AUTOHOMEOMORPHISMS OF THE FINITE POWERS OF THE DOUBLE ARROW.

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Abstract

Let $A$ and $S$ denote the double arrow of Alexandroff and the Sorgenfrey line, respectively. We show that any homeomorphism $h : m A \to m A$ is locally (outside of a nowhere dense set) a product of monotone embeddings $h_i : J_i \subseteq A \to A (i \in m)$ followed by a permutation of the coordinates.

We also prove that the symmetric products $F_m(A)$ are not homogeneous for any $m \geq 2$. This partially solves an open question of A. Arhangel’skii [Ar87]. In contrast, we show that symmetric product $F_2(S)$ is homogeneous.

Keywords: Double arrow, hyperspaces, homogeneous spaces, Sorgenfrey.

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1 INTRODUCTION

A space $X$ is homogeneous if for every $x, y \in X$ there exists a autohomeomorphism $h$ of $X$ such that $h(x) = y$. Several classic results on homogeneity involve the study of the hyperspace $Exp(X)$ (set of closed subsets of $X$) in the Vietoris topology. In this paper we are motivated by the following general question.

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Question 1.1. When is \( \text{Exp}(X) \) homogeneous?

In the seventies, it was shown by Shori and West [SW75] that \( \text{Exp}([0,1]) \) is homeomorphic to the Hilbert’s cube. In particular, is possible that the \( \text{Exp}(X) \) is homogeneous while \( X \) is not. On the other hand, if \( \kappa > \aleph_1 \), then \( \text{Exp}(2^\kappa) \) is not homogeneous (see [Sce76]). Thus, the question of homogeneity of the hyperspace turns out to be quite subtle.

Arhangel’skiǐ in [Ar87] asked the following question (it also appears in [AvM13]).

Question 1.2. Is the hyperspace \( \text{Exp}(A) \) homogeneous?

In this paper we partially answer Question 1.2, by showing that.

Theorem 1.3. The symmetric product \( \mathcal{F}_m(A) \) is not homogeneous for any \( m \geq 2 \).

Where the symmetric product \( \mathcal{F}_m(A) \) is the subspace of \( \text{Exp}(A) \) consisting of all finite, non-empty, subsets of cardinality at most \( m \).

The following result can be seen as a companion of the previous Theorem.

Theorem 1.4. The symmetric product \( \mathcal{F}_2(S) \) is homeomorphic to \( S^2 \). In particular, it is homogeneous.

In the course of proving Theorem 1.3, we study the group of autohomeomorphisms of \( mA \), and obtain the following Theorem which give us a complete picture on the structure of such autohomeomorphisms.

Theorem 1.5. Let \( h : mA \to mA \) be a homeomorphism. Then there is a pairwise disjoint sequence of basic clopen boxes \( U_n := \prod_{j \in m} I_j \) such that \( \bigcup U_n \) is dense in \( mA \) and \( h \restriction U_n = \sigma \circ h^0 \times \cdots \times h^{m-1} \), where each \( h^i : I_i \to A \) is a strictly monotone homeomorphism onto a clopen interval, and \( \sigma \) is a permutation of \( mA \).

The paper is organized as follows. In Section 2 we study the autohomeomorphisms of \( mA \) and give a proof of Theorem 1.5. In Section 3 we give a proof of Theorem 1.3 and in Section 4 a proof of Theorem 1.4. The notation and terminology in this paper is fairly standard. We will use [En89] as a basic reference on topology and [AvM13] as a reference for homogeneity and hyperspaces.

2 AUTOHOMEOMORPHISMS OF \( mA \)

The purpose of this section it to prove Theorem 1.5. It will be convenient to introduce some notation. Let \( A_0 = [0,1] \times \{0\}, A_1 = [0,1] \times \{1\} \) and \( A = A_0 \cup A_1 \). Define the lexicographical ordering \( (a,r) < (b,s) \) if \( a < b \) or \( a = b \) and \( r < s \). The set \( A \) with the order topology is the double arrow space.

