Canonical forms of metric graph eikonal algebra and graph geometry

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Abstract

The algebra of eikonals $\mathfrak{E}$ of a metric graph $\Omega$ is an operator $C^*$-algebra determined by dynamical system with boundary control that describes wave propagation on the graph. In this paper, two canonical block forms (algebraic and geometric) of the algebra $\mathfrak{E}$ are provided for an arbitrary connected locally compact graph. These forms determine some metric graphs (frames) $\mathfrak{F}_a$ and $\mathfrak{F}_g$. Frame $\mathfrak{F}_a$ is determined by the boundary inverse data. Frame $\mathfrak{F}_g$ is related to graph geometry. A class of ordinary graphs is introduced, whose frames are identical: $\mathfrak{F}_a \equiv \mathfrak{F}_g$. The results are supposed to be used in the inverse problem that consists in determination of the graph from its boundary inverse data.

1 Introduction

• The eikonal algebra of the metric graph was introduced in [6] and studied in [3, 4, 9, 10]. It is a noncommutative operator $C^*$-algebra determined by dynamical system that describes the propagation of waves into the graph; the waves are initiated by sources (controls) at the boundary vertices. The interest to such an algebra is motivated by possible applications to inverse problems, in particular for graph reconstruction from boundary spectral and dynamical inverse data. The relation between inverse problems and Banach algebras is a separate topic within the boundary control method (BC-method; [2]).

• In our paper, for an arbitrary connected locally compact graph $\Omega$ and its corresponding eikonal algebra $\mathfrak{E}_\Omega$, we describe two of its canonical block forms — algebraic and geometric. Both forms are derived from the original parametric representation of the algebra $\mathfrak{E}_\Omega$. The algebraic form is known and has been described in details in [1] (see also examples in [3, 9]). The geometric form is a new one.

• Our results are as follows.

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0This work was supported by the Leonhard Euler Mathematical Institute, agreement no. 075-15-2022-289, grant RFBR 18-01-00269, and partially by the Young Mathematicians of Russia contest.
It is shown that both forms determine some metric graphs: frames $\mathfrak{F}^\Sigma_a$ and $\mathfrak{F}^\Sigma_g$ respectively.

The frame $\mathfrak{F}^\Sigma_a$ is the spectrum (the set set of irreducible representations) of $\mathfrak{E}_\Sigma$ factorized by some relation and equipped with relevant coordinates. It is an invariant of the algebra $\mathfrak{E}_\Sigma$: any of its isomorphic copy determines frame $\mathfrak{F}^\Sigma_a$ up to isometry of metric spaces.

The frame $\mathfrak{F}^\Sigma_g$ is directly related to the graph geometry; more precisely, to the shape of the domain of the graph filled with waves. It is the result of factorization (gluing) of this domain according to a relation which has a transparent geometric meaning. Frame $\mathfrak{F}^\Sigma_g$ also gets relevant coordinates which have the same nature as in the case of the frame $\mathfrak{F}^\Sigma_a$.

The concept of ordinary graphs is introduced. Their frames $\mathfrak{F}^\Sigma_a$ and $\mathfrak{F}^\Sigma_g$ are identical: the bijection linking points of frames with the same coordinates turns out to be an isometry.

- Possible applications of our results to inverse problems are discussed in Comments at the end of the paper. So far there are no successful applications, that explains the lack of references to the extensive literature on inverse problems on graphs.

The introductory part of the paper has significant overlaps with the materials of previous papers on the eikonal algebra. This is justified by the wish to make the paper as independent as possible.

2 Waves on graph

Metric graph

- Let $I_j := (0, a_j) = \{s_j \in \mathbb{R} \mid 0 < s_j < a_j < \infty\}, j = 1, \ldots, d$ be finite intervals. The set $S_d := \{0\} \sqcup I_1 \sqcup \cdots \sqcup I_d$ equipped with the metric

$$\tau(x, y) := \begin{cases} |x - y|, & \text{in } x, y \in I_j \\ x + y, & \text{at } x \in I_i, y \in I_j, i \neq j \\ y, & \text{at } x = 0, y \in I_j \\ x, & \text{at } x \in I_i, y = 0 \\ 0, & \text{in } x = y = 0 \end{cases} \quad (2.1)$$

is said to be a $d$-star.

- Metric graph $\Omega$ is a connected metric space which is locally isometric to either a star or an interval. Its interior vertices are points whose (small) neighborhoods are isometric to stars $S_d$ with $d > 2$; the boundary vertices correspond to stars $S_1$ (semi-open intervals). The number $d$ is called the valence of a vertex. The absence of vertices with valence 2 in the further considerations is explained by the fact that 2-star is isometric to an interval. Therefore the part of the graph that is isometric to $S_2$ can be replaced by the appropriate interval. However, below, while considering frames, the vertices with $d = 2$ are used. The edges are the maximal parts of $\Omega$ that do not contain vertices and are isometric to intervals.
Thus, $\Omega = E \sqcup V \sqcup \Gamma$, where $E = \{e_i\}_{i=1}^p$ are edges; $V = \{v_j\}_{j=1}^q$, $v_k$ - internal vertices; $\Gamma = \{\gamma_k\}_{k=1}^n$, $\gamma_k$ - boundary vertices; each vertex $w \in V \sqcup \Gamma$ has a neighborhood in $\Omega$ that is isometric to $S_d$. Further, we assume that the set of boundary vertices is not empty.

The given definition admits the edges of infinite length.

- Let $\tau$ be a metric on $\Omega$, $A \subset \Omega$ a subset. By $\Omega \tau[A] := \{x \in \Omega \mid \tau(x, A) < r\}$ we denote the metric neighborhood of radius $r > 0$ of $A$.

Operators and spaces on graph

- Let us orient each of the edges by a linear order $\prec$ (in one of two possible ways). For the edge $e \in E$, the function $y$ on $\Omega$ and the point $x \in e$, we define the derivative
  \[
  \frac{dy}{de}(x) := \lim_{m \to x} \frac{y(m) - y(x)}{s_m \tau(m,x)},
  \]
  where $s_m = 1$ for $x \prec m$ and $s_m = -1$ for $m \prec x$. Note also that the value of the second derivative $\frac{d^2 y}{de^2}$ is independent from the orientation of the edges.

  Let us choose a vertex $w \in V \cup \Gamma$ and its neighborhood in $\Omega$, which is isometric to a star $S_d$. We say that an edge $e$ is adjacent to $w$ if $w \in e$ (closure in $\Omega$). For each $e$ adjacent to $w$, we define outgoing derivative
  \[
  \frac{dy}{de^+}(w) := \lim_{e \ni m \to w} \frac{y(m) - y(w)}{\tau(m,w)},
  \]
  which does not depend on the orientation. For each vertex $w \in V \cup \Gamma$ and function $y$ we define outgoing flow
  \[
  \Pi_w[y] = \sum_{e \ni w} \frac{dy}{de^+}(w).
  \]

- Consider a real Hilbert space $\mathcal{H} := L_2(\Omega)$ of functions on $\Omega$ with the scalar product
  \[
  (y, u)_{\mathcal{H}} = \int_\Omega yu \, d\tau = \sum_{e \in E} \int_e yu \, d\tau.
  \]
  By $C(\Omega)$ we denote the space of continuous functions with the norm $\|y\| = \sup \|y(\cdot)\|_\Omega$. We assign the function $y$ to the Sobolev class $\mathcal{H}^2(\Omega)$ if $y \in C(\Omega)$ and $\frac{du}{de}, \frac{d^2 y}{de^2} \in L_2(e)$ on each edge $e \in E$.

  Introduce the Kirchhoff class
  \[
  \mathcal{K} := \{y \in \mathcal{H}^2(\Omega) \mid \Pi_v[y] = 0, \; v \in V\}.
  \]
  The Laplace operator on the graph is introduced via the definition
  \[
  \Delta : \mathcal{K} \to \mathcal{K}; \quad \text{Dom} \; \Delta = \mathcal{K}; \quad (\Delta y) \bigg|_e = \frac{d^2 y}{de^2}, \; e \in E. \quad (2.2)
  \]
  It is densely defined, closed, and does not depend on the orientation of the edges.
Dynamical system with boundary control

- The boundary value problem that describes wave propagation in the graph, is of the form

\[
\begin{align*}
  u_{tt} - \Delta u &= 0 \quad \text{in } H, \quad 0 < t < T; \quad (2.3) \\
  u(\cdot, t) &\in \mathcal{H} \quad \text{for } 0 \leq t \leq T; \quad (2.4) \\
  u|_{t=0} = u_t|_{t=0} &= 0 \quad \text{in } \Omega; \quad (2.5) \\
  u &= f \quad \text{on } \Gamma \times [0, T]. \quad (2.6)
\end{align*}
\]

Here \( T > 0 \) is end time; \( f = f(\gamma, t) \) is boundary control; \( u = u^f(x, t) \) is the solution (wave). With \( C^2 \)-smooth (with respect to \( t \)) control \( f \) disappearing near \( t = 0 \), the problem has a unique classical solution \( u^f \).

As follows from the definition (2.2), on each edge \( e \) the solution \( u^f \) satisfies the homogeneous string equation \( u_{tt} - u_{ee} = 0 \). Therefore, the waves propagate from the boundary \( \Gamma \) into \( \Omega \) with unit velocity. As a consequence, if the control operates from a part of the boundary \( \Sigma \subseteq \Gamma \), i.e., \( \text{supp } f \subset \Sigma \times [0, T] \) is satisfied then we have the relation

\[
\text{supp } u^f(\cdot, t) \subset \Omega^f[\Sigma], \quad t > 0. \quad (2.7)
\]

- The control space \( \mathcal{F}^T := L_2(\Gamma \times [0, T]) \) with the scalar product

\[
(f, g)_{\mathcal{F}^T} := \sum_{\gamma \in \Gamma} \int_0^T f(\gamma, t) g(\gamma, t) \, dt
\]

is called an external space of the system (2.3)-(2.6). It is represented in the form

\[
\mathcal{F}^T = \oplus \sum_{\gamma \in \Gamma} \mathcal{F}^T_{\gamma}
\]

as a sum of subspaces \( \mathcal{F}^T_{\gamma} := \{ f \in \mathcal{F}^T \mid \text{supp } \subset \{ \gamma \times [0, T] \} \} \).

The space \( \mathcal{H} = L_2(\Omega) \) is called internal, the waves \( u^f(\cdot, t) \) are its time-dependent elements.

