Abstract. In ESOP 2008, Gulwani and Musuvathi introduced a notion of cover and exploited it to handle infinite-state model checking problems. Motivated by applications to the verification of data-aware processes, we proved in a previous paper that covers are strictly related to model completions, a well-known topic in model theory. In this paper we investigate cover transfer to theory combinations in the disjoint signatures case. We prove that for convex theories, cover algorithms can be transferred to theory combinations under the same hypothesis (equality interpolation property aka strong amalgamation property) needed to transfer quantifier-free interpolation. In the non-convex case, we show by a counterexample that cover may not exist in the combined theories. However, we exhibit a cover transfer algorithm operating also in the non-convex case for special kinds of theory combinations; these combinations (called ‘tame combinations’) concern multi-sorted theories arising in many model-checking applications (in particular, in model-checking applications oriented to data-aware verification).

1 Introduction

The cover of an existential formula \( \exists x \phi(x,y) \) (modulo a first-order theory \( T \)) is the strongest quantifier-free formula \( \psi(y) \) which is implied (modulo \( T \)) by \( \exists x \phi(x,y) \). Covers do not always exist, however they exist in many theories arising in verification, like linear (integer or real) arithmetic, \( EUF \), etc. The usefulness of covers in model-checking was already stressed in [13] and further motivated for data-aware verification in recent papers of ours [6,5,3,8]; it is also concretely witnessed in MCMT implementation since version 2.8.

An important question suggested by the application is the cover transfer problem for combined theories: suppose that covers exist in theories \( T_1, T_2 \), under which conditions do they exist also in the combined theory \( T_1 \cup T_2 \)? In this paper we show that the answer is affirmative in the disjoint signatures convex case under the same hypothesis (namely under the equality interpolation condition) under which quantifier-free interpolation transfers. Thus for convex theories we essentially obtain a necessary and sufficient condition, in the sense explained by Theorem 8 below.

We also prove that if convexity fails, the non-convex equality interpolation property [2] may not be sufficient to ensure covers transfer property (we show that \( EUF \) combined with integer difference logic is a counterexample). Finally, we prove that for
the ‘tame’ multi-sorted theory combinations used in our database-driven applications, covers existence transfers to the combined theory under only the stable infiniteness requirement for the shared sorts.

The main tool employed in our combination result is **Beth definability theorem for primitive formulae** (this theorem is shown to be equivalent to equality interpolation condition in [2]). In order to design a combined cover algorithm, we exploit the equivalence (supplied by Beth Theorem) between implicit and explicit definability. Implicit definability is reformulated, via covers for input theories, at quantifier-free level. Thus the algorithm guesses the implicitly definable variables, it eliminates them via explicit definability and uses the input covers algorithms to eliminate the remaining (non implicitly definable) variables.

Gulwani and Musuvathi in [13] also have a combined cover algorithm for convex, signature disjoint theories. Their algorithm looks quite different from ours; apart from the fact that a full correctness and completeness proof for such an algorithm seems not to have ever been published, we underline that our algorithm works on different hypotheses. In fact, we only need the equality interpolation condition and we show that such an hypothesis is not only sufficient, but also necessary for covers transfer in convex theories (thus our result is at least formally stronger).

### 2 Preliminaries

We adopt the usual first-order syntactic notions of signature, term, atom, (ground) formula, and so on; our signatures are always finite or countable and include equality. We compactly represent a tuple $\langle x_1, \ldots, x_n \rangle$ of variables as $x$. The notation $t(x), \phi(x)$ means that the term $t$, the formula $\phi$ has free variables included in the tuple $x$. This tuple is assumed to be formed by distinct variables; thus we underline that when we write e.g. $\phi(x,y)$, we mean that the tuples $x, y$ are made of distinct variables and are also disjoint from each other.

A formula is said to be universal (resp., existential) if it has the form $\forall x(\phi(x))$ (resp., $\exists x(\phi(x))$), where $\phi$ is quantifier-free. Formulae with no free variables are called sentences. From the semantic side, we use the standard notion of $\Sigma$-structure $M$ and of truth of a formula in a $\Sigma$-structure under a free variables assignment. The support of $M$ is denoted as $|M|$. The interpretation of a (function, predicate) symbol $\sigma$ in $M$ is denoted $\sigma_M$.

A $\Sigma$-theory $T$ is a set of $\Sigma$-sentences; a model of $T$ is a $\Sigma$-structure $M$ where all sentences in $T$ are true. We use the standard notation $T \models \phi$ to say that $\phi$ is true in all models of $T$ for every assignment to the variables occurring free in $\phi$. We say that $\phi$ is $T$-satisfiable iff there is a model $M$ of $T$ and an assignment to the variables occurring free in $\phi$ making $\phi$ true in $M$.

We now focus on the constraint satisfiability problem and quantifier elimination for a theory $T$. A $\Sigma$-formula $\phi$ is a $\Sigma$-constraint (or just a constraint) iff it is a conjunction

---

1 The equality interpolation condition was known to the authors of [13], in fact it was introduced by one of them some years before in [16]. Equality interpolation was extended to the non convex case in [2], where it was also semantically characterized via strong amalgamation property.
of literals. The constraint satisfiability problem for $T$ is the following: we are given a constraint $\phi(x)$ and we are asked whether there exist a model $M$ of $T$ and an assignment $I$ to the free variables $x$ such that $M, I \models \phi(x)$. A theory $T$ has quantifier elimination iff for every formula $\phi(x)$ in the signature of $T$ there is a quantifier-free formula $\phi'(x)$ such that $T \models \phi(x) \leftrightarrow \phi'(x)$. Since we are in a computational logic context, when we speak of quantifier elimination, we assume that it is effective, namely that it comes with an algorithm for computing $\phi'$ out of $\phi$. It is well-known that quantifier elimination holds in case we can eliminate quantifiers from primitive formulae, i.e., formulae of the kind $\exists y \phi(x, y)$, with $\phi$ a constraint.

We recall also some further basic notions. Let $\Sigma$ be a first-order signature. The signature obtained from $\Sigma$ by adding to it a set $g$ of new constants (i.e., 0-ary function symbols) is denoted by $\Sigma^g$. Analogously, given a $\Sigma$-structure $M$, the signature $\Sigma$ can be expanded to a new signature $\Sigma^M := \Sigma \cup \{ \bar{a} \mid a \in |M| \}$ by adding a set of new constants $\bar{a}$ (the name for $a$), one for each element $a$ in the support of $M$, with the convention that two distinct elements are denoted by different “name” constants. $M$ can be expanded to a $\Sigma^M$-structure $\bar{M} := (M, a)_{a \in |M|}$ just interpreting the additional constants over the corresponding elements. From now on, when the meaning is clear from the context, we will freely use the notation $M$ and $\bar{M}$ interchangeably: in particular, given a $\Sigma$-structure $M$ and a $\Sigma$-formula $\phi(x)$ with free variables that are all in $x$, we will write, by abuse of notation, $M \models \phi(a)$ instead of $\bar{M} \models \phi(\bar{a})$.

A $\Sigma$-homomorphism (or, simply, a homomorphism) between two $\Sigma$-structures $M$ and $N$ is a map $\mu : |M| \to |N|$ among the support sets $|M|$ of $M$ and $|N|$ of $N$ satisfying the condition $(M, I) \models \phi \Rightarrow (N, I) \models \phi$ for all $\Sigma^{|M|}$-atoms $\phi$ ($M$ is regarded as a $\Sigma^{|M|}$-structure, by interpreting each additional constant $a \in |M|$ into itself and $N$ is regarded as a $\Sigma^{|M|}$-structure by interpreting each additional constant $a \in |M|$ into $\mu(a)$). In case the last condition holds for all $\Sigma^{|M|}$-literals, the homomorphism $\mu$ is said to be an embedding and if it holds for all first order formulae, the embedding $\mu$ is said to be elementary. If $\mu : M \to N$ is an embedding which is just the identity inclusion $|M| \subseteq |N|$, we say that $M$ is a substructure of $N$ or that $N$ is an extension of $M$. Universal theories can be characterized as those theories $T$ having the property that if $M \models T$ and $\bar{N}$ is a substructure of $M$, then $\bar{N} \models T$ (see [9]). If $M$ is a structure and $X \subseteq |M|$, then there is the smallest substructure of $M$ including $X$ in its support; this is called the substructure generated by $X$. If $X$ is the set of elements of a finite tuple $\bar{a}$, then the substructure generated by $X$ has in its support precisely the $\bar{b} \in |M|$ such that $M \models b = t(\bar{a})$ for some term $t$.

