Strongly Starlike Functions and Related Classes

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Abstract. We consider univalent functions, analytic in the unit disc $|z| < 1$ in the complex plane $\mathbb{C}$ which map $|z| < 1$ onto a domain with some nice property. The purpose of this paper is to find some new conditions for strong starlikeness and some related results.

1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{ z : |z| < 1 \}$ in the complex plane $\mathbb{C}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we denote by

$$\mathcal{H}[a, n] = \{ f \in \mathcal{H} : f(z) = a + a_n z^n + \cdots, a_n \neq 0 \}$$

and

$$\mathcal{A}_n = \{ f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + \cdots \},$$

so $\mathcal{A} = \mathcal{A}_1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{D}$.

A set $E \subset \mathbb{C}$ is said to be starlike with respect to the point $0 \in E$ if and only if the linear segment joining the origin to every other point $w \in E$ lies entirely in $E$, while a set $E$ is said to be convex if and only if it is starlike with respect to each of its points. Let $\mathcal{S}^*$ denote the class of all functions $f \in \mathcal{S}$ such that the set $f(\mathbb{D})$ is starlike with respect to 0. A natural extension of the notion starlike is to be starlike of order $\alpha$. The class $\mathcal{S}_\alpha^*$ of starlike functions of order $\alpha < 1$ may be defined as

$$\mathcal{S}_\alpha^* = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > \alpha, z \in \mathbb{D} \right\}.$$

The class $\mathcal{S}_\alpha^*$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha < 1$

$$\mathcal{K}_\alpha : = \left\{ f \in \mathcal{A} : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in \mathbb{D} \right\}.$$

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\begin{equation}
= \{ f \in \mathcal{A} : zf' \in S_{\alpha}^* \}
\end{equation}
introduced Robertson in [14], see also [5]. If \( \alpha \in [0; 1) \), then a function in either of these sets is univalent, if \( \alpha < 0 \) it may fail to be univalent. In particular we denote \( S_0^* = S^*, K_0 = K \), the classes of starlike and convex functions, respectively. Furthermore, note that if \( f \in K_{\alpha} \) then \( f \in S_{\delta(\alpha)}^* \), see [18], where
\begin{equation}
\delta(\alpha) = \begin{cases} 
\frac{1 - 2\alpha}{2 - 2\alpha - 2} & \text{for } \alpha \neq \frac{1}{2}, \\
\frac{1}{2\log 2} & \text{for } \alpha = \frac{1}{2}.
\end{cases}
\end{equation}
Let \( SS^*(\beta) \) denote the class of strongly starlike functions of order \( \beta, 0 < \beta < 2 \),
\begin{equation}
SS^*(\beta) := \left\{ f \in \mathcal{A} : \left| \arg \frac{zf''(z)}{f'(z)} \right| < \frac{\beta \pi}{2}, \ z \in \mathbb{D} \right\},
\end{equation}
which was introduced in [15] and [1], see also [9]. Furthermore, \( SK(\beta) = \{ f \in \mathcal{A} : zf' \in SS^*(\beta) \} \) denote the class of strongly convex functions of order \( \beta \). Analogously to (1.1), in the work [8] it was proved that if \( \beta \in (0, 1) \) and \( f \in SK(\alpha(\beta)) \), then \( f \in SS(\beta) \), where
\begin{equation}
\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \left( \frac{\beta n(\beta) \sin(\pi(1 - \beta)/2)}{m(\beta) + \beta n(\beta) \cos(\pi(1 - \beta)/2)} \right),
\end{equation}
and where
\begin{align*}
m(\beta) &= (1 + \beta)^{(1 + \beta)/2},
m(\beta) &= (1 - \beta)^{(\beta - 1)/2}.
\end{align*}
The class \( G(\alpha, \gamma) \), \( \gamma > 0, 0 < \alpha \leq 1 \) of \( \gamma \)-strongly starlike functions of order \( \alpha \) consists of functions \( f \in \mathcal{A} \) satisfying
\begin{equation}
\left| \arg \left\{ \left( \frac{zf''(z)}{f'(z)} \right)^{1 - \gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{\gamma} \right\} \right| < \frac{\alpha \pi}{2}, \ z \in \mathbb{D},
\end{equation}
and such that
\begin{equation}
f(z)f'(z) \left( 1 + \frac{zf''(z)}{f'(z)} \right) \neq 0, \ z \in \mathbb{D} \setminus \{ 0 \}.
\end{equation}
Note that Lewandowski, S. Miller and Zlotkiewicz 1974 [6] have introduced the class of \( \gamma \)-starlike functions, denoted here by \( G(1, \gamma) \), which satisfy (1.5) and such that
\begin{equation}
\Re \left\{ \left( \frac{zf'(z)}{f(z)} \right)^{1 - \gamma} \left( 1 + \frac{zf'(z)}{f'(z)} \right)^{\gamma} \right\} > 0, \ z \in \mathbb{D}.
\end{equation}

