ULRICH MODULES OVER CYCLIC QUOTIENT SURFACE SINGULARITIES

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ABSTRACT. Ulrich modules are a certain class of maximal Cohen-Macaulay modules. Their existence and properties are studied in several literatures. However, it is not easy to decide which maximal Cohen-Macaulay module is an Ulrich module. In this paper, we characterize Ulrich modules over cyclic quotient surface singularities by using the notion of special Cohen-Macaulay modules. Also, we investigate the number of indecomposable Ulrich modules for a given cyclic quotient surface singularity. Especially, the number of exceptional curves on the minimal resolution gives boundaries of the number of indecomposable Ulrich modules.

1. INTRODUCTION

Let \((R, m, \mathbb{k})\) be a Cohen-Macaulay (= CM) local ring with \(\text{dim } R = d\). For a finitely generated \(R\)-module \(M\), we say \(M\) is a maximal Cohen-Macaulay (= MCM) \(R\)-module if \(\text{depth } R M = d\). For each MCM \(R\)-module \(M\), we have \(\mu_R(M) \leq e^0_m(M)\) where \(\mu_R(M)\) stands for the number of minimal generators (i.e. \(\mu_R(M) = \dim \mathbb{k} M / m M\)) and \(e^0_m(M)\) is the multiplicity of \(M\) with respect to \(m\). Note that if \(R\) is a domain, then we have \(e^0_m(M) = (\text{rank}_R M) e^0_m(R)\).

An Ulrich module is defined as a module which has the maximum number of generators with respect to the above inequality. So we sometimes call it a maximally generated maximal Cohen-Macaulay (= MGMCM) module after the original terminology [Ulr, BHU]. The name “Ulrich modules” was introduced in [HK].

Definition 1.1 ([Ulr, BHU]). Let \(M\) be an MCM \(R\)-module. We say \(M\) is an Ulrich module if it satisfies \(\mu_R(M) = e^0_m(M)\).

We remark that the above conditions are inherited by direct summands and direct sums. So Ulrich modules are closed under direct summands and direct sums.

The properties of these modules were investigated in the aforementioned references. More geometrically, they are also studied as Ulrich bundles e.g. [ESW, CH1, CH2, CKM]. Recently, this notion is generalized for each non-parameter \(m\)-primary ideal \(I\) as follows [GOTWY1] and it is studied actively (cf. [GOTWY2, GOTWY3]). Namely, we say an MCM \(R\)-module \(M\) is an Ulrich module “with respect to \(I\)” if it satisfies the following conditions:

\[
(1) \ e^0_I(M) = \ell_R(M/IM), \quad (2) \ M/IM \text{ is an } R/I\text{-free module}
\]

where \(e^0_I(M)\) is the multiplicity of \(M\) with respect to \(I\) and \(\ell_R(M/IM)\) stands for the length of the \(R\)-module \(M/IM\). Thus, an Ulrich module with respect to \(m\) is nothing else but an Ulrich module in the sense of Definition1.1 (The condition (2) is automatically if \(I = m\).) In this paper, we only discuss Ulrich modules with respect to \(m\). Also, Ulrich modules appear in an attempt to formulate the notion of “almost Gorenstein rings” [GTT]. Thus, the importance to understand this module has increased. However, even the existence of an Ulrich module for a given CM local ring is still not known in general. Another important problem is to characterize (and to classify) Ulrich modules when a given ring \(R\) has an Ulrich module. For example, we know the existence of such a module for the case where

\[
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\]

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· a two dimensional domain with the infinite field \([BHU]\),
· a CM local ring which has maximal embedding dimension \([BHU]\),
· a strict complete intersection \([HUB]\),
· a Veronese subring of polynomial ring over field of characteristic 0 \([ESW]\) etc.

The characterization problem is also not known for many cases. Therefore, in this paper, we will characterize Ulrich modules over cyclic quotient surface singularities (i.e. toric surfaces). We remark that this singularity is of finite CM representation type (i.e. it has only finitely many non-isomorphic indecomposable MCM modules). Since the number of indecomposable Ulrich modules is finite, we will also consider the number of them. The point is to consider special CM modules (see Definition 2.1). This is another class of MCM modules and closely related with the minimal resolution of a quotient surface singularity as the special McKay correspondence (see Theorem 2.4). Roughly, the number of minimal generators of a special CM module is small. So special CM modules are the opposite of Ulrich modules in that sense. However, those give us the simple description of Ulrich modules as follows. (For further details on terminologies, see later sections.)

Let \( R \) be the invariant subring of \( \mathbb{k}[x,y] \) under the action of a cyclic group \( \mathbb{Z}/n(1,a) \). Then \( M_t = \langle x^iy^j \mid i + ja \equiv t \mod n \rangle \) is an MCM \( R \)-module for \( t = 0, 1, \ldots, n - 1 \). Suppose that \( M_{t_1}, \ldots, M_{t_r} \) are non-free indecomposable special CM \( R \)-modules \( (i_1 > \cdots > i_r) \). We call these subscripts \( (i_1, \cdots, i_r) \) the \( i \)-series.

Then we define the integers \( (d_{1,t}, \cdots, d_{r,t}) \) for each subscript \( t \in [0,n - 1] \) as follows:

\[
\begin{align*}
  t &= d_{1,1}i_1 + h_{1,t}, \\
  h_{1,t} &\in \mathbb{Z}_{>0}, \\
  0 &\leq h_{1,t} < i_1, \\
  h_{u+1,t} &= d_{u+1,1}i_{u+1} + h_{u+1,t}, \\
  h_{u+1,t} &\in \mathbb{Z}_{>0}, \\
  0 &\leq h_{u+1,t} < i_{u+1}, \\
  (u &= 1, \cdots, r - 1), \\
  h_{r,t} &= 0.
\end{align*}
\]

Then, we describe \( t \) as \( t = d_{1,1}i_1 + d_{2,1}i_2 + \cdots + d_{r,1}i_r \) and obtain the following.

**Theorem 1.2** (= Theorem 4.3 and Corollary 4.4). \( M_t \) is an Ulrich \( R \)-module if and only if \( d_{1,1} + d_{2,1} + \cdots + d_{r,1} = e(R) - 1 \) where \( e(R) \) is the multiplicity of \( R \) with respect to the maximal ideal.

We can prove this theorem by combining the Riemann-Roch formula \([Kat]\) and the special McKay correspondence \([Wun1, Wun2]\), and this is reinterpretation of Wunram’s results from the viewpoint of Ulrich modules. Besides such a proof, we will also give another proof based on the Auslander-Reiten theory.

By using this theorem, we can check which MCM \( R \)-module is an Ulrich one. However, it is sometimes tough to compute \( (d_{1,t}, \cdots, d_{r,t}) \) for all \( t = 0, 1, \cdots, n - 1 \). Thus, we will give a sharper characterization of Ulrich modules in terms of the \( i \)-series as follows. The crucial point is to consider good pairs of the \( i \)-series.

**Theorem 1.3** (= Theorem 4.7 and 4.8). Take any sequences of pairs of the \( i \)-series \( (i_{k(1)}, i_{k(1)'}), \cdots, (i_{k(b)}, i_{k(b)'})) \in \mathbb{U} \) (see (7.2)) with \( i_{k(c)'} > i_{k(c+1)} \) for any \( c = 1, 2, \cdots, b - 1 \).

If \( t = n - 1 - \sum_{c=1}^{b} (i_{k(c)} - i_{k(c)'}) \) or \( t = n - 1 \), then \( M_t \) is an Ulrich module.

Conversely, if \( M_t \) is an Ulrich module \( t \neq n - 1 \), then we can take a sequence of pairs \( (i_{k(1)}, i_{k(1)'}), \cdots, (i_{k(b)}, i_{k(b)'})) \in \mathbb{U} \) with \( i_{k(c)'} > i_{k(c+1)} \) for any \( c = 1, 2, \cdots, b - 1 \) and

\[
  t = n - 1 - \sum_{c=1}^{b} (i_{k(c)} - i_{k(c)'}) .
\]

As a corollary, we have the following.
Corollary 1.4 (= Corollary[4,11]). We suppose $M_t$ is an Ulrich module. Then we have $n - a \leq t \leq n - 1$. Furthermore, $M_{a-1}$ and $M_{a-a}$ are actually Ulrich modules.

In particular, this result gives us an upper bound of the number of Ulrich modules. That is, it is less than or equal to $a$. Also, we can obtain other bounds from more geometric informations.

Theorem 1.5 (= Theorem[5,6]). Suppose $R$ is a cyclic quotient surface singularity whose number of irreducible exceptional curves (= that of non-free indecomposable special CM modules) is $r$. Then the number of Ulrich modules $N$ satisfies $r \leq N \leq 2r^{-1}$.

This paper is organized as follows. Firstly, we introduce the notion of special CM modules in Section[2]. Also, we introduce the Auslander-Reiten quiver and visualize the relation among MCM modules. This oriented graph also plays the important role to characterize Ulrich modules. For the convenience of the reader, we discuss these notions for quotient surface singularities in Section[4]. In Section[5] we collect related topics. Especially, we count the number of Ulrich modules for some cases.

