Abstract: We determine complete one-loop beta functions of the multi-scalar four-point couplings in four-dimensional $SU(N)$ gauge theories with $M$ adjoint scalar multiplets. For adjoints scalars, the sign of the one loop gauge coupling beta function depends solely on $M$, vanishing and changing sign precisely at $M = 22$. For the multi-scalar potential at fixed gauge coupling we find several fixed points with different stability properties at large $N$. The analysis crucially involves the full set of four $SU(N)$ and $O(M)$ invariant single trace and double trace couplings. Taking the gauge coupling into account, there are asymptotically free RG flows for $M < 22$ and non-trivial fixed points for $M = 22$ at one loop, while $M > 22$ appears to ruin the UV properties of the theory. Surprisingly, uniquely between $M = 22$ and $M = 21$ the number of fixed flows drops from eight to four in the large $N$ limit. There seems to be something very special about $M = 22$. More speculatively, the $M = 22$ one-loop conformal fixed point theory with $M$ adjoint scalars in $d = 4$ suggests the possibility of an isolated non-supersymmetric, purely bosonic AdS$_{4+1}$×$S^{22−1}$/CFT$_4$ correspondence.

Our example suggests that extending the potential to the complete set of terms allowed by symmetries may lead to real fixed points also in non-supersymmetric theories descending from $\mathcal{N} = 4$ super-Yang-Mills theory.
1 Introduction

Field theories known to behave well at high energies are scale symmetric in this limit and thus approach UV fixed points of the renormalisation group (RG) flow at high energy. If the UV fixed point is trivial, i.e. at vanishing coupling, the theory is asymptotically free, and if the UV fixed point is non-trivial, the theory is instead asymptotically safe [1]. Scalar fields are known to be the most challenging fields for good UV properties in four dimensions, already since the early days of asymptotic freedom [2]. Delicate balancing is needed to make scalar field theories be well behaved at short distances. First and foremost, non-Abelian gauge fields coupling to the scalars are critical for the UV properties of scalar interactions — without gauge fields scalar couplings grow indefinitely at high energies. Furthermore, the improvement requires the number of scalars to be limited relative to the size of the gauge group. Finally, the scalar potential has to have a stable minimum. In contrast, for theories with both bosons and fermions, Yukawa couplings can improve the UV properties without leading to additional constraints. The present work instead searches for qualitatively new purely bosonic conformal field theories in four dimensions.

For the reasons mentioned above, we believe that conformal fixed points in multi-scalar field theory deserve more attention. There is a literature on the beta functions of scalar
gauge theories starting from [2, 3], eg [4]. For particle physics, fermions are obviously important, and discussion on purely scalar gauge theories are often hidden in more general discussions, see eg [5, 6]. Rather than aiming for immediate phenomenological importance, the present work performs a pilot study for more general conformal field theories in four dimensions, which may find applications in phenomenology as well as holography and string theory. Our focus on the most challenging field content, increases the chance that multiple generalisations of our results are possible, eg by adding fermions and Yukawa couplings.

We focus our study on adjoint scalar models, which offer the simplification that all fields, vectors and scalars, transform in the same representation of the gauge group. In addition to being on the critical boundary between small and large matter representations, they may be seen classically as dimensional reductions of higher-dimensional Yang-Mills theories, which are also related to brane physics, string theory and AdS/CFT. Many of the connections to these fields are related to the limit of large $N$, essentially the rank of the gauge group, a limit which is also an important tool for probing the internal field theory dynamics.

This paper considers the multi-scalar potentials of four dimensional conformal versions of scalar $\phi^4$ theory. A stable multi-scalar $\phi^4$ potential may have an isolated critical point at the origin leading to a phase of the theory with the same symmetry as the potential, or it may have flat directions along lines or hyperplanes through the origin opening for breaking of the symmetry of the potential. The potential of $\mathcal{N} = 4$ super-Yang-Mills theory is of the latter kind, while our purely bosonic theory appears to yield potentials with isolated symmetry preserving minima at the origin. There are interesting intermediate cases: non-supersymmetric theories constructed from $\mathcal{N} = 4$ super-Yang-Mills theory by orbifold techniques were initially thought to be conformal field theories, but were later found to have running double trace couplings [7–9] that are induced at one loop and have to be included even if they are not present in the naive classical action. Already earlier, in attempts to find a non-supersymmetric AdS/CFT correspondence from D3 branes in Type 0 string theory [10] such couplings jeopardising conformal symmetry had been found [11]. In non-supersymmetric orbifold models symmetry and conformality is simultaneously lost [12, 13] in the large $N$ limit, apparently without exception, since no real fixed points have been found, only fixed points at complex values of some couplings, corresponding to complex conformal field theories [14]. In this context, double trace couplings are crucial to the breaking of conformal symmetry.

Another interesting class of $d = 4$ gauge theories descending from $\mathcal{N} = 4$ super-Yang-Mills theory was proposed as a candidate non-supersymmetric CFT in [15]. The construction starts from AdS/CFT, and involves an explicit deformation of the action which breaks its original global symmetry. In this case, too, double trace couplings were found to ruin conformal invariance [16], except in certain interesting limits where integrable fishnet models could be constructed [17], which are however also complex, non-unitary, conformal field theories\footnote{In fishnet field theories, double trace deformations figure prominently in the complex fixed point solutions.}.
A suspicion that double trace couplings endanger conformal symmetry is important to follow up. There may be other aspects than just the double trace terms of orbifold theories and deformed $\mathcal{N} = 4$ theories which prevent real fixed points. Indeed, in the present study of bosonic models we find new real one loop fixed points, with fixed real double trace couplings. We believe that the presence of several single trace couplings with non-trivial beta functions is crucial in this mechanism, and may generalise to other examples, perhaps even to the above descendants of $\mathcal{N} = 4$ super-Yang-Mills theory, which could be extended with such couplings.

We find zeros of the one loop beta functions for dimensionless quartic multi-scalar couplings at fixed gauge coupling, i.e. we find fixed points of the RG flow at multi-scalar couplings scaling with the gauge coupling. The gauge coupling is asymptotically free for sufficiently few adjoint scalars, but its one loop beta function vanishes for $M = 22$ scalars\(^2\). Then, the multi-scalar fixed points appear to be lines of fixed points parametrised by an arbitrary gauge coupling, but these lines should be replaced by ordinary fixed points at higher order. A preliminary check of higher loop terms suggests that our one loop results indeed may translate to fixed points at higher order, but a complete analysis should also include a convincing argument that the fixed points are reliably within the perturbative regime. Unfortunately, the present model does not have a free parameter that can ensure that corrections are parametrically small. We hope that our ideas can come to use in such improved models, thus serving to identify interesting new conformal field theories.

We searched for and found one-loop fixed points in one of the theories we examined. This should be viewed in a context where the range of four dimensional fixed point theories is quite limited. Except for the free Gaussian fixed point of asymptotically free gauge theories, there are lines of fixed points in some supersymmetric field theories like $\mathcal{N} = 4$ super-Yang-Mills theory, parametrised by an arbitrary gauge coupling, and two other classes of fixed points discussed in the literature. First, gauge fields and fermions and/or bosons interact weakly at IR fixed points of the Caswell-Banks-Zaks [21, 22] type which do not require matter interactions. Second, asymptotically safe fixed points of gauge theories with fermions which have to be coupled to scalars by Yukawa couplings have been found [23, 24]. Both classes of fixed points require cancellations of one and two loop contributions to the gauge coupling beta function, which are independent of purely scalar interactions. The asymptotically safe class is substantially more complicated and typically involves further cancellations between one and two loop contributions depending on matter couplings. We believe that our model with 22 scalars could be the first example of a new class of four-dimensional conformal field theories. In contrast to the known classes cited above, such models would have vanishing one loop gauge couplings, and require cancellations at three loop order.

