Stochastic Event-triggered Variational Bayesian Filtering

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Abstract—This paper proposes an event-triggered variational Bayesian filter for remote state estimation with unknown and time-varying noise covariances. After presetting multiple nominal process noise covariances and an initial measurement noise covariance, a variational Bayesian method and a fixed-point iteration method are utilized to jointly estimate the posterior state vector and the unknown noise covariances under a stochastic event-triggered mechanism. The proposed algorithm ensures low communication loads and excellent estimation performances for a wide range of unknown noise covariances. Finally, the performance of the proposed algorithm is demonstrated by tracking simulations of a vehicle.

Index Terms—Event-based scheduling; Variational Bayesian; Kalman filter; Remote state estimation.

I. INTRODUCTION

The last few decades have seen a great development of state estimation techniques with their wide applications in navigation, tracking, and sensor networks. Various types of state estimation algorithms have been proposed such as linear filters [1], nonlinear filters [2] [3], particle filters [4], filters with state constraints [5] [7], filters with intermittent observations [8] [9], distributed filters [10], etc.

In practice, considering small batteries equipped by sensors with limited channel bandwidths, an efficient remote estimation system is desirable. In such a system, the sensor decides whether it sends measurement to a remote estimator. Usually, a tradeoff between the communication rate and the estimation performance is necessary. Various event-based communication schemes provide a solution to such problems, which are typically subject to limited transmission resources as depicted in Fig. 1. An event-based sensor data scheduler and a state estimation algorithm were proposed in [12]. However, to drive the minimum mean-squared error (MMSE) estimator, Gaussian approximation was adopted for prior estimates, and an exact MMSE estimation algorithm was designed by exploiting the generalized closed skew normal distribution in [13]. Then, two new types of event-triggered schedules were designed to eliminate an associated approximation problem [14]. Lately, a robust event-triggered remote state estimation algorithm was derived by minimizing a risk-sensitive criterion in [15]. In the presence of packet drops between the sensor and the estimator, the remote state estimation problem was studied, and a suboptimal estimator was proposed in [16]. For general non-Gaussian systems, an event-based transmission scheme was derived for particle filter based on remote state estimation in [17].

However, in practical applications, the measurement and the process noise covariances cannot be precisely calculated, which may even be time-varying. To jointly estimate the state and the time-varying noise covariances, the variational Bayesian method is effective. In [18], a recursive adaptive Kalman filter was proposed by forming a separable variational approximation based on the inverse-Gamma distribution. Later, by combining variational Bayesian and Gaussian filtering methods, a variational Bayesian adaptive algorithm was designed in [19]. However, the above two filters cannot handle the case with unknown process noise covariances. Recently, by using inverse Wishart priors for the predicted error covariance matrix, a variational Bayesian adaptive Kalman filter was presented in [20]. Moreover, a new variational adaptive Kalman filter with the Gaussian-inverse-Wishart mixture distribution was developed in [21]. By modelling the probability density functions of state transition and measurement as Gaussian-Gamma mixture distributions, an adaptive Kalman filter was derived in [22]. Although these variational Bayesian algorithms have good filtering performances, they are costly in performing information transmission from a sensor to a remote estimator. It is therefore desirable to solve the above Bayesian filtering problem with lower transmission costs, which is the objective of the present paper.

Specifically, this paper addresses the event-triggered variational Bayesian estimation problem with unknown and time-varying noise covariances. Our main contributions are briefly summarized as follows:

1) For remote state estimation with unknown and time-varying noise covariances, this paper proposes an event-triggered variational Bayesian filter to jointly estimate the state and the noise covariances. Multiple inverse Wishart priors are utilized for estimating the predicted error covariance with weight combinations being adaptively inferred by the variational Bayesian method.

2) With an event-triggered mechanism, the algorithm ensures excellent and robust performances for a wide range of unknown noise covariances with low communication overhead.

The remainder of this paper is organized as follows. Section II presents preliminaries and the problem formulation. Section III describes the designed variational Bayesian filter. Section IV provides simulations to verify the effectiveness of the proposed algorithm. Section V concludes the investigation.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Preliminaries

Let $P$ be a symmetric positive definite matrix of random variables, and define the inverse Wishart distribution as

$$\text{IW}(P|g,G) = \frac{1}{2^{|g|} |P|^{\frac{n+1}{2}} \Gamma_n\left(\frac{n}{2}\right)} G^{\frac{n}{2}} e^{-\frac{1}{2} tr(GP^{-1})},$$

where $|P|$ is the determinant of $P$, $\Gamma_n(a)$ is the gamma function evaluated at $a$, and $n$ is the dimension of $P$.
where $n$, $q$, and $G$ denote the dimensions of $P$, the degree of freedom (dof), a positive definite scale matrix, respectively, and $\Gamma_n$ is the $n$-th order gamma function.

Let $\lambda$ be the $M$-dimensional binary variable with $\lambda_j \in \{0, 1\}$ such that $\sum_{j=1}^M \lambda_j = 1$. The categorical distribution is defined as

$$\text{Cat}(\lambda | \mu, M) = \prod_{j=1}^M [\mu_j]^{\lambda_j},$$

where $\mu = (\mu_1, ..., \mu_M)$ is subject to constraints $0 \leq \mu_j \leq 1$ and $\sum_{j=1}^M \mu_j = 1$. Moreover, the Dirichlet distribution is defined as

$$\text{Dir}(\mu, \alpha, M) = C(\alpha) \prod_{j=1}^M [\mu_j]^{\alpha_j - 1},$$

where $\alpha = (\alpha_1, ..., \alpha_M)$, $\alpha_j = \sum_{j=1}^M \alpha_j$, $C(\alpha) = \frac{\Gamma(\alpha)}{\prod_{j=1}^M \Gamma(\alpha_j)}$, and $\Gamma(\cdot)$ is the gamma function. The above distributions will be used for the estimation of covariances later.

Variational inference is a method based on optimization to estimate unknown probability densities. Denote the set of all latent unknown variables as $A = \{\theta_1, ..., \theta_N\}$ and the set of all observed variables as $Z = \{Z_1, ..., Z_M\}$. According to the mean field theory [23], the true distribution $p(\Lambda|Z)$ can be approximated by $p(\Lambda|Z) \approx \prod_{k=1}^N q(\theta_k)$, where every $\theta_k$ has its own independent distribution $q(\theta_k)$.

