ON EXTENSION OF GREEN’S OPERATOR ON BOUNDED SMOOTH DOMAINS

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Abstract. We prove a regularity result for Green’s functions that are associated to elliptic second order divergence-type linear PDO’s with coefficients in $C^{1,\alpha}(\Omega)$. Here $\alpha \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2,\alpha}$ domain in dimension $n \geq 3$. The regularity result gives boundary estimates for derivatives up to order $(2 + \alpha)$ and, by using these estimates, we extend the associated Green’s operator to a globally defined singular integral which of Calderón–Zygmund type.

1. INTRODUCTION

1.1. Background. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain satisfying an exterior ball condition, and consider the boundary value problem:

\[
\begin{aligned}
- Lu & = f \in L^2(\Omega), \\
   u & \in W^{1,2}_0(\Omega).
\end{aligned}
\] (1.1)

Here $L$ is a second order partial differential operator, which is of divergence type,

\[
Lu = \sum_{i,j=1}^n \partial_i (a^{ij} \partial_j u)
\] (1.2)

such that the coefficients $a^{ij} \in L^\infty(\Omega)$ are symmetric ($a^{ij} = a^{ji}$) and $L$ is strictly elliptic: there is a constant $\lambda > 0$ such that for almost every $x \in \Omega$, we have

\[
\lambda |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \quad \forall \xi \in \mathbb{R}^n.
\] (1.3)

A prototype is the Laplacian $L = \Delta = \sum_{i=1}^n \partial_i^2$ for which the problem (1.1) in case of domains with only Lipschitz boundary is studied in [JK95].

It is well known that the solution of (1.1) can be expressed in terms of a so called Green’s operator:

\[
u(x) = Gf(x) = \int_{\Omega} G(x, y)f(y)dy, \quad x \in \Omega.
\] (1.4)

The existence of Green’s operator is established in the fundamental paper [GWS2]. In what follows we recapitulate some results therein.

The following existence result is of importance to us.

1.5. Theorem. There exists a unique function $G : \Omega \times \Omega \setminus \{(x, x)\} \to \mathbb{R}$, $G \geq 0$, such that for every $x \in \Omega$,

\[
G(x, \cdot) \in W^{1,1}_0(\Omega) \cap W^{1,2}(\Omega \setminus B(x, r))
\]
and also, if $\varphi \in C_0^\infty(\Omega)$, then
\[ \langle -LG(x,\cdot), \varphi \rangle = \sum_{i,j=1}^n \int_{\Omega} a^{ij}(y) \partial_y G(x,y) \partial_y \varphi(y) dy = \varphi(x), \quad x \in \Omega. \]

The function $G$ is the Green’s function for the operator $-L$. It satisfies $G(x,y) = G(y,x)$ and $G(x,y) \leq C|x-y|^{2-n}$ if $x,y \in \Omega$.

A proof can be found in [GW82]. Since $\Omega$ satisfies, in particular, an exterior cone condition, the Green’s function has Hölder regularity even if the coefficients $a^{ij}$ are only essentially bounded, see [GW82] Theorem 1.8, Theorem 1.9. More regularity is available if one assumes that the coefficients are Dini continuous,

\[ |a^{ij}(x) - a^{ij}(y)| \leq \omega(|x-y|), \quad x,y \in \Omega. \]

Here $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is supposed to be non-decreasing, $\omega(2t) \leq K \omega(t)$ for some $K > 0$ and all $t > 0$ and

\[ \int_0^1 \frac{\omega(t)}{t} dt < \infty. \]

In case the coefficients belong to the space $C^\alpha(\Omega) \supset C^{1,\alpha}(\Omega)$, they satisfy the Dini condition. The following is proven in [GW82] Theorem 3.3.

1.7. Theorem. Assume that (1.3) holds and the coefficients $a^{ij}$ are Dini continuous. Then the Green’s function of the corresponding differential operator $-L$ satisfies the following inequalities for any $x,y \in \Omega$; here $\delta(\cdot) = \text{dist}(\cdot, \partial \Omega)$,

\[ G(x,y) \leq C|x-y|^{2-n} \min \left\{ 1, \frac{\delta(x)}{|x-y|}, \frac{\delta(x)\delta(y)}{|x-y|^2} \right\}; \]

\[ |\nabla_y G(x,y)| \leq C|x-y|^{1-n} \min \left\{ 1, \frac{\delta(x)}{|x-y|} \right\}. \]

Furthermore, the mixed derivatives satisfy $|\nabla_x \nabla_y G(x,y)| \leq C|x-y|^{-n}$, $x,y \in \Omega$.

The following estimate for more regular coefficients is in [Fas98] Theorem 1.8.

1.8. Theorem. Assume that (1.3) holds and the coefficients $a^{ij}$ belong to $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. Then

\[ |\nabla_x^2 \nabla_y G(x,y)| \leq C|x-y|^{-n} \min\{ |x-y|, \delta(x) \} \]

for every $x,y \in \Omega$.

Estimates like above have been used in establishing weak type $(1,1)$ estimates for operators $\nabla^2G$ on bounded and convex domains, $n \geq 3$. In case of Laplacian this is an unpublished result due to Dahlberg, Verchota, and Wolff; a proof can be found in [Fro93]. In case of Lipschitz coefficients similar methods are shown to apply [Fas98]. The weak type estimate is established by utilizing theory of singular integrals in $\mathbb{R}^n$ – first one extends kernel $\nabla_x^2 G(x,y)$ by zero and then proves that extended kernel $K = \chi_{\Omega \times \Omega} \nabla_x^2 G$ satisfies the Hörmander condition

\[ \int_{|x-y| \geq 5|h|} |K(x,y+h) - K(x,y)| dx \leq C \]

with $C$ independent of $y,h \in \mathbb{R}^n$.

In this paper we study when $\nabla^2 G$ extends to a Calderón–Zygmund operator [DJ84]. These are linear operators $T \in \mathcal{L}(L^2(\mathbb{R}^n))$ having a kernel representation

\[ \langle Tf, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x,y) f(y) g(x) dy dx \]
if the supports of \( f, g \in C^\infty_0 \subset \mathcal{S} \) are disjoint. In this connection it is assumed that \( K \) is a so called Calderón–Zygmund kernel: there is \( \delta \in (0, 1] \) such that

\[
|K(x, y)| \leq C|x - y|^{-n}, \quad x, y \in \mathbb{R}^n, \tag{1.10}
\]

and the transposed kernel \( K^t(x, y) = K(y, x) \) also satisfies estimates (1.10). It is easy to verify that \( K \) satisfies the Hörmander condition (1.9), so the operator \( T \) is of weak type \( (1, 1) \), and bounded on \( L^p(\mathbb{R}^n) \) for \( p \in (1, \infty) \).

In light of estimates (1.10) it is a prerequisite for the extension of \( \nabla^2 G \) that one can control the derivatives of \( K = \nabla^2_x G(x, y) \) up to order \( \leq 2 + \delta \) and up to the boundary. To indicate this, assume that \( \Omega \) is convex and bounded. We can estimate the Hölder condition in (1.10) by using the mean-value theorem and Theorem 1.8. Accordingly, there is \( \xi \in [y, y + h] \subset \Omega \) for which

\[
|K(x, y + h) - K(x, y)| \leq |h||\nabla_y K(x, \xi)| \leq C|h||x - y|^{-n} \max\{1, |x - y|, \delta(x)\}.
\]

The upper bound blows up when \( x \) tends to the boundary. In particular, extension of \( K \) to a Calderón–Zygmund kernel is not possible by using these estimates.

Boundary estimates for higher order derivatives of Green’s functions, associated to uniformly elliptic operators of order \( 2m \) on bounded domains, are derived in [Kra67] and later refined in [DS04]. Therein, if \( m = 1 \) (as in our case) and dimension \( n \geq 3 \), the coefficients of \( L \) should belong to the space \( C^7(\overline{\Omega}) \) and \( \partial \Omega \) to the class \( C^3 \).

1.2. Main results. First we establish (order \( 2 + \alpha \)) boundary regularity estimates for Green’s functions under reasonable assumptions. Using these we then extend the corresponding Green’s operator to a Calderón–Zygmund operator.

