Effective Action study of $\mathcal{PT}$-Symmetry Breaking for the non-Hermitian $(i\phi^3)_{6-\epsilon}$ Theory and The Yang-Lee Edge Singularity

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Abstract

We use the effective potential method to study the $\mathcal{PT}$-symmetry breaking of the non-Hermitian $i\phi^3$ field theory in $6-\epsilon$ space-time dimensions. The critical exponents so obtained coincide with the exact values listed in the literature. We showed that at the point of $\mathcal{PT}$-symmetry breaking, the vacuum-vacuum amplitude is certainly zero and the fugacity is one which mimics a Yang-Lee edge singularity in magnetic systems. What makes this work interesting is that it takes into account problems which are always overlooked in the literature for the Yang-Lee model like stability, unitarity and generation of Stokes wedges at space-time dimensions for which divergences occur in the theory. Besides, here we make direct calculation of critical exponents from the dependance of the order parameter on external magnetic field not from the density of zeros of the partition function.

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The paper of Carl Bender and Stefan Boettcher in Ref. [1] sparked what we can call an actual start of the possible acceptance of non-Hermitian theories to play a role in nature’s description. This paper stimulated the interest of many researchers to study the $\mathcal{PT}$-symmetric theories which led to the growing of their applications in different branches in Physics. In fact, the importance of employing such theories to play a role in nature’s description might go back to Dirac [2] who tried to introduce a finite quantum field theory via the introduction of auxiliary fields that turn the theory non-self adjoint. Then Pauli followed Dirac with more details [3]. Lee and Wick then introduced a finite QED [4, 5] via the inclusion of abnormal fields (Lee-Wick fields) which besides of turning the theory non-self adjoint, it makes the theory possessing ghost states. In 2007, the idea has been extended to the standard model via an extension called the Lee-Wick standard model [6] which solves the Hierarchy problem but suffers from the existence of ghost states too. In Ref. [7], we showed that one can use the $\mathcal{PT}$-symmetric tools to overcome the ghost states problem in such theories. In the Lee-Wick standard model however, they add higher derivative terms to the Lagrangian instead of adding Lee-Wick field directly but it has been shown that the Lagrangian can be rewritten in terms of a normal and a Lee-Wick field which in turn shows that the theory is not self-adjoint. Moreover, recently the idea has been extended to gravity and a super-renormailzable gravity has been obtained via the insertion of higher derivative terms [8]. So, it seems that the importance of studying $\mathcal{PT}$-symmetric theories goes beyond considering it as a mathematical training and might bear the solution of existing problems appearing in models that are constrained to Hermitian theories only. However, in space-time dimensions higher than one (quantum field theory), the tools followed to study $\mathcal{PT}$-symmetric quantum problems might not work. For instance, if one generates the Stokes wedges for a theory in one dimension (Quantum mechanics) through the constraint of finite controlling integral [33], the quantum field version of the same theory will not have finite controlling integral due to existence of divergences in the theory. Accordingly, one needs to test a technique that can mimic the selection of a contour in the complex field plane within the Stokes wedges as well as being able to remedy the divergences in the theory. The suggested tool we follow here is the effective action formulation and shall try to test its predictions via comparison with indirect experimental results. The model we select to study in this work is the $\mathcal{PT}$-symmetric $i\phi^3$ theory. In $0 + 1$ space-time dimensions, one can find recent rigorous work in the literature that stressing that model [9–17]. In Higher
space-time dimensions, one always confronted by mainly two problems in studying a $\mathcal{PT}$-symmetric field theory. The first one is the generation of the Stokes wedges of the theory where divergences will prevent us from having a finite integral representing the generating function. The other problem is the calculation of the metric operator in $\mathcal{PT}$-symmetric field theory where it is hard to obtain. As we will see in this work, the effective action technique seems to be the most suitable tool to study a $\mathcal{PT}$-symmetric field theory.

The experimental test of predictions within $\mathcal{PT}$-symmetric theories have been stressed in many articles (for instance Refs. [18–20]). In fact, there is a growing interest in that direction and it seems to be a matter of time to the full acceptance of non-Hermitian theories to share in solving existing problems in the current Hermitian description of a natural phenomena. A theoretical test (or indirect experimental test) can be offered by comparing critical behaviors of a $\mathcal{PT}$-symmetric model with another model that lies in the same class of universality.

