THE DYNKIN DIAGRAM COHOMOLOGY OF FINITE COXETER GROUPS

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Abstract. Let $D$ be a connected graph. The Dynkin complex $CD(A)$ of a $D$–algebra $A$ was introduced by the second author in [TL2] to control the deformations of quasi–Coxeter algebra structures on $A$. In the present paper, we study the cohomology of this complex when $A$ is the group algebra of a Coxeter group $W$ and $D$ is the Dynkin diagram of $W$. We compute this cohomology when $W$ is finite and prove in particular the rigidity of quasi–Coxeter algebra structures on $kW$. For an arbitrary $W$, we compute the top cohomology group and obtain a number of additional partial results when $W$ is affine. Our computations are carried out by filtering the Dynkin complex by the number of vertices of subgraphs of $D$. The corresponding graded complex turns out to be dual to the sum of the Coxeter complexes of all standard, irreducible parabolic subgroups of $W$.

1. Introduction

Let $\mathfrak{g}$ be a complex, semisimple Lie algebra and $D$ the corresponding Dynkin diagram. The notion of quasi–Coxeter algebra of type $D$ was introduced in [TL2] to put the monodromy of the Casimir connection of $\mathfrak{g}$ [MTI TL] and the quantum Weyl group representations arising from the quantum group $U_q\mathfrak{g}$ [Lu] on an equal footing, and allow for their comparison via a suitable deformation complex.

Roughly speaking, a quasi–Coxeter algebra $A$ of type $D$ is an algebra which carries representations of the generalised braid group $B_D$ corresponding to $D$ on its finite–dimensional modules. The deformation theory of such an algebra is controlled by a complex $CD(A)$ concentrated in degrees $0 \leq p \leq |D|$ called the Dynkin complex of $A$.

If $W$ is the Weyl group of $\mathfrak{g}$, the complex group algebra $A = CW$ possesses quasi–Coxeter algebra structures accounting respectively for the representations of $B_D$ coming from the Hecke algebra $\mathcal{H}(W)$ of $W$ and the monodromy of Cherednik’s KZ connection [Ch] (see [TL2 §4] for details).

While in this case these representations may easily be shown to be equivalent, this raises nonetheless the question of computing the Dynkin diagram cohomology of $CW$ and more generally of the group algebra $kW$ of a Coxeter group over an arbitrary ground ring $k$, when the underlying $D$–algebra structure arises from the standard parabolic subgroups of $W$.

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In the present paper, we answer this question when $W$ is finite and $k$ is a field of characteristic 0. Along the way, we also determine the top cohomology groups for arbitrary Coxeter groups and obtain partial results for affine ones.

We carry out our computations by filtering $CD(kW)$ by the number of vertices of subgraphs of $D$. Interestingly, and crucially for us, the associated graded complex $\text{Gr}(CD(kW))$ turns out to be dual to the sum of the Coxeter complexes of all standard, irreducible, parabolic subgroups of $W$. This greatly simplifies the computation of $H^*(CD(kW))$ since the Coxeter complex is acyclic for affine Coxeter groups and has cohomology in one degree only for finite ones.

We turn now to a detailed description of the paper.

In Section 2, we review the definition of the Dynkin complex of a $D$–algebra and define its canonical filtration.

We then consider in Section 3 the case of the group algebra of a Coxeter group. In this case, the Dynkin complex has a direct sum decomposition parametrised by the conjugacy classes of $W$. Further, the graded complex $\text{Gr}(CD(kW))$ determined by the filtration of $CD(kW)$ decomposes as a sum parametrised by connected subgraphs $B$ of $D$. The summand associated to $B$ is given by the morphisms from the Coxeter complex of $W_B$ to $kW_B$ endowed with the adjoint action of $W_B$.

In Section 4, we use the known description of the cohomology of the Coxeter complex for finite and affine Coxeter groups to compute the cohomology of $\text{Gr}(CD(kW))$.

In Section 5, we consider finite Coxeter groups. We compute the induced differential on $H^*(\text{Gr}(CD(kW)))$ and show that the resulting complex is quasi–isomorphic to $CD$.

We apply this result in Section 6 to prove the rigidity of quasi–Coxeter algebra structures on $kW$ when $W$ is finite.

The main result of Section 7 is the construction of a basis of the top Dynkin cohomology of the group algebra of an arbitrary Coxeter group: it is parametrised by cuspidal conjugacy classes such that the centraliser of an element of the class is in the kernel of the sign character.

Section 8 is devoted to the determination of the Dynkin cohomology of finite Coxeter groups. We proceed case by case, by explicit computation for $W$ classical or of type $G_2$ and $F_4$, and by using the program GAP for the remaining exceptional groups. In types $A$ and $B$, we provide a very simple formula for the corresponding generating series which, in type $A$ turns out to be the product of a bosonic and fermionic partition function. We show moreover that, for $W$ classical, the Dynkin cohomology spaces stabilise with the rank of $W$.

In the final Section 9, we describe the part of the Dynkin cohomology of an affine Weyl group corresponding to conjugacy classes of elements of infinite order.
2. D-algebras and the Dynkin complex

This section reviews the definition of D-algebras and of the Dynkin complex. With the exception of §2.4, the material is borrowed from [TL2].

2.1. D-algebras [TL2, §3]. Let D be a connected diagram, that is a non-empty undirected graph with no multiple edges or loops. We denote the set of vertices of D by V(D) and set |D| = |V(D)|. By a subdiagram B ⊆ D we shall mean a non-empty full subgraph of D, that is a graph consisting of a subset V(B) of vertices of D, together with all edges of D joining any two elements of V(B). We will often abusively identify such a B with its set of vertices and write α ∈ B to mean α ∈ V(B). Two subdiagrams B₁, B₂ ⊆ D are orthogonal if they have no vertices in common and no two vertices α₁ ∈ B₁, α₂ ∈ B₂ are joined by an edge in D.

Let k be a fixed commutative ring with unit. By an algebra we shall mean a unital, associative k-algebra. All algebra homomorphisms are assumed to be unital. Recall the following

Definition 2.1. A D-algebra is an algebra A endowed with subalgebras Aₕ for labelled by the connected subdiagrams B of D such that the following holds

- A₀ ⊆ A₁ whenever B ⊆ B'.
- Aₜ and Aₙ commute whenever B and B' are orthogonal.

If B₁, B₂ ⊆ D are subdiagrams with B₁ connected, we denote by Aᵣ, B₂ the centraliser in Aᵣ of the subalgebras Aᵣ, B₁, where B₂ runs over the connected components of B₂.

2.2. The Dynkin complex [TL2, §5]. For any 0 ≤ p ≤ |D|, set

\[ C^p(A) = \bigoplus_{\alpha \subseteq B \subseteq D, |\alpha| = p} A^B\alpha \]

where the sum ranges over all connected subdiagrams B of D and ordered subsets \( \alpha = \{\alpha_1, \ldots, \alpha_p\} \subseteq V(B) \) of cardinality p and, by convention, \( A^B\emptyset = A_B \). We denote the component of a ∈ Cₚ(A) along \( A^B\alpha \) by \( a(B;\alpha) \).

Definition 2.2. The group of Dynkin p-cochains on A is the subspace \( C^p(A) \) of Cₚ(A) of elements a such that

\[ a(B;\sigma\alpha) = (-1)^{\sigma} a(B;\alpha) \]

where, for any \( \sigma \in S_p \), \( \sigma\{\alpha_1, \ldots, \alpha_p\} = \{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)}\} \)

\[ ^1 \text{In [TL2] Dynkin chains are defined with values in any D-bimodule M over A. We shall only need to consider M = A in this paper and therefore restrict attention to this case.} \]
Note that
\[ CD^0(A) = \bigoplus_{B \subseteq D} Z(A_B) \] and \[ CD^n(A) \simeq A_D \]
For \(1 \leq p \leq n - 1\), define a map \(d^p_D : C^p(A) \to C^{p+1}(A)\) by
\[ (d^p_D a)(B; \underline{\alpha}) = \sum_{i=1}^{p+1} (-1)^{i-1} \left( a(B; \underline{\alpha} \setminus \alpha_i) - a(B; \underline{\alpha} \setminus \alpha \setminus \alpha_i) \right) \] (1)
where \(\underline{\alpha} = \{\alpha_1, \ldots, \alpha_{p+1}\}\), \(B; \underline{\alpha} \setminus \alpha_i\) is the connected component of \(B \setminus \alpha_i\) containing \(\alpha \setminus \alpha_i\) if one such exists and the empty set otherwise, and we set \(a(\emptyset; \alpha) = 0\). For \(p = 0\), define \(d^0_D : C^0(A) \to C^1(A)\) by
\[ d^0_D a(B; \alpha) = a_B - a_{B \setminus \alpha} \]
where \(a_{B \setminus \alpha}\) is the sum of \(a_{B_2}\) with \(B_2\) ranging over the connected components of \(B \setminus \alpha\). Finally, set \(d^n_D = 0\). The map \(d_D\) leaves \(CD(A)\) invariant and satisfies \(d^2_D = 0\). The cohomology \(HD(A)\) of \(CD(A)\) with respect to \(d_D\) is called the Dynkin diagram cohomology of \(A\).