Lemma 1: For matrices $A$, $B$, $C$, and $D$ with appropriate dimensions, if $A$ and $E = D - CA^{-1}B$ are nonsingular, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BE^{-1}CA^{-1} & -A^{-1}BE^{-1} \\ -E^{-1}CA^{-1} & E^{-1} \end{bmatrix}. \quad (9)$$

B. Problem Statement

Consider a discrete-time system as follows:

$$x_k = F_k x_{k-1} + \omega_k, \quad (10)$$

$$z_k = H_k x_k + v_k, \quad (11)$$

where $x_k \in \mathbb{R}^n$ is the system state, $F_k \in \mathbb{R}^{n \times n}$ is the state transition matrix, $z_k \in \mathbb{R}^m$ is the measurement, $H_k \in \mathbb{R}^{m \times n}$ is the measurement matrix, and $\omega_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are mutually uncorrelated zero-mean Gaussian noise with covariances $Q_k > 0$ and $R_k > 0$, respectively. The initial state $x_0$ and its estimates $\hat{x}_{0|0}$ obey Gaussian distributions.

In practice, the process and the measurement noise covariances may be unknown and time-varying, which are usually empirically estimated. In this paper, it is assumed that the initial nominal measurement noise covariance and $M$ nominal process noise covariances are described by $\bar{R}_0$ and

$$\bar{Q}_k = \prod_{j=1}^M [\bar{Q}_{j,k}]^{\lambda_{j,k}}, \quad (12)$$

respectively, where $\bar{Q}_{j,k}$ represents the $j$-th nominal process noise covariance at time $k$, and $\lambda_{k} = (\lambda_{1,k}, ..., \lambda_{M,k})$ is subject to the categorical distribution like [2]. Practically, $\lambda_{k}$ is not exactly known.

Remark 1: Under the event-triggered mechanism, the unknown process noise covariances cannot be jointly estimated in a recursive form like the measurement noise covariances by the variational Bayesian method. To solve this problem, the unknown process noise covariances are expected to be estimated by adaptive weight combinations of multiple nominal process noise covariances. From [12], $E\{\bar{Q}_k\} = \sum_{j=1}^M E\{\bar{Q}_{j,k}\} \bar{Q}_{j,k}$, where $E\{\bar{Q}_{j,k}\}$ is utilized as the adaptive weight parameter. Hence, the unknown process noise covariance estimation problem is reformulated as the adaptive weight combination problem of multiple nominal process noise covariances.

In the present framework, when the sensor obtains the measurement $z_k$, to reduce the transmission cost, it first makes the decision whether to send the measurement $z_k$ to a remote estimator. To do so, a binary decision variable $\gamma_k = 1$ or 0 is introduced. Specifically, if the sensor decides to send the measurement, $\gamma_k = 1$; otherwise $\gamma_k = 0$. Here, the information set at time $k$ for the estimator is defined as $\mathcal{I}_k \triangleq \{\gamma_0 z_0, ..., \gamma_k z_k\} \cup \{\gamma_{k+1}, ..., \gamma_N\}$. In practice, closed-loop stochastic event-triggered schedule is adopted to design the decision variable $\gamma_k$ as [15]

$$\gamma_k = \begin{cases} 0, & \zeta \leq \varphi(e_k) \\ 1, & \zeta > \varphi(e_k) \end{cases}, \quad (13)$$

where $\zeta$ is a uniformly distributed random variable over $[0, 1]$ at every step; $e_k = z_k - \hat{z}_{k|k-1}$ is the innovation with the feedback predicted measurement $\hat{z}_{k|k-1} = H_k \hat{x}_{k|k-1}$; $Y_k$ is a positive definite matrix; and $\varphi(e_k)$ is designed as

$$\varphi(e_k) = \exp \left( -\frac{1}{2} e_k^T Y_k e_k \right). \quad (14)$$

The objective of this paper is to design an event-triggered variational Bayesian filter for the sensor system with unknown process and measurement noise covariances. The following tasks will be accomplished:
1) design a variational Bayesian filter in the presence of unknown noise covariances with multiple nominal covariances.
2) design the above filter with a low transmission cost using the event-triggered schedule [13].

III. VARIATIONAL BAYESIAN FILTER

In this section, a novel filtering algorithm is proposed by using the variational Bayesian method, estimating the unknown states $x_k$ under the event-triggered mechanism.

A. Unknown Noise Distributions

First, it follows from (12) that the prior state $\hat{x}_{k|k-1}$ and the nominal predicted error covariance $P_{k|k-1}$ are given by

$$\hat{x}_{k|k-1} = F_{k-1} \hat{x}_{k-1|k-1},$$

and

$$P_{k|k-1} = \prod_{j=1}^{M} [F_{k-1} P_{k-1|k-1} F_{k-1}^T + \bar{Q}_{j,k}]^{\lambda_{j,k}},$$

respectively, where $P_{j,k|k-1} = F_{k-1} P_{k-1|k-1} F_{k-1}^T + \bar{Q}_{j,k}$, and $\lambda_{j,k}$ is given in (12). However, since the noise covariances are unknown, it is not feasible to directly perform (15) and (16) for state estimation. To address this issue, the noise covariances are estimated as follows.

Following [23], the inverse Wishart distribution is utilized as the conjugate prior distribution for the Gaussian distribution with an unknown covariance matrix. In so doing, the distributions of $p(P_{k|k-1}|I_{k-1})$ and $p(R_{k}|I_{k})$ are chosen as

$$p(P_{k|k-1}|\lambda_k, I_{k-1}) = \prod_{j=1}^{M} \left[ 1W(P_{k|k-1}|g_{j,k}|k-1, \hat{G}_{j,k}|k-1) \right]^{\lambda_{j,k}},$$

and

$$p(R_{k}|I_{k-1}) = IW(R_{k-1}|\hat{s}_{k-1|k-1}, \hat{S}_{k-1|k-1}),$$

respectively, where $\hat{g}_{j,k}|k-1$ is a preselected tuning parameter, $\hat{G}_{j,k}|k-1 = g_{j,k}|k-1 P_{j,k|k-1}$, $\hat{s}_{0|0} = S_{0|0}$ is another preselected tuning parameter, and $\hat{S}_{0|0} = S_{0|0} R_{0|0}$. It should be noted that $\lambda_k$ in (17) are unknown.

To obtain the value for $\lambda_k$ in (17), following [21], it is modelled by the categorical distribution, like [3], as

$$p(\lambda_k|\mu_k) = Cat(\lambda_k|\mu_k, M) = \prod_{j=1}^{M} [\mu_{j,k}]^{\lambda_{j,k}}.$$ 

Then, since the conjugate prior distribution of the categorical distribution is the Dirichlet distribution, like [3], as

$$p(\mu_k) = Dir(\mu_k, \alpha_{k|k-1}, M) = C(\alpha_{k|k-1}) \prod_{j=1}^{M} [\mu_{j,k}]^{\alpha_{j,k}-1}.$$ 

To this end, the distributions of the unknown noise covariances have been constructed. Based on these distributions, the posterior estimates will be inferred by the variational Bayesian method, as further discussed in the following.

