On Grosswald’s conjecture on primitive roots

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Abstract
Grosswald’s conjecture is that $g(p)$, the least primitive root modulo $p$, satisfies $g(p) \leq \sqrt{p} - 2$ for all $p > 409$. We make progress towards this conjecture by proving that $g(p) \leq \sqrt{p} - 2$ for all $409 < p < 2.5 \times 10^{15}$ and for all $p > 3.67 \times 10^{71}$.

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1 Introduction

Let $g(p)$ denote the least primitive root of a prime $p$. Burgess [4] showed that $g(p) \ll p^{1/4+\epsilon}$ for any $\epsilon > 0$. This remains the best known bound in general — see [12] for an insightful survey of related problems. Grosswald [7] conjectured that

$$g(p) < \sqrt{p} - 2,$$

for all primes $p > 409$. This has implications for the generators of $\Gamma(p)$, the principal congruence subgroup modulo $p$ of the modular group $\Gamma$ — see [7, §8]. Grosswald verified numerically that (1) is true for all $409 < p \leq 10000$. He also gave an explicit version of Burgess’ bound, thereby proving that $g(p) \leq p^{0.499}$ for all $p > 1 + \exp(\exp(24)) \approx 10^{10^{10}}$.

Using computational and theoretical arguments we improve on Grosswald’s estimate in the following theorem.

**Theorem 1.** Let $g(p)$ denote the least primitive root modulo $p$. Then $g(p) \leq \sqrt{p} - 2$ for all $409 < p < 2.5 \times 10^{15}$ and for all $p > 3.67 \times 10^{71}$.

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The ‘gap’ in Theorem 1 between the ranges of $p$ seems difficult to bridge. The trivial bound $g(p) \leq p$ when combined with the results in Theorem 1 gives the following corollary.

**Corollary 1.** $g(p) \leq 5.19 p^{0.99}$ for all $p$.

The bound in Corollary 1, while weak, appears to be the first bound that holds for all $p$. The remainder of the paper is organised as follows. In §2 we collect the necessary results to make Burgess’ result explicit. This gives a substantial improvement on the upper bound $\exp(\exp(24)) + 1$ given by Grosswald. We introduce a sieving inequality in §3 which enables us to reduce this further. Finally, in §4 we present some computational arguments which complete the proof of Theorem 1, and present some data on two related problems involving primitive roots.

## 2 Explicit versions of Burgess’ bounds

Burgess’s bounds on the character sum

\[ S_H(N) = \sum_{m=N+1}^{N+H} \chi(m) \]

were first made explicit by Grosswald [op. cit.], and were later refined by Booker [3], McGown [10], and, most recently, by Treviño [15]. The following is Theorem 1.7 in [15].

**Theorem 2.** [Treviño] Suppose $\chi$ is a non-principal Dirichlet character modulo $p$ where $p \geq 10^{20}$. Let $N, H \in \mathbb{Z}$ with $H \geq 1$. Fix a positive integer $r \geq 2$. Then there exists a computable constant $C(r)$ such that whenever $H \leq 2 p^{1/2+1/4r}$ we have

\[ |S_H(N)| \leq C(r) H^{1-1/r} p^{3+1} / (\log p)^{1/2}. \]

(2)

We follow Burgess, who, in [4, §6] considers

\[ f(x) = \frac{\phi(p-1)}{p-1} \left\{ 1 + \sum_{d \mid p-1, d > 1} \frac{\mu(d)}{\phi(d)} \sum_{x \equiv d \pmod{p}} \chi_d(x) \right\}, \]

(3)

whence it follows that $f(x) = 1$ if $x$ is a primitive root, and $f(x) = 0$ otherwise. Thus, if $N(H)$ denotes the number of primitive roots in the interval $N+1 \leq x \leq N+H$ we have

\[ N(H) = \sum_{x=N+1, x \equiv 0 \pmod{p}}^{N+H} f(x). \]

Hence, if $H < p$ there is at most one $x \in [N+1, N+H]$ with $x \equiv 0 \pmod{p}$ so that

\[ |N(H) - \sum_{x=N+1}^{N+H} f(x)| \leq 1. \]

We can estimate the sum of $f(x)$ using (2) with $H = \left(1-\frac{2}{p_0}\right) p^{1/2}$. This choice of $H$ guarantees that $H < \sqrt{p} - 2$ for $p > p_0$.

