Leading Order Temporal Asymptotics of the Modified Non-Linear Schrödinger Equation: Solitonless Sector

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Abstract

Using the matrix Riemann-Hilbert factorisation approach for non-linear evolution equations (NLEEs) integrable in the sense of the inverse scattering method, we obtain, in the solitonless sector, the leading-order asymptotics as $t \to \pm \infty$ of the solution to the Cauchy initial-value problem for the modified non-linear Schrödinger equation,

$$i\partial_t u + \frac{1}{2} \partial_x^2 u + |u|^2 u + i s \partial_x (|u|^2 u) = 0, \quad s \in \mathbb{R}_{>0};$$

also obtained are analogous results for two gauge-equivalent NLEEs; in particular, the derivative non-linear Schrödinger equation,

$$i\partial_t q + \partial_x^2 q - i \partial_x (|q|^2 q) = 0.$$

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1 Introduction

The study of the fundamental dynamical processes associated with the propagation of high-power ultrashort pulses in optical fibres is of paramount importance: the non-linear soliton(ic), or near-soliton(ic), operating mode(s) in such systems are very promising; in particular, very high data transmission rates, high noise immunity, and the accessibility of new frequency bands [1]. The classical, mathematical model for non-linear pulse propagation in the picosecond time scale in the anomalous dispersion regime in an isotropic, homogeneous, lossless, non-amplifying, polarisation-preserving single-mode optical fibre is the non-linear Schrödinger equation, NLSE [1, 2]; however, in the subpicosecond-femtosecond time scale, experiments and theories on the propagation of high-power ultrashort pulses in long monomode optical fibres have shown that the NLSE is no longer valid and that additional non-linear terms (dispersive and dissipative) and higher-order linear dispersion should be taken into account [1]. In this case, subpicosecond-femtosecond pulse propagation is described (in dimensionless and normalised form) by the following non-linear evolution equation (NLEE) [1],

\[ i\partial_{\xi}u + \frac{1}{2}\partial_{\tau}^2u + |u|^2u + is\partial_{\tau}(|u|^2u) = -\bar{\Gamma}u + i\tilde{\delta}\partial_{\tau}^3u + \frac{\tau_n}{\tau_0}u\partial_{\tau}(|u|^2), \]  

(1)

where \( u \) is the slowly varying amplitude of the complex field envelope, \( \xi \) is the propagation distance along the fibre length, \( \tau \) is the time measured in a frame of reference moving with the pulse at the group velocity (the retarded frame), \( s (> 0) \) governs the effects due to the intensity dependence of the group velocity (self-steepening), \( \bar{\Gamma} \) is the intrinsic fibre loss, \( \tilde{\delta} \) governs the effects of the third-order linear dispersion, and \( \tau_n/\tau_0 \), where \( \tau_0 \) is the normalised input pulsewidth and \( \tau_n \) is related to the slope of the Raman gain curve (assumed to vary linearly in the vicinity of the mean carrier frequency, \( \omega_0 \)), governs the soliton self-frequency shift (SSFS) effect [3].

Our strategy to the study of (1) will be the following: (i) by setting the right-hand side of (1) equal to zero, we obtain the following equation (integrable in the sense of the inverse scattering method (ISM) [4]),

\[ i\partial_{t}u + \frac{1}{2}\partial_{x}^2u + |u|^2u + is\partial_{x}(|u|^2u) = 0, \]  

(2)

which we hereafter call the modified non-linear Schrödinger equation, MNLSE (note that, the physical variables, \( \xi \) and \( \tau \), have been mapped isomorphically onto the mathematical \( t \) and \( x \) variables, which are standard in the ISM context); and (ii) since \( \bar{\Gamma}, \tilde{\delta}, \) and \( \tau_n/\tau_0 \) are small parameters [1], we treat (1) as a non-integrable perturbation of the MNLSE.

From the above discussion, it is clear that (multi-) soliton solutions of the MNLSE play a pivotal role in the physical context related to optical fibres. However, since practical lasers cannot be designed to excite only the soliton(ic) mode(s), but also excite an entire continuum of linear-like dispersive (radiative) waves, to have physically meaningful and practically representative results, it is necessary to investigate solutions of the MNLSE under general initial (launching, in the optical fibre literature [1]) conditions, without any artificial restrictions and/or constraints, which have both soliton(ic) and non-soliton(ic) (continuum) components; physically, it is towards such a solution that the initial pulse launched into an optical fibre is evolving asymptotically [3]. Also, one may think of general initial conditions (Cauchy data) as a perturbation of the so-called reflectionless potential (soliton(ic)) initial conditions, and, since the action of any perturbation is always more or less non-adiabatic, the soliton(ic) component of the solution will be accompanied by the
appearance of a non-soliton(ic) component \[1\]. From the physical point of view, therefore, it is seminal to understand how the continuum and the soliton(s) interact. For several soliton-bearing equations, e.g., KdV, Landau-Lifshitz, and NLS, and the RMB (reduced Maxwell-Bloch) system, it is well known that the dominant \((O(1))\) asymptotic \((t \to \infty)\) effect of the continuous spectra on the multi-soliton solutions is a shift in phase and position of their constituent solitons \[8\]. The purpose of our study is to derive an explicit functional form for the next-to-leading-order \((O(t^{-1/2}))\) term of the effect of this interaction for the MNLSE. Although there have been several papers devoted to studying soliton solutions of the MNLSE \[8\], to the best of our knowledge, very little, if anything, is still known about its solution(s) for the class of non-reflectionless initial data. As is well known \[8\], the investigation of multi-soliton solutions on the continuum background requires more sophisticated techniques, e.g., asymptotic, than, say, the algebraic methods for the construction of pure soliton solutions. An asymptotic investigation of the aforementioned solution can be divided into two stages: (i) the investigation of the continuum (solitonless) component of the solution \[10\]; and (ii) the inclusion of the soliton component via the application of a “dressing” procedure \[11\] to the continuum background. The purpose of this paper is to carry out, systematically, stage (i) of the above-mentioned asymptotic paradigm (since this phase of the asymptotic procedure is rather technical and long in itself, the completed results for stage (ii) are the subject of a forthcoming article \[12\]). The results obtained in this paper are formulated as Theorems 3.2 and 3.3.

This paper is organised as follows. In Sec. 2, we transform (2) into a gauge-equivalent NLEE (see the first of (4)), and, using only some basic facts concerning its direct scattering theory \[3\], pose the inverse scattering problem as a matrix oscillatory Riemann-Hilbert (RH) factorisation problem. In Sec. 3, we: (i) briefly review the Beals and Coifman formulation \[4\] for the solution of a RH problem on an oriented contour; (ii) give a general outline of the Deift and Zhou procedure \[5\] for obtaining the long-time asymptotics of the solution to the RH problem; and (iii) state our final results as Theorems 3.2 and 3.3.

Section 4, the initial oscillatory RH problem is formulated as an auxiliary RH problem on an augmented contour. In Sec. 5, the auxiliary RH problem is reformulated as an equivalent RH problem on a truncated contour. In Sec. 6, it is shown that, to leading order as \(t \to +\infty\), the solution of the equivalent RH problem on the truncated contour is equal, modulo decreasing terms, to the solution of an explicitly solvable model RH problem on a contour which consists of the disjoint union of three crosses. Finally, in Secs. 7 and 8, the model RH problem is solved, and the asymptotic solution of related, auxiliary NLEEs and the MNLSE are obtained.

2 The Riemann-Hilbert (RH) Problem

We begin by introducing some notation which is used throughout the paper: (i) for \(D\) an unbounded domain of \(\mathbb{R} \cup i\mathbb{R}\), let \(S(D)\) denote the Schwartz class on \(D\), i.e., the class of smooth scalar-valued functions \(f(x)\) on \(D\) which together with all derivatives tend to zero faster than any positive power of \(|x|^{-1}\) for \(|x| \to \infty\); (ii) \(\sigma_3 = \text{diag}(1, -1), \sigma_+ = \left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right), \sigma_- = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right), \text{ and } \sigma_1 = \sigma_- + \sigma_+\); (iii) for a scalar \(\varpi\) and a \(2 \times 2\) matrix \(\Upsilon\), \(\varpi^{\text{ad}(\sigma_3)} \Upsilon \equiv \varpi^{\sigma_3} \Upsilon \varpi^{-\sigma_3}\); (iv) for \(k \in \{1, 2\}, \mathcal{L}_k^{(2 \times 2)}(D) \equiv \{F(\lambda); \lambda \in D, F_{ij}(\lambda) \in \mathcal{L}_k(D), i, j \in \{1, 2\}\}, \text{ where } \mathcal{L}_k(D) \equiv \{f(\lambda); \lambda \in D, ||f||_{\mathcal{L}_k(D)} \equiv (\int_D |f(\lambda)|^k |d\lambda|)^{1/k} < \infty\}, \text{ and } \mathcal{L}_\infty^{(2 \times 2)}(D) \equiv \{G(\lambda); \lambda \in D, ||G_{ij}||_{\mathcal{L}_\infty(D)} \equiv \sup_{\lambda \in D} |G_{ij}(\lambda)| < \infty, i, j \in \{1, 2\}\}, \text{ with the...
norms taken as follows, \( \| \cdot \|_{\mathcal{L}_n^{(2x2)}}(\cdot) \equiv \max_{i,j \in \{1,2\}} \| (\cdot)_{ij} \|_{\mathcal{L}_n(\cdot)}, n \in \{1,2,\infty\}; \) (v) \( \overline{\cdot} \) denotes complex conjugation of \( \cdot \); and (vi) \( \mathbf{1} \) denotes the \( 2 \times 2 \) identity matrix.

**Proposition 2.1** \( [13] \) The necessary and sufficient condition for the compatibility of the following system of linear ODEs (the Lax pair) for arbitrary \( \lambda \in \mathbb{C} \),

\[
\partial_x \Psi(x,t;\lambda) = U(x,t;\lambda)\Psi(x,t;\lambda), \quad \partial_t \Psi(x,t;\lambda) = V(x,t;\lambda)\Psi(x,t;\lambda),
\]

where

\[
U(x,t;\lambda) = \lambda(-i\lambda\sigma_3 + P\sigma_- + Q\sigma_+ - i\frac{1}{2}PQ\sigma_3), \\
V(x,t;\lambda) = 2\lambda^2 U(x,t;\lambda) - i\lambda((\partial_x P)\sigma_- - (\partial_x Q)\sigma_+) + (\frac{1}{4}(PQ)^2 + \frac{1}{4}(P\partial_x Q - Q\partial_x P))\sigma_3,
\]

with \( P(x,t) = \overline{Q(x,t)} \), is that \( Q(x,t) \) (resp. \( P(x,t) \)) satisfies the following NLEE,

\[
i\partial_t Q + \partial_x^2 Q + iQ^2\partial_x \overline{Q} + \frac{1}{2}Q|Q|^4 = 0 \quad (i\partial_t P - \partial_x^2 P + iP^2\partial_x \overline{P} - \frac{1}{2}P|P|^4 = 0).
\]

**Proof.** (4) are the Frobenius compatibility conditions for (3). \( \blacksquare \)

**Proposition 2.2** Let \( Q(x,t) \) be a solution of the first of Eqs. (4). Then there exists a corresponding solution of system (3) such that \( \Psi(x,t;0) \) is a diagonal matrix.

**Proof.** For given \( Q(x,t) \), let \( \hat{\Psi}(x,t;\lambda) \) be a solution of (3) which exists in accordance with Proposition 2.1. Setting \( \lambda = 0 \) in (3), we get that \( \hat{\Psi}(x,t;0) = \exp\{-\frac{\alpha^2}{2}\int_{x_0}^x |Q(\xi,t)|^2 d\xi\} \cdot \mathcal{K} \), for some \( x_0 \in \mathbb{R} \) and non-degenerate matrix \( \mathcal{K} \) which is independent of \( x \) and \( t \). The function \( \hat{\Psi}(x,t;\lambda) \equiv \hat{\Psi}(x,t;\lambda)\mathcal{K}^{-1} \) is the solution of (3) which is diagonal at \( \lambda = 0 \). \( \blacksquare \)

**Proposition 2.3** \( [16] \) Let \( Q(x,t) \) be a solution of the first of Eqs. (4) and \( \Psi(x,t;\lambda) \) the corresponding solution of system (3) given in Proposition 2.2. Set \( \Psi_q(x,t;\lambda) \equiv \Psi^{-1}(x,t;0) \cdot \Psi(x,t;\lambda) \). Then

\[
\partial_x \Psi_q(x,t;\lambda) = U_q(x,t;\lambda)\Psi_q(x,t;\lambda), \quad \partial_t \Psi_q(x,t;\lambda) = V_q(x,t;\lambda)\Psi_q(x,t;\lambda),
\]

where

\[
U_q(x,t;\lambda) = -i\lambda^2 \sigma_3 + \lambda(\overline{\sigma}_- + q\sigma_+), \\
V_q(x,t;\lambda) = \begin{pmatrix} -2i\lambda^4 - i\lambda^2 q & 2\lambda^3 q + i\lambda \partial_x q + \lambda \overline{q}^2 q \\ 2\lambda^3 q - i\lambda \partial_x \overline{q} + \lambda q^2 q & 2i\lambda^4 + i\lambda^2 \overline{q}^2 q \end{pmatrix},
\]

with

\[
q(x,t) \equiv Q(x,t)((\Psi^{-1}(x,t;0))_{11})^2,
\]

is the “Kaup-Newell” Lax pair for the derivative non-linear Schrödinger equation, DNLSE \( [17] \),

\[
i\partial_t q + \partial_x^2 q - i\partial_x(|q|^2 q) = 0.
\]

**Proof.** Differentiating \( \Psi_q(x,t;\lambda) \equiv \Psi^{-1}(x,t;0)\Psi(x,t;\lambda) \) with respect to \( x \) and \( t \) and using the fact that \( \Psi(x,t;0) = \exp\{-\frac{\alpha^2}{2}\int_{x_0}^x |Q(\xi,t)|^2 d\xi\} \), for some \( x_0 \in \mathbb{R} \), and \( \Psi(x,t;\lambda) \) satisfy (3) for \( \lambda = 0 \) and \( \lambda \in \mathbb{C} \backslash \{0\} \), respectively, defining \( q(x,t) \) as in (8), we get that \( \Psi_q(x,t;\lambda) \) satisfies (5), where \( U_q(x,t;\lambda) \) and \( V_q(x,t;\lambda) \) are given by (6) and (7): (9) is the compatibility condition for (5). \( \blacksquare \)
Proposition 2.4 If \( q(x,t) \) is a solution of the DNLSE (Eq. (9)) such that \( q(x,0) \in \mathcal{S}(\mathbb{R}) \), then \( u(x,0) = \frac{1}{\sqrt{2\pi}} \exp\{\frac{i}{\pi}(x - \frac{1}{2})\}q(\frac{1}{2} - x, \frac{1}{2}) \) satisfies the MNLSE (Eq. (2)) with \( u(x,0) \in \mathcal{S}(\mathbb{R}) \).

Proof. Direct substitution. ■

Definition 2.1 For \( Q(x,t) \), as a function of \( x \in \mathcal{S}(\mathbb{R}) \), define the vector functions \( \psi^\pm(x,t; \lambda) \) and \( \phi^\pm(x,t; \lambda) \) as the Jost solutions of the first equation of system (3), namely, \( (\partial_x - U(x,t; \lambda))\psi^\pm(x,t; \lambda) = 0 \) and \( (\partial_x - U(x,t; \lambda))\phi^\pm(x,t; \lambda) = 0 \), with the following asymptotics,

\[
\lim_{x \to +\infty} \psi^+(x,t; \lambda) = (0,1)^T e^{i\lambda x} e^{-2i\lambda t}, \quad \lim_{x \to +\infty} \phi^+(x,t; \lambda) = (0,1)^T e^{i\lambda x} e^{2i\lambda t},
\]

\[
\lim_{x \to -\infty} \psi^-(x,t; \lambda) = (1,0)^T e^{-i\lambda x} e^{-2i\lambda t}, \quad \lim_{x \to -\infty} \phi^-(x,t; \lambda) = (1,0)^T e^{-i\lambda x} e^{2i\lambda t},
\]

where the superscripts \( \pm \) mean \( \Im(\lambda^2) > 0 \), and \( T \) denotes transposition.

