THE GROUND STATE OF THE \( D = 11 \) SUPERMEMBRANE AND MATRIX MODELS ON COMPACT REGIONS

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Abstract. We establish a general framework for the analysis of boundary value problems at zero energy of matrix models on compact regions. This allows us to prove existence and uniqueness of ground state wavefunctions for the mass operator of the \( D = 11 \) regularized supermembrane theory (and therefore the \( N = 16 \) supersymmetric matrix model) on a ball of finite radius. Our results rely on the structure of the associated Dirichlet form and a factorization in terms of the supersymmetric charges. They also rely on the polynomial structure of the potential and various other supersymmetric properties of the system.

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1. Introduction

Physical theories subject to boundary conditions play a crucial role in the study of physical properties at high energies. They are also highly relevant in the study of some condensed matter effects. Recently, supersymmetric boundary conditions have received renewed attention, due to their relation with quantum phase transition at the boundary of topological superconductors. Cf. [1] and [2] for a recent systematic analysis of a class of $N = 2$ supersymmetric theories subject to different boundary conditions.

Matrix models related to the regularization of field theories with boundary conditions have been studied in order to test the AdS/CFT conjecture at finite temperature. In part this is due to its relation in the bulk picture with the black hole microstates in this regime [3]. An analysis of the unbounded matrix models wavefunction related to the $(0+1)$ supersymmetric Yang-Mills ground state has been considered in various other cases, [4]-[7]. In the context of M-theory and for the
regularized supermembrane, this has been studied in [8] and subsequent works. For a matrix model wavefunction perspective see [9]-[14], for different aspects of Lorentz invariance see [15]-[16], for inner solutions see [17] and for asymptotic solutions see [18]-[22].

M-theory as a unification theory should provide a quantum description of $D = 11$ supergravity. In this setting, String Theory is regarded as a perturbative “limit” of M-theory which should include all non-perturbative effects. The $D = 11$ supermembrane describes relevant degrees of freedom of M-theory, because it couples consistently to a $D = 11$ supergravity background without destroying the local fermionic symmetry [23]. This strongly suggests that the ground state of the $D = 11$ supermembrane should correspond to a wavefunction constructed in terms of the $D = 11$ supergravity multiplet.

In spite of several insightful attempts [8]-[22], a construction of the ground state wavefunction of the $D = 11$ supermembrane has remained an elusive open problem since the original analysis performed in [8]. It is well-known that the hamiltonian of the theory formulated on a Minkowski spacetime in the Light Cone Gauge [8] is the sum of two components. One component is associated to the kinematics of the center of mass of the supermembrane described in terms of the zero modes. The other component is the mass operator of the supermembrane which only depends on the non-zero modes.

The ground state wavefunction, $\Psi$, then factors into two parts: $\Psi = \Psi^0 \Psi^\text{non-zero}$. The zero mode wavefunction, $\Psi^0$, is responsible for the planar wave associated to the supergravity supermultiplet. The non-zero mode wavefunction, $\Psi^\text{non-zero}$, must be annihilated by the mass operator of the supermembrane and must be a singlet under $SO(9)$. The latter ensures that the full $\Psi$ is the solution corresponding to the unique $D = 11$ supergravity multiplet.

Rigorous treatments of the spectrum (in particular the ground state of the supermembrane) have been achieved by means of an $SU(N)$ regularization of the theory [8], [24]-[26]. These always involve the quantum mechanics of an $SU(N)$ matrix model which was first introduced in [4], [27] and [28], in a different context. The corresponding hamiltonian is the starting point of the matrix model theory developed in [29].

The regularized $D = 11$ supermembrane was rigorously shown to have a continuous spectrum from zero to infinity in [30]. The compactification on a sector of the target space by itself does not change this property [31]. However, the spectrum becomes purely discrete with finite multiplicities, when the maps describing the regularized hamiltonian satisfy a topological condition [32] corresponding to a non-trivial central charge in the supersymmetric algebra. See [33]-[38] for a
rigourous treatment. The setting developed in [38] also shows that the
BMN matrix model [39] has spectrum purely discrete when considered
beyond its semi-classical limit.

**Aims and scope of the present work.** In this paper we present a
new methodology for examining the ground state wavefunctions of a
class of supersymmetric models on a compact region, subject to Dirich-
let conditions on the boundary. Preliminary results in this direction
were already announced in [40] and [41].

We consider the $SU(N)$ regularized supermembrane theory. The
mass operator is the hamiltonian of the $N = 16$ supersymmetric $SU(N)$
matrix model [29]. The center of mass is allowed to move freely in a
$D = 11$ Minkowski spacetime but the membrane excitations are
restricted to a bounded domain $\Omega$ of dimension $9(N^2 - 1)$ with a smooth
boundary $\partial\Omega$. In order to simplify the exposition, in some of the proofs
presented below we will assume that $\partial\Omega$ is a sphere of finite radius.

In Theorem 6 we establish the existence and uniqueness of ground
state wavefunctions of the mass operator, assuming that their values are
known on $\partial\Omega$. The mass operator is subject to the physical constraints
of the theory which, in the regularized model, generate local $SU(N)$
invariance. In the large $N$ limit, these constraints are associated to
the residual area preserving diffeomorphisms symmetry of the $D = 11$
supermembrane in the Light Cone Gauge. Our proofs depend crucially
on the supersymmetric structure of the mass operator.

Mathematically we consider the homogeneous Dirichlet problem on
$\Omega$ for the hamiltonian, subject to inhomogeneous boundary conditions.
The latter are represented by means of a given datum, $g$, on $\partial\Omega$. We
will show that a unique wavefunction $\Psi$ exists, such that

\[
\begin{cases}
H\Psi = 0 & \text{in } \Omega \\
\Psi = g & \text{on } \partial\Omega \\
\varphi\Psi = 0.
\end{cases}
\]

The linear map $\varphi$ is the operator associated to the $SU(N)$ constraint.

Here and elsewhere we assume that $g \neq 0$. If $g = 0$, the unique sol-
ution turns out to be $\Psi = 0$. The hamiltonian of the supermembrane,
$H$, being a selfadjoint elliptic operator, has a basis of eigenfunctions
vanishing at the boundary and only positive eigenvalues which accu-
mulate at infinity. The ground eigenvalue of this hamiltonian is therefore
positive. Note that the latter is not directly related to the ground state
problem on unbounded domains. We call $\Psi$ the ground state wavefunc-
tion of the mass operator in $\Omega$, since it corresponds to the restriction
to $\Omega$ of the ground state wavefunction of the mass operator in $\mathbb{R}^{9(N^2 - 1)}$.
for \( g \neq 0 \). We will show in Section 5.4 that \( g \) can be chosen so that \( \Psi \) is invariant under \( SO(9) \).

