STATISTICAL HYPERBOLICITY IN GROUPS

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Abstract. In this paper, we introduce a geometric statistic called the sprawl of a group with respect to a generating set, based on the average distance in the word metric between pairs of words of equal length. The sprawl quantifies a certain obstruction to hyperbolicity. Group presentations with maximum sprawl (i.e., without this obstruction) are called statistically hyperbolic. We first relate sprawl to curvature and show that nonelementary hyperbolic groups are statistically hyperbolic, then give some results for products and for certain solvable groups. In free abelian groups, the word metrics asymptotically approach norms induced by convex polytopes, causing the study of sprawl to reduce to a problem in convex geometry. We present an algorithm that computes sprawl exactly for any generating set, thus quantifying the failure of various presentations of $\mathbb{Z}^d$ to be hyperbolic. This leads to a conjecture about the extreme values, with a connection to the classic Mahler conjecture.

1. Introduction

We will define and study a new geometric statistic for groups in this paper, called the sprawl of a group (with respect to a generating set). Sprawl measures the average distance between pairs of points on the spheres in the word metric, normalized by the radius, as the spheres get large. This gives a numerical measure of the asymptotic shape of spheres that can be studied for arbitrary finitely generated groups and locally finite graphs.

To be precise, let

$$E(G, S) := \lim_{n \to \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y),$$

provided this limit exists. Note that since $0 \leq d(x, y) \leq 2n$, the value is always between 0 and 2. By way of interpretation, note that $E = 2$ means that one can almost always pass through the origin when traveling between any two points on the sphere without taking a significant detour. (The name is intended to invoke urban sprawl: a higher value means a lack of significant shortcuts between points on the periphery of the “city.”)

As we will see, this statistic is not quasi-isometry invariant but nonetheless captures interesting features of the large-scale geometry, to be developed in §2. Sprawl has connections to other geometric statistics such as divergence, almost-convexity, and discrete Ricci curvature. After explaining why this statistic detects curvature properties, we show below that non-elementary hyperbolic groups always have $E(G, S) = 2$ for any generating set, so we can think of $2 - E$ as quantifying an obstruction to hyperbolicity in groups. We give some results about sprawl for non-hyperbolic groups, including product groups and some solvable examples (lamplighter graphs).

Free abelian groups and convex geometry. For free abelian groups, there are particularly clear results on the asymptotic shape of spheres and the distribution of their points that allow us to compute the sprawl. As we will review below in §3, the word metrics on $\mathbb{Z}^d$ are close at large scale to certain norms, and the points of the spheres are distributed in a way that tends to a limit measure on the unit sphere in the norm. This allows us to replace an asymptotic computation on large discrete spheres by a finite computation: integrating average distance on a polytope against an appropriate measure. We give an algorithm for performing this calculation for arbitrary $(\mathbb{Z}^d, S)$ in §4.1. Though we can compute sprawl exactly for any finite presentation of $\mathbb{Z}^d$, it is still an interesting problem to find the extremal values over all generating sets. That is, we are studying a group statistic that depends on the choice of generators, but how much can it vary? This becomes a (possibly hard) problem in convex geometry, which we will study below.

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Definition 1. A convex body is a convex set in $\mathbb{R}^d$ with interior. A perimeter is the boundary of a centrally symmetric convex body in $\mathbb{R}^d$.

(To emphasize this point: we are using the word “perimeter” in a special way, which includes the assumption of central symmetry. Accordingly, we assume that our generating sets $S$ for $\mathbb{Z}^d$ are symmetric, so that $S = -S$.)

We will show that a generating set $S$ for $\mathbb{Z}^d$ induces a perimeter $L$ in a very simple way ($L$ is just the boundary of the convex hull of $S$ in $\mathbb{R}^d$) and that the sprawl $E(\mathbb{Z}^d, S) = E(L)$ depends only on $L$. Furthermore $E(L) = E(TL)$ for linear transformations $T$, so sprawl gives an affine geometric invariant: average distance between two points on the perimeter, where both the distance and the measure have natural intrinsic definitions with respect to the shape. We conjecture that the cube and the sphere are the extreme shapes in every dimension, and we give some rigorous evidence for that in [HI]. This would mean, for example, that over all generating sets for $\mathbb{Z}^2$, the values achieved by sprawl are pinched between $4/\pi \approx 1.273$ and $4/3 \approx 1.333$. This extremization problem resembles the well-known Mahler conjecture in convex geometry, a parallel developed in the last section below.

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2. Hyperbolic groups and statistically hyperbolic groups

In a graph, let us adopt the convention that for a real number $r \geq 0$, the notation $S_r$ denotes the metric sphere of radius $|r|$. We will study the Cayley graph as the metric model of a group, adopting the convention that the points of our metric space are the vertices (that is, the elements of the group), endowed with the distance induced by the edges (the word metric). We will write $\beta (r) := \#B_r(e)$ to denote the number of group elements in the closed ball of radius $r$ about the identity (or by translation-invariance, about any other center) in the group.

2.1. Hyperbolicity. A metric space is called $\delta$-hyperbolic (or just hyperbolic, without specifying a value $\delta$) if every geodesic triangle has the property that each side is contained in the $\delta$-neighborhood of the union of the other two sides. In such a space, suppose two geodesic rays share a common endpoint. Then if they become separated by $2\delta$ at time $t$, they must subsequently diverge completely: the two subrays after this separation can be concatenated to form a complete quasigeodesic, because for $t > t_0$, any geodesic segment connecting $\gamma_1(t)$ and $\gamma_2(t)$ must return to a $2\delta$-neighborhood of $\gamma_i(t_0)$. This means that the distance between $\gamma_1(t)$ and $\gamma_2(t)$ is at least $2(t - t_0 - \delta)$. Since we have strong estimates on the distance after the rays stop fellow-traveling, our task for hyperbolic groups will be to get quantitative control of the fellow-traveling.

To illustrate the issues involved in finding the sprawl of a group, first consider the free (nonabelian) group $F_2$ with its standard generating set. (Here and from now on, std will denote the standard generating set for a group). The Cayley graph is a 4-regular tree, and to evaluate the average on the sphere directly, one forms a finite sum by fixing one point on the sphere and then counting the other points of $S_n$ at various distances from the first:

$$\sum_{y \in S_n} d(x, y) = \frac{3}{4} (2n) + \frac{1}{4} \frac{2}{3} (2n - 2) + \frac{1}{4} \frac{3}{2} \frac{2}{3} (2n - 4) + \frac{1}{4} \frac{3}{3} \frac{2}{3} (2n - 6) + \cdots + 0.$$

As $n \to \infty$, this can be evaluated using a geometric sum, and one computes in this way that $E(F_2, \text{std}) = 2$. This argument, however, is sensitive to the choice of generating set. What would happen for some other generating set? Does the $\delta$-hyperbolicity of the model space suffice? The answer is “No” in general. One can easily construct trees with sprawl any number between 0 and 2, trees where sprawl does not exist, and trees where sprawl depends on basepoint. These trees are highly nonhomogeneous and are not quasi-isometric to any group.