Eikonals

Let us choose the boundary vertex \( \gamma \in \Sigma \) and fix a final moment \( t = T \). The set of waves

\[
\mathcal{U}^{s}_{\gamma} := \{ u^f(\cdot, s) \mid f \in \mathcal{F}^T_{\gamma} \} \subset \mathcal{H}, \quad 0 \leq s \leq T
\]

is called reachable (from vertex \( \gamma \), at time \( t = s \)). It can be shown that \( \mathcal{U}^{s}_{\gamma} \) are (closed) subspaces in \( \mathcal{H} \). As \( s \) grows, they expand: \( \mathcal{U}^{s}_{\gamma} \subset \mathcal{U}^{s'}_{\gamma} \) at \( s < s' \). Let \( P^s_{\gamma} \) in \( \mathcal{H} \) be (orthogonal) projectors on subspaces \( \mathcal{U}^{s}_{\gamma} \). The family \( \{ P^s_{\gamma} \mid 0 \leq s \leq T \} \) is continuous with respect to \( s \) and defines an eikonal operator (briefly – eikonal)

\[
E_{\gamma} : \mathcal{H} \to \mathcal{H}, \quad E_{\gamma} := \int_0^T (s + 1) \, dP^s_{\gamma}.
\]
From the definition it follows that $E_\gamma$ is a bounded self-adjoint positive operator. Let $\gamma_T^* := \sup_{\gamma' \in \Gamma} \tau(\gamma, \gamma') \leq \infty$ be the time of filling the graph with waves from the vertex $\gamma$ (see (2.7)). The following statement appears to be true

**Proposition 1** Eikonals $E_\gamma$ satisfies $\text{Ran} E_\gamma = \mathcal{U}_T^*$ and $\text{Ker} E_\gamma = \mathcal{H} \ominus \mathcal{U}_T^*$. At $T < T_\gamma^*$ it has eigenvalue 0 of infinite multiplicity and a simple absolutely continuous spectrum filling the segment $[1, T + 1]$.

Note that existence of at least one edge of infinite length leads to the fact that the spectrum of the eikonal has no lacunas and for any $T < \infty$ the equality

$$\sigma(E_\gamma|_{\mathcal{U}_T}) = \sigma_{ac}(E_\gamma) = [1, T + 1]$$

holds [4]. The existence of lacunae in the spectrum in the case $T_\gamma^* < T < \infty$ for compact graphs is an open question.

### 3 Algebra of eikonals

**On algebras**

The following information about $C^*$-algebras is taken from [1, 13, 8, 12].

- Recall that a Banach algebra $\mathfrak{A}$ with involution $(\cdot)^*$ is called a $C^*$-algebra if

$$||a^* a|| = ||a||^2, \quad a \in \mathfrak{A}$$

holds. In particular, such are the algebras of bounded operators $\mathcal{B}(H)$ in the Hilbert space $H$, in which the role of involution plays the operator conjugation.

All algebras in the paper are operator $C^*$-algebras [12].

The notation $\mathfrak{A} \cong \mathfrak{B}$ will mean that $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are related by $*$-isomorphism (hereafter briefly – isomorphism). Such isomorphism is an isometry.

For the set $S \subset \mathfrak{A}$, by $\vee S$ we denote the minimal $C^*$-(sub)algebra in $\mathfrak{A}$ that contains $S$.

- For the algebra $A = A_1 \oplus \cdots \oplus A_p$, by $\text{pr}_j$ we denote the projections $a_1 \oplus \cdots \oplus a_j \oplus \cdots a_p \mapsto a_j$. Let $C \subset A$ be a subalgebra; the algebras $\text{pr}_j C$ are called the blocks of $C$.

We say that algebra $C$ separates the blocks $A_{i_1}, \ldots, A_{i_m}$ of algebra $A$ if, for any set of elements $a_{i_k} \in A_{i_k}, \ldots, a_{i_m} \in A_{i_m}$, one can find the elements $c_{i_1}, \ldots, c_{i_m}$ in $C$ that satisfy

$$\text{pr}_{i_k} c_{i_k} = \begin{cases} a_{i_k}, & i_k = i_l; \\ 0_{i_l}, & i_k \neq i_l; \end{cases}$$

If algebra $A$ has the blocks that are not separated by algebra $C$, we say that $C$ links blocks of $A$. 

5
A representation of \( C^* \)-algebra \( A \) is a \( * \)-homomorphism \( \pi : A \to \mathfrak{B}(H) \). The equivalence of the representations \( \pi \sim \pi' \) means that \( \iota \pi(a) = \pi'(a) \iota, \ a \in A \), where \( \iota : H \to H' \) is a unitary operator. The representation is irreducible if the operators \( \pi(A) \) share no nonzero invariant subspace in \( H \).

The spectrum of \( C^* \)-algebra \( A \) is the set \( \hat{A} \) of equivalence classes of its irreducible representations. The equivalence class (a point of the spectrum) corresponding to the representation \( \pi \) will be denoted by \( \hat{\pi} \). The spectrum is equipped with the canonical Jacobson topology [8, 12].

Isomorphism of algebras \( u : A \to \mathfrak{B} \) induces the correspondence between the representations \( \pi \mapsto u_* \pi \), \( (u_* \pi)(b) := \pi(u^{-1}(b)) \), which extends to the canonical homeomorphism of the spectra:

\[
\hat{\mathfrak{A}} \ni \hat{\pi} \mapsto u_* \hat{\pi} \in \hat{\mathfrak{B}}, \quad u_* \hat{\pi} := \{ u_* \pi \mid \pi \in \hat{\pi} \}.
\]

**Standard algebras and their spectra**

- By \( M^n \) we denote the algebra of the real \( n \times n \)-matrices viewed as operators in \( \mathbb{R}^n \) and equipped with the corresponding (operator) norm. This algebra is irreducible.

  The \( C^* \)-subalgebra \( \mathfrak{A} \subset M^n \) can also be considered irreducible if \( \mathfrak{A} \cong M^k \), where \( k \leq n \) holds. Such an algebra, in a suitable basis in \( \mathbb{R}^n \), takes a block-diagonal form and is composed of two blocks, one of which is \( M^k \) and the second (if exists) is zero.

**Proposition 2** Any \( C^* \)-subalgebra of the algebra \( M^n \) is isomorphic to a direct sum \( \bigoplus k M^{n_k} \), where \( \sum k n_k \leq n \).

- By \( C([a,b], M^n) \) we denote the algebra of continuous \( M^n \)-valued functions with the norm \( \|\phi\| = \sup_{a \leq t \leq b} \|\phi(t)\|_{M^n} \). By the same symbol we denote the (sub)algebra in \( \mathfrak{B}(L_2([a,b]; \mathbb{R}^n)) \) of the operators that multiply square-integrable \( \mathbb{R}^n \)-valued functions by functions from \( C([a,b], M^n) \). The correspondence \( \phi \mapsto \phi \cdot \) establishes the isomorphism of these algebras.

  The following fact plays an important role (see, e.g., [12]).

**Proposition 3** The representations

\[
\pi_t : C([a,b], M^n) \to M^n; \quad \pi_t(\phi) := \phi(t), \quad a \leq t \leq b \tag{3.1}
\]

are irreducible; their equivalence classes exhaust the spectrum of the algebra \( C([a,b], M^n) \). For any irreducible representation \( \pi \) of the algebra \( C([a,b], M^n) \) there exists a single point \( t \in [a,b] \) such that \( \pi \sim \pi_t \).

Algebra \( C([a,b]; M^n) \) contains subalgebras

\[
\hat{C}([a,b]; M^n) := \{ \phi \in C([a,b]; M^n) \mid \phi(a) \in M_{a}, \ \phi(b) \in M_b \} \tag{3.2}
\]
where $M_a, M_b$ are $C^*$-subalgebras of $M^n$, which we call *boundary algebras*. Proposition 2 implies

\begin{align*}
M_a \cong \bigoplus_{k=1}^{n_a} M^{\kappa_k}, \quad \kappa_1 + \cdots + \kappa_{n_a} \leq n; \\
M_b \cong \bigoplus_{k=1}^{n_b} M^{\lambda_k}, \quad \lambda_1 + \cdots + \lambda_{n_b} \leq n. \quad (3.3)
\end{align*}

In the case $M_a = M_b = M^n$ we have $\hat{C}([a, b]; M^n) = C([a, b]; M^n)$. We call $\hat{C}([a, b]; M^n)$ *standard algebras*.

- The standard algebra spectrum consists of the classes $\pi_t$, $t \in (a, b)$ of irreducible representations of the form (3.1), and the classes $\hat{\pi}_a^k, \hat{\pi}_b^k$, into which the representations of the boundary algebras $M_a, M_b$ (generally speaking, reducible) are decomposed. If, for example, $n_a \geq 2$, then $\pi_a$ decomposes into irreducible representations

\[ \pi_a^k: \phi(a) \mapsto [\phi(a)]^k \in M^{\kappa_k}, \quad (3.4) \]

where $[\ldots]^k$ is the $k$-th block of the block-diagonal matrix in representations (3.3). In this case we say that the representations $\hat{\pi}_a^1, \ldots, \hat{\pi}_a^n$ form a *cluster* in the spectrum of the standard algebra. This term is motivated by the fact that they are inseparable from each other in the Jacobson topology [1]. A similar cluster may exist at the right-hand end at $t = b$. At the same time, all $\hat{\pi}_t$ with different $t \in (a, b)$ are separable from each other and from clusters (see [6], [3]). The spectrum of the algebra $C([a, b]; M^n)$ contains no clusters.

The preceding facts and results are taken from [11, 13, 8, 12].

- Here is a general concept. Let $\sim_0$ be a symmetric reflexive (but, may be, not transitive) relation on a set $X$. Consider on $X$ the relation $\sim$ that is defined by the following rule: $x \sim y$ if there is a finite set of elements $x_1, \ldots, x_n$ such that $x \sim_0 x_1 \sim_0 \cdots \sim_0 x_n \sim_0 y$ holds. This relation will be called the *transitive closure* of the relation $\sim_0$. One can easily see that $\sim$ is an equivalence relation.

### Algebra $E$:

In the rest of the paper, unless otherwise specified, the final moment $T$ in (2.3)–(2.6) is considered fixed and is not always indicated in the notation.

- Let us choose a subset $\Sigma = \{\gamma_1, \ldots, \gamma_N\} \subset \Gamma$; the vertices, which it consists of, are called *controlling*. Due to (2.7) we have

\[ \mathcal{H}_\gamma^s \subset \mathcal{H}_\Sigma^T := \{y \in \mathcal{H} \mid \text{supp } y \subset \Omega^T[\Sigma]\}, \quad \gamma \in \Sigma, \quad 0 \leq s \leq T. \]

As a consequence, for a full reachable set we have the embedding

\[ \mathcal{H}_\Sigma^T := \text{span} \{ \mathcal{H}_\gamma^s \mid \gamma \in \Sigma \} \subset \mathcal{H}_\Sigma^T, \quad 0 \leq s \leq T. \]

Thus, if the controls $f$ act from the vertices $\gamma \in \Sigma$ only, then the natural interior space of the system (2.3)–(2.6) is the subspace $\mathcal{H}_\Sigma^T = L_2(\Omega^T[\Sigma])$. Eikonals $E_\gamma$, which correspond to the vertices of $\gamma \in \Sigma$, are the operators in $\mathcal{H}_\Sigma^T$.

- By the eikonal algebra we call the operator $C^*$-algebra

\[ \mathcal{E}_\Sigma := \bigvee\{E_\gamma \mid \gamma \in \Sigma\} \subset \mathcal{B}(\mathcal{H}_\Sigma^T), \]

where $\mathcal{B}(\cdot)$ is the space of bounded operators.
introduced in [6]. The new results relating to it, which constitute the main subject of our paper, are placed in section 5. Their presentation requires certain preparation, which is done in the remaining part of the section 3 and in the section 4 in which known facts and results are briefly described. The reader can find more details in [6, 4].

