Sub-barrier recollisions and the tunnel exit time delay in strong-field ionization

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Tunneling ionization is characterized by a time delay, observed asymptotically as a specific shift of the photoelectron momentum distribution. This shift corresponds to a negative time delay which is caused by the interference of the sub-barrier recolliding and direct ionization paths, as laid out in [Phys. Rev. Lett. 120, 013201 (2018)]. While a Gedankenexperiment following the peak of the wavefunction considering only the direct ionization path shows a positive tunneling time delay at the tunnel exit, in this paper we investigate the effects of sub-barrier recollisions on the time delay pattern at the tunnel exit. We conclude that the interference of the direct and recolliding trajectories slightly decreases the tunneling time delay at the exit, the latter nevertheless maintaining its sizeable positive value. The relation of the variation of the exit time delay due to the sub-barrier recollisions to the asymptotic momentum shift, and its dependence on the laser field are discussed.

I. INTRODUCTION

The tunneling ionization is at the heart of attoscience [1][2]. The state-of-the-art attoclock technique [3, 4] provides exceptional time resolution of the order of tens of attoseconds via mapping the time to the attoclock offset angle in the angular streaking process. This allows for an experimental investigation of the time delay problem during the quantum tunneling process [5–9]. This fundamental question raised an extensive discussion [5, 44], which can be more appreciated in the context of the general problem of the tunneling time [45–59], recently measured for cold atoms [60].

Two definitions of the tunneling time delay are discussed in strong field ionization. The first is the asymptotic time delay, which is investigated in the attoclock experiments and the second is the time delay near the tunnel exit, which is only observable in a Gedankenexperiment with a virtual detector [61, 62]. The signal of the virtual detector can be derived from the numerical solution of time-dependent Schrödinger equation (TDSE) calculating the current density of the tunneled electron wave packet. The time delay at the tunneling exit calculated numerically with the virtual detector method [9, 12] has been shown to be positive.

The asymptotic time delay is extracted from the photoelectron momentum distribution (PMD) as a shift of PMD with respect to the expected distribution with an assumption of the vanishing time delay. The extraction of the asymptotic time delay can be implemented using the method of the classical backpropagation [19, 22]. In the deep tunneling regime the asymptotic time delay is vanishing [12, 13]. However, near the over-the-barrier ionization (OTBI) regime it is nonnegligible and negative [18, 20]. This negative asymptotic tunneling time delay is explained as arising due to interference of direct and the under-the-barrier recolliding trajectories [29].

Recently, we have investigated the time delay in tunneling ionization using the first-order strong-field approximation (SFA) within the virtual detector approach [63]. The calculation of the SFA wave function based on the direct ionization amplitude showed a positive time delay in the region of the tunnel exit, without invoking recollisions. It has been confirmed that reflections of the electron wavepacket under the tunneling barrier are responsible for this non-zero time delay around the tunnel exit, but not the interference of the direct and recolliding trajectories, the latter being responsible for the asymptotic time delay.

In this paper we continue the investigation of the time delay in tunneling ionization within SFA. Our aim is to analyze the role of the under-barrier recollisions for the tunneling time delay at the tunnel exit. To this end we calculate the wavefunction of the tunneling electron with an accuracy up to the second-order SFA, which includes the recolliding quantum orbits. The Wigner trajectory is constructed corresponding to the peak of the ionized part of the wave function, and the time delay is extracted from the latter. While already the Wigner trajectory based on the first-order SFA wavefunction shows a positive exit time delay [63], here we examine how it is perturbed by the sub-barrier recollisions. Firstly, we employ the same simple model for tunnel-ionization as in the previous study [63], considering a one-dimensional (1D) atom, with an electron bound by a short-range potential, which is ionized by a half-cycle laser pulse. This simple model still contains major features of the tunneling ionization and allows a fully analytical treatment. The 1D treatment is justified as the ionization occurs mainly in the direction of the electric field. The use of a half-cycle laser pulse excludes recollisions via the electron continuum dynamics, that are not related to the tunneling time delay. We consider the regime close to the OTBI regime. In this regime the asymptotic time delay induced by sub-barrier recollisions is not vanishing, in contrast to the deep tunneling regime discussed in [63]. Secondly, we extend the 1D model into three dimensions keeping the short-range character of the binding potential, and show that the qualitative features of the tunneling time delay are maintained in the 3D case.

The structure of the paper is as follows: in Sec. I we introduce the theoretical approach based on the SFA formalism, in particular, the applied low-frequency approximation (LFA). The results for 1D and 3D cases are discussed in Sec. III and our conclusion is given in Sec. IV. Atomic units (a.u.) are used throughout the paper.
II. THEORY

A. Statement of the problem

We consider strong-field ionization of an atom in a unipolar laser field, aiming at the investigation of the tunneling time delay. A short-range potential, \( V(\mathbf{r}) \), is chosen to model an atomic potential. This allows the discussion to focus uniquely on the presence of time delay effects, unlike angular streaking experiments for which long-range Coulomb effects must be accounted. The ionization process is described by the Schrödinger equation

\[
 i \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = (H_0 + H_i) \Psi(\mathbf{r}, t),
\]

with the atomic Hamiltonian

\[
 H_0 = -\frac{\nabla^2}{2} + V(\mathbf{r}),
\]

and the interaction Hamiltonian of the electron with the time-dependent laser field \( \mathbf{E}(t) \) in the length gauge

\[
 H_i = \mathbf{r} \cdot \mathbf{E}(t).
\]