Remark 2 (Sprawl and classical curvature). Moving beyond locally finite graphs and groups, we can define the sprawl for metric spaces that have natural measures on spheres. Instead of counting measure one may
take Hausdorff measure in the appropriate dimension, for example. Thus for a space and measure \((X, \mu)\), we can write

\[
E(X) := \lim_{r \to \infty} \frac{1}{\mu(S_r)^2} \int_{S_r \times S_r} \frac{1}{r} d(x, y) \, d\mu^2.
\]

One can quickly show that the hyperbolic plane (and thus hyperbolic space of any dimension) has \(E = 2\): for two rays making angle \(\theta\) at their common basepoint, \(d(\gamma_1(t), \gamma_2(t)) \geq 2t - c(\theta)\), where \(c(\theta)\) is a constant depending on \(\theta\).

Indeed, it is not hard to identify a relationship between sprawl and curvature: if \(E_r\) is defined to be the average distance between pairs of points on \(S_r\) and \(M_\kappa\) is the model space of constant sectional curvature \(\kappa\), it is easily observed that for every fixed value of \(r\), the values \(E_r(M_\kappa)\) are strictly decreasing in \(\kappa\) (taking \(\kappa \leq \pi^2/4\) so that \(S_r\) is non-empty).

However, a \(\delta\)-hyperbolic space, indeed even a tree, need not have \(E = 2\), and exponential growth of balls or spheres does not suffice. For instance, consider modifying the four-regular tree by choosing one axis and modifying the degree at each vertex in that axis as a function of distance from the origin. Examples constructed in this way can achieve all values \(0 \leq E \leq 2\), and can also have \(E\) not exist or depend on basepoint. Thus to prove that hyperbolic groups have maximal sprawl, it is essential to make use of the homogeneity guaranteed by a transitive group action. We will use this by appealing to a strong result of Michel Coornaert giving definite exponential growth (not just a growth rate but furthermore a bound on the coefficients) for hyperbolic groups.

**Remark 3** (Divergence, almost-convexity, discrete Ricci curvature). Recall that sprawl is measured by computing the distances between pairs of points \(x, y \in S_n\), then taking the average and letting \(n \to \infty\). At least three other geometric statistics also study the geometry of pairs of points in the sphere.

- **Divergence** is measured by minimizing the length of paths between \(x, y \in S_n\) such that the path lies outside of \(B_n\), then taking the sup and letting \(n \to \infty\). This is widely studied for groups, for instance in [11][10][13][7].
- **Almost-convexity** for groups is measured by minimizing the length of paths between \(x, y \in S_n\) such that the path lies inside of \(B_n\). This was defined by Cannon in [3] and further explored in many papers, such as [13][5][4].
- **Ricci curvature for manifolds** is defined by considering infinitesimal spheres at a pair of basepoints, and measuring the average distances between corresponding points on the spheres. If that distance is greater than the distance between basepoints, then the curvature is negative; if smaller, then the curvature is positive; and if equal, then the curvature is zero. **Discrete Ricci curvature** mimics this construction in a manner usable for groups by measuring distances between corresponding points in metric spheres at different basepoints. This was defined by Yann Ollivier in [16] and compared to optimal transport definitions of Cédric Villani and coauthors in [17].

Thus the definition of sprawl gives it a family resemblance to other synthetic curvature conditions that have already proved useful.

Recall that a hyperbolic group is called **elementary** if it is finite or has a finite-index cyclic subgroup.

**Theorem 4.** Let \(G\) be a non-elementary hyperbolic group. Then \(E(G, S) = 2\) for any finite generating set \(S\). (That is, every presentation is statistically hyperbolic.)

**Proof.** Recall that \(z\) is said to be (metrically) **between** \(x\) and \(y\) if \(d(x, z) + d(z, y) = d(x, y)\). A set is between two other sets if there exists a triple of points, one from each of the sets, satisfying the betweenness condition.

Choose any \(0 < \rho < 1\) and \(x \in S_n\), and let \(x'\) be an arbitrary point on \(S_{\rho n}\) between \(e\) and \(x\). We need to bound the number of \(w \in S_n\) such that \(B_{2\delta}(x')\) is between \(e\) and \(w\). But if \(w'\) is a point in \(S_{\rho n}\) between \(e\) and \(w\), then \(d(w', w) = n - |pn|\). That means that the number of such \(w\) is overcounted by \(|B_{2\delta}| \cdot |S_n - |pn||\).

For every point \(v\) of \(S_n\), which is not of this kind, \(d(x, v) \geq 2(n - |pn| - \delta) \geq 2(n - \rho n - \delta)\) because the geodesics from the identity to \(x\) and to \(v\) have \(2\delta\)-divergence by time \(|pn|\). Thus,\n
\[
(*) \quad \sum_{x, y \in S_n} d(x, y) \geq 2(n - \rho n - \delta) \left( |S_n| - |B_{2\delta}| \cdot |S_n - |pn|| \right) \cdot |S_n|.
\]
Now we make use of the homogeneity. Coornaert proved in [5] that for every non-elementary hyperbolic group with fixed generating set, there are bounded coefficients of exponential growth:

\[ \exists c_1, c_2 > 0, \omega > 1 \quad \text{s.t.} \quad c_1 \omega^n \leq \beta(n) \leq c_2 \omega^n \quad \forall n \in \mathbb{N}. \]

It follows from these inequalities that

\[ \frac{|S_n - |\rho n||}{|S_n|} = \frac{\beta(n - |\rho n|) - \beta(n - |\rho n| - 1)}{\beta(n) - \beta(n - 1)} \to 0 \]

as \( n \to \infty \), which together with (†) gives us

\[ E(G, S) = \lim_{n \to \infty} \frac{\sum d(x, y)}{n|S_n|^2} \geq 2(1 - \rho). \]

Since \( 0 < \rho < 1 \) was arbitrary, this means \( E = 2 \). \( \square \)

To quickly clarify the necessity for the non-elementary hypothesis: for \( G = \mathbb{Z} \) and any finite generating set, the spheres of large radius are divided into a positive part and a negative part, each of uniformly bounded diameter. Thus a pair of points has bounded distance with probability \( 1/2 \) and distance boundedly close to \( 2n \) with probability \( 1/2 \). This gives \( E(\mathbb{Z}, S) = 1 \) for all finite generating sets \( S \).

### 2.2. Some statistically hyperbolic groups and spaces.

Here we exhibit several examples of non-hyperbolic groups with statistically hyperbolic presentations. We first consider groups that are direct products with a hyperbolic factor, and then use the results on products to consider Diestel-Leader graphs.

