LEFT-RIGHT CROSSINGS IN THE MILLER-ABRAHAMS
RANDOM RESISTOR NETWORK ON A POISSON POINT
PROCESS

ALESSANDRA FAGGIONATO AND HLAFO ALFIE MIMUN

ABSTRACT. We consider the Miller-Abrahams (MA) random resistor network built on a homogeneous Poisson point process (PPP) on $\mathbb{R}^d$, $d \geq 2$. Points of the PPP are marked by i.i.d. random variables and the MA random resistor network is obtained by plugging an electrical filament between any pair of distinct points in the PPP. The conductivity of the filament between two points decays exponentially in their distance and depends on their marks in a suitable form prescribed by electron transport in amorphous materials. The graph obtained by keeping filaments with conductivity lower bounded by a threshold $\vartheta$ exhibits a phase transition at some $\vartheta_{\text{crit}}$. Under the assumption that the marks are nonnegative (or nonpositive) and bounded, we show that in the supercritical phase the maximal number of vertex-disjoint left-right crossings in a box of size $n$ is lower bounded by $Cn^{d-1}$ apart an event of exponentially small probability. This result is one of the main ingredients entering in the proof of Mott’s law in [4].

Keywords: Poisson point process, Mott variable range hopping, Miller–Abrahams random resistor network, left-right crossings, renormalization.

AMS 2010 Subject Classification: 60G55, 82B43, 82D30

1. Introduction

The Miller-Abrahams (MA) random resistor network has been introduced in [11] to study the anomalous conductivity at low temperature in amorphous materials as doped semiconductors, in the regime of Anderson localization and at low density of impurities. It has been further investigated in the physical literature (cf. [1], [12] and references therein), where percolation properties have been heuristically analyzed. A fundamental target has been to get a more robust derivation of the so called Mott’s law, which is a physical law predicting that at low temperature the conductivity of the above amorphous materials decays in a stretched exponential form as

$$\exp\{-\kappa \beta^{-\frac{\alpha+1}{2(d+1)}}\}$$

(1)

for some constant $\kappa > 0$. Above $\beta$ is the inverse temperature, $d \geq 2$ is the dimension of the medium and $\alpha \geq 0$ is a constant entering in the distribution of the ground state energies of the electron wavefunctions.

This work has been partially supported by the ERC Starting Grant 680275 MALIG and by the Grant PRIN 20155PAWZB "Large Scale Random Structures".
The MA random resistor network is defined from a translation invariant and ergodic simple point process \( \{x_i\} \), marked by i.i.d. random variables \( \{E_{x_i}\} \) with common law \( \nu \). It is obtained as follows. Given a realization \( \{(x_i, E_{x_i})\} \) of the above marked simple point process, we associate to any unordered pair of distinct points \( x_i \neq x_j \) a filament with electrical conductivity

\[
c(x_i, x_j) := \exp \left\{ -\frac{2}{\gamma} |x_i - x_j| - \frac{\beta}{2} \left( |E_{x_i}| + |E_{x_j}| + |E_{x_i} - E_{x_j}| \right) \right\},
\]

where \( \gamma \) is the so-called localization length. The physically relevant distributions \( \nu \) (for inorganic materials) are of the form \( \nu_{\text{phys}}(dE) \propto 1(|E| \leq a_0)|E|^\alpha dE \) with \( \alpha \geq 0 \) and \( a_0 > 0 \).

We call \( \sigma_n(\beta) \) the effective conductivity of the MA random resistor network restricted to the box centered at the origin with radius \( n \). For simplicity we restrict to marked point processes \( \{(x_i, E_i)\} \) with isotropic law. Then, as proved in [3] under suitable assumptions, as \( n \) goes to \( \infty \) a.s. the rescaled effective conductivity \((2n)^{d-4} \sigma_n(\beta)\) converges to a non random finite limit \( \sigma_\infty(\beta) \). In addition, \( \sigma_\infty(\beta) \) equals the diffusion coefficient \( d(\beta) \) of the so-called Mott’s random walk introduced in [7]. The latter is the continuous–time random walk on \( \{x_i\} \) with probability rate for a jump from \( x_i \) to \( x_j \neq x_i \) given by \( c(x_i, x_j) \). As a consequence, Mott’s law can be stated both for the limiting conductivity \( \sigma_\infty(\beta) \) in the MA random resistor network and for the diffusion coefficient \( d(\beta) \) in Mott’s random walk. We recall that, for Mott’s random walk, upper and lower bounds of \( d(\beta) \) in agreement with Mott’s law have been proved in [5] and [7], respectively.

We suppose here that \( \{x_i\} \) is a homogeneous Poisson point process (PPP) with density \( \lambda \). Given \( \vartheta \in (0, 1) \) we denote by \( \text{MA}(\vartheta) \) the subgraph obtained from the MA resistor network by keeping only filaments of conductivity lower bounded by \( \vartheta \). It is known (cf. [2] [1] [3]) that there exists \( \vartheta_{\text{crit}} \in (0, 1) \) such that \( \text{MA}(\vartheta) \) a.s. percolates for \( \vartheta < \vartheta_{\text{crit}} \) and a.s. does not percolate for \( \vartheta > \vartheta_{\text{crit}} \). As discussed in [4], an important tool to rigorously prove Mott’s law and characterize the constant \( \kappa \) in [1] consists in showing for \( \vartheta < \vartheta_{\text{crit}} \) that, apart an event of exponentially small probability, there are in \( \text{MA}(\vartheta) \) at least \( Cn^{d-1} \) disjoint left-right (LR) crossings, i.e. linear chains along the first direction. This is indeed our main result (cf. Theorem 1 in Section 2) under the assumption that the mark distribution \( \nu \) has finite support included in \( [0, +\infty) \) and including the origin. We point out that a positive lower bound of \( \sigma_\infty(\beta) \) can be obtained by standard methods (cf. [10]) when having the above LR crossings property for \( \text{MA}(\vartheta) \) with \( \vartheta \) small enough. In this case a stochastic domination argument would allow to recycle the LR crossings property for supercritical percolation on \( \mathbb{Z}^d \) [8]. On the other hand, to have a fine control on \( \sigma_\infty(\beta) \) as necessary for Mott’s law, one needs the LR crossings property for all \( \vartheta < \vartheta_c \). We also remark that an analysis of the subcritical \( \text{MA}(\vartheta) \) (i.e. with \( \vartheta > \vartheta_c \)) has been provided in [6].

We comment now some technical aspects in the derivation of our contribution. To prove Theorem 1 we first show that it is enough to derive a similar
result (given by Theorem 2 in Section 3) for a suitable random graph \( \mathbb{G}_\ast \) with vertices in \( \varepsilon \mathbb{Z}^d \), defined in terms of i.i.d. random variables parametrized by points in \( \varepsilon \mathbb{Z}^d \) (cf. Section 3). The proof of Theorem 2 is then inspired by the renormalization procedure developed by Grimmett and Marstrand in [9] for site percolation on \( \mathbb{Z}^d \) and by a construction presented in [13] Section 4. We recall that in [9] it is proved that the critical probability of a slab in \( \mathbb{Z}^d \) converges to the critical probability of \( \mathbb{Z}^d \) when the thickness of the slab goes to \(+\infty\). Moreover, in [13] Tanemura studies the left–right crossings in the supercritical Boolean model with deterministic radius.

We point out that the renormalization method developed in [9] does not apply verbatim to our setting. In particular the adaptation of Lemma 6 in [9] to our setting presents several obstacles due to the spatial correlations in the MA resistor network. We use the term “quasi-cluster” since usually these sets are not connected in \( \mathbb{G}_\ast \) and can present some cuts at suitable localized regions. By expressing the PPP of density \( \lambda \) as superposition of two independent PPP’s with density \( \lambda - \delta \) and \( \delta \ll 1 \), respectively, a quasi-cluster is built only by means of points in the PPP with density \( \lambda - \delta \). On the other hand, we will show that, with high probability, when superposing the PPP with density \( \delta \) we will insert a family of points \( x \) with very small mark \( E_x \), which will link with the quasi-cluster, making the resulting set connected in \( \mathbb{G}_\ast \). The quasi-clusters are produced by iterative steps, in which we attempt to enlarge the set. A lower bound of the probability that this attempt is successful, conditioned to the previous steps, is provided in Lemma 6.1, while measurability and the geometric properties of the quasi-clusters are analyzed in Section 7.

We finally comment our assumptions. We point out that the Grimmett-Marstrand method relies on the FKG inequality. Also for the MA resistor network one can introduce a natural ordering of the random objects, but it turns out that the FKG inequality is valid only when the marks are a.s. nonnegative (or nonpositive). In fact, in this case, the term \(|E_{x_i}| + |E_{x_j}| + |E_{x_i} - E_{x_j}|\) in (2) equals \(2 \max\{E_{x_i}, E_{x_j}\}\), and therefore it increases when increasing \( E_{x_i} \) or \( E_{x_j} \). The restriction to marks with a given sign is therefore motivated by the use of the FKG inequality. On the other hand, our results cover mark distributions \( \nu \) of the form \( \nu(dE) \propto 1(0 \leq E \leq a_0)E^\alpha dE \) for \( \alpha \geq 0 \) and \( a_0 > 0 \), which share several scaling properties with the physical distributions \( \nu_{\text{phys}} \). We stress that these scaling properties are relevant in the heuristic derivation of Mott’s law as well as in its rigorous analysis [1]. Our other assumption concerns the choice of the point process \( \{x_i\} \), which is a PPP. From a technical viewpoint, this choice avoids to introduce further spatial dependence in the model. On the other hand, the PPP plays a special role for Mott’s law. Due to (2) one expects that, when \( \beta \gg 1 \), points \( x \) with \( |E_x| \) not small give a negligible contribution to the conductivity. Hence one expects that, asymptotically as \( \beta \to +\infty \), the conductivity is the same as for the Miller-Abrahams resistor network obtained from the set \( \{x_i : |E_{x_i}| \leq E(\beta)\} \) for a suitable function \( E(\beta) \) with
\[ \lim_{\beta \to +\infty} E(\beta) = 0. \]

If in general \( \{x_i\} \) is sampled according to a stationary ergodic point process with finite density \( \rho \), it then follows that the thinned set \( \{x_i : |E_{x_i}| \leq E(\beta)\} \) converges to a PPP with density \( \rho \) when rescaling points as \( x \mapsto \nu(-E(\beta), E(\beta))^{1/d}x \). Hence the PPP should be the emerging point process when \( \beta \to +\infty \). This argument was indeed used in [7] to motivate the universality of Mott’s law.

Outline of the paper. In Section 2 we describe the model and state our main result (cf. Theorem 1). In Section 3 we show how to reduce the problem to a discrete setting (cf. Theorem 2 and Proposition 3.7). In Section 4 we introduce basic geometrical objects and state Proposition 4.6, which we prove in Section 5. In Section 6 we state our main technical tool (cf. Lemma 6.1). In Section 7 we introduce and study our fundamental step in constructing connected subsets (cf. Definition 7.1). In Section 8 we implement the above step in the same spirit of [9] and introduce the concept of occupied site in the renormalized lattice (cf. Definitions 8.8 and 8.11). In Section 9 we prove Propositions 8.9 and 8.12. In Section 10 we comment how to extend the basic construction of Section 8. In Section 11 we prove Theorem 2. In Appendix A we collect some minor technical facts.

2. Model and main results

Given \( \lambda > 0 \) and a probability measure \( \nu \) on \( \mathbb{R} \), we consider the marked Poisson point process (PPP) obtained by taking a homogeneous PPP \( \xi \) of density \( \lambda \) on \( \mathbb{R}^d \) \((d \geq 2)\) and marking each point \( x \in \xi \) independently with a random variable \( E_x \) having distribution \( \nu \) (i.e., conditionally to \( \xi \)), the marks \( (E_x)_{x \in \xi} \) are i.i.d. random variables with distribution \( \nu \). The above marked point process is usually called the \( \nu \)-randomization of the PPP with density \( \lambda \). We call \( \Omega \) the configuration space of the above marked point process and write \( \omega = \{(x, E_x) : x \in \xi\} \) for a generic element in \( \Omega \).

Definition 2.1. Given \( \zeta > 0 \) we associate to \( \omega = \{(x, E_x) : x \in \xi\} \) the graph \( G = G(\lambda, \nu, \zeta) \) with vertex set \( \xi \) and edge set given by the pairs \( \{x, y\} \subset \xi \) with \( x \neq y \) and such that

\[ |x - y| + |E_x| + |E_y| + |E_x - E_y| \leq \zeta. \quad (3) \]

For later use, we point out that, given \( E, E' \in \mathbb{R} \), it holds

\[ |E| + |E'| + |E - E'| = \begin{cases} 2\max(|E|, |E'|) & \text{if } E \cdot E' \geq 0, \\ 2(|E - E'|) & \text{if } E \cdot E' \leq 0. \end{cases} \quad (4) \]

The above graph \( G \) corresponds to the resistor network obtained from the Miller–Abrahams resistor network by keeping only filaments with conductivity lower bounded by \( e^{-\zeta} \) (without loss of generality we have set \( \gamma := 2 \) and \( \beta := 2 \), \( \gamma \) being the localization length and \( \beta \) being the inverse temperature).

Given a generic graph with vertexes in \( \mathbb{R}^d \), one says that it percolates if it has an unbounded connected component. We recall (see [2] [4]) that there
exists a critical length \( \zeta_c(\lambda, \nu) \) such that
\[
\mathbb{P}(G(\lambda, \nu, \zeta) \text{ percolates }) = \begin{cases} 
1 & \text{if } \zeta > \zeta_c(\lambda, \nu), \\
0 & \text{if } \zeta < \zeta_c(\lambda, \nu).
\end{cases}
\]  

**Definition 2.2.** Given \( L > 0 \), a left-right (LR) crossing of the box \([-L, L]^d\) in the graph \( G = G(\lambda, \nu, \zeta) \) is any sequence of distinct points \( x_1, x_2, \ldots, x_n \in \xi \) such that
- \( \{x_i, x_{i+1}\} \in \mathcal{E} \) for all \( i = 1, 2, \ldots, n-1; \)
- \( x_1 \in (-\infty, -L) \times [-L, L]^{d-1}; \)
- \( x_2, x_3, \ldots, x_{n-1} \in [-L, L]^d; \)
- \( x_n \in (L, +\infty) \times [-L, L]^{d-1}. \)

We also define \( R_L(G) \) as the maximal number of vertex-disjoint LR crossings of \([-L, L]^d\) in \( G \).

Our main result is the following one:

**Theorem 1.** Suppose that \( \nu \) has bounded support contained in \([0, +\infty)\) or in \((-\infty, 0]\) and suppose that 0 belongs to the support of \( \nu \). Then, given \( \lambda > 0 \) and \( \zeta > \zeta_c(\lambda, \nu), \) there exist positive constants \( c, c' \) such that
\[
\mathbb{P}(R_L(G) \geq cL^{d-1}) \geq 1 - e^{-cL^{d-1}},
\]
for \( L \) large enough, where \( G = G(\lambda, \nu, \zeta) \).

3. **Discretization**

In this section we show how to reduce the problem of estimating the probability \( \mathbb{P}(R_L(G) \geq cL^{d-1}) \) to a similar problem for a graph with vertexes contained in a lattice.

**Lemma 3.1.** To prove Theorem 1 it is enough to consider the case \( \zeta = 1 > \zeta_c(\lambda, \nu). \)

**Proof.** We fix \( \zeta > \zeta_c(\lambda, \nu) \) and we let \( G \) be as in Theorem 1. The linear map \( x \mapsto \psi(x) := x/\zeta \) gives a graph isomorphism between \( G \) and its image \( G' \).

Note that \( G' \) has the same law of \( G(\lambda', \nu, 1) \), where \( \lambda' := \lambda \zeta^{-d} \). Due to the above isomorphism, we also have that \( \zeta_c(\lambda', \nu) = \zeta_c(\lambda, \nu)/\zeta \) and the condition \( \zeta > \zeta_c(\lambda, \nu) \) reads \( 1 > \zeta_c(\lambda', \nu) \). To conclude it is enough to observe that \( R_L(G) \geq cL^{d-1} \) if and only if \( R_{L'}(G') \geq cL^{d-1}(L')^{d-1} \) where \( L' := L/\zeta \), hence it is enough to focus on \( G' = G(\lambda', \nu, 1) \). \( \square \)

**Warning 3.1.** Due to Lemma 3.1, without any loss of generality, we take once and for all \( \zeta = 1 \) in Theorem 1 and assume that \( \zeta = 1 > \zeta_c(\lambda, \nu). \) In particular, \( G \) will always denote the graph \( G(\lambda, \nu, 1) \). Moreover, we fix once and for all a constant \( C_0 > 0 \) such that \( \nu \) has support inside \([0, C_0]\). By symmetry, the case of nonpositive marks can be treated similarly.

**Lemma 3.2.** There exist \( \lambda_* \in (0, \lambda) \) and \( u_* \in (\zeta_c(\lambda, \nu), 1) \) such that
\[
\mathbb{P}(G(\rho, \nu, u) \text{ percolates }) = 1 \quad \forall \rho \geq \lambda_*, \forall u \geq u_*.
\]
Proof. Let \( \zeta_c := \zeta_c(\lambda, \nu) \). It is trivial to build a coupling such that \( G(\rho, \nu, u) \subset G(\rho', \nu, u') \) if \( \rho \leq \rho' \) and \( u \leq u' \). As a consequence, we only need to show that there exist \( \lambda_* < \lambda \) and \( u_* \in (\zeta_c, 1) \) such that \( P(G(\rho, \nu, u_*) \text{ percolates}) = 1 \). To this aim we fix \( \zeta' \in (\zeta_c, 1) \). Fixed \( \gamma \in (0, 1) \), \( G(\lambda, \nu, \zeta') \) can be described also as the graph with vertex set \( \xi \) given by a PPP with density \( \lambda \) and edge set \( E' \) given by the pairs \( \{x, y\} \subset \xi \) with \( x \neq y \) and

\[
|x/\gamma - y/\gamma| + |E_x| + |E_y| + |E_x - E_y| \\
\leq \zeta'/\gamma - (1/\gamma - 1)(|E_x| + |E_y| + |E_x - E_y|),
\]

where the marks come from the \( \nu \)-randomization of the PPP \( \xi \). Note that the r.h.s. of (8) is bounded from above by \( \zeta'/\gamma - 3C_0(1/\gamma - 1) \), which goes to \( \zeta' \) as \( \gamma \uparrow 1 \). In particular, we can fix \( \gamma \) very near to 1 (from the left) to have \( u_* := \zeta'/\gamma - 3C_0(1/\gamma - 1) \in (\zeta_c, 1) \). We now introduce the graph \( \hat{G} = (\xi, \hat{E}) \) where \( \{x, y\} \subset \hat{E} \) and \( x \neq y \) and

\[
|x/\gamma - y/\gamma| + |E_x| + |E_y| + |E_x - E_y| \leq u_*.
\]

Since the r.h.s. of (8) is bounded by \( u_* \) by our choice of \( \gamma \), \( \hat{G} \) contains \( G(\lambda, \nu, \zeta') \). Since \( P(G(\lambda, \nu, \zeta') \text{ percolates}) = 1 \) by (5), we get that \( P(\hat{G} \text{ percolates}) = 1 \). On the other hand, due to (9), the graph obtained by rescaling \( \hat{G} \) according to the map \( x \mapsto x/\gamma \) has the same law of the graph \( G(\lambda \gamma^d, \nu, u_*) \). Since \( P(\hat{G} \text{ percolates}) = 1 \), we conclude that \( P(G(\lambda \gamma^d, \nu, u_*) \text{ percolates}) = 1 \). It is therefore enough to take \( \lambda_* := \lambda \gamma^d \).