B. Variational Approximation: $\gamma_k = 0$

This subsection studies the case where $z_k$ is not transmitted by the sensor at step $k$ under the event-triggered law [13], which results in the unavailability of $z_k$. Under this circumstance, $x_k$ and $z_k$ are regarded as jointly Gaussian [12]. Hence, they are strongly coupled and have to be jointly estimated. It is worth mentioning that the coupling poses a great challenge to obtain the solutions of $x_k$, $P_{k|k-1}$, and $R_k$ by using the traditional variational Bayesian technique, since $P_{k|k-1}$ and $R_k$ are coupled in the logarithm of the joint probability density function. To address this difficulty, a novel decoupling approach is proposed as follows.

For convenience, denote the set of the unknown variables as $\Lambda_0 \triangleq \{ (x_k, z_k), P_{k|k-1}, R_k, \lambda_k, \mu_k \}$. (19)

In the following, the approximate posterior probability density function (PDF) for every element in $\Lambda_0$ is calculated.

1) The logarithm of joint PDF $p(\Lambda_0, \gamma_k)$: First, the joint PDF $p(\Lambda_0, \gamma_k)$ is factorized as

$$p(\Lambda_0, \gamma_k|I_{k-1}) = p(\gamma_k|z_k, Y_k, I_{k-1}) p(x_k, z_k|P_{k|k-1}, R_k, I_{k-1}) \times p(P_{k|k-1}|\lambda_k) p(\lambda_k|\mu_k) p(\mu_k|P_{k|k-1}, R_k).$$

(20)

Based on the event-triggered scheme [13], one has

$$p(\gamma_k = 0|z_k, Y_k, I_{k-1}) = Pr(\exp(-\frac{1}{2} z_k^T Y_k^{-1} z_k) \geq \xi_k|z_k, Y_k, I_{k-1})$$

$$= \exp(-\frac{1}{2} (z_k - H_k \hat{x}_{k|k-1})^T Y_k (z_k - H_k \hat{x}_{k|k-1})).$$

Define $\phi_k \triangleq [x_k^T, z_k^T]^T$, $[\hat{x}_{k|k-1}^T, \hat{z}_{k|k-1}^T]^T \triangleq E\{ \phi_k|I_{k-1}\}$, and $\Phi_{k|k-1} \triangleq E\{ (\phi_k - \phi_k)(\phi_k - \phi_k)^T | I_{k-1}\}$. Then, $\Phi_{k|k-1}$ is obtained as

$$\Phi_{k|k-1} = \begin{bmatrix} P_{k|k-1} & H_k P_{k|k-1}^T H_k^T + R_k \end{bmatrix}.$$

Hence, the probability density function of $\phi_k$ in (20) is given by

$$p(x_k, z_k|P_{k|k-1}, R_k, I_{k-1}) = N(\phi_k|\Phi_{k|k-1}, \Phi_{k|k-1})$$

$$\exp\left(-\frac{1}{2} \begin{bmatrix} x_k - \hat{x}_{k|k-1} \; z_k - \hat{z}_{k|k-1} \end{bmatrix}^T \Phi_{k|k-1}^{-1} \begin{bmatrix} x_k - \hat{x}_{k|k-1} \\ z_k - \hat{z}_{k|k-1} \end{bmatrix} \right) \frac{(2\pi)^{n+m}/2}{|\Phi_{k|k-1}|^{1/2}}.$$ 

(21)

Using the above distributions, $p(\Lambda_0, \gamma_k|I_{k-1})$ is reformulated as

$$p(\Lambda_0, \gamma_k|I_{k-1}) = p(\gamma_k|z_k, Y_k, I_{k-1}) N(\phi_k|\Phi_{k|k-1}, \Phi_{k|k-1})$$

$$\times \prod_{j=1}^{M} \left[ 1W(P_{k|k-1}|g_{j,k}|k-1, \hat{G}_{j,k}|k-1) \right]^{\lambda_{j,k}}$$

$$\times Cat(\lambda_k|\mu_k, M) Dir(\mu_k, \alpha_{k|k-1}, M)$$

$$\times IW(R_k|\hat{s}_{k|k-1}, \hat{S}_{k|k-1}).$$

(22)
Then, using (6), $\log p(\Lambda_0, \gamma_k | Z_{k-1})$ in (22) is calculated as

$$
\log p(\Lambda_0, \gamma_k | Z_{k-1}) = -\frac{1}{2} (z_k - H_k \tilde{x}_{k|k-1})^T Y_k (z_k - H_k \tilde{x}_{k|k-1}) - 0.5 \log |\Phi_{k|k-1}| +
\sum_{j=1}^{M} \lambda_j \log \mu_{j,k} - \frac{1}{2} (\phi_k - \hat{\phi}_{k|k-1})^T \Phi_{k|k-1}^{-1} (\phi_k - \hat{\phi}_{k|k-1}) +
\sum_{j=1}^{M} \lambda_j \left[ 0.5 \hat{g}_{j,k|k-1} \log (|\hat{G}_{j,k|k-1}|) - 0.5 \operatorname{tr}(\hat{G}_{j,k|k-1} P_{k|k-1}^{\text{rec}}) - 0.5 (g_{j,k|k-1}) \log 2 - \log \Gamma_n (g_{j,k|k-1}/2) + \sum_{j=1}^{M} (\alpha_{j,k|k-1}) \log \mu_{j,k}ight]
- 0.5 (\hat{\delta}_{k|k-1} + m + 1) \log |R_k| - 0.5 \operatorname{tr}(\hat{S}_{k|k-1} R_k^{-1}) + c_{\Lambda_0},
$$

where $c_{\Lambda_0}$ is a constant independent of $\Lambda_0$.

2) **Decoupling of $\Phi_{k|k-1}$**: Each element of $\Lambda_0$ in (22) will be calculated alternatively, and the corresponding posterior distribution should have the same functional form as the prior distribution according to the variational inference theory [24]. However, $P_{k|k-1}$ and $R_k$ are coupled in $\Phi_{k|k-1}$, which poses a challenge in deriving the posterior $x_k, P_{k|k-1}$, and $R_k$ by using (6). In (23), $-0.5 \log |\Phi_{k|k-1}|$ and $-\frac{1}{2} (\phi_k - \hat{\phi}_{k|k-1})^T \Phi_{k|k-1}^{-1} (\phi_k - \hat{\phi}_{k|k-1})$ will be decoupled for $P_{k|k-1}$ and $R_k$. In addition, the expectation of $\Phi_{k|k-1}^{-1}$ is calculated, as follows.

