The Kramers problem for Holway—Shakhov equation

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The analytical solution of a problem on isothermal sliding of rarefied gas along a flat firm surface (the Kramers problem) for Holway—Shakhov equation is presented.

Key words: Kramers problem, Holway—Shakhov equation, division of variables, dispersion matrix–function, characteristic equation, eigen vectors of continuous and attached to continuous spectra, expansion in eigen vectors, boundary value Riemann—Hilbert problem, homogeneous and inhomogeneous boundary value problem, resolvability conditions, slip velocity.

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1. Introduction

One of the first known problems of the kinetic theory, for which the exact solution is received, is the Kramers problem, or the problem about an isothermal flow of gas with sliding. For this problem some its methods of the exact solution are constructed (see, for example, [1], [2], [3], [4], [5], [6]).

Interest to problems of gas flow with sliding is important because its solution allows to calculate, in particular, boundary conditions for the Navier—Stokes equation.

The kinetic Holway—Shakhon equation was already used for solution boundary problems of the kinetic theory (see, [7] - [9]).

In work [10] parametres of Holway—Shakhon equation were expressed through Prandtl number, self-diffusion coefficient and kinematic viscosity.

Let’s notice, that the kinetic ellipsoidal statistical equation of Holway [11] was is applied to the solution of the Smoluchovsky problem about temperature
jump in work \[12\]. In works \[13\] – \[15\] were studied boundary problems of the kinetic theory, in particular, the Kramers problem for binary gases.

In work \[16\] the analytical solution of the Kramers problem has been received for the ellipsoidal statistical equation with frequency of collisions of the molecules, proportional to the module of molecules velocity.

The analytical solution of problems about isothermal and thermal sliding with accommodation boundary conditions has been received in work \[17\].

Later in works \[18\] and \[19\] the Kramers problem was generalized on the case of quantum gases. So, in work \[18\] the case of Fermi gases was considered, and in work \[19\] the case of Bose gases was considered.

In work \[20\] have been entered moment boundary conditions for boundary problems for the rarefied gas. In \[20\] the Kramers problem was solved for 2-moment boundary conditions.

On example of Kramers problem in works \[21\] – \[24\] were effective methods of the approached solution of boundary problems of kinetic theory are developed.

Let the half-space \(x > 0\) is occupied by the one-nuclear rarefied gas, a plane \(yz\) is combined with a wall, gas moves in the axis direction \(y\) with the mass velocity \(u_y(x)\). Far from a wall the constant of a gradient of mass velocity of gas is set

\[
g_v = \left(\frac{du_y(x)}{dx}\right)_{x=+\infty}. \tag{1.1}
\]

The given gradient of mass velocity of gas far from a wall causes so-called sliding of gas along a wall with some unknown speed of sliding, proportional quantity of the gradient of mass speed. In the Kramers problem is required to define this unknown velocity of sliding, to construct function of distribution of gas molecules and to find distribution mass velocity of gas in half-space.

Let’s notice, that the gradient of mass velocity \(g_v\) and the gradient of dimensionless mass velocity \(U_y(x_1) = \sqrt{\beta}u_y(x_1)\) (on dimensionless coordinate \(x_1 = \nu\sqrt{\beta}x\)) are connected by the relation

\[
G_v = \frac{g_v}{\nu}.
\]

Let’s consider, that for the dimensionless gradient \(G_v = g_v/\nu\) the inequality
is carried out

\[ G_v \ll 1. \] \hspace{1cm} (1.2)

The condition (1.2) allows to consider Kramers problem in the linear statement.

From (1.1) follows, that for the mass velocity in half-space \( x > 0 \) far from boundary looks like

\[ u_y(x) = u_0 + g_v x + o(1), \quad x \to +\infty, \] \hspace{1cm} (1.3)

where the quantity \( u_{sl} \) is called as velocity of isothermal slidings along a flat surface. Dimensionless velocity of sliding is equal \( U_{sl} = \sqrt{\beta} u_{sl} \).

As the boundary condition on a wall we will accept the condition purely diffusion reflexion of molecules from a wall

\[ f(x = 0, y, z, v) = f_0(v), \quad v_x > 0. \] \hspace{1cm} (1.4)

Here \( f_0(v) \) is the absolute Maxwellian,

\[ f_0(v) = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left[ -\frac{mv^2}{2kT} \right], \]

i.e. the molecules reflected from a wall have Maxwellian distribution by the velocities.

Concentration of gas and temperature in the Kramers problem about isothermal sliding are considered as constants.

In the Kramers problem it is possible to consider distribution function depending from velocity of molecules \( v \) and one spatial coordinate \( x \). Distribution function we will search in the form

\[ f(x, v) = f_0(v)[1 + h(x, C)]. \] \hspace{1cm} (1.5)

The mass velocity of gas directed along an axis \( y \), looks like

\[ u_y(x) = \frac{1}{n} \int v_y f(x, v) \, d^3v. \] \hspace{1cm} (1.6)

Boundary condition on the wall under the condition diffusion reflexions of molecules from the wall we will receive, if we will substitute decomposition
(1.5) in condition (1.4). As result we receive the first boundary condition (condition on the wall)

\[ h(0, C_x) = 0, \quad C_x > 0. \]  

(1.7)

The second boundary condition (condition "far from the wall") follows from this the fact, that at \( x \to +\infty \) distribution function \( h(x_1, C) \) passes into asymptotic function \( h_{as}(x_1, C) \). Asymptotic function should be solution of the Holway—Shakhov equation.

