Systems of MDS codes from units and idempotents.

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Abstract

Algebraic systems are constructed from which series of maximum distance separable (mds) codes are derived. The methods use unit and idempotent schemes.

1 Introduction

Algebraic coding theory deals with the design of error-correcting and error-detecting codes for the reliable transmission of information across noisy channels. It has many applications, modern communications could not be undertaken without it and much research is still on-going. Coding theory in general makes use of many abstract notions such as fields, group theory, polynomial algebra and areas of discrete mathematics.

A basic reference for coding theory is Blahut [2]. Codes from zero divisors and unit-derived codes in group rings and matrix rings are obtained in [8] and in more detail in [9]. An \((n, r, d)\) (linear) code is a code of length \(n\), dimension \(r\) and distance \(d\). By the Singleton Bound, see for example Theorem 3.2.6 of [2], the maximum \(d\) can have is \((n - r + 1)\) and so an mds (maximum distance separable) code is defined as a code of the form \((n, r, n - r + 1)\) or equivalently a code of the form \((n, n - r, r + 1)\).

Here systems and series of such mds codes are derived. This paper originated from ideas of constructing codes from complete orthogonal sets of idempotents in general rings and in particular in group rings. Constructions of such idempotent systems in general are dealt with in [11] where these are used to construct paraunitary (single and multivariable) matrices which are used in the communications’ areas.

A query to the pub-group forum was answered by Marty Isaacs who brought the results of [7] and a result of Chebotarév to our attention. Using Chebotarév’s result directly and the unit-derived coding method of [8] enables the construction of series of mds codes over \(C\) initially using the Fourier matrices. Results in [7] are then exploited to construct finite fields over which the Chebotarév’s result is true and hence to derive series of mds codes over these finite fields. The paper [7] in addition contains a proof of Chebotarév’s original result and a number of other nice results besides.

In section 4 methods are derived for constructing general codes from complete orthogonal sets of idempotents. Specialising then enables systems of mds codes to be derived over various fields; Chebotarév’s result and the results of [7] are used to show algebraically that the maximum distances are actually attained.

Sets of vectors \(S = \{e_0, e_1, \ldots, e_{n-1}\}\) in \(K^n\) for various fields \(K\) and prime \(n\) are derived such that any \(r\) elements of \(S\) generates an \((n, r, n - r + 1)\) code. For given \(r\) there are \(\binom{n}{r}\) choices for defining such a code from \(S\) and each code is different.

Sets of idempotents matrices \(T = \{E_0, E_1, \ldots, E_{s-1}\}\) in \(K_{n \times n}\) are defined over fields \(K\) such that \(\{E_j \mid j \in J\}\) where \(J \subset I = \{0, 1, 2, \ldots, s - 1\}\) generates an \((n, r)\) code where \(r = \sum_{j \in I} \text{rank} E_j\). In certain cases when \(s = n\) and \(n\) is prime these are shown to be mds codes.

The mds codes derived using idempotents from the cyclic group ring may be considered as those where the Fourier transform has zeros at \(k\) specified locations which need not be consecutive.

One of the features of some of the series of mds codes derived is that these are codes over a finite field \(F_p\), for \(p\) a prime, and modular arithmetic may be used.

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Section 5 considers decoding methods for such codes. As the dimension and distance of a space generated by a subset of $S$ is easily determined, it is then possible to find $t$-error correcting pairs in many of these $(n,r,n-r+1)$ codes for maximum $t$ (that is for $t = \lfloor \frac{n-r}{2} \rfloor$). Now $t$-error correcting pairs were introduced by Duursma and Kötter, and by Pellikaan.

2 Codes from units

Unit-derived codes, as in [8, 9], are defined as follows. Suppose $UV = I$ in $F_{n \times n}$. Divide $U = \begin{pmatrix} A \\ B \end{pmatrix}$ into block matrices where $A$ is an $r \times n$ matrix and $B$ is $(n-r) \times n$. Similarly divide $V$ into blocks $V = \begin{pmatrix} C & D \end{pmatrix}$ where $C$ is an $n \times r$ matrix and $D$ is an $n \times (n-r)$ matrix.

Now $AD = 0$ as $UV = I$. It is easy to show that $A$ generates an $(n,r)$ code and that $D^T$ is a check matrix for this code.

The above method is generalised as follows, see [8, 9] for details. Let the rows of $U$ be denoted by $\{u_1, u_2, \ldots, u_n\}$ and the columns of $V$ denoted by $\{v_1, v_2, \ldots, v_n\}$. Choose $r$ rows $\{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\}$ of $U$ as a generating matrix $A$ which is then of size $r \times n$ and has rank $r$. Let $K = \{1, 2, \ldots, n\}$ and $L = \{i_1, i_2, \ldots, i_r\}$ and $J = (K - M)$. Choose $D$ to be the matrix formed (in any order) from the (column) vectors $S = \{v_j \mid j \in J\}$. Then $D$ has rank $(n-r)$ and is of size $n \times (n-r)$ and $D^T$ is a check matrix for the $(n,r)$ code generated by $A$.

(The $r$ rows of $U$ used to form $A$ are usually taken in their naturally occurring order but this is not necessary. The matrix $D$ can be formed from the column vectors $S$ in any order but the natural order of the elements of $S$ would normally be used.)

These codes are linear but in general are not ideals.

Thus any rows of $U$ may be used as a generator matrix for a code and then corresponding columns of $V$ as indicated give a check matrix. From a single unit of size $n \times n$ there are $\binom{n}{r}$ choices for an $(n,r)$ code and each code is different. The fact that the codes are different follows from the following Lemma 2.1 Define, in a vector space, $\langle X \rangle$ to be the subspace generated by $X$.

Lemma 2.1 Let $T$ be a set of linearly independent vectors and $S \subseteq T,W \subseteq T$. Then $\langle S \rangle \cap \langle W \rangle = \langle S \cap W \rangle$.

Proof: The proof follows directly from the linearity independence of the sets $S$ and $W$.          

Suppose then $UV = I$ in $F_{n \times n}$. Then taking any $r$ rows of $U$ as a generator matrix $U_r$ and then certain defined $(n-r)$ columns of $V$ to give the check matrix $V_{n-r}$ defines an $(n,r)$ code. Let such a code be denoted by $C_r$. If matrix $V$ has the property that the determinant of any square submatrix of $V$ is non-zero then any such code is an mds $(n,r,n-r+1)$ code.

Theorem 2.1 Suppose the determinant of any square submatrix of $V$ is non-zero. Then any such code $C_r$ has distance $(n-r+1)$ and is thus an $(n,r,n-r+1)$ mds code.

Proof: The proof follows from Theorem 3.2.2/Corollary 3.2.3 of [2] as any $(n-r) \times (n-r)$ submatrix of $V$ has non-zero determinant.          

Suppose for example $UV = I$, $U$ has size $101 \times 101$ and we are interested in $(101, 50)$ codes. Choosing any 50 of the rows of $U$ gives such a code and each one is different thus giving $\binom{101}{50}$ such codes. Now $\binom{101}{50}$ is of order $10^{29}$ or $2^{97}$. There exist $\binom{101}{80}$, which is of order of $2^{27}$, high rate code $(101, 80)$ in such a system. If the determinant of any square submatrix of $V$ is non-zero we get of the order of $2^{97}$ mds codes $(101, 50, 52)$ and of the order of $2^{27}$ mds codes $(101, 80, 22)$.

