Dynamics of harmonically-confined systems: some rigorous results

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Abstract

In this paper we consider the dynamics of harmonically-confined atomic gases. We present various general results which are independent of particle statistics, interatomic interactions and dimensionality. Of particular interest is the response of the system to external perturbations which can be either static or dynamic in nature. We prove an extended Harmonic Potential Theorem which is useful in determining the damping of the centre of mass motion when the system is prepared initially in a highly nonequilibrium state. We also study the response of the gas to a dynamic external potential whose position is made to oscillate sinusoidally in a given direction. We show in this case that either the energy absorption rate or the centre of mass dynamics can serve as a probe of the optical conductivity of the system.

Contents

1 Introduction

2 Harmonically confined systems and dipole modes

3 Centre of mass motion in the presence of an external driving force: the Harmonic Potential Theorem

4 Dipole oscillations in the presence of perturbations: extension of the HPT

5 Harmonically confined systems in the presence of an oscillating external potential

6 Conclusions

Appendix A

Appendix B

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1. Introduction

The atomic gases in many cold atom experiments are confined in harmonic traps. An important consequence of this kind of confinement is that, in the absence of any additional external perturbation, the centre of mass of the system oscillates about the centre of the trap in simple harmonic motion without dissipation. This particular collective oscillation is referred to as the centre of mass or dipole mode. According to the generalized Kohn theorem [1, 2], this behaviour is a generic property of a harmonically-confined system in which the interactions between particles depend only on their relative separation, and is independent of other intrinsic properties such as dimensionality, quantum statistics and the state of internal excitation. For these reasons, the undamped dipole oscillation can in fact be used to accurately determine the trapping frequencies [3] in situations where the experimental parameters defining the trapping potential are not known precisely. An additional, but more subtle, implication of such confinement is the content of the so-called Harmonic Potential Theorem (HPT) [4]. In essence, the HPT demonstrates the existence of a class of dynamical many-body states for which the probability density moves without change in shape. This theorem imposes important constraints on the form of approximate theories which deal with the dynamics of harmonically-confined many-body systems [4, 5, 6].

When the harmonicity of the confining potential is compromised, however, the centre of mass is coupled to the internal degrees of freedom and its dynamics becomes sensitive to the intrinsic properties of the system, including the specific form of the particle interactions. For this reason, the dipole oscillation can serve as an experimental diagnostic of various perturbations acting on the system. For instance, several experiments [7, 9, 8] have used dipole oscillations to study the transport of a Bose-condensate through a disordered medium or past a localized impurity. Although the motion of the condensate in these experiments does not lose its collectivity, dissipation does occur and leads to the damping of the centre of mass motion. Another experimental example is provided by the dipole oscillation of a trapped Bose gas in the presence of an optical lattice potential [10, 11, 12, 13]. Here it was found that the dimensionality of the Bose gas plays a critical role in determining the way in which the centre of mass behaves as a function of time.

In all of these experiments, the dipole oscillation of the atomic system is initiated by an abrupt displacement of the trapping potential along a certain axial direction. If the displacement is large, the system begins its evolution in a highly non-equilibrium initial state. It is partly for this reason that much of the theoretical work dealing with the collective dynamics of Bose-condensed systems relies on numerical simulations of the time-dependent Gross-Pitaevskii (GP) equation [14, 15, 16]. One of the goals of this paper is to show that this nonequilibrium dynamics in the presence of the external perturbation can be considered from a different point of view when the system is harmonically confined. By means of an appropriate transformation, one can equivalently think of the system as being driven out of an initial equilibrium state by a dynamic external perturbation oscillating sinusoidally at the frequency of the trap. The availability of this alternate point of view is a consequence of what we refer to as the extended HPT. Its advantage is that the external perturbation can be treated by conventional linear response theory, at least when the perturbation is sufficiently weak. This approach was used effectively in an earlier paper [17] to determine the damping of the centre of mass motion due to a disorder potential.

A second purpose of this paper is to study the response of a harmonically-confined system to an external potential which is made to oscillate at an arbitrary frequency. Our discussion is motivated by a recent proposal [18] to probe the optical conductivity of a cold atomic gas in an optical lattice by shaking the lattice periodically along a certain direction. This is an interesting idea since it provides a method of addressing experimentally the optical conductivity of a system
consisting of neutral atoms. However, the authors of Ref. [18] only considered bosons within a uniform lattice, while in most experiments the atoms are also subjected to a harmonic potential. In this paper, we show that one can also probe directly the optical conductivity of a gas that experiences a combination of a harmonic trapping potential and an arbitrary external potential when the latter is made to oscillate sinusoidally with a small amplitude. This generalization provides a precise link between theoretical calculations of the optical conductivity and possible experimental measurements on harmonically-confined gases in the presence of various external perturbations.

The rest of the paper is organized as follows. In Sec. 2, we provide a basic discussion of the dipole modes of a harmonically-confined system. In Sec. 3, we consider the response of a harmonically-confined system to a time-dependent homogeneous force. An explicit expression for the evolution operator of the system is obtained which motivates the introduction of a rather useful unitary displacement operator. These results are then used to provide an alternative derivation of the HPT. We next consider in Sec. 4 situations in which the system is perturbed by an additional external potential that couples the centre of mass and internal degrees of freedom. Here we present a derivation of the extended HPT. In Sec. 5, we consider the energy absorption rate and centre of mass dynamics of a harmonically-confined gas that is subjected to an oscillating external potential and demonstrate that both aspects serve to probe the optical conductivity of the system. All of our findings are summarized in Sec. 6.

2. Harmonically confined systems and dipole modes

As a preliminary to the development of the HPT and its extension, we discuss in this section the dipole modes of a harmonically-confined system and the underlying physics for the existence of such modes. The dipole modes are the low-lying collective excitations that have frequencies equal to the frequencies of the trap. Unlike other low-lying excitations, the frequencies of these modes are independent of the total number of trapped atoms and the atomic interactions. For Bose-condensed systems, these modes are often discussed in the context of mean-field theory, namely as solutions to the time-dependent GP equation. However, it can be shown rigorously that such dipole modes exist for any harmonically-confined (bosonic or fermionic) system in which interactions depend only on the relative coordinates of the particles. The fundamental reason behind this is that the centre of mass degree of freedom is separable from all the internal degrees of freedom, which implies that there are excitations associated solely with motion of the centre of mass.

To demonstrate this, we consider a harmonically-confined many-body system described by the generic Hamiltonian

$$\hat{H}_0 = \sum_{i=1}^{N} \left( \frac{\hat{p}_i^2}{2m} + V_{\text{trap}}(\hat{r}_i) \right) + \sum_{i<j} v(\hat{r}_i - \hat{r}_j), \quad (1)$$

where the harmonic trapping potential is

$$V_{\text{trap}}(\mathbf{r}) = \frac{1}{2m} \sum_{\mu=x,y,z} \omega_{\mu}^2 \hat{r}_\mu^2. \quad (2)$$

We define the centre of mass co-ordinate $\hat{R} = \frac{1}{N} \sum_{i=1}^{N} \hat{r}_i$ and the total momentum operator of the system $\hat{P} = \sum_i \hat{p}_i$. Since these two operators satisfy the commutation relation $[\hat{R}_\mu, \hat{P}_\nu] = i\hbar \delta_{\mu \nu}$,
they are canonically conjugate variables. Introducing the relative variables \( \hat{r}_r = \hat{r} - \hat{R} \) and \( \hat{p}_r = \hat{p} - \hat{P}/N \) we observe that the Hamiltonian \( \hat{H}_0 \) can be written as

\[
\hat{H}_0 = \hat{H}_{\text{cm}} + \hat{H}_{\text{int}}.
\]

Here

\[
\hat{H}_{\text{cm}} = \frac{\hat{p}^2}{2M} + \frac{1}{2} \sum_{\mu=x,y,z} \omega_\mu^2 \hat{R}_\mu^2
\]

and

\[
\hat{H}_{\text{int}} = \sum_{i=1}^N \left( \frac{\hat{p}_i^2}{2m} + V_{\text{trap}}(\hat{r}_i) \right) + \sum_{\nu < \mu} v(\hat{r}_\nu - \hat{r}_\mu),
\]

where \( M = Nm \) is the total mass of the system. The Hamiltonian for the centre of mass degree of freedom \( \hat{H}_{\text{cm}} \) is that of a harmonic oscillator; \( \hat{H}_{\text{int}} \) is the Hamiltonian determining the internal dynamics of the system. One can check that the centre of mass and relative variables commute, namely

\[
[\hat{R}_\mu, \hat{P}_\nu] = 0; \quad [\hat{P}_\mu, \hat{r}_\nu] = 0.
\]

It follows from these results that the centre of mass Hamiltonian \( \hat{H}_{\text{cm}} \) commutes with \( \hat{H}_{\text{int}} \). This means that the motion of the centre of mass decouples from the internal dynamics of the system.

One simple implication of this decoupling is that the centre of mass exhibits simple harmonic motion at the frequencies of the trap. Using Eq. (4) and (6), the Heisenberg equations of motion for the centre of mass coordinate and the total momentum are

\[
\frac{d\hat{R}_{\mu,1}(t)}{dt} = \frac{1}{i\hbar} [\hat{R}_{\mu,1}(t), \hat{H}_0] = \frac{\hat{P}_{\mu,1}(t)}{M},
\]

\[
\frac{d\hat{P}_{\mu,1}(t)}{dt} = \frac{1}{i\hbar} [\hat{P}_{\mu,1}(t), \hat{H}_0] = -M\omega_\mu^2 \hat{R}_{\mu,1}(t),
\]

where \( \hat{R}_{\mu,1}(t) \equiv e^{i\hat{R}_\mu \hbar / \omega_\mu} \hat{R}_\mu e^{-i\hat{R}_\mu \hbar / \omega_\mu} \) and \( \hat{P}_{\mu,1}(t) \equiv e^{i\hat{R}_\mu \hbar / \omega_\mu} \hat{P}_\mu e^{-i\hat{R}_\mu \hbar / \omega_\mu} \). Equations (7) and (8) lead to the simple harmonic motion equation

\[
\frac{d^2 \hat{R}_{\mu,1}(t)}{dt^2} + \omega_\mu^2 \hat{R}_{\mu,1}(t) = 0.
\]

The formal solution of this equation is

\[
\hat{R}_{\mu,1}(t) = \hat{R}_\mu \cos \omega_\mu t + \frac{\hat{P}_\mu}{M\omega_\mu} \sin \omega_\mu t,
\]

\[
\hat{P}_{\mu,1}(t) = -M\omega_\mu \hat{R}_\mu \sin \omega_\mu t + \hat{P}_\mu \cos \omega_\mu t.
\]

From this we see that the expectation value \( R_\mu(t) = \langle \Psi(t) | \hat{R}_\mu | \Psi(t) \rangle \) for an arbitrary state \( |\Psi(t)\rangle \) evolves in time according to the equation

\[
R_\mu(t) = R_\mu(0) \cos \omega_\mu t + \frac{P_\mu(0)}{M\omega_\mu} \sin \omega_\mu t = A \cos(\omega_\mu t + \phi_0),
\]

\(1\)It should be noted that these relative variables are not independent and are therefore not canonically conjugate.
where the amplitude $A$ and phase angle $\phi_0$ are determined by the initial conditions $R_\mu(0)$ and $P_\mu(0)$. This undamped harmonic oscillation of the centre of mass coordinate is the dipole oscillation we have been referring to.