Throughout this section we will assume that \( \Omega \subset \mathbb{R}^n, n \geq 3 \), is a bounded \( C^{2, \alpha} \) domain, \( \alpha \in (0, 1) \). In particular, this domain satisfies an exterior ball condition. We will also assume that the coefficients \( a^{ij} \) are \( C^{1, \alpha}(\Omega) \) regular, and we will denote

\[
\delta(\cdot) = \text{dist}(\cdot, \partial \Omega).
\]

Here is our boundary regularity result for derivatives up to order \( 2 + \alpha \):

1.11. Theorem. Assume that \( \Omega \subset \mathbb{R}^n \) and the coefficients \( a^{ij} \) are as above. Then, if \( \beta \in \mathbb{N}^n_0 \) satisfies \( |\beta| \leq 2 \), we have

\[
|\partial^\beta_y G(x, y)| \leq C|x - y|^{2-n-|\beta|} \min\left\{1, \frac{\delta(x)}{|x - y|}\right\}, \quad x, y \in \Omega. \tag{1.12}
\]

Furthermore, if \( |\beta| = 2 \), then

\[
|\partial^\beta_y G(x, y + h) - \partial^\beta_y G(x, y)| \leq C|h|^\alpha|x - y|^{-n-\alpha} \min\left\{1, \frac{\delta(x)}{|x - y|}\right\} \tag{1.13}
\]

for every \( x, y, y + h \in \Omega \) satisfying \( |h| \leq |x - y|/2 \).

If \( |\beta| < 2 \), estimate (1.12) is covered in Theorem 1.7. Under further regularity assumptions for the coefficients and domain, estimates like (1.12) for higher order derivatives are established in [DS04, Theorem 12].

The proof of Theorem 1.11 relies on the (known) size estimate

\[
G(x, y) \leq C|x - y|^{2-n} \min\left\{1, \frac{\delta(x)}{|x - y|}\right\}, \quad x, y \in \Omega,
\]

and certain local boundary type Schauder estimates [GT83]. Latter estimates are available on bounded \( C^{2, \alpha} \) smooth domains, and they give us control to the solutions of \( Lu = 0 \) up to the boundary of the domain and for derivatives up to order \( 2 + \alpha \).
We then utilize the refined estimates given in Theorem 1.11 by showing that the Green’s operator extends to an integral operator in \( \mathbb{R}^n \), whose second order partials are Calderón–Zygmund type operators. For this purpose we will invoke various results about weakly singular integral operators on domains whose theory is developed in the thesis [Väh09]. First we invoke following spaces.

1.14. Definition. Let \( \emptyset \neq D \subset \mathbb{R}^n \), \( n \geq 2 \), be a domain. Assume that \( m \in \mathbb{N} \) and \( 0 < \delta < 1 \). The space of smooth kernels, denoted by \( \mathcal{K}^{-m}_D(\delta) \), consists of complex-valued functions \( K \in C^m(D \times D \setminus \{(x, x)\}) \) satisfying

- size-estimate, given \( \alpha, \beta \in \mathbb{N}_0^n \) so that \(|\alpha| + |\beta| \leq m\),
  \[ |\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C_K|x - y|^{m - |\alpha| - |\beta|}, \quad x, y \in D. \]

- Hölder-regularity estimate, given \( \alpha, \beta \in \mathbb{N}_0^n \) so that \(|\alpha| + |\beta| = m\),
  \[ |\partial_x^\alpha \partial_y^\beta K(x, y + h) - \partial_x^\alpha \partial_y^\beta K(x, y)| \leq C_K|h|^\delta|x - y|^{m - \delta}, \quad \text{if } x, y, y + h \in D \text{ satisfy } |h| \leq |x - y|/2. \]

It is important to observe that the order \( m \) derivatives of kernels in \( \mathcal{K}^{-m}_D(\delta) \) are Calderón–Zygmund kernels, that is, they satisfy estimates (1.10). This follows from the case \( D = \mathbb{R}^n \) and \(|\alpha| + |\beta| = m\) in Definition 1.14.

We show that Green’s function belongs to the space \( \mathcal{K}^{-2}_\Omega(\alpha) \). Furthermore, as a main result of this paper, we will establish the following extension theorem.

1.15. Theorem. Let \( \Omega \) and \( a^{ij} \) be as quantified above. Then there exists a smooth kernel \( \hat{G} \in \mathcal{K}^{-2}_\Omega(\alpha) \) such that

\[ \hat{G}|\Omega \times \Omega \setminus \{(x, x)\} = G, \]

and the operators \( \partial^\sigma \hat{G}, \partial^\sigma \hat{G}^* \), \( |\sigma| = 2 \), are Calderón–Zygmund operators. Hence they belong to \( L^2(\Omega) \). Here \( \hat{G} \) denotes the integral operator associated with \( G \), that is,

\[ \hat{G}f(x) = \int_{\mathbb{R}^n} \hat{G}(x, y) f(y) dy, \quad f \in C^\infty_0(\mathbb{R}^n). \]

It follows that the differentiated Green’s operators \( \partial^\sigma \hat{G}, \partial^\sigma \hat{G}^* \), \( |\sigma| = 2 \), are restrictions of Calderón–Zygmund operators to the domain \( \Omega \),

\[ \langle Gf, \partial^\sigma g \rangle = \langle \hat{G}f, \partial^\sigma g \rangle, \quad f, g \in C^\infty_0(\Omega). \]

This paper is organized as follows: in Section 2 we define weakly singular integral operators and invoke their basic properties from [Väh09]. We also prove a sharpening of one of the results in [Väh09]; this sharpening is needed. Section 3 begins with Schauder estimates, taken from [GT83], and it ends with proofs of Theorem 1.11 and Theorem 1.15.

2. Weakly singular integral operators

Here we recapitulate theory of weakly singular integral operators on domains, developed in [Väh09]. First we define bounded \( C^{2,\alpha} \) domains, but also uniform and coplump domains. The latter classes of domains are useful in connecting to theory of weakly singular integrals.
2.1. Classes of domains. The Green’s function will be defined on a bounded $C^{2,\alpha}$ domain. For later purposes we need to verify that such a domain is both uniform and coplump. This type of results are well known, so we only indicate the proofs.

2.1. Definition. A bounded domain $\Omega \neq \emptyset$ in $\mathbb{R}^n$ is of class $C^{2,\alpha}$, $0 < \alpha < 1$, if at each point $\bar{y} \in \partial \Omega$ there is a ball $B = B(\bar{y}, \rho(\bar{y}))$ and a diffeomorphism $\psi : B \to D \subset \mathbb{R}^n$ such that the following conditions (1)–(3) hold:

1. $\psi(B \cap \Omega) \subset \mathbb{R}^n_+$;
2. $\psi(B \cap \partial \Omega) \subset \partial \mathbb{R}^n_+$;
3. $\psi \in C^{2,\alpha}(\bar{B}), \psi^{-1} \in C^{2,\alpha}(\bar{D})$.

In (3) we assume that norms of the diffeomorphisms $\psi$ are uniformly bounded by some constant $K = K(\Omega) > 0$,

$$||\psi||_{C^{k,\alpha'}(\bar{B})} + ||\psi^{-1}||_{C^{k,\alpha'}(\bar{\partial})} \leq K, \quad k + \alpha' \leq 2 + \alpha.$$ 

In particular, the Lipschitz constants of $\psi$ and $\psi^{-1}$ bounded by $K$. We shall say that the diffeomorphism $\psi$ straightens the boundary near $\bar{y}$.

That a bounded domain is of class $C^{2,\alpha}$ is a local property of its boundary. If $\Omega$ is such a domain then, due to compactness of $\partial \Omega$, there exists $\rho > 0$ and a set $\{\bar{y}_1, \ldots, \bar{y}_k\} \subset \partial \Omega$ such that, if $\bar{y} \in \partial \Omega$ is a boundary point, then

$$B(\bar{y}, \rho) \subset B(\bar{y}_\ell, \rho(\bar{y}_\ell))$$

for some $\ell \in \{1, 2, \ldots, k\}$. In the sequel we assume that $\rho$ satisfies $\rho / \text{diam}(\Omega) < 1/4$.

2.2. Definition. A domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is $c$-uniform for $c \geq 1$ if every pair of distinct points $x, y \in \Omega$ can be joined by a $c$-cigar in $\Omega$, that is, there exists a continuum $E \subset \Omega$ containing these two points such that $\text{diam}(E) \leq c|x - y|$ and

$$\min\{|z - x|, |z - y|\} \leq c\text{dist}(z, \partial \Omega)$$

if $z \in E$.

