The large sieve for $2^{O(n^{15/14+o(1)})}$ modulo primes

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Abstract

Let $\lambda$ be a fixed integer, $\lambda \geq 2$. Let $s_n$ be any strictly increasing sequence of positive integers satisfying $s_n \leq n^{15/14+o(1)}$. In this paper we give a version of the large sieve inequality for the sequence $\lambda^{s_n}$. In particular, we prove that for $\pi(X)(1+o(1))$ primes $p$, $p \leq X$, the numbers

$$\lambda^{s_n}, \quad n \leq X(\log X)^{2+\varepsilon}$$

are uniformly distributed modulo $p$.

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1 Notation

Throughout the paper the following notations will be used:

- \( \lambda \) denotes a fixed positive integer, \( \lambda \geq 2 \);
- \( X \) and \( T \) are large parameters, \( T \) is an integer;
- \( \Delta > X^{1/3} \) is a parameter;
- \( s_n, n = 1, 2, \ldots \), is a strictly increasing sequence of positive integers (which may depend on the parameters \( X, T, \Delta \));
- \( \gamma_n, n = 1, 2, \ldots \), are any complex coefficients (which may depend on the parameters \( X, T, \Delta \)) with \( |\gamma_n| \leq 1 \);
- \( p \) and \( q \) always denote prime numbers;
- \( t_p \) denotes the multiplicative order of \( \lambda \) modulo \( p \);
- \( \mathcal{E} = \mathcal{E}(\Delta, X) = \{ p : p \leq X, t_p > \Delta \} \); that is the set of all primes \( p, p \leq X \), with \( t_p > \Delta \);

For integers \( a \) and \( b \), their greatest common divisor is denoted by \( (a, b) \).

Given a set \( \mathcal{X} \) we use \( |\mathcal{X}| \) to denote its cardinality.

As usual, \( \pi(X) \) denotes the number of primes not exceeding \( X \), and \( \tau(n) \) denotes the number of positive integer divisors of \( n \). We also follow the standard abbreviation \( e_m(z) = e^{2\pi iz/m} \).

2 Introduction

Recently, J. Bourgain \[2,3\] has proved that for \( \pi(X)(1 + o(1)) \) primes \( p, p \leq X \), the Mersenne numbers \( M_q = 2^q - 1, q \leq X^{2+\varepsilon} \), are uniformly distributed modulo \( p \) for any given \( \varepsilon > 0 \). Furthermore, he has explicitly described the set of primes \( p \) for which we can be sure that the Mersenne numbers are uniformly distributed modulo \( p \). This set is expressed in terms of certain conditions to the size of the multiplicative order of 2 modulo \( p \), which are satisfied for almost all primes \( p \).
Bourgain’s result is based on his deep work related to nontrivial estimates of double trigonometric sums. The possibility of applications of such estimates to investigate Mersenne numbers in residue classes modulo \( p \) has been first discovered in [1].

An alternative approach, based on the large sieve inequality, has been recently suggested in [9]. From the result of Erdős and Murty [7] we know that the estimate \( t_p > X^{1/2+o(1)} \) holds for almost all primes \( p, p \leq X \). This has been used in [9] to obtain a nontrivial bound for the exponential sum

\[
\max_{(a,p)=1} \left| \sum_{n \in S_N} e^{2\pi i a \lambda n / p} \right|
\]

for \( \pi(X) + o(\pi(X)) \) primes \( p, p \leq X \), provided that \( S_N \subset [1, N] \) is sufficiently dense (that is \( |S_N| > N^{1+o(1)} \)) and \( N \) is of the size \( X^{1+o(1)} \).

The result of [9] does not apply for sparser sets \( S_N \), but it is shown that such results can be obtained conditionally, namely assuming the truth of the Extended Riemann Hypothesis.

In the present paper we provide a new argument which allows to deal with sparse sets \( S_N \) unconditionally. In particular, we obtain equidistribution properties of \( \lambda^n \) (mod \( p \)), \( n \in S_N \) with \( |S_N| > N^{14/15-o(1)} \). We show that further improvement could be obtained if one knows how to complement in appropriate way the set of exponent pairs for Gauss sums obtained by Konyagin.

Furthermore, while the result of [9] only apply for the set of primes \( p \leq X \) with \( t_p > X^{1/2(\log X)^c} \), \( c > 0 \), here our result works when \( t_p > \Delta \), where, depending on how sparse the set \( S \) is, \( \Delta \) varies in \( (X^{1/3+\varepsilon}, X^{1/2+o(1)}) \). This is useful if one is interested in obtaining sharp upper bound estimates for the exceptional set of primes \( p \) in the equidistribution problem of the sequence \( \lambda^n \) (mod \( p \)), \( n \in S_N \).

