Primitive graphs with small exponent and small scrambling index

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Abstract. A connected graph $G$ is primitive provide there is a positive integer $k$ such that for each pair of vertices $u$ and $v$ there is a $uv$-walk of length $k$. The smallest of such positive integer $k$ is the exponent of $G$ and is denoted by $\exp(G)$. The scrambling index of a primitive graph $G$, denoted by $k'(G)$, is the smallest positive integer $k$ such that for each pair of vertices $u$ and $v$ there is a vertex $w$ such that there is a $uw$-walk and a $vw$-walk of length $k$. By an $n$-chainring $CR(n)$ we mean a graph obtained from an $n$-cycle by replacing each edge of the $n$-cycle by a triangle. By a $(q, p)$-dory, $D(q, p)$, we mean a graph with vertex set $V(D(q, p)) = V(P_q \times P_p) \cup \{w_1, w_2\}$ and edge set $E(D(q, p)) = E(P_q \times P_p) \cup \{w_1 - (u_i, v_1) : i = 1, 2, \ldots, q\} \cup \{w_2 - (u_i, v_p) : i = 1, 2, \ldots, q\}$, where $P_n$ is a path on $n$ vertices. We discuss the exponent and scrambling index of an $n$-chainring and $(q, p)$-dory. We present formulae for exponent and scrambling index in terms of their diameter.

1. Introduction
Let $G$ be a simple graph. We follow graph terminologies from [1, 2]. Let $u$ and $v$ be two vertices in a graph $G$. A walk connecting $u$ and $v$ is denoted by $W_{uv}$ or $uv$-walk. A $uv$-path is a $uv$-walk without repeated vertices except possibly $u = v$. A cycle is a closed path. The length of a walk $W_{uv}$ is denoted by $\ell(W_{uv})$. By a triangle we mean a cycle of length three. A walk is even or odd if it is of even or odd length respectively. For a connected graph $G$ the distance $d(u, v)$ of vertices $u$ and $v$ in $G$ is defined to be the length of a shortest $uv$-path in $G$. The diameter of a connected graph $G$, denoted by $\text{diam}(G)$, is defined to be

$$\text{diam}(G) = \max_{u,v} \{d(u, v)\}.$$ 

A connected graph $G$ is said to be primitive if there is a positive integer $k$ such that for each pair of vertices $u$ and $v$ in $G$ there is a $uv$-walk of length $k$. The exponent of a primitive graph $G$, denoted by $\exp(G)$, is the smallest of such positive integer $k$. The scrambling index of a primitive graph $G$ is the smallest positive integer $k$ such that for each pair of vertices $u$ and $v$ there is a $uv$-walk of length $2k$ [3–5]. It is known that a connected graph is primitive if and only if it contains an odd cycle [2]. From definition we have $\exp(G) \geq \text{diam}(G)$. Notice also that if $\text{diam}(G)$ of $G$ is even then by definition $k(G) \geq \text{diam}(G)$. If the $\text{diam}(G)$ is odd, then there is a pair of vertices $u$ and $v$ such that the shortest even $uv$-walk is of length $\text{diam}(G) + 1$. Hence if $\text{diam}(G)$ is odd, we have $k(G) \geq (\text{diam}(G) + 1)/2$. Therefore, for a primitive graph $G$ we find
that $k(G) \geq \left\lceil \frac{\text{diam}(G)}{2} \right\rceil$. Therefore, for a primitive graph $G$ with the smallest cycle of odd length $r \geq 3$

\[ \text{diam}(G) \leq \exp(G) \text{ and } \left\lceil \frac{\text{diam}(G)}{2} \right\rceil \leq k(G). \quad (1) \]

In this paper we discuss two-classes of primitive graph with small exponent and small scrambling, that is, primitive graphs $G$ with $\exp(G) = \text{diam}(G)$ and $k(G) = \left\lceil \frac{\text{diam}(G)}{2} \right\rceil$. For positive integer $n \geq 3$, an $n$-chainring is a graph obtained form an $n$-cycle by replacing each edge of the $n$-cycle by a triangle. More precisely, an $n$-chainring $CR(n)$ is a primitive graph on $2n$ vertices consisting of the the $n$-cycle $v_2 - v_4 - \cdots - v_{2n-2} - v_{2n} - v_2$ and the $2n$-cycle $v_1 - v_2 - v_3 - v_4 - \cdots - v_{2n-1} - v_{2n} - v_1$. The graph of $CR(8)$ is given in Figure 1.