Let us say that a based space \((H, h_0)\) has **definite exponential growth** if the growth function \( \beta(n) \) of balls of radius \( k \) centered at \( h_0 \) in \( H \) satisfies (†). Given a sequence of finite sets \( A_n \), we will say that **almost all** points of \( A_n \) satisfy a property \( (P) \), or that the property has **full measure**, if the subset of elements satisfying \( (P) \) has proportion tending to one as \( n \to \infty \).

The subtlety in analyzing products is that the sphere of radius \( n \) projects to a sphere but to a ball in each factor. Thus we need estimates for distances when points are on spheres of different radii: we can use definite exponential growth in one factor to get control on the difference in radius (so that most of the projection is in an annulus \( A_n \)), and then use hyperbolicity to get the distance estimates. We also need to know that spheres in these annuli are **evenly covered** by which we mean that there is a function \( f_n : \mathbb{N} \to \mathbb{N} \) such that \( \#(\pi^{-1}(h) \cap S_n^X) = f_n(|h|_H) \) for almost all \( h \in A_n^H \).

In the following technical lemma, the reader should imagine that \( H \) is a direct factor of \( X \) and that \( \pi : X \to H \) is coordinate projection. Recall that a **semi-contraction** is a distance non-increasing map.

**Lemma 5** (Annulus lemma). Let \((X, x_0)\) and \((H, h_0)\) be based graphs, suppose \( H \) is \( \delta \)-hyperbolic with definite exponential growth, and fix any \( 0 < \rho < 1 \). Consider the annulus \( A_n^H = B_n^H(h_0) \setminus B_{|\rho n|}^{H_{-1}}(h_0) \) in \( H \) and the sphere \( S_n^X = S_n^X(x_0) \) in \( X \). Let \( \pi : (X, x_0) \to (H, h_0) \) be a semi-contraction, mapping almost all points of \( S_n^X \) into \( A_n^H \) such that spheres in \( A_n \) are evenly covered. Then

\[ \liminf_{n \to \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y) \geq 2\rho. \]

This is proved by showing that when \( \rho n \leq i, j \leq n \), then the average distance between a point in \( S_n^H \) and a point in \( S_n^H \) is bounded below by \( i + j - 2\delta - 2\rho n - c\omega^{-i\rho} \) for a constant \( c \) where \( \omega \) is the growth rate of \( H \), as in (†).

We will apply this lemma to products of the form \( H \times K \) where \( H \) is hyperbolic and \( K \) grows strictly slower—that is, \( K \) has subexponential growth, or has a smaller exponential growth rate. Let us say that a generating set for a product is **split** if every generator projects to the identity in one of the factors.

**Proposition 6** (Products with a dominated factor). Suppose that \( H \) is a non-elementary hyperbolic group, \( K \) is finitely generated, and \( S \) is a split finite generating set for \( H \times K \) such that the growth function of \( H \) dominates the growth function of \( K \) with generators projected to the factors from \( S \). Then \((H \times K, S)\) is a statistically hyperbolic presentation.
Proof. Let $\pi$ be projection to the $H$ factor from $X = H \times K$ and note that 

$$S_k^X = \bigcup_{i=0}^{k} S_i^H \times S_{k-1}^K.$$ 

Thus one easily verifies the hypotheses of the annulus lemma. Letting $\rho \to 1$ gives $E(X, S) = 2$. \hfill $\Box$

Another class of statistically hyperbolic spaces is the *Diestel-Leader graphs*. We describe them briefly here and refer the reader to [13] for a more thorough treatment and some relevant properties. For $m, n \geq 2$, take an $(m + 1)$-valent tree $T_1$ and an $(n + 1)$-valent tree $T_2$. Choose ends and corresponding horofunctions $f_1$ and $f_2$. This gives height functions $h_1 = f_1$ and $h_2 = -f_2$ on the trees. We visualize $T_1$ as “growing up” from its end at height $-\infty$ and $T_2$ as “hanging down” from its end at $+\infty$. The Diestel-Leader graph $DL(m, n)$ is defined to be the subspace of $T_1 \times T_2$ on which $h_1 = h_2$. A height function $h$ is induced on this graph from the tree factors, since their height functions match. Like Cayley graphs, Diestel-Leader graphs have vertex-transitive group actions by isometries, which guarantees that geometric invariants of $DL(m, n)$ do not depend on the choice of basepoint.

These graphs are considered models for solvable geometry: the structure described above is in precise analogy with the geometry of Sol, which has hyperbolic plane factors in the place of trees. Eskin, Fisher, and Whyte [9] exploit this analogy to completely classify Diestel-Leader graphs and spaces with Sol geometry up to quasi-isometry. Furthermore, for $m \geq 2$, $DL(m, m)$ can be realized as Cayley graphs of solvable groups, namely the *lamplighter groups* $F \wr \mathbb{Z}$ where $F$ is a finite group of order $m$.

Denote the coordinate projections by $\pi_i : DL(m, n) \to T_i$ for $i = 1, 2$. A geodesic $\gamma$ in $DL(m, n)$ is said to *turn* if it switches from increasing in height to decreasing in height or vice versa. Geodesics in Diestel-Leader graphs have at most two turns. The following lemma tells us that spheres of large radius in a Diestel-Leader graph are “concentrated in distant heights.”

**Lemma 7** (Concentration in height). Let $X = DL(m, n)$ be a Diestel-Leader graph, $x_0 \in X$ be a basepoint at height $0$, and $0 < \rho < 1$. Denote by $S_k$ the sphere of radius $k$ in $X$ centered at $x_0$. For almost all $x \in S_k$, $\rho k \leq |h(x)| \leq k$. If $m > n$, then for almost all $x \in S_k$, $\rho k \leq h(x) \leq k$.

**Proof.** Assume $m \geq n$ and consider the problem of counting $k$-tuples $(x_1, x_2, \ldots, x_k)$ of vertices of $X$ such that each pair $(x_i, x_{i+1})$ bounds an edge and the concatenation of these edges forms a geodesic in $X$. If we start by choosing $x_1$ to be immediately above $x_0$, then we have $m$ choices, since there are $m$ vertices of $T_1$ immediately above $x_0$. $\pi_1(x_0)$ and only one vertex of $T_2$ above $\pi_2(x_0)$. If $x_2$ is chosen immediately above $x_1$, then there are $m^2$ choices for the pair $(x_1, x_2)$. In general, there are $m^i$ ways to choose a tuple of vertices $(x_1, \ldots, x_i)$ such that $x_{i+1}$ is immediately above $x_i$. Suppose we now choose $x_{i+1}$ below $x_i$. Then $\pi_1(x_{i+1}) = \pi_1(x_{i-1})$, which means that we have lost one of our choices for a vertex in $T_1$. This choice is replaced by the choice of a vertex $\pi_2(x_{i+1})$ immediately below $\pi_2(x_i)$ other than $\pi_2(x_{i-1})$. There are $n - 1$ such possibilities. If we now continue choosing vertices to be decreasing in height, then we continue replacing factors of $m^i$ with factors $k \leq m$. So turns in a geodesic reduce the number of choices and geodesics continue in the same direction for as long as they can before turning. If $m = n$, then the same argument applies if we begin choosing $x_1$ immediately below $x_0$, so geodesics tend to end in heights which are distant in either the positive or the negative direction. If $m > n$, then the above argument shows that geodesics in fact tend to end in high (positive) heights. \hfill $\Box$

**Theorem 8** (Diestel-Leader graphs). For any $m, n \geq 2$, the Diestel-Leader graph $X = DL(m, n)$ is statistically hyperbolic.