First, $\Phi_{k|k-1}$ is factorized as

$$
\Phi_{k|k-1} = \left[ \begin{array}{cc} I & 0 \\ H & I \end{array} \right] \left[ \begin{array}{cc} P_{k|k-1} & 0 \\ 0 & R_k \end{array} \right] \left[ \begin{array}{cc} I & H^T \\ 0 & I \end{array} \right].
$$

Then, the inverse of $\Phi_{k|k-1}$ is computed by

$$
\Phi_{k|k-1}^{-1} = \left[ \begin{array}{cc} I & 0 \\ 0 & H^T \end{array} \right] \left[ \begin{array}{cc} P_{k|k-1}^{-1} & 0 \\ 0 & R_k^{-1} \end{array} \right] \left[ \begin{array}{cc} I & 0 \\ 0 & -H \end{array} \right].
$$

It follows from (24) that the determinant of $\Phi_{k|k-1}$ is obtained as

$$
|\Phi_{k|k-1}| = |P_{k|k-1}| |R_k|.
$$

Hence,

$$
\log |\Phi_{k|k-1}| = \log |P_{k|k-1}| + \log |R_k|.
$$

Now, based on (25) and (21), define a new variable as

$$
\rho_{k|k-1} = \left[ \begin{array}{cc} I & 0 \\ -H & I \end{array} \right] \left[ \begin{array}{c} x_k - \tilde{x}_{k|k-1} \\ z_k - \tilde{z}_{k|k-1} \end{array} \right].
$$

Then, $\rho_{k|k-1} \rho_{k|k-1}^T$ is computed by

$$
\rho_{k|k-1} \rho_{k|k-1}^T = \left[ \begin{array}{cc} I & 0 \\ -H & I \end{array} \right] \left[ \begin{array}{c} (\phi_k - \hat{\phi}_{k|k-1})(\phi_k - \hat{\phi}_{k|k-1})^T \\ -H \end{array} \right] = \left( \rho_{k|k-1} \rho_{k|k-1}^T \right)_{xx} \left( \rho_{k|k-1} \rho_{k|k-1}^T \right)_{zz}.
$$

$$(\rho_{k|k-1} \rho_{k|k-1}^T)_{xx} = (x_k - \tilde{x}_{k|k-1})(x_k - \tilde{x}_{k|k-1})^T \tag{28}$$

and

$$(\rho_{k|k-1} \rho_{k|k-1}^T)_{zz} = (z_k - \tilde{z}_{k|k-1})(z_k - \tilde{z}_{k|k-1})^T \tag{29}$$

respectively.

Combined (28), (30), one has

$$
\frac{1}{2} (\phi_k - \hat{\phi}_{k|k-1})^T \Phi_{k|k-1}^{-1} (\phi_k - \hat{\phi}_{k|k-1}) = \frac{1}{2} \begin{bmatrix} x_k - \tilde{x}_{k|k-1} \\ z_k - \tilde{z}_{k|k-1} \end{bmatrix}^T \Phi_{k|k-1}^{-1} \begin{bmatrix} x_k - \tilde{x}_{k|k-1} \\ z_k - \tilde{z}_{k|k-1} \end{bmatrix} = 0.5 \rho_{k|k-1}^T \left[ \begin{array}{cc} P_{k|k-1}^{-1} & 0 \\ 0 & R_k^{-1} \end{array} \right] \rho_{k|k-1}.
$$

Hence, according to (27) and (31), $P_{k|k-1}$ and $R_k$ are decoupled. Then, the approximate optimal solutions $q(P_{k|k-1})$ and $q(R_k)$ can be calculated by (6), respectively. Furthermore, the posterior distributions are derived in the same form as the prior distribution to provide a recursive filter.

3) **The update of $\phi_k$**: Now, every component in $\Lambda_0$ is computed. Let $\theta = \phi_k$, so that

$$
\log q^{t+1}(\phi_k) = \frac{1}{2} (\phi_k - \hat{\phi}_{k|k-1})^T \Phi_{k|k-1}^{-1} (\phi_k - \hat{\phi}_{k|k-1}) - \frac{1}{2} (\phi_k - \hat{\phi}_{k|k-1})^T \Theta_k^{-1} (\Theta_k^{t+1})^{-1} (\phi_k - \hat{\phi}_{k|k-1}) + c_{\phi_k}
$$

where $c_{\phi_k}$ is a constant independent of $x_k, z_k$, and $\Theta_k^{t+1}$ is given by

$$
\Theta_k^{t+1} = \left[ \begin{array}{cc} P_{k|k-1} + \tilde{P}_{k|k-1} & 0 \\ 0 & \tilde{P}_{k|k-1} \end{array} \right] = \left( \Theta_k^{t} \right)^{-1}.
$$

It follows from (25) that $E^t(\Phi_{k|k-1}^{-1})$ in (33) can be computed by

$$
E^t(\Phi_{k|k-1}^{-1}) = \left[ \begin{array}{cc} I & 0 \\ 0 & H \end{array} \right] \left[ \begin{array}{cc} I & 0 \\ 0 & H \end{array} \right] = \left[ \begin{array}{cc} I & 0 \\ 0 & H \end{array} \right] \left[ \begin{array}{cc} I & 0 \\ 0 & H \end{array} \right] = \left( \Theta_k^{t} \right)^{-1}.
$$

Note that $P_{k|k-1}$ and $R_k$ follow inverse Wishart distributions, and $P_{k|k-1}$ and $R_k$ obey Wishart distributions. Define $(\tilde{P}_{k|k-1})^{-1}$ and $(\tilde{R}_k)^{-1}$ as $E^t(\Phi_{k|k-1}^{-1})$. Further, $E^t(\Phi_{k|k-1}^{-1})$ in (34) can be derived as

$$
E^t(\Phi_{k|k-1}^{-1}) = \left[ \begin{array}{cc} I & 0 \\ 0 & H \end{array} \right] \left[ \begin{array}{cc} I & 0 \\ 0 & H \end{array} \right] = \left( \tilde{P}_{k|k-1} \right)^{-1} + H^T (\tilde{R}_k)^{-1} H_k - H^T (\tilde{R}_k)^{-1} H_k = (\tilde{R}_k)^{-1}.
$$

To this end, $\Theta_k^{t+1}$ can be computed by using Lemma 1. Denote $A_\Theta = (\tilde{P}_{k|k-1} + H_k^T (\tilde{R}_k)^{-1} H_k), B_\Theta = -H_k^T (\tilde{R}_k)^{-1} C_\Theta = -(\tilde{R}_k)^{-1} H_k$, and $D_\Theta = (\tilde{R}_k)^{-1} + Y_k$. Then, $\tilde{P}_{k|k-1}$ and $P_{k|k-1}^{t+1}$ in
can be calculated as
\begin{align}
P_{k|k}^{i+1} &= (A_{\Theta} + B_{\Theta}D_{\Theta}^{-1}C_{\Theta})^{-1} \\
&= ((\hat{P}_{k|k-1}^i)^{-1} + H_{k}^i (\hat{R}_{k}^i)^{-1} - (\hat{R}_{k}^i)^{-1} + Y_k)^{-1} \\
&= ((\hat{P}_{k|k-1}^i)^{-1} + H_{k}^i (R_k + Y_k^{-1})^{-1}H_k^{-1} \\
&= \hat{P}_{k|k-1}^i - \hat{P}_{k|k-1}^i H_k^i (H_k \hat{P}_{k|k-1}^i H_k^i)^{-1} H_k^{-1} + Y_k^{-1} \\
&= (H_k \hat{P}_{k|k-1}^i H_k^i + \hat{R}_k^{-1})^{-1} - Y_k^{-1},
\end{align}
and
\begin{align}
P_{x|x,k|k}^{i+1} &= ((\hat{R}_k)^{-1} - (\hat{R}_k)^{-1} H_k^i (\hat{R}_k)^{-1} \\
&= ((H_k \hat{P}_{k|k-1}^i H_k^i + \hat{R}_k^{-1}) - Y_k^{-1},
\end{align}
respectively.