2. The Kramers problem for Holway—Shakhov equation

For Kramers problem the Holway—Shakhov equation becomes simpler and has the following form \[10\]

\[ C_x \frac{\partial h}{\partial x} + h(x, C) = 2C_y U_y(x) + \gamma C_y \left( C^2 - \frac{5}{2} \right) Q_y(x) + 2\omega C_x C_y P_{xy}(x). \]  

(2.1)

In this equation the variable \( x \) is the dimensionless coordinate, connected with dimensional coordinate \( x_1 \) by the relation \( x = \nu \sqrt{\beta} x_1 \), \( \nu \) is the effective frequency of collisions, \( \beta = m/(2kT) \), \( m \) is the mass of a molecule of gas, \( k \) is the Boltzmann constant, \( T \) is the temperature of gas, constants \( \nu, \gamma \) and \( \omega \) are expressed through Prandtl number \( Pr \), self-diffusion coefficient \( D \) and kinematic viscosity \( \nu_* \) by following equalities \[10\]

\[ \nu = \frac{kT}{mD}, \quad \gamma = \frac{4}{5} \left[ 1 - \left( 1 - \frac{\omega}{2} \right) Pr \right], \quad \omega = 2 \left( 1 - \frac{D}{\nu_*} \right). \]

Besides, in the equation (2.1) \( U_y(x) \) is the dimensionless mass velocity of gas along an axis \( y \),

\[ U_y(x) = \frac{1}{\pi^{3/2}} \int \exp(-C'^2) C'_y h(x, C') d^3 C', \]

\( Q_y(x) \) is the component of the vector of thermal stream along an axis \( y \),

\[ Q_y(x) = \frac{1}{\pi^{3/2}} \int \exp(-C'^2) C'_y \left( C'^2 - \frac{5}{2} \right) h(x, C') d^3 C', \]

and \( P_{xy}(x) \) is the component of viscous pressure tensor,

\[ P_{xy}(x) = \frac{1}{\pi^{3/2}} \int \exp(-C'^2) C'_x C'_y h(x, C') d^3 C', \]
Let’s expand the function $h$ in two directions

$$h(x, C) = C_y h_1(x, C_x) + \gamma C_y \left(C^2 - \frac{5}{2}\right) h_2(x, C_x). \quad (2.2)$$

By means of (2.2) mass velocity is equal

$$U_y(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} \left[h_1(x, \mu') + \gamma \left(\mu'^2 - \frac{1}{2}\right) h_2(x, \mu')\right] \, d\mu',$$

$y$-component of thermal stream vector equals

$$Q_y(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} \left\{ \left(\mu'^2 - \frac{1}{2}\right) h_1(x, \mu') + \gamma \left[\left(\mu'^2 - \frac{1}{2}\right)^2 + 2\right] h_2(x, \mu') \right\} \, d\mu',$$

$xy$-component of viscous pressure tensor equals

$$P_{xy}(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} \mu' \left[h_1(x, \mu') + \gamma \left(\mu'^2 - \frac{1}{2}\right) h_2(x, \mu')\right] \, d\mu'.$$

By means of last three equalities and decomposition (2.2) we conclude, that the equation (2.1) is equivalent to system from two equations

$$\mu \frac{\partial h_1}{\partial x} + h_1(x, \mu) =$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} (1 + \omega \mu \mu') [h_1(x, \mu') + \gamma \left(\mu'^2 - \frac{1}{2}\right) h_2(x, \mu')] \, d\mu'$$

and

$$\mu \frac{\partial h_2}{\partial x} + h_2(x, \mu) =$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} \left\{ \left(\mu'^2 - \frac{1}{2}\right) h_1(x, \mu') + \gamma \left[\left(\mu'^2 - \frac{1}{2}\right)^2 + 2\right] h_2(x, \mu') \right\} \, d\mu'.$$

We introduce vector-column

$$h(x, \mu) = \begin{pmatrix} h_1(x, \mu) \\ h_2(x, \mu) \end{pmatrix}$$

and we transform previous system of equations in vector form

$$\mu \frac{\partial h}{\partial x} + h(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu'^2} K(\mu, \mu') h(x, \mu') \, d\mu'. \quad (2.3)$$
In equation (2.3) $K(\mu, \mu')$ is the kernel of equation,

$$K(\mu, \mu') = \begin{pmatrix} 1 + \omega \mu' & \gamma(\mu'^2 - \frac{1}{2})(1 + \omega \mu') \\ \frac{1}{2}(\mu'^2 - \frac{1}{2}) & \frac{\gamma}{2}[(\mu'^2 - \frac{1}{2})^2 + 2] \end{pmatrix},$$

with determinant

$$\text{det} K(\mu, \mu') = \gamma(1 + \omega \mu').$$

The kernel of equation (2.3) we transform as sum

$$K(\mu, \mu') = K(\mu') + \omega \mu' \begin{pmatrix} 1 & \gamma(\mu'^2 - \frac{1}{2}) \\ 0 & 0 \end{pmatrix},$$

or, in the form

$$K(\mu, \mu') = K(\mu') + \omega \mu' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} K(\mu'),$$

where

$$K(\mu') = \begin{pmatrix} 1 & \gamma(\mu'^2 - \frac{1}{2}) \\ \frac{1}{2}(\mu'^2 - \frac{1}{2}) & \frac{\gamma}{2}[(\mu'^2 - \frac{1}{2})^2 + 2] \end{pmatrix},$$

$$\text{det} K(\mu') = \gamma.$$

Now we will present the equation (2.3) in the form convenient for division variables

$$\mu \frac{\partial h}{\partial x} + h(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} K(\mu') h(x, \mu') d\mu' +$$

$$+ \omega \mu \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} \mu' \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} K(\mu') h(x, \mu') d\mu'. \quad (2.4)$$

