ON ENHANCED DISSIPATION FOR THE BOUSSINESQ EQUATIONS

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ABSTRACT. In this article we consider the stability and damping problem for the 2D Boussinesq equations with partial dissipation near a two parameter family of stationary solutions which includes Couette flow and hydrostatic balance.

In the first part we show that for the linearized problem in an infinite periodic channel the evolution is asymptotically stable if any diffusion coefficient is non-zero. In particular, this imposes weaker conditions than for example vertical diffusion. Furthermore, we study the interaction of shear flow, hydrostatic balance and partial dissipation.

In a second part we adapt the methods used by Bedrossian, Vicol and Wang [BVW16] in the Navier-Stokes problem and combine them with cancellation properties of the Boussinesq equations to establish small data stability and enhanced dissipation results for the nonlinear Boussinesq problem with full dissipation.

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1. INTRODUCTION

The Boussinesq equations are a common model in the study of heat conduction and are given by a coupled system of the Navier-Stokes equations and a diffusion equation for the temperature (see for instance [Tem12, Section 3.5]). In this article

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we specifically consider the two-dimensional incompressible Boussinesq equations on $\mathbb{T} \times \mathbb{R}$ which model a heat-conducting fluid in terms of its velocity field $v$, the pressure $p$ and its temperature density $\theta$:

$$\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p &= \nu_1 \partial_x^2 v + \nu_2 \partial_y^2 v + \begin{pmatrix} 0 \\ \theta \end{pmatrix}, \\
\partial_t \theta + v \cdot \nabla \theta &= \eta_1 \partial_x^2 \theta + \eta_2 \partial_y^2 \theta, \\
\nabla \cdot v &= 0,
\end{align*}$$

$$(t, x, y) \in (0, \infty) \times \mathbb{T} \times \mathbb{R}.\)$$

Here differences in $\theta$ cause the fluid to rise or fall due to buoyancy and the temperature density is advected by the velocity. The diffusion coefficients $\nu_1, \nu_2, \eta_1, \eta_2 \geq 0$ are constants which model viscosity and thermal diffusion and may in general be anisotropic.

In the setting of full dissipation, that is if $\nu_1, \nu_2, \eta_1, \eta_2$ are all bounded below by a common constant $\nu > 0$, global well-posedness results are classical and make use of energy arguments (see for instance the textbook by Teman [Tem12] or the articles [FMT87, CD80]). However, in some physical problems the thermal and viscous diffusivity $\nu_i, \eta_i$ may be of very different orders of magnitude or highly anisotropic. In particular, some coefficients might be much smaller than all others. A natural question thus concerns the problem of partial dissipation where some of the coefficients are allowed to vanish. Here, in a recent article Doering, Wu, Zhao and Zheng [DWZZ18] consider the case without thermal diffusivity, $\nu_1 = \nu_2 > 0$ and $\eta_1 = \eta_2 = 0$. We further mention the works by Titi, Lunasin and Larios, [LT16, LLT13] on vertical dissipation and anisotropic dissipation and the works by Chae, Kim and Nam [CKN99, Cha06] on cases with no viscosity or thermal diffusivity. For further discussion and references, the interested reader is referred to the lecture notes by JH Wu [Wu12]. In all these problems partial dissipation also implies a potential lack of smoothing and hence questions of well-posedness become challenging problems.

In this article we are interested in the behavior of the Boussinesq equations with partial or full dissipation close to the following two parameter family of stationary solutions:

$$v = \begin{pmatrix} \beta y \\ 0 \end{pmatrix}, \quad \theta = \alpha y.\)$$

The case $\beta = 0, \alpha > 0$ is known as hydrostatic balance and shares structural similarities with stratified compressible flow (that is, with a mass density $\rho$ instead of a temperature density $\theta$; see Section 2). The case $\beta = 1, \alpha = 0$ corresponds to a linear shear flow in the Navier-Stokes equations (e.g. between moving plates or as a model for rotating concentric cylinders) and is known as Couette flow. We aim at understanding the asymptotic stability of these solutions, the interaction of hydrostatic balance and shear and, in particular, at obtaining (mixing enhanced) dissipation rates for the partial dissipation case.

Related settings have for instance been studied in the following works:

- Wu, Xu and Zhu [WXZ19] studied the nonlinear Boussinesq-Bénard system near the trivial steady state $(0, 0)$.
- The linearized inviscid Boussinesq problem near Couette flow ($\alpha = 0, \beta = 1$) was considered by Yang and Lin [YL18] in a work on stratified fluids (the
linearized equations of the stratified fluids problem and of the Boussinesq problem are structurally similar).

- In a recent work W. Tao and Wu [TW19] consider the corresponding viscous linear problem with vertical dissipation, $\nu_x = \eta_x = 0$, $\nu_y, \eta_y > 0$ in the half-infinite periodic channel $T \times (0, \infty)$. In Section 2 we revisit this problem for the infinite channel $T \times \mathbb{R}$ with general partial dissipation and with hydrostatic balance. Here the interaction of shear flow, hydrostatic balance and partial dissipation leads to challenging stability problems, while the absence of boundaries simplifies approaches by Fourier methods.

- Since the Boussinesq equations are coupled Navier-Stokes equations, results for the latter are very closely related. In Section 3 we adapt the strategy employed by Bedrossian, Vicol and Wang [BVW16] for the 2D Navier-Stokes equations to the Boussinesq equations. Furthermore, we combine these techniques with cancellation properties of the Boussinesq equations to treat the case of “large” $\alpha$.

In this article we are interested in three main questions:

- How do shear flows, hydrostatic balance and diffusion interact and what (mixing enhanced) damping rates can be obtained?
- How small should perturbations be so that the nonlinear dynamics remain well-approximated by the linear dynamics? Or in other words, can we describe a Sobolev stability threshold for the nonlinear problem as the dissipation coefficients tend to zero?
- How little dissipation is necessary for asymptotic stability results? In particular, we study how vanishing diffusivity coefficients effect decay rates and asymptotic stability results in the linearized problems.

We remark that in the inviscid case the linearized problem is algebraically unstable at the level of the vorticity (see Lemma 2.2). However, it is stable at the level of the velocity (see [YL18]). In this work we focus on the (partially) viscous problem and stability of the vorticity in Sobolev regularity. In view of the results of Bedrossian, Vicol and Masmoudi [BMV16] a further extension to the case of Gevrey regular data seems possible but technically very challenging (see also the comments following Corollary 1.5). In particular, it would have to precisely capture the growth and loss of regularity due to resonances (see [BMM16, DZ19, DM18, Zil20]).

1.1. Main Results. Our first main results concern small data nonlinear asymptotic stability and enhanced dissipation for the setting with shear and with full dissipation $\nu_x = \nu_y > 0$, $\eta_x = \eta_y > 0$. In Theorem 1.1 (later restated as Theorem 3.1) we focus on the setting where $\alpha$ is “small” and adapt the methods of [BMV16] used in the Navier-Stokes problem near Couette flow to the Boussinesq equations. We then combine these methods with energy arguments and cancellations for hydrostatic balance (see [DWZZ18]) to treat the “large” $\alpha$ case in Theorem 1.2 (later restated as Theorem 3.6). Here and in the following results $\omega, \nu$ and $\theta$ denote the perturbation of the two parameter family (1) and if $\beta \neq 0$ we consider coordinates moving with the shear:

$$(x + t\beta y, y, t).$$
Under this change of variables the gradient and Laplacian are given by
\[ \nabla_t = \left( \frac{\partial_x}{\partial t - t \beta \partial_x} \right), \Delta_t = \partial_x^2 + (\partial_y - t \beta \partial_x)^2. \]

**Theorem 1.1.** Let \( N \geq 5 \) and let \( \beta = 1, \epsilon_1 \leq \frac{1}{100} \min(\nu, \eta)^{1/2}, \epsilon_2 \leq \frac{1}{100} \sqrt{\eta} \sqrt{\nu} \epsilon_1 \) and suppose that \( 0 \leq \alpha < \eta^{1/2} L^{1/3} \nu^{2/3} \). Then if \( \|\omega_0\|_{H^N} \leq \epsilon_1 \) and \( \|\theta_0\|_{H^N} \leq \epsilon_2 \), the unique global solution with this initial data satisfies
\[
\|\omega\|_{L^\infty((0, \infty); H^N)} + \nu\|\nabla_t \omega\|_{L^2((0, \infty); H^N)} + \|\nabla_t \Delta_t^{-1} \omega\|_{L^2((0, \infty); H^N)} \leq 8\epsilon_1,
\]
\[
\|\theta\|_{L^\infty((0, \infty); H^N)} + \eta\|\nabla_t \theta\|_{L^2((0, \infty); H^N)} \leq 8\epsilon_2.
\]

**Theorem 1.2.** Let \( \alpha \geq 1 \) and \( \beta = 1 \) and \( \nu > 0 \) and suppose that \( \eta > 2 \). Let further \( (\omega_0, \theta_0) \in H^N \times H^{N+1} \) be given initial data such that
\[
\alpha\|\omega_0\|_{H^N}^2 + \|\nabla \theta_0\|_{H^N}^2 \leq \frac{1}{100} \epsilon_1^2.
\]
Then for all times \( T > 0 \) it holds that
\[
\text{ess-sup}_{0 \leq t \leq T} (\alpha\|\omega(t)\|_{H^N}^2 + \|\nabla_t \theta(t)\|_{H^N}^2)
+ \nu \alpha\|\nabla_t \omega(t)\|_{L^2 H^N}^2
+ \eta\|\nabla_t \theta(t)\|_{L^2 H^N}^2 \leq \epsilon^2.
\]

For a discussion of the assumptions see Section 3.

As a corollary we derive exponential decay rates and enhanced dissipation (later restated as Proposition 4.1).

**Proposition 1.3.** Let \( N, \alpha, \epsilon_1, \epsilon_2 \) be as in Theorem 1.1. Then the nonlinear Boussinesq equations further satisfy
\[
\|\omega(t)\|_{H^N} + \|\theta(t)\|_{H^N} \leq 2C \exp(-\min(\nu, \eta)^{1/3} t/10)(\|\omega_0\|_{H^N} + \|\theta_0\|_{H^N})
\]
for all \( t > 0 \). In particular, we observe dissipation on a time scale \( \min(\nu, \eta)^{-1/3} \) faster than heat flow. We say that the equations exhibit enhanced dissipation.

The nonlinear Boussinesq equations in particular with partial dissipation have been studied in numerous previous works, e.g. [LLT13, LT16, DWZZ18] (see the introduction and Section 3 for a discussion). Our main differences and novelties here are:

- We consider the effects of a linear shear and hydrostatic balance at the same time. In particular, the effects of mixing by a shear flow and the resulting enhanced dissipation of the velocity field and the interaction of shear and hydrostatic balance have, to our knowledge, not previously been studied for the Boussinesq equations.
- Our results concern higher regularity and decay rates near combinations of shear flow and hydrostatic balance. In contrast, well-posedness and asymptotic stability results such as [DWZZ18, LLT13, CKN99] focus on perturbations of \((0,0)\) and energies at the level of \( L^2 \times H^1 \) or make use of energy functionals of the type \( \alpha\|v\|_{L^2}^2 + \|\theta\|_{L^2}^2 \) (for hydrostatic balance without shear).
- In particular, our results are stable under the limit \( \alpha \downarrow 0 \) and incorporate mixing enhanced dissipation rates.
In addition to the nonlinear results obtained in Theorems 1.1 and 1.2 we also study the linearized setting around more general profiles of the form (1), derive finer characterizations of asymptotics and are in particular interested in the effects of partial dissipation.

More precisely, we ask how little dissipation is required for asymptotic stability of the vorticity and the temperature to hold and how this is influenced by shear and hydrostatic balance, respectively. In this context we mention numerous previous works by J. Wu and coauthors on related (sub)settings [TW19, WXZ19] (see Section 2 for a longer discussion). Our main results are collected in Theorem 2.1, which we restate here, and are derived in the subsections of Section 2.

We recall that for \( \beta \neq 0 \) we consider the perturbations in coordinates moving with the shear flow:

\[
\omega(t, x + \beta ty, y), \theta(t, x + t\beta y, y).
\]

**Theorem 1.4.** Consider the linearized problem (5) around the state

\[
v^* = (\beta y, 0), \theta^* = \alpha y,
\]

with initial data \( \omega_0 \in H^N \) and \( \theta_0 \in H^{N+1}, \ N \in \mathbb{N} \).

In the inviscid case the evolution of the vorticity \( \omega \) is unstable in the sense that

\[
\limsup_{t \to \infty} \|\omega(t)\|_{H^{N}} = \infty,
\]

unless \( \alpha > 0, \beta = 0 \) or \( \alpha = 0 \) and \( \partial_x \theta \) is trivial.

If \( \alpha = 0 \) the evolution of the vorticity is asymptotically stable if at least one diffusion coefficient is non-zero. More precisely, for every \( N \in \mathbb{N} \) there exists \( C = C(\eta_x, \eta_y, N) \) such that the temperature density satisfies

\[
\|\theta(t) - \int \theta(t) dx\|_{H^{N}} \leq C \exp \left(-\eta_x t - \eta_y \frac{t^3}{12}\right) \|\theta_0\|_{H^{N}}.
\]

Furthermore, there exists \( C = C(\eta_x, \eta_y, \nu_x, \nu_y, N) \) and a profile \( \omega_1(t) \) (see Theorem 2.4 for a detailed description) such that

\[
\|\omega(t) - \omega_1(t)\|_{H^{N}} \leq C \exp \left(-\max(\eta_y, \nu_y) \frac{t^3}{12} - \max(\eta_x, \nu_x) t\right) (\|\omega_0\|_{H^{N}} + \|\partial_x \theta_0\|_{H^{N}})
\]

and \( \omega_1(t) \) satisfies

\[
\|\omega_1(t)\|_{H^{N}} \leq C(\nu_x, \nu_y, \eta_x, \eta_y) \exp(- \min(\eta_y, \nu_y) \frac{t^3}{12} - \min(\eta_x, \nu_x) t/2) (\|\omega_0\|_{H^{N}} + \|\partial_x \theta_0\|_{H^{N}}).
\]

Finally, let \( \alpha > 0 \) and suppose that at least one of \( \min(\nu_x, \eta_x) \) or \( \min(\nu_y, \eta_y) \) is positive. Then it holds that (see Propositions 2.5 and 2.7)

\[
E(t) := \alpha \left[ \omega(t) \right]_{H^{N}}^2 + \left[\partial_x \theta(t)\right]_{H^{N}}^2 + \left[\partial_y - t\beta \partial_x \theta(t)\right]_{H^{N}}^2.
\]

is bounded uniformly in time. Furthermore, if \( \beta = 1 \) (or more generally \( \beta \neq 0 \)) we obtain the enhanced dissipation estimates:

\[
E(t) \leq C(1 + t^2) \exp \left(-\min(\nu_y, \eta_y) \frac{t^3}{12} - \min(\eta_x, \nu_x) t\right) E(0).
\]
spaces. For simplicity of notation and as an example we state such a corollary for the case \( \alpha > 0, \beta = 0 \) of Theorem 1.4 (see also Proposition 2.5).