Graph parametrization

• A set \( \Lambda = \{x_1, \ldots, x_m\} \subset \Omega T[\Sigma] \) is called determination set, if for any function \( y \in C(\Omega T[\Sigma]) \) and for all \( \gamma \in \Sigma \), the values of functions \( E_\gamma y \) on \( \Lambda \) are determined by the values of \( y \) on \( \Lambda \). Equivalently, the implication takes place

\[
y|_\Lambda = 0 \Rightarrow (E_\gamma y)|_\Lambda = 0, \quad \gamma \in \Sigma.
\]

(3.5)

Every point \( x \in \Omega T[\Sigma] \), with the possible exception of a finite number of so-called critical points, belongs to some (generally speaking, not unique) determination set \( \Lambda[x] \): see [6, 4].

• As it follows from (3.5), the pairs \( \{y|_\Lambda, (E_\gamma y)|_\Lambda\} \) constitute the graph of an operator \( e_\gamma \), which acts in the \( m \)-dimensional space \( l_2(\Lambda) \) of functions (vectors) with the scalar product \( (a, b) = \sum_{k=1}^{m} a(x_k) b(x_k) \). The correspondence \( \pi : E_\gamma \rightarrow e_\gamma \) extends from the generators \( E_\gamma \), \( \gamma \in \Sigma \) to the whole algebra \( \mathcal{E}_\Sigma \) and provides its \( m \)-dimensional representation

\[
\pi : \mathcal{E}_\Sigma \rightarrow \mathcal{B}(l_2(\Lambda)).
\]

(3.6)

Let \( \{\chi_1, \ldots, \chi_m\} \), \( \chi_i(x_k) = \delta_{ik} \) be the basis in \( l_2(\Lambda) \) of indicators of the points forming \( \Lambda \). In this basis, operators \( e_\gamma \) get the matrices \( \tilde{e}_\gamma \). Consequently, the representation (3.6) takes a matrix form:

\[
\pi : \mathcal{E}_\Sigma \rightarrow M^m.
\]

(3.7)

As will be seen later, almost all representations of the eikonal algebra are of the form (3.6), (3.7).

• When the point \( x \in \Lambda[x] \) is smoothly varied, the points \( x_1, \ldots, x_m \) composing \( \Lambda[x] \), vary and cover open intervals (cells) \( \omega_1, \ldots, \omega_m \) of the equal length \( |\omega_k| = \epsilon > 0 \), which are located on the graph edges. The cells form a family

\[
\Phi := \bigcup_{k=1}^{m} \omega_k = \bigcup_{x \in \omega} \Lambda[x],
\]

where \( \omega \) is any cell of the family \( \Phi \).

The entire (sub)graph \( \Omega T[\Sigma] \) that is filled with waves is partitioned into a finite number of families \( \Phi^1, \ldots, \Phi^J \) of this form

\[
\Phi^j := \bigcup_{k=1}^{m_j} \omega^j_k, \quad |\omega^j_k| = \epsilon_j
\]

(3.8)
and a finite set $\Theta$ of the so-called critical points that cover cell junctions:

$$\overline{\Omega^T[\Sigma]} = \left[ \bigcup_{j=1}^{J} \Phi^j \right] \cup \Theta.$$  

As a consequence, there occurs the decomposition

$$L_2(\Omega^T[\Sigma]) = \bigoplus_{j=1}^{J} L_2(\Phi^j). \quad (3.9)$$

- Each family $\Phi^j$ in (3.8) is parameterized as follows. We take a cell $\omega^j_k \subset \Phi^j$ and fix one of its endpoints $c \in \omega^j_k$. The point $x \in \omega^j_k$ gets the parameter $r := \tau(x,c), \; 0 < r < |\omega^j_k|$ and is denoted by $x_k(r)$:

$$x_k(r) := \Lambda[x(r)] \cap \omega^j_k, \quad 0 < r < \epsilon_j.$$  

As $r$ changes, the points $x_1(r), \ldots, x_{m_j}(r)$ that constitute the set $\Lambda[x(r)]$ move coherently along the edges at unit speed and cover the corresponding (parameterized) cells $\omega^j_k$.

The second of two possible parametrizations corresponds to the choice of the other endpoint of the cell $\omega^j_k \subset \Phi^j$. Other $\Phi^j$ are parametrized independently (in one of two possible ways). In what follows, we assume that all families $\Phi^j$ are parametrized.

- These facts have a quite transparent geometrical meaning associated with the so-called hydra $H^T_{\Sigma}$ - a space-time graph defined by the dynamical system (2.3)–(2.6); see [6] for details and illustrations.

Note in addition that the dynamics-related partition of the form (3.8) is not unique: others can be obtained from it, for example, by partitioning the families $\Phi^j$ into families with smaller cells. The most interesting partition is the one that corresponds to minimal (by $m = \#\Lambda$) sets $\Lambda$. The relations between minimality and irreducibility of representations (3.6) are discussed below.

- The parametrization of the graph induces the parametrization of representations (3.10): we have the series

$$\pi_j(r) : \mathfrak{C}_{\Sigma} \to \mathfrak{B}(I_2(\Lambda[x(r)])) \quad x(r) \in \Phi^j, \; 0 < r < \epsilon_j; \; j = 1, \ldots, J. \quad (3.10)$$

It will be revealed later that representations of this type essentially cover the spectrum of the algebra $\mathfrak{C}_{\Sigma}$.

Further, along with (3.10), it is convenient to use concrete matrix-valued representations of the form (3.7). For this purpose let us identify the spaces $I_2(\Lambda[x(r)]) \equiv I_2(\Lambda[x(r')]) =: I^j_2 \cong \mathbb{R}^{m_j}$ for $r, r' \in (0, \epsilon_j)$, the indicator bases in them, and define the appropriate basis $\{\chi_1, \ldots, \chi_{m_j}\}$ in $I^j_2$.  

9
Parametrization of eikonal algebra

Parametrization of the graph determines the parametric form of the algebra $\mathfrak{E}_\Sigma$. Let us describe it according to [6].

- The partition (3.8) corresponds to the set of unitary operators $U^j : L_2(\Phi^j) \rightarrow L_2((0, \epsilon_j); l^j_2)$,

$$(U^j y)(r) := y|_{\Lambda[x(r)]} = \left( \begin{array}{c} y(x_1(r)) \\ \vdots \\ y(x_m(r)) \end{array} \right) \in l^j_2, \quad r \in (0, \epsilon_j); \quad j = 1, \ldots, J, \quad (3.11)$$

that define the unitary operator $U : L_2(\Omega_T[\Sigma]) \rightarrow \bigoplus_{j=1}^J L_2((0, \epsilon_j); l^j_2)$, $U := \bigoplus_{j=1}^J U^j$.

All eikonals $E_{\gamma}$ are reduced by subspaces $L_2(\Phi^j)$ from the decomposition (3.9), which expectedly follows from $E_{\gamma}$-invariance of determination sets: see (3.5). They turn into operators that multiply the elements of the space of parametric representation $\bigoplus_{j=1}^J L_2((0, \epsilon_j); l^j_2)$ by the operator-valued functions

$$UE_{\gamma}U^{-1} = \bigoplus_{j=1}^J \left( \bigoplus_{i=1}^{n_{\gamma j}} \tau_i^{\gamma j}(r) P^i_{\gamma j} \right) \in \bigoplus_{j=1}^J C \left( [0, \epsilon_j]; \mathfrak{B}(l^j_2) \right), \quad \mathfrak{B}(l^j_2) \cong M^{n_{\gamma j}}, \quad (3.12)$$

where each (scalar) function $\tau_i^{\gamma j}$ depends on its argument $r_j \in [0, \epsilon_j]$ and is one of the following forms:

$$\tau_i^{\gamma j}(r) = t_i^{\gamma j} + r \quad \text{or} \quad \tau_i^{\gamma j}(r) = \tilde{t}_i^{\gamma j} - r = (t_i^{\gamma j} + \epsilon_j) - r \quad (3.13)$$

with the constants $t_i^{\gamma j}, \tilde{t}_i^{\gamma j}$:

$$t_i^{\gamma j} := \min_{r \in [0, \epsilon_j]} \tau_i^{\gamma j}, \quad \tilde{t}_i^{\gamma j} := \max_{r \in [0, \epsilon_j]} \tau_i^{\gamma j}.$$ 

The ranges of values (the segments $\tau_i^{\gamma j} = [t_i^{\gamma j}, \tilde{t}_i^{\gamma j}] \subset \mathbb{R}$) for different pairs of indices $i, j$ and $i', j'$ (but for the same $\gamma$) can intersect only by the endpoints (so that, either $t_i^{\gamma j} = t_i'^{\gamma j'}$ or $\tilde{t}_i^{\gamma j} = \tilde{t}_i'^{\gamma j'}$ holds). There is an equality that relates functions $\tau_i^{\gamma j}$ to the absolutely continuous spectrum of eikonals:

$$\sigma_{\text{ac}}(E_{\gamma}) = \bigcup_{j=1}^J \bigcup_{i=1}^{n_{\gamma j}} \text{ran} \tau_i^{\gamma j}.$$ 

The operators $P^i_{\gamma j}$ are one-dimensional projectors in $l^j_2$, mutually orthogonal for each fixed vertex $\gamma$: $P^i_{\gamma j} P^k_{\gamma j} = \delta_{ik} P^i_{\gamma j}$. Remarkably and importantly, in (3.12) they remain constant – they do not depend on $r \in (0, \epsilon_j)$. 

10
The summands

$$[UE\gamma U^{-1}]^j \equiv \sum_{i=1}^{n_{\gamma_j}} x_{\gamma_j}(i)P_{\gamma_j}^i \in C \left([0,\varepsilon_j];\mathfrak{B}(L_2^j)\right)$$

in (3.12) are the blocks of the eikonal $E\gamma$ in the parametric representation.

- From the definition $E_{\Sigma} := \bigvee\{E_{\gamma} | \gamma \in \Sigma\}$ and (3.12) one can get the representation

$$U E_{\Sigma} U^{-1} \subset \bigoplus_{j=1}^J C \left([0,\varepsilon_j];\mathfrak{B}(L_2^j)\right); \quad pr_j U E_{\Sigma} U^{-1} =: [UE_{\Sigma}U^{-1}]^j = \bigvee \left\{[UE\gamma U^{-1}]^j | \gamma \in \Sigma\right\} \subset C \left([0,\varepsilon_j];\mathfrak{B}(L_2^j)\right)$$

which we will refer to as the source parametric form of the eikonal algebra. In the first embedding, it is essential that the algebra $UE_{\Sigma}U^{-1}$ can link blocks of the algebra to the right. The characterization of these connections is the main subject of [4].

Let us repeat again that the above mentioned facts and results are taken from [6, 4].

- Let

$$P_j := \{P_{\gamma_j}^i | i = 1,\ldots,n_{\gamma_j}; \gamma \in \Sigma\}. \quad (3.15)$$

be the set of projectors associated with the block $[UE_{\Sigma}U^{-1}]^j$. The equations (3.14) are clarified as follows:

$$U E_{\Sigma} U^{-1} \subset \bigoplus_{j=1}^J C \left([0,\varepsilon_j];\mathfrak{B}(L_2^j)\right); \quad [UE_{\Sigma}U^{-1}]^j \subset C([0,\varepsilon_j];\mathfrak{B}(L_2^j)); \quad (3.16)$$

where $\mathfrak{B}(L_2^j) \cong M^{m_j}$.

