Morita equivalence of Fedosov star products and deformed Hermitian vector bundles

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Abstract

Based on the usual Fedosov construction of star products for a symplectic manifold $M$ we give a simple geometric construction of a bimodule deformation for the sections of a vector bundle over $M$ starting with a symplectic connection on $M$ and a connection for $E$. In the case of a line bundle this gives a Morita equivalence bimodule where the relation between the characteristic classes of the Morita equivalent star products can be found very easily in this framework. Moreover, we also discuss the case of a Hermitian vector bundle and give a Fedosov construction of the deformation of the Hermitian fiber metric.
1 Introduction

Deformation quantization as introduced in [1] has proved to be an extremely successful framework for the problem of quantization: the existence of the associative deformation of the classical observable algebra, the ‘star product’, is well established for the case of a symplectic phase space [10,12,21] as well as for the more general Poisson case [18]. Moreover, star products have been classified up to equivalence [2, 18, 19] in terms of geometrical data on the phase space. For several physical applications one also needs to represent the deformed observable algebra on a pre-Hilbert space. This led to the development of a representation theory for star products starting with [5]. Recent reviews as well as further references may be found in [11, 13, 23, 24].

Having a reasonable notion for a representation theory, a natural question is whether two star product algebras have the ‘same’ representation theory. This question was made precise in [7] using a notion of Morita equivalence very similar to and in fact generalizing Rieffel’s notion of strong Morita equivalence for $C^*$-algebras [22]. The classification of star products up to Morita equivalence was achieved in [8] for the symplectic case. In the particular case of cotangent bundles it leads to a physical interpretation of Morita equivalence as Dirac’s quantization condition for magnetic charges. Beside this more ‘conservative’ occurrence of Morita equivalence in deformation quantization, Morita equivalence of star products also appears in non-commutative gauge theories, see [14].

The purpose of this paper is to give an alternative and more geometric construction of the deformation of vector bundles $E \to M$ as introduced in [3] which are the basic ingredients for Morita equivalence of star products. Here we shall use a ‘Fedosov-like’ construction of the bimodule structure and give thereby a simple description of Morita equivalence for Fedosov star products.

The paper is organized as follows: first we recall the basic structures needed for Fedosov’s approach to star products in Section 2 and 3, where we also introduce the fiberwise bimodule structure. In the next two sections we show how the usual Fedosov derivatives, which lead to deformations of $C^\infty(M)$ and $\Gamma^\infty(\text{End}(E))$, can be used to obtain also a Fedosov derivative for the vector bundle itself. This will allow to define the deformed bimodule structure as well as an easy identification of the characteristic classes of the involved star products. In Section 6 we demonstrate how a Hermitian fiber metric can be deformed in this framework and Section 7 contains a conclusion with some further questions arising in this context.

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2 Preliminaries on the Fedosov construction

The aim of this section is to recall the basics of Fedosov’s construction and to set up our notation, where we mainly follow [3]. In the following $(M, \omega)$ is a symplectic manifold, $\nabla$ a symplectic torsion-free connection, $E \to M$ a complex vector bundle, and $\nabla^E$ a connection for $E$. By $\text{End}(E) \to M$ we denote the endomorphism bundle of $E$ and $\nabla^\text{End}(E)$ is the induced connection for $\text{End}(E)$ coming from $\nabla^E$. The starting point for the Fedosov construction are the following $\mathbb{C}[[\lambda]]$-modules:

$$\mathcal{W} := \prod_{s=0}^{\infty} \Gamma^\infty(V^sT^*M)[[\lambda]]$$ (2.1)
\[
\mathcal{W} \otimes \Lambda^\bullet := \prod_{s=0}^{\infty} \Gamma^\infty(\sqrt{s}T^* M \otimes \Lambda^s T^* M)[[\lambda]]
\] \hspace{1cm} (2.2)

\[
\mathcal{W} \otimes \Lambda^\bullet \otimes \mathcal{E} := \prod_{s=0}^{\infty} \Gamma^\infty(\sqrt{s}T^* M \otimes \Lambda^s T^* M \otimes \mathcal{E})[[\lambda]]
\] \hspace{1cm} (2.3)

\[
\mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}) := \prod_{s=0}^{\infty} \Gamma^\infty(\sqrt{s}T^* M \otimes \Lambda^s T^* M \otimes \text{End}(\mathcal{E}))[\lambda]
\] \hspace{1cm} (2.4)

Clearly, \(\mathcal{W} \otimes \Lambda^\bullet\) is a super-commutative associative algebra for the (anti-)symmetric tensor product and \(\mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E})\) is associative but non-commutative unless \(E\) is a line bundle. Moreover, \(\mathcal{W} \otimes \Lambda^\bullet \otimes \mathcal{E}\) is a \(\mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E})\) left module and a \(\mathcal{W} \otimes \Lambda^\bullet\) right module in the obvious way and both module actions commute. Thus it is a bimodule. The degree derivations \(\text{deg}_a\), \(\text{deg}_s\), \(\text{deg}_\lambda\) and the total degree \(\text{Deg} = \text{deg}_\lambda + 2\text{deg}_s\) are defined in the usual way and yield (module-) derivations for the (yet still undeformed) fiberwise products. Finally we have the operators \(\delta := (1 \otimes dx^i)i_s(\partial_i)\) and \(\delta^* := (dx^i \otimes 1)i_s(\partial_i)\), acting on \(\mathcal{W} \otimes \Lambda^\bullet\), \(\mathcal{W} \otimes \Lambda^\bullet \otimes \mathcal{E}\) and \(\mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E})\), where we write \((1 \otimes dx^i)\) to emphasize that the one-form \(dx^i\) is considered to be ‘anti-symmetric’ while \((dx^i \otimes 1)\) refers to a ‘symmetric’ one-form. The maps \(i_s(\partial_i)\) and \(i_s(\partial_i)\) are the symmetric and anti-symmetric insertion maps, respectively. One has \(\delta^2 = 0 = (\delta^*)^2\) and \(\delta\delta^* + \delta^*\delta = \text{deg}_a + \text{deg}_s\). Defining the symbol map \(\sigma\) as projection onto the part of symmetric and antisymmetric degree 0 one has