The let \( \pi : \mathbb{A} \to [0,1] \) be the projection onto the first factor \( \pi((x,r)) = x \), we will think of an element of the finite power \( x \in mA \) as function \( x : m \to A \). For any \( a \in A \) we will denote by \( \overline{a} \) the constant sequence \( a \) of arbitrary finite length. Let \( \pi_i : mA \to A \) be the projection onto the \( i \)-coordinate, and let \( h_i = \pi_i \circ h \). Recall that a partial function \( f : A \to A \) is monotone if it is either non-decreasing or non-increasing, and \( f \) is strictly monotone if it is either strictly increasing or strictly decreasing.

We now recall the following result from R. Hernández-Gutiérrrez.
Proposition 2.1 ([HG13] Proposition 3.1). Let \( h : \mathbb{A} \to \mathbb{A} \) be a continuous function, then there exists a pairwise disjoint sequence \( J_n (n \in \omega) \) of clopen intervals such that \( \bigcup_{n \in \omega} J_n \) is dense in \( \mathbb{A} \), and \( h \upharpoonright J_n \) is monotone, for any \( n \in \omega \).

The following proposition tell us how continuous monotone functions look like locally.

Proposition 2.2. Let \( h : \mathbb{A} \to \mathbb{A} \) be a monotone continuous function. Then there is a clopen interval \( J \) such that either \( h \upharpoonright J \) is constant or \( h \upharpoonright J \) is strictly monotone.

Proof. On one hand, if there is a clopen interval \( J \) so that \( h \upharpoonright J \) is a injection, then there is nothing to prove. On the other hand, if there are \( x, y \in \mathbb{A} \) so that \( \pi(x) \neq \pi(y) \) and \( h(x) = h(y) \), then we are similarly finished. If neither of the above alternatives hold, then there exists a dense set \( D \subset \mathbb{A} \times \{0, 1\} \) so that \( h \upharpoonright (\mathbb{A} \setminus D) \) is strictly monotone and \( h(\langle a, 0 \rangle) = h(\langle a, 1 \rangle) \) for any \( \langle a, 0 \rangle, \langle a, 1 \rangle \in \mathbb{A} \). However, such a function cannot be continuous. This finishes the proof of the Lemma.

Definition 2.3. Let \( h : \mathbb{A} \to^m \mathbb{A} \) and \( j_0 \in m \) be given. We say that a clopen interval \( J \) is \( j_0 \)-good for \( h \) if \( h_{j_0} \upharpoonright J \) is strictly monotone and \( h \upharpoonright J \) is constant for any \( j \in m \setminus \{j_0\} \). We say that \( J \) is good for \( h \) if it is \( j_0 \)-good for some \( j_0 \in m \).

Remark 2.4. In other words, \( J \) is \( j_0 \)-good for \( h \) if and only if \( h \) sends \( J \) into a line parallel to the \( j_0 \)-th axis.

The following lemma gives an indication as to why this definition will play a role. It will be used in the verification of Theorem 1.5.

Lemma 2.5. Let \( h : \mathbb{A} \to^m \mathbb{A} \) be an embedding such that \( h''[\mathbb{A}] \) is \( G_\delta \) in \( ^m \mathbb{A} \). Then there exists a pairwise disjoint sequence \( J_n (n \in \omega) \) of clopen intervals such that \( \bigcup_{n \in \omega} J_n \) is dense in \( \mathbb{A} \) and for each \( n \), there is \( j \in m \) so that \( J_n \) is \( j \)-good for \( h \).

Proof. Let \( U \) denote the union of all clopen intervals which are good for \( h \). Since \( \mathbb{A} \) is separable, it suffices to show that \( U \) is dense. In order to get a contradiction, suppose that there is a nonempty clopen interval \( J \) disjoint from \( U \). By going to a clopen sub-interval of \( J \) if necessary, and applying, Proposition 2.1 and Lemma 2.2, \( m \) times, we may assume that \( h_j \upharpoonright J \) is either one-to-one or constant, for any \( j \in m \). Since \( h \) is an embedding there is at least one \( j \in m \), such that \( h_j \) is non-constant (equivalently, strictly monotone). Therefore, we are left to show that there is at most one \( j \in m \) so that \( h_j \) is non-constant.

Claim 2.6. If there exists \( j_0 \neq j_1 \in m \) such that \( h_{j_0} \upharpoonright J \) and \( h_{j_1} \upharpoonright J \) are one-to-one, then \( X = h''[J] \) is not a \( G_\delta \) in \( ^m \mathbb{A} \).

