CONGRUENCES FOR A MOCK MODULAR FORM ON SL$_2$($\mathbb{Z}$) AND
THE SMALLEST PARTS FUNCTION

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Abstract. Using a family of mock modular forms constructed by Zagier, we study the coefficients of a mock modular form of weight $3/2$ on SL$_2$($\mathbb{Z}$) modulo primes $\ell \geq 5$. These coefficients are related to the smallest parts function of Andrews. As an application, we reprove a theorem of Garvan regarding the properties of this function modulo $\ell$. As another application, we show that congruences modulo $\ell$ for the smallest parts function are rare in a precise sense.

1. Introduction

Let $spt(n)$ denote the number of smallest parts in the partitions of $n$. This function has been the subject of much recent research. See, for example, [2, 4] and the references in these papers. The best known arithmetic properties of $spt$ are the congruences of Andrews [6]:

$$spt\left(\frac{5n+1}{24}\right) \equiv 0 \pmod{5},$$
$$spt\left(\frac{7n+1}{24}\right) \equiv 0 \pmod{7},$$
$$spt\left(\frac{13n+1}{24}\right) \equiv 0 \pmod{13}$$

(1.1)

(here $spt(n)$ is defined to be zero if $n$ is not a natural number). Much of the interest in $spt$ arises from the fact that its generating function is related to a distinguished mock modular form. In particular, let $p(n)$ denote the number of partitions of $n$ and define

$$f(z) := \sum a(n)q^{\frac{n}{24}} = -\sum_{n=1}^{\infty} \left(12 spt\left(\frac{n+1}{24}\right) + np\left(\frac{n+1}{24}\right)\right) q^{\frac{n}{24}}$$
$$= q^{\frac{1}{24}} \left(1 - 35q - 130q^2 - 273q^3 + \cdots \right) \quad q := e^{2\pi iz}.$$  

Then $f$ is a mock modular form of weight $3/2$ on SL$_2$($\mathbb{Z}$) (see the next section for details).

Suppose that $\ell \geq 5$ is prime. Improving a result of Bringmann, Garvan and Mahlburg [9], Garvan [10] identified each generating function

$$\sum spt\left(\frac{\ell n+1}{24}\right) q^{\frac{n}{24}} \pmod{\ell}$$

Date: March 16, 2018.

2010 Mathematics Subject Classification. 11F33, 11F37, 11P83.

Key words and phrases. mock modular forms, smallest parts function, modular forms modulo $\ell$.

The first author was supported by a grant from the Simons Foundation (#426145 to Scott Ahlgren). Byungchan Kim was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1A09917344).
as a modular form modulo \( \ell \) of low weight on \( \text{SL}_2(\mathbb{Z}) \). To state his result, we introduce some notation. Let \( \mathbb{Z}_{(\ell)} \) denote the set of \( \ell \)-integral rational numbers, and for each integer \( k \) let \( M_k \) denote the space of modular forms on \( \text{SL}_2(\mathbb{Z}) \) whose coefficients lie in \( \mathbb{Z}_{(\ell)} \). Define the reduction \( \overline{g} \) of \( g = \sum b(n)q^{\frac{n}{24}} \in \mathbb{Z}_{(\ell)}[q^{\frac{1}{24}}] \) coefficientwise, and define \( \overline{M}_k \) as the set of reductions of elements of \( M_k \). Define the Dedekind eta-function
\[
\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),
\]
and for each prime \( \ell \geq 5 \) define
\[
r_\ell \in \{1, \ldots, 23\} \text{ by } r_\ell \equiv -\ell \pmod{24}.
\]
Garvan [11, Corollary 4.2] proved the following

**Theorem 1** (Garvan). Suppose that \( \ell \geq 5 \) is prime. Then
\[
\sum \text{spt} \left( \frac{\ell n + 1}{24} \right) q^{\frac{n}{24}} \in \eta^{\ell} \overline{M}_{-r_\ell+1}.
\]

Note that, since
\[
M_{-r_\ell+1} = \{0\} \iff \ell = 5, 7, 13,
\]
the congruences (1.1) of Andrews follow from this result. Garvan obtains a similar result for the second rank moment \( N_2(n) \), which is defined by \( \text{spt}(n) = np(n) - \frac{1}{2} N_2(n) \). His method involves a careful study of modular forms of level \( \ell \).

