Time Optimal Control of Coupled Qubits Under Non-Stationary Interactions

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In this article, we give a characterization of all the unitary transformations that can be synthesized in a given time for a two-qubit system in the presence of general time varying coupling tensor. This characterization helps to compute the minimum time and the shortest pulse sequence for generating a general two-qubit transformation under non-stationary interactions. The methods presented here can be applied in design of time optimal pulse sequences for transferring coherence and polarization between coupled spins with time varying couplings as in solid state NMR under magic angle spinning.

I. INTRODUCTION

An important question in quantum information science is to determine the minimum time required to perform a quantum computation using a set of physical resources. Since two-qubit gates are the building blocks of quantum information processing, it is of fundamental interest to find the minimal time required to implement a unitary operation on a two-qubit system using the interaction Hamiltonian $H_d$, and the local unitary operations on the two qubits. This problem was studied in [3], where it was shown that any two qubit unitary propagator $U_F$ can be expressed as

$$U_F = U_2 \left( \prod_{k=1}^{4} V_k \exp(-iH_d t_k) V_k^\dagger \right) U_1,$$  \hspace{1cm} (1)

where $U_1, U_2, V_k$ are local unitary transformations and the effective Hamiltonians $V_k H_d V_k^\dagger$ all mutually commute. Under the assumption that the synthesis of local unitaries takes arbitrarily small time, the minimum time to produce a desired $U_F$ is the smallest value of $\sum_{k=1}^{4} t_k$ in equation (1). This characterization of time optimal trajectories is used in [3] to explicitly compute an elegant expression for the minimal time for synthesis of arbitrary unitary transformation of two qubits. Alternative proofs for time optimality have been presented in [7, 11]. There is now a considerable literature on the subject; see for example, [6–14] and references therein.

All these investigations assume that the interaction Hamiltonian $H_d$ is fixed. In this paper, we consider the general problem when $H_d$ varies with time. For example, in solid state NMR [17], the interaction between the spins are varying with time during magic angle spinning when the sample is rotated around an axis making an angle of $\theta_M = \tan^{-1}(\sqrt{2})$ with the static magnetic field $B_0$. As a result the dipolar couplings between nuclear spins that have an orientational dependence of the form $3 \cos^2(\theta) - 1$ averages out ( $\theta$ is the angle of internuclear axis with the static magnetic field), leading to better resolved NMR spectrum [17]. An important problem in multi-dimensional solid state NMR experiments is to find radio-frequency pulse sequence that re-couple desired spins whose interactions are being modulated in time by magic angle spinning. Finding short pulse sequences that transfer polarization or coherence between coupled nuclear spins under time varying interactions is of interest in solid state NMR. In this paper, we give a complete characterization of all the unitary transformations that can be synthesized in a given time for a two-qubit system in presence of general time varying coupling tensor, assuming that the local unitary transformation on two qubits can be performed arbitrarily fast (on a time scale governed by the strength of couplings). From the perspective of quantum control theory, this problem is equivalent to characterizing the reachable set of the Schrödinger equation

$$\dot{U}(t) = -i[H_d(t) + \sum_{j=1}^{m} v_j(t)H_j]U(t),$$  \hspace{1cm} (2)

where $U \in SU(4)$ and $H_d(t)$ is the interaction Hamiltonian that is internal to the system and $\sum_{j=1}^{m} v_j(t)H_j$ is the part of the Hamiltonian that can be externally changed, and generates the local unitary operations. We assume the control parameters $v_j$ are a priori not bounded.

Before stating the main result, we review some background material.

II. BACKGROUND

A. Majorization

For an element $x = (x_1, ..., x_k)^T$ of $\mathbb{R}^k$ we denote by $x^i = (x_1^i, ..., x_k^i)^T$ a permutation of $x$ so that $x_i^i \geq x_j^i$ if $i < j$, where $1 \leq i, j \leq k$.