3. Division of variables. Dispersion matrix – function

Following Euler, we search solutions of the equation (2.4) in the form

$$h_\eta(x, \mu) = \exp(-\frac{x}{\eta}) \Phi(\eta, \mu), \quad \eta \in \mathbb{C}, \quad (3.1)$$
where $\eta$ is the spectral parameter, generally speaking, complex parameter. Substituting (3.1) in the equation (2.4), we receive the characteristic equation

$$(\eta - \mu)\Phi(\eta, \mu) = \frac{1}{\sqrt{\pi}}\eta n(\eta) + \omega \mu \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) m(\eta),$$

(3.2)

in which

$$n(\eta) = \int_{-\infty}^{\infty} e^{-\mu^2} K(\mu') \Phi(\eta, \mu') d\mu',$$  

(3.3)

$$m(\eta) = \int_{-\infty}^{\infty} e^{-\mu^2} \mu' K(\mu') \Phi(\eta, \mu') d\mu'.$$

Multiplying the equation (3.2) at the left on the matrix $e^{-\mu^2} K(\mu)$ and integrating on all real axis, we receive that

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) m(\eta) = \left( \begin{array}{cc} 0 \\ 0 \end{array} \right).$$

Hence, the characteristic equation becomes simpler

$$(\eta - \mu)\Phi(\eta, \mu) = \frac{1}{\sqrt{\pi}}\eta n(\eta).$$

(3.4)

From the equation (3.4) by means of the condition of normalization (3.3) we find eigen vectors of the characteristic equation correspond to continuous spectrum $\sigma_c = (-\infty, +\infty)$

$$\Phi(\eta, \mu) = \left[ \frac{1}{\sqrt{\pi}}\eta P \frac{1}{\eta - \mu} + e^{\eta^2} K^{-1}(\eta) \Lambda(\eta) \delta(\eta - \mu) \right] n(\eta).$$

(3.5)

In (3.5) the symbol $P x^{-1}$ means principal value of integral at integration of expression $x^{-1}$, $\delta(x)$ is the Dirac delta–function, $\Lambda(z)$ is the dispersion matrix–function,

$$\Lambda(z) = E + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} K(\mu)}{\mu - z} d\mu,$$

where $E$ is the unit matrix of the second order.

Let’s present the dispersion matrix–function in the explicit form
\[
\begin{pmatrix}
\lambda_0(z) & \gamma \lambda_0(z)(z^2 - \frac{1}{2}) + \frac{\gamma}{2} \\
\frac{1}{2} \lambda_0(z)(z^2 - \frac{1}{2}) + \frac{\gamma}{4} (z^2 - \frac{1}{2})^2 + 2] \lambda_0(z) + 1 + \frac{\gamma}{4} (z^2 - \frac{9}{2})
\end{pmatrix}.
\]

Here \( \lambda_0(z) \) is the dispersion plasma function,

\[
\lambda_0(z) = 1 + \frac{z}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{\mu - z}.
\]

Let’s present the dispersion matrix–function in the form, linear concerning of the matrix \( K(z) \)

\[
\Lambda(z) = \lambda_0(z) K(z) + A(z),
\]

where

\[
A(z) = \begin{pmatrix}
0 & \frac{\gamma}{2} \\
\frac{1}{4} & 1 + \frac{\gamma}{4} (z^2 - \frac{9}{2})
\end{pmatrix}.
\]

Determinant of the dispersion matrix–functions we name dispersion function. It is easy to see, that

\[
\lambda(z) \equiv \det \Lambda(z) = \gamma \lambda_0^2(z) + \left[ 1 - \frac{\gamma}{4} (z^2 + \frac{7}{2}) \right] \lambda_0(z) - \frac{\gamma}{8}.
\]

Let’s expand dispersion function in asymptotic series in neighbourhood of infinitely remote point

\[
\lambda(z) = \frac{1}{2} \left( \frac{5\gamma}{4} - 1 \right) \frac{1}{z^2} + o\left( \frac{1}{z^2} \right), \quad z \to \infty.
\]

This expansion means, that the discrete spectrum of characteristic equation, consisting of zero of the dispersion function, consists of one infinitely remote point \( z_i = \infty \) with order two. This spectrum is attached to the continuous. This spectrum corresponds to two solutions of the initial equations (2.4)

\[
h^{(1)}(x, \mu) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

and

\[
h^{(1)}(x, \mu) = \left( x - \frac{2}{2 - \omega} \mu \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
4. Boundary Kramers problem

On the condition of Kramers problem the given quantity is the gradient of the mass velocity, given far from the wall

\[ g_v = \left( \frac{du_y(x_1)}{dx_1} \right)_{x_1=+\infty}. \] (4.1)

Let’s consider, that for the dimensionless gradient is carried out inequality \( G_v \ll 1 \) that allows to solve Kramers problem in linear statement.

From the relation (4.1) we see, that far from the wall for mass velocity is fair the asymptotic distribution

\[ u_y(x_1) = u_{sl} + g_v x_1, \quad x_1 \to +\infty. \]

From here and from the equation (2.4) follows, that far from the wall function \( h(x, C) \) has following distribution

\[ h_{as}(x, C) = 2C_y U_{sl} + 2C_y (x - \frac{2}{2 - \omega} C_x) G_v, \quad x \to +\infty. \] (4.2)

It is the linear combination of two partial solution of the initial equation (2.1)

\[ h_1(x, C) = 1 \]

and

\[ h_2(x, C) = x - \frac{2}{2 - \omega} C_x. \]

The distribution (4.2) is the Chapman—Enskog distribution.

By means of (2.2) we will present distribution (4.2) in the vector form

\[ h_{as}(x, \mu) = \left[ 2U_{sl} + \left( x - \frac{2}{2 - \omega} \mu \right) G_v \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \] (4.3)

Decomposition (4.3) represents the linear combination of two discrete (partial) solutions of the equation (2.4), correspond to the spectrum attached to continuous spectrum.

