Covert Communication over Adversarially Jammed Channels

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Abstract

We consider a situation in which a transmitter Alice may wish to communicate with a receiver Bob over an adversarial channel. An active adversary James eavesdrops on their communication over a binary symmetric channel (BSC(\(q\)), and may maliciously flip (up to) a certain fraction \(p\) of their transmitted bits. The communication should be both covert and reliable. Covertness requires that the adversary James should be unable to estimate whether or not Alice is communicating based on his noisy observations, while reliability requires that the receiver Bob should be able to correctly recover Alice’s message with high probability.

Unlike the setting with passive adversaries considered thus far in the literature, we show that reliable covert communication in the presence of actively jamming adversaries requires Alice and Bob to have a shared key (unknown to James). The optimal throughput obtainable depends critically on the size of this key:

- When Alice and Bob’s shared key is less than \(\frac{1}{2} \log(n)\) bits, no communication that is simultaneously covert and reliable is possible. This is true even under reasonable computational assumptions on James, and if his jamming is required to be causal. Conversely, when the shared key is larger than \(6 \log(n)\), the optimal throughput scales as \(\mathcal{O}(\sqrt{n})\) — we explicitly characterize even the constant factor (with matching inner and outer bounds) for a wide range of parameters of interest.
- When Alice and Bob have a large amount (\(\omega(\sqrt{n})\) bits) of shared key, we again present a tight covert capacity characterization for all parameters of interest, and the capacity is independent of the number of bits in the shared key as long as it is larger than \(\omega(\sqrt{n})\) (again regardless of computational or causality assumptions on James).
- When the size of the shared key is “moderate” (belongs to \((\Omega(\log(n)), \mathcal{O}(\sqrt{n}))\), we show an achievable coding scheme as well as an outer bound on the information-theoretically optimal throughput.
- When Alice and Bob’s shared key is \(\mathcal{O}(\sqrt{n} \log(n))\) bits, we develop a computationally efficient coding scheme for Alice/Bob whose throughput is only a constant factor smaller than information-theoretically optimal, and it ensures both covertness and reliability even against a computationally unbounded adversary James.

I. INTRODUCTION

The security of our communication schemes is of significant concern — Big Brother is often watching! While much attention focuses on schemes that aim to hide the content of communication, in many scenarios, the fact of communication should also be kept secret. For example, a secret agent being caught communicating with an accomplice is of potentially drastic consequences — merely ensuring secrecy does not guarantee undetectability. This observation has drawn attention to the problem of covert communication. In a canonical information-theoretic setting for this problem, a transmitter Alice may wish to transmit messages to a receiver Bob over a noisy channel, and remains silent otherwise. James eavesdrops on her transmission through another noisy channel. The communication goals are twofold. Firstly, the communication should be covert, i.e., James should be unable to reliably distinguish whether or not Alice is transmitting. Simultaneously, it should also be reliable, i.e. Bob should be able to correctly estimate Alice’s transmission with a high probability of success. Recent work in the literature \([1]\)–\([10]\) has quite successfully characterized the information-theoretic aspects of this problem, in terms of characterizing the fundamental limits on the total amount of covert communication possible from Alice to Bob. Specifically, it turns out that no more than \(c_{p,q} \sqrt{n}\) bits may be covertly transmitted from Alice to Bob over \(n\) channel uses, where \(c_{p,q}\) is an explicitly characterizable constant depending on the channels from Alice to Bob and James. This sublinear throughput (as opposed to the linear throughput in most communication settings) results from the stringent requirement on Alice’s transmissions imposed by the need to remain covert — she must “whisper”, so to speak (pun intended). Indeed, most of her transmitted codewords must have low Hamming weight (\(\mathcal{O}(\sqrt{n})\)).

Prior information-theoretic work on covert communication largely focuses on random noise to both Bob and James. While such channel models are appropriate for passive eavesdroppers, a truly malicious adversary might wish to also actively disrupt any potential communication even when it is unable to detect if transmission has indeed taken place. To model this scenario, we take in this work a somewhat coding-theoretic view — we let the
channel from Alice to James be probabilistic, but we allow James to try to jam the channel to Bob adversarially, as a function of his noisy observations of Alice’s potential transmissions.

Semi-formally, in our setting, Alice’s channel input is an $n$-length binary vector $X$. The channel from Alice to James is a Binary Symmetric Channel with transition probability $q$ (i.e., BSC$(q)$). James uses his observation $Z$ in two ways — to detect if communication is being attempted via an estimator $\Phi$, and to choose a binary jamming vector $S$ of Hamming weight at most $pn$ — Bob receives the vector $Y = X \oplus S$. We denote the channel from Alice to Bob as ADVC$(p,q)$. When Alice is silent, $X$ must be the all-zeros vector $0$; when Alice is active, the $X$ she transmits may be a function of the message she wishes to transmit. Alice and Bob’s encoding/decoding procedures are known to all parties. We measure covertness via a hypothesis-testing metric — we say that the communication is $(1 - \epsilon_d)$-covert if irrespective of James’ estimator $\Phi$, his probability of false alarm plus his probability of missed detection is always lower-bounded by $1 - \epsilon_d$. Secondly, we require reliability — Bob should be able to reconstruct Alice’s transmission with high probability (w.h.p.) regardless of James’ jamming strategy.

Unfortunately, in our setting this turns out to be impossible — it turns out (as we show in our first main result) that the noise $S$ on Bob’s channel is adversarially chosen (rather than randomly as in the classical setting, e.g. [4]) implies James can ensure any such communication protocol must be either non-covert or unreliable. This is true even if James has computational restrictions, or is required to behave causally [11]. This is in stark contrast to the probabilistic channel setting wherein covert and reliable communication is possible for a wide range of parameters.

Hence, we mildly relax our problem — prior to transmission, Alice and Bob secretly share a $\Delta(n)$-bit randomly generated shared key that is unknown to James. It turns out (as another result in this work shows) that a modest value of $\Delta(n) = 6 \log(n)$ suffices (and $\Delta(n) \geq \frac{1}{2} \log(n)$ is necessary) to instantiate throughput scaling as $r^c_{\Delta(n),\epsilon_d}(p,q)$, for a constant $r^c_{\Delta(n),\epsilon_d}(p,q)$ that we explicitly and tightly characterize (provide matching inner and outer bounds) for a wide range of parameters $\Delta(n)$ (shared key size), $\epsilon_d$ (covertness parameter), and $p$, $q$ (noise parameters on channels from and to James). Hence in these parameter regimes the amount $\Delta(n) \in \mathcal{O}(\log(n))$ of shared key required to initiate reliable and covert communication scales much more gracefully than the amount of communication $\mathcal{O}(\sqrt{n})$ thereby instantiated. When the size of shared key is “moderate” (in the regime $(\Omega(\log(n)), \mathcal{O}(\sqrt{n}))$, we provide inner and outer bounds on the information-theoretically optimal throughput, and a larger amount of shared key in general yields a better inner bound. In contrast, increasing the amount of shared key $\Delta(n)$ leads to diminishing returns in the regime $\Delta(n) \in \omega(\sqrt{n})$ — the optimal throughput possible when $\Delta(n) \leq \omega(\sqrt{n})$ is the same as when $\Delta(n) = \infty$, and we are able to fully characterize this optimal throughput, as our inner and outer bounds match in this regime. Our achievability schemes make no computational or causality assumptions on James.

While the achievability schemes alluded to in the paragraphs above are existential and therefore have high computational complexity for Alice/Bob, when $\Delta(n) \in \Theta(\sqrt{n})$ we demonstrate a communication scheme that has poly$(n)$ computational complexity for Alice/Bob, makes no computational or causality assumptions on James, and achieves within a constant factor of the information-theoretically optimal throughput possible.

A. Related Work & Comparisons

Covert Communication: Bash et al. were the first to study covert communication for Gaussian channels in an information-theoretic setting and demonstrate a square-root law — communication that is simultaneously covert and reliable is possible only when the message length is $\mathcal{O}(\sqrt{n})$ bits [1]. Their work relies on the presence of a large amount of shared key (about $\mathcal{O}(\sqrt{n})$ bits) between Alice and Bob. Subsequently, Che et al. showed that for binary symmetric channels (BSCs), as long as James has a noisier channel than Bob, no shared key is necessary [7–10]. Bloch et al. [4], [12] and Wang et al. [6] then derived tight capacity characterizations for general discrete memoryless channels (DMCs). The work in [4] also showed that the amount of shared key needed when Bob has a noisier channel than James is $\mathcal{O}(\sqrt{n})$. Covert communication over multiple access channels [13] and broadcast channels [14] have also been investigated. Zhang et al. showed the first computationally efficient communication schemes for covert communication in [15]. While prior work on covert communication focuses on random noise channels (e.g., BSCs, AWGNs, and DMCs). To the best of our knowledge, our work is the first to examine covert communication over adversarial channels.

Random noise vs adversarial noise channels: In the non-covert setting, much work has focused on two classes of noisy channels — random noise channels and adversarial noise channels.

The capacities of random noise channels have been fully characterized by Shannon in his seminal work [16]. For instance, the capacity of a BSC$(p)$ is given as $1 - H(p)$, where $H(\cdot)$ is the binary entropy function. On

1 All logarithms in this paper are binary.
the contrary, though many upper and lower bounds (sometimes but not always matching) for a variety of special adversarial jamming models, a tight capacity characterization for general adversarial channels (also called Arbitrarily Varying Channels (AVCs)) in the information-theory literature — see [17] for an excellent survey) is still elusive. One way to classify adversarial models is via the adversary’s knowledge level of the transmitted codeword X. Models of interest include classical/omniscient adversarial model [18]–[20] (full knowledge of X), the myopic adversarial model [21]–[24] (noisy observations of X), the oblivious adversarial model [17], [25], [26] (no knowledge of X) and the causal adversarial model [11], [27], [28] (causal observations of X). Also, the computationally bounded adversary model [29], [30] considers models wherein Alice/Bob/James are all computationally bounded.

**Arbitrarily Varying Channels (AVCs):** At a high level, reliable communication in the model considered in this work is closely related that of communication over an AVC [17], [31], [32] with stringent input constraints. Indeed, (ii) Covertness is not considered as in this work; (iii) Indeed, the stringent requirements in channel inputs enforced due to covertness imply that some of the analytical techniques used in the AVC literature do not translate to our setting; and (iv) No effort is made to consider computational restrictions on Alice/Bob/James, unlike in our work.

1) **Myopic adversaries with shared key:** These are AVC problems first explicitly considered by Sarwate [22] wherein James only observes a noisy version Z of X (for instance through a BSC(q)) before deciding on his jamming vector S. Sarwate [22] provided a tight characterization of the throughput achievable in such settings over general discrete input-output channels in the presence of an unlimited-sized shared key — as such, the model therein has strong connections to the problem we consider. Indeed, the converse we present in Theorem 2 relies heavily on the information-theoretic framework for impossibility results in AVCs in general and [22] in particular.

2) **Myopic adversaries without shared key:** Problems concerning myopic adversaries without shared key between Alice and Bob [23] are considerably more challenging than when shared key is available. However, if the adversary is sufficiently myopic, e.g., the noise q on the BSC(q) to James is strictly larger than the fraction p of bit flips he can impose on the channel to Bob, and there are no constraints on Alice’s transmissions, the capacity of such a channel has been shown to exactly equal that of a BSC(p). Note that the adversarial model of interest in this work is essentially the insufficiently myopic adversarial model due to the stringent constraints on Alice’s X imposed by covertness.

**Remark:** Despite similarities, key features of general results in the AVC literature that distinguish them from those in this work are as follows: (i) An unlimited-sized shared key between Alice and Bob is assumed, as opposed to the careful classification of achievabilities/converses obtained in our work pertaining to differing-sized shared keys; (ii) Covertness is not considered as in this work; (iii) Indeed, the stringent requirements in channel inputs enforced due to covertness imply that some of the analytical techniques used in the AVC literature do not translate to our setting; and (iv) No effort is made to consider computational restrictions on Alice/Bob/James, unlike in our work.

**List decoding:** One of the primitives our achievability schemes rely heavily on is that of list decoding [33]–[35]. Results in this subset of the literature guarantee that even in the presence of omniscient adversaries Bob is able to localize Alice’s transmission to a small (often constant-sized) list at a communication rate approaching that of a symmetrizability condition for AVCs.

**Usage of shared key:** One pathway to achievability schemes for AVCs (e.g. [59]) is to ensure that Bob can list-decode to a small list, and then to use the key shared with Alice to disambiguate this list down to a unique message. There are multiple such schemes in the literature, including computationally efficient schemes [37].

**Permutation-based coding:** Another idea in the literature that has borne multiple dividends (e.g. [36], [38]) in the context of code design for AVCs (especially computationally efficient codes, e.g. [26], [29]) and even in covert communication from a source-resolvability perspective [4], [40] is that of permutation-based coding. Alice and Bob generate a small (polynomially-size) set Π (known also to James) of randomly sampled permutations as part of code-design, and then use their shared key to pick a particular permutation π that is unknown to James. Alice then transmits the codeword π(X), and Bob attempts to decode π−1(Y). In several problems it can be shown that the effect of this permutation π is to “scramble” James’ jamming action, and hence makes him behave essentially like i.i.d. noise. In our work we show that similar ideas work even in the presence of a myopic and computationally unbounded jammer James, and results in a computationally efficient communication scheme for Alice and Bob.
II. Model

We first briefly introduce the notations used throughout this paper. Random variables are denoted by uppercase letters, e.g., $X$, and their realizations are denoted by lowercase letters, e.g., $x$. Sets are denoted by calligraphic letters, e.g., $X$. Vectors of length $n$ are denoted by boldface letters, e.g., $X$ and $x$. The $i$-th locations of $X$ and $x$ are denoted by $X_i$ and $x_i$ respectively.

**Encoder:** Let $n$ denote the blocklength (number of channel uses) of Alice’s communication. Alice’s encoder $\Psi(\ldots)$ takes three inputs: (i) The single bit transmission status $T$: Alice’s silence is denoted by $T = 0$ whereas $T = 1$ denotes that she is active. (ii) The message $M$, which is either 0 (if Alice is silent), or uniformly distributed over $\{1, 2, \ldots, N\}$ (if Alice is active). (iii) The $\Delta(n)$-bit shared key $K$ distributed uniformly over $\{0, 1\}^{\Delta(n)}$. Prior to transmission, only Alice knows the transmission status $T$ and message $M$, and both Alice and Bob know the key $K$. James is a priori ignorant of all three.

If Alice’s transmission status $T$ equals 0 (Alice is silent), then Alice’s encoder $\Psi(0 \ldots)$ must output $X = 0$, a length-$n$ vector comprising entirely of zeros. On the other hand if Alice’s transmission status $T$ equals 1 (Alice is active), then Alice’s encoder $\Psi(1 \ldots)$ may output an arbitrary length-$n$ binary vector $X$. The collection of all outputs of Alice’s encoder $\Psi(\ldots)$ is called the codebook, denoted by $C$. This encoder is known a priori to all parties (Alice, Bob, and James). The rate of the code is defined as $R \triangleq (\log N)/n$, while the relative throughput is defined as $r \triangleq (\log N)/\sqrt{n}$. \[3\]

**James’s observations:** James receives the vector $Z = X \oplus S$, where $S_i$ is a Bernoulli($q$) random variable. Hence James’ observed vector $Z$ is the output of a BSC($q$) channel to which the input is Alice’s transmission $X$. On the basis of this observation $Z$ and his knowledge of Alice’s encoder $\Psi(\ldots)$, James, as described below: (i) estimates Alice’s transmission status $T$, and (ii) generates a jamming vector $S$ to disrupt communication.

**Estimator:** James’ estimator $\Phi(\ldots): \{0, 1\}^n \rightarrow \{0, 1\}$ estimates Alice’s transmission status $T$ as $\hat{T} = \Phi(Z)$. We use a hypothesis-testing metric (defined below) to measure covertness:

\[4\]

In some scenarios in the literature, in addition to the three inputs below, the encoder also incorporates additional private randomness (known a priori only to Alice, but not to Bob or James). Indeed, in some communication scenarios \[27\] it can be shown that the throughput in the presence of such private randomness is strictly higher than in its absence. However, since in this work such types of encoders do not help, we ignore this potential flexibility in code design.

Note that no assumptions are made about any probability distribution on $T$. The relative throughput $r$ is commonly used in the covert communication literature to measure the size $N$ of Alice’s potential message since it can be shown that as the blocklength $n$ grows without bound, $N$ can scale as most as $2^{O(\sqrt{n})}$ (rather than $2^{O(n)}$ as is common in communication scenarios). Hence when $n$ increases without bound, the relative throughput $r$ scales as a constant while the rate $R$ goes to zero.
Definition 1 (Covertness). Let \( \alpha(\Phi) = \Pr_S(\hat{T} = 1|T = 0) \) and \( \beta(\Phi) = \Pr_M,K,S(\hat{T} = 0|T = 1) \) respectively be the probability of false alarm and the probability of missed detection of an estimator \( \Phi \). The communication is said to be \((1 - \epsilon_d)\)-covert if there does not exist an estimator \( \Phi \) such that \( \alpha(\Phi) + \beta(\Phi) < 1 - \epsilon_d \).

For the optimal estimator \( \Phi^* \), \( \alpha(\Phi^*) + \beta(\Phi^*) = \mathcal{V}(p_0(Z),p_1(Z)) \), where \( \mathcal{V}(p_0(Z),p_1(Z)) \) is the variational distance between the two distributions (corresponding to \( T = 0 \) and \( T = 1 \), respectively) on James’ observations \( Z \). In general the computational complexity of implementing the optimal estimator \( \Phi^* \) is high (potentially \( \exp(n) \)); also, analyzing its performance can also be tricky.

**Jamming Function:** As a function of his observation \( Z \) and his knowledge of Alice’s encoding function \( \Psi(\ldots) \) James chooses a jamming function to output a length-\( n \) binary jamming vector \( S \) of Hamming weight at most \( pn \). In general James’ jamming function corresponds to a conditional probability distribution \( p_S|Z,C \) that stochastically maps James’ observations to his jamming vector \( S \). Note that \( p_S|Z,C \) generates an \( n \)-letter distribution over length-\( n \) binary sequences \( S \), given James’ length-\( n \) observation \( Z \), and his knowledge of Alice and Bob’s code \( C \).