The wavefunction \( \Psi \) is the state which has a minimal seminorm among all other states satisfying the constraint and the boundary condition. The seminorm here is the one associated to the inner product defined in terms of the supersymmetric charges, \( Q \) and \( Q^\dagger \), and is given by

\[
(Q_\eta, Q_\lambda)_{L^2(\Omega)} + (Q^\dagger_\eta, Q^\dagger_\lambda)_{L^2(\Omega)} \quad \text{for} \, \eta, \lambda \in H^1(\Omega).
\]

In the subspace \( H^1_0(\Omega) \), this expression defines a norm and is directly related to the Hamiltonian \( H \) through integration by parts. However it is only a seminorm in the full Sobolev space \( H^1(\Omega) \). This is analogous to the Dirichlet principle in Electrostatics. See Remark 2.

Our contribution is a crucial step towards solving the ground state problem on an unbounded domain. The next step is to consider the external problem on the complement of \( \Omega \) and examine the regularity of suitable glueings of the two solutions. Another strategy, which we have already explored from a numerical perspective [38], is to obtain conditions under which one gets a convergent sequence of solutions when the diameter of \( \Omega \) increases to infinity.

We present the specific description of the Hamiltonian associated to the problem (1) in Section 2. Then we turn to a framework which is general in character. In Section 3 we consider existence and uniqueness of the solution for models with global symmetry only. In Section 4 we formulate the criterion for existence and uniqueness of solutions for the more general case of models with gauge symmetry. The latter is the case \( \varphi \neq 0 \) and it requires a few more technical considerations. The framework established in sections 3 and 4 is applicable to a large class of supersymmetric model. The Hamiltonian should be a Schrödinger operator with a polynomial potential, satisfying conditions of strong ellipticity. In Section 5 we go back to (1). We implement the general argumentation in order to show the main result of the paper, Theorem 6. There we consider the case of the complete regularized supermembrane theory including the regularized area preserving constraint. We include a discussion on Lorentz invariance in Section 5.4.

2. Formulation of the problem

The \( D = 11 \) supermembrane is described in terms of the membrane coordinates \( X^m \) and Grassmann coordinates \( \theta_\alpha \). The former is regarded as a vector and the latter as a Majorana spinor. They transform as
scalars under diffeomorphisms on the base manifold. The supermembrane theory in the Light Cone Gauge is invariant under rigid supersymmetry, under rigid $SO(9)$ symmetry and under area preserving diffeomorphisms on the base manifold (the residual gauge symmetry obtained from the original invariance of the action under supermembrane worldvolume diffeomorphisms) once the Light Cone Gauge condition has been imposed.

In [8] the hamiltonian and the wavefunction were expressed according to the symmetry group $SO(9)$, so the above representation of the fields is given explicitly. For convenience the Majorana spinor is represented by linear combinations of elements of the subgroup $SO(7) \times U(1)$. In this way an explicit expansion $\lambda_\alpha$ of the operator associated to the fermionic coordinates in terms of a unique complex spinor of eight components was obtained in [8]. For this purpose, one defines two eigenspinors of $\gamma_9$, called $\theta^\pm$, such that

$$
\gamma_9 \theta^\pm = \pm \theta^\pm.
$$

Then a complex $SO(7)$ spinor satisfies

$$
\lambda^\dagger = 2^{1/4}(\theta^+ - i\theta^-) \quad \text{and} \quad \lambda = 2^{1/4}(\theta^+ + i\theta^-),
$$

where $\lambda^\dagger$ is the fermionic canonical conjugate momentum to $\lambda$.

In a similar manner the bosonic coordinates $X^M$ can be expressed in terms of the representations of $SO(7) \times U(1)$ by means of $(X^m, Z, \overline{Z})$. Here $X^m$ for $m = 1, \ldots, 7$ are the components of a vector in $SO(7)$, and the complex scalars

$$
Z = \frac{1}{\sqrt{2}}(X^8 + iX^9) \quad \text{and} \quad \overline{Z} = \frac{1}{\sqrt{2}}(X^8 - iX^9)
$$

transform under $U(1)$. The corresponding bosonic canonical momenta decouple as a vector in $SO(7)$ of components $P_m$, a complex momentum $\mathcal{P}$ in $U(1)$ and its conjugate $\overline{\mathcal{P}}$. That is $P_M = (P_m, \mathcal{P}, \overline{\mathcal{P}})$ where

$$
\mathcal{P} = \frac{1}{\sqrt{2}}(P^8 - iP^9) \quad \text{and} \quad \overline{\mathcal{P}} = \frac{1}{\sqrt{2}}(P^8 + iP^9).
$$

Once the theory is regularized by means of the group $SU(N)$ [8], [24]-[26], the field operators are labeled by an index $A$ in $SU(N)$. The fields transform in the adjoint representation of the group.

We consider two realizations of the wavefunctions. One of these will be used in the arguments concerning the existence and uniqueness of the ground state wavefunction under an assumption on the kernel of the susy charges. The other one will be used in the proof of this assumption for the $D = 11$ supermembrane.
For the first representation, we consider the fermion Fock space. That is a linear space of dimension $2^{9(N^2-1)}$ which carries an irreducible representation of the Clifford algebra generated by $(\lambda^\dagger + \lambda)$ and $i(\lambda^\dagger - \lambda)$. The Hilbert space of physical states consists of the wavefunctions which take values in the fermion Fock space.

In the second representation the wavefunction comprises elements of a Grassmann algebra generated by $\lambda^A_{\alpha}$ and is expressed as

$$
\Psi(X^A_i, Z^A_i, \lambda^A_{\alpha}) = \sum_{u=0}^{8(N^2-1)} \Phi_{A_1^{\alpha_1} \cdots A_u^{\alpha_u}}(X, Z, \lambda^A_i \lambda^A_{\alpha_1} \cdots \lambda^A_{\alpha_u}).
$$

In the Schrödinger picture, $\lambda^\dagger_{\alpha A} = \frac{\partial}{\partial \lambda^A_{\alpha}}$ is the conjugate momentum to $\lambda^A_{\alpha}$. The coefficient functions $\Phi(X, Z, \lambda)$ lie in the usual $L_2$ space and the norm of the state is given by

$$
\|\Psi\|^2 = \sum_{u=0}^{8(N^2-1)} \frac{1}{u!} \left\| \Phi_{A_1^{\alpha_1} \cdots A_u} \right\|^2.
$$

In [8] it was shown that the zero mode states transform under $SO(9)$ as a $[(44 + 84)_{\text{bos}} \oplus 128_{\text{fer}}]$ representation which corresponds to the massless $D = 11$ supergravity supermultiplet. Then the construction of the ground state wavefunction reduces to finding a non-trivial solution to

$$
H \Psi = 0
$$

where $H = \frac{1}{2}M$ and $\Psi \equiv \Psi^{\text{non-zero}}$ is required to be a singlet under $SO(9)$. Here $M$ is the mass operator of the supermembrane.