We need to introduce some notation since we will deal with several couplings:

- We write \( \text{PPP}(\rho) \) for the Poisson point process with density \( \rho \).
- We write \( \text{PPP}(\rho, \nu) \) for the marked PPP obtained as \( \nu \)-randomization of a \( \text{PPP}(\rho) \).
- We write \( \mathcal{L}(\rho, \nu) \) for the law of \( \inf \{X_1, X_2, \ldots, X_N\} \), where \( (X_n)_{n \geq 1} \) is a sequence of i.i.d. random variables with law \( \nu \) and \( N \) is a Poisson random variable with parameter \( \rho \). When \( \rho = 0 \), the set \( \{X_1, X_2, \ldots, X_N\} \) is given by \( \emptyset \).

Above, and in what follows, we use the convention \( \inf \emptyset := +\infty \).

**Definition 3.3** (Parameters \( \alpha, \varepsilon \)). We fix \( \alpha \) small enough that \( 1 - 10\alpha \geq u_* \) (see Lemma 3.2) and \( \sqrt{d}/\alpha \in \mathbb{N}_+ \). We define \( \varepsilon \) by \( \varepsilon := \alpha/100 \) (note that \( 1/\varepsilon \in \mathbb{N}_+ \)). For each \( z \in \varepsilon \mathbb{Z}^d \) we set \( R_z := z + [0, \varepsilon]^d \).

We fix a positive integer \( K \), very large. In Section 10 we will explain how to choose \( K \).

**Definition 3.4.** Let \( \lambda_* \in (0, \lambda) \) be as in Lemma 3.2. We introduce the following independent random fields defined on a common probability space \( (\Theta, \mathbb{P}) \):

- Let \( (A_z)_{z \in \mathbb{Z}^d} \) be i.i.d. random variables with law \( \mathcal{L}(\lambda_* \varepsilon^d, \nu) \).
- For \( j = 1, 2, \ldots, K \) let \( (T_z^{(j)})_{z \in \mathbb{Z}^d} \) be i.i.d. random variables with law \( \mathcal{L}((\lambda - \lambda_* \varepsilon^d)/K, \nu) \).
Definition 3.5. On the probability space \((\Theta, \mathbb{P})\) we define the graphs \(G_\varepsilon = (\mathbb{V}, \mathbb{E}_\varepsilon)\), \(G = (\mathbb{V}, \mathbb{E})\) and \(G_* = (\mathbb{V}_*, \mathbb{E}_*)\) with vertexes in \(\varepsilon \mathbb{Z}^d\) as follows.

The vertex set \(\mathbb{V}\) is given by \(\mathbb{V} := \{ z \in \varepsilon \mathbb{Z}^d : A_z < +\infty \}\). The edge set \(\mathbb{E}\) is given by the unordered pairs \(\{z, z'\}\) with \(z \neq z'\) in \(\mathbb{V}\) such that

\[ |z - z'| + 2 \max \{ A_z, A_{z'} \} \leq 1 - 2\alpha, \]  

while the edge set \(\mathbb{E}_\varepsilon\) is given by the unordered pairs \(\{z, z'\}\) with \(z \neq z'\) in \(\mathbb{V}\) such that

\[ |z - z'| + 2 \max \{ A_z, A_{z'} \} \leq 1 - 3\alpha. \]  

The vertex set \(\mathbb{V}_*\) is given by \(\mathbb{V}_* := \{ z \in \varepsilon \mathbb{Z}^d : A_z \wedge \min_{1 \leq j \leq K} T_z^{(j)} < +\infty \}\).

The edge set \(\mathbb{E}_*\) is given by the unordered pairs \(\{z, z'\}\) with \(z \neq z'\) in \(\mathbb{V}_*\) and

\[ |z - z'| + 2 \max \{ A_z \wedge \min_{1 \leq j \leq K} T_z^{(j)}, A_{z'} \wedge \min_{1 \leq j \leq K} T_{z'}^{(j)} \} \leq 1 - \alpha. \]  

Trivially \(G_\varepsilon \subset G \subset G_*\). Note also that the graphs \(G\) and \(G_\varepsilon\) depend only on the random field \((A_z)_{z \in \varepsilon \mathbb{Z}^d}\). The graph \(G\) will play an important role in the next sections.

Definition 3.6. Given \(L > 0\), a left-right (LR) crossing of the box \(\Delta_L := [-L - 2, L + 2] \times [-L, L]^{d-1}\) in the graph \(G_*\) is any sequence of distinct vertexes \(x_1, x_2, \ldots, x_n\) of \(G_*\) such that

- \(\{x_i, x_{i+1}\} \in \mathbb{E}_*\) for all \(i = 1, 2, \ldots, n - 1\);
- \(x_1 \in (-\infty, -L - 2) \times [-L, L]^{d-1}\);
- \(x_2, x_3, \ldots, x_{n-1} \in \Delta_L\);
- \(x_n \in (L + 2, +\infty) \times [-L, L]^{d-1}\).

We also define \(R_L(G_*)\) as the maximal number of vertex-disjoint LR crossings of \(\Delta_L\) in \(G_*\).

Theorem 2. Let \(G_*\) be the random graph given in Definition 3.5. Then there exist positive constants \(c, c'\) such that

\[ \mathbb{P} \left( R_L(G_*) \geq cL^{d-1} \right) \geq 1 - e^{-c'L^{d-1}} \]  

for \(L\) large enough.

The rest of the paper will be devoted to the proof of Theorem 2 due to the following fact:

Proposition 3.7. Theorem 2 implies Theorem 1.

An important tool to prove Theorem 2 will be the following:

Lemma 3.8. The graph \(G_\varepsilon\) percolates \(\mathbb{P}\)-a.s.

At this point, we can disregard the original problem and the original random objects. One could start afresh keeping in mind only Definitions 3.3, 3.4, 3.5 and 3.6 and Lemma 3.8. The next sections will be devoted to the proof of Theorem 2.
The proofs of Proposition \ref{prop:ordering} and Lemma \ref{lem:ordering} are postponed to Subsections \ref{sec:proof-of-prop-3.7} and \ref{sec:proof-of-lem-3.8} below, respectively. We end with some observations concerning the FKG inequality.

On the probability space $(\Theta, \mathbb{P})$ we introduce the partial ordering $\preceq$ as follows: given $\theta_1, \theta_2 \in \Theta$ we say that $\theta_1 \preceq \theta_2$ if, for all $z \in \varepsilon \mathbb{Z}^d$ and $j \in \{1, 2, \ldots, K\}$, it holds

$$A_z(\theta_1) \geq A_z(\theta_2), \quad T_z^{(j)}(\theta_1) \geq T_z^{(j)}(\theta_2).$$

We point out that, if $\theta_1 \preceq \theta_2$, then $\mathbb{G}_z(\theta_1) \subset \mathbb{G}_z(\theta_2)$, $\mathbb{G}(\theta_1) \subset \mathbb{G}(\theta_2)$ and $\mathbb{G}_s(\theta_1) \subset \mathbb{G}_s(\theta_2)$. We stress that the above inclusions follow from Definition \ref{def:ordering} and expressions \eqref{eq:ordering}, \eqref{eq:ordering-s}. We have also that the partial ordering $\preceq$ satisfies the FKG inequality: if $F, G$ are increasing events for $\preceq$, then $\mathbb{P}(F \cap G) \geq \mathbb{P}(F)\mathbb{P}(G)$.

3.1. Proof of Proposition \ref{prop:ordering}. We first clarify the relation of the random fields introduced in Definition \ref{def:ordering-b} with the marked PPP($\lambda, \nu$). We observe that a PPP($\lambda, \nu$) can be obtained as follows. Let

\begin{align}
\{(x, E_x) : x \in \sigma\}, \\
\{(x, E_x) : x \in \xi^{(j)}\} \quad j = 1, 2, \ldots, K,
\end{align}

be independent marked PPP’s, respectively with law PPP($\lambda, \nu$) and PPP($\lambda - \lambda_s$)/$K, \nu$). The random sets $\sigma$ and $\xi^{(j)}$, with $1 \leq j \leq K$, are a.s. disjoint. To simplify that notation, at cost to remove an event of probability zero, from now on we suppose that $\sigma$ and $\xi^{(j)}$, with $1 \leq j \leq K$, are disjoint subsets of $\mathbb{R}^d$. Then, setting $\xi := \sigma \cup \bigcup_{j=1}^{K} \xi^{(j)}$, $\{(x, E_x) : x \in \xi\}$ is a PPP($\lambda, \nu$). We define

\begin{align}
B_z := \inf\{E_x : x \in \sigma \cap R_z\}, \quad z \in \varepsilon \mathbb{Z}^d, \\
B_z^{(j)} := \inf\{E_x : x \in \xi^{(j)} \cap R_z\}, \quad z \in \varepsilon \mathbb{Z}^d, \quad j = 1, 2, \ldots, K.
\end{align}

We note that $(B_z)_{z \in \varepsilon \mathbb{Z}^d}$ has the same law of $(A_z)_{z \in \varepsilon \mathbb{Z}^d}$ and $(B_z^{(j)})_{z \in \varepsilon \mathbb{Z}^d}$ has the same law of $(T_z^{(j)})_{z \in \varepsilon \mathbb{Z}^d}$, for $j = 1, 2, \ldots, K$. Moreover the above fields in \eqref{eq:ordering} and \eqref{eq:ordering-s} are independent. Trivially, we have

$$B_z \wedge \min_{1 \leq j \leq K} B_z^{(j)} = \inf\{E_x : x \in \xi \cap R_z\}, \quad z \in \varepsilon \mathbb{Z}^d.$$  \hspace{1cm} (19)

By the above discussion $\mathbb{G}_s$ has the same law of the following graph $\bar{\mathbb{G}}$ built in terms of the marked point processes \eqref{eq:ordering} and \eqref{eq:ordering-s}. The vertex set of $\bar{\mathbb{G}}$ is given by $\{z \in \varepsilon \mathbb{Z}^d : B_z \wedge \min_{1 \leq j \leq K} B_z^{(j)} < +\infty\}$. The edges of $\bar{\mathbb{G}}$ are given by the unordered pairs $\{z, z'\}$ with $z \neq z'$ in the vertex set and

$$|z - z'| + 2 \max\{B_z \wedge \min_{1 \leq j \leq K} B_z^{(j)}, B_{z'} \wedge \min_{1 \leq j \leq K} B_{z'}^{(j)}\} \leq 1 - \alpha.$$  \hspace{1cm} (20)

\begin{align}
\{x, E_x \} : x \in \sigma\}, \\
\{x, E_x \} : x \in \xi^{(j)}\} \quad j = 1, 2, \ldots, K,
\end{align}
Due to (19) for each vertex $z$ of $\bar{G}$ we can fix a point $x(z) \in \xi \cap R_z$ such that $E_{x(z)} = B_z \wedge \min_{1 \leq j \leq K} B_z^{(j)}$. Hence, if $\{z, z'\}$ is an edge of $\bar{G}$, then $x(z)$ and $x(z')$ are defined and it holds $|z - z'| + 2 \max \{E_{x(z)}, E_{x(z')}\} \leq 1 - \alpha$. As $x(z) \in R_z$ it must be $|x(z) - z| \leq \sqrt{d} \varepsilon = \alpha/100$ and, similarly, $|x(z') - z'| \leq \alpha/100$. It then follows that $|x - y| + 2 \max \{E_x, E_y\} \leq 1$ where $x = x(z)$ and $y = x(z')$. This implies that $\{x, y\}$ is an edge of $\mathcal{G}(\lambda, \nu, 1)$ (recall Warning 3.1).

We extend Definition 3.6 to $\bar{G}$ (it is enough to replace $\mathbb{G}_s$ by $\bar{G}$ there). Due to the above discussion, if $z_1, z_2, \ldots, z_n$ is a LR crossing of the box $\Delta_L$ for $\bar{G}$, then we can extract from $x(z_1), x(z_2), \ldots, x(z_n)$ a LR crossing of the box $[-L - 1, L + 1]^d$ for $\mathcal{G}(\lambda, \nu, 1)$ (we use that $\varepsilon < 1$). Since disjointness is preserved, we deduce that $R_{L+1}(\mathcal{G}(\lambda, \nu, 1)) \supseteq \mathbb{R}_L(\bar{G})$. Due to this inequality Theorem 2 implies Theorem 1 (by changing the constants $c, c'$ when moving from Theorem 2 to Theorem 1).

3.2. Proof of Lemma 3.8 Let $\{(x, E_x) : x \in \sigma\}$ be a PPP($\lambda, \nu$) as in (15) and let $(B_z)_{z \in \varepsilon \mathbb{Z}^d}$ be the random field introduced in (17). We recall that $(B_z)_{z \in \varepsilon \mathbb{Z}^d}$ has the same law of $(A_z)_{z \in \varepsilon \mathbb{Z}^d}$. In particular, it is enough to prove that the graph $\bar{G}_1$ percolates a.s., where $\bar{G}_1$ is defined as $\mathbb{G}_2$ with $A_z$ replaced by $B_z$. Take $x \neq y$ in $\sigma$ such that

$$|x - y| + 2 \max \{E_x, E_y\} \leq u_\varepsilon. \quad (21)$$

Equivalently, $\{x, y\}$ is an edge of the graph $\mathcal{G}(\lambda, \nu, u_\varepsilon)$ built by means of the marked PPP $\{(x, E_x) : x \in \sigma\}$. Let $z(x)$ and $z(y)$ be the points in $\varepsilon \mathbb{Z}^d$ such that $x \in R_{z(x)}$ and $y \in R_{z(y)}$. Trivially, $|z(x) - x| \leq \varepsilon \sqrt{d}$, $|z(y) - y| \leq \varepsilon \sqrt{d}$, $B_{z(x)} \subseteq E_x$ and $B_{z(y)} \subseteq E_y$. Then from (21) and Definition 3.3 we get

$$|z(x) - z(y)| + 2 \max \{B_{z(x)}, B_{z(y)}\} \leq u_\varepsilon + 2 \varepsilon \sqrt{d} \leq 1 - 3 \alpha. \quad (22)$$

As a consequence, for each edge $\{x, y\}$ in $\mathcal{G}(\lambda, \nu, u_\varepsilon)$, either we have $z(x) = z(y)$ or we have that $\{z(x), z(y)\}$ is an edge of $\bar{G}_1$. Since $\mathcal{G}(\lambda, \nu, u_\varepsilon)$ a.s. percolates, due to the above observation we conclude that $\bar{G}_1$ a.s. percolates.

4. Basic geometrical objects in the discrete context

In the rest we will often write $\mathbb{P}(E_1, E_2, \ldots, E_n)$ instead of $\mathbb{P}(E_1 \cap E_2 \cap \cdots \cap E_n)$, also for other probabilities. Recall the Definition 3.5 of the graphs $\mathbb{G}_2 = (\mathbb{V}, E_2)$, $\mathbb{G} = (\mathbb{V}, E)$ and $\mathbb{G}_s = (\mathbb{V}_s, E_s)$. We introduce the following conventions:

- Given $x \in \mathbb{V}$ and $C \subseteq \mathbb{V}$ with $x \not\in C$, we say that $x$ is directly connected to $C$ inside $\mathbb{G}$ if there exists $y \in C$ such that $\{x, y\} \in E$.
- Given $A, B, C \subseteq \varepsilon \mathbb{Z}^d$, we say that “$A \leftrightarrow B$ in $\mathbb{G}$” if there exist $x_1, x_2, \ldots, x_k \in C \cap \mathbb{V}$ such that $x_1 \in A$, $x_k \in B$ and $\{x_i, x_{i+1}\} \in E$ for all $i : 1 \leq i < k$.
- Given a bounded set $A \subseteq \mathbb{R}^d$ we say that “$A \leftrightarrow \infty$ for $\mathbb{G}$” if there exists an unbounded path in $\mathbb{G}$ starting at some point in $A$.

Similar definitions hold for the graphs $\mathbb{G}_2 = (\mathbb{V}, E_2)$ and $\mathbb{G}_s = (\mathbb{V}_s, E_s)$.
Given Definition 4.1. For $m \leq n \in \mathbb{N}_+$, $z \in \varepsilon \mathbb{Z}^d$, $\sigma \in \{-1,1\}^d$, $J \in \{1,2,\ldots,d\}$ we define the following sets (see Figure 1-(left))

\[
B(m) := [-m,m]^d \cap \varepsilon \mathbb{Z}^d \text{ and } B(z,m) := z + B(m),
\]

\[
A(n) := \{x \in \varepsilon \mathbb{Z}^d : n - 1 < \|x\|_\infty \leq n\},
\]

\[
T(n) := \{x \in \varepsilon \mathbb{Z}^d : n - 1 < \|x\|_\infty \leq n, 0 \leq x_i \leq x_1 \forall i = 1,2,\ldots,d\},
\]

\[
T_{\sigma,J}(n) := \{x \in \varepsilon \mathbb{Z}^d : n - 1 < \|x\|_\infty \leq n, 0 \leq \sigma_i x_i \leq \sigma_J x_J \forall i = 1,2,\ldots,d\},
\]

\[
T(m,n) := (n + \varepsilon,n + \varepsilon + 2m] \times [0,n]^{d-1} \cap \varepsilon \mathbb{Z}^d.
\]

Note that $T_{1,1}(n) = T(n)$, where $1 := (1,1,\ldots,1)$. The following fact can be easily checked (hence we omit its proof):

**Lemma 4.2.** We have the following properties:

(i) $A(n) = \bigcup_{\sigma \in \{-1,1\}^d} \bigcup_{J=1}^d T_{\sigma,J}(n)$;

(ii) given $(\sigma,J)$ the map $\psi_{\sigma,J}(x_1,x_2,\ldots,x_d) := (y_1,y_2,\ldots,y_d)$, where

\[
y_k := \begin{cases} x_J \sigma_1 & \text{if } k = 1, \\ x_J \sigma_J & \text{if } k = J, \\ x_k \sigma_k & \text{otherwise}, \end{cases}
\]

is an isometry from $T(n)$ to $T_{\sigma,J}(n)$ and it is the identity when $\sigma = 1$ and $J = 1$.