In what follows, we use the Landau symbol ‘\( o \)’, as well as the Vinogradov symbols ‘\( \ll \)’ and ‘\( \gg \)’ in their usual meanings. The implied constants may depend on the small positive quantity \( \varepsilon \), \( \lambda \) and other fixed constants, and also on the choice of the function \( \nu(n) \) (in Corollary [2] below, see also [11]).

### 3 Results

The following statement is the main result of our paper. We recall that \( s_n, \ n = 1, 2 \ldots \), is any sequence of strictly increasing positive integers.
Theorem 1. For any \( L > 0 \) the following bound holds:

\[
\sum_{p \in \mathcal{E}} \frac{1}{\tau(p-1)} \max_{(a,p)=1} \left| \sum_{n \leq T} \gamma_n e_p(a \lambda^{s_n}) \right|^2 \ll (X + s_T X^{1/7} \Delta^{-3/7} L + T L^{-7/4}) XT.
\]

If we optimize the choice of \( L \), then the estimate can be reformulated in the form

\[
\sum_{p \in \mathcal{E}} \frac{1}{\tau(p-1)} \max_{(a,p)=1} \left| \sum_{n \leq T} \gamma_n e_p(a \lambda^{s_n}) \right|^2 \ll (1 + (s^7 T^4 X^{-10} \Delta^{-3})^{1/11}) X^2 T.
\]

As we have already mentioned in the Introduction, for \( \pi(X)(1 + o(1)) \) primes \( p, p \leq X \), the inequality \( t_p > X^{1/2+g(x)} \) holds for any given function \( g(x) = o(1) \).

Let now \( s_n \) satisfy the condition

\[ s_n \leq n^{15/14 + \nu_n}, \quad \lim_{n \to \infty} \nu_n = 0, \tag{1} \]

where \( \nu_n \) is an absolutely fixed sequence (therefore, does not depend on the parameters \( T, X, \Delta \)). Set \( T = [X (\log X)^{2+\varepsilon}] \) and take

\[ L = T^{1/2} |\log T|^{10}, \quad \Delta = T^{1/2} L^7. \]

Obviously, \( L^7 = X^{o(1)}, \Delta = X^{1/2+o(1)} \). Therefore,

\[ |\mathcal{E}| = \pi(X)(1 + o(1)). \]

Incorporating this choice of the parameters in Theorem 1 we obtain

\[
\sum_{p \in \mathcal{E}} \frac{1}{\tau(p-1)} \max_{(a,p)=1} \left| \sum_{n \leq T} \gamma_n e_p(a \lambda^{s_n}) \right|^2
\ll X^2 T + XT^2 (\log T)^{-10} \ll XT^2 (\log T)^{-2-\varepsilon}.
\]

Next, let \( \mathcal{E}' \) be the subset of \( \mathcal{E} \) with \( \tau(p-1) < (\log X)^{1+\varepsilon/2} \). From the Titchmarsh bound

\[ \sum_{p \leq X} \tau(p-1) \ll X \]  

(see for example Theorem 7.1 in Chapter 5 of [14]) it follows that the inequality

\[ \tau(p-1) \ll (\log X)^{1+\varepsilon/2} \]
holds for $\pi(X)(1 + O((\log X)^{−\varepsilon/2}))$ primes $p, p \leq X$. That is, we still have

$$|\mathcal{E}'| = \pi(X)(1 + o(1)).$$

Now, the range of summation over $p$ in the above bound we concise to $\mathcal{E}'$. Then

$$\sum_{p \in \mathcal{E}'} \max_{(a,p)=1} \left| \sum_{n \leq T} \gamma_n e_p(a\lambda^{s_n}) \right|^2 \ll \pi(X)T^2(\log T)^{−\varepsilon/2}.$$

From this, by taking $\gamma_n = 1$, we deduce the following consequence.

**Corollary 2.** Let $s_n$ satisfy the condition (1) and let $T = [X(\log X)^{2+\varepsilon}]$. Then the inequality

$$\max_{(a,p)=1} \left| \sum_{n \leq T} e^{2\pi i a\lambda^{s_n}/p} \right| \ll T(\log T)^{−\varepsilon/5}$$

holds for all primes $p, p \leq X$, except at most $o(\pi(X))$ of them.

We recall that the discrepancy $D$ of a sequence of $N$ points $(x_j)_{j=1}^N$ of the unit interval $[0,1)$ is defined as

$$D = \sup_{0 \leq a,b \leq 1} \left| \frac{A(a,b)}{N} - (b - a) \right|,$$

where $A(a,b)$ is the number of points of this sequence which belong to $[a,b)$.