2.3. Restriction [TL2, §5.2]. Let \(D'\) be a connected subgraph of \(D\). We have a morphism of complexes
\[ \text{Res}_{D'}^{D} : CD(A) \to CD(A_{D'}) \]
\[ A^{B; \underline{\alpha}}_B \ni a \mapsto \begin{cases} a & \text{if } B \subseteq D' \\ 0 & \text{otherwise.} \end{cases} \]

2.4. The canonical filtration on \(CD(A)\). Endow each chain group \(C^p(A)\) with the \(\mathbb{N}\)-grading given by
\[ C^p_q(A) = \bigoplus_{\underline{\alpha} \subseteq B \subseteq D, |\underline{\alpha}| = p, |B| = q} A^{B; \underline{\alpha}}_B \]
where \(p \leq q \leq n\), and set \(CD^p_q(A) = CD^p(A) \cap C^p_q(A)\). Since \(|B^{\underline{\alpha} \setminus \alpha}| < |B|\), the Dynkin differential \(d_D\) maps \(CD^p_q(A)\) to
\[ CD^{p+1}_{\geq q}(A) = \bigoplus_{r=q}^{n} CD^{p+1}_r(A) \]
This gives a decreasing \(\mathbb{N}\)-filtration on the Dynkin complex of \(A\). The \(E_2\)-term of the corresponding spectral sequence is the cohomology of \(CD(A)\) with respect to the differential
\[ (d^p_D a)(B; \underline{\alpha}) = \sum_{i=1}^{p+1} (-1)^{i-1} a(B; \underline{\alpha} \setminus \alpha_i) \] (2)
3. The Dynkin complex of a Coxeter group

3.1. Description. Let \( W \) be an irreducible Coxeter group with system of generators \( S = \{ s_i \}_{i \in I} \) and let \( D \) be the Coxeter graph of \((W, S)\). For any subgraph \( B \subseteq D \) with vertex set \( I_B \subseteq I \), let \( W_B \subseteq W \) be the standard parabolic subgroup generated by \( s_i, i \in I_B \).

Regard the group algebra \( A = kW \) as a \( D \)-algebra by setting \( A_B = kW_B \). By choosing a total order on the vertices of \( D \), we can identify the corresponding Dynkin complex with

\[
CD^p = \bigoplus_{\alpha \subset B \subseteq D, |\alpha| = p} kW_B^{W_B \setminus \alpha}
\]

where \( \alpha \) now ranges over the unordered subsets of \( V(B) \) and the \( W_B \setminus \alpha \)-fixed points in \( kW_B \) are taken with respect to the diagonal (adjoint) action. The Dynkin differential on \( kW_B^{W_B \setminus \alpha} \) is the map

\[
kW_B^{W_B \setminus \alpha} \to \bigoplus_{\beta \in B \setminus \alpha} kW_B^{W_B \setminus (\alpha \cup \beta)} \oplus \bigoplus_{B' \subseteq B \setminus \alpha: B = B' \setminus \beta} kW_{B'}^{W_{B'} \setminus (\alpha \cup \beta)}
\]

given by

\[
\left( \sum_{\beta} (-1)^{\text{pos}(\beta, \alpha \cup \beta)} - 1 \text{can}, \sum_{B', \beta} (-1)^{\text{pos}(\beta, \alpha \cup \beta)} \text{can} \right)
\]

where \( \text{can} \) is the canonical inclusion map and for any \( \beta \in \alpha \subset V(D) \), \( \text{pos}(\beta, \alpha) \in \{1, \ldots, |\beta|\} \) is the position of \( \beta \) relative to the total order on \( \alpha \).

3.2. Decomposition by conjugacy classes. Let \( C \) be the set of conjugacy classes of \( W \). For any \( c \in C \), set

\[
CD_c^p = \bigoplus_{\alpha \subset B \subseteq D, |\alpha| = p} k (W_B \cap c)^{W_B \setminus \alpha}
\]

where, for any set \( X \), \( k(X) \) is the vector space with basis \( X \). Then \( CD_c = \bigoplus_p CD_c^p \) is a subcomplex of \( CD \) and

\[
CD = \bigoplus_{c \in C} CD_c.
\]

We denote the cohomology of the corresponding complex by \( HD_c(kW) \).
3.3. Filtration. When filtered as in §2.4, the associated graded complex \( \text{Gr}(CD) \) (= the \( E_1 \)-term of the spectral sequence) is the sum over all connected subdiagrams \( B \subseteq D \) of the subcomplexes \( CD_B \) given by

\[
CD_B^p = \bigoplus_{\alpha \subseteq B, |\alpha| = p} kW_B^{W_B \backslash \alpha}
\]

\[
\simeq \bigoplus_{\alpha \subseteq B, |\alpha| = p} \text{Hom}_{W_B}(k(W_B/W_B \backslash \alpha), kW_B)
\]  

(3)

Recall that the Coxeter complex \( CC_B \) of \( W_B \) is the (homology) complex \( CC_B^p = \bigoplus_{\alpha \subseteq B, |\alpha| = p} k(W_B/W_B \backslash \alpha) \) with differential given by

\[
\partial_C(wW_B \backslash \alpha) = \sum_{i=1}^p (-1)^{i-1} wW_B(\alpha_1 \cup \ldots \cup \alpha_i)
\]

where \( \alpha = \{\alpha_1, \ldots, \alpha_p\} \) with \( \alpha_1 < \cdots < \alpha_p \). The following immediate result identifies the \( B \)-component \( CD_B \) of \( \text{Gr}(CD) \) with the dual of the Coxeter complex of \( W_B \) with values in \( kW_B \).

**Proposition 3.1.** The isomorphism (3) induces an isomorphism of complexes

\[
CD_B \simeq \text{Hom}_{W_B}(CC_B, kW_B).
\]

4. The cohomology of \( CD_B \)

Assume henceforth that \( k \) is a field of characteristic 0.

4.1. Finite Coxeter groups. Assume in this paragraph that \( W_B \) is finite. Let \( S_B \) be the unit sphere in the Euclidean reflection representation of \( W_B \), and cellulate \( S_B \) by its intersections with the chambers of \( W_B \). Then, the Coxeter complex \( CC_B \) is the cellular homology complex of \( S_B \), reduced and shifted by one [Hu]. Thus, \( H_p(CC_B) \) is zero if \( p < |B| \) and the sign representation \( \varepsilon \) of \( W_B \) otherwise, so that, by Proposition 3.1, we have

\[
H^p(CD_B) \simeq \begin{cases} kW_B^\varepsilon & \text{if } p = |B| \\ 0 & \text{otherwise.} \end{cases}
\]  

(4)

where \( kW_B^\varepsilon \subset kW_B \) is the subspace transforming like the sign representation \( \varepsilon \) of \( W_B \).

Consider \( kW_B^\varepsilon[-|B|] \), a complex concentrated in degree \( |B| \). We have a morphism of complexes \( i_B : kW_B^\varepsilon[-|B|] \to CD_B \) given by the inclusion

\[
kW_B^\varepsilon \hookrightarrow kW_B = kW_B^{W_B \backslash B}
\]

in degree \( |B| \). Let \( \text{Alt}_B : kW_B \to kW_B^\varepsilon \) be the projection given by

\[
\text{Alt}_B(f) = \frac{1}{|W_B|} \sum_{w \in W_B} \varepsilon(w)wfw^{-1}.
\]
Since $\text{Alt}_B$ is zero on $\sum_{\alpha \in B} kW_B^{\varepsilon_{\omega_{vB}}}$, it defines a morphism of complexes $\rho_B : CD_B \to kW_B^{\varepsilon_{\varepsilon_{\varepsilon}}[-|B|]}$. Summarising, we have the following Proposition.

**Proposition 4.1.** If $W_B$ is finite, the maps $i_B : kW_B^{\varepsilon_{\varepsilon_{\varepsilon}}[-|B|]} \to CD_B$ and $\rho_B : CD_B \to kW_B^{\varepsilon_{\varepsilon_{\varepsilon}}[-|B|]}$ are quasi–isomorphisms such that $\rho_B \circ i_B = \text{id}$.

4.2. **Affine Coxeter groups.** Assume now that $W_B$ is an affine Coxeter group, and let $E_B$ be the Euclidean space of dimension $|B| - 1$ cellulated by the alcoves of $W_B$. The Coxeter complex $CC_B$ is the cellular homology complex of $E_B$, reduced and shifted by one [14], and is therefore acyclic. Its terms of positive degree are induced from the trivial representation of finite (parabolic) subgroups of $W_B$ and are therefore projective. Thus, $CC_B$ is a projective resolution of the trivial $W_B$–module. It follows from this, and Proposition 3.1, that

$$H^p(\text{CD}_B) \simeq \begin{cases} \text{Ext}^{p-1}_{W_B}(k, kW_B) & \text{if } p \geq 2 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where $kW_B$ is endowed with the adjoint action of $W_B$.

5. **The $E_2$–term of the Dynkin complex of a Coxeter group**

5.1. **Finite Coxeter groups.** We assume in this section that the Coxeter group $W$ is finite.

Consider the complex

$$HC = \bigoplus_{B \subseteq D} kW_B^{\varepsilon} \quad (6)$$

where $kW_B^{\varepsilon}$ appears in degree $|B|$, endowed with the differential $d^\#$ given by

$$d^\# = \sum_{B'} (-1)^{|B'|} \cdot \text{Alt}_{B'} : kW_B^{\varepsilon} \longrightarrow \bigoplus_{B \subseteq B' \subseteq D \atop |B'| = |B| + 1} kW_B^{\varepsilon} \quad (7)$$

where, given $B' \subseteq D$ with ordered set of vertices $\alpha_1 < \cdots < \alpha_q$ and $B \subset B'$ such that $B' \setminus B = \{\alpha_i\}$, we set $(B'; B) = i$.

Note that this complex is concentrated in degrees $\geq 2$ since, for $|B| = 1$, we have $kW_B^{\varepsilon} = kS_2^{\varepsilon} = 0$.