A unipolar laser pulse is employed to avoid multi-half-cycle interference effects, which could hinder the observation of the time-delay signature. The pulse is linearly polarized and has a Gaussian form:

\[
 \mathbf{E}(t) = -E_0 \exp \left[ -\left( \frac{\omega t}{\gamma} \right)^2 \right] \hat{x},
\]

with the field strength \( E_0 \), and the angular frequency \( \omega \). We are interested in the tunneling ionization regime, when the dimensionless Keldysh parameter \[64\]

\[
 \gamma = \sqrt{\frac{L_p}{2U_p}} \ll 1,
\]

is much smaller than unity. Here, \(-L_p = -\kappa^2/2\) is the atomic binding energy, and \( U_p \) the ponderomotive potential which in a linearly polarized laser field is \( U_p = E_0^2/(4\omega^2) \). Throughout this paper we choose \( \gamma = 0.3 \) and are, therefore, always in the quasistatic ionization regime, while varying the laser field strength and, accordingly, the frequency.

B. The Strong Field Approximation

A formal solution to Eq. (1), is given by the time evolution operator \( U(t, t_i) \) which unitarily evolves the wavefunction according to the full Hamiltonian \( H = H_0 + H_i \), from the initial state \( |\phi(t_i)\rangle \) at time \( t_i \) into the state \( |\phi(t)\rangle = U(t, t_i)|\phi(t_i)\rangle \) at time \( t \). In the interaction picture the unitary time evolution operator obeys the Dyson equation

\[
 U(t, t_i) = U_0(t, t_i) - i \int_{t_i}^{t} dt_1 \ U(t_1, t_i) H_i(t_1) U_0(t_1, t_i),
\]

where \( U_0 \) is the evolution operator corresponding to the Hamiltonian \( H_0 \).

We work within the well-known SFA \[64–66\], which assumes that after ionization the electron dynamics are dominated by the laser field, treating any further interactions with the atomic core perturbatively. That is, the full time evolution operator is iterated with respect to the atomic potential \( V \)[67]:

\[
 U(t, t_i) = U_f(t, t_i) - i \int_{t_i}^{t} dt_2 \ U(t_2, t_i) V U_f(t_2, t_i).
\]

Here \( U_f(t, t_i) \) is the time evolution operator for the electron dynamics purely in the laser field:

\[
 U_f(t, t_i) = \sum_{\mathbf{p}} |\Psi_\mathbf{p}(t_i)\rangle \langle \Psi_\mathbf{p}(t)|,
\]

expressed via Volkov states of the electron in the laser field [68],

\[
 |\Psi_\mathbf{p}(t)\rangle = |\mathbf{p}(t)\rangle \exp [-iS_\mathbf{p}(t)],
\]

with the kinetic momentum \( \mathbf{p}(t) = \mathbf{p} + \mathbf{A}(t) \), the phase \( S_\mathbf{p}(t) = \int_0^t d\tau \mathbf{p}(\tau)^2/2 \), and the laser vector potential \( \mathbf{A}(t) = \int_0^\infty E(\tau) d\tau \); the state \( |\mathbf{p}\rangle = \exp [i\mathbf{p} \cdot \mathbf{r}]/(2\pi)^{d/2} \) is the \( d \)-dimensional plane wave state.

Thus, formally, the electron wavefunction in the SFA is obtained as a truncated series in the evolution operator with respect to the atomic potential \( V \), \( U(t, t_i) = \sum_{n=0}^{N} U^{(n)}(t, t_i) \), and the full electron wavefunction takes the form of a series

\[
 |\Psi(t)\rangle = |\Psi_0(t)\rangle + |\Psi_1(t)\rangle + |\Psi_2(t)\rangle + \mathcal{O}(V^2),
\]

which describes the ionization of the ground state \( |\Psi_0(t)\rangle \) via the direct path,

\[
 |\Psi_1(t)\rangle = -i \int_{t_i}^{t} d\tau_1 \int d\mathbf{p} |\Psi_\mathbf{p}(t_1)\rangle \langle \Psi_\mathbf{p}(t_1)| H_i(t_1) |\psi_0(t_1)\rangle,
\]

and via the recollision path, involving one further interaction with the atomic core, given by

\[
 |\Psi_2(t)\rangle = -i \int_{t_i}^{t} d\tau_1 \int_{t_1}^{t} d\tau_2 \int d\mathbf{p} \int d\mathbf{k} \langle \Psi_\mathbf{k}(t_2) | V | \Psi_\mathbf{k}(t_1) \rangle \langle \Psi_\mathbf{k}(t_1)| H_i(t_1) |\psi_0(t_1)\rangle,
\]

with each further interaction with atomic potential corresponding to a higher order term in \( V \).

