Skorohod Equation and Reflected Backward Stochastic Differential Equations

By Zhongmin Qian and Mingyu Xu
Oxford University and Chinese Academy Sciences

Abstract. By using the Skorohod equation we derive an iteration procedure which allows us to solve a class of reflected backward stochastic differential equations with non-linear resistance induced by the reflected local time. In particular, we present a new method to study the reflected BSDE proposed first by El Karoui et al. [3].

Key words. Brownian motion, local time, optional dual projection, reflected BSDE, Skorohod’s equation.

AMS Classification. 60H10, 60H30, 60J45
1 Introduction

Backward stochastic differential equations (BSDEs in short), in the linear case and a special non-linear case, were first introduced by Bismut [2] in the study of the stochastic maximal principle. General non-linear BSDEs were first considered by Pardoux and Peng [12], who proved the existence and uniqueness of adapted solutions, under smooth square-integrability assumptions on the coefficient and the terminal data, and when the coefficient \( f(t,\omega,y,z) \) is Lipschitz in \((y,z)\) uniformly in \((t,\omega)\). El Karoui, Kapoudjian, Pardoux, Peng and Quenez introduced the notion of reflected BSDE [3], with one continuous lower barrier. More precisely, a solution for such equation associated with a coefficient \( f \), a terminal value \( \xi \), a continuous barrier \( S \) which is modelled by a semimartingale, is a triple \((Y,Z,K)\) of adapted stochastic processes in \(\mathbb{R}^{1+d+1}\), which satisfies the following stochastic integral equation

\[
Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s \tag{1.1}
\]

for any \(0 \leq t \leq T\) and \(Y \geq S\) a.s., \(K\) is non-decreasing continuous, where \(B\) is a \(d\)-dimensional Brownian motion. The role of \(K\) is to push upward the process \(Y\) in a minimal way, while to keep it above \(S\). In this sense it satisfies

\[
\int_0^T (Y_s - S_s) dK_s = 0. \tag{1.2}
\]

In order to prove the existence and uniqueness of the solution, they used first a Picard-type iterative procedure, which requires at each step the solution of an optimal stopping problem. The second approximation is constructed by penalization of the constraint. From then on, many researches have been done to relax the assumptions of reflected BSDE in [3] based on these two main methods. In [11], the case that the coefficient \( f \) are not Lipschitz function with only linear growth has been considered. In [7] monotonicity condition is used instead of the Lipschitz condition, and in [6], [9], [14], these authors studied RBSDE with quadratic increasing condition. In another direction, different barrier conditions have been studied, for example the barrier \( S \) is not continuous, but only right continuous with left limits, (cf. [8]), or is in more general case, cf. [13].

Recently a new type of reflected BSDEs has been introduced by Bank and El Karoui [1] by a variation of Skorohod’s obstacle problem, which is named as variant reflected BSDE, which has been generalized by Ma and Wang in [10]. The formulation of such equation with an optional process \(X\) (as an upper barrier)

\[
Y_t = X_T + \int_t^T f(s,Y_s,Z_s,A_s)ds - \int_t^T Z_s dB_s \text{ and } Y \leq X,
\]

where \(A\) is an increasing process, with \(A_0 = -\infty\), and the flat-off condition holds \(\int_t^T |Y_s - X_s| dA_s = 0\). Here the increasing process \(A\) does not directly act on \(Y\) to push the solution downwards such that \(Y_t \leq X_t\), instead it acts through the generator \(f\) which is decreasing.
in $A$, like a ‘density’ of reflecting force. In [10], it has been proved that the solution in a small-time duration, under some extra conditions, exists and is unique.

However we still do not know much about the increasing process $K$. In all these papers mentioned above, there are few results considering the increasing process $K$. In this paper, we will use the Skorohod equation to represent the increasing process $K$ in terms of the solution $Y$ and $Z$. This representation is ”explicit” showing how the force $K$ pushes the solution $Y$ according to the barrier $S$ and is given by

$$K_t = \max \left[ 0, \max_0 \leq s \leq T \left\{ - \left( \xi + \int_s^T f(r, Y_r, Z_r)dr - S_s - \int_s^T Z_r dB_r \right) \right\} \right] - \max \left[ 0, \max_t \leq s \leq T \left\{ - \left( \xi + \int_s^T f(r, Y_r, Z_r)dr - S_s - \int_s^T Z_r dB_r \right) \right\} \right].$$

Together with the theory of optional dual projections (for details about the general theory, see for example [4]), we construct a new Picard iteration based on this formula and prove that if $(Y, Z, K)$ is the fixed point, then it is the solution of reflected BSDE.

With this approach we are able to consider the following type of reflected BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, K_s)ds + K_T - K_t - \int_t^T Z_s dB_s$$

for $t \leq T$, subject to the constrain that

$$Y_t \geq S_t \text{ and } \int_0^T (Y_t - S_t)dK_t = 0.$$ 

Here the force is still given by an increasing process $K$, which satisfies the flat-off condition, but $K$ also appears in the driver $f$ as a resistance force. If $f$ is decreasing in $K$, then we get an extra force from the Lebesgue integral. If $f$ is increasing in $K$, then through the driver, there is a kind of cancellation of the positive force. So in general case, we can consider this reflected BSDE as an equation with resistance, given by the dependence of the driver on $K$. Since $Y_t \geq S_t$ has to be satisfied, and $Y$ is still square integrable, the extra force from the driver must be controlled is some sense, which is characterized the magnitude of Lipschitz constant in $K$.

The paper is organized as following. We first recall in Section 2 the Tanaka’s formula and Skorohod’s equation to give various formulate for the increasing process $K$. In section 3, we introduce a type of reflected BSDEs with resistance and prove the existence and uniqueness of the solution. In section 4, the uniqueness and some properties of the solution we have constructed are studied.

## 2 Local and reflected local times

Let $B = (B^1, B^2, \ldots, B^d)$ be a Brownian motion of dimension $d$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t \geq 0}$ be the Brownian motion filtration associated with $B$. Let $T > 0$
be a terminal time. Denote by $\mathcal{P}$ the $\sigma$-algebra of predictable sets on $[0, T] \times \Omega$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

For simplicity, we introduce the following spaces of random processes over $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. $L^2(\mathcal{F}_t)$ denotes the space of all $\mathcal{F}_t$-measurable, real random variable such that $\mathbb{E}(\vert \eta \vert^2) < \infty$, $\mathcal{M}^2$ denotes the space of (continuous) square-integrable martingales (up to time $T$), and $\mathcal{H}^2(0, T)$ is the space of $\mathbb{R}^d$-valued predictable process $\psi$ such that $\mathbb{E} \int_0^T \vert \psi(t) \vert^2 \, dt < \infty$. $\mathcal{S}^2(0, T)$ is the space of all continuous semimartingales (with running time $[0, T]$) over $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and $\mathcal{A}^2(0, T)$ the space of all $\mathcal{F}_T$-measurable, continuous and increasing processes with initial zero such that $\mathbb{E}(K_T^2) < \infty$.

