PERIODIC HOMOGENIZATION OF ELLIPTIC SYSTEMS WITH STRATIFIED STRUCTURE

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Abstract. This paper concerns with the quantitative homogenization of second-order elliptic systems with periodic stratified structure in Lipschitz domains. Under the symmetry assumption on coefficient matrix, the sharp \( \mathcal{O}(\varepsilon) \)-convergence rate in \( L^p_0(\Omega) \) with \( p_0 = \frac{2d}{d-1} \) is obtained based on detailed discussions on stratified functions. Without the symmetry assumption, an \( \mathcal{O}(\varepsilon^\sigma) \)-convergence rate is also derived for some \( \sigma < 1 \) by the Meyers estimate. Based on this convergence rate, we establish the uniform interior Lipschitz estimate. The uniform interior \( W^{1,p} \) and Hölder estimates are also obtained by the real variable method.

1. Introduction. Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \), \( d \geq 2 \). Consider the following elliptic system with stratified structure

\[
\begin{aligned}
\mathcal{L}_\varepsilon u_\varepsilon &= f \quad \text{in } \Omega, \\
u_\varepsilon &= g \quad \text{on } \partial\Omega,
\end{aligned}
\]

where

\[
\mathcal{L}_\varepsilon = -\frac{\partial}{\partial x_i} \left[ A^{\alpha\beta}_{ij} \left( x, \rho(x) \frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_j} \right], \quad \varepsilon > 0,
\]

\[
A(x,y) = (A^{\alpha\beta}_{ij}(x,y)) \in C^{0,1}(\overline{\Omega}; L^\infty_{\text{per}}(Y)) \quad \text{with } 1 \leq i,j \leq d, 1 \leq \alpha,\beta \leq m \text{ satisfies}
\]

\[
\| A \|_{L^\infty(\Omega \times \mathbb{R}^n)} \leq \frac{1}{\mu},
\]

\[
A^{\alpha\beta}_{ij}(x,y)\xi_i\xi_j\zeta^\alpha\zeta^\beta \geq \mu|\xi|^2|\zeta|^2 \quad \text{for any } \xi \in \mathbb{R}^d, \zeta \in \mathbb{R}^m, x \in \overline{\Omega} \text{ and a.e. } y \in \mathbb{R}^n.
\]

Here \( \mu > 0, n \leq d, Y = [-1/2, 1/2]^n \) and \( C^{0,1}(\overline{\Omega}; L^\infty_{\text{per}}(Y)) \) denotes the space of real-valued functions \( \phi(x,y) \) defined on \( \overline{\Omega} \times \mathbb{R}^n \), such that \( x \mapsto \phi(x,\cdot) \) is Lipschitz

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from $\Omega$ to $L^\infty(\mathbb{R}^n)$, and $\phi$ is 1-periodic in $y$. Note that the summation convention for repeated indices is used here and throughout the paper. We also assume that $\rho \in C^{1,1}(\Omega; \mathbb{R}^n)$ and

$$
\sum_{i=1}^{d} \left( \sum_{k=1}^{n} \frac{\partial \rho_k(x)}{\partial x_i} \eta_k \right)^2 \geq \mu |\eta|^2 \quad \text{for any } \eta \in \mathbb{R}^n \text{ and } x \in \Omega.
$$

(5)

Under the assumptions above, one can verify that $A(x, \rho(x)/\varepsilon)$ is measurable by the invertibility of the matrix $(\frac{\partial \rho_l}{\partial x_j})$ (see the nondegeneracy of $\rho$ in Section 2.1). Moreover, $L_\varepsilon$ is elliptic, which implies that problem (1) is well-posed.

The systems with stratified structure generally describe materials which cannot be generated by translating one elementary cell repeatedly. This can be found in many branches of biomechanics and engineering. For example, the human heart is formed by biological fibers nearly parallel to the cardiac wall with orientations varying continuously. More mathematical models arising from materials science includes multilayered spherical particles, cylindrical fibers, wavy multilayered composites, wavy radially multilayered composites and so on (see [9, 10, 37, 36] for more details). The layers or the fibers composing these materials are characterized by the parametric equation $\rho(x) = \text{const}$. As an example, the wavy fibered composite where $\rho(x) = (x_2 - H \sin(\frac{2\pi}{L} x_1), 0, x_3)$ is composed by $\sin$-shaped wavy fibers. Moreover, when $\rho(x) = x_1$ specifically, problem (1) is reduced to the directional homogenization, which has been studied widely (see [19, 20, 33, 11, 13] and references therein).

Periodic homogenization of elliptic systems with stratified structure was first introduced in [8]. There the authors considered two cases, where $A(x, \rho(x)/\varepsilon) = A(x, x/\varepsilon)$ and $A(x, \rho(x)/\varepsilon) = A(\rho(x)/\varepsilon)$ separately, and proved the qualitative results for the former case. The work on the latter case was developed in [9, 10] together with some interesting biomechanics and engineering applications. Then the G-convergence for general stratified media was presented in [21], where abstract differential operators with $n = 1$ were discussed via explicit expressions. In [37], the analytical and computational homogenization of stratified elastic materials were studied, especially for the wavy multilayers and wavy fibered composites. Later on, in [36] the authors built a systematic theory of homogenization for heterogeneous elastoplastic composites, including the qualitative results, the computational algorithm and some numerical examples. For more works on related topics, we refer readers to [32] for approximating nonperiodic materials by locally periodic structures and to [15, 14] for approximations by the solutions of heterogeneous multiscale method.

It is known that under assumptions (3)–(5), $L_\varepsilon$ is $G$-convergent to an operator $L_0$, where $L_0$ is a heterogeneous elliptic operator with coefficient $\hat{A}$ defined in Section 2.2. Moreover, the weak limit of $u_\varepsilon$ in $H^1(\Omega; \mathbb{R}^m)$ as $\varepsilon \to 0$ satisfies

$$
\begin{cases}
L_0 u_0 = f & \text{in } \Omega, \\
u_0 = g & \text{on } \partial \Omega.
\end{cases}
$$

(6)

Our main goal is to establish the quantitative homogenization for problem (1), including the convergence rate of $u_\varepsilon$ to $u_0$ and the uniform interior regularity of $u_\varepsilon$.

The first result concerns with the optimal convergence rate of $u_\varepsilon$ to $u_0$ in $L^{p_0}(\Omega)$ with $p_0 = \frac{2d}{2d-1}$. 

Theorem 1.1. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Assume that $A \in C^{0,1}(\overline{\Omega}; L^\infty_{\text{per}}(Y))$ satisfies (3)–(4) and $\rho \in C^{1,1}(\overline{\Omega}; \mathbb{R}^n)$ satisfies (5). Let $u_\varepsilon, u_0$ be the weak solutions to problems (1) and (6) respectively. Suppose further $A = A^*$ and $u_0 \in H^2(\Omega)$. Then for $\varepsilon \leq 1$

$$\|u_\varepsilon - u_0\|_{L^p(\Omega)} \leq C\varepsilon \delta^2 \|u_0\|_{H^2(\Omega)},$$

(7)

where $p_0 = \frac{2d}{d-1}$, $\delta$ is given by (55) and $C$ depends only on $d, m, n, \mu, \Omega, \|\nabla^2 \rho\|_{L^\infty(\Omega)}$ and $\|\nabla_x A\|_{L^\infty(\Omega \times \mathbb{R}^d)}$. If in addition $A \in C^{k,1}(\overline{\Omega}; L^\infty_{\text{per}}(Y)), \rho \in C^{k+1,1}(\overline{\Omega}; \mathbb{R}^n)$ for some $k > \frac{d}{2}$, then for $\varepsilon \leq 1$

$$\|u_\varepsilon - u_0\|_{L^p(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)},$$

(8)

where $C$ depends only on $d, m, n, \mu, k, \Omega, \|\nabla \rho\|_{C^{k+1}(\overline{\Omega})}$ and $\|A\|_{C^{k,1}([0, 1]; L^\infty(\Omega))}$.

Convergence rates in the periodic homogenization of elliptic equations (or systems)

$$- \text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = f \quad \text{in} \ \Omega,$$

(9)

has been largely studied in recent years. We refer readers to [22, 34, 35, 31, 30] for the convergence rate in $L^2(\Omega)$ and [29, 30] for the convergence rate in $L^{p_0}(\Omega)$. Theorem 1.1 extends the corresponding results for system (9) to the general stratified systems. Note that $\delta$ in (7) depends on the correctors and the flux correctors, thereby on the function $\rho$ and the coefficient $A$. For system (9), $\delta$ is trivially bounded and, therefore, the convergence rate in $L^{p_0}(\Omega)$ for system (9) can be deduced directly from Theorem 1.1.

The next two theorems provide uniform interior Lipschitz and $W^{1,p}$ estimates of $u_\varepsilon$.

Theorem 1.2. Assume that $A \in C^{0,1}(\overline{B_R}; L^\infty_{\text{per}}(Y))$ satisfies (3)–(4) and $\rho \in C^{1,1}(\overline{B_R}; \mathbb{R}^n)$ satisfies (5), where $B_R = B(x_0, R)$. Let $u_\varepsilon \in H^1(B_R; \mathbb{R}^m)$ be a solution to $L_\varepsilon u_\varepsilon = f$ in $B_R$, where $f \in L^q(B_R; \mathbb{R}^m)$ for some $q > d$. Suppose further $A$ satisfies the uniformly VMO condition, i.e.,

$$\sup_{x \in \overline{B_R}, 0 < r < t} \frac{1}{r} \int_{B(r)} \left| A(x, y) - \int_{B(z, r)} A(x, \cdot) \right| dy \leq \varrho(t),$$

(10)

for some nondecreasing continuous function $\varrho(t)$ on $[0, 1]$ with $\varrho(0) = 0$. Then for $\varepsilon \leq r \leq R/2$, it holds that

$$(\int_{B_r} |\nabla u_\varepsilon|^2)^{1/2} \leq C \left\{ \frac{1}{R} \left( \int_{B_R} |u_\varepsilon|^2 \right)^{1/2} + R \left( \int_{B_R} |f|^q \right)^{1/q} \right\},$$

(11)

where $C$ depends only on $d, m, n, \mu, q, \varrho(t)$ and

$$\Theta_{B_R} := R\|\nabla^2 \rho\|_{L^\infty(B_R)} + R\|\nabla_x A\|_{L^\infty(B_R \times \mathbb{R}^d)}.$$

(12)

If in addition $A$ is Hölder continuous in $x, y$, i.e.,

$$|A(x_1, y_1) - A(x_2, y_2)| \leq \Lambda_0(|x_1 - x_2| + |y_1 - y_2|)^\gamma_0$$

(13)

for any $x_1, x_2 \in \overline{B_R}, y_1, y_2 \in \mathbb{R}^n$, then

$$|\nabla u_\varepsilon(x_0)| \leq C \left\{ \frac{1}{R} \left( \int_{B_R} |u_\varepsilon|^2 \right)^{1/2} + R \left( \int_{B_R} |f|^q \right)^{1/q} \right\},$$

(14)

for any $\varepsilon > 0$, where $C$ depends only on $d, m, n, \mu, \lambda_0, \tau_0$ and $\Theta_{B_R}$. 
Theorem 1.3. Assume that $A \in C^{0,1}(2B; L^\infty_{\text{per}}(Y))$ satisfies (3)–(4) and $\rho \in C^{1,1}(2B; \mathbb{R}^m)$ satisfies (5), where $B = B(x_0, r)$. Let $u_\varepsilon \in H^1(2B; \mathbb{R}^n)$ be a solution to $\mathcal{L}_\varepsilon u_\varepsilon = f_1 + \text{div}(f_2) \in 2B$. Suppose further $A$ satisfies (10). Then for any $2 < p < \infty$,

$$
\left( \int_B |\nabla u_\varepsilon|^p \right)^{1/p} \leq C_p \left\{ \frac{1}{r} \left( \int_{2B} |u_\varepsilon|^2 \right)^{1/2} + r \left( \int_{2B} |f_1|^p \right)^{1/p} + \left( \int_{2B} |f_2|^p \right)^{1/p} \right\},
$$

where $C_p$ depends only on $d, m, n, \mu, \rho, \varrho$ in (10) and $\Theta_{2B}$ in (12).

Uniform regularity estimates (in $\varepsilon > 0$) are another main issue in quantitative homogenization. It goes back to a series of papers of M. Avellaneda and F. Lin [4, 5, 6, 7], where interior and boundary estimates for problem (9) under Dirichlet condition were established by a compactness method. The corresponding boundary estimates under Neumann condition were obtained by C. Kenig, F. Lin and Z. Shen in [23]. Recently, another scheme for large-scale uniform regularity estimates was formulated in [3] and further developed in [2, 29]. It is based on convergence rates and is effective for both Lipschitz and Hölder estimates. Uniform $W^{1,p}$ estimates in the homogenization of elliptic systems have also been studied largely (see e.g., [12, 27, 28, 16, 17, 39]). Especially, an approach called the real variable method was initiated in [12] and further developed in [27, 28]. It reduces $W^{1,p}$ estimates to weak reverse Hölder inequalities, which can be derived from the large-scale Lipschitz or Hölder estimates.

It is worth noting that the constants $C$ in the regularity estimates of Theorems 1.2 and 1.3 are dependent on the radius $r$ or $R$ of ball $B$ obliquely. We call estimates like these local estimates. This is more or less sharp for the estimates in Theorems 1.2 and 1.3 since $\hat{A}$ is heterogeneous. For this reason, we could not establish Liouville theorems and global size estimates of fundamental solutions for system (1). However, the corresponding estimates for (9) are global and uniform, i.e., the constants $C$ are independent of the radius (see e.g., [30]). In fact, $\Theta_{BR}$ is trivially null in the case of (9). Moreover, condition (10) is also not needed for (9), which is used here to bound the quantities involving correctors and flux correctors (see Lemma 4.7).

The paper is organized as follows. In Section 2, we introduce some preliminaries, including the nondegeneracy of $\rho$, the properties of stratified functions of the form $h(x, \rho(x)/\varepsilon)$, as well as the correctors. Afterwards, we establish the qualitative homogenization for $\mathcal{L}_\varepsilon$ by the Div-Curl Lemma and an approximating argument. In Section 3, the sharp convergence rate in Theorem 1.1 is proved under the symmetry assumption by a general duality scheme (see e.g., [34, 29, 31]) together with further properties of stratified functions. The process is much complicated since the auxiliary functions are double-variable and the matrix of effective coefficients is variable. Then in Section 4, we derive the (uniform) local Lipschitz estimates in Theorem 1.2, based on a weaker $L^2$-convergence rate obtained from the Meyers estimate without the symmetry assumption. The rescaling property of system (1) plays an essential role in these regularity estimates. The Lipschitz estimate is used in Section 5 to obtain the local $W^{1,p}$ estimates in Theorem 1.3 via the real variable method. As a corollary, we also provide the Hölder and $L^\infty$ estimates for $u_\varepsilon$. 
In this paper, we use the notation \( h^{\rho,\varepsilon}(x) = h(x, \rho(x)/\varepsilon) \). Note that for \( h(x, y) \),
\[
\partial_i(h^{\rho,\varepsilon}(x)) = (\partial_{x_i} h)^{\rho,\varepsilon} + \varepsilon^{-1}\partial_i \rho(\partial_{y_i} h)^{\rho,\varepsilon}.
\]
For the sake of simplicity, we write
\[
D_j h(x, y) = \partial_i \rho(x) \partial_{y_i} h(x, y).
\]
Thus,
\[
\partial_i(h^{\rho,\varepsilon}(x)) = (\partial_{x_i} h)^{\rho,\varepsilon}(x) + \varepsilon^{-1}(D_j h)^{\rho,\varepsilon}(x).
\]

We should point out that \( D_j \) is commutative with \( D_i \) as well as the partial derivatives with respect to \( y \), but not commutative with the partial derivatives with respect to \( x \).

