Geometric Estimation of a Potential and Cone Conditions of a Body

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Abstract We investigate a potential obtained as the convolution of a radially symmetric function and the characteristic function of a body (the closure of a bonded open set) with exterior cones. In order to restrict the location of a maximizer of the potential into a smaller closed region contained in the interior of the body, we give an estimate of the potential using the exterior cones of the body. Moreover, we apply the result to the Poisson integral for the upper half-space.

Keywords Hot spot · Poisson integral · Solid angle · Illuminating center · Riesz potential · Hadamard finite part · Renormalization · \( r^{\alpha-m} \)-potential · Minimal unfolded region · Heart · Cone condition

Mathematics Subject Classification 31B25 · 35B38 · 35B50 · 51M16 · 52A40

1 Introduction

Let \( \Omega \) be a body (the closure of a bounded open set) in \( \mathbb{R}^m \). We consider a potential of the form

\[
K_\Omega(x, t) = \int_{\Omega} k(|x - \xi|, t) \, d\xi, \quad x \in \mathbb{R}^m, \ t > 0,
\]

and investigate its spatial maximizers.
When $k(r, t)$ is given by the Gauss kernel, the potential $K_{Ω}(x, t)$ is the solution of the Cauchy problem for the heat equation with initial datum $χ_{Ω}$,

$$W_{Ω}(x, t) = \frac{1}{(4\pi t)^{m/2}} \int_{Ω} \exp \left( -\frac{|x - ξ|^2}{4t} \right) dξ, \quad x \in \mathbb{R}^m, \ t > 0. \quad (1.2)$$

A spatial maximizer of $W_{Ω}$ is called a hot spot of $Ω$ at time $t$.

In [4], Chavel and Karp showed that $Ω$ has (at least) a hot spot for each $t$, that any hot spot belongs to the convex hull of $Ω$, and that the set of hot spots converges to the one-point set of the centroid (center of mass) of $Ω$ as $t$ goes to infinity with respect to the Hausdorff distance. Furthermore, calculating the Hessian of $W_{Ω}(\cdot, t) : \mathbb{R}^m \to \mathbb{R}$, in [9], Jimbo and Sakaguchi indicated that $Ω$ has a unique hot spot whenever $t \geq (\text{diam } Ω)^2/2$. Roughly speaking, the large-time behavior of hot spots was studied in [4, 9].

In contrast, in [10], Karp and Peyerimhoff gave geometric heat comparison criteria and investigated the small-time behavior of hot spots. Roughly speaking, they compared two heat flows for two points in two different bodies by using the distance functions from the complements and showed that any sequence of hot spots of $Ω$ at time $t_ℓ$ converges to an incenter of $Ω$ as $t_ℓ$ tends to zero. Let us review their exact statement as below: let $X$ and $Y$ be bodies in $\mathbb{R}^m$; fix two constants $R > S \geq 0$; let $X' = \{x \in X| \text{dist}(x, X^c) \geq R\}$, and $Y' = \{y \in \mathbb{R}^m| \text{dist}(y, Y^c) \leq S\}$; then, we can choose a small-time $τ$ such that if $0 < t < τ$, then, for any $x \in X'$ and $y \in Y'$, we have $W_X(x, t) > W_Y(y, t)$; taking $X = Y = Ω$ and $R = R_∞(Ω) = \text{max dist}(ξ, Ω^c)$ (the inradius), we can conclude that, for any decreasing sequence $\{t_ℓ\}$ with zero limiting value and any hot spot $h(t_ℓ)$ of $Ω$ at time $t_ℓ$, the distance between $h(t_ℓ)$ and the set of incenters $I_Ω = \{x \in Ω| \text{dist}(x, Ω^c) = R_∞(Ω)\}$ tends to zero as $ℓ$ goes to infinity. We also refer to [13, pp. 2–3] for the small-time behavior of hot spots.

On the other hand, in [21], for the kernel $k$ in (1.1), the author gave a sufficient condition implying the results shown in [4, 9]. As a by-product, for example, his sufficient condition can be applied to the Poisson integral for the upper half-space,

$$P_Ω(x, h) = \frac{2h}{σ_m(S^m)} \int_{Ω} \left( |x - ξ|^2 + h^2 \right)^{-(m+1)/2} dξ, \quad x \in \mathbb{R}^m, \ h > 0, \quad (1.3)$$

where $σ_m$ denotes the $m$-dimensional spherical Lebesgue measure and $S^m$ is the $m$-dimensional unit sphere in $\mathbb{R}^{m+1}$. Precisely, his sufficient condition implies that the function $P_Ω(\cdot, h) : \mathbb{R}^m \to \mathbb{R}$ has a maximizer for each $h$, that any maximizer of $P_Ω(\cdot, h)$ belongs to the convex hull of $Ω$, that the set of maximizers of $P_Ω(\cdot, h)$ converges to the one-point set of the centroid of $Ω$ as $h$ goes to infinity with respect to the Hausdorff distance, and that $P_Ω(\cdot, h)$ has a unique maximizer whenever $h \geq \sqrt{m+2}\text{diam } Ω$.

Here, we remark that the Poisson integral $P_Ω$ satisfies the Laplace equation for the upper half-space,
\[ \left( \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial h^2} \right) P_\Omega(x, h) = 0, \; x \in \mathbb{R}^m, \; h > 0, \]  \hspace{1cm} (1.4) 

and the fractional heat equation,

\[ \left( \frac{\partial}{\partial t} + \sqrt{-\Delta} \right) P_\Omega(x, t) = 0, \; x \in \mathbb{R}^m, \; t > 0, \]  \hspace{1cm} (1.5) 

We have the boundary condition

\[ \lim_{h \to 0^+} P_\Omega(x, h) = \chi_\Omega(x), \; x \in \mathbb{R}^m \setminus \partial \Omega. \]  \hspace{1cm} (1.6) 

In order to understand the geometric meaning of the author’s results on maximizers of \( P_\Omega(\cdot, h) \), let \( A_\Omega = \sigma_m(S^m) P_\Omega/2 \). The function \( A_\Omega \) is obtained in the following manner: let \( x \) be a point in \( \mathbb{R}^m \), and \( h \) a positive constant; define the map

\[ p_{(x,h)} : \Omega \times \{0\} \ni (\xi, 0) \mapsto \frac{(\xi, 0) - (x, h)}{|(\xi, 0) - (x, h)|} \in S^m; \]  \hspace{1cm} (1.7) 

the solid angle of \( \Omega \) at \( (x, h) \) is defined as the \( m \)-dimensional spherical Lebesgue measure of the image \( p_{(x,h)}(\Omega) \) (see Figure 1), and direct calculation shows

\[ \sigma_m \left( p_{(x,h)}(\Omega) \right) = h \int_{\Omega} \left( |x - \xi|^2 + h^2 \right)^{-(m+1)/2} d\xi = A_\Omega(x, h). \]  \hspace{1cm} (1.8) 

In [24], the solid angle of \( \Omega \) at \( (x, h) \) was regarded as the “brightness” of \( \Omega \) having a light source at \( (x, h) \). We call a maximizer of \( A_\Omega(\cdot, h) \) an illuminating center of \( \Omega \) of height \( h \). Thus, the properties of \( P_\Omega \) shown in [21] are understood as the large-height behavior of illuminating centers. In other words, it was shown that the large-parameter behavior of spatial maximizers of \( P_\Omega \) is similar to that of \( W_\Omega \).

**Fig. 1** The solid angle of \( \Omega \) at \((x, h)\)
From such backgrounds, in this paper, in order to compare small-parameter behavior of spatial maximizers of \( P_\Omega \) and \( W_\Omega \), we mainly investigate the small-height behavior of illuminating centers. Informal computations show

\[
\frac{A_\Omega(x, h)}{h} = \int_\Omega \left( |x - \xi|^2 + h^2 \right)^{-\frac{m+1}{2}} d\xi \to \int_\Omega |x - \xi|^{-\frac{m+1}{2}} d\xi \quad (1.9)
\]
as \( h \) tends to \( 0^+ \). But the right-hand side diverges whenever \( x \) is in \( \Omega \). Then, for a point \( x \) in the interior of \( \Omega \), let us consider its Hadamard finite part,

\[
V_\Omega^{(-1)}(x) = \lim_{\varepsilon \to 0^+} \left( \int_{\Omega \setminus B_\varepsilon(x)} |x - \xi|^{-(m+1)} \sigma_{m-1}\left(\frac{S^{m-1}}{\varepsilon}\right) \right) \quad (1.10)
\]
\[
= \int_{\Omega \setminus B_\varepsilon(x)} |x - \xi|^{-(m+1)} - \frac{\sigma_{m-1}\left(\frac{S^{m-1}}{\varepsilon}\right)}{\varepsilon}. \quad (1.11)
\]

Here, we remark that the latter equality (1.11) holds whenever \( 0 < \varepsilon < \text{dist}(x, \Omega^c) \) (see Proposition 2.7).

It is expected that any sequence of illuminating centers of height \( h_\ell \) converges to a maximizer of \( V_\Omega^{(-1)} \) as \( h_\ell \) tends to zero. This expectation comes from the following procedure: let \( \varepsilon > 0 \) be small enough; suppose that, for any small enough \( h > 0 \), any illuminating center is at least \( \varepsilon \) away from the boundary of \( \Omega \); since the Poisson kernel is radially symmetric, the solid angle of \( B_\varepsilon(x) \) at \( (x, h) \) depends only on \( \varepsilon \) and \( h \); decomposing the solid angle function as \( A_\Omega(x, h) = A_{\Omega \setminus B_\varepsilon(x)}(x, h) + A_{B_\varepsilon(x)}(x, h) \), a point \( c(h) \) is an illuminating center if and only if it is a maximizer of \( A_{\Omega \setminus B_\varepsilon(c)}(\cdot, h) \); as \( h \) tends to zero, the kernel \( (|x - \xi|^2 + h^2)^{-\frac{m+1}{2}} \) converges to \( |x - \xi|^{-(m+1)} \) uniformly for \( \xi \) in \( \Omega \setminus B_\varepsilon(x) \); roughly speaking, if the height parameter \( h \) is small enough, then we have

\[
\frac{A_\Omega(x, h)}{h} = \frac{A_{\Omega \setminus B_\varepsilon(x)}(x, h)}{h} + \frac{A_{B_\varepsilon(x)}(x, h)}{h} \approx V_\Omega^{(-1)}(x) + \frac{\sigma_{m-1}\left(\frac{S^{m-1}}{\varepsilon}\right)}{\varepsilon} + \frac{A_{B_\varepsilon(x)}(x, h)}{h} \quad (1.12)
\]

for any point \( x \) in the interior of \( \Omega \) with \( \text{dist}(x, \Omega^c) > \varepsilon \).

In order to formulate the above procedure, we have to give a closed subset in the interior of \( \Omega \) such that it contains all the illuminating centers for any small enough \( h > 0 \). This is because we can use the expression (1.11) of the potential \( V_\Omega^{(-1)} \) only in the interior of \( \Omega \). Namely, in (1.11), we want to take a uniform \( \varepsilon \) for illuminating centers of any small enough height and maximizers of \( V_\Omega^{(-1)} \).

We refer to [1,2,18,19,21,22] for the study of the location of maximizers of a potential. Some authors tried to restrict the location of maximizers of a potential into a smaller region. Using the moving plane argument ([5,23]), all the maximizers of a potential with a radially symmetric strictly decreasing kernel are contained in the minimal unfolded region of \( \Omega \). (The minimal unfolded region is sometimes called
the heart.) But, in general, the minimal unfolded region of \( \Omega \) is not contained in the interior of \( \Omega \) (see Example 2.21). The survey [12] is a good reference for this topic. We also refer to [20] for a recent development of this topic.

In this paper, assuming the uniform interior cone condition for the complement of the body \( \Omega \) (see Definition 2.1) and taking the following three steps, we formulate the above procedure:

**Step 1** We give a constant \( 0 < \tilde{R} < R_\infty(\Omega) \) such that, for any \( x \in \Omega \) with \( \text{dist}(x, \Omega^c) = R_\infty(\Omega) \) and \( y \in \Omega \) with \( \text{dist}(y, \Omega^c) \leq \tilde{R} \), we have \( V_\Omega^{(-1)}(y) < V_\Omega^{(-1)}(x) \). Namely, any maximizer of \( V_\Omega^{(-1)} \) belongs to the inner-parallel body of \( \Omega \) of radius \( \tilde{R} \).

**Step 2** For any constant \( 0 < b < 1 \), there exists a positive \( h_0 \) such that if \( 0 < h < h_0 \), then, for any \( x \in \Omega \) with \( \text{dist}(x, \Omega^c) = R_\infty(\Omega) \) and \( y \in \mathbb{R}^m \) with \( \text{dist}(y, \Omega^c) \leq b\tilde{R} \), we have \( A_\Omega(y, h) < A_\Omega(x, h) \). Namely, if \( h \) is sufficiently small, then any illuminating center belongs to the inner-parallel body of \( \Omega \) of radius \( b\tilde{R} \).

**Step 3** The limit point of any illuminating center of height \( h_\ell \) must be a maximizer of \( V_\Omega^{(-1)} \).

