Fluctuations and magnetoresistance oscillations near the half-filled Landau level

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Abstract

We study magnetoresistance oscillations near the half-filled lowest Landau level ($\nu = 1/2$) that result from the presence of a periodic one-dimensional electrostatic potential using the Dirac composite fermion theory of Son, where the $\nu = 1/2$ state is described by a (2 + 1)-dimensional theory of quantum electrodynamics. Previous work showed that when gauge field fluctuations are neglected, there is a small, but systematic deviation between the locations of the oscillation minima predicted by theory and those observed in experiment [Kamburov et. al., Phys. Rev. Lett. 113, 196801 (2014)]. Here, we study how effects due to gauge fluctuations improves this comparison. Through an approximate large flavor analysis of the Schwinger-Dyson equations for the Dirac composite fermion theory, we argue that gauge field fluctuations dynamically generate a magnetic field-dependent mass for the Dirac composite fermions. We show how this mass results in a shift of the theoretically-obtained oscillation minima towards those found in experiment. We discuss how the temperature-dependent amplitude of these oscillations enables an additional way to measure this mass. At finite temperatures, away from the universal low-energy limit, the behavior of this amplitude may also distinguish the Dirac and Halperin, Lee, and Read composite fermion theories of the half-filled Landau level.
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1 Introduction and summary

In recent years, there has been a renewed debate about the effective description of the non-Fermi liquid state at a half-filled lowest Landau level ($\nu = 1/2$) of the two-dimensional electron gas. The issue concerns how an emergent Landau level particle-hole (PH) symmetry [1, 2], found in electrical Hall transport measurements [3, 4] and numerical experiments [5, 6] at $\nu = 1/2$, is compatible with the traditional theory of the half-filled Landau level of Halperin, Lee, and Read [7]. The HLR theory, which describes electrons at $\nu = 1/2$ in terms of non-relativistic composite fermions in zero effective magnetic field [8, 9], has had tremendous phenomenological success [10]. However, the flux attachment procedure that “converts” electrons into composite fermions is asymmetric in its treatment of electrons and holes [11, 12] and leads to questions like [11]: How do HLR composite fermions produce the Hall effect $\sigma_{xy}^{cf} = -\frac{1}{2} e^2/h$, required by PH symmetry at $\nu = 1/2$, when the effective magnetic field they experience is zero?

Currently, there are two converging lines of thought that point towards a resolution of this issue[1]. The first comes by way of an a priori different composite fermion theory, introduced by Son [14]. In this theory, the half-filled Landau level is described by a $(2 + 1)$-dimensional theory of quantum electrodynamics at finite density. One advantage of this Dirac composite fermion theory is that PH symmetry is a manifest invariance. On the other hand, it has recently been shown that the HLR theory can also produce PH symmetric response, if quenched disorder is properly included in the form of a precisely correlated random chemical potential and magnetic flux (as a result of the flux attachment constraint) [15–17]. In addition, mean-field treatments of both theories, where the fluctuations of the respective emergent gauge fields are ignored, have been found to produce identical predictions away from half-filling [14, 15, 18]. These results suggest that the HLR and Dirac composite fermion theories may belong to the same universality class[2].

Nevertheless, the mean-field predictions of these two theories are not quantitatively consistent with the Weiss magnetoresistance oscillations observed near $\nu = 1/2$ [21]. Prior

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1Ref. [13] studied Friedel oscillations around $\nu = 1/2$ using the Jain wave functions. Lowest-Landau level projected wave functions have identical oscillatory dependence at particle-hole conjugate filling fractions, while un-projected wave functions do not.

2There are two caveats to this discussion. Son and Levin [19] have derived a linear relation between the Hall conductivity and susceptibility that any PH symmetric theory must satisfy. As yet, it is unknown whether the HLR theory is consistent with this relation. In addition, it is unknown how the thermal Hall conductivities of the two theories compare [20].

3Weiss oscillations [22–24] are quantum oscillations that occur in electrical transport in response to the
Figure 1: Weiss oscillations of the Dirac composite fermion theory at fixed electron density $n_e$ and varying magnetic field $B$ about half-filling $B_{1/2}$. The blue curve corresponds to Dirac composite fermion mean-field theory [18]. The orange curve includes the effects of a Dirac composite fermion mass $m \propto |B - 4\pi n_e|^{1/3}B^{1/6}$, induced by gauge fluctuations, studied in this paper. The vertical black lines correspond to the oscillation minima observed experimentally [21]. Associating a magnetic field $B_d = \hbar/ed^2$ with the modulation period $d$, the result above corresponds to the choice: $B_d/B_{1/2} = 0.001$ and $k_B T = 0.3\sqrt{2B_{1/2}}$.

Work [18] found that the locations of the oscillation minima found experimentally and those predicted by Dirac composite fermion mean-field theory (as well as HLR mean-field theory) differ by as much as 2%, with the difference decreasing for minima that lie closer to $\nu = 1/2$. While this discrepancy may be small, it is systematic: any particular observed oscillation minimum is shifted away from the corresponding minimum predicted by mean-field theory inwards towards half-filling (see Fig. 1).

Here, we show how fluctuation effects in the Dirac composite fermion theory, due to the interaction between the Dirac composite fermions and an emergent gauge field, can improve the agreement between theory and experiment. To do this, we study how fluctuations modify combined effects of a transverse magnetic field and an externally-imposed periodic scalar potential. By an appropriate choice of experimental parameters, the locations of the prominent Weiss oscillation minima can be tuned to lie nearer to $\nu = 1/2$ than traditional Shubnikov de Haas oscillations and can thereby provide a more sensitive measure of the $\nu = 1/2$ state.
the Dirac composite fermion mean-field Hamiltonian. To the order that we work, we find that the Dirac composite fermion chemical potential is unaffected by fluctuations, however, the fermions obtain a dynamically-generated mass $m \propto |B - 4\pi n_e|^{1/3} B^{1/6}$, where $B > 0$ is the external magnetic field and $n_e$ is the electron density. While unbroken PH symmetry prevents a mass term at $\nu = 1/2$, such a term is to be expected away from half-filling when fluctuations are included. We determine the form of the dynamically-generated mass through an approximate large $N$ flavor analysis of the Schwinger-Dyson equations [25] for the Dirac composite fermion theory. We use the Dirac composite fermion propagator, obtained within this analysis, to specify the input parameters of the mean-field Hamiltonian. We can then readily follow our previous analysis [18] to determine the Weiss oscillations of the “fluctuation-improved” Dirac composite fermion mean-field theory. The results are summarized in Fig. 1. Evidently, inclusion of this mass provides remarkably good agreement between experiment and theory.

In addition, we comment upon the finite-temperature behavior of quantum oscillations near $\nu = 1/2$. This behavior is interesting to consider because at finite temperatures, away from the long wavelength limit, differences in the HLR and Dirac composite fermion theories may appear. Previously, it has been argued theoretically [7, 26] and experimentally supported [27] that HLR composite fermions have an effective mass proportional to the characteristic scale of the Coulomb energy, $\sqrt{B}$. While this leads to qualitative agreement between the finite-temperature behavior of the Shubnikov de Haas oscillations of the two theories, the temperature dependence of the Weiss oscillations appear to differ.

The remaining sections are organized as follows. In 2 we review the Dirac composite fermion theory. In 3 we analyze the Schwinger-Dyson equations and obtain an approximate solution. We use the chemical potential and mass of the corrected Dirac composite fermion propagator as input parameters for the analysis of Weiss oscillations of the “fluctuation-improved” Dirac composite fermion mean-field theory in 4. We discuss the consequences of this analysis and also comment upon how such behavior might be found in the HLR theory in 5 and conclude in 6. Appendix A contains details of a few calculations summarized in the main text.
2 Dirac composite fermions: review

Electrons in the lowest Landau level near half-filling can be described by a Lagrangian of a 2-component Dirac electron $\Psi_e$ [14]:

$$L_e = \overline{\Psi}_e \gamma^\alpha (i\partial_\alpha + A_\alpha) \Psi_e - m_e \overline{\Psi}_e \Psi_e + \frac{1}{8\pi} \epsilon^{\alpha\beta\sigma} A_\alpha \partial_\beta A_\sigma + \ldots,$$

(2.1)

where $A_\alpha$ with $\alpha \in \{0, 1, 2\}$ is the background electromagnetic gauge field, $\overline{\Psi}_e = \Psi_e^* \gamma^0$, the $\gamma$ matrices $\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^1$, $\gamma^2 = i\sigma^2$ satisfy the Clifford algebra $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$ with $\eta^{0\beta} = \text{diag}(+1, -1, -1)$, and the anti-symmetric symbol $\epsilon^{012} = 1$. The benefit of the Dirac formulation is that the limit of infinite cyclotron energy $\omega_c = B/m$ can be smoothly achieved at fixed external magnetic field $B = \partial_1 A_2 - \partial_2 A_1 > 0$ by taking the electron mass $m_e \to 0$. The \ldots include additional interactions, e.g., the Coulomb interaction and coupling to disorder, that we exclude in this discussion. Throughout, we simplify expressions by setting $e = \hbar = c = 1$.

The electron density,

$$n_e = \Psi_e^* \Psi_e + \frac{B}{4\pi},$$

(2.2)

Consequently, when $\nu \equiv 2\pi n_e/B = 1/2$, the Dirac electrons half-fill the zeroth Landau level. For $m_e = 0$ and $\nu = 1/2$, the Dirac Lagrangian is invariant under the anti-unitary ($i \mapsto -i$) PH transformation that takes $(t, x, y) \mapsto (-t, x, y)$,

$$\Psi_e \mapsto -\gamma^0 \Psi_e^*,$$

$$A_0, A_1, A_2) \mapsto (-A_0, A_1, A_2),$$

(2.3)

and shifts the Lagrangian by a filled Landau level $L_e \mapsto L_e + \frac{1}{4\pi} \epsilon^{\alpha\beta\sigma} A_\alpha \partial_\beta A_\sigma$.