Let’s formulate boundary conditions in the Kramers problem under the condition of diffusion reflexion of molecules from the wall

\[ h(0, \mu) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mu > 0, \] (4.4)
\[ h(x, \mu) = h_{as}(x, \mu) + o(1), \quad x \to +\infty. \quad (4.5) \]

So, boundary Kramers problem consists in finding of such solutions of the equation (2.4) which satisfies boundary conditions (4.4) and (4.5).

The solution of the problem (2.4), (4.4) and (4.5) we search in the form of the sum of the partial solutions of the attached spectrum and integral on the continuous spectrum of continuous eigen solutions

\[ h(x, \mu) = h_{as}(x, \mu) + \int_0^\infty \exp(-\frac{x}{\eta})\Phi(\eta, \mu) d\eta. \quad (4.6) \]

Let’s present the solution (4.6) in the explicit form

\[ h(x, \mu) = h_{as}(x, \mu) + \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-\frac{x}{\eta})\frac{\eta n(\eta)}{\eta - \mu} d\eta + \exp(\mu^2 - \frac{x}{\mu})K^{-1}(\mu)\Lambda(\mu)n(\mu). \quad (4.7) \]

Unknown members in expansion (4.6) or (4.7) are dimensionless velocity of sliding \( U_{sl} \) (coefficient of the attached spectrum) and vector–function \( n(\eta) \) (coefficient of the continuous spectrum).

Expansion (4.7) automatically satisfies to the condition (4.5). Substituting (4.7) in the condition (4.4), we receive the one-side vector singular integral equation with Cauchy kernel

\[ h_{as}(0, \mu) + \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\eta n(\eta)}{\eta - \mu} d\eta + e^{\mu^2}K^{-1}(\mu)\Lambda(\mu)n(\mu) = 0, \quad \mu > 0. \quad (4.8) \]

We introduce the matrix \( P(z) = K^{-1}(z)\Lambda(z) \). It is easy find that

\[ P(z) = \lambda_0(z)E + B(z), \]

where

\[ B(z) = \left( \begin{array}{cc} -\frac{1}{4}(z^2 - \frac{1}{2}) & (\gamma - 1)z^2 + \frac{1}{2} \\ \frac{1}{4\gamma} & \frac{1}{\gamma} - 1 \end{array} \right), \quad \det B(z) = -\frac{1}{8}. \]
For the matrix $P(z)$ formulas Sokhotsky are carried out

$$P^+(\mu) - P^-(\mu) = 2\sqrt{\pi i \mu} e^{-\mu^2}, \quad -\infty < \mu < +\infty,$$

$$\frac{P^+(\mu) + P^-(\mu)}{2} = P(\mu), \quad P(\mu) = \lambda_0(\mu)E + B(\mu).$$

The singular integral equation (4.8) we transform to the vector boundary condition

$$P^+(\mu)[h_{as}(0, \mu) + N^+(\mu)] = P^-(\mu)[h_{as}(0, \mu) + N^-(\mu)], \quad \mu > 0. \quad (4.9)$$

In (4.9) we have entered the new auxiliary vector–function

$$N(z) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\eta m(\eta) d\eta}{\eta - z}, \quad (4.10)$$

for which Sokhotsky formulas are carried out

$$N^+(\mu) - N^-(\mu) = 2\sqrt{\pi i \mu} n(\mu), \quad \mu > 0, \quad (4.10a)$$

$$\frac{1}{2}[N^+(\mu) + N^-(\mu)] = N(\mu), \quad \mu > 0.$$

Boundary condition (4.9) we will present in the form of the non-uniform vector boundary value Riemann–Hilbert problem with matrix coefficient

$$N^+(\mu) = G(\mu)N^-(\mu) + [G(\mu) - E]h_{as}(0, \mu), \quad \mu > 0. \quad (4.11)$$

Coefficient of the problem (4.11) is the matrix–function

$$G(\mu) = [P^+(\mu)]^{-1}P^-(\mu) = [\Lambda^+(\mu)]\Lambda^-(\mu).$$

5. Homogeneous vector boundary value Riemann–Hilbert problem

For solution of the problem (4.11) at first we will solve the corresponding homogeneous vector boundary value problem

$$X^+(\mu) = G(\mu)X^-(\mu), \quad \mu > 0. \quad (5.1)$$

where the unknown matrix–function $X(z)$ is analytic in complex planes with a cut along the real positive half-axis.
For solution of this problem reduction to diagonal form the matrix $P(z)$ is required. For this purpose reduction to diagonal form the matrix $B(z)$ is required.

Eigen numbers (functions) of matrix $B(z)$ are equal

$$
\mu_{1,2}(z) = -\frac{1}{8}(z^2 + \frac{7}{2} - \frac{4}{\gamma}) \pm \frac{1}{8} r(z).
$$

Here

$$
r(z) = \sqrt{q(z)}, \quad q(z) = \left(z^2 + \frac{7}{2} - \frac{4}{\gamma}\right),
$$

where low index 1 corresponds to sign plus, and index 2 corresponds to sign minus.