In [Vái09, Definition 1.13] we pose a definition which is based on rectifiable paths, but this is equivalent to the definition given above [Vái88].

2.3. Definition. A domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is $c$-coplump, $c \geq 1$, if for all $x \in \mathbb{R}^n \setminus \Omega$ and $0 < r < \text{diam}(\mathbb{R}^n \setminus \Omega)$ there is $z \in B(x, r)$ with $B(z, r/c) \subset \mathbb{R}^n \setminus \Omega$.

Uniformity and coplumpness are local properties of the boundary of the domain. This is easily seen for the coplumpness, and we omit the precise formulation. For uniformity we invoke the following theorem, due to J. Väisälä [Vái88, Theorem 4.1].

2.4. Theorem. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain and that $c \geq 1$, $0 < r < \text{diam}(\Omega)$. Suppose also that if $z \in \partial \Omega$, then every pair of points in $\Omega \cap B(z, r)$ can be joined by a $c$-cigar in $\Omega$. Then $\Omega$ is $c_1$-uniform with $c_1 = 40c^2\text{diam}(\Omega)/r$.

Here is a simple consequence of the locality of all definitions given above.

2.5. Theorem. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{2,\alpha}$ domain. Then

- $\Omega$ is $c$-uniform with $c = 200c_n^3K^6\text{diam}(\Omega)/\rho$, where $c_n$ is any constant such that $\mathbb{R}^n_+$ is $c_n$-uniform.
- $\Omega$ is $c$-coplump for $c = 8K^3\text{diam}(\Omega)/\rho$.

We omit the proof. It relies on the bi-Lipschitz bound $K$ of the diffeomorphisms straightening the boundary near the boundary points. Hence the result holds true if we only assume that $\Omega$ is a bounded Lipschitz domain, that is, the mappings $\psi$ are only assumed to be bi-Lipschitz.
2.2. Standard kernels and their regularity. A so called standard kernel space furnishes an approximation theoretic approach to smooth kernels if the underlying domain is uniform. This space is defined in terms of integral averages of differences, and its advantages include that it is – a priori – easier to verify that a given kernel is standard than smooth.

The difference operators \( y \mapsto \Delta^\ell_h(f, D, y) : \mathbb{R}^n \to \mathbb{C} \) are parametrized by \( \ell \in \mathbb{N} \), \( h \in \mathbb{R}^n \), and \( D \subset \mathbb{R}^n \). These operate on functions \( f : D \to \mathbb{C} \) according to the rule

\[
\Delta^\ell_h(f, D, y) = \begin{cases} 
\sum_{k=0}^{\ell} (-1)^{\ell+k} (\frac{\ell}{k}) f(y + kh), & \text{if } \{y, y+h, \ldots, y+\ell h\} \subset D, \\
0, & \text{otherwise}.
\end{cases}
\]

2.6. Definition. Let \( \emptyset \neq \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a domain. Let \( m \in \mathbb{N} \) and \( 0 < \delta < 1 \). The space of standard kernels, denoted by \( K^m_\Omega(\delta) \), consists of continuous functions \( K : \Omega \times \Omega \setminus \{(x, x)\} \to \mathbb{C} \) satisfying

- kernel size estimate

\[
|K(x, y)| \leq C_K |x - y|^{m-n}, \quad x, y \in \Omega,
\]

- semilocal integral estimate

\[
\sup_{|h| \leq \text{diam}(B)} \frac{1}{|B|^{1+(m+\delta)/n}} \int_B |\Delta^{m+1}_h(K(x, \cdot), B, y)| dy \leq C_K |x - y^B|^{-n-\delta},
\]

if \( x \in \Omega \) and \( B \subset \subset \Omega \) is a ball, centered at \( y^B \) so that \( C_K \text{diam}(B) \leq |x - y^B| \). We also assume the estimate (2.8) with \( K(x, \cdot) \) replaced by \( K(\cdot, x) \).

Notice that we use balls in the definition of standard kernels instead of cubes, but this difference compared to the definition given in [Väh09] is not important. Here is a result stating that standard kernels are smooth if the underlying domain is uniform.

2.9. Theorem. Let \( \Omega \subset \mathbb{R}^n \) be a uniform domain and \( 0 < m < n \) and \( 0 < \delta < 1 \). Then the classes of standard and smooth kernels coincide, that is,

\[
K^m_\Omega(\delta) = K^{-m}_\Omega(\delta).
\]

As a consequence, if \( K \in K^m_\Omega(\delta) \) and \( \alpha, \beta \in \mathbb{N}_0^\mathbb{N} \) satisfy \( |\alpha| + |\beta| = m \), then

\[
\partial_x^\alpha \partial_y^\beta K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\} \to \mathbb{C}
\]

is a Calderón–Zygmund kernel.

A related result – where the sharp Hölder exponent is missing – is proven in [Väh09, Chapter 4] by using approximation theoretic approach. In what follows we modify that approach, and thereby provide a proof for Theorem 2.9. Generally speaking, the proof is based on so called dyadic resolution of unity on uniform domains. Let us explain what we mean by this: In case \( \Omega = \mathbb{R}^n \), we can fix one bump function \( \varphi \in C^\infty_0(\mathbb{R}^n) \) so that \( \int_{\mathbb{R}^n} \varphi(x) dx = 1 \) and define \( \varphi_j = 2^{jn} \varphi(2^j \cdot) \) for \( j \in \mathbb{Z} \). This results in the decomposition

\[
(2.10) \quad f(x) = \int_{\mathbb{R}^n} f(y) \varphi\ell(y - x) dy + \sum_{j=\ell+1}^{\infty} \int_{\mathbb{R}^n} f(y) (\varphi_j - \varphi_{j-1})(y - x) dy , \quad \ell \in \mathbb{Z},
\]

assuming, say, that \( f \) is continuous at \( x \).

In proper domains the difficulties lie in modifying this construction such that the supports of the bump functions are included in the domain and the coarseness parameter \( \ell \) is independent of \( \delta(x) \). To indicate the difficulties, one expects vanishing
moments from the difference of two consecutive bump functions in order to induce cancellation. There are also certain geometric properties that the modification should preserve.

We begin the proof of Theorem 2.9 with invoking a dyadic resolution of a given kernel $K \in K_{\Gamma \ast}^m(\delta)$ in a uniform domain $\Omega \subset \mathbb{R}^n$ [Väh09, p. 69]. Let $x_0, y_0 \in \Omega$ be distinct points and let $\ell = \ell(x_0, y_0)$ be defined by $2^{-\ell} < |x_0 - y_0|/16 < 2^{-(\ell+1)}$. Then we let $\{\varphi_{\sigma,2^{-\ell}}\}_{j \geq \ell}, \{\varphi_{\rho,2^{-\ell}}\}_{j \geq \ell}$ be $m$-regular bump functions along so called quasihyperbolic geodesics $\sigma : x_0 \sim y_0$ and $\rho = \sigma^{-1} : y_0 \sim x_0$. These bump functions approximate the Dirac’s delta at the origin and satisfy supp $\varphi_{\sigma,2^{-\ell}}(-x) \subset \Omega$ if $x$ is close to $x_0$. The geometry of uniform domains plays a crucial role in the construction of these functions.