3 Chebotarëv’s Theorem

Let $\alpha$ be a primitive $n^{th}$ root of unity in a field $K$ in which the inverse of $n$ exists. The Fourier $n \times n$ matrix $F_n$ over $K$ is
3.1 Fourier

We are grateful to Marty Isaacs for bringing the following result of Chebotarëv and the paper [7] to our attention. A proof of this Chebotarëv theorem may be found in [4] and proofs also appear in the expository paper of P. Stevenhagen and H. W. Lenstra [15]; paper [5] contains a relatively short proof. There are several other proofs in the literature some of which are referred to in [15]. A proof of the Theorem is also contained in [7] and this paper contains many nice related results and results related to fields in general (and not just \( \mathbb{C} \)).

3.1.1 Example

Let \( \omega \) be a primitive 7\(^{th} \) root of 1 in \( \mathbb{C} \). Consider \( F_7 = \)

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 \\
1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^2 & \omega^5 \\
1 & \omega^4 & \omega^5 & \omega^2 & \omega^6 & \omega^3 & \omega \\
1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 \\
1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \\
\end{pmatrix}
\]
Let \( C_4 \) be the code generated by the following matrix: 
\[
A = \begin{pmatrix}
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^5 & \omega^3 \\
1 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 & \omega^5 \\
1 & \omega^6 & \omega^5 & \omega^3 & \omega^2 & \omega & \omega^4
\end{pmatrix}.
\]

A has rank 4. A check matrix for \( C_4 \) is 
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \\
1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4
\end{pmatrix}.
\]
This has rank 3.

The code \( C_4 \) is a \((7, 4, 4)\) code. Indeed, \( \binom{7}{4} = 35 \) different such codes may be derived from \( F_7 \).

### 3.2 Finite fields

In the finite field case it is not true in all cases when \( n \) is prime that the Fourier matrix \( F_n \), when it exists, has non-zero determinant of each square submatrix. The purpose now is find finite fields \( K \) and primes \( p \) such the Fourier \( F_p \) matrix over \( K \) has non-zero determinant of each square submatrix. In order that the Fourier \( p \times p \) matrix over \( K \) should exist, it is necessary that \( \text{char} K \neq p \), and \( p/(q-1) \) where \( q \) is the order of the field \( K \).

Say a square matrix \( M \) over the field \( K \) has the Chebotarëv property if the determinant of any square submatrix is non-zero. By \([7]\) if the characteristic of \( K \) is 0, the Fourier matrix \( F_n \) over \( K \) has the Chebotarëv property for a prime \( n \).

See the paper \([7]\) for details on the following. \( F[G] \) denotes the group ring of the group \( G \) over the field \( F \). Let \( z \) be a generator for the cyclic group \( G \) of order a prime \( p \). Each vector \( v \in F[G] \) is uniquely in the form \( f(z) \), where \( f \in F[X] \) and \( \deg f < p \). The quantity \( t = t(v) \) which is \( \text{supp}(v) \) is exactly the number of non-zero coefficients in the polynomial \( f \) and this number is written as \( t(f) \).

Now \( d(v) \) denotes the dimension of the space generated by \( v \).

As shown in \([7]\) if \( K \) is a field containing a primitive \( p^{th} \) root of unity, then the conclusion of Chebotarëv’s theorem over \( K \) is equivalent to the assertion that \( t(v) + d(v) > p \) for all choices of nonzero vectors \( v \in K[X] \). In \([7]\) cases of finite fields and primes \( p \) with \( t(v) + d(v) \leq p \) were found and this enabled the authors of \([7]\) to find examples where Chebotarëv’s theorem fails in prime characteristic.

The following theorem of \([7]\) gives necessary and sufficient conditions for this failure to occur, where the conditions are expressed in terms of the polynomial ring \( K[X] \).

**Theorem 3.3** (Goldstein, Guralnick, Isaacs \([7]\), (6.3) Theorem). Let \( G = \langle z \rangle \) be a group of prime order \( p \) and suppose that \( v \in K[G] \) is nonzero, where \( K \) is an arbitrary field. Write \( v = f(z) \), where \( f \in K[X] \) and \( \deg f < p \). Then \( t(v) + d(v) \leq p \) if and only if \( t(f) \leq \deg h \), where \( h(X) = \gcd(X^p - 1, f(X)) \).

It is worth noting that the examples given in the paper \([7]\) (pages 4035–6) for which Chebotarëv Theorem fails use (distinct) primes \( p, q \) in which the order of \( q \) mod \( p \) is less than \( \phi(p) = p - 1 \); this should be compared with Theorem \(3.3\) below. We are interested in finite fields \( K \) and primes \( p \) for which the Fourier matrix over \( K \) exists and satisfies the Chebotarëv condition.

The paper \([7]\) argues as follows to show that for each prime \( p \), there are only finitely many characteristics where Chebotarëv can fail: “Consider the determinants of all square submatrices of the complex matrix \( [c_{ij}] \), as in Theorem \(3.3\). These are algebraic integers, and they are nonzero by Chebotarëv’s theorem, and so their norms are nonzero rational integers. It should be reasonably clear that the characteristics where the conclusion of Chebotarëv’s can fail are exactly the primes that divide at least one of these integers, and clearly, there are just finitely many such primes.”

### 3.3 Fourier matrix over finite fields

In order to construct the Fourier matrix \( F_p \) over \( GF(q) \) it is necessary that \( p/(q-1) \). For given unequal primes \( p, t \) by Fermat’s little Theorem \( p/(t^{\phi(p)} - 1) \). As \( p \) is prime, \( \phi(p) = p - 1 \). For given unequal primes \( p, t \) there is a field \( GF(t^d) \) such that \( p/(t^d - 1) \) and the Fourier matrix \( F_p \) exists over this field.

Let \( p, q \) be unequal primes and \( K = GF(q^{\phi(p)}) \). Then \( p/(q^{\phi(p)} - 1) \) and the Fourier matrix \( F_p \) exists over \( K \).
Lemma 3.1 Let $p, q$ be unequal primes. Suppose the order of $q$ mod $p$ is $\phi(p)$. Then $(x^{p-1} + x^{p-2} + \ldots + x + 1)$ is irreducible over $GF(q)$.

Proof: It is known that the cyclotomic polynomial $\Phi_n(x)$ factors over a finite field $GF(q)$ into irreducible polynomials of degree $r$ where $r$ is the order of $q$ mod $n$. Here $\Phi_p(x) = x^{p-1} + x^{p-2} + \ldots + x + 1$ and $r = \phi(p) = p - 1 = \deg(\Phi_p(x))$ and so $\Phi_p(x)$ is irreducible.

Theorem 3.4 Let $p, q$ be unequal primes and $K = GF(q^{\phi(p)})$. Suppose the order of $q$ mod $p$ is $\phi(p)$ and (hence) that $(x^{p-1} + x^{p-2} + \ldots + x + 1)$ is irreducible over $GF(q) = Z_q$. Then the Fourier matrix $F_p$ exists over $K$ and satisfies the Chebotaev condition.

Proof: It has already been noted that $F_p$ exists.

Now $GF(q^{\phi(p)}) \cong GF(q)[\alpha] \cong Z_q[\alpha] \cong \frac{Z_p[x]}{(f(x))}$ where $\alpha$ is the cofactor $x + (f(x))$.

