Dimension of the Center of a Brauer Configuration Algebra

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Abstract

We consider an arbitrary algebra of the class of Brauer configuration algebras and calculate the dimension of the center by determining a $K$-basis.

1 Introduction

Brauer configuration algebras were recently added to the mathematical literature by E. Green and S. Schroll in [GS] as a generalization of Brauer graph algebras. As the authors mention in [GS, Introduction] this class of algebras is a new class of mostly wild algebras whose additional structure arises from a combinatorial data, called Brauer configuration. Brauer configuration algebras are a generalization of Brauer graph algebras, in the sense that every Brauer graph is a Brauer configuration and every Brauer graph algebra is a Brauer configuration algebra.

Brauer graph algebras are so well-studied and understood because the combinatorial data of the underlying Brauer graph encodes many aspects of the representation theory of a Brauer graph algebra. For example, the Yoneda algebra [GSST], or group actions and coverings [GSS], just to mention a few. So, the expectation is that the Brauer configuration will encode the representation theory of Brauer configuration algebras.

2 Preliminaries and Notation

We will give a quick review of the definition of a Brauer configuration and its associated Brauer configuration algebra. We refer the reader to [GS, Sections 1 and 2] for complete details and many examples.

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We begin with a tuple $\Gamma = (\Gamma_0, \Gamma_1, \mu)$ where $\Gamma_0$ is a finite set whose elements we call vertices, $\mu : \Gamma_0 \rightarrow \mathbb{Z}_{>0}$ a function, called a multiplicity function of $\Gamma$, and $\Gamma_1$ is a finite collection of labeled finite sets of vertices where repetitions are allowed. We can also say that $\Gamma_1$ is a finite collection of finite labeled multisets whose elements are in $\Gamma_0$. We call the elements of $\Gamma_1$ polygons. Given a polygon $V$, we call its elements the vertices in $V$ and define the valency of $\alpha$, denoted by val($\alpha$), to be the value $\sum_{V \in \Gamma_1} \text{occ}(\alpha, V)$.

An orientation for $\Gamma$ is a choice, for each vertex $\alpha \in \Gamma_0$, of a cyclic ordering of the polygons in which $\alpha$ occurs as a vertex, including repetitions. To be more precise, for each $\alpha \in \Gamma_0$ such that val($\alpha$) = $t > 1$ or $\mu(\alpha) > 1$, let $V_1, \ldots, V_t$ be the list of polygons in which $\alpha$ occurs as a vertex, with a polygon $V$ occurring occ($\alpha, V$) times in the list, that is $V$ occurs the number of times $\alpha$ occurs as a vertex in $V$. The cyclic order at vertex $\alpha$ is obtained by linearly ordering the list, say $V_{i_1} < \cdots < V_{i_t}$ and by adding $V_{i_t} < V_{i_1}$. We observe that any cyclic permutation of a chosen cyclic ordering at vertex $\alpha$ can represent the same ordering. That is, if $V_1 < \cdots < V_t$ is the chosen cyclic ordering at vertex $\alpha$, so is a cyclic permutation such as $V_2 < V_3 < \cdots < V_t < V_1$ or $V_3 < V_4 < \cdots < V_t < V_1 < V_2$.

**Definition 2.1.** A Brauer configuration is a tuple $\Gamma = (\Gamma_0, \Gamma_1, \mu, o)$, where $\Gamma_0$ is a set of vertices, $\Gamma_1$ a set of polygons, $\mu$ is a multiplicity function, and $o$ is an orientation for $\Gamma$, such that the following conditions hold

C1. Every vertex in $\Gamma_0$ is a vertex in at least one polygon in $\Gamma_1$.

C2. Every polygon in $\Gamma_1$ has at least two vertices (which can be the same).

C3. Every polygon in $\Gamma_1$ has at least one vertex $\alpha$ such that val($\alpha$)$\mu(\alpha) > 1$.

We now define some special sets formed by vertices of a Brauer configuration. They will be used to simplify the notation in the process of the computations. For the Brauer configuration $\Gamma$ we define the following special subsets of $\Gamma_0$.

$$\mathcal{T}_\Gamma = \{ \alpha \in \Gamma_0 | \mu(\alpha)\text{val}(\alpha) = 1 \},$$  
$$\mathcal{C}_\Gamma = \{ \alpha \in \Gamma_0 | \text{val}(\alpha) = 1 \text{ and } \mu(\alpha) > 1 \},$$  
$$\mathcal{D}_\Gamma = \{ \alpha \in \Gamma_0 | \text{val}(\alpha) > 1 \}.$$  

Any element of $\mathcal{T}_\Gamma$ is called a truncated vertex of $\Gamma$ (see [GS, Definition 1.3]). Thus, we call the set $\Gamma_0 \setminus \mathcal{T}_\Gamma$ the collection of the non-truncated vertices. We can decompose the set $\mathcal{D}_\Gamma$ into the following sets.

$$\mathcal{A}_\Gamma = \{ \alpha \in \mathcal{D}_\Gamma | \mu(\alpha) > 1 \},$$  
$$\mathcal{B}_\Gamma = \{ \alpha \in \mathcal{D}_\Gamma | \mu(\alpha) = 1 \}.$$  

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It is clear that $\Gamma_0 \setminus \mathcal{T} = \mathcal{E}_\Gamma \cup \mathcal{T} = \mathcal{A}_\Gamma \cup \mathcal{B}_\Gamma \cup \mathcal{C}_\Gamma$.

Let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma)$ be a Brauer configuration. For $\alpha \in \Gamma_0 \setminus \mathcal{T}_\Gamma$ consider the list of polygons $V$ containing $\alpha$ such that $V$ occurs in this list $\text{occ}(\alpha, V)$ times. We know that the orientation $\sigma$ provides a cyclic ordering of this list. We call such a cyclically ordered list the successor sequence at $\alpha$. Suppose that $\alpha \in \mathcal{T}_\Gamma$ and that $V_1 < \cdots < V_t$ is the successor sequence at $\alpha$ (here $\text{val}(\alpha) = t$). Then we say that $V_{i+1}$ is the successor of $V_i$ at $\alpha$, for $1 \leq i \leq t$, where $V_{t+1} = V_1$. Note that for the case $\alpha \in \mathcal{E}_\Gamma$ there exists a unique polygon $V$ such that $\alpha \in V$, then the successor sequence at $\alpha$ is just $V$.

Given $\Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma)$ we define the quiver $\mathcal{Q}_\Gamma$ as follows. The vertex set $\{v_1, \ldots, v_m\}$ of $\mathcal{Q}_\Gamma$ is in a bijection with the set of polygons $\{V_1, \ldots, V_m\}$ in $\Gamma_1$, noting that there is exactly one vertex in $\mathcal{Q}_\Gamma$ for every polygon in $\Gamma_1$. We call $v_l$ (resp. $V_l$) the vertex (resp. polygon) associated to the unique polygon where $\alpha \in \mathcal{C}_\Gamma$ and suppose that $\text{occ}(\alpha, V_l) = 1$. In order to define the arrows in $\mathcal{Q}_\Gamma$, we use the successor sequences. For each $\alpha \in \Gamma_0 \setminus \mathcal{T}_\Gamma$, and each successor $V'$ of $V$ at $\alpha$, there is an arrow from $v$ to $v'$ in $\mathcal{Q}_\Gamma$, where $v$ and $v'$ are the vertices in $\mathcal{Q}_\Gamma$ associated to the polygons $V$ and $V'$ in $\Gamma_1$, respectively.

We should note that $V'$ can be a successor of $V$ more than once at a given vertex of $\Gamma_0$. We also note that $V'$ can be a successor of $V$ at more than one vertex of $\Gamma_0$, and for each such occurrence there is an arrow from $v$ to $v'$. In particular, $\mathcal{Q}_\Gamma$ may have multiple arrows from $v$ to $v'$. We will see an example of this (Example 3.1).

For each $\alpha \in \mathcal{D}_\Gamma$ with successor sequence $V_{i_1} < \cdots < V_{\text{val}(\alpha)}$ we have a corresponding collection of arrows in $\mathcal{Q}_\Gamma$:

$$
\begin{align*}
&v_{i_1} \xrightarrow{a_{i_2}^{(\alpha)}} v_{i_2} \xrightarrow{a_{i_3}^{(\alpha)}} \cdots \xrightarrow{a_{\text{val}(\alpha)}^{(\alpha)-1}} v_{\text{val}(\alpha)} \xrightarrow{a_{i_{1}}^{(\alpha)}} v_{i_1}.
\end{align*}
$$

(1)

Let $C_l = a_{j_1}^{(\alpha)} \cdots a_{j_t}^{(\alpha)} \cdots a_{\text{val}(\alpha)}^{(\alpha)} a_{j_t}^{(\alpha)} \cdots a_{j_1}^{(\alpha)}$ be the oriented cycle in $\mathcal{Q}_\Gamma$, for $1 \leq l \leq \text{val}(\alpha)$. We call any of these cycles a special $\alpha$-cycle. We observe that when $\alpha \in \mathcal{E}_\Gamma$, we have just one special $\alpha$-cycle, which is a loop at the vertex in $\mathcal{Q}_\Gamma$ associated to the unique polygon where $\alpha$ containing $\alpha$. Now fix a polygon $V$ in $\Gamma_1$ and suppose that $\text{occ}(\alpha, V) = t \geq 1$. Then there are $t$ indices $l_1, \ldots, l_t$ such that $V = V_{l_r}$ for every $1 \leq r \leq t$. We call any of the cycles $C_{l_1}, \ldots, C_{l_t}$ a special $\alpha$-cycle at $\alpha$, and we denote the collection of these cycles in $\mathcal{Q}_\Gamma$ by $\mathcal{C}_{(\alpha)}^v$. Note that $|\mathcal{C}_{(\alpha)}^v| = \text{occ}(\alpha, V)$. And if we define $\mathcal{Y}_\alpha = \{V \in \Gamma_1 \mid \alpha \in V\}$, the set of polygons containing $\alpha$, we define

$$
\mathcal{C}_\alpha := \bigcup_{V \in \mathcal{Y}_\alpha} \mathcal{C}_{(\alpha)}^v.
$$

(2)
This is the collection of all special \( \alpha \)-cycles. For the particular case \( \alpha \in C_\Gamma \) we define \( C C(\alpha) \) as the set consisting of the unique loop associated to the vertex \( \alpha \), and by \( Y(\alpha) \) the set consisting of the only polygon containing \( \alpha \).

From these observations and definitions we have that for any \( \alpha \in \Gamma_0 \setminus \mathcal{T}_\Gamma \)

\[
|CC(\alpha)| = \sum_{V \in Y(\alpha)} \text{occ}(\alpha, V)
= \sum_{V \in \Gamma_1} \text{occ}(\alpha, V)
= \text{val}(\alpha).
\] (3)

Before giving the definition of a Brauer configuration algebra we need one last definition. Let \( CC \) be the set of special cycles in \( Q_\Gamma \), that is

\[
CC := \bigcup_{\alpha \in \Gamma_0 \setminus \mathcal{T}_\Gamma} CC(\alpha)
\] (4)

and let \( f : CC \to (Q_\Gamma)_1 \) be the map which sends a special cycle to its first arrow.