Throughout this paper, we will use \( C \) to denote positive constants which may depend on \( d, m, n, \mu \) if unindicated. It should be understood that \( C \) may differ from each other even in the same line. For simplicity, we may omit the superscripts \( \alpha, \beta \) if it’s clear to understand. We also use the notation \( \int_E f := (1/|E|) \int_E f \) for the integral average of \( f \) over \( E \).

2. Preliminaries and qualitative results. In this section, we will present some useful lemmas and establish the qualitative homogenization for problem (1). Some technical facts on \( \rho \) are stated in Section 2.1, which are essential in our discussions. Then we introduce the correctors in Section 2.2 and prove the qualitative homogenization briefly in Section 2.3.

2.1. Nondegeneracy of \( \rho \). First of all, since \( A(x, \frac{\rho(x)}{\varepsilon}) = A(x, \frac{\rho(x)}{\varepsilon}/L_\rho) \), we can assume that \( |\nabla \rho(x)| \leq 1 \) for any \( x \in \Omega \) throughout the paper, even if this would affect some other parameters.

For fixed \( x_0 \in \Omega \), it follows from (5) that the rank of the \( d \times n \) matrix \( (\partial \rho/\partial x_i)(x_0) \) is \( n \). Therefore, if \( n = d \), \( (\partial \rho/\partial x_i)(x_0) \) is nondegenerate and the inverse function theorem implies that \( \rho \) is a diffeomorphism in a neighbourhood \( U(x_0) \) of \( x_0 \).

If \( n < d \), we can define \( \tilde{\rho}_1, \ldots, \tilde{\rho}_d \) locally near \( x_0 \) as linear functions such that \( \tilde{\rho}(x) = (\tilde{\rho}_1(x), \ldots, \tilde{\rho}_d(x), \rho_{n+1}(x), \ldots, \rho_d(x)) \) is a diffeomorphism in a neighbourhood \( U(x_0) \) of \( x_0 \) and
\[
\sum_{k=1}^d \left( \sum_{i=1}^d \frac{\partial \tilde{\rho}_k(x)}{\partial x_i} \eta_i \right)^2 \geq \mu |\eta|^2 \quad \text{for any } \eta \in \mathbb{R}^d \text{ and } x \in U(x_0).
\]

In this case, we can regard functions on \( \mathbb{R}^n \) (for example, \( A(x, \cdot) \)) as functions on \( \mathbb{R}^d \) which are independent of \( y_{n+1}, \ldots, y_d \).

We will apply this idea when a process is conducted locally in a neighbourhood \( U \), that is, we assume that \( d = n \) and \( \rho \) is a diffeomorphism in \( U \). Note that by the inverse function theorem, \( \text{diam}(U) \) depends only on the continuity modulus of \( \nabla \rho \) and \( \|\nabla(\rho^{-1})\|_{L^\infty(\Omega)} \).

Lemma 2.1. Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \). Suppose \( \rho \in C^1(\Omega, \mathbb{R}^n) \) satisfies (5) and \( h(y) : \mathbb{R}^n \to \mathbb{R} \) is 1-periodic. Then there exists \( \varepsilon_0 > 0 \), depending only on \( d, \mu \) and the continuity modulus of \( \nabla \rho \), such that for \( \varepsilon \leq \min\{\varepsilon_0, \text{diam}(\Omega)\} \),
\[
\int_\Omega |h(\rho(x)/\varepsilon)| dx \leq C|\Omega| \int_Y |h(y)| dy,
\]
where \( C \) depends only on \( d, \mu \) and the Lipschitz character of \( \Omega \).
Proof. Note that we have assumed that $d = n$ and $\rho$ is a diffeomorphism locally. Since $\overline{\Omega}$ is compact, there exist finite subsets $U_i \subset \overline{\Omega}$, $i = 1, \ldots, N$, such that $\rho$ is a diffeomorphism on each $U_i$ and

$$\overline{\Omega} \subset \bigcup_{i=1}^{N} U_i, \quad \sum_{i=1}^{N} |U_i| \leq C|\Omega|, \quad \text{diam}(U_i) = \min\{c_i, \text{diam}(\Omega)\},$$

where $c_i$ depends only on the continuity modulus of $\nabla \rho$ and $\|\nabla(\rho^{-1})\|_{L^\infty(\Omega)}$, and $C$ depends only on the Lipschitz character of $\Omega$. Therefore,

$$\int_{\Omega} |h(\rho(x)/\varepsilon)| dx \leq \sum_{i=1}^{N} \int_{U_i} |h(\rho(x)/\varepsilon)| dx = \sum_{i=1}^{N} \int_{\rho(U_i)} |h(y/\varepsilon)||\det J \rho^{-1}| dy. \quad (20)$$

Note that (18) implies that $|\det J \rho| \geq \mu^{\frac{d}{2}}$, which, together with the inverse function theorem, yields that

$$|\nabla(\rho^{-1})| = |(\nabla \rho)^{-1}| \leq C \mu^{-\frac{d}{2}} |\nabla \rho|^{d-1} \leq C \mu^{-\frac{d}{2}}. \quad (21)$$

Moreover, for each $i$, 

$$C \text{diam}(U_i) \leq \text{diam}(\rho(U_i)) \leq \text{diam}(U_i). \quad (22)$$

Let $\varepsilon_0 = \min\{c_i\}$, which depends only on $d, \mu$ and the continuity modulus of $\nabla \rho$. By combining (19)–(22) and the periodicity of $h$, we obtain for $\varepsilon \leq \min\{\text{diam}(\rho(U_i))\} = \min\{\varepsilon_0, \text{diam}(\Omega)\}$,

$$\int_{\Omega} |h(\rho(x)/\varepsilon)| dx \leq C \sum_{i=1}^{N} |\text{diam}(\rho(U_i))|^d \int_{Y} |h(y)| dy$$

$$\leq C \sum_{i=1}^{N} |U_i| \int_{Y} |h(y)| dy$$

$$\leq C|\Omega| \int_{Y} |h(y)| dy,$$

where $C$ depends only on $d, \mu$ and the Lipschitz character of $\Omega$. The proof is completed.  

Remark 1. If $\rho \in C^{1,1}(\overline{\Omega}; \mathbb{R}^n)$, then the continuity modulus of $\nabla \rho$ can be controlled by $\|\nabla^2 \rho\|_{L^\infty(\Omega)}$. In this case, $\varepsilon_0$ depends only on $d, \mu$ and $\|\nabla^2 \rho\|_{L^\infty(\Omega)}$.

2.2. Correctors. Now we introduce the correctors. Let

$$\tilde{A}^{\alpha\beta}_{kl}(x, y) = A^{\alpha\beta}_{ij}(x, y) \frac{\partial \rho_k}{\partial x_i}(x) \frac{\partial \rho_l}{\partial x_j}(x).$$

According to (3), (4) and (5), it’s easy to show that

$$|\tilde{A}(x, y)| \leq \frac{1}{\mu} \quad \text{and} \quad \tilde{A}^{\alpha\beta}_{kl}(x, y) \xi_k \xi_l \xi^\alpha \xi^\beta \geq \mu^2 |\xi|^2 |\xi|^2$$

for any $\xi \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^m$, $x \in \overline{\Omega}$ and a.e. $y \in \mathbb{R}^n$. For $|\beta| = m$, $1 \leq j \leq d$, the matrix of correctors $\chi_j^{\beta}(x, y)$ is given by the following system

$$\begin{cases}
\mathcal{L}^\alpha_y \chi_j^{\beta}(x, y) = \partial_{y_k} (A^{\alpha\beta}_{ij}(x, y) \frac{\partial \rho_k}{\partial x_i}(x)) \quad \text{in } \mathbb{R}^n, \\
\chi_j^{\beta}(x, y) \text{ is 1-periodic in } y \text{ and } \int_{Y} \chi_j^{\beta}(x, y) dy = 0 \text{ for any } x \in \overline{\Omega}.
\end{cases} \quad (23)$$
Recall that $D_i h(x,y) = \partial_i \rho(x) \partial_{y_i} h(x,y)$. Thus $\chi^\beta_j$ satisfies
\begin{equation}
- D_k (A^\alpha_{kj}(x,y) D_l \chi^\beta_j(x,y)) = D_i A^\alpha_{ij}(x,y).
\end{equation}
Denote
\begin{equation}
\hat{\chi}^\alpha_{ij}(x) = \int_Y A^\alpha_{ij}(x,y) + A^\alpha_{ik}(x,y) D_k \chi^\beta_j(x,y) dy.
\end{equation}
Obviously, $\hat{\chi}$ is heterogeneous. We will show later that $\hat{\chi}$ is in fact the matrix of effective coefficients for $A(x, \rho(x)/\varepsilon)$.

By the equation of $\chi^\beta_j$, we have the following estimates.

**Lemma 2.2.** Suppose $A \in C^{0,1}(\Omega; L^\infty_{per}(Y))$ satisfies (3)–(4) and $\rho \in C^{1,1}(\Omega; \mathbb{R}^n)$ satisfies (5).

i) Then there exists $p > 2$, depending only on $\mu$, such that for any $x \in \overline{\Omega},$
\begin{equation}
\|\chi(x, \cdot)\|_{W^{1,p}(Y)} \leq C, \\
\|\nabla_x \chi(x, \cdot)\|_{W^{1,2,p}(Y)} \leq C \{\|\nabla^2 \rho(x)\| + \|\nabla_x A(x, \cdot)\|_{L^\infty(Y)}\},
\end{equation}
and for any $x_1, x_2 \in \overline{\Omega},$
\begin{equation}
\|\chi_j(x_1, \cdot) - \chi_j(x_2, \cdot)\|_{W^{1,1,2,p}(Y)} \leq C \{\|\nabla \rho(x_1) - \nabla \rho(x_2)\| + \|A(x_1, \cdot) - A(x_2, \cdot)\|_{L^\infty(Y)}\},
\end{equation}
where $C$ depends only on $\mu$. Consequently, $\chi \in C^{0,1}(\overline{\Omega}; W^{1,p}(Y))$ with
\begin{equation}
\|\chi\|_{C^{0,1}(\Omega; W^{1,p}(Y))} \leq C,
\end{equation}
where $C$ depends only on $\mu$, $\|\nabla \rho\|_{C^{0,1}(\Omega; \mathbb{R}^n)}$ and $\|A\|_{C^{0,1}(\Omega; L^\infty(Y))}$.

ii) If in addition $A \in C^{k-1,1}(\Omega; L^\infty_{per}(Y))$, $\rho \in C^{k,1}(\Omega)$ for some $k \geq 1$, then $\chi \in C^{k-1,1}(\overline{\Omega}; W^{1,p}(Y))$ with
\begin{equation}
\|\chi\|_{C^{k-1,1}(\Omega; W^{1,p}(Y))} \leq C,
\end{equation}
where $p$ is given in case (i) and $C$ depends only on $\mu$, $\|\nabla \rho\|_{C^{k-1,1}(\Omega; \mathbb{R}^n)}$ and $\|A\|_{C^{k-1,1}(\Omega; L^\infty(Y))}$.

iii) If $A$ satisfies the uniformly VMO condition (10), then estimates (26) and (27) hold for any $1 < p < \infty$ with $C$ depending only on $\mu, p$ and $g(t)$.

**Proof.** To prove estimates (26) and (27), one can apply the reverse Hölder inequality (see e.g. [18]) to the equations of $\chi^\beta_j$, $\nabla_x \chi^\beta_j$, as well as the equation of $\chi^\beta_j(x_1, \cdot) - \chi^\beta_j(x_2, \cdot)$, which is derived by taking differences. Similar argument also gives estimate (28). Finally, case (iii) follows from $W^{1,p}$ estimates of elliptic systems with VMO coefficients. \hfill \Box

**Corollary 1.** Under the assumptions of (3)–(5), we have $\hat{\chi}^\alpha_{ij}(x) \in C^{0,1}(\overline{\Omega})$. If in addition $A \in C^{k-1,1}(\Omega; L^\infty_{per}(Y))$ and $\rho \in C^{k,1}(\Omega)$ for some $k \geq 1$, then $\hat{\chi}^\alpha_{ij}(x) \in C^{k-1,1}(\overline{\Omega})$.

Moreover, by replacing $A$ with its adjoint $A^*$, we can define $\chi^*_{ij} \beta$ and $\hat{\chi}^*_{ij}$ similarly. Let
\begin{equation}
\overline{a}_{per}(u,v)(x) := \int_Y \hat{\chi}^\alpha_{kj}(x,y) \frac{\partial u^\alpha}{\partial y_j} \frac{\partial v^\alpha}{\partial y_k} dy = \int_Y A^\alpha_{ij}(x,y) D_j u^\alpha D_i v^\alpha dy.
\end{equation}
for \( u, v \in H^1_{\text{per}}(\Omega; \mathbb{R}^m) \) and denote \( P_k^\gamma(y) = y_k e^\gamma, \ e^\gamma = (0, \ldots, 1, \ldots, 0) \) with 1 at the \( \gamma \)th position. Direct computations yields that

\[
\widehat{A}^{\alpha \beta}_{ij}(x) \frac{\partial p_i}{\partial x_j}(x) \frac{\partial p_j}{\partial x_i}(x) = \widehat{\alpha}_{\text{per}}(P^\beta_1 + \chi^\gamma_j \frac{\partial p_i}{\partial x_j}, P^\gamma_k) \\
= \widehat{\alpha}_{\text{per}}(P^\beta_1 + \chi^\gamma_j \frac{\partial p_i}{\partial x_j}, P^\gamma_k) + \widehat{\alpha}_{\text{per}}(P^\beta_1 + \chi^\gamma_j \frac{\partial p_i}{\partial x_j}, \chi^\gamma_i \frac{\partial p_k}{\partial x_i}) \\
= \widehat{\alpha}_{\text{per}}(P^\beta_1 + \chi^\gamma_j \frac{\partial p_i}{\partial x_j}, P^\gamma_k + \chi^\gamma_i \frac{\partial p_k}{\partial x_i}),
\]

where we have used the equation of \( \chi^\beta_j \) in the second step. Therefore, it follows from the invertibility of the matrix \( (\frac{\partial p_i}{\partial x_j}) \) (see the nondegeneracy of \( \rho \) in Section 2.1) that \( \widehat{A}^* = \widehat{A}^* \). Similarly, one can show that \( \widehat{A} \) satisfies the ellipticity condition.

Next we present more properties of \( \widehat{A} \), which are useful in our research next. By the definition of \( \widehat{A} \) and Lemma 2.2, we have

\[
|\widehat{A}(x_1) - \widehat{A}(x_2)| \leq C[|\nabla \rho(x_1) - \nabla \rho(x_2)| + \|A(x_1, \cdot) - A(x_2, \cdot)\|_{L^\infty(Y)}]
\]

for any \( x_1, x_2 \in \Omega \), where \( C \) depends only on \( \mu \). Thus, the conditions on \( \widehat{A} \), such as VMO condition and Hölder continuity condition, can be reduced to the corresponding conditions on \( \nabla \rho \) and \( A \), which are involved in the estimates of \( u_0 \).

Furthermore, note that \( \widehat{A} \) is a \( C^{0,1} \)-tensor defined on \( \overline{\Omega} \). The lemma below extends \( \widehat{A} \) to the whole space.

**Lemma 2.3.** The matrix \( \widehat{A} \) on \( \overline{\Omega} \) can be extended onto \( \mathbb{R}^d \) preserving the Lipschitz property, the boundedness property and the ellipticity condition.