In this paper we take a different approach. For each prime \( \ell \), define the function
\[
f_\ell := \eta \left( f - \left( -\frac{24}{\ell} \right) \Theta^{\ell-1} f \right),
\]
where \( \Theta \) is the operator defined by the derivative \( \Theta := \frac{1}{2\pi i} \frac{d}{dz} \). Using a family of mock modular forms constructed by Zagier [17], we will prove that each \( f_\ell \) is congruent to a modular form of low weight on \( \text{SL}_2(\mathbb{Z}) \).

**Theorem 2.** Suppose that \( \ell \geq 5 \) is prime. Then we have
\[
\overline{f}_\ell \in \overline{S}_{\ell+1},
\]
where \( S_k \) denotes the space of cusp forms of weight \( k \).

As an application of Theorem 2 we deduce Garvan’s Theorem 1 as a corollary. As another application, we show that congruences (1.1) of the type found by Andrews are exceedingly rare. For \( \ell \geq 5 \) we say that \( \text{spt} \) has a congruence at \( \ell \) if
\[
\text{spt} \left( \frac{\ell n + 1}{24} \right) \equiv 0 \pmod{\ell} \text{ for all } n,
\]
and we define
\[
w := \limsup_{X \to \infty} \frac{\# \{ \ell \leq X : \text{spt has a congruence at } \ell \} \cdot X/\log X}{X}.
\]
In [3] it was shown that the partition function has a congruence at \( \ell \) only if \( \ell = 5, 7 \) or 11. For the \( \text{spt} \) function we can prove

**Theorem 3.** We have \( w = 0 \). Moreover, for \( \ell < 10^{11} \), \( \text{spt} \) has a congruence only at \( \ell = 5, 7 \) and 13.
The bound $10^{11}$ is obtained from a few hours of computation using the first 50 coefficients of $f$ as described in Section 5 and could easily be improved.

Our method uses the properties of a family of mock modular forms on $SL_2(\mathbb{Z})$ introduced by Zagier [17] together with the theory of modular forms modulo $\ell$. We begin in the next section by describing these mock modular forms and developing the necessary background before turning to the proofs in the following sections.

2. Background

By work of Bringmann [8] and Zagier [17, §6] (see also [2, §3] for example) it is known that

$$f(z) = \sum a(n)q^n = -\sum_{n=-1}^{\infty} \left( 12 \text{ spt} \left( \frac{n+1}{24} \right) + np \left( \frac{n+1}{24} \right) \right) q^n$$

(2.1)

is a mock modular form of weight $3/2$ on $SL_2(\mathbb{Z})$ whose multiplier is conjugate to that of the eta-function. Recall [16] that the $n$-th Rankin-Cohen bracket is given by

$$[[g, h]]_n := \sum_{r=0}^{n} (-1)^r \binom{n+k_1-1}{n-r} \binom{n+k_2-1}{r} \Theta^r g \Theta^{n-r} h,$$

where $k_1$ and $k_2$ are the respective weights of the modular forms $g$ and $h$.

Let $E_2$ be the usual quasi-modular Eisenstein series of weight 2 and for even $k \geq 2$ define

$$F_k := \sum_{n \neq 0} (-1)^n \frac{-3}{n-1} n^{k-1} q^{(n+1)/6} = -\sum_{r \geq s > 0} \frac{12}{r^2 - s^2} s^{k-1} q^{rs/6}.$$

Zagier [17, §6] described a family of mock modular forms on $SL_2(\mathbb{Z})$ in every even weight.

Proposition 4. We have

1. $12 F_2 + f \eta = E_2$.

2. For all $n \geq 1$, the function

$$12 F_{2n+2} + 24^n \binom{2n}{n}^{-1} [f, \eta]_n$$

is a modular form of weight $2n + 2$ on $SL_2(\mathbb{Z})$.