**Proposition 2.2.** Let \( \Gamma = (\Gamma_0, \Gamma_1, \mu, o) \) be a Brauer configuration with associated quiver \( Q \). Then

\[
|Q_1| = \sum_{\alpha \in \Gamma_0 \setminus \mathcal{T}_\Gamma} \text{val}(\alpha),
\]

and for each \( a \in Q_1 \) there exists a unique \( \alpha \in \Gamma_0 \setminus \mathcal{T}_\Gamma \) such that \( a \) occurs once in any of the special cycles in \( CC(\alpha) \).

**Proof.** By the properties stated for special cycles in [GS, Section 2.3] we have

- \( \{CC(\alpha) | \alpha \in \Gamma_0 \setminus \mathcal{T}_\Gamma \} \) is a partition of \( CC \);
- the map \( f \) defined above is a bijective function.

Then by the bijection of \( f \) and the equalities in (3) and (4), we obtain

\[
|Q_1| = \sum_{\alpha \in \Gamma_0 \setminus \mathcal{T}_\Gamma} \text{val}(\alpha).
\]

This value also coincides with the number of special cycles. The second assertion follows from the bijection of \( f \) and the fact that any two special cycles in \( CC(\alpha) \) are cyclic permutations of one another (see (F’3) and (F’4) in [GS, Section 2.3]). \( \square \)

**2.1 Definition of a Brauer configuration algebra**

We define a set of elements \( \rho_\Gamma \) in \( KQ_\Gamma \) which will generate the ideal of relations \( I_\Gamma \) of the Brauer configuration algebra associated to the Brauer configuration \( \Gamma \). The set of relations \( \rho_\Gamma \) is defined by the following three
types of relations:

Relations of type one. It is the subset of $KQ_\Gamma$

$$\bigcup_{V \in \Gamma_1} \left( \bigcup_{\alpha, \beta \in V \setminus T} \left\{ C^{\mu(\alpha)} - D^{\mu(\beta)} \mid C \in \mathcal{C}_v^{\alpha}, D \in \mathcal{C}_v^{\beta} \right\} \right).$$

Relations of type two. It is the subset of $KQ_\Gamma$

$$\bigcup_{\alpha \in \Gamma_0 \setminus T} \left\{ C^{\mu(\alpha)} f(C) \mid C \in \mathcal{C}_v^{\alpha} \right\}.$$

Relations of type three. It is the set of all quadratic monomial relations of the form $ab$ in $KQ_\Gamma$ where $ab$ is not a subpath of any special cycle.

**Definition 2.3.** Let $K$ be a field and $\Gamma$ a Brauer configuration. The Brauer configuration algebra $\Lambda_\Gamma$ associated to $\Gamma$ is defined to be $KQ_\Gamma/I_\Gamma$, where $Q_\Gamma$ is the quiver associated to $\Gamma$ and $I_\Gamma$ is the ideal in $KQ_\Gamma$ generated by the set of relations $\rho_\Gamma$ of type one, two and three.

We show now the following proposition.

**Proposition 2.4.** Let $\Lambda$ be the Brauer configuration algebra associated to the connected Brauer configuration $\Gamma$. Then $\text{rad}^2(\Lambda) \neq 0$.

**Proof.** Let $\Gamma = (\Gamma_0, \Gamma_1, \mu, o)$ be a connected Brauer configuration, and let $\Lambda = Q/I$ be the induced Brauer configuration algebra. By [GS, Proposition 3.5] we have that $\Lambda$ is an indecomposable algebra. Now, we assume that $\text{val}(\alpha) < 2$, for all $\alpha \in \Gamma_0$. Because of the connectedness of $\Gamma$ we have that necessarily $\Gamma_1 = \{V\}$, then the induced quiver $Q$ has an only one vertex $v$, which is the vertex associated to $V$. Then $Q$ has the form

![Diagram](attachment:image.png)

where the number of loops at $v$ coincides with the number of nontruncated vertices in $V$. By condition C3 of Definition 2.1 there exists $\alpha \in V$ such that $\mu(\alpha) \geq 2$. If $a$ is the loop in $Q$ associated to $\alpha$, then the class of $a^2$ in $\Lambda$ is a non zero element of $\text{rad}^2(\Lambda)$ by [GS, Proposition 3.3].

Now, we assume that there exists a vertex $\alpha \in \Gamma_0$ such that $\text{val}(\alpha) \geq 2$. Then we have that any special $\alpha$-cycle in $Q$ contains at least two consecutive arrows. Hence by [GS, Proposition 3.3] we have again that $\text{rad}^2(\Lambda) \neq 0$. \qed
3 Basis in $v\Lambda v$.

Let $\Gamma$ be a Brauer configuration and $\Lambda$ the induced Brauer configuration algebra. Before giving the set of elements of a $K$-basis of $v\Lambda v$ for the vertex $v$ in $Q$ associated to the polygon $V$, we introduce a simple graphical method which allows us to recognize all the possible elements, in terms of a basis, that live in $v\Lambda v$. We start with an example.

Example 3.1. Let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma)$ be a Brauer configuration and let $Q$ be its induced quiver. We suppose that in the configuration $\Gamma$ we have a non-truncated vertex $\alpha$ which has a successor sequence given by

$$\alpha : V < V < V < V < W < W < W < W.$$  

We see at this case that $\text{occ}(\alpha, V) = \text{occ}(\alpha, W) = 5$. In particular, we also see that, for example, $V$ is the successor of $V$ at $\alpha$ four times. So having in mind this successor sequence we can say that attached to the quiver $Q$ we will have the subquiver.

$$a_8^{(\alpha)} \quad a_6^{(\alpha)}$$

$$a_5^{(\alpha)} \quad a_7^{(\alpha)}$$

$$a_4^{(\alpha)} \quad a_9^{(\alpha)}$$

$$a_3^{(\alpha)} \quad a_2^{(\alpha)} \quad a_1^{(\alpha)}$$

It is easy to see the special $\alpha$-cycles at $v$ and $w$. But there are many more elements that are in the space $v\Lambda v$ and they are not special cycles. This is not easily seen from the previous graphical expression. So the idea is to look at this quiver in a different way. We open up the entire cycle like this.
If we are only interested to look for oriented cycles at the vertex $v$ of the quiver, we simply erase the $w$’s and consider the $v$’s.

Now consider the following elements

$$q_{1}^{(\alpha,v)} = a_{1}^{(\alpha)}, \quad q_{2}^{(\alpha,v)} = a_{2}^{(\alpha)}, \quad q_{3}^{(\alpha,v)} = a_{3}^{(\alpha)},$$
$$q_{4}^{(\alpha,v)} = a_{4}^{(\alpha)}, \quad q_{5}^{(\alpha,v)} = a_{5}^{(\alpha)}a_{6}^{(\alpha)}\cdots a_{10}^{(\alpha)}$$

Each of these elements is an oriented cycle at $v$. By defining $q_{6}^{(\alpha,v)} = q_{1}^{(\alpha,v)}$ and reducing subindices modulo $\text{occ}(\alpha, V) = 5$, we also have that all elements in the set

$$\bigcup_{r=1}^{4} \{ q_{l}^{(\alpha,v)} \cdots q_{l+r-1}^{(\alpha,v)} \mid 1 \leq l \leq 5 \}$$
are oriented cycles at $v$. As we can see there are many oriented cycles at the vertex $v$ which are not special cycles. We can do the same to find oriented cycles for the vertex $w$. These ideas can be generalized.

### 3.1 Non-special cycles

We see in [GS, Proposition 3.3] that for a Brauer configuration algebra $\Lambda$ induced by the configuration $\Gamma$, the elements of a basis of $\Lambda$ are all prefixes of the elements $C^{\mu(\alpha)}$, where $C \in \mathcal{C}(\alpha)$ and $\alpha \in \Gamma_0 \setminus \mathcal{T}$. We will now compute a vector space basis for $v\Lambda v$, where $v$ is the vertex in the induced quiver associated to the polygon $V$.

For $\alpha \in \mathcal{T}$ fixed let $V_{i_1} < \cdots < V_{\text{val}(\alpha)}$ be its successor sequence, then in the quiver $Q$ we have a sequence of arrows

\[
v_{i_1} \xrightarrow{a_{j_1}^{(\alpha)}} v_{i_2} \xrightarrow{a_{j_2}^{(\alpha)}} \cdots \xrightarrow{a_{j_{\text{val}(\alpha)-1}}^{(\alpha)}} v_{\text{val}(\alpha)} \xrightarrow{a_{j_{\text{val}(\alpha)}}^{(\alpha)}} v_{i_1}.
\]

This sequence can be represented in a cyclic way. The sequentially composition of these arrows is a oriented cycle in $Q$. Let’s put the arrows and vertices in a cyclic drawing, like this.

![Cyclic Drawing](image)

Figure 3:

Now for a polygon $V \in \Gamma_1$ we suppose that $\text{occ}(\alpha, V) > 1$ then there are indices $l_1, \ldots, l_{\text{occ}(\alpha, V)}$ such that $v = v_{i_t}$ for each $1 \leq t \leq \text{occ}(\alpha, V)$, and where $v$ is the vertex in $Q$ associated to $V$. From Figure 3 we are going to derive another one. So, in Figure 3 we delete the vertices that are not equal to $v$, as also the arrows. We obtain the following figure.
Now fix one of the vertices $v$, any vertex you want, and label it as $1st \ v$; in counter clockwise consider the next one $v$ and label it as $2nd \ v$, and then the next one and label it as $3rd \ v$, and so on. At each occurrence of these labeled $v$’s we define two types of oriented cycles.

- At the $1st \ v$ we have $C^{\alpha,v}_1$ as the special $\alpha$-cycle at $v$ corresponding to the $1st \ v$, and $q_1^{(\alpha,v)}$, the oriented cycle obtained by composing all the arrows between the $1st \ v$ and the $2nd \ v$.

- At the $2nd \ v$ we have $C^{\alpha,v}_2$ as the special $\alpha$-cycle at $v$ corresponding to the $2nd \ v$, and $q_2^{(\alpha,v)}$, the oriented cycle obtained by composing all the arrows between the $2nd \ v$ and the $3rd \ v$.

- And so on, ... 

What we obtain after this finite procedure is the following figure.
For $\alpha \in \mathcal{D}_{\Gamma}$ and $V \in \mathcal{V}_{(\alpha)}$ such that $\text{occ}(\alpha, V) > 1$, we define the non-special $\alpha$-cycles at $v$ to be the oriented cycles $q^{(\alpha, v)}_1, \ldots, q^{(\alpha, v)}_{\text{occ}(\alpha, V)}$ in $\mathcal{Q}$. We denote these collection by $\neg
abla_{(\alpha)}^v$. From the same construction we see that $|\nabla_{(\alpha)}^v| = |\neg
abla_{(\alpha)}^v| = \text{occ}(\alpha, V)$. When $\text{occ}(\alpha, V) = 1$ we say that $\neg
abla_{(\alpha)}^v = \emptyset$. We have the following properties.