### 4 First canonical form

In [4], the source parametric form of the eikonal algebra is transformed to some canonical form that represents $E_{\Sigma}$ as a sum of independent standard algebras of the type (3.2). Let us briefly describe the transformation procedure.

**Partitioning into blocks**

- On each set of projectors $P_j$ we introduce the relation $\overset{\text{nort}}{\sim} \otimes 0$ ("not orthogonal") by the rule: $P_{\gamma_j}^i \overset{\text{nort}}{\sim} 0 P_{\gamma_j}^{i'}$ if $P_{\gamma_j}^i P_{\gamma_j}^{i'} \neq 0$. Then we define its transitive closure $\overset{\text{nort}}{\sim}$.

  Let $[P]$ be the equivalence class of the projector $P \in P_j$ with respect to $\overset{\text{nort}}{\sim}$. It can be shown [4] that the splitting

$$P_j = [P_{j_1}^1] \cup \ldots \cup [P_{j_p}^p] \quad (4.1)$$
corresponds to the decomposition of the algebra $\mathcal{P}^j$ into \textit{unreducible} blocks $\mathcal{P}^j_p$:

$$\mathcal{P}^j = \bigoplus_{p=1}^{p_j} \mathcal{P}^j_p,$$

(4.2)

where $\mathcal{P}^j_p := \vee \{P \}^j_p \cong \mathbb{M}^j_{\kappa_j^p}, \quad \kappa_1^j + \cdots + \kappa_{p_j}^j \leq m_j$.

- In accordance with (4.1), eikonal blocks in (3.16) are decomposed into sub-blocks:

$$[U E_\gamma U^{-1}]^j_j \sum_{p=1}^{p_j} [U E_\gamma U^{-1}]^j_j, \quad [U E_\gamma U^{-1}]^j_j := \sum_{P^j \in [P]^j_p} \tau_\gamma^j(P^j) \in \mathbb{C}(\{0, \epsilon_j\}; \mathcal{P}^j_p),$$

(4.3)

and the eikonal algebra satisfies

$$U \mathcal{E}_U^{-1} \subset \bigoplus_{j=1}^{J} \sum_{p=1}^{p_j} \mathbb{C}(\{0, \epsilon_j\}; \mathcal{P}^j_p).$$

Simplifying the notations, let us go to a through numbering of blocks, algebras and parameters:

$$[U E_\gamma U^{-1}]^1_1, \ldots, [U E_\gamma U^{-1}]^1_{p_1}, \ldots, \ldots ; [U E_\gamma U^{-1}]^j_j, \ldots, [U E_\gamma U^{-1}]^J_J, \text{ } \gamma \in \Sigma; \quad [P]^1, \ldots, [P]^j, \ldots, [P]^J, \quad \mathcal{P}^1, \ldots, \mathcal{P}^J,$$

and rewrite the last relation in the form

$$U \mathcal{E}_U^{-1} \subset \bigoplus_{l=1}^{L} \mathbb{C}(\{0, \epsilon_l\}; \mathcal{P}_l); \quad \text{pr}_l U \mathcal{E}_U^{-1} =: [U \mathcal{E}_U^{-1}]_l =$$

$$= \vee \{ [U E_\gamma U^{-1}]_l \mid \gamma \in \Sigma \}, \quad [U E_\gamma U^{-1}]_l = \sum_{P^k_l \in [P]^k_l} \tau_\gamma^k(P^k_l) \in \mathbb{C}(\{0, \epsilon_l\}; \mathcal{P}_l),$$

(4.4)

with \textit{irreducible} $\mathcal{P}_l$.

\subsection*{Connecting the blocks}

The algebra $U \mathcal{E}_U^{-1}$ in (3.14) consists of new (with respect to (3.14)) blocks $[U \mathcal{E}_U^{-1}]_l := \vee \{ [U E_\gamma U^{-1}]_l \mid \gamma \in \Sigma \}$. A further transformation of the decomposition (4.4) is possible: it consists in merging some of these blocks. Let us briefly describe the corresponding procedure; see [3] for details.
Let us choose an element \( e \in \mathfrak{E}_\Sigma \). Turning to the parametric form in accordance with (4.4) we have the decomposition:

\[
U e U^{-1} = \bigoplus_{l=1}^{L} [U e U^{-1}]_l, \quad [U e U^{-1}]_l \in C([0, \epsilon_l]; \mathfrak{P}_l).
\]

The irreducibility of algebras \( \mathfrak{P}_l \) implies the irreducibility of representations of the eikonal algebra of the form

\[
\pi^l_\gamma : \mathfrak{E}_\Sigma \to \mathfrak{P}_l, \quad \pi^l_\gamma (e) := [U e U^{-1}]_l (r), \quad 0 < r < \epsilon_l
\]

(See [4] and Proposition 3). At the same time, the boundary representations

\[
\rho_\gamma^l \mathfrak{E}_\Sigma \to \mathfrak{P}_l; \quad \rho_\gamma^- (e) := [U e U^{-1}]_l (0), \quad \rho_\gamma^+ (e) := [U e U^{-1}]_l (\epsilon_l), \quad (4.5)
\]

may, in general, turn out to be reducible. In the same time, the representations \( \rho^-_\gamma \) and \( \rho^+_\gamma \) corresponding to the same block are obviously not equivalent. The reason for this is that, due to the monotonicity of the functions \( \tau^l_{\gamma l} \) (see (3.13)), the eikonal matrices \( \sum_{k=1}^{n_{\gamma l}} \tau^l_{\gamma l}(0) P^l_{\gamma l} \) and \( \sum_{k=1}^{n_{\gamma l}} \tau^l_{\gamma l}(\epsilon_l) P^l_{\gamma l} \) do have distinct sets of eigenvalues \( \{ \tau^l_{\gamma l}(0) \}_{k=1}^{n_{\gamma l}} \neq \{ \tau^l_{\gamma l}(\epsilon_l) \}_{k=1}^{n_{\gamma l}} \), which excludes equivalence.

We say that the blocks \( [U \mathfrak{E}_\Sigma U^{-1}]_l \) and \( [U \mathfrak{E}_\Sigma U^{-1}]_l \) can be connected, if there exist representations \( \rho \in \{ \rho^l_{\gamma l}, \rho^l_{\gamma l} \} \) and \( \rho' \in \{ \rho^l_{\gamma l}, \rho^l_{\gamma l} \} \) which are equivalent: \( \rho \sim \rho' \). It can be shown [4, 10] that the complete set of blocks \( \{ [U \mathfrak{E}_\Sigma U^{-1}]_l \}_{l=1, \ldots, L} \) uniquely divides into chains of connectable ones, whereas the order of the latter in each chain is also uniquely determined.

Let the blocks \( [U \mathfrak{E}_\Sigma U^{-1}]_{l_1}, \ldots, [U \mathfrak{E}_\Sigma U^{-1}]_{l_n} \) form a chain of connected blocks with the relations \( \rho_{l_1}^+ \sim \rho_{l_2}^+, \rho_{l_2}^+ \sim \rho_{l_3}^+, \ldots, \rho_{l_{n-1}}^+ \sim \rho_{l_n}^+ \) (for another order of chain connections, the consideration is quite similar). Then there is a set of isomorphisms \( \mathfrak{Y}_{\gamma l_{l+1}} : \mathfrak{P}_{l+1} \to \mathfrak{P}_l \), which satisfy the relations

\[
[U E_l U^{-1}]_{l+1} (\epsilon_{l+1}) = \mathfrak{Y}_{\gamma l_{l+1}} \left( [U E_l U^{-1}]_{l+1} (0) \right), \quad \gamma \in \Sigma.
\]

Denote

\[
\mathfrak{Y}_{\gamma l_1} := \mathfrak{Y}_{l_1 l_2} \cdots \mathfrak{Y}_{l_{n-1} l_n}, \quad r_i (r) := r - \epsilon_{l_1} - \cdots - \epsilon_{l_{i-1}}, \quad i = 2, \ldots, n.
\]

By the union of the chain of blocks \( [U \mathfrak{E}_\Sigma U^{-1}]_{l_1}, \ldots, [U \mathfrak{E}_\Sigma U^{-1}]_{l_n} \) we name the algebra \( [U E_{\gamma l} U^{-1}]_{l_1 \cdots l_n} \) defined by the equality

\[
[U \mathfrak{E}_\Sigma U^{-1}]_{l_1 \cdots l_n} := \bigvee \{ E^1_{\gamma l_1} \cdots E^n_{\gamma l_n} | \gamma \in \Sigma \} \subset C([0, \epsilon_{l_1} + \cdots + \epsilon_{l_n}]; \mathfrak{P}_{l_1}) \quad (4.6)
\]

with matrix-functions

\[
E^1_{\gamma l_1} \cdots E^n_{\gamma l_n} (r) := \begin{cases} 
[U E_{\gamma l_1} U^{-1}]_{l_1} (r), & r \in [0, \epsilon_{l_1}); \\
\mathfrak{Y}_{l_1 l_2} [U E_{\gamma l_2} U^{-1}]_{l_2} (r_2 (r)), & r_2 (r) \in [0, \epsilon_{l_2}); \\
\cdots \\
\mathfrak{Y}_{l_1 \cdots l_n} [U E_{\gamma l_n} U^{-1}]_{l_n} (r_n (r)), & r_n (r) \in [0, \epsilon_{l_n}].
\end{cases}
\]
These functions are the parts of eikonals $E_\gamma$ (in parametric representation), corresponding to the new (enlarged) block formed by the union of blocks $[UE_\Sigma U^{-1} l_i]$, \ldots, $[UE_\Sigma U^{-1} l_n]$. It is also easy to observe that the merging of blocks leads to the representation

$$E_{l_1 \cdots l_n}(r) = \sum_{k=1}^{n_\gamma} \tau_{\gamma}^k(r) P_k,$$

in which $n_\gamma := n_{l_1} = \cdots = n_{l_n}$, $P_k := P_{\gamma l_1}^k$, and the functions $\tau_{\gamma}^k$ are the extensions of linear functions $\tau_{\gamma l_1}^k$, $k = 1, \ldots, n_\gamma$ to the larger segment $[0, \epsilon_{l_1} + \cdots + \epsilon_{l_n}]$. Thus the possible distinction between the union $[UE_\Sigma U^{-1} l_1 \cdots l_n]$ and the algebra, in which it is embedded (see (4.6)), is again that the elements of the union may satisfy additional conditions at the endpoints of the total segment, while the elements of the algebra $C((0, \epsilon_{l_1} + \cdots + \epsilon_{l_n}); \mathfrak{P}_{l_i})$ do not have them.