Let $\mathcal{M}$ be a $\Sigma$-structure. The diagram of $\mathcal{M}$, written $\Delta_{\Sigma}(\mathcal{M})$ (or just $\Delta(\mathcal{M})$), is the set of ground $\Sigma^{\mathcal{M}}$-literals that are true in $\mathcal{M}$. An easy but important result, called Robinson Diagram Lemma [9], says that, given any $\Sigma$-structure $\bar{N}$, the embeddings $\mu : \mathcal{M} \to \bar{N}$ are in bijective correspondence with expansions of $\bar{N}$ to $\Sigma^{\mathcal{M}}$-structures which are models of $\Delta_{\Sigma}(\mathcal{M})$. The expansions and the embeddings are related in the obvious way: $\bar{a}$ is interpreted as $\mu(a)$.
3 Covers and Model Completions

We report the notion of cover taken from [13] and also the basic results proved in [7].

Fix a theory $T$ and an existential formula $\exists e \phi(e, y)$: call a residue of $\exists e \phi(e, y)$ any quantifier-free formula belonging to the set of quantifier-free formulae $Res(\exists e \phi) = \{ \theta(y, z) \mid T \models \phi(e, y) \rightarrow \theta(y, z) \}$. A quantifier-free formula $\psi(y)$ is said to be a $T$-cover (or, simply, a cover) of $\exists e \phi(e, y)$ iff $\psi(y) \in Res(\exists e \phi)$ and $\psi(y)$ implies (modulo $T$) all the other formulae in $Res(\exists e \phi)$. The following Lemma [7] (to be widely used throughout the paper) supplies a semantic counterpart to the notion of a cover:

**Lemma 1.** A formula $\psi(y)$ is a $T$-cover of $\exists e \phi(e, y)$ iff it satisfies the following two conditions: (i) $T \models \forall y (\exists e \phi(e, y) \rightarrow \psi(y))$; (ii) for every model $M$ of $T$, for every tuple of elements $a$ from the support of $M$ such that $M \models \psi(a)$ it is possible to find another model $N$ of $T$ such that $N$ embeds into $M$ and $N \models \exists e \phi(e, a)$.

We underline that, since our language is at most countable, we can assume that the models $M, N$ from (ii) above are at most countable too, by a Löwenheim-Skolem argument.

We say that a theory $T$ has uniform quantifier-free interpolation iff every existential formula $\exists e \phi(e, y)$ (equivalently, every primitive formula $\exists e \phi(e, y)$) has a $T$-cover.

It is clear that if $T$ has uniform quantifier-free interpolation, then it has ordinary quantifier-free interpolation [2], in the sense that if we have $T \models \phi(e, y) \rightarrow \phi'(y, z)$ (for quantifier-free formulae $\phi, \phi'$), then there is a quantifier-free formula $\theta(y, z)$ such that $T \models \phi(e, y) \rightarrow \theta(y, z)$ and $T \models \theta(y) \rightarrow \phi'(y, z)$. In fact, if $T$ has uniform quantifier-free interpolation, then the interpolant $\theta$ is independent on $\phi'$ (the same $\theta(y, z)$ can be used as interpolant for all entailments $T \models \phi(e, y) \rightarrow \phi'(y, z)$, varying $\phi'$).

We say that a universal theory $T$ has a model completion iff there is a stronger theory $T' \supseteq T$ (still within the same signature $\Sigma$ of $T$) such that (i) every $\Sigma$-constraint that is satisfiable in a model of $T$ is satisfiable in a model of $T'$; (ii) $T'$ eliminates quantifiers. Other equivalent definitions are possible [9]: for instance, (i) is equivalent to the fact that $T$ and $T'$ prove the same universal formulae or again to the fact that every model of $T$ can be embedded into a model of $T'$. We recall that the model completion, if it exists, is unique and that its existence implies the amalgamation property for $T$ [9]. The relationship between uniform interpolation in a propositional logic and model completion of the equational theory of the variety algebraizing it was extensively studied in [12].

In the context of first order theories, we prove an even more direct connection:

**Theorem 1.** Suppose that $T$ is a universal theory. Then $T$ has a model completion $T'$ iff $T$ has uniform quantifier-free interpolation. If this happens, $T'$ is axiomatized by the infinitely many sentences $\forall y (\exists e \phi(e, y) \rightarrow \exists e \phi(e, y))$, where $\exists e \phi(e, y)$ is a primitive formula and $\psi$ is a cover of it.

The proof (via Lemma 1 by iterating a chain construction) is in [3] (see also [4]).

4 Strong Amalgamation and Equality Interpolation

We report here the main results from [2] and from old literature like [11] concerning amalgamation, strong amalgamation and quantifier-free interpolation (some definitions
and results are slightly simplified because we restrict them to the case of universal theories).

**Definition 1.** A universal theory $T$ has the amalgamation property iff whenever we are given models $M_1$ and $M_2$ of $T$ and a common substructure $M_0$ of them, there exists a further model $M$ of $T$ endowed with embeddings $\mu_1 : M_1 \rightarrow M$ and $\mu_2 : M_2 \rightarrow M$ whose restrictions to $|M_0|$ coincide.

A universal theory $T$ has the strong amalgamation property if the above embeddings $\mu_1, \mu_2$ and the above model $M$ can be chosen so to satisfy the following additional condition: if for some $m_1, m_2$ we have $\mu_1(m_1) = \mu_2(m_2)$, then there exists an element $a$ in $|M_0|$ such that $m_1 = a = m_2$.

Amalgamation and strong amalgamation are strictly related to quantifier-free interpolation and to combined quantifier-free interpolation, as the results below show:

**Theorem 2.** [1] A universal theory $T$ has the amalgamation property iff it admits quantifier-free interpolants.

A theory $T$ is stably infinite iff every $T$-satisfiable constraint is satisfiable in an infinite model of $T$. The following Lemma comes from a trivial compactness argument:

**Lemma 2.** If $T$ is stably infinite, then every finite or countable model $M$ of $T$ can be embedded in a model $N$ of $T$ such that $|N| \setminus |M|$ is countable.

**Proof.** Consider $T \cup \Delta(M) \cup \{c_i \neq a \mid a \in |M| \}_i \cup \{c_i \neq c_j \}_i \neq j$, where $\{c_i\}_i$ is a countable set of fresh constants: by the Diagram Lemma and the downward Löwenheim-Skolem theorem [9], it is sufficient to show that this set is consistent. Suppose not; then by compactness $T \cup \Delta_0 \cup \Delta_1 \cup \{c_i \neq c_j \}_i \neq j$ is not satisfiable, for a finite subset $\Delta_0$ of $\Delta(M)$ and a finite subset $\Delta_1$ of $\{c_i \neq a \mid a \in |M| \}_i$. However, this is a contradiction because by stable infiniteness, $\Delta_0$ (being satisfiable in $M$) is satisfiable in an infinite model of $T$.

**Theorem 3.** [2] Let $T_1$ and $T_2$ be two universal stably infinite theories over disjoint signatures $\Sigma_1$ and $\Sigma_2$. If both $T_1$ and $T_2$ have the strong amalgamation property, then so does $T_1 \cup T_2$ (hence, in particular, the quantifier-free interpolation property transfers from $T_1$ and $T_2$ to their combination $T_1 \cup T_2$).

There is a converse of the previous result; for a signature $\Sigma$, let us call $\text{EUF}(\Sigma)$ the pure equality theory over the signature $\Sigma$ (this theory is easily seen to have the strong amalgamation property).

**Theorem 4.** [2] Let $T$ be a stably infinite universal theory and let $\Sigma$ be a signature disjoint from the signature of $T$ containing at least a unary predicate symbol. Then, $T \cup \text{EUF}(\Sigma)$ has quantifier-free interpolation iff $T$ has the strong amalgamation property.

According to Theorem 2, the amalgamation property has a syntactic counterpart (namely quantifier-free interpolation); there is an analogous syntactic characterization for strong amalgamation [2], which can be exploited in the design of combined interpolation algorithms [2]. Since such a condition is rather complex, we report it here only for the simplified case of convex theories, where it coincides with a well-known condition.