**Lemma 1.1.** (Nunokawa [8]) Let \( q(z) \) be in \( \mathcal{H}[1, m] \) with \( q(z) \neq 0 \). If there exists a point \( z_0, |z_0| < 1 \), such that \( |\arg \{ q(z_0) \}| < \pi \gamma /2 \) for \( |z| < |z_0| \) and \( |\arg \{ q(z_0) \}| = \pi \gamma /2 \) for some \( \gamma \in (0, 2) \), then we have
\begin{equation}
\frac{z_0q'(z_0)}{q(z_0)} = \frac{2ik}{\pi} \arg \{ q(z_0) \},
\end{equation}
for some \( k \geq m(a + a^{-1})/2 \geq (a + a^{-1})/2 \), where \( \{ q(z_0) \}^{1/\gamma} = \pm ia \), and \( a > 0 \).
It should be remarked that for interesting related results associated with Lemma 2.1, the authors may refer to [7]. In this paper we consider some problems related to Lemma 2.1, which may be called as Nunokawa’s lemma.

2 Main result

**Lemma 2.1.** Let \( \beta \) be in \((0, 1)\). Then we have \( \forall \beta \in (0, 1) : \)

\[
\arg \left\{ e^{i\pi(1-\beta)/2} \left[ \left( \frac{1 + \beta}{1 - \beta} \right)^{(1-\beta)/2} + \left( \frac{1 + \beta}{1 - \beta} \right)^{(1+\beta)/2} \right] - 1 \right\} \leq \pi - \frac{\pi \beta}{2}. \tag{2.1}
\]

**Proof.** Observe that

\[
\forall \beta \in (0, 1) : 1 < \left( \frac{1 + \beta}{1 - \beta} \right)^{(1-\beta)/2} + \left( \frac{1 + \beta}{1 - \beta} \right)^{(1+\beta)/2} < 2.
\]

Therefore, a geometrical observation shows that for all \( \beta \in (0, 1) \), we have

\[
\arg \left\{ e^{i\pi(1-\beta)/2} \left[ \left( \frac{1 + \beta}{1 - \beta} \right)^{(1-\beta)/2} + \left( \frac{1 + \beta}{1 - \beta} \right)^{(1+\beta)/2} \right] - 1 \right\} \\
\leq \arg \left\{ e^{i\pi(1-\beta)/2} - 1 \right\} \\
= \frac{\pi}{2} + \frac{(1 - \beta)\pi}{4} \\
< \pi - \frac{\pi \beta}{2}.
\]

\( \square \)

**Theorem 2.1.** Let \( \beta \) be in \((0, 1)\), \( p(z) \in \mathcal{H}[1, 1] \), and suppose that

\[
\left| \arg \left\{ \frac{zp'(z)}{p(z) - p(z)} \right\} \right| > \frac{\pi \alpha(\beta)}{2}, \quad z \in \mathbb{D}, \tag{2.2}
\]

then we have

\[
|\arg \{ p(z) \}| < \frac{\pi \beta}{2}, \quad z \in \mathbb{D}, \tag{2.3}
\]

where

\[
\tan \frac{\pi \alpha(\beta)}{2} = \tan \frac{\pi \beta}{2} + \frac{\beta}{(1 - \beta) \cos(\pi \beta/2)} \left( \frac{1 - \beta}{1 + \beta} \right)^{(1+\beta)/2}. \tag{2.4}
\]

