Prethermalization in confined spin chains

Stefan Birnkammer,1,2 Alvise Bastianello,1,2 and Michael Knap1,2

1Department of Physics, Technical University of Munich, 85748 Garching, Germany
2Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, D-80799 München, Germany

Unconventional nonequilibrium phases with restricted correlation spreading and slow entanglement growth have been proposed to emerge in systems with confined excitations, calling their thermalization dynamics into question. Here, we investigate the many-body dynamics of a confined Ising spin chain, in which domain walls in the ordered phase form bound states reminiscent of mesons. We show that the thermalization dynamics after a quantum quench exhibits multiple stages with well separated time scales. The system first relaxes towards a prethermal state, described by a Gibbs ensemble with conserved meson number. The prethermal state arises from rare events in which mesons are created in close vicinity, leading to an avalanche of scattering events. Only at much later times a true thermal equilibrium is achieved in which the meson number conservation is violated by a mechanism akin to the Schwinger effect.

Quantum many-body systems far from thermal equilibrium arise naturally in a variety of disciplines, ranging from cosmology to condensed matter physics. In recent years, there has been an intense focus on investigating the dynamical evolution of quantum many-body systems which are well isolated from their environment [1, 2]. This research has been fueled by the progress in engineering coherent and interacting quantum many-body systems which made it possible to experimentally study unconventional relaxation dynamics. A recent interest is to explore phenomena from high-energy physics with synthetic quantum systems in a controlled way; for example lattice gauge theories have been realized [3–9] and phenomena akin to quark confinement have been explored [3, 6, 10–12].

An archetypical model to study confinement phenomena in condensed matter settings is the Ising model with both transverse and longitudinal magnetic fields [13–18]. In this model, domain walls—interpreted as quarks—are pairwise confined into mesons by a weak longitudinal field; see Fig. 1 (a). Confinement strongly affects the relaxation dynamics of the system, leading to unconventional spreading of correlations and slow entanglement growth [19–21]. A key feature of the model is the long lifetime of mesons, ascribed to a strong suppression of the Schwinger mechanism [22–27], which creates new quarks from the energy stored in the confining force and vice versa. Hence, except for some fine-tuned regimes [28–30], mesons are stable excitations. Due to the approximate conservation of the meson number, various exotic dynamical phenomena have been proposed, including Wannier-Stark localization [31–33], time crystals [34], quantum scars [35, 36]. Even though the realization of these phenomena does not require particular fine tuning, they arise in a regime in which interactions between mesons are extremely unlikely. The few-meson scattering has been recently considered [28, 29, 37, 38], but so far, apart from special limits [39], the full many-body dynamics of confined systems has not been addressed. The Ising model with longitudinal and transverse fields is non-integrable [14] and features a Wigner-Dyson level statistics of the eigenenergies [40], hence one would expect on general grounds [41–43] that the system thermalizes at late times and interactions between mesons can become relevant. Given this wealth of unconventional nonequilibrium phenomena and the discrepancy with the expected thermalization in non-integrable models, it is important to un-
derstand the mechanisms of relaxation and their timescales.

In this work, we investigate the relaxation dynamics of the Ising model with both transverse and longitudinal fields following a quantum quench. Two scenarios could be envisioned for the thermalization process. The first one is that the Schwinger effect, leading to a violation of the meson number conservation, could be the only responsible mechanism for equilibration, causing an extremely slow thermalization dynamics. A more exciting, second scenario involves an intermediate thermalization of the mesons themselves. Here, we show that indeed the second scenario is realized. Generic states first relax to a Gibbs ensemble in which the meson number is conserved up to extremely long times; Fig. 1 (b) and (c). We show that relaxation to this state is activated through rare events in which two mesons are produced in their vicinity, initiating an avalanche of scattering events. This prethermal state can then be understood as a dilute thermal gas of mesons with conserved meson density. Only at exponentially long times, the Schwinger mechanism causes a full thermalization of the system coupling sectors with different number of mesons.

Model and protocol.— The Ising chain with both transverse and longitudinal fields is described by the Hamiltonian

$$H = - \sum_j [\hat{\sigma}^z_{j+1} \hat{\sigma}^z_j + h_\perp \hat{\sigma}^x_j + h_\parallel \hat{\sigma}^y_j].$$

In the pure transverse-field regime ($h_\parallel = 0$) the model is equivalent to non-interacting fermions and exhibits spontaneous $\mathbb{Z}_2$-symmetry breaking for $|h_\perp| < 1$ in the thermodynamic limit. For $h_\perp \to 0$, the two degenerate ground states $|\text{GS}_\pm\rangle$ are simple product states of maximally positive/negative magnetization, which are renormalized for finite transverse field, such that $\langle \text{GS}_+ | \hat{\sigma}^z_j | \text{GS}_+ \rangle = \pm \bar{\sigma}$, with $\bar{\sigma} = (1 - h_\perp^2)^{1/8}$ [44]. In this phase, the fermionic modes are interpreted as (dressed) domain walls (or kinks) relating the two vacua and are thus of topological nature. A small longitudinal field $h_\perp > 0$ lifts the ground state degeneracy, leading to a low-energy “true vacuum” and a high-energy “false vacuum,” and induces a pairwise linear potential $\propto 2h_\parallel \bar{\sigma}$ between kinks; Fig. 1 (a).

We consider the following quantum quench [19]: The system is initialized for $h_\parallel = 0$ in one of the two degenerate ground states (specifically, we select $\langle \hat{\sigma}^z \rangle > 0$) and then brought out of equilibrium by suddenly changing both the transverse and the longitudinal field components. Building on the knowledge of quenches in the transverse field only [45], one can argue that fermions are locally produced in pairs with opposite momenta [19], each of them having a dispersion $\epsilon(k) = 2\sqrt{(\cos k - h_\perp)^2 + \sin^2 k}$. However, pairs of fermions are then confined due to the finite longitudinal field $h_\parallel \neq 0$. For weak quenches, very few excitations are produced and, due to translational invariance, mesons are mostly initialized at rest and are well-isolated. Their stability is guaranteed by the strong suppression of fermion number-changing processes. In the case of small transverse field ($|h_\perp| < 1/3$) two fermions cannot energetically couple to the four-fermion sector without using the energy stored in the false-vacuum string. Hence, this process resembles the false vacuum decay, whose lifetime has been shown to scale exponentially with $h_\perp^{-1}$ [22]. Even in the less restricted regime where the scattering of two fermions into four is energetically allowed ($1/3 < |h_\perp| < 1$), the cross section is induced by the weak longitudinal term, leading to a meson lifetime that scales algebraically in the longitudinal field $h_\parallel^{-3}$ [46]. To confirm this expectation, we perform tensor network simulations [47–49] based on the TenPy library [49] of the quantum quench and compute the meson density $\rho$; Fig. 1 (c). In the limit of small $h_\parallel$ the meson number is conserved on the numerically accessible time scales.

Excitation spectrum and thermodynamics.— Assuming that the meson number is conserved, we now study the thermodynamics of a gas of mesons, which is expected to describe the prethermal state. In the dilute regime, the mean-free path is much larger than the typical meson length. In a first approximation, we therefore neglect the effects that the size of the meson has on the thermodynamics. A convenient starting point is the semiclassical limit of a single meson, in which one treats the two fermions as point-like particles with coordinates $(x_{1,2}, k_{1,2})$ governed by the classical Hamiltonian

$$\mathcal{H} = \epsilon(k_1) + \epsilon(k_2) + 2h_\perp \bar{\sigma}|x_1 - x_2|.$$

The semiclassical approximation holds when interactions cannot resolve the discreteness of the underlying lattice, i.e., for $h_\perp \ll 1$. Hence, the position of the particle $x_{1,2}$ is a continuous variable. In the reduced two-body problem, the total momentum $k = k_1 + k_2$ of a meson is conserved, thus the dynamics of the relative coordinates $(q = (k_1 - k_2)/2, x = \ldots)$.
which are self-consistently determined by the meson-meson scattering leads to a prethermal Gibbs equilibrium. We consider a chain of length \( L \) that is tackled by transforming to action-angle variables \( (J, \phi) \) \[50\], where \( J \equiv \oint \mathcal{H}_{0}(q, x) \, dq \) labels the phase-space orbits of the classical motion and \( \phi \) is a periodic variable \( \phi \in [0, 1] \), leading to \( P(E, k) = e^{-\beta(E - \mu)} \int \frac{dj}{(2\pi)^{2}} \delta(\mathcal{E}(J, k) - E) \). Leaving the classical limit, the energy levels become quantized according to the Bohr-Sommerfeld rule \( \pi \sqrt{E_{j}^{n}} = \hbar \) \[15\]. Deep in the quantum regime, the classical Hamiltonian (2) can be directly promoted to a quantum Hamiltonian and explicitly diagonalized \[21\].