Denote

$$\Omega(x_0, y_0) = B(x_0, r(x_0) \wedge (|x_0 - y_0|/4b)), \quad r(x_0) = \text{dist}(x_0, \partial \Omega)/4b,$$

where the constant $b \geq 1$ depending on the geometry of the domain is given in Lemma [Väh09] Lemma 4.16. Then, if $j, k \geq \ell = \ell(x_0, y_0)$ and $(x, y) \in \Omega(x_0, y_0) \times \Omega(y_0, x_0) \subset \Omega \times \Omega$, we denote

$$(2.11) \quad K_{j}^{\sigma,\rho}(x, y) = \int_{\Omega} \varphi_{\sigma,2^{-j}}(\alpha - x) \int_{\Omega} K(\alpha, \omega) \varphi_{\rho,2^{-j}}(\omega - y) d\omega d\alpha.$$

Due to continuity of the kernel in the domain $\Omega \times \Omega \setminus \{(x, x)\}$, we have the following decomposition, valid for the points $(x, y) \in \Omega(x_0, y_0) \times \Omega(y_0, x_0)$,

$$(2.12) \quad K(x, y) = \lim_{j \to \infty} K_{j}^{\sigma,\rho}(x, y) = K_{\ell}^{\sigma,\rho}(x, y) + \sum_{j = \ell + 1}^{\infty} (K_{j}^{\sigma,\rho} - K_{j-1}^{\sigma,\rho})(x, y).$$

For notational purposes it is convenient to denote $K_{j}^{\sigma,\rho} = K_{\ell}^{\sigma,\rho}$ and express the differences inside the summation in the following way. Given $j > \ell$ and points $x, y$ as above, we also denote

$$K_{j}^{\sigma,\rho}(x, y) := K_{j}^{\sigma,\rho}(x, y) - K_{\ell}^{\sigma,\rho}(x, y)$$

$$= \int_{\Omega} (\varphi_{\sigma,2^{-j}} - \varphi_{\sigma,2^{-j+1}})(\alpha - x) \int_{\Omega} K(\alpha, \omega) \varphi_{\rho,2^{-j}}(\omega - y) d\omega d\alpha$$

$$+ \int_{\Omega} \varphi_{\sigma,2^{-j+1}}(\alpha - x) \int_{\Omega} K(\alpha, \omega) (\varphi_{\rho,2^{-j}} - \varphi_{\rho,2^{-j+1}})(\omega - y) d\omega d\alpha$$

$$=: \mu_{j}^{\sigma,\rho}(x, y) + \nu_{j}^{\sigma,\rho}(x, y).$$

As a consequence, we can write

$$(2.14) \quad K(x, y) = \sum_{j = \ell}^{\infty} K_{j}^{\sigma,\rho}(x, y), \quad (x, y) \in \Omega(x_0, y_0) \times \Omega(y_0, x_0).$$

The proof of the following auxiliary lemma is essentially the same as the proof of [Väh09] Lemma 4.33. One of the important ingredients is that the differences of consecutive bump functions satisfy

$$\int_{\mathbb{R}^n} x^\alpha \psi_{\alpha,j}(x) dx = 0 = \int_{\mathbb{R}^n} x^\alpha \psi_{\rho,j}(x) dx, \quad |\alpha| \leq m,$$

and one of these differences appear in the definition of both $\mu_{j}^{\sigma,\rho}$ and $\nu_{j}^{\sigma,\rho}$. This allows one to connect to the integral estimate (2.8), satisfied by standard kernels and their transposes.
2.15. **Lemma.** Let \( \Omega \subset \mathbb{R}^n \) be a uniform domain and \( T \in \text{SK}^{-m}_\Omega(\delta) \) be associated with a kernel \( K \in \mathcal{K}^{-m}_\Omega(\delta) \) that is decomposed as in (2.14). Let \( j \geq \ell = \ell(x_0, y_0) \). Then the summands in this decomposition enjoy the regularity

\[
\kappa^\sigma_\rho_j \in C^\infty(\Omega(x_0, y_0) \times \Omega(y_0, x_0))
\]

and, if \( \alpha, \beta \in \mathbb{N}_0^n \) and \( (x, y) \in \Omega(x_0, y_0) \times \Omega(y_0, x_0) \), they satisfy the estimate

\[
|\partial^\alpha_x \partial^\beta_y \kappa^\sigma_\rho_j(x, y)| \leq C 2^{j(|\alpha| + |\beta| - m - \delta)} |x - y|^{-n - \delta},
\]

where the constant \( C \) depends at most on \( n, m, \alpha, \beta, K, \Omega \).

We have performed all the preparations, and we can proceed to the actual proof of Theorem 2.9. It will be a straightforward modification of the proof of Theorem 2.13. Applying the mean value theorem and Lemma 2.15 we find a point

\[
(2.16)
\]

Using the triangle inequality and Lemma 2.15, we also have the estimate

\[
(2.17)
\]

The Weierstrass M-test, combined with the identity (2.14), shows that

\[
K|\{\Omega(x_0, y_0) \times \Omega(y_0, x_0)\} = \sum_{j=\ell}^{\infty} \kappa^\sigma_\rho_j \in C^m(\Omega(x_0, y_0) \times \Omega(y_0, x_0))
\]

and the series can be differentiated termwise up to the order \( m \). As a consequence of this identity we have the regularity \( K \in C^m(\Omega \times \Omega \setminus \{(x, x)\}) \) and, by using (2.17), we also have the estimate

\[
|\partial^\alpha_x \partial^\beta_y K(x_0, y_0)| \leq C |x_0 - y_0|^{-n - |\alpha| - |\beta|}, \quad |\alpha| + |\beta| \leq m,
\]

which is the required size-estimate for smooth kernels.

We turn to the required Hölder-regularity estimates. Due to symmetry it suffices to consider differences in the first \( \mathbb{R}^n \)-variable only. To begin with consider the situation, where \( x_0, y_0 \in \Omega \) are distinct points, \( |\alpha| + |\beta| = m \), and \( h \in \mathbb{R}^n \) is close to \( x_0 \) so that \( x_0 + h \in \Omega(x_0, y_0) \). Fix \( j_0 \in \mathbb{Z} \) such that \( 2^{-j_0} < |h| \leq 2^{-j_0 + 1} \).

Fix \( j \geq \ell = \ell(x_0, y_0) \) and denote

\[
\Delta^j_h(\partial^\alpha_x \partial^\beta_y \kappa^\sigma_\rho_j(\cdot, y_0), x_0) = \partial^\alpha_x \partial^\beta_y \kappa^\sigma_\rho_j(x_0 + h, y_0) - \partial^\alpha_x \partial^\beta_y \kappa^\sigma_\rho_j(x_0, y_0).
\]

Applying the mean value theorem and Lemma 2.15 we find a point \( \xi \in \mathbb{R}^n \), belonging to the line segment \( [x_0, x_0 + h] \subset \Omega(x_0, y_0) \), so that \( |\xi - y_0| \geq |x_0 - y_0|/2 \) and

\[
(2.20)
\]

Using the triangle inequality and Lemma 2.15 we also have the estimate

\[
(2.21)
\]
By using these, we have
\[
\sum_{j=\ell}^{\infty} |\partial_x^\alpha \partial_y^\beta \kappa_j^{\sigma,\rho}(x_0 + h, y_0) - \partial_x^\alpha \partial_y^\beta \kappa_j^{\sigma,\rho}(x_0, y_0)|
\leq C |x_0 - y_0|^{-n-\delta}
\begin{align*}
\sum_{j=-\infty}^{j_0} 2^{j(1-\delta)} + \sum_{j=j_0}^{\infty} 2^{-j\delta}
\leq C |h|^{\delta} |x_0 - y_0|^{-n-\delta}.
\end{align*}
\] (2.22)

It follows that
\[
|\partial_x^\alpha \partial_y^\beta K(x_0 + h, y_0) - \partial_x^\alpha \partial_y^\beta K(x_0, y_0)| \leq C |h|^{\delta} |x_0 - y_0|^{-n-\delta}.
\] if \(x_0 + h \in \Omega(x_0, y_0)\).

What comes next is to establish this estimate for more general \(h\)'s. This argument proceeds precisely as in [Väh09 p. 78], and we omit the details which are based on the uniformity of the domain.

2.3. Extension of kernels. Following theorem gives an extension result for smooth kernels defined on uniform domains. According to Theorem [2.9] it equally well gives an extension theorem for standard kernels.

2.24. Theorem. Let \(\Omega \subset \mathbb{R}^n\) be a uniform domain, \(0 < m < n\), \(0 < \delta < 1\), and \(K \in \mathcal{K}^{-m}(\delta)\) be a smooth kernel. Then there exists \(\tilde{K} \in \mathcal{K}^{-m}(\delta)\) such that
\[
\tilde{K}|\Omega \times \Omega \setminus \{(x, x)\} = K.
\]
In words, \(K\) has an extension to a smooth kernel \(\tilde{K} : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\} \rightarrow \mathbb{C}\).

The proof of this theorem is available in [Väh09, Theorem 5.56]. To describe it briefly, one decomposes smooth kernels by using a partition of unity, subordinate to the Whitney decomposition of the open set \(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\}\). This results in a characterization of smooth kernels in terms of a certain atomic decomposition. In particular, the extension problem reduces to the Hölder extension of individual kernel atoms, which are (a priori) Hölder functions defined on \(\Omega \times \Omega\).