**Proof.** Let $S_k$ denote the sphere of radius $k$ centered at a point $x_0$ of height zero and let $\rho$ be fixed. We will begin by considering the case where $m > n$. $\pi_1 : X \to T_1$ is a semi-contraction, since $\pi_1$ takes paths in $X$ to paths in $T_1$ while preserving their length. By the previous lemma, a full-measure subset $U_k$ of $S_k$ lies above height $\rho k$. A similar argument can be used to show that the same is true for $T_1$, and that a full-measure subset $V_k$ of the annulus $A_k = B_k \setminus B_{\rho k}$ centered at $\pi_1(x_0)$ lies above height $\rho k$. In fact, $V_k = \pi_1(U_k)$. In order to apply the Annulus Lemma, we just need show the even covering condition. Suppose $y \in V_k$ with $d = d(y, \pi_1(x_0))$, and let $\gamma$ be a geodesic in $X$ of length $k$ starting at $x_0$ and ending at a point of $\pi_1^{-1}(y)$. Then $\gamma_{[0, d]}$ is a path from $\pi_1(x_0)$ to $y$. Such a path may initially decrease in height, and so choices are
made in the $T_2$ coordinate. But since $y$ is above height zero, $\gamma$ must then come back up and any choices made in $T_2$ for the initial portion of $\gamma$ will have no effect on where $\gamma$ ends. The only significant choices in $T_2$ for $\gamma$ occur after $\pi_1 \gamma$ passes $y$ and turns around again. This final downward portion of $\gamma$ has length $(k - d)/2$. So $k$ and $d$ must have the same parity and the number of points in the preimage of $y$ is a function of $d$. Thus we apply the annulus lemma to get $E(X) = 2$.

If $m = n$, then above argument shows that the average distance between a pair of points in $S_k$ above height 0 is close to 2$k$. By symmetry, it follows that the average distance between a pair of points in $S_k$ below height 0 is also close to 2$k$. But a significant proportion of pairs $x, y \in S_k$ will have the property that $h(x) > 0$ and $h(y) < 0$. By the previous lemma we may again assume that $h(x)$ is close to $k$ and that $h(y)$ is close to $-k$. So the difference in heights, a lower bound on distance, is close to 2$k$. □

When $m = n$, the Diestel-Leader graph $DL(m, m)$ can be realized as the Cayley graph of the lamplighter group $Z_m \wr Z$ for a certain natural generating set (or, more generally, $F \wr Z$ for any finite group $F$ of order $m$). These interesting solvable groups are not nilpotent and they are not finitely presented.

**Corollary 9 (Lamplighter groups).** The lamplighter groups $Z_m \wr Z$ have statistically hyperbolic presentations.

Finally, besides Euclidean space itself, the symmetric spaces of noncompact type also have $E < 2$, but because of the facts above it would be natural to expect that groups of non-uniform exponential growth need not have $E = 2$.

### 3. Reducing from free abelian groups to convex geometry

In the free abelian groups $\mathbb{Z}^d$, studying the large-scale metric geometry is greatly aided by the natural embedding in $\mathbb{R}^d$. It is known that the finite word metrics on $\mathbb{Z}^d$ are asymptotic to norms on $\mathbb{R}^d$ (originally due to Burago [2], and shown by an elementary geometric argument in [9]), so that these norms can be thought of as **limit metrics** coming from group theory. Recall that any convex, centrally symmetric body in $\mathbb{R}^d$ induces a **Minkowski norm**, namely the norm for which that convex body is the unit ball. If a generating set for $\mathbb{Z}^d$ is called $S$, let $|w|$ denote the length of $w \in \mathbb{Z}^d$ in the word metric, and let $L$ be the boundary of the convex hull of $S$ in $\mathbb{R}^d$. Then the Minkowski norm $\| \cdot \|_L$ having $L$ as its unit sphere is the limit metric, in the sense that there is a constant $K$ depending on $S$ such that

$$\|w\|_L \leq |w| \leq \|w\|_L + K$$

for all $w \in \mathbb{Z}^d$. This limit shape $L$ describes the asymptotic shape of spheres in the sense that $\frac{1}{n} S_n \to L$ (say as a Gromov-Hausdorff limit).

In an earlier paper, we proved counting results for spheres in word metrics on $\mathbb{Z}^d$, showing that counting measure on the discrete spheres $S_n$ converges to the **cone measure** $\mu_L$ on $L$, as pictured in Figure 11. The case of that theorem that is useful for us here states that

$$\lim_{n \to \infty} \frac{1}{|S_n|^2} \sum_{x, y \in S_n} \frac{1}{n} d(x, y) = \int_{L^2} \|x - y\|_L^2 d\mu_L(x, y).$$

(The original theorem addresses more general averaging problems.) Thus it follows immediately that

$$E(\mathbb{Z}^d, S) = \int_{L^2} \|x - y\|_L^2 d\mu_L^2(x, y)$$

for all finite generating sets $S$.

We can define the sprawl of any perimeter $L$ by the right-hand side, which we can denote by $E(L)$, measuring average distance between points of $L$ as measured in its intrinsic geometry. We remark that $E(L) = E(TL)$ for any linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$, since both the norm and the measure push forward under linear transformation. That is, $\|Tx - Ty\|_{TL} = \|x - y\|_L$, and $d\mu_{TL}(Tx) = d\mu_L(x)$.

One immediate consequence of the reduction to convex geometry is that $E(\mathbb{Z}^d, S)$ is always greater than 1/2.

**Proposition 10.** $E(L) > \frac{1}{2}$ for all perimeters $L$ in $\mathbb{R}^d$. 
Figure 1. Six arcs are shown in red in this figure, each having cone measure 1/14; in other words, all of the colored regions have 1/14 as much area as the convex body they are in. In the square and the hexagon, all sides have equal measure. On the other hand, for this octagon generated by the chess-knight moves \{ (±2, ±1), (±1, ±2) \}, the measure of its two types of sides (shown with green and blue) is in the ratio 4 : 3. Cone measure is defined on any perimeter, and in particular it is uniform on the circle.

