A BLACK–SCHOLES MODEL WITH LONG MEMORY

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Abstract. This note develops a stochastic model of asset volatility. The volatility obeys a continuous–time autoregressive equation. Conditions under which the process is asymptotically stationary and possesses long memory are characterised. Connections with the class of ARCH(∞) processes are sketched.

1. Introduction

In this paper we consider the autocorrelation of the volatility of the following stochastic functional differential equation:

\[
\begin{align*}
\frac{dX(t)}{dt} &= \left(\sigma + \beta \left(\int_{-\tau}^{0} X(t+v)\lambda(dv) - \int_{0}^{t} X(t-s)\kappa(ds)\right)\right)dB(t), \quad t \geq 0; \\
X(t) &= 0, \quad t \in [-\tau, 0].
\end{align*}
\]

1.1

Here, the volatility is the process \( V \) such that \( dX(t) = V(t)dB(t) \). The form of equation (1.1) is in part motivated by models of volatility in financial mathematics in which some traders use past information about the market to determine their investment strategies. As indicated in the next section, this leads us to assume that \( \lambda \) and \( \kappa \) are finite measures without singular parts such that

\[
\lambda[0, \infty) = \int_{0}^{\infty} \kappa(ds) = \kappa[0, \infty).
\]

In the context of this work, we show that \( X \) can be thought of as de–trended market returns, and hence (1.1) leads automatically to a Black–Scholes type model with memory.

In financial markets, it is of practical interest to determine whether market returns or other important indicators, such as the volatility \( V \), possess predictable components. Therefore, under the condition (1.2), we give necessary and sufficient conditions under which \( V \) is an asymptotically weakly stationary process, with non–trivial limiting autocovariance function. We do this by establishing that \( V \) is a continuous–time analogue of solutions of stochastic difference equations structurally related to the class of ARCH(∞) processes.

It is also of interest to see whether such processes in finance possess long memory or long range dependence (ARCH), in the usual sense that the limiting autocovariance function \( \gamma \) of \( V \) has the property

\[
\int_{0}^{\infty} \gamma(s)ds = +\infty.
\]

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2. Motivation from finance

Let \( S = \{ S(t) : t \geq 0 \} \) be the stock price of a single risky asset whose evolution is governed by

\[
dS(t) = \mu S(t) \, dt + S(t) \, dX(t), \quad t \geq 0; \quad S(0) = s_0 > 0,
\]

and \( X \) obeys (1.1). This means that shares in the stock started trading at time \( t = 0 \). In what follows, we assume that \( \lambda \) and \( \kappa \) have finite total variation, which implies that there is a unique continuous adapted processes \( X \) which satisfies (1.1) and which is moreover a semimartingale. Therefore there is a unique positive continuous adapted processes \( X \) which satisfies (2.1). It is reasonable to call the process \( V = \{ V(t) : t \geq 0 \} \) which is defined by

\[
X(t) = \int_0^t V(s) \, dB(s), \quad t \geq 0,
\]

the volatility of the stock price because from (2.1) and (1.1) we have

\[
dS(t) = \mu S(t) \, dt + V(t)S(t) \, dB(t), \quad t \geq 0.
\]

We now motivate the form of (1.1), and in particular begin by explaining the economic interpretation of \( X \). The cumulative return \( R = \{ R(t) : t \geq 0 \} \) on the stock is defined by the identity \( dS(t) = S(t) \, dR(t) \) for \( t \geq 0 \) and \( R(0) = 0 \). From this, (2.1) and (2.2) we see that \( X \) is the de-trended cumulative return, because \( X(t) = R(t) - \mu t \) for \( t \geq 0 \).

At time \( t \geq 0 \), traders in the market take a weighted average of the de-trended returns over the last \( \tau > 0 \) units of time, giving a short-run indicator of returns \( \int_{t-\tau}^t X(t+s)\lambda(ds) \). They also form a long-run indicator of returns by taking a weighted average of the de-trended returns over the entire history of the asset, according to \( \int_0^\infty X(t-s)\kappa(ds) \). Since there is no trading before time \( t = 0 \), we set \( X(t) = 0 \) for \( t \leq 0 \), so that the long-term indicator is also given by \( \int_0^\infty X(t-s)\kappa(ds) \). In order that the indicators represent weighted averages with the same weight, we require that \( \lambda \) and \( \kappa \) obey (1.2). The traders believe that these indicators signal that the market is far from equilibrium whenever the indicators differ significantly, and this causes the traders to trade greater amounts of the stock. It has been observed in real financial markets that the volume of trade is positively correlated with the volatility of the asset (see e.g. [9], [19] and the references therein), which leads to the simple model that the volatility depends linearly on the trading volume, which itself depends on the difference between these indicators. By this reasoning, we arrive at

\[
V(t) = \sigma + \beta \left( \int_0^t X(t+v)\lambda(\, dv) - \int_0^t X(t-s)\kappa(\, ds) \right), \quad t \geq 0.
\]

Therefore, using (2.3) and (2.2), we see that \( X \) obeys (1.1), because \( X \) is identically zero on \((-\infty, 0] \). It can be readily shown for \( \Delta > \delta \geq 0 \) that

\[
\text{Cov}(R(t+\delta) - R(t), R(t+\delta + \Delta) - R(t+\Delta)) = 0, \quad t \geq 0,
\]
so the δ-returns over non-overlapping time intervals are uncorrelated at all time horizons. Hence the market is efficient in the sense of Fama, see e.g. [11].