If $K \in \mathcal{A}^2(0, T)$, then the optional projection $K^\circ$ (which is a right continuous modification of $t \to \mathbb{E}(K_t | \mathcal{F}_t)$) and the dual optional projection $K^\pound$ of $K$ exist. The dual optional projection $K^\pound$ is continuous and increasing with initial zero, while the optional projection $K^\circ$ is right continuous but not necessary increasing. Their difference $N = K^\circ - K^\pound$ is a martingale which must be continuous. Hence the optional projection $K^\circ$ is continuous as well. Moreover $K$ to $K^\circ$ is a contraction in $L^2$-norm.

The reflected backward stochastic differential equation (RBSDE or reflected BSDE in short) considered in El Karoui, et al. \cite{3} is a stochastic integral equation

\begin{equation}
Y_t = \xi + \int_t^T f(s, Y_s, Z_s) \, ds + K_T - K_t - \int_t^T Z_s \, dB_s \tag{2.1}
\end{equation}

for $t \leq T$, subject to the constrain that

\begin{equation}
Y_t \geq S_t \quad \text{and} \quad \int_0^T (Y_t - S_t) \, dK_t = 0, \tag{2.2}
\end{equation}

where $S$ is a continuous semimartingale such that $\sup_{t \leq T} S_t^+$ is square integrable, and $\xi \in \mathcal{L}^2(\mathcal{F}_T)$, which are given data. $f$ is called the (non-linear) driver of the reflected BSDE (2.1), which is global Lipschitz in $(Y, Z)$ uniformly in $t \in [0, T]$ and $\omega \in \Omega$.

By a solution $(Y, Z, K)$ of the terminal problem \((2.1) - (2.2)\) we mean that $Y \in \mathcal{S}^2(0, T)$, $K \in \mathcal{A}^2(0, T)$ and $K$ is optional, and $Z \in \mathcal{H}^2_d(0, T)$, which satisfies the stochastic integral equations \((2.1)\) with time $t$ running from 0 to $T$.

The constrain \((2.2)\) implies that $\xi - S_T$ must be non-negative, and the second condition in \((2.2)\) says that $K$ has no charge on \(\{t \in [0, T] : Y_t > S_t\}\) and increases only on \(\{t : Y_t = S_t\}\), which is equivalent to say that $\int_0^t 1_{\{Y_s - S_s = 0\}} \, dK_s = K_t$ for $0 \leq t \leq T$.

Since

\begin{equation}
Y_0 = \xi + \int_0^T f(s, Y_s, Z_s) \, ds + K_T - \int_0^T Z_s \, dB_s \tag{2.3}
\end{equation}

so that

\begin{equation}
Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) \, ds - K_t + \int_0^t Z_s \, dB_s \tag{2.4}
\end{equation}

and therefore the martingale part of $Y$ is $M_t = \int_0^t Z_s \, dB_s$. 

4
Our first task is to give two representations of the increasing process \( K \) in terms of sample paths \( Y \) and \( Z \), by using the Tanaka formula and the Skorohod equation respectively. These representations formulate are well known in stochastic analysis, we however employ them to the study of existence and uniqueness for a class of reflected backward stochastic differential equations with resistance.

### 2.1 Tanaka’s formula

We wish to interpret the increasing process \( K \) as a time-reversed local time, so that \( K \) will be called the reflected local time of \( Y \) at \( S \). To this end, we introduce the following notation: if \( X \) is a continuous semimartingale, then \( L^X \) denotes the local time of the continuous semimartingale \( X - S \) at zero. That is, \( L^X \) is defined by the Tanaka formula

\[
|X_t - S_t| = |X_0 - S_0| + \int_0^t \text{sgn}(X_s - S_s)d(X_s - S_s) + 2L^X_t
\]  
(2.5)

where \( \text{sgn}(r) = -1 \) for \( r \leq 0 \) and \( \text{sgn}(r) = 1 \) for \( r > 0 \). According to Tanaka’s formula

\[
(X_t - S_t)^- = (X_0 - S_0)^- - \int_0^t 1_{\{X_s \leq S_s\}}d(X_s - S_s) + L^X_t.
\]  
(2.6)

**Proposition 2.1** Suppose that \( Y_t = \int_0^t Z_s dB_s + V_t \) and \( S_t = \int_0^t \sigma_s dB_s + A_t \) are two continuous semimartingales, where \( V \) and \( A \) are continuous, adapted with finite variations, and suppose that \( Y \geq S \), then

\[
L^Y_t = \int_0^t 1_{\{Y_s = S_s\}}d(V_s - A_s)
\]  
(2.7)

and

\[
1_{\{Y_t = S_t\}}(Z_t - \sigma_t) = 0.
\]  
(2.8)

**Proof.** Since \( Y - S \geq 0 \), so that \( (Y - S)^- = 0 \). According to the Tanaka formula

\[
(Y_t - S_t)^- = (Y_0 - S_0)^- - \int_0^t 1_{\{Y_s \leq S_s\}}d(Y_s - S_s) + L^Y_t
\]

so that

\[
L^Y_t = \int_0^t 1_{\{Y_s = S_s\}}d(Y_s - S_s)
\]

\[
= \int_0^t 1_{\{Y_s = S_s\}}d(V_s - A_s) + \int_0^t 1_{\{Y_s = S_s\}}(Z_s - \sigma_s)dB_s.
\]

The martingale part must be zero as \( L \) is increasing, which yields (2.8), and therefore (2.7) follows as well. ■

The following lemma demonstrates in some sense \( K \) is the time inverse of the local time of \( Y - S \).
Corollary 2.2 Assume that $Y \geq S$ are two continuous semimartingale,

$$Y_t = Y_0 - \int_0^t f_s ds - K_t + \int_0^t Z_s dB_s$$

(2.9)

and $S = N + A$ ($N$ is the martingale part of $S$ and $A$ is its variation part), where $(f_t)_{t \in [0,T]}$ is optional and $E\int_0^T f_s^2 ds < \infty$, $Z \in \mathcal{H}^d(0,T)$, $Y_0 \in \mathcal{L}^2(F_0)$, $K \in \mathcal{A}^2(0,T)$ is adapted, such that $\int_0^t 1_{\{Y_s = S_s\}} dK_s = K_t$. Then

$$K_t = - \int_0^t 1_{\{Y_s = S_s\}} f_s ds - \int_0^t 1_{\{Y_s = S_s\}} dA_s - L^Y_t$$

(2.10)

and

$$K_t = - \int_0^t 1_{\{Y_s = S_s\}} f_s ds - \int_0^t 1_{\{Y_s = S_s\}} dY_s$$

$$+ \int_0^t 1_{\{Y_s = S_s\}} dN_s,$$

(2.11)

where $N$ is the martingale part of $S$.