The centre of mass excitations are conveniently described by defining the centre of mass annihilation and creation operators

$$\hat{a}_\mu = \sqrt{\frac{M\omega_\mu}{2\hbar}}(\hat{R}_\mu + \frac{i}{M\omega_\mu}\hat{P}_\mu); \quad \hat{a}_\mu^\dagger = \sqrt{\frac{M\omega_\mu}{2\hbar}}(\hat{R}_\mu - \frac{i}{M\omega_\mu}\hat{P}_\mu).$$

(13)

In terms of these operators, the Hamiltonian $\hat{H}_0$ takes the form

$$\hat{H}_0 = \sum_\mu \hbar \omega_\mu (\hat{a}_\mu^\dagger \hat{a}_\mu + \frac{1}{2}) + \hat{H}_{\text{int}}.$$  

(14)

From Eq. (6) and the definition in Eq. (13), we observe that $\hat{a}_\mu$ and $\hat{a}_\mu^\dagger$ commute with $\hat{H}_{\text{int}}$. Taking $|\Psi_\alpha\rangle$ to be an eigenstate state of $\hat{H}_0$ with energy $E_\alpha$, we find that the state $|\Psi\rangle = \frac{1}{\sqrt{n!}} (\hat{a}_\mu^\dagger)^n |\Psi_\alpha\rangle$ satisfies

$$\hat{H}_0 |\Psi\rangle = (E_\alpha + n\hbar \omega_\mu) |\Psi\rangle,$$

(15)

that is, $|\Psi\rangle$ remains an eigenstate of $\hat{H}_0$ with energy $E = E_\alpha + n\hbar \omega_\mu$; the application of $\hat{a}_\mu^\dagger$ creates a quantum of excitation of the centre of mass oscillation with energy $\hbar \omega_\mu$.

Finally, we mention the well known fact [19] that a simple harmonic oscillator has wave packet quantum states that move harmonically without any change in shape. These states are the so-called coherent states. Since the centre of mass degree of freedom is effectively a harmonic oscillator, analogous states also exist for the many-body system described by the Hamiltonian $\hat{H}_0$ in Eq. (1). This property is encapsulated by the HPT discussed in the next section.

3. Centre of mass motion in the presence of an external driving force: the Harmonic Potential Theorem

The separability of the centre of mass and internal degrees of freedom leads to excitations which are associated solely with the motion of the centre of mass. We now discuss a further implication of this property, namely the existence of a class of dynamical many-body states for which the probability density moves without change in shape. As shown by Dobson in his proof of the Harmonic Potential Theorem [4], this behaviour can occur not only for systems described by the Hamiltonian in Eq. (1), but also when the system is subjected to an arbitrary time-dependent, but spatially homogeneous, force. In this situation, the system is described by the Hamiltonian

$$\hat{H}_F(t) = \hat{H}_0 - \mathbf{F}(t) \cdot \sum_{i=1}^N \hat{r}_i = \hat{H}_0 - N\mathbf{F}(t) \cdot \hat{R}.$$  

(16)

In the presence of the force $\mathbf{F}(t)$, the system is still subjected at any instant of time to a purely harmonic, albeit time-dependent, confining potential. It is in this sense that we use the phrase "purely harmonic confinement" to distinguish this situation from those we consider later in which additional perturbing potentials are also present. Before dealing with these situations, we first determine the dynamical state $|\Psi(t)\rangle$ which evolves in time according to the Hamiltonian $\hat{H}_F(t)$. This result will then be used to provide an alternative derivation of the HPT.
To arrive at this result we have noted that

$$\Psi(t) = \mathcal{U}(t)\Psi(0),$$

where the unitary evolution operator $\mathcal{U}(t)$ satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \mathcal{U}(t) = \hat{H}(t)\mathcal{U}(t)$$

with the initial condition $\mathcal{U}(0) = \mathbb{I}$. To determine $\mathcal{U}(t)$, we go to the interaction picture and define

$$\hat{\mathcal{U}}(t) = e^{i\hat{H}_I t/\hbar} \mathcal{U}(t),$$

which also has the initial condition $\hat{\mathcal{U}}(0) = \mathbb{I}$. This evolution operator satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \hat{\mathcal{U}}(t) = -N \sum \mu F_\mu(t) \hat{R}_\mu(\mu') \hat{\mathcal{U}}(t),$$

where $\hat{R}_{\mu,\mu'}(t)$ is given explicitly in Eq. (10). The formal solution of Eq. (20) can be written as

$$\hat{\mathcal{U}}_I(t) = \lim_{N_r \to \infty} \prod_{t=0}^{N_r-1} e^{\int_{\Delta t}^{t} \mathcal{H}(t') dt'} \hat{R}_{\mu,\mu'}(t'),$$

where $\Delta t = t/N_r$. The second line follows from the fact that $\hat{R}_{\mu,\mu'}(t)$ and $\hat{R}_{\nu,\nu'}(t')$ commute when $\mu \neq \nu$.

The product of operators in Eq. (21) can be evaluated recursively. Starting from the right and making use of the Baker-Hausdorff formula\(^2\) we find

$$\exp \left( \frac{i}{\hbar} N F_\mu(\Delta t) \hat{R}_{\mu,\mu'}(\Delta t) \Delta t \right) \exp \left( \frac{i}{\hbar} N F_\mu(0) \hat{R}_{\mu,\mu'}(0) \Delta t \right)$$

$$= \exp \left( \frac{i}{\hbar} N \left[ F_\mu(\Delta t) \hat{R}_{\mu,\mu'}(\Delta t) + F_\mu(0) \hat{R}_{\mu,\mu'}(0) \right] \Delta t \right) \exp \left( -\frac{\hbar}{2m \omega_\mu} F_\mu(\Delta t) F_\mu(0) \sin \omega_\mu \Delta t (\Delta t)^2 \right).$$

To arrive at this result we have noted that

$$[\hat{R}_{\mu,\mu'}(t), \hat{R}_{\mu,\mu'}(t')] = \frac{\hbar}{N \omega_\mu} \sin \omega_\mu (t' - t),$$

which is obtained using Eq. (10). After repeating these steps $j - 1$ times, we must consider in the next step

$$\exp \left( \frac{i}{\hbar} N F_\mu(j\Delta t) \hat{R}_{\mu,\mu'}(j\Delta t) \Delta t \right) \exp \left( \frac{i}{\hbar} N \sum_{k=0}^{j-1} F_\mu(k\Delta t) \hat{R}_{\mu,\mu'}(k\Delta t) \Delta t \right)$$

$$= \exp \left( \frac{i}{\hbar} N \sum_{k=0}^{j-1} F_\mu(k\Delta t) \hat{R}_{\mu,\mu'}(k\Delta t) \Delta t \right) \exp \left( -\frac{\hbar}{2m \omega_\mu} F_\mu(j\Delta t) \sum_{k=0}^{j-1} F_\mu(k\Delta t) \sin \omega_\mu (j\Delta t - k\Delta t)(\Delta t)^2 \right).$$

\(^2\)The Baker-Hausdorff formula states that $e^{A} e^{B} = e^{A + \frac{[A, B]}{2}}$ if $A$ and $B$ commute with $C = [A, B]$. 

We will make use of these transformation properties in the following. The arbitrary state, the operator \( \hat{R} \) by

\[
\text{where } x = \text{is a position vector and } p = \text{a momentum vector. This operator can be viewed as a generalization of the usual translation operator } \exp[-i x \cdot \hat{p}/\hbar] \text{[19]. When applied to some arbitrary state, the operator } \hat{T}(x, p) \text{ has the effect of shifting the state by } x \text{ in position space and by } p/N \text{ in momentum space. To see this, we make use of the Baker-Hausdorff formula to obtain}
\]

\[
\hat{T}(x, p) = \exp\left\{ -\frac{i}{\hbar} x \cdot p \right\} \exp\left\{ \frac{i}{\hbar} p \cdot \hat{R} \right\} \exp\left\{ -\frac{i}{\hbar} x \cdot \hat{p} \right\}.
\]

Defining the state \( |\Psi'\rangle = \hat{T}(x, p) |\Psi\rangle \) and using Eq. (27), one finds

\[
|\Psi'(r_1, \cdots, r_N) = (r_1, \cdots, r_N) |\Psi')
\]

\[
= \exp\left\{ -\frac{i}{\hbar} p \cdot (R - x/2) \right\} |\Psi(r_1 - x, \cdots, r_N - x)\rangle,
\]

where \( R = \sum_{i=1}^{N} r_i/N \). We thus have \( |\Psi'(r_1, \cdots, r_N)\rangle = |\Psi(r_1, \cdots, r_N)\rangle \). Similarly, by interchanging the final two exponentials in Eq. (27), one can show that the momentum-space wavefunction is

\[
|\Psi'(p_1, \cdots, p_N) = (p_1, \cdots, p_N) |\Psi')
\]

\[
= \exp\left\{ -\frac{i}{\hbar} x \cdot (P - p/2) \right\} |\tilde{\Psi}(p_1 - p/N, \cdots, p_N - p/N)\rangle,
\]

where \( P = \sum_{i=1}^{N} p_i \). Thus \( |\tilde{\Psi}(p_1, \cdots, p_N)\rangle = |\Psi'(p_1 - p/N, \cdots, p_N - p/N)\rangle \), which implies that the total momentum of the state is boosted by \( p \). Furthermore, it is straightforward to demonstrate the operator displacement properties

\[
\hat{R}(x) \sum_{i=1}^{N} f(\hat{r}_i) \hat{T}(x, p) = \sum_{i=1}^{N} f(\hat{r}_i + x),
\]

\[
\hat{P}(p) \sum_{i=1}^{N} f(\hat{p}_i) \hat{T}(x, p) = \sum_{i=1}^{N} f(\hat{p}_i + p/N).
\]

We will make use of these transformation properties in the following.
We now consider the dynamical evolution of the system when prepared in the initial state
\[ | \Psi(0) \rangle = \hat{T}(x, p)|\Phi \rangle, \]
where $|\Phi \rangle$ is an arbitrary many-body state. As discussed above, this initial state is simply the state $|\Phi \rangle$ translated rigidly in position space through the vector $x$ and given a total momentum boost of $p$. At the end of this section we shall explain how such an initial state can be realized in cold atom experiments.