Now let $D(p,X)$ denote the discrepancy of the fractional parts

$$\{\lambda^{s_n}/p\}, \quad n \leq X(\log X)^{2+\varepsilon},$$

where $s_n$ satisfies the condition (1). According to the well-known Erdős-Turán relation between the discrepancy and the associated exponential sums (see [3], or alternatively one can use Theorem 4 of [4]), we derive from Corollary 2 that for $\pi(X)(1 + o(1))$ primes $p, p \leq X$, the following bound holds with some $\varepsilon_1 > 0$:

$$D(p,X) \ll (\log X)^{−\varepsilon_1}.$$

In other words, the numbers

$$\lambda^{s_n}, \quad n \leq X(\log X)^{2+\varepsilon}$$

are uniformly distributed modulo $p$ for any given $\varepsilon > 0$. In particular, one can take $s_n = \lfloor q_n^c \rfloor$, where $1 \leq c \leq 15/14$ and $q_n$ denotes the $n$-th prime number.

The following Theorem is an analogy of Theorem 1 where the range of summation over $n$ now depends on $p$. 

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Theorem 3. Let $T_p, p \in \mathcal{E}$, be any positive integers with $T_p \leq T$ and let $\mathcal{E}_1 \subset \mathcal{E}$. For any positive numbers $L$ and $K$ the following bound holds:

$$
\sum_{p \in \mathcal{E}_1} \frac{1}{\tau(p-1)} \max_{(a,p)=1} \left| \sum_{n \leq T_p} \gamma_n e_p(a \lambda^* n) \right|^2 \ll (X + s_T X^{1/7} \Delta^{-3/7} L + TL^{-7/4}) XT(\log K)^2 + \frac{T^2}{K^2} \sum_{p \in \mathcal{E}_1} \frac{1}{\tau(p-1)} .
$$

Taking $\mathcal{E}_1 = \mathcal{E}$ and $K = T$ and observing that the last term never dominates, we see that Theorem 3 extends Theorem 1 to more general sums at the cost of the slight factor $(\log T)^2$. In some applications one can further relax this factor by special choices of $\mathcal{E}_1$ and $K$.

One may want to have an explicit estimate for $|\mathcal{E}|$, where

$$
\mathcal{E} = \{ p : p \leq X, p \notin \mathcal{E} \}.
$$

In this connection we remark that the argument given in [3] immediately shows the inequality

$$
|\mathcal{E}| \ll \frac{\Delta^2}{\log \Delta}.
$$

Indeed

$$
\prod_{p \in \mathcal{E}} p \left| \prod_{k \leq \Delta} (\lambda^k - 1) ,
$$

Therefore, if $\omega(n)$ denotes the number of prime divisors of $n$, then we have

$$
|\mathcal{E}| \ll \omega \left( \prod_{k \leq \Delta} (\lambda^k - 1) \right) \ll \frac{\Delta^2}{\log \Delta} ,
$$

where we have used the well known bound $\omega(n) \ll (\log n)(\log \log n)^{-1}$.

For $\Delta = X^{1/2 + o(1)}$ one can use the results of [11].

4 Lemmas

We need the version of the large sieve inequality applied to our situation (recall that $|\gamma_n| \leq 1$).
Lemma 4. For any $K \geq 1$ the following estimate holds:

$$\sum_{k \leq K} \sum_{1 \leq c \leq k} \left| \sum_{n \leq T} \gamma_n e_k(c s_n) \right|^2 \ll (K^2 + s_T)T.$$ 

For the proof, see for example, [5, pp. 153-154].

We also recall the following bound of Heath-Brown and Konyagin [10].

Lemma 5. Let an integer $\theta$ be of multiplicative order $t$ modulo $p$. Then the following bound holds:

$$\max_{(a, p) = 1} \left| \sum_{z=1}^t \varepsilon_p(a \theta^z) \right| \ll \min\{p^{1/2}, p^{1/4}t^{3/8}, p^{1/8}t^{5/8}\}.$$ 

Instead of Lemma 5 one can use the bound due to Bourgain-Konyagin [4], which however does not improve our final results.

5 Proof of Theorem

If $L \leq 1$, then the estimate of Theorem 1 becomes trivial. Therefore, we will suppose that $L > 1$.

Denote

$$\sigma_p(a) = \sum_{n \leq T} \gamma_n e_p(a \lambda^s_n).$$

We have

$$\sigma_p(a) = \sum_{x=1}^{t_p} \sum_{n \leq T \atop t_p | s_n - x} \gamma_n e_p(a \lambda^s_n) = \frac{1}{t_p} \sum_{x=1}^{t_p} \sum_{b=1}^{t_p} \sum_{n \leq T} \gamma_n e_{t_p}(b(s_n - x)) e_p(a \lambda^x).$$