We have corrected a typographical error in the first statement. The proof, which uses holomorphic projection, is not given in [17]. A sketch of a proof of the first assertion is described in [1]. If $F = f + f^-$ is the completion of the mock modular form $f$, then for $n \geq 1$ we have

$$\pi_{\text{hol}}([F, \eta]_n) = [f, \eta]_n + \pi_{\text{hol}}([f^-, \eta]_n) \in M_{2n+2}.$$

A description of the holomorphic projection of Rankin-Cohen brackets in weight $(3/2, 1/2)$ is given by Mertens [12, §5] (for these weights there is quite a bit of simplification). Zagier’s result follows from computing $\pi_{\text{hol}}([f^-, \eta]_n)$ explicitly in terms of $F_{2n+2}$.

Finally, we require some basic facts from the theory of modular forms modulo $\ell$. Each $g \in M_k$ has a filtration defined by

$$w(g) := \inf \{ k' : g \in M_{k'} \}. $$
Define the $U$-operator by its action on $q$-series:

$$\left(\sum b(n)q^{\frac{n^2}{24}}\right) | U_\ell := \sum b(\ell n)q^{\frac{n^2}{24}}.$$  

These facts about filtrations can be found in [13] and [14, §2.2].

**Lemma 5.** If $g \in M_k$ then the following are true.

1. $w(g) \equiv k \pmod{\ell - 1}$.
2. $\Theta g \in M_k + \ell + 1$.
3. $w(\Theta g) \leq w(g) + \ell + 1$, with equality if and only if $w(g) \not\equiv 0 \pmod{\ell}$.
4. $w(g|U_\ell) \leq \ell + \frac{w(g)-1}{\ell}$.

3. **Proof of Theorem 2**

From the definitions (2.1) and (1.3) we have

$$f_\ell \equiv \eta \left(\sum a(n) \left(1 - \frac{n}{\ell}\right) q^{\frac{n^2}{24}}\right) \pmod{\ell}. \quad (3.1)$$

We have $E_2 \equiv E_{\ell+1} \pmod{\ell}$ and $F_2 \equiv F_{\ell+1} \pmod{\ell}$. Set

$$c(n) := 24^n \binom{2n}{n}^{-1}.$$  

From Proposition 4 it follows that

$$f\eta - c\left(\frac{\ell - 1}{2}\right)[f, \eta]_{\frac{\ell - 1}{2}} \in M_{\ell+1}.$$  

(3.2)

From the definition we have

$$[f, \eta]_{\frac{\ell - 1}{2}} := \sum_{r=0}^{\frac{\ell - 1}{2}} (-1)^r \binom{\ell/2}{(\ell - 1)/2 - r} \binom{\ell/2 - 1}{r} \Theta^r f \Theta^{\frac{\ell - 1}{2} - r} \eta.$$  

We have

$$\binom{\ell/2}{(\ell - 1)/2 - r} \equiv 0 \pmod{\ell}, \quad 0 \leq r < \frac{\ell - 1}{2}$$

and

$$\binom{\ell - 1}{(\ell - 1)/2} \equiv \binom{\ell - 2}{1/2} \equiv \binom{\ell - 1}{\ell} \pmod{\ell}.$$  

Therefore

$$c\left(\frac{\ell - 1}{2}\right)[f, \eta]_{\frac{\ell - 1}{2}} \equiv \left(\frac{-24}{\ell}\right) \Theta^{\frac{\ell - 1}{2}} f \cdot \eta \pmod{\ell},$$  

and Theorem 2 follows from (3.2), after noting that

$$\left(\frac{-24}{\ell}\right) \Theta^{\frac{\ell - 1}{2}} f \equiv \eta^{-1/24} + O\left(q^{\frac{21}{24}}\right) \pmod{\ell}. \quad (3.3)$$
4. Deduction of Theorem 1

We define \( g_\ell \) by

\[
g_\ell := f_\ell \Delta^{\ell^2 - 1},
\]

(4.1)

so that \( \overline{g_\ell} \in \overline{S}_{\ell + 1 + \frac{\ell - 1}{2}} \). Using Theorem 2 and Lemma 5 we find that

\[
w(g_\ell | U_\ell) \leq \ell + 1 + \frac{\ell^2 - 1}{2\ell} \leq \frac{3}{2} \ell + 1.
\]

Since \( w(g_\ell) \equiv 2 \pmod{\ell - 1} \), we conclude that

\[
\overline{g_\ell} | U_\ell = \overline{f} | U_\ell \cdot \eta^\ell \in \overline{S}_{\ell + 1}.
\]

Finally, we find that the \( q \)-expansion has the form

\[
\overline{g_\ell} | U_\ell = c \cdot q^r + \cdots
\]

for some \( c \). Therefore

\[
\overline{g_\ell} | U_\ell \in \Delta^r \overline{M}_{\ell + 1 - r}.
\]

and Theorem 1 follows.