Notations. Since we will focus on Ulrich modules (with respect to $m$), we use the notations $e(M) := e^0_m(M)$ for simplicity. We denote the $R$-dual (resp. the canonical dual) functor by $(-)^* := \text{Hom}_R(-, R)$ (resp. $(-)^\dagger := \text{Hom}_R(-, \omega_R)$). Also, we denote the syzygy functor by $\Omega(-)$. Sometimes we denote $\text{CM}(R)$ to be the category of maximal Cohen-Macaulay modules and $\text{add}_e(M)$ to be the full subcategory consisting of direct summands of finite direct sums of some copies of $M$.

2. Preliminaries

Throughout this paper, we assume $k$ is an algebraically closed field. Although the main body of this paper is devoted to the case of cyclic groups, we will start a discussion in more general settings. Namely, let $G$ be a finite subgroup of $GL(2, k)$ which contains no pseudo-reflections. We assume that the order of $G$ is invertible in $k$. In the rest of this paper, let $S := k[[x, y]]$ be a power series ring and denote the invariant subring of $S$ under the action of $G$ by $R := S^G$.

Moreover, let $V_0 = k, V_1, \cdots, V_n$ be the full set of non-isomorphic irreducible representations of $G$ and set $M_t := (S \otimes_k V_t)^G$ ($t = 0, 1, \cdots, n$). It is well known that $R$ is of finite CM representation type and finitely many indecomposable MCMs are $\{R \cong M_0, M_1, \cdots, M_n\}$.

2.1. Special Cohen-Macaulay modules. In this subsection, we introduce the notion of special CM modules. As we will show below, these MCM modules are compatible with the geometry. Special CM modules have the properties opposite to Ulrich modules from the viewpoint of the number of minimal generators. However, they play the crucial role when we determine the Ulrich modules. Firstly, we recall the definition of special CM modules.

Definition 2.1 ([Wun2]). For an MCM $R$-module $M$, we say $M$ is special if $(M \otimes_R \omega_R)/\text{tor}$ is also an MCM $R$-module.

Remark 2.2. From the definition, if $R$ is Gorenstein (i.e. $G \subset SL(2, k)$ [Wat]), then every MCM module is special.

Special CM modules appear when we try to extend the classical McKay correspondence to a finite subgroup $G \subset GL(2, k)$. For a finite subgroup $G \subset SL(2, k)$, the original McKay correspondence says that there is a one-to-one correspondence between non-free indecomposable
MCM $R$-modules (equivalently, non-trivial irreducible representations of $G$) and irreducible exceptional curves on the minimal resolution of $\text{Spec } R$. This brilliant correspondence collapse if we consider a finite subgroup $G \subset \text{GL}(2, \mathbb{C})$. Indeed, we have more indecomposable MCM modules than exceptional curves. So J. Wunram introduced the notion of special CM modules. By choosing indecomposable special ones from all MCM modules, we again have a one-to-one correspondence between non-free indecomposable “special” MCM $R$-modules and irreducible exceptional curves ([Wun2] (see Theorem 2.4). Moreover, the original McKay correspondence is recovered from the special one (see Remark 2.2). For more details, see the next subsection and references [Wun1, Wun2, Ish, Ito, Rie].

Definition 2.1 is the original one. There are now several characterizations of special CM modules. For example, the following conditions are manageable in our context.

**Proposition 2.3.** ([IW 2.7 and 3.6]) Suppose that $M$ is an MCM $R$-module. Then the following are equivalent.

(a) $M$ is a special CM module,
(b) $\text{Ext}_R^i(M, R) = 0$,
(c) $(\Omega M)^* \cong M$.

Suppose $M$ is a special CM $R$-module, then we have the following exact sequence by the condition (c).

$$0 \to M^* \cong \Omega M \to R^{\mu_R(M)} \to M \to 0.$$  

Thus, we have $\mu_R(M) = 2 \text{rank}_R M$. The converse is true if $\text{rank}_R M = 1$ (cf. [Wun2 Theorem 2.1] [GOTWY2 Lemma 4.6]). If $\text{rank}_R M > 1$, the converse is no longer true (cf. [Nak] Example A.5) and [IW]). As we will see later, every MCM module over cyclic quotient surface singularities has rank one. Thus, the structure of special CM module is quite simple. (More precise description is given in Theorem 3.3)

**2.2. Special McKay correspondence.** As we mentioned, special CM modules are compatible with the geometry. Thus we will introduce terminologies in the geometric side and show the relationship between special CM modules and geometrical objects.

Let $\pi : X \to \text{Spec } R$ be the minimal resolution of singularities and $E := \pi^{-1}(\mathfrak{m})$ be the exceptional divisor. We decompose $E = \bigcup_{i=1}^r E_i$ into irreducible components and define the set of cycles supported on $E$:

$$\mathcal{C} = \left\{ \sum_{i=1}^r a_i E_i \mid a_i \in \mathbb{Z} \right\}.$$  

Also, we can impose a partial order $\leq$ on $\mathcal{C}$. That is, $Z \leq Z'$ if every coefficient of $E_i$ in $Z' - Z$ is non-negative ($Z, Z' \in \mathcal{C}$). We say a cycle $Z = \sum_{i=1}^r a_i E_i$ is positive if $Z \geq 0$ and $Z \neq 0$. (So we denote it by $Z > 0$.) We call a positive cycle $Z = \sum_{i=1}^r a_i E_i$ anti-nef if $Z \cdot E_i \leq 0$ for all $i = 1, \ldots, r$. Here, $Z \cdot Z'$ means the intersection number of $Z$ and $Z'$ ($Z, Z' \in \mathcal{C}$). If $Z = Z'$, the self-intersection number of $Z$ is denoted by $Z^2$. We define the fundamental cycle $Z_0$ as the unique smallest element of anti-nef cycles. There is an algorithm to determine $Z_0$ by [Lau].

The following is famous as the special McKay correspondence.

**Theorem 2.4 ([Wun2]).** For any $i$, there is a unique indecomposable MCM $R$-module $M_i$ (up to isomorphism) such that $H^1(M_i) = 0$ and $c_1(M_i) \cdot E_j = \delta_{ij}$ for $1 \leq i, j \leq r$ where $\tilde{M}_i = \pi^*(M_i)/\text{tor}$ and $c_1(\tilde{M}_i)$ stands for the first Chern class of $\tilde{M}_i$ and $(-)^\vee = \text{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$. These MCM modules $M_1, \ldots, M_r$ are precisely indecomposable non-free special CM modules in the sense of Definition 2.1 and $\text{rank}_R M_i = c_1(\tilde{M}_i) \cdot Z_0$. 

When we consider Ulrich modules, the multiplicity \( e(M) = (\text{rank}_R M) e(R) \) is important. It is known that the multiplicity \( e(R) \) is computed by the self-intersection number of the fundamental cycle \( Z_0 \) as follows [Art].

**Proposition 2.5.** Let the notations be the same as before. Then we have \( e(R) = -Z_0^2 \).

### 2.3. Auslander-Reiten theory.

In the previous subsection, we saw that the multiplicity is computed by the fundamental cycle. In addition, we have to investigate the number of minimal generators to determine an Ulrich module. In Section 4, we will use the Auslander-Reiten quiver to understand minimal generators. Moreover, the functor \( \tau \) so-called “the Auslander-Reiten translation” will give an Ulrich module from a special CM module (see Proposition 2.7 and 4.1). Thus, we show some results of Auslander-Reiten theory in this subsection. For the brevity, we only discuss the case of the invariant subring \( R \). For more details, see [Aus1, Aus2, Yos, LW].

**Definition 2.6.** (Auslander-Reiten sequence). Let \( R \) be the same as above and \( M, N \) be indecomposable MCM \( R \)-modules. We call a non split short exact sequence

\[
0 \to N \xrightarrow{f} L \xrightarrow{g} M \to 0
\]

the Auslander-Reiten (\( = \text{AR} \)) sequence ending in \( M \) if for any MCM module \( X \) and for any morphism \( \phi : X \to M \) which is not a split surjection there exists \( \hat{\phi} : X \to L \) such that \( \phi = g \circ \hat{\phi} \).

Since \( R \) is an isolated singularity, the AR sequence ending in \( M \) actually exists where \( M \) is a non-free indecomposable MCM \( R \)-module [Aus2], and it is unique up to isomorphism. In our situation, we can construct the AR sequence by using the Koszul complex over \( S \) and a natural representation \( V \) of \( G \). (see [Yos, Chapter 10].)

In the case where \( t \neq 0 \), the AR sequence ending in \( M_t \) is

\[
0 \to (S \otimes_k (\wedge^2 V \otimes_k V_t))^G \to (S \otimes_k (V \otimes_k V_t))^G \to M_t = (S \otimes_k V_t)^G \to 0.
\]

In the case where \( t = 0 \), there is the following sequence and it is called the fundamental sequence of \( R \).

\[
0 \to \omega_R = (S \otimes_k \wedge^2 V)^G \to (S \otimes_k V)^G \to R = S^G \to k \to 0.
\]

We call the left term of these sequences the Auslander-Reiten (\( = \text{AR} \)) translation and we denote it by \( \tau(M_t) \). Sometimes we denote the next term of \( \tau(M_t) \) by \( E_{M_t} \). It is known that the AR translation \( \tau \) is obtained via the functors

\[
\tau : \text{CM}(R) \xrightarrow{(-)^*} \text{CM}(R) \xrightarrow{-} \text{CM}(R).
\]

By applying these functors to special CM modules, we have some Ulrich modules.