---

\footnote{For the results of this paper to be correct, the notion of structure (and of course that of substructure) should encompass the case of structures with empty domains. Readers feeling uncomfortable with empty domains can assume that signatures always contain an individual constant.}
previously introduced in \[16\]. Recall that a theory $T$ is *convex* iff for every constraint $\delta$, if $T \vdash \delta \rightarrow \bigwedge_{i=1}^{n} x_i = y_i$, then $T \vdash \delta \rightarrow x_i = y_i$ holds for some $i \in \{1, \ldots, n\}$.

**Definition 2.** A convex universal theory $T$ is *equality interpolating* iff for every pair $y_1, y_2$ of variables and for every pair of constraints $\delta_1(x, \bar{z}, y_1), \delta_2(x, \bar{z}, y_2)$ such that

$$T \vdash \delta_1(x, \bar{z}, y_1) \land \delta_2(x, \bar{z}, y_2) \rightarrow y_1 = y_2$$

there exists a term $t(x)$ such that

$$T \vdash \delta_1(x, \bar{z}, y_1) \land \delta_2(x, \bar{z}, y_2) \rightarrow y_1 = t \land y_2 = t.$$  

\[1\]

**Theorem 5.** \[2\] A convex universal theory $T$ having quantifier-free interpolation has the strong amalgamation property iff it is equality interpolating.  

In conclusion, if $T_1, T_2$ are universal, convex, stably infinite\[7\] have disjoint signatures and have quantifier-free interpolation, in order for $T_1 \cup T_2$ to have quantifier-free interpolation too, we need to ask that both $T_1$ and $T_2$ are equality interpolating; in view of Theorem\[4\] this condition is not only sufficient but also (almost) necessary.

In \[2\] a long list of universal strongly amalgamable (i.e. both quantifier-free interpolating and equality interpolating) theories is given, comprising both convex and non-convex theories. The list includes $\mathcal{EUF}(\Sigma)$, recursive data theories, as well as linear (both integer and real) arithmetics. For linear arithmetics (and fragments of its), it is essential to make a very careful choice of the signature, see again \[2\] (especially Subsection 4.1) for details. All the above theories admit a model completion (which coincides with the theory itself in case the theory admits quantifier elimination).

The equality interpolating property in a convex theory $T$ can be equivalently characterized using Beth definability as follows. Consider a primitive formula $\exists x \phi(x, \bar{z}, y)$ (here $\phi$ is a conjunction of literals); we say that $\exists x \phi(x, \bar{z}, y)$ *implicitly defines* $y$ in $T$ iff the following formula

$$\forall y \forall y' (\exists x \phi(x, \bar{z}, y) \land \exists x \phi(x, \bar{z}, y') \rightarrow y = y')$$

is $T$-valid. We say that $\exists x \phi(x, \bar{z}, y)$ *explicitly defines* $y$ in $T$ iff there is a term $t(x)$ such that the formula

$$\forall y (\exists x \phi(x, \bar{z}, y) \rightarrow y = t(x))$$

is $T$-valid.

For future use, we notice that, by trivial logical manipulations, the formulae \[3\] and \[4\] are logically equivalent to

$$\forall y \forall y' \forall z (\phi(x, \bar{z}, y) \land \phi(x, \bar{z}, y') \rightarrow y = y') .$$

and to

$$\forall y \forall z (\phi(x, \bar{z}, y) \rightarrow y = t(x))$$

respectively (we shall use such equivalences without explicit mention).

\[3\] A convex theory $T$ is ‘almost’ stably infinite in the sense that every constraint which is $T$-satisfiable in a $T$-model whose support has at least two elements is satisfiable also in an infinite $T$-model. The one-element model can be used to build counterexamples, though: e.g. the theory of Boolean algebras is convex (like any other universal Horn theory) but the constraint $x = 0 \land x = 1$ is only satisfiable in the degenerate one-element Boolean algebra.
We say that a theory $T$ has the *Beth definability property for primitive formulae* iff whenever a primitive formula $\exists z \phi(x, z, y)$ implicitly defines the variable $y$ then it also explicitly defines it.

**Theorem 6.** [2] A convex theory $T$ having quantifier-free interpolation is equality interpolating iff it has the Beth definability property for primitive formulae.

**Proof.** We recall the easy proof of the left-to-right side (this is the only side we need in this paper). Suppose that $T$ is equality interpolating and that

$$T \vdash \phi(x, z, y) \land \phi(x, z', y') \rightarrow y = y'$$

then there is a term $t(x)$ such that

$$T \vdash \phi(x, z, y) \land \phi(x, z', y') \rightarrow y = t(x) \land y' = t(x).$$

Replacing $z', y$ by $z, y$ via a substitution, we get precisely (6).

\[ \square \]

## 5 Convex Theories

We now collect some useful facts concerning convex theories. We fix for this section a convex, stably infinite, strongly amalgamable universal theory $T$ admitting a model completion $T^*$. We let $\Sigma$ be the signature of $T$. We fix also a $\Sigma$-constraint $\phi(x, y)$, where we assume that $y = y_1, \ldots, y_n$ (recall that the tuple $x$ is disjoint from the tuple $y$ according to our conventions from Section [2]).

For $i = 1, \ldots, n$, we let the formula $\text{ImplDef}^T_{\phi, y_i}(x)$ be the quantifier-free formula equivalent in $T^*$ to the formula

$$\forall y \forall y' (\phi(x, y) \land \phi(x, y') \rightarrow y_i = y'_i)$$

where the $y'$ are renamed copies of the $y_i$.

**Lemma 3.** Suppose that we are given a model $\mathcal{M}$ of $T$ and elements $a$ from the support of $\mathcal{M}$ such that $\mathcal{M} \not\models \text{ImplDef}^T_{\phi, y_i}(a)$ for all $i = 1, \ldots, n$. Then there exists an extension $\mathcal{N}$ of $\mathcal{M}$ such that for some $b \in [\mathcal{N}] \setminus [\mathcal{M}]$ we have $\mathcal{N} \models \phi(a, b)$.

**Proof.** By strong amalgamability, we can freely assume that $\mathcal{M}$ is generated, as a $\Sigma$-structure, by the $a$. What we need is to prove the consistency of $T \cup \Delta(\mathcal{M})$ with the set of ground sentences

$${\{ \phi(a, b) \} \cup \{ b_i \neq t(a) \}, b_i}$$

where $t(x)$ varies over $\Sigma(x)$-terms, the $b = b_1, \ldots, b_n$ are fresh constants and $i$ vary over $1, \ldots, n$. By convexity, this set is inconsistent iff there exist a term $t(x)$ and $i = 1, \ldots, n$ such that

$$T \cup \Delta(\mathcal{M}) \vdash \phi(a, y) \rightarrow y_i = t.$$  

This however implies that $T \cup \Delta(\mathcal{M})$ has the formula

$$\forall y \forall y' (\phi(a, y) \land \phi(a, y') \rightarrow y_i = y'_i)$$

as a logical consequence. If we now embed $\mathcal{M}$ into a model $\mathcal{N}$ of $T^*$, we have that $\mathcal{N} \models \text{ImplDef}^T_{\phi, y_i}(a)$, which is in contrast to $\mathcal{M} \not\models \text{ImplDef}^T_{\phi, y_i}(a)$ (because $\mathcal{M}$ is a substructure of $\mathcal{N}$ and $\text{ImplDef}^T_{\phi, y_i}(a)$ is quantifier-free).

\[ \square \]
Lemma 4. Let $L_1(x) \lor \cdots \lor L_k(x)$ be the disjunctive normal form (DNF) of $\text{ImplDef}_{\phi,\Sigma}(x)$. Then, for every $j = 1, \ldots, k$, there is a $\Sigma$-term $t_{ij}(x)$ such that

$$T \vdash L_{ij}(x) \land \phi(y, y') \rightarrow y_i = t_{ij}. \tag{8}$$

As a consequence, a formula of the kind $\text{ImplDef}_{\phi,\Sigma}(x) \land \exists y (\phi(x, y) \land \phi')$ is equivalent (modulo $T$) to the formula

$$\bigvee_{j=1}^{k_i} \exists y (y_i = t_{ij} \land L_{ij}(x) \land \phi(x, y) \land \phi') \tag{9}$$

Proof. We have that $(\bigvee_j L_{ij}) \leftrightarrow \text{ImplDef}_{\phi,\Sigma}(x)$ is a tautology, hence from the definition of $\text{ImplDef}_{\phi,\Sigma}(x)$, we have that

$$T^* \vdash L_{ij}(x) \rightarrow \forall y \forall y' (\phi(x, y) \land \phi(x, y') \rightarrow y_i = y_i') \tag{10}$$

however this formula is trivially equivalent to a universal formula, hence since $T$ and $T^*$ prove the universal formulae, we get

$$T \vdash L_{ij}(x) \land \phi(x, y) \land \phi(x, y') \rightarrow y_i = y_i' \tag{11}$$

Using Beth definability property (Theorem 6), we get (8), as required, for some terms $t_{ij}(x)$. Finally, (9) follows from (8) by trivial logical manipulations.