Proof. Let \( X \subseteq \bigcap_{n \in \omega} U_n \), where each \( U_n \) is an open set. Since \( X \) is compact, we may assume, that \( U_n = \bigcup_{k_n} \prod_{j \in m} I_{n,s}^j \), where \( I_{n,s}^j \) are clopen intervals. Let \( A \) be equal to

\[ \{ \pi(x) : \exists n \in \omega \exists i \leq k_n, j \in \{j_0, j_1\} \{x \in \{\min(T_{n,s}^j), \max(T_{n,s}^j)\}) \} \]

Fix \( x \in X \) so that \( \pi(x(i_0)) \) does not belong to \( A \), this is possible as \( A \) is countable and \( h_{i_0} \upharpoonright J \) is an injection. We claim that both points

\[ x \upharpoonright (m \setminus \{i_0\}) \cup \{\{i_0, \langle \pi(x), 0 \rangle \} \text{ and } x \upharpoonright (m \setminus \{i_0\}) \cup \{\{i_0, \langle \pi(x), 1 \rangle \} \text{ are distinct.} \]

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belong to $\bigcap_{n \in \omega} U_n$. However, only one of them belongs to $X$ as $h_{j_1}$ is injective, which is a contradiction. In order to prove the claim, fix $N \in \omega$. As $x \in X \subseteq \bigcap_{n \in \omega} U_n$. We can find $i \in k_n$ so that $x \in \prod_{j \in m} I^i_{n,j}$. Note that, $\langle \pi(x(i_0)), 0 \rangle$ and $\langle \pi(x(i_0)), 1 \rangle$, both belong to $I^i_N$, as neither of them are the maximum nor the minimum. Thus, $x \upharpoonright (m \setminus \{i_0\}) \cup \{i_0, \langle \pi(x), 0 \rangle\}$ and $x \upharpoonright (m \setminus \{i_0\}) \cup \{i_0, \langle \pi(x), 1 \rangle\}$ belong to $U_N$ as required. This finishes the proof of the Claim.

Observe that since $h''[\mathcal{A}]$ is a $G_\delta$ in $^m \mathcal{A}$ and $J$ is clopen in $\mathcal{A}$, it follows that $X$ is also a $G_\delta$ in $^m \mathcal{A}$, which contradicts the previous Claim.

We are now ready to prove the main Theorem of the section.

**Theorem 2.7.** Let $h: ^m \mathcal{A} \to ^m \mathcal{A}$ be a homeomorphism. Then there is a pairwise disjoint sequence of basic clopen boxes $U_n := \prod_{j \in m} I^i_n (n \in \omega)$ such that $U_n$ is dense in $^m \mathcal{A}$ and $h \upharpoonright U_n = \sigma \circ h^0 \times \cdots \times h^{m-1}$, where each $h^i: I^i_n \to \mathcal{A}$ is an strictly monotone homeomorphism onto a clopen interval, and $\sigma$ is a permutation of $^m \mathcal{A}$.

**Proof.** Since $^m \mathcal{A}$ is separable, and every clopen box is homeomorphic to $^m \mathcal{A}$ via a product of strictly increasing homeomorphisms. Thus, it suffices to show that there is a clopen box so that $h$ restricted to it is as desired. For each $a \in ^{m-1} \mathcal{A}$ and $i \in m$, define the line $E_{a,i} = \{ x \in ^m \mathcal{A} : x \upharpoonright (m \setminus \{i\}) = a \}$, and define an embedding $e_{a,i}: \mathcal{A} \to ^m \mathcal{A}$ by $e_{a,i}(p) = a \cup \{(i, p)\}$. Note that $h''[E_{a,i}]$ is a $G_\delta$ in $^m \mathcal{A}$, since $E_{a,i}$ is $G_\delta$ in $^m \mathcal{A}$, and $h$ is a homeomorphism. We will recursively construct, for $j \in m$, clopen boxes $V_j := \prod_{i \in m} J^i_j \subseteq ^m \mathcal{A}$ and functions $\sigma_j: \{0, \ldots, j\} \to m$ such that

i. $V_{j+1} \subset V_j$ for $j \in m - 1$.

ii. For each $i < j$ and $a \in \pi_{m-\{i\}} \{V_j\}$, $I^i_j$ is $a(i)$-good for $h \circ e_{a,i}$.

iii. $\sigma_{j+1} \upharpoonright j = \sigma_j$ and $\sigma_i$ is injective for $j \in m - 1$.