We next show how Eq. (25) together with Eq. (32) leads to the HPT. We have
\[ | \Psi(t) \rangle = e^{-i\hat{H}t/\hbar} \hat{T}(t)|\Psi(0)\rangle, \]
\[ = \left( e^{-i\hat{H}t/\hbar} \hat{T}(x, p) e^{i\hat{H}t/\hbar} \right) e^{-i\hat{H}t/\hbar}|\Phi \rangle. \]

Here, the product $\hat{T}(t) \hat{T}(x, p)$ can again be evaluated using the Baker-Hausdorff formula. We find
\[ \hat{T}(t) \hat{T}(x, p) = e^{-iS(t)/\hbar} \exp \left\{ \frac{i}{\hbar} \left[ p \cdot \hat{R} - x \cdot \hat{P} + \int_0^t dt' NF(t') \cdot \hat{R}_i(t') \right] \right\}, \]
where the phase $S(t)$ is
\[ S(t) = -\chi(t) - \frac{N}{2\mu\omega} \int_0^t dt' \int_0^t dt'' F_\mu(t') F_\mu(t'') \sin \omega_\mu(t' - t''), \]
with
\[ \chi(t) = \frac{1}{2} \sum_\mu \int_0^t dt' NF_\mu(t') \left[ x_\mu \cos \omega_\mu t' + \frac{p_\mu}{M\omega_\mu} \sin \omega_\mu t' \right]. \]

Substituting Eq. (34) into Eq. (33), we find
\[ | \Psi(t) \rangle = e^{-iS(t)/\hbar} \exp \left\{ \frac{i}{\hbar} \left[ p \cdot \hat{R}(-t) - x \cdot \hat{P}(-t) + \int_0^t dt' NF(t') \cdot \hat{R}_i(t' - t) \right] \right\} e^{-i\hat{H}t/\hbar}|\Phi \rangle, \]
where Eqs. (10) and (11) are used to obtain the final result. Here $x_\mu(t)$ and $p_\mu(t)$ are given by
\[ x_\mu(t) = x_\mu(0) + \frac{N}{M\omega_\mu} \int_0^t dt' \sin \omega_\mu(t - t') F_\mu(t'), \]
\[ p_\mu(t) = p_\mu(0) + N \int_0^t dt' \cos \omega_\mu(t - t') F_\mu(t'), \]
where
\[ x_\mu(0) = x_\mu \cos \omega_\mu t + \frac{p_\mu}{M\omega_\mu} \sin \omega_\mu t, \]
\[ p_\mu(0) = -M\omega_\mu x_\mu \sin \omega_\mu t + p_\mu \cos \omega_\mu t. \]

From these expressions we see that $x_\mu(t)$ is simply the solution of the forced harmonic oscillator equation
\[ M\ddot{x}_\mu(t) + M\omega_\mu^2 x_\mu(t) = NF_\mu(t) \]
with the initial conditions $x_\mu(0) = x_\mu$ and $\dot{x}_\mu(0) = p_\mu/M$. Likewise, $x_{0\mu}(0)$ is the solution in the absence of the force, again with the initial conditions $x_{0\mu}(0) = x_\mu$ and $\dot{x}_{0\mu}(0) = p_\mu/M$. In terms of the solution of Eq. (42), the phase $S(t)$ can be simplified as

$$S(t) = -\frac{N}{2} \int_0^t dt' \mathbf{F}(t') \cdot \mathbf{x}(t').$$

(43)

Equivalently, this can be written as

$$S(t) = \int_0^t dt' \sum_\mu \left[ \frac{1}{2} M \dot{x}_\mu^2(t') - \frac{1}{2} M \omega_\mu^2 \dot{x}_\mu^2(t') \right],$$

(44)

which is the classical action of a harmonic oscillator.

Equation (47) shows that the evolution of the initial state in Eq. (32) can be considered as taking place in two steps. First, the state $\psi(t) \equiv e^{-i\hat{H}_0 t/\hbar} \psi(0)$ is then displaced by $\hat{T}(\mathbf{x}(t), \mathbf{p}(t))$ to generate, apart from a phase factor, the final dynamical state of interest. Since the operator $\hat{T}(\mathbf{x}(t), \mathbf{p}(t))$ shifts a state in position space by $\mathbf{x}(t)$, the wave function corresponding to the state $\psi(t)$ in Eq. (37) is

$$\psi(t) = \psi(0) e^{i\hat{H}_0 t/\hbar} \rho_0 e^{-i\hat{H}_0 t/\hbar} \hat{T}(\mathbf{x}(t), \mathbf{p}(t)).$$

Comparing Eqs. (33) and (37), we have the operator identity

$$\hat{U}(t) \mathbf{F}(t', \mathbf{p}) = e^{iS(t', \mathbf{p})} \hat{T}(\mathbf{x}(t), \mathbf{p}(t)) e^{-iS(t', \mathbf{p})} \hat{U}(t).$$

(48)

where Eqs. (38) and (39) define the time evolution of the displacement operator. We thus find

$$\rho(t) = \hat{T}(\mathbf{x}(t), \mathbf{p}(t)) e^{-i\hat{H}_0 t/\hbar} \rho_0 e^{i\hat{H}_0 t/\hbar} \hat{T}^\dagger(\mathbf{x}(t), \mathbf{p}(t)).$$

(49)

This density matrix corresponds to each state $|\xi_k\rangle$ evolving freely for a time $t$ and then being displaced along the forced oscillator trajectory. If the states $|\xi_k\rangle$ in $\rho_0$ are in fact eigenstates $|\xi_n\rangle$ of $\hat{H}_0$, we have the simpler result

$$\rho(t) = \hat{T}(\mathbf{x}(t), \mathbf{p}(t)) \rho_0 \hat{T}^\dagger(\mathbf{x}(t), \mathbf{p}(t)).$$

(50)
With this density matrix, the time-dependent density of the system is

\[
n(r, t) = \text{Tr}[\hat{n}(t)\hat{n}(r)] = \text{Tr}[\hat{\mathcal{T}}(x(t), p(t))\hat{n}(r)] = \text{Tr}[\hat{\mathcal{T}}(x(t), p(t))\hat{n}(r)\hat{\mathcal{T}}(x(t), p(t))].
\]  

(51)

Recalling that \(\hat{n}(r) = \sum_{n=1}^{N} \delta(\hat{r} - r)\) and using Eq. (30), we find

\[
n(r, t) = \text{Tr}[\hat{\rho}_0\hat{n}(r - x(t))]
= n_0(r - x(t)),
\]

(52)

where \(n_0(r) = \text{Tr}[\hat{\rho}_0\hat{n}(r)]\) is the density of the system before its displacement. We thus see that the density of the system experiences the same kind of rigid motion in the density matrix description as it does for a pure state.

In the rest of this section, we consider the special case in which the force is absent. We then find from Eq. (37) that the dynamical state of the system at time \(t\) is given by

\[
|\Psi(t)\rangle = \hat{\mathcal{T}}(x_0(t), p_0(t)) e^{-i\hat{H}_0 t/\hbar} |\Phi\rangle.
\]

(53)

where \(x_0(t)\) and \(p_0(t)\) are given by Eqs. (40) and (41). This result is of course consistent with the general result in Eq. (12) for the dynamics of the centre of mass coordinate. Eq. (53) also implies that the displacement operator evolves according to

\[
\hat{\mathcal{T}}(x_0(t), p_0(t)) = e^{-i\hat{\mathcal{H}}_{\alpha z} t/\hbar} \hat{\mathcal{T}}(x, p) e^{i\hat{\mathcal{H}}_{\alpha z} t/\hbar}
\]

(54)

which is the force-free analogue of Eq. (45).

As we alluded to earlier, one can also understand the result in Eq. (53) from the perspective of coherent states. For simplicity, we take \(x = \hat{z}z_0\) and \(p = \hat{z}p_0\) in Eq. (32). Using Eq. (13), the initial state in Eq. (32) can be written as

\[
|\Psi(0)\rangle = e^{-i\hat{\mathcal{H}}_{\alpha z} \gamma\hat{\mathcal{A}}_{\alpha}^\dagger} |\Psi_{\alpha}\rangle,
\]

(55)

where \(\gamma = z_0 \sqrt{M\hbar \omega_z/2\hbar} - ip_0 / \sqrt{2M\hbar \omega_z}\). We recognize this state as the analogue of a coherent state of a simple harmonic oscillator [19]. Its dynamics is then given by

\[
|\Psi(t)\rangle = e^{-i\hat{\mathcal{H}}_{\alpha z} t/\hbar} e^{-i\hat{\mathcal{H}}_{\alpha z} \gamma\hat{\mathcal{A}}_{\alpha}^\dagger} |\Psi_{\alpha}\rangle
= e^{-i\hat{\mathcal{H}}_{\alpha z} t/\hbar} e^{-i\gamma \hat{\mathcal{A}}_{\alpha}^\dagger(\hat{\mathcal{A}}_{\alpha}^\dagger)^{(-1)}} |\Psi_{\alpha}\rangle,
\]

(56)

where

\[
\hat{\mathcal{A}}_{\alpha}(t) \equiv e^{i\hat{\mathcal{H}}_{\alpha z} t/\hbar} \hat{\mathcal{A}}_{\alpha} e^{-i\hat{\mathcal{H}}_{\alpha z} t/\hbar} = e^{-i\omega_z x(t) \hat{\mathcal{A}}_{\alpha}}.
\]

(57)

Substituting Eq. (57) into Eq. (56) and using Eqs. (14), we recover Eq. (53) where \(x_0(0) = \hat{z}z_0\) and \(p_0(0) = \hat{z}p_0\).

Finally we discuss the experimental realization of the initial state in Eq. (32). The basic idea is to initiate the oscillatory motion of a harmonically-confined system by a sudden displacement of the trapping potential. To be specific, we take the state of the undisplaced potential to be the eigenstate \(|\Psi_{\alpha}\rangle\). If the trapping potential is then displaced in the \(z\)-direction by an amount \(-z\), the state of the system relative to the shifted potential is given by

\[
|\Psi(0)\rangle = \hat{\mathcal{T}}(x = \hat{z}z, p = 0) |\Psi_{\alpha}\rangle.
\]

(58)
At a later time $t_0$, the state of the system according to Eq. (53) is

$$|\Psi(t_0)\rangle = e^{-i\hat{H}_0 t_0} \hat{T} \left( x_0(t_0), p_0(t_0) \right)|\Psi_0\rangle,$$

(59)

where, from Eqs. (40) and (41), $x_0(t_0) = z \cos \omega_z t_0 \hat{z}$ and $p_0(t_0) = -M \omega_z z \sin \omega_z t_0 \hat{z}$. With an appropriate choice of the initial displacement $z$ and time $t_0$, we can achieve the initial conditions $x_0(t_0) = z_0 \hat{z}$ and $p_0(t_0) = p_0 \hat{z}$. A more elaborate sequence of displacements of the trap in different directions can in principle be used to achieve arbitrary initial conditions.

4. Dipole oscillations in the presence of perturbations: extension of the HPT

The theoretical development in this section is motivated by several recent experiments [7, 8, 9] which studied the centre of mass dynamics of trapped Bose condensates in the presence of a disorder potential. The disorder in these experiments is an example of an external perturbation which couples the centre of mass and internal degrees of freedom. As a result of this coupling, the energy associated with the centre of mass motion is transferred to internal excitations, in other words, mechanical energy is converted into ‘heat’. One can think of the external perturbation as effectively exerting a drag force on the centre of mass which leads to a damped oscillation. Although the external potential acting on the system can be quite arbitrary, for illustration purposes we will occasionally visualize it as a disorder potential in the following.