The matrix transforming the matrix $B(z)$ to diagonal form, has the following form

$$
S(z) = \begin{pmatrix}
\frac{1}{\gamma} & \frac{1}{\gamma} \\
\frac{1}{4\gamma} & \frac{1}{4\gamma}
\end{pmatrix}, \quad \det S(z) = \frac{r(z)}{16\gamma},
$$

More low the return matrix is required to us also

$$
S^{-1}(z) = \frac{4}{r(z)} \begin{pmatrix}
1 & \frac{\gamma}{2} \left[ r(z) + \left(z^2 - \frac{9}{2} + \frac{4}{\gamma}\right)\right] \\
-1 & \frac{\gamma}{2} \left[ r(z) - \left(z^2 - \frac{9}{2} + \frac{4}{\gamma}\right)\right]
\end{pmatrix}.
$$

For the solution of the problem (5.1) we search in the form

$$
X(z) = S(z)U(z)S^{-1}(z), \quad (5.2)
$$

where $U(z)$ is the new unknown diagonal matrix,

$$
U(z) = \begin{pmatrix}
U_1(z) & 0 \\
0 & U_2(z)
\end{pmatrix} = \text{diag} \{U_1(z), U_2(z)\}.
$$

Matrix $S(z)$ and return to it contain the radical $r(z)$, representing square root from the polynom of the fourth degree $q(z)$. Hence, these matrixes have four branching points. These points are polynom zero $q(z)$:

$$a(\gamma), \quad \bar{a}(\gamma), \quad -a(\gamma), \quad -\bar{a}(\gamma),$$
where $a(\gamma)$ is the zero laying in the first quarter,

$$a(\gamma) = \sqrt{\frac{4}{\gamma} - \frac{7}{2}} + i\sqrt{8}.$$ 

Additional cuts we will spend as follows. We will connect points of branching with infinitely remote point following beams

$$\Gamma_1(\gamma) = [a(\gamma), +\infty + i\sqrt{8}], \quad \Gamma_2(\gamma) = [-\bar{a}(\gamma), -\infty + i\sqrt{8}],$$
$$\Gamma_3(\gamma) = [\bar{a}(\gamma), +\infty - i\sqrt{8}], \quad \Gamma_4(\gamma) = [-a(\gamma), -\infty - i\sqrt{8}].$$

Let’s unite these cuts, having entered the designation

$$\Gamma(\gamma) = \bigcup_{j=1}^{4} \Gamma_j(\gamma).$$

Matrixes $S(z)$ and $S^{-1}(z)$ are analytical in all complex planes with the cut lengthways $\Gamma(\gamma)$. It means, that we search unequivocal analytical matrix $X(z)$ in domain $\mathbb{C} \setminus \left( \Gamma(\gamma) \cup \mathbb{R} \right)$.

Substituting (5.2) in (5.1), we receive the following matrix boundary value problem

$$\Omega^+(\mu)U^+(\mu) = \Omega^-(\mu)U^-(\mu), \quad \mu > 0,$$  

where the matrix $\Omega(z)$ is entered by following equality

$$\Omega(z) = S^{-1}(z)P(z)S(z) = \lambda_0(z)E + S^{-1}(z)B(z)S(z) =$$

$$= \lambda_0(z)E + \text{diag} \{\mu_1(z), \mu_2(z)\} = \text{diag} \{\Omega_1(z), \Omega_2(z)\},$$

where

$$\Omega_j(z) = \lambda_0(z) + \mu_1(z) = \lambda_0(z) - \frac{1}{8} \left( z^2 + \frac{7}{2} - \frac{4}{\gamma} \right) \pm \frac{1}{8} \sqrt{q(z)}, \quad j = 1, 2,$$

and $j = 1$ corresponds to the sign plus, and $j = 2$ corresponds to the sign minus.

Considering, that the structure of matrix $X(z)$ contains radicals $\pm r(z)$, changing the sign at transition on opposite through additional cuts $\Gamma_j(\gamma) \ (j = 1, 2, 3, 4)$, for unambiguity of a matrix $X(z)$ should be demanded, that on coast of additional cuts boundary values of the matrix $X(z)$ from above and from below were equal

$$X^+(\tau) = X^-(\tau), \quad \tau \in \Gamma(\gamma),$$

(5.4)
or, that all the same,

\[ U^+(\tau)T(\tau) = T(\tau)U^-(\tau), \quad \tau \in \Gamma(\gamma), \quad (5.5) \]

where the matrix \( T(\tau) \) is defined on the additional cut and has the following form

\[ T(\tau) = \left[ S^+(\tau) \right]^{-1}S^-(\tau), \quad \tau \in \Gamma(\gamma). \]

Rectilinear calculations show, that a matrix \( T(\tau) \) is constant

\[ T(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

The matrix boundary value problem (5.3) is equivalent to two scalar boundary value problems on the basic cut

\[ U_j^+(\mu) = \Omega^{-}_j(\mu) \Omega^+_j(\mu) U_j^-(\mu), \quad j = 1, 2, \quad \mu > 0. \]

Noticing, that

\[ \Omega^-_j(\mu) = \Omega^+_j(\mu), \quad -\infty < \mu < +\infty, \]

and considering augmentation of arguments \( \theta_j(\mu) = \arg \Omega^+_j(\mu) \) on the semi-axis \([0, +\infty]\), we will rewrite these problems in the following form

\[ U_1^+(\mu) = \exp(-2i\theta_1(\mu))U_1^-(\mu), \quad \mu > 0, \quad (5.6) \]

and

\[ U_2^+(\mu) = \exp(-2i[\theta_2(\mu) - \pi])U_2^-(\mu), \quad \mu > 0, \quad (5.7) \]

Let’s notice, that problems (5.4) (or (5.5)) are not reduced to scalar problems, and are essentially vector boundary value problems rather in respect of vector \( U(z) = \{U_1(z), U_2(z)\} \). By means of the found matrix \( T \) let’s rewrite these problems in the form of two vector boundary value problems

\[ U_1^+(\tau) = U_2^-(\tau), \quad \tau \in \Gamma(\gamma), \quad (5.8) \]

\[ U_1^-(\tau) = U_2^+(\tau), \quad \tau \in \Gamma(\gamma). \quad (5.9) \]

Now the basic difficulty consists in search of such solution

\[ U(z) = \{U_1(z), U_2(z)\}, \]
which would satisfy simultaneously to four boundary value problems (5.6) – (5.9).