Moreover, the above argument can be extended to a general case. Precisely, we give the same estimate as in the first step of the Hadamard finite part of the Riesz potential,

\[
V_\Omega^{(\alpha)}(x) = \begin{cases} 
\lim_{\varepsilon \to 0^+} \left( \int_{\Omega \setminus B_\varepsilon(x)} |x - \xi|^{\alpha-m} d\xi - \sigma_{m-1} \left( \frac{S^{m-1}}{-\alpha} \right) \right) & (\alpha < 0), \\
\lim_{\varepsilon \to 0^+} \left( \int_{\Omega \setminus B_\varepsilon(x)} |x - \xi|^{-m} d\xi - \sigma_{m-1} \left( S^{m-1} \right) \log \frac{1}{\varepsilon} \right) & (\alpha = 0).
\end{cases}
\]

(1.13)

Also, we give the same estimate as in the second step of the potential of the form (1.1). In other words, our main result in this paper is the estimate of a potential like the second step, and, as its by-product, we derive the small-height behavior of illuminating centers.

Throughout this paper, \( \overset{\circ}{X}, \bar{X}, X^c, R_\infty(X) \), and \( \text{diam} X \) denote the interior, the closure, the complement, the inradius, and the diameter of a set \( X \) in \( \mathbb{R}^m \), respectively. For a set \( X \) in \( \mathbb{R}^m \) and a positive constant \( \rho \), the symbol \( X \sim \rho B^m \) denotes the inner-parallel body of \( X \) of radius \( \rho \), that is, \( X \sim \rho B^m = \{ x \in X \mid \text{dist}(x, X^c) \geq \rho \} \). For a set \( X \) in \( \mathbb{R}^m \), a positive constant \( \rho \), and a point \( p \) in \( \mathbb{R}^m \), we use the notation \( \rho X + p = \{ \rho x + p \mid x \in X \} \). In particular, we denote by \( B_\rho(x) = \rho B^m + x \) the \( m \)-dimensional closed ball of radius \( \rho \) and centered at \( x \). Let \( S^{m-1} \) be the boundary of \( B^m \) (the \( m-1 \)-dimensional unit sphere). We denote a point \( x \) in \( \mathbb{R}^m \) by \( X = (x_1, \ldots, x_m) \). The standard orthonormal frame of \( \mathbb{R}^m \) is \( \{ e_1, \ldots, e_m \} \). The \( N \)-dimensional spherical Lebesgue measure is denoted by \( \sigma_N \). In particular, the symbol \( \sigma \) is used in the case of \( N = m - 1 \), for short.
2 Preliminaries

2.1 Cone Conditions

Let us prepare the cone conditions which are related to the complexity of the boundary of a body. Throughout this paper, we understand that $C$ is an open cone of vertex $x$, axis direction $v$, aperture angle $\kappa$, and height $\delta$ if $C$ is given as

$$C = \left\{ x + \rho Rv \mid 0 < \rho < \delta, \ R \in SO(m), \ Rv \cdot v > \cos \frac{\kappa}{2} \right\},$$

(2.1)

where $SO(m)$ denotes the special orthogonal group of $\mathbb{R}^m$. We remark that, in this paper, every cone is a spherical cone (whose base is not flat).

**Definition 2.1** An open set $U$ in $\mathbb{R}^m$ satisfies the *uniform interior cone condition* if there exists an open cone $C$ in $\mathbb{R}^m$ such that, for each point $x \in U$, we can take an open cone of vertex $x$ contained in $U$ and congruent to $C$.

**Definition 2.2** An open set $U$ in $\mathbb{R}^m$ satisfies the *uniform boundary inner cone condition* if there exists an open cone $C$ in $\mathbb{R}^m$ such that, for each point $x \in \partial U$, we can take an open cone of vertex $x$ contained in $U$ and congruent to $C$.

The proof of the following Lemma is due to Professor Hiroaki Aikawa.

**Lemma 2.3** Let $U$ be an open set in $\mathbb{R}^m$. If $U$ satisfies the uniform interior cone condition for an open cone $C$ of aperture angle $\kappa$ and height $\delta$, then it also satisfies the uniform boundary inner cone condition for the cone $C$.

**Proof** Fix a point $x$ on the boundary of $U$. For each natural number $n$, we take a point $\xi^n$ from $B_{1/n}(x) \cap U$. Thanks to the uniform interior cone condition of $U$, we can take an open cone $C(\xi^n)$ of vertex $\xi^n$ contained in $U$ and congruent to $C$. Let $v^n$ be the axis direction of $C(\xi^n)$. Since the unit sphere $S^{m-1}$ is compact in $\mathbb{R}^m$, we may assume that the sequence $\{v^n\}$ converges to a direction $v$.

Let $C(x)$ be the open cone of vertex $x$ and axis direction $v$ congruent to $C$. We show that $C(x)$ is contained in $U$. Suppose that $C(x)$ is not contained in $U$. We take a point from $C(x) \cap U^c$. The point can be expressed as $x + \rho Rv$ for some $0 < \rho < \delta$ and rotation matrix $R$ with $Rv \cdot v > \cos(\kappa/2)$. We remark that the point $\xi^n + \rho Rv^n$ is in $C(\xi^n)$. Since $C(\xi^n)$ is contained in $U$ for any $n$, we have

$$| (x + \rho Rv) - (\xi^n + \rho Rv^n) | \geq \text{dist} \left( \xi^n + \rho Rv^n, U^c \right) \geq \min \left\{ \delta - \rho, \ \rho \sin \left( \frac{\kappa}{2} - \theta \right) \right\},$$

where $\theta = \arccos(Rv \cdot v)$. On the other hand, for any large enough $n$,

$$| (x + \rho Rv) - (\xi^n + \rho Rv^n) | < \frac{1}{2} \min \left\{ \delta - \rho, \ \rho \sin \left( \frac{\kappa}{2} - \theta \right) \right\},$$

which is a contradiction. \hfill $\square$
Remark 2.4 Let $\Omega$ be a body (the closure of a bounded open set) in $\mathbb{R}^m$. Regarding a half-space as a cone of aperture angle $\pi$ and height $+\infty$, $\Omega$ is convex if and only if the complement of $\Omega$ satisfies the uniform boundary inner cone condition of aperture angle $\pi$ and height $+\infty$.

Remark 2.5 The converse statement of Lemma 2.3 does not always hold. Precisely, there is an open set $U$ in $\mathbb{R}^m$ such that it satisfies the uniform boundary inner cone condition for some open cone and does not satisfy the uniform interior cone condition for any open cone. The open set

$$U = B^2 \setminus \left( \left\{ (x_1, x_2) \mid x_1^2 + \left( x_2 - \frac{1}{2} \right)^2 = \frac{1}{4}, \quad 0 \leq x_1 \leq \frac{1}{2}, \quad x_2 \leq \frac{1}{2} \right\} \cup \left\{ (x_1, 0) \mid 0 \leq x_1 \leq \frac{1}{2} \right\} \right)$$

is such an example. (see Figure 2).

The author would like to express his gratitude to Professor Hiroaki Aikawa for informing him of this example.

The counterexample of the converse statement of Lemma 2.3 given in Remark 2.5 is not the interior of any body. The author would like to give the following problem:

**Problem 2.6** Let $U$ be the interior of a body in $\mathbb{R}^m$. Does the uniform boundary inner cone condition of $U$ imply the uniform interior cone condition of $U$?

### 2.2 Renormalization of the Riesz Potential

Let $\Omega$ be a body (the closure of a bounded open set) in $\mathbb{R}^m$. We consider the *Riesz potential* of $\Omega$ of order $0 < \alpha < m$,

$$V_\Omega^{(\alpha)}(x) = \int_{\Omega} |x - \xi|^\alpha \, d\xi, \quad x \in \mathbb{R}^m. \quad (2.2)$$

We remark that if $\alpha \leq 0$, then the above integral diverges for any interior point $x$ of $\Omega$. In [18], O’Hara extended the potential $V_\Omega^{(\alpha)}$ to the case of $\alpha \leq 0$ by using the same
renormalizing process as in the definition of his energy of knots introduced in [16,17]. Precisely, for $\alpha \leq 0$ and $x \in \overset{\circ}{\Omega}$, define the renormalization of the Riesz potential

$$V_\Omega^{(\alpha)}(x) = \begin{cases} \lim_{\varepsilon \to 0^+} \left( \int_{\Omega \setminus B_\varepsilon(x)} |x - \xi|^{\alpha-m} d\xi - \sigma \left( \frac{S^{m-1}}{-\alpha} \right) \varepsilon^\alpha \right) & (\alpha < 0), \\
\lim_{\varepsilon \to 0^+} \left( \int_{\Omega \setminus B_\varepsilon(x)} |x - \xi|^{-m} d\xi - \sigma \left( \frac{S^{m-1}}{-\alpha} \right) \log \frac{1}{\varepsilon} \right) & (\alpha = 0),
\end{cases}$$

(2.3)

and we call it the $r^{\alpha-m}$-potential of order $\alpha$ in what follows. Here, for $\alpha \leq 0$ and $x \in \overset{\circ}{\Omega}^c$, we define the potential $V_\Omega^{(\alpha)}(x)$ as the usual Riesz potential, that is,

$$V_\Omega^{(\alpha)}(x) = \int_{\Omega} |x - \xi|^{\alpha-m} d\xi, \quad \alpha \leq 0, \ x \in \overset{\circ}{\Omega}^c. \quad (2.4)$$

Let us prepare some terminologies and properties of $V_\Omega^{(\alpha)}$ from [18].

**Proposition 2.7** ([18, Proposition 2.5]) *Let $\Omega$ be a body in $\mathbb{R}^m$. For $\alpha \leq 0$ and $x \in \overset{\circ}{\Omega}$, we have*

$$V_\Omega^{(\alpha)}(x) = \begin{cases} \int_{\overset{\circ}{\Omega} \setminus B_\varepsilon(x)} |x - \xi|^{\alpha-m} d\xi - \sigma \left( \frac{S^{m-1}}{-\alpha} \right) \varepsilon^\alpha & (\alpha < 0), \\
\int_{\overset{\circ}{\Omega} \setminus B_\varepsilon(x)} |x - \xi|^{-m} d\xi - \sigma \left( \frac{S^{m-1}}{-\alpha} \right) \log \frac{1}{\varepsilon} & (\alpha = 0)
\end{cases}$$

whenever $\varepsilon < \text{dist}(x, \overset{\circ}{\Omega}^c)$. Moreover, for $\alpha < 0$ and $x \in \overset{\circ}{\Omega}$, we have

$$V_\Omega^{(\alpha)}(x) = -\int_{\overset{\circ}{\Omega}^c} |x - \xi|^{\alpha-m} d\xi.$$
Thanks to Proposition 2.8 and Lemma 2.9, for \( \alpha \leq 0 \), the restriction of \( V^{(\alpha)}(\Omega) \) to the interior of \( \Omega \) has a maximizer.

**Theorem 2.10** ([18, Theorem 3.5]) Let \( \Omega \) be a body in \( \mathbb{R}^m \). Suppose that the complement of \( \Omega \) satisfies the uniform boundary inner cone condition. For \( \alpha \leq 0 \), the restriction of \( V^{(\alpha)}(\Omega) \) to the interior of \( \Omega \) has a maximizer.

**Definition 2.11** ([18, Definition 3.1]) Let \( \Omega \) be a body in \( \mathbb{R}^m \). An interior point \( c \) of \( \Omega \) is called an \( r^{\alpha-m} \)-center of \( \Omega \) if it gives the maximum value of the restriction of \( V^{(\alpha)}(\Omega) \) to the interior of \( \Omega \). Let us denote the set of \( r^{\alpha-m} \)-centers by \( V^{(\alpha)}_\Omega(\Omega) \), that is,

\[
V^{(\alpha)}_\Omega(\Omega) = \{ c \in \Omega \mid c \in V^{(\alpha)}(\Omega) = \max_{x \in \Omega} V^{(\alpha)}(\Omega)(x) \},
\]

**Remark 2.12** The name of a maximizer of \( V^{(\alpha)}(\Omega) \), \( r^{\alpha-m} \)-center, is originated in [14]. Moszyńska defined a radial center of a star-shaped body \( A \) as a maximizer of a function of the form

\[
\Phi_A(x) = \int_{S^{m-1}} \varphi(\rho_{A-x}(v)) d\sigma(v), \quad x \in \text{Ker } A := \{ \xi \in A \mid \forall \eta \in A, \xi\eta \subset A \},
\]

where \( \rho_{A-x}(v) = \max\{\lambda \geq 0 \mid x + \lambda v \in A\} \) is the radial function of \( A \) with respect to \( x \), and \( \xi\eta \) denotes the line segment from \( \xi \) to \( \eta \). If \( \varphi(r) = r^{\alpha}/\alpha \) \( (0 < \alpha < m) \), then we have

\[
\Phi_A(x) = \int_{S^{m-1}} \left( \int_0^{\rho_{A-x}(v)} r^{\alpha-m} r^{m-1} dr \right) d\sigma(v) = V^{(\alpha)}_A(x).
\]

Her motivation comes from the study on the intersection body of a star-shaped body. Intersection bodies were introduced by Lutwak in [11] to give an affirmative answer to Busemann and Petty’s problem [3]. The intersection body of a star-shaped body \( A \) is defined by the radial function as \( \rho_{IA}(v) = \text{Vol}_{m-1}(A \cap v^\perp) \). Thus, the definition depends on the position of the origin. In [14], Moszyńska looked for an optimal position of the origin (see also [15, Part III]).

We refer to [6] for the physical meaning of the study on centers of a body. The uniqueness of a radial center was discussed in [7,14] but the investigation in [7] has an error, and it was pointed out in [18,22].

Using the form of the potential \( V^{(\alpha)}_\Omega \) in Remark 2.12, one can show the concavity of \( V^{(\alpha)}_\Omega \) if \( \Omega \) is convex.