**Decoder:** Bob receives the length-\( n \) binary vector \( Y = X \oplus S \), and then applies his decoding function \( \Gamma(\ldots) : \{0,1\}^n \times \{0,1\}^n \to \{0\} \cup \{1,2,\ldots,N\} \) to produce his message reconstruction \( \hat{M} \) from his observed vector \( Y \) and the shared key \( K \).

**Probability of decoding error:** Bob’s probability of error is defined as

\[
P_e(\Psi,\Gamma) \triangleq \max_{p|0,c} \left( \Pr_{K,S,S}(\hat{M} \neq 0|T = 0) + \Pr_{M,K,S,S}(\hat{M} \neq M|T = 1) \right).
\]

**Remarks:**

a) Note that the probability as defined in (1) is maximized over the \( n \)-letter distribution \( p_S|Z,C \). This is to indicate that there may (or may not) be a stochastic component to the jamming function James uses to generate \( S \) from his observation \( Z \). Hence we include an averaging over \( S \).

b) Note that the \( P_e \) defined in (1) is the average probability of error (averaged over both the message and the shared key). A stronger metric (which is not studied in this work) is the maximum probability of error, which would be the probability that for each message the corresponding probability of error is small.

**Code:** Alice and Bob’s code comprise of the encoder decoder pair \( (\Psi,\Gamma) \).

**Achievable robust relative throughput/Robust covert capacity:** For any \( 0 < p,q < 1/2 \), \( \Delta(n) \geq 0 \), and sufficiently small \( \epsilon_d \), a relative throughput \( r_{\Delta(n),\epsilon_d}(p,q) \) is said to be achievable if there exists an infinite sequence of codes \( (\Psi_n,\Gamma_n) \) with \( \Delta(n) \) bits of shared key such that each of the codes in the sequence has relative throughput at least \( r_{\Delta(n),\epsilon_d}(p,q) \), \( \lim_{n \to \infty} P_e(\Psi_n,\Gamma_n) = 0 \), and ensures the communication is \((1 - \epsilon_d)\)-covert. Then the robust covert capacity \( r^*_{\Delta(n),\epsilon_d}(p,q) \) is defined as the supremum over all possible achievable relative throughputs.

**Positive throughput region:** For any \( \Delta(n) \) and sufficiently small \( \epsilon_d \), the positive throughput region \( \mathcal{R}^+_{\Delta(n),\epsilon_d}(p,q) \) is defined as a collection of values \( (p,q) \) such that the robust covert capacity \( r^*_{\Delta(n),\epsilon_d}(p,q) \) is positive.

### III. MAIN RESULTS

We now summarize the main contributions of this work. There are at least two types of estimators and jamming functions James can use, each of which results in a non-trivial restriction on the reliable and covert throughput obtainable from Alice to Bob. Perhaps surprisingly, there is a unified achievability scheme that Alice and Bob can use that meets these constraints for a wide range of parameters of interest, and thereby shows that these types of estimators/jamming functions are in some sense optimal from James’ perspective.

- **Weight-detector:** This estimator (with computational complexity \( O(n) \)) merely computes the Hamming weight of the observed \( Z \), and if this is significantly higher than expected \( (qn + c_\ell \sqrt{n}) \) for some constant \( c_\ell \), then James estimates \( \hat{T} = 1 \), i.e., that Alice is indeed transmitting.

- **Hypothesis-testing estimator:** James first computes two distributions \( p_0(Z) \) and \( p_1(Z) \), which respectively correspond to the distributions of \( Z \) when \( T = 0 \) and \( T = 1 \). The optimal estimator \( \Phi^* \) outputs \( \hat{T} = 1 \) if \( p_1(z) \geq p_0(z) \), and outputs \( \hat{T} = 0 \) if \( p_0(z) > p_1(z) \). Note that this estimator potentially has computational complexity \( \exp(n) \) for James.

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5 Note that even if James ignores the knowledge of \( Z \), a naive estimator \( \hat{\Phi} \) (which always outputs \( T = 0 \) or \( T = 1 \)) can also guarantee \( \alpha(\Phi) + \beta(\Phi) = 1 \). Therefore, the definition for covert communication implies that James’ optimal estimator \( \Phi^* \) cannot be much better than the naive estimator \( \hat{\Phi} \).

6 The two terms correspond to Bob’s decoder making an error in each of two scenarios: when Alice is silent, and when she is active.

7 Even though this estimator is a sub-optimal proxy to the Hypothesis-testing estimator, it has been shown in \cite{4, 7} to be “good enough” from James’ perspective, in the sense that it constrains Alice’s throughput to the same extent as does the Hypothesis-testing estimator, which is known \cite{12} to be optimal.
• **Oblivious jamming:** This jamming strategy ignores James’ channel observations $Z$, and chooses $S$ as a sum of multiple (at most $O(\sqrt{n})$) potentially valid transmissions from Alice. Since Bob’s observation is a sum of Alice’s transmission $X$ and James’ state vector $S$, this jamming strategy attempts to confuse Bob as to what Alice truly transmitted. Note that this jamming strategy can be implemented by James causally, with computation complexity at most $\sqrt{n}$ times the computational complexity of Alice’s encoder. This converse is presented in Theorem 1.

• **Myopic jamming:** In this jamming strategy, even if Alice’s transmission is covert and hence James is unsure whether or not Alice is active, James nonetheless uses his observations in $Z$ to guess which channel uses correspond to potential 1’s in Alice’s transmitted codeword if she indeed is active. He then preferentially flips these bits — specifically, if $Z_i = 1$ then he flips the corresponding $X_i$, w.p. about $p/q$, but if $Z_i = 0$ he doesn’t flip $X_i$. Note that this jamming strategy can be implemented by James causally, with computation complexity linear in $n$. This converse is presented in Theorem 2.

To facilitate the statement of our results, we define the following notation. For any channel parameters $p, q \in [0, 1/2)$ and any sufficiently small covertness parameter $\epsilon_d > 0$, the following definitions help characterize upper and lower bounds on the throughput.

**Definition 2** (Weight normalized mutual information and code-weight parameter). The weight normalized mutual information for Bob and James are respectively defined as

$$I_B(p, q) \triangleq \frac{p(q-1)}{q} \log \left( \frac{q-p+pq}{p^2(1-q)} \right) + \log \left( \frac{q-p+pq}{pq} \right), \quad \text{and}$$

$$I_J(q) \triangleq (1-2q) \log \left( \frac{1-q}{q} \right).$$

Let the $Q$-function equal $Q(x) = \frac{1}{2\pi} \int_{x}^{\infty} \exp \left( -u^2/2 \right) du$. The parameter $t(q, \epsilon_d)$ equals

$$t(q, \epsilon_d) \triangleq \frac{2\sqrt{q(1-q)}}{1-2q} \cdot Q^{-1} \left( 1 - \frac{\epsilon_d}{2} \right),$$

The parameter $t(q, \epsilon_d)$ is independent of the blocklength $n$, corresponding to the average weight of our codewords. Roughly speaking, “most” codewords have Hamming weight about $t(q, \epsilon_d)\sqrt{n}$. Following the techniques in [4]–[6] it has been optimized to be as large as possible while still ensuring $(1-\epsilon_d)$ covertness. The quantity $I_J(q)$ denotes the mutual information (times the normalization $t(q, \epsilon_d)\sqrt{n}$) corresponding to the BSC($q$) from Alice to James, given the code-weight parameter $t(q, \epsilon_d)$ on $X$, derived by taking the appropriate Taylor series expansion of the mutual information between $X$ and $Z$. The quantity $I_B(p, q)$ denotes the mutual information (times the normalization $t(q, \epsilon_d)\sqrt{n}$) of the worst i.i.d. channel inducible from Alice to Bob due to an i.i.d. myopic jamming strategy employed by James. As outlined in Theorem 2 this corresponds to the asymmetric channel arising from James flipping $X_i$ with probability approximately $p/q$ within the support of $Z$, and 0 outside — hence James concentrates his bit-flip power in bits he observes to be likelier to correspond to actual transmissions from Alice. While mutual information from Alice to Bob in the presence of such an i.i.d. jamming strategy clearly serves as an outer bound on Alice’s achievable throughput, it is perhaps more surprising that this is also achievable by our codes in a wide range of parameter regimes (corresponding to the achievable positive throughput region presented in Theorem 3 below).

**A. Impossibility of covert communication with $\Delta(n) < \frac{1}{2} \log(n)$**

When the amount of shared key is less than $\frac{1}{2} \log n$, if James employs a weight-detector with an appropriate threshold, combined with an oblivious jamming strategy, it turns out that he can ensure that the probability of decoding error is bounded away from zero for any non-trivial covert codebook. Roughly speaking, since Alice’s codebook comprises mostly of low-weight codewords, James is able to confuse Bob by choosing a jamming vector that comprises of the sum of multiple potential codewords — “spoofs” — Bob is unable to disambiguate Alice’s true $X$ from among the cacophony of spoofs. In the following, let $\Phi_{p, \rho}$ denote the weight-detector with threshold $\rho$, and $p^{(ob)}_{\Phi_{p, \rho}}$, $C_n$ denote the oblivious jamming strategy. The following theorem makes the above claim precise, and the proof of Theorem 1 can be found in Section IV.

**Theorem 1.** Let $\epsilon_d \in (0, 1)$ and $\Delta(n) < \frac{1}{2} \log(n)$. For every sequence of codes $\{C_n\}$ of blocklength $n$, message length $\log N = nr$, and encoding complexity $f_C(n)$, at least one of the following is true:
1) \((C_n \text{ is not covert})\) There exists a detector \(\Phi\) with computational complexity \(O(n)\) such that \(\alpha(\Phi) + \beta(\Phi) < 1 - \epsilon_d\). In particular, \(\Phi\) can be chosen to be the weight-detector \(\Phi_p\) for an appropriately set threshold \(\rho\).

2) \((C_n \text{ is not reliable})\) There exists a constant \(\eta = \eta(\epsilon_d, p, q)\) and causal jamming strategy \(p_{S/Z,C}\) with computational complexity \(O(\sqrt{n} f_c(n))\), such that Bob's probability of decoding error when \(T = 1\) is bounded from below as

\[
\Pr_{M, K, S, S} (M \neq \hat{M}|T = 1) \geq 1 - \max \left\{ \frac{2\Delta(n)}{N}, \frac{2\Delta(n)\eta}{\sqrt{n}} \right\}. \tag{5}
\]

In particular, \(p_{S/Z,C}\) may be chosen as the oblivious jamming strategy \(p_{S/Z,C}^{(ob)}\).

Remark 1. The lower bound on the probability of error in (5) is valid for all values of \(\Delta(n)\). However, it is non-vanishing only if \(\Delta(n) < \frac{1}{2} \log(n)\).

B. An upper bound on the robust covert capacity for any \(\Delta(n)\)

Next, we obtain an upper bound on the robust covert capacity that holds regardless of the amount of shared key available. Our strategy here is to bound the throughput of any simultaneously covert and reliable code by first showing that the average Hamming weight of codewords from such a code must be bounded from above by an appropriate function of the covertness parameter. Next, since the transmitted message must also be reliably decoded under all jamming strategies, this gives an upper bound on the number of distinct messages possible. In order to get a bound on the average weight of the codewords, we analyze the detection probabilities with respect to the weight-detector \(\Phi_p\) with a threshold \(\rho\) that depends on the codebook. To bound the number of codewords, we analyze Bob's reliability with respect to the mutual information \(t(q, \epsilon_d)I_B(p, q)\sqrt{n}\) of the channel induced under James' myopic jamming strategy \(p_{S/Z,C}^{(my)}\). The proof of Theorem 2 can be found in Section V.

**Theorem 2.** Let \(\epsilon_d \in (0, 1)\). For every sequence \(\{\Delta(n)\}\),

1) \(\Delta(n) \geq \frac{\epsilon_d}{2}\) and \(\epsilon_d \geq 0\) (corresponds to the region below the blue dashed line in Figure 2).

2) \(\Delta(n) \geq \frac{\epsilon_d}{2}\) and \(\epsilon_d \geq 0\) (corresponds to the region above the blue dashed line in Figure 2).

Further, when \(0 < p < q < 1/2\) (resp. \(0 < q \leq p < 1/2\) and \(q > r > 0\)), if \(\{C_n\}\) is a sequence of codes with relative throughput \(r > t(q, \epsilon_d)I_B(p, q)\) (resp. \(r > 0\)), then there exists \(\epsilon > 0\) such that at least one of the following is true for infinitely many values of \(n\):

1) \((C_n \text{ is not covert})\) There exists a detector \(\Phi\) with computational complexity \(\Theta(n)\) such that \(\alpha(\Phi) + \beta(\Phi) < 1 - \epsilon_d\).

In particular, \(\Phi\) can be chosen to be the weight-detector \(\Phi_p\) for an appropriately set threshold \(\rho\).

2) \((C_n \text{ is not reliable})\) There exists a causal jamming strategy \(p_{S/Z,C}\) with computational complexity \(\Theta(n)\) under which Bob's probability of decoding error when \(T = 1\) is bounded from below as

\[
\Pr_{M, K, S, S} (M \neq \hat{M}|T = 1) \geq \epsilon.
\]

In particular, \(p_{S/Z,C}\) may be chosen to be the myopic jamming strategy \(p_{S/Z,C}^{(my)}\).

C. Achievability of covert communication with \(\Delta(n) \geq 6 \log(n)\)

Next, we give an achievability result based on low-weight random codes and list decoding. The crux of our proof is a novel myopic list decoding lemma described in the Introduction, and formally presented in Claims 8 and 9 in Section VI-D. This lemma first demonstrates that for the parameter regime under consideration, with high probability, from James' perspective there are multiple (roughly \(\exp(\sqrt{n})\)) equally plausible transmissions by Alice — hence James has a large "uncertainty set". It then shows that, averaged over the uncertainty set, regardless of
James’ specific choice of jamming vector $S$, very few codewords $X$ in James’ uncertainty set are “killed” by $S$, i.e. if Bob attempts to list-decode the corresponding $Y = X \oplus S$, his list-size is “too large” (larger than some polynomial — say $n^2$). Hence, w.h.p. over the randomness in which $X$ in the uncertainty set is instantiated, James is unable to force too large a list on Bob. To complete the argument we show that the dominant error-event (among $p<q$) is one which has zero channel capacity even if $p>q$.

As in the presence of a passive adversary (rather than an actively jamming adversary), the dominant error-event is one which has zero channel capacity even if $p>q$, i.e., the main channel from Alice to Bob has more bit-flips than the channel from Alice to James. This is due to the fact that when $p>q$, among the classes of channels James can induce from Alice to Bob is one which has zero channel capacity even if $p<1/2$.

For any $\Delta(n) \in (\Omega(\log(n)), o(\sqrt{n}))$, as shown via Theorem 3, robust covert communication is possible above the red curve. When $\Delta(n) \in \Omega(\sqrt{n})$, as shown via Theorem 3, the achievable positive throughput region increases. The two black curves delineate the achievable positive throughput regions for $\Delta(n) = 0.015 \sqrt{n}$ and $0.03 \sqrt{n}$ respectively. The achievable positive throughput regions for each corresponding $\Delta(n)$ are now above the respective black curves. (3) Regardless of the value of shared key $\Delta(n)$, no robust covert communication is possible below the blue dashed line corresponding to $p=q$. This is in contrast to “classical” covert communication in the presence of a passive adversary (rather than an actively jamming adversary), wherein increasing amounts of shared key allow for covert communication even when $p>q$, i.e., the main channel from Alice to Bob has more bit-flips than the channel from Alice to James. This is due to the fact that when $p>q$, among the classes of channels James can induce from Alice to Bob is one which has zero channel capacity even if $p<1/2$.

James’ specific choice of jamming vector $S$, very few codewords $X$ in James’ uncertainty set are “killed” by $S$, i.e. if Bob attempts to list-decode the corresponding $Y = X \oplus S$, his list-size is “too large” (larger than some polynomial — say $n^2$). Hence, w.h.p. over the randomness in which $X$ in the uncertainty set is instantiated, James is unable to force too large a list on Bob. To complete the argument we show that the dominant error-event (among all joint distributions James can induce between $Z$ and $S$) corresponds to James behaving in the i.i.d. manner specific in the myopic jamming strategy. Bob is then able to use the $O(\log(n))$-sized shared key to disambiguate the list down to a unique element via a hashing scheme. While we do not have a full characterization of the positive throughput region when $\Delta(n) \in (\Omega(\log(n)), o(\sqrt{n}))$, we present below the achievable positive throughput regions $R_{(\Delta(n))}^+(p, q)$ corresponding to the parameter regime where our codes have positive throughput. The achievable positive throughput regions $R_{\Delta(n),\epsilon_d}^+(p, q)$ are subsets of the true positive throughput regions $R_{\Delta(n),\epsilon_d}^+(p, q)$.

**Theorem 3.** Let $\epsilon_d \in (0, 1)$ and let the shared key length $\Delta(n) \geq 6 \log(n)$. For three different regimes of $\Delta(n)$, the achievable positive throughput regions $R_{\Delta(n),\epsilon_d}^+(p, q)$ are given as

1) **Small key:** $R_{\Delta(n),\epsilon_d}^+(p, q) \triangleq \{(p, q) : p < q \text{ and } I_B(p, q) > I_J(q)\}$ if $\Delta(n) \in (\Omega(\log(n)), o(\sqrt{n}))$.

2) **Moderate-sized key:** $R_{\Delta(n),\epsilon_d}^+(p, q) \triangleq \{(p, q) : p < q \text{ and } I_B(p, q) + \sigma > I_J(q)\}$ if $\Delta(n) = \sigma \sqrt{n}$ for a constant $\sigma > 0$.

3) **Large key:** $R_{\Delta(n),\epsilon_d}^+(p, q) \triangleq \{(p, q) : p < q\}$ if $\Delta(n) \in \omega(\sqrt{n})$.

For any $\Delta(n)$ and $(p, q) \in R_{\Delta(n),\epsilon_d}^+(p, q)$, the maximum achievable relative throughput $r_{\Delta(n),\epsilon_d}^+(p, q)$ meets the outer bound (in Theorem 3) on robust covert capacity, i.e., $r_{\Delta(n),\epsilon_d}^+(p, q) = t(q, \epsilon_d) I_B(p, q) = r_{\Delta(n),\epsilon_d}^+(p, q)$, and both encoding and decoding may be performed with complexity $exp\Omega(\sqrt{n})$.