For $\omega$ a primitive $p^t$th root of 1 in $\mathbb{C}$, $Z[\omega] \cong \frac{Z[x]}{(f(y))}$. This gives the natural map $Z[\omega] \cong \frac{Z[x]}{(f(y))} \rightarrow \frac{Z_p[x]}{(f(x))} = GF(q)[\alpha]$. The kernel of this map is polynomials of degree less than $p$ in $y$ in which each coefficient is divisible by $p$.

This mapping may be extended $\frac{Z[\omega]}{(f(y))} \rightarrow \frac{Z_p[x]}{(f(x))}[z]$.

Suppose now $g(z) \in \frac{Z_p[x]}{(f(x))}[z]$ satisfies $\deg g < p$ and let $h(z) = \gcd(g(z), z^p - 1)$. Consider then $\hat{g}(z) \in \frac{Z_p[y]}{(f(y))}[z] \subset \frac{Z[\omega]}{(f(y))}[z]$ with the pre-image of the coefficients of $g(z)$ as the coefficients of $\hat{g}(z)$. Let then $(\hat{g}(z), z^p - 1) = \gcd(\hat{g}(z), z^p - 1)$. Now by Theorem 3.3 $t(\hat{g}) > \deg \hat{h}$.

Let $\frac{Z[y]}{(f(y))} = Z[\omega]$ where $\omega$ is a primitive $p^t$th root of 1 in $\mathbb{C}$ and $\frac{Z_p[y]}{(f(y))} = Z_q[\alpha]$ where $\alpha$ is a primitive $p^t$th root of 1 in $Z_q$.

Now $z^p - 1 = \prod_{i=0}^{p-1}(z - \omega^i)$ in $Z[\omega]$ and $z^p - 1 = \prod_{i=0}^{p-1}(z - \omega^i)$ in $GF(q^{p-1}) = Z_q[\alpha]$.

Thus in $Z_p[\alpha]$, $\gcd(g(z), z^p - 1) = \prod_{j \in J}(z - \alpha^j) = h(z)$ where $J$ is a proper subset of $I = \{0, 1, \ldots, p - 1\}$. In $Z[\omega]$, $\gcd(\hat{g}(z), z^p - 1) = \prod_{j \in J}(z - \omega^j) = \hat{h}(z)$.

Hence $\deg \hat{h}(z) = \deg h(z)$. Thus it is seen that since $t(\hat{g}(z)) = t(g(z))$ and $\deg \hat{h}(z) = \deg h(z)$ and $t(\hat{g}) > \deg h$ that $t(g(z)) > \deg h(z)$. Hence by Theorem 3.3 the Fourier matrix $F_p$ over $GF(q^{\phi(p)})$ satisfies Chebotaev’s condition.

Thus fields $GF(q^{\phi(p)})$ with $p, q$ unequal primes where the order of $q$ mod $p$ is $\phi(p) = p - 1$, and (hence) where $x^{p-1} + x^{p-2} + \ldots + 1$ is irreducible over $GF(q)$ is such that the Fourier matrix $F_p$ over $GF(q^{\phi(p)})$ satisfies the Chebotaev property. There are clearly many such examples and particular ones are given in section 3.5.

3.4 Germain type

A prime $p$ is a Germain prime if $(2p + 1)$ is also a prime. A safe prime is one of the form $(2p + 1)$ where $p$ is prime.

Proposition 3.1 Suppose $p$ and $q = (2p + 1)$ are primes. Then the Fourier matrix $F_p$ exists over $GF(q)$ and satisfies the Chebotaev condition.

Proof: Now $p/(q - 1)$ and the order of $q$ mod $p$ is 1. Let $\alpha$ be an element of order 2 $p = q - 1$ in $GF(q)$. Now $\alpha^2$ has order $p$ and the Fourier matrix $F_p$ over $GF(q)$ then exists and can be constructed from powers of $\alpha^2$. Let $f(x)$ be a polynomial of degree less than $p$ and consider $\gcd((x^p - 1), f(x)) = h(x)$ in $GF(q)$. Now in $GF(q)$, $x^p - 1 = \prod_{i=0}^{p-1}(x - \alpha^i)$ as each $\alpha^i, 0 \leq i \leq (p - 1)$ is a root of $x^p - 1$. Hence $h(x) = \gcd((x^p - 1), f(x)) = \prod_{j \in J}(x - \alpha^j)$ where $J \subseteq \{0, 1, \ldots, p - 1\}$. Let $\omega$ be a primitive $p^th$ root of 1. Consider $(x^p - 1), f(x)$ as polynomials in $Z[x]$. Now $t(f)$ in $GF(q)$ is the same as $t(f)$ in $Z[x]$. Then $\gcd((x^p - 1), f(x)) = h(x)$ satisfies $t(f) > \deg(h(x))$ as elements in $Z[x]$. Now $h(x) = \prod_{j \in J}(x - \omega^j)$ for $J \subseteq \{0, 1, \ldots, p - 1\}$. Now $J = J$ and so $\deg h(x)$ in $\mathbb{C}[x]$ must be the same as $\deg(h(x))$ in $GF(q)[x]$. Hence the Fourier matrix $F_p$ over $GF(q)$ satisfies the Chebotaev condition.

The Fourier matrices in these cases are particularly nice as they consist of integers modulo a prime $q$.
3.5 Examples

A Computer Algebra system such as GAP [6], MAPLE or MATLAB is useful for calculations.

A circulant matrix is a matrix of the form
\[
\begin{pmatrix}
  a_0 & a_1 & \cdots & a_{n-1} \\
  a_{n-1} & a_0 & \cdots & a_{n-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_1 & a_2 & \cdots & a_0
\end{pmatrix}
\]

Thus \( \text{circ}(a_0, a_1, \ldots, a_{n-1}) \) will denote the circulant matrix with first row \((a_0, a_1, \ldots, a_{n-1})\).

3.5.1 \( GF(2^r) \)

1. \( GF(2^2) \): The order of 2 mod 3 is 2 and \((x^2 + x + 1)\) is irreducible over \( GF(2) \). Thus \( F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix} \) has the Chebotarëv property where \( \omega \) is a primitive 3rd root of unity in \( GF(4) \). This gives \( \binom{3}{2} = 3 \) codes of type (3, 2, 2).

2. \( GF(2^4) \). The order of 2 mod 5 is 4 and \((x^4 + x^3 + x^2 + x + 1)\) is irreducible over \( GF(2) \). Hence by Theorem 3.4 the Fourier matrix \( F_5 \) exists over \( GF(2^4) \) and satisfies Chebotarëv’s condition that every square submatrix has determinant non-zero. Consider \( F_5 \).

Let \( \alpha \) be a primitive element and define \( \omega = \alpha^3 \). Then \( F_5 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ \omega^2 & \omega^4 & \omega & \omega^4 \\ \omega^3 & \omega^4 & \omega & \omega^4 \end{pmatrix} \) has the determinant of every submatrix non-zero. We can use \( F_5 \) to define maximal distance separable (mds) codes over \( GF(16) \). So for example choosing 3 of the rows to get a generator matrix and then use the other two corresponding of \( F^* \) as check matrix gives \( (5, 3, 3) \) codes. In total this gives \( \binom{5}{3} = 10 \) different \( (5, 3, 3) \) codes.