A harmonically-confined system in the presence of an additional external potential $V_{\text{ext}}(r)$ is governed by the Hamiltonian

$$\hat{H} = \hat{H}_0 + \sum_{i=1}^{N} V_{\text{ext}}(\hat{r}_i) \equiv \hat{H}_0 + \hat{V}_{\text{ext}}.$$

(60)

To investigate the dynamics of the centre of mass, we consider the Heisenberg equations of motion (the Heisenberg operators here are defined with respect to the full Hamiltonian $\hat{H}$)

$$\frac{d\hat{R}_\mu(t)}{dt} = \frac{1}{i\hbar}[\hat{R}_\mu(t), \hat{H}] = \frac{\hat{P}_\mu(t)}{M},$$

(61)

$$\frac{d\hat{P}_\mu(t)}{dt} = \frac{1}{i\hbar}[\hat{P}_\mu(t), \hat{H}] = -M \omega^2 \hat{R}_\mu(t) + \hat{F}_\mu(t),$$

(62)

where

$$\hat{F}_\mu = -\sum_{i=1}^{N} \frac{\partial V_{\text{ext}}(\hat{r}_i)}{\partial \hat{r}_{i,\mu}}$$

(63)

is the $\mu$-component of the external force operator. Equations (61) and (62) then lead to

$$\frac{d^2 \hat{R}_\mu}{dt^2} + \omega^2 \hat{R}_\mu(t) = \frac{\hat{F}_\mu(t)}{M}.$$

(64)

Taking the expectation value of both sides of Eq. (64) with respect to the initial state $|\Psi(0)\rangle$, we find that the $z$-component of the centre of mass position satisfies the equation

$$\frac{d^2 Z(t)}{dt^2} + \omega^2 Z(t) = \frac{F(t)}{M},$$

(65)
The physical situation of interest is one in which the dynamics is initiated by suddenly shifting section. For this reason, we will consider an initial state of the form given in Eq. (32). The state in the harmonic trapping potential at some instant of time as discussed at the end of the previous section. To obtain this result we have used Eqs. (54) and (30). The above equation can be interpreted as the interaction picture evolution of the state \( \langle \Phi | \hat{I} | \Psi(0) \rangle \) which has the initial value \( |\Phi \rangle \). We thus see that the state \( |\Psi(t)\rangle \) is governed by the time-dependent Schrödinger equation

\[
\hat{I} \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle.
\]

The dynamical evolution of the state \( |\Psi(t)\rangle \) is governed by the time-dependent Schrödinger equation

\[
i \hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle.
\]

The physical situation of interest is one in which the dynamics is initiated by suddenly shifting the harmonic trapping potential at some instant of time as discussed at the end of the previous section. For this reason, we will consider an initial state of the form given in Eq. (32). The state \( |\Phi\rangle \) being displaced, will be specified later when we consider various experimental protocols for its preparation. The following development does not depend on the specific choice of the state \( |\Phi\rangle \).

In the interaction picture, the state \( |\Psi(t)\rangle \equiv e^{i\hat{H}_0 t/\hbar} |\Psi(t)\rangle \) satisfies the equation

\[
i \hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = e^{i\hat{H}_0 t/\hbar} \hat{V}_{\text{ext}} e^{-i\hat{H}_0 t/\hbar} |\Psi(t)\rangle.
\]

We now define the state

\[
|\tilde{\Psi}_1(t)\rangle \equiv \hat{T}^\dagger(x, p)|\Psi_1(t)\rangle
\]

which has the initial value \( |\tilde{\Psi}_1(0)\rangle \equiv |\Phi\rangle \). This state satisfies the equation

\[
i \hbar \frac{\partial}{\partial t} |\tilde{\Psi}_1(t)\rangle = \hat{T}^\dagger(x, p) e^{i\hat{H}_0 t/\hbar} \hat{V}_{\text{ext}} e^{-i\hat{H}_0 t/\hbar} \hat{T}(x, p) |\tilde{\Psi}_1(t)\rangle
\]

\[
= e^{i\hat{H}_0 t/\hbar} \hat{T}^\dagger(x_0(t), p_0(t)) \hat{V}_{\text{ext}} \hat{T}(x_0(t), p_0(t)) e^{-i\hat{H}_0 t/\hbar} |\tilde{\Psi}_1(t)\rangle
\]

\[
= e^{i\hat{H}_0 t/\hbar} \sum_{i=1}^N \hat{V}_{\text{ext}}(\tilde{r}_i + x_0(t)) e^{-i\hat{H}_0 t/\hbar} |\tilde{\Psi}_1(t)\rangle.
\]

To obtain this result we have used Eqs. 54 and 40. The above equation can be interpreted as the interaction picture evolution of the state \( |\tilde{\Psi}(t)\rangle \equiv e^{i\hat{H}_0 t/\hbar} |\Psi_1(t)\rangle \) which satisfies the equation

\[
i \hbar \frac{\partial}{\partial t} |\tilde{\Psi}(t)\rangle = \left[ \hat{H}_0 + \sum_{i=1}^N \hat{V}_{\text{ext}}(\tilde{r}_i + x_0(t)) \right] |\tilde{\Psi}(t)\rangle \equiv \hat{H}(t) |\tilde{\Psi}(t)\rangle,
\]

with the initial condition \( |\tilde{\Psi}(0)\rangle \equiv |\Phi\rangle \). The states \( |\tilde{\Psi}(t)\rangle \) and \( |\Psi(t)\rangle \) are related by

\[
|\Psi(t)\rangle \equiv \hat{T}(x_0(t), p_0(t)) |\tilde{\Psi}(t)\rangle.
\]

We thus see that the state \( |\Psi(t)\rangle \) of interest, which evolves from \( \hat{T}(x, p) |\Phi\rangle \) according to the stationary Hamiltonian \( \hat{H}_0 \), can be obtained by a displacement of the state \( |\tilde{\Psi}(t)\rangle \) via Eq. (72). The latter state corresponds to a different physical situation in which the system starts in the state \( |\Phi\rangle \) and then evolves in the presence of a dynamic potential oscillating at the trap frequency. Equation (72) is the main result of this section. It in fact reduces to the result given in Eq. 53 obtained in the context of the HPT when \( \hat{V}_{\text{ext}} \equiv 0 \). For this reason, we refer to it as the extended HPT. Its utility will become clear in the subsequent discussion.
Figure 1: Left panel: (a) The condensate, originally in equilibrium with the unshifted trap (dashed) and the external potential, begins to oscillate about the centre of the shifted trap (solid). (b) The condensate, originally in equilibrium with the trap and external potential, is driven by an oscillating external potential. Right panel: (a) At time $t < t_0$, the condensate is in its ground state with respect to the trapping potential indicated by the dashed curve. At $t = t_0 < 0$, the trapping potential is suddenly shifted to the origin, initiating the free oscillation of the condensate. (b) At $t = 0$, the condensate arrives at $x = \hat{z} \cos \omega z t_0$ with the velocity $v = \hat{z} \omega z t_0 \sin \omega z t_0$ and the external potential is suddenly switched on. For $t > 0$, the condensate evolves according to the Hamiltonian in Eq. (60). (c) The external potential has a velocity $-v$ at $t = 0$ and subsequently oscillates according to the free centre of mass motion. The condensate, initially stationary, begins to respond to the dynamic external potential.
To give a concrete example of these general ideas, we consider the effect of a disorder potential on the dynamics of a harmonically-confined Bose-condensed gas. Fig. 1 illustrates two possible protocols for the initiation of the dynamics [8, 9]. In the left panel (a), we start with the condensate initially in its ground state with respect to the trapping potential (dashed curve) and the external (disorder) potential. At $t = 0$, the trapping potential is suddenly shifted to the origin (solid curve) which initiates the centre of mass oscillation. The initial state $|\Psi(0)\rangle$ is the ground state of the condensate in the total potential

$$V_{\text{trap}}(r-x) + V_{\text{ext}}(r),$$

where $x = z_0 \hat{z}$. This state can be expressed as

$$|\Psi(0)\rangle = \hat{T}(x, p = 0)|\Phi_0\rangle,$$

where $|\Phi_0\rangle$ is the ground state in the total potential

$$V_{\text{trap}}(r) + V_{\text{ext}}(r + x).$$

This latter potential together with the state $|\Phi_0\rangle$ is illustrated in (b) of the left panel. The initial state $|\Phi_0\rangle$ evolves into the state $|\tilde{\Psi}(t)\rangle$ according to Eq. (71).

In the situation illustrated in the right panel of Fig. 1, the condensate starts off in the ground state of the harmonic potential which is shifted to the origin at some instant of time (a). The state is then allowed to evolve freely for some interval of time after which the state of the condensate is given by

$$|\Psi(0)\rangle = \tilde{T}(x, p)|\Phi_0\rangle,$$

where $|\Phi_0\rangle$ is now the ground state of $\hat{H}_0$ (with the harmonic potential $V_{\text{trap}}(r)$ centred on the origin) and the displacement operator $\tilde{T}(x, p)$ determines the position and velocity of the condensate at the time $t = 0$ as shown in (b) of the right panel of Fig. 1. At this instant, the disorder potential is switched on and the system evolves according to the Hamiltonian $\hat{H}$.

We thus see that in both scenarios illustrated in Fig. 1 the initial state can be expressed in the form shown in Eq. (32). Although the two initial states are different, the subsequent evolution for $t > 0$ (left panel (a) and right panel (b)) takes place in both cases according to the Hamiltonian $\hat{H}$ to generate the state $|\Psi(t)\rangle$. This state is related to the state $|\tilde{\Psi}(t)\rangle$ through Eq. (72). This relationship implies that the force appearing in Eq. (65) can be expressed as

$$F(t) = \langle \Psi(t)|\tilde{F}_z|\Psi(t)\rangle$$

$$= \langle \tilde{\Psi}(t)\tilde{T}^\dagger(x_0(t), p_0(t))\tilde{F}_z\tilde{T}(x_0(t), p_0(t))|\tilde{\Psi}(t)\rangle$$

$$= \langle \tilde{\Psi}(t)|\tilde{F}_z(t)|\tilde{\Psi}(t)\rangle$$

$$\equiv \tilde{F}(t),$$

where we have used Eq. (50) to obtain the force operator

$$\tilde{F}_z(t) = -\sum_{i=1}^{N} \frac{\partial V_{\text{ext}}(\hat{r}_i + x(t))}{\partial \hat{z}_i}$$

corresponding to the oscillating disorder potential. That the cloud experiences the same force due to the disorder in these two situations is by no means obvious and is a consequence of the validity.
of the extended HPT. This equivalence was exploited in our earlier work [17] to determine the
disorder-induced damping in the limit of a weak disorder potential. Since the state \(|\hat{\Psi}(t)\rangle\) starts
off at \(t = 0\) in the ground state \(|\Phi_0\rangle\) (either with or without the disorder potential), the effect of
a weak disorder potential can be accounted for using conventional linear response theory. On
the other hand, the initial state in the \(|\Psi(t)\rangle\) dynamics is a highly excited state of the harmonic
trapping potential and conventional linear response theory cannot be applied in this case.

