Generalized $d$-Koszul Modules

NING BIAN, YU YE, PU ZHANG∗

Abstract. Generalized $d$-Koszul modules are introduced to solve an open problem: the odd Ext-module $E^\text{odd}(M)$ of a $d$-Koszul module $M$ over a $d$-Koszul algebra $\Lambda$ is a Koszul module over the even Yoneda algebra $E^\text{ev}(\Lambda)$.

Introduction

For an integer $d \geq 2$, a $d$-Koszul algebra was introduced and studied by R. Berger [B1], and developed by E. L. Green et al. [GMMZ] to the nonlocal case. If $d = 2$ it is the usual Koszul algebra. This class of generalized Koszul structures turns out to be important for example in theory of the Artin-Shelter algebras, the Calabi-Yau algebras, and the Yang-Mills algebras (see e.g. [B1], [B2], [CD]).

Let $\Lambda$ be a $d$-Koszul algebra and $M$ a $d$-Koszul $\Lambda$-module. It was shown in Theorem 6.1 of [GMMZ] that the even Ext-algebra $E^\text{ev}(\Lambda)$ is a Koszul algebra and the even Ext-module $E^\text{ev}(M)$ is a Koszul $E^\text{ev}(\Lambda)$-module. This generalizes the corresponding result of J. Backelin and R. Fröberg [BF] on the local Koszul algebras. An open problem was raised by E. L. Green et al. [GMMZ], Section 6: Is the odd Ext-module $E^\text{odd}(M)$ also a Koszul module over $E^\text{ev}(\Lambda)$? E. N. Marcos and R. Martínez-Villa [MM] proved that this is the case if the orthogonal algebra $\Lambda'$ is also a $d$-Koszul algebra. However, in general $\Lambda'$ is not a $d$-Koszul algebra (see [B1]; also Example 2 in [MM]). So the problem remains to be open.

In this paper we introduce the so-called generalized $d$-Koszul modules. This is a natural class of graded modules. For example, the syzygies of a $d$-Koszul module are generalized $d$-Koszul modules up to shifts. Also for each $i$, $J^i M$ is a generalized $d$-Koszul module up to shift, where $M$ is a generalized $d$-Koszul module over a $d$-Koszul algebra, and $J$ is the graded Jacobson radical.

Our main result is as follows.

Main Theorem. Let $\Lambda$ be a $d$-Koszul algebra, and $M$ a generalized $d$-Koszul $\Lambda$-module. Then $E^\text{ev}(M)$ is a Koszul module over the Koszul algebra $E^\text{ev}(\Lambda)$.

As a consequence, we have

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∗The corresponding author.

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mathbn@hotmail.com, yeyu@ustc.edu.cn, pzhang@sjtu.edu.cn.
Corollary. Let $\Lambda$ be a d-Koszul algebra, and $M$ a d-Koszul $\Lambda$-module. Then $E^{\text{odd}}(M)$ is a Koszul module over the Koszul algebra $E^{\text{ev}}(\Lambda)$.

This answers in the affirmative the open problem mentioned above.

1. Preliminaries

We fix the notations and recall some facts frequently used later. For the details we refer to [BGS], [GM], and [GMMZ].

1.1. Throughout $\Lambda$ is a standardly graded algebra over a field $k$ (see [GM], p.250), i.e., $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ is a positively graded $k$-algebra satisfying the following three conditions:

(i) $\Lambda_0 = k^r$ for some integer $r \geq 1$,
(ii) $\dim_k \Lambda_i < \infty$, $\forall i \geq 0$,
(iii) $\Lambda_i \Lambda_j = \Lambda_{i+j}$, $\forall i,j \geq 0$.

A left graded $\Lambda$-module $M$ is a $\Lambda$-module together with a decomposition of $k$-spaces $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $\Lambda_i M_j \subseteq M_{i+j}$, $\forall i,j \in \mathbb{Z}$. Let $M$ and $N$ be graded $\Lambda$-modules.

A $\Lambda$-homomorphism $f : M \to N$ is a graded homomorphism if $f(M_i) \subseteq N_i$, $\forall i \in \mathbb{Z}$. For $M \in \text{Gr}(\Lambda)$, let $M[n]$ denote the graded module with $M[n]_i = M_{i-n}$. Let $\text{A-Mod}$ be the category of the left $\Lambda$-modules, $\text{Gr}(\Lambda)$ the category of the left graded $\Lambda$-modules and graded homomorphisms, and $\text{gr}(\Lambda)$ the full subcategory of $\text{Gr}(\Lambda)$ consisting of finitely generated $\Lambda$-modules. Then $\text{A-Mod}$ and $\text{Gr}(\Lambda)$ are abelian categories; and $\text{gr}(\Lambda)$ is abelian if $\Lambda$ is noetherian. Let $\text{Hom}_{\text{Gr}(\Lambda)}$ and $\text{Ext}^i_{\text{Gr}(\Lambda)}$ denote the homomorphisms and extensions in $\text{Gr}(\Lambda)$, as opposed to the usual $\text{Hom}_\Lambda$ and $\text{Ext}^i_\Lambda$ in $\text{A-Mod}$.