---

*Softly, by James’ song [43].*
The proof of Theorem 3 is included in Section VII. For any $0 < p < q < 1/2$, to achieve relative throughput $t(q, \epsilon_d) I_B(p, q)$, the minimum size of the shared key is $\Delta(n) = O(\log(n)) + [t(q, \epsilon_d)(I(q) - I_B(p, q))]^+ \sqrt{n}$, where $x^+ \triangleq \max(0, x)$. The intuition behind this scaling of $\Delta(n)$ is as follows — when the BSC channel from Alice to James is worse (has lower mutual information) than the worst channel he can instantiate from Alice to Bob, then $O(\log(n))$ bits of common randomness suffice for our scheme to work. Conversely, if James can make the channel from Alice to Bob to be worse than the channel to him, then Alice and Bob need a larger shared key (equaling at least the mutual information difference between the two channels) to cause James’ uncertainty set to be large enough for the myopic list decoding lemma (Claims 8 and 9) to hold. Structurally this phenomenon in the presence of an active adversary is intriguingly reminiscent of the phenomenon observed in [4] showing that covert communication in the presence of a passive adversary is possible if and only if the key-rate exceeds the normalized mutual information difference between the main channel and the eavesdropped channel.

Motivated by our results in Theorems 2 and 3 note that $\Delta(n) = \omega(\sqrt{n})$ is the same as $\Delta(n) = \infty$, since both achievable positive throughput regime and robust covert capacity are independent of $\Delta(n)$ as long as it is larger than $\omega(\sqrt{n})$. Figure 2 graphically represents the numerics of Theorems 2 and 3. Comparing Theorems 2 and 3, it is clear that, in general, the achievability and the converse may not match when $\Delta(n) \in (\Omega(\log(n)), o(\sqrt{n}))$ or $\Delta(n) = \sigma \sqrt{n}$ (for some small $\sigma > 0$). We believe that the gap is due to restricting our attention to computationally efficient detection and jamming strategies for James in our converse proof. In particular, if we use the Kullback-Leibler divergence, instead of the total variation distance to measure our covertness, and allow an optimal estimator $\Phi^*$ (which may be computationally inefficient), our achievable rates in Theorem 3 turn out to be optimal in all regimes.

D. Computationally efficient codes with $\Delta(n) \in \Omega(\sqrt{n} \log n)$

This result presents computationally efficient encoding and decoding schemes when the amount of shared key is $\Omega(\sqrt{n} \log n)$. Consider the Binary Asymmetric Channel (BAC) with input $X \in \{0, 1\}$, output $Y \in \{0, 1\}$, and the bit flip probabilities $Pr(Y = 1|X = 0) = p$ and $Pr(Y = 0|X = 1) = (1-q)p/q$. This corresponds to the channel from Alice and Bob caused by the myopic jamming strategy. Let $C_{BAC}(p, q) = \max_{\mu(X)} I(X;Y)$ denote the capacity of this channel, and let Bernoulli($\rho^*$) be the input distribution that achieves the maximum value of $I(X;Y)$.

**Theorem 4.** Let $\epsilon_d, \epsilon \in (0, 1)$, $0 < p < q < 1/2$, and $r < \frac{t(q, \epsilon_d)}{p^*} C_{BAC}(p, q)$. There exists a sequence of codes $\{C_n\}$ of blocklength $n$ and relative throughput $r$, and $n_0$ such that for every $n \geq n_0$,

1. $C_n$ is $(1-\epsilon_d)$-covert.
2. $C_n$ ensures the probability of error $P_\epsilon \leq \epsilon$.
3. $C_n$ can be encoded and decoded with $\text{poly}(n)$ complexity.

The proof of Theorem 4 can be found in Section VII. This scheme works via the permutation-based coding described in the Introduction — Alice permutes her codeword (of Hamming weight $t(q, \epsilon_d)\sqrt{n}$) uniformly at random among all length-$n$ binary sequences of Hamming weight $t(q, \epsilon_d)\sqrt{n}$ using her shared key. As argued in [4], such a source-resolvability scheme results in covertness against James. Also, as argued in [36], [39] such codes also work well to scramble James’ bit-flips, and make his actions behave in an i.i.d. manner.

Note that the above theorem does not place any computational restrictions on James. If we further allow James to be of at most polynomial complexity, the amount of shared key needed can be significantly relaxed under the existence of Cryptographic Pseudorandom Generators (PRG). In particular, a small length ($O(n^\xi)$ for any $\xi > 0$) shared key, can be used in conjunction with a PRG to generate the $O(\sqrt{n} \log n)$ length pseudorandom shared key $K$. Using the code from Theorem 4, we argue that if James can distinguish $T = 0$ from $T = 1$ in polynomial time, he should also be able to distinguish a truly random $K$ from a pseudorandom $K$ in polynomial time. This leads to the following corollary.

**Corollary 1.** Let $\epsilon_d, \epsilon, \xi \in (0, 1)$, $0 < p < q < 1/2$, and $r < \frac{t(q, \epsilon_d)}{p^*} C_{BAC}(p, q)$. Let $\Delta(n) \in \Omega(n^\xi)$. There exists a sequence of codes $\{C_n\}$ of blocklength $n$ and relative throughput $r$, and $n_0$ such that for every $n \geq n_0$,

1. For every polynomial time detector $\Phi$, $\alpha(\Phi) + \beta(\Phi) > 1 - \epsilon_d$.
2. $C_n$ ensures the probability of error $P_\epsilon \leq \epsilon$.
3. $C_n$ can be encoded and decoded with $\text{poly}(n)$ complexity.

E. Graphical representation of capacities

Figures 3-4 give a graphical representation of the robust covert capacities.
(a) The two sets of curves show robust covert capacities as functions of $q$ for two fixed values of $p$ ($p = 0.15$ and $p = 0.3$) and different amounts of shared key $\Delta(n)$ and $q = 0.4$ and different amounts of shared key $\Delta(n)$ ($\Delta(n) = o(\sqrt{n})$, $\Delta(n) = 0.015\sqrt{n}$, and $\Delta(n) = \infty$). ($\Delta(n) = o(\sqrt{n})$, $\Delta(n) = c(\sqrt{n})$, and $\Delta(n) = \infty$).

Fig. 3: Since the robust covert capacity region would require a three dimensional plot ($p$ and $q$ along the $x$ and $y$ axes respectively, and the relative throughput along the $z$ axis) that is hard to digest, we instead present here cross-sections of our partial characterization of the capacity region. The plot in Figure 3a shows inner and outer bounds on the optimal relative throughput curves for two values of $p$ ($p = 0.15$ and $p = 0.3$), and that in Figure 3b shows the corresponding curves for two values of $q$ ($q = 0.25$ and $q = 0.4$), and the covertness parameter $\epsilon_d = 0.02$ for each of these curves. For each of these values of $p$ or $q$, the blue dashed curves indicate outer bounds on the robust covert capacity, and indeed, these are attainable via matching achievability schemes when $\Delta(n) \in o(\sqrt{n})$ — i.e., effectively unlimited-sized shared keys. As alluded to in the achievable positive throughput region plot in Figure 2, note the impact of increasing values of $\Delta(n)$ — the achievable positive throughput region increases, and the corresponding throughput achievable by our coding scheme in Theorem 3 tracks the blue curve corresponding to having unbounded shared keys. The red curve corresponds to the relative throughput attainable by our coding scheme for any value of $\Delta(n) \in \Omega(\sqrt{n})$, $o(\sqrt{n})$), and the black curve corresponds to the attainable relative throughput for $\Delta(n) = 0.015\sqrt{n}$ in Figure 3a and $\Delta(n) = 0.01\sqrt{n}$ in Figure 3b.

(b) The two sets of curves show robust covert capacities as functions of $p$ for two fixed values of $q$ ($q = 0.15$ and $q = 0.3$) and different amounts of shared key $\Delta(n)$ and $p = 0.4$ and different amounts of shared key $\Delta(n)$ ($\Delta(n) = o(\sqrt{n})$, $\Delta(n) = 0.015\sqrt{n}$, and $\Delta(n) = \infty$). ($\Delta(n) = o(\sqrt{n})$, $\Delta(n) = c(\sqrt{n})$, and $\Delta(n) = \infty$).

Fig. 4: The covertness parameter $\epsilon_d$ also has an impact on the robust covert capacity — as shown in Figures 4a and 4b, increasing $\epsilon_d$ increases the robust covert capacity, since Alice’s codebook can comprise of somewhat “heavier” codewords.
### Table II: Table of parameters

| Symbol | Description | Equality/Range | Section |
|--------|-------------|----------------|---------|
| $M$    | Message     | $M \in \{1, 2, \ldots, N\}$ | Section II |
| $T$    | Transmission status | $T \in \{0, 1\}$ | Section II |
| $K$    | Shared key | $K \in \{0, 1\}^\Delta(n)$ | Section II |
| $p$    | ADVC($p$) – channel from Alice to Bob | $0 \leq p \leq 0.5$ | Section II |
| $q$    | BSC($q$) – channel from Alice to James | $0 \leq q < 0.5$ | Section II |
| $\epsilon_d$ | Covertness parameter | $\epsilon_d > 0$ | Section II |
| $\Delta(n)$ | Size of shared key | N/A | Section II |
| $t(q, \epsilon_d)$ | Code-weight design parameter | $t(q, \epsilon_d) = \frac{2\sqrt{q(1-q)}}{1-2q} \cdot Q^{-1}(\frac{1-\epsilon_d}{2})$ | Section III |
| $\rho$ | Normalized code-weight design parameter | $\rho = t(q, \epsilon_d)/\sqrt{n}$ | Section III |
| $\mathcal{C}$ | Code | N/A | Section II |
| $\Psi$ | Alice’s encoder | N/A | Section II |
| $\Gamma$ | Bob’s decoder | N/A | Section II |
| $\Phi$ | James’ estimator | N/A | Section II |
| $\alpha(\Phi)$ | Probability of false alarm | $\alpha(\Phi) = Pr_\Phi(T = 1|T = 0)$ | Section II |
| $\beta(\Phi)$ | Probability of missed detection | $\beta(\Phi) = Pr_\Phi(T = 0|T = 1)$ | Section II |
| $X$ | Codeword | $X \in \{0, 1\}$ | Section II |
| $Z$ | James’ received vector | $Z \in \{0, 1\}^n$ | Section II |
| $S$ | James’ jamming vector | $S \in \{0, 1\}^n$ | Section II |
| $Y$ | Bob’s received vector | $Y \in \{0, 1\}^n$ | Section II |
| $R$ | Rate | $R = (\log N)/n$ | Section II |
| $r$ | Relative throughput | $r = (\log N)/\sqrt{n}$ | Section II |
| $R^\beta(\Delta(n), \epsilon_d)(p, q)$ | Positive throughput region | N/A | Section II |
| $R^\alpha(\Delta(n), \epsilon_d)(p, q)$ | Achievable positive throughput region | N/A | Section II |
| $p_{\text{in}}(Z)$ | Innocent distribution of $Z$ ($T = 0$) | N/A | Section II |
| $p_{\text{in}}(Z)$ | Active distribution of $Z$ ($T = 1$) | N/A | Section II |
| $E_c(p_1(Z))$ | Ensemble-averaged active distribution | N/A | Section II |
| $T_\Psi(q)$ | Weight normalized mutual information | N/A | Section II |
| $T_\Phi(p, q)$ | Weight normalized mutual information | N/A | Section II |
| $A^\Phi$ | $n$-letter typical set of $X$ | N/A | Section VI-A |
| $A^\Psi$ | $n$-letter typical set of $X$ | N/A | Section VI-A |
| $f^{ij}$ | Fraction of pair $(i, j)$ in $(x, y)$ | N/A | Section VI-A |
| $f^{ij}$ | Fraction of pair $(i, j)$ in $(x, z)$ | N/A | Section VI-A |
| $f_X$ | $n$-letter conditionally typical set of $X$ | N/A | Section VI-A |
| $J_X \cup J_Z$ | $n$-letter conditional type class of $X$ | N/A | Section VI-A |
| $J_n$ | Set of typical fractional Hamming weight | N/A | Section VI-A |

### IV. Proof of Theorem

We now show that if $\Delta(n) < \frac{1}{2} \log(n)$, the probability of error is bounded from below by $1 - \max \left\{ \frac{2^\Delta(n)}{N}, \frac{2^\Delta(n)}{N} \right\}$, for some constant $\eta$ independent of $n$. First note that due to the covertness constraint, most of the codewords have Hamming weight $O(\sqrt{n})$, otherwise Alice’s transmission status can be detected by James’ weight-detector. Since James is able to flip $O(n)$ bits, he can apply an oblivious jamming strategy — generate his jamming vector by selecting $O(\sqrt{n})$ codewords. Since the number of possible values of shared key is $2^{\Delta(n)} < O(\sqrt{n})$, he can select codewords in the following way (without loss of generality we assume Alice chooses $x(m_0, k_0)$ — the codeword corresponding to message $m_0$ and shared key $k_0$ — is transmitted):

1. For each value of $k$, James randomly chooses $b = \min \left\{ \frac{O(\sqrt{n})}{2^\Delta(n)}, \frac{N}{2} \right\}$ messages $m_1, m_2, \ldots, m_b$, and use Alice’s encoding function to obtain codewords $x(m_1, k), x(m_2, k), \ldots, x(m_b, k)$.

2. Let $S_k$ be the set with elements $x(m_1, k), x(m_2, k), \ldots, x(m_b, k)$. James’ jamming vector $s$ equals $\oplus_k \oplus_{S_k} x$, i.e., the binary additions of all selected codewords. Note that Bob’s received vector $y$ equals the binary addition of $x(m_0, k_0)$ and $s$.

Now let’s focus on the set $S_{k_0}$. We define a modified set $\hat{S}_{k_0} = S_{k_0} \setminus x(m_0, k_0)$ if $x(m_0, k_0) \in S_{k_0}$, and $\hat{S}_{k_0} = \{x(m_1, k_0), x(m_2, k_0), \ldots, x(m_b, k_0), x(m_0, k_0)\}$ if $x(m_0, k_0) \notin S_{k_0}$. We assume there is an oracle that reveals to Bob the value of $k_0$, the set $S_{k_0}$, and all $S_k$ for $k \neq k_0$ selected by James, and the fact that whether or not Alice’s true codeword in the set $S_{k_0}$. Note that the oracle only strengthens Bob, since he can recover the
received vector from the oracle revealed information. Thus, Bob’s probability of decoding error with the knowledge of the oracle information is no larger than that without it. If \( x(m_0, k_0) \in S_{k_0} \), from Bob’s point of view the true message is uniformly distributed over the set \( S_{k_0} \), since he cannot distinguish the following \((b + 1)\) equally likely events

- \( \mathcal{E}_{m_0} \): Alice transmits \( x(m_0, k_0) \) and James selects \( \{x(m_1, k_0), x(m_2, k_0), \ldots, x(m_b, k_0)\} \).
- \( \mathcal{E}_{m_i} (i \neq 0) \): Alice transmits \( x(m_i, k_0) \) and James selects \( \{x(m_0, k_0), x(m_1, k_0), \ldots, x(m_b, k_0)\} \setminus x(m_i, k_0) \).

Similarly, if \( x(m_0, k_0) \notin S_{k_0} \), from Bob’s point of view the true message is uniformly distributed over the set \( S_{k_0} \).

These imply the probability of decoding error (when \( T = 1 \)) is bounded from below by \( 1 - \max \left\{ \frac{2^{A(n)\eta}}{\sqrt{n}}, \frac{2}{N} \right\} \), for some \( \eta > 0 \).

V. PROOF OF THEOREM

Consider any code \( C \) of relative throughput \( r \) that ensures \((1 - \epsilon_d)\)-covertness and the average probability of error \( P_e \leq \epsilon_n \) (where \( \epsilon_n \to 0 \) as \( n \to \infty \)). We first find an upper bound on the maximum weight of codewords in a suitable subcode and then, use it find a bound on the maximum number of messages that can be reliably decoded.

To get an upper bound in the average weight of codewords, we employ an analysis based on weight detection as in [5, 7]. Specializing [5, Lemma 12] to our setting, we obtain that for any \( \gamma \in (0, 1) \), there exists a subset \( C_\gamma \) of \( C \) such that

1. \( |C_\gamma| \geq \gamma |C| \)
2. there is a constant \( C \) such that
   \[
   \rho(C_\gamma) \leq \frac{1}{\sqrt{n}} \max_{x \in C_\gamma} wt(x) \leq \frac{2\sqrt{q(1-q)}}{1-2q} Q^{-1} \left( \frac{1-\epsilon_d}{2} - \frac{C}{\sqrt{n}} - \gamma \right).
   \]

Next, consider the jamming strategy \( p_{S|Z,C}^{(m)} \) described as follows. For each \( i \in \{1, 2, \ldots, n\} \), James does not flip bit \( X_i \) if the corresponding \( Z_i \) he observes equals 0, and flips with probability approximately \( \frac{p}{q} \) if the corresponding \( Z_i \) he observes equals 1. This ensures that his bit-flips are stochastically distributed in the support of the \( Z \) vector. Since the \( Z \) vector is correlated with Alice’s transmission \( X \) via a \( BSC(q) \), this ensures that James’ jamming vector \( S \) is likelier to flip 1’s in \( X \) to 0’s, than it is to flip 0’s in \( X \) to 1’s.

More precisely, let \( \nu = n^{-1/3} \) be a slackness parameter. For any \( i \in \{1, 2, \ldots, n\} \),

\[
S_i = \begin{cases} 
0, & \text{with probability } 1 \text{ if } Z_i = 0, \\
0, & \text{with probability } 1 - \frac{(1-\nu)p}{q} \text{ if } Z_i = 1, \\
1, & \text{with probability } \frac{(1-\nu)p}{q} \text{ if } Z_i = 1, 
\end{cases}
\]

Note that generating \( S \) in the i.i.d. manner specified above may in general result in James’ exceeding his jamming budget \( pn \). However, by setting the slackness parameter \( \nu = n^{-1/3} \), we ensure with probability at least \( 1 - \exp(-\mathcal{O}(n^{1/3})) \), the Hamming weight of \( S \) is bounded from above by \( pn \). This jamming strategy induces a Binary Asymmetric Channel from Alice to Bob with channel transition probabilities

\[
\begin{align*}
p_{Y|X}(Y_i = 0|X_i = 0) &= 1 - p(1-\nu), & p_{Y|X}(Y_i = 1|X_i = 0) &= p(1-\nu), \\
p_{Y|X}(Y_i = 0|X_i = 1) &= \frac{(1-q)p}{q} (1-\nu), & p_{Y|X}(Y_i = 1|X_i = 1) &= 1 - \frac{(1-q)p}{q} (1-\nu).
\end{align*}
\]

Note that \( p_{Y|X}(Y_i = 0|X_i = 1) > p_{Y|X}(Y_i = 1|X_i = 0) \) since \( q < 1/2 \) — i.e., as mentioned above, the probability of a bit-flip is higher when \( X_i = 1 \), than when \( X_i = 0 \).

