An analogue of Hawking radiation in the quantum Hall effect

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Abstract
The edge of a droplet of lowest Landau-level quantum Hall fluid provides a model for a 1 + 1 dimensional chiral Dirac fermion. We use this fact to construct an analogue model for a chiral fermion in a space-time geometry possessing an event horizon. We show that the analogue horizon emits particles and holes with a thermal spectrum. Each emitted quasiparticle is correlated with an opposite-energy partner on the other side of the horizon. Once we trace out these ‘unobservable’ partners, we are left with a thermal density matrix. For typical quantum Hall device parameters, the predicted ‘Hawking’ temperature is about 2 K.

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(Some figures may appear in colour only in the online journal)

1. Introduction

There are several apparently distinct explanations for the origin of black hole radiation. In his original account [1] Hawking kept track of what one means by a ‘particle’ as a wavefunction propagates through the background geometry of the collapsing star. A field theory derivation using the trace anomaly in the energy momentum tensor was given by Christensen and Fulling [2]. More recently Robinson and Wilczek [3] and others [4, 5] have applied the full two-dimensional gravitational anomaly in the region near the horizon. Yet another route obtains the Hawking radiation from quantum tunnelling across the horizon [6, 7]. (For a review of the tunnelling approach see [8].)

Given these alternative derivations, it is reasonable to ask just what is required for a horizon to emit thermal radiation. Is gravity really necessary? This question has led to the study of analogues of black holes and event horizons in other areas of wave propagation. The first such analogue was the acoustic black hole proposed by Unruh, who discovered that the wave equation for sound in a background fluid flow was equivalent to the wave equation for a scalar field in a curved space-time [9]. The subject has now developed extensively, with
gravity and Hawking radiation analogues being proposed and constructed in quantum-fluids, optics, and solid-state devices. For reviews with an extensive list of references see [10, 11].

The present paper proposes a conceptually simple, and possibly experimentally realizable, condensed matter model of quantum mode propagation in which an event horizon emits thermal radiation. The analogue space-time is flat, but consists of two causally disconnected halves. It is therefore a member of the general class of condensed-matter event horizons discussed by Volovik in [12], although with the population inversion observed in [13]. Our model exploits the interpretation of the edge-modes of a filling fraction $\nu = 1$ quantum Hall system as a massless chiral Dirac fermion whose local ‘speed of light’ is determined by the potential that confines the Hall fluid, and is therefore subject to external control.

In the next section we describe the model in the language of first-quantized tunnelling. In the third section we adopt a second-quantized formalism so as to obtain a Bogoliubov transformation between two natural bases for the system. This allows us to display the physical ‘vacuum’ as a coherent superposition of particle–hole pairs that are entangled across the horizon. Just as the Minkowski pure-state vacuum is a thermal mixed state when seen by a Rindler co-ordinate observer [17, 18], our pure-state vacuum appears thermal when we trace out the ‘unobservable’ over-the-horizon member of each pair. A fourth section contains a discussion of how our model relates to analogue gravity, and finally we explore the experimental feasibility of detecting the predicted thermal radiation in realistic quantum Hall devices.

2. A quantum Hall effect event horizon

We begin with a finite region occupied by a two-dimensional electron gas (2DEG) in a large perpendicular magnetic field $B$ that has been adjusted so that the gas is in the filling fraction $\nu = 1$ quantum Hall phase. A filling fraction of unity means that only the lowest Landau level is occupied, and that all the electron spins are parallel. As in any quantum Hall phase, there are no low energy degrees of freedom in the interior of the 2DEG. In the present case the energy gap arises because changing anything within the fluid requires either exciting an electron to a higher Landau level or overturning a spin, and these processes demand a minimum energy cost—the gap between the Landau levels, or the Zeeman splitting respectively. Low energy states do exist at the edge of the electron droplet however. In the $\nu = 1$ phase these edge states can be accurately modelled by a 1 + 1 dimensional chiral electron with energy–momentum dispersion $\epsilon(k) = ck$ [14–16]. The analogue speed of light is $v_{\text{edge}} = E/B$, where $eE$ is given by the gradient of the potential $V(x, y) = eA_0(x, y)$ that confines the electron gas to the finite region. (Classically the electrons move in small cyclotron orbits whose centres drift along the equipotentials at speed $E/B$. Quantum mechanically any local perturbation of the edge-mode occupation number is advected at this speed by the Hall current that flows perpendicular to the in-plane confining field.) In a typical 2DEG the electrons are trapped at the planar interface (a heterojunction) between thin layers of semiconductors, such as Al$_x$Ga$_{1-x}$As with differing values of the doping level $x$. The in-plane confining potential $V(x, y)$ can be controlled by means of means of electrodes (top gates) placed on the surface of the semiconductors, and so the effective speed of light seen by the edge-mode electrons can be manipulated so as to vary with position.