Consider the application $\phi : CD \to HC$ given by

$$kW_B^{\varepsilon_{\varepsilon_{\varepsilon}}} \ni a \mapsto \text{Alt}_B(a). \quad (8)$$

**Proposition 5.1.** The application $\phi$ is a quasi–isomorphism of complexes.

**Proof.** Let $a \in kW_B^{\varepsilon_{\varepsilon_{\varepsilon}}}$. We have $\phi d_D(a) = 0 = \phi(a)$ if $\alpha \neq B$. On the other hand, if $\alpha = B$, we have

$$\phi d_D(a) = \sum_{B \subseteq B' \subseteq D \atop |B'| = |B| + 1} (-1)^{\text{pos}(B' \setminus B', B')} \text{Alt}(a) = d_D \phi(a).$$

so that $\phi$ is compatible with the differential.
Consider the filtration on $HC$ given by $(HC_{\geq q})^p = HC^p$ for $p \geq q$ and $(HC_{\geq q})^p = 0$ for $p < q$. The morphism $\phi$ is a morphism of filtered complexes. Via the canonical isomorphisms of §3, the induced morphism $\bar{\phi}_p : CD_{\geq p}/CD_{\geq p+1} \to HC_{\geq p}/HC_{\geq p+1}$ becomes the sum over connected subdiagrams $B$ of $D$ of cardinality $p$ of the morphisms $\rho_B$ of §4.1. It follows that $\bar{\phi}_p$ is a quasi–isomorphism by Proposition 4.1. The Proposition follows.

Let $D'$ be a connected subgraph of $D$ and $HC''$ the corresponding complex. Via the isomorphisms $\phi$ above, the restriction map of §2.3 becomes

$$\text{Res}_{D'}^D : HC \to HC''$$

$$kW_B^\varepsilon \ni a \mapsto \begin{cases} a & \text{if } B \subset D' \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 5.2.** It seems an interesting problem to determine whether the complex (6)–(7) is the cellular cochain of a CW–complex or the Morse complex of a smooth manifold naturally associated to $W$.

### 5.2. Finite part.

Assume now that $W$ is an arbitrary Coxeter group. We proceed as in §5.1 for the subspace of $CD$ corresponding to subdiagrams $B \subseteq D$ such that $W_B$ is finite.

Let $CD_{\inf} \subset CD$ be the subcomplex

$$CD_{\inf} = \bigoplus_{B \subseteq D : |W_B| = \infty} CD_B$$

Let

$$HC_f = \bigoplus_{B \subseteq D : |W_B| < \infty} kW_B^\varepsilon$$

where $B$ runs over connected subdiagrams of $D$ which are Dynkin. Here, $kW_B^\varepsilon$ is in degree $|B|$, endowed with the differential $d^\#$ given by

$$\sum_{B'} (-1)^{|(B';B)|} \cdot \text{Alt}_B : kW_B^\varepsilon \to \bigoplus_{B \subset B' : |B'| = |B| + 1} kW_{B'}^\varepsilon$$

Consider the application

$$\phi : CD \to HC_f$$

$$kW_B^{W_B^\geq 2} \ni a \mapsto \begin{cases} \text{Alt}_B(a) & \text{if } \alpha = B \text{ and } |W_B| < \infty \\ 0 & \text{otherwise.} \end{cases}$$

As in Proposition 5.1, one checks that $\phi$ is a morphism of complexes.

**Proposition 5.3.** There is a distinguished triangle

$$CD_{\inf} \xrightarrow{\text{can}} CD \xrightarrow{\phi} HC_f \xrightarrow{\sim}$$
Proof. One shows as in Proposition 5.1 that the map $CD/CD_{inf} \rightarrow HC_f$ induced by $\phi$ is a quasi–isomorphism.

6. RIGIDITY OF QUASI–COXETER ALGEBRA STRUCTURES ON $kW$

We apply below the results of Section 5 to show that quasi–Coxeter algebra structures on $kW$ are rigid if $W$ is finite. We begin by reviewing the definition of quasi–Coxeter algebras and their deformations.

6.1. Quasi–Coxeter algebras [TL2 §3]. Let $R$ be a commutative ring with unit. Recall that a quasi–Coxeter algebra structure on $RW$ is given by endowing it with the following data

- **Local monodromies:** for each $i \in I$, an invertible element $S_i \in RW_{\alpha_i} \cong \mathbb{Z}_2$.
- **Elementary associators:** for each connected subdiagram $B \subseteq D$ and vertices $\alpha_i \neq \alpha_j \in B$, an invertible element $\Phi(B;\alpha_i,\alpha_j) \in RW$ satisfying the following axioms
  
  - **Orientation:**
    $$\Phi(B;\alpha_j,\alpha_i) = \Phi^{-1}(B;\alpha_i,\alpha_j)$$
  
  - **Support:**
    $$\Phi(B;\alpha_i,\alpha_j) \in RW_B^{B\setminus\{\alpha_i,\alpha_j\}}$$
  
  - **Braid relations:** for any connected subdiagram $B \subseteq D$ consisting of two vertices $\alpha_i, \alpha_j$ such that the order $m_{ij}$ of $s_is_j \in W$ is finite, the following holds:
    $$\text{Ad}(\Phi(B;\alpha_i,\alpha_j))(S_i) \cdot S_j \cdots = S_j \cdot \text{Ad}(\Phi(B;\alpha_i,\alpha_j))(S_i) \cdots$$

    where the number of factors on each side is equal to $m_{ij}$.

as well as an additional axiom called the generalised pentagon relations, see [TL2 §3.17].

The above axioms are designed so that the elements $S_i$ and $\Phi(B;\alpha_i,\alpha_j)$ define a representation of the Tits braid group $B_W$ on any $W$–module, with isomorphic quasi–Coxeter algebra structures yielding equivalent representations of $B_W$, see [TL2 §3.14].

6.2. Deformations of quasi–Coxeter algebra structures [TL2 §5]. Let now $R = k[[h]]$ be the ring of formal power series in a variable $h$. Let

$$\{(S_i), \{\Phi(B;\alpha_i,\alpha_j)\}\} \quad \text{and} \quad \{(S_i'), \{\Phi'(B;\alpha_i,\alpha_j)\}\}$$

be quasi–Coxeter algebra structures on $RW = kW[[h]]$ such that, mod $h$

$$\Phi(B;\alpha_i,\alpha_j) = 1 = \Phi'(B;\alpha_i,\alpha_j)$$

Assume further that $S_i = S_i'$ for any $\alpha_i \in D$, and that the two structures coincide mod $h^n$ for some $n \geq 1$, that is that

$$\Phi'(B;\alpha_i,\alpha_j) = \Phi(B;\alpha_i,\alpha_j) + h^n \varphi(B;\alpha_i,\alpha_j) \mod h^{n+1}$$

for any $\alpha_i \neq \alpha_j \in B \subseteq D$, where $\varphi(B;\alpha_i,\alpha_j) \in kW_B^{B\setminus\{\alpha_i,\alpha_j\}}$. 
Then, by [TL2, Thm. 5.22], \( \varphi = \{ \varphi(B; \alpha_i, \alpha_j) \} \) is a 2–cocycle in the Dynkin complex \( CD(kW) \) and the two structures are isomorphic mod \( \hbar^{n+1} \) if, and only if, \( \varphi \) is a coboundary.

6.3. Rigidity. Let \( \{ S_i \}_{i \in I} \) be elements such that
\[
S_i \in kW_{\alpha_i}[h] \quad \text{and} \quad S_i = s_i \mod h
\]

**Theorem 6.1.** Assume that \( W \) is finite. Then there exists, up to isomorphism, at most one quasi–Coxeter algebra structure on \( kW[h] \) with local monodromies given by the elements \( S_i \) and associators equal to 1 mod \( h \).

**Proof.** Let \( \varphi \in CD^2(kW) \) be the infinitesimal defined by (10). The linearisation of the braid relations (9) reads, for any connected subdiagram \( B \subseteq D \) with vertex set \( \{ \alpha_i, \alpha_j \} \)
\[
\text{Alt}_B(\varphi(B; \alpha_i, \alpha_j)) = 0
\]
where \( \text{Alt}_B : kW_B \to kW_B^\varepsilon \) is the antisymmetrisation operator.

The image of \( \varphi \) in \( HC^2(kW) = \bigoplus_{B \subseteq D : |B| = 2} kW_B^\varepsilon \) via the morphism (8) is therefore zero so that \( [\varphi] = 0 \) in \( HD^2(kW) \) by Proposition 3.1. \( \square \)

6.4. When \( k = \mathbb{C} \), one can endow \( kW[h] \) with two quasi–Coxeter algebra structures having local monodromies
\[
S_i = s_i \cdot \exp(\pi \sqrt{-1} k_{\alpha_i} h s_i)
\]
where \( q_{\alpha_i} \in \mathbb{C} \) are a set of complex weights invariant under \( W \) [TL2, §4]. The first structure comes from the standard one on the Iwahori–Hecke algebra \( \mathcal{H}W \) obtained by quotienting the group algebra of the braid group \( B_W \) by the quadratic relations
\[
(S_i - q_i)(S_i + q_i^{-1}) = 0
\]
where \( q_i = \exp(2\pi \sqrt{-1} k_{\alpha_i} h) \), The second one underlies the monodromy of Cherednik’s rational KZ connection [Ch].

By [TL2, §4.2.2], these two structures are isomorphic. Theorem 6.1 strengthens this result by showing that there are no other such structures with the above local monodromies.

7. Top dimensional Dynkin diagram cohomology of Coxeter groups

7.1. Sign–coinvariants of \( kW \). Let \( W \) be a Coxeter group with system of generators \( S = \{ s_i \}_{i \in I} \). Let \( V \) be a \( W \)–module and set
\[
\nabla = \sum_i V^{s_i} \quad \text{and} \quad V_\varepsilon = V/\nabla
\]

**Proposition 7.1.**

(1) \( \nabla \) is invariant under \( W \).
For any \( v \in V \) and \( w \in W \),
\[
w v = \varepsilon(w)v \mod V
\]
where \( \varepsilon \) is the sign character of \( W \). Thus, \( W \) acts on \( V_\varepsilon \) by \( \varepsilon \).

If \( V = \bigoplus_j V_j \) is a direct sum of \( W \)-submodules, then \( V_\varepsilon = \bigoplus_j V_j \varepsilon \).