The parameter β represents the sensitivity of these trend-following traders; if it is large, the traders are sensitive and have a large impact on the price dynamics. If no such traders were present, then β = 0, and S obeys the classical Black–Scholes stochastic differential equation.

In the case when β ≠ 0, our model of asset price evolution depends on the path of the price process, and the returns follow a stochastic functional differential equation (SFDE). Other models of financial markets where price evolution is described by a SFDE include [1, 4, 2, 6, 16].

3. Mathematical preliminaries

By M(I) we denote the set of all signed σ-finite Borel measures on I ⊆ ₌ with values in ₌. Let |κ| and ∥κ∥ denote the variation and the total variation of a measure κ ∈ M(I) respectively.

Let (Ω, F, P) be a complete probability space equipped with a filtration F = (F_t)_{t≥0}, and let B = {B(t) : t ≥ 0} be a one-dimensional Brownian motion on this probability space. Let D denote the set of all adapted càdlàg processes. By H* we denote the set of all F-adapted càdlàg processes X = {X(t) : t ≥ 0}, satisfying

\[ \mathbb{E} \left( \max_{0 ≤ t ≤ T} |X(t)|^2 \right) < +∞, \quad \text{for every } T > 0. \]

Let σ, β ∈ ₌. Suppose that the measure κ has finite total variation. Then, the following stochastic differential equation has a unique strong solution.

\[ dX(t) = \left( \sigma + \beta \left( \int_0^t X(t-s)κ(ds) \right) \right) dB(t), \quad t ≥ 0; \quad X(0) = 0. \]

To see this, we introduce the well–defined operator F : D → D

\[ F(1_{[0,t]}X) = \beta \int_0^t X(t-s)κ(ds), \quad t ≥ 0. \]

The process F is functional Lipschitz with F(0) = 0 in the sense of [18, p. 250], since it satisfies for two càdlàg processes X, Y

\[ |F(1_{[0,t]}X) - F(1_{[0,t]}Y)| ≤ |β||κ||sup_{s ≤ t}|X(s) - Y(s)|, \]

almost surely for each t ≥ 0. Hence, the equation (3.2) fulfills all the assumptions of Lemma V.2, in [18] and has a unique strong solution. We refer to this process X as the solution of (3.2). Moreover, by Doob’s inequality (cf. [18, Theorem I.20]) the solution X belongs to H*.

We denote the spaces of real–valued integrable and continuous functions by L^1(0,∞) and C([0,∞); ₌) respectively. Then, L^1_{loc}(0,∞) denotes the space of all Lebesgue measurable functions, whose restrictions to compact subsets of ₌ belong to L^1.

We write f ∼ g for x → x_0 ∈ ₌ ∪ {±∞} if \( \lim_{x \to x_0} f(x)/g(x) = 1 \). A function L : [0,∞) → (0,∞) is slowly varying at infinity if \( \lim_{t \to \infty} L(tx)/L(t) = 1 \) holds for all x > 0. A function f varies regularly with index α ∈ ₌, f ∈ RV_∞(α), if it is of the form f(t) = t^α L(t) with L slowly varying, see e.g. [12, Ch. VIII.8].

3.1. Assumptions on Equation (1.1). Next, we give some concrete assumptions under which (1.1) has a well-defined and unique solution, and introduce some useful notation. Let σ, β ∈ ₌, τ ∈ ₌+. Consider the stochastic differential equation with delay given by (1.1). We assume that the two measures κ ∈ M(τ) and λ ∈ M([−τ,0]) are decomposable into absolutely continuous and discrete parts:
there exist Lebesgue integrable functions \( k \in L^1[0, \infty), \ell \in L^1[-\tau, 0] \), real-valued sequences \((\kappa_j)_{j \in \mathbb{N}}, (\lambda_j)_{0 \leq j \leq N}, N > 0 \), and monotone increasing positive sequences \((\rho_j)_{j \in \mathbb{N}}, (\tau_j)_{0 \leq j \leq N}, \tau_0 \leq \tau \), so that

\[
\kappa(ds) = \sum_{j=0}^{\infty} \kappa_j 1_{(\rho_j, \rho_j]}(ds) + k(ds) \, ds,
\]

\[
\lambda(ds) = \sum_{j=0}^{N} \lambda_j 1_{(-\tau_j, \tau_j]}(ds) + \ell(s) \, ds.
\]