Away from the dilute regime mesons should be treated as hard-rod like extended objects and their thermodynamics needs to be suitably modified. To this end, we consider mesons as hard-rods of fixed length \( \ell(J, k) \), the latter being the meson length averaged over one oscillation period. Within this assumption, \( P(E, k) \) gets modified as

\[
P(E, k) = \frac{1}{1 - \rho M} e^{-\beta(E - \mu)} \int \frac{dJ}{(2\pi)^{2}} \delta(\mathcal{E}(J, k) - E) e^{-\rho M(1 - \rho M)^{-1}}
\]

with \( \rho \) the meson density and \( M \) the average meson length, which are self-consistently determined by \( P(E, k) \); see also supplementary material \[51\]. The meson coverage \( \rho M \) is connected to the magnetization of the Ising chain as \( \rho M = 1/2 - \rho \). Prethermalization of quantum mesons.— In order to show that meson-meson scattering leads to a prethermal Gibbs ensemble, we numerically calculate the time evolution in the subspace with a fixed number of mesons using exact diagonalization. We consider a chain of length \( L \) with periodic boundary conditions, and focus on the limit \( 0 < h_{\parallel} \ll 1 \) where fermions can be identified with domain walls. In this regime, \( \sigma \to 1 \) and the confinement strength is determined by \( h_{\perp} / h_{\parallel} \).

We initialize the state in the form of moving wave packets and probe relaxation by tracking the meson momentum distribution; Fig. 2. Whereas for two mesons, energy and momentum conservation inhibits thermalization, see supplementary material \[51\], for three mesons we observe the relaxation to the prethermal Gibbs ensemble; Eq. (3). Two-body scattering processes between different energy bands are responsible for the thermalization; Fig. 2 (b). For wave packets which are initialized with energies below the second band thermalization is largely suppressed, as two-body collisions become elastic due to momentum-energy conservation and three-body scattering events are unlikely; Fig. 2 (a). We provide additional details on the thermalization in the supplementary material \[51\].

Prethermalization through rare events.— Equipped with the meson conservation, the thermodynamics of the prethermal state, and the quantum thermalization of a few mesons, we now study the full quench protocol. In order to access large system sizes and time scales, we use the Truncated Wigner Approach \[52\] on the quantum dynamics projected in the fermion number conserving sector. In order to study the relaxation dynamics, a precise knowledge of the excitation content of the initial state is crucial. The quantum quench of both the longitudinal and the transverse field excites dilute pairs of fermions with opposite momenta \( (k, -k) \) at density \( n(k) \), which can be computed from the quench parameters \[19, 21\]. These pairs of fermions are then confined into mesons by the longitudinal field according to Eq. (2).

For small quenches within the ferromagnetic phase, the density of mesons is low. Typically, mesons are excited far apart and are thus isolated and at rest. In this scenario, inter-meson scattering and thermalization seems impossible. However, considering only the typical behavior is misleading, as the probability of creating two nearby mesons is never strictly zero. To obtain a rough estimate, we consider the maximum size \( d_{\text{max}} \) that a meson can have when fermions are initially created at the same position, which is given by \( d_{\text{max}} = 4h_{\parallel} / (h_{\parallel} \sigma) \), and compare it with the meson density \( \rho \). On a finite volume \( L \), the probability \( P(L) \) that \( N = L \rho \) randomly distributed particles are placed at distance larger than \( d_{\text{max}} \) is \( P(L) = \frac{1}{2\pi} \prod_{j=1}^{N} (L - j d_{\text{max}}) \approx e^{-L \sigma d_{\text{max}}/2} \). No matter how small the excitation density \( \sigma \) is, eventually in the thermodynamic limit the probability that all the excited mesons are far apart vanishes. Crucially, the rare near-by mesons scatter and acquire a finite velocity. These moving mesons subsequently trigger an avalanche, that hits the surrounding mesons, and initiates prethermalization; see Fig. 3 (a) for a typical meson state. 

![FIG. 3. Prethermalization through rare events. (a) Rare events in which mesons are in close vicinity lead to an avalanche effect activating dynamics in the entire meson ensemble. (b) For comparably small values of \( h_{\parallel} \) semiclassical results for the average meson coverage \( \rho M \) agree well with exact quantum evolution obtained from tensor network techniques. (Inset) The semiclassical analysis reveals relaxation towards a prethermal plateau (red), which is distinct from the thermal state in the absence of meson conservation (green dashed). Side panels: Relaxation of the semiclassical ensemble is also reflected in the decay of the the momentum distribution \( P(k) \) at \( k = 0 \).](image-url)
The full numerical results agree with this picture; the late time relaxation of the average meson length $M$ to the prethermal plateau is well-described by the phenomenological prediction $\langle O(t) \rangle = O_{\text{PreTh}} + \Delta F(vd/\rho^2)$. The quantity $vd$ is obtained from a fit to the data. (b) The normalized one-meson phase space occupation relaxes to a prethermal ensemble (prethermal: red continuous line; thermal: green dashed line; numerics: blue shaded area). Finite-density corrections are captured by the hard-rods approximation and cause an additional peak in the energy distribution $P(E)$ (bottom). The relative difference in the meson densities between the thermal and prethermal ensemble $\Delta \rho = (\rho_{\text{PreTh}} - \rho_{\text{Th}})/\rho_{\text{Th}}$ are $\Delta \rho = 0.16$ and $\Delta \rho = O(10^{-3})$ for $h_\parallel = 0.015$ (top) and $h_\parallel = 0.001$ (bottom), respectively.

The density dependence of the prethermalization timescale $t_{\text{PreTh}}$ can be understood as follows. Initially, the configuration consists of large regions of average size $\sim (\rho^2 d_{\text{max}}^2)^{-1}$ with mesons at rest separated by growing thermalizing domains. Hence, we estimate prethermalizing regions to cover the whole system on a typical time $t^* \sim (\rho^2 d_{\text{max}} v)^{-1}$, where $v$ is a typical velocity. Once all mesons are set in motion, two-body inelastic scatterings drive the relaxation to the system on a timescale $t^{**} \sim (yv)^{-1}$. At low excitation density, $t^* \gg t^{**}$, hence $t_{\text{PreTh}} \sim (\rho^2 d_{\text{max}} v)^{-1}$. This estimate is consistent with the full semiclassical computation.

Building on this approximation, we can understand the relaxation of local observables by assuming prethermalizing regions contribute with the thermal values $O_{\text{PreTh}}$, while regions with static mesons retain the initial value $O_0$ (after a short dephasing time). Hence, $\langle O(t) \rangle$ follows the average growth of thermalizing regions leading to: $\langle O(t) \rangle = O_{\text{PreTh}} + [O_0 - O_{\text{PreTh}}]F(tv d^2)$, with $F(\tau) = \int_0^\infty ds e^{-\tau s}(1 - \tau/s)$ and $d$ is a typical meson size $d \sim d_{\text{max}}$; see supplementary material [51]. The full numerical results agree with this picture; Fig. 4 (a). Since $d_{\text{max}} \propto h_{\parallel}^{-1}$, smaller longitudinal fields lead to a shorter prethermalization time scale for the same meson density $\rho$. Even in the less favorable case where $h_\parallel$ is kept constant and only $h_\parallel$ is quenched (i.e., only the small longitudinal field is ultimately responsible of creating fermionic excitations), we find $\rho^2 \propto h_{\parallel}^{-1}$ [51]. Hence, there is in any case a separation of scales between the prethermalization time $t_{\text{PreTh}} \propto h_{\parallel}^{-1}$ and the violation of meson-number conservation $t_{\text{Th}} \sim \exp[(\ldots)h_{\parallel}^{-1}]$, consistently ensuring the existence of the prethermal regime for a large class of quenches.