To construct an analogue of an event horizon we arrange for the boundary of the 2DEG to lie along the $y$ axis, with the region $x < 0$ occupied by the gas, and the region $x > 0$ empty. Now assume that the ‘confining’ potential $V$ is of the form

$$V(x, y) = \lambda xy.$$ (1)
Figure 1. The 2DEG black-hole analogue. The shaded region is the 2DEG. The lines indicate the semiclassical electron orbits (dashed when mostly unoccupied). The low energy excitations near the boundary at $x = 0$ constitute the quantum system in which we will find Hawking radiation. This radiation is illustrated by three correlated pairs of electrons and holes moving in opposite directions inside and outside the black hole.

(The reason for the ‘…’ around ‘confining’ will become clear in a moment.) The classical drift velocity is now

$$v_{\text{drift}} = \frac{1}{eB} \left( -\frac{\partial V}{\partial y}, \frac{\partial V}{\partial x} \right) = \frac{\lambda}{eB} (-x, y). \tag{2}$$

The electrons move along equipotentials $V(x, y) = \epsilon$, which, for this potential, are rectangular hyperbolæ that have the $x$ and $y$ axes as asymptotes (see figure 1). In particular, the electrons at the edge of our 2DEG move vertically along the $y$ axis at velocity

$$v_{\text{edge}} = \frac{1}{eB} \frac{\partial V}{\partial y} = \frac{\lambda}{eB} y, \tag{3}$$

and so the local ‘speed of light’ depends on $y$. Observe that the edge modes in the regions $y > 0$ and $y < 0$ move in opposite directions—away from $y = 0$. The these two regions are therefore causally disconnected, being separated by an analogue event horizon at $y = 0$.

The price we pay for the event horizon is that the electrons in the occupied region with $y < 0$ (the interior of the black hole) are in a state of population inversion—indeed they are ‘anti-confined’. In the absence of the magnetic field the electrons would rapidly fall into one of the lower energy quadrants. Because of the strong magnetic field, however, and in the absence of inelastic or tunnelling processes, they are constrained to stay on their hyperbolic classical orbits. (The issue of as to how we can establish such an anti-confined state is discussed in section 5.)

The inherent instability of the ‘vacuum’ in the analogue black hole interior is similar to that in the model of [13] and also to what happens inside to an outgoing mode at a Schwarzschild horizon. Here the $r$ and $t$ co-ordinates exchange roles from spacelike to timelike, and so a
mode that appears positive frequency outside the horizon analytically continues to a mode that is negative frequency inside. It is this population inversion that lies behind the tunnelling interpretation of Hawking radiation [6].

Except for the case $\epsilon = 0$, each of the classical equipotentials $\lambda_{xy} = \epsilon$ consists of two disconnected branches, and initially all the particles lie on only one of these branches. The branches for small $\epsilon$ approach each other near the origin. There is therefore a non-zero amplitude for a particle to tunnel from one branch to the other of the same energy. This tunnelling leads to electrons and holes being emitted from the event horizon.

To calculate the tunnelling amplitude, we will assume that the magnetic field is large enough that we can ignore all Landau levels except the lowest. The lowest Landau level (LLL) approximation is very natural as it is this situation that the excitations near the edge of a quantum Hall droplet are most cleanly identified with those of a 1 + 1 dimensional chiral Fermi field $\hat{\psi}(x, t)$ with Hamiltonian

$$\hat{H} = \int_{-\infty}^{\infty} v_{\text{edge}}(y) \hat{\psi}^\dagger (-i \partial_y) \hat{\psi} \, dy.$$  

(4)

In this picture the interior of the 2DEG corresponds to the filled Dirac sea.

We take the magnetic vector potential $(A_x, A_y)$ for the uniform field $B$ to be in the symmetric gauge where $A_x = By/2, A_y = -Bx/2$. The LLL wave-functions are then of the form

$$\psi(x, y) = \exp \left\{ -\frac{1}{4} eB |z|^2 \right\} \psi(z),$$

(5)

where $z = x + iy$. All quantum information resides in the holomorphic factor $\psi(z)$, and we will refer to this factor as the LLL ‘wave-function.’ We therefore regard the LLL Hilbert space as a Bargmann–Fock space of finite-norm holomorphic functions with inner product

$$\langle \phi, \chi \rangle = \int d^2z e^{-eB|z|^2/2} \bar{\psi}(z) \chi(z), \quad d^2z \equiv \frac{1}{2i} d\bar{z} \wedge dz = dx \wedge dy.$$  

(6)

Bear in mind however that the LLL wavefunction should be multiplied by $\exp \left\{ -\frac{1}{4} eB |z|^2 \right\}$ before plotting probability densities or computing currents.