3. \( GF(2^6) \): Now 7/\((2^6 - 1)\) and so the Fourier \( F_7 \) exists over \( GF(2^6) \). However \((x^3 + x + 1)\), which is ‘missing a term’, is a factor of \((x^7 - 1)\) and so \( F_7 \) does not satisfy Chebotarëv’s condition. Here the order of 2 mod 7 is 3 and \((x^3 + x + 1)\) is irreducible over \( GF(2) \).

4. \( GF(2^{10}) \). The order of 2 mod 11 is \( \phi(11) = 10 \) and \((x^{10} + x^9 + \ldots + x + 1)\) is irreducible over \( GF(2) \). Thus by Theorem 3.4 the Fourier \( F_{11} \) over \( GF(2^{10}) \) has the Chebotarëv property and mds codes may be constructed from it. For example \( \binom{11}{2} = 330 \) mds codes (of rate \( \frac{10}{11} \)) may be constructed over \( GF(2^{10}) \) and each of these is 2-error correcting.

5. \( GF(2^{12}) \): The order of 2 mod 13 is \( \phi(13) = 12 \) so by lemma 3.4 \((x^{12} + x^{11} + \ldots + 1)\) is irreducible over \( GF(2) \). Thus by Theorem 3.4 \( F_{13} \) over \( GF(2^{12}) \) exists and satisfies Chebotarëv’s condition. So for example this enables the construction of \( \binom{13}{7} = 1716 \) (different) codes of type \((13, 7, 7)\) in \( GF(2^{12}) \) which are then 3-error correcting.

\( \vdots \)

3.5.2 \( GF(3^r) \):

1. \( GF(3^4) \): The order of 3 mod 5 is \( \phi(5) = 4 \) and so the polynomial \((x^4 + x^3 + x^2 + x + 1)\) is irreducible over \( GF(3) \). The Fourier matrix \( F_5 \) over \( GF(3^4) \) exists and has the Chebotarëv property by Theorem 3.4 from which mds codes can be constructed.

2. \( GF(3^6) \): The order of 3 mod 7 is 6 and \((x^6 + x^5 + \ldots + x + 1)\) is irreducible over \( GF(3) \). Hence by Theorem 3.4 \( F_7 \) exists and satisfies Chebotarëv’s condition. This enables the construction of mds codes from \( F_7 \). For example \( \binom{7}{3} = 35 \) mds codes (7, 3, 5) codes may be formed in \( GF(3^6) \).

3. \( GF(3^{10}) \): The order of 3 mod 17 is 16 and \((x^{16} + x^{15} + \ldots + x + 1)\) is irreducible over \( GF(3) \). Hence by Theorem 3.4 \( F_{17} \) satisfies Chebotarëv’s condition. This enables the formation of mds codes from
3.5.4 \( GF(5^2) \):

1. \( GF(5^2) \): The order of 5 mod 3 is 2 and \((x^2 + x + 1)\) is irreducible in \(GF(5)\). Thus the Fourier \( F_3 \) exists in \( GF(5^2) \) and has Chebotarëv property.

2. \( GF(5^5) \): The order of 5 mod 7 is 6 and \((x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)\) is irreducible in \(GF(5)\). Thus the Fourier matrix \( F_7 \) over \( GF(5^6) \) exists and satisfies Chebotarëv’s property. Hence for example it may be used to construct \( \binom{7}{3} = 35 \) different mds codes over \( GF(5^6) \) and indeed \( \binom{7}{3} = 2^1 \) different \( (7,5,3) \) codes over \( GF(5^6) \).

3.5.4 \( GF(7^4) \):

1. \( GF(7^4) \): The order of 7 mod 5 is 4 and \((x^4 + x^3 + x^2 + x + 1)\) is irreducible over \(GF(7)\). Hence by Theorem 3.4 \( F_5 \) exists over \( GF(7^4) \) and satisfies Chebotarëv’s condition. Hence mds codes may be constructed from \( F_5 \).

2. \( GF(7^{10}) \): The order of 7 mod 11 is 10 and \((x^{10} + x^9 + \ldots + x + 1)\) is irreducible over \(GF(7)\). Thus by Theorem 3.4 \( F_{11} \) exists over \( GF(7^{10}) \) and satisfies Chebotarëv’s condition.

3.5.5 \( GF(11^7) \):

1. \( GF(11) \): Here 5/(11 − 1) and so the Fourier matrix \( F_5 \) exists over \( GF(11) \). Theorem 3.4 cannot be applied as the irreducible factors of \((x^5 − 1)\) in \(GF(11)\) are \(\{x − 1, x − α^2, x − α^4, x − α^6, x − α^8\}\), where \(α\) is a primitive element in \(GF(11)\). (This \(α\) can be chosen to be 2 as the order of 2 mod 11 is 10.) However 5 is a Germain prime (with safe prime 11 = 5 × 2 + 1 and so the Proposition 3.1 may be applied.

Thus the Fourier \( F_5 \) over \( GF(11) \) has the Chebotarëv property. From this mds codes many be constructed. Here 2 is a primitive root and so \(2^4 = 4\) has order 5. Thus then

\[
F_5 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 4 & 4^2 & 4^3 & 4^4 \\
1 & 4^2 & 4^3 & 4^4 & 4^3 \\
1 & 4^3 & 4^4 & 4^3 & 4^2 \\
1 & 4^4 & 4^3 & 4^2 & 4
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 4 & 5 & 9 & 3 \\
1 & 5 & 3 & 4 & 9 \\
1 & 9 & 4 & 3 & 5 \\
1 & 3 & 9 & 5 & 4
\end{pmatrix}
\]

is a Fourier matrix over \( GF(11) \) which has the Chebotarëv property. This gives for example \( \binom{5}{10} = 10 \) mds codes (5,3,3) over \( Z_{11} \) which are 1-error correcting.

2. \( GF(23) \). Here \( p = 11 \) is a Germain prime with safe prime \( q = 2p + 1 = 23 \). The Fourier matrix \( F_{11} \) exists over \( GF(23) \) exists and by Proposition 3.1 it satisfies the Chebotarëv condition. In \( GF(23) \) a primitive element is 5 and so \(5^2 = 2\) is an element of order 11 from which the Fourier matrix \( F_{11} \) over \( GF(23) \) can be constructed. This gives

\[
F_{11} = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 2 & 2^2 & \ldots & 2^{10} \\
1 & 2^2 & 2^4 & \ldots & 2^{20} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2^{10} & 2^{20} & \ldots & 2^{100}
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 2 & 4 & \ldots & 12 \\
1 & 4 & 14 & \ldots & 6 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 12 & 6 & \ldots & 2
\end{pmatrix}
\]

3. \( GF(11^3) \): Now \(7/(11^3 − 1)\) and \((x^7 − 1)\) has irreducible factors \((x − 1), (x^3 + α^4 x^2 + α^2 x − 1), (x^3 + α^7 x^2 + α^9 x − 1)\) over \(GF(11)\) where \(α\) is primitive. As pointed out in \[7\], \( F_7 \) over \(GF(11^3)\) does not have the the Chebotarëv property.

4. A large example: Consider \( GF(227) \). The Fourier matrix \( F_{113} \) exists over \( GF(227) \) and by Proposition 3.1 satisfies the Chebotarëv property since 113 is a Germain prime with matching safe prime 227. This for example enables the construction of \( \binom{113}{57} \) (different) mds codes (113, 57, 57) codes over
$\mathbb{Z}_{227}$. The number $\binom{113}{57}$ is of order $10^{32}$ or $2^{109}$. Also for example $\binom{113}{99}$ high rate mds (113,99,15) codes may be constructed over $\mathbb{Z}_{227}$ which are $7$-error correcting. This number $\binom{113}{99}$ is of order $10^{17}$ or $2^{57}$. 

