Weyl ordering rule and new Lie bracket of quantum mechanics

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Abstract

The product of quantum mechanics is defined as the ordinary multiplication followed by the application of superoperator that orders involved operators. The operator version of Poisson bracket is defined being the Lie bracket which substitutes commutator in the von Neumann equation. These result in obstruction free quantization, with the ordering rule which coincides with Weyl ordering rule.

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I Introduction

Quantization may roughly be thought as transition from classical to quantum description of a system. What is its precise definition is not generally agreed since many of related questions are still opened. With different motivations, quantization was approached in different manners, see [5] and references therein. Even in this case, the ordering problem is not trivial. The most important problem regarding the quantization of $\mathbb{R}^{2n}$ is in that it is not known how should the algebraic and Lie algebraic multiplications of quantum mechanics be realized in unambiguous and consistent way.

In [3, 7] one finds a short review of different propositions of symmetrized product. In [7] it was shown that there given propositions for symmetrized product start to differ for quadratic monomials in $\hat{q}$ and $\hat{p}$. In [8] there is a critical discussion of many ordering rules introduced in there cited references. In [3, 1, 4, 2] and references therein, it was found that algebraic and Lie algebraic structures of quantum mechanics are interrelated in such a way that obstructions result in quantization which manifest themselves through some contradictions in a formalism. A detailed derivation of the Lie bracket of quantum mechanics in relation to quantization one can find in [6]. (Needless to say, as the standard choice of the Lie bracket of quantum mechanics appears the commutator divided by $i\hbar$.) Because of the mentioned contradictions, one can conclude, see [2], that the problem of quantization is impossible or, as was noticed in [5], that some subtler symmetrization rule is necessary.

In this article we shall define the symmetrized product of quantum mechanics. Then, by using this product, we shall be able to propose new Lie bracket of quantum mechanics. It will be the operator version of Poisson bracket. Since in the classical mechanics this bracket is used in equation of motion - the Liouville equation, we shall propose the reformulation of dynamical equation of quantum mechanics. It will be shown that the von Neumann - Schrödinger equation can be seen as the operator version of Liouville equation. In this
way quantum and classical mechanics will appear to be the same regarding
the algebraic and Lie algebraic aspect and the dynamical equation. The same
realization of these crucial elements of the theories will allow us to propose
the quantization of $\mathbb{R}^{2n}$ that is free of obstruction. Finally, we will show that
the ordering rule given by our quantization coincide with Weyl ordering rule.
This fact is proved in Theorem 5.

II The symmetrizer and symmetrized product

Let us introduce basic notations and notions. Let $\Omega_{n,m}^2(\hat{q}, \hat{p}) = \Omega_{n,m}^2$, be the
permutation group (with repetition) of $n$ examples of $\hat{q}$ and $m$ of $\hat{p}$. $\mathcal{U}(\mathcal{H}_3)$
will denote the universal enveloping algebra of Heisenberg algebra generated
by $\{\hat{q}, \hat{p}, \hat{I}\}$ and with the only nontrivial commutator

$$[\hat{q}, \hat{p}] = i\hbar \hat{I}. \quad (1)$$

If we introduce, the operator $\hat{h} = -i\hbar \hat{I}$, then because of (1) it is clear that the
set

$${\mathcal{B}} = \{ \hat{q}^n \hat{p}^m \hat{h}^p \mid n, m, p \in \mathbb{N}_0 \}$$

is a PBW-basis of $\mathcal{U}(\mathcal{H}_3)$. We will use the following notations $\hat{\mathcal{U}}(\mathcal{H}_3)$, for subspace of $\mathcal{U}(\mathcal{H}_3)$ generated by $\hat{p}$’s and $\hat{q}$’s;
by $\hat{\mathcal{U}}(\mathcal{H}_3)_{n,m}$ we will denote the subspace of $\mathcal{U}(\mathcal{H}_3)$ generated by all possible
monomials with $n$ $\hat{q}$’s and $m$ $\hat{p}$’s, i.e., an arbitrary expression of the form

$$\hat{q}^{n_1} \hat{p}^{m_1} \cdots \hat{q}^{n_s} \hat{p}^{m_s},$$

where $n_i \geq 0$, $m_j \geq 0$ and $\sum_{i=1}^s n_i = n$, $\sum_{i=1}^s m_i = m$.

