Nonlinear estimates for traveling wave solutions of reaction diffusion equations

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Abstract
In this paper we will establish nonlinear a priori lower and upper bounds for the solutions to a large class of equations which arise from the study of traveling wave solutions of reaction–diffusion equations, and we will apply our nonlinear bounds to the Lotka–Volterra system of two and four competing species as examples. The idea used in a series of papers by the first author et al. for the establishment of the linear N-barrier maximum principle will also be used in the proof.

Keywords Traveling wave solutions · Reaction diffusion equations · Lotka–Volterra system

Mathematics Subject Classification 35B50 · 35C07 · 35K57

1 Introduction

The present paper is devoted to nonlinear a priori upper and lower bounds for the solutions \( u_i = u_i(x) : \mathbb{R} \mapsto [0, \infty) \), \( i = 1, \ldots, n \) to the following boundary value problem of \( n \) equations

\[
\begin{align*}
\left\{ \begin{array}{l}
d_i (u_i)_{xx} + \theta (u_i)_x + u_i^{l_i} f_i (u_1, u_2, \ldots, u_n) = 0, \quad x \in \mathbb{R}, \quad i = 1, 2, \ldots, n, \\
(u_1, u_2, \ldots, u_n)(-\infty) = e_-, \quad (u_1, u_2, \ldots, u_n)(\infty) = e_+.
\end{array} \right.
\end{align*}
\]

(1)

In the above, \( d_i, l_i > 0, \theta \in \mathbb{R} \) are parameters, \( f_i \in C^1([0, \infty)^n) \) are given functions and the boundary values \( e_-, e_+ \) take value in the following constant equilibria set

\[ e_- = (u_{1n}, u_{2n}, \ldots, u_{nn}), \quad e_+ = (u_{1f}, u_{2f}, \ldots, u_{nf}). \]

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\begin{equation}
\left\{(u_1, \ldots, u_n) \mid u_i^I f_i(u_1, \ldots, u_n) = 0, \quad u_i \geq 0, \quad \forall i = 1, \ldots, n\right\}.
\end{equation}

Equations (1) arise from the study of traveling waves solutions of reaction–diffusion equations (see [16, 18]). A series of papers [2–7] by Hung et al. have contributed to the linear (N-barrier) maximum principle for the n Eq. (1), and in particular the lower and upper bounds for any linear combination of the solutions

$$\sum_{i=1}^{n} \alpha_i u_i(x), \quad \forall (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}^+)^n$$

have been established in terms of the parameters \(d_i, l_i, \theta\) in (1).

Here we aim to derive nonlinear estimates for the polynomials of the solutions:

$$\prod_{i=1}^{n} (u_i(x) + k_i^{a_i}), \quad \forall (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}^+)^n$$

for some \(k_i \geq 0\), which is related to the diversity indices of the species in ecology: \(D^q = (\sum_{i=1}^{n} (u_i)^q)^{1/(1-q)}, \quad q \in (1, \infty)\).

Observe that when either \(e_- = (0, \ldots, 0)\) or \(e_- = (0, \ldots, 0)\), the trivial lower bound of \(\prod_{i=1}^{n} (u_i(x) + k_i^{a_i})\) is \(\prod_{i=1}^{n} k_i^{a_i}\). For \(k_i > 0\), the following lower bound for the upper solutions of (1) holds.

**Theorem 1 (Lower bound)** Suppose that \((u_i(x))_{i=1}^{n} \in (C^2(\mathbb{R}))^n\) with \(u_i(x) \geq 0, \quad \forall i = 1, \ldots, n\) is an upper solution of (1):

\begin{equation}
\begin{cases}
d_i(u_i)_{xx} + \theta(u_i)_x + u_i^I f_i(u_1, u_2, \ldots, u_n) \leq 0, & x \in \mathbb{R}, \quad i = 1, 2, \ldots, n, \\
(u_1, u_2, \ldots, u_n)(-\infty) = e_-, \quad (u_1, u_2, \ldots, u_n)(\infty) = e_+.
\end{cases}
\end{equation}

and that there exist \((u_i^n)_{i=1}^{n} \in (\mathbb{R}^+)^n\) such that

\begin{equation}
f_i(u_1, \ldots, u_n) \geq 0, \quad \text{for all} \quad (u_1, \ldots, u_n) \in \mathcal{R} := \{(u_i)_{i=1}^{n} \in ([0, \infty))^n \mid \sum_{i=1}^{n} \frac{u_i}{U_i} \leq 1\}.
\end{equation}

