Lectures on nonlinear integrable equations and their solutions

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Abstract

This is an introductory course on nonlinear integrable partial differential and differential-difference equations based on lectures given for students of Moscow Institute of Physics and Technology and Higher School of Economics. The typical examples of Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP) and Toda lattice equations are studied in detail. We give a detailed description of the Lax representation of these equations and their hierarchies in terms of pseudo-differential or pseudo-difference operators and also of different classes of the solutions including famous soliton solutions. The formulation in terms of tau-function and Hirota bilinear differential and difference equations is also discussed. Finally, we give a representation of tau-functions as vacuum expectation values of certain operators composed of free fermions.

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1 Introduction remarks

Before sixties of XX century the list of problems of classical and quantum physics that admit exact solution in one or another form was very short and included just a few examples (some problems on tops in classical mechanics, Ising model, Heisenberg model). They seemed to be exotic exceptions.

In the sixties-seventies the situation drastically changed: large families of non-trivial exactly solvable models were discovered in that time and the basic principles underlying their construction and integrability were understood. Herewith it has appeared that many of them unexpectedly emerge in very different physical contexts and are directly related to the structure of our world.

What types of integrable systems are known

There are several important types:

- Models with a small number of degrees of freedom (integrable cases of tops).
- Systems of $N$ interacting particles in one dimension: the Calogero-Moser model and all its relatives (classical and quantum).
- Nonlinear partial differential equations as well as difference equations: the Korteweg-de Vries equation (KdV), the nonlinear Schrodinger equation (NLS), the sine-Gordon equation (SG), the Toda chain and the 2D Toda lattice, the Benjamin-Ono equation (BO), the Kadomtsev-Petviashvili equation (KP) and many others.
- Models of statistical mechanics on 2D lattice: Ising model, six- and eight-vertex models and their generalizations.
- Integrable models of quantum physics on 1D lattice: spin chains of XXX, XXZ and XYZ type and their various generalizations.
- Integrable models of quantum field theory in $1 + 1$ dimensions: one-dimensional bose-gas with point-like interaction (quantum NLS equation), the Thirring model, the sine-Gordon model, ...

This division is rather conditional and the list is not complete. There are deep and beautiful connections between the different types of integrable models. For example, poles of singular solutions to integrable partial differential equations move as particles of integrable many-body systems of the Calogero-Moser type. Another example is that the connection between classical and quantum systems is not exhausted by their correspondence in the classical limit. It appears that classical integrable systems emerge in quantum integrable problems even at $\hbar \neq 0$.

In what sense one should understand integrability of a system

Here are several possible meanings of integrability:
- A possibility to integrate the equation (i.e. eliminate all derivatives).
- Existence of a complete set of integrals of motion in involution (the Liouville integrability).
- Presence of a large set of exact solutions and a possibility to express the answers through known elementary or special functions.
- A possibility to reduce the problem to a solution of a finite system of algebraic or integral equations.
- Presence of rich symmetries and interesting algebraic or analytic structures.

The different types of integrable systems mentioned above provide examples to all these meanings of the notion of integrability.

**Nonlinear integrable partial differential equations: some more introductory words**

In these lectures we will be concerned with classical integrable models described by nonlinear partial differential equations. In the first approximation, our world is described by *linear equations*, since a response to a small perturbation is usually proportional to the perturbation. The main content of the standard mathematical physics courses is essentially the theory of three linear partial differential equations: the Laplace equation, the heat equation and the wave equation. Their fundamental role is explained by their exceptional universality.

However, the nonlinear corrections, though small, can lead to significant outcomes. During last few decades it was understood that

a) A correct account of nonlinear effects leads to *integrable* equations which have the same high degree of universality as equations of linear mathematical physics;

b) These nonlinear equations possess rich hidden symmetries which allow one to effectively construct large classes of exact solutions (for example, solutions of soliton type).

Each of the nonlinear integrable equations generates an infinite chain of compatible "higher" equations, *the hierarchy*. Besides, all known integrable equations (KdV, NLS, SG, Toda chain, KP and many others) are close relatives. From a general point of view all of them are contained (as limiting and particular cases, reductions, equivalent forms obtained by a change of variables) in *one* universal equation – *the bilinear difference Hirota equation*.

**The aim of these lectures**

These notes are based on lectures given for students of Moscow Institute of Physics and Technology (MIPT) and Higher School of Economics (HSE) in 2009-2017. The lectures were intended for an audience unfamiliar with the subject and were aimed at an initial
introduction to it. There are a number of exercises and problems in the text which are an essential part of the course. Exercises are very simple; problems are somewhat more complicated.

We start from the KdV equation (section 2) which is the most familiar and well-studied example of integrable nonlinear equation. After acquaintance with the Lax representation of the KdV equation we study its (abelian) symmetries and introduce the whole infinite hierarchy of compatible “higher” KdV equations. A crucial role in the constructions is played by the common solution of the auxiliary linear problems, the Baker-Akhiezer function or $\psi$-function. Using properties of the $\psi$-function, the family of soliton solutions is constructed. We also discuss the non-abelian symmetries of the KdV equation, their stationary points and their relation to the Painlèvé equations.

Section 3 is devoted to the more general infinite hierarchy – the KP hierarchy. We introduce its Lax representation (in terms of pseudo-differential operators) and zero-curvature (Zakharov-Shabat) representation (in terms of differential operators) as well as the dressing operator formalism. Again, the role of the $\psi$-function is crucial. We construct the solutions of the soliton type and discuss other types of solutions. Towards the end of this section, we approach the concept of tau-function, the general solution to the whole KP hierarchy. We prove the bilinear identity, the formulas for the $\psi$-function in terms of tau-function and the bilinear generating equation for the KP hierarchy (the Hirota-Miwa equation).

In section 4 we briefly discuss even more general hierarchy then the KP one – the 2D Toda lattice (2DTL) hierarchy. The commutation representation for it is constructed with the help of difference and pseudo-difference operators. The main objects we introduce here are similar to the case of the KP hierarchy: the Lax operators, the dressing operators, the $\psi$-function, the tau-function.

In section 5 we present an approach to nonlinear integrable equations from the point of view of quantum field theory. Namely, a remarkable fact is that the tau-function can be realized as vacuum expectation value of certain quantum field operators composed of free fermions. The equations of the KP hierarchy obeyed by tau-function are then basically consequences of the Wick’s theorem. In the formalism of free fermions, the construction of tau-functions for the KP, modified KP and 2D Toda hierarchies becomes simple and natural.

We should note that some standard material is out of the scope of the lectures. In particular, we do not even mention the inverse scattering method. Our approach is mostly algebraic.

Comments on the literature

The literature on the subject is enormous. The list of references [1]–[12] includes only books and papers that had the greatest impact on the content of these notes and/or were recommended to students for an additional and further reading. We do not cite any literature in the main text.
2 The KdV equation

The KdV equation was suggested in 1895 for description of waves on shallow water. Propagation of waves in nonlinear media with dispersion in the case of general position is also described by the KdV equation.

The motivation is as follows. The wave equation $u_{tt} = c_0^2 u_{xx}$ has general solution in the form of superposition of waves propagating to the left and to the right with velocity $c_0$: $u(x, t) = f(x - c_0 t) + g(x + c_0 t)$. Let us consider the left-moving wave; it satisfies the first order equation $u_t = c_0 u_x$. Nonlinear effects make the wave velocity dependent on the amplitude. In the first order this dependence is linear: $c_0(u) = c_0 + \alpha u + \ldots$. Therefore, the nonlinearity yields the additional term $\alpha uu_x$ while dispersion yields the term with third order derivative:

$$ u_t = c_0 u_x \quad \rightarrow \quad u_t = c_0 u_x + \alpha uu_x + \beta u_{xxx} $$

(in the presence of the second order derivative the dynamics becomes dissipative). The coefficients in front of the correction terms may be small but if $u$ is large and/or changes rapidly, these terms become significant. In the frame moving with velocity $c_0$ we get the KdV equation

$$ u_t = \alpha uu_x + \beta u_{xxx}. \quad (2.1) $$

In the limiting case $\alpha = 0$ (the linear approximation) the equation is solved by the Fourier transform. At $\beta = 0$ (the dispersionless approximation) we get the Hopf equation $u_t = \alpha uu_x$ which (as any equation of the form $u_t = V(u)u_x$) can be solved by the method of characteristics. The general solution is written in an implicit form $x + c_0 t = f(u)$ with arbitrary function $f$. Remarkably, in the general case $\alpha \neq 0, \beta \neq 0$ equation $\text{(2.1)}$ can be also integrated but using very different methods.

Rescaling the variables $x, t, u$, one can get rid of the coefficients $\alpha, \beta$. For reasons which will be more clear later, we fix them as follows: $\alpha = 3/2, \beta = 1/4$, so the KdV equation acquires the form

$$ 4u_t = 6uu_x + u_{xxx} \quad (2.2) $$

which we call canonical.

2.1 An example of exact solution: the traveling wave (one-soliton solution)

One can try to find solutions to the KdV equation in the form of a traveling wave: $u(x, t) = f(x + ct)$ with $c > 0$. Substituting this ansatz to the equation $\text{(2.2)}$, we get the ordinary differential equation $4cf' = 6ff' + f'''$ in which one derivative can be eliminated:

$$ 4cf = 3f^2 + f'' + C_1. $$

Multiplying this equation by $f'$, we see that one more derivative can be eliminated with the result

$$ f'^2 = 4cf^2 - 2f^3 - 2C_1f - C_2, $$
where $C_1, C_2$ are constants. This first order ordinary differential equation can be integrated:

$$x - x_0 = \int f \frac{dy}{\sqrt{4cy^2 - 2y^3 - 2C_1y - C_2}}.$$ 

In the general case $C_1 \neq 0$, $C_2 \neq 0$ it is an elliptic integral. It simplifies if we assume that the function $f$ with all its derivatives tends to zero as $x \to \pm \infty$, then $C_1 = C_2 = 0$ and the integral can be taken in elementary functions. The result is

$$2\sqrt{c}(x - x_0) = \log\left(\frac{2\sqrt{c} - \sqrt{4c - 2f}}{2\sqrt{c} + \sqrt{4c - 2f}}\right)$$

or

$$f(x) = \frac{2c}{\cosh^2(\sqrt{c}(x - x_0))}.$$ 

Therefore, the solution to the KdV equation is of the form

$$u(x, t) = \frac{2c}{\cosh^2(\sqrt{c}(x - x_0 + ct))}, \quad (2.3)$$

where $c$ is an arbitrary positive real parameter. This is the famous one-soliton solution. Note that the soliton excitation propagates with velocity which is greater than the velocity of sound (because the solution (2.3) is already written in the frame moving with velocity of sound). The velocity of the soliton is proportional to its amplitude.

We stress that the existence of traveling wave exact solutions is a common feature of all evolution equations of the form $u_t = F[u, u_x, \ldots]$ and the possibility to find one-soliton solution is by no means a characteristic property of the KdV equation. The KdV equation is really distinguished by the property that it has multi-soliton solutions.

**Problem.** Find traveling wave solutions of the modified KdV equation $4v_t = -6v^2v_x + v_{xxx}$.

**Problem.** Find solutions of the form $\varphi(x, t) = f(x - ct)$ to the sine-Gordon equation

$$\varphi_{tt} - \varphi_{xx} + \frac{m^2}{\beta} \sin(\beta \varphi) = 0$$

such that $f(\infty) - f(-\infty) = 2\pi/\beta$.

### 2.2 The Lax representation

**Proposition.** The KdV equation (2.2) is equivalent to the operator relation

$$\partial_t L = [A, L], \quad (2.4)$$

where $L = \partial^2 + u$, $A = \partial^3 + \frac{3}{2} u \partial + \frac{3}{2} u_x$, $\partial := \partial/\partial x$.

The proof is a direct computation. Equation (2.4) is called the Lax equation or Lax representation (for KdV) while $L$ is called the Lax operator. The Lax equation can be also written in the form $[\partial_t - A, L] = 0$. Note that $L$ is a Hermitean operator ($L^\dagger = L$).
while $A$ is antihermitean ($A^\dagger = -A$), where the conjugation is defined as $\partial^\dagger = -\partial$, $\partial f^\dagger = f$, $(AB)^\dagger = B^\dagger A^\dagger$.

**Exercise.** Prove that the Lax equation for $L$ implies the Lax equation for $L^n$: $\partial_t L^n = [A, L^n]$ for any positive integer $n$.

The Lax equation means that $L(t) = U(t)L(0)U^{-1}(t)$, where $U$ is some operator with the property $A = \partial_t U U^{-1}$. Hence we have the important corollary:

**Spectrum of L does not depend on time**

i.e., it is an integral of motion. Accordingly, det$(\lambda - L)$, tr $(\lambda - L)^{-1}$ are time-independent. Expanding these quantities in powers of $\lambda$, one in principle can construct an infinite set of integrals of motion. Practically, there are two problems in this way: a) the meaning of det and tr for operators in functional space should be made more precise, b) it is not clear how to find local integrals of motion, i.e. integrals such that their densities in any point depend only on values of the function $u$ and its derivatives with respect to $x$ at this point.

**Remark.** The Lax representation is not unique. For example, the Lax equation (2.4) is equivalent to the KdV equation in the form $2u_t + 3uu_x + u_{xxx} = 0$ for the operators $L = \partial^4 + 2u\partial^2 + u_x\partial$, $A = \partial^3 + \frac{3}{2}u\partial$.

### 2.3 Symmetries and conservation laws

By symmetry of a differential equation $\partial_t u = K[u]$ we understand an equation $\partial_t u = R[u]$ such that evolutions in $t$ and $s$ commute: $\partial_s K[u] = \partial_t R[u]$. (Here $K[u]$ and $R[u]$ are differential polynomials of $u$, i.e. polynomials of $u$ and its $x$-derivatives.) This means that any solution $u(x, t)$ of the first equation can be extended to a function $u(x, t, s)$ in such a way that at any fixed $s$ it is a solution to the first equation (and at any fixed $t$ it is a solution of the second equation), and $u(x, t, 0) = u(x, t)$.

If the coefficients of the differential polynomial $K[u]$ do not depend on $x$ and $t$, the equation $\partial_t u = K[u]$ always has two trivial symmetries: shifts of the variables $x$ and $t$.

**Exercise.** Represent these trivial symmetries in the differential form $\partial_s u = R[u]$.

**Problem.** Find any non-trivial symmetry of the equation $\partial_t u = uu_x$.

It turns out that the KdV equation has infinitely many non-trivial symmetries. They can be found using the technique of pseudo-differential operators.

#### 2.3.1 Pseudo-differential operators

A pseudo-differential operator is a series of the form $\sum_{k=0}^{\infty} v_k \partial^{N-k}$, where $v_k$ are functions and the operator $\partial$ has the following standard commutation rule with arbitrary function: $\partial f = f' + f\partial$. Multiplying both sides of this equality by $\partial^{-1}$ from the left and from the right, one can understand it as a rule of commutation of $\partial^{-1}$ with a function: $\partial^{-1} f = f\partial^{-1} - \partial^{-1} f' \partial^{-1}$. The multiple application of this rule yields:

$$\partial^{-1} f = f\partial^{-1} - f'\partial^{-2} + f''\partial^{-3} + \ldots$$
Pseudo-differential operators can be multiplied as Laurent series taking into account that the symbol $\partial$ does not commute with coefficient functions. For example,

$$(1 + f\partial^{-1})(1 + g\partial^{-1}) = 1 + (f + g)\partial^{-1} + fg\partial^{-2} - fg'\partial^{-3} + fg''\partial^{-4} + \ldots$$

**Remark.** For brevity we write $\partial f$ for composition of the operator of multiplication by the function $f$ and the differential operator $\partial$. We hope that this will not lead to a confusion. The composition is usually written as $\partial \circ f$ but pedantic use of this notation, in our opinion, makes it difficult to read formulas.

**Problem.** For any two functions $f, g$ prove the following identities in the algebra of pseudo-differential operators:

a) $$(\partial - g)^{-1}f = \sum_{n=0}^{\infty} (-1)^n f^{(n)}(\partial - g)^{-n-1},$$

b) $$e^{-f\partial^{-1}}e^f = (\partial + f')^{-1},$$

c) $$\partial^n f = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} \partial^{n-k}, \quad n \geq 0,$$

d) $$\partial^{-n} f = \sum_{k \geq 0} (-1)^k \binom{k+n-1}{k} f^{(k)} \partial^{-n-k}, \quad n > 0.$$  

Here $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. Note that formulas c) and d) can be unified by extending the definition of the binomial coefficient to arbitrary complex numbers $n$ as

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{1 \cdot 2 \cdot 3 \ldots \cdot k},$$

then for integer $n < 0$ $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$. For the proof of c) and d) one may use induction in $n$.

Given a pseudo-differential operator $P = \sum_{k=0}^{\infty} v_k \partial^{N-k}$, we call $N$ the order of it. Let $P_+$ be its differential part (i.e. sum of the terms with non-negative powers of $\partial$): $P_+ = \sum_{k=0}^{N} v_k \partial^{N-k}$, then $P_- = P - P_+$ is sum of the terms with negative powers.

Let us introduce the important notion of residue of the pseudo-differential operator $P = \sum_{k=0}^{\infty} v_k \partial^{N-k}$: $${\text{res}} P = v_{-1}.$$  

We note that any operator $P$ can be written putting powers of the symbol $\partial$ both from the right and from the left of coefficient functions:

$$P = \sum_{k} v_k \partial^k = \sum_{k} \partial^k v_k.$$
Exercise. Prove that the residue does not depend on the way of writing, i.e. \( \text{res } P = v_{-1} = \tilde{v}_{-1} \). (For other coefficient functions this is in general not the case.)

In what follows we will need an important property of the residue: for any two pseudo-differential operators the residue of their commutator is a full derivative.

**Lemma.** For any two pseudo-differential operators \( P,Q \)

\[
\text{res } ([P,Q]) = \partial C,
\]

where \( C \) is a differential polynomial of coefficients of the operators \( P,Q \), i.e. a linear combination of these coefficients and their \( x \)-derivatives of any order.

It is enough to check this statement in the case \( P = f\partial^n, Q = g\partial^{-m} \), where \( n > m > 0 \). Using the results of problems c) and d), one can conclude that the residue of the commutator is proportional to

\[
fg^{(n-m+1)} + (-1)^{n-m}gf^{(n-m+1)} = \partial \left( \sum_{k=0}^{n-m} (-1)^k f^{(k)} g^{(n-m-k)} \right),
\]

i.e., it is a full derivative. It is also possible to give an inductive proof which does not use the results of the exercises: assuming that the residue of the commutator \( f\partial^n g\partial^{-m} \) is a full derivative for all \( n \) and some \( m \) (for example, this is easy to check at \( m = 1 \)), to derive from this that the same statement holds for \( m \to m + 1 \).

**Problem.** Give a detailed proof of the lemma.

The operation of conjugation (defined as \( \partial^\dagger = -\partial, f^\dagger = f, (AB)^\dagger = B^\dagger A^\dagger \)) can be extended to pseudo-differential operators:

\[
\left( \sum_{k=0}^{\infty} v_k \partial^{N-k} \right)^\dagger = \sum_{k=0}^{\infty} (-\partial)^{N-k} v_k.
\] (2.5)

The following technical lemma, which connects the notions of the operator and usual residue, turns out to be useful in many cases.

**Lemma.** Let \( P = \sum_j p_j \partial^j \) and \( Q = \sum_j q_j \partial^j \) be two pseudo-differential operators, then

\[
\text{res}_\partial (PQ^\dagger) = \text{res}_z \left[ (Pe^{xz}) (Qe^{-xz}) \right],
\] (2.6)

where by \( \text{res}_\partial \) and \( \text{res}_z \) we denote the operator and usual residue (the coefficient of the Laurent series at \( z^{-1} \)) respectively. (It is implied that the rule of acting by the operator \( \partial^n \) to the exponential function \( \partial^n e^{xz} = z^n e^{xz} \) is extended also to negative values of \( n \), i.e., for example, \( \partial^{-1} e^{xz} = z^{-1} e^{xz} \).

Indeed,

\[
\text{res}_z \left[ (Pe^{xz}) (Qe^{-xz}) \right] = \text{res}_z \left( \sum_i p_i z^i \sum_j q_j (-z)^j \right) = \sum_{i+j=-1} (-1)^i p_i q_j,
\]

and the same expression is obtained after writing what is \( \text{res}_\partial (PQ^\dagger) \):

\[
\text{res}_\partial (PQ^\dagger) = \text{res}_\partial \left( \sum_{i,j} p_i \partial^i (-\partial)^j q_j \right) = \sum_{i+j=-1} (-1)^j p_i q_j.
\]
The pseudo-differential operators allow one to take the square root of \( L = \partial^2 + u \):

\[
(\partial^2 + u)^{1/2} = \partial + \frac{u}{2} \partial^{-1} - \frac{u_x}{4} \partial^{-2} + \frac{u_{xx} - u^2}{8} \partial^{-3} - \frac{u_{xxx} - 6uu_x}{16} \partial^{-4} + \ldots
\]  

(2.7)

Note that if the symbol \( \partial \) commutes with \( u \) (for example, if \( u = \text{const} \)), then derivatives in the right hand side vanish and one obtains the usual Laurent series for the function \( \sqrt{\partial^2 + u} \). One can also define any half-integer powers of the operator \( L \): \( L^{n/2} = (L^{1/2})^n \).

It is easy to see that all of them commute with \( L \).

**Problem** (unsolved). Prove that the “even” coefficients \( u_{2k}, k \geq 1 \) in the expansion \((\partial^2 + u)^{1/2} = \partial + \sum_{m \geq 1} u_k \partial^{-m}\) are full derivatives (see (2.7)).

### 2.3.2 The KdV hierarchy

It is easy to check that \( A = (L^{3/2})_+, \quad \frac{3}{2} uu_x + \frac{1}{4} u_{xxx} = 2\partial \text{res} L^{3/2} \), so the Lax equation can be written as \( \partial_t L = [(L^{3/2})_+, L] \), and the KdV equation acquires the form \( \partial_t u = 2\partial \text{res} L^{3/2} \). In these terms, all its symmetries are written in a unified way: instead of power \( 3/2 \) it is enough to take arbitrary other half-integer power of the operator \( L \).

**Proposition.** The Lax equations

\[
\partial_t L = [A_k, L], \quad A_k = (L^{k/2})_+
\]

for any odd \( k \geq 1 \) generate the equations

\[
\partial_t u = 2\partial \text{res} L^{k/2},
\]

(2.9)

which are symmetries of the KdV equation.

The set of equations (2.9) is called the KdV hierarchy. The variables \( t_k \) are called times. Any equation of the form \( \partial_t u = \sum_j c_j \partial \text{res} L^{j/2} \), where \( c_j \) are arbitrary constants, also belongs to the hierarchy. The KdV equation itself is obtained at \( k = 3 \) if we identify \( t_3 = t \). At \( k = 1 \) we have \( u_{t_1} = u_x \); this allow us to identify \( t_1 \) with \( x + c \). Here are the first three equations of the hierarchy:

\[
\begin{align*}
    u_{t_1} &= u_x, \\
    4u_{t_3} &= 6uu_x + u_{xxx}, \\
    16u_{t_5} &= 30u^2 u_x + 20u_x u_{xx} + 10u_{xxx} + u_{xxxxx}.
\end{align*}
\]

For the proof of the proposition we should check two facts: first, that in any Lax equation the right hand side is actually the operator of multiplication by a function (and that it has the form \( 2\partial \text{res} L^{k/2} \)), and, second, that \( \partial_t \text{res} L^{3/2} = \partial_{t_3} \text{res} L^{k/2} \). The first fact follows from the equality

\[
[(L^{k/2})_+, L] = -[(L^{k/2})_-, L]
\]

(2.10)
(the left hand side is a purely differential operator while the right hand side contains only non-positive powers of \( \partial \), hence we have the operator of multiplication by a function in both sides). The second statement holds true even in a more general form \( \partial_t \text{res} L^{m/2} = \partial_m \text{res} L^{k/2} \) for any pair of times \( t_k, t_m \). It is proved by the following chain of equalities:

\[
\partial_t \text{res} L^{m/2} = \text{res} (\partial_t \text{res} L^{m/2}) = \text{res} \left( [(L^{k/2})_+, (L^{m/2})_-] \right) \\
= \text{res} \left( [(L^{k/2})_+, (L^{m/2})_-] \right) = \text{res} \left( [L^{k/2}, (L^{m/2})_-] \right) \\
= \text{res} \left( [L^{m/2}_+, L^{k/2}] \right) = \text{res} (\partial_m L^{k/2}) = \partial_m \text{res} L^{k/2}.
\]

Here we use the fact that \( \text{res} \left( [(L^{k/2})_+, (L^{m/2})_-] \right) = \text{res} \left( [L^{k/2}, (L^{m/2})_-] \right) = 0 \). Besides, at the second step we implicitly assumed that the Lax equation for \( L \) implies the Lax equation for its half-integer powers. Strictly speaking, this is not obvious and requires a proof. Clearly, it is enough to check this for the pseudo-differential operator \( L^{1/2} \).

**Lemma.** It follows from the Lax equation \( \partial_t L = [A, L] \) that \( \partial_t L^{1/2} = [A, L^{1/2}] \).

For brevity denote \( L^{1/2} = \mathcal{L} \), then from \( \partial_t \mathcal{L}^2 = [A, \mathcal{L}^2] \) we find \( (\dot{\mathcal{L}} - [A, \mathcal{L}]) \mathcal{L} + \mathcal{L}(\dot{\mathcal{L}} - [A, \mathcal{L}]) = 0 \), where dot denotes the \( t \)-derivative. Using the same argument as above (see (2.10)), it is easy to see that the operator \( \dot{\mathcal{L}} - [A, \mathcal{L}] := P_+ = p_1 \partial^{-1} + p_2 \partial^{-2} + \ldots \) contains only negative powers of \( \partial \). Since \( \mathcal{L} \) has the form \( \mathcal{L} = \partial + O(\partial^{-1}) \), from \( P_- \mathcal{L} + \mathcal{L} P_- = 0 \) it follows that \( p_1 = 0 \), i.e. that \( P_+ \) is actually of the form \( P_+ = P_- \partial^{-1} \). Substituting it in this form into the equality mentioned above, we obtain \( \dot{P} \mathcal{L} + \mathcal{L} \dot{P}_- = 0 \), where \( \dot{\mathcal{L}} = -1 \mathcal{L} \partial = \partial + O(\partial^{-1}) \), hence it follows in the same way that \( p_2 = 0 \). Repeating this process, we find that all coefficients of the operator \( P_- = \dot{\mathcal{L}} - [A, \mathcal{L}] \) are equal to 0, i.e., \( \mathcal{L} = [A, \mathcal{L}] \), what is the statement of the lemma.

Hence the KdV equation has infinitely many *commuting* symmetries, i.e. such that any two symmetries from this set are symmetries for each other. It turns out that the KdV equation has a larger set of symmetries. It includes also those which do not possess the commutativity property (so-called additional or non-abelian symmetries). They will be discussed later in these notes.

### 2.3.3 Integrals of motion

The infinite number of symmetries implies the existence of an infinite set of integrals of motion (conservation laws).

**Proposition.** The quantities

\[
I_j = \int \text{res } L^{j/2} \, dx \tag{2.11}
\]

at \( j \geq 1 \) are integrals of motion for all equations of the KdV hierarchy.

(Obviously, \( I_{2l} = 0 \), so only integrals with odd indices are non-trivial.) The proof is based on the Lax representation:

\[
\partial_t I_j = \int \text{res} (\partial_t L^{j/2}) \, dx = \int \text{res} \left( [(L^{j/2})_+, L^{j/2}] \right) \, dx.
\]
Now we recall that \( \text{res}[P, Q] \) for any two pseudo-differential operators \( P, Q \) is a full derivative. Therefore, in the case of rapidly decreasing or periodic solutions we obtain \( \partial_t I_j = 0 \).

### 2.3.4 The Gelfand-Dickey coefficients and their properties

The densities of integrals of motion \( R_j \equiv \text{res} L^{j/2} \) are called the Gelfand-Dickey coefficients. There is a simple recurrence formula for them. In order to derive it, we write

\[
(L^{j/2})_- = R_j \partial^{-1} + S_j \partial^{-2} + T_j \partial^{-3} + \ldots ,
\]

then

\[
\begin{align*}
R_{j+2} &= \text{res} L^{j+1} = \text{res} \left( (\partial^2 + u) L^{j/2} \right) \\
&= \text{res} (\partial^2(R_j \partial^{-1} + S_j \partial^{-2} + T_j \partial^{-3}) + uR_j \partial^{-1}) \\
&= R''_j + uR_j + 2S'_j + T_j.
\end{align*}
\]

Now calculate

\[
-[(L^{j/2})_-, L] = -[R_j \partial^{-1} + S_j \partial^{-2} + T_j \partial^{-3} + \ldots , \partial^2 + u] = 2R'_j + (R''_j + 2S'_j) \partial^{-1} + (u'R_j + S''_j + 2T'_j) \partial^{-2} + \ldots
\]

and use equality (2.10). One can see from it that the operator \([(L^{j/2})_-, L] \) does not contain negative powers of \( \partial \), i.e., the following identities hold:

\[
R''_j + 2S'_j = 0,
\]
\[
u'R_j + S''_j + 2T'_j = 0.
\]

Expressing \( S_j, T_j \) through \( R_j \), we find the recurrence relation from the formula for \( R_{j+2} \) written above:

\[
4R'_{j+2} = R'''_j + 4uR'_j + 2u'R_j.
\]

One can rewrite it in the form

\[
4R'_{j+2} = (\partial^2 + 4u + 2u' \partial^{-1})R'_j
\]

or

\[
4R_{j+2} = (\partial^2 + 2u + 2\partial^{-1}u \partial)R_j.
\]

It is convenient to start with \( j = -1 \) putting \( R_{-1} = 1 \). The recurrence relation connects \( R_j \) with odd indices; all \( R_{2l} \) are equal to 0. The generating function

\[
R(z) = \sum_{n=1}^{\infty} R_n z^{-n-2} = z^{-1} + R_1 z^{-3} + \ldots
\]

allows one to represent the recurrence relation in the form of the linear differential equation

\[
R'''(z) + 4(u - z^2)R'(z) + 2u'R(z) = 0.
\]

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Problem. Prove that $R(z)$ satisfies the nonlinear differential equation

$$2R(z)R''(z) - (R'(z))^2 + 4(u - z^2)R^2(z) + 4 = 0. \quad (2.17)$$

Remark. The function $R(z)$ has the meaning of diagonal of the kernel of the operator $(\partial^2 + u - z^2)^{-1}$, i.e., $R(z) = R(z; x, x)$, where $R(z; x, x)$ is the kernel of the resolvent of the operator $\partial^2 + u$. Thus the generating function of integrals of motion $\int R(z) \, dx$ should be understood as $\text{tr} (\partial^2 + u - z^2)^{-1}$, which is in agreement with conservation of this quantity by virtue of the Lax equation.

The operator $\Lambda = \partial^2 + 2u + 2\partial^{-1}u\partial$ in the right hand side of (2.14) has a name. It is called the recursion operator. It allows one to formally solve the recurrence relation writing

$$R_{2j-1} = 2^{-2j} \Lambda^j \cdot 1, \quad j = 0, 1, 2, \ldots \quad (2.18)$$

Here are the first few $R_j$’s:

- $2R_1 = u$,
- $8R_3 = 3u^2 + u''$,
- $32R_5 = 10u^3 + 5u'^2 + 10uu'' + u'''$.

We remind the reader that the higher KdV equations have the form $\partial_t u = 2R'_j$.

Problem. Prove that

$$A_m = (L^{m/2})_+ = \sum_{j=0}^{m-1} \left( R_{2j-1} \partial - \frac{1}{2} R'_{2j-1} \right) L^{-j}, \quad m = 1, 3, 5 \ldots \quad (2.19)$$

(Hint: first prove the recurrence relation $A_{m+2} = A_m L + R_m \partial - \frac{1}{2} R'_m$.)

The coefficients $R_j$ have an important property: for any $j, l$ the product $R_j R'_l$ is a full derivative, i.e. there exists a differential polynomial $P_{j,l}$ such that

$$R_j R'_l = P'_{j,l}. \quad (2.20)$$

This statement can be proved by induction with the help of the recurrence relation (2.12). Namely, assume that this property holds for some (odd) $j$ for all $l$ (for example, this is obvious at $j = -1$); then the recurrence relation implies that this property holds for all $l$ and the next odd $j$.

Problem. Prove the property (2.20).

In particular, this property guaranties that the operator $\partial^{-1}$ in (2.14) is always applied to a full derivative of a differential polynomial: $u \partial R_j = 2R_1 R'_j = 2P'_{1,j}$, and so the result of its action is the differential polynomial $2P_{1,j}$.