**Proof.** If there exists a point \( z_0, \ |z_0| < 1 \), such that \( |\arg \{ p(z) \}| < \pi \beta/2 \) for \( |z| < |z_0| \) and \( |\arg \{ p(z_0) \}| = \pi \beta/2 \) for some \( 0 < \beta < 1 \), then by Lemma 1.1 we have

\[
\frac{z_0p'(z_0)}{p(z_0)} = 2i k \frac{\arg \{ p(z_0) \}}{\pi}, \tag{2.5}
\]
for some
\[ k \geq (a + a^{-1})/2, \]
where
\[ \{p(z_0)\}^{1/\beta} = \pm ia \]
and \( a > 0 \). Then, for the case \( p(z_0) = a^\beta (\cos(\pi \beta/2) + i \sin(\pi \beta/2)) \), we have from (2.5)
\[ \frac{z_0p'(z_0)}{p^2(z_0)} - 1 = e^{i\pi(1-\beta)/2} \beta k \frac{1}{a^\beta} - 1. \] (2.7)
Note that
\[ \beta k \frac{1}{a^\beta} \geq \frac{\beta}{2}(a^{1-\beta} + a^{-1-\beta}). \]
Let us put
\[ g(a) = \frac{1}{2}(a^{1-\beta} + a^{-1-\beta}), \quad a > 0. \]
Then we have
\[ g'(a) = \frac{1}{2}((1 - \beta)a^{-\beta} - (1 + \beta)a^{-2+\beta}), \]
and so, \( g(a) \) takes its minimum value at
\[ a = \sqrt{(1 + \beta)/(1 - \beta)}. \] (2.8)
Furthermore, for the function
\[ H(\beta) = \left( \frac{1 + \beta}{1 - \beta} \right)^{(1-\beta)/2} + \left( \frac{1 + \beta}{1 - \beta} \right)^{-(1+\beta)/2}, \quad \beta \in (0, 1), \]
the \( H'(\beta) \) decreases from 0, when \( \beta \in (0, 1) \). Therefore, \( H(\beta) \) is a decreasing function, from 2 to 1:
\[ \lim_{\beta \to 1^-} = \left( \frac{1 + \beta}{1 - \beta} \right)^{(1-\beta)/2} + \left( \frac{1 + \beta}{1 - \beta} \right)^{-(1+\beta)/2} = 1. \]
Therefore, from (2.7) and from (2.8), we have
\[ \arg \left\{ \frac{z_0p'(z_0)}{p^2(z_0)} - 1 \right\} = \arg \left\{ e^{i\pi(1-\beta)/2} \beta k \frac{1}{a^\beta} - 1 \right\} \]
\[ \leq \arg \left\{ e^{i\pi(1-\beta)/2} \left( \left( \frac{1 + \beta}{1 - \beta} \right)^{(1-\beta)/2} + \left( \frac{1 + \beta}{1 - \beta} \right)^{-(1+\beta)/2} \right) - 1 \right\} \]
\[ \leq \pi - \frac{\pi \beta}{2} \] (2.9)
because of Lemma 2.1. We also have
\[ \frac{z_0p'(z_0)}{p(z_0)} - p(z_0) = p(z_0) \left\{ \frac{z_0p'(z_0)}{p^2(z_0)} - 1 \right\} \]
= (ia)^β \left( iβk \frac{1}{(ia)^β} - 1 \right) \\
= a^β e^{iπβ/2} \left( e^{iπ(1-β)/2} βk \frac{1}{a^β} - 1 \right).

Hence we may write

\[ \arg \left\{ \frac{z_0 p'(z_0)}{p(z_0)} - p(z_0) \right\} = \arg \left\{ p(z_0) \left\{ \frac{z_0 p'(z_0)}{p^2(z_0)} - 1 \right\} \right\} \]

\[ = \arg \left\{ p(z_0) \right\} + \arg \left\{ \frac{z_0 p'(z_0)}{p^2(z_0)} - 1 \right\} \]

\[ = \frac{πβ}{2} + \arg \left\{ e^{iπ(1-β)/2} \frac{βk}{a^β} - 1 \right\} \]

\[ \leq \frac{πβ}{2} + \arg \left\{ e^{iπ(1-β)/2} \left( \frac{1 + β}{1 - β} \right)^{(1-β)/2} + \left( \frac{1 + β}{1 - β} \right)^{-(1+β)/2} \right\} - 1 \]

\[ = \frac{πβ}{2} + \tan^{-1} \left( \frac{β}{1 - β} \right)^{(1+β)/2} \left( \frac{1 + β}{1 - β} \right)^{(1-β)/2} \sin \frac{π(1-β)}{2} \]