**Proposition 3.2.** Let $\Gamma$ be a Brauer configuration with induced quiver $\mathcal{Q}$ and $\alpha \in \mathcal{D}_{\Gamma}$. $V \in \mathcal{V}_{(\alpha)}$ such that $\text{occ}(\alpha, V) > 1$, and $v$ the vertex in $\mathcal{Q}$ associated to the polygon $V$. Then reducing all the subindices modulo $\text{occ}(\alpha, V)$ we have

1. $C^{(\alpha, v)}_j = q^{(\alpha, v)}_j \cdots q^{(\alpha, v)}_{j+\text{occ}(\alpha, V)-1}$, $1 \leq j \leq \text{occ}(\alpha, V)$.

2. $\left(C^{(\alpha, v)}_j\right)^l q^{(\alpha, v)}_j = q^{(\alpha, v)}_j \left(C^{(\alpha, v)}_{j+1}\right)^l$, $\forall l \geq 0$; $1 \leq j \leq \text{occ}(\alpha, V)$.

3. $\bigcup_{k=1}^{\text{occ}(\alpha, V)-1} \{q^{(\alpha, v)}_l \cdots q^{(\alpha, v)}_{l+k-1} | 1 \leq l \leq \text{occ}(\alpha, V)\} \subset v\mathcal{Q}v$, where $v\mathcal{Q}v$ is the collection of all the oriented cycles in $\mathcal{Q}$ with initial and final vertex $v$.

**Proof.** It follows using the Figure 5. \qed

Given a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma)$ we say that the polygon $V \in \Gamma_1$ is a $d$-gon if the number of vertices appearing in $V$ is $d$. We say that the configuration $\Gamma$ is reduced if and only if every polygon $V \in \Gamma_1$ satisfies one of the following conditions:

1. $V \cap \mathcal{T}_{\Gamma} = \emptyset$.

2. If $V \cap \mathcal{T}_{\Gamma} \neq \emptyset$, then $V$ is a 2-gon with only one truncated vertex.

Let $\Lambda = K\mathcal{Q}/I$ be the Brauer configuration algebra associated to a reduced Brauer configuration $\Gamma$. Denote by $\pi : K\mathcal{Q} \to \Lambda$ the canonical surjection. When no confusion may arise we denote $\pi(x)$ by $\overline{x}$, for $x \in K\mathcal{Q}$. From the definition of relations of type one in $K\mathcal{Q}$ we see that for a $V \in \Gamma_1$ and any $\alpha, \beta \in V \setminus \mathcal{T}_{\Gamma}$

$$\overline{C^{(\alpha)}_{\mu(\alpha)}} = \overline{D^{(\beta)}}, \forall C \in \nabla^{(\alpha)}_v, \forall D \in \nabla^{(\beta)}_v.$$ 

So, let $\alpha$ be any nontruncated vertex in $V$, and let $C \in \nabla^{(\alpha)}_v$. We denote by $C^{(V)}$ any representative of the equivalence class

$$\left\{C^{(\mu(\alpha))} + x | x \in I_{\Gamma}\right\}.$$

Then, if $\beta \in V$ is another nontruncated vertex we have

$$C^{(V)} = \overline{D^{(\mu(\beta))}}, \forall D \in \nabla^{(\beta)}_v.$$ (7)
We observe that the definition of \( C(V) \) depends only on the polygon \( V \).

Now we calculate an explicit \( K \)-basis for the vectorial space \( v\Lambda v \) where \( v \) is the vertex in \( Q \) associated to the polygon \( V \in \Gamma_1 \).

**Proposition 3.3.** Let \( \Lambda = KQ/I \) be the Brauer configuration algebra induced by \( \Gamma \). Then for any \( V \in \Gamma_1 \)

\[
\dim_K v\Lambda v = 2 + \sum_{\alpha \in \widehat{V}} \operatorname{occ}(\alpha, V)(\operatorname{occ}(\alpha, V)\mu(\alpha) - 1),
\]

where \( \widehat{V} = V \cap \Gamma_0 \).

**Proof.** For the polygon \( V \in \Gamma_1 \) define the following sets of vertices

\[
\begin{align*}
\hat{V} &= V \cap (\mathcal{A}_1 \cup \mathcal{C}_1), \\
\mathcal{A}^V_{>1} &= \{ \alpha \in V \cap \mathcal{A}_1 | \operatorname{occ}(\alpha, V) > 1 \}, \\
\mathcal{B}^V_{>1} &= \{ \beta \in V \cap \mathcal{B}_1 | \operatorname{occ}(\beta, V) > 1 \}.
\end{align*}
\]

Now in the induced Brauer configuration algebra \( \Lambda \) define the following subsets of the \( K \)-space

\[
\begin{align*}
L_1 &= \bigcup_{\alpha \in \hat{V}} \{ C \in \mathbb{C}^{(\alpha, V)}_i | 1 \leq j \leq \mu(\alpha) - 1 \}, \\
L_2 &= \bigcup_{\alpha \in \mathcal{A}^V_{>1}} \left( \bigcup_{k=1}^{\operatorname{occ}(\alpha, V) - 1} \left\{ \frac{q^j_{(\alpha, v)}}{q^j_{(\alpha, V)}} \cdots \frac{q^j_{(\alpha, V)_{k-1}}}{q^j_{(\alpha, V)_{k-1}}} | 0 \leq j \leq \mu(\alpha) - 1, 1 \leq l \leq \operatorname{occ}(\alpha, V) \right\} \right), \\
L_3 &= \bigcup_{\beta \in \mathcal{B}^V_{>1}} \{ q^j_{(\beta, v)} \cdots q^j_{(\beta, V)_{k-1}} | 1 \leq l \leq \operatorname{occ}(\beta, V), 1 \leq k \leq \operatorname{occ}(\beta, V) - 1 \}.
\end{align*}
\]

By using Proposition 3.2 we have that the set \( L_1 \cup L_2 \cup L_3 \cup \{ \overline{\mathbb{V}}, C(V) \} \) is a \( K \)-basis of \( v\Lambda v \). From the same definition of the collections \( L_i \)'s, we see that

\[
\begin{align*}
|L_1| &= \sum_{\alpha \in \widehat{V}} \operatorname{occ}(\alpha, V)(\mu(\alpha) - 1) + \sum_{\gamma \in V \cap \mathcal{C}_1} (\mu(\gamma) - 1), \\
|L_2| &= \sum_{\alpha \in \mathcal{A}^V_{>1}} \operatorname{occ}(\alpha, V)(\operatorname{occ}(\alpha, V) - 1)\mu(\alpha), \\
|L_3| &= \sum_{\beta \in \mathcal{B}^V_{>1}} \operatorname{occ}(\beta, V)(\operatorname{occ}(\beta, V) - 1),
\end{align*}
\]

but the expressions in (8) and (10) are respectively equal to

\[
\begin{align*}
\sum_{\alpha \in \widehat{V}} \operatorname{occ}(\alpha, V)(\operatorname{occ}(\alpha, V) - 1)\mu(\alpha), \\
\sum_{\beta \in \widehat{V}} \operatorname{occ}(\beta, V)(\operatorname{occ}(\beta, V) - 1).
\end{align*}
\]
Finally using the expressions (8), (11) and (12) we obtain

\[
\dim_K v \Lambda v = 2 + \sum_{\alpha \in V \cap A} \text{occ}(\alpha, V) (\mu(\alpha) - 1) + \sum_{\gamma \in V \cap \mathcal{B}} \text{occ}(\gamma, V) (\mu(\gamma) - 1)
\]

\[
= 2 + \sum_{\alpha \in V \cap A} \text{occ}(\alpha, V) (\mu(\alpha) - 1) + \sum_{\beta \in V \cap \mathcal{B}} \text{occ}(\beta, V) (\mu(\beta) - 1)
\]

\[
= 2 + \sum_{\gamma \in V \cap \mathcal{C}} \text{occ}(\gamma, V) (\mu(\gamma) - 1) \quad (13)
\]

Now if \( \alpha \in V \) is such that \( \alpha \in \mathcal{B}_T \) then \( \text{val}(\alpha) = \text{occ}(\alpha, V) = 1 \) so \( \text{occ}(\alpha, V) \mu(\alpha) = 1 \), then the expression in (13) remains equal to

\[
2 + \sum_{\alpha \in V \cap \mathcal{B}_0} \text{occ}(\alpha, V) (\mu(\alpha) - 1).
\]

\[\square\]

### 3.2 Induced Boundary Maps

Let \( \Lambda \) be a finite dimensional \( K \)-algebra (not necessarily a Brauer configuration algebra), and \( \Lambda^e = \Lambda^{\text{op}} \otimes_K \Lambda \) the enveloping algebra of \( \Lambda \). Consider a \( \Lambda^e \)-projective resolution of \( \Lambda \)

\[
P : \cdots \rightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} \Lambda \rightarrow 0 \quad (14)
\]

Applying the functor \( \text{Hom}_{\Lambda^e}(\cdot, \Lambda) \) to \( P \) in (14) we obtain the complex

\[
0 \rightarrow \text{Hom}_{\Lambda^e}(P_0, \Lambda) \xrightarrow{\hat{d}_1} \text{Hom}_{\Lambda^e}(P_1, \Lambda) \xrightarrow{\hat{d}_2} \cdots \xrightarrow{\hat{d}_n} \text{Hom}_{\Lambda^e}(P_n, \Lambda) \xrightarrow{\hat{d}_{n+1}} \cdots
\]

where \( \hat{d}_n \) is de boundary map induced by \( d_n \) for all \( n \geq 1 \). From this complex the \( n \)-th Hochschild cohomology group of \( \Lambda \) is defined by \( \text{HH}^n(\Lambda) = \ker d_{n+1}/\text{im} \hat{d}_n \). We see that \( \text{HH}^n(\Lambda) = \text{Ext}^n_{\Lambda^e}(\Lambda, \Lambda) \).

