Soliton model of dark matter and natural inflation

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Abstract. Axion-like scalar fields with their periodic potential are analyzed as a toy model of dark matter halos. Here, solitons collision of the well-known kinks in (1+1) spacetime dimensions are mapped to the localized lumps of the Lane-Emden (LE) truncation. Recent attempts to generalize this mapping to (2+1)D or even (3+1)D collision are related to an intrinsic inelastic effect during relativistic soliton collisions.

An indirect method of determining the stability of (3+1)D soliton or boson stars (BSs) is displayed without resorting to explicit solutions. Instead, the scalar part of the Lagrangian is mapped into a function \( L(R) \) of the scalar curvature, such that a BS is effectively described by a higher order gravity model. This higher order Lagrangian exhibits minima which are correlated with the transition point from stable to unstable configurations. In the case of an axion BS, the function \( L(R) \) exhibits a combination of swallow tail cusps well-known from catastrophe theory.

1. Introduction

Baryons account for a rather small fraction of the observed total matter of the Universe. The major part of it is a mysterious dark matter (DM) and dark energy (DE), both interacting gravitationally. Without lack of generality we can limit ourselves to a scalar field \( \phi \), at times advocated as ‘fuzzy’ DM [30]. Head-on collisions of massive galaxy clusters, like those occurring in the so-called bullet cluster [9] are a challenge for the cold dark matter (CDM) paradigm.

Axion-like particles [15] have been probed as DM candidates. In particular, previous investigations [5, 6] whether or not two dark matter halos pulling towards one another can be modeled via soliton-type collisions\(^1\) are continued here.

Since solitons are rather stable entities, they behave effectively like colliding particles, i.e. after leaving the interaction region, where they may deform due to a temporally inelastic relativistic mechanism, they ultimately return to their original shapes and velocities.

Recently, our earlier proposal [23, 25] has been adopted [3] that DM may be composed of a gas of ‘axion mini clusters’ or mini axion stars.

\(^1\) This paper is dedicated to the memory of Victor M. Villanueva, with whom EWM had a fruitful collaboration [4] on inflation.
2. Soliton collisions

Let us depart from the nonlinear Klein-Gordon (NLKG) equation

$$\Box \phi = \frac{\partial U(\phi)}{\partial \phi},$$

where $$\Box := \nabla \cdot \nabla - \partial^2_{x}/c^2 \partial_{t}^2$$ is the Lorentz-invariant wave operator of d’Alembert.

In quantum chromodynamics (QCD), axions with inertial mass $$m$$ are self-interacting via the effective [14] periodic potential

$$U_a(\phi) = \frac{m^4}{\lambda} \left[ 1 - \cos \left( \frac{\sqrt{\lambda} m \phi}{m} \right) \right] \approx U_{LE}(\phi) - \frac{\lambda}{4!} \phi^4 - \cdots$$

which deviates from the Lane-Emden (LE) potential $$U_{LE}(\phi) = m^2 \phi^2 \left( 1 - \chi \phi^4 \right) / 2$$ at higher order of $$|\phi|$$.

In two dimensions (2D), the sine-Gordon equation results as a well known example. The first term of the Taylor series of the potential (2) corresponds to the mass in the Klein-Gordon equation; the next one would give rise to the famous $$\phi^4$$-theory. Combining the first and third term can be reckoned as the modified $$\phi^6$$-theory of LE, and so on [13]. By plotting some of the potentials we obtained an approximated view of the field-theoretical ‘vacuum’ at the origin, cf. Fig. 1.

For constructing multi-solitons, the well-established Bäcklund transformation (BT) [28, 26], is employed.