From the supersymmetric algebra, it follows that the hamiltonian can be express in terms of the supercharges as

$$
H = \frac{1}{16} \{Q_\alpha, Q^\dagger_\alpha\}.
$$

The physical subspace of solutions is given by the kernel of the first class constraint $\varphi^A$. That is

$$
\varphi^A \Psi = 0.
$$

The supercharges associated to modes invariant under $SO(7) \times U(1)$ are given explicitly [8] by

$$
Q_\alpha = \left\{-i \Gamma^i_{\alpha \beta} \partial X^i + \frac{1}{2} f_{ABC} X^B_i X^C_j \Gamma^{ij}_{\alpha \beta} - f_{ABC} Z^B Z^C \delta_{\alpha \beta} \right\} \lambda^A_{\beta}
$$

$$
+ \sqrt{2} \left\{ \delta_{\alpha \beta} \partial Z^A + if_{ABC} X^B_i Z^C \Gamma^{i}_{\alpha \beta} \right\} \partial \lambda^A_{\beta}
$$
and

\[
Q^\dagger = \left\{ i\Gamma^{i\alpha \beta} \partial X^A_i + \frac{1}{2} f_{ABC} X^B_i X^C_j \Gamma^{ij} \Gamma^{i\alpha \beta} + f_{ABC} Z^B \bar{Z}^C \delta_{\alpha \beta} \right\} \partial \lambda^A_{\beta}
+ \sqrt{2} \left\{ -\delta_{\alpha \beta} \partial Z^A + i f_{ABC} X^B_i Z^A \Gamma^{i\alpha \beta} \right\} \lambda^A_{\beta}.
\]

The corresponding superalgebra satisfies

\[
\{Q_\alpha, Q_\beta\} = 2\sqrt{2} \delta_{\alpha \beta} \bar{Z}^A \varphi_A,
\]

\[
\{Q_\alpha^\dagger, Q_\beta^\dagger\} = 2\sqrt{2} \delta_{\alpha \beta} Z^A \varphi_A,
\]

\[
\{Q_\alpha, Q_\beta^\dagger\} = 2\delta_{\alpha \beta} H - 2i \Gamma^{i\alpha \beta} X^A_i \varphi_A.
\]

The hamiltonian associated to the the regularized mass operator of the supermembrane is

\[
H = \frac{1}{2} M = -\Delta + V_B + V_F
\]

\[
\Delta = \frac{1}{2} \frac{\partial^2}{\partial X^A_i \partial X^A_i} + \frac{1}{2} \frac{\partial^2}{\partial Z^A \partial \bar{Z}^A}
\]

\[
V_B = \frac{1}{4} f_{AB} f_{CDE} \left\{ X^A_i X^B_i X^C_j X^D_j + 4 X^A_i Z^B_j X^C_j \bar{Z}^D_j + 2 Z^A \bar{Z}^B \bar{Z}^C \bar{Z}^D \right\}
\]

\[
V_F = i f_{ABC} X^A_i \lambda^{B\alpha}_C \Gamma^{i\alpha \beta} \partial_{\lambda^A_{\beta}} + \frac{1}{\sqrt{2}} f_{ABC} \left( Z^A \lambda^B C - Z^A \frac{\partial}{\partial \lambda_{\alpha B}} \frac{\partial}{\partial \lambda_{\alpha C}} \right).
\]

The generators of the local SU(N) symmetry are

\[
\varphi^A = f^{ABC} \left( X^B_i \partial_{X^C_i} + Z^B \partial_{Z^C} + \bar{Z}_B \partial_{\bar{Z}^C} + \lambda^B \partial_{\lambda^C} \right).
\]

They annihilate the physical states.

The hamiltonian \( H \) is a positive operator. It annihilates \( \Psi \) on the physical subspace, if and only if \( \Psi \) is a singlet under supersymmetry. In such a case, \( Q_\alpha \Psi = 0 \) and \( Q_\alpha^\dagger \Psi = 0 \).

The latter ensures that the wavefunction is massless, however it does not guarantee that the ground state wavefunction is the corresponding supermultiplet associated to supergravity. This is the case only when \( \Psi \) is a singlet under SO(9).

### 3. Matrix models with global symmetries

The framework of the Dirichlet boundary value problem associated to the hamiltonian \( H \) above provides a valuable insight into the problem of existence for its ground state. In this section we describe a rigorous setting which can be applied to the analysis of the ground state of
a wide variety of matrix models on compact domains, given suitable boundary data. The presence of an area preserving constraint will be addressed in Section 4.

Below $\Omega \subset \mathbb{R}^{9(N^2 - 1)}$ will be a bounded open set whose boundary $\partial \Omega$ is of class $C^\infty$. We will sometimes assume that $\Omega$ is a ball, but not for the time being. Here and elsewhere $L_2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$, are the corresponding Lebesgue and Sobolev Hilbert spaces of fields in $\Omega$ with $d \leq 9(N^2 - 1)$ components. We will denote the inner product of $L^2(\Omega)$ by $(\cdot, \cdot)$. Here $d$ is usually a large integer. The space $H^1_0(\Omega)$ is the completion in the norm of $H^1(\Omega)$ of the subspace $C^\infty_c(\Omega)$, the functions ($d$ components also) with support a compact subset of $\Omega$.

As it is customary in the theory of Lebesgue spaces, the symbol “$\in$” below will always mean “a representative in the equivalent class up to a set of measure 0 belongs to”.

3.1. Conditions on the Hamiltonian. Let

$$\mathcal{H} = -\nabla^2 + V$$

be a Schrödinger operator for a matrix potential $V = V^\dagger \in C^\infty(\Omega)$. Then $\mathcal{H}$ is strongly elliptic [43, Chapter 7]. Let

$$\text{Dom}(\mathcal{H}) = H^1_0(\Omega) \cap H^2(\Omega).$$

Then

$$\mathcal{H} : \text{Dom}(\mathcal{H}) \longrightarrow L^2(\Omega)$$

is a selfadjoint operator. Indeed the Dirichlet Laplacian on $\Omega$

$$\nabla^2 : \text{Dom}(\mathcal{H}) \longrightarrow L^2(\Omega)$$

is selfadjoint and $V : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a bounded selfadjoint operator.

Everywhere below we will assume that $\mathcal{H}$ is the Hamiltonian of a supersymmetric theory. By that we mean that it satisfies the identity

$$\mathcal{H} = \frac{1}{2}(QQ^\dagger + Q^\dagger Q) = [Q, Q^\dagger],$$

where $Q$ is the supercharge operator. This supercharge operator is rigorously defined as follows. It is a first order supersymmetric linear differential operator

$$Q, Q^\dagger : H^1(\Omega) \longrightarrow L^2(\Omega)$$

such that

$$Q, Q^\dagger : H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow H^1(\Omega).$$

This ensures that (S) above holds true at the rigorous level. Note that we do not impose here any condition on whether $Q$ or $Q^\dagger$ are closed.
operators, however $H$ should be selfadjoint on Dom$(H)$. Below we will not need the former.