\[
\delta^{-1}\delta^{-1} + \delta^{-1}\delta + \sigma = \text{id},
\] \hspace{1cm} (2.5)

where \(\delta^{-1}a = \frac{1}{k+l}\delta^*a\) for \(\text{deg}_s a = ka\), \(\text{deg}_s a = la\) with \(k + l \neq 0\) and \(\delta^{-1}a = 0\) else.

We shall need a slight generalization of \(\mathcal{W}\) in the following. Let \(\hat{\mathcal{W}}\) denote the space of formal Laurent series in the total degree \(\text{Deg}\) such that \(a \in \hat{\mathcal{W}}\) is of the form \(a = \sum_{r=N}^{\infty} a^{(r)}\) with \(N \in \mathbb{Z}\) and each \(a^{(r)}\) may contain arbitrarily high negative powers of \(\lambda\) as long as they are compensated by the symmetric degree such that still \(\text{Deg} a^{(r)} = ra^{(r)}\). It is clear that all the algebraic operations are still defined on \(\mathcal{W}, \hat{\mathcal{W}} \otimes \Lambda^\bullet, \hat{\mathcal{W}} \otimes \Lambda^\bullet \otimes \mathcal{E},\) and \(\hat{\mathcal{W}} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E})\), respectively, and obey the same algebraic identities. Note that now \(\sigma\) maps \(\hat{\mathcal{W}} \otimes \Lambda^\bullet\) onto the space \(C^\infty(M)((\lambda))\) of formal Laurent series in \(\lambda\).

The connections \(\nabla\) and \(\nabla^E\) extend to super-derivations of anti-symmetric degree +1

\[
D : \mathcal{W} \otimes \Lambda^\bullet \rightarrow \mathcal{W} \otimes \Lambda^{\bullet + 1}
\] \hspace{1cm} (2.6)

\[
D^E : \mathcal{W} \otimes \Lambda^\bullet \otimes \mathcal{E} \rightarrow \mathcal{W} \otimes \Lambda^{\bullet + 1} \otimes \mathcal{E}
\] \hspace{1cm} (2.7)

\[
D' : \mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}) \rightarrow \mathcal{W} \otimes \Lambda^{\bullet + 1} \otimes \text{End}(\mathcal{E})
\] \hspace{1cm} (2.8)

in the usual way. In particular we have the compatibility conditions

\[
D^E(a\Psi) = (D'a)\Psi + (-1)^{\text{deg}_sa}a(D^E\Psi) \quad \text{and} \quad D^E(\Psi b) = (D^E\Psi)b + (-1)^{\text{deg}_s\Psi}\Psi(Db)
\] \hspace{1cm} (2.9)

for \(a \in \mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}), \Psi \in \mathcal{W} \otimes \Lambda^\bullet \otimes \mathcal{E},\) and \(b \in \mathcal{W} \otimes \Lambda^\bullet\). A straightforward computation shows that

\[
[\delta, D] = 0, \quad [\delta, D^E] = 0, \quad \text{and} \quad [\delta, D'] = 0.
\] \hspace{1cm} (2.10)
The symplectic form $\omega$ can be viewed as element in $W \otimes \Lambda^2$ but we can also view it as $\tilde{\omega} \in W \otimes \Lambda^1$ of symmetric degree 1 and antisymmetric degree 1. Then we obtain

$$\delta \tilde{\omega} = 2\omega, \quad \delta^{-1}\tilde{\omega} = 0, \quad \text{and} \quad D\tilde{\omega} = 0,$$

(2.11)

since $\nabla$ is a symplectic connection. Next we compute the squares of the covariant derivatives where we obtain curvature contributions. We define $R \in W \otimes \Lambda^2$ of symmetric degree 2 by

$$R = \frac{1}{4} \omega_{lm} R^m_{kil} dx^l \otimes dx^i \wedge dx^j$$

(2.12)

and $R^E \in W \otimes \Lambda^2 \otimes \mathrm{End}(E)$ of symmetric degree 0 by

$$R^E = \frac{1}{2} dx^i \wedge dx^j \otimes R^E_{ij},$$

(2.13)

where $R^m_{kil}$ are the components of the curvature of $\nabla$ and $R^E_{ij} = R^E(\partial_i, \partial_j)$ are the curvature endomorphisms of $\nabla^E$. Clearly the above local formulas define global objects. A straightforward computation gives the following lemma:

**Lemma 1** One has the following relations:

$$D^2 = -\frac{1}{2} R^m_{kil} dx^k \otimes dx^i \wedge dx^j i_s(\partial_m)$$

(3.1)

$$(D^E)^2 = -\frac{1}{2} R^m_{kil} dx^k \otimes dx^i \wedge dx^j i_s(\partial_m) + R^E$$

(2.14)

$$(D')^2 = -\frac{1}{2} R^m_{kil} dx^k \otimes dx^i \wedge dx^j i_s(\partial_m) + [R^E, \cdot]$$