Then we have for any \((k_i)_{i=1}^{n} \in (\mathbb{R}^+)^n\), and \((\alpha_i)_{i=1}^{n} \in (\mathbb{R}^+)^n\),

\begin{equation}
\prod_{i=1}^{n} (u_i(x) + k_i)^{d_i^{a_i}} \geq e^{\lambda_1}, \quad x \in \mathbb{R},
\end{equation}

where

\begin{equation}
4\lambda_1 = \min_{1 \leq j \leq n} \left(\eta d_j + \sum_{i=1, i \neq j}^{n} \alpha_i (d_i - d_j) \ln k_i\right), \quad (6a)
\end{equation}

\begin{equation}
\eta = \min_{1 \leq j \leq n} \frac{1}{d_j} \left(\lambda_2 - \sum_{i=1, i \neq j}^{n} \alpha_i (d_i - d_j) \ln k_i\right), \quad (6b)
\end{equation}
Nonlinear estimates for reaction diffusion equations

\[ \lambda_2 = \min_{1 \leq i \leq n} \left( \alpha_i d_j \ln(u_j + k_j) + \sum_{i=1,i \neq j}^{n} \alpha_i d_i \ln k_i \right). \]  

(6c)

Remark 2 (Equal diffusion) When \( d_i = d \) for all \( i = 1, 2, \ldots, n \), then

\[ \lambda_1 = \min_{1 \leq i \leq n} \left( \alpha_i \ln(u_i + k_i) + \sum_{i=1,i \neq j}^{n} \alpha_i \ln k_i \right) d = \lambda_2 = d \eta, \]

and the lower bound (5) becomes

\[ \prod_{i=1}^{n} (u_i(x) + k_i)^{\alpha_i} \geq \min_{1 \leq i \leq n} \left( (u_i + k_i)^{\alpha_i} \prod_{i \neq j}^{n} k_i^{\alpha_j} \right), \quad x \in \mathbb{R}. \]

If furthermore \( \alpha_i = \alpha, \forall i = 1, \ldots, n \), then the inequality of arithmetic and geometric averages yields

\[ \sum_{i=1}^{n} (u_i + k_i)^{\alpha} \geq n \left( \prod_{i=1}^{n} (u_i + k_i)^{\alpha} \right)^{\frac{1}{n}} \geq n \min_{1 \leq i \leq n} \left( (u_i + k_i)^{\alpha} \prod_{i \neq j}^{n} k_i^{\alpha_j} \right)^{\frac{1}{n}}. \]

On the other hand, we can find an upper bound of \( \prod_{i=1}^{n} (u_i(x))^{\alpha} \) for the lower solutions of (1).

Theorem 3 (Upper bound) Suppose that \( (u_i(x))^{\alpha} \in (C^2(\mathbb{R}))^n \) with \( u_i(x) \geq 0 \) \( \forall i = 1, \ldots, n \) is a lower solution of (1):

\[
\begin{align*}
    d_i(u_i)_{xx} + \theta(u_i)_{x} + u_i f_i(u_1, u_2, \ldots, u_n) & \geq 0, \quad x \in \mathbb{R}, \quad i = 1, 2, \ldots, n, \\
    (u_1, u_2, \ldots, u_n)(-\infty) & = \mathbf{e}_-, \quad (u_1, u_2, \ldots, u_n)(\infty) = \mathbf{e}_+. \tag{7}
\end{align*}
\]

and that there exist \( \bar{u}_i > 0, i = 1, \ldots, n \), such that

\[ f_i(u_1, \ldots, u_n) \leq 0, \quad \text{for all } (u_1, \ldots, u_n) \in \mathcal{R} := \left\{ (u_i)_{i=1}^{n} \in ([0, \infty))^n \mid \sum_{i=1}^{n} \frac{u_i}{\bar{u}_i} \geq 1 \right\}. \tag{8} \]

Then we have for any \( m_i \geq 1 \) and \( \alpha_i > 0 \) \( (i = 1, 2, \ldots, n) \)

\[ \sum_{i=1}^{n} \alpha_i (u_i(x))^{m_i} \leq \left( \max_{1 \leq i \leq n} \alpha_i (\bar{u}_i)^{m_i} \right) \frac{\max d_i}{\min d_i}, \quad x \in \mathbb{R}, \tag{9} \]

and hence

\[ \prod_{i=1}^{n} (u_i(x))^{m_i/n} \leq \max_{1 \leq i \leq n} \alpha_i \frac{\max d_i}{\min d_i}, \quad x \in \mathbb{R}. \tag{10} \]
In particular, when $\alpha_i = \alpha$ for all $i = 1, \ldots, n$, (10) becomes
\[
\prod_{i=1}^{n} (u_i(x))^{m_i/n} \leq \frac{\max u_i^{m_i}}{1 \leq i \leq n} \frac{\max d_i}{1 \leq i \leq n}, \quad x \in \mathbb{R}.
\] (11)

In order to prove Theorem 1, we will first rewrite the system (3) into the system for the new unknowns $(U_i)_{i=1}^n := (\ln(u_i + k_i))^{m_i}_{i=1}$. Then we will follow the ideas in [2–7] to establish the lower bound for the linear combination of $(U_i)_{i=1}^n$, which implies the nonlinear lower bound (5) correspondingly. Similarly, we will consider the new unknowns $(U_i)_{i=1}^n := (u^{m_i}_{i=1})$ to establish the upper bound (9). The proofs will be found in Sect. 2.