The canonical interpretation of the above term, Eq. (12), is that it corresponds to the event of an ionized electron being driven by the oscillating laser field back towards the core and scattering from it. However, in the case where the electron is not driven back towards its parent ion, in particular, in the case of the applied unipolar laser field, this term takes on a different meaning. In the quantum orbit picture, this additional perturbative interaction with the atomic core is described as a recollision in the complex (imaginary) time during the sub-barrier dynamics [69, 70].
Such processes were dubbed “under-the-barrier recollisions” in Ref. [29], where it was shown that, for ionization near the threshold for over-the-tunnelling-barrier ionization (OTBI), the interference between the direct ionization terms and under-the-barrier-recollision terms (i.e. between the first order term in the SFA, Eq. (11) and higher orders in the SFA, e.g. Eq. (12)) produced a measurable shift in the resulting asymptotic photoelectron momentum distribution (PMD).

The aim of the current work is to investigate in which extent the time delay around the classical tunnel exit is altered due to the under-the-barrier recollisions, i.e. due to the higher order terms in the SFA. To this end the time dependent wave function $\Psi_0(r,t)$ is calculated and the time delay around the classical tunnel exit is deduced via the maximum in time, $t$, of the spatial probability $|\Psi_0(r,t)|^2$ for a given coordinate, $r$.

For comparison we also consider the asymptotic PMD $|M(p)|^2$ measurable at a detector. This is calculated by the projection of the wave function

$$m(p,t) = \langle \Psi_p(t) | \Psi(t) \rangle,$$

on the free electron Volkov state $|\Psi_p\rangle$, in the limit of asymptotic times:

$$M(p) = \lim_{t \to \infty} m(p,t).$$

In terms of the SFA wave function of Eq. (10), the amplitude, $m(p,t) = m_1(p,t) + m_2(p,t) + ...$, with $m_1(p,t) = \langle \Psi_p(t) | \Psi_1(t) \rangle$, and $m_2(p,t) = \langle \Psi_p(t) | \Psi_2(t) \rangle$, which generates the asymptotic PMD as a similar perturbation series

$$M(p) = M_1(p) + M_2(p) + ...,$$

where the ground state contribution vanishes due to the orthogonality of the bound and free electron states.

### C. The Low Frequency Approximation

In the second-order SFA, recollisions are treated in the first-order Born approximation, which is inaccurate for moderate energy electrons $|p| \sim k$. For the under-the-barrier recollisions, given by Eq. (12), exactly this condition applies and so the recollision treatment must be improved.

In order to do this, we make use of LFA [71, 72]. In the LFA, the recollision matrix element in the Born approximation is replaced by the exact $T(p)$-matrix for the field free scattering off the zero-range potential:

$$\langle \Psi_p(t_2) | V | \Psi_k(t_2) \rangle \to \langle \Psi_p(t_2) | T(p + A(t_2)) | \Psi_k(t_2) \rangle.$$

In the LFA, the amplitude including recollisions with the core is then approximated by the integral:

$$\langle \Psi_p(t) \rangle = -\int_{t_1}^{t_2} dt_1 \int_{t_1}^{t_2} dt_2 \int dp \int dk \langle \Psi_p(t) \rangle \times \langle \Psi_p(t_2) | T(p + A(t_2)) | \Psi_k(t_2) \rangle \langle \Psi_k(t_1) | H(t_1) | \phi_0(t_1) \rangle.$$

We simplify more LFA by correcting the second-order SFA by the so-called LFA-factor. This factor provides an analytical estimate of the effect of the LFA in the quasistatic limit. By considering a constant electric field $E = E_0\hat{x}$, we derive the LFA-factor via the ratio of the second-order momentum amplitude in the LFA to that in the second-order SFA:

$$T_{LFA} = \frac{m_2|_{LFA}}{m_2|_{SFA}}.$$  (18)

This is a relatively straightforward calculation, details of which can be found in Appendix A. In one dimension this ratio is calculated to be

$$T_{LFA}^{(1D)} = 1 - \sqrt{\frac{\pi k^3}{2 E_0}}.$$  (19)

In three dimensions, the calculation contains additional integrations on the transversal coordinate and momentum. Under the assumption that the final momentum lays in the polarization axis $p_x = p_z = 0$, we obtain a similar scaling

$$T_{LFA}^{(3D)} = \sqrt{\frac{\pi k^3}{2 E_0}}.$$  (20)

The LFA-factor is then inserted as a prefactor into the time dependent second-order SFA amplitude and the latter is calculated numerically in the quasi-static regime of $\gamma = 0.3$.

### D. The electron wavefunction in momentum space

The electron wavefunction in momentum space $m(p,t) = m_1(p,t) + m_2(p,t)$ reads in LFA:

$$m_1(p,t) = \int_{t_1}^{t_2} dt_1 \langle \Psi_p(t_1) | H(t_1) | \phi_0(t_1) \rangle.$$  (21)

$$m_2(p,t) = T_{LFA} \int_{t_1}^{t_2} dt_1 \int_{t_1}^{t_2} dt_2 \int_{-\infty}^{\infty} dk \langle \Psi_p(t_2) | V(r) | \Psi_k(t_2) \rangle \times \langle \Psi_k(t_1) | H(t_1) | \phi_0(t_1) \rangle.$$  (22)