Definition 1 (majorization) A vector $x \in \mathbb{R}^k$ is ma-
jorized by a vector \( y \in \mathbb{R}^K \) (denoted \( x \prec y \)), if
\[
\sum_{j=1}^{k} x_j^i \leq \sum_{j=1}^{k} y_j^j
\]
for \( k = 1, \ldots, D - 1 \), and the inequality holds with equality when \( k = D \).

**Proposition 1** \( x \prec y \) iff \( x \) lies in the convex hull of \( y \) and all its permutations \( P_i y \), where \( P_i \) are permutation matrices.

**Proposition 2 (Schur, Horn [1, 2])** For an element \( \lambda = (\lambda_1, ..., \lambda_n)^T \), let \( D_\lambda \) be a diagonal matrix with \( (\lambda_1, ..., \lambda_n) \) as its diagonal entries, let \( a = (a_1, ..., a_n)^T \) be the diagonal entries of matrix \( A = K^T D_\lambda K \), where \( K \in SO(n) \). Then \( a \prec \lambda \). Conversely for any vector \( a \prec \lambda \), there exists a \( K \in SO(n) \), such that \( (a_1, ..., a_n)^T \) are the diagonal entries of \( A = K^T D_\lambda K \).

Following [3, 5], for an element \( x = (x_1, x_2, x_3)^T \) of \( \mathbb{R}^3 \), we introduce the vector \( \hat{x} = (|x_1|, |x_2|, |x_3|)^T \), and define the \( s \)-order version \( x^s \) of \( x \) by setting \( x_1^s = \hat{x}_1^s \), \( x_2^s = \hat{x}_2^s \), \( x_3^s = \text{sgn}(x_1 x_2 x_3) \hat{x}_3^s \).

**Definition 2 ([3–7])** The vector \( x \in \mathbb{R}^3 \) is \( s \)-majorized by \( y \in \mathbb{R}^3 \) (denoted \( x \prec_s y \)) if
\[
\begin{align*}
&x_1^s \leq y_1^s \\
&x_1^s + x_2^s + x_3^s \leq y_1^s + y_2^s + y_3^s \\
&x_1^s + x_2^s - x_3^s \leq y_1^s + y_2^s - y_3^s
\end{align*}
\]

**B. Canonical Decomposition**

An arbitrary two-qubit Hamiltonian can be parameterized
\[
H_d(t) = I \otimes (\hat{a}(t) \cdot \sigma) + (\hat{b}(t) \cdot \sigma) \otimes I + \sum_{i,j} M_{ij}(t) \sigma_i \otimes \sigma_j
\]
where \( i, j \in \{x, y, z\} \) and \( \hat{a} = (a_x, a_y, a_z), \hat{b} = (b_x, b_y, b_z) \) are real 3-vectors, \( M \) is a 3 by 3 real matrix, and \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is the vector of Pauli operators.

Let \( H'_d(t) \) be the non-local part of \( H_d(t) \), i.e.,
\[
H'_d(t) = \sum_{i,j} M_{ij}(t) \sigma_i \otimes \sigma_j.
\]

Since we assume that the local units can be generated in arbitrarily small time, all the unitaries transformations that can be synthesized in a given time under \( H_d(t) \) can also be synthesized under \( H'_d(t) \) and vice versa [3]. We therefore consider \( H_d(t) \) and \( H'_d(t) \) are interchangeable resources under fast local unitaries. From now on we assume \( H_d(t) \) has only non-local terms.

