Analysis of the hidden-charm tetraquark mass spectrum with the QCD sum rules

Zhi-Gang Wang
Department of Physics, North China Electric Power University, Baoding 071003, P. R. China

Abstract

In this article, we take the pseudoscalar, scalar, axialvector, vector, tensor (anti)diquark operators as the basic constituents, and construct the scalar, axialvector and tensor tetraquark currents to study the mass spectrum of the ground state hidden-charm tetraquark states with the QCD sum rules in a comprehensive way. We revisit the assignments of the $X$, $Y$, $Z$ states, such as the $X(3860)$, $X(3872)$, $X(3940)$, $X(4160)$, $Z_c(3900)$, $Z_c(4020)$, $Z_c(4050)$, $Z_c(4055)$, $Z_c(4100)$, $Z_c(4200)$, $Z_c(4250)$, $Z_c(4430)$, $Z_c(4600)$, etc in the scenario of tetraquark states based on the QCD sum rules.

PACS number: 12.39.Mk, 12.38.Lg
Key words: Tetraquark state, QCD sum rules

1 Introduction

In 2003, the Belle collaboration observed a narrow charmonium-like state $X(3872)$ in the $\pi^+\pi^-J/\psi$ mass spectrum in the exclusive decays $B^\pm \rightarrow K^\pm\pi^+\pi^-J/\psi$, which cannot be accommodated in the conventional two quark model as the $\chi_c^+'$ state with $J^{PC}=1^{++}$. Thereafter, about twenty charmonium-like states were observed by the BaBar, Belle, BESIII, CDF, CMS, D0, LHCb collaborations, which cannot be accommodated in the conventional two quark model, and are denoted as the $X$, $Y$ and $Z$ states now, some are still needed confirmation and the quantum numbers have not been established yet. In Table 1 we list out the masses, widths and $J^{PC}$ of the $X$, $Y$, $Z$ states in the $c\bar{c}$ region from the Particle Data Group. In 2018, the LHCb collaboration observed an evidence for an exotic $\eta_c\pi^-$ resonant state (now it referred to as $Z_c(4100)$) in the $B^0 \rightarrow \eta_cK^+\pi^-$ decays with the significance of more than three standard deviations, the possible spin-parity assignments are $J^P=0^+$ and $1^−$ [3]. We add the $Z_c(4100)$ in Table 1.

There have seen several possible interpretations for those $X$, $Y$ and $Z$ states, such as the tetraquark states, hadronic molecular states, dynamically generated resonances, hadroquarkonium, kinematical effects, cusp effects, virtual states, etc using the phenomenological approaches (potential quark models), effective field theories for QCD (such as heavy quark effective field theory, nonrelativistic QCD, potential nonrelativistic QCD, Born-Oppeinner approximation, chiral unitary models), QCD sum rules, lattice QCD, etc. For comprehensive reviews, one can consult Refs. [4, 5, 6, 7, 8, 9, 10, 11, 12]. In the present work, we will focus on the tetraquark interpretations.

The QCD sum rules is a powerful theoretical approach in studying the hadron properties, and has been applied extensively to calculate the masses, decay constants, form-factors, hadronic coupling constants, etc. In 2016, R. D. Matheus et al took the $X(3872)$ as the $J^{PC}=1^{++}$ diquark-antidiquark type tetraquark state, and studied its mass with the QCD sum rules by carrying out the operator product expansion up to the vacuum condensates of dimension 8. Thereafter the QCD sum rules became a powerful theoretical approach in studying the masses and widths of the $X$, $Y$ and $Z$ states, irrespective of the hidden-charm (or hidden-bottom) tetraquark states or hadronic molecular states [10, 17, 18, 19, 20, 21, 22, 23, 24, 25].

In the QCD sum rules for the hidden-charm (or hidden-bottom) tetraquark states and molecular states, the integrals

$$\int_{4m_Q^2(\mu)}^{s_0} ds \rho_{QCD}(s, \mu) \exp \left(-\frac{s}{T^2}\right), \quad (1)$$

1 E-mail: zgwang@aliyun.com.
| State  | M (MeV)          | Γ (MeV)       | $J^{PC}$         | Process           | experiment          |
|--------|------------------|---------------|-------------------|-------------------|---------------------|
| $X(3860)$ | $3862^{+26}_{-32}^{+30}_{-10}$ | $201^{+124}_{-67}^{+88}$ | 0++               | $e^+e^- \rightarrow J/\psi DD$ | Belle               |
| $X(3872)$ | $3871.69 \pm 0.17$ | $< 1.2$       | 1++               | $B \rightarrow K\pi^+\pi^- J/\psi$ | Belle               |
| $X(3915)$ | $3918.4 \pm 1.9$   | $20 \pm 5$    | 0/2++             | $B \rightarrow K J/\psi \omega$ | Belle, BaBar        |
| $X(3940)$ | $3942^{+7}_{-6}^{+8}$ | $37^{+26}_{-15}^{+8}$ | ???               | $e^+e^- \rightarrow J/\psi DD^*$ | Belle               |
| $X(4140)$ | $4146.8 \pm 2.4$   | $22^{+8}_{-7}$  | 1++               | $B^+ \rightarrow J/\psi \phi K^+$ | CDF, D0, LHCb       |
| $X(4160)$ | $4156^{+25}_{-20}^{+15}$ | $139^{+111}_{-61}^{+21}$ | ???               | $e^+e^- \rightarrow J/\psi D^+ D^*$ | Belle               |
| $X(4274)$ | $4274^{+6}_{-6}$     | $49 \pm 12$    | 1++               | $B^+ \rightarrow J/\psi \phi K^+$ | CDF, LHCb           |
| $X(4350)$ | $4350.6^{+4.6}_{-5.1}^{+0.7}$ | $13^{+18}_{-9}^{+4}$ | ?+                | $e^+e^- \rightarrow \phi J/\psi$ | Belle               |
| $X(4500)$ | $4506 \pm 11_{-12}^{+12}$ | $92 \pm 21_{-20}^{+21}$ | 0++               | $B^+ \rightarrow J/\psi \phi K^+$ | LHCb               |
| $X(4700)$ | $4704 \pm 10_{-12}^{+13}$ | $120 \pm 31_{-23}^{+12}$ | 0++               | $B^+ \rightarrow J/\psi \phi K^+$ | LHCb               |
| $Y(4220)$ | $4218^{+3}_{-4}$     | $59^{+12}_{-10}$ | 1−                | $e^+e^- \rightarrow h_c \pi^+ \pi^-$ | BESIII             |
| $Y(4360)$ | $4368 \pm 13$       | $96 \pm 7$     | 1−                | $e^+e^- \rightarrow \pi^+ \pi^- \psi'$ | BESIII             |
| $Y(4390)$ | $4391.5^{+6.3}_{-6.8}^{+0.7}$ | $139.5^{+16.2}_{-20.6}^{+0.6}$ | 1−                | $e^+e^- \rightarrow h_c \pi^+ \pi^-$ | BESIII             |
| $Y(4660)$ | $4643 \pm 9$        | $72 \pm 11$    | 1−                | $e^+e^- \rightarrow \pi^+ \pi^- \psi'$ | BESIII             |
| $Z_c(3900)$ | $3887.2 \pm 2.3$    | $28.2 \pm 2.6$ | 1−                | $Y(4260) \rightarrow J/\psi \pi^+ \pi^-$ | BESIII, BaBar      |
| $Z_c(4020)$ | $4024.1 \pm 1.9$    | $13 \pm 5$     | ?−                | $e^+e^- \rightarrow \pi^+ \pi^- h_c$ | BESIII             |
| $Z_c(4050)$ | $4051 \pm 14_{-2}^{+20}_{-41}$ | $82^{+21}_{-17}^{+47}_{-22}$ | ?+                | $B^0 \rightarrow K^- \pi^+ \chi_{c1}$ | Belle               |
| $Z_c(4055)$ | $4054 \pm 3 \pm 1$ | $45 \pm 11 \pm 6$ | ?−                | $e^+e^- \rightarrow \pi^+ \pi^- \psi'$ | BESIII             |
| $Z_c(4100)$ | $4096 \pm 20_{-12}^{+18}_{-2}^{+18}$ | $152 \pm 58_{-13}^{+40}_{-35}$ | 0+/1−              | $B^0 \rightarrow K^+ \pi^- \eta_c$ | LHCb               |
| $Z_c(4200)$ | $4196_{-29}^{+34}^{+17}_{-13}$ | $370 \pm 70_{-132}^{+50}_{-70}$ | 1−                | $B^0 \rightarrow K^- \pi^+ J/\psi$ | Belle               |
| $Z_c(4250)$ | $4248_{-29}^{+44}^{+35}_{-35}$ | $177 \pm 34_{-16}^{+116}_{-51}$ | 1+                | $B^0 \rightarrow K^- \pi^+ \chi_{c1}$ | Belle               |
| $Z_c(4430)$ | $4478_{-18}^{+15}$   | $181 \pm 31$   | 1+                | $B \rightarrow K^- \pi^+ \psi'$ | Belle, LHCb         |

Table 1: The masses, widths and $J^{PC}$ of the $X$, $Y$ and $Z$ states in the $c\bar{c}$ region from the Particle Data Group except for the $Z_c(4100)$. 
are sensitive to the heavy quark masses $m_Q(\mu)$, more precisely speaking, the integrals are sensitive to the energy scales $\mu$, where the $\rho_{QCD}(s, \mu)$ are the QCD spectral densities, the $T^2$ are the Borel parameters, and the $s_0$ are the continuum threshold parameters. In Ref.\[17\], we tentatively assign the $X(3872)$ and $Z_c(3900)$ as the diquark-antidiquark type axialvector tetraquark states, and study them with the QCD sum rules in details, and explore the energy scale dependence of the QCD sum rules for the hidden-charm tetraquark states for the first time \[17\]. In Ref.\[19\], we suggest a formula,

$$
\mu = \sqrt{M_{X/Y/Z}^2 - (2M_c)^2}, \quad (2)
$$

with the effective heavy mass $M_c$ to determine the optimal energy scales, which can be applied to study the hidden-bottom tetraquark states directly with the simple replacement $M_c \rightarrow M_b$. \[20\].

