Searching for a modification of a conformally flat metric in the five-dimensional braneworld model that localizes scalar and vector fields

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Abstract. A modification of the Randall-Sundrum metric model in the form of conformally flat metric was proposed some years ago. The proposed model shows better localization properties than the original Randall-Sundrum model. However, this new model still does not meet the expectation that all fundamental fields can be localized for one type of warp factor, e.g. for a decreasing warp factor. In this research we propose to add a factor $f(r)$ in the extra coordinate part of the above modified Randall-Sundrum metric and derive conditions for localization for scalar and vector fields. $r$ is the extra coordinate. We then look for some functions $f(r)$ that fulfill the localization conditions for both scalar and vector fields. Examples of the function $f(r)$ under investigation are $e^{Ar}$, $e^{-Ar-Br^2}$, $e^{-Ai/r}$, $Ar$, $Ar + B + r$, and $\sinh(\sqrt{A}r)$. We found the expected function, that is $e^{Ar}$, which might be used as a further discussion in fields localization.

1. Introduction

The concept of an extra dimension starts from the concept by Kaluza (1921), later elaborated by Klein (1926) [1], which aims to unite gravitational and electromagnetic interactions into one fundamental law. The basic concept of this theory is the postulate of compact extra-dimensions, and introducing gravity in (1+4)-dimensional spacetime as manifestations of gravitational, electromagnetic and scalar fields. After the standard model of particle physics, the Kaluza-Klein extra dimension concept is still used and inspired several new theories, e.g. string theory, which still assumes the same regarding extra dimensions. One of the main problems in physics is the hierarchy problem. One model that try to solve is the braneworld model. The braneworld scenario represents standard model of particles (except gravitons) in a 3-brane or hyporsurface embedded in higher dimensional spacetime. Among the extra dimension models developed over the last few decades are the ADD model [2-5], and the warped extra dimension Randall-Sundrum (RS) models [6-8]. Here we consider in the RS model, which has conformally flat branes. There are two types of the models, i.e. the RS model which has two branes (Planck brane at $y = 0$ and TeV brane at $y = 2\pi R$), and the RS model which has one brane only at $y = 0$. In the RS model, the standard model TeV particles can not be localized properly on the 3-brane.

Then, an alternative model of RS has been proposed by modifying the metric of the extra dimension which conformally flat, called the MRS model [9,10]. Interesting facts from the MRS model compared to the RS model are that the massive scalar fields can be localized on the brane...
for both decreasing and increasing warp factors, and that massless vector fields can be localized only for a decreasing warp factor and for a constant solution. Nevertheless, the MRS model still does not meet the expectation that all matter fields are localized on the brane, including interacting fields [11-13]. Thus, the existence of a braneworld model that is able to localize all fields is still under question [14]. This is a worth searching.

As a development of the MRS model, we will then look for whether there is a spacetime geometry that has better localization properties than the RS models. In this study, we modify the metric of the MRS model by adding a factor $f(r)$ to the extra coordinate, and provide scalar and vector localization analysis generally. Then, we consider some $f(r)$ functions explicitly to be applied to this model.

2. Conformally Flat Metric in 5-Dimensional Braneworld Model

2.1. The Randall-Sundrum background

The Randall-Sundrum model is the 5-dimensional braneworld model consisting of (1+3)-dimensional known spacetime and one extra dimension $r$. The general metric of RS is written as follows [9]

$$ds^2 = g_{MN}dx^M dx^N = a^2(r)\eta_{\mu\nu}dx^\mu dx^\nu - b^2(r)dr^2,$$

where the 4-dimensional Minkowskian metric is $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. For RS model, the warp factor $a(r) = e^{-kr}$ and $b(r) = 1$. Meanwhile in MRS model, the warp factor $a(r) = b(r) = e^{-kr}$, by taking coordinate transformation in extra dimension from the RS model.

