UNIFORM APPROXIMATION OF SOME DIRICHLET SERIES
BY PARTIAL PRODUCTS OF EULER TYPE

I.SH.JABBAROV

Dedicated to the memory of professor Voronin S. M.

Abstract. In the work it is gotten a uniform approximation of Dirichlet series defined by Euler product by partial products of Euler type in circles placed in the right half of the critical strip. As a consequence the analog of the Riemann Hypothesis is proven.

1. Introduction.

Appearance of Dirichlet series and understanding of their fundamental role in Analytical Number Theory is connected with L.Euler’s name. In 1748 Euler ([1]) entered the zeta-function \( \zeta(s) = \sum n^{-s} \), considering it as a function of real variable \( s \), and proved an important identity:

\[
\zeta(s) = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}, \quad s > 1;
\]

here the product is taken over all prime numbers. This product is called to be Euler product.

In 1837 using and developing Euler’s ideas, L.Dirichlet gave generalization of the theorem of Euclid for arithmetic progressions, considering L -functions. Investigations of Dirichlet showed an importance of studying of Dirichlet series defined by more general Euler products.

After coming into the world of essential Riemann’s work [2] it stood clear that deeper results of the theory of prime numbers are connected with analytical properties of Dirichlet series of complex variable.

Many of analytical properties of Dirichlet series were studied by using of various their finite approximations. For example, some questions of the theory of Dirichlet series, connected with mean values, an order, or a density distribution of zeroes, investigated in the works [3-8] by using of approximations by partial sums. G. Bohr and E. Landau were first who applied partial products of Euler type to investigate the zero distribution of the zeta function ([6-8]). In the works [9-16], S. M. Voronin, developing the method of G. Bohr and E. Landau, used special type of partial products in the questions of distribution of zeroes and non-zero values of L-functions in the critical strip. In compliance with the Universality Theory of S. M. Voronin, every analytical function, non vanishing in and on the circle \( |s| = r < 1/4 \), can be
approximated by finite products of the form \( \prod_p \left( 1 - e^{-2\pi i \theta_p p^{-s}} \right)^{-1} \) where \( p \) takes values from some finite set of prime numbers.

In the present work we show that the Dirichlet series with the Euler product, having analytical continuation to the critical strip without singularities, in some natural conditions, can be approximated by partial products of Euler type in the circles of the critical strip, where the primes, over which are taken the products, are distributed by a suitable way (see formulation of the theorem below) (see [29,30]). The family of such series includes many of widely used Dirichlet series as the zeta – function, Dirichlet L – functions, or L-functions of some algebraic extensions with the commutative Galois groups and etc.

Let we are given with a following infinite product over all prime numbers \( p \): 

\[
F(s) = \prod_p f_p(p^{-s}),
\]

where \( f_p(z) \) is a rational function of a variable \( z \) having not poles in the circle \(|z| < 1|\),

\[
f_p(z) = 1 + \sum_{m=1}^{\infty} a_p^m z^m,
\]

and for any positive small \( \varepsilon \) the inequality \( |a_p^m| \leq c(\varepsilon) p^{\varepsilon} ; c(\varepsilon) \geq 1 \) is satisfied uniformly by \( p \).

**Theorem.** Let the function \( F(s) \) to have not singularities in the half plane \( \sigma > 1/2 \), with exception of finite number of poles on the line \( \sigma = 1 \), and every factor of the product (1) to have not zeroes in the half plane \( \sigma > 1/2 \). Suppose that for any small positive number \( \lambda \) there exist constants \( c_0 = c_0(\lambda) > 0 \) and \( h_0 = h_0(\lambda) > 0 \), satisfying, for any \( h > h_0 \), the following inequality:

\[
\sum_{h < p \leq h(1 + \log^{-10} h)} |a_p^1|^2 p^{- (1 - \lambda)} \geq c_0(\lambda) h^{\lambda/4}.
\]

If, for \( 1/2 < \sigma_0 < 1 \), in the circle \(|s - \sigma_0| \leq r < r_0 = \min(1 - \sigma_0, \sigma_0 - 1/2) \), \( F(s + it_0) \) has not zeros for some real \( t_0 \), then there exists sequence \( \{\theta_n\} \), \( \theta_n \in \Omega = [0,1] \times [0,1] \times \cdots \) and a sequence \( \{m_n\} \) of integers that

\[
\lim_{n \to \infty} F_n(s + it, \theta_n) = F(s + it + it_0),
\]

for every real \( t \), uniformly by \( s \), in this circle; here \( \theta_n = (\theta_p^m) \), and

\[
(1.2) \quad F_n(s + it, \theta_n) = \prod_{p \leq m_n} f_p(e^{-2\pi i (\theta_p^m + \gamma_p)} p^{-s-it}); \quad \gamma_p = \frac{t_0 \log p}{2\pi}.
\]

If we put now \( a_p^m = 1 \) for all natural \( m \) and prime \( p \) we get the zeta function. From the theorem of Valle-Poussin C. J. [see, 4, p. 59] it follows that the all conditions of the theorem, formulated above, are satisfied for the Riemann zeta function with \( \sigma_0 = 3/4 \) and \( t_0 = 0 \). Following by [30] we prove the analog of the Riemann Hypothesis for the function \( F(s) \).

**Corollary.** The analog of the Riemann Hypothesis is true: \( F(s) \neq 0 \), when \( 1/2 < \sigma < 1 \).
2. Additional statements.

Lemma 1. Let a series of analytical functions
\[ \sum_{n=1}^{\infty} f_n(s) \]
be given in one-connected domain \( G \) of a complex \( s \)-plane and converges absolutely almost everywhere in \( G \) in the Lebesgue sense and the function
\[ \Phi(\sigma, t) = \sum_{n=1}^{\infty} |f_n(s)| \]
is a summable function in \( G \). Then, the given series converges uniformly in any compact subdomain of \( G \); particularly, the sum of this series is an analytical function in \( G \).

Proof. It is enough to show that the theorem is true for any rectangular area in \( G \). Let \( C \) be a rectangle in \( G \) and \( C' \) another rectangle inside \( C \) and their sides are parallel to the co-ordinate axes. We can assume that on a contour of these rectangles the series converges almost everywhere, according to the theorem of Fubini (see [17, p. 208]). Let \( \Phi_0(s) = \Phi_0(\sigma, t) \) to be the sum of given series at points of convergence. Under the theorem of Lebesgue on a bounded convergence (see [20, p. 293]), we have:
\[
(2\pi)^{-1} \int_C \frac{\Phi_0(s)}{s - \xi} ds = \sum_{n=1}^{\infty} (2\pi i)^{-1} \int_C \frac{f_n(s)}{s - \xi} ds,
\]
where the integrals are taken in Lebesgue sense. As on the right part of the last equality the integrals are existing in the Riemann sense also, then by applying Couchy’s formula and denoting the left side of the last equality by \( \Phi_1(\xi) \) we get, for any point \( \xi \) on or in the contour of \( C' \):
\[
\Phi_1(\xi) = (2\pi)^{-1} \int_C \frac{\Phi_0(s)}{s - \xi} ds = \sum_{n=1}^{\infty} f_n(\xi)
\]
(certainly, \( \Phi_1(\xi) = \Phi_0(\xi) \) almost everywhere). Further, the series can be estimated in \( C' \) by a following way
\[
|f_n(\xi)| \leq (2\pi)^{-1} \int_C \frac{|f_n(s)|}{|s - \xi|} ds \leq (2\pi \delta)^{-1} \int_C |f_n(s)| |ds|,
\]
when \( \delta \) designates the minimum of distances between the sides of \( C \) and \( C' \). The series
\[
\sum_{n=1}^{\infty} \int_C |f_n(s)| |ds|
\]
converges, in the consent with the theorem of Lebesgue on monotone convergence (see [20, p. 290]). Hence, the series \( \sum_{n=1}^{\infty} f_n(\xi) \) converges uniformly in the inside of \( C' \). The lemma 1 is proved.

Let’s enter the notion of Hardy space.

Definition 1. The set of functions \( f(s) \) defined for \( |s| < R \) and analytical in this area for which
\[
\|f\|^2 = \lim_{r \to R} \iint_{|s|<r} |f(s)|^2 \, d\sigma dt < \infty; s = \sigma + it
\]
is called the Hardy space and is designated as \( H^2_2(R), R > 0 \).

Obviously, Hardy space is a real linear space in which is possible to enter a scalar product of functions by means of the equality

\[
(f(s), g(s)) = \text{Re} \iint_{|s| \leq R} f(s)\overline{g(s)} \, d\sigma dt.
\]

Using the entered scalar product, we will prove that Hardy space is a real Hilbert space.