**Canonical form**

- By performing all possible chain unions, we present the algebra of eikonals as a sum of blocks of the form $[UE_\Sigma U^{-1} l_1 \cdots l_n]$, which are no longer connectable and are (in a relevant sense) independent [4]. Using the relations $\mathfrak{P}_{l_i} \cong M^{\kappa_i}$ we obtain that the result of such "reformatting" of the source parametric representation is the following statement which is the main subject of [4]:

**Theorem 1** There exists an isomorphism $I$ which provides the algebra $E_\Sigma$ and its generators-eikonals representations

$$IE_\Sigma = \bigoplus_{l_1} \mathfrak{C}([0, \epsilon_{l_1}]; M^{\kappa_{l_1}}); \quad IE_\gamma = \bigoplus_{l_1} \left[ \sum_{l_1}^{n_{l_1}} \sum_{k=1}^{n_\gamma} \tau_{\gamma l_1}^k P_k \right], \quad \gamma \in \Sigma.$$  (4.7)

Here $\tau_{\gamma l_1}^k$ are linear functions of $r_1 \in [0, \epsilon_{l_1}]$ such that $\left| \frac{d\tau_{\gamma l_1}^k}{dr_1} \right| = 1$. Their ranges $\psi_{\gamma l_1}^k := \text{ran} \tau_{\gamma l_1}^k$ are segments of length $\epsilon_{l_1}$, which can only share common endpoints for the same $\gamma$ and distinct $k,l$. In this case, for all $\gamma \in \Sigma$ the equality

$$\sigma_{ac}(E_\gamma) = \bigcup_{l_1} \bigcup_{k=1}^{n_{l_1}} \psi_{\gamma l_1}^k.$$

holds. The matrices $P_k^{\kappa_{l_1}} \in M^{\kappa_{l_1}}$ are one-dimensional projectors, mutually orthogonal for each $\gamma$ and such that $\forall \{P_k^{\kappa_{l_1}} \mid k = 1, \ldots, n_{l_1}; \gamma \in \Sigma \} = M^{\kappa_{l_1}}$.

The isomorphic copy $IE_\Sigma$ of the algebra $E_\Sigma$ we will call its first canonical form.

The passage to this form reveals the block structure of the eikonal algebra.

The representation of the algebra in the form (4.7) is not unique, but it can be shown that any two of such representations differ from each other only by block numbering, their parameterization (direction of change of $r_1$) and by replacements $P_{\gamma l_1}^k \rightarrow I_l P_{\gamma l_1}^k$ where $I_l : M^{\kappa_{l_1}} \rightarrow M^{\kappa_{l_1}}$ is an isomorphism. The functions $\tau_{\gamma l_1}^k$ are the same in all representations, i.e. they are invariants of
the algebra $\mathcal{E}_\Sigma$. Later on, this will allow us to use them as coordinates on the spectrum of the eikonal algebra.

- Let us recall that all considerations are performed under the assumption that the final time moment $t = T$ in the dynamical system (2.3)–(2.6) is fixed. As it increases, the structure of representations (4.7) changes. Significant changes occur at those $T$ at which the waves propagating from the controlling vertices of $\gamma \in \Sigma$ capture new (internal or boundary) vertices: see [3, 9]. The evolution of eikonal algebra over time is a separate interesting topic.

**Coordinates on the spectrum**

- Due to (4.7) and Proposition 3, the spectrum of the algebra $\mathcal{E}_\Sigma$ is the union of the spectra of individual standard algebras $\hat{C}([0,\varepsilon_l]; \mathbb{M}^{\kappa_l})$:

$$\mathcal{E}_\Sigma = \mathcal{S}_1 \sqcup \ldots \sqcup \mathcal{S}_L$$

(see (3.2), (3.4)). Each component (segment) $\mathcal{S}_l$ consists of the set (interval) $\text{int} \mathcal{S}_l$ containing points which have neighborhoods, homeomorphic to open intervals in $\mathbb{R}$ (we call them internal), and two boundaries $\mathcal{K}_l^-$ and $\mathcal{K}_l^+$:

$$\mathcal{S}_l = \mathcal{K}_l^- \sqcup \text{int} \mathcal{S}_l \sqcup \mathcal{K}_l^+$$

The sets $\text{int} \mathcal{S}_l$ are homeomorphic to the corresponding intervals $(0, \varepsilon_l)$. Through $\text{int} \mathcal{E}_\Sigma$ we denote the set of all internal points of the spectrum. Boundaries $\mathcal{K}_l^\pm$ consist of a finite number of points. We say that a boundary is a cluster if it contains more than one point. The points forming the cluster are inseparable from each other in Jacobson’s topology [1]. The intervals $\text{int} \mathcal{S}_l$ can be metricized. As can be easily seen from the second expression in (4.7), each point $\hat{\pi} \in \text{int} \mathcal{S}_l$ corresponds to a unique value of the parameter $r \in (0, \varepsilon_l)$. The definition

$$\delta(\hat{\pi}, \hat{\pi}') := |r - r'|, \quad \hat{\pi}, \hat{\pi}' \in \text{int} \mathcal{S}_l$$

provides the natural metric on the interval. One can also determine the distance between a boundary point and an interior point by continuity. However, in this case the distances (4.8) between points in the same cluster will be zero, since they all correspond to the same $r = 0$ or $r = \varepsilon_l$.

- The following well-known fact motivates the further considerations. Let $\mathfrak{A}$ be a commutative Banach algebra with a finite number of generators $E_1, \ldots, E_n$, $\hat{\mathfrak{A}}$ is its spectrum consisting of homomorphisms (characters) $\pi : \mathfrak{A} \to \mathbb{C}$. Then the correspondence

$$\hat{\mathfrak{A}} \ni \pi \mapsto \{\pi(E_1), \ldots, \pi(E_n)\} \in \mathbb{C}^n$$

provides coordinates on the spectrum (see, e.g., [13]).

Let us provide an analogue of such coordinates on the spectrum of the eikonal algebra. Technically it is more complicated, which is to be expected since $\mathcal{E}_\Sigma$ is noncommutative.
Let \( \hat{\pi} \in \hat{E}_\Sigma \) be an arbitrary point of the spectrum of the eikonal algebra and let \( \pi \in \hat{\pi} \) be some of its representatives (irreducible representation in a Hilbert space \( H_\pi \)). For each vertex \( \gamma \in \Sigma \) we define the operator \( e_\gamma(\pi) \) by

\[
e_\gamma(\pi) := \pi(E_\gamma) \in \mathfrak{B}(H_\pi);
\]
let \( \sigma^+(e_\gamma(\pi)) \) be the set of its positive eigenvalues. In this case there is equality

\[
\sigma^+(e_\gamma(\pi)) = \sigma^+(e_\gamma(\pi'))
\]
for any representations \( \pi, \pi' \in \hat{\pi} \) from the same equivalence class.

Let \( \hat{\pi} \in \text{int}\hat{E}_\Sigma \) be an interior point of the spectrum. For it, we define \( \gamma \)-coordinates by the equality

\[
\sigma_\gamma(\hat{\pi}) := \sigma^+(e_\gamma(\pi)), \quad \pi \in \hat{\pi};
\]
the correctness of the definition comes from (4.10). From the second expression in (4.7) and the properties of its included functions \( \tau_{nk}^k \), it is easy to see that the selected point \( \hat{\pi} \) corresponds to a unique number \( l \) and parameter \( r \in (0, \varepsilon_l) \) such that the following equality holds

\[
\sigma_\gamma(\hat{\pi}) = \{ \tau_{1\gamma}^1(r), \ldots, \tau_{nl}^{n_l}(r) \}
\]
For the boundary points \( \hat{\pi} \in K_+^\pm \) we take by continuity:

\[
\sigma_\gamma(\hat{\pi}) := \left\{ \lim_{r \to c} \tau_{1\gamma}^1(r), \ldots, \lim_{r \to c} \tau_{nl}^{n_l}(r) \right\}, \quad c = 0, \varepsilon_l.
\]
Note that if the boundary set is a cluster, then all its points will be assigned the same numerical set (4.11).

The correspondence

\[
\hat{E}_\Sigma \ni \hat{\pi} \leftrightarrow \{ \sigma_\gamma(\hat{\pi}) | \gamma \in \Sigma \}
\]
is proposed as a generalization (4.9) for the case of noncommutative eikonal algebras.

It should be mentioned that the sets in the right-hand side of (4.12) are not coordinates in the rigorous sense: as noted, they distinguish internal points of the spectrum but do not distinguish points belonging to the same cluster. Nevertheless, they are useful because they provide true coordinates on the set \( \text{int}\hat{E}_\Sigma \), i.e., on the main part of the spectrum of algebra \( \hat{E}_\Sigma \).

**Frame** \( \hat{E}_\Sigma \)

Here we introduce some equivalence relation for points of \( \hat{E}_\Sigma \). Factorization (gluing) of the spectrum by this relation will turn it into a graph.

- Define on \( \hat{E}_\Sigma \) the relation \( \sim_0 \) by the following rule: \( \hat{\pi} \sim_0 \hat{\pi}' \) if there exists a vertex \( \gamma \in \Sigma \) such that \( \sigma_\gamma(\hat{\pi}) \cap \sigma_\gamma(\hat{\pi}') \neq \emptyset \). Let \( \sim \) be the transitive closure of \( \sim_0 \) and \( [\hat{\pi}] \) be the equivalence class of the spectrum point \( \hat{\pi} \).
Proposition 4 Let $\hat{\pi} \in \hat{E}_\Sigma$ be a point of the spectrum, $[\hat{\pi}]$ is its equivalence class. Then:
1. if $\hat{\pi} \in \text{int} \hat{E}_\Sigma$ is an interior point, then $[\hat{\pi}] = \{\hat{\pi}\}$, i.e., its equivalence class is limited to the point itself;
2. if $\hat{\pi} \in K_i^\pm$ is a point of a boundary set (possibly a cluster), then there is an embedding $K_i^\pm \subset [\hat{\pi}] \subset \hat{E}_\Sigma \setminus \text{int} \hat{E}_\Sigma$.

Part 1 easily follows from the properties of the functions $\tau_k^\pm$ (see Theorem 1), namely, the disjunction of their ranges $\psi_k^{\pm}$. The embedding in Part 2 follows directly from the definition of equivalence, and the difference $[\hat{\pi}] \setminus K_i^\pm$ can consist of the points of another boundary sets $K_i^\pm$ that got into class $[\hat{\pi}]$ during factorization.

We say the factor-space $F^{a}_\Sigma := \hat{E}_\Sigma / \sim$ to be an algebraic frame of the domain $\Omega^T[\Sigma]$. By $\text{proj} : \hat{E}_\Sigma \rightarrow F^{a}_\Sigma$ we denote the canonical projection. The spectrum is equipped with the Jacobson topology and hence there is a canonical factor-topology on $F^{a}_\Sigma$.

Proposition 4 implies that the passage from the spectrum $\hat{E}_\Sigma$ to the frame $F^{a}_\Sigma$ is reduced to connecting some segments $S_l$ by identifying points of their boundary sets $K_i^\pm$. The clusters, which make the spectrum non-Hausdorff space, are glued into points during factorization. As a consequence, the space $F^{a}_\Sigma$ turns out to be Hausdorff, and each of its components is homeomorphic to some graph whose edges are $\text{proj} (\text{int} S_l)$ and its vertices are the points, formed by gluing some of the boundaries together:

$$F^{a}_\Sigma = \mathcal{E}^a \sqcup \mathcal{W}^a; \quad \mathcal{E}^a = \{\text{proj} (\text{int} S_1), \ldots, \text{proj} (\text{int} S_L)\}, \quad \mathcal{W}^a = \{w_1, \ldots, w_p\}, \quad w_k = \text{proj} K_{i_1}^{\alpha_k} = \ldots = \text{proj} K_{i_k}^{\alpha_k}, \quad \alpha_k \in \{-, +\}.$$ 

Note that the passage from spectrum to frame may result in appearance of the valency 2 vertices.