![Figure 1. The Graph of CR(8).](image)

For positive integers $p$ and $q$, let $P_q$ be a path on $q$ vertices $\{u_1, u_2, \ldots, u_q\}$ and $P_p$ be a path on $p$ vertices $\{v_1, v_2, \ldots, v_p\}$. By a $(q,p)$-dory, $D(q,p)$, we mean a graph with vertex set $V(D(q,p)) = V(P_q \times P_p) \cup \{w_1, w_2\}$ and edge set $E(D(q,p)) = E(P_q \times P_p) \cup \{(w_1, v_1) : i = 1, 2, \ldots, q\} \cup \{(w_2, u_i, v_p) : i = 1, 2, \ldots, q\}$.

In Section 2, we discuss properties of $uv$-walk especially $uv$-walk in an $n$-chainring. In Section 3, we discuss the exponent and scrambling index of $n$-chainring. In Section 4, we discuss the exponent and scrambling index of $(q,p)$-dory.

### 2. Properties of Walks

We discuss some properties of $uv$-walk necessary for our discussion.

**Proposition 1.** Let $G$ be a graph and let $W_{uv}$ be a $uv$-walk of length $\ell(W_{uv})$ in $G$. If $k$ is a positive integer such that $k \geq \ell(W_{uv})$ and $k \equiv \ell(W_{uv}) \mod 2$, then there is a $uv$-walk of length $k$ in $G$.

**Proof.** Let $W_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_i - v_{i+1} - \cdots - v_m = v$ be a $uv$-walk of length $\ell(W_{uv})$. Since $k \geq \ell(W_{uv})$ and $k \equiv \ell(W_{uv}) \mod 2$, there is a nonnegative integer $t$ such that $k - \ell(W_{uv}) = 2t$. Then the walk that starts at $u = v_0$, moves to $v_{i+1}$ along the walk $u = v_0 - v_1 - v_2 - \cdots - v_i - v_{i+1}$ and then moves $t$ times around the closed walk $v_{i+1} - v_i - v_{i+1}$ and finally moves to $v$ along the walk $v_{i+1} - \cdots - v_m = v$ is a $uv$-walk of length $k = \ell(W_{uv}) + 2t$. \qed
Proposition 2. Let $n$ be integer such that $n \geq 4$. Then for any pair of vertices $u$ and $v$ in $CR(n)$ there is a $uv$-path of length $d(u, v) + 1$.

Proof. Let $u$ and $v$ be any two vertices in $CR(n)$ and let the path

$$P_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_i - v_{i+1} - \cdots - v_\ell = v$$

be a $uv$-path of length $d(u, v) = \ell$. Since every edge $v_i - v_{i+1}$ of the path $P_{uv}$ lies on a triangle, then there is a vertex $v_\ell$ such that the closed path $v_i - v_\ell - v_{i+1} - v_i$ is a triangle. This implies the $uv$-path

$$P_{uv} : u = v_0 - v_1 - v_2 - \cdots - v_i - v_\ell - v_{i+1} - \cdots - v_\ell = v$$

is a $uv$-path of length $d(u, v) + 1$. \hfill \Box

3. Exponent and scrambling index of $n$-chainring

We discuss the exponent and scrambling index of $CR(n)$. We first present formulae for exponent and scrambling index in term of the diameter of $CR(n)$ and then present formulae that depends on $n$.

Theorem 3. Let $n$ be a positive integer with $n \geq 4$. Then $\exp(CR(n)) = \diam(CR(n))$.

Proof. From (1) we have $\exp(CR(n)) \geq \diam(CR(n))$. It remains to show that $\exp(CR(n)) \leq \diam(CR(n))$. For each pair of vertices $u$ and $v$ we show that there exists a $uv$-walk of length $\diam(CR(n))$. Notice that for each pair of vertices $u$ and $v$ there is a $uv$-path $P_{uv}$ of length $d(u, v)$. If $d(u, v) = \diam(CR(n)) \mod 2$, then by Proposition 1 the path $P_{uv}$ can be extended to a walk $W_{uv}$ of length $\diam(CR(n))$. If $d(u, v) \neq \diam(CR(n)) \mod 2$, then by Proposition 2 there is a path $P'_{uv}$ of length $\ell(P'_{uv}) = d(u, v) + 1$. We now have $\ell(P'_{uv}) = \diam(CR(n))$. Hence Proposition 1 guarantees that the path $P'_{uv}$ can be extended to a walk $W_{uv}$ of length $\diam(CR(n))$. \hfill \Box

Corollary 4. Let $n$ be a positive integer with $n \geq 4$. Then

$$\exp(CR(n)) = \begin{cases} (n+1)/2, & \text{if } n \text{ is odd} \\ (n+2)/2, & \text{if } n \text{ is even}. \end{cases}$$

Proof. If $n$ is odd, the $\diam(G) = d(v_1, v_n) = (n+1)/2$. If $n$ is even, the $\diam(G) = d(v_1, v_{n+1}) = (n+2)/2$. \hfill \Box

Theorem 5. Let $n$ be a positive integer such that $n \geq 3$. Then $k(CR(n)) = \left\lfloor \frac{\diam(CR(n))}{2} \right\rfloor$.