**Definition 1.** Let $\mathcal{U}$ be a $\mathbb{C}$—associative algebra with unit generated by
$\{\hat{p}, \hat{q}, \hat{h}, \hat{\rho}, \frac{\partial}{\partial \hat{\rho}}\}$. **Symmetrizer $S$** is a linear map, $S : \mathcal{U} \rightarrow \mathcal{U}$, defined as follows: $\forall \hat{A}_{n,m} \in \hat{\mathcal{U}}(\mathcal{H}_3)_{n,m}$

$$S(\hat{A}_{n,m}) = \left( \begin{array}{c} n + m \\ n \end{array} \right)^{-1} \sum_{\sigma(\hat{q}, \hat{p}) \in \Omega_{n,m}^2} \sigma(\hat{q}, \hat{p}) \quad (2)$$
\[ S(\hat{A}_{n,m} \cdot \hat{\rho}) = \hat{A}_{n,m} \cdot \hat{\rho}, \quad S(\hat{\rho} \cdot \hat{A}_{n,m}) = \hat{\rho} \cdot \hat{A}_{n,m}, \]  
\[ S(\hat{\rho} \cdot \frac{\partial \hat{\rho}}{\partial \hat{r}}) = \hat{\rho} \cdot \frac{\partial \hat{\rho}}{\partial \hat{r}}, \]  
\[ S(\frac{\partial \hat{\rho}}{\partial \hat{r}} \cdot \hat{A}_{n,m}) = \left( n + m + 1 \right) \sum_{\sigma(\hat{q}, \hat{p}) \in \Omega_{n,m}^1} \sigma(\hat{q}, \hat{p}, \frac{\partial \hat{\rho}}{\partial \hat{r}}), \]

**Remark.**
(a) Because of (1) and (2), it is clear that \( S(\hat{B}) = 0 \), for every monomial \( \hat{B} \in \hat{U}(\mathcal{H}_3) \) which has \( \hat{\hbar} \) as a factor.

(b) From the above definition it is clear that \( S \) maps all \( \hat{U}(\mathcal{H}_3)_{n,m} \) onto

\[ \left( n + m + 1 \right) \sum_{\sigma(\hat{q}, \hat{p}) \in \Omega_{n,m}^2} \sigma(\hat{q}, \hat{p}). \]

(c) Since all different combinations of involved \( \hat{q} \)'s and \( \hat{p} \)'s appear on the RHS of (2), the symmetrized product of two symmetrized monomials, which are the Hermitian, obviously will be invariant under the Hermitian conjugation.

(d) From the physical reasons the action of operator \( S \) on another elements of \( \hat{U} \) are not important for us (here).

Definition of symmetrizer enables us to introduce symmetrized product in the following way.

**Definition 2.** For any two \( \hat{A}, \hat{B} \in \hat{U}(\mathcal{H}_3) \), the symmetrized product is composition of ordinary multiplication and application of symmetrizer:

\[ \hat{A} \circ \hat{B} = S(\hat{A} \cdot \hat{B}). \]  

**Example.**
The symmetrized product of the square of operator of coordinate and the square of operator of momentum is:

\[ \hat{q}^2 \circ \hat{p}^2 = S(\hat{q}^2 \hat{p}^2) = \frac{1}{6} (\hat{q}^2 \hat{p}^2 + \hat{q} \hat{p} \hat{q} \hat{p} + \hat{q} \hat{p}^2 \hat{q} + \hat{p} \hat{q}^2 \hat{p} + \hat{p} \hat{q} \hat{p} \hat{q} + \hat{p}^2 \hat{q}^2). \]

The formulas, specially (6) and more general (2), are the most 'natural' (with respect to symmetry) generalization of famous Dirac's symmetrized product given by \( \frac{1}{2} (\hat{q} \hat{p} + \hat{p} \hat{q}) \).
**Proposition 1.** Symmetrized product $\circ$ is a commutative map.