**Lemma 2.** The Hardy space \( H^2_2(R) \), together with the entered scalar product (3), is a real Hilbert space.

**Proof.** It is enough to prove that any fundamental sequence \( (f_n(s))_{n \geq 1} \) converges to some analytical function \( f(s) \in H^2_2(R) \). As the sequence is fundamental, there exist such a sequence of natural numbers \( (n_k)_{k \geq 0} \) that for any natural \( k \) we have:

\[
\|f_{n_k} - f_{n_k-1}\| \leq 2^{-k}.
\]

Let's consider a series of analytical functions

\[
f_{n_0} + \sum_{k=1}^\infty (f_{n_k} - f_{n_k-1}).
\]

We will prove first that it converges uniformly in any closed circle, lying in the circle \( |s| < R \). According to the definition of a norm, we have:

\[
\|f(s)\|^2 = \iint_{|s|<R} |f(s)|^2 \, d\sigma dt,
\]

possible, in improper meaning of definition of the norm. Then, designating

\[
g(s) = \sum_{k=1}^\infty |f_{n_k}(s) - f_{n_k-1}(s)|,
\]

we receive:

\[
\iint_{|s|<R} g(s) \, d\sigma dt \leq \sum_{k=1}^\infty \left( \pi R^2 \iint_{|s|<R} |f_{n_k} - f_{n_k-1}|^2 \, d\sigma dt \right)^{1/2}
\leq \sqrt{\pi R} \sum_{k=1}^\infty 2^{-k} < +\infty.
\]

Hence, the function \( g(s) \) is a summable function of variables \( \sigma, t \), then the lemma 1 is applicable. Applying the lemma 1, we receive, that the series \( f_{n_0} + \sum_{k=1}^\infty (f_{n_k} - f_{n_k-1}) \) converges uniformly in any circle \( |s| \leq r < R \). Then the subsequence \( (f_{n_k}(s))_{k \geq 1} \) converges to some analytical function \( \varphi(s) \). As the sequence is fundamental, for any \( \varepsilon > 0 \) it can be found \( n_0 \) such that for any natural \( m > n_0 \) the inequality
\[
\int_{|s|<R} |\varphi(s) - f_m(s)|^2 \, d\sigma dt < \varepsilon
\]

holds. Let \( r < R \) be any real number. Then using an inequality of [19, p. 345], one can receive

\[ r^2 |\varphi(s) - f_m(s)|^2 \leq \pi^{-1} \int_{|s|<R} |\varphi(s) - f_m(s)|^2 \, d\sigma dt < \varepsilon / \pi, \]

for any \( s, |s| \leq r \). As \( \varepsilon \) is any, then from here it follows the convergence of the sequence \( (f_m(s))_{m \geq 1} \) to \( \varphi(s) \). As,

\[
\int_{|s|<R} |\varphi(s)|^2 \, d\sigma dt \leq \int_{|s|<R} |f_{\alpha_0}(s)|^2 \, d\sigma dt + \int_{|s|<R} |g(s)|^2 \, d\sigma dt < +\infty,
\]

then \( \varphi(s) \in H_{2}^{(r)} \) and, therefore the considered space is complete. The lemma 2 is proved.

The following is a variant of S. M. Voronin’s lemma proved by him in [4] for the zeta – functions, and it is one of the basic arguments of our work.

**Lemma 3.** Let \( 0 < r < 1 / 4 \) and \( g(s) \) is an analytical function in circle \( |s| \leq r \), being continuous and does not vanishing on the circle \( |s| \leq r \). Then, for any \( \varepsilon > 0 \) and \( y > 2 \) there exist a finite set of prime numbers \( M \), containing all of primes \( p \), \( p \leq y \) for which the following inequality is fair:

\[
\max_{|s| \leq r} \left| g(s) - F_M \left( s + 3/4; \theta \right) \right| \leq \varepsilon,
\]

for some \( \theta = (\theta_p)_{p \in M} \), with \( \theta_p = \theta_0^p \) being numbers set beforehand from the interval \([0,1] \), for \( p \leq y \); the function \( F_M \left( s + 3/4; \theta \right) \) is given by the equality

\[
F_M(s + 3/4; \theta) = \prod_{p \in M} f_p \left( e^{-2\pi i \theta_p} p^{-s-3/4} \right).
\]

**Proof.** The proof of the lemma 3 we will spend by the method of the work [4] of Voronin S. M. As \( g(s) \) is an analytical function in the circle \( |s| \leq r \), then we will consider an auxiliary function \( g(s/\gamma^2) \) \( (\gamma > 1, \gamma^2 r < 1/4) \) which for any \( \varepsilon > 0 \), at some \( \gamma \), satisfies the inequality \( \max_{|s| \leq r} |g(s) - g(s/\gamma^2)| < \varepsilon \). Therefore, it is enough to prove the statement of the lemma 3 for the function \( g(s/\gamma^2) \) in circle \( |s| \leq r \). The advantage is consisted in that the function \( g(s/\gamma^2) \) belongs to the space \( H_{2}^{(\gamma^2)} \) (a circle has a radius, greater than \( r \), which is important for our subsequent reasoning). Not breaking, therefore, a generality, we believe that the function \( g(s) \) is an analytical in the circle \( |s| \leq r \gamma^2 \) and we will consider the space \( H_{2}^{(\gamma^2)} \).

The function \( \log g(s) \), on the conditions of the theorem, has no singularities in the circle \( |s| \leq r \gamma \). Therefore, it is enough to prove an existence of such an element \( \theta \), satisfying the conditions of the lemma 3, that

\[
\max_{|s| \leq r} |\log g(s) - \log F_M(s + 3/4; \theta)| \leq \varepsilon.
\]

A series (2) of the work [4, p.241] we define as

\[
u_k(s) = \log f_p \left( e^{-2\pi i (\varphi_k + \theta_k)} p_k^{-s-3/4} \right),
\]
supposing $2\pi \varphi_k$ to be an argument of the coefficient $a^1_{p_k}$. We have:

\[ u_k(s) = \log(1 + a^1_{p_k} e^{-2\pi i(\varphi_k + \theta_k)} x^s p_k^{-s-3/4}) + \log(1 + a^1_{p_k} e^{-2\pi i(\varphi_k + \theta_k)} x^s p_k^{-s-3/4})^{-1} \times f_p(e^{-2\pi i(\varphi_k + \theta_k)} p_k^{-s-3/4}), \]

and using decomposition of the logarithmic function into power series, we get

\[ u_k(s) = a^1_{p_k} e^{-2\pi i(\varphi_k + \theta_k)} p_k^{-s-3/4} + \nu_k(s), \tag{2.2} \]

for every $k$, being big enough. So, for any $\varepsilon > 0$

\[ \nu_k(s) = O(p_k^{2s+2r-3/2}) + \log \left( 1 + \sum_{m=2}^{\infty} b_m (e^{-2\pi i(\varphi_k + \theta_k)} p_k^{-s-3/4} m)^m \right), \]

and the factors $b_n$ are defined by a following equality:

\[ b_m = a_p^m - a_p^{m-1} a_p + a_p^{m-2} (a_p^1)^2 - \cdots + (-1)^{m-2} a_p^2 (a_p^1)^{m-2}. \]

We have

\[ |b_m| \leq (m-1)c^{m-1}(\delta)p^\varepsilon m. \]

As $r < 1/4$, we can take $\varepsilon > 0$ such that the inequality $4\varepsilon + 2r - 3/2 < -1$ was satisfied. Then, definition of $u_m(s)$ and (4), together with the last inequality, shows that the series

\[ \sum_{m=1}^{\infty} \eta_m(s): \eta_m(s) = a^1_{p_m} e^{-2\pi i(\varphi_m + \theta_m)} \]

differs from the series $\sum u_m(s)$ by an absolutely converging series. Really, since $m \leq 2^m$, then

\[ \sum_{m=2}^{\infty} |b_m| p^{-m(\varepsilon+r-3/4)} \leq \sum_{m=2}^{\infty} 2^{m-1} c^{m-2}(\varepsilon)p^{m(\varepsilon+r-3/4)} \leq \]

\[ \frac{2c(\varepsilon)p^{2s+2r-3/2}}{1 - 2c(\varepsilon)p^{s+r-3/4}} \leq 4c(\varepsilon)p^{-2s-1}, \]

if $p$ is so large that $2c(\varepsilon)p^{-3/4} \leq 1/2$. Therefore, the series $\sum |\nu_k|$ converges.

Now, it is enough for us to show that for any $\varphi(s) \in H^{(r)}_2$ ($0 < r < 1$ is any) there exists some subseries of (5) converging to $\varphi(s)$. In particular, taking $n$ equal to the greatest value of $k$ for which $p_k \leq y$, we admit

\[ \varphi(s) = \log g(s) - \sum_{k > n} (u_k(s) - \eta_k(s)) - \sum_{k \leq n} u_k(s). \]

Considering the last remark, we will find some permutation of $\sum_{k > n} \eta_k(s)$ converging to $\varphi(s)$ (clearly, any permutation of the series $\sum_{k > n} (u_k(s) - \eta_k(s))$ converges to the same sum uniformly). Then for any $\varepsilon$ there will be found such a set of indexes $k \in M$ that
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\[
\max_{|s| \leq R} \left| \varphi(s) - \sum_{k \in M, p_k > y} \eta_k(s) \right| \leq \varepsilon/2.
\]

Let \( q(s) = \sum_{k=n+1}^{\infty} (u_k(s) - \eta_k(s)) \). As this series converges absolutely it is possible to select mentioned above set \( M \) so that the following relationship was carried out

\[
\left| q(s) - \sum_{k \in M, k > n} (u_k(s) - \eta_k(s)) \right| \leq \varepsilon/2.
\]

Then we will receive:

\[
\left| \varphi(s) - \sum_{k \in M, p_k > y} \eta_k(s) \right| = \left| \log g(s) - \sum_{n \in M} u_n(s) \right| \leq \varepsilon.
\]

Thereby, the proof of the lemma 3 will be finished. At \( k > n \) we set \( \theta_k = \rho(k)/4 \) and \( \rho(k) \) will be defined below. For \( k \leq n \) we take \( \theta_k + \varphi_k = 0 \). Then, for \( k > n \), we have:

\[
(\eta_k(s), \varphi(s)) = \left| a_{p_k}^1 \right| Re \int_{|s| \leq R} e^{-2\pi i \rho(k)/4} p_k^{-(s+3/4)} \overline{\varphi(s)} d\sigma dt = \]

\[
= Re \left[ \left| a_{p_k}^1 \right| e^{-2\pi i \rho(k)/4} \Delta (\log p_k) \right],
\]

by denoting

\[
\Delta(x) = \iint_{|s| \leq R} e^{-x(s+3/4)} \overline{\varphi(s)} d\sigma dt.
\]

Writing \( R = \gamma r \), consider the space \( H_2^{(R)} \). Then,

\[
\| \eta_k(s) \|^2 = \iint_{|s| \leq R} \left| e^{-2\pi i \theta_k} p_k^{-s-3/4} \right|^2 d\sigma dt \leq \pi R^2 p_k^{2r-3/2}.
\]

Hence,

\[
\sum_{k=1}^{\infty} \| \eta_k(s) \|^2 \leq \pi R^2 \sum_{k=1}^{\infty} p_k^{2r-3/2} < +\infty,
\]

the first condition of the theorem 1, §6 of an appendix of [4] is executed.