There is also a recurrence relation of a different type for the Gelfand-Dickey coefficients:

$$\frac{\delta}{\delta u} \int R_m \, dx = \frac{m}{2} R_{m-2}, \quad (2.21)$$

where in the left hand side we have the variational derivative of $m$th integral of motion $I_m = \int R_m \, dx$. This relation immediately follows from the fact that under $\int$ res the
variation of the operator $L$ elevated to the power $m/2$ can be handled as variation of the usual power function:

$$\int \text{res} \left( \delta \left( L^{\frac{m}{2}} \right) \right) dx = \frac{m}{2} \int \text{res} \left( L^{\frac{m-2}{2}} \delta L \right) dx.$$ 

Since $\delta L = \delta u$, the relation (2.21) is now obvious.

**Problem.** Give a detailed derivation of relation (2.21).

### 2.4 Hamiltonian formulation

The relation (2.21) allows one to write $m$th equation of the KdV hierarchy in the form

$$\partial_{t_m} u = \frac{4}{m+2} \frac{d}{dx} \frac{\delta I_{m+2}}{\delta u}. \quad (2.22)$$

Clearly, this equation has the standard Hamiltonian form $\partial_{t_m} u = \{u, H_m\}$ with the Hamiltonian $H_m = \frac{4}{m+2} I_{m+2}$ and the Poisson bracket defined on functionals $F, G$ of $u$ in the following way:

$$\{F, G\} = \int \delta F \frac{d}{\delta u} \frac{\delta G}{\delta u} dx. \quad (2.23)$$

It is easy to see that the integrals $I_m$ are in involution:

$$\{I_m, I_n\} = \int \delta I_m \frac{d}{dx} \frac{\delta I_n}{\delta u} dx = \frac{mn}{4} \int R_{m-2} R_{n-2}' dx = 0$$

by virtue of (2.20).

**Problem.** Check that the bracket (2.23) satisfies the Jacobi identity.

If in the $m$th KdV equation $\partial_{t_m} u = 2 \partial R_m$ one first expresses $R_m$ through $R_{m-2}$ by means of the recurrence operator ($4R_m = \Lambda R_{m-2}$), and then use relation (2.21), one obtains another Hamiltonian formulation of the same equation:

$$\partial_{t_m} u = \frac{1}{m} \frac{d}{dx} \frac{\delta I_m}{\delta u}, \quad \frac{d}{dx} \Lambda = \partial^3 + 4u\partial + 2u'. \quad (2.24)$$

Now the Hamiltonian is $\frac{1}{m} I_m$, and the Poisson bracket is given by

$$\{F, G\}_2 = \int \delta F \frac{d}{\delta u} \frac{\Lambda \delta G}{\delta u} dx. \quad (2.25)$$

It is easy to check that it is antisymmetric, but the proof of the Jacobi identity for it is non-trivial.

The bracket (2.23) (which can be naturally denoted as $\{ , \}_1$) and the bracket (2.25) define respectively first and second Hamiltonian structures of the KdV equation. It is easy to see that the integrals of motion $I_m$ are in involution with respect to the both brackets. It can be also verified that these brackets are compatible, i.e., any their linear combination $\{ , \}_1 + \lambda \{ , \}_2$ with constant $\lambda$ is also a Poisson bracket.
2.5 Auxiliary linear problems and \( \psi \)-function

2.5.1 The Baker-Akhiezer function

The Lax equation (2.4) is the compatibility condition of overdetermined system of linear differential equations (auxiliary linear problems)

\[
\begin{align*}
L\psi & = z^2 \psi, \\
\partial_t \psi & = A\psi.
\end{align*}
\] (2.26)

Indeed, taking the \( t \)-derivative of the first equation and substituting the second one, one gets \( (\partial_t L + [L, A])\psi = 0 \). The compatibility means existence of a large set of common solutions. It then follows that the operator \( \partial_t L + [L, A] \) should be equal to 0.

The solutions \( \psi \) can be found in the form of a series in \( z \):

\[
\psi = e^{zx+z^3t} \left( 1 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \ldots \right),
\] (2.27)

where \( \xi_i \) depend only on \( x \) (and on \( t \)). The functions \( \xi_i \) can be expressed through \( u \) by substitution of the series for \( \psi \) into the equation \( (\partial^2 + u)\psi = z^2 \psi \). For example,

\[
\begin{align*}
2\xi'_1 + u & = 0, \\
2\xi'_2 + \xi''_1 + \xi_1 u & = 0
\end{align*}
\] (2.28)

and so on (the recurrence relations for \( i \geq 2 \) are \( 2\xi'_i + \xi''_{i-1} + \xi_{i-1} u = 0 \)).

In a similar way, any higher KdV equation is the compatibility condition of the linear problems

\[
\begin{align*}
L\psi & = z^2 \psi, \\
\partial_{tm} \psi & = A_m \psi
\end{align*}
\] (2.29)

with the same operator \( L \) and \( A_m = (L^{n/2})_+ \). Their common solution in this case has the form

\[
\psi = e^{zx+z^3t_3+z^5t_5+\ldots} \left( 1 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \ldots \right).
\] (2.30)

So far it is only some formal series.

The function \( \psi \) regarded as a function of the “spectral parameter” \( z \) is called the Baker-Akhiezer function. Strictly speaking, it should be called formal Baker-Akhiezer function because it is not yet a function but only a formal series. However, we will not emphasize this distinction. The Baker-Akhiezer function plays a fundamental role in the theory of the KdV equation (and other soliton equations). It serves as a basic tool for construction of exact solutions. Namely, the integration scheme of the KdV equation will consist in constructing a family of solutions for \( \psi \). They will be already well-defined functions of \( z \) whose expansion around \( \infty \) will be of the form (2.30). The desired solution \( u \) will be then found using the formula \( u = -2\partial_x \xi_1 \).

**Exercise.** Prove that \( \partial_{tn} \xi_1 = -R_n \), where \( R_n \) is the Gelfand-Dickey coefficient.
2.5.2 Integrals of motion from the $\psi$-function

The Baker-Akhiezer function allows one to find an infinite set of integrals of motion.

**Proposition.** The function

$$\chi := \partial \log \psi - z = \sum_{j=1}^{\infty} \chi_j z^{-j}$$

is a generating function of densities of integrals of motion, i.e. $\partial_t \int \chi_j dx = 0$ for all $j \geq 1$.

Indeed, the function $\chi$ satisfies the equation of the Riccati type

$$\chi^2 + \chi x + 2z\chi + u = 0,$$

hence its coefficients can be recursively expressed through $u$, $u_x$, $u_{xx}$, ..., for example:

$$\chi_1 = -\frac{1}{2}u, \quad \chi_2 = \frac{1}{4}u_x.$$  

On the other hand,

$$\partial_t \chi = \partial \partial_t \log \psi = \partial \left( \frac{\partial_t \psi}{\psi} \right) = \partial \left( \frac{A\psi}{\psi} \right).$$

But the function $A\psi/\psi$ is expressed through $\chi$, $u$ and their $x$-derivatives (because $\partial^3 \psi/\psi$ is expressed through $\chi$ and its derivatives). Therefore, the expansion coefficients of this expression are differential polynomials of $u$. The derivative in the right hand side implies that $\partial_t \int \chi dx = 0$ (for rapidly decreasing and periodic functions $u$).

Note that the non-trivial integrals of motion only come from $\chi_j$ with odd indices. All $\chi_{2n}$ are full derivatives. Indeed, writing the Riccati equations for $\chi(\pm z)$ and subtracting them, we get

$$(\chi(z) + \chi(-z))(\chi(z) - \chi(-z)) + (\chi(z) - \chi(-z)) + 2z(\chi(z) + \chi(-z)) = 0,$$

hence $\chi(z) + \chi(-z) = -\partial \log(\chi(z) - \chi(-z) + 2z)$ is the full derivative.

A natural question is how the integrals $\int \chi_j dx$ are connected with the integrals $I_j$ given by equation (2.8). Let $\psi(z)$ be a solution of the equation $(\partial^2 + u)\psi = z^2\psi$, then the second solution is $\psi(-z)$, and their Wronskian

$$\psi(-z)\psi_x(z) - \psi(z)\psi_x(-z) := W(z)$$

does not depend on $x$. Dividing both sides by $\psi(z)\psi(-z)$, we get

$$2z + \chi(z) - \chi(-z) = \frac{W(z)}{\psi(z)\psi(-z)},$$

where in the left hand side we have the generating function of densities of non-trivial integrals $\chi_j$ (with odd indices). Next, it is not difficult to see that $\psi(z)\psi(-z)$ satisfies the same third order equation (2.16) as $R(z)$. Since both functions have the same structure of expansions in powers of $z$, they may differ only by a $x$-independent common factor which can be found from the limit $x \to \infty$ (assuming that $u \to 0$ as $x \to \infty$). Taking into
account that $\chi(z) \to 0$ as $x \to \infty$, (2.33) implies that $\psi(z)\psi(-z)$ tends to $W(z)/(2z)$ and, therefore,

$$R(z) = \frac{2\psi(z)\psi(-z)}{W(z)}. \quad (2.34)$$

We see that the generating functions of densities of the integrals $\int x_j dx$ and $I_j$ at odd $j$ are connected by the relation

$$\left(z + \frac{\chi(z) - \chi(-z)}{2}\right) R(z) = 1.$$  

Expanding both sides of (2.34) in powers of $z$, we can write:

$$R_l = 2 \text{res}_{z=\infty} \left( \frac{z^{l+1}}{W(z)} \psi(z)\psi(-z) \right), \quad (2.35)$$

where the residue is understood as the coefficient in front of $z^{-1}$. Note that this equality is a consequence of a more general relation

$$\left(L^{m/2}\right)_l = 2 \text{res}_{z=\infty} \left( \frac{z^{m+1}}{W(z)} \psi(z)\psi(-z) \right).$$  

(2.36)

We will not prove it here (it follows from even more general relation which is proved in the section on the KP hierarchy).

### 2.5.3 The mKdV equation

Let us show how the $\psi$-function allows one to pass from the KdV equation to the so-called modified KdV equation (mKdV). Put $v = \partial \log \psi$, then equations $\psi_{xx} + uv = \lambda \psi$ and $\psi_t = \psi_{xxx} + \frac{3}{2}u \psi_x + \frac{3}{4}u_x \psi$ (the spectral parameter is denoted here by $\lambda$) can be written in the form

$$u = \lambda - v^2 - v_x \quad (2.37)$$

and

$$v_t = \partial_x \left( \frac{\psi_{xxx} + \frac{3}{2}u \psi_x + \frac{3}{4}u_x \psi}{\psi} \right). \quad (2.38)$$

The relation (2.37) is called the Miura transformation (usually with $\lambda = 0$). Using the Miura transformation and obvious identities

$$\frac{\psi_{xx}}{\psi} = v_x + v^2, \quad \frac{\psi_{xxx}}{\psi} = v_{xx} + 3vv_x + v^3,$$

one can represent the relation (2.38) as an equation for $v$: $v_t = \frac{1}{4} \partial_x (v_{xx} - 2v^3 + 6\lambda v)$ or

$$4v_t = -6v^2v_x + v_{xxx} + 6\lambda v_x, \quad (2.39)$$

which is called the mKdV equation (usually without the last term).

**Exercise.** Let $u$ and $v$ be connected by the Miura transformation (2.37). Show that

$$-(4u_t - 6uu_x - u_{xxx}) = (\partial_x + 2v)(4v_t + 6v^2v_x - v_{xxx} - 6\lambda v_x). \quad (2.40)$$
2.6 Construction of solutions to the KdV equation with the help of the $\psi$-function

2.6.1 The basic lemma

Let us present simple but important lemma of a technical nature on which the construction of exact solutions is based.

Lemma. For the function $\psi$ of the form (2.27) the following formal equalities hold:

\[
(\partial^2 - z^2 + u)\psi = O(z^{-1})e^{zx+z^3t},
\]

\[
(\partial_t + \frac{1}{2} \partial^3 - \frac{3}{2} z^2 \partial + \frac{3}{4} u_x)\psi = O(z^{-1})e^{zx+z^3t},
\]

where the function $u = u(x,t)$ can be found form vanishing of the coefficients at non-negative powers of $z$: $u = -2\xi_{1,x}$.

The proof is a direct verification. The meaning and use of this statement is demonstrated below. It allows one to construct exact solutions to the KdV equation using methods of linear algebra. Assume that the space of $\psi$-functions of the form (2.27) (defined by imposing certain requirements on analytic properties of these functions as functions of the complex variable $z$) is one-dimensional, i.e. there is only one such function up to multiplication by a constant. Assume also that the operators in the left hand sides of the formal equalities preserve this space. Then the form of the right hand sides implies that they are equal to zero identically and not only up to the terms $O(z^{-1})e^{zx+z^3t}$. In its turn, this means that the function $\psi$ for all $z$ is a common solution to the pair of linear problems (2.26). The compatibility of these linear problems implies the KdV equation for $u = -2\partial\xi_1$.

2.6.2 One-soliton solution

We begin with the simplest example. Let us consider the space of functions $\psi = \psi(z)$ which are meromorphic everywhere except infinity and such that

a) The function $\psi e^{-zx-z^3t}$ is regular at $z = \infty$;

b) The function $\psi$ has the only simple pole at $z = 0$ and holomorphic everywhere else except $\infty$;

c) In some point $p \in \mathbb{C}$ the relation $\psi(p) = \psi(-p)$ holds for all $x, t$.

The variables $x, t$ and the point $p$ are regarded here as fixed parameters. Clearly, such functions form a linear space. It is easy to see that if the parameters are in general position, then the dimension of this space is equal to 1, i.e., there is only one such function (up to multiplication by a constant). Indeed, such $\psi$ has the form

\[
\psi = e^{zx+z^3t} \left( b_0 + \frac{b_1}{z} \right)
\]

with some $b_0, b_1$ but condition c) fixes the ratio $b_1/b_0$, and thus only the common factor remains arbitrary.
Next, it is not difficult to convince oneself that the operators in the left hand sides of (2.41) preserve the linear space of functions defined above. It is enough to check that for any choice of the function \( u \)

\[
(\partial^2 - z^2 + u)\psi = O(1)e^{zx+z^3t},
\]

and that the left hand sides at \( z = p \) and \( z = -p \) are the same. The first is checked by a direct calculation and the second is obvious from the fact that the left hand sides contain \( z^2 \) only. Thus in the right hand sides we have functions from the same linear space; denote them by \( \psi_1 \) and \( \psi_2 \). According to the lemma, at \( u = -2\xi_{1,x} \) the right hand sides behave actually as \( O(1/z)e^{zx+z^3t} \). This means that the functions \( \psi_1 e^{-zx-z^3t} \) and \( \psi_2 e^{-zx-z^3t} \) vanish at infinity. The uniqueness then implies \( \psi_1 = 0 \), i.e.,

\[
(\partial^2 + u - z^2)\psi = 0,
\]

Substituting \( z^2 \psi \) from the first equality to the second one, we come to the pair of linear problems (2.20) together with the explicitly found family of common solutions. Their compatibility guarantees that \( u = -2\xi_{1,x} \) is a solution to the KdV equation.

It remains to find the solution explicitly. For the function

\[
\psi = e^{zx+z^3t}\left(1 + \frac{\xi_1}{z}\right)
\]

the condition \( \psi(p) = \psi(-p) \) is equivalent to the linear equation \( e^{2px+2p^3t}(p + \xi_1) = p - \xi_1 \) for \( \xi_1 \), hence \( \xi_1 = -p \tanh(px + p^3t) \) and, therefore,

\[
u(x,t) = \frac{2p^2}{\cosh^2(px + p^3t)}
\]

(the same formula as (2.3) after identification \( c = p^2 \)). Note that \( \xi_1 = -\partial_x \log \cosh(px + p^3t) \) and hence equation (2.42) can be written in the form

\[
u(x,t) = 2\partial_x^2 \log \cosh(px + p^3t)
\]

The one-soliton solution of the whole KdV hierarchy is given by the same formula in which instead of \( px + p^3t \) under \( \cosh \) we have \( px + p^3t_3 + p^5t_5 + \ldots \)

**Remark.** Instead of functions with a pole at 0 one can consider functions with a pole at an arbitrary point \( a \in \mathbb{C} \) and with a more general condition \( \psi(p) = \alpha \psi(-p) \), where \( \alpha \) is an arbitrary nonzero constant.

**Problem.** Show that the function \( \psi = e^{zx+z^3t}\left(1 + \frac{\xi_1}{z-a}\right) \) with the condition \( \psi(p) = \alpha \psi(-p) \) leads to the one-soliton solution of the similar form (2.42) which differs from it only by a shift \( x \rightarrow x + x_0 \) (here \( x_0 \) is generally speaking a complex number) and express \( x_0 \) through \( a \) and \( \alpha \).

From the point of view of the Schrödinger equation with the potential \(-u\), i.e., \(-\partial^2 \psi - u\psi = E\psi\), the one-soliton solution is remarkable in that the corresponding potential well has only one bound state with energy \( E = -p^2 \) and the states belonging to the continuous spectrum with energy \( E = -z^2 \) at purely imaginary \( z \) have zero reflection coefficient. The propagation through the potential well brings only the phase shift equal to \( \arg \frac{z}{z+p} \).
2.6.3 Multisoliton solutions

Consider the linear space of functions \( \psi = \psi(z) \) which are meromorphic everywhere except infinity and such that

a) The function \( \psi e^{-2x-z^3t} \) is regular at \( z = \infty \);

b) The function \( \psi \) has no more than \( N \) poles (counted with multiplicities) at some marked points of the complex plane and holomorphic everywhere else in the complex plane except at \( \infty \);

c) At \( N \) distinct points \( p_j \in \mathbb{C} \) the relations \( \psi(p_j) = \alpha_j \psi(-p_j), \ j = 1, 2, \ldots, N, \) hold.

For simplicity consider the case when all the poles are concentrated at the point \( z = 0 \), i.e., there is a pole of multiplicity not greater than \( N \) in this point and no other poles. Then \( \psi \) can be represented in the form

\[
\psi = e^{zx+z^3t} \left( b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots + \frac{b_N}{z^N} \right).
\]

The space of such functions has dimension \( N + 1 \). Similarly to the case of just one pole, \( N \) linear conditions \( \psi(p_j) = \alpha_j \psi(-p_j) \) for the coefficients \( b_j \) make this space one-dimensional. The operators in the left hand sides of (2.41) preserve it. Therefore, if we normalize \( \psi \) by the condition that the coefficient in front of \( z^0 \) is 1, as in (2.27), then

\[
\psi = e^{zx+z^3t} \left( 1 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \ldots + \frac{\xi_N}{z^N} \right),
\]

then \( u = -2\xi_{1,x} \) is going to be a solution of the KdV equation.

Let us find this solution explicitly. The conditions \( \psi(p_j) = \alpha_j \psi(-p_j) \) are equivalent to the following system of linear equations for \( \xi_j \):

\[
\sum_{j=1}^{N} M_{ij} \xi_j = -M_{i0},
\]

where \( M_{ij} \) has the form \( M_{ij} = p_i^{-j} e^{p_i x + p_i^3 t} - \alpha_i(-p_i)^{-j} e^{-p_i x - p_i^3 t} \). The Kramer’s rule yields

\[
\xi_1 = -\frac{\det M_{ij}^{(0)}}{\det M_{ij}} , \quad i, j = 1, 2, \ldots, N,
\]

where the matrix \( M_{ij}^{(0)} \) differs from \( M_{ij} \) by the change of the first column \( M_{i1} \) to \( M_{i0} \) (\( i \) numbers the rows). Since \( \partial_x M_{ij} = M_{i,j-1} \), we have \( \det M_{ij}^{(0)} = \partial_x \det M_{ij} \), and \( \xi_1 = -\partial_x \log \det M_{ij} \), so that \( u = 2\partial_x^2 \log \det M_{ij} \). We have thus obtained the family of solutions

\[
u = 2\partial_x^2 \log \tau, \quad (2.43)
\]

where

\[
\tau = \det_{1 \leq i, j \leq N} \left( p_i^{-j} e^{p_i x + p_i^3 t} - \alpha_i(-p_i)^{-j} e^{-p_i x - p_i^3 t} \right). \quad (2.44)
\]
The solution of the whole hierarchy is given by the same formula with the change \( p_i x + p_i^2 t \rightarrow p_i x + p_i^3 t_3 + p_i^5 t_5 + \ldots \) Note that all \( \alpha_i \)’s can be “hidden” in suitably chosen initial values of the times \( t_j \), so from the point of view of the hierarchy the true parameters of the solution are only the \( p_i \)’s.

This solution is called the \( N \)-soliton solution and \( p_i \) are called momenta of the solitons. The function \( \tau = \tau(x, t_3, t_5, \ldots) \) is called the \textit{tau-function}. It plays a fundamental role not only in the theory of the KdV equation but also in the theory of all other integrable equations. It turns out that any exact solution of the KdV hierarchy (not only the \( N \)-soliton solution) can be represented as (multiplied by 2) second logarithmic derivative of the determinant of some matrix or operator (in general case infinite-dimensional). This determinant is the tau-function.

Since the solution is expressed through the second logarithmic derivative of \( \tau \), the tau-functions which differ from each other only by a factor of the form \( C e^{ax} \), with constant \( C \) and \( a \) are equivalent.

It can be shown that the potentials in the Schrodinger equation corresponding to the \( N \)-soliton solutions with real \( p_i \) are reflectionless and have exactly \( N \) bound states with energies \( E_i = -p_i^2 \). In the quantum-mechanical interpretation, the conditions \( \psi(p_j) = \alpha_j \psi(-p_j) \) mean that at the points of the discrete spectrum there is only one linearly independent eigenfunction of the Schrodinger operator.

**Problem.** Find the phase shift of the wave function for scattering on the potential corresponding to the 2-soliton solution.

Let us point out two other useful forms of the soliton tau-function. One of them is the following determinant of the \( N \times N \) matrix:

\[
\tau = \det_{1 \leq i, j \leq N} \left( \delta_{ij} + \frac{2 \beta_i p_i}{p_i + p_j} e^{2p_i x + 2p_i^3 t_3 + 2p_i^5 t_5 + \ldots} \right). 
\] (2.45)

Here \( \beta_i \) are some parameters (which are similar to \( \alpha_i \)). They also can be eliminated by a suitable shift of times.

**Problem.** Prove the equivalence of the determinant representations (2.44) and (2.45).

Expanding the determinant (2.45), we get the following formula:

\[
\tau = \sum_{\{\epsilon_1, \ldots, \epsilon_N\} \in \mathbb{Z}^N} \prod_{i < j}^{N} \left( \frac{p_i - p_j}{p_i + p_j} \right)^{2 \epsilon_i \epsilon_j} \prod_{k=1}^{N} \left( \beta_k e^{2p_k x + 2p_k^3 t_3 + \ldots} \right)^{\epsilon_k}. 
\] (2.46)

Here the sum is taken over all sets of \( N \) numbers \( \epsilon_i \) taking values 0, 1, so that the sum contains \( 2^N \) terms. For example, at \( N = 2 \) we have:

\[
\tau = 1 + \beta_1 e^{2p_1 x} + \beta_2 e^{2p_2 x} + \beta_1 \beta_2 \left( \frac{p_1 - p_2}{p_1 + p_2} \right)^2 e^{2(p_1 + p_2) x},
\]

where only the terms containing \( x \) in the exponential functions are left for simplicity.

The multisoliton solutions are stationary points for higher commuting symmetries.

**Proposition.** Any \( N \)-soliton solution satisfies the ordinary differential equation (a higher stationary KdV)

\[
\sum_i c_i R_i[u] = 0, \quad (2.47)
\]
where \( R_i \) are the Gelfand-Dickey coefficients, with some choice of the constants \( c_i \).

For example, the one-soliton solution (2.42) satisfies the equation \( R_3 - p^2 R_1 = 0 \) or \( 3u^2 + u_{xx} - 4p^2 u = 0 \). This is obvious from the fact that the solution depends on the combination \( x + pt \) and, therefore, \( u_t = p^2 u_x \). The general proof can be carried out using the explicit formulas (2.43), (2.44) and the fact that in the rapidly decreasing case equation (2.47) is equivalent to \( \sum c_i \partial_t \tau = 0 \). It is convenient to use the expression (2.46) for the tau-function.

Problem. Prove that the 2-soliton solution satisfies the equation \( R_5[u] + c_3 R_3[u] + c_1 R_1[u] = 0 \) and find \( c_3, c_1 \).

Remark. Equations (2.47) (called Novikov’s equations) besides rapidly decreasing solutions of the soliton type have a large family of periodic and quasiperiodic solutions. The corresponding potentials in the Schrödinger operators are distinguished by the fact that these operators have only a finite number of forbidden (unstable) bands in the spectrum. These solutions are expressed through the Riemann theta-functions.

### 2.7 Commutation representation of the KdV hierarchy by \( 2 \times 2 \) matrices

There is an alternative commutation representation of the KdV hierarchy which is realized in usual \( 2 \times 2 \) matrices depending on an additional complex parameter (it is called the spectral parameter). It is independent of the technique of pseudo-differential operators and in some cases is more convenient.

#### 2.7.1 Zero curvature representation

The second order equation \((\partial^2 + u)\psi = z^2 \psi\) can be rewritten as a vector first order equation

\[
\partial \Psi = U_1(\lambda) \Psi,
\]

where \( \lambda = z^2 \) is the spectral parameter,

\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad U_1(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}
\]

and \( \psi_1 = \psi, \ \psi_2 = \psi' = \partial \psi \). In order to find a similar representation for the equation \( \partial_m \psi = A_m \psi \), we use the formula (2.19) and write:

\[
\partial_m \psi = A_m \psi = \sum_{j=0}^{m-1} \left( R_{2j-1} \partial - \frac{1}{2} R'_{2j-1} \right) \lambda^{\frac{m-1}{2}} \psi = -\frac{1}{2} R'_{m-2}(\lambda) \psi + R_{m-2}(\lambda) \psi'
\]

where the standard notation

\[
R_m(\lambda) := \sum_{j=0}^{m+1} R_{2j-1} \lambda^{\frac{m+1}{2}} \psi, \quad m = -1, 1, 3, \ldots
\]
for “incomplete generating functions” of the Gelfand-Dickey coefficients is introduced. For example,
\[R_{-1}(\lambda) = 1,\]
\[R_1(\lambda) = \lambda + R_1,\]
\[R_3(\lambda) = \lambda^2 + \lambda R_1 + R_3\] and so on.
The recurrence relation has the form \(R_{m+2}(\lambda) = \lambda R_m(\lambda) + R_{m+2}.\) It is also easy to find that
\[\partial_t m \psi = \left[(\lambda - u)R_{m-2}(\lambda) - \frac{1}{2}R''_{m-2}(\lambda)\right] \psi + \frac{1}{2}R'_{m-2}(\lambda)\psi'.\]
Unifying this with the previously found formula for \(\partial_t m \psi,\) it is possible to represent the equation \(\partial_t m \psi = A_m \psi\) in the vector form similar to (2.48):
\[\partial_t m \Psi = U_m(\lambda) \Psi,\] (2.49)
where
\[U_m(\lambda) = \begin{pmatrix} -\frac{1}{2}R''_{m-2}(\lambda) & R_{m-2}(\lambda) \\ (\lambda - u)R_{m-2}(\lambda) - \frac{1}{2}R''_{m-2}(\lambda) & \frac{1}{2}R'_{m-2}(\lambda) \end{pmatrix}.\]
Note that at \(m = 1\) this equation coincides with (2.48). At \(m = 3\) we have
\[U_3(\lambda) = \frac{1}{4} \begin{pmatrix} -u' & 4\lambda + 2u \\ 4\lambda^2 - 2\lambda u - (2u^2 + u'') & u' \end{pmatrix}.\]
In the general case \(U_m(\lambda)\) is a matrix polynomial of \(\lambda\) of degree \(\frac{1}{2}(m+1).\) We have rewritten the auxiliary linear problems as matrix equations of first order.

The compatibility condition of the problems (2.48) and (2.49) can be written in the form \([\partial_t m - U_m(\lambda), \partial_t 1 - U_1(\lambda)] = 0\) or
\[\partial_t m U_1(\lambda) - \partial_t 1 U_m(\lambda) + [U_1(\lambda), U_m(\lambda)] = 0\] (2.50)
for all \(\lambda,\) which yields the \(m\)th equation of the KdV hierarchy.

**Exercise.** Check this statement by a direct calculation.

The representation of the KdV hierarchy in the form (2.50) is called the zero curvature (or Zakharov-Shabat) representation. Note that the more general relations of the same type
\[\partial_t m U_n(\lambda) - \partial_t n U_m(\lambda) + [U_n(\lambda), U_m(\lambda)] = 0\] (2.51)
hold true for all \(m, n \geq 1.\)

### 2.7.2 Zero curvature representation, another gauge

The linear problems (2.49) can be “gauge transformed”:
\[\Psi \rightarrow \mathcal{W}\Psi, \quad U_m(\lambda) \rightarrow \mathcal{W}U_m(\lambda)\mathcal{W}^{-1},\]
where the matrix $W$ does not depend on all $t_k$'s (but can depend on $\lambda$). Such transformation means that when passing from the second order equation $(\partial^2 + u)\psi = z^2\psi$ to the first order vector equation the components of the vector $\Psi$ are not $\psi$ and $\psi'$ but their linear combinations with coefficients which may depend on $z$.

For example, consider the choice $\tilde{\psi}_1 = \psi$, $\tilde{\psi}_2 = \psi' - z\psi$. The new vector $\tilde{\Psi}$ is connected with $\Psi$ as follows:

\[
\begin{pmatrix}
\psi_1 \\
\psi_2 
\end{pmatrix}
= \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}
\begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2 
\end{pmatrix}.
\]

Ten the linear problems acquire the form

\[
\partial_t m \tilde{\Psi} = \tilde{U}_m(z) \tilde{\Psi}, \quad m = 1, 3, \ldots
\]  

(2.52)

where $z$ plays the role of the spectral parameter and

\[
\tilde{U}_m(z) = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} U_m(z^2) \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}
\]

are matrix polynomials of $z$ of degree $m$:

\[
\tilde{U}_m(z) = \begin{pmatrix}
zR_{m-2}(z^2) - \frac{1}{2}R'_{m-2}(z^2) & R_{m-2}(z^2) \\
zR'_{m-2}(z^2) - uR_{m-2}(z^2) - \frac{1}{2}R''_{m-2}(z^2) & -zR_{m-2}(z^2) + \frac{1}{2}R'_{m-2}(z^2)
\end{pmatrix}.
\]

Here are the first two matrices:

\[
\tilde{U}_1(z) = \begin{pmatrix} z & 1 \\ -u & -z \end{pmatrix}
\]  

(2.53)

\[
\tilde{U}_3(z) = \frac{1}{4} \begin{pmatrix}
4z^3 + 2uz - u' & 4z^2 + 2u \\
-4uz^2 + 2u'z - (2u^2 + u'') & -4z^3 - 2uz + u'
\end{pmatrix}.
\]  

(2.54)

This commutation representation suggests possible generalization of the KdV theory to other evolution equations: the integrable equation should be the compatibility condition of the linear problems (2.52) with some matrices $V_j(z)$ which are rational functions of the spectral parameter $z$. In particular, a minimal generalization of the KdV theory corresponds to the choice

\[
V_1(z) = \begin{pmatrix} z & v \\ -u & -z \end{pmatrix},
\]

where $u, v$ are some functions included in the equation (at $v = 1$ we come back to KdV). The form of the other matrices should be chosen in such a way that the corresponding zero curvature equation be equivalent to some evolution partial differential equations for $u$ and $v$. This is the way to obtain the mKdV equation, the nonlinear Schrodinger equation, the sine-Gordon equation and others.

### 2.7.3 Spectral curve

The zero curvature representation becomes especially useful in the case when the solution is stationary with respect to some higher flow or their combination. Take, for example,
\[ \partial_t u := \sum_{m} c_m \partial_{t_m} u = 0 \] with some constants \( c_m \), then \( \partial_t \Psi = U(\lambda)\Psi \) with the matrix \( U(\lambda) = \sum_{m} c_m U_m(\lambda) \). Taking linear combination of equations (2.51) with the coefficients \( c_m \) and imposing the condition \( \partial_t U_n(\lambda) = 0 \), we get the Lax type equation for \( U(\lambda) \):

\[
\partial_t U(\lambda) = [U_n(\lambda), U(\lambda)]. \tag{2.55}
\]

This equation implies that the characteristic polynomial of the matrix \( U(\lambda) \) does not depend on the times \( t_n \) for all \( \lambda \). In other words, the algebraic curve defined by the equation \( \text{det}(\mu + U(\lambda)) = 0 \) is an integral of motion for all equations of the hierarchy.