The relationship in Eq. (72) shows that there is an intimate connection between the two very
distinct physical situations depicted in Fig. 1 (left panel, (a) or right panel, (b) and left panel, (b)
or right panel, (c)). In the first, the system starts in a highly excited state in which the condensate
is displaced from the minimum of the harmonic trap. The energy of this state is given by

\[
E(0) = \langle \Psi(0) | \hat{H} | \Psi(0) \rangle = \langle \Phi_0 | \hat{T}^\dagger \hat{p} \hat{H} \hat{T} (\mathbf{x}, \mathbf{p}) | \Phi_0 \rangle.
\]  

(79)

Using the properties of the displacement operator in Eqs. (30) and (31), we have

\[
\hat{T}^\dagger (\mathbf{x}, \mathbf{p}) \hat{H} \hat{T} (\mathbf{x}, \mathbf{p}) = \hat{H}_0 + \sum_{i=1}^{N} V_{\text{ext}}(r_i + x) + \frac{1}{M} \mathbf{p} \cdot \hat{P} + \sum_{\mu} M \omega_{\mu}^2 x_{\mu} + E_{\text{cm}},
\]

(80)

where the centre of mass energy is defined as

\[
E_{\text{cm}} = \frac{\hbar^2}{2M} + \frac{1}{2} \sum_{\mu} M \omega_{\mu}^2 x_{\mu}^2.
\]

(81)

Thus the initial energy is

\[
E(0) = E_0 + E_{\text{cm}} + \sum_{\mu} M \omega_{\mu}^2 x_{\mu}^2 (\Phi_0 | \hat{R}_\mu | \Phi_0),
\]

(82)

with \(E_0 = \langle \Phi_0 | \hat{H}_0 + \sum_{i=1}^{N} V_{\text{ext}}(r_i + x) | \Phi_0 \rangle\). We again note that \(|\Phi_0\rangle\) denotes different states for the
two scenarios depicted Fig. 11 but for both we have \(\langle \Phi_0 | \hat{P} | \Phi_0 \rangle = 0\). The initial state then evolves
in the presence of the static external potential according to Eq. (67) and during this evolution, the
total energy of the system is conserved.

In the alternative point of view (Fig. 11: left panel, (b) or right panel, (c)), the condensate
starts out near the minimum of the harmonic trap and is driven by a dynamic external potential
which oscillates at the trap frequency. In view of Eq. (72), the energy is given by

\[
\dot{E}(t) = \langle \Psi(t) | \dot{\hat{H}}(t) | \Psi(t) \rangle
= \langle \Psi(t) | \hat{T} (\mathbf{x}_0(t), \mathbf{p}_0(t) \hat{H} \hat{T}^\dagger (\mathbf{x}_0(t), \mathbf{p}_0(t)) | \Psi(t) \rangle
= E(0) + E_{\text{cm}} - \frac{1}{M} \mathbf{p}_0(t) \cdot \langle \Psi(t) | \hat{P} | \Psi(t) \rangle - M \sum_{\mu} \omega_{\mu}^2 x_{\mu}^2 (\Psi(t) | \hat{R}_\mu | \Psi(t)).
\]

(83)

Here, the dynamic perturbation continually excites the condensate and the total energy increases
as a function of time from the initial value \(\dot{E}(0) = E_0\). Provided the external potential provides
a coupling between the centre of mass and internal degrees of freedom, one would expect on
physical grounds that the expectation value of the total momentum \(\hat{P}\) in the state \(|\Psi(t)\rangle\) should
tend to zero at long times. By the same token, the expectation value of \(\hat{R}_\mu\) should tend to some
constant limiting value. In this case, the time average of \(\dot{E}(t)\) tends to a finite limiting value of
$E(0) + E_{\text{cm}}$. The important conclusion is that the energy does not increase indefinitely as a result of the dynamic perturbation and suggests that the system approaches a steady state.

This interpretation of the long time behaviour is supported by the time dependence of the centre of mass coordinate. In view of Eq. (72), we have

$$\tilde{Z}(t) = \langle \tilde{\Psi}(t) | \tilde{R}_z | \tilde{\Psi}(t) \rangle = \langle \tilde{\Psi}(t) | \tilde{T}(\tilde{x}_0(t), \tilde{p}_0(t)) \tilde{R}_z \tilde{T}^{-1}(\tilde{x}_0(t), \tilde{p}_0(t)) | \tilde{\Psi}(t) \rangle = Z(t) - x_{0z}(t),$$

(84)

where $x_{0z}(t)$ is given by Eq. (49). In Fig. 1 (left panel (b) or right panel (c)) we have placed a marker (filled dot) tied to the external potential that initially coincides with the origin. The position of this marker is $z_M(t) = x_{0c}(0) - x_{0z}(0)$ and the position of the centre of mass relative to it is $\tilde{Z}(t) - z_M(t) = Z(t) - x_{0z}(0)$. With the assumption that $\lim_{t \to \infty} Z(t) = Z_{\infty}$, we thus conclude that the condensate in Fig. 1 (left panel (b) or right panel (c)) moves synchronously with the external potential at long times with the centre of mass located at $Z_{\infty} - x_{0z}(0)$ relative to the marker.

The final position $Z_{\infty}$ of the centre of mass in Fig. 1 (left panel (a) or right panel (b)) depends on the details of the external perturbation. For the example of a weak disorder potential, one expects $|\tilde{\Psi}(t)|$ to approach a quasi-equilibrium state which is centred on the minimum of the harmonic potential, i.e., $Z_{\infty} \approx 0$. However, if the disorder is very strong, the initial state can be localized by the disorder potential and $Z_{\infty} \approx x_{0z}(0)$. In this case, the centre of mass is pinned to the position of the marker in Fig. 1 (left panel (b) or right panel (c)) so that $\tilde{Z}(t) \approx z_M(t)$. This motion gives a centre of mass energy $M\omega^2z_M(t)^2/2 + Mz_M(t)^2/2 = 2E_{\text{cm}}, - M\omega^2x_{0c}(0)x_{0z}(t)$. The oscillatory term in this result corresponds to the $\mu = z$ contribution from the last term in Eq. (83). Its time average is of course zero.

We conclude this section with a few general comments. The dissipation of the harmonically-confined condensate in the presence of an external perturbation is analogous to the dissipation experienced by a uniform superfluid moving past an impurity with a constant velocity. Galilean invariance allows one to consider the latter situation from the equivalent point of view of the impurity moving with a constant velocity through a stationary superfluid. Although a Galilean transformation does not apply to the harmonically trapped gas, the situation depicted in Fig. 1 (left panel (b) or right panel (c)) is analogous to the moving impurity in that the external (disorder) potential is moving relative to the gas which, at least initially, is stationary. However, it would be incorrect to think of these situations as having arisen by means of a transformation to a frame of reference in which the centre of mass, say, is at rest. Such a frame would not oscillate freely at the trap frequency as the external potential is required to do according to the extended HPT.

Although the extended HPT was motivated by other considerations, it is worth pointing out that Eqs. (67) and (71) are in fact related by means of a coordinate transformation. We interpret the state vector $|\tilde{\Psi}(t)|$ satisfying Eq. (71) as the state in the ‘laboratory’ frame of reference. We now imagine making a transformation to the non-inertial frame of reference in which $V_{\text{ext}}(r +
$x_0(t)$ is stationary [21]. In this frame of reference, the trapping potential is of course non-stationary. Nevertheless, when the harmonic trapping potential is transformed, one finds that the state vector in the non-inertial frame is the solution of Eq. (67) provided $x_0(t)$ is given by Eq. (40). It is only in this circumstance that the non-inertial forces generated by the transformation from the laboratory to the non-inertial frame are eliminated, with the evolution of the state vector in the non-inertial frame being governed simply by $\hat{H}$. That Eq. (72) provides the relation between the state vectors in the two frames of reference is a direct consequence of the fact that the trapping potential is harmonic.

5. Harmonically confined systems in the presence of an oscillating external potential

In the course of the derivation of the extended HPT, we encountered the state $|\tilde{\Psi}(t)\rangle$ which satisfies the Schrödinger equation (71). This state begins in the state $|\Phi\rangle$ at $t = 0$ and is subjected to a dynamic perturbing potential. One would usually expect a dynamic perturbation of this kind to lead to a continuous energy absorption, but as discussed earlier, for the special case in which the perturbation oscillates at the trap frequency, the time-averaged energy absorption rate eventually goes to zero. This would not be the case if the perturbing potential oscillated at some arbitrary frequency. In this section, we examine this more general situation. As a concrete example, one might consider the effect of an oscillating optical lattice on a harmonically-confined system. Oscillations of the lattice in a certain range of frequency can lead to suppression of the tunnelling and the so-called dynamically induced phase transition [22, 23]. Here, we show that a calculation of the energy absorption or centre of mass position provides a probe of the optical conductivity of the system when the amplitude of the oscillating perturbing potential is small. This analysis extends the results of earlier work [18] to the case of a harmonically-confined system subjected to an arbitrary oscillating external potential.

The physical problem of interest is described by the Hamiltonian

$$\hat{H}(t) = \hat{H}_0 + \sum_{i=1}^{N} V_{\text{ext}}(\hat{r}_i - r_0(t)), \quad (85)$$

where $r_0(t)$ is an arbitrary, time-dependent displacement vector. We refer to this kind of external potential as a ‘shaking’ perturbation. If $r_0(t)$ is small on the length scale of variations of the external potential, the Hamiltonian can be expanded as

$$\hat{H}(t) = \hat{H} + \hat{H}^\prime(t), \quad (86)$$

where $\hat{H}$ is given in Eq. (60) and the perturbation is

$$\hat{H}^\prime(t) = -\sum_{i=1}^{N} \nabla V_{\text{ext}}(\hat{r}_i) \cdot r_0(t) = -\int d\mathbf{r} \nabla V_{\text{ext}}(\mathbf{r}) \cdot r_0(t)\hat{n}(\mathbf{r}). \quad (87)$$

Although the perturbation is seen to couple to the density operator, the coupling has the very special form of the gradient of the external potential. This allows the perturbation to be expressed in terms of the commutator relation

$$\hat{H}^\prime(t) = \sum_{\mu} n_{\mu}(t) \frac{m}{\hbar} [\hat{J}_\mu, \sum_{i=1}^{N} V_{\text{ext}}(\hat{r}_i)], \quad (88)$$
where \( J_\mu = \hat{P}_\mu / m \) is the total current operator. Expressing \( \sum_{i=1}^{N} V_{\text{ext}}(\hat{r}_i) \) in terms of \( \hat{H} \) and noting that \( J_\mu \) commutes with the total kinetic energy and interatomic interactions, we find

\[
\hat{H}'(t) = \sum_{\mu} r_{0\mu}(t) \frac{m}{i\hbar} [\hat{J}_\mu, \hat{H}] + Nm \sum_{\mu} \omega_\mu^2 r_{0\mu}(t) \hat{\mathcal{R}}_\mu.
\]

The second term on the right hand side comes from the commutator of \( \hat{J}_\mu \) and \( \sum_{i=1}^{N} V_{\text{ext}}(\hat{r}_i) \). This form of the perturbation provides a direct route to the current response of the system. In the following, it is convenient to write the perturbation as

\[
\hat{H}'(t) \equiv \sum_{\mu} r_{0\mu}(t) \hat{A}_\mu,
\]

where

\[
\hat{A}_\mu = \frac{m}{i\hbar} [\hat{J}_\mu, \hat{H}],
\]

and

\[
\hat{\mathcal{R}}_\mu = Nm\omega_\mu^2 \hat{\mathcal{R}}_\mu.
\]

The total energy of the system in the presence of the perturbation is \( \hat{E}(t) = \langle \hat{\Psi}(t) | \hat{H}(t) | \hat{\Psi}(t) \rangle \), where \( | \hat{\Psi}(t) \rangle \) is the state of the system at time \( t \). The tilde notation is used since the Hamiltonian in Eq. (85) is analogous to the Hamiltonian in Eq. (71). The rate of energy absorption is given quite generally by

\[
\frac{d\hat{E}}{dt} = \langle \hat{\Psi}(t) | \frac{d\hat{H}'}{dt} | \hat{\Psi}(t) \rangle
\]