Since we need only consider square-free divisors $d$ in the outer sum in (3), and since there are $\phi(d)$ characters $\chi_d$, we arrive at the following theorem.
Theorem 3. We have $g(p) < \sqrt{p} - 2$ for $p > p_0$ provided that

$$p^{\frac{s+1}{2}} > C(r) \left( 1 - \frac{2}{\log p} \right)^{-1/r} \left( \log p \right)^{1/2r} \left( 2^{\omega(p-1)} - 1 \right).$$

(4)

The exponent on the left side of (4) is maximised when $r = 2$. We rearrange (4) accordingly to show that we require

$$\frac{p}{\log p} > C(2)^{16}(0.99)^{-8} \left( 2^{\omega(p-1)} - 1 \right)^{16}, \quad (p > 10^{20}).$$

(5)

With $C(2) = 3.5751$ as in [15, Table 3], we see that (5) is true whenever $\omega(p-1) \geq 17984$. Hence we need only consider $\omega(p-1) \leq 17983$. Solving for $p$ in (5) we find we need only consider $p < 10^{86650}$, which is much less than $10^{10^{19}}$. We reduce this upper bound substantially by introducing a sieving inequality in the next section.

3. A sieving inequality

Let $e$ be an even divisor of $p-1$. Let $\text{Rad}(n)$ denote the product of the distinct prime divisors of $n$. If $\text{Rad}(e) = \text{Rad}(p-1)$, then set $s = 0$ and $\delta = 1$. Otherwise, if $\text{Rad}(e) < \text{Rad}(p-1)$, let $p_1, \ldots, p_s$, $s \geq 1$, be the primes dividing $p-1$ but not $e$ and set $\delta = 1 - \sum_{i=1}^{s} p_i^{-1}$. In practice, it is essential to choose $e$ so that $\delta > 0$.

Again let $e$ be an even divisor of $p-1$. An integer $x$ (indivisible by $p$) will be called $e$-free if, for any divisor $d$ of $e$, (with $d > 1$), the congruence $x \equiv y^d \pmod{p}$ is insoluble. With this terminology, a primitive root is $(p-1)$-free. Given $N$ and $H$ let $N_e(H)$ be the number of integers $x$ in the range $N + 1 \leq x \leq N + H$ that are indivisible by $p$ and such that $x$ is $e$-free.

Lemma 1. Suppose $e$ is an even divisor of $p-1$. Then, in the above notation,

$$N_{p-1}(H) \geq \sum_{i=1}^{s} N_{p_{i}e}(H) - (s-1)N_{e}(H).$$

(6)

Hence

$$N_{p-1}(H) \geq \sum_{i=1}^{s} \left[ N_{p_{i}e}(H) - \theta(p_{i})N_{e}(H) \right] + \delta N_{e}(H).$$

(7)

Proof. For a given $e$-free integer $x$, the right side of (6) contributes 1 if $x$ is additionally $p_{i}$-free, and otherwise contributes a non-positive quantity. We deduce (7) by rearranging (6) bearing in mind the definitions for $\theta(p_{i})$ and $\delta$. \qed

Given the divisor $e$ of $p-1$, we extend the definition of $f(x)$ to $f_{e}(x)$, where

$$f_{e}(x) = \theta(e) \left\{ 1 + \sum_{d|e,d > 1} \frac{\mu(d)}{\phi(d)} \sum_{d|e} \chi_d(x) \right\},$$

and where $\theta(e) = \frac{\phi(e)}{e}$. Hence $f_{e}(x) = 1$ if $x$ is $e$-free, and $f_{e}(x) = 0$ otherwise. Thus,

$$N_{e}(H) = \sum_{x=N+1,x \not\equiv 0 \pmod{p}}^{N+H} f_{e}(x).$$
It follows from Theorem 2 that, under the constraints of that theorem,
\[
N_e(H) \geq \theta(e) \left( H - (W(e) - 1)C(r)H^{1-1/r}p^{\frac{r+1}{4r}}(\log p)^{\frac{1}{2r}} \right),
\]
where \( W(e) = 2^{\omega(e)} \) is the number of square-free divisors of \( e \).