4 Codes from complete orthogonal sets of idempotents

4.1 Notation

Let $R$ be a ring with identity $1_R = 1$. In general 1 will denote the identity of the system under consideration. A complete family of orthogonal idempotents is a set $\{e_1, e_2, \ldots, e_k\}$ in $R$ such that

(i) $e_i \neq 0$ and $e_i^2 = e_i$, $1 \leq i \leq k$;

(ii) If $i \neq j$ then $e_i e_j = 0$;

(iii) $1 = e_1 + e_2 + \ldots + e_k$.

The idempotent $e_i$ is said to be primitive if it cannot be written as $e_i = e_i' + e_i''$ where $e_i', e_i''$ are idempotents such that $e_i' e_i'' \neq 0$ and $e_i' e_i'' = 0$. A set of idempotents is said to be primitive if each idempotent in the set is primitive.

Methods for constructing complete orthogonal sets of idempotents are derived in [10]. Such sets always exist in $FG$, the group ring over a field $F$, when $\text{char} F \not| |G|$. See [12] for properties of group rings and related definitions. These idempotent sets are related to the representation theory of $FG$. Other methods for constructing complete orthogonal sets of matrices such as from orthonormal bases are considered in [11].

4.2 Rank

Lemma 4.1 Suppose $\{E_1, E_2, \ldots, E_s\}$ is a set of orthogonal idempotent matrices. Then $\text{rank}(E_1 + E_2 + \ldots + E_s) = \text{tr}(E_1 + E_2 + \ldots + E_s) = \text{tr}E_1 + \text{tr}E_2 + \ldots + \text{tr}E_s = \text{rank}E_1 + \text{rank}E_2 + \ldots + \text{rank}E_s$.

Proof: It is known that $\text{rank} A = \text{tr} A$ for an idempotent matrix, see for example [1], and so $\text{rank} E_i = \text{tr} E_i$ for each $i$. If $\{E, F, G\}$ is a set of orthogonal idempotent matrices so is $\{E + F, G\}$. From this it follows (by induction) that $\text{rank}(E_1 +E_2 + \ldots + E_s) = \text{tr}(E_1 + E_2 + \ldots + E_s) = \text{tr}E_1 + \text{tr}E_2 + \ldots + \text{tr}E_s = \text{rank}E_1 + \text{rank}E_2 + \ldots + \text{rank}E_s$.

Corollary 4.1 $\text{rank}(E_{i_1} + E_{i_2} + \ldots + E_{i_k}) = \text{rank}E_{i_1} + \text{rank}E_{i_2} + \ldots + \text{rank}E_{i_k}$ for $i_j \in \{1, 2, \ldots, s\}$, $i_j \neq i_l$.

4.3 The codes

Let $\{E_1, E_2, \ldots, E_k\}$ be a complete orthogonal set of idempotents in $F_{n \times n}$ and suppose $\text{rank} E_i = r_i$ with then $\sum_{i=1}^k r_i = n$. Let $I = \{1, 2, \ldots, k\}$ and suppose $J \subseteq I$. Then by Lemma 4.1 $\text{rank}(\sum_{j \in J} E_j) = \sum_{j \in J} \text{rank}(E_j)$.

Let $G = (E_1 + E_2 + \ldots + E_k)$ with $s < k$ and $H = (E_{s+1} + \ldots + E_k)$. Let $r = \text{rank} G = (r_1 + r_2 + \ldots + r_s)$, and then $(n - r) = \text{rank} H = (r_{s+1} + r_{s+2} + \ldots + r_k) = (n - r)$. Note that $GH = 0$.

Let $C_s$ denote the code with generator matrix $G$ and check matrix $H^T$. Then $C_s$ is an $(n, r)$ code.

Lemma 4.2 Let $A \in F_{n \times n}$. Then $AH = 0$ if and only if $AE_i = 0$ for $i = s + 1, s + 2, \ldots, k$.

Proof: Suppose $AH = 0$. Multiply through on the right by $E_i$ for $s + 1 \leq i \leq k$. Then $AE_i = 0$ as $E_i E_k = E_i$ and $E_i E_j = 0$ for $i \neq j$. On the other hand if $AE_i$ for $i = s + 1, s + 2, \ldots, k$ then clearly $AH = 0$.

Any $s$ elements of $\{E_1, E_2, \ldots, E_k\}$ can be used as a generator matrix and then the other $(k - s)$ elements give the check matrix. The ranks are determined by the ranks of the elements chosen. Any complete orthogonal set of idempotents may be used and the reader is referred to [11] for general constructions of these. Here we stick to cases related to the idempotents in the cyclic group ring.
Now suppose \( S = \{E_1, E_2, \ldots, E_n\} \) is a complete orthogonal set of idempotents in \( K_{n \times n} \) where each \( E_i \) has rank 1. In this case it can be seen that choosing \( r \) elements gives a \((n, r)\) code with the generator matrix given by the sum of these \( r \) elements and the check matrix given by \((\text{the transpose of the sum of the other} \ (n-r) \text{ elements})\). Each choice of the \( r \) elements gives a different \((n, r)\) code so the set-up gives \( \binom{n}{r} \) different \((n, r)\) codes.

### 4.4 Distances attained

Suppose now that \( S = \{E_1, E_2, \ldots, E_n\} \) is a complete orthogonal set of idempotents in \( F_{n \times n} \) for a field \( F \). Let \( F_n \) be the \( n \times n \) matrix consisting of the first columns of each of \( \{E_1, E_2, \ldots, E_n\} \).

Let \( G \) be the matrix consisting of the sum of \( r \) elements of \( S \) and let \( H \) be the sum of the other \((n-r)\) elements of \( S \). Then as explained in section 4.3 this defines an \((n, r)\) code say \( C_r \) with generator matrix \( G \) and check matrix \( H^T \).

**Theorem 4.1** Suppose the determinant of any square submatrix of \( F_n \) is non-zero. Then any such code \( C_r \) has distance \((n-r+1)\) and is thus an \((n, r, n-r+1)\) MDS code.

**Proof:** Suppose \( u = (u_1, u_2, \ldots, u_n) \in C_r \) has support at most \((n-r)\). Thus \( u \) has entry 0 in \( r \) places. Suppose \( u \) has entry 0 except (possibly) at places \( \{u_{k_1}, u_{k_2}, \ldots, u_{k_{n-r}}\} \). Define \( \hat{u} = (u_{k_1}, u_{k_2}, \ldots, u_{k_{n-r}}) \).

Let \( H = E_{j_1} + E_{j_2} + \ldots + E_{j_{n-r}} \). Then \( uH = 0 \) and so by Lemma 4.2, \( uE_{j_i} = 0 \) for \( i = 1, 2, \ldots, (n-r) \).

Let the \( k^{th} \) entry of the column of \( E_{j_i} \) be denoted by \( E_{j_{i_k}} \). Then \( \sum_{i=1}^{n-r} u_{k_i} E_{j_{i_k}} = 0 \) for \( i = 1, 2, \ldots, (n-r) \).