The bridge between $\mathcal{PT}$-symmetric theories and Ising model has been offered by Fisher [23, 24] who identified the effective action of the Magnetization by a Landau-Ginzberg theory given by [27]:

$$S = \int dx^d \left( \frac{1}{2} (\partial_\mu \phi)^2 + i (h - h_c) \phi + i g \phi^3 \right),$$

where $h$ represents the external magnetic field. This effective field theory has reproduced the critical behavior associated with the Yang-Lee edge singularity [25, 26]. In fact, at the continuum limit, the zeroth of the partition function in the fugacity complex plane can touch the real axis and the intersection with the real axis represents a critical point. At that limit, the zeros are more dense and the density of zeros follows a critical formula from which one can extract the critical exponent $\sigma = \frac{1}{3}$ [39].

The Yang-Lee model in Eq. (1) has been extensively studied in the literature regarding the edge singularity [27, 30] but most of the studies always skip important problems like unitarity and the needed $\mathcal{PT}$-symmetric boundary conditions. Unitarity can be solved via the insertion of the metric operator $\eta$ such that the physical amplitudes associated with an operator $\hat{O}$ take the form $\langle \hat{O} \rangle = \langle \psi \vert \eta \hat{O} \vert \psi \rangle$ [31]. Such formulation leads to a Hilbert space for which orthonormality is satisfied as well as unitarity is guaranteed. However, the metric operator is always hard to get for a quantum field theory but it has been argued by Jones that certain techniques can be used to obtain physical amplitudes for which the metric operator is automatically implemented [32]. Regarding the other problem, in order to quantize a
\(\mathcal{PT}\)-symmetric theory one needs to subject it to a boundary condition in the complex-field plane. The boundary condition is set by selecting a complex contour in the complex-field plane over which the controlling integral \(\int D\phi \exp (-\int dx^d i\phi^3 (x))\) exists \([33]\). In fact, it is always stated that this integral is given by the multiplication of infinite number of the one dimensional integral \(\int_c d\phi_i \exp (-i\phi^3 (x_i))\). Now, if one selects a contour over which the low dimensional integral \(\int_c d\phi_i \exp (-i\phi^3 (x_i))\) is set finite, this does not guarantee that the full integral \(\int D\phi \exp (-\int dx^d i\phi^3 (x))\) is finite. This is a manifestation of the existence of divergences in quantum field theories at relatively high space-time dimensions. Accordingly claiming that the Yang-Lee model in space-time dimensions higher than two is a real-line theory is not assured \([40]\). So the challenge in studying the Yang-Lee model in \(6 - \epsilon\) space-time dimensions is to use an algorithm that overcomes all of these problems and at the end can be tested via comparison of its predictions of the critical exponents with those found in the literature. This is the main aim of our work here.

In our study we shall use path integral formulation of quantum field theory. The point is that the partition function in statistical models has its counterpart in quantum field theory where in path integral formulation of a quantum field theory it is the generating functional (the vacuum persistent amplitude) which is given by \([34]\):

\[
Z(J) = \int D\phi \exp \left( i\int d^dx (\mathcal{L} [\phi] + iJ\phi) \right).
\] (2)

In Euclidean space it takes a form \(Z(J) = \exp (-S_E)\), where \(S_E\) is the Euclidean form of the action. The critical phenomena in a \(\mathcal{PT}\)-symmetric quantum field model can be investigated by monitoring the \(\mathcal{PT}\)-symmetry breaking which is expected to be associated with a zero of \(Z(J)\) or more specifically with a Yang-Lee Edge Singularity. Instead of studying the critical phenomena via the density of zeros of the partition function we shall study the behavior of the order parameter versus the coupling \(J\) where as we will see in this work that \(\mathcal{PT}\)-symmetry breaking occurs for the \(i\phi^3\) field theory at a zero of \(Z(J)\) and exact critical exponents are extracted from our calculations using path integral formulation of the theory. In fact, at the continuum limit the zeros can touch the real axis in the complex fugacity plane, a fact that will be reflected clearly in our calculations. Another point for following the path integral formulation to investigate the \(\mathcal{PT}\)-symmetry breaking of the Yang-Lee model is that the effective action technique associated with it can account for the selection of a complex contour in the complex field plane but with well known methods to cure the
divergences.