2.2 Skorohod’s equation

The most useful form for $K$ to our study is however the representation formula given by the Skorohod equation.

Again we assume that $Y \geq S$ are two continuous semimartingales, and $Y$ is given by (2.9). Let $y_t = Y_{T-t} - S_{T-t}$, $L_t = K_T - K_{T-t}$ and

$$x_t = \int_{T-t}^T f_s ds - \int_{T-t}^T Z_s dB_s + S_T - S_{T-t}.$$  

(2.12)

Then $L_0 = 0$, $t \rightarrow L_t$ increases only on $\{t: y_t = 0\}$, $y_t \geq 0$, $\eta = Y_T - S_T \geq 0$, $x_0 = 0$, and

$$y_t = \eta + x_t + L_t.$$  

(2.13)

According to Skorohod’s equation (Lemma 6.14, page 210 in [3], with the convention that $x_t = x_T$, $y_t = y_T$ and $L_t = L_T$ for $t \geq T$)  

$$L_t = \max \left[ 0, \max_{0 \leq s \leq t} \left\{ - (\eta + x_s) \right\} \right], \ \forall t \geq 0.$$  

(2.14)

That is

$$L_t = \max \left[ 0, \max_{T-t \leq s \leq T} \left\{ - \left( Y_T + \int_s^T f_r dr - S_s - \int_s^T Z_r dB_r \right) \right\} \right]$$

(2.15)

for $0 \leq t \leq T$, and we may recover $K_t = L_T - L_{T-t}$ to obtain

$$K_t = \max \left[ 0, \max_{0 \leq s \leq T} \left\{ - \left( Y_T + \int_s^T f_r dr - S_s - \int_s^T Z_r dB_r \right) \right\} \right]$$

$$- \max \left[ 0, \max_{t \leq s \leq T} \left\{ - \left( Y_T + \int_s^T f_r dr - S_s - \int_s^T Z_r dB_r \right) \right\} \right].$$

(2.16)
3 Reflected BSDE with resistance

The representation formula (2.16) for the reflected local time $K$ may be used to study a class of reflected backward stochastic differential equations with non-linear resistance caused by the reflected local time $K$. In this paper we study the following stochastic integral equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, K_s) ds + K_T - K_t - \int_t^T Z_s dB_s$$

for $t \leq T$, subject to the constrain that

$$Y_t \geq S_t \quad \text{and} \quad \int_0^T (Y_t - S_t) dK_t = 0,$$

where $S$ is a continuous semimartingale such that $\sup_{t \leq T} S_t^+$ is square integrable, and $\xi \in L^2(\mathcal{F}_T)$, which are given data.

Assume that $f$ is global Lipschitz continuous

$$|f(s, y, z, k) - f(s, y', z', k')| \leq C_1(|y - y'| + |z - z'|) + C_2|k - k'|$$

for all $y, y', z, z', k, k'$, where $C_1$ and $C_2$ are two constants, and $\mathbb{E} \int_0^T f^0(t)^2 dt < \infty$, where

$$f^0(t) \equiv f(t, 0, 0, 0).$$

By a solution triple $(Y, Z, K)$ of the terminal problem (3.1) we mean that $Y \in \mathcal{S}^2(0, T)$, $K \in \mathcal{A}^2(0, T)$ and $K$ is optional, and $Z \in \mathcal{H}^2(0, T)$, which satisfies the stochastic integral equations (3.1) with time $t$ running from 0 to $T$.

An additional feature over the reflected BSDE (2.1) is the dependence of the driver with respect to the reflected local time $K$. The integral equation (3.1) is not local in time, since $K$ will be path dependent over the whole range $[0, T]$. This is the reason why we have to require the Lipschitz constant $C_2$ in (3.3) to be small.

According to (2.16), if $(Y, Z, K)$ is a solution of the problem (3.1)-(3.2), then we must have

$$K_t = \max \left\{ \max_{0 \leq s \leq T} \left\{ - \left( \xi - S_s + \int_s^T f(r, Y_r, Z_r, K_r)dr - \int_s^T Z_r dB_r \right) \right\}, 0 \right\}$$

which is already noticed in [3]. What is new here is the use of this formula to build a proper Picard iteration associated with (3.1), which is described in the following section.

**Remark 3.1** For the reflected BSDE with resistance (3.1), we can still write the solution $Y$ as the value process of an optimal stopping problem following the same arguments of Proposition 2.3 in [3], noticing that $K$ is continuous in $t$. However, here it is not a standard optimal stopping problem, and we can not get the solution given by Snell envelope, which is the essential step of Picard’s iteration used in [3].
3.1 Constructing Picard’s iteration

We show the existence of a unique solution by constructing an appropriate (non)-linear mapping defined by the stochastic integral equation (3.1), so that the unique solution is given as its fixed point.

Let us develop the iteration procedure as following. Suppose $Y \in \mathcal{S}^2(0,T)$, $Z \in \mathcal{H}^2_d(0,T)$ and $K \in \mathcal{A}^2(0,T)$, $Y \geq S$. After iteration once we will obtain $(\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathcal{S}^2(0,T) \times \mathcal{H}^2_d(0,T) \times \mathcal{A}^2(0,T)$, and $\tilde{Z}.B$ is the martingale part of $\tilde{Y}$. Thus we can assume from the beginning, without losing generality, that $M_t - M_0 = \int_0^T Z_s dB_s$ is the martingale part of $Y$, although we prefer consider three processes $Y, Z, K$ as independent variables. According to (3.5) we first define

$$\tilde{K}_t = \max \left[ 0, \max_{0 \leq s \leq T} \left\{ - \left( \xi + \int_s^T f(r, Y_r, Z_r, K_r^\flat)dr - S_s - \int_s^T Z_r dB_r \right) \right\} \right] - \max \left[ 0, \max_{t \leq s \leq T} \left\{ - \left( \xi + \int_s^T f(r, Y_r, Z_r, K_r^\flat)dr - S_s - \int_s^T Z_r dB_r \right) \right\} \right] \quad (3.6)$$

where $K$ is replaced by the optional projection of $K$, as we do not assume $K$ is optional, but we want to ensure the arguments in the driver $f$ are optional.