The action of $z$ on the LLL wavefunction is by simple multiplication, but multiplication by $\bar{z}$ takes us out of the space of holomorphic functions. As a consequence the corresponding quantum operator has matrix elements that take us from one Landau level to another. The LLL approximation consists of ignoring these inter-level matrix elements and replacing $z$ by $P_0 \bar{z} P_0$ where $P_0$ is the projection operator onto the space of LLL eigenstates. An integration by parts

$$\langle \phi, \bar{z} \chi \rangle = \int d^2z \exp[ -eBz/2] \bar{\psi}(z) \frac{\partial}{\partial z} \chi(z)$$

$$= \int d^2z \left( \frac{2}{eB} \frac{\partial}{\partial z} \exp[ -eBz/2] \right) \bar{\psi}(z) \chi(z)$$

$$= \int d^2z \exp[ -eBz/2] \bar{\psi}(z) \left( \frac{2}{eB} \frac{\partial}{\partial z} \chi(z) \right),$$  

(7)

shows that, within the Lowest Landau level, the matrix elements of $\bar{z}$ coincide with those of

$$\bar{z} = \frac{2}{eB} \frac{\partial}{\partial z}.$$  

(8)

which is the adjoint (Hermitian conjugate) of the operator $z$ with respect to the Bargmann–Fock inner product.

For our potential

$$\lambda_{xy} = \frac{\lambda}{4i} (z^2 - \bar{z}^2)$$

(9)
the first-quantized eigenvalue problem
\[
\left[-\frac{1}{2m} \left( \frac{\partial}{\partial x} - i e A_x \right) - \frac{1}{2m} \left( \frac{\partial}{\partial y} - i e A_y \right) + V(x, y) \right] \psi = \epsilon \psi
\] (10)
therefore becomes
\[
\left( \frac{1}{e^2 B^2} \frac{d^2}{dz^2} - \frac{z^2}{4} \right) \psi(z) = -i \frac{\epsilon}{\lambda} \psi(z).
\] (11)
Only the potential appears in the this equation as the LLL wavefunctions are annihilated by the electron kinetic energy operator. A rescaling gives us a standard form of Weber’s equation (See [19] section 16.5, or [20] chapter 19.):
\[
\left( \frac{d^2}{d\zeta^2} - \frac{\zeta^2}{4} \right) f(\zeta) = af(\zeta),
\] (12)
with
\[
a = -i e \left( \frac{eB}{E} \right) f(\zeta), \quad \zeta = \sqrt{eBz} = \frac{z}{\ell_{\text{mag}}}.
\] (13)
For simplicity we now set \( \lambda = eB = 1 \). We can always restore the general parameters by scaling the units of length and energy.
If \( \varphi(\epsilon, z) \) is a solution of
\[
\left( \frac{d^2}{d\zeta^2} - \frac{\zeta^2}{4} \right) \varphi(\epsilon, \zeta) = -i \epsilon \varphi(\epsilon, \zeta)
\] (14)
then so are \( \varphi(\epsilon, -z), \varphi(-\epsilon, iz) \) and \( \varphi(-\epsilon, -iz) \). At most two of these solutions can be linearly independent.
A fundamental pair of independent solutions is
\[
y_1(\epsilon, z) = e^{-z^2/4} {}_1F_1 \left( \frac{1}{4} - \frac{i\epsilon}{2}, \frac{1}{2}, \frac{z^2}{4} \right),
\]
\[
y_2(\epsilon, z) = z e^{-z^2/4} {}_1F_1 \left( \frac{3}{4} - \frac{i\epsilon}{2}, \frac{3}{2}, \frac{z^2}{4} \right).
\] (15)
Here \( {}_1F_1(a, b, z) \) is the confluent hypergeometric function. These functions are even and odd, respectively, under \( z \leftrightarrow -z \). After multiplication by \( \exp[-|z|^2/4] \) the resulting wavefunctions are localized on the semiclassical orbits, which form the two disconnected branches of the rectangular hyperbola \( xy = \epsilon \) (see figure 2). These solutions to the LLL potential have been studied in connection with Riemann hypothesis [21].
More useful to us is the solution of (14) given by the parabolic cylinder function
\[
U_{-i\epsilon}(z) \equiv D_{-i^{1/2}}(z)
\]
\[
= 2^{-i(\epsilon/2+1/4)} \frac{1}{\sqrt{\pi}} \left( \cos \left[ \pi \left( \frac{1}{4} - \frac{i\epsilon}{2} \right) \right] \Gamma \left( \frac{1}{4} + \frac{i\epsilon}{4} \right) \right) y_1(\epsilon, z)
\]
\[
-\sqrt{2} \sin \left[ \pi \left( \frac{1}{4} - \frac{i\epsilon}{2} \right) \right] \Gamma \left( \frac{3}{4} + \frac{i\epsilon}{2} \right) y_2(\epsilon, z)
\]
\[
= \frac{e^{-z^2/4}}{\Gamma \left( \frac{1}{2} - i\epsilon \right)} \int_0^\infty r^{-i\epsilon-1/2} e^{-r^2-z^2} dr.
\] (16)
Here \( D_n(z) \) is Whittaker and Watson’s notation for their parabolic cylinder function [19], and \( U_n(z) \) is the now more common notation used by Abramowitz and Stegun [20]. The essential properties of \( U_n(z) \) are that it is an entire function, and that it decays rapidly as \( x \to +\infty \) for any real or complex \( n \).
Figure 2. Left figure: a density plot of the absolute value of the even function $\exp\left(-\frac{|z|^2}{4}\right)y_1(x, y)$ for the case $\epsilon = -10$. Right figure: a density plot of the absolute value of the odd function $\exp\left(-\frac{|z|^2}{4}\right)y_2(x, y)$ for $\epsilon = -2$.