Suppose now that Alice uses the code \( C_\gamma \) with \( \gamma = \max\{\sqrt{\eta}, \exp(-n^{1/2-\epsilon})\} \) for some \( \epsilon > 0 \). Note that the average probability of error with \( C_\gamma \), denoted by \( \epsilon_n' \), is bounded from above as

\[
\epsilon_n' \leq \epsilon_n / \gamma = \max\{\sqrt{\eta}, \exp(-n^{1/2-\epsilon})\},
\]

since \( |C_\gamma| \leq |C| \epsilon_n \) if Bob simply employs the decoding rule for \( C \). Note that \( \epsilon_n' \to 0 \) as \( n \to \infty \). Let \( \hat{X} \sim \text{Bernoulli}(\rho(C)/\sqrt{n}) \) and let \( Y \) be the random variable corresponding to the output of the above BAC with input \( \hat{X} \). By standard information inequalities, we bound the relative throughput of \( C_\gamma \) as follows.

\[
r\sqrt{n} - \log \gamma = H(M) = I(M; YK) + H(M|YK)
\]
Inequality (7) follows from the Fano’s inequality. Inequality (8) holds since $I(M; Y | K) = I(M; K) + I(M; Y) - I(K; Y)$. Similarly, inequality (9) holds since $I(Y | K) = I(Y; K) - I(Y; K)$. Inequality (10) is due to the Data processing inequality. (11) is due to the concavity of mutual information with respect to the marginal distributions. Hence we have

$$r \leq \frac{1}{1 - \epsilon'_n} \sqrt{n} I(\bar{X}; \bar{Y}) + \frac{1 + \log \gamma}{(1 - \epsilon'_n) \sqrt{n}},$$

where $\frac{1 + \log \gamma}{(1 - \epsilon'_n) \sqrt{n}}$ goes to zero for sufficiently large $n$, since $\gamma = \max\{\sqrt{n}, \exp(-n^{\frac{1}{2} - \epsilon})\}$. The mutual information $I(\bar{X}; \bar{Y})$ can be approximated as

$$\sqrt{n} I(\bar{X}; \bar{Y}) = \sqrt{n} (H(\bar{Y}) - H(\bar{Y} | \bar{X})) $$

$$= \rho(C_\gamma) \frac{(1 - p)(q - p + pq)}{q} \log \left( \frac{(1 - p)(q - p + pq)}{p^2(1 - q)} \right) + \rho(C_\gamma) \log \left( \frac{q - p + pq}{pq} \right),$$

$$n \rightarrow \infty \quad t(q, \epsilon_d) \cdot I_B(p, q).$$

Hence, we obtain that $r \leq t(q, \epsilon_d) \cdot I_B(p, q)$. Note that $I_B(p, q) = 0$ when $p = q$, and thus, $r = 0$ whenever $p \geq q$.

VI. PROOF OF THEOREM 3

As presented in Theorem 3 if $I_f(q) < I_B(p, q)$, the optimal throughput $t(q, \epsilon_d) \cdot I_B(p, q)$ is achievable with only $O(\log(n))$ bits of shared key. We first introduce important definitions and our coding schemes in subsections VI-A and VI-B respectively, and then formally present the detailed proofs in subsections VI-C and VI-D. In subsection VI-E we also discuss how to achieve the optimal throughput $t(q, \epsilon_d) \cdot I_B(p, q)$ with a large amount of shared key when $(p, q)$ satisfies $I_f(q) > I_B(p, q) > 0$.

A. Definitions of typical sets and type classes

The fractional Hamming weight of $x$ and $z$ are respectively denoted by

$$f^r_x(x) \triangleq \frac{w_H(x)}{n},$$

$$f^r_z(z) \triangleq \frac{w_H(z)}{n}.$$

Since the codeword $X$ is generated i.i.d. according to Bernoulli($\rho$), we define the $n$-letter typical set of $X$ as

$$\mathcal{A}_X \triangleq \{ x \in \{0, 1\}^n : f^r_x(x) \in [\rho \cdot (1 + \Delta^r_1)] \}.$$  

As mentioned earlier, the ensemble-average active distribution of $Z$ is a Binomial($n, \rho \ast q$) distribution, where the binary convolution $\rho \ast q \triangleq \rho(1 - q) + q(1 - \rho)$. Thus, we define the $n$-letter typical set of $Z$ when Alice is transmitting ($T = 1$) as

$$\mathcal{A}_Z \triangleq \{ z \in \{0, 1\}^n : f^r_z(z) \in [(\rho \ast q) \cdot (1 + \Delta^r_1)] \}.$$  

By choosing $\Delta^r_1$ and $\Delta^r_\epsilon$ carefully, we ensure the typical sets $\mathcal{A}_X$ and $\mathcal{A}_Z$ are high probability set, and are also as “narrow” as possible. Throughout this work we set $\Delta^r_1 = 1$ and $\Delta^r_\epsilon = n^{-1/4}$. Next, we denote the fraction of pair $(i, j)$ in $(x, y)$ by

$$f^{xy}_{ij}(x, y) \triangleq \frac{|k \in \{1, \ldots, n\} : (x_k, y_k) = (i, j)|}{n},$$

where $i \in \{0, 1\}$ and $j \in \{0, 1\}$.
and similarly, the fraction of pair \((i, j)\) in \((x, z)\) is denoted by
\[
 f_{ij}^{xz}(x, z) \triangleq \frac{|k \in \{1, \ldots, n\} : (x_k, z_k) = (i, j)|}{n}, \quad \text{where } i \in \{0, 1\} \text{ and } j \in \{0, 1\}.
\]

(18)

Moreover, given a fixed \(z\), we define the \(n\)-letter conditionally typical set of \(X\) as
\[
 A_{X|z} \triangleq \left\{ x \in \{0, 1\}^n : \left. f_{10}^{xz}(x, z) \in \left[ p q (1 \pm \Delta_{10}^{xz}) \right] \cap f_{11}^{xz}(x, z) \in \left[ p (1 - q) (1 \pm \Delta_{11}^{xz}) \right] \right\}.
\]

(19)

and the \(n\)-letter conditional type class of \(X\) as
\[
 T_{X|(f_{10}^{xz}, f_{11}^{xz})} \triangleq \left\{ x \in \{0, 1\}^n : \left. f_{10}^{xz}(x, z) = (1, 0) \right\} = n f_{10}^{xz} \right\},
\]

(20)

where \(\Delta_{10}^{xz} = \Delta_{11}^{xz} = n^{-1/8}\). It is worth noting that the conditionally typical set can be represented as the union of many conditional type classes, i.e.,
\[
 A_{X|z} = \bigcup_{(f_{10}^{xz}, f_{11}^{xz}) \in \mathcal{F}^xz_n} T_{X|(f_{10}^{xz}, f_{11}^{xz})},
\]

(21)

where \(\mathcal{F}^xz_n\) is the set of typical fractional Hamming weight, and is defined as
\[
\mathcal{F}^xz_n \triangleq \left\{ (f_{10}^{xz}, f_{11}^{xz}) : \begin{array}{l}
 f_{10}^{xz} \in \left[ p q (1 \pm \Delta_{10}^{xz}) \right] \\
 f_{11}^{xz} \in \left[ p (1 - q) (1 \pm \Delta_{11}^{xz}) \right].
\end{array} \right\}.
\]

(22)

B. Code design

**Polynomial hash function:** Alice and Bob have \(6 \log(n)\) bits of shared key \(K\), which is partitioned into two equal parts \(K_1\) and \(K_2\). Each part of the key contains \(3 \log(n)\) bits, and hence can be viewed as an element of finite field \(\mathbb{F}_{n^3}\). The message \(M\) can also be partitioned into \(3 \log(n)\) sized small chunks \(M_1, M_2, \ldots, M_l\), where \(l = r \sqrt{n} / (3 \log(n))\). Likewise, each message chunk \(M_i\) is also viewed as an element of \(\mathbb{F}_{n^3}\). The polynomial hash function is defined as
\[
 h_K(M) = K_2 + \sum_{i=1}^{l} K_1 M_i,
\]

(23)

where the additions and the multiplications are operated over \(\mathbb{F}_{n^3}\). Note that this usage of shared key is distinct from the manner in which shared key is used in a wiretap secrecy setting. In particular, in a wiretap secrecy setting, it is highly unlikely that a single codeword could correspond to many message-key pairs, while in our constructions, each codeword corresponds to multiple different message-key pairs. This property is critical since it ensures part of the shared key (\(K_1\) in this work) is uniformly distributed from James’ perspective even if he gains some information from his received vector, and this uniformity helps us to analyze the list decoding argument.

**Codebook generation:** Let the relative throughput \(r = t(q, \epsilon_d) I_B(p, q) - \delta\), where \(\delta > 0\). For each message-hash pair \((m, h) \in \{0, 1\}^r \sqrt{n} \times \{0, 1\}^{3 \log(n)}\), we generate a length-\(n\) codeword \(x_{mh}\) according to \(p_X \sim \text{i.i.d. Bernoulli}(\rho)\). For different message-hash pairs, the codewords are generated independently. The codebook is a collection of \(x_{mh}, \forall (m, h)\).

**Encoder:** To transmit a message \(m\), Alice computes a hash \(h_k(m)\) using the polynomial hash function and the shared key \(k\). She then encodes the pair \((m, h_k(m))\) to the codeword \(x_{mh_k(m)}\). In the following we abbreviate \((m, h_k(m))\) and \(x_{mh_k(m)}\) as \((m, h(m))\) and \(x_{mh}(m)\) respectively.

**Decoding rule:** Given a received vector \(y\), the list decoder \(\mathcal{L}(y)\) contains all the codewords satisfying the following constraints:
\[
 \mathcal{L}(y) \triangleq \left\{ x : \begin{array}{l}
 n f_{10}^{xy}(x, y) < \rho n \left( \frac{p(1-q)}{q} \right) (1+\epsilon_1) \\
 n f_{11}^{xy}(x, y) > \rho n \left( 1 - \frac{p(1-q)}{q} \right) (1-\epsilon_2)
\end{array} \right\}.
\]

(24)

In this work we set \(\epsilon_1 = \frac{1}{\log(n)}\) and \(\epsilon_2 = \frac{p - pq}{(q-p+pq) \log(n)}\), and explain the reason for such choices in subsection VI-D. The decoding rule is as follows:

1) Output all the codewords satisfying the list decoding rule (24) to \(\mathcal{L}(y)\).
2) For each $x_{mh} \in \mathcal{L}(y)$, keep it in the list if $x_{mh}$ is consistent with the shared key $k$, i.e., $h = k_2 + \sum_{i=1}^{k_1} m_i$, and throw it out otherwise.

3) Decode $\hat{M} = m$ if there is exactly one codeword $x_{mh(m)}$ left in the list. Decode $\hat{M} = 0$ if there is no codeword left in the list. Declare an error if there is more than one codeword left in the list.

**Decoding error events:** When Alice is active ($T = 1$), the three possible error events are as follows (without loss of generality, suppose $\hat{M} = m$ and hence $x_{mh(m)}$ is transmitted):

- $E_1$: The transmitted codeword $x_{mh(m)}$ does not fall into the list $\mathcal{L}(y)$.
- $E_2$: The number of codewords $x' \neq x_{mh(m)}$ falling into the list $\mathcal{L}(y)$ is greater than $n^2$.
- $E_3$: There exists $x' \neq x_{mh(m)}$ such that $x' \in \mathcal{L}(y)$ and $x'$ is consistent with $k$.

Generally speaking, the list decoder should contain the correct codeword $x_{mh(m)}$, and also keep the size as small as possible (no larger than $n^2$). The corresponding error events are $E_1$ and $E_2$. Secondly, even if the list decoder does not make an error, one still need to worry the situation in which more than one codeword in $\mathcal{L}(y)$ is consistent with the shared key $k$. In particular, the true codeword $x_{mh(m)}$ is definitely consistent with $k$, so we hope none of the other codewords $x' \neq x_{mh(m)}$ are consistent with $k$, which corresponds to $E_3$.

When Alice is silent ($T = 0$), a decoding error occurs if there exists a codeword $x \in \mathcal{L}(y)$ such that $x$ is consistent with $k$, hence the two possible error events are

- $E_4$: The total number of codewords $x \in \mathcal{C}$ falling into the list $\mathcal{L}(y)$ is greater than $n^2$.
- $E_5$: There exists $x \in \mathcal{L}(y)$ such that $x$ is consistent with $k$.

**C. Proof of covertness**

The proof of covertness essentially connects to the analysis of the distributions of James’ channel outputs $\mathbf{Z}$. Recall that by Definition 1 the code is $(1 - \epsilon_d)$-covert if there is no estimator $\hat{\Phi}$ such that $\alpha(\Phi) + \beta(\Phi) < 1 - \epsilon_d$. Let $p_0(\mathbf{Z})$ be the $n$-letter innocent distribution of James’ channel output $\mathbf{Z}$ when Alice is silent ($T = 0$), and $p_1(\mathbf{Z})$ be the $n$-letter active distribution of James’ channel output $\mathbf{Z}$ when Alice is transmitting ($T = 1$). A standard statistical arguments \[41\] shows that the optimal estimator $\hat{\Phi}$ satisfies $\alpha(\hat{\Phi}) + \beta(\hat{\Phi}) = 1 - \mathbb{V}(p_0(\mathbf{Z}), p_1(\mathbf{Z}))$, where $\mathbb{V}(\cdot, \cdot)$ is the variational distance between two distributions.

**Claim 1.** To prove $(1 - \epsilon_d)$-covertness, it suffice to show $\mathbb{V}(p_0(\mathbf{Z}), p_1(\mathbf{Z})) \leq \frac{1}{2} \sum_x |p_0(x) - p_1(x)| \leq \epsilon_d$.

The $n$-letter innocent distribution $p_0(\mathbf{Z})$ is a Binomial($n, q$) distribution, with

$$p_0(z) \triangleq q^{wt_H(z)}(1 - q)^{(n - wt_H(z))}, \forall z \in \{0, 1\}^n.$$  \tag{25}

The $n$-letter active distribution $p_1(\mathbf{Z})$ is averaged over all the codewords, and is denoted by

$$p_1(z) \triangleq \sum_{x \in \mathcal{C}} p(x)p_{Z|x}(z|x) = \frac{1}{|\mathcal{C}|} \sum_{x \in \mathcal{C}} q^{d_H(x,z)}(1 - q)^{(n - d_H(x,z))}, \forall z \in \{0, 1\}^n,$$  \tag{26}

where $d_H(\cdot, \cdot)$ is the Hamming distance between two vectors. Note that each of the codeword in $\mathcal{C}$ is equally likely to be transmitted, i.e., $p(x) = 1/|\mathcal{C}|$. Equation (26) follows from the total probability theorem and the fact that $p_{Z|x}$ depends only on the Hamming distance between $x$ and $z$. Note that the active distribution $p_1(\mathbf{Z})$ has complicated dependence on the codebook $\mathcal{C}$, hence for the purpose of analysis, we also define an $n$-letter ensemble-averaged active distribution $E_\mathcal{C}(p_1(\mathbf{Z}))$ as

$$E_\mathcal{C}(p_1(\mathbf{Z})) \triangleq \sum_{\mathcal{C}} P(\mathcal{C}) \sum_{x \in \mathcal{C}} p(x)p_{Z|x}(z|x) = (\rho + q)^{wt_H(z)}(1 - \rho * q)^{(n - wt_H(z))}, \forall z \in \{0, 1\}^n,$$  \tag{27}

The distribution $E_\mathcal{C}(p_1(\mathbf{Z}))$ is essentially the active distribution $p_1(\mathbf{Z})$ averaged over all the possible codebooks. As derived in \[27\], $E_\mathcal{C}(p_1(\mathbf{Z}))$ is a Binomial($n, \rho + q$) distribution, which can be viewed as passing the all-zero codeword through two successive BSCs with crossover probabilities respectively $\rho$ and $q$. For convenience, we abbreviate $p_0(\mathbf{Z}), p_1(\mathbf{Z}), E_\mathcal{C}(p_1(\mathbf{Z}))$ as $p_0, p_1, E_\mathcal{C}(p_1)$ respectively, when the meaning is clear from the context.