**Proof:** It follows from

$$(\hat{q}^a \circ \hat{p}^b) \circ (\hat{q}^c \circ \hat{p}^d) = S(S(\hat{q}^a \cdot \hat{p}^b) \cdot S(\hat{q}^c \cdot \hat{p}^d)) = \hat{q}^{a+c} \circ \hat{p}^{b+d}.$$  \hfill (7)

The ordered product of $f(\hat{r})$ and $\hat{p}$ has to be the one half of the anti-commutator of these two (this appears in the Hamiltonian of charged particle in the electromagnetic field), see [10]. Since with our proposal of symmetrized product we do not want to contradict the well-known facts of standard quantum mechanics, we have to prove the following proposition.

**Proposition 2.** In the algebra $\mathcal{U}$ the following properties hold:

(a) $$\frac{1}{2} \left( \sum_{j=0}^{n} c_j \hat{q}^j \hat{p} + \hat{p} \sum_{j=0}^{n} c_j \hat{q}^j \right) = \sum_{j=0}^{n} c_j \hat{q}^j \circ \hat{p},$$  \hfill (8)

(b) $$\frac{\partial}{\partial \hat{q}} \sum_{j,k \geq 0} c_{jk} \hat{q}^j \circ \hat{p}^k = \sum_{j,k \geq 0} j c_{jk} \hat{q}^{j-1} \circ \hat{p}^k,$$

(c) $$\frac{\partial}{\partial \hat{p}} \sum_{j,k \geq 0} c_{jk} \hat{q}^j \circ \hat{p}^k = \sum_{j,k \geq 0} k c_{jk} \hat{q}^j \circ \hat{p}^{k-1}.$$  \hfill (10)

**Proof.** (a) The LHS can be transformed into:

$$\sum_{j=0}^{n} c_j \hat{q}^j \hat{p} - \frac{i \hbar}{2} \frac{\partial}{\partial \hat{q}} \sum_{j=0}^{n} c_j \hat{q}^j.$$

For the RHS it holds:

$$\sum_{j=0}^{n} c_j \frac{1}{j+1} (\hat{p} \hat{q}^j + \hat{q} \hat{p} \hat{q}^{j-1} + \cdots + \hat{q}^{j-1} \hat{p} \hat{q} + \hat{q}^j \hat{p})$$

$$= \sum_{j=0}^{n} c_j \frac{1}{j+1} (\hat{q} \hat{p} - i \hbar j \hat{q}^{j-1} + \hat{q} (\hat{q}^{j-1} \hat{p} - i \hbar (j-1) \hat{q}^{j-2}) + \cdots + \hat{q}^{j-1} \hat{q} \hat{p} - i \hbar \hat{q}^{j-2})$$

$$\times (\hat{q} \hat{p} - i \hbar) + \hat{q}^{j-1} \hat{p}) = \sum_{j=0}^{n} c_j \hat{q}^j \hat{p} - i \hbar \sum_{j=0}^{n} c_j \frac{1}{j+1} (j + (j-1) + \cdots + 1) \hat{q}^{j-1}$$
\[
\sum_{j=0}^{n} c_j \hat{q}^j \hat{p} - \frac{i\hbar}{2} \sum_{j=0}^{n} c_j j \hat{q}^{j-1} = \sum_{j=0}^{n} c_j \hat{q}^j \hat{p} - \frac{i\hbar}{2} \frac{\partial}{\partial \hat{q}} \sum_{j=0}^{n} c_j \hat{q}^j.
\]

Therefore, both sides are equal.