Figure 3. A contour and a density plot of $\exp\left(-\frac{|z|^2}{4}\right)U_{-\epsilon}(z)$ for the case $\epsilon = -0.5$. Particles enter from the left and the beam divides between down-going and weaker tunnelled edge-mode wave and the up-going and stronger direct edge-mode wave.

The solution $U_{-\epsilon}(z)$ describes particles moving in from the left (the occupied region) in the lower left quadrant if $\epsilon > 0$ and the upper left quadrant if $\epsilon < 0$. They mostly remain in that quadrant, but there is some probability of tunnelling to the other branch of the hyperbola (see figure 3). If $\epsilon > 0$ the result is that a tunnelled positive energy particle is emitted by the black hole, leaving a negative energy hole (i.e. the absence of positive energy particle) inside the event horizon. If $\epsilon < 0$ then a positive energy hole (the absence of a negative energy particle) is emitted by the black hole leaving a negative energy particle inside the event horizon.
Figure 4. The ‘in’ wavefunctions: (a) $\psi^{(\infty, +)}(z)$ for $\epsilon < 0$, (b) $\psi^{(\infty, +)}(z)$ for $\epsilon > 0$, (c) $\psi^{(\infty, -)}(z)$ for $\epsilon > 0$, (d) $\psi^{(\infty, -)}(z)$ for $\epsilon < 0$. In each case the incoming wave divides between two outgoing waves.

Along with the solution $U_{-i\epsilon}(z)$ we have the solutions $U_{-i\epsilon}(-z)$ and $U_{i\epsilon}(iz)$ and $U_{i\epsilon}(-iz)$. We will find use for all of these solutions, as they describe motion with different boundary conditions (see figures 4 and 5).

To discover the relative amplitudes of the direct and tunnelled waves we can use the asymptotic expansion

$$e^{-|z|^2/4} U_{-i\epsilon}(z) \sim e^{-|z|^2/4 - i^2/4} z^{i - 1/2} \left[ 1 + O \left( \frac{1}{z^2} \right) \right], \quad |\arg(z)| < 3/4. \quad (17)$$

Near the y axis this reduces to

$$\psi(x, y) \sim (\text{gauge phase}) e^{-x^2/2} \frac{1}{\sqrt{y}} \exp[i\epsilon \ln|y| - \text{sgn}(y)\epsilon\pi/2]. \quad (18)$$

The ratio of tunnelled to direct amplitude is therefore exactly $\exp[-\pi \epsilon]$.

We can confirm this result by using the identity

$$U_{-i\epsilon}(z) = \Gamma \left( \frac{1}{2} + i\epsilon \right) \sqrt{2\pi} [e^{-i\pi/4} e^{-i\pi/4} U_{i\epsilon}(iz) + e^{i\pi/2} e^{i\pi/4} U_{i\epsilon}(-iz)] \quad (19)$$

together with the fact that $U_{-i\epsilon}(z)$ tends rapidly to zero in the right half-plane. Thus, if $R$ is positive

$$U_{-i\epsilon}(iR) = \Gamma \left( \frac{1}{2} + i\epsilon \right) \sqrt{2\pi} [e^{-i\pi/4} e^{-i\pi/4} U_{i\epsilon}(-R) + e^{i\pi/2} e^{i\pi/4} U_{i\epsilon}(R)]$$

$$\sim \Gamma \left( \frac{1}{2} + i\epsilon \right) e^{-i\pi/2} e^{-i\pi/4} U_{i\epsilon}(-R) \quad (20)$$
Figure 5. The ‘out’ wavefunctions: (a) $\psi_{\text{out,ext}}(z)$ for $\epsilon > 0$, (b) $\psi_{\text{out,ext}}(z)$ for $\epsilon < 0$, (c) $\psi_{\text{out,int}}(z)$ for $\epsilon > 0$, (d) $\psi_{\text{out,int}}(z)$ for $\epsilon < 0$. In each case two weaker incoming waves assemble the outgoing wave.