This curve is called the spectral curve because it is closely connected with the spectrum of the Schrodinger operator \( \partial^2 + u \). Since \( \text{tr} U(\lambda) = 0 \), the equation of the spectral curve has the form \( \mu^2 = \text{det} U(\lambda) = 0 \). For example, if \( U(\lambda) = U_3(\lambda) \), then the spectral curve is the elliptic curve

\[
\mu^2 = \lambda^3 - \frac{3u^2 + u''}{4} \lambda - \frac{4u^3 - u'^2 + 2uu''}{16}.
\]

**Problem.** Verify directly that this spectral curve is an integral of motion for all equations of the hierarchy, i.e. that \( \partial_{t_k} (3u^2 + u'') = \partial_{t_k} (4u^3 - u'^2 + 2uu'') = 0 \) for all \( t_k \).

In the general case \( U(\lambda) = c_m U_m(\lambda) + c_{m-2} U_{m-2}(\lambda) + \ldots + c_1 U_1(\lambda), c_m \neq 0 \), the curve is hyperelliptic. The equation of the curve is of the form \( \mu^2 = P_m(\lambda) \), where \( P_m(\lambda) \) is a polynomial, \( \deg P_m(\lambda) = m \).

### 2.8 Nonabelian symmetries

#### 2.8.1 The Galilean transformation and the similarity transformation

We start with a very simple statement.

**Proposition.** The KdV equation preserves its form under the transformations

\[
\begin{align*}
    u &\rightarrow \lambda^{-2} u - 2a\lambda^{-1}, \\
x &\rightarrow \lambda x + 3a\lambda^2 t, \\
t &\rightarrow \lambda^3 t
\end{align*}
\]

with arbitrary constants \( \lambda, a \).

In other words, if \( u(x,t) \) satisfies the KdV equation (2.2), then \( \tilde{u} \) defined as a function of \( \tilde{x}, \tilde{t} \) by the equalities

\[
\begin{align*}
    \tilde{u} &\rightarrow \lambda^{-2} u - 2a\lambda^{-1}, \\
    \tilde{x} &\rightarrow \lambda x + 3a\lambda^2 t, \\
    \tilde{t} &\rightarrow \lambda^3 t
\end{align*}
\]

satisfies the same equation \( 4\tilde{u}_{\tilde{t}} = 6\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} \). Even simpler, one can say that the transformation

\[
u(x,t) \rightarrow \lambda^{-2} u \left( \lambda^{-1} x - 3a\lambda^{-2} t, \lambda^{-3} t \right) - 2a\lambda^{-1}
\]

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sends a solution to another solution. This can be checked by a direct substitution.

The transformation with \( \lambda = 1, \ a \neq 0 \), i.e.

\[
u(x,t) \rightarrow \nu(x - 3at, t) - 2a
\]
is called the Galilean transformation. In the case when \( \lambda \neq 1, \ a = 0 \), i.e.

\[
u(x,t) \rightarrow \lambda^{-2} \nu(\lambda^{-1}x, \lambda^{-3}t)
\]
we have a similarity transformation. For example, the Galilean transformation of the one-soliton solution (2.42) is

\[
u = \frac{2p^2}{\cosh^2(px + (p^3 - 3ap)t)} - 2a
\]
while the similarity transformation of this solution is

\[
u = \frac{2(p/\lambda)^2}{\cosh^2((p/\lambda)x + (p/\lambda)^3t)}.
\]

In the latter case the form of the solution is preserved with \( p \rightarrow p/\lambda \).

In the infinitesimal form the Galilean and similarity transformations have the form

\[
u \rightarrow \nu + \left( \frac{3}{2} tu_x + 1 \right) \varepsilon,
\]

\[
u \rightarrow \nu + (3 tu_t + xu_x + 2u) \varepsilon,
\]
where \( \varepsilon \) is the small parameter of the transformation. One may introduce the corresponding “times” \( s_{-1}, \ s_1 \) and represent the infinitesimal transformations in the form of the differential equations

\[
\frac{\partial \nu}{\partial s_{-1}} = \frac{3}{2} tu_x + 1, \tag{2.57}
\]

\[
\frac{\partial \nu}{\partial s_1} = \frac{3}{4} t(6uu_x + u_{xxx}) + xu_x + 2u.
\]

They are symmetries of the KdV equation in the sense of section 2.3. This fact can be checked directly by calculation of the derivatives \( \partial_{s_i} (\partial_t \nu) \) and \( \partial_t (\partial_{s_i} \nu) \) for \( i = -1, 1 \). However, equations (2.57) are not symmetries for each other: the derivatives \( \partial_{s_{-1}} (\partial_{s_1} \nu) \) and \( \partial_{s_1} (\partial_{s_{-1}} \nu) \) are not equal. This is why these symmetries are called nonabelian.

**Exercise.** Calculate \( \partial_{s_{-1}} (\partial_{s_1} \nu) - \partial_{s_1} (\partial_{s_{-1}} \nu) \).

The characteristic feature of nonabelian symmetries is the form of the right hand sides of (2.57): they are non differential polynomials of \( \nu \) but contain \( x, t \) explicitly. In this respect, they differ from the previously discussed commuting symmetries.
2.8.2 Infinite series of nonabelian symmetries

It turns out that the KdV equation has an infinite series of nonabelian symmetries. They can be compactly written in the form

\[
\frac{\partial u}{\partial s_m} = 2^{-m+1/2} \partial x \Lambda^{m+1/2} \left( \frac{3}{2} tu + x \right), \quad m = -1, 1, 3, \ldots
\]  

(2.58)

where \( \Lambda = \partial^2 + 2u + 2\partial^{-1}u\partial \) is the recursion operator. It is easy to check that the first two symmetries coincide with (2.57). The other ones do not have such a simple form. The symmetries (2.58) can be extended to symmetries of the whole hierarchy if instead of \( \frac{3}{2} tu + x \) in the right hand side one substitutes

\[
S := \sum_{n \geq 1} nt_n R_{n-2} = x + 3t_3 R_1 + 5t_5 R_3 + \ldots
\]

2.8.3 Stationary points of nonabelian symmetries: examples

It is interesting to find solutions which are stationary points of nonabelian symmetries. For example, the solution invariant with respect to the Galilean transformation in which all \( t_j \) starting from \( t_5 \) are equal to 0 should satisfy the condition \( \partial u/\partial s_{-1} = \frac{3}{2} tu_x + 1 = 0 \), hence

\[
u(x, t) = -\frac{2x}{3t}.
\]

(2.59)

This is probably the simplest solution of the KdV equation which is not constant in each of the variables. If one puts all higher times equal to 0 starting from \( t_7 \) while \( t_5 \neq 0 \) (it is convenient to put \( t_5 = 2/5 \)), then the stationarity condition \( \partial u/\partial s_{-1} = 0 \) acquires the form \( 3u^2 + u_{xx} + 6tu + 4x = 0 \). In this case the solution can not be expressed through elementary functions. The best one can do is to express it through a solution of the Painlevé I. Let \( f(x) \) be a solution of Painlevé I

\[
3f^2 + f_{xx} + 4x = 0,
\]

then an easy calculation shows that

\[
u(x, t) = f \left( x - \frac{3}{4} t^2 \right) - t
\]

satisfies the stationarity condition and the KdV equation. Note that the Painlevé I equation (more precisely, its \( x \)-derivative) can be represented as the commutation relation \([L, A] = 1\). This is so-called “string equation”, which became popular in the beginning of nineties of the last century in connection with the attempts to construct the theory of 2D quantum gravity based on the model of random matrices.

2.8.4 Digression on Painlevé equations

The appearance of the Painlevé equation in connection with the KdV equation illustrates a rather general fact: ordinary differential equations that are obtained as reductions of integrable partial differential equations also have certain “good” properties which distinguish them from all others. Namely, they have so-called Painlevé property.
To formulate it, we should introduce the notion of critical point of solution to ordinary differential equation: a singular point is called critical if it is not a pole of arbitrary integer order. In other words, critical points are ramification points (algebraic and logarithmic) and essential singularities. Consider the full set of solutions to a differential equation; their critical points can be divided in two groups: those which depend only on the equation itself and does not depend on the choice of solution (immovable singularities) and those which depend on the integration constants (movable singularities). A differential equation has the Painlevé property if all its solutions have only immovable critical points.

As it follows from the theory of linear ordinary differential equations, all linear equations have the Painlevé property. Simple examples show that nonlinear equations may or may not possess the Painlevé property ($y' = y^2$ and $y' = y^3$ respectively). For equations of first and second order which a linear in the highest derivative all equations possessing the Painlevé property are known. For first order equations of the form $y' = F(y, x)$ the answer is simple (Fuchs, 1884): only generalized Riccati equations

$$y' = f_2(x)y^2 + f_1(x)y + f_0(x)$$

do not have movable critical points.

For second order equations of the form $y'' = G(y, y', x)$ the answer was obtained in the beginning of XX century in the works of Painlevé, Fuchs and Gambier. There are 50 equations of this form which do not have movable critical points. They can be reduced either to equations integrable in known elementary or special functions or to one of 6 canonical equations which in general can not be integrated in known functions. These six equations are now called Painlevé equations:

$$\begin{align*}
P_I: & \quad y'' = 6y^2 + x \\
P_{II}: & \quad y'' = 2y^3 + xy + \alpha \\
P_{III}: & \quad y'' = \frac{y'^2}{y} - \frac{y'}{x} + x^{-1}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
P_{IV}: & \quad y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y} \\
P_V: & \quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{x} + \frac{y(y-1)^2}{x^2}\left(\frac{\beta}{y^2} + \frac{\gamma}{(y-1)^2} + \frac{\delta(x(y+1))}{(y-1)^3}\right) \\
P_{VI}: & \quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' \\
& \quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{y-x}\right)
\end{align*}$$

There is an extensive literature devoted to analysis of these equations and properties of their solutions. There are still open questions in the theory of Painlevé equations.
Besides, the Painléve equations, as other fundamental objects, sometimes appear in rather unexpected contexts.

In the context of integrable partial differential equations the Painléve equations emerge as self-similar reductions of two-dimensional equations.
3 The KP hierarchy

The theory of the KdV equation allows generalizations in different directions. One of them is to take the Lax operator to be an $N$th order differential operator rather than a second order operator. For example, at $N = 3$ one can consider the operator $L = \partial^3 + u\partial + w$ with two independent coefficients $u, w$ which depend on $x$ and all the times $t_j$, with the dynamics being determined by the Lax equations $\partial_t L = [(L^j/3)_+, L]$. It can be shown that each of the Lax equations defines a well-defined system of evolution equations for the functions $u, w$, and the equations obtained in this way are symmetries for each other. At arbitrary $N > 2$ the situation is similar, with the only difference that the generators of the flows are $(L^j/N)_+$, and the equations are written for $N - 1$ unknown function. All these hierarchies (generalized KdV hierarchies of order $N$) can be embedded in one big hierarchy (in some sense “the biggest” one) the definition of which already does not depend on the number $N$. It is called the Kadomtsev-Petviashvili (KP) hierarchy.

To make such embedding possible and natural, it is necessary to “equalize in rights” the Lax operators for different $N$, i.e. represent them as elements of one common algebra. The way how to do this was suggested mainly in the works of the Japanese school (Sato, Jimbo, Miwa). The idea is to work not with the Lax operators themselves but with roots of $N$th degree from them. All of them are pseudo-differential operators of first order and satisfy the same Lax equations. After that one may forget about their origin and extend the Lax equations to first order pseudo-differential operators of general form.

3.1 The Lax equations

From now on by the Lax operator the pseudo-differential operator of the form

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \ldots$$

(3.1)

will be understood. The coefficients $u_i$ are in general independent functions of $x$ and the times $t_j$ with integer $j \geq 1$. The operator $L$ is subject to the Lax equations

$$\partial_t L = [(L^j)_+, L], \quad j = 1, 2, 3, \ldots$$

(3.2)

Each of them defines an infinite system of evolution equations for the infinite set of functions $u_i$: $\partial_t u_i = \mathcal{P}_{ij}(\{u_i\})$, where $\mathcal{P}_{ij}(\{u_i\})$ are differential polynomials of $u_i$. For example, comparing of the coefficients in front of $\partial^{-1}$ in both sides yields the equations

$$\partial_t u_1 = \partial_x \text{res } L^j$$

(3.3)

which are similar in this form to the equations of the KdV hierarchy. However, these equations are not closed because the right hand sides contain also the functions $u_2, u_3, \ldots$. A closed system includes also equations for $\partial_t u_2, \partial_t u_3$ and so on, which are obtained by comparing the coefficients in front of higher degrees of the operator $\partial^{-1}$.

The Lax equation at $j = 1$ tells us that $\partial_t L = [\partial, L]$, or $\partial_t u_1 = \partial_x u_1$, which allows one to identify $t_1$ with $x$. In other words, the evolution in the time $t_1$ is simply a shift of the argument $x$: $u_i(x) \rightarrow u_i(x + t_1)$. 

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Remark. The KdV hierarchy is obtained after imposing the condition that $L^2$ is a purely differential operator, i.e. does not contain negative powers of $\partial$: $(L^2)_- = 0$ or $(L^2)_+ = L^2$ (in the section devoted to the KdV equation it was this differential operator $L^2$ which was denoted by $L$ and was called the Lax operator). This condition makes the functions $u_i$ dependent; there is only one independent function among them: $u = 2u_1$, and all other $u_i$ with $i \geq 2$ are expressed as differential polynomials of $u$. Since $(L^2_m)_+ = L^2_m$ for all positive integer $m$, $(L^2_m)_+$ commutes with $L$ and all flows with even orders are trivial in this case.

3.2 Zero curvature representation

There is another (equivalent) representation of the KP hierarchy in which the Lax operator does not participate explicitly. In this approach the main objects are differential operators $(L^j)_{\pm}$. For brevity we denote $A_j = (L^j)_{\pm}$. For example, $A_1 = \partial$, $A_2 = \partial^2 + 2u_1$.

Exercise. Find $A_3 = (L^3)_{\pm}$.

Proposition. The Lax equations (3.2) imply the equations

$$\partial_{t_m} A_n - \partial_{t_n} A_m - [A_m, A_n] = 0$$

for all $m, n \geq 1$.

For the proof we first note that the Lax equations imply $\partial_{t_m} L^n = [A_m, L^n]$ for all $n$, then

$$\partial_{t_m} (L^n)_+ - \partial_{t_n} (L^m)_+ - [A_m, A_n]$$

$$= \left(\partial_{t_m} L^n - \partial_{t_n} L^m - [A_m, A_n]\right)_+$$

$$= \left([A_m, L^n] - [A_n, L^m] - [A_m, A_n]\right)_+$$

$$= \left([A_m, L^n - A_n] - [A_n, L^m]\right)_+$$

$$= \left([L^m, (L^n)_-] + [L^m, (L^n)_+]\right)_+ = \left([L^m, L^n]\right)_+ = 0.$$  

The converse is also true: from the full set of equations (3.4) the Lax equations follow. Clearly, equation (3.4) is equivalent to the commutation relation $[\partial_{t_m} - A_m, \partial_{t_n} - A_n] = 0$.

Exercise. Check that equations (3.4) can be rewritten in the form

$$\partial_{t_m} (L^n)_- - \partial_{t_n} (L^m)_- + [(L^m)_-, (L^n)_-] = 0.$$  

By analogy with (2.51), the representation of the KP hierarchy in the form (3.4) or (3.6) is called the zero curvature (or Zakharov-Shabat) representation. Each of the zero curvature equations generates a closed system of a finite number of equations for a finite number of unknown functions which, however, does not have the evolution form. It contains derivatives with respect to three times: $x = t_1, t_m, t_n$. (Unlike the matrix
zero curvature equation (2.50), equations (3.4) at \( n = 1 \) are identities, so in non-trivial equations three times rather than two will participate.) If \( n > m \), then we have a system of \( n - 1 \) equations for unknown functions \( u_1, u_2, \ldots, u_{n-1} \). These systems are called equations of the KP hierarchy.

The simplest non-trivial example corresponds to the choice \( m = 2, n = 3 \). Denote \( t_1 = x, t_2 = y, t_3 = t, u = 2u_1, w = u_2 \), then from (3.4) we obtain the system

\[
\begin{cases}
4w_x = 3u_y + 3u_{xx}, \\
u_t - w_y = \frac{3}{2} uu_x + u_{xxx} - w_{xx}.
\end{cases}
\]

Excluding \( w \), one obtains the closed equation for \( u \):

\[3u_{yy} = \left(4u_t - 6uu_x - u_{xxx}\right)_x, \quad (3.7)\]

which is called the KP equation. It was suggested in 1970. In general case the system obtained from (3.4) cannot be reduced to a single equation.

**Remark.** In physical literature equation (3.7) is called KP2; the equation KP1 is obtained by the change \( y \to iy \). Properties of solutions and physical applications in these two cases are very different. We will not discuss them since we are mainly interested in algebraic structures.

### 3.3 Symmetries and conservation laws

Similarly to the KdV case, all equations of the KP hierarchy are symmetries for each other. In other words, all vector fields \( \partial_{t_j} \) commute and the quantities \( u_i \) are functions not only on \( x \) but also on all the times \( t_j \) simultaneously. This can be easily verified:

\[
\begin{align*}
\partial_{t_m}(\partial_{t_n} L) - \partial_{t_n}(\partial_{t_m} L) &= \partial_{t_m}[A_n, L] - \partial_{t_n}[A_m, L] \\
&= [\partial_{t_m} A_n - \partial_{t_n} A_m, L] + [A_n, \partial_{t_m} L] - [A_m, \partial_{t_n} L] \\
&= [[A_m, A_n], L] + [A_n, [A_m, L]] - [A_m, [A_n, L]] = 0.
\end{align*}
\]

When passing to the last line, we have used the zero curvature equation. The expression in the last line vanishes identically after opening brackets in the commutators.

One of the consequences of commutativity of the vector fields \( \partial_{t_j} \) are the relations

\[\partial_{t_m} \text{res} L^n = \partial_{t_n} \text{res} L^m \quad (3.8)\]

which literally generalize the corresponding formulas for KdV. One can prove them independently by a calculation similar to (3.5) under the sign res rather than \((\ldots)_+\):

\[
\partial_{t_m} \text{res} L^n - \partial_{t_n} \text{res} L^m = \text{res} \left( \partial_{t_m} L^n - \partial_{t_n} L^m - [A_m, A_n] \right) \quad \text{and so on.}
\]

Besides, these relations immediately follow from (3.6).
The integrals of motion are constructed in the same way as in the KdV case. Now all
the integrals are non-trivial (not only those with odd numbers).

**Proposition.** The quantities

\[ I_j = \int \text{res} L^j \, dx \quad (3.9) \]

at \( j \geq 1 \) are integrals of motion for all equations of the KP hierarchy: \( \partial_{t_n} I_j = 0 \).

The proof is again based on the Lax equations:

\[ \partial_{t_n} I_j = \int \text{res} (\partial_{t_n} L^j) \, dx = \int \text{res} ([A_n, L^j]) \, dx. \]

Since \( \text{res} [P, Q] \) is a full derivative for any \( P, Q \), we conclude that in the rapidly decreasing
and periodic cases \( \partial_{t_n} I_j = 0 \).

Note that equation (3.8) implies that densities of integrals of motion are time de-
rivatives of one and the same function \( v \): \( \text{res} L^j = \partial_j v \). The fact that the residue of
commutator is a full derivative in \( x = t_1 \) implies that \( v \) can be represented as derivative
of some function \( f \):

\[ \partial_{t_m} \text{res} L^n = \text{res} [(L^n)_+, L^n] = \partial_{t_m} \partial_{t_n} v = \partial_{t_1} \partial_{t_m} \partial_{t_n} f, \]

and also

\[ \text{res} L^n = \partial_{t_1} \partial_{t_n} f. \]

The function \( f \) will play an important role. We will see that it is logarithm of the
tau-function.

### 3.4 Dressing operator

The Lax and zero curvature representations admit a nice reformulation in terms of so-
called dressing operator \( K \). This operator could be introduced also for KdV but the
construction becomes really useful for the KP hierarchy which is free of any constraints
like \((L^N)_- = 0\).

The dressing operator is the pseudo-differential operator of the form

\[ K = 1 + \xi_1 \partial^{-1} + \xi_2 \partial^{-2} + \ldots \]

such that

\[ L = K \partial K^{-1}. \quad (3.10) \]

This equality is said to be the “dressing” of the operator \( \partial \) by the operator \( K \). The
operator \( L \) is the result of the dressing. Obviously, \( L^m = K \partial^m K^{-1} \). Note that \( K \) is
defined up to multiplication from the right to an operator of the form \( 1 + \sum_{k \geq 1} a_k \partial^{-k} \)
with constant coefficients \( a_k \).

**Proposition.** Assume that the dressing operator satisfies the equations

\[ \partial_{t_n} K = -(K \partial^n K^{-1})_- K, \quad (3.11) \]

then \( L = K \partial K^{-1} \) satisfies the Lax equations of the KP hierarchy.
Exercise. Prove this proposition by a direct calculation and check that (3.11) can be written in the form

\[ \partial_t n \mathcal{K} = (L^n)_{+} \mathcal{K} - K \partial^n. \]

By a similar calculation one can prove that the vector fields \( \partial_t j \) defined by (3.11) commute not only when they act to coefficients of the operator \( L \) but also when they act to coefficients of the operator \( K \) (which, generally speaking, are not differential polynomials of \( u_i \)):

\[ \partial_t m (\partial_t n \mathcal{K}) = \partial_t n (\partial_t m \mathcal{K}). \]

Since

\[ K(\partial_t m - \partial^m) K^{-1} = \partial_t m - (\partial_t m \mathcal{K}) K^{-1} - L^m = \partial_t m + (K \partial^m K^{-1})_+ - L^m = \partial_t m - A_m, \]

the Lax and zero curvature equations can be represented in the form

\[ K[\partial_t m - \partial^m, \partial] K^{-1} = 0, \]

\[ K[\partial_t m - \partial^m, \partial_t n - \partial^n] K^{-1} = 0. \]

One can say that they are obtained by dressing the obvious relations \([\partial_t m - \partial^m, \partial] = 0\) and \([\partial_t m - \partial^m, \partial_t n - \partial^n] = 0\).

### 3.5 Linear problems and Baker-Akhiezer function

The zero curvature equation (3.4) is the compatibility condition of the linear problems

\[ \begin{cases} 
\partial_t m \psi = A_m \psi, \\
\partial_t n \psi = A_n \psi.
\end{cases} \tag{3.12} \]

As before, compatibility means existence of a large set of common solutions. The solution can be again found as a series in a spectral parameter \( z \), which now does not enter the linear problems explicitly. For brevity we denote

\[ \xi(t, z) = xz + t_2 z^2 + t_3 z^3 + \ldots \tag{3.13} \]

Let us find the solution of (3.12) in the form

\[ \psi = \left(1 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \ldots \right) e^{\xi(t, z)}, \tag{3.14} \]

where the coefficients \( \xi_i \) depend only on \( x \) (and on \( t_j \)). One can add to the system the equation \( L \psi = z \psi \), which contains the spectral parameter explicitly.

It is easy to see that common solutions to the system (3.12) are constructed by application of the dressing operator \( K \) to the function \( e^{\xi(t, z)} \):

\[ \psi = K e^{\xi(t, z)} = \left(1 + \xi_1 \partial^{-1} + \xi_2 \partial^{-2} + \ldots \right) e^{\xi(t, z)} \tag{3.15} \]

(\( \partial^{-1} \) acts to the exponential function according to the rule \( \partial^{-1} e^{xz} = z^{-1} e^{xz} \)). Indeed,

\[
\partial_t m \psi = z^m K e^{\xi(t, z)} + \partial_t m K e^{\xi(t, z)} = (K \partial^m - (K \partial^m K^{-1})_+ K) e^{\xi(t, z)}
\]

\[
= (K \partial^m K^{-1} - (K \partial^m K^{-1})_-) K e^{\xi(t, z)} = (L^m - (L^m)_-) \psi = A_m \psi.
\]
Together with the dressing operator $K$ it is useful to consider the formally conjugate operator $K^\dagger = 1 - \partial^{-1}\xi_1 + \partial^{-2}\xi_2 - \ldots$ and construct the adjoint Baker-Akhiezer function

$$\psi^* = (K^\dagger)^{-1} e^{-\xi(t,z)}.$$  

(3.16)

It has the form

$$\psi^* = \left(1 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \ldots \right) e^{-\xi(t,z)}$$

(here the star does not mean the complex conjugation!) and satisfies the system of compatible linear problems

$$\left\{ \begin{array}{l} \partial_t \psi^* = -A^\dagger_m \psi^*, \\
L^\dagger \psi^* = z \psi^*. \end{array} \right.$$  

(3.17)

**Problem.** Prove that in the KdV case (\((L^2)_- = 0\)) \(\psi^*(z) = 2z\psi(-z)/W(z)\), where \(W\) is the Wronskian of the functions \(\psi(z)\) and \(\psi(-z)\) (see (2.32)).

Let us prove a formula for \((L^m)_-\) through \(\psi\) and \(\psi^*:\)

$$\quad (L^m)_- = \text{res}_z \left( z^m \psi(z) \partial^{-1} \psi^*(z) \right).$$  

(3.18)

Here \(\partial^{-1} \psi^*(z)\) in the right hand side is understood as composition of operators (and not as the result of action of \(\partial^{-1}\) to \(\psi^*(z)\)). Since we will deal with both ordinary and operator residues, let us denote them by \(\text{res}_z\) and \(\text{res}_\theta\) respectively (\(\text{res}_z\) is the coefficient in front of \(z^{-1}\)). It is obvious that

$$\quad (L^m)_- = \sum_{l \geq 0} \text{res}_\theta \left( z^m (\partial^l)^{\dagger} \right) \partial^{-l-1} = \sum_{l \geq 0} \text{res}_\theta \left( \partial^m K^{-1} \partial^l \right) \partial^{-l-1}.$$  

Now, in order to transform the operator residue into the ordinary one, we use the previously proved lemma which states that for any two pseudo-differential operators \(P, Q\) the relation \(\text{res}_z [(Pe^{xz}) (Qe^{-xz})] = \text{res}_\theta (PQ^\dagger)\) holds. We write, continuing the equality:

$$\quad (L^m)_- = \sum_{l \geq 0} \text{res}_z \left[ \left( K \partial^m e^{\xi(t,z)} \right) \left( (-\partial)^l (K^\dagger)^{-1} e^{-\xi(t,z)} \right) \right] \partial^{-l-1}$$

(Here the operator \((-\partial)^l\) acts to what stands from the right of it.) Rewriting the right hand side in terms of the Baker-Akhiezer function and its adjoint, we will have:

$$\quad (L^m)_- = \sum_{l \geq 0} \text{res}_z \left[ (L^m \psi(z)) (\partial^l \psi^*(z)) \right] (-1)^l \partial^{-l-1} = \sum_{l \geq 0} \text{res}_z \left[ z^m \psi(z) \partial^l \psi^*(z) \right] (-1)^l \partial^{-l-1}.$$  

Finally, using the commutation relation \(\partial^{-1} f = \sum_{l \geq 0} (-1)^l f^{(l)} \partial^{-l-1}\), we arrive at (3.18).

The reduction to KdV gives equation (2.36).

The key for construction of solutions to the KP equation is the following technical lemma (we use the notation \(t_1 = x, t_2 = y, t_3 = t, u = 2u_1\)).

**Lemma (simple but important).** For the function \(\psi\) of the form

$$\quad \psi = e^{xz + z^2y + z^3t} \left( 1 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \ldots \right)$$  

(3.19)
the formal equalities
\[
(-\partial_y + \partial^2 + u)\psi = O(z^{-1})e^{zx+z^2y+z^3t},
\]
\[
(-\partial_t + \partial^3 + \frac{3}{2}u\partial + w)\psi = O(z^{-1})e^{zx+z^2y+z^3t}
\]
hold, where the functions \(u, w\) are found from the condition that coefficients at non-negative powers of \(z\) vanish:
\[
u = -2\xi_{1x},
\]
\[
w = 3\xi_{1x} - 3\xi_{1xx} - 3\xi_{2x}.
\]
The meaning of this lemma is similar to the corresponding statement about the \(\psi\)-function of the KdV equation. The proof is a direct calculation.

This lemma can be generalized in two directions: a) one may allow the function (in fact the formal series) \(\psi\) to have, on the background of the essential singularity, a pole at \(\infty\) of order \(n\), in the lemma can be extended to the whole hierarchy.

**Lemma (generalization of the previous one).** For the function \(\psi\) of the form
\[
\psi = z^n Ke^{\xi(t,z)} = \left(1 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \ldots\right) z^n e^{\xi(t,z)}
\]
built with the help of the dressing operator \(K = 1 + \xi_{1}\partial^{-1} + \xi_{2}\partial^{-2} + \ldots\), the formal equalities
\[
(\partial_{t_k} - A_k)\psi = O(z^{n-1})e^{\xi(t,z)}
\]
hold, where the coefficient functions of the differential operators \(A_k\) are differential polynomials of \(\xi_1, \xi_2, \ldots\). Their explicit form is determined from the equalities \(A_k = (K\partial^{k}K^{-1})_+\).

The proof which uses the technique of dressing operators is very simple. We have:
\[
(\partial_{t_m} - A_m)\psi = \left((\partial_{t_m}K)K^{-1} + z^m - A_m\right)\psi = \left((\partial_{t_m}K)K^{-1} + L^m - (L^m)_+\right)\psi.
\]
It is clear that the operator \((\partial_{t_m}K)K^{-1}+L^m-(L^m)_+ = (\partial_{t_m}K)K^{-1}+(L^m)_-\) has the form \(O(\partial^{-1})\), so acting to the exponential function it gives the factor \(O(z^{-1})\).

### 3.6 Solutions of the KP hierarchy
#### 3.6.1 Soliton solutions

The construction of soliton solutions is similar to the corresponding construction for KdV. Consider the linear space of functions \(\psi = \psi(z)\) meromorphic everywhere except at infinity and such that

a) The function \(\psi e^{-zx-z^2y-z^3t}\) is regular at \(z = \infty\);
b) The function \( \psi \) has not more than \( N \) poles (counted with multiplicities) at some marked points of the complex plane and holomorphic everywhere else except \( \infty \).

c) For \( N \) pairs of distinct points \( p_j, q_j \in \mathbb{C} \) the relations \( \psi(p_j) = \alpha_j \psi(q_j), \ j = 1, 2, \ldots, N \) hold.

Similarly to the KdV case, this space is one-dimensional and the operators in the left hand sides of (3.20) preserve it. For simplicity we consider the case when all poles are concentrated at \( z = 0 \). The function \( \psi \) can be found in the form

\[
\psi = e^{\xi(t,z)} \left( 1 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \ldots + \frac{\xi_N}{z^N} \right),
\]

then \( u = -2\xi_{1,x} \) is a solution to the KP equation.

Let us find this solution explicitly. The conditions \( \psi(p_j) = \alpha_j \psi(q_j) \) are equivalent to the following system of linear equations for \( \xi_j \):

\[
\sum_{j=1}^{N} M_{ij} \xi_j = -M_{i0},
\]

where \( M_{ij} = p_i^{-1} e^{\xi(t,p_i)} - \alpha_i q_i^{-1} e^{\xi(t,q_i)} \). The Kramer’s rule yields

\[
\xi_1 = -\frac{\det M_{ij}^{(0)}}{\det M_{ij}}, \quad i, j = 1, 2, \ldots, N,
\]

where the matrix \( M_{ij}^{(0)} \) differs from \( M_{ij} \) by the change of the first column \( M_{i1} \) to \( M_{i0} \).

Since \( \partial_x M_{ij} = M_{i,j-1} \), we have \( \det M_{ij}^{(0)} = \partial_x \det M_{ij} \), and \( \xi_1 = -\partial_x \log \det M_{ij} \), so that \( u = 2\partial^2_x \log \det M_{ij} \). We have obtained a family of solutions which are expressed by the same formula (2.43) with the tau-function

\[
\tau = \frac{\det M_{ij}}{\det M_{ij}} \left( p_i^{-1} e^{\xi(t,p_i)} - \alpha_i q_i^{-1} e^{\xi(t,q_i)} \right),
\]

(3.24)

As before, we have got solutions of the whole hierarchy. All \( \alpha_i \)'s can be again “hidden” in initial values of the times \( t_j \), so the real parameters are only \( p_i \) and \( q_i \) (2\( N \) parameters).