(b) Due to the linearity of partial derivations, it is enough to prove this equation for monomials. In \(\hat{q}^j \circ \hat{p}^k\) there are \(\frac{(j+k)!}{j!k!}\) different sequences in the sum. Each sequence contains \(j\) operators of coordinate and \(k\) operators of momentum. Partial derivation with respect to \(\hat{q}\) produce \(j\) new terms from each of these sequences, so there are \(j \frac{(j+k)!}{j!k!}\) terms in the sum after this derivation. Each of these new terms is the sequence of \(j - 1\) operators of coordinate and \(k\) operators of momentum, as is needed for \(\hat{q}^{j-1} \circ \hat{p}^k\). Since the number of different combinations of \(j - 1\) operators of coordinate and \(k\) operators of momentum is less than \(j \frac{(j+k)!}{j!k!}\), many of these new sequences are the same. Each of \(\frac{(j-1+k)!}{(j-1)!k!}\) different sequences needed for \(\hat{q}^{j-1} \circ \hat{p}^k\) appears \(j + k\) times among the new terms and in this way the multiplicative factor \(j \frac{1k!}{(j+k)!}\), standing in front of the sum and coming from \(\hat{q}^j \circ \hat{p}^k\) is regularized. So, the proper multiplicative factor needed for \(\hat{q}^{j-1} \circ \hat{p}^k\) is gained.

(c) Similarly to the previous case. \(\diamond\)

III The symmetrized Poisson bracket

Using the symmetrized product in the previous section we can introduce the corresponding Poisson bracket.

**Definition 3.** The *symmetrized Poisson bracket* of \(\hat{A}, \hat{B} \in \hat{U}(\mathcal{H}_3)\) is given by:

\[
\{\hat{A}, \hat{B}\}_s = \frac{\partial \hat{A}}{\partial \hat{q}} \circ \frac{\partial \hat{B}}{\partial \hat{p}} - \frac{\partial \hat{A}}{\partial \hat{p}} \circ \frac{\partial \hat{B}}{\partial \hat{q}}.
\]

The most important properties of symmetrized Poisson bracket are content of the following
Proposition 3. (a) The symmetrized Poisson bracket is the Lie bracket of quantum mechanical symmetrized observables.
(b) The symmetrized Poisson bracket is a derivative:
\[
\{\hat{A}, \hat{B} \circ \hat{C}\}_S = \{\hat{A}, \hat{B}\}_S \circ \hat{C} + \hat{B} \circ \{\hat{A}, \hat{C}\}_S.
\] (12)

Proof. That \{ , \}_S is linear holds due to the fact that the partial derivations and application of symmetrizer are linear operations. That it is anti-symmetric follows from the commutativity of symmetrized product. The confirmation of the Jacobi identity can rest on the analogy between algebraic products of quantum and classical mechanics and their relations with partial derivations. Each step of the calculation in the case of operators \(\hat{q}^{n'} \circ \hat{p}^{m'}\), \(\hat{q}^{n''} \circ \hat{p}^{m''}\) and \(\hat{q}^{n'''} \circ \hat{p}^{m'''}\) has the corresponding one in the c-number case for \(q^{n'} \cdot p^{m'}\), \(q^{n''} \cdot p^{m''}\) and \(q^{n'''} \cdot p^{m'''}\) which satisfy the Jacobi identity. Due to the linearity of \{ , \}_S, this identity holds for polynomials and analytical functions of \(\hat{q}\) and \(\hat{p}\) with these two operators multiplied according to the symmetrized product.

(b) Confirmation is trivial due to the one-to-one relation with the c-number case. 

Since the Poisson bracket is crucial part of the Liouville equation, it does not come as surprise that we want to consider the question whether it is possible to reexpress the dynamical equation of quantum mechanics. But, before addressing this topic, let us remark that the general state of quantum mechanical system \(\hat{\rho}\) can be expressed via \(\hat{q}\) and \(\hat{p}\). Details are given in Appendix A. If it is seen as \(\rho(\hat{q}, \hat{p}, t)\), this operator can be derived with respect to \(\hat{q}\) and \(\hat{p}\) directly, as it is done in the case of probability distribution \(\rho\) that describes state of classical mechanical system.