(2.15)

The curvature elements fulfill the Bianchi identities

$$\delta R = 0, \quad \delta R^E = 0 \quad \text{and} \quad DR = 0, \quad D'R^E = 0.$$  

(2.16)

In the following we shall sometimes consider $W \otimes \Lambda^k$ as sub-algebra of $W \otimes \Lambda^k \otimes \mathrm{End}(E)$. Hence $R \in W \otimes \Lambda^k \otimes \mathrm{End}(E)$ satisfies $D'R = 0$, too. Moreover, for an antisymmetric $k$-form $\Omega \in \Lambda^k \subseteq W \otimes \Lambda^k$ we simply have

$$D\Omega = D'\Omega = d\Omega.$$  

(2.17)

3 The fiberwise deformations

For $f, g \in \hat{W}$ one defines the fiberwise Weyl product by

$$f \circ g = \mu \circ e^{\frac{i}{2} \Lambda^{kl} i_s(\partial_k) \otimes i_s(\partial_l)} f \otimes g,$$

(3.1)

where $\mu$ is the undeformed fiberwise product and $\Lambda^{kl} = -\omega^{kl}$ are the components of the Poisson tensor. We extend this product to $\hat{W} \otimes \Lambda^k$ and $\hat{W} \otimes \Lambda^k \otimes \mathrm{End}(E)$ in the obvious way. Finally we define

$$(f \otimes \alpha \otimes A) \circ (g \otimes \beta \otimes s) := f \circ g \otimes \alpha \wedge \beta \otimes As$$

(2.18)

$$\quad (g \otimes \beta \otimes s) \circ (h \otimes \gamma) := g \circ h \otimes \beta \wedge \gamma \otimes s$$

(3.2)

(3.3)

for $f, g, h \in \hat{W}, \alpha, \beta \in \Lambda^k, A \in \Gamma^\infty(\mathrm{End}(E))$, and $s \in \Gamma^\infty(E)$ and extend this by linearity to a left and right action of $\hat{W} \otimes \Lambda^k \otimes \mathrm{End}(E)$ and $\hat{W} \otimes \Lambda^k$ on $\hat{W} \otimes \Lambda^k \otimes \Lambda^k$, respectively. Then the following is obvious:
Lemma 2  The fiberwise Weyl product is globally well-defined and makes \( \hat{\mathcal{W}}, \mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}) \) into associative algebras with subalgebras \( \mathcal{W}, \mathcal{W} \otimes \Lambda^\bullet, \) and \( \mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}) \), respectively. Moreover, \( \hat{\mathcal{W}} \otimes \Lambda^\bullet \otimes \mathcal{E} \) becomes a bimodule for \( \hat{\mathcal{W}} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}) \) and \( \mathcal{W} \otimes \Lambda^\bullet \) and \( \mathcal{W} \otimes \Lambda^\bullet \otimes \mathcal{E} \) is a bimodule for \( \mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}) \) and \( \mathcal{W} \otimes \Lambda^\bullet \).

Moreover, \( \circ \) is still (formally) \( \text{deg}_a \) and \( \text{Deg} \)-graded, i.e. \( \text{deg}_a \) and \( \text{Deg} \) are still derivations. The (super-) center with respect to \( \circ \) is now given by the anti-symmetric forms \( \Lambda^\bullet \). Moreover, a simple computation gives that

\[
-\delta = \frac{i}{\lambda} \text{ad}(\bar{\omega})
\]

is an inner \( \circ \)-derivation. As we have chosen \( \nabla \) to be symplectic it turns out that \( D \) as well as \( D' \) are still derivations of \( \circ \). Moreover, \( D^E \) is a module derivation in the sense that

\[
D^E(a \circ \Psi) = (D'a) \circ \Psi + (-1)^{\text{deg}_a a} a \circ (D^E \Psi)
\]

\[
D^E(\Psi \circ b) = (D^E \Psi) \circ b + (-1)^{\text{deg}_a \Psi} \Psi \circ (Db)
\]

for all \( a \in \hat{\mathcal{W}} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}), \Psi \in \hat{\mathcal{W}} \otimes \Lambda^\bullet \otimes \mathcal{E}, \) and \( b \in \hat{\mathcal{W}} \otimes \Lambda^\bullet \) which is the quantum analog of (2.4).

The squares of the covariant derivatives turn out to be inner derivations with respect to \( \circ \):

Lemma 3  With \( R \) and \( R^E \) as in (2.13) and (2.13) we have

\[
D^2 = \frac{i}{\lambda} \text{ad}(R) \quad \text{and} \quad (D')^2 = \frac{i}{\lambda} \text{ad}(R - iR^E)
\]

\[
(D^E)^2 = \frac{i}{\lambda} \text{ad}(R) + R^E.
\]

Here we note that \( R \) can act from the left as well as from the right on \( \hat{\mathcal{W}} \otimes \Lambda^\bullet \otimes \mathcal{E} \) whence \( \text{ad}(R) \) is meaningful, while \( R^E \) simply acts by \( \circ \)-left multiplication. The proof for (3.7) can be found in Fedosov’s book [12, Sect. 5.3] and (3.8) is an easy computation using Lemma 1. Note that in (3.7) the curvature \( R^E \) appears with an additional power of \( \lambda \). This will play a major role later.