Let’s proceed as follows. We will multiply and will divide against each other boundary value problems (5.6) and (5.7). Then we find the logarithm of the received problems, and the second of them we will divide term by term on radical \( r(z) \). We receive, that

\[
\ln[U_1(\mu)U_2(\mu)]^+ - \ln[U_1(\mu)U_2(\mu)]^- = -2i[\theta_1(\mu) + \theta_2(\mu) - \pi], \quad \mu > 0,
\]

and

\[
\frac{1}{r(\mu)} \ln \left[ \frac{U_1(\mu)}{U_2(\mu)} \right]^+ - \frac{1}{r(\mu)} \ln \left[ \frac{U_1(\mu)}{U_2(\mu)} \right]^-= -2i\frac{\theta_1(\mu) - \theta_2(\mu) + \pi}{r(\mu)}, \quad \mu > 0.
\]

These problems as problems "on jump", have the following solutions

\[
\ln \left[ U_1(z)U_2(z) \right] = -\frac{1}{\pi} \int_0^\infty \frac{\theta_1(\mu) + \theta_2(\mu) - \pi}{\mu - z} d\mu,
\]

and

\[
\frac{1}{r(z)} \ln \left[ \frac{U_1(z)}{U_2(z)} \right] = -\frac{1}{\pi} \int_0^\infty \frac{\theta_1(\mu) - \theta_2(\mu) + \pi}{r(\mu)(\mu - z)} d\mu.
\]

From two last equalities we receive

\[
U_1(z)U_2(z) = \exp(-2A(z))
\]

and

\[
\frac{U_1(z)}{U_2(z)} = \exp(-2r(z)B(z)),
\]

where

\[
A(z) = \frac{1}{2\pi} \int_0^\infty \frac{\theta_1(\mu) + \theta_2(\mu) - \pi}{\mu - z} d\mu,
\]

and

\[
B(z) = \frac{1}{2\pi} \int_0^\infty \frac{\theta_1(\mu) - \theta_2(\mu) + \pi}{r(\mu)(\mu - z)} d\mu.
\]

Hence, having designated the received solution through

\[
U^\circ(z) = \{U_1^\circ(z), U_2^\circ(z)\},
\]
we will write

$$U_1(z) = \exp(-A(z) - r(z)B(z)),$$

and

$$U_2(z) = \exp(-A(z) + r(z)B(z)).$$

It is easy to check up, that these functions are the solution at once all four boundary value problems (5.9) – (5.9). However, the received solution has one basic lack, which is essential singularity in infinitely remote point.

For its elimination we search functions $U_j(z) (j = 1, 2)$ in the form

$$U_1(z) = U_1^0(z) \cdot \varphi(z)$$

and

$$U_2(z) = U_2^0(z) \cdot \frac{1}{\varphi(z)},$$

where function $\varphi(z)$ is analitical in the complex plane out of additional cuts $\Gamma(\gamma)$ (with the essential singularity in infinitely remote point). Thus boundary value conditions (5.6) and (5.7) are carried out automatically, and boundary value conditions (5.8) and (5.9) are carried out then and only when on additional cuts is carried out condition

$$\varphi^+(\tau) = \frac{1}{\varphi^-(\tau)}, \quad \tau \in \Gamma(\gamma).$$

As the solution of this nonlinear boundary value problem we take function

$$\varphi(z) = \exp(r(z)R(z)),$$

where

$$R(z) = \int_{0}^{\mu_0} \frac{d\tau}{r(\tau)(\tau - z)}.$$

Here the point $\mu_0 \in (0, +\infty)$ is more low defined unequivocally.

Without the proof we will inform (see, for example, [24]), that the point $\mu_0$ is the unique solution of the special case Jacobi problem of inverse for elliptic integrals:

$$\frac{1}{2\pi} \int_{0}^{\infty} \frac{\theta_1(\mu) - \theta_2(\mu) + \pi}{r(\mu)} d\mu = \int_{0}^{\mu_0} \frac{d\tau}{r(\tau)}.$$  \hspace{1cm} (5.10)
It is possible to check up, that functions

\[ U_1(z) = \exp \left[ -A(z) - r(z)\left( B(z) - R(z) \right) \right] \]  

and

\[ U_2(z) = \exp \left[ -A(z) + r(z)\left( B(z) - R(z) \right) \right] \]

are solutions at once all four boundary value problems (5.6) – (5.9) and also have no essential singularity in infinitely removed point at performance of the condition (5.10).

It is possible to show, that function \( U_1(z) \) has a simple pole in the origin of coordinates and simple zero in the point \( \mu_0 \), and function \( U_2(z) \) is limited in the origin of coordinates and does not disappear, and has simple pole in the point \( \mu_0 \).

So, factor–matrix \( X(z) \) is constructed and has following elements

\[
X_{11}(z) = \frac{U_1(z) + U_2(z)}{2} - \frac{z^2 - \frac{9}{2} + \frac{4}{\gamma} U_1(z) - U_2(z)}{r(z)},
\]

\[
X_{12}(z) = \frac{2}{r(z)} \left[ 2z^2(\gamma - 1) + 1 \right] (U_1(z) - U_2(z)),
\]

\[
X_{21}(z) = \frac{U_1(z) - U_2(z)}{\gamma r(z)},
\]

\[
X_{22}(z) = \frac{U_1(z) + U_2(z)}{2} + \frac{z^2 - \frac{9}{2} + \frac{4}{\gamma} U_1(z) - U_2(z)}{r(z)}.
\]

On construction, the matrix–function \( X(z) \) satisfies to the condition (5.1), is analytical everywhere in \( \mathbb{C} \setminus [0, +\infty) \) with possible special points in polynom zero \( q(z) \). But thanking coincidence of functions \( U_1(z) \) and \( U_2(z) \) in polynom zero \( q(z) \) matrix \( X(z) \) is analytical and in these points. The matrix determinant \( X(z) \) does not degenerate everywhere in \( \mathbb{C} \setminus [0, +\infty) \) also has a pole of the first order in the origin of coordinates.