In Fig. 4 (b) we study the semiclassical prethermal regime for different confining strengths, but the same average density and energy. We observe that weakly-bound mesons strongly influence the phase space distribution by i) introducing a momentum-dependent cutoff in the energy, which is ultimately caused by the fact that the average meson length is bounded by the mean free path, and ii) the probability distribution is squeezed to the boundaries of the allowed phase space. A consequence of this is the emergence of a peak in the energy distribution corresponding to the Brillouin zone boundaries. This effect is captured by our hard-rods approximation. When the prethermal distribution is compared with the thermal expectation, more mesons with higher energy are thermally excited. Hence, fewer thermal mesons are needed to account for energy conservation and the final thermal state is reached by fusing small mesons into larger ones, by the reverse process of the Schwinger effect; Fig. 1 (b). The difference between the prethermal and thermal state is reduced at higher meson densities, where the hard-rods correction penalizes large mesons. See also the supplementary material [51] for additional analysis of different quenches and observables.

By virtue of the simple underlying kinetic mechanism, the validity of our study is expected beyond the classical realm to hold in the quantum case as well, with an additional refinement. As previously mentioned, thermalization is activated by two-body scattering between different energy bands. Hence, the estimate of $t_{\text{PreTh}}$ should be corrected considering that only a fraction of $\rho$ is contributing to the inelastic scattering.

Discussion & Outlook.— Confined spin chains exhibit an intriguing multi-stage thermalization dynamics. We show that the Schwinger mechanism is not responsible for activating transport, but rather rare events in which two mesons are generated in their vicinity lead to a prethermal regime, that can be understood as a thermal gas of mesons. The different mechanism ensures the separation of timescales and the existence of a prethermal regime. The prethermalization time can be greatly reduced by considering quench protocols that create mesons with non-zero velocity. This, for example, can be realized with spatially-modulated pulses of the transverse field [51, 53].

Several intriguing questions remain for future research. It will be interesting to understand how this scenario is modified in other realizations of confined dynamics, such as the experimentally realized long-range Ising chain [11]. The long-range interactions can be envisaged to affect the approximation of dilute mesons, rendering prethermalization faster on the one
hand, but making the approximation of the prethermal regime as a thermal gas of non-interacting mesons unreliable on the other hand. Furthermore, it would be interesting to address scenarios where the violation of the meson number conservation is not negligible and must be properly considered. Can one observe and describe the drift to the thermal regime in such cases? A kinetic theory would require a quantitative understanding of meson creation-annihilation processes beyond the estimates discussed in this work.

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1. **THE WEAKLY CONFINED TRANSVERSE ISING CHAIN**

For the sake of completeness, in this section we summarize the basics of the Ising spin chain in a weak longitudinal field, discussing the dynamics projected in the subspace with a fixed number of fermions and the quench dynamics in the protocol of interest.

*The Ising chain in pure transverse field* — As a starting point, we consider first the Ising chain in a pure transverse field \((h_\perp = 0\) in Eq. (1)). This is a well-known exactly solvable model equivalent to free fermions, see e.g. Ref. [45] for an extensive discussion. Spinless fermions \(\hat{c}_i, \hat{c}^\dagger_j\) are defined through a Jordan-Wigner transformation

\[
\hat{\sigma}_x^i + i\hat{\sigma}_y^i = \exp\left(i\pi \sum_{i<j} \hat{c}^\dagger_i \hat{c}_j \right).
\]

(S1)

The Ising Hamiltonian commutes with the parity operator

\[
\hat{P} = \prod_j \hat{\sigma}_x^j,
\]

thus the Hilbert space splits into two disconnected parts of different parity \(\hat{P}\) = \pm 1: the spinless fermions obey different boundary conditions depending on the parity sector, but this subtlety can be ignored in the infinite volume limit. The transverse Ising Hamiltonian is then diagonalized in the Fourier basis after a Bogoliubov rotation

\[
\begin{align*}
\hat{c}_j &= \int \frac{dk}{\sqrt{2\pi}} \alpha(k) \hat{\gamma}_{\alpha}^\dagger(k) + \text{const.} \\
\hat{c}^\dagger_j &= \int \frac{dk}{\sqrt{2\pi}} \alpha(-k) \hat{\gamma}_{\alpha}^\dagger(-k), \\
U_{\theta_k} &= \begin{pmatrix}
\cos \theta_k & i \sin \theta_k \\
i \sin \theta_k & \cos \theta_k
\end{pmatrix},
\end{align*}
\]

(S2)

where the modes \(\hat{\gamma}(k)\) obey the canonical anticommutation rules \(\{\hat{\gamma}(k), \hat{\gamma}^\dagger(q)\} = \delta(k - q)\). The angle \(\theta_k\) parametrizes the Bogoliubov rotation and the choice

\[
\theta_k = -\frac{1}{2i} \log \left( \frac{h_\perp - e^{ik}}{(\cos k - h_\perp)^2 + \sin^2 k} \right)
\]

(S3)
diagonalizes the Hamiltonian \(\hat{H} = \int dk \epsilon(k) \hat{\gamma}^\dagger(k) \hat{\gamma}(k) + \text{const.} \) with \(\epsilon(k) = 2\sqrt{(\cos k - h_\perp)^2 + \sin^2 k}\). In the fermion language, the Hilbert space can be described as a Fock space built by acting with the creation operators on the vacuum \(|0\rangle\) (defined as \(\hat{\gamma}(k)|0\rangle = 0\)), which is also identified with the ground state of the chain. Homogeneous quantum quenches in the transverse Ising has been first addressed in Ref. [45]: in this framework, one initializes the system in the ground state for a certain transverse field \(\tilde{h}_\perp\) and brings the system out of equilibrium by changing \(\tilde{h}_\perp \rightarrow h_\perp\). By using that the initial state is annihilated by the modes that
diagonalize the prequench Hamiltonian and that the pre- and post-quench modes are connected through a Bogoliubov rotation, one can write the initial state in a simple squeezed form

$$|\psi\rangle \propto \exp\left[\int_0^{\pi} dk \, K(k) \hat{\gamma}^\dagger(k) \hat{\gamma}(-k)\right]|0\rangle,$$

(S4)

with the wavefunction $K(k)$ determined by the difference of the pre and post quench Bogoliubov angles $K(k) = -i \tan(\theta_k^\text{post} - \theta_k^\text{pre})$. By similar means, in Section 4 we will determine the initial state obtained after a staggered pulse in the transverse magnetization.

**Effects of weak confinement: the projected dynamics** — We now consider the activation of a non-trivial longitudinal field $h_L$. Following Ref. [15], one expresses the full Hamiltonian in the basis of the modes of the transverse part. Hence, let $\{|k_i\rangle_{i=1}^N\}$ be a multi-fermionic state: apart from a non-essential constant offset, the Hamiltonian reads

$$\hat{H}[\{k_i\}_{i=1}^N] = \left(\sum_{i=1}^N \epsilon(k_i)\right) |\{k_i\}_{i=1}^N\rangle + h_L \sum_{M=1}^\infty \frac{1}{M!} \int_0^{\pi} \frac{dM}{2\pi} \frac{M}{M} 2\pi \delta \left(\sum_{i=1}^N \{q_j\}_{j=1}^M \sum_{i=1}^N k_i \right) |\{q_j\}_{j=1}^M \rangle |\{q_j\}_{j=1}^M \langle \{k_i\}_{i=1}^N\rangle\rangle.$$

(S5)

The matrix elements of the longitudinal magnetization $\langle\{q_j\}_{j=1}^M \mid \hat{\sigma}_z^\dagger \mid \{k_i\}_{i=1}^N\rangle$ —also called form factors— are known, and, by means of a repeated use of the Wick theorem, are entirely encoded in the two-fermion form factors $\langle k_i k_2 | \hat{\sigma}_z^\dagger | 0\rangle$, $\langle 0 | \hat{\sigma}_z^\dagger | k_1, k_2\rangle$ and $\langle k_1 | \hat{\sigma}_z^\dagger | k_2\rangle$, see Ref. [15] for further details. The Hamiltonian (S5) is valid for arbitrary values of the longitudinal field $h_L$. However, in the limit of weak confinement $|h_L| \ll \min_k \epsilon(k)$, important simplifications can be invoked. As we mentioned in the main text, the effective dynamics conserves the number of fermions on very large time scales, hence we can project the dynamics of Eq. (S5) in the number-conserving space. Secondly, the interaction in the number-conserving sector splits into a long-range linear potential (inducing the confinement) plus short range corrections [15] (see also Ref. [21]). Short range corrections are negligible when compared with the linear term (and are furthermore linearly suppressed for $h_L$ small). With these considerations, in the two particle sector one reaches the simple Hamiltonian

$$\hat{H}_{2pt} = \int \frac{dk_1 dk_2}{2\pi} \epsilon(k_1) |k_1, k_2\rangle |k_1, k_2\rangle + \sum_{j_1, j_2} 2h_L \sigma_1 j_1 - j_2 |j_1, j_2\rangle \langle j_1, j_2\rangle,$$