We are going to define $\tilde{M}$ and $\tilde{Y}$. The natural way to define $\tilde{Y}$ is to take the right-hand side of (3.1) which is

$$\hat{Y}_t = \xi + \int_t^T f(s, Y_s, Z_s, K_s^\flat)ds + K_T - K_t - \int_t^T Z_s dB_s. \quad (3.7)$$

$\hat{Y}$ is however not not necessary adapted. Therefore we define $\tilde{Y}$ to be its optional projection $\hat{Y}^\flat$. That is,

$$\tilde{Y}_t = \mathbb{E} \left\{ \xi + \int_t^T f(s, Y_s, Z_s, K_s^\flat)ds + \tilde{K}_T - \tilde{K}_t - \int_t^T Z_s dB_s \ \bigg| \ F_t \right\}$$

$$\tilde{Y}_t = \mathbb{E} \left\{ \xi + \int_t^T f(s, Y_s, Z_s, K_s^\flat)ds + \tilde{K}_T - \tilde{K}_t \ \bigg| \ F_t \right\}. \quad (3.8)$$

According to Skorohod’s equation, $\hat{Y} \geq S$, so is $\tilde{Y}$. Therefore the mapping $Y \to \tilde{Y}$ preserves the constraint $\hat{Y} \geq S$. Moreover $\tilde{K}$ increases only on $\{ t : \hat{Y}_t - S_t = 0 \}$, which does not necessarily coincide with level set $\{ t : \hat{Y}_t - S_t = 0 \}$.

On the other hand, according to (3.8), the semimartingale decomposition of $\hat{Y}$ is given by

$$\hat{Y}_t = \mathbb{E} \left\{ \xi + \hat{K}_T + \int_0^T f(s, Y_s, Z_s, K_s^\flat)ds \ \bigg| \ F_t \right\} - \tilde{N}_t$$

$$\hat{Y}_t = \mathbb{E} \left\{ \xi + \hat{K}_T + \int_0^T f(s, Y_s, Z_s, K_s^\flat)ds \ \bigg| \ F_t \right\} - \tilde{N}_t - \int_0^t f(s, Y_s, Z_s, K_s^\flat)ds. \quad (3.9)$$
where $\tilde{N}_t = K_t^0 - K_t^♭$ is a continuous martingale. Therefore the martingale part of $\tilde{Y}$ is given by

$$\tilde{M}_t = \mathbb{E}\left\{ \xi + \tilde{K}_T + \int_0^T f(s, Y_s, Z_s, K_s^0)ds \mid \mathcal{F}_t \right\} - \tilde{N}_t$$

which in turn defines the density predictable process $\tilde{Z}$ by Itô’s martingale representation

$$\tilde{M}_t - \tilde{M}_0 = \int_0^t \tilde{Z}_s dB_s,$$

so that

$$\tilde{Y}_t = \xi + \int_0^T f(s, Y_s, Z_s, K_s^0)ds + K_T^0 - \tilde{K}_t^♭ - \int_0^T \tilde{Z}_s dB_s. \eqno{(3.11)}$$

It is clear that from the definition and the Lipschitz condition (3.3)

$$(\tilde{Y}, \tilde{Z}, \tilde{K}) \in S^2(0, T) \times H^2_d(0, T) \times A^2(0, T).$$

The mapping $\mathcal{L} : (Y, Z, K) \to (\tilde{Y}, \tilde{Z}, \tilde{K})$ is thus well defined.

It seems reasonable to have the optional dual projection $K^o$ in place of the optional projection $K^♭$ in defining $\tilde{Y}$ by (3.9). The reason we prefer the optional projection lies in the fact that $X \to X^♭$ is a contraction in $L^p$-space, but $X \to X^o$ is not.

**Proposition 3.2** If $(Y, Z, K)$ is a fixed point of $\mathcal{L}$, then $(Y, Z, K)$ is a solution the reflected BSDE (3.1)-(3.2).

**Proof.** Suppose $(Y, Z, K)$ is a fixed point of the non-linear mapping $\mathcal{L}$, so that

$$M_t = \mathbb{E}\left\{ \xi + \int_0^T f(s, Y_s, Z_s, K_s^0)ds + K_T - K_t \mid \mathcal{F}_t \right\} + K_t^o,$$

and

$$Y_t = \mathbb{E}\left\{ \xi + \int_t^T f(s, Y_s, Z_s, K_s^0)ds + K_T - K_t \mid \mathcal{F}_t \right\}$$

where

$$K_t = \max_{0 \leq s \leq T} \left\{ - \left( \xi + \int_s^T f(r, Y_r, Z_r, K_r^0)dr - S_r - \int_s^T Z_r dB_r \right) \right\} - \max_{t \leq s \leq T} \left\{ - \left( \xi + \int_s^T f(r, Y_r, Z_r, K_r^0)dr - S_r - \int_s^T Z_r dB_r \right) \right\}. $$

Then $Y_T = \xi$ and

$$Y_t = M_t - K_t^o - \int_0^t f(s, Y_s, Z_s, K_s^0)ds,$$

so that

$$\xi - Y_t = \int_t^T Z_s dB_s - (K_T^o - K_t^♭) - \int_t^T f(s, Y_s, Z_s, K_s^0)ds.$$

According to the uniqueness of the Skorohod’s equation, it follows that $K^o = K$. therefore $K$ is adapted, and $K = K^♭ = K^o$. That completes the proof. ■
3.2 Main estimates

Let us develop several a priori estimates for $\mathcal{L}(Y, Z, K) = (\tilde{Y}, \tilde{Z}, \tilde{K})$. We begin with an elementary fact:

**Lemma 3.3** Let $\varphi, \psi$ be two continuous paths in $\mathbb{R}^1$. Then

$$\left| \sup_{s \leq t} \varphi_s - \sup_{s \leq t} \psi_s \right| \leq \sup_{s \leq t} |\varphi_s - \psi_s|.$$ 

**Proof.** $\delta = \sup_{s \leq t} |\varphi_s - \psi_s|$. Then

$$\varphi_s \leq \psi_s + \delta \leq \sup_{s \leq t} \psi_s + \delta$$

for any $s \leq t$, so that $\sup_{s \leq t} \varphi_s \leq \sup_{s \leq t} \psi_s + \delta$. Similarly $\sup_{s \leq t} \psi_s \leq \sup_{s \leq t} \varphi_s + \delta$. $\Box$

Suppose $(Y, Z, K), (Y', Z', K') \in S^2 \times H^2 \times A^2$ such that $Y_T = Y'_T = \xi$, and $Y \geq S, Y' \geq S$.