**Proposition 3 (Canonical Decomposition [3, 4])** Any two-qubit non-local Hamiltonian \( H \) can be written in the form
\[
H = (A \otimes B)^\dagger (\theta_1 H_1 \sigma_x \otimes \sigma_x + \theta_2 H_2 \sigma_y \otimes \sigma_y + \theta_3 H_3 \sigma_z \otimes \sigma_z)(A \otimes B)
\]
and any two-qubit unitary \( U \in SU(4) \) may be written in the form
\[
U = (A_1 \otimes B_1) e^{-i(\theta_1 H_1 \sigma_x \otimes \sigma_x + \theta_2 H_2 \sigma_y \otimes \sigma_y + \theta_3 H_3 \sigma_z \otimes \sigma_z)}(A_2 \otimes B_2)
\]
where \( A_1, A_2, B_1, B_2 \) are single-qubit unitaries, and
\[
\frac{\pi}{4} \geq \theta_1 H_1 \geq \theta_2 H_2 \geq \theta_3 H_3 \geq 0
\]
We call \( \theta_1 H_1 \sigma_x \otimes \sigma_x + \theta_2 H_2 \sigma_y \otimes \sigma_y + \theta_3 H_3 \sigma_z \otimes \sigma_z \) the canonical form of \( H \) and \( U \) respectively, and \( \theta_1 H_1 \) and \( \theta_2 H_2 \) the canonical parameters of \( H \) and \( U \) respectively. For a 3-vector \( \vec{\beta} \), we denote
\[
H_{\vec{\beta}} = \beta_1 \sigma_x \otimes \sigma_x + \beta_2 \sigma_y \otimes \sigma_y + \beta_3 \sigma_z \otimes \sigma_z
\]
\[
U_{\vec{\beta}} = e^{-i(\beta_1 \sigma_x \otimes \sigma_x + \beta_2 \sigma_y \otimes \sigma_y + \beta_3 \sigma_z \otimes \sigma_z)}
\]

**C. Magic Basis**

The magic basis is a vector space basis for two-qubit pure states:
\[
\begin{align*}
|00\rangle + |11\rangle & : \frac{i}{\sqrt{2}} \langle 00\rangle - |11\rangle \\
|01\rangle + |10\rangle & : \frac{1}{\sqrt{2}} \langle 01\rangle - |10\rangle
\end{align*}
\]
The basis change from the standard basis \( \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \) to the magic basis is given by \( Q^{-1} \), where
\[
Q = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & i \\
0 & i & 1 & 0 \\
0 & i & -1 & 0 \\
1 & 0 & 0 & -i
\end{pmatrix}
\]
For elements \( U \in SU(4) \) the map \( U \to Q^{-1} U Q \) reflects the isomorphism between \( SU(2) \otimes SU(2) \) and \( SO(4) \) [10]. When expressed in the magic basis, canonical form Hamiltonian and unitaries are diagonal. In magic basis, the canonical decomposition takes the form \( H_d = K D_H K \), \( U = RD_H S \), where \( K \), \( R \) and \( S \) are real orthogonal matrices, and \( D_H, D_U \) are diagonal matrices. The diagonal elements of \( D_H \) and \( D_U \) are easily written in the terms of the canonical form parameters \( \theta_i, i \in \{H, U\} \). Define
\[
\begin{align*}
\varphi_1 &= \theta_1^i + \theta_2^i - \theta_3^i, \\
\varphi_2 &= \theta_1^i - \theta_2^i + \theta_3^i, \\
\varphi_3 &= \theta_2^i - \theta_1^i + \theta_3^i, \\
\varphi_4 &= \theta_1^i - \theta_2^i - \theta_3^i
\end{align*}
\]
The diagonal elements of $D_H$ are $(\varphi_1^H, \varphi_2^H, \varphi_3^H, \varphi_4^H)$ and the diagonal elements of $D_U$ are $(e^{-i\varphi_1^U}, e^{-i\varphi_2^U}, e^{-i\varphi_3^U}, e^{-i\varphi_4^U})$. Under choice of magic basis, $H_d$ is real symmetric and $(\varphi_1^H, \varphi_2^H, \varphi_3^H, \varphi_4^H)$ are its eigenvalues. Eq. (8) together with Eq. (10) implies that

$$\varphi_1^H \geq \varphi_2^H \geq \varphi_3^H \geq \varphi_4^H.$$ 

**Proposition 4** (8) Let $\bar{\alpha}$ and $\bar{\beta}$ be two real s-ordered 3-vectors, let $\bar{\lambda}$ and $\bar{\mu}$ be the 4-vectors related to $\bar{\alpha}$ and $\bar{\beta}$ respectively via (10), then $\bar{\lambda} \prec \bar{\mu}$ if $\bar{\alpha} \prec_s \bar{\beta}$.