**Corollary 1.5.** Let \( \alpha > 0, \beta = 0 \) and let \( \nu_x, \eta_x, \nu_y, \eta_y \geq 0 \) be given. Suppose that \( \omega_0, \theta_0 \) are in the Gevrey class \( G_2 \), that is there exists \( R_1, R_2 > 0 \) such that for all \( j \in \mathbb{N} \)

\[
\|\omega_0\|_{H^j} \leq R_2^{1+j}(j!)^{2j}, \\
\|\nabla \theta_0\|_{H^j} \leq R_2^{1+j}(j!)^{2j}.
\]

Then there exist \( R_3, R_4 > 0 \), which depend on \( R_1, R_2 \) and \( \alpha \) such that the solution of the linearized Boussinesq equations \( (\omega, \theta) \) satisfies

\[
\|\omega(t)\|_{H^j} \leq R_3^{1+j}(j!)^{2j}, \\
\|\nabla \theta(t)\|_{H^j} \leq R_4^{1+j}(j!)^{2j}.
\]

for all times \( t > 0 \) and all \( j \in \mathbb{N} \).

**Proof of Corollary 1.5.** By Theorem 1.4 we know that

\[
\alpha \|\omega(t)\|_{H^N}^2 + \|\nabla \omega(t)\|_{H^N}^2 \\
\leq c(t)(\alpha \|\omega_0\|_{H^N}^2 + \|\nabla \omega_0\|_{H^N}^2) \\
\leq c(t)(\alpha R_2^{2N} + R_2^{2N})(j!)^4,
\]

where

\[
c(t) = \exp\left( -\min(\nu_x, \eta_x)t - (\min(\nu_y, \eta_y)\frac{t^3}{8}) \right) \leq 1.
\]

Thus we may choose

\[
\sqrt[\frac{N}{2}]{c(t)(R_1^{2N} + \alpha^{-1}R_2^{2N})}, \\
\sqrt[\frac{N}{2}]{c(t)(\alpha R_1^{2N} + R_2^{2N})},
\]

which can be controlled in terms of \( 2 \max(\alpha, \alpha^{-1}) \max(R_1, R_2) \).

We remark that more generally it suffices to establish a bound of the form

\[
\| (\omega(t), \theta(t)) \|_{H^N \times H^{N+1}} \leq c^N \| (\omega_0, \theta_0) \|_{H^N \times H^{N+1}},
\]

for some constant \( c \) independent of \( N \) (see Section 2 for several estimates of this type). Furthermore, due to the change of coordinates associated with \( \beta \), in the general case one may obtain estimates of the form

\[
\| \theta(t) \|_{H^N} \leq \| \nabla \theta(t) \|_{H^N} \leq c^N \| \nabla \theta_0 \|_{H^N} \leq c^N \| \theta_0 \|_{H^{N+1}},
\]

which “lose” one derivative. For this reason general Gevrey estimates either need to track spaces more precisely or allow for losses in \( R \) or the Gevrey class exponent with time. As this is not a focus of the article, we opted to only state a simple result.

The remainder of the article is organized as follows:

- In Section 1.2 we introduce notational conventions used throughout the article.
• In Section 2 we consider the linearized problem around the two parameter family (1). Here a particular focus is placed on the problem of partial dissipation and we show that if even just one dissipation coefficient is non-trivial asymptotic stability results hold. Furthermore we discuss how the interaction of shear flow and hydrostatic balance influence (mixing enhanced) dissipation rates.

• In Section 3 we discuss the nonlinear problem with full dissipation. In a first result we adapt the approach Bedrossian, Vicol and Wang [BVW16] used for the Navier-Stokes problem to the Boussinesq equations and establish stability in Sobolev regularity for small data and small slope $\alpha \geq 0$ of the hydrostatic balance. We then combine these tools with additional cancellation properties of the Boussinesq equations with hydrostatic balance (see [DWZZ18]) to treat the case of “large” $\alpha$.

• In Section 4 we show that the nonlinear stability results of Section 3 combined with the estimates on the linear problem obtained in Section 2 yield nonlinear (enhanced) dissipation estimates.

1.2. Notation. In the study of both the linearized and nonlinear Boussinesq equations we make extensive use of the Fourier transform. We denote the Fourier transform of a function $f(x, y) \in L^2(T \times \mathbb{R})$ by

$$\hat{f}(k, \xi) := (\mathcal{F}_{x,y} f)(k, \xi)$$

with $k \in \mathbb{Z}$ being discrete and $\xi \in \mathbb{R}$.

In our analysis the $x$-average, $k = 0$, plays a distinct role in that it might be conserved or decay slower than its $L^2$-orthogonal complement. We thus denote

$$f_\equiv(y) = \int_0^1 f(x, y) dx$$

and its complement

$$f_\not\equiv(x, y) = f(x, y) - f_\equiv(y).$$

As related notation in Section 3 we split a nonlinear sum of integrals $T$ into contributions $T^\equiv$ involving the $x$-average of the velocity (which is a shear flow) and its complement $T^\not\equiv$.

In this article our main object of interest is the evolution of perturbations around the stationary states given by the two parameter family (1). Hence, with slight abuse of notation we use $\omega, v$ and $\theta$ to refer to the perturbation of the vorticity, velocity and temperature (instead of the full solution). Similarly, when $\beta \neq 0$ it is natural to work in coordinates moving with the flow and consider

$$\omega(t, x + t\beta y, y), v(t, x + t\beta y, y), \theta(t, x + t\beta y, y),$$

as well as Sobolev spaces with respect to these coordinates.

In Section 3 we consider spaces of the form $L^p((0, T); H^N)$ or $L^p((0, \infty), H^N)$, which we abbreviate as $L^p_t H^N$. In some asymptotic estimates we denote universal constants, which do not depend on the quantities under consideration, by $C > 0$. These constants may change from line to line. Similarly, we write $a \ll b$ if there exists a small universal constant ($C \leq \frac{1}{100}$ for our purposes) such that $|a| \leq C|b|$. 


2. THE LINEARIZED PROBLEM AROUND COUETTE FLOW AND HYDROSTATIC BALANCE

In this section we consider the linearized two-dimensional Boussinesq equations on $T \times \mathbb{R}$ near the two-parameter family of stationary solutions

$$v = (\beta y, 0), \theta = \alpha y$$

and with possibly partial dissipation:

$$\begin{align*}
\partial_t \omega + \beta y \partial_x \omega &= \nu_x \partial_{xx} \omega + \nu_y \partial_{yy} \omega + \partial_x \theta, \\
\partial_t \theta + \beta y \partial_x \theta + \alpha \partial_x \Delta^{-1} \omega &= \eta_x \partial_{xx} \theta + \eta_y \partial_{yy} \theta,
\end{align*}$$

(5) \hspace{1cm} (t, x, y) \in \mathbb{R} \times T \times \mathbb{R},

$$\nu_x, \nu_y, \eta_x, \eta_y \geq 0,$$

$$\alpha \geq 0, \beta \in \mathbb{R}.$$

In a recent work L. Tao and J. Wu [TW19] considered the related (sub)case of the linearization of the Boussinesq equations with vertical dissipation in both vorticity and temperature

$$\eta_x = 0, \eta_y > 0, \nu_x = 0, \nu_y > 0,$$

for $\beta = 1, \alpha = 0$ in the periodic half-space $T \times (0, \infty)$ with Neumann boundary conditions.

In our setting, on the one hand, the interaction of non-trivial shear $\beta \neq 0$ and non-trivial balance $\alpha > 0$ and allowing for more diffusion coefficients to vanish allows for a multitude of different dynamics and stability results and proves very challenging in the case of full generality. On the one hand, as this setting does not possess boundaries, Fourier methods can be more easily used and allow for a fine, optimal descriptions of asymptotic behavior. In particular, we can clearly isolate the effects of each diffusion parameter and show that in this setting it suffices to impose even weaker conditions on the diffusivity parameters than in [DWZZ18] or [TW19]: Only a single parameter needs to be non-zero.

**Theorem 2.1.** Consider the linearized problem (5) around the state

$$v = (\beta y, 0), \theta = \alpha y,$$

with initial data $\omega_0 \in H^N$ and $\theta_0 \in H^{N+1}$, $N \in \mathbb{N}$.

In the inviscid case the evolution of the vorticity $\omega$ is unstable in the sense that

$$\limsup_{t \to \infty} \|\omega_0\|_{H^N} = \infty,$$

unless $\alpha > 0, \beta = 0$ or $\alpha = 0$ and $\partial_x \theta$ is trivial.

If $\alpha = 0$ the evolution of the vorticity is asymptotically stable if at least one diffusion coefficient is non-zero. More precisely, for every $N \in \mathbb{N}$ there exists $C = C(\eta_x, \eta_y, N)$ such that the temperature density satisfies

$$\|\theta(x)\|_{H^N} \leq \exp \left(-\eta_x t - \eta_y t^3 \frac{3}{12}\right) \|\theta_0\|_{H^N}$$

Furthermore, there exists $C = C(\eta_x, \eta_y, \nu_x, \nu_y, N)$ and a profile $\omega_1(t)$ (see Theorem 2.4) such that

$$\|\omega(t) - \omega_1(t)\|_{H^N} \leq C \exp \left(-\max(\eta_y, \nu_y) t^3 \frac{12}{3} - \max(\eta_x, \nu_x) t\right) \left(\|\omega_0\|_{H^N} + \|\partial_x \theta_0\|_{H^N}\right)$$
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and \( \omega_1(t) \) satisfies

\[
\|\omega_1(t)\|_{H^N} \leq C(\nu_x, \nu_y, \eta_x, \eta_y) \exp \left( -\min(\eta_y, \nu_y) \frac{t^3}{12} - \min(\eta_x, \nu_x) t/2 \right) \left( \|\omega_0\|_{H^N} + \|\partial_x \theta_0\|_{H^N} \right).
\]

Finally, let \( \alpha > 0 \) and suppose that at least one of \( \min(\nu_x, \eta_x) \) or \( \min(\nu_y, \eta_y) \) is positive. Then it holds that (see Propositions 2.5 and 2.7)

\[
E(t) := \alpha \|\omega(t)\|_{L^2}^2 + \|\partial_x \theta(t)\|_{L^2}^2 + \|\partial_y - t\beta \partial_x \theta(t)\|_{L^2}^2.
\]

is bounded uniformly in time. Furthermore, if \( \beta = 1 \) (or more generally \( \beta \neq 0 \)) we obtain the enhanced dissipation estimates:

\[
E(t) \leq C(1 + t^2) \exp \left( -\min(\nu_y, \eta_y) \frac{t^3}{12} - \min(\eta_x, \nu_x) t \right) E(0).
\]

In order to introduce methods and techniques, we first consider some exceptional cases, such as \( \alpha = 0 \) in Sections 2.1 and 2.2 and \( \alpha > 0, \beta = 0 \) in Section 2.3.1. The setting with both effects \( \alpha > 0, \beta = 1 \) is then considered in Section 2.3.2. Finally, we revisit these results in Section 4 to establish decay rates for the nonlinear problem with small data.

2.1. The Inviscid Case. In this section we consider the inviscid problem \( \eta_x = \eta_y = \nu_x = \nu_y = 0 \) with \( \alpha \geq 0 \) and \( \beta \in \mathbb{R} \) to study the interaction between shear and hydrostatic balance.

As a first simple model setting we consider the case of homogeneous temperature, \( (\alpha = 0) \) and an affine flow \( (\beta \in \mathbb{R}) \). Here we obtain a simple, explicit solution and in particular observe that the evolution is linearly algebraically unstable at the level of the vorticity but the density \( \theta \) is stable, as is the velocity.

**Lemma 2.2.** Consider the inviscid linearized problem (5) with \( \alpha = 0 \) and \( \beta \in \mathbb{R} \) on \( T \times \mathbb{R} \) (or \( T \times I \))

\[
\begin{align*}
\partial_t \omega + \beta y \partial_x \omega &= \partial_x \theta, \\
\partial_t \theta + \beta y \partial_x \theta &= 0, \\
\omega|_{t=0} &= \omega_0, \theta|_{t=0} = \theta_0,
\end{align*}
\]

with initial data

\[
(\omega, \eta)|_{t=0} = (\omega_0, \theta) \in H^N \times H^{N+1}.
\]

It has the following explicit solution:

\[
\begin{align*}
\omega(t, x, y) &= \omega_0(x - \beta ty, y) + t \partial_x \theta_0(x - \beta ty, y), \\
\theta(t, x, y) &= \theta_0(x - \beta ty, y).
\end{align*}
\]

In particular, \( \theta(t, x + \beta ty, y) \) is stationary and hence stable and the velocity field satisfies

\[
\|v(t, x, y)\|_{L^2} \leq \|\omega_0\|_{L^2} + C \frac{1}{\beta} \|\nabla \theta_0\|_{L^2}.
\]

The velocity is stable stable in \( L^2 \) for any \( \beta \neq 0 \).