Proof. Fix an arbitrary point \( x \in L \), and denote by \( Q \) the convex body of which \( L \) is boundary. The points of \( L \) whose distance from \( x \) is less than one are those contained in \( Q + x \), the translated copy of \( Q \) centered at \( x \). Since \( L + x \) contains 0 and \( Q \) is convex, there is a hyperplane \( P \) through 0 which does not intersect the interior of \( Q + x \). So the interior of \( Q + x \) is on one side of \( P \), and by central symmetry, half of the cone measure lies on each side of \( P \). Thus the average distance on \( L \) from \( x \) is \( \geq (1/2)(1) \). To obtain the strict inequality, just note that the distance from \( x \) to \(-x\) is always 2 and so a small neighborhood of \(-x\) contributes an amount near 2 to the average. □

4. Sprawl in the plane

From the work above, we have reduced the group calculation \( E(\mathbb{Z}^d, S) \) to the convex geometry calculation \( E(L) \). In this section we study this convex geometry in dimension 2, by first introducing an algorithm for evaluating \( E(L) \). This algorithm can be given to a computer (which we did, producing a great deal of experimental evidence for the conjectures to follow) but can also be used to produce precise formulas, such as those given below for the regular polygons.

4.1. Cutline algorithm. To compute the sprawl of a polygon, we can average the expected distances between pairs of sides. Pick two sides \( \sigma \) and \( \tau \) of \( L \) and parametrize each of them (say clockwise) by \([0,1]\); then the distance in the \( L \)-norm from \( \sigma(s) \) to \( \tau(t) \) is piecewise linear. Thus for an appropriate triangulation of the parameter space, average-distance is a linear function on each triangle. We outline here a method for triangulating, which we call the cutline algorithm for computing the sprawl of a polygon. We note that the algorithm generalizes straightforwardly to higher dimensions.

Fix \( \sigma \) and \( \tau \). Find the sector of angles at which the sides “see” each other—that is, the interval of arguments obtained by vectors from \( \sigma(s) \) to \( \tau(t) \) as in the first picture in Figure 2. Considering the same sector of angles viewed from the origin, as in the second picture, mark the angles that point in vertex directions in this sector (shown as a dashed line).

Figure 2. A depiction of the algorithm for finding the average distance between sides \( \sigma \) and \( \tau \).

For each vertex direction \( \theta \), consider the line \( T \subset [0,1] \times [0,1] \) of times at which the vector between the sides points in the vertex direction; the corresponding chords form a trapezoid as in the third picture. For each trapezoid, record the lengths of its bases, marked in the figure as \( l_0 \) and \( l_1 \). (In general, for direction \( \theta_i \), these are the largest and smallest values of \( d(\sigma(s), \tau(t)) \) for \( (s,t) \in T_i \), and can be denoted \( l_{i0} \) and \( l_{i1} \).)
Let $d_{00}, d_{01}, d_{10}, d_{11}$ be the four distances between an endpoint of $\sigma$ and an endpoint of $\tau$ (measured in the $L$-norm), with $d_{ij} = d(\sigma(i), \tau(j))$.

Next, consider the unit square formed by the parameters $[0, 1] \times [0, 1]$. All the distances between points on the two chosen sides of the polygon can be recorded by a real-valued function on this square. To find the average distance between sides $\sigma$ and $\tau$, we only need to integrate that function over the square (using Lebesgue measure because the cone measure is proportional to arclength on each side; the proper weights will be restored below). Since the function is piecewise linear, it will suffice to know its values at the points of a triangulation that is fine enough that the function is linear on each triangle.

For each vertex direction $\theta_i$, the corresponding times $T_i$ cut out a straight segment across the square, which we will call a cutline. The values at the corners of the square are the $d_{ij}$ and the values at the endpoints of the cutlines are the $t_{ij}$. If the cutlines do not triangulate the square, add dummy cutlines as needed (between these same points, so requiring no further distance calculations) to complete a triangulation. One such dummy cutline is shown in the figure.

Now the average distance between a point on side $\sigma$ and a point on side $\tau$ can be read off of this parameter square by just knowing the values at the vertices of the triangles: for each triangle, average the values at its vertices, and then sum those averages over all the triangles, weighted by the areas of the triangles. Thus let $E_{ij}$ denote the average distance between $\sigma_i$ and $\sigma_j$. Let $w_i$ be the weight of the $i$th side in the cone measure: $w_i = \mu_i(\sigma_i)$. Then, finally, the average distance between all pairs of points on the polygon can be written as the weighted average:

$$E(L) = \frac{\sum_{i,j} w_i w_j E_{ij}}{\sum_{i,j} w_i w_j}.$$

4.2. Values. By applying the cutline algorithm, we find formulas for the sprawls of regular polygons. We note that the regular hexagon is equivalent by linear transformation to the hexagon with vertices $\pm(1,0), \pm(1,1), \pm(0,1)$, which is the limit set for the generating set $S = \pm\{e_1, e_2, e_1 + e_2\}$. For regular polygons with at least 8 sides, however, they are not exactly realized by word metrics on $\mathbb{Z}^2$.

**Proposition 11.** Let $P_k$ be the regular $k$-gon and let $S^1$ be the round unit circle. Then

$$E(\mathbb{Z}^2, \pm\{e_1, e_2\}) = E(P_k) = \frac{4}{3};$$

$$E(\mathbb{Z}^2, \pm\{e_1, e_2, e_1 + e_2\}) = E(P_6) = \frac{23}{18} < \frac{4}{3};$$

$$E(P_k) = \frac{1 + 2\sqrt{2}}{3} < \frac{23}{18};$$

$$E(P_2) = \begin{cases} \frac{4}{\pi} \cdot \left( \frac{\pi}{x} \tan(\frac{\pi}{x}) + \frac{1}{3} \frac{\pi}{x} \tan(\frac{\pi}{x}) \right), & x \in 4\mathbb{N}, \\ \frac{4}{\pi} \cdot \left( \frac{\pi}{x} \sin(\frac{\pi}{x}) - \frac{1}{6} \frac{\pi}{x} \sin(\frac{\pi}{x}) \right), & x \in 4\mathbb{N} + 2, \end{cases}$$

$$E(S^1) = \frac{4}{\pi}.$$

**Proposition 11** shown in Figure 8 below, shows of the nontrivial dependence of sprawl on the choice of generating set. Since the word metrics of a group $G$ with respect to finite generating sets $S, S'$ are quasi-isometric, we see that sprawl is not a quasi-isometry invariant.

To prove the formula for regular polygons, one can set $a_j$ for the average distance from $\sigma_1$ to $\sigma_j$ and re-express that using the chordlengths $\ell_i = d(v_i, v_j)$, by the cutline algorithm. The $\ell_i$ themselves can then be written as trigonometric functions of $\pi/x$. Trigonometric identities finish the proof, since $E(P_2)$ is the weighted average of the $a_j$.

We note that the formulas for regular polygons each converge quickly to $4/\pi$, and track close together. Writing $E_{4N}(x)$ for a function whose values agree with $E(P_2)$ when $x \in 4\mathbb{N}$, and likewise $E_{4N+2}(x)$, we have:

$$E_{4N}(x) - \frac{4}{\pi} \sim \frac{16\pi^3}{45x^4}, \quad E_{4N+2}(x) - \frac{4}{\pi} \sim \frac{17\pi^3}{90x^4}, \quad E_{4N}(x) - E_{4N+2}(x) \sim \frac{\pi^3}{6x^4}.$$
Corollary 12. A dense subset of the interval \([4/\pi, 4/3]\) is contained in the set \(\{E(Z^2, S) : \text{gensets } S\}\).