In the scenario of tetraquark states, the $Y$ states, i.e. the $Y(4220), Y(4360), Y(4390), Y(4660)$, can be assigned to be the diquark-antidiquark type tetraquark states. In Ref.\[23\], we introduce a relative $P$-wave between the diquark and antidiquark operators explicitly in constructing the tetraquark currents to study the vector tetraquark states with the QCD sum rules systematically, and obtain the lowest vector tetraquark masses up to now, which support assigning the $Y(4220/4260), Y(4320/4360), Y(4390)$ and $Z_c(4250)$ to be the vector hidden-charm tetraquark states. While novel analysis of the masses and widths of the vector hidden-charm tetraquark states without a relative $P$-wave between the diquark and antidiquark constituents indicate that the $Y(4660)$ can be assigned to be a $[s\bar{c}]\rho[\bar{s}\bar{c}]A - [s\bar{c}]A[\bar{s}\bar{c}]\rho$ type tetraquark state \[24\]. In those studies, the energy scale formula and modified energy scale formula play an important role in enhancing the pole contributions and in improving the convergent behavior of the operator product expansion.

The $X$ states $X(4140), X(4274), X(4350), X(4500)$ and $X(4700)$ are observed in the $J/\psi\phi$ mass spectrum, if their dominant Fock components are tetraquark states, their quark constituents are $\bar{c}\bar{c}s\bar{s}$ rather than $\bar{c}\bar{c}qq$. The QCD sum rules support assigning the $X(3915), X(4140)$, $X(4274), X(4500)$ and $X(4700)$ to be the diquark-antidiquark type tetraquark states \[26\], \[27\], and the decay $X(3915) \rightarrow J/\psi\phi \rightarrow J/\psi\omega$ can take place through the $\phi-\omega$ mixing \[28\].

The $X$ states $X(3860), X(3872), X(3940)$ and $X(4160)$ are observed in the final states $D\bar{D}, D\bar{D}^*, D^+\bar{D}, D^*\bar{D}^*$ or $J/\psi\pi\pi$, if their dominant Fock components are tetraquark states, their constituents are $\bar{c}\bar{c}qq$.

The $Z$ states $Z_c(3900), Z_c(4020), Z_c(4050), Z_c(4055), Z_c(4100), Z_c(4200), Z_c(4250)$ and $Z_c(4430)$ have non-zero electric charge, which prevent them from being the conventional two-quark mesons, they are excellent candidates for the tetraquark states. Those $Z_c$ states have been studied with the QCD sum rules in one way or the other \[10\], \[17\], \[19\], \[22\], \[23\], \[29\].

We usually take the diquarks in color antitriplet as the basic building blocks to construct the tetraquark states. The diquarks operators $\varepsilon^{abc}q^T C T q'_d$ have five structures in Dirac spinor space, where $CT = C\gamma_5$, $C, C\gamma_\mu\gamma_5, C\gamma_\mu$ and $C\sigma_{\mu\nu}$ (or $C\sigma_{\mu\nu}\gamma_5$) for the scalar ($S$), pseudoscalar ($P$), vector ($V$), axialvector ($A$) and tensor ($T$) diquarks, respectively, the $a, b, c$ are color indexes. The tensor diquark states have both $J^P = 1^+$ and $1^-$ components, we project out the $1^+$ and $1^-$ components explicitly, and denote the corresponding axialvector and vector diquarks as $A$ and $V$, respectively.

In this article, we take the scalar, pseudoscalar, axialvector, vector and tensor diquark operators as the basic building blocks to construct the tetraquark currents to study the mass spectrum of the hidden-charm tetraquark states with the QCD sum rules in a comprehensive way, and recalculate the vacuum condensates up to dimension 10 in the operator product expansion and preform updated analysis, and revisit the assignments of the $X$ and $Z$ states in the scenario of tetraquark states. Those hidden-charm tetraquark states may be observed at the BESIII, LHCb, Belle II, CEPC, FCC, ILC in the future, and shed light on the nature of the exotic $X, Y, Z$ states.

The article is arranged as follows: we derive the QCD sum rules for the masses and pole residues of the hidden-charm tetraquark states in section 2; in section 3, we present the numerical results and discussions; section 4 is reserved for our conclusion.
2 QCD sum rules for the hidden-charm tetraquark states

We write down the two-point correlation functions \( \Pi(p) \), \( \Pi_{\mu\nu}(p) \) and \( \Pi_{\mu\nu\alpha\beta}(p) \) in the QCD sum rules,

\[
\Pi(p) = i \int d^4xe^{ipx} \langle 0 | T \left\{ J(x) J^\dagger(0) \right\} \rangle 0 ,
\]
\[
\Pi_{\mu\nu}(p) = i \int d^4xe^{ipx} \langle 0 | T \left\{ J_\mu(x) J_\nu^\dagger(0) \right\} \rangle 0 ,
\]
\[
\Pi_{\mu\nu\alpha\beta}(p) = i \int d^4xe^{ipx} \langle 0 | T \left\{ J_{\mu\nu}(x) J_{\alpha\beta}^\dagger(0) \right\} \rangle 0 ,
\]

where the currents \( J(x) = J_{SS}(x), J_{AA}(x), J_{\bar{A}\bar{A}}(x), J_{VV}(x), J_{\bar{V}\bar{V}}(x), J_{PP}(x), J_\mu(x) = J_{SA}(x), J_{\bar{S}A}(x), J_{PV}(x), J_{\bar{P}V}(x), J_{AP}(x), J_\mu(x) = J_{SA}(x), J_{\bar{S}A}(x), J_{PV}(x), J_{\bar{P}V}(x), J_{AP}(x), \)

\[
J_{SS}(x) = \varepsilon^{ijk} \gamma^m \gamma^5 c^k(x) d^m(x) \gamma_5 \bar{c} T_j(x), \\
J_{AA}(x) = \varepsilon^{ijk} \gamma^m \gamma^5 c^k(x) d^m(x) \gamma_\mu \bar{c} T_j(x), \\
J_{\bar{A}\bar{A}}(x) = \varepsilon^{ijk} \gamma^m \gamma^5 c^k(x) d^m(x) \gamma^\mu \bar{c} T_j(x), \\
J_{VV}(x) = \varepsilon^{ijk} \gamma^m \gamma^5 c^k(x) d^m(x) \gamma^\mu \bar{c} T_j(x), \\
J_{\bar{V}\bar{V}}(x) = \varepsilon^{ijk} \gamma^m \gamma^5 c^k(x) d^m(x) \gamma^\mu \bar{c} T_j(x), \\
J_{PP}(x) = \varepsilon^{ijk} \gamma^m \gamma^5 c^k(x) d^m(x) \gamma^\mu \bar{c} T_j(x),
\]