At \( N = 1 \) we obtain one-soliton solution: \( \tau = p^{-1} e^{\xi(t,p)} - \alpha q^{-1} e^{\xi(t,q)} \),

\[
u = -\frac{2pq(p-q)^2 e^{\xi(t,p)+\xi(t,q)}}{(qe^{\xi(t,p)} - \alpha p e^{\xi(t,q)})^2} = -\frac{(p-q)^2}{2 \sinh^2 \left( \frac{1}{2} (\xi(t,p) - \xi(t,q) + \varphi) \right)},
\]

where \( \varphi = \log \left( \frac{\alpha}{\alpha p} \right) \). Putting \( \alpha = -q/p \), \( t_1 = t_5 = \ldots = 0 \), we write this solution in the form

\[
u(x,y,t) = \frac{(p-q)^2}{2 \cosh^2 \left( \frac{1}{2} (p-q)x + \frac{1}{2} (p^2-q^2)y + \frac{1}{2} (p^3-q^3)t \right)}.
\]

(3.25)

In the case \( q = -p \) the dependence on \( y = t_2 \) (and on all even times) disappears, and this formula reproduces the one-soliton solution of the KdV equation (2.42). Note that \( u(x,y,t) \) defined by (3.25) exponentially decreases in the plane \((x,y)\) in all directions except the direction along the line \( x + (p+q)y = 0 \), and so it can not be regarded as a
2D soliton in the physical sense. For this reason the solutions found in this section are sometimes called soliton-like.

We point out the equivalent determinant representation of the multi-soliton tau-function:

$$
\tau = \det_{1 \leq i, j \leq N} \left( \delta_{ij} + \frac{\beta_i(p_i - q_i)}{p_i - q_j} e^{\xi(t,p_i) - \xi(t,q_i)} \right).
$$

(3.26)

Expanding the determinant, using the known expression for the Cauchy determinant

$$
\psi
$$

3.6.2 Soliton-like solutions of general form and solutions depending on functional parameters

The construction described above can be extended to a much larger class of solutions. For this it is enough to notice that all arguments go through if instead of $N$ linear conditions for coefficients of the $\psi$-function one imposes more general conditions

$$
\sum_l A_{ll} \psi(p_l^{(t)}) = 0
$$

with some (generally speaking, rectangular) matrix $A$.

A more precise formulation is as follows.

a) Fix some points $p_m^{(j)} \in \mathbb{C}$ which are assumed to be distinct. It is convenient to regard them as divided in $N_c$ “clusters”, so that $j = 1, 2, \ldots, N_c$ numbers the clusters, and $m$ numbers points inside the clusters. Let the number of pints in $j$th cluster be $M_j > 1$. Formally the case $M_j = 1$ is also possible but not very interesting. To each cluster (with the number $j$) assign a rectangular matrix $A_m^{(j,\alpha)}$, where $m = 1, \ldots, M_j$ and $\alpha = 1, \ldots, \mu_j$ with some $1 \leq \mu_j \leq M_j$. Put $N = \sum_{j=1}^{N_c} \mu_j$. This $N$ has the meaning of the number of solitons in the solution.

b) Fix $N$ distinct points $z_1, z_2, \ldots, z_N \in \mathbb{C}$ not coinciding with $p_m^{(j)}$ and consider the Baker-Akhiezer function of the form

$$
\psi(t, z) = \left( 1 + \sum_{k=1}^{N} \frac{w_k(t)}{z - z_k} \right) e^{\xi(t,z)},
$$

(3.27)
with $N$ linear conditions
\[
\sum_{m=1}^{M_i} A^{(j)}_{\alpha,m} \psi(t, p^{(j)}_m) = 0
\]
(where $j = 1, \ldots, N_c$, $\alpha = 1, \ldots, \mu_j$).

It is easy to see that the lemma can be applied to this more general situation, and we obtain a large class of solutions which are sometimes also called soliton-like. If the number of points in the clusters is greater than 2, these solutions do not have analogues in the KdV theory.

One may generalize this even further and impose the conditions of the form
\[
\int \rho_i(p) \psi(p) d\mu(p) = 0
\]
with some measure $d\mu$ and functions $\rho_i(p)$ in the complex plane. More precisely, consider the linear space of functions $\psi = \psi(z)$ with the same analytic properties a), b) as above but instead of condition c) impose
\[ c' \quad \text{For } N \text{ different fixed functions } \rho_i(z) \text{ and some measure } d\mu(z) \text{ in the complex plane the relations}
\[
\int \rho_i(z) \psi(z) d\mu(z) = 0, \quad i = 1, 2, \ldots, N
\]
hold.

It is easy to see that this space is one-dimensional in the general case and the operators in the right hand sides of (3.20) preserve it. The procedure again reduces to solving a linear system with the result
\[
\tau(t) = \det_{1 \leq i, j \leq N} \int z^{-j} \rho_i(z) e^{\xi(t,z)} d\mu(z).
\] (3.28)

It is assumed that the measure is such that the integral converges. The functions $\rho_i(z)$ (and the measure $d\mu$) are functional parameters of the solution. Note that such solutions exist only for the KP hierarchy. In the KdV case the general linear condition of the type $c'$ is broken under the action of the operators in the left hand sides of (2.41).

It is clear that the soliton-like solutions correspond to the case when $\rho_i(z)$ is a linear combination of $\delta$-functions concentrated in the points $p^{(j)}_m$.

### 3.6.3 Rational solutions

Tending the points in each cluster to each other and choosing $A^{(j)}_{\alpha,m}$ in a special way, in the limit one can obtain solutions whose tau-function is a polynomial of $x$ and all $t_j$, and $u$ is thus a rational function. Such solutions are called rational. They can be constructed in the same way as soliton-like solutions if one considers the linear space of functions $\psi = \psi(z)$ with the same analytic properties a) and b) and
\[ c'' \quad \text{In } N \text{ distinct points } p_i \text{ of the complex plane the relations}
\[
\sum_{m=0}^{M_i} a_{im} \partial^m_z \psi(z) \bigg|_{z=p_i} = 0, \quad i = 1, 2, \ldots, N
\]
hold for all $t_j$.  

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Here $a_{im}$ are some constants (parameters of the solution together with the points $p_i$). In general some points $p_i$ may coincide which means that more than one condition is imposed in one point.

A physically meaningful example of such solution can be obtained by putting $q_1 = p_1 + \epsilon, q_2 = p_2 + \epsilon, \beta_1 = \beta_2 = -1$ in the two-soliton solution and tending $\epsilon \to 0$. Setting $p_1 = p, p_2 = -\overline{p}$ and taking $y$ to be purely imaginary (i.e. redefining $y \to iy$ thus passing from KP2 to KP1), we obtain in the limit

$$\tau = |x + 2ipy + 3p^2t|^2 + \frac{1}{(\overline{p} + \overline{p})^2}. $$

This tau-function describes the motion of a bell-shaped excitation in the plane $(x, y)$. This is a two-dimensional soliton in the physical sense. The equation KP2 does not have solutions of this type.

**Problem.** Find the linear conditions for the $\psi$-function of the form $c''$) for this solution.

Let us give an explicit description of rational solutions which are obtained after imposing the condition $c''$) to the Baker-Akhiezer function

$$\psi(t, z) = z^N e^{\xi(t, z)} \left( 1 + \frac{\xi_1(t)}{z} + \ldots + \frac{\xi_N(t)}{z^N} \right) \quad (3.29)$$

(in this case it is convenient to place all poles at $\infty$). The conditions $c''$) are equivalent to the system of linear equations

$$A_i(N, t) + \sum_{k=1}^{N} A_i(N - k, t) \xi_k(t) = 0, \quad (3.30)$$

where

$$A_i(n, t) = \sum_{m=0}^{M_i} a_{im} \partial_z^n \left( z^n e^{\xi(t, z)} \right) \bigg|_{z=p_i} \quad (3.31)$$

These functions are polynomials of $t_j$ multiplied by exponential factors $e^{\xi(t, p_i)}$. Solving the system by the Kramer’s rule, we write the answer for $\psi$:

$$\psi(t, z) = e^{\xi(t, z)} (\tau(t))^{-1} \begin{vmatrix} z^N & z^{N-1} & \ldots & 1 \\ A_1(N, t) & A_1(N-1, t) & \ldots & A_1(0, t) \\ \vdots & \vdots & \ddots & \vdots \\ A_N(N, t) & A_N(N-1, t) & \ldots & A_N(0, t) \end{vmatrix}, \quad (3.32)$$

where

$$\tau(t) = \begin{vmatrix} A_1(N - 1, t) & \ldots & A_1(0, t) \\ \vdots & \ddots & \vdots \\ A_N(N - 1, t) & \ldots & A_N(0, t) \end{vmatrix} = \det_{1 \leq i, j \leq N} A_i(N - j, t). \quad (3.33)$$

The result of the next problem implies that $\tau$ is the tau-function for this class of solutions and $u = -2\xi_1' = 2\partial^2 \log \tau$.

**Problem.** Check that $\xi_1(t) = -\partial_t \log \tau(t)$ (hint: use the identity $\partial_t A_i(n, t) = A_i(n + 1, t))$. 

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A further degeneration of rational solutions can be obtained by merging the points \( p_i \) and tending them to 0. The solutions remain rational and admit an explicit description in terms of Schur polynomials.

Let us define the polynomials \( h_j = h_j(t) \) of the times \( t_i \) using the expansion
\[
e^{\xi(t,z)} = \sum_{l=0}^{\infty} h_l(t) z^l. \tag{3.34}\]
These are the simplest Schur polynomials. For example, \( h_0 = 1, h_1 = t_1, h_2 = t_2 + \frac{1}{2} t_1^2, \)
\( h_3 = t_3 + \frac{1}{2} t_1 t_2 + \frac{1}{6} t_1^3 \) and so on. Here is the general formula:
\[
h_k(t) = \sum_{k_1 + 2k_2 + \ldots + k_l = k} \sum_{k_1, \ldots, k_l \geq 1} \prod_{i=1}^{l} t_i^{k_i} \prod_{i=1}^{l} \frac{k_i!}{k_1! \ldots k_l!}. \tag{3.35}\]
It is convenient to put \( h_k(t) = 0 \) at \( k < 0. \)

**Problem.** Prove the identities
\[
\partial_t h_k(t) = h_{k-1}(t), \tag{3.36}
\]
\[
\sum_{j=1}^{n} j t_j h_{n-j}(t) = nh_n(t), \quad n \geq 1. \tag{3.37}
\]
It turns out that any polynomial \( h_j(t) \) is the tau-function of the KP hierarchy, i.e. \( u(t) = 2 \partial_t^2 \log h_j(t) \) is a solution. The general Schur polynomials can be defined using Young diagrams \( \lambda \) of arbitrary form. A Young diagram is a set of positive integer numbers \( \lambda_1, \lambda_2, \ldots, \lambda_n \) such that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) (\( \lambda_i \) are lengths of rows of the diagram \( \lambda \)). The Schur polynomial corresponding to the Young diagram \( \lambda \) is
\[
s_{\lambda}(t) = \det_{1 \leq i,j \leq n} h_{\lambda_i-i+j}(t). \tag{3.38}\]
The polynomial \( h_j \) corresponds to the diagram consisting of just one row of length \( j \).

All Schur polynomials \( s_{\lambda}(t) \) are tau-functions of the KP hierarchy. They are maximally degenerate solutions. Their characterization with the help of the Baker-Akhiezer function \( \psi \) of the form \([3.29]\) is as follows. Fix a sequence of non-negative integer numbers \( n_i \) such that \( n_1 > n_2 > \ldots > n_{N-1} \geq 0 \) and impose \( N \) conditions
\[
\partial_z^{n_i} \psi(t, z) \big|_{z=0} = 0.
\]
They are equivalent to the following system of linear equations for the coefficients \( \xi_j \):
\[
\sum_{j=1}^{N} h_{n_i+j-N}(t) \xi_j(t) = -h_{n_i-N}(t), \quad i = 1, \ldots, N. \tag{3.39}\]
Instead of \( n_i \) introduce the sequence \( \lambda_i \) according to the formula \( \lambda_i = n_i+i-N \). Obviously, \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0 \), so that they define a Young diagram. In terms of the numbers \( \lambda_i \) the system is rewritten as
\[
\sum_{j=1}^{N} h_{\lambda_i-i+j}(t) \xi_j(t) = -h_{\lambda_i-i}(t), \quad i = 1, \ldots, N.
\]
The Kramer’s rule yields

\[
\xi_1(t) = - \frac{\begin{vmatrix}
  h_{\lambda_1-1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1-1+N} \\
  h_{\lambda_2-2} & h_{\lambda_2} & \cdots & h_{\lambda_2-2+N} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{\lambda_N-N-1} & h_{\lambda_N-N+2} & \cdots & h_{\lambda_N} \\
\end{vmatrix}}{\begin{vmatrix}
  h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1-1+N} \\
  h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2-2+N} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{\lambda_N-N+1} & h_{\lambda_N-N+2} & \cdots & h_{\lambda_N} \\
\end{vmatrix}} = -\partial_x \log \det_{i,j=1,\ldots,N} h_{\lambda_i-i+j}(t). \tag{3.40}
\]

**Problem.** Give a detailed derivation of formulas (3.39) and (3.40).

### 3.6.4 Dynamics of poles of rational solutions to the KP equation and the Calogero-Moser system of particles

As we have seen, the tau-function for rational solutions of the KP equation is a polynomial in \(x = t_1\):

\[
\tau = C \prod_{j=1}^N (x - x_j(t))
\]

Its roots \(x_j\) (assumed to be distinct) are poles of the function

\[
u(t) = 2\partial_x^2 \log \tau(t) = -\sum_{j=1}^N \frac{2}{(x - x_j(t))^2}. \tag{3.41}\]

They are functions of \(t_2, t_3, \ldots\). (The construction of rational solutions given above implies that these solutions are rational functions of all the times.) The problem about dynamics of poles of rational solutions to the KP equation was solved by Krichever in 1978. It turns out that the dynamical equations for \(x_j\) coincide with equations of motion of the integrable \(N\)-body system on the line with pairwise interaction. The interaction potential is proportional to the inverse square of the distance between particles. This is the (rational) Calogero-Moser system.

The method suggested by Krichever consists in the substitution of the pole expression (3.41) for \(u\) not into the nonlinear equation but into the auxiliary linear problem for the Baker-Akhiezer function \(\psi\). This allows one to separate the variables from the very beginning and to obtain the Lax representation for the Calogero-Moser system. For this, one should derive or guess the corresponding "pole ansatz" for \(\psi\).

Let us describe the main points of the derivation of equations of motion for \(x_j\). For simplicity we will follow only the dynamics with respect to the time \(t_2\), which in this section will be denoted by \(t\). All higher times will be put equal to 0. The linear equation for \(\psi\) has the form

\[
\partial_t \psi = \partial_x^2 \psi + u \psi, \tag{3.42}
\]

where \(u\) is the sum of pole terms (3.41). The function \(\psi\) will be found in the form

\[
\psi = e^{xz+tz^2} \left( c_0(z) + \sum_{i=1}^N \frac{c_i(z,t)}{x - x_i(t)} \right) \tag{3.43}
\]
with some $x$-independent coefficients $c_i$. The fact that it should have simple poles at the points $x_i$ follows from (3.32) (the denominator of this expression is the tau-function). Substituting $\psi$ in this form into (3.42), we obtain

$$e^{-xz+tz^2}(\partial_t - \partial_x^2) \left[ e^{xz+tz^2} \left( c_0 + \sum_{i=1}^N \frac{c_i}{x-x_i} \right) + 2 \left( \sum_{i=1}^N \frac{1}{(x-x_i)^2} \right) \left( c_0 + \sum_{i=1}^N \frac{c_i}{x-x_i} \right) \right] = 0.$$ 

The left hand side is a rational function of $x$ with first and second order poles at $x = x_i$ (possible poles of third order cancel identically), which is equal to 0 at infinity. Therefore, it is enough to cancel all the poles. Equating the coefficients at each pole to zero, we obtain the following system of $2N$ linear equations for the coefficients $c_1, \ldots, c_N$:

$$\begin{aligned}
(\dot{x}_i + 2z)c_i + 2 \sum_{k \neq i} \frac{c_k}{x_i - x_k} &= -2c_0 \quad \text{(cancellation of second order poles)}, \\
\dot{c}_i + 2c_i \sum_{k \neq i} \frac{1}{(x_i - x_k)^2} - 2 \sum_{k \neq i} \frac{c_k}{(x_i - x_k)^2} &= 0 \quad \text{(cancellation of first order poles)},
\end{aligned}$$

where $i = 1, \ldots, N$. The coefficient $c_0$ can be put equal to a constant (for example, 1), since it affects only the common factor of the $\psi$-function. These equations can be compactly written in the matrix form:

$$\begin{aligned}
(\mathcal{L} - 2zI)\mathbf{c} &= c_0(z)\mathbf{1}, \\
\dot{\mathbf{c}} &= \mathcal{M}\mathbf{c},
\end{aligned}$$

(3.44)

where $I$ is the unity matrix, $\mathbf{c} = (c_1, c_2, \ldots, c_N)^T$, $\mathbf{1} = (1,1,\ldots,1)^T$ are $N$-component vectors and $N \times N$ matrices $\mathcal{L} = \mathcal{L}(t)$, $\mathcal{M} = \mathcal{M}(t)$ have the form

$$\mathcal{L}_{ik} = -2p_i\delta_{ik} - 2 \frac{1 - \delta_{ik}}{x_i - x_k}, \quad p_i := \frac{1}{2} \dot{x}_i,$$  

(3.45)

$$\mathcal{M}_{ik} = -\delta_{ik} \sum_{j \neq i} \frac{2}{(x_i - x_j)^2} + \frac{2(1 - \delta_{ik})}{(x_i - x_k)^2}.$$  

(3.46)

The system (3.44) is overdetermined. The compatibility condition is

$$\dot{\mathcal{L}} = [\mathcal{M}, \mathcal{L}].$$

(3.47)

A calculation shows that the non-diagonal elements of the matrices on the left and right hand sides equal identically while equality of diagonal elements yields the equations

$$\ddot{x}_i = -8 \sum_{j \neq i} \frac{1}{(x_i - x_j)^3}.$$  

(3.48)

These are equations of motion of the Calogero-Moser system. The matrices $\mathcal{L}$, $\mathcal{M}$ form the Lax pair; $\mathcal{L}$ is the Lax matrix for this system. As it is seen from (3.47), the time evolution preserves the spectrum of the Lax matrix. The coefficients $J_k$ of its characteristic polynomial

$$\det \left( 2zI - \mathcal{L}(t) \right) = \sum_{k=0}^n J_k z^{n-k}$$  

(3.49)
are integrals of motion.

Introduce the matrix \( X = \text{diag} \left( x_1, x_2, \ldots, x_N \right) \). It is easy to check that the matrices \( X, L \) satisfy the commutation relation

\[
[X, L] = I - 1 \otimes 1^T
\]

(3.50)

(here \( 1 \otimes 1^r \) is the \( N \times N \) matrix of rank 1 with all elements equal to 1).

The function \( \psi \) (3.43) is found from the solution of the linear equations (3.44) in the following form:

\[
\psi = c_0(z) e^{x(t - z)} \left( 1 - 1^T (x - X(t))^{-1} (z - L(t))^{-1} 1 \right).
\]

(3.51)

The equations of motion of the Calogero-Moser system can be written in the Hamiltonian form

\[
\left( \begin{array}{c}
\dot{x}_i \\
\dot{p}_i
\end{array} \right) = \left( \begin{array}{c}
\frac{\partial_p \mathcal{H}_2}{\partial_x} \\
-\frac{\partial_x \mathcal{H}_2}{\partial_p}
\end{array} \right)
\]

(3.52)

with the canonical variables \( p_i, x_i \) and the Hamiltonian

\[
\mathcal{H}_2 = \frac{1}{4} \text{tr} L^2 = \sum_i p_i^2 - \sum_{i<j} \frac{2}{(x_i - x_j)^2}.
\]

(3.53)

The connection of the pole dynamics with the Calogero-Moser system can be extended to the whole KP hierarchy (Shiota’s result). Namely, the higher times dynamics is given by the Hamiltonian equations

\[
\left( \begin{array}{c}
\frac{\partial_t x_i}{\partial t} \\
\frac{\partial_t p_i}{\partial t}
\end{array} \right) = \left( \begin{array}{c}
\frac{\partial_p \mathcal{H}_k}{\partial_x} \\
-\frac{\partial_x \mathcal{H}_k}{\partial_p}
\end{array} \right), \quad \mathcal{H}_k = 2^{-k} \text{tr} L^k,
\]

(3.54)

where \( \mathcal{H}_k \) are higher integrals of motion (Hamiltonians) of the Calogero-Moser system. It can be shown that they are in involution. This is in agreement with commutativity of the KP flows.

Note that in the case of KdV the correspondence with the Calogero-Moser system means that the particles in the system with the Hamiltonian \( \mathcal{H}_2 \) do not move, i.e. they are in equilibrium with respect to this dynamics while the \( t_3 \)-dynamics is generated by the Hamiltonian \( \mathcal{H}_3 \). The set of all such equilibrium positions is called locus. For rational solutions the locus is not empty only if \( N = n(n + 1)/2 \) with positive integer \( n \).

**Problem.** Solve the KdV equation \( 4u_t = 6uu_x + u_{xxx} \) with the initial condition \( u(x, 0) = -6/x^2 \).

Finally, we give an explicit determinant formula for the tau-function. Let \( X_0 = X(0) \) be the diagonal matrix \( X_0 = \text{diag}(x_1(0), x_2(0), \ldots, x_n(0)) \) and let \( L_0 \) be the Lax matrix (3.45) taken at \( t = 0 \). It can be shown that the tau-function is given by the formula

\[
\tau(t) = \det_{N \times N} \left( xI - X_0 + \sum_{k \geq 1} kt_k L_0^{k-1} \right).
\]

(3.55)
3.6.5 Trigonometric solutions

Now we come back to the non-degenerate soliton-like solutions. Let all \( p_i, q_i \) be real. Then the constructed solutions \( u(x) \) exponentially decrease as \( x \to \pm \infty \) along the real axis and oscillate along the imaginary axis. At \( N > 1 \) the solution in general does not have any definite period along the imaginary axis (because the numbers \( p_i - q_i \) and \( p_j - q_j \) are in general incommensurate). Nevertheless, among \( N \)-soliton solutions of the KP equation there is an \( N \)-parametric family of solutions such that they have a period \( 2\pi L \) along the imaginary axis. For this it is enough for the parameters \( p_i, q_i \) to be connected by the constraints \( q_i = p_i + 2\pi / L \) which make the number of free parameters equal to \( N \) (not counting the period itself). Equation (3.27) then implies that the tau-function as a function of \( x \) will be a polynomial of \( e^{2\pi x / L} \) of degree \( N \). Multiplying it by a inessential common factor, one can represent it in the form

\[
\tau = C \prod_{j=1}^{N} \sinh \frac{\pi(x - x_j(t))}{L},
\]

(3.56)

where the zeros \( x_j \) depend on all the times starting from \( t_2 \): \( x_j = x_j(t_2, t_3, \ldots) \). For \( u \) we get the pole expansion

\[
u = -2 \sum_{j=1}^{N} \frac{(\pi / L)^2}{\sinh^2 \pi(x - x_j(t))/L}.
\]

(3.57)

We call solutions of this form trigonometric.

The remarkable connection with the Calogero-Moser model discussed above is generalized to the trigonometric solutions (and even to elliptic solutions, see below). If the function of the form (3.57) satisfies the KP equation, then its poles \( x_j \) as functions of \( t_2 \) move according to the equations of motion of the system of \( N \) particles on the line with the Hamiltonian

\[
H_2 = \sum_{j=1}^{N} p_j^2 - 2 \sum_{i<j}^{N} \frac{(\pi / L)^2}{\sinh^2 \pi(x_i - x_j)/L}
\]

(3.58)

which is the trigonometric version of the Calogero-Moser system (the Sutherland model).

**Problem.** Solve the KdV equation \( 4u_t = 6uu_x + u_{xxx} \) with the initial condition

\[
u(x, 0) = \frac{6p^2}{\cosh^2 px}.
\]

3.6.6 Elliptic solutions

Elliptic (double periodic in the complex plane) solutions to the KP equation were studied by Krichever in 1980 who showed that their poles move according to the equations of motion of the Calogero-Moser system with the elliptic potential.

The double periodicity means the existence of two periods \( 2\omega, 2\omega' \in \mathbb{C} \) such that \( \text{Im}(\omega'/\omega) > 0 \): \( u(x + 2\omega) = u(x) \), \( u(x + 2\omega') = u(x) \). For such solutions the tau-function is an “elliptic quasi-polynomial” in the variable \( x \):

\[
\tau = e^{Q(x, t_2, t_3, \ldots)} \prod_{i=1}^{N} \sigma(x - x_i(t))
\]

(3.59)
where $Q(x, t_2, t_3, \ldots)$ is a quadratic form in the times $t_i$ and

$$
\sigma(x) = \sigma(x|\omega, \omega') = x \prod_{s \neq 0} \left(1 - \frac{x}{s}\right) e^{x \frac{s}{2} + \frac{x^2}{2}}, \quad s = 2\omega m + 2\omega' m' \quad \text{with integer } m, m',
$$

is the Weierstrass $\sigma$-function with the quasi-periods $2\omega$, $2\omega'$. It is connected with the Weierstrass $\zeta$- and $\varphi$-functions by the formulas

$$
\zeta(x) = \sigma'(x)/\sigma(x), \quad \varphi(x) = -\zeta'(x) = -\partial_x^2 \log \sigma(x).
$$

We set $Q = cx^2 + \ldots$ with some constant $c$. Correspondingly, the function $u = 2\partial_x^2 \log \tau$ is an elliptic function with double poles at the points $x_i$:

$$
u = -2 \sum_{i=1}^N \varphi(x - x_i) + 4c. \quad (3.60)
$$

The poles depend on the times $t_2, t_3$.

According to Krichever’s method, the basic tool for studying $t_2$-dynamics of poles is the auxiliary linear problem

$$
\partial_{t_2} \psi = \partial_x^2 \psi + u \psi. \quad (3.61)
$$

Since the coefficient function $u$ is double-periodic, one can find double-Bloch solutions $\psi(x)$, i.e., solutions such that $\psi(x + 2\omega) = b \psi(x)$, $\psi(x + 2\omega') = b' \psi(x)$ with some Bloch multipliers $b, b'$. The ansatz for the $\psi$-function is

$$
\psi = e^{xz + t_2 z^2} \sum_{i=1}^N c_i \Phi(x - x_i, \lambda), \quad (3.62)
$$

where the coefficients $c_i$ do not depend on $x$. Here the function

$$
\Phi(x, \lambda) = \frac{\sigma(x + \lambda)}{\sigma(\lambda) \sigma(x)} e^{-\zeta(\lambda)x} \quad (3.63)
$$

has a simple pole at $x = 0$ ($\zeta$ is the Weierstrass $\zeta$-function). The expansion of $\Phi$ as $x \to 0$ is

$$
\Phi(x, \lambda) = x^{-1} - \frac{1}{2} \varphi(\lambda)x - \frac{1}{6} \varphi'(\lambda)x^2 + \ldots, \quad x \to 0.
$$

The parameter $\lambda$ is a spectral parameter. Using the quasiperiodicity properties of the function $\Phi$,

$$
\Phi(x + 2\omega, \lambda) = e^{2(\zeta(\omega)\lambda - \zeta(\lambda)\omega)} \Phi(x, \lambda),
$$

$$
\Phi(x + 2\omega', \lambda) = e^{2(\zeta(\omega')\lambda - \zeta(\lambda)\omega')} \Phi(x, \lambda),
$$

one concludes that $\psi$ given by (3.62) is indeed a double-Bloch function with Bloch multipliers

$$
b = e^{2(\omega z + \zeta(\omega)\lambda - \zeta(\lambda)\omega)}, \quad b' = e^{2(\omega' z + \zeta(\omega')\lambda - \zeta(\lambda)\omega')}.
$$

We will often suppress the second argument of $\Phi$ writing simply $\Phi(x) = \Phi(x, \lambda)$. We will also need the $x$-derivatives $\Phi'(x, \lambda) = \partial_x \Phi(x, \lambda)$, $\Phi''(x, \lambda) = \partial_x^2 \Phi(x, \lambda)$.

Substituting (3.62) into (3.61) with $u$ given by (3.60), we get:

$$
- \sum_i \dot{c}_i \Phi(x - x_i) + \sum_i c_i \dot{x}_i \Phi'(x - x_i) + 2z \sum_i c_i \Phi'(x - x_i) + \sum_i c_i \Phi''(x - x_i)
$$

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where dot means the \( t_2 \)-derivative. Different terms of this expression have poles at \( x = x_i \). The highest poles are of third order but it is easy to see that they cancel identically. The conditions of cancellation of second and first order poles have the form

\[
c_i \dot{x}_i = -2z c_i - 2 \sum_{j \neq i} c_j \Phi(x_i - x_j),
\]

\[
\dot{c}_i = (4c + \varphi(\lambda))c_i - 2 \sum_{j \neq i} c_j \Phi'(x_i - x_j) - 2c_i \sum_{j \neq i} \varphi(x_i - x_j).
\]

Introducing \( N \times N \) matrices

\[
\mathcal{L}_{ij} = -\delta_{ij} \dot{x}_i - 2(1 - \delta_{ij})\Phi(x_i - x_j),
\]

\[
\mathcal{M}_{ij} = \delta_{ij}(\varphi(\lambda) + 4c) - 2\delta_{ij} \sum_{k \neq i} \varphi(x_i - x_k) - 2(1 - \delta_{ij})\Phi'(x_i - x_j),
\]

we can write the above conditions as a system of linear equations for the vector \( \mathbf{c} = (c_1, \ldots, c_N)^T \):

\[
\left\{
\begin{array}{l}
\mathcal{L}\mathbf{c} = 2z\mathbf{c} \\
\dot{\mathbf{c}} = \mathcal{M}\mathbf{c}.
\end{array}
\right.
\]

The compatibility condition is

\[
\dot{\mathcal{L}} + [\mathcal{L}, \mathcal{M}] = 0.
\]

Using certain identities for the \( \Phi \)-function, one can see that it is equivalent to the equations of motion

\[
\ddot{x}_i = 4 \sum_{k \neq i} \varphi'(x_i - x_k).
\]

The Hamiltonian is

\[
\mathcal{H}_2 = \sum_i p_i^2 - 2 \sum_{i<j} \varphi(x_i - x_j).
\]

This is the elliptic version of the Calogero-Moser model. We have obtained it together with its Lax representation (3.69) with the matrices \( \mathcal{L}, \mathcal{M} \) depending on the spectral parameter \( \lambda \).

### 3.6.7 Algebro-geometric (finite gap) solutions

The soliton-like solutions are degenerate cases of a more general family of solutions which are associated with algebraic curves (Riemann surfaces) according to Krichever’s construction. These solutions are called algebro-geometric. In general they are quasi-periodic in all the times.

The main building block of the algebro-geometric solutions is the Riemann theta-function given by the absolutely convergent series

\[
\Theta(\vec{z}|T) = \sum_{\vec{n} \in \mathbb{Z}^g} \exp \left( \pi i (T\vec{n}, \vec{n}) + 2\pi i (\vec{n}, \vec{z}) \right).
\]
Here $\vec{z} = (z_1, \ldots, z_g)^T$ is a complex vector, the brackets denote the Euclidean scalar product $(\vec{x}, \vec{y}) = \sum_{i=1}^{g} x_i y_i$ and $T$ is a complex $g \times g$ symmetric matrix with positively definite imaginary part (called the Riemann matrix).

The tau-function of the algebro-geometric solutions can be found in the form

$$\tau(t) = e^{Q(t)} \Theta \left( \vec{Z}_0 + x \vec{U}_1 + t_2 \vec{U}_2 + t_3 \vec{U}_3 + \ldots \right)^T,$$

(3.73)

where $Q(t)$ is a quadratic form of the times $t_k$ and $\vec{Z}_0, \vec{U}_k$ are some $g$-dimensional constant vectors. However, in general (3.73) is not a solution (i.e. it is not the tau-function) if the vectors $U_k$ and the Riemann matrix $T$ are arbitrary. For (3.73) to be the tau-function these parameters have to obey some constraints. The necessary relations between the parameters can be in principle found by a direct substitution of the expression (3.73) into the KP equation. In practice this method effectively works for small $g$ only.