**Proof.** This is deduced from a result of [38]. It says that, if \( U \subset \mathbb{R}^n \) and \( f : U \to \mathbb{R}^n \) is a Lipschitz function, then \( f \) can be extended onto \( \mathbb{R}^n \) preserving the Lipschitz property with the range contained in the closed convex hull of \( f(U) \). Thus, by regarding \( \Omega \) as a subset of \( \mathbb{R}^{d \times m^2} \), we can extend \( \widehat{A} \) on \( \mathbb{R}^d \) preserving the Lipschitz property. The boundedness property and the ellipticity condition follow from the fact that the value of \( \widehat{A} \) at each point outside \( \Omega \) is a limit of convex combinations of the points in \( \widehat{A}(\Omega) \).

\[\square\]

### 2.3. Qualitative homogenization

Lemmas 2.4 and 2.5 below concern with the properties of \( h^{\rho, \varepsilon}(x) \). Lemma 2.4 extends the results in [8, 1], while Lemma 2.5 presents the weak convergence of \( h^{\rho, \varepsilon}(x) \).

**Lemma 2.4.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \). Suppose \( h(x, y) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R} \) is 1-periodic in \( y \) and \( \rho \in C^1(\overline{\Omega}; \mathbb{R}^n) \) satisfies (5). Let \( \varepsilon_0 \) be given in Lemma 2.1.

i) If \( h \in W^{1,p}(\Omega; L^p_{\text{per}}(Y)) \) with \( p > d \), then for any \( 1 \leq q \leq p \) and \( \varepsilon \leq \min\{\varepsilon_0, \text{diam}(\Omega)\} \),

\[
\|h(x, \rho(x)/\varepsilon)\|_{L^q(\Omega)} \leq C\|h(x, y)\|_{W^{1,p}(\Omega; L^p_{\text{per}}(Y))},
\]

where \( C \) depends only on \( d, \mu, p, \Omega \).

ii) If \( h \in L^q(\Omega; W^{1,p}_{\text{per}}(Y)) \) with \( p > d \) and \( q \geq 1 \), then for any \( \varepsilon > 0 \),

\[
\|h(x, \rho(x)/\varepsilon)\|_{L^q(\Omega)} \leq C\|h(x, y)\|_{L^q(\Omega; W^{1,p}_{\text{per}}(Y))},
\]

where \( C \) depends only on \( d \) and \( p \).
Proof. Let us prove (30) under the assumption that $h(x, y)$ is smooth. The result for $h \in W^{1,p}(\Omega; L^p_{\text{per}}(Y))$ follows from a standard density argument. By Sobolev imbedding theorem, for $x \in \Omega$ and any $y \in \mathbb{R}^d$, we have for $p > d$

$$|h(x, y)| \leq C \left( \int_{\Omega} |\nabla_x h(z, y)|^p dz \right)^{1/p} + C \int_{\Omega} |h(z, y)| dz,$$

(32)

where $C$ depends only on $\Omega$ and $p$. By setting $y = \rho(x)/\varepsilon$ in (32) and taking $L^q$-norm over $\Omega$, we get

$$\int_{\Omega} |h(x, \rho(x)/\varepsilon)|^q dx \leq C \int_{\Omega} \left[ \left( \int_{\Omega} |\nabla_x h(z, \rho(x)/\varepsilon)|^p dz \right)^{\frac{q}{p}} + \left( \int_{\Omega} |h(z, \rho(x)/\varepsilon)| dz \right)^q \right] dx$$

$$\leq C \int_{\Omega} \left[ \left( \int_{\Omega} |\nabla_x h(z, y)|^p dz \right)^{\frac{q}{p}} + \left( \int_{\Omega} |h(z, y)| dz \right)^q \right] dy,$$

where $C$ depends on $d, \mu, \Omega, p$, and we have used Lemma 2.1 in the second inequality. This implies (30) directly.

The proof of (ii) is similar and much easier. Instead of (32), one may start with the inequality

$$|h(x, y)| \leq C \left( \int_Y |\nabla_y h(x, z)|^p dz \right)^{1/p} + C \left( \int_Y |h(x, z)|^{2q} dz \right)^{1/2},$$

(33)

which is insured by Sobolev imbedding theorem and the periodicity of $h(x, \cdot)$.

\begin{lemma}
Let $\Omega$ be a bounded Lipschitz domain. Suppose $h(x, y) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is $1$-periodic in $y$ and $\rho \in C^{1,1}(\overline{\Omega}; \mathbb{R}^n)$ satisfies (5). If $h \in W^{1,p}(\Omega; L^p_{\text{per}}(Y))$ or $h \in L^2(\Omega; W^{1,p}_{\text{per}}(Y))$ with $p > d$, then

$$h(x, \rho(x)/\varepsilon) \rightharpoonup h(x, \cdot) \text{ weakly in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$  

(34)

\end{lemma}

Proof. First we prove (34) for $h \in W^{1,p}(\Omega; L^p_{\text{per}}(Y))$. By considering $h(x, y) - \int_Y h(x, \cdot)$, we may assume that $\int_Y h(x, \cdot) = 0$ for any $x \in \Omega$. Let $u(x, \cdot) \in H^2_{\text{per}}(Y)$ be a 1-periodic function such that $L^p_y u(x, y) = h(x, y)$ in $Y$, where $L^p_y = \partial_{y_1}(\partial_i \rho_k(x) \partial_i \rho_l(x) \partial_l \rho_j(x) \partial_j y_i) = D_i D_j$ (recall that $D_i h(x, y) = \partial_i \rho_k(x) \partial_i \rho_l(x) \partial_l \rho_j(x) \partial_j y_i$). By setting $g_i(x, y) = D_i u(x, y)$, we have $h(x, y) = D_i g_i(x, y)$. Note that for any fixed $x$, by (5), $L^p_y$ is a scalar elliptic operator with symmetric, constant coefficients. Hence, $\|g(x, y)\|_{W^{1,p}(\Omega; L^p_{\text{per}}(Y))} \leq C\|h(x, y)\|_{W^{1,p}(\Omega; L^p_{\text{per}}(Y))}$, where $C$ depends on $\|\nabla^2 \rho\|_{L^\infty(\Omega)}$. It follows from Lemma 2.4 that

$$\|g(x, \rho(x)/\varepsilon)\|_{L^2(\Omega)} + \|\nabla_x g(x, \rho(x)/\varepsilon)\|_{L^2(\Omega)}$$

$$\leq C\|g(x, y)\|_{W^{1,p}(\Omega; L^p_{\text{per}}(Y))} + C\|\nabla_x h(x, y)\|_{L^2(\Omega; W^{1,p}_{\text{per}}(Y))}$$

$$\leq C\|h(x, y)\|_{W^{1,p}(\Omega; L^p_{\text{per}}(Y))}.$$  

(35)

By (17),

$$h(x, \rho(x)/\varepsilon) = \varepsilon \partial_i (g_i(x, \rho(x)/\varepsilon)) - \varepsilon \partial_x g_i(x, \rho(x)/\varepsilon).$$

Thus, for any $\varphi \in C^1_0(\Omega)$,

$$\int_\Omega h(x, \rho(x)/\varepsilon) \varphi(x) dx = -\varepsilon \int_\Omega g_i(x, \rho(x)/\varepsilon) \partial_i \varphi - \varepsilon \int_\Omega \partial_x g_i(x, \rho(x)/\varepsilon) \varphi(x) dx$$

$$\to 0 \text{ as } \varepsilon \to 0,$$
Moreover, by the result proved above, we have
\[ h_k \to h \text{ in } L^2(\Omega; W^{1,p}_{\text{per}}(Y)) \text{ as } k \to \infty. \]

Thus, by Lemma 2.4, as \( k \to \infty \),
\[
\|h_k(x, \rho(x)/\varepsilon) - h(x, \rho(x)/\varepsilon)\|_{L^2(\Omega)} \leq C\|h_k(x, y) - h(x, y)\|_{L^2(\Omega; W^{1,p}_{\text{per}}(Y))} \to 0. \tag{36}
\]

Moreover, by the result proved above, we have
\[
h_k(x, \rho(x)/\varepsilon) \to \int_Y h_k(x, \cdot) \ast \varphi_k \text{ weakly in } L^2(\Omega) \tag{37}
\]
as \( \varepsilon \to 0 \). Combining (36) and (37), we obtain (34). The proof is completed. \( \square \)

Now we are prepared to consider the qualitative homogenization of problem (1).

We will show that \( L_0 = L_{0,\Omega}^\varepsilon = -\text{div}(\hat{A}(x) \nabla) \) is the homogenized operator to \( L_\varepsilon \).

**Theorem 2.6.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \). Assume that \( A \in C^0(\overline{\Omega}; L^\infty_{\text{per}}(Y)) \) satisfies (3)--(4) and \( \rho \in C^{1,1}(\overline{\Omega}; \mathbb{R}^n) \) satisfies (5). Suppose \( L_\varepsilon(u_\varepsilon) = f \) in \( \Omega \) and \( u_\varepsilon \rightharpoonup u_0 \) weakly in \( H^1(\Omega; \mathbb{R}^m) \) as \( \varepsilon \to 0 \). Then \( u_0 \) is a solution of \( L_0 u_0 = f \) in \( \Omega \).

**Proof.** We split the proof into two steps. In step 1, we focus on the case \( A(x, y) \in C^{0,1}(\overline{\Omega}; C^\infty_{\text{per}}(Y)) \). Step 2 is devoted to removing the smoothness condition.

**Step 1.** Suppose now \( A(x, y) \in C^{0,1}(\overline{\Omega}; C^\infty_{\text{per}}(Y)) \) and \( A^{\rho,\varepsilon}(x) \nabla u_\varepsilon \rightharpoonup H \) weakly in \( L^2(\Omega; \mathbb{R}^{m \times d}) \). We will show \( H = \hat{A} \nabla u_0 \). Then \( \text{div}H = \text{div}(\hat{A} \nabla u_0) = -f \) in \( \Omega \).

Let \( \chi_k^\gamma(x, y) \) denote the correctors associated to the matrix \( A^* \) and \( \varphi \in C^0(\Omega) \). Note that \( \chi_k^\gamma(x, y) \) is smooth in \( y \). Consider the identity
\[
\int_\Omega A_{ij}^{\alpha\beta}(x, \rho(x)/\varepsilon) \frac{\partial u^\varepsilon_i}{\partial x_j} \left\{ \delta_{ik} \delta^{\alpha\gamma} + D_i \chi_k^{\ast\alpha\gamma}(x, \rho(x)/\varepsilon) \right\} \varphi(x) dx
= \int_\Omega \frac{\partial u^\varepsilon_i}{\partial x_i} A_{ij}^{\alpha\beta}(x, \rho(x)/\varepsilon) \left\{ \delta_{jk} \delta^{\beta\gamma} + D_j \chi_k^{\ast\beta\gamma}(x, \rho(x)/\varepsilon) \right\} \varphi(x) dx. \tag{38}
\]

For the left hand side,
\[
D_i \chi_k^{\ast\alpha\gamma}(x, \rho(x)/\varepsilon) = \varepsilon \partial_{x_i} \left[ \chi_k^{\ast\alpha\gamma}(x, \rho(x)/\varepsilon) - \varepsilon \partial_{x_i} \chi_k^{\ast\alpha\gamma}(x, \rho(x)/\varepsilon), \right]
\]
which, by virtue of Lemma 2.5 and the Div-Curl Lemma, implies that the l.h.s. of (38) converges to \( \int_\Omega H_k^\gamma \varphi \) as \( \varepsilon \to 0 \). On the other hand, thanks to Lemma 2.5,
\[
A_{ij}^{\ast\alpha\beta}(x, \rho(x)/\varepsilon) \left\{ \delta_{jk} \delta^{\beta\gamma} + D_j \chi_k^{\ast\beta\gamma}(x, \rho(x)/\varepsilon) \right\} \rightharpoonup (A^*)_{ik}^{\alpha\gamma}(x) \text{ weakly in } L^2(\Omega) \tag{39}
\]
as $\varepsilon \to 0$. Moreover,

$$\text{div}\left\{ A_{ij}^\alpha \beta (x, \rho(x) / \varepsilon) \left[ \delta_{jk} \delta^{\beta \gamma} + D_j \chi_k^\gamma (x, \rho(x) / \varepsilon) \right] \right\}$$

$$= \varepsilon^{-1} \left\{ D_j A_{ik}^{\alpha \gamma} (x, y) + \tilde{L}_y \chi_k^\gamma (x, y) \right\} \bigg|_{y=\rho(x)/\varepsilon}$$

$$+ \partial_x \left\{ A_{ik}^{\alpha \gamma} (x, y) + A_{ij}^{\alpha \beta} (x, y) D_j \chi_k^\gamma (x, y) \right\} \bigg|_{y=\rho(x)/\varepsilon}$$

$$- \partial_x \left\{ \int_Y A_{ik}^{\alpha \gamma} (x, \cdot) + A_{ij}^{\alpha \beta} (x, \cdot) D_j \chi_k^\gamma (x, \cdot) \right\} \text{weakly in } L^2(\Omega), \text{ as } \varepsilon \to 0,$$

where we have used the equation of $\chi_k^\gamma$ in the second step as well as Lemma 2.5 in the last step. Thus, by applying the Div-Curl Lemma and taking (39)–(40) into consideration, we conclude that the limit of the r.h.s. of (38) is

$$\int_{\Omega} \frac{\partial u_0^\alpha}{\partial x_i} \tilde{A}_i A_{\alpha \gamma} \varphi dx = \int_{\Omega} \tilde{A}_i A_{\alpha \gamma} \varphi dx,$$

where we have also used the fact that $\tilde{A}^\gamma = \tilde{A}^\ast$. Consequently, we obtain

$$\int_{\Omega} H_k^\gamma \varphi = \int_{\Omega} \tilde{A}_i A_{\alpha \gamma} \varphi dx,$$

which, together with the arbitrariness of $\varphi$ in $C^0_0(\Omega)$, shows that $H = \tilde{A} \nabla u_0$.

**Step 2.** For general $A$, we can find a sequence of $A_k(x, y) \in C^{0,1}(\bar{\Omega}; C^\infty_0(Y))$ such that $A_k$ satisfies the same conditions as $A$ and

$$\| A_k - A \|_{L^p_{per}(Y; C^0(\bar{\Omega}))} \leq 1/k,$$

(41)

where $p$ is to be determined and can be arbitrarily large. By Lemma 2.1, (41) tells us that

$$\| A_k^\rho - A^\rho \|_{L^p(\Omega)} \leq C \| A_k - A \|_{L^p_{per}(Y; C^0(\bar{\Omega}))} \leq C/k.$$

(42)

For each $\varepsilon$, let $v_{k,\varepsilon} \in L^2(\Omega)$ and $v_{k,\varepsilon} = u_\varepsilon$ on $\partial \Omega$, where $A_k^{\rho \varepsilon} = -\text{div}(A_k^{\rho \varepsilon}(x) \nabla)$, then, by a duality argument and the reverse Hölder inequality, there exist constants $p$ and $C$, depending only on $\mu, m, \Omega$, such that,

$$\| u_\varepsilon - v_{k,\varepsilon} \|_{L^2(\Omega)} \leq C \| A_k^{\rho \varepsilon} - A^\rho \varepsilon \|_{L^p(\Omega)} \| \nabla u_\varepsilon \|_{L^2(\Omega)} \leq C \| A_k^{\rho \varepsilon} - A^\rho \varepsilon \|_{L^p(\Omega)},$$

(43)

(see the proof of Theorem 1.4 in [40] for details). We can assume that $p$ in (42) coincides with the one in (43). By virtue of Step 1, it’s not hard to show that $v_{k,\varepsilon} \to v_{k,0}$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$, and by (42)–(43),

$$\| u_0 - v_{k,0} \|_{L^2(\Omega)} \leq C \| A_k - A \|_{L^p_{per}(Y; C^0(\bar{\Omega}))} \leq C/k,$$

(44)

where $v_{k,0}$ is the unique solution of $L_{0}^\tilde{A}(v_{k,0}) = f$ in $\Omega$ and $v_{k,0} = u_0$ on $\partial \Omega$. Moreover, by using (26), one can verify that for any $x$,

$$| \tilde{A}_k(x) - \tilde{A}(x) | \leq C/k,$$

which together with the equations of $v_{k,0}$ implies that any weak limit of $v_{k,0}$ in $H^1(\Omega)$ satisfies

$$\begin{cases} L_0^\tilde{A}(v) = f & \text{in } \Omega, \\ v = u_0 & \text{on } \partial \Omega. \end{cases}$$

(45)
3. Convergence rates.