Further, additional mathematical operations are performed to derive a concise form of $P_{x|x,k|k}^{i+1}$, as follows:
\begin{align}
P_{x|x,k|k}^{i+1} &= ((\hat{P}_{k|k-1}^i)^{-1} + H_{k}^i (\hat{R}_{k}^i)^{-1} - (\hat{R}_{k}^i)^{-1} + Y_k)^{-1} \\
&= ((H_k \hat{P}_{k|k-1}^i H_k^i + \hat{R}_k^{-1}) - Y_k^{-1},
\end{align}

respectively.

5) The update of $R_k$: Let $\theta = R_k$, so that
\begin{align}
\log q^{i+1}(R_k) &= -0.5E^{i+1}\{\text{tr}(\rho_k|_{k-1} T_k|_{k-1})x_k R_k^{-1}\} \\
&= -0.5(\hat{s}_{k|k-1} + m + 2\log |R_k|) + c_R \\
&= -0.5(\hat{s}_{k|k-1} + m + 2\log |R_k|) + c_R,
\end{align}
where $c_R$ is a constant independent of $R_k$, and
\begin{align}
B_k^{i+1} &= E^{i+1}\{(\hat{P}_k|_{k-1}) x_k^T\} \\
&= E^{i+1}\{H_k(x_k - \hat{x}_k|_{k-1})^T H_k^T \\
&= (z_k - \hat{z}_k|_{k-1}) (x_k - \hat{x}_k|_{k-1})^T \\
&= H_k \hat{P}_k^{i+1} H_k^T - (H_k \hat{P}_k^{i+1} H_k^T + H_k \hat{P}_k^{i+1} H_k^T)^{-1} \\
&= \hat{s}_{k|k-1} + 1 \\
&= \hat{s}_{k|k-1} + 1 + B_k^{i+1},
\end{align}
Hence, $\hat{s}_{k|k}$ and $\hat{S}_{k|k}$ are updated as
\begin{align}
\hat{s}_{k|k} &= \hat{s}_{k|k-1} + 1 \\
\hat{S}_{k|k} &= \hat{S}_{k|k-1} + B_k^{i+1},
\end{align}
For further iteration, define
\begin{align}
\hat{P}_k^{i+1} = E^{i+1}\{P_{k|k-1}^i\} = \hat{G}_k^{i+1}/\hat{g}^{i+1}.
\end{align}

6) The update of $\lambda_k$ and $\mu_k$: Let $\theta = \lambda_k$, and it follows from
\begin{align}
\log q^{i+1}(\lambda_k) = \sum_{j=1}^{M} E^{i}\{\lambda_{j,k}\} + c_\lambda,
\end{align}
where $c_\lambda$ is a constant independent of $\lambda_k$, and
\begin{align}
\tau^{i+1}_k &= 0.5\hat{g}_{j,k|k-1} \log(\hat{G}_{j,k|k-1}) - 0.5\hat{G}_{j,k|k-1} E^{i+1}\{P_{k|k-1}^i|\} \\
&= 0.5\hat{g}_{j,k|k-1} \log(\hat{G}_{j,k|k-1}) - 0.5(\mu_{j,k} + 1) E^{i+1}\{\log(\hat{P}_{k|k-1})\} \\
&= 0.5(\mu_{j,k} + 1) E^{i+1}\{\log(\hat{P}_{k|k-1})\} - \mu_{j,k} \log(\hat{G}_{j,k|k-1})/2, \\
E^{i+1}\{\log(\hat{P}_{k|k-1})\} &= \log(\hat{G}_{k|k}) - \tau \log(\hat{G}_{k|k})/2, \\
&= \psi(\alpha_{j,k}) - \psi\left(\sum_{j=1}^{M} \alpha_{j,k}\right),
\end{align}
in which $\psi(\cdot)$ is the digamma function.

Define a new variable
\begin{align}
\chi^{i+1}_k &= \exp(\tau^{i+1}_k + E^{i}\{\log(\mu_{j,k})\}),
\end{align}
Based on the categorical distribution, $\hat{\chi}^{i+1}_k$ is updated by
\begin{align}
\hat{\chi}^{i+1}_k &= \chi^{i+1}_k / \sum_{j=1}^{M} \chi^{i+1}_j.
\end{align}
Similarly, according to the Dirichlet distribution, $\alpha_{k+1}^{i+1}$ is updated by
\[
\alpha_{k+1}^{i+1} = \alpha_{k|k-1} + E_{k+1}^{i+1}(\lambda_k),
\]
where $E_{k+1}^{i+1}(\lambda_k) = \tilde{\chi}_{k+1}^{i+1}$.

C. Variational Approximation: $\gamma_k = 1$

This subsection studies the case where $z_k$ is transmitted by the sensor at step $k$, i.e., $\gamma_k = 1$. The following unknown parameters will be jointly estimated:
\[
\Lambda_{i} = \{x_k, P_k | k - 1, R_k, \lambda_k, \mu_k\}.
\]
(41)
The joint pdf $p(\Lambda_{i}, z_k, \gamma_k)$ is factorized as
\[
p(\Lambda_{i}, z_k, \gamma_k | x_k) = p(\gamma_k | x_k, z_k, x_k, \mathcal{Y}_k, \mathcal{Z}_k-1)p(x_k | P_k | k-1, \mathcal{Z}_k-1) 
\times p(z_k | x_k, R_k, \mathcal{Z}_k-1)p(\Lambda_{i} | P_k | k-1, \mathcal{Z}_k-1) 
\times p(\Lambda_{i} | \mu_k)p(\mu_k)p(R_k),
\]
where $p(\gamma_k = 1 | z_k, x_k, y_k, \mathcal{Z}_k-1) = 1 - \exp(-\frac{1}{2}c^T_y \mathcal{Y}_k c_y)$.

The techniques are similar to that in the above subsection, and the results about $P_k | k-1, \lambda_k, \mu_k$ are the same as that for the case of $\gamma_k = 0$ situation. Here, the results about $x_k$ and $R_k$ are given.