We are not interested, at the moment, in a complete \( \Lambda^e \)-projective resolution of \( \Lambda \). In fact, we are interested in just a small segment of a particular projective resolution of \( \Lambda \), where \( \Lambda = K \mathcal{Q}/I \) with \( \mathcal{Q} \) a finite quiver and \( I \) an admissible ideal of \( K \mathcal{Q} \). For this we use the construction in [GN]. Let’s consider the following segment of the minimal \( \Lambda^e \)-projective resolution of \( \Lambda = K \mathcal{Q}/I \)

\[
P_1 \xrightarrow{d_1} P_0 \xrightarrow{g} \Lambda \rightarrow 0, \quad (15)
\]
and where the $\Lambda$-bimodules $P_0, P_1$ are given by

\[
P_0 = \bigoplus_{v \in Q_0} \Lambda v \otimes v\Lambda,
\]
\[
P_1 = \bigoplus_{a \in Q_1} \Lambda s(a) \otimes t(a)\Lambda,
\]

and the $\Lambda$-bimodule homomorphisms $g$ and $d_1$ are defined by

\[
P_0 g : v \otimes_v v \mapsto v,
\]
\[
P_1 d_1 : s(a) \otimes_a t(a) \mapsto s(a) \otimes_s a - \bar{a} \otimes t(a) t(a),
\]

Here $\alpha_1 \otimes_v \beta_1$ means the element in $P_0$ which has all its entries equal to zero except the $v$-entry which is equal to $\alpha_1 \otimes \beta_1$. We have the same meaning for $P_1$, i.e., $\alpha_2 \otimes_a \beta_2$ denotes the element in $P_1$ which has all its entries equal to zero except the $a$-entry which is equal to $\alpha_2 \otimes \beta_2$.

Applying $\text{Hom}_{\Lambda^e}(\cdot, \Lambda)$ to the complex in (15) we obtain

\[
0 \longrightarrow \text{Hom}_{\Lambda^e}(P_0, \Lambda) \xrightarrow{\hat{d}_1} \text{Hom}_{\Lambda^e}(P_1, \Lambda).
\]

(16)

It is well known that $\text{HH}^0(\Lambda) = \ker \hat{d}_1 = Z(\Lambda)$, where $Z(\Lambda)$ denotes the center of $\Lambda$. We are not going to make the calculation of $\ker \hat{d}_1$ directly from the homomorphism $\hat{d}_1$. We use the ideas in [BM] and [B] to do this.

We know that there exist natural isomorphisms of vector spaces

\[
\text{Hom}_{\Lambda^e}(P_0, \Lambda) \cong \bigoplus_{v \in Q_0} v\Lambda v,
\]
\[
\text{Hom}_{\Lambda^e}(P_1, \Lambda) \cong \bigoplus_{a \in Q_1} s(a)\Lambda t(a).
\]

Let $v$ be a vertex in the quiver $Q$, $R_0 = \bigoplus_{v \in Q_0} v\Lambda v$ and let $\pi'_v : R_0 \to v\Lambda v$ be the respective natural projection onto the component $v$. For $\chi \in R_0$ we denote by $\chi_v$ the element in $v\Lambda v$ given by

\[
\chi_v := \pi'_v(\chi).
\]

(17)

Now we give the morphism of vector spaces

\[
d_1^* : \bigoplus_{v \in Q_0} v\Lambda v \longrightarrow \bigoplus_{a \in Q_1} s(a)\Lambda t(a)
\]

defined by

\[
d_1^*(\chi) = (\chi_{s(a)} \bar{a} - \bar{a} \chi_{t(a)})_{a \in Q_1},
\]

(18)
for any $\chi \in \bigsqcup_{v \in Q_0} v\Lambda v$.

It is not difficult to prove that the complex in (16) can be identified with

$$0 \longrightarrow \prod_{v \in Q_0} v\Lambda v \xrightarrow{d_1^*} \prod_{a \in Q_1} s(a)\Lambda t(a),$$  \hspace{1cm} (18)

then it follows that $Z(\Lambda) \cong \text{ker} d_1^*$. So, we will work with the morphism of $k$-spaces $d_1^*$, instead of $\hat{d}_1$, which is easier to handle.

4 Dimension of the center of Brauer configuration algebras.

4.1 Characteristic elements.

For each $\alpha \in \Gamma_0 \setminus \mathcal{R}_\Gamma$ of the Brauer configuration $\Gamma$ we denote by $C(\alpha)$ the element in the induced Brauer configuration algebra $\Lambda$ given by

$$C(\alpha) := \sum_{C \in \mathcal{C}(\alpha)} C.$$  \hspace{1cm} (19)

Remember that for a polygon $V \in \Gamma_1$ we denote by $C(V)$ the element in $\Lambda$ defined by the property in (7).

**Proposition 4.1.** Let $\Lambda$ be the algebra induced by the Brauer configuration $\Gamma$. Then

1. $\{ C(V) \mid V \in \Gamma_1 \} \subset \text{soc}_{\Lambda^e} \Lambda$.

2. For $\alpha \in \mathcal{R}_\Gamma$ let $C, D \in \mathcal{C}(\alpha)$ be special cycles.
   - (a) If $C \neq D$ then $\overline{C} \overline{D} = \overline{D} \overline{C} = 0$.
   - (b) If $C = D$ and $\alpha \in \mathcal{R}_\Gamma$ then $\overline{C}^2 = 0$.

3. For any $\alpha \in \Gamma_0 \setminus \mathcal{R}_\Gamma$,
   - (a) $C(\alpha)^{\mu(\alpha)} = \sum_{V \in \mathcal{V}(\alpha)} \text{occ}(\alpha, V) C(V)$;
   - (b) $\overline{C}^{\mu(\alpha)+1} = 0$, for all $C \in \mathcal{C}(\alpha)$.

**Proof.** It follows immediately from the definition of type two relations.

2 If a pair of two distinct elements from $\mathcal{C}(\alpha)$ are cycles starting at different vertices, then their respective product in $\Lambda$ must be zero. Now if $C, D \in \mathcal{C}(\alpha)$ for some $V \in \mathcal{V}(\alpha)$, and $C \neq D$, then their product $CD$ must contain a relation of type three and $\overline{C} \overline{D} = 0$ in $\Lambda$. We have the same for the product $DC$. If $\alpha \in \mathcal{R}_\Gamma$ and $C$ is any cycle in $\mathcal{C}(\alpha)$ we have that $C^2$
contains a relation of type two, i.e., $C^2 = 0$ in $\Lambda$.

3a We see that if $\alpha \in C_\Gamma$ the affirmation holds. Now, if $\alpha \in D_\Gamma$ then using what we proved in 2 and the property in (7) we have

$$C(\alpha)^{\mu(\alpha)} = \sum_{C \in \mathcal{E}(\alpha)} C^{\mu(\alpha)}$$

$$= \sum_{V \in \mathcal{V}(\alpha)} \left( \sum_{C \in \mathcal{E}(\alpha)} C^{(V)} \right)$$

$$= \sum_{V \in \mathcal{V}(\alpha)} \left( \sum_{C \in \mathcal{E}(\alpha)} 1_K C^{(V)} \right)$$

$$= \sum_{V \in \mathcal{V}(\alpha)} \text{occ}(\alpha, V)C^{(V)}.$$  

3b It follows from the definition of type two relations.

**Observation 4.2.** Using the statement 2 of Proposition 4.1 we have that for any $\alpha \in \Gamma_0 \setminus \mathcal{R}_\Gamma$ and any $j \geq 1$

$$C(\alpha)^j = \left( \sum_{C \in \mathcal{E}(\alpha)} C \right)^j = \sum_{C \in \mathcal{E}(\alpha)} C^j.$$  

(20)

For $\alpha \in \Gamma_0 \setminus \mathcal{R}_\Gamma$ let $\mathcal{A}^{\mathcal{A}}(\alpha)$ denote the subset of $Q_1$ defined by

$\mathcal{A}^{\mathcal{A}}(\alpha) := \{ a \in Q_1 | a \ is \ contained \ in \ a \ special \ \alpha \text{-cycle} \}.$  

(21)

We call the collection $\mathcal{A}^{\mathcal{A}}(\alpha)$ the *set of arrows associated to $\alpha$*. Let $h : \mathcal{R}_\Gamma \rightarrow \{0, 1\}$ be the map defined by

$$h(\alpha) = \begin{cases} 
0, & \mathcal{A}^{\mathcal{A}}(\alpha) \text{ contains no loops;} \\
1, & \mathcal{A}^{\mathcal{A}}(\alpha) \text{ contains loops.}
\end{cases}$$  

(22)

It is easy to check that

$$\alpha \in h^{-1}(1) \iff \exists V \in \mathcal{V}(\alpha); \text{occ}(\alpha, V) > 1 \text{ and } \exists 1 \leq s \leq \text{occ}(\alpha, V); q_s^{(\alpha, V)} \in Q_1.$$  

(23)

For $\alpha \in h^{-1}(1)$ let $\mathcal{M}^{(\alpha)}$ be the set of ordered pairs defined by

$$\mathcal{M}^{(\alpha)} = \{(V, s) | V \text{ and } s \text{ satisfies (23)}\}.$$  

(24)
Each pair \((V, s) \in \mathcal{M}(\alpha)\) is associated to a unique loop in \(Q\), the non-special \(\alpha\)-cycle \(q_{s}^{(\alpha,v)}\) in \((23)\). For \((V, s) \in \mathcal{M}(\alpha)\) let \(D_{V,s}^{(\alpha)}\) be the cycle in \(Q\) defined by

\[
D_{V,s}^{(\alpha)} = \left\{ \begin{array}{ll}
(C_{s+1}^{(\alpha,v)})^{\mu(\alpha)-1} q_{s+1}^{(\alpha,v)} \cdots q_{s-1}^{(\alpha,v)}, & \alpha \in \mathcal{A}; \\
q_{s+1}^{(\alpha,v)} \cdots q_{s-1}^{(\alpha,v)}, & \alpha \in \mathcal{B}. \end{array} \right.
\]

(25)

In \((25)\) the composition of cycles \(q_{s+1}^{(\alpha,v)} \cdots q_{s-1}^{(\alpha,v)}\) means

\[
q_{s+1}^{(\alpha,v)} \cdots q_{\text{occ}(\alpha,v)}^{(\alpha,v)} q_{1}^{(\alpha,v)} \cdots q_{s-1}^{(\alpha,v)}.
\]

We call \(D_{V,s}^{(\alpha)}\) a central mixed \(\alpha\)-cycle at the vertex \(v\) in \(Q\). We denote the collection of these cycles by \(\Psi_{(\alpha)}^{v}\), i.e,

\[
\Psi_{(\alpha)}^{v} = \left\{ D_{V,s}^{(\alpha)} \mid (V, s) \in \mathcal{M}(\alpha) \right\}.
\]

(26)

We recall that a path \(p\) is a prefix of the path \(q\) in the quiver \(Q\) if the first arrow of \(p\) coincides with the first arrow of \(q\) and \(p\) is a subpath of \(q\). The following lemma is very useful.

**Lemma 4.3.** Let \(\Lambda\) be a Brauer configuration algebra associated to the Brauer configuration \(\Gamma\). Let \(x\) be an element in \(\Lambda\) that satisfies the following property

\[
\forall \alpha \in \Gamma_0 \setminus \mathcal{F}, \forall C \in \mathcal{C}C(\alpha)
\]

\[
p \text{ is a prefix of } C^{\mu(\alpha)} \implies \overline{p}x = x\overline{p}.
\]

Then \(x \in Z(\Lambda)\).