To simplify the writing, we will denote frame points by $\pi := [\hat{\pi}]$. Let’s introduce $\gamma-$coordinates:

$$\sigma_\gamma(\pi) := \bigcup_{\hat{\pi} \in \pi} \sigma_\gamma(\hat{\pi}), \quad \pi \in F^{a}_\Sigma$$

(the union of numerical sets on a common numerical axis) and define the coordinates on the whole frame by the rule

$$F^{a}_\Sigma \ni \pi \mapsto \{\sigma_\gamma(\pi) | \gamma \in \Sigma\}. \quad (4.13)$$

From the Proposition 4 one can easily conclude that the sets in the right part (4.13) distinguish all frame points, i.e. they are the proper coordinates on $F^{a}_\Sigma$.

As follows from the Proposition 4 the projection $\text{proj}$ acts injectively on the interior points of the spectrum. This allows us to metricize the edges of the frame by the rule

$$\Delta(\pi, \pi') := \delta (\text{proj}^{-1}(\pi), \text{proj}^{-1}(\pi'))$$
(see (4.8)), and then, by analogy with the metric (2.1) on the stars, extend the
\[ \Delta \]
metric to the frame vertices. As a result, the whole frame \( \mathfrak{F}a \Sigma \) turns out to be
a (possibly non-connected) compact metric graph.

- The result of the previous considerations is a remarkable fact: the part of
the graph \( \Omega \) filled with waves, by the scheme
\[
\Omega^T[\Sigma] \rightarrow C \Sigma \rightarrow \hat{C} \Sigma \rightarrow \mathfrak{F}a \Sigma,
\]
is canonically mapped to some metric canonically coordinated graph that is the
frame \( \mathfrak{F}a \Sigma \) extracted from the algebra \( C \Sigma \). As we can easily see from Theorem
1 and the remarks below it, this construction is an invariant of the eikonal
algebra: the frames corresponding to different versions of the representation
(4.7) are isometric.

**Functional model**

- Let the representation (4.7) be fixed. Recall that then each point of the
spectrum \( \hat{\pi} \in \hat{C} \Sigma \) is associated with a certain value of the parameter \( r_{\hat{\pi}} \) taking
values in the segments \([0, \epsilon_1]\). The correspondence \( \hat{\pi} \rightarrow r_{\hat{\pi}} \) is injective on the set
of interior points \( \text{int} C \Sigma \).

Let \( [Ie]_l(\cdot) \in \mathcal{C}([0, \epsilon_1]; M^\kappa l) \) be the \( l \)-th block of the element \( Ie \) in the form
(4.7). For each point \( \hat{\pi} \in \text{int} \mathcal{S} \) we choose a representative \( \pi \in \hat{\pi} \) such that
equality \( \pi(e) = [Ie]_l(r_{\hat{\pi}}) \) holds. For the points of the boundary sets \( K^+ l \) we fix
their numbering \( K^+ l = \{ \hat{\pi}^+ 1, \ldots, \hat{\pi}^+ m_l^+ \} \) and representatives \( \pi^+ k \in \hat{\pi}^+ k \) such that
the equality \( \oplus_{k=1}^{m_l^+} \pi^+ k(e) = [Ie]_l(c^+), \quad c^+ = \epsilon_1, \quad c^- = 0 \) holds. Then, matrix-
functions \( Ie \) are transferred to the spectrum by the rule
\[
e(\hat{\pi}) := \pi(e), \quad \pi \in \hat{\pi} \in \hat{C} \Sigma,
\]
where \( \pi \) is a representative of \( \hat{\pi} \) defined above.

These functions are then transferred from the spectrum to the frame \( \mathfrak{F}a \Sigma \) by
the rule
\[
e(\pi) := \begin{cases} e(\text{proj}^{-1}(\pi)), & \pi \in \mathfrak{F}a; \\ \oplus_{\hat{\pi}_k \in \text{proj}^{-1}(\pi)} e(\hat{\pi}_k), & \pi \in \mathfrak{W}^a. \end{cases}
\] (4.14)

In the second line in (4.14) there is a matrix composed of blocks arranged in
some order. The order is not crucial, but it is assumed that for each \( \pi \in \mathfrak{W}^a \)
the order is fixed.

As a result, the eikonal algebra is realized as an algebra of of matrix-valued
functions on an algebraic frame. These functions are continuous on its edges
and, generally speaking, discontinuous at vertices.

- The above construction can be interpreted as a \( \mathbb{C}^* \)-algebra bundle over the
base \( \mathfrak{F}a \Sigma \), and the functions (4.14) as its sections \([7, 11]\). It can be shown that
these sections have an additional property semicontinuity: see \([11]\).
5 Second canonical form

Converting the eikonal algebra to the second form starts from the same source parametric form \(3.14\), \(3.16\).

Partitioning into blocks

- Recall that the projectors \(P_{i\gamma j}\) in the source parametric form are one-dimensional operators which act in \(l_{2}^j\). In the indicator basis they take the form

\[
P_{i\gamma j} = \langle \cdot, \beta_{i\gamma j}^i \rangle l_{2}^j, \quad \text{where} \quad \beta_{i\gamma j}^i = \sum_{l=1}^{m_j} \beta_{\gamma j}^l \chi_k \in l_{2}^j, \quad (\beta_{i\gamma j}^i, \beta_{\gamma j}^k)_{l_{2}^j} = \delta_{ik},
\]

and get the matrices \(\tilde{p}_{i\gamma j} = \{\beta_{i\gamma j}^l \beta_{\gamma j}^l \} \in M^{m_j}. \) Just like the projectors \(P_{i\gamma j}\) the vectors \(\beta_{i\gamma j}^i \in l_{2}^j\) that represent them, do not depend on \(r_j\). If we consider the elements of the space \(l_{2}^j\) as numerical functions on the set \(\Lambda[x(r)] \subset \Phi^j\) (see \(3.11\)), then for each vector its support \(\text{supp} \beta_{i\gamma j}^i \subset \Lambda[x(r)]\) is defined.

The definition is correct because both the projectors \(P_{i\gamma j}\), and the vectors \(\beta_{i\gamma j}^i\) associated with them, do not depend on \(r_j \in (0, \epsilon_j)\). Let us note the equivalence

\[
\text{supp} \beta_{i\gamma j}^i \cap \text{supp} \beta_{i'\gamma j'}^i \neq \emptyset \iff \sum_{l=1}^{m_j} |\beta_{i\gamma j}^l| |\beta_{i'\gamma j'}^l| \neq 0, \quad \gamma, \gamma' \in \Sigma
\]

and put by definition: \(\text{supp} P_{i\gamma j} := \text{supp} \beta_{i\gamma j}^i\).

- Now, on each set of projectors \(P^j\) (see \(3.15\)), we define the relation \(\text{supp} \sim_0\) by the following rule: \(P_{i\gamma j}^i \sim_0 P_{i'\gamma j'}^i\) if \(\text{supp} P_{i\gamma j}^i \cap \text{supp} P_{i'\gamma j'}^i \neq \emptyset\). By \(\text{supp} \sim\) we denote its transitive closure.

Let \(\langle P \rangle\) be the equivalence class of the projector \(P \in P^j\) with respect to \(\text{supp} \sim\). From the definition of the relation \(\text{supp} \sim\) it easily follows that the partition

\[
\mathbb{P}^j = \langle P \rangle_1^j \cup \cdots \cup \langle P \rangle_q^j
\]

leads to the decomposition of the algebra \(\mathbb{P}^j = \vee \mathbb{P}^j\) into orthogonal blocks:

\[
\mathbb{P}^j = \bigoplus_{q=1}^{q_j} \Omega_q^j, \quad \Omega_q^j := \langle P \rangle_q^j.
\]

- It is useful to compare the representations \(5.1\) and \(5.2\) with \(4.1\) and \(4.2\): if the algebras \(\mathbb{P}^j \cong M^{s_q^j}\) are irreducible, then the irreducibility of \(\Omega_q^j\) is generally speaking not guaranteed. At the same time, one can show \([10]\) that the representation \(5.1\) corresponds to the partition

\[
\Lambda[x(r)] = \Lambda_1^j[x(r)] \cup \cdots \cup \Lambda_{q_j}^j[x(r)] \subset \Phi^j
\]
on minimal (by number of points) determination sets \( \Lambda^*_q \{ x(r) \} = \{ x^{i,q}_1(r), \ldots, x^{i,q}_{s_q}(r) \} \), i.e., on the smallest sets having the property (3.5). Correspondingly, there is a decomposition of families into subfamilies:

\[
\Phi^j = \Phi^j_1 \cup \ldots \cup \Phi^j_{q_j}, \quad \Phi^j_q = \bigcup_{0 < r < \epsilon_j} \Lambda^*_q \{ x(r) \} = \bigcup_{s=1}^{s_q} \omega^j_k \{ x \}, \quad \omega^j_k = \bigcup_{0 < r < \epsilon_j} x^{j,q}_k \{ x \}.
\]

(5.3)

With the latter in mind, we can say that the partition (5.1), unlike (4.1), has a geometric meaning: it corresponds to a graph partition

\[
\Omega^T[\Sigma] = \left[ \bigcup_{j=1}^J \Phi^j_q \right] \cup \Theta.
\]

In this case, by construction of the representation (5.3), the following is fulfilled

\[
L_2(\Omega^T[\Sigma]) = \bigoplus_{j=1}^J \bigoplus_{q=1}^{q_j} L_2(\Phi^j_q); \quad E_\gamma L_2(\Phi^j_q) \subset L_2(\Phi^j_q), \quad \gamma \in \Sigma.
\]

(5.4)

- According to (5.4), the blocks of eikons in (3.10) are decomposed into subblocks:

\[
[U E_\gamma U^{-1}]^{j} = \bigoplus_{q=1}^{q_j} (U E_\gamma U^{-1})^{j} q; \quad (U E_\gamma U^{-1})^{j} q = \sum_{P_{\gamma,j} \in (P)^j_q} \tau^{j}_{\gamma} P_{\gamma,j} \in C \left( [0, \epsilon_j]; \Omega^j_q \right),
\]

and for the eikonal algebra we have the following relation

\[
U \in \mathcal{E}^1 U^{-1} \subset \bigoplus_{j=1}^J \left[ \bigoplus_{q=1}^{q_j} C \left( [0, \epsilon_j]; \Omega^j_q \right) \right].
\]