In the 1D case the first-order amplitude can be expressed as

$$m_1^{(1D)}(p,t) = \int_{t_1}^{t_2} dt_1 \int_{-\infty}^{\infty} dx_1 \tilde{m}_1^{(1D)}(p,x_1,t_1),$$  (23)

where for convenience we define the integrand of the matrix element

$$\tilde{m}_1^{(1D)}(p,x_1,t_1) = \Psi_p^*(x_1,t_1) H(t_1) \phi_0(x_1,t_1).$$  (24)
The second-order amplitude can be simplified via analytical integration of the recollision coordinate $x_2$ in the first matrix element and the intermediate momentum $k$, which yields

$$m_2^{(1D)}(p, t) = \int_{t_1}^{t_2} dt_1 \int_{t_1}^{t_1} dt_2 \int_{-\infty}^{\infty} dx_1 \tilde{\mu}_2^{(1D)}(p, x_1, t_1, t_2)$$

(25)

with

$$\tilde{\mu}_2^{(1D)}(p, x_1, t_1, t_2) = \mathcal{T}_L D^{1D} \left[ \frac{2\pi}{i(t_2 - t_1)} \right] \times \hat{\Psi}_2(0, t_2) \hat{\Psi}_2(0, t_1) \tilde{\mu}_1^{(1D)}(k_x, x_1, t_1).$$

(26)

where $k_x = (x_1 - \alpha(t_2) + \alpha(t_1))/(t_2 - t_1)$ and $\alpha(t) = \int_0^t d\tau A(\tau)$. Both amplitudes now are expressed via the integral

$$I_1 = \int_{-\infty}^{\infty} dx_1 \exp \left[ a_i x_1 + b_i x_1^2 - \kappa |x_1| \right].$$

(27)

with corresponding coefficients $a_i$ and $b_i$ for $i = 1, 2$, shown in Table I.

|     | $I_1$    | $I_2$    |
|-----|----------|----------|
| $a_i$ | $-i(p + A(t_1))$ | $-i(k_x + A(t_1))$ |
| $b_i$ | $0$ | $\frac{i}{2(t_2 - t_1)}$ |

Table I. Coefficients of the integral Eq. (27) for the first- and second-order SFA. In the above $k_x = -\alpha(t_2) + \alpha(t_1))/(t_2 - t_1)$

The integral of Eq. (27) has an analytic solution

$$I_1 = \frac{\sqrt{\pi} e^{-\frac{(a_1)^2}{4(-b_1)^2}}}{4(-b_1)^{\frac{3}{2}}} \left\{ \frac{a_1}{a_1 - \kappa} \left[ 1 + \text{erf} \left( \frac{a_1 - \kappa}{\sqrt{-b_1}} \right) \right] + (a_1 + \kappa) \left( 1 - \text{erf} \left( \frac{a_1 + \kappa}{\sqrt{-b_1}} \right) \right) \right\}$$

(28)

$$\tilde{\mu}_2^{(3D)}(p_x, x_1, t_1, t_2) = -\int dy_1 dy_2 dz_1 dz_2 \tilde{\mu}_2^{(1D)}(p_x, x_1, t_1, t_2)$$

(30)

with the following correction factor to the 1D case:

$$c_2(x_1, t_1, t_2) = \frac{1}{i(t_2 - t_1)^2 + |x_1|^2}.$$ 

(37)

In the consequent $x_1$-integration in the second-order amplitude a typical value of $|x_1| \sim 1/\kappa$ in $c_2(x_1)$ is assumed [73], after which the integration is carried out analytically. This choice is
justified because $x_s$ is the $x_1$-saddle point of the product of 1D bound state wavefunction $\sim \exp[-\kappa |x_1|]$ with the interaction Hamiltonian $H_I \sim E_0 (t)x_1$.

Thus, the total amplitude (in the $p_s$ plane) in three dimensions can be calculated from the amplitude in one dimension, using Eqs. (24), (26):

$$m_1^{(3D)}(p_s, t) = \int dt_1 m_1^{(1D)}(p_s, 0, t_1) I_1(t_1),$$

(38)

$$m_2^{(3D)}(p_s, t) = \int dt_1 \int dt_2 m_2^{(1D)}(p_s, 0, t_1, t_2) I_2(t_1, t_2),$$

(39)

### E. The electron wavefunction in coordinate space

#### 1. 1D wavefunction

The coordinate wavefunction can be represented straightforwardly via a 1D-Fourier transformation using the SFA amplitudes $m_1$ and $m_2$ as follows:

$$\Psi_{\alpha}(x, t, t_1) = \int_{-\infty}^{\infty} dp m_1(p, t) \Psi_{\alpha}(x, t),$$

(40)

$$\Psi_{\tau}(x, t, t_1) = \int_{-\infty}^{\infty} dp m_2(p, t) \Psi_{\alpha}(x, t),$$

(41)

The momentum integration is performed by SPA yielding an extra factor:

$$\Psi_{\alpha}(x, t, t_1) = \int_{t_1}^{t} dt_1 \int_{-\infty}^{\infty} dx_1 \tilde{m}_1^{(1D)}(p_{s}, x_1, t_1) \sqrt{\frac{2\pi}{i(t-t_1)}} \Psi_{\alpha}(x, t),$$

(42)

$$\Psi_{\tau}(x, t, t_1) = \int_{t_1}^{t} dt_1 \int_{t_1}^{t} dt_2 \int_{-\infty}^{\infty} dx_1 \tilde{m}_2^{(1D)}(p_{s}, x_1, t_1, t_2) \sqrt{\frac{2\pi}{i(t-t_2)}} \Psi_{\alpha}(x, t),$$