\[ - \frac{1}{1 + \frac{β}{1 - β}} \left( \frac{1 - β}{1 + β} \right)^{(1-β)/2} \cos \frac{π(1-β)}{2} \]

\[ = \frac{πβ}{2} + \tan^{-1} \left( \frac{β}{1 - β} \right)^{(1+β)/2} \frac{π(1-β)}{m(β) + βn(β) \cos \frac{π(1-β)}{2}} \]

\[ \text{where} \]

\[ m(β) = (1 + β)^{(1+β)/2}, \quad n(β) = (1 - β)^{(β-1)/2}. \]

This contradicts the hypothesis (2.2) and therefore, it completes the proof of Theorem 2.1.

For \( β = 1/2 \) Theorem 2.1 becomes the following corollary.

**Corollary 2.2.** Let \( p(z) \) be in \( H[1, 1] \), and suppose that

\[ \left| \arg \left\{ \frac{zp'(z)}{p(z)} - p(z) \right\} \right| > \frac{π}{2} \tan^{-1}(1 + \sqrt{108}) \]

\[ = \frac{π}{2} (0.85 \ldots) = 1.3383 \ldots, \quad z \in \mathbb{D} \]

then we have

\[ |\arg \{ p(z)\}| < \frac{π}{4}, \quad z \in \mathbb{D}. \]

For \( p(z) = zf''(z)/f(z) \), \( f(z) \in \mathcal{A} \), Theorem 2.1 becomes the following corollary.
Corollary 2.3. Let \( \beta \) be in \((0, 1)\). Assume that \( f(z) \in A \) and \( zf'(z)/f(z) \in H \) in \( \mathbb{D} \), and that

\[
\left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} \right\} \right| > \frac{\pi \alpha(\beta)}{2}, \quad z \in \mathbb{D},
\]

then we have

\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi \beta}{2}, \quad z \in \mathbb{D},
\]

where \( \alpha(\beta) \) is defined in (2.4). This means that \( f(z) \in SS^*(\beta) \) or \( f(z) \) is a strongly starlike function of order \( \beta \), see (1.2).

Corollary 2.2 and 2.3 give together the following result.

Corollary 2.4. Assume that \( f(z) \in A \) and \( zf'(z)/f(z) \in H \) in \( \mathbb{D} \), and that

\[
\left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} - \frac{2zf'(z)}{f(z)} \right\} \right| > \frac{\pi 2 \tan^{-1} (1 + \sqrt{108})}{2} = \frac{\pi}{2} (0.85 \ldots) = 1.3383 \ldots, \quad z \in \mathbb{D},
\]

then we have

\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{4}, \quad z \in \mathbb{D},
\]

or \( f(z) \) is a strongly starlike function of order 1/2, see (1.2).

Corollary 2.5. Assume that \( f(z) \in A \) and \( zf'(z)/f(z) \in H \) in \( \mathbb{D} \), and that

\[
\left| \arg \left\{ \sqrt{\frac{zf'(z)}{f(z)}} - \frac{zf''(z)}{f'(z)} \right\} \right| > \frac{\pi \alpha(\beta)}{2}, \quad z \in \mathbb{D},
\]

then we have

\[
\left| \arg \left\{ \sqrt{1 + \frac{zf''(z)}{f'(z)}} \right\} \right| < \frac{\pi \beta}{2}, \quad z \in \mathbb{D},
\]

where \( \alpha(\beta) \) is defined in (2.4), or \( f(z) \) is a 1/2-strongly starlike functions of order \( \beta \), see (1.6).

Theorem 2.6. [12] If \( f(z) \) is analytic, \( h(z) \) is convex univalent in \( \mathbb{D} \), and that

\[
\left| \arg \left\{ \frac{f'(z)}{h'(z)} \right\} \right| \leq \frac{\alpha \pi}{2}, \quad z \in \mathbb{D} \quad (2.10)
\]

for some \( \alpha \), \( 0 \leq \alpha \leq 1 \), then

\[
\left| \arg \left\{ \frac{f(z_2) - f(z_1)}{h(z_2) - h(z_1)} \right\} \right| \leq \frac{\alpha \pi}{2} \quad (2.11)
\]