Before we start our calculation we give a brief introduction for $\mathcal{PT}$-symmetric Hamiltonians. A Hamiltonian is said to be $\mathcal{PT}$-symmetric if it is invariant under the application of both parity ($x \to -x$) and time reversal operations. A $\mathcal{PT}$-symmetric and non-Hermitian Hamiltonian can have real spectrum as long as the $\mathcal{PT}$-symmetry is exact. In changing parameters of that theory one can break this symmetry and after a point of level crossing the energy spectrum appears in complex conjugate pairs \[9, 10, 17\]. Moreover, at the critical point the wave function is self-orthogonal. For the purpose of quantization, such theories follows certain boundary conditions either in the complex-x plane or the complex-field plane.

The usual recipe for the quantization of a $\mathcal{PT}$-symmetric theory is to subject it to the boundary condition $\psi(x) \to 0$ as $|x| \to 0$ in the complex $x-$plane. Such condition selects regions in the complex plane called Stokes wedges from which a contour is selected over which the calculations are carried out. However, for the study of quantum field $\mathcal{PT}$-symmetric theory, it would be better to follow path integral formulation since it has its own boundary condition\[33\]. In such case, the Stokes wedges are generated by demanding the existence of the integral $\int d\phi \exp(-V(\phi))$, where $\phi$ is the scalar field while $V(\phi)$ is the classical potential. In other words one needs $V(\phi) \to \infty$ as $|\phi| \to \infty$ in the complex-field plane. Without loss of generality, one selects contours over which the classical potential is bounded from below as a functional of $|\phi(x)|$. In higher dimensions the dominant integral in the partition function is an infinite multiplications of the integral $\int d\phi \exp(-V(\phi))$ and it is not guaranteed that the whole functional integral is finite for a contour selected by the necessity of the existence of the the one dimensional integral $\int d\phi \exp(-V(\phi))$. Moreover, exact calculations in quantum field theory are always beyond achievement and thus one resorts to perturbative or non-perturbative methods. So instead of selecting a contour over which to make exact functional integrations (which is hard for most of quantum field models), one can follow the well-known effective action regime of calculations \[34\]. The corresponding effective potential is satisfying the bounded from below condition and thus can mimic the boundary condition mentioned above. In this situation the complex field has a real part that depends on position while the imaginary part is represented by the classical field (vacuum condensate). The equations obtained by demanding the effective potential to be bounded from below might have many solutions for the vacuum condensate which represent different phases of the theory. In fact, making the field $\phi(x)$ goes to $\phi(x) + v$,
where $v$ is the classical field is a kind of a point canonical transformations like the selection of a complex contour in the complex plane. In quantum field theory, canonical transformations can be inequivalent and thus represent different phases of the theory [35]. So, in this work, we apply the effective action method to study the Yang-Lee model of the $\mathcal{PT}$-symmetric $i\phi^3$ field theory in $6 - \epsilon$ space-time dimensions. Other reasons behind such selection are that the effective action formalism implements the metric in its calculations and the similarity of the vacuum persistence amplitude to the partition function in quantum statistical systems which enables us to study the critical behavior of the model. Up to the best of our knowledge, there is no other technique used in literature to study the Yang-Lee model and take-caring of the metric and respects the boundary condition used to study $\mathcal{PT}$-symmetric theories.

To start, consider the Lagrangian density of the $\mathcal{PT}$-symmetric $i\phi^3$ field given by the form:

$$L[\phi] = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2(x) - \frac{ig}{3} \phi^3(x) + iJ \phi(x),$$  

(3)

where $m$ is the bare mass of the field, $g$ is the coupling constant and $J$ is a source term that resembles the external magnetic field in magnetic systems. Here we put the source $iJ$ to make the theory $\mathcal{PT}$-symmetric. The vacuum persistence amplitude is given in Eq. (2). The vacuum energy $E[J]$ can be introduced via the relation $Z(J) = \exp (-iE[J])$ while the effective action $\Gamma(v)$ can be obtained through the Legendre transformation of the form [34]:

$$\Gamma(v) = -E[J] - i \int d^4 y J(y) v(y),$$  

(4)

where $v$ is the vacuum condensate. In assuming that the vacuum is translational invariant then the effective potential can be introduced as $V_{eff} = -\frac{\Gamma(v)}{VT}$, where $VT$ is the size of the space-time region over which the functional integral is to go over. Following the steps in Ref. [34], one can obtain the one-loop effective potential as:

$$V_{eff}(v) = \left[ V(\phi) + \frac{i}{2} \left( \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left( \frac{\partial^2 V(\phi)}{\partial \phi^2} \right)^{\frac{d}{2}} \right) \right]_{\phi=v},$$  