(S6)

where the potential term is more conveniently expressed in the real basis $|j_1, j_2\rangle \equiv \int \frac{dk_1 dk_2}{2\pi} e^{ik_1 j_1 + ik_2 j_2} |k_1, k_2\rangle$. The expression can be directly generalized beyond the two fermions sector. We notice that for small transverse fields, the kinetic energy reads $\epsilon(k) = 2 - 2h_L \cos(k) + \ldots$ and the kinetic term becomes a simple nearest-neighbor hopping. For the sake of simplicity, in Fig. 2 of the main text we consider this limit, where in addition fermions can be identified with sharp domain walls in the Ising chain. The more general case of finite $h_L$ can be also addressed in a similar way. When considering the quench protocol of interest, the state Eq. (S4) can be used as the initial condition of the dynamics: this proposal has been originally made in Ref. [19]. However, while at late time the Schwinger mechanism is extremely suppressed, it has been recently understood that the activation of a longitudinal field non-trivially affects the number of fermions [21]. This additional production of excitations ceases after a short time transient and effectively renormalizes the wavefunction

$$K(k) \to K(k) = -i \tan(\theta_k^\text{post} - \theta_k^\text{pre}) - i h_L v(k)/\epsilon^2(k),$$

(S7)

with $v(k) = \partial_k \epsilon(k)$ being the group velocity. While this renormalization can be important to check meson conservation for relatively strong longitudinal field, its effect is negligible for the parameter choice of Figs. 3 and 4 and it is thus neglected.

**Quantization of the mesonic states** — Mesons have internal degrees of freedom associated with the different energy levels of the two-particle problem described by $\hat{H}_{2pt}$ (S6). In the limit of weak transverse field $\epsilon(k) = 2 - 2h_L \cos(k) + \mathcal{O}(h_L^2)$ the eigenfunctions and energies can be exactly computed by means of Bessel functions [15]. For finite transverse field, a simple analytical solution is not available, but $\hat{H}_{2pt}$ can be easily numerically diagonalized (see eg. Ref. [21]). Nonetheless, the limit where the longitudinal field is much weaker than the transverse one $h_L \ll h_L$ is amenable of semiclassical methods. Already in the seminal paper [19], the semiclassical quantization of meson energies has been observed to be in very good agreement with numerical data, even far from the extreme limit $h_L \ll h_L$. Therefore, further motivated by our goal to describe the prethermal phase observed in the classical regime, we briefly review the semiclassical quantization of mesonic masses [15]: the pure classical limit is then recovered in the limit of vanishing longitudinal field, where the quantized energies merge in a continuum.

In order to find the momentum-dependent energy levels $E(J, k)$, let us address the classical two-fermion problem associated with Eq. (S6). It is convenient to consider the center-of-mass and the relative coordinates, $(X, k)$ and $(x, q)$ respectively. For this choice of coordinates the 2–particle classical Hamiltonian takes the simple form

$$H_{2pt}(k, q, X, x) = \epsilon(k/2 + q) + \epsilon(k/2 - q) + \chi |x| \equiv \omega(q, k) + \chi |x|$$

(S8)
where we introduced the notation $\chi = 2\hbar_1 \bar{\sigma}$ and $\omega(k, q) = \epsilon(k/2 + q) + \epsilon(k/2 - q)$. The total momentum $k$ is conserved.

Generally $q(t)$ and $x(t)$ describe an oscillating motion associated to the breathing of the fermion bound state. The latter is formally captured by

$$
q(t) = \begin{cases} 
q_a(E, k) - \chi t & t \in [0, t_1] \\
q_b(E, k) + \chi (t - t_1) & t \in [t_1, 2t_1]
\end{cases}
$$

$$
x(t) = \begin{cases} 
\chi^{-1}(E - \omega(k, q(t))) & t \in [0, t_1] \\
-\chi^{-1}(E - \omega(k, q(t))) & t \in [t_1, 2t_1]
\end{cases}
$$

(S9)

Here we introduced the points $(q_a, q_b)$ as the turning points of the classical problem, where the total energy of the relative motion is stored in kinetic energy, i.e., the relative distance of the fermions vanishes. These turning points are reached after multiples of the time period $t_1 = (q_a - q_b)/\chi$. Depending on the choice for $k$ the function $\omega(k, q)$ will either resemble the shape of a single or double-well potential in $q$. This entails two fundamentally different cases for the turning points $q_a, q_b$. For small values of $k, \omega(k, q)$ looks like a single well potential symmetric in $q \rightarrow -q$. The turning points will consequential share this reflection symmetry around $q = 0$ and satisfy the constraint $q_b = -q_a$ independent of the choice of $E$. For larger values of $k$ we, however, find a different behavior. In this case $\omega(k, q)$ can be interpreted as a double-well potential and the motion can be stuck within a single-well for sufficiently small energies $E < \omega(k, q = 0)$. In consequence, the relation $q_b = -q_a$ is no longer valid and the motion will no longer be symmetric under the reflection $q \rightarrow -q$.

In order to determine the mesonic energy bands $E(J, k)$ we apply a semiclassical Bohr-Sommerfeld quantization to the set of conjugate variables $(q(t), x(t))$ given by

$$
J = \oint dx \, q(x) = 2\pi (n - \frac{1}{2}) \quad \text{with} \quad n \in \mathbb{N}.
$$

(S10)

Using the functional form of $q(t)$ and $x(t)$ of (S9) we thus find

$$
J = 2\chi^{-1}E(J, k)(q_b - q_a) - 2\chi^{-1} \int_{q_a}^{q_b} dq \, \omega(k, q)
$$

(S11)

However, due to the Pauli exclusion principle $n$ is forced to be even when the two fermions can come in contact. We thus find two different conditions referring to the cases of a symmetric motion ($n$ even integer) and a motion stuck within a single well of the double-well potential ($n$ integer). For a more detailed derivation of this quantization procedure we refer to Ref. [15]. Solving Eq. (S11) numerically equips us with all information required to describe the thermodynamics of the prethermal state, as we now discuss.

2. THERMODYNAMICS FOR MESONIC SYSTEMS

In the extremely dilute regime, the typical size of a meson is negligible with respect to their relative distance and their thermodynamics can be approximated as if they were point-like particles. However, mesons are extended objects and their size gives non-negligible contributions moving aside from the extreme dilute scenario. Therefore, we now aim to a better treatment where mesons are considered as extended objects with a fixed length $(J, k)$ which in first approximation can be taken as the average magnetization. Despite the fact this is a rather crude approximation (for example, the size of the meson oscillates in time), it nicely captures features beyond the extreme dilute scenario. Hence, we now discuss the thermodynamics of a gas of hard-rods, where the length of the meson depends on its momentum $k$ and internal energy level $J$. The momentum $k$ is quantized in units of $2\pi/L$, but we are eventually interested in the thermodynamic limit and replace summations with integrals. Even though we wish to address directly the semiclassical limit, considering the correct momentum quantization is necessary to obtain the correct phase-space normalization of the thermal curves. In the limit of a weakly interacting many-meson system all thermodynamic information is contained in the grand-canonical partition function

$$
Z = \sum_{\{\rho_J(k)\}_{J,k}} e^{S} e^{-L\beta \sum_{J,k} \rho_J(k)(E(J, k) - \mu)}
$$

(S12)

where $\rho_J(k)$ is the density of mesons with quantum numbers $(J, k)$ and energies $E(J, k)$. In this notation, $J$ is the classical action variable that is discretized in units of $2\pi$ in the quantum case, due to the Bohr-Sommerfeld quantization condition. The summation over the densities should be interpreted in a path integral sense, but it is useful to think about it as a discrete object first and take the limit at the end. The system size is $L$. We also introduce the number of mesons with quantum number $(J, k)$ as $N(J, k) = Ldk \rho_J(k)$, with $dk$ the size of a small momentum cell.