If we assume that the perturbation is turned on at \( t = t_0 \) and that the system starts out in the ground state \( | \Phi_0 \rangle \) of \( \hat{H} \), the energy absorption rate in the linear response regime is given by

\[
\frac{d\hat{E}}{dt} = \sum_{\mu, j} \overline{r}_{0\mu}(t) \langle \Phi_0 | \hat{A}_\mu | \Phi_0 \rangle - \sum_{\mu, j} \overline{r}_{0\mu}(t) \int_{-\infty}^{\infty} dt' \chi_{\mu, j}(t' - t) r_{0j}(t')
\]

where we have defined the retarded response functions

\[
\chi_{\mu, j}(t' - t) \equiv \frac{1}{\hbar} \langle \Phi_0 | [\hat{A}_\mu(t), \hat{A}_j(t')] | \Phi_0 \rangle.
\]

The Heisenberg operators appearing in the response function are \( \hat{A}_\mu(t) = e^{i\hat{H}/\hbar} \hat{A}_\mu e^{-i\hat{H}/\hbar} \). By introducing the correlation function

\[
\phi_{\mu, j}(t' - t) \equiv \frac{1}{2\hbar} \langle \Phi_0 | [\hat{A}_\mu(t), \hat{A}_j(t')] | \Phi_0 \rangle,
\]

the response function can be written as

\[
\chi_{\mu, j}(t' - t) = 2i\phi(t' - t) \phi_{\mu, j}(t' - t).
\]

We now consider the energy absorption rate for the special case of a monochromatic displacement, namely \( r_{0\mu}(t) = \theta(t - t_0)(r_{0\mu} e^{-i\omega t} + r_{0\mu}^* e^{i\omega t})/2 \). Taking the limit \( t_0 \to -\infty \) and averaging
Eq. (94) over one period \( T = 2\pi/\omega \), we find

\[
\frac{d\hat{E}}{dt} = \frac{1}{T} \int_0^T \frac{d\hat{E}}{dt} \, dt = \frac{\omega}{2} \sum_{\mu, \nu} \hat{R}_{\mu} \phi_{\mu, \nu}^*(\omega) r_{\nu},
\]

(98)

where \( \phi_{\mu, \nu}(\omega) \), the Fourier transform of Eq. (96), has the spectral representation (here, \( \hat{H}(\Phi_m) = E_m(\Phi_m) \))

\[
\phi_{\mu, \nu}(\omega) = \pi \sum_n \left\{ \langle \Phi_0 | \hat{A}_{\mu} | \Phi_n \rangle \langle \Phi_n | \hat{A}_{\nu} | \Phi_0 \rangle \delta(h\omega - E_n) - \langle \Phi_0 | \hat{A}_{\nu} | \Phi_n \rangle \langle \Phi_n | \hat{A}_{\mu} | \Phi_0 \rangle \delta(h\omega + E_n) \right\}.
\]

(99)

Since \( \phi_{\mu, \nu}^*(\omega) = \phi_{\nu, \mu}(\omega) \), it is clear that Eq. (98) is manifestly real. Furthermore, time-reversal symmetry implies that \( \phi_{\mu, \nu}(\omega) = \phi_{\nu, \mu}(\omega) \) and hence that \( \phi_{\mu, \nu}(\omega) \) is real. In this case, \( \text{Im} \phi_{\mu, \nu}(\omega) = \phi_{\mu, \nu}(\omega) \) and \( \text{Re} \phi_{\mu, \nu}(\omega) \) is obtained from \( \phi_{\mu, \nu}(\omega) \) by a Kramers-Kronig relation \( \left[ 20 \right] \).

From Eqs. (97) and (92), we find

\[
\langle \Phi_m | \hat{A}_{\mu} | \Phi_n \rangle = \frac{m}{i\hbar} E_{nm} \langle \Phi_m | \hat{J}_{\mu} | \Phi_n \rangle
\]

(100)

and

\[
\langle \Phi_m | \hat{A}_{\nu} | \Phi_n \rangle = \frac{i\hbar m \omega^2}{E_{nm}} \langle \Phi_m | \hat{J}_{\nu} | \Phi_n \rangle.
\]

(101)

The latter equation follows from the identity

\[
[\hat{R}_\mu, \hat{H}] = \frac{i\hbar}{N} \hat{J}_\mu.
\]

(102)

In the above equations, \( E_{nm} = E_n - E_m \). We see that all the required matrix elements can be expressed in terms of those of the total current operator.

Using Eqs. (100) and (101) in Eq. (99), we obtain

\[
\phi_{1\mu, 1\nu}(\omega) = m^2 \omega^2 \text{Im} \Pi_{\mu\nu}(\omega),
\]

(103)

\[
\phi_{1\mu, 2\nu}(\omega) = -m^2 \omega^2 \text{Im} \Pi_{\mu\nu}(\omega),
\]

(104)

\[
\phi_{2\mu, 1\nu}(\omega) = -m^2 \omega^2 \text{Im} \Pi_{\mu\nu}(\omega),
\]

(105)

\[
\phi_{2\mu, 2\nu}(\omega) = m^2 \frac{\omega^2 \omega^2}{\omega^2} \text{Im} \Pi_{\mu\nu}(\omega),
\]

(106)

where

\[
\text{Im} \Pi_{\mu\nu}(\omega) = \pi \sum_n \left\{ \langle \Phi_0 | \hat{J}_{\mu} | \Phi_n \rangle \langle \Phi_n | \hat{J}_{\nu} | \Phi_0 \rangle \delta(h\omega - E_n) - \langle \Phi_0 | \hat{J}_{\nu} | \Phi_n \rangle \langle \Phi_n | \hat{J}_{\mu} | \Phi_0 \rangle \delta(h\omega + E_n) \right\}.
\]

(107)

This quantity is the imaginary part of the Fourier transform of the current-current response function

\[
\Pi_{\mu\nu}(t - t') = i \frac{\hbar}{\pi} \theta(t - t') \langle \Phi_0 | [\hat{J}_\mu(t), \hat{J}_\nu(t')] | \Phi_0 \rangle.
\]

(108)
It defines the real part of the optical conductivity according to (24)

\[ \text{Re} \sigma_{\mu \nu}(\omega) = \frac{1}{\omega} \text{Im} \Pi_{\mu \nu}(\omega). \tag{109} \]

We emphasize that this is the optical conductivity of the system described by the Hamiltonian \( \hat{H} \) in Eq. (60) which includes both the harmonic and external potentials.

Inserting Eqs. (103)-(106) into Eq. (98), we thus find

\[ \frac{d \tilde{E}}{dt} = \frac{m^2 \omega^3}{2} \sum_{\mu \nu} r_{0 \mu}^* r_{0 \nu} \left( 1 - \frac{\omega^2}{\omega^2 + \omega^2} \right) \left( 1 - \frac{\omega^2}{\omega^2 + \omega^2} \right) \text{Im} \Pi_{\mu \nu}(\omega). \tag{110} \]

This is a general result valid for any harmonically-confined system in the presence of an arbitrary oscillating external potential. We thus see that the time-averaged energy absorption rate in the presence of harmonic confinement is proportional to the optical conductivity. If the oscillation is restricted to the \( z \)-direction, i.e. \( r_0(t) = z_0 \hat{z} \sin \omega t \), the energy absorption rate becomes

\[ \frac{d \tilde{E}}{dt} = \frac{1}{2} \frac{m^2 \omega^3}{\omega^2} \sum_{\mu} \left[ \sin \omega t \sum_j \text{Re} \chi_{2 \mu, j}(\omega) - \cos \omega t \sum_j \text{Im} \chi_{2 \mu, j}(\omega) \right]. \tag{111} \]

In the absence of harmonic confinement (\( \omega \to 0 \)), Eq. (111) reduces to the result given in Ref. [18] which was derived for the special case of an oscillating uniform optical lattice. An alternative derivation of this limiting result is provided in Appendix A; this derivation points out a shortcoming of the original derivation in Ref. [18].

It is interesting to observe that the energy absorption rate in Eq. (111) vanishes if the external perturbation is oscillating at the trapping frequency \( \omega = \omega_z \). This is in fact an exact result and is not a consequence of the perturbative analysis. If we take \( r_0(t) = x_0(t) \hat{z} \) where \( x_0(t) \) is given in Eq. (40), the Hamiltonian in Eq. (85) takes the form of the Hamiltonian in Eq. (71). According to the extended HPT, the time-averaged energy absorption rate goes to zero at long times (see Eq. (83) and the discussion thereafter). This result applies for any amplitude of the oscillating potential and in particular, accounts for the second order result in Eq. (111). An alternative proof is given in Appendix B where we consider the oscillation of the external potential to be turned on adiabatically. If the oscillation occurs at the frequency \( \omega = \omega_z \), we show that the initial state of the system, \( |\Phi_0\rangle \), evolves into a state which moves together with the oscillating potential. This dynamical state has a vanishing time-averaged energy absorption rate which is consistent with the \( \omega = \omega_z \) result in Eq. (111).

Eq. (110) or (111) shows that information about the current-current correlation function can in principle be accessed via the energy absorption rate. However, the latter is not a quantity that is easily measured. On the other hand, a measurement of the centre of mass motion is relatively straightforward and provides an alternative means of probing the current-current correlation function. For simplicity, we consider a perturbation shaking in the \( z \)-direction. The relevant centre of mass coordinate is \( Z(t) \equiv |\Psi(t)| \langle \hat{R}_z | \Psi(t) \rangle \) which, in linear response, is given by

\[ Z(t) = \frac{1}{N \omega z} \sum_j \int_{-\infty}^{\infty} dt' \chi_{2z, j}(t - t') z_0 \sin \omega t' \]

\[ = \frac{z_0}{N \omega z} \left[ \sin \omega t \sum_j \text{Re} \chi_{2z, j}(\omega) - \cos \omega t \sum_j \text{Im} \chi_{2z, j}(\omega) \right]. \tag{112} \]
We see that $Z(t)$ oscillates at the frequency $\omega$ with some phase lag relative to the oscillating external potential. If we compare this with the expected experimental centre of mass trajectory $Z_{\text{exp}}(t) = Z_0 \sin(\omega t - \varphi)$, we see immediately that

$$
\sum_j \text{Im}\chi_{zz,j}(\omega) = \frac{Z_0}{z_0} N m \omega^2 z \sin \varphi.
$$

(113)

Using Eqs. (105) and (106), we find

$$
\text{Im} \Pi_{zz}(\omega) = \frac{Z_0}{z_0} N \sin \varphi \frac{m}{N m (\omega^2 / \omega^2 - 1)}.
$$

(114)

This result indicates that the current-current correlation function can be obtained experimentally through a measurement of the centre of mass oscillation amplitude $Z_0$ and phase lag $\varphi$. This kind of measurement is thus a direct probe of the optical conductivity.

6. Conclusions

In this paper we have studied various aspects of the dynamics of harmonically-confined atomic systems. The results we have obtained are of a general nature and have a broad applicability to trapped atomic gases. We have focused, in particular, on the effect that various external perturbations have on the dynamical evolution of the many-body wavefunction. In the case of a perturbation that couples solely to the centre of mass, we were able to obtain an explicit expression for the Schrödinger evolution operator; Dobson’s Harmonic Potential Theorem follows naturally from this result.