2.4. T1 theorem for restricted WSIO’s. A standard kernel \(K \in \mathcal{K}^{-m}(\delta)\) gives a rise to a weakly singular integral operator (abbreviated WSIO). Such operators emerge in connection with elliptic PDE’s on domains, and their derivatives of order \(m\) should be thought of as singular integral operators. The boundedness properties of WSIO’s are treated in [Väh09].

Here is the formal definition for WSIO’s: an integral operator \(T\) is associated with a standard kernel if there exists \(\tilde{K} \in \mathcal{K}^{-m}(\delta)\), \(m \in \{1, 2, \ldots, n - 1\}\), such that
\[
Tf(x) = \int_{\Omega} K(x, y) f(y) dy, \quad x \in \Omega, \quad f \in C_0^\infty(\Omega).
\]
We denote this by \(T \in \mathcal{S}K^{-m}(\Omega)\). The following theorem describes the boundedness properties of globally defined WSIO’s when restricted to a suitable domain. A proof is in [Väh09, Theorem 3.118].

2.25. Theorem. Let \(\emptyset \neq \Omega \subset \mathbb{R}^n\) be a c-cplump domain such that either \(\Omega = \mathbb{R}^n\) or \(\text{diam}(\mathbb{R}^n \setminus \Omega) = \infty\). Let \(T \in \mathcal{S}K^{-m}(\delta)\), where \(0 < m < n\) and \(0 < \delta < 1\). Then the following two conditions are equivalent
- \(T\chi_\Omega, T^*\chi_\Omega \in f^{m, 2}(\Omega)\),
- \(\partial^{\sigma} T, \partial^{\sigma} T^* \in L(L^2(\Omega))\) if \(|\sigma| = m\).
Furthermore, if these conditions hold true, then there exists \( \hat{T} \in \text{SK}_{\mathbb{R}^n}(\delta) \) whose associated kernel coincides to that of \( T \) on \( \Omega \times \Omega \setminus \{(x,x)\} \),

\[
\langle Tf, g \rangle = \langle \hat{T}f, g \rangle, \quad f, g \in C_0^\infty(\Omega),
\]

(2.26)

the operator \( \hat{T} \) satisfies two equivalent conditions above with \( \Omega = \mathbb{R}^n \), and operators \( \partial^\sigma \hat{T} \) and \( \partial^\sigma \hat{T}^* \), \( |\sigma| = 2 \), are Calderón–Zygmund operators.

The space \( \dot{f}_{\infty}^{m,2}(\Omega) \) is a BMO-type Sobolev space, see [Väh09, Definition 3.47]. Hence Theorem 2.25 is similar to the well known result about Calderón–Zygmund type operators: \( T_1 \) theorem due to David and Journé [DJ84], in which the characterizing conditions for \( L^2 \) boundedness include that \( T_1, T_1^* \in \text{BMO}(\mathbb{R}^n) \).

The boundedness properties of restricted operators can be used to study WSIO’s that are (a priori) defined on domains. Indeed, by using Theorem 2.24 we can first extend a given operator \( T \in \text{SK}_{\mathbb{R}^n}(\delta) \) – which is associated to a kernel \( K \in \text{K}_{\mathbb{R}^n}(\delta) \) – to a globally defined operator \( \hat{T} \in \text{SK}_{\mathbb{R}^n}(\delta) \) if the underlying domain is uniform. This extension is given by the formula

\[
\hat{T}f(x) = \int_{\mathbb{R}^n} \hat{K}(x, y)f(y)dy, \quad x \in \mathbb{R}^n, \quad f \in C_0^\infty(\mathbb{R}^n).
\]

Because the extended kernel \( \hat{K} \) coincides with the kernel \( K \) in \( \Omega \times \Omega \setminus \{(x,x)\} \), we see that \( \partial^\sigma \hat{T} \in \text{L}(L^p(\Omega)) \) if, and only if, \( \partial^\sigma T \in \text{L}(L^p(\Omega)) \).

The proof of Theorem 2.25 depends on certain reflected paraproduct operators whose role is twofold: they are used in a reduction as in the proof of David and Journé, but they also modify the associated kernel \( K \) outside of the product domain \( \Omega \times \Omega \) to reach a Calderón–Zygmund operator \( \hat{T} \) whose second order partials are bounded on \( L^2(\mathbb{R}^n) \). The novelty lies in this modification procedure where certain boundary terms are treated by using coplumpness.

3. PROOF OF MAIN RESULTS

Theorem 1.11 is proven by using Schauder theory. A byproduct is that the Green’s function satisfies the standard kernel estimates, and the Green’s operator is WSIO associated to a standard kernel. Finally the machinery in Section 2 is invoked to finish the proof of Theorem 1.15.

3.1. Schauder estimates on \( C^{2,\alpha} \) domains. We invoke certain estimates for the solutions of second order elliptic equations. These are classical Schauder estimates involving a (possibly empty) boundary portion [GT83].

3.1. Definition. An open set \( D \subset \mathbb{R}^n \) will be said to have a boundary partion \( T \subset \partial D \) (of class \( C^{2,\alpha} \)) if at each point \( \tilde{y} \in T \) there is a ball \( B \subset B(\tilde{y}, \rho) \) which is centered at \( \tilde{y} \), which satisfies \( B \cap \partial D \subset T \), and in which the conditions (1)–(3) of Definition 2.1 are satisfied.

We invoke the following notation from [GT83] pp. 95–96. Let \( D \) be an open set in \( \mathbb{R}^n \) with a boundary partion \( T \) of class \( C^{2,\alpha} \). For \( x, y \in \Omega \), we set

\[
\delta_x = \text{dist}(x, \partial D \setminus T), \quad \delta_{x,y} = \delta_x \wedge \delta_y.
\]
For bounded continuous functions $u \in C(D)$ we define $|u|_{0,D} = \sup_{x \in D} |u(x)|$ and, for functions $u \in C^{k,\alpha}(D \cup T)$, we define
\[
[u]_{k,0;D \cup T}^* = \begin{cases} 
[u]_{k;D \cup T} = \sup_{x \in D, |\beta| = k} \delta_x^k |\partial^\beta u(x)|, & k = 0, 1, \ldots 
\end{cases}
\]
\[
[u]_{k,\alpha;D \cup T}^* = \sup_{x,y \in D, |\beta| = k} \delta_{x,y}^{k+\alpha} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|}{|x - y|^\alpha}, & 0 < \alpha \leq 1;
\]
\[
[u]_{k,0;D \cup T}^* = \sum_{j=0}^k [u]_{j;D \cup T}^*; 
[u]_{k,\alpha;D \cup T}^* = [u]_{k;D \cup T}^* + [u]_{k,\alpha;D \cup T}^*;
\]
\[
[u]_{0,\alpha;D \cup T}^{(k)} = \sup_{x \in D} \delta_x^k |u(x)| + \sup_{x,y \in D} \delta_{x,y}^{k+\alpha} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.
\]

In case $T = \emptyset$ we denote $D \cup T = D$ in the definitions above.

We rely on the following local boundary estimate, see [GT83, Theorem 6.2.] and [GT83, Lemma 6.4.].

3.2. Lemma. Assume that $D \subset \mathbb{R}^n$ is a proper open subset of $\mathbb{R}^n_+$ with (possibly empty) boundary portion $T \subset \partial \mathbb{R}^n_+ \cap \partial D$. Suppose that $\tilde{u} \in C^{2,\alpha}(D \cup T)$ is a bounded solution in $D$ of
\[
\tilde{L}\tilde{u}(x) = \sum_{i,j=1}^n \tilde{a}^{ij}(x) \partial_{ij} \tilde{u}(x) + \sum_{i=1}^n \tilde{b}^i(x) \partial_i \tilde{u}(x) = 0, \quad x \in D,
\]
and it satisfies the boundary condition $\tilde{u} \equiv 0$ on $T$. We also assume that $\tilde{L}$ is strictly elliptic in the sense that there exists a positive constant $\tilde{\lambda} > 0$ such that
\[
\sum_{i,j=1}^n \tilde{a}^{ij}(x) \xi_i \xi_j \geq \tilde{\lambda} |\xi|^2, \quad x \in D, \quad \xi \in \mathbb{R}^n.
\]
Furthermore, if $i,j \in \{1,2,\ldots,n\}$, the coefficients are assumed to satisfy $\tilde{a}^{ij} = \tilde{a}^{ji}$ and
\[
|\tilde{a}^{ij}(0)|_{0,\alpha;D \cup T} + |\tilde{b}^i(1)|_{0,\alpha;D \cup T} \leq \tilde{\Lambda}.
\]
Under these assumptions, we have the estimate
\[
\|\tilde{u}\|_{2,\alpha;D \cup T} \leq C \|\tilde{u}\|_{0,D},
\]
where $C = C(n,\alpha,\tilde{\lambda},\tilde{\Lambda})$.