Similarly, for any prime divisor \( l \) of \( p - 1 \) not dividing \( e \),
\[
\left| N_{le}(H) - \left( 1 - \frac{1}{l} \right) N_e(H) \right| \leq \theta(e)W(e)C(r)H^{1-1/r}p^{\frac{r+1}{4r}}(\log p)^{\frac{1}{2r}},
\]
where the factor \( W(e) \) arises from the expression \( W(le) - W(e) \).

Now apply (8) and (9) to (7) to obtain
\[
N_{p-1}(H) \geq \delta \theta(e)H - C(r)H^{1-1/r}p^{\frac{r+1}{4r}}(\log p)^{\frac{1}{2r}}W(e) \left( \delta + \sum_{i=1}^{s} \frac{1}{p_i} \right).
\]

Since \( \sum_{i=1}^{s} \frac{1}{p_i} = s - 1 + \delta \), this yields
\[
N_{p-1}(H) \geq \delta \theta(e) \left\{ H - W(e)C(r)H^{1-1/r}p^{\frac{r+1}{4r}}(\log p)^{\frac{1}{2r}} \left( \frac{s-1}{\delta} + 2 \right) \right\}.
\]

As in §2, we take \( H = \left( 1 - \frac{2}{p_0^2} \right)p^{\frac{1}{4}} \) and \( r = 2 \) in (10). This proves the following refinement of Theorem 3.

**Theorem 4.** Let \( e \) be an even divisor of \( p - 1 \) and \( s, \delta \) as in Lemma 1 with \( \delta > 0 \). We have \( g(p) < \sqrt{p} - 2 \) for \( p > p_0 \) provided that
\[
\theta(e) \left( 1 - \frac{2}{p_0^2} \right)p^{\frac{1}{4}} \left( 1 - C(2) \left( 1 - \frac{2}{p_0^2} \right)^{-\frac{1}{2}}(\log p)^{\frac{1}{2}} \left( \frac{s-1}{\delta} + 2 \right) W(e)p^{\frac{1}{4r}} \right) > 0.
\]

We can rearrange (11) to show that our criterion becomes
\[
\frac{p}{\log^4 p} > C(2)^{16} \left( 1 - \frac{2}{p_0^2} \right)^{-8} \left\{ \left( \frac{s-1}{\delta} + 2 \right) 2^{n-s} \right\}^{16}.
\]

We consider (12) for \( \omega(p - 1) = n \leq 17983 \). By making the choice of \( s \) for \( n \) given in Table 1 we verify (12) for all \( n \geq 42 \).

We are left with those \( p \) satisfying \( \omega(p - 1) \leq 41 \). When \( \omega(p - 1) = n = 41 \) we choose \( s = 37 \) to minimise the right-side of (12). This shows that Grosswald’s conjecture is satisfied provided that
\[
\frac{p}{\log^4 p} > 4.97 \times 10^{62}.
\]

Solving (13) for \( p \) gives \( p > 3.67 \times 10^{71} \). It is tempting to try to remove the \( \omega(p - 1) = 41 \) case by enumerating possible primes as in [5]. Since \( p - 1 > p_1 \cdots p_{41} \) we seek the number of solutions of
\[
2.98 \times 10^{70} \leq p \leq 3.67 \times 10^{71}, \quad p \text{ prime}, \quad \omega(p - 1) = 41.
\]