Let now \( \varphi(s) \in H_2^{(R)} \) be arbitrary element of the space with condition \( \| \varphi(s) \|^2 = 1 \). Let \( \varphi(s) \) to have a following expansion into a power series in the circle \( |s| \leq R \):

\[
\varphi(s) = \sum_{n=0}^{\infty} \alpha_n s^n.
\]

Then,

\[
1 = \iint_{|s| \leq R} \left| \sum_{n=0}^{\infty} \alpha_n s^n \right|^2 d\sigma dt.
\]

Exchange the variables under the integral by formulas: \( \sigma = r \cos \varphi, \ t = r \sin \varphi, \ r \leq R, 0 \leq \varphi < 2\pi \). Then,
\[ 1 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha_n \tilde{\alpha}_m \int_0^R r^{n+m+1} \int_0^{2\pi} (\cos 2\pi(n-m)\varphi + i \sin 2\pi(n-m)\varphi) \, d\varphi. \]

The inner integral is equal to 0, when \( m \neq n \), and \( 2\pi \) otherwise. Hence,

\[ \pi \sum_{n=0}^{\infty} |\alpha_n|^2 R^{2n+2}(n+1)^{-1} = 1. \]

Let’s prove now that there is a point \( \theta \), non dependent on the function \( \varphi(s) \), such that the series \( \sum_{k=1}^{\infty} (\eta_k(s), \varphi(s)) \) converges after of some permutation of its members. We have

\[ (\eta_k(s), \varphi(s)) = -\text{Re} \int \int_{|s| \leq R} e^{-2\pi i \theta_k p_k - s - 3/4} \overline{\varphi(s)} d\sigma dt = \text{Re}[-e^{-2\pi i \theta_k} \Delta(\log p_k)]. \]

It is possible to represent the function \( \Delta(x) \) by a following way:

\[ \Delta(x) = e^{-3x/4} \int \int_{|s| \leq R} \frac{\sum_{n=0}^{\infty} (-s)^n/n!}{\left( \sum_{n=0}^{\infty} \alpha_n s^n \right)} d\sigma dt = \]

\[ = \pi R^2 e^{-3x/4} \sum_{n=0}^{\infty} (-1)^n \tilde{\alpha}_n x^n R^{2n}/(n+1)! = \pi R^2 e^{-3x/4} \sum_{n=0}^{\infty} \beta_n (xR)^n / n!, \]

by denoting \( \beta_n = (-1)^n R^n \tilde{\alpha}_n / (n+1) \). From (6) one may conclude:

\[ \sum_{n=1}^{\infty} |\beta_n|^2 \leq 1. \]

Hence, \( |\beta_n| \leq 1 \), and, therefore, the function

\[ (2.5) \quad H(u) = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} u^m \]

will be an entire function. So,

\[ \Delta(x) = \pi R^2 e^{-3x/4} H(xR). \]

Let’s prove that for any \( \delta > 0 \) there will be found tending to the infinity sequence \( u_1, u_2, \ldots \), satisfying the inequality

\[ (2.6) \quad |H(u_j)| > c e^{-(1+2\delta)u_j}. \]

Let’s admit an opposite, i.e. let there exists a positive number \( \delta < 1 \) such that at some \( A > 0 \), being big enough, the following inequality

\[ |H(u)| \leq A e^{-(1+2\delta)u} \]

is executed for all \( u \geq 0 \); in this case we have:

\[ \left| e^{(1+\delta)u} H(u) \right| \leq A e^{-\delta |u|} ; u \geq 0. \]
From proved above for \( u < 0 \) one receives:

\[
|H(u)| \leq \sum_{n=0}^{\infty} |u|^n / n! = e^{-u}.
\]

Then we have

\[
e^{(1+\delta)u} H(u) \leq e^{\delta u} \leq e^{-|u|}.
\]

Consequently, the integral below is existing:

\[
\int_{-\infty}^{\infty} \left| e^{(1+\delta)u} H(u) \right|^2 du.
\]

As, the function (7) is an entire function of exponential type, then the function \( e^{(1+\delta)u} H(u) \) will be such one also and belong to the class \( E^\sigma \) (see [4, p. 408]), with \( \sigma < 3 \). Then under the theorem of Paley - Wiener (see in the same place), it is existing a finitary function \( h(\xi) \in L_2(-3, 3) \) such that

\[
e^{(1+\delta)u} F(u) = \int_{-3}^{3} h(\xi) e^{iu\xi} d\xi.
\]

Taking converse transformation, we find:

\[
h(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( e^{(1+\delta)u} H(u) \right) e^{-iu\xi} du.
\]

From the found above estimations it follows that this integral converges absolutely and uniformly in the strip \( |Im\xi| < \delta/2 \) and, consequently, represents an analytical function in this strip, which contradicts finitaryness of \( f(\xi) \). The received contradiction proves an existence of a sequence of points with the condition (8).

Denoting \( x_j = u_j / R \), on the basis of (8), we can assert that

\[
|\Delta(x_j)| > ce^{-3x_j/4} |H(x_j)R| \geq ce^{-3x_j/4} e^{-(1+2\delta)x_j} R = ce^{-x_j(R+2\delta R+3/4)}.
\]

If \( \delta > 0 \) is sufficiently small, then \( R + 2\delta R + 3/4 < 1 \) and, hence, there is \( \delta_0 > 0 \) such that

\[
|\Delta(x_j)| > e^{-(1-\delta_0)x_j}.
\]

Let’s consider the function \( \Delta(x) \) on the segment \( [x_j - 1, x_j + 1] \). Following by [4], we denote \( N = [x_j] + 1 \). From the estimation for factors \( \beta_n \) it follows the inequality:

\[
\left| \sum_{n=N^2+1}^{\infty} \frac{\beta_n(xR)^n}{n!} \right| \leq \sum_{n=N^2+1}^{\infty} \frac{(xR)^n}{n!} \leq \frac{(xR)^N}{(N^2)!} \sum_{n=0}^{\infty} \frac{(xR)^n}{n!} \leq \frac{(xR)^N}{(N^2)!} e^N.
\]

If \( n, m \geq 0 \) are integers, then \((n + m)! = n!(n + 1) \cdots (n + m) \geq n!m!\). For big enough natural \( m \) one has under Stirling formula:

\[
m! = \Gamma(m + 1) \geq e^m \log m - m = (m/e)^m.
\]

Hence,
Let's consider the first possibility. Let 
\[ x \]

Therefore, for any \( j \) or, \[ j \]

According to (8), we receive an inequality
\[ \Delta(x) = \pi R^2 \sum_{n=0}^{N^2} \frac{(-3x/4)^n}{n!} \sum_{n=0}^{N^2} \frac{\beta_n}{n!} (xR)^n + O(e^{-x_j}) = \sum_{n=0}^{N^4} a_n x^n + O(e^{-x_j}) \]

for any \( j = 1, 2, \ldots \) Let \( a_n = b_n + ic_n, b_n, c_n \in R \). Then,
\[ \Delta(x) = \sum_{n=0}^{N^4} b_n x^n + i \sum_{n=0}^{N^4} c_n x^n + O(e^{-x_j}) \]

Therefore, for any \( j \), at least, one of the following inequalities is executed:
\[ \max_{|x-x_j| \leq 1} \left| \sum_{n=0}^{N^4} b_n x^n \right| > 0.1 e^{-(1-\delta_0)x_j} \]

or,
\[ \max_{|x-x_j| \leq 1} \left| \sum_{n=0}^{N^4} c_n x^n \right| > 0.1 e^{-(1-\delta_0)x_j} \]

Let's consider the first possibility. Let \( x_0 \) be the point at which the maximum of modulus is reached. We will designate by \( \tau_j \) a segment which lies in the interval \([x_j-1, x_j+1]\], containing the point \( x_0 \), and each point \( x \) of \( \tau_j \) satisfies the inequality:
\[ |g(x)| \geq 0.1 |g(x_0)| . \]

Let for definiteness \( g(x_0) < 0 \); \( g(x) = \sum_{n=0}^{N^4} b_n x^n \). If
\[ \tau_j \neq [x_j-1, x_j+1], \]

(the case of coincidence of intervals is trivial) then there will be a point \( x_1 \in \tau_j \), for which
\[ |g(x_1)| \leq 0.1 |g(x_0)| . \]

Now we have:
that primes, satisfying the condition $h < p$.

Under the theorem of Lagrange, there exist a point $y_j \in \tau_j$ such that

$$|g(y_j) - (x_1 - x_0)| \geq 0.5 |g(x_0)|.$$ 

Applying the theorem, 9, §2 of the appendix of [4], we find:

$$|g'(y_j)(x_1 - x_0)| \geq 0.5 |g(x_0)|.$$ 

So, the interval $\tau_j$ has a length not less than $0.5x_j^{-8}$. Accepting $h = e^{x_j}$, we notice that $[h, h(1 + \log^{-1} h)] \subseteq [e^\alpha, e^{\alpha + \beta}]$. From the condition (2) it follows that the set of primes, satisfying the condition $h < p \leq h(1 + \log^{-1} h)$ can be distributed among subsets $P_1, P_2, P_3, P_4$ for each of which the following inequality is fulfilled

$$\sum_{p \in P_i, \lambda < \delta} |a_p^{1/2}| p^{-(1-\lambda)} \geq 0.1c_0(\lambda)h^\lambda/4, i = 1, 2, 3, 4.$$ 

To prove the last statement, at first, we divide the set of primes with $h < p \leq h(1 + \log^{-1} h)$ into fore subsets $P'_1, P'_2, P'_3, P'_4$ arbitrarily. Then, for one of subsets, say for the $P'_1$, we will have

$$\sum_{p \in P'_1} |a_p^{1/2}| p^{-(1-\lambda)} \geq 0.25c_0(\lambda)h^{\lambda/4}.$$ 

Now we return to the union $P'_2 \cup P'_3 \cup P'_4$ the primes, corresponding to addends of the last sum, omitting at the same time addends, consequently, until it is not less than $0.2c_0(\lambda)h^{\lambda/4}$. The last returned addend has a bound $|a_p^{1/2}| p^{-(1-\lambda)} \leq s(\varepsilon)p^{\varepsilon + \lambda - 1}$. So, denoting by $P_1$ the set of remaining in the previous sum primes, for great values of $h_0$, we have

$$\sum_{p \in P_1} |a_p^{1/2}| p^{-(1-\lambda)} \geq 0.1c_0(\lambda)h^{\lambda/4}.$$ 

At the same time

$$\sum_{p \in P'} |a_p^{1/2}| p^{-(1-\lambda)} \geq 0.8c_0(\lambda)h^{\lambda/4},$$

where the sum over the set $P'$ contains the addends not belonging into the subset $P_1$. Continuing the same reasoning we construct suitable subsets.