As a consequence, we obtain the following local boundary estimate for curved boundaries.

3.3. Lemma. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain and $0 < S \leq \rho$. Let $\bar{y} \in \partial \Omega$ and
\[
B = B(\bar{y}, S) \cap \Omega, \quad T = B(\bar{y}, S) \cap \partial \Omega \subset \partial B.
\]
Let $u \in C^{2,\alpha}(\bar{B})$ satisfy $Lu = 0$ on $B$, where $L$ is defined in (1.2), and assume that $u \equiv 0$ on $T$. Then
\[
[u]_{2,\alpha;B \cup T} \leq C[u]_{0,B},
\]
where $C = C(\alpha, K, \Omega, L)$. 

Proof. By Definition 2.1 there is a neighborhood $N$ of $\bar{y}$ such that $B(\bar{y}, S) \subset N$, and a diffeomorphism $\psi$ defined on $N$ that straightens the boundary near $\bar{y}$. Denote

$$D' = \psi(B) \subset \mathbb{R}^n, \quad T' = \psi(T) \subset \partial \mathbb{R}^n \cap \partial D'.$$

Then $T'$ is a boundary portion of $D'$. Under mapping $y = \psi(x) = (\psi_1(x), \ldots, \psi_n(x))$, $x \in \bar{B}$, let $\tilde{u}(y) = u(x)$ and $\tilde{L}\tilde{u}(y) = Lu(x) = 0$, where

$$\tilde{L}\tilde{u}(y) = \sum_{i,j=1}^{n} \tilde{a}^{ij}(y)\partial_{ij}\tilde{u}(y) + \sum_{i=1}^{n} \tilde{b}^{i}(y)\partial_{i}\tilde{u}(y) = 0, \quad y = \psi(x) \in D',$$

and

$$\tilde{a}^{ij}(y) = \sum_{r,s=1}^{n} a^{rs}(x)\partial_{r}\psi_1(x)\partial_{s}\psi_j(x);$$

$$\tilde{b}^{i}(y) = \sum_{r,s=1}^{n} a^{rs}(x)\partial_{rs}\psi_1(x) + \sum_{r=1}^{n} \left( \sum_{s=1}^{n} \partial_{s}a^{sr}(x) \right)\partial_{r}\psi_1(x).$$

A straightforward computation using (1.3) shows that, for $y \in D'$,

$$C(K, n)\lambda|\xi|^2 \leq \lambda|\nabla(\xi \cdot \psi)(x)|^2 \leq \sum_{r,s=1}^{n} a^{rs}(x)\partial_{r}(\xi \cdot \psi)(x)\partial_{s}(\xi \cdot \psi)(x) = \sum_{i,j=1}^{n} \tilde{a}^{ij}(y)\xi_i\xi_j, \quad \xi \in \mathbb{R}^n,$$

for a constant $C(n, K) > 0$, depending on $n$ and on $K$. Because $a^{ij} \in C^{1,\alpha}(\overline{\Omega})$ and $\text{diam}(D') \leq K\text{diam}(B) \leq K\text{diam}(\Omega)$, we also have

$$|\tilde{a}^{ij}^{(0)}|_{0,\alpha;D' \cup T'} + |\tilde{b}^{i(1)}|_{0,\alpha;D' \cup T'} \leq C(\alpha, K, \Omega, L).$$

Furthermore, we have $\tilde{u} = u \circ \psi^{-1} \in C^{2,\alpha}(\overline{D'})$, so that the conditions of Lemma 3.2 are satisfied for the equation $L\tilde{u} = 0$ in $D'$ with the boundary portion $T'$. Therefore we have

$$|u|_{2,\alpha;B \cup T'} \leq C(\alpha, K, \Omega)|\tilde{u}|_{2,\alpha;D' \cup T'}^{*} \leq C(\alpha, K, L)|\tilde{u}|_{0,D'}^{*} = C(\alpha, K, L)|u|_{0,B}^{*},$$

where in the first inequality can be found in [GT83] p. 96] This is the required estimate. \hfill \Box

3.2. Proof of Theorem 1.11. First we establish qualitative $C^{2,\alpha}$ estimates for the Green’s function up to the boundary. In this connection we advance quite rapidly, providing citations to the required regularity results. Then we proceed to quantitative estimates, where the prior Schauder estimates are used. The proof of Theorem 1.11 is finished by local-to-global type Hölder estimate.

To begin with, by using Theorem 1.7 we find that the function

$$u = G(x, \cdot), \quad x \in \Omega,$$

belongs to $C(\partial \Omega \cup \Omega \setminus \{x\})$ if we define $u(y) = 0$ in points $y \in \partial \Omega$. The required exterior sphere condition is satisfied in our situation: by using the implicit function theorem, it follows that $\partial \Omega$ can be locally represented as graph of a $C^{2,\alpha}$ function. Then the exterior sphere property follows from [AKSZ07] Lemma 2.2.
Denote $\Omega(r) = \Omega \setminus B(x, r)$ if $r > 0$ is so small that $B(x, r) \subseteq \Omega$. According to Theorem \[1.3\] function $u$ belongs to the Sobolev space $W^{1,2}(\Omega(r))$, and it is a weak solution to the equation $Lu = 0$ in the domain $\Omega(r)$. That is, it satisfies
\[
\int_{\Omega(r)} a^{ij}(y) \partial_j u(y) \partial_i \varphi(y) dy = 0, \quad \varphi \in C^1_0(\Omega(r)).
\]
The coefficients $a^{ij}$ belong to the space $C^{1,\alpha}(\overline{\Omega})$. In particular, they are bounded and uniformly Lipschitz continuous in $\Omega$. Hence, by using (1.3) and Theorem \[GT83, \text{Theorem 8.8}\], we find that $u \in W^{2,2}_\text{loc}(\Omega(r))$. Furthermore $u$ is a strong solution to the equation $Lu = 0$:
\[
Lu = \sum_{i,j=1}^n a^{ij} \partial_j u + \sum_{i=1}^n \left( \sum_{j=1}^n \partial_j a^{ij} \right) \partial_i u = 0
\]
almost everywhere in $\Omega(r)$. Since the coefficients in (3.4) belong to $C^{0,\alpha}(\overline{\Omega(r)})$, we have the regularity $u \in C^{2,\alpha}(\Omega(r))$ by Theorem \[GT83, \text{Theorem 9.19}\]. From the discussion above it is now clear that $u \in C(\Omega(r)) \cap C^{2,\alpha}(\Omega(r))$ is a classical solution to the equation in $\Omega(r)$ with boundary values $u = 0$ in the $C^{2,\alpha}$ boundary portion $\partial \Omega \subseteq \partial \Omega(r)$. By using \[GT83, \text{Lemma 6.18}\] we then deduce that $u \in C^{2,\alpha}(\Omega(r) \cup \partial \Omega)$ and, because $r > 0$ was arbitrary, we find that $u \in C^{2,\alpha}(\Omega \setminus \{x\} \cup \partial \Omega)$ satisfies the equation (3.4) pointwise in $\Omega \setminus \{x\}$.
To conclude from above and by using Theorem \[1.7\] the Green’s function has the following properties in case $\Omega \subset \mathbb{R}^n, n \geq 3$, is a bounded $C^{2,\alpha}$ domain and the coefficients $a^{ij}$ belong to the space $C^{1,\alpha}(\overline{\Omega})$:
\[
\begin{cases}
L \{G(x, \cdot)\}(y) = 0, & x \in \Omega, y \in \Omega \setminus \{x\}; \\
G(x, \cdot) \in C^{2,\alpha}(\partial \Omega \cup \Omega \setminus \{x\}); & x \in \Omega; \\
G(x, y) = 0, & x \in \Omega, y \in \partial \Omega; \\
G(x, y) \leq C_L |x - y|^{2-n} \min \{1, \delta(x) |x - y|^{-1}\}, & x \in \Omega, y \in \Omega \setminus \{x\}.
\end{cases}
\]
We are ready for the main parts of Theorem \[1.1\].