Proposition 2.5 The Jost solutions (Definition 2.1) have the following properties: (i) \( \psi^\pm(x,t; \lambda) = a^\pm(\lambda,t) \phi^\pm(x,t; \lambda) + b^\pm(\lambda,t) \hat{\phi}^\pm(x,t; \lambda) \), \( \Im(\lambda^2) = 0 \), where \( a^\pm(\lambda,t) = W(\hat{\phi}^\pm(x,t; \lambda), \psi^\pm(x,t; \lambda)) \), \( b^\pm(\lambda,t) = W(\phi^\pm(x,t; \lambda), \psi^\pm(x,t; \lambda)) \), and \( W(f,g) \) is the Wronskian of \( f \) and \( g \), \( W(f,g) \equiv f_1g_2 - f_2g_1 \); (ii) if \( \hat{\psi}^\pm(x,t; \lambda) \) and \( \hat{\phi}^\pm(x,t; \lambda) \) satisfy the second equation of system (3) as well, i.e., \( (\partial_t - V(x,t; \lambda))\hat{\psi}^\pm(x,t; \lambda) = 0 \) and \( (\partial_t - V(x,t; \lambda))\hat{\phi}^\pm(x,t; \lambda) = 0 \), then \( a^\pm(\lambda,t) = a^\pm(\lambda,0) \equiv a^\pm(\lambda) \), and \( b^\pm(\lambda,t) = b^\pm(\lambda,0) e^{4i\lambda t} \equiv b^\pm(\lambda) e^{4i\lambda t} \); (iii) for \( \Im(\lambda^2) = 0 \), \( a^+(\lambda) a^-(\lambda) + b^+(\lambda) b^-(\lambda) = 1 \), \( a^+(\lambda,t) = \frac{a^+(\lambda,0)}{e^{4i\lambda t} - a^-(\lambda,0)} \), \( b^+(\lambda,t) = -\frac{b^-(\lambda,0)}{e^{4i\lambda t} - a^-(\lambda,0)} \), and \( \frac{1}{2} \ln \left( \frac{a^+(\lambda,0)}{a^-(\lambda,0)} \right) = -\int_\hat{\Gamma} \frac{\mu \ln(1 + r^+(\mu) r^-(\mu))}{(\mu^2 - \lambda^2)} \frac{d\mu}{2\pi i} \), with \( r^\pm(\lambda,t) = b^\pm(\lambda,t)/a^\pm(\lambda,t) \equiv r^\pm(\lambda) e^{4i\lambda t} \), and \( \hat{\Gamma} = \{ \lambda; \Im(\lambda^2) = 0 \} \) (oriented as in Fig. 1); (iv) \( a^\pm(\lambda) \) are analytically continuable to \( \Im(\lambda^2) < 0 \) with

\[
\begin{array}{c|c|c}
\lambda(\lambda^2) < 0 & \lambda(\lambda^2) = 0 & \lambda(\lambda^2) > 0 \\
\hline
- \hspace{2cm} & + \hspace{2cm} & - \\
\end{array}
\]

Figure 1: Continuous spectrum \( \hat{\Gamma} \).

the following integral representation, \( \pm \ln(a^\pm(\lambda)) = -\int_\hat{\Gamma} \frac{\mu \ln(1 + r^+(\mu) r^-(\mu))}{(\mu^2 - \lambda^2)} \frac{d\mu}{2\pi i} \); and (v) for \( \lambda \to \infty \), \( \Im(\lambda^2) > 0 \), \( a^+(\lambda,t) = 1 + \mathcal{O}(\lambda^{-2}) \),

\[
\psi^+(x,t; \lambda)e^{-i\lambda x} e^{-2i\lambda t} = \left( \frac{Q(x,t)}{2\pi x} \right)^T + \mathcal{O}(\lambda^{-2}),
\]

\[
\phi^+(x,t; \lambda)e^{i\lambda x} e^{2i\lambda t} = \left( 1, \frac{p(x,t)}{2\pi x} \right)^T + \mathcal{O}(\lambda^{-2}),
\]
and, for \(\lambda \to \infty\), \(\Im(\lambda^2) < 0\), \(a^- (\lambda, t) = 1 + \mathcal{O}(\lambda^{-2})\),

\[
\psi^- (x, t; \lambda) e^{i\lambda^2 x + 2i\lambda^4 t} = \left(1, \frac{p(x, t)}{2\lambda}\right)^T + \mathcal{O}(\lambda^{-2}),
\]

\[
\phi^- (x, t; \lambda) e^{-i\lambda^2 x - 2i\lambda^4 t} = -\left(\frac{Q(x, t)}{2\lambda^2}, 1\right)^T + \mathcal{O}(\lambda^{-2}).
\]

**Proof.** See [18]. ■

**Remark 2.1.** We now adopt a convention which is adhered to sensus strictu throughout the paper: for each segment of an oriented contour, according to the given orientation, the “+” side is to the left and the “−” side is to the right as one traverses the contour in the direction of the orientation; hence, \((\cdot)_+\) and \((\cdot)_-\) denote, respectively, the non-tangential limits (boundary values) of \((\cdot)\) on an oriented contour from the “+” (left) side and “−” (right) side (this should not be confused with the \(\pm\) superscripts).

As a result of Propositions 2.1–2.4, for the asymptotic analysis of the solution to the MNLSE, it is enough to investigate asymptotically the solutions of (4) and the corresponding function \(\Psi(x, t; 0)\) (Proposition 2.2): the main tool for this analysis is the following matrix RH factorisation problem.

**Lemma 2.1** Let \(Q(x, t)\), as a function of \(x\), \(\in \mathcal{S}(\mathbb{R})\). Set \(\Psi(x, t; \lambda) \equiv m(x, t; \lambda) \exp\{-i(\lambda^2 x + 2\lambda^4 t)s_3\}\). Define: (i) for \(\Im(\lambda^2) > 0\),

\[
m(x, t; \lambda) \equiv \begin{pmatrix}
\phi^+ (x, t; \lambda) & \psi^+ (x, t; \lambda) \\
\phi^+(x, t; \lambda) & \psi^+(x, t; \lambda)
\end{pmatrix} \exp\{i(\lambda^2 x + 2\lambda^4 t)s_3\};
\]

(ii) for \(\Im(\lambda^2) < 0\),

\[
m(x, t; \lambda) \equiv \begin{pmatrix}
\phi^- (x, t; \lambda) & \psi^- (x, t; \lambda) \\
\phi^-(x, t; \lambda) & \psi^-(x, t; \lambda)
\end{pmatrix} \exp\{i(\lambda^2 x + 2\lambda^4 t)s_3\};
\]

and (iii) \(r(\lambda) \equiv r^-(\lambda)\). Then the \(2 \times 2\) matrix-valued function \(m(x, t; \lambda)\) (\(\det(m(x, t; \lambda)) = 1\)) solves the following RH problem:

a. \(m(x, t; \lambda)\) is holomorphic \(\forall \lambda \in \mathbb{C} \setminus \hat{\Gamma}\);

b. \(m(x, t; \lambda)\) satisfies the following jump conditions,

\[
m_+(x, t; \lambda) = m_-(x, t; \lambda) \exp\{-i(\lambda^2 x + 2\lambda^4 t)\text{ad}(s_3)\} G(\lambda), \quad \lambda \in \hat{\Gamma},
\]

where

\[
G(\lambda) = \begin{pmatrix}
1 - \overline{r(\lambda)} r(\lambda) & r(\lambda) \\
-\overline{r(\lambda)} & 1
\end{pmatrix},
\]

\(r(\lambda) \in \mathcal{S}(\hat{\Gamma})\), and \(r(-\lambda) = -r(\lambda)\);

c. as \(\lambda \to \infty\), \(\lambda \in \mathbb{C} \setminus \hat{\Gamma}\),

\[
m(x, t; \lambda) = 1 + \mathcal{O}(\lambda^{-1}).
\]

**Proof.** Follows from the definitions of \(m(x, t; \lambda)\) for \(\Im(\lambda^2) \geq 0\) given in the Lemma and the properties of the Jost solutions and scattering data for (3) given in Definition 2.1 and Proposition 2.5: for details, see [19]. ■
Lemma 2.2 Let \( |r|_{L^\infty(\Gamma)} = \sup_{\lambda \in \Gamma} |r(\lambda)| < 1 \). Then: (i) the RH problem formulated in Lemma 2.1 is uniquely solvable; (ii) \( \Psi(x, t; \lambda) \equiv m(x, t; \lambda) \exp \{ -i(\lambda^2 x + 2\lambda^4 t)\sigma_3 \} \) is the solution of system (3) with

\[
Q(x, t) \equiv 2i \lim_{\lambda \to \infty} (\lambda m(x, t; \lambda))_{12} \quad \text{and} \quad P(x, t) \equiv -2i \lim_{\lambda \to \infty} (\lambda m(x, t; \lambda))_{21};
\]

(iii) the functions \( Q(x, t) \) and \( P(x, t) \) defined by Eqs. (10) satisfy Eqs. (4), and

\[
q(x, t) \equiv Q(x, t)((m^{-1}(x, t; 0))_{11})^2
\]

satisfies the DNLS (Eq. (9)); and (iv) \( m(x, t; \lambda) \) possesses the following symmetry reductions, \( m(x, t; -\lambda) = \sigma_3 m(x, t; \lambda) \sigma_3 \) and \( m(x, t; \lambda) = \sigma_1 m(x, t; \lambda) \sigma_1 \).

Proof. The solvability of the RH problem \( \forall t \) follows from the condition \( |r|_{L^\infty(\Gamma)} < 1 \) and the discussions in [13, 24] (see, also, Zhou’s skew Schwarz reflection invariant symmetry arguments [21, 22]). The fact that \( q(x, t) \) defined by (11) satisfies the DNLS follows from Proposition 2.3.

3 Solution Procedure and Summary of Results

Prior to describing the Deift and Zhou procedure [13] for obtaining the leading-order temporal asymptotics of (4), let us briefly review the Beals and Coifman formulation [14] for the solution of a RH problem on an oriented contour \( \Xi \). The RH problem (in the absence of a discrete spectrum) on \( \Xi \) consists of finding a \( 2 \times 2 \) matrix-valued function \( m(\lambda) \) such that: (i) \( m(\lambda) \) is holomorphic \( \forall \lambda \in \mathbb{C} \setminus \Xi \); (ii) \( m_+(\lambda) = m_-(\lambda)v(\lambda) \) \( \forall \lambda \in \Xi \), for some jump matrix \( v(\lambda): \Xi \to \text{Mat}(2, \mathbb{C}) \); and (iii) as \( \lambda \to \infty \), \( \lambda \in \mathbb{C} \setminus \Xi, m(\lambda) \to I \). Writing the jump matrix, \( v(\lambda) \), in the following factored form, \( v(\lambda) \equiv (I - w_{-}(\lambda))^{-1}(I + w_{+}(\lambda)) \) \( \forall \lambda \in \Xi \), where \( w_{\pm}(\lambda) \in L^2(\Xi) \cap L^{2}(\Xi) \), with \( ||w||_{L^2(\Xi)} \equiv ||w||_{L^2(\Xi)} \equiv ||w||_{L^{2}(\Xi)} + ||w||_{L^{2}(\Xi)} \) are off-diagonal upper/lower triangular matrices, denote \( w = w_{+} + w_{-} \), and introduce the operator \( C_w \) on \( L^2(\Xi) \) as follows,

\[
C_w f \equiv C_+(f w_-) + C_-(f w_+),
\]

where \( f \in L^2(\Xi) \), and \( C_\pm: L^2(\Xi) \to L^2(\Xi) \) denote the Cauchy operators (bounded monomorphisms),

\[
(C_\pm f)(\lambda) = \lim_{\lambda' \to \pm \lambda} \int_\Xi \frac{f(\xi)}{\xi - \lambda'} d\xi / 2\pi i.
\]

Theorem 3.1 ([14]) If \( \mu(\lambda) = m_+(\lambda)(I - w_{+}(\lambda)) = m_-(\lambda)(I + w_{-}(\lambda)) \) \( \in L^2(\Xi) + L^{\infty}(\Xi) \) solves the following linear integral equation,

\[
(\text{Id} - C_w)\mu = I,
\]

where \( \text{Id} \) is the identity operator on \( L^2(\Xi) + L^{\infty}(\Xi) \), then the solution of the RH problem for \( m(\lambda) \) is

\[
m(\lambda) = I + \int_\Xi \frac{\mu(\xi)w(\xi)}{(\xi - \lambda)} d\xi / 2\pi i, \quad \lambda \in \mathbb{C} \setminus \Xi.
\]
From Theorem 3.1 and (10), we obtain the following integral representation for \( P(x, t) \) (\( Q(x, t) = \overline{P(x, t)} \)) in terms of the resolvent kernel of the linear singular integral equation (13),

\[
P(x, t) = -i \left( [\sigma_3, \int_{\hat{\Gamma}} ((\mathbf{I} - C_{w_+, t})^{-1})(\xi)w_{x,t}(\xi)\frac{d\xi}{2\pi i}] \right)_{21}
\]

(see Sec. 5), which can be written in the following, simpler form,

\[
P(x, t) = \frac{1}{\pi} \int_{\Gamma} (\mu(x, t; \xi))_{22} (w_-(\xi))_{21} e^{2i\theta(\xi)} d\xi, \quad \text{where} \quad \theta(\lambda) \equiv 2\lambda^2(\lambda^2 - 2\lambda_0^2),
\]

with real, first-order (distinct) stationary phase points, \( \{0, \pm \lambda_0\} = \{0, \pm \frac{i}{2} \} \) (\( \{0, \pm \lambda_0\} = \{\lambda'; \partial_\lambda \theta(\lambda)|_{\lambda=\lambda'} = 0\} \)).

At this point, the natural tendency would be to apply the method of steepest descents \([23]\) to the integral in (15); however, even though \( \theta(\lambda) \) is known and \( w_-(\lambda) \) can be characterised completely, no \textit{a priori} explicit information regarding the analytical properties of the resolvent kernel, \( \mu(x, t; \lambda) \), is available. To eschew this manifest difficulty, Deift and Zhou \([15]\), based on the earlier, seminal work of Beals and Coifman \([14]\), introduced a novel non-linear analogue of the classical steepest-descent method and showed that, as \( t \to +\infty \), via a sequence of transformations which convert the original (full) RH problem (Lemma 2.1) into an equivalent, auxiliary RH problem with jump matrix \( v' \) of the form

\[
v' = v_{\text{model}} + v_{\text{error}},
\]

where \( v_{\text{model}} \) denotes the jump matrix for an explicitly solvable model RH problem and \( v_{\text{error}} \) contains only terms which are decreasing as \( t \to +\infty \), modulo some estimates, the solution of the original RH problem converges to the solution of the model RH problem. More precisely, the sequence of transformations which constitute the Deift and Zhou procedure \([15]\) can be summarised as follows (even though the delineation of the procedure is general enough, the notation is specific to this paper):

\begin{enumerate}[(i)]
  
  \item based on the classical method of steepest descents \([23]\) for decomposing the complex plane of the spectral parameter \( \lambda \) according to the signature of \( \Re(it\theta(\lambda)) \), deform the original oscillatory RH problem to an auxiliary RH problem formulated on an (oriented) augmented contour \( \Sigma \) in such a way that the respective jump matrices on \( \hat{\Gamma} \subset \Sigma \) and the finite “triangular” parts of \( \Sigma \setminus \hat{\Gamma} \), away from the neighbourhood of the real, first-order stationary phase points, \( \{0, \pm \lambda_0\} \), converge (with respect to appropriately defined norms) to the identity (I), i.e., rewrite the original undulatory RH problem as an auxiliary RH problem on the paths of steepest descents with upper/lower triangular jump matrices whose structures depend on the signature of \( \Re(it\theta(\lambda)) \) (see Sec. 4);
  
  \item reduce the contour \( \Sigma \) to a truncated contour \( \Sigma' \) which can be written as the disjoint union of three crosses, \( \Sigma' = \Sigma_{A'} \cup \Sigma_{B'} \cup \Sigma_{C'} \) (according to the number of real, first-order stationary phase points: see Sec. 5);
  
  \item analysing the higher order interaction between the three disjoint crosses, \( \Sigma_{A'}, \Sigma_{B'}, \) and \( \Sigma_{C'} \), and noting that, as \( t \to \pm \infty \), they are negligible (with respect to certain norms), separate out the contributions of the disjoint crosses in \( \Sigma' \) and show that the solution \( P(x, t) \) \( (Q(x, t) = \overline{P(x, t)} \) can be written as the linear superposition of the contributions of the various disjoint crosses (see Sec. 6); and
  
  \item localise the jump matrices of the most rapidly descended RH problems on the disjoint crosses to the neighbourhood of the real, first-order stationary phase points, \( \{0, \pm \lambda_0\} \),
\end{enumerate}
and, under suitable scalings of the spectral parameter, reduce the respective RH problems to RH problems on \( \mathbb{R} \) with constant jump matrices which can be solved explicitly (see Sec. 7).

In this paper, we prove that the application of the aforementioned procedure leads to the following results.

**Remark 3.1.** To facilitate the reading of the results stated in Theorems 3.2 and 3.3 (see below), as well as those which appear throughout the paper, the following preamble is necessary: (i) \( M \in \mathbb{R}_{>0} \) is a fixed constant; (ii) the “symbols” (“notations”) \( \xi \) and \( c^S \), respectively, which appear in the various error estimates are to be understood as follows, \( \xi \equiv \xi(\lambda_0) \in L^\infty(\mathbb{R}_{>M}) \), and \( c^S \equiv c^S(\lambda_0) \in S(\mathbb{R}_{>M}) \); (iii) even though the “symbols” \( \xi \) and \( c^S \) appearing in the various error estimates are not equal and should properly be denoted as \( \xi_1(\lambda_0), \xi_2(\lambda_0) \), etc., the simplified “notations” \( \xi \) and \( c^S \) are retained throughout since the main concern in this paper is not their explicit (functional) \( \lambda_0 \)-dependence, but rather, the explicit class(es) to which they belong; and (iv) with the exception of Theorems 3.2 and 3.3 (see below), the arguments of the “symbols” appearing in the error estimations will be suppressed.

**Remark 3.2.** In Theorems 3.2 and 3.3 (see below), one should keep everywhere the upper signs as \( t \to +\infty \) and the lower signs as \( t \to -\infty \).