Suppose we have constructed $V_j, \sigma_j$ for $j < k \leq m$. By applying Lemma 2.5 to the map $h \circ e_{a,k} \upharpoonright I^k_{-1}$, we can find for each $a \in A := \pi_{m-\{k\}} \{V_{k-1}\}$, rationals $q_a, r_a \in Q$ and an integer $j_a,k$ such that $\{[q_a, 1), (r_a, 0)]\} \subseteq I^i_{k-1}$ is $j_a,k$-good for $h \circ e_{a,k}$. Since $A$ is a Baire space, there exists $j_a, q, r$ such that $A_{j_a,q,r} := \{ x \in A : q_a = q, r_a = r, j_a,0 = j_a \}$ is dense in some clopen box $V := \prod_{j \in m} \{0 \times \cdots \times J_{m-1} \} \subseteq A$.

**Claim 2.8.** The interval $\{[q, 1), (r, 0)\}$ is $j_a$-good, for $h \circ e_{a,k}$ for any $x \in V$.

**Proof.** We must show that $h_1 \circ e_{x,k} \upharpoonright \{[q, 1), (r, 0)\}$ is constant for any $j \in m \setminus \{j_0\}$. In order to do so, pick $j \in m \setminus \{j_0\}$, $x \in V$ and $s, t \in ([q, 1), (r, 0)]$. Choose a sequence $x_n$ of elements of $A_{j_0,q,r}$ converging to $x$, this is possible as $A_{j_0,q,r}$ is dense in $V$. Notice that $e_{x,n,k}(s)$ and $e_{x,n,k}(t)$ converge to $e_{x,k}(s)$ and $e_{x,k}(t)$, respectively. By assumption, we have that $h_1(e_{x,n,k}(s)) = \lim_{n \to \infty} h_1(e_{x,n,k}(s)) = \lim_{n \to \infty} h_1(e_{x,n,k}(t)) = h_1(e_{x,k}(t))$. Define $V_k = J_0 \times \cdots \times J_{k-1} \times ([q, 1), (r, 0)] \times J_{k+1} \times \cdots \times J_{m-1}$ and $\sigma_k = \sigma_{k-1} \cup \{(k, j_0)\}$. 

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It follows from the previous claim that properties i. and ii. hold and also clearly \( \sigma_k \) extends \( \sigma_{k-1} \). Hence, we are left to show that \( \sigma_k \) is injective. Suppose for a contradiction that \( \sigma_k(i_0) = \sigma_k(i_1) \) for some \( i_0 \neq i_1 \). Pick \( a \neq b \in V_k \) so that
\[
a^{i_0} := a \mid m - \{i_0\} = b \mid m - \{i_0\} := b^{i_0}, \text{ and let } a^{i_1} = a \mid m - \{i_1\}, b^{i_1} = b \mid m - \{i_1\}.
\]
Notice that \( |E_{a^{i_0},i_0} \cap E_{a^{i_1},i_1}| = 1 \). Hence, it follows that \( h''(E_{a^{i_0},i_0} \cap V_k) \subset E_{h(a)^{i_0},i_0} \) and also \( h''(E_{a^{i_1},i_1} \cap V_k) \subset E_{h(a)^{i_1},i_1} \). By assumption, \( E_{h(a)^{i_0},i_0} = E_{h(a)^{i_1},i_1} \). However, this would imply that \( h''(E_{a^{i_0},i_0} \cap V_k) \cap h''(E_{a^{i_1},i_1} \cap V_k) = \emptyset \), which contradicts the fact that \( |E_{a^{i_0},i_0} \cap E_{a^{i_1},i_1} \cap V_k| = 1 \). Finally, let \( \sigma = \sigma_{m-1} \) and let \( h^j = h_j \circ \sigma_{\ell} \mid I_j \) for some fixed \( a \in V_k \) and \( j \in m \). It follows from our construction that \( h \mid V_k = \sigma \circ h^0 \times \cdots \times h^m \) as required.

The previous Theorem give us an a posteriori explanation of why the space \( mA \) is not countable dense homogeneous for any \( m \geq 2 \). A fact first observed by Arhangel’skii and van Mill [AvM13] for \( m = 1 \) and by Hernández-Gutiérrez in the case \( m \geq 2 \).