4  The Fedosov derivatives

The main idea of Fedosov’s construction is to realize \( C^\infty(M)[[\lambda]] \) and \( \Gamma^\infty(\text{End}(\mathcal{E}))[\lambda] \) as kernels of derivatives \( \mathcal{D} \) and \( \mathcal{D}' \) of the fiberwise Weyl algebras \( \mathcal{W} \otimes \Lambda^\bullet \) and \( \mathcal{W} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}) \), respectively. The Ansatz for \( \mathcal{D}' \) (and analogously for \( \mathcal{D} \)) is

\[
\mathcal{D}' = -\delta + D' + \frac{i}{\lambda} \text{ad}(r'),
\]

where \( r' \in \hat{\mathcal{W}} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}) \) is an element with total degree \( \geq 3 \) and anti-symmetric degree 1.

Note that \( \frac{i}{\lambda} \text{ad}(r') \) raises the \( \text{Deg} \)-degree by at least +1 but it may lower the \( \lambda \)-degree even if \( r' \) does not contain negative powers of \( \lambda \) as in \( \hat{\mathcal{W}} \otimes \Lambda^\bullet \otimes \text{End}(\mathcal{E}) \) the undeformed product is not super-commutative. For the square of \( \mathcal{D}' \) as in (4.1) one has

\[
(\mathcal{D}')^2 = \frac{i}{\lambda} \text{ad} \left( -\omega - \delta r' + R - i\lambda R^E + D'r' + \frac{i}{\lambda} r' \circ r' \right),
\]

where \( \omega = R - \delta r' \).
as a straightforward computation shows. Moreover, the ‘curvature’ of $\mathcal{D}'$ satisfies the Bianchi identity

$$
\mathcal{D}' \left( -\omega + \delta r' + R - i\lambda R^E + D'r' + \frac{i}{\lambda} r' \circ r' \right) = 0.
$$

(4.3)

Now if we want a ‘flat connection’ $(\mathcal{D}')^2 = 0$ then the curvature has to be a central element, i.e. a formal power series of two-forms $\Omega = \sum_{i=1}^{\infty} \lambda^i \Omega_i$. Then (4.3) implies that necessarily $d\Omega = 0$. The following theorem of Fedosov ensures that we can find such a $r'$ for any given choice of $\Omega$.

**Theorem 1 (Fedosov [12, Sect. 5.3])** Let $\Omega = \sum_{i=1}^{\infty} \lambda^i \Omega_i$ be a closed two-form. Then there exists a unique $r' \in \hat{\mathcal{W}} \otimes \mathcal{A} \otimes \text{End}(\mathcal{E})$ with anti-symmetric degree +1 and total degree $\geq 3$ such that

$$
\delta r' = R - i\lambda R^E + D'r' + \frac{i}{\lambda} r' \circ r' + \Omega \quad \text{and} \quad \delta^{-1} r' = 0.
$$

(4.4)

In this case $(\mathcal{D}')^2 = 0$.

**Remark 1** A priori, the recursion for finding $r'$ works only in $\hat{\mathcal{W}}$ and not in $\mathcal{W}$ as the undeformed product of $\mathcal{W} \otimes \mathcal{A} \otimes \text{End}(\mathcal{E})$ is already non-commutative whence $\frac{1}{\lambda} r' \circ r'$ may generate negative powers of $\lambda$. However, the crucial element causing the non-commutativity is $-i\lambda R^E$ which comes with an additional power of $\lambda$. Then proof that $r'$ does not contain negative powers of $\lambda$ can be done by induction using the following recursion formulas, see e.g. [4].

**Lemma 4** The element $r'$ can be obtained recursively with respect to the total degree $\text{Deg}$ by

$$
(r')^{(3)} = \delta^{-1} \left( R - i\lambda R^E + \lambda \Omega_1 \right)
$$

$$
(r')^{(k+3)} = \delta^{-1} \left( D'(r')^{(k+2)} + \frac{i}{\lambda} \sum_{l=1}^{k-1} (r')^{(l+2)} \circ (r')^{(k+2-l)} + \left\{ \begin{array}{ll} \lambda^{k/2+1} \Omega_{k/2+1} & \text{k even} \\ 0 & \text{k odd} \end{array} \right\} \right),
$$

(4.5)

(4.6)

where $r' = \sum_{k=3}^{\infty} (r')^{(k)}$. Moreover, $r' \in \mathcal{W} \otimes \mathcal{A} \otimes \text{End}(\mathcal{E})$.

In a second step one computes the kernel of $\mathcal{D}'$. It turns out that the kernel is in bijection to the sections $\Gamma^\infty(\text{End}(\mathcal{E}))[\lambda]$.

**Theorem 2 (Fedosov [12, Sect. 5.3])** The map $\sigma : \ker \mathcal{D}' \cap \ker \text{deg}_a \to \Gamma^\infty(\text{End}(\mathcal{E}))[\lambda]$ is a $\mathbb{C}((\lambda))$-linear and $\lambda$-adically continuous bijection.

The inverse is denoted by $\tau'$ and referred to as the Fedosov-Taylor series as it is the quantum analog of the formal Taylor series. For the Fedosov-Taylor series one has the following recursion formula, see also [4].

**Lemma 5** The Fedosov-Taylor series of a section $A \in \Gamma^\infty(\text{End}(\mathcal{E}))$ can be obtained recursively with respect to the total degree $\text{Deg}$ by

$$
\tau'(A)^{(0)} = A
$$

$$
\tau'(A)^{(k+1)} = \delta^{-1} \left( D' \tau'(A)^{(k)} + \frac{i}{\lambda} \sum_{l=1}^{k-1} \text{ad} \left( (r')^{(l+2)} \right) \tau'(A)^{(k-l)} \right),
$$

(4.7)

(4.8)

where $\tau'(A) = \sum_{k=0}^{\infty} \tau'(A)^{(k)}$. Moreover, $\tau(A) \in \mathcal{W} \otimes \mathcal{A} \otimes \text{End}(\mathcal{E})$. 