We take $\rho(k) = i - 1$ for every $p_k \in P_i$. Therefore, if $\lambda < \delta_0/2$, then, denoting by $\gamma_j$ a corresponding interval of variance of numbers $\log p_k$ we have:

$$\sum_{p_k \in P_i, \log p_k \in \gamma_j} Re \left[ |a_{p_k}^{1/2}| e^{-2\pi i \rho(k)/4} \Delta(\log p_k) \right] >> e^{-(1-\delta_0)x_j}x_i^{(1-\lambda)x_j} >> e^{\delta_0 x_j/2}.$$ 

Similarly, we can prove an inequality
\[ - \sum_{p_k \in P_3} Re \left[ \left( a_{p_k}^1 \right) e^{-2\pi i\rho(k)/4} \Delta (\log p_k) \right] \gg e^{-(1-\delta_0)x_j} e^{(1-\lambda)x_j} \gg e^{\delta_0 x_j/2}. \]

As it was noted above, the condition (8) is satisfied for unbounded sequence \((u_j)\). Therefore, the intervals \(\tau_j\) can be taken disjoint. Thus, from the found above one deduces the existence of an infinite set of indexes \(j\), satisfying the conditions

\[ \sum_{\log p_k \in \tau_j, p_k \in P_v} (\eta_k(s), \varphi(s)) \gg e^{\delta_0 x_j/2}; \quad v = 0, 2, \]

and an infinite set of other values \(j\), for which

\[ - \sum_{\log p_k \in \tau_j, p_k \in P_v} (\eta_k(s), \varphi(s)) \gg e^{\delta_0 x_j/2}; \quad v = 1, 3. \]

Further, from proved above estimations we conclude, that

\[ |\Delta(x)| \leq \pi R^2 e^{-x/2}. \]

So, \(|(\eta_k(s), \varphi(s))| \to 0\), when \(k \to \infty\). Hence, the series

\[ \sum_{n=1}^{\infty} (\eta_k(s), \varphi(s)) \]

contains two subseries diverging, accordingly, to \(+\infty\) and to \(-\infty\), not having the common components. Then, some permutation of the series

\[ \sum_{n=1}^{\infty} (\eta_k(s), \varphi(s)) \]

converges conditionally. Therefore, by the theorem 1, §6 of [4], there is a permutation of the series \(\sum_{p_n > y} u_n(s)\), converging to \(\varphi(s) - \sum_{p_n \leq y} u_n(s)\) uniformly. Taking long enough partial sum, we receive the necessary result. The lemma 3 is proved.

**Note 1.** The statement of the lemma 3 remains invariable, if we consider instead of the circle \(|s - 3/4| \leq r < 1/4\) any circle of a kind \(|s - \sigma_0| \leq r < \rho_0, 1/2 < \sigma_0 < 1\).

3. **The basic auxiliary result.**

Let \(\omega \in \Omega, \Sigma(\omega) = \{\sigma \omega | \sigma \in \Sigma\} \) and \(\Sigma'(\omega)\) designates the closed set of all limit points of the sequence \(\Sigma(\omega)\). For real \(t\) we denote \(\{t\Lambda\} = \{\{t\lambda_n\}\}\), where \(\Lambda = (\lambda_n)\). We will assume, that \(\mu\) designates the product of the linear Lebesgue measures in \([0, 1]; \mu = m \times m \times \cdots\). In the set \(\Omega\) it is possible to define Tikhonov’s metric by following equality

\[ d(x, y) = \sum_{n=1}^{\infty} e^{1-n} |x_n - y_n|. \]

**Note 2.** For completeness, we shall show that in the cube \(\Omega\) a regular measure may be constructed by using of open sets. At first we define the volume of the sphere of a radius \(r > 0\)

\[ B(0, r) = \{E \in \Omega | d(x, 0) < r\}. \]
Since $|x_n| \leq 1$, then for the natural number $N$ we have

$$\sum_{n=N+1}^{\infty} e^{1-n}|x_n| \leq e^{-N} \sum_{n=0}^{\infty} e^{-n} < e^{1-N}.$$  

Taking arbitrarily small real number $\varepsilon > 0$ we get

$$\sum_{n=1}^{N} e^{1-n}|x_n| \leq d(x, 0) \leq \sum_{n=1}^{N} e^{1-n}|x_n| + \varepsilon$$

when $N \geq \log e\varepsilon^{-1}$. Therefore,

$$B_N(0, r - \varepsilon) \times [0, 1] \times \cdots \subset B_N(0, r) \subset B_N(0, r) \times [0, 1] \times \cdots,$$

where $B_N(0, r)$ denotes the projection of the sphere $B(0, r)$ into the subspace of first $N$ coordinate axis. Then, for the volume $\mu_N(r)$ of the sphere $B_N(0, r)$, we have (see [18, p.319])

$$\mu_N(r) - \mu_N(r - \varepsilon) = \int_{r - \varepsilon \leq \sum_{n=1}^{N} e^{1-n}|x_n| \leq r} dx_1 \cdots dx_N$$

$$= 2 \sum_{n=1}^{N} e^{1-n}|u_n| \int_{r - \varepsilon \leq u \leq r} du \int_{\sum_{n=1}^{N} e^{1-n}u_n = u} ds ||\nabla|| \leq$$

$$\leq \varepsilon 2^N \int_{M} ds ||\nabla||,$$

and the last integral is an surface integral over the surface $M$ defined by the equation

$$\sum_{n=1}^{N} e^{1-n}u_n = u, \quad 0 \leq u_k \leq 1;$$

here $\nabla$ is a gradient of the linear function on the left side of the latest equality, i.e.

$$||\nabla|| = \sqrt{1 + e^{-2} + \cdots + e^{2-2N}}.$$  

Defining $uI$ from (9) we get

$$\int_{M} \frac{ds}{||\nabla||} \leq \int_{0}^{1} \cdots \int_{0}^{1} du_2 \cdots du_N = 1.$$  

So, we have

$$\mu_N(r) - \mu_N(r - \varepsilon) \leq \varepsilon 2^N.$$  

By taking the greatest $N$, satisfying the condition $N \geq \log e\varepsilon^{-1},$ i.e. $N = [\log e\varepsilon^{-1}] + 1$, we may write $\varepsilon \leq e^{2-N}$. Then from (10) it is follows that

$$\mu_N(r) - \mu_N(r - \varepsilon) \leq 2^N e^{2-N} \rightarrow 0$$

as $N \rightarrow \infty$, or as $\varepsilon \rightarrow 0$. Since the sequence $(\mu_N(r))$ is monotonically decreasing, then

$$B_{N+1}(0, r) \subset B_N(0, r) \times [0, 1].$$
So, it is bounded with the lower bound \( \mu_{N_0}(r/2) \), with \( N_0 = \lceil \log 2er^{-1} \rceil + 1 \). Therefore, there exists a limit

\[
\lim_{\varepsilon \to 0} B_N(0, r - \varepsilon) = \lim_{N \to \infty} B_N(0, r) = \mu(r)
\]

which we receive as a measure of the sphere \( B(0, r) \).

On this bases it may be introduced the measure in the \( \Omega \), by known way, by using of open spheres. An open sphere in the \( \Omega \) we define as an intersection \( \Omega \cap B(\theta, r) \).

The elementary set we define as a set being gotten by finite number of operations of unionize, taking differences or complements of open spheres. The outer and inner measures could be introduced by known way (see [17,26]). This measure will be, as it seen from the reasoning above, a regular measure. As it is clear (see [17, p. 182]), every measurable set, in the meaning of introduced measure, is measurable in the meaning of product Lebesgue measure also. Main difference between this measure and Haar or product measures is studied in [31]. Now for us it is enough that every set of zero measure can be overlapped by enumerable union of spheres with the arbitrarily small total measure.

**Definition 2.** Let \( \sigma : N \to N \) is any one-one mapping of the set of natural numbers. If there will be natural number \( m \) such, that \( \sigma(n) = n \) for any \( n > m \) then we will say that \( \sigma \) is a finite permutation. Subset \( A \subset \Omega \) we will call finite-symmetric if for any element \( \theta = (\theta_n) \in A \) and any finite permutation \( \sigma \) one has \( \sigma\theta = (\theta_{\sigma(n)}) \in A \).

The following lemma is a result of the work [28, p. 46].

**Lemma 4.** Let \( A \subset \Omega \) the is a finite-symmetric subset of zero measure and \( \Lambda = (\lambda_n) \) is an unbounded, monotonically increasing sequence of positive real numbers any finite subfamily of elements of which are linearly independent over the field of rational numbers. Let \( B \supset A \) any open subset in Tikhonov’s metric with \( \mu(B) < \varepsilon \) and

\[
E_0 = \{0 \leq t \leq 1 \mid \{t\lambda \} \in A \land \Sigma' \{t\lambda \} \subset B \}.
\]

Then, \( m(E_0) \leq 6c\varepsilon \) where \( c \) is absolute constant and \( m \) designates the Lebesgue measure.