**Theorem 3.2** For \( ||r||_{L^\infty(\Gamma)} < 1 \), let \( m(x,t; \lambda) \) be the solution of the RH problem formulated in Lemma 2.1 and \( Q(x,t) \) and \( q(x,t) \) be defined by Eqs. (10) and (11), respectively. Then as \( t \to \pm \infty \) and \( x \to \mp \infty \) such that \( \lambda_0 \equiv \frac{1}{2} \sqrt{\frac{\pi}{T}} > M \),

\[
Q(x,t) = \sqrt{\nu(\lambda_0)} \exp \left\{ i \left( \theta^+(\lambda_0) + \hat{\Phi}^+(\lambda_0; t) \right) \right\} + O \left( \frac{\nu(\lambda_0) \ln |t|}{\lambda_0 t} \right),
\]

\[
q(x,t) = \sqrt{\nu(\lambda_0)} \exp \left\{ i \left( \theta^+(\lambda_0) + \hat{\Phi}^+(\lambda_0; t) \right) \right\} + O \left( \frac{\nu(\lambda_0) \ln |t|}{\lambda_0 t} \right),
\]

where

\[
\nu(z) = -\frac{1}{2\pi} \ln(1 - |r(z)|^2),
\]

\[
\theta^+(z) = \frac{1}{2} \int_0^x \ln |\mu|^2 - z^2 | d \ln(1 - |r(\mu)|^2) - \frac{1}{2} \int_0^\infty \ln |\mu|^2 + z^2 | d \ln(1 + |r(\mu)|^2),
\]

\[
\theta^-(z) = \frac{1}{2} \int_x^\infty \ln |\mu|^2 - z^2 | d \ln(1 - |r(\mu)|^2),
\]

\[
\hat{\Phi}^+(\lambda_0; t) = 4\lambda_0^2 t + \nu(\lambda_0) \ln |t| + \frac{2\pi}{\nu} \pm \arg \Gamma(i\nu(\lambda_0)) + \arg r(\lambda_0) + 3\nu(\lambda_0) \ln 2,
\]

\[
\hat{\Phi}^-(\lambda_0; t) = \hat{\Phi}^+(\lambda_0; t) + \frac{2\pi}{\nu} \int_{\lambda_0}^x \frac{\ln(1 - |r(\mu)|^2)}{\mu} d\mu - \frac{2\pi}{\nu} \int_{\lambda_0}^\infty \frac{\ln(1 + |r(\mu)|^2)}{\mu} d\mu.
\]

and, as \( t \to \pm \infty \) and \( x \to \pm \infty \) such that \( \lambda_0 \equiv \frac{1}{2} \sqrt{\frac{\pi}{T}} > M \),

\[
Q(x,t) = \sqrt{\nu(i\lambda_0)} \exp \left\{ i \left( \phi^+(\lambda_0) + \hat{\Phi}^+(\lambda_0; t) \right) \right\} + O \left( \frac{\nu(i\lambda_0) \ln |t|}{\lambda_0 t} \right),
\]

\[
q(x,t) = \sqrt{\nu(i\lambda_0)} \exp \left\{ i \left( \phi^+(\lambda_0) + \hat{\Phi}^+(\lambda_0; t) \right) \right\} + O \left( \frac{\nu(i\lambda_0) \ln |t|}{\lambda_0 t} \right),
\]

where

\[
\nu(iz) = -\frac{1}{2\pi} \ln(1 + |r(iz)|^2),
\]

\[
\phi^+(z) = \frac{1}{2} \int_0^x \ln |\mu|^2 - z^2 | d \ln(1 + |r(\mu)|^2) - \frac{1}{2} \int_0^\infty \ln |\mu|^2 + z^2 | d \ln(1 - |r(\mu)|^2),
\]
where
\( \hat{\Theta}^{\pm}(\lambda'_0; t) = 4(\lambda'_0)^4 t \mp \nu(i\lambda'_0) \ln |t| + \hat{\epsilon}_\pm \pm \arg \Gamma(i\nu(i\lambda'_0)) + \arg r(i\lambda'_0) \mp 3\nu(i\lambda'_0) \ln 2, \quad \) (29)\n
\( \hat{\Theta}^+(\lambda'_0; t) = \hat{\Theta}^+(\lambda_0; t) + \frac{2}{\pi} \int_{\lambda'_0}^{\infty} \frac{\ln(1+|r(\mu)|^2)}{\mu} d\mu - \frac{2}{\pi} \int_0^{\infty} \frac{\ln(1+|r(\mu)|^2)}{\mu} d\mu, \quad \) (30)\n
\( \hat{\Theta}^{-}(\lambda'_0; t) = \hat{\Theta}^-(\lambda_0; t) + \frac{2}{\pi} \int_{\lambda'_0}^{\infty} \frac{\ln(1+|r(\mu)|^2)}{\mu} d\mu, \quad \) (31)\n
\( \hat{\epsilon}_\pm = (2 \pm 1)\pi/4, \) and \( \Gamma(\cdot) \) is the gamma function \( [24] \).

\textbf{Theorem 3.3} For \( ||r||_{C^\infty(\Gamma)} < 1 \), let \( m(x, t; \lambda) \) be the solution of the RH problem formulated in Lemma 2.1 and \( u(x, t) \), the solution of the MNLSE, be given as in Proposition 2.4 in terms of the function \( q(x, t) \) given in Theorem 3.2. Then as \( t \to \pm \infty \) and \( x \to \pm \infty \) such that \( \hat{\lambda}_0 \equiv \sqrt{\frac{1}{2} \left( \frac{1}{s} - \frac{1}{t} \right)} > M \) and \( \frac{1}{s} > \frac{1}{t} \), \( s \in \mathbb{R}_{>0} \),

\( u(x, t) = \sqrt{\pm \frac{\nu(i\lambda_0)}{2(\lambda'_0)^2 \sin t}} \exp \left\{ i \left( \theta^\pm(\lambda_0) + \tilde{\Theta}^\pm(\lambda_0; t) \right) \right\} + \mathcal{O} \left( \frac{\theta(\lambda_0) \ln |t|}{\lambda_0 t} \right), \quad \) (32)\n
where

\[ \tilde{\Theta}^+(\lambda_0; t) = 2 \left( \hat{\lambda}_0^2 + \frac{1}{2s} \right)^2 t - \nu(\hat{\lambda}_0) \ln t - \frac{3\pi}{4} + \arg \Gamma(i\nu(\hat{\lambda}_0)) + \arg r(\hat{\lambda}_0) \]

\[ - 2\nu(\hat{\lambda}_0) \ln 2 + \frac{2}{\pi} \int_{\lambda'_0}^{\lambda_0} \ln(1+|r(\mu)|^2) d\mu - \frac{2}{\pi} \int_0^{\infty} \frac{\ln(1+|r(\mu)|^2)}{\mu} d\mu, \quad \) (33)\n
\[ \tilde{\Theta}^-(\lambda_0; t) = 2 \left( \hat{\lambda}_0^2 + \frac{1}{2s} \right)^2 t + \nu(\hat{\lambda}_0) \ln |t| + \frac{3\pi}{4} - \arg \Gamma(i\nu(\hat{\lambda}_0)) + \arg r(\hat{\lambda}_0) \]

\[ + 2\nu(\hat{\lambda}_0) \ln 2 + \frac{2}{\pi} \int_{\lambda'_0}^{\infty} \ln(1+|r(\mu)|^2) d\mu, \quad \) (34)\n
and, as \( t \to \pm \infty \) and \( x \to \mp \infty \) or \( \pm \infty \) such that \( \hat{\lambda}_0' \equiv \sqrt{\frac{1}{2} \left( \frac{1}{s} - \frac{1}{t} \right)} > M \) and \( \frac{1}{s} < \frac{1}{t} \), \( s \in \mathbb{R}_{>0} \),

\( u(x, t) = \sqrt{\pm \frac{\nu(i\lambda_0)}{2(\lambda'_0)^2 \sin t}} \exp \left\{ i \left( \phi^\pm(\lambda'_0) + \hat{\Theta}^\pm(\lambda_0; t) \right) \right\} + \mathcal{O} \left( \frac{\theta(\lambda'_0) \ln |t|}{\lambda'_0 t} \right), \quad \) (35)\n
where

\[ \hat{\Theta}^+(\lambda'_0; t) = 2 \left( (\lambda'_0)^2 - \frac{1}{2s} \right)^2 t - \nu(i\lambda'_0) \ln t + \frac{3\pi}{4} + \arg \Gamma(i\nu(\lambda'_0)) + \arg r(i\lambda'_0) \]

\[ - 2\nu(i\lambda'_0) \ln 2 + \frac{2}{\pi} \int_{\lambda'_0}^{\infty} \frac{\ln(1+|r(\mu)|^2)}{\mu} d\mu - \frac{2}{\pi} \int_0^{\infty} \frac{\ln(1+|r(\mu)|^2)}{\mu} d\mu, \quad \) (36)\n
\[ \hat{\Theta}^-(\lambda'_0; t) = 2 \left( (\lambda'_0)^2 - \frac{1}{2s} \right)^2 t + \nu(i\lambda'_0) \ln |t| + \frac{\pi}{4} - \arg \Gamma(i\nu(\lambda'_0)) + \arg r(i\lambda'_0) \]

\[ + 2\nu(i\lambda'_0) \ln 2 + \frac{2}{\pi} \int_{\lambda'_0}^{\infty} \ln(1+|r(\mu)|^2) d\mu, \quad \) (37)\n
and \( \nu(\cdot) \), \( \theta^\pm(\cdot) \), \( \nu(i\cdot) \), and \( \phi^\pm(\cdot) \) are given in Theorem 3.2, Eqs. (18)–(20) and (26)–(28).
Remark 3.3. The results presented in Theorem 3.3 give the asymptotic expansion of \( u(x,t) \) in the domain \( x/t \neq 1/s \). In the case \( t \to \pm \infty \) and \( x/t = 1/s \), \( u(x,t) \) can be written as follows,

\[
u(x,t) = \frac{u_0}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right),
\]

where \( u_0 \in \mathbb{C} \setminus \{0\} \) is some constant: the determination of \( u_0 \) requires special consideration and will be presented elsewhere.

Remark 3.4. Some recent results concerning the asymptotic form of solutions as \( t \to \pm \infty \) of the DNLSE were written in terms of the so-called final states (the final-value problem) [25]; however, unlike the present article, the authors in [25] did not treat the Cauchy initial-value problem.

Remark 3.5. In the proof of Theorems 3.2 and 3.3, according to the above-formulated scheme, we omit the details of intermediate estimations which are analogous to those in [15]: in corresponding places, we make exact references to the appropriate assertions in [15], and [19], where these details may be found; moreover, for \( Q(x,t) \), \( q(x,t) \), and \( u(x,t) \), one should consider actually four different cases, depending on the quadrant in the \((x,t)\)-plane. In the subsequent sections, we present only the proof for the function \( Q(x,t) \) in the case \( t \to +\infty \) and \( x \to -\infty \) such that \( \lambda_0 = \frac{1}{2} \sqrt{-\frac{2}{t^4}} > M \); the results for the remaining domains of the \((x,t)\)-plane are obtained in an analogous manner. The corresponding results for the functions \( q(x,t) \) and \( u(x,t) \) are obtained from Propositions 2.3 and 2.4.

4 The Augmented RH Problem

As explained in the previous section, we begin with the decomposition of the complex \( \lambda \)-plane according to the signature of the real part of the phase of the conjugating exponential of the oscillatory RH problem (Lemma 2.1), \( \Re(it\theta(\lambda)) \), where \( \theta(\lambda) = 2\lambda^2(\lambda^2 - 2\lambda_0^2), \lambda_0^2 = -\frac{2}{4t} > 0, \) and \( t > 0 \) (see Fig. 2).

\[
\begin{align*}
\Re(it\theta(\lambda)) &< 0 & \Re(it\theta(\lambda)) &> 0 \\
\Re(it\theta(\lambda)) &> 0 & \Re(it\theta(\lambda)) &< 0
\end{align*}
\]

Remark 4.1. Note that, we have reoriented the contour \( \tilde{\Gamma} \) in accordance with the signature of \( \Re(it\theta(\lambda)) \) and the “sign” convention of Remark 2.1.

The main purpose of this section is to reformulate the original RH problem (Lemma 2.1) as an equivalent RH problem (see Lemma 4.1) on the augmented contour \( \Sigma \) (see Fig. 3),

\[
\Sigma = L \cup T \cup \tilde{\Gamma},
\]
where $L \equiv L^0 \cup L^1$, and
\[
L^0 = \{ \lambda; \lambda = \lambda_0 + \frac{\lambda_0 u}{2} e^{\frac{i\theta}{2}}, u \in (-\infty, \sqrt{2}] \} \cup \{ \lambda; \lambda = -\lambda_0 + \frac{\lambda_0 u}{2} e^{-\frac{i\theta}{2}}, u \in (-\infty, \sqrt{2}] \},
\]
\[
L^1 = \{ \lambda; \lambda = \frac{\lambda_0 u}{2} e^{\frac{i\theta}{2}}, u \in \mathbb{R} \}.
\]

Remark 4.2. Actually, for the following analysis, the augmented contour $\Sigma$ can be chosen in different ways, not necessarily consisting of straight lines: its important characteristic is the position of the contour $L \cup \overline{L}$ with respect to the lines $\Re(it\theta(\lambda)) = 0$.

In order to define the conjugation matrices on $\Sigma$ and exploit the analyses in [14, 15], we need to formulate two technical propositions: the first concerns the triangular factorisation of the conjugation matrices of the original RH problem (Lemma 2.1), and the second pertains to a special decomposition for the reflection coefficient, $r(\lambda)$. Write the jump matrices of the original RH problem (Lemma 2.1) in upper/lower triangular form for $\lambda \in (-\infty, -\lambda_0) \cup (+\lambda_0, +\infty)$ as
\[
e^{-it\theta(\lambda)\text{ad}(\sigma_3)} G(\lambda) = e^{-it\theta(\lambda)\text{ad}(\sigma_3)} \begin{pmatrix} 1 & r(\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -r(\lambda) & 1 \end{pmatrix},
\]
and, in lower/upper triangular form for $\lambda \in (-\lambda_0, +\lambda_0) \cup (-i\infty, +i\infty)$ as
\[
e^{-it\theta(\lambda)\text{ad}(\sigma_3)} G(\lambda) = e^{-it\theta(\lambda)\text{ad}(\sigma_3)} \begin{pmatrix} 1 & 0 \\ -r(\lambda)(1 - r(\lambda)r(\lambda))^{-1} & 1 \end{pmatrix} 
\times \begin{pmatrix} (1 - r(\lambda)r(\lambda)) & 0 \\ 0 & (1 - r(\lambda)r(\lambda))^{-1} \end{pmatrix} \begin{pmatrix} 1 & r(\lambda)(1 - r(\lambda)r(\lambda))^{-1} \\ 0 & 1 \end{pmatrix}.
\]

To eliminate the diagonal matrix between the lower/upper triangular factors in (39), we, according to the scheme in [15], introduce the auxiliary function $\delta(\lambda)$ which solves the following discontinuous scalar RH problem:
\[
\delta_{\pm}(\lambda) = \begin{cases} 
\delta_{-}(\lambda) = \delta(\lambda), & \lambda \in (-\infty, -\lambda_0) \cup (+\lambda_0, +\infty), \\
\delta_{-}(\lambda)(1 - r(\lambda)r(\lambda)), & \lambda \in (-\lambda_0, +\lambda_0) \cup (-i\infty, +i\infty),
\end{cases}
\]
\[
\delta(\lambda) \to 1 \text{ as } \lambda \to \infty.
\]
Proposition 4.1 The unique solution of the above RH problem (40) can be written as
\[
\delta(\lambda) = \left(\left(\frac{\lambda - \lambda_0}{\chi} \right) \left(\frac{\lambda + \lambda_0}{\chi} \right)\right)^{\nu} e^{\chi^+(\lambda)} e^{\chi^-(\lambda)} e^{\widehat{\chi}^+(\lambda)} e^{\widehat{\chi}^-(\lambda)},
\]
where \(\nu \equiv \nu(\lambda_0)\) is given in Theorem 3.2, Eq. (18), and
\[
\chi^\pm(\lambda) = \frac{1}{2\pi} \int_{L_0^\pm} \frac{r(\mu)}{r(\mu) - r(\lambda)} d\mu, \quad \widehat{\chi}^\pm(\lambda) = \int_{\pm \infty}^{0} \frac{\ln(1 - r(\mu))}{(r(\mu) - r(\lambda))^2} d\mu.
\]
in Eq. (41), \((\lambda - \mu)^{\pm 2\nu} \equiv \exp\{\pm 2\nu \ln(\lambda - \mu)\},\) where the principal branch of the logarithmic functions, \(\ln(\lambda - \mu), \mu \in \{0, \pm \lambda_0\},\) has been chosen, with the branch cuts along \((-\infty, \mu),\) \(\ln(\lambda - \mu) \equiv \ln|\lambda - \mu| + i\arg(\lambda - \mu),\) \(\arg(\lambda - \mu) \in (-\pi, \pi);\) moreover, the function \(\delta(\lambda)\) possesses the following properties,
\[
\delta(\lambda) = (\delta(\lambda))^{-1} = (\delta(-\lambda))^{-1} \Rightarrow \delta(\lambda) = \delta(-\lambda),
\]
\[
|\delta^\pm(\lambda)|^2 \leq \left(1 - \sup_{\mu \in \widehat{\Gamma}} r(\mu) r(\mu)^{-1}\right) < \infty \quad \forall \ \lambda \in \widehat{\Gamma},
\]
and \(\|\delta(\cdot)^{\pm 1}\|_{L^\infty(\mathbb{C})} \equiv \sup_{\lambda \in \mathbb{C}} |\delta(\lambda)|^{\pm 1} < \infty.\)