For each divisor $d | t_p$ we collect together the values of $b$ with $(b, t_p) = d$. Thus

$$\sigma_p(a) = \frac{1}{t_p} \sum_{d | t_p} \sum_{x=1}^{t_p} \sum_{c \leq t_p / d \atop (c, t_p / d) = 1} \sum_{n \leq T} \gamma_n e_{t_p/d}(c(s_n - x)) e_p(a \lambda^x).$$
We treat the cases of big and small values of $d$ separately. For big values of $d$ we will enjoy the summation over $x$ in a proper way to get sufficient to our purposes cancellations. The small values of $d$ are treated in a different way. Thus, we define 

$$v_p = t_p^{4/7} p^{1/7}$$

and set

$$R_1 = \max_{(a,p)=1} \left| \sum_{d \geq Ltv_p} \sum_{x=1}^{t_p} \sum_{c \leq t_p/d} \sum_{n \leq T} \gamma_n e_{tp/d} (c(s_n - x)) e_p (a \lambda^x) \right|, \quad (3)$$

$$R_2 = \max_{(a,p)=1} \left| \sum_{d < Ltv_p} \sum_{x=1}^{t_p} \sum_{c \leq t_p/d} \sum_{n \leq T} \gamma_n e_{tp/d} (c(s_n - x)) e_p (a \lambda^x) \right|. \quad (4)$$

Then

$$\max_{(a,p)=1} |\sigma_p(a)| \leq R_1 + R_2.$$ 

In particular,

$$\sum_{p \in \mathcal{E}} \frac{1}{\tau(p-1)} \max_{(a,p)=1} |\sigma_p(a)|^2 \leq \sum_{p \in \mathcal{E}} \frac{R_1^2}{\tau(p-1)} + \sum_{p \in \mathcal{E}} \frac{R_2^2}{\tau(p-1)}. \quad (5)$$

Our aim is to estimate the sums on the right hand side of (5).

To estimate $R_1$ we divide the interval of summation over $x$ to progressions of the form $y + z t_p/d$, $1 \leq y \leq t_p/d$, $1 \leq z \leq d$. Thus

$$R_1 = \max_{(a,p)=1} \left| \sum_{d \mid t_p} \sum_{y=1}^{t_p/d} \sum_{z=1}^{d} \sum_{c \leq t_p/d} \sum_{n \leq T} \gamma_n e_{tp/d} (c(s_n - y)) e_p (a \lambda^y \lambda^{zt_p/d}) \right|,$$

whence

$$R_1 \ll \frac{1}{t_p} \sum_{d \mid t_p} \sum_{y=1}^{t_p/d} \sum_{c \leq t_p/d} \sum_{n \leq T} \gamma_n e_{tp/d} (c(s_n - y)) \max_{(a,p)=1} \sum_{z=1}^{d} e_p (a \lambda^y \lambda^{zt_p/d}).$$

8
The sum over $z$ is estimated by Lemma 5. Since $\lambda^{t_p/d}$ is an element of multiplicative order $d$, then from Lemma 5 we derive

$$\max_{(a,p)=1} \left| \sum_{z=1}^{d} e_p(a\lambda^y\lambda^{zt_p/d}) \right| \ll p^{1/8} d^{5/8}.$$ 

Therefore,

$$R_1 \ll \sum_{d|t_p, d \geq L v_p} p^{1/8} d^{5/8} R_3, \quad (6)$$

where

$$R_3 = \frac{1}{t_p} \sum_{y=1}^{t_p/d} \sum_{c \leq t_p/d} \sum_{n \leq T} \gamma_n e_{t_p/d}(c(s_n - y)) \left| \sum_{c \leq t_p/d} \sum_{n \leq T} \gamma_n e_{t_p/d}(c(s_n - y)) \right|^2.$$

Next, applying the Cauchy inequality we obtain

$$R_3^2 \ll \frac{1}{dt_p} \sum_{y=1}^{t_p/d} \sum_{c_1 \leq t_p/d} \sum_{c_2 \leq t_p/d} \sum_{n_1 \leq T} \sum_{n_2 \leq T} \gamma_{n_1} e_{t_p/d}(c_1(s_{n_1} - y) - c_2(s_{n_2} - y)).$$

Observe that

$$\sum_{y=1}^{t_p/d} e_{t_p/d}(-c_1 y + c_2 y) = \begin{cases} \frac{t_p}{d}, & \text{if } c_1 \equiv c_2 \pmod{t_p/d}, \\ 0, & \text{if } c_1 \not\equiv c_2 \pmod{t_p/d}. \end{cases}$$

Hence,

$$R_3^2 \ll \frac{1}{d^2} \sum_{c_1 \leq t_p/d} \sum_{n_1 \leq T} \sum_{c_2 \leq t_p/d} \sum_{n_2 \leq T} \gamma_{n_1} \gamma_{n_2} e_{t_p/d}(c(s_{n_1} - s_{n_2})).$$

Estimating trivially the sums over $c$, $n_1$ and $n_2$ we obtain

$$R_3^2 \ll \frac{t_p}{d^2 T^2}.$$
Substituting this in (6), we derive that
\[ R_1^2 \ll \tau(p-1)T^2 \sum_{\substack{d|t_p \atop d \geq Lv_p}} \frac{p^{1/4}t_p}{d^{7/4}}. \]