1) The update of $x_k$: Let $\theta = x_k$, and $\tilde{x}_{k|k-1}^{i+1}$ is updated by
\[
\tilde{x}_{k|k-1}^{i+1} = \tilde{x}_{k|k-1} + K_{k|k-1}^{i+1}(z_k - H_k \tilde{x}_{k|k-1})
\]
\[
K_{k|k-1}^{i+1} = \tilde{P}_{k|k-1}^{i+1} K_{k|k-1}^{i+1} \tilde{P}_{k|k-1}^{i+1} + R_k
\]
\[
P_{k|k}^{i+1} = \tilde{P}_{k|k-1}^{i+1} - \tilde{P}_{k|k-1}^{i+1} \tilde{P}^T_{k|k-1} (H_k \tilde{P}_{k|k-1}^{i+1} H_k^T + \tilde{R}_k)^{-1} H_k \tilde{P}_{k|k-1}^{i+1},
\]
(42)
2) The update of $R_k$: Let $\theta = R_k$, and $\tilde{S}_{k|k-1}^{i+1}$ and $\tilde{S}_{k|k-1}^{i+1}$ are updated by
\[
\tilde{S}_{k|k-1}^{i+1} = \tilde{S}_{k|k-1} + B_{k+k}
\]
where
\[
B_{k+k} = (z_k - H_k \tilde{x}_{k|k-1})(z_k - H_k \tilde{x}_{k|k-1})^T + H_k P_{k|k}^{i+1} H_k^T,
\]

D. Prior Parameters

This subsection provides a designing method for the prior parameters $\alpha_{k|k-1}, \tilde{s}_{k|k-1}$, and $\tilde{S}_{k|k-1}$. Similarly to [20], the prior parameters $\alpha_{k|k-1}, \tilde{s}_{k|k-1}$, and $\tilde{S}_{k|k-1}$ are chosen as
\[
\alpha_{k|k-1} = \rho \alpha_{k-1|k-1}, \tilde{s}_{k|k-1} = \rho \tilde{s}_{k-1|k-1}, \tilde{S}_{k|k-1} = \rho \tilde{S}_{k-1|k-1},
\]
(43)
where $\rho$ is the forgetting factor over $[0, 1]$.

Before the iteration steps, some parameters are initialized as
\[
\tilde{x}_{0|k-1}^0 = \tilde{x}_{k-1}, \chi_k^0 = 0, G_k^0 = 0, \tilde{G}_k^0 = 0,
\]
\[
g_k^0 = \sum_{j=1}^{M} x_j \tilde{g}_j | k-1, G_k^0 = \sum_{j=1}^{M} \chi_j \tilde{G}_j | k-1,
\]
\[
\tilde{P}_{k|k-1}^0 = \tilde{S}_{k|k-1}^0 = \tilde{S}_{k|k-1}^0.
\]
(44)

Now, based on the above results, the event-triggered variational Bayesian filter (ETVBF) is summarized as Algorithm 1. It is observed that only $x_k$ and $R_k$ are directly affected, when the measurement information is not received by the estimator. Then, other parameters are influenced by $x_k$ and $R_k$.

E. Discussions

In the algorithm, there exist several parameters that need to be selected and designed, i.e., $\tilde{g}_j | k-1, Q_j, \alpha_0, \rho, R_0, \tilde{s}_0 | 0, N$, and $\delta$. In the following, the influence of these parameters, the selections of these parameters, and the performances are discussed.

1) The influence of parameters:

For parameters $\tilde{g}_j | k-1$ and $Q_j$, based on Algorithm 1, $\tilde{P}_{k|k-1}^{i+1}$ can be formulated as
\[
\tilde{P}_{k|k-1}^{i+1} = \sum_{j=1}^{M} \tilde{X}_j \tilde{g}_j | k-1 + A_{k+1}^{i+1},
\]
(45)
where $A_{k+1}^{i+1}$ is a weighted sum of $P_{j, k+1} | k-1, j \in M$ and $A_{k+1}^{i+1}$. The bigger the $\tilde{g}_j | k-1$ is, the more the prior information $P_{j, k+1} | k-1$ is introduced into $\tilde{P}_{k|k-1}^{i+1}$.

b) $Q_j$: When $\tilde{g}_j | k-1, j \in M$, are set as the same value $\tilde{g}_{k|k-1}$, the adaptive parameter $\tilde{X}_j | k$ is subject to

Algorithm 1 Event-triggered Variational Bayesian Filter (ETVBF)

Input: $\tilde{x}_{k-1|k-1}, P_k | k-1, \tilde{s}_{k-1|k-1}, \tilde{S}_{k-1|k-1}, \alpha_{k-1|k-1}, \tilde{g}_j | k-1, Q_j, \rho, N,

Predicted state:
\[
\tilde{x}_{k|k-1} = F_{k-1} \tilde{x}_{k-1|k-1},
\]
\[
P_{j, k|k-1} = F_{k-1} P_{j, k-1} | k-1 F_{k-1}^T + Q_j, \tilde{g}_j | k-1 = \tilde{g}_{k|k-1} - F_{k-1} P_{j, k-1} | k-1 F_{k-1}^T + \tilde{g}_{k|k-1}
\]

Updated state:

Initialization:

Initialized as (43) and (44).

Update $x_z$:

If $\gamma_k = 0$,
\[
\tilde{x}_{k|k-1} = \tilde{x}_{k|k-1}, \tilde{z}_{k|k-1} = H_k \tilde{x}_{k|k-1},
\]
\[
P_{k|k}^{i+1}, P_{zz|k}^{i+1}, \text{ and } P_{z|z|k}^{i+1} \text{ are calculated as (35), (36), and (37), respectively.}
\]

Else if $\gamma_k = 1$,
\[
P_{k|k}^{i+1}, P_{zz|k}^{i+1}, \text{ and } P_{z|z|k}^{i+1} \text{ are updated as (42).}
\]

Update $\Lambda_k$:

\[
E_{i+1}^{i+1}(\log \tilde{g}_j | k) = \psi(\alpha_k | k) - \psi(\sum_{j=1}^{M} \alpha_j | k),
\]
\[
E_{i+1}^{i+1}(\log (|P_{k|k}|)) = \log(G_k | k) - n \log 2 - (0.5 g_k | k).
\]

Update $\alpha_k$:

\[
\alpha_{k|k} = \alpha_{k|k-1} + \chi_k | k
\]
If $||\tilde{z}_{k|k-1} - \tilde{x}_{k|k-1}||/||\tilde{x}_{k|k-1}|| \leq \delta$, terminate iteration.