Theorem 4. Let \(\hat{H} = \sum_i c_i \hat{q}^{n_i} \circ \hat{p}^{m_i}\) be a Hamiltonian, then the following relations hold
\[
(a) \quad \{\hat{H}, \hat{\rho}\}_S = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}],
\] (13)
Proof. (a) Due to the linearity of symmetrized Poisson bracket and commutator, this equation holds if it holds for \( \hat{H} = \hat{q}^n \circ \hat{p}^m \). The LHS of (13) in the case of monomial, after partial derivations of \( \hat{H} \) and multiplications, becomes:

\[
\frac{n!m!}{(n+m)!}(\hat{q}^{n-1}\hat{p}^m \frac{\partial \hat{\rho}}{\partial \hat{p}} + \cdots + \frac{\partial \hat{\rho}}{\partial \hat{p}} \hat{p}^m \hat{q}^{n-1} - (\hat{q}^n \hat{p}^{m-1} \frac{\partial \hat{\rho}}{\partial \hat{q}} + \cdots + \frac{\partial \hat{\rho}}{\partial \hat{q}} \hat{p}^{m-1} \hat{q}^n)).
\]  

(15)

After substituting \( \frac{\partial \hat{\rho}}{\partial \hat{p}} \) with \( \frac{1}{\hbar} [\hat{q}, \hat{\rho}] \) and \( -\frac{\partial \hat{\rho}}{\partial \hat{q}} \) with \( \frac{1}{\hbar} [\hat{p}, \hat{\rho}] \), (15) can be simplified. Some terms in this expression are of the form \( \hat{A} \hat{q} \hat{q} (\hat{q} \hat{\rho}) \hat{q} \hat{B} \), where \( \hat{A}, \hat{B} \) represent (different) sequences of \( \hat{q} \)'s and \( \hat{p} \)'s and \( (\hat{q} \hat{\rho}) \) means that these two come from the commutator \([\hat{q}, \hat{\rho}]\). In transformed (15) terms \( -\hat{A} \hat{q} (\hat{q} \hat{\rho}) \hat{B} \), where the minus sign comes from the commutator, certainly appear as well. So, these terms mutually cancel each other. This holds for all other forms of terms except for those where \( \hat{\rho} \) stands at the beginning or at the end of the sequence. Consequently, (15) is equal to:

\[
\frac{1}{\hbar} \frac{n!m!}{(n+m)!} ((\hat{q}^n \hat{p}^m + \cdots + \hat{p}^m \hat{q}^n) \hat{\rho} - \hat{\rho} (\hat{q}^n \hat{p}^m + \cdots + \hat{p}^m \hat{q}^n)),
\]

which is nothing else than the RHS of (13) for the considered monomial.

(b) Directly follows from (a). 

\(\Box\)

Remark. The equation (14) is the dynamical equation of quantum mechanics. Obviously, von Neumann equation (14) is symmetrized version of the Liouville equation.

From the above given, it follows that one can propose quantization which is, we believe, unambiguous, \(i.e.,\) obstruction free \textit{in toto}.

Definition 4. Let the algebra of variables of a classical mechanical system be generated by \( 1, q \) and \( p \), then quantization is transition to operator formulation defined in the following way:

\[
1, q, p \longrightarrow \hat{h}, \hat{q}, \hat{p}, \quad (16)
\]
Remark. The Hamiltonian function of classical mechanical system:

\[ H(q, p) = \sum_i c_i q^{a_i} \cdot p^{b_i}, \]

is mapped in the Hamiltonian

\[ H(\hat{q}, \hat{p}) = \sum_i c_i \hat{q}^{a_i} \circ \hat{p}^{b_i}, \]

of quantum mechanical system and dynamical equations of these theories are related in the following way:

\[ \frac{\partial \rho(q, p, t)}{\partial t} = \{H(q, p), \rho(q, p, t)\} \rightarrow \frac{\partial \rho(\hat{q}, \hat{p}, t)}{\partial t} = \{H(\hat{q}, \hat{p}), \rho(\hat{q}, \hat{p}, t)\} \]

IV Symmetrized product and Weyl ordering rule

It is well known that the Weyl ordering rule is given by the following formula

\[ \mathcal{O}(q^n p^m) = \frac{1}{2^n} \sum_{i=0}^{n} \hat{q}^{n-i} \hat{p}^{m} \hat{q}^{i}, \]

and also it is known that Weyl quantization is obstruction free.