Let \( T_i \) denote the column \((E_{j_{i_1}}, E_{j_{i_2}}, \ldots, E_{j_{i_{n-r}}})^T \). Then this says that \( \hat{u}T_i = 0 \) for \( i = 1, 2, \ldots, (n-r) \). Hence \( \hat{u}(T_1, T_2, \ldots, T_{n-r}) = 0 \).

Let \( A \) be the \((n-r) \times (n-r)\) matrix \((T_1, T_2, \ldots, T_{n-r}) \). This is a square submatrix of \( F_n \) and so its determinant is non-zero. Hence \( \hat{u} = 0 \) and so \( u = 0 \).

\( \square \)

### 4.5 Cyclic case

Let \( N = \{E_0, E_1, \ldots, E_{n-1}\} \) be the primitive orthogonal complete set of idempotents obtained from the cyclic group \( C_n \) of order \( n \) in \( \mathbb{C} \). Take \( E_i = \text{circ}(\omega^i, \omega^{2i}, \ldots, \omega^{(n-1)i}) \) where \( \omega \) is a primitive \( n^{th} \) root of 1.

Let \( C_r \) be the code with generator matrix \( G = (E_0 + E_1 + \ldots + E_{r-1}) \) and check matrix \( (\text{the transpose of}) \ H = E_r + E_{r+1} + \ldots + E_{n-1} \). Then \( G \) has rank \( r \) and \( H \) has rank \((n-r)\) by Lemma 4.1 and so \( C_r \) is a \((n, r)\) code. The first \( r \) rows of \( G \) are independent and the first \((n-r)\) rows of \( H \) are independent by results in \([3, 9]\). Hence the first \( r \) rows of \( G \) can be taken as the generator matrix. Similarly the first \((n-r)\) rows of \( H^T \) can be taken as the check matrix of \( C_r \).

More generally choose the sum of any \( r \) of \( S = \{E_0, E_2, \ldots, E_{n-1}\} \) to form a generator matrix \( G_r \) of a code \( C_r \) of size \((n, r)\) and the sum of the remaining \((n-r)\) elements give the matrix \( H_{n-r} \), where \( H_{n-r}^T \) is a check matrix. As explained the first \( r \) rows of \( G_r \) are linearly independent and these may be taken as the generator matrix of this cyclic code and the first \((n-r)\) rows of \( H_{n-r}^T \) may be taken as a check matrix.

**Theorem 4.2** Suppose \( n \) is prime. Then the distance of \( C_r \) is \((n-r+1)\).

**Proof:** The proof follows from Theorems 4.1 and 3.1.

\( \square \)

The codes constructed in this section are cyclic codes and are also ideals in the group ring of the cyclic group.

Note that using these orthogonal sets of idempotents it is then easy to construct MDS codes over \( \mathbb{R} \) by combining complex conjugate idempotents when constructing the generator matrix. This is illustrated in the following examples. By using complete orthogonal sets of idempotents in \( \mathbb{Q}_{n \times n} \), codes over \( \mathbb{Q} \) may be obtained.
4.5.1 Codes from idempotents, examples

Consider from $\mathbb{C}C_5$ the following complete orthogonal set of idempotents giving \( \{E_0, E_1, E_2, E_3, E_4\} \) with
\[
E_0 = \frac{1}{\sqrt{5}} \text{circ}(1, 1, 1, 1), \quad E_1 = \frac{1}{\sqrt{5}} \text{circ}(1, \omega, \omega^2, \omega^3, \omega^4), \quad E_2 = \frac{1}{\sqrt{5}} \text{circ}(1, \omega^2, \omega^4, \omega, \omega^3), \\
E_3 = \frac{1}{\sqrt{5}} \text{circ}(1, \omega^3, \omega, \omega^4, \omega^2), \quad E_4 = \frac{1}{\sqrt{5}} \text{circ}(1, \omega^4, \omega^3, \omega^2, \omega).
\]

If we choose \( U = (E_0 + E_1 + E_2) \) as a generator matrix of a code \( C \) then \( V = (E_3 + E_4) \) gives the check matrix \( V^T \) of \( C \). By Theorem 4.2 this code is a \((5, 3, 3)\) code. The first three rows of \( U \) are linearly independent and constitute the generator matrix. The first two columns of \( V \) are linearly independent and any \( 2 \times 2 \) submatrix has det \( \neq 0 \) which gives the distance 3. The generator matrix is
\[
U = (E_0 + E_1 + E_2) = \frac{1}{\sqrt{5}} \text{circ}(3, 1 + \omega + \omega^2, 1 + \omega^2 + \omega^4, 1 + \omega^3 + \omega, 1 + \omega^2 + \omega^3).
\]

Suppose we wish to generate a real \((5, 3, 3)\) code from \( \{E_0, E_1, E_2, E_3, E_4\} \). It is noted that \( \{E_1, E_4\} \) and \( \{E_2, E_3\} \) consist of pairs whose sums are real and that \( E_0 \) is real. Consider \( G = (E_0 + E_1 + E_4) \) as the generator matrix and \( H = (E_2 + E_3) \) as the transpose of the check matrix. Both \( G \) and \( H \) are real and thus get a real \((5, 3, 3)\) code.

4.6 Over finite fields

We may now use Theorem 4.1 and analogies of Theorem 3.4 and Proposition 3.1 to construct series of cyclic mds codes over finite fields. Note that if \( \{F_1, F_2, \ldots, F_k\} \) are cyclic (orthogonal) idempotent matrices and rank \( F_1 + \text{rank} F_2 + \ldots + \text{rank} F_k = r \) then also \( G = (F_1 + F_2 + \ldots + F_k) \) is circulant and the first \( r \) rows of \( G \) are linearly independent; this follows for example from [5]. Thus generator and check matrices of the \((n, r)\) codes produced are obtained from the \((n \times n)\) matrices by using the first \( r \) rows for the check matrix of the (natural) generator matrix and the first \((n - r)\) rows of the check matrix.

Consider then as in Section 3.3 two unequal prime \( p, q \) and \( GF(q^{\phi(p)}) \) where the order of \( q \) mod \( p \) is \( \phi(p) \) and \( x^{p-1} + x^{p-2} + \ldots + x + 1 \) is irreducible over \( GF(q) \). In these cases by Theorem 4.3 the Fourier matrix \( F_p \) over \( GF(q^{\phi(p)}) \) exists and satisfies Chebotarev’s condition.

**Theorem 4.3** Let \( p, q \) be unequal primes and \( K = GF(q^{\phi(p)}) \). Suppose the order of \( q \) mod \( p \) is \( \phi(P) \) and (hence) that \( (x^{p-1} + x^{p-2} + \ldots + x + 1) \) is irreducible over \( GF(q) \). Let \( \omega \) be a primitive \( p^{th} \) root of 1 in \( K \). Define, \( (in K_{max}) \), \( E_i = \frac{1}{p} \text{circ}(1, \omega^i, \omega^{2i}, \ldots, \omega^{(p-1)i}) \) for \( i = 0, 1, \ldots, (p - 1) \). Then \( S = \{E_0, E_1, \ldots, E_{p-1}\} \) is a complete orthogonal set of idempotents (each \( E_i \) has rank 1) and the codes produced using any subset of \( S \) are cyclic mds codes.