Intriguingly, the $M = 22$ one-loop conformal fixed point theory in $d = 4$ is a bosonic version of $\mathcal{N} = 4$ super-Yang-Mills theory with 6 adjoint scalars, which describes the low energy limit of open superstrings on a 3-brane in the critical dimension $10 = 1 + 3 + 6$.

\(^2\)This is apparently an old result [18]. We thank Igor Klebanov for bringing this paper and the related [19] on 26-dimensional Yang-Mills theory and bosonic strings to our attention as well as the recent discussion in [20] which is thought-provoking.
Standard bosonic 3-branes in the critical dimension $26 = 1 + 3 + 22$ are more complicated, partly due to tachyons [25], but the similarity of the counting suggests a role for a conformal gauge theory with $M = 22$ in bosonic 3-branes. Due to its symmetries an $O(22)$ symmetric conformal four-dimensional gauge theory would also suggest an $\text{AdS}_{4+1} \times S^{22-1}$ holographic interpretation\(^3\).

The paper is organised as follows. The overview in section 2 summarises the main ideas and results and also includes our discussion of them. Our conventions and the models to be studied are specified in section 3. One loop beta functions from a natural and specific example Lagrangian are studied concretely in section 4, explicitly demonstrating the need to complete the model to get the closed RG equations obtained in section 5, where large $N$ limits are also taken. Finally, the RG flow, the fixed point equations, the fixed flows and the fixed points are discussed in section 6.

## 2 Overview

The basic idea of the present work is to assume unbroken symmetry of the renormalised action of adjoint multi-scalar theories, and investigate the RG flow and fixed points in the symmetric sector. In classical scalar gauge theory the Lagrangian is a single trace operator, schematically

$$\mathcal{L}_{\text{st}} = N \text{Tr} \left[ F^2 + (D\Phi)^2 + f_1\Phi^4 \right],$$

where $F, \Phi, D$ are Yang-Mills field strength, scalar field and gauge covariant derivative, and we have rescaled the fields represented as $N \times N$ matrices as well as the couplings to facilitate a large $N$ limit. It has been stressed repeatedly, see for example [8] and [12], which inspired this paragraph, that double trace terms are of the same order at large $N$, and cannot be neglected in this limit. They lack an explicit factor of $N$ but there is instead an additional sum of $N$ terms in the second trace. The renormalisable terms with the same symmetries include also these double trace terms,

$$\mathcal{L}_{\text{dt}} = f_2 (\text{Tr} \Phi^2)^2,$$

but no higher terms. We explicitly generate double trace terms in our models in a calculation of the beta function in section 4. Hence, for renormalisability and large $N$ considerations,

$$\mathcal{L} = \mathcal{L}_{\text{st}} + \mathcal{L}_{\text{dt}}$$

is the complete Lagrangian, and $f_1$ and $f_2$ are fixed in the large $N$ limit. In the rest of the paper we will instead write such large $N$ couplings as standard couplings with explicit factors of $N$.

One loop beta functions of the scalar interactions receive three kinds of contributions: repeated four point interactions illustrated by figure 1, external leg corrections, and induced four point interactions. The latter two terms are universal independently of the details of

\(^3\)For relevant ideas in similar directions, see [26–28]. In particular, non-trivial fixed points of Yang-Mills theory with 22 adjoint scalars are mentioned in [26], although double trace interactions are not included.
the scalar four point interaction, and we first focus on the repeated four point interactions, which have an interesting structure.

The invariant terms in the scalar potential generally include several single trace terms and several double trace terms, leading to beta functions coupling their RG equations. Temporarily disregarding the gauge interactions, and assuming that the Lagrangian
\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\lambda = \mathcal{L}_0 + \lambda_n g^n(\Phi)
\] (2.1)
is closed under renormalisation, the RG equations will be governed by the beta functions
\[
\beta_k = \lambda m \lambda_n g^m_{\lambda n} + \mathcal{O}(\lambda^3).
\] (2.2)
Since there is one beta function for each coupling constant, the \( g^m_{\lambda n} \) symmetric coefficients encode the effects of the operators \( g^m(\Phi) \) and \( g^n(\Phi) \) on the presence of the operator \( g^k(\Phi) \) in the renormalised action.

Intuitively, these beta functions represent the effect on totally symmetric scalar interactions of once repeated totally symmetric scalar interactions, allowing the coefficients \( g^m_{\lambda n} \) to be extracted from pure algebra. Writing a totally symmetric quartic term in the potential as
\[
\lambda \bar{A} \bar{B} \bar{C} \bar{D} \phi \bar{A} \bar{B} \bar{C} \bar{D}
\]
the repeated interactions of figure 1 are proportional to symmetrised double contractions of rank four tensors,
\[
\lambda \bar{A} \bar{B} \bar{C} \bar{D} \lambda \bar{C} \bar{D} \bar{E} \bar{F}.
\] As described in section 5 the coefficients \( g^m_{\lambda n} \) are effectively structure constants of a commutative algebra, also in the technical sense of the word. In the large \( N \) limit the structure simplifies considerably and we find a hierarchy between single and double trace couplings. While we have worked out this structure only for a special example, it seems likely that it, and the simplifications in the large \( N \) limit, are more generally valid and useful.

2.1 The gauge coupling beta function, fixed flows and fixed points

When the gauge coupling dependent terms in the multi-scalar beta functions are included the one loop beta functions have a special property: the one-loop beta functions \( \beta_\lambda \) are homogeneous quadratic polynomials in \( \lambda \) and \( \alpha = g^2 \). These beta functions may then vanish for certain ratios of \( \lambda \) and \( \alpha \). Solutions with fixed gauge coupling, \( \beta_\alpha = 0 \), are then actual fixed points. If instead the gauge coupling is asymptotically free, the equations \( \beta_{\lambda/g^2} = 0 \) determine fixed flows with fixed ratios of coupling constants, on which \( \lambda \) are also asymptotically free. More generally, the space of all \( \lambda \) values is split into a totally asymptotically free region and its complement, where some \( \lambda \) will instead grow in the UV. Since the \( \lambda \) follow \( \alpha \) on fixed flows, the fixed flows lie in the totally asymptotically free region or on its boundary. In the body of the paper we find four real valued fixed flows or eight real valued fixed lines at one loop order, depending on the number of scalar fields.

Our work focuses on the scalar coupling, but at this point the running of the gauge coupling becomes important. For \( SU(N) \) gauge theory with \( M \) scalars in the representation \( R \), the one loop gauge coupling beta function is
\[
\beta_g = - \frac{g^3}{16\pi^2} \left\{ \frac{11}{3} C_2(G) - \frac{M}{6} \frac{d(R)}{r} C_2(R) \right\},
\] (2.3)
with \(d(R)\) the dimension of \(R\), \(C_2\) the quadratic Casimir, \(G\) the adjoint representation and \(r = d(G)\) the rank of the group. For adjoint scalars we get

\[
\beta_g = -\frac{g^3}{16\pi^2} \frac{22 - M}{6} C_2(G). \tag{2.4}
\]

Thus, the sign or vanishing of the gauge coupling beta function solely depends on the number of adjoint multiplets. The gauge coupling is asymptotically free for \(M < 22\), leading to the fixed flow structure described above. One loop scale invariance for \(M = 22\) is particularly intriguing in light of the relation of adjoint scalar gauge theories to D branes, AdS/CFT and string theory. With the spacetime dimension \(d = 4\) of the gauge theory, we have \(d + M = 26\), the critical dimension of bosonic string theory.