**Corollary 2.9 ([HG13]).** The space \( mA \) is not countable dense homogeneous for any \( m \geq 1 \).

### 3 NON-HOMOGENEITY OF \( F_m(A) \)

In this section we prove Theorem 1.3. It will be convenient to introduce some notation.

Let \( \Delta_m = \{ x \in mA : \forall i \in m - 1 (x(i) \leq x(i + 1)) \} \), let \( \rho: \Delta_m \to F_m(A) \) be the map given by \( \rho(x) = (x(0), \ldots, x(m - 1)) \) and let \( \sim \) denote the equivalence relation on \( \Delta_m \) defined by \( x \sim y \) if and only if \( \rho(x) = \rho(y) \). Finally, let \( q: \Delta_m \to \Delta_m/ \sim \) be the quotient map, we will sometimes write \([x]\), instead of \( q(x)\), to represent the equivalence class. We consider \( \Delta_m/ \sim \) as a topological space with the quotient topology.

The following classical fact give us a more geometric representation of \( F_m(A) \).

**Proposition 3.1 ([Ga54]).** The map \( \tilde{\rho}: \Delta_m/ \sim \to F_m(A) \) given by \( \tilde{\rho}([x]) = \rho(x) \) is a homeomorphism.

The next proposition is straightforward and it is left to the reader.

**Proposition 3.2.** Every clopen subset of \( mA \) is homeomorphic to \( mA \).

**Lemma 3.3.** If \( F_m(A) \) is homogeneous, then it is homeomorphic to \( mA \).

**Proof.** Suppose \( F_m(A) \) is homogeneous, then there is an autohomeomorphism \( h: \Delta_m/ \sim \to \Delta_m/ \sim \) such that \( h([x]) = [x] \), where \( x_0 < x_1 < \cdots < x_{m-1} \). On one hand, notice that, if \( J_0 < \cdots < J_{m-1} \) is a sequence of pairwise disjoint clopen intervals with \( x_i \in J_i \) for \( i \in m \), then \( q \mid \prod_{i \in m} J_i: \prod_{i \in m} J_i \to \Delta_m/ \sim \) is an homeomorphism.

On the other hand, observe that for any \( 0 < \epsilon < 1 \) the clopen cube \( m[0, \epsilon] \) is a saturated neighborhood of \( \overline{0} \) such that \( q''(m[0, \epsilon]) \) is homeomorphic to \( \Delta_m/ \sim \). Since \( h \) is continuous there is an \( \epsilon > 0 \) such that \( h''(m[0, \epsilon])/ \sim \subset \prod_{i \in m} J_i \). It follows, from the previous Lemma, that \( mA \cong \prod_{i \in m} J_i \cong h''(m[0, \epsilon]/ \sim) \cong m[0, \epsilon]/ \sim \cong \Delta_m/ \sim \).

**Theorem 3.4.** The hyperspace \( F_m(A) \) is not homogeneous for any \( m \geq 2 \).
Proof. Suppose for a contradiction that there is a homeomorphism \( h : \Delta_m/\sim \to^{m} \mathbb{A}, \) and let \( \Gamma = \{[x] \in \Delta_m/\sim : x \in \mathbb{A} \}. \) Observe that \( \Gamma \) is homeomorphic to \( \mathbb{A} \) and it is not \( G_\delta \) in \( \Delta_m/\sim \) as \( q^{-1}(\Gamma) = \{(x, \ldots, x) \in m A : x \in \mathbb{A} \} \) is not a \( G_\delta \) in \( \Delta_m \) (by Lemma 2.5). We now consider the embedding \( \alpha : \mathbb{A} \to^{m} \mathbb{A} \) given by \( \alpha(x) = h([x]) \). Since \( h''(\Gamma) \) is not a \( G_\delta \) in \( \mathbb{A} \), it follows, again by Lemma 2.5, that there exits \( j_0 \neq j_1 \in m \) and a clopen interval \( J \) such that \( \alpha_0 := \pi_{j_0} \circ \alpha \) and \( \alpha_1 := \pi_{j_1} \circ \alpha \) are strictly monotone restricted to \( J \). We will assume that both \( \alpha_0 \mid J, \alpha_1 \mid J \) are strictly increasing, as the other cases are analogous.