The proof follows from the definitions.

### III. RESULT

The main result of this paper is as follows:

**Theorem 1** Let $\bar{\theta}^H(t)$ be the canonical parameters of $H_d(t)$ in (2) and $\bar{\theta}^U(T) = \int_0^T \theta^H(t)dt$, where the integration is performed for each entry of the vector. All the unitary operators that can be generated within time $T$ with $H_d(t)$ and fast local unitaries are given by the set

$$\mathcal{R}(T) = \{K_1U_{\bar{\beta}}K_2K_1, K_2 \in SU(2) \otimes SU(2), \bar{\beta} \prec_s \bar{\theta}(T)\}$$

**Remark 1** We prove this theorem by using the choice of magic basis. In this basis, $\{-iH_j\}$ are skew-symmetric matrices and generate the group SO(4). The interaction Hamiltonian $H_d$ can be expressed as $H_d = K^T D_\chi K$, where $K \in SO(4)$ and $D_\chi$ a diagonal matrix with diagonal entries $\lambda$ related to $\bar{\theta}^U$ via (10). Let $\bar{\gamma}(T) = \int_0^T \lambda(t)dt$, then in the basicıng series

$$\mathcal{R}(T) = \{Re^{-iD_\beta}S|R, S \in SO(4), \bar{\beta} \prec \bar{\gamma}(T)\}$$

**Proof:** Under the choice of magic basis, we can write

$$U(t) = R(t)A(t)S(t),$$

where $R(t), S(t) \in SO(4)$ and $A(t)$ be diagonal matrix. Assumption of fast local unitaries imply we can generate SO(4) instantly, so it suffices to prove all we can generate for the $A$ part is $e^{-iD_\beta}$,

$$\bar{\beta} \prec \bar{\gamma}(T).$$

Assume $U(t) = R(t)A(t)S(t)$ is a trajectory of Eq. (2), then $A(t) = R^T(t)U(t)S^T(t)$ and $\dot{A}(t) = R^T(t)U(t)S^T(t) + R^T(t)\dot{U}(t)S^T(t) + R^T(t)U(t)\dot{S}^T(t)$. Let $R^T(t) = r(t)R(t)$ and $S^T(t) = S^T(t)s(t)$, substituting for $\dot{U}(t)$, we get $R^T(t)\dot{U}(t)S^T(t) = R^T(t)[-iH_d(t) - i\sum_{j=1}^m v_j(t)H_j]R(t)(R^T(t)U(t)S^T(t))$. Using $H_d(t) = K^T(t)D_\chi K(t)$ we get $R^T(t)\dot{U}(t)S^T(t) = R^T(t)[-iK^T(t)D_\chi K(t) - i\sum_{j=1}^m v_j(t)H_j]R(t)(A(t))$. Let $P(t) = K(t)R(t)$ and denote $h(t) = R^T(t)[-i\sum_{j=1}^m v_j(t)H_j]R(t)$. Equation for evolution of $A(t)$ then takes the form $\dot{A}(t) = r(t)A(t) + [P^T(t)D_\chi(t)P(t)]A(t) + h(t)A(t) + A(t)s(t)$.

$$r(t)A(t) + [P^T(t)D_\chi(t)P(t)]A(t) + h(t)A(t) + A(t)s(t).$$

(11)