Proof. There is a continuous path \(L_t\) through the space of perimeters that starts with the circle and ends with the square. The sprawl passes through all values from \(4/\pi\) to \(4/3\) along the path.

Any such perimeter \(L_t\) can be approximated arbitrarily closely by a rational polygon, which can be rescaled to an integer polygon without changing \(E\). The sprawl of a polygon is continuous in the coordinates of its vertices, and \(E\) of the approximants approaches \(E\) of the original body. Finally, the set of integer vertices can be completed to a generating set without changing \(E\), since the sprawl only depends on the extreme vertices. \(\square\)

4.3. Hexagons. Let \(H_{x,y}\) be the hexagon with vertices \(v_1 = (x, y), v_2 = (1, 1), v_3 = (-1, 1)\), where \(x \geq 1, y \geq 0, \) and \(x + y \leq 2\). Thus \(H_{1,0}\) is a square (realized as a degenerate hexagon) and \(H_{2,0}\) is a linear transform of the regular hexagon, giving

\[
E(H_{1,0}) = \frac{4}{3}, \quad E(H_{2,0}) = \frac{23}{18}.
\]

Lemma 13 (Parametrizing hexagons). Every convex, centrally symmetric hexagon is equivalent by a linear transformation to some \(H_{x,y}\).

Proof. Take a hexagon with vertices \(v_1, v_2, v_3, -v_1, -v_2, -v_3\). We can always find a linear transformation sending \(v_2 \mapsto (1,1)\) and \(v_3 \mapsto (-1,1)\). This reduces the parameter space to \(\{v_1 = (x, y) : x \geq 1, -1 \leq y \leq 1\}\). Also, without loss of generality, we have \(x + |y| \leq 2\); otherwise, change the choice of \(v_2, v_3\), as in Figure 4. Finally, up to reflection in one of the coordinate axes, we can assume \(y \geq 0\). \(\square\)

Applying the algorithm sketched above, we can compute the side-pair averages, and obtain the following formula for a hexagon parametrized as above.
Thus we have reduced the task of bounding the sprawl of hexagons to a calculus exercise (which we omit): verifying that in the domain defined by \( x \geq 1, y \geq 0, \) and \( x + y \leq 2, \) this quantity takes values between 23/18 and 4/3.

This establishes the following statement:

**Theorem 14** (Sprawls of hexagons and three-generator presentations).

\[
\{ E(H) : \text{ hexagons } H \} = \left[ \frac{23}{18}, \frac{4}{3} \right].
\]

Thus, \( \frac{23}{18} \leq E(\mathbb{Z}^2, S) \leq \frac{4}{3} \) whenever \( |S| \leq 6. \)

This provides evidence, taken together with the fast convergence for sprawls of regular polyhedra towards \( 4/\pi, \) for the following conjecture.

**Conjecture 15** (Sprawl Conjecture for \( d = 2 \)). The circle and the square are the extreme cases for all perimeters in \( \mathbb{R}^2. \) That is,

\[
\{ E(L) : \text{ perimeters } L \subset \mathbb{R}^2 \} = \left[ \frac{4}{\pi}, \frac{4}{3} \right].
\]

Further evidence is given in the next section, where the sphere and cube are shown to be sharp bounds asymptotically as \( d \to \infty. \)

## 5. Sprawl in \( d \) dimensions: The not-so-flatness of \( \mathbb{Z}^d \)

In higher dimensions, the computation of expected distance between two points becomes quite intuitive for the sphere and the cube. Suppose \( d \) is very large. For the round unit sphere \( \text{Sphere}_d \subset \mathbb{R}^d, \) which induces the Euclidean metric as its Minkowski norm, take one point to be at the north pole without loss of generality. Then concentration of measure phenomena ensure that the second point is almost surely on the equator, so the Euclidean metric as its Minkowski norm, take one point to be at the north pole without loss of generality.

In this case, the distance computation is performed by sampling the random variable \( |x_i - y_i|, \) which ranges between 0 and 2, a total of \( d \) times. For very large \( d, \) we should expect this supremum to tend to 2. This reasoning predicts that \( E(\text{Sphere}_d) \to \sqrt{2} \) and \( E(\text{Cube}_d) \to 2; \) the former can be approximated and the latter can be exactly realized by a word metric. What about the group \( \mathbb{Z}^d \) with its standard generating set? In dimension 2, this is isometric to the cube metric, but that is no longer true for \( d > 2. \) In dimension 3, the limit shape for the standard word metric is an octahedron, and more generally in dimension \( d \) it is the join of \( d \) copies of \( S^0, \) called an orthoplex (or cross-polytope). We will derive the answer below, finding that \( E(\mathbb{Z}^d, \text{std}) \to \frac{3}{2}. \)

By way of interpretation, this says that a cubical generating set gives \( \mathbb{Z}^d \) more and more hyperbolic-like geometry as \( d \) gets large, while the standard word metric is bounded uniformly away (see Figure 5). We are accustomed to describing the group \( \mathbb{Z}^d \) as “flat” because it is quasi-isometric to Euclidean space. However, using this statistic that gives a finer measure of large-scale curvature, we see that the standard generators give more of a hyperbolic character to the group, and that there exist generators for large \( d \) which make the geometry a good deal closer to hyperbolic than flat.

In the computations below, recall that for natural numbers \( n, \) the double factorial \( n!! \) denotes the product of all the natural numbers up to \( n \) that have the same parity:

\[
n!! = \prod_{i \leq n, \text{ even}} (n - 2i).
\]

Double factorials will occur in the calculations, but they can be re-expressed in two cases:

\[
(2n)!! = 2^n \cdot n! ; \quad (2n + 1)!! = \frac{(2n + 1)!}{2^n \cdot n!}.
\]

To get rates of approach, we use an approximation for \( n! \) that goes one term beyond Stirling’s formula:

\[
n! = \sqrt{2\pi n} \left( \frac{n}{e} \right)^n \left( 1 + \frac{1}{12n} + O\left( \frac{1}{n^2} \right) \right).
\]
5.1. The sphere.