**Proof.** This follows from \([GS, \text{Proposition 3.3}].\)

Let \(\Gamma = (\Gamma_0, \Gamma_1, \mu, \circ)\) be a reduced Brauer configuration with induced quiver \(Q\) and associated Brauer configuration algebra \(\Lambda = KQ/I\). For \(\alpha \in \mathcal{C}_I\) let \(\beta\) be a nontruncated vertex such that \(\beta \neq \alpha\) and let \(p\) be a prefix of \(C^{\mu(\beta)}\), for some \(C \in \mathcal{C}C(\alpha)\). Let \(a^{(\alpha)}\) be the unique element in \(\mathcal{C}C(\alpha)\), then \(a^{(\alpha)}\) is a loop. If \(pa^{(\alpha)} \neq 0\) and \(a^{(\alpha)}p = 0\) in \(KQ\) then \(pa^{(\alpha)}\) contains a relation of type three and hence \(pa^{(\alpha)} = a^{(\alpha)}p = 0\). In the same way, if \(a^{(\alpha)}p \neq 0\) but \(pa^{(\alpha)} = 0\) then \(a^{(\alpha)}p\) also contains a relation of type three and hence \(a^{(\alpha)}p = pa^{(\alpha)} = 0\). Now, if \(a^{(\alpha)}p \neq 0\) and \(pa^{(\alpha)} \neq 0\) then \(p\) must be an oriented cycle and hence both \(a^{(\alpha)}p\) and \(pa^{(\alpha)}\) contain a relation of type three, and also in this case \(a^{(\alpha)}p = pa^{(\alpha)} = 0\). Now, if \(\beta = \alpha\) then \(p\) must be a prefix of \((a^{(\alpha)})^{\mu(\alpha)}\) and obviously \(p\) commutes with \(a^{(\alpha)}\).

**Proposition 4.4.** Let \(\Gamma = (\Gamma_0, \Gamma_1, \mu, \circ)\) be a reduced Brauer configuration with associated Brauer configuration algebra \(\Lambda\). Then
1. \[ \left\{ C^{(V)} \mid V \in \Gamma_1 \right\} \subset Z(\Lambda); \]

2. \[ \bigcup_{\alpha \in \mathcal{C}} \left\{ \bar{a} \mid a \in \mathcal{A} \mathcal{A}_{\alpha} \right\} \subset Z(\Lambda); \]

3. \[ \bigcup_{\alpha \in h^{-1}(1)} \left\{ \overline{D^{(\alpha)}_{V,s}} \mid (V, s) \in \mathcal{M}^{(\alpha)} \right\} \subset Z(\Lambda). \]

Proof. 1. It is obvious by Proposition 4.1.1.

2. It follows from the previous observations and Lemma 4.3.

3. Let \( \alpha \in h^{-1}(1) \) and \((V, s) \in \mathcal{M}^{(\alpha)}\) be an ordered pair that satisfies (23). So, we have that the non-special cycle \( q^{(\alpha,v)} \) is a loop and the elements \( C^{(\alpha,v)}_s, C^{(\alpha,v)}_{s+1} \) and \( q^{(\alpha,v)}_s \) can be graphically represented as

![Figure 6:](image)

Let \( D \) be the element in \( \Psi^{v}_{(\alpha)} \) associated to the pair \((V, s)\). Let \( \beta \in \Gamma_0 \setminus \mathcal{I} \) and \( p \) be a prefix of \( C^{(\mu(\beta))} \) for some \( C \in \mathcal{C} \mathcal{C}_{(\beta)} \). Suppose that \( \alpha \neq \beta \). If \( Dp \neq 0 \) and \( pD = 0 \) in \( KQ \) then the path \( Dp \) contains a relation of type three, and hence \( Dp = pD = 0 \) in \( \Lambda \). In a similar way we have that if \( Dp = 0 \) but \( pD \neq 0 \) in \( KQ \), then \( \overline{pD} = Dp = 0 \) in \( \Lambda \). If \( pD \neq 0 \) and \( Dp \neq 0 \) then \( p \) must be a cycle, but also in this case both \( pD \) and \( Dp \) contain a relation of type three, and hence \( \overline{pD} = Dp = 0 \). Suppose now that \( \alpha = \beta \), and without loss of generality assume that \( \alpha \in \mathcal{A} \) (in the case \( \alpha \in \mathcal{B} \) the reasoning is analogous). It is sufficient to consider only the case when \( p \) is a prefix of \( \left( C^{(\alpha,v)}_s \right)^{\mu(\alpha)} \). If \( \ell(p) > 1 \) then is easy to see that \( Dp \) contains a
relation of type two, and if \( pD \neq 0 \) in \( KQ \) then also contains a relation of type two. Hence \( \overline{Dp} = \overline{pD} = 0 \). If \( \ell(p) = 1 \) then necessarily \( p = q_s^{(\alpha,v)} \), and by Proposition 5.2 we obtain

\[
\overline{Dp} = \left( C_s^{(\alpha,v)} \right)^{\mu(\alpha)-1} q_{s+1}^{(\alpha,v)} \cdots q_{s-1}^{(\alpha,v)} q_s^{(\alpha,v)}
\]

(27)

\[
\overline{pD} = \left( C_s^{(\alpha,v)} \right)^{\mu(\alpha)}.
\]

(28)

Note that \( \left( C_s^{(\alpha,v)} \right)^{\mu(\alpha)} - \left( C_s^{(\alpha,v)} \right)^{\mu(\alpha)} \) is a relation of type three, then (27) and (28) are equal. By Lemma 4.3 we can conclude that \( \overline{D} \) is in \( Z(\Lambda) \).  

\[ \square \]

### 4.2 Calculating the dimension of the center.

We start this section with two lemmas.

**Lemma 4.5.** Let \( \Lambda \) be the Brauer configuration algebra associated to the reduced and connected Brauer configuration \( \Gamma \). For each \( a \in Q_1 \) there exists a unique \( \alpha \in \Gamma_0 \setminus \mathcal{T} \) such that

1. If \( \alpha \notin \mathcal{B}_\Gamma \) then there exist a unique pair \( (C, C') \) of elements in \( \mathcal{CC}_{(\alpha)} \) such that \( \overline{Ca} \neq 0 \) and \( \overline{aC'} \neq 0 \).

2. If \( \alpha \in \mathcal{B}_\Gamma \) then \( \overline{Da} = \overline{aE} = 0 \), for any \( D, E \in \mathcal{CC} \).

**Proof.** 1 By Proposition 2.2 there exists a unique \( \alpha \in \Gamma_0 \setminus \mathcal{T} \) such that \( a \) occurs once in any of the special cycles in \( \mathcal{CC}_{(\alpha)} \). Let \( (C, C') \) be the pair of elements of \( \mathcal{CC}_{(\alpha)} \) defined by

- \( a \) is the first arrow of \( C \);
- \( a \) is the last arrow of \( C' \).

Then \( (C, C') \) is the required pair.

2 It follows from the type two and type three relations.  

If \( \alpha \in \mathcal{T} \) it follows from the statement of Lemma 4.3 that, without lost of generality, we can assume that \( a = a_i^{(\alpha)} \), where \( 1 \leq i \leq \text{val}(\alpha) \), as in the sequence in (1). Then the special cycles \( C \) and \( C' \) are given respectively by

\[
C = a_i^{(\alpha)} \cdots a_1^{(\alpha)},
\]

\[
C' = a_{i+1}^{(\alpha)} \cdots a_{i}^{(\alpha)},
\]

(29)
where $\alpha_1^{(a)}, \ldots, \alpha_{\text{val}(a)}^{(a)}$ is the collection of arrows induced by the successor sequence of the vertex $\alpha$. We have the following lemma.

**Lemma 4.6.** Assuming all the hypothesis of Lemma 4.5 for $a \in Q_1$ let $\alpha \in \Gamma_0 \setminus \mathcal{R}$ be the unique vertex such that $\alpha \notin \mathcal{R}$ and let $C, C' \in \mathcal{C}^{(a)}_\Gamma$ be the unique special cycles associated to $\alpha$ and $a$ as set up in (29). For any positive integer $l$ we have that

$$C^l a = a C^l.$$

**Proof.** The proof is straightforward and is left to the reader. We just mention that when the vertex $\alpha$ of Lemma 4.5 belongs to $\Gamma$ necessarily the arrow $a$ is a loop, and so $a = C = C'$.

Let $\Lambda = KQ/I$ be the Brauer configuration algebra associated to the reduced Brauer configuration $\Gamma$, and assume that $\Lambda$ is indecomposable and $\text{rad}^2(\Lambda) \neq 0$. Let $\gamma$ be an element of the $K$-space $\bigoplus_{v \in Q_0} v \Lambda v$. For a vertex $v$ in $Q$ let $\chi_v$ denote the element defined in (17). By Proposition 3.3 and its proof we have that the element $\chi_v$ of $v \Lambda v$ can be expressed by

$$\chi_v = x(v)_v + \sum_{\alpha \in V} \left( \sum_{j=1}^{\mu(\alpha)-1} \left( \sum_{C \in \mathcal{C}_{(\alpha)}^j} y(\alpha,j,f(C)^v) \right) \right) + z(v) C(v) + \sum_{\alpha \in \mathcal{C}_{(\alpha)}^{\bar{v}}} \left( \sum_{k=1}^{\text{occ}(\alpha,v)-1} \left( \sum_{l=1}^{\mu(\alpha)-1} \left( \sum_{f \in C_{(\alpha)}^l} y(\alpha,j,f(C)^{\alpha,v}) \chi_v \right) \right) \right)$$

$$+ \sum_{\beta \in \mathcal{C}_{(\beta)}^{\bar{v}}} \left( \sum_{k=1}^{\text{occ}(\beta,v)-1} \left( \sum_{l=1}^{\mu(\beta)-1} \left( \sum_{f \in C_{(\beta)}^l} y(\beta,j,f(C)^{\beta,v}) \right) \right) \right)$$

$$+ \sum_{\gamma \in \mathcal{C}_{(\gamma)}^{\bar{v}}} \left( \sum_{k=1}^{\text{occ}(\gamma,v)-1} \left( \sum_{l=1}^{\mu(\gamma)-1} \left( \sum_{f \in C_{(\gamma)}^l} y(\gamma,j,f(C)^{\gamma,v}) \right) \right) \right)$$

$$+ \sum_{\delta \in \mathcal{C}_{(\delta)}^{\bar{v}}} \left( \sum_{k=1}^{\text{occ}(\delta,v)-1} \left( \sum_{l=1}^{\mu(\delta)-1} \left( \sum_{f \in C_{(\delta)}^l} y(\delta,j,f(C)^{\delta,v}) \right) \right) \right)$$

where $x(v), z(v), y(\alpha,j,f(C)^v), y(\alpha,j,f(C)^{\alpha,v}), y(\beta,j,f(C)^{\beta,v}), y(\gamma,j,f(C)^{\gamma,v}), y(\delta,j,f(C)^{\delta,v})$ are scalars in the field $K$, and and $g : -\mathcal{C} \rightarrow (Q)_1$ the map that sends a non-special cycle to its first arrow, with

$$\mathcal{C}_{(\alpha)}^- = \bigcup_{v \in \mathcal{V}^-} \mathcal{C}_{(\alpha)}^v$$

for every $\alpha \in \mathcal{R}$, and

$$\mathcal{C} = \bigcup_{\alpha \in \mathcal{R}} \mathcal{C}_{(\alpha)}^-.$$

If $d_1^\alpha : \bigoplus_{v \in Q_0} v \Lambda v \rightarrow \bigoplus_{v \in Q_1} s(a) \Lambda t(a)$ is the $\Lambda$-bimodule homomorphism defined in Subsection 3.2 we have that the image of $\chi$ under $d_1^\alpha$ can be computed as

$$d_1^\alpha(\chi) = (\chi_{s(a)} - a \chi_{t(a)} - a \chi_{t(a)})_{a \in Q_1}. \tag{31}$$

Let $a$ be an arrow of the quiver $Q$, and let $\alpha$ be the unique vertex in Lemma 4.3 associated to the arrow $a$. For the vertex $\alpha$ we have the following possible cases.
(a) $\alpha \in \mathcal{A}_\Gamma$;
(b) $\alpha \in \mathcal{B}_\Gamma$;
(c) $\alpha \in \mathcal{C}_\Gamma$.