**Proposition 2.7.** Suppose \( M \) is a non-free special CM module over \( R \). Then we have

1. \( M^* \) is an Ulrich \( R \)-module,
2. \( \tau(M) \) is also an Ulrich \( R \)-module.

**Proof.**

1. By Proposition 2.3, \( M^* \) is the syzygy of an MCM \( R \)-module. Thus, it is an Ulrich \( R \)-module by [GOTWY1, Lemma 4.2].

2. By [GOTWY1, Theorem 5.1], the canonical dual of an Ulrich module is also an Ulrich module. Combining with (1), we have the conclusion.

Thus, we can obtain some Ulrich modules from special ones. However, there exists Ulrich modules which don’t come from this operation.

Next, we prepare some notions to define the Auslander-Reiten quiver.
**Definition 2.8** (Irreducible morphism). Let $M$ and $N$ be indecomposable MCM $R$-modules. We define the submodule $\text{rad}_R(M, N) \subset \text{Hom}_R(M, N)$ as the set of morphisms which is not an isomorphism.

In addition, we define the submodule $\text{rad}_R^2(M, N) \subset \text{Hom}_R(M, N)$. The submodule $\text{rad}_R^2(M, N)$ consists of morphisms $\psi : M \to N$ such that $\psi$ decomposes as $\psi = g \circ f$, where $f \in \text{rad}_R(M, X)$, $g \in \text{rad}_R(X, N)$ and $X$ is an MCM $R$-module. We call a morphism $\psi : M \to N$ irreducible if $\psi \in \text{rad}_R(M, N) \setminus \text{rad}_R^2(M, N)$.

Set the $\mathbb{k}$-vector space $\text{Irr}_R(M, N) := \text{rad}_R(M, N) / \text{rad}_R^2(M, N)$. By using these notions, we define the Auslander-Reiten quiver.

**Definition 2.9** (Auslander-Reiten quiver). The Auslander-Reiten (= AR) quiver of $R$ is an oriented graph whose vertices are indecomposable MCM $R$-modules $R, M_1, \cdots, M_n$ and we draw $\dim \text{Irr}_R(M_s, M_t)$ arrows from $M_s$ to $M_t$ $(s, t = 0, 1, \cdots, n)$.

For a quotient surface singularity $R = \mathcal{O}_{\mathbb{C}^2}/\langle \alpha \rangle$, the shape of the AR quiver of $R$ coincides with that of the McKay quiver of $G$ [Aus1], and such quivers are described in [AR]. We will give an example of the AR quiver in Example 3.8.

### 3. Special Cohen-Macaulay Modules for Cyclic Cases

Until now, we introduced a special CM module in general situation. In this section, we focus on the case of cyclic quotient surface singularities and apply the former results.

We suppose that $G$ is a cyclic group as follows.

$$G := \langle \sigma = \left( \begin{array}{cc} \zeta_n^a & 0 \\ 0 & \zeta_n \end{array} \right) \rangle,$$

where $\zeta_n$ is a primitive $n$-th root of unity, $1 \leq a \leq n - 1$, and $\gcd(a, n) = 1$ and assume that $n$ is invertible in $\mathbb{k}$. This cyclic group $G$ is denoted by $\mathbb{Z}/(1, a)$. Since $G$ is an abelian group, every irreducible representation of $G$ is one dimensional and described as

$$V_t : \sigma \mapsto \zeta_n^{-t} \quad (t = 0, 1, \cdots, n - 1).$$

Then we set,

$$M_t := (S \otimes_{\mathbb{k}} V_t)^G = \left\langle x^j y^i \left| \; i + ja \equiv t \pmod{n} \right. \right\rangle, \quad (t = 0, 1, \cdots, n - 1).$$

Then, these $M_t$ are MCM modules over $R$ and rank $M_t = 1$. Note that $R$ is of finite CM representation type and CM($R$) = $\text{add}_R(R \oplus \bigoplus_{t=1}^{n-1} M_t)$.

For a cyclic group $G = \mathbb{Z}/(1, a)$, we can determine special CM modules by using the following combinatorial data. As the first step, we consider the Hirzebruch-Jung continued fraction expansion of $n/a$, that is,

$$\frac{n}{a} = \alpha_1 - \frac{1}{\alpha_2 - \frac{1}{\cdots - \frac{1}{\alpha_r}}} := [\alpha_1, \alpha_2, \cdots, \alpha_r],$$

and then we define the notion of $i$-series and $j$-series (cf. [Wem], [Wun1]).

**Definition 3.1.** For $n/a = [\alpha_1, \alpha_2, \cdots, \alpha_r]$, the $i$-series and the $j$-series are defined as follows.

$$i_0 = n, \quad i_1 = a, \quad i_t = \alpha_{t-1} i_{t-1} - i_{t-2} \quad (t = 2, \cdots, r + 1),$$

$$j_0 = 0, \quad j_1 = 1, \quad j_t = \alpha_{t-1} j_{t-1} - j_{t-2} \quad (t = 2, \cdots, r + 1).$$
Remark 3.2. From the construction method, it is easy to see
\begin{align*}
&\cdot \ i_t \equiv j_t a \pmod{n}, \\
&\cdot \ i_0 = n > i_1 = a > i_2 > \cdots > i_r = 1 > i_{r+1} = 0, \\
&\cdot \ j_0 = 0 < j_1 = 1 < j_2 = \alpha_1 < \cdots < j_r < j_{r+1} = n.
\end{align*}

By using the \(i\)-series and the \(j\)-series, we can characterize special CM \(R\)-modules.

Theorem 3.3 \([\text{Wun1}]\). For a cyclic group \(G = \mathbb{Z}/n\mathbb{Z}\) with \(n/\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_r]\), \(M_t (t = 1, \cdots, r)\) and \(R\) are precisely special CM modules over \(R\). Furthermore, the minimal generators of \(M_t\) are \(x^i\) and \(y^j\) for \(i = 1, \cdots, r\).

Example 3.4. Suppose \(G = \mathbb{Z}/n\mathbb{Z}\) is a cyclic group of order \(n\) (Dynkin type \(A_{n-1}\)). The Hirzebruch-Jung continued fraction expansion of \(n/(n-1)\) is
\[
\frac{n}{n-1} = 2 - \frac{1}{2 - \frac{1}{\cdots - 1/2}} = [2, 2, \cdots, 2],
\]
and the \(i\)-series and the \(j\)-series are
\[
i_0 = n, \ i_1 = n-1, \ i_2 = n-2, \ \cdots, \ i_{n-1} = 1, \ i_n = 0, \\
j_0 = 0, \ j_1 = 1, \ j_2 = 2, \ \cdots, \ j_{n-1} = n-1, \ j_n = n.
\]
Therefore, every MCM module is special (cf. Remark 2.2). Since \(e(R) = 2\), every non-free MCM \(R\)-module is also an Ulrich module. This kind of property holds in more general situation (cf. \([\text{GOTWY2}]\) Theorem 5.2, \([\text{HK}]\) Corollary 1.4).

Example 3.5. Let \(G = \mathbb{Z}/12\mathbb{Z}\) be a cyclic group of order 12. The Hirzebruch-Jung continued fraction expansion of 12/7 is
\[
\frac{12}{7} = 2 - \frac{1}{4 - 1/2} = [2, 4, 2],
\]
and the \(i\)-series and the \(j\)-series are obtained as follows.
\[
i_0 = 12, \ i_1 = 7, \ i_2 = 2, \ i_3 = 1, \ i_4 = 0, \\
j_0 = 0, \ j_1 = 1, \ j_2 = 2, \ j_3 = 7, \ j_4 = 12.
\]
Thus, the special CM modules are \(M_7, M_2, M_1, R\) and they take the form
\[
M_7 = Rx^3 + Ry, \quad M_2 = Rx^2 + Ry^2, \quad M_1 = Rx + Ry^3.
\]

From Theorem 2.4 there is a one-to-one correspondence between non-free indecomposable special CM modules and irreducible exceptional curves. The dual graph of the minimal resolution of singularity \(X \to \text{Spec}(R)\) is also obtained by the Hirzebruch-Jung continued fraction expansion:

\[
\begin{array}{ccccc}
E_t & E_{t+1} & \cdots & E_r \\
-\alpha_t & -\alpha_{t+1} & \cdots & -\alpha_r
\end{array}
\]

Here, an including number in each circle is the self-intersection number of the corresponding exceptional curve. The fundamental cycle is \(Z_0 = \sum_{t=1}^r E_t\). Furthermore, we have \(e(R) = \alpha_1 + \cdots + \alpha_r - 2(r-1)\) by Proposition 2.5.