2.2. Metric modification

As a modification, we give an additional function $f(r)$ in the warped extra coordinate factor. So, the metric is

$$ds^2 = e^{-2kr}\left(\eta_{\mu\nu}dx^\mu dx^\nu - f^2(r)dr^2\right).$$

According to general metric form (1), the warp factor $a(r) = e^{-kr}$ and $b(r) = e^{-kr}f(r)$. Then, we choose the factor $b(r)$ to be a convergent function, for fields localization purposes. From metric (2), we derive the energy-momentum tensors for the brane and the bulk, respectively, are written as follows

$$T_{\mu\nu} = \frac{\eta_{\mu\nu}}{\kappa^2} \left(-\frac{3}{f^3}(k^2f + kf' - 2kf\delta(r)) + e^{-2kr}\lambda\right), \quad T_{rr} = \frac{1}{\kappa^2}(6k^2 - e^{-2kr}f^2\lambda),$$

where $\kappa \equiv c^2/\sqrt{8\pi G}$ is a constant, $\delta(r)$ is Dirac delta function which shows that any TeV fields confined on a thin brane located at $r$, and $\lambda$ is a cosmological constant in 5-dimension which represents the energy density in the spacetime. These two equations define any matter that occupies in the spacetime with the metric (2).

3. Fields Localization

3.1. Scalar field

Action of a complex scalar field $\Phi(x^\mu, r)$ is

$$S_0 = \int d^3x \sqrt{g}g^{MN}(\partial_M \Phi^*)(\partial_N \Phi)$$

$$= \int_0^\infty dr \sqrt{g}\chi_0^*\chi_0 g^{\mu\nu} \int d^3x (\partial_\mu \varphi^*)(\partial_\nu \varphi) + \int_0^\infty dr \sqrt{gg^{\tau\tau}}(\partial_r \chi_0^*)(\partial_r \chi_0) \int d^4x \varphi^* \varphi,$$

where $\Phi(x^\mu, r)$ has been separated into the brane and bulk parts, $\Phi(x^\mu, r) = \varphi(x^\mu)\chi_0(r)$. The bulk has the range of $[-\infty, +\infty]$, and as in the RS and MRS models we consider positive $r$ only.
The first term corresponds to normalization of the field, while the second term defines the mass of \( \varphi(x^\mu) \). Metric (2) gives localization conditions

\[
\int_0^\infty dr\, e^{-3kr} f(r)\chi_0'(r)\chi_0(r) = N_0 = 1; \\
- \int_0^\infty dr\, e^{-3kr} \frac{f'(r)}{f(r)} (\partial_r \chi_0^*) (\partial_r \chi_0) = -M_0^2,
\]

where \( M_0 = m_\varphi \) is the mass of \( \varphi \), the scalar field defined on the brane. If the normalization \( N_0 \) and mass \( M_0^2 \) get finite values, then the scalar field is said to be localized on the brane. Using the least action principle of the action (4), one yields dynamical equations for brane and bulk parts, respectively, as follows

\[
\eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + m^2 \varphi = 0,
\]

\[
\partial_r^2 \chi_0 - \left( 3k + f'f^{-1} \right) \partial_r \chi_0 + m^2 f^2 \chi_0 = 0,
\]

where \( m \) is the TeV scale particle mass, and consider the dynamics equation in the bulk, from the metric (2). The prime means derivative with respect to the \( r \). A solution of \( \chi_0 \) is needed to analyze field localization.

Consider a massless scalar field, \( m = 0 \), the dynamics equation (8) gives solution and its derivative to \( r \) as follows

\[
\chi_0(r) = b_0 \int \exp \left\{ \int \left( 3k + f'f^{-1} \right) dr \right\} dr + c_0, \quad \partial_r \chi_0(r) = b_0 \int \exp \left\{ \int \left( 3k + f'f^{-1} \right) dr \right\} dr + c_0, \quad (9)
\]

where \( b_0 \) dan \( c_0 \) are constant. For a spacetime that has a metric (2), the normalization (5) and mass (6) then be written as follows

\[
N_0 = \int_0^\infty dr e^{-3kr} \left[ b_0^2 \left\{ \int \exp \left( \int \left( 3k + \frac{f'}{f} \right) dr \right) dr \right\}^2 + 2b_0c_0 \int \exp \left( \int \left( 3k + \frac{f'}{f} \right) dr \right) dr + c_0^2 \right], \quad (10)
\]

\[
M_0^2 = b_0^2 \int_0^\infty dr f(r)e^{3kr}. \quad (11)
\]

In this case, the mass of the particle should be consistent at \( m = 0 \), so the mass in equation (11) should be zero. However, the integrand in the mass equation is divergent for \( k > 0 \). The choices are taking the constant \( b_0 = 0 \) or \( f(r) = 0 \). But, if we take \( f(r) = 0 \) the Randall-Sundrum metric changes to a 4-dimensional Minkowski metric, since there are no extra coordinates and the warp factor \( a^2(r) = 1 \). Therefore, the constant \( b_0 = 0 \) is chosen. On the other hand, for an increasing warp factor, \( k < 0 \), the integrand mass equation converges and produces a nonzero value, \( M_0 = b_0/\sqrt{3k} \). But again, we will use \( b = 0 \) for the mass consistency. So, we get the localization conditions

\[
N_0 = \frac{c_0^2}{b_0^2} \int_0^\infty dr f(r)e^{-3kr}, \quad M_0^2 = 0. \quad (12)
\]

Then, we can determine any \( f(r) \) functions in which massless scalar fields can be localized.