**Proof.** Let \( \varepsilon \) is any small positive number. As numbers \( \lambda_n \) are linearly independent, for any finite permutation \( \sigma \), one has \( \{t_1\lambda_n\} \neq \{t_2\lambda_{\sigma(n)}\} \) when \( t_1 \neq t_2 \).

Really, otherwise we would receive the equality \( \{t_1\lambda_s\} = \{t_2\lambda_s\} \), for a big enough natural \( s \), i.e. \( (t_1 - t_2)\lambda_s = k, k \in \mathbb{Z} \). Further, writing down the same equality for some other whole \( r > s \), we at some whole \( k_1 \) get the relation

\[
k_1/\lambda_r - k/\lambda_s = k_1\lambda_s - k\lambda_r \lambda_s = 0
\]

which contradicts the linear independence of numbers \( \lambda_n \). Hence, for any pair of various numbers \( t_1 \) and \( t_2 \) one has \( \{t_1\lambda_n\} \notin \{\{t_2\lambda_{\sigma(n)}\} \mid \sigma \in \Sigma \} \).

By the conditions, there exist a family of open spheres \( B_1, B_2, \ldots \) (in Tikhonov’s metrics) such that each sphere does not contain any other sphere from this family (the sphere, containing in other one can be omitted) and

\[
A \subset B \subset \bigcup_{j=1}^{\infty} B_j, \sum \mu(B_j) < 1.5\varepsilon.
\]
Now we take some permutation $\sigma \in \Sigma$ defined by the equalities $\sigma(1) = n_1, \ldots, \sigma(k) = n_k$ where natural numbers are taken as below. At first we take $N$ such, that

$$\mu(B'_N) < 2\varepsilon_1$$

where $B'_N$ is a projection of the sphere $B_1$ into the subspace of first $N$ co-ordinate axes and $\mu(B_1) = \varepsilon_1$. Let $B'_N$ be enclosed into the union of cubes with edge $\delta$ and a total measure not exceeding $3\varepsilon_1$. We will put $k = N$ and define numbers $n_1, \ldots, n_k$, using following inequalities

$$\lambda n_1 > 1, \lambda n_2 < (1/4)\delta\lambda n_1, \lambda n_3 < (1/4)\delta\lambda n_2, \ldots, \lambda n_k < (1/4)\delta\lambda n_{k-1}, \delta < 0.1.$$ 

Now we take any cube with edge $\delta$ and with the center in some point $(\alpha_m)_{1 \leq m \leq k}$. Then the point $(\{t\lambda_{n_m}\})$ belongs to this cube, if

$$|\{t\lambda_{n_m}\} - \alpha_m| \leq \frac{\delta}{2}.$$ 

Since the interval $(\alpha_m - \delta/2, \alpha_m + \delta/2)$ has a length $< 0.1$ then the real numbers $t\lambda_{n_m}$, fractional parts of which lie in this interval have one and the same integral parts during continuous variation of $t$. So at $m = 1$, for some whole $r$, one has:

$$\frac{r + \alpha_1 - \delta/2}{\lambda n_1} \leq t \leq \frac{r + \alpha_1 + \delta/2}{\lambda n_1}.$$ 

The measure of a connected set of such $t$ does not exceed the size $\delta\lambda_{n_1}^{-1}$. The number of such intervals corresponding to different values of $r = \lfloor t\lambda n_1 \rfloor$ does not exceed

$$\lfloor \lambda n_1 \rfloor + 2 \leq \lambda n_1 + 2.$$ 

So, the total measure of intervals satisfying (12) at $m = 1$ is less or equal to

$$(\lambda n_1 + 2)\delta\lambda_{n_1}^{-1} \leq (1 + 2\lambda_{n_1}^{-1})\delta.$$ 

Consider the case $m = 2$. Taking one of intervals of a view (12) we will have

$$\frac{s + \alpha_2 - \delta/2}{\lambda n_2} \leq t \leq \frac{s + \alpha_2 + \delta/2}{\lambda n_2},$$ 

with some $s = \lfloor t\lambda_{n_2} \rfloor \leq \lambda n_2$. As we consider the condition (12) for values $m = 1$ and $m = 2$ simultaneously, we should estimate a total measure of intervals (14) which have nonempty intersections with intervals of a kind (13), using conditions (11). Every interval of a kind (14) is placed in the one interval with the length $\lambda_{n_2}^{-1}$ only (on the end points of this interval $t\lambda n_2$ takes consecutive integral values), corresponding one and the same value of $s$. The number of intervals with the length $\lambda_{n_2}^{-1}$, having a nonempty intersection with one fixed interval of a kind (13), does not exceed the size

$$\lfloor \delta\lambda_{n_2}^{-1}\lambda n_2 \rfloor + 2 \leq \delta\lambda_{n_2}^{-1}\lambda n_2 + 2.$$ 

So, the measure of values $t$ for which intervals (14) have a nonempty intersections only with one of intervals of a kind (13) is bounded by the value $(2 + \delta\lambda_{n_2}^{-1}\lambda n_2)\delta\lambda_{n_2}^{-1}$. Since, the number of intervals (13) is no more than $\lambda n_1 + 2$, then the measure of a
set of values \( t \) for which the condition (12) at both \( m = 1 \) and \( m = 2 \) are satisfied simultaneously, will be less or equal than

\[
(\lambda_{n_1} + 2)(2 + \delta \lambda_{n_1}^{-1} \lambda_{n_2}) \delta \lambda_{n_2}^{-1}.
\]

It is possible to continue this reasoning considering all of conditions of a kind

\[
\frac{l + \alpha - \delta/2}{\lambda_{n_m}} \leq t \leq \frac{l + \alpha + \delta/2}{\lambda_{n_m}}, \quad m = 1, \ldots, k.
\]

Then we find the following estimation for the measure \( m(\delta) \) of a set of those \( t \) for which the points \( \{t \lambda_{n_m}\} \) located in the given cube with the edge \( \delta \):

\[
m(\delta) \leq (2 + \lambda_{n_1})(2 + \delta \lambda_{n_1}^{-1} \lambda_{n_2}) \cdots (2 + \delta \lambda_{n_{k-1}}^{-1} \lambda_{n_k}) \delta \lambda_{n_k}^{-1} \leq \delta^k \prod_{m=1}^{\infty} (1 + 2m^{-2})
\]

Summarizing over all such cubes, we receive the final estimation of a kind \( \leq 3c \varepsilon_1 \) for the measure of a set of those \( t \) for which \( \{t \lambda_{n_m}\} \in B_1 \) with the absolute constant \( c = \prod_{m=1}^{\infty} (1 + 2m^{-2}) \).

We notice, that the sequence \( \Lambda = (\lambda_n) \) satisfying the conditions (11) defined above depends on \( \delta \). We, for each sphere \( B_k \), will fix some sequence \( \Lambda_k \), using conditions (11). Considering all such spheres we designate \( \Lambda_0 = \{ \Lambda_k | k = 1, 2, \ldots \} \).

Let’s prove that for any point \( t \in E_0 \) the set \( \Sigma(\{t \Lambda\}) \) is contained in the finite union \( \bigcup_{k \leq n} B_k \) for some \( n \). Really, let at some \( t \in E_0 \) all members of the sequence \( \Sigma(\{t \Lambda\}) \) does not contained in the union \( \bigcup_{k \leq n} B_k \), for any natural \( n \). Two cases are possible: 1) there will be a point \( \bar{\theta} \in \Sigma(\{t \Lambda\}) \) belonging to infinite number of spheres \( B_k \); 2) there will be a sequence of elements \( \bar{\theta}_j, j \in \Sigma(\{t \Lambda\}) \) which does not contained in any finite union of spheres \( B_k \). We shall consider both possibilities separately and shall prove that they lead to the contradiction.

1) Let \( \bar{\theta} \in B_{k_1}, B_{k_2}, B_{k_3}, \ldots \) are all spheres to which the element \( \bar{\theta} \) belongs. We shall denote \( d \) the distance from \( \bar{\theta} \) to the bound of \( B_{k_j} \). As \( B_{k_j} \) is open set, then \( d > 0 \). Let \( B_k \) be any sphere of radius \( < d/2 \) from the list above, containing the point \( \bar{\theta} \). From the told it follows that the sphere \( B_k \) should contained in the sphere \( B_{k_j} \). But it contradicts the agreement accepted above.

2) Let \( \bar{\theta} \) be some limit point of the sequence \( (\bar{\theta}_j) \). According to the condition of the lemma 3 \( \bar{\theta} \in B_s \) for some. Let \( d \) denotes the distance from \( \bar{\theta} \) to the bound of \( B_{s_j} \). As \( \bar{\theta} \) is a limit point, then a sphere with the center in the point \( \bar{\theta} \) and radius \( d/4 \) contains an infinite set of members of the sequence \( (\bar{\theta}_j) \), say members \( \bar{\theta}_{j_1}, \bar{\theta}_{j_2}, \ldots \). According to 1), each point of this sequence can belong only to finite number of spheres. So the specified sequence will be contained in a union of infinite subfamily of spheres \( B_k \). Among them will be found infinitely many number of spheres having radius \( < d/4 \). All of them, then, should contained in the sphere \( B_s \). The received contradiction excludes the case 2) also.

So, for any \( t \in E_0 \) it will be found such \( n \) for which \( \Sigma(\{t \Lambda\}) \subset \bigcup_{k \leq n} B_k \). From here it follows that the set \( E_0 \) can be represented as a union of subsets \( E_k, k = 1, 2, \ldots \), where
For any $t \in E_0|\Sigma(t\Lambda) \subset \bigcup_{s \leq k} B_s$.