Son [14] conjectured that $L_e$ is dual to the Dirac composite fermion Lagrangian,

$$L = \overline{\psi} \gamma^\alpha (i\partial_\alpha + a_\alpha) \psi - m \overline{\psi} \psi - \frac{1}{4\pi} \epsilon^{\alpha\beta\sigma} a_\alpha \partial_\beta A_\sigma + \frac{1}{8\pi} \epsilon^{\alpha\beta\sigma} A_\alpha \partial_\beta A_\sigma + \ldots,$$

(2.4)

where $\psi$ is the electrically-neutral Dirac composite fermion, $a_\alpha$ is a dynamical or emergent $U(1)$ gauge field, and $m \propto m_e$ is the Dirac composite fermion mass. $A_\alpha$ remains a non-dynamical gauge field, whose primary role in $L$ is to determine how electromagnetism enters the Dirac composite fermion theory. As before, the \ldots represent additional interactions and couplings to disorder. In addition, the \ldots can include higher-order terms involving the gauge field, e.g., a Maxwell term $-\frac{1}{2} f_{\alpha\beta}^2$ for $a_\alpha$. 5
The $a_0$ equation of motion implies the Dirac composite fermion density:\footnote{This equation is valid at weak coupling. At strong coupling, the right-hand side of Eq. (2.5) generally receives corrections from the \ldots and should be replaced by $-\frac{\delta \mathcal{L}}{\delta a_0} + \psi^\dagger \psi$.}

$$\psi^\dagger \psi = \frac{B}{4\pi}. \quad (2.5)$$

In the Dirac composite fermion theory, the electron density,

$$n_e = \frac{1}{4\pi} (-b + B), \quad (2.6)$$

where the effective magnetic field $b = \partial_1 a_2 - \partial_2 a_1$. Thus, the statement of the duality is that the half-filled lowest Landau level is alternatively described by a $(2 + 1)$-dimensional quantum electrodynamics theory of a Dirac composite fermion in zero effective field at a density fixed by the external magnetic field. By now, there is a great deal of evidence for the duality between $\mathcal{L}$ and $\mathcal{L}_e$, e.g., [6, 28–34]). See [35] for a recent review and for additional references.

In the Dirac composite fermion theory, the PH transformation takes $(t, x, y) \mapsto (-t, x, y),$

$$\psi \mapsto \gamma^2 \psi,$$

$$(a_0, a_1, a_2) \mapsto (a_0, -a_1, -a_2),$$

$$(A_0, A_1, A_2) \mapsto (-A_0, A_1, A_2), \quad (2.7)$$

and shifts the Lagrangian by a filled Landau level. Intuitively, the PH transformation acts on the dynamical fields of $\mathcal{L}$ like a time-reversal transformation. As such, PH symmetry requires $m = 0$ and forbids a Chern-Simons term for $a_\alpha$. So long as $m = 0$ and $\nu = 1/2$, the Dirac composite fermion theory is invariant under the PH transformation of Eq. (2.7).

Away from half-filling, PH symmetry is necessarily broken. Consequently, we can no longer exclude any PH breaking term allowed by symmetry. In particular, we generally expect a Dirac mass to be induced by fluctuations. If PH symmetry is not spontaneously broken at half-filling, we expect any dynamically-generated mass to be a regular function of $b = B - 4\pi n_e \neq 0$. In the next section, we study the Schwinger-Dyson equations to determine how $m$ depends on $b$ within a controlled expansion where the number of fermion flavors $N \to \infty$. We study the effects of $m$ on the Weiss oscillations of the Dirac composite fermion theory in §4.
3 Dynamical mass generation in an effective magnetic field

In the theory of a free Dirac fermion at zero density, a uniform magnetic field sources a vacuum expectation value for the mass operator. Short-ranged attractive interactions can then induce a non-zero mass term in its effective Lagrangian [36]. Our goal in this section is to determine how a similar phenomenon occurs in the Dirac composite fermion theory. We first summarize the relevant formalism that we use to study Dirac composite fermions in a uniform effective magnetic field \( b \). Then, we analyze the Schwinger-Dyson equations for the Dirac composite fermion theory away from half-filling when the fluctuations of the emergent gauge field \( a_\alpha \) about a uniform \( b \neq 0 \) are considered.

3.1 Dirac fermions in a magnetic field

Beginning with the works of Schwinger [37] and Ritus [38], there have been a number of studies on the effects of a background magnetic field on quantum electrodynamics in various dimensions. In this paper, we rely most heavily on Refs. [39–41]. See Ref. [42] for an excellent introduction to this formalism and for additional references.

At tree-level, the time-ordered real-space propagator \( G_0(x, y) \) for a massive Dirac fermion in a uniform magnetic field \((\vec{a}_0, \vec{a}_1, \vec{a}_2) = (0, 0, bx_1)\) can be written in the form,

\[
G_0(x, y) = e^{i\Phi(x, y)} \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}_\alpha(x-y)} G_0(p),
\]

(3.1)

where the Schwinger phase,

\[
\Phi(x, y) = -\frac{b}{2} (x_2 - y_2)(x_1 + y_1).
\]

(3.2)

The tree-level pseudo-momentum-space propagator,

\[
-iG_0(p) = i \int_0^\infty dse \left[ (p_0 + \mu_0 + i\epsilon p_0)^2 - m_0^2 + i\delta - \frac{p_1^2 + p_2^2}{bs} \tan(bs) \right]^2 \left[ (p_0 + \mu_0)\gamma^0 \tan(bs) + p_i \gamma^i \tan^2(bs) \right].
\]

(3.3)

where the pseudo-momenta \( p = (p_0, p_1, p_2) \) are analogous to the conserved momenta in a translationally-invariant system, \( \mu_0 \) is a bare chemical potential, \( m_0 \) is a constant bare mass, \( \epsilon p_0 = \text{sign}(p_0)\epsilon \) with the infinitesimal \( \epsilon > 0 \) ensures the Feynman pole prescription is satisfied, \( \delta > 0 \) is an infinitesimal included for convergence of the \( s \) integral, and \( \mathbb{I} \) is the \( 2 \times 2 \) identity matrix. To linear order in \( b \):

\[
-iG_0(p) \equiv \frac{(p_0 + \mu_0\delta_{\alpha,0})\gamma^\alpha + m_0\mathbb{I}}{(p_0 + \mu_0 + i\epsilon p_0)^2 - p_1^2 - m_0^2} + b \frac{(p_0 + \mu_0)\mathbb{I} + m_0\gamma^0}{((p_0 + \mu_0 + i\epsilon p_0)^2 - p_1^2 - m_0^2)^2} + O(b^2).
\]

(3.4)
Because we are interested in applying this formalism to the Dirac composite fermion theory when \(|b|/\mu_0^2 \ll 1\), near \(\nu = 1/2\), we drop all \(O(b^2)\) and higher terms in the pseudo-momentum-space propagator. For convenience, we use \(G_0(p)\) to denote the linear expansion in Eq. (3.4) with higher order in \(b\) terms excluded.

The tree-level inverse propagator \(G^{-1}_0(x, y)\) by definition satisfies

\[
\int d^3 y \ G^{-1}_0(x, y) G_0(y, z) = \delta^{(3)}(x - z). \tag{3.5}
\]

It takes a particularly simple form:

\[
iG^{-1}_0(x, y) = e^{i\Phi(x, y)} \int \frac{d^3 p}{(2\pi)^3} e^{ip_\alpha (x - y)\alpha} \left((p_\alpha + \mu_0 \delta_{\alpha, 0})\gamma^\alpha - m_0\mathbb{I}\right). \tag{3.6}
\]

In contrast to \(G_0(x, y)\), the magnetic field dependence is entirely parameterized by the Schwinger phase in \(G^{-1}_0(x, y)\). Both the propagator and its inverse are obtained after performing an infinite sum over all Landau levels. The benefit of the forms for \(G_0(x, y)\) and its inverse in Eqs. (3.1) and (3.6) is that they allow for the straightforward expansion about their translationally-invariant forms at \(b = 0\). See [10] for further discussion of \(G_0(x, y)\) and \(G^{-1}_0(x, y)\).

We use the following ansatz for the exact real-space propagator:

\[
G(x, y) = e^{i\Phi(x, y)} \int \frac{d^3 p}{(2\pi)^3} e^{ip_\alpha (x - y)\alpha} G(p). \tag{3.7}
\]

For the exact pseudo-momentum propagator \(G(p)\), we write

\[
-iG(p) = -iG^{(0)}(p) - iG^{(1)}(p), \tag{3.8}
\]

where

\[
-iG^{(0)}(p) = \frac{(p_\alpha + \mu_0 \delta_{\alpha, 0} - \Sigma_\alpha(p))\gamma^\alpha + \Sigma_m(p)\mathbb{I}}{(p_0 + \mu_0 - \Sigma_0(p) + i\epsilon_{p_0})^2 - (p_i - \Sigma_i(p))^2 - \Sigma^2_m(p)}, \tag{3.9}
\]

\[
-iG^{(1)}(p) = b \frac{(p_0 + \mu_0 - \Sigma_0(p))^i + \Sigma_m(p)\gamma^0}{\left((p_0 + \mu_0 - \Sigma_0(p) + i\epsilon_{p_0})^2 - (p_i - \Sigma_i(p))^2 - \Sigma^2_m(p)\right)^2}. \tag{3.10}
\]

In contrast to the tree-level pseudo-momentum propagator, \(G_0(p)\), both \(G^{(0)}(p)\) and \(G^{(1)}(p)\) are expected to depend upon \(b\) through the self-energies \(\Sigma_m(p)\) and \(\Sigma_\alpha(p)\), in addition to the explicit linear dependence that appears in \(G^{(1)}(p)\). We write the exact inverse propagator as

\[
iG^{-1}(x, y) = e^{i\Phi(x, y)} \int \frac{d^3 p}{(2\pi)^3} e^{ip_\alpha (x - y)\alpha} \left((p_\alpha + \mu_0 \delta_{\alpha, 0} - \Sigma_\alpha(p))\gamma^\alpha - \Sigma_m(p)\mathbb{I}\right). \tag{3.11}
\]
In both $G(p)$ and $G^{-1}(p)$, we have set the bare mass $m_0 = 0$; this corresponds to assuming unbroken PH symmetry at $\nu = 1/2$. Our ansatze for the exact propagator and its inverse are motivated by the tree-level expressions to which they reduce when the self-energies are set to zero. The ansatze for the exact propagator and its inverse are simplifications of that which symmetry allows for a Dirac fermion in a magnetic field [10]. Nevertheless, our ansatze are consistent to leading order in a $1/N$ analysis of the Schwinger-Dyson equations described in the next section.