For a massive mode, we still have not found whether there is an exact solution to the dynamics equation or not. Maybe the Sturm-Liouville problem can be used to get alternative solutions, which we do not review here. Next, after the following section, we consider some \( f(r) \) functions will be tried which allow for an exact solution to be obtained from the massive scalar field dynamics equation, and then use the solution to search localization properties.
3.2. Vector field

Consider a gauge field in a 5-dimensional spacetime, $A_M(x^N) = (A_{\mu}(x^M), A_r)$, where as in the MRS model $A_r$ is a constant. Action for the field $A_M(x^N)$ is

$$S_1 = -\frac{1}{4} \int d^5x \sqrt{g} F_{MN} g^{RS} F_{MR} F_{NS},$$

where $F_{MN} = \partial_M A_N - \partial_N A_M$ is field strength tensor in 5-dimensional spacetime, and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is defined as the field strength in 4-dimension. From the action (13) and metric (2), we get the localization conditions

$$\int_0^\infty dr e^{-kr} f(r) \eta^{\mu\nu} \eta^{\rho\sigma} c^2(r) = N_1 \eta^{\mu\nu} \eta^{\rho\sigma},$$

$$\int_0^\infty d^4r e^{-kr} f^{-1}(r) \eta^{\mu\nu}(\partial_r c(r))^2 = m_1^2 \eta^{\mu\nu},$$

where $N_1$ is a normalization constant and $m_1$ is mass of the vector field $a^\mu$. They should have finite values, so that the vector field can be localized. Using the least action principle, $\delta S_1 = 0$, then we obtain the dynamical equation of vector field,

$$c(r) \partial_\mu (\sqrt{g} g^{\mu\nu} g^{R\rho} f_{\nu\rho}) + \partial_r \left( \sqrt{g} g^{rr} g^{R\rho} a_\sigma(x^\mu) \partial_r c(r) \right) = 0,$$

which is equivalent to

$$c(r) e^{-kr} f(r) \partial_\mu f^{\mu\rho} - \partial_r \left( e^{-kr} f^{-1}(r) a^\rho(x^\mu) \partial_r c(r) \right) = 0,$$

where we know that $\sqrt{g} g^{\mu\nu} g^{\rho\sigma} = b(r) \eta^{\mu\nu} \eta^{\rho\sigma} = e^{-kr} f(r) \eta^{\mu\nu} \eta^{\rho\sigma}$, $\sqrt{g} g^{rr} g^{\rho\sigma} = -a^2(r) b^{-1}(r) \eta^{\rho\sigma} = e^{-kr} f^{-1}(r) \eta^{\rho\sigma}$.

Massless vector fields satisfy Maxwell equation, $\partial_\mu f^{\mu\nu} = 0$, so from equation (17) and metric (2) we obtain dynamics equation of massless vector field,

$$\partial_r^2 c(r) - (k + f'f^{-1}) \partial_r c(r) = 0.$$

This equation is similar to the massless scalar field dynamics equation (8). The solution of equation (18) is then

$$c(r) = b_1 \int \exp \left\{ \int (k + f'f^{-1}) \, dr \right\} \, dr + c_1, \quad \partial_r c(r) = b_1 \exp \left\{ \int (k + f'f^{-1}) \, dr \right\},$$

where $b_1$ and $c_1$ are constants. The localization conditions in equation (14) and (15), respectively, as follows

$$N_1 = \int_0^\infty d^4r e^{-kr} f \left[ b_1^2 \left\{ \int \exp \left( \int (k + f'f^{-1}) \, dr \right) \, dr \right\}^2 + 2b_1c_1 \int \exp \left( \int (k + f'f^{-1}) \, dr \right) \, dr + c_1^2 \right],$$

$$m_1^2 = b_1^2 \int_0^\infty d^4r f(r) e^{kr}. $$

If we compare the localization conditions between scalar field in equations (10)-(11), and vector fields in equations (20)-(21), we can see the similarity between them. The only difference is the
constant $k$. So, we will provide the same analysis for the vector fields. Then, we can conclude that the massless vector field can be localized by using a constant solution, $c(r) = c_1$.