**Definition 4.3.** Given $z \in \varepsilon \mathbb{Z}^d$ and $m \in \mathbb{N}_+$, we say that $B(z,m)$ is a seed if $B(z,m) \subset \mathcal{V}$ and $A_x \leq \alpha/100$ for all $x \in B(z,m)$.

**Definition 4.4.** Given $m \leq n \in \mathbb{N}_+$, $K(m,n)$ is given by the points $x \in T(n)$ which are directly connected inside $\mathcal{G}$ to a seed contained in $T(m,n)$. Equivalently, $K(m,n)$ is given by the points $x \in \mathcal{V} \cap T(n)$ such that, for some $z \in \varepsilon \mathbb{Z}^d$, the box $B(z,m) \subset T(m,n)$ is a seed and $\exists y \in B(z,m)$ with $\{x,y\} \in \mathcal{E}$.

**Lemma 4.5.** If $B(z,m)$ is a seed, then $B(z,m)$ is a connected subset in the graph $\mathcal{G}$.

**Proof.** Recall that $B(z,m) \subset \mathcal{V}$ since $B(z,m)$ is a seed. Let $x,y$ be points in $B(z,m)$ with $|x - y| = \varepsilon$. Since $\varepsilon = \alpha/100\sqrt{d}$, we get $|x - y| \leq \alpha/100$. 

![Figure 1](image-url)
By definition of seed, we have $|A_x|, |A_y| \leq \alpha/100$. Then trivially $|x - y| + 2\max\{A_x, A_y\} \leq 3\alpha/100$. By Definition 3.3 it holds $1 - 10\alpha \geq u_\ast > 0$, hence $\alpha < 0.1$ and therefore $3\alpha/100 < 1 - 2\alpha$. This proves that $\{x, y\} \in \mathbb{E}$ for any $x, y$ in $B(z, m)$ with $|x - y| = \varepsilon$. It is trivial to conclude.

\[\square\]

**Proposition 4.6.** Given $\eta \in (0, 1)$, there exist positive integers $m = m(\eta)$ and $n = n(\eta)$ such that $m > 2, 2m < n, 2m|n$ and

$$\mathbb{P}(B(m) \leftrightarrow K(m, n) \text{ in } B(n) \text{ for } \mathbb{G}) > 1 - \eta.$$  

(23)

5. PROOF OF PROPOSITION 4.6

Recall Definition 4.1. The following lemma and its proof are inspired by [9, Lemma 3] and its proof.

**Lemma 5.1.** Let $m$ and $n$ be positive integers such that $n > m$. Let $U_n$ be the set of points $x \in A(n)$ such that $B(m) \leftrightarrow x$ in $B(n)$ for $\mathbb{G}$ and

$$d(x, B(n)^c) + 2A_x \leq 1 - 3\alpha,$$

(24)

where $d(\cdot, \cdot)$ denotes the Euclidean distance. Then, for each integer $k$, it holds

$$\sum_{n=m+1}^{\infty} \mathbb{P}(|U_n| < k, B(m) \leftrightarrow \infty \text{ for } \mathbb{G}) < e^{c(d)\lambda, k}.$$  

(25)

for a positive constant $c(d)$ depending only on the dimension.

**Proof.** We claim that the event $\{B(m) \leftrightarrow \infty \text{ for } \mathbb{G}\} \implies \text{that } |U_n| \geq 1$. To prove our claim we observe that, since the edges in $\mathbb{G}$ have length at most $1 - 3\alpha$, the event $\{B(m) \leftrightarrow \infty \text{ for } \mathbb{G}\}$ implies that there exists $x \in A(n)$ such that $B(m) \leftrightarrow x$ in $B(n)$ for $\mathbb{G}$ and $\{x, y\} \in \mathbb{E}$ for some $y \in B(n)^c \cap \mathbb{V}$. Indeed, it is enough to take any path from $B(m)$ to $\infty$ for $\mathbb{G}$ and define $y$ as the first visited point in $B(n)^c$ and $x$ as the point visited before $y$. Note that the property $\{x, y\} \in \mathbb{E}$ implies (24) by (11). Hence $x \in U_n$. This concludes the proof of our claim. Due to the above claim we have

$$\mathbb{P}(|U_n| < k, B(m) \leftrightarrow \infty \text{ for } \mathbb{G}) \leq \mathbb{P}(1 \leq |U_n| < k).$$  

(26)

We now want to estimate $\mathbb{P}(U_{n+1} = \emptyset \mid 1 \leq |U_n| < k)$ from below (the result will be given in (28) below).

For each $x \in U_n$ we denote by $I_{n+1}(x)$ the set of points $y$ in $A(n+1)$ such that $|x - y| \leq 1 - 3\alpha$. We call $G_n$ the event that $\mathbb{V}$ has no points in $\cup_{x \in U_n} I_{n+1}(x)$. We now claim that $G_n \subset \{U_{n+1} = \emptyset\}$. To prove our claim let $z$ be in $U_{n+1}$. Then there is a path in $\mathbb{G}$ from $z$ to some point in $B(m)$ visiting only points in $B(n+1)$. We call $v$ the last point in the path inside $A(n+1) + x$ and the next point in the path. Then $x \in A(n)$ and all the points visited by the path after $x$ are in $B(n)$. Hence, $B(m) \leftrightarrow x$ in $B(n)$ for $\mathbb{G}$. Moreover, since $\{x, v\} \in \mathbb{E}$, property (24) is verified. Then $x \in U_n$ and $\mathbb{V}$ has some point (indeed $v$) in $I_{n+1}(x)$. In particular, we have shown that, if $U_{n+1} \neq \emptyset$, then $G_n$ does not occur, thus proving our claim.
Recall that the graph $G_2$ depends only on the random field $(A_z)_{z \in \mathbb{Z}^d}$ and that $\mathbb{P}(A_z = +\infty) = e^{-\lambda z \varepsilon}$ for any $z \in \varepsilon \mathbb{Z}^d$. We call $\mathcal{F}_n$ the $\sigma$-algebra generated by the random variables $A_z$ with $z \in B(n)$. Note that the set $\bigcup_{x \in U_n} I_{n+1}(x)$ and the event $\{1 \leq |U_n| < k\}$ are $\mathcal{F}_n$-measurable. Moreover, on the event $\{1 \leq |U_n| < k\}$, the set $\bigcup_{x \in U_n} I_{n+1}(x)$ has cardinality bounded by $c(d)k \varepsilon^{-d}$, where $c(d)$ is a positive constant depending only on $d$. By the independence of the $A_z$'s we conclude that that $\mathbb{P}$-a.s. on the event $\{1 \leq |U_n| < k\}$ it holds

$$\mathbb{P}(G_n | \mathcal{F}_n) = \mathbb{P}(A_z = +\infty \ \forall z \in \bigcup_{x \in U_n} I_{n+1}(x) | \mathcal{F}_n) \geq \mathbb{P}(A_0 = +\infty)^{c(d)k \varepsilon^{-d}} = e^{-c(d)\lambda k}.$$  

(27)

Hence, since $G_n \subset \{U_{n+1} = \emptyset\}$, by (27) we conclude that

$$\mathbb{P}(U_{n+1} = \emptyset | 1 \leq |U_n| < k) \geq \mathbb{P}(G_n | 1 \leq |U_n| < k) \geq \exp\{-c(d)\lambda k\}.$$  

(28)

As a byproduct of (26) and (28) we get

$$e^{-c(d)\lambda k} \mathbb{P}(|U_n| < k, B(m) \leftrightarrow \infty \text{ for } G_2) \leq e^{-c(d)\lambda k} \mathbb{P}(1 \leq |U_n| < k) \leq \mathbb{P}(U_{n+1} = \emptyset | 1 \leq |U_n| < k) \mathbb{P}(1 \leq |U_n| < k) = \mathbb{P}(U_{n+1} = \emptyset, 1 \leq |U_n| < k).$$  

(29)

Since the events $\{U_{n+1} = \emptyset, 1 \leq |U_n| < k\}$ are disjoint, we get (25). $\square$

We now present the analogous of [9, Lemma 4].

**Lemma 5.2.** Let $w := 2^d d$ and call $V_n$ the set of points $x \in T(n)$ satisfying (24) and such that $B(m) \leftrightarrow x$ in $B(n)$ for $G_2$. Then, for any $\ell \in \mathbb{N}$, it holds

$$\liminf_{n \to \infty} \mathbb{P}(|V_n| \geq \ell) \geq 1 - \mathbb{P}(B(m) \not\leftrightarrow \infty \text{ for } G_2)^{1/w}.$$  

(30)

**Proof.** Let $\sigma, J$ be as in Definition 4.4. If in the definition of $V_n$ we take $T_{\sigma,J}(n)$ instead of $T(n)$, then we call $V_{\sigma,J,n}$ the resulting set. Note that $V_{1,1,n} = V_n$. By Lemma 4.2(i) we get that $|U_n| \leq \sum_{(\sigma,J)} |V_{\sigma,J,n}|$, hence

$$\{|U_n| < w \ell\} \supset \cap_{(\sigma,J)} \{|V_{\sigma,J,n}| < \ell\}.$$  

(31)

By the FKG inequality and since each event $\{|V_{\sigma,J,n}| < \ell\}$ is decreasing, and by the isometries given in Lemma 4.2(ii), we have

$$\mathbb{P}(|U_n| < w \ell) \geq \prod_{(\sigma,J)} \mathbb{P}(|V_{\sigma,J,n}| < \ell) = \mathbb{P}(|V_n| < \ell)^w.$$  

The above bound implies that $\mathbb{P}(|V_n| \geq \ell) \geq 1 - \mathbb{P}(|U_n| < w \ell)^{1/w}$. On the other hand we have

$$\mathbb{P}(|U_n| < w \ell) \leq \mathbb{P}(|U_n| < w \ell, B(m) \leftrightarrow \infty \text{ for } G_2) + \mathbb{P}(B(m) \not\leftrightarrow \infty \text{ for } G_2)$$

(32)

and by Lemma 5.1 the first term in the r.h.s. goes to zero as $n \to \infty$, thus implying the thesis. $\square$

We can finally give the proof of Proposition 4.6.
Proof of Proposition 4.6. By Lemma 3.8, \( \mathbb{G}_z \) percolates \( \mathbb{P} \)-a.s., hence we can fix an integer \( m > 2 \) such that
\[
\mathbb{P}(B(m) \nleftrightarrow \infty \text{ for } \mathbb{G}_z) < (\eta/2)^w, \quad w := d2^d.
\]
Then, by Lemma 5.2, for any \( \ell \in \mathbb{N} \) we have
\[
\liminf_{n \to \infty} \mathbb{P}(|V_n| \geq \ell) \geq 1 - \mathbb{P}(B(m) \nleftrightarrow \infty \text{ for } \mathbb{G}_z)^{1/w} - 1 - \eta/2. \tag{33}
\]
We set \( \rho := \mathbb{P}(B(m) \text{ is a seed}) \in (0, 1) \) and fix an integer \( M \) large enough that \( (1 - \rho)^M < \eta/2 \). We set \( \ell := (2m)^{d-1}3^{d-1}M\varepsilon^{-d} \) and, by (33), we can fix \( n \) large enough that \( \mathbb{P}(|V_n| \geq \ell) > 1 - \eta/2, 2m < n \) and \( 2m \mid n \).

Since \( 2m \mid n \) we can partition \( [0,n]^{d-1} \) in non–overlapping \((d-1)\)–dimensional closed boxes \( D_i^\ell, \ i \in \mathcal{I} \), of side length \( 2m \) (by “non–overlapping” we mean that the interior parts are disjoint). We set \( D_i := D_i^\ell \cap \varepsilon \mathbb{Z}^d \). Note that \( T(n) \subset \bigcup_{i \in \mathcal{I}} (n-1, n] \times D_i \) and \( T(m, n) = \bigcup_{i \in \mathcal{I}} [(n + \varepsilon, n + \varepsilon + 2m] \cap \varepsilon \mathbb{Z}) \times D_i \).

By construction, any set \( (n-1, n] \times D_i \) contains at most \( (2m)^{d-1} \varepsilon^{-d} \) points \( x \in T(n) \). Since \( \ell = (2m)^{d-1}3^{d-1}M\varepsilon^{-d} \), the event \( \{|V_n| \geq \ell\} \) implies that there exists \( \mathcal{I}_x \subset \mathcal{I} \) with \( |\mathcal{I}_x| \geq 3^{d-1}M \) fulfilling the following property: for any \( k \in \mathcal{I}_x \) there exists \( x \in V_n \) with \( x \in (n-1, n] \times D_k \). We can choose univocally \( \mathcal{I}_x \) by defining it as the set of the first \((\text{w.r.t. the lexicographic order}) \) \( M \) indexes \( k \in \mathcal{I} \) satisfying the above property. We now thin \( \mathcal{I}_x \) since we want to deal with disjoint sets \( D_k \)’s. To this aim we observe that each \( D_k \) can intersect at most \( 3^{d-1} - 1 \) other sets of the form \( D_{k’} \). Hence, there must exists \( \mathcal{I}_x \subset \mathcal{I}_x \) such that \( D_k \cap D_{k’} = \emptyset \) for any \( k \neq k’ \) in \( \mathcal{I}_x \) and such that \( |\mathcal{I}_x| = M \) (again \( \mathcal{I}_x \) can be fixed deterministically by using the lexicographic order). We introduce the events
\[
G_k := \{(n + \varepsilon, n + \varepsilon + 2m] \cap \varepsilon \mathbb{Z}) \times D_k \text{ is a seed}\}. \tag{35}
\]
We claim that
\[
\mathbb{P}\left( \{|V_n| \geq \ell\} \cap (\bigcup_{k \in \mathcal{I}_x} G_k) \right) \geq 1 - \eta. \tag{36}
\]
To this aim we call \( \mathcal{F}_n \) the \( \sigma \)-algebra generated by the r.v.’s \( A_x \) with \( z \in B(n) \). We observe that the event \( \{|V_n| \geq \ell\} \) belongs to \( \mathcal{F}_n \), the set \( \mathcal{I}_x \) is \( \mathcal{F}_n \)-measurable and w.r.t. \( \mathbb{P}(\mathcal{F}_n) \) the events \( \{G_k : k \in \mathcal{I}_x\} \) are independent (recall that \( D_k \cap D_{k’} = \emptyset \) for any \( k \neq k’ \) in \( \mathcal{I}_x \) and each \( G_k \) has probability \( \rho := \mathbb{P}(B(m) \text{ is a seed}) \). Hence, \( \mathbb{P} \)-a.s. on the event \( \{|V_n| \geq \ell\} \) we can bound
\[
\mathbb{P}(\cup_{k \in \mathcal{I}_x} G_k | \mathcal{F}_n) \geq 1 - (1 - \rho)^M > 1 - \eta/2. \tag{37}
\]
Note that the last bound follows from our choice of \( M \). Since, by our choice of \( n \), \( \mathbb{P}(|V_n| \geq \ell) > 1 - \eta/2 \), we conclude that the l.h.s. of (36) is lower bounded by \((1 - \eta/2)^2 > 1 - \eta \). This concludes the proof of (36).

Let us now suppose that \( |V_n| \geq \ell \) and that the event \( G_k \) takes place for some \( k \in \mathcal{I}_x \). We claim that necessarily \( B(m) \leftrightarrow K(m, n) \) in \( B(n) \) for \( \mathbb{G} \). Note that the above claim and (36) lead to (23). We prove our claim. As discussed before (35), since \( k \in \mathcal{I}_x \) there exists \( x \in V_n \cap ((n-1, n] \times D_k) \). Let \( S \) be the seed \( (n + \varepsilon, n + \varepsilon + 2m] \cap \varepsilon \mathbb{Z}) \times D_k \). By definition of \( V_n \),
\[
d(x, B(n)^c) + 2A_x \leq 1 - 3\alpha \quad \text{and} \quad B(m) \leftrightarrow x \text{ in } B(n) \text{ for } \mathbb{G}_z.
\]
Call \( x’ \) the point in
\( \partial B(n) \) such that \( |x-x'| = d(x, B(n)^c) \). Note that \( x' \in \{n\} \times D_k \) as \( V_n \subset T(n) \). Let \( y := x' + \varepsilon e_1 \). Then \( y \in S \) and therefore \( A_y \leq \alpha/100 \) (as \( S \) is a seed) and \( |x'-y| = \varepsilon \leq \alpha/100 \). Then we have
\[
|x-y| + 2\max\{A_x, A_y\} \leq |x-x'| + |x'-y| + 2A_x + 2A_y
\]
\[
\leq d(x, B(n)^c) + \alpha/100 + 2A_x + \alpha/50 \leq 1 - 3\alpha/3 + 3\alpha/100 \leq 1 - 2\alpha .
\]
We have therefore shown that \( B(m) \leftrightarrow x \) in \( B(n) \) for \( G \) for some \( x \in T(n) \) with \( \{x, y\} \in E \) for some \( y \in S \). As a consequence, \( x \in K(m, n) \). Since \( G_2 \subset G \), we get that \( B(m) \leftrightarrow K(m, n) \) in \( B(n) \) for \( G \).

\section{The Fundamental Lemma}

Given a finite set \( R \subset \varepsilon \mathbb{Z}^d \), we define the non–random boundary set
\[
\partial R := \{y \in \varepsilon \mathbb{Z}^d \setminus R : d(y, R) \leq 1 - 2\alpha\} ,
\]
where \( d(\cdot, \cdot) \) denotes the Euclidean distance. To avoid ambiguity, we point out that in what follows the set \( \partial R \cap B(n) \) has to be thought of as \( (\partial R) \cap B(n) \) and not as \( \partial (R \cap B(n)) \).