In all our concrete examples, the theory $T$ has decidable quantifier-free fragment (namely it is decidable whether a quantifier-free formula is a logical consequence of $T$ or not), thus the terms $t_{ij}$ mentioned in Lemma 4 can be computed just by enumerating all possible $\Sigma(x)$-terms: the computation terminates, because the above proof shows that the appropriate terms always exist. However, this is terribly inefficient and, from a practical point of view, one needs to have at disposal dedicated algorithms to find the required equality interpolating terms.

6 The Convex Combined Cover Algorithm

Let us now fix two theories $T_1, T_2$ over disjoint signatures $\Sigma_1, \Sigma_2$. We assume that both of them satisfy the assumptions from the previous section, meaning that they are convex, stably infinite, strongly amalgamable, universal and admit model completions $T_1^*, T_2^*$ respectively. We shall supply a cover algorithm for $T_1 \cup T_2$ (thus proving that $T_1 \cup T_2$ has a model completion too).

We need to compute a cover for $\exists e \phi(x, e)$, where $\phi$ is a conjunction of $\Sigma_1 \cup \Sigma_2$-literals. By applying rewriting purification steps like

$$\phi \implies \exists d (d = t \land \phi(d/t))$$

(where $d$ is a fresh variable and $t$ is a pure term, i.e. it is either a $\Sigma_1$- or a $\Sigma_2$-term), we can assume that our formula $\phi$ is of the kind $\phi_1 \land \phi_2$, where $\phi_1$ is a $\Sigma_1$-formula and $\phi_2$ is a $\Sigma_2$-formula. Thus we need to compute a cover for a formula of the kind

$$\exists e (\phi_1(x, e) \land \phi_2(x, e)) \tag{10}$$

where $\phi_i$ is a conjunction of $\Sigma_i$-literals ($i = 1, 2$). We also assume that both $\phi_1$ and $\phi_2$ contain the literals $e_i \neq e_j$ (for $i \neq j$) as a conjunct: this can be achieved by guess-
ing a partition of the $e$ and by replacing each $e_i$ with the representative element of its equivalence class$^4$.

To manipulate formulae, our algorithm employs acyclic explicit definitions as follows. When we write $\text{ExplDef}(\bar{z}, \bar{x})$ (where $\bar{z}, \bar{x}$ are tuples of distinct variables), we mean any formula of the kind (let $\bar{z} := \bar{z}_1 \ldots \bar{z}_m$)

$$\bigwedge_{j=1}^m z_j = t_i,$$

where the term $t_i$ is pure and only the variables $z_1, \ldots, z_{i-1}, x$ can occur in it. When we assert a formula like $\exists \bar{z} (\text{ExplDef}(\bar{z}, \bar{x}) \land \varphi(\bar{z}, \bar{x}))$, we are in fact in the condition of recursively eliminating the variables $\bar{z}$ from it via terms containing only the parameters $\bar{x}$ (the ‘explicit definitions’ $z_i = t_i$ are in fact arranged acyclically).

A working formula is a formula of the kind

$$\exists \bar{z} (\text{ExplDef}(\bar{z}, \bar{x}) \land \exists \bar{z} (\varphi_1(\bar{x}, \bar{z}, \bar{e}) \land \varphi_2(\bar{x}, \bar{z}, \bar{e})))$$, \hspace{1cm} (11)

where $\varphi_1$ is a conjunction of $\Sigma_1$-literals and $\varphi_2$ is a conjunction of $\Sigma_2$-literals. The variables $\bar{x}$ are called parameters, the variables $\bar{z}$ are called defined variables and the variables $\bar{e}$ (truly) existential variables. The parameters do not change during the execution of the algorithm. We assume that $\varphi_1, \varphi_2$ in a working formula (11) always contain the literals $e_i \neq e_j$ (for distinct $e_i, e_j$ from $\bar{e}$) as a conjunct.

In our starting formula (10), there are no defined variables. However, if via some syntactic check it happens that some of the existential variables can be recognized as defined, then it is useful to display them as such (this observation may avoid redundant cases - leading to inconsistent disjuncts - in the computations below).

A working formula like (11) is said to be terminal iff for every existential variable $e_i \in \bar{e}$ we have that

$$T_1 \vdash \varphi_1 \rightarrow \neg \text{ImplDef}_{\varphi_1, e_i}^T(\bar{x}, \bar{z}) \quad \text{and} \quad T_2 \vdash \varphi_2 \rightarrow \neg \text{ImplDef}_{\varphi_2, e_i}^T(\bar{x}, \bar{z})$$.

(12)

Roughly speaking, we can say that in a terminal working formula, all variables which are not parameters are either explicitly definable or recognized as not implicitly definable by both theories; of course, a working formula with no existential variables is terminal.

**Lemma 5.** Every working formula is equivalent (modulo $T_1 \cup T_2$) to a disjunction of terminal working formulae.

**Proof.** To compute the required terminal working formulae, it is sufficient to apply the following non-deterministic procedure (the output is the disjunction of all possible outcomes). Notice that the following formula is trivially a tautology:

$$\left(\bigwedge_{e_i \in \bar{e}} \neg \text{ImplDef}_{\varphi_1, e_i}^T(\bar{x}, \bar{z}) \land \bigwedge_{e_i \in \bar{e}} \neg \text{ImplDef}_{\varphi_2, e_i}^T(\bar{x}, \bar{z})\right) \lor$$

$$\lor \bigvee_{e_i \in \bar{e}} \text{ImplDef}_{\varphi_1, e_i}^T(\bar{x}, \bar{z}) \lor \bigvee_{e_i \in \bar{e}} \text{ImplDef}_{\varphi_2, e_i}^T(\bar{x}, \bar{z})$$

$^4$ Usually, for efficiency reasons, when combining constraint satisfiability algorithms in convex theories, instead of guessing a partition, entailed equalities are propagated between the two theories. Here however, as already strongly pointed out in [13], equality propagation is not sufficient and one needs more complicated mechanisms like propagation of conditional equalities or specific case-splitting rules. Thus, in the end, we preferred to introduce the algorithm in a plain way via partitions - this is easier to explain and more transparent.
The nondeterministic procedure applies one of the following alternatives.

(1) Update \( \psi_1 \) by adding it a disjunct from the DNF of \( \bigwedge_{e_i \in e} \neg \text{ImplDef}_{T_i}^\psi (x; z) \) and \( \psi_2 \) by adding a disjunct from the DNF of \( \bigwedge_{e_i \in e} \neg \text{ImplDef}_{T_2}^\psi (x; z) \);

(2i) Select \( e_i \in e \) and \( h \in \{1, 2\} \); then update \( \psi_h \) by adding it a disjunct \( \bar{L}_{ij} \) from the DNF of \( \text{ImplDef}_{W_h}^{\bar{a}, e} (x; z) \); the equality \( e_i = t_{ij} \) (where \( t_{ij} \) is the term mentioned in Lemma 4) is added to \( \text{ExpDef}(\bar{z}; \bar{x}) \); the variable \( e_i \) becomes in this way part of the defined variables.

If alternative (1) is chosen, the procedure stops, otherwise it is recursively applied again and again (we have one truly existential variable less after applying alternative (2i), so we eventually terminate).

Thus we are left to the problem of computing a cover of a terminal working formula; this is an easily solvable problem:

**Proposition 1.** A cover of a terminal working formula \( \psi \) can be obtained just by unravelling the explicit definitions of the variables \( z \) from the formula

\[
\exists z (\text{ExpDef}(\bar{z}; \bar{x}) \land \theta_1(x, z) \land \theta_2(x, z))
\]

where \( \theta_1(x, z) \) is the \( T_1 \)-cover of \( \exists z \psi_1(x, z) \) and \( \theta_2(x, z) \) is the \( T_2 \)-cover of \( \exists z \psi_2(x, z) \).