Notice that $r(t), s(t)$, and $h(t)$ are in so(4) (skew symmetric matrices of dimension 4) and hence their diagonal entries are all zero. When multiplied by a diagonal matrix $A(t)$, the diagonal entries remain zero. Therefore in the evolution equation of $A(t)$, these terms must sum to zero and we can discard these terms. We get

$$\dot{A}(t) = D_{-\bar{\mu}(t)}A(t)$$

(12)

where $\bar{\mu}(t)$ is the diagonal entries of $P^T(t)D_\chi(t)P(t)$. Since we can generate elements of SO(4) in arbitrarily small time, $R(t)$ and hence $P(t)$ can take value of any element in SO(4) and from Proposition 4, $\bar{\mu}(t)$ can take any element of the set $\{\bar{\mu}(t) | \bar{\mu}(t) \prec \bar{\lambda}(t)\}$. From Eq. (12), we get $A(T) = e^{-iD_\beta}$, where $\bar{\beta} = \int_0^T \bar{\mu}(t)dt = \bar{\lambda}(t)$. We first prove $\bar{\beta} \prec \bar{\gamma}(T)$, and then show that $\bar{\beta}$ can take on the values of any vector majorized by $\bar{\gamma}(T)$.

$$\sum_{j=1}^k \gamma_j^2(T) = \sum_{j=1}^k \int_0^T \lambda_j^2(t)dt$$

(13)

$$\sum_{j=1}^k \beta_j^2 = \sum_{j=1}^k \int_0^T \mu_{\sigma(j)}(t)dt$$

(14)

where $\sigma$ is some permutations and $k = 1, 2, 3, 4$. On subtracting Eq. (14) from Eq. (13), we get

$$\sum_{j=1}^k \gamma_j^2(T) - \sum_{j=1}^k \beta_j^2 = \int_0^T \sum_{j=1}^k \lambda_j^2(t) - \sum_{j=1}^k \mu_{\sigma(j)}(t)dt.$$ (15)

Since $\mu(t) \prec \lambda(t)$, $\sum_{j=1}^k \lambda_j^2(t) - \sum_{j=1}^k \mu_{\sigma(j)}(t) \geq 0$, and from Eq. (15), $\sum_{j=1}^k \gamma_j^2(T) - \sum_{j=1}^k \beta_j^2 \geq 0$. Obviously when $k = 4$, both terms equal 0, the equality holds, so $\bar{\beta} \prec \bar{\gamma}(T)$.

We now prove $\bar{\beta}$ can take on the values of all the vectors majorized by $\bar{\gamma}(T)$, which is the convex hull of $\bar{\gamma}(T)$ and all its permutations. If we take $R(t) = K^T(t)$, then $\bar{\beta} = \bar{\gamma}(T)$. It is also easy to see $\bar{\beta}$ can take all the permutations of $\bar{\gamma}(T)$, so we just need to prove that the vectors $\bar{\beta}$ can reach is a convex set. Let $\alpha \in [0, 1]$,

$$\bar{\beta}_1 = \int_0^T \mu_1(t)dt$$

(16)

$$\bar{\beta}_2 = \int_0^T \mu_2(t)dt$$

(17)
then
\[ \alpha \vec{\beta}_1 + (1 - \alpha) \vec{\beta}_2 = \int_0^T \alpha \vec{\mu}_1(t) + (1 - \alpha) \vec{\mu}_2(t) \, dt \] (18)
but \( \alpha \vec{\mu}_1(t) + (1 - \alpha) \vec{\mu}_2(t) \neq \vec{\lambda}(t) \), so \( \alpha \vec{\beta}_1 + (1 - \alpha) \vec{\beta}_2 \) can also be achieved.

Q.E.D

Given these theorems, we can compute the minimum time needed to generate any unitary operator \( U \) in \( SU(4) \) with \( H_d(t) \) and fast local unitaries.

**Theorem 2** Using the Hamiltonian \( H_d(t) \) and fast local unitaries, a two-qubit gate \( U \) can be generated within time \( T \) iff there exists a vector \( \vec{n} = (n_1, n_2, n_3) \) of integers, such that \( \vec{\beta}_\vec{n} = \vec{\theta} U + \frac{\pi}{4} \vec{n} \) satisfies
\[
\vec{\beta}_\vec{n} \sim_s \int_0^T \vec{\theta} H_d(t) \, dt
\]
where \( \vec{\theta} U \) and \( \vec{\theta} H_d(t) \) are the canonical parameters of \( U \) and \( H_d(t) \) respectively. The minimum time required to simulate \( U \) is given by the minimum value of \( T \geq 0 \) such that either
\[
\vec{\beta}_{(0,0,0)} \sim_s \int_0^T \vec{\theta} H_d(t) \, dt
\] (19)
or
\[
\vec{\beta}_{(-1,0,0)} \sim_s \int_0^T \vec{\theta} H_d(t) \, dt
\] (20)
holds.