**Theorem 2.13** ([18, Theorem 3.12]) Let \( \Omega \) be a convex body in \( \mathbb{R}^m \). For \( \alpha \leq 1 \), the potential \( V^{(\alpha)}_\Omega \) is strictly concave on \( \Omega \). In particular, \( \Omega \) has a unique \( r^{\alpha-m} \)-center.
2.3 Properties of the Solid Angle Function $A_\Omega$

Let $\Omega$ be the closure of an open set in $\mathbb{R}^m$. We consider the solid angle of $\Omega$ at $(x, h) \in \mathbb{R}^m \times (0, +\infty)$. From its definition mentioned in the introduction, we can show the following properties:

$$A_{\mathbb{R}^m}(x, h) = \frac{\sigma (S^m)}{2}, \ x \in \mathbb{R}^m, \ h > 0,$$

$$(2.5)$$

$$\lim_{h \to 0^+} A_\Omega(x, h) = \frac{\sigma (S^m)}{2} \chi_\Omega(x), \ x \in \mathbb{R}^m \setminus \partial \Omega.$$

$$(2.6)$$

In [21, 22], the author investigated properties of the solid angle function $A_\Omega$. Let us prepare some terminologies and properties of $A_\Omega$ from [21].

**Proposition 2.14** ([21, Proposition 5.16]) Let $\Omega$ be a body in $\mathbb{R}^m$. For any $h > 0$, the solid angle function $A_\Omega(\cdot, h)$ has a maximizer, and all of them are contained in the convex hull of $\Omega$.

**Definition 2.15** ([21, Definition 5.23]) Let $\Omega$ be a body in $\mathbb{R}^m$. A point $c$ is called an illuminating center of $\Omega$ of height $h$ if it gives the maximum value of $A_\Omega(\cdot, h)$. Let us denote the set of illuminating centers by $A_\Omega(h)$, that is,

$$A_\Omega(h) = \left\{ c \in \mathbb{R}^m \mid A_\Omega(c, h) = \max_{x \in \mathbb{R}^m} A_\Omega(x, h) \right\}.$$

The gradient of $A_\Omega(\cdot, h)$ vanishes at a point $x$ if and only if the point $x$ satisfies the equation

$$x = \int_\Omega \left( |x - \xi|^2 + h^2 \right)^{-(m+3)/2} \frac{\xi d\xi}{\int_\Omega \left( |x - \xi|^2 + h^2 \right)^{-(m+3)/2} d\xi},$$

$$(2.7)$$

which implies the following proposition:

**Proposition 2.16** ([21, Proposition 5.19]) Let $\Omega$ be a body in $\mathbb{R}^m$. The set of illuminating centers converges to the one-point set of the centroid of $\Omega$ as $h$ goes to infinity with respect to the Hausdorff distance.

The small-height behavior of illuminating centers will be investigated in Proposition 5.8.

2.4 Properties of a Potential with a Radially Symmetric Kernel

Let $\Omega$ be a body (the closure of a bounded open set) in $\mathbb{R}^m$. We consider a potential of the form

$$K_\Omega(x) = \int_\Omega k (|x - \xi|) d\xi, \ x \in \mathbb{R}^m.$$

$$(2.8)$$
We understand that the kernel $k$ satisfies the condition $(C^0_{\beta})$ for a positive $\beta$ if $k$ is continuous on the interval $(0, +\infty)$, and if

\[
k(r) = \begin{cases} 
O(r^{\beta-m}) & (\beta < m), \\
O(\log r) & (\beta = m), \\
O(1) & (\beta > m)
\end{cases} \tag{2.9}
\]

as $r$ tends to $0^+$. If $k$ satisfies the condition $(C^0_{\beta})$ for some $\beta > 0$, then $k$ is locally integrable on $\mathbb{R}^m$, and the compactness of $\Omega$ guarantees that the potential $K_\Omega = k(|\cdot|) * \chi_\Omega$ is continuous on $\mathbb{R}^m$.

**Proposition 2.17** ([21, Proposition 3.2]) Let $\Omega$ be a body in $\mathbb{R}^m$. If $k$ is strictly decreasing and satisfies the condition $(C^0_{\beta})$ for some $\beta > 0$, then the potential $K_\Omega$ has a maximizer, and all of them are contained in the convex hull of $\Omega$.

**Definition 2.18** ([21, Definition 3.3]) Let $\Omega$ be a body in $\mathbb{R}^m$. A point $c$ is called a $k$-center of $\Omega$ if it gives the maximum value of $K_\Omega$. We denote the set of $k$-centers of $\Omega$ by $K_\Omega$, that is,

\[K_\Omega = \left\{ c \in \mathbb{R}^m \mid K_\Omega(c) = \max_{x \in \mathbb{R}^m} K_\Omega(x) \right\}.
\]

For the potential $K_\Omega(x, t)$ defined in (1.1), we call a maximizer of $K_\Omega(\cdot, t)$ a $k$-center at time $t$.

**Proposition 2.19** Let $\Omega$ be a convex body in $\mathbb{R}^m$. Let $\Omega'$ be a convex body contained in the interior of $\Omega$. Put

\[
d(\Omega, \Omega') = \inf \left\{ |z - w| \mid z \in \partial \Omega, \ w \in \Omega' \right\},
\]

\[
D(\Omega, \Omega') = \sup \left\{ |z - w| \mid z \in \partial \Omega, \ w \in \Omega' \right\}.
\]

Let $k$ be positive and satisfy the condition $(C^0_{\beta})$ for some $\beta > 0$. If $k(r)r^{m-1}$ is decreasing for $r \in [d(\Omega, \Omega'), D(\Omega, \Omega')]$, then $K_\Omega$ is strictly concave in $\Omega'$.

The idea of the proof of Proposition 2.19 is based on [14, Theorem 3.1].

**Proof** We take two distinct points $x$ and $y$ from $\Omega'$. From the convexity of $\Omega$, we have

\[
\rho_{\Omega-(x+y)/2}(v) \geq \frac{\rho_{\Omega-x}(v) + \rho_{\Omega-y}(v)}{2}
\]

for any $v \in S^{m-1}$. Moreover, using the Brunn–Minkowski inequality and the dual Brunn–Minkowski inequality, one can show the existence of a subset $S$ of $S^{m-1}$ with positive measure such that the above inequality strictly holds for any $v \in S$. This fact is due to the proof of [14, Theorem 3.1].
Put
\[
a = a(x, y, v) = \min \{ \rho_{\Omega, x}(v), \rho_{\Omega, y}(v) \},
b = b(x, y, v) = \max \{ \rho_{\Omega, x}(v), \rho_{\Omega, y}(v) \}.
\]

Using polar coordinates, we have
\[
2K\left( \frac{x + y}{2} \right) - (K\Omega(x) + K\Omega(y))
= \int_{S^{m-1}} \left( 2 \int_0^{\rho_{\Omega, x}(v)/2} (r) r^{m-1} dr - \int_0^{\rho_{\Omega, y}(v)} (r) r^{m-1} dr \right) d\sigma(v)
> \int_{S^{m-1}} \left( 2 \int_0^{(a+b)/2} (r) r^{m-1} dr - \int_0^{b} (r) r^{m-1} dr \right) d\sigma(v)
= \int_{S^{m-1}} \left( \int_a^{(a+b)/2} (r) r^{m-1} dr - \int_0^{b} (r) r^{m-1} dr \right) d\sigma(v)
= \int_{S^{m-1}} \left( \int_a^{(a+b)/2} (r) r^{m-1} - k \left( r + \frac{b - a}{2} \right) \left( r + \frac{b - a}{2} \right)^{m-1} \right) dr \right) d\sigma(v).
\]

The decreasing behavior of \( k(r)r^{m-1} \) implies the non-negativity of the above integral. \( \square \)

2.5 The Minimal Unfolded Region of a Body

Let \( \Omega \) be a body (the closure of a bounded open set) in \( \mathbb{R}^m \). Using radial symmetry of the kernels of the potentials mentioned in the previous subsections, we can restrict the region containing those centers. We introduce the restricted region and its properties from \([2,18,21]\) (see also \([1,19,22]\)).

![Fig. 3](image-url)  
**Fig. 3** Folding a body \( \Omega \) in the manner in Definition 2.20

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Definition 2.20 ([18, Definition 3.3]) Let \( v \) be a direction in the unit sphere \( S^{m-1} \), and \( b \) a real parameter. Let \( \text{Refl}_{v,b} \) be the reflection of \( \mathbb{R}^m \) in the hyperplane \( \{ z \in \mathbb{R}^m | z \cdot v = b \} \). Put

\[
\Omega^+_{v,b} = \Omega \cap \{ z \in \mathbb{R}^m | z \cdot v \geq b \} . \\
l(v) = \min \{ a \in \mathbb{R} | \forall b \geq a, \ \text{Refl}_{v,b} \left( \Omega^+_{v,b} \right) \subset \Omega \}
\]

(see Figure 3). Define the minimal unfolded region of \( \Omega \) by

\[
Uf(\Omega) = \bigcap_{v \in S^{m-1}} \{ z \in \mathbb{R}^m | z \cdot v \leq l(v) \} .
\]

Example 2.21 ([2, Lemma 5], [18, Example 3.4])

1. The minimal unfolded region of the disjoint union of three disks is the triangle with vertices at the centers of the disks (see Figure 4).
2. The minimal unfolded region of an acute triangle is surrounded by the mid-perpendiculars of edges and the bisectors of angles (see Figure 5).
3. The minimal unfolded region of an obtuse triangle is surrounded by the largest edge, its mid-perpendicular, and the bisectors of angles (see Figure 6). We remark that the minimal unfolded region of \( \Omega \) is not always contained in the interior of the convex hull of \( \Omega \) even if \( \Omega \) is convex.

Remark 2.22 ([2, Proposition 1], [18, p. 381])

1. The centroid (center of mass) of \( \Omega \) is contained in \( Uf(\Omega) \). Hence \( Uf(\Omega) \) is not empty.
Fig. 6  The minimal unfolded region of an obtuse triangle

(2) $U_f(\Omega)$ is compact and convex.
(3) $U_f(\Omega)$ is contained in the convex hull of $\Omega$.

Using the \textit{moving plane method} ([5,23]), we can restrict the location of the centers prepared in the previous subsections.

\textbf{Proposition 2.23} Let $\Omega$ be a body in $\mathbb{R}^m$. For $\alpha \leq 0$, any possibility of an $r^{\alpha-m}$-center of $\Omega$ belongs to the minimal unfolded region of $\Omega$.

\textbf{Proof} Let $x$ be an interior point of $\Omega$ in the complement of the minimal unfolded region of $\Omega$. We show that the point $x$ is not an $r^{\alpha-m}$-center of $\Omega$.

We can choose a direction $v \in S^{m-1}$ with $l(v) < x \cdot v$. Let $b = (l(v) + x \cdot v)/2$. Then, $x \in \Omega^+_v$, the region $\text{Refl}_{v,b}(\Omega^+_v)$ is contained in $\Omega$, and $\Omega \setminus (\Omega^+_v \cup \text{Refl}_{v,b}(\Omega^+_v))$ has an interior point.

Let $x' = \text{Refl}_{v,b}(x)$. We choose a small enough $\varepsilon > 0$ so that the ball $B_\varepsilon(x)$ is contained in the interior of $\Omega$. Then, we have the following properties:

$$\int_{\Omega^+_v \setminus B_\varepsilon(x)} |x - \xi|^{\alpha-m} d\xi = \int_{\text{Refl}_{v,b}(\Omega^+_v) \setminus B_\varepsilon(x')} |x' - \xi|^{\alpha-m} d\xi,$$
$$\int_{\text{Refl}_{v,b}(\Omega^+_v)} |x - \xi|^{\alpha-m} d\xi = \int_{\Omega^+_v} |x' - \xi|^{\alpha-m} d\xi.$$

Furthermore, for any point $\xi \in \Omega \setminus (\Omega^+_v \cup \text{Refl}_{v,b}(\Omega^+_v))$, we have $|x - \xi|^{\alpha-m} < |x' - \xi|^{\alpha-m}$. Hence, by Proposition 2.7, we obtain

$$V^{(\alpha)}_{\Omega}(x) - V^{(\alpha)}_{\Omega}(x') = \int_{\Omega \setminus B_\varepsilon(x)} |x - \xi|^{\alpha-m} d\xi - \int_{\Omega \setminus B_\varepsilon(x')} |x' - \xi|^{\alpha-m} d\xi$$
$$= \left( \int_{\Omega^+_v \setminus B_\varepsilon(x)} + \int_{\text{Refl}_{v,b}(\Omega^+_v)} + \int_{\Omega \setminus (\Omega^+_v \cup \text{Refl}_{v,b}(\Omega^+_v))} \right) |x - \xi|^{\alpha-m} d\xi$$
$$- \left( \int_{\text{Refl}_{v,b}(\Omega^+_v) \setminus B_\varepsilon(x')} + \int_{\Omega^+_v} + \int_{\Omega \setminus (\Omega^+_v \cup \text{Refl}_{v,b}(\Omega^+_v))} \right) |x' - \xi|^{\alpha-m} d\xi$$
$$< 0,$$

which completes the proof. \hfill \Box
Remark 2.24 In [18, Theorem 3.5], O’Hara showed that any possibility of a critical point of $V^{(\alpha)}_{\Omega}$ belongs to the minimal unfolded region of $\Omega$ when $\Omega$ has a piecewise $C^1$ boundary. As shown in Proposition 2.23, we can weaken the assumption for $\Omega$ if we investigate the location of maximizers of $V^{(\alpha)}_{\Omega}$.

Remark 2.25 Since there is a body such that $\Omega \subsetneq \text{conv} \Omega = Uf(\Omega)$, Proposition 2.23 is a slight modification of the location of possibilities of $r^{\alpha-m}$-centers. For example, in Figure 7, let $\Omega$ be the body surrounded by the curve drawn with the solid line.