For more general external perturbations, the centre of mass and internal degrees of freedom are coupled and dissipation of the centre of mass motion sets in. We have proved an extension of the HPT which demonstrates that this dissipative dynamics can be considered from two distinct points of view. In the first, the evolution of an initial nonequilibrium state takes place in the presence of a static external potential. On the other hand, one can equivalently think of the evolution as taking place in the presence of an external potential that itself moves according to the trajectory of a harmonically-confined particle. Here, the trapped atomic cloud starts off in an initial equilibrium state and is then continually excited by a dynamic perturbation. This latter point of view has the advantage that the calculation of the damping of the centre of mass motion can be addressed by means of linear response theory when the perturbation is weak [17].

We next considered the response of a harmonically-confined system to a ‘shaking’ potential. For a weak perturbation, this response is directly related to current-current correlations and hence the optical conductivity. Our result for the energy absorption is a generalization of one obtained previously [18]. We have also shown that the optical conductivity can be probed by measuring the trajectory of the centre of mass itself. This may in fact be the most feasible way of determining the optical conductivity experimentally.

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21
Appendix A.

In this appendix we give a different derivation of the result in Eq. (111) for the special case of no harmonic confinement ($\omega_{ij} = 0$). The Hamiltonian in this case is

$$\hat{H}(t) = \sum_{i=1}^{N} \left[ \frac{\hat{p}_i^2}{2m} + V_{\text{ext}}(\hat{x}_i, \hat{y}_i, \hat{z}_i - z_0(t)) \right] + \sum_{i<j} v(\hat{r}_i - \hat{r}_j), \quad (A.1)$$

where we allow the displacement in the $z$-direction to have an arbitrary time dependence. A Hamiltonian of this kind was considered in Ref. [18] for the case in which $V_{\text{ext}}$ corresponds to a uniform optical lattice. In this context, the displacement $z_0(t)$ provides a phase modulation of the lattice potential.

We observe that the Hamiltonian can be expressed as

$$\hat{H}(t) = \hat{U}(t)^\dagger \hat{H} \hat{U}(t), \quad (A.2)$$

where $\hat{H}$ is the Hamiltonian in Eq. (A.1) with $z_0(t) \equiv 0$, and $\hat{U}(t)$ is the translation operator

$$\hat{U}(t) = \exp \left[ iz_0(t)\hat{P}_z / \hbar \right] = \exp \left[ imz_0(t)\hat{J}_z / \hbar \right]. \quad (A.3)$$

The dynamic state of the system evolves according to the Schrödinger equation

$$i\hbar \frac{d|\Psi(t)\rangle}{dt} = \hat{H}(t)|\Psi(t)\rangle = \hat{U}(t)^\dagger \hat{H} \hat{U}(t)|\Psi(t)\rangle. \quad (A.4)$$

Defining the state

$$|\tilde{\Psi}(t)\rangle = |\Psi(t)\rangle,$$ \quad (A.5)

we find that $|\tilde{\Psi}(t)\rangle$ satisfies the equation

$$i\hbar \frac{d|\tilde{\Psi}(t)\rangle}{dt} = \tilde{H}(t)|\tilde{\Psi}(t)\rangle,$$ \quad (A.6)

where

$$\tilde{H}(t) = \hat{H} - mz_0(t)\hat{J}_z. \quad (A.7)$$

We see that the Hamiltonian governing the evolution of the state $|\tilde{\Psi}(t)\rangle$ contains a perturbation proportional to the total current operator. It should be emphasized that $|\tilde{\Psi}(t)\rangle$ is not the state of the system as seen in the non-inertial frame of reference in which the external potential is stationary. To obtain the state in this frame of reference one must apply a momentum boost in addition to the spatial displacement provided by $\hat{U}(t)$ [21].

The total energy of the system is given by

$$E(t) = \langle \Psi(t)|\hat{H}(t)|\Psi(t)\rangle$$

$$= \langle \tilde{\Psi}(t)|\tilde{H}|\tilde{\Psi}(t)\rangle. \quad (A.8)$$

Using Eq. (A.6) and Eq. (A.3), we find that the energy absorption rate is given by

$$\frac{dE}{dt} = \frac{1}{i\hbar} \langle \Psi(t)|[\hat{H}, \hat{H}(t)]|\Psi(t)\rangle$$

$$= -mz_0(t) \frac{1}{i\hbar} \langle \Psi(t)|[\hat{H}, \hat{J}_z]|\Psi(t)\rangle. \quad (A.9)$$
Introducing the interaction picture state vector \(|\hat{\Psi}(t)\rangle \equiv \exp(i\hat{H}t/\hbar)|\Psi(t)\rangle\), we have

\[
\frac{dE}{dt} = -\frac{m\dot{z}_0(t)}{\hbar}\langle\hat{\Psi}(t)|[\hat{H}, \hat{J}_z(t)]|\hat{\Psi}(t)\rangle,
\]

(A.10)

where \(\hat{J}_z(t) \equiv \exp(i\hat{H}t/\hbar)\hat{J}_z\exp(-i\hat{H}t/\hbar)\). The state \(|\hat{\Psi}(t)\rangle\) evolves according to

\[
\hbar\frac{d|\hat{\Psi}(t)\rangle}{dt} = -m\dot{z}_0(t)|\hat{J}_z(t)|\hat{\Psi}(t)\rangle.
\]

(A.11)

First order perturbation theory gives

\[
|\hat{\Psi}(t)\rangle \simeq |\Phi_0\rangle - \frac{m}{\hbar} \int_0^t dt' \dot{z}_0(t')\hat{J}_z(t')|\Phi_0\rangle,
\]

(A.12)

where we assume that \(z_0(t) \equiv 0\) for \(t \leq t_0\) and that \(|\Phi_0\rangle\) is the ground state of \(\hat{H}\). Substituting Eq. (A.12) into Eq. (A.10) we have

\[
\frac{dE}{dt} = -\frac{m}{\hbar}\dot{z}_0(t)|\Phi_0\rangle[[\hat{H}, \hat{J}_z(t)]|\Phi_0\rangle + \left(\frac{m}{\hbar}\right)^2 \dot{z}_0(t) \int_0^t \int_0^\infty dt' \Pi_{z\z}(t-t')\dot{z}_0(t').
\]

(A.13)

The first term on the right hand side of this equation vanishes since \(|\Phi_0\rangle\) is the ground state of \(\hat{H}\). Using

\[
\hbar\frac{d}{dt}\hat{J}_z(t) = [\hat{J}_z(t), \hat{H}],
\]

(A.14)

Eq. (A.13) can be written as

\[
\frac{dE}{dt} = -m^2\dot{z}_0(t) \int_0^t \int_0^\infty dt' \Pi_{z\z}(t-t')\dot{z}_0(t'),
\]

(A.15)

where \(\Pi_{z\z}(t-t')\) is defined in Eq. (108). This is a general result for any displacement \(z_0(t)\) that vanishes for \(t \leq t_0\).

For the case of a sinusoidal perturbation, \(z_0(t) = z_0 \sin \omega t\), we can take the limit \(t_0 \rightarrow -\infty\) and obtain

\[
\frac{dE}{dt} = m^2 z_0^2 \omega^3 \cos \omega t \left[ -\sin \omega t \Re \Pi_{z\z}(t) + \cos \omega t \Im \Pi_{z\z}(t) \right].
\]

(A.16)

Averaging this expression over one period, we obtain

\[
\overline{\frac{dE}{dt}} = \frac{1}{T} \int_0^T \frac{dE}{dt} dt = \frac{1}{2} m^2 z_0^2 \omega^3 \Im \Pi_{z\z}(\omega),
\]

(A.17)

which is the result given in Eq. (111) for the case of \(\omega_z = 0\).

We now point out that the above is not in fact the derivation given in Ref. [18]. Instead of the correct expression for the energy given in Eq. (A.13), the authors of Ref. [18] take the energy...
of the system to be \( \dot{E}(t) = \langle \tilde{\Psi}(t)|\tilde{H}(t)|\tilde{\Psi}(t) \rangle \), where \( \tilde{\Psi}(t) \) is the solution of Eq. (A.6). The energy absorption rate in Ref. \[18\] is then defined to be

\[
\frac{d\dot{E}}{dt} = \langle \tilde{\Psi}(t)|\frac{\partial \tilde{H}(t)}{\partial t}|\tilde{\Psi}(t) \rangle.
\]

(A.18)

With \( \tilde{H}(t) \) given by Eq. (A.7), one has

\[
\frac{d\dot{E}}{dt} = -m\dot{z}_0(t)\langle \tilde{\Psi}(t)|\hat{J}_z|\tilde{\Psi}(t) \rangle = -m\dot{z}_0(t)\langle \tilde{\Psi}(t)|\hat{J}_z(t)|\tilde{\Psi}(t) \rangle.
\]

(A.19)

Substituting Eq. (A.12) into this result, one finds

\[
\frac{d\dot{E}}{dt} = -m\frac{z_0}{\omega}(t)\int_0^\infty d\tau \Pi_{zz}(t-t\tau)\dot{z}_0(t\tau),
\]

(A.20)

which differs from the correct result in Eq. (A.15). For the sinusoidal displacement, we have

\[
\frac{d\dot{E}}{dt} = m\frac{z_0^2}{\omega^3}\sin \omega t \left[ \cos \omega t \text{Re} \Pi_{zz}(\omega) + \sin \omega t \text{Im} \Pi_{zz}(\omega) \right],
\]

(A.21)

which clearly has a different time dependence from \( \frac{dE}{dt} \) in Eq. (A.16). However, when averaged over one period, the energy absorption rate is

\[
\frac{d\dot{E}}{dt} = \frac{1}{2}m\frac{z_0^2}{\omega^3}\text{Im} \Pi_{zz}(\omega),
\]

(A.22)

which is the same as \( \frac{dE}{dt} \). Thus the approach adopted in Ref. [18] does indeed yield the correct time-averaged energy absorption rate for a sinusoidal displacement. However, this is not true for other forms of the displacement. For example, for the displacement \( z_0(t) = \theta(t)v_0t \), which corresponds to the external potential moving with a constant velocity \( v_0 \) for \( t > 0 \), Eq. (A.18) gives

\[
\frac{d\dot{E}}{dt} = \langle \tilde{\Psi}(t)|(-m\dot{z}_0(t)\hat{J}_z)|\tilde{\Psi}(t) \rangle = -mv_0\delta(t)\langle \tilde{\Psi}(0)|\hat{J}_z|\tilde{\Psi}(0) \rangle = 0
\]

(A.23)

since the initial state is one in which there is no current. This conclusion also follows from Eq. (A.20). On the other hand, Eq. (A.15) gives

\[
\frac{dE}{dt} = m\frac{z_0^2}{\omega^3}d\frac{d}{dt} \int_0^\infty d\tau \Pi_{zz}(t-t\tau) = m\frac{z_0^2}{\omega^3}\Pi_{zz}(t),
\]

(A.24)

which is a non-zero result. This shows that Eq. (A.20) cannot be the correct result for the energy absorption rate in general.
Appendix B.