\[
J_{SA}(x) = \frac{\varepsilon^{ijk} \gamma^m}{\sqrt{2}} \left[ u^T_j(x) \gamma_5 c^k(x) \gamma_\mu \bar{c} T_j(x) - u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) \right],
\]
\[
J_{\bar{S}A}(x) = \frac{\varepsilon^{ijk} \gamma^m}{\sqrt{2}} \left[ u^T_j(x) \gamma_5 c^k(x) \gamma_\mu \bar{c} T_j(x) - u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) \right],
\]
\[
J_{PV}(x) = \frac{\varepsilon^{ijk} \gamma^m}{\sqrt{2}} \left[ u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) - u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) \right],
\]
\[
J_{\bar{P}V}(x) = \frac{\varepsilon^{ijk} \gamma^m}{\sqrt{2}} \left[ u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) - u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) \right],
\]
\[
J_{AP}(x) = \frac{\varepsilon^{ijk} \gamma^m}{\sqrt{2}} \left[ u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) - u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) \right],
\]
\[
J_{\bar{P}A}(x) = \frac{\varepsilon^{ijk} \gamma^m}{\sqrt{2}} \left[ u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) - u^T_j(x) \gamma_5 c^k(x) \gamma^\mu \bar{c} T_j(x) \right].
\]
\[
\begin{align*}
J_{+\mu}^{SA}(x) &= \frac{\varepsilon^{ijk} e_{imn}}{\sqrt{2}} \left[ u^{Tj}(x) C \gamma_5 c^k(x) \bar{d}^m(x) \gamma_\mu C \bar{c}^T(x) + u^{Tj}(x) C \gamma_\mu c^k(x) \bar{d}^m(x) \gamma_5 C \bar{c}^T(x) \right], \\
J_{+\mu}^{AA}(x) &= \frac{\varepsilon^{ijk} e_{imn}}{\sqrt{2}} \left[ u^{Tj}(x) C \gamma_5 c^k(x) \bar{d}^m(x) \sigma_{\mu \nu} C \bar{c}^T(x) + u^{Tj}(x) C \sigma_{\mu \nu} c^k(x) \bar{d}^m(x) \gamma_5 C \bar{c}^T(x) \right], \\
J_{+\mu}^{PV}(x) &= \frac{\varepsilon^{ijk} e_{imn}}{\sqrt{2}} \left[ u^{Tj}(x) C \sigma_{\mu \nu} \gamma_5 c^k(x) \bar{d}^m(x) \gamma_\nu C \bar{c}^T(x) - u^{Tj}(x) C \gamma_\nu \gamma_5 c^k(x) \bar{d}^m(x) \sigma_{\mu \nu} C \bar{c}^T(x) \right], \\
J_{+\mu}^{AA}(x) &= \frac{\varepsilon^{ijk} e_{imn}}{\sqrt{2}} \left[ u^{Tj}(x) C \gamma_5 c^k(x) \bar{d}^m(x) \gamma_\nu C \bar{c}^T(x) + u^{Tj}(x) C \gamma_\nu \gamma_5 c^k(x) \bar{d}^m(x) \gamma_\mu C \bar{c}^T(x) \right], \\
J_{+\mu}^{VV}(x) &= \frac{\varepsilon^{ijk} e_{imn}}{\sqrt{2}} \left[ u^{Tj}(x) C \gamma_\mu \gamma_5 c^k(x) \bar{d}^m(x) \gamma_5 C \bar{c}^T(x) + u^{Tj}(x) C \gamma_5 \gamma_\mu c^k(x) \bar{d}^m(x) \gamma_\gamma C \bar{c}^T(x) \right]. \\
\end{align*}
\]  

(6)

\[
\sigma_{\mu \nu}^t = \frac{1}{2} \left[ \gamma_\mu^t, \gamma_\nu^t \right], \quad \sigma_{\mu \nu}^v = \frac{1}{2} \left[ \gamma_\mu^v, \gamma_\nu^v \right], \quad \gamma_{\mu}^v = \gamma \cdot t_{\mu}, \quad \gamma_{\mu}^t = \gamma = \gamma \cdot t_{\mu}, \quad t^v = (1, \vec{0}), \quad \text{the } i, j, k, m, n \text{ are color indexes, the } C \text{ is the charge conjugation matrix, the subscripts } \pm \text{ denote the positive charge conjugation and negative charge conjugation, respectively, the superscripts or subscripts } P, S, A(A) \text{ and } V(V) \text{ denote the pseudoscalar, scalar, axialvector and vector diquark and antidiquark operators, respectively.} \quad [17, 20, 26, 27, 29].
\]

Under parity transform \( \hat{P} \), the current operators have the properties,
\[
\hat{P} J(x) \hat{P}^{-1} = + J(\bar{x}), \\
\hat{P} J_{\mu}(x) \hat{P}^{-1} = - J_{\mu}(\bar{x}), \\
\hat{P} J_{\mu}^{SA}(x) \hat{P}^{-1} = - J_{\mu}^{\mu S} \bar{A}(\bar{x}), \\
\hat{P} J_{\mu}^{AA/VV}(x) \hat{P}^{-1} = + J_{\mu}^{\mu AA/VV}(\bar{x}),
\]

(7)

where \( x^\mu = (t, \vec{x}) \) and \( \bar{x}^\mu = (t, -\vec{x}) \), and we have neglected other superscripts and subscripts of the current operators.

The current operators \( J(x), J_{\mu}(x) \) and \( J_{\mu \nu}(x) \) have the symbolic quark structure \( \bar{c} \bar{c} u d \) with the isospin \( I = 1 \) and \( I_3 = 1 \), other currents in the isospin multiplets can be constructed analogously, for example, we can write down the corresponding isospin singlet current for the \( J_{SS}(x) \) directly,
\[
J_{SS}^{I=0}(x) = \frac{\varepsilon^{ijk} e_{imn}}{\sqrt{2}} \left[ u^{Tj}(x) C \gamma_5 c^k(x) \bar{u}^m(x) \gamma_5 C \bar{c}^T(x) + d^{Tj}(x) C \gamma_5 c^k(x) \bar{d}^m(x) \gamma_5 C \bar{c}^T(x) \right].
\]

(8)

In the isospin limit, the current operators with the symbolic quark structures \( \bar{c} \bar{c} u d, \bar{c} \bar{c} u d, \bar{c} \bar{c} \bar{u} \bar{d} + \bar{d} d \sqrt{2} \), \( \bar{c} \bar{c} \bar{u} \bar{d} + \bar{d} d \sqrt{2} \) couple potentially to the hidden-charm tetraquark states with degenerate masses, the current operators with \( I = 1 \) and 0 lead to the same QCD sum rules. Thereafter, we will denote the \( Z_c \) states as the isospin triplet, and the \( X \) states as the isospin singlet,
\[
Z_c : \quad \bar{c} \bar{c} u d, \quad \bar{c} \bar{c} u d, \quad \bar{c} \bar{c} \bar{u} \bar{d} - \bar{d} d \sqrt{2}, \\
X : \quad \bar{c} \bar{c} \bar{u} + \bar{d} d \sqrt{2}.
\]

(9)
The current operators with the symbolic quark structures $\bar{c}c\frac{uu-dd}{\sqrt{2}}$ and $\bar{c}d\frac{uu+dd}{\sqrt{2}}$ have definite charge conjugation. In this article, we will assume that the $\bar{c}cd\bar{u}$ type tetraquark states have the same charge conjugation as their neutral charge partners.

Under charge conjugation transform $\hat{C}$, the currents $J(x)$, $J_{\mu}(x)$ and $J_{\mu\nu}(x)$ have the properties,

\[
\hat{C}J(x)\hat{C}^{-1} = +J(x)_{|_{u+d}}, \\
\hat{C}J_{\pm,\mu}(x)\hat{C}^{-1} = \pm J_{\pm,\mu}(x)_{|_{u+d}}, \\
\hat{C}J_{\pm,\mu\nu}(x)\hat{C}^{-1} = \pm J_{\pm,\mu\nu}(x)_{|_{u+d}},
\]

where we have neglected other superscripts and subscripts of the current operators.

At the hadron side, we insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators $J(x)$, $J_{\mu}(x)$ and $J_{\mu\nu}(x)$ into the correlation functions $\Pi(p)$, $\Pi_{\mu\nu}(p)$ and $\Pi_{\mu\nu\alpha\beta}(p)$ to obtain the hadronic representation $[13]$ $[14]$, and isolate the ground state.
state hidden-charm tetraquark contributions,

\[
\Pi(p) = \frac{\lambda_{Z^+}^2}{M_{Z^+}^2 - p^2} + \cdots \\
= \Pi_+(p^2),
\]

\[
\Pi_{\mu\nu}(p) = \frac{\lambda_{Z^+}^2}{M_{Z^+}^2 - p^2} \left( -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) + \cdots \\
= \Pi_+(p^2) \left( -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) + \cdots,
\]

\[
\Pi_{\mu\nu\alpha\beta}^{AA,-}(p) = \frac{\lambda_{Z^+}^2}{M_{Z^+}^2 - p^2} \left( p^2 g_{\mu\alpha} g_{\nu\beta} - p^2 g_{\mu\beta} g_{\nu\alpha} - g_{\mu\alpha} p_\beta p_\alpha - g_{\mu\beta} p_\alpha p_\alpha + g_{\nu\alpha} p_\beta p_\alpha + g_{\nu\beta} p_\alpha p_\alpha \right) \\
+ \frac{\lambda_{Z^-}^2}{M_{Z^-}^2 - p^2} \left( -g_{\mu\alpha} p_\beta p_\alpha - g_{\mu\beta} p_\alpha p_\alpha + g_{\nu\alpha} p_\beta p_\alpha + g_{\nu\beta} p_\alpha p_\alpha \right) + \cdots \\
= \tilde{\Pi}_+(p^2) \left( p^2 g_{\mu\alpha} g_{\nu\beta} - p^2 g_{\mu\beta} g_{\nu\alpha} - g_{\mu\alpha} p_\beta p_\alpha - g_{\mu\beta} p_\alpha p_\alpha + g_{\nu\alpha} p_\beta p_\alpha + g_{\nu\beta} p_\alpha p_\alpha \right) \\
+ \tilde{\Pi}_-(p^2) \left( -g_{\mu\alpha} p_\beta p_\alpha - g_{\mu\beta} p_\alpha p_\alpha + g_{\nu\alpha} p_\beta p_\alpha + g_{\nu\beta} p_\alpha p_\alpha \right),
\]