and

$$U_{-i\epsilon}(-iR) = \frac{\Gamma\left(\frac{1}{2} + i\epsilon\right)}{\sqrt{2\pi}} \left[ e^{i\pi/2} e^{-i\pi/4} U_{\text{in}}(R) + e^{i\pi/2} e^{i\pi/4} U_{\text{in}}(-R) \right] \sim \frac{\Gamma\left(\frac{1}{2} + i\epsilon\right)}{\sqrt{2\pi}} e^{i\pi/2} e^{i\pi/4} U_{\text{in}}(-R).$$

(21)

The direct and tunnelling amplitudes therefore have magnitude

$$|d(\epsilon)| = \left| \frac{\Gamma\left(\frac{1}{2} + i\epsilon\right)}{\sqrt{2\pi}} e^{i\pi/2} \right|$$

$$|t(\epsilon)| = \left| \frac{\Gamma\left(\frac{1}{2} + i\epsilon\right)}{\sqrt{2\pi}} e^{-i\pi/2} \right|.$$  

(22)

Note that $|d(\epsilon)|^2 + |t(\epsilon)|^2 = 1$ because $\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$ gives us

$$\left| \frac{\Gamma\left(\frac{1}{2} + i\epsilon\right)}{\sqrt{2\pi}} \right|^2 = \frac{2\pi}{e^{\pi\epsilon} + e^{-\pi\epsilon}}.$$  

(23)

The occupation probability of an outgoing particle or hole state with energy $\epsilon$ is therefore

$$P(\epsilon) = \frac{1}{1 + \exp[2\pi\epsilon]}. \quad (24)$$

The chiral edge states emerging from the event horizon are therefore thermal with

$$T = \frac{1}{2\pi}. \quad (25)$$
or

\[ k_B T = \frac{\hbar \lambda}{2\pi eB} \]  

(26)

once we restore parameters and units. Comparison of this with the usual Hawking radiation formula

\[ k_B T_{\text{Hawking}} = \frac{\hbar \kappa}{2\pi} \]  

(27)

indicates that the ‘surface gravity’ \( \kappa \) of our analogue black hole is the edge-velocity acceleration

\[ \kappa = \frac{\lambda}{eB} = \left. \frac{dv_{\text{edge}}}{dy} \right|_{\text{horizon}} \]  

(28)

3. Second quantization, mode expansions, and a Bogoliubov transformation

The space of LLL functions (5) does not contain the delta function. Its place is taken by a reproducing kernel

\[ \{x_1, y_1 | x_2, y_2\} \overset{\text{def}}{=} \frac{1}{2\pi} \exp \left\{ -\frac{1}{4} |z_1|^2 - \frac{1}{4} |z_2|^2 + \frac{1}{2} \overline{z_1} z_2 \right\}. \]  

(29)

If \( \psi(x,y) \) is of the form (5) then

\[ \int d^2z \psi(x_1, y_1) \{x_1, y_1 | x_2, y_2\} = \psi(x_2, y_2). \]  

(30)

In particular \( \{x_0, y_0 | x_1, y_1\} \), considered as a function of \((x_1, y_1)\), is of this form, so the kernel reproduces itself:

\[ \int d^2z \{x_0, y_0 | x_1, y_1\} \{x_1, y_1 | x_2, y_2\} = \{x_0, y_0 | x_2, y_2\}. \]  

(31)

When we expand out the second quantized LLL field operator in terms of the discrete set of normalized eigenmodes \( z^n / \sqrt{2\pi 2^n n!} \) for the potential

\[ V(x, y) = \frac{1}{2} (x^2 + y^2) = \frac{1}{2} z^2 \mapsto z \frac{d}{dz} \]  

(32)

we find

\[ \hat{\psi}(x_1, y_1) = \sum_{n=0}^{\infty} \hat{a}_n \frac{1}{\sqrt{2\pi 2^n n!}} z^n e^{-|z|^2/4}. \]  

(33)

Here the operators \( \hat{a}_n, \hat{a}_m^\dagger \) obey

\[ [\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}, \]  

(34)

and the usual canonical anticommutation relation \( \{\hat{\psi}(x), \hat{\psi}(x')\} = \delta(x - x') \) for the field is replaced by

\[ \{\hat{\psi}(x_1, y_1), \hat{\psi}(x_2, y_2)\} = \{x_1, y_1 | x_2, y_2\}. \]  

(35)

If we retain only the holomorphic factors, then we have

\[ \{\hat{\psi}(z_1), \hat{\psi}(z_2)\} = \frac{1}{2\pi} \exp \left\{ \frac{1}{2} \overline{z_1} z_2 \right\}. \]  

(36)

We can also expand in a continuous set of eigenfunctions. For example, we can make use of energy \( \epsilon \) eigenfunctions for the potential \( V(x, y) = x \). These are

\[ \varphi_\epsilon(z) = \frac{1}{\pi^{1/4}} \exp \left\{ \epsilon z - \frac{1}{4} \overline{z}^2 - \frac{1}{2} \epsilon^2 \right\}. \]  

(37)
They have been normalized so that
\[ \langle \phi_{\epsilon}, \phi_{\epsilon'} \rangle = \frac{2\pi}{2\pi} \delta(\epsilon - \epsilon') \]  
(38)
The holomorphic field operator is then
\[ \hat{\psi}(z) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \hat{a}_\epsilon \phi_\epsilon(z) \]  
(39)
with
\[ \{\hat{a}_\epsilon, \hat{a}^\dagger_{\epsilon'}\} = 2\pi \delta(\epsilon - \epsilon') \]  
(40)
We easily confirm that (39) still satisfies (36).