Proof. It is well known that the scalar RH problem for \(\delta(\lambda)\) stated in (40) can be solved explicitly (see, for example, [26, 27]): in our case, the solution is given in (41) and (42). As a consequence of (41) and (42), one proves the symmetry properties in (43): from (40), using (43), one deduces inequalities (44); therefore, from the maximum modulus principle and the fact that \(\delta(\lambda)\) has no zeros for \(\lambda \in \mathbb{C},\) one gets that \(\delta(\lambda) \in L^\infty(\mathbb{C}).\) ■

The conjugation matrices for the RH problem on the augmented contour \(\Sigma\) should, of course, be written in terms of the matrix elements of the original RH problem on \(\widehat{\Gamma}\) (Lemma 2.1); but, since the reflection coefficient, \(r(\lambda),\) does not, in general, have analytical continuation off \(\widehat{\Gamma},\) we, following [13], decompose it as the sum of an analytically continuable part and a negligible non-analytic remainder. To formulate an exact result which we use later, let us define: (i)
\[
\rho(\lambda) \equiv \begin{cases} r(\lambda)(1 - r(\lambda)r(\lambda))^{-1}, & \lambda \in (-\lambda_0, +\lambda_0) \cup (-i\infty, +i\infty), \\ -r(\lambda), & \lambda \in (-\infty, -\lambda_0) \cup (+\lambda_0, +\infty); \end{cases}
\]
and (ii) the contour \(L_\delta = L_\delta^0 \cup L_\delta^1,\) where
\[
L_\delta^0 = \{\lambda; \lambda = \lambda_0 + \frac{\lambda_0 u}{2} e^{\frac{3\pi}{2} i}, u \in [\delta, \sqrt{2}]\} \cup \{\lambda; \lambda = -\lambda_0 + \frac{\lambda_0 u}{2} e^{\frac{3\pi}{2} i}, u \in [\delta, \sqrt{2}]\},
\]
\[
L_\delta^1 = \{\lambda; \lambda = \lambda_0 + \frac{\lambda_0 u}{2} e^{\frac{3\pi}{2} i}, u \in [-\sqrt{2}, -\delta) \cup (\delta, \sqrt{2})\}.
\]

Proposition 4.2 For each \(l \in \mathbb{Z}_{\geq 1},\) there exists a decomposition of the function \(\rho(\lambda)\) in Eq. (45),
\[
\rho(\lambda) = h_I(\lambda) + R(\lambda) + h_{II}(\lambda), \quad \lambda \in \widehat{\Gamma},
\]
such that \(h_I(\lambda)\) is analytic on \(\widehat{\Gamma}\) (generally, it has no analytic continuation off \(\widehat{\Gamma}\)), \(R(\lambda)\) is a piecewise-rational function such that \(\frac{d^j r(\lambda)}{d\lambda^j}|_{\lambda \in \delta} = \frac{d^j R(\lambda)}{d\lambda^j}|_{\lambda \in \delta},\) \(0 \leq j \leq 12l + 1,\) where \(\delta \equiv \{0, \pm \lambda_0\},\) and \(h_{II}(\lambda)\) has an analytic continuation to \(L;\) moreover, in the domain \(\lambda_0 > M,\) the following estimates are valid as \(t \to +\infty,\)
\[
|e^{-2i\theta(\lambda)} h_I(\lambda)| \leq \frac{C}{(1 + |\lambda_0|)(\lambda_0^2)^l}, \quad \lambda \in \widehat{\Gamma}, \quad |e^{-2i\theta(\lambda)} h_{II}(\lambda)| \leq \frac{C}{(1 + |\lambda_0|)(\lambda_0^2)^l}, \quad \lambda \in L,
\]
\[
|e^{-2i\theta(\lambda)} R(\lambda)| \leq \exp\{-2\lambda_0^4 \bar{\delta}^2 t\}, \quad \lambda \in L_\delta,
\]
and \(\bar{\delta} \in \mathbb{R}_{> 0}\) is sufficiently small.
Proof. Rather technical: proceed analogously as in the proof of Proposition 1.92 in [15] by expanding \( \rho(\lambda) \) in terms of a rational polynomial approximation in the neighbourhood of the real, first-order stationary phase points, \( \{0, \pm \lambda_0\} \), and show that, for \( l \in \mathbb{Z}_{\geq 1} \), 

\[ |e^{-2it\theta(\lambda)}h_I(\lambda)| \leq \frac{\delta}{(1+|\lambda|)^{2|z|}}, \quad \lambda \in \bar{\Gamma}, \quad |e^{-2it\theta(\lambda)}h_{II}(\lambda)| \leq \frac{\delta}{(1+|\lambda|)^{2|z|}}, \quad \lambda \in L, \]

and 

\[ |e^{-2it\theta(\lambda)}R(\lambda)| \leq \epsilon \exp\{-\tfrac{1}{2}\lambda_0^2\delta^2|x|\}, \quad \lambda \in L_{\delta}. \]

Using the relation \( |x| = 4\lambda_0^2 t \), one obtains the result stated in the Proposition: for details, see [19]. \( \Box \)

**Lemma 4.1** Let \( m(x, t; \lambda) \) be the solution of the RH problem formulated in Lemma 2.1. Set \( m^\Delta(x, t; \lambda) \equiv m(x, t; \lambda)(\Delta(\lambda))^{-1} \), where

\[ \Delta(\lambda) \equiv (\delta(\lambda))^{\sigma_3}, \quad (47) \]

and \( \delta(\lambda) \) is given in Proposition 4.1. Define

\[
m^\ast(x, t; \lambda) \equiv \begin{cases} m^\Delta(x, t; \lambda), & \lambda \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4, \\ m^\Delta(x, t; \lambda)(I-(w^\ast_{x,t,\delta})^{-1}), & \lambda \in \Omega_5 \cup \Omega_6 \cup \Omega_7 \cup \Omega_8 \cup \Omega_9 \cup \Omega_{10}, \\ m^\Delta(x, t; \lambda)(I+(w^\ast_{x,t,\delta})^{-1}), & \lambda \in \Omega_{11} \cup \Omega_{12} \cup \Omega_{13} \cup \Omega_{14} \cup \Omega_{15} \cup \Omega_{16}. \end{cases}
\]

Then \( m^\ast(x, t; \lambda) \) solves the following (augmented) RH problem on \( \Sigma \),

\[ m^\ast_{\pm}(x, t; \lambda) = m^\ast_{\pm}(x, t; \lambda)v^\ast_{x,t,\delta}(\lambda), \quad \lambda \in \Sigma, \]

\[ m^\ast(x, t; \lambda) \to I \quad \text{as} \quad \lambda \to \infty, \quad (48) \]

where

\[ v^\ast_{x,t,\delta}(\lambda) \equiv (I-(w^\ast_{x,t,\delta})^{-1})(I+(w^\ast_{x,t,\delta})) = \begin{cases} (I-(w^0_{x,t,\delta})^{-1})(I+(w^0_{x,t,\delta})), & \lambda \in \bar{\Gamma}, \\ (I+(w^0_{x,t,\delta})^{-1}), & \lambda \in L, \\ (I-(w^0_{x,t,\delta})^{-1}), & \lambda \in \mathcal{T}, \end{cases} \quad (49) \]

and

\[ (w^{(0,a)}_{\pm})_{x,t,\delta} = (\delta_{\pm}(\lambda))^{ad(\sigma_3)} \exp \{-it\theta(\lambda) ad(\sigma_3)\} w^{(0,a)}_{\pm}, \quad (50) \]

\[ w^0_+ = h_I(\lambda) \sigma_+, \quad w^0_+ = (h_{II}(\lambda) R(\lambda)) \sigma_+, \quad (51) \]

\[ w^0_- = -h_I(\lambda) \sigma_-, \quad w^0_- = -(h_{II}(\lambda) + R(\lambda)) \sigma_- \quad (52) \]

**Proof.** In terms of the function \( m^\Delta(x, t; \lambda) \) defined in the Lemma, the original oscillatory RH problem (Lemma 2.1) can be rewritten in the following form,

\[ m^\Delta_{\pm}(x, t; \lambda) = m^\Delta_{\pm}(x, t; \lambda)(I-(w_{x,t,\delta})^{-1})(I+(w_{x,t,\delta})), \quad \lambda \in \bar{\Gamma}, \quad (53) \]

where

\[ (w_{x,t,\delta}) = (\delta_{\pm}(\lambda))^{ad(\sigma_3)} \exp \{-it\theta(\lambda) ad(\sigma_3)\} w_{\pm}, \quad (54) \]

and

\[ w_+ = \rho(\lambda) \sigma_+, \quad w_- = -\overline{\rho(\lambda)} \sigma_- \quad (55) \]

Defining \( m^\ast(x, t; \lambda) \) as in the Lemma, one arrives, as a consequence of Proposition 4.2, (47), and (50)–(55), to the RH problem stated in the Lemma. \( \Box \)
5 RH Problem on the Truncated Contour

The goal of this section is to get rid (asymptotically) of the contribution of the functions \(h_I(\lambda)\) and \(h_{II}(\lambda)\) (Proposition 4.2) to the conjugation matrices of the augmented RH problem: simultaneously, it is proved that, as \(t \to +\infty\), under the condition \(\lambda_0 > M\), the contribution to the asymptotics of the function \(P(x,t)\) coming from \(\hat{\Gamma}\) and the finite “triangular” segments \(L_\delta^\gamma\) and \(\overline{L_\delta^\gamma}\), respectively, are negligible; hence, we derive the RH problem on the truncated contour \(\Sigma\) (see Fig. 4).

\[
\Sigma' = \Sigma \setminus (L_\delta^\gamma \cup \overline{L_\delta^\gamma} \cup \hat{\Gamma}).
\]

Instead of (10), from Lemma 4.1, one can write

\[
P(x,t) = i \lim_{\lambda \to \infty} (\lambda | \sigma_3, m^\sharp(x,t;\lambda)|)_{21}, \quad Q(x,t) = \overline{P(x,t)}; \tag{56}
\]

whence, following the Beals and Coifman formulation (Theorem 3.1), by setting

\[
w^\sharp_{x,t,\delta} = (w^\sharp_+)_{x,t,\delta} + (w^\sharp_-)_{x,t,\delta},
\]

where \((w^\sharp_\pm)_{x,t,\delta}\) are given in (49)–(52), one gets the following integral representation for \(P(x,t)\),

\[
P(x,t) = -i \left( \int_{\Sigma} [\sigma_3, ((\text{Id} - C_{w^\sharp_{x,t,\delta}})^{-1})] (\xi) w^\sharp_\delta(x,t,\delta)(\xi) \frac{d\xi}{2\pi i} \right)_{21}, \tag{57}
\]

where \((\text{Id} - C_{w^\sharp_{x,t,\delta}})^{-1}\) is invertible as an operator in \(L_2^{(2 \times 2)}(\Sigma) + L_\infty^{(2 \times 2)}(\Sigma)\). To reduce the RH problem on \(\Sigma\) to the one on the contour \(\Sigma'\), we need to estimate corrections which arise as a result of the corresponding reduction of the integration contour in (57). To do this, we decompose \(w^\sharp_{x,t,\delta}\) as

\[
w^\sharp_{x,t,\delta} = w^e + w', \quad w^e \equiv w^a + w^b + w^c; \tag{58}
\]

(i) \(w^a = w^\sharp_{x,t,\delta}|_{\hat{\Gamma}}\) is the restriction of \(w^\sharp_{x,t,\delta}\) to \(\hat{\Gamma}\) and consists of the contribution to \(w^\sharp_{x,t,\delta}\) from \(h_I(\lambda)\) and \(h_{II}(\lambda)\); (ii) \(w^b\) has support on \(L \cup \overline{\mathcal{L}}\) and consists of the contribution to \(w^\sharp_{x,t,\delta}\) from \(h_{II}(\lambda)\) and \(h_{II}(\lambda)\); and (iii) \(w^c\) has support on \(L_\delta \cup \overline{L_\delta}\) and consists of the contribution
to $w_{x,t,\delta}^\sharp$ from $R(\lambda)$ and $\overline{R}(\lambda)$. The main idea of the estimation of the integral in (57) will be (based on Proposition 4.2) to show that, as $t \to +\infty$ ($x/t \sim O(1)$), $w'_{x,t} \to 0$ and $w^\delta \to 0$, in the sense of appropriately defined operator norms, and, therefore, to “lump” the contribution to $w_{x,t,\delta}^\sharp$ from $R(\lambda)$ and $\overline{R}(\lambda)$ into the factor $w'$, which is supported on $\Sigma'$, and show that it encapsulates the leading order asymptotics.

**Lemma 5.1** For arbitrary $l \in \mathbb{Z}_{\geq 1}$ and sufficiently small $\delta \in \mathbb{R}_{>0}$, as $t \to +\infty$ such that $\lambda_0 > M$, the operator $N$ acts in $L_2^{(2\times 2)}(\cdot)$. The main idea of the estimation of the integral in (57) will be that $w'_{x,t} \to 0$, in the sense of appropriately defined operator norms, and, therefore, to “lump” the contribution to $w_{x,t,\delta}^\sharp$ from $R(\lambda)$ and $\overline{R}(\lambda)$ into the factor $w'$, which is supported on $\Sigma'$, and show that it encapsulates the leading order asymptotics.

**Lemma 5.2** As $t \to +\infty$ such that $\lambda_0 > M$, $(\text{Id} - C_{w_{x,t,\delta}^\sharp})^{-1} \in \mathcal{N}(\Sigma) \Leftrightarrow (\text{Id} - C_{w'})^{-1} \in \mathcal{N}(\Sigma)$.

**Proof.** Consequence of the following inequality, $||C_{w_{x,t,\delta}^\sharp} - C_{w'}||_{\mathcal{N}(\Sigma)} \leq ||w^\delta||_{L_2^{(2\times 2)}(\Sigma)}$, the fact that $||w^\delta||_{L_2^{(2\times 2)}(\Sigma)} \leq C|x|^{-l} \leq C(\lambda_0 t)^{-l}$ (Lemma 5.1), and the second resolvent identity. \[ \square \]

**Remark 5.1.** Actually, the operator $(\text{Id} - C_{w'})^{-1}$ acts in $L_2^{(2\times 2)}(\Sigma) + L_\infty^{(2\times 2)}(\Sigma)$, and the operator $C_{w_{x,t,\delta}^\sharp}$ acts from $L_2^{(2\times 2)}(\Sigma) + L_\infty^{(2\times 2)}(\Sigma)$ into $L_2^{(2\times 2)}(\Sigma)$: we consider their restrictions to $L_2^{(2\times 2)}(\Sigma)$.

**Proposition 5.1** If $(\text{Id} - C_{w'})^{-1} \in \mathcal{N}(\Sigma)$, then for arbitrary $l \in \mathbb{Z}_{\geq 1}$, as $t \to +\infty$ such that $\lambda_0 > M$,

$$P(x,t) = -i \left( \int_{\Sigma} [\sigma_3, \left( (\text{Id} - C_{w'})^{-1}\right)(\xi)w'(\xi)] \frac{d\xi}{2\pi i} \right) + O \left( \frac{C}{(\lambda_0 t)^l} \right).$$

**Proof.** From the second resolvent identity, one can derive the following expression (see Eq. (2.27) in \cite{13}):

$$f_{\Sigma}((\text{Id} - C_{w_{x,t,\delta}^\sharp})^{-1}(\xi)w_{x,t,\delta}^\sharp(\xi)) \frac{d\xi}{2\pi i} = f_{\Sigma}((\text{Id} - C_{w'})^{-1}(\xi)w'(\xi)) \frac{d\xi}{2\pi i} + I + II + III + IV,$$

where

$$I \equiv f_{\Sigma} w^\delta(\xi) \frac{d\xi}{2\pi i}, \quad IV \equiv f_{\Sigma}((\text{Id} - C_{w'})^{-1}C_{w'}((\text{Id} - C_{w_{x,t,\delta}^\sharp})^{-1})(\xi)w_{x,t,\delta}^\sharp(\xi)) \frac{d\xi}{2\pi i},$$

$$II \equiv f_{\Sigma}((\text{Id} - C_{w'})^{-1}(C_{w'}\xi)(\xi)w_{x,t,\delta}^\sharp(\xi)) \frac{d\xi}{2\pi i}, \quad III \equiv f_{\Sigma}((\text{Id} - C_{w'})^{-1}(C_{w'}\xi)) \frac{d\xi}{2\pi i},$$
The terms $I$, $II$, $III$, and $IV$ have, respectively, for $l \in \mathbb{Z}_{\geq 1}$, the following estimate, $O(\frac{\lambda}{t})$ (hence, using the relation $|x| = 4 \lambda_0 t$, $O(\lambda_0^2 t^{-1})$). Let us prove, for example, the last estimate:

$$|IV| \leq \| (\mathbf{1} - C_w) C_w \|_{L^2(\Sigma)}^2 \leq \| (\mathbf{1} - C_w) C_w \|_{L^2(\Sigma)} \| (\mathbf{1} - C_w) C_w \|_{L^2(\Sigma)} \leq C \| (\mathbf{1} - C_w) \|_{L^2(\Sigma)} \| (\mathbf{1} - C_w) C_w \|_{L^2(\Sigma)} \leq C \| (\mathbf{1} - C_w) \|_{L^2(\Sigma)} \| (\mathbf{1} - C_w) C_w \|_{L^2(\Sigma)}^2.$$ 

From Lemma 5.1, the arithmetic mean inequality, and the Cauchy-Schwarz inequality, one shows that,

$$\left( \| w_{x,t,\delta} \|_{L^2(\Sigma)} \right)^2 \leq 4(\| w^a \|_{L^2(\Sigma)}^2) + 4(\| w^b \|_{L^2(\Sigma)}^2) + 4(\| w^c \|_{L^2(\Sigma)}^2) \leq C,$$

whence, estimate $|IV|$ follows from the fact that $\| w^a \|_{L^2(\Sigma)} \leq C \lambda^2 (\lambda_0 t)^{-1}$ ((58) and Lemma 5.1). 