Let us prove the following key a priori estimate about $\mathcal{L}$. Let $(\tilde{Y}, \tilde{Z}, \tilde{K}) = \mathcal{L}(Y, Z, K)$ and $(\tilde{Y}', \tilde{Z}', \tilde{K}') = \mathcal{L}(Y', Z', K')$. Let $\alpha \geq 0$ to be chosen late, and let $\tilde{D}_t = e^{\alpha t} |Y_t - Y'_t|^2$ and $\tilde{D}_t = e^{\alpha t} |\tilde{Y}_t - \tilde{Y}'_t|^2$.

**Proposition 3.4** Suppose $f$ satisfies the Lipschitz condition (3.3). Then for any $\alpha \geq 0$, $\varepsilon > 0$ and $\varepsilon' > 0$ we have

$$\mathbb{E} \left( \tilde{D}_0 \right) \leq -(\alpha - \varepsilon C_1 - \varepsilon' C_2) ||\tilde{Y} - \tilde{Y}'||^2_\alpha - ||\tilde{Z} - \tilde{Z}'||^2_\alpha$$

$$+ \frac{2C_1}{\varepsilon} \left( ||Y - Y'||^2_\alpha + ||Z - Z'||^2_\alpha \right)$$

$$+ \frac{2C_2}{\varepsilon'} ||K^b - K'^b||^2_\alpha$$

(3.12)

where $C_1, C_2$ are the Lipschitz constants appearing in (3.3).

**Proof.** According to (3.9)

$$\tilde{Y}_t - \tilde{Y}'_t = \left( \tilde{M}_t - \tilde{M}'_t \right) - (\tilde{K}^p_t - \tilde{K}'^p_t)$$

$$- \int_0^t \left( f(s, Y_s, Z_s, K^p_s) - f(s, Y'_s, Z'_s, K'^p_s) \right) ds$$

(3.13)
where $\tilde{M}$ (resp. $\tilde{M}'$) is the martingale part of $\tilde{Y}$ (resp. $\tilde{Y}'$), given by (3.10), so that by Itô’s formula,

\[
\tilde{D}_t = -\int_t^T e^{\alpha s} d\left(\tilde{Y}_s - \tilde{Y}'_s\right)^2 - \alpha \int_t^T e^{\alpha s} \left(\tilde{Y}_s - \tilde{Y}'_s\right)^2 ds
\]
\[
= -\int_t^T e^{\alpha s} d\langle\tilde{M} - \tilde{M}'\rangle_s - \alpha \int_t^T e^{\alpha s} \left(\tilde{Y}_s - \tilde{Y}'_s\right)^2 ds
\]
\[
- \int_t^T 2e^{\alpha s} \left(\tilde{Y}_s - \tilde{Y}'_s\right) d\left(\tilde{Y}_s - \tilde{Y}'_s\right)
\]
\[
= -\alpha \int_t^T \tilde{D}_s ds - \int_t^T e^{\alpha s} d\langle\tilde{M} - \tilde{M}'\rangle_s - 2 \int_t^T e^{\alpha s} \left(\tilde{Y}_s - \tilde{Y}'_s\right) d\left(\tilde{M}_s - \tilde{M}'_s\right)
\]
\[
+ 2 \int_t^T e^{\alpha s} \left(\tilde{Y}_s - \tilde{Y}'_s\right) d\left(\tilde{K}_s^\alpha - \tilde{K}'_s^\alpha\right)
\]
\[
+ 2 \int_t^T e^{\alpha s} \left(\tilde{Y}_s - \tilde{Y}'_s\right) (f(s, Y_s, Z_s, K_s^\varphi) - f(s, Y'_s, Z'_s, K'^\varphi_s)) ds.
\] (3.14)

Taking expectation to obtain

\[
\mathbb{E}\tilde{D}_t = -\alpha \int_t^T \mathbb{E}\left(\tilde{D}_s\right) ds - \mathbb{E} \int_t^T e^{\alpha s} d\langle\tilde{M} - \tilde{M}'\rangle_s
\]
\[
+ 2\mathbb{E} \int_t^T e^{\alpha s} \left(\tilde{Y}_s - \tilde{Y}'_s\right) d\left(\tilde{K}_s^\alpha - \tilde{K}'_s^\alpha\right)
\]
\[
+ 2 \int_t^T \mathbb{E} \left\{e^{\alpha s} \left(\tilde{Y}_s - \tilde{Y}'_s\right) [f(s, Y_s, Z_s, K_s^\varphi) - f(s, Y'_s, Z'_s, K'^\varphi_s)]\right\} ds,
\] (3.15)

where we have used the fact that

\[
\mathbb{E} \int_t^T \varphi_s d\left(\tilde{K}_s^\alpha - \tilde{K}'_s^\alpha\right) = \mathbb{E} \int_t^T \varphi_s d\left(\tilde{K}_s - \tilde{K}'_s\right)
\]

for an optional process $\varphi$. Now we use an important observation due to [3], that is,

\[
\mathbb{E} \int_t^T e^{\alpha s} \left(\tilde{Y}_s - \tilde{Y}'_s\right) d\left(\tilde{K}_s - \tilde{K}'_s\right)
\]
\[
= \mathbb{E} \int_t^T e^{\alpha s} (\tilde{Y}_s - S_s) d\tilde{K}_s + \mathbb{E} \int_t^T e^{\alpha s} (\tilde{Y}'_s - S_s) d\tilde{K}'_s
\]
\[
- \mathbb{E} \int_t^T e^{\alpha s} (\tilde{Y}_s - S_s) d\tilde{K}'_s - \mathbb{E} \int_t^T e^{\alpha s} (\tilde{Y}'_s - S_s) d\tilde{K}_s
\]
\[
\leq \mathbb{E} \int_t^T e^{\alpha s} (\tilde{Y}_s - S_s) d\tilde{K}_s + \mathbb{E} \int_t^T e^{\alpha s} (\tilde{Y}'_s - S_s) d\tilde{K}'_s.
\]
Moreover, according to Skorohod’s equation, $\tilde{K}$ increases only on $\{ s : \tilde{Y}_s - S_s = 0 \}$ so that

$$E \int_t^T e^{\alpha s}(\tilde{Y}_s - S_s) d\tilde{K}_s = 0.$$  

Since $\tilde{Y}$ is the optional projection, and $\tilde{K}^o$ is the dual optional projection of $\tilde{K}$, therefore

$$E \int_t^T e^{\alpha s}(\tilde{Y}_s - S_s) d\tilde{K}_s = E \int_t^T e^{\alpha s}(\tilde{Y}_s - S_s) d\tilde{K}_s^o = E \left( \int_t^T e^{\alpha s}(\tilde{Y}_s - S_s) d\tilde{K}_s \right)^o.$$  