6
From the two lemmas we observe that it is sufficient to stay within the framework of formal power series in \( \lambda \); we do not need the extension to \( \hat{W} \otimes \Lambda^* \otimes \text{End}(E) \) \( a \text{ posteriori} \). Note however, that in the original recursion it is not so obvious that we do not produce negative \( \lambda \)-powers.

Since the kernel of a super-derivation is a sub-algebra one can pull-back the fiberwise Weyl product \( \circ \) of \( W \otimes \Lambda^* \otimes \text{End}(E) \) to \( \Gamma^\infty(\text{End}(E))[[\lambda]] \) by means of \( \sigma \) and \( \tau' \). Hence we obtain an associative deformation

\[
A \ast' B = \sigma(\tau'(A) \circ \tau'(B))
\]

for \( A, B \in \Gamma^\infty(\text{End}(E))[[\lambda]] \).

Of course the same line of argument can be applied to \( W \otimes \Lambda^* \) itself without endomorphism-valued elements. In this case we obtain a unique \( r \in W \otimes \Lambda^1 \) of total degree \( \geq 3 \) such that

\[
\delta r = R + D r + \frac{i}{\lambda} r \circ r + \Omega \quad \text{and} \quad \delta^{-1} r = 0
\]

with corresponding Fedosov derivative \( D = -\delta + D + \frac{i}{\lambda} \text{ad}(r) \) and \( D^2 = 0 \). Note that we have used the same closed two-form \( \Omega \) as in (4.4) to specify \( r \) and thus \( D \). Again one has a Fedosov-Taylor series

\[
\tau : C^\infty(M)[[\lambda]] \to \ker D \cap \ker \text{deg}_a \quad \text{with} \quad \sigma(\tau(f)) = f
\]

and a corresponding associative deformation

\[
f \ast g = \sigma(\tau(f) \circ \tau(g)).
\]

It turns out that this is actually a star product. Up to now we have just recalled Fedosov’s original construction of the star product \( \ast \) and the associative deformation \( \ast' \) of \( \Gamma^\infty(\text{End}(E)) \).

For later use we shall consider the two elements \( r \) and \( r' \) more closely:

**Lemma 6** The classical limits of \( r \) and \( r' \) coincide whence

\[
r^E := \frac{i}{\lambda}(r' - r) \in W \otimes \Lambda^* \otimes \text{End}(E)
\]

does not contain negative powers of \( \lambda \). Moreover, \( r^E \) is uniquely determined by

\[
\delta r^E = R^E + D' r^E + \frac{i}{\lambda} \text{ad}(r)r^E + r^E \circ r^E \quad \text{and} \quad \delta^{-1} r^E = 0.
\]

In particular, \( r^E = 0 \) if and only if \( \nabla r^E \) is flat, i.e. \( R^E = 0 \).

**Proof:** First note that \( D' r = D r \). Comparing the recursion formulas for \( r \) and \( r' \) we see that their difference has to be of order \( \lambda \). Moreover, (4.14) follows directly from (4.13) and (4.10) since we have used the same \( \Omega \). Finally, any element \( a \) of anti-symmetric degree +1 is uniquely determined by specifying \( \delta a \) and \( \delta^{-1} a \) according to (2.4). Clearly \( r^E = 0 \) implies \( R^E = 0 \) as this is the component of \( r^E \) of total degree 1. Conversely, if \( R^E = 0 \) then \( r^E = 0 \) follows as in this case it is the unique solution of (4.14). \( \square \)
5 The bimodule structure and Morita equivalence

From the construction of $\star$ and $\star'$ it is easy to guess how one can deform the classical bimodule structure of $\Gamma^\infty(E)$. We just have to find a suitable Fedosov derivative $D^E$ with Fedosov-Taylor series $\tau^E$ for $W\otimes A^\bullet\otimes \mathcal{E}$, too. This can indeed be done, we even have already all pieces present and do not have to start a new recursion. We define $D^E : W\otimes A^\bullet\otimes \mathcal{E} \to W\otimes A^\bullet+1\otimes \mathcal{E}$ by

$$D^E = -\delta + D^E + \frac{i}{\lambda} \text{ad}(r) + r^E,$$

(5.1)

where $r$ and $r^E$ are given as before. Note that $\text{ad}(r)$ is well-defined as $r$ can act form left and right. The element $r^E$ is understood to act by $\circ$-left multiplication. Finally note, that $\text{ad}(r)\Psi$ is always of order $\lambda$, whence $D^E$ does not produce negative powers of $\lambda$.