The chemical potential $\mu$ ensures the meson number conservation present in the prethermal regime. The summation included in Eq. (S12) takes into account all possible ways to distribute a given number of mesons in the system. While the summation
over \( \{ \rho_J(k) \}_{J,k} \) spans the possible populations for each quantum number, the entropic factor \( e^S \) counts the possible spatial arrangements of the mesons for a given population distribution. The entropic term is sensitive to the length of the mesons: as anticipated, let \( \ell(J,k) \) be the effective length of a meson, which we approximate as a constant. To simplify notation, we group the two quantum numbers in a single one \((J,k) \rightarrow \eta\), hence \( \ell(J,k) \rightarrow \ell_{\eta} \). This identification takes us of hard-rods of different lengths \( \ell_{\eta} \) with \( N_\eta \) particles for each species. For the sake of simplicity, we now choose to neglect the discreteness of the lattice and treat the position of a meson as a continuous variable. This approximation is valid either in the small density limit where finite-volume effects are negligible, or in the limit where mesons are large compared with the unit cell. Therefore, we expect corrections can be important only in the extreme limit of very dense and tight mesons.

To compute the partition function, we first introduce an ordering in the particle species: mesons with quantum number \( \eta \) will be contained in an interval of length \( L_\eta \), where \( L = \sum_\eta L_\eta \). The ordered partition function due to the spatial degrees of freedom is readily computed as (we assume \( \eta \) runs from 1 to \( m \))

\[
Z_m^{(ord)} = \int_0^L dL_1 dL_2 \ldots dL_m \delta \left( L - \sum_i L_i \right) \prod_{r=1}^m Z_1(L_r, N_r) = \int_{L_{min}}^{L_{max}} dL_1 \int_{L_{min}}^{L_{max}} dL_2 \ldots \int_{L_{min}}^{L_{max}} \prod_{r=1}^m Z_1(L_r, N_r)
\]

\[
= \frac{1}{(N_1 + N_2 + \ldots + N_m)!} \prod_{r=1}^m (L - N_1 \ell_1 - N_2 \ell_2 - \ldots - N_m \ell_m)^{N_r}.
\]

(S13)

Here we introduced the minimal and maximal length of subsystem \( r \) as \( L_r^{(min)} = N_r \ell_r \) and \( L_r^{(max)} = L - \sum_{\eta \neq r} L_\eta - \sum_{\eta \neq r} N_\eta \ell_\eta \), respectively, to make the constraint of the \( \delta \)-distribution explicit. The first, hereby, just takes into account that the subsystem \( r \) containing \( N_r \) particles of length \( \ell_r \) can not be compressed further than \( L_r^{(min)} \). The upper bound results from the opposite situation. Having already fixed the subsystem sizes \( L_1, \ldots, L_{r-1} \) we can compress the remaining subsystems to their minimal configuration and find \( L_r^{(max)} \). For the last equality of Eq. (S13) we used the well-known result for the partition function of a simple gas of a single species of hard-rods \( Z_1(L, N) = [(L - N \ell)]^{N}/N! \). Since the different particles are grouped into subsystems the partition function of Eq. (S13) does not take into account reordering of the individual meson species. This can, nevertheless, be accounted for by a combinatorial factor

\[
Z_m = \frac{(N_1 + N_2 + \ldots + N_m)(N_1 + N_2 + \ldots + N_{m-1}) \ldots (N_1 + N_2)}{N_m} Z_m^{(ord)}
\]

\[
= \prod_{r=1}^m \frac{1}{N_r!} (L - N_1 \ell_1 - N_2 \ell_2 - \ldots - N_m \ell_m)^{N_1 + N_2 + \ldots + N_m}.
\]

(S14)

When computing (S13), we did not take into account the momentum quantization which allows for a reshuffle of the particles within the single momentum cell of width \( dk \). Due to quantization, such a cell has \( \frac{2\pi}{k} \) available quantum numbers. We now consider the problem of arranging \( N_r \) indistinguishable particles on \( \frac{L}{k} \) sites. We neglect the case of double occupancies, which is equivalent to consider a classical statistics (in contrast with Fermi or Bose statistics). In this case, the number of possible arrangements is \( \frac{(\frac{L}{k})^N}{N_r!} = \frac{L^N}{k^N} \). Now, we argue that the prefactor \( L^{N_r} \) has already been taken into account in Eq. (S13), while the factor \( 1/N_r! \) has already been accounted when considering (S14). Hence, one is left with the additional phase-space contribution \( \left( \frac{dk}{2\pi} \right)^{N_r} \). We can now finally identify the entropic term as

\[
S = \log \left( Z_m \prod_r \left( \frac{dk}{2\pi} \right)^{N_r} \right)
\]

(S15)

In the hypothesis that \( N_r \) is large and using a Stirling approximation for the factorial, one finds

\[
S/L \approx \sum_{r=1}^m dk \rho_r \left[ \ln \left( \frac{1 - dk \rho_1 \ell_1 - dk \rho_2 \ell_2 - \ldots - dk \rho_m \ell_m}{2\pi \rho_r} \right) + 1 \right].
\]

(S16)

By the extensivity of the entropic term, in the thermodynamic limit \( L \rightarrow \infty \) the partition function localizes to the saddle point of the free energy \( \mathcal{F} = -\beta^{-1} \log Z \)

\[
-\beta^{-1} \mathcal{F} = \sum_J dk \rho_J(k) \left[ \ln \left( 1 - \sum_J dq \ell(J', q) \rho_J(q) \right) - \ln(2 \pi \rho_J(k)) + 1 - \beta(\mathcal{E}(J,k) - \mu) \right]
\]

(S17)

Finally, imposing \( \delta \mathcal{F}/\delta \rho_J(k) = 0 \) we find

\[
\rho_J(k) = \frac{1 - \rho M}{2\pi} \exp \left[ -\beta(\mathcal{E}(J,k) - \mu) - \frac{\rho \ell(J,k)}{1 - \rho M} \right]
\]

(S18)
We benchmark our predictions from thermodynamic analysis for a system of point particles (PP) as well as for an ensemble of hard-rods (HR) against the results of semiclassical simulations. (a)-(c) We display the energy distribution \( P(E) \) of the prethermal steady state for different densities \( \rho \) in a system characterized by a longitudinal and transverse field \((h_1, h_\perp) = (0.015, 0.2)\). The initial energy per meson is thereby on average given by \( E = 3.73737 \). We find that the finite-volume correction included in the HR treatment (red solid curves) improves the thermodynamic description of the semiclassical data points (green markers) compared to the PP description (blue dashed curves). (d)-(f) The prethermal state of hard rods captures the momentum distribution \( P(k) \) especially well for densities \( \rho /\text{unit} 0.2 \). For very high densities \( \rho \approx 0.4 \) deviations become apparent.

Where the meson coverage \( \rho_M \), the total density \( \rho \) and the mean energy \( E \) are

(i) \[ \rho = \sum_J \int dk \rho_J(k) \]  

(ii) \[ E = \sum_J \int dk \mathcal{E}(J,k) \rho_J(k) \]  

(iii) \[ \rho M = \sum_J \int dk \ell(J,k) \rho_J(k). \]

The meson density and coverage must be computed self consistently with (S18). On prethermal states, thus enforcing the meson density conservation, the chemical potential \( \mu \) and inverse temperature \( \beta \) must be chosen to match the initial density \( \rho \) and energy \( E \). On the other hand, thermal ensembles fix only the energy \( E \) and \( \mu = 0 \). It is worth to make the momentum-energy probability distribution of a meson \( P(E, k) \) in the classical case. We start considering \( P(E, k) \) in the quantum regime writing it as

\[ P(E, k) = \sum_J \rho_J(k) \delta(E - \mathcal{E}(J,k)), \]

with \( \delta \) a Dirac delta. In the semiclassical regime, we can replace the summation over \( J \) with an integral \( \sum_J \to \int \frac{dJ}{2\pi} \), where the factor \( 2\pi \) comes from the Born-Sommerfeld quantization condition \( J = 2\pi(n - 1/2) \)

\[ P(E, k) = \frac{1}{(2\pi)^2} e^{-\beta(E-\mu)} \int dJ \delta(E - \mathcal{E}(J,k)) e^{\frac{\rho M(J,k)}{\beta}}. \]

Notice that in the small density limit \( \rho \to 0 \), the coverage \( \rho M \) vanishes as well and one recovers the usual thermal distribution of non-interacting particles.