Similarly, we can expand the LLL field operator in terms of complete sets of parabolic cylinder functions. There are two distinct ways of doing this. Begin by defining
\[ \phi_{\epsilon}(\text{in}, +) = \frac{1}{\pi^{1/4}} U_{\epsilon} \Gamma\left(\frac{1}{2} - i\epsilon\right) \]  
(41)
\[ \phi_{\epsilon}(\text{in}, -) = \frac{1}{\pi^{1/4}} U_{\epsilon} \Gamma\left(\frac{1}{2} - i\epsilon\right) \]  
(42)
The label ‘in’ designates that these wave-functions describe states that have a simple description prior to their opportunity for tunnelling (see figure 4). These ‘in’ functions have been normalized so that
\[ \langle \phi_{\epsilon}(\text{in}, \alpha'), \phi_{\epsilon'}(\text{in}, \alpha) \rangle = \frac{2\pi}{2\pi} \delta(\epsilon - \epsilon') \delta_{\alpha, \alpha'} \]  
(43)
and they obey the LLL completeness relation
\[ \sum_{\alpha} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \phi_{\epsilon}(\text{in}, \alpha) \phi_{\epsilon'}(\text{in}, \alpha) = \frac{1}{2\pi} \exp\left\{ \frac{1}{2} \frac{\epsilon_1}{\epsilon_2} \right\} \]  
(44)
(Both normalization and completeness are easily established from the integral expression in the last line of (16).) Then we can set
\[ \hat{\psi}(z) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \left( \hat{b}_{\epsilon}(\text{in})^\dagger \phi_{\epsilon}(\text{in}, +) + \hat{a}_{\epsilon}(\text{in}) \phi_{\epsilon}(\text{in}, -) \right) \]  
(45)
The ‘in’ vacuum is the appropriate many-body state for our initial conditions. It is characterized physically by the condition that no particle is approaching the 2DEG from the empty single-particle states to the right, and that all the single-particle states incoming from the left are occupied. It is characterized mathematically by the conditions
\[ \hat{a}_{\epsilon}(0, \text{in}) = 0 = \hat{b}_{\epsilon}(0, \text{in}), \quad \forall \epsilon. \]  
(46)
The second set of functions is
\[ \phi_{\epsilon}(\text{out, ext})(z) = \frac{1}{\pi^{1/4}} \Gamma(1/2 + i\epsilon) U_{\epsilon}(iz), \]  
(47)
\[ \phi_{\epsilon}(\text{out, int})(z) = \frac{1}{\pi^{1/4}} \Gamma(1/2 + i\epsilon) U_{\epsilon}(-iz). \]  
(48)
They are also orthogonal
\[ \langle \phi_{\epsilon}(\text{out, } \alpha), \phi_{\epsilon'}(\text{out, } \alpha') \rangle = 2\pi \delta(\epsilon - \epsilon') \delta_{\alpha, \alpha'}, \quad \alpha = \text{int, ext}, \]  
(49)
and obey the LLL completeness relation
\[ \sum_{\alpha} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \phi_{\epsilon}(\text{out, } \alpha)(z_1) \phi_{\epsilon}(\text{out, } \alpha)(z_2) = \frac{1}{2\pi} \exp\left\{ \frac{1}{2} \frac{\epsilon_1}{\epsilon_2} \right\} \]  
(50)
The labels ‘ext’ and ‘int’ indicate that the functions live mostly in the exterior \((y > 0)\) and interior \((y < 0)\) of the black hole. They decay rapidly in the other region (see figure 5). In terms of these new functions we have

\[
\hat{\psi}(z) = \sum_a \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \hat{\sigma}_e^{(\text{out},a)}(\epsilon) \phi_i^{(\text{out},a)}(z)
\]  

(51)

The ‘out’ operators \(\hat{\sigma}_e^{(\text{out},a)}\) and \((\hat{\sigma}_e^{(\text{out},a)})^\dagger\) create and annihilate particles that are simply described as excitations over the asymptotic naive vacuum in which every state in the region \(x < 0\) is filled and every state in \(x > 0\) is empty. For \(\epsilon > 0\) the operator \(\hat{\sigma}_e^{(\text{out},\text{ext})}\) annihilates a positive energy particle in the asymptotic region \(y > 0\) outside the black hole. For \(\epsilon < 0\) it annihilates a particle in the 2DEG and so creates a positive energy hole in the same region.