(43)

where $p_{s1} = (x-x_1-\alpha(t)+\alpha(t_1))/(t-t_1)$, and $p_{s2} = (-x-\alpha(t_2)+\alpha(t))/(t_2-t_2)$. The coordinate integration can be represented again by the functions $I_1$, with the coefficients $a_i$ in the integral of Eq. (27) are now given by those of Table II.

| $I_1$ | $I_2$ |
|-------|-------|
| $a_i$ | $-i(\tilde{p}_{s1} + A(t_1))$ |
| $b_i$ | $i/2(t-t_1)$ |
|       | $i/2(t_2-t_1)$ |

Table II. Coefficients of the integral Eq. (27) for the first- and second-order SFA wavefunction in the coordinate representation. In the above $\tilde{p}_{s1} = (x-\alpha(t)+\alpha(t_1))/(t-t_1)$, $\tilde{k}_{s2} = -(\alpha(t_2)+\alpha(t))/(t_2-t_1)$.

#### 2. 3D wavefunction

In the 3D case we use the wavefunction in a mixed representation $\Psi(x, p_{s}, p_{s}, t)$ to derive the Wigner trajectory, choosing the most probable values for the transverse momentum $p_{s1} = 0$:

$$\Psi_{\alpha}(x, p_{s}, t) \big|_{p_{s1}=0} = \int_{-\infty}^{\infty} dp_{s1} m_1^{(3D)}(p_{s1}, t) \Psi_{\alpha}(x, t),$$

(44)

and $p_{s2} = (-\alpha(t_2)+\alpha(t))/(t_2-t_1)$.

The calculations similar to the 1D case provide:

$$\Psi_{\alpha}(x, p_{s}, t) \big|_{p_{s1}=0} = \int_{-\infty}^{\infty} dp_{s1} m_1^{(3D)}(p_{s1}, t) \Psi_{\alpha}(x, t),$$

(46)

$$\Psi_{\tau}(x, p_{s}, t) \big|_{p_{s1}=0} = \int_{-\infty}^{\infty} dp_{s1} m_2^{(3D)}(p_{s1}, t) \Psi_{\alpha}(x, t),$$

(47)

Further, the probability distribution at the exit, $x = x_{e}$, is calculated via the wavefunction $|\Psi(x_{e}, p_{s} = 0, p_{s} = 0, t)|^2$. Here, the average exit coordinate $x_{e}$ is obtained by averaging over the tunneling probability (Keldysh-exponent), similar to Ref. [63]:

$$x_{e} = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{dE}{|E|} \exp[-\frac{2\pi}{|E|}]$$

(50)

### III. DISCUSSION

#### A. The Wigner Trajectory

The dynamics of the laser driven electron during strong-field ionization are described by the SFA wavefunction $\Psi_{\alpha}(x, t) + \Psi_{\tau}(x, t)$. The Wigner trajectory $\tau_{\alpha}(x)$ is derived using the probability $P(x, t) = |\Psi_{\alpha}(x, t) + \Psi_{\tau}(x, t)|^2$ of the laser driven
The positive tunnelling time delay at the exit is reduced by an under-the-barrier recollision tends to a negative asymptotic lowed by classical motion), while the trajectory containing delay with respect to the “simple man” model (tunnelling, fol-
tum). The direct trajectory shows vanishing asymptotic time
increase in probability flux implies reduced hindrance of the
current by accounting for an additional ionization channel. An
of ionization through a recollision increases the probability
sub-barrier recollision, which can be intuitively explained
as the reflections hinder the wavepacket crossing the barrier.

![Figure 1](image.png)

**Figure 1.** Wigner trajectories in the 1D case: (orange triangles) the trajectory $\tau(x)$ calculated via the 2SFA wave function including the direct and sub-barrier recolliding paths; (blue circles) the trajectory $\tau(x)$ calculated via the 1SFA wave function including only the direct ionization path. The shaded area indicates regions under the barrier, i.e. smaller than the tunnel exit coordinate $x_e$ given by Eq. (50). The field strength $E_0 = 0.15$ a.u. is below the OTBI threshold.

wavepacket in the following way: for each fixed space point $x$, $\tau_W(x)$ corresponds to the peak of the probability $P(x, t)$. For comparison, we additionally consider an analogous “direct ionization” Wigner trajectory, $\tau_{W0}(x)$, calculated only using the maximum of the direct ionization probability $P(x, t) = |\Psi_i(x, t)|^2$. These Wigner trajectories for the field strength $E_0 = 0.15$ a.u. are shown in Fig. 1 beginning at the typical value $|x_e| \sim 1/\kappa = 1$ a.u., which is the $x_1$-saddle point of the product of the 1D bound state $\sim \exp[-\kappa|x_1|]$ with the interaction Hamiltonian $H_0 = E_0(t)x_1$. The field value is chosen not to exceed, but to be close to the threshold for OTBI, when the tunneling time delay is significant.

Both the direct ionization Wigner trajectory, $\tau_{W0}(x)$, and the one via the full ionization amplitude including a recollision, $\tau_W(x)$, show a positive time delay at the tunnel exit compared to the peak of the laser field. However, the recollision under the barrier acts to reduce the time delay slightly. In Ref. [63], we have shown that the possible time delay of the direct ionization path arises due to reflections inside the barrier, and it is positive as the reflections hinder the wavepacket crossing the barrier. The positive tunnelling time delay at the exit is reduced by the sub-barrier recollision, which can be intuitively explained by the additional positive probability current induced by the recollision. That is, accounting for the additional possibility of ionization through a recollision increases the probability current by accounting for an additional ionization channel. An increase in probability flux implies reduced hindrance of the tunnelling wavefunction which in turn implies a smaller time delay.