A quick computer check shows that there are 329 different primes that could appear in the factorisation of \( p - 1 \). While it may be possible to enumerate all such products satisfying (14), this would, at best, eliminate the \( n = 41 \) case only. We have not pursued such an enumeration.
Table 1: Choices of $s$ for various ranges of $\omega(p - 1) = n$ such that (12) holds.

| Range of $\omega(p - 1) = n$ | $s$ |
|-----------------------------|-----|
| [800, 17983]                | 750 |
| [400, 799]                  | 300 |
| [200, 399]                  | 180 |
| [105, 199]                  | 105 |
| [72, 104]                   | 68  |
| [55, 71]                    | 52  |
| [47, 54]                    | 44  |
| [43, 46]                    | 40  |
| 42                          | 38  |

4 Computational results

The computational part of Theorem 1 was proved in the following way. The interval $[2, 10^{15}]$ was subdivided into consecutive sub-intervals of manageable size (each with $2^{20}$ integers). An efficient segmented Eratosthenes sieve (see [2] and [13, §1.1]) was then used to identify all primes in each interval. For each prime $p$ that was found, a second Eratosthenes sieve, modified to yield complete factorizations [6, §3.2.4], was used to find the factorization of $(p - 1)/2$. Since the least primitive root modulo $p$ cannot be of the form $a^b$ with $a > 0$ and $b > 1$, i.e., it cannot be a perfect power, the integers $2, 3, 5, 6, 7, 10, \ldots$, were tried one at a time until a primitive root was found.

With $c$ as a candidate primitive root, the first test was to check if $c^{(p-1)/2} \equiv -1 \pmod{p}$. This was efficiently done using the quadratic reciprocity law data from known tables. If this test failed the next $c$ candidate was tried. Otherwise, for each odd prime factor $q$ of $(p - 1)/2$ it was checked whether $c^{(p-1)/q} \neq 1 \pmod{p}$. The next $c$ candidate was tried if one of these tests failed. These tests were efficiently done by performing all modular arithmetic using the Montgomery method [11]. Since the “probability” of failure of an individual test is $1/q$, the odd factors $q$ were sorted in increasing order before performing these tests. Note that $g(p)$ is equal to the first $c$ that passes all tests.

Instead of checking (1) directly for each prime up to $2.5 \times 10^{15}$, the record-holder values of $g(p)$, i.e., values of $g(p)$ such that $g(p') < g(p)$ for all $p' < p$, were computed, as these are of independent interest [1] and can be used to check (1) indirectly. The computation required a total time of about 3 one-core years, and took about one month to finish on nine computers (each with 4 cores) of one computer lab of the Electronics, Telecommunications, and Informatics Department of the University of Aveiro. Table 2 presents all $g(p)$ record-holders that were found up to $2.5 \times 10^{15}$. It extends and corrects one entry of Table 2 of [14], which is a summary of computations up to $4 \times 10^{10}$.

The largest $g(p)$ record-holder in Table 2 that does not satisfy (1) is 21, corresponding to $p = 409$. Thus, up to $2.5 \times 10^{15}$, the largest $p$ for which (1) is possibly false satisfies $\sqrt{p} - 2 \leq 21$, i.e., $p < 529$. It turns out, as already verified by Grosswald, that the last failure of (1) occurs for $p = 409$.

An analysis similar to the one described above was also performed for least prime primitive roots $\hat{g}(p)$, and for least negative primitive roots $h(p)$. The least negative primitive root modulo $p$ is equal to the negative integer, least in absolute value, that is a primitive root.
Table 2: $g(p)$ record-holders with $p < 2.5 \times 10^{15}$.