Example 3.6. Let \(G\) be the same as Example 3.5. In this case, the dual graph is

\[
\begin{array}{ccc}
E_7 & E_2 & E_1 \\
-2 & -4 & -2
\end{array}
\]
and the fundamental cycle is \(Z_0 = E_7 + E_2 + E_1\). Thus, we have the multiplicity \(e(R) = -Z_0^2 = 4\).
Also, we consider the AR quiver for the cyclic cases. For a cyclic quotient surface singularity $R$, the AR sequence ending in $M_t$ ($t \neq 0$) is
\[
0 \longrightarrow M_{t-a-1} \longrightarrow M_{t-1} \oplus M_{t-a} \longrightarrow M_t \longrightarrow 0.
\]
For the case where $t = 0$, we have the fundamental sequence of $R$;
\[
0 \longrightarrow \omega_R \longrightarrow M_{-1} \oplus M_{-a} \longrightarrow R \longrightarrow k \longrightarrow 0.
\]
Thus, $E_{M_t} = M_{t-1} \oplus M_{t-a}$ and $\tau(M_t) = M_{t-a-1}$ for $t = 0, 1, \cdots, n-1$.

Remark 3.7. It is known that $\dim_k \text{Irr}_R(M_s, M_t)$ is equal to the multiplicity of $M_s$ in the decomposition of $E_{M_t}$. From (3.1) and (3.2), we have $\dim_k \text{Irr}_R(M_{t-1}, M_t) = 1$ and $\dim_k \text{Irr}_R(M_{t-a}, M_t) = 1$ for $t = 0, 1, \cdots, n-1$. We can take a morphism $x$ (resp. $y$) as a basis of $\text{Irr}_R(M_{t-1}, M_t)$ (resp. $\text{Irr}_R(M_{t-a}, M_t)$).

Example 3.8. Let $G = \frac{1}{12}(1, 7)$ be a cyclic group of order 12 (cf. Example 3.5 and 3.6). Then the AR quiver of $R = S^G$ is the following. For simplicity, we only describe subscripts as vertices.

4. ULRICH MODULES FOR CYCLIC CASES

We start this section with the following proposition.

Proposition 4.1. Let the notation be the same as Section 3. For a non-free special MCM $R$-module $M_i$, MCM modules $M_{r-i}$ and $M_{i-a-1}$ are Ulrich modules.

Proof. Since $M_{i} \cong M_{r-i}$ and $\tau(M_{i}) \cong M_{i-a-1}$, it follows from Proposition 2.7. \qed

From this proposition, we can obtain some Ulrich modules. However, there exists Ulrich modules which don’t take the form as in Proposition 4.1. In order to determine all of them, we will show the relationship between the multiplicity $e(M_i) = e(R)$ and the number of minimal generators $\mu_R(M_t)$ in terms of the $i$-series. As a conclusion, we characterize Ulrich $R$-modules. To state the theorem, we prepare some notations.

For the $i$-series $(i_1, \cdots, i_r)$ associated with $\frac{1}{t}(1, a)$ and for any $t \in [0, n-1]$, there are unique non-negative integers $d_{1,t}, \cdots, d_{r,t} \in \mathbb{Z}_{\geq 0}$ such that
\[
\begin{align*}
t &= d_{1,t} i_1 + h_{1,t}, & h_{1,t} &\in \mathbb{Z}_{\geq 0}, & 0 \leq h_{1,t} < i_1, \\
h_{u+1,t} &= d_{u+1,t} i_{u+1} + h_{u+1,t}, & h_{u+1,t} &\in \mathbb{Z}_{\geq 0}, & 0 \leq h_{u+1,t} < i_{u+1}, & (u = 1, \cdots, r-1), \\
n_{r,t} &= 0.
\end{align*}
\]
Thus, we describe $t$ as follows.
\[
  t = d_{1,t}i_1 + d_{2,t}i_2 + \cdots + d_{r,t}i_r
  = \left(\frac{i_1 + \cdots + i_d}{d_{1,t}}\right) + \left(\frac{i_2 + \cdots + i_r}{d_{2,t}}\right) + \cdots + \left(\frac{i_r + \cdots + i_r}{d_{r,t}}\right).
\]

If a situation is clear, we will denote simply $d_{u,t}$ by $d_u$. This sequence $(d_{1,t}, \cdots, d_{r,t}) \in (\mathbb{Z}_{\geq 0})^r$ is characterized as follows. We will use this lemma heavily in the future.

**Lemma 4.2.** ([Wun1, Lemma 1]) A sequence $(d_1, \cdots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ is obtained from the description
\[
  t = d_1i_1 + d_2i_2 + \cdots + d ri_r
\]
for some subscript $t = 0, 1, \cdots, n - 1$ if and only if a sequence satisfies the following two condition.
- $0 \leq d_u \leq \alpha_u - 1$ for every $u = 1, \cdots, r$.
- If $d_u = \alpha_u - 1$ and $d_v = \alpha_v - 1$ ($u < v$), then there exists $w$ such that $u < w < v$ and $d'_w \leq \alpha_w - 3$.

So we are now in a position to state the theorem.

**Theorem 4.3.** Let the notation be the same as above. Then we have
\[
  \mu_R(M_t) = d_1 + d_2 + \cdots + d_r + 1.
\]

To this theorem, we will give two kinds of proofs (geometric one and representation theoretic one). The geometric proof is quite simple and it says the above formula is a reinterpretation of special McKay correspondence from the viewpoint of Ulrich modules. However, the authors believe that the method used in another one will give us a new aspect for this subject (e.g. Remark 4.6). Therefore, we note both of them.

**Geometric proof of Theorem 4.3** From the Kato’s Riemann-Roch formula [Kat], we have
\[
  \mu_R(M_t) = 1 + c_1(\widetilde{M}_t) \cdot (E_{i_1} + \cdots + E_{i_r}).
\]
Also, $c_1(\widetilde{M}_t) \cdot E_{i_u} = d_{u,t}$ [Wun2]. So we have the conclusion. \(\square\)

Before moving to the representation theoretic proof, we give a crucial observation. Firstly, we will identify a minimal generator of $M_t$ with a certain path from $R$ to $M_t$ on the AR quiver. Since $M_t \cong \text{Hom}_R(R, M_t)$, a path from $R$ to $M_t$ corresponds with an element of $M_t$.

For example, in the AR quiver of Example 3.8, we can find a path
\[
  0 \xrightarrow{x} 1 \xrightarrow{y} 8 \xrightarrow{x} 9 \xrightarrow{x} 10 \xrightarrow{y} 5.
\]
A unit $1 \in R$ maps on $x^3y^2 \in M_5$ under the above path.

Note that if a given path from $R$ to $M_t$ factor through $R$ (≠ the starting point), then its image will be in $mM_t$. Thus, by Nakayama’s lemma, such a path does not correspond to a minimal generator of $M_t$.

\[
  mM_t \cong \{ R \xrightarrow{\text{non-split}} R^\oplus m \rightarrow M_t \}
\]
So we identify a minimal generator of $M_t$ with a path from $R$ to $M_t$ which doesn’t factor through a free module except the starting point. (Note that each arrow on the AR quiver is not split.) Thus, we will count such paths on the AR quiver.
**Representation theoretic proof of Theorem 4.3**  We write the AR quiver $Q$ as the form of the translation quiver $\mathbb{Z}Q$, see Figure 1 (it is the repetition of the AR quiver). Here, each diagram $\xymatrix{a & \ar[l]_c \ar[r]_b & d}$ corresponds to the AR sequence ending in $M_b$ ($b \neq 0$):

$$0 \to M_c \to M_d \oplus M_b \to M_r \to 0,$$

and the fundamental sequence of $R$. These diagrams are commute $\xymatrix{\star & \ar[l]_c \ar[r]_b & }$ from Remark 3.7.

\[\begin{array}{cccc}
\cdots & 0 & a & \cdots \\
\cdots & n-1 & a-1 & \cdots \\
\cdots & n-2 & a-2 & \cdots \\
& \ddots & \ddots & \ddots \\
\cdots & 2-a & 2-a & 2-a \\
\cdots & 1-2a & 1-a & 1 \\
\cdots & -2a & -a & 0 \\
& \ddots & \ddots & \ddots \\
\end{array}\]

**Figure 1.**

From this quiver, we extract an appropriate part which implies paths from $R$ to $M_t$ corresponding to minimal generators of $M_t$. Such paths takes the form like Figure 2. Here, we assume grayed areas don’t contain $R(=0)$ (otherwise we can divide those areas into smaller ones). Indeed, a vertex 0 which is located at outside of Figure 2 certainly go through free modules on the way to $M_t$. Thus, we may only consider the paths from $R(=0)$ to $M_t(=t)$ appearing in Figure 2. Furthermore, the number of vertex 0 appearing in Figure 2 coincides with $\mu_R(M_t)$ and we see that the rightmost vertical arrows are divided into $\mu_R(M_t) - 1$ blocks. We have to remark that vertices described by $\star$ and $\star_1$ are special CM modules because the number of minimal generators of them is two (see Theorem 3.3 and the discussion following Proposition 2.3). From now on, we will show this division corresponds to the integers $(d_1, d_2, \cdots, d_r)$.