Let $I$ be a subset of $\mathbb{Z}$, and $M \in \text{Gr}(\Lambda)$. $M$ is generated in degrees in $I$, if $M = \Lambda(\bigoplus_{j \in I} M_j)$; $M$ is generated in degree $i$ if $M = \Lambda M_j$; $M$ is supported above degree $n$ if $M_j = 0$ for $j < n$; and $M$ is concentrated in degrees in $I$ if $M_i = 0$ for $i \notin I$.

Let $J$ be the ideal $\bigoplus_{i \geq 1} \Lambda_i$ of $\Lambda$. The trivial $\Lambda$-module $\Lambda_0$ is the lift of the $\Lambda_0$-module $\Lambda_0$ via the $k$-algebra homomorphism $\Lambda \to \Lambda/J = \Lambda_0$. It is a graded $\Lambda$-module concentrated in degree 0. We need the following well-known fact.

Lemma 1.1. Let $M \in \text{Gr}(\Lambda)$, and $I$ be a subset of $\mathbb{Z}$. If $\text{Hom}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin I$, then $M$ is generated in degrees in $I$.

Proof. For the convenience of the reader we include a justification. Put $L := M/\Lambda(\bigoplus_{j \in I} M_j)$.

If $L \neq 0$, then $L/JL \neq 0$. While $L/JL$ is a graded module over the semisimple algebra $\Lambda_0$, it follows that $\text{Hom}_{\text{Gr}(\Lambda)}(L/JL, \Lambda_0[j]) \neq 0$ for some $j \notin I$, and hence $\text{Hom}_{\text{Gr}(\Lambda)}(M, \Lambda_0[j]) \neq 0$ for some $j \notin I$, contrary to the assumption. ■

1.2. Denote by $E(\Lambda)$ the Ext-algebra $\bigoplus_{i \geq 0} \text{Ext}_\Lambda^i(\Lambda_0, \Lambda_0)$, with the multiplication given by the Yoneda product. We also consider the even Ext-algebra $E^{\text{ev}}(\Lambda) := \bigoplus_{i \geq 0} \text{Ext}_\Lambda^i(\Lambda_0, \Lambda_0)$,
which is a positively graded algebra with grading $E^v(\Lambda)_n := \text{Ext}^2_\Lambda(\Lambda_0, \Lambda_0)$. For a $\Lambda$-module $M$, let $E(M)$ be the graded $E(\Lambda)$-module $\bigoplus_{i \geq 0} \text{Ext}^i_\Lambda(M, \Lambda_0)$. We also consider the even Ext-module $E^v(M) := \bigoplus_{n \geq 0} \text{Ext}^{2n}_\Lambda(M, \Lambda_0)$ over $E(\Lambda)$, and the odd Ext-module $E^{\text{odd}}(M) := \bigoplus_{n \geq 0} \text{Ext}^{2n+1}_\Lambda(M, \Lambda_0)$ over $E^v(\Lambda)$: they are graded modules with gradings

$$E^v(M)_n := \text{Ext}^{2n}_\Lambda(M, \Lambda_0), \quad \text{and} \quad E^{\text{odd}}(M)_n := \text{Ext}^{2n+1}_\Lambda(M, \Lambda_0), \quad \forall \ n \geq 0.$$

Every graded $\Lambda$-module $M$ has a graded projective resolution

$$(1) \quad Q^* : \cdots \to Q^i \to \cdots \to Q^1 \to Q^0 \to M \to 0.$$  

If each $Q^i$ is finitely generated, then we say that $Q^*$ is a finitely generated graded projective resolution of $M$. If $M \in \text{gr}(\Lambda)$, then $M$ admits a minimal graded projective resolution (1) in the sense that $\text{Im}(Q^i \to Q^{i-1}) \subseteq Q^{i-1}, \ \forall \ i \geq 1$ (see Propositions 2.3 and 2.4 in [GM]).

If $M \in \text{gr}(\Lambda)$, then for each $N \in \text{Gr}(\Lambda)$, $\text{Hom}_\Lambda(M, N)$ is a graded $k$-space with the shift-grading: $\text{Hom}_\Lambda(M, N) = \text{Hom}_{\text{Gr}(\Lambda)}(M, N[i])$, i.e., $\text{Hom}_\Lambda(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{Gr}(\Lambda)}(M, N[i])$.

If $M$ has a finitely generated graded projective resolution, then for each $N \in \text{Gr}(\Lambda)$ and each $n \geq 1$, $\text{Ext}^n_\Lambda(M, N)$ is a graded $k$-space with the shift grading: $\text{Ext}^n_\Lambda(M, N)_i = \text{Ext}^n_{\text{Gr}(\Lambda)}(M, N[i])$, i.e., $\text{Ext}^n_\Lambda(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Ext}^n_{\text{Gr}(\Lambda)}(M, N[i])$.

Fix a minimal graded projective resolution of the trivial $\Lambda$-module $\Lambda_0$:

$$(2) \quad P^* : \cdots \to P^n \to \cdots \to P^1 \to P^0 \to \Lambda_0 \to 0.$$  

We need the following fact.