However, the evolution of \( \omega(t, x, y) \) and \( \omega(t, x + \beta ty, y) \) is unstable in any positive Sobolev norm unless \( \partial_x \theta_0 \) is trivial.
We interpret this to say that a shear $\beta y$ has a stabilizing effect on the velocity and that $\partial_x \theta_0$ has a destabilizing effect on $\omega$. Such a stability result for the velocity has previously been obtained by Lin and Yang [YL18] in a work on the linearized inviscid, stratified Euler equations around $u = (y, 0), \rho = e^{-\gamma y}$ (which yield a very similar equation). However, in view of the nonlinear problem considered in Section 3 we here emphasize the instability of the vorticity due to $\partial_x \theta_0$. Our question in the following is then how much dissipation is required to restore asymptotic stability of the vorticity (see Theorem 2.4).

**Proof of Lemma 2.2.** In the Lagrangian coordinates $(x - ty, y)$ the system reads
\[
\frac{d}{dt} \omega = \partial_x \theta, \\
\frac{d}{dt} \theta = 0.
\]
One observes that the explicit solution of this system is given by
\[
\theta = \theta_0, \\
\omega = \omega_0 + t\partial_x \theta_0.
\]
The result of the lemma then follows by expressing these solutions in Eulerian coordinates. Concerning the stability estimate of the velocity, we note that $t\partial_x \theta_0(x - ty, y) = (-\frac{d}{dy} + \partial_y)\theta_0(x - ty, y)$.

Since the velocity corresponds to gaining one derivative compared to the vorticity, we may thus absorb the $\frac{d}{dy}$ and hence obtain a uniform bound. However, we remark that while $\omega$ only depends on $\partial_x \theta_0$, not the full gradient, in this estimate of the velocity we require control of $\partial_y \theta_0$ as well. $\square$

In the following lemma we consider the effect of affine hydrostatic balance $\alpha > 0$. The positive sign here corresponds to hotter fluid being on top. If this is inverted the solution is known to be unstable [DWZZ18, Theorem 1.4 (3)]. Here, if there is no shear ($\beta = 0$) the hydrostatic balance serves to stabilize the dynamics of the vorticity. However, if $\beta \neq 0$ the evolution of the vorticity is still algebraically unstable with a rate depending on $\alpha$ and $\beta$.

**Lemma 2.3.** Consider the inviscid problem (5) with $\alpha > 0$ and initial data $(\omega_0, \theta_0) \in H^N \times H^{N+1}$.

If there is no shear, $\beta = 0$, then the evolution
\[
(\omega_0, \theta_0) \mapsto (\omega(t), \theta(t))
\]
is stable as a map on $H^N \times H^{N+1}$ for any $N \geq 0$. More precisely, for every $\alpha > 0$ and every $t \geq 0$ it holds that
\[
\|\omega(t)\|_{H^N} \leq \|\omega_0\|_{H^N} + \frac{1}{\sqrt{\alpha}}\|\nabla \theta_0\|_{H^N}, \\
\|\theta(t)\|_{H^{N+1}} \leq \sqrt{\alpha}\|\omega_0\|_{H^N} + \|\nabla \theta_0\|_{H^{N+1}}.
\]

If there is shear, $\beta \neq 0$, then the evolution of
\[
\omega(t, x + \beta ty, y), \theta(t, x + \beta ty, y)
\]
is unstable in $H^N \times H^{N+1}$ with an algebraic growth rate $t^\gamma$ as $t \to \infty$. Here $\gamma$ depends on $\alpha$ and $\beta$.

**Proof.** The case without shear: In the case $\beta = 0$ the equation reduces to

$$\partial_t \omega = \partial_x \theta,$$
$$\partial_t \theta = \alpha \partial_x \Delta^{-1} \omega.$$

Taking a Fourier transform in both $x$ and $y$ we obtain a two-dimensional constant coefficient ODE system at each frequency:

$$\partial_t \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} 0 & ik \\ \frac{\alpha}{k^2 + \xi^2} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix}.$$

This then has the explicit solution

$$\begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix} (t) = \begin{pmatrix} \cos(\frac{k\sqrt{\alpha}}{\sqrt{k^2 + \xi^2}} t) & i\frac{\sqrt{k^2 + \xi^2}}{\sqrt{\alpha}} \sin(\frac{k\sqrt{\alpha}}{\sqrt{k^2 + \xi^2}} t) \\ i\frac{\sqrt{\alpha}}{\sqrt{k^2 + \xi^2}} \sin(\frac{k\sqrt{\alpha}}{\sqrt{k^2 + \xi^2}} t) & \cos(\frac{k\sqrt{\alpha}}{\sqrt{k^2 + \xi^2}} t) \end{pmatrix} \begin{pmatrix} \tilde{\omega}_0 \\ \tilde{\theta}_0 \end{pmatrix}.$$

In particular, we observe that $\sqrt{k^2 + \xi^2} \tilde{\theta}_0$ loses one derivative in $x$ and $y$ as opposed to just $\partial_x \theta_0$ in the $\alpha = 0$ case. In contrast $\frac{\sqrt{\alpha}}{\sqrt{k^2 + \xi^2}}$ gains one derivative.

We remark that as $\alpha \downarrow 0$ we recover the growth by $t$ as in Lemma 2.2.

The case with shear: If $\beta \neq 0$ we may consider a rescaling of $y \mapsto \beta y$ to obtain:

$$\partial_t \omega + y \partial_x \omega = \partial_x \theta,$$
$$\partial_t \theta + y \partial_x \theta + \alpha \partial_x (\partial_x^2 + \beta^2 \partial_y^2)^{-1} \omega = 0.$$

In view of stability properties of the flow by $y \partial_x$ we further change to coordinates $(x+ty, y)$ (or $(x+\beta ty, y)$ in the original coordinates). In these coordinates a Fourier transform then leads to the following time-dependent ODE system:

$$\partial_t \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} 0 \frac{k}{\beta y} & ik \\ \frac{\alpha}{k^2 + \beta^2 (\xi - k t)} & 0 \end{pmatrix} \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix}.$$

We note that for $k = 0$ this system is trivial. In the following thus let $k \neq 0$ be arbitrary but fixed.

As the matrix is time-dependent, we cannot anymore use a matrix exponential function to solve it. Instead we follow a similar approach as in a prior work on fluid echoes in Euler’s equations [DZ19] and consider a corresponding second order ODE system. Indeed, since $ik$ does not depend on $t$ we observe that the equation for $\partial_t^2 \tilde{\omega}$ decouples and is given by a Schrödinger problem with potential:

$$\partial_t^2 \tilde{\omega} + \frac{\alpha}{\beta^2 k^2 + (\xi - k t)^2} \tilde{\omega} = 0. \tag{6}$$

We remark that we may recover

$$\tilde{\theta} = \frac{1}{ik} \partial_t \tilde{\omega}.$$

in terms of $\partial_t \omega$. Thus it suffices to understand how $\omega$ and $\partial_t \omega$ evolve under the equation (6).

Shifting in time by $\frac{\xi}{k}$, problem (6) becomes independent of $k$:

$$\partial_t^2 \tilde{\omega} + \alpha \frac{1}{1 + (\beta t)^2} \tilde{\omega} = 0.$$
We then further rescale time by $t \mapsto \frac{1}{\beta} t =: \tau$, which yields
\[
\partial^2_{\tau} \tilde{\omega} + \alpha \beta^2 \frac{1}{1 + \tau^2} \tilde{\omega} = 0.
\]
For simplicity of notation in the following we consider the special case $\beta = 1$ and again use $t$ for the time variable. However, by the above scaling argument this is no loss of generality.

This problem then has an explicit solution in terms of hypergeometric functions of the second kind (see the NIST Digital Library of Mathematical Functions [DLMF], Chapter 15):
\[
\omega = C_1 F\left(\frac{1}{4} - \frac{1}{4} \sqrt{1 - 4\alpha}, -\frac{1}{4} + \frac{1}{4} \sqrt{1 - 4\alpha}, \frac{1}{2}, -t^2\right)
+ C_2 t F\left(\frac{1}{4} - \frac{1}{4} \sqrt{1 - 4\alpha}, \frac{1}{4} - \frac{1}{4} \sqrt{1 - 4\alpha}, \frac{3}{2}, -t^2\right).
\]
In particular, we note that asymptotically (see Chapter 15.8 in [DLMF])
\[
F\left(a, b, c; z\right) \sim c_1 z^{-a} (1 + \mathcal{O}(z^{-1})) + c_2 z^{-b} (1 + \mathcal{O}(z^{-1})).
\]
as $z = -t^2$ tends to $-\infty$. Since
\[
-a = \frac{1}{4} + \frac{1}{4} \sqrt{1 - 4\alpha}
\]
has positive real part for $\alpha \neq 0$ we conclude that the evolution for $\omega$ is algebraically unstable.

In these introductory results we have seen that $\beta \neq 0$ and $\alpha > 0$ introduce competing (de)stabilizing effects and that the evolution of the vorticity in the inviscid problem is generally unstable. In the following we investigate whether stability can be restored by dissipation and if so how much dissipation is required. Here we first consider the case $\alpha = 0$ in Section 2.2 and then $\alpha > 0$ in Section 2.3.

2.2. The Homogeneous, Partial Dissipation Case. In this section we consider the problem of homogeneous hydrostatic balance, $\alpha = 0$, and shear flow, $\beta \in \mathbb{R}$, with partial dissipation. The case of affine balance, $\alpha > 0$, is studied in Section 2.3. Due to the constant coefficient structure and the absence of boundary terms we here can construct (semi-)explicit solutions and thus clearly identify the effects of each dissipation coefficient.

Problems of partial dissipation naturally appear as limiting cases where for instance vertical and horizontal length scales are of very different magnitude or either thermal or viscous effects are considered dominant. In particular, we mention the work of Doering, Wu, Zhao and Zheng [DWZZ18] on the nonlinear problem without buoyancy diffusion ($\alpha > 0$, $\beta = 0$, $\eta_x = \eta_y = 0$, $\nu_x, \nu_y > 0$) and the work by L. Tao and Wu [TW19] on the linearized problem with shear and vertical diffusion ($\alpha = 0$, $\beta = 1$, $\eta_x = \nu_x = 0$, $\eta_y, \nu_y > 0$).

In the following we consider the linear problem (5) with $\alpha = 0$ and in particular show that if at least just one of the diffusivity coefficients is positive then the problem is asymptotically stable. Moreover, if at least one of $\eta_y$ and $\nu_y$ is positive the problem exhibits enhanced dissipation, that is damping on faster time scales than might be expected for heat flow. Thus, in this setting we can hence show directly that milder assumptions are sufficient.
Theorem 2.4. Consider the linearized Boussinesq problem (5) for $\alpha = 0, \beta = 1$ with $\omega_0, \theta_0, \partial_x \theta_0 \in H^N$, $N > 0$, and suppose that at least one of $\eta_x, \eta_y, \nu_x, \nu_y \geq 0$ is non-trivial. Then the evolution is asymptotically stable in the following sense.

The $x$-averages $\omega_x(t, y), \theta_x(t, y)$ (see Section 1.2 for a summary of notation) satisfy the one-dimensional heat equation with diffusivity $\nu_y, \eta_y$, respectively. In particular, they are stable in $H^N$ and decay as time tends to infinity.

Next consider the orthogonal complement or by linearity assume that $\omega_x(0) = 0 = \theta_x(0)$. Then it holds that for every $j \leq N$, there exists $C = C(N, \eta_x, \eta_y)$ such that

$$\|\theta_{x'}(t)\|_{H^j} \leq C \exp(-\eta_x t - \eta_y t^3/12)\|\theta_{0,x'}\|_{H^j}.$$ 

Thus the evolution of the temperature is stable, exponentially decreasing if $\eta_x > 0$ and exhibits enhanced dissipation if $\eta_y > 0$. Furthermore, there exists $\omega_1 = \omega_1(t, \omega_0, \theta_0, \nu, \eta) \in L^1_{loc}(\mathbb{R}, H^N)$ and $C = C(\nu_x, \nu_y, \eta_x, \eta_y, N)$ such that

$$\|\omega(t) - \omega_1(t)\|_{H^N} \leq C \exp(-\max(\eta_y, \nu_y) k^2 t^3 - \max(\eta_x, \nu_x) k^2 t) (\|\omega_0\|_{H^N} + \|\partial_x \theta_0\|_{H^N}).$$

and

$$\|\omega_1(t)\|_{H^N} \leq \min\left(t, \frac{1}{\nu_x k^2}, \frac{1}{\sqrt{\nu_y k^2}}, \frac{1}{\eta_x k^2}, \frac{1}{\sqrt{\eta_y k^2}}\right) \exp\left(-\frac{1}{12} \min(\eta_y, \nu_y) k^2 t^3 - \frac{1}{2} \min(\eta_x, \nu_x) k^2 t\right) (\|\omega_0\|_{H^N} + \|\theta_0\|_{H^{N+1}}).$$

Thus $\omega - \omega_1$ is stable for any choice of diffusivity parameters. The function $\omega_1$ is stable in time if at least one diffusion coefficient is non-zero and grows linearly if all are zero.

We remark that if at least one of the vertical diffusion coefficients $\nu_y, \eta_y$ is positive then $\omega - \omega_1$ exhibits enhanced dissipation on the time scale $\max(\eta_y, \nu_y)^{-1/3}$. In contrast $\omega_1$ only exhibits (enhanced) dissipation if pairs of diffusion coefficients are positive, but is uniformly bounded if at least one coefficient is non-zero. As we have seen in Lemma 2.2, in the inviscid limit $\omega_1$ grows linearly in $t$.