We have also found another special property of the \(M = 22\) model. For integers \(M \geq 22\) we find eight numerical solutions to the fixed points equations, while integers \(M < 22\) only yield four numerical roots. To the resolution we have checked it looks like there is a critical non-integer \(M\) between 21 and 22 where four roots coalesce two real double roots, just before these four roots all go complex.

2.2 The relation to D branes and AdS/CFT

A \((p+1)\)-dimensional \(SU(N)\) gauge theory with \((D-p-1)\) adjoint scalars can be regarded as the low energy limit of an open string theory defined on \(N\) Dp branes, if the fermion content is appropriate for the specific string theory. For example, maximally supersymmetric Yang-Mills theory in \(d\) dimensions has \((10 - d)\) adjoint scalars. In string theories with tachyons, like the standard bosonic string theory, this picture is only formal. Much more work is required to make sense of calculations in this framework and, indeed, D branes with open string tachyons have been studied extensively [25]. It is not clear to us how our result is related to earlier work on D branes with tachyons, but it suggests that it may be possible to decouple the open string tachyon and make sense of bosonic D3-branes in the critical dimension. Branes would then have the massless field content of adjoint 22-scalar Yang-Mills theory. A more direct use of the adjoint scalar gauge theories would be in a non-supersymmetric AdS/CFT.

The standard AdS/CFT example arose from maximally supersymmetric D3 branes [29–31], but has since got a life of its own. It is plausible that large \(N\) conformal field theories more generally can be interpreted as a weakly coupled gravity theory. In the large \(\'t\) Hooft coupling limit they may even approach classical gravity coupled to standard field theories. One-loop perturbative fixed points for \(M = 22\) should correspond to small \(\'t\) Hooft coupling examples of bosonic, and thus non-supersymmetric, AdS_5/CFT_4. The model we studied is like a naive bosonic D3 brane theory without tachyons, but the AdS/CFT framework can be considered without the brane interpretation. For this scenario to work, at least one fixed point has to survive at three loop order, and higher loops only perturb the fixed point.

2.3 Comparison to orbifold constructions

Orbifold models [32] were proposed as promising models for non-supersymmetric AdS/CFT but were found to have tachyon instabilities in the bulk and corresponding problems with
breaking of conformal invariance in the boundary [7–9]. No models have been found with fixed lines through the origin. These models share many features with our model: double trace operators are generally induced, and one-loop fixed point equations form a system of second order polynomial equations depending on the gauge coupling. Thus, one might have expected similar results in our case. However, the one loop RG equations for the scalar couplings yield several real fixed points, which are actually fixed lines through the origin if two loops terms and higher in $\beta_g$ are ignored.

One notable distinction between the models is that technically orbifold models are special descendants of the very special $\mathcal{N} = 4$ super-Yang-Mills theory and give results to all orders in the single trace gauge coupling, while our present result is only a one-loop result in the gauge coupling. So called inheritance arguments [33] relate the running of single trace couplings in orbifold models to the gauge couplings in maximally supersymmetric Yang-Mills theory, which do not run. It is logically possible that the real fixed points are one loop accidents in the bosonic model, but one would generally expect less symmetric models to reveal its properties at low orders, rather than requiring higher order calculations. Alternatively, the different ways of completing the set of couplings to close under the RG flow may be significant. The standard orbifold model calculations make the minimal assumption which permits inheritance arguments: excluding all new single trace couplings and double trace couplings built from single trace operators present in the original un-orbifolded theory, while our calculation allow all couplings consistent with the continuous symmetries. It would be interesting to relax the assumptions in orbifold theories.

3 Adjoint multi-scalar gauge theory

In this section we introduce our models of choice. Since we are ultimately interested in the UV definition of these renormalisable $d = 4$ theories we consider only quartic interactions between scalars. The gauged versions of these multi-scalar theories are obtained in the standard way by substituting ordinary derivatives in the kinetic terms of the action with gauge covariant derivatives and adding the Yang-Mills action term for the non-Abelian vector fields.

We consider $SU(N)$ Yang-Mills theory coupled to $M$ adjoint scalars $\phi_{aA}$ labeled by $a = 1, \ldots, M$, where the adjoint index $A$ runs over $A = 1, \ldots, N^2 - 1$. We will also use the symbolic multi-indices $\bar{A} \equiv aA$ and $\phi_{\bar{A}} \equiv \phi_{aA}$. A dimensionless four-point coupling between scalars produce a quartic potential term in the Lagrangian,

$$\lambda_{\bar{A}\bar{B}\bar{C}\bar{D}} \phi_{\bar{A}} \phi_{\bar{B}} \phi_{\bar{C}} \phi_{\bar{D}} = \lambda_{aAbBcCdD} \phi_a \phi_b \phi_c \phi_d .$$

We assume this coupling between $M$ $SU(N)$ adjoint representation scalars to be invariant also under a global $O(M)$ symmetry. Furthermore, it is totally symmetric in permutations of the four multi-indices.

The space of invariants can be found by representing the scalars as $M N \times N$ matrix valued fields $\Phi_a \equiv \phi_{aA} T_A$, where $T_A$ represents the Lie algebra generators and the coefficients $\phi_{aA}$ are real valued\footnote{We note that the present theory is closely related to multi-matrix theories of Hermitean matrices.}. Then, four independent quartic invariants can be formed by...
contracting indices and tracing over products of matrices:

\[ g^{2s}(\Phi) = \text{Tr} \Phi_a \Phi_a \text{Tr} \Phi_b \Phi_b \]
\[ g^{2m}(\Phi) = \text{Tr} \Phi_a \Phi_b \text{Tr} \Phi_a \Phi_b \]
\[ g^{1s}(\Phi) = \text{Tr} \Phi_a \Phi_a \Phi_b \Phi_b \]
\[ g^{1m}(\Phi) = \text{Tr} \Phi_a \Phi_b \Phi_a \Phi_b \]

The 1 or 2 label of the polynomials \( g(\Phi) \) refer to whether the operator is single or double trace, respectively, while \( s \) indicates that the operator is built from the global singlet \( \Phi_a \Phi_a \), and \( m \) indicates that the operator is built from \( \Phi_a \Phi_b \) which is a matrix in global indices. The expressions in terms of \( \phi_{aA} \) components will be given below.

We will first investigate a natural \( SU(N) \otimes O(M) \) invariant potential which is obtained by noticing that scalar multiplets in the adjoint representation are very much like components of the vector potential, except for the gauge transformation properties. The similarity is maximised by taking the form of the potential from the quartic term in the Yang-Mills action. The resulting potential is

\[ \lambda_{\text{YM}} \phi_A \phi_B \phi_C \phi_D = \lambda_{\text{YM}} \phi_a \phi_b \phi_c \phi_d 
= -\lambda \text{Tr} [\Phi_a, \Phi_b] [\Phi_a, \Phi_b] = -i^2 \lambda f_{AB}^C \phi_a \phi_b f_{A'B'}^C \phi_a \phi_b' 
= \lambda f_{AB}^E f_{CD}^E \phi_a \phi_b \phi_c \phi_d = \lambda \delta_{ac} \delta_{bd} f_{AB}^E f_{CD}^E \phi_a \phi_b \phi_c \phi_d, \]

which is also the potential of the scalars obtained by dimensional reduction of higher dimensional pure Yang-Mills theory. Hence,

\[ \lambda_{\text{YM}}^{\text{NM}} \phi_A \phi_B \phi_C \phi_D = \lambda \delta_{ac} \delta_{bd} f_{AB}^E f_{CD}^E \]