In general, the self-energies $\Sigma_m(p)$ and $\Sigma_\alpha(p)$ are non-trivial functions of the pseudo-momenta $p$. For sufficiently small $|b|/\mu_0^2$, we expect the low-energy dynamics to be dominated by fluctuations about the chemical potential $\mu_0$. In terms of the pseudomomenta $p$, this corresponds to evaluating the self-energies at the Fermi surface defined by $\mu_0$. Thus, in order to simplify our analysis below, we replace the self-energies as follows:

$$\Sigma_m(p_{FS} + \delta p) \mapsto \Sigma_m(p_{FS}), \quad (3.12)$$

$$\Sigma_\alpha(p_{FS} + \delta p) \mapsto \delta \mu_0 \Sigma_0(p_{FS}) + \delta p_\alpha \Sigma'_\alpha(p_{FS}), \quad (3.13)$$

where $p_{FS} = (0, p_i)$ with $p_i^2 = \mu_0^2$, $|\delta p_\alpha| \ll \mu$, $\Sigma'_\alpha(p_{FS}) = \partial_{p_\alpha} \Sigma_\alpha(p = p_{FS})$, and there is no sum over $\alpha$ in Eq. (3.13).

We define the dynamically-generated mass,

$$m = \frac{\Sigma_m(p_{FS})}{1 - \Sigma_0(p_{FS})} \equiv \frac{\Sigma_m}{1 - \Sigma_0}. \quad (3.14)$$

In addition, there will generally be adjustments to the chemical potential that respond to any corrections to the Dirac composite fermion density and/or dynamically-generated mass. Thus, we define the quantum-corrected chemical potential,

$$\mu = \frac{\mu_0 - \Sigma_0}{1 - \Sigma_0}. \quad (3.15)$$

We now study whether $m$ is generated dynamically and how $\mu$ is corrected by fluctuations of $a_\alpha$ when $b \neq 0$ in the Dirac composite fermion theory.

### 3.2 Schwinger-Dyson equations: setup

The Schwinger-Dyson equations [25] are a set of coupled integral equations that relate the exact fermion and gauge field propagators to one another by way of the exact cubic interaction vertex $\Gamma^{\alpha}$ coupling the Dirac composite fermion current to $a_\alpha$. We will not solve the equations exactly; rather, we seek an approximate solution that one obtains within a
large flavor generalization of the Dirac composite fermion theory. We hope this approximate solution reflects the qualitative behavior of the Dirac composite fermion theory.

Specifically, we consider the Lagrangian,

\[ \mathcal{L}_N = \bar{\psi}_n \gamma^\alpha (i \partial_\alpha + a_\alpha) \psi_n - \frac{N}{4\pi} \epsilon^{\alpha\beta\sigma} a_\alpha \partial_\beta A_\sigma + \frac{N}{8\pi} \epsilon^{\alpha\beta\sigma} A_\alpha \partial_\beta A_\sigma, \] (3.16)

where the different fermion flavors are labeled by \( n = 1, \ldots, N \). When \( N = 1 \), we recover the Dirac composite fermion theory. The analog of Eq. (2.6) now becomes

\[ n e = \frac{N}{4\pi} (B - b). \]

Thus, in our large \( N \) theory, half-filling means \( \nu = N/2 \). To make contact with the formalism of §3.1, we introduce a \( SU(N) \)-invariant chemical potential \( \mu_0 = \sqrt{B} \) (corresponding to a tree-level Dirac composite fermion density proportional to \( B > 0 \)) and we factor out the uniform effective magnetic field \((\bar{a}_0, \bar{a}_1, \bar{a}_2) = (0, 0, bx_1)\) that is generated away from half-filling from the dynamical fluctuations of the emergent gauge field. Setting \( A_\alpha = 0 \), Eq. (3.16) becomes

\[ \mathcal{L}_N = \bar{\psi}_n \gamma^\alpha (i \partial_\alpha + \bar{a}_\alpha + a_\alpha) \psi_n + \mu_0 \bar{\psi}_n \psi_n. \] (3.17)

This is the large \( N \) theory that we analyze.

To leading order in \( 1/N \), the Schwinger-Dyson equations for \( \mathcal{L}_N \) in Eq. (3.17) take the form:

\[ iG^{-1}(x, y) - iG_0^{-1}(x, y) = \gamma^\alpha G(x, y) \gamma^\beta \Pi^{-1}_{\alpha\beta}(x - y), \] (3.18)

\[ i\Pi_{\alpha\beta}(x - y) = N \text{tr} \left[ \gamma^\alpha G(x, y) \gamma^\beta G(y, x) \right], \] (3.19)

where \( \Pi_{\alpha\beta}(x - y) \) is the exact gauge field self-energy and we have used a fermion propagator \( G_{n,n'}(x, y) = G(x, y) \delta_{n,n'} \) that is diagonal in flavor space. (The factor of \( N \) in Eq. (3.19) arises from the \( N \) flavors in the fermion loop.) Notice that the gauge field propagator used in Eq. (3.18) is the inverse of its self-energy. This is a result of the absence of a kinetic term for \( a_\alpha \) in \( \mathcal{L}_N \). Within the random phase approximation, the Ward identity implies that there are no corrections to the cubic interaction vertex when \( b = 0 \). Any such corrections must be higher-order in \( 1/N \) and so to leading order, we have replaced the exact vertex \( \Gamma^\alpha \) by its tree-level value \( \gamma^\alpha \). We expect a non-zero effective magnetic field to soften the effects of the emergent gauge field and that this approximation will only improve for \( b \neq 0 \).

Defining the Fourier transform \( \Pi_{\alpha\beta}(p) \),

\[ \Pi_{\alpha\beta}(y - x) = \int \frac{d^3 p}{(2\pi)^3} e^{ip_\sigma(x - y)^\sigma} \Pi_{\alpha\beta}(p), \] (3.20)

5Furthermore, there are no corrections to this vertex if the Dirac composite fermion is given a non-zero bare mass \( m_0^2 = \mu_0^2 \) at \( b = 0 \). We thank N. Rombes and S. Chakravarty for correspondence on this point.
the Schwinger-Dyson equations \((3.18)\) and \((3.18)\) become \[40\]

\[
i\Sigma_\alpha(q)\gamma^\alpha + i\Sigma_m(q)\mathbb{I} = \int \frac{d^3p}{(2\pi)^3} \gamma^\alpha G(p + q)\gamma^\beta \Pi_{\alpha\beta}^{-1}(p),
\]

\[
i\Pi^{\alpha\beta}(\delta q) = N\int \frac{d^3p}{(2\pi)^3} \text{tr} \left[\gamma^\alpha G(p)\gamma^\beta G(p + \delta q)\right],
\]

where \(q = q_{FS} + \delta q_\alpha\) with \(|\delta q_\alpha| \ll \mu\) parameterizes the deviation away from the Fermi surface. It is these equations we aim to approximately solve.

In our analysis of Eqs. \((3.21)\) and \((3.22)\) at \(b \neq 0\), we set \(\Sigma_\alpha = 0\) in the expression for \(G(p)\). The resulting Schwinger-Dyson equations dramatically simplify and we are able to determine a self-consistent solution for \(\Pi^{\alpha\beta}\) and \(\Sigma_m\). More explicitly, we consider the large \(N\) expansion for the fermion self-energies,

\[
\Sigma_\alpha = \Sigma^{(1)}_\alpha + \Sigma^{(2)}_\alpha + \ldots,
\]

\[
\Sigma_m = \Sigma^{(1)}_m + \Sigma^{(2)}_m + \ldots.
\]

(3.23)

To leading order, i.e., ignoring all \(\Sigma^{(i)}_\alpha, \Sigma^{(i)}_m\) with \(i \geq 2\), we take \(\Sigma_\alpha = \Sigma^{(1)}_\alpha = 0\) and \(\Sigma_m = \Sigma^{(1)}_m\). Because there is no perturbative correction to \(\Sigma_m\) at leading order in \(1/N\), the solution we find for \(\Sigma_m\) is necessarily non-perturbative. With the solutions for \(\Sigma_m\) and \(\Pi^{\alpha\beta}\), we then calculate the leading perturbative corrections to \(\Sigma_\alpha = \Sigma^{(2)}_\alpha\) and verify that \(\Sigma_\alpha \rightarrow 0\) as \(N \rightarrow \infty\). This allows us to find the dynamically-generated mass \(m = \Sigma^{(1)}_m\) (to the order that we work) using Eq. \((3.14)\), the corrected chemical potential \(\mu = \mu_0\) using Eq. \((3.15)\), and to estimate where our approximation of setting \(\Sigma_\alpha = 0\) in the propagator \(G(p)\) is expected to be valid.

### 3.3 Gauge field self-energy

There is no kinetic term for the gauge field in the tree-level Lagrangian \(\mathcal{L}_N\). Even if a Maxwell term for \(a_\alpha\) were added to \(\mathcal{L}_N\), its effects would be subdominant at low energies. Instead, the low-energy dynamics of the gauge field are generated by fluctuations of the Dirac composite fermion. Thus, we first study Eq. \((3.22)\) to determine the leading contribution to the gauge field propagator and then consider Eq. \((3.21)\) for the fermion self-energy.