Moreover, \( \kappa \) and \( \lambda \) satisfy

\[
k \in L^1[0, \infty), \quad \sum_{j=0}^{\infty} |\kappa_j| < \infty; \quad \int_{-\tau}^{0} \kappa(ds) = \int_{-\tau}^{0} \lambda(ds).
\]

The last equality in equation (3.5) can be written as

\[
\int_{0}^{\infty} k(s) \, ds + \sum_{j=0}^{\infty} \kappa_j = \int_{-\tau}^{0} \ell(s) \, ds + \sum_{j=0}^{N} \lambda_j.
\]

Then, as stated in Section 3, (1.1) has a unique strong solution. Moreover, this solution obeys \( X(t) = 0 \) for all \( t \geq 0 \) almost surely. For this reason, we suppose that \( \sigma \neq 0 \) and \( \beta \neq 0 \).

By hypothesis we see that \( K : [0, \infty) \to \mathbb{R} \) given by

\[
K(x) = -\int_{-\tau}^{-\tau \wedge x} \lambda(ds) + \int_{x}^{\infty} \kappa(ds), \quad x \geq 0,
\]

is well-defined. Moreover, \( K \) can be written as

\[
K(x) = \begin{cases}
\sum_{(j: \rho_j \geq x)} \kappa_j + \int_{x}^{\infty} k(s) \, ds, & x \geq \tau, \\
-\sum_{(j: \tau_j \geq x)} \lambda_j - \int_{-\tau}^{-\tau \wedge x} \ell(s) \, ds + \sum_{(j: \kappa_j \geq x)} \kappa_j + \int_{x}^{\infty} k(s) \, ds, & x < \tau.
\end{cases}
\]

4. Autoregression of the Volatility Process

We introduce the well-defined process \( V = \{V(t) : t \geq 0\} \)

\[
V(t) = \sigma + \beta \left( \int_{-\tau}^{0} X(t + v) \lambda( dv) - \int_{0}^{t} X(t - s) \kappa(ds) \right), \quad t \geq 0.
\]

Since \( X \in \mathcal{H}^* \) and by (3.5) we have that \( V \in \mathcal{H}^* \) and \( \mathbb{E}[V(t)] = \sigma, t \geq 0 \). By (1.1) and (4.1) we have \( dX(t) = V(t) \, dB(t) \). Therefore \( V \) is the volatility process. We see also that

\[
\mathbb{E}[X(t)] = 0, \quad \mathbb{E}[X^2(t)] = \int_{0}^{t} \mathbb{E}[V^2(s)] \, ds, \quad t \geq 0.
\]

We show that \( V \) obeys a linear stochastic integral equation, and deduce that \( t \mapsto \mathbb{E}[V^2(t)] \) satisfies a linear Volterra integral equation.

**Proposition 1.** Suppose that \( \kappa \) obeys (3.5). Then \( V \) defined by (4.1) obeys

\[
V(t) = \sigma + \beta \int_{0}^{t} K(t - s) V(s) \, dB(s), \quad t \geq 0.
\]

Moreover,

\[
\mathbb{E}[V^2(t)] = \sigma^2 + \beta^2 \int_{0}^{t} K^2(t - s) \mathbb{E}[V^2(s)] \, ds, \quad t \geq 0.
\]
Proposition 2. Suppose that $\kappa$ obeys (3.5) and that $K$ obeys $K \in L^2(0, \infty)$. Suppose that $V$ is defined by (3.1). If

$$\beta^2 \int_0^\infty K^2(s) \, ds < 1,$$

then

$$\lim_{t \to \infty} \mathbb{E}[V(t)^2] = \frac{\sigma^2}{1 - \beta^2 \int_0^\infty K^2(s) \, ds}. $$

The fact that $\lim_{t \to \infty} \mathbb{E}[V^2(t)]$ is always greater than $\sigma^2$ shows that the presence of the trend following speculators increases market volatility relative to the level $\sigma^2$, which would be obtained in their absence (where $\beta = 0$). In other words, the presence of these traders makes the market more risky, and leads to greater fluctuations. This is similar to findings of [10], in which the presence of noise traders increases the risk for informed investors.

Proof. From representation (3.4) we see that satisfies $K \in L^1_{\text{loc}}(0, \infty)$. Therefore, Theorem 2.3.1 in [15] applies and there exists a unique solution $r \in L^2_{\text{loc}}(0, \infty)$ of

$$r(t) = \beta^2 K^2(t) + \beta^2 \int_0^t K^2(t-s) r(s) \, ds, \quad t \geq 0. $$