Since \( v_p = t_p^{4/7}p^{1/7} \), then
\[ R_1^2 \ll \tau(p-1)^2T^2 \frac{p^{1/4}t_p}{(Lv_p)^{7/4}} = \tau(p-1)^2T^2L^{-7/4}, \]
whence
\[ \frac{R_1^2}{\tau(p-1)} \ll \tau(p-1)T^2L^{-7/4}. \]

Application of the Titchmarsh estimate (2) yields
\[ \sum_{p \in \mathcal{E}} \frac{R_1^2}{\tau(p-1)} \ll XT^2L^{-7/4}. \tag{7} \]

Now we proceed to treat \( R_2 \). From (4) we have
\[ R_2 \leq \frac{1}{t_p} \sum_{\substack{d|t_p \atop d < Lv_p}} \sum_{x=1}^{t_p} \left| \sum_{\substack{c \leq t_p/d \atop (c,t_p/d) = 1}} \gamma_n e_{t_p/d}(c(s_n - x)) \right|. \]

We apply the Cauchy inequality to the sums over \( d \) and \( x \) and then obtain
\[ R_2^2 \ll \tau(p-1) \sum_{\substack{d|t_p \atop d < Lv_p}} \sum_{x=1}^{t_p} \left| \sum_{\substack{c \leq t_p/d \atop (c,t_p/d) = 1}} \sum_{n \leq T} \gamma_n e_{t_p/d}(c(s_n - x)) \right|^2, \]
whence
\[ \frac{R_2^2}{\tau(p-1)} \ll \frac{1}{t_p} \sum_{\substack{d|t_p \atop d < Lv_p}} \sum_{x=1}^{t_p} \sum_{\substack{c_1 \leq t_p/d \atop (c_1,t_p/d) = 1}} \sum_{c_2 \leq t_p/d} \sum_{n_1 \leq T} \sum_{n_2 \leq T} \gamma_{n_1} \gamma_{n_2} e_{t_p/d}(c_1(s_{n_1} - x) - c_2(s_{n_2} - x)). \]
The summation over $x$ guarantees that $c_1 = c_2$. Therefore,
\[
\frac{R_2^2}{\tau(p-1)} \ll \sum_{d \mid t_p, \, d < L \nu_p (c, t_p/d) = 1} \sum_{c \leq t_p/d} \sum_{n_1 \leq T} \sum_{n_2 \leq T} \gamma_{n_1} \gamma_{n_2} e_{t_p/d}(c(s_{n_1} - s_{n_2})),
\]
whence
\[
\frac{R_2^2}{\tau(p-1)} \ll \sum_{d \mid t_p, \, d < L \nu_p (c, t_p/d) = 1} \sum_{c \leq t_p/d} \left| \sum_{n \leq T} \gamma_n e_{t_p/d}(c s_n) \right|^2.
\]

Summing up both sides of this bound over $p \in E$, we obtain
\[
\sum_{p \in E} \frac{R_2^2}{\tau(p-1)} \ll \sum_{p \in E} \sum_{d \mid t_p, \, d < L \nu_p (c, t_p/d) = 1} \sum_{c \leq t_p/d} \left| \sum_{n \leq T} \gamma_n e_{t_p/d}(c s_n) \right|^2.
\]

We divide the interval $(\Delta, X]$ into disjoint subintervals $(X_j, X_{j+1}]$, where
\[
X_1 = \Delta, \quad X_{j+1} = \min\{2X_j, X\}.
\]

Denote by $E_j$ the subset of $E$ such that $t_p \in (X_j, X_{j+1}]$ for any $p \in E_j$. Next, define
\[
V_j = 2X_j^{4/7} X^{1/7}
\]
and observe that $V_j$ does not depend on $p$, and $V_j \geq v_p$ for any $p \in E_j$. Thus,
\[
\sum_{p \in E} \frac{R_2^2}{\tau(p-1)} \ll \sum_{j} \sum_{p \in E_j} \sum_{d \mid t_p, \, d < L \nu_j (c, t_p/d) = 1} \sum_{c \leq t_p/d} \left| \sum_{n \leq T} \gamma_n e_{t_p/d}(c s_n) \right|^2.
\]

We remember that $j \ll \log X, 2^j X_1 \ll X$ and
\[
\Delta \leq X_j < X_{j+1} \leq 2X_j \leq 2X.
\]