So,

$$E_0 = \bigcup_{k=1}^{\infty} E_k; \quad E_k \subset E_{k+1} (k \geq 1).$$

Further, $m(E_0) = \lim_{k \to \infty} m(E_k)$, in agreement with [42, p. 368]. As the set $E_k$ is a finite symmetrical, then the measure of a set of values $t$, interesting us, is possible to estimate by using of any sequence $\Lambda_k$, since, as it has been shown above, the sets $\Sigma(\{t\Lambda\})$ for different values of $t$ have empty intersection. So,

$$m(E_k) \leq \limsup_{k' \in \Delta_0} m(E_k(\Lambda')),$$

where $E_k(\Lambda') = \{t \in E_k|\{t\Lambda'\} \in \bigcup_{s \leq k} B_s\}$. Hence,

$$m(E_k(\Lambda')) \leq \sum_{s \leq k} m(E^{(s)}(\Lambda')),$$

where $E^{(s)}(\Lambda') = \{t \in E_0|\{t\Lambda'\} \in B_s\}$. Applying the inequality found above, we receive:

$$m(E_k(\Lambda')) \leq 6c(\varepsilon_1 + \cdots + \varepsilon_k).$$

This result is invariant for all $\Lambda' = \Lambda_r$ beginning from some natural $r = r(k)$. Taking limsup, as $k \to \infty$, we receive the demanded result. The proof of the lemma 4 is finished.

4. Local approximation.

**Lemma 5.** Let the conditions of the theorem be executed. Then there exist consequences of points $(\theta_k)$ $(\theta_k \in \Omega)$ and natural numbers $(m_k)$ such, that

$$\lim_{k \to \infty} F_k(\sigma_0 + s, \theta_k) = F(s_0 + s), s_0 = \sigma_0 + it_0$$

uniformly by $s$ in the circle $|s| \leq r < r_0$.

**Proof.** Let $y > 2$ be a whole positive number which will be precisely defined below. We set

$$y_0 = y, \quad y_1 = 2y_0, \ldots, y_m = 2y_{m-1} = 2^m y_0, \ldots.$$  

From the lemma 2 it follows that for given $\varepsilon$ and an integer $y > 2$ there exist a finite set $M_1$ of primes such that $M_1$ contains all of prime numbers $p, p \leq y$ and

$$\max_{|s| \leq r} |F(s_0 + s) - \eta_1(s_1)| \leq \varepsilon; \quad \eta_1(s_1) = \prod_{p \in M_1} f_p(e^{-2\pi i (\theta_0 + \gamma_p)p - s_1}), s_1 = \sigma_0 + s;$$

besides $\theta_0 = 0$ and $\gamma_p = (t_0/2\pi) \log p$, when $p \leq y$, and $\gamma_p = 0$ if $p > y$. Now we designate

$$h_1(s_1; \theta) = F_1(s_1; \theta) \cdot \eta_1^{-1}(s_1) - 1,$$
\[
F_1(s_1; \theta) = \prod_{p \leq m_1} f_p \left( e^{-2\pi i (\theta_p + \gamma_p) p^{-s_1}} \right); \\
\theta_p = \theta_p^0, \text{ when } p \in M_1 \text{ and } m_1 = \max_{m \in M_1} m. \text{ If } r + \delta + 2\lambda < r_0, \text{ then}
\]

\[
\int_{\Omega_1} \left( \int_{|s| \leq r + \delta + \lambda} |h_1(s_1; \theta)|^2 \, d\sigma \, dt \right) \, d\theta \leq \int_{|s| \leq r + \delta + \lambda} \left( \int_{\Omega_1} |h_1(s_1; \theta)|^2 \, d\theta \right) \, d\sigma \, dt \leq \\
\leq \pi(r + \delta + \lambda)^2 \max_{|s| \leq r + \delta + \lambda} \int_{\Omega_1} \left| \sum_{n>y} a_n(\theta) n^{-s_1 - it_0} \right|^2 \, d\theta;
\]

here the summation under the sign of integral is taken over a set of such natural numbers \( n \) the canonical factorization of which contains only primes \( p \) with conditions \( p \notin M_1, p \leq m_1 \):

\[
a_n(\theta) = \prod_{p|n} a_p^{\alpha_p} |e^{2\pi i \alpha_p \theta_p}| \quad n = \prod p^{\alpha_p},
\]

and \( \Omega_1 \) means projection of \( \Omega \) into the subspace of co-ordinate axes \( \theta_p, p \notin M_1 \).

By using of orthogonality of the system of functions \( e^{2\pi i n \theta}, r = 1, 2, \ldots \) we get

\[
\int_{\Omega_1} \left( \int_{|s| \leq r + \delta + \lambda} |h_1(s_1; \theta)|^2 \, d\sigma \, dt \right) \, d\theta \leq \pi(r + \delta + \lambda)^2 \sum_{n>y} a_n^2 n^{2r + 2\delta + 2\lambda - 2\sigma_0} \leq \\
\leq \frac{4e^2(\lambda)(r + \delta + \lambda)^2}{1 - 2\sigma_0 - 2r - 2\delta} \eta^{1 + 4\lambda + 2r - 2\sigma_0 + 2\delta}.
\]

Then there will be found a point \( \theta_1' = (\theta_p)_{p \notin M_1} \) such that

\[
\int_{|s| \leq r + \delta + \lambda} |h_1(s_1; \theta_1')|^2 \, d\sigma \, dt \leq \frac{4e^2(\lambda)(r + \delta + \lambda)^2}{1 - 2\sigma_0 - 2r - 2\delta} \eta^{1 + 4\lambda + 2r - 2\sigma_0 + 2\delta},
\]

or

\[
\max_{|s| \leq r} |h_1(s_1; \theta_1')| \leq \sqrt{2}(\delta + \lambda)^{-1} \left( \frac{1}{2\pi} \int_{|s| \leq r + \delta + \lambda} |h_1(s_1; \theta_1')|^2 \, d\sigma \, dt \right)^{1/2} \leq c_1(\delta, \lambda) \eta^{1/2 + 2\lambda + \delta + r - \sigma_0},
\]

(see [19, p. 345]) with a constant \( c_1(\delta, \lambda) > 0 \). Then, designating \( \theta_1 = (\theta_0, \theta_1'), \theta_0 = (\theta_p')_{p \in M_1} \), we will have

\[
\max_{|s| \leq r} \{|F(s_1 + it_0) - F_1(s_1; \theta_1)|\} \\
\leq \max_{|s| \leq r} \{|F(s_1 + it_0) - \eta_1(s_1)| + |\eta_1(s_1) : |h_1(s_1; \theta_1')|\} \leq \\
\leq \varepsilon + (A + 1)c_1(\delta, \lambda) \eta_0^{1/2 + r + 2\lambda + \delta - \sigma_0}; \quad y_0 = y,
\]

only if \( y_0 \) satisfies the condition

\[
(A + 1)c_1(\delta, \lambda) \eta_0^{1/2 + r + 2\lambda + \delta - \sigma_0} \leq \varepsilon; A = \max_{|s| \leq r} |F(s_1 + it_0)|.
\]
We replace now \( \varepsilon \) by \( \varepsilon/2 \). There is a finite set of primes \( M_2 \), containing the all of prime numbers \( \leq 2y_0 = y_1 \) and satisfying, according to the lemma 3, an inequality

\[
\max_{|s| \leq \varepsilon} |F(s_1 + it_0) - \eta_2(s_1)| \leq \varepsilon/2;
\]

here

\[
\eta_2(s_1) = \prod_{p \in M_2} f_p(e^{-2\pi i (\theta_1^p + \gamma_p)} p^{-s_1}),
\]

\( \theta_1^p = 0 \wedge \gamma_p = (t_0/2\pi) \log p \) when \( p \leq y_1 \), and \( \gamma_p = 0 \) when \( p > y_1 \). Similarly to performed above, we find \( \theta_2' \in \Omega_2 \) (here \( \Omega_2 \) is a projection of \( \Omega \) into the subspace of coordinate axes \( \theta_p, p \notin M_2 \)) such that

\[
\max_{|s| \leq \varepsilon} |F(s_1 + it_0) - F(s_1; \theta_2')| \leq \varepsilon; \ \theta_2' = (\theta_1, \theta_2').
\]

Really,

\[
|F_2(s_1; \theta) - \eta_2(s_1)| = |\eta_2(s_1)| \cdot |h_2(s_1; \theta)|; h_1(s_2; \theta) = F_2(s_1; \theta) \cdot \eta_2^{-1}(s_1) - 1.
\]

Now taking mean values, we receive

\[
\max_{|s| \leq \varepsilon} |h_2(s_1; \theta_2')| \leq \sqrt{2}(\delta + \lambda)^{-1} \left( \frac{1}{2\pi} \int_{|s| \leq r + \delta + \lambda} |h_2(s_1; \theta_2')|^2 \, d\sigma \, dt \right)^{1/2}
\]

\[
\leq c_1(\delta, \lambda)(2y_0)^{1/2 + r + 2\lambda + \delta - \sigma_0}.
\]

Therefore,

\[
\max_{|s| \leq \varepsilon} \{ |F(s_1 + it_0) - F_2(s_1; \theta_1)| \} \leq \\
\max_{|s| \leq \varepsilon} \{ |F(s_1 + it_0) - \eta_2(s_1)| + |\eta_2(s_1)| \cdot |h_2(s_1; \theta_2')| \} \leq \\
\leq \varepsilon/2 + (A + 1)c_1(\delta, \lambda)(2y_0)^{1/2 + r + 2\lambda + \delta - \sigma_0} \leq 2 \cdot 2^{1/2 + r + 2\lambda + \delta - \sigma_0} \varepsilon; \ \theta_2 = (\theta_1, \theta_2').
\]

Repeating similar reasoning, for every \( k > 1 \), one finds \( \theta_{k+1} = (\theta_k, \theta_{k+1}') \in \Omega \), with \( \theta_k = (\theta_k^p)_{p \in M_{k+1}} \) such that \( \theta_k^p = 0 \wedge \gamma_p = (t_0/2\pi) \log p \) when \( p \leq y_k \), and \( \gamma_p = 0 \) at \( p > y_k \), for which

\[
\max_{|s| \leq \varepsilon} |F(s_1 + it_0) - F_{k+1}(s_1; \theta_{k+1})| \leq 2^{1 + k(1/2 + r + 2\lambda + \delta - \sigma_0)} \varepsilon,
\]

where

\[
F_{k+1}(s_1; \theta) = \prod_{p \leq m_{k+1}} f_p(e^{-2\pi i (\theta_1^p + \gamma_p)} p^{-s_1}); \ m_{k+1} = \max_{m \in M_{k+1}} m.
\]

Since \( 1/2 + r + 2\lambda + \delta - \sigma_0 < 0 \), then uniformly by \( s \), \( |s| \leq r \)

\[
\lim_{k \to \infty} F_k(s_1; \theta_k) = F(s_1 + it_0).
\]

The lemma 5 is proved.
5. Proof of the theorem.