Since $\tilde{K}$ increases only on $\{ s : \tilde{Y}_s - S_s = 0 \}$, so that $\int_t^T e^{\alpha s}(\tilde{Y}_s - S_s) d\tilde{K}_s = 0$ and therefore $E \int_t^T e^{\alpha s}(\tilde{Y}_s - S_s) d\tilde{K}_s = 0$. Similarly $E \int_t^T e^{\alpha s}(\tilde{Y}_s' - S_s) d\tilde{K}_s' = 0$. Hence

$$E \int_t^T e^{\alpha s} (\tilde{Y}_s - \tilde{Y}_s') d(\tilde{K}_s - \tilde{K}_s') \leq 0.$$  

Putting this estimate into (3.15) to obtain

$$E \left( \tilde{D}_t \right) \leq -\alpha \int_t^T E \left( \tilde{D}_s \right) ds - E \int_t^T e^{\alpha s} d\langle \tilde{M} - \tilde{M}' \rangle_s + 2 \int_t^T e^{\alpha s} E \left\{ \left( \tilde{Y}_s - \tilde{Y}_s' \right) \left[ f(s, Y_s, Z_s, K_s^b) - f(s, Y_s', Z_s', K_s'^b) \right] \right\} ds. \ (3.16)$$  

We use the global Lipschitz continuity of $f$ to handle the last integral on the right-hand side of (3.16), the method however is standard. Indeed

$$E \left( \tilde{D}_t \right) \leq -\alpha \int_t^T E \left( \tilde{D}_s \right) ds - E \int_t^T e^{\alpha s} |\tilde{Z}_s - \tilde{Z}_s'|^2 ds + 2C_1 \int_t^T e^{\alpha s} E \left( |\tilde{Y}_s - \tilde{Y}_s'| (|Y_s - Y_s'| + |Z_s - Z_s'|) \right) ds + 2C_2 \int_t^T e^{\alpha s} E \left( |\tilde{Y}_s - \tilde{Y}_s'| |K_s^b - K_s'^b| \right) ds \leq -(\alpha - \varepsilon C_1 - \varepsilon' C_2) \int_t^T E \left( \tilde{D}_s \right) ds - E \int_t^T e^{\alpha s} |\tilde{Z}_s - \tilde{Z}_s'|^2 ds + \frac{2C_1}{\varepsilon} \int_t^T e^{\alpha s} (|Y_s - Y_s'|^2 + |Z_s - Z_s'|^2) ds + \frac{2C_2}{\varepsilon'} \int_t^T e^{\alpha s} |K_s^b - K_s'^b|^2 ds. \ (3.17)$$  

Choosing $t = 0$ we deduce the required estimate. ■

The next estimate is also essential in this article.
Proposition 3.5 We have

\[ ||K - K'||_{\infty}^2 \leq (24TC_1^2 + 4C_3) \left(||Y - Y'||_0^2 + ||Z - Z'||_0^2\right) + 24TC_2^2 ||K - K'||_{\infty}^2 \]  

(3.18)

where \( ||K - K'||_{\infty}^2 = \sup_{0 \leq t \leq T} \mathbb{E}|K_s - K'_s|^2 \), where \( C_3 \) is the constant appearing in the Burkholder inequality.

Proof. Recall that

\[
\tilde{K}_t = \max\left[0, \max_{0 \leq s \leq T} \left\{ -\left( \xi + \int_s^T f(r, Y_r, Z_r, K_r)dr - S_s - \int_s^T Z_r dB_r \right) \right\} \right]
\]

which yields that

\[
|\tilde{K}_t - \tilde{K}'_t|^2 \leq 4T \int_0^T |f(s, Y_s, Z_s, K_s) - f(s, Y'_s, Z'_s, K'_s)|^2 ds
\]

\[
+4 \sup_{0 \leq s \leq T} \int_s^T (Z_r - Z'_r) dB_r \bigg|^2
\]

\[
\leq 24TC_1^2 \int_0^T \left( |Y_s - Y'_s|^2 + |Z_s - Z'_s|^2 \right) ds
\]

\[
+24TC_2^2 \int_0^T |K_s - K'_s|^2 ds + 4 \sup_{0 \leq s \leq T} \int_s^T (Z_r - Z'_r) dB_r \bigg|^2
\]

so that

\[
\mathbb{E}|\tilde{K}_t - \tilde{K}'_t|^2 \leq (24TC_1^2 + 4C_3) \mathbb{E} \int_0^T \left( |Y_s - Y'_s|^2 + |Z_s - Z'_s|^2 \right) ds
\]

\[
+24TC_1^2 \int_0^T \mathbb{E}|K_s - K'_s|^2 ds
\]

\[
\leq (24TC_1^2 + 4C_3) \mathbb{E} \int_0^T \left( |Y_s - Y'_s|^2 + |Z_s - Z'_s|^2 \right) ds
\]

\[
+24TC_2^2 \int_0^T \mathbb{E}|K_s - K'_s|^2 ds
\]

which implies (3.18). □

3.3 Existence theorem

We are now in a position to show the existence of a solution to (3.1)-(3.2).
Theorem 3.6 There is a constant \( C_0 > 0 \) depending only on \( C_1 \) such that if \( C_2 T \leq C_0 \), where \( C_2 \) is the Lipschitz constant appearing in (3.3), then there is a unique solution \((Y, Z, K)\) to the problem (3.1)-(3.2). Moreover the reversed local time satisfies (3.5). If \( C_2 = 0 \) that is the driver \( f \) does not depend on \( K \), then there is no restriction on \( T \).

Proof. Let \( \alpha \geq 0 \) and \( \beta > 0 \) to be chosen late, and define

\[
|| (Y, Z, K) - (Y', Z', K') ||_{\alpha, \beta}^2 = || Y - Y' ||_\alpha^2 + || Z - Z' ||_\alpha^2 + \beta || K - K' ||_\infty^2.
\]

Let \((\tilde{Y}, \tilde{Z}, \tilde{K}) = \mathcal{L}(Y, Z, K)\) and \((\tilde{Y}', \tilde{Z}', \tilde{K}') = \mathcal{L}(Y', Z', K')\). Then

\[
||K^b - K'^b||_{\alpha}^2 \leq \frac{e^{\alpha T} - 1}{\alpha} || K - K' ||_\infty^2
\]

so that, together with (3.12) (in which choose \( \alpha - \varepsilon C_1 - \varepsilon' C_2 = 1 \))

\[
|| \tilde{Y} - \tilde{Y}' ||_\alpha^2 + || \tilde{Z} - \tilde{Z}' ||_\alpha^2 + \beta || \tilde{K} - \tilde{K}' ||_\infty^2
\]

\[
\leq \left( 24 TC_1^2 + 4 C_3 \right) \beta + \frac{2C_1}{\varepsilon} \left( || Y - Y' ||_0^2 + || Z - Z' ||_0^2 \right)
\]

\[+ \left( \frac{24 C_2^2}{\beta} T^2 + \frac{2 C_2 e^{\alpha T} - 1}{\varepsilon' \beta} \right) \beta || K - K' ||_\infty^2. \tag{3.19}
\]