Claim 3.5. There is a countable subset \( C \subseteq \pi''[J] \) such that \( \pi(\alpha_{i_0}(\langle a, 0 \rangle)) = \pi(\alpha_{i_1}(\langle a, 1 \rangle)) \) and \( \pi(\alpha_{i_1}(\langle a, 0 \rangle)) = \pi(\alpha_{i_1}(\langle a, 1 \rangle)) \) for any \( a \in \pi''[J] \setminus C \).

Proof. Let \( C_k = \{a \in \pi''[J] : \alpha_{i_k}(\langle a, 0 \rangle) < \pi(\alpha_{i_0}(\langle a, 1 \rangle)) \} \) for \( k \in 2 \). For each \( a \in C_k \), pick a rational \( q_a \) such that \( \alpha_{i_k}(\langle a, 0 \rangle) < q_a < \pi(\alpha_{i_0}(\langle a, 1 \rangle)) \). Observe that since \( \alpha_{i_k} \) is strictly monotone, the map \( f : C_k \to C \) given by \( f(\alpha) = q_a \), is one-to-one. Thus, \( C = C_0 \cup C_1 \) is countable as desired.

For each \( a \in A := \pi''[J] \setminus C \). Let \( P_a^+ = \alpha(\langle a, 0 \rangle) \), \( Q_a^+ = \alpha(\langle a, 1 \rangle) \) and let

\[
P_a^+ = \alpha(\langle a, 0 \rangle) \setminus \{\alpha_{i_0}(\langle a, 0 \rangle) \} \cup \{\pi(\alpha_{i_0}(\langle a, 0 \rangle)) \}
\]

and

\[
Q_a^- = \alpha(\langle a, 1 \rangle) \setminus \{\alpha_{i_1}(\langle a, 0 \rangle) \} \cup \{\pi(\alpha_{i_1}(\langle a, 0 \rangle)) \}.
\]

Pick an element \( [x_a] \) belonging to

\[
h^{-1}(\{P_a^+, P_a^-, Q_a^+, Q_a^-\}) \setminus \{\tilde{\rho}^{-1}(\langle 0, 0 \rangle), \tilde{\rho}^{-1}(\langle 0, 1 \rangle), \tilde{\rho}^{-1}(\langle 1, 0 \rangle), \tilde{\rho}^{-1}(\langle 1, 1 \rangle)\}.
\]

Observe that, by our choice of \( x_a \), for any \( x \sim x_a \) there is a \( j \in m \) so that \( \pi(x(j)) \neq a \). By successively refining \( A \), we can find an uncountable subset \( B \subseteq A \), a natural number \( N \) so that \( q^{-1}(\{x_a\}) = \{x_a' : i \in N\} \), an \( N \)-tuple \( (j_0, \ldots, j_{N-1}) \in m^N \), a rational number \( q \in \mathbb{Q} \) such that \( x_a' = x_a''(j_i) \) for all \( i, j \in N \) and \( q \in \pi(x_a'(j_i)) \). If \( h(x_a) = P_a^+ \) (the case \( h(x_a) = Q_a^- \) is analogous), for any \( a \in B \). Consider the clopen cubes \( U := \langle [q, 1, 1, 0] \rangle \) and \( V := \langle [q, 0, 1, 0] \rangle \). Since both cubes are saturated we have that \( U := q''[U] \) and \( V := q''[V] \) form a clopen partition of \( \Delta_m/\sim \). Observe that \( X := \{\langle a, 0 \rangle : a \in B \} \subset U \) and \( Y := \{\langle x_a \rangle : a \in B \} \subset V \), since \( B \) is infinite (uncountable) and \( \Delta/m/\sim \) is compact, it follows that its accumulation points are non-empty and disjoint. However, this contradicts the fact that \( h''(X) = \{P_a^- : a \in B \} \) and \( h''(Y) = \{P_a^+ : a \in B \} \) have the same set of accumulation points. This finishes the proof of the Theorem.

It would be interesting to see if the above theorem can be extended to the hyperspace of all non-empty finite subsets \( \pi(\mathbb{A}) \).

Question 3.6. Is the hyperspace \( \pi(\mathbb{A}) \) homogeneous?

\(^2\)We are using the convention that \([a, b] = \min(a, b), \max(a, b)\).
4 HOMOGENEITY OF $\mathcal{F}_2(\mathbb{S})$.