As examples to illustrate our main result, we use the Lotka–Volterra system of two and four competing species to conclude with Sect. 1. This example provides an intuitive idea of the construction of the N-barrier in multi-species cases.

To illustrate Theorem 3 for the case $n = 2$, we use the Lotka–Volterra system of two competing species coupled with Dirichlet boundary conditions:
\[
\begin{align*}
    d_1 u_{xx} + \theta u_x + u(1 - u - a_1 v) &= 0, & x \in \mathbb{R}, \\
    d_2 v_{xx} + \theta v_x + \kappa v(1 - a_2 u - v) &= 0, & x \in \mathbb{R}, \\
    (u, v)(-\infty) &= e_j, & (u, v)(+\infty) = e_j,
\end{align*}
\] (12)

where $a_1, a_2, \kappa > 0$ are constants. In (12), the constant equilibria are $e_1 = (0, 0)$, $e_2 = (1, 0)$, $e_3 = (0, 1)$ and $e_4 = (u^*, v^*)$, where
\[
(u^*, v^*) = \left( \frac{1 - a_1}{1 - a_1 a_2}, \frac{1 - a_2}{1 - a_1 a_2} \right)
\]
is the intersection of the two straight lines $1 - u - a_1 v = 0$ and $1 - a_2 u - v = 0$ whenever it exists. We call the solution $(u(x), v(x))$ of (12) an $(e_j, e_j)$-wave.

Tang and Fife [17], and Ahmad and Lazer [1] established the existence of the $(e_1, e_4)$-waves. Kan-on [10, 11], Fei and Carr [8], Leung et al. [15], and Leung and Feng [14] proved the existence of the $(e_2, e_3)$-waves using different approaches. $(e_2, e_3)$-waves were studied for instance, by Kanel and Zhou [13], Kanel [12], and Hou and Leung [9].

For the above-mentioned $(e_1, e_4)$-waves, $(e_2, e_3)$-waves, and $(e_2, e_4)$-waves, we show a lower and an upper bounds of $u(x)v(x)$ by Theorems 1 and 3 respectively. To this end, let
\[
\begin{align*}
    u &= \min \left( 1, \frac{1}{a_2} \right), & \bar{u} &= \max \left( 1, \frac{1}{a_2} \right), \\
    v &= \min \left( 1, \frac{1}{a_1} \right), & \bar{v} &= \max \left( 1, \frac{1}{a_1} \right),
\end{align*}
\]
then the hypothesis (4) and (8) are satisfied. According to (10), letting $a_1 = a_2 = m_1 = m_2 = 1$ leads to
\[
\sqrt{u(x)v(x)} \leq \frac{1}{2} \max (\bar{u}, \bar{v}) \frac{\max (d_1, d_2)}{\min (d_1, d_2)}, \quad x \in \mathbb{R}
\]  
(13)

or

\[
u(x)v(x) \leq \frac{1}{4} (\max (\bar{u}, \bar{v}))^2 \left(\frac{\max (d_1, d_2)}{\min (d_1, d_2)}\right)^2, \quad x \in \mathbb{R}.
\]  
(14)

For the equal diffusion case \(d_1 = d_2 = 1\) with the bistable condition \(a_1, a_2 > 1\), (14) is simplified to

\[
u(x)v(x) \leq \frac{1}{4}, \quad x \in \mathbb{R}.
\]  
(15)

On the other hand, we can apply Theorem 3 to the Lotka–Volterra system of multi-species. By using the hyperbolic tangent method, we have in Theorem 5.1. [6] constructed exact solutions of the following Lotka–Volterra system of 4-species

\[
d_i(u_i)_{xx} + \theta (u_i)_x + u_i \left(\sigma_i - \sum_{j=1}^{4} c_{ij} u_j\right) = 0, \quad x \in \mathbb{R}, \quad i = 1, \ldots, 4.
\]  
(16)