**Lemma 1.2. ([GMMZ], Lemma 3.2)** Let $M$ be a graded module supported above degree 0 with a minimal graded projective resolution (1). For any integer $n \geq 1$, if $P^n$ in (2) is supported above degree $s$, then so is $Q^n$.

1.3. For the theory of the Koszul algebras and the Koszul modules we refer to A. Beilinson, V. Ginzburg and W. Soergel [BGS], and E. L. Green and R. Martínez-Villa [GM].

**Definition 1.3 ([GMMZ], [MM]).** Let $d \geq 2$ be an integer. A graded $\Lambda$-module $M$ is a $d$-Koszul module if $M$ admits a finitely generated graded projective resolution (1) such that each $Q^i$ is generated in degree $\delta(i)$, where

$$\delta(i) := \begin{cases} \ nd, & \text{if } i \ equal \ 2n, \\ nd + 1, & \text{if } i \ equal \ 2n + 1. \end{cases}$$

If the trivial $\Lambda$-module $\Lambda_0$ is a $d$-Koszul module, then we call $\Lambda$ a $d$-Koszul algebra.

**Theorem 1.4. ([GMMZ], Theorem 6.1)** Let $\Lambda$ be a $d$-Koszul algebra and $M$ a $d$-Koszul $\Lambda$-module. Then $E^v(\Lambda)$ is a Koszul algebra, and $E^v(M)$ is a Koszul $E^v(\Lambda)$-module.
2. Generalized $d$-Koszul modules

2.1. Let $d \geq 2$ be an integer. For each integer $i \geq 0$ we assign a subset $\Delta(i)$ of $\mathbb{N}_0$ as

$$\Delta(i) := \begin{cases} \{nd\}, & \text{if } i = 2n; \\ \{nd + 1, \ldots, nd + d - 1\}, & \text{if } i = 2n + 1. \end{cases}$$

**Definition 2.1.** A graded $\Lambda$-module $M$ is called a generalized $d$-Koszul module if $M$ admits a finitely generated graded projective resolution $Q^\bullet$ such that each $Q^i$ is generated in degrees in $\Delta(i)$, i.e., $Q^i = \Lambda(\bigoplus_{j \in \Delta(i)} Q^i_j)$, $i \geq 0$.

**Remark 2.2.**

(i) As remarked by Beilinson-Ginzburg-Soergel [BGS] (p.476) in the Koszul situation, $Q^\bullet$ in Definition 2.1 is unique up to isomorphism. More precisely, if $L^\bullet$ is another graded projective resolution of $M$ such that each $L^i$ is also generated in degrees in $\Delta(i)$ (it is not assumed to be finitely generated), then $L^\bullet \cong Q^\bullet$ as complexes. In fact, $L^\bullet$ is homotopy equivalent to $Q^\bullet$; while any chain maps $Q^\bullet \to Q^\bullet$ and $L^\bullet \to L^\bullet$, which respect the grading on $Q^i$ and $L^i$ and are homotopic to zero must themselves be zero (since any element in $\Delta(i)$ is strictly smaller than any element in $\Delta(i + 1)$, and $Q^i$ and $L^i$ are both generated in degrees in $\Delta(i)$). It follows that $L^\bullet \cong Q^\bullet$ as complexes.

(ii) We emphasize that, as in the $d$-Koszul situation, here $Q^i$ is also required to be finitely generated: it is for the application of the shift grading on $\text{Ext}^n_\Lambda(M,-)$.

(iii) If $M$ is a generalized $d$-Koszul module, then such a graded projective resolution $Q^\bullet$ in the definition is minimal, and each syzygy $\Omega^i(M)$ is a graded $\Lambda$-module finitely generated in degrees in $\Delta(i)$. In particular, $M$ is finitely generated in degree 0.

(iv) A $d$-Koszul module is always generalized $d$-Koszul; and a generalized $2$-Koszul module is a finitely generated Koszul module (if $\Lambda$ is noetherian, then a generalized $2$-Koszul $\Lambda$-module is exactly a finitely generated Koszul $\Lambda$-module).

**Example 2.3.** Let $A$ be the algebra given by the quiver

$$\begin{array}{cccc}
\alpha & 1 & \beta & 2 \\
\downarrow & \swarrow & \searrow & \downarrow \\
& 3 & & \\
\end{array}$$

with relations $\alpha^3$, $\gamma \beta \alpha$. Then the simple (left) module $S(1)$ has a minimal graded projective resolution

$$\cdots \to P(1)[4] \oplus P(2)[5] \to P(1)[3] \oplus P(3)[3] \to P(1)[1] \oplus P(2)[1] \to P(1) \to S(1) \to 0$$

where $\Omega^4 S(1) = (\Omega^3 S(1))[3]$. Thus $S(1)$ is a generalized $3$-Koszul $A$-module. Since $Q^3 = P(1)[4] \oplus P(2)[5]$ is generated in degrees 4 and 5, but not generated in degree 4, it follows that $S(3)$ is not a $3$-Koszul $A$-module (by an argument in Remark 2.2(i)).
2.2. We have the following characterization for a $d$-Koszul module and for a generalized $d$-Koszul module, which is the corresponding version of Proposition 2.14.2 in Belinson - Ginzburg - Soergel [BGS] for the Koszul modules.