Since $K^2$ is nonnegative, the iteration method in the proof of theorem 2.3.1 in [15] yields, that the resolvent $r$ is also nonnegative. By Theorem 2.3.5 in the same book, the process $\mathbb{E}[V^2(t)]$ is continuous and satisfies

$$\mathbb{E}[V^2(t)] = \sigma^2 + \int_0^t r(t-s) \sigma^2 \, ds = \sigma^2 \left( 1 + \int_0^t r(s) \, ds \right).$$
Therefore we have \( \int_0^T r(t) \, dt \leq \alpha + \alpha \int_0^T r(s) \, ds \). Since \( \alpha \in (0, 1) \) we have \( \int_0^T r(t) \, dt \leq \alpha/(1 - \alpha) \), for all \( T \geq 0 \). Therefore we have \( r \in L^1(0, \infty) \). Since \( \alpha \) converges as \( T \to \infty \) and \( r \in L^1(0, \infty) \) we have that

\[
\lim_{T \to \infty} \int_0^T a(T - s) r(s) \, ds = \alpha \int_0^\infty r(s) \, ds.
\]

Therefore we have \( \int_0^\infty r(s) \, ds = \alpha + \alpha \int_0^\infty r(s) \, ds \), from which we infer

\[
\int_0^\infty r(s) \, ds = \frac{\alpha}{1 - \alpha} = \frac{\beta^2 \int_0^\infty K^2(s) \, ds}{1 - \beta^2 \int_0^\infty K^2(s) \, ds}.
\]

Therefore we have

\[
\lim_{t \to \infty} \mathbb{E}[V^2(t)] = \sigma^2 \left( 1 + \int_0^\infty r(s) \, ds \right) = \sigma^2 \left( 1 + \frac{\beta^2 \int_0^\infty K^2(s) \, ds}{1 - \beta^2 \int_0^\infty K^2(s) \, ds} \right)
\]

which confirms the result. \( \square \)

Remark 1. In the case \( K \not\in L^2(0, \infty) \), the solution of (4.3) obeys \( \lim_{t \to \infty} \mathbb{E}[V(t)^2] = +\infty \). To see this notice first, by (4.3), that \( \mathbb{E}[V^2(t)] \geq \sigma^2 \) for all \( t \geq 0 \). Therefore by (4.3), we have

\[
\mathbb{E}[V^2(t)] = \sigma^2 + \beta^2 \int_0^t K^2(t - s) \mathbb{E}[V^2(s)] \, ds \geq \sigma^2 + \sigma^2 \beta^2 \int_0^t K^2(s) \, ds.
\]

Therefore as \( t \to \infty \) and \( K \not\in L^2(0, \infty) \), we have that \( \mathbb{E}[V^2(t)] \to \infty \) as \( t \to \infty \).

5. Asymptotic stationarity of \( V \) and Long Memory in \( V \)

In our next result, we show that \( K \in L^2(0, \infty) \) and \( K \) obeying (4.3) are necessary conditions for \( V \) to be asymptotically stationary. To fix terminology, we say that a real scalar process \( U = \{U(t) : t \geq 0\} \) is (weakly) asymptotically stationary if there exists \( \theta \in \mathbb{R} \) and a function \( \gamma : [0, \infty) \to \mathbb{R} \) such that \( \lim_{t \to \infty} \mathbb{E}[U(t)] = \theta \) and \( \lim_{t \to \infty} \text{Cov}(U(t), U(t + \Delta)) = \gamma(\Delta) \) for each \( \Delta \geq 0 \).

Lemma 1. Suppose that \( K \) obeys (4.3). Suppose that \( V \) defined by (4.1) is asymptotically stationary. Then \( K \in L^2(0, \infty) \) and \( K \) obeys (4.3).

Proof. Since \( V \) is asymptotically stationary, it follows that there is a finite \( g \geq 0 \) such that \( g := \lim_{t \to \infty} \text{Cov}(V(t), V(t)) \). Since \( \mathbb{E}[V(t)] = \sigma \) for all \( t \geq 0 \), we have that there is an \( a \in \mathbb{R} \) such that

\[
a := \lim_{t \to \infty} \mathbb{E}[V(t)^2] = \lim_{t \to \infty} \left\{ \text{Cov}(V(t), V(t)) + \mathbb{E}[V(t)^2] \right\} = g + \sigma^2.
\]

If \( K \not\in L^2(0, \infty) \), we have by Remark 1 that \( \lim_{t \to \infty} \mathbb{E}[V(t)^2] = +\infty \), which contradicts (5.1). Therefore we must have that \( K \in L^2(0, \infty) \).
Since $V$ obeys (5.1), we see that $a \geq \sigma^2 > 0$. Since $K \in L^2(0, \infty)$ we have
\[
\lim_{t \to \infty} \int_0^t K^2(t - s)E[V^2(s)] \, ds = \int_0^\infty K^2(s) \, ds = a.
\]
Therefore from (4.3) we have that
\[
a = \sigma^2 + \beta^2 \int_0^\infty K^2(s) \, ds.
\]
Since $a > 0$, we must have $\beta^2 \int_0^{\infty} K^2(s) \, ds < 1$, as required.

