Confinement, Chiral Symmetry Breaking and Continuum Limits in Quantum Link Models

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Using the example of compact $U(1)$ lattice gauge theory we argue that quantum link models can be used to reproduce the physics of conventional Hamiltonian lattice gauge theories. In addition to the usual gauge coupling $g$, these models have a new parameter $j$ which naturally cuts-off large electric flux quanta on each link while preserving exact $U(1)$ gauge invariance. The $j \to \infty$ limit recovers the conventional Hamiltonian. At strong couplings, the theory shows confinement and chiral symmetry breaking for all non-trivial values of $j$. The phase diagram of the $3+1$ dimensional theory suggests that a coulomb phase is present at large but finite $j$. Setting $g = 0$, a new approach to the physics of compact $U(1)$ gauge theory on the lattice emerges. In this case the parameter $j$ takes over the role of the gauge coupling, and $j \to \infty$ describes free photons.

1. INTRODUCTION

Hamiltonian lattice gauge theories were proposed a long time ago\cite{1}. Unfortunately, a class of extensions\cite{2,3}, which we refer to as quantum link models, were left unexplored. These models suggest an interesting alternative for tackling gauge theories. For example they give new insight into cluster algorithms for gauge theories\cite{4}. For a review of the subject look at \cite{5}.

An interesting question which we wish to address here is how the new formulations can reproduce the physics of the conventional formulations. We consider this question in the context of compact $U(1)$ gauge theory. In particular, we show that the new formulations show confinement and chiral symmetry breaking at strong couplings. At weak couplings, we argue that these models can have a coulomb phase and thus can also describe massless photons. The presence of these two different phases shows that quantum link models can also exhibit rich and complex dynamics.

The conventional Kogut-Susskind Hamiltonian for $U(1)$ lattice gauge theory is given by

\begin{equation}
H = \frac{g^2}{2} \sum_{x,\mu} E_{x,\mu}^2 - \frac{1}{2g^2} \sum_{P} (U_P + U_P^\dagger) \tag{1}
\end{equation}

where $E_{x,\mu}$ and $U_{x,\mu}$ are operators on each link. $U_P$ is the plaquette operator given by $U_P = U_{x,\mu} U_{x+\mu,\nu} U_{x+\nu,\mu} U_{x,\nu}$. The electric field operator $E$ and the gauge field operator $U$ on each link obey the commutation relations, $[E, U] = U$, $[E, U^\dagger] = -U^\dagger$, and $[U, U^\dagger] = 0$. These commutation relations cannot be realized by operators in a finite dimensional space. For example, the eigenvalues $m$ of $E$ are integers, $m = -\infty, -1, 0, 1, \ldots, \infty$. Further, $U$ also can be diagonalized, although not simultaneously with $E$. Its eigenvalues, exp$(iA)$ with $A = (-\pi, \pi)$, are the usual $U(1)$ phases.

However, it is not necessary to insist that $[U, U^\dagger] = 0$ to preserve the gauge invariance of the Hamiltonian, since it is the commutation relations between $E$ and $U$ or $U^\dagger$ that make $H$ gauge invariant. In particular by demanding that $[U, U^\dagger] = 2E/|j(j+1)|$, for a positive integer $j$, we can extend the Kogut-Susskind Hamiltonian to a class of gauge invariant Hamiltonians $H^{(j)}$ dependent on a new parameter $j$. In the $j \to \infty$ limit the conventional Hamiltonian $H$ is regained. For finite $j$, $E$ and $U$ can be represented by spin-$j$ operators and we will call them $E^{(j)}$ and $U^{(j)}$ instead. $E^{(j)}$ has eigenvalues between $-j$ and $j$ and $U^{(j)}$ is no longer diagonalizable for finite $j$. We will argue below that, even with these constraints interesting qualitative features of the theory are reproduced at finite $j$. 

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2. STRONG COUPLINGS

The new class of Hamiltonians \( H^{(j)} \) are formulated on a Hilbert space where the magnitude of the electric flux is bounded by \( j \) units on each link. At strong couplings, since large electric flux states are naturally suppressed, \( H^{(j)} \) and \( H \) will describe qualitatively similar physics. The dynamics of conventional lattice gauge theories at strong couplings produce confinement and chiral symmetry breaking. We illustrate how these phenomena emerge in quantum link models using the \((1+1)\) dimensional lattice Schwinger model constructed with staggered fermions. Similar results in higher dimensions can also be derived with some work.

Lattice Schwinger model with staggered fermions in the Hamiltonian formulation at strong couplings was first studied by Banks et. al., and a more accurate analysis was presented recently by Berruto et.al., \[\text{[6]}\]. We extend the latter results to the quantum link model whose Hamiltonian is given by

\[
H^{(j)}_{\text{Schw}} = \sum_{x=0}^{N-1} \left[ \frac{g^2 a}{2} (E_x^{(j)} - (-1)^x \frac{1}{4})^2 - (-1)^x m (\psi_x^\dagger \psi_x - \frac{1}{2}) \right] - \frac{i t}{2a} \sum_{x=0}^{N-1} \left( \psi_{x+1}^\dagger U_x^{(j)} \psi_x - \psi_x^\dagger (U_x^{(j)})_x^\dagger \psi_{x+1} \right)
\]

where \( N \) is even. Ignoring the quantum link nature of \( E^{(j)} \) and \( U^{(j)} \), the model has been studied by Berruto et.al. We refer the readers to their article for a more complete discussion and justifications for the above form of the Hamiltonian. The gauge generator

\[
G_x = E_x - E_{x-1} + \psi_x^\dagger \psi_x - \frac{1}{2}(1 + (-1)^x)
\]

\[(2)\]
can be used to define gauge invariant states in the Hilbert space.