Output: $\tilde{x}_{k|k} = \tilde{x}_{k|k-1}, P_k | k = P_{k|k}^{i+1}, \tilde{s}_k | k = \tilde{s}_{k|k}^{i+1}, \tilde{S}_k | k = \tilde{S}_{k|k}^{i+1}, \alpha_{k|k} = \alpha_{k|k-1}^{i+1}$.
\[ \sum_{j=1}^{M} Y_{i,j}^k = 1, 0 \leq Y_{i,j}^k \leq 1, \] which means that a convex combination of multiple nominal process noise covariances \( \overline{Q}_{j,k} \), \( j \in M \), is adaptively achieved. Multiple nominal process noise covariances are characterized by a range of the true covariance, and their accuracy requirements are reduced compared to the single covariance case. This condition is easier to satisfy and verify.

For parameters  \( \rho, R_0, \) and \( s_{0,0} \), similar to \([20]\), it can be obtained that
\[ \bar{R}_{k+1} = \frac{\eta(\rho,k) \bar{R}_{k-1} + B_{k+1}}{\eta(\rho,k) + 1}, \tag{46} \]
\[ \bar{R}_{k} = \left( \prod_{i=1}^{k-1} q_i \right) \bar{R}_0 + \sum_{i=1}^{k-1} \left( \prod_{j=i+1}^{k-1} q_j \right) \bar{B}_i, \tag{47} \]
where \( \eta(\rho,k) = \rho^k s_{0,0} + \frac{k \rho}{\rho + 1}, \rho \in (0,1], q_k = \frac{\eta(\rho,k)}{\eta(\rho,k) + 1}, \bar{B}_k = -\frac{\rho \bar{B}_{k+1}}{\eta(\rho,k) + 1}, \) and \( B_{k+1} \) represents \( B_k \) at the loop termination step.

c) For \( \text{ETVBF} \), \( B_{k+1} \) in \((46)\) is different under different event-triggered situations, and \( \eta(\rho,k) \) plays a role in balancing a weighted sum of \( \bar{R}_{k-1} \) and \( B_{k+1} \). Moreover, \( \eta(\rho,k) \) is a monotone increasing function of \( \rho \) as \( k \) approaches infinity, and \( \lim_{k \to \infty} \eta(\rho,k) = 1 - \rho \). Hence, the smaller \( \rho \) is, the more information the \( B_{k+1} \) is introduced into \( \bar{R}_{k+1} \).

d) \( \bar{R}_0 \): Based on \((47)\), it can be concluded that the effect of the past information about \( \bar{R}_0 \) and \( \bar{B}_k \) exponentially decays as the time increases.

e) \( N \) and \( \delta \): Based on the variational inference theory \((6)\) and the fixed-point iteration \((6) \[a(25)\]), a more accurate approximate optimal solution can be obtained by increasing the iteration number \( N \). By setting a reasonable loop termination condition \( \delta \), the iteration can be terminated early with guaranteed accuracy.

2) The selections of parameters: For parameters \( \hat{g}_{j,k} \) and \( \hat{s}_{0,0} \), they can be selected within a wide range, and it is suggested that they are tuned based on the specific system. For \( \bar{Q}_{j,k} \), to obtain better performances, it is suggested that multiple nominal process noise covariances \( \bar{Q}_{i,j} \) are selected such that the true process noise covariance \( Q_{i,j} \) satisfies \( \min(Q_{i,j}, j \in M) \leq Q_{i,j} \leq \max(Q_{i,j}, j \in M) \). For \( s_{0,0} \), it can be set as \( 1 \times M \) since it will be adaptively adjusted. For \( \rho \), the forgetting factor is suggested to be selected as \( \rho \in [0.94, 1] \) to obtain a good performance, as shown in the simulation part. For \( N \) and \( \delta \), they are selected as a big number and a sufficiently small number, respectively, to guarantee the iteration performance. It is worth mentioning that the effectiveness of such selections is verified in the simulation part.

3) Performance Analysis: The estimation performance has a strong correlation with the deviation between the nominal value and the true value, such as \( \bar{R}_0 \) and \( R_0 \). When the deviation is smaller, the proposed algorithm has a better performance, as shown in Fig. \(6\) in Simulations. The parameters \( N \) and \( \delta \) have a main influence on the estimation performance, and they have been discussed above. A concept “numerical stability” from \([20] \[a(25)\] is introduced to guarantee the feasibility of the proposed algorithm, i.e., the covariance matrices must be positive definite when the algorithm is in the numerical iteration process. Since \( \bar{Q}_{j,k} > 0, \bar{R}_0 > 0, \bar{A}_{k+1} > 0, \) and \( \bar{B}_{k+1} > 0 \), it follows that \( \bar{P}_{k+1} > 0 \) and \( \bar{R}_{k+1} > 0 \). Thus, \( \text{ETVBF} \) is numerically stable.

**Proposition 1:** For systems \([10] \[a(11)\] under Algorithm \([1]\), the estimation error \( e_{k|i} = \hat{x}_{k|i} - x_k \) is Gaussian.

**Proof.** In Algorithm \([1]\) denote the measurement update error and the predicted error as \( e_{k|i} = x_{k|i} - x_k \) and \( e_{k|i-1} = x_{k|i-1} - x_k \), respectively. When \( \gamma = 0 \), one has \( e_{k|i} = \hat{x}_{k|i-1} - x_k = e_{k|i-1} \). When \( \gamma = 1 \), one has \( e_{k|i} = (I - K_{k+1}^i H_k)(\hat{x}_{k|i-1} - x_k) + K_{k+1}^i \nu_k = (I - K_{k+1}^i H_k)e_{k|i-1} + K_{k+1}^i \nu_k \). Additionally, \( e_{k|i-1} = F_k \hat{x}_{k|i-1} - x_k = F_k e_{k|i-1} - x_k \). Due to the Gaussianity of \( e_{k|0} = \hat{x}_{0} - x_0, \) \( \omega_{k-1} \), and \( \nu_k \), the error \( e_{k|i} \) is Gaussian as can be verified indutively.

### IV. Simulations

This section shows some simulations and comparison with existing algorithms to verify the effectiveness of the proposed \( \text{ETVBF} \).

A tracking problem for a vehicle is considered as in \([20]\). The dynamical system is described by \((10) \) and \((11) \) with system matrices
\[ F_k = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} \tag{48} \]
and
\[ H_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{49} \]
respectively, where \( T = 1 \) s. The state dimension and the measurement dimension are \( n = 4 \) and \( m = 2 \), respectively. The true process and true measurement noise covariances are set as
\[ Q_k = \begin{bmatrix} T^3/3 & 0 & T^2/2 & 0 \\ 0 & T^3/3 & 0 & T^2/2 \\ T^2/2 & 0 & T & 0 \\ 0 & T^2/2 & 0 & T \end{bmatrix} \]
and
\[ R_k = \begin{bmatrix} 100 + 50 \cos((\pi k)/T_1) \\ 0.5 \end{bmatrix}, \tag{50} \]
respectively, where \( T_1 = 500 \).