In this section we prove Theorem 1.4. It is worth mentioning that our proof is based on work of Bennett, Burke and Lutzer [BBL12] and we will also borrow some of its notation.

First of all, since the Sorgenfrey line is homeomorphic to $[0, 1]$ with the subspace topology, we will assume that the $\mathbb{S} = [0, 1]$. A Sorgenfrey rectangle denote any set of the form $[a, b] \times [c, d]$ where $a, b, c, d \in [0, 1]$; $a < b$ and $c < d$. By the Euclidean closure of such a rectangle we mean its closure in the euclidean topology of $\mathbb{S}$. Let $\Delta_2 = \{(x, y) \in \mathbb{S}^2 : x \leq y\}$ and let $\Delta$ be the diagonal of $\mathbb{S}$.

For each $k \geq 1$, let $L_k$ be the straight line joining the points $(0, \frac{k}{3})$ and $(1, 1)$.

**Proposition 4.1 ([BBL12]).** There is a countable collection $\mathcal{T}$ of pairwise disjoint Sorgenfrey rectangles such that:

1. $\bigcup \mathcal{T} = \Delta_2 \setminus \Delta$;
2. for each $T \in \mathcal{T}$, the Euclidean closure of $T$ is disjoint from $\Delta$;
3. for each $x \in [0, 1]$ the set $\{T \in \mathcal{T} : T \cap \{\{x\} \times [x, 1]\} \neq \emptyset\}$ is infinite and can be indexed as $\{T_m : m \geq 1\}$ in such a way that for all $k$ points of $T_k$ lie above points of $T_{k+1}$.
4. for each $T \in \mathcal{T}$, there is a $k \geq 1$ such that $T$ is between $L_k$ and $L_{k+2}$.

It worth pointing out that clause (4) is a consequence of the proof in [BBL12].

Let $T$ be an element of the partition $\mathcal{T}$. If $T = [a, b] \times [c, d]$, then define $T^U = [a, b] \times [\frac{c + d}{2}, d]$, $T^L = [a, b] \times [\frac{c + d}{2}, c]$ and $T^S = [c, d] \times [a, b]$. Finally, let $T^S = \{T^S : T \in \mathcal{T}\}$.

**Remark 4.2.** Note that $T^U$ is the upper half of $T$, $T^L$ is the lower half of $T$ and $T^S$ is the reflection of $T$ across the diagonal $\Delta$.

**Theorem 4.3.** The spaces $\Delta_2$ and $\mathbb{S}^2$ are homeomorphic.

**Proof.** We shall define a homeomorphism $h : \Delta_2 \to \mathbb{S}^2$. First, we will define the function $h$. After that we will prove that $h$ is 1-1 and onto. Finally, we will prove that $h$ and its inverse are continuous.

For each $T = [a, b] \times [c, d] \in \mathcal{T}$, we consider the following homeomorphisms

$$h_{T^U, T^S} : T^U \to T^S \text{ and } h_{T^L, T} : T^L \to T$$

given by $h_{T^U, T^S}(x, y) = (2y - d, x)$ and $h_{T^L, T}(x, y) = (x, 2y - c)$, respectively. Let

$$h := Id_\Delta \cup \bigcup_{T \in \mathcal{T}} \left( h_{T^L, T} \cup h_{T^U, T^S} \right),$$

where $Id_\Delta$ represents the identity function restricted to the diagonal. Since $\Delta_2 = \Delta \cup \bigcup_{T \in \mathcal{T}} (T^U \cup T^L)$ and $\mathbb{S}^2 = \Delta \cup \bigcup_{T \in \mathcal{T}} (T \cup T^S)$, it follows that $h : \Delta_2 \to \mathbb{S}^2$ is a bijection. Notice that $h \mid (\Delta_2 \setminus \Delta)$ and $h^{-1} \mid (\mathbb{S}^2 \setminus \Delta)$ are continuous, as $h_{T^U, T^S}$ and $h_{T^L, T}$ are homeomorphism between clopen subspaces.