3.1. ε-smoothing operators. Let $S_{\varepsilon}(\cdot)$ be the so-called ε-smoothing operator. That is,

$$S_{\varepsilon}(f)(x) = \int_{\mathbb{R}^d} f(x-y)\varphi_{\varepsilon}(y)dy,$$

where $\varphi_{\varepsilon}(y) = \varepsilon^{-d}\varphi(y/\varepsilon)$, $\varphi \in C^\infty_0(B(0,1/2))$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi dx = 1$.

Lemma 3.1. For $1 \leq p < \infty$,

$$\|g(x)S_{\varepsilon}(f)(x)\|_{L^p(\Omega)} \leq C\|g\|_{S^p(\Omega)}\|f\|_{L^p(\Omega^c)},$$

where $\|g\|_{S^p(\Omega)} = \sup_{y \in \Omega} \left( \int_{\Omega \cap B(y, 2\varepsilon)} |g(x)|^p dx \right)^{1/p}$ and $\Omega^c = \{ x \in \mathbb{R}^d : \text{dist}(x, \Omega) < \varepsilon \}$.

Proof.\[\begin{align*}
\int_{\Omega} |g(x)S_{\varepsilon}(f)(x)|^p dx &= \int_{\Omega} g(x) \left( \int_{\mathbb{R}^d} \varphi_{\varepsilon}(x-y)f(y)dy \right)^p dx \\
&\leq \int_{\Omega} |g(x)|^p \left( \int_{\mathbb{R}^d} \varphi_{\varepsilon}(x-y)dy \right)^p \left( \int_{\mathbb{R}^d} \varphi_{\varepsilon}(x-y)|f(y)|^p dy \right) dx \\
&\leq C \sup_{y \in \Omega} \frac{1}{\varepsilon^d} \int_{\Omega \cap B(y, 2\varepsilon)} |g(x)|^p dx \int_{\Omega^c} |f(y)|^p dy.
\end{align*}\]

Lemma 3.2 (See [29]). Suppose $f \in W^{1,p}(\mathbb{R}^d)$ for some $1 \leq p \leq \infty$. Then

$$\|S_{\varepsilon}(f) - f\|_{L^p(\mathbb{R}^d)} \leq \varepsilon\|\nabla f\|_{L^p(\mathbb{R}^d)}.$$\

Moreover, if $q = \frac{2d}{d+1}$,

$$\|S_{\varepsilon}(f)\|_{L^q(\mathbb{R}^d)} \leq C\varepsilon^{-1/2}\|f\|_{L^q(\mathbb{R}^d)}, \quad \|S_{\varepsilon}(f) - f\|_{L^q(\mathbb{R}^d)} \leq C\varepsilon^{1/2}\|\nabla f\|_{L^q(\mathbb{R}^d)},$$

where $C$ depends only on $d$.

Lemma 3.3 (see [30]). Let $q = \frac{2d}{d+1}$ and $f \in W^{1,q}(\Omega)$. Then

$$\|f\|_{L^q(\partial\Omega)} \leq C1^{1/2}\|f\|_{W^{1,q}(\Omega)}, \quad \|f\|_{L^q(\partial\Omega)} \leq C\|f\|_{W^{1,q}(\Omega)},$$

where $\Omega = \{ x \in \Omega : \text{dist}(x, \partial\Omega) < t \}$ and $C$ depends only on $\Omega$.

3.2. Convergence rates. Now we turn to the problem of convergence rate. We start with the equation of $u_\varepsilon - u_0$, where $L_\varepsilon u_\varepsilon = f$ in $\Omega$, $\varepsilon \geq 0$. By calculation,

$$L_\varepsilon(u_\varepsilon - u_0) = -\text{div}[(A(\cdot) - A^{\varepsilon}(\cdot))(\nabla u_0 - K_\varepsilon(\nabla u_0))]$$

$$-\text{div}[(A(\cdot) - A^{\varepsilon}(\cdot))K_\varepsilon(\nabla u_0)],$$

where $K_\varepsilon(\cdot)$ is a linear operator to be determined. Observing that informally

$A^{\varepsilon}_{ij} + A^{\varepsilon}_{ik}(D_k\chi_j)^{\varepsilon} \to \tilde{A}_{ij} \quad \text{weakly in } L^2(\Omega),$\]

we write

$$L_\varepsilon(u_\varepsilon - u_0) = -\text{div}[(\tilde{A} - A^{\varepsilon})(\nabla u_0 - K_\varepsilon(\nabla u_0))] + \partial_i[(B_{ij})^{\varepsilon}(\cdot)K_\varepsilon(\partial_j u_0)].$$
where $D_k$ is given by (16) and
\[ D_{ij}^{\alpha\beta}(x, y) = A_{ij}^{\alpha\beta}(x, y) + A_{ik}^\gamma(x, y)D_k\chi_j^\gamma(x, y) - \hat{A}_{ij}^{\alpha\beta}(x). \] (46)

For the last term,
\[
- \partial_i[A_{ik}^{\rho}(x)(D_k\chi_j^\rho(x))K_x(\partial_j u_0)] = \mathcal{L}_e(\varepsilon\chi_j^{\rho}(x)K_x(\partial_j u_0)) \\
+ \partial_i[A_{ik}^{\rho}(x)\varepsilon\chi_j^{\rho}(x)\partial_k K_x(\partial_j u_0)] + \partial_i[A_{ik}^{\rho}(x)\varepsilon(\partial_x \chi_j)^{\rho}(x)K_x(\partial_j u_0)].
\]

Thus, by defining
\[ w_{\varepsilon} = u_{\varepsilon} - u_0 - \varepsilon\chi_j^{\rho}(x)K_x(\partial_j u_0), \] (47)

we obtain
\[
\mathcal{L}_e(w_{\varepsilon}) = -\text{div}[(\hat{A}(x) - A^{\rho}(x))(\nabla u_0 - K_x(\nabla u_0))] + \partial_i[B_{ij}^{\alpha}(x)\varepsilon\chi_j^{\rho}(x)K_x(\partial_j u_0)] \\
+ \partial_i[A_{ik}^{\rho}(x)\varepsilon\chi_j^{\rho}(x)\partial_k K_x(\partial_j u_0)] + \partial_i[A_{ik}^{\rho}(x)\varepsilon(\partial_x \chi_j)^{\rho}(x)K_x(\partial_j u_0)]. \] (48)

Next we introduce the flux correctors. For fixed $\varepsilon$, consider the equation
\[
\begin{align*}
\mathcal{L}_{y}^\varepsilon(\phi_{ij}^{\varepsilon}(x, y)) &= B_{ij}^{\beta}(x, y) \quad \text{in } Y \\
\phi_{ij}^{\varepsilon}(x, y) \text{ is 1-periodic in } y \text{ and } \int_Y \phi(x, \cdot) &= 0,
\end{align*} \] (49)

where $\mathcal{L}_{y}^\varepsilon = D_iD_i$ and $D_i$ is defined by (16). Recall that for any fixed $x$, by (5), $\mathcal{L}_{y}^\varepsilon$ is a scalar elliptic operator with symmetric, constant coefficients. Therefore, there exists a unique solution $\phi_{ij}(x, \cdot)$ satisfying
\[ ||\phi(x, \cdot)||_{W^{2,p}_v(Y)} \leq C ||B(x, \cdot)||_{L^p_v(Y)}. \]

We define the flux corrector as $\mathfrak{B}_{kij}(x, y) = D_k\phi_{ij}(x, y) - D_i\phi_{kj}(x, y)$ which is skew-symmetric. By (17),
\[
\begin{align*}
\partial_k[\mathfrak{B}_{kij}(x, \rho(x)/\varepsilon)] &= \partial_x [\mathfrak{B}_{kij}(x, \rho(x)/\varepsilon)] + \varepsilon^{-1}D_k\mathfrak{B}_{kij}(x, \rho(x)/\varepsilon) \\
&= \partial_x [\mathfrak{B}_{kij}(x, \rho(x)/\varepsilon)] + \varepsilon^{-1}B_{ij}(x, \rho(x)/\varepsilon) - \varepsilon^{-1}D_kD_i\phi_{kj}(x, \rho(x)/\varepsilon).
\end{align*} \] (50)

In view of equation (24), for any $\alpha, \beta$,
\[
\mathcal{L}_{y}^\varepsilon[D_k\phi_{kj}^{\alpha\beta}(x, \cdot)] = D_kB_{kj}^{\alpha\beta}(x, \cdot) = 0,
\]

which, together with the periodicity in $y$, implies that $D_k\phi_{kj}(x, \cdot)$ is constant in $y$. Therefore,
\[ D_kD_i\phi_{kj}(x, y) = 0 \quad \text{for any } x, y. \]

By taking this into (50), we obtain
\[ B_{ij}(x, \rho(x)/\varepsilon) = \varepsilon\partial_k[\mathfrak{B}_{kij}(x, \rho(x)/\varepsilon)] - \varepsilon\partial_x [\mathfrak{B}_{kij}(x, \rho(x)/\varepsilon)]. \]

Thus,
\[
\begin{align*}
\partial_i[B_{ij}^{\rho}(x)K_x(\partial_j u_0)] &= \partial_i[\varepsilon\partial_k [\mathfrak{B}_{kij}^{\rho}(x)]K_x(\partial_j u_0)] - \partial_i[\varepsilon(\partial_x \mathfrak{B}_{kij})^{\rho}(x)K_x(\partial_j u_0)] \\
&= -\partial_i[\varepsilon\mathfrak{B}_{kij}^{\rho}(x)\partial_k K_x(\partial_j u_0)] - \partial_i[\varepsilon(\partial_x \mathfrak{B}_{kij})^{\rho}(x)K_x(\partial_j u_0)],
\end{align*} \] (51)

where we have used the skew-symmetry of $\mathfrak{B}$ in the last equality.

By taking (51) into (48), we finally get
\[ \mathcal{L}_e(w_{\varepsilon}) = -\text{div}[(\hat{A}(x) - A^{\rho}(x))(\nabla u_0 - K_x(\nabla u_0))] - \partial_i[\varepsilon\mathfrak{B}_{kij}^{\rho}(x)\partial_k K_x(\partial_j u_0)] \]
For the second term, by using Lemma 3.1 and the definition of $K$, thanks to (52), we have

$$- \partial_t \{ \varepsilon (\partial_x B_{kij})^\rho,\varepsilon (x) K_{k}(\partial_x u_0) \} + \partial_t \{ \varepsilon A^\rho,\varepsilon_{k}(x) \chi_j^\rho,\varepsilon (x) \partial_k K_{j}(\partial_x u_0) \} + \partial_t \{ \varepsilon A^\rho,\varepsilon_{k}(x)(\partial_x \chi_j)^\rho,\varepsilon (x) K_{j}(\partial_x u_0) \}. \tag{52}$$

In the following, suppose $\eta_\epsilon \in C_0^\infty (\Omega)$ is a cut-off function satisfying

$$0 \leq \eta_\epsilon \leq 1, \| \nabla \eta_\epsilon \| \leq C/\varepsilon,$$

$$\eta_\epsilon = 0 \text{ on } \Omega_{3\varepsilon}, \eta_\epsilon = 1 \text{ on } \Omega \setminus \Omega_{4\varepsilon}. \tag{53}$$

**Lemma 3.4.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Assume that $A \in C^{0,1}[\Omega; L_{per}^\infty (Y)]$ satisfies (3)–(4) and $\rho \in C^{1,1}[\Omega; \mathbb{R}^n]$ satisfies (5). Let $w_\epsilon$ be given by (47) and $K_{\epsilon}(u) = S_\epsilon^2 (u) \eta_\epsilon$. Then for any $\psi \in H_0^1 (\Omega; \mathbb{R}^m)$,

$$\left| \int_{\Omega} A^\rho,\varepsilon (x) \nabla w_\epsilon \cdot \nabla \psi dx \right| \leq C \| \nabla u_0 - S_\epsilon(\nabla u_0) \|_{L^2(\Omega; \mathbb{R}^m)} \| \nabla \psi \|_{L^2(\Omega)} + C \delta \{ \| \nabla u_0 \|_{L^2(\Omega; \mathbb{R}^m)} \| \nabla \psi \|_{L^2(\Omega; \mathbb{R}^m)} + \varepsilon \| S_\epsilon(\nabla u_0) \|_{L^2(\Omega; \mathbb{R}^m)} \| \nabla \psi \|_{L^2(\Omega)} \}, \tag{54}$$

where $C$ depends only on $\mu$ and

$$\delta := 1 + \| \rho_\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \nabla \rho_\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \nabla \rho_\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \rho_\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \rho_\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \rho_\varepsilon \|_{S_\epsilon^2 (\Omega)} \tag{55}$$

with $2^* = \frac{2d}{d-2}$.

**Proof.** Thanks to (52), we have

$$\left| \int_{\Omega} A^\rho,\varepsilon (x) \nabla w_\epsilon \cdot \nabla \psi dx \right| \leq C \| \nabla u_0 - K_{\epsilon}(\nabla u_0) \|_{L^2(\Omega; \mathbb{R}^m)} \| \nabla \psi \|_{L^2(\Omega)} + \varepsilon \left( \int_{\Omega} \| \nabla x \mathbf{B}^\rho,\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \nabla x \mathbf{B}^\rho,\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \nabla x \mathbf{B}^\rho,\varepsilon \|_{S_\epsilon^2 (\Omega)} \right) \| K_{\epsilon}(\nabla u_0) \|_{L^2(\Omega)} \| \nabla \psi \|_{L^2(\Omega)}. \tag{56}$$

The first term in (56) can be bounded by

$$C \| \nabla u_0 \|_{L^2(\Omega; \mathbb{R}^m)} \| \nabla \psi \|_{L^2(\Omega)} + C \| \nabla u_0 - S_\epsilon(\nabla u_0) \|_{L^2(\Omega; \mathbb{R}^m)} \| \nabla \psi \|_{L^2(\Omega)}.$$ 

For the second term, by using Lemma 3.1 and the definition of $K_{\epsilon}$, we have

$$\varepsilon \left( \int_{\Omega} \| \nabla x \mathbf{B}^\rho,\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \nabla x \mathbf{B}^\rho,\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \nabla x \mathbf{B}^\rho,\varepsilon \|_{S_\epsilon^2 (\Omega)} \right) \| K_{\epsilon}(\nabla u_0) \|_{L^2(\Omega; \mathbb{R}^m)} \| \nabla \psi \|_{L^2(\Omega)}.$$

Similarly, the third term can be dominated by

$$C \| \nabla x \mathbf{B}^\rho,\varepsilon \|_{S_\epsilon^2 (\Omega)} + \| \nabla \rho_\varepsilon \|_{S_\epsilon^2 (\Omega)} \cdot \| S_\epsilon(\nabla u_0) \|_{L^2(\Omega; \mathbb{R}^m)} \| \nabla \psi \|_{L^2(\Omega)}.$$ 

Note that $\| \nabla \rho_\varepsilon \|_{S_\epsilon^2 (\Omega)} \leq \| \nabla \rho_\varepsilon \|_{S_\epsilon^2 (\Omega)}$. Then we can obtain (54) by combining the estimates above.