Finally, we wish to explicitly discuss how \( \ell(J,k) \) is computed: as we already mention, we estimate the length of the hard rod approximation of the meson with its average length. In the classical limit, this amounts to computing the time average of
the distance between the two fermions within one period of the oscillation. In the quantum regime, instead, \( \ell(J, k) \) is computed by considering the quantum expectation value of the relative distance \(|x|\) \((S8)\) on the energy eigenstate. Of course, the two definitions coincide in the semiclassical limit.

To further emphasize the importance of the finite volume corrections, we refer to Fig. S1 providing a benchmark of our thermodynamic predictions against semiclassical simulations. The semiclassical data for the energy distribution \( P(E) \) of the prethermal steady state shown in Fig. S1 (a) - (c) reveals that systems of higher meson densities \( \rho \) in fact exhibit increased meson occupancy towards the edges of the Brillouin zone \((E \approx 4.0)\), where mesons show their smallest average lengths. While the conventional thermodynamic ansatz for point particles fails in reproducing this feature of the steady state, including finite volume corrections indeed allows us to capture this effect. This is further supported by results for the momentum distributions \( P(k) \) of the same systems, as illustrated in Fig. S1 (d) - (f).

### 3. THERMALIZATION BY RARE EVENTS: EVOLUTION OF OBSERVABLES

In this section, we derive the late-time prediction for the time evolution of local observables as triggered by rare events. As anticipated, building on simple arguments we envisage (see main text for notation)

\[
\langle O(t) \rangle = O_{\text{PreTh}} + \Delta O F(t_v \rho^2 d); \quad F(\tau) = \int_0^\infty ds e^{-s(1-\tau/s)},
\]

where \( O \) is a local observable and \( O_{\text{PreTh}} \) the final prethermal value and \( \Delta O = O_0 - O_{\text{PreTh}} \), with \( O_0 \) the initial value of the observable (after a short dephasing time). The argument is the following. As we discussed in the main text, rare events are initially spaced on a distance \( \sim \rho^2 d_{\text{max}} \). Prethermalization occurs when the scrambling regions, started by the rare events, cover the whole system. Within a growing scrambling region, the system drifts towards the prethermal regime: two-body collisions are responsible for prethermalization, setting a typical timescale \( \sim (\rho v)^{-1} \). At late times the system began to approach the prethermal regime when the scrambling regions grow enough to cover the distance between rare events, thus their extension must be \( \sim (d_{\text{max}} \rho^2)^{-1} \). This is expected to happen on a timescale \( \sim (\nu d_{\text{max}} \rho^2)^{-1} \). Hence, in the low density limit, there is a scale separation and we can approximate the expectation value of local observable within the relaxing region as thermal. This leads to the following simple model

\[
\langle O(t) \rangle = O_{\text{PreTh}} + \Delta O \int_0^\infty dD P(D) \frac{D - 2vt}{D} \theta(D - 2vt),
\]

where \( \theta(x) \) is the Heaviside theta function \( \theta(x > 0) = 1 \) and zero otherwise, \( D \) is the distance between two rare events which is distributed with probability distribution \( P(D) \). By its very definition, \( O_0 \) can be computed as the late-time limit of the single meson approximation, since outside of the scrambling region the mesons are not interacting. Finally, the term \( \frac{D - 2vt}{D} \) is nothing else than the portion of frozen region that remained after the thermalizing region propagated with velocity \( \nu \) inside of it. The last step is now to estimate \( P \). In the main text we have already computed the probability that, within a system of size \( L \), there are no rare events. In the computation, we used the maximum extension of a meson \( d_{\text{max}} \) as an upperbound, but a better estimate is obtained using the average size of the excited mesons, which we call \( d \). Hence, the probability that within an interval \( L \) there are no rare events is \( P(L) = e^{-L\rho^2 d/2} \). Of course, \( P(L) \) and \( P(D) \) are easily related as \( P(L) = \int_L^\infty dD P(D) \), leading to \( P(D) = \frac{\rho^2 d}{2} e^{D - D\rho^2 d/2} \). Eq. (S24) then immediately follows by plugging this result in Eq. (S25).

### 4. INITIALIZING MOVING MESONS BY MODULATED PULSES OF THE TRANSVERSE FIELD

The prethermalization time scale can be strongly reduced when mesons are initialized with a finite velocity, as we demonstrated in Fig. 2. This scenario can be achieved through a different state preparation, by replacing the homogeneous quench in the transverse field with a modulated pulse on a period of \( n \) sites \([53]\), i.e., \( H_{\text{Pulse}} = -\delta(t) \sum_j h_j \sigma_j \), where \( h_{j+n} = h_j \). While fermions are still excited in pairs \((k, k')\), the periodic modulation breaks translation invariance thus giving a non-trivial momentum to the meson \( k + k' = 2\pi j/n \), with \( j \) an integer. While this strategy allows us to maintain the dilute meson approximation, it creates two-body collisions right from the beginning, thus promoting scrambling. In particular, let us assume the state is initially prepared in the ground state \( |0\rangle \) of the transverse-field Ising model \( \hat{H}_r = -\sum_j \sigma_j \sigma_{j+1} + h_1 \sigma_j^z \), then we consider a pulse Hamiltonian in the form \( H_{\text{Pulse}} = -\delta(t) \sum_j h_j \sigma_j^z \), with \( h_j \) being a periodic modulation with period \( n \), i.e. \( h_j = h_{j+n} \). After the pulse application, the state evolved into \( |\psi\rangle \equiv e^{i\sum_j h_j \sigma_j^z} |0\rangle \) which we now characterize. After the pulse, the longitudinal field is activated and \( |\psi\rangle \) is used as the initial state for the confining dynamics: we assume the regime of weak confinement, hence we neglect meson production caused by the activation of the longitudinal field \( h_{1||} \). As a first step, we express the pulse in the fermionic basis and...
eventually in the modes of the transverse Ising Hamiltonian, by Eq. (S2). For the sake of a more compact notation, we define the non-rotated fermionic modes in the Fourier basis as \( \hat{\alpha}(k) = \cos \theta_k \hat{\gamma}(k) + i \sin \theta_k \hat{\gamma}^\dagger(-k) \), where \( \{ \hat{\alpha}(k), \hat{\alpha}^\dagger(q) \} = \delta(k - q) \). We write the pulse as
\[
- \sum_j h_j \sigma^+_j = \sum_j 2h_j c^+_j c_j + \text{const.} = \int_{-\pi}^{\pi} \frac{dkdq}{2\pi} \left( \sum_j 2h_j e^{ij(k-q)} \right) \hat{\alpha}^\dagger(q) \hat{\alpha}(k) + \text{const.} = \int_{-\pi}^{\pi} dk dq 2\hat{h}(k-q) \delta(e^{in(k-q)} - 1) \hat{\alpha}^\dagger(q) \hat{\alpha}(k) + \text{const.} = \sum_{j,j'} \int_{-\pi/n}^{\pi/n} dk \frac{2}{n} \hat{h}(2\pi(j-j')) \hat{\alpha}^\dagger(k + 2\pi j/n) \hat{\alpha}(k + 2\pi j/n) + \text{const.}.
\]
(S26)

Above, we then use the periodicity of \( h_j \) to extract a delta function in the momentum space and defined \( \hat{h}(k) \equiv \sum_{j=0}^{n-1} e^{ikj} h_j \). We now use this expression to show that
\[
\text{We write the pulse as}
\]

\[
\text{and explore prethermalization.}
\]

\[
\left. \right| \psi \rangle = e^{i \sum h_j \sigma_j^+} |0\rangle = N \exp \left[ \frac{1}{2} \sum_{j,j'} \int_{-\pi/n}^{\pi/n} dk M_{j,j'}(k) \hat{\gamma}^\dagger(k + j2\pi/n) \hat{\gamma}(k + j'2\pi/n) \right] |0\rangle.
\]
(S27)

Above, \( M_{j,j'}(k) \) is a \( k \)-dependent \( n \times n \) complex matrix. Notice that, due to the fermionic anticommutation relations, it holds \( M_{j,j'}(k) = M^\dagger_{n-j',n-j}(-k) \). Eq. (S27) generalizes the state (S4) to the case of fermions pairwise excited, but with pairs possibly having non-zero total momentum.