In the same manner as in Proposition 2.23, we can restrict the location of $k$-centers of $\Omega$ into the minimal unfolded region of $\Omega$.

**Proposition 2.26** ([21, Proposition 4.9]) Let $\Omega$ be a body in $\mathbb{R}^m$. If $k$ is strictly decreasing and satisfies the condition $(C^0_\beta)$ for some $\beta > 0$, then any $k$-center of $\Omega$ belongs to the minimal unfolded region of $\Omega$.

Remark 2.27 We refer to [8,18] for the location of $r^{\alpha-m}$-centers. Herburt showed that the (unique) $r^{1-m}$-center of a smooth convex body $A$ belongs to the interior of $A$. O’Hara extended her result to the case where $0 < \alpha \leq 1$ and $\Omega$ is not smooth (see [18, Lemma 2.15]).

For $\alpha \leq 0$, we discuss the location of $r^{\alpha-m}$-centers in Theorem 3.9 and Corollary 3.10. For $\alpha > 1$, any $r^{\alpha-m}$-center of a body $\Omega$ belongs to the intersection $Uf(\Omega) \cap (\text{conv} \Omega)^\circ$. This statement follows from the fact that, for a boundary point $x$ of $\text{conv} \Omega$ and the unit outer normal field $n$ of $\text{conv} \Omega$, the derivative

$$\frac{\partial V^{(\alpha)}_{\Omega}}{\partial n(x)}(x) = (\alpha - m) \int_{\Omega} r^{\alpha-m-2}(x - y) \cdot n(x) dy$$

does not vanish.

Herburt and O’Hara’s results do not follow from the same argument as in the case of $\alpha > 1$. This is because the potential $V^{(\alpha)}_{\Omega}$ is not differentiable at any boundary point of $\Omega$. Also, the minimal unfolded region of $\Omega$ touches the boundary of $\Omega$ in general (see Example 2.21 (3) and Figure 6).
3 Estimation of an $r^{α−m}$-Potential

Let $Ω$ be a body (the closure of a bonded open set) in $\mathbb{R}^m$. By Proposition 2.23, the set of $r^{α−m}$-centers ($α \leq 0$) of $Ω$ is contained in the minimal unfolded region of $Ω$. But, by Lemma 2.9, it is expected that any $r^{α−m}$-center does not exist “near” the boundary of $Ω$. For example, when $Ω$ is an obtuse triangle in $\mathbb{R}^2$, the minimal unfolded region of $Ω$ touches the boundary of $Ω$, but it is expected that any $r^{α−2}$-center belongs to a smaller closed region contained in the interior of the minimal unfolded region of $Ω$. Let us show that the expectation is true when the complement of $Ω$ satisfies the uniform boundary inner cone condition.

Let $C(x; \kappa, \delta)$ be an open cone of vertex $x$, axis direction $e_1$, aperture angle $0 < \kappa \leq \pi$, and height $0 < \delta \leq +\infty$, that is,

$$C(x) = C(x; \kappa, \delta) = \left\{ \rho \begin{bmatrix} \cos \phi_1 \\ \sin \phi_1 \cos \phi_2 \\ \vdots \\ \sin \phi_1 \cdots \sin \phi_{m-2} \cos \phi_{m-1} \\ \sin \phi_1 \cdots \sin \phi_{m-2} \sin \phi_{m-1} \end{bmatrix} + x \mid 0 < \rho < \delta, \phi \in \left( -\frac{\kappa}{2}, \frac{\kappa}{2} \right) \times [0, \pi]^{m-2} \right\},$$

(3.1)

where $\phi = (\phi_1, \ldots, \phi_{m-1})$. Let $\text{Rot}_{1m}(\theta)$ denote the rotation in the plane $\text{Span}\{e_1, e_m\}$ of angle $\theta$, that is,

$$\text{Rot}_{1m}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \vdots & \vdots \\ \sin \theta & \cos \theta \end{bmatrix}.$$  

(3.2)

Let

$$C_\theta(x) = C_\theta(x; \kappa, \delta) = \text{Rot}_{1m}(\theta) C(0; \kappa, \delta) + x.$$  

(3.3)

The following remark plays an important role in the proof of Lemmas 3.3 – 3.5.

**Remark 3.1** (1) Let $Φ$ be an increasing and concave function defined on $(0, +\infty)$ ($(0, +\infty)$), and $0 \leq (\text{resp.} \leq) f_1 \leq f_2 \leq f_3 \leq f_4$. If $f_2 - f_1 \geq f_4 - f_3$, then we have $Φ(f_2) - Φ(f_1) \geq Φ(f_4) - Φ(f_3)$.

(2) Let $Φ$ be a decreasing and convex function defined on $(0, +\infty)$ ($(0, +\infty)$), and $0 \leq (\text{resp.} \leq) f_1 \leq f_2 \leq f_3 \leq f_4$. If $f_2 - f_1 \geq f_4 - f_3$, then we have $Φ(f_1) - Φ(f_2) \geq Φ(f_3) - Φ(f_4)$.

**Lemma 3.2** Let $0 < \kappa \leq \pi$, $0 \leq \theta \leq (\pi - \kappa)/2$, $\phi \in [-\kappa/2, \kappa/2] \times [0, \pi]^{m−2}$, and

$$f(\theta, \phi) = \cos \theta \cos \phi_1 - \sin \theta \sin \phi_1 \cdots \sin \phi_{m-1}.$$  

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Fig. 8 The position of $\xi(\rho, \theta, \phi)$

(1) We have $0 \leq f(\theta, \phi) \leq 1$.
(2) If $0 \leq \phi_1 \leq \kappa/2$, then we have $f(0, \phi) \geq f(\theta, \phi)$.
(3) We have $f(0, \phi) - f(\theta, \phi) \geq f(\theta, \tilde{\phi}) - f(0, \tilde{\phi})$, where $\tilde{\phi} = (-\phi_1, \phi_2, \ldots, \phi_{m-1})$.

Proof (1) The inequality $-\kappa/2 \leq \theta + \phi_1 \leq \pi/2$ implies $\cos(\theta + \phi_1) \geq 0$ and the conclusion.

(2) Direct computation shows
\[
 f(0, \phi) - f(\theta, \phi) = \cos \phi_1 (1 - \cos \theta) + \sin \theta \sin \phi_1 \cdots \sin \phi_{m-1}
\]
which is non-negative if $0 \leq \phi_1 \leq \kappa/2$.

(3) Direct computation shows
\[
(f(0, \phi) - f(\theta, \phi)) - (f(\theta, \tilde{\phi}) - f(0, \tilde{\phi})) = 2 \cos \phi_1 (1 - \cos \theta)
\]
which is non-negative.

Lemma 3.3 Let $0 < \kappa \leq \pi$, $\rho > 0$, $0 \leq \theta \leq (\pi - \kappa)/2$, $\phi \in [-\kappa/2, \kappa/2] \times [0, \pi]^{m-2}$, and $R > 0$. Put
\[
\xi(\rho, \theta, \phi) = \rho \text{Rot}_m(\theta) \begin{pmatrix} \cos \phi_1 \\ \sin \phi_1 \cos \phi_2 \\ \vdots \\ \sin \phi_1 \cdots \sin \phi_{m-2} \cos \phi_{m-1} \\ \sin \phi_1 \cdots \sin \phi_{m-2} \sin \phi_{m-1} \end{pmatrix} + Re_1.
\]

If $0 \leq \phi_1 \leq \kappa/2$, then we have
\[
|\xi(\rho, \phi)| - |\xi(\rho, \phi)| \geq |\xi(\rho, \tilde{\phi})| - |\xi(\rho, \phi)|
\]
where $\tilde{\phi} = (-\phi_1, \phi_2, \ldots, \phi_{m-1})$ (see Figure 8).

Proof Let $f(\theta, \phi)$ be as in Lemma 3.2. Direct computation shows
\[
|\xi(\rho, \theta, \phi)| = \sqrt{\rho^2 + R^2 + 2\rho R f(\theta, \phi)}.
\]
From Lemma 3.2 (2), if $0 \leq \phi_1 \leq \kappa/2$, then we have $|\xi(\rho, 0, \phi)| \geq |\xi(\rho, \theta, \phi)|$. Thus, in the case of $|\xi(\rho, \theta, \phi)| \leq |\xi(\rho, 0, \phi)|$, we obtain the conclusion.

Let us consider the case of $|\xi(\rho, \theta, \bar{\phi})| \leq |\xi(\rho, 0, \bar{\phi})|$. We remark

$$f(\theta, \phi) \leq f(0, \phi) = f(0, \bar{\phi}) \leq f(\theta, \bar{\phi})$$

if $0 \leq \phi_1 \leq \kappa/2$. Since the function $f \mapsto \sqrt{\rho^2 + R^2 + 2\rho R f}$ is increasing and concave, Remark 3.1 (1) and Lemma 3.2 (3) imply the conclusion. $\square$

**Lemma 3.4** Let $\alpha < m$, $0 < \kappa \leq \pi$, $0 < \delta \leq +\infty$, $0 \leq \theta \leq (\pi - \kappa)/2$, and $0 < R < D$. Put

$$U_\theta = \text{Rot}_{1m}(\theta) \left( C(Re_1) \cap B_D(0) \cap \{\xi_m \geq 0\} - Re_1 \right) + Re_1,$$

$$L_\theta = \text{Rot}_{1m}(\theta) \left( C(Re_1) \cap B_D(0) \cap \{\xi_m \leq 0\} - Re_1 \right) + Re_1,$$

where $C(Re_1) = C(Re_1; \kappa, \delta)$ is the cone defined in (3.1). Then, we have

$$\int_{U_\theta \cup L_\theta} |\xi|^\alpha - m d\xi \leq \int_{U_\theta \cup L_\theta} |\xi|^\alpha - m d\xi.$$

**Proof** Let $f(\theta, \phi)$ and $\xi(\rho, \theta, \phi)$ be as in Lemmas 3.2 and 3.3, respectively. Let

$$\tilde{\rho}(\theta, \phi) = -R f(\theta, \phi) + \sqrt{D^2 + R^2 \left(f(\theta, \phi)^2 - 1\right)}.$$ (3.4)

This is the positive root of the equation $|\xi(\rho, \theta, \phi)| = D$ with respect to $\rho$. We remark that $U_\theta$ and $L_\theta$ are expressed as follows:

$$U_\theta = \{\xi(\rho, \theta, \phi) \mid 0 \leq \rho \leq \min\{\delta, \tilde{\rho}(0, \phi)\}, \phi \in [0, \kappa/2] \times [0, \pi]^{m-2}\},$$

$$L_\theta = \{\xi(\rho, \theta, \bar{\phi}) \mid 0 \leq \rho \leq \min\{\delta, \tilde{\rho}(0, \phi)\}, \phi \in [0, \kappa/2] \times [0, \pi]^{m-2}\}.$$

Let

$$\Delta^\alpha(\rho, \theta, \phi) = (|\xi(\rho, \theta, \phi)|\alpha - m - |\xi(\rho, 0, \phi)|\alpha - m) - (|\xi(\rho, \theta, \bar{\phi})|\alpha - m - |\xi(\rho, \theta, \bar{\phi})|\alpha - m).$$

Using polar coordinates, we have

$$\left(\int_{U_\theta} - \int_{U_\theta}\right) |\xi|^\alpha - m d\xi - \left(\int_{L_\theta} - \int_{L_\theta}\right) |\xi|^\alpha - m d\xi$$

$$= \int_{[0, \kappa/2] \times [0, \pi]^{m-2}} \left(\int_0^{\min\{\delta, \tilde{\rho}(0, \phi)\}} \Delta^\alpha(\rho, \theta, \phi) \rho^{m-1} d\rho\right) \sin^{m-2} \phi_1 \cdots \sin \phi_{m-2} d\phi.$$ 

Thus, it is sufficient to show the non-negativity of $\Delta^\alpha(\rho, \theta, \phi)$. ☢ Springer
If $|\xi(\rho, \theta, \phi)| \leq |\xi(\rho, 0, \phi)|$, then the strictly decreasing behavior of the function $r \mapsto r^{\alpha-m}$ implies the non-negativity of $\Delta^\alpha(\rho, \theta, \phi)$.

Let us consider the case of $|\xi(\rho, \theta, \phi)| \geq |\xi(\rho, 0, \phi)|$. We remark

$$|\xi (\rho, \theta, \phi)| \leq |\xi (\rho, 0, \phi)| = |\xi (\rho, 0, \phi)| \leq |\xi (\rho, \theta, \phi)|.$$

Thanks to Remark 3.1 (2), Lemma 3.3 implies the non-negativity of $\Delta^\alpha(\rho, \theta, \phi)$. □

**Lemma 3.5** Let $0 < \kappa \leq \pi$, $0 \leq \theta \leq (\pi - \kappa)/2$, $\phi \in [-\kappa/2, \kappa/2] \times [0, \pi]^{m-2}$, and $0 < R < D$. Let $\bar{\rho}$ be as in (3.4).

1. We have $D - R = -\bar{\rho}(\theta, \phi) \leq \sqrt{D^2 - R^2}$.
2. If $0 \leq \phi_1 \leq \kappa/2$, then we have $\bar{\rho}(0, \phi) \leq \bar{\rho}(\theta, \phi)$.
3. If $0 \leq \phi_1 \leq \kappa/2$, then we have $\bar{\rho}(\theta, \phi) - \bar{\rho}(0, \phi) \geq \bar{\rho}(0, \phi) - \bar{\rho}(\theta, \phi)$.