Observe that for different primes $p, p \in E_j$, the corresponding values of $t_p$ do not have to be different. For a given $r \in (X_j, X_{j+1}]$ denote by $s(r)$ the number of all primes $p, p \in E_j$, for which $t_p = r$. Since $p - 1 \equiv 0 \pmod{r}$, then
\[
s(r) \leq X/r \leq X/X_j.
\]
Therefore,

\[
\sum_{p \in \mathcal{E}} \frac{R_p^2}{\tau(p-1)} \ll \sum_{j} \frac{X}{X_j} \sum_{r \in [X_j, X_{j+1}]} \sum_{d < LV_j \text{ s.t. } d \equiv 1 \pmod{r/d}} \sum_{c \leq r/d} \left| \sum_{n \leq T} \gamma_n e_{r/d}(c s_n) \right|^2.
\]

Changing the order of summation over \( r \) and \( d \) we deduce

\[
\sum_{p \in \mathcal{E}} \frac{R_p^2}{\tau(p-1)} \ll \sum_{j} \frac{X}{X_j} \sum_{d < LV_j} F_j(d), \tag{8}
\]

where

\[
F_j(d) = \sum_{r \in [X_j, X_{j+1}]} \sum_{1 \leq c \leq r/d \text{ s.t. } c \equiv 1 \pmod{r/d}} \left| \sum_{n \leq T} \gamma_n e_{r/d}(c s_n) \right|^2.
\]

To estimate \( F_j(d) \) we apply the large sieve inequality given in Lemma 4. Then

\[
F_j(d) \ll (X_j^2 d^{-2} + s_T) T.
\]

Inserting this bound into (8), we obtain

\[
\sum_{p \in \mathcal{E}} \frac{R_p^2}{\tau(p-1)} \ll \sum_{j} \frac{X}{X_j} \sum_{d < LV_j} (X_j^2 d^{-2} + s_T) T,
\]

whence

\[
\sum_{p \in \mathcal{E}} \frac{R_p^2}{\tau(p-1)} \ll \sum_{j} X (X_j + s_T V_j L X_j^{-1}) T.
\]

Since \( V_j = 2X_j^{4/7} X^{1/7} \), we have

\[
\sum_{p \in \mathcal{E}} \frac{R_p^2}{\tau(p-1)} \ll XT \left( \sum_{j} X_j + s_T L X^{1/7} \sum_{j} X_j^{-3/7} \right).
\]
Finally, from the definition of $X_j$ we know that
$$
\sum_j X_j \ll X, \quad \sum_j X_j^{-3/7} \ll \Delta^{-3/7}.
$$

Therefore,
$$
\sum_{p \in \mathcal{E}} \frac{R_p^2}{\tau(p-1)} \ll XT(X + sT X^{1/7} \Delta^{-3/7} L).
$$

(9)

Theorem 4 now follows from (5), (7) and (9).

6 Proof of Theorem 3

For $K \leq 10$ the estimate of Theorem 3 is trivial. Therefore, we will suppose that $K > 10$.

Set $M = [T/K]$. Without loss of generality we may assume that for $n \geq 1$,
$$
\gamma_{T+n} = 0, \quad s_{T+n} = s_T + n.
$$

Applying the shifting argument we obtain
$$
\left| \sum_{n=1}^{T_p} \gamma_n e_p(a\lambda^{s_n}) \right|^2 \ll \frac{1}{(M+1)^2} \left| \sum_{n=1}^{T_p} \sum_{r=0}^{M} \gamma_{n+r} e_p(a\lambda^{s_{n+r}}) \right|^2 + \frac{T^2}{K^2}.
$$

(10)

Further, we have
$$
\sum_{n=1}^{T_p} \sum_{r=0}^{M} \gamma_{n+r} e_p(a\lambda^{s_{n+r}}) = \frac{1}{2T+1} \sum_{b=-T}^{T} \sum_{m=1}^{2T} \sum_{n=1}^{M} \sum_{r=0}^{T_p} \gamma_m e^{2\pi i \beta(n+r-m)}/2T+1} e_p(a\lambda^{s_m}).
$$

(11)

By the Cauchy inequality,
$$
\left( \sum_{0<|b|\leq T} \left| \sum_{n=1}^{T_p} \sum_{r=0}^{M} e^{2\pi i \beta(n+r)}/2T+1} e_p(a\lambda^{s_n}) \right| \right)^2 \ll
$$
$$
\left( \sum_{0<|b|\leq T} \left| \sum_{n=1}^{T_p} \sum_{r=0}^{M} e^{2\pi i \beta(n+r)}/2T+1} \right| \right) \times
$$
$$
\left( \sum_{0<|b|\leq T} \left| \sum_{n=1}^{T_p} \sum_{r=0}^{M} e^{2\pi i \beta(n+r)}/2T+1} \right| \right) \times
$$
$$
\left( \sum_{0<|b|\leq T} \left| \sum_{n=1}^{T_p} \sum_{r=0}^{M} e^{2\pi i \beta(n+r)}/2T+1} e_p(a\lambda^{s_n}) \right| \right)^2.
$$
Hence, using