**Lemma 3.5.** Suppose the assumptions of Lemma 3.4 hold. Then for any $\psi \in H_0^1 (\Omega; \mathbb{R}^m)$,

$$\left| \int_{\Omega} A^\rho,\varepsilon (x) \nabla w_\epsilon \cdot \nabla \psi dx \right| \leq C \{ \| \nabla u_0 - S_\epsilon(\nabla u_0) \|_{L^2(\Omega; \mathbb{R}^m)} + \| \nabla u_0 \|_{L^2(\Omega; \mathbb{R}^m)} \} \| \nabla \psi \|_{L^2(\Omega)} \tag{57}$$

+ $\varepsilon \| S_\epsilon(\nabla^2 u_0) \|_{L^2(\Omega; \mathbb{R}^m)} + \| \nabla u_0 \|_{L^2(\Omega; \mathbb{R}^m)} \| \nabla \psi \|_{L^2(\Omega)}$. 


and
\[ \left| \int_{\Omega} A^{\varepsilon}(x) \nabla w_{\varepsilon} \cdot \nabla \psi \, dx \right| \leq C \delta \{ \varepsilon^{1/2} \| \nabla \psi \|_{L^2(\Omega_{2\varepsilon})} + \varepsilon \| \nabla \psi \|_{L^2(\Omega)} \} \| \nabla u_0 \|_{H^1(\Omega)}, \] (58)
where \( C \) depends only on \( \mu \) and \( \Omega \).

**Proof.** (57) and (58) follow from (54). Note that Lemmas 3.2 and 3.3 have been used to obtain (58).

**Theorem 3.6.** Suppose the assumptions of Lemma 3.4 hold. Then
\[ \| w_{\varepsilon} \|_{H^1_0(\Omega)} \leq C \varepsilon^{1/2} \| u_0 \|_{H^2(\Omega)}, \] (59)
where \( C \) depends only on \( \mu \) and \( \Omega \). If in addition \( A = A^* \), then
\[ \| w_{\varepsilon} \|_{H^1_0(\Omega)} \leq C \varepsilon^{1/2} \delta \left\{ \| g \|_{H^1(\partial \Omega)} + \| f \|_{L^{q_0}(\Omega)} \right\}, \] (60)
where \( q_0 = \frac{2d}{d+1} \) and \( C \) depends only on \( \mu, \Omega, \| \nabla^2 \rho \|_{L^\infty(\Omega)} \) and \( \| \nabla_x A \|_{L^\infty(\Omega \times Y)} \).

**Proof.** Setting \( \psi = w_{\varepsilon} \) in (58), we get (59). To prove (60), set \( \psi = w_{\varepsilon} \) in (57) and it’s sufficient to bound \( \| \nabla u_0 - S_\varepsilon(\nabla u_0) \|_{L^2(\Omega_{5\varepsilon} \setminus \Omega_{2\varepsilon})}, \| \nabla u_0 \|_{L^2(\Omega_{3\varepsilon})}, \| S_\varepsilon(\nabla^2 u_0) \|_{L^2(\Omega_{3\varepsilon} \setminus \Omega_{2\varepsilon})} \) and \( \| \nabla u_0 \|_{L^2(\Omega_{2\varepsilon})} \).

Evidently,
\[ \| \nabla u_0 \|_{L^2(\Omega_{3\varepsilon} \setminus \Omega_{2\varepsilon})} \leq \| \nabla u_0 \|_{L^2(\Omega_{3\varepsilon})} \leq C \left\{ \| g \|_{H^1(\partial \Omega)} + \| f \|_{L^{q_0}(\Omega)} \right\}. \] (61)

Now choose a ball \( D \) such that \( \Omega \subset D \), \( \text{diam}(\Omega) = \text{diam}(D) \), and consider the solution \( u_1 \) of
\[ \begin{cases} L_0 u_1 = f & \text{in } D, \\ u_1 = 0 & \text{on } \partial D, \end{cases} \]
where \( \hat{A} \) has been extended onto \( \mathbb{R}^d \) as a Lipschitz matrix by Lemma 2.3 and \( f \) has been extended to 0 outside \( \Omega \). By standard estimates of elliptic systems with Lipschitz coefficients in smooth domains, for any \( 1 < p < \infty \),
\[ \| u_1 \|_{W^{2,p}(D)} \leq C \| f \|_{L^p(\Omega)}, \] (62)
where \( C \) depends only on \( \mu, p, \text{diam}(\Omega) \) and \( \| \nabla \hat{A} \|_{L^\infty(\Omega)} \). Moreover, by Lemma 3.3
\[ \| u_1 \|_{H^1(\partial \Omega)} \leq C \| u_1 \|_{W^{2,0}(\Omega)} \leq C \| f \|_{L^{q_0}(\Omega)}. \] (63)

On the other hand, by setting \( u_2(x) = u_0(x) - u_1(x) \), we know that
\[ \begin{cases} L_0 u_2 = L_0 u_0 - L_0 u_1 = 0 & \text{in } \Omega, \\ u_2 = g - u_1 & \text{on } \partial \Omega. \end{cases} \]

Since \( A = A^* \), we have \( (\hat{A})^* = \hat{A}^* = \hat{A} \). This allows us to apply the nontangential maximal function estimates to elliptic systems with symmetric \( C^\alpha \)-coefficients in Lipschitz domains (see e.g. [24, 30]) to obtain
\[ \| \mathcal{M}(\nabla u_2) \|_{L^2(\partial \Omega)} \leq C \| u_2 \|_{H^1(\partial \Omega)} \leq C \| g \|_{H^1(\partial \Omega)} + C \| f \|_{L^{q_0}(\Omega)}, \] (64)
where \( C \) depends only on \( \mu, \Omega \) and the Hölder character of \( \hat{A} \), \( \mathcal{M}(\nabla u_2) \) denotes the nontangential maximal function of \( \nabla u_2 \) and (63) was used in the last step. By combining (63) and (64), we get
\[ \| \nabla u_0 \|_{L^2(\Omega_{2\varepsilon})} \leq C \varepsilon^{1/2} \left\{ \| g \|_{H^1(\partial \Omega)} + \| f \|_{L^{q_0}(\Omega)} \right\}. \] (65)
Moreover, by the interior $H^2$ estimates for elliptic systems with Lipschitz coefficients,
\[
\int_{B(y, c\tilde{\delta}(y))} |\nabla^2 u_2(x)|^2 \, dx \leq \frac{C}{|\tilde{\delta}(y)|^2} \int_{B(y, 2c\tilde{\delta}(y))} |\nabla u_2(x)|^2 \, dx,
\]
where $\tilde{\delta}(y) := \text{dist}(y, \partial \Omega)$, $c$ is a very small constant and $C$ depends only on $\mu, \text{diam}(\Omega), \|\nabla A\|_{L^\infty(\Omega)}$. Note that for any $x \in B(y, c\tilde{\delta}(y))$, $\tilde{\delta}(y) \sim \tilde{\delta}(x)$. By integrating both sides in $y$ over $\Omega \setminus \Omega_\varepsilon$, we obtain
\[
\int_{\Omega \setminus \Omega_\varepsilon} |\nabla^2 u_2(x)|^2 \, dx \leq C \int_{\Omega \setminus \Omega_\varepsilon} \frac{1}{|\tilde{\delta}(y)|^2} \int_{B(y, 2c\tilde{\delta}(y))} |\nabla u_2(x)|^2 \, dx \, dy
\]
\[
\leq C \varepsilon^{-1} \|\mathcal{M}(\nabla u_2)\|^2_{L^2(\Omega)}
\]
\[
\leq C \varepsilon^{-1} \left\{ \|g\|_{H^1(\partial \Omega)} + \|f\|_{L^8(\Omega)} \right\}^2.
\]
Hence,
\[
\|S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \leq C \varepsilon^{-1/2} \|\nabla^2 u_1\|_{L^6(\mathbb{R}^d)} + C \|\nabla^2 u_2\|_{L^2(\Omega \setminus \Omega_\varepsilon)}
\]
\[
\leq C \varepsilon^{-1/2} \left\{ \|g\|_{H^1(\partial \Omega)} + \|f\|_{L^8(\Omega)} \right\},
\]
where we have used Lemma 3.2 for the first step and (62) for the second step.

Lastly, by Lemma 3.2 and the fact that (see e.g. [26]),
\[
\|u - S_\varepsilon(u)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})} \leq C \{\varepsilon \|\nabla u\|_{L^2(\Omega \setminus \Omega_\varepsilon)} + \|u\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})}\},
\]
we have
\[
\|\nabla^2 u_0 - S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})}
\]
\[
\leq C \varepsilon^{1/2} \|\nabla^2 u_1\|_{L^6(\mathbb{R}^d)} + C \{\varepsilon \|\nabla^2 u_2\|_{L^2(\Omega \setminus \Omega_\varepsilon)} + \|\nabla u_2\|_{L^2(\Omega \setminus \Omega_{2\varepsilon})}\}
\]
\[
\leq C \varepsilon^{1/2} \left\{ \|g\|_{H^1(\partial \Omega)} + \|f\|_{L^8(\Omega)} \right\},
\]
where we have also used (62), (64) and (66).

As a result, by taking estimates (61), (65), (67) and (69) into (57) with $\psi = w_\varepsilon$, we get (60). Also observing that by (29)
\[
\|\nabla A\|_{L^\infty(\Omega)} \leq C \|\nabla^2 \rho\|_{L^\infty(\Omega)} + \|\nabla_x A\|_{L^\infty(\Omega \times Y)},
\]
we know $C$ in (60) depends only on $\mu, \Omega, \|\nabla^2 \rho\|_{L^\infty(\Omega)}$ and $\|\nabla_x A\|_{L^\infty(\Omega \times Y)}$. The proof is completed.

**Proof of (7) in Theorem 1.1.** Note that
\[
\|\varepsilon \chi_{\rho,\varepsilon}^\rho(x)K_\varepsilon(\partial_j u_0)\|_{L^{p_0}(\Omega)} \leq C\varepsilon \|\chi_{\rho,\varepsilon}^\rho\|_{L^{p_0}(\Omega)} \|\nabla u_0\|_{L^{p_0}(\Omega)}
\]
\[
\leq C\varepsilon \delta \|u_0\|_{H^2(\Omega)},
\]
since $p_0 < 2^*$. Thus, it’s sufficient to bound $\|w_\varepsilon\|_{L^{p_0}(\Omega)}$.

For any $F \in C^\infty_0(\Omega; \mathbb{R}^m)$, let $v_\varepsilon \in H^1_0(\Omega; \mathbb{R}^m)(\varepsilon \geq 0)$ be the weak solution to
\[
\begin{cases}
\mathcal{L}_\varepsilon v_\varepsilon = F & \text{in } \Omega, \\
v_\varepsilon = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Define
\[
\varpi_\varepsilon = v_\varepsilon - v_0 - \varepsilon \chi_{\rho,\varepsilon}^\rho K_\varepsilon(\partial_j v_0).
\]

Then
\[
\left| \int_\Omega w_\varepsilon \cdot F \, dx \right| = \left| \int_\Omega A^{\rho,\varepsilon}(x) \nabla w_\varepsilon \cdot \nabla v_\varepsilon \, dx \right|
\]
Lemma 3.7. Suppose \( s \) satisfies \( \mathcal{A} \). By (59) and (60), it is easy to see that
\[
\eta = \left| \int_\Omega A^{p,e}(x) \nabla w_x \cdot \nabla \varphi \, dx \right| + \left| \int_\Omega A^{p,e}(x) \nabla w_x \cdot \nabla v_0 \, dx \right|
\]
\[
+ \left| \int_\Omega A^{p,e}(x) \nabla w_x \cdot \nabla (\varepsilon \chi^{p,e}_j K_e(\partial_j v_0)) \, dx \right|
\]
\[
= I_1 + I_2 + I_3. \tag{70}
\]
By (59) and (60), it is easy to see that
\[
I_1 \leq C \varepsilon^{d/2} \| u_0 \|_{H^2(\Omega)} \| F \|_{L^0(\Omega)}, \tag{71}
\]
where \( q_0 = \frac{2d}{d+1} \). To bound \( I_2 \), we use (58) to obtain
\[
I_2 \leq C \delta \left( \varepsilon^{1/2} \| \nabla v_0 \|_{L^0(\Omega_{2\varepsilon})} + \varepsilon \| \nabla v_0 \|_{L^2(\Omega)} \right) \| \nabla u_0 \|_{H^1(\Omega)}
\]
\[
\leq C \delta \| F \|_{L^0(\Omega)} \| u_0 \|_{H^2(\Omega)}, \tag{72}
\]
where we have also used the estimates (61) and (65) for \( v_0 \). Again, by (58),
\[
I_3 \leq C \delta \left( \varepsilon^{1/2} \| \nabla (\varepsilon \chi^{p,e}_j K_e(\partial_j v_0)) \|_{L^0(\Omega_{2\varepsilon})} \right.
\]
\[
+ \varepsilon \| \nabla (\varepsilon \chi^{p,e}_j K_e(\partial_j v_0)) \|_{L^2(\Omega)} \| \nabla u_0 \|_{H^1(\Omega)}
\]
\[
\leq C \delta \left( \varepsilon^{1/2} \| \nabla (\varepsilon \chi^{p,e}_j K_e(\partial_j v_0(1 - \eta_2)) \|_{L^2(\Omega)} \right.
\]
\[
+ \varepsilon \| \nabla (\varepsilon \chi^{p,e}_j K_e(\partial_j v_0)) \|_{L^2(\Omega)} \| \nabla u_0 \|_{H^1(\Omega)}
\]
where \( \eta_2 \) is defined as (53). Note that by Lemma 3.1,
\[
\| \nabla (\varepsilon \chi^{p,e}_j K_e(\partial_j v_0)) \|_{L^2(\Omega)}
\]
\[
\leq C \delta \| F \|_{L^0(\Omega)}, \tag{73}
\]
where we have used estimate (61) for \( v_0 \) in the second inequality. Similarly,
\[
\| \nabla (\varepsilon \chi^{p,e}_j K_e(\partial_j v_0(1 - \eta_2)) \|_{L^2(\Omega)}
\]
\[
\leq C \delta \| F \|_{L^0(\Omega)}
\]
Thus,
\[
I_3 \leq C \delta \| F \|_{L^0(\Omega)} \| u_0 \|_{H^2(\Omega)}. \tag{73}
\]
In view of the estimates of \( I_1, I_2, I_3 \) and (70), we obtain
\[
\left| \int_\Omega w_x \cdot F \, dx \right| \leq C \delta \| F \|_{L^0(\Omega)} \| u_0 \|_{H^2(\Omega)},
\]
which, by duality, gives the desired result. The proof is completed. \( \square \)

Similar to Lemma 2.2, we have the following lemma for \( \phi \) (see (49)).