On the theorem’s conditions, there exists a real $t_0$ such that the function $F(s + it_0)$ has no zeros in the circle $|s - \sigma_0| \leq r < r_0 = \min(1 - \sigma_0, \sigma_0 - 1/2)$ at some $1/2 < \sigma_0 < 1$ (in the notations of the lemma 5, $r + \delta + 2\lambda < r_0$). Now we will consider the integrals

$$B_k = \int_\Omega \left( \int_{|s| \leq r} |F_{k+1} (s_1; \theta_{k+1} + \theta) - F_k (s_1; \theta_k + \theta)| \, d\sigma d\tau \right) d\theta,$$

where $k = 0, 1, \ldots$, where we accept $F_0 (s_1, \theta_0 + \theta) = 0$, if $k = 0$. Applying Schwartz’s inequality and changing the order of the integration, we find as above (denote $\rho = \pi (2^k y)$):

$$B_k^2 \leq 4\pi \rho^2 \int_{|s| \leq r} d\sigma d\tau \int_{[0, 1]^p} \left| \prod_{p \leq 2^k - 1} f_p (e^{-2\pi \gamma_p p^{-s-i\tau}}) \right|^2 \prod_{p \leq 2^k - 1} d\theta_p \times$$

$$\left( \prod_{2^k - 1 < p \leq 2^k} f_p \left( e^{-2\pi i (\theta_{k+1} + \theta_p)} \right) - 1 \right) \prod_{2^k - 1 < p \leq 2^k} d\theta_p \leq$$

$$c(\lambda, \delta) \sum_{n > 2^{k-1} y_0} n^{\lambda + 2r + 2\delta - 2\sigma_0} \leq$$

$$c(\lambda, \delta) (2^{k-1} y_0)^{1 + 4\lambda + 2r + 2\delta - 2\sigma_0}; c(\lambda, \sigma) > 0..$$

As $1 + r + \delta + 2\lambda - \sigma_0 < 0$, then from this estimation it follows the convergence of the series below almost everywhere (i.e. for all $\theta \in \Omega_0$, where $\Omega_0$ is a subset of full measure, and set $A = \Omega \setminus \Omega_0$ is finite-symmetrical):

$$\sum_{k=1}^\infty \int_{|s| \leq r} |F_k (s + \sigma_0, \theta_k + \theta) - F_{k-1} (s + \sigma_0, \theta_{k-1} + \theta)| \, d\sigma d\tau; s = \sigma + it.$$

According to Yegorov’s theorem (see [40, p. 166]) this series converges uniformly in the outside of some open set $\Omega (\varepsilon), \mu (\Omega (\varepsilon)) \leq \varepsilon$ for every given $\varepsilon > 0$. Put $\Omega' = \bigcap \Omega (\varepsilon)$, we can assume, that $\mu (\Omega' \setminus \Omega_0') = 0$ and the set $A \cup \Omega_0'$ is finite-symmetrical (otherwise it is possible to take the set of all finite permutations of all its elements).

There will be found some enumerable family of spheres $B_r$ with a total measure, not exceeding $\varepsilon$, the union of which contains the set $A \cup \Omega_0'$. For every natural $n$ we define the set $\Sigma'_n (t \Lambda)$ as a set of all limit points of the sequence $\Sigma_n (\omega) = \{ \sigma \omega | \sigma \in \Sigma \land \sigma(1) = 1 \land \cdots \land \sigma(n) = n \}$. Let

$$B^{(n)} = \{ t \{ t \Lambda \} \in A \land \sum_{n} \lambda \{ t \Lambda \} \subset \bigcup_{r=1}^\infty B_r, \lambda_n = (1/2\pi) \log p_n n = 1, 2, \ldots \}$$

For every $t$ the sequence $\sum_{r=1}^{t} \{ t \Lambda \}$ is a subsequence of the sequence $\sum_{n} \{ t \Lambda \}$. Therefore, $\sum_{r=1}^{t} \{ t \Lambda \} \subset \sum_{n} \{ t \Lambda \}$ and we have $B^{(n)} \subset B^{(n+1)}$. Then we have an inequality $m(B) \leq \sup m(B^{(n)})$, denoting $B = \bigcup_{n} B^{(n)}$. 

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Let’s estimate \( m(B^{(n)}) \). The set \( \sum'_{n}(\{t\Lambda\}) \) is a closed set. Clearly, if we will ”truncate” sequences \( \{t\Lambda\} \), leaving only components \( \{t\lambda_n\} \) with indexes greater than \( n \) and will denote the truncated sequence as \( \{t\Lambda\}' \), \( \Omega \), then the set \( \sum'(\{t\Lambda\}') \) also will be closed. Now we consider the products \([0, 1]^n \times \{\{t\Lambda\}'\}\) (external brackets designate the set of one element) for every \( t \). We have

\[
\{t\Lambda\} \in [0, 1]^n \times \{\{t\Lambda\}'\} \subset A.
\]

(The example below shows that from the feasibility of the last relationship it does not follow the equality \( A = \Omega \). Let \( I = [0, 1]; U = [0; 1/2]; V = [1/2; 1] \) and

\[
X_0 = U \times U \times \ldots, X_1 = V \times U \times \ldots,
\]

\[
X_2 = I \times V \times U \times \ldots, X_{s+1} = I^s \times V \times U \times \ldots, \ldots.
\]

Clearly, that \( \mu(X_s) = 0 \) for alls. Let

\[
X = \bigcup_{s=0}^{\infty} X_s.
\]

So, we have \( X = [0, 1]^s \times X \) for any naturals. Then \( \mu(X) = 0 \) and \( X \neq \Omega \).

Let \( (\theta_1, \ldots, \theta_n) \in [0, 1]^n \). There exist a neighborhood \( V \subset [0, 1]^n \) of this point such that \( (\theta_1, \ldots, \theta_n, \{t\Lambda\}') \in V \times W \subset \bigcup_r B_r \), for some neighborhood \( W \) of the point \( \{t\Lambda\}' \). Since the set \([0, 1]^n\) is closed, then they can be found a finite number of open sets \( V \) the union of which contain \([0, 1]^n\). The intersection of corresponding open sets \( W \), being an open set, contains the point \( \{t\Lambda\}' \). Therefore, we have

\[
[0, 1]^n \times \{\{t\Lambda\}'\} \subset \bigcup_V \bigcap W = [0, 1]^n \times \bigcap W \subset \bigcup_{r \in R} B_r,
\]

for each considered point \( t \). The similar relationship is fair in the case when the point \( \{t\Lambda\} \) would be replaced by any limit point \( \bar{\omega} \) of the sequence \( \Sigma(\{t\Lambda\}) \) also, because \( \bar{\omega} \in B_r \). If one denotes by \( B' \) the union of all open sets of a kind \( \bigcap_{r \in R} B'_r \), corresponding to every possible values of \( t \) and of a limit point \( \bar{\omega} \), then we shall receive the relation

\[
\{t\Lambda\} \in [0, 1]^n \times \{\{t\Lambda\}'\} \subset A \subset [0, 1]^n \times B' \subset \bigcup_{r=1}^{\infty} B_r,
\]

for each considered values of \( t \) and

\[
\{\bar{\omega}\} \in [0, 1]^n \times \{\bar{\omega}'\} \subset A \subset [0, 1]^n \times B' \subset \bigcup_{r=1}^{\infty} B_r,
\]

for each limit point \( \bar{\omega} \). From this it follows the inequality \( \mu'(B') \leq \varepsilon \) where \( \mu' \) means an outer measure. The set \( B' \) is open and \( \Sigma'(\{t\Lambda\}') \subset B' \). Now we can apply the lemma 3 and receive an estimation \( \text{m}(B^{(n)}) \leq 6\varepsilon \). Thus, we havem \( \text{m}(B) \leq 6\varepsilon \).

Let \( t \notin B \). Then, \( t \notin B^{(n)} \) for every \( n = y_k, k = 1, 2, 3, \ldots \). Consequently, for every \( k \), there is a such limit point \( \bar{\omega}_k \in \Omega \setminus \bigcup_r B_r \) of the sequence \( \sum_{n}(\{t\Lambda\}) \) for which the series
\[
\sum_{l=1}^{\infty} \int_{|s| \leq r} \left| F_l(\sigma_0 + s; \bar{\theta}_l + \omega_k) - F_{l-1}(\sigma_0 + s; \bar{\theta}_{l-1} + \omega_k) \right| d\sigma d\tau
\]

converges. As the set \( \Omega \setminus \bigcup_r B_r \) is closed, the limit point \( \mathfrak{F} = \{ \ell \Lambda \} \) of the sequence \( (\omega_k) \) will belong to the set \( \Omega \setminus \bigcup_r B_r \). Therefore, the series

\[
(5.1) \sum_{l=1}^{\infty} \int_{|s| \leq r} \left| F_l(s + \sigma_0; \theta_l + i\ell \Lambda) - F_{l-1}(s + \sigma_0; \theta_{l-1} + i\ell \Lambda) \right| d\sigma d\tau
\]

converges. So the last series converges for all \( t \), with exception of values \( t \) from some set of measure, not exceeding \( 12\varepsilon \). Owing to randomness \( \varepsilon \), last result shows convergence of (15) for almost all \( t \) (clearly, that condition \( 0 \leq t \leq 1 \) can be lowered now). Then, by the lemma 1, for \( \delta_0 < 1 \), taken by any way, the sequence

\[
F_k(s + \sigma_0; \theta_k + i\ell \Lambda)
\]

converges in the circle \( |s| \leq r\delta_0(\delta_0 < 1) \) uniformly, for all such \( t \), to some analytical function \( f(s + \sigma_0; t) \):

\[
\lim_{k \to \infty} F_k(s + \sigma_0 + it; \theta_k) = f(s + \sigma_0; t).
\]

Despite the received result, we cannot use \( t \) as a variable as the left and right parts of this equality can differ, each from other, by the arguments (the right part is defined as a limit of the sequence (16) where \( t \) enters into the expression, containing discontinuous function). Hence, the principle of analytical continuation cannot be applied. To finish the theorem proof we take any great real number \( \mathcal{T} \).

