The largest linear space of operators satisfying the Daugavet Equation in $L_1$

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Abstract
We find the largest linear space of bounded linear operators on $L_1(Ω)$ that being restricted to any $L_1(A), A ∈ Ω$, satisfy the Daugavet equation.

1 Introduction.
Let $(Ω, Σ, μ)$ be an arbitrary measure space without atoms of infinite measure. Let also $Σ^+ = \{A ∈ Σ : μ(A) > 0\}$. If $A ∈ Σ^+, L_1(A)$ stands for the space of (classes of) $μ$-integrable functions supported on $A$. If $T$ is a bounded linear operator on $L_1(Ω)$ and $A ∈ Σ^+$, we denote by $T_A$ the restriction of $T$ onto $L_1(A)$. Finally, $ℒ(L_1(Ω))$ denotes the space of all bounded linear operators on $L_1(Ω)$.

The purpose of this note is to give an explicit description of the largest linear space $\mathcal{M}$ of operators $T ∈ ℒ(L_1(Ω))$ satisfying the following identity:

\[ \|Id_A + T_A\| = 1 + \|T_A\|, \]

for any set $A ∈ Σ^+$. 
Identity (1) is known as the Daugavet equation and is investigated in a series of works (see [4] and [6] for recent results and further references). It was first discovered by Babenko and Pichugov ([1]) that all the compact operators on $L_1[0,1]$ satisfy (1), if $A = [0,1]$. Later, Holub proved the same result for the weakly compact operators on an arbitrary atomless $L_1(\Omega)$ (see [3]). Plichko and Popov in their work [5] found much broader (in case of atomless $\mu$) linear class of so-called narrow operators satisfying the Daugavet equation, and in fact their proof works for operators from $L_1(A)$ to $L_1(\Omega)$, whenever $A \in \Sigma^+$.

So, finding the largest class of such operators naturally completes this line of results.

2 Main result.

In the sequel it is convenient to denote $\Sigma^+_A = \{B : B \subset A, B \in \Sigma^+\}$, whenever $A \in \Sigma^+$.

We define $\mathcal{M}$ as the set of all operators $T \in \mathcal{L}(L_1(\Omega))$ that meet the following condition:

for every $\varepsilon > 0$ and $A \in \Sigma^+$ there is a $B \in \Sigma^+_A$ with $\mu(B) < \infty$ such that

\[
\left\| \chi_B \cdot T \left( \frac{\chi_B}{\mu(B)} \right) \right\| < \varepsilon.
\]

This condition simply means that the operator $T$ can shift sufficiently many functions from their supports.

Let us state our main result.

**Theorem 1** Every linear set of operators satisfying (1) for any $A \in \Sigma^+$ is contained in $\mathcal{M}$, and $\mathcal{M}$ is itself a closed linear space consisting of such operators.

The main ingredient in the proof of this theorem is the following proposition.

**Proposition 2** For an operator $T \in \mathcal{L}(L_1(\Omega))$ the following conditions are equivalent:

(i) $T$ and $-T$ satisfy (1) for all $A \in \Sigma^+$;
(ii) For every $\varepsilon > 0$ and $A \in \Sigma^+$ there is an $A' \in \Sigma^+_A$ such that if $B \in \Sigma^+_A$ then we can find a $B' \in \Sigma^+_B$ with the following properties:

a) $\left\| \frac{\chi_B}{\mu(B)} - \frac{\chi_B}{\mu(B)} \right\| < \varepsilon,$

b) $\left\| \chi_{B'} \cdot T \left( \frac{\chi_B}{\mu(B)} \right) \right\| < \varepsilon;$

(iii) $T \in \mathcal{M}.$

Proof. (i) implies (ii). We begin with the following observation.

Suppose $S : L_1(A) \rightarrow L_1(\Omega)$ is a bounded linear operator, then any given $\varepsilon > 0$ there is a set $A_1 \in \Sigma^+_A$ with $\mu(A_1) < \infty$ such that for every non-negative function $f \in S(L_1(A))$ we have $\|Sf\| > \|S\| - \varepsilon.$

Indeed, we can assume that $\mu(A) < \infty$ and choose $g^* \in S(L_1(\Omega))$ so that $\|S^*g^*\| > \|S\| - \varepsilon.$ Then, regarding $S^*g^*$ as an element of $L_\infty(A)$ we find a set $A_1 \in \Sigma^+_A$ with $\theta S^*g^*(A_1) \subset (\|S\| - \varepsilon, \|S\|],$ where $\theta$ is a sign. Now, if $f \in S(L_1(A)), f \geq 0$ and supp$(f) \subset A_1$, then $\|Sf\| > \theta g^*(Sf) = \theta S^*g^*(f) > \|S\| - \varepsilon,$ from where the observation follows.