for all \( z_1 \in \mathbb{D} \), and \( z_2 \in \mathbb{D} \).
Notice that (2.10) implies that \( f(z) \) satisfies Ozaki’s univalence condition [11], or that \( f(z) \) strongly close-to-convex of order \( \alpha \) with respect to \( g \). Recall [13], that \( f(z) \in A \) is said to be in the class \( C(\alpha) \) of strongly close-to-convex functions of order \( \alpha \), \( 0 < \alpha \leq 1 \), if and only if there exist \( g \in K, \varphi \in \mathbb{R} \), such that

\[
\left| \arg \left\{ \frac{f'(z)}{e^{i\varphi}g'(z)} \right\} \right| < \frac{\pi \alpha}{2}, \quad z \in \mathbb{U}.
\] (2.12)

If \( \alpha = 1 \), then \( C(\alpha) \) becomes the well known class of close-to-convex functions, Kaplan [3]. Functions defined by (2.12) with \( \varphi = 0, \alpha = 1 \) where considered earlier by Ozaki [11], see also Umezawa [16, 17]. Moreover, Lewandowski [4, 5] defined the class of functions \( f \in A \) for which the complement of \( f(\mathbb{U}) \) with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski’s class is identical with the Kaplan’s class \( C(1) \). It is worthy to note that \( f \in A \) satisfies the condition (2.11) with some convex univalent \( g \) and \( \alpha = 1 \) if and only if \( f \) is close-to-convex function, see [2, p. 31].

**Theorem 2.6** was improved to the following.

**Theorem 2.7.** [10] If \( f(z) \) is analytic, \( g(z) \) is convex univalent in \( |z| < 1 \), \( f'(0) = g'(0) \) and

\[
\left| \arg \left\{ \frac{f'(z)}{g'(z)} \right\} \right| \leq \frac{\alpha \pi}{2}, \quad \text{in } |z| < 1,
\] (2.13)

for some \( \alpha, 0 \leq \alpha \leq 1 \), then

\[
\left| \arg \left\{ \frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)} \right\} \right| < \alpha \left( \frac{\pi}{2} - \log 2 \right), \quad |z_1| < 1, \quad |z_2| < 1,
\] (2.14)

for all \( z_1 \in \mathbb{D} \), and \( z_2 \in \mathbb{D} \), where \( \pi/2 - \log 2 = 0.877649147 \ldots \)

In this paper, we will improve Theorem 2.6 by adding some conditions.

**Theorem 2.8.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \), \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) be analytic in \( \mathbb{D} \), and suppose that

\[
\frac{g(z)}{zg'(z)} < \frac{1 + \delta z}{1 - \delta z}, \quad z \in \mathbb{D}
\] (2.15)

for some \( \delta, 0 < \delta \leq 1 \), and

\[
\left| \arg \left\{ \frac{f'(z)}{g'(z)} \right\} \right| < \frac{\pi}{2} \left\{ \alpha + \frac{2}{\pi} \tan^{-1} \frac{\alpha(1 - \delta)^2}{1 + 2\alpha\delta - \delta^2} \right\}, \quad z \in \mathbb{D},
\] (2.16)

and

\[
\tan \frac{\alpha \pi}{2} \leq \frac{(1 - \delta)^2}{2\delta}
\] (2.17)

for some \( \alpha, 0 < \alpha < 1 \). Then we have

\[
\left| \arg \left\{ \frac{f(z)}{g(z)} \right\} \right| \leq \frac{\alpha \pi}{2}, \quad z \in \mathbb{D}.
\] (2.18)
Proof. Let us put
\[ p(z) = \frac{f(z)}{g(z)}, \quad p(0) = 1, \quad z \in \mathbb{D}. \]  
(2.19)

Then it follows that
\[ f(z) = p(z)g(z), \quad f'(z) = g'(z)p(z) + g(z)p'(z), \]
and
\[ \frac{f'(z)}{g'(z)} = p(z) + \frac{g(z)}{g'(z)}p'(z) = p(z)\left(1 + \frac{g(z)}{zg'(z)} \frac{zp'(z)}{p(z)}\right). \]  
(2.20)

From the hypothesis, we have
\[ 1 - \delta \frac{1}{1 + \delta} < \Re \left\{ \frac{g(z)}{zg'(z)} \right\} < 1 + \delta \frac{1}{1 - \delta}, \quad z \in \mathbb{D} \]  
(2.21)
and
\[ -2\delta \frac{1}{1 - \delta^2} < \Im \left\{ \frac{g(z)}{zg'(z)} \right\} < 2\delta \frac{1}{1 - \delta^2}, \quad z \in \mathbb{D}. \]  
(2.22)