In particular, when \(\theta = \frac{9}{2}, d_1 = \frac{1}{2}, d_2 = \frac{1}{2}, d_3 = \frac{5}{12}, d_4 = \frac{5}{12}, \sigma_1 = 1, \sigma_2 = 24, \sigma_3 = \frac{76}{3}, \sigma_4 = \frac{34}{3}, c_{11} = \frac{1}{2}, c_{12} = 1, c_{13} = 1, c_{14} = 1, c_{21} = 7, c_{22} = 2, c_{23} = \frac{3}{5}, c_{24} = \frac{3}{5}, c_{31} = 7, c_{32} = 3, c_{33} = 2, c_{34} = 2, c_{41} = 2, c_{42} = 3, c_{43} = 2, c_{44} = 2\), (16) admits a solution of the form

\[
\begin{align*}
u_1(x) &= \left(1 + \tanh x\right), \quad x \in \mathbb{R}, \\
u_2(x) &= 3 \left(1 - \tanh x\right)^2, \quad x \in \mathbb{R}, \\
u_3(x) &= \left(1 + \tanh x\right) \left(1 - \tanh x\right)^2, \quad x \in \mathbb{R}, \\
u_4(x) &= \left(1 - \tanh x\right) \left(1 + \tanh x\right)^2, \quad x \in \mathbb{R}.
\end{align*}
\]  
(17)

In this case, we find that

- \(u_1(x) u_2(x) u_3(x) u_4(x) = 3 (1 - \tanh x)^5 (1 + \tanh x)^4\) gives

\[
\max_{x \in \mathbb{R}} \left(u_1(x) u_2(x) u_3(x) u_4(x)\right) \approx 3.17175
\]

- According to Theorem 3, we have

\[
\left(u_1(x) u_2(x) u_3(x) u_4(x)\right)^{1/4} \leq \frac{\max_{1 \leq i \leq 4} \bar{u}_i}{4} \frac{\max_{1 \leq i \leq 4} d_i}{\min_{1 \leq i \leq 4} d_i} = \frac{76}{3} \frac{\frac{5}{12}}{\frac{3}{4}} = \frac{95}{18}, \quad x \in \mathbb{R}.
\]  
(18)
We see in this case that the a priori upper bound is obviously not optimal. This is due to the fact that Theorem 3 (also Theorem 1) is applicable for quite general non-linearity \( f_i(u_1, u_2, \ldots, u_n) \). A further study would be to improve the a priori bounds established in Theorems 3 and 1.

2 Proofs of Theorems 1 and 3

**Proof of Theorem 1** We first rewrite the inequality \( d_i(u_i)'' + \theta(u_i)' + u_i f_i \leq 0 \) in (3). If \( u(x) \geq 0 \), then for any \( k > 0 \), a straightforward calculation gives

\[
\frac{(\ln(u(x) + k))'}{u(x) + k} = \frac{u'(x)}{u(x) + k},
\]

\[
\frac{(\ln(u(x) + k))''}{u(x) + k} = \frac{u''(x)}{u(x) + k} - \frac{(u'(x))^2}{(u(x) + k)^2}.
\]

Hence we divide the inequality by \( u_i + k_i > 0 \) with \( k_i > 0 \) to arrive at

\[
d_i(\ln(u_i + k_i))'' + d_i \frac{(u_i')^2}{(u_i + k_i)^2} + \theta(\ln(u_i + k_i))' + \frac{u_i f_i}{u_i + k_i} \leq 0.
\]

Thus \( (U_i)_{i=1}^n : = (\ln(u_i + k_i))_{i=1}^n \) satisfies the following inequalities:

\[
d_i U_i'' + \theta U_i' + \frac{u_i f_i}{u_i + k_i} \leq 0, \quad i = 1, \ldots, n.
\]

(19)

For any \( (\alpha_i)_{i=1}^n \in (\mathbb{R}^+)^n \), let

\[
p(x) = \sum_{i=1}^n \alpha_i U_i, \quad q(x) = \sum_{i=1}^n \alpha_i d_i U_i,
\]

then the above inequality (19) reads as

\[
q'' + \theta q' + \frac{\alpha_i u_i f_i}{u_i + k_i} \leq 0, \quad F := \sum_{i=1}^n \frac{\alpha_i u_i f_i}{u_i + k_i} (u_1, \ldots, u_n).
\]

(20)

We are going to derive a lower bound for

\[
q = \sum_{i=1}^n \alpha_i d_i U_i = \sum_{i=1}^n \alpha_i d_i \ln(u_i(x) + k_i),
\]

and hence a lower bound for \( \prod_{i=1}^n (u_i + k_i)^{\alpha_i} \). The idea is similar as in the papers [2–7], namely we are going to determine three parameters

\[
\lambda_1, \quad \eta, \quad \lambda_2
\]

to construct an N-barrier consisting of three hypersurfaces
Nonlinear estimates for reaction diffusion equations