**Lemma 2.4.** Let $M$ be a graded $\Lambda$-module with a finitely generated graded projective resolution. Then

(i) $M$ is $d$-Koszul if and only if $\Ext^i_{\Lambda}(M, \Lambda_0) = 0$, $\forall j \neq \delta(i)$.

(ii) $M$ is generalized $d$-Koszul if and only if $\Ext^i_{\Lambda}(M, \Lambda_0)$ is concentrated in degrees in $\Delta(i)$, with the shift grading, i.e., $\Ext^i_{\Lambda}(M, \Lambda_0) = 0$, $\forall j \notin \Delta(i)$.

**Proof.** They can be similarly proved as Proposition 2.14.2 in [BGS]. For the convenience of the reader we include a justification of (ii).

Assume that $M$ is generalized $d$-Koszul. Then $M$ has a graded projective resolution $Q^*$ such that each $Q^i$ is generated in degrees in $\Delta(i)$, and $\Ext^i_{\Gr(\Lambda)}(M, \Lambda_0[j])$ is the $i$-th cohomology group of the complex $\Hom_{\Gr(\Lambda)}(Q^*, \Lambda_0[j])$. Since $Q^i$ is generated in degrees in $\Delta(i)$, and $\Lambda_0[j]$ is concentrated in degree $j$, it follows that $\Hom_{\Gr(\Lambda)}(Q^i, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$, and hence $\Ext^i_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$.

Conversely, assume that $\Ext^i_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(i)$. We construct inductively a graded projective resolution $L^*$ of $M$ such that each $L^i$ is generated in degrees in $\Delta(i)$. Since $\Ext^i_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \neq 0$, by Lemma 1.1, $M$ is generated in degree 0, and hence we have a surjective graded $\Lambda$-homomorphism $L^0 \to M$ such that $L^0$ is generated in degree 0. Denote by $K^1$ its kernel. Then $\Hom_{\Gr(\Lambda)}(K^1, \Lambda_0[j]) = \Ext^1_{\Gr(\Lambda)}(M, \Lambda_0[j]) = 0$ for $j \notin \Delta(1)$, and hence by Lemma 1.1, $K^1$ is generated in degrees in $\Delta(1)$. Thus we have a surjective graded $\Lambda$-homomorphism $L^1 \to K^1$ such that $L^1$ is generated in degrees in $\Delta(1)$. Repeating this process we are done.

By assumption we have already a finitely generated graded projective resolution $Q^*$. By the argument in Remark 2.2(i), there are chain maps $\bar{f} : L^* \to Q^*$ and $g : Q^* \to L^*$ such that $g\bar{f} = \Id_{L^*}$, which means that $L^*$ is a direct summand of $Q^*$. Thus $L^*$ is also a finitely generated resolution. By definition $M$ is generalized $d$-Koszul.

2.3. For a $d$-Koszul module $M$, in general $\Omega^i M$ and $J^i M$ are not $d$-Koszul modules, up to shifts (see Proposition 5.2 in [GMMZ] for some special cases); however, they turn out to be generalized $d$-Koszul, after proper shifts. In the rest of this section we precisely state and prove these results, which will be important in the proof of Main Theorem and Corollary.

**Lemma 2.5.** Let $M$ be a $d$-Koszul $\Lambda$-module. Then

(i) $E^{\text{odd}}(M) \cong E^{\text{ev}}(\Omega M)$ as graded $E(\Lambda)$-modules.

(ii) $(\Omega^i M)[−\delta(i)]$ is a generalized $d$-Koszul module for each $i \geq 0$.

**Proof.** (i) By definition we have an isomorphism of graded $E(\Lambda)$-modules

$$E^{\text{odd}}(M) = \bigoplus_{n \geq 0} \Ext^i_{\Lambda}(M, \Lambda_0) \cong \bigoplus_{n \geq 0} \Ext^i_{\Lambda}(\Omega M, \Lambda_0) = E^{\text{ev}}(\Omega M).$$
(ii) Taking a graded projective resolution $Q^i$ of $M$ such that each $Q^i$ is finitely generated in degrees in $\delta(i)$, we see that $(\Omega^i M)[-\delta(i)]$ has a graded projective resolution

$$L^\bullet : \cdots \to L^i \to \cdots \to L^1 \to L^0 \to (\Omega^i M)[-\delta(i)] \to 0$$

where $L^j = Q^{i+j}[-\delta(i)]$ is finitely generated in degree $\delta(i+j)-\delta(i)$ for $j \geq 0$.

If $i$ is even, then $L^j$ is generated in degree $\delta(j)$. That is, $(\Omega^i M)[-\delta(i)]$ is a $d$-Koszul module, and hence a generalized $d$-Koszul module.

Assume that $i$ is odd. Then $L^j$ is generated in degree $nd$ if $j = 2n$, and $L^j$ is generated in degree $nd + d - 1$ if $j = 2n + 1$. By definition $(\Omega^i M)[-\delta(i)]$ is generalized $d$-Koszul. □

**Theorem 2.6.** Let $\Lambda$ be a $d$-Koszul algebra and $M$ a generalized $d$-Koszul $\Lambda$-module. Then

(i) $(J^i M)[-i]$ is generalized $d$-Koszul for each $i \geq 1$.