Remark 2. Perusal of the proof of Lemma 1 shows that if $\kappa$ obeys (3.3), $K$ is defined by (3.7), and
\[
(5.2) \quad \lim_{t \to \infty} \int_0^t K^2(t - s)E[V^2(s)] \, ds = \int_0^\infty K^2(s) \, ds = \beta.
\]
then $K \in L^2(0, \infty)$ and (4.4) holds. Therefore, by this remark and Proposition 2, we see that $K$ obeying $K \in L^2(0, \infty)$ and (4.4) is equivalent to (5.2), and that both imply that the limit is equal to $\sigma^2/(1 - \beta^2 \int_0^{\infty} K^2(s) \, ds)$.

In our next result, we show that the conditions imposed on $K$ in Proposition 2 are necessary and sufficient for $V$ to be asymptotically stationary. Moreover, we determine a formula for the limiting autocovariance function of $V$.

**Theorem 1.** Suppose that $\kappa$ obeys (3.3). Then the following statements are equivalent.

(A) $K \in L^2(0, \infty)$ and $K$ obeys (4.1); 
(B) The process $V$ defined by (4.1) is asymptotically stationary.

Moreover, both imply that the function $\gamma : [0, \infty) \to \mathbb{R}$ given by
\[
\gamma(\Delta) = \beta^2 \frac{\sigma^2}{1 - \beta^2 \int_0^{\infty} K^2(s) \, ds} \cdot \int_0^\infty K(s) K(s + \Delta) \, ds, \quad \Delta \geq 0,
\]
is well-defined, and that $E[V(t)] = \sigma$, for all $t \geq 0$,
\[
(5.4) \quad \lim_{t \to \infty} \text{Cov}(V(t), V(t + \Delta)) = \gamma(\Delta), \quad \text{for all } \Delta \geq 0.
\]

**Proof.** In Lemma 1 we have shown that statement (B) implies statement (A). Suppose statement (A) holds. Let $t \geq 0$. Since $V$ obeys (4.1), we have
\[
V(t) = E[V(t)] + \beta \int_0^t K(t - s) V(s) \, dB(s).
\]
Therefore it follows with $V \in \mathcal{H}^\star$ and $K \in L^2(0, \infty)$ that
\[
\text{Cov}(V(t), V(t + \Delta)) = E \left[ \beta \int_0^t K(t - s) V(s) \, dB(s) \cdot \beta \int_0^{t+\Delta} K(t + \Delta - s) V(s) \, dB(s) \right].
\]
Since $\Delta \geq 0$, for each $t \geq 0$ fixed we have
\[
\text{Cov}(V(t), V(t + \Delta)) = \beta^2 \int_0^t K(t - s) K(t + \Delta - s) E[V^2(s)] \, ds.
\]
For $\tau \geq 0$ define $K_\Delta(\tau) = K(\tau + \Delta)$. Then
\[
(5.5) \quad \text{Cov}(V(t), V(t + \Delta)) = \beta^2 \int_0^t K_\Delta(t - s) E[V^2(s)] \, ds.
\]
Since $2|x| \leq x^2 + y^2$ for all $x, y \in \mathbb{R}$, we have $0 \leq |K_\Delta(\tau)| \leq 1/2 K^2(\tau) + 1/2 K^2(\tau + \Delta)$. Since $K \in L^2(0, \infty)$, it follows that $K_\Delta \in L^1([0, \infty); [0, \infty))$. By Proposition 2 we have that $t \mapsto E[V^2(t)]$ obeys (5.5).
Therefore it follows that
\[ \lim_{t \to \infty} \int_0^t K_{\Delta}(t-s)E[V^2(s)]\,ds = \int_0^\infty K_{\Delta}(s)\,ds \cdot \lim_{t \to \infty} E[V^2(t)], \]
which, by (5.5) and (5.3) implies (5.4).

Therefore we have that there is a function \( \gamma \), defined by (5.3), such that (5.4) holds true. Furthermore we have that \( E[V(t)] = \sigma \). Thus \( V \) is asymptotically stationary, which proves (B). Hence (A) and (B) are equivalent. Moreover, we have shown that \( E[V(t)] \) and \( \text{Cov}(V(t), V(t+\Delta)) \) have the desired properties. \( \square \)

In our next result we show that \( V \) has short memory or long memory according as to whether \( K \) is integrable or not.

**Theorem 2.** Suppose that \( \kappa \) obeys (5.5) and that \( K \) satisfies \( K \in L^2(0, \infty) \) and obeys (1.1).

(a) If \( \kappa \) obeys \( \int_0^\infty s|\kappa|(ds) < \infty \), then \( \gamma \) defined by (5.3) obeys \( \int_0^\infty |\gamma(\Delta)|\,d\Delta < +\infty \).

(b) If \( \kappa \) obeys \( \int_0^\infty s|\kappa|(ds) = +\infty \), and \( \kappa \) is a non–negative measure, then \( \gamma \) defined by (5.3) obeys \( \int_0^\infty |\gamma(\Delta)|\,d\Delta = +\infty \).