(5)

where $d$ is the dimension of the space-time and $\Gamma$ here (not to be confused with the effective action $\Gamma(v)$) is the gamma function. For the theory under consideration, $V_{eff}(v)$ (it is the vacuum energy density) takes the form:

$$V_{eff}(v) = \frac{1}{2} m^2 v^2 - \frac{g}{3} (iv)^3 + iJv + \frac{i}{2} \left( \frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \left( m^2 + 2igv \right)^{\frac{d}{2}} \right).$$  

(6)
The effective action satisfies the condition $\frac{\delta V(v)}{\delta (v)} = 0$ or equivalently the effective potential above satisfies the relation:

$$\frac{\partial V_{\text{eff}}(v)}{\partial v} = 0. \quad (7)$$

The renormalized mass $M$ is also given by:

$$\frac{\partial^2 V_{\text{eff}}(v)}{\partial v^2} = M^2. \quad (8)$$

As long as the renormalized mass is chosen positive so these two equations define a minimum of the effective potential and thus can be considered a stable one. These conditions are the counter part of the boundary condition used in $\mathcal{PT}$-symmetric quantum mechanics and the condition that the integral $\int_C d\phi \exp (-V(\phi))$ do exist on a complex contour $C$ in the complex $\phi$ plane. Thus the effective action formalism bears the spirit of the boundary condition in both $\mathcal{PT}$-symmetric quantum mechanics and quantum field theory. However, the effective action can be turned finite for a normalized $\mathcal{PT}$-symmetric field theory via well known procedures. This property gives the effective potential a privilege for the study of $\mathcal{PT}$-symmetric field theory.

Let us go back to the effective potential in Eq.(6). In $6 - \epsilon$ space-time dimension, this effective potential is divergent and a regularization scheme is needed followed by a renormalization procedure. To do that we expand in powers of $\epsilon = d - 6$ to get:

$$V_{\text{eff}} \rightarrow -\frac{1}{384\pi^3} \frac{(m^2 + 2igv)^3}{d - 6} + \frac{1}{3} igv^3 + \frac{1}{2} m^2 v^2 + iJv - \frac{1}{768} \left(\frac{2igv + m^2}{\gamma - 2\ln 2 - \ln \pi + \ln (2igv + m^2)}\right) \frac{11}{6} + O(d - 6). \quad (9)$$

In using the modified minimal subtraction method ($\overline{MS}$) and introducing a renormalization mass scale $\mu$, one gets:

$$V_{\text{eff}} = \frac{1}{2} m^2 v^2 - \frac{g}{3} (iv)^3 + iJv - \frac{1}{768\pi^3} (2igv + m^2)^3 \ln \left(\frac{2igv + m^2}{\mu^2}\right). \quad (10)$$

In applying the condition $\frac{\partial V_{\text{eff}}}{\partial v} = 0$, we get the relation:

$$m^2 v + igv^2 + iJ - \frac{1}{384\pi^3} g \left(3 \ln \frac{1}{\mu^2} (m^2 + 2igv) + 1\right) (m^2 + 2igv)^2 = 0, \quad (11)$$
while the mass renormalization condition gives the equation:

\[ m^2 + 2igv + \frac{1}{192\pi^3}g^2 \left(6\ln\frac{1}{\mu} (m^2 + 2igv) + 5\right) (m^2 + 2igv) = \mu^2, \]  

(12)

where with no loss of generality we set the renormalized mass \( M \) to coincide with the renormalization scale \( \mu \).