\[ \left| \sum_{n=1}^{T_p} e^{2\pi i \frac{b_n}{2T+1}} \right| \ll \frac{T}{|b|}, \]

we obtain the bound

\[ T^2 \left( \sum_{0<|b|\leq T} \frac{|S(b)|}{|b|} \right) \left( \sum_{b=1}^{T} \frac{|S(b)|}{|b|} \right) \left( \sum_{m=1}^{T} \left| \gamma_m e^{2\pi i \frac{b_m}{2T+1}} e_p(a\lambda^{s_m}) \right|^2 \right) \ll \]

where

\[ S(b) = \sum_{r=0}^{M} e^{2\pi i \frac{b_r}{2T+1}}. \] (12)

Combining this with (10) and (11), we deduce

\[ \left| \sum_{n=1}^{T_p} \gamma_n e_p(a\lambda^{s_n}) \right|^2 \ll \]

\[ \frac{1}{(M+1)^2} \left( \sum_{0<|b|\leq T} \frac{|S(b)|}{|b|} \right) \left( \sum_{0<|b|\leq T} \frac{|S(b)|}{|b|} \right) \left( \sum_{m=1}^{T} \left| \gamma_m e^{2\pi i \frac{b_m}{2T+1}} e_p(a\lambda^{s_m}) \right|^2 \right) \]

\[ + \left| \sum_{m=1}^{T} \gamma_m e_p(a\lambda^{s_m}) \right|^2 + \frac{T^2}{K^2}. \]

Now we take maximum over \( a \), \( (a, p) = 1 \), and observe that the maximum of sums is not greater than the sum of maximums. We then divide the estimate
by $\tau(p-1)$ and perform the summation over $p \in \mathcal{E}_1$. This yields

$$
\sum_{p \in \mathcal{E}_1} \frac{1}{\tau(p-1)} \max_{(a,p)=1} \left| \sum_{n=1}^{T_p} \gamma_n e_p(a\lambda^{s_n}) \right|^2 \ll
$$

$$
\frac{1}{(M+1)^2} \left( \sum_{0 \leq |b| \leq T} \frac{|S(b)|}{|b|} \right) \times
$$

$$
\left( \sum_{0 \leq |b| \leq T} \frac{|S(b)|}{|b|} \sum_{p \in \mathcal{E}_1} \frac{1}{\tau(p-1)} \max_{(a,p)=1} \left| \sum_{m=1}^{T} \gamma_m e^{2\pi i b_m T + r} e_p(a\lambda^{s_m}) \right|^2 \right)
$$

$$
+ \sum_{p \in \mathcal{E}_1} \frac{1}{\tau(p-1)} \max_{(a,p)=1} \left| \sum_{m=1}^{T} \gamma_m e_p(a\lambda^{s_m}) \right|^2 + \frac{T^2}{K^2} \sum_{p \in \mathcal{E}_1} \frac{1}{\tau(p-1)}
$$

For each $b$ to the sum

$$
\sum_{p \in \mathcal{E}_1} \frac{1}{\tau(p-1)} \max_{(a,p)=1} \left| \sum_{m=1}^{T} \gamma_m e^{2\pi i b_m T + r} e_p(a\lambda^{s_m}) \right|^2
$$

we apply Theorem 1 with $\gamma_n$ substituted by $\gamma_n e^{2\pi i b_n T + r}$. Thus,

$$
\sum_{p \in \mathcal{E}_1} \frac{1}{\tau(p-1)} \max_{(a,p)=1} \left| \sum_{n=1}^{T_p} \gamma_n e_p(a\lambda^{s_n}) \right|^2 \ll
$$

$$
\left( \frac{1}{(M+1)^2} \left( \sum_{b=1}^{T} \frac{|S(b)|}{b} \right)^2 + 1 \right) (X + s_T X^{1/7} \Delta^{-3/7} L + T L^{-7/4}) XT
$$

$$
+ \frac{T^2}{K^2} \sum_{p \in \mathcal{E}_1} \frac{1}{\tau(p-1)}
$$

Now it remains to prove that

$$
\sum_{b=1}^{T} \frac{|S(b)|}{b} \ll (M+1) \log K.
$$

To this end, choose $\ell = [\log K]$ and use the Holder inequality to obtain

$$
\sum_{b=1}^{T} \frac{|S(b)|}{b} \leq \left( \sum_{b=1}^{2T+1} \frac{1}{b^{2\ell/(2\ell-1)}} \right)^{1-1/2\ell} \left( \sum_{b=1}^{2T+1} |S(b)|^{2\ell} \right)^{1/2\ell}.
$$

(13)
Next, we have
\[
\sum_{b=1}^{2T+1} \frac{1}{b^{2\ell/(2\ell-1)}} \ll \int_{1}^{\infty} x^{-1-(2\ell-1)^{-1}} \, dx = 2\ell - 1 \ll \log K. \quad (14)
\]