So, homogeneous boundary value Riemann–Hilbert problem (5.1) is completely solved.

6. Inhomogeneous vector boundary value Riemann–Hilbert problem and its conditions of resolvability
By means of the factorization problem (5.1) we will reduce the problem (4.9) to vector boundary value Riemann–Hilbert problem "on zero jump"

\[
\begin{bmatrix}
X^+(\mu)
\end{bmatrix}^{-1} \begin{bmatrix}
N^+(\mu) + h_{as}(0, \mu)
\end{bmatrix} = \\
\begin{bmatrix}
X^-(\mu)
\end{bmatrix}^{-1} \begin{bmatrix}
N^-(\mu) + h_{as}(0, \mu)
\end{bmatrix}, \quad \mu > 0. \tag{6.1}
\]

Here

\[
X^{-1}(z) = S(z)U^{-1}(z)S^{-1}(z).
\]

Considering behaviour of matrixes and vectors entering into the boundary condition (6.1) in the complex plane, we will write its common solution

\[
X(z) = -h_{as}(0, z) + X(z)\Phi(z), \tag{6.2}
\]

where

\[
\Phi(z) = \begin{pmatrix}
\Phi_1(z) \\
\Phi_2(z)
\end{pmatrix} = \begin{pmatrix}
\alpha_1 z + \alpha_0 + \frac{\alpha_{-1}}{z - \mu_0} \\
\beta_1 z + \beta_0 + \frac{\beta_{-1}}{z - \mu_0}
\end{pmatrix}, \tag{6.3}
\]

and in expression (6.3) all coefficients \( \alpha_j, \beta_j \) \((j = -1, 0, 1)\) are arbitrary constants.

The received solution (6.2) has following singularities: the pole the first order in the origin of coordinates \( z = 0 \), the pole of the second order in the point \( z = \mu_0 \) and the pole of the first order in the point \( z = \infty \).

That the vector \( N(z) \), defined by equality (6.2), was possible to accept as auxiliary function \( N(z) \), entered above by equality (4.10), we will eliminate the specified singularities for the account choice of free parametres of the solution (6.2) and the unknown coefficient of attached spectrum \( U_{sl} \).

Let’s rewrite the solution (6.2) in the explicit form

\[
N(z) = -\left(U_{sl} - \frac{2G_{\nu}}{2 - \omega}z\right) \begin{pmatrix}
1 \\
0
\end{pmatrix} + S(z)U(z)S^{-1}(z)\Phi(z). \tag{6.4}
\]

Let’s find asymptotic of matrix \( X(z) \) in the vicinity infinitely remote point

\[
X(z) = \begin{pmatrix}
U_2(z) & 4(\gamma - 1)[U_1(z) - U_2(z)] \\
0 & U_1(z)
\end{pmatrix} + o\left(\frac{1}{z}\right), \quad z \to \infty. \tag{6.5}
\]
Let’s expand functions $A(z)$, $B(z)$ and $R(z)$ into Laurent series in vicinities of infinitely remote point

$$
A(z) = \frac{A_{-1}}{z} + \frac{A_{-2}}{z^2} + \cdots, \quad z \to \infty,
$$

$$
B(z) = \frac{B_{-1}}{z} + \frac{B_{-2}}{z^2} + \cdots, \quad z \to \infty,
$$

$$
R(z) = \frac{R_{-1}}{z} + \frac{R_{-2}}{z^2} + \cdots, \quad z \to \infty.
$$

Here

$$
A_{-k} = -\frac{1}{2\pi} \int_0^\infty \tau^{k-1} \left[ \theta_1(\tau) + \theta_2(\tau) - \pi \right] d\tau, \quad k = 1, 2, \cdots,
$$

$$
B_{-k} = -\frac{1}{2\pi} \int_0^\infty \tau^{k-1} \frac{\theta_1(\tau) - \theta_2(\tau) + \pi}{r(\tau)} d\tau, \quad k = 1, 2, \cdots,
$$

$$
R_{-k} = -\int_0^{\mu_0} \frac{\tau^{k-1}}{r(\tau)} d\tau, \quad k = 1, 2, \cdots.
$$

Now, considering equalities (5.11) and (5.12), it is easy to expand into Laurent series functions $U_1(z)$ and $U_2(z)$

$$
U_1(z) = p_0 \left( 1 + \frac{p_{-1}}{z} + \cdots \right), \quad z \to \infty,
$$

$$
U_2(z) = q_0 \left( 1 + \frac{q_{-1}}{z} + \cdots \right), \quad z \to \infty.
$$