Proof. (i) Since \( V_{s_i} = (1 + s_i)V \), we have
\[
s_j V_{s_i} = s_j(1 + s_i)V \subset (1 + s_j)(1 + s_i)V + (1 + s_i)V \subset V_{s_j} + V_{s_i},
\]
(ii) Write \( w = s_{i_1} \cdots s_{i_\ell} \). Then, for any \( v \in V \)
\[
w v = s_{i_1} \cdots s_{i_\ell} v
\]
\[
= -s_{i_2} \cdots s_{i_\ell} v + (1 + s_{i_1})s_{i_2} \cdots s_{i_\ell} v
\]
\[
= (-1)^{\ell} v + \sum_{j=1}^{\ell} (-1)^{j-1} (1 + s_{i_j})s_{i_{j+1}} \cdots s_{i_\ell} v
\]
(iii) is clear.

Remark 7.2. Proposition 7.1 also follows from the fact that \( V_{s_i} = (1 + s_i)V \) so that \( V_\varepsilon \simeq \varepsilon \otimes_{kW} V \).

Let now \( V = kW \) endowed with the conjugation action of \( W \) and let \( C \) be the set of conjugacy classes of \( W \). For any \( c \in C \), choose \( w_c \in c \) and let \( A_{w_c} \) be the image of \( w_c \in kW \) in \( kW_\varepsilon \). Let \( C_W(w_c) \) be the centraliser of \( w_c \) in \( W \).

Proposition 7.3.

(1) For any Coxeter group \( W \), we have \( A_{w_c} \neq 0 \) when \( \varepsilon(C_W(w_c)) = 1 \) and
\[
kW_\varepsilon = \bigoplus_{c \in C: \varepsilon(C_W(w_c))=1} kA_{w_c}
\]
(2) If \( W \) is finite, then given \( c \) such that \( \varepsilon(C_W(w_c)) = 1 \), the element \( A^{w_c} = \sum_{w' \in W/C_W(w_c)} \varepsilon(w')w'w_cw'^{-1} \) is non-zero and
\[
kW_\varepsilon = \bigoplus_{c \in C: \varepsilon(C_W(w_c))=1} kA_{w_c}
\]

Proof. (i) Since \( kW = \bigoplus_{c \in C} F_c \), where \( F_c = k(c) \) is the subspace spanned by elements of \( c \), Proposition 7.1 yields \( kW_\varepsilon = \bigoplus_{c \in C} (F_c)_\varepsilon \). Since \( W \) acts transitively on \( c \), it follows from (ii) of Proposition 7.1 that \( (F_c)_\varepsilon \) is spanned by \( A_{w_c} \) and therefore at most one-dimensional. If \( w \) centralises \( w_c \), then, by (ii) of Proposition 7.1,
\[
A_{w_c} = A_{ww_cw'^{-1}} = wA_{w_c} = \varepsilon(w)A_{w_c}
\]
so that \( A_{w_c} \) is zero if the sign character is not trivial on the centraliser of \( w_c \). Conversely, if \( \varepsilon(C_W(w_c)) = 1 \), the assignement \( \varepsilon(w_c) \) consistently defines a non–zero linear form on \( F \) which descends to \( (F)_\varepsilon \) so that \( A_{w_c} \neq 0 \). (ii) readily follows from (i) and the fact that if \( W \) is finite, \( V_\varepsilon \cong V_\varepsilon' \) for any \( W \)–module \( V \).

Consider \( c \in C \) such that \( \varepsilon(C_W(w_c)) = 1 \). Then, \( A_{w_c} \) depends only on \( c \) and we put \( A_c = A_{w_c} \). Similarly, when \( W \) is finite we put \( A_c = A_{w_c} \).

7.2. Top cohomology. Assume now that \( W \) is irreducible of rank \( n \) and let \( D \) be its Coxeter graph. Let \( \mathcal{P} \) be the collection of proper, maximal connected subdiagrams \( B \) of \( D \) and \( kW_\mathcal{P} \subset kW \) the span of \( kW_B \) as \( B \) varies in \( \mathcal{P} \). For any \( c \in C \), let \( A_c \) be the class of \( A_{w_c} \) in \( kW_\varepsilon/kW_\varepsilon \cap kW_\varepsilon \).

**Proposition 7.4.** We have a decomposition in one-dimensional subspaces

\[
H^D_n(kW) = \bigoplus_{c \in C: \varepsilon(C_W(w_c)) = 1, c \cap kW_\varepsilon = \emptyset, \forall B \in \mathcal{P}} kW_\varepsilon/kA_c.
\]

**Proof.** The top degree Dynkin differential \( d_D^n \) is zero. Moreover,

\[
d_D^{n-1} = \sum_{i=1}^n (-1)^{i-1} a_{(D,D\setminus\alpha_i)} - a_{(C_D^{\alpha_i};D\setminus\alpha_i)\varepsilon}.
\]

Since \( a_{(D,D\setminus\alpha_i)} \in kW_\varepsilon \) and \( a_{(C_D^{\alpha_i};D\setminus\alpha_i)\varepsilon} \in kW_\varepsilon \),

\[
\text{Im} d_D^{n-1} = \sum_{i=1}^n kW_\varepsilon + \sum_{B \in \mathcal{P}} kW_B
\]

The result now follows from Proposition 7.3.

8. Finite Coxeter groups

8.1. Finite Coxeter groups of rank 2. Let \( W = I_2(m), m \geq 3 \), be the Coxeter group with generators \( s, t \) and relations \( s^2 = 1 = t^2 \) and \( (st)^m = 1 \). For \( p = 0, \ldots, m-1 \), let \( c^p \) be the conjugacy class of \( (st)^p \).

**Proposition 8.1.**

\[
kW_\varepsilon = \bigoplus_{p=1}^{\left\lfloor \frac{m-1}{2} \right\rfloor} kW \epsilon^p
\]

**Proof.** The only conjugacy classes involved in the decomposition of proposition 7.3 are those of words in \( s, t \) of even length and therefore those of the powers of \( st \) and \( ts \). Since \( ts = s(st)s \) we need only consider the classes \( c^p \), \( p = 1, \ldots, m-1 \). Moreover, since \( (st)^{m-p} = (ts)^p = s(st)^p s \), \( c^p = c^{m-p} \) and we may restrict our attention to \( 1 \leq p \leq m/2 \). Finally,
since for \( m \) even \((st)^{m/2}\) is central in \( W \), the only possible relevant values of \( p \) are \( 1, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor \). The proposition follows from the fact that for \( p = 1, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor \), the centraliser of \((st)^p\) in \( W \) is generated by \( st \). □

Since the differential \( d^\#(7) \) is zero for \( W \) of rank 2, Proposition 8.1 implies the following

**Theorem 8.2.** For any \( m \geq 3 \),

\[
\dim HD^p(kl_2(m)) = \begin{cases} 
0 & \text{if } p = 0 \\
0 & \text{if } p = 1 \\
\left\lfloor \frac{m-1}{2} \right\rfloor & \text{if } p = 2 \\
0 & \text{if } p \geq 3
\end{cases}
\]

8.2. **Type \( A_n \).** Let \( W = \mathfrak{S}_{n+1} \) be the Weyl group of type \( A_n \), \( n \geq 1 \). The conjugacy classes in \( W \) are parametrised by partitions \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of \( n+1 \), with \( c^\lambda \) the class of the product of cycles

\[
\tau^\lambda = (12 \cdots \lambda_1)(\lambda_1 + 1 \lambda_1 + 2 \cdots \lambda_1 + \lambda_2) \cdots (\sum_{i=1}^{k-1} \lambda_i + 1 \cdots \sum_{i=1}^k \lambda_i)
\]

8.2.1. For any \( m \in \mathbb{N}^* \), let

\[
\mathcal{O}_m = \{ \lambda \vdash m | \lambda_i < 2n+1, \forall i \text{ and } \lambda_i \neq \lambda_j, \forall i \neq j \}
\]

be the set of partitions of \( m \) consisting of odd, distinct parts.

**Proposition 8.3.**

\[
k\mathfrak{S}_{n+1} = \bigoplus_{\lambda \in \mathcal{O}_{n+1}} kA^\lambda
\]

**Proof.** Since \( \varepsilon(m m + 1 \cdots m + p - 1) = (-1)^{p-1} \), the only conjugacy classes involved in the decomposition of Proposition 7.3 are those such that each \( \lambda_i \) is odd. Moreover, since the product

\[
(m m + 1 \cdots m + p - 1)(m' m' + 1 \cdots m' + p - 1)
\]

of two disjoint cycles of equal length is centralised by \( \pi = (m m') \cdots (m + p - 1 m' + p - 1) \) and \( \varepsilon(\pi) = (-1)^p \), the \( \lambda_i \)'s must all be distinct. When this last condition is fulfilled, any element centralising \( \tau^\lambda \) is of the form

\[
\zeta = (12 \cdots \lambda_1)^{m_1}(\lambda_1 + 1 \lambda_1 + 2 \cdots \lambda_1 + \lambda_2)^{m_2} \cdots (\sum_{i=1}^{k-1} \lambda_i + 1 \cdots \sum_{i=1}^k \lambda_i)^{m_k}
\]

for some \( 0 \leq m_i \leq \lambda_i - 1 \). Since \( \varepsilon(\zeta) = \prod_i ((-1)^{\lambda_i - 1})^{m_i} \), the partitions arising in the decomposition of Proposition 7.3 are exactly those in \( \mathcal{O}_{n+1} \). □
8.2.2. Identify the connected subdiagrams of the Coxeter graph $D$ of $W$ with the subintervals of $[1, n]$ having integral endpoints so that $W_{[i, j]} \simeq \mathcal{S}_{j-i+2}$. For any $2 \leq p \leq n$ and $1 \leq i \leq n - p + 1$, let

$$A^\lambda_{[i, i+p-1]} = \text{Alt}_{[i, i+p-1]}(\tau^\lambda_{[i, i+p-1]}) \in kW_{[i, i+p-1]}^\varepsilon$$

be the generator corresponding to $\lambda \in O_{p+1}$. We shall need the following

Lemma 8.4.

$$\text{Alt}_{[i, i+p]}(A^\lambda_{[i, i+p-1]}) = \begin{cases} A^{\lambda \cup \{1\}}_{[i, i+p]} & \text{if } 1 \notin \lambda \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{Alt}_{[i-1, i+p-1]}(A^\lambda_{[i, i+p-1]}) = \begin{cases} (-1)^{p+1}A^{\lambda \cup \{1\}}_{[i-1, i+p-1]} & \text{if } 1 \notin \lambda \\ 0 & \text{otherwise} \end{cases}$$

provided $i \leq n - p$ and $i \geq 2$ respectively.