In general the relations between the parameters can be implemented if one starts with a smooth algebraic curve (a Riemann surface) $C$ of genus $g$ with a marked point which we call $\infty$. Topologically the surface of genus $g$ is a sphere with $g$ handles. In the space spanned by closed cycles one can choose a basis of cycles (closed contours) $a_1, \ldots, a_g$ and $b_1, \ldots, b_g$ with the following intersections:

$$a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}, \quad i, j = 1, \ldots, g.$$  

As is known, there are $g$ linear independent holomorphic differentials $\Omega_k$ on the curve of genus $g$. We normalize them by the condition $\oint_{a_i} \Omega_k = \delta_{ik}$. Then the $b$-periods form the matrix of periods

$$T_{ik} = \oint_{b_i} \Omega_k$$

(3.74)

which is the Riemann matrix. Let $z^{-1}$ be a local parameter in a neighborhood of the marked point $\infty$. Consider the meromorphic differentials $\Omega^{(k)}$ which have the only high order pole on $C$ at $\infty$ with the principle parts $\Omega^{(k)} = dz^k + O(z^{-2})$, $z \to \infty$. Let us normalize them by the conditions $\oint_{a_i} \Omega^{(k)} = 0$.

Proposition. If the matrix $T$ in (3.73) is the matrix of periods of holomorphic differentials on $C$ (3.74) and the vectors $\vec{U}_k = (U_{k,1}, \ldots, U_{k,g})^T$ are given by

$$U_{k,i} = \oint_{b_i} \Omega^{(k)},$$

(3.75)

then $\tau(t)$ given by (3.73) is the tau-function of the KP hierarchy.

The proof is based on properties of Baker-Akhiezer functions on Riemann surfaces.

The set of Riemann matrices has $\frac{1}{2} g (g + 1)$ complex parameters. A fundamental result in the theory of Riemann surfaces states that the set of period matrices for $g \geq 2$ has $3g - 3$ complex parameters. At $g = 2$ and $g = 3$ these two numbers are equal but for $g \geq 4$ $3g - 3$ is strictly less than $\frac{1}{2} g (g + 1)$. The famous Schottky problem is to characterize the matrices of periods of holomorphic differentials on Riemann surfaces among all Riemann matrices. A solution to the Schottky problem is suggested by the Novikov’s conjecture (proved by Shiota in 1986): matrices of periods (coming from a
Riemann surface) are precisely those Riemann matrices for which the expression (3.73) gives a solution to the KP equation.

For the KdV equation $\vec{U}_2 = 0$. In this case the curve $\Gamma$ should be hyperelliptic of genus $g$ with a ramification point at $\infty$, i.e. given by an equation of the form $y^2 = R_{2g+1}(x)$ with a polynomial $R_{2g+1}(x)$ of degree $2g + 1$. The functions $u = 2\partial_x^2 \log \Theta(\vec{Z}_0 + x\vec{U}_1 + t_3\vec{U}_3 + \ldots | T) + c$ constructed from hyperelliptic curves as functions of $x$ have a rather special property: the corresponding Schrödinger operator with the potential $-u$ has only finite number of unstable bands (gaps) in the spectrum. That is why the algebro-geometric solutions are sometimes called finite-gap solutions.

### 3.6.8 General solution to the KP hierarchy: non-local $\bar{\partial}$-problem

A general method to define the function $\psi$ in such a way that the asymptotic equalities (3.20) would imply the exact ones consists in posing the so-called non-local $\bar{\partial}$-problem ($\bar{\partial} \equiv \partial_z$). Let the function $\psi = \psi(z)$ be of the form

$$\psi(z) = e^{\xi(t,z)}w(z), \quad w(z) = 1 + \frac{\xi_1}{z} + \frac{\xi_2}{z^2} + \ldots$$

as $z \to \infty$. Impose the equation

$$\partial_z \psi(z) = \int K(z, \zeta)\psi(\zeta)d^2\zeta,$$  

where the integration in general goes over all complex plane with the measure $d^2z \equiv d(\text{Re}~z)d(\text{Im}~z)$, and $K(z, \zeta)$ is some kernel function which does not depend on the times $t_i$. Assume that the function $K(z, \zeta)$ in each of the variables does not vanish only in some compact domain, then outside this domain, when $|z|$ is sufficiently large, one can require the holomorphic asymptotics (3.76). The function $K(z, \zeta)$ can be a generalized function (a distribution), i.e. it may contain delta-functions concentrated in points or on contours. Equation (3.77) is called the non-local $\bar{\partial}$-problem (non-local because the right hand side contains the integral). Assume that this problem has a unique solution (up to a constant factor). Then we are in the situation discussed above and $u = -2\xi_1'$ will satisfy the KP equation. It is hard to formulate general conditions on the kernel when there is a unique solution. This problem should be solved by a separate investigation in each concrete case.

The integro-differential equation (3.77) can be reduced to an integral equation for the function $w = \psi e^{-\xi(t,z)}$. Let us represent the $\bar{\partial}$-problem in the form

$$\partial_z w(z) = \int K_t(z, \zeta)w(\zeta)d^2\zeta,$$  

where the kernel $K_t(z, \zeta)$ depends on times: $K_t(z, \zeta) = e^{\xi(t,\zeta) - \xi(t,z)}K(z, \zeta)$. Using the formal equality $\partial_z (1/z) = \pi\delta^{(2)}(z)$, we can write

$$w(z) = 1 + \frac{1}{\pi} \int \frac{\partial_\zeta w(\zeta)d^2\zeta}{z - \zeta}$$

and then (3.78) becomes the integral equation

$$w(z) = 1 + \frac{1}{\pi} \int \int \frac{K_t(\zeta, \xi)w(\xi)}{z - \zeta} d^2\zeta d^2\xi.$$  

(3.79)
Remark. In the case \( K(z, \zeta) = K(z) \delta^{(2)}(z + \zeta) \) one gets solutions of the KdV equation; the \( \bar{\partial} \)-problem acquires the form
\[
\partial_z \psi(z) = K(z) \psi(-z). 
\]
(3.80)

As an example, consider the soliton-like solutions. Let the kernel be concentrated at a finite number of points:
\[
K(z, \zeta) = -\pi \sum_{j=1}^{N} \tilde{\beta}_j \delta^{(2)}(z - q_j) \delta^{(2)}(\zeta - p_j),
\]
(3.81)
then the integral equation (3.79) will be of the form
\[
w(z) = 1 - \sum_{j=1}^{N} \tilde{\beta}_j e^{\xi(t,p_j) - \xi(t,q_j)} w_j.
\]
(3.82)
from which we should find
\[
\xi_1 = - \sum_{j=1}^{N} \tilde{\beta}_j e^{\xi(t,p_j) - \xi(t,q_j)} w_j.
\]
The solution consists in application of the Kramer’s rule. It gives \( u = 2\partial_z^2 \log \tau \) with the tau-function (3.26) and \( \tilde{\beta}_j = (p_j - q_j) \beta_j \).

Problem. Find the kernel of the \( \bar{\partial} \)-problem corresponding to the solutions depending on functional parameters.

3.7 Additional (nonabelian) symmetries of the KP hierarchy

Nonabelian symmetries of the KP hierarchy are richer than in the case of KdV. In order to define them in a general form, we need an extended Lax formalism. Let us introduce a new operator, \( M \), which satisfies the same Lax equations and forms a canonical pair with \( L \), i.e. their commutator is equal to 1. (This is the so-called Orlov-Shulman operator.) The simplest way to define it is to use dressing by the operator \( K \).

Note that besides \( \partial \) there is yet another operator commuting with \( \partial_n - \partial^n \); it is
\[
\Gamma = \sum_{k=1}^{\infty} k t_k \partial^{k-1} = x + 2t_2 \partial + 3t_3 \partial^2 + \ldots
\]
The operator \( \partial \) acts to the exponential function \( e^{\xi(t,z)} \) as multiplication by \( z \) while \( \Gamma \) acts as \( z \)-derivative:
\[
\partial e^{\xi(t,z)} = z e^{\xi(t,z)}, \quad \Gamma e^{\xi(t,z)} = \partial_z e^{\xi(t,z)}.
\]
(3.83)
We recall that dressing of the obvious commutation relation \( [\partial, \partial_n - \partial^n] = 0 \) by the operator \( K \) gives the “equations of motion” for the Lax operator. In a similar way, dressing of the relation \( [\Gamma, \partial_n - \partial^n] = 0 \) gives \( [M, \partial_n - A_n] = 0 \) or
\[
\partial_n M = [A_n, M], \quad A_n = (L^n)_+,
\] (3.84)
where \( M = K\Gamma K^{-1} \) is the Orlov-Shulman operator.

**Exercise.** Check (3.84) by a direct calculation.

The Orlov-Shulman operator can be represented as a series in powers of the Lax operator:
\[
M = \sum_{k \geq 2} kt_k L^{k-1} + x + \sum_{k \geq 1} v_k L^{-k-1},
\] (3.85)
where the “tail” of the expansion (negative powers of \( L \)) is obtained from the dressing \( KxK^{-1} \) and \( v_k \) depend on \( x \) and all higher times. It is clear that the Lax equation of the type (3.84) holds for any function of \( L \) and \( M \), in particular:
\[
\partial_n (M^m L^l) = [A_n, M^m L^l].
\] (3.86)

Acting by the dressing operator from the left to both sides of relations (3.83), we find how \( L \) and \( M \) act to the Baker-Akiezer function:
\[
L\psi = z\psi, \quad M\psi = \partial_z \psi.
\] (3.87)

Finally, dressing of the obvious relation \( [\partial, \Gamma] = 1 \) yields
\[
[L, M] = 1,
\] (3.88)
so \( L \) and \( M \) is indeed the canonical pair. The same is seen from (3.87).

Now everything is ready for introducing of the additional symmetries. Let \( s_{lm} \) be parameters of the symmetries (now they are numbered by two indices) and \( \partial_{s_{lm}} \) be the corresponding vector fields. Consider the equations
\[
\partial_{s_{lm}} L = -[(M^m L^l)_-, L],
\] (3.89)
which define flows on the space of operators \( L \). Obviously, \( \partial_{s_{00}} = \partial_{t_n} \).

**Proposition.** Equations (3.89) are symmetries of the KP hierarchy.

It is easy to check that both sides of (3.89) are pseudo-differential operators of order \(-1\), and thus the flows are well-defined. To see that they are symmetries, one should calculate \( X := [\partial_{s_{lm}}, \partial_{t_n}] L = \partial_{s_{lm}} (\partial_{t_n} L) - \partial_{t_n} (\partial_{s_{lm}} L) \) and show that \( X = 0 \). We have:
\[
X = \partial_{s_{lm}} (\partial_{t_n} L) - \partial_{t_n} (\partial_{s_{lm}} L)
\]
\[
= -\partial_{s_{lm}} [(L^n)_-, L] + \partial_{t_n} [(M^m L^l)_-, L]
\]
\[
= -[(L^n)_-, \partial_{s_{lm}} L] - [(M^m L^l)_-, L] + [\partial_{t_n} (M^m L^l)_-, L] + [(M^m L^l)_-, \partial_{t_n} L]
\]
\[
= [(L^n)_-, [(M^m L^l)_-, L]] - [\partial_{s_{lm}} (L^n)_- - \partial_{t_n} (M^m L^l)_-, L] - [(M^m L^l)_-, [(L^n)_-, L]]
\]
\[
= [\partial_{t_n} (M^m L^l)_- - \partial_{s_{lm}} (L^n)_- + [(L^n)_-, (M^m L^l)_-], L].
\]
When passing to the last line, we have used the Jacobi identity for the double commutator. Calculate now \( \partial_{s_{lm}} (L^n)_- \) with the help of the definition (3.89) and show that the operator inside the commutator is equal to 0, i.e.

\[ \partial_{t_n} (M^m L^l)_- - \partial_{s_{lm}} (L^n)_- + [(L^n)_-, (M^m L^l)_-] = 0. \]

(3.90)

Indeed,

\[ \partial_{s_{lm}} (L^n)_- = -\left( [(M^m L^l)_-, L^n] \right)_- \]

\[ = -[(M^m L^l)_-, (L^n)_-] - \left( [M^m L^l, (L^n)_+] \right)_- \]

\[ = -[(M^m L^l)_-, (L^n)_-] + \partial_{t_n} (M^m L^l), \]

which is equivalent to (3.90).

We have shown that the vector fields given by (3.89) are symmetries of the KP hierarchy. However, they do not commute and thus they are not symmetries for each other. In general these symmetries are non-local, i.e. contain integrals of \( u \) over \( x \).

The doubly infinite family of symmetries constructed above includes three-parametric set of symmetries which generalize the KdV symmetries (2.56). For the KP equation they are as follows. If \( u(x, y, t) \) satisfies the KP equation (3.7), then \( \tilde{u} \), defined as a function of \( \tilde{x}, \tilde{y}, \tilde{t} \) by the equalities

\[
\begin{align*}
\tilde{u} &= \lambda^{-2}u - 2a\lambda^{-1}, \\
\tilde{x} &= \lambda x + 2b\lambda y + (3a\lambda^2 + 3b^2\lambda)t, \\
\tilde{y} &= \lambda^2 y + 3b\lambda^2 t, \\
\tilde{t} &= \lambda^3 t
\end{align*}
\]

(3.91)

satisfies the same equation. In other words, the transformation

\[ u(x, y, t) \rightarrow \lambda^{-2}u \left( \lambda^{-1} x - 2b\lambda^{-2} y - (3a\lambda^{-2} - 3b^2\lambda^{-3})t, \lambda^{-2} y - 3b\lambda^{-3} t, \lambda^{-3} t \right) - 2a\lambda^{-1} \]

sends a solution to another solution. (This can be checked by a direct substitution.) At \( \lambda = 1, b = 0, a \neq 0 \) we have the Galilean transformation. It generates by the vector field

\[ \partial_{s_{-1,1}} L = -[(ML^{-1})_-, L]. \]

Problem. Find vector fields of the type (3.89) which generate transformations (3.91) with \( \lambda = 1, b \neq 0, a = 0 \) and \( \lambda \neq 1, b = 0, a = 0 \).

3.8 Tau-function of the KP hierarchy

So far the tau-function occasionally appeared as a convenient auxiliary object in the discussion of soliton solutions. In fact the tau-function plays a fundamental role in the theory of the KP hierarchy (and other integrable hierarchies). Passing to the tau-function, regarded as a dependent variable, allows one to formulate the KP hierarchy as
an infinite set of compatible equations for one function rather than an infinite number of them, as in the original formulation. In terms of the tau-function, all equations of the KP hierarchy become bilinear and they can be encoded in one "generating equation", which is known as the difference Hirota equation. However, it is not an easy task to derive all that starting from the Lax representation. Introducing the very concept of the tau-function requires some preparation.

3.8.1 The bilinear identity

We start from a reformulation of the KP hierarchy in terms of the Baker-Akhiezer function and its adjoint.

As before, together with the operator residue \( \text{res} = \text{res}_0 \) we will consider the ordinary residue \( \text{res}_z \) of the Laurent series \( \sum_j a_j z^j \) defined as \( \text{res}_z (\sum_j a_j z^j) = a_{-1} \).

**Proposition.** For the \( \psi \)-function of any solution to the KP hierarchy the following bilinear identity holds:

\[
\text{res}_z \left( \psi(t, z) \psi^*(t', z) \right) = 0, \tag{3.92}
\]

where \( t = \{t_j\}, \ t' = \{t'_j\} \) are two arbitrary sets of times.

For the proof we use the previously proved lemma which states that for any pseudo-differential operators \( P, Q \) the equality

\[
\text{res}_z \left[ (Pe^{xz}) (Qe^{-xz}) \right] = \text{res}_0 (PQ^\dagger)
\]

holds true. Assume that the function \( \psi(t', z) \) can be obtained from \( \psi(t, z) \) as a Taylor series in \( t'_i - t_i \). Then it is enough to prove (3.92) for any terms of this series. Note that the \( t_i \)-derivatives with \( i \geq 2 \) are expressed linearly through derivatives with respect to \( t_1 = x \) by virtue of (3.12). Hence it is enough to show that for all \( n \geq 0 \)

\[
\text{res}_z \left( (\partial^n \psi(t, z)) \psi^*(t, z) \right) = 0.
\]

The latter is easily checked using the above lemma:

\[
\text{res}_z (\partial^n \psi \psi^*) = \text{res}_z \left( \partial^n Ke^{\xi(t,z)}(K^\dagger)^{-1}e^{-\xi(t,z)} \right) = \text{res}_0 \left( \partial^n K K^{-1} \right) = \text{res}_0 \partial^n = 0.
\]

The inverse statement is also true: let \( \psi, \psi^* \) be the series of the form

\[
\psi = e^{\xi(t,z)} \left( 1 + \xi_1 z^{-1} + \xi_2 z^{-2} + \ldots \right), \quad \psi^* = e^{-\xi(t,z)} \left( 1 + \xi_1^* z^{-1} + \xi_2^* z^{-2} + \ldots \right),
\]

where the coefficients \( \xi_j, \xi_j^* \) are some functions of \( t_i \), and for all sets of times \( t_i' = t_i \) \ref{3.92} holds; then there exists the \( L \)-operator of the form \( \ref{3.1} \) satisfying all the Lax equations and the Baker-Akhiezer function for it coincides with \( \psi \) (and the adjoint function coincides with \( \psi^* \)).

**Problem.** For the tau-function \( \tau = t_2 + \frac{1}{2} t_1^2 \) find \( \psi, \psi^* \) and check the bilinear identity.

**Problem.** Check the bilinear identity for the functions \( \psi, \psi^* \) corresponding to the one-soliton solution.

Let us explain in more detail how one should understand the residue in the bilinear identity. We have \( \psi(t, z) = e^{\xi(t,z)} w(t, z), \ \psi^*(t, z) = e^{-\xi(t,z)} w^*(t, z) \), where the functions
are expanded in series in inverse powers of and thus they are regular at $\infty$. The bilinear identity acquires the form

$$\text{res}_z \left( e^{\xi(t-t',z)} w(t, z) w^*(t', z) \right) = 0.$$  

(3.93)

One should expand the function $e^{\xi(t-t',z)}$ in positive powers of $z$, the functions $w(t, z)$, $w^*(t', z)$ in negative powers of $z$, multiply these series, extract the coefficient in front of $z^{-1}$ and equate it to zero. This is equivalent to vanishing of the contour integral

$$\oint_{\mathcal{C}} e^{\xi(t-t',z)} w(t, z) w^*(t', z) dz = 0,$$

(3.94)

where the contour $\mathcal{C}$ should encircle all singularities of the function $e^{\xi(t-t',z)}$ and should not encircle singularities of the function $w(t, z)w^*(t', z)$. In particular, if only a finite number of the times are not equal to 0, then the function $e^{\xi(t-t',z)}$ is regular everywhere except $\infty$, where it has an essential singularity. In this case the contour $\mathcal{C}$ is a circle of radius $R$ for sufficiently large $R$.

### 3.8.2 “Japanese” formula for the Baker-Akhiezer function

One of the most important results of the Japanese school (Sato, Jimbo, Miwa and others) is the remarkable formula for the Baker-Akhiezer function through the tau-function. (This formula simultaneously serves as a definition of the latter.)

**Theorem.** Let $\psi$ be the Baker-Akhiezer function of the KP hierarchy, then there exists a function $\tau(t_1, t_2, t_3, \ldots)$ such that

$$\psi(t, z) = e^{\xi(t(z))} \frac{\tau(t_1 - \frac{1}{2}, t_2 - \frac{1}{2}, t_3 - \frac{1}{2}, \ldots)}{\tau(t_1, t_2, t_3, \ldots)}.$$  

(3.95)

There is also a similar formula for the adjoint Baker-Akhiezer function through the same tau-function:

$$\psi^*(t, z) = e^{-\xi(t(z))} \frac{\tau(t_1 + \frac{1}{2}, t_2 + \frac{1}{2}, t_3 + \frac{1}{2}, \ldots)}{\tau(t_1, t_2, t_3, \ldots)}.$$  

(3.96)

Equation (3.95) can be written in a different form. Write $\psi(t, z) = e^{\xi(t,z)} w(t, z)$ and take the logarithmic derivative with respect to $z$ of both sides of (3.95). We get

$$\partial_z \log w(t, z) = \sum_{m \geq 1} \frac{\partial}{\partial t_m} \log w(t, z) z^{-m-1} + \sum_{m \geq 1} \frac{\partial}{\partial t_m} \tau z^{-m-1}$$

which is equivalent to the relations

$$\frac{\partial}{\partial t_n} \log \tau = \text{res}_z \left( z^n (\partial_z - \partial(z)) \log w(t, z) \right),$$  

(3.97)

where we use the notation

$$\partial(z) := \sum_{j \geq 1} z^{-j-1} \frac{\partial}{\partial t_j}.$$
Therefore, for the proof of existence of the tau-function it is enough to prove that the expression
\[ \text{res}_z \left( z^n \left( \partial_z - \partial(z) \right) \partial_m \log w(t, z) \right) \]
is symmetric under permutation of \( m, n \).

The proof is based on the bilinear identity. Below we present its main points.

Putting \( t'_j = t_j - \zeta^{-j}/j \) in the bilinear identity, we write it in the form
\[ \text{res}_z \left( \psi(t, z) \psi^*(t - [\zeta^{-1}], z) \right) = 0, \] (3.98)
where we have introduced the convenient notation
\[ F(t \pm [z]) \equiv F(t_1 \pm z, t_2 \pm z^2/2, t_3 \pm z^3/3, \ldots). \]

After the substitutions \( \psi(t, z) = e^{\xi(t,z)}w(t, z), \psi^*(t, z) = e^{-\xi(t,z)}w^*(t, z) \) equation \[ (3.98) \]
reads
\[ \text{res}_z \left( \frac{w(t, z) w^*(t - [\zeta^{-1}], z)}{1 - z/\zeta} \right) = 0. \]

It is easy to check that for any series \( f(z) = 1 + \sum_{j \geq 1} f_j z^{-j} \) the identity
\[ \text{res}_z \left( \frac{f(z)}{1 - z/\zeta} \right) = \sum_{j \geq 1} f_j \zeta^{1-j} = \zeta(f(\zeta) - 1) \] (3.99)
holds. Applying it to the previous equality, we get the relation between \( w \) and \( w^* \):
\[ w(t, z) w^*(t - [z^{-1}], z) = 1. \] (3.100)

Similarly, from the bilinear identity \( \text{res}_z \left( \psi(t, z) \psi^*(t - [\zeta_1^{-1}] - [\zeta_2^{-1}], z) \right) = 0 \) rewritten in the form
\[ \text{res}_z \left( \frac{w(t, z) w^*(t - [\zeta_1^{-1}] - [\zeta_2^{-1}], z)}{(1 - z/\zeta_1)(1 - z/\zeta_2)} \right) = 0 \]
it follows that
\[ w(t, \zeta_1) w^*(t - [\zeta_1^{-1}] - [\zeta_2^{-1}], \zeta_1) = w(t, \zeta_2) w^*(t - [\zeta_1^{-1}] - [\zeta_2^{-1}], \zeta_2), \]
where we have used the identity
\[ \frac{1/\zeta_1 - 1/\zeta_2}{(1 - z/\zeta_1)(1 - z/\zeta_2)} = \frac{1}{\zeta_1(1 - z/\zeta_1)} - \frac{1}{\zeta_2(1 - z/\zeta_2)} \]
and equation \[ (3.99) \]. Now one can express \( w^* \) through \( w \) with the help of \[ (3.99) \] and put \( \zeta_1 = z, \zeta_2 = \zeta \). The result is
\[ \frac{w(t, z)}{w(t - [\zeta^{-1}], z)} = \frac{w(t, \zeta)}{w(t - [z^{-1}], \zeta)}. \] (3.101)

Let us take logarithm of this equality and apply the operator \( \partial_z - \partial(z) \). We get:
\[ (\partial_z - \partial(z)) \log w(t, z) - (\partial_z - \partial(z)) \log w(t - [\zeta^{-1}], z) = -\partial(z) \log w(t, \zeta). \] (3.102)
For brevity denote \( Y_n(t) := \text{res}_z \left( z^n (\partial_z - \partial(t)) \log w(t, z) \right) \). Multiplying both sides of (3.102) by \( z^n \) and taking the residue, we have

\[
Y_n(t) - Y_n(t - [\zeta^{-1}]) = -\partial_t \log w(t, \zeta).
\]

After differentiating with respect to \( t_m \) and subtracting the similar equality with interchanged \( m, n \), this yields

\[
\partial_{t_m} Y_n(t) - \partial_{t_n} Y_m(t) = \partial_{t_m} Y_n(t - [\zeta^{-1}]) - \partial_{t_n} Y_m(t - [\zeta^{-1}]).
\]

Let us denote the left hand side by \( F_{mn}(t) \). Expand \( F_{mn}(t - [\zeta^{-1}]) \) in powers of \( \zeta \): 

\[
F_{mn}(t - [\zeta^{-1}]) = F_{mn}(t) - \zeta^{-1} \partial_t F_{mn}(t) + \frac{1}{2!} \zeta^{-2} (\partial_t^2 F_{mn}(t) - \partial_{t^2} F_{mn}(t)) + \ldots.
\]

Since \( F_{mn}(t - [\zeta^{-1}]) = F_{mn}(t) \) for all \( t_k \) and for all solutions, the comparison of coefficients at \( \zeta^{-1} \) implies that \( \partial_t F_{mn}(t) = 0 \) for all \( t \), i.e., \( F_{mn} \) does not depend on \( t_1 \). Next, comparing the coefficients at \( \zeta^{-2} \) we similarly find that \( F_{mn} \) does not depend on \( t_2 \). Repeating this argument, we conclude that \( F_{mn} \) does not depend on all times, i.e., it is a constant. For the trivial solution \( u_i = 0 \) the constant is 0. Since \( F_{mn} \) is a differential polynomial of \( u_i \), this constant is equal to 0 for any solution. Thus we have shown that \( \partial_{t_m} Y_n(t) = \partial_{t_n} Y_m(t) \), which implies existence of the tau-function and validity of equations (3.95), (3.97). Equation (3.96) is proved in a similar way.

### 3.8.3 Equations of the KP hierarchy in the Hirota form

One can express \( \psi \) and \( \psi^* \) through the tau-function using the “Japanese” formulas. This yields the bilinear relation for the tau-function:

\[
\text{res}_z \left( \tau(t - [z^{-1}]) \tau(t' + [z^{-1}]) e^{\xi(t-t',z)} \right) = 0, \tag{3.103}
\]

or

\[
\oint_C e^{\xi(t-t',z)} \tau(t - [z^{-1}]) \tau(t' + [z^{-1}]) dz = 0 \tag{3.104}
\]

with the convention about the integration contour discussed above. This relation is equivalent to an infinite system of bilinear differential equations for the tau-function. They are obtained by equating to 0 the expansion coefficients in the Taylor series for the left hand side in \( t' - t \). Technically this expansion is conveniently made if one substitutes \( t_i - T_i \) and \( t_i + T_i \) for \( t_i \) and \( t_i' \) respectively:

\[
\text{res}_z \left[ \tau(t - T - [z^{-1}]) \tau(t + T + [z^{-1}]) e^{-\xi(T,z)} \right]
= \text{res}_z \left[ e^{\xi(\partial_T z^{-1})} (\tau(t - T) \tau(t + T)) e^{-\xi(T,z)} \right]
= \text{res}_z \left[ \sum_{j \geq 0} z^{-j} h_j(\partial_T)(\tau(t - T) \tau(t + T)) \sum_{l \geq 0} z^l h_l(-2T) \right]
= \sum_{j \geq 0} h_j(-2T) h_{j+1}(\partial_T) \tau(t - T) \tau(t + T) = 0.
\]
In the second line, the shift by \([z^{-1}]\) is represented as action of the exponential function of the differential operator

\[
\xi(\hat{\partial}_T, z^{-1}) = \sum_{j \geq 1} \frac{z^{-j}}{j} \partial_{T_j}
\]

(the notation \(\hat{\partial}_T = \{\partial_{T_1}, \frac{1}{z} \partial_{T_2}, \frac{1}{z} \partial_{T_3}, \ldots\} \) is used). In the third line, the exponential function is expanded with the help of the Schur polynomials (3.34). The latter equality can be also written in the form

\[
\sum_{j \geq 0} h_j(-2T)h_{j+1}(\hat{\partial}_X) e^{\sum_{l \geq 1} T_l \partial_{X_l} \tau(t - X) \tau(t + X)} \bigg|_{X_m = 0} = 0.
\]

Using the symbol \(D_i\) of the “Hirota derivative” defined by the rule

\[
P(D) f(t) \cdot g(t) := P(\partial_X)(f(t - X)g(t + X)) \bigg|_{X = 0}
\]

for any polynomial \(P(D)\) of \(D_i\), we can write it in the form

\[
\sum_{j \geq 0} h_j(-2T)h_{j+1}(\hat{\partial}_X) e^{\sum_{l \geq 1} T_l D_l \tau(t) \cdot \tau(t)} = 0.
\]

This relation contains, in the encoded form, all bilinear Hirota equations for the KP hierarchy. The first non-trivial equation obtained as a result of expanding in the series in \(T_i\) has the form

\[
\left( D_1^4 + 3D_2^2 - 4D_1 D_3 \right) \tau \cdot \tau = 0.
\]

**Exercise.** Check that equation (3.106) after the substitution \(u = 2\partial^2 \log \tau\) turns into the KP equation (3.7).

**Problem.** Prove that if \(\tau(t)\) is the tau-function of the KP hierarchy (a solution to all bilinear Hirota equations), then \(\tau(-t)\) is also the tau-function.

### 3.9 Bilinear difference Hirota equation

#### 3.9.1 The Hirota-Miwa equation

In the bilinear relation (3.103) put \(t' = t - [\lambda_1^{-1}] - [\lambda_2^{-1}] - [\lambda_3^{-1}]\), where \(\lambda_{1,2,3}\) are arbitrary complex parameters, i.e., put

\[
t' = t - \frac{\lambda_1^{-k}}{k} - \frac{\lambda_2^{-k}}{k} - \frac{\lambda_3^{-k}}{k}.
\]

Then the bilinear relation reads

\[
\text{res}_{z} \left( \frac{\tau(t - [z^{-1}]) \tau(t - [\lambda_1^{-1}] - [\lambda_2^{-1}] - [\lambda_3^{-1}] + [z^{-1}])}{(1 - z/\lambda_1)(1 - z/\lambda_2)(1 - z/\lambda_3)} \right) = 0.
\]

To use identity (3.99), we represent the product of the pole factors as a sum of poles:

\[
(1/\lambda_1 - 1/\lambda_2)(1/\lambda_1 - 1/\lambda_3)(1/\lambda_2 - 1/\lambda_3) = \frac{1/\lambda_2 - 1/\lambda_3}{1 - z/\lambda_1} + (231) + (312),
\]

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where the last two terms are obtained from the first one by cyclic permutations of indices $(1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1)$ and $(1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2)$. Using (3.99), we obtain the fundamental equation

\[
(\lambda_2 - \lambda_3) \tau(t - [\lambda_1^{-1}]) \tau(t - [\lambda_2^{-1}] - [\lambda_3^{-1}])
+ (\lambda_3 - \lambda_1) \tau(t - [\lambda_2^{-1}]) \tau(t - [\lambda_3^{-1}] - [\lambda_1^{-1}])
+ (\lambda_1 - \lambda_2) \tau(t - [\lambda_3^{-1}]) \tau(t - [\lambda_1^{-1}] - [\lambda_2^{-1}]) = 0
\]  

(3.107)

satisfied by the tau-function of the KP hierarchy. This is the Hirota-Miwa equation.

**Problem.** From the bilinear relation derive the equation

\[
(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_3) \tau(t - [\lambda_0^{-1}] - [\lambda_1^{-1}]) \tau(t - [\lambda_2^{-1}] - [\lambda_3^{-1}]) + (231) + (312) = 0
\]  

(3.108)

and show that it is equivalent to (3.107).

**Problem.** From the bilinear relation derive the equation

\[
\partial_t \log \frac{\tau(t + [\lambda_1^{-1}])}{\tau(t + [\lambda_2^{-1}])} = (\lambda_2 - \lambda_1) \left( \frac{\tau(t) \tau(t + [\lambda_1^{-1}] + [\lambda_2^{-1}])}{\tau(t + [\lambda_1^{-1}]) \tau(t + [\lambda_2^{-1}])} - 1 \right)
\]  

(3.109)

and show that it is equivalent to (3.107).

**Problem.** Prove that the tau-function of the KP hierarchy satisfies the equation

\[
\prod_{1 \leq i < j \leq m} (\lambda_j - \lambda_i) \tau(t + \sum_{a=1}^{m} [\lambda_a^{-1}])^{m-1}(t) = \det_{1 \leq i, k \leq m} ((\lambda_j - \partial_t)^{k-1} \tau(t + [\lambda_j^{-1}])).
\]  

(3.110)

(Hint: use induction in $m$.)