**Proof:** It is easy to check that \( S \) is a complete orthogonal set of idempotents in \( GF(q^{\phi(p)}) \). Then the first rows of the elements of \( S \) constitute (a multiple of) the rows of the Fourier matrix \( F_p \). The result then follows from Theorem 4.1. \( \square \)

**Proposition 4.1** Suppose \( p \) and \( q = 2p + 1 \) are primes and that \( \omega \) is a primitive \( p^{th} \) root of 1 in \( K = GF(q) \). Define \( (in K) \), \( E_i = \frac{1}{p} \text{circ}(1, \omega^i, \omega^{2i}, \ldots, \omega^{(p-1)i}) \) for \( i = 0, 1, \ldots, (p - 1) \). Then \( S = \{E_0, E_1, \ldots, E_{p-1}\} \) is a complete orthogonal set of idempotents. Further the codes produced using any subset of \( S \) are cyclic mds codes.

The constructions are fairly general and examples are easy to construct. Similar examples to those of section 3.3 from the point of view of orthogonal sets of idempotents may be derived. A small selection of corresponding examples to those of 3.5 are given below with details omitted.

4.6.1 Examples in finite fields

1. \( GF(2^2) \): Let \( \omega \) be a primitive 3rd root of 1 in \( GF(2^2) \). A complete orthogonal set of idempotents is \( S = \{E_0 = \text{circ}(1, 1, 1), E_1 = \text{circ}(1, \omega, \omega^2), E_2 = \text{circ}(1, \omega^2, \omega)\} \). The first rows of this set gives a non-zero multiple of the Fourier matrix \( F_3 \) which has the Chebotarev property. Thus choosing any subset of \( S \) as a generator matrix determines an mds code and each such code is cyclic. This gives for example \((\frac{3}{2}) = 3\) cyclic codes of type \((3, 2, 2)\).
2. \(GF(2^4)\). Let \(\omega\) be a primitive \(5^{th}\) root of unity in \(GF(2^4)\). Consider the complete orthogonal set of idempotents \(S = \{E_0 = \text{circ}(1,1,1,1), E_1 = \text{circ}(1,\omega,\omega^2,\omega^3), E_2 = \text{circ}(1,\omega^2,\omega^3,\omega), E_3 = \text{circ}(1,\omega^3,\omega,\omega^2), E_4 = \text{circ}(1,\omega^4,\omega^3,\omega^2,\omega)\} \).

The first rows of \(\{E_0, E_1, E_2, E_3, E_4\}\) determine a non-zero multiple of the Fourier matrix \(F_5\) over \(GF(2^4)\).

So for example choosing the sum of 3 of the elements of \(S\) gives a \((5,3,3)\) code and this gives \((\frac{5}{3}) = 10\) different cyclic \((5,3,3)\) codes.

3. \(GF(2^{10})\). Let \(E_i = \text{circ}(1,\omega^i,\omega^{2i}, \ldots, \omega^{10i})\) where \(\omega\) is a primitive \(11^{th}\) root of unity in \(GF(2^{10})\) and \(S = \{E_0, E_1, \ldots, E_{10}\}\). Using the first rows of \(E_0, E_1, \ldots, E_{10}\) constitutes the the Fourier \(F_{11}\) over \(GF(2^{10})\). Now by section 3.3 this \(F_{11}\) has the Chebotarëv property and hence codes formed using sums of elements from \(S\) are mds codes which are also cyclic.

4. \(GF(2^{12})\). Let \(E_i = \text{circ}(1,\omega^i,\omega^{2i}, \ldots, \omega^{12i})\) where \(\omega\) is a primitive \(13^{th}\) root of unity in \(GF(2^{12})\) and \(S = \{E_0, E_1, \ldots, E_{12}\}\). Using the first rows of \(E_0, E_1, \ldots, E_{12}\) constitutes a multiple of the Fourier \(F_{13}\) over \(GF(2^{12})\). Now by Section 3.3 this \(F_{13}\) has the Chebotarëv property and hence codes formed using sums of elements from \(S\) are mds codes and these are also cyclic.

5. \(GF(3^4)\): Construct \(E_i = \text{circ}(1,\omega^i,\omega^{2i}, \ldots, \omega^{4i})\) where \(\omega\) is a primitive \(5^{th}\) root of unity in \(GF(3^4)\) and let \(S = \{E_0, E_1, E_2, E_3, E_4\}\). Then the first rows of these constitute a multiple of the Fourier matrix \(F_5\) over \(GF(3^4)\) which has noted in section 3.3 has the Chebotarëv property. Thus codes formed using subsets of \(S\) are mds cyclic codes.

6. \(GF(3^7)\): Construct \(E_i = \text{circ}(1,\omega^i,\omega^{2i}, \ldots, \omega^{7i})\) where \(\omega\) is a primitive \(7^{th}\) root of unity in \(GF(3^7)\) and let \(S = \{E_0, E_1, E_2, E_3, E_4, E_5, E_6\}\). The first rows of \(\{E_0, E_1, \ldots, E_6\}\) constitute a multiple of the Fourier matrix \(F_7\) over \(GF(3^7)\) which has noted in section 3.3 has the Chebotarëv property. Thus codes formed from subsets of \(S\) are mds codes.

7. \(GF(3^{10})\): Mds cyclic codes may be obtained from \(\{E_0, E_1, \ldots, E_{16}\}\); details are omitted. For example we may obtain \((\frac{17}{9}) = 24310\) mds cyclic \((17,9,9)\) codes.

Further (cyclic) examples may be obtained similar to those in section 3.3 using \(GF(5^r), GF(7^r), GF(11^r)\), and so on.

### 4.7 Equality

The question arises in this case as to whether or not the codes produced from idempotents in the group ring of the cyclic group are the same as the (corresponding) unit-derived ones in section 2 using rows of the Fourier matrix. It may be shown that they have the same check matrix (the details are omitted) and so they are equal but this is not obvious from the way they are constructed and going from one generator matrix to another is not easy. Each presentation has its own advantages.

### 5 Decoding

A minor variation of the Peterson-Gorenstein-Zierler algorithm, see [2] Chapter 6 for details, may be used for codes where the chosen rows of the Fourier matrix as in section 2 are consecutive. The details are not included as better and more efficient decoding algorithms exist as shown below. Cyclic codes may be decoded by any general technique for decoding cyclic codes.

Error Locating and Error Correcting pairs were introduced in [3] and [14] and this is the approach taken here.

For vectors \(u = (u_0, u_1, \ldots, u_{n-1})\) and \(v = (v_0, v_1, \ldots, v_{n-1})\) define \(u \ast v = (u_0v_0, u_1v_1, \ldots, u_{n-1}v_{n-1})\). For subspaces \(U, V\) define \(U \ast V = \{u \ast v | u \in U, v \in V\}\).

Let \(C^\perp\) denote the orthogonal complement of \(C\), \(k(C)\) the dimension of \(C\) and \(d(C)\) denote the (minimum) distance of \(C\). Let \(U, V, C\) be linear codes over a field \(K\). Say \((U, V)\) is a \(t\)-error locating pair.
for C if (i) $U \ast V \subseteq C^\perp$, (ii) $k(U) > t$, (iii) $d(V^\perp) > t$. If further (iv) $d(C) + d(U) > n$, where $n$ denotes the code length of $C$, then say $(U, V)$ is a $t$-error correcting pair for $C$.

We now show how $t$-error correcting pairs may be constructed for many of the $(n, r, n - r + 1)$ codes as described by the unit-derived method of section 2 with $2t = n - r$.