\[
\Pi_{\mu\nu\alpha\beta}^{AA,+}(p) = \frac{\lambda_{Z^-}^2}{M_{Z^-}^2 - p^2} \left( \frac{g_{\mu\alpha} g_{\nu\beta} + \tilde{g}_{\mu\beta} g_{\nu\alpha}}{2} - \frac{\tilde{g}_{\mu\nu} g_{\beta\alpha}}{3} \right) + \cdots,
\]

\[
\Pi_{\mu\nu\alpha\beta}^{AA,\pm}(p) = \frac{\lambda_{Z^\pm}^2}{M_{Z^\pm}^2 - p^2} \left( \frac{g_{\mu\alpha} g_{\nu\beta} + \tilde{g}_{\mu\beta} g_{\nu\alpha}}{2} - \frac{\tilde{g}_{\mu\nu} g_{\beta\alpha}}{3} \right) + \cdots,
\]

where \( \tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \). We add the superscripts \( \pm \) in the correlation functions \( \Pi_{\mu\nu\alpha\beta}^{AA,-}(p) \), \( \Pi_{\mu\nu\alpha\beta}^{AA,\pm}(p) \) and \( \Pi_{\mu\nu\alpha\beta}^{AA,+}(p) \) to denote the positive and negative charge conjugation, respectively, and the superscripts (subscripts) \( \pm \) in the hidden-charm tetraquark states \( Z_{\pm}^\pm \) (the \( \Pi_+ (p^2) \) and \( \Pi_- (p^2) \) components of the correlation functions) to denote the positive and negative parity contributions, respectively. The correlation functions \( \Pi_{\mu\nu\alpha\beta}^{VV,\pm}(p) \) and \( \Pi_{\mu\nu\alpha\beta}^{AA,\pm}(p) \) have the same tensor structures, we work the explicit expressions of the \( \Pi_{\mu\nu\alpha\beta}^{VV,\pm}(p) \) for simplicity. The pole residues \( \lambda_{Z^\pm} \) or \( \lambda_{Z^\pm} \) current-tetraquark coupling constants are defined by

\[
\langle 0| J(0) Z_{\pm}^+(p) \rangle = \lambda_{Z^\pm}, \\
\langle 0| J_{\mu}(0) Z_{\pm}^+(p) \rangle = \lambda_{Z^\pm} \epsilon_{\mu}, \\
\langle 0| J_{\pm,\mu}^{AA,\pm}(0) Z_{\pm}^+(p) \rangle = \frac{\lambda_{Z^\pm}^2}{M_{Z^\pm}^2 - p^2} \epsilon_{\mu\nu} \epsilon_{\alpha\beta} p^\beta, \\
\langle 0| J_{\pm,\mu}^{AA,\pm}(0) Z_{\pm}^+(p) \rangle = \frac{\lambda_{Z^\pm}^2}{M_{Z^\pm}^2} \epsilon_{\mu\nu} \epsilon_{\alpha\beta} p^\beta, \\
\langle 0| J_{\pm,\mu}^{AA,\pm}(0) Z_{\pm}^-(p) \rangle = \frac{\lambda_{Z^-}}{M_{Z^-}} \epsilon_{\mu\nu} \epsilon_{\alpha\beta} p^\beta, \\
\langle 0| J_{\pm,\mu}^{AA,\pm}(0) Z_{\pm}^-(p) \rangle = \frac{\lambda_{Z^-}}{M_{Z^-}} \epsilon_{\mu\nu} \epsilon_{\alpha\beta} p^\beta, \\
\langle 0| J_{\pm,\mu}^{AA,\pm}(0) Z_{\pm}^+(p) \rangle = \lambda_{Z^+} \epsilon_{\mu}, \\
\langle 0| J_{\pm,\mu}^{AA,\pm}(0) Z_{\pm}^-(p) \rangle = \lambda_{Z^+} \epsilon_{\mu}.
\]
the $\varepsilon_{\mu/\alpha}$ and $\varepsilon_{\mu\nu}$ are the polarization vectors. In this article, we choose the components $\Pi_+(p^2)$ and $p^2\Pi_+(p^2)$ to study the scalar, axialvector and tensor hidden-charm tetraquark states with the QCD sum rules. In Table 2 we present the quark structures and corresponding interpolating currents for the hidden-charm tetraquark states.

At the QCD side, we carry out the operator product expansion for the correlation functions $\Pi(p)$, $\Pi_{\mu\nu}(p)$ and $\Pi_{\mu\nu\alpha\beta}(p)$ up to the vacuum condensates of dimension 10 consistently, and take into account the vacuum condensates $\langle \bar{q}q \rangle$, $\langle \bar{t}g_{\mu\nu}G \rangle$, $\langle \bar{q}q \rangle^2$, $g_s^2\langle \bar{q}q \rangle^2$, $\langle \bar{q}q \rangle \langle \bar{q}g_{\mu\nu}G \rangle$, $\langle \bar{q}g_{\mu\nu}G \rangle^2$ and $\langle \bar{q}q \rangle^2(\bar{s}G\bar{s})$, and obtain the QCD spectral densities $\rho(s)$ through dispersion relation. The contributions of the terms $g_s^2\langle \bar{q}q \rangle^2$ are tiny and neglected in most of the QCD sum rules, as they are not associated with the Borel parameters of the forms $\frac{1}{T^2}$, $\frac{1}{T}$, $\frac{1}{T^3}$, ..., which amplify themselves at small values of the $T^2$ and play an important role in determining the Borel windows. There are terms of the forms $\langle \bar{q}g_{\mu\nu}q \rangle$ and $\langle \bar{q}g_{\mu\nu}s \rangle$ in the full light quark propagators [17], which absorb the gluons emitted from the other quark lines to form $\langle \bar{q}g_sG_{\alpha\beta}^\mu t_{\mu\nu}^a\bar{s}\sigma_{\mu\nu}q \rangle$ and $\langle \bar{q}g_{\mu\nu}qg_{\mu\nu}\bar{D}_\alpha t_{\alpha\beta}^a \rangle$ to make contributions to the mixed condensate and four-quark condensate $\langle \bar{q}\bar{q}g_{\mu\nu}Gq \rangle$ and $g_s^2\langle \bar{q}q \rangle^2$, respectively. The four-quark condensate $g_s^2\langle \bar{q}q \rangle^2$ comes from the terms $\langle \bar{q}g_{\mu\nu}t_{\mu\nu}^aG_{\alpha\beta}s \rangle$, $\langle \bar{q}g_{\mu\nu}D_\alpha t_{\alpha\beta}^a \rangle$ and $\langle \bar{q}g_{\mu\nu}D_\alpha t_{\alpha\beta}^a \rangle$ rather than comes from the perturbative $O(\alpha_s)$ corrections for the four-quark condensate $\langle \bar{q}q \rangle^2$, where $D_\alpha = \partial_\alpha - ig_sG_{\alpha\nu}t_\nu$, $t_\mu = \frac{\lambda_\mu}{2}$, the $\lambda_\mu$ is the Gell-Mann matrix. The strong coupling constant $\alpha_s(\mu)$ appears at the tree level, which is energy scale dependent. One can consult Ref. [17] for the technical details. Furthermore, we recalculate the higher dimensional vacuum condensates using the formula $t_{ij}^a t_{mn}^a = -\frac{3}{4} \delta_{ij}\delta_{mn} + \frac{1}{4} \delta_{ij}\delta_{mn}$, and obtain slightly different expressions compared to the old calculations.

As we have obtained both the hadron spectral representations and the QCD spectral representations, now we match the hadron side with the QCD side of the components $\Pi_+(p^2)$ and $p^2\Pi_+(p^2)$ of the correlation functions $\Pi(p)$, $\Pi_{\mu\nu}(p)$ and $\Pi_{\mu\nu\alpha\beta}(p)$ below the continuum thresholds $s_0$ and perform Borel transform with respect to $P^2 = -p^2$ to obtain the QCD sum rules:

$$\lambda_2^+ \exp \left( -\frac{M_2^+}{T^2} \right) = \int_{4m_c^2}^{t_0} ds \rho(s) \exp \left( -\frac{s}{T^2} \right). \quad (13)$$

The explicit expressions of the QCD spectral densities $\rho(s)$ are available upon request by contacting me via E-mail.