Choose \( \varepsilon = 8C_1, \varepsilon' = 1, \alpha = 1 + 8C_1^2 + C_2 \) and \( \beta = \frac{1}{16(4TC_1^2 + 1)} \). Then there is a number \( C_0 > 0 \) such that if \( C_2 T \leq C_0 \), we have

\[
12C_2^2 T^2 + C_2 \frac{e^{(1+8C_1^2+C_2)T} - 1}{1 + 8C_1^2 + C_2} \leq \frac{1}{48 (6TC_1^2 + C_3)}
\]

so that

\[
\frac{24 C_2^2}{\beta} T^2 + \frac{2 C_2 e^{\alpha T} - 1}{\varepsilon' \beta} \leq \frac{1}{2}.
\]

Hence

\[
|| (\tilde{Y}, \tilde{Z}, \tilde{K}) - (\tilde{Y}', \tilde{Z}', \tilde{K}') ||_{\alpha, \beta} \leq \frac{1}{\sqrt{2}} || (Y, Z, K) - (Y', Z', K') ||_{\alpha, \beta}, \tag{3.20}
\]

so there is a fixed point \((Y, Z, K)\), which is clearly a solution according to Proposition 3.2.

3.4 Continuous dependence and uniqueness

Here we study the continuous dependence result of the solution of this reflected equation with respect to the parameters, which will lead to the uniqueness of the solution immediately. First we consider following a priori estimation.
Proposition 3.7 Under the same assumptions in Theorem 3.6. Suppose \((Y, Z, K)\) to be the solution of reflected BSDE \((3.1)\), then there exists a constant \(C\) depending only on \(C_1\) and \(C_2 T\), such that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} Y_t^2 + \int_0^T |Z_s|^2 ds + K_T^2 \right) \leq C \mathbb{E} \left( \xi^2 + \int_0^T (f_t^0)^2 dt + \left( \sup_{0 \leq t \leq T} S_t^+ \right)^2 \right). \tag{3.21}
\]

Proof. Applying Itô’s formula to \(Y_t^2\), and taking expectation, then

\[
\mathbb{E}[Y_t^2 + \int_t^T |Z_s|^2 ds] = \mathbb{E}[\xi^2 + 2 \int_t^T Y_s f(s, Y_s, Z_s, K_s) ds + 2 \int_t^T Y_s dK_s]
\]

as \(\int_0^T (Y_s - S_s) dK_s = 0\). By (3.3), we have

\[
\mathbb{E}[Y_t^2 + \int_t^T |Z_s|^2 ds] \leq \mathbb{E} \left( \xi^2 + 2 \int_t^T S_s dK_s \right) + 2 \mathbb{E} \int_t^T |Y_s| \left( |f^0_s| + C_1 (|Y_s| + |Z_s|) + C_2 |K_s| \right) ds \\
\leq \mathbb{E} \left( \xi^2 + \int_t^T |f^0_s|^2 ds + \alpha \left( \sup_{0 \leq t \leq T} S_t^+ \right)^2 \right) + \mathbb{E} \int_t^T |Z_s|^2 ds \\
+ (2C_1 + C_2 + 2) \mathbb{E} \int_t^T |Y_s|^2 ds + (C_2^2 T + \frac{1}{\alpha}) \mathbb{E} K_T^2.
\]

By using Gronwell’s inequality, we get

\[
\mathbb{E}[Y_t^2] \leq e^{(2C_1+C_2^2+2)T} \mathbb{E} \left[ \xi^2 + \int_t^T |f^0_s|^2 ds + \alpha \left( \sup_{0 \leq t \leq T} S_t^+ \right)^2 \right] \\
+ (C_2^2 T + \frac{1}{\alpha}) e^{(2C_1+C_2^2+2)T} \mathbb{E} K_T^2
\]

and

\[
\mathbb{E} \int_t^T |Z_s|^2 ds \leq C_T \mathbb{E} \left[ \xi^2 + \int_t^T |f^0_s|^2 ds + \alpha \left( \sup_{0 \leq t \leq T} S_t^+ \right)^2 + (C_2^2 T + \frac{1}{\alpha}) K_T^2 \right]
\]

where

\[
C_T = (2C_1 + C_1^2 + 2) e^{(2C_1+C_2^2+2)T} T + 1.
\]

Since

\[
K_T = Y_0 - \xi - \int_0^T f(s, Y_s, Z_s, K_s) ds + \int_0^T Z_s dB_s
\]
so
\[
\mathbb{E}K_T^2 \leq C\mathbb{E} \left( \xi^2 + \int_t^T \left| f_s^0 \right|^2 ds \right) + 8C_1^2 \mathbb{E} \int_t^T \left| Y_s \right|^2 ds \\
+ (8C_1^2 + 8) \mathbb{E} \int_t^T \left| Z_s \right|^2 ds + 8C_2^2 T \mathbb{E}K_T^2 \\
\leq C\mathbb{E} \left( \xi^2 + \int_t^T \left| f_s^0 \right|^2 ds + \alpha \left( \sup_{0 \leq t \leq T} S_t^+ \right)^2 \right) \\
+ \left( C_4 C_2^2 T^2 + 4C_2^2 T + \frac{C_4}{\alpha} \right) \mathbb{E}K_T^2
\]

where \( C_4 = (8C_1^2 + (8C_1^2 + 8)(2C_1 + C_1^2 + 2)) e^{(2C_1+C_1^2+2)T} \) which is independent of \( C_2 \), so that if \( C_2 T \) small enough, and choosing \( \alpha \) big enough, we can ensure that \( C_4 C_2^2 T^2 + 4C_2^2 T + \frac{C_4}{\alpha} < 1 \), so that

\[
\mathbb{E} \left[ Y_t^2 + \int_0^T |Z_s|^2 ds + K_T \right] \leq C\mathbb{E} \left[ \xi^2 + \int_0^T (f_t^0)^2 dt \right] + \left( \sup_{0 \leq t \leq T} S_t^+ \right)^2,
\]

with some constant \( C \). \( (3.21) \) follows by the use of the Burkholder inequality. ■