**Remark 3.** In the case that \( \kappa \) obeys \( \int_0^\infty s|\kappa|(ds) < \infty \) it follows that \( K \in L^1(0, \infty) \cap C([0, \infty); \mathbb{R}) \) and that \( K(t) \to 0 \) as \( t \to \infty \). Therefore we have automatically that \( K \in L^2(0, \infty) \).

**Remark 4.** Part (b) of the theorem still holds in the case when \( \kappa \) is a non–positive measure, by an almost identical argument. One implication of this fact is that the sensitivity parameter \( \beta \) can be negative in (1.1). In terms of modelling, therefore, it is the magnitude of the difference between the short and long run indicators that matters, rather than the difference itself.

**Proof.** If \( \int_0^\infty s|\kappa|(ds) < \infty \), then \( K \in L^1(0, \infty) \):

\[ \int_0^\infty |K(x)|\,dx \leq \int_0^\infty s|\kappa|(ds) + \int_{-\tau}^0 (-s)|\lambda|(ds) \]
\[ \leq \int_0^\infty s|\kappa|(ds) + \tau\|\lambda\|. \]

Set \( c := \beta^2\sigma^2/(1 - \beta^2 \int_0^\infty K^2(s)\,ds) \). By the definition of \( \gamma \), we have
\[ \int_0^\infty |\gamma(\Delta)|\,d\Delta = c \int_0^\infty \left| \int_0^\infty K(s)K(s+\Delta)\,ds \right|\,d\Delta \]
\[ = c \int_0^\infty |K(s)| \int_s^\infty |K(u)|\,du\,ds. \]

Since \( K \in L^1(0, \infty) \), it follows that
\[ \int_0^\infty |\gamma(\Delta)|\,d\Delta \leq c \int_0^\infty |K(s)| \int_0^\infty |K(u)|\,du\,ds = c \left( \int_0^\infty |K(s)|\,ds \right)^2, \]
so \( \gamma \) is in \( L^1(0, \infty) \).

To prove the second part, since \( \kappa \) is non–negative, it follows that \( K(t) \) is non–increasing and non–negative for \( t \geq \tau \). Let \( f \) be defined by \( f(\Delta) := \int_{[\tau]}^\infty K(s)K(s+\Delta)\,ds \), where \( [x] \) denotes the smallest integer not less than \( x \). Then \( f \) is non–negative and non–increasing. Hence \( f \) (and therefore \( \gamma \)) is not integrable if and only if
\[ \sum_{n=0}^\infty f(n) = \sum_{n=0}^\infty \int_{[\tau]}^\infty K(s)K(s+n)\,ds \]
is infinite. Since $K$ is non-negative for $t \geq \tau$, we have

$$\sum_{n=0}^{\infty} \int_{[\tau]}^\infty K(s)K(s+n)\,ds = \sum_{n=0}^{\infty} \sum_{j=\lfloor \tau \rfloor}^{j+1} K(s)K(s+n)\,ds.$$  \hfill (5.7)

Following the steps of (5.6), we see that $K$ is not integrable. Since it is non-negative and non-increasing for $t \geq \tau$, we have that $\sum_{j=\lfloor \tau \rfloor}^{\infty} K(j) = \infty$. Therefore

$$\sum_{n=0}^{\infty} \int_{[\tau]}^\infty K(s)K(s+n)\,ds = \sum_{n=0}^{\infty} \sum_{j=\lfloor \tau \rfloor}^{\infty} K(s)K(s+n)\,ds$$

$$\geq \sum_{n=0}^{\infty} \sum_{j=\lfloor \tau \rfloor}^{\infty} K(j+1)K(j+n+1)$$

$$= \sum_{j=\lfloor \tau \rfloor+1}^{\infty} K(j) \sum_{l=j}^{\infty} K(l) = \sum_{l=\lfloor \tau \rfloor+1}^{\infty} K(l) \sum_{j=\lfloor \tau \rfloor+1}^{\infty} K(j).$$

Since $\sum_{j=\lfloor \tau \rfloor}^{\infty} K(j) = \infty$ there is an $N > 0$ such that $\sum_{j=\lfloor \tau \rfloor+1}^{\infty} K(j) \geq 1$ for all $l \geq N$. Therefore as $K$ is non-negative for $t \geq \tau$, we have

$$\sum_{n=0}^{\infty} \int_{[\tau]}^\infty K(s)K(s+n)\,ds \geq \sum_{l=\lfloor \tau \rfloor+1}^{\infty} K(l) \sum_{j=\lfloor \tau \rfloor+1}^{\infty} K(j) \geq \sum_{l=N}^{\infty} K(l) = +\infty.$$  \hfill \hfill (5.8)

Hence $\sum_{n=0}^{\infty} f(n) = +\infty$, as required. \hfill \hfill □

6. **Exact convergence rates for regularly varying weight–functions**

In the previous section, we gave conditions under which $\gamma$ is either integrable or non-integrable, but did not establish the pointwise rate of decay of $\gamma(\Delta)$ as $\Delta \to \infty$.