**Proof.** We prove that for every \( T_1 \cup T_2 \)-model \( M \), for every tuple \( \bar{a}, e \) from \( |M| \) such that \( M \models \theta_1(\bar{a}, e) \land \theta_2(\bar{a}, e) \) there is an extension \( N \) of \( M \) such \( N \) is still a model of \( T_1 \cup T_2 \) and \( N \models \exists z (\psi_1(\bar{a}, e) \land \psi_2(\bar{a}, e)) \) (this is actually more than what we need, because according to Lemma 5 we need this only for the case where \( e \) is the unique tuple satisfying \( M \models \text{ExpDef}(\bar{e}; \bar{a}) \)). By a Löwenheim-Skolem argument, since our languages are countable, we can suppose that \( M \) is at most countable and actually that it is countable by stable infiniteness of our theories, see Lemma 2 (the fact that \( T_1 \cup T_2 \) is stably infinite in case both \( T_1, T_2 \) are such, comes from the proof of Nelson-Oppen combination result, see [14, 15, 10]).

According to the conditions (12) and the definition of a cover (notice that the formulae \( \neg \text{ImplDef}_{W_{1, e}}^\psi (x; z) \) do not contain the \( e \) and are quantifier-free) we have that

\( T_1 \models \theta_1 \rightarrow \neg \text{ImplDef}_{W_{1, e}}^\psi (x; z) \) and \( T_2 \models \theta_2 \rightarrow \neg \text{ImplDef}_{W_{2, e}}^\psi (x; z) \)

(for every \( e_i \in e \)). Thus, since \( M \models \neg \text{ImplDef}_{W_{1, e}}^\psi (x; z) \) and \( M \models \neg \text{ImplDef}_{W_{2, e}}^\psi (x; z) \) holds for every \( e_i \in e \), we can apply Lemma 3 and conclude that there exist a \( T_1 \)-model \( N_1 \) and a \( T_2 \)-model \( N_2 \) such that \( N_1 \models \psi_1(\bar{a}, e_1) \) and \( N_2 \models \psi_2(\bar{a}, e_2) \) for tuples \( \bar{b}_1 \in |N_1| \) and \( \bar{b}_2 \in |N_2| \), both disjoint from \( |M| \). By a Löwenheim-Skolem argument, we can suppose that \( N_1, N_2 \) are countable and by Lemma 2 even that they are both countable extensions of \( M \).

The tuples \( \bar{b}_1 \) and \( \bar{b}_2 \) have equal length because the \( \psi_1, \psi_2 \) from our working formulae entail \( e_i \neq e_j \), where \( e_i, e_j \) are different existential variables. Thus there is a bijection \( \iota : |N_1| \rightarrow |N_2| \) fixing all elements in \( M \) and mapping componentwise the \( \bar{b}_1 \) onto the \( \bar{b}_2 \). But this means that, exactly as it happens in the proof of the completeness of the Nelson-Oppen combination procedure, the \( \Sigma_2 \)-structure on \( N_2 \) can be moved back via \( \iota^{-1} \) to \( |N_1| \) in such a way that the \( \Sigma_2 \)-substructure from \( M \) is fixed and in such a way that the tuple \( \bar{b}_2 \) is mapped to the tuple \( \bar{b}_1 \). In this way, \( N_1 \) becomes a \( \Sigma_1 \cup \Sigma_2 \)-structure.
which is a model of \( T_1 \cup T_2 \) and which is such that \( \mathcal{N}_1 \models \psi_1(a, c, b_1) \land \psi_2(a, c, b_2) \), as required.

From Lemma 5 and Proposition 1 we immediately get

**Theorem 7.** Let \( T_1, T_2 \) be convex, stably infinite, strongly amalgamable, universal theories admitting a model completion. Then \( T_1 \cup T_2 \) admits a model completion too. Covers in \( T_1 \cup T_2 \) can be effectively computed as shown above.

Notice that the input cover algorithms in the above combined cover computation algorithm are used not only in the final step described in Proposition 1 but also every time we need to compute a formula \( \text{ImplDef}_{\psi_i, e_i}^{f_i}(x, z) \): according to its definition, this formula is obtained by eliminating quantifiers in \( T_i^* \) from (7) (this is done via a cover computation, reading \( \lor \) as \( \neg \exists \neg \)). In practice, implicit definability is a rather sparse phenomenon, so that in many concrete cases \( \text{ImplDef}_{\psi_i, e_i}^{f_i}(x, z) \) is trivially equivalent to \( \perp \) (in such cases, Step (2i) above can obviously be disregarded).

**Example 1.** Our results apply for instance to the case where \( T_1 \) is \( \mathcal{EUF}(\Sigma) \) (where \( \Sigma \) just contains the unary symbol \( f \)) and \( T_2 \) is linear real arithmetic. We recall that covers are computed in real arithmetic by quantifier elimination, whereas for \( \mathcal{EUF} \) one can apply the superposition-based algorithm from [7]. Let us compute the cover of

\[
\exists e_1 \cdots \exists e_4 \left( e_1 = f(x_1) \land e_2 = f(x_2) \land e_3 \land f(e_4) = x_1 \land x_1 + e_1 \leq e_3 \land e_3 \leq x_2 + e_2 \land e_4 = x_2 + e_3 \right)
\]

This is the following formula

\[
[x_2 = 0 \land f(x_1) = x_1 \land x_1 \leq 0 \land x_1 \leq f(0)] \lor [x_1 + f(x_1) < x_2 + f(x_2) \land x_2 \neq 0] \lor
\lor [x_2 \neq 0 \land x_1 + f(x_1) = x_2 + f(x_2) \land f(2x_2 + f(x_2)) = x_1 \land f(x_1 + f(x_1)) = x_1 + f(x_1)]
\]

(see the Appendix A for a fully detailed analysis). This example shows that combined cover computations may introduce highly impure terms.

The following result shows that equality interpolating is a necessary condition for our combination result:

**Theorem 8.** Let \( T \) be a convex, stably infinite, universal theory admitting a model completion and let \( \Sigma \) be a signature disjoint from the signature of \( T \) containing at least a unary predicate symbol. Then \( T \cup \mathcal{EUF}(\Sigma) \) admits a model completion iff \( T \) is equality interpolating.

**Proof.** The sufficiency comes from Theorem 7 together with the fact that \( \mathcal{EUF}(\Sigma) \) is trivially universal, convex, stably infinite, has a model completion [7] and is strongly amalgamable [2]. The necessity is because of Theorems 4, 5 together with the well-known fact [9] that a universal theory admitting a model completion has the amalgamation property.

### 7 The Non-Convex Case: a Counterexample

In this section, we show that the convexity hypothesis cannot be dropped from Theorems 7, 8 by giving a suitable counterexample. We make use of basic facts about ultrapowers (see [9] for the essential information we need).
We take as $T_1$ integer difference logic $IDL$, i.e. the theory of integer numbers under the unary operations of successor and predecessor, the constant 0 and the strict order relation $<$. This is stably infinite, strongly amalgamable, universal and has quantifier elimination (thus it coincides with its own model completion), see [2]. However it is not convex. As $T_2$, we take $EUF(\Sigma_f)$, where $\Sigma_f$ has just one unary free function symbol $f$ (this $f$ is supposed not to belong to the signature of $T_1$).

**Proposition 2.** Let $T_1, T_2$ be as above; the formula
\[ \exists e \ (0 < e \wedge e < x \wedge f(e) = 0) \] (16)
does not have a cover in $T_1 \cup T_2$.

*Proof.* Suppose that (16) has a cover $\phi(x)$. This means (according to Lemma [1]) that for every model $M$ of $T_1 \cup T_2$ and for every element $a \in |M|$ such that $M \models \phi(a)$, there is an extension $\mathcal{N}$ of $M$ such that $\mathcal{N} \models \exists e \ (0 < e \wedge e < a \wedge f(e) = 0)$.

Consider the model $M$, so specified: the support of $M$ is the set of the integers, the symbols from the signature of $T_1$ are interpreted in the standard way and the symbol $f$ is interpreted so that 0 is not in the image of $f$. Let $a_k$ be the number $k > 0$ (it is an element from the support of $M$). Clearly it is not possible to expand $M$ so that $\exists e \ (0 < e \wedge e < a_k \wedge f(e) = 0)$ becomes true: all elements in the interval $(0, k)$ are definable as iterated successors of 0 and this interval cannot be enlarged in a superstructure. We conclude that $M \not\models \neg \phi(a_k)$.