**Proposition 16.** The sphere induces the $\ell^2$ metric on $\mathbb{R}^d$. The formula for the sprawl of the sphere is given in the following closed form:

$$E(\text{Sphere}_d) = \frac{2^{d-1}}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2}d)^2}{\Gamma(d - \frac{3}{2})}.$$ 

Thus, $E(\text{Sphere}_d) \to \sqrt{2}$ as $d \to \infty$, with

$$\sqrt{2} - E(\text{Sphere}_d) \sim \frac{1}{8d}.$$

**Proof.** Recall that, where $A_k$ denotes the surface area of $S^k$ (so that $A_1 = 2\pi$ and $A_2 = 4\pi$), there is a recursive formula given by $A_k = \int_0^\pi A_{k-1} \sin^{k-1}(\theta) \, d\theta$. The distance between two points on the sphere that subtend an angle $\theta$ at the origin is $\sqrt{2 - 2\cos \theta}$. Then we find that the $A_k$ terms cancel out, giving

$$E(\text{Sphere}_d) = \frac{\int_0^\pi \sqrt{2 - 2\cos \theta} \cdot \sin^{d-2}(\theta) \, d\theta}{\int_0^\pi \sin^{d-2}(\theta) \, d\theta},$$

which can be computed explicitly.

Let

$$a_n = \int_0^\pi \sqrt{2 - 2\cos \theta} \sin^n \theta \, d\theta, \quad b_n = \int_0^\pi \sin^n \theta \, d\theta,$$

so that $E(\text{Sphere}_d) = a_{d-2}/b_{d-2}$.

Integrating by parts gives $b_{n+2} = \frac{n+1}{n+2} b_n$, so since $b_0 = \pi$ and $b_1 = 2$, we get $b_n = c_n \frac{(n-1)!!}{n!}$, with $c_n = \pi$ if $n$ is even and 2 if $n$ is odd.

Change of variables and integration by parts gives the recursion $a_{n+1} = 2n+2 \frac{n+1}{2n+3} a_n$. Since $a_0 = 4$, this gives $a_n = 4 \frac{(2n)!!}{(2n+1)!!}$.

Combining and re-indexing with $d = n + 2$, we get

$$E(\text{Sphere}_d) = \frac{a_{d-2}}{b_{d-2}} = c_d \frac{(2d-4)!! (d-2)!!}{(2d-3)!! (d-3)!!}$$

where $c_d$ is $4/\pi$ if $d$ is even, and 2 if $d$ is odd. Re-expressing the double factorials completes the proof. Note that the use of the gamma function enables us to drop the dependence on parity of $d$ because $\Gamma(z)$ is an integer for whole numbers $z$ but has $\sqrt{\pi}$ in the denominator for half-integers $z$. \qed

5.2. The cube.

**Proposition 17.** The cube is the limit shape for $\mathbb{Z}^d$ with a nonstandard generating set $\{\pm e_1 \cdots \pm e_d\}$, and it induces the $\ell^\infty$ metric on $\mathbb{R}^d$. The formula for the sprawl of the cube is given in the following closed form:

$$E(\text{Cube}_d) = \frac{2d + 2}{d} - \left( \frac{2d + 1}{2d^2} \right) \left( \frac{4^d d!^2}{(2d)!^2} \right).$$

Thus, $E(\text{Cube}_d) \to 2$ as $d \to \infty$, with

$$2 - E(\text{Cube}_d) \sim \frac{\sqrt{\pi}}{\sqrt{d}}.$$

**Proof.** Let $x_i$ and $y_i$ be independently distributed uniformly on the interval $I = [-1, 1]$. We will use these random variables to compute the sprawl for $\text{Cube}_d$, which we identify with the $(d-1)$-complex in $\mathbb{R}^d$ with vertices $(\pm 1, \ldots, \pm 1)$. To fix notation: $\text{Cube}_1$ is a pair of points on the line and $\text{Cube}_2$ is a square in the plane. $\text{Cube}_d$ has $2d$ top-dimensional facets, each a copy of $I^{d-1}$. Note that each facet is the locus of points satisfying $x_i = c$ for $c = \pm 1$. It has exactly one opposite face ($x_i = -c$), and all the others are adjacent since the defining equations can be simultaneously satisfied. For a point in $\mathbb{R}^d$ to be in $\text{Cube}_d$, all coordinates must be in $I$, and at least one of its coordinates must be $\pm 1$.

We compute

$$P(|x_i - y_i| < r) = \frac{4r - r^2}{4} \quad \text{and} \quad P(|1 - y_i|) < r = \frac{r}{2}.$$
by considering the uniform measure on the square $I^2$ and calculating the portion of the area between the
lines $x - y = r$ and $y - x = r$ in the first case, and above the line $y = r$ in the second. From this we get
cumulative distribution functions

\[ F_{\text{same}}(r) = P(d_{\infty}(x, y) < r : x, y \text{ on same face}) = \frac{(4r - r^2)^{d-1}}{4^{d-1}}; \]

\[ F_{\text{adj}}(r) = P(d_{\infty}(x, y) < r : x, y \text{ on adjacent faces}) = \frac{(4r - r^2)^{d-2}}{4^{d-2}} \left( \frac{r}{2} \right)^2. \]

To find expectations, we integrate $\int_0^2 rF'(r) \, dr$.

The $d$-cube has $2d$ faces, so if $x$ is placed randomly, then the probability that $y$ is on the same face or on
the opposite face is $1/2d$ in each case, while all of the other $2d - 2$ faces are in the adjacent case. Recalling
that the distance between any two points on opposite faces is 2, we get

\[ E(\text{Cube}_d) = \frac{1}{d} + 1 \cdot \int_0^2 rF'_{\text{same}}(r) \, dr + (2d - 2) \cdot \int_0^2 rF'_{\text{adj}}(r) \, dr. \]

From this and some algebraic manipulation we derive

\[ E(\text{Cube}_d) = \frac{1}{d} + \frac{d-1}{4^{d-1}d} \left[ 2 \int_0^{d-1} r^{d-1}(4-r)^{d-2} \, dr + (d-1) \int_0^{d-1} r^d(4-r)^{d-2} \, dr + (2d - d) \int_0^{d+1} r^{d+1}(4-r)^{d-3} \, dr \right]. \]

Let’s let $I_{n,n} = \int_0^2 r^n (4-r)^n \, dr$. Integration by parts and some further manipulations will give recursive
formulas, for instance

\[ I_{n,n} = 2^{2n+1} \frac{(2n)!}{(2n+1)!} \]

which simplifies to $I_{n,n} = 2^{2n+1} \frac{(2n)!}{(2n+1)!}$ since $I_{0,0} = 2$.

The $I_{n+1,n}$, $I_{n+2,n}$, and $I_{n+4,n}$ are derived similarly, from which we find

\[ E(\text{Cube}_d) = 2 + \left( 2 + \frac{1}{d} \right) \frac{(2d-2)!}{(2d-1)!} + \frac{2}{d}. \]

Re-expressing the double factorials completes the proof.