Lemma 3.7. Suppose \( A \in C^{0,1}(\Omega; L^\infty_{per}(Y)) \) satisfies (3)–(4) and \( \rho \in C^{1,1}(\Omega; \mathbb{R}^n) \) satisfies (5).

i) Then for any \( x \in \Omega \),
\[
\| \phi(x, \cdot) \|_{W^{2,p}(Y)} \leq C,
\]
\[
\| \nabla \phi(x, \cdot) \|_{W^{2,p}(Y)} \leq C \{ | \nabla^2 \rho(x) | + \| \nabla A(x, \cdot) \|_{L^\infty(Y)} \}, \tag{74}
\]
and for any \(x_1, x_2 \in \overline{\Omega}\),
\[
\|\phi(x_1, \cdot) - \phi(x_2, \cdot)\|_{W^{2,p}(Y)} \leq C\{|\nabla \rho(x_1) - \nabla \rho(x_2)|
+ \|A(x_1, \cdot) - A(x_2, \cdot)\|_{L^\infty(Y)}\},
\]
(75)
where \(p\) is given by case (i) in Lemma 2.2 and \(C\) depends only on \(\mu\).

i) If in addition \(A \in C^{k-1,1}(\overline{\Omega}; L^\infty_{\text{per}}(Y))\), \(\rho \in C^{k,1}(\overline{\Omega})\) for some \(k \geq 1\), then \(\phi \in C^{k-1,1}(\overline{\Omega}; W^{2,p}_{\text{per}}(Y))\) with
\[
\|\phi\|_{C^{k-1,1}(\overline{\Omega}; W^{2,p}_{\text{per}}(Y))} \leq C,
\]
(76)
where \(p\) is given in case (i) and \(C\) depends only on \(\mu, \|\nabla \rho\|_{C^{k-1,1}(\overline{\Omega})}\) and \(\|A\|_{C^{k-1,1}(\overline{\Omega}; L^\infty(Y))}\).

ii) If \(A\) satisfies the uniformly VMO condition (10), then estimates (74) and (75) hold for any \(1 < p < \infty\) with \(C\) depending only on \(\mu, p\) and \(\rho(t)\).

Proof. By the definition of \(B(x, y)\) and Lemma 2.2, one can verify that, for any \(x \in \overline{\Omega}\),
\[
\|B(x, \cdot)\|_{L^p(Y)} \leq C, \quad \|\nabla_x B(x, \cdot)\|_{L^p(Y)} \leq C\{|\nabla^2 \rho(x)| + \|\nabla_x A(x, \cdot)\|_{L^\infty(Y)}\},
\]
and for any \(x_1, x_2 \in \overline{\Omega}\),
\[
\|B(x_1, \cdot) - B(x_2, \cdot)\|_{L^p(Y)} \leq C\{|\nabla \rho(x_1) - \nabla \rho(x_2)| + \|A(x_1, \cdot) - A(x_2, \cdot)\|_{L^\infty(Y)}\}.
\]
Then Lemma 3.7 can be derived in the same way as Lemma 2.2. We omit the details.

Lemma 3.8. Let \(\Omega\) be a bounded Lipschitz domain. Suppose \(h(x, y) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}\) is \(1\)-periodic in \(y\) and \(\rho \in C^1(\overline{\Omega}; \mathbb{R}^n)\) satisfies (5). Let \(\varepsilon_0\) be given in Lemma 2.1.

i) If \(h \in W^{k,\infty}(\Omega; L^q_{\text{per}}(Y))\) with \(k > \frac{d}{2}, q \geq 2\), then for any \(\varepsilon \leq \varepsilon_0\),
\[
\|h(x, \rho(x)/\varepsilon)\|_{S^2(\Omega)} \leq C\|h(x, y)\|_{W^{k,\infty}(\Omega; L^q_{\text{per}}(Y))},
\]
(77)
where \(C\) depends only on \(d, \mu, k\) and the Lipschitz character of \(\Omega\).

ii) If \(h \in W^{1,\infty}(\Omega; L^p_{\text{per}}(Y))\) with \(p > d\), then for any \(2 \leq q \leq p\) and \(\varepsilon \leq \varepsilon_0\),
\[
\|h(x, \rho(x)/\varepsilon)\|_{S^2(\Omega)} \leq C\|h(x, y)\|_{W^{1,\infty}(\Omega; L^p_{\text{per}}(Y))},
\]
(78)
where \(C\) depends only on \(d, \mu, p\) and the Lipschitz character of \(\Omega\).

iii) If \(h \in L^\infty(\Omega; W^{1,p}_{\text{per}}(Y))\) with \(p > d\), then for any \(q \geq 1\) and \(\varepsilon > 0\),
\[
\|h(x, \rho(x)/\varepsilon)\|_{S^2(\Omega)} \leq C\|h(x, y)\|_{L^\infty(\Omega; W^{1,p}_{\text{per}}(Y))},
\]
(79)
where \(C\) depends only on \(d\) and \(p\).

Proof. Thanks to Sobolev imbedding theorem, for a.e. \(x \in \Omega \cap B(y, 2\varepsilon)\),
\[
|h(x, \rho(x)/\varepsilon)| \leq C\left(\int_{\Omega \cap B(y, 2\varepsilon)} |\nabla^2_x h(z, \rho(x)/\varepsilon)|^2 dz\right)^{1/2}
+ C\left(\int_{\Omega \cap B(y, 2\varepsilon)} |h(z, \rho(x)/\varepsilon)|^2 dz\right)^{1/2},
\]
(80)
where \(C\) depends only on \(k\) and the Lipschitz character of \(\Omega\). By taking the \(L^q\)-average of (80) over \(\Omega \cap B(y, 2\varepsilon)\), we obtain
\[
\left(\int_{\Omega \cap B(y, 2\varepsilon)} |h(x, \rho(x)/\varepsilon)|^q dx\right)^{1/q}
\]
Remark 2. Proof of character of $\Omega$. By taking the supremum of (33) respectively. We omit the details. Thus, we conclude that $\delta$ depends only on $d, \mu, k$ and the Lipschitz character of $\Omega$. By taking the supremum of $y$ over $\Omega$, we get case (i).

Cases (ii) and (iii) are obtained by similar ideas and the inequalities as (32) and (33) respectively. We omit the details. \hfill $\square$

Proof of (8) in Theorem 1.1. It suffices to bound $\delta$. Since $A \in C^{k,1}(\overline{\Omega}; L^\infty_{\text{per}}(Y))$ and $\rho \in C^{k+1,1}(\overline{\Omega})$, by Lemmas 2.2 and 3.7, we know that $\chi \in C^{k,1}(\overline{\Omega}; H^1_{\text{per}}(Y))$ and $\phi \in C^{k,1}(\overline{\Omega}; H^2_{\text{per}}(Y))$. Thus, by lemma 3.8, for any $\varepsilon \leq \varepsilon_0$

$$\|\chi^{\rho,\varepsilon}\|_{S^2_2(\Omega)} + \|\nabla_x \chi^{\rho,\varepsilon}\|_{S^2_2(\Omega)} + \|\nabla_y \chi^{\rho,\varepsilon}\|_{S^2_2(\Omega)} + \|\nabla_y \phi^{\rho,\varepsilon}\|_{S^2_2(\Omega)} \leq C.$$ 

Moreover, by the definition of $\mathfrak{B}$,

$$\|\mathfrak{B}^{\rho,\varepsilon}\|_{S^2_2(\Omega)} \leq C\|\nabla \phi^{\rho,\varepsilon}\|_{S^2_2(\Omega)},$$

$$\|\nabla_x \mathfrak{B}^{\rho,\varepsilon}\|_{S^2_2(\Omega)} \leq C\|\nabla_x \nabla_y \phi^{\rho,\varepsilon}\|_{S^2_2(\Omega)} + C\|\nabla_y \phi^{\rho,\varepsilon}\|_{C(\overline{\Omega})} \|\nabla_y \phi^{\rho,\varepsilon}\|_{S^2_2(\Omega)}.$$ 

Thus, we conclude that $\delta \leq C$ for $\varepsilon \leq \varepsilon_0$. If $\varepsilon_0 < 1$, the case where $\varepsilon_0 < \varepsilon \leq 1$ is trivial. The proof is completed. \hfill $\square$

Remark 2. The additional conditions on $A$ and $\rho$ can be replaced by the uniformly VMO condition of $A$ without changing (8). To do this, one need to make use of (iii) in Lemmas 2.2, 3.7 and (ii)–(iii) in Lemma 3.8.

4. Interior Lipschitz estimates. This section is devoted to the interior Lipschitz estimates for $u_\varepsilon$.

We first study the rescaling and translation properties for the problem $\mathcal{L}_\varepsilon(u_\varepsilon) = f$ in $\Omega$. By setting $v_\varepsilon(x) = u_\varepsilon(rx + x_0)$, we have

$$-\text{div}(A(rx + x_0, \rho(rx + x_0)/\varepsilon) \nabla v_\varepsilon) = F \quad \text{in } \Omega_{r,x_0},$$

where $F(x) = r^2 f(rx + x_0)$ and $\Omega_{r,x_0} = \{(x - x_0)/r : x \in \Omega\}$. Denote by $\mathcal{L}_{\varepsilon}^{A,\rho}$ the elliptic operator associated to $A, \rho$ and $\varepsilon$, that is,

$$\mathcal{L}_{\varepsilon}^{A,\rho} = -\text{div}(A(x, \rho(x)/\varepsilon) \nabla).$$

Then $v_\varepsilon$ satisfies

$$\mathcal{L}_{r,x_0}^{A,\rho}(v_\varepsilon) = F \quad \text{in } \Omega_{r,x_0},$$

where $A_{r,x_0}(x, y) = A(rx + x_0, y), \rho_{r,x_0}(x) = \rho(rx + x_0)/r$. Note that $\rho_{r,x_0}$ satisfies condition (5) with the same $\mu$. For system (81), we can define the correctors $\chi_{r,x_0}$ as (23) and the homogenized matrix $\hat{A}_{r,x_0}$ as (25) with $A, \rho$ replaced by $A_{r,x_0}, \rho_{r,x_0}$ respectively. It’s easy to verify that $\chi_{r,x_0}(x, y) = \chi(rx + x_0, y)$ and $\hat{A}_{r,x_0}(x) = \hat{A}(rx + x_0)$. Moreover, one can also define the associated flux corrector $\mathfrak{B}_{r,x_0}$ and function $\phi_{r,x_0}$ which satisfy $\mathfrak{B}_{r,x_0}(x, y) = \mathfrak{B}(rx + x_0, y)$ and $\phi_{r,x_0}(x, y) = \phi(rx + x_0, y)$.

Now we turn to the Lipschitz estimates for $u_\varepsilon$ by using a scheme of regularity estimates of large scale (see e.g. [3, 2, 29]). It is based on the $O(\varepsilon^d)$-convergence rate of $u_\varepsilon$ to $u_0$. Recall that the convergence rate in Section 3 was obtained under the assumption that $\mathcal{A} = \mathcal{A}^\ast$. Thus, to avoid the symmetry assumption in this section, we shall establish another $O(\varepsilon^d)$-convergence rate.
Lemma 4.1. Suppose the assumptions of Lemma 3.4 hold. Then
\[ \|\nabla w_\varepsilon\|_{L^2(\Omega)} \leq C \delta \left\{ \|\nabla u_0\|_{L^2(\Omega_{2\varepsilon})} + \varepsilon \|\nabla^2 u_0\|_{L^2(\Omega_{2\varepsilon})} + \varepsilon \|\nabla u_0\|_{L^2(\Omega_{2\varepsilon})} \right\}, \] (82)
where \( 2^* = \frac{2d}{d-2} \), \( C \) depends only on \( \mu \) and
\[ \delta = \delta_\Omega := 1 + \| (\nabla_\varphi \rho)_{\rho,\varepsilon} \|_{S^2(\Omega)} + \| \lambda_{\rho,\varepsilon} \|_{S^2(\Omega)} + \| (\nabla_\varphi \rho)_{\rho,\varepsilon} \|_{S^2(\Omega)} + \| (\nabla^2 \rho)_{\rho,\varepsilon} \|_{L^4(\Omega)} + \| (\nabla x \varphi \rho)_{\rho,\varepsilon} \|_{L^4(\Omega)}. \] (83)

Proof. First of all, observing that
\[ |\nabla x \mathcal{B}| \leq C |\nabla^2 \rho| |\nabla_\varphi \rho| + C |\nabla x \nabla_\varphi \rho|, \]
we obtain that the third term in (56) can be dominated by
\[ C \varepsilon |\nabla^2 \rho|_{L^2(\Omega)} \| (\nabla_\varphi \rho)_{\rho,\varepsilon} \|_{S^2(\Omega)} \| S_\varepsilon(\nabla u_0)\|_{L^2(\Omega_{2\varepsilon})} \| \nabla \psi\|_{L^2(\Omega)} \]
\[ + C \varepsilon \| (\nabla x \nabla_\varphi \rho)_{\rho,\varepsilon} \|_{L^2(\Omega)} + \| (\nabla x \chi_{\rho,\varepsilon} \|_{L^2(\Omega)} \| K_\varepsilon(\nabla u_0)\|_{L^2(\Omega)} \| \nabla \psi\|_{L^2(\Omega)}, \]
where we have used Hölder’s inequality and Lemma 3.1. Besides, we have
\[ \|\mathcal{B}_{\rho,\varepsilon}\|_{S^2(\Omega)} \leq \| (\nabla_\varphi \rho)_{\rho,\varepsilon} \|_{S^2(\Omega)}. \]
Thus, similar to Lemma 3.4, it follows that
\[ \left| \int_\Omega A_{\rho,\varepsilon}(x) \nabla w_\varepsilon \cdot \nabla \psi dx \right| \leq C \delta \left\{ \|\nabla u_0 - S_\varepsilon(\nabla u_0)\|_{L^2(\Omega_{2\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{2\varepsilon})} \right\} \]
\[ + \varepsilon \| S_\varepsilon(\nabla^2 u_0)\|_{L^2(\Omega_{2\varepsilon})} + \varepsilon \| S_\varepsilon(\nabla u_0)\|_{L^2(\Omega_{2\varepsilon})} \| \nabla \psi\|_{L^2(\Omega)}. \] (84)

By setting \( \psi = w_\varepsilon \) in (84) and applying (68), we obtain (82). \( \square \)

Remark 3. One can verify that \( \delta \) is scaling-invariant, i.e., \( \delta_{\Omega_{\varepsilon},\varepsilon} = \delta_\Omega \), where \( \delta_{\Omega_{\varepsilon},\varepsilon} \) is the quantity defined as (83) for system (81). This is why we rewrite (54) into (84).

Lemma 4.2. Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \). Assume that \( A \in C^{0,1}(\overline{\Omega}, \mathbb{R}^n) \) satisfies (3)–(4) and \( \rho \in C^{1,1}(\overline{\Omega}, \mathbb{R}^n) \) satisfies (5). Let \( u_\varepsilon (\varepsilon \geq 0) \) be the solution of \( \mathcal{L}_\varepsilon u_\varepsilon = f \) in \( \Omega \) and \( u_\varepsilon = g \) on \( \partial \Omega \). Then there exists \( \sigma > 0 \), depending only on \( \mu \) and the Lipschitz character of \( \Omega \), such that for any \( 0 < \varepsilon \leq \delta \)
\[ \| u_\varepsilon - u_0 \|_{L^2(\Omega)} \leq C \varepsilon \delta \left\{ 1 + \|\nabla^2 \rho\|_{L^\infty(\Omega)} + \|\nabla x A\|_{L^\infty(\Omega \times Y)} \right\} \cdot \left\{ \|f\|_{L^2(\Omega)} + \|g\|_{H^1(\partial \Omega)} \right\}, \] (85)
where \( \delta \) is given by (83) and \( C \) depends only on \( \mu \) and \( \Omega \).