Gauge invariance and PH symmetry constrain the form of \(\Pi^{\alpha\beta}(p)\):

\[
\Pi^{\alpha\beta}(p) = \Pi^{\alpha\beta}_{\text{even}}(p) + \Pi^{\alpha\beta}_{\text{odd}}(p),
\]

(3.24)

\[6\] At \(b = 0\), the low energy and \(N \rightarrow \infty\) limits do not commute \[45\][47]. We leave a careful analysis of this when \(b \neq 0\) for a future study.
where $\Pi^{\alpha\beta}_{\text{even}}(p)$ is even under a PH transformation and $\Pi^{\alpha\beta}_{\text{odd}}(p)$ is odd under a PH transformation. To leading order in $b$, we substitute $G(p) = G^{(0)}(p)$ into Eq. (3.22):

$$i\Pi^{\alpha\beta}_{\text{even}}(\delta q) + i\Pi^{\alpha\beta}_{\text{odd}}(\delta q) = N\int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \gamma^\alpha G^{(0)}(p) \gamma^\beta G^{(0)}(p + \delta q) \right].$$

(3.25)

Below, we focus on the effects of $\Pi^{\alpha\beta}_{\text{odd}}(p)$ and ignore any contributions from $\Pi^{\alpha\beta}_{\text{even}}(p)$ to the fermion self-energy (see [48] for an explicit expression for $\Pi^{\alpha\beta}_{\text{even}}$). We believe $\Pi^{\alpha\beta}_{\text{odd}}(p)$ provides the dominant source for the generation of a PH symmetry-breaking mass $m$. The logic behind this assertion, which leads to a drastic simplification of the analysis, is the following. At $b = 0$, unbroken PH symmetry forces $\Pi^{\alpha\beta}_{\text{odd}}(p) = 0$. For $b \neq 0$, a non-zero $\Pi^{\alpha\beta}_{\text{odd}}(p)$ is allowed and we show below that $\Pi^{\alpha\beta}_{\text{odd}}(p) \propto \Sigma_m$, which vanishes at $b = 0$. Thus, there is a relative enhancement by $\Sigma^{-1}_m$ of $(\Pi^{\alpha\beta}_{\text{odd}})(p)$ relative to $(\Pi^{\alpha\beta}_{\text{even}})^{-1}(p)$. It would be interesting to carry out a critical analysis of this assumption.

Writing $\Pi^{\alpha\beta}_{\text{odd}}(p) = i\epsilon^{\alpha\beta\sigma} p_\sigma \Pi^{\alpha\beta}_{\text{odd}}(p)$, we compute:

$$\epsilon^{\alpha\beta\sigma} \delta q_\sigma \Pi^{\alpha\beta}_{\text{odd}}(\delta q) = -N \left\{ \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \gamma^\alpha G^{(0)}(p) \gamma^\beta G^{(0)}(p + \delta q) \right] \right\}_{\text{odd}},$$

(3.26)

where $\{\cdot\}_{\text{odd}}$ indicates the odd term is isolated. To leading order in the derivative expansion, i.e., $\Pi^{\alpha\beta}_{\text{odd}}(\delta q = 0)$, we find

$$\Pi^{\alpha\beta}_{\text{odd}}(0) = \frac{N}{4\pi} \left( \Theta(|\Sigma_m| - \mu_0) \frac{\Sigma_m}{|\Sigma_m|} + \Theta(\mu_0 - |\Sigma_m|) \frac{\Sigma_m}{\mu_0} \right),$$

(3.27)

where $\Theta(x)$ is the step function. Since we are in the regime where $\mu_0 > |\Sigma_m|$, Eq. (3.27) implies an effective Chern-Simons term for $a_\alpha$ with level,

$$k = \frac{N \Sigma_m}{2 \mu_0},$$

(3.28)

is generated if $\Sigma_m \neq 0$. (This non-quantized Chern-Simons level is reminiscent of the anomalous Hall effect [49].) Such a Chern-Simons term is consistent with the broken PH symmetry reflected by any non-zero $\Sigma_m$.

To obtain the effective gauge field propagator, we add the covariant gauge fixing term $-\frac{1}{2} q^\alpha q^\beta$ to $\Pi^{\alpha\beta}_{\text{odd}}(q)$ (thinking of $\Pi^{\alpha\beta}_{\text{odd}}(q)$ as the only term in an effective Lagrangian for $a_\alpha$) and invert. Choosing Feynman gauge $\xi = 0$, we obtain the effective gauge field propagator:

$$\left( \Pi^{\alpha\beta}_{\text{odd}}(q) \right)^{-1} = \frac{2\pi \epsilon^{\alpha\beta\sigma} q_\sigma}{k q^2},$$

(3.29)

where $k$ is given in Eq. (3.28).
3.4 Fermion self-energy

We now study Eq. (3.21) for the $\Sigma_m$ and $\Sigma_0$ components of the Dirac composite fermion self-energy using the effective gauge field propagator in Eq. (3.29).

3.4.1 $\Sigma_m$

Taking the trace of both sides of Eq. (3.21) and setting $\delta q_\alpha = 0$, we find:

$$i\Sigma_m(q_{FS}) = iM^{(0)}(q_{FS}) + iM^{(1)}(q_{FS}),$$

(3.30)

where

$$iM^{(0)}(q_{FS}) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \text{tr} \left[ \gamma^\alpha G^{(0)}(p + q_{FS}) \gamma^\beta \left( \frac{2\pi}{k} \epsilon_{\alpha\beta\sigma} p^\sigma \right) \right],$$

(3.31)

$$iM^{(1)}(q_{FS}) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \text{tr} \left[ \gamma^\alpha G^{(1)}(p + q_{FS}) \gamma^\beta \left( \frac{2\pi}{k} \epsilon_{\alpha\beta\sigma} p^\sigma \right) \right],$$

(3.32)

and $G^{(0)}(p)$ and $G^{(1)}(p)$ are given in Eqs. (3.9) and (3.10). Recall that we set $\Sigma_\alpha = 0$ and only retain $\Sigma_m$ when using $G^{(0)}(p)$ and $G^{(1)}(p)$ to evaluate $M^{(0)}$ and $M^{(1)}$. The details of our evaluation of $M^{(0)}$ and $M^{(1)}$ are given in Appendix A. Here, we quote the results:

$$M^{(0)} = -\frac{2\mu_0 \text{sign}(\Sigma_m)}{N},$$

(3.33)

$$M^{(1)} = \frac{2}{3} \frac{b\mu_0^2}{N |\Sigma_m|^3}.$$  

(3.34)

Using the above expressions in Eq. (3.30) we can solve for $\Sigma_m$ as:

$$\Sigma_m = -\frac{2\mu_0 \text{sign}(\Sigma_m)}{N} + \frac{2}{3} \frac{b\mu_0^2}{N |\Sigma_m|^3}.$$  

(3.35)

When $b = 0$, the only solution is $\Sigma_m = 0$, consistent with our expectation that PH symmetry is unbroken at $\nu = 1/2$. Dimensional analysis and $1/N$ scaling implies

$$\Sigma_m = \frac{\mu_0}{N} f \left( \frac{bN^3}{\mu_0^2} \right).$$

(3.36)

We find that $f \left( \frac{bN^3}{\mu_0^2} \right)$ has the following asymptotics: for fixed $|b|/\mu_0^2$,

$$\lim_{N \to \infty} \frac{1}{N} f \left( \frac{bN^3}{\mu_0^2} \right) = 0;$$

(3.37)
while for fixed $N$,
\[
\lim_{|b|/\mu_0^2 \to 0} f \left( \frac{bN^3}{\mu_0^2} \right) = 0. \tag{3.38}
\]

Unfortunately, it is not possible to find a simple, explicit expression that is faithful to the two limits in Eqs. (3.37) and (3.38): the two limits, $\lim_{x \to \infty} f(x)$ and $\lim_{x \to 0} f(x)$, are incompatible. For large $N > 10$ and fixed $|b|/\mu_0^2 \sim 10^{-1}$, we can fit
\[
\Sigma_m = \mu_0 \text{sign}(b) \left( \frac{|b|}{\mu_0^2 N} \right)^{1/4} \left[ c_1 + c_2 \left( \frac{\mu_0^2}{|b|N^3} \right)^{1/4} + \ldots \right], \tag{3.39}
\]
where $c_1 \approx 0.9$, $c_2 \approx -0.5$, and the $\ldots$ are suppressed as $N \to \infty$. On the other hand, for $|b|/\mu_0^2 \ll 1$ and fixed $N$, $\Sigma_m$ has the expansion:
\[
\Sigma_m = \mu_0 \text{sign}(b) \left( \frac{|b|}{\mu_0^2} \right)^{1/3} \left[ c_3 + c_4 \left( \frac{|b|N^3}{\mu_0^2} \right)^{1/3} + \ldots \right], \tag{3.40}
\]
where $c_3 \approx 0.69$, $c_4 \approx -0.08$, and the $\ldots$ vanish as $|b|/\mu_0^2 \to 0$.

### 3.4.2 $\Sigma_0$

We now consider $\Sigma_0$. This allows us to calculate $\Sigma'_0$ and the shift of the tree-level chemical potential $\mu_0$. We find that the leading correction to $\Sigma'_0 \ll 1$ as $N \to \infty$. We have checked that the other components of $\Sigma_\alpha$ are likewise suppressed at large $N$; as such and because they do not enter our subsequent calculations, we will not discuss them further. This justifies our setting of $\Sigma_\alpha = 0$ in the fermion propagator $G(p)$ in our large $N$ analysis.