Now we consider the following continuous dependence theorem

**Theorem 3.8** Under the same assumptions in Theorem 3.6. Suppose \((Y^i, Z^i, K^i), (i = 1, 2)\) to be the solution of reflected BSDE \( (3.1) \) with parameters \((\xi^i, f^i, S^i)\), respectively. Set

\[
\Delta Y = Y^1 - Y^2, \quad \Delta Z = Z^1 - Z^2, \quad \Delta K = K^1 - K^2, \\
\Delta \xi = \xi^1 - \xi^2, \quad \Delta f = f^1 - f^2, \quad \Delta S = S^1 - S^2.
\]

Then

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |\Delta Y_t|^2 + \int_0^T |\Delta Z_s|^2 ds + |\Delta K_T| \right) \\
\leq C\mathbb{E} \left( \Delta \xi^2 + \int_0^T \left| \Delta f(t, Y_t^1, Z_t^1, K_t^1) \right|^2 dt \right) \\
+C\Psi_T \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} |\Delta S_t| \right)^2 \right]^{1/2}
\]

where \( C \) depends only on \( C_1 \) and \( C_2 T \), and

\[
\Psi_T = \mathbb{E} \left[ |\xi|^2 + |\xi|^2 \right] + \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} (S_t^1) \right)^2 + \left( \sup_{0 \leq t \leq T} (S_t^2) \right)^2 \right] \\
+ \mathbb{E} \int_0^T \left| f^1(t, 0, 0, 0) \right|^2 + \left| f^2(t, 0, 0, 0) \right|^2 dt.
\]
Proof. Applying Itô’s formula to $|\triangle Y_t|^2$, then taking expectation, and using the fact that $\int_t^T (\triangle Y_s - \triangle S_s)d(\triangle K_s) \leq 0$, we get

$$
\begin{align*}
\mathbb{E} \left( |\triangle Y_t|^2 + \int_t^T |\triangle Z_s|^2 ds \right) \\
= \mathbb{E} \left( \triangle \xi^2 + 2 \int_t^T \triangle Y_s \triangle f(s, Y_s^1, Z_s^1, K_s^1) ds \right) \\
+ 2 \mathbb{E} \left( \int_t^T \triangle Y_s \left( f^2(s, Y_s^1, Z_s^1, K_s^1) - f^2(s, Y_s^2, Z_s^2, K_s^2) \right) ds \right) \\
+ 2 \mathbb{E} \int_t^T \triangle S_s d(\triangle K_s).
\end{align*}
$$

By (3.3), we get

$$
\mathbb{E} |\triangle Y_t|^2 \leq \mathbb{E} \left( \triangle \xi^2 + \int_t^T |\triangle f(s, Y_s^1, Z_s^1, K_s^1)|^2 ds \right) \\
+ (2 + 2C_1 + C_2^2) \mathbb{E} \int_t^T |\triangle Y_s|^2 ds + C_2^2 T \mathbb{E}[\triangle K_T^2] \\
+ \mathbb{E} \sup_{0 \leq t \leq T} \left( |\triangle S_t| (K_T^1 + K_T^2) \right).
$$

Now we are in the same situation as in the previous proposition, with similar arguments, as the Lipschitz constant $C_2$ is small, the result thus follows immediately. ■

It follows from the continuous dependence result, the solution of reflected BSDE (3.1)-(3.2) is unique.

Acknowledgements. This research was carried out while the second author was visiting the Mathematics Institute, Oxford, and the Oxford-Man Institute, partially supported by EPSRC grant EP/F029578/1 and an LMS scheme 2 grant. The second author is partly supported by NSF Youth Grant 10901154/A0110.

References

[1] Peter Bank and Nicole El Karoui, A stochastic representation theorem with applications to optimization and obstacle problems, Ann. Probab. 32 (2004), no. 1B, 1030–1067. MR 2044673 (2005a:60046)

[2] Jean-Michel Bismut, Théorie probabiliste du contrôle des diffusions, Mem. Amer. Math. Soc. 4 (1976), no. 167, xiii+130. MR 0453161 (56 #11428)

[3] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M. C. Quenez, Reflected solutions of backward SDE’s, and related obstacle problems for PDE’s, Ann. Probab. 25 (1997), no. 2, 702–737. MR 1434123 (98k:60096)
[4] S. W. He, J. G. Wang and J. A. Yan, Semimartingales and Stochastic Calculus, CRC Press and Science Press, 1992.

[5] Ioannis Karatzas and Steven E. Shreve, Brownian motion and stochastic calculus, second ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR 1121940 (92h:60127)

[6] M. Kobylanski, J. P. Lepeltier, M. C. Quenez, and S. Torres, Reflected BSDE with superlinear quadratic coefficient, Probab. Math. Statist. 22 (2002), no. 1, Acta Univ. Wratislav. No. 2409, 51–83. MR 1944142 (2003k:60134)

[7] J.-P. Lepeltier, A. Matoussi, and M. Xu, Reflected backward stochastic differential equations under monotonicity and general increasing growth conditions, Adv. in Appl. Probab. 37 (2005), no. 1, 134–159. MR 2135157 (2006c:60070)

[8] J.-P. Lepeltier and M. Xu, Penalization method for reflected backward stochastic differential equations with one r.c.l.l. barrier, Statist. Probab. Lett. 75 (2005), no. 1, 58–66. MR 2185610 (2006g:60085)

[9] Lepeltier, J.-P. and Xu, M. (2007) Reflected BSDE with quadratic growth and unbounded terminal value. On AxXiv: 0711.06191.

[10] Jin Ma and Yusun Wang, On variant reflected backward SDEs, with applications, J. Appl. Math. Stoch. Anal. (2009), Art. ID 854768, 26. MR 2511615 (2010g:60139)

[11] Anis Matoussi, Reflected solutions of backward stochastic differential equations with continuous coefficient, Statist. Probab. Lett. 34 (1997), no. 4, 347–354. MR 1467440 (98e:60097)

[12] É. Pardoux and S. G. Peng, Adapted solution of a backward stochastic differential equation, Systems Control Lett. 14 (1990), no. 1, 55–61. MR 1037747 (91e:60171)

[13] Shige Peng and Mingyu Xu, The smallest $g$-supermartingale and reflected BSDE with single and double $L^2$ obstacles, Ann. Inst. H. Poincaré Probab. Statist. 41 (2005), no. 3, 605–630. MR 2139035 (2006e:60057)

[14] Mingyu Xu, Backward stochastic differential equations with reflection and weak assumptions on the coefficients, Stochastic Process. Appl. 118 (2008), no. 6, 968–980. MR 2418253 (2009f:60074)

Zhongmin Qian, Mathematical Institute, University of Oxford, Oxford OX1 3LB, England
Email: qianz@maths.ox.ac.uk

Mingyu Xu, Key Laboratory of Random Complex Structures and Data Science, Academy of Mathematics & Systems Science, CAS, Beijing
Email: xumy@amss.ac.cn