For the arrow $a$ let $V, W$ be polygons of the configuration $\Gamma$ associated to the vertices $v, w$ of the quiver $Q$ respectively, and such that $s(a) = v$ and $t(a) = w$. If $\alpha \in \mathcal{D}_\Gamma$ we can suppose that $a = a_1^{(\alpha)}$ for some $1 \leq i \leq \text{val}(\alpha)$, and the unique cycles $C, C'$ of Lemma 4.5 are given respectively by

$$C_s^{(\alpha,v)} = a_1^{(\alpha)} \cdots a_{i-1}^{(\alpha)},$$
$$C_r^{(\alpha,w)} = a_{i+1}^{(\alpha)} \cdots a_i^{(\alpha)},$$

where $1 \leq s \leq \text{occ}(\alpha, V)$ and $1 \leq r \leq \text{occ}(\alpha, W)$. For the understanding of some further computations we will represent graphically the location of $C_s^{(\alpha,v)}$ and $C_r^{(\alpha,w)}$ by

![Diagram](image)

Figure 7:

(32) Case $\alpha \in \mathcal{A}_\Gamma$: Using expression (30) for $\chi_v$ and $\chi_w$ and Lemma 4.5 we have that the $a$-entry in (31) is equal to

$$\left( x_v^{(a)} - x_w^{(a)} \right) \bar{a} + \sum_{j=1}^{\mu(\alpha)-1} y_j^{(a)} \left( C_s^{(\alpha,v)} \right)^j \bar{a}$$
$$+ \sum_{l=1}^{\text{occ}(\alpha,V)-1} \left( \sum_{j=0}^{\mu(\alpha)-1} y_j^{(a)} C_r^{(\alpha,w), \text{occ}(\alpha,V)-l} \right) \left( C_s^{(\alpha,v)} \right)^j \bar{a}$$
$$- \left( \sum_{k=1}^{\text{occ}(\alpha,W)} \left( \sum_{j=0}^{\mu(\alpha)-1} y_j^{(a)} C_r^{(\alpha,w)} \right)^j q_{l+k-1}^{(\alpha,v)} \bar{a} \right) + \sum_{j=1}^{\mu(\alpha)-1} y_j^{(a)} \left( C_r^{(\alpha,w)} \right)^j \bar{a}$$

(32)
By Proposition 3.2.2 and Lemma 4.6 we have
\[
\left( C_s^{(a,v)} \right)^j q_{s+1}^{(a,v)} \cdots q_{s+\text{occ}(a,V)-1}^{(a,v)} a = q_{s+1}^{(a,v)} \cdots q_{s+\text{occ}(a,V)-1}^{(a,v)} \left( C_s^{(a,v)} \right)^j a,
\]
for all \( 0 \leq j \leq \mu(a) - 1 \), and all \( 1 \leq l \leq \text{occ}(a,V) - 1 \). So applying these equalities in (32) and then collecting similar terms we obtain
\[
\left( x^{(v)} - y^{(v)} \right) a + \sum_{l=1}^{\mu(a)-1} \left( \sum_{j=0}^{\mu(a)-1} y^{(a)}_{j,j+1,l} q_{s+1}^{(a,v)} \cdots q_{s+\text{occ}(a,V)-1}^{(a,v)} \left( C_s^{(a,v)} \right)^j a \right)
\]
\[
+ \sum_{j=1}^{\mu(a)-1} \left( \sum_{j=0}^{\mu(a)-1} y^{(a)}_{j,j+1,l} \left( C_s^{(a,v)} \right)^j a \right) - \left( \sum_{k=1}^{\mu(a)-1} \left( \sum_{j=0}^{\mu(a)-1} y^{(a)}_{j,k+1,l} \left( C_s^{(a,v)} \right)^j a \right) \right) q_{s+1}^{(a,v)} \cdots q_{s+\text{occ}(a,V)-1}^{(a,v)}
\]
\[
= \sum_{j=0}^{\mu(a)-2} \left( \sum_{j=0}^{\mu(a)-1} y^{(a)}_{j,j+1,l} \left( C_s^{(a,v)} \right)^j a \right) q_{s+1}^{(a,v)} \cdots q_{s+\text{occ}(a,V)-1}^{(a,v)} + \sum_{j=0}^{\mu(a)-2} \left( \sum_{j=0}^{\mu(a)-1} y^{(a)}_{j,k+1,l} \left( C_s^{(a,v)} \right)^j a \right) q_{s+1}^{(a,v)} \cdots q_{s+\text{occ}(a,V)-1}^{(a,v)}
\]
(33)

Now, if \( a = a_{k+1}^{(a)} \) is a loop then we obtain that \( q_s^{(a,v)} = a \), \( v = w \), \( s = s + 1 \), and by Figure 1 \( f(C_s^{(a,v)}) = a_{k+1}^{(a)} \). For the particular values \( l = 1 \) and \( k = \text{occ}(a,V) - 1 \) we can regroup terms in (33) which have same scalars. Then in (33) appears the expression
\[
\sum_{j=0}^{\mu(a)-1} y^{(a)}_{j,a_{k+1}^{(a)},l} \left( C_s^{(a,v)} \right)^j a
\]
(34)

The right hand side of the equality in (34) holds because

- \( C_{s+1}^{(a,v)} = q_{s+1}^{(a,v)} \cdots q_{s-1}^{(a,v)} a \) and \( C_s^{(a,v)} = a q_{s+1}^{(a,v)} \cdots q_{s-1}^{(a,v)} \) by Proposition 3.2.1
- \( \left( C_{s+1}^{(a,v)} \right)^{\mu(a)} = \left( C_s^{(a,v)} \right)^{\mu(a)} \) by type one relations.

In conclusion, when \( a = a_{k+1}^{(a)} \) is a loop (33) is equal to
\[
\sum_{j=0}^{\mu(a)-1} \left( y^{(a)}_{j,a_{k+1}^{(a)},l} \left( C_s^{(a,v)} \right)^j a \right)
\]
\[
+ \sum_{j=0}^{\mu(a)-2} \left( \sum_{j=0}^{\mu(a)-1} y^{(a)}_{j,k+1,l} \left( C_s^{(a,v)} \right)^j a \right) q_{s+1}^{(a,v)} \cdots q_{s+\text{occ}(a,V)-1}^{(a,v)}
\]
\[
- \sum_{j=0}^{\mu(a)-2} \left( \sum_{j=0}^{\mu(a)-1} y^{(a)}_{j,k+1,l} \left( C_s^{(a,v)} \right)^j a \right) q_{s+1}^{(a,v)} \cdots q_{s+\text{occ}(a,V)-1}^{(a,v)}
\]
\[
+ \sum_{j=0}^{\mu(a)-2} \left( \sum_{j=0}^{\mu(a)-1} y^{(a)}_{j,k+1,l} \left( C_s^{(a,v)} \right)^j a \right) q_{s+1}^{(a,v)} \cdots q_{s+\text{occ}(a,V)-1}^{(a,v)}
\]
(35)
Note that the expression obtained in (35) was expected. One of the missing scalars in (35) is exactly the same associated to the class of the elements defined in (25) of Subsection 4.1, which is associated to the loop $q_s^{(a,v)} = a_i^{(a)}$ (the element associated to the scalar $y_{\alpha,1}^{(a-1)}(\alpha,v)-1$). We showed in Proposition 4.4.2 that this type of elements are in the center of the algebra.

Case $\alpha \in \mathcal{B}$: Using again (30) to compute $\chi_v$ and $\chi_w$ and Lemma 4.3 we have that the $a$-entry in (31), in this case, is equal to

$$\left(x^{(a)} - x^{(w)} \right) \tilde{a} + \sum_{l=1}^{\text{occ}(a,V)-1} y^{(a)}_{\alpha,1+1,\text{occ}(a,V)-l} \cdot q^{(a,v)}_{\alpha+1} \cdots q^{(a,v)}_{\alpha+\text{occ}(a,V)-1} \tilde{a} - \sum_{k=1}^{\text{occ}(a,W)-1} y^{(a)}_{\alpha,1+1,k} \cdot \bar{q}^{(a,v)}_{\alpha+1} \cdots \bar{q}^{(a,v)}_{\alpha+k-1}$$

Also in this case, if $a = a_i^{(a)}$ is a loop then $q_s^{(a,v)} = a, v = w, r = s + 1$ and $g(q_s^{(a,v)}) = a_i^{(a)}$. For the particular values $l = 1$ and $k = \text{occ}(\alpha, V) - 1$ we have that in (36) appears the expression

$$y^{(a)}_{\alpha,1+1,\text{occ}(a,V)-1} \cdot q^{(a,v)}_{\alpha+1} \cdots q^{(a,v)}_{\alpha+\text{occ}(a,V)-1} \cdot q^{(a,v)}_{\alpha+1} \cdots q^{(a,v)}_{\alpha+k-1}$$

This is due the fact that $\bar{C}_{s+1}^{(a,v)} = C_s^{(a,v)}$ and Proposition 3.2.1. So, when $a = a_1^{(a)}$ is a loop then (36) is equal to

$$\sum_{l=2}^{\text{occ}(a,V)-1} y^{(a)}_{\alpha,1+1,\text{occ}(a,V)-l} \cdot q^{(a,v)}_{\alpha+1} \cdots q^{(a,v)}_{\alpha+\text{occ}(a,V)-1} - \sum_{k=1}^{\text{occ}(a,W)-1} y^{(a)}_{\alpha,1+1,k} \cdot \bar{q}^{(a,v)}_{\alpha+1} \cdots \bar{q}^{(a,v)}_{\alpha+k-1}$$

As in the case (3), the expression obtained in (36) was expected. The missing scalar, $y^{(a)}_{\alpha,1+1,\text{occ}(a,V)-1}$, is associated to the class of elements defined in (35), which belong to the center of the algebra.