To evaluate the leading non-zero correction to $\Sigma_0$ that one obtains when $G(p) = G^{(0)}(p)$, we multiply both sides of Eq. (3.21) by $\gamma^0$ on the left, and take the trace to find:
\[
i \Sigma_0(q) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \gamma^0 \gamma^\alpha G^{(0)}(p + q) \gamma^\beta \left( \frac{2\pi}{k} \frac{\epsilon_{\alpha\beta\gamma}q^\gamma}{p^2} \right) \right], \tag{3.41}
\]
where $q^\alpha = q^\alpha_{\text{FS}} + q_0 \delta^\alpha_0$. As detailed in Appendix A, we find the leading non-zero correction $\Sigma_0 = \Sigma^{(2)}_0$ (see Eq. (3.23)) for $|q_0|/\mu_0 \ll \Sigma_m^2/\mu_0^2$,
\[
i \Sigma_0(q_{\text{FS}}) = -i \frac{2\mu_0}{3N|\Sigma_m|} (q_0 + \mu_0) \tag{3.42}
\]
Using Eq. (3.39) for $\Sigma_m$ in the above, one finds $\Sigma_0 \propto \Sigma'_0 \propto N^{-3/4}$ which vanishes as $N \to \infty$ for finite $|b|/\mu_0^2$. Next-order terms in $\Sigma_\alpha$ are obtained by self-consistently solving the Schwinger-Dyson equations with propagators corrected by the leading self-energy corrections. We leave a detailed study for future work. Because $\Sigma_m$ vanishes at half-filling, we may only ignore $\Sigma'_0$ for sufficiently large $|b|/\mu_0^2$ at large $N$. 

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3.4.3 Dynamically-generated mass and corrected chemical potential

We are now ready to evaluate Eq. (3.14) for the dynamically-generated mass. We use Eq. (3.40) to extrapolate our large $N$ solution for $\Sigma_m$ to $N = 1$ and Eq. (3.23) to evaluate

$$m = \frac{\Sigma_m^{(1)}}{1 - \Sigma_0^{(1)}} \approx 0.69 \text{sign}(b)|b|^{1/3}B^{1/6},$$

(3.43)

where we plugged in $\mu_0 = \sqrt{B}$. The specific behavior of the mass $m$, away from $\nu = 1/2$, depends on whether the electron density $n_e$ or external magnetic field $B$ is varied. At fixed $B$, the magnitude of $m$ is symmetric if $n_e$ is varied about half-filling; on the other hand, $|m|$ is asymmetric for fixed $n_e$ and varying $B$. Using Eqs. (3.15) and (3.23), the corrected chemical potential,

$$\mu = \mu_0 - \frac{\Sigma_m^{(1)}}{1 - \Sigma_0^{(1)}} = \sqrt{B}.$$

(3.44)

This result and Eq. (3.42) imply that the leading fluctuation-induced shifts in the Dirac composite fermion density (due to gauge field fluctuations with an effective Chern-Simons propagator) and mass cancel in such a way that the chemical potential is unaffected. Of course, there is no reason to trust these results when $N = 1$; we hope these expressions capture the leading qualitative effects of fluctuations within the Dirac composite fermion theory.

In our analysis of the Weiss oscillations in the next section, we ignore all higher-order in $1/N$ corrections and assume that a mass term is the dominant correction to the Dirac composite fermion mean-field Hamiltonian away from $\nu = 1/2$. The chemical potential for this fluctuation-improved mean-field Hamiltonian will be taken to be $\mu = \sqrt{B}$.

4 Weiss oscillations of massive Dirac composite fermions

Following earlier work [18, 50–52], we now study the effect of the field-dependent mass of Eq. (3.43) on the Weiss oscillations near $\nu = 1/2$ using the fluctuation-improved Dirac composite fermion mean-field theory. We find that a non-zero mass results in an inward shift of the locations of the oscillation minima toward half-filling.

---

7 This philosophy can be summarized as follows: set $N = 1$ and hope for the best.
4.1 Setup

We are interested in determining the quantum oscillations in the electrical resistivity near \( \nu = 1/2 \) that result from a one-dimensional periodic scalar potential. In the Dirac composite fermion theory, the dc electrical conductivity,

\[
\sigma_{ij} = \frac{1}{4\pi} \left( \epsilon_{ij} - \frac{1}{2} \epsilon_{ik}(\sigma^\psi)_{kl}^{-1} \epsilon_{lj} \right),
\]

where the (dimensionless) dc Dirac composite fermion conductivity

\[
\sigma^\psi_{ij} = \lim_{q_0 \to 0} \frac{\langle \overline{\psi} \gamma_i \psi(-q_0) \overline{\psi} \gamma_j \psi(q_0) \rangle_{iq_0}}{iq_0}.
\]

Thus, the longitudinal electrical resistivity,

\[
\rho_{ii} \propto |\epsilon_{ij}| \sigma^\psi_{jj},
\]

where there is no sum over repeated indices. When a one-dimensional periodic scalar potential, \( A_0 = V \cos(Kx_1) \) with \( K = 2\pi/d \), is applied to the electronic system, the \( a_2 \) equation of motion following from the Dirac composite fermion Lagrangian (2.4) implies

\[
\overline{\psi} \gamma^2 \psi = -\frac{KV}{4\pi} \sin(Kx).
\]

We accommodate this current modulation within Dirac composite fermion mean-field theory by turning on a modulated perturbation to the emergent vector potential,

\[
\delta \vec{a} = \left( 0, W \sin(Kx_1) \right),
\]

where \( W = W(V) \) vanishes when \( V = 0 \). (Fluctuations will also generate a modulation in the Dirac composite fermion chemical potential; we ignore such effects here.) Putting together Eqs. (4.3) and (4.5), our goal in this section is to determine the correction to \( \sigma^\psi_{jj} \) and, therefore \( \rho_{ii} \), due to \( \delta \vec{a} \),

\[
\Delta \rho_{ii} \propto |\epsilon_{ij}| \Delta \sigma^\psi_{jj},
\]

In Dirac composite fermion mean-field theory, corrected by Eq. (3.43), the calculation of \( \Delta \sigma^\psi_{ij} \) simplifies to the determination of the conductivity of a free massive Dirac fermion. We use the Kubo formula [53] to find the conductivity correction:

\[
\Delta \sigma^\psi_{ij} = \frac{1}{L_1 L_2} \Sigma_M \left( \partial_{E_M} f_D(E_M) \right) \tau(E_M) v_i^M v_j^M,
\]

This equality is true at weak coupling; at strong coupling, \( \langle \overline{\psi} \gamma_i \psi(-q_0) \overline{\psi} \gamma_j \psi(q_0) \rangle \) should be replaced by the exact gauge field \( a_\alpha \) self-energy, evaluated at \( q_1 = q_2 = 0 \).
where $L_1$ ($L_2$) is the length of the system in the $x_1$-direction ($x_2$-direction), $\beta^{-1} = T$ is the temperature, $M$ denotes the quantum numbers of the single-particle states, $f_D^{-1}(E) = 1 + \exp(\beta(E - \mu))$ is the Fermi-Dirac distribution function with chemical potential $\mu = \sqrt{B}$, $\tau(E_M)$ is the scattering time for states at energy $E_M$, and $v_i^M = \partial_{p_i}E_M$ is the velocity correction in the $x_i$-direction of the state $M$ due to the periodic vector potential. Here, we have set the Fermi velocity to unity. Assuming constant $\tau(E) = \tau \neq 0$, we only need to calculate how the energies $E_M$ are affected by $\delta \vec{a}$, which in turn will determine the velocities $v_i^M$. We will show that the leading correction in $W$ to $E_M$ only contributes to $v_{i2}^M$. Calling $x_1 = x$ and $x_2 = y$, this implies the dominant correction is to $\Delta \rho_{xx} \propto \Delta \sigma_{yy}$. There are generally additional oscillations in $\rho_{yy}$ and $\rho_{xy}$, however, their amplitudes are typically less prominent and so we concentrate on $\Delta \rho_{xx}$ here.

### 4.2 Dirac composite fermion Weiss oscillations

The Dirac composite fermion mean-field Hamiltonian, corrected by Eq. (3.43),

$$H = \vec{\sigma} \cdot \left( \frac{\partial}{\partial \vec{x}} + \vec{a} \right) + m \sigma_3,$$

where

$$\vec{a} = \left( 0, bx_1 + W \sin(Kx_1) \right).$$

To zeroth order in $W$, $H$ has the particle spectrum,

$$E_n^{(0)} = \begin{cases} \sqrt{2n|b| + m^2}, & n = 1, 2, \ldots, \\ |m|, & n = 0. \end{cases}$$

with the corresponding eigenfunctions,

$$\psi_n^{(p_2x_2)}(\vec{x}) = \begin{cases} N e^{ip_2x_2} \left( \frac{-i\Phi_{n-1}(\frac{x_1+x_b}{L_b})}{\sqrt{2n|b| - m}} \Phi_{n}(\frac{x_1+x_b}{L_b}) \right) & \text{for } n = 1, 2, \ldots, \\ N e^{ip_2x_2} \left( \frac{0}{\Phi_{0}(\frac{x_1+x_b}{L_b})} \right) & \text{for } n = 0, \end{cases}$$

where the normalization constant,

$$N = \sqrt{\frac{n|b|}{L_b L_y(m^2 + 2n|b| - m\sqrt{m^2 + 2n|b|})}}.$$
\( k_2 \in \frac{2\pi}{L_2} \) is the momentum along the \( x_2 \)-direction (\( L_2 \to \infty \)), \( x_b(p_2) \equiv x_b = p_2 l_b^2, l_b^{-1} = |b| \), and \( \Phi_n(z) = \frac{e^{-z^2/2}}{\sqrt{2^{n!}\pi}} H_n(z) \) for the \( n \)-th Hermite polynomial \( H_n(z) \). Thus, the states are labeled by \( M = (n, p_2) \). We are interested in how the periodic vector potential in Eq. [4.5] lifts the degeneracy of the flat Landau level spectrum and contributes to the velocity \( v_i^M \).

