Phase Transition in the Two-Dimensional Gauge Glass

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The two-dimensional XY gauge glass, which describes disordered superconducting grains in strong magnetic fields, is investigated, with regard to the possibility of a glass transition. We compute the glass susceptibility and the correlation function of the system via extensive numerical simulations and perform the finite-size scaling analysis. This gives strong evidence for a finite-temperature transition, which is expected to be of a novel type.

PACS numbers: 74.50.+r, 64.60.Cn, 74.60.Ge

The gauge glass model, which was originally proposed as a generalization of the spin-glass model \[1\], has attracted much attention in relation to the vortex-glass phase of high-$T_c$ superconductors \[2\]. In three dimensions, the XY gauge glass model is believed to exhibit a finite-temperature glass transition \[3,4\], in agreement with experimental evidence for the vortex-glass phase at finite temperatures \[5\]. In two dimensions, on the other hand, there has been controversy as to the existence of a finite-temperature transition. Equilibrium studies of several quantities such as the defect-wall energy \[6\] and the root-mean-square current \[7\] have suggested the absence of ordering at finite temperatures; this appears to be consistent with experiment, where no evidence for the glassy phase at finite temperatures has been found \[8\]. Indeed the gauge-glass order parameter has been shown analytically to be zero at any finite temperature \[9\]. However, the helicity modulus computed via Monte Carlo simulations indicates a signal of a glassy phase at low but non-zero temperatures \[10\]. Dynamical simulations also give conflicting results: Whereas earlier study of the current-voltage characteristics has shown a strong evidence for the possibility of glass ordering at finite temperatures \[11\], later one has been interpreted to be consistent with the zero-temperature transition \[12\]. However, the lowest temperature considered in Ref. \[12\] is apparently higher than the estimated transition temperature $T_c \approx 0.15$ (in units of the coupling energy $J$) \[11\]. Indeed the recent study of relaxation dynamics has indicated a finite transition temperature $T_c \approx 0.22$ \[13\].

It should be noted here that the absence of an ordered phase does not necessarily imply the absence of a phase transition, as we will discuss later. We thus believe that the presence/absence of a finite-temperature transition in two-dimensions as well as its nature is still inconclusive. For the resolution of this, careful analysis of the behavior should be performed at sufficiently low temperatures.

In this work we investigate the two-dimensional XY gauge glass via extensive numerical calculations, with particular attention to the possibility of a finite-temperature glass transition. We adopt the equilibration test method in Refs. \[14,15\] to obtain the equilibrium and to determine the equilibration time, up to the system size $L = 48$. From the obtained equilibrium configuration, we compute the glass susceptibility and examine it by the finite-size scaling analysis \[14\]. This reveals remarkable divergent behavior at low but non-zero temperatures, providing strong evidence for the finite-temperature transition. It is discussed how the presence of such a finite-temperature transition can be reconciled with the results of existing studies.

We consider the standard XY gauge glass model on a square $L \times L$ lattice, which is described by the Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j - A_{ij}),$$

where $J$ is the coupling energy between nearest-neighboring grains, $\phi_i$ is the phase of the order parameter of the grain at site $i \equiv (x,y)$ $(x,y = 1,2,\cdots,L)$, and the bond angle $A_{ij}$'s are taken to be quenched random variables distributed uniformly on the interval $[0,2\pi)$. The presence of a glass transition in the system can be conveniently described by the divergence of the glass susceptibility, given by $\chi_G = \sum_{ij} G_{ij}$ with the correlation function of the glass order parameter

$$G_{ij} \equiv \left| \left< e^{i(\phi_i - \phi_j)} \right> \right|^2,$$

where $[\cdots]$ denotes the disorder average and $\langle \cdots \rangle$ the thermal average. In the limit the distance between the two grains $i$ and $j$ becomes large, the correlation function reduces to $q^2$, the square of the Edwards-Anderson glass order parameter

$$q \equiv \left| \left< e^{i\phi_i} \right> \right|^2.$$

In an infinite system the glass susceptibility is expected to display the critical behavior

$$\chi_G \sim (T - T_c)^{-\gamma}$$

with the scaling relation among the exponents.
$\gamma = (2 - \eta) \nu,$  \hspace{1cm} (5)

where $\eta$ describes the power law decay of the correlation at $T_c$ and $\nu$ the divergence of the correlation length $\xi$:

$\xi \sim (T - T_c)^{-\nu},$  \hspace{1cm} (6)

as the temperature $T$ approaches $T_c$. In a system of size $L$, the glass susceptibility has the finite-size-scaling form \[\chi\]

$\chi_G = L^{2-\gamma} (L^{1/\nu} (T - T_c))$  \hspace{1cm} (7)

with the appropriate scaling function $\chi$, which can be examined via extensive simulations at various temperatures and sizes.

In the simulation concerning equilibrium properties, equilibration is an important issue. We follow Ref. [15] to have a criterion for equilibration, and consider two replicas $\alpha$ and $\beta$ of the system with the same realization of disorder. The susceptibility can be calculated from the overlap between the two replicas, according to

$\chi_G(t_0) = \frac{1}{Nt_0} \left[ \sum_{t=1}^{t_0} \left| \sum_j e^{i(t\phi^\alpha_j + t\phi^\beta_j)} \right|^2 \right]$,  \hspace{1cm} (8)

where $N \equiv L^2$ is the number of spins and time $t$ is measured in units of the Monte Carlo sweep (MCS). Note also that the time-dependent four-spin-correlation function defined as

$\chi_G(t_0) = \frac{1}{N} \left[ \left| \sum_j e^{i(t\phi_j(t_0))} \right|^2 \right]$  \hspace{1cm} (9)

converges to the glass susceptibility $\chi_G$ in the limit $t_0 \to \infty$. Thus the equilibration time $\tau$ can be estimated as the value of $t_0$ at which the two expressions, Eqs. (8) and (9), give coincident results. The expected behavior of the equilibration time $\tau$ is

$\tau \propto \xi^z \propto (T - T_c)^{-\nu z}$  \hspace{1cm} (10)

in an infinite system near $T_c$, with the dynamical exponent $z$; in a finite system at $T = T_c$, it is expected

$\tau \propto L^z.$  \hspace{1cm} (11)

Equations (10) and (11) naturally lead to the finite-size scaling form \[\tau\]

$\tau = L^z \bar{\tau}(L^{1/\nu}(T - T_c)).$  \hspace{1cm} (12)

To examine the behavior of the glass susceptibility, we have performed extensive simulations at several temperatures, ranging from $T = 0.2$ to $T = 1.0$. After the equilibration time $\tau$ estimated as above, the glass susceptibility has been measured according to Eq. (6) except for that the data have been taken during sufficiently large time intervals, from three to ten times of the equilibration time. Namely, $t_0$ has been chosen to be from $3\tau$ to $10\tau$, depending on the size. We have also performed independent runs with up to 16000 different disorder configurations, over which the disorder average has been taken. Since the different disorder realizations give statistically independent thermal averages, the statistical error can be estimated from the standard deviation of the results for different samples. Here the number of the disorder realizations to get sufficiently reliable statistics for the equilibration time turns out much larger than that for the glass susceptibility. Thus it has taken much more averages to obtain reliable data for the equilibration time than those for the glass susceptibility. In this way, for example, the equilibration time at $T = 0.3$ has been estimated to vary from $(7 \pm 1) \times 10^4$ MCS ($L = 4$) to $(2 \pm 0.1) \times 10^6$ MCS ($L = 24$); it increases rapidly as the temperature is lowered, reaching at $T = 0.25$ the value $(1.5 \pm 0.4) \times 10^6$ for $L = 12$.