We know that $\|Id_A + T_A\| = 1 + \|T_A\|.$ By scaling, without loss of generality we can and do assume that $\|T_A\| = 1.$ So there is an $A_1 \in \Sigma^+_A$ with $\mu(A_1) < \infty$ such that

$$
\left\| \frac{\chi_B}{\mu(B)} + T \left( \frac{\chi_B}{\mu(B)} \right) \right\| > 2 - \varepsilon,
$$
whenever $B \in \Sigma^+_A.$ We also know that $\|Id_{A_1} - T_{A_1}\| = 1 + \|T_{A_1}\| > 2 - \varepsilon.$

Thus there exists an $A' \in \Sigma^+_A$ such that

$$
\left\| \frac{\chi_B}{\mu(B)} - T \left( \frac{\chi_B}{\mu(B)} \right) \right\| > 2 - \varepsilon,
$$
whenever $B \in \Sigma^+_A.$

We prove that $A'$ is the desired set.

To this end, let us fix $B \in \Sigma^+_A.$ It follows from (3), (4) and a theorem of Dor [2] that there are two disjoint measurable sets $\Omega_1$ and $\Omega_2$ in $\Omega$ such that

$$
\int_{\Omega_1} T \left( \frac{\chi_B}{\mu(B)} \right)(t)dt > (1 - \varepsilon)^2,
$$
and

$$
\int_{\Omega_2} \frac{\chi_B}{\mu(B)}(t)dt > (1 - \varepsilon)^2.
$$
The last inequality implies
\[
\mu(B \cap \Omega_1) = \mu(B) \int_{B \cap \Omega_1} \frac{X_B}{\mu(B)}(t) dt < \mu(B) \int_{\Omega \setminus \Omega_2} \frac{X_B}{\mu(B)}(t) dt
\]
\[
< (1 - (1 - \varepsilon)^2) \mu(B) = (2\varepsilon - \varepsilon^2) \mu(B).
\]

Let us put \(B' = B \setminus \Omega_1\) and show that \(B'\) meets conditions a) and b).

First,
\[
\left\| \frac{X_{B'}}{\mu(B')} - \frac{X_B}{\mu(B)} \right\| = \int_{\Omega} \left| \frac{X_{B'}}{\mu(B')} - \frac{X_{B'}}{\mu(B)} + \frac{X_{B'}}{\mu(B)} - \frac{X_B}{\mu(B)} \right| (t) dt
\]
\[
\leq 1 - \frac{\mu(B')}{\mu(B)} + \frac{\mu(B \cap \Omega_1)}{\mu(B)} = 2\frac{\mu(B \cap \Omega_1)}{\mu(B)},
\]
and taking into account (6), we obtain
\[
\left\| \frac{X_{B'}}{\mu(B')} - \frac{X_B}{\mu(B)} \right\| < 2(2\varepsilon - \varepsilon^2). \tag{7}
\]

Second, from (5), (7) and \(\|T_A\| = 1\) it follows that
\[
\left\| \chi_{B'} \cdot T \left( \frac{X_{B'}}{\mu(B')} \right) \right\| = \int_{B'} \left| T \left( \frac{X_{B'}}{\mu(B')} \right) \right| (t) dt
\]
\[
< \int_{B'} \left| T \left( \frac{X_B}{\mu(B)} \right) \right| (t) dt + 2(2\varepsilon - \varepsilon^2)
\]
\[
\leq \int_{\Omega \setminus \Omega_1} \left| T \left( \frac{X_B}{\mu(B)} \right) \right| (t) dt + 2(2\varepsilon - \varepsilon^2)
\]
\[
\leq 3(2\varepsilon - \varepsilon^2).
\]

In view of arbitrariness of \(\varepsilon\), this gives the desired result.