Proof of Theorem 2.4. We recall that the linearized Boussinesq problem (5) is given by

$$\partial_t \omega + y \partial_x \omega = \nu_x \partial_{xx} \omega + \nu_y \partial_{yy} \omega + \partial_x \theta,$$

$$\partial_t \theta + y \partial_x \theta = \eta_x \partial_{xx} \theta + \eta_y \partial_{yy} \theta,$$

$$(t, x, y) \in \mathbb{R} \times T \times \mathbb{R}. $$

After changing to coordinates $(x + ty, y)$ moving with the flow we obtain constant coefficient but time-dependent differential operators on the right-hand-side. It is therefore natural to consider an equivalent formulation by means of the Fourier transform.

Let $k, \xi$ denote the Fourier variables with respect to $x, y$ and define

$$f(t, k, \xi) = \hat{\omega}(t, k, \xi + kt),$$

$$g(t, k, \xi) = \hat{\theta}(t, k, \xi + kt).$$
Then the system \((5)\) can be equivalently expressed as

\[
\begin{align*}
\partial_t f(t, k, \xi) &= -\nu_x k^2 f(t, k, \xi) - \nu_y (\xi - kt)^2 f(t, k, \xi) + ikg(t, k, \xi), \\
\partial_t g(t, k, \xi) &= -\eta_x k^2 g(t, k, \xi) - \eta_y (\xi - kt)^2 g(t, k, \xi).
\end{align*}
\]  

(7)

Note that \(f(0) = \tilde{\omega}_0, \ g(0) = \tilde{\theta}_0\). We in particular observe that this problem decouples with respect to the spatial frequencies \(k, \xi\) and that (only for this \(\alpha = 0\) case) the evolution equation for \(g\) decouples from the equation for \(f\).

If \(k = 0\) the system simplifies to

\[
\begin{align*}
\partial_t f(t, 0, \xi) &= -\nu_y \xi^2 f(t, 0, \xi), \\
\partial_t g(t, 0, \xi) &= -\eta_y \xi^2 g(t, 0, \xi),
\end{align*}
\]

which has the explicit solutions

\[
\begin{align*}
f(t, 0, \xi) &= \exp(-\nu_y t \xi^2) \tilde{\omega}_0(0, \xi), \\
g(t, 0, \xi) &= \exp(-\eta_y t \xi^2) \tilde{\theta}_0(0, \xi).
\end{align*}
\]

In particular, both quantities are stable in any Sobolev norm and decay at heat flow rates if \(\eta_y\) or \(\nu_y\) are positive, respectively.

Let next \(k \neq 0\) be arbitrary but fixed. We may then explicitly compute \(g(t)\) as

\[
g(t, k, \eta) = \exp\left(-\eta_x k^2 t - \eta_y \int_0^t (\xi - k\tau)^2 d\tau\right) \theta_0(k, \xi).
\]

In particular, we observe that if \(\eta_x > 0\) we obtain exponential decay. If \(\eta_y > 0\) we may compute

\[
\int_0^t (\xi - k\tau)^2 d\tau = \frac{1}{3k}((kt - \xi)^3 + \xi^3) = \frac{k^3t^3 - 3k^2t^2\xi + 3kt\xi^2}{3k}.
\]

We note that for fixed \(k\) and \(t\) this is a quadratic function in \(\xi\) with positive leading coefficient \(t\) and attains its minimum for

\[-3k^2t^2 + 6kt\xi = 0 \iff \xi = \frac{kt}{2}.\]

Hence, for any \(\xi\) it holds that

\[
\int_0^t (\xi - k\tau)^2 d\tau \geq \frac{1}{3k}((kt/2)^3 + (kt/2)^3) = \frac{k^2t^3}{12}.
\]

Thus, it follows that \(g(t)\) satisfies the pointwise estimate

\[
|g(t, k, \eta)| \leq \exp\left(-\eta_x k^2 t - \eta_y \frac{k^2t^3}{12}\right) |\theta_0(k, \xi)|.
\]

Hence, \(g\) is stable in any Sobolev norm and exhibits exponential decay if \(\eta_x > 0\) and enhanced decay if \(\eta_y > 0\).
Let us next consider \( f(t) \). We may express \( f \) using the following integral formula

\[
f(t, k, \xi) = \exp\left(-\nu_x k^2 t - \nu_y \int_0^t (\xi - k\tau)^2 \, d\tau\right) \omega_0(k, \xi)
+ \int_0^t \exp\left(-\nu_x k^2 (t - s) - \nu_y \int_s^t (\xi - k\tau)^2 \, d\tau\right)
\times \exp\left(-\eta_x k^2 s - \eta_y \int_0^s (\xi - k\tau)^2 \, d\tau\right) \, i k \theta_0(k, \xi) \, ds
=: f_1(t, k, \xi) - f_2(t, k, \xi).
\]

The first contribution

\[
f_1(t, k, \xi) = \exp\left(-\nu_x k^2 t - \nu_y \int_0^t (\xi - k\tau)^2 \, d\tau\right) \omega_0(k, \xi)
\]

is again stable for any choice of \( \nu_x, \nu_y \geq 0 \) and exhibits (enhanced) dissipation if either coefficient is positive. Let us thus focus on the second contribution \( f_2 \). Again estimating

\[
\int_0^s (\xi - k\tau)^2 \, d\tau \geq \frac{k^2 s^3}{12}
\]

from below we readily see that the integral

\[
\int_0^t \exp\left(-\nu_x k^2 (t - s) - \nu_y \int_s^t (\xi - k\tau)^2 \, d\tau\right)
\times \exp\left(-\eta_x k^2 s - \eta_y \int_0^s (\xi - k\tau)^2 \, d\tau\right) \, ds
\]

is bounded by a universal constant times

\[
\min\left(t, \frac{1}{\nu_x k^2}, \frac{1}{\nu_y k^2}, \frac{1}{\eta_x k^2}, \frac{1}{\eta_y k^2}\right).
\]

In particular, if any coefficient is positive this integral is bounded. However, if several diffusion coefficients are zero, then this integral need not converge to zero as time tends to infinity. Thus, in order to obtain uniform decay estimates we separately account for asymptotic behavior in terms of a function \( \omega_1 \).

**Defining \( \omega_1 \):** In order to introduce ideas, let us first consider a special case. If \( \nu_x = \nu_y = 0 \), we observe that as \( t \to \infty \)

\[
\int_0^t \exp\left(-\eta_x k^2 s - \eta_y \int_0^s (\xi - k\tau)^2 \, d\tau\right) \, ds \, i k \theta_0(k, \xi)
\]

converges to a nontrivial limit. We thus define \( \omega_1 \) as this limit, which for this case has the following explicit formula:

\[
\omega_1 := \int_0^\infty \exp\left(-\eta_x k^2 s - \eta_y \int_0^s (\xi - k\tau)^2 \, d\tau\right) \, ds \, i k \theta_0(k, \xi).
\]

We then observe that the difference

\[
\omega_1 - f_2(t) = \int_t^\infty \exp\left(-\eta_x k^2 s - \eta_y \int_0^s (\xi - k\tau)^2 \, d\tau\right) \, ds \, i k \theta_0(k, \xi).
\]
is bounded by
\[
\min \left( \frac{1}{\eta_x} k^2 \exp(-k^2 t), \frac{1}{\sqrt{k^2 \eta_y}} \exp\left(-\nu_y k^2 t^3 / 8\right) \right) |k\tilde{\theta}_0(k, \xi)|
\]
and hence exhibits (enhanced) decay.

More generally, we define \(\omega_1(t)\) to capture the slowest decay (in the above example that is no decay since \(\nu_x = \nu_y = 0\)). We therefore split
\[
\begin{align*}
\exp\left(-\nu_x k^2 (t-s) - \eta_x k^2 s\right) \\
= \exp(-\eta_x k^2 t) \exp(-\nu_x k^2 t - \eta_x k^2 s) \\
= \exp(-\nu_x k^2 t) \exp(-\eta_x k^2 (t-s)) \\
= \exp(-\nu_x k^2 t) \exp(-\eta_x k^2 s),
\end{align*}
\]
and hence exhibits (enhanced) decay.

More generally, we define \(\omega_1(t)\) to capture the slowest decay (in the above example that is no decay since \(\nu_x = \nu_y = 0\)). We therefore split
\[
\begin{align*}
\exp\left(-\nu_x \int_s^t (\xi - k\tau)^2 - \eta_x \int_0^s (\xi - k\tau)^2\right) \\
= \exp(-\eta_x \int_s^t (\xi - k\tau)^2) \exp\left(-\nu_x \int_s^t (\xi - k\tau)^2\right) \\
= \exp\left(-\nu_x \int_s^t (\xi - k\tau)^2\right) \exp\left(-\nu_x \int_0^s (\xi - k\tau)^2\right).
\end{align*}
\]
Here the first factor is independent of \(s\). Hence, for instance for \(\nu_x \leq \eta_x, \nu_y \leq \eta_y\) we may write
\[
(13) \quad f_2(t) = \exp(-\nu_x k^2 t) \exp\left(-\nu_x \int_0^t (\xi - k\tau)^2\right)
\]
\[
\times \int_0^t \exp(-\eta_x (\xi - k\tau)^2) \exp\left(-\nu_x \int_0^s (\xi - k\tau)^2\right) ds \, ik\tilde{\theta}_0(k, \xi).
\]
If both \(\eta_x = \nu_x\) and \(\eta_y = \nu_y\) the inner integral simplifies to \(t\) and \(f_2(t)\) decays exponentially. In this case we simply set \(\omega_1(t) := f_2(t)\). In the following we restrict to the case where at least one pair is not equal.

Then the inner integral is uniformly bounded by a uniform constant times
\[
\min\left(\frac{1}{k^2 |\eta_x - \nu_x|}, \frac{1}{\sqrt{k^2 |\eta_y - \nu_y|}}\right).
\]
We therefore aim to define \(\omega_1(t)\) by passing to the limit \(t \to \infty\) in the inner integral of equation (13). We distinguish the following four cases:

If \(\nu_x \leq \eta_x, \nu_y \leq \eta_y\) we define
\[
\omega_1(t) := \exp(-\nu_x k^2 t) \exp(-\nu_x \int_0^t (\xi - k\tau)^2)
\]
\[
\times \int_0^\infty \exp(-(\eta_x - \nu_x) k^2 s) \exp\left(-(\eta_y - \nu_y) \int_0^s (\xi - k\tau)^2\right) ds \, ik\tilde{\theta}_0(k, \xi).
\]
and observe that
\[
|f_2(t) - \omega_1(t)| = \exp(-\nu_x k^2 t) \exp(-\nu_y t) \int_0^t (\xi - k\tau)^2 \\
\times \int_0^\infty \exp(-\eta_x - \nu_x k^2 s) \exp\left(-\left(\eta_y - \nu_y \int_0^s (\xi - k\tau)^2 ds\right) \right) \exp\left(-\left(\eta_y - \nu_y \int_0^t (\xi - k\tau)^2 ds\right) \right) ds |k\tilde{\theta}_0(k, \xi)| \\
\leq C \exp(-\nu_x k^2 t) \exp(-\nu_y t) \int_0^t (\xi - k\tau)^2 \\
\exp\left(-\left(\eta_x - \nu_x k^2 t - (\eta_y - \nu_y) k^2 t^3/12\right) \right) |k\tilde{\theta}_0(k, \xi)|
\]

exhibits (enhanced) dissipation with the larger of the coefficients.

If \( \nu_x \geq \eta_x \), \( \nu_y \geq \eta_y \) we similarly define
\[
\omega_1(t) := \exp(-\eta_x k^2 t) \exp(-\eta_y t) \int_0^t (\xi - k\tau)^2 \\
\times \int_0^\infty \exp(-\eta_x - \nu_x k^2 \sigma) \exp\left(-\left(\eta_y - \nu_y \int_0^\sigma (\xi - k\tau)^2 ds\right) \right) ds ik\tilde{\theta}_0(k, \xi),
\]
where we introduced the change of variables \( s = t - \sigma \) and extended the domain of integration from \( \sigma \in [0, t] \) to \( \sigma \in [0, \infty) \). We remark that here the inner integral still depends on \( t \). By an analogous calculation we then again observe that \( f_2(t) - \omega_1(t) \) exhibits (enhanced) dissipation with the larger of the coefficients.

Finally, for \( \eta_x \leq \nu_x \) and \( \eta_y \geq \nu_y \) we define
\[
\omega_1(t) := \exp(-\eta_x k^2 t) \exp\left(-\nu_y t \int_0^t (\xi - k\tau)^2\right) \\
\times \int_{-\infty}^\infty \exp\left(-\eta_x - \nu_x k^2 \min(s, 0)\right) - (\eta_y - \nu_y) \int_{\min(s, t)}^t (\xi - k\tau)^2 ds \, ik\tilde{\theta}_0(k, \xi),
\]
and analogously for \( \eta_x \geq \nu_x \), \( \eta_y \leq \nu_y \).

\[
\end{proof}
\]

2.3. The Effects of Hydrostatic Balance. In the previous Section 2.2 we have shown that in the case \( \alpha = 0 \) very weak partial dissipation (just one non-zero coefficient) is sufficient to obtain asymptotic stability and decay rates and that the vorticity can be decomposed into a slower (or not all) decaying part \( \omega_1 \) and a fast decaying part \( \omega - \omega_1 \). For that setting we could exploit that the equation for \( \theta \) decouples and that we can thus first solve for \( \theta(t) \) and subsequently for \( \omega(t) \).

If \( \alpha > 0 \) this decoupling structure is lost and we obtain the following system at each Fourier frequency:

\[
\partial_t \left(\frac{\omega}{\theta}\right) = \left(\frac{-\nu_x k^2 - \nu_y (\xi - \beta k t)^2}{\kappa^2 + \mu (\xi - \beta k t)}\right) \omega - \eta_x k^2 - \eta_y (\xi - \beta k t) \frac{ik}{\kappa} \left(\frac{\omega}{\theta}\right).
\]

We note that if \( \beta \neq 0 \) all coefficients except \( ik \) are time-dependent, which makes this problem very challenging. As a first step we hence discuss the setting without shear, \( \beta = 0 \), and introduce two methods of proof. The first method is an adaptation of energy methods commonly used in the nonlinear problem and second, more precise result constructs explicit solutions.
2.3.1. The Case without Shear. In this section we consider the linearized problem around \((\omega, \theta) = (0, \alpha y)\) with \(\alpha > 0\). The corresponding nonlinear problem has been studied in [DWZZ18] for the setting of full dissipation and of vertical dissipation.