Far from the exit, both trajectories approach the classical trajectory (i.e. the trajectory of a classical electron appearing at $x_e$ at the peak of the laser field, with a vanishing momentum). The direct trajectory shows vanishing asymptotic time delay with respect to the “simple man” model (tunnelling, followed by classical motion), while the trajectory containing an under-the-barrier recollision tends to a negative asymptotic time delay with respect to the simple man model. The latter is in accordance with the previous result of Ref. [29].

### B. Time Delay Dependence on Field Strength

The dependence of the time delay at the exit, $\delta t_{exit}$, on the laser field strength, $E_0$, is shown in Fig. 2. With larger fields, the time delay decreases, which was already established for direct ionization in Ref. [63].

However, the effect of recollisions on the time delay, i.e., the difference of the time delay between the direct and recolliding trajectories,

$$\delta t_{exit}^{(r)} = \delta t_{exit}(2SFA) - \delta t_{exit}(1SFA),$$

in this case increases by absolute value, as shown in Fig. 2.

Thus, on the one hand, the sub-barrier recollision decreases the positive time delay at the tunnel exit, i.e. has a negative contribution to the exit time delay. On the other hand, it is known [29, 74] that the sub-barrier recollision also induces a shift of the peak

$$\delta p_{asym} = \max_t[|M_1(p) + M_2(p)|^2] - \max_t[|M_1(p)|^2]$$

of the asymptotic PMD, corresponding to an asymptotic negative time delay

$$\delta t_{asym} = -\delta p_{asym}/E_0.$$  

$$\delta t_{asym} = -\delta p_{asym}/E_0.$$  

We illustrate the latter in Fig. 3, where the asymptotic PMD, $|M_1(p) + M_2(p)|^2$, via the SFA up to first and second orders is shown for a field strength $E_0 = 0.15$ a.u.. A positive moment shift $\delta p \approx 0.08$ a.u. (corresponding to the negative time delay $\delta t_{asym} \approx -0.53$ a.u.) is observed in the peak of asymptotic momentum, which is directly attributable to the under-the-barrier recollision.

We can give an estimate of the scaling of $\delta t_{asym}$ with respect to the field strength by calculating the Wigner time delay [48] of an electron in an adiabatic field [12, 75]. The Wigner delay corresponds to the energy derivative,

$$\delta t_{asym} = \frac{\partial \ln(\Psi_\kappa)}{\partial I_p},$$

of the electron wavefunction in a static field, which in the SFA reads

$$\Psi_\kappa \sim \exp[-\kappa^2/(3E_0)] + i T_{LFA}^{(1D)} \exp[-\kappa^2/E_0].$$

The first term above corresponds to the direct tunneling amplitude, and the second to the sub-barrier recollision. Thus, a straightforward calculation (recalling the binding energy $-I_p = -\kappa^2/2$) yields

$$\delta t_{asym} \sim \frac{e^{2\pi i \kappa^2}}{\kappa^2} \left[-3 \pi \left(\frac{k^3}{E_0}\right) + \sqrt{2\pi} \left(\frac{k^3}{E_0}\right) \right]$$

(56)
Table III. Scaling of tunnelling times and asymptotic momentum w.r.t field strength, $E_0$, for $E_0 \ll \kappa^3$, for three different models of ionization in one dimension (the classical simple man model, direct ionization using the first order SFA, and ionization including one recollision using the second order SFA). These estimates provide upper or lower bounds for these measures which are indicated above as characteristic values. Here $\tau_0 = (\frac{\kappa^3}{E_0})^{1/2}$, as in Eq. (57). For the scaling via 1SFA see Ref. [63].

\[
\begin{array}{cccc}
\text{exit time delay} & \text{Simple man model} & 1\text{SFA} & 2\text{SFA} \\
& & E_0^{-2/3} & E_0^{-2/3} - \tau_0 \exp\left(\frac{-\kappa^3}{E_0}\right) \\
\text{asymptotic time delay} & 0 & \tau_0 \exp\left(\frac{-\kappa^3}{E_0}\right) & \leq 10 \text{ a.u.} \\
\text{asymptotic momentum shift} & 0 & 0 & \leq 0.4 \text{ a.u.} \\
\end{array}
\]

Figure 2. Time delay at the tunnel exit vs the laser field: (a) in the 1D case, (b) in the 3D case, using the first order SFA (blue circles), as well as the second order SFA (orange triangles). Both time delays $\delta t_{exit} = \max_0[\mathcal{P}(x_e, t)] - \max_i[E(t)]$, are calculated as the peak of the temporal probability distribution at the tunnel exit $x_e$, as laid out in the text. In the 2SFA (orange triangles), the probability $\mathcal{P}(x, t) = |\Psi_r(x, t) + \Psi_i(x, t)|^2$ includes the effects of recollisions, whereas as in the 1SFA the probability distribution $\mathcal{P}_r(x, t) = |\Psi_r(x, t)|^2$ accounts only for direct ionization. The difference of these two delays, $|\delta t_{exit}^{2\text{SFA}} - \delta t_{exit}^{1\text{SFA}}|$ increases with field strength, an effect directly attributable to the under-the-barrier recollision.