We set $i_s = \max \{ i_r \in i\text{-series} \mid d_i \neq 0 \}$, then we can find the vertex $i_s$ on the rightmost vertical column in Figure 2. From this position, we will follow vertices to the left direction and if we arrive at a vertex 0, then we stop there (see Figure 3). Since $M_{i_s}$ is a special CM module, the length of the vertical (resp. horizontal) path from 0 to $i_s$ in Figure 3 is $i_s$ (resp. $j_s$) by Theorem 3.3. From the selecting method of $i_s$, we have $\star_1 \leq i_s$. If $\star_1 < i_s$, we have Figure 4 by Remark 3.2 and see that a path from the lower vertex 0 go through the upper one. Thus, this contradicts the choice of $\star_1$. It follows that $i_s$ coincides with $\star_1$. After that, we replace $d_s$ by $d_s - 1$. If $d_s \neq 0$, then we repeat the same operation to the vertical column starting at the second
rightmost vertex 0. Repeating these processes to the other vertical columns in order until $d_s$ become 0, we have $d_t$ blocks of length $i_s$.

Next, we set $i'_s = \max \{ i_t \ in \ i\text{-series} \ | \ d_t \neq 0, \ i_t < i_s \}$ and apply the same process to $i'_s$. Repeating the above processes until we arrive at the top row, we can see the number of divided blocks in Figure 2 is equal to $d_1 + d_2 + \cdots + d_r$. \hfill \Box
Since \( e(R) = e(M_t) \) and \( \mu_R(M_t) \leq e(M_t) \), we may set \( \mu_R(M_t) = e(R) - s \) where \( 0 \leq s \leq e(R) - 1 \). The next corollary immediately follows from the theorem. By this corollary, we can determine which \( M_t \) is Ulrich module for a given cyclic quotient surface singularity.

**Corollary 4.4.** Let the notation be the same as above. Then

\[
\mu_R(M_t) = e(R) - s \iff d_1 + d_2 + \cdots + d_r = e(R) - (s + 1)
\]

for \( s = 0, 1, \cdots, e(R) - 1 \).

In particular, an MCM \( R \)-module \( M_t \) is Ulrich if and only if \( d_1 + d_2 + \cdots + d_r = e(R) - 1 \).

**Example 4.5.** Let \( G = \mathbb{Z}/12\mathbb{Z} \) be a cyclic group of order 12 (cf. Example 3.5, 3.6 and 3.8). In this case, non-free special CM modules are \( M_7, M_2, M_1 \) and \( e(R) = 4 \).

So we obtain the following division of each subscript into integers appearing in the \( i \)-series.

\[
11 = 7 + 2 + 2 \quad 7 = 7 \quad 3 = 2 + 1 \\
10 = 7 + 2 + 1 \quad 6 = 2 + 2 + 2 \quad 2 = 2 \\
9 = 7 + 2 \quad 5 = 2 + 2 + 1 \quad 1 = 1 \\
8 = 7 + 1 \quad 4 = 2 + 2
\]

Therefore, Ulrich modules are \( M_{11}, M_{10}, M_6, \) and \( M_5 \). For example, paths in the AR quiver which correspond to minimal generators of \( M_{10} \) are described as follows:

```
0 \rightarrow 7 \rightarrow \cdots \rightarrow 6 \rightarrow 1 \rightarrow 8 \rightarrow 3 \rightarrow 10
  \quad |       |       |       |
0 \rightarrow 7 \rightarrow 2 \rightarrow 9
  \quad |       |       |
0 \rightarrow 7
  \quad |       |
6
  |   |
0
```

**Remark 4.6.** The method used in the representation theoretic proof enables us to determine Ulrich modules for other quotient surface singularities. For example, see [Nak] Example 3.6 and A.5.

In this way, we can check which MCM \( R \)-module \( M_t \) is an Ulrich one. However, if the order of \( G \) is large enough, then a process to obtain the sequence \( (d_{1,t}, \cdots, d_{r,t}) \) for every \( t = 0, 1, \cdots, n - 1 \) will be tough (although it is not difficult). Therefore we will show another characterization of Ulrich modules in terms of the \( i \)-series. Firstly, for each subscript \( t = 0, 1, \cdots, n - 1 \), we decompose it as in Lemma 4.2.

\[
t = d_{1,t}i_1 + d_{2,t}i_2 + \cdots + d_{r,t}i_r.
\]  \hfill (4.1)

Then, for each subscript \( t = 0, 1, \cdots, n - 1 \), we define a subset of the \( i \)-series as follows.

\[
I_t := \{i_s \mid d_{s,t} \neq 0 \text{ in the decomposition (4.1)}\}.
\]
In order to characterize Ulrich modules, we need $I_{n-1}$. Since we can decompose $n-1$ as
\[
-\quad
\]
\[
= (\alpha_1 - 1)i_1 + (\alpha_2 - 2)i_2 + \cdots + (\alpha_{r-1} - 2)i_{r-1} + (\alpha_r - i_r - i_{r+1} - 1) \\
= (\alpha_1 - 1)i_1 + (\alpha_2 - 2)i_2 + \cdots + (\alpha_{r-1} - 2)i_{r-1} + (\alpha_r - i_r),
\]
we have
\[
I_{n-1} = \{i_1\} \cup \{i_s | \alpha_s > 2 \text{ and } 2 \leq s \leq r\}.
\]
Here, since the sum of coefficient is $(\alpha_1 - 1) + \sum_{u=2}^{r}(\alpha_u - 2) = \alpha_1 + \cdots + \alpha_r - 2r + 1 = e(R) - 1$, $M_{n-1}$ is an Ulrich module. (This also come from Proposition 4.1 because from the definition of the $i$-series, we have $i_r = 1$.) Then we define pairs of integers appearing in the $i$-series:
\[
U := \{(i_s, i_u) | i_s \in I_{n-1} \text{ and } i_s > i_u \text{ (equivalently } u > s)\}. \quad (4.2)
\]
We emphasize that the determination method of $U$ is the core of the characterization of Ulrich modules.

We are now ready to state the theorem.

**Theorem 4.7.** Consider any sequences of pairs $(i_{k(1)}, i_{k(1)'})$, $\cdots$, $(i_{k(b)}, i_{k(b)'}) \in U$ which satisfy $i_{k(c)'} > i_{k(c+1)}$ for any $c = 1, 2, \cdots, b - 1$. If $t = n - 1 - \sum_{c=1}^{b}(i_{k(c)} - i_{k(c)'})$ or $t = n - 1$, then $M_t$ is an Ulrich module.

**Proof.** We already know that $M_{n-1}$ is an Ulrich module. Thus, we will consider the other case. We take a pair $(i_{k(1)}, i_{k(1)'}) \in U$, then $n - 1 - (i_{k(1)} - i_{k(1)'})$ is deformed as follows.
\[
\begin{align*}
(\text{if } k(1) = 1) & = \sum_{v=1}^{k(1)'-1}(\alpha_v - 2)i_v + (\alpha_{k(1)' - 1}i_{k(1)'}) + \sum_{v=k(1)'-1}^{r}(\alpha_v - 2)i_v, \\
(\text{if } k(1) \neq 1) & = (\alpha_1 - 1)i_1 + \sum_{v=2}^{k(1)-1}((\alpha_v - 2)i_v + (\alpha_{k(1) - 3}i_{k(1)}) \\
& + \sum_{v=k(1)-1}^{k(1)'-1}(\alpha_v - 2)i_v + (\alpha_{k(1)' - 1}i_{k(1)'}) + \sum_{v=k(1)'-1}^{r}(\alpha_v - 2)i_v.
\end{align*}
\]
In this decomposition, the coefficients of the $i$-series satisfy the conditions as in Lemma 4.2 and the sum of them is equal to $\alpha_1 + \cdots + \alpha_r - 2r + 1 = e(R) - 1$. Therefore, $M_{n-1 - (i_{k(1)} - i_{k(1)'})}$ is an Ulrich module by Corollary 4.4. Then we take a pair $(i_{k(2)}, i_{k(2)'}) \in U$ with $i_{k(2)'} > i_{k(2)}$. By the same argument, we can show that $M_t$ is an Ulrich module for $t = n - 1 - (i_{k(1)} - i_{k(1)'}) - (i_{k(2)} - i_{k(2)'})$. Repeating these processes, we have the conclusion. \hfill $\Box$

**Theorem 4.8.** Conversely, if $M_t$ is an Ulrich module ($t \neq n - 1$), then we can take a sequence of pairs $(i_{k(1)}, i_{k(1)'})$, $\cdots$, $(i_{k(b)}, i_{k(b)'}) \in U$ with $i_{k(c)'} > i_{k(c+1)}$ for any $c = 1, 2, \cdots, b - 1$ and
\[
t = n - 1 - \sum_{c=1}^{b}(i_{k(c)} - i_{k(c)'})
\]
Proof. Suppose \( M_r \) is an Ulrich module and describe \( t = d_1i_1 + d_2i_2 + \cdots + d_r i_r \) as in Lemma 4.2. Recall that \( n - 1 = (\alpha_1 - 1)i_1 + (\alpha_2 - 2)i_2 + \cdots + (\alpha_r - 2)i_r \). Then we set the integers
\[
(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r) := (d_1 - (\alpha_1 - 1), d_2 - (\alpha_2 - 2), \cdots, d_r - (\alpha_r - 2)).
\]
Since \( t \neq n - 1 \), \( (\varepsilon_1, \cdots, \varepsilon_r) \neq (0, \cdots, 0) \). By Lemma 4.9 and 4.10, we can take elements of \( U \) as in the statement.

So we have to show the following two lemmas.