Proof. The first identity follows from the fact that under the inclusion $\mathcal{S}_{[i, i+p-1]} \subset \mathcal{S}_{[i, i+p]}$, $\tau^\lambda_{[i, i+p-1]}$ is mapped to $\tau^{\lambda \cup \{1\}}_{[i, i+p]}$. The second one follows from the first and the fact that, in $\mathcal{S}_{[i-1, i+p-1]}$,

$$\tau^\lambda_{[i, i+p-1]} = \text{Ad}((i - 1 \ i \cdot \cdot \cdot \ i + p))\tau^\lambda_{[i-1, i+p-2]}$$

and $\varepsilon(i - 1 \ i \cdot \cdot \cdot \ i + p) = (-1)^{p+1}$. \qed

8.2.3. Label the nodes of $D$ as in [BG Planche I] and order them as $\alpha_1 < \cdots < \alpha_n$. For any $p = 1, \ldots, n$, let

$$d^\#_p : \bigoplus_{i=1}^{n-p+1} kW_{[i, i+p-1]}^\varepsilon \longrightarrow \bigoplus_{j=1}^{n-p} kW_{[j, j+p]}^\varepsilon$$

be the differential of $HC$, where the right–hand side is understood to be 0 if $p = n$. Set

$$O^*_p = \{ \lambda \in O_{p+1} | 1 \notin \lambda \}$$

(14)

Proposition 8.5. For any $p = 2, \ldots, n$,

$$\text{Im} d^\#_{p-1} = \bigoplus_{i=1}^{n-p+1} \bigoplus_{\lambda \in O_{p+1} : 1 \notin \lambda} kA^\lambda_{[i, i+p-1]}$$

(15)

$$\text{Ker} d^\#_p = \text{Im} d^\#_{p-1} \oplus \bigoplus_{\lambda \in O^*_p} kB^\lambda$$

(16)

where $B^\lambda = \sum_{i=1}^{n-p+1} A^\lambda_{[i, i+p-1]}$. 


Proof. Since

\[(i - 1, i + p - 2); [i, i + p - 2]) = 1 \text{ and } (i, i + p - 1); [i, i + p - 2]) = p\]

Lemma 8.4 yields, for any \(\lambda \in O_p\) and \(i = 1, \ldots, n - p + 2\),

\[d_{i-1} A_{i+1, i+2}^{\lambda} = \delta_{i \geq 2} A_{i, i+1, i+2}^{\lambda(1)} - \delta_{i \leq n-p+1} A_{i, i+1, i+2}^{\lambda(1)}\]  (17)

It follows that, for any \(\lambda \in O_p^*\),

\[d_{i-1} \left(\sum_{j=1}^{i} A_{i,j+1, i+2}^{\lambda}\right) = (-1)^p A_{i, i+1, i+2}^{\lambda(1)}\]  (18)

which yields (15). It also follows from (17) that, for \(\lambda \in O_p^*\), \(B^\lambda\) is the unique linear combination of \(A_{i, i+1, i+2}^{\lambda}\) such that \(d_{i-1} B^\lambda = 0\), which yields (16). \(\square\)

8.2.4.

**Theorem 8.6.** For any \(0 \leq p \leq n\),

\[
\dim HD^p(kS_{n+1})_{\lambda} = \begin{cases} 1 & \text{if } \lambda = \lambda' \cup \{1^{n-p}\} \text{ and } \lambda' \in O_{p+1}^* \\ 0 & \text{otherwise} \end{cases} \tag{19}
\]

Proof. (19) holds for \(p \geq 2\) by proposition 8.5 and therefore for \(p \geq 0\) since \(\dim HD^i(kS_{n+1}) = 0 = |O_{i+1}^*|\) for \(i = 0, 1\). \(\square\)

8.2.5. Generating function.

**Theorem 8.7.** Set

\[
\chi^A(q, t) = \sum_{n \geq 1, p \geq 0} q^n t^p \dim HD^p(kS_{n+1})
\]

Then,

\[
\chi^A(q, t) = \frac{1}{1-q} \prod_{d \geq 1} (1 + (qt)^{2d+1}) - 1
\]

Proof. Since \(HD^p(kS_{n+1}) = 0\) if \(p = 0\) or \(p \geq n + 1\), we have

\[
\chi^A(q, t) = \sum_{n \geq 1, p \geq 1} q^n t^p \dim HD^p(kS_{n+1})
\]

\[
= \sum_{p \geq 1} \frac{t^p q^p}{1-q} |O_{p+1}^*|
\]

\[
= \frac{1}{1-q} \prod_{d \geq 1} (1 + (qt)^{2d+1}) - 1
\]
where the last identity follows from the fact that
\[ \sum_{m \geq 1} z^m |O_m^*| = \sum_{m \geq 2} z^m |O_m^*| = (1 + z^3)(1 + z^5)(1 + z^7) \cdots - 1 \] (20)

\[ \square \]

**Remark 8.8.** Up to a multiplication by \( qt \), the generating function \( \chi^A \) is the product a Fermionic partition function by that of a one-dimensional harmonic oscillator. It would be interesting to know whether the direct sum
\[ \bigoplus_{n \geq 1, p \geq 0} HD^p(kG_{n+1}) \]
possesses a natural action of an infinite-dimensional Clifford algebra similar in spirit to that on the cohomology of the Hilbert schemes of points on a surface \( \overline{G} \overline{a} \).

**8.3. Type \( B_n \).** Let now \( W = \mathfrak{S}_n \ltimes \mathbb{Z}_2^n \) be the Weyl group of type \( B_n \), \( n \geq 2 \), and denote the generators of \( \mathbb{Z}_2^n \) by \( \varepsilon_i \), \( i = 1, \ldots, n \). The conjugacy classes in \( W \) are parametrised by ordered pairs of partitions \( (\lambda, \mu) \) such that \( |\lambda| + |\mu| = n \), where \( |\lambda| = \sum_i \lambda_i \) [GP, Prop. 3.4.7]. The class \( c(\lambda, \mu) \) is that of the product \( \tau^\lambda \tilde{\tau}^\mu \), where
\[ \tau^\lambda = (1 2 \cdots \lambda_1) \cdots (|\lambda| - \lambda_k + 1 \cdots |\lambda|) \] (21)
and
\[ \tilde{\tau}^\mu = (|\lambda| + 1 \cdots |\lambda| + \mu_1) \varepsilon_{|\lambda|+\mu_1} \cdots (n - \mu_\ell + 1 \cdots n) \varepsilon_n \] (22)

**Proposition 8.9.**
\[ k \left( \mathfrak{S}_n \ltimes \mathbb{Z}_2^n \right)^{\varepsilon} = \begin{cases} \bigoplus_{\nu \vdash n/2} kA^{c(0, 2\nu)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \]

**Proof.** A necessary condition for a conjugacy class \( c(\lambda, \mu) \) to contribute to the decomposition of Proposition 7.3 is that the \( \lambda_i \) be odd and the \( \mu_j \) even since \( \varepsilon(\varepsilon_i) = -1 \). Since \( (m \cdots m + 1 \cdots m + p - 1) \) is centralised by \( \varepsilon_m \cdots \varepsilon_{m+p-1} \) and \( \varepsilon(\varepsilon_m \cdots \varepsilon_{m+p-1}) = (-1)^p \), \( \lambda \) must in fact be the empty partition. In particular, \( n \) must be even. \( \square \)

**Lemma 8.10.** The centraliser of \( (1 \cdots p) \varepsilon_p \) in \( \mathfrak{S}_p \ltimes \mathbb{Z}_2^p \) is the group \( \mathbb{Z}_{2p} \) generated by \( (1 \cdots p) \varepsilon_p \).

\[ ^2 \text{this also follows from the fact that the central element } \zeta = \varepsilon_1 \cdots \varepsilon_n \text{ is of sign } (-1)^n \text{ so that } \]
\[ kW^{\varepsilon} \subseteq \{ f \in kW \mid \zeta f \zeta = (-1)^n f \} \]
is zero if \( n \) is odd.
Proof. If $w \in G_p \times \mathbb{Z}_2^n$ centralises $(1 \cdots p)\varepsilon_p$, its projection in $G_p$ centralises $(1 \cdots p)$ and is therefore equal to $(1 \cdots p)^s$ for some $0 \leq s \leq p - 1$. Thus $(1 \cdots p)\varepsilon_p^s w$ centralises $(1 \cdots p)\varepsilon_p$ and lies in $\mathbb{Z}_2^n$ from which it readily follows that it is either equal to 1 or to $\varepsilon_1 \cdots \varepsilon_p = ((1 \cdots p)\varepsilon_p)^p$. □

**Proposition 8.11.** For any $0 \leq p \leq n$, we have

$$\dim HD^p(kW_{B_n})_{\chi, \mu} = \begin{cases} 1 & \text{if } (\mu = 0 \text{ and } \lambda = \lambda' \cup \{1^n-p-1\} \text{ with } \lambda' \in O_{n+1}^*) \\ \text{or } (\lambda = (1^n-p) \text{ and } \mu = 2\nu \text{ with } \nu \vdash p/2) \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\dim HD^p(W_{B_n}) = \dim HD^p(k\mathfrak{S}_n) + \delta_{p \in 2\mathbb{N}} \cdot P(p/2)$$

where $P$ is the partition function.