Recall Definition 3.4 Since the support of \( \nu \) contains zero, the constant \( \gamma := \mathbb{P}(T_0^{(j)} \leq \alpha/100) \) is strictly positive.

\begin{lemma}
Fix \( \varepsilon' \in (0,1) \). Then there exist positive integers \( m \) and \( n \), with \( m > 2, 2m < n \) and \( 2m|n \), satisfying the following property.

Consider the following sets (see Figure 2):
- Let \( R \) be a finite subset of \( \varepsilon \mathbb{Z}^d \) satisfying
  \[
  B(m) \subset R , \quad (R \cup \partial R) \cap (T(n) \cup T(m, n)) = \emptyset .
  \]
- For any \( x \in R \cup \partial R \), let \( \Lambda(x) \) be a subset of \( \{1,2,\ldots,K\} \). We suppose that there exists \( k_* \in \{1,2,\ldots,K\} \) such that
  \[
  k_* \notin \cup_{x \in D} \Lambda(x) ,
  \]
where \( D \subset \varepsilon \mathbb{Z}^d \) is defined as
  \[
  D := (\partial R \cap B(n))
  \cup \{x \in R : \exists y \in \partial R \cap B(n) \text{ with } |x-y| \leq 1 - 2\alpha\} .
  \]

Consider the following events:
- Let \( H \) be any event in the \( \sigma \)-algebra \( \mathcal{F} \) generated by the random variables \( (A_x)_{x \in R \cup \partial R} \) and \( (T_j^{(j)})_{x \in R \cup \partial R, j \in \Lambda(x)} \).
- Let \( G \) be the event that there exists a string \( (z_0, z_1, z_2, \ldots, z_{\ell}) \) in \( \mathbb{V} \) such that
  \[
  \begin{align*}
  (P1) & \ z_0 \in R ; \\
  (P2) & \ z_1 \in \partial R \cap B(n) ; \\
  (P3) & \ z_2, \ldots, z_{\ell} \in B(n) \setminus (R \cup \partial R) ; \\
  (P4) & \ z_2, \ldots, z_{\ell} \text{ is a path in } G ; \\
  (P5) & \ z_{\ell} \in K(m, n) ; \\
  (P6) & \ T_{z_0}^{(k_*)} \leq \alpha/100 \text{ and } T_{z_1}^{(k_*)} \leq \alpha/100 ;
  \end{align*}
  \]
\end{lemma}
Figure 2. $\partial R$ is the very dark grey contour. $R$ is given by the light/dark grey region around the origin. $D$ is the dark grey subset of $R$.

(P7) $|z_0 - z_1| \leq 1 - 2\alpha$;
(P8) $|z_1 - z_2| + 2A_{z_2} \leq 1 - 2\alpha$.

Then $\mathbb{P}(G \mid H) \geq 1 - \varepsilon'$.

We point out that the above properties (P6), (P7), (P8) (which can appear a little exotic now) will be crucial to derive the $G_\ast$–connectivity issue stated in Lemma 7.3 in Section 7. Indeed, although $(z_0, z_1, z_2, \ldots, z_\ell)$ could be not a path in $G$, one can prove that it is a path in $G_\ast$ (in Lemma 7.3 we will state and prove the $G_\ast$–connectivity property in the form relevant for our applications).

6.1. Proof of Lemma 6.1. Recall that $\gamma := \mathbb{P}(T_0^{(j)} \leq \alpha/100) > 0$. We can fix a positive constant $c(d) \leq 2$ such that the ball $\{y \in \mathbb{R}^d : |y| \leq 2\}$ contains at most $c(d)\varepsilon^{-d}$ points of $\varepsilon \mathbb{Z}^d$. We then choose $t$ large enough that $(1 - \gamma^2)(te^{d/c(d)})^{-1} \leq \varepsilon'/2$. Afterwards we choose $\eta > 0$ small enough so that $(1 - p)^{-t}\eta \leq \varepsilon'/2$, where

$$p := \mathbb{P}(A_x < +\infty) = 1 - \exp\{-\lambda_x \varepsilon^d\} < 1.$$  \hspace{1cm} (43)

Then we take $m = m(\eta)$ and $n = n(\eta)$ as in Proposition 4.6. In particular, (23) holds and moreover

$$[1 - (1 - p)^{-t}\eta][1 - (1 - \gamma^2)(te^{d/c(d)})^{-1}] \geq (1 - \varepsilon'/2)^2 > 1 - \varepsilon'.$$  \hspace{1cm} (44)

Remark 6.2. As $\eta \leq \varepsilon'/2$, from (23) we get that

$$\mathbb{P}(B(m) \leftrightarrow K(m, n) \text{ in } B(n) \text{ for } \mathbb{G}) > 1 - \varepsilon'.$$  \hspace{1cm} (45)

This will be used in other sections.

Lemma 6.3. In the same context of Lemma 6.1 let

$V_R := \{x \in \partial R \cap B(n) : \exists y \in B(n) \setminus (R \cup \partial R) \text{ such that} \}

|x - y| + 2A_y \leq 1 - 2\alpha \text{ and }$

$$\{y\} \leftrightarrow K(m, n) \text{ in } B(n) \setminus (R \cup \partial R) \text{ for } \mathbb{G} \}.$$

Then we have (recall (43))

$$\mathbb{P}(|V_R| > t) \geq 1 - (1 - p)^{-t}\eta.$$  \hspace{1cm} (46)
We postpone the proof of Lemma 6.3 to Subsection 6.2.

**Remark 6.4.** The random set $V_R$ depends only on $A_x$ with $x \in B(n) \setminus (R \cup \partial R)$ and $A_x$ with $x \in T(m, n)$. Indeed, to determine $K(m, n)$, one needs to know the seeds inside $T(m, n)$.

Given $x \in \partial R$ we define $x*$ as the minimal (w.r.t. lexicographic order) point $y \in R$ such that $|x - y| \leq 1 - 2\alpha$. Note that $x*$ exists for any $x \in V_R$ since $V_R \subset \partial R$. Let us show that $F \subset G$, where

$$F := \{ \exists x \in V_R \text{ with } T_{x^*}^{(k_*)} \leq \alpha/100, T_{x^*}^{(k_*)} \leq \alpha/100 \}.$$

To this aim, suppose the event $F$ to be fulfilled and take $x \in V_R$ with $T_{x^*}^{(k_*)} \leq \alpha/100$ and $T_{x^*}^{(k_*)} \leq \alpha/100$. Since $x \in V_R$, by definition of $V_R$ there exists $y \in B(n) \setminus (R \cup \partial R)$ such that $|x - y| + 2A_y \leq 1 - 2\alpha$ and there exists a path $(y, z_3, z_4, \ldots, z_\ell)$ inside $G$ connecting $y$ to $K(m, n)$ with vertexes in $B(n) \setminus (R \cup \partial R)$. We set $z_0 := x, z_1 := x, z_2 := y$. Then the event $G$ is satisfied by the string $(z_0, z_1, \ldots, z_\ell)$. This proves that $F \subset G$.

Since $F \subset G$ we can estimate

$$\mathbb{P}(G \mid H) \geq \mathbb{P}(|V_R| > t, F \mid H) = \sum_{B \subset \partial R \cap B(n): |B| > t} \mathbb{P}(V_R = B, F_B \mid H),$$

where

$$F_B := \{ \exists x \in B \text{ with } T_{x^*}^{(k_*)} \leq \alpha/100, T_{x^*}^{(k_*)} \leq \alpha/100 \}.$$

The event $F_B$ is determined by the random variables $\{T_{x^*}^{(k_*)}\}_{x \in D}$. In particular (cf. Remark 6.4) the event $\{V_R = B\} \cap F_B$ is determined by

$$\begin{cases} T_{x^*}^{(k_*)} & \text{with } x \in D, \\ A_x & \text{with } x \in B(n) \setminus (R \cup \partial R) \text{ and with } x \in T(m, n). \end{cases}$$

Since by assumption $H$ is $\mathcal{F}$–measurable, and due to conditions (40) and (41), we conclude that the event $\{V_R = B\} \cap F_B$ and $H$ are independent. Hence

$$\mathbb{P}(V_R = B, F_B \mid H) = \mathbb{P}(V_R = B, F_B).$$

In particular, coming back to (47), we have

$$\mathbb{P}(G \mid H) \geq \sum_{B \subset \partial R \cap B(n): |B| > t} \mathbb{P}(V_R = B, F_B).$$

To deal with $\mathbb{P}(V_R = B, F_B)$ we observe that the events $\{V_R = B\}$ and $F_B$ are independent (see Remark 6.4), hence we get

$$\mathbb{P}(V_R = B, F_B) = \mathbb{P}(V_R = B)\mathbb{P}(F_B).$$

It remains to lower bound $\mathbb{P}(F_B)$. We first show that there exists a subset $\tilde{B} \subset B$ such that

$$|\tilde{B}| \geq |B|\varepsilon^d/c(d) - 1$$

and such that all points of the form $x$ or $x*$, with $x \in \tilde{B}$, are distinct. We recall that the positive constant $c(d)$ has been introduced at the beginning of
Subsection 6.1. To build the above set \( \hat{B} \) we recall that \( B \subset \partial R \) and that, for any \( x \in \hat{B} \), it holds \( |x - x_*| \leq 1 - 2\alpha \) and \( x_* \in R \). As a consequence, given \( x, x' \in B \), \( x_* \) and \( x'_* \) are distinct if \( |x - x'| \geq 2 \) and moreover any point of the form \( x_* \), with \( x \in B \) cannot coincide with a point in \( B \). Hence it is enough to exhibit a subset \( \hat{B} \subset B \) satisfying (50) and such that all points in \( \hat{B} \) have reciprocal distance at least 2. We know that the ball \( \mathbb{B} \) of radius 2 contains at most \( c(d)\varepsilon^{-d} \) points of \( \varepsilon \mathbb{Z}^d \). The set \( \hat{B} \) is then built as follows: choose a point \( a_1 \) in \( B_1 := B \) and define \( B_2 := B_1 \setminus (a_1 + \mathbb{B}) \), then choose a point \( a_2 \in B_2 \) and define \( B_3 := B_2 \setminus (a_2 + \mathbb{B}) \) and so on until possible (each \( a_k \) can be chosen as the minimal point w.r.t. the lexicographic order). We call \( \hat{B} := \{a_1, a_2, \ldots, a_s\} \) the set of chosen points. Since \( |B_k| \geq |B| - (k-1)c(d)\varepsilon^{-d} \), we get that \( s = |\hat{B}| \) is bounded from below by the maximal integer \( k \) such that \( |B| > (k-1)c(d)\varepsilon^{-d} \), i.e. \( |B|\varepsilon^d/c(d) > k-1 \). Hence, \( s = |\hat{B}| \geq \lceil |B|\varepsilon^d/c(d) \rceil \).

By the above observations, \( \hat{B} \) fulfills the desired properties.

Using \( \hat{B} \) and independence, we have
\[
\mathbb{P}(F_B) = 1 - \mathbb{P}(\bigcap_{x \in B} \{T^{(k_x)}_x \leq \alpha/100, T^{(k_x)}_{x_*} \leq \alpha/100\})^c \\
\geq 1 - \mathbb{P}(\bigcap_{x \in \hat{B}} \{T^{(k_x)}_x \leq \alpha/100, T^{(k_x)}_{x_*} \leq \alpha/100\})^c \\
= 1 - \prod_{x \in \hat{B}} (1 - \mathbb{P}(T^{(k_x)}_x \leq \alpha/100) \mathbb{P}(T^{(k_x)}_{x_*} \leq \alpha/100)) = 1 - (1 - \gamma^2)^{|\hat{B}|}.
\]

As a byproduct of (48), (49), (50) and (51) and finally using (46) in Lemma 6.3 we get
\[
\mathbb{P}(G \mid H) \geq \sum_{B \subset \partial R \cap B(n): \mid B \mid > t} \mathbb{P}(V_R = B) \left( 1 - (1 - \gamma^2)^{|\hat{B}|} \right) \\
\geq \left( 1 - (1 - \gamma^2)^{(te^d/c(d)-1)} \right) \sum_{B \subset \partial R \cap B(n): \mid B \mid > t} \mathbb{P}(V_R = B) = \left( 1 - (1 - \gamma^2)^{(te^d/c(d)-1)} \right) \mathbb{P}(|V_R| > t) \\
\geq \left( 1 - (1 - \gamma^2)^{(te^d/c(d)-1)} \right) [1 - (1 - p)^{-t}\eta].
\]

Finally, using (44) we conclude the proof of Lemma 6.1.

6.2. Proof of Lemma 6.3. Suppose that \( B(m) \leftrightarrow K(m, n) \) in \( B(n) \) for \( G \). Take a path \((x_0, x_1, \ldots, x_k)\) from \( B(m) \) to \( K(m, n) \) inside \( G \) with all vertexes \( x_i \) in \( B(n) \). Recall that \( K(m, n) \subset T(n) \) and \( R \cup \partial R \) is disjoint from \( T(n) \) by (40). In particular, \( R \cup \partial R \) is disjoint from \( K(m, n) \). Since \( B(m) \subset R \), the path starts at \( R \). Let \( x_r \) be the last point of the path contained in \( R \). Since \( R \) is disjoint from \( K(m, n) \) and \( x_k \in K(m, n) \), it must be \( r < k \). Necessarily, \( x_{r+1} \in \partial R \). Call \( x_\ell \) the last point of the path contained in \( \partial R \). It must be \( \ell < k \) since \( \partial R \) is disjoint from \( K(m, n) \supset x_k \). We claim that \( x_\ell \in V_R \) and \( A_{x_\ell} < +\infty \).
To prove our claim we observe that the last property follows from the fact that all points $x_0,x_1,\ldots,x_k$ are in $\mathbb{V}$. Recall that these points are also in $B(n)$. Hence $x_t \in \partial R \cap B(n)$. Since $\{x_t,x_{t+1}\} \in E$, we have $|x-y| + 2A_y \leq 1 - 2\alpha$ with $x := x_t$ and $y := x_{t+1}$. Finally, it remains to observe that $(x_{t+1},\ldots,x_k)$ is a path connecting $x_{t+1}$ to $x_k \in K(m,n)$ in $B(n) \setminus (R \cup \partial R)$ for $\mathbb{G}$. Hence, $x_t \in V_R$.

We have proved that if $B(m) \leftrightarrow K(m,n)$ in $B(n)$ for $\mathbb{G}$, then $V_R$ contains at least a vertex of $\mathbb{V}$. As a byproduct with (23) (see the first paragraph of Subsection 6.1) we therefore have

$$\eta > \mathbb{P}(B(m) \not\leftrightarrow K(m,n) \text{ in } B(n) \text{ for } \mathbb{G}) \geq \mathbb{P}(V_R \cap \mathbb{V} = \emptyset),$$

(53)

On the other hand, we can bound

$$\mathbb{P}(V_R \cap \mathbb{V} = \emptyset) \geq \mathbb{P}(V_R \cap \mathbb{V} = \emptyset, |V_R| \leq t).$$

(54)

Note that $V_R$ and $(A_x)_{x \in \partial R}$ are independent (see Remark 6.4). Hence

$$\mathbb{P}(V_R \cap \mathbb{V} = \emptyset, |V_R| \leq t) = \sum_{B \subseteq \partial R: |B| \leq t} \mathbb{P}(V_R = B, A_x = +\infty \forall x \in B)$$

$$= \sum_{B \subseteq \partial R: |B| \leq t} \mathbb{P}(V_R = B)\mathbb{P}(A_x = +\infty \forall x \in B)$$

$$= \sum_{B \subseteq \partial R: |B| \leq t} \mathbb{P}(V_R = B)(1-p)^{|B|}$$

(55)

$$\geq \sum_{B \subseteq \partial R: |B| \leq t} \mathbb{P}(V_R = B)(1-p)^t = \mathbb{P}(|V_R| \leq t)(1-p)^t.$$ 

By combining (53), (54) and (55) we get that $\eta \geq \mathbb{P}(|V_R| \leq t)(1-p)^t$, which is equivalent to (46).

7. The sets $E[C,B,B',i]$ and $F[C,B,B',i]$

In the next sections we will iteratively construct random subsets of $\varepsilon\mathbb{Z}^d$ sharing the property to be connected in $\mathbb{G}_s$. We isolate here the fundamental building procedure.

**Definition 7.1.** Given three sets $C,B,B' \subset \varepsilon\mathbb{Z}^d$ and given $i \in \{1,2,\ldots,K\}$, we define the random subsets $E,F \subset \varepsilon\mathbb{Z}^d$ as follows:

- $E$ is given by the points $z_1$ in $B \cap \partial C$ such that $T_1^{(i)} \leq \alpha/100$ and there exists $z_0 \in C$ with $|z_0 - z_1| \leq 1 - 2\alpha$ and $T_0^{(i)} \leq \alpha/100$;
- $F$ is given by the points $z \in B'$ such that there exists a path $(z_2,\ldots,z_k)$ inside $\mathbb{G}$ where $z_k = z$, all points $z_2,\ldots,z_k$ are in $B' \setminus (C \cup \partial C)$ and $|z_1 - z_2| + 2A_{z_2} \leq 1 - 2\alpha$ for some $z_1 \in E$.

To stress the dependence from $C,B,B',i$, we will also write $E[C,B,B',i]$ and $F[C,B,B',i]$. 
Lemma 7.2. Let $E = E[C, B, B', i]$ and $F = F[C, B, B', i]$ be as in Definition 7.1. Let $\hat{E}, \hat{F} \subset \varepsilon \mathbb{Z}^d$.

(i) If $\hat{E}, \hat{F}$ do not satisfy

$$\hat{E} \cap \hat{F} = \emptyset, \quad \hat{E} \subset (B \cap \partial C), \quad \hat{F} \subset B' \setminus (C \cup \partial C), \quad (56)$$

then the event $\{E = \hat{E}\} \cap \{F = \hat{F}\}$ is impossible.

(ii) If $\hat{E}, \hat{F}$ satisfy (56), then the event $\{E = \hat{E}\} \cap \{F = \hat{F}\}$ belongs to the $\sigma$–algebra generated by

- $T^{(i)}_z$ with $z \in (B \cap \partial C) \cup D$, where
  $$D := \{x \in C : \exists y \in B \cap \partial C \text{ with } |x - y| \leq 1 - 2\alpha\};$$
- $A_z$ with $z$ belonging to some of the following sets:
  $$\hat{F}, \quad (B' \setminus (C \cup \partial C)) \cap \partial \hat{F}, \quad (B' \setminus (C \cup \partial C)) \cap \partial \hat{E}.$$  

(iii) As a consequence, given $R \subset \varepsilon \mathbb{Z}^d$, the event $\{E \cup F = R\}$ belongs to the $\sigma$–algebra generated by $\{T^{(i)}_z : z \in (B \cap \partial C) \cup D\} \cup \{A_z : z \in R \cup \partial R\}$.