As a first method of proof in Proposition 2.5 we adapt energy arguments which are well-known for the nonlinear problem (see [DWZZ18, LT16, LLT13]) to this linear setting. This approach has the benefit of a very simple and robust structure. However, it does not precisely capture the effects of the various diffusion coefficients.

As a second method in Proposition 2.6 we hence derive explicit solutions of the ODE systems in Fourier variables. Here we crucially exploit the lack of shear and hence time-independence of the coefficients.

**Proposition 2.5.** Let \(\alpha > 0\) and \(\nu_x, \nu_y, \eta_x, \eta_y \geq 0\) be given. Then for any initial data \((\omega_0, \theta) \in H^N \times H^{N+1}\) the solution \(\omega, \theta\) of the linearized problem

\[
\partial_t \omega = \nu_x \partial_x^2 \omega + \nu_y \partial_y^2 \omega + \partial_x \theta,
\]

\[
\partial_t \theta + \alpha \partial_x \Delta^{-1} \omega = \eta_x \partial_x^2 \theta + \eta_y \partial_y^2 \theta.
\]

is stable and satisfies

\[
\frac{d}{dt}(\alpha\|\omega\|^2_{H^N} + \|\nabla \theta\|^2_{H^N}) + \nu_x \|\partial_x \omega\|^2_{H^N} + \nu_y \|\partial_y \omega\|^2_{H^N} + \eta_x \|\partial_x \theta\|^2_{H^N} + \eta_y \|\partial_y \theta\|^2_{H^N} = 0
\]

**Proof.** We note that all differential operators in (15) are linear and involve constant coefficients. Hence, the problem decouples in frequency and we may without loss of generality restrict to \(N = 0\) and studying single modes \((k, \xi)\). Here, the \(x\)-average decouples and evolves by heat flow, so we further restrict to analyzing \(k \neq 0\).

Then after a Fourier transform we obtain

\[
\partial_t \begin{pmatrix} \omega \\ \theta \end{pmatrix} = \begin{pmatrix} -\nu_x k^2 - \nu_y \xi^2 & ik \\ ik \sqrt{\frac{k \alpha}{k^2 + \xi^2}} & \eta_x k^2 - \eta_y \xi^2 \end{pmatrix} \begin{pmatrix} \omega \\ \theta \end{pmatrix}.
\]

As we discuss in Proposition 2.6 this constant coefficient ODE system can be solved explicitly by means of the matrix exponential function. However, for this proposition we instead use an energy argument which exploits anti-symmetry: If we multiply \(\theta\) by \(\sqrt{k^2 + \xi^2}\) and \(\omega\) by \(\sqrt{\alpha}\) our system reads

\[
\partial_t \begin{pmatrix} \sqrt{\alpha} \omega \\ \sqrt{k^2 + \xi^2} \theta \end{pmatrix} = \begin{pmatrix} -\nu_x k^2 - \nu_y \xi^2 & \frac{ik \sqrt{\alpha}}{\sqrt{k^2 + \xi^2}} \\ \frac{ik \sqrt{\alpha}}{\sqrt{k^2 + \xi^2}} & \eta_x k^2 - \eta_y \xi^2 \end{pmatrix} \begin{pmatrix} \sqrt{\alpha} \omega \\ \sqrt{k^2 + \xi^2} \theta \end{pmatrix}.
\]

The off-diagonal matrix entries are the same and purely imaginary. Therefore, they cancel when considering \(M + M^T\) and

\[
\frac{d}{dt}(|\sqrt{\alpha} \omega|^2 + |\sqrt{k^2 + \xi^2} \theta|^2) = -(\nu_x k^2 + \nu_y \xi^2)|\sqrt{\alpha} \omega|^2 - (\eta_x k^2 + \eta_y \xi^2)|\sqrt{k^2 + \xi^2} \theta|^2.
\]

This energy functional is hence non-increasing and we obtain decay estimates in terms of \(\min(\nu_x, \eta_x)k^2\) and \(\min(\nu_y, \eta_y)\xi^2\). We further remark that after multiplying by \(\sqrt{k^2 + \xi^2}\), this is equivalent to an estimate on the velocity and density

\[
\alpha \|u\|^2_{H^N} + \|\theta\|^2_{H^N},
\]

see [DWZZ18] for a nonlinear analogous estimate.

As an alternative, more fragile but also more precise approach, we may compute explicit solution operators in Fourier variables.
Proposition 2.6. Let $\alpha > 0$ and $\nu_x, \nu_y, \eta_x, \eta_y \geq 0$ be given. Then for any initial data $(\omega_0, \theta_0) \in H^N \times H^{N+1}$ of the linearized problem (15) is stable. Furthermore, for every frequency $(k, \xi)$ and $\alpha \neq k^2 + \xi^2 \left( \frac{\eta_x - \nu_x}{2} k^2 + \frac{\eta_y - \nu_y}{2} \xi^2 \right)^2 =: \alpha^*$, there exists a basis $(v_1, v_2)$ and constants
\[
\lambda_{1,2} = -\frac{\eta_x + \nu_x}{2} k^2 - \frac{\eta_y + \nu_y}{2} \xi^2 \pm \sqrt{\left( \frac{\eta_x - \nu_x}{2} k^2 + \frac{\eta_y - \nu_y}{2} \xi^2 \right)^2 - \alpha^* k^2 + \xi^2},
\]
such that in this basis the evolution of $(\tilde{\omega}, \tilde{\theta})$ is given by
\[
\begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{pmatrix}.
\]
We in particular observe that for all $\alpha \in (0, \alpha^*)$ it holds that
\[
\lambda_1, \lambda_2 < 0
\]
and for $\alpha > \alpha^*$
\[
\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = -\frac{\eta_x + \nu_x}{2} k^2 - \frac{\eta_y + \nu_y}{2} \xi^2.
\]

Proof of Proposition 2.6. We recall that equation (15) is equivalent to the ODE system (16)
\[
\partial_t \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} -\nu_x k^2 - \nu_y \xi^2 & ik \\ \frac{i k \eta_x}{k^2 + \xi^2} & -\eta_x k^2 - \eta_y \xi^2 \end{pmatrix} \begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix},
\]
at each frequency $(k, \xi)$. We denote the coefficient matrix as
\[
M = \begin{pmatrix} -\nu_x k^2 - \nu_y \xi^2 & ik \\ \frac{i k \eta_x}{k^2 + \xi^2} & -\eta_x k^2 - \eta_y \xi^2 \end{pmatrix}.
\]
Since $M$ is time-independent, we obtain a solution in terms of the matrix exponential function:
\[
\begin{pmatrix} \tilde{\omega} \\ \tilde{\theta} \end{pmatrix} = \exp(tM) \begin{pmatrix} \omega_0 \\ \theta_0 \end{pmatrix}.
\]
It thus remains to explicitly compute the matrix exponential $\exp(tM)$. We recall that the eigenvalues of a $2 \times 2$ matrix are given by the roots of the characteristic polynomial
\[
\lambda^2 - \text{tr}(M) \lambda + \text{det}(M).
\]
We thus obtain
\[
\lambda_{1,2} = \frac{1}{2} \text{tr}(M) \pm \sqrt{\left( \frac{\text{tr}(M)}{2} \right)^2 - \text{det}(M)}
\]
\[
= -\frac{\eta_x + \nu_x}{2} k^2 - \frac{\eta_y + \nu_y}{2} \xi^2 \pm \sqrt{\left( \frac{\eta_x - \nu_x}{2} k^2 + \frac{\eta_y - \nu_y}{2} \xi^2 \right)^2 - \alpha^* k^2 + \xi^2},
\]
where we used that \((a^2 + d^2) - ad + bc = (a - d)^2 + bc\). For simplicity of notation let us denote

\[
r = \sqrt{\left(\frac{\eta_x - \nu_x}{2} - \alpha \frac{k^2}{k^2 + \xi^2}\right)^2 - \alpha \frac{k^2}{k^2 + \xi^2}}
\]

Then corresponding eigenvectors are given by

\[
\left(\frac{k^2 + \xi^2}{k^2} \left(i \left(\frac{\eta_x - \nu_x}{2} + \frac{\eta_y - \nu_y}{2} \xi^2\right) \pm r\right)\right)
\]

and

\[
\exp(tM) = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \exp(t\lambda_1) & 0 \\ 0 & \exp(t\lambda_2) \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1},
\]

if \(\lambda_1 \neq \lambda_2\). This slightly degenerates if \(\alpha = \alpha^*\) (if \(r = 0\)) with a cyclic subspace and growth with a factor \(t\). We omit this case for brevity.

It remains to discuss the size of the eigenvalues. We note that if

\[
\alpha > \frac{k^2 + \xi^2}{k^2} \left(\frac{\eta_x - \nu_x}{2} + \frac{\eta_y - \nu_y}{2} \xi^2\right)^2 =: \alpha^*,
\]

then \(r\) is strictly imaginary and

\[
\Re(\lambda_1) = \Re(\lambda_2) = -\frac{\eta_x + \nu_x}{2} k^2 - \frac{\eta_y + \nu_y}{2} \xi^2.
\]

In particular,

\[
|e^{\lambda_1 t}| = |e^{\lambda_2 t}| = \exp\left(-t \left(\frac{\eta_x - \nu_x}{2} k^2 + \frac{\eta_y - \nu_y}{2} \xi^2\right)\right)
\]

both decay exponentially even if only some of the dissipation coefficients are non-zero.

Moreover, if \(0 < \alpha < \alpha^*\) the eigenvalues \(\lambda_1, \lambda_2\) are distinct and real-valued and

\[
\left(\frac{\eta_x - \nu_x}{2} k^2 + \frac{\eta_y - \nu_y}{2} \xi^2\right)^2 - \alpha \frac{k^2}{k^2 + \xi^2} < \left|\frac{\eta_x - \nu_x}{2} k^2 + \frac{\eta_y - \nu_y}{2} \xi^2\right| \leq \left|\frac{\eta_x - \nu_x}{2} k^2 + \frac{\eta_y - \nu_y}{2} \xi^2\right|.
\]

Therefore

\[
\lambda_2 > \frac{\min(\eta_x, \nu_x)}{2} k^2 + \frac{\min(\eta_y, \nu_y)}{2} \xi^2
\]

is positive even if multiple dissipation coefficients are zero. \(\Box\)

This explicit solution shows that the dependence of sharp decay rates on the parameters is more subtle than captured in Proposition 2.5. However, in the case \(\beta \neq 0\) of the following section explicit solutions become infeasible to compute and we hence rely on more robust but less precise energy arguments.
2.3.2. On the Interaction of Shear and Hydrostatic balance. In this section we consider the linearized problem with \( \alpha > 0 \) and \( \beta = 1 \) and with partial dissipation:

\[
\partial_t \left( \omega \begin{pmatrix} \omega \\ \theta \end{pmatrix} \right) = \begin{pmatrix} -\nu_x k^2 - \nu_y (\xi - kt)^2 \\ ik \frac{i k \omega}{k^2 + (\xi - kt)^2} \end{pmatrix} \begin{pmatrix} \omega \\ \theta \end{pmatrix}.
\]

Unlike the setting studied in Section 2.2 here the evolution of \( \theta \) does not decouple anymore and most coefficients are time-dependent. Therefore, this problem cannot be solved explicitly by means of a matrix exponential function and also does not easily decouple into second order equations as in Section 2.1.

Instead, we aim at adapting the energy method discussed in Proposition 2.5 of Section 2.2 to this setting.

**Proposition 2.7.** Let \( \omega, \theta \) be a solution of the problem (18). Then it holds that

\[
\alpha \|\omega(t)\|^2_{H^N} + \|\partial_x \omega - t \partial_x \theta \|^2_{H^N} \leq C (1 + t^2) \exp(-\min(\nu_x, \nu_y)t - \min(\nu_y, \eta_y)t^3/12)(\alpha \|\omega_0\|_{H^N} + \|\nabla \theta_0\|_{H^N}).
\]

In particular, if at least one of \( \min(\nu_x, \nu_y) \) or \( \min(\nu_y, \eta_y) \) is positive (that is, pairs of entries are non-zero), the system is asymptotically stable.

Furthermore, it holds that

\[
\alpha \|\nu\|^2_{H^N} + \|\theta\|^2_{H^N} \leq C \exp(-\min(\nu_x, \nu_y)t - \min(\nu_y, \eta_y)t^3/8)(\alpha \|\omega_0\|_{H^{N+1}} + \|\nabla \theta_0\|_{H^N}).
\]

Thus we may trade higher regularity of \( \omega_0 \) for a uniform bound on the velocity.