Consider now an ultrapower $\Pi_D M$ of $M$ modulo a non principal ultrafilter and let $a$ be the equivalence class of the tuple $\langle a_k \rangle$; by the fundamental Los theorem [9], $\Pi_D M \models \neg \phi(a)$. However (contradiction), now it is possible to extend $\Pi_D M$ to a superstructure $\mathcal{N}$ such that $\mathcal{N} \models \exists e \ (0 < e \wedge e < a \wedge f(e) = 0)$: in fact, $a$ is bigger in $\Pi_D M$ than all standard numbers, so if we take a further nonprincipal ultrapower $\mathcal{N}$ of $\Pi_D M$, it is well possible to change in it the evaluation of $f(b)$ for some $b < a$ and set it to 0 (in fact, as it is easily seen, there are elements $b \in |\mathcal{N}|$ less than $a$ but not in the support of $\Pi_D M$).

Even if in general covers do not exists for combination of nonconvex theories, it would be interesting to see under what conditions one can decide whether a given cover exists and, in the affirmative case, to compute it.

8 Tame Combinations

So far, we only analyzed the mono-sorted case. However, many interesting examples arising in model-checking verification are multi-sorted: this is the case of array-based systems [11] and in particular of the array-based system used in data-aware verification [6],[4]. The above examples suggest restrictions on the theories to be combined other than convexity, in particular they suggest restrictions that make sense in a multi-sorted context.

Most definitions we gave in Section [2] have straightforward natural extensions to the multi-sorted case (we leave the reader to formulate them). A little care is needed however for the disjoint signatures requirement. Let $T_1, T_2$ be multisorted theories in the signatures $\Sigma_1, \Sigma_2$: the disjointness requirement for $\Sigma_1$ and $\Sigma_2$ can be formulated in this context by saying that the only function or relation symbols in $\Sigma_1 \cap \Sigma_2$ are the equality predicates over the common sorts in $\Sigma_1 \cap \Sigma_2$. We want to strengthen this requirement:
we say that the combination $T_1 \cup T_2$ is tame iff the sorts in $\Sigma_1 \cap \Sigma_2$ can only be the codomain sort (and not a domain sort) of a symbol from $\Sigma_1$ other than an equality predicate. In other word, if a relation or a function symbol has as among its domain sorts a sort from $\Sigma_1 \cap \Sigma_2$, then this symbol is from $\Sigma_2$ (and not from $\Sigma_1$, unless it is the equality predicate).

Tame combinations arise in infinite-state model-checking (in fact, the definition is suggested by this application domain), where signatures can be split into a signature $\Sigma_2$ for 'data' and a signature $\Sigma_1$ for 'data containers', see [6], [3].

Notice that the notion of a tame combination is not symmetric in $T_1$ and $T_2$: to see this, notice that if the sorts of $\Sigma_1$ are included in the sorts of $\Sigma_2$, then $T_1$ must be a pure equality theory (but this is not the case if we swap $T_1$ with $T_2$). The combination of $\mathcal{IDL}$ and $\mathcal{EUF}(\Sigma)$ used in the counterexample of section 7 is not tame: even if we formulate $\mathcal{EUF}(\Sigma)$ as a two-sorted theory, the unique sort of $\mathcal{IDL}$ must be a sort of $\mathcal{EUF}(\Sigma)$ too, as witnessed by the impure atom $f(e) = 0$ in the formula (16). Because of this, for the combination to be tame, $\mathcal{IDL}$ should play the role of $T_2$ (the arithmetic operation symbols are defined on a shared sort); however, the unary function symbol $f \in \Sigma$ has a shared sort as domain sort, so the combination is not tame anyway.

In a tame combination, an atomic formula $A$ can only be of two kinds: (1) we say that $A$ is of the first kind iff the sorts of its root predicate are from $\Sigma_1 \setminus \Sigma_2$; (2) we say that $A$ is of the second kind iff the sorts of its root predicate are from $\Sigma_2$. We use the roman letters $e, x, \ldots$ for variables ranging over sorts in $\Sigma_1 \setminus \Sigma_2$ and the greek letters $\eta, \xi, \ldots$ for variables ranging over sorts in $\Sigma_2$. Thus, if we want to display free variables, atoms of the first kind can be represented as $A(e, x, \ldots)$, whereas atoms of the second kind can be represented as $A(\eta, \xi, \ldots, t(e, x, \ldots), \ldots)$, where $t$ are $\Sigma_1$-terms.

Suppose not that $T_1 \cup T_2$ is a tame combination and that $T_1, T_2$ are universal theories admitting model completions $T'_1, T'_2$. We propose the following algorithm to compute the cover of a primitive formula; the latter must be of the kind

$$\exists \exists \eta (\phi(e, x) \land \psi(\eta, \xi, t(e, x)))$$

(17)

where $\phi$ is a $\Sigma_1$-conjunction of literals, $\psi$ is a conjunction of $\Sigma_2$-literals and the $t$ are $\Sigma_1$-terms. The algorithm has three steps.

(i) We flatten (17) and get

$$\exists \exists \eta' \exists \eta'' (\phi(e, x) \land \eta' = t(e, x) \land \psi(\eta, \xi, \eta''))$$

(18)

where the $\eta'$ are fresh variables abstracting out the $t$ and $\eta'' = t(e, x)$ is a componentwise conjunction of equalities.

(ii) We apply the cover algorithm of $T_1$ to the formula

$$\exists \exists \eta (\phi(e, x) \land \eta' = t(e, x))$$

(19)

this gives as a result a formula $\phi'(e, \eta')$ that we put in DNF. A disjunct of $\phi$ will have the form $\phi_1(e, x) \land \phi_2(\eta', t'(x))$ after separation of the literals of the first and of the second kind. We pick such a disjunct $\phi_1(e, x) \land \phi_2(\eta', t'(x))$ of the DNF of $\phi'(e, \eta')$ and update our current primitive formula to

$$\exists \exists \exists (\xi' = t'(x) \land (\exists \exists \exists (\phi_1(e, x) \land \phi_2(\eta', \xi') \land \psi(\eta, \xi, \eta'))))$$

(20)

(this step is nondeterministic: in the end we shall output the disjunction of all possible outcomes). Here again the $\xi'$ are fresh variables abstracting out the terms $t'$. 13
Notice that, according to the definition of a tame combination, \( \phi_2(\eta', \xi') \) must be a conjunction of equalities and disequalities between variable terms, because it is a \( \Sigma_1 \)-formula (it comes from a \( T_1 \)-cover computation) and \( \eta', \xi' \) are variables of \( \Sigma_2 \)-sorts.

(iii) We apply the cover algorithm of \( T_2 \) to the formula

\[
\exists \eta \exists \eta' (\phi_2(\eta', \xi') \land \psi(\eta, \xi, \eta'))
\]

this gives as a result a formula \( \psi_1(\xi, \eta') \). We update our current formula to

\[
\exists \xi' (\xi' = \xi(\chi) \land \phi_1(\chi) \land \psi_1(\xi, \eta'))
\]

and finally to the equivalent quantifier-free formula

\[
\phi_1(\chi) \land \psi_1(\xi, \eta')(\chi)
\]

We now show that the above algorithm is correct under very mild hypotheses. We need some technical facts about stably infinite theories in a multi-sorted context. We say that a multi-sorted theory \( T \) is stably infinite with respect to a subset \( S \) of its signature iff every \( T \)-satisfiable constraint is satisfiable in a model \( M \) where, for every \( S \subseteq S \), the set \( SM \) (namely the interpretation of the sort \( S \) in \( M \)) is infinite. Next Lemma is a light generalization of Lemma 2 and is proved in the same way:

**Lemma 6.** Let \( T \) be stably infinite with respect to a subset \( S \) of the set of sorts of the signature of \( T \). Let \( M \) be a model of \( T \) and let, for every \( S \subseteq S \), \( X_S \) be a superset of \( SM \). Then there is an extension \( N \) of \( M \) such that for all \( S \subseteq S \) we have \( SN \supseteq X_S \).