Additionally we will assume that the supercharge operator satisfies the condition

$$(K) \quad \ker(Q|_{H_0^1(\Omega)}) \cap \ker(Q^\dagger|_{H_0^1(\Omega)}) = \{0\}. $$

That is

$$Q\psi = 0 \quad \text{and} \quad Q^\dagger\psi = 0 \quad \text{for} \quad \psi \in H_0^1(\Omega) \quad \Rightarrow \quad \psi = 0.$$ 

This will be satisfied by the hamiltonian $H$ and by the hamiltonian considered in Section 3.6. It is also true for other interesting cases, see for example [41, 40].

All the results reported here depend strongly on the condition $(K)$. The following lemma highlights the role played by this assumption in relation to the boundary value problem associated to $\mathfrak{H}$. The identity $(\mathfrak{S})$ is crucial for determining existence and uniqueness of the solutions of $(\mathfrak{S})$.

**Lemma 1.** Let $\psi \in \text{Dom}(\mathfrak{S})$ be such that $\mathfrak{S}\psi = 0$. Then $\psi = 0$.

**Proof.** If $\psi$ is as in the hypothesis, then

$$0 = (\psi, \mathfrak{S}\psi) = \frac{1}{2}(\psi, QQ^\dagger\psi) + \frac{1}{2}(\psi, Q^\dagger Q\psi)$$

$$= \frac{1}{2} \left( \|Q\psi\|_{L_2(\Omega)}^2 + \|Q^\dagger\psi\|_{L_2(\Omega)}^2 \right).$$

Hence $\psi \in \ker(Q|_{H_0^1(\Omega)}) \cap \ker(Q^\dagger|_{H_0^1(\Omega)})$ and $\psi = 0$ according to $(K)$. □

That is, $(K)$ always renders

$$\ker \mathfrak{S} = \{0\},$$

on a supersymmetric hamiltonian $(S)$.  

3.2. **The Dirichlet form.** Define the strongly elliptic Dirichlet form (of order one) associated to $\mathfrak{S}$,

$$\mathfrak{D}(\phi, \psi) = (\nabla \phi, \nabla \psi) + (\phi, V\psi).$$

Then

$$\mathfrak{D} : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{C}$$

is a non-negative coercive closed quadratic form.

That the form $\mathfrak{D}$ is coercive means that for suitable constants $C > 0$ and $\lambda \geq 0$ (sufficiently large),

$$\mathfrak{D}(\psi, \psi) \geq C \|\psi\|_{H_0^1(\Omega)}^2 - \lambda \|\psi\|_{L_2(\Omega)}^2 \quad \text{for all} \quad \psi \in H_0^1(\Omega).$$
In the current setting, the inequality is valid for $C = 1$ and $\lambda = 1 + \Lambda$, where $\Lambda$ is a lower bound of the minimum eigenvalue of the potential on $\Omega$.

That the form $\mathcal{D}$ is non-negative can be seen as follows. For all $\psi \in \text{Dom}(\mathcal{H})$,

$$\mathcal{D}(\psi, \psi) = \|Q\psi\|_{L^2(\Omega)}^2 + \|Q^t\psi\|_{L^2(\Omega)}^2 \geq 0.$$  \hfill (7)

See (6). As $\text{Dom}(\mathcal{H})$ is a core (in the form sense) for $\mathcal{D}$, then (7) also holds true for all $\psi \in \mathcal{H}^1_0(\Omega)$.

By virtue of Lemma 1, it then follows that the ground eigenvalue of $\mathcal{H}$ is strictly positive. Note that $\psi \in \text{Dom}(\mathcal{H})$ necessarily vanishes at $\partial \Omega$.

3.3. The boundary value problems. The Dirichlet problems associated the operator $\mathcal{H}$ can be re-written in terms of $\mathcal{D}$. Here we assume that the data, $g \in H^2(\Omega)$ and $f = (\nabla^2 - V)g \in L^2(\Omega)$, are given.

The homogeneous Dirichlet problem with inhomogeneous boundary conditions associated to $\mathcal{H}$ is formulated as follows. Find $\Psi \in H^2(\Omega)$ such that

$$\begin{cases} (-\nabla^2 + V)\Psi = 0 & \text{in } \Omega \\ \Psi = g & \text{on } \partial \Omega. \end{cases}$$  \hfill (DI)

This is related to the inhomogeneous Dirichlet problem with homogeneous boundary conditions. Find $\Phi \in H^2(\Omega)$ such that

$$\begin{cases} (-\nabla^2 + V)\Phi = f & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial \Omega. \end{cases}$$  \hfill (DH)

The weak formulation of the latter is the weak inhomogeneous Dirichlet problem with homogeneous boundary conditions. Find $\Phi \in H^1_0(\Omega)$ such that

$$\mathcal{D}(\phi, \Phi) = (\phi, f) \text{ for all } \phi \in H^1_0(\Omega).$$  \hfill (WH)

By setting $\Psi = \Phi + g$ we see that (DI) and (DH) are equivalent. Clearly a solution of (DH) would also be a (weak) solution of (WH). Moreover, a solution $\Phi$ of (WH) originally in $H^1_0(\Omega)$ will also be in $H^2(\Omega)$ and would satisfy (DH) (see below for the precise statement).

In passing from (DI) to (WH) the boundary condition $\Psi = g$ on $\partial \Omega$ has been replaced by the condition $\Phi \in H^1_0(\Omega)$.

Let $\delta(\xi)$ be the seminorm associated to the inner product (2),

$$\delta^2(\xi) = \|Q\xi\|_{L^2(\Omega)}^2 + \|Q^t\xi\|_{L^2(\Omega)}^2,$$
for $\xi \in H^1(\Omega)$. The solution $\Psi$ to the inhomogeneous Dirichlet problem (DI) is the state that minimises this seminorm, among all other states $\xi \in H^1(\Omega)$ satisfying the same boundary condition $\xi = g$ on $\partial \Omega$. Indeed,

$$\xi - \Psi = \eta \in H^1_0(\Omega)$$

and

$$\delta^2(\xi) = \delta^2(\Psi) + \delta^2(\eta) \geq \delta^2(\Psi),$$

since

$$(Q\Psi, Q\eta)_{L^2(\Omega)} + (Q^\dagger \Psi, Q^\dagger \eta)_{L^2(\Omega)} = ((-\nabla^2 + V)\Psi, \eta)_{L^2(\Omega)} = 0.$$  

Below we will show that this minimising state $\Psi$ exists and is unique. Note that $\delta^2(\eta) = 0$ implies $\eta = 0$, because $\eta \in H^1_0(\Omega)$.