| $g(p)$ | $p$ | $g(p)$ | $p$ | $g(p)$ | $p$ |
|--------|-----|--------|-----|--------|-----|
| 2      | 3   | 69     | 110881 | 179   | 6064561441 |
| 3      | 7   | 73     | 760321 | 194   | 7111268641  |
| 5      | 23  | 94     | 5109721| 197   | 9470788801  |
| 6      | 41  | 97     | 17551561| 227   | 28725635761 |
| 7      | 71  | 101    | 29418841| 229   | 108709927561|
| 19     | 191 | 107    | 33358081| 263   | 386681163961|
| 21     | 409 | 111    | 45024841| 281   | 1990614824641|
| 23     | 2161| 113    | 90441961| 293   | 44384069747161|
| 31     | 5881| 127    | 184254841| 335   | 89637484042681|
| 37     | 36721| 137   | 324013369| 347   | 358973066123281|
| 38     | 55441| 151   | 831143041| 359   | 2069304073407481|
| 44     | 71761| 164   | 1685283601|

modulo $p$. It cannot be of the form $-a^b$ with $a > 0$ and $b > 2$, and is equal to $-g(p)$ if $p \equiv 1 \pmod{4}$. It was found that $\hat{g}(p) < \sqrt{p} - 2$ for $2791 < p < 2.5 \times 10^{15}$, and that $-h(p) < \sqrt{p} - 2$ for $409 < p < 10^{15}$. We remark that little is known about either $\hat{g}(p)$ or $h(p)$ — the reader may consult [9] for more details.

5 Conclusion

It appears difficult to resolve completely Grosswald’s conjecture. Table 3 in [15] indicates that one may hope to reduce the size of $C(2)$ further by taking a larger value of $p^0$. However, this appears at present not to give an improvement for our purposes.

An alternative approach is to use a smoothed version of Burgess’ bounds, in the same way that a smoothed Pólya–Vinogradov inequality was used in [8].

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References

[1] E. Bach. Comments on search procedures for primitive roots. *Math. Comp.*, 66(220):1719–1727, 1997.

[2] C. Bays and R. H. Hudson. The segmented sieve of Eratosthenes and primes in arithmetic progressions to $10^{12}$. *Nordisk Tidskr. Informationsbehandling (BIT)*, 17(2):121–127, June 1977.
[3] A. R. Booker. Quadratic class numbers and character sums. *Math. Comp.*, 75(255):1481–1492, 2006.

[4] D. A. Burgess. On character sums and primitive roots. *Proc. London Math. Soc.*, 12(3):179–192, 1962.

[5] S. D. Cohen, T. Oliveira e Silva, and T. S. Trudgian. A proof of the conjecture of Cohen and Mullen on sums of primitive roots. *Math. Comp.* To appear.

[6] R. Crandall and C. Pomerance. *Prime Numbers: A Computational Perspective*. Springer, New York, 2002 (second edition).

[7] E. Grosswald. On Burgess’ bound for primitive roots modulo primes and an application to $\Gamma(p)$. *Amer. J. Math.*, 103(6):1171–1183, 1981.

[8] M. Levin, C. Pomerance, and K. Soundararajan. Fixed points for discrete logarithms. *Lecture Notes in Comput. Sci.*, 6197:6–15, 2010.

[9] G. Martin. The least prime primitive root and the shifted sieve. *Acta Arith.*, 80(3):277–288, 1997.

[10] K. J. McGown. Norm-Euclidean cyclic fields of prime degree. *Int. J. Number Theory*, 8(1):227–254, 2012.

[11] P. L. Montgomery. Modular multiplication without trial division. *Math. Comp.*, 44(170):519–521, 1985.

[12] P. Moree. Artin’s primitive root conjecture — a survey. *Integers*, 12(6):1305–1416, 2012.

[13] T. Oliveira e Silva, S. Herzog, and S. Pardi. Empirical verification of the even Goldbach conjecture and computation of prime gaps up to $4 \cdot 10^{18}$. *Math. Comp.*, 83(288):2033–2060, 2014. Published electronically on November 18, 2013.

[14] A. Paszkiewicz and A. Schinzel. Numerical calculation of the density of prime numbers with a given least primitive root. *Math. Comp.*, 71(240):1781–1797, 2002.

[15] E. Treviño. The Burgess inequality and the least $k$-th power non-residue. *Int. J. Number Theory*, to appear, 2015.