In this section we will prove that our symmetrized product and Weyl ordering rule are the same. This fact explains why Weyl quantization is so symmetric. More precisely we will prove,
Theorem 5. The quantizations given by the following formulas

\[ O(q^n p^m) = \frac{1}{2^n} \sum_{i=0}^{n} \hat{q}^{n-i} \hat{p}^i \hat{q}^i, \]  

(19)

\[ S(q^n p^m) = \binom{n+m}{n}^{-1} \sum_{\sigma(\hat{q}, \hat{p}) \in \Omega_{n,m}^2} \sigma(\hat{q}, \hat{p}) \]  

(20)

are the same.

Proof. We will show that the right sides of the formulas (19) and (20) have same coefficients in the standard basis of \( \hat{U}(\mathcal{H}_3) \). Firstly, let us start with the following identities:

\[ \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}, \quad k \leq n, \]  

(21)

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{k}{j} j! = 2^{n-j} n (n-1) \cdots (n-j+1), \quad j \in \mathbb{N}_0. \]  

(22)

The relation (1) is equivalent with the following relation

\[ \hat{p} \hat{q} = \hat{q} \hat{p} + \hat{h}, \]  

(23)

where, as we introduced before, \( \hat{h} = -i \hbar \hat{I} \). Then using the relation (23), by induction one can show the following identities.

Lemma 5.1. \( \forall m, n \in \mathbb{N} \) holds

\[ (i1) \quad \hat{p}^m \hat{q} = \hat{q} \hat{p}^m + m \hat{p}^{m-1} \hat{h}, \]  

(24)

\[ (i2) \quad \hat{p}^m \hat{q}^n = \sum_{k=0}^{n \wedge m} \binom{n}{k} \binom{m}{k} k! \hat{q}^{n-k} \hat{p}^{m-k} \hat{h}^k. \]  

(25)

where \( n \wedge m = \min(n, m) \).

Let us now find, using (21), (22) and above lemma, the coordinates of \( O(q^n p^m) \) in standard basis of \( \hat{U}(\mathcal{H}_3) \). We have

\[ O(q^n p^m) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \hat{q}^{n-k} (\hat{p}^m \hat{q}^k) \]
\[
\sum_{k=0}^{n} \binom{n}{k} \hat{q}^{n-k} \left( \sum_{j=0}^{k \land m} \binom{k}{j} \frac{m!}{j!} \hat{p}^{m-j} \hat{h}^{j} \right) = \frac{1}{2^n} \sum_{j=0}^{n \land m} \left( \sum_{k=0}^{n} \binom{n}{k} \binom{k}{j} \frac{m!}{j!} \hat{p}^{m-j} \hat{h}^{j} \right) = \frac{1}{2^n} \sum_{j=0}^{n \land m} \binom{n}{j} \binom{m}{j} 2^{n-j} \hat{q}^{n-j} \hat{p}^{m-j} \hat{h}^{j} = 1.
\]

If we introduce
\[
\alpha_{n,m}^{j} = \binom{n}{j} \binom{m}{j} \frac{j!}{2^j},
\]
then we have

**Lemma 5.2.** \(\forall m, n \in \mathbb{N}, i \leq n, \) the following relations hold

(i1) \[
\sum_{k=0}^{n} \binom{n+m-k}{m} = \binom{n+m+1}{n},
\]

(i2) \[
\sum_{k=1}^{n} k \binom{n+m-k}{m} = \binom{n+m+1}{n-1},
\]