\[
\hat{\sigma}_e^{(\text{out},\text{ext})}|0, \text{out}, \text{ext}\rangle = 0, \quad \epsilon > 0,
\]  

(52)

\[
(\hat{\sigma}_e^{(\text{out},\text{ext})})^\dagger |0, \text{out}, \text{ext}\rangle = 0, \quad \epsilon < 0,
\]  

(53)

\[
\hat{\sigma}_e^{(\text{out},\text{int})}|0, \text{out}, \text{int}\rangle = 0, \quad \epsilon < 0,
\]  

(54)

Comparing the two expressions for \(\hat{\psi}(z)\) gives us the Bogoliubov transformation

\[
\hat{a}_e^{(\text{in})} = \frac{\Gamma(\frac{i}{2} - i\epsilon)}{\sqrt{2\pi}} \left[ e^{-\epsilon \pi/2} e^{i\pi/4} \hat{\sigma}_e^{(\text{out},\text{int})} + e^{\epsilon \pi/2} e^{i\pi/4} \hat{\sigma}_e^{(\text{out},\text{ext})}\right],
\]  

(55)

\[
\hat{b}_e^{(\text{in})} = \frac{\Gamma(\frac{i}{2} + i\epsilon)}{\sqrt{2\pi}} \left[ e^{-\epsilon \pi/2} e^{-i\pi/4} (\hat{\sigma}_e^{(\text{out},\text{ext})})^\dagger + e^{\epsilon \pi/2} e^{-i\pi/4} (\hat{\sigma}_e^{(\text{out},\text{int})})^\dagger\right].
\]  

(56)

Similarly

\[
\hat{a}_e^{(\text{out},\text{int})} = \frac{\Gamma(\frac{i}{2} + i\epsilon)}{\sqrt{2\pi}} \left[ e^{\epsilon \pi/2} e^{-i\pi/4} (\hat{\sigma}_e^{(\text{in})})^\dagger + e^{-\epsilon \pi/2} e^{i\pi/4} \hat{\sigma}_e^{(\text{in})}\right],
\]  

(57)

\[
\hat{a}_e^{(\text{out},\text{ext})} = \frac{\Gamma(\frac{i}{2} + i\epsilon)}{\sqrt{2\pi}} \left[ e^{\epsilon \pi/2} e^{-i\pi/4} \hat{\sigma}_e^{(\text{in})} + e^{-\epsilon \pi/2} e^{i\pi/4} (\hat{\sigma}_e^{(\text{in})})^\dagger\right].
\]  

(58)

From the Bogoliubov transformation and the mathematical characterization of \(|0, \text{in}\rangle\) we find that

\[
|0, \text{in}\rangle = N \exp \left\{ \int_0^{\infty} e^{-|\epsilon|} \left[ (\hat{a}_e^{(\text{out},\text{ext})})^\dagger \hat{a}_e^{(\text{out},\text{ext})} + (\hat{a}_e^{(\text{out},\text{ext})})^\dagger \hat{a}_e^{(\text{out},\text{ext})}\right] \frac{d\epsilon}{2\pi}\right\} |0, \text{out}\rangle,
\]  

(59)

where \(N\) is a normalization factor. We have therefore exhibited the physical ground state as a sea of particle–hole pairs correlated between the interior and exterior regions. We now have the same formal situation as described in [18]. If we trace out the ‘unobservable’ interior of the black hole, we end up with density matrix is of the form

\[
\hat{\rho} = \sum_i e^{-2\pi |\epsilon(i)|} |i, \text{ext}\rangle \otimes |i, \text{ext}\rangle,
\]  

(60)

where \(i\) labels the many-body state whose energy is \(\epsilon(i)\). However, unlike the situation in the Unruh–Rindler vacuum [17, 18] our system contains genuine radiation rather than a thermal bath. This is because the chiral character of the particles means that they can only flow outwards.
4. Theory discussion

The chiral edge-mode fermions are massless and so move along null geodesics in the effective space-time geometry. If \( t \) is the laboratory time co-ordinate and \( y \) the physical distance along the edge of the 2DEG, then effective space-time metric in which the chiral edge-mode fermions move must be

\[
ds^2 = \frac{1}{v_{\text{edge}}^2(y)} \, dy^2 - dt^2. \tag{61}
\]

In particular the coefficient of \( dt^2 \) must be (minus) unity as there is no gravitational redshift when we use the physical laboratory time. The quantization of chiral fermions in the more general background metric

\[
ds^2 = e^\phi \left( \frac{1}{v_{\text{edge}}^2(y)} \, dy^2 - dt^2 \right) \tag{62}
\]

has been carried out in [22] who show that the conformal factor \( \phi \) does not appear in the Hamiltonian, which therefore coincides with our equation (4). (The conformal factor does affect the high-energy cutoff and therefore the gravitational anomaly, but this effect is not important to us.) The authors of [22] did not consider the effect of an event horizon.