determines the four point coupling completely, although it is not totally symmetric under permutations of the four double indices. Of course, only the totally symmetric part contributes to the potential, since \( \phi_a \phi_b \phi_c \phi_d \) is totally symmetric. The totally symmetric four-point coupling is obtained by symmetrisation over all permutations \( s \) of double indices,

\[ \lambda_{\text{YM}}^{\text{NM}} \phi_A \phi_B \phi_C \phi_D = \frac{1}{24} \sum_{s(\bar{A} \bar{B} \bar{C} \bar{D})} \lambda_{\text{YM}}^{\text{NM}} \phi_{\bar{a} \bar{b} \bar{c} \bar{d}} = \frac{\lambda}{6} S_{6, \bar{a}, \bar{b}, \bar{c}, \bar{d}} [\delta_{ac} \delta_{bd} f_{AB}^E f_{CD}^E], \]

where we have defined a symmetriser \( S_{n, \bar{A} \bar{B} \bar{C} \bar{D}} \) that acts on a four-index tensor \( T_{\bar{A} \bar{B} \bar{C} \bar{D}} \) to make a fully symmetric four-index tensor by adding the minimal number of terms necessary. \( n \) is the total number of terms in the resulting tensor. We can treat each pair of indices \( \bar{A} = aA \) as one multi-index here because the symmetriser acts identically on \( a \) and \( A \). The terminology is the same as the one used in [34].\(^5\) The rank four tensor \( \lambda^{YM} \) is invariant under a 4 element subgroup of permutations simultaneously swapping \( (\bar{A}, \bar{B}) \leftrightarrow (\bar{C}, \bar{D}) \) or \( (\bar{A}, \bar{C}) \leftrightarrow (\bar{B}, \bar{D}) \). The remaining transformations map \( \lambda^{YM} \) into an orbit of 6 = 24/4 different terms.

\(^5\)For example, when symmetrising \( \delta_{ac} \delta_{bd} \lambda_{\bar{A} \bar{B} \bar{C} \bar{D}} \) with \( S_{6, \bar{A} \bar{B} \bar{C} \bar{D}} \) we add one copy of each of the five other terms in the orbit under permutations, each with a unit coefficient.
The following form effectuates the representation as an orbit.

\[ \lambda_{aAbBcCdD} = \frac{\lambda}{6} S_{aAbBcCdD} \left[ \delta_{ac} \delta_{bd} f^E_{AB} f^F_{CD} \right] \]  

\[ = \frac{\lambda}{6} \left[ \delta_{ac} \delta_{bd} f^E_{AB} f^E_{CD} + \delta_{bc} \delta_{ad} f^E_{BA} f^E_{CD} + \delta_{ca} \delta_{bd} f^E_{CB} f^E_{AD} \right] + \delta_{dc} \delta_{ba} f^E_{DB} f^E_{CA} + \delta_{cd} \delta_{ab} f^E_{DA} f^E_{CB} \]  

(3.6)  

We may verify the symmetry under all transpositions, which implies the total symmetry under all permutations.

4 A one-loop quartic potential and beta functions

As an important example, we now describe explicitly the calculation of one loop beta functions for multi-scalar four-point couplings of the Yang-Mills type described in the previous section 3. We first state the general one loop results from the literature, and then specialise to the calculation of a specific four-point coupling, with the required group theory and index manipulations, also illustrating it in terms of diagrams. The description of the calculation is intentionally technical, to introduce tensor structures that are helpful in the complete calculation in the next section 5. The result demonstrates that the Yang-Mills type scalar coupling is insufficient. It is not stable under renormalisation: In general the scalar coupling runs, but more drastically a double trace term as well as a new single trace term is generated.

In a theory with tree level scalar potential of Yang-Mills type \( \lambda \text{Tr} [\Phi_a, \Phi_b] [\Phi_b, \Phi_a] \), see eq. (3.3), the one-loop beta function for the quartic scalar coupling can be taken from general results \([35, 36]\) for scalar four-point couplings \( \lambda_{aAbBcCdD} = \lambda_{aAbBcCdD} \), here expressed explicitly with one global and one gauge index for each external leg. It is

\[ \beta_{aAbBcCdD} = \beta_{aAbBcCdD} = \frac{1}{(4\pi)^2} \left( \Lambda^2_{aAbBcCdD} - 3g^2\Lambda^S_{aAbBcCdD} + 3g^4A_{aAbBcCdD} \right), \]  

(4.1)  

where

\[ \Lambda^2_{aAbBcCdD} = \frac{1}{8} \sum_{\text{perms}} \lambda_{aAbBcCdD} \]  

\[ \Lambda^S_{aAbBcCdD} = \sum_i C_2(i) \lambda_{aAbBcCdD} \]  

\[ A_{aAbBcCdD} = \frac{1}{8} \sum_{\text{perms}} \{ \theta^I, \theta^I \}_{aAbB} \{ \theta^I, \theta^I \}_{cCdD}. \]  

(4.2)  

\( C_2(i) \) is the quadratic Casimir for external leg \( i, i = aA, bB, cC, dD, \) \([35]\) and \( g \) is the gauge coupling. That \( \theta^C_{aAbB} = if^C_{AB} \delta_{ab} \), where \( f^C_{AB} \) are the structure constants of the gauge group, means that \( \theta^C_{aAbB} \) represent the gauge group on \( M \) independent adjoint scalars (labeled by \( a \)). The first term \( \Lambda^2 \) in eq. \((4.1)\) represents the repeated scalar interactions in figure 1 and the second one \( g^2\Lambda^S \) describes self-interactions via gauge bosons on the external legs of the four-point coupling, figure 2a. The last term \( g^4A \) represents the induced scalar four-vertex from box and crossed box exchanges of two gauge bosons, see figures 2e and 2f.
Figure 1: The three permutations of repeated four-point interactions contributing to $\Lambda^2$. Each interaction is depicted as an empty dot. The crossing lines in the last diagram do not represent an interaction.

4.1 Calculation for the specific YM type tree level coupling

The quartic scalar coupling we want to study is single-trace, totally symmetric and can be derived from a $\lambda \text{Tr} [\Phi_a, \Phi_b] [\Phi_b, \Phi_a]$ term in the action, see eq. (3.3). Explicitly, the symmetric tensor structure coupling four scalars is taken from eq. (3.5) and (3.6) and reads

$$
\lambda_{aAbCcdD}^Y = \frac{\lambda}{6} \left[ \delta_{ac}\delta_{bd} f_{AB}^E f_{CD}^E + \delta_{bc}\delta_{ad} f_{BA}^E f_{CD}^E + \delta_{ca}\delta_{bd} f_{CB}^E f_{AD}^E \\
+ \delta_{dc}\delta_{ba} f_{DB}^E f_{CA}^E + \delta_{cb}\delta_{ad} f_{CA}^E f_{BD}^E + \delta_{dc}\delta_{ab} f_{DA}^E f_{CB}^E \right] \tag{4.3}
$$

