ into account. Furthermore, it is often considered to be too pessimistic (conservative) as the probability of occurrence of bad scenarios may be small. Our approach overcome these drawbacks. By specifying the possibility distribution in $U$, as an upper bound on the unknown probability distribution, we provide additional information which can be utilized to improve the quality of computed solutions. Furthermore, following the idea of light robustness, we allow decision makers to control the price of robustness of the solutions. It is important that the proposed model can be solved in polynomial time if the underlying deterministic counterpart is polynomially solvable. In particular, this is true for uncertain linear programming problems and some uncertain combinatorial optimization problems (shortest path, minimum spanning tree, minimum assignment, etc.)

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