For a massive vector fields, it satisfies Proca equation, $\partial_\mu f^{\mu\nu} + m^2 a^{\nu} = 0$. According to this Proca equation and metric (2), the general dynamics equation (17) becomes

$$\partial_r c(r) - \left(k + f' f^{-1}\right) \partial_r c(r) + m^2 f^2 c(r) = 0.$$  \hfill (22)

Similar to the previous problem, that exact solution to this equation has not yet been found. Then, we try any functions $f(r)$ that can localize the massive vector field.

4. Some Choices of $f(r)$

After we obtained the general formulation of localization conditions, then we try to choose several forms of $f(r)$ and search for localization properties.

(i) $f(r) = e^{Ar}$

For a given $f(r) = e^{Ar}$, where $A$ is a constant, we obtain the localization conditions for the massless scalar field as follows,

$$N_0 = c_0^2 \int_0^\infty dr e^{-(3k-A)r} = \frac{c_0^2}{3k-A}, \quad M_0^2 = 0.$$  \hfill (23)

for positive $k$ and $A$. As we choose the constant $b_0 = 0$, so the mass $M_0^2$ is always zero for all $f(r)$. The localization condition is obtained as a constant $c_0 = \pm \sqrt{3k-A}$. Then for the massive scalar field, from the equation of motion (8), the solution is

$$\chi_0(r) = b_0 \frac{e^{\frac{1}{2}(r(3k+A))}}{\pi} \left( -\frac{3k}{A} \right) J_{\frac{3k}{A}+1} \left( \frac{2\sqrt{e^{Ar}m^2}}{A} \right) + c_0 e^{\frac{1}{2}(r(3k+A))} \Gamma \left( \frac{3k}{A} + 2 \right) J_{\frac{3k}{A}+1} \left( \frac{2\sqrt{e^{Ar}m^2}}{A} \right),$$

(24)

where $\Gamma(x)$ is the gamma function, and $J_n(z)$ is the Bessel function of the first kind. In this case, we can not calculate integrals of $N_0$ and $M_0^2$ analytically. The point is the integrand must be convergent to obtain massive scalar field localization.

For massless vector field, we obtain the localization conditions as the constants $b_1 = 0$ and $c_1 = \pm \sqrt{k-A}$. For the massive mode, the solution $c(r)$ is similar in equation (24) which the warp constant $3k$ is replaced by $k$.

(ii) $f(r) = e^{-Ar-Br^2}$

For a given $f(r) = e^{-Ar-Br^2}$, where $A$ and $B$ are positive constants, we obtain the localization conditions for the massless scalar field as follows,

$$N_0 = c_0^2 \int_0^\infty dr e^{-(3k+A)r-Br^2} = \frac{c_0^2}{2} \sqrt{\frac{\pi}{B}} \exp \left( \frac{(3k+A)^2}{4B} \right) \left[ 1 - \text{erf} \left( \frac{3k+A}{2\sqrt{B}} \right) \right],$$

(25)

for $3k + A > 0$. The erf($x$) is the error function which has the form $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. While the massless vector field, the localization conditions are

$$N_1 = \frac{c_0^2}{2} \sqrt{\frac{\pi}{B}} \exp \left( \frac{(k+A)^2}{4B} \right) \left[ 1 - \text{erf} \left( \frac{k+A}{2\sqrt{B}} \right) \right], \quad k + A > 0, \quad \text{and} \quad m_1^2 = 0. $$

(26)

(iii) $f(r) = e^{-A/r}$

For a given $f(r) = e^{-A/r}$, where $A$ is a positive constant, we obtain the localization conditions for the massless scalar field as follows,

$$N_0 = c_0^2 \int_0^\infty dr e^{-(A/r)-3kr} = 2c_0^2 \sqrt{\frac{A}{3k}} K_1(2\sqrt{3Ak}) ; \quad k > 0.$$  \hfill (27)
The $K_n(x)$ is a modified Bessel function of the second kind (or sometimes called the Basset function). For the massless vector field, the localization conditions are

$$N_1 = 2c_0^2 \sqrt{\frac{A}{k}} K_1(2\sqrt{Ak}); \ k > 0, \quad \text{and} \ m_1^2 = 0.$$  \hfill (28)