Proof. From (1) we have $k(CR(n)) \geq \left\lfloor \frac{\diam(CR(n))}{2} \right\rfloor$. It remains to show that $k(CR(n)) \leq \left\lfloor \frac{\diam(CR(n))}{2} \right\rfloor$.

If the $\diam(CR(n))$ is even, then by Proposition 1 for each pair of vertices $u$ and $v$ there is a $uv$-walk of length $\diam(CR(n))$. Thus we conclude that $k(CR(n)) \leq \diam(CR(n))/2$. If the $\diam(CR(n))$ is odd, then the shortest even walk connecting $u_0$ and $v_0$ is of length $\diam(CR(n)) + 1$. Notice that for every pair of vertices $u$ and $v$, $d(u, v) \leq \diam(CR(n))$. Proposition 1 and Proposition 2 imply that for each pair of vertices $u$ and $v$ there is an even $uv$-walk of length $\diam(CR(n)) + 1$. Hence $k(CR(n)) \leq \frac{\diam(CR(n)) + 1}{2}$. We now conclude that $k(CR(n)) \leq \left\lfloor \frac{\diam(CR(n))}{2} \right\rfloor$.

We now conclude that $k(CR(n)) = \left\lfloor \frac{\diam(CR(n))}{2} \right\rfloor$. \hfill \Box
Corollary 6. For positive integer \( n \geq 3 \), \( k(CR(n)) = \lceil n/4 \rceil + 1 \).

Proof. Suppose \( n \) is even. Then the \( \text{diam}(CR(n)) = (n+2)/2 \) and is obtained by the \( v_1v_{n+1} \)-path \( v_1 - v_2 - v_4 - \cdots - v_{n-1} - v_n \). If \( n \equiv 0 \) mod 4, then \( \text{diam}(CR(n)) = (4m+2)/2 \) for some positive integer \( m \). By Theorem 5 we have

\[
k(CR(n)) = \left\lceil \frac{\text{diam}(CR(n))}{2} \right\rceil = \left\lceil \frac{4m+2}{4} \right\rceil = \lceil m/2 \rceil + 1.
\]

If \( n \equiv 2 \) mod 4, then \( \text{diam}(CR(n)) = (4m+4)/2 \) for some positive integer \( m \). By Theorem 5 we have

\[
k(CR(n)) = \left\lceil \frac{\text{diam}(CR(n))}{2} \right\rceil = \left\lceil \frac{4m+4}{4} \right\rceil = \lceil m/2 \rceil + 1.
\]

Suppose now that \( n \) is odd. Then the \( \text{diam}(CR(n)) = (n+1)/2 \) and is obtained by the \( v_1v_{n-1} \)-path \( v_1 - v_2 - v_4 - \cdots - v_{n-1} - v_n \). If \( n \equiv 1 \) mod 4, then \( \text{diam}(CR(n)) = (4m+2)/2 \) for some positive integer \( m \). By Theorem 5 we have

\[
k(CR(n)) = \left\lceil \frac{\text{diam}(CR(n))}{2} \right\rceil = \left\lceil \frac{4m+2}{4} \right\rceil = \lceil m/2 \rceil + 1.
\]

If \( n \equiv 3 \) mod 4, then \( \text{diam}(CR(n)) = (4m+4)/2 \) for some positive integer \( m \). By Theorem 5 we have

\[
k(CR(n)) = \left\lceil \frac{\text{diam}(CR(n))}{2} \right\rceil = \left\lceil \frac{4m+4}{4} \right\rceil = \lceil m/2 \rceil + 1.
\]

From (2), (3), (4) and (5) we have \( k(CR(n)) = \lceil n/4 \rceil + 1 \).