**Proof.** Let us expand the signature of \( T \) with the set \( C \) of fresh constants (we take one constant for every \( c \in X_S \)). We need to prove the \( T \)-consistency of \( \Delta(M) \) with a the set \( D \) of disequalities asserting that all \( c \in C \) are different from each other and from the names of the elements of the support of \( M \). By compactness, it is sufficient to ensure the \( T \)-consistency of \( \Delta_0 \cup D_0 \), where \( \Delta_0 \) and \( D_0 \) are finite subsets of \( \Delta(M) \) and \( D \), respectively. Since \( M \models \Delta_0 \), this set is \( T \)-consistent and hence it is satisfied in a \( T \)-model \( M' \) where all the sorts in \( S \) are interpreted as infinite sets; in such \( M' \), it is trivially seen that we can interpret also the constants occurring in \( D_0 \) so as to make \( D_0 \) true too.

**Lemma 7.** Let \( T_1, T_2 \) be universal signature disjoint theories which are stably infinite with respect to the set of shared sorts (we let \( \Sigma_1 \) be the signature of \( T_1 \) and \( \Sigma_2 \) be the signature of \( T_2 \)). Let \( M_0 \) be a model of \( T_1 \cup T_2 \) and let \( M_1 \) be a model of \( T_1 \) extending the \( \Sigma_1 \)-reduct of \( M_0 \). Then there exists a model \( N \) of \( T_1 \cup T_2 \), extending \( M_0 \) as a \( \Sigma_1 \cup \Sigma_2 \)-structure and whose \( \Sigma_1 \)-reduct extends \( M_1 \).

**Proof.** Using the previous lemma, build a chain of models \( M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \) such that for all \( i \), \( M_{2i} \) is a model of \( T_2 \), \( M_{2i+1} \) is a model of \( T_1 \) and \( M_{2i+2} \) is a \( \Sigma_2 \)-extension of \( M_{2i} \), whereas \( M_{2i+3} \) is a \( \Sigma_2 \)-extension of \( M_{2i+1} \). The union over this chain of models will be the desired \( N \).

We are now ready for the main result of this section:

**Theorem 9.** Let \( T_1 \cup T_2 \) be a tame combination of two universal theories admitting a model completion. If \( T_1, T_2 \) are also stably infinite with respect to their shared sorts, then \( T_1 \cup T_2 \) has a model completion. Covers in \( T_1 \cup T_2 \) can be computed as shown in the above three-steps algorithm.
Proof. Since condition (i) of Lemma 1 is trivially true, we need only to check condition (ii), namely that given a $T_1 \cup T_2$-model $M$ and elements $a, b$ from its support such that $M \models \phi_1(a) \land \psi_1(b, t'(a))$ as in (22), there is an extension $N$ of $M$ such that (17) is true in $N$ when evaluating $x$ over $a$ and $\xi$ over $b$.

If we let $b'$ be the tuple such that $M \models b' = t'(a)$, then we have $M \models b' = t'(a) \land \phi_1(a) \land \psi_1(b, b')$. Since $\psi_1(\xi, \xi')$ is the $T_2$-cover of (21), the $\Sigma_2$-reduct of $M$ embeds into a $T_2$-model where (21) is true under the evaluation of the $\xi$ as the $b$. By Lemma 7, this model can be embedded into a $T_1 \cup T_2$-model $M'$ in such a way that $M'$ is an extension of $M$ and that $M' \models b' = t'(a) \land \phi_1(a) \land \phi_2(c', b') \land \psi(c, b, c')$ for some $c, c'$. Since $\phi_1(\xi) \land \phi_2(\eta', t'(\xi))$ implies the $T_1$-cover of (19) and $M' \models \phi_1(a) \land \phi_2(c', t'(a))$, then the $\Sigma_1$-reduct of $M'$ can be expanded to a $T_1$-model where (19) is true when evaluating the $x, \eta'$ to the $c, c'$. Again by Lemma 7, this model can be expanded to a $T_1 \cup T_2$-model $N$ such that $N$ is an extension of $M'$ (hence also of $M$) and $N \models \phi(a', a) \land \psi(c', c, b', c)$, that is $N \models \phi(a', a) \land \psi(c, b, t(a', a))$. This means that $N \models \exists c \exists \eta(\phi(c, a) \land \psi(\eta, b, t(c, a)))$, as desired. $\dashv$

9 Conclusions and Future Work

In database-driven verification [6, 5, 3] one uses tame combinations $T_1 \cup T_2$, where $T_1$ is a multi-sorted version of $\Sigma$ in a signature $\Sigma$ containing only unary function symbols and relations of any arity. In this context, quantifier elimination in $T_1$ for primitive formulae is quadratic in complexity. Model-checkers like MCMT represent sets of reachable states by using conjunctions of literals and primitive formulae to which quantifier elimination should be applied arise from preimage computations. Now, in this context, if all relation symbols are at most binary, then quantifier elimination in $T_1$ produces conjunctions of literals out of primitive formulae, thus step (ii) in the above algorithm becomes deterministic and the only reason why the algorithm may become expensive (i.e. non polynomial) lies in the final quantifier elimination step for $T_2$. The latter might be extremely expensive if substantial arithmetic is involved, but it might still be efficiently handled in practical cases where only very limited arithmetic (e.g. difference bound constraints like $x - y \leq n$ or $x \leq n$, where $n$ is a constant) is involved. This is why we feel that our algorithm for covers in tame combinations can be really useful in the applications. This is confirmed by our first experiments with version 2.9 of MCMT, where such algorithm has been implemented.

From the theoretical point of view, the main challenge seems to consist in finding sufficient condition for existence of covers in combination of non-convex theories: in fact, we know from Section 7 that the non-convex version of the equality interpolation property [2] is not enough for this purpose.

References

1. P. D. Bacsich. Amalgamation properties and interpolation theorems for equational theories. *Algebra Universalis*, 5:45–55, 1975.
2. R. Bruttomesso, S. Ghilardi, and S. Ranise. Quantifier-free interpolation in combinations of equality interpolating theories. *ACM Trans. Comput. Log.*, 15(1):5:1–5:34, 2014.
3. D. Calvanese, S. Ghilardi, A. Gianola, M. Montali, and A. Rivkin. SMT-based verification of data-aware processes: a model-theoretic approach. *Mathematical Structures in Computer Science*. To appear.

4. D. Calvanese, S. Ghilardi, A. Gianola, M. Montali, and A. Rivkin. Quantifier elimination for database driven verification. *CoRR*, abs/1806.09686, 2018.

5. D. Calvanese, S. Ghilardi, A. Gianola, M. Montali, and A. Rivkin. Formal modeling and SMT-based parameterized verification of data-aware BPMN. In *Proc. of BPM*, volume 11675 of *LNCS*. Springer, 2019.

6. D. Calvanese, S. Ghilardi, A. Gianola, M. Montali, and A. Rivkin. From model completeness to verification of data aware processes. In *Description Logic, Theory Combination, and All That*, volume 11560 of *LNCS*. Springer, 2019.

7. D. Calvanese, S. Ghilardi, A. Gianola, M. Montali, and A. Rivkin. Model completeness, covers and superposition. In *Proc. of CADE*, volume 11716 of *LNCS (LNAI)*. Springer, 2019.

8. D. Calvanese, S. Ghilardi, A. Gianola, M. Montali, and A. Rivkin. Verification of data-aware processes: Challenges and opportunities for automated reasoning. In *Proc. of ARCADE*. EPTCS, 2019.

9. C.-C. Chang and J. H. Keisler. *Model Theory*. North-Holland Publishing Co., Amsterdam-London, third edition, 1990.

10. S. Ghilardi. Model theoretic methods in combined constraint satisfiability. *J. Autom. Reasoning*, 33(3-4):221–249, 2004.

11. S. Ghilardi and S. Ranise. Backward reachability of array-based systems by SMT solving: Termination and invariant synthesis. *Logical Methods in Computer Science*, 6(4), 2010.

12. S. Ghilardi and M. Zawadowski. *Sheaves, games, and model completions*, volume 14 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2002. A categorical approach to nonclassical propositional logics.

13. S. Gulwani and M. Musuvathi. Cover algorithms and their combination. In *Proc. of ESOP, Held as Part of ETAPS*, pages 193–207, 2008.

14. G. Nelson and D. C. Oppen. Fast decision procedures based on congruence closure. *J. ACM*, 27(2):356–364, 1980.

15. C. Tinelli and M. T. Harandi. A new correctness proof of the Nelson-Oppen combination procedure. In *Proc. of FroCoS 1996*, pages 103–119, 1996.