**Theorem 3** The Fedosov derivative $D^E$ satisfies

$$D^E(a \circ \Psi) = (D^E a) \circ \Psi + (-1)^{\text{deg}_a} a \circ (D^E \Psi)$$

(5.2)

$$D^E(\Psi \circ b) = (D^E \Psi) \circ b + (-1)^{\text{deg}_a} \Psi \circ (D^E b)$$

(5.3)

as well as $(D^E)^2 = 0$. Moreover,

$$\sigma : \ker D^E \cap \ker \text{deg}_a \to \Gamma^\infty(E)[[\lambda]]$$

(5.4)

is a $\mathbb{C}[[\lambda]]$-linear bijection with inverse denoted by $\tau^E$. The Fedosov-Taylor series $\tau^E(s)$ of a section $s \in \Gamma^\infty(E)$ can be obtained recursively with respect to the total degree by

$$\tau^E(s)^{(0)} = s$$

(5.5)

$$\tau^E(s)^{(k+1)} = \delta^{-1} \left( D^E \tau^E(s)^{(k)} + \sum_{l=1}^{k} \left( \frac{i}{\lambda} \text{ad}(r^{(l+2)}) + (r^E)^{(l)} \right) \tau^E(s)^{(k-l)} \right)$$

(5.6)

where $\tau^E(s) = \sum_{k=0}^{\infty} \tau^E(s)^{(k)}$.

**Proof:** The equations (5.2), (5.3) and $(D^E)^2 = 0$ are just straightforward computations using the relations between $r$ and $r^E$ as well as the results from Section 3. The crucial point is that we have used the same $\Omega$ for $r'$ and $r$. Now let $s \in \Gamma^\infty(E)[[\lambda]]$ be given and define the operator

$$T_s \Psi := s + \delta^{-1} \left( D^E \Psi + \frac{i}{\lambda} \text{ad}(r)\Psi + r^E \circ \Psi \right)$$

(5.7)

for $\Psi \in W\otimes A^\bullet\otimes \mathcal{E}$ of anti-symmetric degree 0. It immediately follows that $T_s$ is strictly contracting in the complete metric space $W\otimes A^0\otimes \mathcal{E}$, whence the ultra-metric is defined by means of the Deg-degree. Thus $T_s$ has a unique fixed point, denoted by $\tau^E(s)$, by Banach’s fixed point theorem, see e.g. [3, App. A]. One also finds that $s \mapsto \tau^E(s)$ is $\mathbb{C}[[\lambda]]$-linear. From (5.5) it follows that $\delta^{-1} D^E \tau^E(s) = 0$ and with $(D^E)^2 = 0$ it follows that $\delta D^E \tau^E(s) = (D^E + \frac{i}{\lambda} \text{ad}(r) + r^E) \tau^E(s)$ whence by (2.5) we have

$$D^E \tau^E(s) = \delta^{-1} \left( D^E + \frac{i}{\lambda} \text{ad}(r) + r^E \right) D^E \tau^E(s).$$

(5.8)

Thus $D^E \tau^E(s)$ turns out to be the fixed point of a strictly contracting linear operator, whence $D^E \tau^E(s) = 0$. On the other hand, let $D^E \Psi = 0$. Then with (2.5) it follows that $\Psi$ is the fixed point of $T_{\sigma}(\Psi)$ whence $\Psi = \tau^E(\sigma(\Psi))$. Finally, the recursion formulas follow immediately from $\tau^E(s) = T_{\sigma} \tau^E(s)$. \[\square\]
Corollary 1 The sections \( \Gamma^\infty(E)[[\lambda]] \) become a bimodule for \( \star' \) and \( \star \) by

\[
A \cdot' s = \sigma(\tau'(A) \circ \tau^E(s)) \quad \text{and} \quad s \cdot f = \sigma(\tau^E(s) \circ \tau(f)),
\]

where \( A \in \Gamma^\infty(\text{End}(E))[[\lambda]], \ s \in \Gamma^\infty(E)[[\lambda]], \) and \( f \in C^\infty(M)[[\lambda]]. \)

Let us now focus on the case of a line bundle \( E = L \rightarrow M. \) In this case \( \Gamma^\infty(\text{End}(L))[[\lambda]] = C^\infty(M)[[\lambda]] \) whence the product \( \star' \) is defined for functions on \( M. \) In fact, it turns out to be a star product as well.

By definition, the Fedosov class of a Fedosov star product is the deRham cohomology class of the curvature of the corresponding Fedosov derivative, i.e.

\[
F(\star) = \left[ \omega + \delta r - R - Dr - \frac{i}{\lambda} r \circ r \right] = [\omega] + [\Omega],
\]

if \( \star \) is obtained from \( \mathcal{D} = -\delta + D + \frac{i}{\lambda} \text{ad}(r). \) Thus, according to (4.4), the class of \( \star' \) is given by

\[
F(\star') = \left[ \omega + \delta r' - R - Dr' - \frac{i}{\lambda} r' \circ r' \right] = [\omega] + [\Omega] + i\lambda[R^L],
\]

where \( R^L \) is the curvature of \( \nabla^L. \) But this is just the Chern class of \( L \) whence

\[
F(\star') = F(\star) - 2\pi \lambda c_1(L).
\]

Taking into account that the characteristic class \( c(\star) \) of a Fedosov star product is given by \( \frac{1}{\lambda} F(\star), \) see e.g. the discussion in [20], we have the following corollary:

Corollary 2 In case of a line bundle \( E = L \rightarrow M, \) the characteristic classes of \( \star' \) and \( \star \) are related by

\[
c(\star') = c(\star) + 2\pi i c_1(L).
\]

Remark 2 This is of course to be expected from [3, Thm. 3.1] as the bimodule structure \( \cdot' \) and \( \cdot \) on \( \Gamma^\infty(L)[[\lambda]] \) is exactly a Morita equivalence bimodule for the two star products \( \star' \) and \( \star. \) The remarkable point is that the computation of \( c(\star') \) is almost a triviality in the Fedosov framework, compared to the Čech cohomological computation in [3].