**Proof** Let $f(\theta, \phi)$ be as in Lemma 3.2, and

$$\Phi(f) = -Rf + \sqrt{D^2 + R^2 (f^2 - 1)}.$$

We remark that $\bar{\rho}(\theta, \phi) = \Phi(f(\theta, \phi))$. Direct computations show the following properties of $\Phi$:

$$\Phi'(f) = -\frac{R\Phi(f)}{\sqrt{D^2 + R^2 (f^2 - 1)}} < 0,$$

$$\Phi''(f) = \frac{R^2\Phi(f)}{(D^2 + R^2 (f^2 - 1))^{3/2}} \left( \sqrt{D^2 + R^2 (f^2 - 1)} + Rf \right) > 0.$$

1. From Lemma 3.2 (1), the decreasing behavior of $\Phi$ implies

$$D - R = \Phi(1) \leq \Phi(f(\theta, \phi)) \leq \Phi(0) = \sqrt{D^2 - R^2}.$$

2. From Lemma 3.2 (2), the decreasing behavior of $\Phi$ implies

$$\bar{\rho}(0, \phi) = \Phi(f(0, \phi)) \leq \Phi(f(\theta, \phi)) = \bar{\rho}(\theta, \phi)$$

if $0 \leq \phi_1 \leq \kappa/2$.

3. Thanks to (2), in the case of $f(0, \phi) \geq f(\theta, \phi)$, the decreasing behavior of $\Phi$ implies the conclusion. Let us consider the case of $f(0, \phi) \leq f(\theta, \phi)$. Then, we have

$$f(\theta, \phi) \leq f(0, \phi) = f(0, \phi) \leq f(\theta, \phi).$$

Thanks to Remark 3.1 (2), Lemma 3.2 (3) implies the conclusion. □
Lemma 3.6 Let $\alpha < m$, $0 < \kappa \leq \pi$, $0 < \delta \leq +\infty$, and $0 < R < D$. If $\delta \leq D - R$ or $\delta \geq \sqrt{D^2 - R^2}$, then we have

$$
\min_{0 \leq \theta \leq (\pi - \kappa)/2} \int_{C_\theta(Re_1) \cap B_D(0)} |\xi|^{\alpha - m} d\xi = \int_{C(Re_1) \cap B_D(0)} |\xi|^{\alpha - m} d\xi,
$$

where $C_\theta(Re_1) = C_\theta(Re_1; \kappa, \delta)$ is the cone defined in (3.3).

Proof Let $\xi(\rho, \theta, \phi)$ and $\bar{\rho}(\theta, \phi)$ be as in Lemma 3.3 and (3.4), respectively.

From Lemma 3.5 (1), if $\delta \leq D - R$, then $C_\theta(Re_1) \cap B_D(0) = U_\theta \cup L_\theta$, where $U_\theta$ and $L_\theta$ are defined in Lemma 3.4. Hence Lemma 3.4 implies the conclusion in this case.

Let us consider the case of $\delta \geq \sqrt{D^2 - R^2}$. From Lemma 3.5 (1), in this case, we have $\bar{\rho}(\theta, \phi) \leq \delta$ for any $(\theta, \phi)$. Using polar coordinates, we have the following identities:

$$
\int_{C_\theta(Re_1) \cap B_D(0)} |\xi|^{\alpha - m} d\xi
$$

$$
= \int_{[0,\kappa/2] \times [0,\pi]} (\int_0^1 |\xi(\rho, \theta, \phi)|^{\alpha - m - 1} \rho^m d\rho) \sin^{m-2} \phi_1 \cdots \sin \phi_{m-2} d\phi,
$$

$$
+ \int_{U_\theta \cup L_\theta} |\xi|^{\alpha - m} d\xi
$$

$$
= \int_{[0,\kappa/2] \times [0,\pi]} (\int_0^1 |\xi(\rho, \theta, \phi)|^{\alpha - m - 1} \rho^m d\rho) \sin^{m-2} \phi_1 \cdots \sin \phi_{m-2} d\phi,
$$

(see Fig. 9).

Thus, it is sufficient to show the non-negativity of

$$
\int_0^1 |\xi(\rho, \theta, \phi)|^{\alpha - m - 1} \rho^m d\rho + \int_0^1 |\xi(\rho, \theta, \bar{\phi})|^{\alpha - m - 1} \rho^m d\rho
$$

$$
- \int_0^1 |\xi(\rho, \theta, \phi)|^{\alpha - m - 1} \rho^m d\rho - \int_0^1 |\xi(\rho, \theta, \bar{\phi})|^{\alpha - m - 1} \rho^m d\rho
$$

$$
= \int_{\tilde{\rho}(0,\phi)} |\xi(\rho, \theta, \phi)|^{\alpha - m - 1} \rho^m d\rho - \int_{\tilde{\rho}(0,\bar{\phi})} |\xi(\rho, \theta, \bar{\phi})|^{\alpha - m - 1} \rho^m d\rho.
$$
when \( \tilde{\rho}(0, \tilde{\phi}) \geq \tilde{\rho}(\theta, \tilde{\phi}) \) (see also Lemma 3.5 (2)). We remark that, from the definition of \( \tilde{\rho}(\theta, \phi) \), we have

\[
| \xi (\tilde{\rho}(\theta, \phi), \theta, \phi) | = | \xi (\tilde{\rho}(0, \phi), \theta, \phi) | = D.
\]

Since the function \( \rho \mapsto | \xi (\rho, \theta, \phi) | \) is increasing, we obtain

\[
\int_{\tilde{\rho}(0, \phi)}^{\tilde{\rho}(\theta, \phi)} | \xi (\rho, \theta, \phi) |^{\alpha-m} \rho^{m-1} d\rho = \int_{\tilde{\rho}(0, \phi)}^{\tilde{\rho}(\theta, \phi)} | \xi (\rho, \theta, \phi) |^{\alpha-m} \rho^{m-1} d\rho
\geq D^{\alpha-m} \tilde{\rho}(0, \phi)^{m-1} \left( (\tilde{\rho}(\theta, \phi) - \tilde{\rho}(0, \phi)) - (\tilde{\rho}(0, \phi) - \tilde{\rho}(\theta, \phi)) \right)
\geq 0,
\]

where the last inequality follows from Lemma 3.5 (3).

Remark 3.7 Let \( I \) be an open interval in \( \mathbb{R} \), \( J \) a bounded and closed interval in \( \mathbb{R} \), \( F \) a continuous function defined on \( I \times J \), and

\[
\tilde{F} : I \ni R \mapsto \min_{\theta \in J} F(R, \theta) \in \mathbb{R}.
\]

Then, \( \tilde{F} \) is continuous on \( I \).

We give the proof of Remark 3.7 for readers’ convenience.

Proof Fix an arbitrary \( R \in I \). Let \( \{R_n\} \) be a sequence converging to \( R \). We show that \( \tilde{F}(R_n) \) converges to \( \tilde{F}(R) \).

From the continuity of \( F \) and the compactness of \( J \), there are \( \theta_n \in J \) and \( \theta_n \in J \) such that \( \tilde{F}(R) = F(R, \theta_n) \) and \( \tilde{F}(R_n) = F(R_n, \theta_n) \). Since \( J \) is compact in \( \mathbb{R} \), we may assume that \( \theta_n \) converges to a point \( \theta_0 \in J \). By the minimality of \( F(R_n, \theta_n) \) as the function \( \theta \mapsto F(R_n, \theta) \), we have \( F(R_n, \theta_n) \leq F(R_n, \theta_0) \). The continuity of \( F \) implies \( F(R, \theta_0) \leq F(R, \theta_n) \). By the minimality of \( F(R, \theta_n) \) as the function
Let $0 < \theta < \pi$, $0 < \delta \leq +\infty$, and $0 < R_0 < D$. Define the function

$$E(R) = E(R; \alpha, \kappa, \delta, D, R_0)$$

where $C_\theta(Re_1) = C_\theta(Re_1; \kappa, \delta)$ is the cone defined in (3.3).

\begin{enumerate}
\item The function $E$ is strictly decreasing.
\item There exists a unique positive constant $\tilde{R} = \tilde{R}(\alpha, \kappa, \delta, D, R_0)$ such that $E(R) > 0$ if $R < \tilde{R}$, and that $E(R) < 0$ if $R > \tilde{R}$. In particular, $\tilde{R}$ is the unique root of $E$.
\item We have $\tilde{R} < R_0$.
\end{enumerate}

Proof (1) Let $0 < R_1 < R_2$. We denote by $\theta_j$ an angle giving the minimum value in the definition of $E(R_j)$. The strictly decreasing behavior of the function $r \mapsto r^{\alpha - m}$ implies

$$E(R_1) = \left( \int_{C_\theta(Re_1) \cap B_D(0)} - \int_{B_D(0) \setminus B_{R_0}(0)} \right) |\xi|^{\alpha - m} d\xi$$

$$> \left( \int_{C_\theta(Re_2) \cap B_D(0)} - \int_{B_D(0) \setminus B_{R_0}(0)} \right) |\xi|^{\alpha - m} d\xi$$

$$\geq \left( \int_{C_\theta(Re_2) \cap B_D(0)} - \int_{B_D(0) \setminus B_{R_0}(0)} \right) |\xi|^{\alpha - m} d\xi$$

$$= E(R_2).$$

(2) First, we remark that $E(R)$ is negative for $R \geq R_0$. This is because, for any $0 \leq \theta \leq (\pi - \kappa)/2$ and $R \geq R_0$, $C_\theta(Re_1) \cap B_D(0)$ is contained in the annulus $B_D(0) \setminus B_{R_0}(0)$.

Next, we show that $E(R)$ diverges to $+\infty$ as $R$ tends to $0^+$. We take a small enough $\varepsilon > 0$ so that $\varepsilon \delta < D - R_0$. Then, for any $0 \leq \theta \leq (\pi - \kappa)/2$ and $R \leq R_0$, the small cone $\varepsilon C_\theta(Re_1)$ is contained in the ball $B_D(0)$. From Lemma 3.6, we have

$$\min_{0 \leq \theta \leq (\pi - \kappa)/2} \int_{\varepsilon C_\theta(Re_1) \cap B_D(0)} |\xi|^{\alpha - m} d\xi = \int_{\varepsilon C(Re_1) \cap B_D(0)} |\xi|^{\alpha - m} d\xi.$$
Therefore, Lemma 2.9 implies

\[ E(R) \geq \left( \int_{\mathcal{C}(R_1)} \left( \int_{B_D(0)} - \int_{B_D(0) \setminus B_{R_0}(0)} \right) |\xi|^\alpha d\xi \right) \to +\infty \]

as \( R \) tends to \( 0^+ \).

From Remark 3.7, the function \( E \) is continuous. Hence we obtain the existence and uniqueness of a root of \( E \).

(3) The statement was shown in the proof of (2) as \( E(R) \) is negative for \( R \geq R_0 \). □

**Theorem 3.9** Let \( \alpha \leq 0 \). Let \( X \) and \( Y \) be bodies in \( \mathbb{R}^m \). Suppose that the complement of \( Y \) satisfies the uniform boundary inner cone condition of aperture angle \( \kappa \) and height \( \delta \). Let \( 0 < R_0 \leq R_\infty(X) \), and \( \tilde{R} = \tilde{R}(\alpha, \kappa, \delta, \text{diam } Y, R_0) \) be as in Lemma 3.8. For any points \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \) with \( \text{dist}(y, Y^c) \leq \tilde{R} \), we have \( V_Y^{(x)}(y) < V_X^{(x)}(x) \).

**Proof** If \( R_0 \) is not smaller than the diameter of \( Y \), then we have

\[ V_X^{(x)}(x) \geq V_{B_{R_0}(x)}^{(x)} = V_{B_{\text{diam } Y}(0)}^{(x)} \geq V_{B_{\text{diam } Y}(y)}^{(x)} = V_{Y^c}^{(x)}(y) > V_Y^{(x)}(y) \]

for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \). Let us consider the case where \( R_0 \) is smaller than the diameter of \( Y \).

Fix an interior point \( y \) of \( Y \). Let \( y' \) be a boundary point of \( Y \) with \( |y - y'| = \text{dist}(y, Y^c) \). From the uniform boundary inner cone condition of the complement of \( Y \), there is a direction \( v(y) \) such that we can take an open cone of vertex \( y' \), axis direction \( v(y) \), aperture angle \( \kappa \), and height \( \delta \). Let \( \theta(y) \) be the angle between \( (y' - y)/|y' - y| \) and \( v(y) \). There is a motion \( g \) with the following properties:

\[ \frac{g \cdot (y' - y)}{|y' - y|} = e_1, \quad \frac{g \cdot (v(y) - v(y))(y' - y)/|y' - y|}{|v(y) - v(y)(y' - y)/|y' - y||} = e_m, \]

where \( v(y)(y' - y)/|y' - y| \) denotes the orthogonal projection of \( v(y) \) to \( (y' - y)/|y' - y| \). Then, we have

\[ g \cdot (Y - y) + y \subset B_{\text{diam } Y}(y) \setminus C_{\theta(y)} \left( y + \text{dist} \left( y, Y^c \right) e_1 \right), \]

where \( C_{\theta(y)}(y + \text{dist}(y, Y^c)e_1) = C_{\theta(y)}(y + \text{dist}(y, Y^c)e_1; \kappa, \delta) \) is the cone defined in (3.3) (see Figure 10).