The strong coupling ground state energy of the Hamiltonian is given by \( E_0 = N g^2 a / 32 - m N / 2 \). The energy \( E \) of a state with two heavy charges separated by a distance of \( R \), is given by \( (E - E_0) / (g^2 a) = 2m / (g^2 a) + R + 1 / 2 \), which shows the linear confining potential. It is possible to obtain the pseudo-scalar(\( \pi \)) and scalar(\( \sigma \)) masses in the strong coupling expansion. If the expansion parameter is taken to be \( \epsilon = t / 2 g^2 a^2 \), and \( \mu = m / g^2 a \), up to fourth order we get

\[
\frac{E_\pi}{g^2 a} = \frac{E_0}{g^2 a} + \frac{1}{4} + 2\mu + \epsilon^2 \frac{8}{1 + 8\mu} - \epsilon^4 \frac{512 + p(j)(64 + 512\mu)}{(1 + 8\mu)^3}
\]

\[
\frac{E_\sigma}{g^2 a} = \frac{E_0}{g^2 a} + \frac{1}{4} + 2\mu + \epsilon^2 \frac{24}{1 + 8\mu} - \epsilon^4 \frac{1536 + p(j)(64 + 512\mu)}{(1 + 8\mu)^3}
\]

\[
\frac{E_0}{g^2 a} = \frac{N}{32} - \frac{\mu N}{2} - \epsilon^2 \frac{4N}{1 + 8\mu} + \epsilon^4 \frac{192N}{(1 + 8\mu)^3}
\]

The value of \( j \) begins to enter only at the fourth order in the expansion, through \( p(j) = (j - 1)(j + 2)/(j(j + 1)) \), which clearly shows the minor role of \( j \). Chiral symmetry is also broken, and can be seen by a non-zero value of the condensate. Up to fourth order

\[
\langle \psi \psi \rangle = \frac{1}{2a} \left[ 1 - \epsilon^2 \frac{32}{(1 + 8\mu)^2} + \epsilon^4 \frac{1536}{(1 + 8\mu)^4} \right]
\]

\[(4)\]

which shows that now \( j \) enters the result only at higher orders. In the \( j \rightarrow \infty \) limit, the above results match with \[\text{[6]}\]. On general grounds it can be argued that the minor role of \( j \) at strong couplings is a universal feature and extends to higher dimensions.

3. COULOMB PHASE

In 3+1 dimensions the conventional compact \( U(1) \) gauge theory without matter fields has interesting phase structure. The Hamiltonian is given by eq.\[\text{[6]}\]. The theory has a confined phase at strong couplings and a coulomb phase at weak couplings. By extending the Hamiltonian to \( H^{(j)} \), we can study the phase diagram in the \( j \) vs. \( 1/g^2 \) plane. Based on results from the previous section we can expect that the strong coupling phase will extend to small \( j \). On the other hand, disallowing large electric flux states can naturally make the theory strongly coupled even at small \( g \). Thus the weak coupling phase may not be accessible at small \( g \). A simple plausible phase diagram is given in figure. \[\text{[6]}\]
The conjectured phase diagram of 3+1 dimensional $U(1)$ quantum link model. The point C refers to the theory with free photons.

The point C represents the fixed point describing free photons and usually approached along the $j = \infty$ axis in the conventional formulation. However, the phase diagram, if correct suggests that we can also approach it along the $g = 0$ axis. Setting $g = 0$ in $H^{(j)}$ we obtain the quantum link Hamiltonian in d spatial dimensions, given by

$$H^{(j)}(g = 0) = \sum_p \left[ U_P^{(j)} + (U_P^{(j)})^\dagger \right]$$

where $j$ is the only free parameter. We expect that in the $j \to \infty$ limit this Hamiltonian describes free photons. To see this let us consider the large $j$ expansion. We first write

$$U = \left( 1 + \frac{E}{j(j + 1)} - \frac{E^2}{j(j + 1)} \right)^{1/2} e^{iA}$$

where, $[A, E] = i$, which reproduces the necessary commutation relations. We then substitute $U$ in eq. (3) to obtain a low energy, large $j$ effective (tree-level) Hamiltonian,

$$H_{\text{eff}} = \frac{2}{\sqrt{j(j + 1)}} \left( \frac{c^2}{2\sqrt{j(j + 1)}} \sum_{x,\mu} E_{x,\mu}^2 + \frac{\sqrt{j(j + 1)}}{2} \sum_{x,\mu} B_{x,\mu}^2 \right)$$

where $B$ is the magnetic field with the usual definition and $c^2 = 2(d - 1)$ is the speed of light. $H_{\text{eff}}$ shows that $j$ takes over the role of the coupling through the relation $g_{\text{eff}}^2 \sim 1/\sqrt{j(j + 1)}$. As expected smaller $j$ makes the theory effectively strongly coupled. The results of the large $j$ expansion gives further credibility to the conjecture of the phase diagram shown in figure 1. Since $g_c \sim 1$ in the conventional formulation, based on the tree level relation between $g_{\text{eff}}$ and $j$, we may expect that $j_c$ is of order one. Perhaps $j \sim 1$ or 2 will already describe a coulomb phase! The presence of such a phase shows the potential of quantum link models to describe long range physics and continuum limits.

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