**Proof.** We recall that the problem under consideration is given by the following time-dependent system of ODEs:

\[
\partial_t \left( \omega \begin{pmatrix} \omega \\ \theta \end{pmatrix} \right) = \begin{pmatrix} -\nu_x k^2 - \nu_y (\xi - kt)^2 \\ ik \frac{i k \omega}{k^2 + (\xi - kt)^2} \end{pmatrix} \begin{pmatrix} \omega \\ \theta \end{pmatrix}.
\]

As the coefficient matrix does not exhibit anti-symmetry in this formulation, we aim to use a change of basis similar to the one of Section 2.3.1. That is, we consider

\[
\begin{pmatrix} \sqrt{\alpha} \omega \\ \sqrt{k^2 + (\xi - kt)^2} \theta \end{pmatrix}.
\]

Here we obtain an additional correction term involving

\[
\partial_t \sqrt{k^2 + (\xi - kt)^2} = \frac{k(k t - \xi)}{\sqrt{k^2 + (\xi - kt)^2}}.
\]

Inserting this ansatz into the equation (18) we obtain the following system:

\[
\partial_t \left( \sqrt{\alpha} \omega \begin{pmatrix} \sqrt{\alpha} \omega \\ \sqrt{k^2 + (\xi - kt)^2} \theta \end{pmatrix} \right) = \begin{pmatrix} -\nu_x k^2 - \nu_y (\xi - kt)^2 \\ ik \frac{i k \sqrt{\alpha} \omega}{\sqrt{k^2 + (\xi - kt)^2}} \end{pmatrix} \begin{pmatrix} \sqrt{\alpha} \omega \\ \sqrt{k^2 + (\xi - kt)^2} \theta \end{pmatrix}.
\]

As the off-diagonal entries are identical and purely imaginary, we deduce that

\[
\partial_t (\alpha |\omega|^2 + (k^2 + (\xi - kt)^2)|\theta|^2) = (-\nu_x k^2 - \nu_y (\xi - kt)^2)\alpha |\omega|^2 + \left( -\alpha \eta_x k^2 - \alpha \eta_y (\xi - kt)^2 + \frac{t - \frac{\xi}{k}}{1 + (t - \frac{\xi}{k})^2} \right) (k^2 + (\xi - kt)^2)|\theta|^2.
\]
The right-hand-side thus contains terms yielding exponential decay due to dissipation (see Section 2.2) as well as possible algebraic growth due to
$$
\exp \left( \int_0^T \max(0, \frac{t - \xi}{1 + (t - \xi)_k^2}) dt \right)
= \exp \left( \int_0^T 1_{t > \frac{\xi}{k}} \frac{1}{2} \ln(1 + (t - \xi)_k^2) \right) = 1_{T > \frac{\xi}{k} > 0} \sqrt{1 + (T - \frac{\xi}{k})^2} \leq \sqrt{1 + T^2}.
$$
We remark that here we could pass to the positive part since negative contributions are beneficial in energy estimates. Combining both bounds we obtain the desired result.

We may repeat the same argument for
$$
\frac{\sqrt{T}}{\sqrt{k^2 + (\xi - kt)^2}} \hat{\omega}
$$
However, here
$$
\frac{\xi - kt}{\sqrt{k^2 + (\xi - kt)^2}} = \frac{(\xi - kt)k}{(\sqrt{k^2 + (\xi - kt)^2})^3}
$$
has an opposite sign (it grows until \( t = \frac{\xi}{k} \) and decreases afterwards). In particular,
$$
\int_0^\infty \max \left( 0, \frac{\xi - t}{1 + (\frac{\xi}{k} - t)^2} \right) dt \leq C \left( 1 + \left( \frac{\xi}{k} \right)^2 \right)
$$
corresponds to a loss of two derivatives compared to \( \frac{1}{\sqrt{k^2 + \xi^2}} \omega_0 \) and thus one derivative of \( \omega_0 \).

We remark that due to the less explicit structure of the solutions, these results are less optimal than those of previous sections. However, they serve to highlight how the interaction of shear and hydrostatic balance introduces a stronger coupling between the vorticity and temperature.

3. The Nonlinear Full Dissipation Case

In this section we consider the nonlinear, viscous Boussinesq problem. We remark that questions of well-posedness or asymptotic stability for partial dissipation problems here are very challenging and for instance considered in [DWZZ18, LT16, LLT13] or [Cha06, CKN99]. For this reason we instead consider the full dissipation case and aim to obtain a more precise description of asymptotic behavior near the stationary solutions
$$
\omega = \beta, \ v = (\beta y, 0), \ \theta = \alpha y,
$$
which combine both shear flow and hydrostatic balance.

Here we consider two distinct cases. If \( \alpha \geq 0 \) is “small”, we adapt the methods developed by Bedrossian, Vicol and Wang [BVW16] for the 2D Navier-Stokes equations near Couette flow to the Boussinesq setting (see Section 3.1). See also the recent work by Luo [Luo], who adapt these methods to the hyperviscosity equations near Couette flow. As a second case we consider the setting where \( \alpha > 0 \) is “large” and combined with a shear \( \beta = 1 \) (see Section 3.2). There we combine
We explain how the stability results we obtained can be used to derive BVW16 is of limited use. Instead we aim to show that for sufficiently small initial data this system of equations has a unique global solution with this initial data satisfies \[ v = \nabla^\perp \Delta^{-1} \omega. \]

Given a stationary solution of the form (19) we consider the equation for perturbations \( \omega = \beta + \omega^*, v = (\beta y, 0) + v^*, \theta = \alpha y + \theta^* \):

\[
\begin{align*}
\partial_t \omega^* + \beta y \partial_x \omega^* + v^* \cdot \nabla \omega^* &= \nu \Delta \omega^* + \partial_x \theta^*, \\
\partial_t \theta^* + \beta y \partial_x \theta^* + v^* \cdot \nabla \theta^* - \alpha v_x^* &= \eta \Delta \theta^*.
\end{align*}
\]

We view this problem as a modification of the transport equation \( \partial_t + \beta y \partial_x \) and with slight abuse of notation reuse \( \omega, v, \theta \) to denote

\[
\begin{align*}
\omega(t, x, y) &:= \omega^*(t, x + t\beta y, y), \\
v(t, x, y) &:= v^*(t, x + t\beta y, y), \\
\theta(t, x, y) &:= \theta^*(t, x + t\beta y, y),
\end{align*}
\]

and define

\[
\nabla_t = (\partial_x, \partial_y - t\beta \partial_x), \quad \Delta_t = (\partial_x^2 + (\partial_y - t\beta \partial_x^2)).
\]

With these conventions the nonlinear Boussinesq equations read:

\[
\begin{align*}
\partial_t \omega + \nabla_t \Delta_t^{-1} \omega \cdot \nabla_t \omega &= \nu \Delta_t \omega + \partial_x \theta, \\
\partial_t \theta + \nabla_t \Delta_t^{-1} \omega \cdot \nabla_t \theta &= \eta \Delta_t \theta + \alpha \partial_x \Delta_t^{-1} \omega.
\end{align*}
\]

We then aim to show that for sufficiently small initial data this system of equations is asymptotically stable and \( \omega, \theta \) converge to zero as \( t \to \infty \) at enhanced dissipation rates. Here we first consider the question of stability in the setting where \( \alpha \) is “small” in Section 3.1. Subsequently we discuss the setting of \( \alpha > 1 \) in Section 3.2. Finally, in Section 4 we explain how the stability results we obtained can be used to derive enhanced dissipation rates.

### 3.1. Shear and Small Hydrostatic Balance

In this section we consider the nonlinear asymptotic stability for the case when \( \alpha \) is “small”. In this case a quantity such as \( \|\omega\|_{H^N}^2 + \|\nabla \theta\|_{H^N}^2 \) considered in Section 2.3 is of limited use. Instead we aim to exploit shearing behavior for \( \beta \neq 0 \) following the bootstrap/multiplier approach employed in [BVW16, Section 2] for the Navier-Stokes problem near Couette flow with relatively minor changes. The case of “large” \( \alpha \) is considered in Theorem 3.6. For simplicity of notation we in the following consider the case \( \beta = 1 \).

**Theorem 3.1.** Let \( N \geq 5, \beta = 1 \) and let \( \epsilon_1 \ll \min(\nu, \eta)^{1/2}, \epsilon_2 \ll \sqrt[4]{\nu} \epsilon_1 \) and suppose that \( 0 \leq \alpha < \nu^{1/2} \nu^{1/2} \epsilon_1^2 \). Then if \( \|\omega_0\|_{H^N} \leq \epsilon_1 \) and \( \|\theta_0\|_{H^N} \leq \epsilon_2 \), the unique global solution with this initial data satisfies

\[
\begin{align*}
\|\omega\|_{L_t^\infty H^N}^2 + \nu \|\nabla_t \omega\|_{L_t^2 H^N}^2 + \|\nabla_t \Delta_t^{-1} \omega\|_{L_t^2 H^N}^2 &\leq 8 \epsilon_1^2, \\
\|\theta\|_{L_t^\infty H^N}^2 + \eta \|\nabla_t \theta\|_{L_t^2 H^N}^2 &\leq 8 \epsilon_2.
\end{align*}
\]
Remark 1. • Here we study the regime of “small” $\alpha$, where stabilizing by mixing is dominant. In contrast, if $\alpha$ is “large” we may make use of (higher regularity analogues) of conserved quantities, which we discuss in Section 3.2.

• A constraint of the form $\epsilon_1 \ll \nu^{1/2}$ is also imposed in [BVW16]. In view of instabilities in the inviscid setting some constraint of this type is likely necessary, though weaker asymptotic stability results may persist under weaker constraints [DZ19, DM18]. The constraints imposed on $\epsilon_1, \alpha$ and, in particular, $\epsilon_2$ are probably quite far from optimal but allow for a relatively simple proof. See Proposition 3.5 for details.

• We remark that global well-posedness results, also for larger data and partial dissipation, have been already previously obtained in several works by other methods, for example [DWZZ18, LT16, LLT13, CKN99, Cha06]. This method’s focus instead lies on establishing stability of the two parameter family, as well as damping and convergence rates (see Section 4). The convergence rates are derived in Section 4 as a corollary of this theorem’s bounds.

• Instead of bounds relating $\epsilon_1, \epsilon_2$ and $\alpha$, we could for instance denote $\epsilon_1 = \epsilon$ and require $\epsilon_2 = \epsilon^2$, $\alpha < \epsilon$.

• In view of the existing well-posedness results we just referenced and the linear results of Section 2.3.2 the constraint on $\alpha$ here is probably far from optimal. However, it allows us to treat the effects of hydrostatic balance perturbatively.

We make use of the following multiplier constructed in [BVW16].

Lemma 3.2 ([BVW16]). There exists a Fourier multiplier $M(t, k, \xi)$ with the following properties:

\begin{align}
M(0, k, \xi) &= M(t, 0, \xi) = 1, \\
1 &\geq M(t, k, \xi) \geq c, \\
\frac{\dot{M}}{M} &\geq \frac{|k|}{k^2 + |\xi - kt|^2} \text{ for } k \neq 0, \\
\left| \frac{\partial_k M(k, \xi)}{M(k, \xi)} \right| &\lesssim \frac{1}{|k|} \text{ for } k \neq 0, \text{ uniformly in } \xi, \\
1 &\lesssim \nu^{-1/6} (\sqrt{-\dot{M}M(t, k, \xi)} + \nu^{1/2} |k, \xi - kt|),
\end{align}

\begin{align}
\sqrt{-\dot{M}M(t, k, \eta)} &\lesssim (\eta - \xi) \sqrt{-\dot{M}M(t, k, \xi)}.
\end{align}

For later reference we also recall from [BVW16] that (26) implies that for any function $f$ it holds that

\begin{align}
\|f\|_{L^2 H^N} &\lesssim \nu^{-1/6} (\|\nabla_t A f\|_{L^2 L^2} + \|\sqrt{-\dot{M}M(D)^N f}\|_{L^2 L^2}).
\end{align}

Given $M$ we are ready to define the main quantities of our proof:

Definition 3.1. Let $N \in \mathbb{N}, N \geq 5$ be given and define the Fourier multiplier

\begin{align}
A = M(D)^N,
\end{align}

where $D = i\nabla$ is the Fourier multiplier $(k, \xi)$. 
We then define two energies:

\begin{align}
E_{\omega}(T) &:= \|A\omega\|_{L^\infty L^2((0,T))}^2 + \nu \|\nabla_{t} A\omega\|_{L^2 L^2((0,T))}^2 + \|\sqrt{-M M} \langle D \rangle_{N} \omega\|_{L^2 L^2((0,T))}^2, \\
E_{\theta}(T) &:= \|A\theta\|_{L^\infty L^2((0,T))}^2 + \eta \|\nabla_{t} A\theta\|_{L^2 L^2((0,T))}^2 + \|\sqrt{-M M} \langle D \rangle_{N} \theta\|_{L^2 L^2((0,T))}^2.
\end{align}

As $M$ is comparable to 1, we in the following may replace $(21)$ by the estimates

\begin{align}
E_{\omega}(T) &\leq 8 \epsilon_1^2, \\
E_{\theta}(T) &\leq 8 \epsilon_2^2.
\end{align}

We then follow a classic bootstrap approach (e.g. see $\text{BM15}$):

- By local well-posedness there exists at least some small time $T > 0$ for which (32) and (33) hold. This is established in Proposition 3.3.
- Since these are closed conditions, there exists some maximal time $T_*$ for which (32) and (33) hold. Suppose for the sake of contradiction that $T_* < \infty$. Then we show in Proposition 3.4 that on $(0,T_*)$ improved estimates with $4\epsilon_1^2$ and $4\epsilon_2^2$ hold.
- Therefore, by local continuity (32) and (33) remain true for an additional small time. Hence, $T_* < \infty$ was not maximal, which contradicts the assumption of the previous step. Therefore the maximal time has to have been infinity, which concludes the proof.

**Proposition 3.3.** Let $0 < \epsilon_1 < \min(\nu, \eta)^{1/2}$ and $0 < \epsilon_2 < \sqrt{\nu \eta} \epsilon_1$ and $N \in \mathbb{N}, N \geq 5$.