In dimensionless light-cone coordinates $$\xi := \frac{1}{2}(\tilde{x} + \tilde{c} \tilde{t})$$ and $$\eta := \frac{1}{2}(\tilde{x} - \tilde{c} \tilde{t})$$, the sine-Gordon (sG) equation acquires the form

$$\theta_{\xi \eta} = \sin \theta$$

and is CPT invariant. In a moving frame, with $$\gamma := 1/\sqrt{1 - v^2/c^2}$$ as Lorentz factor, it has the exact kink solution

$$\theta = 4C \arctan \left[ \exp \gamma (\tilde{x} - v \tilde{t}) \right].$$
for $C = 1$ and anti-kink for $C = -1$. Since its spatial derivative
\[ \theta_x = 2\gamma C \text{sech} \left[ \gamma \left( \tilde{x} - \nu \tilde{t} \right) \right] \] (5)
becomes localized and square-integrable, its absolute value will facilitate a subsequent comparison with the scattering behavior of solitons or lumps regarded as Bose-Einstein condensates [19] of DM.

Bianchi’s permutability theorem of BTs provides a ‘nonlinear superposition’ principle
\[ \tan \left( \frac{\theta_3 - \theta_0}{4} \right) = B \tan \left( \frac{\theta_1 - \theta_2}{4} \right), \] (6)
where $B$ is a common or ‘average’ relativistic velocity of the superposed solitons.

The CPT invariance of our relativistic KG equation allows us to distinguish solitons from anti-solitons: In the case of the collision of two kinks (instead of a collision of a kink and its CP odd anti-kink, as in Ref. [5]) the trivial seed solution $\theta_0 = 0$ leads to the exact solution
\[ \theta_{kk} = 4 \arctan \left[ K(\zeta_1, \zeta_2) \right], \] (7)
where the kinetic factor
\[ K(\zeta_1, \zeta_2) : = B \frac{\exp(\zeta_1) + C \exp(\zeta_2)}{\exp(\zeta_1 + \zeta_2) - C} \] (8)
\[ \simeq \exp(\zeta_1 + \gamma_1 \delta_1) + C \exp(\zeta_2 + \gamma_2 \delta_2) \]
depends on the initial velocities and the inverse Lorentz transformations $\zeta_i := \gamma_i(\tilde{x} + v_i \tilde{t})$. 

Figure 2. (Color online) Kink-kink collision monitored via the absolute value of its spatial derivative.
At large separations from the interaction region, cf. Fig. 2, the solution (7) clearly decouples asymptotically into a (non-interacting) kink–kink or kink–antikink pair [8] distinguished by the sign $C = \pm 1$ of the topological charge. The global stability of solitons has been analyzed by Kusmartsev [16].

A generalization to (2+1) D has been attempted in Ref. [7], following [12]. Although thereby an exact integrability has been lost, certain mappings from solitons to 3D lumps of the Lane-Emden type are instrumental.

2.1. Boson star stability without solution

The stability analysis of neutron and boson stars has in common that first solutions have to be determined. Afterwards, the influence of small perturbations of these solutions can be investigated; if the perturbation stays small the star is called stable.

The Lagrangian density of gravitationally coupled complex scalar field $\Phi$ reads

$$L_{BS} = \frac{\sqrt{g}}{2\kappa} \left\{ R + \kappa \left[ g^{\mu\nu} (\partial_{\mu} \Phi^{*}) (\partial_{\nu} \Phi) - 2 U (|\Phi|^{2}) \right] \right\}. \quad (9)$$

Here $\kappa = 8\pi G$ is the gravitational constant in natural units, $g$ the determinant of the metric $g_{\mu\nu}$, $R$ the curvature scalar with Tolman’s sign convention [36].

As is well-known, the dynamics of a complex scalar $\Phi := \phi_{1} + i\phi_{2}$ of the boson star (BS) model [21, 31] as an alternative to black holes (BHs) is invariant under the global $O(2) \simeq U(1)$ symmetry $\Phi \to \Phi \exp(i\theta)$, where the phase $\theta$ is a constant.

Mathematically, this is equivalent to two real scalar fields $\phi_{1}$ and $\phi_{2}$, self-interacting merely via $|\Phi|^{2} = \phi_{1}^{2} + \phi_{2}^{2}$. Consequently, the BS Lagrangian can be ‘separated’ into two parts $\tilde{L}_{1} := \tilde{R}/2 + (\partial\phi_{1})^{2} - U(\phi_{1})$ and $\tilde{L}_{2} := \tilde{R}/2 + (\partial\phi_{2})^{2} - U(\phi_{2})$, which are of the same form.