Proof. Observing that \( w_\varepsilon \in H^1_0(\Omega) \), and by Lemma 3.1
\[ \| \varepsilon \chi_{\rho,\varepsilon}^p K_\varepsilon(\partial u_0) \|_{L^2(\Omega)} \leq C \varepsilon \| \chi_{\rho,\varepsilon}^p \|_{S^2(\Omega)} \| S_\varepsilon(\nabla u_0)\|_{L^2(\Omega_{2\varepsilon})} \]
\[ \leq C \varepsilon \| \chi_{\rho,\varepsilon}^p \|_{S^2(\Omega)} \| \nabla u_0\|_{L^2(\Omega_{2\varepsilon})}, \]
we obtain from Lemma 4.1 that
\[ \| u_\varepsilon - u_0 \|_{L^2(\Omega)} \leq C \delta \left\{ \|\nabla u_0\|_{L^2(\Omega_{2\varepsilon})} + \|\nabla^2 u_0\|_{L^2(\Omega_{2\varepsilon})} + \|\nabla u_0\|_{L^2(\Omega_{2\varepsilon})} \right\}, \] (86)
where \( C \) depends only on \( \mu \) and \( \Omega \). Note that by the Meyers estimate [25, 18], there exists \( p > 2 \), depending only on \( \mu \) and the Lipschitz character of \( \Omega \), such that
\[ \|\nabla u_0\|_{L^p(\Omega)} \leq C \left\{ \|f\|_{L^2(\Omega)} + \|g\|_{H^1(\partial \Omega)} \right\}, \] (87)
where \( C \) depends only on \( \mu \) and \( \Omega \). For the first term on the r.h.s. of (86),
\[ \|\nabla u_0\|_{L^2(\Omega_{2\varepsilon})} \leq C \varepsilon^{\frac{1}{2} - \frac{n}{p}} \|\nabla u_0\|_{L^p(\Omega)}. \]
where $C$. Without loss of generality, we assume that only on $\mu$
where $L$ such that $B$
Lemma 4.3. Suppose $(\n\nL_0\text{\hat{u}}_0)\in H^{1}(\Omega)$, we obtain

$$\int_{\Omega\setminus\Omega_{\varepsilon}}|\nabla^2 u_0|^2 \leq C \int_{\Omega\setminus\Omega_{\varepsilon}}|\nabla u_0|^2 dy + C \int_{\Omega} |\nabla u_0|^2 dy + C \int_{\Omega} |f|^2 dy$$

where $\hat{\delta}(y) := \text{dist}(y, \partial \Omega)$, $c$ is a very small constant and $C$ depends only on $\mu$. Note that for any $x \in B(y, c\hat{\delta}(y))$, $\hat{\delta}(y) \sim \hat{\delta}(x)$. By integrating both sides in $y$ over $\Omega \setminus \Omega_{\varepsilon}$, we obtain

$$\int_{\Omega \setminus \Omega_{\varepsilon}} |\nabla^2 u_0|^2 \leq C \int_{\Omega} |\nabla u_0|^2 dy + C \int_{\Omega} |f|^2 dy$$

where $s > 1$. By setting $p = 2s > 2$, we get that

$$\|\nabla^2 u_0\|_{L^2(\Omega \setminus \Omega_{\varepsilon})} \leq C \varepsilon^{-\frac{1}{2}} (1 + \|\nabla \hat{A}\|_{L^\infty(\Omega)})\|\nabla u_0\|_{L^p(\Omega)} + C \|f\|_{L^2(\Omega)}$$

For the last term, by Sobolev imbedding theorem,

$$\|\nabla u_0\|_{L^{2^*}(\Omega \setminus \Omega_{\varepsilon})} \leq C \|\nabla u_0\|_{H^1(\Omega \setminus \Omega_{\varepsilon})}$$

where (87) and (89) were used in the second inequality. In view of (29), (86) and (88)–(90), we get (85) with $\sigma = \frac{1}{2} - \frac{1}{p}$ immediately.

Lemma 4.3. Suppose $u_{\varepsilon} \in H^1(B_{2r}; \mathbb{R}^m)$ is a solution to $L_{\varepsilon} u_{\varepsilon} = f$ in $B_{2r}$, where $B_{2r} = B(x_0, 2r)$, $x_0 \in \mathbb{R}^d$, $0 < \varepsilon \leq r$. Then there exists a function $u_0 \in H^1(B_{r}; \mathbb{R}^m)$ such that $L_0 u_0 = f$ in $B_r$ and for any $b \in \mathbb{R}^m$

$$\left( \int_{B_r} |u_{\varepsilon} - u_0|^2 \right)^{1/2} \leq C \left[ \left( \int_{B_{2r}} |\nabla u_{\varepsilon}|^2 + |u_{\varepsilon}|^2 \right)^{1/2} + r^2 \left( \int_{B_{2r}} |f|^2 \right)^{1/2} \right]$$

where $\hat{\delta}_{B_{2r}}$ is given by (83) with $\Omega = B_{2r}$, $\theta_{B_{2r}}$ is given by (12), $\sigma$ and $C$ depend only on $\mu$ and $d$.

Proof. Without loss of generality, we assume that $b = 0$. We first consider the case $r = 1$ and $x_0 = 0$. By Caccioppoli’s inequality,

$$\int_{B_{1/2}} \left| \nabla u_{\varepsilon} \right|^2 + |u_{\varepsilon}|^2 \leq C \int_{B_{2}} \left| \nabla u_{\varepsilon} \right|^2 + C \int_{B_{2}} |f|^2,$$

where $C$ depends only on $\mu$. According to the co-area formula, there exists $t \in [5/4, 3/2]$ such that

$$\int_{\partial B_t} \left| \nabla u_{\varepsilon} \right|^2 + \int_{\partial B_t} |u_{\varepsilon}|^2 \leq C \int_{B_2} |u_{\varepsilon}|^2 + C \int_{B_2} |f|^2.$$

Moreover, by the interior $H^2$ estimates for elliptic systems,
Now let $u_0$ be the solution to $L_0 u_0 = f$ in $B_t$, $u_0 = u_0$ on $\partial B_t$. By Lemma 4.2,
$$\|u_0 - u_0\|_{L^2(B_t)} \leq C \varepsilon \delta_{B_t}(1 + \Theta_{B_t})\{\|u_0\|_{H^1(\partial B_t)} + \|f\|_{L^2(B_t)}\},$$
where $C$ depends only on $\mu$ and $d$. This, together with (92), implies that
$$\|u_0 - u_0\|_{L^2(B_t)} \leq C \|u_0 - u_0\|_{L^2(B_t)} \leq C \varepsilon \delta_{B_t}(1 + \Theta_{B_t})\{\|u_0\|_{L^2(B_t)} + \|f\|_{L^2(B_t)}\},$$
which is exactly (91).

For general $r$ and $x_0$, let $v_r(x) = u_0(rx + x_0)$. Then $v_r$ satisfies $L_{r_0}(\varepsilon/\tau) v_r = 0$ in $B_{2r}$, where $A_r(x, y) = A(rx + x_0, y), \rho_r(x_0) = \rho(rx + x_0)/r$ and $f(r^2(x + x_0)) = r^2 f(rx + x_0)$. By the argument above, there exists a function $u_0 \in H^1(B_r; \mathbb{R}^m)$ such that $L_{r_0}(\varepsilon/\tau)(u_0) = F$ in $B_1$ and
$$\left(\int_{B_1} |v_r - v_0|^2\right)^{1/2} \leq C \left(\frac{\varepsilon}{r}\right) \frac{\delta_{r_0}(1 + \Theta_{B_2})}{\delta_{B_2}} \left\{\int_{B_2} |v_r|^2 + \int_{B_2} |F|^2\right\}^{1/2}, \tag{93}$$
where $L_{r_0}(\varepsilon/\tau) = -\text{div}(\hat{A}_{r,x_0} \nabla), \delta_{r,x_0}$ is given by (83) with $\chi$ and $\phi$ replaced by $\chi_{r,x_0}$ and $\phi_{r,x_0}$ respectively, and
$$\Theta_{B_2} = ||\nabla^2 \rho_{r,x_0}||_{L^\infty(B_2)} + ||\nabla x A_{r,x_0}||_{L^\infty(B_2 \times Y)}.$$ Let $u_0(x) = u_0((x - x_0)/r)$. It is easy to verify that $u_0 \in H^1(B_r; \mathbb{R}^m)$ satisfies $L_0 u_0 = f$ in $B_r$. Then (91) follows from (93) and Remark 3. The proof is completed.

**Lemma 4.4.** Let $B_r = B(x_0, r)$ be a ball in $\mathbb{R}^d$ and $u_0 \in H^1(B_r; \mathbb{R}^m)$ be a solution to $L_0 u_0 = f$ in $B_r$, where $f \in L^q(B_r; \mathbb{R}^m)$ with $q > d$. Define for $0 < t \leq r$,
$$G(t; u_0) = \frac{1}{t} \inf_{P \in \mathcal{P}} \left\{\left(\int_{B_t} |u_0 - P|^2\right)^{1/2} + t^2 \left(\int_{B_t} |f|^q\right)^{1/q}\right\},$$
where $\mathcal{P}$ denotes the set of $\mathbb{R}^m$-valued polynomials of degree at most 1. Then there exists $\theta \in (0, 1/8)$, depending only on $d, m, \mu, q, \Theta_{B_r}$ (in fact, the upper bound of $\Theta_{B_r}$), such that,
$$G(\theta r; u_0) \leq \frac{1}{2} G(r; u_0).$$

**Proof.** First we assume that $x_0 = 0$ and $r = 1$. By choosing $P(x) = u_0(0) + \nabla u_0(0) \cdot x$, we have
$$G(\theta; u_0) \leq C \theta^\alpha ||\nabla u_0||_{C^\alpha(B_1)} + \theta \left(\int_{B_1} |f|^q\right)^{1/q}$$
$$\leq C \theta^\alpha \|u_0 - P\|_{L^2(B_1)} + \|f\|_{L^q(B_1)} + \theta \left(\int_{B_1} |f|^q\right)^{1/q}$$
$$\leq C \theta^\alpha \left(\int_{B_1} |u_0 - P|^2\right)^{1/2} + C \theta^\alpha \left(\int_{B_1} |f|^q\right)^{1/q}$$
for any $P \in \mathcal{P}$, where $\alpha = 1 - \frac{d}{q}, C$ depends only on $\mu, ||\hat{A}||_{C^\alpha(B_1)}$ and we have used the $C^{1,\alpha}$ estimates for elliptic systems with $C^\alpha$ coefficients in the second inequality. By setting $\theta \in (0, 1/8)$ small enough, we obtain the desired result. Note that we also used the fact that
$$||\hat{A}||_{C^\alpha(B_1)} \leq ||\nabla \hat{A}||_{L^\infty(B_1)} \leq C ||\nabla^2 \rho||_{L^\infty(B_1)} + ||\nabla x A||_{L^\infty(B_1 \times Y)}.$$
For any fixed $\theta$, one can do rescaling as usual. Then the same result holds with $\theta$ depending on $\|\nabla \tilde{A} \cdot x_0\|_{L^\infty(B_1)}$. Note that

\[
\|\nabla \tilde{A} \cdot x_0\|_{L^\infty(B_1)} = r \|\nabla \tilde{A}\|_{L^\infty(B_r)} \leq C r \|\nabla^2 \rho\|_{L^\infty(B_r)} + \|\nabla \cdot A\|_{L^\infty(B_r \times Y)} = C \Theta_{B_r}.
\]

This completes the proof.

**Lemma 4.5.** Let $u_\varepsilon \in H^1(B_1; \mathbb{R}^m)$ be a solution to $\mathcal{L}_\varepsilon u_\varepsilon = f$ in $B_1$, where $f \in L^q(B_1; \mathbb{R}^m)$ for some $q > d$ and $\varepsilon \in (0, 1/4)$. For $0 < r \leq 1$, define

\[
H(r) = \frac{1}{r} \inf_{P \in P} \left\{ \left( \int_{B_r} |u_\varepsilon - P|^2 \right)^{1/2} + r^2 \left( \int_{B_r} |f|^q \right)^{1/q} \right\},
\]

\[
\Phi(r) = \frac{1}{r} \inf_{b \in \mathbb{R}^m} \left\{ \left( \int_{B_r} |u_\varepsilon - b|^2 \right)^{1/2} + r^2 \left( \int_{B_r} |f|^2 \right)^{1/2} \right\}.
\]

Then there exists $\theta \in (0, 1/2)$, depending only on $d, m, \mu, q, \Theta_{B_1}$, such that,

\[
H(\theta r) \leq \frac{1}{2} H(r) + C \left( \frac{\varepsilon}{r} \right)^{\sigma} \delta_{B_{2r}} \Phi(2r),
\]

for any $r \in [\varepsilon, 1/2]$, where $C$ depends only on $d, m, \mu, q$ and $\Theta_{B_1}$.

**Proof.** For any fixed $r \in [\varepsilon, 1/2]$, let $u_0$ be the solution to $\mathcal{L}_0 u_0 = f$ in $B_r$ given in Lemma 4.3. Note that $\Theta_{B_r}$ is increasing in $r$. Thus, by Lemma 4.4, there exists $\theta$ depending only on $d, m, \mu, q, \Theta_{B_1}$, such that

\[
H(r) \leq \frac{1}{1 \theta r} \left( \int_{B_{\theta r}} |u_\varepsilon - u_0|^2 \right)^{1/2} + G(\theta r; u_0)
\]

\[
\leq \frac{1}{\theta r} \left( \int_{B_{\theta r}} |u_\varepsilon - u_0|^2 \right)^{1/2} + \frac{1}{2} G(r; u_0)
\]

\[
\leq \frac{1}{\theta r} \left( \int_{B_{\theta r}} |u_\varepsilon - u_0|^2 \right)^{1/2} + C \left( \int_{B_{\theta r}} |u_\varepsilon - u_0|^2 \right)^{1/2} + \frac{1}{2} H(r)
\]

\[
\leq C \left( \frac{\varepsilon}{r} \right)^{\sigma} \delta_{B_{2r}} \frac{1}{r} \left\{ \left( \int_{B_{2r}} |u_\varepsilon - b|^2 \right)^{1/2} + r^2 \left( \int_{B_{2r}} |f|^2 \right)^{1/2} \right\} + \frac{1}{2} H(r)
\]

for any $b \in \mathbb{R}^m$, where we have used Lemma 4.3 in the last step. This is exactly what we desired. \qed

The following lemma was proved in [29].

**Lemma 4.6.** Let $H(r)$ and $h(r)$ be two nonnegative continuous functions on the interval $(0, 1]$ and let $\varepsilon \in (0, 1/4)$. Assume that

\[
\max_{r \leq t \leq 2r} H(t) \leq C_0 H(2r), \quad \max_{r \leq t, s \leq 2r} |h(t) - h(s)| \leq C_0 H(2r),
\]

for any $r \in [\varepsilon, 1/2]$, and also

\[
H(\theta r) \leq \frac{1}{2} H(r) + C_0 \omega(\varepsilon/r) \{ H(2r) + h(2r) \},
\]

for any $r \in [\varepsilon, 1/2]$, where $\theta \in (0, 1/4)$ and $\omega$ is a nonnegative increasing function on $[0, 1]$ such that $\omega(0) = 0$ and

\[
\int_0^1 \frac{\omega(s)}{s} \, d\ll s < \infty.
\]

Then

\[
\max_{\varepsilon \leq r \leq 1} \left\{ H(r) + h(r) \right\} \leq C \{ H(1) + h(1) \},
\]

(98)
where $C$ depends only on $C_0$, $\theta$ and $\omega$.

**Lemma 4.7.** Suppose $A \in C^{0,1}(\overline{B_R}; L^\infty_{\text{loc}}(Y))$ satisfies (3)-(4) and $\rho \in C^{1,1}(\overline{B_R}; \mathbb{R}^n)$ satisfies (5), where $B_R = B(x_0, R)$. Suppose further $A$ satisfies the uniformly VMO condition (10). Then

$$\delta_{B_R} \leq C(1 + \|\nabla^2 \rho\|_{L^4(B_R)} + \|\nabla x A\|_{L^4(B_R; L^2(Y))}),$$

where $C$ depends only on $d, \mu$ and $g(t)$.