Lemma 4.9. One has \( \varepsilon_1 \in \{-1, 0\} \) and \( \varepsilon_u \in \{-1, 0, 1\} \) for \( u = 2, \cdots, r \).

Proof. By Lemma 4.2, we have \( 0 \leq d_u \leq \alpha_u - 1 \) for \( u \in [1, r] \). So \( \varepsilon_1 \leq 0 \) and \( \varepsilon_u \leq 1 \) for \( u \in [2, r] \). Since \( M_{n - 1} \) and \( M_r \) are Ulrich modules, we have \( \varepsilon_1 + \cdots + \varepsilon_r = 0 \) by Corollary 4.4.

Case 1. Assume \( \varepsilon_1 \leq -2 \). Since \( \varepsilon_1 + \cdots + \varepsilon_r = 0 \), there are \(-\varepsilon_1(\geq 2)\) components which satisfy \( \varepsilon_u = 1 \), \( u \in [2, r] \). Set \( k := -\varepsilon_1 \) and suppose such components are
\[
\varepsilon_u = \varepsilon_{u_2} = \cdots = \varepsilon_{u_k} = 1 \quad (u_1 < u_2 < \cdots < u_k).
\]
Here, we may assume that for any \( j \in [1, k - 1] \) there is no component \( \varepsilon_{u'} = 1 \) such that \( u_j < u' < u_{j+1} \). Then, by Lemma 4.2, there exists a subscript \( v \) such that \( u_1 < v < u_2 \) and \( \varepsilon_v \leq -1 \). So there is a subscript \( u_{k+1} \) with \( u_k < u_{k+1} \) such that \( \varepsilon_{u_{k+1}} = 1 \) because \( \varepsilon_1 + \cdots + \varepsilon_r = 0 \). We again use Lemma 4.2 and there is a subscript \( v' \) such that \( u_k < v' < u_{k+1} \) and \( \varepsilon_{v'} \leq -1 \). These processes will continue infinitely, but the sequence \( (\varepsilon_1, \cdots, \varepsilon_r) \) is finite. Thus, we have \( \varepsilon_1 \in \{-1, 0\} \).

Case 2. Assume that there exists \( u \in [2, r] \) such that \( \varepsilon_u \leq -2 \). As we did in Case 1, set \( k := -\varepsilon_u(\geq 2) \) and \( \varepsilon_{u_1} = \varepsilon_{u_2} = \cdots = \varepsilon_{u_k} = 1 \).

If \( \varepsilon_1 = 0 \), then \( d_1 = \alpha_1 - 1 \). Thus, the sequence \( (d_1, \cdots, d_r) \) is of the form
\[
(\alpha_1 - 1, \boxed{A}, \alpha_{u_1} - 1, \boxed{B}, \alpha_{u_2} - 1, \cdots).
\]
By Lemma 4.2, we can find \( d_v \)'s which satisfy \( d_v \leq \alpha_v - 3 \) in the both part of \( A \) and \( B \). Even if one is the above \( d_v \) with \( e_u = d_u - (\alpha_u - 2) \leq -2 \), the other one leads us to the conclusion that there is a subscript \( u_{k+1} \) with \( u_k < u_{k+1} \) such that \( \varepsilon_{u_{k+1}} = 1 \). In the same way as Case 1, we have the contradiction.

If \( \varepsilon_1 = -1 \), then there is a subscript \( u_{k+1} \) with \( u_k < u_{k+1} \) such that \( \varepsilon_{u_{k+1}} = 1 \) as well because \( \varepsilon_1 + \cdots + \varepsilon_r = 0 \). Similarly, we have the contradiction.

As the consequence, \( \varepsilon_u \in \{-1, 0, 1\} \) for \( u \in [2, r] \).

Lemma 4.10. Let \( (\varepsilon'_1, \varepsilon'_2, \cdots, \varepsilon'_\ell) \in \{-1, 0\}^\ell \) be the subsequence of \( (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r) \) removing every 0 components from \( (\varepsilon_1, \cdots, \varepsilon_r) \). Then \( (\varepsilon'_1, \cdots, \varepsilon'_\ell) \) takes the alternate form as
\[
(-1, +1, -1, +1, \cdots, -1, +1).
\]

Proof. By the definition, we have \( \varepsilon'_1 + \cdots + \varepsilon'_\ell = 0 \) and the number of \(+1\) appearing in \( (\varepsilon'_1, \cdots, \varepsilon'_\ell) \) coincides with that of \(-1\).

Case 1. If \( \varepsilon_1 = -1 \), then \( \varepsilon_1 = \varepsilon'_1 = -1 \). Assume the sequence \( (\varepsilon'_1, \cdots, \varepsilon'_\ell) \) is not alternate. Then we can find a part appearing \(+1\) continuously. This contradicts Lemma 4.2.

Case 2. If \( \varepsilon_1 = 0 \), then \( \varepsilon_1 \neq \varepsilon'_1 \) and \( d_1 = \alpha_1 - 1 \). So we set \( \varepsilon_p = \varepsilon'_1 \in \{-1, 1\} \).

If \( \varepsilon_p = \varepsilon'_1 = 1 \), then there exists a subscript \( q \) with \( 1 < q < p \) such that \( \varepsilon_q = -1 \) by Lemma 4.2. This contradicts the definition of \( \varepsilon_p \). Therefore we have \( \varepsilon_p = \varepsilon'_1 = -1 \).

Assume the sequence \( (\varepsilon'_1, \cdots, \varepsilon'_\ell) \) is not alternate. Then we will run into the contradiction by the same reason as in Case 1.

□
Corollary 4.11. We suppose $M_i$ is an Ulrich module. Then we have $n - a \leq t \leq n - 1$. Furthermore, $M_{n-1}$ and $M_{n-a}$ are actually Ulrich modules.

Proof. By Theorem 4.8 we describe $t$ as $t = n - 1 - \sum_{c=1}^{b} (i_{(c)} - i'_{(c)})$ where

\[(i_{(1)}, i'_{(1)}), \ldots, (i_{(b)}, i'_{(b)}) \in U \text{ with } i_{(c)} > i_{(c+1)} \text{ for any } c = 1, 2, \ldots, b - 1.\]

So we have

\[\sum_{c=1}^{b} (i_{(c)} - i'_{(c)}) = i_{(1)} - (i_{(1)} - i_{(2)}) - \cdots - (i_{(b-1)} - i_{(b)}) - i_{(b)}' \leq i_1 - i_r = a - 1.\]

Also, $M_{n-1}$ and $M_{n-a}$ are Ulrich modules because $M_1$ and $M_a$ are special CM modules (see Proposition 4.1).

Example 4.12. Suppose $G = \frac{158}{159}(1, 57)$. Then we have $\frac{158}{57} = [3, 5, 2, 3, 3]$ and $i_1 = 57, i_2 = 13, i_3 = 8, i_4 = 3, i_5 = 1$. Since $\mathbb{I}_{n-1} = \mathbb{I}_{157} = \{i_1, i_2, i_4, i_5\}$, we obtain

\[U = \{(i_1, i_2), (i_1, i_3), (i_1, i_4), (i_1, i_5), (i_2, i_3), (i_2, i_4), (i_2, i_5), (i_4, i_5)\}.\]

By the following table and Theorem 4.7 and 4.8 we see that Ulrich modules are only $M_{101}, M_{103}, M_{106}, M_{108}, M_{111}, M_{113}, M_{145}, M_{147}, M_{150}, M_{152}, M_{155}$ and $M_{157}$.

| pairs $(i_{(1)}, i'_{(1)}), \ldots, (i_{(b)}, i'_{(b)})$ | $t = n - 1 - \sum_{c=1}^{b} (i_{(c)} - i'_{(c)})$ |
|-----------------------------------------------------|----------------------------------|
| $(i_1, i_2) = (57, 13)$                              | 157 - (57 - 13) = 113            |
| $(i_1, i_3) = (57, 8)$                               | 157 - (57 - 8) = 108             |
| $(i_1, i_4) = (57, 3)$                               | 157 - (57 - 3) = 103             |
| $(i_1, i_5) = (57, 1)$                               | 157 - (57 - 1) = 101             |
| $(i_2, i_3) = (13, 8)$                               | 157 - (13 - 8) = 152             |
| $(i_2, i_4) = (13, 3)$                               | 157 - (13 - 3) = 147             |
| $(i_2, i_5) = (13, 1)$                               | 157 - (13 - 1) = 145             |
| $(i_4, i_5) = (3, 1)$                                | 157 - (3 - 1) = 155              |
| $\{(i_1, i_2), (i_4, i_5)\}$                        | 157 - (57 - 13) - (3 - 1) = 111  |
| $\{(i_1, i_3), (i_4, i_5)\}$                        | 157 - (57 - 8) - (3 - 1) = 106   |
| $\{(i_2, i_3), (i_4, i_5)\}$                        | 157 - (13 - 8) - (3 - 1) = 150   |

5. Further topics

In this section, we will consider the following question.

Question 5.1. For a cyclic quotient surface singularity $R$, fix the integer $1 \leq m \leq e(R)$.

How many indecomposable MCM modules which satisfy $\mu_R(M_i) = m$ are there? In particular, how many indecomposable Ulrich modules are there? ??