As considered values \( t \) are everywhere dense in the segment \([-T, T]\), the union of circles \( C(t) = \{ \sigma_0 + it + s : |s| \leq r\delta_0 \} \) contains the rectangle

\[
\sigma_0 - r\delta_0^2 \leq \text{Re}(s + \sigma_0) \leq \sigma_0 + r\delta_0^2, -T \leq \text{Im}(s + 3/4) \leq T
\]

in which conditions of the lemma 1 are executed for the series

\[
(5.3) F_1(s + \sigma_0; \theta_1) + (F_2(s + \sigma_0; \theta_2) - F_1(s + \sigma_0; \theta_1)) + \ldots
\]

Hence, on the lemma 1, this series defines an analytical function, in the considered rectangle, which coincides with \( F(s_0 + s) \) in the circle \( C(0) \). To apply the principle of analytical continuation we take one-connected open domain where both of the functions \( \log F_\ast(s) \) and \( \log F(s + s_0) \) are regular (here function \( F_\ast(s) \) is the sum of the series (17)). Let \( \rho_1, \ldots, \rho_L \) to designate all possible zeros of the function \( F(s_0 + s) \) in the considered rectangle the contour of which does not contain zeros of the function \( F(s_0 + s) \). We will take cuts through the segments \( 1/2 \leq \text{Res} \leq \text{Re}\rho_l \), \( \text{Im}s = \text{Im}\rho_l, l = 1, \ldots, L \). In the open domain of the considered rectangle, not containing specified segments, the functions \( \log F_\ast(s) \) and \( \log F(s + s_0) \) are regular. Therefore, in this domain the equality \( \log F_\ast(s) = \log F(s + s_0) \) holds. Then, the equality \( F_\ast(s) = F(s + s_0) \) is executed in all open area defined above. Now we receive justice of the relation \( F_\ast(s) = F(s + s_0) \) in the all rectangle (without cuts) where both functions are regular. The theorem is proved.
6. Proof of the consequence.

The conclusion of the consequence based on the theorem of Rouché (see [19, p. 137]). Let \( t \) be any real number. We shall prove that for any \( 0 < r' < 3/4 \) in the domain, bounded by the circle \( C' = \{ s \mid |s - \sigma_0 - it| = r' \} \), the function \( F(s) \) has not zeroes. Since there are only a finite set of zeroes, satisfying the condition \( |s - \sigma_0 - it| \leq r < 3/4 \) then we may take \( r > r' \) such that the circle \( C = \{ s \mid |s - \sigma_0 - it| = r \} \) did not contain zeroes of \( F(s) \). Let

\[
m = \min_{s \in C} |F(s)|.\]

Since the \( C \) is a compact set, clearly \( m > 0 \). Under the theorem, there exist \( n = n(t) \) such that the following inequality is executed on or in the circle \( C \):

\[
|F(s) - F_n(s; \bar{\theta}_n)| \leq 0.25m.
\]

Then, on contour \( C \) the following inequality is true:

\[
|F(s) - F_n(s; \bar{\theta}_n)| < |F(s)|.
\]

Then, from the theorem of Rouché it follows that the functions \( F(s) \) and \( F_n(s; \bar{\theta}_n) \) have an identical number of zeroes inside \( C \). But, the function \( F_n(s; \bar{\theta}_n) \) has not zeroes in the circle \( C \). Hence, \( F(s) \) also has not zeroes in the circle \( C \). As \( t \) is any, from the last we conclude that the strip \( -r < Re s - 3/4 < r \) for any \( 0 < r < 1/4 \) is free from the zeroes of the function \( F(s) \). Obviously, for any \( 1/4 > \lambda > 0 \) there exist a segment \( [1/2 + \lambda + i\tau, 1 - \lambda + i\tau] \) not containing zeros of \( F(s) \). This segment can be covered by finite number of circles, not containing zeros of \( F(s) \). Applying to each circle proved above, we receive the strips free from the zeros of \( F(s) \) the union of which contains the strip \( 1/2 + \lambda < Re s < 1 - \lambda \). As \( \lambda \) is any positive number, then the statement of the theorem is proved.

References
1. L. Euler. Introduction to the analyses of infinitesimals. – . :ONTI, 1936. (rus).
2. B. Riemann. On the number of prime numbers not exceeding a given quality. //Compositions.–. : GIZ, 1948 – p. 216 – 224(rus).
3. E. C. Titchmarsh. Theory of Riemann Zeta – function. . : IL, 1953(rus).
4. S. M. Voronin, A.A. Karatsuba. The Riemann Zeta–function. M: fiz. mat. lit. ,1994, 376 p. (rus).
5. . . . ., 1983, (rus).
6. H.L. Montgomery. Topics in multiplicative number theory.,1974(rus).
7. H.Davenport. Multiplicative number theory. .Nauka 1971(rus).
8. K. Chandrasekharan. Arithmetical functions. . Nauka, 1975., 270 pp. (rus).
9. S.M. Voronin. On The distribution of non – zero values of the Riemann Zeta function. Labors. MIAS – 1972 – v. 128, p. 153-175, (rus).
10. S.M. Voronin. On an differential independence of \( \zeta \) – function. Reports of AS USSR – 1973, v. 209, 6, pp.1264 – 1266, (rus).
11. S.M. Voronin. On an differential independence of Dirichlet’s L– functions. ct rith. – 1975 , v. VII – pp. 493 – 509.
12. S.M. Voronin. The theorem on “universality” of the Riemann zeta – function. Bulletin Acad. Sci. of USSR mat.ser. – 1975 – v. 39, 3 – pp. 475 – 486, (rus).
13. S.M. Voronin. On the zeroes of zeta – functions of quadratic forms. Labors of MIAS – 1976 – v. 142 – pp. 135 – 147 (rus).
14. S.M. Voronin. Analytical properties of Dirichlet generating functions of arithmetical objects: Diss. . . . . D – r of fiz. – mat. sci. MIAS USSR – . , 1977 – 90p, (rus).
15. S.M. Voronin. On an zeroes of some Dirichlet series, lying on the critical line. Bull. Acad. Sci. of USSR mat.ser. . – 1980 – v. 44 1 – pp.63-91 (rus).
16. S.M. Voronin. On the distribution of zeroes of some Dirichlet series. Labor. MIAS – 1984 – v. 163 – pp. 74 – 77, (rus).
17. N.Dunford and J.T.Schwartz. Linear operators. Part I: General theory. . . PFL,1962, 896 .
18. R.Differential and integral calculus. : Nauka, 1967.
19. E. C. Titchmarsh. Theory of function. . : GITTL, 1951.506 pp.(rus).
20. W. Rudin. Principles of mathematical analysis. . : Mir.1976.319 pp. (rus).
21. Hewitt E. and Ross K. Abstrakt Harmonic Analysis. v.1, Nauka, 1975.
22. B. Bagchi. A joint universality theorem for Dirichlet L –functions. Math. Zeit., 1982,v. 181, p.

319 – 335.

23. A. Laurinchikas. Limit Theorems for the Riemann Zeta-Function, Kluwer, Dordrecht, 1996.
24. .. On zeros of linear combinations of numbers . Lit. Mat. collec., 1986, v.26, 3, p.468-477.
25. A.Zigmund. Trigonometrical series. v. 2., : Mir, 1965.
26. V. I. Bogachev. Measure Theory. Springer-Verlag Berlin Heidelberg 2007, v. 1-2.
27.Zbl0664.10023<http://www.zentralblatt-math.org/zmath/en>

Dzhabbarov, I.Sh. <http://www.zentralblatt-math.org/zmath/en/mmath/en>
Dzhabbarov%2C+I%2A> Mean values of Dirichlet L-functions on short closed intervals of the critical line and their applications. (Russian. English summary) Izv. Akad. Nauk Az. SSR, Ser. Fiz.-Tehk.Mat. Nauk <http://www.zentralblatt-math.org/zmath/en/journals/search/?an=00001590> 1988, No.1, 3-9 (1988).

MSC2000: *11M06 <http://www.zentralblatt-math.org/zmath/en/search/?q=cc:*11M06>,
Reviewer:J.Kaczorowski<http://www.zentralblatt-math.org/zmath/en/search/?q=rv:J.Kaczorowski>.

28. Dzhabbarov I. Sh. On Ergodic Hypothesis. Euler International Mathematical Institute. Topology, Geometry and Dynamics: Rokhlin Memorial. Short abstracts of an international meeting held on January 11-16, 2010. St. Petersburg, 2010, p. 46-48.
29. Dzhabbarov I. Sh. Uniform approximation of Dirichlet series by partial products of Euler type. International conference “Approximation theory”. Abstracts (Saint-Petersburg, 6-8 may, 2010), St. Petersburg, 2010, p. 117-119.
30. Dzhabbarov I. Sh. The Riemann Hypothesis. arXiv:1006.0381v1, 2010.
31. Dzhabbarov I. Sh. On the connection between measure and metric in infinite dimensional space. International conference on Differential Equations and Dynamical Systems. Abstracts. Suzdal (Russia) July 2-7, 2010, Moscow, 2010, p. 213-214.

AZ2000, Shah Ismail Hatai avenue, 187, Ganja State University, Azerbaijan
E-mail address: jabbarovish@rambler.ru