(i3) \[
\sum_{k=i}^{n} \binom{k}{i} \binom{m+k}{m} = \binom{n+m+1}{n} \binom{n}{i} \frac{m+1}{m+1+i},
\]

(i4) \[
\sum_{k=0}^{n} \binom{n-k+m}{m} \left( \alpha_{j+1}^{n-k,m} + (n-k-j) \alpha_{j}^{n-k,m} \right) = \binom{m+n+1}{m} \alpha_{j+1}^{n,m+1}.
\]
Proof of Lemma 5.2. (i1)-(i3) by induction. For (i4), firstly we check the following relation,

\[ \alpha_{j+1}^{n-k,m} + (n - k - j) \alpha_j^{n-k,m} = \frac{m + j + 2}{2^{j+1}} \binom{n - k}{j+1} m(m-1) \cdots (m-j+1), \]

then we have

\[
\sum_{k=0}^{n} \binom{n - k + m}{m} \left( \alpha_{j+1}^{n-k,m} + (n - k - j) \alpha_j^{n-k,m} \right) = \\
\frac{m + j + 2}{2^{j+1}} m(m-1) \cdots (m-j+1) \sum_{k=0}^{n} \binom{n - k + m}{m} \left( \alpha_{j+1}^{n-k,m} + (n - k - j) \alpha_j^{n-k,m} \right) = \\
\frac{m + j + 2}{2^{j+1}} m(m-1) \cdots (m-j+1) \sum_{k=0}^{n} \binom{k + m}{m} \left( \alpha_{j+1}^{n-k,m} + (n - k - j) \alpha_j^{n-k,m} \right) = \\
\frac{m(m-1) \cdots (m-j+1) (m+j+2) (m+n+1)!}{2^{j+1} (j+1)! m! (n-j)!} = \\
\binom{m + n + 1}{m} \alpha_{j+1}^{n,m+1}.
\]

Let us show that \( S(q^n p^m) \) is equal to (26). For \( m = 0 \) and all \( n \) it is clear that formula holds. If we assume that it is true for some \( m \in \mathbb{N} \) and arbitrary \( n \in \mathbb{N}_0 \), then using Lemma 5.2., we have

\[
S(q^n p^{m+1}) = \binom{n + m + 1}{n}^{-1} \sum_{\sigma(\hat{q}, \hat{p})}^{\sigma(q, p) \in \Omega_{n,m+1}}^{\tau} \sigma(\hat{q}, \hat{p}) = \\
\binom{n + m + 1}{n}^{-1} \sum_{n_0 \geq 0}^{\sum_{i=0}^{m+1} n_i = m+1} \hat{q}^{n_0} (\hat{p} \hat{q}^{n_1}) \cdots (\hat{p} \hat{q}^{n_{m+1}}) = \\
\binom{n + m + 1}{n}^{-1} \sum_{n_0=0}^{n} \hat{q}^{n_0} \hat{p} \sum_{n_i \geq 0}^{\sum_{i=1}^{m+1} n_i = m+1} \hat{q}^{n_1} (\hat{p} \hat{q}^{n_2}) \cdots (\hat{p} \hat{q}^{n_{m+1}}) = \\
\binom{n + m + 1}{n}^{-1} \sum_{n_0=0}^{n} \hat{q}^{n_0} \hat{p} \left( \binom{n - n_0 + m}{m} \right)
\]
\[
\times \sum_{j=0}^{(n-n_0)^m} \alpha_j^{n-n_0,m} \hat{q}^{n-n_0-j} \hat{p}^{m-j} \hat{h}^j
\]

\[
= \binom{n + m + 1}{n}^{-1} \sum_{n_0=0}^{n} \binom{n - n_0 + m}{m} \sum_{j=0}^{(n-n_0)^m} \alpha_j^{n-n_0,m} \hat{q}^{n-j} \hat{p}^{m+1-j} \hat{h}^j
\]

\[
= \binom{n + m + 1}{n}^{-1} n^{\wedge(m+1)} \sum_{n_0=0}^{n} \binom{n - n_0 + m}{m} \times (\alpha_{j+1}^{n-n_0,m} + (n-n_0-j) \alpha_j^{n-n_0,m}) \hat{q}^{n-j-1} \hat{p}^{m+1-j} \hat{h}^j
\]

\[
= \sum_{j=0}^{n^{\wedge(m+1)}} \alpha_j^{n,m+1} \hat{q}^{n-j} \hat{p}^{m+1-j} \hat{h}^j.
\]