For simplicity, we denote the number of indecomposable MCM modules $M_i$ which satisfies $\mu_R(M_i) = m$ by $N_m$. Namely,

\[N_m = \#\{M_i \in \text{CM}(R) \mid \mu_R(M_i) = m\}.\]

5.1. The number of minimal generators for each MCM modules. Firstly, we show that there actually exists an MCM $R$-module which satisfies $\mu_R(M_i) = m$ for any $m = 1, \cdots, e(R)$. That is, $N_m \geq 1$ for any $m = 1, \cdots, e(R)$ (see Proposition 5.3).

Proposition 5.2. Fix an integer $m = 1, \cdots, e(R)$. Assume there exists an MCM $R$-module $M_i$ such that $\mu_R(M_i) = m$ and $t$ is described as $t = a_1i_1 + a_2i_2 + \cdots + a_ri_r$. Then for every $\ell = m, m - 1, \cdots, 1$, there is an MCM $R$-module $M_i'$ such that $\mu_R(M_i') = \ell$.
Proof. Taking \( i_u \in \mathcal{I}_t \), we have \( \mu_R(M_{t-i_u}) = m - 1 \) by Theorem 4.3. Similarly we take \( i_v \in \mathcal{I}_{t-i_u} \) and have \( \mu_R(M_{t-i_u-i_v}) = m - 2 \). Since \( d_{1,t} + d_{2,t} + \cdots + d_{r,t} = m - 1 \) by the hypothesis, we can repeat the above process \( m - 1 \) times.

\[ \square \]

**Proposition 5.3.** For every integer \( m = 1, \cdots, e(R) \), there is an MCM \( R \)-module \( M_t \) such that \( \mu_R(M_t) = m \).

**Proof.** Since there exists an Ulrich module, we apply Proposition 5.2 to \( m = e(R) \) and have the conclusion. \[ \square \]

From these results we have the following relation among some classes of MCM \( R \)-modules.

**Corollary 5.4.** Let \( R \) be a cyclic quotient surface singularity. Then

1. If \( e(R) = 2 \), \( CM(R) = SCM(R) = \mathbb{add}(R) \sqcup UCM(R) \) (cf. Example 3.4),
2. If \( e(R) = 3 \), \( CM(R) = SCM(R) \sqcup UCM(R) \),
3. If \( e(R) > 3 \), \( CM(R) \nsubseteq SCM(R) \sqcup UCM(R) \).

where \( SCM(R) \) (resp. \( UCM(R) \)) is the full subcategory of \( CM(R) \) consisting of special (resp. Ulrich) CM \( R \)-modules.

**Remark 5.5.** These are typical results for cyclic quotient surface singularities.

1. Proposition 5.3 doesn’t hold in a higher dimension. For example, we consider the action of \( G = \langle \text{diag}(-1, -1, -1) \rangle \) on \( S = k[x, y, z] \). Then the invariant subring \( R = S^G \) is of finite CM representation type and finitely many indecomposable MCMs are \( R \), \( \omega_R \) and \( \Omega \omega_R \) (cf. [Yos, LW]). Also, we have \( e(R) = 4 \) but \( \mu_R(\omega_R) = 3 \) and \( \mu_R(\Omega \omega_R) = 8 \).
2. Corollary 5.4 (2) doesn’t hold for non-cyclic cases. For example, let \( R \) be the invariant subring as in [Nak, Example 3.6]. Note that \( e(R) = 3 \). We can find some indecomposable MCM \( R \)-modules which are neither special CM modules nor Ulrich modules (see [Nak, Example A.5] and [LW]).

### 5.2. The number of Ulrich modules

In the previous subsection, we investigated the number \( N_m \) and showed \( N_m \geq 1 \) for any \( m = 1, \cdots, e(R) \). In this subsection, we will focus on \( N_{e(R)} \), that is, the number of Ulrich modules.

Firstly, we should remark that Corollary 4.11 gives an upper bound of \( N_{e(R)} \). Namely, we have \( N_{e(R)} \leq a \). Next, we will give another other bounds in terms of the number of irreducible exceptional curves.

**Theorem 5.6.** Suppose \( R \) is a cyclic quotient surface singularity whose number of irreducible exceptional curves (= that of non-free indecomposable special CM modules) is \( r \):

\[ \alpha_1 < -\alpha_2 < \cdots < -\alpha_r \]

Then we have \( r \leq N_{e(R)} \leq 2^{r-1} \). Especially, \( N_{e(R)} = 2^{r-1} \) holds only if \( \alpha_u > 2 \) for every \( u = 2, \cdots, r - 1 \), and \( N_{e(R)} = r \) holds only if \( \alpha_2 = \cdots = \alpha_{r-1} = 2 \).

**Proof.** By Theorem 4.7 and 4.8, \( M_t \) is an Ulrich module if and only if \( t = n - 1 \) or \( t \) is described by a sequence of elements which satisfy

\[ (\clubsuit) \quad (i_{k(1)}, i_{k(1)'}) \cdots, (i_{k(c)}, i_{k(c)'}) \in U \text{ with } i_{k(c)'} > i_{k(c+1)} \text{ for any } c = 1, 2, \cdots, b - 1. \]

Note that if we take different sequences of elements in \( U \) which satisfy \( (\clubsuit) \), then corresponding subscripts are also different, because the sequence \( (d_{1,t}, \cdots, d_{r,t}) \) as in Lemma 4.2 is unique for each subscript \( t \). Thus, \( N_{e(R)} = 1 \) is equal to the number of sequences satisfying the condition \( (\clubsuit) \). Therefore we may show the maximal (resp. minimal) number of such sequences is equal to \( 2^{r-1} - 1 \) (resp. \( r - 1 \)). Clearly, we should consider the case where \( \mathcal{I}_{n-1} = \{i_1, \cdots, i_r\} \) to obtain the upper bound of \( N_{e(R)} \). (Notice that the element \( i_r \) doesn’t influence the number of elements in \( U \).)
Furthermore, we denote the set of sequences of $U^k$ which satisfies (a) by $\mathcal{U}^r$, and the number of elements in $\mathcal{U}^r$ by $\#\mathcal{U}^r$. In this situation, we may show $\#\mathcal{U}^r = 2^r - 1$ and the following inductive argument asserts the conclusion.

The case where $k = r$ is easy. ($R$ is a Veronese subring and $U^r = \emptyset$.) Assume we have $\#\mathcal{U}^r = 2^r - 1$ for $k = 2, \ldots, r$. Then we can obtain elements in $\mathcal{U}^r$ as follows.

- $(i_1, i_2), (i_1, i_3), \ldots, (i_1, i_r)$,
- elements in $\mathcal{U}^{r-2}$,
- combine $(i_1, i_2)$ and elements in $\mathcal{U}^{r-3}$,
- \quad ... \quad
- combine $(i_1, i_{r-3})$ and elements in $\mathcal{U}^2$,
- combine $(i_1, i_{r-2})$ and elements in $\mathcal{U}^1 = \{(i_{r-1}, i_r)\}$.

By the hypothesis, the number of these elements is less than or equal to $(r - 1) + (2^{r-2} - 1) + (2^{r-3} - 1) + \ldots + (2^1 - 1) = 2^{r-2} + 2^{r-3} + \ldots + 2 + 1 = 2^r - 1$. In order to obtain the lower bound, we consider the case where $\mathcal{I}_{n-1} = \{i_1\}$, and it is easy to see $N_{c(R)} = r$. \hfill \Box

**Remark 5.7.** We could obtain two upper bounds $N_{c(R)} \leq a$ or $2^r - 1$, but it depends on a case whether one is a better bound.

For the case where $r$ is small, we can compute $N_{c(R)}$ explicitly. (Check the bounds of $N_{c(R)}$ for the following examples.)

**Example 5.8.** Suppose $R$ is a cyclic quotient surface singularity whose dual graph of the following form $C$.

1. The case where

$$C: \begin{array}{c}
\alpha \\
\beta \\
\end{array} \quad (\alpha, \beta \geq 2).$$

Then we have $N_{c(R)} = 2$.

2. The case where

$$C: \begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\end{array} \quad (\alpha, \gamma \geq 2).$$

(2-1) If $\beta = 2$, then we have $N_{c(R)} = 3$.

(2-2) If $\beta > 2$, then we have $N_{c(R)} = 4$.

3. The case where

$$C: \begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta \\
\end{array} \quad (\alpha, \beta, \gamma, \delta \geq 2).$$

(3-1) If $\beta = 2, \gamma = 2$, then we have $N_{c(R)} = 4$.

(3-2) If $\beta = 2, \gamma > 2$, then we have $N_{c(R)} = 6$.

(3-3) If $\beta > 2, \gamma = 2$, then we have $N_{c(R)} = 6$.

(3-4) If $\beta > 2, \gamma > 2$, then we have $N_{c(R)} = 8$.

**Proof.** We only show the case (3-2). The other cases are similar.

Let $i_1, \ldots, i_4$ be the $i$-series corresponding to each exceptional curve. Since $\beta = 2$ and $\gamma > 2$, we have $\mathcal{I}_{n-1} = \{i_1, i_3\}$. The statement follows from Theorem 4.7 and 4.8 because we can take the following pairs: $\{(i_1, i_2)\}, \{(i_1, i_3)\}, \{(i_2, i_4)\}, \{(i_3, i_4)\}$, $\{(i_1, i_2), (i_3, i_4)\}$.
5.3. Examples. We finish this paper with some special examples. In particular, we determine the number \( N_m \) completely. From Theorem 4.3, special CM modules behave like a “basis”. Thus, we can identify each MCM \( R \)-module \( M \) with the lattice point \((d_1, t, \cdots, d_r, t) \in \mathbb{Z}^r\).