\( Q_1 := \{(u_i)_i \mid q = \lambda_1 \}, \quad P := \{(u_i)_i \mid p = \eta \}, \quad Q_2 := \{(u_i)_i \mid q = \lambda_2 \}, \)

such that the following inclusion relations hold:

\[
Q_1 := \{(u_i)_i \in ([0, \infty])^n \mid q \leq \lambda_1 \} \subset P := \{(u_i)_i \in ([0, \infty])^n \mid p \leq \eta \}
\]

\[
\subset Q_2 := \{(u_i)_i \in ([0, \infty])^n \mid q \leq \lambda_2 \}
\]

\[
\subset \mathcal{R} = \{(u_i)_i \in ([0, \infty])^n \mid \sum_{i=1}^n \frac{u_i}{u_j} \leq 1 \}.
\]

It will turn out that if \( \lambda_1, \eta, \) and \( \lambda_2 \) are given respectively by (6a), (6b), and (6c), then \( \lambda_1 \) determines a lower bound of \( q(x) \): \( q(x) \geq \lambda_1 \), which is exactly (5).

More precisely, we follow the steps as in [2–7] to determine \( \lambda_2, \eta, \lambda_1 \) such that the above inclusion relations \( Q_1 \subset P \subset Q_2 \subset \mathcal{R} \) hold:

(i) **Determine \( \lambda_2 \)** We first find the \( u_j \)-intercept of \( \sum_{i=1}^n \alpha_i d_i \ln(u_i + k_i) = \lambda_2 \). Solving \( u_j \) from

\[
\alpha_j d_j \ln(u_j + k_j) + \sum_{i=1, i \neq j}^n \alpha_i d_i \ln k_i = \lambda_2,
\]

we find the \( u_j \)-intercept is given by

\[
u_{2,j} = e^{\frac{\lambda_2 - \sum_{i=1, i \neq j}^n \alpha_i d_i \ln k_i}{\alpha_j d_j}} - k_j, \quad j = 1, 2, \ldots, n.
\]

Since we need \( Q_2 \subset \mathcal{R} \), this yields that the condition \( u_{2,j} \leq u_j \) for all \( j = 1, 2, \ldots, n \) needs to be true, i.e.

\[
e^{\frac{\lambda_2 - \sum_{i=1, i \neq j}^n \alpha_i d_i \ln k_i}{\alpha_j d_j}} - k_j \leq u_j, \quad \forall j = 1, 2, \ldots, n.
\]

Solving \( \lambda_2 \) from (23) leads to

\[
\lambda_2 \leq \alpha_j d_j \ln(u_j + k_j) + \sum_{i=1, i \neq j}^n \alpha_i d_i \ln k_i, \quad \forall j = 1, 2, \ldots, n.
\]

So we choose \( \lambda_2 \) as shown in (6c), i.e.

\[
\lambda_2 = \min_{1 \leq j \leq n} \left( \alpha_j d_j \ln(u_j + k_j) + \sum_{i=1, i \neq j}^n \alpha_i d_i \ln k_i \right).
\]

(ii) **Determine \( \eta \)** We first find the \( u_j \)-intercept of \( \sum_{i=1}^n \alpha_i \ln(u_i + k_i) = \eta \). Solving \( u_j \) from

\[
\sum_{i=1}^n \alpha_i \ln(u_i + k_i) = \eta.
\]
\[ a_j \ln(u_j + k_j) + \sum_{i=1, i \neq j}^{n} \alpha_i \ln k_i = \eta, \quad (25) \]

we find the \( u_j \)-intercept is given by
\[ u_{0,j} = e^{-\eta - \sum_{i=1, i \neq j}^{n} \alpha_i \ln k_i} - k_j, \quad j = 1, 2, \ldots, n. \quad (26) \]

Since we need \( P \subset Q_2 \), this yields that the condition \( u_{0,j} \leq u_{2,j} \) for all \( j = 1, 2, \ldots, n \) needs to be true, i.e.
\[ e^{-\sum_{i=1, i \neq j}^{n} \alpha_i \ln k_i} - k_j \leq e^{-\sum_{i=1, i \neq j}^{n} \alpha_i \ln k_i} - k_j, \quad \forall j = 1, 2, \ldots, n. \quad (27) \]

Solving \( \eta \) from (27) leads to
\[ \eta \leq \frac{1}{d_j} \left( \frac{\lambda_2 - \sum_{i=1, i \neq j}^{n} \alpha_i (d_i - d_j) \ln k_i}{\eta} \right), \quad \forall j = 1, 2, \ldots, n. \quad (28) \]

So we choose \( \eta \) as shown in (6b), i.e.
\[ \eta = \min_{1 \leq j \leq n} \left( \frac{\lambda_2 - \sum_{i=1, i \neq j}^{n} \alpha_i (d_i - d_j) \ln k_i}{d_j} \right). \]