\[
\delta t_{exit} = \max_0[\mathcal{P}(x_e, t)] - \max_i[E(t)]
\]

A similar derivation yields a three dimensional estimate $\delta t_{exit}^{3\text{D}} \sim (E_0/\kappa^3)$, when the 3D SFA wavefunction is employed, $\psi_{i, \kappa}^{(3\text{D})}\sim \exp[-\kappa^3/(3E_0)] + i c_0(0, t_{s1}, t_{s2}) \mathcal{T}_{LFA}^{(3\text{D})} \exp[-\kappa^3/E_0]$. Here, $t_{s1} = 3 t_{s2} = 3i k/E_0$ are the saddle points of time integration in a constant field, for details see Ref. [29].

These estimates are in good qualitative accordance with the time dependent SFA results shown in Fig. [4].

C. Relationship Between Asymptotic and Exit Delays

It is interesting to see whether there is a relationship between these two time delays, $\delta t_{asym}$ and $\delta t_{exit}$. This question is particularly relevant to attosecond streaking techniques which attempt to extract information on the tunnelling process from...
measurements at distances much greater than the atomic scale. This question is analyzed in Fig. 4, comparing these two time delays.

The qualitative features of the exit time delay are similar in both the 1D and 3D cases. The exit time delay is positive and decreases with larger fields. The sub-barrier recollision reduces the exit time delay [Fig. 2], and the sub-barrier recollision effect increases with the field [Fig. 3]. The value of the time delay in the 3D case is smaller than in 1D, because of the decreasing contribution of the recolliding wave packet which spreads in three dimensions for the case of a short-range atomic potential.

IV. SUMMARY AND CONCLUSION

We have considered the tunneling time delay of an electron in strong-field ionization in a unipolar time-dependent laser field, accounting for under-the-tunneling-barrier processes. The electron wave function within a simplified model of ionization, with a short-range atomic potential, has been calculated analytically using SFA. We considered the direct ionization (via the first-order SFA), and the full ionization amplitudes including the direct tunneling path and the path with the sub-barrier recollision (via the second-order SFA). Employing the wavefunction in its spatial representation, we derived the Wigner trajectory near the tunnel exit. The Wigner trajectory shows a positive time delay near the tunnel exit both with and without the under-barrier processes. However, we find that when one accounts for sub-barrier recollisions the time delay at the exit is decreased slightly, see the summary in Table III.

As is known from Ref. [29], the interference of the direct and sub-barrier recolliding paths induces an asymptotic momentum distribution shift, which is equivalent to a negative time delay with respect to the simple man model. We found a relationship between the change of the exit time delay due to sub-barrier recollisions and the asymptotic time delay. Furthermore, we proved that these time delays are equal in the tunneling regime, as expected because of the same origin related to the effect of the sub-barrier recollision. The field dependence of these time delays is also obtained.

We provided also the 3D generalization of our results. The features of the tunneling time delay were shown to be similar to those in one dimension.

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Appendix A: The LFA factor

1. 1D LFA Factor

We derive the LFA factor in the 1D case. To this end we compare the second-order SFA amplitude with that of LFA in the quasistatic limit, namely, in a static field. The second-order SFA amplitude in a constant field \( F(t) = E_0 \) has the following structure in the 1D case after performing the coordinate integrations analytically:

\[
m^{(1D)}_{2SFA} = \int_{-\infty}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \int dk \mathcal{P}_{2SFA} \exp \left[ -i \frac{p^2}{2} (t - t_2) - ip(\alpha(t) - \alpha(t_2)) \right] \\
- i \frac{k^2}{2} (t_2 - t_1) - ik(\alpha(t_2) - \alpha(t_1)) + ib(t_1) + i \frac{k^2}{2} t_1 ,
\]

where \( \alpha(t) = E_0 t^2 / 2 \) and \( \beta(t) = E_0^3 t^3 / 6 \). In the above, \( \mathcal{P}_{2SFA} = -k^{5/2} / (\sqrt{2\pi})^3 \) is the pre-exponential factor of the second-order SFA, which we is weakly dependent on the integration variables. Without loss of generality we can set \( p = 0 \)
and simplify. Then, using SPA for the \( k \)- and \( t_1 \)-integrations yields

\[
m_{2SFA}^{(1D)} = \int_{t_1}^{\infty} dt_2 \frac{2\pi}{k} \mathcal{P}_{2SFA} \exp \left[ -\frac{k^3}{E_0} - \frac{E_0 k}{2} \left( t_2 - i \frac{k}{E_0} \right)^2 \right],
\]

where the exponent is already expanded quadratically around the saddle point in \( t_2 \). The integration contour in Eq. (A2) consists of two parts: 1) from \( t_1 \) to \( i k/E_0 \), and 2) from \( i k/E_0 \) to \( \infty \). The integration along the first part of the contour gives the direct ionization amplitude \( m_1 \), see Appendix [B] cf. [76], which is dropped because in this section our aim is to derive the LFA factor for the recollision amplitude. Then, the integral along the second part of the contour yields

\[
m_{2SFA}^{(1D)} = \mathcal{P}_{2SFA} \frac{(2\pi)^{3/2}}{2k \sqrt{\pi E_0}} \exp \left[ -\frac{k^3}{E_0} \right].
\]