A good test for our calculations can be achieved via comparison of the critical exponents with those known in the literature. So before we try to solve the above equations for the classical field \( v \) and the renormalized mass \( \mu \) we need to remind ourselves by the critical behavior of a magnetic (Ising model) system undergoing a phase transition. At the critical isotherm, the magnetization \( M \) has power law dependence of the external field \( J \) as \( M \sim J^{\delta} \) \[23\] and the correlation length follows the formula \( \zeta_{gap} \sim J^{-\nu} \) \[36\]. The free energy per unit volume has a mass dimension \( d \) and thus it has a relation of the form \( \zeta_{gap}^{-d} \sim J^{-d\nu} \) \[37\]. In Ref.\[23\], it has pointed out that the Yang-Lee model is the Landau-Ginzburg approximation that can describe the critical behavior of the ferromagnetic materials. To check the validity of our results we will try to obtain the critical exponents for \( \mathcal{PT} \)-symmetric \( i\phi^3 \). In fact, the critical isotherm is equivalent to set the parameter \( m \) to zero and then we solve Eqs.\((11,12)\) to get:

\[ v = \pm i \frac{J^{\frac{1}{2}}}{\sqrt{g + \frac{1}{96} \frac{1}{\pi^3} g^3 + \frac{1}{32\pi^3} g^3 \ln \frac{g}{(2\pi)^3} \text{LambertW} \left( \frac{32 \pi^3}{g^2 e^\frac{5}{6} e^{\frac{32 \pi^3}{9^2}} \right)}} \],

\[ \mu = \sqrt{iv} \sqrt{1 \frac{1}{16} \frac{1}{\pi^3} g^3 \text{LambertW} \left( \frac{32 \pi^3}{g^2 e^\frac{5}{6} e^{\frac{32 \pi^3}{9^2}} \right)}} \],

(13)

where \( \text{LambertW} \) is the Lambert-W function satisfying \( W(x) \exp W(x) = x \). Since the renormalized mass is real, then the vacuum condensate takes the negative imaginary sign only as expected or

\[ v = -i \frac{J^{\frac{1}{2}}}{\sqrt{g + \frac{1}{96} \frac{1}{\pi^3} g^3 + \frac{1}{32\pi^3} g^3 \ln \frac{g}{(2\pi)^3} \text{LambertW} \left( \frac{32 \pi^3}{g^2 e^\frac{5}{6} e^{\frac{32 \pi^3}{9^2}} \right)}} \].

(14)

The counter part of magnetization here is the classical field \( v \) and \( J \) plays the role of external magnetic field. The correlation length is represented by inverse of mass gap \( \mu \) while the free energy is represented by the effective potential which is equivalent to vacuum energy \( E_0 \):

\[ E_0 = iJv - \frac{g}{3} (iv)^3 + \frac{1}{96} \frac{i}{\pi^3} g^3 v^3 \ln \frac{2igv}{\mu^2} \sim J^{\frac{3}{2}}. \]

(15)
From Eq. (14), one can realize that the critical exponent $\sigma = \frac{1}{3} = \frac{1}{2}$ while from Eq. (13) it gives $\nu_c = \frac{1}{4}$. These are exactly the critical exponents obtained in Ref. [23]. Moreover, from the scaling of the free energy we should have $E_0 \sim \mu^d \sim J^{\frac{2}{3}} = J^{\frac{1}{2}}$ which is exactly what Eq. (15) gives.

The theory undergoes a phase transition at the point of $\mathcal{PT}$-symmetry breaching and thus the critical point should be associated by level crossing. To check this in our calculations, consider the lowest energy levels $E_0$ and $E_1 = E_0 + \mu$. We plot them in Fig. 1 as functions of the external magnetic field $J$ and it is very clear that at $J = 0$, $\mathcal{PT}$-symmetry is broken. For the model under consideration this point is supposed to be a Yang-Lee edge singularity where the zeros of the Partition function touches the real axis of the complex fugacity. To check that this is reflected in our calculations consider the vacuum to vacuum amplitude given by:

$$Z(J) = \exp (-iE[J]) = \exp (-iE_0(\mathcal{VT})) .$$

(16)

Since we renormalized the theory at a scale $\mu$ so the size of $\mathcal{VT}$ is $\frac{1}{\mu^d}$. The factor $i$ comes from the fact that we used Wick rotation in Feynman diagram calculation. In other words

$$Z(J) = \exp \left(-\frac{E_0}{\mu^6}\right) .$$

(17)

This form does not explicitly depend on $J$ since from Eq. (13) we obtain

$$Z = \exp \left(-\frac{256}{9} \pi^6 \frac{3 \text{LambertW} \left( \frac{32 \pi^3}{g^2} \exp \left( \frac{1.5 g^2 + 192 \pi^3}{g^2} \right) \right) - 9 \ln 2 - 1}{g^6 \left( \text{LambertW} \left( \frac{32 \pi^3}{g^2} \exp \left( \frac{1.5 g^2 + 192 \pi^3}{g^2} \right) \right) \right)^3} \right) .$$