In this appendix we provide an alternative explanation for why the energy absorption rate in Eq. (111) vanishes when \( \omega = \omega_c \). To this end, we write the Hamiltonian in Eq. (85) as

\[
\hat{H}_{ad}(t) = \hat{H} + e^{i\eta} \hat{H}'(t, \lambda)
\]

where

\[
\hat{H}'(t, \lambda) = \sum_{i=1}^{N} \left[ V_{\text{ext}}(\hat{r}_i - \lambda \hat{x}(t)) - V_{\text{ext}}(\hat{r}_i) \right].
\]

The perturbation \( \hat{H}'(t, \lambda) \) is turned on adiabatically via the parameter \( \eta \). For \( t \to -\infty, \hat{H}_{ad}(t) \) reduces to \( \hat{H} \), while for \( \eta \to 0 \) one recovers the Hamiltonian \( \hat{H}(t) \) in Eq. (85). The parameter \( \lambda \) is introduced as an ordering parameter in the perturbation analysis of \( \hat{H}'(t, \lambda) \) and is set to unity at the end of the calculation. The displacement will be taken to have the specific form \( \hat{x}(t) = \hat{z}_0 \sin \omega_c t \). We observe that \( \hat{H}'(t, \lambda) \) can be written as

\[
\hat{H}'(t, \lambda) = \hat{T}(\lambda \hat{x}(t), \lambda \hat{p}(t)) \hat{V}_{\text{ext}} \hat{T}^+(\lambda \hat{x}(t), \lambda \hat{p}(t)) - \hat{V}_{\text{ext}},
\]

where \( \hat{V}_{\text{ext}} = \sum_{i=1}^{N} V_{\text{ext}}(\hat{r}_i) \). Although the required displacement of \( \hat{V}_{\text{ext}} \) can be generated with any \( \hat{p}(t) \), we make the choice \( \hat{p}(t) = M \hat{x}(t)/\hbar \) to ensure that the displacement operator evolves in time according to Eq. (53). This property will be shown to be crucial in the derivation of our final result.

We now construct the dynamical state in the interaction picture which reduces to \( |\Phi_0\rangle \), the ground state of \( \hat{H} \), in the \( t \to -\infty \) limit. This state is given by

\[
|\Psi(t)\rangle = \hat{U}(t, \lambda) |\Phi_0\rangle,
\]

where the evolution operator satisfies the integral equation

\[
\hat{U}(t, \lambda) = 1 + \frac{1}{i \hbar} \int_{-\infty}^{t} dt' e^{i \hat{H}'(t', \lambda)t'/\hbar} \hat{U}(t', \lambda).
\]

Here

\[
\hat{H}'(t, \lambda) = e^{i \hat{H}'(t, \lambda)t/\hbar} e^{-i \hat{H}'(t, \lambda)t/\hbar}
\]

\[
= \hat{T}(\lambda \hat{x}(t), \lambda \hat{p}(t)) \hat{V}_{\text{ext},1}(t) \hat{T}^+(\lambda \hat{x}(t), \lambda \hat{p}(t)) - \hat{V}_{\text{ext},1}(t),
\]

where \( \hat{V}_{\text{ext},1}(t) = e^{i \hat{H}'(t, \lambda)t/\hbar} \hat{V}_{\text{ext}} e^{-i \hat{H}'(t, \lambda)t/\hbar} \) and

\[
\hat{T}(\lambda \hat{x}(t), \lambda \hat{p}(t)) \equiv e^{i \hat{H}'(t, \lambda)t/\hbar} \hat{T}(\lambda \hat{x}(t), \lambda \hat{p}(t)) e^{-i \hat{H}'(t, \lambda)t/\hbar}
\]

\[
= \exp \left\{ \frac{i}{\hbar} \left( \hat{p}(t) \cdot \hat{\mathbf{R}}(t) - \hat{x}(t) \cdot \hat{\mathbf{P}}_1(t) \right) \right\}
\]

\[
\equiv \exp \left\{ \frac{i}{\hbar} \hat{A}_1(t) \right\},
\]

with

\[
\hat{A}_1(t) = \hat{p}(t) \cdot \hat{\mathbf{R}} - \hat{x}(t) \cdot \hat{\mathbf{P}}.
\]
We now evaluate \( \hat{\mathcal{U}}_1(t, \lambda) \) explicitly. Expanding this operator in powers of \( \lambda \), we have

\[
\hat{\mathcal{U}}_1(t, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{\mathcal{U}}_1^{(n)}(t, 0) \lambda^n, \tag{B.9}
\]

where

\[
\hat{\mathcal{U}}_1^{(n)}(t, 0) \equiv \left. \frac{\partial^n}{\partial \lambda^n} \hat{\mathcal{U}}_1(t, \lambda) \right|_{\lambda=0}. \tag{B.10}
\]

Similarly we have

\[
\hat{\mathcal{H}}_1(t, \lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{\mathcal{H}}_1^{(n)}(t, 0) \lambda^n, \tag{B.11}
\]

where

\[
\hat{\mathcal{H}}_1^{(n)}(t, 0) \equiv \left. \frac{\partial^n}{\partial \lambda^n} \hat{\mathcal{H}}_1(t, \lambda) \right|_{\lambda=0}. \tag{B.12}
\]

Using Eq. (B.6) in Eq. (B.12), we obtain

\[
\hat{\mathcal{H}}_1^{(0)}(t, 0) = 0, \tag{B.13}
\]

\[
\hat{\mathcal{H}}_1^{(1)}(t, 0) = i \hbar \{ \hat{A}_1(t), \hat{V}_{ext}(t) \}, \tag{B.14}
\]

and for \( n > 1 \),

\[
\hat{\mathcal{H}}_1^{(n)}(t, 0) = i \hbar [ \hat{A}_1(t), \hat{\mathcal{H}}_1^{(n-1)}(t, 0) ]. \tag{B.15}
\]

Substituting Eqs. (B.9) and (B.11) into Eq. (B.5) and comparing like powers of \( \lambda \), we find

\[
\hat{\mathcal{U}}_1^{(n)}(t, 0) = \frac{1}{\hbar} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \int_{-\infty}^{t} dt_1 e^{\rho_n(t, t_1) \hat{\mathcal{H}}_1^{(m)}(t_1, 0) \hat{\mathcal{U}}_1^{(n-m)}(t_1, 0)} \tag{B.16}
\]

Let us first consider \( \hat{\mathcal{U}}_1^{(1)}(t, 0) \). From Eq. (B.14) we have

\[
\hat{\mathcal{H}}_1^{(1)}(t, 0) = -\frac{im_0(t)}{\hbar} e^{i\hat{\mathcal{H}}_1^t} \left[ \hat{J}_z(t), \hat{V}_{ext}(t) \right] e^{-i\hat{\mathcal{H}}_1^t} / \hbar
\]

\[
= -\frac{im_0(t)}{\hbar} e^{i\hat{\mathcal{H}}_1^t} \left[ \hat{J}_z \hat{H} - \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} - \sum_{i<j} v(\hat{r}_i - \hat{r}_j) - \sum_{i=1}^{N} V_u(\hat{r}_i) \right] e^{-i\hat{\mathcal{H}}_1^t} / \hbar
\]

\[
= m_0(t) \left\{ \frac{1}{\hbar} \left[ \hat{J}_z \hat{H} + N\omega^2 \hat{R}_{2z}(t) \right] \right\}
\]

\[
= N m_0(t) \left( \frac{\partial^2}{\partial \hat{R}_{2z}} \hat{R}_{2z}(t) + \omega^2 \hat{R}_{2z}(t) \right). \tag{B.17}
\]

Using Eq. (B.3) and \( \omega(t) = \omega_0 \sin \omega t \), we find

\[
\hat{\mathcal{H}}_1^{(1)}(t, 0) = -\frac{\partial}{\partial t} \hat{A}_1(t). \tag{B.18}
\]
This result is only true when \( p(t) = Mdx(t)/dt \), as assumed. Inserting Eq. \( (B.18) \) into Eq. \( (B.15) \) for \( n = 1 \), we have

\[
\hat{U}_1^{(1)}(t, 0) = \frac{1}{i\hbar} \int_{-\infty}^{t} dt_1 e^{i\theta_1} \hat{H}_1^{(1)}(0, t_1)
= \frac{i}{\hbar} \hat{A}_1(t) + O(\eta),
\]

where \( O(\eta) \) denotes terms that vanish in the \( \eta \to 0 \) limit.

Repeating this calculation for \( n = 2 \), we find

\[
\hat{U}_1^{(2)}(t, 0) = \frac{1}{i\hbar} \int_{-\infty}^{t} dt_1 e^{i\theta_1} \{ \hat{H}_1^{(2)}(t_1, 0)\hat{U}_1^{(0)}(t_1, 0) + 2\hat{H}_1^{(1)}(t_1, 0)\hat{U}_1^{(1)}(t_1, 0) \}
= \frac{1}{\hbar^2} \int_{-\infty}^{t} dt_1 e^{i\theta_1} \left\{ i \hat{A}_1(t_1), \frac{\partial}{\partial t} \hat{A}_1(t_1) \right\}
= \frac{1}{\hbar^2} \int_{-\infty}^{t} dt_1 e^{i\theta_1} \frac{\partial \hat{A}_1^2(t_1)}{\partial t_1}
= \left( \frac{i}{\hbar} \hat{A}_1(t) \right)^2 + O(\eta)
\]

The results for \( n = 1 \) and \( n = 2 \) suggest that

\[
\hat{U}_1^{(n)}(0, t) = \left( \frac{i}{\hbar} \hat{A}_1(t) \right)^n + O(\eta)
\]

for all \( n \geq 1 \). This in fact can be proven by induction. Without presenting the details, we thus find that in the \( \eta \to 0 \) limit,

\[
\hat{U}_1(t, \lambda) = \exp \left\{ \frac{i}{\hbar} \hat{A}_1(t) \right\}
= e^{i\hat{H}_1 t} \hat{T}(\lambda x(t), \lambda p(t))e^{-i\hat{H}_1 t}.
\]

The dynamic state of interest is thus given by

\[
|\Psi(t)\rangle = e^{-i\hat{H}_1 t/\hbar} |\Psi_0(t)\rangle
= e^{-i\hat{H}_1 t/\hbar} \hat{U}_1(t, \lambda) |\Phi_0\rangle
= e^{-i\hat{H}_0 t/\hbar} \hat{T}(\lambda x(t), \lambda p(t)) |\Phi_0\rangle.
\]

We have thus proved that, after the perturbation is switched on adiabatically, the final state is the ground state of \( \hat{H} \) oscillating together with the external potential.

The energy of the system in the state \(|\Psi(t)\rangle\) is

\[
E(t, \lambda) = \langle \Psi(t) | \hat{H}_0 + \sum_{i=1}^{N} V_{\text{ext}}(\hat{r}_i - \lambda \hat{z}_0(t)) |\Psi(t)\rangle.
\]

This energy can be obtained from Eq. \( (83) \) with the transcription \( x_0(t) \to -\lambda \hat{z}_0(t) \hat{z} \), giving

\[
E(t, \lambda) = E_0 + \lambda \hat{P}_x E_x + \lambda \langle \Phi_0 | \hat{P}_x | \Phi_0 \rangle \hat{z}_0^2(t) + \lambda M \omega_c^2 \langle \Phi_0 | \hat{R}_z | \Phi_0 \rangle \hat{z}_0(t).
\]
The terms in Eq. (B.25) linear in $\lambda$ oscillate harmonically at the frequency $\omega_z$ and do not contribute to the time-averaged energy absorption rate. We thus find

$$\frac{\partial E(t, \lambda)}{\partial t} = 0.$$  \hspace{1cm} (B.26)

This result is true to all orders in $\lambda$ and in particular, demonstrates that the linear response energy absorption rate (of order $\lambda^2$) vanishes when $\omega = \omega_z$.

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