We derive Eq. (13) with respect to $\tau = \frac{1}{T^2}$, and obtain the QCD sum rules for the masses of the scalar, axialvector and tensor hidden-charm tetraquark states $Z_c$ or $X$,

$$M_{Z_c}^2 = - \int_{4m_c^2}^{t_0} ds \frac{d}{ds} \rho(s) \exp \left( -\tau s \right). \quad (14)$$

### 3 Numerical results and discussions

In this article, we neglect the small $u$ and $d$ quark masses. The heavy quark $(M_S)$ mass $m_c(\mu)$ and the vacuum condensates depend on the energy scale $\mu$, so the QCD spectral densities $\rho(s, \mu)$ depend on the energy scale $\mu$, the thresholds $4m_c^2(\mu)$ also depend on the energy scale $\mu$, we cannot extract the masses of the hidden-charm tetraquark states from the energy scale independent QCD sum rules, and have to choose the ideal energy scales to extract the tetraquark masses. The
energy-scale dependence of the input parameters at the QCD side can be written as

\[
\langle \bar{q}q \rangle (\mu) = \langle q\bar{q} \rangle (1\text{GeV}) \left[ \frac{\alpha_s (1\text{GeV})}{\alpha_s (\mu)} \right] \frac{12}{2\pi f}, \\
\langle \bar{q}g_s \sigma Gq \rangle (\mu) = \langle q\bar{q} \rangle (1\text{GeV}) \left[ \frac{\alpha_s (1\text{GeV})}{\alpha_s (\mu)} \right] \frac{2}{2\pi f}, \\
m_c (\mu) = m_c (m_c) \left[ \frac{\alpha_s (\mu)}{\alpha_s (m_c)} \right] \frac{12}{2\pi f}, \\
\alpha_s (\mu) = \frac{1}{b_0 t} \left[ 1 - \frac{b_1 \log t}{b_0^2} + \frac{b_2^2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2} \right],
\]

from the renormalization group equation, where \( t = \log \frac{\mu^2}{\Lambda_{QCD}^2} \), \( b_0 = \frac{33 - 2n_f}{12\pi} \), \( b_1 = \frac{153 - 19n_f}{24\pi} \), \( b_2 = \frac{2857 - 5633n_f + 332n_f^2}{12\pi} \), \( \Lambda_{QCD} = 210 \text{ MeV}, \text{ 292 MeV and 332 MeV for the flavors } n_f = 5, 4 \text{ and } 3 \), respectively \[2, 30\]. In the present work, as the \( c \)-quark is involved, we take the flavor \( n_f = 4 \).

At the initial points, we take the standard values of the vacuum condensates \( \langle \bar{q}q \rangle = -0.24 \pm 0.01 \text{ GeV}^3 \), \( \langle \bar{q}g_s \sigma Gq \rangle = m_0^2 \langle \bar{q}q \rangle \), \( m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2 \), \( \left\langle \frac{\alpha_s \sigma G}{\pi} \right\rangle = (0.33 \text{ GeV})^4 \) at the energy scale \( \mu = 1 \text{ GeV} \) \[13, 14, 15\], and take the \( \overline{\text{MS}} \) mass \( m_c (m_c) = (1.275 \pm 0.025) \text{ GeV} \) from the Particle Data Group \[2\].

The hidden-bottom or hidden-charm tetraquark states can be described by a double-well-potential model, the heavy quark \( Q \) (heavy antiquark \( \bar{Q} \)) serves as a static well potential and attracts the light quark \( q \) (light antiquark \( \bar{q} \)) to form a diquark (antidiquark) in the color antitriplet (triplet) channel. The diquark-antidiquark type tetraquark states, which are excellent candidates for the \( X, Y, Z \) states, are characterized by the effective heavy quark mass \( M_Q \) and the virtuality \( V = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2} \). If we choose the energy scale \[19, 20\],

\[
\mu^2 = V^2 = O(T^2),
\]

then we obtain the formula \( \mu = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2} \) to determine the ideal energy scales for the QCD sum rules. At the ideal energy scales, we can enhance the pole contributions at the hadron side remarkably and improve the convergent behaviors of the operator product expansion at the QCD side remarkably. In the present work, we use the energy scale formula \( \mu = \sqrt{M_{X/Y/Z}^2 - (2M_c)^2} \) with the updated effective \( c \)-quark mass \( M_c = 1.82 \text{ GeV} \) to determine the ideal energy scales for the QCD sum rules \[31\].

At the QCD side, we carry out the operator product expansion at the large space-like region,

\[
x_0 \sim |\vec{x}| \sim \frac{1}{\sqrt{P^2}} \ll \frac{1}{\Lambda_{QCD}},
\]

the anomalous dimensions \( \gamma_{ij} \) of the interpolating currents \( J(x), J_\mu(x) \) and \( J_{\mu\nu}(x) \) lead to a factor multiplying the correlation functions at the QCD side \[32\]. After the Borel transformation, the factor is changed to

\[
\left[ \frac{\alpha_s (P^2)}{\alpha_s (\mu^2)} \right]^{2\gamma_{ij}} \rightarrow \left[ \frac{\alpha_s (T^2)}{\alpha_s (\mu^2)} \right]^{2\gamma_{ij}}.
\]

If we choose \( \mu^2 = O(T^2) \), the factor is neglectful. In fact, the present approach weakens the energy scale dependence of the QCD sum rules, although we cannot obtain energy scale independent QCD sum rules.

The continuum threshold parameters are not completely free parameters, and cannot be determined by the QCD sum rules themselves completely. We often consult the experimental data in
choosing the continuum threshold parameters. The $Z_c(4430)$ can be assigned to be the first radial excitation of the $Z_c(3900)$ according to the analogous decays,

$$Z_c^+(3900) \rightarrow J/\psi \pi^+, \hspace{1cm} Z_c^+(4430) \rightarrow J/\psi \pi^+,$$

and the analogous mass gaps $M_{Z_c(4430)} - M_{Z_c(3900)} = 591$ MeV and $M_{\psi'} - M_{J/\psi} = 589$ MeV from the Particle Data Group \[2\] 33 34 35. Thereafter, we will use the superscripts $\pm$ to denote the electric charges. The energy gap $M_{X(4500)} - M_{X(3915)} = 588$ MeV from the Particle Data Group \[2\], the $X(3915)$ and $X(4500)$ can be assigned as the ground state and the first radial excited state of the axialvector-diquark-axialvector-antidiquark type scalar $csc\bar{s}$ tetraquark states \[25\], the QCD sum rules also support such assignments \[26\]. Recently, the LHCb collaboration performed an angular analysis of the $B^0 \rightarrow J/\psi K^+\pi^-$ decays, and observed two possible structures near $m(J/\psi \pi^-) = 4200$ MeV and 4600 MeV, respectively \[36\]. There have been two tentative assignments for the structure $Z_c(4600)$, the vector tetraquark state with $J^{PC} = 1^{--}$ \[37\] and the first radial excited axialvector tetraquark state with $J^{PC} = 1^{+-}$ \[38\] 39. If the dominant Fock component of the $Z_c(4600)$ is an axialvector tetraquark state, the energy gap between the ground state $Z_c(4020)$ and the first radial excited state $Z_c(4600)$ is about $M_{Z_c(4600)} - M_{Z_c(4020)} = 576$ MeV from the Particle Data Group \[2\].

In the present work, we tentatively choose the continuum threshold parameters as $s_0 = M_Z + 0.58/0.59$ GeV and vary the continuum threshold parameters $s_0$ and Borel parameters $T^2$ to satisfy the following four criteria:

- Pole dominance at the hadron side;
- Convergence of the operator product expansion;
- Appearance of the Borel platforms;
- Satisfying the energy scale formula, via try and error.

The pole dominance at the hadron side and convergence of the operator product expansion at the QCD side are two basic criteria for the QCD sum rules, we should satisfy the two basic criteria to obtain reliable QCD sum rules. Furthermore, we should obtain Borel platforms to avoid additional uncertainties originate from the Borel parameters. The pole contributions (PC) or ground state tetraquark contributions are defined by

$$\text{PC} = \frac{\int_{4m^2}^{s_0} d\rho(s) \exp\left(-\frac{s}{T^2}\right)}{\int_{4m^2}^{\infty} d\rho(s) \exp\left(-\frac{s}{T^2}\right)},$$

while the contributions of the vacuum condensates $D(n)$ of dimension $n$ are defined by

$$D(n) = \frac{\int_{4m^2}^{s_0} d\rho_n(s) \exp\left(-\frac{s}{T^2}\right)}{\int_{4m^2}^{\infty} d\rho(s) \exp\left(-\frac{s}{T^2}\right)},$$

as we only study the ground state contributions. At the QCD side of the correlation functions $\Pi(p), \Pi_{\mu\nu}(p)$ and $\Pi_{\mu\nu\alpha\beta}(p)$, there are four full quark propagators, i.e. two heavy quark propagators and two light quark propagators. If each heavy quark line emits a gluon and each light quark line contributes a quark pair, we obtain a operator $GG\bar{u}\bar{d}$ of dimension 10, so we should calculate the vacuum condensates at least up to dimension 10 to judge the convergent behavior of the operator product expansion. The vacuum condensates $\langle qg, \sigma Gq \rangle^2$ and $\langle q\bar{q} \rangle^2 \langle \alpha G \sqrt{\pi} \rangle^2$ are associated with $\frac{1}{T^2}$, $\frac{1}{T^2}$ and $\frac{1}{T^2}$, they play an important role in determining the Borel windows, although in the Borel windows, they are of minor importance. In the present work, we require the contributions $|D(10)| \sim 1\%$ at the Borel windows.