Let us now show that, in the sense of appropriately defined operator norms, one may always choose to delete (or add) a portion of a contour(s) on which the jump is $I^+$ and $\mathbf{1}$, estimate $5.1$).

Definition 5.2 Let: (i) $R_{\Sigma}: E_{L^2(\Sigma)} \to E_{L^2(\Sigma')}$ denote the restriction map; (ii) $I_{\Sigma}: E_{L^2(\Sigma)} \to E_{L^2(\Sigma')}$ denote the embedding; (iii) $C_{w^a}: E_{L^2(\Sigma)} \to E_{L^2(\Sigma')}$ denote the operator in Eq. (12) with $w \leftrightarrow w'$; (iv) $C_{w^b}: E_{L^2(\Sigma)} \to E_{L^2(\Sigma')}$ denote the operator in Eq. (12) with $w \leftrightarrow w'\big|_{\Sigma}$; (v) $C_{w^c}: E_{L^2(\Sigma)} \to E_{L^2(\Sigma')}$ denote the restriction of $C_{w^a}$ to $E_{L^2(\Sigma')}$; (vi) $\mathbf{1}_{\Sigma}$ and $\mathbf{1}_{\Sigma'}$ denote, respectively, the identity operators on $E_{L^2(\Sigma)}$ and $E_{L^2(\Sigma')}$. 

Lemma 5.3

$$C_{w^a} C_{w^b} C_{w^c} = C_{w^b} C_{w^a} C_{w^c},$$

$$(\mathbf{1}_{\Sigma'} - C_{w^a})^{-1} = R_{\Sigma'} (\mathbf{1}_{\Sigma} - C_{w^a})^{-1} I_{\Sigma'} \Sigma,$$

$$(\mathbf{1}_{\Sigma'} - C_{w^b})^{-1} = C_{w^b} (\mathbf{1}_{\Sigma} - C_{w^a})^{-1} R_{\Sigma'}. \quad (60)$$

Proof. See Lemma 2.56 in [15]. 

Proposition 5.2 If $(\mathbf{1}_{\Sigma'} - C_{w^a}) \in \mathcal{N}(\Sigma)$, then for arbitrary $l \in \mathbb{Z}_{\geq 1}$, as $t \to +\infty$ such that $\lambda_0 > M$,

$$P(x,t) = -i \left( \int_{\Sigma'} (\mathbf{1}_{\Sigma'} - C_{w^a})^{-1} I((\xi) w'((\xi)) \frac{d\xi}{2\pi i} \right) + O\left( \frac{\lambda^2}{(\lambda_0^2 t)^l} \right). \quad (61)$$

Proof. The boundedness of $\| (\mathbf{1}_{\Sigma'} - C_{w^a})^{-1} \|_{\mathcal{N}(\Sigma')}$ follows from the assertion of the Lemma and identity (60): the remainder is a consequence of Proposition 5.1. 

From Proposition 5.2, it is clear that the asymptotic expansion (to $O\left( \frac{\lambda^2}{(\lambda_0^2 t)^l} \right)$, $l \in \mathbb{Z}_{\geq 1}$) can be constructed by means of the following RH problem on the contour $\Sigma'$:

$$m_{\Sigma'}(x,t;\lambda) = m_{\Sigma'}(x,t;\lambda) w_{x,t,\delta}(\lambda), \quad \lambda \in \Sigma',$$

$$m_{\Sigma'}(x,t;\lambda) \to 1 \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in \mathbb{C} \setminus \Sigma'.$$  

(62)
where \( v_{x,t,\delta}^{\Sigma}(\lambda) = (I - (w_{x,t}^{\Sigma})_{x,t,\delta})^{-1}(I + (w_{x,t}^{\Sigma})_{x,t,\delta}) \), with

\[
(w_{x}^{\Sigma})_{x,t,\delta} = (\delta(\lambda))^{\text{ad}(\sigma_3)}e^{-it\theta(\lambda)\text{ad}(\sigma_3)}R(\lambda)\sigma_+ \quad \text{and} \quad (w_{x,t}^{\Sigma})_{x,t,\delta} = 0, \quad \lambda \in L \setminus L_{\delta}, \quad \delta(\lambda) \equiv \text{ad}(\sigma_3)R(\lambda)\sigma_+ \quad \lambda \in \mathbb{C} \setminus \Sigma, \quad (63)
\]

Denote \((w_{x,t}^{\Sigma})_{x,t,\delta} = (w_{x}^{\Sigma})_{x,t,\delta} + (w_{t}^{\Sigma})_{x,t,\delta}, so that \((w_{x,t}^{\Sigma})_{x,t,\delta} = w'_{\Sigma}; then, according to Theorem 3.1, the solution of RH problem (62) has the following integral representation,

\[
m^{\Sigma}(x,t;\lambda) = I + \int_{\Sigma^{'}} \frac{\mu^{\Sigma}(x,t;\xi)(w_{x,t}^{\Sigma}(\xi))_{x,t,\delta}}{(\xi - \lambda)} \frac{d\xi}{2\pi i}, \quad \lambda \in \mathbb{C} \setminus \Sigma,
\]

where \( \mu^{\Sigma}(x,t;\lambda) \equiv (I - C_{\Sigma})^{-1} \).

**Remark 5.2.** In (64), \( R(\lambda) \) is the same piecewise-rational function \( R(\lambda) \) appearing in (63), except with the complex conjugated coefficients.

## 6 RH Problem on the Disjoint Crosses

In this section, we make a further simplification of the RH problem on the truncated contour \( \Sigma' \) by reducing it to the one which is stated on the three disjoint crosses, \( \Sigma_A, \Sigma_B', \) and \( \Sigma_C' \), and prove that the leading term of the asymptotic expansion for \( P(x,t) \) (Proposition 5.2, (61)) can be written as the sum of three terms corresponding to the solutions of three auxiliary RH problems, each of which is set on one of the crosses; moreover, the solution of the latter RH problem can be presented in terms of an exactly solvable model matrix RH problem, which is studied in the next section. At the end of this section, we also prove the basic bound on \( (I - C_{\Sigma})^{-1} \) (Proposition 5.2).

Let us prepare the notations which are needed for exact formulations. Write \( \Sigma' \) as the disjoint union of the three crosses, \( \Sigma_A, \Sigma_B', \) and \( \Sigma_C' \), extend the contours \( \Sigma_A, \Sigma_B', \) and \( \Sigma_C' \) (with orientations unchanged) to the following ones,

\[
\hat{\Sigma}_A = \{\lambda; \lambda = -\lambda_0 + \frac{\lambda_0}{2}e^{\pm it}, \ u \in \mathbb{R}\}, \quad \hat{\Sigma}_B = \{\lambda; \lambda = \lambda_0 + \frac{\lambda_0}{2}e^{\pm it}, \ u \in \mathbb{R}\},
\]

\[
\hat{\Sigma}_C = \{\lambda; \lambda = \frac{\lambda_0}{2}e^{\pm it}, \ u \in \mathbb{R}\},
\]

and define by \( \Sigma_A, \Sigma_B, \) and \( \Sigma_C, \) respectively, the contours \{\( \lambda; \lambda = \frac{\lambda_0}{2}e^{\pm it}, \ u \in \mathbb{R}\) oriented inward as in \( \Sigma_A \) and \( \hat{\Sigma}_A \), inward as in \( \Sigma_B' \) and \( \hat{\Sigma}_B \), and inward/outward as in \( \Sigma_C' \) and \( \hat{\Sigma}_C \).

For \( k \in \{A, B, C\} \), introduce the following operators,

\[
N_k: L^2(\hat{\Sigma}_k') \to L^2(\Sigma_k), \quad f(\lambda) \mapsto (N_k f)(\lambda) = f(\lambda_k + \varepsilon_k), \quad (65)
\]

where

\[
\lambda_A = -\lambda_0, \quad \lambda_B = \lambda_0, \quad \lambda_C = 0, \quad \varepsilon_A = \varepsilon_B = \lambda(16\lambda_0^2 t)^{-1/2}, \quad \varepsilon_C = \lambda(8\lambda_0^2 t)^{-1/2}. \quad (66)
\]

Considering the action of the operators \( N_k \) on \( \delta(\lambda)e^{-it\theta(\lambda)} \), we find that, for \( k \in \{A, B, C\} \), \( I_A \equiv (-\infty, \lambda_A), \ I_B \equiv (\lambda_B, +\infty), \) and \( I_C \equiv (-\lambda_0, +\lambda_0), \)

\[
N_k \{\delta(\lambda)e^{-it\theta(\lambda)}\} = \delta_k^0 \delta_k^0(\lambda), \quad \mathbb{R}(\lambda) \in I_k,
\]

where, as a result of the second of (15), (41), and (42),

\[
\delta_k^0 = (16\lambda_0^4 t)^{-1} \exp \left\{ 2i\lambda_0^4 t + \sum_{m \in M} \chi_m(\lambda_0) \right\}, \quad l \in \{A, B\}, \quad \delta_k^0 = \exp \left\{ \sum_{m \in M} \chi_m(0) \right\}, \quad (67)
\]

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\[ \delta^1_l(\lambda) = \frac{(\lambda \rho_0)^{\nu}(\varepsilon_l + 2\lambda_0)^{\nu}}{(\varepsilon_l + \lambda_0)^{2\nu}} \exp \left\{ -\frac{i\lambda^2}{2} (1 + \frac{\varepsilon_l}{\lambda_0})^2 + \sum_{m \in M} (\chi_m (\lambda_l + \varepsilon_l) - \chi_m (\lambda_l)) \right\}, \]  
(68)

\[ \delta^1_C(\lambda) = (\varepsilon_0^2 - \lambda_0^2)^{\nu} \exp \left\{ \frac{i\lambda^2}{2} (1 - \frac{4\varepsilon_0^2}{\lambda_0}) + \sum_{m \in M} (\chi_m (\varepsilon_C) - \chi_m (0)) \right\}, \]  
(69)

with \( M \equiv \{ A, B, +, - \} \), and

\[ \chi_k(\lambda_l) = \frac{i}{2\pi} \int_0^{\lambda_l} \ln \left( \frac{1 - |\mu|^2}{1 - \lambda_l |\mu|^2} \right) \frac{d\mu}{(\mu - \lambda_l)}, \quad \chi_{\pm}(\lambda_l) = \frac{i}{2\pi} \int_{\pm\infty}^{0} \ln |i\mu - \lambda_l| d\ln (1 + |\mu|^2), \]  
(70)

\[ \chi_k(0) = \frac{i}{2\pi} \int_0^{\lambda_0} \ln |\mu| d\ln (1 - |\mu|^2), \quad \chi_{\pm}(0) = \int_{\pm\infty}^{0} \frac{\ln (1 + |\mu|^2)}{\mu} \frac{d\mu}{2\pi i}. \]

Set

\[ \Delta^0_k = (\delta^0_k)^{\sigma_3}, \]  
(71)

and let \( \tilde{\Delta}^0_k \) denote right multiplication by \( \Delta^0_k \):

\[ \tilde{\Delta}^0_k \phi \equiv \phi \Delta^0_k. \]  
(72)

Denote

\[ w^{k'}(\lambda) = \begin{cases} (w^{\Sigma'})_{x, t, \delta}, & \lambda \in \Sigma_{k'}, \\ 0, & \lambda \in \Sigma' \setminus \Sigma_{k'} \end{cases} \quad \text{and} \quad \tilde{w}^{k'}(\lambda) = \begin{cases} w^{k'}, & \lambda \in \Sigma_{k'}, \\ 0, & \lambda \in \tilde{\Sigma}_k \setminus \Sigma_{k'} \end{cases}. \]  
(73)

According to this,

\[ (w^{\Sigma'})_{x, t, \delta} = \sum_{k \in \{ A, B, C \}} w^{k'}, \quad C^{\Sigma'}_{(w^{B'})_{x, t, \delta}} = \sum_{k \in \{ A, B, C \}} C^{\Sigma'}_{w^{k'}} \equiv \sum_{k \in \{ A, B, C \}} C^{\Sigma'}_{w^{k'}}: \]  
(74)

hereafter, we do not introduce a special notation for \( w^{k'}|_{\Sigma_{k'}} \). Let us prove some technical results concerning the operators \( C^{\Sigma'}_{w^{k'}} \) and \( C^{\Sigma'}_{\tilde{w}^{k'}} \).

**Proposition 6.1** For \( k \in \{ A, B, C \} \),

\[ C^{\Sigma'}_{\tilde{w}^{k'}} = (N_k)^{-1}(\Delta^0_k)^{-1}C^{\Sigma'}_{w^{k'}}(\Delta^0_k)N_k, \quad w^{k'} \equiv (\Delta^0_k)^{-1}(N_k \tilde{w}^{k'})(\Delta^0_k), \]  
(75)

where

\[ C^{\Sigma_k}_{w^{k'}}|_{L_2(2 \times 2)(L_k)} = -C_+((R(\varepsilon_k + \lambda_k))(\delta^1_k(\lambda)^{-2}\sigma_-)), \]  
\[ C^{\Sigma_k}_{\tilde{w}^{k'}}|_{L_2(2 \times 2)(L_k)} = C_-((R(\varepsilon_k + \lambda_k))(\delta^1_k(\lambda)^2\sigma_+)). \]  
(76)

Here, the rays \( L_k \) are defined as follows,

\[ L_l \equiv \{ \lambda; \lambda = \frac{\lambda_0}{2}(16\lambda_0^2t)^{1/2} e^{\frac{\pi i}{4}}, \ u \in (-\varepsilon, +\infty) \}, \quad l \in \{ A, B \}, \]  
\[ L_C \equiv \{ \lambda; \lambda = \frac{\lambda_0}{2}(8\lambda_0^2t)^{1/2} e^{\frac{\pi i}{4}}, \ u \in \mathbb{R} \}, \]

so that \( \Sigma_{k'} = L_k \cup \overline{L_k} \).

**Proof.** We consider the case \( k = B \): the cases \( k = A \) and \( C \) follow in an analogous manner. Since, from the first of (67), \( [\delta^0_B] = 1 \), it follows from the definition of the operator \( \tilde{\Delta}^0_B \) in (72) that \( (\tilde{\Delta}^0_B)^\dagger = (\Delta^0_B)^{-1} \), where \( \dagger \) denotes Hermitian conjugation \((\Delta^0_B \text{ is a unitary operator})\). Changing variables \((65)\), recalling \((12)\), using \((71)\) and \((72)\), as well as the unitarity of \( \tilde{\Delta}^0_B \), one obtains (75), where \( C^{\Sigma_B}_{w^{B'}} = C_{(\Delta^0_B)^{-1}(N_B \tilde{w}^{B'})(\Delta^0_B)} \)

\[ = C_+((\Delta^0_B)^{-1}(N_B \tilde{w}^{B'})(\Delta^0_B)) + C_-((\Delta^0_B)^{-1}(N_B \tilde{w}^{B'})(\Delta^0_B)). \]

Using \((63)\) and \((64)\), one shows that \((\Delta^0_B)^{-1}(N_B \tilde{w}^{B'})(\Delta^0_B) = 0 \) on \( L_B \) and \((\Delta^0_B)^{-1}(N_B \tilde{w}^{B'})(\Delta^0_B) = 0 \) on \( L_B \), so that \((76)\) are valid. \( \blacksquare \)
Lemma 6.1 Let \( \kappa \in (0,1) \), \( p(B) = -p(A) = 1 \), \( p(C) = -\lambda_0^2/(2\lambda) \), \( \text{sgn}(A) = \text{sgn}(B) = -\text{sgn}(C) = 1 \). Then \( \forall \lambda \in L_k \subset \Sigma_k \), as \( t \to +\infty \) such that \( \lambda_0 > M \),

\[
|R(\varepsilon_k + \lambda_k)(\delta_k^1(\lambda)) - 2 - R(\lambda_0^k)(2\lambda p(k))^{-2i\text{sgn}(k)}| \leq \frac{c(k)\ln t}{\sqrt{\lambda_0^t}} e^{-2\kappa \lambda_0^3 t u^2},
\]

and \( \forall \lambda \in L_k \subset \Sigma_k \),

\[
|R(\varepsilon_k + \lambda_k)(\delta_k^1(\lambda)) - 2 - R(\lambda_0^k)(2\lambda p(k))^{-2i\text{sgn}(k)}| \leq \frac{c(k)\ln t}{\sqrt{\lambda_0^t}} e^{-2\kappa \lambda_0^3 t u^2},
\]

where \( L_k \) (resp. \( \overline{L}_k \)), \( k \in \{A, B, C\} \), are defined in Proposition 6.1, \( u \in (-\varepsilon, +\infty) \), with \( 0 < \varepsilon < \sqrt{2} \), \( R(\lambda_0^+) = \lim_{\Re(\lambda) \downarrow \lambda_0} R(\lambda) = -r(\lambda_0), R(\lambda_0^-) = \lim_{\Re(\lambda) \uparrow \lambda_0} R(\lambda) = r(\lambda_0)(1 - |r(\lambda_0)|^2)^{-1}, \)

\[
R(\lambda_0^+) = \lim_{\Re(\lambda) \downarrow \lambda_0} R(\lambda) = -r(\lambda_0)(1 - |r(\lambda_0)|^2)^{-1}, \quad R(\lambda_0^-) = \lim_{\Re(\lambda) \uparrow \lambda_0} R(\lambda) = r(\lambda_0),
\]

\( R(0^+) = 0 \) since \( r(0) = r(\pm 0) = 0 \), \( c(B) = e^S, c(A) = e^S, \) and \( c(C) = e \).