Proof. Item (i) is trivial and Item (iii) follows from Items (i) and (ii). Let us assume (56) and prove Item (ii). We claim that $\{E = \hat{E}\} \cap \{F = \hat{F}\}$ equals $\{E = \hat{E}\} \cap W$, where $W$ is the event that

(a) for any $z \in \hat{F}$ there are points $z_2, \ldots, z_{k-1}, z_k = z$ in $\hat{F}$ such that $|z_i - z_{i+1}| + 2\max\{A_{z_i}, A_{z_{i+1}}\} \leq 1 - 2\alpha$ for $i = 2, \ldots, k - 1$ and such that $|z_1 - z_2| + 2A_{z_2} \leq 1 - 2\alpha$ for some $z_1 \in \hat{E}$;

(b) if $z \in B' \setminus (C \cup \partial C)$ is such that $\exists z' \in \hat{E}$ with $|z - z'| + 2A_z \leq 1 - 2\alpha$, then $z \in \hat{F}$;

(c) for any $z \in (B' \setminus (C \cup \partial C)) \cap \partial \hat{F}$ there is no $z' \in \hat{E}$ such that $|z - z'| + 2\max\{A_z, A_{z'}\} \leq 1 - 2\alpha$.

Before proving our claim, we observe that it allows to conclude the proof of the lemma. Indeed, as the point $z$ appearing in Item (b) must be in $\partial \hat{E}$, the event $\{E = \hat{E}\} \cap W$ belongs to the $\sigma$–algebra in Item (ii) of the lemma due to the explicit description given above.

It remains to derive our claim. Due to Item (a), the event $\{E = \hat{E}\} \cap W$ implies that $\{E = \hat{E}\} \cap \{F \supset \hat{F}\}$. On the other hand, suppose that the event $\{E = \hat{E}\} \cap \{F \supset \hat{F}\}$ takes place and let $z \in F$. By Definition 7.1 there exists a path $(z_2, \ldots, z_k)$ inside $G$ where $z_k = z$, all points $z_2, \ldots, z_k$ are in $B' \setminus (C \cup \partial C)$ and $|z_i - z_{i+1}| + 2A_{z_{i+1}} \leq 1 - 2\alpha$ for some $z_i \in \hat{E}$. By Item (b), $z_2 \in \hat{F}$. Let $j$ be the maximal index in $\{2, 3, \ldots, k\}$ such that $z_2, z_3, \ldots, z_j \in \hat{F}$. Suppose that $j < k$. As $\{z_j, z_{j+1}\} \in E$, we get that $z_{j+1} \in \partial \hat{F}$. Since $z_j \in \hat{F}$, $z_{j+1} \in (B'\setminus(C\cup\partial C)) \cap \partial \hat{F}$ and $\{z_j, z_{j+1}\} \in E$, we get a contradiction with Item (c). Then, it must be $j = k$, thus implying that $z = z_j$ and therefore $z \in \hat{F}$. Up to now, we have proved that $\{E = \hat{E}\} \cap W \subset \{E = \hat{E}\} \cap \{F = \hat{F}\}$. We observe that, given $z, z_1, z_2, \ldots, z_k$ as in Definition 7.1, it must be $z_2, \ldots, z_k \in F$. This
observation and the above Items (a), (b), (c) imply the opposite inclusion 
\( \{ E = \hat{E} \} \cap \{ F = \hat{E} \} \subset \{ E = \hat{E} \} \cap W. \)

\[ \square \]

**Lemma 7.3.** Given sets \( C, B, B' \subset \varepsilon \mathbb{Z}^d \) and an index \( i \in \{ 1, 2, \ldots, K \} \), we define \( E := E[C, B, B', i] \) and \( F := F[C, B, B', i] \). If \( C \subset \mathbb{V}_* \) is connected in the graph \( \mathbb{G}_* = (\mathbb{V}_*, \mathbb{E}_*) \), then the set \( C' := C \cup E \cup F \) is contained in \( \mathbb{V}_* \) and is connected in the graph \( \mathbb{G}_*. \)

**Proof.** Recall Definition 3.5. If \( z \in E \), then \( T_z(i) < +\infty \) and therefore \( z \in \mathbb{V}_*. \)
If \( z \in F \), then \( z \in \mathbb{V} \) (by definition of \( F \)) and therefore \( z \in \mathbb{V}_*. \) This implies that \( E, F \subset \mathbb{V}_* \), hence \( C' \subset \mathbb{V}_*. \)

Since \( C \) is connected in \( \mathbb{G}_* \) and since \( \mathbb{G} \subset \mathbb{G}_* \) (in particular the string \((z_2, \ldots, z_k)\) appearing in the definition of \( F \) is a path in \( \mathbb{G}_* \)), to prove the connectivity of \( C' \) in \( \mathbb{G}_* \) it is enough to show the following:

(i) if \( z_0, z_1 \in \mathbb{V}_* \) satisfy \( T_z(i) \leq \alpha/100 \), \( T_{z_1}(i) \leq \alpha/100 \) and \( |z_0 - z_1| \leq 1 - 2\alpha \), then \( \{ z_0, z_1 \} \in \mathbb{E}_* \);

(ii) if \( z_1, z_2 \in \mathbb{V}_* \) satisfy \( T_z(i) \leq \alpha/100 \) and \( |z_1 - z_2| + 2A_{z_2} \leq 1 - 2\alpha \), then \( \{ z_1, z_2 \} \in \mathbb{E}_* \).

Using the assumptions of Item (i) we get
\[
|z_1 - z_0| + 2 \max \{ A_{z_1} \wedge \min_{1 \leq j \leq K} T_z^{(j)} \}, A_{z_0} \wedge \min_{1 \leq j \leq K} T_z^{(j)} \} \leq |z_1 - z_0| + 2 \max \{ T_z(i), T_z(i) \} \leq |z_1 - z_0| + 2T_z(i) + 2T_z(i) \leq 1 - 2\alpha + \alpha/50 + \alpha/50 < 1 - \alpha. \tag{57}
\]

Using the assumptions of Item (ii) we get
\[
|z_1 - z_2| + 2 \max \{ A_{z_1} \wedge \min_{1 \leq j \leq K} T_z^{(j)} \}, A_{z_2} \wedge \min_{1 \leq j \leq K} T_z^{(j)} \} \leq |z_1 - z_2| + 2 \max \{ T_z(i), A_{z_2} \} \leq |z_1 - z_2| + 2A_{z_2} + 2T_z(i) \leq 1 - 2\alpha + \alpha/50 < 1 - \alpha. \tag{58}
\]

The thesis then follows from Definition 3.5 \( \square \)

8. Success–events \( S_0, \ldots, S_{11} \)

The construction presented in this section is inspired by \[9\].

We let \( N := n + m + \varepsilon \). From now on \( \varepsilon' \in (0, 1) \) is fixed and we choose \( m, n \) as in Lemma 6.1. The precise value of \( \varepsilon' \) will be chosen in Section 11.

Let \( e_1, e_2, \ldots, e_d \) be the canonical basis of \( \mathbb{R}^d \). We denote by \( L_1, L_2, L_3, L_4 \) the isometries of \( \mathbb{R}^d \) given respectively by \( 1, \theta, \theta^2, \theta^3 \), where \( 1 \) is the identity and \( \theta \) is the unique rotation such that \( \theta(e_1) = e_2, \theta(e_2) = -e_1, \theta(e_i) = e_i \) for all \( i = 3, \ldots, d \). We define \( B'_0 \subset \varepsilon \mathbb{Z}^d \) as
\[
B'_0 := B(n) \cup \left( \bigcup_{j=1}^4 L_j(T(m, n)) \right). \tag{59}
\]

For \( j = 1, 2, 3, 4 \) we call \( K^{(j)}(m, n) \) the random set of points defined similarly to \( K(m, n) \) (cf. Definition 4.4) but with \( T(m, n) \) and \( T(n) \) replaced by \( L_j(T(m, n)) \) and \( L_j(T(n)) \), respectively.
Figure 3. Left: the set $C_1$ when $S_1$ occurs. Right: The set $C_2$ when $S_1 \cap S_2$ occurs. Points in $C_1$ correspond to circles, while points in $C_2 \setminus C_1$ correspond to triangles.

**Definition 8.1.** We define $S_0$ as the success-event that $B(m)$ is a seed. We define $C_1$ as the set of points $x \in B'_0$ such that

$$\{x\} \leftrightarrow B(m) \text{ in } B'_0 \text{ for } G.$$  

Furthermore, we denote by $S_1$ the success-event that $C_1$ contains a point of $K^{(j)}(m,n)$ for each $j = 1, 2, 3, 4$.

We refer to Figure 3–(left) for an example of the set $C_1$ when $S_1$ occurs. We note that the event $S_0$ implies that $B(m) \subset V$, hence $B(m) \subset C_1$.

**Remark 8.2.** If the event $S_0$ occurs, then $C_1$ is a connected subset of $G$ (and therefore of $G_*$) by Lemma 4.5.

If the event $S_1$ takes place, for $i = 1, 2, 3, 4$ we define $b^{(i)}$ as the minimal (w.r.t. the lexicographic order) point $z$ in $\varepsilon\mathbb{Z}^d$ such that $B(z,m)$ is a seed contained in $C_1 \cap L_i(T(m,n))$. We point out that such a seed exists by Lemma 4.5 and the definition of $S_1$. If $S_i$ ($1 \leq i \leq 4$) does not take place, we set $b^{(i)} := 0$ (just to complete the definition, this case will be irrelevant). It is simple to check that, when $S_1$ takes place,

$$b_1^{(1)} = N \text{ and } b_j^{(1)} \in [m, n - m] \text{ for } 2 \leq j \leq d,$$

where $b_a^{(1)}$ denotes the $a$–th coordinate of $b^{(1)}$. Similar formulas hold for $b^{(i)}$, $i = 2, 3, 4$.

Below, for $i = 5, \ldots, 10$, we will iteratively define points $b^{(i)}$. Moreover, for $i \in \{1, 2, \ldots, 10\} \setminus \{7\}$, we will iteratively define sets $T_i(n)$ and $T_i(m,n)$ obtained from $T(n)$ and $T(m,n)$ by an $i$–parametrized orthogonal map. Apart the case $i = 7$, many objects will be defined similarly. Hence, we isolate some special definitions to which we will refer in what follows. We stress that we collect these generic definitions below, but we will apply them only when describing the construction step by step in the next subsections.
(i)-Definition 8.3. Given \( b^{(i)} \), \( T_i(n) \) and \( T_i(m,n) \), we define \( K_i(m,n) \) as the set of points \( x \in b^{(i)} + T_i(n) \) which are directly connected inside \( \mathcal{G} \) to a seed contained in \( b^{(i)} + T_i(m,n) \). Moreover, we define \( B'_i := b^{(i)} + (B(n) \cup T_i(m,n)) \).

(i)-Definition 8.4. We set

\[
E_i := E[C_i, B(b^{(i)}, n), B'_i, i], \\
F_i := F[C_i, B(b^{(i)}, n), B'_i, i], \\
C_{i+1} := C_i \cup E_i \cup F_i.
\]

(i)-Definition 8.5. We call \( S_{i+1} \) the success event that \( C_{i+1} \) contains at least one vertex in \( K_i(m,n) \).

(i)-Definition 8.6. We say that property \( p_i \) is satisfied if the sets \( C_i \cup \partial C_i \) and \( b^{(i)} + (T_i(n) \cup T_i(m,n)) \) are disjoint.

In several steps below we will claim without further comments that property \( p_i \) is satisfied. This property will correspond to the second property in \( \text{(40)} \) in the applications of Lemma 6.1 in Section 9, which (when not immediate) will be checked in Section 9 and Appendices A.0.1, A.0.2.

Remark 8.7. If \( S_{i+1} \) occurs, then \( C_{i+1} \) contains a point \( x \in b^{(i)} + T_i(n) \) which is directly connected inside \( \mathcal{G} \) to a seed \( B(z, m) \) contained in \( b^{(i)} + T_i(m,n) \). Let us suppose that also property \( p_i \) in Definition 8.6 is satisfied. Then \( (C_i \cup E_i) \subset (C_i \cup \partial C_i) \) does not intersect \( b^{(i)} + T_i(n) \), thus implying that \( x \in F_i \).

Since the above seed \( B(z, m) \) is contained in \( b^{(i)} + T_i(m,n) \), which is contained in \( B'_i \setminus (C_i \cup \partial C_i) \) due to property \( p_i \), by Lemma 4.3 and Definition 7.7 we conclude that \( F_i \subset C_{i+1} \) contains the above seed \( B(z, m) \).

We now continue with the construction of increasing clusters and success-events. Figure 4 will be useful to locate objects.

8.1. Case \( 1 \leq i \leq 4 \). We define

\[
T^*(m,n) := f(T(m,n)) \quad \text{and} \quad T^*(n) := f(T(n)),
\]

where \( f : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is the isometry \( f(x_1, x_2, \ldots, x_d) := (x_1, -x_2, \ldots, -x_d) \) (see Figure 1). For \( 1 \leq i \leq 4 \), we define \( T_i(n) := L_i(T^*(n)) \) and \( T_i(m,n) = L_i(T^*(m,n)) \). We iteratively apply (i)–Definition 8.3, (i)–Definition 8.4, (i)–Definition 8.5 and (i)–Definition 8.6 for \( i = 1, 2, 3, 4 \). In particular, this defines the sets \( E_1, F_1, C_2, \ldots, E_4, F_4, C_5 \) and the success-events \( S_2, \ldots, S_5 \).

Definition 8.8. We say that the origin \( 0 \in \mathbb{Z}^d \) is occupied if the event \( \bigcap_{i=0}^5 S_i \) takes place.

Proposition 8.9. If \( 0 \) is occupied, then the sets \( C_1, C_2, \ldots, C_5 \) are connected in \( \mathcal{G}_4 \). Moreover, \( P(0 \text{~is~occupied} \mid S_0) \geq 1 - 8\varepsilon' \).

We postpone the proof of the above proposition to Section 9.

If the origin is non occupied, then there is no interest to proceed with further extensions of \( C_5 \) and we stop our construction. Hence, from now on we assume
Figure 4. Colored small boxes are the seeds $B(m)$ and $B(b^{(i)}, m)$, while bigger boxes are given by $B(n)$ and $B(b^{(i)}, n)$.

that 0 is occupied. In order to shorten the presentation, we will define geometric objects only in the successful cases relevant to continue the construction (in the other cases, the definition can be chosen arbitrarily).

When the origin is occupied the set $C_5$ intersects the box $B(2Nv, N)$ with $v = \pm e_1, \pm e_2$. Indeed, for $1 \leq i \leq 4$, one can iteratively prove that property $p_i$ holds due to (60) and the analogous formulas for $b^{(2)}$, $b^{(3)}$, $b^{(4)}$. Hence, by Remark 8.7, the event $S_0 \cap \cdots \cap S_{i+1}$ implies that $F_i \subset C_{i+1}$ contains a seed inside $b^{(i)} + L_i(T^*(m, n))$ (see Figure 3–(right) for $i = 1$). One can easily check that $b^{(i)} + L_i(T^*(m, n)) \subset B(2Nl(e_1), N)$. For example, by (60), if $x \in b^{(1)} + L_1(T^*(m, n))$, then $x_1 \in [2N - m, 2N + m] \subset 2N + [-N, N]$ and $x_j \in [b_j^{(1)} - n, b_j^{(1)}] \subset [m - n, m - n - m] \subset [-N, N]$ for $2 \leq j \leq d$.

We fix a unitary vector, that we take equal to $e_1$ without loss of generality, and we explain how we attempt to extend $C_5$ in the direction $e_1$. First we define $b^{(5)}$ as the minimal point $z \in \mathbb{Z}^d$ such that $B(z, m)$ is a seed contained in $C_5 \cap (b^{(1)} + T^*(m, n))$. By the above discussion, $b^{(1)} + T^*(m, n) \subset B(2Ne_1, N)$. By (60) and since $b_j^{(5)} \in [b_j^{(1)} - n, b_j^{(1)}]$ for $j \neq 1$, we get for $i = 5$

$$b_i^{(i)} = (i - 3)N, \quad b_j^{(i)} \in [-n + m, n - m] \text{ for } j \neq 1.$$  \hfill (62)

8.2. Case $i = 5$. We assume that $S_0 \cap \cdots \cap S_5$ occurs.

Given $a \in \mathbb{R}^d$, $g(|a|) : \mathbb{R}^d \to \mathbb{R}^d$ is the isometry 

$$g(x|a) := (x_1, -\text{sgn}(a_2)x_2, \ldots, -\text{sgn}(a_d)x_d)$$  \hfill (63)
and \(\text{sgn}(\cdot)\) is the sign function, with the convention that \(\text{sgn}(0) = +1\). We set \(T_5(m,n) := g(T(m,n)|b^{(5)})\) and \(T_5(n) := g(T(n)|b^{(5)})\). We apply (i)–Definition 8.3, (i)–Definition 8.4, (i)–Definition 8.5 and (i)–Definition 8.6 for \(i = 5\). In particular this defines \(E_5, F_5, C_6, S_6\). It is simple to check that property \(p_5\) is satisfied. If also \(S_6\) occurs, by Remark 8.7 we can define \(b^{(6)}\) as the minimal point in \(\varepsilon\mathbb{Z}^d\) such that \(B(z,m)\) is a seed contained in \(C_6 \cap (b^{(5)} + T_5(m,n))\).

Let us localize some objects. Due to Claim A.1 in Appendix A, we define \(T_i, B_i\) such that \(T_i B_i\) occurs, by Remark 8.7 we can define \(b^{(6)}\) as minimal point in \(\varepsilon\mathbb{Z}^d\) such that \(B(z,m)\) is a seed contained in \(C_6 \cap (b^{(5)} + T_5(m,n))\).

Let us localize some objects. Due to Claim A.1 in Appendix A, we define \(T_i, B_i\) such that \(T_i B_i\) occurs, by Remark 8.7 we can define \(b^{(6)}\) as minimal point in \(\varepsilon\mathbb{Z}^d\) such that \(B(z,m)\) is a seed contained in \(C_6 \cap (b^{(5)} + T_5(m,n))\).