Finally, we obtain the Borel windows, continuum threshold parameters, energy scales of the QCD spectral densities, pole contributions, and the contributions of the vacuum condensates of
At the QCD side, the contributions of the vacuum condensates of dimension 10 are $|\langle D(10) \rangle| \leq 1\%$ or $\ll 1\%$ except for $|\langle D(10) \rangle| < 2\%$ for the $[uc]_V[dc]_V - [uc]_V[dc]_V$ tetraquark state with $J^{PC} = 1^{++}$, the convergent behavior of the operator product expansion is very good.

We take into account all the uncertainties of the input parameters and obtain the masses and pole residues of the scalar, axialvector, tensor hidden-charm tetraquark states, which are shown explicitly in Table 3. From Tables 3–4, we can see that the energy scale formula $\mu = \sqrt{M_X^2 - (2M_C)^2}$ is well satisfied. In Table 4, we plot the masses of the $[uc]_S[dc]_A - [uc]_A[dc]_S$ and $[uc]_S[dc]_A + [uc]_A[dc]_S$ axialvector tetraquark states with variations of the Borel parameters at much larger ranges than the Borel widows as an example. From the figure, we can see that there appear platforms in the Borel windows indeed. Now the four criteria of the QCD sum rules are all satisfied, and we expect to make reliable predictions.

In Table 5, we present the possible assignments of the ground state hidden-charm tetraquark states. There are one scalar tetraquark candidate with $J^{PC} = 0^{++}$ for the $X(3860)$, one scalar tetraquark candidate with $J^{PC} = 0^{++}$ for the $X(3915)$, one axialvector tetraquark candidate with $J^{PC} = 1^{++}$ for the $X(3872)$, one axialvector tetraquark candidate with $J^{PC} = 1^{++}$ for the $Z_c(3900)$, one axialvector tetraquark candidate with $J^{PC} = 1^{++}$ for the $Z_c(4600)$, three axialvector tetraquark candidates with $J^{PC} = 1^{++}$ for the $Z_c(4020)$ and $Z_c(4055)$, two axialvector tetraquark candidates with $J^{PC} = 1^{++}$ for the $Z_c(4050)$, one tensor tetraquark candidate with $J^{PC} = 2^{++}$ for the $Z_c(4050)$.

In 2008, the Belle collaboration observed two resonance-like structures, which are known as the $Z_c(4050)$ and $Z_c(4250)$ now, in the $\pi^+\chi_c(1)$ mass spectrum in the decays $\bar{B}^0 \rightarrow K^-\pi^+\chi_c(1)$ with a statistical significance exceeds 5$\sigma$. In 2014, the Belle collaboration observed an evidence for the $Z_c(4055)$ in the $\pi^+\psi'$ mass spectrum in the decays $Y(4360) \rightarrow \pi^+\pi^-\psi'$ with a statistical significance of 3.5$\sigma$. The $Z_c^+(4050)$, $Z_c^\pm(4055)$ and $Z_c^\pm(4250)$ have not been confirmed by other experiments yet.

The $Z^+_c(4050)$, $Z^-_c(4055)$ and $Z^\pm_+(4250)$ are not necessary to have the definite charge conjugation $C = +, -$ and $+$, respectively, as they are not the charge conjugation eigenstates. The possible quantum numbers of the $Z^+_c(4050)$ and $Z^+_c(4250)$ are $J^P = 0^+, 1^+, 1^+$ and $2^+$. From Table 3, we can see that there is no candidate for the $Z^+_c(4250)$ in the case of the spin-parity $J^P = 0^+, 1^+$ and $2^+$. In Ref. [23], we observe that the $Z_c(4250)$ can be assigned to be a vector tetraquark state with a relative P-wave between the diquark and antidiquark constituents based on predictions of the QCD sum rules.

From Table 5, we can see that the lowest tetraquark state with $J^{PC} = 1^{+-}$ has a mass about 3.9 GeV, if the energy gap between the ground state and the first radial excited state is about $0.5 - 0.6$ GeV, the first radial excitation of the axialvector tetraquark state has a mass about $4.4 - 4.5$ GeV, which happens to coincide with the experimental mass of the $Z_c(4430)$, see Table 1. The $Z_c(3900)$ and $Z_c(4430)$ can be assigned to be the ground state and the first radial excited axialvector tetraquark states $J^{PC} = 1^{+-}$, respectively [33, 34, 35], the QCD sum rules supports such assignments [35]. The QCD sum rules also support assigning the $Z_c(4600)$ to be the first radial excited state of the $Z_c(4020)$ [33, 34].

The $Z_c(4200)$ has the $J^{PC} = 1^{++}$, there is no room to accommodate it in the scenario of tetraquark state. The pure axialvector tetraquark states have the masses about 3.9 GeV, 4.0 GeV, 4.7 GeV and 5.5 GeV, however, a mixing $[uc]_S[dc]_A - [uc]_A[dc]_S$ or $[uc]_A[dc]_A$ or $[uc]_S[dc]_A - [uc]_A[dc]_S$ or $[uc]_S[dc]_A + [uc]_A[dc]_S + [uc]_V[dc]_V$ or $[uc]_V[dc]_V$ or $[uc]_V[dc]_V$ or $[uc]_V[dc]_V$ or $[uc]_V[dc]_V$ or $[uc]_V[dc]_V$ or $[uc]_V[dc]_V$ or $[uc]_V[dc]_V$ type axialvector tetraquark state may be have a mass about 4.2 GeV. In Ref. [42], we observe that the $Z_c(4200)$ can be assigned to be the color octet-octet type molecule-like state by calculating the mass and width with the QCD sum rules.

If the $Z_c(4100)$ has the $J^{PC} = 0^{++}$, it cannot be a pure scalar tetraquark candidate. In Ref. [23], we observe that if we introduce the mixing effects, a mixing $[uc]_A[dc]_A + [uc]_V[dc]_V$ type scalar
tetraquark state can have a mass about 4.1 GeV. From Table 5 we can see that a mixing $|uc\rangle_S|dc\rangle_S$ or $|uc\rangle_A|dc\rangle_A$ or $(|uc\rangle_S|dc\rangle_S + |uc\rangle_V|dc\rangle_V)$ or $(|uc\rangle_V|dc\rangle_V)\rho$ type scalar tetraquark state with a suitable mixing angle can have a mass about 4.1 GeV.

In Refs. [23, 24], we observe that the lowest vector tetraquark state from the QCD sum rules has a mass about 4.24 ± 0.10 GeV, which lies above the $Z_c(4100)$, the $Z_c(4100)$ is unlikely to be a vector tetraquark state.

There is no room to accommodate the $X(3940)$ and $X(4160)$ in the scenario of tetraquark states, they may be the conventional $\eta_c(3S)$ and $\eta_c(4S)$ with $J^{PC} = 0^{-+}$, respectively [13].

The hidden-charm tetraquark masses obtained in the present work can be confronted to the experimental data at the BESIII, LHCb, Belle II, CEPC, FCC, ILC in the future, and shed light on the nature of the exotic $X$, $Y$, $Z$ particles.