**Proof.** By applying Lemmas 2.2, 3.7 and 3.8, we have for any $q \geq 1$

$$\|x^{\rho,\varepsilon}\|_{S^1_{\text{per}}(B_R)} \leq C\|\chi(x, y)\|_{L^\infty(B_R; W^{1,q}_{\text{per}}(Y))} \leq C,$$

$$\|\nabla_y \phi(x, y)\|_{L^\infty(B_R; W^{1,q}_{\text{per}}(Y))} \leq C,$$

where $p = d + 1$ and $C$ depends only on $d, \mu$ and $g(t)$. Moreover, by virtue of Lemmas 2.4, 2.2 and 3.7,

$$\|\nabla_y \phi\|_{L^d(B_R)} \leq C\|\nabla_y \phi(x, y)\|_{L^d(B_R; W^{1,q}_{\text{per}}(Y))} \leq C\|\nabla^2 \rho(x)\|_{L^d(B_R)} + \|\nabla x A\|_{L^d(B_R; L^2(Y))},$$

$$\|\nabla_y \phi\|_{L^d(B_R)} \leq C\|\nabla y \phi(x, y)\|_{L^d(B_R; W^{1,q}_{\text{per}}(Y))} \leq C\|\nabla^2 \rho(x)\|_{L^d(B_R)} + \|\nabla x A\|_{L^d(B_R; L^2(Y))},$$

where $C$ depends only on $d, \mu$ and $g(t)$. Thus, we have

$$\delta_{B_R} \leq C(1 + \|\nabla^2 \rho\|_{L^d(B_R)} + \|\nabla x A\|_{L^d(B_R; L^2(Y))}),$$

where $C$ depends only on $d, \mu$ and $g(t)$. This completes the proof.

**Proof of Theorem 1.2.** We first consider the case $R = 1$ and $x_0 = 0$. Since the case $r \in [1/4, 1/2]$ is trivial, we also assume that $\varepsilon \in (0, 1/4)$. Let $H(r), \Phi(r)$ be defined as in Lemma 4.5 and $\omega(t) = t^\varepsilon$ which satisfies (97). Denote by $P_r$ the polynomial achieving the infimum in (94), i.e., $P_r$ satisfies

$$H(r) = \frac{1}{r} \left\{ \left( \int_{B_r} |u_\varepsilon - P_r|^2 \right)^{1/2} + r^2 \left( \int_{B_r} |f|^q \right)^{1/q} \right\},$$

and let $h(r) = |\nabla P_r|$. Note that $\nabla P_r$ is a constant matrix.

For $t \in [r, 2r]$, it is evident that $H(t) \leq C H(2r)$. Note that for $t, s \in [r, 2r],$

$$|\nabla(P_t - P_s)| \leq C \left( \int_{B_r} |P_t - P_s|^2 \right)^{1/2},$$

$$\leq C \left( \int_{B_r} |u_\varepsilon - P_t|^2 \right)^{1/2} + C \left( \int_{B_r} |u_\varepsilon - P_s|^2 \right)^{1/2},$$

$$\leq C \left( \int_{B_s} |u_\varepsilon - P_t|^2 \right)^{1/2} + C \left( \int_{B_s} |u_\varepsilon - P_s|^2 \right)^{1/2},$$

$$\leq C \{H(t) + H(s)\} \leq C H(2r).$$

We obtain that

$$\max_{r \leq t, s \leq 2r} |h(t) - h(s)| \leq C H(2r).$$

Thus, condition (95) is satisfied.
Moreover, by Poincaré’s inequality, we have
\[ \Phi(2r) \leq H(2r) + \frac{1}{r} \inf_{b \in \mathbb{R}^m} \left( \int_{B_2r} |P_{2r} - b|^2 \right)^{1/2} \leq H(2r) + Ch(2r), \]
which together with Lemmas 4.5 and 4.7 implies condition (96) with \( \theta, C_0 \) depending only on \( d, m, \mu, q, \phi(t) \) and \( \Theta_{B_1} \).

Therefore, by Lemma 4.6, it holds that
\[
\frac{1}{r} \inf_{b \in \mathbb{R}^m} \left( \int_{B_r} |u_{\varepsilon} - b|^2 \right)^{1/2} \leq H(r) + \frac{1}{r} \inf_{b \in \mathbb{R}^m} \left( \int_{B_r} |P_r - b|^2 \right)^{1/2} \\
\leq C\{H(r) + h(r)\} \\
\leq C\{H(1) + h(1)\} \\
\leq C\{\left( \int_{B_1} |u_{\varepsilon}|^2 \right)^{1/2} + \left( \int_{B_1} |f|^q \right)^{1/q} \},
\]
for any \( r \in [\varepsilon, 1/2] \), where in the last step we have used the observation that
\[
h(1) \leq C\left( \int_{B_1} |P_1|^2 \right)^{1/2} \\
\leq C\left( \int_{B_1} |u_{\varepsilon} - P_1|^2 \right)^{1/2} + C\left( \int_{B_1} |u_{\varepsilon}|^2 \right)^{1/2} \\
\leq CH(1) + C\left( \int_{B_1} |u_{\varepsilon}|^2 \right)^{1/2}.
\]
The desired estimate for the case \( R = 1 \) and \( x_0 = 0 \) now follows from (99) by Caccioppoli’s inequality. The general case can be reduced to this special case by rescaling and translation. We omit the details here.

Next we turn to (14). By (13), we have for \( \varepsilon \geq 1/2 \)
\[
|A^{\rho\varepsilon}(x_1) - A^{\rho\varepsilon}(x_2)| \lesssim A_0|x_1 - x_2|^{\gamma_0}.
\]
Thus, we can assume that \( 0 < \varepsilon < 1/2 \), since the case \( \varepsilon \geq 1/2 \) follows from the \( C^{1,\alpha} \) estimates for elliptic systems with Hölder continuous coefficients. Without loss of generality, we also assume that \( R = 1, x_0 = 0 \).

To cope with the case \( 0 < \varepsilon < 1/2 \), we apply the blowup argument. Set \( v(x) = \varepsilon^{-1}u_{\varepsilon}(\varepsilon x) \). Then \( L_1^{A^{\rho\varepsilon}}(v) = F \) in \( B_1 \), where \( A_\varepsilon(x, y) = A(\varepsilon x, y), \rho_\varepsilon(x) = \varepsilon^{-1}\rho(\varepsilon x), F(x) = \varepsilon f(\varepsilon x) \). One can verify that \( A_\varepsilon(x, \rho_\varepsilon(x)) \) is also Hölder continuous. By the \( C^{1,\alpha} \) estimate for elliptic systems,
\[
|\nabla v(0)| \leq C\left\{ \left( \int_{B_1} |\nabla v|^2 \right)^{1/2} + \left( \int_{B_1} |F|^q \right)^{1/q} \right\} \\
\leq C\left\{ \left( \int_{B_1} |\nabla u_{\varepsilon}|^2 \right)^{1/2} + \varepsilon^{1-d/q} \left( \int_{B_1} |f|^q \right)^{1/q} \right\},
\]
which, together with (11) for \( r = \varepsilon \) and the fact that \( \nabla v(0) = \nabla u_{\varepsilon}(0) \), yields (14). The proof is completed.

5. \( W^{1,p} \) estimates. In this section, we establish the uniform interior \( W^{1,p} \) estimates under the additional assumption that \( A \) satisfies the uniformly VMO condition.
Lemma 5.1. Suppose \( A \in C^{0,1}(\mathbb{R}^n) \) satisfies (3)–(4) and \( \rho \in C^{1,1}(\mathbb{R}^n) \) satisfies (5), where \( B = B(x_0, r) \). Suppose further \( A \) satisfies (10). Let \( u_\varepsilon \in H^1(2B; \mathbb{R}^m) \) be a solution to \( \mathcal{L}_\varepsilon u_\varepsilon = 0 \) in \( 2B \). Then for any \( 2 < p < \infty \),

\[
\left( \int_{2B} |\nabla u_\varepsilon|^p \right)^{1/p} \leq C_p \left( \int_{2B} |\nabla u_\varepsilon|^2 \right)^{1/2},
\]

where \( C_p \) depends only on \( p, \Theta_2B \) and \( \rho(t) \) in (10).

Proof. Without loss of generality, we assume that \( x_0 = 0 \) and \( r = 1 \). Note that when \( \varepsilon \geq 1/4 \), we have for any \( z \in 2B \) and \( s < t \),

\[
\begin{align*}
&\int_{B(z,s)\cap 2B} |A^{\rho,\varepsilon}(x) - A^{\rho,\varepsilon}(x)| dx \\
&\quad \leq 2 \int_{B(z,s)\cap 2B} |A^{\rho,\varepsilon}(x) - A(z, \frac{\rho(x)}{\varepsilon})| dx \\
&\quad + \int_{B(z,s)\cap 2B} A(z, \frac{\rho(x)}{\varepsilon}) - \int_{B(z,s)\cap 2B} A(z, \frac{\rho(x)}{\varepsilon})| dx \\
&\quad \leq 2\varepsilon \left\| \nabla A \right\|_{L^\infty(2B \times \mathbb{R}^d)} + C \rho(t),
\end{align*}
\]

which implies that \( A^{\rho,\varepsilon} \in \text{VMO} \) uniformly. Thus, if \( \varepsilon \geq 1/4 \), (100) follows from the classical \( W^{1,p} \) estimates for elliptic systems with VMO coefficients.

For \( 0 < \varepsilon < 1/4 \), set \( v(x) = u_\varepsilon(\varepsilon x) \). Then \( \mathcal{L}_1^{A^{\rho,\varepsilon}}(v) = 0 \) in \( B(0, 2/\varepsilon) \), which implies that

\[
\left( \int_{B(0,1/2)} |\nabla v|^p \right)^{1/p} \leq C \left( \int_{B(0,1)} |\nabla v|^2 \right)^{1/2},
\]

where we have used the fact that \( A_\varepsilon(\rho_\varepsilon(x)) \in \text{VMO} \) uniformly. By a change of variables, this leads to

\[
\left( \int_{B(0,\varepsilon/2)} |\nabla u_\varepsilon|^p \right)^{1/p} \leq C \left( \int_{B(0,\varepsilon)} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left( \int_{B(0,2)} |\nabla u_\varepsilon|^2 \right)^{1/2},
\]

where we have used (11) in the last step and \( C \) depends only on \( p, \rho(t) \) and \( \Theta_{B(0,2)} \).

Similarly, for any \( y \in B(0, 1) \),

\[
\left( \int_{B(y,\varepsilon/2)} |\nabla u_\varepsilon|^p \right)^{1/p} \leq C \left( \int_{B(0,2)} |\nabla u_\varepsilon|^2 \right)^{1/2}.
\]

Thus, it follows that

\[
\int_{B(y,\varepsilon/2)} |\nabla u_\varepsilon|^p \leq C \varepsilon^d \left\| \nabla u_\varepsilon \right\|_{L^2(B(0,2))}^p,
\]

which yields estimate (100) by integrating in \( y \) over \( B(0,1) \).

The next theorem is a real variable argument formulated in [12, 27, 28].

Theorem 5.2. Let \( q > 2 \), \( F \in L^2(4B_0) \) and \( f \in L^p(4B_0) \) for some \( 2 < p < q \), where \( B_0 \) is a ball in \( \mathbb{R}^d \). Suppose that for each ball \( B \subset 2B_0 \) with \( |B| < c_0|B_0| \), there exists two measurable functions \( F_B \) and \( R_B \) on \( 2B \) such that \( |F| \leq |F_B| + |R_B| \) on \( 2B \), and

\[
\left( \int_{2B} |R_B|^q \right)^{1/q} \leq C_1 \left\{ \left( \int_{4B} |F|^2 \right)^{1/2} + \sup_{B \subset B' \subset 4B_0} \left( \int_{B'} |f|^2 \right)^{1/2} \right\},
\]

(101)
\[
\left( \int_{2B} |F_B|^2 \right)^{1/2} \leq C_2 \sup_{B \subset B' \subset 4B_0} \left( \int_{2B'} |f|^2 \right)^{1/2},
\]
\(102\)
where \(C_1, C_2 > 0\) and \(0 < c_0 < 1\). Then \(F \in L^p(B_0)\) and
\[
\left( \int_{B_0} |F|^p \right)^{1/p} \leq C \left\{ \left( \int_{4B_0} |F|^2 \right)^{1/2} + \left( \int_{4B_0} |f|^p \right)^{1/p} \right\},
\]
where \(C\) depends only on \(d, C_1, C_2, c_0, p\) and \(q\).

**Proof of Theorem 1.3.** By rescaling we can assume that \(r = 1\). We use Theorem 5.2 to establish (15). For each ball \(B\) with \(4B \subset 2B_0\), we decompose \(u_\varepsilon\) as \(u_\varepsilon = v_\varepsilon + w_\varepsilon\) on \(2B\), where \(v_\varepsilon \in H_0^1(4B; \mathbb{R}^m)\) is the solution to \(\mathcal{L}_\varepsilon v_\varepsilon = f_1 + \text{div}(f_2)\) in \(4B\) and \(w_\varepsilon\) is the solution to \(\mathcal{L}_\varepsilon w_\varepsilon = 0\) in \(4B\). Set \(q = p + 1\),
\[
F = |\nabla u_\varepsilon|, \quad F_B = |\nabla v_\varepsilon|, \quad R_B = |\nabla w_\varepsilon| \quad \text{and} \quad f = |f_1| + |f_2|.
\]
Clearly, \(F \leq F_B + R_B\) on \(2B\). And (102) follows from the standard energy estimates. Moreover, by using Lemma 5.1,
\[
\left( \int_B |R_B|^q \right)^{1/q} \leq C \left( \int_{2B} |\nabla w_\varepsilon|^2 \right)^{1/2}
\leq C \left\{ \left( \int_{2B} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left( \int_{2B} |\nabla v_\varepsilon|^2 \right)^{1/2} \right\}
\leq C \left\{ \left( \int_{4B} |F|^2 \right)^{1/2} + \left( \int_{4B} |f_1|^2 \right)^{1/2} + \left( \int_{4B} |f_2|^2 \right)^{1/2} \right\},
\]
which is exactly (101). Thanks to Theorem 5.2, we obtain (15). This completes the proof. \(\square\)

The interior \(W^{1,p}\) estimate in Theorem 1.3 gives the following interior Hölder and \(L^\infty\) estimates by Sobolev imbedding.

**Corollary 2.** Suppose the assumptions of Theorem 1.3 hold. Let \(u_\varepsilon \in H^1(2B; \mathbb{R}^m)\) be a weak solution to
\[
\mathcal{L}_\varepsilon u_\varepsilon = f_1 + \text{div}(f_2) \quad \text{in} \ 2B,
\]
where \(B = B(x_0, r)\), and \(f_1, f_2 \in L^p(2B; \mathbb{R}^m)\) for some \(p > d\). Then for any \(x, y \in B\)
\[
|u_\varepsilon(x) - u_\varepsilon(y)| \leq C \left( \frac{|x - y|}{r} \right)^{1 - \frac{d}{p}} \left\{ \left( \int_{2B} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left( \int_{2B} |f_1|^p \right)^{1/p} \right.
+ \left. r \left( \int_{2B} |f_2|^p \right)^{1/p} \right\},
\]
\[
\|u_\varepsilon\|_{L^\infty(B)} \leq C \left\{ \left( \int_{2B} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left( \int_{2B} |f_1|^p \right)^{1/p} + r \left( \int_{2B} |f_2|^p \right)^{1/p} \right\},
\]
where \(C\) depends only on \(p, g(t)\) in (10) and \(\Theta_{2B}\) in (12).

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