3.6. \textbf{Lemma.} The following size-estimate is valid for disjoint points $x, y \in \Omega$,
\[
|\partial_y^\beta G(x, y)| \leq C |x - y|^{2-n-|\beta|} \min \left\{1, \frac{\delta(x)}{|x - y|}\right\}, \quad |\beta| \leq 2.
\]
If, in addition $|\beta| = 2$ and $y + h \in B(y, \text{dist}(y, \partial \Omega) \land \rho |x - y|/8 \text{diam}(\Omega))$, then
\[
|\partial_y^\beta G(x, y + h) - \partial_y^\beta G(x, y)| \leq C|h|^\alpha |x - y|^{-n-\alpha} \min \left\{1, \frac{\delta(x)}{|x - y|}\right\}.
\]
Here $C = C(\alpha, K, \Omega, L)$.

\textbf{Proof.} Throughout the proof $C$ denotes a constant, which depends at most on the parameters $\alpha, K, \Omega, L$. We also denote by
\[
\Gamma_\Omega(x, y) := |x - y|^{2-n} \min \{1, \delta(x) |x - y|^{-1}\}
\]
the upper bound in (3.5). Let $x_0, y_0 \in \Omega$. We will treat the following cases
\[
\frac{|x_0 - y_0|}{\text{dist}(y_0, \partial \Omega)} \leq \frac{4 \text{diam}(\Omega)}{\rho}, \quad \frac{|x_0 - y_0|}{\text{dist}(y_0, \partial \Omega)} \geq \frac{4 \text{diam}(\Omega)}{\rho}
\]
separately.
We begin with the first case in (3.9), which is equivalent to that
\[
(3.10) \quad \frac{\rho}{4\text{diam}(\Omega)} |x_0 - y_0| \leq \text{dist}(y_0, \partial \Omega).
\]
Denote \( R = \rho|x_0 - y_0|/8\text{diam}. \) We claim that
\[
(3.11) \quad B(y_0, 2R) \subset \Omega, \quad \text{dist}(x_0, B(y_0, 2R)) > \frac{15}{16}|x_0 - y_0| > 0.
\]
Notice that the inclusion in (3.11) follows from that, if \( y \in B(y_0, 2R) \), then
\[
|y - y_0| < 2R = \frac{\rho}{4\text{diam}(\Omega)} |x_0 - y_0| \leq \text{dist}(y_0, \partial \Omega)
\]
by using (3.10). Furthermore, by using estimate \( \rho/\text{diam}(\Omega) < 1/4 \), we get
\[
|x_0 - y| \geq |x_0 - y_0| - |y_0 - y| > |x_0 - y_0| - 2R = |x_0 - y_0| - \frac{\rho}{4\text{diam}(\Omega)} |x_0 - y_0|
\]
\[
> |x_0 - y_0| - \frac{1}{16}|x_0 - y_0| = \frac{15}{16}|x_0 - y_0|.
\]
The inequality in (3.11) follows by minimizing the left-hand side over \( y \in B(y_0, 2R) \).

Notice that \( D := B(y_0, 2R) \subset \Omega \setminus \{x_0\} \) by (3.11). By using (3.5), we find that the function \( u = G(x_0, \cdot) \) satisfies \( Lu = 0 \) in \( D \) and \( |u(y)| \leq C \Gamma_\Omega(x_0, y) \) for \( y \in D \). Without loss of generality, we can assume that \( D \subset \mathbb{R}^n_+ \). Hence, by using Lemma 3.2 with \( D \subset \mathbb{R}^n_+ \) and \( T = \emptyset \), we obtain the important estimate
\[
[u]^*_{2,\Omega,D} + [u]_{2,\alpha,D}^* = [u]_{2,\alpha,D}^* \leq C |u|_{0,D} = C \sup_{y \in D} |u(y)| = C \sup_{y \in D} |G(x_0, y)| \leq C \sup_{y \in D} \Gamma_\Omega(x_0, y_0) \leq C \Gamma_\Omega(x_0, y_0).
\]

It remains to collect the implications of this strong estimate. The first consequence of (3.12) is that, if \( y \in B(y_0, R) \subset D \) and \( |\beta| \leq 2 \), we have
\[
\bar{\partial}_y^{|\beta|} \bar{\partial}_y^{|\beta|} G(x_0, y) = \bar{\partial}_y^{|\beta|} \bar{\partial}_y^{|\beta|} u(y) \leq \sup_{z \in D, |\gamma| = |\beta|} \bar{\partial}_z^{|\beta|} |\partial^\gamma u(z)| = [u]_{|\beta|;D}^*,
\]
\[
(3.13) \quad \leq \sum_{j=0}^{2} [u]^*_{j;D} = |u|^*_{2;D} \leq C \Gamma_\Omega(x_0, y_0).
\]

By the inclusion in (3.11), we have \( \bar{\partial}_y = \text{dist}(y, \partial D) \geq R \geq C|x_0 - y_0| \) given that \( y \in B(y_0, R) \). Hence estimate (3.13) implies that
\[
|\partial^\beta \bar{\partial}_y G(x_0, y) | \leq C \Gamma_\Omega(x_0, y_0)|x_0 - y_0|^{-|\beta|}, \quad y \in B(y_0, R).
\]

In the special case \( y = y_0 \) this implies the required estimate (3.7).

Next assume that \( |\beta| = 2 \) and \( y_0 + h \in B(y_0, R) \subset D \). Then, by (3.12), we have
\[
\min \{\delta_{y_0}, \bar{\delta}_{y_0+h}\} 2^{2+\alpha} \frac{|\partial^\beta \bar{\partial}_y G(x_0, y_0 + h) - \partial^\beta \bar{\partial}_y G(x_0, y_0)|}{|h|^\alpha}
\]
\[
= \min \{\bar{\delta}_{y_0}, \bar{\delta}_{y_0+h}\} 2^{2+\alpha} \frac{|\partial^\beta u(y_0 + h) - \partial^\beta u(y_0)|}{|h|^\alpha}
\]
\[
\leq \sup_{z, w \in D, |\beta| = 2} \left[ \min \{\delta_z, \delta_w\} 2^{2+\alpha} \frac{|\partial^\beta u(z) - \partial^\beta u(w)|}{|z - w|^\alpha} \right] = [u]^*_{2,\alpha,D} \leq C \Gamma_\Omega(x_0, y_0).
\]

As above, by using (3.11), we have the estimate
\[
\min \{\delta_{y_0}, \bar{\delta}_{y_0+h}\} \geq R = C|x_0 - y_0|.
\]
since \( y_0, y_0 + h \in B(y_0, R) \). As a consequence, we obtain the estimate
\[
|\partial^\beta G(x_0, y_0 + h) - \partial^\beta G(x_0, y_0)| \leq C|h|^\alpha \Gamma(x_0, y_0) |x_0 - y_0|^{-2-\alpha},
\]
which clearly suffices for (3.8). This concludes the first case in (3.9).

Next, we proceed to the second case in (3.9). This is a boundary estimate, where we assume that
\[
(3.14) \quad \frac{\rho}{4 \text{diam}(\Omega)} |x_0 - y_0| \geq \text{dist}(y_0, \partial \Omega) = \delta(y_0).
\]
Denote \( S = \rho |x_0 - y_0| / \text{diam}(\Omega) \) and fix a point \( \bar{y} \in \partial \Omega \) such that \( |\bar{y} - y_0| = \delta(y_0) \). We begin by claiming that the following auxiliary estimates
\[
(3.15) \quad \text{dist}(x_0, B(\bar{y}, S)) \geq \frac{1}{2} |x_0 - y_0|, \quad \text{dist}(B(y_0, \delta(y_0)), \mathbb{R}^n \setminus B(\bar{y}, S)) \geq S/2.
\]
hold true. Indeed, the first estimate (3.15) follows from that, if \( z \in B(\bar{y}, S) \), then
\[
|x_0 - z| \geq |x_0 - y_0| - |y_0 - \bar{y}| - |\bar{y} - z|
\]
\[
\geq |x_0 - y_0| - \delta(y_0) - S \geq |x_0 - y_0| - \frac{1}{2} |x_0 - y_0| \geq \frac{1}{2} |x_0 - y_0|
\]
because \( 2\rho/\text{diam}(\Omega) < 1/2 \), so that \( \delta(y_0) + S \leq 2\rho |x_0 - y_0| / \text{diam}(\Omega) \leq |x_0 - y_0|/2 \).
For the second estimate in (3.15), we fix \( w \in B(y_0, \delta(y_0)) \). Then
\[
|w - \bar{y}| \leq |w - y_0| + |y_0 - \bar{y}| \leq 2\delta(y_0) \leq S/2.
\]
If also \( z \in \mathbb{R}^n \setminus B(\bar{y}, S) \), then
\[
|z - w| = |z - \bar{y} + \bar{y} - w| \geq |z - \bar{y}| - |\bar{y} - w| \geq S/2.
\]
It remains to infimize the left-hand side over \( z \) and \( w \).