### 3.9.2 The $T$-system and the $Y$-system

The Hirota-Miwa equation can be read as a difference equation for the function

\[
\tau(p_1, p_2, p_3) := \tau\left(t - p_1[\lambda_1^{-1}] - p_2[\lambda_2^{-1}] - p_3[\lambda_3^{-1}]\right)
\]

(the arguments in the right hand side are $t_k - (p_1\lambda_1^{-k} + p_2\lambda_2^{-k} + p_3\lambda_3^{-k})/k$). In this form, it is called the bilinear difference Hirota equation:

\[
(\lambda_2 - \lambda_3) \tau(p_1 + 1, p_2, p_3) \tau(p_1 + 1, p_2 + 1, p_3 + 1)
+ (\lambda_3 - \lambda_1) \tau(p_1, p_2 + 1, p_3) \tau(p_1 + 1, p_2, p_3 + 1)
+ (\lambda_1 - \lambda_2) \tau(p_1, p_2, p_3 + 1) \tau(p_1 + 1, p_2 + 1, p_3) = 0
\]

(3.111)

If one forgets about the origin of the function $\tau(p_1, p_2, p_3)$, then this equation can be understood as an equation for a function of discrete variables $p_j$ assuming integer values or as a difference equation for a function of real or complex variables $p_j$. It is the
integrable difference analogue of the KP equation. One can see that it is much more symmetric than the KP equation itself.

Equation (3.111) can be represented in different equivalent forms. Let us point out two of them.

The linear change of variables \( x_1 = p_2 + p_3, \ x_2 = p_1 + p_3, \ x_3 = p_1 + p_2 \) brings equation (3.111) to the form

\[
z_1 T(x_1 + 1) T(x_1 - 1) + z_2 T(x_2 + 1) T(x_2 - 1) + z_3 T(x_3 + 1) T(x_3 - 1) = 0, \quad (3.112)
\]

where \( T(x_1, x_2, x_3) = \tau(p_1, p_2, p_3) \) provided that \( x_i \) and \( p_i \) are connected by the above linear relations. For brevity in (3.112) the arguments which are not shifted are not written explicitly, i.e., for example, \( T(x_1 + 1) = T(x_1 + 1, x_2, x_3) \) and so on. The constants \( z_i \) are subject to the relation \( z_1 + z_2 + z_3 = 0 \) (then \( T = 1 \) is the simplest solution); however, one can consider them as arbitrary or even equal to 1, which is achieved by the simple transformation of the \( T \)-function

\[
T(x_1, x_2, x_3) \longrightarrow z_1^{-x_1^2/2} z_2^{-x_2^2/2} z_3^{-x_3^2/2} T(x_1, x_2, x_3).
\]

It is this form in which this equation was introduced by Hirota in 1981. It was suggested as a universal discretization of soliton equations. Sometimes it is called the \( T \)-system (in analogy with independently suggested \( Y \)-system, which is a consequence of (3.112), see below). The majority (if not all) of known soliton equations are obtained form (3.112) by various reductions, limits, changes of variables.

Note that if \( T(x_1, x_2, x_3) \) is a solution to (3.112), then

\[
f_0(x_1 + x_2 + x_3) f_1(-x_1 + x_2 + x_3) f_2(x_1 - x_2 + x_3) f_3(x_1 + x_2 - x_3) T(x_1, x_2, x_3),
\]

where \( f_i \) are arbitrary functions, is also a solution. Introduce the new function

\[
Y(x_1, x_2, x_3) = \frac{T(x_1, x_2, x_3 + 1) T(x_1, x_2, x_3 - 1)}{T(x_1 + 1, x_2, x_3) T(x_1 - 1, x_2, x_3)} \quad (3.113)
\]

which remains invariant under multiplication of the \( T \)-function by \( f_i \), as above. The Hirota equation for \( T \) implies the following equation for \( Y \):

\[
Y(x_1, x_2 + 1, x_3) Y(x_1, x_2 - 1, x_3) = \frac{(1 + Y(x_1, x_2, x_3 + 1))(1 + Y(x_1, x_2, x_3 - 1))}{(1 + Y^{-1}(x_1 + 1, x_2, x_3))(1 + Y^{-1}(x_1 - 1, x_2, x_3))} \quad (3.114)
\]

which is called the \( Y \)-system.

### 3.9.3 Auxiliary linear problems for the Hirota equation

Let us come back to equation (3.111) and denote for brevity

\[
\tau(p_1 + 1, p_2, p_3) := \tau_1, \quad \tau(p_1, p_2 + 1, p_3) := \tau_2, \quad \tau(p_1 + 1, p_2 + 1, p_3) := \tau_{12}, \quad \ldots
\]

and \( \tau(p_1 + 2, p_2, p_3) := \tau_{11} \) and so on. Let \( \alpha \beta \gamma \) be any cyclic permutation of 123. Consider the following system of three linear equations for the function \( \psi = \psi(p_1, p_2, p_3) \):

\[
\left( e^{\theta_\alpha} + z_\gamma \frac{\tau \tau_{\alpha \beta}}{\tau_\alpha \tau_\beta} \right) \psi = e^{\theta_\beta} \psi, \quad \alpha \beta \gamma = 123, \ 231, \ 312, \quad (3.115)
\]
where $z_{\alpha}$ are some parameters and $\partial_{\alpha} \equiv \partial/\partial_{p_{\alpha}}$. It is not difficult to see that the compatibility of these equations implies the Hirota equation for $\tau$:

$$z_{1}\tau_{123} + z_{2}\tau_{13} + z_{3}\tau_{12} = 0.$$  \hspace{1cm} (3.116)

Indeed, consider, for example, the second and the third equations and rewrite them in the form

$$\psi(p_1 + 1) = \psi(p_3 + 1) + z_2 \frac{\tau_{13}}{\tau_1 \tau_3} \psi,$$

$$\psi(p_2 + 1) = \psi(p_3 + 1) - z_1 \frac{\tau_{23}}{\tau_2 \tau_3} \psi.$$  

These equations allow one to represent the function $\psi(p_1 + 1, p_2 + 1)$ as a linear combination of $\psi(p_3)$, $\psi(p_3 + 1)$, $\psi(p_3 + 2)$ in two different ways. Compatibility of the linear problems means that the results must coincide. Equating to each other the expressions obtained in this way, we see that the terms proportional to $\psi(p_3)$ and $\psi(p_3 + 2)$ cancel identically while the terms proportional to $\psi(p_3 + 1)$ give a non-trivial relation (provided that $\psi(p_3 + 1)$ is not identically zero):

$$z_{1}\tau_{123} \psi_{1} + z_{2}\tau_{13} \psi_{2} + z_{3}\tau_{12} \psi_{3} = 0.$$  \hspace{1cm} (3.118)

Remark. Generally speaking, compatibility of linear problems follows from existence of a continuous family of common solutions. In our case the coefficient functions in the difference operators are such that the compatibility is equivalent to the presence of at least one non-trivial solution.

Introduce the function $\varphi = \psi \tau$, then the linear problems acquire the form

$$\tau_{\gamma} \varphi_{\beta} - \tau_{\beta} \varphi_{\gamma} + z_{\alpha} \tau_{\beta \gamma} \varphi = 0, \hspace{1cm} \alpha \beta \gamma = 123, 231, 312.$$  \hspace{1cm} (3.117)

From the first and the second equations we get

$$\varphi_{2} = \frac{\tau_{13}^2 \varphi_{1} + z_{1}\tau_{13} \varphi}{\varphi_3}, \hspace{1cm} \varphi_{1} = \frac{\tau_{3} \varphi_{2} + z_{2}\tau_{13} \varphi}{\varphi_3}.$$  

Substituting this into the Hirota equation, we obtain yet another linear problem compatible with the other three:

$$z_{1}\tau_{123} \varphi_{1} + z_{2}\tau_{13} \varphi_{2} + z_{3}\tau_{12} \varphi_{3} = 0.$$  \hspace{1cm} (3.118)
All four linear problems can be unified into one matrix equation

\[
\begin{pmatrix}
0 & \tau_3 & -\tau_2 & z_1\tau_{23} \\
-\tau_3 & 0 & \tau_1 & z_2\tau_{13} \\
\tau_2 & -\tau_1 & 0 & z_3\tau_{12} \\
-z_1\tau_{23} & -z_2\tau_{13} & -z_3\tau_{12} & 0
\end{pmatrix}
\begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi
\end{pmatrix} = 0.
\]  
(3.119)

The determinant of the antisymmetric matrix in the left hand side is equal to \((z_1\tau_{23} + z_2\tau_{13} + z_3\tau_{12})^2\). It vanishes if \(\tau\) satisfies the Hirota equation, with the rank of the matrix in this case being equal to 2, hence only two of the four equations are linearly independent.

The system (3.117) can be treated as a system of linear equations for \(\tau\) with the coefficients \(\varphi\). Shifting the variables \(p_\beta \rightarrow p_\beta - 1, p_\gamma \rightarrow p_\gamma - 1\) and changing their sign, \(p_{1,2,3} \rightarrow -p_{1,2,3}\), we see that the form of this system is the same. Since the Hirota equation is invariant under simultaneous change of signs of all variables, the compatibility condition gives the Hirota equation for \(\varphi\) of the same form:

\[
z_1\varphi_{23}\varphi_1 + z_2\varphi_{13}\varphi_2 + z_3\varphi_{12}\varphi_3 = 0.
\]  
(3.120)

Let us pass to the new variables \(x_1, x_2, x_3\) according to the formulas

\[
p_1 = \frac{1}{2}(-\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3),
\]

\[
p_2 = \frac{1}{2}(\varepsilon_1 x_1 - \varepsilon_2 x_2 + \varepsilon_3 x_3),
\]

\[
p_3 = \frac{1}{2}(\varepsilon_1 x_1 + \varepsilon_2 x_2 - \varepsilon_3 x_3).
\]

Here \(\varepsilon_\alpha = \pm 1\) is some fixed set of signs (there are \(2^3 = 8\) possible choices). The inverse transformation is given by

\[
x_1 = \varepsilon_1(p_2 + p_3), \quad x_2 = \varepsilon_2(p_1 + p_3), \quad x_3 = \varepsilon_3(p_1 + p_2).
\]

Introduce the functions \(T(x_1, x_2, x_3) = \tau(p_1, p_2, p_3)\) and \(F(x_1, x_2, x_3) = \varphi(p_1, p_2, p_3)\), with the variables \(x_\alpha\) and \(p_\alpha\) being connected by the above formulas. In the new variables, the system of four linear problems has the form

\[
\begin{pmatrix}
0 & T_{12} & -T_{13} & z_1T_{1123} \\
-T_{12} & 0 & T_{23} & z_2T_{1223} \\
T_{13} & -T_{23} & 0 & z_3T_{1233} \\
-z_1T_{1123} & -z_2T_{1223} & -z_3T_{1233} & 0
\end{pmatrix}
\begin{pmatrix}
F_{23} \\
F_{13} \\
F_{12} \\
F
\end{pmatrix} = 0,
\]  
(3.121)

where \(T_1 \equiv T(x_1+\varepsilon_1, x_2, x_3), T_{12} \equiv T(x_1+\varepsilon_1, x_2+\varepsilon_2, x_3), T_{1123} \equiv T(x_1+2\varepsilon_1, x_2+\varepsilon_2, x_3+\varepsilon_3),\) and so on (and similarly for \(F\)).
Compatibility of these linear problems implies the Hirota equation

\[ z_1 T_{1123} T_{23} + z_2 T_{1223} T_{13} + z_3 T_{1233} T_{12} = 0. \]

After the shift of the variables \( x_\alpha \to x_\alpha - \varepsilon_\alpha \) we get the equation

\[ z_1 T(x_1+\varepsilon_1, x_2, x_3) T(x_1-\varepsilon_1, x_2, x_3) + z_2 T(x_1, x_2+\varepsilon_2, x_3) T(x_1, x_2-\varepsilon_2, x_3) \]
\[ + z_3 T(x_1, x_2, x_3+\varepsilon_3) T(x_1, x_2, x_3-\varepsilon_3) = 0. \]

Note that its form does not depend on the choice of signs \( \varepsilon_\alpha \) and coincides with (3.112). The systems of linear equations (3.121) give difference Backlund transformations for it. However, only four of them (corresponding, for example, to the choices \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1, -\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1, \varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = 1, \varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1 \)) are really different because the simultaneous change of signs means passing to the conjugate system of linear problems in which the roles of \( T \) and \( F \) are interchanged.

### 3.9.4 Continuous (dispersionless) limit of the Hirota equation

The bilinear difference Hirota equation admits different continuous limits. As we know, one of them leads to the KP equation with the whole hierarchy. Here we describe another continuous limit which leads to the dispersionless analogue of the KP hierarchy.

We begin with introducing a small parameter \( \epsilon \) and redefine the times and the tau-function in the following way:

\[ t_k = \frac{T_k}{\epsilon}, \quad \tau(T) = \tau_\epsilon(T). \]  

(3.122)

Introduce also the differential operator

\[ D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{T_k} \]  

(3.123)

which is a “generating function” of the vector fields \( \partial_{T_k} \). Equation (3.107) will acquire the form

\[ (\lambda_2 - \lambda_3) \left( e^{-\epsilon D(\lambda_1)} \tau_\epsilon \right) \left( e^{- \epsilon (D(\lambda_2)+D(\lambda_3))} \tau_\epsilon \right) + (231) + (312) = 0. \]  

(3.124)

The parameter \( \epsilon \) plays the role of the lattice spacing. The dispersionless limit is well-defined on the class of solutions for which a finite limit

\[ F(T) = \lim_{\epsilon \to 0} \epsilon^2 \log \tau_\epsilon(T) \]  

(3.125)

exists, i.e., such that the tau-function \( \tau_\epsilon(T) \) behaves in the limit \( \epsilon \to 0 \) as \( e^{F(T)/\epsilon^2} \), where \( F(T) \) is some function of the variables \( T_k \). If the limit exists, it is sometimes called the dispersionless tau-function although actually it is the limit of its re-scaled logarithm. The tau-function itself in this limit does not make sense because the expression \( \tau_\epsilon = e^{F/\epsilon^2} \) is not defined at \( \epsilon = 0 \). Note that the soliton-like solutions do not have the dispersionless limit (the very existence of a soliton is the effect due to non-zero dispersion).
Problem. Give an example of a solution which has the dispersionless limit.

Substituting $\tau_e = e^{F/\epsilon^2}$ into equation (3.124), we get

$$(\lambda_2 - \lambda_3)e^{e^{-2}e^{-D(\lambda_1)}F}e^{e^{-2}e^{-D(\lambda_2) + D(\lambda_3)}F} + (31) + (312) = 0.$$ 

The limit $\epsilon \to 0$ yields the following equation for $F$:

$$(\lambda_1 - \lambda_2)e^{D(\lambda_1)D(\lambda_2)F} + (\lambda_2 - \lambda_3)e^{D(\lambda_2)D(\lambda_3)F} + (\lambda_3 - \lambda_1)e^{D(\lambda_1)D(\lambda_3)F} = 0,$$ 

which is called the dispersionless analogue of the KP hierarchy in the Hirota form.

Exercise. Give a detailed derivation of equation (3.126).

Tending $\lambda_3 \to \infty$ we will have:

$$((\lambda_1 - \lambda_2)\left(1 - e^{D(\lambda_1)D(\lambda_2)F}\right) = \left(D(\lambda_1) - D(\lambda_2)\right)\partial_1 F$$

or

$$D(\lambda_1)D(\lambda_2)F = \log \frac{p(\lambda_1) - p(\lambda_2)}{\lambda_1 - \lambda_2},$$

where

$$p(\lambda) = \lambda - \sum_{k \geq 1} \frac{\lambda^{-k}}{k} F_{1k}, \quad F_{ik} := \partial_{\lambda_i} \partial_{\lambda_k} F.$$ 

In fact the relation (3.127) is equivalent to (3.126), since the latter is obtained from the former by applying $\exp$, multiplying by $\lambda_1 - \lambda_2$ and summing three such equations (for the pairs $\{\lambda_1, \lambda_2\}$, $\{\lambda_2, \lambda_3\}$ and $\{\lambda_3, \lambda_1\}$). It is seen from (3.127) that these relations express $F_{ij}$ with $i, j \geq 2$ through $F_{1j}$, $j \geq 1$. 

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4 Toda lattice hierarchy

The Toda lattice hierarchy, or, more precisely, 2D Toda lattice hierarchy (2DTL) is “twice as large” as the KP hierarchy. It has the double infinite set of times
\[ \{ \ldots t_{-3}, t_{-2}, t_{-1}, t_0, t_1, t_2, t_3, \ldots \} , \]
with the zeroth time, \( t_0 = n \), taking integer values. The KP hierarchy can be embedded into the 2DTL one in the sense that when all the non-positive times are frozen, the dependent variables as functions of the other times satisfy the equations of the KP hierarchy. In this section we briefly discuss the main points of the theory of the 2DTL hierarchy.

4.1 Commutation representations of the 2DTL hierarchy

The 2DTL hierarchy has two Lax operators, \( L \) and \( \bar{L} \) (we use the same notation as for the Lax operator of the KP hierarchy and hope that this will not lead to a confusion because they do not mix). They are pseudo-difference operators, i.e. infinite series in powers of the shift operator \( e^{\partial_n} \) acting on a function \( f(n) \) as \( e^{\pm \partial_n} f(n) = f(n \pm 1) \):
\begin{align*}
L &= e^{\partial_n} + u_0 + u_1 e^{-\partial_n} + u_2 e^{-2\partial_n} + \ldots , \\
\bar{L} &= ce^{-\partial_n} + \bar{u}_0 + \bar{u}_1 e^{\partial_n} + \bar{u}_2 e^{2\partial_n} + \ldots .
\end{align*}
(4.1)

Here the coefficients \( c, u_i, \bar{u}_i \) are functions of \( n \) and all the higher times (\( \bar{u}_i \) is not a complex conjugate of \( u_i \)).

On the algebra of pseudo-difference operators we have two truncation operations, \( (\ldots)_{\geq 0} \) and \( (\ldots)_{< 0} \) defined by
\[ (\sum_{k \in \mathbb{Z}} a_k e^{k\partial_n})_{\geq 0} = \sum_{k \geq 0} a_k e^{k\partial_n}, \quad (\sum_{k \in \mathbb{Z}} a_k e^{k\partial_n})_{< 0} = \sum_{k < 0} a_k e^{k\partial_n}. \]

The 2DTL hierarchy is the system of differential-difference equations for the coefficient functions of the Lax operators that follow from the Lax equations
\[ \partial_j L = [B_j, L], \quad \partial_j \bar{L} = [B_j, \bar{L}], \quad j = \pm 1, \pm 2, \ldots , \]
(4.2)
where the difference operators \( B_j \) are
\[ B_j = (L^j)_{\geq 0}, \quad B_{-j} = (L^j)_{< 0}, \quad j \geq 1. \]
(4.3)
For example, \( B_1 = e^{\partial_n} + u_0(n) \), \( B_{-1} = c(n) e^{-\partial_n} \).

As in the KP case, the Lax representation is equivalent to the zero curvature equations
\[ \partial_j B_k - \partial_k B_j - [B_j, B_k] = 0, \quad j, k = \pm 1, \pm 2, \ldots . \]
(4.4)

**Exercise.** Show that the zero curvature equation at \( j = 1, k = -1 \) gives the system of equations
\[ \begin{cases} 
\partial_t \log c(n) = u_0(n) - u_0(n - 1), \\
\partial_{t-1} u_0(n) = c(n) - c(n + 1).
\end{cases} \]
(4.5)

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Excluding $u_0$ from (4.5), one obtains the following closed second order differential-difference equation for $c(n)$:

$$\partial_t \partial_{t-1} \log c(n) = 2c(n) - c(n+1) - c(n-1) \quad (4.6)$$

which is one of the forms of the 2D Toda equation. In terms of the function $\varphi_n$ introduced through the relation $c(n) = e^{\varphi_n - \varphi_{n-1}}$ it acquires the most familiar form

$$\partial_t \partial_{t-1} \varphi_n = e^{\varphi_{n+1} - \varphi_n} - e^{\varphi_n - \varphi_{n-1}}. \quad (4.7)$$

Let us point out two important reductions of the 2DTL equation. The Toda chain equation is the 1D reduction of the 2DTL such that $(\partial_t + \partial_{t-1}) c(n) = (\partial_t + \partial_{t-1}) u_0(n) = 0$. The Toda chain equation reads

$$\partial_{t-1}^2 \varphi_n = e^{\varphi_n + 1} - e^{\varphi_n - \varphi_{n-1}}. \quad (4.8)$$

The 2-periodic constraint $\varphi_{n+2} = \varphi_n$ leads to the sine-Gordon equation in “light cone coordinates”

$$\partial_t \partial_{t-1} \varphi = 4 \sin \varphi \quad (4.9)$$

for $\varphi = i(\varphi_0 - \varphi_1)$.

The form of $L, \bar{L}$ can be made more symmetric if one applies a “gauge transformation”

$$L \mapsto L^G = G^{-1} L G, \quad \bar{L} \mapsto \bar{L}^G = G^{-1} \bar{L} G,$$

with a function $G(n) = e^{\alpha \varphi_n}$. At $\alpha = \frac{1}{2}$ we have the gauge-transformed operators

$$L = u_{-1} e^{\varphi_n} + u_0 + u_1 e^{-\varphi_n} + u_2 e^{-2\varphi_n} + \ldots,$$

$$\bar{L} = \bar{u}_{-1} e^{-\varphi_n} + \bar{u}_0 + \bar{u}_1 e^{\varphi_n} + \bar{u}_2 e^{2\varphi_n} + \ldots$$

with $u_{-1}(n) = \bar{u}_{-1}(n+1)$ and

$$B_j = (L^j)_{>0} + \frac{1}{2} (L^j)_0, \quad B_{-j} = (\bar{L}^j)_{<0} + \frac{1}{2} (\bar{L}^j)_0,$$

with the obvious definition of the truncation operations $(\ldots)_{>0}$ and $(\ldots)_0$. In what follows we use the original $\alpha = 0$ gauge.

### 4.2 Conserved quantities

We call the residue $\text{res}_{e^\varphi}$ of a pseudo-difference operator $P = \sum_k p_k(n) e^{k\varphi_n}$ the coefficient $p_0$: $\text{res}_{e^\varphi}P = (P)_0 = p_0$. The following proposition is a difference counterpart of the corresponding property of the pseudo-differential operators.

**Proposition.** The residue of the commutator of pseudo-difference operators is a “full difference”:

$$\text{res}_{e^\varphi}[P, Q] = \Delta C, \quad \Delta := e^{\varphi_n} - 1,$$

where $C$ is a polynomial of coefficients of the operators $P, Q$. 


The proof is elementary.

The densities of conserved quantities for the 2DTL hierarchy are \( \text{res}_{e^\partial} L^k = (L^k)_0 \), \( k \geq 1 \), i.e., the conserved quantities are

\[
J_k = \sum_{n=-\infty}^{\infty} \text{res}_{e^\partial} L^k.
\]

Indeed,

\[
\partial_t J_k = \partial_t \sum_{n=-\infty}^{\infty} \text{res}_{e^\partial} (L^k) = \sum_{n=-\infty}^{\infty} \text{res}_{e^\partial} [B_k, \{L^k\}] = \sum_{n=-\infty}^{\infty} \Delta C(n)
\]

which is zero for rapidly decreasing and periodic solutions. A similar series of conserved quantities exists for the second Lax operator \( \bar{L} \).

### 4.3 Dressing operators and \( \psi \)-functions

Similarly to the KP case, the dressing operators are pseudo-difference operators of the form

\[
W = 1 + w_1 e^{-\partial_0} + w_2 e^{-2\partial_0} + \ldots,
\]

\[
\bar{W} = \bar{w}_0 + \bar{w}_1 e^{\partial_0} + \bar{w}_2 e^{2\partial_0} + \ldots
\]

such that

\[
L = W e^{\partial_0} W^{-1}, \quad \bar{L} = \bar{W} e^{-\partial_0} \bar{W}^{-1}.
\]

The equations of motion for the dressing operators are

\[
\partial_t W = B_j W - W e^{j\partial_0}, \quad \partial_{t_j} W = B_{-j} W, \quad j \geq 1,
\]

\[
\partial_{t_{-j}} \bar{W} = B_{-j} \bar{W} - W e^{-j\partial_0}, \quad \partial_{t_j} \bar{W} = B_j \bar{W}, \quad j \geq 1,
\]

Accordingly, there are two \( \psi \)-functions, \( \psi \) and \( \bar{\psi} \), depending on the spectral parameter \( z \), which are obtained by applying the dressing operators to the functions \( z^n e^{\xi(t_+, z)} \) and \( z^n e^{\xi(t_-, z^{-1})} \):

\[
\psi = W z^n e^{\xi(t_+, z)} = z^n e^{\xi(t_+, z)} \left( 1 + w_1 z^{-1} + w_2 z^{-2} + \ldots \right),
\]

\[
\bar{\psi} = \bar{W} z^n e^{\xi(t_-, z^{-1})} = z^n e^{\xi(t_-, z^{-1})} \left( \bar{w}_0 + \bar{w}_1 z + \bar{w}_2 z^2 + \ldots \right),
\]

where \( t_{\pm} = \{t_{\pm 1}, t_{\pm 2}, t_{\pm 3}, \ldots \} \). Here \( \psi \) should be understood as a series around \( z = \infty \) while \( \bar{\psi} \) as a series around \( z = 0 \). In fact for algebro-geometric solutions \( \psi \) and \( \bar{\psi} \) are expansions around \( z = \infty \) and \( z = 0 \) of one and the same function on a Riemann surface (the Baker-Akhiezer function). The functions \( \psi, \bar{\psi} \) satisfy the linear equations

\[
\partial_j \psi = B_j \psi, \quad \partial_j \bar{\psi} = B_j \bar{\psi}, \quad j = \pm 1, \pm 2, \ldots
\]

and

\[
L \psi = z \psi, \quad \bar{L} \bar{\psi} = z^{-1} \bar{\psi}.
\]
Their compatibility is equivalent to the zero curvature equations and the Lax equations. The simplest linear equations are

\[ \partial_t \psi(n) = \psi(n + 1) + u_0(n)\psi(n), \quad \partial_{t_{-1}} \psi(n) = c(n)\psi(n - 1). \quad (4.17) \]

Along with the \( \psi \)-functions \( \psi, \bar{\psi} \), one may introduce the adjoint functions \( \psi^*, \bar{\psi}^* \):

\[ \psi^* = (W^\dagger)^{-1}z^{-n}e^{-\xi(t_+, z)}, \quad \bar{\psi}^* = (\bar{W}^\dagger)^{-1}z^{-n}e^{-\xi(t_+, z)}, \quad (4.18) \]

where \( (f(n)e^{k\partial_n})^\dagger = e^{-k\partial_n}f(n) \).

### 4.4 The tau-function of the 2DTL hierarchy

The tau-function \( \tau_n(t_+, t_-) \) of the 2DTL hierarchy can be introduced by the formulas similar to (3.95):

\[ \psi(n) = z^n e^{\xi(t_+, z)} \frac{\tau_n(t_+ - [z^{-1}], t_-)}{\tau_n(t_+, t_-)}, \quad \psi^*(n) = z^{-n} e^{-\xi(t_+, z)} \frac{\tau_n(t_+ + [z^{-1}], t_-)}{\tau_n(t_+, t_-)}, \quad (4.19) \]

\[ \bar{\psi}(n) = z^n e^{\xi(t_-, z^{-1})} \frac{\tau_{n+1}(t_+, t_- - [z])}{\tau_n(t_+, t_-)}, \quad \bar{\psi}^*(n) = z^{-n} e^{-\xi(t_-, z^{-1})} \frac{\tau_{n-1}(t_+, t_- + [z])}{\tau_n(t_+, t_-)}. \quad (4.20) \]

All equations of the 2DTL hierarchy are encoded in the bilinear relation

\[ \oint_{\mathbb{C}} z^{n-n'}e^{\xi(t_+-t'_+, z)}\tau_n(t_+ - [z^{-1}], t_-)\tau_{n'}(t'_+ + [z^{-1}], t'_-) \, dz \]

\[ = \oint_{\mathbb{C}} z^{-n+n'}e^{\xi(t_--t'_-, z)}\tau_{n+1}(t_+, t_- - [z^{-1}])\tau_{n'-1}(t'_+, t'_- + [z^{-1}]) \, z^{-2} \, dz \quad (4.21) \]

valid for any sets \( t_+, t_-, t'_+, t'_- \) and integers \( n, n' \). Note that at \( n = n' \), \( t_- = t'_- \) the right hand side vanishes and the bilinear relation reduces to the one for the KP hierarchy (3.104) for the set of “positive” times. Similarly, at \( n - n' = -2 \), \( t_+ = t'_+ \) the left hand side vanishes and we get the bilinear relation for the KP hierarchy for the set of “negative” times.

Taking \( n = n' \), \( t'_+ = t_+ - [a^{-1}] \), \( t'_- = t_- - [b^{-1}] \) and noting that \( e^{\xi(t_+-t'_+, z)} = \left(1 - \frac{z}{a}\right)^{-1} \), \( e^{\xi(t_--t'_-, z)} = \left(1 - \frac{z}{b}\right)^{-1} \), one finds from (4.21) with the help of residue calculus:

\[ \tau_n(t_+ - [a^{-1}], t_-)\tau_n(t_+, t_- - [b^{-1}]) - \tau_n(t_+, t_-)\tau_n(t_+ - [a^{-1}], t_- - [b^{-1}]) \]

\[ = (ab)^{-1}\tau_{n+1}(t_+, t_- - [b^{-1}])\tau_{n-1}(t_+ - [a^{-1}], t_-). \quad (4.22) \]

The simplest tau-function obeying this equation is

\[ \tau_n(t_+, t_-) = \exp\left(-\sum_{k \geq 1} kt_k t_{-k}\right). \]

It corresponds to the trivial solution.
The 2D Toda equation itself is obtained from (4.22) as a coefficient at \((ab)^{-1}\) in the expansion as \(a, b \to \infty\):
\[
\partial_t \partial_{\bar{t}} \log \tau_n = - \frac{\tau_n + 1}{\tau_n^2} \frac{\tau_n + 1}{\tau_n} \tau_n - 1 \tau_n^2.
\]
(4.23)

The original variables \(c(n), u_0(n)\) are connected with the tau-function by the formulas:
\[
c(n) = \frac{\tau_n + 1}{\tau_n^2} \tau_n - 1 \tau_n, \quad u_0(n) = \partial_{\bar{t}} \log \frac{\tau_n + 1}{\tau_n}.
\]
(4.24)

### 4.5 Solutions to the 2DTL hierarchy

The 2DTL hierarchy has different classes of solutions which are analogues of the corresponding classes for the KP hierarchy. Below we give some details on the soliton solutions and elliptic solutions.

#### 4.5.1 Soliton solutions

The soliton solutions to the KP hierarchy can be extended to the 2DTL hierarchy. Here we present the 2DTL analogues of the determinant representations (3.24) and (3.26). It is convenient to redefine the tau-function:
\[
\tau'_n(t_+, t_-) = \exp \left( \sum_{k \geq 1} k t_k t_{-k} \right) \tau_n(t_+, t_-).
\]

The analogue of (3.24) is
\[
\tau'_n(t_+, t_-) = \det_{N \times N} \left( \xi(t_+, q_i) + \xi(t_-, q_i^{-1}) q_i^{n-j} + b_i \xi(t_+, p_i) + \xi(t_-, p_i^{-1}) p_i^{n-j} \right).
\]
(4.25)

The analogue of (3.26) is
\[
\tau'_n(t_+, t_-) = \det_{N \times N} \left( \delta_{ij} + \frac{a_k q_k}{q_k - p_i} \left( \frac{p_i}{q_k} \right)^n e^{\xi(t_+, p_i) - \xi(t_+, q_k)} + \xi(t_-, p_i^{-1}) - \xi(t_-, q_k^{-1}) \right).
\]
(4.26)

These two forms are in fact equivalent (i.e. differ by an exponential function of a linear form in times and by a constant factor). The structure of the right hand side of (4.26) is as follows:
\[
\tau'_n(t_+, t_-) = 1 + \sum_i e^{n_i} + \sum_{i < j} c_{ij} e^{n_i + n_j} + \sum_{i < j < k} c_{ijk} c_{jik} e^{n_i + n_j + n_k} + \ldots,
\]
(4.27)

where
\[
e^{n_i} = \frac{a_i q_i}{q_i - p_i} \left( \frac{p_i}{q_k} \right)^n e^{\xi(t_+, p_i) - \xi(t_+, q_k)} + \xi(t_-, p_i^{-1}) - \xi(t_-, q_k^{-1}),
\]
\[
c_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)}.
\]

The more general soliton-like solutions like the ones discussed in section 3.6.2 also exist.
4.5.2 Elliptic solutions and the Ruijsenaars-Schneider system

In this section we study solutions to the Toda lattice equation for which \(c(n), u_0(n)\) are elliptic (double-periodic) functions of \(x = n\eta\) (\(\eta\) is an arbitrary parameter). We denote them by \(c(x), u_0(x)\). For such solutions the tau-function has the form

\[
\tau(x) = Ce^{cx^2 + \gamma x t_1 + \gamma t_1} \prod_{i=1}^{N} \sigma(x - x_i),
\]

then

\[
c(x) = e^{2\gamma t_1} \prod_{k} \frac{\sigma(x - x_k + \eta)\sigma(x - x_k - \eta)}{\sigma^2(x - x_k)},
\]

\[
u_0(x) = \sum_{k} \hat{x}_k \left(\zeta(x - x_k) - \zeta(x - x_k + \eta)\right) + r\eta,
\]

where \(\hat{x}_k = \partial_{x_k} x_k\).