(ii) For each $n \geq 1$ we have $k$-isomorphisms

$$\text{Ext}^{2n-1}_{Gr(\Lambda)}(J^i M, \Lambda_0[nd]) \cong \text{Ext}_{Gr(\Lambda)}^{2n-1}(J^2 M, \Lambda_0[nd]) \cong \cdots \cong \text{Ext}_{Gr(\Lambda)}^{2n-1}(J^{2n-1} M, \Lambda_0[nd]).$$

**Proof.** (i) It suffices to prove that $(J^i M)[-1]$ is generalized $d$-Koszul. Since $M = \bigoplus_{i \geq 0} M_i$ is finitely generated in degree 0, $J M = \bigoplus_{i \geq 1} M_i$ is finitely generated in degree 1.

We first prove the following claim: $J M[-1]$ admits a graded projective resolution $Q^i$ such that $Q^i$ is generated in degrees in $\Delta(i)$. By the proof of Lemma 2.4(ii), it suffices for each $n \geq 0$ to prove that $\text{Ext}_{Gr(\Lambda)}^{2n}(J^i M, \Lambda_0[j]) = \text{Ext}_{Gr(\Lambda)}^{2n}(J^2 M, \Lambda_0[j+1]) = 0$, $\forall j \neq nd$, and that $\text{Ext}_{Gr(\Lambda)}^{2n+1}(J^i M, \Lambda_0[j+1]) = 0$, $\forall j \notin \Delta(2n+1) = \{nd+1, \ldots, nd+d-1\}$.

Applying $\text{Hom}_{Gr(\Lambda)}(-, \Lambda_0[j+1])$ to the graded exact sequence $0 \to J M \to M \to M/J M \to 0$ we get the following exact sequence of $k$-spaces

$$\text{Ext}_{Gr(\Lambda)}^{2n}(J^i M, \Lambda_0[j+1]) \to \text{Ext}_{Gr(\Lambda)}^{2n}(J^2 M, \Lambda_0[j+1]) \to \text{Ext}_{Gr(\Lambda)}^{2n+1}(J^i M, \Lambda_0[j+1]) \\
\to \text{Ext}_{Gr(\Lambda)}^{2n+1}(J^2 M, \Lambda_0[j+1]) \to \text{Ext}_{Gr(\Lambda)}^{2n+1}(J^i M, \Lambda_0[j+1]) \to \text{Ext}_{Gr(\Lambda)}^{2n+1}(J^{i+1} M, \Lambda_0[j+1]).$$

Since $\Lambda$ is a $d$-Koszul algebra, $P^{2n}$ in (2) is supported above degrees $nd$, and hence by Lemma 1.2 $Q^{2n}$ is supported above degrees $nd$, where $Q^i$ is a minimal graded projective resolution of $J M[-1]$. Thus $\text{Ext}_{Gr(\Lambda)}^{2n}(J^i M, \Lambda_0[j+1]) = 0$ for $j < nd$. Similarly, $\text{Ext}_{Gr(\Lambda)}^{2n+1}(J^i M, \Lambda_0[j+1]) = 0$ for $j < nd + 1$.

Since $M$ is generalized $d$-Koszul, by Lemma 2.4(ii), $\text{Ext}_{Gr(\Lambda)}^{2n}(M, \Lambda_0[j+1]) = 0$ if $j \neq nd - 1$, and $\text{Ext}_{Gr(\Lambda)}^{2n+1}(M, \Lambda_0[j+1]) = 0$ if $j \notin \{nd, \ldots, nd+d-2\}$.

Note that $M/J M$ is a $\Lambda/J$-module and $\Lambda/J = \Lambda_0$ is a semisimple algebra. Thus $M/J M$ is a direct summand of a finite direct sum of copies of the trivial $\Lambda$-module $\Lambda_0$. In particular, $M/J M$ is a $d$-Koszul module. By Lemma 2.4(i), $\text{Ext}_{Gr(\Lambda)}^{2n+1}(M/J M, \Lambda_0[j+1]) = 0$ if $j \neq nd$, and $\text{Ext}_{Gr(\Lambda)}^{2n+1}(M/J M, \Lambda_0[j+1]) = 0$ if $j \neq (n+1)d - 1$.

Now if $j \neq nd$, then by the exact sequence above we have the following exact sequence

$$\text{Ext}_{Gr(\Lambda)}^{2n}(M, \Lambda_0[j+1]) \to \text{Ext}_{Gr(\Lambda)}^{2n}(J^i M, \Lambda_0[j+1]) \to \text{Ext}_{Gr(\Lambda)}^{2n+1}(J^i M, \Lambda_0[j+1]) = 0.$$
where if \( j \neq nd - 1 \) then \( \text{Ext}^{2n}_{\text{Gr}(A)}(M, \Lambda_0[j + 1]) = 0 \), and hence \( \text{Ext}^{2n}_{\text{Gr}(A)}(JM, \Lambda_0[j + 1]) = 0 \); and if \( j = nd - 1 < nd \), then we already know \( \text{Ext}^{2n}_{\text{Gr}(A)}(JM, \Lambda_0[j + 1]) = 0 \).