To show Eq. (S27) it is convenient to introduce a fictitious parameter \( \lambda \) and considering \( |\psi_\lambda\rangle = e^{i\lambda \sum h_j \sigma_j^+} |0\rangle \), by showing infinitesimal \( \lambda \)-changes move within the parametrization (S27) with a \( \lambda \)-dependent wavefunction \( M_{j,j'}^\lambda(k) \) and normalization \( N_\lambda \). By introducing the \( \lambda \)-dependence in Eq. (S27) and taking the derivative, one obtains
\[
\partial_\lambda |\psi_\lambda\rangle = \left( \frac{\partial_\lambda N_\lambda}{N_\lambda} + \frac{1}{2} \sum_{j,j'} \int_{-\pi/n}^{\pi/n} dk \partial_\lambda M_{j,j'}^\lambda(k) \hat{\gamma}^\dagger(k + j2\pi/n) \hat{\gamma}(k + j'2\pi/n) \right) |\psi_\lambda\rangle.
\]
(S28)

On the other hand, by the very definition one has \( \partial_\lambda |\psi_\lambda\rangle = i \sum h_j \sigma_j^+ |\psi_\lambda\rangle \). By repeatedly using the fermionic commutation relation on Eq. (S27), the action of \( \hat{\alpha}(k + 2\pi j/n) \) on \( |\psi_\lambda\rangle \) is computed as
\[
\hat{\alpha}(k + 2\pi j/n) |\psi_\lambda\rangle = \left[ \sum s \cos \theta_{k+2\pi j/n} M_{s,\delta}^\lambda(k) \hat{\gamma}^\dagger(-k - s2\pi/n) + i \sin \theta_{k+2\pi j/n} \hat{\gamma}^\dagger(-k - 2\pi j/n) \right] |\psi_\lambda\rangle.
\]
(S29)

Further applying \( \hat{\alpha}^\dagger(k + 2\pi j'/n) \) on the so-obtained state, one can proceed by the same methods and the action of Eq. (S26) on \( |\psi_\lambda\rangle \) is easily obtained. After some simple, but tedious calculations it is shown that the gaussian form (S27) closes the equations if \( M_{j,j'}^\lambda \) satisfies
\[
i \partial_\lambda M_{j,j'}^\lambda(k) = \sum_{s,s'} \frac{2}{n} \hat{h}(2\pi(s - s')) \left[ \cos \theta_{k+2\pi s'/n} M_{s,\delta}^\lambda(k) + i \sin \theta_{k+2\pi s'/n} \delta_{s,s'} \right] \left[ -i \sin \theta_{k+2\pi s'/n} M_{s',\delta'}^\lambda(k) + \cos \theta_{k+2\pi s'/n} \delta_{s',s'} \right] + \left[ k \rightarrow -k, j \rightarrow -j', j' \rightarrow -j \right],
\]
(S30)

where the term \( [k \rightarrow -k, j \rightarrow -j', j' \rightarrow -j] \) is obtained from the first through the proper index replacement. Finally, the desired matrix \( M_{j,j'}^\lambda(k) \) is computed by integrating the above equation up to \( \lambda = 1 \). Albeit a close analytical expression is hard to obtain, the \( n \times n \) matrix equation (S26) can be easily numerically integrated. Furthermore, in the problem at hand we are mostly interested in the regime of dilute mesons, i.e. when \( h_j \) is small. Eq. (S30) is easily solved at the leading order in \( h_j \) obtaining
\[
i M_{j,j'}^\lambda(k) = \frac{2}{n} \hat{h}(2\pi(j - j')) i \sin \theta_{k+2\pi j/n} \cos \theta_{k+2\pi j'/n} + \frac{2}{n} \hat{h}(2\pi(j' - j)) i \sin \theta_{-k+2\pi j'/n} \cos \theta_{-k+2\pi j/n} + O(h^2).
\]
(S31)

5. Exact Diagonalization in the Few Kinks Subspace

Building on the stability of the fermions for exponentially long times, one can project the Hamiltonian (S6) within the few-fermions sector. In this way, by exact numerical integration of the few-fermion wavefunction, we can access very long timescales and explore prethermalization.
Two-meson dynamics. Analogously to the three-meson scenario considered in the main text, we illustrate the results for dynamics of two mesons in a system of \( L = 100 \) sites and confinement field \( h_1/h_0 = 0.1 \). We consider two distinct initial states, where the mesons are initialized with energies below (a) or above (b) the second band of the single meson spectrum \( E(J, k) \). The momentum distribution \( P(k) \) of the meson with positive initial momentum is evaluated (empty dot in the plot of dispersion bands). (a) A double-peak structure centered at the initial momenta of the mesons \( k_0 = \pm \pi/4 \) survives even at late times. This characteristic disappears in (b) for initial meson momenta of \( (k_0 = \pm 3\pi/4) \), where after a time transient various momenta are occupied. Crucially, for both initial states, (a) and (b), the system does not attain the prethermal state (red line) because scattering events involve only two mesons. This fact clearly distinguishes the scenario of two mesons from the case of three mesons discussed in the main text.

While this strategy can be applied for arbitrary values of the transverse field by considering \((S6)\) (generalized to many fermions), here, we focus on the small transverse field limit where the Hamiltonian is further simplified. In this regime, the kinetic part reduces to nearest-neighbor hopping and the fermions are equivalently describing domain walls. Hence, let \( \Psi(j_1, j_2, ..., j_{2n-1}, j_{2n}) \) be the wave function labeling the state with domain walls between the lattice sites \( j_i - 1 \) and \( j_i \) and, without loss of generality, we consider the ordering \( j_1 < j_2 < ... < j_{2n} \), and periodic boundary conditions are assumed. Furthermore, we consider the false vacuum to be between the kinks \( j_{2i-1} \) and \( j_{2i} \). On this wave function, the Hamiltonian acts as

\[
\hat{H}_{\text{Kinks}} \Psi(j_1, ..., j_{2n}) = \sum_{i=1}^{2n} -h_i [\Psi(j_1, ..., j_i + 1, ..., j_{2n}) + \Psi(j_1, ..., j_i - 1, ..., j_{2n})] + \sum_{i=1}^{n} \chi |j_{2i-1} - j_{2i}| \Psi(j_1, ..., j_{2n}). \tag{S32}
\]

Above, we neglect an overall unimportant constant and the hopping term should respect the hard core constraint \( j_1 < j_2 < ... < j_{2n} \). Since we are mostly interested in the scattering among mesons, we consider a translational invariant scenario: this allows us to further enhance the performance of the approach by removing a degree of freedom. For the sake of simplicity, we consider the case of global zero momentum, but the same method can be applied to the general case. It is convenient to use the position of the first domain wall as a reference coordinate and introduce new variables \( s_i = j_{i+1} - j_1 \). We denote with \( \Phi(s_1, ..., s_{2n-1}) \) the wavefunction in the relative coordinates. In this case, the dynamics is

\[
\hat{H}_{\text{Kinks, zero momentum}} \Phi(s_1, ..., s_{2n-1}) = -h_1 [\Phi(s_1 + 1, ..., s_{2n-1} + 1) - \Phi(s_1 - 1, ..., s_{2n-1} - 1)] + \sum_{i=1}^{2n-1} -h_i [\Phi(s_1, ..., s_i + 1, ..., s_{2n-1}) + \Phi(s_1, ..., s_i - 1, ..., s_{2n-1})] + \sum_{i=0}^{n-1} \chi |s_{2i+1} - s_{2i}| \Phi(s_1, ..., s_{2n-1}). \tag{S33}
\]

Above, the first term accounts for the hopping of the first domain wall, which equivalently shift of one site all the relative distances \( s_i \). From the knowledge of the wavefunction \( \Phi \), several observables of interest can be computed. First, the total magnetization is directly connected to the length of the mesons, since the part of the chain enclosed within one meson lays in the false vacuum

\[
\sum_{j=1}^{L} (S_j^z) = L/2 - \sum_{s_1} \left[ s_1 + \sum_{i=1}^{n-1} |s_{2i+1} - s_{2i}| \right] |\Phi(s_1, ..., s_{2n-1})|^2. \tag{S34}
\]
FIG. S3. Comparison of two and three meson scenario. Dynamics in a system of \( L = 100 \) sites containing (a) two mesons with momenta \( k_0 = \pm \pi/2 \) and (b) three mesons with momenta \( k_0 = 0, \pm \pi/2 \), respectively. The underlying confinement field is \( h_1/h_1 = 0.1 \). The meson coverage \( \rho_M \) reveals a fundamental difference between systems of (c) two and (d) three mesons. Whereas an ensemble of three mesons relaxes to a prethermal configuration (red dashed and solid lines) provided the initial energy per meson is above the second band of the single meson spectrum, we solely find relaxation to a non-thermal state for the two meson system. (e) - (f) This observation is supported by studies of the length distribution of mesons \( P(\ell) \). (g) - (h) Even though the meson densities are different in the two cases, the number of scattering events is similar. Therefore, an absence of thermalization in the two-meson scenario, due to a reduced number of scattering events can be ruled out.