Proof. We prove inequality (77) for \( k = B \): the other results follow in an analogous manner. By using (68), for \( \lambda \in L_B \subset \Sigma_B \) and \( \kappa \in (0,1) \), write

\[
R(\lambda_0 + \varepsilon_B)(\delta_B^1(\lambda))^{-2} - R(\lambda_0^+)(2\lambda)^{-2i\lambda} = e^{\frac{i\lambda^2}{2}} (I + II + III),
\]

where

\[
I = (2\lambda)^{-2i\lambda} e^{i(1-\frac{\varepsilon}{2})\lambda^2} \left( R(\lambda_0 + \varepsilon_B) - R(\lambda_0^+) \right),
\]

\[
II = (2\lambda)^{-2i\lambda} e^{i(1-\frac{\varepsilon}{2})\lambda^2} \left( \frac{\lambda_0(\lambda_0 + \varepsilon_B/2)}{(\lambda_0 + \varepsilon_B)^2} \right)^{-2i\lambda} - 1,
\]

\[
III = (2\lambda)^{-2i\lambda} e^{i(1-\frac{\varepsilon}{2})\lambda^2} \frac{\lambda_0(\lambda_0 + \varepsilon_B/2)}{(\lambda_0 + \varepsilon_B)^2} \left( \frac{\lambda_0(\lambda_0 + \varepsilon_B/2)}{(\lambda_0 + \varepsilon_B)^2} \right)^{-2i\lambda} \left( 1 - \sum_{m \in \mathcal{M}} \frac{\chi_m(\lambda_0 + \varepsilon_B) - \chi_m(\lambda_0)}{\lambda_0} \right),
\]

and, in the last formula, \( \mathcal{Z} \equiv i\lambda^2 \left( \frac{\varepsilon_B^2}{4\lambda_0^2} + \frac{\varepsilon_B}{\lambda_0} \right) - 2 \sum_{m \in \mathcal{M}} (\chi_m(\lambda_0 + \varepsilon_B) - \chi_m(\lambda_0)). \) Note that

\[
|\exp\{i\lambda^2\} | = |\exp\{-2\kappa \lambda_0^3 t u^2\} |,
\]

which gives the exponential factor in (77). The terms \( I, II, \) and \( III \) can be estimated in the following way.

\[
|I| \leq \left| (2\lambda)^{-2i\lambda} \right| \left| e^{i(1-\frac{\varepsilon}{2})\lambda^2} \right| \varepsilon_B \sup_{\lambda \in L_B} \left| \frac{\partial \lambda R(\lambda)}{\lambda} \right| \leq e^{\frac{2\pi}{4\lambda_0^2 t}} \frac{|\lambda|^2}{\sqrt{16\lambda_0^2 t}} \left| \frac{\lambda_0}{\lambda} \right| \left| \frac{\lambda_0}{\lambda_0 + \varepsilon_B} \right| \left| \frac{\lambda_0}{\lambda_0 + \varepsilon_B} \right| \left| II_1 + II_2 \right|,
\]

where \( \xi \) is independent of \( \lambda \).

\[
|II| \leq e^{\frac{2\pi}{4\lambda_0^2 t}} \left| \frac{\lambda_0}{\lambda_0 + \varepsilon_B} \right| \left| \frac{\lambda_0}{\lambda_0 + \varepsilon_B} \right| \left| \frac{\lambda_0}{\lambda_0 + \varepsilon_B} \right| \left| II_1 + II_2 \right|,
\]

where

\[
II_1 = \left( \frac{\lambda_0(\lambda_0 + \varepsilon_B/2)}{(\lambda_0 + \varepsilon_B)^2} \right)^{-2i\lambda} \left\{ 1 - \left( \frac{\lambda_0}{\lambda_0 + \varepsilon_B} \right)^{-2i\lambda} \right\},
\]

\[
II_2 = \left( \frac{\lambda_0}{\lambda_0 + \varepsilon_B} \right)^{-2i\lambda} - 1.
\]

To estimate \( |II_2| \), one proceeds as follows:

\[
e^{-\left(1 - \frac{\varepsilon}{2}\right)|\lambda|^2} |II_2| \leq e^{-\left(1 - \frac{\varepsilon}{2}\right)|\lambda|^2} \left| \int_1^{1 + \varepsilon_B/\lambda_0} \xi^{4i\lambda - 1}(4i\lambda) d\xi \right|
\]

\[
\leq 4\varepsilon e^{-\left(1 - \frac{\varepsilon}{2}\right)|\lambda|^2} \sup_{\xi \in [0,1]} |\xi^{2i\lambda - 1}|; \xi = 1 + \frac{s}{\lambda_0} \varepsilon_B, s \in [0,1], \frac{\varepsilon_B}{\lambda_0} \leq \frac{\varepsilon}{\lambda_0} \xi^{-1/2},
\]

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since $|q^{2|\nu-1|}| \leq \sqrt{q} \exp(-2\nu \arg(\xi))$ and $-\frac{3\pi}{q} < \arg(\xi) < \frac{\pi}{q}$: in this estimation, one uses the fact that $|\xi| \geq \sqrt{q}$ and $0 < \nu \leq \nu_{\max} \equiv -\frac{1}{2\pi} \ln(1-\frac{\im\mu}{\im\mu} r(\mu) r(\mu)) < \infty$. Since the first term on the right-hand side of the equation for $II_1$ is bounded, namely,

$$\left| \left( \frac{\lambda_0(\lambda_0 + \epsilon B/2)}{(\lambda_0 + \epsilon B)^2} \right)^{-2\nu} \right| \leq e^{\frac{2\pi}{q}},$$

one gets an analogous estimate for $|II_1|$.

$$|III| \leq e^{\pi \nu} \left\| \frac{\sqrt{q}\ln(L_B)}{\mathcal{L}^\infty(L_B)} e^{-(1-\frac{q}{2})}|\lambda|^2 \sup_{0 \leq s \leq 1} \left| \frac{d}{ds} e^s z \right| \leq C e^{-(1-\frac{q}{2})}|\lambda|^2 |z| \sup_{0 \leq s \leq 1} |e^s z|$$

$$\leq C e^{-(1-\frac{q}{2})|\lambda|^2} |z| \leq C e^{-(1-\frac{q}{2})|\lambda|^2} \left( \frac{\lambda^4}{64\lambda_0^4 t} + \frac{\lambda^3}{\sqrt{16\lambda_0^4 t}} + O \left( \frac{c^2 \ln(t)}{\sqrt{\lambda_0^4 t}} \right) \right),$$

where one uses the boundedness of $\exp\left\{ \sum_{m \in M} (\chi_m(\lambda_0 + \epsilon B) - \chi_m(\lambda_0)) \right\}$, which follows from (44), (67), and (68). The term $O(c^2 (\lambda_0 t)^{-1/2} \ln(t))$ appears in $|III|$ due to the estimation $\chi_l(\lambda_0 + \epsilon B) - \chi_l(\lambda_0), l \in \{A, B\}$, which can be obtained by using the Lipschitz property of the function $\ln(\lambda_0 + \epsilon B)$, $|\mu| < \lambda_0$, and integrating by parts the first of (70): analogous estimations for $\chi_\pm(\lambda_0 + \epsilon B) - \chi_{\pm}(\lambda_0)$ are $O(c^2 (\lambda_0 t)^{-1/2})$.

**Proposition 6.2** For general operators $C_{\omega k'}$, $k \in \{1, 2, \ldots, N\}$, if $(\text{Id}_{\Sigma'} - C_{\omega k'})^{-1}$ exist, then

$$(\text{Id}_{\Sigma'} + \sum_{1 \leq \alpha \leq N} C_{\omega k'}(\text{Id}_{\Sigma'} - C_{\omega k'})^{-1})(\text{Id}_{\Sigma'} - \sum_{1 \leq \beta \leq N} C_{\omega k'}) = \text{Id}_{\Sigma'} - \sum_{1 \leq \alpha \leq N} \sum_{1 \leq \beta \leq N} (1 - \delta_{\alpha\beta})(\text{Id}_{\Sigma'} - C_{\omega k'})^{-1} C_{\omega k'} C_{\omega k'},$$

and

$$(\text{Id}_{\Sigma'} - \sum_{1 \leq \beta \leq N} C_{\omega k'})(\text{Id}_{\Sigma'} + \sum_{1 \leq \alpha \leq N} \sum_{1 \leq \beta \leq N} (1 - \delta_{\alpha\beta}) C_{\omega k'} C_{\omega k'}(\text{Id}_{\Sigma'} - C_{\omega k'})^{-1}) = \text{Id}_{\Sigma'} - \sum_{1 \leq \alpha \leq N} \sum_{1 \leq \beta \leq N} (1 - \delta_{\alpha\beta}) C_{\omega k'} C_{\omega k'} C_{\omega k'}(\text{Id}_{\Sigma'} - C_{\omega k'})^{-1},$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

**Proof.** Assumption of existence of general operators $(\text{Id}_{\Sigma'} - C_{\omega k'})^{-1}$, $k \in \{1, 2, \ldots, N\}$, induction, and a straightforward application of the second resolvent identity. ■

**Lemma 6.2** For $\alpha \neq \beta \in \{A', B', C'\}$, as $t \to +\infty$ such that $\lambda_0 > M$, $\|C_{\omega k'}(\Sigma')\|_{\mathcal{N}(\Sigma')} \leq \frac{C}{\lambda_0 t}, \|C_{\omega k'}(\Sigma')\|_{\mathcal{L}^{(2,2)}(\Sigma')} \leq \frac{C}{(\lambda_0 t)^{1/4} \sqrt{\lambda_0 t}}$.

**Proof.** Analogous to Lemma 3.5 in [13]. ■

**Lemma 6.3** If, for $k \in \{A, B, C\}$, $(\text{Id}_{\Sigma'} - C_{\omega k'})^{-1} \in \mathcal{N}(\Sigma')$, then as $t \to +\infty$ such that $\lambda_0 > M$, $P(x, t) = -i \sum_{k \in \{A, B, C\}} \left( \int_{\Sigma'} \left[ \sigma_3 \left( (\text{Id}_{\Sigma'} - C_{\omega k'})^{-1} \right) \right]_{21} + O(\lambda^4_{0}) \right)$. (79)
Proof. Using Proposition 6.2 and the second resolvent identity, one writes
\[
(Id_{\Sigma'}) - \sum_{k \in \{A,B,C\}} C_{\omega k'}^{\Sigma'} = D_{\Sigma'} + E_{\Sigma'}(Id_{\Sigma'} - \bar{E}_{\Sigma'})^{-1},
\]
where
\[
D_{\Sigma'} \equiv Id_{\Sigma'} + \sum_{k \in \{A,B,C\}} C_{\omega k'}^{\Sigma'} (Id_{\Sigma'} - C_{\omega k'}^{\Sigma'})^{-1},
\]
\[
E_{\Sigma'} = \sum_{\alpha, \beta \in \{A,B,C\}} (1 - \delta_{\alpha \beta}) C_{\omega \alpha}^{\Sigma'} (Id_{\Sigma'} - C_{\omega \beta}^{\Sigma'})^{-1}.
\]
Since, from Lemma 5.1, \( ||C_{\omega k'}^{\Sigma'}||_{\mathcal{N}(\Sigma')} \leq ||w^{k'}||_{L^2(\Sigma')} \leq \frac{c}{(\lambda_0^2 t)^{1/4}}, \) and, by assumption \( ||(Id_{\Sigma'} - C_{\omega k'}^{\Sigma'})^{-1}||_{\mathcal{N}(\Sigma')} < \infty, \) via Lemma 6.2, one finds that, as \( t \to +\infty \) such that \( \lambda_0 > M, \)
\[
||D_{\Sigma'}||_{\mathcal{N}(\Sigma')} \leq \mathfrak{c} \quad \text{and} \quad ||(Id_{\Sigma'} - E_{\Sigma'})^{-1}||_{\mathcal{N}(\Sigma')} \leq \mathfrak{c}
\]
Taking account of the second resolvent identity, one also finds that,
\[
||E_{\Sigma'}||_{L^2(\Sigma')} \leq \sum_{\alpha, \beta \in \{A,B,C\}} (1 - \delta_{\alpha \beta}) ||C_{\omega \alpha}^{\Sigma'} (Id_{\Sigma'} - C_{\omega \beta}^{\Sigma'})^{-1}||_{L^2(\Sigma')} + \sum_{\alpha, \beta \in \{A,B,C\}} (1 - \delta_{\alpha \beta}) ||C_{\omega \alpha}^{\Sigma'} (Id_{\Sigma'} - C_{\omega \beta}^{\Sigma'})^{-1}||_{L^2(\Sigma')}||C_{\omega \beta}^{\Sigma'}||_{L^2(\Sigma')}.
\]
Noting from Lemma 5.1 that,
\[
||C_{\omega k'}^{\Sigma'}||_{L^2(\Sigma')} \leq ||w^{k'}||_{L^2(\Sigma')} \leq \frac{c}{(\lambda_0^2 t)^{1/4}},
\]
one finds, using Lemma 6.2, and applying once more the second resolvent identity,
\[
||E_{\Sigma'}||_{L^2(\Sigma')} \leq \frac{c}{\lambda_0^{3/2}}.
\]
Now, from the Cauchy-Schwarz inequality and Lemma 5.1,
\[
||E_{\Sigma'} w^{\Sigma'}||_{L^1(\Sigma')} \leq ||E_{\Sigma'}||_{L^2(\Sigma')} ||w^{\Sigma'}||_{L^2(\Sigma')} \leq \frac{c}{\lambda_0^{3/2} (\lambda_0^2 t)^{1/4}} \leq \frac{\mathfrak{c}}{\lambda_0^2};
\]
hence, recalling (81), \( (Id_{\Sigma'} - C_{\omega k'}^{\Sigma'})^{-1} \in \mathcal{N}(\Sigma'). \) From Proposition 5.2 and the above estimates as \( t \to +\infty \) such that \( \lambda_0 > M, \)
\[
\int_{\Sigma'} ((Id_{\Sigma'} - C_{\omega k'}^{\Sigma'})^{-1} I) w^{\Sigma'}(\xi) d\xi = \int_{\Sigma'} (D_{\Sigma'} I) w^{\Sigma'}(\xi) d\xi + O(\frac{\mathfrak{c}}{\lambda_0 t}) + O\left(\frac{\mathfrak{c}}{(\lambda_0^2 t)^{1/4}}\right), \quad (81)
\]
for arbitrary \( l \in \mathbb{Z}_{\geq 1}. \) Recalling that \( w^{\Sigma'} = \sum_{k \in \{A,B,C\}} w^{k'}, \) the integral on the right-hand side of (81) can be written as follows:
\[
\int_{\Sigma'} (D_{\Sigma'} I) w^{\Sigma'}(\xi) d\xi = \int_{\Sigma'} (Id_{\Sigma'} \sum_{k \in \{A,B,C\}} w^{k'} + \sum_{\alpha, \beta \in \{A,B,C\}} C_{\omega \alpha}^{\Sigma'} (Id_{\Sigma'} - C_{\omega \beta}^{\Sigma'})^{-1} I w^{\Sigma'}(\xi) d\xi.
\]
To estimate the right-hand side of (82), consider, say, the following integral, \( \int_{\Sigma'} (C_{\omega A'}^{\Sigma'} (Id_{\Sigma'} - C_{\omega A'}^{\Sigma'})^{-1} I) w^{\Sigma'}(\xi) d\xi: \)
\[
| \int_{\Sigma'} (C_{\omega A'}^{\Sigma'} (Id_{\Sigma'} - C_{\omega A'}^{\Sigma'})^{-1} I) w^{\Sigma'}(\xi) d\xi | \leq | \int_{\Sigma'} (C_{\omega A'}^{\Sigma'} I) (\xi) w^{\Sigma'}(\xi) d\xi |.
\]
\[ + \int_{\Sigma'} \left( C_{w_{A'}}^\Sigma \frac{I}{I} - C_{w_{A'}}^\Sigma \right)^{-1} C_{w_{A'}}^\Sigma \frac{I}{I} \right) w(B) (\xi) d\xi \leq \int_{\Sigma'} \left( \int \frac{w^B (\eta) d\eta}{2\pi i} \right) w(B) (\xi) d\xi \]
\[ + \int_{\Sigma'} \left( \int \frac{C_{w_{A'}}^\Sigma w_{A'}^\Sigma \frac{I}{I} (\eta) w_{A'}^\Sigma (\eta) \frac{d\eta}{2\pi i}}{2\pi \text{dist}(\Sigma_A', \Sigma_B')} \right) w(B) (\xi) d\xi \leq \frac{\|w^A\|_{L^1_1(\Sigma_A')} \|w^B\|_{L^1_1(\Sigma_B')}}{2\pi \text{dist}(\Sigma_A', \Sigma_B')} \]
\[ + \left( \frac{\|C_{w_{A'}}^\Sigma w_{A'}^\Sigma \|_{L^2_{2}(\Sigma_A')} \|w^A\|_{L^2_{2}(\Sigma_A')} \|w^B\|_{L^2_{2}(\Sigma_B')}}{2\pi \text{dist}(\Sigma_A', \Sigma_B')} \right) \]
\[ \leq \frac{\xi_{00}}{\lambda_0} \|w^A\|_{L^2_{2}(\Sigma_A')} \|w^B\|_{L^2_{2}(\Sigma_B')} \]
\[ + \frac{\xi_{00}}{\lambda_0} \|I_{w_{A'}}^\Sigma - C_{w_{A'}}^\Sigma\|^{-1} \|N(\Sigma_{A'})\| \|C_{w_{A'}}^\Sigma \|_{L^2_{2}(\Sigma_A')} \|w^A\|_{L^2_{2}(\Sigma_A')} \|w^B\|_{L^2_{2}(\Sigma_B')} \]