**Remark 2.** The electrostatic field in the vacuum fulfils an analogous property. The electrostatic energy $E$ of an electrostatic potential $\xi$ on a bounded domain $\Omega$ is given by

$$E = \int_\Omega \nabla \xi \cdot \nabla \xi.$$  

This is a seminorm in $H^1(\Omega)$. The harmonic potential is the one that minimises $E$ among all other potentials satisfying the same boundary condition on $\partial \Omega$.

### 3.4. Existence and uniqueness of solutions

As we shall see next, for regular data as above, (DI) and (DH) are always solvable uniquely.

**Lemma 3.** Let $g \in H^2(\Omega)$. There always exists a unique solution $\Phi \in H^1_0(\Omega) \cap H^2(\Omega)$ to the equation (DH) and a corresponding unique solution $\Psi = \Phi + g \in H^2(\Omega)$ to the equation (DI).

**Proof.** Recall that $\mathcal{D}$ is a non-negative selfadjoint Dirichlet form of order 1. Let

$$\mathcal{K} = \{ \xi \in H^1_0(\Omega) : \mathcal{D}(\varphi, \xi) = 0 \text{ for all } \varphi \in H^1_0(\Omega) \}.$$  

By virtue of regularity results for strongly elliptic Dirichlet forms [43, Theorem (7.32)], it follows that $\mathcal{K} \subset H^2(\Omega)$ and in fact $\mathcal{K} = \ker(\mathcal{H})$. Then, according to Lemma [1], $\mathcal{K} = \{0\}$. Hence, by virtue of [43, Theorem (7.21)], there exists $\Phi \in H^1_0(\Omega)$ such that (WH) holds true. Moreover, once again from [43, Theorem (7.32)], in fact $\Phi \in H^2(\Omega)$. Thus $\Phi$ is also a solution to (DH) and $\Psi = \Phi + g$ a solution to (DI).

Now, suppose that $\Phi_1$ and $\Phi_2$ are two solutions for (DH) (they are both in $H^2(\Omega) \cap H^1_0(\Omega)$). Then $\Phi_1 - \Phi_2 \in \text{Dom} \mathcal{H}$ satisfies $\mathcal{H}(\Phi_1 - \Phi_2) = 0$. By Lemma [1] we then have $\Phi_1 = \Phi_2$. 

$\square$
3.5. **Pointwise regularity at the boundary.** A crucial observation on the regularity properties of supercharge operators of first order is now in place. This observation is independent of the assumption (K), however the potential should be smooth.

**Lemma 4.** Let \( \tilde{Q} : H^1(\Omega) \to L^2(\Omega) \) be a (generic) supercharge operator of first order. Suppose additionally that \( \Omega \) is a \( C^\infty \) domain. If \( \tilde{Q}\Phi = 0 \) and \( \tilde{Q}^\dagger\Phi = 0 \) for \( \Phi \in H^1_0(\Omega) \), then \( \Phi \in C^\infty(\overline{\Omega}) \) and

\[
\tilde{Q}\Phi(x) = \tilde{Q}^\dagger\Phi(x) = 0 \quad \text{for all} \quad x \in \overline{\Omega}.
\]

**Proof.** Let \( \Phi \) be as in the hypothesis. By virtue of classical bootstrap arguments and the Sobolev Lemma \[43\], it follows that \( \Phi \in C^\infty(\overline{\Omega}) \). That is \( \Phi \) is smooth in the domain up to the boundary. Thus also \( \tilde{Q}\Phi \) and \( \tilde{Q}^\dagger\Phi \) lie in \( C^\infty(\overline{\Omega}) \). Hence \( \tilde{Q}\Phi(x) = \tilde{Q}^\dagger\Phi(x) = 0 \) for all \( x \in \overline{\Omega} \). \( \square \)

3.6. **A toy model example.** Consider a version of the toy model introduced in \[30\] on a compact region. In \[42\] it was shown that this model has no zero eigenvalue for the non-compact problem. By combining the supersymmetric structure of the hamiltonian shown below with Lemma 3, it follows that the solutions to the problems (DI) and (DH) exist and are unique in this case.

Let

\[
\mathcal{H} = p_x^2 + p_y^2 + x^2y^2 + x\sigma_3 + y\sigma_1
\]

where \( \sigma_i \) are the Pauli matrices. The supersymmetric charges in this case are

\[
Q = Q^t = \begin{pmatrix} -xy & i\partial_x - \partial_y \\ i\partial_x - \partial_y & xy \end{pmatrix}.
\]

The wavefunctions are such that

\[(8) \quad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \quad \text{and} \quad \Phi = 0 \quad \text{on} \quad \partial\Omega.
\]

We firstly show that the condition (K) is valid for the supersymmetric charges. Let \( \Phi \in H^1_0(\Omega) \) be such that

\[(9) \quad Q\Phi = Q^\dagger\Phi = 0 \quad \text{in} \quad \Omega.
\]

According to Lemma 4, this condition holds true pointwise up to the boundary of \( \Omega \). Let \( x \in \partial\Omega \) and denote by \( n_1, n_2 \) the components of the normal to \( \partial\Omega \) at \( x \). The tangent to \( \partial\Omega \) at \( x \) is then \( (n_2, -n_1) \), and we must have

\[
(n_2\partial_x - n_1\partial_y)\Phi(x) = 0.
\]

The solutions are regular, so we can extend them continuously up to the boundary. Then (9) yields

\[
(i\partial_x + \partial_y)\Phi_2(x) = (i\partial_x - \partial_y)\Phi_1(x) = 0
\]
pointwise. Since \((n_1, n_2) \neq 0\), if \(n_2 \neq 0\)

\[
\begin{cases}
(1 + i \frac{n_1}{n_2}) \partial_y \Phi_2(x) = 0 \quad \Rightarrow \quad \partial_y \Phi_2(x) = 0 \text{ and } \partial_x \Phi_2(x) = 0 \\
(-1 + i \frac{n_1}{n_2}) \partial_y \Phi_1(x) = 0 \quad \Rightarrow \quad \partial_y \Phi_1(x) = 0 \text{ and } \partial_x \Phi_1(x) = 0.
\end{cases}
\]

A similar conclusion is obtained for \(n_1 \neq 0\). Hence \(\Phi\) and \(\partial_n \Phi\) must vanish on \(\partial \Omega\). By virtue of the Cauchy-Kovalevskaya Theorem [43], from the fact that the potential is analytic, we conclude that \(\Phi = 0\) pointwise in \(\bar{\Omega}\). This yields (K).

Similar arguments can be used to prove existence and uniqueness of the solution for the \(SU(N)\) truncated model of the \(D = 11\) supermembrane considered in [8], as it only contains global symmetries. See [40].