4. Exponent and scrambling index of \((q,p)\)-dory

Let \( P_q \) be a path on \( q \) vertices \( \{u_1, u_2, \ldots, u_q\} \) and \( P_p \) be a path on \( p \) vertices \( \{v_1, v_2, \ldots, v_p\} \). By a \((q,p)\)-dory, \( D(q,p) \), we mean a graph with vertex set \( V(D(q,p)) = V(P_q \times P_p) \cup \{w_1, w_2\} \) and edge set \( E(D(q,p)) = E(P_q \times P_p) \cup \{w_1 - (u_i, v_1) : i = 1, 2, \ldots, q\} \cup \{w_2 - (u_i, v_p) : i = 1, 2, \ldots, q\} \).

Since \( D(q,p) \) is connected and contains triangles, \( D(q,p) \) is primitive. We also note that if \( p \geq q \), then \( \text{diam}(D(q,p)) = d(w_1, w_2) = p + 1 \).

Theorem 7. Let \( p \) and \( q \) be positive integers such that \( p \geq q \). Then \( \exp(D(q,p)) = p + 1 \).

Proof. We note from (1) that \( \exp(D(q,p)) \geq \text{diam}(D(q,p)) = p + 1 \). It remains to show that \( \exp(D(q,p)) \leq p + 1 \). We show that for each pair of vertices \( x_0 \) and \( y_0 \) in \( D(q,p) \) there is a walk connecting \( x_0 \) and \( y_0 \) of length \( p + 1 \).

If \( d(x_0, y_0) \equiv p + 1 \) mod 2, then by Proposition 1 there is a \( x_0, y_0 \)-walk of length \( p + 1 \). It remains to consider the case where \( d(x_0, y_0) \not\equiv p + 1 \) mod 2.

Let \( P_{x_0,y_0} \) be the \( x_0, y_0 \)-path of length \( \ell(P_{x_0,y_0}) = d(x_0, y_0) \). If one end vertex of \( P_{x_0,y_0} \) is \( w_1 \) or \( w_2 \), then there is a path \( P_{x_0,y_0} \) such that \( \ell(P_{x_0,y_0}) = d(x_0, y_0) + 1 \equiv p + 1 \). We now assume
that the end vertices of $P_{x_0,y_0}$ are not $w_1$ or $w_2$. We consider three cases.

Case 1. The vertices $x_0 = (u_i, v_j)$ and $y_0 = (u_i, v_k)$ for some $1 \leq i < q$ and $1 < j < k \leq p$. We note that $d(x_0, y_0) = k - j \equiv p + 1 \pmod{2}$. We assume without loss of generality that $p - k \geq j - 1$. The walk that starts at $(u_i, v_j)$ moves to $(u_i, v_1)$ along the path of length $(j - 1)$, then moves one times around the triangle $(u_i, v_1) - w_1 - (u_{i-1}, v_1) - (u_i, v_1)$, and finally moves to $(u_i, v_k)$ along the path of length $k - 1$ is a $x_0, y_0$-walk of length $k + j + 1$. We note that $k + j + 1 = 2(j - 1) + k - j + 3$. Since $k - j \not\equiv p + 1 \pmod{2}$, we have $k + j + 1 \equiv p + 1 \pmod{2}$. Moreover since $(j - 1) \leq n - k$, we have $k + j + 1 \leq p + 2$. Since $k + j + 1 \equiv p + 1 \pmod{2}$, then $k + j + 1 \leq p + 1$.

Case 2. The vertices $x_0 = (u_i, v_k)$ and $y_0 = (u_j, v_k)$ for some $1 \leq k \leq p$ and $1 \leq i < j \leq q$. We note that $d(x_0, y_0) = j - i \not\equiv p + 1 \pmod{2}$. Assume without loss of generality that $p - k > k - 1$. If $(j - i)$ is even, then the walk that starts at $(u_i, v_k)$ moves to $(u_i, v_1)$ along the path of length $k - 1$, then moves to $(u_j, v_1)$ along the path $(u_i, v_1) - (u_{i+1}, v_1) - w_1 - (u_j, v_1)$ of length 3, and finally moves to $(u_j, v_k)$ along the path of length $k - 1$, is a $x_0, y_0$-path of length $2(k - 1) + 3$. Since $2(k - 1) + 3 \not\equiv j - i \pmod{2}$, we have $2(k - 1) + 3 \equiv p + 1 \pmod{2}$. We note that $n - k > k - 1$. Therefore $2(k - 1) + 3 \leq p + 2$. Since $2(k - 1) + 3 \equiv p + 1 \pmod{2}$, we conclude that $2(k - 1) + 3 \leq p + 1$.