In this section, the performances of \( \text{ETVBF} \) in Algorithm \([1]\) a closed-loop stochastic event-triggered Kalman filter (CLSET-KF) in \([24]\), and a variational Bayesian filter (VBF, without the triggering mechanism in Algorithm \([1]\)) a variational Bayesian based adaptive Kalman filter (VBAKF) in \([20]\), and a variational Bayesian adaptive Kalman filter with Gaussian-Gamma mixture (VBAKF-GGM) in \([22]\) are compared.

The nominal initial measurement noise covariance is set as \( \bar{R}_0 = r I_2 \) and the event-triggered parameter is set as \( y_k = y_1 l_2 \), where \( r \) and \( y \) are the noise scale factor and the event-triggered scale factor, respectively. The step internal \( N_{\text{step}} \) of \( k \) and the total iteration number \( N \) are chosen as \( 150 \) and \( 50 \), respectively. The initial state is \( x_0 = \begin{bmatrix} 100, 100, 10, 10 \end{bmatrix}^T \), and the initial estimation error covariance is \( P_0 = 100 I_4 \). Then, the initial state estimate is given by \( \hat{x}_{0|0} = \bar{X}(x_0, \bar{P}_{0|0}) \). The dof parameters are set as \( \hat{g}_{k|i} = 10 \times 1 \times M \) and \( \hat{s}_{0,0} = 5 \). The forgetting factor and the initial concentration parameter are selected as \( \rho = 0.997 \) and \( \alpha_0 = 1 \times 1 \times M \), respectively. For every case, Monte Carlo simulation experiments with \( N_{\text{MC}} = 500 \) are performed. The mixture term is selected as \( M = 5 \), and the nominal process noise covariances are set as \( Q_{k,1} = I_{4}, Q_{k,2} = 2I_{4}, Q_{k,3} = 3I_{4}, Q_{k,4} = 9I_{4}, Q_{k,5} = 10I_{4} \). For \( \text{VBAKF-GGM} \), the shape parameters are set as \( \alpha_0 = e_0 = [10, 10, 10, 10] \), and the rate parameters are set as \( b_0 = f_0 = [10, 100, 1000, 10000] \). For \( \text{CLSET-KF}, \text{VBAKF}, \) and \( \text{VBAKF-GGM} \), the nominal process noise covariance is set as \( Q_k = 4 I_4 \).

The root-mean-square error (RMSE) is utilized to evaluate the performance of the algorithm:
\[
\text{RMSE} = \sqrt{\frac{1}{nN_{\text{MC}}N_{\text{step}}} \sum_{k=1}^{N_{\text{MC}}} \sum_{j=1}^{N_{\text{step}}} \sum_{n=1}^{n} (\hat{x}_{k,j}(l) - x_{k,j}(l))^2},
\]
where \( n, N_{\text{MC}}, N_{\text{step}} \) denote the dimension of the state, the total Monte Carlo experiment number, and the step internal, respectively.
and \( \hat{x}_{k|j}(l) \) and \( x_{k|j}(l) \) represent the \( l \)-th component of the vector \( \hat{x}_{k|j} \) and the vector \( x_{k|j} \) at the \( k \)-th step in the \( j \)-th trial experiment, respectively. Similarly, the communication rate \( \gamma_{\text{trail}} \) is defined as

\[
\gamma_{\text{trail}} = \frac{1}{N_{MC} N_{\text{step}}} \sum_{k=1}^{N_{MC}} \sum_{j=1}^{N_{\text{step}}} \gamma_{k,j},
\]

where \( \gamma_{k,j} \) denotes \( \gamma_k \) of the \( j \)-th trial experiment at step \( k \).

First, under the different event-triggered communication rates, the estimation performances are compared among five algorithms. The event-triggered communication rates are dominated by \( Y_k \). Hence, the event-triggered scale factor \( y \) is set as \( 0.0005 : 0.0005 : 0.1 \), and the noise covariance scale factor is set as \( r = 150 \). Fig. 2 and Fig. 3 show the RMSE of the five algorithms and the event-triggered communication rates of ETVBF and CLSET-KF, respectively. To show the event-triggered performance, denote \( t_l \) as the \( l \)-th event-triggered time instant, and \( t_{l+1} - t_l \) as the interval between two adjacent event-triggered time instants. Fig. 4 presents the scheduling sequences of ETVBF and CLSET-KF under the event-triggered mechanism with \( y = 0.0005 \) and \( r = 150 \). Fig. 5 shows the average iteration numbers of ETVBF, VBF, and VBAKF-GGM with \( y = 0.0005 \) and \( r = 150 \).

1) **ETVBF** has a better performance than CLSET-KF all the time, while CLSET-KF needs higher communication rates than ETVBF. VBAKF-GGW has the worst performance, since it may be specially designed in the presence of outliers.

2) As \( y \) increases, the communication rates of ETVBF and CLSET-KF increase, and their performances become better.

**VBF** and **VBAKF** have the best performance at a high transmission cost. As \( y \) increases, the performance of ETVBF approaches that of VBF.

3) The proposed ETVBF needs the least average iteration number, and VBF needs a little larger average iteration number than ETVBF. As these filters converge, the average iteration numbers decrease gradually.

Next, with different nominal measurement noise covariances \( \bar{R}_0 \), the estimation performances are compared among the five algorithms. Similarly, the noise scale \( r \) is set as \( 10 : 10 : 300 \), and the event-triggered scale parameter is set as \( y = 0.015 \). Fig. 6 and Fig. 7 illustrate the RMSE of the five algorithms and communication rates of ETVBF and CLSET-KF, respectively.

1) When the error between the nominal measurement noise covariance \( r I_2 \) and the true noise covariance is small, all algorithms have good performances. As the error gets larger, the performances are degraded. ETVBF and VBF have better and stabler performances for different noise covariances.

2) As the nominal measurement noise covariance increases, the communication rates of ETVBF and CLSET-KF increase.

Finally, under different forgetting factors \( \rho \), the estimation performances of ETVBF are compared. Fig. 8 shows the RMSE of ETVBF under the forgetting factors \( \rho = 0.92, 0.94, 0.96, 0.98, 1.00 \). When \( \rho < 0.94 \), the RMSE of ETVBF shows a significant decline. Hence, it is suggested that the forgetting factor is selected as \( \rho \in [0.94, 1] \).

In summary, ETVBF possesses excellent and robust performances.
In this paper, an event-triggered variational Bayesian filter is proposed for systems with unknown and time-varying noise covariances. The state vector, the predicted error covariance, and the unknown measurement noise covariance are jointly estimated. Simulations show excellent and robust performances of the proposed algorithm. The event-triggered variational Bayesian filter in the nonlinear setting will be considered in the future.

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