First order perturbation theory gives the energy level corrections,

\[
E_n^{(1)} = W \frac{\sqrt{2n}}{Kl_b} \left[ \sqrt{\frac{2n|b|}{m^2 + 2n|b|}} \right] \cos(Kx_b) e^{-z/2} \left[ L_{n-1}(z) - L_n(z) \right],
\]

where \( L_n(z) \) is the \( n \)-th Laguerre polynomial, \( z = K^2 l_b^2 / 2 \), and terms suppressed as \( L_1, L_2 \to \infty \) have been dropped. Thus, to leading order, \( v_i^{n,p_2} \to 0 \) and

\[
v_2^{n,p_2} = \frac{\partial E_n^{(1)}(p_2)}{\partial p_2} = -Wl_b \sqrt{2n} \left[ \sqrt{\frac{2n|b|}{m^2 + 2n|b|}} \right] \sin(Kx_b) e^{-z/2} \left[ L_{n-1}(z) - L_n(z) \right].
\]

We substitute these \( v_i^{n,p_2} \) into the Kubo formula (4.7) to find \( \Delta \sigma_{yy}^{\psi} \). To perform the integral over \( p_2 \), we approximate the Fermi-Dirac distribution function by substituting in the zeroth order energies \( E_n^{(0)} \) (which are independent of \( p_2 \)). Thus, we obtain the periodic potential correction to the Dirac composite fermion conductivity:

\[
\Delta \sigma_{yy}^{\psi} \approx W^2 \tilde{\beta} \sum_{n=0}^{\infty} \left( \frac{2n|b|}{m^2 + 2n|b|} \right) \frac{n \exp(\beta(E_n^{(0)} - \mu))}{1 + \exp(\beta(E_n^{(0)} - \mu))} e^{-z/2} \left[ L_{n-1}(z) - L_n(z) \right]^2,
\]

where \( \tilde{\beta} \) has absorbed non-universal \( \mathcal{O}(1) \) constants.

\( \Delta \sigma_{yy}^{\psi} \) in Eq. [4.12] exhibits both Shubnikov de Haas oscillations (for large \( |b| \)) and Weiss oscillations (for smaller \( |b| \)). We are interested in extracting an analytic expression that approximates Eq. [4.12] at low temperatures near \( \nu = 1/2 \), following the earlier analysis in [54]. In the weak field limit, \( |b|/\mu^2 \ll 1 \), a large number of Landau levels are filled \((n \to \infty)\) and the dominant contribution to \( \Delta \sigma_{yy}^{\psi} \) at low temperatures arises from states with energies near \( \mu \). Thus, we express the Laguerre polynomials \( L_n \) as,

\[
L_n(z) \to \frac{e^{z/2} \cos \left( \frac{2\sqrt{nz} - \pi}{4} \right)}{(\pi^2 n z)^{1/4}} + \mathcal{O}(\frac{1}{n^{3/4}}).
\]

Next, we take the continuum approximation for the summation over \( n \) by substituting

\[
n \to \frac{l_b^2}{2} \left( E^2 - m^2 \right), \quad \sum_n \to l_b^2 \int E dE,
\]

Finite dissipation has already been assumed in using a finite, non-zero scattering time \( \tau \) in our calculation of the oscillatory component of \( \rho_{xx} \).
into Eq. (4.12):

\[
\Delta \sigma_{yy}^{\psi} = C \int_{-\infty}^{\infty} dE \frac{\beta e^{\beta(E-\mu)}}{(1+e^{\beta(E-\mu)})^2} \sin^2 \left( l_b^2 K \sqrt{E^2 - m^2} - \frac{\pi}{4} \right). 
\]

(4.14)

where \( C = W^2 \tilde{\tau} l_b^2 K \) and we have approximated \( 2n|b|/(m^2+2n|b|) \) by unity. (The substitution for \( n \) is motivated by the zeroth order expression for the energy of the Dirac composite fermion Landau levels.) Anticipating that at sufficiently low temperatures the integrand in Eq. (4.14) is dominated by “energies” \( E \) near the Fermi energy \( \mu \), we write:

\[
E = \mu + sT
\]

(4.15)

so that Eq. (4.14) becomes for \( |s|T \ll \mu = \sqrt{B} \):

\[
\Delta \sigma_{yy}^{\psi} = C \int_{-\infty}^{\infty} ds \frac{e^s}{(1+e^s)^2} \sin^2 \left( l_b^2 K \sqrt{B - m^2} + \frac{sTl_b^2 K}{\sqrt{1-m^2/B}} - \frac{\pi}{4} \right). 
\]

(4.16)

Performing the integral over \( s \), we find the Weiss oscillations (see Eq. (4.6)):

\[
\Delta \rho_{xx} \propto 1 - \frac{T/T_D}{\sinh(T/T_D)} \left[ 1 - 2 \sin^2 \left( \frac{2\pi l_b^2 \sqrt{B-m^2}}{d} - \frac{\pi}{4} \right) \right],
\]

(4.17)

where

\[
T_D^{-1} = \frac{4\pi^2 l_b^2}{d} \frac{1}{\sqrt{1-m^2/B}},
\]

(4.18)

we have substituted \( K = 2\pi/d, l_b^2 = |b|^{-1} \), and the proportionality constant is controlled by the longitudinal resistivity at \( \nu = 1/2 \).

Eq. (4.17) constitutes the primary result of this section. The minima of \( \Delta \rho_{xx} \) occur when

\[
\frac{1}{|b|} = \frac{d}{2\sqrt{B-m^2}} \left( p + \frac{1}{4} \right), \quad p = 1, 2, 3, \ldots,
\]

(4.19)

where \( m \) is given in Eq. (3.43). For either fixed electron density \( n_e \) or fixed external field \( B \), the locations of the oscillation minima for a given \( p \) (either \( B(p) \) or \( n_e(p) \)) are shifted inwards towards \( \nu = 1/2 \). This is shown in Fig. 1 for fixed \( n_e \) and in Fig. 2 for fixed \( B \). The magnitude of this shift is symmetric for fixed \( B \), but asymmetric for fixed \( n_e \), given the form of the mass in Eq. (3.43). Mass dependence also appears in the temperature-dependent prefactor \( \frac{T/T_D}{\sinh(T/T_D)} \). In principle, this mass dependence could be extracted from the finite-temperature scaling of \( \Delta \rho_{xx} \) at the oscillation extrema.
Figure 2: Weiss oscillations of the Dirac composite fermion theory at fixed magnetic field $B$ and varying electron density $n_e$ about half-filling $n_{1/2}$. The blue curve corresponds to Dirac composite fermion mean-field theory \cite{18}. The orange curve includes the effects of a Dirac composite fermion mass $m \propto |B - 4\pi n_e|^{1/3} B^{1/6}$, induced by gauge fluctuations. The vertical black lines correspond to the oscillation minima observed experimentally \cite{21}. Associating a magnetic field $B_d = \hbar/ed^2$ with the modulation period $d$, the result above corresponds to the choice: $B_d/B_{1/2} = 0.001$ and $k_B T = 0.3 \sqrt{2B_{1/2}}$. 
5 Discussion

5.1 Shubnikov de Haas oscillations of the HLR theory

In [27], Shayegan et al. found the Shubnikov de Haas (SdH) oscillations near half-filling to be well described over two orders of magnitude in temperature by the formula,

\[ \frac{\Delta \rho_{xx}}{\rho_0} \propto \frac{\xi_{NR}}{\sinh(\xi_{NR})} \cos(2\pi \nu - \pi), \] (5.1)

where \( \xi_{NR} = \frac{2\pi^2 T}{\omega_c} \), \( \omega_c = |b|/m^* \), \( m^* \) is an effective mass, \( \nu \) is the electron filling fraction, and \( \rho_0 \) is the longitudinal resistivity at half-filling (measured at the lowest accessible temperature). Recall that we are using units where \( k_B = \hbar = e = c = 1 \). Interpreted within the HLR composite fermion framework, \( m^* \) corresponds to the composite fermion effective mass and \( b \) is the effective magnetic field the composite fermions experience. Furthermore, Shayegan et al. found that \( m^* \propto \sqrt{B} \) for sufficiently large \( |b| \) and that \( m^* \) appeared to diverge for smaller \( |b| \). (Note that these experiments were performed without any background periodic potential and so no Weiss oscillations were present.) The \( \sqrt{B} \) scaling of the composite fermion effective mass is consistent with the theoretical expectation [7, 26] that the composite fermion mass scale is determined entirely by the characteristic energy of the Coulomb interaction.

Applying previous treatments of SdH oscillations in graphene [55] to the Dirac composite fermion theory, the temperature dependence of the SdH oscillations is controlled by

\[ \frac{\Delta \rho_{xx}}{\rho_0} \propto \frac{\xi_D}{\sinh(\xi_D)}, \] (5.2)

where \( \xi_D = \frac{2\pi^2 T \sqrt{B}}{|b|} \). Thus, \( \xi_{NR} \propto \xi_D \), if we assume \( m^* \propto \sqrt{B} \). Consequently, the Dirac composite fermion theory accounts for the scaling of the observed temperature dependence with \( \sqrt{B} \). We cannot account for the divergences attributed to \( m^* \) in our treatment at finite \( |b|/\mu^2 \).

5.2 Comparison to the finite-temperature behavior of the Weiss oscillations of the HLR theory

In [18], it was shown that the locations of the Weiss oscillation minima obtained from Dirac and HLR composite fermion mean-field theories coincide to 0.002%. This result provides evidence that the two composite fermion theories may belong to the same universality class. However, the (possible) equivalence of the two theories only occurs at long distances and so the finite-temperature behavior of the two theories will generally differ.
In HLR mean-field theory, the temperature dependence of the Weiss oscillations enters in the factor \[5.4\],

\[
\Delta \rho_{xx} \propto \frac{T/T_{NR}}{\sinh(T/T_{NR})},
\]

(5.3)

where the characteristic temperature scale,

\[
T_{NR}^{-1} = \frac{4\pi l_b^2}{\sqrt{4\pi n_e}}.
\]

(5.4)

Assuming the effective mass \(m^* \propto \sqrt{B}\), the characteristic temperatures \(T_D\) and \(T_{NR}\) generally have very different behaviors as functions of \(B\) and \(n_e\).\(^{10}\) It would be interesting to compare these behaviors with experiment.