Due to the above bounds and since \(b^{(6)}_1 = 3N\), we get (62) for \(i = 6\).

8.3. Case \(i = 6\). We assume that the event \(S_0 \cap \cdots \cap S_6\) occurs.

We define \(T_6(m,n) := g(T(m,n)|b^{(6)})\) and \(T_6(n) := g(T(n)|b^{(6)})\). We apply (i)–Definition 8.3, (i)–Definition 8.4, (i)–Definition 8.5 and (i)–Definition 8.6 for \(i = 6\). In particular, this defines \(E_6, F_6, C_7, S_7\). Property \(p_6\) is satisfied. If also \(S_7\) occurs, by Remark 8.7 we can define \(b^{(7)}\) as minimal point in \(\varepsilon\mathbb{Z}^d\) such that \(B(z,m)\) is a seed in \(b^{(6)} + T_6(m,n)\).

Let us localize some objects. Due to Claim A.1 in Appendix A, we define \(T_i, B_i\) such that \(T_i B_i\) occurs, by Remark 8.7 we can define \(b^{(6)}\) as minimal point in \(\varepsilon\mathbb{Z}^d\) such that \(B(z,m)\) is a seed contained in \(C_6 \cap (b^{(5)} + T_5(m,n))\).

8.4. Case \(i = 7\). We assume that the event \(S_0 \cap \cdots \cap S_7\) occurs. The idea now is to connect the cluster \(C_7\) to seeds adjacent to the remaining three faces of the cube \(b^{(7)} + B(n)\) in directions \(e_1\) and \(\pm e_2\). To this aim we set

\[
\begin{align*}
\hat{T}_1(n) &:= g(T(n)|b^{(7)}) \quad \hat{T}_1(m,n) := g(T(m,n)|b^{(7)}) \\
\hat{T}_2(n) &:= (h \circ \theta)(\hat{T}_1(n)) \quad \hat{T}_2(m,n) := (h \circ \theta)(\hat{T}_1(m,n)) \\
\hat{T}_3(n) &:= (h \circ \theta^3)(\hat{T}_1(n)) \quad \hat{T}_3(m,n) := (h \circ \theta^3)(\hat{T}_1(m,n))
\end{align*}
\]

where \(\theta\) is the rotation introduced before (59) and the map \(h\) is defined as \(h(x_1, x_2, \ldots, x_n) := (|x_1|, x_2, \ldots, x_n)\).

By these definitions, points in \(b^{(7)} + \hat{T}_j(m,n)\) have first coordinate lower bounded by \(4N = b^{(7)}\). We also set

\[
B'_j := b^{(7)} + (B(n) \cup \bigcup_{j=1,2,3} \hat{T}_j(m,n))
\]

For \(j = 1, 2, 3\) we call \(K^{(6+j)}(m,n)\) the set of points \(x \in b^{(7)} + \hat{T}_j(n)\) which are directly connected inside \(G\) to a seed contained in \(b^{(7)} + \hat{T}_j(m,n)\).

We apply only (i)–Definition 8.4 with \(i = 7\) to define \(E_7, F_7, C_8\).
Definition 8.10. We call $S_8$ the success-event that $C_8$ contains at least one vertex inside $K^{(t+2)}$ for all $j = 1, 2, 3$. When $S_8$ occurs, for $j = 1, 2, 3$ we define $b_j^{(t+2)}$ as the minimal point in $\varepsilon \mathbb{Z}^d$ such that $b_j^{(t+2)} + B(m)$ is a seed in $C_8 \cap (b_j^{(t)} + \tilde{T}_j(m, n))$. The existence of such a seed can be derived by the same arguments of Remark 8.7 since $C_7 \cup \partial C_7$ and $b_j^{(t)} + (\tilde{T}_j(n) \cup \tilde{T}_j(m, n))$ are disjoint for $j = 1, 2, 3$ (as shown in Section 9.6).

Let us localize the above objects. Due to (62) for $i = 7$ and reasoning as in (64), we get that (62) holds also for $i = 8$. For $i = 9, 10$ we have

$$b_i^{(i)} \in 4N + [m, n - m], \quad \begin{cases} b_9^{(9)} = b_9^{(7)} + N, \\ b_9^{(10)} = b_9^{(7)} - N, \\ b_i^{(i)} \in [-n + m, n - m] \text{ for } j \geq 3 \end{cases} \tag{66}$$

(for $j \geq 3$ one has to argue as in (64)). Moreover, due to Claim A.2 in Appendix A, if $S_0 \cap \cdots \cap S_8$ occurs, then $b^{(i)} + \tilde{T}_i(m, n) \subset B(5N, m, n)\cap B(4N, m, n) \cap B(4N, m, n)$ and $b^{(i)} + \tilde{T}_i(m, n) \subset B(4N, m, n) \cap B(4N, m, n)$. In particular, the same inclusions hold for the seeds $b^{(8)} + B(m)$, $b^{(9)} + B(m)$ and $b^{(10)} + B(m)$, respectively.

8.5. Cases $i = 8, 9, 10$. We assume that $S_0 \cap \cdots \cap S_8$ occurs. We define

$$T_8(n) := g(T(n) | b^{(8)}) \quad T_8(m, n) := g(T(m, n) | b^{(8)})$$
$$T_9(n) := g'(T_1(n) | b^{(9)}) \quad T_9(m, n) := g'(T_2(m, n) | b^{(9)})$$
$$T_{10}(n) := g'(T_3(n) | b^{(10)}) \quad T_{10}(m, n) := g'(T_4(m, n) | b^{(10)})$$

where $g'(x | a) := (-\text{sgn}(a_1)x_1, x_2, -\text{sgn}(a_3)x_3, \ldots, -\text{sgn}(a_d)x_d)$.

By (i)–Definition 8.3 for $i = 8, 9, 10$ we define $K_i(m, n)$ and $B_i'$. By (i)–Definition 8.4 for $i = 8, 9, 10$ we define the sets $E_8$, $F_8$, $C_9, \ldots, E_10$, $F_{10}$, $C_{11}$. By (i)–Definition 8.5 for $i = 8, 9, 10$ we define the success-events $S_9$, $S_{10}$, $S_{11}$.

Let $i = 8, 9, 10$. We apply (i)–Definition 8.6. It is simple to check that property $p_i$ holds. If also $S_{i+1}$ holds, then (cf. Remark 8.7) $C_{i+1}$ contains a seed $B(z, m)$ inside $b^{(i)} + T_i(m, n)$.

Definition 8.11. Knowing that the origin $0 \in \mathbb{Z}^d$ is occupied, we say that the site $e_1$ is linked to 0 and occupied if the event $\cap_{i=1}^{11} S_i$ takes place.

Proposition 8.12. If $0$ is occupied and $e_1$ is linked to 0 and occupied, then the sets $C_0, C_2, \ldots, C_{11}$ are connected in $\mathbb{G}_*$. Moreover,

$$\mathbb{P}(e_1 \text{ is linked to 0 and occupied} \mid S_0 \cap \{0 \text{ is occupied}\}) \geq 1 - 8\varepsilon'. \tag{67}$$

We postpone the proof of the above proposition to Section 9.

9. Proof of Propositions 8.9 and 8.12

By iteratively applying Lemma 7.3 and using Remark 8.2 as starting point, we get that $C_2, \ldots, C_{11}$ are connected subsets in $\mathbb{G}_*$, if the associated success-events are satisfied. The lower bounds $\mathbb{P}(0 \text{ is occupied} \mid S_0) \geq 1 - 8\varepsilon'$ and (67)
follow from the inequalities

\[ \mathbb{P}(S_{i+1}|S_0, S_1, \ldots, S_i) \geq \begin{cases} 
1 - \varepsilon' & \text{for } i \in \{1, \ldots, 10\} \setminus \{0, 7\}, \\
1 - 4\varepsilon' & \text{for } i = 0, \\
1 - 3\varepsilon' & \text{for } i = 7,
\end{cases} \tag{68} \]

by applying the chain rule and the Bernoulli’s inequality (i.e. \((1 - \delta)^k \geq 1 - \delta k\) for all \(k \in \mathbb{N}\) and \(\delta \in [0, 1]\)). Apart the case \(i = 0\), the proof of (68) can be obtained by applying Lemma 6.1. Below we will treat in detail the cases \(i = 0, 1, 2\). For the other cases we will give some comments, and show the validity of conditions (40) and (41) in Lemma 6.1. In what follows we will introduce points \(\tilde{b}_1, \tilde{b}_2, \ldots\). We stress that the subindex \(k\) in \(\tilde{b}_k\) does not refer to the \(k\)-th coordinate. We write \((b_k)_a\) for the \(a\)-th coordinate of \(\tilde{b}_k\).

9.1. Proof of (68) with \(i = 0\). We want to show that \(\mathbb{P}(S_i|S_0) \geq 1 - 4\varepsilon'\).

Since \(S_0\) and \(S_1\) are increasing events w.r.t. \(\preceq\), by the FKG inequality (see Section 3) we have \(\mathbb{P}(S_1|S_0) \geq \mathbb{P}(S_1)\). To show that \(\mathbb{P}(S_1) \geq 1 - 4\varepsilon'\), we note that the event \(W_j := \{B(m) \leftrightarrow K^{(j)}(m, n)\} \cap B(n)\) for \(\mathcal{G}\) implies that \(C_1\) contains a point of \(K^{(j)}(m, n)\). Hence, \(\cap_{j=1}^4 W_j \subset S_1\) and therefore (see Remark 6.2) \(\mathbb{P}(S_1) \leq \mathbb{P}\left(\bigcup_{j=1}^4 W_j^{(j)}\right) \leq 4\varepsilon'\).

9.2. Proof of (68) with \(i = 1\). We want to show that \(\mathbb{P}(S_2|S_0, S_1) \geq 1 - \varepsilon'\).

Lemma 9.1. Given \(B(m) \subset R_1 \subset B'_0\), the event \(S_0 \cap \{C_1 = R_1\}\) is determined by the random variables \(\{A_x\}_{x \in R_1 \cup \partial R_1}\).

Proof. The claim is trivially true for the event \(S_0\). It is therefore enough to show that, if \(S_0\) takes place, then the event \(\{C_1 = R_1\}\) is equivalent to the following: (i) for any \(x \in R_1\) there is a path from \(x\) to \(B(m)\) inside \(R_1\) for \(\mathcal{G}\) and (ii) any \(x \in \partial R_1 \cap B'_0\) is not directly connected to \(R_1\) in \(\mathcal{G}\), i.e. there is no \(y \in R_1\) such that \(\{x, y\} \in \mathcal{E}\). Trivially the event \(\{C_1 = R_1\}\) implies (i) and (ii). On the other hand, let us suppose that (i) and (ii) are satisfied, in addition to \(S_0\). Then (i) implies that \(R_1 \subset C_1\). Take, by contradiction, \(x \in C_1 \setminus R_1\). By definition of \(C_1\) there exists a path from \(x\) to \(B(m)\) in \(B'_0\) for \(\mathcal{G}\). Since \(x \notin R_1\) and \(B(m) \subset R_1\), there exists a last point \(x'\) in \(R_1^c\) visited by the path. Since the path ends in \(B(m) \subset R_1\), after \(x'\) the path visits another point \(y\) which must belong to \(R_1\). Hence we have \(\{x', y\} \in \mathcal{E}\) (and therefore \(x' \in \partial R_1 \cap B'_0\)) and \(y \in R_1\), thus contradicting (ii). \(\square\)

We proceed with the proof that \(\mathbb{P}(S_2|S_0, S_1) \geq 1 - \varepsilon'\) by applying Lemma 6.1. Recall that \(T_1(n) = T^*(n)\), \(T_1(m, n) = T^*(m, n)\) and recall Definition 8.3 of \(K_1(m, n)\). We can write

\[ \mathbb{P}(S_2|S_0, S_1) = \sum_{R_1, b_1} \mathbb{P}(S_2|S_0, S_1, C_1 = R_1, b^{(1)} = \tilde{b}_1) \mathbb{P}(C_1 = R_1, b^{(1)} = \tilde{b}_1|S_0, S_1), \tag{69} \]
where in the above sum $R_1 \subset \varepsilon \mathbb{Z}^d$ and $\tilde{b}_1 \in \varepsilon \mathbb{Z}^d$ are taken such that $\mathbb{P}(S_0, S_1, C_1 = R_1, b^{(1)} = \tilde{b}_1) > 0$. We now apply Lemma 6.1 (with the origin there replaced by $\tilde{b}_1$) to lower bound $\mathbb{P}(S_2|H_1)$ by $1 - \varepsilon'$, where $H_1 := \{ S_0, S_1, C_1 = R_1, b^{(1)} = \tilde{b}_1 \}$.

We first check condition (40). Note that $B(m) \subset R_1 \subset B_0'$ and $B(\tilde{b}_1, m) \subset R_1 \cap T(m, n)$ as $\mathbb{P}(H_1) > 0$. Hence we have $(R_1 \cup \partial R_1) \subset (B_0' \cup \partial B_0')$. We point out that, given $x \in B_0' \cup \partial B_0'$, it must be $x_1 \leq n + \varepsilon + 2m + 1 - 2\alpha$. On the other hand, given $x \in \tilde{b}_1 + (T^*(n) \cup T^*(m, n))$, it must be $x_1 \geq 2n + m + \varepsilon - 1$. As $2m < n$ and $2m|n$, we have $n \geq 4m$ and therefore $n > m + 2$. Hence $x$ cannot belong to both sets. In particular, we have the analogous of (40), i.e., $B(\tilde{b}_1, m) \subset R_1$ and $(R_1 \cup \partial R_1) \cap (\tilde{b}_1 + (T^*(n) \cup T^*(m, n))) = \emptyset$. Condition (41) is trivially satisfied by taking $\Lambda_1(x) := \emptyset$ for all $x \in R_1 \cup \partial R_1$ and $k_s = 1$.

We now prove that $H_1$ belongs to the $\sigma$–algebra $\mathcal{F}_1$ generated by $(A_x)_{x \in R_1 \cup \partial R_1}$. Due to Lemma 9.1, the event $S_0 \cap \{ C_1 = R_1 \}$ is determined by $\{ A_x \}_{x \in R_1 \cup \partial R_1}$. If the event $S_0 \cap \{ C_1 = R_1 \}$ takes place, then the event $S_1 \cap \{ b^{(1)} = \tilde{b}_1 \}$ becomes equivalent to the following: (1) $B(\tilde{b}_1, m)$ is a seed and, for any other seed $B(z, m) \subset R_1 \cap T(m, n)$, $\tilde{b}_1$ is lexicographically smaller than $z$; (2) for any $j = 2, 3, 4$ the set $R_1$ contains a point $x \in L_j(T(n))$ directly connected for $G$ to a seed contained in $R_1 \cap L_j(T(m, n))$. Note that in Item (2) we have used Lemma 4.5, thus implying that if a seed $B(z, m)$ is directly connected for $G$ to a point in $R_1$, then any point of $B(z, m)$ is connected for $G$ to $x$, and therefore $B(z, m) \subset R_1$ as $C_1 = R_1$. The above properties (1), (2) can be checked when knowing $\{ A_x \}_{x \in R_1 \cup \partial R_1}$. Hence, $H_1$ belongs to the $\sigma$–algebra $\mathcal{F}_1$.

Due to the above observations, we can apply Lemma 6.1 with conditional event $H_1$, $\tilde{b}_1$ as new origin, $R_1$ as new set $R$, $\Lambda_1(x) := \emptyset$ for any $x \in R_1 \cup \partial R_1$ as new function $\Lambda(x)$ and $k_s := 1$. We get that $\mathbb{P}(G_1|H_1) \geq 1 - \varepsilon'$, where $G_1$ is the event corresponding to the event $G$ appearing in Lemma 6.1 (replacing also $K(m, n)$ by $K_1(m, n)$). To show that $\mathbb{P}(S_2|H_1) \geq 1 - \varepsilon'$, and therefore that $\mathbb{P}(S_2|S_0, S_1) \geq 1 - \varepsilon'$ by (69), it is enough to show that $G_1 \cap H_1 \subset S_2 \cap H_1$. To this aim let us suppose that $G_1 \cap H_1$ takes place. Let (P1),...,(P8) be the properties entering in the definition of $G$ in Lemma 6.1, when replacing $R$, $B(n)$ and $K(m, n)$ by $R_1$, $B(\tilde{b}_1, n)$ and $K_1(m, n)$, respectively. To get the thesis it is enough to show that $z \in C_2$ since $z \in K_1(m, n)$ by (P5). Note that by $H_1$, (P1), (P2), (P6) and (P7) we have that $z_0 \in C_1$ and $z_1 \in E_1$, while by $H_1$, (P3), (P4) and (P8) we get that $z_2, \ldots, z_t \in F_1$. Since $C_2 := C_1 \cup E_1 \cup F_1$, we have that $z \in C_2$.

9.3. Generalized notation. In order to define objects once and for all, given $2 \leq i \leq 10$ and given sets $R_1, R_2, \ldots, R_i$ and points $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_i$ we set

$$H_i := \left( \bigcap_{k=0}^{i} S_k \right) \cap \left( \bigcap_{k=1}^{i} \{ C_k = R_k \} \right) \cap \left( \bigcap_{k=1}^{i} \{ b^{(k)} = \tilde{b}_k \} \right), \quad (70)$$

$$\Lambda_i(x) := \{ k : 1 \leq k \leq i - 1, \ x \in B(\tilde{b}_k, n + 1) \}, \quad \forall x \in R_i \cup \partial R_i, \quad (71)$$

$$\mathcal{F}_i := \sigma \left( \{ A_x \}_{x \in R_i \cup \partial R_i}, \{ T_x^{(k)} \}_{x \in R_i \cup \partial R_i, k \in \Lambda_i(x)} \right). \quad (72)$$
Note that the above definitions cover also the objects $H_1, \Lambda_1, F_1$ introduced in Section 9.2.

For later use, it will be convenient to write also $H_i[R_1, \ldots, R_i; \tilde{b}_1, \ldots, \tilde{b}_i]$ (instead of $H_i$) to stress the dependence on $R_1, \ldots, R_i, b_1, \ldots, b_i$.