Later we will invoke Lemma 3.3. For this purpose, we denote \( B = B(\bar{y}, \bar{S}) \cap \Omega \) and \( T = B(\bar{y}, S) \cap \partial \Omega \subset \partial B \). Notice that, by (3.15),
\[
B(y_0, \delta(y_0)) = B(y_0, \delta(y_0)) \cap \Omega \subset B.
\]
Denote \( \overline{\delta}_y = \text{dist}(y, \partial B \setminus T) \). Then \( \partial B \setminus T \subset \mathbb{R}^n \setminus B(\bar{y}, S) \) so that, for \( y \in B(y_0, \delta(y_0)) \), we have
\[
(3.16) \quad \overline{\delta}_y \geq \text{dist}(B(y_0, \delta(y_0)), \partial B \setminus T) \geq \text{dist}(B(y_0, \delta(y_0)), \mathbb{R}^n \setminus B(\bar{y}, S)) \quad \geq S/2 = C|x_0 - y_0|.
\]
by (3.15). The function \( u = G(x_0, \cdot) \) satisfies \( Lu = 0 \) in \( B \subset \subset \bar{\Omega} \setminus \{x_0\} \), which is seen by using both (3.15) and (3.5). It also satisfies the boundary condition \( u \equiv 0 \) on \( T \). As a consequence of Lemma 3.3 and estimate \( |u(y)| \leq C_L |x_0 - y|^{2-n} \) combined with (3.15), we have
\[
(3.17) \quad |u|_{2,B,T}^* + [u]_{2,\alpha;B,T}^* \leq C|u|_{0,B} \leq C \sup_{y \in B} \Gamma_\Omega(x_0, y) \leq C \Gamma_\Omega(x_0, y_0).
\]
The first consequence of (3.17) is that, if \( y \in B(y_0, \delta(y_0)) \subset B \) and \( |\beta| \leq 2 \), we have
\[
|\partial^\beta u| \leq |\partial^\beta G(x_0, y_0)| = |\partial^\beta G(x_0, y_0)| \leq |u|_{2,B,T}^* \leq C \Gamma_\Omega(x_0, y_0).
\]
Taking estimate (3.16) into account, we have
\[
|\partial^\beta G(x_0, y, y_0)| \leq C \Gamma_\Omega(x_0, y_0)|x_0 - y_0|^{-|\beta|}, \quad y \in B(y_0, \delta(y_0)).
\]
In the special case \( y = y_0 \) this yields (3.17).

Next, if \( y_0 + h \in B(y_0, \delta(y_0)) \subset B \) and \( |\beta| = 2 \), then by using (3.17) we have
\[
\min \{ \delta(y_0), \overline{\delta}(y_0 + h) \}^{2+\alpha} |\partial^\beta G(x_0, y_0 + h) - \partial^\beta G(x_0, y_0)| \leq [u]_{2,\alpha;B,T}^* \leq C \Gamma_\Omega(x_0, y_0).
\]
On the other hand, by using (3.10), we have \(\min\{\delta_{y_0}, \tilde{\delta}_{y_0+h}\} \geq C|x_0 - y_0|\). As a consequence, we find that

\[
|\partial^\beta_y G(x_0, y_0 + h) - \partial^\beta_y G(x_0, y_0)| \leq C|h|^{\alpha} \Gamma_\Omega(x_0, y_0)|x_0 - y_0|^{-2-\alpha},
\]

which clearly suffices for (3.8). \(\square\)

To conclude the proof of Theorem 1.15 we still need the global Hölder estimate (1.13) where, in contrast to Lemma 3.6, point \(y + h\) can not be restricted to the ball \(B(y, \delta(y) \wedge \rho|x - y|/8\text{diam}(\Omega))\). Proof proceeds as in [Väh09, p. 78] with minor modifications, and we omit the details that are based on uniformity.

3.3. **Proof of Theorem 1.15.** The main step is to show that \(G \in K^{-2}_\Omega(\alpha)\). By using (3.7) we see that the first required estimate (2.7) holds true, that is,

\[
|G(x, y)| \leq C_G |x - y|^{2-n}
\]

if \(x, y \in \Omega\) are distinct points. Next we verify (2.8). For this purpose we fix \(x \in \Omega\) and let \(B = B(y^B, r) \subset \subset \Omega\) be a ball which satisfies the condition

\[
\frac{8r\text{diam}(\Omega)}{\rho} \leq |x - y^B|.
\]

Then, in particular, we have

\[
B(y^B, r) \subset B(y^B, \text{dist}(y^B, \partial \Omega) \wedge \rho|x - y^B|/8\text{diam}(\Omega)).
\]

Hence, if \(\{y, \ldots, y + 3h\} \subset B \subset \subset \Omega\), using an integral representation [Väh09, p.102] for the second order differences, we find that

\[
|\Delta^3_h(G(x, \cdot), B, y)| \lesssim |h|^2 \sup_{\theta \in [0,2]; |\alpha|=2} \left\{ |\partial^\alpha_y G(x, y + (1 + \theta)h) - \partial^\alpha_y G(x, y + \theta h)| \right\}.
\]

Notice that \(y + \theta h, y + (1 + \theta)h, y^B \in B\) inside the supremum so that, by invoking Lemma 3.6, we find that

\[
|\Delta^3_h(G(x, \cdot), B, y)| \lesssim |h|^2 \text{diam}(B)^{\alpha} |x - y^B|^{-n-\delta} \leq \text{diam}(B)^{2+\alpha} |x - y^B|^{-n-\alpha}.
\]

Integrating this inequality over \(y \in B\) shows that \(G\) satisfies the estimate (2.8). Because \(G\) is symmetric, we also find that the kernel \((x, y) \mapsto G(y, x) = G(x, y)\) also satisfies this condition.

All in all, we have shown that \(G \in K^{-2}_\Omega(\alpha)\). Also, by Theorem 2.5, the bounded \(C^{2,\alpha}\) domain \(\Omega\) is uniform. Hence Theorem 2.9 applies and, as a consequence, we have the following.

3.18. **Theorem.** Let \(\Omega \subset \mathbb{R}^n, n \geq 3\), be a bounded \(C^{2,\alpha}\) domain and assume that the coefficients \(a^{ij}\) of \(L\) belong to \(C^{1,\alpha}(\overline{\Omega})\). Then Green’s function \(G\) of \(L\) in \(\Omega\) is a smooth kernel, that is, \(G \in K^{-2}_\Omega(\alpha)\).

By combining theorems 3.18 and 2.24, we see that there exists a globally defined smooth kernel \(G \in K^{-2}_\mathbb{R^n}(\alpha) = K^{-2}_\mathbb{R^n}(\alpha)\) such that

\[
\tilde{G} |\Omega \times \Omega \setminus \{(x, x)\}| = G.
\]

Next we invoke Theorem 2.5 for the conclusion that \(\Omega\) is c-coplump for some \(c \geq 1\). Also, since \(\Omega\) is bounded, we have \(\text{diam}(\mathbb{R}^n \setminus \Omega) = \infty\). Furthermore, the operators \(\partial^\sigma G = \partial^\sigma G^*, |\sigma| = 2\), are bounded on \(L^2(\Omega)\) [GT83, Theorem 8.12]. Hence the assumptions of Theorem 2.25 hold true and, by invoking it, we finish the proof Theorem 1.15.
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