**Proof.** Identify the connected subdiagrams of the Coxeter graph $D$ of $W$ with the subintervals of $[1, n]$ having integral endpoints so that

$$W_{[i,j]} \simeq \begin{cases} \mathfrak{S}_{j-i+2} & \text{if } j \leq n - 1 \\ \mathfrak{S}_{n-i+1} \times \mathbb{Z}^{n-i+1} & \text{if } j = n \end{cases}$$

For $1 \leq p \leq n - 1$ and $1 \leq i \leq n - p$, let

$$A_{[i,i+p-1]}^\lambda = kW_{[i,i+p-1]}$$

be the generator corresponding to $\lambda \in O_{p+1}$, as in (13). By Proposition 8.9, $\text{Alt}_{[i,n]}(A_{[i,n-1]}^\lambda) = 0$. Since, in addition $kW_{[n-p+1,n]}^\varepsilon$ is zero whenever $p$ is odd and of dimension $|\nu \vdash p/2| = P(p/2)$ when $p$ is even, the complex $(\bigoplus_{B \subseteq D} kW_{B}^\varepsilon; d^\#)$ of (5) decomposes as the direct sum of the corresponding complex for $\mathfrak{S}_n$ and a complex concentrated in positive, even degrees with chain groups of dimension $\delta_{p \in 2\mathbb{N}} \cdot P(p/2)$. □

**Theorem 8.12.** Set $\chi^B(q,t) = \sum_{n \geq 2, p \geq 0} q^n t^p \dim HD^p(kW_{B_n})$. Then,

$$\chi^B(q,t) = \frac{\prod_{d \geq 1}(1 + (qt)^{2d+1}) - 1}{(1-q)t} + \prod_{d \geq 1}(1 - (qt)^{2d})^{-1} - 1$$

**Proof.** This follows from Proposition 8.11, Theorem 8.7 and the fact that

$$\sum_{m \geq 0} z^m P(m) = \prod_{d \geq 1}(1 - z^d)^{-1}$$

□

8.4. **Type $D_n$.** Let $\mathbb{Z}_{2,+}^n \subset \mathbb{Z}_2^n$ be the kernel of the sign character and $W = \mathfrak{S}_n \times \mathbb{Z}_{2,+}^n$ the Weyl group of type $D_n$, $n \geq 3$. The conjugacy classes in $W$ fall into two types [GP] Prop. 3.4.12:

- **type I.** These are labelled by ordered pairs $(\lambda, \mu)$ of partitions such that $|\lambda| + |\mu| = n$ and the number of parts $|\mu|$ of $\mu$ is even. The corresponding class $c^{(\lambda, \mu)}$ is that of the product $\tau^\lambda \tau^\mu$, where $\tau^\lambda$, $\tau^\mu$ are given by (21), (22).
type II. These are labelled by partitions $\lambda$ of $n$ all of whose parts are even, with $c^{\lambda,\mu}$ the class of

$$
\tau^{\lambda,\mu} = (1 2 \cdots \lambda_1) \cdots (n - \lambda_{k-1} - \lambda_k + 1 \cdots n - \lambda_k)(n - \lambda_{k} + 1 \cdots n)\varepsilon_{n-1}\varepsilon_n
$$

8.4.1.

Proposition 8.13.

$$
\left(k \mathfrak{S}_n \ltimes \mathbb{Z}_2^m\right)^c = \bigoplus_{\substack{0 \leq m \leq n, \\ \lambda \in \mathcal{O}_m, \\ \mu \in \mathcal{O}_{n-m}: \\ |\mu| \in 2\mathbb{N}}} kA^{c^{(\lambda,\mu)}_c} \oplus \begin{cases} 
\bigoplus_{\nu+n/2: \\ |\nu| \in 2\mathbb{N}} kA^{c^{(0,\nu)}_c} & \text{if } n \text{ is even} \\
0 & \text{if } n \text{ is odd}
\end{cases}
$$

(23)

Proof. Conjugacy classes of type II do not contribute to the decomposition of Proposition 7.3 since $\tau^{\lambda,\mu}$ is centralised by

$$
w = (n - \lambda_k + 1 \cdots n)\varepsilon_{n-1}\varepsilon_n
$$

and $\varepsilon(w) = (-1)^{\lambda_k-1} = -1$. Consider now a class $c^{(\lambda,\mu)}$ of type I which contributes to the decomposition of Proposition 7.3. As in Proposition 8.3, the $\lambda_i$ must be odd and distinct if $\lambda$ is nonempty. Moreover, since for any $1 \leq i < j \leq \ell$

$$
w_{i,j} = (|\lambda| + \sum_{a=1}^{i-1} \mu_a + 1 \cdots |\lambda| + \sum_{a=1}^{i} \mu_a)\varepsilon_{|\lambda|+\sum_{a=1}^{i} \mu_a} \\
\cdot (|\lambda| + \sum_{a=1}^{j-1} \mu_a + 1 \cdots |\lambda| + \sum_{a=1}^{j} \mu_a)\varepsilon_{|\lambda|+\sum_{a=1}^{j} \mu_a} \in W
$$

centralises $\tau^{\lambda,\tilde{\mu}}$ and $\varepsilon(w_{i,j}) = (-1)^{\mu_i+\mu_j}$, all $\mu_i$ must be of the same parity. Finally, since for $\lambda_1$ odd,

$$
w = (1 2 \cdots \lambda_1)\varepsilon_1 \cdots \varepsilon_{\lambda_1} \cdot (|\lambda| + 1 \cdots |\lambda| + \mu_1)\varepsilon_{|\lambda|+\mu_1}
$$

lies in $W$, centralises $\tau^{\lambda,\mu}$ and $\varepsilon(w) = (-1)^{\mu_1-1}$, all $\mu_i$ must be odd, and therefore distinct, if $\lambda$ is nonempty.

There remains to show that if $c$ is a conjugacy class appearing on the right-hand side of (23), then $\varepsilon$ is trivial on the centraliser of any element of $c$. If $c = c^{(0,\mu)}$ with all $\mu_i$ even, the centraliser of $\tau^{\mu}$ in $\mathbb{Z}_2^m \ltimes \mathfrak{S}_n$ lies in the kernel of $\varepsilon$ by Proposition 8.3. A fortiori, this is true in $\mathbb{Z}_2^m \ltimes \mathfrak{S}_n$. If, on the other hand, $c = c^{(\lambda,\mu)}$, where $\lambda$ and $\mu$ are either empty or consist of odd, distinct parts, it follows from Lemma 8.14 below that the component $\sigma \in \mathfrak{S}_n$ of any $w \in W$ centralising $\tau^{\lambda,\mu}$ is of the form

$$
(1 2 \cdots \lambda_1)^{m_1} \cdots (|\lambda| - \lambda_k + 1 \cdots |\lambda|)^{m_k} \\
\cdot (|\lambda| + 1 \cdots |\lambda| + \mu_1)^{m_1} \cdots (n - \mu_\ell + 1 \cdots n)^{m_\ell}
$$

so that $\varepsilon(w) = \varepsilon(\sigma) = 1$. \qed
Lemma 8.14. The centraliser of $(1 \cdots p)(p+1 \cdots 2p) \in S_{2p} \ltimes \mathbb{Z}_2^{2p}$ is the product of the centralisers of $(1 \cdots p)$ and $(p+1 \cdots 2p)$ in $S_{(1,\ldots,p)} \ltimes \mathbb{Z}_2^p$ and $S_{(p+1,\ldots,2p)} \ltimes \mathbb{Z}_2^p$ respectively.

8.4.2. Label now the nodes of the Dynkin diagram $D$ of $W$ as in [Bo, Planche IV], so that $\alpha_{n-2}$ is the trivalent node of $D$ if $n \geq 4$, and order them as $\alpha_1 < \cdots < \alpha_{n-2} < \alpha_{n-1} < \alpha_n$. For any $1 \leq i \leq j \leq n-2$, let $[i,j] \subset D$ be the connected subdiagram with vertices $\alpha_i,\ldots,\alpha_j$. For any $i = 1,\ldots,n-2$, let $B_i^{\pm} \subset D$ be the connected subdiagrams with vertices $\alpha_i,\ldots,\alpha_{n-2}$ and $\alpha_{n-1}$ (resp. $\alpha_n$) and $B_i^Y \subset D$ the subdiagram with vertices $\alpha_i,\ldots,\alpha_n$. Thus,

$$W_{[i,j]} \simeq S_{j-i+2}, \quad W_{B_i^\pm} \simeq S_{n-i+1} \quad \text{and} \quad W_{B_i^Y} \simeq S_{n-i+1} \ltimes \mathbb{Z}_2^{n-i+1}$$

Let $\sigma \in \text{Aut}(W)$ be the involution induced by fixing the nodes $\alpha_1,\ldots,\alpha_{n-2}$ and permuting $\alpha_{n-1}$ and $\alpha_n$ so that $\sigma(W_{B_i^+}) = W_{B_i^-}$. For any $\lambda \in O_{n-i+1}$, set $A_{B_i^\pm}^\lambda = \sigma A_{B_i^\pm}^\lambda$.