Figure 1 shows the behavior of the obtained glass susceptibility $\chi_G$ with the temperature $T$ for system sizes $L = 4, 6, 8, 12, 24$, and 48. In all cases, periodic boundary conditions have been employed, and the data points without error bars indicate that the errors estimated by the standard deviations are smaller than the size of the symbols. It is indeed observed that, as $L$ is increased, the data points apparently approach the dashed line representing $(T - T_c)^{-\gamma}$ with $\gamma$ obtained from the finite-size-scaling method below. The finite-size scaling analysis of the data is displayed in the inset of Fig. 1, where the data collapse nicely to the finite-size-scaling form given by Eq. (7). The corresponding exponents and the transition temperature are obtained:

$\eta = 0.30 \pm 0.05,$  \hspace{1cm} (13)

$\nu = 1.14 \pm 0.07,$  \hspace{1cm} (13)

$T_c = 0.22 \pm 0.02,$  \hspace{1cm} (13)

which agree with the results of Ref. [13]. The errors have been estimated by the size of the region beyond which the data do not scale well. The scaling law in Eq. (7) then gives the susceptibility exponent

$\gamma = 1.93 \pm 0.08.$  \hspace{1cm} (14)

We have also tried to fit the data to the finite-size scaling form with $T_c = 0$, and observed that the high-temperature data $(T \gtrsim 0.35)$ appear to fit also the $T_c = 0$ scaling, in agreement with Ref. [12]. At lower temperatures, however, systematic deviation from the scaling has been revealed, which becomes conspicuous for large sizes ($L = 24, 48$) \[\chi\]. Note that such low-temperature regions, which are crucial in differentiating the two scal-
nings \((T_c = 0 \text{ and } T_c \neq 0)\), have not been probed properly in previous studies concluding a zero-temperature transition. Moreover, even for small sizes such zero-temperature scaling yields \(\eta\) very close to zero, which is rather unlikely in view of the large ground-state degeneracy expected in the system \([14]\). We also perform the finite-size scaling analysis of the equilibration time. In this case, assuming the scaling form in Eq. (12), we use the values of the exponent \(\nu\) and the transition temperature \(T_c\) in Eq. (13), and estimate the dynamic exponent \(z\). Figure 2 shows that such scaling is indeed reasonable, with the dynamic exponent estimated as

\[ z = 2.4 \pm 0.3. \]  

In contrast, these equilibration time data hardly fit the scaling with \(T_c = 0\), displaying marked deviation at \(T \lesssim 0.4\). \([17]\)

We now examine the implications of such a finite-temperature transition, and first consider the scaling behavior of the defect-wall energy, which is defined to be the difference in the ground-state energy upon changing the boundary conditions along one direction from periodic to antiperiodic \(\{\phi_i^{(0)}\}\). The defect-wall energy fluctuates from sample to sample with zero mean, and its typical value, which may be taken as the average of the absolute value over quenched randomness, scales with the system size \(L\) according to

\[ \Delta E \propto L^\theta, \]  

in the asymptotic domain \((L \to \infty)\). \([8]\). The sign of the exponent \(\theta\) then determines the presence or absence of long-range order at finite temperatures: Whereas for \(\theta\) positive, the system displays rigidity or ordering, the negative value of \(\theta\) obtained numerically in two dimensions \([4]\), implies ubiquity of long-wavelength fluctuations and thus the absence of order. Here it should be stressed that the absence of (long-range) order does not necessarily correspond to the absence of a finite-temperature transition, as the Berezinskii-Kosterlitz-Thouless (BKT) transition and the associated algebraic order present an example \([19]\). In fact it is easy to show that in any two-dimensional system described by the Hamiltonian \([4]\), gapless spin-wave excitations prevent \(\theta\) from having a positive value: Suppose that the ground-state energy in the periodic boundary conditions (PBC), \(E_p\), is smaller than that in the antiperiodic boundary conditions (APBC). From the ground-state configuration \(\{\phi_i^{(0)}\}\) in the PBC, we can make a configuration satisfying the APBC by rotating the phases according to \(\phi_i^{(0)} \to \phi_i^{(0)} + \pi/L\). It is then obvious that the energy \(E\) of the new configuration is not smaller than the ground-state energy \(E_p\) in the same APBC. In the limit \(L \to \infty\), upon expanding the energy \(E\) of the new configuration around the ground-state configuration \(\{\phi_i^{(0)}\}\), we obtain the energy change (in units of \(J\)) \(\hat{E} - E_p = O(L^0)\), and thus the desired relation

\[ E_a - E_p \leq \hat{E} - E_p = O(L^0), \]  

which shows that \(\theta\) cannot be positive. For \(E_a\) smaller than \(E_p\), similar argument starting from the ground-state configuration in the APBC then yields \(E_p - E_a \leq O(L^0)\). Accordingly, it may be the case that the negative value of \(\theta\) results from spin-wave excitations, which destroy long-range order but maintain criticality.