We will investigate the dynamics of poles as functions of the time \(t_1\). To this end, it is enough to solve the first linear problem in (4.17)

\[
\partial_{t_1} \psi(x) = \psi(x + \eta) + u_0(x) \psi(x)
\]

with \(u_0(x)\) as above and the following pole ansatz for the \(\psi\)-function similar to (3.62):

\[
\psi = z^{x/\eta} e^{t_1 z} \sum_{i=1}^{N} c_i \Phi(x - x_i, \lambda).
\]

Here \(\Phi\) is the same function (3.63) as in section 3.6.6. Substituting (4.29) into (4.17), we get:

\[
z \sum_{i} c_i \Phi(x - x_i) + \sum_{i} \hat{c}_i \Phi(x - x_i) - \sum_{i} c_i \hat{x}_i \Phi'(x - x_i) = z \sum_{i} c_i \Phi(x - x_i + \eta)
\]

\[
+ \left(\sum_{k} \hat{x}_k \left(\zeta(x - x_k) - \zeta(x - x_k + \eta)\right) + r\eta\right) \sum_{i} c_i \Phi(x - x_i).
\]

The second order poles at \(x = x_i\) cancel identically. The cancellation of first order poles at \(x = x_i\) and \(x = x_i - \eta\) leads to the conditions

\[
\begin{align*}
zc_i + \hat{c}_i &= r\eta c_i + \hat{x}_i \sum_{k \neq i} c_k \Phi(x - x_k) + c_i \sum_{k \neq i} \hat{x}_k \left(\zeta(x_i - x_k) - \zeta(x_i - x_k + \eta)\right) \\
zc_i - \hat{x}_i \sum_{k} c_k \Phi(x_i - x_k - \eta) &= 0.
\end{align*}
\]

In the matrix form they read

\[
\begin{align*}
\mathcal{L} c &= z c \\
\dot{c} &= \mathcal{M} c
\end{align*}
\]

with the \(N \times N\) matrices \(\mathcal{L} = \hat{X} A^-, \mathcal{M} = r\eta I + \hat{X} A - \hat{X} A^- + D^0 - D^+\), where \(A\) is the off-diagonal matrix \(A_{ij} = (1 - \delta_{ij}) \Phi(x_i - x_j)\) and

\[
A_{ij}^- = \Phi(x_i - x_j - \eta), \quad D^\pm_{ij} = \delta_{ij} \sum_{k \neq i} \hat{x}_k \zeta(x_i - x_k \pm \eta), \quad D^0_{ij} = \delta_{ij} \sum_{k \neq i} \hat{x}_k \zeta(x_i - x_k).
\]
The compatibility condition of the linear system (4.30) is \( \dot{\mathcal{L}} + [\mathcal{L}, \mathcal{M}] = 0 \). A direct calculation shows that
\[
\dot{\mathcal{L}} + [\mathcal{L}, \mathcal{M}] = (\ddot{X} \dot{X}^{-1} + D^+ + D^- - 2D^0) \mathcal{L},
\]
so the compatibility condition is \( \ddot{X} \dot{X}^{-1} + D^+ + D^- - 2D^0 = 0 \), which implies equations of motion
\[
\ddot{x}_i = - \sum_{k \neq i} \dot{x}_i \dot{x}_k \left( \zeta(x_i - x_k + \eta) + \zeta(x_i - x_k - \eta) - 2\zeta(x_i - x_k) \right)
\]
(4.31)
\[
= \sum_{k \neq i} \dot{x}_i \dot{x}_k \frac{\varphi'(x_i - x_k)}{\varphi(\eta) - \varphi(x_i - x_k)}
\]

Together with their Lax representation. These are equations of motion for the elliptic Ruijsenaars-Schneider \( N \)-body system (a relativistic generalization of the Calogero-Moser system).

It can be directly verified that the Ruijsenaars-Schneider system is Hamiltonian with the Hamiltonian
\[
\mathcal{H} = \sum_i e^{p_i} \prod_{k \neq i} \frac{\sigma(x_i - x_k + \eta)}{\sigma(x_i - x_k)},
\]
(4.32)

where \( p_i, x_i \) are canonical variables. Clearly,
\[
\mathcal{H} = \sum_i \dot{x}_i = \text{const } \text{tr } \mathcal{L}.
\]
(4.33)

It can be also shown that the \( t_{-1} \)-dynamics of poles leads to the same equations of motion (4.31).
5 Tau-functions as vacuum expectation values of fermionic operators

The tau-functions among all functions of infinitely many variables are characterized by the property that they satisfy the set of bilinear Hirota equations. An outstanding discovery of Japanese school (Sato, Jimbo, Miwa and others) is another, equivalent, characterization of tau-functions as vacuum expectation values of special quantum field operators composed of free fermions.

5.1 Fermionic operators

Let us introduce the fermionic operators $\psi_n, \psi^*_n$, $n \in \mathbb{Z}$ with the standard (anti)commutation relations $[\psi_n, \psi_m] = [\psi^*_n, \psi^*_m] = 0$, $[\psi_n, \psi^*_m] = \delta_{mn}$. They generate the infinite dimensional Clifford algebra. We will also use their Fourier transforms

$$\psi(z) = \sum_{k \in \mathbb{Z}} \psi_k z^k, \quad \psi^*(z) = \sum_{k \in \mathbb{Z}} \psi^*_k z^{-k} \quad (5.1)$$

which have the meaning of Fermi fields in the complex plane of the variable $z$. (We hope that the standard notation $\psi(z)$ for the Fermi field will not lead to a confusion with the Baker-Akhiezer function $\psi(t, z)$.)

From the fact that the anti-commutator of any linear combinations of the fermionic operators is a number, it follows that the commutator of any bilinear expressions in $\psi_n$ and $\psi^*_n$ is again bilinear in $\psi_n$ and $\psi^*_n$. For example,

$$[\psi_m \psi^*_n, \psi_{m'} \psi^*_{n'}] = \delta_{nm'} \psi_m \psi^*_{n'} - \delta_{mn'} \psi_{m'} \psi^*_n.$$

We see that the expressions $\psi_m \psi^*_n$ commute in the same way as matrices $E^{(mn)}$ with matrix elements $E_{ij}^{(mn)} = \delta_{im} \delta_{jn}$, which are generators of the algebra $gl(\infty)$ of infinite matrices with only finite (but arbitrary) number of nonzero elements. More generally, consider a bilinear expression

$$X_A = \sum_{ij} A_{ij} \psi_i \psi^*_j \quad (5.2)$$

with some matrix $A$, then $[X_A, X_B] = X_{[A,B]}$, and

$$[X_A, \psi_n] = \sum_i A_{in} \psi_i, \quad [X_A, \psi^*_n] = - \sum_i A_{ni} \psi^*_i.$$

In order to find the adjoint action of $e^{X_A}$ on fermions, we apply the useful formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \ldots \quad (5.3)$$

which is valid for any two operators $A, B$.

**Problem.** Prove (5.3). (Hint: consider $C(t) = e^{tA} B e^{-tA}$ and show that the Taylor expansions in $t$ of the both sides of (5.3) coincide.)
Using (5.3), we get:
\[ e^{X_A} \psi_n e^{-X_A} = \sum_i \psi_i R_{in}, \]  
\[ e^{X_A} \psi_n^* e^{-X_A} = \sum_i (R^{-1})_{ni} \psi_i^*, \] (5.4)
where the matrix \( R \) is connected with \( A \) by the relation \( R = e^A \).

Thus exponential functions of expressions bilinear in \( \psi_n \) and \( \psi_n^* \) possess a rather special property: the result of their adjoint action on linear combinations of \( \psi_n \)'s (or \( \psi_n^* \)'s) are again linear. One can say that elements \( G \) of the form
\[ G = \exp \left( \sum_{ij} A_{ij} \psi_i \psi_j^* \right) \] (5.5)
perform a linear transformation in the space of fermionic operators:
\[ G \psi_n = \sum_i \psi_i G R_{in}, \]
\[ \psi_n^* G = \sum_i G \psi_i^* R_{ni}. \] (5.6)
Note that the operator \( \psi_n^* \) transforms with the matrix which is inverse to the transposed matrix of the transformation for \( \psi_n \).

**Problem.** Show that the operators (5.5) obey the group composition law: \( e^{X_A} e^{X_B} = e^{X_C} \), where the matrix \( C \) is defined by the relation \( e^C = e^A e^B \).

We will call \( G \) of the form (5.5) an element of the Clifford group \( \mathcal{G}_{\text{Cliff}} \) (this name is not quite precise because the Clifford algebra consists of all, not only bilinear, combinations of the fermionic operators; however, this name was used in the original papers by Sato, Jimbo, Miwa and others). The Clifford group is isomorphic to the infinite dimensional group \( GL(\infty) \). With some reservations, the definition of the Clifford group can be extended to matrices \( A_{ij} \) in (5.3) having infinite number of nonzero elements (but such that \( A_{ij} = 0 \) for sufficiently large \( |i - j| \)).

### 5.2 The space of states and the basis

Introduce the vacuum state \(|0\rangle \) (“Dirac sea”), in which all one-fermion states with negative (positive) \( n \) are free (occupied):
\[ \psi_n |0\rangle = 0, \quad n < 0; \quad \psi_n^* |0\rangle = 0, \quad n \geq 0 \]
(for brevity we call indices \( n \geq 0 \) positive). With respect to this vacuum, \( \psi_n \) with \( n < 0 \) and \( \psi_n^* \) with \( n \geq 0 \) are annihilation operators while \( \psi_n^* \) with \( n < 0 \) and \( \psi_n \) with \( n \geq 0 \) are creation operators of a particle and a hole respectively. One can also define the “shifted” Dirac vacuum \(|n\rangle\):
\[ |n\rangle = \begin{cases} 
\psi_{n-1} \ldots \psi_1 \psi_0 |0\rangle, & n > 0 \\
\psi_n^* \ldots \psi_{-2} \psi_{-1}^* |0\rangle, & n < 0.
\end{cases} \]
The dual vacuum state (vector from the dual Hilbert space) is such that
\[ \langle 0 | \psi^*_n = 0, \quad n < 0; \quad \langle 0 | \psi_n = 0, \quad n \geq 0 \]
and
\[ \langle n | = \begin{cases} \langle 0 | \psi^*_0 \psi^*_1 \cdots \psi^*_n, & n > 0 \\ \langle 0 | \psi^*_1 \psi^*_2 \cdots \psi^*_n, & n < 0. \end{cases} \]
We have:
\[ \psi_m | n \rangle = 0, \quad m < n; \quad \psi^*_m | n \rangle = 0, \quad m \geq n, \]
\[ \langle n | \psi_m = 0, \quad m \geq n; \quad \langle n | \psi^*_m = 0, \quad m < n \]
and also
\[ \psi_n | n \rangle = | n + 1 \rangle, \quad \psi^*_n | n + 1 \rangle = | n \rangle, \]
\[ \langle n + 1 | \psi_n = \langle n | \quad \langle n | \psi^*_n = \langle n + 1 |. \]

As a basis of the Hilbert space \( \mathcal{H}_F \) of states in the theory of free fermions one can take all states which are obtained from the vacuum \( |0\rangle \) by application of a finite number of creation operators (of particles and holes). Let a particle (created by \( \psi^*_n \)) carry the charge \( -1 \), and a hole (created by \( \psi_n \)) carry the charge \( +1 \). Then all basis states have a definite charge equal to the difference between the numbers of holes and particles.

Let us give a more precise definition. The basis states \( |\lambda, n\rangle \) are parametrized by the integer number \( n \) and the Young diagram \( \lambda \) in the following way. Let \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) \( \ell = \ell(\lambda) \) be a Young diagram with nonzero rows \( \lambda_1, \ldots, \lambda_\ell \). In the Frobenius notation \( \lambda = (\alpha | \beta) = (\alpha_1, \ldots, \alpha_{d(\lambda)} | \beta_1, \ldots, \beta_{d(\lambda)}) \), where \( d(\lambda) \) is the number of boxes on the main diagonal and \( \alpha_i = \lambda_i - i, \beta_i = \lambda'_i - i \). Here \( \lambda' \) is the transposed diagram (reflected with respect to the main diagonal). Then the basis states and their dual are defined as
\[ |\lambda, n\rangle := \psi^*_{\lambda_{i-1}} \cdots \psi^*_{\lambda_{d(\lambda)-1}} \psi_{\lambda d(\lambda)} \cdots \psi_{\lambda+\alpha_1} | n\rangle, \]
\[ \langle \lambda, n | := \langle n | \psi^*_{\lambda_{d(\lambda)}} \cdots \psi^*_{\lambda_{d(\lambda)+1}} \psi_{\lambda_{d(\lambda)}+\alpha_1} \cdots \psi_{\lambda_{d(\lambda)+\beta_1}}. \]

The state \( |\lambda, n\rangle \) has charge \( n \). For the empty diagram we put \( \langle \emptyset, n | = \langle n |, \quad |\emptyset, n\rangle = | n\rangle \).

### 5.3 Vacuum expectation values

The vacuum expectation value \( \langle 0 | \cdots | 0 \rangle \) is the Hermitian linear form on the Clifford algebra. It is defined by the following properties: \( \langle 0 | 0 \rangle = 1, \langle 0 | \psi_n \psi_m | 0 \rangle = \langle 0 | \psi^*_n \psi^*_m | 0 \rangle = 0 \) for all \( m, n \), and
\[ \langle 0 | \psi_n \psi^*_m | 0 \rangle = \delta_{mn} \text{ for } m < 0, \quad \langle 0 | \psi_n \psi^*_m | 0 \rangle = 0 \text{ for } m \geq 0. \]

The vacuum expectation value of an operator with nonzero charge is equal to 0.

**Exercise.** Using the commutation relations for the fermionic operators and the definition of the shifted vacuum, show that \( \langle n | n \rangle = 1 \) and
\[ \langle n | \psi_i \psi^*_j | n \rangle = \delta_{ij} \text{ at } j < n, \quad \langle n | \psi_i \psi^*_j | n \rangle = 0 \text{ at } j \geq n. \]
for all \(i, j\) and \(n\).

The scalar product in the fermionic Fock space \(\mathcal{H}_F\) is introduced as the vacuum expectation value of product of the operators creating the states from the vacuum. The basis vectors (5.7) are then orthonormal:

\[
\langle \lambda, n | \mu, m \rangle = \delta_{mn} \delta_{\lambda \mu}.
\]

This can be seen moving the operators \(\psi_{n-\beta_i-1}\) to the right, taking into account that the sequences \(\alpha_1, \alpha_2, \ldots, \alpha_d\) and \(\beta_1, \beta_2, \ldots, \beta_d\) are strictly decreasing.

In general vacuum expectation values of products of fermion operators are given by the Wick theorem. Let \(v_i = \sum_j v_{ij} \psi_j\) (respectively, \(w_i^* = \sum_j w_{ij}^* \psi_j^*\)) be arbitrary linear combinations of the operators \(\psi_j\) (respectively, \(\psi_j^*\)). In the simplest form the Wick theorem states that

\[
\langle n | v_1 \ldots v_m w_m^* \ldots w_1^* | n \rangle = \det_{i,j=1,\ldots,m} \langle n | v_i w_j^* | n \rangle,
\]

\[
\langle n | w_1^* \ldots w_m^* v_m \ldots v_1 | n \rangle = \det_{i,j=1,\ldots,m} \langle n | w_i^* v_j | n \rangle.
\]

This can be proved by induction. We will not prove this now because later we will prove a more general statement.

For the fermion fields \(\psi(z), \psi^*(z)\) we have:

\[
\langle n | \psi^*(\zeta) \psi(z) | n \rangle = \sum_{j,k} \zeta^{-j} z^k \langle n | \psi_j^* \psi_k | n \rangle = \sum_{k \geq n} (z/\zeta)^k = \frac{z^n \zeta^{1-n}}{\zeta - z}
\]

(assuming that \(|\zeta| > |z|\)) and

\[
\langle n | \psi(z) \psi^*(\zeta) | n \rangle = \sum_{j,k} \zeta^{-j} z^k \langle n | \psi_j \psi_k^* | n \rangle = \sum_{k < n} (z/\zeta)^k = \frac{z^n \zeta^{1-n}}{z - \zeta}
\]

(assuming that \(|z| > |\zeta|\)).

**Problem.** Prove the formula

\[
\langle n | \psi^*(\zeta_1) \ldots \psi^*(\zeta_m) \psi(z_m) \ldots \psi(z_1) | n \rangle = \prod_{l=1}^m (z_l/\zeta_l)^n \cdot \det_{i,j} \frac{\zeta_i}{\zeta_i - z_j} = \frac{\prod_{i < i'} (z_i - z_{i'}) \prod_{j > j'} (\zeta_j - \zeta_{j'})}{\prod_{i,j} (\zeta_i - z_j)} \prod_l z_l^n \zeta_l^{1-n}.
\]

### 5.4 Normal ordering

Here we introduce the useful operation of normal ordering. For this, it is necessary to fix a vacuum state. The normal ordering \(\cdot(\ldots)\cdot\) with respect to the Dirac vacuum \(|0\rangle\) is defined as follows: all annihilation operators are moved to the right, all creation operators are moved to the left, taking into account that each time the positions of
two neighboring fermionic operators are interchanged the sign factor \((-1)\) appears. For example: \(\psi_1^* \psi_1 = -\psi_1^* \psi_1\), \(\psi_{-1} \psi_0 = -\psi_0 \psi_{-1}\), \(\psi_2 \psi_1^* \psi_1 \psi_{-2}^* = \psi_2 \psi_1^* \psi_{-2}^* \psi_1\), and so on.

**Problem.** Prove the identity \(e^{\alpha \psi_k \psi_k^*} = 1 + (e^\alpha - 1)\psi_k \psi_k^* = e^{(e^\alpha - 1)\psi_k \psi_k^*}\), \(k \geq 0\).

Under the sign of normal ordering, all fermionic operators, both \(\psi_j\) and \(\psi_j^*\), anti-commute. We stress that the relations of the Clifford algebra are *not valid* under the sign of normal ordering, i.e., for example, \(\psi_1^* \psi_1 \neq (1 - \psi_1^* \psi_1)\).

Using the normal ordering, one can introduce the charge operator \(Q\):

\[
Q = \sum_{k \in \mathbb{Z}} : \psi_k \psi_k^* :
\]

(5.9)

This operator counts the charge of a state: \(Q |\lambda, n\rangle = n |\lambda, n\rangle\), and thus \(\langle \mu, m | Q |\lambda, n\rangle = n \delta_{nm} \delta_{\lambda \mu}\) (without normal ordering this matrix element is ill-defined!). The operator \(Q\) has commutation relations \([Q, \psi_n] = \psi_n\), \([Q, \psi_n^*] = -\psi_n^*\) which mean that \(\psi_n^*\) have charges \(\pm 1\). More generally, we say that an element \(X\) of the Clifford algebra has charge \(q\) if \([Q, X] = qX\).

The definition of normal ordering is closely connected with vacuum expectation value.

**Exercise.** Check that \(\psi_k^* \psi_1 = \psi_k^* \psi_1 - \langle 0 | \psi_k^* \psi_1 | 0 \rangle\).

More generally, for any linear combinations \(f_0, f_1, \ldots, f_m\) of fermionic operators \(\psi_i, \psi_j^*\) the recurrence formula

\[
f_0 : f_1 f_2 \cdots f_m : = : f_0 f_1 f_2 \cdots f_m : + \sum_{j=1}^{m} (-1)^{j-1} \langle 0 | f_0 f_j | 0 \rangle : f_1 f_2 \cdots f_j \cdots f_m :
\]

(5.10)

holds, where \(f_j\) means that this operator should be omitted. This relation allows one to express normally ordered monomials with any number of fermionic operators as linear combinations of monomials without normal ordering and vice versa.

**Problem.** Prove the identity \(e^{\alpha \psi_k \psi_k^*} = e^{(e^\alpha - 1)\psi_k \psi_k^*}\), \(k \geq 0\).

In a similar way, one can define normal ordering with respect to any vacuum. For example, one can consider the empty vacuum \(|\infty\rangle\). With respect to this vacuum, all \(\psi_j\)’s are annihilation operators and all \(\psi_j^*\)’s are creation operators. The corresponding normal ordering will be denoted by \(\times\) \((\cdots)\). Examples: \(\times \psi_m^* \psi_n \times = \psi_m^* \psi_n\), \(\times \psi_n \psi_m^* \times = -\psi_m^* \psi_n\) and

\[
\times \exp \left(\sum_{i,k} B_{ik} \psi_i^* \psi_k \right) \times = 1 + \sum_{i,k} B_{ik} \psi_i^* \psi_k + \frac{1}{2!} \sum_{i,j,k,k'} B_{ik} B_{i'k'} \psi_i^* \psi_{i'} \psi_k \psi_{k'} + \cdots
\]

(5.11)

5.5 Group and quasigroup elements of the Clifford algebra

5.5.1 Group elements

Bilinear combinations \(\sum_{mn} b_{mn} \psi_m^* \psi_n\) with certain conditions on the matrix \(b = (b_{mn})\) form an infinite dimensional Lie algebra. Exponentiating them, one gets an infinite dimensional group, one of the versions of \(GL(\infty)\). Elements of this group can be represented in the form

\[
G = \exp \left(\sum_{i,k \in \mathbb{Z}} b_{ik} \psi_i^* \psi_k \right).
\]

(5.12)
The inverse element has the same form with the matrix \((-b_{ik})\).

As it was already mentioned, the group elements \([5.12]\) have a rather special property: the adjoint action of such element preserves the linear space spanned by the fermionic operators \(\psi_n\) and the same is true for \(\psi_n^*\). More precisely, we have:

\[
G\psi_n^*G^{-1} = \sum_l \psi_l^* R_{ln}, \quad G\psi_n G^{-1} = \sum_l (R^{-1})_{nl} \psi_l
\]
or

\[
G\psi_n^* = \sum_l R_{ln} \psi_l^* G, \quad \psi_n G = \sum_l R_{nl} G \psi_l
\]  
(5.13)

with some matrix \(R = (R_{nl})\).

**Problem.** Prove that \(R = e^b\).

Hence it is clear that the product of group elements is an element of the same form:

\[
\exp\left(\sum_{i,k \in \mathbb{Z}} b_{ik}' \psi_i^* \psi_k \right) \exp\left(\sum_{i,k \in \mathbb{Z}} b_{ik} \psi_i^* \psi_k \right) = \exp\left(\sum_{i,k \in \mathbb{Z}} b_{ik}' \psi_i^* \psi_k \right),
\]
(5.14)

where \(e^be^b = e^{b''}\). It is also clear that the multiplication of \(G\) of the form \([5.12]\) by arbitrary complex number preserves the characteristic property \([5.13]\). From the fact that the center of the Clifford algebra is the field of complex numbers \(\mathbb{C}\) it follows that two group elements \(G, G'\) with the same matrix of linear transformation \(R\) can differ by a \(c\)-number \(c \in \mathbb{C}\) only: \(G' = cG\).

Group elements can be also represented as normally ordered exponential functions of bilinear forms. For example, it is easy to check that \(G = \exp e^{B_{ik} \psi_i^* \psi_k \times} (\text{here and below}} \text{the summation over repeated indices is implied})\) satisfies the first relation in \([5.13]\) with \(R_{ln} = \delta_{ln} + B_{ln}\): \(= \psi_n^* \times e^{B_{ik} \psi_i^* \psi_k \times} \).

\[
\exp \left(\sum_{i,k \in \mathbb{Z}} b_{ik}' \psi_i^* \psi_k \right) \exp \left(\sum_{i,k \in \mathbb{Z}} b_{ik} \psi_i^* \psi_k \right) = \left(1 + B_{a1b1} \psi_{a1}^* \psi_{b1} + \frac{1}{2} B_{a1b1} B_{a2b2} \psi_{a1}^* \psi_{b1} \psi_{a2} \psi_{b2} + \ldots \right) \psi_n^*
\]

\[
= \psi_n^* \times e^{B_{ik} \psi_i^* \psi_k \times} + B_{a1n} \psi_{a1}^* + B_{a1n} B_{a2b2} \psi_{a1}^* \psi_{a2} \psi_{b2} + \frac{1}{2} B_{a1n} B_{a2b2} B_{a3b3} \psi_{a1}^* \psi_{a2} \psi_{a3} \psi_{b3} \psi_{b2} + \ldots
\]

\[
= \psi_n^* \times e^{B_{ik} \psi_i^* \psi_k \times} + B_{a1n} \psi_{a1}^* \times e^{B_{ik} \psi_i^* \psi_k \times} = \left(\delta_{an} + B_{an}\right) \psi_{an}^* \times e^{B_{ik} \psi_i^* \psi_k \times}.
\]

**Exercise.** Prove the second relation in \([5.13]\).

In a similar way, one can show that \(\exp (b_{ik} \psi_i^* \psi_k) = \exp \left((e^b - I)_{ik} \psi_i^* \psi_k\right)\) \(\times\),

(5.15)

where \(I\) is the unity matrix. The composition law has the form

\[
\exp \left(B_{ik} \psi_i^* \psi_k\right) \exp \left(B_{ik} \psi_i^* \psi_k\right) = \exp \left((B + B' + B')_ {ik} \psi_i^* \psi_k\right) \times
\]
(5.16)

This directly follows from the composition law \([5.14]\) and the relation \(B = e^b - I\).

Let us prove another useful formula which allows one to represent the group element as a normal ordered exponent with respect to different vacua:

\[
\exp \left(B_{ik} \psi_i^* \psi_k\right) = \det (I + P_+) \exp \left(A_{ik} \psi_i^* \psi_k\right)
\]
(5.17)
or, equivalently,

$$\exp(A \psi^*_i \psi_k) \cdot \det(I - P A) \cdot \exp(B \psi^*_i \psi_k)$$

(5.18)

Here $P_+$ is the projector to the space of positive modes ($P_{ik} = \delta_{ik}$ at $i, k \geq 0$ and 0 otherwise), and the matrices $A, B$ are connected by the relations

$$B - A = AP_+B, \quad \text{i.e.,} \quad B = (I - AP_+)A \quad \text{or} \quad A = (I + P_+)B^{-1}.$$  

(5.19)

For the proof we first notice that we can write $\exp(A \psi^*_i \psi_k)$ as a composition of three operators:

$$\exp(A \psi^*_i \psi_k) = \exp(A_{ab} \psi^*_i \psi_b) \cdot \exp(A_{ac} \psi^*_a \psi_c) \cdot \exp(A_{bc} \psi^*_b \psi_c),$$

where in the right hand side the summation over repeated positive indices $a, b$ (and negative $\bar{a}, \bar{b}$) is implied. (summation over repeated $i, k$ in the left hand side is over all integers). The operator $G_1$ contains only creation operators while $G_3$ contains only annihilation operators. Note also that the two operators under the sign of normal ordering commute with each other. It is not difficult to find the linear transformations performed by $G_1, G_2, G_3$. For $G_1, G_3$ this is especially easy:

$$\psi_n G_1 = G_1 \begin{cases} \psi_n + A_{ab} \psi_b, & n < 0, \\ \psi_n, & n \geq 0, \end{cases} \quad \psi_n G_3 = G_3 \begin{cases} \psi_n, & n < 0, \\ \psi_n + A_{ab} \psi_b, & n \geq 0. \end{cases}$$

For $G_2$ we write $G_2 = \exp(G_2^+ G_2^-)$ with $G_2^+ = \exp(A_{ab} \psi^*_a \psi_b)$, and $G_2^- = \exp(A_{ab} \psi^*_a \psi_b)$. Moving $\psi_n$ through this element, one can ignore either $G_2^+$ or $G_2^-$, depending on whether $n$ is negative or positive. The remaining calculations are similar to the above calculation with the normally ordered expression $\psi^*_i (\ldots) \psi_i$. It yields:

$$\begin{cases} \psi_n G_2 = G_2(\psi_n + A_{ab} \psi_b), & n < 0, \\ G_2 \psi_n = (\psi_n - A_{ab} \psi_b) G_2, & n \geq 0. \end{cases}$$

It is useful to rewrite these transformations in the block-matrix form:

$$\begin{pmatrix} \psi_{-} G_1 \\ \psi_{+} G_1 \end{pmatrix} = \begin{pmatrix} I & A_{-}^+ \\ 0 & I \end{pmatrix} \begin{pmatrix} G_1 \psi_{-} \\ G_1 \psi_{+} \end{pmatrix},$$

$$\begin{pmatrix} \psi_{-} G_3 \\ \psi_{+} G_3 \end{pmatrix} = \begin{pmatrix} I & 0 \\ A_{-}^+ & I \end{pmatrix} \begin{pmatrix} G_3 \psi_{-} \\ G_3 \psi_{+} \end{pmatrix},$$

$$\begin{pmatrix} \psi_{-} G_2 \\ \psi_{+} G_2 \end{pmatrix} = \begin{pmatrix} I + A_{-}^+ & 0 \\ 0 & (I - A_{-}^+)^{-1} \end{pmatrix} \begin{pmatrix} G_2 \psi_{-} \\ G_2 \psi_{+} \end{pmatrix}$$

(assuming that the matrix $I - A_{-}^+$ is invertible). In this notation $P_+ = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$. The full transformation matrix is obtained as the product of these three:

$$R = \begin{pmatrix} I & A_{-}^+ \\ 0 & I \end{pmatrix} \begin{pmatrix} I + A_{-}^+ & 0 \\ 0 & (I - A_{-}^+)^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ A_{-}^+ & I \end{pmatrix},$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} A_{-}^+ + A_{-}^+(I - A_{-}^+)^{-1}A_{-}^+ & A_{-}^+(I - A_{-}^+)^{-1}A_{-}^- \\ (I - A_{-}^+)^{-1}A_{-}^- & (I - A_{-}^+)^{-1}A_{-}^- \end{pmatrix}.\]
It can be checked that the second matrix in the last line (which is \( R - I = B \)) is precisely \((I - AP_+)^{-1} A\) in agreement with (5.19). It remains to find the scalar multiplier in (5.18). Let us find the vacuum expectation value of the both sides with respect to the bare (empty) vacuum. Then we should show that

\[
\langle \infty | \exp \left( A_{ik} \psi_i^* \psi_k \right) | \infty \rangle = \det(I - P_+ A).
\]

Using the representation of the operator in the left hand side as composition of three, we have:

\[
\langle \infty | e^{A_{ik} \psi_i^* \psi_k} | \infty \rangle = \langle \infty | e^{-A_{ik} \psi_i^* \psi_k} | \infty \rangle
\]

\[
= \sum_{k \geq 0} \frac{(-1)^k}{k!} A_{ai_1} \cdots A_{ai_k} \langle \infty | \psi_{i_1} \cdots \psi_{i_k} \psi_{a_1}^* \cdots \psi_{a_k}^* | \infty \rangle
\]

\[
= \sum_{k \geq 0} \frac{(-1)^k}{k!} \sum_{a_1, \ldots, a_k \geq 0} \begin{vmatrix} A_{ai_1} & A_{ai_2} & \cdots & A_{ai_k} \\ A_{a_1 a_1} & A_{a_1 a_2} & \cdots & A_{a_1 a_k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{a_k a_1} & A_{a_k a_2} & \cdots & A_{a_k a_k} \end{vmatrix}
\]

\[
= \det(I - A^+) = \det(I - P_+ A).
\]

### 5.5.2 Quasigroup elements

The normal ordering allow one to represent in the form (5.4) not only group elements but also some not invertible elements of the Clifford algebra which obey the property

\[
G \psi_n^* = \sum_l R_{ln} \psi_l^* G, \quad \psi_n G = \sum_l R_{nl} G \psi_l, \quad (5.20)
\]

or

\[
\psi_n^* G = \sum_l R'_{ln} G \psi_l^*, \quad G \psi_n = \sum_l R'_{nl} \psi_l G \quad (5.21)
\]

with some (not necessarily invertible) matrices \( R, R' \) (for non-invertible elements only one pair of these relations is satisfied). We call an element \( G \) of the Clifford algebra a quasigroup element if the commutation relations (5.20) or (5.21) hold. If the matrix \( R \) is not invertible, then \( G \) is also not invertible. In this case it can not be written in the exponential form (5.12) but can be written as a normally ordered exponent.