In our case \( v_{\text{edge}}(y) = \kappa y \), \( \kappa = \lambda/eB \), and a change to an exterior tortoise co-ordinate \( y_* = \kappa^{-1} \ln(y) \) in (61) leads to

\[
ds^2 = dy_*^2 - dt^2. \tag{63}
\]

The new coordinates reveal that our space-time is flat, but the singularity at the horizon is not removed. It has been pushed to \( y_* = -\infty \), and the interior of the black hole has become invisible. A superfluid system with this metric and event horizon was studied by Volovik in [12]. He uses a WKB analytic continuation method to compute the Bogoliubov coefficients, and finds the same Hawking temperature as our present calculation, but his non-chiral system has no actual radiation. The agreement in the temperature is perhaps not surprising. It must be obvious from looking at the classical trajectories of our particles that there is some connection between our 2DEG problem and that of Landau–Zener tunnelling through an avoided level crossing. Indeed, although the physics is superficially different, the Landau–Zener time-dependent Schrödinger equation is solved using the same families of parabolic cylinder functions that we have used [23], and it is well known that an analytically continued form of the WKB approximation obtains the correct asymptotic Landau–Zener tunnelling probabilities [24].

The most remarkable property of the present model is that the emitted radiation is exactly thermal. There is no immediately obvious reason why the mathematical properties of the parabolic cylinder functions should lead to this result. In a real black hole the emitted radiation is modified by grey-body factors in dimensions greater than two, but that the hole can only be in equilibrium with radiation at \( T_{\text{Hawking}} \) follows from the geometry of the Euclidean section of space-time being asymptotically periodic in imaginary time [25]. Does our space-time geometry tacitly force a Euclidean temporal periodicity?

We can write

\[
ds^2 = \frac{1}{\kappa^2 y^2} (dy^2 - y^2 d(\kappa t)^2) \tag{64}
\]

so, up to the conformal factor \( \kappa^{-2} y^{-2} \), the metric is that of Rindler space whose Euclidean section \( t \mapsto i\tau \) has metric

\[
d_{\text{Rindler}}^2 = y^2 d(\kappa \tau)^2 + dy^2. \tag{65}
\]
The absence of a conical singularity at \( y = 0 \) in the manifold described by (65) requires identifying \( \kappa \tau \sim \kappa \tau + 2\pi \) and so implies a temperature \( T = \hbar \kappa / 2\pi \)—which is exactly what the tunnelling calculation gives. However, given that it blows up at the point of interest, it seems questionable to ignore the conformal factor, making this argument at most suggestive.

5. Experimental feasibility

The analysis of our model requires only standard quantum mechanics, and therefore hardly needs experimental verification. It is nonetheless worthwhile to ask whether the model is more than a pretty conceptual idea. Can the proposed quantum thermal radiation actually be created and detected with current quantum Hall technology?

We would require a 2DEG hetero-junction equipped with four top gates—one for each of the four co-ordinate quadrants in figure 1. Initially the two gates overlying the \( x < 0 \) would be held at a positive voltage, and those over the region \( x > 0 \) would be held at a negative voltage sufficient to deplete the 2DEG below and adjacent to them. The gate electrodes should be separated by a micron or so. The region shown in figure 1 is then the micron-scale gate-free area at the intersection of the gaps.

At time \( t = 0 \) the polarity of the two gates over \( y < 0 \) will be reversed. In the gate-free region the potential will now vary as \( V(x, y) \propto xy \), so establishing the initial conditions of our \( |0, \text{in}\rangle \). We now have a short time to look for the emitted edge-state particles and holes. How long we have depends on two parameters, only one of which is under our control. The controllable parameter is the time taken for the edge state excitations to travel around the perimeter of electrodes and re-enter (and pollute) the tunnelling region. This time depends on the size of the electrodes and the edge-wave velocity, but it is hard to imagine making longer than a microsecond or so. The uncontrollable parameter is the inelastic relaxation time that determines how long it takes for the 2DEG electrons to shed phonons and fall into the lower energy quadrant. There appears to be no experiments measuring this relaxation time.

The second issue is the realizable temperature. If we assume that the electrodes are held at \( \pm 1V \) and are separated by \( 1\mu \) the edge mode velocity \( v_{\text{edge}} = E/B \) will be \( 10^6 \text{ ms}^{-1} \) at \( B = 2T \). The direction of edge-mode propagation will reverse in the \( 1\mu \) spacing between the \( \pm y \) electrodes giving an effective surface gravity of \( \kappa = 2 \times 10^{12} \text{ s} \) and an analogue Hawking temperature of 2.4 K. This temperature is larger than the mK scale of typical quantum Hall experiments, and should be detectable.

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