Proposition 8.15. The following holds for $3 \leq p \leq n$,

$$\text{Im} A_{p-1}^{\#} = \bigoplus_{\lambda \in O_{p+1}: i=1}^{n-p-1} kA_{[i,i+p-1]}^\lambda \bigoplus_{\lambda \in O_{p+1}: \lambda \neq \lambda} V^\lambda \bigoplus_{1 \leq m \leq p, \pi \in O_{m-1}, \mu \in O_{m+1}, 1 \leq \lambda \neq \mu \neq \pi} kA_{B_i^m}^\lambda \bigoplus_{1 \leq \lambda \neq \mu \neq \pi} kA_{B_i^m}^\lambda$$

where the first summand only arises if $p \leq n - 2$ and

$$V^\lambda = \begin{cases} \langle A_{B_{n-p}}^\lambda + A_{B_{n-p}^-}^\lambda, A_{B_{n-p}^+}^\lambda - A_{B_{n-p}^+}^\lambda, A_{B_{n-p}^-}^\lambda + A_{B_{n-p}^-}^\lambda \rangle & \text{if } p \leq n - 1 \\ kA_{B_i^Y}^\lambda & \text{if } p = n \end{cases}$$

Proof. By [18], the image of the restriction of $d_{p-1}^{\#}$ to

$$\bigoplus_{j=1}^{n-p-1} kW_{[j,j+p-2]}$$

is the span of the generators $A_{[i,i+p-1]}^\lambda; \ i = 1,\ldots,n-p-1$, as $\lambda$ runs through the elements of $O_{p+1}$ containing 1. This accounts for the first summand in (24). Further, since

$$([n-p-1,n-2];[n-p,n-2]) = 1 \quad \text{and} \quad ([B_{n-p}^\pm;n-p,n-2]) = p$$
Lemma 8.4 yields, for any $\overline{\lambda} \in \mathcal{O}_p$, 
\[
d_{p-1}^\#(A_{[n-p,n-2]}^\overline{\lambda}) = (-1)^p \cdot \delta_1 \varepsilon_\overline{\lambda} \left( -A_{n-p-1,n-2}^{\overline{\lambda}_i} + A_{B_{n-p}^+}^{\overline{\lambda}_i(1)} + A_{B_{n-p}^-}^{\overline{\lambda}_i(1)} \right)
\]
Thus, if $p \leq n - 1$ and $\lambda \in \mathcal{O}_{p+1}$ contains 1, 
\[
A_{B_{n-p}^+}^{\lambda} + A_{B_{n-p}^-}^{\lambda} \in \text{Im} \ d_{p-1}^\#
\]
To proceed, we need the following

**Lemma 8.16.** For any $\overline{\lambda} \in \mathcal{O}_p$, 
\[
\text{Alt}_{B_{n-p+1}^Y}(A_{B_{n-p+1}^+}^{\overline{\lambda}}) = A_{B_{n-p+1}^Y}^{(\overline{\lambda},0)}
\]

*Proof.* The "+" identity follows from the fact that, under the inclusion $W_{B_{n-p+1}^+} \subset W_{B_{n-p+1}^Y}$, $\tau_{\overline{\lambda}}^{B_{n-p+1}}$ is mapped to $\tau_{B_{n-p+1}^Y}^{\overline{\lambda}}$. The "-" one follows by applying the automorphism $\sigma$ and noticing that $A_{B_{n-p+1}^Y}^{(\overline{\lambda},0)}$ is fixed by $\sigma$. Indeed if $1 \in \overline{\lambda}$, $\tau_{B_{n-p+1}^Y}^{\overline{\lambda}}$ lies in $W_{[n-p+1,n-2]}$ and is therefore fixed by $\sigma$.

If on the other hand $1 \notin \overline{\lambda}$, the cycle $(n - \overline{\lambda}_k + 1 \cdots n)$ is the product $s_n \overline{\lambda}_k + 1 \cdots s_{n-1}$, so that 
\[
\sigma(n - \overline{\lambda}_k + 1 \cdots n) = s_{n-\overline{\lambda}_k+1} \cdots s_{n-2} s_n
\]
\[
= (n - \overline{\lambda}_k + 1 \cdots n) \varepsilon_{n-1} \varepsilon_n
\]
\[
= \text{Ad}(\varepsilon_{n-\overline{\lambda}_k+1} \cdots \varepsilon_{n-1})(n - \overline{\lambda}_k + 1 \cdots n)
\]
whence 
\[
\sigma A_{B_{n-p+1}^Y}^{(\overline{\lambda},0)} = \text{Alt}_{B_{n-p+1}^Y}( \text{Ad}(\varepsilon_{n-\overline{\lambda}_k+1} \cdots \varepsilon_{n-1}) \tau_{B_{n-p+1}^Y}^{\overline{\lambda}} ) = A_{B_{n-p+1}^Y}^{(\overline{\lambda},0)}
\]
since $\varepsilon_{n-\overline{\lambda}_k+1} \cdots \varepsilon_{n-1} = (-1)^{\overline{\lambda}_k - 1}$ and $\overline{\lambda}_k$ is odd. \hfill $\square$

Since 
\[
(B_{n-p}^+; B_{n-p}^+; B_{n-p+1}^+; B_{n-p+1}^-) = 1, \quad (B_{n-p+1}^Y; B_{n-p+1}^+; B_{n-p}^-) = p \quad \text{and} \quad (B_{n-p+1}^Y; B_{n-p+1}^-) = p - 1
\]
lemmas 8.4 and 8.16 imply that, for any $\overline{\lambda} \in \mathcal{O}_p$, 
\[
d_{p-1}^\#(A_{B_{n-p+1}^Y}^{\overline{\lambda}}) = (-1)^p \left( -\delta_1 \varepsilon_\overline{\lambda} \cdot A_{B_{n-p}^+}^{\overline{\lambda}_i(1)} \pm A_{B_{n-p}^-}^{\overline{\lambda}_i(1)} \right)
\]
Choosing $\overline{\lambda} \in \mathcal{O}_p$ such that $1 \notin \overline{\lambda}$ in (27) and using (26) accounts for the second summand in (24). To conclude, we need the following straightforward consequence of Lemma 8.3

**Lemma 8.17.** For any $4 \leq p \leq n$,
\[
\text{Alt}_{B_{n-p+1}^Y}(A_{B_{n-p+1}^+}^{(\lambda,\mu)}) = \begin{cases} 
0 & \text{if } 1 \in \lambda \text{ or } \mu_i \in 2\mathbb{N}^* \forall i \\
(-1)^{|\lambda|} \cdot A_{B_{n-p+1}^Y}^{(\lambda,\mu)} & \text{otherwise}
\end{cases}
\]
Since $d_{p-1}^\#(f) = -\text{Alt}_{B_{n-p+1}}^\varepsilon(f)$ for any $f \in kW_{B_{n-p+2}}^\varepsilon$, Lemma 8.17 accounts for the third summand in (24).

8.4.3.

**Proposition 8.18.** The following holds for any $2 \leq p \leq n$,

$$\text{Ker } d_p^\# = \text{Im } d_{p-1}^\# \oplus \bigoplus_{\nu \vdash p/2; [\nu] \in 2N} kA_{B_{n-p+1}}^{c(0,2\nu)} \oplus \bigoplus_{\lambda \in O_{p+1}^*} k\tilde{B}_\lambda \quad \text{if } p \leq n-1$$

$$+ \bigoplus_{0 \leq m < n, \mu \in O_{p+1}^*} kA_{B_1^\varepsilon}^{c(\lambda,\mu)} \quad \text{if } p = n$$

where the second summand only arises if $p$ is even and greater or equal to 3, $O_q^* = \{ \lambda \in O_q | 1 \notin \lambda \}$ for $q \geq 1$, $O_0^* = \{ \emptyset \}$ and

$$\tilde{B}_\lambda = \sum_{i=1}^{n-p-1} A_{[i,i+p-1]}^\lambda + A_{B_{n-p}}^\lambda + A_{B_{n-p}}^\lambda$$

**Proof.** For $p \geq 3$ even, the subspace spanned by $A_{B_{n-p+1}}^{c(0,2\nu)}$, $\nu \vdash p/2$, lies in $\text{Ker } d_p^\#$ by lemma 8.17 and is in direct sum with $\text{Im } d_{p-1}^\#$ by proposition 8.15. This accounts for the second summand in (28). Let now $f \in \text{Ker } d_p^\#$ be such that all components of $f_{B_{n-p+1}}$ of type $c(0,2\nu)$, $\nu \vdash p/2$, are zero. If $p = n$, $f$ lies in the span of the elements $A_{B_1^\varepsilon}^{c(\lambda,\mu)}$, $(\lambda,\mu) \in O_m \times O_{n-m}$ and $0 \leq m \leq n$. Since $d_n^\# = 0$ and

$$\text{Im } d_{n-1}^\# = \bigoplus_{\lambda \in O_n^*} kA_{B_1^\varepsilon}^{c(\lambda,\emptyset)} \oplus \bigoplus_{1 \leq m \leq n, \lambda \in O_m, \mu \in O_{n-m}; \nu \in 2N} kA_{B_1^\varepsilon}^{c(\lambda,\mu)}$$

by Proposition 8.15 (28) holds for $p = n$. Assume now that $p \leq n-1$. By lemmas 8.16, 8.17 the restriction of $d_p^\#$ is injective on the span of the elements $A_{B_{n-p+1}}^{c(\lambda,\mu)}$ where $(\lambda,\mu) \in O_m \times O_{p-m}$, $0 \leq m \leq p$, are such that $\lambda$ does not contain 1 and $\mu$ is nonempty. The corresponding components of $f$ are therefore zero. Since $A_{B_{n-p+1}}^{c(\lambda,\mu)}$ lies in $\text{Im } d_{p-1}^\#$ if $1 \in \lambda$ by Lemma 8.17 we may therefore assume that $f_{B_{n-p+1}}$ only has components along $A_{B_{n-p+1}}^{c(\lambda,\emptyset)}$, $\lambda \in O_p^*$. Working modulo the first summand of $\text{Im } d_{p-1}^\#$ given by Proposition
we may further assume that \( f \) lies in the span of
\[
\bigoplus_{\lambda \in \mathcal{O}_{p+1}^*} \bigoplus_{i=1}^{n-p-1} kA^{c^\lambda}_{[i,i+p-1]} \bigoplus_{\lambda \in \mathcal{O}_{p+1}} \left( kA^{c^\lambda}_{B_{n-p}^+} \oplus kA^{c^\lambda}_{B_{n-p}^-} \right) \bigoplus_{\lambda \in \mathcal{O}_p} kA^{c(\lambda,\emptyset)}_{B_{n-p+1}}
\]
\[
\bigoplus_{\lambda \in \mathcal{O}_{p+1}: 1 \in \lambda} \left( kA^{c^\lambda}_{B_{n-p}^+} \oplus kA^{c^\lambda}_{B_{n-p}^-} \oplus kA^{c(\lambda\{1\},\emptyset)}_{B_{n-p+1}} \right)
\]