(iii) \textbf{Determine} \( \lambda_1 \). Replacing \( \lambda_2 \) by \( \lambda_1 \) in step (i), we can find the \( u_j \)-intercept of
\[ \sum_{i=1}^{n} \alpha_i d_i \ln(u_i + k_i) = \lambda_1 \] is given by
\[ u_{1,j} = e^{-\frac{\lambda_1 - \sum_{i=1, i \neq j}^{n} \alpha_i d_i \ln k_i}{\eta d_j}} - k_j, \quad j = 1, 2, \ldots, n. \quad (29) \]

Since we need \( Q_1 \subset P \), this yields that the condition \( u_{1,j} \leq u_{0,j} \) for all \( j = 1, 2, \ldots, n \) needs to be true, i.e.
\[ e^{-\frac{\lambda_1 - \sum_{i=1, i \neq j}^{n} \alpha_i d_i \ln k_i}{\eta d_j}} - k_j \leq e^{-\sum_{i=1, i \neq j}^{n} \alpha_i \ln k_i} - k_j, \quad \forall j = 1, 2, \ldots, n. \quad (30) \]

Solving \( \lambda_1 \) from (30) leads to
\[ \lambda_1 \leq \eta d_j + \sum_{i=1, i \neq j}^{n} \alpha_i (d_i - d_j) \ln k_i, \quad \forall j = 1, 2, \ldots, n. \quad (31) \]

So we choose \( \lambda_1 \) as shown in (6a), i.e.
\[ \lambda_1 = \min_{1 \leq j \leq n} \left( \eta d_j + \sum_{i=1, i \neq j}^{n} \alpha_i (d_i - d_j) \ln k_i \right). \]

Steps (i)–(iii) complete the construction of the N-barrier.
We now show $q(x) \geq \lambda_1$, $x \in \mathbb{R}$ by a contradiction argument. Suppose by contradiction that there exists $z \in \mathbb{R}$ such that $q(z) < \lambda_1$. Since $u_i(x) \in C^2(\mathbb{R})$ $(i = 1, \ldots, n)$ and $(u_1, u_2, \ldots, u_n)_{\pm\infty} = e_{\pm}$, we may assume $\min_{x \in \mathbb{R}} q(x) = q(z)$. We denote respectively by $z_2$ and $z_1$ the first points at which the solution trajectory $\{u_i(x)\}_{i=1}^n | x \in \mathbb{R} \}$ intersects the hypersurface $Q_2$ when $x$ moves from $z$ towards $\infty$ and $-\infty$. For the case where $\theta \leq 0$, we integrate (20) with respect to $x$ from $z_1$ to $z$ and obtain

$$q'(z) - q'(z_1) + \theta (p(z) - p(z_1)) + \int_{z_1}^{z} F(u_1(x), \ldots, u_n(x)) \, dx \leq 0. \quad (32)$$

We also have the following facts from the construction of the hypersurfaces $Q_1, Q_2, P$:

- $q'(z) = 0$ because of $\min_{x \in \mathbb{R}} q(x) = q(z)$;
- $q(z_1) = \lambda_2$ because of $(u_i(z_1))_{i=1}^n \in Q_2$.
- $q'(z_1) < 0$ because $z_1$ is the first point for $q(x)$ taking the value $\lambda_2$ when $x$ moves from $z$ to $-\infty$, such that $q(z_1 + \delta) > \lambda_2$ for $z - z_1 > \delta > 0$;
- $p(z) < \eta$ since $(u_i(z))_{i=1}^n$ is below the hypersurface $P$;
- $p(z_1) > \eta$ since $(u_i(z_1))_{i=1}^n$ is above the hypersurface $P$;
- $F(u_1(x), \ldots, u_n(x)) = \sum_{i=1}^n \frac{a_i u_i}{u_i + k_i} f_i(u_1, \ldots, u_n) \geq 0$, $\forall x \in [z_1, z]$. Indeed, since $(u_i(z_1))_{i=1}^n \in Q_2 \subset Q_2 \subset \mathbb{R}$ and $(u_i(z))_{i=1}^n \in Q_1 \subset \mathbb{R}$, we derive that $F(u_1(x), \ldots, u_n(x))|_{x \in [z_1, z]} \geq 0$ by the hypothesis (4).