Therefore, we obtain

\[ V_Y^{(x)}(y) < V_{B_{\text{diam } Y}(y) \setminus \left( C_{\theta(y)}(y + \text{dist}(y, Y^c)e_1) \right)}^{(x)}(y) = V_{B_{\text{diam } Y}(0) \setminus \left( C_{\theta(y)}(\text{dist}(y, Y^c)e_1) \right)}^{(x)}(0). \]

For any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \), the body \( X \) contains the ball \( B_{R_0}(x) \), and we obtain

\[ V_X^{(x)}(x) \geq V_{B_{R_0}(x)}^{(x)}(x) = V_{B_{R_0}(0)}^{(x)}(0). \]
Hence, for any points $x \in X$ with $\operatorname{dist}(x, X^c) \geq R_0$ and $y \in Y$ with $\operatorname{dist}(y, Y^c) \leq \tilde{R}$, we get

$$V_X^{(\alpha)}(x) - V_Y^{(\alpha)}(y) > V_{B_{R_0}(0)}^{(\alpha)}(0) - V_{B_{\operatorname{diam}(Y)}(0) \setminus C(Y) \cap B_{\tilde{Y}}(y)}^{(\alpha)}(0)$$

$$= \left( \int_{C(Y) \cap B_{\tilde{Y}}(y)} - \int_{B_{\operatorname{diam}(Y)}(0) \setminus B_{R_0}(0)} \right) |\xi|^{\alpha-m} d\xi$$

$$\geq E \left( \operatorname{dist}(y, Y^c) \right)$$

$$\geq 0,$$

where $E = E(\cdot; \alpha, \kappa, \delta, \operatorname{diam}Y, R_0)$ is defined in Lemma 3.8, that is, we used the conclusion in Lemma 3.8 with $D = \operatorname{diam}Y$.

\begin{corollary}
Let $\alpha \leq 0$. Let $\Omega$ be a body in $\mathbb{R}^m$ whose complement satisfies the uniform boundary inner cone condition of aperture angle $\kappa$ and height $\delta$. Any $r^{\alpha-m}$-center of $\Omega$ belongs to the intersection $\Omega \sim \tilde{R}B^m \cap Uf(\Omega)$, where $\tilde{R} = \tilde{R}(\alpha, \kappa, \delta, \operatorname{diam} \Omega, R_\infty(\Omega))$ is given in Lemma 3.8.
\end{corollary}

\begin{example}
Let $\alpha \leq 0$. For $0 \leq \varepsilon \leq 1$, let

$$\Omega_\varepsilon = \left( [-3, -1] \times B_1^{m-1} \right) \cup \left( [-1, 1] \times \varepsilon B_1^{m-1} \right) \cup \left( [1, 3] \times B_1^{m-1} \right).$$

We take an open cone $C$ of aperture angle $\kappa$ and height $\delta$ such that the complement of $\Omega_0$ satisfies the uniform interior cone condition for $C$. Then, for any $0 < \varepsilon \leq 1$, the complement of the body $\Omega_\varepsilon$ satisfies the uniform interior cone condition for $C$.

We remark $\operatorname{diam} \Omega_\varepsilon = 2\sqrt{10}$ and $R_\infty(\Omega_\varepsilon) = 1$ for any $0 \leq \varepsilon \leq 1$. Let $\tilde{R} = \tilde{R}(\alpha, \kappa, \delta, 2\sqrt{10}, 1)$ be as in Lemma 3.8, and fix $0 < \varepsilon < \tilde{R}$.

Since $Uf(\Omega) = [-2, 2] \times \{0\}^{m-1}$ and $\tilde{R} < 1$, Corollary 3.10 implies that any $r^{\alpha-m}$-center of the body $\Omega_\varepsilon$ belongs to the disjoint union of the intervals

$$\left( [-2, -1 - \sqrt{\tilde{R}^2 - \varepsilon^2}] \cup [1 + \sqrt{\tilde{R}^2 - \varepsilon^2}, 2) \right) \times \{0\}^{m-1}.$$
Radial symmetry of the kernel of $V_{\Omega_\epsilon}^{(\alpha)}$ guarantees that each interval has an $r^{\alpha-m}$-center. In particular, the potential $V_{\Omega_\epsilon}^{(\alpha)}$ has at least two maximizers.

Example 3.12 Let $\alpha \leq 0$, and $\Omega = B_3(0) \setminus B_1(0)$. We take an open cone $C$ of aperture angle $\kappa$ and height $\delta$ such that the complement of $\Omega$ satisfies the uniform boundary inner cone condition for $C$. We remark $\text{diam} \Omega_1 = 6$ and $R_\infty(\Omega_1) = 1$. Let $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(\alpha, \kappa, \delta, 6, 1)$ be as in Lemma 3.8. Since $U \mathcal{U}(\Omega_1) = B_2(0)$ and $\tilde{\mathcal{R}} < 1$, Corollary 3.10 implies that any $r^{\alpha-m}$-center belongs to the annulus $B_2(0) \setminus B_1(0)$.

Radial symmetry of $V_{\Omega_\epsilon}^{(\alpha)}$ guarantees the existence of a positive constant $1 + \tilde{\mathcal{R}} \leq \rho \leq 2$ such that the set of $r^{\alpha-m}$-centers contains the sphere $\rho S^{m-1}$.

4 Estimation of a Potential with a Summable Kernel

Let $\Omega$ be a body (the closure of a bounded open set) in $\mathbb{R}^m$. In this section, we estimate a potential of the form

$$K_{\Omega}^{(\alpha)}(x, t) = \int_\Omega k_\alpha(|x - \xi|, t) d\xi, \quad x \in \mathbb{R}^m, \quad t > 0. \quad (4.1)$$

Assumption 4.1 In the sequel for the kernel $k_\alpha$ in (4.1), we will assume from time to time some (or all) of the following conditions:

1. There is a pair of positive functions $(\psi, \tilde{k}_\alpha)$ such that the kernel $k_\alpha$ is expressed as $k_\alpha(r, t) = \psi(t)\tilde{k}_\alpha(r, t)$, and that, as $t$ tends to $0^+$, $\tilde{k}_\alpha(r, t)$ compactly converges to $r^{\alpha-m}$ for $r \in (0, +\infty)$.
2. For any $t > 0$, there is a positive $\beta$ such that $\tilde{k}_\alpha(\cdot, t)$ satisfies the condition $(C^0_\beta)$.
3. For any $t > 0$, $\tilde{k}_\alpha(\cdot, t)$ is strictly decreasing.
4. For each positive $t$, we have

$$\int_{\mathbb{R}^m} k_\alpha(|\xi|, t) d\xi = 1.$$  
5. For any positive $\rho$, we have

$$\lim_{t \to 0^+} \int_{\mathbb{R}^m \setminus B_\rho(0)} k_\alpha(|\xi|, t) d\xi = 0.$$

We remark that the condition (1) in Assumption 4.1 implies the positivity of the kernel $k_\alpha$. The condition (2) will be used for the continuity of $K_{\Omega}^{(\alpha)}$ and the existence of $k_\alpha$-centers. Usually, a radially symmetric kernel is said to be summable if it satisfies the conditions (4) and (5).

Lemma 4.2 Let $\alpha \leq 0$. Suppose that $k_\alpha$ satisfies the conditions (1) and (2) in Assumption 4.1. Let $X$ and $Y$ be bodies in $\mathbb{R}^m$. Suppose that the complement of $Y$ satisfies
the uniform boundary inner cone condition of aperture angle \( \kappa \) and height \( \delta \). Let \( 0 < R_0 \leq R_\infty (X) \), and \( \tilde{R} = \tilde{R}(\alpha, \kappa, \delta, \text{diam} \, Y, R_0) \) be given in Lemma 3.8. For any \( 0 < b < 1 \), there exists a positive \( \tau_1 \) such that if \( 0 < t < \tau_1 \), then, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \) with \( (b/2)\tilde{R} \leq \text{dist}(y, Y^c) \leq b\tilde{R} \), we have

\[
K_X^{(\alpha)}(x, t) < K_Y^{(\alpha)}(y, t) < K_{B_{\text{diam} \, Y}(0) \setminus C_{\theta(y)}(\text{dist}(y, Y^c)e_1)}^{(\alpha)}(0, t),
\]

where \( C_{\theta(y)}(\text{dist}(y, Y^c)e_1) = C_{\theta(y)}(\text{dist}(y, Y^c)e_1; \kappa, \delta) \) is the cone defined in (3.3).

By the condition (1) in Assumption 4.1, there exists a positive constant \( \tau_1 \) such that if \( 0 < t < \tau_1 \), then, for any \((b/2)\tilde{R} \leq r \leq \text{diam} \, Y \), we have

\[
\left| \tilde{k}_\alpha(r, t) - r^{\alpha - m} \right| \leq \frac{E(b\tilde{R})}{2 \left( \text{Vol} \left( C((b/2)\tilde{R}e_1; \pi, +\infty) \cap B_{\text{diam} \, Y}(0) \right) + \text{Vol} \left( B_{\text{diam} \, Y}(0) \setminus B_{R_0}(0) \right) \right)}
\]

where \( E = E(\cdot; \alpha, \kappa, \delta, \text{diam} \, Y, R_0) \) is defined in Lemma 3.8. Since, for any \( y \in Y \) with \((b/2)\tilde{R} \leq \text{dist}(y, Y^c) \), the cone \( C_{\theta(y)}(\text{dist}(y, Y^c)e_1) \) is contained in the half-space \( C((b/2)\tilde{R}e_1; \pi, +\infty) \), if \( 0 < t < \tau_1 \), then, for any \( y \in Y \) with \((b/2)\tilde{R} \leq \text{dist}(y, Y^c) \), we obtain

\[
\left| \left( \int_{C_{\theta(y)}(\text{dist}(y, Y^c)e_1) \cap B_{\text{diam} \, Y}(0)} - \int_{B_{\text{diam} \, Y}(0) \setminus B_{R_0}(0)} \right) (\tilde{k}_\alpha (| \xi |, t) - | \xi |^{\alpha - m}) d\xi \right|
\]

\[
\leq \left( \int_{C_{\theta(y)}(\text{dist}(y, Y^c)e_1) \cap B_{\text{diam} \, Y}(0)} + \int_{B_{\text{diam} \, Y}(0) \setminus B_{R_0}(0)} \right) \left| \tilde{k}_\alpha (| \xi |, t) - | \xi |^{\alpha - m} \right| d\xi
\]

\[
< \frac{1}{2} E(b\tilde{R}).
\]

Hence if \( 0 < t < \tau_1 \), then, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \) with \((b/2)\tilde{R} \leq \text{dist}(y, Y^c) \leq b\tilde{R} \), we obtain

\[
\frac{K_X^{(\alpha)}(x, t) - K_Y^{(\alpha)}(y, t)}{\psi(t)} > \frac{K^{(\alpha)}_{B_{R_0}(0)}(0, t) - K_{B_{\text{diam} \, Y}(0) \setminus C_{\theta(y)}(\text{dist}(y, Y^c)e_1)}^{(\alpha)}(0, t)}{\psi(t)}
\]

\[
= \left( \int_{C_{\theta(y)}(\text{dist}(y, Y^c)e_1) \cap B_{\text{diam} \, Y}(0)} - \int_{B_{\text{diam} \, Y}(0) \setminus B_{R_0}(0)} \right) \tilde{k}_\alpha (| \xi |, t) d\xi
\]
Lemma 4.3 Let \( \alpha, k_{\alpha}, X, Y, R_0, \bar{R}, b, \) and \( \tau_1 \) be as in Lemma 4.2. Moreover, we assume the condition (3) in Assumption 4.1. If \( 0 < t < \tau_1 \), then, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \) with \( \text{dist}(y, Y^c) \leq (b/2)\bar{R} \), we have \( K_Y^{(\alpha)}(y, t) < K_X^{(\alpha)}(x, t) \).

Proof In the same manner as in Theorem 3.9, for any point \( y \in Y \), we can choose a constant \( 0 \leq \theta(y) \leq (\pi - \kappa)/2 \) such that, for each \( t \), we have

\[
K_Y^{(\alpha)}(y, t) < K^{(\alpha)}_{B_{\text{diam}} Y(0) \setminus C_{\theta(y)}(\text{dist}(y, Y^c)e_1)}(0, t),
\]

where \( C_{\theta(y)}(\text{dist}(y, Y^c)e_1) = C_{\theta(y)}(\text{dist}(y, Y^c)e_1; \kappa, \delta) \) is the cone defined in (3.3). Thus, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \) with \( \text{dist}(y, Y^c) \leq (b/2)\bar{R} \), we obtain

\[
\frac{K_X^{(\alpha)}(x, t) - K_Y^{(\alpha)}(y, t)}{\psi(t)} > \frac{K^{(\alpha)}_{B_{R_0}(0)}(0, t) - K^{(\alpha)}_{B_{\text{diam}} Y(0) \setminus C_{\theta(y)}(\text{dist}(y, Y^c)e_1)}(0, t)}{\psi(t)}
\]

\[
= \left( \int_{C_{\theta(y)}(\text{dist}(y, Y^c)e_1) \cap B_{\text{diam}} Y(0) - \int_{B_{\text{diam}} Y(0) \setminus B_{R_0}(0)} \right) \bar{k}_{\alpha} (| \xi | , t) d\xi
\]

\[
> \left( \int_{C_{\theta(y)}((b/2)\bar{R} + \text{dist}(y, Y^c))e_1) \cap B_{\text{diam}} Y(0) - \int_{B_{\text{diam}} Y(0) \setminus B_{R_0}(0)} \right) \bar{k}_{\alpha} (| \xi | , t) d\xi
\]

\[
> 0.
\]

Here, the second inequality follows from the strictly decreasing behavior of \( k_{\alpha}(. , t) \). The third inequality was shown in Lemma 4.2.

\[ \square \]

Proposition 4.4 Let \( \alpha, k_{\alpha}, X, Y, R_0, \bar{R}, b, \) and \( \tau_1 \) be as in Lemma 4.3. If \( 0 < t < \tau_1 \), then, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \) with \( \text{dist}(y, Y^c) \leq b\bar{R} \), we have \( K_Y^{(\alpha)}(y, t) < K_X^{(\alpha)}(x, t) \).