To prove $(1 - \epsilon_d)$-covertness, it suffices to show $\mathbb{V}(p_0, p_1) \leq \epsilon_d$, by Claim 1. Applying the triangle inequality, we obtain that $\mathbb{V}(p_0, p_1) \leq \mathbb{V}(p_0, E_\mathcal{C}(p_1)) + \mathbb{V}(E_\mathcal{C}(p_1), p_1)$. Lemmas 1 and 2 below respectively establish that

- $\lim_{n \to \infty} \mathbb{V}(p_0, E_\mathcal{C}(p_1)) \leq \epsilon_d$.
- $\lim_{n \to \infty} \mathbb{V}(E_\mathcal{C}(p_1), p_1) = 0$ with high probability over the code design.
Lemma 1. By setting the “code-weight design” parameter \( t = \frac{2\sqrt{q(1-q)}}{1-2q} Q^{-1} \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \), the variational distance between the innocent distribution \( p_0 \) and the ensemble-averaged active distribution \( \mathbb{E}_C(p_1) \) is given as

\[
\lim_{n \to \infty} \mathbb{V}(p_0, \mathbb{E}_C(p_1)) = \epsilon_d.
\]

Proof: As shown in [44, Lemma 9], the variational distance can be expressed as

\[
\mathbb{V}(\mathbb{E}_C(p_1), p_1) = 1 - 2Q \left( \frac{t(1-2q)}{2\sqrt{q(1-q)}} \right) + \mathcal{O} \left( \frac{1}{\sqrt{n}} \right),
\]

by applying the Berry-Esseen theorem. Since the “code-weight” design parameter \( t \) is set to equal \( \frac{2\sqrt{q(1-q)}}{1-2q} Q^{-1} \left( \frac{1}{2} - \frac{\epsilon}{2} \right) \), we obtain

\[
\lim_{n \to \infty} \mathbb{V}(p_0, \mathbb{E}_C(p_1)) = \epsilon_d.
\]

In addition to studying the averaged performance of the random code ensemble, we also need to guarantee the specific performance of a randomly chosen code does not deviate the average performance “too much”, i.e., with high probability over the code design, \( \mathbb{V}(\mathbb{E}_C(p_1), p_1) \) goes to 0 as \( n \) goes to infinity. Note that

\[
\mathbb{V}(\mathbb{E}_C(p_1), p_1) = \frac{1}{2} \sum_{z \in \{0,1\}^n} \left| \sum_{C} p_C(C) \sum_{x \in C} p_{Z|X}(z|x)p(x) - \sum_{x \in C} p_{Z|X}(z|x)p(x) \right|
\]

\[
\leq \frac{1}{2} \sum_{z \in A_Z} \left| \sum_{C} p_C(C) \sum_{x \in C \cap A_X|_z} p_{Z|X}(z|x)p(x) - \sum_{x \in C \cap A_X|_z} p_{Z|X}(z|x)p(x) \right|
\]

\[
+ \frac{1}{2} \sum_{z \in A_Z} \left| \sum_{C} p_C(C) \sum_{x \in C \cap A_X|_z} p_{Z|X}(z|x)p(x) - \sum_{x \in C \cap A_X|_z} p_{Z|X}(z|x)p(x) \right|
\]

\[
\leq \frac{1}{2} \sum_{z \in A_Z} \left| \sum_{C} p_C(C) \sum_{x \in C \cap A_X|_z} p_{Z|X}(z|x)p(x) - \sum_{x \in C \cap A_X|_z} p_{Z|X}(z|x)p(x) \right|
\]

\[
+ \frac{1}{2} \mathbb{E}_C \left( \sum_{z \in A_Z} \sum_{x \in C \cap A_X|_z} p_{Z|X}(z|x)p(x) \right) + \frac{1}{2} \mathbb{E}_C \left( \sum_{z \notin A_Z} p_1(z) \right)
\]

\[
+ \frac{1}{2} \sum_{z \in A_Z} \sum_{x \in C \cap A_X|_z} p_{Z|X}(z|x)p(x) + \frac{1}{2} \sum_{z \notin A_Z} p_1(z).
\]

The variational distance \( \mathbb{V}(\mathbb{E}_C(p_1), p_1) \) is divided into three terms — term (31) corresponds to the variational distance contributed by typical \( z \) and conditionally typical \( x \), term (32) corresponds to the variational distance contributed by typical \( z \) and conditionally atypical \( x \), and term (33) corresponds to the variational distance contributed by atypical \( z \). Since typical \( z \) and conditionally typical \( x \) covers almost all the probability mass, a primary goal is to bound term (34), which is the same as term (31). Also, we simply use triangle inequalities to bound the atypical events in (35) and (36). Before bounding these terms, we first prove the following claim, which is widely used throughout this work.

Claim 2. For any typical \( z \) and any conditionally typical type class \( T_{X|z}(f_{10}^{z}, f_{11}^{z}) \), the expected number of codewords falling into \( T_{X|z}(f_{10}^{z}, f_{11}^{z}) \) is super-polynomially large, i.e., \( \mathbb{E}_C \left( |x : x \in \mathcal{C} \cap T_{X|z}(f_{10}^{z}, f_{11}^{z})| \right) = 2^{\Omega(\sqrt{\pi})} \), where \( c = r - t \cdot I_f(q) > 0 \). With probability at least \( 1 - \exp(-2^{\Omega(\sqrt{n})}) \) over the code design, a randomly chosen code \( C \) satisfies

\[
\left| \left| x : x \in \mathcal{C} \cap T_{X|z}(f_{10}^{z}, f_{11}^{z}) \right| - \mathbb{E}_C \left( \left| x : x \in \mathcal{C} \cap T_{X|z}(f_{10}^{z}, f_{11}^{z}) \right| \right) \right| < \exp(-n^{\frac{1}{2}}) \mathbb{E}_C \left( \left| x : x \in \mathcal{C} \cap T_{X|z}(f_{10}^{z}, f_{11}^{z}) \right| \right).
\]
Proof: Note that the expected number of codewords falling into a type class equals the probability of a codeword falling into the type class times the total number of codewords. By a counting argument, the probability is given as

\[
\Pr_c \left( X \in T_{X|z}(f_{T0}^{zz}, f_{T1}^{zz}) \right) = \left( \frac{n(f_{T0}^{zz} + f_{T1}^{zz})}{n f_{T1}^{zz}} \right) \rho^{n f_{T1}^{zz}} (1 - \rho)^{n f_{T0}^{zz}}.
\]

We first bound the term in (34) (corresponding to typical \( z \), \( \rho = 1 - 2q \log \left( \frac{1 - q}{q} \right) + O(n^{-3/4}), \( D(\mathbb{X} | \rho) = O(1). \)

As first shown in [7], for any typical \( z \) and any conditionally typical \( x \), \( i.e., (f_{T0}^{zz}, f_{T1}^{zz}) \in F_z^{zz} \),

\[
\mathbb{P}(x; z) = \rho(1 - 2q) \log \left( \frac{1 - q}{q} \right) + O(n^{-3/4}),
\]

\[
\mathbb{D}(x \parallel \rho) = O(1).
\]

Therefore, the expected number of codewords falling into \( T_{X|z}(f_{T0}^{zz}, f_{T1}^{zz}) \) is bounded from below as

\[
\mu_1 \triangleq \mathbb{E}_C \left| \{ x : x \in C \cap T_{X|z}(f_{T0}^{zz}, f_{T1}^{zz}) \} \right| \geq 2^{r \sqrt{n}} \cdot 2^{-(1 - 2q) \log((1 - q)/q) \sqrt{n} + o(\sqrt{n})}
\]

Since \( r > I_J(q) \) in the positive throughput regime, the expected number of codewords falling into is at least \( 2^{r \sqrt{n}} \), where \( c = r - I_J(q) > 0. \) By using the Chernoff bound, we prove that with probability at least \( 1 - 2 \exp \left( -\frac{1}{3} \epsilon_A^2 \mu_1 \right) \), a randomly chosen code satisfies

\[
\mathbb{E}_C \left| \{ x : x \in C \cap T_{X|z}(f_{T0}^{zz}, f_{T1}^{zz}) \} \right| < \epsilon_A \mathbb{E}_C \left| \{ x : x \in C \cap T_{X|z}(f_{T0}^{zz}, f_{T1}^{zz}) \} \right|.
\]

By setting \( \epsilon_A = \exp(-n^{3/4}) \), we complete the proof of Claim 2.

Lemma 2. With probability at least \( 1 - \exp(-2O(\sqrt{n})) \) over the code design, a randomly chosen code \( C \) ensures that \( \Pr(\mathbb{E}_C(p_1), p_1 \leq \exp \left( -O \left( n^{3/4} \right) \right). \)

Proof: We first bound the term in (34) (corresponding to typical \( z \) and conditionally typical \( x \)) in the following. The key step is to decompose the the conditionally typical set \( A_{X|z} \) into conditionally typical type classes \( T_{X|z}(f_{T0}^{zz}, f_{T1}^{zz}) \), where \( (f_{T0}^{zz}, f_{T1}^{zz}) \in F_z^{zz} \). For any typical \( z \), we have

\[
\sum_{C} \Pr(C) \sum_{x \in C \cap A_{X|z}} p_{Z|X}(z|x) p(x) \leq \sum_{C} \Pr(C) \sum_{(f_{T0}^{zz}, f_{T1}^{zz}) \in F_z^{zz}} \sum_{x \in C \cap T_{X|z}(f_{T0}^{zz}, f_{T1}^{zz})} p_{Z|X}(z|x) p(x)
\]

\[
= \sum_{C} \Pr(C) \sum_{(f_{T0}^{zz}, f_{T1}^{zz}) \in F_z^{zz}} p_{Z|X}(z|x) p(x) \cdot \mathbb{E}_C \left| \{ x : x \in C \cap T_{X|z}(f_{T0}^{zz}, f_{T1}^{zz}) \} \right|
\]

The reason why we make the type classes decomposition in (49) is that all the codewords in the same type class \( T_{X|z}(f_{T0}^{zz}, f_{T1}^{zz}) \) have same Hamming distance from \( z \), hence the channel transition probability \( p_{Z|X}(z|x) \) is the
same, and the summation problem can be converted to a counting problem in (50). Combining (51) and (34), we have

$$\frac{1}{2} \sum_{z \in \mathcal{A}_z} \left| \sum_{c} p_c(C) \sum_{x \in \mathcal{C} \cap \mathcal{A}_x} p_{Z|X}(z|x)p(x) - \sum_{c} \sum_{x \in \mathcal{C} \cap \mathcal{A}_x} p_{Z|X}(z|x)p(x) \right|$$

(52)

$$= \frac{1}{2} \sum_{z \in \mathcal{A}_z} \sum_{(f_i^{(z)}, f_i^{(z)}) \in \mathcal{F}_z} p_{Z|X}(z|x)p(x) \cdot \left| \{x : x \in \mathcal{C} \cap \mathcal{T}_r \mathcal{C} | (f_i^{(z)}, f_i^{(z)})\} - \mathbb{E}_C \left[ \{x : x \in \mathcal{C} \cap \mathcal{T}_r \mathcal{C} | (f_i^{(z)}, f_i^{(z)})\} \right] \right|$$

(53)

$$< \frac{1}{2} \sum_{z \in \mathcal{A}_z} \sum_{(f_i^{(z)}, f_i^{(z)}) \in \mathcal{F}_z} p_{Z|X}(z|x)p(x) \cdot \exp (-n^{\frac{1}{2}}) \cdot \mathbb{E}_C \left[ \{x : x \in \mathcal{C} \cap \mathcal{T}_r \mathcal{C} | (f_i^{(z)}, f_i^{(z)})\} \right]$$

(54)

$$= \frac{\exp (-n^{\frac{1}{2}})}{2} \sum_c p_c(C) \sum_{z \in \mathcal{A}_z} \sum_{x \in \mathcal{C} \cap \mathcal{T}_r \mathcal{C} | (f_i^{(z)}, f_i^{(z)})} p_{Z|X}(z|x)p(x)$$

(55)

$$\leq \frac{\exp (-n^{\frac{1}{2}})}{2} \sum_c p_c(C) \sum_{z \in \mathcal{A}_z} \sum_{x \in \mathcal{C}} p_{Z|X}(z|x)p(x)$$

(56)

$$= \frac{\exp (-n^{\frac{1}{2}})}{2}$$

(57)

Inequality (54) follows from Claim 3 and holds with probability at least $1 - \exp (-2^{O(\sqrt{n})})$. Note that in order to obtain (54), one needs to take a union bound over exponentially many typical $z$ and conditionally typical type class $\mathcal{T}_r \mathcal{C} | (f_i^{(z)}, f_i^{(z)})$, which is valid since $1 - \exp (-2^{O(\sqrt{n})})$ is super-exponentially close to one. We exchange the order of summation in equation (55), and relax the constraints on $z$ and $x$ to obtain inequality (56).

After showing that the variational distance $\mathbb{V}(\mathbb{E}_C(p_1), p_1)$ contributed by typical $z$ and conditionally typical $x$ goes to 0 asymptotically, we then bound the atypical events (as shown in (55) and (56)) in Claims 3-5, with proofs deferred to Appendix A.

Claim 3 (Second term in (55)). $\frac{1}{2} \mathbb{E}_C \left( \sum_{z \notin \mathcal{A}_z} p_1(z) \right) \leq \exp \left( -O(\sqrt{n}) \right)$.

Claim 4 (First term in (55)). $\frac{1}{2} \mathbb{E}_C \left( \sum_{z \in \mathcal{A}_z} \sum_{x \notin \mathcal{C} \cap \mathcal{A}_x} p_{Z|X}(z|x)p(x) \right) \leq \exp \left( -O \left( n^{\frac{1}{2}} \right) \right)$.

Claim 5 (Term in (56)). With probability at least $1 - \exp (-2^{O(\sqrt{n})})$ over the code design, a randomly chosen code $C$ satisfies

$$\frac{1}{2} \sum_{z \in \mathcal{A}_z} \sum_{x \in \mathcal{C} \cap \mathcal{A}_x} p_{Z|X}(z|x)p(x) + \frac{1}{2} \sum_{z \notin \mathcal{A}_z} p_1(z) \leq \exp \left( -O \left( n^{\frac{1}{2}} \right) \right).$$

(58)

Equipped with (57) and Claims 3-5, we are able to bound the variational distance between $p_1$ and $\mathbb{E}_C(p_1)$ from above as

$$\mathbb{V}(\mathbb{E}_C(p_1), p_1)$$

(59)

$$\leq \frac{1}{2} \sum_{z \in \mathcal{A}_z} \left| \sum_{c} p_c(C) \sum_{x \in \mathcal{C} \cap \mathcal{A}_x} p_{Z|X}(z|x)p(x) - \sum_{c} \sum_{x \in \mathcal{C} \cap \mathcal{A}_x} p_{Z|X}(z|x)p(x) \right|$$

(60)

$$+ \frac{1}{2} \mathbb{E}_C \left( \sum_{z \in \mathcal{A}_z} \sum_{x \in \mathcal{C} \cap \mathcal{A}_x} p_{Z|X}(z|x)p(x) \right) + \frac{1}{2} \mathbb{E}_C \left( \sum_{z \notin \mathcal{A}_z} p_1(z) \right)$$

(61)

$$+ \frac{1}{2} \sum_{z \in \mathcal{A}_z} \sum_{x \in \mathcal{C} \cap \mathcal{A}_x} p_{Z|X}(z|x)p(x) + \frac{1}{2} \sum_{z \notin \mathcal{A}_z} p_1(z).$$

(62)

w.h.p. $\frac{1}{2} \exp (-n^{\frac{1}{2}}) + \exp (-O(\sqrt{n})) + \exp \left( -O \left( n^{\frac{1}{2}} \right) \right) + \exp \left( -O \left( n^{\frac{1}{2}} \right) \right)$

(63)

$$= \exp \left( -O \left( n^{\frac{1}{2}} \right) \right),$$

(64)
where equations (60)-(62) are derived in (34)-(36), and equation (63) holds with probability at least \(1 - \exp(-2^{O(\sqrt{n})}).\) This completes the proof of Lemma 2, as well as the proof of covertexness.

D. Proof of reliability

Suppose a codeword \(x_{mh(m)}\) (corresponding to the message \(m\)) is transmitted. After passing through a BSC\((\eta)\), a noisy version \(z\) of \(x_{mh(m)}\) is received by James. We now consider the error events \(E_1, E_2, E_3\) when Alice is active \((T = 1)\). The following three lemmas establish the reliability.

**Lemma 3.** With probability at least \(1 - \exp(-O(n^{1/4}))\) over the code design, a randomly chosen code \(C\) ensures that the error event \(E_1\) occurs with probability at most \(3\exp(-n^{1/8})\).

**Lemma 4.** With probability at least \(1 - \exp(-O(\sqrt{n}))\) over the code design, a randomly chosen code \(C\) ensures that the error event \(E_2\) occurs with probability at most \(\exp(-n^{1/4})\).

**Lemma 5.** The error event \(E_3\) occurs with probability (over the common randomness \(K\)) at most \(O\left(\frac{1}{\sqrt{n} \log(n)}\right)\).

1) Proof of Lemma 3: Recall that the error event \(E_1\) occurs if the transmitted codeword \(x_{mh(m)}\) does not fall into the list under the jamming vector \(s\), i.e., \(x_{mh(m)} \notin \mathcal{L}(x_{mh(m)} + s)\). In the following we prove that for any shared key \(k\), the probability of error event \(E_1\) is bounded from above by \(3\exp(-n^{1/8})\), which is a stronger criterion than that averages over all \(k\). Note that for a fixed \(k\), each message \(m\) maps to a unique codeword \(x_{mh(m)}\). The probability of error event \(E_1\) is given as

\[
P_c(1)(C) = \max_{p_0, z, c} \left\{ \sum_m p_M(m) \sum_z p_{z|x}(z|x_{mh(m)}) \sum_s p_{s|z,c}(s|z,c) \cdot I\{x_{mh(m)} \notin \mathcal{L}(x_{mh(m)} + s)\} \right\}
\]

\[
\leq \max_{p_0, z, c} \left\{ \frac{1}{N} \sum_{z \in \mathcal{A}_z} \sum_{m: x_{mh(m)} \in \mathcal{A}_x} p_{z|x}(z|x_{mh(m)}) \sum_s p_{s|z,c}(s|z,c) \cdot I\{x_{mh(m)} \notin \mathcal{L}(x_{mh(m)} + s)\} \right\}
\]

\[
+ \max_{p_0, z, c} \left\{ \frac{1}{N} \sum_{z \in \mathcal{A}_z} \sum_{m: x_{mh(m)} \notin \mathcal{A}_x} p_{z|x}(z|x_{mh(m)}) \sum_s p_{s|z,c}(s|z,c) \cdot I\{x_{mh(m)} \notin \mathcal{L}(x_{mh(m)} + s)\} \right\}
\]

In (67), we separately consider typical \(z\) and atypical \(z\), and conditionally typical codewords and conditionally atypical codewords. Inequality (69) follows since the indicator function \(I(\cdot)\) is always bounded from above by one. The following two claims shows that the two atypical events goes to 0 as \(n\) grows without bound, and we defer the proofs to Appendix B.