In this section, we address this question. First, consider measures $\kappa$ and $\lambda$ satisfying

$$\kappa(ds) = k(s)\,ds, \quad \lambda(ds) = \lambda_0 1_{(0)}(ds),$$

where $k$ is a continuous integrable kernel, $k \in L^{1}(0, \infty) \cap C((0, \infty); (0, \infty))$, and $\lambda_0 = \int_0^\infty k(s)\,ds$. In this case equation (1.1) reads

$$dX(t) = \left(\sigma + \beta \left(\lambda_0 X(t) - \int_0^t X(t-s)k(s)\,ds\right)\right)\,dB(t), \quad t \geq 0;$$

$$X(t) = 0, \quad t \in [-\tau, 0].$$

We consider kernels of the form $k(t) \sim L(t)t^{-\alpha-1}$ as $t \to \infty$, for certain positive $\alpha$, where $L$ is a slowly varying function.

**Corollary 1.** Suppose that $k \in RV_\infty(-1-\alpha)$ with $1/2 < \alpha < 1$ and $K$ satisfies (1.4). Then, the function $\gamma$ defined by (1.3) is not integrable and satisfies

$$\gamma(\Delta) \sim \frac{\beta^2}{1 - \beta^2} \frac{\sigma^2}{\int_0^\infty K^2(s)\,ds} \frac{\Gamma(2\alpha - 1)\Gamma(1 - \alpha)}{\Gamma(\alpha + 1)\alpha} \Delta^{1-2\alpha} L(\Delta)^2, \quad \Delta \to \infty.$$  \hfill (6.2)

Since $-1 < 1 - 2\alpha < 0$, the function $\gamma$ is obviously non-integrable.

**Proof.** By Karamata’s Theorem (see e.g. [5, Theorem 1.5.11]), the function $K$, defined as in (5.7), satisfies

$$K(x) = \int_x^\infty k(s)\,ds \sim \frac{1}{\alpha} L(x)x^{-\alpha}, \quad x \to \infty.$$

Since \( k \) is non-negative, \( K \) is also non-negative and non-increasing. Hence, following the steps of the proof of Theorem 7.1 in [3] we obtain

\[
\lim_{\Delta \to \infty} \frac{\gamma(\Delta)}{\Delta K(\Delta)^2} = \beta^2 \frac{\sigma^2}{1 - \beta^2} \int_0^\infty K^2(s) \, ds \frac{\Gamma(2\alpha - 1)\Gamma(1 - \alpha)}{\Gamma(\alpha)}.
\]

Now, since \( \Delta K(\Delta)^2 \sim 1/\alpha^2 L(\Delta)^2 \Delta^{1-2\alpha} \) for \( \Delta \to \infty \), (6.2) follows immediately from (6.3) and (5.3).

**Example 3.** Let \( k(t) := 1/(1+t)^{1+\alpha} \) for \( 1/2 < \alpha < 1 \). Then,

\[
K(x) = \frac{1}{\alpha} \frac{1}{(1+x)^\alpha} \quad \text{and} \quad \int_0^\infty K(x)^2 \, dx = \frac{1}{\alpha^2} \frac{1}{2\alpha - 1}.
\]

If moreover \( \beta^2 < \alpha^2(2\alpha - 1) \) holds true, then assumptions of Corollary 1 are satisfied and we obtain

\[
\gamma(\Delta) \sim \frac{\beta^2 \sigma^2}{\alpha^2(2\alpha - 1) - \beta^2} \frac{\Gamma(2\alpha - 1)\Gamma(1 - \alpha)}{\Gamma(\alpha)} \Delta^{1-2\alpha}, \quad \Delta \to \infty.
\]

We can also determine the rate of decay of \( \gamma \) when \( \alpha > 1 \).

**Corollary 2.** Suppose that \( k \in RV_{\infty}(-1-\alpha) \) with \( \alpha > 1 \) and \( K \) satisfies (4.4). Then, the function \( \gamma \) defined by (5.3) is integrable and satisfies

\[
\gamma(\Delta) \sim \frac{\beta^2 \sigma^2}{1 - \beta^2} \int_0^\infty K^2(s) \, ds \frac{\Gamma(2\alpha - 1)\Gamma(1 - \alpha)}{\Gamma(\alpha)} \Delta^{1-2\alpha}, \quad \Delta \to \infty.
\]

**Proof.** Since \( \alpha > 1 \), it is clear that \( \gamma \) is integrable. The proof of the asymptotic behaviour of \( \gamma \) follows from Theorem 5.2 in [3]. \( \square \)

Whereas in the case \( \alpha < 1/2 \) \( K \) does not satisfy \( K \in L^2(0, \infty) \), the case \( \alpha = 1/2 \) turns out to be critical: depending on the properties of the slowly varying function \( L \), both \( K \not\in L^2(0, \infty) \) as well as \( K \in L^2(0, \infty) \) is possible. In the latter case, we can achieve arbitrary slow decay rates of the autocovariance function.