Simplifying the notations, we will proceed to a through numbering:

\[
\Phi^1, \ldots, \Phi^{1}_{q_1}, \ldots; \quad \Phi^{j}, \ldots, \Phi^{j}_{q_j} \rightarrow \Phi_1, \ldots, \Phi_M;
\]

\[
\Lambda^1, \ldots, \Lambda^{1}_{q_1}, \ldots; \quad \Lambda^{j}, \ldots, \Lambda^{j}_{q_j} \rightarrow \Lambda_1, \ldots, \Lambda_M;
\]

\[
\langle P \rangle^{1}, \ldots, \langle P \rangle^{1}_{q_1}, \ldots; \langle P \rangle^{j}, \ldots, \langle P \rangle^{j}_{q_j} \rightarrow \langle P \rangle_1, \ldots, \langle P \rangle_M;
\]

\[
(U E_\gamma U^{-1})^{1}, \ldots, (U E_\gamma U^{-1})^{1}_{q_1}, \ldots; \quad (U E_\gamma U^{-1})^{j}, \ldots, (U E_\gamma U^{-1})^{j}_{q_j} \rightarrow \langle U E_\gamma U^{-1} \rangle_1, \ldots, \langle U E_\gamma U^{-1} \rangle_M, \quad \gamma \in \Sigma;
\]

\[
\Omega^1, \ldots, \Omega^{1}_{q_1}, \ldots; \quad \Omega^{j}, \ldots, \Omega^{j}_{q_j} \rightarrow \Omega_1, \ldots, \Omega_M;
\]

\[
e_1, \ldots, e_{m_1}; \ldots; \epsilon_{j-m_j}, \ldots, \epsilon_j \rightarrow e_1, \ldots, e_M.
\]

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In the new notation we have:

$$\Omega^T[\Sigma] = \bigcup_{l=1}^{M} \Phi_l \cup \Theta; \quad \Phi_l = \bigcup_{0<r<e_l} \Lambda_l[x(r)] = \bigcup_{s=1}^{m_l} \omega_{ls}, \quad \Lambda_l[x(r)] = \{x^l_1(r), \ldots, x^l_{m_l}(r)\}, \quad \omega_{ls} = \bigcup_{0<r<e_l} x^l_s(r); \quad \text{(5.6)}$$

$$U \mathcal{E}_\Sigma U^{-1} \subset \bigoplus_{l=1}^{M} C([0, e_l]; \Omega_l); \quad \text{pr}_l U \mathcal{E}_\Sigma U^{-1} =: \langle U \mathcal{E}_\Sigma U^{-1} \rangle_l = \bigvee \{\langle UE_s U^{-1} \rangle_l \mid \gamma \in \Sigma\}, \quad \langle UE_s U^{-1} \rangle_l = \sum_{\gamma^{-l}_{sl}(r) \in (P)} \tau^{l}_{sl}(\gamma) P^l_{s l} \in C([0, e_l]; \Omega_l). \quad \text{(5.7)}$$

**Connecting the families**

Connecting families is a procedure similar to the connecting blocks in the passage to the first form. The important difference is that now all the steps remain related to the geometry and, as a result, the second form will correspond to a new partition of the graph $\Omega^T[\Sigma]$ into families $\Phi$. The new families, generally speaking, consist of larger cells and are distinguished by the fact that they are formed by minimal $\Lambda_l[x(r)]$ under variations of the parameter $r$: see (5.6).

- As $r$ varies, the sets $\Lambda_l[x(r)] = \{x^l_1(r), \ldots, x^l_{m_l}(r)\}$ continuously vary their position on the graph. When the parameter tends to the boundary values 0 or $e_l$, they go to the limiting sets $\Lambda_l[x(0)]$, $\Lambda_l[x(e_l)] \subset \Theta'$ consisting of boundaries $x^l_1(0)$, $x^l_s(e_l)$ of the cells $\omega_{ls} \subset \Phi_l$.

For the elements $U \mathcal{E}_\Sigma U^{-1} \in \langle U \mathcal{E}_\Sigma U^{-1} \rangle_l \subset C([0, e_l]; \Omega_l)$ we define the boundary representations $\rho_l^u \mathcal{E}_\Sigma \to \Omega_l$:

$$\rho_l^u(e) := \langle U e U^{-1} \rangle_l (0), \quad \rho_l^u(e) := \langle U e U^{-1} \rangle_l (e); \quad l = 1, \ldots, M.$$  

Just like representations (4.5), they can generally be reducible, and representations $\rho_l^u$ that correspond to the same block, are not equivalent.

- Let $\Phi_l$ and $\Phi'_l$ be two families with equal number of cells: $m_l = m_{l'} =: m_{ll'}$. We say that these families are connectable if

1. under some parametrization of them, $\Lambda_l[x(e_l)] = \Lambda_{l'}[x(0)]$ is fulfilled, where $\# \Lambda_l[x(e_l)] = \# \Lambda_{l'}[x(0)] = m_{ll'}$;

2. under a suitable choice of cell numbering, the bijection $\Lambda_l[x(e_l)] \leftrightarrow \Lambda_{l'}[x(0)]$, given by the condition $x^l_s(e_l) = x^l_s(0), \quad s = 1, \ldots, m_{ll'}$, is well-defined. It determines the unitary operator $V_{ll'} : \text{I}_2 \to \text{I}_2, \quad V_{ll'} x^l_s = x^l_s, \quad s = 1, \ldots, m_{ll'}$ which links the boundary representations: $V_{ll'} \rho_l^u = \rho_{l'} V_{ll'}$ (and thus establishes their equivalence).

In this case, the cells $\omega_{ls} \subset \Phi_l$ and $\omega_{l's} \subset \Phi_{l'}$ are coupled in pairs on the graph through common boundaries $x^l_s(e_l) = x^{l'}_s(0)$. Such a connection of cells will be denoted by $\omega_{ls} \leftrightarrow \omega_{l's}$. It is not difficult to see that the complete set of families $\{\Phi_l \mid l = 1, \ldots, M\}$ breaks down into chains of connectable ones.
Let $\Phi_1, \ldots, \Phi_n$ be a chain of connectable families and the relations $\rho_1^+ \sim \rho_1^-$, $\rho_2^+ \sim \rho_2^-$, $\ldots$, $\rho_n^{+} \sim \rho_n^-$ hold (the considerations are quite similar for the other order of connections in the chain). Thus a chain of cells is established:

$$\omega_s^1 \leftrightarrow \omega_s^2 \leftrightarrow \ldots \leftrightarrow \omega_s^{n-1} \leftrightarrow \omega_s^n, \quad s = 1, \ldots, m_1, \ldots, m_n$$

where $m_1, \ldots, m_n := m_1 = \cdots = m_n$. Let us define the cells $\omega_s^{i_1 \ldots i_n}$ by the equalities:

$$\omega_s^{i_1 \ldots i_n} := \omega_s^{i_1} \cup \omega_s^{i_2} \cup \ldots \cup \omega_s^{i_{n-1}} \cup \omega_s^{i_n}, \quad s = 1, \ldots, m_1, \ldots, m_n.$$

The set $\Phi_{i_1, \ldots, i_n}$ defined by

$$\Phi_{i_1, \ldots, i_n} := \bigcup_{s=1}^{m_1 \ldots m_n} \omega_s^{i_1 \ldots i_n},$$

is said to be the union of the families $\Phi_{i_1, \ldots, i_n}$.

One can easily see that $\Phi_{i_1, \ldots, i_n}$ is also a family, and invariance of spaces $L_2(\Phi)$ for eikonal in (5.4) implies invariance $E_\gamma L_2(\Phi_{i_1, \ldots, i_n}) \subset L_2(\Phi_{i_1, \ldots, i_n})$, $\gamma \in \Sigma$. Searching the connecting procedure, it is easy to verify that the values of the functions $\tau_{\gamma j}$, which correspond to the blocks $(UE_\gamma U^{-1})_{i_1}, \ldots, (UE_\gamma U^{-1})_{i_n}$ in the eikonal representation (5.5), are properly connected on the cell boundaries $\omega_{1s}, \ldots, \omega_{ns}$, so that they form linear functions $\tau_{\gamma s}$ on the new cells $\omega_{s1 \ldots s_n}$. In the mean time, equivalence of boundary representations implies that the projectors $P_{\gamma j}$ corresponding to blocks $(UE_\gamma U^{-1})_{i_1}, \ldots, (UE_\gamma U^{-1})_{i_n}$, are sequentially intertwined by operators $V_{i_1, i_{n+1}}$. Hence, they have the same matrices $\rho_{\gamma 1, \ldots, \gamma n}$ in the corresponding indicator bases.

As a result, in all previous representations, the chain of blocks $(UE_\gamma U^{-1})_{i_1}, \ldots, (UE_\gamma U^{-1})_{i_n}$ is replaced by one block $(UE_\gamma U^{-1})_{i_1, \ldots, i_n}$—their union. Repeating this procedure for all chains of connectable blocks, we arrive at a partition of the wave-filled domain $\Omega^T[\Sigma]$ into families that no longer admit connections:

$$\Omega^T[\Sigma] = \bigcup_{j=1}^{M} \Phi'_j \cup \Theta'; \quad \Phi'_j = \bigcup_{0<r<r'_j} \Lambda'_j[x(r)] = \bigcup_{s=1}^{m'_s} \omega'_s, \quad \Lambda'_j[x(r)] = \{x'^1(r), \ldots, x'^{m'_s}(r)\}, \quad \omega'_s = \bigcup_{0<r<r'_s} x'^s(r). \quad (5.8)$$

Here $\Lambda'_j[x(r)]$ are the minimal determination sets, which gives reason to call the partition (5.8) optimal.

As one can easily see, the optimal partition reduces eikonal: $E_\gamma L_2(\Phi'_j) \subset L_2(\Phi'_j)$ holds for all $\gamma \in \Sigma$.

**Geometric form**

The previous considerations are summarized in the form of the following analogue of Theorem 1.
Theorem 2  The optimal partition corresponds to the representation of the eikonal algebra in the form

\[
U \mathcal{E}_2 U^{-1} \subset \bigoplus_{l=1}^M C ([0, \epsilon_l]; \Omega'_l); \quad \text{pr}_l U \mathcal{E}_2 U^{-1} = \langle U \mathcal{E}_2 U^{-1} \rangle'_l = \\
= \bigvee \left\{ \langle U E_\gamma U^{-1} \rangle'_l \mid \gamma \in \Sigma \right\}, \quad \langle U E_\gamma U^{-1} \rangle'_l = \sum_{s=1}^{n'_l} \tau^s_{\gamma l}(r) P^s_{\gamma l}, \\
U E_\gamma U^{-1} = \bigoplus_{l=1}^M \langle U E_\gamma U^{-1} \rangle'_l
\]

with the blocks \( \langle U \mathcal{E}_2 U^{-1} \rangle'_l \subset C ([0, \epsilon'_l]; \Omega'_l) \). The functions \( \tau^s_{\gamma l} \) are linear functions of \( r_l \in [0, \epsilon'_l] \) such that \( \left| \frac{d \tau^s_{\gamma l}}{dr_l} \right| = 1 \). Their ranges \( \xi^s_{\gamma l} := \text{ran} \tau^s_{\gamma l} \) are segments of the length \( \epsilon'_l \), which may have (for the same \( \gamma \) and different \( s,l \)) only common endpoints. In this case, for all \( \gamma \in \Sigma \) the following equality holds

\[
\sigma_{ac}(E_\gamma) = \bigcup_{l=1}^M \bigcup_{s=1}^{n'_l} \xi^s_{\gamma l}.
\]

The matrices \( P^s_{\gamma l} \in \Omega'_l \) are one-dimensional projectors, pairwise orthogonal for each \( \gamma \) and such that \( \bigvee \{ P^s_{\gamma l} \mid k = 1, \ldots, n'_l; \gamma \in \Sigma \} = \Omega'_l \) holds.