Here

$$
p_0 = \exp\left[ -(B_{-2} - R_{-2}) \right], \quad q_0 = \frac{1}{p_0},
$$

$$
p_{-1} = -A_{-1} - (B_{-3} - R_{-3}),
$$

$$
q_{-1} = -A_{-1} + (B_{-3} - R_{-3}).
$$

In terms of Laurent coefficients of functions $B(z)$ and $R(z)$ the problem of Jacobi inverse (5.10) can be presented in the form simple equality

$$
B_{-1} = R_{-1}. \quad (6.6)
$$

According to asymptotic equality (6.5), we have:

$$
X(z) = X_0 + X_{-1} \frac{1}{z} + o\left(\frac{1}{z}\right), \quad z \to \infty. \quad (6.7)
$$
Here
\[
X_0 = \begin{pmatrix} q_0 & 4(\gamma - 1)(p_0 - q_0) \\ 0 & p_0 \end{pmatrix}
\]
and
\[
X_{-1} = \begin{pmatrix} q_0q_{-1} & 4(\gamma - 1)(p_0p_{-1} - q_0q_{-1}) \\ 0 & p_0p_{-1} \end{pmatrix}.
\]

On the basis of (6.7) we will write out decomposition top and bottom elements of common solution in the vicinity infinitely removed point
\[
N_1(z) = -2 \left(U_{sl} - \frac{2G_v}{2 - \omega} z\right) + X_{11}(z) \left(\alpha_1 z + \alpha_0 + \frac{\alpha_{-1}}{z - \mu_0}\right) +
+ X_{12}(z) \left(\beta_1 z + \beta_0 + \frac{\beta_{-1}}{z - \mu_0}\right) + O\left(\frac{1}{z}\right),
\]

(6.8)
\[
N_2(z) = X_{22}(z) \left(\beta_1 z + \beta_0 + \frac{\beta_{-1}}{z - \mu_0}\right) + O\left(\frac{1}{z}\right).
\]

(6.9)

From decomposition (6.8) and (6.9) taking into account the previous equalities it is visible, that
\[
\beta_0 = \beta_1 = 0,
\]
and
\[
\alpha_1 = -\frac{4G_v}{2 - \omega} \cdot \frac{1}{q_0},
\]

(6.10)
\[
U_{sl} = -\frac{2G_v}{2 - \omega} \cdot \frac{q_{-1}}{q_0} + \frac{1}{q_0} \alpha_0.
\]

(6.11)

The pole of the second order in the point \(\mu_0\) has the bottom element of the common solution. Hence, for elimination of this pole it is necessary and sufficient to demand that it was carried out equality:
\[
S_{21}^{-1}(z)\Phi_1(z) + S_{22}^{-1}(z)\Phi_2(z) = O(z - \mu_0), \quad z \to \mu_0.
\]

Let’s write down this condition in the explicit form
\[
-\left(\alpha_1 z + \alpha_0 + \frac{\alpha_{-1}}{z - \mu_0}\right) + \frac{\gamma}{2} \left(r(z) - \left(z^2 - \frac{9}{2} + \frac{4}{\gamma}\right)\right) \frac{\beta_{-1}}{z - \mu_0} = O(z - \mu_0), z \to \mu_0.
\]
From here we find, that
\[ \alpha_1 = \alpha(\mu_0)\beta_{-1}, \]
and
\[ \alpha_1\mu_0 + \alpha_0 = \alpha'(\mu_0)\beta_{-1}. \]  
(6.12)

Here
\[ \alpha(z) = \frac{\gamma}{2} \left[ r(z) - \left( \frac{z^2}{2} - \frac{9}{2} + \frac{4}{\gamma} \right) \right], \]
\[ \alpha'(z) = \frac{\gamma}{2} z \left[ \frac{3z^2 + 2(7 - 8/\gamma)}{2r'(z)} - 2 \right]. \]

For pole elimination in zero it is necessary and sufficient to demand, that equality was carried out
\[ S_{11}^{-1}(z)\Phi_1(z) + S_{12}^{-1}(z)\Phi_2(z) = O(z), \quad z \to 0. \]

From this relation we find that
\[ \beta_{-1} = \frac{\mu_0\alpha_0}{\alpha(\mu_0) + \delta}, \]  
(6.13)

where
\[ \delta = \frac{\gamma}{2} \left[ r(0) - \frac{9}{2} + \frac{4}{\gamma} \right], \]
\[ r(0) = \sqrt{\frac{81}{9} + \frac{16}{\gamma^2} - \frac{28}{\gamma}} = \sqrt{\left( \frac{7}{2} - \frac{4}{\gamma} \right)^2 + 8} = \sqrt{\left( \frac{9}{2} - \frac{4}{\gamma} \right)^2 + \frac{8}{\gamma}}. \]

Substituting (6.13) in (6.12), we find
\[ \alpha_0 = -\frac{\alpha_1\mu_0(\alpha(\mu_0) + \delta)}{\alpha(\mu_0) + \delta - \mu_0\alpha'(\mu_0)}. \]  
(6.14)

Now on the basis of (6.11) by means of (6.10) and (6.14) we found required velocity of sliding (the coefficient correspond to the attached spectrum)
\[ U_{sl} = -\frac{2G_v}{2 - \omega} \left[ q_1 - \frac{\mu_0(\alpha(\mu_0) + \delta)}{\alpha(\mu_0) + \delta - \mu_0\alpha'(\mu_0)} \right]. \]  
(6.15)

The coefficient \( n(\mu) \), correspond to the continuous spectrum, we find from equality (4.10a):
\[ n(\mu) = \frac{1}{2\sqrt{\pi}} \frac{1}{i\mu} \left[ N^+(\mu) - N^-(\mu) \right] = \]
\[ \frac{1}{2\sqrt{\pi}i\mu} [X^+(\mu) - X^-(\mu)] \Phi(\mu). \] (6.16)

So, all coefficients of decomposition (4.6) are found. Coefficient of continuous spectrum is given by equality (6.16), and coefficient of the spectrum attached to continuous, it is given by equality (6.15).

7. Conclusion

In the present work the analytical solution of the Kramers problem about isothermal sliding for the Holway—Shakhov equation is constructed.

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