The proof follows the treatment in [6].

**Proof:** Recall that all commutators \([\sigma_j \otimes \sigma_j, \sigma_k \otimes \sigma_k]\) vanish, and that \( \exp(-i \frac{\pi}{2} \sigma_j \otimes \sigma_j) = -i \sigma_j \otimes \sigma_j \) is a local gate. This implies that \( \vec{\theta} U + \frac{\pi}{4} \vec{n} \) represents all vectors compatible with the gate \( U \). It is straightforward to check from Eq. (4) that for any two vectors \( \vec{x} \) and \( \vec{y} \), with components \( x_1 \geq x_2 \geq |x_3|, y_1 \geq y_2 \geq |y_3| \), if \( y_1 \geq 3x_1 \), then \( \vec{x} \ll \vec{y} \). By definition \( \vec{\theta} U \geq 0 \), if some component \( n_j \) of \( \vec{n} \) fulfills \( |n_j| \geq 1 \), then the maximal component of the reordered version of \( \vec{\theta} U + \frac{\pi}{4} \vec{n} \) is at least \( \frac{3\pi}{4} \), which implies \( \vec{\theta} U \ll \vec{\theta} U + \frac{\pi}{4} \vec{n} \). Therefore we can restrict our attention to vectors \( \vec{n} \) with \( |n_j| \leq 1 \). A case by case check shows that for \( \vec{n} \in \{(-1, -1, -1), (0, -1, 0), (0, 0, 1), (0, 1, 0)\} \), \( \vec{\theta} U + \frac{\pi}{2} (-1, 0, 0) \ll \vec{\theta} U + \frac{\pi}{2} \vec{n} \) and for the remaining vectors \( \vec{n}, \vec{\theta} U + \frac{\pi}{2} (0, 0, 0) \ll \vec{\theta} U + \frac{\pi}{2} \vec{n} \). Thus the result follows.

**IV. EXAMPLE**

We now work an explicit example on finding the minimum time to synthesize a desired unitary under time varying couplings. Assume the interaction \( H_d(t) \) takes the form \( D(t)(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - 2\sigma_z \otimes \sigma_z) \). We compute the minimum time to generate a swap gate corresponding to the unitary transformation \( U = \exp(-i\frac{\pi}{2} \sigma_z \otimes \sigma_x + \sigma_y \otimes + \sigma_z \otimes \sigma_x) \).

To fix ideas, consider the case when \( D(t) \) is constant, say \( D > 0 \). The canonical parameters of \( H_d(t) \) and \( U \) are \( D(2, 1, -1) \) and \( \frac{\pi}{4}(1, 1, 1) \) respectively. The minimum time to generate \( U \) is the minimum \( T \) that satisfies \( \frac{\pi}{4}(1, 1, 1) \ll_s DT(2, 1, -1) \) or \( \frac{\pi}{4}(1, 1, 1) \ll_s DT(2, 1, -1) \), which is \( \frac{3\pi}{4} T \). The strategy to generate \( U \) is to use selective excitation on first spin preparing an effective Hamiltonian \( D(-\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + 2\sigma_z \otimes \sigma_z) \), which evolves \( \frac{\pi}{16} T \) units of time. This is followed by evolution of effective Hamiltonians \( D(-2\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \) and \( D(-\sigma_x \otimes \sigma_x + 2\sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \) for \( \frac{\pi}{16} T \) units of time each. In the end we apply a local unitaries \( e^{-i\frac{\pi}{4} \sigma_y \otimes \sigma_x} = -i \sigma_x \otimes \sigma_x \).