If there exists a point \( z_0 \) with \( |z_0| < 1 \), such that \( |\arg\{p(z)\}| < \pi \alpha/2 \) for \( |z| < |z_0| \) and \( |\arg\{p(z_0)\}| = \pi \alpha/2 \) for some \( 0 < \alpha < 1 \), then by Lemma 2.1 we have
\[ \frac{z_0p'(z_0)}{p(z_0)} = \frac{2ik}{\pi} \arg\{p(z_0)\}, \]  
(2.23)
for some
\[ k \geq (a + a^{-1})/2, \]
where
\[ \{p(z_0)\}^{1/\alpha} = \pm ia \]
and \( a > 0 \). Then, for the case \( p(z_0) = a^\alpha(\cos(\pi \alpha/2) + i \sin(\pi \alpha/2)) \), we have from (2.21)
\[ \arg\left\{ \frac{f'(z_0)}{g'(z_0)} \right\} = \arg\left\{ p(z_0)\left(1 + \frac{g(z_0)}{z_0g'(z_0)} \frac{zp'(z_0)}{p(z_0)}\right) \right\} \]
Notice that if we assume \( \arg\{w\} \in [0, \pi] \), then under the hypothesis of Theorem 2.8, we may apply the formula
\[ \arg\left\{ p(z_0)\left(1 + \frac{g(z_0)}{z_0g'(z_0)} \frac{zp'(z_0)}{p(z_0)}\right) \right\} = \arg\{p(z_0)\} + \arg\left\{ 1 + \frac{g(z_0)}{z_0g'(z_0)} \frac{zp'(z_0)}{p(z_0)}\right\}. \]
It is because of that
\[ \arg\{p(z_0)\} \in [0, \pi], \]
\[ \arg\left\{ 1 + \frac{g(z_0)}{z_0g'(z_0)} \frac{zp'(z_0)}{p(z_0)}\right\} \in [0, \pi], \]
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$$\arg \{p(z_0)\} + \arg \left\{ 1 + \frac{g(z_0)}{z_0 g'(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} \right\} \in [0, \pi], \quad (2.24)$$

where (2.24) holds because of (2.17), namely

$$\arg \{p(z_0)\} + \arg \left\{ 1 + \frac{g(z_0)}{z_0 g'(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} \right\} \in [0, \pi]$$

\[ \Leftrightarrow \frac{\alpha \pi}{2} + \pi - \tan^{-1} \frac{\alpha}{1 + \delta} \in [0, \pi] \]

\[ \Leftrightarrow \frac{\alpha \pi}{2} + \pi - \tan^{-1} \frac{\alpha}{1 + \delta} \in [0, \pi] \]

\[ \Leftrightarrow 0 \leq \frac{\alpha \pi}{2} + \pi - \tan^{-1} \frac{\alpha}{1 + \delta} \leq \pi \]

\[ \Leftrightarrow 0 \leq \tan^{-1} \frac{1 - \delta}{2 \delta} - \frac{\alpha \pi}{2} \leq \pi \]

\[ \Leftrightarrow \frac{\alpha \pi}{2} \leq \tan^{-1} \frac{1 - \delta}{2 \delta} \]

\[ \Leftrightarrow \frac{\alpha \pi}{2} \leq \tan^{-1} \left( \frac{1 - \delta}{2 \delta} \right)^2 \]

\[ \Leftrightarrow \tan \frac{\alpha \pi}{2} \leq \frac{(1 - \delta)^2}{2 \delta}. \]

Therefore, we have

$$\arg \left\{ \frac{f'(z_0)}{g'(z_0)} \right\} = \arg \{p(z_0)\} + \arg \left\{ 1 + \frac{g(z_0)}{z_0 g'(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} \right\}$$

$$= \frac{\alpha \pi}{2} + \arg \left\{ 1 + i \alpha k \frac{g(z_0)}{z_0 g'(z_0)} \right\}. \quad (2.25)$$