Lemma 4.5 Suppose that \( k_{\alpha} \) satisfies the conditions (4) and (5) in Assumption 4.1. Let \( C_{\theta}(x; \kappa, \delta) \) be the cone defined in (3.3). For any \( \theta \), we have

\[
\lim_{t \to 0^+} \int_{C_{\theta}(0; \kappa, \delta)} k_{\alpha} (| \xi | , t) d\xi = \frac{\sigma(C(0; \kappa, 1) \cap S^{m-1})}{\sigma(S^{m-1})}.
\]

\[ \square \]
We remark that the conditions (4) and (5) imply

$$\lim_{t \to 0^+} \int_{B_\delta(0)} k_\alpha (|\xi|, t) \, d\xi = \lim_{t \to 0^+} \left( \int_{\mathbb{R}^m} - \int_{\mathbb{R}^m \setminus B_\delta(0)} \right) k_\alpha (|\xi|, t) \, d\xi = 1.$$ 

Since we have

$$\text{Vol} (C_{\theta} (0; \kappa, \delta)) = \frac{\sigma (C(0; \kappa, 1) \cap S^{m-1})}{\sigma (S^{m-1})} \text{Vol} (B_\delta(0)),$$

the rotation invariance of our potential implies

$$\int_{C_{\theta}(0; \kappa, \delta)} k_\alpha (|\xi|, t) \, d\xi = \frac{\sigma (C(0; \kappa, 1) \cap S^{m-1})}{\sigma (S^{m-1})} \int_{B_\delta(0)} k_\alpha (|\xi|, t) \, d\xi \rightarrow \frac{\sigma (C(0; \kappa, 1) \cap S^{m-1})}{\sigma (S^{m-1})}$$

as \( t \) tends to \( 0^+ \).

**Proposition 4.6** Suppose that \( k_\alpha \) is positive and satisfies the conditions (4) and (5) in Assumption 4.1. Let \( X \) and \( Y \) be bodies in \( \mathbb{R}^m \). Suppose that the complement of \( Y \) satisfies the uniform interior cone condition of aperture angle \( \kappa \) and height \( \delta \). Let \( 0 < R_0 \leq R_\infty (X) \). There exists a positive \( \tau_2 \) such that if \( 0 < t < \tau_2 \), then, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y^c \), we have \( K_Y^{(\alpha)} (y, t) < K_X^{(\alpha)} (x, t) \).

**Proof** By the assumption of \( k_\alpha \), we can choose a positive constant \( \tau_{21} \) such that if \( 0 < t < \tau_{21} \), then, for any point \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \), we have

$$K_X^{(\alpha)} (x, t) \geq K_{B_{R_0}(0)}^{(\alpha)} (0, t) > 1 - \frac{\sigma (C(0; \kappa, 1) \cap S^{m-1})}{2\sigma (S^{m-1})},$$

where the cone \( C(0; \kappa, 1) \) is defined in (3.1).

On the other hand, we can choose a positive constant \( \tau_{22} \) such that if \( 0 < t < \tau_{22} \), then, for any \( y \in Y^c \), the uniform interior cone condition of \( Y^c \) and Lemma 4.5 imply

$$K_Y^{(\alpha)} (y, t) \leq K_{\mathbb{R}^m \setminus C(0; \kappa, \delta)}^{(\alpha)} (0, t) = 1 - K_{C(0; \kappa, \delta)}^{(\alpha)} (0, t) < 1 - \frac{\sigma (C(0; \kappa, 1) \cap S^{m-1})}{2\sigma (S^{m-1})}.$$ 

Taking \( \tau_2 = \min \{ \tau_{21}, \tau_{22} \} \), the proof is completed. \( \square \)

**Theorem 4.7** Let \( \alpha < 0 \). Suppose that \( k_\alpha \) satisfies all the conditions in Assumption 4.1. Let \( X \) and \( Y \) be bodies in \( \mathbb{R}^m \). Suppose that the complement of \( Y \) satisfies the uniform interior cone condition of aperture angle \( \kappa \) and height \( \delta \). Let \( 0 < R_0 \leq R_\infty (X) \), and \( \tilde{R} = \tilde{R}(\alpha, \kappa, \delta, \text{diam} Y, R_0) \) be given in Lemma 3.8. For any \( 0 < b < 1 \), there exists a positive \( \tau \) such that if \( 0 < t < \tau \), then, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in \mathbb{R}^m \) with \( \text{dist}(y, Y^c) \leq b\tilde{R} \), we have \( K_Y^{(\alpha)} (y, t) < K_X^{(\alpha)} (x, t) \).
Proof Thanks to Lemma 2.3, the complement of $Y$ satisfies the uniform boundary inner cone condition of aperture angle $\kappa$ and height $\delta$. Let $\tau_1$ and $\tau_2$ be as in Propositions 4.4 and 4.6, respectively. Taking $\tau = \min\{\tau_1, \tau_2\}$, we obtain the conclusion. □

Corollary 4.8 Let $\alpha \leq 0$ and $k_\alpha$ be as in Theorem 4.7. Let $\Omega$ be a body in $\mathbb{R}^m$ whose complement satisfies the uniform interior cone condition of aperture angle $\kappa$ and height $\delta$. Let $\tilde{R} = \tilde{R}(\alpha, \kappa, \delta, \text{diam } \Omega, R_\infty(\Omega))$ be given in Lemma 3.8. For any $0 < b < 1$, there exists a positive constant $\tau$ such that if $0 < t < \tau$, then any $k_\alpha$-center of $\Omega$ at time $t$ is contained in the intersection $(\Omega \sim b\tilde{R}B^m) \cap U_f(\Omega)$.

Proof Thanks to Proposition 2.26, all the $k_\alpha$-centers are contained in the minimal unfolded region of $\Omega$. Hence, combining Theorem 4.7, we obtain the conclusion. □

Example 4.9 Let $\alpha \leq 0$. Suppose that $k_\alpha$ satisfies all the conditions in Assumption 4.1. Let $\varepsilon, \Omega_\varepsilon$, $C$, and $\tilde{R}$ be as in Example 3.11. Fix $a > b > 1$ and an $0 < \varepsilon < b\tilde{R}$.

Corollary 4.8 guarantees the existence of a positive constant $\tau$ such that if $0 < t < \tau$, then any $k_\alpha$-center of the body $\Omega_\varepsilon$ at time $t$ belongs to the disjoint union of the intervals

$$\left(-2, -1 - \sqrt{(b\tilde{R})^2 - \varepsilon^2}\right) \cup \left[1 + \sqrt{(b\tilde{R})^2 - \varepsilon^2}, 2\right) \times \{0\}^{m-1}.$$ 

Radial symmetry of the kernel of $K_{\Omega_\varepsilon}^{(\alpha)}(\cdot, t)$ guarantees that each interval has an $k_\alpha$-center. In particular, the potential $K_{\Omega_\varepsilon}^{(\alpha)}(\cdot, t)$ has at least two maximizers for any sufficiently small $t$.

Example 4.10 Let $\alpha \leq 0$. Suppose that $k_\alpha$ satisfies all the conditions in Assumption 4.1. Let $\Omega$, $C$, and $\tilde{R}$ be as in Example 3.12. Fix $a > b > 1$.

Corollary 4.8 guarantees the existence of a positive constant $\tau$ such that if $0 < t < \tau$, then any $k_\alpha$-center of $\Omega$ at time $t$ belongs to the annulus $B_2(0) \setminus B_{1+b\tilde{R}}(0)$. Radial symmetry of $K_{\Omega}^{(\alpha)}(\cdot, t)$ guarantees the existence of a positive constant $1 + b\tilde{R} \leq \rho(t) \leq 2$ such that the set of $k_\alpha$-centers of $\Omega$ at time $t$ contains the sphere $\rho(t)S^{m-1}$ for any sufficiently small $t$.

Corollary 4.11 Let $\alpha \leq 0$ and $k_\alpha$ be as in Theorem 4.7. Let $\Omega$ be a convex body in $\mathbb{R}^m$. Let $\tilde{R} = \tilde{R}(\alpha, \pi, +\infty, \text{diam } \Omega, R_\infty(\Omega))$ be given in Lemma 3.8. Let $0 < b < 1$, and $\Omega' = (\Omega \sim b\tilde{R}B^m) \cap U_f(\Omega)$. Suppose the existence of a positive constant $\tau'$ such that, for any $0 < t < \tau'$, $k_\alpha(r, t)r^{m-1}$ is decreasing on the interval $[d(\Omega, \Omega'), D(\Omega, \Omega')]$ with respect to $r$. There exists a positive constant $\tau \leq \tau'$ such that, for any $0 < t < \tau$, $\Omega$ has a unique $k_\alpha$-center at time $t$.

Proof Since $(\Omega \sim b\tilde{R}B^m) \cap U_f(\Omega)$ is convex and contained in the interior of $\Omega$, Propositions 2.19 guarantees the conclusion. □

Theorem 4.12 Let $\alpha \leq 0$. Suppose that $k_\alpha$ satisfies all the conditions in Assumption 4.1. Let $\Omega$ be a body in $\mathbb{R}^m$ whose complement satisfies the uniform interior cone condition of aperture angle $\kappa$ and height $\delta$. For any decreasing sequence $\{t_\ell\}$ with zero
limiting value and any \( k_\alpha \)-center \( c_\alpha (t_\ell) \) at time \( t_\ell \), the distance between \( c_\alpha (t_\ell) \) and the set of \( r^{\alpha - m} \)-centers tends to zero as \( \ell \) goes to \( +\infty \).

**Proof** Thanks to Corollary 4.8, we may assume that any \( k_\alpha \)-center at time \( t_\ell \) belongs to the inner-parallel body of \( \Omega \) of radius \( (1/2) \tilde{R} \), where \( \tilde{R} = \tilde{R}(\alpha, \kappa, \delta, \text{diam } \Omega, R_\infty(\Omega)) \) is given in Lemma 3.8. Since the inner-parallel body is compact, without loss of generality, we assume that \( \{ c_\alpha (t_\ell) \} \) converges to a point \( c_\alpha \). In order to show that \( c_\alpha \) is an \( r^{\alpha - m} \)-center of \( \Omega \), we assume that \( c_\alpha \) is not any \( r^{\alpha - m} \)-center, and let us derive a contradiction.

Fix an arbitrary \( 0 < \varepsilon < (1/2) \tilde{R} \). Then, for any point \( x \) in the inner-parallel body of \( \Omega \) of radius \( (1/2) \tilde{R} \), we have

\[
K^{(\alpha)}_\Omega (x, t_\ell) = \left( \int_{\Omega \setminus B_\varepsilon (x)} + \int_{B_\varepsilon (x)} \right) k_\alpha (| x - \xi |, t_\ell) d\xi
\]

\[
= \int_{\Omega \setminus B_\varepsilon (x)} k_\alpha (| x - \xi |, t_\ell) d\xi + \sigma \left( S^{m-1} \right) \int_0^\varepsilon k_\alpha (r, t_\ell) r^{m-1} dr.
\]

Therefore, the maximum value of \( K^{(\alpha)}_\Omega (\cdot, t_\ell) \) is attained at \( c_\alpha (t_\ell) \) if and only if that of the function

\[
\Omega \sim \frac{1}{2} \tilde{R} B^m \ni x \mapsto \int_{\Omega \setminus B_\varepsilon (x)} \tilde{k}_\alpha (| x - \xi |, t_\ell) d\xi \in \mathbb{R}
\]

is attained at \( c_\alpha (t_\ell) \).

Let \( p \) be an \( r^{\alpha - m} \)-center of \( \Omega \). We remark that, by Corollary 3.10, \( p \) is in the inner-parallel body of \( \Omega \) of radius \( \tilde{R} \).