Let \( j \notin \Delta(2n + 1) = \{ nd + 1, \cdots , nd + d - 1 \} \). Then by the exact sequence above we have the following exact sequence

\[
\text{Ext}^{2n+1}_{\text{Gr}(A)}(M, \Lambda_0[j + 1]) \rightarrow \text{Ext}^{2n+1}_{\text{Gr}(A)}(JM, \Lambda_0[j + 1]) \rightarrow \text{Ext}^{2n+1}_{\text{Gr}(A)}(M/JM, \Lambda_0[j + 1]) = 0,
\]

where if \( j \notin \{ nd, \cdots , nd + d - 2 \} \) then \( \text{Ext}^{2n+1}_{\text{Gr}(A)}(M, \Lambda_0[j + 1]) = 0 \), and hence \( \text{Ext}^{2n+1}_{\text{Gr}(A)}(JM, \Lambda_0[j + 1]) = 0 \); and if \( j \in \{ nd, \cdots , nd + d - 2 \} \), then \( j = nd < nd + 1 \), and in this case we already know \( \text{Ext}^{2n+1}_{\text{Gr}(A)}(JM, \Lambda_0[j + 1]) = 0 \). This proves the claim.

Since \( M/JM \) is a d-Koszul module, \( M/JM \) has a finitely generated graded projective resolution, say \( Q^*_i \), such that \( Q^*_i \) is generated in degrees in \( \delta(i) \). By the graded version of the Horseshoe Lemma, we get a graded projective resolution \( Q^*_i \) of \( M \), such that \( Q^*_i = Q_i \oplus Q^*_{i+1} \) for each \( i \). Thus \( Q^*_i \) is also generated in degrees in \( \Delta(i) \). Since \( M \) is a generalized d-Koszul module, by Remark 2.2(i), we know that \( Q^*_i \) is finitely generated, and hence \( Q^*_i \) is finitely generated. By definition \( JM[-1] \) is generalized d-Koszul.

(ii) Let \( d \geq 3 \). Applying \( \text{Hom}_A(-, \Lambda_0) \) to the graded exact sequence \( 0 \rightarrow J^2M \rightarrow JM \rightarrow JM/JM \rightarrow 0 \), we get the following exact sequence

\[
\text{Ext}^{2n-1}_A(JM/J^2M, \Lambda_0) \rightarrow \text{Ext}^{2n-1}_A(JM, \Lambda_0) \rightarrow \text{Ext}^{2n-1}_A(J^2M, \Lambda_0) \rightarrow \text{Ext}^{2n}_A(JM/J^2M, \Lambda_0).
\]

Since \( (JM/J^2M)[-1] \) is d-Koszul, by Lemma 2.4(i), \( \text{Ext}^{2n-1}_A(JM/J^2M, \Lambda_0[j]) = 0 \) if \( j \neq nd - d + 2 \), and \( \text{Ext}^{2n}_A(JM/J^2M, \Lambda_0[j]) = 0 \) if \( j \neq nd + 1 \). Taking the nd-th homogeneous components of the exact sequence above, we obtain that \( \text{Ext}^{2n-1}_{\text{Gr}(A)}(JM, \Lambda_0[nd]) \cong \text{Ext}^{2n-1}_{\text{Gr}(A)}(J^2M, \Lambda_0[nd]) \). Repeating the process one gets (ii).

3. Proofs of Main Theorem and Corollary

3.1. We begin with a lemma, which seems to be of independent interest.

**Lemma 3.1.** Let \( A \) be an arbitrary Koszul algebra and \( \mathcal{C} \) a full subcategory of \( \text{Gr}(A) \). Suppose that for any \( X \in \mathcal{C} \), there exist exact sequences in \( \text{Gr}(A) \)

\[
(3) \quad 0 \rightarrow \Omega \rightarrow P^0 \rightarrow X \rightarrow 0,
\]

\[
(4) \quad 0 \rightarrow X'' \rightarrow X' \rightarrow \Omega[-1] \rightarrow 0,
\]

such that \( P^0 \) is a graded projective \( A \)-module generated in degree 0 and \( X', X'' \in \mathcal{C} \). Then all modules in \( \mathcal{C} \) are Koszul \( A \)-modules.

**Proof.** By Proposition 2.14.2 in Beilinson - Ginzburg - Soergel [BGS], it suffices to prove that for each \( X \in \mathcal{C} \), \( \text{Ext}^1_{\text{Gr}(A)}(X, \Lambda_0[j]) = 0 \) unless \( j = i \). We use induction on \( i \).