6 The case of a Hermitian vector bundle

Consider now a Hermitian fiber metric \( h \) for \( E \) and assume that \( \nabla^E \) is compatible with \( h. \) In this case \( \Gamma^\infty(\text{End}(E)) \) has a natural \( \ast \)-involution defined by \( h(As, A's') = h(s, A's') \) for \( A \in \Gamma^\infty(\text{End}(E)) \) and \( s, s' \in \Gamma^\infty(E). \) Thus we can extend this \( \ast \)-involution, together with the complex conjugation, to a super-\( \ast \)-involution of \( \mathbb{W} \otimes \Lambda^\ast \otimes \text{End}(E) \) and \( \mathbb{W} \otimes \Lambda^\ast, \) respectively. It is well-known that the fiberwise Weyl-product \( \circ \) is compatible with this \( \ast \)-involution, i.e. \( (a \circ b)^\ast = (-1)^{deg_a a \ast deg_b b^\ast} a^\ast \circ b^\ast \) for all \( a, b \in \mathbb{W} \otimes \Lambda^\ast \otimes \text{End}(E). \) By the unique characterization of \( r' \) and \( r \) by (4.4) and (1.14), respectively, the following lemma is straightforward, see also [4, Lem. 3.3].

Lemma 7 Let \( \Omega = \overline{\Omega} \) be a real formal two-form. Then

\[
(r')^\ast = r', \quad r = r, \quad (r^E)^\ast = -r^E,
\]

(6.1)
as well as
\[(D' a)^* = D' a^*, \quad \text{and} \quad (D b) = D b \quad (6.2)\]
for \(a \in \mathcal{W} \otimes \Lambda \otimes \text{End}(\mathcal{E})\) and \(b \in \mathcal{W} \otimes \Lambda^*\). Moreover,
\[\tau'(A^*) = (\tau'(A))^*, \quad \tau(f) = \overline{\tau(f)}, \quad (6.3)\]
whence \(\star'\) and \(\star\) are Hermitian deformations, i.e.
\[(A \star' B)^* = B^* \star' A^* \quad \text{and} \quad f \star g = g \star f.\quad (6.4)\]

Let us assume \(\Omega = \overline{\Omega}\) for the following. Then in a next step we extend the fiber metric to \(\mathcal{W} \otimes \Lambda \otimes \text{End}(\mathcal{E})\) with values in \(\mathcal{W} \otimes \Lambda^*\) by defining
\[H(f \otimes s, g \otimes s') = H(f \circ (s \otimes s'), g \circ (s \otimes s'))\quad (6.5)\]
and extending this by sesquilinearity. Note that we can write \(h(s, s')\) on any side of the tensor product or of the \(\circ\)-product as it is a function. The following properties are immediate:

Lemma 8 Let \(a \in \mathcal{W} \otimes \Lambda \otimes \text{End}(\mathcal{E})\), \(\Psi, \Psi' \in \mathcal{W} \otimes \Lambda^*\), and \(b \in \mathcal{W} \otimes \Lambda^*\). Then
\[H(a \circ \Psi, \Psi') = H(\Psi, a^* \circ \Psi') \quad \text{and} \quad H(\Psi, \Psi' \circ b) = H(\Psi, \Psi') \circ b\quad (6.6)\]
as well as
\[H(\Psi, \Psi') = \overline{H(\Psi', \Psi)}.\quad (6.7)\]

Using Lemma 7 and 8 as well as the properties of \(D^E\) as in (5.2) and (5.3) we obtain by a simple computation the following compatibility
\[D(\tau^E(a)) = \tau^E(a^*) \quad \text{and} \quad \tau^E(b) = \overline{\tau^E(b)}\quad (6.8)\]
where \(\Psi, \Psi' \in \mathcal{W} \otimes \Lambda^*\). Clearly this can be seen as a direct analog of the compatibility of \(\nabla^E\) and \(h\). As a consequence we can define a deformed Hermitian metric \(h\) by
\[h(s, s') = \sigma(h(\tau^E(s), \tau^E(s'))).\quad (6.9)\]

Theorem 4 The map \(h\) is \(C[[\lambda]]\)-sesquilinear and satisfies
\[h(s, s') = \overline{h(s', s)} \quad \text{and} \quad h(s, s) \geq 0 \quad (6.10)\]
\[h(s, s' \cdot f) = h(s, s') \star f\quad (6.11)\]
\[h(A \star' s, s') = h(s, A^* \star' s')\quad (6.12)\]
for all \(s, s' \in \Gamma^\infty(\mathcal{E})[[\lambda]], f \in C^\infty(M)[[\lambda]]\) and \(A \in \Gamma^\infty(\text{End}(\mathcal{E}))[[\lambda]]\).

Proof: The only non-trivial statement is the positivity \(h(s, s) \geq 0\) which follows from [6, Prop. 2.8]. The other statements are straightforward computations using (6.6), (6.7), and (6.8). □
7 Conclusion and further questions

Let us conclude with a few remarks on this approach to Morita equivalence of Fedosov star products. First we would like to point out that Corollary 2, together with the fact that every star product is equivalent to some Fedosov star product, implies that the condition (5.13) is actually sufficient for Morita equivalence. To prove that (5.13) is also necessary, one has to go beyond Fedosov’s construction as within this construction it is not directly clear that any bimodule deformation is equivalent to the one we described here. This fact was shown in [6, 8].