To compute the one-loop beta function for the quartic coupling $\lambda_{aAbCcdD}^Y$ we have used birdtrack techniques, as established in [37, 38] and reviewed in [39]. We are interested in the large $N$ limit, and study rank four tensors, with the indices $\bar{A} \bar{B} \bar{C} \bar{D}$. To ensure that we know when two tensors are equal or different, it is helpful to write all tensors in a basis of linearly independent tensors. For colour structures, there exists a trace basis for tensors with four adjoint indices, formed from traces of products of generators $T_A, T_B, T_C$ and $T_D$ in the fundamental representation,

$$
\begin{align*}
\text{Tr} T_A T_B T_C T_D & \quad \text{Tr} T_A T_C T_D T_B & \quad \text{Tr} T_A T_D T_B T_C \\
\text{Tr} T_A T_B T_C T_D & \quad \text{Tr} T_A T_C T_D T_B & \quad \text{Tr} T_A T_D T_B T_C \\
\text{Tr} T_A T_D T_C T_B & \quad \text{Tr} T_A T_B T_D T_C & \quad \text{Tr} T_A T_C T_B T_D \tag{4.4}
\end{align*}
$$

which however in general is over-complete. To display its origin we have written the first row in terms of traces, but it can be simplified by using our normalisation $\text{Tr} T_A T_B = \delta_{AB}$. Fortunately, it has been shown that for $N \geq 4$ the trace basis elements become linearly independent and form a proper basis. Although this basis is not orthonormal, it is thus practical for large $N$, also because there is a straightforward algorithm for converting expressions to the trace basis. See [39] for a discussion in birdtrack language.
The tree level coupling $\lambda_{YM}$ can be expressed in the trace basis by using the identity $[T_A, T_B] = i f^{E}_{AB} T_E$:

$$
\lambda^\text{YM}_{aAbBcCdD} = -\frac{\lambda}{6} \left[ (\text{Tr} T_A T_B T_C T_D + \text{Tr} T_A T_D T_C T_B)(2\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}) + (\text{Tr} T_A T_B T_D T_C + \text{Tr} T_A T_C T_D T_B)(2\delta_{ad}\delta_{bc} - \delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd}) + (\text{Tr} T_A T_C T_D T_B + \text{Tr} T_A T_D T_B T_C)(2\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}) \right].
$$

At the one-loop level we need to apply more complicated identities and therefore it is useful to convert from structure constant expressions to the trace basis using the birdtrack formalism. In our calculation we found this formalism, which is simply a way to represent contractions of indices, convenient to use, but it is not necessary. The same calculation can be done using the regular index identities directly.

---

**Figure 2**: One-loop colour structure diagrams. The structure constant is denoted by a filled dot and four-point interactions by an empty dot. Note that in (f) the crossed lines do not denote an interaction.
In short, the birdtrack formalism is a diagrammatic notation used to represent index notation. In a birdtrack diagram an index is represented by an external line. Vertices represent the object with indices, for example a vector, and internal lines, i.e., contractions, represent Kronecker deltas. This notation enables us to represent complicated index identities in diagram form. In practice, we turn index notation expressions for $SU(N)$ and its adjoint representation into birdtrack diagrams, manipulate them, and interpret the diagrams in the trace basis. The tree level coupling $\lambda_{YM}$, see eq. (4.3), contains terms with the colour structure given by for example $f^{E}_{AB}f^{E}_{CD}$, whose diagram is illustrated in figure 3.

Regarding the one-loop beta function, to start with we have $\Lambda_{S}aAbBcCdD$, see eq. (4.2), whose diagrams are of the form seen in figure 2a. Each external leg has a loop corresponding to the quadratic Casimir $C_{2}(i)$, which for the adjoint representation in $SU(N)$ is $C_{2}(i) = 2N$. The sum is over the four external legs and therefore

$$\Lambda_{S}^{S}aAbBcCdD = 8N\lambda_{aAbBcCdD}. \quad (4.6)$$

Next, we look at the terms $\Lambda_{aAbBcCdD}^{2}$ and $A_{aAbBcCdD}$, see eq. (4.2). We expand them using eq. (4.3) and the identity $\theta^{C}_{aAbB} = i f^{C}_{AB} \delta_{ab}$. The expressions we obtain have terms of the form

$$\delta_{ac}\delta_{bd} f_{AB}^{G} f_{EF}^{H} f_{CD}^{I}. \quad (4.7)$$

Each term, after simplifying, contains two Kronecker deltas and four structure constants. Terms in $\Lambda_{aAbBcCdD}^{2}$ that contain sums over the $O(M)$ indices produce factors of $M$, for example $\delta_{ab}\delta_{ef}\delta_{cd} = M\delta_{ab}\delta_{cd}$. The colour structure of eq. (4.7) is shown in figure 2b.

These five diagrams were manipulated as birdtrack diagrams and interpreted in the trace basis. Diagrams 2b, 2c and 2d all reduce to figure 3, with different factors of $N$, which in birdtracks is given by figure 4(f) in [38]. Diagram 2e is shown as birdtrack diagram in chapter 2 of [37] and diagram 2f can be written as a linear combination of diagrams 2d and 2e. In effect, the calculation produces the expressions in the trace basis which we explain in the next section.

$^{6}$The normalisation factor is $T_{R} = 1$, $\text{Tr}(T_{A}T_{B}) = \delta_{AB}T_{R}$. 

---

Figure 3: Tree level colour structure diagram. The Lie algebra structure constant is represented by a filled dot and the adjoint representation by lines.
4.2 Result for the specific YM type tree level coupling

When combining all the terms, the one-loop beta functions are

\[ 16\pi^2 \beta_{ABCD} = S_{3,ABCD} \left[ \left( \frac{3g^4 + \frac{\lambda^2}{36}(M-1)}{8} \right) \delta_{ab}\delta_{cd} \left( \hat{\lambda}_{2s,ABCD} + \hat{\lambda}_{2m,ABCD} \right) \right. \]

\[ + \left( \frac{\lambda^2 N(M + 3)}{144} - 4g^2 N\lambda + \frac{3}{4}g^4 N \right) \delta_{ab}\delta_{cd} \hat{\lambda}_{1s,ABCD} \]

\[ - \left( \frac{\lambda^2 N}{18} - 8g^2 N\lambda \right) \delta_{ab}\delta_{cd} \hat{\lambda}_{1m,ABCD} \] (4.8)

in terms of double trace and single trace colour structures

\[ \hat{\lambda}_{2s,ABCD} = \delta_{AB}\delta_{CD} = \text{Tr} T_A T_B \text{Tr} T_C T_D, \]

\[ \hat{\lambda}_{2m,ABCD} = \delta_{AD}\delta_{BC} + \delta_{AC}\delta_{BD} = \text{Tr} T_A T_D \text{Tr} T_B T_C + \text{Tr} T_A T_C \text{Tr} T_B T_D, \]

\[ \hat{\lambda}_{1s,ABCD} = \text{Tr} T_A T_B T_C T_D + \text{Tr} T_A T_D T_C T_B + \text{Tr} T_A T_B T_D + \text{Tr} T_A T_C T_D T_B, \] (4.9)

\[ \hat{\lambda}_{1m,ABCD} = \text{Tr} T_A T_C T_B T_D + \text{Tr} T_A T_D T_C T_B, \]

corresponding to the invariant polynomials in (3.2) by symmetrisation after adding \( O(M) \) factors.