since 2\pi \text{dist}(\Sigma_A', \Sigma_B') \geq \lambda_0/\xi_{00}$. Hence, by Lemma 6.2 and the assumption that $||I_{w_{A'}}^\Sigma - C_{w_{A'}}^\Sigma\|^{-1} \|N(\Sigma_{A'})\| < \infty$, one gets, for $\alpha \neq \beta \in \{A, B, C\}$, the following estimation:
\[ |\int_{\Sigma'} \left( C_{w_{A'}}^\Sigma \frac{I}{I} - C_{w_{A'}}^\Sigma \right)^{-1} I(\xi) w^\beta (\xi) d\xi| \leq \frac{\xi_{00}}{\lambda_0} \|w^\beta\|_{L^1_1(\Sigma')}^2 \]
\[ + \frac{\xi_{00}}{\lambda_0} \|I_{w_{A'}}^\Sigma - C_{w_{A'}}^\Sigma\|^{-1} \|N(\Sigma_{A'})\| \|C_{w_{A'}}^\Sigma\|_{L^2_{2}(\Sigma_A')} \|w^\beta\|_{L^2_{2}(\Sigma_A')}^2 \|w^\beta\|_{L^1_1(\Sigma')} \]

therefore, applying the second resolvent identity to the right-hand side of (82), one finds that
\[ \int_{\Sigma'} \left( I_{w_{A'}}^\Sigma - C_{w_{A'}}^\Sigma \right)^{-1} I(\xi) w^\beta (\xi) d\xi = \sum_{k \in \{A, B, C\}} \sum_{\nu \in \{1, 2\}} \int_{\Sigma'} \left( I_{w_{A'}}^\Sigma - C_{w_{A'}}^\Sigma \right)^{-1} I(\xi) w^\beta (\xi) d\xi \]
\[ + O\left( \frac{\xi}{\lambda_0} \right) + O\left( \frac{\xi}{\lambda_0^2} \right) \]
(83)

for arbitrary $l \in \mathbb{Z}_{\geq 1}$. Now, from Definition 5.2 and Lemma 5.3 (identity (60)),
\[ \left( I_{w_{A'}}^\Sigma - C_{w_{A'}}^\Sigma \right)^{-1} = R_{\Sigma_{k'}} \left( I_{w_{A'}}^\Sigma - C_{w_{A'}}^\Sigma \right)^{-1} \mathbb{1}_{\Sigma_{k'} \to \Sigma} \]
(84)
hence, substituting identity (84) into (83), and recalling (80) and (81), the proof is complete.
\[ \square \]

**Lemma 6.4** For $k \in \{A, B, C\}$, \( \left( I_{w_{A'}}^\Sigma - C_{w_{A'}}^\Sigma \right)^{-1} \in \mathcal{N}(\Sigma_{k'}) \).

**Remark 6.1.** This result was proved in [13]: here, we briefly reproduce this proof since we need a model RH problem which arises in it in order to obtain the explicit asymptotic formulae presented in Theorem 3.2.

**Proof.** Consider, say, the case $k = B$: the cases $k = A$ and $C$ follow in an analogous manner. Define the function (see Fig. 5(a))
\[ w_{\pm}^B (\lambda) \equiv ((\Delta_0^B) - (N_B w_{\pm}^B) (\Delta_0^B)) (\lambda) = \pm \hat{R} (\lambda_0) (2\lambda)^{\pm 2i\nu} e^{\pm i\lambda^2} \sigma_{\pm} \]
where $\hat{w}_{B}^\pm (\lambda)$ is given in (73), and
\[ \hat{R} (\lambda_0) = \begin{cases} R(\lambda_0^1), & \lambda \in \Sigma_2^1, \\ R(\lambda_0^3), & \lambda \in \Sigma_3^1, \end{cases} \] and
\[ \hat{R}^{-1} (\lambda_0) = \begin{cases} \frac{R(\lambda_0^1)}{R(\lambda_0^3)}, & \lambda \in \Sigma_2^1, \\ \frac{R(\lambda_0^3)}{R(\lambda_0^1)}, & \lambda \in \Sigma_3^1. \end{cases} \]
Now, defining as usual $w^B = w^B_+ + w^B_-$, and using Lemma 6.1, one finds that
\[ \|w^B - w^B_\pm\|_N = ||(\Delta_0^B) - (N_B \hat{w}_B^\pm) (\Delta_0^B) - (\Delta_0^B) - (N_B \hat{w}_B^\pm) (\Delta_0^B)||_N \leq \frac{c^2 \ln t}{\sqrt{\lambda_0^2 t^4}} \]
(85)
where \( R \equiv L_p^{(2 \times 2)}(\Sigma_B), p \in \{1, 2, \infty\} \). Hence, as \( t \to +\infty \) such that \( \lambda_0 > M \),

\[
||C_{w}^{\Sigma B} - C_{wB}^{\Sigma B}||_{L_2^{(2 \times 2)}(\Sigma_B)} \leq \frac{e^{\delta \ln t}}{\sqrt{\lambda_0 t}},
\]

and, consequently (from the second resolvent identity), one sees that, for sufficiently large \( t \), \( (\text{Id}_{\Sigma_B} - C_{wB}^{\Sigma B})^{-1} \in \mathcal{N}(\Sigma_B) \Rightarrow (\text{Id}_{\Sigma_B} - C_{wB}^{\Sigma B})^{-1} \in \mathcal{N}(\Sigma_B) \). Reorient \( \Sigma_B \) as in Fig. 5(b) and define \( w^{B,r}(\lambda) = w^+_{B,r}(\lambda) + w^-_{B,r}(\lambda) \), where

\[
w^+_{B,r}(\lambda) = w^+_{B,r}(\lambda), \quad \Re(\lambda) > 0, \quad \text{and} \quad w^-_{B,r}(\lambda) = -w^-_{B,r}(\lambda), \quad \Re(\lambda) < 0,
\]

so that one can consider the operator \( C_{wB}^{\Sigma B,r} = C_{+}(w^-_{B,r}) + C_{-}(w^+_{B,r}) \), where the Cauchy operators are now taken with respect to \( \Sigma_{B,r} \). Extend \( \Sigma_{B,r} \to \Sigma_{ex} \equiv \Sigma_{B,r} \cup \mathbb{R} \), with the orientation in Fig. 5(c), and set

\[
w^{ex}(\lambda) \equiv \begin{cases} w^{B,r}(\lambda), & \lambda \in \Sigma_{B,r}, \\ 0, & \lambda \in \Sigma_{ex} \setminus \Sigma_{B,r}. \end{cases}
\]

From Lemma 5.3, it follows that \( (\text{Id}_{\Sigma_{ex}} - C_{w^{ex}}^{\Sigma B})^{-1} \in \mathcal{N}(\Sigma_{ex}) \Rightarrow (\text{Id}_{\Sigma_{B,r}} - C_{wB}^{\Sigma B})^{-1} \in \mathcal{N}(\Sigma_{B,r}) \Rightarrow (\text{Id}_{\Sigma_{B}} - C_{wB}^{\Sigma B})^{-1} \in \mathcal{N}(\Sigma_{B}) \). Define a piecewise analytic \( 2 \times 2 \) matrix-valued function \( \phi(\lambda) \) as follows,

\[
\phi(\lambda) \equiv \begin{cases} (2\lambda)^{-iv_{\sigma_3}}, & \lambda \in \Omega_5^5 \cup \Omega_5^6, \\ (2\lambda)^{-iv_{\sigma_3}}(I - w^{ext}(\lambda))^{-1}, & \lambda \in \Omega_4^5 \cup \Omega_4^6, \\ (2\lambda)^{-iv_{\sigma_3}}(I + w^{ext}(\lambda))^{-1}, & \lambda \in \Omega_3^5 \cup \Omega_3^6, \end{cases}
\]

where \( I \pm w^{ext}(\lambda) \) are, respectively, the analytic continuation of \( I \pm w^{ex}(\lambda) \) into \( \Omega_4^5, \Omega_5^5, \Omega_4^6, \) and \( \Omega_3^6 \). From the estimates above for \( w^+_{B,r}(\lambda) \), one finds that,

\[
w^{ex}(\lambda) \to 0 \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in \Omega_4^5 \cup \Omega_4^6,
\]

\[
w^{ext}(\lambda) \to 0 \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in \Omega_3^5 \cup \Omega_3^6,
\]

For \( \lambda \in \Sigma_{ex} \), set \( V^{\pm,\phi}(\lambda) = \phi_{-}(\lambda)(I - w^{ex})^{-1}(I \pm w^{ext})\phi_{-}^{-1}(\lambda) \equiv (I - w^{ex,\phi})^{-1}(I \pm w^{ext,\phi}) \), where \( \phi_{\pm}(\lambda) \) are the non-tangential limits of \( \phi(\lambda) \) as \( \lambda \) approaches \( \Sigma_{ex} \) from the \( \pm \) side, respectively (Remark 2.1). For \( \lambda \in \Sigma_{B,r} \), \( V^{\pm,\phi}(\lambda) = 1 \), and, for \( \lambda \in \Sigma_{ex} \setminus \Sigma_{B,r} \), \( V^{\pm,\phi}(\lambda) = \phi_{-}(\lambda)\phi_{-}^{-1}(\lambda) \), whence, one gets that,

\[
V^{\pm,\phi}(\lambda) = \exp\left\{\frac{-i\lambda^2}{2}\text{ad}(\sigma_3)\right\}v(\lambda_0), \quad \Re(\lambda) < 0,
\]

\[
V^{\pm,\phi}(\lambda) = \exp\left\{\frac{-i\lambda^2}{2}\text{ad}(\sigma_3)\right\}v(\lambda_0), \quad \Re(\lambda) > 0,
\]

Figure 5: (a) \( \Sigma_B \); (b) \( \Sigma_{B,r} \); and (c) \( \Sigma_{ex} \equiv \Sigma_{B,r} \cup \mathbb{R} \).
where
\[ v(\lambda_0) = \begin{pmatrix} 1 - |r(\lambda_0)|^2 & r(\lambda_0) \\ -r(\lambda_0) & 1 \end{pmatrix}, \quad \text{det}(v(\lambda_0)) = 1. \tag{90} \]

Hence, the jump matrix, \( V^{e,\phi}(\lambda) \), can be characterised as follows:
\[ V^{e,\phi}(\lambda) = \begin{cases} I, \quad \lambda \in \Sigma_{B,r}, \\ \exp\{ -i\frac{\lambda^2}{2} \text{ad}(\sigma_3) \} v(\lambda_0), \quad \lambda \in \Sigma_{ex} \setminus \Sigma_{B,r}. \end{cases} \]

On \( \mathbb{R} \), one has that,
\[ V^{e,\phi}(\lambda) = (1 - w^{ex,\phi}_-)^{-1}(1 + w^{ex,\phi}_+) = (I - r(\lambda_0)e^{i\lambda^2 \sigma_3})(I + r(\lambda_0)e^{-i\lambda^2 \sigma_3}). \]

Set \( C_{e,\phi} = C_+(-w^{ex,\phi}_-) + C_-(w^{ex,\phi}_+ \right) \) as the associated operator on \( \Sigma_{ex} \), with \( w^{ex,\phi} = w^{ex,\phi}_+ + w^{ex,\phi}_- \). By Lemma 5.3, the boundedness of \( C_{e,\phi} \), i.e., \( ||(\text{Id}_{\Sigma_{ex}} - C_{e,\phi})^{-1}||_{\mathcal{N}(\Sigma_{ex})} \), follows from the boundedness of the operator \( C_{w^{ex,\phi}} : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R}) \) associated with the restriction of \( w^{ex,\phi} \) to \( \mathbb{R} \); but \( ||C_{w^{ex,\phi}||_{\mathcal{N}(\mathbb{R})}} \leq \sup_{\lambda \in \mathbb{R}} |r(\lambda_0)| \exp(-i\lambda^2) | \leq ||r||_{\mathcal{L}^\infty(\mathbb{R})} < 1, \)
and hence, by the second resolvent identity,
\[ \|(\text{Id}_{\mathbb{R}} - C_{w^{ex,\phi}})\|^{-1}_{\mathcal{N}(\mathbb{R})} \leq (1 - ||r||_{\mathcal{L}^\infty(\mathbb{R})})^{-1} \Rightarrow \]
\[ (\text{Id}_{\mathbb{R}} - C_{w^{ex,\phi}})^{-1} \in \mathcal{N}(\mathbb{R}) \Rightarrow (\text{Id}_{\Sigma_{ex}} - C_{\Sigma_{ex}})^{-1} \in \mathcal{N}(\Sigma_{ex}). \]

This completes the proof. \( \blacksquare \)

### 7 Model RH Problem

In this section, we reduce the evaluation of the integrals in Lemma 6.3 to three RH problems on \( \mathbb{R} \) which can be solved explicitly.

For \( k \in \{ A, B, C \} \), define
\[ m^k_0(\lambda) = 1 + \int_{\Sigma_k} \frac{(\text{Id}_{\Sigma_k} - C_{w^{k,0}})^{-1}(\xi)w^{k,0}(\xi)}{\xi - \lambda} \frac{d\xi}{2\pi i}, \quad \lambda \in \mathbb{C} \setminus \Sigma_k. \]

Now, from Theorem 3.1, we find that \( m^k_0(\lambda) \) solves the following RH problem,
\[ m^k_0(\lambda) = m^k_0(\lambda)(I - w^{k,0}_-)^{-1}(I + w^{k,0}_+), \quad \lambda \in \Sigma_k, \]
\[ m^k_0(\lambda) = I - \frac{m^k_0}{\lambda} + \mathcal{O}(\lambda^{-2}) \text{ as } \lambda \to \infty, \quad \lambda \in \mathbb{C} \setminus \Sigma_k. \]

Substituting into (79) of Lemma 6.3 inequalities (85) and (86) (and their analogues for \( \Sigma_A \) and \( \Sigma_C \)), we obtain
\[ P(x,t) = \frac{i}{\sqrt{4\pi t}} \left( (\delta_0^A)^{-2}(m_1^{A,0})_{21} + (\delta_0^B)^{-2}(m_2^{B,0})_{21} + \sqrt{2}(\delta_0^C)^{-2}(m_1^{C,0})_{21} \right) + \mathcal{O}\left( \frac{\text{int} t}{\lambda_0^4} \right). \tag{91} \]

We consider in detail only case \( B \). Let us introduce the function
\[ D(\lambda) = m^{B,0}(\lambda)(\phi(\lambda))^{-1}e^{-i\frac{\lambda^2}{4} \sigma_3}, \quad \lambda \in \mathbb{C} \setminus \Sigma_{ex}, \]
where \( \phi(\lambda) \) is defined in (87), and notice that it is holomorphic in \( \mathbb{C} \setminus \mathbb{R} \); in particular, it has no jumps across \( \Sigma_{ex} \setminus \mathbb{R} \). Across \( \mathbb{R} \), oriented from \(-\infty\) to \(+\infty\), using (87)–(90), one finds that \( \mathcal{D}(\lambda) \) solves the following RH problem:

\[
\mathcal{D}_+(\lambda) = \mathcal{D}_-(\lambda)e(\lambda_0), \quad \lambda \in \mathbb{R},
\]

\[
\mathcal{D}(\lambda) = ((I - \frac{m_1^{p_0}}{\lambda}) + \mathcal{O}(\lambda^{-2}))(2\lambda)^{i\nu}e^{\frac{i\lambda^2}{\lambda} - \sigma_3} \quad \text{as} \quad \lambda \to \infty, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (93)
\]