The sequence (3) implies that \( X \) is generated in degree 0, and hence \( \text{Hom}_{\text{Gr}(A)}(X, \Lambda_0[j]) = 0 \) unless \( j = 0 \). The sequence (4) implies that \( \Omega \) is a graded \( A \)-module and is generated in degree 1, since \( X' \in \mathcal{C} \) is generated in degree 0. By \( \text{Ext}^1_{\text{Gr}(A)}(X, \Lambda_0[j]) \cong \text{Hom}_{\text{Gr}(A)}(\Omega, \Lambda_0[j]) \), we see that \( \text{Ext}^1_{\text{Gr}(A)}(X, \Lambda_0[j]) = 0 \) unless \( j = 1 \).
Let $n \geq 1$. Assume that for each $X \in \mathcal{C}$ and for each positive integer $i$ with $i \leq n$, $\text{Ext}^{i}_{\text{Gr}(A)}(X, A_{0}[j]) = 0$ unless $j = i$. The exact sequence (4) implies the following exact sequence for every integer $j$
\[
\text{Ext}^{n-1}_{\text{Gr}(A)}(X'', A_{0}[j]) \to \text{Ext}^{n}_{\text{Gr}(A)}(\Omega[\mathcal{l}-1], A_{0}[j]) \to \text{Ext}^{n}_{\text{Gr}(A)}(X', A_{0}[j]).
\]
By the inductive hypothesis, we have $\text{Ext}^{n-1}_{\text{Gr}(A)}(X'', A_{0}[j]) = 0$ unless $j = n - 1$, and $\text{Ext}^{n}_{\text{Gr}(A)}(X', A_{0}[j]) = 0$ unless $j = n$. Let $Q^{n}$ be the minimal graded projective resolution of $\Omega[-1]$ (it exists since $\Omega \subseteq \mathcal{P}^{0}$ is supported above 0). By Lemma 4.3, $Q^{n}$ is supported above degree $n$, which implies $\text{Ext}^{n}_{\text{Gr}(A)}(\Omega[\mathcal{l}-1], A_{0}[j]) = 0$ for $j < n$. It follows from the exact sequence above that $\text{Ext}^{n}_{\text{Gr}(A)}(\Omega[\mathcal{l}-1], A_{0}[j]) = 0$ unless $j = n$. Thus $\text{Ext}^{n+1}_{\text{Gr}(A)}(X, A_{0}[j]) = \text{Ext}^{n}_{\text{Gr}(A)}(\Omega, A_{0}[j]) = \text{Ext}^{n}_{\text{Gr}(A)}(\Omega[-1], A_{0}[j-1]) = 0$ unless $j = n + 1$. This completes the proof.

3.2. Proof of Main Theorem. By Theorem 1.4, $E^{\text{ev}}(\Lambda)$ is a Koszul algebra. Put
\[
\mathcal{C} := \{E^{\text{ev}}(N) \in \text{Gr}(E^{\text{ev}}(\Lambda)) \mid N \text{ is a generalized } d\text{-Koszul } \Lambda\text{-module}\}.
\]
It suffices to prove that all the conditions in Lemma 3.1 are satisfied.

The graded exact sequence $0 \to JN \to N \to N/JN \to 0$ induces the following exact sequence of graded $k$-spaces for each $n \geq 0$
\[
(5) \text{Ext}^{2n-1}_{\Lambda}(N, A_{0}) \to \text{Ext}^{2n-1}_{\Lambda}(JN, A_{0}) \to \text{Ext}^{2n}_{\Lambda}(N/JN, A_{0}) \to \text{Ext}^{2n}_{\Lambda}(N, A_{0}) \to \text{Ext}^{2n}_{\Lambda}(JN, A_{0}).
\]
Since $N$ and $JN[-1]$ are generalized $d$-Koszul, by Lemma 2.4(ii), we have $\text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(N, A_{0}[nd]) = 0 = \text{Ext}^{2n}_{\text{Gr}(\Lambda)}(JN, A_{0}[nd])$. Taking the $nd$-th homogeneous components of (5) we get the following exact sequence for each $n \geq 0$
\[
(6) \quad 0 \to \text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(JN, A_{0}[nd]) \to \text{Ext}^{2n}_{\text{Gr}(\Lambda)}(N/JN, A_{0}[nd]) \to \text{Ext}^{2n}_{\text{Gr}(\Lambda)}(N, A_{0}[nd]) \to 0.
\]
Since $N$ is generalized $d$-Koszul and $N/JN$ is $d$-Koszul, by Lemma 2.4, we have
\[
E^{\text{ev}}(N/JN) = \bigoplus_{n \geq 0} \text{Ext}^{2n}_{\text{Gr}(\Lambda)}(N/JN, A_{0}[nd]), \quad E^{\text{ev}}(N) = \bigoplus_{n \geq 0} \text{Ext}^{2n}_{\text{Gr}(\Lambda)}(N, A_{0}[nd]).
\]
By taking direct sum of (6), we get the following short exact sequence in $\text{Gr}(E^{\text{ev}}(\Lambda))$:
\[
(7) \quad 0 \to \Omega \to E^{\text{ev}}(N/JN) \to E^{\text{ev}}(N) \to 0,
\]
where $\Omega := \bigoplus_{n \geq 0} \text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(JN, A_{0}[nd])$. In particular, $\Omega$ is a graded $E^{\text{ev}}(\Lambda)$-module with grading $\Omega_{n} := \text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(JN, A_{0}[nd])$. (One can also prove this directly as follows: since $\Lambda$ is $d$-Koszul algebra, it follows from Lemma 2.4(i) that
\[
\text{Ext}^{2n}_{\Lambda}(A_{0}, A_{0}) \text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(JN, A_{0}[nd]) = \text{Ext}^{2n}_{\text{Gr}(\Lambda)}(A_{0}, A_{0}[nd]) \text{Ext}^{2n-1}_{\text{Gr}(\Lambda)}(JN, A_{0}[nd]) \subseteq \text{Ext}^{2(n+m)-1}_{\text{Gr}(\Lambda)}(JN, A_{0}[(n + m)d]).
\]
By Theorem 1.4, $E^{\text{ev}}(N/JN)$ is a Koszul $E^{\text{ev}}(\Lambda)$-module, in particular it is generated in degree 0. Since $N/JN$ is a direct summand of finite direct sum of copies of the trivial $\Lambda$-module $A_{0}$, $E^{\text{ev}}(N/JN)$ is a projective $E^{\text{ev}}(\Lambda)$-module.
Similarly, the graded exact sequence 0 → \( J^d N \) → \( J^{d-1} N \) → \( J^{d-1} N/J^d N \) → 0 induces the following exact sequence of graded \( k \)-spaces for each \( n ≥ 0 \)
\[
\text{Ext}_A^{2n}(J^{d-1} N, \Lambda_0) → \text{Ext}_A^{2n}(J^d N, \Lambda_0) → \text{Ext}_A^{2n+1}(J^{d-1} N/J^d N, \Lambda_0) → \text{Ext}_A^{2n+1}(J^{d-1} N, \Lambda_0) → \text{Ext}_A^{2n+1}(J^d N, \Lambda_0).
\]