Suppose $K$ is a field which contains a primitive $n^{th}$ root of unity $\omega$ such that the inverse of $n$ exists in $K$. Define $e_0 = (1, 1, \ldots, 1, e_1 = (1, \omega, \omega^2, \ldots, \omega^{n-1}), \ldots, e_{n-1} = (1, \omega^{n-1}, \omega^{2(n-1)}, \ldots, \omega^{(n-1)(n-1)})$.

The set $S = \{e_0, e_1, \ldots, e_{n-1}\}$ is a basis for $K^n$ as it consists of the rows of the Fourier matrix and so $S$ is a set of $n$ linearly independent vectors in $K^n$.

The dot/scalar product of $u, v$ for vectors $u, v$ in $K^n$ is denoted by $u \cdot v$.

**Lemma 5.1** $e_i \ast e_j = e_{i+j}$ where $i + j$ is interpreted mod $n$.

**Proof:** $e_i \ast e_j = (1, \omega^i, \omega^{2i}, \ldots, \omega^{(n-1)i}) \ast (1, \omega^j, \omega^{2j}, \ldots, \omega^{(n-1)j}) = (1, \omega^{i+j}, \omega^{2(i+j)}, \ldots, \omega^{(n-1)(i+j)}) = e_{i+j}$. □

**Lemma 5.2** Suppose $U = \langle u_1, u_2, \ldots, u_k \rangle, V = \langle v_1, v_2, \ldots, v_s \rangle$ for vectors $u_i, v_j$. Then $U \ast V \subseteq \langle u_i \ast v_j \mid 1 \leq i \leq k, 1 \leq j \leq s \rangle$.

**Lemma 5.3** Let $I = \{0, 1, 2, \ldots, n - 1\}$ and $J \subseteq I$. Consider $C = \langle e_j \mid j \in J \rangle$. Define $\bar{J} = \{n - j \mod n \mid j \in J\}$ and $K = (I - \bar{J})$. Then $C^\perp = \langle e_k \mid k \in K \rangle$.

**Proof:** This follows since $e_i \cdot e_j = 0$ if and only if $j = n - i \mod n$. □

Suppose now the Fourier matrix with rows $e_i, 0 \leq i \leq n - 1$ has the Chebotarev property that the determinant of any submatrix is non-zero. Then as pointed out the code generated by any $r$ of the vectors $S$ is an $(n, r, n - r + 1)$ code.

We now construct $t$-error correcting pairs for many of these codes with maximum $t$.

Suppose the $r$ vectors of $S$ are chosen consecutively as $\{e_i, e_{i+1}, \ldots, e_{i+r-1}\}$ to form a code where suffices are interpreted mod $n$. We shall show that in this case how to construct a (nice) $t$-error correcting pair, $2t = n - r$. We do this in the case of the code $C$ generated by $\{e_0, e_1, \ldots, e_{r-1}\}$; the other cases are similar. From Lemma 5.3 it is seen that $\langle e_1, e_2, \ldots, e_{n-r-1} \rangle \subseteq C^\perp$. Set $U = \langle e_0, \ldots, e_t \rangle$. The dimension of $U$ is $k(U) = t + 1 > t$ (as $\langle e_0, e_1, \ldots, e_t \rangle$ is linearly independent). Set $V = \langle e_1, \ldots, e_t \rangle$. Then $V^\perp = \langle e_0, e_1, \ldots, e_{n-t-1} \rangle$. Now $V^\perp$ is a $(n, n - t - t + 1)$ code and so $d(V^\perp) > t$. Now by Lemma 5.1 and Lemma 5.2 $U \ast V \subseteq \langle e_1, e_2, \ldots, e_{2r-1} \rangle = \langle e_1, e_{r+1}, \ldots, e_{n-r-1} \rangle \subseteq C^\perp$. Thus conditions (i),(ii),(iii) are satisfied for the pair $(U, V)$. Now $U$ is a $(n, t + 1, n - t)$ code and so $d(U) = n - t$. Hence $d(C) + d(U) = (n - r + 1) + (n - t) = 2n - r - t + 1 = n + (n - r) - t + 1 = n + 2t - t + 1 = n + t + 1 > n$. Thus condition (iv) is satisfied for the pair $(U, V)$ and so $(U, V)$ is a $t$-error correcting pair.

Similarly it is also possible to construct (nice) $t$-error correcting pairs when the $\{e_1, e_2, \ldots, e_r\}$ has other structures such as when the consecutive differences $i_{j+1} - i_j$ are constant. The problem of getting $t$-error correcting pairs for a more general $\{e_1, e_2, \ldots, e_r\}$ is left open.

As an example consider the Fourier matrix $F_{11}$ over $K = GF(23)$ constructed in section 3.5. Use $\omega = 5^2 = 2$ which is a primitive $11^{th}$ root of unity in $K$. Let the rows of $F_{11}$ be denoted by $\{e_0, e_1, \ldots, e_{10}\}$ and let $C_7$ be the (11, 7, 5) code generated by the first 7 rows of $F_{11}$. We now define a 2-error correcting pair $(U, V)$ as follows.

Define $U = \langle e_0, e_1, e_2 \rangle$ and $V = \langle e_1, e_2 \rangle$. Then (i) $U \ast V \subseteq \langle e_1, e_2, e_3, e_4 \rangle \subseteq C_7^\perp$, (ii) $U$ has dimension 3, (iii) $V^\perp$ has distance 3, (iv) $d(C_7) + d(U) = 5 + 9 > 11$. Thus $(U, V)$ is a 2-error correcting pair. The matrices $M(U), M(V)$ of $U, V$ respectively are as follows:

$$M(U) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{10} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{20} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 & 9 & 18 & 13 & 3 & 6 & 12 \\ 1 & 4 & 16 & 18 & 3 & 12 & 2 & 8 & 9 & 13 & 6 \end{pmatrix}.$$  

$$M(V) = \begin{pmatrix} 1 & \omega & \omega^2 & \ldots & \omega^{10} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{20} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 8 & 16 & 9 & 18 & 13 & 3 & 6 & 12 \\ 1 & 4 & 16 & 18 & 3 & 12 & 2 & 8 & 9 & 13 & 6 \end{pmatrix}.$$ 

Similarly 2-error correcting pairs may be obtained for any code generated by $\{e_1, e_{i+1}, e_{i+2}, e_{i+3}, e_{i+4}, e_{i+5}, e_{i+6}\}$. (The suffices should be taken mod 11.) For the code generated
by \{e_0, e_2, e_4, e_6, e_{10}, e_1\} (which have difference of 2 in the consecutive suffices) the pair \((U, V)\) with 
\[U = \langle e_0, e_2, e_4 \rangle, V = \langle e_2, e_4 \rangle\] is a 2-error correcting pair.

In a similar manner 3-error correcting pairs may be obtained for the \((11, 5, 7)\) code generated by any 
\{e_i, e_{i+1}, e_{i+2}, e_{i+3}, e_{i+4}\} or more generally for any \{e_i, e_{i+j}, e_{i+2j}, e_{i+3j}, e_{i+4j}\} with \(1 \leq j \leq 10\). For example if \(C = \langle e_0, e_2, e_4, e_6 \rangle\) then \(U = \langle e_0, e_2, e_4, e_6 \rangle, V = \langle e_2, e_4, e_6 \rangle\) constitute a 3-error correcting pair \((U, V)\).

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