However, much more information is contained in the probability distribution of the meson length

\[
P_{\text{Length}}(\ell) = \frac{1}{N_{\text{mes}}} \sum_{\{s_i\}} \left[ \delta(\ell - s_1) + \sum_{i=1}^{n-1} \delta(\ell - |s_{2i+1} - s_{2i}|) \right] |\Phi(s_1, ..., s_{2n-1})|^2,
\]

where \( \delta \) is a Kronecker delta distribution. Nonetheless, our primary tool to assess prethermalization is the momentum distribution of the mesons. In this case, particular care should be taken when passing from the original coordinates to the relative ones. Let us consider the density matrix in the momentum space defined as

\[
\rho(k_1, k_2, ..., q_1, q_2, ...) = \sum_{\{j_i\}, \{j'_i\}} e^{i\sum_i (k_i-j_i-q_i)} \Psi(j_1, j_2, ...) \Psi^*(j'_1, j'_2, ...).
\]

Then, we wish to target the momentum distribution of the first meson \( P(k) \), defined as

\[
P(k) = \sum_{\{k_i\}} \delta(k_1 + k_2 - k) \rho(k_1, k_2, ..., k_1, k_2, ...) = \int \frac{d\omega}{2\pi} \sum_{\{k_i\}} e^{i\omega(k_1+k_2-k)} \rho(k_1, k_2, ..., k_1, k_2, ...).
\]

The integral representation of the Dirac \( \delta \) distribution is particularly convenient for carrying out the straightforward but lengthy calculations. By plugging the definition (S36) in the above equation and, within the total zero momentum sector, passing to the relative coordinates one finally obtains

\[
P(k) = \sum_{\{s_i\}} e^{ik s_1} \Phi(s_2 - s_1, s_3 - s_1, ..., s_{2n} - s_1) \Phi^*(s_2 - s_1, s_3, ..., s_{2n}).
\]

In the main text, we use the momentum distribution to analyze relaxation and prethermalization of initial wavepacket configurations. To this end, we initialize states with two (\( n = 2 \)) or three mesons (\( n = 3 \)) in the form of gaussian wavepackets with tunable initial momenta. The functional form of the initial meson state reads

\[
\Phi(s_1, ..., s_{2n}) = \prod_{i=0}^{n-1} \left\{ \phi_{K_i}(s_{2i+1} - s_{2i}) e^{iK_i \frac{2s_{2i+1} + s_{2i}}{2}} \right\} W_{\sigma, \Xi}(s_2, s_2, s_{2i+2}, s_{2i+3}),
\]
where above $W_{\sigma,X}$ is a gaussian wavepacket for the relative distance between to consecutive mesons

$$W_{\sigma,X}(s_{2i}, s_{2i+1}; s_{2i+2}, s_{2i+3}) \propto \exp \left[ -\left( \frac{(s_{2i} + s_{2i+1}) - (s_{2i+2} + s_{2i+3})}{\sigma^2} \right)^2 \right]$$

(S40)

and the wavefunction $\phi_K(s)$ is the mesonic wavefunction of the lowest dispersion band obtained by numerically diagonalizing (S32) in the two-fermion sector with total momentum $K$. Furthermore, we insert a cutoff $\phi_K(|s| < \lambda_c) = 0$ (and similarly in the gaussian wavepackets) to ensure the initial state is correctly ordered. We checked our results to be cutoff-independent.

Further results of the scattering dynamics are provided in figures S2 and S3.

6. THE TENSOR NETWORK SIMULATIONS

In the main text, we relied on tensor network simulations to emphasize the conservation of the meson number during the quantum evolution. Whereas time evolution can be carried out using the standard method of time-evolving block decimation (TEBD) [48, 49], measurements of the meson number are more subtle. This is based on the non-local character of the Jordan-Wigner transformation relating the fermionic basis to the conventional spin basis of tensor network simulations. In the following we briefly outline how the mesonic number operator can be embedded efficiently in tensor network formalism.

In the operator basis of fermionic creators and annihilators ($\gamma^\dagger_k, \gamma_k$), diagonalizing the transverse field Ising model in absence of a longitudinal field $h_1$, the mesonic number operator $N_{\text{mes}}$ is given as

$$2N_{\text{mes}} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \gamma_k^\dagger \gamma_k = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[ \cos \theta_k \alpha_k^\dagger - i \sin \theta_k \alpha_{-k}^\dagger \right] \left[ \cos \theta_k \alpha_k - i \sin \theta_k \alpha_{-k} \right]$$

$$= \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left( \cos^2 \theta_k \alpha_k^\dagger \alpha_k + \sin^2 \theta_k \alpha_{-k}^\dagger \alpha_{-k} + i \sin \theta_k \cos \theta_k \left( \alpha_{-k} \alpha_k - \alpha_k^\dagger \alpha_{-k}^\dagger \right) \right)$$

$$= \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left( \cos^2 \theta_k - \sin^2 \theta_k \right) \alpha_k^\dagger \alpha_k + \delta(0) \sin^2 \theta_k + i \sin \theta_k \cos \theta_k \left( \alpha_{-k} \alpha_k - \alpha_k^\dagger \alpha_{-k}^\dagger \right),$$

(S41)

where the relation between the modes $\gamma$ and the fermions in Fourier space $\alpha(k)$ is given in Eq. (S2).

The obtained relation in terms of $\alpha_k^\dagger, \alpha_k$ can then be normal ordered using the anticommutation relations of the fermionic creators and annihilators $\alpha_k, \alpha^\dagger_k$. The arising $\delta$-distribution gives us a factor of the system size, i.e. $\delta(0) = L$, leading to a diverging contribution in the thermodynamic limit. After tedious but straightforward algebra we find the following expression for the mesonic number operator

$$N_{\text{mes}} = \frac{1}{2} \sum_j \left\{ \sum_{\ell} \left[ f_1(\ell) c_{j+\ell}^\dagger c_j + \frac{1}{2} f_2(\ell) (c_{j+\ell} c_j + c_j^\dagger c_{j+\ell}) \right] + \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sin^2 \theta_k \right\}. $$

(S42)

The constant term in the sum thereby corresponds to the extensive contribution of $\delta(0)$ of Eq. (S41). In Eq. (S42) we introduced the functions $f_1(\ell), f_2(\ell)$ encoding the non-locality of the Jordan-Wigner mapping

$$f_1(\ell) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i \ell k} \cos 2\theta_k \quad \quad f_2(\ell) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i \ell k} \sin 2\theta_k.$$ 

(S43)

Finally we can also invert the Jordan-Wigner mapping to obtain the expression in the spin basis

$$c_{j+\ell}^\dagger c_j = \sigma_{j+\ell}^\dagger e^{i \pi \sum_{i<j} \sigma_i^\dagger \sigma_i} e^{i \pi \sum_{i \geq j} \sigma_i^\dagger \sigma_i} \sigma_j^\dagger = \sigma_{j+\ell}^\dagger \left( \prod_{i=j}^{j+\ell-1} \sigma_i^\dagger \right) \sigma_j^\dagger$$

$$c_{j+\ell} c_j = \sigma_{j+\ell} e^{i \pi \sum_{i<j} \sigma_i \sigma_i^\dagger} e^{i \pi \sum_{i \geq j} \sigma_i \sigma_i^\dagger} = \sigma_{j+\ell} \left( \prod_{i=j}^{j+\ell-1} \sigma_i \right) \sigma_j,$$

(S44)

where we used the identity $\sigma_i^\dagger = e^{i \pi \sigma_i^\dagger \sigma_i}$. The number operator $N_{\text{mes}}$ is hence given by a sum of non-local string operators, whose weights are given by $f_1(\ell), f_2(\ell)$, respectively. Since $N_{\text{mes}}$ contains in general very long-ranged terms an efficient representation in terms of a MPO strongly depends on the functional form of $f_1(\ell), f_2(\ell)$. For small values of the transverse field $h_1$, we find that both $f_1(\ell)$ and $f_2(\ell)$ can be approximated by an exponential decay for $l > 0$. This enables us to make use of the efficient representation of MPOs with coefficients exponentially decaying with distance discussed e.g. in Ref. [48].
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