**Claim 6 (First term in (69)).** With probability at least \(1 - \exp(-O(n^{1/4}))\) over the code design,

\[
\frac{1}{N} \sum_{z \in \mathcal{A}_z} \sum_{m: x_{mh(m)} \notin \mathcal{A}_x} p_{z|x}(z|x_{mh(m)}) < \exp(-n^{1/8})
\]

**Claim 7 (Second term in (69)).** With probability at least \(1 - \exp(-O(\sqrt{n}))\) over the code design,

\[
\frac{1}{N} \sum_m \sum_{z \notin \mathcal{A}_z} p_{z|x}(z|x_{mh(m)}) < \exp(-n^{1/4})
\]
From now on we consider the typical event — a typical $z$ is received and a conditionally typical $x$ is transmitted. From James’ perspective, all the codewords that are conditionally typical with respect to $z$ are “plausible”. We introduce an oracle which helps James by revealing the type class $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$ that the transmitted codeword $x_{mh(m)}$ lies in. If our coding scheme is proven to be reliable against the stronger adversary, it will also succeed against the original adversary. Given the oracle revealed information, James’ uncertainty is reduced since he knows only the codewords in $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$ are likely to be the transmitted codeword. However, in Claim 11 we show that James’ remaining uncertainty is still very high — the number of codewords in $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$ is super-polynomially large. We skip the proof here since it follows directly from Claim 2.

**Claim 8.** For any typical $z$ and any conditionally typical type class $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$, the expected number of codewords falling into $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$ is super-polynomially large, i.e., $E_C\left(\abs{m : x_{mh(m)} \in T_{X|z}(f^{xz}_{10}, f^{xz}_{11})}\right) = 2^{c\sqrt{n}}$, where $c = r - t \cdot I_f(q) > 0$. With probability at least $1 - \exp(-2^{O(\sqrt{n})})$ over the code design, a randomly chosen code $C$ satisfies

$$\abs{m : x_{mh(m)} \in T_{X|z}(f^{xz}_{10}, f^{xz}_{11})} > \left(1 - \exp(-n^{1/4})\right) \cdot 2^{c\sqrt{n}}.$$ 

Note that the error event $E_1$ happens if the transmitted codeword $x_{mh(m)}$ is not contained in the list. To reliable decode, $x_{mh(m)}$ should be “robust” under James’ malicious jamming vector $s$.

**Definition 3.** A codeword $x$ (transmitted from Alice to Bob) is killed by a jamming vector $s$ if $x$ is pushed out of the list decoder by $s$, i.e., $x \notin \mathcal{L}(x + s)$.

If magically James is able to find a jamming vector $s$ such that each of the codeword in $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$ is killed by $s$, the error event $E_1$ will occur with probability one, since the transmitted codeword belongs to $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$. Fortunately, Claim 8 shows that with high probability over the code design, no matter which $s$ James chooses, only a decaying fraction of codewords in $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$ are killed by $s$. Since the probability of the error event $E_1$ is averaged over all the codewords to be transmitted, and each codeword is transmitted with equal probability a priori, we are able to prove that a decaying fraction of codewords are killed implies the probability of $E_1$ is a decaying function of $n$ (in Claim 10). Finally, one needs to take a union bound over all typical $z$ and conditionally typical type class $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$.

**Claim 9 (Myopic list decoding lemma).** For any typical $z$ and any conditionally typical type class $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$, for any fixed jamming vector $s \in \{0, 1\}^n$, with probability at least $1 - \exp\left(-2^{O(\sqrt{n})}\right)$ over the code design,

$$\abs{m : x_{mh(m)} \in T_{X|z}(f^{xz}_{10}, f^{xz}_{11}) \cap x_{mh(m)} \notin \mathcal{L}(x + s)} < \exp(-n^{1/4}) \cdot 2^{c\sqrt{n}}.$$ 

**Proof:** The key step is to calculate the probability that a randomly generated codeword $x$ falls into the the type class $T_{X|z}(f^{xz}_{10}, f^{xz}_{11})$, and is simultaneously killed by a jamming vector $s$. One can argue that this probability is maximized when the support of $s$ is entirely inside the support of $z$. We now fix a typical $z$ and a worst-case...
jamming vector \( s \) satisfying \( |\text{supp}(z) \cap \text{supp}(s)| = pn \). Recall that a codeword \( x \) falls into the list \( \mathcal{L}(y) \) (or \( \mathcal{L}(x + s) \)) if it satisfies the decoding rule

\[
\begin{cases}
    n f_{10}^{xy}(x, y) < \rho n \left( \frac{p(1-q)}{q} \right)(1 + \varepsilon_1) , \\
    n f_{11}^{xy}(x, y) > \rho n \left( 1 - \frac{p(1-q)}{q} \right)(1 - \varepsilon_2).
\end{cases}
\]  

(70)

Note that \( f_{10}^{xy}(x, y) = f_{11}^{xy}(x, s) \) and \( f_{10}^{xy}(x, y) = f_{10}^{xy}(x, s) \) (as illustrated in Figure 6), hence the constraint in (70) is equivalent to

\[
\begin{cases}
    n f_{11}^{xs}(x, s) < \rho n \left( \frac{p(1-q)}{q} \right)(1 + \varepsilon_1), \\
    n f_{10}^{xs}(x, s) > \rho n \left( 1 - \frac{p(1-q)}{q} \right)(1 - \varepsilon_2).
\end{cases}
\]  

(71)

We further notice that \( f_{10}^{xs}(x, s) = f_{11}^{xs}(x, z) - f_{11}^{xs}(x, s) + f_{10}^{xs}(x, z) \), and \( f_{10}^{xs}(x, z) \), \( f_{11}^{xs}(x, z) \) are tightly concentrated since \( z \in \mathcal{A}_2 \) and \( x \in \mathcal{A}_X \). By setting \( \varepsilon_1 = \frac{1}{\log(n)} \) and \( \varepsilon_2 = \frac{p - p q}{(q - p + pq) \log(n)} \), the constraints in (71) is also equivalent to

\[
\begin{cases}
    n f_{11}^{xs}(x, s) < \rho n \left( \frac{p(1-q)}{q} \right)(1 + \frac{1}{\log(n)}), \\
    n f_{10}^{xs}(x, s) < \rho n \left( \frac{p(1-q)}{q} \right)(1 + \frac{1}{\log(n)}) + O(n^{-1/8}).
\end{cases}
\]  

(72)

Without loss of correctness, we ignore the lower order term \( O(n^{-1/8}) \) in (72) in the following analysis. Let \( i_0 = \rho n \left( \frac{p(1-q)}{q} \right)(1 + \frac{1}{\log(n)}) \) be the minimum amount of intersections between \( x \) and \( s \) such that \( x \) is killed by \( s \). A codeword \( x \) does not fall into the list \( \mathcal{L}(x + s) \) if \( i_0 \leq n f_{11}^{xs}(x, s) \leq n f_{11}^{xs} \). For a randomly generated codewords \( X \), the probability that it falls into the type class \( T_{X|s}(f_{10}^{xs}, f_{11}^{xs}) \) and does not fall into the list \( \mathcal{L}(x + s) \) is bounded from above as

\[
\begin{align*}
    \Pr_X(X \in T_{X|s}(f_{10}^{xs}, f_{11}^{xs}) \cap X \notin \mathcal{L}(X + s)) \\
    = \Pr_X(X \in T_{X|s}(f_{10}^{xs}, f_{11}^{xs})) \cdot \Pr_X(X \notin \mathcal{L}(X + s)|X \in T_{X|s}(f_{10}^{xs}, f_{11}^{xs})) \\
    = 2^{-I_2(q)n} + O(n^{1/4}). \sum_{i=i_0}^{n f_{11}^{xs}} \binom{pn}{i} \binom{n f_{10}^{xs} + f_{11}^{xs} - i}{i} \binom{n f_{10}^{xs}}{n f_{10}^{xs} + f_{11}^{xs}} \\
    = 2^{-I_2(q)n} + O(n^{1/4}). \sum_{i=i_0}^{n f_{11}^{xs}} \binom{pn}{i} \binom{n f_{10}^{xs} + f_{11}^{xs} - i}{i} \binom{n f_{10}^{xs}}{n f_{10}^{xs} + f_{11}^{xs}} \\
    = 2^{-I_2(q)n} + O(n^{1/4}). \
\end{align*}
\]  

(73)
The last step follows from the geometric sequence which means the function \( g(i) \) in the following:

\[
 g(i) = \binom{pn}{i} \left( \frac{f_{11}^{\frac{p+1}{2}}}{f_{11}^{\frac{p+1}{2}} - t} \right).
\]

To find the maximum value of \( g(i) \) when \( 0 \leq i \leq n f_{11}^{\frac{p+1}{2}} \), we calculate the ratio between the two successive terms in the following:

\[
 \frac{g(i+1)}{g(i)} = \frac{\binom{pn}{i+1} \left( \frac{f_{11}^{\frac{p+1}{2}}}{f_{11}^{\frac{p+1}{2}} - 1} \right)}{\binom{pn}{i} \left( \frac{f_{11}^{\frac{p+1}{2}}}{f_{11}^{\frac{p+1}{2}} - 1} \right)} = \frac{(pn-i) \left( n f_{11}^{\frac{p+1}{2}} \right)}{(i+1) \left( n f_{11}^{\frac{p+1}{2}} - i \right)}.
\]

Let \( \phi \triangleq \frac{pn f_{11}^{\frac{p+1}{2}}}{n f_{11}^{\frac{p+1}{2}} + pn - 1} \). It turns out that \( g(i+1)/g(i) > 1 \) when \( i < \phi \), and \( g(i+1)/g(i) < 1 \) when \( i > \phi \), which means the function \( g(i) \) achieves its maximum when \( i = \lfloor \phi \rfloor \). Note that the parameter \( \phi \) itself depends on \( f_{01}^{\frac{p+1}{2}} \) and \( f_{11}^{\frac{p+1}{2}} \), i.e., the particular type class. One can prove that for typical \( z \) and conditionally typical type class \( T_{X|z}(f_{01}^{\frac{p+1}{2}}, f_{11}^{\frac{p+1}{2}}) \), the maximum value of \( \phi \) is always bounded from above as

\[
 \phi \leq \frac{\rho n p (1 - q)}{q} \left( 1 + n^{-1/s} \right) \triangleq \phi_{\text{max}}.
\]

Note that as \( n \) grows without bound, \( i_0 \) is larger than \( \phi_{\text{max}} \), hence \( g(i_0) \) is always smaller than \( g(\phi_{\text{max}}) \). On the other hand, \( g(i_0) \) is always greater than \( g(i) \), for any \( i > i_0 \). We now bound the second term in (78) as

\[
 \frac{\sum_{i=i_0}^{n f_{11}^{\frac{p+1}{2}}} g(i)}{\sum_{j=0}^{n f_{11}^{\frac{p+1}{2}}} g(j)} \leq \frac{\sum_{i=i_0}^{n f_{11}^{\frac{p+1}{2}}} g(i)}{\phi_{\text{max}}} \leq \frac{g(i_0)}{g(\phi_{\text{max}})} \cdot \log(n).
\]

The last step follows from the geometric sequence

\[
 \sum_{i=i_0}^{n f_{11}^{\frac{p+1}{2}}} g(i) = \sum_{i=i_0}^{\infty} g(i) = g(i_0) + g(i_0) \frac{g(i_0 + 1)}{g(i_0)} + \ldots + \frac{g(i_0)}{g(i_0)} \frac{g(i_0 + 1)}{g(i_0)} + \ldots \leq g(i_0) + g(i_0) \frac{g(i_0 + 1)}{g(i_0)} + \ldots + \frac{g(i_0)}{g(i_0)} \frac{g(i_0 + 1)}{g(i_0)} \frac{1}{1 - g(i_0 + 1)/g(i_0)} \leq g(i_0) \cdot \log(n),
\]

where inequality \( 83 \) holds since \( g(i+1)/g(i) \) is monotonically decreasing, and inequality \( 86 \) follows from the fact \( g(i_0 + 1)/g(i_0) \leq 1 - 1/\log(n) \).

To calculate the ratio between \( g(i_0) \) and \( g(\phi_{\text{max}}) \), we need to introduce an interpolation point \( \phi' \triangleq \frac{\rho n p (1 - q)}{q} \left( 1 + \frac{1}{(\log(n))^2} \right) \).

Note that \( g(\phi_{\text{max}}) \geq g(\phi') \) since \( \phi' > \phi_{\text{max}} \) and \( g(i) \) is monotonically decreasing when \( i > \phi_{\text{max}} \). Now we consider the ratio between \( g(i_0) \) and \( g(\phi_{\text{max}}) \):

\[
 \frac{g(i_0)}{g(\phi_{\text{max}})} \leq \frac{g(i_0)}{g(\phi')} = \frac{g(\phi' + 1)}{g(\phi')} \frac{g(\phi' + 2)}{g(\phi')} \frac{g(\phi' + 3)}{g(\phi')} \ldots \frac{g(i_0)}{g(i_0 - 1)} \leq \left( \frac{g(\phi' + 1)}{g(\phi')} \right)^{i_0 - \phi'}.
\]
for some constant $c_1 > 0$. Inequality \[(89)\] follows since $g(\phi' + 1)/g(\phi') \leq 1 - 1/(\log(n))^2$. Using the approximation $\lim_{n \to \infty} (1 + 1/n)^n = 1/e$, as $n$ grows without bound, we obtain

\[
\frac{g(i_0)}{g(\phi_{\text{max}})} \leq e^{-c_1 \sqrt{n}/(\log(n))^3},
\]

By combining \[(78)\], \[(82)\] and \[(91)\], we finally show that

\[
\Pr(X \in \mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz}) \cap X \notin \mathcal{L}(X + s)) \leq 2^{-\left(t - I_J(q) + c_2/(\log(n))^3\right)\sqrt{\pi + O(n^{1/4})}},
\]

where $c_2 = c_1 \ln 2$. Without loss of correctness, we ignore the lower order term to simplify the following analysis. The expected number of codewords falling into the type class $\mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz})$, and is simultaneously killed by $s$ equals

\[
\mu_2 \triangleq E_c (|m : X_{mh}(m) \in \mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz}) \cap X_{mh}(m) \notin \mathcal{L}(X_{mh}(m) + s)|)
\]

\[
= 2^{c \sqrt{\pi}} \cdot \Pr_c (X \in \mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz}) \cap X \notin \mathcal{L}(X + s))
\]

\[
\leq 2^{c \sqrt{\pi}} \cdot \exp\left(-\frac{\epsilon_1 \cdot 2^{-c \sqrt{\pi}}}{\epsilon_2} - 1\right) \mu_2,
\]

where $c = r - t \cdot I_J(q) > 0$. The probability that more than $\epsilon_1 \cdot 2^{-c \sqrt{\pi}}$ messages that falls into the type class $\mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz})$ as well as being killed by $s$ is bounded from above as

\[
\Pr_c (|m : X_{mh}(m) \in \mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz}) \cap X_{mh}(m) \notin \mathcal{L}(X_{mh}(m) + s)| \geq \epsilon_1 \cdot 2^{-c \sqrt{\pi}})
\]

\[
= \Pr_c (|m : X_{mh}(m) \in \mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz}) \cap X_{mh}(m) \notin \mathcal{L}(X_{mh}(m) + s)| \geq \left(1 + \epsilon_1 \cdot 2^{-c \sqrt{\pi}} - 1\right) \mu_2)
\]

\[
\leq \exp\left(-\frac{\epsilon_1 \cdot 2^{-c \sqrt{\pi}} - 1}{\mu_2} - 1\right) \mu_2
\]

\[
\leq \exp\left(-\frac{2^{-c \sqrt{\pi}}}{3} \left(1 - 2^{-c \sqrt{\pi}}/(\log(n))^3\right)\right).
\]

By setting $\epsilon_1 = \exp(-n^{1/4})$, we have

\[
\Pr_c (|m : X_{mh}(m) \in \mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz}) \cap X_{mh}(m) \notin \mathcal{L}(X_{mh}(m) + s)| < \exp(-n^{1/4}) \cdot 2^{-c \sqrt{\pi}} \geq 1 - \exp\left(-2^{O(\sqrt{\pi})}\right).
\]

\(\Box\)

**Claim 10.** With probability at least $1 - \exp(-2^{O(\sqrt{\pi})})$ over the code design, a randomly chosen code $C$ satisfies

\[
\max_{p_{S|z,c}} \left\{ \frac{1}{N} \sum_{z \in A_Z} \sum_{m : X_{mh}(m) \in A_{X|z}} p_{Z|X}(z|X_{mh}(m)) \sum_s p_{S|Z|C}(s|z,c) \cdot \mathbb{1}\{X_{mh}(m) \notin \mathcal{L}(X_{mh}(m) + s)\} \right\} \leq \exp(-n^{1/4} + 1).
\]

**Proof:** By Claims \([9]\) and \([10]\) a randomly chosen code $C$ satisfies

\[
\frac{|m : X_{mh}(m) \in \mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz}) \cap X_{mh}(m) \notin \mathcal{L}(X + s)|}{|m : X_{mh}(m) \in \mathcal{T}_{X|z}(f_{10}^{xz}, f_{11}^{xz})|} \leq \frac{\exp(-n^{1/4}) \cdot 2^{-c \sqrt{\pi}}}{1 - \exp(-n^{1/4}) \cdot 2^{-c \sqrt{\pi}}}
\]

\[
\leq \exp(-n^{1/4} + 1),
\]

with probability at least $1 - \exp(-2^{O(\sqrt{\pi})})$ over the code design. We now turn to analyze the term in \([68]\). For any jamming strategy $p_{S|Z|C}$, we have
\[
\frac{1}{N} \sum_{z \in \mathcal{L}} \sum_{m : x_{mh(m)} \in \mathcal{L}(z)} \sum_{s} p_{z|x} \left( z | x_{mh(m)} \right) \cdot p_{s|z,c} (s | z, c) \cdot I\{x_{mh(m)} \not\in \mathcal{L}(x_{mh(m)} + s)\} \\
= \frac{1}{N} \sum_{z \in \mathcal{L}} \sum_{f_{10}, f_{11} \in F_{n}} \sum_{m : x_{mh(m)} \in \mathcal{T}_{10} (f_{10}, f_{11})} p_{z|x} \left( z | x_{mh(m)} \right) \cdot p_{s|z,c} (s | z, c) \cdot I\{x_{mh(m)} \not\in \mathcal{L}(x_{mh(m)} + s)\} \\
= \frac{1}{N} \sum_{z \in \mathcal{L}} \sum_{f_{10}, f_{11} \in F_{n}} p_{z|x} \left( z | \mathcal{T}_{10} (f_{10}, f_{11}) \right) \sum_{s} p_{s|z,c} (s | z, c) \\
\cdot \left| m : x_{mh(m)} \in \mathcal{T}_{10} (f_{10}, f_{11}) \cap x_{mh(m)} \not\in \mathcal{L}(x_{mh(m)} + s) \right| \\
\leq \exp(-n^{\frac{1}{4}} + 1) \sum_{z \in \mathcal{L}} \sum_{f_{10}, f_{11} \in F_{n}} p_{z|x} \left( z | \mathcal{T}_{10} (f_{10}, f_{11}) \right) \sum_{s} p_{s|z,c} (s | z, c) \cdot \left| m : x_{mh(m)} \in \mathcal{T}_{10} (f_{10}, f_{11}) \right| \\
= \exp(-n^{\frac{1}{4}} + 1) \cdot \frac{1}{N} \sum_{z \in \mathcal{L}} \sum_{f_{10}, f_{11} \in F_{n}} \sum_{m : x_{mh(m)} \in \mathcal{T}_{10} (f_{10}, f_{11})} p_{z|x} \left( z | x_{mh(m)} \right) \\
\leq \exp(-n^{\frac{1}{4}} + 1) \cdot \frac{1}{N} \sum_{m} \sum_{z} p_{z|x} \left( z | x_{mh(m)} \right) \\
= \exp(-n^{\frac{1}{4}} + 1). \\
\] (102) (103) (104) (105) (106) (107) (108)

Inequality 103 follows from 100, and holds with probability at least \(1 - \exp(-O(\sqrt{n}))\) over the code design. Note that in 105 we need to take a union bound over exponentially many \(z, s\) and \((f_{10}^{\pm}, f_{11}^{\pm})\), which is valid since \(1 - \exp(-2^O(\sqrt{n}))\) is super-exponentially large. Equation 106 follows since \(\sum_{s} p_{s|z,c} (s | z, c) = 1\), and inequality 107 is obtained by relaxing the constraints on \(z\) and \(m\). Note that equations 102-108 holds for arbitrary jamming strategy \(p_{s|z,c}\), hence Claim 10 is proved.