**Example.** Let \( \Psi, \Phi^* \) be arbitrary linear combinations of the fermionic operators \( \psi_n, \psi_n^* \) respectively. Consider the element

\[
G = e^{\beta \Phi^* \Psi} = \chi e^{\alpha \Phi^* \Psi \chi} = 1 + \alpha \Phi^* \Psi = 1 + \alpha \gamma - \alpha \Psi \Phi^*,
\]

where \( \gamma := \langle \infty | \Psi \Phi^* | \infty \rangle \) and \( \alpha, \beta \) are connected by the relation \( e^{\gamma \beta} = 1 + \gamma \alpha \). For almost all \( \alpha \) the element \( G \) is invertible and the two representations (with and without normal ordering) are absolutely equivalent. However, at \( \alpha = -1/\gamma \) (assuming that \( \gamma \neq 0 \)) \( G = \chi e^{\alpha \Phi^* \Psi \chi} \) becomes non-invertible and acquires the form

\[
G = \frac{\Psi \Phi^*}{\langle \infty | \Psi \Phi^* | \infty \rangle}.
\]
5.6 The basic bilinear relation

It easily follows from (5.20) or (5.21) that any quasigroup element satisfies the commutation relation

\[
\sum_{k \in \mathbb{Z}} \psi_k G \otimes \psi_k^* G = \sum_{k \in \mathbb{Z}} G \psi_k \otimes G \psi_k^* \tag{5.22}
\]

which we call the basic bilinear relation (BBR). It means that \( G \otimes G \) commutes with \( \sum_k \psi_k \otimes \psi_k^* \). In terms of matrix elements, the BBR states that

\[
\sum_{k \in \mathbb{Z}} \langle U | \psi_k G | V \rangle \langle U' | \psi_k^* G | V' \rangle = \sum_{k \in \mathbb{Z}} \langle U | G \psi_k | V \rangle \langle U' | G \psi_k^* | V' \rangle \tag{5.23}
\]

for any states \( |V\rangle, |V'\rangle, \langle U\rangle, \langle U'\rangle \) from the spaces \( \mathcal{H}_F \) and its dual. Indeed, substituting (5.20) (or (5.21)) instead of \( \psi_k G \) and \( G \psi_k^* \) in the right and left hand sides of (5.22), we get an identity.

It turns out that the elements \( G \) satisfying the BBR, are not exhausted by normally ordered exponents. For example, it is easy to check that \( G = \psi_n \) and \( G = \psi_n^* \) satisfy (5.22) but they are not normally ordered exponents of anything. Besides, the element \( G = \psi_n \) does not perform a linear transformation of the space with basis \( \psi_k^* \) neither of the form (5.20) nor (5.21). The class of such examples can be significantly enlarged.

Exercise. Let \( \Psi, \Phi^* \) be arbitrary linear combinations of \( \psi_n, \psi_n^* \) respectively. Check that \( G = \Psi \) and \( G = \Phi^* \) satisfy the condition (5.22).

Let us point out two general properties of elements satisfying the BBR.

Proposition. Elements of the Clifford algebra satisfying the BBR (5.22) form a semigroup: if \( G \) and \( G' \) satisfy it, then \( GG' \) does.

This is obvious from (5.22).

Proposition. All solutions to the BBR (5.22) have a definite charge, i.e., \([Q, G] = qG\) with some integer \( q \).

The proof is omitted.

The BBR in the form (5.22) or (5.23) is the basis for what follows. We will extend the notion of quasigroup elements calling quasigroup elements \( all \ solutions \ to \ the \ BBR \). For simplicity we will work, as a rule, with elements \( G \) having zero charge (for example, with normally ordered exponents) but all main statements can be easily extended to the general case.

5.7 The generalized Wick theorem

Vacuum expectation values of products of fermionic operators with insertions of quasigroup elements obey some special properties which are described by the generalized Wick theorem. Let \( v_i = \sum_j v_{ij} \psi_j \) be an arbitrary linear combination of the operators \( \psi_j \) and \( w_i^* = \sum_j w_{ij}^* \psi_j^* \) be an arbitrary linear combination of the operators \( \psi_j^* \).

Proposition (the generalized Wick theorem). Let \( G, G' \) be any two quasigroup elements with zero charge. Then for any \( v_j, w_i^* \) and arbitrary \( n \) such that \( \langle n | G' G | n \rangle \neq 0 \) the
following identity holds:
\[
\frac{\langle n| G'v_1 \ldots v_m w^*_m \ldots w^*_1 G |n \rangle}{\langle n| G'G |n \rangle} = \det_{i,j=1,\ldots,m} \frac{\langle n| G'v_j w^*_j G |n \rangle}{\langle n| G'G |n \rangle}.
\] (5.24)

This statement can be proved by induction. Assume that the identity (5.24) holds for some \(m \geq 1\) (obviously, this is true for \(m = 1\)). Set
\[
\langle U \rangle = \langle n| G'w^*_1, \quad \langle U' \rangle = \langle n| G'v_1 v_2 \ldots v_{m+1} w^*_m \ldots w^*_2 G |n \rangle, \quad |V \rangle = |V' \rangle = |n \rangle.
\]
Substituting this into the BBR (5.23), we see that its right hand side vanishes identically since either \(\psi_k |n \rangle = 0\) or \(\psi^*_k |n \rangle = 0\), and thus
\[
\sum_k \langle n| G'w^*_1 \psi_k G |n \rangle \langle n| G'v_1 \ldots v_{m+1} w^*_m \ldots w^*_2 \psi^*_k G |n \rangle = 0.
\]

Substitute \(w^*_1 \psi_k = w^*_1 k - \psi_k w^*_1\) in the first multiplier and move \(\psi^*_k\) in the second multiplier through the chain of operators \(w^*_j\)'s. In the left hand side, we get
\[
\langle n| G'G |n \rangle \langle n| G'v_1 \ldots v_{m+1} w^*_m \ldots w^*_1 G |n \rangle
- (-1)^m \sum_k \langle n| G'\psi_k w^*_1 G |n \rangle \langle n| G'v_1 \ldots v_{m+1} \psi_k^* w^*_m \ldots w^*_2 G |n \rangle.
\]

Now move \(\psi^*_k\) to the left through the chain of operators \(v_j\) and use at each step the relation \(v_j \psi^*_k = v_{jk} - \psi^*_k v_j\). As a result, we obtain
\[
\langle n| G'G |n \rangle \langle n| G'v_1 \ldots v_{m+1} w^*_m \ldots w^*_1 G |n \rangle
+ \sum_{j=1}^{m+1} (-1)^j \langle n| G'v_j w^*_1 G |n \rangle \langle n| G'v_1 \ldots v_j \ldots w^*_m \ldots w^*_2 G |n \rangle
+ \sum_k \langle n| G'\psi_k w^*_1 G |n \rangle \langle n| G'\psi_k^* v_1 \ldots v_{m+1} w^*_m \ldots w^*_2 G |n \rangle.
\]

In the last line we can again use the BBR, to bring the last line to the form
\[
\sum_k \langle n| \psi_k G'w^*_1 G |n \rangle \langle n| \psi^*_k G'v_1 \ldots v_{m+1} w^*_m \ldots w^*_2 G |n \rangle.
\]

This is equal to 0 since again either \(\langle n| \psi_k = 0\) or \(\langle n| \psi^*_k = 0\). Therefore, we come to the relation
\[
\langle n| G'G |n \rangle \langle n| G'v_1 \ldots v_{m+1} w^*_m \ldots w^*_1 G |n \rangle
+ \sum_{j=1}^{m+1} (-1)^j \langle n| G'v_j w^*_1 G |n \rangle \langle n| G'v_1 \ldots v_j \ldots v_{m+1} w^*_m \ldots w^*_2 G |n \rangle = 0
\]
or
\[
\frac{\langle n| G'v_1 \ldots v_{m+1} w^*_m \ldots w^*_1 G |n \rangle}{\langle n| G'G |n \rangle} = \sum_{j=1}^{m+1} (-1)^{j-1} \frac{\langle n| G'v_j w^*_1 G |n \rangle}{\langle n| G'G |n \rangle} \frac{\langle n| G'v_1 \ldots v_{j} \ldots v_{m+1} w^*_m \ldots w^*_2 G |n \rangle}{\langle n| G'G |n \rangle}.
\]
By the assumption of induction, the second ratio in the right hand side is given by the determinant of the $m \times m$ matrix written above. Now notice that the expression in the right hand side is the expansion of the determinant of the corresponding $(m+1) \times (m+1)$ matrix in the first column and thus the proposition is proved.

5.8 The operators $e^{J_+}$

For applications to integrable hierarchies, especially important are group elements of a special form, which we introduce in this section.

Consider the operators

$$J_k = \sum_{j \in \mathbb{Z}} \psi_j \psi_{j+k}^* = \text{res}_z (\psi(z) z^{k-1} \psi^*(z) \cdot)$$

(5.25)

which are Fourier modes of the current operator $J(z) = \psi(z) \psi^*(z) \cdot$. The normal ordering in (5.25) is essential only at $k = 0$, and in this case $J_0$ coincides with the charge operator $Q$ (5.9). If $k \neq 0$, then the normal ordering can be omitted:

$$J_k = \sum_{j \in \mathbb{Z}} \psi_j \psi_{j+k}^*, \quad k \neq 0.$$  

(5.26)

These operators have the form (5.2) with the matrix $A_{ij} = \delta_{i,j-k}$. This matrix has the infinite nonzero diagonal, and one should be careful when working with formal expressions containing infinite sums (see the example of calculation of the commutator below).

Exercise. Show that $[J_k, \psi_m] = \psi_{m-k}$, $[J_k, \psi_m^*] = -\psi_{m+k}^*$.

First consider the operators $J_k$ with positive $k$. Put

$$J_+ = J_+ (t_+) = \sum_{k \geq 1} t_k J_k$$

(5.27)

with arbitrary parameters $t_k$ (they will be identified with the times of the KP hierarchy), the whole set of which will be denoted as $t_+ = \{t_1, t_2, \ldots\}$ or simply $t$ if this does not lead to a confusion.

Exercise. Verify that $J_+ (t_+) \mid 0 \rangle = 0$.

It is easy to check that all $J_k$ with $k \geq 1$ commute with each other and the fermionic fields $\psi(z)$, $\psi^*(z)$ transform diagonally under the action of $e^{J_+}$:

$$e^{J_+ (t)} \psi(z) e^{-J_+ (t)} = e^{\xi(t,z)} \psi(z),$$

$$e^{J_+ (t)} \psi^*(z) e^{-J_+ (t)} = e^{-\xi(t,z)} \psi^*(z).$$

(5.28)

Here we use the previously introduced notation $\xi(t,z) = \sum_{j \geq 1} t_j z^j$. Then it is obvious that the operators $\psi_n$, $\psi_n^*$ transform as follows:

$$e^{J_+ (t)} \psi_n e^{-J_+ (t)} = \sum_{k \geq 0} \psi_{n-k} h_k (t),$$

$$e^{J_+ (t)} \psi_n^* e^{-J_+ (t)} = \sum_{k \geq 0} \psi_{n+k}^* h_k (-t),$$

(5.29)
where the Schur polynomials \( h_k \) are defined by (3.34).

In a similar way, introduce the operator

\[
J = J(t) = \sum_{k \geq 1} t_k J_k
\]

with arbitrary parameters \( t_k \). In what follows, to avoid many signs \( \pm \), we will write simply \( t \) for any half-infinite set of times (implying \( t_+ \) or \( t_- \)).

As in the case of \( J_+ \), it is easy to check that all \( J_k \) with \( k \leq -1 \) commute with each other and the fermionic fields \( \psi(z), \psi^*(z) \) transform diagonally:

\[
e^{J_-(t)} \psi(z) e^{-J_-(t)} = e^{\xi(t, \xi^{-1})} \psi(z),
\]

\[
e^{J_-(t)} \psi^*(z) e^{-J_-(t)} = e^{-\xi(t, \xi^{-1})} \psi^*(z),
\]

which is equivalent to the following action on \( \psi_n, \psi_n^* \):

\[
e^{J_-(t)} \psi_n e^{-J_-(t)} = \sum_{k \geq 0} \psi_{n+k} h_k(t),
\]

\[
e^{J_-(t)} \psi_n^* e^{-J_-(t)} = \sum_{k \geq 0} \psi_{n-k}^* h_k(-t).
\]

Let us find the commutator \([J_k, J_l]\):

\[
[J_k, J_l] = \sum_j [J_k, \psi_j \psi_{j+l}^*] = \sum_j \left( [J_k, \psi_j] \psi_{j+l}^* + \psi_j [J_k, \psi_{j+l}^*] \right).
\]

Using the formulas obtained above, we will have:

\[
[J_k, J_l] = \sum_j \left( \psi_{j-k} \psi_{j+l}^* - \psi_j \psi_{j+k+l}^* \right) \neq 0.
\]

Formally one could shift the summation index in the first sum \( j \to j + k \) and get 0. To check the result, calculate \( \langle 0 | [J_k, J_{-k}] | 0 \rangle \) with \( k > 0 \) “by hands”:

\[
\langle 0 | [J_k, J_{-k}] | 0 \rangle = \langle 0 | J_k J_{-k} | 0 \rangle = \sum_{-k \leq j < 0} \sum_{0 \leq l < k} \langle 0 | \psi_j \psi_{j+k}^* \psi_{l-k}^* | 0 \rangle = \sum_{l=0}^{k-1} \delta_{k-l} = k \neq 0. \quad (!)
\]

Where is the mistake? The matter is that the shift of the summation index is legal only if \( k + l \neq 0 \). In this case the sum of operators in the right hand side is well-defined (and is indeed equal to 0) because its matrix elements between any basis states contain only finite number of terms. If \( k + l = 0 \), then an infinite sum appears which requires an additional definition. In this case it is necessary first to rewrite the right hand side in terms of normally ordered expressions:

\[
[J_k, J_{-k}] = \sum_j \left( \psi_{j-k} \psi_{j+k}^* - \psi_j \psi_{j+k}^* \right) : + \sum_j \left( \theta(j < k) - \theta(j < 0) \right).
\]

Here \( \theta(j < k) = 1 \) if \( j < k \) and 0 otherwise. The normally ordered expressions are well-defined and now the summation index in the first sum can be safely shifted. The remaining terms give the commutation rule for the Fourier modes of the current operator:

\[
[J_k, J_l] = k \delta_{k+l,0}.
\]
It is identical to the commutation rule of bosonic operators.

**Exercise.** Prove the formulas

\[ [J_+(t_+), J_-(t_-)] = \sum_{k \geq 1} k t_k t_{-k}, \]

\[ e^{J_+(t_+)} e^{J_-(t_-)} = \exp \left( \sum_{k \geq 1} k t_k t_{-k} \right) e^{J_-(t_-)} e^{J_+(t_+)} \].

(5.35)

**Exercise.** Calculate \( \langle 0 | e^{J_+(t)} \times e^{(\psi_{-1} - \psi_1)(\psi^*_1 - \psi^*_3)} \times |0 \rangle \).

**Problem.** Find \( e^{tH_1} \psi(z) e^{-tH_1} \) and \( e^{tH_1} \psi^*(z) e^{-tH_1} \), where \( H_1 = \sum_{k \in \mathbb{Z}} k \delta_k \psi_k \psi^*_k \); and calculate \( \langle n | e^{J_+(t_+)} e^{tH_1} e^{-J_-(t_-)} |n \rangle \).

**5.9 Boson-fermion correspondence**

**5.9.1 Bosonization rules**

As we have just seen, the Fourier modes \( J_k \) of the current operator have the same commutation relations as the oscillator modes of operators of free bosonic field: \( [J_k, J_l] = k \delta_{k+l,0} \), i.e. \( J_k \) and \( J_{-k}/k \) are canonically conjugated. The operator \( J_0 = Q \) plays a special role. Introduce the operator \( P \) canonically conjugated to \( Q \); the operator \( e^P \) is the operator of shift of the index:

\[ e^P \psi_n e^{-P} = \psi_{n+1}, \quad e^P \psi^*_n e^{-P} = \psi^*_{n+1}. \]

On the vacuum states \( e^\pm P |n\rangle = |n \pm 1\rangle \). One can check that this definition is indeed equivalent to the commutation relation \( [Q, P] = 1 \).

Introduce the chiral bosonic field

\[ \phi(z) = \sum_{k>0} \frac{J_k}{k} z^k + P + Q \log z - \sum_{k>0} \frac{J_k}{k} z^{-k} \]

\( = J_-([z]) + P + J_0 \log z - J_+([z^{-1}]). \)

In the last line we use the notation \( J_\pm([z]) = J_{\pm 1} z + \frac{1}{2} J_{\pm 2} z^2 + \frac{1}{3} J_{\pm 3} z^3 + \ldots \). We have \( z \partial_z \phi(z) = J(z) \) or

\[ \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} J(z) \frac{dz}{z}. \]

The operators \( J_{\pm k} \) \( k > 0 \) annihilate the right vacuum (they are bosonic annihilation operators) while \( J_{-k} \) with \( k > 0 \) annihilate the left vacuum (they are bosonic creation operators). Introduce the bosonic normal ordering \( ^\ast (...) \) with the same rule that all creation operators are moved to the left and all annihilation operators are moved to the right and consider normally ordered exponents of the bosonic fields:

\[ ^\ast e^{\phi(z)} = e^{J_-([z])} e^{P} z Q e^{-J_+(|[z^{-1}])}, \]

\[ ^\ast e^{-\phi(z)} = e^{-J_-(|[z])} z^{-Q} e^{-P} e^{J_+(|[z^{-1}])} \]

(5.37)
(the normal ordering affects only the operators with $J_k \neq 0$). From (5.35) it follows that
\[ e^{J_{\pm}(t)} \cdot e^{\phi(z)} \cdot e^{-J_{\pm}(t)} = e^{\xi(t,z^{\pm 1})} \cdot e^{\phi(z)} \cdot, \]
\[ e^{J_{\pm}(t)} \cdot e^{-\phi(z)} \cdot e^{-J_{\pm}(t)} = e^{-\xi(t,z^{\pm 1})} \cdot e^{-\phi(z)} \cdot. \]

These formulas tell us that $\cdot e^{\pm \phi(z)} \cdot$ behave as the fermionic fields $\psi(z), \psi^*(z)$. Moreover, it can be shown that all matrix elements of the operators $\cdot e^{\pm \phi(z)} \cdot$ and $\psi(z), \psi^*(z)$ between all basis states coincide. Therefore, one can identify
\[ \psi(z) = \cdot e^{\phi(z)} \cdot, \quad \psi^*(z) = \cdot e^{-\phi(z)} \cdot. \tag{5.38} \]

We will not give a complete proof and will just compare the matrix elements between the states $\langle n | e^{J_+(t_+)} e^{\phi(z)} e^{J_-(t_-)} | n - 1 \rangle$ (obviously, the matrix elements of the operators in question are nonzero only if the charge of the right state is less than the charge of the left one by 1). In fact this is enough if to know that these states are “generating functions” of the basis states (the so called coherent states) but we will not prove this. We want to show that
\[ \langle n | e^{J_+(t_+)} e^{\phi(z)} e^{J_-(t_-)} | n - 1 \rangle = \langle n | e^{J_+(t_+)} \psi(z) e^{J_-(t_-)} | n - 1 \rangle \]
for all sets of times $t_+, t_-$. A simple direct calculation using the definition (5.37), the commutation relation (5.35) and the relations (5.28), (5.31) shows that the both sides are equal to
\[ z^{n-1} \exp(\xi(t_+, z) - \xi(t_-, z^{-1}) + \sum_{k \geq 1} k t_k t_{-k}). \]

Let us check that the bosonization rules (5.38) imply that
\[ \cdot \psi(z) \psi^*(z) \cdot = z \partial_z \phi(z). \tag{5.39} \]

In this sense they agree with (5.36). We write $\psi(z_2) \psi^*(z_1) = \cdot e^{\phi(z_2)} \cdot e^{-\phi(z_1)}$ and transform both sides so that they would contain normally ordered expressions:
\[ \psi(z_2) \psi^*(z_1) = \cdot \psi(z_2) \psi^*(z_1) \cdot + \frac{z_1}{z_2 - z_1}, \]
\[ e^{\phi(z_2)} \cdot e^{-\phi(z_1)} = \frac{z_1}{z_2 - z_1} e^{\phi(z_2) - \phi(z_1)} \cdot. \]

Putting $z_1 = z, z_2 = z_1 + \varepsilon$, we get
\[ \frac{z}{\varepsilon} \left( \cdot e^{\phi(z_2) - \phi(z_1)} \cdot - 1 \right) = \cdot \psi(z + \varepsilon) \psi^*(z) \cdot, \]
which coincides with (5.39) in the limit $\varepsilon \to 0$.

When applied to the vacuum states, the operators (5.37) simplify because either $e^{J_-(z)}$ or $e^{J_+(z^{-1})}$ acts to vacuum trivially. This leads to the following simplified bosonization rules:
\[ \langle n | \psi(z) = z^{n-1} \langle n - 1 | e^{-J_+(z^{-1})}, \]
\[ \langle n | \psi^*(z) = z^{-n} \langle n + 1 | e^{J_+(z^{-1})} \tag{5.40} \]
for the left vacuum and
\[ \psi(z) |n\rangle = z^n e^{J_(\{z\})} |n+1\rangle \]
\[ \psi^*(z) |n\rangle = z^{-n+1} e^{-J_(\{z\})} |n-1\rangle \] (5.41)
for the right vacuum.

**Exercise.** Show that
\[ \langle n | \psi^*(\zeta) \psi(z) = \frac{z^n \zeta^{1-n}}{\zeta - z} \langle n | e^{J+([\zeta^{-1}] - [z^{-1}])}. \]
(5.42)

**Problem.** With the help of bosonization rules prove that
\[ \langle n | \psi(z_1) \ldots \psi(z_m) = (z_1 \ldots z_m)^{n-m} \prod_{i < j} (z_i - z_j) \langle n-m | e^{-J_+([z_1^{-1}]) - \ldots - J_+([z_m^{-1}])}, \]
\[ \langle n | \psi^*(z_1) \ldots \psi^*(z_m) = (z_1 \ldots z_m)^{n-m+1} \prod_{i < j} (z_i - z_j) \langle n+m | e^{J_1([z_1^{-1}]) + \ldots + J_+([z_m^{-1}])} \]
and derive similar formulas for the right vacuum.

### 5.9.2 Vertex operators

The bosonization rules can be represented in a different form which is based on the explicit realization of the bosonic Fock space \( \mathcal{H}_B \) as the space of polynomials of infinite number of variables \( t_1, t_2, t_3, \ldots \). More precisely, consider the space
\[ \mathcal{H}_B = \mathbb{C}[w, w^{-1}, t_1, t_2, t_3, \ldots] = \bigoplus_{l \in \mathbb{Z}} w^l \mathbb{C}[t_1, t_2, t_3, \ldots], \]
where we have added an extra variable \( w \) to take into account fermionic states with different charge, and construct the map \( \Phi : \mathcal{H}_F \rightarrow \mathcal{H}_B \) defined for an arbitrary vector \( |U\rangle \in \mathcal{H}_F \) as follows:
\[ \Phi(|U\rangle) = \sum_{l \in \mathbb{Z}} w^l \langle l | e^{J_+(t)} |U\rangle. \] (5.44)

If the state \( |U\rangle \) has a definite charge \( m \), then the sum contains only one term with \( l = m \). The fermionic creation and annihilation operators turn into some operators acting in the space of functions of \( w \) and \( t_i \).

**Proposition.** For the modes of the current there is the correspondence
\[ \Phi(J_k |U\rangle) = \begin{cases} \partial_{t_k} \Phi(|U\rangle), & k > 0, \\ w \partial_{w} \Phi(|U\rangle), & k = 0, \\ -kt_{-k} \Phi(|U\rangle), & k < 0. \end{cases} \] (5.45)

the first two formulas are obvious from the definition (5.44). The third one follows from the commutation relation \( e^{J_+(t)} J_{-k} = (J_{-k} + kt_k) e^{J_+(t)} (k > 0) \) obtained with the help of (5.3).
Introduce vertex operators by the formulas

\[ X(z) = \exp \left( \sum_{j \geq 1} t_j z^j \right) \exp \left( - \sum_{j \geq 1} \frac{1}{j z^j} \partial_j \right) e^P z^Q, \]

\[ X^*(z) = \exp \left( - \sum_{j \geq 1} t_j z^j \right) \exp \left( \sum_{j \geq 1} \frac{1}{j z^j} \partial_j \right) z^{-Q} e^{-P}. \] (5.46)

Here the operators \( P, Q \) are defined by the action on functions of \( w \):

\[ e^P f(w) = w f(w), \quad z^Q f(w) = f(zw). \]

(it is easy to check that \( [Q, P] = 1 \)). Using the shorthand notation introduced above, one can write

\[ X(z) = e^{\xi(t,z)} e^{-\xi(\partial,z^{-1})} e^P z^Q, \]

\[ X^*(z) = e^{-\xi(t,z)} e^{\xi(\partial,z^{-1})} z^{-Q} e^{-P}. \] (5.47)

**Proposition.** Under the boson-fermion correspondence, the fermionic fields \( \psi(z), \psi^*(z) \) are represented by the vertex operators \( X(z), X^*(z) \):

\[ \Phi(\psi(z) | U) = X(z) \Phi(|U\rangle), \]

\[ \Phi(\psi^*(z) | U) = X^*(z) \Phi(|U\rangle). \]

For the proof of the first equality we write separately the left and right hand sides and compare them:

\[ \Phi(\psi(z) | U) = \sum_n w^n \langle n | e^{J_+ (t)} \psi(z) | U \rangle = e^{\xi(t,z)} \sum_n w^n \langle n | \psi(z) e^{J_+ (t)} | U \rangle, \]

\[ X(z) \Phi(|U\rangle) = e^{\xi(t,z)} \sum_n z^{n-1} w^n \langle n-1 | e^{-J_+ ([z])} e^{J_+ (t)} | U \rangle. \]

It remains to use the bosonization rules (5.40). The second equality is proved in a similar way.

**5.10 Tau-functions as vacuum expectation values**

Here is the main statement connecting the theory of free fermions constructed above with the theory of integrable hierarchies. If \( G \) is an arbitrary quasigroup element of the Clifford algebra (for simplicity, with zero charge), then the vacuum expectation value

\[ \tau(t) = \langle 0 | e^{J_+ (t)} G | 0 \rangle \] (5.48)

is the tau-function of the KP hierarchy and the ratios of the expectation values

\[ \psi(t, z) = \frac{\langle 1 | e^{J_+ (t)} \psi(z) G | 0 \rangle}{\langle 0 | e^{J_+ (t)} G | 0 \rangle}, \]

\[ \psi^*(t, z) = \frac{\langle -1 | e^{J_+ (t)} \psi^*(z) G | 0 \rangle}{z \langle 0 | e^{J_+ (t)} G | 0 \rangle}. \] (5.49)
are the Baker-Akhiezer function and its adjoint.

First of all let us show that these formulas agree with (3.95), (3.96). Indeed, moving the operators $\psi, \psi^*$ in the numerator to the left and using formulas (5.40), we see that the Baker-Akhiezer functions $\psi(t, z)$ and $\psi^*(t, z)$ are connected with $\tau$ by the “Japanese” formulas (3.95), (3.96). It remains to show that the function $\tau(t)$ defined by (5.48) satisfies the bilinear relation (3.103), i.e.

$$\text{res}_z \left[ e^{\xi(t-t', z)} \langle 0 | e^{-J_+([z^{-1}])} e^{J_+(t')} G | 0 \rangle \langle 0 | e^{J_+([z^{-1}])} e^{J_+(t')} G | 0 \rangle \right] = 0.$$  

Using equations (5.40), it is easy to see that it is equivalent to the identity (5.23), in which one should put $\langle U \rangle = \langle 1 | e^{J_+(t)}, \langle U' \rangle = \langle \langle -1 \rangle | e^{J_+(t')} \rangle$.

More generally, there are the following classes of tau-functions (which generalize each other):

- **Tau-functions of the KP hierarchy depending on the times $t = \{t_1, t_2, \ldots\}$:**
  $$\tau(t) = \langle 0 | e^{J_+(t)} G | 0 \rangle.$$  

- **Tau-functions of the modified KP hierarchy (mKP):**
  $$\tau_n(t) = \langle n | e^{J_+(t)} G | n \rangle.$$  

The equations of the mKP hierarchy are differential-difference equations including shifts of the variable $n = t_0$ (the “zeroth time”). At any fixed $n$ the tau-function (5.51) solves the KP hierarchy.

- **Tau-functions of the 2DTL hierarchy:**
  $$\tau_n(t_+, t_-) = \langle n | e^{J_+(t_+)} G e^{-J_-(t_-)} | n \rangle.$$  

This is the most general tau-function which can be constructed using the one-component fermions. At fixed $n$, $t_-$ this formula gives the tau-function of the KP hierarchy (as a function of $t_+$).

Consider the tau-function of the mKP hierarchy. Let us show that it satisfies a bilinear relation which is a direct consequence of the BBR in the form (5.23). Set $\langle V \rangle = \langle n \rangle, \langle V' \rangle = \langle n' \rangle$ with $n \geq n'$, where $\langle n \rangle$ and $\langle n' \rangle$ are two shifted Dirac vacua. We have

$$\sum_{k \in \mathbb{Z}} \langle U | \psi_k G | n \rangle \langle U' | \psi_k^* G | n' \rangle = 0$$  

since either $\psi_k$ or $\psi_k^*$ annihilates the right vacuum in each term of the right hand side of (5.23). Now set $\langle U \rangle = \langle n + 1 | e^{J_+(t)} \rangle, \langle U' \rangle = \langle n' - 1 \rangle | e^{J_+(t')} \rangle$ and write

$$0 = \sum_k \langle n + 1 | e^{J_+(t)} \psi_k G | n \rangle \langle n' - 1 \rangle | e^{J_+(t')} \psi_k^* G | n' \rangle$$

$$= \text{res}_z \left[ z^{-1} \langle n + 1 | e^{J_+(t)} \psi(z) G | n \rangle \langle n' - 1 \rangle | e^{J_+(t')} \psi^*(z) G | n' \rangle \right]$$

$$= \text{res}_z \left[ e^{\xi(t-t', z)} z^{n-n'} \langle n | e^{J_+(t-[z^{-1}])} G | n \rangle \langle n' | e^{J_+(t'+[z^{-1}])} G | n' \rangle \right],$$
where we have used the commutation relations of the Fermi operators with $e^{J_+(t)}$ and the bosonization rules (5.40). (As before, $\text{res}_z$ means the coefficient in front of $z^{-1}$ in the Laurent series.) We thus obtain that the identity
\[ \oint_{C} z^{n-n'} e^{\xi(t-t',z)} \tau_n(t - [z^{-1}]) \tau_{n'}(t' + [z^{-1}]) dz = 0 \] (5.54)
holds for all $t, t'$ and $n \geq n'$. At $n = n'$ (5.54) turns into the bilinear relation for the tau-function of the KP hierarchy:
\[ \oint_{C} e^{\xi(t-t',z)} \tau(t - [z^{-1}]) \tau(t' + [z^{-1}]) dz = 0. \] (5.55)
At $n = n' + 1$ (5.54) gives the bilinear relation for the tau-function of the mKP hierarchy:
\[ \oint_{C} z e^{\xi(t-t',z)} \tau_{n+1}(t - [z^{-1}]) \tau_n(t' + [z^{-1}]) dz = 0. \] (5.56)

**Example.** The $N$-soliton tau-function (4.25) is obtained in the fermionic formalism as follows. Introduce the fermionic operators $\Psi_i(p_i, q_i) = \psi(q_i) + b_i \psi(p_i)$. They are group-like elements with charge +1. Therefore,
\[ \tau_n(t_+, t_-) = \langle n | e^{J_+(t_+, t_-, z)} \Psi_1(p_1, q_1) \ldots \Psi_N(p_N, q_N) e^{-J_-(t_+, t_-, z)} | n-N \rangle \]
is the tau-function of the 2DTL hierarchy. It coincides with (4.25).

**Problem.** Show that
\[ G = \sum_{k=1}^{N} a_k \psi^*(q_k) \psi(p_k) \]
in (5.52) leads to the $N$-soliton tau-function in the form (4.26).

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