9.4. Proof of \textit{(68)} with $i = 2$. We want to show that $\mathbb{P}(S_3 | S_0, S_1, S_2) \geq 1 - \varepsilon'$. To this aim we write

$$\mathbb{P}(S_3 | S_0, S_1, S_2) = \mathbb{P}(S_3 | H_2[R_1, R_2; \tilde{b}_1, \tilde{b}_2]) \mathbb{P}(H_2[R_1, R_2; \tilde{b}_1, \tilde{b}_2]),$$

with $i = 2$. We want to show that $\mathbb{P}(S_3 | H_2[R_1, R_2; \tilde{b}_1, \tilde{b}_2]) \geq 1 - \varepsilon'$. To do this, we apply Lemma 6.1 with $k_s := 2$ and $B(m), B(n), T(n, m), K(m, n), R, H$ and $\Lambda(x)$ replaced by $B(\tilde{b}_2, m), B(\tilde{b}_2, n), \tilde{b}_2 + T_2(n), \tilde{b}_2 + T_2(m, n), K_2(m, n), R_2, H_2$ and $\Lambda_2(x)$, respectively (cf. (71)).

Let us check condition (40). As $\mathbb{P}(H_2) > 0$ we have $B(\tilde{b}_2, m) \subset R_2 \cap T_2(m, n)$ and $R_1 \subset R_2 \subset B(m) \cup B_1$. As a consequence, $(R_2 \cup \partial R_2) \subset (B(m) \cup \partial B(m) \cup B_1 \cup \partial B_1)$. We point out that $x_2 \leq n + \varepsilon + 2m + 1$ for $x \in B(m) \cup \partial B(m)$ and $x_2 \leq 2n - m + 1$ if $x \in B_1 \cup \partial B_1$. (cf. (60)). On the other hand, given $x \in \tilde{b}_2 + T_2(n) \cup T_2(m, n)$, it must be $x_2 \geq 2n + m + \varepsilon - 1$. Since $n > m + 2$, $x$ cannot belong to both sets and therefore $(R_2 \cup \partial R_2) \cap (\tilde{b}_2 + (T_2(n) \cup T_2(m, n))) = \emptyset$.

Condition (41) is trivially satisfied. It remains to check that $H_2 \in F_2$. To this aim we write $H_2 = H_1 \cap S_2 \cap \{C_2 = R_2\} \cap \{b^{(2)} = \tilde{b}_2\}$, where $H_1 = H_1[R_1, \tilde{b}_1]$. We already know that $H_1 \in F_1$ (see Section 9.2). By Definition 8.4 $H_1 \cap \{C_2 = R_2\}$ equals $H_1 \cap W$, where $W := \{R_1 \cup E[R_1, B(\tilde{b}^{(1)}), B_1, 1] \cup F[R_1, B(\tilde{b}^{(1)}), B_1, 1] = R_2\}$, and this event belongs to the $\sigma$-algebra $F_2$ due to Lemma 7.2. By Remark 6.1 and since property p1 is trivially satisfied, when $S_2$ occurs any seed $B(z, m)$ in $b^{(1)} + T_1(m, n)$ directly connected for $G$ to a point $x \in K_1(m, n) \cap C_2$ is itself contained in $C_2$. In particular, under the event $\{C_2 = R_2\} \cap \{b^{(1)} = \tilde{b}_1\}$, the event $S_2$ equals the event that $R_2$ contains a point $x \in \widetilde{b}_1 + T_1(n)$ and a seed $B(z, m)$ contained in $R_1 \cap T_2(m, n)$, which are directly connected for $G$, and this last event belongs to $F_2$. Finally, we observe that, in the above intersection of events leading to $H_2$, we can also replace $\{b^{(2)} = \tilde{b}_2\}$ by the event that $B(\tilde{b}_2, m)$ is a seed and, for any other seed $B(z, m)$ contained in $R_1 \cap T_2(m, n)$, $\tilde{b}_2$ is lexicographically smaller than $z$. Recalling that $B(\tilde{b}_2, m) \subset R_2$, this concludes the proof that $H_2 \in F_2$.

We can finally apply Lemma 6.1 with the replacements mentioned above. Calling $G_3$ the event of Lemma 6.1 adapted to our context, we conclude that $\mathbb{P}(G_3 | H_2) \geq 1 - \varepsilon'$. To conclude it is enough to observe that $G_3 \cap H_2 \subset S_3 \cap H_2$. (cf. the arguments presented at the end of Section 9.2).

9.5. Proof of \textit{(68)} with $i \geq 3$ and $i \neq 7$. The proof follows the main arguments presented for the case $i = 2$. One has to condition similarly to (73).
and afterwards apply Lemma 6.1 with \(k_* := i\) and \(B(m), B(n), T(n), T(n, m), K(m, n), R, H\) and \(\Lambda(x)\) replaced by \(B(\hat{b}_i, m), B(\hat{b}_i, n), \hat{b}_i + T_i(n), \hat{b}_i + T_i(m, n), K_i(m, n), R_i, H_i\) and \(\Lambda_i(x)\), respectively. The fact that \(H_i \in \mathcal{F}_i\) can be obtained as for the case \(i = 2\) by writing \(H_i = H_{i-1} \cap S_i \cap \{C_i = R_i\} \cap \{b^{(i)} = \hat{b}_i\}\) and using the iterative result that \(H_{i-1} \in \mathcal{F}_{i-1}\) together with Lemma 7.2. Condition (41) is satisfied also since in step \(i = 7\) the new random variables \(T^{(j)}_k\) involved in the construction will have index \(j = 7\). To check (40) is straightforward but cumbersome and we refer to Appendix A.0.1 for the details.

9.6. Proof of (68) with \(i = 7\). For \(j = 1, 2, 3\), we define the event \(W_j := \{C_8 \text{ contains at least one vertex inside } K^{(6+j)}\}\). Then \(S_8 = \bigcap_{j=1}^3 W_j\) and

\[
\mathbb{P}(S_8 | S_1, S_2, \cdots, S_7) \geq 1 - \sum_{j=1}^3 \mathbb{P}(W_j^c | S_1, S_2, \cdots, S_7). \tag{74}
\]

Hence, we only need to show that \(\mathbb{P}(W_j | S_1, S_2, \cdots, S_7) \geq 1 - \varepsilon'\). We use Lemma 6.1 and the same strategy used in the previous steps. Recall (70), (71), (72). We have to lower bound by \(1 - \varepsilon'\) the conditional probability \(\mathbb{P}(W_j | H_7)\) when \(\mathbb{P}(H_7) > 0\). To this aim we apply Lemma 6.1 with \(R := R_7, \Lambda(x) := \Lambda_7(x)\) and with \(B(n), T(n), T(m, n), k_*\) replaced by \(B(\hat{b}_7, n), \hat{b}_7 + T_7(n), \hat{b}_7 + T_7(m, n)\) and 7, respectively. The validity of (41) is trivial. To check (40) is straightforward but cumbersome and we refer to Appendix A.0.2 for the details.

10. Extended construction by success-events

In Section 8 we have explained how to check if the origin is occupied and (in affirmative case) if \(e_1\) is occupied and linked to the origin. These will be the two basic steps in the algorithmic construction presented in Section 11. There we will start with a point, which we take now equal to the origin, and we will iteratively define an increasing set \(X\) of occupied and linked points, by means of success-events, until the algorithm stops. In general when \(X \neq \emptyset\) we will provide in Section 11 a rule to decide if we have to stop the algorithm or not. If the algorithm is not stopped, the rule will also indicate how to choose points \(v \not\in X\) and \(w \in X\) such that \(\|v - w\|_1 = 1\). Roughly, the algorithm is structured as follows. First we check if the origin is occupied according to Definition 8.8. If it is not occupied, then we end the algorithm with output \(X := \{0\}\), otherwise we temporary set \(X := \{0\}\) and apply the above rule. Suppose the algorithm is not stopped by the rule. In this case, as \(X = \{0\}\), necessarily \(w = 0\) and \(v\) is nearest neighbor to the origin. We therefore check if \(v\) is occupied and linked to the origin according to Definition 8.11 (with \(e_1\) replaced by \(v\)). If this happens, then we update the value of \(X\) by temporary setting \(X := \{0, v\}\), otherwise we do not update the set \(X\). At this point, we apply again the above rule and proceed as before continuing iteratively in this way. We stress that the rule will definitely stop the algorithm.
We point out that, in Section 8, in order to decide if the origin is occupied or not we reveal only random variables associated to points in \( \bigcup_{k=0}^{4} B'_k \). We note that the region \( \bigcup_{k=0}^{4} B'_k \) is included in \( B(2N+m) \) (cf. Fig. 4). If the origin is occupied, in order to decide if e.g. \( e_1 \) is occupied and linked to the origin, we reveal only random variables associated to points in \( \bigcup_{k=5}^{10} B'_k \). The region \( \bigcup_{k=5}^{10} B'_k \) is included in \( B(4Ne_1, 2N+n) \) (cf. Fig. 4 and (62) for \( i = 7 \)), while \( B'_5 \) is included in \( B(b_5, N) \subset B(2Ne_1, 2n) \subset B(2Ne_1, 2N) \) (cf. (62) with \( i = 5 \)). When extending the construction by the algorithm mentioned above and described in Section 11, since we explore uniformly bounded regions, by taking \( K \) large enough in Definition 3.4 we can iteratively apply Lemma 6.1 assuring condition (41) to be fulfilled simply by using some index \( k^* \in \{1, 2, \ldots, K\} \) not already used in the region under exploration.

We point out another relevant issue when adapting the steps of Section 8 to the extended construction of Section 11. In order to check if \( e_1 \) is occupied and linked to the origin, in Section 8 we have considered also the success-events \( S_9, S_{10}, S_{11} \). These success-events are thought in order to assure the presence of seeds localized in \( b^{(i)} + T_i(m,n) \) for \( i = 8, 9, 10, \) which would allow to continue the construction in direction \( e_1, e_2 \) and \( -e_2 \), respectively. In the extended construction, when we need to check if a vertex \( v \) is occupied and linked to some vertex \( w \), we remove the success-events associated to seeds which would direct the construction towards a box already explored. For example, suppose that 0 is occupied and \( e_1 \) is occupied and linked to 0. Suppose that the rule requires now to check if \( e_2 \) is occupied and linked to 0. We do this by success-events similar to the events \( S_2, S_3, \ldots, S_{11} \) of Section 8, now in direction \( e_2 \). Suppose now that the rule requires to check if \( e_2 + e_1 \) is occupied and link to \( e_2 \). We do this by success-events similar to the events \( S_2, S_3, \ldots, S_{10} \) of Section 8. Note that the analogous of \( S_{11} \) has been removed since the region around \( 4Ne_1 \) has already been explored.

In Section 11 after constructing the set \( X \), we construct iteratively other sets \( X' \) by a similar procedure. In order to lower bound the conditional probability as in (68) one can anyway apply Lemma 6.1 as done in Section 9. We also point out that in Section 11 we first check the occupation of the starting points of the \( X' \)-type sets (the points analogous to the origin for \( X \)) and afterwards proceed with the construction described above. The final result is the same.

At the end, similarly to Propositions 8.9 and 8.12, conditioned to the previous construction, the probability that the first point in \( X' \) is occupied is lower bounded by \( 1 - 8\varepsilon' \), and that a point \( v \) is occupied and linked to a given point \( w \) of the built set \( X' \) is lower bounded by \( 1 - 8\varepsilon' \).

11. Proof of Theorem 2 in Section 3

Having recovered results on the renormalized lattice similar to the ones in [9], the proof of Theorem 2 follows the main strategy developed in [13 Section 4] with suitable modifications and extensions.

Let \( p_c(2) \) be the critical probability for the 2-dimensional site percolation. We take \( \varepsilon' \) small enough that \( 1 - 8\varepsilon' \geq \frac{3}{4} > p_c(2) \). We first show that it
is enough to deal with 2-dimensional slices. To this aim recall that $\Delta_L = [-L - 2, L + 2] \times [-L, L]^{d-1}$. We introduce the set $V(a, r) := a + [-r, r]^{d-2}$, we denote by $\mathbb{L}$ the set $4N\mathbb{Z}^{d-2}$ and, for each $z \in \mathbb{L}$, we consider the slice $\Delta(z) := ([L - 2, L + 2] \times [-L, L] \times V(z, 4N)) \cap \mathbb{Z}^d$.

Note that, when varying $z \in \mathbb{L}$, the above slices are disjoint and that $\Delta_L$ contains at least $|2L/8N|^{d-2} \geq c_0L^{d-2}$ slices of the above form.

We denote by $R_L$ the maximal number of vertex-disjoint LR crossings of $\Delta_L$ in $\mathbb{G}_s$ which are included in the slice $\Delta(0)$. We claim that there exist positive constants $c_1, c_2$ such that, for $L$ large enough, it holds

$$\mathbb{P}(R_L \geq c_1L) \geq 1 - e^{-c_2L}.$$  

(75)

Let us first show that (75) implies Theorem 2. To this aim we assume (75). By translation invariance and independence (cf. Definition 3.4) the number of disjoint slices $\Delta(z) \subset \Delta_L$ including at least $c_1L$ vertex-disjoint LR crossings of $\Delta_L$ for $\mathbb{G}_s$ stochastically dominates a binomial random variable $Y$ with parameters $n \asymp c_0L^{d-2}$ and $p := 1 - e^{-c_2L}$. Setting $\delta := e^{-c_2L}$ we get

$$\mathbb{P}(R_L < n/2) \leq \mathbb{P}(Y \leq \delta) \leq \delta - \frac{\delta^2}{2} \mathbb{E}[\delta] = \delta - \frac{\delta^2}{2}[\delta p + 1 - p] = \delta - \frac{\delta^2}{2}[\delta^2 + \delta] = 2^{o(1+o(1))}L^{d-2} e^{-c_0c_2(1+o(1))L^{d-1}/2},$$

thus implying (14) in Theorem 2.

It remains now to prove (75). In order to have a notation close to the one in [13] Section 4), we consider the box

$$\Lambda := ([0, M + 1] \times [0, M - 1]) \cap \mathbb{Z}^2,$$

where $M$ will be linearly related to $L$ as explained at the end.

Let $(x_1, x_2, \ldots, x_n)$ be a string of points in $\Lambda$, such that the set $\{x_1, x_2, \ldots, x_n\}$ is connected when thinking to $\Lambda$ as a graph with edges $\{x_i - x_j\}$ with $|x_i - x_j| = 1$. We introduce a total order on $\Delta\{x_1, \ldots, x_n\}$ (in general, given $A \subset \mathbb{Z}^2$, $\Delta A := \{y \in \mathbb{Z}^2 \setminus A : |x - y| = 1 \text{ for some } x \in A\}$). We have to modify the definition in [13] Section 4) which is restricted there to the case that $(x_1, x_2, \ldots, x_n)$ is a path in $\mathbb{Z}^2$. For later use, it is more convenient to describe the ordering from the largest to the smallest element. We denote by $\Psi$ the anticlockwise rotation of $\pi/2$ around the origin in $\mathbb{R}^2$ (in particular, $\Psi(e_1) = e_2$ and $\Psi(e_2) = -e_1$). We first introduce an order $\prec_k$ on the sites in $\mathbb{Z}^2$ neighboring $x_k$ as follows. Putting $x_0 := x_1 - e_1$, for $k = 1, 2, \ldots, n$ we set $x_k + \Psi(v) \succ_k x_k + \Psi^2(v) \succ_k x_k + \Psi^3(v) \succ_k x_k + \Psi^4(v) = x_{a(k)}$, where $v := x_{a(k)} - x_k$ and $a(k) := \max\{j : 0 \leq j \leq n \text{ and } |x_k - x_j| = 1\}$. The order on $\Delta\{x_1, \ldots, x_n\}$ is obtained as follows. The largest elements are the sites of $\Delta\{x_1, \ldots, x_n\}$ neighboring $x_n$ (if any), ordered according to $\succ_n$. The next elements, in decreasing order, are the sites $\Delta\{x_1, \ldots, x_n\}$ neighboring $x_{n-1}$ but not $x_n$ (if any), ordered according to $\succ_{n-1}$. As so on, in the sense that in the generic step one has to consider the elements of $\Delta\{x_1, \ldots, x_n\}$ neighboring $x_k$ but not $x_{k+1}, \ldots, x_n$ (if any), ordered according to $\succ_k$. 
Let $F_0$ be the event

$$F_0 := \{B(4N,x,m) \text{ is a seed } \forall x \in \Lambda \text{ with } x = (0, s) \text{ for some } s\}.$$ 

We now define a random field $\zeta = (\zeta(x) : x \in \Lambda)$ with $\zeta(x) \in \{0,1\}$ on the probability space $(\Theta, \mathcal{Q})$ where $\mathcal{Q} := \mathbb{P}(\cdot|F_0)$ (cf. Definition 3.4).

To define the field $\zeta$, we have to build the sets $C_j^s = (E_j^s, F_j^s)$, with $s \in \{0,1,\ldots,M-1\}$ and $j = 1, 2, \ldots, M^2$, and the sites $x_j^s$ such that $E_j^s \cup F_j^s = \{x_1^s, x_2^s, \ldots, x_{j+1}^s\}$. The construction will fulfill the following properties: $E_j^s$ will be a connected subset of $\Lambda$; $(E_j^s, F_j^s)$ will be obtained from $(E_j^{s-1}, F_j^{s-1})$ by adding exactly a point (called $x_j^{s+1}$) either to $E_j^s$ or to $F_j^s$; $\zeta \equiv 1$ on $E_j^s$ and $\zeta \equiv 0$ on $F_j^s$.