If $(j - i)$ is odd, then the walk that starts at $(u_i, v_k)$ moves to $(u_i, v_1)$ along the path of length $k - 1$, then moves to $(u_j, v_1)$ along the path $(u_i, v_1) - w_1 - (u_j, v_1)$ of length 2, and finally moves to $(u_j, v_k)$ along the path of length $k - 1$, is a $x_0, y_0$-path of length $2(k - 1) + 2$. Since $2(k - 1) + 2 \not\equiv j - i \pmod{2}$, we have $2(k - 1) + 2 \equiv p + 1 \pmod{2}$. We note that $n - k > k - 1$. Therefore $2(k - 1) + 2 \leq p + 1$.

Case 3. The vertices $x_0 = (u_i, v_j)$ and $y_0 = (u_r, v_s)$ for some $1 \leq j < s \leq p$ and $1 \leq i < r \leq q$. Notice that $d(x_0, y_0) = (r - i) + (s - j) \not\equiv p + 1 \pmod{2}$. Without loss of generality we assume that $j - 1 \leq p - s$. If $(s - j) \equiv p + 1 \pmod{2}$, then the walk that starts at $(u_i, v_j)$ moves to $(u_i, v_1)$ along the path of length $(j - 1)$, then moves to $(u_r, v_1)$ along the path $(u_i, v_1) - w_1 - (u_r, v_1)$, and finally moves to $(u_r, v_s)$ along the path of length $(j - 1) + (s - j)$ is a $x_0, y_0$-path of length $2(j - 1) + 2 + (s - j) \equiv p + 1 \pmod{2}$. Since $j - 1 \leq p - s$, then $2(j - 1) + 2 + (s - j) \leq p + 1$.

If $(s - j) \not\equiv p + 1 \pmod{2}$, then the walk that starts at $(u_i, v_j)$ moves to $(u_i, v_1)$ along the path of length $(j - 1)$, then moves to $(u_i, v_1)$ along the path $(u_i, v_1) - (u_{i+1}, v_1) - w_1 - (u_r, v_1)$, and finally moves to $(u_r, v_s)$ along the path of length $(j - 1) + (s - j)$ is a $x_0, y_0$-path of length $2(j - 1) + 3 + (s - j) \equiv p + 1 \pmod{2}$. Since $j - 1 \leq p - s$, then $2(j - 1) + 3 + (s - j) \leq p + 2$.

Since $2(j - 1) + 3 + (s - j) \equiv p + 1 \pmod{2}$, we conclude that $2(j - 1) + 3 + (s - j) \leq p + 1$.

Therefore, for each pair of vertices $x_0$ and $y_0$ there is a $x_0, y_0$-walk $W_{x_0,y_0}$ of length $\ell(W_{x_0,y_0}) \leq p + 1$. Proposition 1 guarantees that for each for each pair of vertices $x_0$ and $y_0$ there is a $x_0, y_0$-walk $W_{x_0,y_0}$ of length $\ell(W_{x_0,y_0}) = p + 1$. Hence $\exp(D(q,p)) \leq p + 1$. \hfill \Box

Theorem 8. Let $p$ and $q$ be positive integers such that $p \geq q$. Then $k(D(q,p)) = \left\lceil \frac{p+1}{2} \right\rceil$.

Proof. From (1) we have $k(D(q,p)) \geq \left\lceil \frac{\text{diam}(D(q,p))}{2} \right\rceil = \left\lceil \frac{p+1}{2} \right\rceil$. It remains to show that $k(D(q,p)) \leq \left\lceil \frac{p+1}{2} \right\rceil$.

If $p + 1$ is even, then by Proposition 1 for each pair of vertices $x_0$ and $y_0$ there is a $x_0, y_0$-walk of even length $p + 1$. Thus we conclude that $k(D(q,p)) \leq (p + 1)/2$.

If the $p + 1$ is odd, then the shortest even walk connecting $w_1$ and $w_2$ is of length $p + 2$. Notice that from Theorem 7 for every pair of vertices $x_0$ and $y_0$ there is a $x_0 y_0$-walk of length $p + 1$. This implies for each each pair of vertices $x_0$ and $y_0$ there is a $x_0 y_0$-walk of even length $p + 2$. Hence if $p + 1$ is odd, $k(D(q,p)) \leq (p + 2)/2$.
Therefore we now conclude that $k(D(q,p)) \leq \left\lceil \frac{p+1}{2} \right\rceil$. Hence we now have $k(D(q,p)) = \left\lceil \frac{p+1}{2} \right\rceil$.

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