We hence have the following inequality from the above facts when $\theta \leq 0$

$$q'(z) - q'(z_1) + \theta (p(z) - p(z_1)) + \int_{z_1}^{z} F(u_1(x), \ldots, u_n(x)) \, dx > 0,$$

which contradicts (32). Therefore when $\theta \leq 0$, $q(x) \geq \lambda_1$ for $x \in \mathbb{R}$. For the case where $\theta \geq 0$, we simply integrate (20) with respect to $x$ from $z$ to $z_2$ to arrive at

$$q'(z_2) - q'(z) + \theta (p(z_2) - p(z)) + \int_{z}^{z_2} F(u_1(x), \ldots, u_n(x)) \, dx \leq 0.$$

Then we apply the facts that $q'(z_2) > 0$, $q'(z) = 0$, $p(z_2) > \eta$, $p(z) < \eta$ and $F(u_1(x), \ldots, u_n(x))|_{x \in [z, z_2]} \geq 0$, as well as a similar contradiction argument as above, to derive $q \geq \lambda_1$.

**Proof of Theorem 3** We prove Theorem 3 in a similar manner to the proof of Theorem 1. We first rewrite the inequality $d_i (u_i)'' + \theta (u_i)' + u_i^j f_i \geq 0$ in (7). A straightforward calculation shows
\[(u^m)' = mu^{m-1}u',
\]
\[(u^m)'' = m(m - 1)u^{m-2}(u')^2 + u^{m-1}u''(x)).\]

Hence we multiply the inequality by \(m_i u^{m-1}(x)\) to arrive at
\[d_i(u_i^m)'' - d_i m_i (m_i - 1) u_i^{m_i - 2}(u_i')^2 + \theta (u_i^m)' + m_i u_i^{m_i - 1}u_i f_i \geq 0.\]

For notational simplicity, we will adopt the same notations as in the proof of Theorem 1. Since \(u_i \geq 0, \forall i = 1, \ldots, n\), for any \((m_i)_{i=1}^n \in ([1, \infty))^n\), the vector field \((U_i)^n_{i=1} := (u_i^m)_{i=1}^n\) satisfies the following inequalities
\[d_i U_i'' + \theta U_i' + m_i u_i^{m_i - 1}u_i f_i \geq 0, \quad \forall i = 1, \ldots, n.\]

For any \((\alpha_i)_{i=1}^n \in (\mathbb{R}^+)^n\), \(p(x) = \sum_{i=1}^n \alpha_i U_i\) and \(q(x) = \sum_{i=1}^n \alpha_i d_i U_i\) satisfy
\[q'' + \theta p' + F \geq 0, \quad F := \sum_{i=1}^n \alpha_i m_i u_i^{m_i - 1}u_i f_i (u_1, u_2, \ldots, u_n).\]

We are going to show the upper bound \(q \leq \lambda_1\) by employing the N-barrier method as in the proof of Proposition 1. That is, we are going to construct the three hyperellipsoids
\[Q_1 := \{(u_i)_{i=1}^n \mid q = \lambda_1\}, \quad P := \{(u_i)_{i=1}^n \mid p = \eta\}, \quad Q_2 := \{(u_i)_{i=1}^n \mid q = \lambda_2\},\]
such that the following inclusion relations hold:
\[\cup Q_1 := \{(u_i)_{i=1}^n \in ([0, \infty))^n \mid q \geq \lambda_1\} \supset \mathcal{P} := \{(u_i)_{i=1}^n \in ([0, \infty))^n \mid p \geq \eta\}\]
\[\supset \mathcal{Q}_2 := \{(u_i)_{i=1}^n \in ([0, \infty))^n \mid q \geq \lambda_2\}\]
\[\supset \mathcal{R} := \left\{ (u_i)_{i=1}^n \in ([0, \infty))^n \mid \sum_{i=1}^n \frac{u_i}{\bar{u}_i} \geq 1 \right\},\]
and the upper bound \(q \leq \lambda_1\) follows by a contradiction argument. More precisely, we take
\[\lambda_2 = \max_{1 \leq i \leq n} \alpha_i d_i (\bar{u}_i)^{m_i},\]
such that the \(u_j\)-intercept of the hyperellipsoid \(Q_2\)
\[u_{2j} = \left( \frac{\lambda_2}{\alpha_j d_j} \right)^{1/m_j} \geq \bar{u}_j, \quad j = 1, 2, \ldots, n.\]

Then we take
\[\eta = \frac{\lambda_2}{\min_{1 \leq i \leq n} d_i},\]
such that the \(u_j\)-intercept of the hyperellipsoid \(P\).
Nonlinear estimates for reaction diffusion equations

\[ u_{0,j} = \left( \frac{\eta}{\alpha_j} \right)^{1/m_j} \geq u_{2,j}, \quad j = 1, 2, \ldots, n. \]

Finally we take

\[ \lambda_1 = \eta \max_{1 \leq i \leq n} d_i \]

such that the \( u_j \)-intercept of the hyperellipsoid \( Q_1 \)

\[ u_{1,j} = \left( \frac{\lambda_1}{\alpha_j d_j} \right)^{1/m_j} \geq u_{0,j}, \quad j = 1, 2, \ldots, n. \]