Suppose that the initial data $\omega_0, \theta_0$ satisfies

\begin{align}
\|\omega_0\|_{H^N}^2 &\leq \frac{1}{10} \epsilon_1^2, \\
\|\theta_0\|_{H^N}^2 &\leq \frac{1}{10} \epsilon_2^2.
\end{align}

Then there exists a (maximal) time $T > 0$ such that (32) and (33) hold:

\begin{align}
E_{\omega}(T) &\leq 8 \epsilon_1^2, \\
E_{\theta}(T) &\leq 8 \epsilon_2^2.
\end{align}

**Proof.** By classical local well-posedness results for the Navier-Stokes equations (see $\text{BVW16}$) and for the Boussinesq equations (see $\text{Tem12}$, Section 3.5) for a sufficiently small time $T > 0$ we obtain the existence of a solution $(\omega(t), \theta(t))$ with

\[ \|\omega(t)\|_{H^N} \leq \epsilon_1, \|\theta(t)\|_{H^N} \leq \epsilon_2, \]

for all $0 < t \leq T$. Further using the dissipative structure to control $\nabla_{t} \omega$ and $\nabla_{t} \theta$ and possibly choosing $T$ even smaller, we thus may estimate $E_{\omega}(T)$ and $E_{\theta}(T)$ as claimed. \( \square \)

Given this positive time, we next show that the estimates (32), (33) actually hold for all times.
Proposition 3.4. Suppose that for $T > 0$ the estimates (32) and (33) hold and let $\epsilon_1, \epsilon_2$ be as in Theorem 3.1. Then the following improved estimates hold

\begin{align}
E_\omega(T) &\leq 4\epsilon_1^2, \\
E_\theta(T) &\leq 4\epsilon_2^2.
\end{align}

Before proving Proposition 3.4, let us discuss how this allows us to establish Theorem 3.1.

Proof of Theorem 3.1. By Proposition 3.3 there exists a positive time $T > 0$ such that the estimates (32) and (33) hold. Since these are closed conditions, we may take $0 < T^* \leq \infty$ to be the maximal time such that (10) holds. If $T^* = \infty$ this implies the results of Theorem 3.1. Thus, suppose for the sake of contradiction that $T^*$ is finite. Then by Proposition 3.4, on $(0, T^*)$ the improved estimates (36) hold. By local well-posedness and continuity arguments as in the proof of Proposition 3.3 there then exists a time $T_2 > T^*$ (possibly only very slightly larger) such that the solutions exists at least until time $T_2$ and the energies satisfy $E_\omega(T_2) - E_\omega(T^*) < \epsilon_1^2$ and $E_\theta(T_2) - E_\theta(T^*) < \epsilon_2^2$. But by (36) this implies that that also at the larger time $T_2$, the estimates (32) and (33) are satisfied and $T^*$ is therefore not maximal. This contradiction thus shows that $T^* = \infty$, which concludes the proof.

It thus remains to prove Proposition 3.4.

Proof of Proposition 3.4. Let $T > 0$ be a given time such that

\begin{align}
E_\omega(T) &:= \|A\omega\|^2_{L^2((0,T))} + \nu\|\nabla_t A\omega\|^2_{L^2((0,T))} + \|\sqrt{-MM(D)^N}\omega\|^2_{L^2(0,T)} \leq 8\epsilon_1^2, \\
E_\theta(T) &:= \|A\theta\|^2_{L^2((0,T))} + \eta\|\nabla_t A\theta\|^2_{L^2((0,T))} + \|\sqrt{-MM(D)^N}\theta\|^2_{L^2(0,T)} \leq 8\epsilon_2^2.
\end{align}

Then by testing the Boussinesq equation (20) with $A\omega$ and $A\theta$ we observe that

\begin{align*}
\partial_t\|A\omega\|^2_{L^2}/2 + \nu\|\nabla_t A\omega\|^2_{L^2} + \|\sqrt{-MM(D)^N}\omega\|^2_{L^2} &= -\int A(u \cdot \nabla\omega)A\omega + \int A(\partial_x \theta)A\omega, \\
\partial_t\|A\theta\|^2_{L^2}/2 + \eta\|\nabla_t A\theta\|^2_{L^2} + \|\sqrt{-MM(D)^N}\theta\|^2_{L^2} &= -\int A(u \cdot \nabla\theta)A\theta - \alpha \int A(\partial_x \Delta_t^{-1}\omega)A\theta.
\end{align*}

Here we used that $A$ is a Fourier multiplier and hence commutes with derivatives, which greatly simplifies calculations (for related problems for flows other than Couette see [WZZ17, CZZ19]).

Integrating in time, it follows that

\begin{align*}
E_\omega(T) &= \|A\omega_0\|^2_{L^2} - \int A(u \cdot \nabla\omega)A\omega + \int A(\partial_x \theta)A\omega \\
&= : \|A\omega_0\|^2_{L^2} + T_\omega + T_{\omega\theta}, \\
E_\theta(T) &= \|A\theta_0\|^2_{L^2} - \int A(u \cdot \nabla\theta)A\theta - \alpha \int A(\partial_x \Delta_t^{-1}\omega)A\theta \\
&= : \|A\theta_0\|^2_{L^2} + T_\theta + T_{\alpha}.
\end{align*}

Since the initial data by assumption satisfies

\begin{align*}
\frac{\|A\omega_0\|^2_{L^2}}{2} &< \epsilon_1^2, \\
\frac{\|A\theta_0\|^2_{L^2}}{2} &< \epsilon_2^2,
\end{align*}
it remains to estimate $T_\omega, T_{\omega,\theta}, T_\theta$ and $T_\alpha$.

We phrase these bounds as a proposition.

**Proposition 3.5.** Let $T > 0$ and suppose that (32) and (33) hold. Then the following estimates hold:

\begin{align}
T_\omega &\leq \varepsilon_1^3 \nu^{-1/2} + \varepsilon_1^3 \nu^{-1/3}, \\
T_\theta &\leq \varepsilon_2^2 \varepsilon_1 \eta^{-1/2} + \varepsilon_2^2 \varepsilon_1 \nu^{-1/3}, \\
T_{\theta,\omega} &\leq \frac{\varepsilon_1 \varepsilon_2}{\sqrt{\nu \eta}}, \\
T_\alpha &\leq \alpha \eta^{-1/2} \varepsilon_2 \nu^{-1/3} \varepsilon_1. 
\end{align}

These estimates allow us to conclude the proof of Proposition 3.4: Since $\varepsilon_1 \leq \min(\nu, \eta)^{1/2}$, $\varepsilon_2 \leq \sqrt{\nu \eta} \varepsilon_1$ and $\alpha < \eta^{1/2} \nu^{1/3} \frac{\varepsilon_1}{\varepsilon_2}$ it follows that $T_\omega \leq \varepsilon_1^3$, $T_{\omega,\theta} \leq \varepsilon_1^3$, $T_\theta \leq \varepsilon_2^2$, $T_\alpha \leq \varepsilon_2^2$.

This in turn implies that

\begin{align}
E_\omega(T) &\leq \varepsilon_1^3 + \varepsilon_1^2 + \varepsilon_1^2 = 3 \varepsilon_1^2 < 8 \varepsilon_1^2, \\
E_\theta(T) &\leq \varepsilon_2^2 + \varepsilon_2^2 + \varepsilon_2^2 < 8 \varepsilon_2^2.
\end{align}

Thus, we observe an improvement over the bounds (32) and (33), which concludes the proof of this proposition and hence allows us to close the bootstrap argument for Theorem 3.1. \qed

It remains to prove Proposition 3.5.

**Proof of Proposition 3.5.** We remark that $T_{\theta,\omega}$ and $T_\alpha$ have a quadratic structure as opposed to the cubic structure of $T_\theta$ and $T_\omega$. Hence, the additional smallness compared to $8 \varepsilon_1^2$ or $8 \varepsilon_2^2$ in these two cases is achieved by requiring that $\varepsilon_2$ is much smaller than $\varepsilon_1$ and that $\alpha$ is small compared to the quotient $\frac{\varepsilon_1}{\varepsilon_2}$.

Estimating $T_{\theta,\omega}$: Since $\partial_x \theta$ possesses a vanishing $x$-average, we may use Hölder’s inequality and Poincaré’s inequality to estimate

\[
T_{\theta,\omega} = \int_0^T \langle A\omega, A\partial_x \theta \rangle \leq \|A\omega\|_{L^2 L^2} \|\partial_x A\theta\|_{L^2 L^2} \\
\leq \|\nabla_t A\omega\|_{L^2 L^2} \|\nabla_t A\theta\|_{L^2 L^2} \\
\leq \frac{\varepsilon_1 \varepsilon_2}{\sqrt{\nu \sqrt{\eta}}}
\]

Estimating $T_\alpha$: We recall that

\[
T_\alpha = \alpha \int_0^T \langle A\theta, A\partial_x \Delta_t^{-1} \omega \rangle dt.
\]

Using Hölder’s and Poincaré’s inequality we control this by

\[
\alpha \|\nabla A\theta\|_{L^2 L^2} \|\partial_x \Delta_t^{-1} \omega\|_{L^2 H^8} \\
\leq \alpha \eta^{-1/2} \varepsilon_2 \nu^{-1/3} \varepsilon_1.
\]

We remark that here is where we use that $\alpha$ is “small”. An alternative approach for $\alpha$ “large” is discussed in Section 3.2.
Estimating $\mathcal{T}_\omega$ and $\mathcal{T}_\theta$: The estimate for $\mathcal{T}_\omega$ has been established in [BVW16]. Its proof further extends to the case of $\mathcal{T}_\theta$ with minor modifications. In the interest of readability we include it below.

We recall that

$$\mathcal{T}_\omega = -\int_0^T A(v \cdot \nabla \omega) A \omega$$

Since the shear flow component of the velocity field, that is the $x$-average $(\nabla^1_t \Delta^{-1} \omega) = \partial_y^{-1} \omega =: v_m$, decays slower, we split $\mathcal{T}_\omega$ into a contribution involving the shear and a contribution involving its $L^2$-orthogonal complement:

$$\mathcal{T}_\omega = \int_0^T \langle A \omega, A(v \cdot \partial_x \omega) \rangle + \int_0^T \langle A \omega, A(v \cdot \nabla \omega) \rangle = \mathcal{T}_\omega^- + \mathcal{T}_\omega^\perp$$

For $\mathcal{T}_\omega^\perp$ we easily estimate by

$$\|A \omega\|_{L^\infty L^2} \|v \cdot \nabla \omega\|_{L^2 H^N} \leq \epsilon_1 \epsilon_1 \frac{\epsilon_1}{\sqrt{D}} = \epsilon_1^3 \nu^{-1/2}. \tag{42}$$

In order to estimate $\mathcal{T}_\omega^-$ we make use of some cancellations. We note that $\partial_x v_m = 0$ and hence

$$\langle A \omega, v_m \partial_x A \omega \rangle = 0.$$  

We therefore obtain a commutator

$$\mathcal{T}_\omega^- = \int_0^T \langle A \omega, (A(v \cdot \partial_x \omega) - v_0 \partial_x A \omega \cdot)\rangle dt.$$  

By Parseval’s theorem we express the inner $L^2$ integral as

$$\sum_k \iint a(t, k, \xi) \tilde{\omega}(k, \xi) (a(k, \xi) - a(k, \xi - \zeta)) \tilde{\omega}(k, \xi - \zeta) d\xi d\zeta$$

By the properties of $A$ (and $M$) Bedrossian, Vicol and Wang deduce (see (2.17) and (2.18) in [BVW16]) that

$$|A(k, \xi) - A(k, \xi - \zeta)| \leq ((1 + k^2 + (\xi - \zeta)^2)^{N/2} + (1 + k^2 + \zeta^2)^{N/2}) \text{[eta]}.$$  

We note that the factor $|\zeta|$ cancels with $\tilde{\omega}(\zeta) = -i\zeta^{-1} \tilde{\omega}(0, \zeta)$ and hence obtain that

$$|\mathcal{T}_\omega^-| \leq C \sum_{k \neq 0} \iint ((1 + k^2 + (\xi - \zeta)^2)^{N/2} + (1 + k^2 + \zeta^2)^{N/2}) |\tilde{\omega}(0, \zeta) \tilde{\omega}(k, \xi - \zeta)| |A(k, \xi) \tilde{\omega}(k, \xi)| d\xi d\zeta.$$  

It thus follows that

$$|\mathcal{T}_\omega^-| \leq C \|\omega_m\|_{L^\infty H^N} \|\omega \cdot \nabla \omega\|_{L^2 H^N}.$$  

As noted in (28) following the introduction of the multiplier $M$, the last term can be estimate in terms of $\nu^{-1/3} E_\omega$ and therefore

$$\mathcal{T}_\omega^- \leq \epsilon_1 \nu^{-1/3} \epsilon_1^2. \tag{43}$$

Combining the estimate (42) for $\mathcal{T}_\omega^\perp$ and (43) for $\mathcal{T}_\omega^-$ then concludes the proof for $\mathcal{T}_\omega$. 

We next consider $\mathcal{T}_\theta$ and analogously split into a contribution involving the shear and one involving its complement:

$$
\mathcal{T}_\theta = \int_0^T \langle A \omega, (A(u \cdot \partial_x \theta) - u \cdot \partial_x A \theta) \rangle + \int_0^T \langle A \theta, A(u \cdot \nabla \theta) \rangle
$$

$$=: \mathcal{T}_\theta^\rho + \mathcal{T}_\theta^\omega.
$$

By the same argument as for $\mathcal{T}_\omega$ we may estimate

$$
|\mathcal{T}_\theta^\rho| \leq C \omega = \|L^\infty H^N \| \theta^\rho \| L^2 H^N \leq \epsilon_1 \eta^{-1/2} \nu^{-1/3} \epsilon_2^2
$$

and

$$
|\mathcal{T}_\theta^\omega| \leq \|\nabla \Delta^{-1} \omega = \|L^2 H^N \| \theta^\omega \| L^2 H^N \| A \theta \| L^\infty H^N \leq \epsilon_2^2 \eta^{-1/3} \nu^{-1/3}.
$$

This concludes the proof. \(\square\)

### 3.2. Large Hydrostatic Balance and Shear

In Section 3 we considered the nonlinear problem with $\alpha > 0$ “small” as a perturbation of the Navier-Stokes problem. In contrast in Section 2.3 for the linearized problem we exploited $\alpha$ to make use of classical energy methods used for the hydrostatic balance case (without shear) and treated the shear $\beta y$ as a correction. Our aim in the following is to combine both methods to establish (asymptotic) stability also for large $\alpha$ and $\beta = 1$ (after rescaling).