Thanks to the condition (1) in Assumption 4.1, there exists an \( L_1 \in \mathbb{N} \) such that, for any \( \ell \geq L_1 \) and \( \varepsilon \leq r \leq \text{diam } \Omega \), we have

\[
| \tilde{k}_\alpha (r, t_\ell) - r^{\alpha - m} | < \frac{V_\Omega^{(\alpha)} (p) - V_\Omega^{(\alpha)} (c_\alpha)}{6 \text{Vol}(\Omega)}.
\]

Thus, if \( \ell \geq L_1 \), then we have

\[
\left| \int_{\Omega \setminus B_\varepsilon (x)} \tilde{k}_\alpha (| x - \xi |, t_\ell) - \int_{\Omega \setminus B_\varepsilon (x)} | x - \xi |^{\alpha - m} d\xi \right| < \frac{V_\Omega^{(\alpha)} (p) - V_\Omega^{(\alpha)} (c_\alpha)}{6}
\]

for any \( x \in \Omega \sim (1/2) \tilde{R} B^m \). By Proposition 2.8, there exists an \( L_2 \in \mathbb{N} \) such that, for any \( \ell \geq L_2 \), we have

\[
\left| V_\Omega^{(\alpha)} (c_\alpha (t_\ell)) - V_\Omega^{(\alpha)} (c_\alpha) \right| < \frac{V_\Omega^{(\alpha)} (p) - V_\Omega^{(\alpha)} (c_\alpha)}{6}.
\]

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Let $L = \max\{L_1, L_2\}$. If $\ell \geq L$, then we obtain

$$
\left| \int_{\Omega \setminus B_\varepsilon(c_n(t_\ell))} \tilde{k}_\alpha (| c_\alpha (t_\ell) - \xi |, t_\ell) d\xi - \int_{\Omega \setminus B_\varepsilon(c_\alpha)} | c_\alpha - \xi |^\alpha d\xi \right| \\
\leq \left| \int_{\Omega \setminus B_\varepsilon(c_\alpha(t_\ell))} \tilde{k}_\alpha (| c_\alpha (t_\ell) - \xi |, t_\ell) d\xi - \int_{\Omega \setminus B_\varepsilon(c_\alpha)} | c_\alpha - \xi |^\alpha d\xi \right| \\
+ \left| V_{\Omega}^{(\alpha)}(c_\alpha(t_\ell)) - V_{\Omega}^{(\alpha)}(c_\alpha) \right| \\
< \frac{V_{\Omega}^{(\alpha)}(p) - V_{\Omega}^{(\alpha)}(c_\alpha)}{3}
$$

(see also Proposition 2.7). Hence we obtain

$$
0 \leq \int_{\Omega \setminus B_\varepsilon(c_\alpha(t_\ell))} \tilde{k}_\alpha (| c_\alpha (t_\ell) - \xi |, t_\ell) d\xi - \int_{\Omega \setminus B_\varepsilon(p)} \tilde{k}_\alpha (| p - \xi |, t_\ell) d\xi \\
< \left( V_{\Omega}^{(\alpha)}(c_\alpha) + \frac{V_{\Omega}^{(\alpha)}(p) - V_{\Omega}^{(\alpha)}(c_\alpha)}{3} \right) - \left( V_{\Omega}^{(\alpha)}(p) - \frac{V_{\Omega}^{(\alpha)}(p) - V_{\Omega}^{(\alpha)}(c_\alpha)}{6} \right) \\
= -\frac{V_{\Omega}^{(\alpha)}(p) - V_{\Omega}^{(\alpha)}(c_\alpha)}{2},
$$

which contradicts to the maximality of $V_{\Omega}^{(\alpha)}(p)$.

\[ \square \]

**Corollary 4.13** Let $\alpha \leq 0$ and $k_\alpha$ be as in Theorem 4.12. Let $\Omega$ be a convex body. The set of $k_\alpha$-centers at time $t$ converges to the set of $r^{\alpha-m}$-centers as $t$ tends to $0^+$ with respect to the Hausdorff distance.

**Proof** Theorem 2.13 guarantees the uniqueness of an $r^{\alpha-m}$-center of $\Omega$. Hence Theorem 4.12 implies the conclusion.

\[ \square \]

## 5 Applications to the Poisson Integral

Let $\Omega$ be a body (the closure of a bounded open set) in $\mathbb{R}^m$. In this section, we apply the results in the previous section to the Poisson integral for the upper half-space. In other words, we consider the small-height behavior of illuminating centers of a body.

For the Poisson integral, the kernel in (4.1) is given by

$$
k_{-1}(r, h) = \psi(h) \bar{k}_{-1}(r, h) = \frac{2h}{\sigma_m(S^m)} \left( r^2 + h^2 \right)^{-(m+1)/2}. \quad (5.1)
$$

It is easy to check that the kernel (5.1) satisfies the conditions (1), (2), and (3) in Assumption 4.1. From (2.5) and (2.6), the kernel (5.1) satisfies the conditions (4) and (5) in Assumption 4.1.

**Proposition 5.1** Let $X$ and $Y$ be bodies in $\mathbb{R}^m$. Suppose that the complement of $Y$ satisfies the uniform boundary inner cone condition of aperture angle $\kappa$ and height
Let \( 0 < R_0 \leq R_\infty(X) \), and \( \tilde{R} = \tilde{R}(-1, \kappa, \delta, \text{diam } Y, R_0) \) be given in Lemma 3.8. For any \( 0 < b < 1 \), there exists a positive \( h_1 \) such that if \( 0 < h < h_1 \), then, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \) with \( \text{dist}(y, Y^c) \leq b \tilde{R} \), we have \( A_Y(y, h) < A_X(x, h) \).

(This fact follows from Proposition 4.4.)

**Proposition 5.2** Let \( X \) and \( Y \) be bodies in \( \mathbb{R}^m \). Suppose that the complement of \( Y \) satisfies the uniform interior cone condition of aperture angle \( \kappa \) and height \( \delta \). Let \( 0 < R_0 \leq R_\infty(X) \). There exists a positive \( h_2 \) such that if \( 0 < h < h_2 \), then, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \), we have \( A_Y(y, h) < A_X(x, h) \).

(This fact follows from Proposition 4.6.)

**Proposition 5.3** Let \( X \) and \( Y \) be bodies in \( \mathbb{R}^m \). Suppose that the complement of \( Y \) satisfies the uniform interior cone condition of aperture angle \( \kappa \) and height \( \delta \). Let \( 0 < R_0 \leq R_\infty(X) \), and \( \tilde{R} = \tilde{R}(-1, \kappa, \delta, \text{diam } Y, R_0) \) be given in Lemma 3.8. For any \( 0 < b < 1 \), there exists a positive \( h_0 \) such that if \( 0 < h < h_0 \), then, for any \( x \in X \) with \( \text{dist}(x, X^c) \geq R_0 \) and \( y \in Y \) with \( \text{dist}(y, Y^c) \leq b \tilde{R} \), we have \( A_Y(y, h) < A_X(x, h) \).

(This fact follows from Theorem 4.7.)

**Corollary 5.4** Let \( \Omega \) be a body in \( \mathbb{R}^m \) whose complement satisfies the uniform interior cone condition of aperture angle \( \kappa \) and height \( \delta \). Let \( \tilde{R} = \tilde{R}(-1, \kappa, \delta, \text{diam } \Omega, R_\infty(\Omega)) \) be given in Lemma 3.8. For any \( 0 < b < 1 \), there exists a positive \( h_0 \) such that, for any \( 0 < h < h_0 \), any illuminating center of \( \Omega \) of height \( h \) is contained in the intersection \( (\Omega \sim b \tilde{R} \mathbb{B}^m) \cap Uf(\Omega) \).

(This fact follows from Corollary 4.8.)

**Example 5.5** Let \( \epsilon, \Omega_\epsilon \), and \( C \) be as in Example 3.11. Let \( \tilde{R} = \tilde{R}(-1, \kappa, \delta, 2\sqrt{10}, 1) \) be as in Lemma 3.8. Fix a \( 0 < b < 1 \) and \( 0 < \epsilon < b \tilde{R} \).

Corollary 5.4 guarantees the existence of a positive constant \( h_0 \) such that if \( 0 < h < h_0 \), then any illuminating center of the body \( \Omega_\epsilon \) of height \( h \) belongs to the disjoint union of the intervals

\[
\left[ -2, -1 - \sqrt{\left( b \tilde{R} \right)^2 - \epsilon^2} \right] \cup \left[ 1 + \sqrt{\left( b \tilde{R} \right)^2 - \epsilon^2}, 2 \right) \times \{0\}^{m-1}.
\]

Radial symmetry of the Poisson kernel guarantees that each interval has an illuminating center. In particular, the Poisson integral \( P_{\Omega_\epsilon}(\cdot, h) \) has at least two maximizers for any sufficiently small \( h \).

**Example 5.6** Let \( \Omega \) and \( C \) be as in Example 3.12. Let \( \tilde{R} = \tilde{R}(-1, \kappa, \delta, 6, 1) \) be as in Lemma 3.8. Fix a \( 0 < b < 1 \).

Corollary 5.4 guarantees the existence of a positive constant \( h_0 \) such that if \( 0 < h < h_0 \), then any illuminating center of \( \Omega \) of height \( h \) belongs to the annulus \( B_2(0) \setminus \tilde{B}_{1+b \tilde{R}}(0) \). Radial symmetry of the Poisson integral guarantees the existence of a positive constant \( 1 + b \tilde{R} \leq \rho(h) \leq 2 \) such that the set of illuminating centers of \( \Omega \) of height \( h \) contains the sphere \( \rho(h)S^{m-1} \) for any sufficiently small \( h \).
Corollary 5.7 Let $\Omega$ be a convex body in $\mathbb{R}^m$. Let $\tilde{R} = \tilde{R}(-1, \pi, +\infty, \text{diam } \Omega, R_\infty(\Omega))$ be given in Lemma 3.8. Let $0 < b < 1$, and $\Omega' = (\Omega \sim b\tilde{R}B^m) \cap Uf(\Omega)$. There exists a positive constant $h_0$ such that, for any $0 < h < h_0$, $\Omega$ has a unique illuminating center of height $h$.

Proof. We can directly show that if $h \leq \sqrt{(m-1)/2d(\Omega, \Omega')}$, then the function $r \mapsto (r^2 + h^2)^{-1/2}r^{m-1}$ is decreasing for $d(\Omega, \Omega') \leq r \leq D(\Omega, \Omega')$. Let $h'_0 = \sup\{h > 0| A_{\Omega}(h') \subset \Omega' \forall h' < h\}$. Taking $h_0 = \min\{\sqrt{(m-1)/2d(\Omega, \Omega')}, h'_0\}$, Corollary 4.11 implies the conclusion. $$\square$$

Proposition 5.8 Let $\Omega$ be a body in $\mathbb{R}^m$ whose complement satisfies the uniform interior cone condition of aperture angle $\kappa$ and height $\delta$. For any decreasing sequence $\{h_\ell\}$ with zero limiting value and any illuminating center $c(h_\ell)$ of height $h_\ell$, the distance between $c(h_\ell)$ and the set of $r^{-(m-1)}$-centers tends to zero as $\ell$ goes to $+\infty$.

(This fact follows from Theorem 4.12.)

Corollary 5.9 Let $\Omega$ be a convex body in $\mathbb{R}^m$. The set of illuminating centers of height $h$ converges to the set of $r^{-(m-1)}$-centers as $h$ tends to $0^+$ with respect to the Hausdorff distance.

(This fact follows from Corollary 4.13.)

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Appendix: A Lower Bound for $\tilde{R}(-1, \pi, +\infty, \text{diam } \Omega, R_0)$

Let $\Omega$ be a convex body in $\mathbb{R}^m$. Thanks to the convexity of $\Omega$, we can take the uniform boundary inner cone of the complement of $\Omega$ as a half-space. Let $0 < R_0 < \text{diam } \Omega$. In this appendix, we give a lower bound for $\tilde{R} = \tilde{R}(-1, \pi, +\infty, \text{diam } \Omega, R_0)$ given in Lemma 3.8 with $\alpha = -1, \kappa = \pi, \delta = +\infty$, and $D = \text{diam } \Omega$. Let us estimate the root of the function

$$E(R) = \left(\int_{C(Re_1; \pi, +\infty) \cap B_{\text{diam } \Omega}(0)} - \int_{B_{\text{diam } \Omega}(0) \cap B_{R_0}(0)}\right) |\xi|^{-(m+1)} d\xi. \tag{6.1}$$

Here, we remark that, from Lemma 3.8 (3), if $R \geq R_0$, then $E(R) < 0$. Thus we may assume $R < R_0$ in what follows.

Let $\phi(R) = \arccos(R/\text{diam } \Omega)$. Using polar coordinates, we obtain

$$E(R) = \sigma_{m-2} \left(S^{m-2}\right) \int_0^{\arccos(R)} \tfrac{1}{R/\cos \theta} \left(\int_{R/\cos \theta}^{\text{diam } \Omega} r^{-2} dr\right) \sin^{m-2} \theta d\theta$$

$$- \sigma_{m-2} \left(S^{m-2}\right) \int_0^{\pi} \left(\int_{R_0}^{\text{diam } \Omega} r^{-2} dr\right) \sin^{m-2} \theta d\theta$$
\[
\begin{align*}
= & \frac{\sigma_{m-2}(S^m - 2)}{R} \left( \frac{\sin^{m-1} \varphi(R)}{m - 1} \right) + \frac{R}{\text{diam } \Omega} \int_0^{\pi} \sin^{m-2} \theta d\theta \\
& - \frac{R}{R_0} \int_0^{\pi} \sin^{m-2} \theta d\theta \\
=: & \frac{\sigma_{m-2}(S^m - 2)}{R} f(R).
\end{align*}
\]

(6.2)

Direct computation shows the following properties:

\[
\begin{align*}
\frac{d}{dR} f(R) &= \frac{1}{\text{diam } \Omega} \int_{\varphi(R)}^{\pi} \sin^{m-2} \theta d\theta - \frac{1}{R_0} \int_0^{\pi} \sin^{m-2} \theta d\theta < 0, \\
\frac{d^2}{dR^2} f(R) &= -\frac{\varphi'(R)}{\text{diam } \Omega} \sin^{m-2} \varphi(R) > 0.
\end{align*}
\]

(6.3) \hspace{1cm} (6.4)

Since we have

\[
f'(0) = \left( \frac{1}{\text{diam } \Omega} - \frac{2}{R_0} \right) \int_0^{\pi/2} \sin^{m-2} \theta d\theta,
\]

(6.5)

we obtain

\[
\tilde{R}(-1, \pi, +\infty, \text{diam } \Omega, R_0) > -\frac{f(0)}{f'(0)}
\]

\[
= \frac{1}{(m - 1) \left( \frac{2}{R_0} - \frac{1}{\text{diam } \Omega} \right) \int_0^{\pi/2} \sin^{m-2} \theta d\theta}
\]

\[
\geq \frac{R_0}{2(m - 1) \int_0^{\pi/2} \sin^{m-2} \theta d\theta}.
\]

(6.6)

For example, in the case of \( m = 2 \), the above lower bound coincides with \( R_0/\pi \approx 0.3183R_0 \).

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