Besides, from the definition of \( S(b) \), see (12), it follows
\[
\sum_{b=1}^{2T+1} |S(b)|^{2\ell} = (2T + 1)J, \quad (15)
\]
where \( J \) denotes the number of solutions to the congruence
\[
\sum_{i=1}^{\ell} x_i \equiv \sum_{i=1}^{\ell} y_i \pmod{2T+1}, \quad 0 \leq x_i, y_j \leq M.
\]

Since \( M < T \), then the trivial estimate gives \( J \leq (M + 1)^{2\ell-1} \). Besides, \( T < K(M + 1) \). Therefore,
\[
\sum_{b=1}^{2T+1} |S(b)|^{2\ell} \ll K(M + 1)^{2\ell},
\]
whence, in view of (13)–(15), we conclude that
\[
\sum_{b=1}^{T} \frac{|S(b)|}{b} \ll (\log K)(M + 1)K^{1/(2\ell)} \ll (M + 1) \log K.
\]

7 Exponent pairs for Gauss sums

We remark that if in Lemma 5 we have the bound
\[
\max_{(a,p)=1} \left| \sum_{z=1}^{t} e_p(a\theta z) \right| \ll p^{\alpha t^\beta} \quad (16)
\]
with \( 0 \leq \alpha, \beta \leq 1 \), then the right hand side of the estimate of Theorem 1 can be substituted by
\[
(X + s_T X^{\frac{2a}{5-2\alpha}} \Delta^{-\frac{2-2\beta}{5-2\alpha}} L + TL^{-3+2\beta})XT.
\]
In particular, Corollary 2 takes place for the sequence $s_n$ satisfying the condition

$$s_T \leq T^{1 + \frac{1 - 2\alpha - \beta}{3 - 2\beta} + o(1)}.$$ 

Define $\mathcal{K}$ to be the set of all ordered pairs $\{\alpha, \beta\}$ with $0 \leq \alpha, \beta \leq 1$ and satisfying the property (16). Konyagin has proved that the set $\mathcal{K}$ contains the pair $\{\alpha_n, \beta_n\}$ defined as

$$\alpha_n = \frac{1}{2n^2}, \quad \beta_n = 1 - \frac{2}{n^2} + \frac{1}{2n-1n^2}$$

for any positive integer $n$. Furthermore, $\mathcal{K}$ also contains the pair $\{\alpha'_n, \beta'_n\}$ given by

$$\alpha'_n = \frac{1}{2n(n+1)}, \quad \beta'_n = 1 - \frac{2}{n(n+1)} + \frac{3}{2n+1n(n+1)}.$$

We now define the function $f : \mathcal{K} \to \mathbb{R}$ by

$$f(x, y) = 1 + \frac{1 - 2x - y}{3 - 2y}.$$ 

The problem is to find the value of $f(x, y)$ as big as possible. The result of the present paper corresponds to the pair $\{\alpha_2, \beta_2\}$ (which is due to Heath-Brown and Konyagin). Other pairs give less precise bounds. Next, we note that $\mathcal{K}$ is a convex set. That is, if

$$\{\alpha, \beta\} \in \mathcal{K}, \quad \{\alpha', \beta'\} \in \mathcal{K},$$

then for any $x$, $0 \leq x \leq 1$,

$$\{x\alpha + (1-x)\alpha', x\beta + (1-x)\beta'\} \in \mathcal{K}.$$ 

However, this property applied to any two given pairs, in particular to the pairs due to Konyagin, is not sufficient to get further improvements, and it would be very interesting, similar to the set of exponent pairs, to have more non-trivial properties of $\mathcal{K}$. The truth of the conjecture of Montgomery, Vaughan and Wooley would imply

$$\{\varepsilon, 1/2 + \varepsilon\} \in \mathcal{K},$$

which can be considered as an analogy of the exponent pair hypothesis.

Finally, we remark that the method we have applied leads to the following generalization of our main result.
Theorem 6. For any positive integer $N$, any $L > 0$, any pair $\{\alpha, \beta\} \in K$ and any complex coefficients $\delta_n$, $1 \leq n \leq N$, the following bound holds:

$$\sum_{p \in \mathcal{E}} \frac{1}{\tau(p-1)} \max_{(a, p) = 1} \left| \sum_{n=1}^{N} \delta_n e_p(a\lambda^n) \right|^2 \ll X(X + NX^{\frac{2a}{2a-2\beta}} \Delta^{-\frac{2-2\beta}{3-2\beta}} L) \sum_{n=1}^{N} |\delta_n|^2 + XL^{-3+2\beta} \left( \sum_{n=1}^{N} |\delta_n| \right)^2,$$

where the implied constant depends only on the pair $\{\alpha, \beta\}$.

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