Combining (35), (36), and (37), we have

\[ \lambda_1 = \left( \frac{\max_{1 \leq i \leq n} \alpha_i d_i (\bar{u}_i)^{m_i}}{\min_{1 \leq i \leq n} d_i} \right)^{1/n} \]

(38)

We follow exactly the same contradiction argument to prove \( q(x) \leq \lambda_1 \) for \( x \in \mathbb{R} \) as in the proof of Theorem 1, which is omitted here. Since \( (\alpha_i)_{i=1}^n \in (\mathbb{R}^+)^n \) is arbitrary, \( q(x) = \sum_{i=1}^n \alpha_i d_i (u_i(x))^{m_i} \leq \lambda_1 \) implies the upper bound (9). Now we use the inequality of arithmetic and geometric means to obtain

\[ \sum_{i=1}^n \alpha_i (u_i(x))^{m_i} \geq n \left( \prod_{i=1}^n \alpha_i (u_i(x))^{m_i} \right)^{1/n} \geq n \left( \prod_{i=1}^n \alpha_i \right)^{1/n} \left( \prod_{i=1}^n (u_i(x))^{m_i} \right)^{1/n}, \]

(39)

which together with (9) yields (10).

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References

1. Ahmad, S., Lazer, A.C.: An elementary approach to traveling front solutions to a system of \( N \) competition-diffusion equations. Nonlinear Anal. 16, 893–901 (1991)
2. Chen, C.-C., Hsiao, T.-Y., Hung, L.-C.: Discrete n-barrier maximum principle for a lattice dynamical system arising in competition models. Discrete Contin. Dyn. Syst. A 40(1), 153–187 (2020)
3. Chen, C.-C., Hung, L.-C.: A maximum principle for diffusive lotka-volterra systems of two competing species. J. Differ. Equ. 261, 4573–4592 (2016)
4. Chen, C.-C., Hung, L.-C.: Nonexistence of traveling wave solutions, exact and semi-exact traveling wave solutions for diffusive Lotka–Volterra systems of three competing species. Commun. Pure Appl. Anal. 15, 1451–1469 (2016)
5. Chen, C.-C., Hung, L.-C.: An n-barrier maximum principle for elliptic systems arising from the study of traveling waves in reaction–diffusion systems. Discrete Contin. Dyn. Syst. B 22, 1–19 (2017)
6. Chen, C.-C., Hung, L.-C., Lai, C.-C.: An n-barrier maximum principle for autonomous systems of n species and its application to problems arising from population dynamics. Commun. Pure Appl. Anal. 18, 33–50 (2019)
7. Chen, C.-C., Hung, L.-C., Liu, H.-F.: N-barrier maximum principle for degenerate elliptic systems and its application. Discrete Contin. Dyn. Syst. A 38, 791–821 (2018)
8. Fei, N., Carr, J.: Existence of travelling waves with their minimal speed for a diffusing Lotka–Volterra system. Nonlinear Anal. Real World Appl. 4, 503–524 (2003)
9. Hou, X., Leung, A.W.: Traveling wave solutions for a competitive reaction–diffusion system and their asymptotics. Nonlinear Anal. Real World Appl. 9, 2196–2213 (2008)
10. Kan-on, Y.: Parameter dependence of propagation speed of travelling waves for competition-diffusion equations. SIAM J. Math. Anal. 26, 340–363 (1995)
11. Kan-on, Y.: Fisher wave fronts for the Lotka–Volterra competition model with diffusion. Nonlinear Anal. 28, 145–164 (1997)
12. Kanel, J.I.: On the wave front solution of a competition-diffusion system in population dynamics. Nonlinear Anal. 65, 301–320 (2006)
13. Kanel, J.I., Zhou, L.: Existence of wave front solutions and estimates of wave speed for a competition-diffusion system. Nonlinear Anal. 27, 579–587 (1996)
14. Leung, A.W., Hou, X., Feng, W.: Traveling wave solutions for Lotka–Volterra system re-visited. Discrete Contin. Dyn. Syst. B 15, 171–196 (2011)
15. Leung, A.W., Hou, X., Li, Y.: Exclusive traveling waves for competitive reaction–diffusion systems and their stabilities. J. Math. Anal. Appl. 338, 902–924 (2008)
16. Murray, J.D.: Mathematical Biology, vol. 19, 2nd edn. Springer, Berlin (1993). Biomathematics
17. Tang, M., Fife, P.: Propagating fronts for competing species equations with diffusion. Arch. Ration. Mech. Anal. 73, 69–77 (1980)
18. Volpert, A.I., Volpert, V.A., Volpert, V.A.: Traveling wave solutions of parabolic systems, vol. 140 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI. Translated from the Russian manuscript by James F. Heyda (1994)

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