Case $\alpha \in \mathcal{C}$: In this case the arrow $a$ is necessarily a loop, and by Proposition 4.4.2 the $a$-entry in (31) is equal to zero.

Observation 4.7. We give some remarks about the appearing terms in expressions (33), (35), (36) and (38). When the arrow $a$ is not a loop, the collection

$$\left\{ \begin{array}{c} \sum_{l=0}^{\text{occ}(a,V)-1} y^{(a)}_{\alpha,1+1,\text{occ}(a,V)-l} \cdot q^{(a,v)}_{\alpha+1} \cdots q^{(a,v)}_{\alpha+\text{occ}(a,V)-1} \cdot \bar{a} \cdot \bar{q}^{(a,v)}_{\alpha+1} \cdots \bar{q}^{(a,v)}_{\alpha+k-1} \\ \text{with } 0 \leq j \leq \text{occ}(a,V) - 1, 1, 1 \leq k \leq \text{occ}(a,W) - 1 \end{array} \right\}$$

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is a linearly independent subset of the $K$-space $\Lambda$. This is due the fact that the collection
\[
\left\{ q_{e+l}^{(a,v)} \cdots q_{e-1}^{(a,v)} \left( C_s^{(a,v)} \right)^j \bar{a} \mid 0 \leq j \leq \mu(a) - 1, 1 \leq l \leq \operatorname{occ}(a,V) - 1 \right\}
\] (40)
when it is not empty, is formed by paths strictly contained in $\left( C_s^{(a,v)} \right)^{\mu(a)} a$ which don’t start at $a$ but they do finish at $a$; whereas the collection
\[
\left\{ C_s^{(a,v)} \bar{q}_{j+1}^{(a,v)} \cdots q_{e+1}^{(a,v)} \left( C_s^{(a,v)} \right)^j \bar{a} \mid 0 \leq j \leq \mu(a) - 1, 1 \leq k \leq \operatorname{occ}(a,W) - 1 \right\}
\] (41)
when it is not empty, is formed by prefixes of $\left( C_s^{(a,v)} \right)^{\mu(a)} a$ which start at $a$ but they don’t finish at $a$. Particularly, the intersection of (40) and (41) is empty. Elements of these sets correspond to the appearing terms in (33) and (30). When the arrow $a$ is a loop and $\alpha \in \mathcal{A}_1$, then the union of the sets
\[
\left\{ C_s^{(a,v)} \bar{a} \mid 1 \leq j \leq \mu(a) - 1 \right\},
\] (42)
\[
\left\{ q_{e+l}^{(a,v)} \cdots q_{e-1}^{(a,v)} \left( C_s^{(a,v)} \right)^j \bar{a} \mid 1 \leq l \leq \operatorname{occ}(a,V) - 1, 0 \leq j \leq \mu(a) - 1 \right\},
\] (43)
\[
\left\{ C_s^{(a,v)} \bar{q}_{j+1}^{(a,v)} \cdots q_{e+1}^{(a,v)} \left( C_s^{(a,v)} \right)^j \bar{a} \mid 1 \leq k \leq \operatorname{occ}(a,V) - 1, 1 \leq j \leq \mu(a) - 1 \right\},
\] (44)
\[
\left\{ \left( C_s^{(a,v)} \right)^j - \left( C_s^{(a,v)} \right)^{j+1} \mid 0 \leq j \leq \mu(a) - 2 \right\},
\] (45)
is a linearly independent subset of the $K$-space $\Lambda$. Elements of this set correspond to the appearing terms in (47). And when $\alpha \in \mathcal{A}_1$ also the set
\[
\left\{ q_{e+l}^{(a,v)} \cdots q_{e-1}^{(a,v)}, q_{e+1}^{(a,v)} \cdots q_{e+k}^{(a,v)} \mid 1 \leq l \leq \operatorname{occ}(a,V) - 1, 1 \leq k < \operatorname{occ}(a,V) - 1 \right\}
\] (46)
is a linearly independent subset of the $K$-space $\Lambda$, and the elements in this set correspond to the appearing terms in (35).

Now, let $\chi$ be our initial element in $\prod_{v \in Q_0} v\Lambda v$ and we suppose that $\chi \in \ker d_1^*$. Then we have
\[
\chi_{s(a)} \bar{a} - \bar{a} \chi_{t(a)} = 0, \forall a \in Q_1.
\]
From expressions in (33) and (36) we can say that
\[
x^{(s(a))} - x^{(t(a))} = 0, \text{ for all } a \in Q_1.
\] (47)
Because we are assuming that $\Lambda$ is indecomposable it follows that the induced quiver $Q$ is connected; since $Q$ is formed by special cycles the expression in (47) gives us that
\[
x^{(v)} = x^{(w)}, \text{ for all } v, w \in Q_0.
\] (48)
But by (30) we see that this implies that $\sum_{v \in Q_0} v = 1_\Lambda$ is in the center of $\Lambda$. 

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Now, let \( \alpha \in \mathcal{A} \) and \( \mathcal{A}(\alpha) = \{ a_1^{(\alpha)}, \ldots, a_{\text{val}(\alpha)}^{(\alpha)} \} \) be the collection of arrows associated to \( \alpha \), such that \( t(a_i^{(\alpha)}) = s(a_{i+1}^{(\alpha)}) \) for each \( 1 \leq i \leq \text{val}(\alpha) \), and where \( a_{\text{val}(\alpha)+1}^{(\alpha)} = a_1^{(\alpha)} \). For an arbitrary arrow \( a_i^{(\alpha)} \) in \( \mathcal{A}(\alpha) \), by looking at (33) and (35) we see that if \( a_i^{(\alpha)} \) is either a loop or not a loop we obtain the system of linear equations

\[
y_j^{(\alpha)} - y_{j,a_i^{(\alpha)}}^{(a)} = 0, \quad \text{for all } 1 \leq j \leq \mu(\alpha) - 1.
\]

That is, by considering each arrow in \( \mathcal{A}(\alpha) \) what we obtain is the following system of linear equations

\[
y_{1,a_1^{(\alpha)}}^{(\alpha)} - y_{1,a_2^{(\alpha)}}^{(\alpha)} = 0, \quad \ldots, \quad y_{1,a_{\text{val}(\alpha)}^{(\alpha)}}^{(\alpha)} - y_{1,a_{\text{val}(\alpha)+1}^{(\alpha)}}^{(\alpha)} = 0
\]

\[
y_{2,a_1^{(\alpha)}}^{(\alpha)} - y_{2,a_2^{(\alpha)}}^{(\alpha)} = 0, \quad \ldots, \quad y_{2,a_{\text{val}(\alpha)}^{(\alpha)}}^{(\alpha)} - y_{2,a_{\text{val}(\alpha)+1}^{(\alpha)}}^{(\alpha)} = 0
\]

\[
\vdots
\]

\[
y_{\mu(\alpha)-1,a_1^{(\alpha)}}^{(\alpha)} - y_{\mu(\alpha)-1,a_2^{(\alpha)}}^{(\alpha)} = 0, \quad \ldots, \quad y_{\mu(\alpha)-1,a_{\text{val}(\alpha)}^{(\alpha)}}^{(\alpha)} - y_{\mu(\alpha)-1,a_{\text{val}(\alpha)+1}^{(\alpha)}}^{(\alpha)} = 0
\]

Then we have that this linear system gives us

\[
y_j^{(\alpha)} = y_j^{(\alpha)}_{j,a_i^{(\alpha)}} = y_j^{(\alpha)}_{j,a_{i+1}^{(\alpha)}}, \quad \forall 1 \leq j \leq \mu(\alpha) - 1;
\]

\[
y_i^{(\alpha)} = y_i^{(\alpha)}_{j,a_i^{(\alpha)}}, \quad \forall 1 \leq i \leq \text{val}(\alpha).
\]

So, by returning to the expression (30) we see that

\[
\sum_{j=1}^{\mu(\alpha)-1} \left( \sum_{C \in \mathcal{C}(\alpha)} y_j^{(\alpha)} C \right) = \sum_{j=1}^{\mu(\alpha)-1} y_j^{(\alpha)} \left( \sum_{C \in \mathcal{C}(\alpha)} C^j \right)
\]

is an element of the center of the algebra, and by Observation 4.2 this element is equal to

\[
\sum_{j=1}^{\mu(\alpha)-1} y_j^{(\alpha)} C(\alpha)^j.
\]

That is, for each \( \alpha \in \mathcal{A} \) we have that the element in (52) is in the center of the algebra.

**Observation 4.8.** If \( \alpha \in \mathcal{Z} \), then there exists a unique polygon \( V \) such that \( \mathcal{V}(\alpha) = \{ V \} \) and \( \mathcal{A}(\alpha) = \{ a^{(\alpha)} \} \). We proved in Proposition 4.4.2 that \( a^{(\alpha)} \in Z(\Lambda) \), then \( C(\alpha)^j = (a^{(\alpha)})^j \in Z(\Lambda) \), for every \( 1 \leq j \leq \mu(\alpha) - 1 \). So, for each \( \alpha \in \mathcal{Z} \), we have that

\[
\bigcup_{\alpha \in \mathcal{Z}} \{ C(\alpha)^j \mid 1 \leq j \leq \mu(\alpha) - 1 \}
\]

is contained in \( Z(\Lambda) \).
Recall that we are assuming that $\chi \in \ker d^*_1$. Consider the case $\alpha \in \mathcal{A}_\Gamma$ with $\mathcal{A}_\Gamma(\alpha) = \{a^{(\alpha)}_1, \ldots, a^{(\alpha)}_{\val(\alpha)}\}$ the collection of arrows associated to $\alpha$. By Observation 4.7 and expressions (33) and (35) we have that for each $1 \leq i \leq \val(\alpha)$

- if $a^{(\alpha)}_i$ is not a loop then
  \[ y^{(\alpha)}_{j, a^{(\alpha)}_i, k} = 0, \quad \forall 0 \leq j \leq \mu(\alpha) - 1; \quad \forall 1 \leq k \leq \occ(\alpha, W) - 1; \]

- if $a^{(\alpha)}_i$ is a loop then
  \[ y^{(\alpha)}_{j, a^{(\alpha)}_i, k} = 0, \quad \forall 0 \leq j \leq \mu(\alpha) - 1; \quad \forall 1 \leq k < \occ(\alpha, W) - 1; \]

and

\[ y^{(\alpha)}_{j, a^{(\alpha)}_{i+1}, \occ(\alpha, V) - 1} = 0, \quad \forall 0 \leq j \leq \mu(\alpha) - 2. \]

From this we see that the only scalars that are not necessarily zero are those of the form

\[ y^{(\alpha)}_{\mu(\alpha) - 1, a^{(\alpha)}_{i+1}, \occ(\alpha, V) - 1} \]

when $a^{(\alpha)}_i$ is a loop, which are associated to the oriented cycles defined in (25). Now, if $\alpha \in \mathcal{B}_\Gamma$ then by Observation 4.7 and expressions (36) and (38) we have that for each $1 \leq i \leq \val(\alpha)$

- if $a^{(\alpha)}_i$ is not a loop
  \[ y^{(\alpha)}_{a^{(\alpha)}_i, k} = 0, \quad \forall 1 \leq k \leq \occ(\alpha, W) - 1; \]

- if $a^{(\alpha)}_i$ is a loop
  \[ y^{(\alpha)}_{a^{(\alpha)}_i, k} = 0, \quad \forall 1 \leq k < \occ(\alpha, V) - 1. \]

Also at this case we see that the only scalars that are not necessarily equal to zero are those of the form

\[ y^{(\alpha)}_{a^{(\alpha)}_{i+1}, \occ(\alpha, V) - 1} \]

when $a^{(\alpha)}_i$ is a loop, which also are associated to the cycles defined in (26).