We will refer to the representation (5.9) as geometric form of the eikonal algebra.

Frame \( \mathfrak{g}_\Sigma \)

Further considerations deal with the optimal partition (5.8) and the corresponding geometric form (5.9). Simplifying the notation, we remove the primes: \( \Phi'_l =: \Phi_l \), \( \Lambda'_l =: \Lambda_l \), etc.

- Each point \( x = x(r) \in \Phi_l \) is an element of its minimal set \( \Lambda_l[x(r)] = \{ x'_l(r), \ldots, x'_m_l(r) \} \). Each family \( \Phi_l \) possesses the boundaries

\[
\Lambda^-_l := \lim_{r \to 0} \Lambda_l[x(r)], \quad \Lambda^+_l := \lim_{r \to \epsilon_l} \Lambda_l[x(r)].
\]

From (5.8) we have:

\[
\Theta = \bigcup_{l=1}^M [\Lambda^-_l \cup \Lambda^+_l]. \tag{5.10}
\]

On the set of all boundaries \( \{ \Lambda = \Lambda^+_l \mid l = 1, \ldots, M \} \) we introduce the relation \( \sim_0; \Lambda \sim_0 \Lambda' \text{ if } \Lambda \cap \Lambda' \neq \emptyset \). Let \( \sim \) be the transitive closure of \( \sim_0 \) and \( [\Lambda] \) be the equivalence class of \( \Lambda \). The representation (5.10) is transformed to a partition:

\[
\Theta = [\Lambda]_1 \cup \cdots \cup [\Lambda]_K.
\]
The set of classes
\[ W^g := \{w_1, \ldots, w_K\} , \quad w_k := [\Lambda]_k \]
will play the role of the set of vertices of the frame to be constructed. Let us mention that there are also possible vertices of valence 2.

- Let us say that a family \( \Phi_l \) is adjacent to a vertex \( w = [\Lambda] \) if at least one of its boundaries \( \Lambda^\pm_l \) lies in \( [\Lambda] \). By identifying the points \( x_1^l(r), \ldots, x_m^l(r) \) and making up the set \( \Lambda[x(r)] \subset \Phi_l \), we turn the family into an edge
\[ \lambda_l := \{\lambda_l(r) | 0 < r < \epsilon_l\} , \quad \lambda_i(r) := x_1^l(r) \equiv \cdots \equiv x_m^l(r), \]
adjacent to the vertex \( w \). Each vertex has its own set of adjacent edges \( \lambda_1 \), \ldots, \( \lambda_d_w \); the number \( d_w \) is its valency. By \( \mathcal{E}^g := \{\lambda_1, \ldots, \lambda_M\} \) we denote the set of edges.

The edges are metricized: \( \text{dist}(\lambda_l(r), \lambda_l(r')) := |r - r'| \), and equipped with \( \gamma \)-coordinates:
\[ \sigma_\gamma(\lambda(r)) := \{\tau_\gamma^l(r), \ldots, \tau_\gamma^{m_l}(r)\} , \quad \gamma \in \Sigma \quad (5.11) \]
(see (5.9)).

- The set \( \mathfrak{F}^g := W^g \cup \mathcal{E}^g \) is called the geometric frame of \( \Omega^T[\Sigma] \).

Using the same trick (2.1) that equips stars with the metric, the metric from the edges extends to the whole frame, turning it into a metric graph.

The edges are equipped with coordinates (5.11); for the vertices we put
\[ \sigma_\gamma(w) := \bigcup_{\lambda_i \text{ adjacent to } w} \lim_{\lambda_i(r) \to w} \sigma_\gamma(\lambda(r)), \quad \gamma \in \Sigma \]
(the union of numerical sets on a common numerical axis).

As a result, the frame \( \mathfrak{F}^g \) is endowed with coordinates:
\[ \mathfrak{F}^g \ni \lambda \mapsto \{\sigma_\gamma(\lambda) | \gamma \in \Sigma\}. \quad (5.12) \]

- In quite the same way as in the construction of the functional model on the frame \( \mathfrak{F}^g \), the matrix-valued functions which constite the algebra \( \mathcal{U} \mathfrak{E} \mathcal{U}^{-1} \subset \bigoplus_{l=1}^M C([0, \epsilon_l]\Omega_l) \) (see (5.9)), can be transferred to the frame \( \mathfrak{F}^g \) and thus one obtains the second functional model of eikonal algebra. It corresponds to the optimal partition of the domain \( \Omega^T[\Sigma] \).

6 Ordinary graphs

Identity of frames

- Searching the passage from the source parametric form (3.14) to the canonical forms, it is easy to recognize that the possible difference between them is due to the difference in the partitions (4.1) and (5.1). The definitions follow to \( [P]^j_q \subset \langle P\rangle^j_q \), so the difference is possible if and only if at least one of the classes \( \langle P\rangle^j_q \) in (5.1) admits a non-trivial decomposition with respect to \( ^n_{\text{ort}} \).
Definition 1. We say that the domain $\Omega^T[\Sigma]$ is an ordinary graph if the relations $\sim^{\text{not}}$ and $\sim^{\text{supp}}$ are equivalent on all sets $P_j$, $j = 1, \ldots, J$, what is equivalent to the identity of decompositions (4.1) and (5.1) (identities of classes $[P]_q^j = \langle P \rangle_q^j$ for all $q = 1, \ldots, p_j = q_j$).

It seems that a special "tuning" of the graph $\Omega^T[\Sigma]$ parameters (lengths of edges) is required to break ordinariness; in particular, a proper choice of the value $T$. Therefore, it is probably reasonable to speak of ordinariness as a generic case.

The following statement is valid:

**Theorem 3** Let the domain $\Omega^T[\Sigma]$ be an ordinary graph. Then the correspondence between the frames $\mathfrak{F}_a \ni \pi \leftrightarrow \lambda \in \mathfrak{F}_g \Sigma$ defined by the relation

$$\sigma_\gamma(\pi) = \sigma_\gamma(\lambda), \quad \gamma \in \Sigma$$

(6.1)

turns out to be isometry of the frames (as metric spaces).

**Proof** Ordinariness is equivalent to identity of classes in (4.1) and (5.1): $[P]_q^j = \langle P \rangle_q^j$. The identity leads to equality of algebras $P_l = Q_l$ and their simultaneous irreducibility. Thus, the representations (4.4) and (5.7) are identical.

Using the representations (4.4) and (5.7), we can construct isometric copies of frames $\mathfrak{F}_a \Sigma$ and $\mathfrak{F}_g \Sigma$. Let us describe this procedure for $\mathfrak{F}_a \Sigma$; for $\mathfrak{F}_g \Sigma$ all considerations are similar.

With each block $[U \Sigma U^{-1}]_l$ we associate an open interval $\omega_l := \{x(r) := r \mid r \in (0, \epsilon_l)\} \subset \mathbb{R}$ equipped with $\gamma-$coordinates $\sigma(x(r)) := \{\tau^k_{\gamma l}(r) \mid \gamma, l, k \text{ are such that } P^k_{\gamma l} \in [P]_l \text{ holds}\}$.

Let us also consider its closure $\overline{\omega_l}$, where the coordinates of two endpoints $q^+_l, q^-_l \in \overline{\omega_l} \setminus \omega_l$ are defined by continuity. Consider the set of all interval endpoints $\Upsilon := \{q^+_l, q^-_l \mid l = 1, \ldots, M\}$ and introduce the relation $\sim_0$ on it by the rule: $q^+_l \sim_0 q^-_{l'}$ if $\sigma_\gamma(q^+_l) \cap \sigma_\gamma(q^-_{l'}) \neq \emptyset$ for some $\gamma \in \Sigma$, where $c, c' \in \{-, +\}$. By $\sim$ we denote the transitive closure of the relation $\sim_0$. Then the set $\Upsilon$ can be represented as a disjunct union of equivalence classes $[q]_k$ under this relation:

$$\Upsilon = [q]_1 \cup \cdots \cup [q]_K.$$ 

Consider $w_k := [q]_k$; the elements of the set $W := \{w_1, \ldots, w_K\}$ will be called vertices, the elements of the set $E := \{\omega_1, \ldots, \omega_M\}$ - edges, and the set $\mathfrak{F}_a \Sigma := W \cup E$ - an auxiliary algebraic frame. For the points of the edges, the $\gamma-$coordinates were defined above, and for the vertices we put

$$\sigma_\gamma(w) := \bigcup_{q \in w} \sigma_\gamma(q).$$
Note that the frame $\tilde{F}_{\Sigma}^g$ may differ from $\tilde{F}_{\Sigma}^a$ only by the presence of additional valency 2 vertices.

The auxiliary geometric frame $\tilde{F}_{\Sigma}^g$ is constructed quite analogously. It is easy to see that if $\Omega^T[\Sigma]$ is ordinary, then the auxiliary frames are identical: $\tilde{F}_{\Sigma}^a = \tilde{F}_{\Sigma}^g$.

In the mean time, as is easy to see, each of the auxiliary frames is isometric to the original one (algebraic or geometric), whereas the isometries are set up by bijections defined by equality of coordinates (6.1). Thus, the ordinariness of the domain $\Omega^T[\Sigma]$ does lead to isometry between the algebraic and geometric frames $\tilde{F}_{\Sigma}^a$ and $\tilde{F}_{\Sigma}^g$.

□

Commentary

• Using standard techniques of the Boundary Control method [2], it can be shown that the traditional inverse problem data define some isomorphic copy $[E_{\Sigma}]^c$ of the algebra $E_{\Sigma}$. As a consequence, these data determine an isometric copy $[\tilde{F}_{\Sigma}^g]^c$ of the frame $\tilde{F}_{\Sigma}^g$. If the domain $\Omega^T[\Sigma]$ filled with waves is an ordinary graph, then we have an isometric copy $[\tilde{F}_{\Sigma}^a]^c$ of the frame $\tilde{F}_{\Sigma}^a$. Just like this frame itself, its copy corresponds to the optimal partition and, hence, contains information about the graph $\Omega^T[\Sigma]$ geometry. We can try to obtain this information by the following scheme:

$$\text{Inverse problem data} \Rightarrow [E_{\Sigma}]^c \Rightarrow [\tilde{F}_{\Sigma}^g]^c \equiv [\tilde{F}_{\Sigma}^a]^c \Rightarrow \Omega^T[\Sigma]$$

The fundamental question that sets the direction for further investigation is to what extent the copy $[\tilde{F}_{\Sigma}^g]^c$ determines this geometry. To put it in simpler terms, can one "unglue" $\tilde{F}_{\Sigma}^g$ into $\Omega^T[\Sigma]$? The question is open and seems to be rather complicated.

• The question on the validity of the following hypothesis stated in [6] also remains open. In the known examples, each vertex $v$ covered by the waves from (at least) two controlling vertices, i.e. such that $v \in \Omega^T[\gamma] \cap \Omega^T[\gamma']$ is satisfied, corresponds to a cluster in the spectrum $E_{\Sigma}$. Is it always valid? Also, is it possible to detect the presence or absence of cycles in $\Omega^T[\Sigma]$ from the spectrum $E_{\Sigma}$ (frame $\tilde{F}_{\Sigma}^g$)?

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