Now consider the time-dependent case, which models the variation of coupling strength between homo-nuclear spins under magic angle spinning [17]. The dipolar interaction strength \( D(t) \) during magic angle spinning varies in time as \( D(t) = D \frac{\cos(\theta(t)) - 1}{4} \), where \( \theta(t) \) is the angle intermolecular vector makes with the \( B_0 \) field. The angle \( \theta(t) \) changes as the sample is being rotated around an axis making an angle \( \theta_M = \tan^{-1}(\sqrt{2}) \) with the \( B_0 \) field. Let \( \beta \) denote the angle intermolecular axis makes with the magic angle axis. Then we can express \( \theta(t) \) as
\[
\cos(\theta(t)) = \cos(\beta) \cos(\theta_M) + \sin(\beta) \cos(\omega t) \sin(\theta_M),
\]
where \( \omega \) is the spinning frequency. \( D(t) \) is then a periodic function. We choose \( \beta = \frac{\pi}{4} \) and plot modulation of \( D(t) \) in figure 1. Each period of \( D(t) \) can be divided into two parts, \( \{D(t) \leq 0\} \cup \{D(t) > 0\} \). Let \( S_1, S_2 \) denote the area of these two parts respectively, i.e., \( S_1 = -\int_{D(t) \leq 0} D(t) \, dt \), \( S_2 = \int_{D(t) > 0} D(t) \, dt \). We find that \( S_1 = S_2 = \frac{3\pi}{16} D \).

The canonical parameters for \( H_d(t) \) are
\[
\begin{cases}
D(t)(2, 1, -1) \text{ for } D(t) \geq 0 \\
-D(t)(2, 1, 1) \text{ for } D(t) < 0
\end{cases}
\]
i.e., \( (2|D(t)|, |D(t)|, -D(t)) \). Using theorem 2, we get the minimum time to generate \( U \) is the smallest \( T \) that satisfies
\[
3\pi \leq \int_0^T 3|D(t)| - D(t) \, dt
\]
when \( \omega \gg D \), \( \int_0^T 3|D(t)| - D(t) \, dt \) is approximately \( n(2S_1 + 4S_2) \), where \( n \) is the number of periods of \( D(t) \) within time \( T \), so the minimum \( n = \left\lceil \frac{3\pi}{2\pi \omega} + \frac{4\pi \omega}{16D} \right\rceil = \left\lceil \frac{2652\omega}{16D} \right\rceil \) and the minimum time \( T \) is approximately \( 2\pi D \). The pulse sequence prepares effective Hamiltonians \((-\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + 2\sigma_z \otimes \sigma_z), (-2\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \) and \((-\sigma_x \otimes \sigma_x + 2\sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \) for \( n/3 \) periods each, in the part of the period when
FIG. 1: The figure shows the modulation of the coupling strength $D(t)$ as function of time for $\beta = \frac{\pi}{4}$.

$D(t) > 0$. Similarly, we prepare effective Hamiltonians $(-\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + 2\sigma_z \otimes \sigma_z)$, $(-2\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$ for $n/3$ periods each, in the part of the period when $D(t) < 0$. As before, we apply a local rotation $e^{-i\frac{\pi}{4} \sigma_x \otimes \sigma_x}$ in the end.

V. CONCLUSION

In this paper, we studied the problem of time-optimal synthesis of a unitary transformation for coupled qubits under non-stationary interactions. Under the assumption that local unitary transformations can be synthesized arbitrarily fast, we characterized the time optimal trajectories and the minimal time to prepare a general two qubit rotation under general time varying coupling tensor. These results generalize the results presented in [3, 6, 7] for stationary coupling Hamiltonians to the non-stationary case. The problem considered in this paper was motivated by design of time optimal pulse sequences for controlling coupled spin dynamics in solid state NMR spectroscopy, where couplings between spins are modulated in time due to magic angle spinning. The results presented here are of fundamental interest and may find applications in some implementations of quantum information processing.

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