### 5.3 Gauge field fluctuation effects in the HLR theory when \(b \neq 0\)

Recent works \[15–17\] have shown that PH symmetric electrical response in the HLR theory relies on properly including quenched disorder. This means that the HLR composite fermions should propagate in a disordered background where a random chemical potential \(V(x)\) and magnetic flux \(b(x)\) are precisely correlated as

\[
b(x) = -2m^*V(x).
\]

(5.5)

When a periodic scalar potential is applied to the electronic system in order to study Weiss oscillations, Eq. (5.5) implies a periodic vector potential is also generated. The combination of scalar and vector potential modulations whose magnitudes are related by Eq. (5.5) is crucial to the observation that the Weiss oscillations of the Dirac and HLR mean-field theories coincide.

Numerical calculation \[16\] indicates that PH symmetric electrical response is no longer found at half-filling when Eq. (5.5) is violated by tuning the proportionality between \(V(x)\) and \(b(x)\) away from the value \(-2m^*\). We speculate that fluctuations in the HLR theory lead to a violation of Eq. (5.5) when \(b \neq 0\), which in turn will affect the resulting Weiss oscillation via a different proportionality between the scalar and vector potential modulations.

### 6 Conclusion

In this paper, we found an approximate solution to the Schwinger-Dyson equations of the Dirac composite fermion theory for the half-filled Landau level when the effective magnetic

\(^{10}\)Possible fluctuation effects in the HLR theory have not been accounted for here so the comparison should be made cautiously.
field \( b = B - 4\pi n_e \neq 0 \), where \( n_e \) is the electron density and \( B > 0 \) is the external magnetic field. We found that fluctuations of the emergent gauge field generate a Dirac composite fermion mass \( m \propto |b|^{1/3}B^{1/6} \), reflective of the broken particle-hole symmetry away from half-filling. The Dirac composite fermion propagator, obtained within this analysis, then defines input parameters, the chemical potential (which was found to be unaffected by fluctuations at leading order) and dynamically-generated mass, for a study of the Weiss oscillations produced by the “fluctuation-improved” Dirac composite fermion mean-field theory. We showed how a non-zero mass can improve the comparison between theory and experiment. We then showed how a non-zero Dirac mass modifies the finite-temperature amplitude of the Weiss oscillations near half-filling. This finite-temperature behavior could be tested experimentally and might be used to distinguish the HLR and Dirac composite fermion theories away from the deep IR limit.

We only considered the effects of periodic vector potential modulations in the Dirac composite fermion theory (as a result of the applied scalar potential modulation on the electronic system). Gauge field fluctuations should also induce modulations in the Dirac composite fermion chemical potential. It would be interesting to determine the nature of these Dirac composite fermion chemical potential modulations and to understand their effects on the Weiss oscillations near half-filling, in combination with the vector potential modulation.

As we approach half-filling by taking the effective magnetic field \( |b| \to 0 \), our approximate solutions to the Schwinger-Dyson equations break down. Landau damping of the gauge field and the ensuing IR dominant self-energy corrections to the Dirac composite fermion propagator \cite{45,47} are expected to be important to the low-energy dynamics. However, these corrections were ignored in our treatment. The formalism outlined in \( \S 3.1 \) may provide a useful technical tool for addressing these effects. In any such treatment, it would be interesting to consider how modifications to the gauge field propagator, due to the nature of the assumed electron-electron interactions, affects the analysis and possible solutions to the Schwinger-Dyson equations. This analysis might also shed light on how the corrections to Dirac composite mean-field theory depend upon microscopic considerations such as the degree of sample disorder and effective electron-electron interaction.

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A  Integrals

In this appendix, we give details for the calculations of the gauge and fermion self-energy integrals quoted in the main text.

A.1  Gauge field self-energy

We begin with the gauge field self-energy given in §3.3. We are interested in computing the PH odd component of the gauge field self-energy $\Pi_{\text{odd}}$:

$$\Pi^{\alpha\beta}(q) = \Pi^{\alpha\beta}_{\text{even}}(q) + i\epsilon^{\alpha\beta\tau}q_\tau\Pi_{\text{odd}}(q).$$  \hspace{1cm} (A.1)

To leading order in $b$, we substitute $G(p) = G^{(0)}(p)$ from Eq. (3.9) with $\Sigma_\alpha = 0$ for $\alpha \in \{0,1,2\}$ into Eq. (3.22):

$$i\epsilon^{\alpha\beta\tau}q_\tau\Pi_{\text{odd}}(q) = N\left\{\int \frac{d^3p}{(2\pi)^3} \text{tr}\left[\gamma^\alpha G^{(0)}(p)\gamma^\beta G^{(0)}(p + q)\right]\right\}_{\text{odd}}$$

$$= -N\left\{\int \frac{d^3p}{(2\pi)^3} \text{tr}\left[\gamma^\alpha i(\gamma^\sigma(p_\sigma + \mu_0\delta_{\sigma,0}) + \Sigma_m)\gamma^\beta\right.ight.$$

$$\left.\times \frac{i(\gamma^\tau(p_\tau + q_\tau + \mu_0\delta_{\tau,0}) + \Sigma_m)}{(p_0 + q_0 + \mu_0)^2 - (p_\tau + q_\tau)^2 - \Sigma_m^2}\right]\right\}_{\text{odd}}.  \hspace{1cm} (A.2)$$

We have suppressed the $i\epsilon_{p_0}$ factor in Eq. (3.9) that defines the Feynman contour for the Minkowski-signature $p_0$ integration because we will evaluate the above integral in Euclidean signature. In subsequent sections of this appendix, we will likewise suppress the $i\epsilon_{p_0}$ factor for the same reason without further comment. Recall that the factor of $N$ arises from the fermion loop over $N$ flavors of Dirac composite fermions and that $\mu_0 > 0$.

To leading order in the derivative expansion, i.e., $\Pi_{\text{odd}}(q = 0)$, the expression for $\Pi_{\text{odd}}(0)$ simplifies to

$$\Pi_{\text{odd}}(0) = -2iN\Sigma_m\int \frac{d^3p}{(2\pi)^3} \frac{1}{(p_0 + \mu_0)^2 - p_i^2 - \Sigma_m^2)^2}.  \hspace{1cm} (A.3)$$

Here, we have used the trace identities,

$$\text{tr}\left[\gamma^\alpha \gamma^\beta\right] = 2\eta^{\alpha\beta};$$

$$\text{tr}\left[\gamma^\alpha \gamma^\beta \gamma^\tau\right] = -2i\epsilon^{\alpha\beta\tau}.  \hspace{1cm} (A.4)$$
To compute this integral, we first Wick rotate, $p_0 \mapsto i(p_E)_3$ and $d^3p \mapsto id^3p_E$, and then sequentially integrate over $(p_E)_3$ and the spatial momenta $(p_E)_i$ $(i = 1, 2)$ to find:

$$\Pi_{\text{odd}}(0) = 2N\Sigma_m \int \frac{d^3p_E}{(2\pi)^3} \frac{1}{(i(p_E)_3 + \mu_0)^2 - (p_E)_i^2 - \Sigma_m^2}$$

$$= 2N\Sigma_m \int \frac{d^3p_E}{(2\pi)^3} \frac{1}{((p_E)_3 - \omega_+)^2 - (p_E)_3 - \omega_-)^2}$$

$$= \frac{N\Sigma_m}{2} \int \frac{d^3p_E}{(2\pi)^3} \left( (\Theta(|\Sigma_m| - \mu_0) + \Theta(\mu_0 - |\Sigma_m|)\Theta(|(p_E)_3| - \sqrt{\mu_0^2 - \Sigma_m^2}) \right) \frac{1}{(\frac{(p_E)_3}{|p_E|} + \Sigma_m^2)^{3/2}}$$

$$= \frac{N}{4\pi} \left( \Theta(|\Sigma_m| - \mu_0) \frac{\Sigma_m}{|\Sigma_m|} + \Theta(\mu_0 - |\Sigma_m|) \frac{\Sigma_m}{\mu_0} \right),$$

(A.5)

where the step function $\Theta(|(p_E)_3| - \sqrt{\mu_0^2 - \Sigma_m^2})$ in the third line ensures the double poles $\omega_{\pm} = i(\mu_0 \pm \sqrt{(p_E)_3^2 + \Sigma_m^2})$ occur on opposite sides of the real $(p_E)_3$ axis. Eq. (A.5) implies that, for $\mu_0 > |\Sigma_m| > 0$, the gauge field obtains a correction to its propagator that corresponds to an effective Chern-Simons term with level,

$$k = \frac{N \Sigma_m}{2 \mu_0}. \quad \text{(A.6)}$$

A.2 Fermion self-energy

Next, we calculate the fermion self-energies $\Sigma_m$ and $\Sigma_0$ quoted in §3.4.