5. Proof of Theorem 3

Let \( \ell \geq 5 \) be prime and let \( f \) and \( f_\ell \) be defined as in (2.1) and (3.1). We begin with a proposition (this can also be deduced from [13, Thm. 1.1], [4, Thm. 1.2] or [5, Cor. 3.2]).

Proposition 6. If \( f | U_\ell \equiv 0 \pmod{\ell} \) then \( f_\ell \equiv 0 \pmod{\ell} \).

Proof. Suppose that \( f | U_\ell \equiv 0 \pmod{\ell} \). Then \( f \equiv \Theta^\ell f \pmod{\ell} \). Using this with (3.1) and (4.1) we obtain

\[
\Theta^\ell g_\ell \equiv -\left( \frac{-24}{\ell} \right) g_\ell \pmod{\ell}.
\]

In particular we have

\[
w(\Theta^\ell g_\ell) = w(g_\ell).
\]

By Theorem 2 we have \( \overline{g_\ell} \in \overline{S}_{\ell^2 + 1 + \ell} \). If it were the case that

\[
w(g_\ell) = \frac{\ell^2 + 1}{2} + \ell \equiv \frac{\ell + 1}{2} \pmod{\ell},
\]

then Lemma 5 would give the contradiction

\[
w(\Theta^\ell g_\ell) = \frac{\ell^2 + 1}{2} + \ell + \frac{\ell - 1}{2} (\ell + 1) \neq w(g_\ell).
\]

It follows that

\[
w(g_\ell) \leq 2 + \frac{\ell^2 - 1}{2}.
\]

By (3.3) we have

\[
\overline{f_\ell} = c \cdot q^{r + \frac{23}{24}} + \cdots
\]

for some \( c \). Since \( \dim S_k \leq \frac{\ell^2 - 1}{24} \) for \( k \leq 2 + \frac{\ell^2 - 1}{2} \), it follows that

\[
g_\ell \equiv 0 \pmod{\ell}.
\]

□
Finally, we prove Theorem 3.

Proof of Theorem 3. Given a prime \( \ell \), it follows from Proposition 6 that if there is an integer \( n \equiv 23 \pmod{24} \) such that

\[
\left( \frac{\ell}{n} \right) \neq 1 \quad \text{and} \quad a(n) \equiv 0 \pmod{\ell},
\]  

then \( \text{spt} \) does not have a congruence at \( \ell \). For each \( \ell < 10^{11} \) other than 5, 7, and 13, we find an integer \( n \) satisfying (5.1) among the first 50 candidates; this gives the second assertion of Theorem 3.

To prove the first assertion, fix a positive integer \( N \), and let \( p_1, \ldots, p_N \) be the first \( N \) primes which are \( \equiv 23 \pmod{24} \). Let \( E_N \) be the finite set of primes \( \ell \geq 5 \) which divide \( \prod_{1 \leq j \leq N} a(p_j) \). From (5.1), we see that if \( \text{spt} \) has a congruence at \( \ell \), then either \( \ell \in E_N \) or \( \ell \) is in the set \( Q_N \) defined by the quadratic conditions

\[
\left( \frac{\ell}{p_j} \right) = 1, \quad 1 \leq j \leq N.
\]  

(5.2)

It follows that

\[
\# \{ \ell \leq X : \text{spt has a congruence at } \ell \} \leq \#E_N + \# \{ \ell \leq X : \ell \in Q_N \} \sim \frac{1}{2^N} \frac{X}{\log X}.
\]

Therefore \( w \leq \frac{1}{2^N} \). The theorem follows. 

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