The beta functions (4.8) above inform us about the running of all possible scalar couplings compatible with the \( SU(N) \) and \( O(M) \) symmetries. The special \( \lambda_{YM}^{YM} \) coupling could have had special quantum properties protecting it, but our result shows that this is not the case. First, note that the tree level coupling \( \lambda_{YM}^{YM} \) coupling is proportional to the single-trace coupling \( \hat{\lambda}_{1s,ABCD} - 2\hat{\lambda}_{1m,ABCD} \),

\[ \lambda_{YM}^{YM} = \frac{\lambda}{6} S_{3,ABCD} \left[ \delta_{ab}\delta_{cd}(\hat{\lambda}_{1s,ABCD} - 2\hat{\lambda}_{1m,ABCD}) \right] . \] (4.10)

To compare the RG dependence of the couplings with the tree level coupling, the beta function in eq. (4.8) is then written as

\[ 16\pi^2 \beta_{ABCD} = S_{3,ABCD} \left[ \left( \frac{\lambda^2 N(M + 7)}{288} - 4g^2 N\lambda + \frac{3}{8}g^4 N \right) \delta_{ab}\delta_{cd} (\hat{\lambda}_{1s,ABCD} - 2\hat{\lambda}_{1m,ABCD}) \right. \]

\[ + \left( \frac{\lambda^2 N(M - 1)}{288} + \frac{3}{8}g^4 N \right) \delta_{ab}\delta_{cd} (\hat{\lambda}_{1s,ABCD} + 2\hat{\lambda}_{1m,ABCD}) \]

\[ + \left( 3g^4 + \frac{\lambda^2}{36}(M - 1) \right) \delta_{ab}\delta_{cd} \left( \hat{\lambda}_{2s,ABCD} + \hat{\lambda}_{2m,ABCD} \right) \] (4.11)

We observe that the tree level coupling will in general run, and that two new terms have been induced, one single-trace and one double-trace term. This means that the one-loop analysis has to be redone with a larger set of couplings introduced at tree level, closed in the sense that the resulting beta functions do not signal that new couplings are induced by the RG flow.
We end this section by rewriting the beta function (4.8) in yet another way, to prepare for the general one loop analysis, by introducing the completely symmetric global-colour tensors

\[
L^{2s} = S_3, AB, CD [\delta_{ab} \delta_{cd} \text{Tr} T_AT_B \text{Tr} T_CT_D]
\]

\[
L^{2m} = S_3, AB, CD [\delta_{ab} \delta_{cd} (\text{Tr} T_AT_C \text{Tr} T_BT_D + \text{Tr} T_AT_D \text{Tr} T_BT_C)]
\]

\[
L^{1s} = S_3, AB, CD [\delta_{ab} \delta_{cd} (\text{Tr} T_AT_BT_CT_D + \text{Tr} T_AT_BT_CT_B + \text{Tr} T_AT_BT_CT_D + \text{Tr} T_AT_BT_CT_B)]
\]

\[
L^{1m} = S_3, AB, CD [\delta_{ab} \delta_{cd} (\text{Tr} T_AT_CT_BT_D + \text{Tr} T_AT_BT_BT_C)],
\]

which encode the full tensor structures of the corresponding four invariant operators \(g^{1/2s/m}(\Phi)\) listed in eq. (3.2). For the present case, we find

\[
16\pi^2 \beta_{\bar{A}, \bar{B}, \bar{C}, \bar{D}} = \left(3g^4 + \frac{\lambda^2}{36}(M - 1)\right) (L^{2s} + L^{2m})
\]

\[
+ \left(\frac{\lambda^2 N(M + 3)}{144} - 4g^2 N\lambda + \frac{3}{4}g^4 N\right) L^{1s}
\]

\[
- \left(\frac{\lambda^2 N}{18} - 8g^2 N\lambda\right) L^{1m},
\]

which encodes the one loop corrections from a \(\lambda Y^M\) scalar coupling, but which has to be completed, since couplings which actually run were assumed to vanish.

5 The complete one-loop scalar potential and beta functions

5.1 Completion of one-loop scalar couplings and products of tensor structures

New couplings may be induced from loop corrections. For the theory to be renormalisable any term that is induced by a divergence in a loop calculation should be included in the action. The resulting beta function equations for the quartic couplings are then closed.

We include the most general couplings consistent with the assumption of unbroken symmetries. This means that higher loop corrections will typically deform the beta functions but cannot change the number and structure of the RG equations. Asking that the couplings are symmetric under both \(SU(N)\) and \(O(M)\) leads to the four possible quartic interaction terms, \(g^n(\Phi)\) of eq. (3.2).

Now, we can explain the structures we use to organise one-loop renormalisation. The Lagrangian

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\lambda = \mathcal{L}_0 + \lambda_n g^n(\Phi)
\]

is closed under renormalisation: no other couplings are needed in the RG equations, which will be governed by the beta functions

\[
\beta_k = \lambda_m \lambda_n g^{mn}_k + \mathcal{O}(\lambda^3).
\]
It is natural to describe the one loop beta functions in terms of a product of tensor structures $G^m$ defined by contractions precisely matching those in the beta function, cf (4.2) and figure 1. The coefficients $g^m_{kn}$ are then structure constants of a commutative algebra with product

$$G^m \circ G^n = cg^m_k G^k$$

in terms of the basis elements $G^m$. Note, however, that the product is in general not associative. Thus, using a basis $G^m$ of tensor structures, such that the quartic polynomials $g^m(\Phi) = G^m(\Phi, \Phi, \Phi, \Phi)$, the product encodes the $g = 0$ beta functions

$$\frac{d\lambda_k}{dt} G^k = \beta_k G^k = \lambda_m \lambda_n g^m_{kn} G^k = \frac{1}{c} \lambda_m \lambda_n G^m \circ G^n ,$$

where the constant $c$ can be fixed by checking one of the beta functions. To summarise, the $\circ$ product gives the symmetrised rank four tensor resulting from contracting two indices in a product of two symmetric rank four tensors.

### 5.2 Four point tensor structures at large $N$

The large $N$ limit is important in gauge theory dynamics and also establishes relations to string theory and gravity, for instance via the AdS/CFT correspondence. We are therefore interested in the fate of fixed points and the RG flow in the large $N$ limit. As usual, large $N$ perturbation theory is then best formulated in terms of $1/N$ expansion with terms that are functions of one 't Hooft coupling for each coupling in standard perturbation theory.

The one-loop beta function, eq. (4.13), contains four independent tensor structures (4.12) in double indices $A = aA$, which we normalise at large $N$ to

$$G^{2s} \equiv \frac{1}{N^2} S_{3,ABCD}[\delta_{ab}\delta_{cd}\delta_{ABCD}] = \frac{1}{N^2} S_{3,\overline{A}\overline{B}\overline{C}\overline{D}}[\delta_{ab}\delta_{cd}\text{Tr}T_A T_B \text{Tr}T_C T_D]$$

$$G^{2m} \equiv \frac{1}{N^2} S_{3,\overline{A}\overline{B}\overline{C}\overline{D}}[\delta_{ab}\delta_{cd}(\delta_{AC}\delta_{BD} + \delta_{AD}\delta_{BC})]$$

$$= \frac{1}{N^2} S_{3,\overline{A}\overline{B}\overline{C}\overline{D}}[\delta_{ab}\delta_{cd}(\text{Tr}T_A T_C\text{Tr}T_B T_D + \text{Tr}T_A T_D\text{Tr}T_B T_C)]$$

$$G^{1s} \equiv \frac{1}{2N} S_{4,\overline{A}\overline{B}\overline{C}\overline{D}}[\delta_{ab}\delta_{cd}(\text{Tr}T_A T_B T_C T_D + \text{Tr}T_A T_D T_C T_B + \text{Tr}T_A T_B T_D T_C + \text{Tr}T_A T_C T_D T_B)]$$

$$G^{1m} \equiv \frac{1}{2N} S_{4,\overline{A}\overline{B}\overline{C}\overline{D}}[\delta_{ab}\delta_{cd}(\text{Tr}T_A T_C T_D T_B + \text{Tr}T_A T_D T_B T_C)].$$

The scaling with $N$ ensures that the large $N$ limit of the algebra of the $G^a$ is independent of $N$, so that any remaining explicit $N$ dependences combine with the standard coupling constants to large $N$ 't Hooft couplings. This means that the structure constants $g^k_{mn}$ in the algebra (5.3) and the beta functions (5.4) are $N$ independent when the $\lambda_k$ and the corresponding $\beta_k$ are replaced by the correctly $N$-scaled coupling constants. We will find the scalings below in section 5.4.