Nevertheless, the above construction is very geometric and almost explicit which makes it attractive to consider the following questions:

i.) The whole construction depends functorially on $\nabla$ and $\nabla_E$, say for the choice $\Omega = 0$. Thus having a group acting on $M$ with a lift to an action on the vector bundle $E$ such that $\nabla$ and $\nabla_E$ are preserved, this will lead to invariant $\ast$, $\ast'$, $\ast$, $\ast'$. A more detailed investigation of this problem will be subject to a future project.

ii.) On a Kähler manifold one has a Fedosov construction for a star product of Wick type (separation of variables), see [4, 15, 16]. Here one should be able to find a similar notion of ‘separation of variables’ for $\ast$ and $\ast'$ for the case of holomorphic vector bundles. Moreover, even in the case of almost Kähler manifolds [17], one can try to understand the characteristic classes of the star products of Wick type as coming from a particular bimodule deformation. Again this will be discussed in a forthcoming project.

iii.) Finally, we hope that this construction can be transferred to the case of Poisson manifolds using the ‘Fedosov-like’ approach of Cattaneo, Felder and Tomassini [5] in order to give more insight in the classification of star products up to Morita equivalence on Poisson manifolds. It may also give a alternative, global description of the appearance of Morita equivalence in non-commutative field theories as discussed in [14].

References

[1] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation Theory and Quantization. Ann. Phys. 111 (1978), 61–151.

[2] Bertelson, M., Cahen, M., Gutt, S.: Equivalence of Star Products. Class. Quantum Grav. 14 (1997), A93–A107.

[3] Bordemann, M., Neumaier, N., Waldmann, S.: Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion, traces, and applications. J. Geom. Phys. 29 (1999), 199–234.

[4] Bordemann, M., Waldmann, S.: A Fedosov Star Product of Wick Type for Kähler Manifolds. Lett. Math. Phys. 41 (1997), 243–253.

[5] Bordemann, M., Waldmann, S.: Formal GNS Construction and States in Deformation Quantization. Commun. Math. Phys. 195 (1998), 549–583.

[6] Bursztyn, H., Waldmann, S.: Deformation Quantization of Hermitian Vector Bundles. Lett. Math. Phys. 53 (2000), 349–365.

[7] Bursztyn, H., Waldmann, S.: Algebraic Rieffel Induction, Formal Morita Equivalence and Applications to Deformation Quantization. J. Geom. Phys. 37 (2001), 307–364.

[8] Bursztyn, H., Waldmann, S.: The characteristic classes of Morita equivalence of star products on symplectic manifolds. Preprint Freiburg FR-THEP 2001/09, math.QA/0106178 (June 2001).
[9] Cattaneo, A. S., Felder, G., Tomassini, L.: Fedosov connections on jet bundles and deformation quantization. Preprint [math.QA/0111290] (November 2001).

[10] DeWilde, M., Lecomte, P. B. A.: Existence of Star-Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds. Lett. Math. Phys. 7 (1983), 487–496.

[11] Dito, G., Sternheimer, D.: Deformation quantization: genesis, developments and metamorphoses. Preprint (2001). Contribution to the Proceedings for the 68ème Rencontre entre Physiciens Theoriciens et Mathématiciens on Deformation Quantization. Strasbourg, 31. 05. 2001 – 02. 06. 2001.

[12] Fedosov, B. V.: Deformation Quantization and Index Theory. Akademie Verlag, Berlin, 1996.

[13] Gutt, S.: Variations on deformation quantization. In: Dito, G., Sternheimer, D. (eds.): Conférence Moshé Flato 1999. Quantization, Deformations, and Symmetries, Mathematical Physics Studies no. 21. 217–254. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[14] Jurco, B., Schupp, P., Wess, J.: Noncommutative line bundle and Morita equivalence. Preprint [hep-th/0106116] (June 2001).

[15] Karabegov, A. V.: Deformation Quantization with Separation of Variables on a Kähler Manifold. Commun. Math. Phys. 180 (1996), 745–755.

[16] Karabegov, A. V.: On Fedosov’s approach to Deformation Quantization with Separation of Variables. In: Dito, G., Sternheimer, D. (eds.): Conférence Moshé Flato 1999. Quantization, Deformations, and Symmetries, Mathematical Physics Studies no. 22. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[17] Karabegov, A. V., Schlichenmaier, M.: Almost-Kähler Deformation Quantization. Lett. Math. Phys. 57 (2001), 135–148.

[18] Kontsevich, M.: Deformation Quantization of Poisson Manifolds, I. Preprint [q-alg/9709040] (September 1997).

[19] Nest, R., Tsygan, B.: Algebraic Index Theorem. Commun. Math. Phys. 172 (1995), 223–262.

[20] Neumaier, N.: Local $\nu$-Euler Derivations and Deligne’s Characteristic Class of Fedosov Star Products. Preprint Freiburg FR-THEP-99/3, [math.QA/9905176] (May 1999).

[21] Omori, H., Maeda, Y., Yoshioka, A.: Weyl Manifolds and Deformation Quantization. Adv. Math. 85 (1991), 224–255.

[22] Rieffel, M. A.: Morita equivalence for C*-algebras and W*-algebras. J. Pure. Appl. Math. 5 (1974), 51–96.

[23] Waldmann, S.: On the Representation Theory of Deformation Quantization. Preprint Freiburg FR-THEP 2001/10, [math.QA/0107112] (July 2001). Contribution to the Proceedings for the 68ème Rencontre entre Physiciens Theoriciens et Mathématiciens on Deformation Quantization. Strasbourg, 31. 05. 2001 – 02. 06. 2001.

[24] Weinstein, A.: Deformation Quantization. Séminaire Bourbaki 46ème année 789 (1994).