Note that by Theorem 2.6(i), \( J^{d-1} N[-(d-1)] \) and \( J^d N[-d] \) are generalized \( d \)-Koszul \( \Lambda \)-modules, and that \( (J^{d-1} N/J^d N)[-1] \) is a \( d \)-Koszul module. Taking the \( (n+1)d \)-th homogeneous components, and by the same arguments we get another exact sequence in \( \text{Gr}(E^\text{ev}(\Lambda)) \):
\[
0 → \bigoplus_{n ≥ 0} \text{Ext}_A^{2n}(J^d N, \Lambda_0[(n+1)d]) → \bigoplus_{n ≥ 0} \text{Ext}_A^{2n+1}(J^{d-1} N/J^d N, \Lambda_0[(n+1)d]) → \bigoplus_{n ≥ 0} \text{Ext}_A^{2n+1}(J^{d-1} N, \Lambda_0[(n+1)d]) → 0,
\]
or equivalently,
\[
0 → E^\text{ev}(J^d N) → E^\text{odd}(J^{d-1} N/J^d N) → \Omega[-1] → 0,
\]
where
\[
\Omega[-1] = \bigoplus_{n ≥ 0} \text{Ext}_A^{2n-1}(JN, \Lambda_0[nd])[-1] = \bigoplus_{n ≥ 0} \text{Ext}_A^{2n+1}(JN, \Lambda_0[(n+1)d])
\]
\[
≈ \bigoplus_{n ≥ 0} \text{Ext}_A^{2n+1}(J^{d-1} N, \Lambda_0[(n+1)d]),
\]
where the last isomorphism follows from Theorem 2.6(ii).

Since \( (J^{d-1} N/J^d N)[-1] \) is \( d \)-Koszul, by Lemma 2.5(ii), \( \Omega(J^{d-1} N/J^d N)[-d] \) is generalized \( d \)-Koszul, and by Lemma 3.1(i), we have
\[
E^\text{odd}(J^{d-1} N/J^d N) = E^\text{odd}((J^{d-1} N/J^d N)[-1]) = E^\text{ev}(\Omega(J^{d-1} N/J^d N)[-d]),
\]
from which we see \( E^\text{odd}(J^{d-1} N/J^d N) ∈ C \).

Since \( N \) is generalized \( d \)-Koszul, by Theorem 2.6(i), \( (J^d N)[-d] \) is generalized \( d \)-Koszul. Thus \( E^\text{ev}(J^d N) = E^\text{ev}((J^d N)[-d]) ∈ C \).

Now (7) and (8) shows that all the conditions in Lemma 3.1 are satisfied. This completes the proof.

3.3. Proof of Corollary. By Lemma 2.5(ii), \( (\Omega M)[-1] \) is a generalized \( d \)-Koszul module. It follows from Main Theorem that \( E^\text{ev}(\Omega M)[-1] \) is a Koszul \( E^\text{ev}(\Lambda) \)-module. Therefore by Lemma 2.5(i), \( E^\text{odd}(M) ≃ E^\text{ev}(\Omega M) = E^\text{ev}((\Omega M)[-1]) \) is a Koszul \( E^\text{ev}(\Lambda) \)-module.

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N. Bian: Dept. Math., Shanghai Jiao Tong University, Shanghai 200240, P. R. China
Dept. Math., Shandong University of Technology, Zibo 255049, P. R. China

Y. Ye: Dept. Math., University of Science and Technology of China, Hefei 230026, P. R. China

P. Zhang: Dept. Math., Shanghai Jiao Tong University, Shanghai 200240, P. R. China