Finally getting back to expression in (30) and applying each of the previous computations, if $\chi$ is in $\ker d^*_1$ then $\chi$ must be a linear combination of the elements of the following sets
\[ \{ 1_\Lambda \} \]
\[ \bigcup_{\alpha \in A_0} \{ C(\alpha)^j \mid 1 \leq j \leq \mu(\alpha) - 1 \} \]
\[ \{ C(V) \mid V \in \Gamma_1 \} \]
\[ \bigcup_{\alpha \in h^{-1}(1)} \left\{ \overline{D^{(\alpha)}_{V,s}} \mid (V, s) \in \mathcal{M}^{(\alpha)} \right\} \]

where \( h \) is the map defined in (22) and \( \mathcal{M}^{(\alpha)} \) the set defined in (24).

The union of all sets above is disjoint and it is a subset of the \( K \)-basis of \( \prod_{v \in Q_0} v \Lambda v \) generating \( \ker d^*_1 \). Hence is a \( K \)-basis of \( Z(\Lambda) \).

For the induced quiver \( Q \) of the Brauer configuration \( \Gamma \), let \( \#\text{Loops}(Q) \) denote the number of loops in the quiver \( Q \), then it is not difficult to prove that
\[ \#\text{Loops}(Q) = \sum_{\alpha \in h^{-1}(1)} \left( \sum_{V \in \Gamma(\alpha)} \left| \Psi^{v}_{(\alpha)} \right| \right) + |\mathcal{C}_1| \]. \quad (54)

See (24), (25) and (26) to check. We can say that the dimension of \( Z(\Lambda) \) is equal to
\[ 1 + \sum_{\alpha \in B_1 \cup T_1} (\mu(\alpha) - 1) + |\Gamma_1| + \sum_{\alpha \in h^{-1}(1)} \left( \sum_{V \in \Gamma(\alpha)} \left| \Psi^{v}_{(\alpha)} \right| \right). \]

Using the fact that \( \mu(\alpha) - 1 = 0 \) for every \( \alpha \in B_1 \cup T_1 \) and the expression in (54), the dimension can be expressed as
\[ 1 + \sum_{\alpha \in B_1 \cup T_1} (\mu(\alpha) - 1) + |\Gamma_1| + \#\text{Loops}(Q) - |\mathcal{C}_1| \]
\[ = 1 + \sum_{\alpha \in \Gamma_0} (\mu(\alpha) - 1) + |\Gamma_1| + \#\text{Loops}(Q) - |\mathcal{C}_1| \]
\[ = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#\text{Loops}(Q) - |\mathcal{C}_1| \]

Finally we obtain the desired result.

**Theorem 4.9.** Let \( \Lambda = KQ/I \) be the Brauer configuration algebra associated to the connected and reduced Brauer configuration \( \Gamma \). Then
\[ \dim_K Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#\text{Loops}(Q) - |\mathcal{C}_1|, \]

where \( \mathcal{C}_1 = \{ \gamma \in \Gamma_0 \mid \text{val}(\gamma) = 1 \text{ and } \mu(\gamma) > 1 \} \).
Definition 4.10. Let \( \Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma) \) be a Brauer configuration. We say that \( \Gamma \) is a **Brauer graph** if each polygon in \( \Gamma_1 \) is a 2-gon. The induced algebra by \( \Gamma \) is called a **Brauer graph algebra**.

If the Brauer graph \( \Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma) \) satisfies that the induced graph \((\Gamma_0, \Gamma_1)\) is a tree, then we say that \( \Gamma \) is a **Brauer tree** and the induced Brauer graph algebra a **Brauer tree algebra**.

Let \( \Gamma \) be a Brauer tree and suppose that the induced graph \((\Gamma_0, \Gamma_1)\) is different of \(\bigcirc\) \(\text{(55)}\). Hence in \((\Gamma_0, \Gamma_1)\) there are no parallel edges and no loops. Now, if \(Q\) is the induced quiver by \(\Gamma\), then the only possible loops in \(Q\) are those induced by the vertices in \(\mathcal{C}_\Gamma\). This implies that

\[
\#\text{Loops}(Q) = |\mathcal{C}_\Gamma|.
\]

We have the following corollary.

**Corollary 4.11.** Let \( \Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma) \) be a Brauer tree such that \((\Gamma_0, \Gamma_1)\) is not \(\text{(55)}\). If \( \Lambda \) is the induced Brauer tree algebra, then

\[
\dim_K Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0|.
\]

**Observation 4.12.** By convention the algebra \( k[x]/(x^2) \) is considered as a Brauer graph algebra. However, there is no Brauer configuration that induces this algebra.

5 Some examples

In this section we consider some examples to apply the formula obtained previously.

**Definition 5.1.** Let \( \Gamma = (\Gamma_0, \Gamma_1, \mu, \sigma) \) be a Brauer configuration. We say that \( \Gamma \) is a **Brauer graph** if each polygon in \( \Gamma_1 \) is a 2-gon. The induced algebra by \( \Gamma \) is called a **Brauer graph algebra**.

**Example 5.2.** Consider the unoriented graph

\[
\begin{align*}
\text{(56)} & \quad V_1 \\
V_2 & \quad 2 \\
V_3 & \quad 1 \\
V_4 & \quad 4 \\
V_2 & \quad 3 \\
\end{align*}
\]
We can represent this unoriented graph by the ordered pair \((\Gamma_0, \Gamma_1)\) where 
\[
\Gamma_0 = \{1, 2, 3, 4\}, \quad \Gamma_1 = \{V_1, V_2, V_3, V_4\}
\]
and 
\[
V_1 = \{1, 3\}, \quad V_3 = \{1, 2\},
\]
\[
V_2 = \{2, 3\}, \quad V_4 = \{1, 4\}.
\]

If we define the multiplicity function as \(\mu \equiv 1\), and the orientation \(o\) for the nontruncated vertices given by 
\[
1 : V_1 < V_3 < V_4,
\]
\[
2 : V_3 < V_2,
\]
\[
3 : V_1 < V_2,
\]
we have that the tuple \(\Gamma = (\Gamma_0, \Gamma_1, \mu, o)\) is a Brauer graph. Its induced quiver \(Q\) is given by

\[
\text{(57)}
\]

In this example we can see that \(\Gamma_0 = \mathcal{B}_\Gamma\), hence if \(\Lambda\) is the induced Brauer graph algebra we have that the value of the dimension of the center of \(\Lambda\) is equal to
\[
\dim_K Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#Loops(Q) - |\mathcal{C}_\Gamma|
\]
\[
= 1 + 4 + 4 - 4 + 0 - 0
\]
\[
= 5
\]
and the elements of the basis of the center are given by

- \(1_\Lambda\)
  \[
  C^{(V_1)} = a_1^{(1)} a_2^{(1)} a_3^{(1)} = a_1^{(3)} a_2^{(3)},
  \]
  \[
  C^{(V_2)} = a_2^{(3)} a_1^{(3)} = a_2^{(2)} a_1^{(2)},
  \]
  \[
  C^{(V_3)} = a_2^{(1)} a_3^{(1)} a_1^{(1)} = a_2^{(2)} a_1^{(2)},
  \]
  \[
  C^{(V_4)} = a_3^{(1)} a_1^{(1)} a_2^{(1)}.
  \]

**Example 5.3.** In [ST] the authors calculated the Hochschild cohomology ring for the Brauer graph algebra whose Brauer graph is given by a cycle
with \( m \geq 1 \) edges and \( m \) vertices, and the multiplicity function is equal to \( N \) at each vertex. We can represent this Brauer graph with the following Brauer configuration. First we assume that \( m \geq 2 \). Let \( \Gamma = (\Gamma_0, \Gamma_1, \mu, o) \) be the Brauer configuration given by

\[
\begin{align*}
\Gamma_0 &= \mathbb{Z}_m \\
\Gamma_1 &= \{V_i \mid i \in \mathbb{Z}_m\}
\end{align*}
\]

with \( V_i = \{i, i+1\} \), where the multiplicity function is \( \mu \equiv N \) and the orientation \( o \) is given by the successor sequences

\[
i : V_{i-1} < V_i,
\]

for each \( i \in \mathbb{Z}_m \). At this configuration we have that

\[
\mathcal{C}_\Gamma = \{i \in \Gamma_0 \mid \text{val}(i) = 1 \text{ and } \mu(i) > 1\} = \emptyset,
\]

and in the induced quiver \( Q \) we also have that \( \#\text{Loops}(Q) = 0 \). If \( \Lambda \) is the induced Brauer configuration algebra we have that the dimension of the center is

\[
\dim_K Z(\Lambda) = 1 + \sum_{i \in \Gamma_0} \mu(i) + |\Gamma_1| - |\Gamma_0| + \#\text{Loops}(Q) - |\mathcal{C}_\Gamma|
\]

\[
= 1 + Nm + m - m + 0 - 0
\]

\[
= 1 + Nm
\]

which is the same value in [ST, Theorem 3.1]. Now, when \( m = 1 \) the Brauer configuration is simply \( (\{1\}, V = \{1,1\}, \mu \equiv N, o) \), where the orientation \( o \) is given by the successor sequence \( 1 : V < V \). We have that the induced quiver \( Q \) is

\[
\begin{array}{c}
\circ \quad a \\
\circ \quad b \\
\circ \quad a
\end{array}
\]

In this case we have that \( \#\text{Loops}(Q) = 2 \) and \( \mathcal{C}_\Gamma = \emptyset \), then if \( \Lambda \) is the induced Brauer graph algebra we have that the dimension value of the center of \( \Lambda \) is equal to

\[
\dim_K Z(\Lambda) = 1 + N + 1 - 1 + 2 - 0
\]

\[
= N + 3
\]

which coincides with the value given in [ST, Theorem 7.1].
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