It is obvious that (iii) follows from (ii).

Let us finally prove that (iii) implies (i). Since \(\mathcal{M}\) is stable under scalar multiplication, it is sufficient to prove (ii) only for \(T\). To this end, we fix an arbitrary \(A \in \Sigma^+\) and as above for any given \(\varepsilon > 0\) we find an \(A' \in \Sigma^+_A\) with \(\mu(A') < \infty\) such that for every \(B \in \Sigma^+_A\),
\[
\left\| T \left( \frac{X_{B'}}{\mu(B')} \right) \right\| > \|T_A\| - \varepsilon. \tag{8}
\]
By condition (4), there is a \(B_0 \in \Sigma^+_A\) such that
\[
\left\| \chi_{B_0} \cdot T \left( \frac{X_{B_0}}{\mu(B_0)} \right) \right\| < \varepsilon. \tag{9}
\]
This means that \(\chi_{B_0} \cdot T \left( \frac{X_{B_0}}{\mu(B_0)} \right)\) are almost disjoint functions, and as a consequence we have the following estimate:
\[\|Id_A + T_A\| \geq \|\frac{\chi_{B_0}}{\mu(B_0)} + T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right)\| \]

\[= \int_{B_0} \left|\frac{\chi_{B_0}}{\mu(B_0)} + T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right)\right| \left(t\right)dt + \int_\Omega \left|T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right)\right| \left(t\right)dt \]

\[-\int_{B_0} \left|T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right)\right| \left(t\right)dt \]

\[> 1 - \varepsilon + \|T_A\| - \varepsilon - \varepsilon = 1 + \|T_A\| - 3\varepsilon.\]

This finishes the proof. \[\square\]

Now we are in a position to prove our main result.

**Proof of Theorem 4.**

Proposition 2 easily implies that \(M\) is largest and consists of operators satisfying (1) for all \(A \in \Sigma^+\). \(M\) is obviously closed and stable under scaling. So, the only thing we have to prove is that if operators \(U\) and \(V\) belong to \(M\), then their sum belong to \(M\) too. To show this, we check condition (ii) of Proposition 2 for \(U + V\). Further on, we assume that \(\|V\| \leq 1\).

Indeed, let \(A \in \Sigma^+\) and \(\varepsilon > 0\) be arbitrary. Applying Proposition 2 to the operator \(U\) we find a set \(A' \in \Sigma^+_A\) as in condition (ii). Then, by the same proposition applied to \(V\) we find a set \(A'' \in \Sigma^+_A\) with the correspondent properties. To show that \(A''\) is the required set, suppose \(B \in \Sigma^+_A\). By the choice of \(A''\) there is a \(B' \in \Sigma^+_B\) such that

\[\left|\frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)}\right| < \frac{\varepsilon}{4}\]  \hspace{1cm} (8)

and

\[\left|\chi_{B'} \cdot V\left(\frac{\chi_{B'}}{\mu(B')}\right)\right| < \frac{\varepsilon}{4}.\]  \hspace{1cm} (9)

Since \(B' \subset A'\), by the analogous property of \(A'\), there is a \(B'' \in \Sigma^+_B\) with

\[\left|\frac{\chi_{B''}}{\mu(B'')} - \frac{\chi_{B'}}{\mu(B')}\right| < \frac{\varepsilon}{4}\]  \hspace{1cm} (10)

and

\[\left|\chi_{B''} \cdot U\left(\frac{\chi_{B''}}{\mu(B'')}\right)\right| < \frac{\varepsilon}{2}.\]
From (8) and (10) we get \[ \left\| \frac{\chi_{B''}}{\mu(B''')} - \frac{\chi_{B}}{\mu(B)} \right\| < \varepsilon. \] So, if we prove that
\[ \left\| \chi_{B''} \cdot V \left( \frac{\chi_{B''}}{\mu(B''')} \right) \right\| < \frac{\varepsilon}{2}, \]
then
\[ \left\| \chi_{B''} \cdot (V + U) \left( \frac{\chi_{B''}}{\mu(B''')} \right) \right\| < \varepsilon, \]
and we are done. But this easily follows from (9), (10) and the facts that \( \|V\| \leq 1 \) and \( B'' \subset B' \).

The proof is completed. \( \square \)

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