We know that

$$\frac{g(z_0)}{z_0 g'(z_0)}$$

lies in a rectangle described by (2.21), (2.22) in the right half-plane. Therefore,

$$\left\{ i \alpha \frac{g(z_0)}{z_0 g'(z_0)} \right\}$$

lies in a rectangle in the upper half-plane and

$$-\alpha \frac{2 \delta}{1 - \delta^2} < \Re \left\{ i \alpha \frac{g(z)}{z g'(z)} \right\} < -\alpha \frac{2 \delta}{1 - \delta^2}, \quad z \in \mathbb{D} \quad (2.26)$$

and

$$\alpha \frac{1 - \delta}{1 + \delta} < \Im \left\{ i \alpha \frac{g(z)}{z g'(z)} \right\} < \alpha \frac{1 + \delta}{1 - \delta}, \quad z \in \mathbb{D}. \quad (2.27)$$
This gives
\[ 1 - \alpha \frac{2\delta}{1 - \delta^2} < \Re \left\{ 1 + i\alpha \frac{g(z)}{zg'(z)} \right\} < 1 + \alpha \frac{2\delta}{1 - \delta^2}, \quad z \in \mathbb{D} \]
and
\[ \alpha \frac{1 - \delta}{1 + \delta} < \Im \left\{ 1 + i\alpha \frac{g(z)}{zg'(z)} \right\} < \alpha \frac{1 + \delta}{1 - \delta}, \quad z \in \mathbb{D}, \]
and so
\[ \tan^{-1} \frac{\alpha \frac{1 - \delta}{1 + \delta}}{1 + \alpha \frac{2\delta}{1 - \delta^2}} \leq \arg \left\{ 1 + i\alpha k \frac{g(z_0)}{z_0g'(z_0)} \right\}. \tag{2.28} \]

Inequality (2.28) together with (2.25) contradicts condition (2.16). For the case \( \arg \{p(z_0)\} = -\pi \alpha /2 \), applying the same method as above we can get
\[ \arg \left\{ \frac{f(z_0)}{g(z_0)} \right\} \leq \frac{\pi}{2} \left\{ \alpha + \frac{2}{\pi} \tan^{-1} \frac{\alpha (1 - \delta)^2}{1 + 2\alpha \delta - \delta^2} \right\}. \]
This is also contradictions hypothesis and therefore it completes the proof of Theorem 2.8. \( \square \)

Observe that condition (2.15) satisfies the function
\[ g(z) = \frac{z}{(1 + \delta z)^2} \]
so Theorem 2.8 becomes the following corollary.

**Corollary 2.9.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \), \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) be analytic in \( \mathbb{D} \), and suppose that
\[ \left| \arg \left\{ \frac{f'(z)}{(1 + \delta z)^3} \right\} \right| < \frac{\pi}{2} \left\{ \alpha + \frac{2}{\pi} \tan^{-1} \frac{\alpha (1 - \delta)^2}{1 + 2\alpha \delta - \delta^2} \right\}, \quad z \in \mathbb{D}, \tag{2.29} \]
and
\[ \tan \frac{\alpha \pi}{2} \leq \frac{(1 - \delta)^2}{2\delta} \tag{2.30} \]
for some \( \alpha, 0 < \alpha < 1 \). Then we have
\[ \left| \arg \left\{ \frac{(1 + \delta z)^2 f(z)}{z} \right\} \right| \leq \frac{\alpha \pi}{2}, \quad z \in \mathbb{D}. \tag{2.31} \]

For \( \alpha = 1/2 \) and \( \delta = 2 - \sqrt{3} \) Theorem 2.8 becomes the following corollary.

**Corollary 2.10.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \), \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) be analytic in \( \mathbb{D} \), and suppose that
\[ \frac{g(z)}{zg'(z)} < \frac{1 + (2 - \sqrt{3}) z}{1 - (2 - \sqrt{3}) z}, \quad z \in \mathbb{D} \]
and
\[ \arg \left\{ \frac{f'(z)}{g'(z)} \right\} \leq \frac{\pi}{2} \left( \frac{1}{2} + \frac{2}{\pi} \tan^{-1} \frac{2\sqrt{3} - 1}{11} \right) = \frac{\pi}{2} \cdot 0.64 \ldots, \quad z \in \mathbb{D}, \]

then we have
\[ \arg \left\{ \frac{f(z)}{g(z)} \right\} \leq \frac{\pi}{4}, \quad z \in \mathbb{D}. \]

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