(18)

This form goes to zero as $g$ goes to zero too. So to check if at the critical point ($J = 0$), the partition function has a zero, one needs check that $g \to 0$ as $J \to 0$. To do that we need to know the scale behavior of the coupling $g$. In fact, we know the scale behavior of the coupling $J$ where it goes to zero as $\mu \to 0$ while to know the scale dependance of the renormalized coupling $g$ we need to obtain the Beta function of the theory. However, the Beta function can be concluded from the one for the Hermitian $\phi^3$ where it has been obtained in Ref. [38]. Unlike the Hermitian theory, the Beta function for the $\mathcal{PT}$-symmetric $i\phi^3$ is positive and up to one loop is given by:

$$\beta = \mu \frac{dg}{d\mu} = \frac{3 g^3}{512 \pi^2} .$$

(19)
Being positive, the $\beta$–function tells us that the theory has an infra-red fixed point or equivalently $g \to 0$ as $\mu \to 0$. This analysis assures that at the critical point ($\mu = 0$, $J = 0$ and $g = 0$) the partition function is zero too. Since this zero exists at $J = 0$ thus the fugacity is real ($\text{fugacity } z = 1$). This shows that the critical point at which the $\mathcal{PT}$-symmetry is broken is in fact a Yang-Lee Edge Singularity in the sense that it resembles a zero of the partition function that touches the real Fugacity axis as in magnetic systems.

![Energy Levels](image)

**FIG. 1:** The first two energy levels $E_0$ and $E_1$ versus the external magnetic field $J$ for the massless $\mathcal{PT}$-symmetric $i\phi^3$ theory in $6 - \epsilon$ space-time dimensions.

To conclude, We have shown that the effective action technique is the most suitable algorithm (up to the best of our knowledge) in the literature to study $\mathcal{PT}$-symmetric field theories specially in a relatively high space-time dimension. The point is that for high space-time dimensions the controlling factor used to generate the Stokes wedges of the theory in low dimensions is no longer working. In fact for space-time dimensions for which the theory is renormalizable we are confronted by the existence of divergences which means that the integral defining the generating functional can not be finite for wedges selected by the one
dimensional controlling factor. So the field is in a need to discover a technique that mimics the boundary conditions applied to $\mathcal{PT}$-symmetric quantum mechanics or low dimensional quantum field theory and can get rid of divergences as well. We have illustrated that the effective potential bears these features. Then we applied it to the normalizable Yang-Lee model in $6 - \epsilon$ dimensions where we can find predictions from studying the Yang-Lee edge singularity of the model in the literature and thus offers a way to test the effective potential predictions for a $\mathcal{PT}$-symmetric field theory.

To check the validity of our calculations, We obtained the critical exponents of the $\mathcal{PT}$-symmetric $i\phi^3$ theory in $6 - \epsilon$ space-time dimensions and found them coincides with the exact ones listed in Ref. \cite{23}. Moreover, we proved all the expected features associated with $\mathcal{PT}$-symmetry breaking. For instance, we showed a level crossing at the critical point for the first two levels of the theory. Besides, we have shown that at the point of $\mathcal{PT}$- breaking the fugacity is real (in fact equal one) which means that it is a Yang-Lee edge singularity where there is a zero of the partition function that touches the real fugacity axis.

The renormalization group flow of the parameters in the theory has been investigated. The coupling $J$ has a positive mass dimension and is expected to go to zero as the mass scale $\mu$ goes to zero which has been reflected clearly in our calculations. Likewise the vacuum condensate and vacuum energy both have positive mass dimensions and have shown to go to zero as $\mu$ goes to zero too. For the renormalized coupling $g$ on the other hand, it is dimensionless and for its scale behavior one needs to know the $\beta$-function of the theory. Since the $\beta$-function is positive, the theory has an IR- fixed point or in other words $g \to 0$ as $\mu \to 0$ too. This fact was needed to show that the partition function $Z(J)$ goes to zero at the critical point.
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[39] See Eq. (14.5.3) and Eq. (14.5.3) in Ref. [27]

[40] In future, We plane to study if there are available complex contours over which the full controlling factor integral exists. If so, one might obtain a finite field theory via boundary conditions in the complex field plane.