4. Systems with gauge symmetry

Let us now consider supersymmetric theories with a constraint. For this purpose, assume that there exist subspaces \(\mathcal{X}, \mathcal{Y} \subseteq \mathcal{H}^2(\Omega)\) such that the decomposition \(Q = Q_X + Q_Y\) is compatible with the following conditions.

\(a\) \(\mathcal{X} \perp \mathcal{Y}\), meaning

\[(x, y) = 0 \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}.\]

\(b\) \(\mathcal{L}_2(\Omega) = \bar{\mathcal{X}} + \bar{\mathcal{Y}}\).

\(c\) \(\mathcal{H}^1(\Omega) = (\bar{\mathcal{X}} \cap \mathcal{H}^1(\Omega)) + (\bar{\mathcal{Y}} \cap \mathcal{H}^1(\Omega))\).

\(d\) \(\mathcal{H}^2(\Omega) = \mathcal{X} + \mathcal{Y}\).

\(e\)

\[
Q_X : \bar{\mathcal{X}} \cap \mathcal{H}^1(\Omega) \rightarrow \bar{\mathcal{X}} \quad \text{and} \quad Q_Y : \bar{\mathcal{Y}} \cap \mathcal{H}^1(\Omega) \rightarrow \bar{\mathcal{Y}}.
\]

\(f\)

\[
Q_X : \mathcal{X} \cap \mathcal{H}_0^1(\Omega) \rightarrow \bar{\mathcal{X}} \cap \mathcal{H}^1(\Omega) \quad \text{and} \quad Q_Y : \mathcal{Y} \cap \mathcal{H}_0^1(\Omega) \rightarrow \bar{\mathcal{Y}} \cap \mathcal{H}^1(\Omega).
\]

Here and everywhere below, the over-bar on top of subspaces, e.g. \(\bar{\mathcal{X}}\), always refers to the closure in the norm of \(\mathcal{L}_2(\Omega)\).

Since

\[(Qx, y) = 0 \quad \text{and} \quad (Qy, x) = 0 \quad \text{for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y},\]
then also $Q^\dagger = Q^\dagger_{X} + Q^\dagger_{Y}$ is compatible with the analogues to $[e]$ and $[f]$. Hence, since

$$QQ^\dagger X \perp Y \quad \text{and} \quad Q^\dagger QY \perp X,$$

it follows that

$$Q_X Q^\dagger_Y = Q^\dagger_Y Q_X = Q^\dagger_X Q_Y = Q^\dagger_Y Q_X = 0.$$  

Thus

$$\mathcal{H} = \mathcal{H}_X + \mathcal{H}_Y$$

where

$$\mathcal{H}_X = \frac{1}{2}(Q_X Q^\dagger_X + Q^\dagger_X Q_X) \quad \mathcal{H}_Y = \frac{1}{2}(Q_Y Q^\dagger_Y + Q^\dagger_Y Q_Y)$$

$$\text{Dom}(\mathcal{H}_X) = X \cap H^1_0(\Omega) \quad \text{Dom}(\mathcal{H}_Y) = Y \cap H^1_0(\Omega).$$

Both $\mathcal{H}_X$ and $\mathcal{H}_Y$ are selfadjoint operators. The property $[f]$ ensures that the subspaces in (11) are mapped correctly.

The Dirichlet form associated to $\mathcal{H}$ can be written as

$$D(\phi, \psi) = D_X(\phi_X, \psi_X) + D_Y(\phi_Y, \psi_Y)$$

where $\phi_X, \psi_X \in X \cap H^1_0(\Omega)$ and $\phi_Y, \psi_Y \in Y \cap H^1_0(\Omega)$. These subspaces are the domains of closure of these forms, and they are the quadratic forms associated to $\mathcal{H}_X$ and $\mathcal{H}_Y$ respectively.

Additionally we now consider a condition which is similar, but weaker, than the condition $[K]$ from the previous section. We only assume that

$$(K_X) \quad Q_X \psi = Q^\dagger_X \psi = 0 \quad \text{for} \quad \psi \in X \cap H^1_0(\Omega) \quad \Rightarrow \quad \psi = 0.$$  

That is, the supercharge operator satisfies an analogue to $[K]$, but only in the constraint subspace $X$. Note that $[K]$ implies $[K_X]$, but not conversely. This is fulfilled by the hamiltonian of the $D = 11$ supermembrane and other interesting supersymmetric matrix models.

4.1. The constrained boundary value problems. Set data: $g \in X$ and $f = (\nabla^2 - V)g$. According to the property $[f]$ it follows that $f \in X$.

Consider the constrained versions of $[DI]$, $[DH]$ and $[WH]$.

$$\begin{aligned}
(DI_X) & \quad \begin{cases}
-\nabla^2 + V \Psi = 0 \quad & \text{in } \Omega \\
\Psi = g \quad & \text{on } \partial \Omega \\
\Psi \in X,
\end{cases} \\
(DH_X) & \quad \begin{cases}
-\nabla^2 + V \Phi = f \quad & \text{in } \Omega \\
\Phi = 0 \quad & \text{on } \partial \Omega \\
\Phi \in X
\end{cases}
\end{aligned}$$
and

\[ (WH_X) \quad \mathfrak{D}_X(\phi, \Phi) = (\phi, f) \text{ for all } \phi \in \mathcal{X} \cap \mathcal{H}^1_0(\Omega). \]

**Lemma 5.** Let \( g \in \mathcal{X} \) and \( f = (\nabla^2 - V)g \in \overline{\mathcal{X}} \). There always exists a solution \( \Phi \in \mathcal{H}^1_0(\Omega) \cap \mathcal{X} \) to \((DH_X)\) which is unique. The corresponding solution \( \Psi = \Phi + g \in \mathcal{X} \) to \((DI_X)\) exists and is also unique.

**Proof.** We first show that \((WH_X)\) has a solution. Let \( \tilde{\mathcal{X}}_0 = \mathcal{X} \cap \mathcal{H}^1_0(\Omega) \).

By virtue of the assumption \((K_X)\) and the fact that \( \mathfrak{D}_X(\phi_X, \phi_X) = \frac{1}{2}(\|Q_X \phi_X\|^2 + \|Q^\dagger_X \phi_X\|^2) \), it follows that \( \tilde{\mathcal{X}}_0 \) is a Hilbert space with respect to the inner product \( \mathfrak{D}_X(\phi_X, \eta_X) \) (for \( \eta_X, \phi_X \in \tilde{\mathcal{X}}_0 \)). This is the same as saying that the Dirichlet form \( \mathfrak{D}_X \) is closed and positive. The Lax-Milgram Theorem \([44, \S 6.2]\) ensures the existence of a solution \( \Phi \in \tilde{\mathcal{X}}_0 \) for \((WH_X)\).