By combining Claim 6, Claim 7, and Claim 10, we finally prove that with probability at least \(1 - \exp(-O(n^{1/4}))\) over the code design, a randomly chosen code \(C\) ensures the probability of the error event \(E_1\) is bounded from above as

\[
P_{e_1} (C) \leq \exp(-n^{1/8}) + \exp(-n^{1/4}) + \exp(-n^{1/2} + 1) \leq 3 \exp(-n^{1/8}). \\
\] (109)

This completes the proof of Lemma 3.

2) Proof of Lemma 4

Lemma 4. With probability at least 1 - \(\exp(-O(\sqrt{n})\)) over the code design, a randomly chosen code \(C\) ensures that the error event \(E_2\) occurs with probability at most \(\exp(-n^{1/4})\).

Recall that the error event \(E_2\) occurs if more than \(n^2\) codewords in \(C\) falls into the list \(\mathcal{L}(y)\). In Claim 11 we consider a typical \(x_{mh(m)}\) is transmitted and an arbitrary jamming vector \(s\) with \(wt_H(s) \leq pn\) is generated by James. To bound the total number of codewords falling into the list \(\mathcal{L}(y)\), we focus on the rest of the codebook \(C \setminus x_{mh(m)}\).

Claim 11. Fix a typical transmitted codeword \(x_{mh(m)}\) and a jamming vector \(s\). With probability at least 1 - \(\exp(-O(n^{5/2}))\) over the code design, the number of codewords \(x' \neq x_{mh(m)}\) falling into the list \(\mathcal{L}(x_{mh(m)} + s)\) is bounded from above by \(n^2\).

Proof: The Hamming weight of Bob's received vector \(y = x_{mh(m)} + s\) satisfies

\[
wt_H(y) = wt_H(x_{mh(m)} + s) \leq wt_H(x_{mh(m)}) + wt_H(s) \leq 2pm + pn, \\
\]
since \(wt_H(s) \leq pn\), \(wt_H(x_{mh(m)}) \leq 2pm\) for typical \(x_{mh(m)}\), and the intersection between \(x_{mh(m)}\) and \(s\) is greater than zero. For any codeword \(X_{m'h'}\) where \((m', h') \neq (m, h(m))\), \(X_{m'h'} \in \mathcal{L}(y)\) if and only if

\[
\begin{align*}
\left\{ n_{f_{10}^{XY}} (X_{m'h'}, y) < pm \left( \frac{p(1-q)}{q} \right) (1 + \varepsilon_1), \\
\left\{ n_{f_{11}^{XY}} (X_{m'h'}, y) > pm \left( 1 - \frac{p(1-q)}{q} \right) (1 - \varepsilon_2). \\
\right. \\
\end{align*}
\]

Since the probability of the event that \(n_{f_{10}^{XY}} (X_{m'h'}, y) \leq pm \left( \frac{p(1-q)}{q} \right) (1 + \varepsilon_1)\) is less than \(\exp(-O(n^{5/2}))\), the claim is proved.

-proved
Note that the complement of the support of $y$ has size greater than $(1 - p)n - 2pm$, and hence we have
\[ \mathbb{E}_{X_{m', h'}} \left( n_{10}^{XY}(X_{m', h'}, y) \right) \geq \rho \left( (1 - p)n - 2pm \right), \]
since each bit of $X_{m', h'}$ is generated i.i.d. according to Bernoulli($\rho$). Let $\kappa_1 = 1 - \frac{(p - pq)(1 + \varepsilon_1)}{q - pq + 2pq}$. By the Chernoff bound, we have
\[ \Pr_{X_{m', h'}} \left( n_{10}^{XY}(X_{m', h'}, y) < p\left(1 - \frac{p(1 - q)}{q} \right) \right) \leq 2^{-\frac{\varepsilon_1}{\rho n}} + 2^{-\frac{\varepsilon_1}{2n - 2pm}}, \]
(110)
\[ \Pr_{X_{m', h'}} \left( n_{10}^{XY}(X_{m', h'}, y) > p\left(1 - \frac{p(1 - q)}{q} \right) \right) \geq 2^{-\frac{\varepsilon_1}{\rho n}} + 2^{-\frac{\varepsilon_1}{2n - 2pm}}, \]
(111)
\[ \Pr_{X_{m', h'}} \left( n_{10}^{XY}(X_{m', h'}, y) < p\left(1 - \frac{p(1 - q)}{q} \right) \right) = 2^{-\frac{\varepsilon_1}{\rho n}} \cdot 2^{-\frac{\varepsilon_1}{2n - 2pm}}. \]
(112)
Similarly, since $w_{II}(y) \leq 2pm + pn$, we have
\[ \mathbb{E}_{X_{m', h'}} \left( n_{11}^{XY}(X_{m', h'}, y) \right) \leq \rho \left( 2pm + pn \right). \]
Let $\kappa_2 = \frac{(q - p + pq)(1 - \varepsilon_2)}{q(2p + q)}$. By the Chernoff bound, we have
\[ \Pr_{X_{m', h'}} \left( n_{11}^{XY}(X_{m', h'}, y) > p\left(1 - \frac{p(1 - q)}{q} \right) \right) \leq 2^{-\frac{\varepsilon_2}{\rho n}} + 2^{-\frac{\varepsilon_2}{2n - 2pm}}, \]
(113)
\[ \Pr_{X_{m', h'}} \left( n_{11}^{XY}(X_{m', h'}, y) < p\left(1 - \frac{p(1 - q)}{q} \right) \right) \geq 2^{-\frac{\varepsilon_2}{\rho n}} + 2^{-\frac{\varepsilon_2}{2n - 2pm}}, \]
(114)
\[ \Pr_{X_{m', h'}} \left( n_{11}^{XY}(X_{m', h'}, y) < p\left(1 - \frac{p(1 - q)}{q} \right) \right) = 2^{-\frac{\varepsilon_2}{\rho n}} \cdot 2^{-\frac{\varepsilon_2}{2n - 2pm}}. \]
(115)
By combining inequalities (112) and (115), we are able to show that the probability of another codeword $X_{m', h'}$ falling into the list $\mathcal{L}(y)$ is given as
\[ \Pr_{X_{m', h'}} \left( X_{m', h'} \in \mathcal{L}(y) \right) \]
\[ = \Pr_{X_{m', h'}} \left( n_{10}^{XY}(X_{m', h'}, y) < p\left(1 - \frac{p(1 - q)}{q} \right) \right) \cdot \Pr_{X_{m', h'}} \left( n_{11}^{XY}(X_{m', h'}, y) > p\left(1 - \frac{p(1 - q)}{q} \right) \right) \]
\[ \leq 2^{-\frac{\varepsilon_1}{\rho n}} \cdot 2^{-\frac{\varepsilon_2}{2n - 2pm}}. \]
(116)
since the two events (the number of ones of $X_{m', h'}$ inside the support of $y$ and outside the support of $y$) are independent. On expectation, the total number of codewords (except the transmitted codeword $x_{mh(m)}$) falling into the list $\mathcal{L}(y)$ equals $2^{t \cdot I_B(p, q) \cdot \sqrt{n}}$, which is super-polynomially small since $r < t \cdot I_B(p, q)$. Therefore, we are able to use a counting argument to characterize the probability that more than $n^2$ codewords falling into the list $\mathcal{L}(y)$ as below. As long as $r < t \cdot I_B(p, q)$, we have
\[ \Pr_{C \setminus x_{mh(m)}} \left( \left| X \in \mathcal{C} \setminus x_{mh(m)} : X \in \mathcal{L}(y) \right| \geq n^2 \right) \]
\[ \leq \sum_{i = n^2}^{2^{r \cdot \sqrt{n}}} \Pr_{C \setminus x_{mh(m)}} \left( \left| X \in \mathcal{C} \setminus x_{mh(m)} : X \in \mathcal{L}(y) \right| = i \right) \]
\[ \leq \sum_{i = n^2}^{2^{r \cdot \sqrt{n}}} \left( \binom{2^{r \cdot \sqrt{n}}}{i} \right) \left( 2^{t \cdot I_B(p, q) \cdot \sqrt{n}} \right)^i \left( 1 - 2^{t \cdot I_B(p, q) \cdot \sqrt{n}} \right) \left( 2^{r \cdot \sqrt{n}} - i \right) \]
\[ \leq 2^{r \cdot \sqrt{n}} \left( \binom{2^{r \cdot \sqrt{n}}}{n^2} \right) \left( 2^{t \cdot I_B(p, q) \cdot \sqrt{n}} \right)^n \]
\[ \leq 2^{r \cdot \sqrt{n}} \left( \frac{e \cdot 2^{r \cdot \sqrt{n}}}{n^2} \right)^{n^2} \left( 2^{t \cdot I_B(p, q) \cdot \sqrt{n}} \right)^{n^2} \]
\[ = 2^{r \cdot \sqrt{n}} \left( \frac{e \cdot 2^{r \cdot t \cdot I_B(p, q) \cdot \sqrt{n}}}{n^2} \right)^{n^2} \]
\[ = \exp(-O(n^{5/2})]. \]
(123)
Inequality (129) follows since \( i = n^2 \) maximizes the probability in (118), and we bound the number of summations from above by \( 2^{n^2}. \) Inequality (121) holds since \( (\frac{n}{k})^k \leq (\frac{en}{k})^k. \) Finally, we obtain (123) since \( r < t \cdot I_B(p, q). \)

In the following we also consider the atypical events and formally prove that with high probability over the code design, a randomly chosen code \( C \) ensures the error events \( E_2 \) goes to zero. The error events \( E_2 \) is given as:

\[
P_e^{(2)} = \max_{p_S, z, c} \left\{ \sum_m p_M(m) \sum_z p_{z|x}(z|x_{mh}(m)) \sum_s p_{s|z, c}(s|z, C) \cdot \mathbb{I}\{ |x \in C \setminus x_{mh}(m) : x \in \mathcal{L}(y) | \geq n^2 \} \right\}
\]

\[
= \max_{p_S, z, c} \left\{ \frac{1}{N} \sum_m \sum_z p_{z|x}(z|x_{mh}(m)) \sum_s p_{s|z, c}(s|z, C) \cdot \mathbb{I}\{ |x \in C \setminus x_{mh}(m) : x \in \mathcal{L}(y) | \geq n^2 \} \right\}
\]

Note that regardless of \( p_{s|z, c}. \)

\[
\mathbb{E}_C \left[ \frac{1}{N} \sum_m \sum_z p_{z|x}(z|x_{mh}(m)) \sum_s p_{s|z, c}(s|z, C) \cdot \mathbb{I}\{ |x \in C \setminus x_{mh}(m) : x \in \mathcal{L}(y) | \geq n^2 \} \right]
\]

(124)

\[
= \mathbb{E}_C \left[ \sum_z p_{z|x}(z|x_{mh}(m)) \sum_s p_{s|z, c}(s|z, C) \cdot \mathbb{I}\{ |x \in C \setminus x_{mh}(m) : x \in \mathcal{L}(y) | \geq n^2 \} \right]
\]

(125)

\[
= \sum_{x_{mh}(m)} p_{x}(x_{mh}(m)) \sum_s p_{c \setminus x}(C \setminus x_{mh}(m)) \sum_z p_{z|x}(z|x_{mh}(m))
\]

(126)

\[
\leq \sum_{x_{mh}(m) \in A_X} p_{x}(x_{mh}(m)) \sum_s p_{c \setminus x}(C \setminus x_{mh}(m)) \cdot \mathbb{I}\{ |x \in C \setminus x_{mh}(m) : x \in \mathcal{L}(y) | \geq n^2 \}
\]

(127)

\[
\leq \sum_{x_{mh}(m) \in A_X} p_{x}(x_{mh}(m)) \sum_s \Pr_{C \setminus x}( |x \in C \setminus x_{mh}(m) : x \in \mathcal{L}(y) | \geq n^2 ) + \sum_{x_{mh}(m) \notin A_X} p_{x}(x_{mh}(m))
\]

(128)

\[
\text{w.h.p.} \leq \left( \sum_s \exp(-O(n^{5/2})) \right) + \exp \left( -\frac{1}{3} t(q, \epsilon_d) \sqrt{n} \right)
\]

(129)

\[
= \exp(-O(\sqrt{n})).
\]

(130)

Equation (125) is obtained by noting that for each message \( m, \) the averaged probability of error (over the code design) is the same. The notation \( p_{c \setminus x}(C \setminus x_{mh}(m)) \) in (126) represents the probability of generating a code \( C \) excluding the transmitted codeword \( x_{mh}(m). \) In (127), we again consider the transmitted codeword \( x_{mh}(m) \) to be either typical or atypical, and

- For atypical \( x_{mh}(m), \) we simply bound the indicator function \( \mathbb{I}(.) \) from above by one.
- For typical \( x_{mh}(m), \) we bound the probability \( p_{s|z, c}(s|z, C) \) from above by one, and then interchange the order of summations.

Inequality (129) follows from Claim (11) (which is valid for all typical transmitted codewords), and holds with probability at least \( 1 - \exp(-O(n^{5/2})). \) Finally, by noting that (130) holds for every possible jamming strategy \( p_{s|z, c}, \) the Markov’s inequality yields

\[
\Pr \left( P_e^{(2)} \geq \exp(-n^{1/4}) \right) \leq \exp(-O(\sqrt{n})).
\]

(131)

This completes the proof of Lemma 4. \( \square \)

It is worth noting that the probability of error event \( E_4 \) (corresponding to Alice is silent) can also be bounded in a similarly manner (the reasons will be clear in the detailed proof), hence we skip the analysis of \( E_4 \) here.

3) Analysis of error event \( E_3: \)

**Lemma 5.** The error event \( E_3 \) occurs with probability (over the common randomness \( K \)) at most \( O\left( \frac{1}{\sqrt{n \log(n)}} \right) \).

The error event \( E_3 \) occurs if there exists \( x' \neq x_{mh}(m) \) such that \( x' \in \mathcal{L} \) and \( x' \) is consistent with \( K. \) Recall that we partition the \( 6 \log(n) \)-bits common randomness \( K \) into two equal parts \( K_1 \) and \( K_2, \) and partition the message
coding and decoding grow super-polynomially with the blocklength \( n \). In subsection VI-B we first define the polynomial hash function of \( K_1, K_2 \) and \( M \) as

\[
h = K_2 + \sum_{i=1}^{l} K_1^i M_i.
\]

From James’ perspective the shared key \( K \) is uniformly distributed \( a \ p r i o r i \). However, once James receives the vector \( z \) the distribution may not necessarily be uniform. Though he gains some information from the observation \( z \), James is unable to decode which message/codeword is transmitted. To simplify our analysis, we now assume there’s another oracle which reveals extra information to James — the message-hash pair \( (m, h) \) being transmitted. Once James knows \( (m, h) \), he certainly knows the shared key satisfies \( h = K_2 + \sum_{i=1}^{l} K_1^i m_i \). However, the number of \( (K_1, K_2) \) pairs satisfying this equation is exactly \( n^3 \), and each pair contains a distinct \( K_1 \), hence from James’ perspective the first part of the key, \( K_1 \), is still uniformly distributed.

Alice’s transmission pair \( (m, h) \) is consistent with the shared key \( K \) by default. For any message-hash pair \( (m', h') \neq (m, h) \), the probability that \( (m', h') \) is consistent with the common randomness \( K \) equals

\[
Pr_K(h' = K_2 + \sum_{i=1}^{l} K_1^i m'_i \mid h = K_2 + \sum_{i=1}^{l} K_1^i m_i) = Pr_K(h' - h = \sum_{i=1}^{l} K_1^i (m'_i - m_i)) \leq l/n^3,
\]

by applying the Schwartz–Zippel lemma. Since the number of codewords \( x' \neq x_{mh(m)} \) in the list \( L \) is at most \( n^2 \), by taking a union bound, one can prove that with probability at least \( 1 - O(1/(\sqrt{n} \log(n))) \) over the common randomness \( K \), none of these codewords are consistent with \( K \).

Note that the analysis of error event \( \mathcal{E}_3 \) is exactly the same as that of \( \mathcal{E}_3 \) since it does not depend on whether or not Alice is transmitting, hence we also skip the analysis of \( \mathcal{E}_3 \) here.