Now we calculate the corresponding LFA amplitude. In the 1D LFA, the SFA pre-exponential factor \( \mathcal{P}_{2SFA} \) is replaced by

\[
\mathcal{P}_{LFA}^{(1D)}(t_2) = \frac{E_0 t_2}{E_0 t_2 - ik} \mathcal{P}_{2SFA},
\]

see [77], and we consequently arrive at

\[
m_{LFA}^{(1D)} = \int_{t_1}^{\infty} dt_2 \frac{2\pi}{k} \mathcal{P}_{LFA}(t_2) \exp \left[ -\frac{k^3}{E_0} - \frac{E_0 k}{2} \left( t_2 - i \frac{k}{E_0} \right)^2 \right].
\]

The latter is calculated analytically in the same way as that of Eq. (A2), yielding

\[
m_{LFA}^{(1D)} = \int_{t_1}^{\infty} dt_2 \frac{\pi}{k} \mathcal{P}_{LFA}(t_2) \exp \left[ -\frac{k^3}{E_0} - \frac{E_0 k}{2} \left( t_2 - i \frac{k}{E_0} \right)^2 \right] = \left( 1 - \sqrt{\frac{\pi k^3}{2 E_0}} \right) m_{2SFA}^{(1D)}.
\]

Thus, we derive the LFA factor in the 1D case:

\[
\mathcal{T}_{LFA}^{(1D)} = 1 - \sqrt{\frac{\pi k^3}{2 E_0}},
\]

which corrects the SFA recollision amplitude as it incorporates the exact scattering amplitude in the SFA recollision matrix element.

### 2. 3D LFA factor

In the LFA for a 3D system with a short-range potential, the SFA pre-exponential factor \( \mathcal{P}_{2SFA} \) is replaced by

\[
\mathcal{P}_{LFA} = -\frac{ik}{E_0 t_2 - ik} \mathcal{P}_{2SFA},
\]

see [77]. A calculation similar to 1D gives the following expression for the LFA recollision amplitude in the 3D case:

\[
m_{LFA}^{(3D)} = \sqrt{\frac{\pi k^3}{2 E_0}} m_{2SFA}^{(3D)}.
\]

Thus providing the LFA correction factor in the 3D case:

\[
\mathcal{T}_{LFA}^{(3D)} = \sqrt{\frac{\pi k^3}{2 E_0}}.
\]

### Appendix B: Calculation of the integral along the vertical contour

The V-SFA momentum amplitude in one dimension in a constant field reads after the two coordinate integrations

\[
m_{2SFA}^{(1D)} = \int_{t_1}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \frac{d\kappa}{2 \sqrt{\pi E_0} \sqrt{t_2 - t_1}} \exp \left[ -i \frac{k^2}{2} \left( t_2 - t_1 \right) - ik \left( \alpha(t_2) - \alpha(t_1) \right) + i\beta(t_1) + i \frac{k^2}{2} t_1 \right],
\]

where \( p = 0 \) was used. We carry out firstly the integration over the intermediate momentum \( k \) by SPA and arrive at

\[
m_{2SFA}^{(1D)} = \int_{t_1}^{\infty} dt_1 \frac{\pi}{k} \mathcal{P}_{2SFA} \frac{\sqrt{2\pi}}{\sqrt{t_2 - t_1}} \exp \left[ -i \frac{k^2}{2} \left( t_2 - t_1 \right) - ik \left( \alpha(t_2) - \alpha(t_1) \right) + i\beta(t_1) + i \frac{k^2}{2} t_1 \right]
\]

with \( k_s = -\left( \alpha(t_2) - \alpha(t_1) \right) / (t_2 - t_1) \). The following \( t_2 \)-integral consists of a vertical contour from \( t_1 \) to \( t_2 \), and a horizontal contour from \( t_2 \) to \( \infty \). In this section we want to estimate the first which has its dominant contribution in the region around \( t_2 = t_1 \). We therefore expand the integrand in \( t_2 \) around \( t_1 \):

\[
m_{2SFA}^{(1D)} = -\int_{t_1}^{\infty} dt_1 \frac{\pi}{k} \mathcal{P}_{2SFA} \frac{\sqrt{2\pi}}{\sqrt{t_2 - t_1}} \exp \left[ -i \frac{k^2}{2} (t_2 - t_1) t_1^2 + i \frac{k^2}{6} t_1^3 \right] \]

and integrate analytically

\[
m_{2SFA}^{(1D)} = \int_{t_1}^{\infty} dt_1 \frac{\pi}{k} \mathcal{P}_{2SFA} \frac{\sqrt{2\pi}}{\sqrt{t_2 - t_1}} \exp \left[ -i \frac{k^2}{6} (t_2 - t_1)^2 + i \frac{k^2}{3} t_1 \right].
\]

The final integral is again evaluated via SPA, where the exponent is expanded quadratically at the saddle point \( t_{1,s} = i k / E_0 \) and the latter is inserted into the preexponential. With these approximations we derive the direct ionization amplitude

\[
m_{2SFA}^{(1D)} = \frac{ik}{\sqrt{E_0}} \exp \left[ -\frac{k^3}{3 E_0} \right].
\]
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