We will now show that $h \mid \Delta$ is continuous. Let $(x, x) \in \Delta$ and $(x_n, y_n)(n \in \omega)$ be a sequence that converges to $(x, x)$. We may assume, without lose of generality, that
The form \((x, x)\) images converges to \((x, x)\) of the subsequences. If there are infinitely many points of the first type, then their images have the form \((2y_n - d_n, x_n)\), where \(T_n = [a_n, b_n] \times [c_n, d_n]\) is the element of \(T\) that contains \((x_n, y_n)\). Since \(y_n\) converges to \(x\) and \(x < 2y_n - d_n < y_n\), we have that \(\lim_{n \in A_1} h(x_n, y_n) = \lim_{n \in A_1} (2y_n - d_n, x_n) = (x, x)\).

If there are infinitely many points of the second type, then their images have the form \((x_n, 2y_n - c_n)\), where \(T_n = [a_n, b_n] \times [c_n, d_n]\) is the element of \(T\) that contains \((x_n, y_n)\). Since \(2y_n - c_n < d_n\), it is sufficient to prove that:

**Claim 4.4.** If \(\lim_{n \in A_2} (x_n, y_n) = (x, x)\), then \(\lim_{n \in A_2} (x_n, d_n) = (x, x)\).

Proof. Let \(V = [x, x + \epsilon] \cap \Delta_2\) be a given open neighborhood of \((x, x)\). Fix \(k\) such that \(\frac{1}{2k} < \frac{x}{2}\). Since \((x_n, y_n)\) converges to \((x, x)\), we can find \(N\) so that \((x_n, y_n) \in [x, x + \frac{1}{2k}]^2\) and it is below the line \(L_{k+2}\) for all \(n \geq N\). We are left to show that \((x_n, d_n)\) converges to \((x, x)\) for all \(n \geq N\). Let \(n \geq N\) be given. Since \((x_n, y_n)\) is below the line \(L_{k+2}\) and every rectangle \(T \in T\) is between two lines \(L_\ell\) and \(L_{\ell+2}\) for some \(\ell\). It follows that \(T_n = [a_n, b_n] \times [c_n, d_n]\) is below \(L_k\). Hence, \(d_n - c_n < \frac{x}{2k} < \frac{x}{2}\) and \(x < y_n < x + \frac{x}{2}\). Therefore, \((x_n, d_n) \in V\) as required.

We are left to show that the inverse \(h^{-1}\) is continuous on the diagonal. Let \((x, x) \in \Delta\) be a given and let \((x_n, y_n) (n \in \omega)\) be a sequence that converges to \((x, x)\). We may assume, without lose of generality, that \(x < x_n, y < y_n\) for any \(n \in \omega\). Separately, consider three subsequences, namely, those on the diagonal, those above the line \(y = x\), and those in the sets of the form \([c, d] \times [a, b]\). Since \((x_n, y_n)\) converges to \((x, x)\) and \(x < y_n < x + \frac{x}{2}\), we have that \(\lim_{n \in A_1} (x_n, \frac{y_n + d_n}{2}) = (x, x)\).

If there are infinitely many points of the third type, then their images have the form \((y_n, \frac{a_n + d_n}{2})\) where \(T^S_n = [c_n, d_n] \times [a_n, b_n]\) are the elements of \(T\) that contains \((x_n, y_n)\). Since \(y_n\) converges to \(x\) and \(x < \frac{a_n + d_n}{2} < y_n\), we have that \(\lim_{n \in A_1} (x_n, \frac{y_n + d_n}{2}) = (x, x)\).

We are left to show that \((x_n, y_n) \in V\) for all \(n \geq N\). We are left to show that \((y_n, d_n) \in V\) for all \(n \geq N\). Let \(n \geq N\) be given. Since \((x_n, y_n)\) is above the line \(L_\ell\) and \(L_{\ell+2}\) for some \(\ell\). It follows that \(T_n = [c_n, d_n] \times [a_n, b_n]\) is above \(L_k\). Hence, \(d_n - c_n < \frac{x}{2k} < \frac{x}{2}\) and \(x < y_n < x + \frac{x}{2}\). Therefore, \((y_n, d_n) \in V\) as required. 

\[\square\]
This concludes the proof of the Theorem. □

As a corollary we obtain.

**Corollary 4.6.** \( \mathcal{F}_2(S) \) is homogeneous.

**Proof.** By Proposition 4.1, \( \mathcal{F}_2(S) \) is homeomorphic to \( \Delta_2 \). Since \( S^2 \) is homogeneous and \( \Delta_2 \) is homeomorphic to it, by the previous theorem, the result holds. □

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