To examine this possibility, we have generated ground states in the PBC and in the APBC via extensive simulations for several system sizes (up to \(L = 24\)), which has confirmed the value of \(\theta\) obtained in the existing studies on small sizes \([5]\). We have then carefully compared the two corresponding ground-state configurations, one in the PBC and the other in the APBC, and found only small difference in the total number of vortices. In particular the relative difference appears to decrease with the system size \([4]\), indicating that no (bulk) vortex excitation is involved. This suggests that the ubiquitous fluctuations implied by the negative value of \(\theta\) are indeed of the spin-wave type rather than of the vortex one \([20]\). Such spin-wave excitations should lead to the algebraic decay of the correlation function in Eq. (2) and to the vanishing Edwards-Anderson glass order parameter defined in Eq. (3) at all finite temperatures, thus consistent with the result of Ref. \([1]\). It is thus strongly suggested that the system displays quasi-long-range glass order below the finite transition temperature \([21]\), characterized by the algebraic decay of the glass correlation function.

We have thus computed the glass correlation function via large-scale simulations for the system size \(L = 48\) and display the obtained behavior in Fig. 3, where the difference in the behavior at the two temperatures \(T = 0.15\) and \(0.50\) is spectacular: While the dashed line at \(T = 0.50\) represents the least-square fit to the usual exponential decay \(r^{-\eta}e^{-r/\xi}\) with \(\eta = 0.31\) and \(\xi = 1.90\), the algebraic decay at \(T = 0.15\) is manifested by the least-square fit (represented by the dotted line) to \(r^{-\eta}\) with \(\eta = 0.27\). The decrease of the exponent \(\eta\) with temperature, from the value \(0.30\) (at \(T_c\)), reflects the criticality of the system below \(T_c\), confirming quasi-long-range glass order at low temperatures. The fit of the data at \(T = 0.15\) to the exponential-decay form has been tried, only to yield very large deviations as well as inconsistent values of parameters \(\eta\) and \(\xi\). We have also computed the correlation length at various temperatures above \(T_c\) and found perfect scaling to the form in Eq. (3) with the values of \(T_c\) and \(\nu\) given by Eq. (13) (see the inset).

We finally point out that the existence of the finite-temperature glass transition is not inconsistent with the experimental results. The temperature scale in the gauge-glass model may be obtained from the correspondence between the model temperature \(T = 1.15\) (in units of \(J\)) and the real temperature \(T = 12\)K in experiment.
Accordingly, the transition temperature estimated here corresponds to approximately 2.0K; such a low-temperature regime was not probed in Ref. [8] and it would be interesting to investigate the low-temperature regime experimentally.

We would like to thank B. Kim, D. Kim, S. Ryu, and D. Stroud for helpful discussions, and acknowledge the partial support from the Korea Research Foundation and from the Korea Science and Engineering Foundation. The numerical works have been performed on the cray T3E at SERI and on the SP2 supercomputer system at ERCC.

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This is to be contrasted with the Ising system, where the ubiquitous fluctuations (for $\theta < 0$) are domain walls and no criticality is allowed.

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FIG. 3. Behavior of the correlation function with distance \( r \) at two temperatures, \( T = 0.15 \) and 0.50. Algebraic decay is obvious at \( T = 0.15 \). Inset shows correlation length \( \xi \) versus temperature \( T - T_c \) in the log-log scale with \( T_c = 0.22 \pm 0.01 \). The slope of the solid line is fixed to the value \(-\nu = -1.14\).