This completes the proof of Theorem 5.

\diamondsuit

V Conclusion

The symmetrized product and symmetrized Poisson bracket we have defined in a way that they in complete imitate the appropriate operations in the c-number case. If there is some equation for classical variables, then the same equation holds for the quantum counterparts. Consequence of this is that there are, we believe, neither algebraic nor Lie algebraic contradictions in quantum mechanics based on these operations since no such contradictions appear in classical mechanics. Moreover, we have shown that classical and quantum mechanics are not just similar from the point of view of algebra, Lie algebra and dynamical equation, but have the same realizations of these important features.

Also, in the previous section we showed that our quantization implies the same ordering rule as the Weyl quantization. This fact and obvious symmetricity of our quantization explain why Weyl quantization is so symmetric.
Appendix A

With the help of:

\[ |q'\rangle\langle q'| = \int \delta(q - q')|q\rangle\langle dq = \delta(\hat{q} - q'), \]

and:

\[ |q''\rangle\langle q'| = e^{\frac{i}{\hbar}(q'' - q')\hat{p}}|q'\rangle\langle q'| = e^{\frac{i}{\hbar}(q'' - q')\hat{p}} \cdot \delta(\hat{q} - q'), \]

one immediately finds that the operator corresponding to general pure state \( |\psi\rangle = \int \psi(q)|q\rangle\langle dq \) can be expressed as:

\[ |\psi\rangle\langle \psi| = \int \int \psi(q)\psi^*(q')|q\rangle\langle q'|dqdq' = \]

\[ = \int \int \psi(q)\psi^*(q')e^{\frac{i}{\hbar}(q - q')\hat{p}} \cdot \delta(\hat{q} - q')dqdq'. \]

Since \( \hat{\rho} = \sum w_i|\psi_i\rangle\langle \psi_i| \), one concludes that all states of quantum mechanical system can be expressed via operators of coordinate and momentum.

References

[1] Arens, R., and Babit, D., ” Algebraic difficulties of preserving dynamical relations when forming quantum-mechanical operators,” J. Math. Phys. 6, 1071-1075 (1965).

[2] Chernoff, P. R., ” Irreducible representations of infinite dimensional transformation groups and Lie algebras,” J. Funct. Anal. 130, 255-282 (1995).

[3] Cohen, L., ” Generalized phase-space distribution functions”, J. Math. Phys. 7, 781-786 (1966).
[4] Gotay, M. J., ”Functorial geometric quantization and Van Hove’s theorem,” *Int. J. Theor. Phys.* **19**, 139-161 (1980).

[5] Gotay, M. J., *et al.*, ”Obstruction results in quantization theory,” *J. Non. Sci.* **6**, 469-703 (1996).

[6] Joseph, A., ”Derivations of Lie brackets and canonical quantization,” *Commun. Math. Phys.* **17**, 210-232 (1970).

[7] Kerner, E. H., and Sutcliffe, W. G., ”Unique Hamiltonian operators via Feynman path integrals,” *J. Math. Phys.* **11**, 391-393 (1970).

[8] Shewell, J. R., ”On the formation of quantum-mechanical operators,” *Am. J. Phys.* **27**, 16-21 (1959).

[9] Temple, G., ”The fundamental paradox of quantum theory,” *Nature*, **135** (1935), 957-957.

[10] Messiah, A., ”Quantum mechanics I,” North-Holland, Amsterdam, (1961).