### 5.3 Products of tensor structures and one-loop beta functions at large $N$

In the large $N$ limit the sets $\{G^{2s}\}$ and $\{G^{2s}, G^{2m}\}$ each span a closed sub-algebra, ie their product does not generate an element outside of the set, and they each form an ideal. For a
potential in an ideal subspace, no perturbation in the algebra will take the RG flow outside the ideal. In the case of \( \{G^{2s}\} \), the product \( G^{2s} \circ G^n \propto G^{2s} \) for all \( n \). For the set \( \{G^{2s}, G^{2m}\} \) the products are\(^7\)

\[
\begin{align*}
G^{2s} \circ G^{2s} &= 2M G^{2s} \\
G^{2m} \circ G^{2m} &= 4G^{2m} \\
G^{2s} \circ G^{2m} &= 4G^{2s}.
\end{align*}
\]

The set \( \{G^{2s}, G^{2m}, G^{1s}\} \) is also a closed sub-algebra at large \( N \), but it is not an ideal since \( \{G^{1m}, G^{1s}\} \) generates a contribution to \( G^{1m} \),

\[
\begin{align*}
G^{1s} \circ G^{1s} &= 4(M + 3)G^{2s} + 4G^{2m} + 2(M + 3)G^{1s} \\
G^{2s} \circ G^{1s} &= 4(M + 1)G^{2s} \\
G^{2m} \circ G^{1s} &= 8G^{2s} + 4G^{2m} \\
G^{1m} \circ G^{1m} &= (M + 2)G^{2m} + 2G^{1s} \\
G^{1m} \circ G^{1s} &= 4G^{2s} + 4G^{2m} + 2G^{1s} + 4G^{1m} \\
G^{2s} \circ G^{1m} &= 4G^{2s} \\
G^{2m} \circ G^{1m} &= 4G^{2m}.
\end{align*}
\]

Temporarily just assuming that the \( \lambda_n \) and the \( \beta_n \) refer to the proper 't Hooft couplings, the one loop the beta functions (5.4) can be read off from the products above:

\[
\begin{align*}
c\beta_{2s} &= 2M \lambda_{2s}^2 + 8\lambda_{2s} \lambda_{2m} + 8\lambda_{2s} \lambda_{1m} + 8(M + 1)\lambda_{2s} \lambda_{1s} + 16\lambda_{2m} \lambda_{1s} + 8\lambda_{1m} \lambda_{1s} + 4(M + 3)\lambda_{1s}^2 \\
c\beta_{2m} &= 4\lambda_{2m}^2 + 8\lambda_{2m} \lambda_{1m} + 8\lambda_{2m} \lambda_{1s} + (M + 2)\lambda_{1m}^2 + 8\lambda_{1m} \lambda_{1s} + 4\lambda_{1s}^2 \\
c\beta_{1s} &= 2\lambda_{1m}^2 + 4\lambda_{1m} \lambda_{1s} + 2(M + 3)\lambda_{1s}^2 \\
c\beta_{1m} &= 8\lambda_{1m} \lambda_{1s}.
\end{align*}
\]

\[5.4\] One-loop beta functions for YM type multi-scalar theory

We can now test the general beta functions by specialising them to the particular case we studied in section 4.1. In the calculation of \( \beta_{ABCD} \), the Lagrangian was perturbed by \( \lambda_{ABCD}^{YM} \Phi_A \Phi_B \Phi_C \Phi_D \), eq. (4.5). In terms of \( G^n \),

\[
\lambda_{ABCD}^{YM} \equiv \frac{\lambda N}{3} (G^{1s} - 2G^{1m}) ,
\]

\[5.9\]

cf Eqs (4.10) and (5.5). Therefore, the \( \lambda^2 \)-contribution to the beta function of the coupling \( \lambda_{ABCD}^{YM} \equiv \lambda_m G^n \) should be proportional to the totally symmetric product

\[
(G^{1s} - 2G^{1m}) \circ (G^{1s} - 2G^{1m}) = 4(M - 1)(G^{2s} + G^{2m}) - 16G^{1m} + 2(M + 3)G^{1s} .
\]

We find

\[
\frac{1}{4} \lambda_{ABCD}^{YM} \circ \frac{1}{4} \lambda_{ABCD}^{YM} = \frac{\lambda^2 N^2}{36} (M - 1)(G^{2s} + G^{2m}) - \frac{\lambda^2 N^2}{9} G^{1m} + \frac{\lambda^2 N^2}{72} (M + 3)G^{1s} .
\]

\[^7\text{The equations (5.6) and (5.7) are all in the large } N \text{ limit.}\]
which implies that the previous result for the beta function, eq. (4.8), rewritten in terms of $G^2_s, G^2_m, G^1_m$ and $G^1_s$ and evaluated for $g = 0$, is summarised by the ♦ product

$$\beta_{YM} \big|_{g=0} = \frac{1}{\pi} \lambda^{YM} \ast \frac{1}{\pi} \lambda^{YM}. \quad (5.12)$$

We have thereby determined the constant $c = \pi^2$ in (5.4).

Similarly, the one loop beta function for general $g \neq 0$ can also be described in terms of the large $N$ normalised tensor structures. From eq. (4.13),

$$16\pi^2 \beta_{ABCD} = \left( \frac{1}{36} (M - 1)(\lambda N)^2 + 3(\lambda^2 N)^2 \right) (G^{2s} + G^{2m}) - \frac{1}{9} G^{1m}(\lambda N)^2$$

$$+ \frac{(M + 3)}{72} G^{1s}(\lambda N)^2 + 16(\lambda^2 N)G^{1m}(\lambda N) - 8(\lambda^2 N)G^{1s}(\lambda N) + \frac{3}{2} (\lambda^2 N)^2 G^{1s}, \quad (5.13)$$

at large $N$, explicitly demonstrating the appearance of the expected 't Hooft couplings. The beta function on the left hand side written as

$$\beta = N\beta_{1m}G^{1m} + N\beta_{1s}G^{1s} + N^2\beta_{1m}G^{1m} + N^2\beta_{2s}G^{2s} \quad (5.14)$$

exposes the large $N$ scaling also of double trace 't Hooft couplings, and thus gives closed RG equations in terms of $\lambda_{1m}N, \lambda_{1s}N, \lambda_{2m}N^2, \lambda_{2s}N^2$ and $g^2N$. Finally, we obtain the generalised 't Hooft coupling RG equations on the form

$$\pi^2 \beta_k G^k = G \ast G - a g^2 N G + (g^2 N)^2 G^{(ind)}, \quad (5.15)$$

in terms of $\beta = \beta_k G^k$, an arbitrary symmetric tensor structure $G = \lambda_k G^k$, where we have simply absorbed the appropriate $N$ factors in the couplings. The numerical constant $a$ comes from gluon loops on the external legs and the tensor structure $G^{(ind)}$ is universally induced from exchange of two gluons, as explained below.