(iv) $f(r) = Ar$

For a given $f(r) = Ar$, we obtain the localization conditions for the massless scalar field as follows,

$$N_0 = c_0^2 A \int_0^\infty dr e^{-3kr} = \frac{A c_0^2}{9k^2}; \ k > 0.$$  \hfill (29)

So, we get $c_0 = \pm 3k/\sqrt{A}$. While for the massless vector field, we obtain the localization conditions as $c_1 = \pm k/\sqrt{A}$, and $m_1 = 0$.

(v) $f(r) = Ar + Br^2$

For a given $f(r) = Ar + Br^2$, we obtain the localization conditions for the massless scalar field as follows,

$$N_0 = c_0^2 \int_0^\infty dr (A|r| + Br^2) e^{-3kr} = c_0^2 \left( \frac{A}{9k^2} + \frac{2B}{27k^2} \right); \ k > 0.$$  \hfill (30)

So, we get $c_0 = \pm \left( \frac{A}{9k^2} + \frac{2B}{27k^2} \right)^{-1/2}$. And for massless vector field, the localization conditions require $c_1 = \pm \left( \frac{A}{27} + \frac{2B}{k^2} \right)^{-1/2}$.

(vi) $f(r) = \sinh Ar$

For a given $f(r) = \sinh Ar$, we obtain the localization conditions for the massless scalar field as follows,

$$N_0 = c_0^2 \int_0^\infty dr \sinh(Ar)e^{-3kr} = \frac{c_0^2 A}{9k^2 + A^2}; \ 3|k| > |A|,$$  \hfill (31)

So, we get $c_0 = \pm \frac{1}{3} \sqrt{3k^2 + A^2}$. For massless vector field, the localization conditions require $c_1 = \pm \frac{1}{3} \sqrt{k^2 + A^2}$.

We can see from several $f(r)$ forms, that the scalar field can be localized in case of massless scalar particle. Meanwhile, for the massive case, we still have a problem in determining field exact solutions, except of the first $f(r)$ case. The aim for the massive mode, that the localization conditions $N_0$ and $M_0^2$ will be finite if we can prove that the integrand is convergent. However, we can conclude that $f(r) = e^{Ar}$ can be used as a candidate to explain further related to fields localization on the brane.

5. Conclusion

In this paper, we searched for a modification of a conformally flat metric in the five-dimensional braneworld model that localizes scalar and vector fields. We give a general form of $f(r)$ to the extra coordinates $r$ metric component, then analyze whether the metric can provides localization of scalar and vector fields completely. Here, scalar fields are reviewed in two cases, they are massless mode and massive mode. From the explanation above, massless scalar fields can be localized to the brane only by taking the constant $b_0 = 0$. The constant solution $\chi_0 = c_0$ is chosen to localize the scalar field. If we select arbitrary constant $b_0 \neq 0$, it will give an inconsistent mass value, $M_0 \neq 0$. Then, the second one is a massive scalar field. The first step is choosing the field solution $\chi_0(r)$ which converges to the extra coordinate axis. Unfortunately, we
have not got an exact solution for the massive fields. Furthermore, we considered the massless and massive vector fields which satisfy the Maxwell’s and Proca’s equations, respectively. For massless vector fields, it is also reviewed only in a constant solution, $c(r) = c_1$. The nonconstant solution will always produce divergent integrals for arbitrary $f(r)$, and we obtained an infinite value for normalization and mass. Whereas for the massive vector field, we still encounter similar problem in determining the solutions.

In general, for the braneworld model with this modification, any kind of $f(r)$ factor will provide similar localization conditions for each scalar field (massless and massive modes) and vector field (massless and massive modes), except for $f(r) = e^{Ar}$ which the massive scalar solution can be determined. It can be used as a candidate to explain further about fields localization on the brane. Physically, the Randall-Sundrum model states that the Planck scale brane can be generated from the TeV scale brane with an increasing exponential factor. If we look at the form of $b(r)$ function for each of the given $f(r)$, we can see that the only exponential $f(r)$ models have the possibility that the Planck scale can be generated from the TeV scale. Other than exponential $f(r)$, they give a factor of $b(r)$ that returns to its original value, $b(r) = 0$, or even becomes divergent along the $r$ coordinate.

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