Let \( \lambda \in \mathcal{O}_{p+1}^* \). It readily follows from (16) and (27) that \( \tilde{B}_{\lambda} \in \text{Ker } d^p_\# \). Next, applying (16) to the \( c^\lambda \)-components of \( f \) along \( kW_{[i,i+p-1]}^\varepsilon \), \( i = 1, \ldots, n - p - 1 \) and \( kW_{B_{n-p}^+}^\varepsilon \), we see that these components are equal to \( a_{\lambda}A^{c^\lambda}_{[i,i+p-1]} \) and \( a_{\lambda}A^{c^\lambda}_{B_{n-p}^+} \) respectively, for some constant \( a_{\lambda} \). Thus, subtracting \( a_{\lambda}\tilde{B}_{\lambda} \) to \( f \), we may assume that all these components are equal to zero and therefore that \( f \) lies in
\[
\bigoplus_{\lambda \in \mathcal{O}_{p+1}: 1 \in \lambda} \left( kA^{c^\lambda}_{B_{n-p}^+} \oplus kA^{c^\lambda}_{B_{n-p}^-} \oplus kA^{c(\lambda\{1\},\emptyset)}_{B_{n-p+1}} \right)
\]

By (27) and Lemma 8.17 any solution of \( d^p_#f = 0 \) with values in (30) lies in the subspace \( \bigoplus_{\lambda \in \mathcal{O}_{p+1}: 1 \in \lambda} V^\lambda \) defined by (25) and therefore in \( \text{Im } d^p_# \). □

8.4.4. **Theorem 8.19.** For any \( 0 \leq p < n \), we have
\[
\dim HD^p(kW_{D_n})_c = \begin{cases} 
1 & \text{if } (c = c^{(1^{n-p}, 2\nu)}) \text{ for some } \nu \vdash p/2 \text{ with } [\nu] \in 2\mathbb{N} \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
\dim HD^n(kW_{D_n})_c = \begin{cases} 
1 & \text{if } c = c^{\lambda, \mu} \text{ with } \lambda \in \mathcal{O}_m^* \text{ and } \mu \in \mathcal{O}_{n-m} \text{ and } [\mu] \in 2\mathbb{N} \\
0 & \text{otherwise}
\end{cases}
\]

8.5. **Stabilisation.** Assume that \( W \) is of classical type \( X = A, B \) or \( D \). Then, using the description of the Dynkin cohomology of \( kW \) given in this section, one readily checks that for \( n \geq m \), the restriction map on Dynkin diagram cohomology defined in \( \text{Res}^{X_n}_{X_m} : HD^p(kW_{X_n}) \to HD^p(kW_{X_m}) \)
is an isomorphism for \( p \leq m \).

8.6. **Exceptional groups.**

8.6.1. **G_2.** The Weyl group of type \( G_2 = I(6) \) was treated in Section 8.1.
8.6.2. $F_4$. Let $W$ be the Weyl group of type $F_4$ and label the connected subdiagrams of $D$ by the subintervals of $[1, 4]$ with integral endpoints. Thus, $W_{[1,3]}$ and $W_{[2,4]}$ are of type $B_3$, $W_{[1,2]}$ and $W_{[3,4]}$ are of type $A_2$, and $W_{[2,3]}$ is of type $B_2$. It follows from Propositions 8.3 and 8.9 that 

$$kW_{[1,3]} \varepsilon \simeq k$$

$$kW_{[2,4]} \varepsilon = 0$$

$$kW_{[1,2]} \varepsilon \simeq k$$

$$kW_{[3,4]} \varepsilon \simeq k$$

The differential $d^\#$ of $HC$ is therefore equal to zero, so that 

$$\text{dim } HD^p(kW) = \begin{cases} 
0 & \text{if } p = 0 \\
0 & \text{if } p = 1 \\
3 & \text{if } p = 2 \\
0 & \text{if } p = 3 \\
\text{dim } kW \varepsilon & \text{if } p = 4 \\
0 & \text{if } p \geq 5
\end{cases}$$

8.6.3. $GAP$ calculations. For the groups $H_3$, $H_4$, $E_6$, $E_7$ and $E_8$, the dimension of the cohomology spaces of the complex $HC$ can be readily computed with the computer algebra package $[\text{GAP}]$. For each $B$, we enumerate conjugacy classes of $W_B$ and select those whose elements have a centraliser in the kernel of $\varepsilon$. Then, we compute the matrices corresponding to the differentials and determine their rank. We indicate the results in the following table, where the columns provide $\text{dim } HD^i(kW)$, $i = 2, \ldots, n$.

| $i$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|
| $E_6$ | 1 | 0 | 2 | 0 | 4 |
| $E_7$ | 1 | 0 | 2 | 0 | 7 | 0 |
| $E_8$ | 1 | 0 | 2 | 0 | 6 | 1 | 17 |
| $F_4$ | 3 | 0 | 5 |
| $G_2$ | 2 |
| $H_3$ | 3 | 0 |
| $H_4$ | 3 | 0 | 16 |

9. Affine Coxeter groups

The methods of Sections 4.2 and 5.2 provide a partial computation of the Dynkin cohomology of affine Weyl groups. Let $D_0$ be a finite crystallographic Dynkin diagram with $n$ vertices and let $D$ be its completion. Let $V$ be the reflection representation of $W_0 = W_{D_0}$ and let $Q \subset V$ be the coroot lattice, so that $W \simeq Q \rtimes W_0$.

We have canonical Serre duality isomorphisms

$$\text{Ext}^i_{kQ}(k, M) \simeq \text{Ext}^{n-i}_{kQ}(M, k)^* \otimes \Lambda^n V^*$$
for any finitely generated $kQ$–module $M$ and any integer $i$. Thus, given a finitely generated $kW$–module $M$ and an integer $i$, we have isomorphisms

$$\text{Ext}^i_{kW}(k, M) \cong \text{Ext}^i_{kQ}(k, M)^{W_0} \cong \left(\text{Ext}^{n-i}_{kQ}(M, \varepsilon)^*\right)^{W_0} \cong \text{Ext}^{n-i}_{kW}(M, \varepsilon)^*$$

(31)

Let us describe the conjugacy classes of $W$. Given $v \in W_0$, let $Q_v \subseteq Q$ be the sublattice given by

$$Q_v = \{x - v(x) \mid x \in Q\}$$

**Proposition 9.1.** Representatives of conjugacy classes of $W$ are given by elements $tv$ where $v$ runs over representatives of conjugacy classes of $W_0$ and $t$ runs over representatives of $(Q/Q_v)/C_{W_0}(v)$.

**Proof.** Given $t, t' \in Q$ and $v, v' \in W_0$, if $tv$ and $t'v'$ are conjugate in $W$, then $v$ and $v'$ are conjugate in $W_0$. Let then $g = \tau x$ with $\tau \in Q$ and $x \in W_0$ be such that $gtvg^{-1} = t'v$. Then, $x \in C_{W_0}(v)$. We have $gtvg^{-1} = \tau x t x^{-1} \cdot \tau \cdot v \tau^{-1}v^{-1} \cdot v$ and the proposition follows. \qed

Let $c$ be a conjugacy class of $W$, let $w \in c$ and let $\bar{w}$ be the image of $w$ in $W_0$. The quotient $\overline{C_W(w)} = C_W(w)/Q^w$ is a subgroup of $C_{W_0}(w)$. Denoting as customary the vector space spanned by the elements of $c$ by $k(c)$, we have

$$\text{Ext}^i_{kW}(k, k(\{c\})) \cong \text{Ext}^{n-i}_{kW}(k, k(\{c\}), \varepsilon)^*$$

$$\cong \text{Ext}^{n-i}_{kC_W(w)}(k, k)^*$$

$$\cong \left(\text{Ext}^{n-i}_{kC_W(w)}(k, k)^* \otimes \varepsilon\right)\overline{C_W(w)}$$

$$\cong \left(\Lambda^{n-i}(V^w) \otimes \varepsilon\right)\overline{C_W(w)},$$

where the first isomorphism uses (31), the second one Frobenius reciprocity and the last one the fact that, given a finitely–generated free abelian group $L$, we have the Koszul isomorphism $H^*(L, k) \cong \Lambda^*(L^* \otimes_Z k)$.

**Theorem 9.2.** If the elements in $c$ have infinite order, then

$$HD^i_c \cong \left(\Lambda^{n+i-1}(V^\bar{w}) \otimes \varepsilon\right)\overline{C_W(w)}.$$ 

**Proof.** The assumption on $c$ shows that $c \cap W_B = \emptyset$ for any proper subset $B$ of $D$. Thus, the subcomplex $CD_c$ defined in Section 3.2 is concentrated in degree $n$. The theorem now follows from (1.2) and the isomorphisms above. \qed

**Remark 9.3.** If the elements of $c$ have finite order, then $c$ has a non empty intersection with $W_B$ for some proper connected subdiagram $B$ of $D$. In that case, there is a distinguished triangle

$$\bigoplus_i \left(\Lambda^{n-i+1}(V^\bar{w}) \otimes \varepsilon\right)\overline{C_W(w)}[-i] \to CD_c \to HC_{f_c} \to$$
where $HC_{f_c}$ is the subcomplex of $HC_f$ given by $HC_{f_c}^i = \bigoplus_B (c \cap W_B)^\epsilon$ and $B$ runs over the proper subdiagrams of $D$ of size $i$.

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