Note that \( \Phi \in \mathcal{X} \), because of \( \Phi \in \overline{\mathcal{X}} \). Then, since

\[ \mathfrak{D}(\phi, \Phi) = \mathfrak{D}_X(\phi_X, \Phi) + \mathfrak{D}_Y(\phi_Y, 0) = 0, \]

it turns out that \( \Phi \) is also a solution of \((WH)\). By repeating the same steps as in the proof of Lemma 3 (which applies on physical states), we get that \( \Phi \in \mathcal{H}^2(\Omega) \cap \overline{\mathcal{X}} = \mathcal{X} \) and that \( \Phi \) is a solution of \((DH)\). Hence it is a solution of \((DH_X)\).

The rest of the lemma is shown from a similar argument as the one presented in Lemma 3. \( \square \)

### 4.2. Operator realization of the constraint.

The subspace \( \mathcal{X} \) is generally related to an operator constraint. Suppose that there exists a first order differential operator

\[ G : \mathcal{H}^1(\Omega) \longrightarrow \mathcal{L}_2(\Omega) \]

such that

a) \( G : \mathcal{H}^2(\Omega) \longrightarrow \mathcal{H}^1(\Omega) \)

b) \( GQ\psi = QG\psi \) and \( GQ^\dagger\psi = Q^\dagger G\psi \) for all \( \psi \in \mathcal{X} \).

Then \( \mathcal{X} \) can be any subspace of \( \mathcal{H}^2(\Omega) \) such that

\[ \mathcal{X} \subseteq \ker(G) \subseteq \overline{\mathcal{X}}. \]

In this case, the complementary subspace \( \mathcal{Y} \) is such that its closure \( \overline{\mathcal{Y}} \) is the orthogonal complement of \( \overline{\mathcal{X}} \) in \( \mathcal{L}_2(\Omega) \). This ensures the validity of \([b][f]\).

Observe that regularity up to the boundary (Lemma 4) is still holds true in the constrained setting. A particular application of the Lemma 5 covers the \( SU(2) \) model examined in \([41]\).
5. The ground state of the $D=11$ supermembrane

This section contains one of the main contributions of this paper. Consider the supermembrane theory on a $D=11$ Minkowski spacetime introduced in Section 2. We now determine the existence and uniqueness of a groundstate of this theory restricted to a compact region. The relevance of this setting is twofold. On the one hand, it is a problem of physical interest by itself due to its potential implications in M-theory. On the other hand, it is a crucial step towards the solution of the ground state problem on an unbounded domain. As we are in the presence of a gauge constraint, we resource to the framework of Section 4.

Let $H$ be the hamiltonian given by the expression (5). Let $\Omega$ be a ball in $\mathbb{R}^{9(N^2-1)}$ of radius $R > 0$. The boundary, $\partial \Omega$, is a sphere of dimension $9(N^2-1)-1$ with the same radius. Consider the $SO(7) \times U(1)$ decomposition as described in Section 2. The coordinates are $(X^A_i, Z^A, \overline{Z}^A, \lambda^A_\alpha)$ where $A$ is the $SU(N)$ index. The radial coordinate $\rho$ is defined by

$$\rho^2 = (X^A_i)^2 + 2\overline{Z}Z.$$

This radial coordinate is invariant under the symmetry generated by the first class constraint. That is, the generators of local $SU(N)$ transformations. Consequently the constraint imposes no restriction to the normal derivative on the border, $\partial_\rho \Psi|_{\partial \Omega}$.

5.1. Validity of $\mathcal{K}$. Let us verify the condition $\mathcal{K}$ for the supersymmetric charges. Assume that

$$Q_\alpha \psi = 0 \quad \text{and} \quad Q^\dagger_\alpha \psi = 0 \quad \text{in} \quad \Omega$$

for $\psi \in \mathcal{H}^2(\Omega) \cap \mathcal{H}^1_0(\Omega)$. We wish to prove that $\psi = 0$ in $\Omega$.

Regularity at the boundary (Lemma 4), allows us to extend the restriction on $\frac{\partial \psi}{\partial \rho}$ arising from (12) smoothly to the boundary. The conditions $Q_\alpha \psi|_{\partial \Omega} = 0$ and $Q^\dagger_\alpha \psi|_{\partial \Omega} = 0$ for the $SU(N)$ regularized supermembrane found in [8], now evaluated on the boundary where $\psi = 0$, are

$$Q_8 \psi = \sqrt{2} \partial Z_8 \partial \lambda^A_8 \psi - i \Gamma^j_{8j} \partial X^A_i \lambda^A_j \psi = 0 \quad (13)$$
$$Q^\dagger_8 \psi = -\sqrt{2} \partial \overline{Z}^A_8 \lambda^A_8 \psi + i \Gamma^j_{8j} \partial X^A_i \partial \lambda^A_j \psi = 0 \quad (14)$$
$$Q_j \psi = -i \Gamma^j_{8j} \lambda^A_8 \partial X^A_i \psi - i \Gamma^j_{jk} \lambda^A_j \partial X^A_k \psi + \sqrt{2} \partial Z_8 \partial \lambda^A_8 \psi = 0 \quad (15)$$
$$Q^\dagger_j \psi = +i \Gamma^j_{8j} \partial X^A_i \partial \lambda^A_8 \psi + i \Gamma^j_{jk} \partial X^A_i \partial \lambda^A_k \psi - \sqrt{2} \partial \overline{Z}^A_8 \lambda^A_j \psi = 0 \quad (16).$$
Here $\Gamma^i_j = -i\delta^i_j$ and $\Gamma^i_{jk} = iC^i_{ijk}$, where the $C^i_{ijk}$ are the structure constants of the octonion algebra.

In order to verify (14) we only need (13) and (14), as the remaining identities (15) and (16) will be identically satisfied. These equations are only valid at $\partial\Omega$ and they pertain the normal derivative of $\psi$ there. The $9(N^2 - 1) - 1$ remaining angular derivatives are tangential derivatives vanishing at the boundary. That is,

$$\partial_{\alpha_m}\psi|_{\partial\Omega} = 0 \qquad \text{for} \quad m = 1, \ldots, 9(N^2 - 1) - 1.$$

Consider the derivative with respect to $\rho^2$. Observe that

$$(17) \quad \partial_{Z^A}\psi|_{\partial\Omega} = 2Z^A\frac{\partial\psi}{\partial\rho^2}|_{\partial\Omega} \quad \text{and} \quad \partial_{X^A_i}\psi|_{\partial\Omega} = 2X^A_i\frac{\partial\psi}{\partial\rho^2}|_{\partial\Omega}.$$

Write $\partial_{\rho^2}\psi \equiv \psi_{\rho^2}$. Then (13) and (14) reduce to

$$(18) \quad Q^8\psi = \sqrt{2}Z^A\partial_{\lambda^A}\psi_{\rho^2} + (X^A_i\lambda^A_i)\psi_{\rho^2} = 0$$

and

$$(19) \quad Q^8\psi = -\sqrt{2}Z^A\lambda^A_8\psi_{\rho^2} - X^A_i\partial_{\lambda^A_i}\psi_{\rho^2} = 0.$$