16. G. Yorsh and M. Musuvathi. A combination method for generating interpolants. In *Proc. of CADE-20*, LNCS, pages 353–368. 2005.
A Appendix

In this Appendix we give a detailed pedantic account of the computations required to solve Example 1.

Formula (13) is already purified. Notice also that the variables $e_1, e_2$ are in fact already explicitly defined (only $e_3, e_4$ are truly existential variables).

We first make the partition guessing. There is no need to involve defined variables into the partition guessing, hence we need to consider only two partitions; they are described by the following formulae:

$$E_1(e_3, e_4) \equiv e_3 \neq e_4$$
$$E_2(e_3, e_4) \equiv e_3 = e_4$$

We first analyze the case of $E_1$. The formulae $\psi_1$ and $\psi_2$ to which we need to apply exhaustively Step (1) and Step (2i) of our algorithm are:

$$\psi_1 \equiv f(e_3) = e_3 \land f(e_4) = x_1 \land e_3 \neq e_4$$
$$\psi_2 \equiv x_1 + e_1 \leq e_3 \land e_3 \leq x_2 + e_2 \land e_4 = x_2 + e_3 \land e_3 \neq e_4$$

We first compute the implicit definability formulae for the truly existential variables with respect to both $T_1$ and $T_2$.

- We first consider $\text{ImplDef}^{T_1}_{\psi_1, e_3}(x, z)$. Here we show that the cover of the negation of formula (7) is equivalent to $\top$ (so that $\text{ImplDef}^{T_1}_{\psi_1, e_3}(x, z)$ is equivalent to $\bot$). We must quantify over truly existential variables and their dualities, thus we need to compute the cover of

$$f(e_3') = e_3' \land f(e_3) = e_3 \land f(e_4) = x_1 \land f(e_4) = x_1 \land e_3 \neq e_4 \land e_3' \neq e_4'$$

This is a saturated set according to the superposition based procedure of [7], hence the result is $\top$, as claimed.

- The formula $\text{ImplDef}^{T_1}_{\psi_1, e_4}(x, z)$ is also equivalent to $\bot$, by the same argument as above.

- To compute $\text{ImplDef}^{T_2}_{\psi_2, e_3}(x, z)$ we use Fourier-Motzkin quantifier elimination. We need to eliminate the variables $e_3, e_3', e_4, e_4'$ (intended as existentially quantified variables) from

$$x_1 + e_1 \leq e_3' \leq x_2 + e_2 \land x_1 + e_1 \leq e_3 \leq x_2 + e_2 \land e_4' = x_2 + e_3' \land$$
$$\land e_4 = x_2 + e_3 \land e_3 \neq e_4 \land e_3' \neq e_4' \land e_3 \neq e_4$$

This gives $x_1 + e_1 \neq x_2 + e_2 \land x_2 \neq 0$, so that $\text{ImplDef}^{T_2}_{\psi_2, e_1}(x, z)$ is $x_1 + e_1 = x_2 + e_2 \land x_2 \neq 0$. The corresponding equality interpolating term for $e_3$ is $x_1 + e_1$.

- The formula $\text{ImplDef}^{T_2}_{\psi_2, e_4}(x, z)$ is also equivalent to $x_1 + e_1 = x_2 + e_2 \land x_2 \neq 0$ and the equality interpolating term for $e_4$ is $x_1 + e_1 + x_2$.

So, if we apply Step 1 we get

$$\exists e_1 \cdots \exists e_4 \left( e_1 = f(x_1) \land e_2 = f(x_2) \land \right. \left. f(e_3) = e_3 \land f(e_4) = x_1 \land e_3 \neq e_4 \land \right. \left. x_1 + e_1 \leq e_3 \land e_3 \leq x_2 + e_2 \land e_4 = x_2 + e_3 \land x_1 + e_1 \neq x_2 + e_2 \right) \tag{23}$$

The literal $x_2 \neq 0$ is entailed by $\psi_2$, so we can simplify it to $\top$ in $\text{ImplDef}^{T_2}_{\psi_2, e_1}(x, z)$ and $\text{ImplDef}^{T_2}_{\psi_2, e_4}(x, z)$.  

17
If we apply Step (2i) (for i=3), we get (after removing implied equalities)

\[ \exists e_1 \cdots \exists e_4 \left( \begin{array}{l}
  e_1 = f(x_1) \land e_2 = f(x_2) \land e_3 = x_1 + e_1 \\
  \land f(e_3) = e_3 \land f(e_4) = x_1 \land e_3 \neq e_4 \\
  \land e_4 = x_2 + e_3 \land x_1 + e_1 = x_2 + e_2
\end{array} \right) \]  

(24)

Step (2i) (for i=4) gives a formula logically equivalent to (24). Notice that (24) is terminal too, because all existential variables are now explicitly defined (this is a side-effect of the fact that \( e_3 \) has been moved to the defined variables). Thus the exhaustive application of Steps (1) and (2i) is concluded.

Applying the final step of Proposition [1] to (24) is quite easy: it is sufficient to unravel the acyclic definitions. The result, after little simplification, is

\[ x_2 \neq 0 \land x_1 + f(x_1) = x_2 + f(x_2) \land f(x_2 + f(x_1 + f(x_1))) = x_1 \land f(x_1 + f(x_1)) = x_1 + f(x_1); \]

this can be further simplified to

\[ x_2 \neq 0 \land x_1 + f(x_1) = x_2 + f(x_2) \land f(2x_2 + f(x_2)) = x_1 \land f(x_1 + f(x_1)) = x_1 + f(x_1); \]

(25)

As to formula (23), we need to apply the final cover computations mentioned in Proposition [1]. The formulae \( \psi_1 \) and \( \psi_2 \) are now

\[ \psi_1' \equiv f(e_3) = e_3 \land f(e_4) = x_1 \land e_3 \neq e_4 \]
\[ \psi_2' \equiv x_1 + e_1 \leq e_3 \land x_2 \land e_4 = x_2 + e_3 \land x_1 + e_1 \neq x_2 + e_2 \land e_3 \neq e_4 \]

The \( T_1 \)-cover of \( \psi_1' \) is \( \top \). For the \( T_2 \)-cover of \( \psi_2' \), eliminating, with Fourier-Motzkin the variables \( e_4 \) and \( e_1 \), we get

\[ x_1 + e_1 < x_2 + e_2 \land x_2 \neq 0 \]

which becomes

\[ x_1 + f(x_1) < x_2 + f(x_2) \land x_2 \neq 0 \]

(26)
after unravelling the explicit definitions of \( e_1, e_2 \). Thus, the analysis of the case of the partition \( E_1 \) gives, as a result, the disjunction of (25) and (26).

We now analyze the case of \( E_2 \). Before proceeding, we replace \( e_4 \) with \( e_3 \) (since \( E_2 \) precisely asserts that these two variables coincide); our formulae \( \psi_1 \) and \( \psi_2 \) become

\[ \psi_1'' \equiv f(e_3) = e_3 \land f(e_3) = x_1 \]
\[ \psi_2'' \equiv x_1 + e_1 \leq e_3 \land e_3 \leq x_2 + e_2 \land 0 = x_2 \]

From \( \psi_1'' \) we deduce \( e_3 = x_1 \), thus we can move \( e_3 \) to the explicitly defined variables. In this way we get the terminal working formula

\[ \exists e_1 \cdots \exists e_3 \left( \begin{array}{l}
  e_1 = f(x_1) \land e_2 = f(x_2) \land e_3 = x_1 \\
  \land f(e_3) = e_3 \land f(e_3) = x_1 \land \\
  \land x_1 + e_1 \leq e_3 \land e_3 \leq x_2 + e_2 \land 0 = x_2
\end{array} \right) \]

(27)

Unravelling the explicit definitions, we get (after exhaustive simplifications)

\[ x_2 = 0 \land f(x_1) = x_1 \land x_1 \leq 0 \land x_1 \leq f(0) \]

(28)

---

\[ ^6 \text{This avoids useless calculations: the implicit definability condition for variables having an entailed explicit definition is obviously } \top, \text{ so making case split on it produces either tautological consequences or inconsistencies.} \]
Now, the disjunction of (25), (26) and (28) is precisely the final result (15) claimed in Section 6. This concludes our detailed analysis of Example 1.