Here, we further adapt the previous bootstrap approach to consider an energy of the form

$$
\alpha \|A \omega\|_{H^N}^2 + \langle A \theta, (-\partial_x^2 - (\partial_y - t \partial_x)^2) A \theta \rangle_{L^2}.
$$

**Theorem 3.6.** Let $\alpha \geq 1$ and $\beta = 1$ and $\nu > 0$ and suppose that $\eta > 2$. Let further $(\omega_0, \theta_0) \in H^N \times H^{N+1}$ be given initial data such that

$$
\alpha \|\omega_0\|_{H^N}^2 + \|\nabla \theta_0\|_{H^N}^2 \ll \epsilon^2.
$$

Then for all times $T > 0$ it holds that

$$
\text{ess-sup}_{0 \leq t \leq T} (\alpha \|A \omega(t)\|_{L^2}^2 + \langle A \theta(t), (-\partial_x^2 - (\partial_y - t \partial_x)^2) A \theta(t) \rangle_{L^2})
$$

$$
+ \nu \int_0^T \alpha \|\nabla \omega(t)\|_{H^N}^2 \, dt
$$

$$
+ \eta \int_0^T \langle A \theta(t), (-\partial_x^2 - (\partial_y - t \partial_x)^2) A \theta(t) \rangle_{L^2} \, dt \leq \epsilon^2.
$$

(45)

We remark that lower bound on $\eta$ is very restrictive, but allows use to treat the time-dependence of $(\partial_y - t \partial_x)^2$ perturbatively. In the general case $\beta \in \mathbb{R}$ this restriction would read $\eta \gg \beta$ and thus requires that thermal dissipation dominates the shear.

**Proof of Theorem 3.6.** Similarly to the proof of Theorem 3.1 we begin by considering the time-derivative of equation (44). We compute

$$
\frac{d}{dt} \alpha \|A \omega\|_{H^N}^2 + \|\sqrt{-A} A \omega\|_{L^2} + \nu \|\nabla \omega\|_{L^2}
$$

$$= \alpha \langle A \omega, A(v \cdot \nabla \omega) \rangle
$$

$$+ \alpha \langle A \omega, A \partial_x \theta \rangle,
$$
and
\[
\frac{d}{dt} \langle A\theta, (-\partial_x^2 - (\partial_y - t \partial_x)^2)A\theta \rangle_{L^2} + \| \sqrt{-\partial_x^2 - (\partial_y - t \partial_x)^2} \sqrt{-AA\theta} \|_{L^2}^2 \\
+ \eta \| \nabla_t \sqrt{-\partial_x^2 - (\partial_y - t \partial_x)^2} AA\theta \|_{L^2}^2 \\
= \alpha \langle A\theta, (-\partial_x^2 - (\partial_y - t \partial_x)^2)A\partial_x(-\partial_x^2 - (\partial_y - t \partial_x)^2)^{-1} \omega \rangle \\
+ \langle A\theta, (-\partial_x^2 - (\partial_y - t \partial_x)^2)A(v \cdot \nabla_t \theta) \rangle \\
+ \langle A\theta, -2\partial_x(\partial_y - t \partial_x)A\theta \rangle.
\]

Since $A$ is a Fourier multiplier and hence commutes with $(-\partial_x^2 - (\partial_y - t \partial_x)^2)$, we observe that the contributions
\[
\alpha \langle A\omega, A\partial_x \theta \rangle
\]
and
\[
\alpha \langle A\omega, A\partial_x \theta \rangle
\]
cancel out.

Integrating from 0 to $T$ as in the proof of Theorem 3.1, in our bootstrap approach we thus have to control three contributions:
\[
\mathcal{T}_\omega := \int_0^T \alpha \langle A\omega, A(v \cdot \nabla \omega) \rangle,
\]
\[
\mathcal{T}_\theta := \int_0^T \langle A\theta, (-\partial_x^2 - (\partial_y - t \partial_x)^2)A(v \cdot \nabla_t \theta) \rangle,
\]
and
\[
\mathcal{T}_A := \int_0^T \langle A\theta, -2\partial_x(\partial_y - t \partial_x)A\theta \rangle.
\]

The first contribution $\mathcal{T}_\omega$ can be controlled in exactly the same way as in the proof of Proposition 3.5:
\[
\mathcal{T}_\omega \leq \alpha \| A\omega \|_{L^\infty L^2} \| u_\# \|_{L^2 HN} \| \nabla \omega \|_{L^2 HN} + C\alpha \| \omega_\# \|_{L^\infty HN} \| \omega_\# \|_{L^2 HN}^2 \\
\leq \epsilon^3 \left( \frac{\nu^{-1/2}}{\sqrt{\alpha}} + \frac{\nu^{-1/3}}{\sqrt{\alpha}} \right).
\]

The contribution $\mathcal{T}_A$ can be absorbed into
\[
\eta \| \nabla_t \sqrt{-\partial_x^2 - (\partial_y - t \partial_x)^2} AA\theta \|_{L^2 L^2}^2
\]
by using that $\eta \geq 2$.

Finally, for the contribution $\mathcal{T}_\theta$ we follow the same strategy of proof as in Proposition 3.5. We again split $\mathcal{T}_\theta$ into contributions due to $v_\#$ and $v_\#$. For
\[
\mathcal{T}_\theta := \int_0^T \int_0^T \langle A\theta, (-\partial_x^2 - (\partial_y - t \partial_x)^2)A(v_\# \cdot \nabla_t \theta) \rangle
\]
we may estimate by
\[
\| \sqrt{-\partial_x^2 - (\partial_y - t \partial_x)^2} AA\theta \|_{L^\infty L^2} \| v_\# \|_{L^2 HN} \| \nabla_t \sqrt{-\partial_x^2 - (\partial_y - t \partial_x)^2} \theta \|_{L^2 HN} \\
+ \| \sqrt{-\partial_x^2 - (\partial_y - t \partial_x)^2} AA\theta \|_{L^\infty L^2} \| \omega_\# \|_{L^2 HN} \| \sqrt{-\partial_x^2 - (\partial_y - t \partial_x)^2} \omega_\# \|_{L^2 HN} \\
\leq \epsilon \nu^{-1/3} \eta^{-1/2} + \epsilon \nu^{-1/2} \eta^{-1/2}.
\]
Compared to the setting of Theorem 3.1 we thus lose more powers of $\nu$ and $\eta$.

For the last contribution

$$\mathcal{T}_\theta = \int_0^T \int_0^T \langle A \theta, (-\partial_x^2 - (\partial_y - t \partial_z)^2) A(v \partial_x \theta) \rangle,$$

we again use Parseval’s theorem to obtain a cancellation for the contributions by $\theta_-$. Next, we integrate $-\partial_x^2 - (\partial_y - t \partial_z)^2$ by parts once and use the product rule to split

$$\partial_x A(v \partial_x \theta) = A((\partial_x v \partial_x \theta) + A(v \partial_x \partial_x \theta),$$

and

$$(\partial_y - t \partial_z) A(v \partial_x \theta) = A(((\partial_y - t \partial_x) v \partial_x) \partial_x \theta) + A(v \partial_x \partial_y - t \partial_x \partial_x \theta).$$

For the first terms we bound by

$$\| \sqrt{-\partial_x^2 - (\partial_y - t \partial_z)^2} A \theta \|_{L^2 H^N}^2 \| \omega \|_{L^\infty H^N} \| \nabla_t \theta \|_{L^2 H^N} \leq \epsilon \epsilon \sqrt{\alpha} \epsilon.$$

For the second terms we argue exactly as in the proof of Proposition 3.5 with $\partial_x \theta$ or $(\partial_y - t \partial_z) \theta$ in place of $\theta$, which yields a bound by

$$\| \omega \|_{L^\infty H^N} \| \sqrt{\partial_x^2 + (\partial_y - t \partial_z)^2} \theta \|_{L^2 H^N}^2 \leq \epsilon_3 \epsilon^{-1/2} \epsilon^{-1/3}.$$

\[\square\]

4. FROM BOUNDS TO DECAY

In Theorem 3.1 in Section 3 we have shown that the nonlinear Boussinesq equations satisfy energy estimates of the form

$$\omega, \theta \in L^\infty_t H^N,$$

$$\nabla_t \omega, \nabla_t \theta \in L^2_t H^N.$$  

Hence, we know that the solutions stay bounded and their gradients are integrable in time. However, integrability does not by itself imply any decay (consider for example a series of thinner and thinner step functions) and even if one additionally requires uniform continuity it only implies convergence to zero but yields no rate.

In the following we make use of additional bounds on the semigroup associated with the linearized operator to deduce decay estimates.

**Proposition 4.1.** Let $N, \alpha, \epsilon_1, \epsilon_2$ be as in Theorem 3.1. Additionally suppose you know the following two estimates:

- The evolution semigroup $S(\cdot, \cdot)$ of the linearized problem satisfies
  $$\| S(t, \tau) \|_{H^N \times H^{N+1} \rightarrow H^N \times H^{N+1}} \leq C \exp(-C \gamma (t - \tau))$$
  for any $t \geq \tau \geq 0$ and some $\gamma > 0$. (This is established in Section 2).
- Due to (enhanced) dissipation $\omega \in L^2_t H^{N+1}$ and we have the following estimate:
  $$\| \omega \|_{L^2 H^{N+1}} \ll \frac{1}{\alpha}$$
  (This follows by Theorem 3.1)
Then the nonlinear Boussinesq equations further satisfy
\[ \| \omega(t) \|_{H^N} + \| \theta(t) \|_{H^N} \leq 2C \exp(-C\gamma t/2)(\| \omega_0 \|_{H^N} + \| \theta_0 \|_{H^N}) \]
for all \( t > 0 \). In particular, we may choose \( \gamma = \min(\nu, \eta)^{1/3} \) and thus observe dissipation on a time scale faster than heat flow, that is enhanced dissipation.

We remark that the linearized problem around Couette flow decays with a rate \( \exp(-C(\nu^{1/3}t)^3) \) (see Section 2.2), which we may estimate from above by
\[ \exp(C) \exp(-C\nu^{1/3}t) \]
since \( t^3 \geq t - 1 \) for all \( t \geq 0 \). To the author’s knowledge it is not known whether the nonlinear Navier-Stokes problem exhibits the same faster \( \exp(-C(\nu^{1/3}t)^3) \) decay instead of the exponential decay by \( \exp(-C\nu^{1/3}t) \).

**Proof of Proposition 4.1.** In order to prove Proposition 4.1 we again use a bootstrap approach. For this purpose we interpret the nonlinear problem as a forced linear problem:
\[ \begin{align*}
\partial_t \omega + \nu \Delta \omega + \partial_x \theta &= -u \cdot \nabla_t \omega =: f, \\
\partial_t \theta + \eta \Delta \theta &= -u \cdot \nabla \theta =: g.
\end{align*} \]
Denoting the semigroup of the linearized problem by \( S(\cdot, \cdot) \), we obtain the integral equation
\[ (\omega(t), \theta(t)) = S(t, 0)(\omega_0, \theta_0) + \int_0^t S(t, \tau)(f(\tau), g(\tau))d\tau. \]
By assumption on the decay rate of the semi-group the first contribution can be estimated by
\[ \exp(-\gamma t)(\| \omega_0 \|_{H^N} + \| \theta_0 \|_{H^N}). \]
For the nonlinear contribution we derive a first, rough estimate by using that
\[ \begin{align*}
\| f \|_{H^N} &\leq \| \omega \|_{H^N} \| \nabla_t \omega \|_{H^N}, \\
\| g \|_{H^{N+1}} &\leq \| \omega \|_{H^N} \| \nabla \theta \|_{H^{N+1}}.
\end{align*} \]
It then follows that at least for very small times the nonlinear contribution is bounded by \( \epsilon^2 \) and as a consequence for these small times
\[ \|(\omega(t), \theta(t))\|_{H^N} \leq 2 \exp(-\gamma t/2)\epsilon. \]
We next argue by a bootstrap iteration that the estimate (53) holds for all times. Thus suppose that (53) holds for \( 0 \leq t \leq T \) and assume for the sake of contradiction that \( T < \infty \) is maximal. The first contribution in (50) is bounded by
\[ \exp(-\gamma t)\epsilon \]
and thus both small and fast decaying. We hence focus on the contribution by the nonlinearity. Here we combine the combine the decay estimate of \( S(t, \tau) \), (51) and
(53) to estimate
\[
\left\| \int_{t}^{T} S(t, \tau)(f(\tau), g(\tau))d\tau \right\| d\tau
\]
\[
\leq C \int_{0}^{t} \exp(-\gamma(t-\tau))(\|\omega(\tau)\|_{H^N} \|\nabla t \omega(\tau)\|_{H^N} + \|\omega(\tau)\|_{H^N} \|\nabla t \theta(\tau)\|_{H^N})d\tau
\]
\[
\leq C^2 \int_{0}^{t} \exp(-\gamma(t-\tau)) \exp(-\gamma t/2)(\|\nabla t \omega(\tau)\|_{H^N} + \|\nabla t \theta(\tau)\|_{H^N})d\tau
\]
\[
\leq C^2 \epsilon \exp(-\gamma t/2) \int_{0}^{t} \exp(-\gamma(t-\tau)/2)(\|\nabla t \omega(\tau)\|_{H^N} + \|\nabla t \theta(\tau)\|_{H^N})d\tau.
\]

We then use the $L^2$ integrability assumption on $(\|\nabla t \omega(\tau)\|_{H^N} + \|\nabla t \theta(\tau)\|_{H^N})$ and the Cauchy-Schwarz inequality to further bound this by
\[
C^2 \epsilon^2 \frac{(\mu^{-1/2} + \eta^{-1/2})}{\sqrt{\gamma}} \exp(-\gamma t/2).
\]

By the assumption on $\epsilon$ this is smaller than
\[
\epsilon \exp(-\gamma t/2) < 2\epsilon \exp(-\gamma t/2).
\]

Thus equality in (53) is not attained for $t = T$, which contradicts the maximality of $T$. Therefore, the maximal time is infinity, which concludes the proof. □

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