E. Achievability scheme with large \( \Delta(n) \)

For any values of \( (p, q) \) such that \( I_J(q) > I_B(p, q) > 0 \), the optimal throughput \( t(q, \epsilon) \cdot I_B(p, q) \) is achievable with \( O(\log(n)) + t(q, \epsilon)(I_J(q) - I_B(p, q))\sqrt{n} \) bits of shared key. We highlight the difference between this achievability scheme and the scheme introduced in subsection VI-B. The public codebook \( C \) contains roughly \( 2^{t(q, \epsilon)(I_J(q) - I_B(p, q))\sqrt{n}} \) sub-codebooks, and each of size roughly \( 2^{t(q, \epsilon)(I_J(q) - I_B(p, q))\sqrt{n}} \). The extra \( t(q, \epsilon)(I_J(q) - I_B(p, q))\sqrt{n} \) bits of shared key \( K \) is used for Alice and Bob to select which sub-codebook is used during transmission, and the selected codebook is kept secret from James. Note that each sub-codebook also contains \( O(\log(n)) \) bits of shared key and it is critical for list decoding. The size of each sub-codebook is small enough so that Bob can decode reliably (as proved in subsection VI-D). On the other hand, from James’ perspective the size of the whole codebook (contains more than \( 2^{t(q, \epsilon)(I_J(q) - I_B(p, q))\sqrt{n}} \) codewords) is also sufficiently large, since he does not know which sub-codebook is used. Therefore, as proved in subsection VI-C the covertness requirement can also be guaranteed.

VII. Computational Efficiency Achievability Scheme

The achievability scheme in Section VI relies on the random codes, and the computational complexities of encoding and decoding grow super-polynomially with the blocklength \( n \). In this Section we develop a computationally efficient coding scheme which is proven to be both covert and reliable.

A. Permutation-based coding scheme

Recall that for covert communication, the average Hamming weights of codewords should be \( O(\sqrt{n}) \), and the rate \( R \) scales as \( O(1/\sqrt{n}) \). Instead of generating a low-weight (or low-rate) code directly, we now generate a code \( \hat{C} \) of length \( d\sqrt{n} \) and rate \( \hat{R} \), where \( d \) and \( R \) scale as constants, hence the fraction of ones in the code also scales as a constant. The code \( \hat{C} \) consists of \( 2^{dR\sqrt{n}} \) codewords \( \hat{X} \). To satisfy the covertness constraint, Alice uses a uniformly distributed shared key \( \Pi_1 \) to select \( d\sqrt{n} \) locations (out of \( n \) locations), and insert \( \hat{X} \) (of length \( d\sqrt{n} \) bits) into these locations. The key \( \Pi_1 \) of size \( \log \left( \binom{d\sqrt{n}}{n} \right) \) is \( O(\sqrt{n} \log(n)) \) only is shared between Alice and Bob. All the locations except for those \( d\sqrt{n} \) locations selected by \( \Pi_1 \) comprising entirely of zeros. For the purpose of analysis, we introduce another \( \log ((d\sqrt{n})!) = O(\sqrt{n} \log(n)) \) bits of shared key \( \Pi_2 \) (between Alice and Bob), which is used to permute the \( d\sqrt{n} \) bits (inside the codewords \( \hat{X} \)). A length-\( n \) sequence \( \hat{X} = \Pi_2(\Pi_1(\hat{X})) \) is obtained after the two permutations. Note that to guarantee covertness, the parameter \( d \), which determines the length of \( \hat{C} \), cannot

\footnote{Note that \( K_2 \) may or may not be uniformly distributed from James’ perspective}
Fig. 7: A polynomial time algorithm $D_1$ that can distinguish $K = \mathcal{P}$ and $K = \mathcal{R}$ based on the assumptions on the estimator $\Phi$.

Fig. 8: A polynomial time algorithm $D_2$ that can distinguish $K = \mathcal{P}$ and $K = \mathcal{R}$ based on the assumptions on the decoder $\Gamma$.

be arbitrarily large (a larger $d$ implies a larger average Hamming weight of $\hat{X}$). It turns out that as long as $d$ is bounded from above as $d \leq \frac{(q,\epsilon_d)}{p} = \frac{2\sqrt{q(1-q)}}{p(1-2q)} \cdot Q^{-1} \left( \frac{1-\epsilon}{2} \right)$, $(1-\epsilon_d)$-covertness can be guaranteed.

In the following let’s focus on the selected $d\sqrt{n}$ locations corresponding to $\hat{X}$. Since $\Pi_1$ and $\Pi_2$ are uniformly distributed from James’ perspective, he seems unlikely to estimate the $d\sqrt{n}$ locations selected by $\Pi_1$. Nevertheless, James can perform the BAC-type jamming on the length-$n$ sequence $X$ (since it only depends on his observation $Z$) – flip $\hat{X}_i$ with probability approximately $p/q$ if $Z_i = 1$, and does not flip $\hat{X}_i$ if $Z_i = 0$ (as described in Section V). Effectively, this attack roughly flips $p$ fraction of zeros and $(1-q)p/q$ fraction of ones of $\hat{X}$, irrespective of the permutations $\Pi_1$ and $\Pi_2$ that Alice and Bob use.

One may notice that even if James does not know any information about $\Pi_1$ a priori, he is still able to learn something non-trivial from $z$. For example, if the channel from Alice to James is a noiseless channel, James will know exactly the locations corresponding to the support of $z$ must be selected by $\Pi_1$ (though he is unable to infer all the $d\sqrt{n}$ locations selected by $\Pi_1$). If the channel to James is a BSC$(q)$, he can still learn some information about $\Pi_1$, though not as much as the noiseless case. However, by a carefully designed oracle-based argument, we are able to show that there are super-polynomially many “plausible” permutations $\Pi_1 \times \Pi_2$ from James’ perspective, and each with equal probability. Under specific permutations $\pi_1$ and $\pi_2$, a jamming vector $s$ is said to outperform the BAC-type jamming if $s$ flips more than $p(1+\epsilon)$ fraction of zeros or more than $(1-q)p(1+\epsilon)/q$ fraction of ones in $X$. We are able to show that no matter which jamming vector $s$ James chooses, only under an extremely small fraction of plausible permutations, $s$ will outperform the BAC-type jamming. This implies that with high probability over the permutations, the channel between Alice and Bob is not worse than a binary asymmetric channel with $Pr(Y = 1|X = 0) = p(1+\epsilon)$ and $Pr(Y = 0|X = 1) = (1-q)p(1+\epsilon)/q$. Therefore, the length-$d\sqrt{n}$ code $\hat{C}$ should satisfy the following properties:

1) $\hat{C}$ is robust to the binary asymmetric channel described above, and also achieves the channel capacity $C_{BAC}(p, q)$.
2) The encoding and decoding of $\hat{C}$ is computationally efficient.

The existence of such $\hat{C}$ is guaranteed by Forney’s concatenated codes, which provides a generic computationally efficient code construction for arbitrary discrete memoryless channels. Hence the optimal relative throughput of the permutation-based coding scheme equals $dR = \frac{(q,\epsilon_d)}{p} \cdot C_{BAC}(p, q)$.

B. Computational assumptions on the adversary

As is usual in the standard “cryptographic” setting, the computational power of the adversary is always restricted to be polynomial in the blocklength $n$. Here we also make this assumption on James, by introducing a pseudorandom
generator (PRG) and reusing the coding scheme in subsection VII-A, we can hope for a computationally efficient coding scheme with much less amount of shared key. We rely in the “standard” assumption that PRGs exist, i.e., there exists an efficiently computable function \( g : \{0, 1\}^n \to \{0, 1\}^{\sqrt{n}} \) such that if \( U \sim \text{Unif}\{\{0, 1\}^n\} \) and \( K \sim \text{Unif}\{\{0, 1\}^{\sqrt{n}}\} \), then for all polynomial time computable functions \( A : \{0, 1\}^{\sqrt{n}} \to \{0, 1\} \),

\[
\left| \Pr_U(A(g(U)) = 1) - \Pr_K(A(K) = 1) \right| \in o(1).
\]

We say \( K = \mathcal{P} \) if the \( \Delta(n) \) bits of shared key comes from the output of a PRG with a length-\( n^\xi \) seed (\( \forall \xi > 0 \)), and \( K = \mathcal{R} \) if the \( \Delta(n) \) bits of shared key is truly uniformly distributed. As stated in Definition 1 (1-\( \epsilon_d \))-covertness requires that

\[
\forall \Phi, \Pr(\hat{T} = 1|T = 0) + \Pr(\hat{T} = 0|T = 1, K = \mathcal{P}) \geq 1 - \epsilon_d. \tag{134}
\]

By using the triangle inequality, it suffices to show that

\[
\begin{cases}
\forall \Phi, \Pr(\hat{T} = 1|T = 1, K = \mathcal{R}) - \Pr(\hat{T} = 1|T = 1, K = \mathcal{P}) \leq \delta, \text{ where } \delta \ll \epsilon_d, \\
\forall \Phi, \Pr(\hat{T} = 1|T = 0) + \Pr(\hat{T} = 0|T = 1, K = \mathcal{R}) \geq 1 - (\epsilon_d - \delta).
\end{cases} \tag{135}
\]

Note that the second condition is satisfied if the coding scheme in subsection VII-A is covert, since the shared key used there is truly uniformly distributed. Hence it suffices to focus on the first condition in equation (135) only.

Suppose there is a polynomial time estimator \( \Phi \) such that \( \Pr(\hat{T} = 1|T = 1, K = \mathcal{R}) - \Pr(\hat{T} = 1|T = 1, K = \mathcal{P}) > \delta \), i.e., \( K \) is more likely to equal \( \mathcal{R} \) if \( \Phi \) outputs \( \hat{T} = 1 \), then James is able to design a polynomial time algorithm \( D_1 \) to distinguish \( \mathcal{P} \) and \( \mathcal{R} \) with \( f_1(\delta) \) advantage (for some function \( f_1(.) \)), by generating an artificial system that contains message, encoder, channel and estimator \( \Phi \) (as shown in figure 7). The algorithm \( D_1 \) is valid since both the encoder and the estimator are efficient. The algorithm \( D_1 \) decides \( K = \mathcal{P} \) if \( \Phi \) outputs \( \hat{T} = 1 \), and decides \( K = \mathcal{R} \) if \( \Phi \) outputs \( \hat{T} = 0 \).

On the other hand, we also need to show that using PRG does not increase the probability of error “too much”. Suppose there is a polynomial time decoder \( \Gamma \) satisfying

\[
\Pr_{M, K, S, S}(M \neq \hat{M}|K = \mathcal{P}) - \Pr_{M, K, S, S}(M \neq \hat{M}|K = \mathcal{R}) \geq \delta.
\]

Similarly, Bob is able to design a polynomial time algorithm \( D_2 \) to distinguish \( \mathcal{P} \) and \( \mathcal{R} \) with \( f_2(\delta) \) advantage (for some function \( f_2(.) \)), by generating an artificial system as illustrated in Figure 8 which is also valid since the encoder and the decoder \( \Gamma \) are efficient. The algorithm \( D_2 \) outputs \( K = \mathcal{R} \) if \( M = M \), and outputs \( K = \mathcal{P} \) if \( M \neq M \). The assumptions on the decoder \( \Gamma \) implies the polynomial time algorithm \( D_2 \) can distinguish \( K = \mathcal{P} \) or \( K = \mathcal{R} \) with \( f_2(\delta) \) advantage (for some function \( f_2(.) \)). Therefore, we conclude that for any polynomial time decoder \( \Gamma \), the

\[
\Pr_{M, K, S, S}(M \neq \hat{M}|K = \mathcal{P}) - \Pr_{M, K, S, S}(M \neq \hat{M}|K = \mathcal{R}) \leq \delta, \tag{136}
\]

and the probability of error does not increase significantly.

**APPENDIX A**

**PROOF OF CLAIMS 3-5 (COVERTNESS)**

**Claim 3** (Second term in (15)). \( \frac{1}{2} \mathbb{E}_C \left( \sum_{z \notin A_Z} p_1(z) \right) \leq \exp \left( -O(\sqrt{n}) \right) \).

**Proof:** Note that the ensemble-averaged active distribution \( \mathbb{E}_C(p_1) \) is essentially a Binomial(\( n, \rho \ast q \)) distribution, since each bit of the codeword \( X \) is generated i.i.d. according to Bernoulli(\( \rho \)), and then passes through a BSC(\( q \)). By applying the Chernoff bound, we show that

\[
\frac{1}{2} \mathbb{E}_C \left( \sum_{z \notin A_Z} p_1(z) \right) = \frac{1}{2} \Pr_{M, K, S}(Z \notin A_Z) = \frac{1}{2} \Pr_{M, K, S} \left( f_1^T(Z) \notin (\rho \ast q) \cdot (1 \pm n^{-\frac{1}{2}}) \right) \leq \exp \left( -\frac{1}{3}(\rho \ast q)\sqrt{n} \right). \tag{139}
\]
We simplify the notation in (141) since the expectation due to the triangle inequality, while (151) follows from Claims 3 and 4. Therefore, \( C \) (Term in (36))

\[
\frac{1}{2} \mathbb{E}_C \left( \sum_{z \in A_z} \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) \right) = \frac{1}{2} \mathbb{E}_C \left( \sum_{z} p_{Z|X}(z|x) \mathbb{I} \{ (z \in A_z) \cap (x \in A_{X|x}) \} \right)
\]

(140)

\[
= \frac{1}{2} \mathbb{E}_C \left( \sum_{z} \sum_{x \in \{0,1\}} p_z(x)p_{Z|X}(z|x) \mathbb{I} \{ (z \in A_z) \cap (x \in A_{X|x}) \} \right)
\]

(141)

\[
\leq \frac{1}{2} \Pr_{XZ} (Z \in A_z \cap X \in A_{X|x})
\]

(143)

\[
\leq \frac{1}{2} \Pr_{XZ} \left( f_{10}^{\epsilon}(X,Z) \notin \rho(q(1 \pm n^{-\frac{1}{2}})) \right)
\]

(145)

\[
\leq \exp \left( -\frac{q \cdot t(q,\epsilon)}{3} n^{\frac{1}{2}} \right) + \exp \left( -\frac{(1 - q) \cdot t(q,\epsilon)}{3} n^{\frac{1}{2}} \right).
\]

(146)

We simplify the notation in (141) since the expectation \( \mathbb{E}_C(.) \) for each codeword \( x \in C \) is exactly the same, and the number of codewords is \( |C| \). It is worth noting that \( p_x(x) \) in (142) is the probability of generating a codeword \( x \) (which follows from an i.i.d. Bernoulli(\( \rho \)) distribution), instead of the probability of a codeword being transmitted. Equation (145) follows from the definition of the conditionally typical set \( A_{X|x} \) in equation (19), while (146) is due to the Chernoff bound.

Claim 5 (Term in (36)). With probability at least \( 1 - \exp(-2^{O(\sqrt{n})}) \) over the code design, a randomly chosen code \( C \) satisfies

\[
\frac{1}{2} \sum_{x \in A_z} \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) + \frac{1}{2} \sum_{x \notin A_z} p_1(z) \leq \exp \left( -O\left(n^{\frac{1}{2}}\right) \right).
\]

(58)

Proof: Note that

\[
\frac{1}{2} \exp(-n^{\frac{1}{2}})
\]

(147)

\[
\text{w.h.p.} \quad \frac{1}{2} \sum_{x \in A_z} \mathbb{E}_C \left( \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) - \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) \right)
\]

(148)

\[
\geq \frac{1}{2} \mathbb{E}_C \left( \sum_{x \in A_z} \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) - \frac{1}{2} \sum_{x \in A_z} \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) \right)
\]

(149)

\[
= \frac{1}{2} \left( 1 - \mathbb{E}_C \left( \sum_{x \in A_z} \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) \right) - \mathbb{E}_C \left( \sum_{x \notin A_z} p_1(z) \right) \right) \geq -\exp \left( -O(\sqrt{n}) \right) + \frac{1}{2} \left( 1 - \sum_{x \in A_z} \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) \right)
\]

(150)

Inequality (148) follows from equation (57), and holds with probability at least \( 1 - \exp(-2^{O(\sqrt{n})}) \). Inequality (149) is due to the triangle inequality, while (151) follows from Claims 3 and 4. Therefore,

\[
\frac{1}{2} \sum_{x \in A_z} \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) + \frac{1}{2} \sum_{x \notin A_z} p_1(z) = \frac{1}{2} \left( 1 - \sum_{x \in A_z} \sum_{x \in C \setminus A_{X|x}} p_{Z|X}(z|x)p(x) \right)
\]

(152)
The expected value is bounded from above as

\[
\Pr \left( \frac{1}{N} \sum_{z \in A_Z} \sum_{x_{m(h(m)}} \notin A_X | a \right) \leq \exp \left( -n^{1/4} \right) \quad \text{(153)}
\]

which directly follows from the proof of Claim 3 in Appendix A. Finally, by applying the Markov’s inequality, we have

\[
\Pr \left( \frac{1}{N} \sum_{z \in A_Z} \sum_{x_{m(h(m)}} \notin A_X | a \right) \leq \exp \left( -n^{1/4} \right) \quad \text{(155)}
\]

where inequality (153) holds with probability at least \(1 - \exp(-O(\sqrt{n}))\) over the code design.

\[\square\]

**APPENDIX B**

**Proofs of Claims 6 and 7 (Reliability)**

Claim 6 (First term in (69). With probability at least \(1 - \exp(-O(n^{1/4}))\) over the code design,

\[
\Pr \left( \frac{1}{N} \sum_{z \in A_Z} \sum_{x_{m(h(m)}} \notin A_X | a \right) \leq \exp \left( -n^{1/4} \right) \quad \text{(155)}
\]

which directly follows from the proof of Claim 3 in Appendix A. Finally, by applying the Markov’s inequality, we have

\[
\Pr \left( \frac{1}{N} \sum_{z \in A_Z} \sum_{x_{m(h(m)}} \notin A_X | a \right) \leq \exp \left( -n^{1/4} \right) \quad \text{(156)}
\]

Claim 7 (Second term in (69). With probability at least \(1 - \exp(-O(\sqrt{n}))\) over the code design,

\[
\Pr \left( \frac{1}{N} \sum_{z \in A_Z} \sum_{x_{m(h(m)}} \notin A_X | a \right) \leq \exp \left( -n^{1/4} \right) \quad \text{(157)}
\]

which follows from the proof of Claim 3 in Appendix A. By the Markov’s inequality, we have

\[
\Pr \left( \frac{1}{N} \sum_{z \in A_Z} \sum_{x_{m(h(m)}} \notin A_X | a \right) \leq \exp \left( -n^{1/4} \right) \quad \text{(158)}
\]

\[\square\]

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