In what follows, the index $s$ will vary in $\{0,1,\ldots,M-1\}$. We also set $x_1^s := (0, s)$. We build the sets $C_0^0, C_1^1, \ldots, C_{M^2-1}^{M-1}$ as follows. We say that $x_1^s$ is occupied if the analogous of Definition 8.8, with removed success-events $S_3, S_4, S_5$, is fulfilled. If the point $x_1^s$ is occupied, then we set

$$\zeta(x_1^s) := 1 \text{ and } C_1^s := (E_1^s, F_1^s) := (\{x_1^s\}, \emptyset),$$

otherwise we set

$$\zeta(x_1^s) := 0 \text{ and } C_1^s := (E_1^s, F_1^s) := (\emptyset, \{x_1^s\}).$$

We then define iteratively

$$C_2^0, C_3^0, \ldots, C_M^0, C_2^1, C_3^1, \ldots, C_{M^2-1}^1, C_2^{M-1}, C_3^{M-1}, \ldots, C_{M^2-1}^{M-1}$$

as follows. If $E_1^s = \emptyset$, then we set $C_j^s := C_j^s$ for all $j : 2 \leq j \leq M^2$. We restrict now to the case $E_1^s \neq \emptyset$. Suppose we have defined all the sets preceding $C_j^{s+1}$ in the above string \[(78)\] and we want to define $C_j^{s+1}$. We call $W_j$ the points of $\Lambda$ involved in the construction up to this moment, i.e.

$$W_j := \{x_r^s : 0 \leq k \leq M-1\} \cup \{x_r^s : 0 \leq s' < s, 1 \leq r \leq M^2\} \cup \{x_r^s : 1 \leq r \leq j\}.$$ 

As already mentioned, it must be $E_0^s \subset E_1^s \subset \cdots \subset E_j^s$ and at each inclusion either the two sets are equal or the second one is obtained from the first one by adding exactly a point. We then write $E_j^s$ for the non–empty string obtained by ordering the points of $E_j^s$ according to the chronological order with which they have been added. Equivalently, $x_a^s$ precedes $x_b^s$ in $E_j^s$ if $a < b$. Hence the total order in $\Delta \bar{E}_j^s$ is well defined. We call $P_j^s$ the following property: $E_j^s$ is disjoint from the right vertical face of $\Lambda$, i.e. $E_j^s \cap (\{M+1\} \times \{0,1,\ldots,M-1\}) = \emptyset$, and $(\Lambda \cap \Delta \bar{E}_j^s) \setminus W_j^s \neq \emptyset$. If property $P_j^s$ is satisfied, then we denote by $x_j^{s+1}$ the last element of $(\Lambda \cap \Delta \bar{E}_j^s) \setminus W_j^s$. We define $k$ as the largest integer $k$ such that $x_k^s \in E_j^s$ and $|x_j^{s+1} - x_k^s| = 1$. If $x_j^{s+1}$ is occupied and linked to $x_k^s$ (cf. Definition 8.11 and Section 10), then we set

$$\zeta(x_{j+1}^s) := 1 \text{ and } C_{j+1}^s := (E_j^s \cup \{x_{j+1}^s\}, F_j^s),$$

otherwise we set

$$\zeta(x_{j+1}^s) := 0 \text{ and } C_{j+1}^s := (E_j^s, F_j^s \cup \{x_{j+1}^s\}).$$
On the other hand, if property $P^*_j$ is not verified, then we set $x_{j+1}^* := x_j^*$ (hence $\zeta(x_{j+1}^*)$ has already been defined) and $C_{j+1}^* := C_j^*$.

It is possible that the set $\bigcup_{s=0}^{M-1} \bigcup_{j=1}^{M^2} (E^*_j \cup F^*_j)$ does not fill all $\Lambda$. In this case we set $\zeta \equiv 0$ on the remaining points. This completes the definition of the random field $\zeta$.

Above we have constructed the sets $C_j^*$ in the following order: $C_1^0, C_1^1, \ldots, C_1^{M-1}, C_2^0, C_2^1, \ldots, C_2^{M-1}, C_3^0, C_3^1, \ldots, C_3^{M-1}, \ldots, C_{M^2-1}, \ldots, C_{M^2}$. By the results of Section 8 and the discussion in Section 10, at every step the probability to add a point to a set of the form $E^*_s$, bounded by $1 - 8c' \geq 3/4$.

Call $N_M$ the maximal number of vertex-disjoint LR crossings of the box $\Lambda$ for $\zeta$ (here crossings are the standard ones for percolation on $\mathbb{Z}^d$ [8]). Note that $N_M$ also equals the number of indexes $s \in \{0, 1, \ldots, M-1\}$ such that $E^*_s$ intersect the right vertical face of $\Lambda$. By establishing a stochastic domination on a 2–dimensional site percolation in the same spirit of [9] Lemma 1 (cf. [13] Lemma 4.1) and using the above lower bound on the conditional probability to add a point to a set of the form $E^*_s$, one obtains that $N_M$ stochastically dominates the corresponding number in a site percolation of parameter $p = 3/4 > p_c(2)$. Hence there exist $c_3, c_4 > 0$ such that $\mathcal{Q}(N_M \geq c_3 M) \geq 1 - e^{-c_4 M}$ for $M$ large enough [8].

In the rest we derive (75) from the above bound on $\mathcal{Q}(N_M \geq c_3 M)$. Due to the translation invariance of $\mathbb{P}$, it is enough to prove (75) with $\Delta(0)$ replaced by $\Delta'(0) := \{(m + 1, 2L + 5 + m) \times [0, 2L] \times [-4N, 4N]^{d-2} \} \cap \epsilon \mathbb{Z}^d$. We take $M$ as the minimal integer such that $(M + 1)4N > 2L + 5 + m + N$. Without loss of generality, when referring to the LR crossings of the box $\Lambda$ for $\zeta$ we restrict to crossings such that only the first and the last points intersect the vertical faces of $\Lambda$ (which would not change the random number $N_M$). We fix a set $\Gamma'$ of vertex–disjoint LR crossings of $\Lambda$ for $\zeta$ with cardinality $N_M$. Then we define $\Gamma$ as the set of paths $(x_1, x_2, \ldots, x_k)$ in $\Gamma'$ such that $x_i$ has second coordinate in $[1, M-2]$ for each $i$. Note that, since $\Lambda$ is bidimensional, $|\Gamma| \geq |\Gamma'| - 2$. Given $x \in \mathbb{Z}^d$ we set $\bar{x} := (x, 0, 0, \ldots, 0) \in \mathbb{Z}^d$.

Take a LR crossing $(x_1, x_2, \ldots, x_k)$ in $\Gamma$. By the discussion in the previous sections, we get that there is a path $\gamma$ in $\mathbb{G}_s$ from $B(4N\bar{x}_1, m)$ to $B(4N\bar{x}_k, N)$ without self-intersections. Moreover, this path is included in the region $\mathcal{R}$ obtained as union of the boxes $B(4N\bar{x}_i, 3N)$ with $i = 1, \ldots, k$ (see Fig. 4 and the second paragraph in Section 10). We point out that the second coordinate of any point in $B(4N\bar{x}_i, 3N)$ is in $[4N - 3N, 4N(M - 2) + 3N] \subset [0, 2L]$ due to the definition of $\Gamma$ and since $4N M \leq 2L + 5 + m + N$ by the minimality of $M$. In addition, the box $B(4N\bar{x}_1, m)$ lies in the halfspace $\{(z_1, \ldots, z_d) \in \epsilon \mathbb{Z}^d : z_1 \leq m\}$, while the box $B(4N\bar{x}_k, N)$ lies in the halfspace $\{(z_1, \ldots, z_d) \in \epsilon \mathbb{Z}^d : z_1 > 2L + 5 + m\}$ (by our choice of $M$). As a consequence we can extract from the above path $\gamma$ a new path $\bar{\gamma}$ for $\mathbb{G}_s$ lying in $\mathcal{R} \cap \{(z_1, \ldots, z_d) \in \epsilon \mathbb{Z}^d : m \leq z_1 \leq 2L + 5 + m, 0 \leq z_2 \leq 2L\} \subset \Delta(0)'$. At cost to further refine $\bar{\gamma}$ we have that $\bar{\gamma}$ is a LR crossing of $\Delta(0)'$. Moreover, due to the dimension 2, there is an integer $\ell$ such that every path $\bar{\gamma}$ can share some vertex with
at most \( \ell \) paths \( \tilde{\gamma}' \) with \( \gamma' \in \Gamma \). Since \( M \geq c_5 L \), by the above observations we have proved that the event \( \{ N_M \geq c_5 M \} \) implies the event \( F \) that \( \Delta(0)' \) has at least \( c_6 L \) vertex–disjoint LR crossings for \( G_\ast \). Hence, by our bound on \( Q(N_M \geq c_5 M) \) and since \( M \leq c_7 L \), we get that \( Q(F) \geq 1 - e^{-c_8 L} \). Since edges in \( G_\ast \) have length bounded by 1, the event \( F \) does not depend on the vertexes of \( G_\ast \) in \( \bigcup_{s \geq 0} B(4N\tilde{x}_s^1, m) \), neither on the edges exiting from the above region. In particular, \( F \) and \( F_0 \) are independent, thus implying that \( \mathbb{P}(F) = \mathbb{P}(F|F_0) = Q(F) \geq 1 - e^{-c_8 L} \), and in particular \( j^{(7)} \) is verified.

**Appendix A. Locations**

**Claim A.1.** Given \( i \in \{5, 6\} \), if \( S_1 \cap \cdots \cap S_i \) occurs, then \( b^{(i)} + T_i(m, n) \subset B((i - 2)N\epsilon_1, N) \).

**Proof.** Let \( x \in (b^{(i)} + T_i(m, n)) \). Then by \( (62) \) \( x_1 \in (b_1^{(i)} + N + [-m, m]) \) if \( i = 3 \). \( N + [-m, m] \subset (i - 2)N + [-N, N] \). Using again \( (62) \), for \( j \geq 2 \) we get the following cases: (a) if \( b_1^{(i)} \geq 0 \), then \( x_j \in [b_j^{(i)} - n, b_j^{(i)}] \subset [-n, n] \cap [-N, N] \); (b) if \( b_1^{(i)} < 0 \), then \( x_j \in [b_j^{(i)}, b_j^{(i)} + n] \subset [-n, n] \cap [-N, N] \). \( \square \)

**Claim A.2.** If \( S_1 \cap \cdots \cap S_8 \) occurs, then \( b^{(7)} + \tilde{T}_1(m, n) \subset B(5N\epsilon_1, N), b^{(7)} + \tilde{T}_2(m, n) \subset B(4N\epsilon_1 + N\epsilon_2, N) \) and \( b^{(7)} + \tilde{T}_3(m, n) \subset B(4N\epsilon_1 - N\epsilon_2, N) \).

**Proof.** The inclusion concerning \( b^{(7)} + \tilde{T}_1(m, n) \) can be derived as in the proof of Claim A.1. Consider now \( x \in (b^{(7)} + \tilde{T}_r(m, n)) \) for \( r \in \{2, 3\} \). Then by \( (62) \) with \( i = 7 \) we get that \( x_1 \in b_1^{(7)} + [0, n] \subset 4N + [-N, N] \) and \( x_2 \in b_2^{(7)} + N + [-m, m] \subset N + [-n, n] \) for \( r = 2 \); (b) \( x_3 \in b_2^{(7)} - N + [-m, m] \subset -N + [-n, n] \) for \( r = 3 \). Using \( (62) \) with \( i = 7 \) one finally gets that \( |x_j| \leq N \) for \( j \geq 3 \), as in the proof of Claim A.1. \( \square \)

**A.0.1. Validity of condition (40) in Section 9.3.** The inclusion in (40) is trivially satisfied in all steps. We concentrate only on the second property in (40), concerning disjointness. The validity for \( i = 3, 4 \) can be checked as done for \( i = 2 \).

To proceed we first recall that (62) holds for \( i = 5, 6, 7, 8 \). To get the disjointness in (40) for \( i \in \{5, 6, 8\} \) we argue as follows. We observe that \( R_i \cup \partial R_i \subset \bigcup_{k=0}^{i-1} (B_k' \cup \partial B_k') \) and points in \( \bigcup_{k=0}^{i-1} (B_k' \cup \partial B_k') \) have their first coordinate not bigger than \( (i - 3)N + m + 1 \) (cf. Fig. 4) (60) and (62) for \( i - 1 \) instead of \( i \). On the other hand, points in \( \tilde{b}_i + (T_i(n) \cup T_i(m, n)) \) have their first coordinate not smaller than \( (i - 3)N + n - 1 \) (cf. (62)). Since \( n - m > 2 \) we get that \( \tilde{b}_i + (T_i(n) \cup T_i(m, n)) \) and \( R_i \cup \partial R_i \) are disjoint for \( i \in \{5, 6, 8\} \).

For \( i \in \{9, 10\} \) we write as follows. We write \( \tilde{b}_k \) for the \( j \)-th coordinate of \( \tilde{b}_k \). Note that \( |\tilde{b}_1| + |\tilde{b}_2| \leq n - m \). Hence, by (60), \( \tilde{b}_2 + n - 1 = \tilde{b}_2 + N + n - 1 \geq (\tilde{b}_2)2 - n + m + N + n - 1 = (\tilde{b}_2)2 + m + N - 1 > (\tilde{b}_2)2 + N \). Similarly, we have \( (\tilde{b}_10)2 - n + 1 < (\tilde{b}_6)2 - N \). By (60), for \( x \in \tilde{b}_i + (T_i(n) \cup T_i(m, n)) \), we have \( x_1 \geq \tilde{b}_1 - n + m \geq 4N + m - n \). By the previous observations we have that \( x_2 \geq (\tilde{b}_1)2 + n - 1 > (\tilde{b}_6)2 + N \) if \( i = 9 \) and that \( x_2 \leq (\tilde{b}_{10})2 - n + 1 <
On the other hand $R_i \cup \partial R_i \subset \bigcup_{k=0}^{i-1}(B_k' \cup \partial B_k')$. Let us fix $x \in \bigcup_{k=0}^{i-1}(B_k' \cup \partial B_k')$ such that $x_1 \geq 4N + m - n$. If $i = 9$ it must be (cf. Fig. 4) that $x_1 \geq (\tilde{b}_6)_2 + n + 1$ or $x_2 \geq (\tilde{b}_0)_2 + m + 1$. If $i = 10$ it must be $x_2 \geq (\tilde{b}_6)_2 - n - 1$ or $x_2 \geq (\tilde{b}_{10})_2 - m - 1$. By comparing the above bounds we get that $\tilde{b}_1 + (\tilde{T}_1(n) \cup T_1(m,n))$ and $R_i \cup \partial R_i \subset \bigcup_{k=0}^{i-1}(B_k' \cup \partial B_k')$ are disjoint for $i = 9, 10$.

A.0.2. Validity of condition $[40]$ in Section $[9.6]$. The disjointness of $R_7 \cup \partial R_7$ and $\tilde{b}_7 + (\tilde{T}_2(n) \cup \tilde{T}_3(m,n))$ is similar. Suppose first that $(\tilde{b}_6)_2 \geq 0$. Then $(\tilde{b}_7)_2 \in (\tilde{b}_6)_2 + [-n, 0]$. By construction, if $x \in R_7$ with $x_1 \geq 4N - m$, then $x \in \tilde{b}_6 + T_6(m, n)$ and therefore $x_2 \geq (\tilde{b}_6)_2 - n$. Take now $y \in \tilde{b}_7 + (\tilde{T}_3(n) \cup \tilde{T}_3(m,n))$. Then $y_1 \geq (\tilde{b}_7)_1 + 4N$ and $y_2 \leq (\tilde{b}_7)_2 - n + 1 \leq (\tilde{b}_6)_2 - m + 1$. Suppose by contradiction that $y \in R_7 \cup \partial R_7$. Then there exists $x \in R_7$ such that $|x - y| < 1$. This implies that $x_1 \geq y_1 - 1 \geq 4N - 1$. Hence, by the initial observations, $x_2 \geq (\tilde{b}_6)_2 - n$. This last bound, together with $y_2 \leq (\tilde{b}_6)_2 - m + 1 + 1$ and $|x_2 - y_2| \leq 1$, leads to a contradiction as $m > 2$.

Suppose that $(\tilde{b}_6)_2 < 0$. Then $\tilde{(b)}_7 \subset (\tilde{b}_6)_2 + [m, n - m]$. By construction, if $x \in R_7$ with $x_1 \geq 4N - m$, then $x \in \tilde{b}_6 + T_6(m, n)$ and therefore $x_2 \leq (\tilde{b}_6)_2 + n$. Take $y \in \tilde{b}_7 + (\tilde{T}_3(n) \cup \tilde{T}_3(m,n))$. Then $y_1 \geq (\tilde{b}_7)_1 + 4N$ and $y_2 \geq (\tilde{b}_7)_2 + n - 1 \geq (\tilde{b}_6)_2 + m + n - 1$. Suppose by contradiction that $y \in R_7 \cup \partial R_7$. Then there exists $x \in R_7$ such that $|x - y| < 1$. This implies that $x_1 \geq y_1 - 1 \geq 4N - 1$. Hence, by the initial observations, $x_2 \leq (\tilde{b}_6)_2 + n$. This last bound, together with $y_2 \geq (\tilde{b}_6)_2 + m + n - 1$ and $|x_2 - y_2| \leq 1$, leads to a contradiction as $m > 2$.

References

[1] V. Ambegoakar, B.I. Halperin, J.S. Langer. Hopping conductivity in disordered systems. Phys. Rev B 4, 2612-2620 (1971).
[2] A. Faggionato. Mott’s law for the effective conductance of the Miller–Abrahams random resistor network. Preprint (2017).
[3] A. Faggionato. Miller–Abrahams random resistor network. Mott random walk and 2-scale homogenization. Forthcoming.
[4] A. Faggionato. Mott’s law for Miller–Abrahams random resistor network and for Mott random walk. In preparation.
[5] A. Faggionato, P. Mathieu. Mott law as upper bound for a random walk in a random environment. Comm. Math. Phys. 281, 263–286 (2008).
[6] A. Faggionato, H.A. Mimun. Connection probabilities in Poisson random graphs with uniformly bounded edges. ALEA 16 463–486 (2019).
[7] A. Faggionato, H. Schulz-Baldes, D. Spehner. Mott law as lower bound for a random walk in a random environment. Comm. Math. Phys., 263, 21–64 (2006).
[8] G. Grimmett, Percolation. Second edition. Die Grundlehren der mathematischen Wissenschaften 321. Springer Verlag, Berlin, 1999.
[9] G.R. Grimmett, J.M. Marstrand. *The supercritical phase of percolation is well behaved.* Proc. R. Soc. Lond. A **430**, 439–457 (1990).
[10] H. Kesten. *Percolation theory for mathematicians.* Birkhäuser, 1982.
[11] A. Miller, E. Abrahams. *Impurity Conduction at Low Concentrations.* Phys. Rev. **120**, 745–755 (1960).
[12] M. Pollak, M. Ortuño, A. Frydman. *The electron glass.* First edition, Cambridge University Press, United Kingdom, 2013.
[13] H. Tanemura. *Behavior of the Supercritical Phase of a Continuum Percolation Model on \( \mathbb{R}^d \).* J. Appl. Prob. **30**, 382-396 (1993).

**Alessandra Faggionato.** Dipartimento di Matematica, Università di Roma ‘La Sapienza’ P.le Aldo Moro 2, 00185 Roma, Italy  
*E-mail address:* faggiona@mat.uniroma1.it

**Hlafo Alfie Mimun.** Dipartimento di Matematica, Università di Roma ‘La Sapienza’ P.le Aldo Moro 2, 00185 Roma, Italy  
*E-mail address:* mimun@mat.uniroma1.it