Applying $\frac{\partial}{\partial\lambda^A_8}$ to $Q^8\psi$, gives

$$(20) \quad X^A_j\partial_{\lambda^A_i}(Z^B\partial_{\psi_{\rho^2}}) - \sqrt{2}Z^A\psi_{\rho^2} + \sqrt{2}(Z^A\lambda^A_8)(\overline{Z}^B\partial_{\lambda^A_i}\psi_{\rho^2}) = 0.$$

Replacing (18) into (20), then gives

$$(21) \quad -X^A_j\partial_{\lambda^A_i}(\frac{1}{\sqrt{2}}(X^B_i\lambda^B_i)\psi_{\rho^2}) - \sqrt{2}Z^A\psi_{\rho^2} + \sqrt{2}(Z^A\lambda^A_8)(\overline{Z}^B\partial_{\lambda^A_i}\psi_{\rho^2}) = 0.$$

Thus

$$(22) \quad -((X^A_jX^A_j + 2Z^A\overline{Z}^A)\psi_{\rho^2} + (X^B_i\lambda^B_i)X^A_j\partial_{\lambda^A_i}\psi_{\rho^2} + 2(Z^A\lambda^A_8)(\overline{Z}^B\partial_{\lambda^A_i}\psi_{\rho^2}) = 0.$$

Now, on $\partial\Omega$,

$$(X^A_j)^2 + 2Z^A\overline{Z}^A = R^2.$$

Then

$$R^2\psi_{\rho^2}|_{\partial\Omega} = 0$$

for $R^2 \neq 0$. Thus

$$\psi_{\rho^2}|_{\partial\Omega} = 0.$$

By virtue of the Cauchy-Kovalevskaya Theorem, it then follows that $\psi = 0$ is the unique solution in a neighborhood of the boundary. Moreover, since the potential is analytic on $\Omega$ this solution can be extended
uniquely to the whole ball Ω. Thus, the supercharges \( Q_\alpha \) and \( Q^\dagger_\alpha \) indeed satisfy the condition (K).

5.2. The constraint. We now define the subspace \( \mathcal{X} \subset H^2(\Omega) \) associated to the constraint (4), which is required in the framework of Section 4. Let
\[
(23) \quad \mathcal{X} = \ker \varphi^A \cap H^2.
\]
As
\[
\varphi^A : H^1(\Omega) \longrightarrow L_2(\Omega)
\]
is such that
\[
\varphi^A : H^2(\Omega) \longrightarrow H^1(\Omega),
\]
and also
\[
\varphi^A Q \Psi = Q \varphi^A \Psi \quad \text{and} \quad \varphi^A Q^\dagger \Psi = Q^\dagger \varphi^A \Psi \quad \text{for all} \quad \Psi \in \mathcal{X},
\]
we gather that \((a)\) in Section 4 hold true.

5.3. Existence and uniqueness of the ground state. Since (K) is fulfilled, then also (K\( \mathcal{X} \)) is fulfilled. The following main result is a direct consequence of Lemma 4.

**Theorem 6.** Given \( g \in \mathcal{X} \) on \( \overline{\Omega} \), there always exists a unique solution \( \Phi \) to the problem \((DH\mathcal{X})\) associated with the \( D = 11 \) supermembrane (and the \( N = 16 \) supersymmetric matrix model), which lies in the space \( H^0_0(\Omega) \cap \mathcal{X} \). The corresponding solution \( \Psi = \Phi + g \in \mathcal{X} \) to the problem \((DI\mathcal{X})\) also exists and is also unique.

5.4. Invariance of the solution under \( SO(9) \). Denote by \( J \) the generators of the algebra of \( SO(9) \). Then \( J \) commute with \( H \). Assume that \( g \) is a singlet under \( SO(9) \). That is \( Jg = 0 \). Then the solution \( \Phi \) to \((DH\mathcal{X})\) is also a singlet under \( SO(9) \).

Indeed, firstly observe that
\[
(\xi, Jg) = 0 \quad \text{for all} \quad \xi \in C^\infty_c(\Omega).
\]
Hence
\[
(J^\dagger \xi, H\Phi) = (J^\dagger \xi, -Hg) = -(H\xi, Jg) = 0,
\]
and so
\[
(H\xi, J\Phi) = 0 \quad \text{for all} \quad \xi \in C^\infty_c(\Omega).
\]
Now, recall that the hamiltonian \( H \) is positive definite. Then, given any \( \eta \in C^\infty_c(\Omega) \), there always exists a unique \( \xi \in C^\infty_c(\Omega) \) such that \( H\xi = \eta \). Note that \( \xi \) turns out to be \( C^\infty \) by similar arguments as in the proof of Lemma 4. Therefore
\[
(\eta, J\Phi) = 0 \quad \text{for all} \quad \eta \in C^\infty_c(\Omega).
\]
But, since $C_c^\infty(\Omega)$ is a dense subspace of $L_2(\Omega)$, it follows that in fact

$$(\eta, J\Phi) = 0 \quad \text{for all } \eta \in L_2(\Omega).$$

Hence $J\Phi = 0$ as claimed. Since $\Psi = \Phi + g$, consequently

$$J\Psi = J\Phi + Jg = 0.$$

6. Conclusions

We show that the ground state wavefunction for the mass operator of the regularized $D = 11$ supermembrane theory on a bounded smooth domain exists and is unique. Under regularity of the boundary, which is a given datum, this solution corresponds to a singlet under $SO(9)$, the residual Lorentz invariance. The results presented rely on general rigorous arguments formulated in sections \[ \text{and} \] These are valid in the context of supersymmetric theories for a Schrödinger Hamiltonian with a polynomial potential.

Supersymmetry plays a crucial role in all the results presented in this paper. The framework of sections \[ and \] provide a new approach in the context of matrix models which allows characterizing the ground state wavefunction for a wide variety of supersymmetric matrix models. The potential should be a polynomial. The physical theory may or may not possess gauge symmetry. A novel feature here is a simplification of the treatment of the gauge constraint. There is no need to solve it explicitly, as it is enough to set it as a subspace of the configuration space.

Our contribution is a crucial step towards the solution of the ground state wavefunction problem on an unbounded domain. The next step in a possible strategy would be to solve the external problem on the complement of $\Omega$ and glue together the two solutions. Another possibility would be to examine a suitable limit of the solution as the diameter of the ball increases. The quest for the ground state of the $D = 11$ supermembrane and $N = 16$ supersymmetric matrix model—which is expected to be a multiplet of the $D = 11$ supergravity—is a fundamental step towards the quantization of the theories, and in a more general context is fundamental in the quantization of M-theory.

The methods presented above might also have an impact in other areas of physics. These include the study of AdS/CFT black holes, compact Yang-Mills matrix models and other M-theory characterizations.
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