Inasmuch such gravitational objects are rather inaccessible in our Universe, recently the dynamics of rotating boson stars have been experimentally probed via optical analogues [27].

3. Stability

Alternatively, a bifurcation diagram, commonly mass versus particle number, is constructed from the solutions: Cusps in such diagrams determine the onset of instability.

Commonly, the kind of matter in the star determines through its equation of state the point of instability. For a soliton or boson star (BS), the scalar part of the Lagrangian can be transformed into a purely higher order gravity theory [33], a gravitational Lagrangian which is nonlinear in the scalar curvature. As a consequence, the very vivid image of scalar matter as a complex field in the center gets lost. But one gains a method to determine the onset of instability.

In the spherically symmetric case, we have shown via catastrophe theory [16, 20, 32] that these boson stars have a stable branch with a wide range of masses and radii. Moreover, rotating boson stars have been calculated which show a rather surprising behavior in view of their conserved angular momentum and their toroidal shape [34, 18].

Moreover, a correlation to real scalar fields as applied in inflationary theories can be exhibited as well [24]. To produce an inflationary phase in the very early universe, instead of using real scalar fields, one can also use higher-order gravity theories [35]. The underlying reason is that both descriptions are conformally equivalent to each other as is recapitulated here.

3.1. Legendre map to a higher-order curvature Lagrangian

In a n-dimensional spacetime, let us assume that the general nonlinear Lagrangian density

$$L = L(R) \sqrt{|g|}, \quad (10)$$
solely depends on the scalar curvature \( R \). Then the corresponding field momentum ("Legendre map")

\[
P = \frac{\delta L}{\delta R} = L'(R)
\]

induces the Legendre transformation

\[
L \rightarrow H = \left( R \frac{\delta L}{\delta R} - L \right) \sqrt{|g|} = (RP - L) \sqrt{|g|}
\]

of the corresponding densities.

After a conformal mapping to an Einstein frame, satisfying \( \sqrt{|g|} \rightarrow \sqrt{\tilde{|g|}} = \frac{P^{n/(2-n)}}{2-n} \sqrt{|g|} \), there arises in \( n > 2 \) dimensions an effective potential \( u \) via the Wagoner type [37] transformation

\[
u(P) = \frac{P^{n/(2-n)}}{2-n} (RP - L).
\]

Observe that a fixed point of this Wagoner transformation requires a vanishing Hamiltonian, i.e. \( H(R) := RP - L = 0 \) of the original higher–order Lagrangian. This condition would lead us back to the Hilbert–Einstein Lagrangian \( L_{HE} = R/\kappa \) with vanishing self-interacting.

In view of the definition \( (11) \), the “master equation" \( (13) \) is highly nonlinear and governs the reconstruction of the higher order Lagrangian \( L(R) \) from a given potential \( u \) or vice versa. Instead of solving it directly, via Lagrange multipliers, or the method of of Helmholtz, one can invert \( (13) \) in order to recover the scalar curvature

\[
R = \frac{\partial}{\partial P} \left[ \frac{P^n}{n-2} u(P) \right],
\]

with the reparametrized Hamiltonian \( H(P) := \frac{P^n}{n-2} u(P) \) playing the role of a generating functional of the Legendre transformation \( (12) \) from the original Lagrangian \( (9) \) to the general nonlinear curvature scalar Lagrangian \( L = L(R) \). Then the corresponding higher–order Lagrangian

\[
L = \frac{1}{2} \frac{\partial}{\partial P} \left[ \frac{P^{2(n-2)}}{(n-2)} u(P) \right],
\]

will facilitate a visualization of the bifurcations of the systems.