We can take the pole residues $\lambda_Z$ as the basic input parameters to study the two-body strong decays of those hidden-charm tetraquark states,

$$
\begin{align*}
Z^c_1(1^{-+}) & \rightarrow \pi^\pm J/\psi, \pi^\pm \psi', \pi^\pm h_c, \rho^\pm \eta_c, (D\bar{D})^\pm, (D^*\bar{D})^\pm, (D^*\bar{D}^*)^\pm, \\
Z^c_1(0^{++}) & \rightarrow \pi^\pm \eta_c, \pi^\pm \chi_{c1}, \rho^\pm J/\psi, \rho^\pm \psi', (D\bar{D})^\pm, (D^*\bar{D})^\pm, (D^*\bar{D}^*)^\pm, \\
Z^c_1(2^{++}) & \rightarrow \pi^\pm \eta_c, \pi^\pm \chi_{c1}, \rho^\pm J/\psi, \rho^\pm \psi', (D\bar{D})^\pm, (D^*\bar{D})^\pm, (D^*\bar{D}^*)^\pm,
\end{align*}
$$

$$
\begin{align*}
Z^0_1(1^{-+}) & \rightarrow \pi^0 J/\psi, \pi^0 \psi', \pi^0 h_c, \rho^0 \eta_c, (D\bar{D})^0, (D^*\bar{D})^0, (D^*\bar{D}^*)^0, \\
Z^0_1(0^{++}) & \rightarrow \pi^0 \eta_c, \pi^0 \chi_{c1}, \rho^0 J/\psi, \rho^0 \psi', (D\bar{D})^0, (D^*\bar{D})^0, (D^*\bar{D}^*)^0, \\
Z^0_1(2^{++}) & \rightarrow \pi^0 \eta_c, \pi^0 \chi_{c1}, \rho^0 J/\psi, \rho^0 \psi', (D\bar{D})^0, (D^*\bar{D})^0, (D^*\bar{D}^*)^0,
\end{align*}
$$

$$
\begin{align*}
X(1^{++}) & \rightarrow \eta J/\psi, \eta \psi', \eta h_c, \omega \eta_c, (D\bar{D})^*, (D^*\bar{D})^0, (D^*\bar{D}^*)^0, \\
X(0^{++}) & \rightarrow \eta \eta_c, \eta \chi_{c1}, \omega J/\psi, \omega \psi', (D\bar{D})^0, (D^*\bar{D})^0, (D^*\bar{D}^*)^0, \\
X(1^{++}) & \rightarrow \eta \chi_{c1}, \omega J/\psi, \omega \psi', (D\bar{D})^*, (D^*\bar{D})^0, (D^*\bar{D}^*)^0, \\
X(2^{++}) & \rightarrow \eta \eta_c, \eta \chi_{c1}, \omega J/\psi, \omega \psi', (D\bar{D})^0, (D^*\bar{D})^0, (D^*\bar{D}^*)^0,
\end{align*}
$$

with the three-point QCD sum rules or the light-cone QCD sum rules, and obtain the partial decay widths to diagnose the nature of the $Z_c$ and $X$ states. For example, we tentatively assign the $Z^c_1(3900)$ to be the $|uc\rangle_S|dc\rangle_A$ or $|uc\rangle_A|dc\rangle_S$ type axialvector tetraquark state, study the two-body strong decays $Z^c_1(3900) \rightarrow J/\psi \pi^+$, $\eta_c \rho^+$, $D^+ \bar{D}^{*0}$, $\bar{D}^0 D^{*+}$ with the QCD sum rules based on solid quark-hadron duality, and produce the experimental value of the total width [14]. Experimentally, the BESIII collaboration measured the ratios of the partial widths of the decays $Z_c(3900/4020) \rightarrow \eta_c \rho$, $J/\psi \pi$ at the 90% C.L. [15].

4 Conclusion

In this article, we take the pseudoscalar, scalar, axialvector, vector, tensor charmed (anti) diquark operators as the basic constituents, and construct the scalar, axialvector and tensor hidden-charm tetraquark currents to study the mass spectrum of the ground state hidden-charm tetraquark states with the QCD sum rules in a comprehensive way. In calculations, we carry out the operator product expansion up to the vacuum condensates of dimension 10 to obtain the QCD spectral densities, and use the energy scale formula $\mu = \sqrt{M^2_{X/Y/Z} - (2M_c)^2}$ to determine the ideal energy scales. The present predictions support assigning the $X(3860)$ to be the $[qc]_S[\bar{qc}]_S$ type scalar tetraquark state with $J^{PC} = 0^{++}$, assigning the $X(3915)$ to be the $[qc]_A[\bar{qc}]_A$ type scalar tetraquark state with $J^{PC} = 0^{++}$, assigning the $X(3872)$ to be the $[qc]_S[\bar{qc}]_A + [qc]_A[\bar{qc}]_S$ type axialvector tetraquark state.
Table 3: The Borel parameters, continuum threshold parameters, energy scales of the QCD spectral densities, pole contributions, and the contributions of the vacuum condensates of dimension 10 for the ground state hidden-charm tetraquark states.

| $Z_{uc}(X_{uc})$ | $J^{PC}$ | $T^2$(GeV$^2$) | $\sqrt{s_{0}}$(GeV) | $\mu$(GeV) | pole | $|D|$(10$^{-2}$) |
|------------------|---------|----------------|-------------------|-----------|------|----------------|
| $[uc]_{S}[dc]_{S}$ | 0$^{++}$ | 2.7 - 3.1 | 4.40 ± 0.10 | 1.3 | (40 - 63)% | < 1% |
| $[uc]_{A}[dc]_{A}$ | 0$^{++}$ | 2.8 - 3.2 | 4.52 ± 0.10 | 1.5 | (40 - 63)% | ≤ 1% |
| $[uc]_{\overline{A}}[dc]_{\overline{A}}$ | 0$^{++}$ | 3.1 - 3.5 | 4.55 ± 0.10 | 1.6 | (42 - 62)% | < 1% |
| $[uc]_{V}[dc]_{V}$ | 0$^{++}$ | 3.7 - 4.1 | 5.22 ± 0.10 | 2.9 | (41 - 60)% | < 1% |
| $[uc]_{\overline{V}}[dc]_{V}$ | 0$^{++}$ | 4.9 - 5.7 | 5.90 ± 0.10 | 3.9 | (41 - 61)% | < 1% |
| $[uc]_{p}[dc]_{p}$ | 0$^{++}$ | 5.2 - 6.0 | 6.03 ± 0.10 | 4.1 | (40 - 60)% | < 1% |
| $[uc]_{S}[dc]_{A} - [uc]_{A}[dc]_{S}$ | 1$^{+-}$ | 2.7 - 3.1 | 4.40 ± 0.10 | 1.4 | (40 - 63)% | < 1% |
| $[uc]_{A}[dc]_{A} - [uc]_{\overline{A}}[dc]_{\overline{A}}$ | 1$^{+-}$ | 3.3 - 3.7 | 4.60 ± 0.10 | 1.7 | (40 - 59)% | < 1% |
| $[uc]_{S}[dc]_{\overline{A}} - [uc]_{\overline{A}}[dc]_{S}$ | 1$^{+-}$ | 3.3 - 3.7 | 4.60 ± 0.10 | 1.7 | (40 - 59)% | < 1% |
| $[uc]_{V}[dc]_{V} + [uc]_{\overline{V}}[dc]_{\overline{V}}$ | 1$^{+-}$ | 3.7 - 4.1 | 5.25 ± 0.10 | 2.9 | (41 - 60)% | < 1% |
| $[uc]_{V}[dc]_{p} - [uc]_{\overline{V}}[dc]_{p}$ | 1$^{+-}$ | 5.1 - 5.9 | 6.00 ± 0.10 | 4.1 | (41 - 60)% | < 1% |

Figure 1: The masses of the $[uc]_{S}[dc]_{A} - [uc]_{A}[dc]_{S}$(I) and $[uc]_{S}[dc]_{A} + [uc]_{A}[dc]_{S}$(II) axialvector tetraquark states with variations of the Borel parameters $T^2$. 
Table 4: The masses and pole residues of the ground state hidden-charm tetraquark states.

| $Z_c(X_c)$                     | $J^{PC}$ | $M_Z$(GeV)   | $\lambda_Z$(GeV$^2$) |
|--------------------------------|----------|--------------|----------------------|
| $[uc]_S[dc]_S$                 | 0++      | 3.88 ± 0.09  | (2.07 ± 0.35) × 10^{-2} |
| $[uc]_A[dc]_A$                 | 0++      | 3.95 ± 0.09  | (4.49 ± 0.77) × 10^{-2} |
| $[uc]_\bar{A}[dc]_{\bar{A}}$  | 0++      | 3.98 ± 0.08  | (4.30 ± 0.63) × 10^{-2} |
| $[uc]_V[dc]_V$                 | 0++      | 4.65 ± 0.09  | (1.35 ± 0.22) × 10^{-1} |
| $[uc]_{\bar{V}}[dc]_{\bar{V}}$| 0++      | 5.35 ± 0.09  | (4.87 ± 0.51) × 10^{-1} |
| $[uc]_P[dc]_P$                 | 0++      | 5.49 ± 0.09  | (2.11 ± 0.21) × 10^{-1} |
| $[uc]_S[dc]_A - [uc]_A[dc]_S$ | 1++      | 3.90 ± 0.08  | (2.09 ± 0.33) × 10^{-2} |
| $[uc]_A[dc]_A - [uc]_{\bar{A}}[dc]_{\bar{A}}$ | 1++ | 4.02 ± 0.09  | (3.00 ± 0.45) × 10^{-2} |
| $[uc]_A[dc]_A - [uc]_{\bar{A}}[dc]_{\bar{A}}$ | 1++ | 4.01 ± 0.09  | (3.02 ± 0.45) × 10^{-2} |
| $[uc]_V[dc]_V + [uc]_{\bar{V}}[dc]_{\bar{V}}$ | 1++ | 4.66 ± 0.10  | (1.18 ± 0.21) × 10^{-1} |
| $[uc]_P[dc]_V + [uc]_{\bar{V}}[dc]_{\bar{V}}$ | 1++ | 5.46 ± 0.09  | (1.72 ± 0.17) × 10^{-1} |
| $[uc]_S[dc]_A + [uc]_A[dc]_S$ | 1+++     | 3.91 ± 0.08  | (2.10 ± 0.34) × 10^{-2} |
| $[uc]_S[dc]_A + [uc]_A[dc]_S$ | 1+++     | 4.02 ± 0.09  | (3.01 ± 0.45) × 10^{-2} |
| $[uc]_V[dc]_V - [uc]_{\bar{V}}[dc]_{\bar{V}}$ | 1+++ | 4.08 ± 0.09  | (3.67 ± 0.67) × 10^{-2} |
| $[uc]_A[dc]_A + [uc]_{\bar{A}}[dc]_{\bar{A}}$ | 1+++ | 5.19 ± 0.09  | (2.12 ± 0.24) × 10^{-1} |
| $[uc]_P[dc]_V - [uc]_{\bar{V}}[dc]_{\bar{V}}$ | 1+++ | 5.46 ± 0.09  | (1.89 ± 0.19) × 10^{-1} |