Since, according to (92) and (93), \( \det(\mathcal{D}(\lambda)) \) is a holomorphic, bounded in \( \mathbb{C} \) function, it is, by Liouville’s theorem, a constant. From (92), one finds that, \( \det(\mathcal{D}(\lambda)) = 1 \). By differentiating (93), we see that \( \partial_\lambda \mathcal{D}(\lambda) \) also solves (92), so that \( \partial_\lambda \mathcal{D}(\lambda)(\mathcal{D}(\lambda))^{-1} \) is an entire function of \( \lambda \). Using (93),

\[
\partial_\lambda \mathcal{D}(\lambda)(\mathcal{D}(\lambda))^{-1} = -i\lambda \sigma_3 - i[\sigma_3, m_1^{p_0}] + \mathcal{O}(\lambda^{-1}).
\]

Applying Liouville’s theorem to the left-hand side of the last equation, we arrive at the following ODE for \( \mathcal{D}(\lambda) \),

\[
(\partial_\lambda \mathcal{D} + i\lambda \sigma_3 \mathcal{D}) = \beta \mathcal{D}, \quad (94)
\]

where

\[
\beta = -i[\sigma_3, m_1^{p_0}] = \beta_{21} \sigma_- + \beta_{12} \sigma_+;
\]

hence,

\[
(m_1^{p_0})_{21} = -\frac{i}{2} \beta_{21}.
\]

It is well known \[26, 27\] that the fundamental solution of (94) can be written in terms of the parabolic-cylinder function, \( D_a(\cdot) \) \[24\]; in particular,

\[
\begin{align*}
\mathcal{D}_{+11}(\lambda) &= 2^{\frac{3}{2}}e^{-\frac{3i\nu}{2}}D_{i\nu}(\sqrt{2\lambda}e^{\frac{i\pi}{4}}), \\
\mathcal{D}_{+22}(\lambda) &= 2^{-\frac{1}{2}}e^{\frac{3i\nu}{4}}D_{-i\nu}(\sqrt{2\lambda}e^{-\frac{i\pi}{4}}), \\
\mathcal{D}_{+12}(\lambda) &= (\beta_{21})^{-1}2^{\frac{3}{2}}e^{\frac{3i\nu}{4}}\{\partial_\lambda D_{-i\nu}(\sqrt{2\lambda}e^{-\frac{i\pi}{4}}) - i\lambda D_{-i\nu}(\sqrt{2\lambda}e^{-\frac{i\pi}{4}})\}, \\
\mathcal{D}_{+21}(\lambda) &= (\beta_{12})^{-1}2^{\frac{3}{2}}e^{\frac{3i\nu}{4}}\{\partial_\lambda D_{i\nu}(\sqrt{2\lambda}e^{\frac{i\pi}{4}}) + i\lambda D_{i\nu}(\sqrt{2\lambda}e^{\frac{i\pi}{4}})\},
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{D}_{-11}(\lambda) &= 2^{\frac{3}{2}}e^{\frac{3i\nu}{4}}D_{i\nu}(\sqrt{2\lambda}e^{\frac{i\pi}{4}}), \\
\mathcal{D}_{-22}(\lambda) &= 2^{-\frac{1}{2}}e^{\frac{3i\nu}{4}}D_{-i\nu}(\sqrt{2\lambda}e^{-\frac{i\pi}{4}}), \\
\mathcal{D}_{-12}(\lambda) &= (\beta_{21})^{-1}2^{\frac{3}{2}}e^{\frac{3i\nu}{4}}\{\partial_\lambda D_{i\nu}(\sqrt{2\lambda}e^{-\frac{i\pi}{4}}) - i\lambda D_{i\nu}(\sqrt{2\lambda}e^{-\frac{i\pi}{4}})\}, \\
\mathcal{D}_{-21}(\lambda) &= (\beta_{12})^{-1}2^{\frac{3}{2}}e^{\frac{3i\nu}{4}}\{\partial_\lambda D_{-i\nu}(\sqrt{2\lambda}e^{\frac{i\pi}{4}}) + i\lambda D_{-i\nu}(\sqrt{2\lambda}e^{\frac{i\pi}{4}})\},
\end{align*}
\]

where \( \beta_{12} \beta_{21} = 2\nu \). Now, substituting (95) and (96) into (92), we find that

\[
-\beta_{12}^2(\lambda_0) = 2^\nu e^{\frac{3i\nu}{4}}W(D_{i\nu}(\sqrt{2\lambda}e^{\frac{i\pi}{4}}), D_{-i\nu}(\sqrt{2\lambda}e^{-\frac{i\pi}{4}})),
\]

where \( W(f, g) \) \( \equiv f(\lambda)\frac{dg(\lambda)}{d\lambda} - g(\lambda)\frac{df(\lambda)}{d\lambda} \) is the Wronskian of \( f = D_{i\nu}(\sqrt{2\lambda}e^{\frac{i\pi}{4}}) \) and \( g = D_{-i\nu}(\sqrt{2\lambda}e^{-\frac{i\pi}{4}}) \),

\[
W(D_{i\nu}(\sqrt{2\lambda}e^{\frac{i\pi}{4}}), D_{-i\nu}(\sqrt{2\lambda}e^{-\frac{i\pi}{4}})) = \frac{2\sqrt{\pi e^{\frac{3i\nu}{4}}}}{\Gamma(i\nu r(\lambda_0))},
\]

and \( \Gamma(\cdot) \) is the gamma function \[24\]; hence,

\[
(m_1^{p_0})_{21} = \frac{2^{-i\nu}\sqrt{\pi e^{\frac{3i\nu}{4}}}}{\Gamma(i\nu r(\lambda_0))}. \quad (97)
\]
From the $\sigma_3$ symmetry reduction for $m(\lambda)$, i.e., $m(-\lambda) = \sigma_3 m(\lambda) \sigma_3$, we have that,

$$
(m_1^{A0})_{21} = (m_1^{B0})_{21}. \quad (98)
$$

In the case $C$, $r(0) = r(i0) = 0$, so that the corresponding model RH problem (92)–(93) ($\lambda_0 = 0$) has identity conjugation matrix, $v(0) = I$, whence,

$$
(m_1^{C0})_{21} = 0. \quad (99)
$$

Hence, substituting (71), (97), (98), and (99) into (91), and recalling that $Q(x, t)$, and Proposition 2.5, and the definitions of

$$
\text{asymptotic expansion of } Q(x, t)
$$

we obtain (16), (18), (19), and (21) in Theorem 3.2.

8 Asymptotic Evaluation of $m(x, t; 0)$

In order to complete the proofs for $q(x, t)$ and $u(x, t)$ given in Theorems 3.2 and 3.3, we need to evaluate $m(x, t; 0)$ asymptotically (Lemma 2.1, (11)).

**Proposition 8.1** If $Q(x, 0) \in \mathcal{S}(\mathbb{R})$, then $m(x, t; 0) = \exp\{-\frac{ie\pi}{2} \int_{x_0}^{x} |Q(\xi, t)|^2 d\xi\}$.

**Proof.** From the proof of Proposition 2.2, it follows that, $m(x, t; 0) = \Psi(x, t; 0) = \exp\{-\frac{ie\pi}{2} \int_{x_0}^{x} |Q(\xi, t)|^2 d\xi\}$, for some $x_0 \in \mathbb{R}$. Since $Q(x, 0) \in \mathcal{S}(\mathbb{R})$, it follows from Definition 2.1, Proposition 2.5, and the definitions of $m(x, t; \lambda)$ for $\mathcal{R} \lambda^2 \geq 0$ given in Lemma 2.1 that, $\lim_{x \to \pm \infty} m(x, t; 0) = 1$: note also that, $m_+(x, t; 0) = m_-(x, t; 0) \equiv m(x, t; 0)$, since $G(0) = I$ (Lemma 2.1); hence, $\exp\{-\frac{ie\pi}{2} \int_{x_0}^{x} |Q(\xi, t)|^2 d\xi\} = I$. \[\blacksquare\]

**Proposition 8.2**

$$
(||Q||_{L^2(\mathbb{R})})^2 = \frac{2}{\pi} \left[ \int_{t}^{\infty} \ln(1+|r(\mu)|^2) d\mu - \int_{-\infty}^{t} \ln(1-|r(\mu)|^2) d\mu \right].
$$

**Proof.** For $Q(x, 0) \in \mathcal{S}(\mathbb{R})$, from Definition 2.1, Proposition 2.5, and the definitions of $m(x, t; \lambda)$ for $\mathcal{R} \lambda^2 \geq 0$ given in Lemma 2.1, one shows that, $\lim_{x \to \pm \infty} m(x, t; 0) = (a^-)(0)^{\sigma_3}$, where $a^- \equiv \exp\{\int_{0}^{t} \ln(1+r(\mu-r(\mu)) \frac{d\mu}{\pi}) \}$: the proof now follows from Proposition 2.5 ($r(\mu) = -\frac{1}{(r(\mu))}$), Lemma 2.1 ($r(\mu) \equiv r(\mu)$), and Proposition 8.1. \[\blacksquare\]

**Lemma 8.1** As $t \to +\infty$ and $x \to -\infty$ such that $\lambda_0 \equiv \frac{1}{2} \sqrt{-\frac{x}{t}} > M$,

$$
((m^{-1}(x, t; 0))_{11})^2 = \exp\{\frac{2i}{\pi} \int_{0}^{t} \ln(1-|r(\mu)|^2) d\mu - \int_{0}^{t} \ln(1+|r(\mu)|^2) d\mu \} + O\left(\frac{e^{\ln t}}{\sqrt{t}}\right).
$$

**Proof.** Writing $\int_{t}^{\infty} |Q(\xi, t)|^2 d\xi = -(||Q||_{L^2(\mathbb{R})})^2 + \int_{-\infty}^{t} |Q(\xi, t)|^2 d\xi$, the proof follows from the asymptotic expansion of $Q(x, t)$ given in Theorem 3.2, (16), (18), (19), and (21), the fact that $r(\lambda) \in \mathcal{S}(\mathbb{R})$, and Proposition 8.2. \[\blacksquare\]

**Remark 8.1.** It is possible to prove the estimate in Lemma 8.1, $O\left(\frac{e^{\ln t}}{\sqrt{t}}\right)$, without reference to the conserved quantity in Proposition 8.2 by using only the asymptotic results for $Q(x, t)$: this fact was first pointed out by Ablowitz and Segur in connection with their studies of the NLSE. To follow this paradigm, one needs the asymptotic expansion of $Q(x, t)$ when $\lambda_0 \geq 0$; however, we did not prove the aforementioned asymptotics for $Q(x, t)$ (Remark 3.3).

**Corollary 8.1** As $t \to +\infty$ and $x \to -\infty$ such that $\lambda_0 \equiv \frac{1}{2} \sqrt{-\frac{x}{t}} > M$, the solution of the DNLSE is given in Theorem 3.2, Eqs. (17)–(19) and (22).
Proof. Consequence of (11) in Lemma 2.1 and Lemma 8.1. ■

Corollary 8.2 As $t \to +\infty$ and $x \to +\infty$ such that

$$\tilde{\lambda}_0 \equiv \sqrt{\frac{1}{2} \left( \frac{x}{T} - \frac{1}{s} \right)} > M \quad \text{and} \quad \frac{x}{T} > \frac{1}{s}, \quad s \in \mathbb{R}_{>0},$$

the solution of the MNLSE is given in Theorem 3.3, Eqs. (32) and (33).

Proof. Follows from Proposition 2.4 and Corollary 8.1. ■

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References

[1] Kodama, Y., 1985, J. Stat. Phys. 39, 597–614. Kodama, Y. and Hasegawa, A., 1987, IEEE J. Quantum Electron. QE-23, 510–524. Agrawal, G. P., 1989, Nonlinear Fiber Optics (Academic: New York). Taylor, J. R., ed., 1992, Optical Solitons - Theory and Experiment, Cambridge Studies in Modern Optics, Vol. 10 (CUP: Cambridge). Haus, H. A., 1993, Proc. IEEE 81, 970–983. Hasegawa, A. and Kodama, Y., 1995, Solitons in Optical Communications, Oxford Series in Optical and Imaging Sciences, No. 7 (OUP: Oxford).

[2] Faddeev, L. D. and Takhtajan, L. A., 1987, Hamiltonian Methods in the Theory of Solitons (Springer-Verlag: Berlin).

[3] Gordon, J. P., 1986, Opt. Lett. 11, 662–664. Mitschke, F. M. and Mollenauer, L. F., 1986, Opt. Lett. 11, 659–661. Agrawal, G. P., 1990, Opt. Lett. 15, 224–226.

[4] Wadati, M., Konno, K. and Ichikawa, Y., 1979, J. Phys. Soc. Japan 46, 1965–1966. Eichhorn, H., 1985, Inverse Problems 1, 193–198. Nijhof, J. H. B. and Roelofs, G. H. M., 1992, J. Phys. A: Math. Gen. 25, 2403–2416.

[5] Chbat, M. W., Prucnal, P. R., Islam, M. N., Soccolich, C. E. and Gordon, J. P., 1993, J. Opt. Soc. Am. B 10, 1386–1395. Lundquist, P. B., Andersen, D. R. and Swartzlander, G. A., Jr., 1995, J. Opt. Soc. Am. B 12, 698–703.

[6] Mamyshev, P. V., Wiley, P. G. J., Wilson, J., Stegeman, G. I., Semenov, V. A., Dianov, E. M. and Miroshnichenko, S. I., 1993, Phys. Rev. Lett. 71, 73–76.

[7] Schuur, P. C., 1986, Asymptotic Analysis of Soliton Problems, Lecture Notes in Mathematics, Vol. 1232 (Springer-Verlag: Berlin). Bikbaev, R. F., 1988, Theor. Math. Phys. 77, 1117–1123. Fokas, A. S. and Its, A. R., 1992, Phys. Rev. Lett. 68, 3117–3120. Rybin, A. and Timonen, J., 1993, J. Phys. A: Math. Gen. 26, 3869–3882.

[8] Chen, Z. and Huang, N., 1989, Phys. Lett. A 142, 31–35. Vysloukh, V. A. and Cherednik, I. V., 1989, Theor. Math. Phys. 78, 24–31. Chen, Z. and Huang, N., 1990, Phys. Rev. A 41, 4066–4069. Chen, Z., 1991, Commun. Theor. Phys. 15, 271–276. Liu, S.-L. and Wang, W.-Z., 1993, Phys. Rev. E 48, 3054–3059. Doktorov, E. V. and Shchesnovich, V. S., 1995, J. Math. Phys. 36, 7009–7023.

[9] Segur, H. and Ablowitz, M. J., 1976, J. Math. Phys. 17, 710–713. Segur, H., 1976, J. Math. Phys. 17, 714–716. Ablowitz, M. J. and Segur, H., 1981, Solitons and the Inverse Scattering Transform (SIAM: Philadelphia).

[10] Manakov, S. V., 1974, Sov. Phys. JETP 38, 693–696. Zakharov, V. E. and Manakov, S. V., 1976, Sov. Phys. JETP 44, 106–112. Its, A. R., 1981, Sov. Math. Dokl. 24, 452–456.

[11] Zakharov, V. E. and Shabat, A. B., 1979, Funct. Anal. Appl. 13, 166–174. Beals, R., Deift, P. and Tomei, C., 1988, Direct and Inverse Scattering on the Line, Mathematical Surveys and Monographs, No. 28 (AMS: Providence).

[12] Kitaev, A. V. and Vartanian, A. H., “Asymptotics of the Modified Non-Linear Schrödinger Equation”, to be submitted.

[13] Kitaev, A. V., 1985, Theor. Math. Phys. 64, 878–894.
[14] Beals, R. and Coifman, R. R., 1984, Comm. Pure Appl. Math. 37, 39–90.

[15] Deift, P. and Zhou, X., 1993, Ann. Math. 137, 295–368.

[16] Bikbaev, R. F., private communication.

[17] Kaup, D. J. and Newell, A. C., 1978, J. Math. Phys. 19, 798–801.

[18] Gerdzhikov, V. S., Ivanov, M. I. and Kulish, P. P., 1980, Theor. Math. Phys. 44, 784–795.

[19] Vartanian, A. H., Ph.D. Thesis (in preparation), Steklov Mathematical Institute (St. Petersburg).

[20] Lee, J.-H., 1989, Trans. Amer. Math. Soc. 314, 107–118.

[21] Zhou, X., 1995, J. Diff. Eqn. 115, 277–303.

[22] Zhou, X., 1989, Comm. Pure Appl. Math. 42, 895–938. Zhou, X., 1989, SIAM J. Math. Anal. 20, 966–986.

[23] Murray, J. D., 1984, Asymptotic Analysis, Applied Mathematical Sciences, Vol. 48 (Springer-Verlag: New York).

[24] Gradshteyn, I. S. and Ryzhik, I. M., 1994, Tables of Integrals, Series, and Products, 5th ed., Jeffrey, A., ed. (Academic Press: San Diego).

[25] Hayashi, N. and Ozawa, T., 1994, Math. Anal. 298, 557–576.

[26] Ablowitz, M. J. and Clarkson, P. A., 1991, Solitons, Nonlinear Evolution Equations and Inverse Scattering, LMS, No. 149 (CUP: Cambridge).

[27] Novikov, S. P., Manakov, S. V., Pitaevskii, L. P. and Zakharov, V. E., 1984, Theory of Solitons: The Inverse Scattering Method (Plenum: New York).