If we relate the dimensionless conformal factor \( \Omega = 2\kappa P \) to a real scalar field

\[
\phi = \sqrt{\frac{\omega}{\kappa}} \ln(2\kappa P),
\]

where

\[
\omega := \frac{n - 1}{n - 2}
\]

is a Brans-Dicke (BD) type parameter, the following parametric reconstruction\(^2\) of the higher–order Lagrangian \( L(R) \) from the self–interacting scalar potential \( U(\phi) = (2\kappa)^{2/(2-n)} u(P) \) arises:

\[
R = \exp \left( 2 \sqrt{\frac{\kappa}{\omega}} \phi \right) \left[ \frac{n}{n-2} U(\phi) + \sqrt{\frac{\kappa}{\omega}} \frac{dU}{d\phi} \right],
\]

and

\[
L = \frac{1}{2\kappa} \exp \left( \frac{n}{n-2} \phi \right) \left[ \frac{2}{n-2} U(\phi) + \sqrt{\frac{\kappa}{\omega}} \frac{dU}{d\phi} \right].
\]

\(^2\) Note that Eq. (6.7) of Ref. [4] is misprinted and that our potential differs by a factor \( 2\kappa \).
Here the scalar field $\phi$ merely plays the role of a control parameter in the Whitney surface, cf. Arnol’d [2].

The form of this surface with its local valleys and mountains is qualitatively shown in Fig. 3 in the case of a BD type toy model $L(\phi, R) = \phi R + \phi^3$. There the curve of local maxima and minima is parameterized via $(R = -3\phi^2, L = -2\phi^3)$ which projects for $R < 0$ to a semi-cubic real cusp $L = \pm 2i/3\sqrt{3}R^{3/2}$ in the right $(R, L)$ plane.

4. Natural inflation from an axion type potential?

In QCD, after integrating out the fermion fields, its generating functional including a topological Pontrjagin term for the Yang-Mills gauge fields induces an effective axion potential

$$U = \Lambda_{\text{QCD}}^4 [1 - \cos(\phi/f_\phi)].$$

(20)

This potential displays a ‘shift symmetry’ with a period of $2\pi f_\phi$, has a minimum at $\phi = 0$, as required, and leads to the induced axion mass of $m_\phi = \Lambda_{\text{QCD}}^2 / f_\phi$. An effective quintaxion may
Figure 4. The axion gives rise again to a swallow tail catastrophe in the Lagrangian $L(R)$, with the choice of $\Lambda_{\text{QCD}} = 1$, $\kappa = 1$, and $f_\phi = 1$.

also be induced by spacetime torsion [22] with a symmetry breaking decay constant $f_\phi \propto 1/\sqrt{2\kappa}$ around the Planck scale.

Within natural inflation [11], such a potential has been proposed for an axion coupling constant close to the Planck scale. For simplicity, let us assume in the following that $f_\phi \simeq \sqrt{3/(2\kappa)}$. Then, following the general prescription (18)-(19), we find the reparametrized solution

$$R = \exp(x) \Lambda_{\text{QCD}}^4 \left[ -2 \cos(x) + \sin(x) \right],$$  \hspace{1cm} (21)

$$L = \frac{1}{2\kappa} \exp(2x) \Lambda_{\text{QCD}}^4 \left[ -2 \cos(x) + \sin(x) \right].$$  \hspace{1cm} (22)

The resulting nonlinear Lagrangian $L(R)$ exhibits a swallow tail catastrophe, see its projection in Fig. 4.

The energetically lowest branch induces an effective cosmological constant $\Lambda_{\text{eff}} = \Lambda_{\text{QCD}}^4 \exp(2x_0) \sin(x_0)/2$, giving rise to a kind of ‘triple unification’ of axion-type pseudo-scalars, DM and DE.

5. Discussion

Lumps or solitons in 3D need to be stabilized via their self-generated gravity [17], similarly as in the case of colliding boson stars.

Such hypothetical objects, rather compact but with an exponential decreasing radial tail, have been advocated as alternatives [1] for the standard scenario of colliding black holes (BHs) producing the ringdown gravitational wave event GW150914. However, in the related time window no gamma rays [29] have been observed by the INTEGRAL satellite, despite the rather large mass of about 30 $M_\odot$ of the colliding objects.

Quite recently, the supermassive central engine of the center of Milky Way has been reanalyzed [10], unfortunately ignoring, however, the known counter examples of Wheeler’s "No Hair conjecture" for rotating BHs in a scalar field environment, cf. Ref. [18].

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