Table 5: The possible assignments of the ground state hidden-charm tetraquark states, the isospin limit is implied.
state with $J^{PC} = 1^{++}$, assigning the $Z_c(3900)$ to be the $[uc]_S[dc]_A - [uc]_A[dc]_S$ type axialvector tetraquark state with $J^{PC} = 1^{+-}$, assigning the $Z_c(4020)$ and $Z_c(4055)$ to be the $[uc]_A[dc]_A$ type, $[uc]_S[dc]_A - [uc]_A[dc]_S$ type or $[uc]_A[dc]_A - [uc]_A[dc]_\bar{A}$ type axialvector tetraquark states with $J^{PC} = 1^{+-}$, assigning the $Z_c(4600)$ to be the $[uc]_V[dc]_V + [uc]_V[dc]_\bar{V}$ type axialvector tetraquark state with $J^{PC} = 1^{+-}$, or the first radial excited state of the $Z_c(4020)$, assigning the $Z_c(4050)$ to be the $[uc]_S[dc]_A + [uc]_A[dc]_S$ type or $[uc]_A[dc]_A - [uc]_A[dc]_\bar{A}$ type axialvector tetraquark state with $J^{PC} = 1^{++}$ or $[uc]_A[dc]_A$ type tensor tetraquark state with $J^{PC} = 2^{++}$, assigning the $Z_c(4430)$ to be the first radial excited state of the $Z_c(3900)$. More experimental data and theoretical work are still needed to make unambiguous assignments. There is no room to accommodate the $X(3940)$, $X(4160)$, $Z_c(4100)$, $Z_c(4200)$ in the scenario of tetraquark states. The $X(3940)$ and $X(4160)$ may be the conventional $\eta_c(3S)$ and $\eta_c(4S)$ state with $J^{PC} = 0^{--}$, respectively. While the $Z_c(4100)$ may be a mixing axialvector tetraquark state with $J^{PC} = 0^{++}$, the $Z_c(4200)$ may be axialvector molecule-like state with $J^{PC} = 1^{--}$. The predicted tetraquark masses can be confronted to the experimental data in the future at the BESIII, LHCb, Belle II, CEPC, FCC, ILC.

Acknowledgements

This work is supported by National Natural Science Foundation, Grant Number 11775079.

References

[1] S. K. Choi et al, Phys. Rev. Lett. 91 (2003) 262001.
[2] M. Tanabashi et al, Phys. Rev. D98 (2018) 030001.
[3] R. Aaij et al, Eur. Phys. J. C78 (2018) 1019.
[4] H. X. Chen, W. Chen, X. Liu and S. L. Zhu, Phys. Rept. 639 (2016) 1.
[5] R. F. Lebed, R. E. Mitchell and E. S. Swanson, Prog. Part. Nucl. Phys. 93 (2017) 143.
[6] A. Esposito, A. Pilloni and A. D. Polosa, Phys. Rept. 668 (2017) 1.
[7] F. K. Guo, C. Hanhart, U. G. Meissner, Q. Wang, Q. Zhao and B. S. Zou, Rev. Mod. Phys. 90 (2018) 015004.
[8] A. Ali, J. S. Lange and S. Stone, Prog. Part. Nucl. Phys. 97 (2017) 123.
[9] S. L. Olsen, T. Skwarnicki and D. Zieminska, Rev. Mod. Phys. 90 (2018) 015003.
[10] R. M. Albuquerque, J. M. Dias, K. P. Khemchandani, A. M. Torres, F. S. Navarra, M. Nielsen and C. M. Zanetti, J. Phys. G46 (2019) 093002.
[11] Y. R. Liu, H. X. Chen, W. Chen, X. Liu and S. L. Zhu, Prog. Part. Nucl. Phys. 107 (2019) 237.
[12] N. Brambilla, S. Eidelman, C. Hanhart, A. Nefediev, C. P. Shen, C. E. Thomas, A. Vairo and C. Z. Yuan, arXiv:1907.07583.
[13] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B147 (1979) 385; Nucl. Phys. B147 (1979) 448.
[14] L. J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rept. 127 (1985) 1.
[15] P. Colangelo and A. Khodjamirian, hep-ph/0010175.
[16] R. D. Matheus, S. Narison, M. Nielsen and J. M. Richard, Phys. Rev. D75 (2007) 014005.
[17] Z. G. Wang and T. Huang, Phys. Rev. D89 (2014) 054019.
[18] J. R. Zhang and M. Q. Huang, Phys. Rev. D83 (2011) 036005; J. R. Zhang, Phys. Rev. D87 (2013) 116004.
[19] Z. G. Wang, Eur. Phys. J. C74 (2014) 2874.
[20] Z. G. Wang and T. Huang, Nucl. Phys. A930 (2014) 63; Z. G. Wang, Eur. Phys. J. C79 (2019) 489.
[21] W. Chen and S. L. Zhu, Phys. Rev. D83 (2011) 034010.
[22] C. F. Qiao and L. Tang, Eur. Phys. J. C74 (2014) 2810.
[23] Z. G. Wang, Eur. Phys. J. C78 (2018) 933; Z. G. Wang, Eur. Phys. J. C79 (2019) 29.
[24] Z. G. Wang, Eur. Phys. J. C78 (2018) 518; Z. G. Wang, Eur. Phys. J. C79 (2019) 184.
[25] S. S. Agaev, K. Azizi and H. Sundu, Phys. Rev. D93 (2016) 074002; H. Sundu, S. S. Agaev and K. Azizi, Eur. Phys. J. C79 (2019) 215.
[26] Z. G. Wang, Eur. Phys. J. C77 (2017) 78; Z. G. Wang, Eur. Phys. J. A53 (2017) 19.
[27] Z. G. Wang and Z. Y. Di, Eur. Phys. J. C79 (2019) 72; Z. G. Wang, arXiv:1812.04503
[28] R. F. Lebed and A. D. Polosa, Phys. Rev. D93 (2016) 094024.
[29] Z. G. Wang, arXiv:1903.03468
[30] S. Narison and R. Tarrach, Phys. Lett. 125 B (1983) 217.
[31] Z. G. Wang, Eur. Phys. J. C76 (2016) 387.
[32] B. L. Ioffe, Nucl. Phys. B188 (1981) 317.
[33] L. Maiani, F. Piccinini, A. D. Polosa and V. Riquer, Phys. Rev. D89 (2014) 114010.
[34] M. Nielsen and F. S. Navarra, Mod. Phys. Lett. A29 (2014) 1430005.
[35] Z. G. Wang, Commun. Theor. Phys. 63 (2015) 325.
[36] R. Aaij et al, Phys. Rev. Lett. 122 (2019) 152002.
[37] Z. G. Wang, Int. J. Mod. Phys. A34 (2019) 1950110.
[38] H. X. Chen and W. Chen, Phys. Rev. D99 (2019) 074022.
[39] Z. G. Wang, arXiv:1901.10741
[40] R. Mizuk et al, Phys. Rev. D78 (2008) 072004.
[41] X. L. Wang et al, Phys. Rev. D91 (2015) 112007.
[42] Z. G. Wang, Int. J. Mod. Phys. A30 (2015) 1550168.
[43] J. L. Rosner, AIP Conf. Proc. 815 (2006) 218; K. T. Chao, Phys. Lett. B661 (2008) 348.
[44] Z. G. Wang and J. X. Zhang, Eur. Phys. J. C78 (2018) 14.
[45] C. Z. Yuan, Int. J. Mod. Phys. A33 (2018) 1830018.