We begin with $\Sigma_m$. Taking the trace of both sides of Eq. (3.21) and setting $\delta q_\alpha = 0$, we find:

$$i\Sigma_m(q_{\text{FS}}) = iM^{(0)}(q_{\text{FS}}) + iM^{(1)}(q_{\text{FS}}), \quad \text{(A.7)}$$

where

$$iM^{(0)}(q_{\text{FS}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \gamma^\alpha G^{(0)}(p + q_{\text{FS}}) \gamma^\beta \frac{2\pi \epsilon_\alpha\beta\sigma p^2}{k} \right], \quad \text{(A.8)}$$

$$iM^{(1)}(q_{\text{FS}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \gamma^\alpha G^{(1)}(p + q_{\text{FS}}) \gamma^\beta \frac{2\pi \epsilon_\alpha\beta\sigma p^2}{k} \right], \quad \text{(A.9)}$$

$G^{(0)}(p)$ and $G^{(1)}(p)$ are given in Eqs. (3.9) and (3.10), $k$ is given in Eq. (A.6), and $q_{\text{FS}} = (0, \mu_0 \hat{n})$ for the unit vector $\hat{n}$ (e.g., $\hat{n} = (\cos(\varphi), \sin(\varphi))$ where $\varphi$ parameterizes a point on the Fermi surface) normal to the (assumed) spherical Fermi surface. As before, we set $\Sigma_\alpha = 0$ for $\alpha \in \{0, 1, 2\}$ and only retain $\Sigma_m$ when using $G^{(0)}(p)$ and $G^{(1)}(p)$ to evaluate $M^{(0)}$ and
$\mathcal{M}^{(1)}$, as well as $\Sigma_0$ below. It is convenient to define $Q = (\mu_0, \mu_0 \hat{n})$ so that

$$i\mathcal{M}^{(0)}(q_{\text{FS}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \gamma^\alpha \left( i(\gamma^\sigma (p + Q)_\sigma + \Sigma_m) \right) \gamma^\beta \left( \frac{2\pi}{k} \epsilon_{\alpha\beta\tau} p^\tau \right) \right], \quad (A.10)$$

$$i\mathcal{M}^{(1)}(q_{\text{FS}}) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \gamma^\alpha \left( ib[(p + Q)_0 + \gamma^0 \Sigma_m) \right) \gamma^\beta \left( \frac{2\pi}{k} \epsilon_{\alpha\beta\tau} p^\tau \right) \right]. \quad (A.11)$$

We first consider $\mathcal{M}^{(0)} = \mathcal{M}^{(0)}(q_{\text{FS}})$. Using the trace identities in Eq. (A.4), we find

$$i\mathcal{M}^{(0)} = \frac{\pi i}{k} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \gamma^\alpha \left( \frac{(\gamma^\sigma (p + Q)_\sigma + \Sigma_m)}{(p + Q)^2 - \Sigma_m^2} \right) \gamma^\beta \left( \frac{\epsilon_{\alpha\beta\tau} p^\tau}{p^2} \right) \right]$$

$$= -\frac{4\pi}{k} \int \frac{d^3p}{(2\pi)^3} \frac{(p + Q)_\sigma p^\sigma}{(p^2 + 2p \cdot Q x + Q^2 x - \Sigma_m^2 x^2)}$$

$$= -\frac{4\pi}{k} \int \frac{d^3\ell}{(2\pi)^3} \int_0^1 dx \frac{\ell^2 + \ell \cdot Q(1 - 2x) - x(1 - x)Q^2}{(\ell^2 + Q^2 x(1 - x) - \Sigma_m^2 x^2)}$$

$$= -\frac{4\pi}{k} \int \frac{d^3\ell}{(2\pi)^3} \int_0^1 dx \frac{\ell^2}{(\ell^2 - \Sigma_m^2 x^2)}, \quad (A.13)$$

where we evaluated $Q^2 = 0$ and dropped the linear in $\ell$ term in the third line since it vanishes upon integration over $\ell$. Next, we Wick rotate by taking $\ell_0 \mapsto i(\ell_E)_3$, $\ell^2 \mapsto -\ell_E^2$, and $d^3\ell \mapsto i d^3\ell_E$, integrate over $\ell_E$ via dimensional regularization, and finally integrate over $x$:

$$i\mathcal{M}^{(0)} = \frac{4\pi i}{k} \int \frac{d^3\ell_E}{(2\pi)^3} \int_0^1 dx \frac{\ell_E^2}{(\ell_E^2 + \Sigma_m^2 x^2)}$$

$$= -\frac{12\pi^3/2 |\Sigma_m|}{k(4\pi)^{3/2}} \int_0^1 dx \, x^{1/2}$$

$$= -\frac{|\Sigma_m|}{k}$$

$$= -i \frac{2\mu_0 \text{sign}(\Sigma_m)}{N}, \quad (A.14)$$

where we substituted in the Chern-Simons level given in Eq. (A.6) in the final line.

Next, consider $\mathcal{M}^{(1)} = \mathcal{M}^{(1)}(q_{\text{FS}})$. Using the trace identities in Eq. (A.4), we find

$$i\mathcal{M}^{(1)} = -\frac{4\pi b |\Sigma_m|}{k} \int \frac{d^3p}{(2\pi)^3} \frac{p_0}{((p + Q)^2 - \Sigma_m^2 p^2)}$$

$$= -\frac{4\pi b |\Sigma_m|}{k} I(\Sigma_m^2, Q). \quad (A.15)$$
With the help of the formal identity,

\[ I(\Sigma^2_m, Q) = -\partial_{\Sigma^2_m} J(\Sigma^2_m, Q) = -\partial_{\Sigma^2_m} \int \frac{d^3p}{(2\pi)^3} \frac{p_0}{((p + Q)^2 - \Sigma^2_m)p^2}, \]  

(A.16)

we rewrite

\[ iM^{(1)} = \frac{4\pi b \Sigma^2_m \partial_{\Sigma^2_m}}{k} \int \frac{d^3p}{(2\pi)^3} \frac{p_0}{((p + Q)^2 - \Sigma^2_m)p^2}. \]  

(A.17)

This integral has the same basic form as the one we encountered in calculating \( M(0) \) and so we will follow the same steps as before: combine denominators with the Feynman parameter \( x \), shift the integration \( \ell_\alpha = p_\alpha + Q_\alpha x \), and substitute in \( Q_0 = \mu_0 \) and \( Q^2 = 0 \):

\[ iM^{(1)} = -\frac{4\pi b \Sigma^2_m \mu_0}{k} \partial_{\Sigma^2_m} \int \frac{d^3\ell}{(2\pi)^3} \int_0^1 dx \frac{x}{(\ell^2 - \Sigma^2_m x)^2} \]  

(A.18)

Next, we Wick rotate by taking \( \ell_0 \mapsto i(\ell_E)_3 \), integrate over \( \ell_E \) via dimensional regularization, integrate over \( x \), take the derivative with respect to \( \Sigma^2_m \), and then evaluate \( k = \frac{N \Sigma_m}{2 \mu_0} \):

\[ iM^{(1)} = -i \frac{4\pi b \Sigma^2_m \mu_0}{k} \partial_{\Sigma^2_m} \int \frac{d^3\ell}{(2\pi)^3} \int_0^1 dx \frac{x}{(\ell^2_E + \Sigma^2_m x)^2} \]
\[ = -i \frac{b \Sigma^2_m \mu_0}{k} \partial_{\Sigma^2_m} \frac{1}{(\Sigma^2_m)^{1/2}} \int_0^1 dx \ x^{1/2} \]
\[ = i \frac{2}{3} \frac{b \mu_0^2}{N|\Sigma_m|^3}. \]  

(A.19)

Finally, we calculate \( \Sigma_0(q_{FS}) \) and \( \Sigma_0'(q_{FS}) \), which we obtain from evaluating the derivative with respect to \( q_0 \) of \( \Sigma_0(P) \) at the Fermi surface:

\[ i\Sigma_0(P) = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[ \gamma^0 \gamma^\alpha G^{(0)}(p + P) \gamma^\beta \left( \frac{2\pi \epsilon_{\alpha\beta\sigma} p^\sigma}{k} \right) \right], \]  

(A.20)

where \( P = (q_0 + \mu_0, \mu_0 \hat{n}) \). First, we note that

\[ \text{tr} [\gamma^0 \gamma^\alpha \gamma^\sigma \gamma^\beta](p + P)_\sigma p^\tau \epsilon_{\alpha\beta\tau} = 2(\eta^{0\alpha} \eta^{\sigma\beta} - \eta^{0\sigma} \eta^{\alpha\beta} + \eta^{0\beta} \eta^{\sigma\alpha})(p + P)_\sigma p^\tau \epsilon_{\alpha\beta\tau} \]
\[ = (p + P)^\beta p^\tau \epsilon_{0\beta\tau} + (p + P)^\alpha p^\tau \epsilon_{\alpha0\tau} \]
\[ = 0. \]  

(A.21)

Therefore, only the term proportional to \( \Sigma_m \) in the numerator of \( G^{(0)} \) contributes. Using the trace identities in Eq. (A.4), we find

\[ i\Sigma_0(P) = \frac{4\pi \Sigma_m}{k} \int \frac{d^3p}{(2\pi)^3} \frac{p_0}{((p + P)^2 - \Sigma^2_m)p^2}. \]  

(A.22)
As above, we combine denominators, shift the integration variable \( \ell_\alpha = p_\alpha + P_\alpha x \), and drop any linear in \( \ell \) terms in the numerator:

\[
i \Sigma_0(P) = -\frac{4\pi \Sigma_m(q_0 + \mu_0)}{k} \int \frac{d^3 \ell}{(2\pi)^3} \int_0^1 dx \frac{x}{(\ell^2 + x(1-x)P^2 - \Sigma_m^2 x^2)^2}.
\]

(A.23)

We assume \( \Sigma_m^2 > |P^2| \approx |2\mu q_0| \). Wick rotating \( \ell_0 \mapsto i(\ell_E)_3 \) and sequentially performing the \( \ell_E \) and \( x \) integrals, we find:

\[
i \Sigma_0(P) = -i \frac{4\pi \Sigma_m(q_0 + \mu_0)}{k} \int \frac{d^3 \ell_E}{(2\pi)^3} \int_0^1 dx \frac{x}{(\ell_E^2 - x(1-x)(2\mu_0 q_0 + \Sigma_m^2 x^2)^2}.
\]

(A.24)

Taking the derivative of \( \Sigma_0(P) \) with respect to \( q_0 \), evaluating at \( q = (0, \mu_0 \hat{n}) \), and retaining only the first term \( (\mu_0 q_0 \ll |\Sigma_m|^2) \), we obtain

\[
i \Sigma_0'(q_{FS}) = -i \frac{2\mu_0}{3N|\Sigma_m|}.
\]

(A.25)

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