**Corollary 3.** (1) Suppose that \( k(t) = L(t)t^{-3/2}, \ t \geq 0 \), with a slowly varying function \( L \). Then, \( K \in L^2(0, \infty) \) if and only if

\[
\int_1^\infty \frac{L(t)^2}{t} \, dt < \infty.
\]

Moreover, if (6.5) holds true and \( K \) additionally satisfies (4.4), then

\[
\gamma(\Delta) \sim \frac{\Delta \sigma^2}{1 - \beta^2} \int_0^\infty K^2(s) \, ds \int_\Delta^{\infty} \frac{L(s)^2}{s} \, ds, \quad \Delta \to \infty.
\]

(2) Suppose that \( f \) is in \( C^1((0, \infty); (0, \infty)) \), \( f(t) \to 0 \) as \( t \to \infty \) and that \(-f' \in RV_{\infty}(-1)\). Then \( f \in RV_{\infty}(0) \) and there exists \( L \in RV_{\infty}(0) \) which satisfies (6.5) and

\[
\int_\Delta^{\infty} \frac{L(s)^2}{s} \, ds \sim f(\Delta), \quad \Delta \to \infty.
\]

**Proof.** The proof follows from Theorem 3.6 and Corollary 3.7 in [3]. \( \square \)
7. Connection with ARCH(∞) processes

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space equipped with a filtration \(\mathcal{F} = \{\mathcal{F}_n : n \in \mathbb{N}\}\). Let us now consider the discrete version of equation (1.1). Let \(\sigma > 0, \alpha > 0\) and suppose \(X = \{X_n : n \in \mathbb{N}\}\) satisfies

\[
X_{n+1} - X_n = \left(\sigma + \beta \left(\alpha X_n - \sum_{j=1}^{n} a_{n-j} X_j\right)\right)\xi_{n+1} =: V_{n+1}\xi_{n+1}, \quad X_0 = 0,
\]

where \((a_n)_{n \geq 0}\) is a summable non-negative sequence, \(a = \sum_{j=0}^{\infty} a_j\) and \(\xi = \{\xi_n : n \in \mathbb{N}\}\) is a sequence of \(\mathbb{F}\)-adapted independent, identically distributed random variables with \(\mathbb{E}(\xi_n) = 0, \mathbb{E}(\xi_n^2) = 1\) for all \(n \in \mathbb{N}\). By \(U\) we denote the uncorrelated process \(U_n := V_n\xi_n, n \geq 1\) and its conditional variance process satisfies

\[
\text{Var}(U_{n+1}|\mathcal{F}_n) = \mathbb{E}(U_{n+1}^2|\mathcal{F}_n) = V_{n+1}^2, \quad n \in \mathbb{N}.
\]

Moreover, using \(X_n = \sum_{j=1}^{n} X_j - X_{j-1} = \sum_{j=1}^{n} V_j\xi_j\), we see that \(V_n, n \in \mathbb{N}\) is given by

\[
V_{n+1} = \sigma + \beta \sum_{j=1}^{n} \left(a - \sum_{i=j}^{n} a_{n-i}\right)U_j = \sigma + \beta \sum_{j=1}^{n} K_{n-j}U_j = \sigma + \beta \sum_{j=1}^{n} K_{n-j}V_j\xi_j,
\]

where \((K_n)_{n \in \mathbb{N}}\) is a non-negative and non-increasing sequence which is given by

\[
K_n := \sum_{j=n}^{\infty} a_j, n \in \mathbb{N}.
\]

Therefore

\[
U_{n+1} = \left(\sigma + \beta \sum_{j=1}^{n} K_{n-j}U_j\right)\xi_{n+1}, \quad n \geq 1.
\]

It can be seen that this governing equation for \(U\) is similar in structure to that describing the dynamics of ARCH(\(\infty\)) processes (see [13, 17, 20]). However, \(U\) is not an ARCH(\(\infty\)) process because it is the squares of volatility that obeys an autoregressive equation of the form (7.2) in the ARCH(\(\infty\)) case, whereas here it is merely the process \(U\) itself in (7.2) or (7.1).

For ARCH(\(\infty\)) processes, under conditions that imply the weak stationarity of \(U^2\) (see [20]), the rate of decay of the autocovariance function of \(U^2\) can be determined (see [17, 13, 14, 20]). It has been shown that if \(U^2\) is stationary, then its autocovariance function must be summable.

However, by applying to equation (7.2) the methods of this paper, it can be shown that the conditions that \((K_n)_{n \in \mathbb{N}}\) is square–summable, and \(\beta^2 \sum_{j=0}^{\infty} K_j^2 < 1\) are equivalent to the asymptotic stationarity of \(V\). Moreover, the function \(\Delta \mapsto \lim_{n \to \infty} \text{Cov}(V_n, V_{n+\Delta})\) is not summable over \(\mathbb{N}\) if and only if \(\sum_{j=0}^{\infty} ja_j = \infty\).

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