6 Fixed point potentials of adjoint multi-scalar Yang-Mills theory

Gauging the $SU(N)$ symmetry and introducing corresponding non-abelian gauge fields leaves any multi-scalar potential unchanged at tree level. At one loop, in addition to the second order scalar coupling terms, diagrams with vectors yield new terms of the same kind as in section 4. Repeated exchange of gauge fields — $A$ in (4.1) — induce a $O(g^4)$ term. Gauge field loops on external legs again contribute with the same multiplicative factor for all tree level couplings $\lambda_i$ producing $O(g^2 \lambda_i)$ terms to the corresponding $\beta_{\lambda_i} - \Delta S$ in (4.1). From the these modifications of the beta functions we get the RG equations (5.15) which
decompose into the component equations

\[
\pi^2 \frac{d\lambda_{2s}}{dt} = 2M\lambda_{2s}^2 + 8\lambda_{2s}\lambda_{2m} + 8\lambda_{2s}\lambda_{1m} + 8(M + 1)\lambda_{2s}\lambda_{1s} + 16\lambda_{2m}\lambda_{1s} + 8\lambda_{1m}\lambda_{1s} + 4(M + 3)\lambda_{1s}^2 - ag^2N\lambda_{2s} + 2(g^2N)^2\mu^{(ind)}
\]

\[
\pi^2 \frac{d\lambda_{2m}}{dt} = 4\lambda_{2m}^2 + 8\lambda_{2m}\lambda_{1m} + 8\lambda_{2m}\lambda_{1s} + (M + 2)\lambda_{1m}^2 + 8\lambda_{1m}\lambda_{1s} + 4\lambda_{1s}^2 - ag^2N\lambda_{2m} + 2(g^2N)^2\mu^{(ind)}
\]

\[
\pi^2 \frac{d\lambda_{1m}}{dt} = 8\lambda_{1m}\lambda_{1s} - ag^2N\lambda_{1m}
\]

\[
\pi^2 \frac{d\lambda_{1s}}{dt} = 2\lambda_{1m}^2 + 4\lambda_{1m}\lambda_{1s} + 2(M + 3)\lambda_{1s}^2 - ag^2N\lambda_{1s} + (g^2N)^2\mu^{(ind)}.
\]

(6.1)

To determine the coefficients \( a \) and \( \mu^{(ind)} \) we set \( \lambda_{1m} = -2\lambda_{1s} = -2\lambda \) and the other \( \lambda_i = 0 \) as in the tree level action of section 4 and compare with eq. (4.8), implying that \( a = 1152 \) and \( \mu^{(ind)} = 216 \) independently of \( M \).

The large \( N \) RG equations also depend on the gauge coupling \( g \) and the integer \( M \) — the number of adjoint multiplets. The \( g \) dependence is however trivial, as can be seen from the \( g^2 \) independent equations for \( \mu_i \equiv \lambda_i/g^2 \):

\[
0 = 2M\mu_{2s}^2 + 8\mu_{2s}\mu_{2m} + 8\mu_{2s}\mu_{1m} + 8(M + 1)\mu_{2s}\mu_{1s} + 16\mu_{2m}\mu_{1s} + 8\mu_{1m}\mu_{1s} + 4(M + 3)\mu_{1s}^2 - (a - b)\mu_{2s} + 2\mu^{(ind)}
\]

\[
0 = 4\mu_{2m}^2 + 8\mu_{2m}\mu_{1m} + 8\mu_{2m}\mu_{1s} + (M + 2)\mu_{1m}^2 + 8\mu_{1m}\mu_{1s} + 4\mu_{1s}^2 - (a - b)\mu_{2m} + 2\mu^{(ind)}
\]

\[
0 = 8\mu_{1m}\mu_{1s} - (a - b)\mu_{1m}
\]

\[
0 = 2\mu_{1m}^2 + 4\mu_{1m}\mu_{1s} + 2(M + 3)\mu_{1s}^2 - (a - b)\mu_{1s} + \mu^{(ind)}.
\]

\( b \propto (22 - M) \)

(6.2)

Note that the replacement \( a \to a - b \) is the sole consequence for the fixed point equations of considering the relative couplings \( \mu_i = \frac{\lambda_i}{g^2} \) rather than the absolute couplings \( \lambda_i \). We now summarise the solutions to the fixed point equations at large \( N \). Specifically, at large \( N \) the equations have a special hierarchic structure that simplifies the solution of the equations. It seems related to the algebraic structures we described above. The third and the fourth equations only involve the single trace couplings, which can be determined first. The other two equations then determine the double trace coupling, first \( \mu_{2m} \) and then \( \mu_{2s} \).
Numerically, we find eight real fixed points for $M = 22$.

| Fixed point | $\mu_{1m}$ | $\mu_{1s}$ | $\mu_{2m}$ | $\mu_{2s}$ |
|-------------|------------|------------|------------|------------|
| $f_{+,---}$ | 0.228509  | 2.62932   | -36.1581   |            |
| $f_{+,-++}$ | 0.228509  | 2.62932   | -33.6966   |            |
| $f_{+,-+-}$ | 0.228509  | 239.669   | -57.7071   |            |
| $f_{+,-++}$ | 0.228509  | 239.669   | -55.2457   |            |
| $f_{-,---}$ | 0.189051  | 0.376109  | 0.398211   |            |
| $f_{-,--+}$ | 0.189051  | 0.376109  | 24.9246    |            |
| $f_{-,--}$  | 0.189051  | 287.246   | -25.6809   |            |
| $f_{-,+++}$ | 0.189051  | 287.246   | -1.15442   |            |

The eight fixed points $f$ have been labelled by UV/IR stable directions (-/+). One fact stands out clearly in this table. The real fixed points all have vanishing $\mu_{1m}$, ie they all involve a single trace potential of the form $g_{1s}(\Phi) = \text{Tr} \Phi_a \Phi_a \Phi_b \Phi_b$ which is quite different from a potential of form $\text{Tr} \Phi_a \Phi_a \Phi_b - \text{Tr} \Phi_a \Phi_b \text{Tr} \Phi_a \Phi_b$ corresponding to the dimensionally reduced Yang-Mills theory potential we studied in section 4.1. One might wonder if this could have been anticipated on general grounds, at least at large $N$\(^8\). Since the fixed point equations do not favour any particular coupling, the answer has to come from the structure of the large $N$ beta functions, and we do not have a complete explanation, but we note that a vanishing $\mu_{1m}$ is consistent with the closed sub-algebra of tensor structures mentioned just before eq. (5.7): At large $N$, the RG equations of the $\mu_{1s}$ with the two double trace coefficients close among themselves.

The fixed points move with $M$. Numerically, the first four real fixed points $f_+$, present for $M \geq 22$, disappear for integer $M < 22$. This is quite intriguing, because $M = 22$ is precisely where the gauge coupling vanishes at one-loop. Thus, for $M > 22$ the UV limit is problematic, while there are eight interesting fixed lines (with arbitrary $g^2$) in the one-loop approximation for $M = 22$, which at best is going to give eight isolated non-trivial fixed points at higher orders. At $f_--$ we would then have a completely UV stable (UV safe) non-trivial fixed point and a well defined UV complete theory. It remains to be seen if any of these points are under perturbative control.

For $M < 22$, the four $f_-$ fixed points turn into fixed flows, meaning that the ratios of scalar couplings to $g^2$ are fixed, but $g^2$ is asymptotically free. Formally there are Banks-Zaks non-trivial fixed points for all $5 < M < 22$, but the fixed points are only reliably perturbative for $M$ close to the upper end of the range. Note also that the Banks-Zaks fixed points associated to $f_-$ have at least two IR unstable directions, requiring (allowed) fine tuning for the definition of the corresponding theories, or leading generic IR flows to infinity, if the one-loop result is any guide. At $f_--$ we have a completely asymptotically free fixed point and a well defined UV complete theory. The other fixed points all have UV unstable directions, which means that only special trajectories reach them in the UV limit to define a UV theory with couplings corresponding to the fixed point.

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