5.3. The orthoplex.

**Proposition 18.** The orthoplex is the limit shape for $\mathbb{Z}^d$ with its standard generating set $\pm \{e_1\}$, and it
induces the $\ell^1$ metric on $\mathbb{R}^d$. The formula for the sprawl of the orthoplex is given in the following closed
form:

\[ E(\text{Orth}_d) = \frac{3d - 2}{2d - 1}. \]

Thus, $E(\text{Orth}_d) \to \frac{3}{2}$ as $d \to \infty$, with

\[ \frac{3}{2} - E(\text{Orth}_d) \sim \frac{1}{4d}. \]

**Proof.** First note that by symmetry, the expectation of $\|x-y\|_1$ is equal to $d$ times the expectation of $|x_1-y_1|$. Thus

\[ E(\text{Orth}_d) = d \int_{I^2} |x_1 - y_1| \, d\mu(x_1) \, d\mu(y_1), \]

where $d\mu$ is the measure induced by $\mu$ on a single coordinate axis of $\mathbb{R}^d$. That measure is given by

\[ d\mu(x_1) = \frac{(1 - |x_1|)^{d-2}}{(d-2)!} \, dx_1, \]

as can be verified by considering how much volume the orthoplex has at height $x_1$. We can renormalize to a
probability measure by taking $\nu = \frac{(d-1)!}{2} \mu$, so that $\int_{\text{Orth}_d} \, d\nu^2 = 1$. Thus we are calculating

\[ E(\text{Orth}_d) = d \int_{I^2} |x_1 - y_1| \, d\nu^2 = \frac{d(d-1)^2}{4} \int_{I^2} |x - y| \cdot (1 - |x|)^{d-2} (1 - |y|)^{d-2} \, dx \, dy. \]
But again by symmetry, this is just
\[ 2d(d-1)^2 \int_0^1 x(1-x)^{d-2} \int_y^x (1-y)^{d-2} dy \, dx. \]
Evaluating in \( y \) and then performing light manipulation gives us
\[ 2d(d-1) \left[ \int_0^1 x(1-x)^{d-2} dx - \int_0^1 x(1-x)^{2d-3} dx \right] = 2d(d-1) \left[ \frac{1}{(d-1)d} - \frac{1}{(2d-2)(2d-1)} \right] = \frac{3d-2}{2d-1}, \]
as desired.

\[ \square \]

**Figure 5.** Ranges of sprawls: \([E(\text{Sphere}_d), E(\text{Cube}_d)]\) is shown for \(d = 2, 3, 4, 5, 100, \infty\).

### 5.4. The range of sprawls and the Mahler conjecture.

By rational approximation of convex bodies (as in the proof of Corollary 12), we find that a dense subset of the interval \([E(\text{Sphere}_d), E(\text{Cube}_d)]\) is contained in the set of values realized by groups, so
\[ [E(\text{Sphere}_d), E(\text{Cube}_d)] \subseteq \{ E(\mathbb{Z}^d, S) : \text{gensets } S \}. \]

We conclude by conjecturing that this is everything.

**Conjecture 19** (Sprawl Conjecture). *The sphere and the cube are the extremes for the sprawl. That is,\[
\{ E(L) : \text{perimeters } L \subset \mathbb{R}^d \} = [E(\text{Sphere}_d), E(\text{Cube}_d)].
\]*

This conjecture would complete the description for free abelian groups of the dependence of this curvature statistic on the generating set, showing the values to be “pinched” as in Figure 5.

A similar conjecture could be formulated for the balls instead of the spheres: consider the average distance statistic for convex, centrally symmetric \(\Omega \subset \mathbb{R}^d\) defined by
\[ \hat{E}(\Omega) := \frac{\int_{\Omega^2} ||x-y||_{\Omega} \, d\text{Vol}^2}{(\text{Vol } \Omega)^2}. \]

Here, it is known (by the Brascamp-Lieb-Luttinger inequality 12 Thm 1) that \(\hat{E}\) is minimized by (round) balls and ellipsoids, but the question of verifying that it is maximized by cubes is open.

Some evidence for the Sprawl Conjecture can be found in the high-dimensional asymptotics. Because \(E \leq 2\) always, it is immediate that
\[ \lim_{d \to \infty} \sup \{ E(L) \} = \lim_{d \to \infty} E(\text{Cube}_d) = 2. \]

Arias-de-Reyna, Ball, and Villa consider \(\hat{E}(\Omega)\) and prove that for almost all pairs of points in \(\Omega \times \Omega\), the distance is greater than \(\sqrt{2}(1 - \epsilon) \) 11 Thm 1. As they note, the points in the ball become concentrated in its boundary as \(d \to \infty\). This shows that the \(E(\text{Sphere}_d)\) is a lower bound for sprawl asymptotically, i.e.,
\[ \lim_{d \to \infty} \inf \{ E(L) \} = \lim_{d \to \infty} E(\text{Sphere}_d) = \sqrt{2}. \]
The Sprawl Conjecture resembles another well-studied problem in convex geometry. For a convex, centrally symmetric body $\Omega$, define its polar body by

$$\Omega^\circ := \{ x \in \mathbb{R}^d : x \cdot y \leq 1 \quad \forall y \in \Omega \}. $$

Thus for instance, the sphere is its own polar body in every dimension, $(\Omega^\circ)^\circ = \Omega$, and $(\text{Orth}_d)^\circ = \text{Cube}_d$. The Mahler volume of $\Omega$ is defined to be $M(\Omega) = \text{Vol}(\Omega) \cdot \text{Vol}(\Omega^\circ)$. Let us also say that for any set $A = -A$, we write $M(A)$ for the Mahler volume of the convex hull of $A$. Then, just as for the sprawl, this is a statistic that is continuous in $\Omega$ and invariant under linear transformations; it has been described as measuring the “roundness” of the convex body. Mahler conjectured in 1939 that the extremes in every dimension were realized by the sphere and the cube. Santaló proved in 1949 that the spheres did indeed realize the upper bound on Mahler volume, but the lower bound is still an open problem, despite some interesting recent progress by Kuperberg and others.

Above, we have staked out the point of view that, like the Mahler volume and other affine isoperimetric invariants, sprawl is measuring a quality of roundness versus pointiness of the shape $L$. Inspecting the estimates for sprawls of regular polygons derived after Proposition \ref{prop:regular-polygons} shows something surprising: there is no point after which sprawl decreases monotonically as the number of sides in the polygon increases. Thus, regular polygons with $4k - 2$ sides are a bit “rounder” than regular polygons with $4k$ sides (for all $k \geq 4$), even though they have fewer sides. On the other hand, as measured by Mahler volume the roundness of regular polygons increases monotonically in the number of sides.

Finally, we note that the average distance between two points on the round sphere is precisely equal to the constant $\frac{2}{\pi}$ that Kuperberg uses to state the inequality in his \cite[Corollary 1.6]{Kuperberg}, where it is described as “a monotonic factor that begins at $4/\pi$ and converges to $\sqrt{2}$.” Recognizing the geometric meaning of this constant allows his result to be rephrased as

$$M(\Omega) \geq \left( \frac{\pi}{2} \right)^d \cdot E(\text{Sphere}_d) \cdot M(\text{Cube}_d).$$

The fact that this general inequality for Mahler volume should be so simply stated involving the sprawl is, we hope, intriguing.

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