**Example 5.9.** Suppose \( G = \frac{1}{23}(1, 6) \) and \( R = \mathbb{k}[[x, y]]^G \). Then \( \frac{23}{6} = 4 - \frac{1}{6} = [4, 6] \) and \( e(R) = 8 \). The i-series are \( i_1 = 6, i_2 = 1 \).

In this situation, we identify each subscript \( t = 0, 1, \cdots, 22 \) with the lattice point \((d_1, d_2, t) \in \mathbb{Z}^r\).

Example 5.10. Take integers \( \alpha, \beta \geq 2 \). Suppose \( G = \frac{1}{n}(1, a) \) which satisfies \( n/a = \alpha - \frac{1}{\beta} \). Then \( R = S^G \) is the cyclic quotient surface singularity whose dual graph is

\[
\begin{array}{c}
\text{–} \alpha \\
\text{–} \beta
\end{array}
\]

and \( e(R) = \alpha + \beta - 2 \). The i-series are \( i_1 = \beta, i_2 = 1 \).

If \( \alpha \leq \beta \), we have

\[
\begin{align*}
N_{\alpha+\beta-2} &= 2, & N_{\beta-1} &= \alpha, & N_{\alpha-2} &= \alpha - 2, \\
N_{\alpha+\beta-3} &= 3, & \vdots & & \vdots \\
& \vdots & N_\alpha &= \alpha, & N_2 &= 2, \\
N_\beta &= \alpha, & N_{\alpha-1} &= \alpha - 1, & N_1 &= 1.
\end{align*}
\]

The case \( \alpha \geq \beta \) is similar. (replace \( \alpha \) by \( \beta \) and vice versa.)
Proof. By the above figure of lattice points, we easily count the number of desired MCM modules. □

We can also determine $N_m$ for the following situation.

**Example 5.11.** Consider a cyclic quotient surface singularity $R$ whose dual graph is

```
\begin{array}{c}
\alpha \\
A - 1
\end{array}
```

where $\alpha \geq 2$ and $A, B \geq 1$. Then $e(R) = \alpha$ and we have the following.

\[
\begin{align*}
N_\alpha &= AB, & N_3 &= AB, \\
N_{\alpha-1} &= AB, & N_2 &= A + B - 1, \\
\vdots &= & N_1 &= 1, \\
N_4 &= AB,
\end{align*}
\]

**Proof.** Let $i_1, \ldots, i_{A-1}, i_A, i_{A+1}, \ldots, i_{A+B-1}$ be the $i$-series corresponding to each exceptional curve. Especially, $i_2$ corresponds to the exceptional curve whose self-intersection number is $-\alpha$. Thus, we have $I_{n-1} = \{i_1, i_A\}$ and $U = \{(i_1, i_2), \ldots, (i_1, i_{A+B-1}), (i_A, i_{A+1}), \ldots, (i_A, i_{A+B-1})\}$. It is easy to see $i_A = B, i_{A+1} = B - 1, \ldots, i_{A+B-1} = 1$. Therefore, we can see an MCM $R$-module $M_I$ whose subscript is appearing in the following table is an Ulrich module by Theorem 4.7 and 4.8

\[
\begin{align*}
n - 1, &\quad n - 1 - (i_1 - i_2), \quad \ldots \quad n - 1 - (i_1 - i_{A-1}), \quad n - 1 - (i_1 - i_A), \\
n - 2 = n - 1 - (i_A - i_{A+1}), &\quad n - 2 - (i_1 - i_2), \quad \ldots \quad n - 2 - (i_1 - i_{A-1}), \quad n - 1 - (i_1 - i_{A+1}), \\
\vdots &\quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
n - B = n - 1 - (i_A - i_{A+B-1}), &\quad n - B - (i_1 - i_2), \quad \ldots \quad n - B - (i_1 - i_{A-1}), \quad n - 1 - (i_1 - i_{A+B-1}).
\end{align*}
\]

Thus, we obtain $N_\alpha = AB$. By the same arguments as in Proposition 5.2, we can determine $N_m$ for $m = 3, 4, \ldots, \alpha - 1$. The value of $N_2$ follows from the special McKay correspondence. □

**Remark 5.12.** If we could obtain $N_m$ for $m = 1, \ldots, e(R)$ as in Example 5.10 and 5.11, then we can compute the Hilbert-Kunz multiplicity $e_{HK}(R)$. It is a numerical invariant in positive characteristic and is obtained by the formula $e_{HK}(R) = \frac{1}{n} \sum_{t=0}^{n} \mu_R(M_t)$ in our situation (see Nak Appendix). Thus, let $R$ be as in Example 5.10 (resp. Example 5.11). Then we have $e_{HK}(R) = \frac{1}{2}(\alpha \beta - 2)(\alpha + 1) + 2$ (resp. $e_{HK}(R) = 1/2(AB(\alpha - 2)(\alpha + 3) + 4(A + B) - 2)$).

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**REFERENCES**

[Art] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136.

[Aus1] M. Auslander, *Rational singularities and almost split sequences*, Trans. Amer. Math. Soc. **293** (1986), no. 2, 511–531.

[Aus2] M. Auslander, *Isolated singularities and existence of almost split sequences*, Proc. ICRA IV, Springer Lecture Notes in Math. **1178** (1986), 194–241.

[AR] M. Auslander and I. Reiten, *McKay quivers and extended Dynkin diagrams*, Trans. Amer. Math. Soc. **293** (1986), no. 1, 293–301.

[BHU] J. Brennan, J. Herzog, and B. Ulrich, *Maximally generated Cohen-Macaulay modules*, Math. Scand. **61** (1987), 181–203.
[CH1] M. Casanellas and R. Hartshorne, ACM bundles on cubic surfaces, J. Eur. Math. Soc. (JEMS) 13 (2011), no. 3, 709–731.

[CH2] M. Casanellas and R. Hartshorne, Stable Ulrich bundles, Internat. J. Math. 23 (2012), no. 8, 1250083, 50 pp, with an appendix by F. Geiss and F.-O. Schreyer.

[CKM] E. Coskun, R. Kulkarni and Y. Mustopa, The geometry of Ulrich bundles on del Pezzo surfaces, J. Algebra 375 (2013), 280–301.

[ESW] D. Eisenbud, F.-O. Schreyer and J. Weymann, Resultants and Chow forms via Exterior Syzygies, J. Amer. Math. Soc. 16 (2003), 537–579.

[GOTWY1] S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe and K. Yoshida, Ulrich ideals and modules, Math. Proc. Cambridge Philos. Soc. 156 (2014), 137–166.

[GOTWY2] S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe and K. Yoshida, Ulrich ideals and modules over two-dimensional rational singularities, to appear in Nagoya Math. J., arXiv:1307.2093.

[GOTWY3] S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe and K. Yoshida, Ulrich ideals and modules for simple singularities, in preparation.

[GT] S. Goto, R. Takahashi and N. Taniguchi, Almost Gorenstein rings –towards a theory of higher dimension–, to appear in J. Pure Appl. Algebra, arXiv:1403.3599.

[HK] J. Herzog and M. Kühn, Maximal Cohen-Macaulay modules over Gorenstein rings and Bourbaki-sequences, Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math., 11, North-Holland, Amsterdam (1987), 65–92.

[HUB] J. Herzog, B. Ulrich and J. Backelin, Linear maximal Cohen-Macaulay modules over strict complete intersections, J. Pure Appl. Algebra, 71 (1991), 187–202.

[Hun] C. Huneke, Hilbert-Kunz Multiplicity and the F-signature, Commutative algebra. Expository papers dedicated to David Eisenbud on the occasion of his 65th birthday, Springer-Verlag, New York, 485–525.

[Ish] A. Ishii, On the McKay correspondence for a finite small subgroup of GL(2, C), J. Reine Angew. Math. 549 (2002), 221–233.

[Ito] Y. Ito, Special McKay correspondence, Sémin. Congr. 6 (2002), 213–225.

[IW] O. Iyama and M. Wemyss, The classification of special Cohen Macaulay modules, Math. Z. 265 (2010), no. 1, 41–83.

[Kat] M. Kato, Riemann-Roch Theorem for strongly convex manifolds of dimension 2, Math. Ann. 222 (1976), 243–250.

[Lau] H. Laufer, On rational singularities, Amer. J. Math. 94 (1972), 597–608.

[LeW] G. Leuschke and R. Wiegand, Cohen-Macaulay Representations, vol. 181 of Mathematical Surveys and Monographs, American Mathematical Society (2012).

[Mon] P. Monsky, The Hilbert-Kunz function, Math. Ann. 263 (1983), 43–49.

[Nak] Y. Nakajima, Dual F-signature of Cohen-Macaulay modules over rational double points, arXiv:1407.5230.

[Rie] O. Riemenschneider, Special representations and the two-dimensional McKay correspondence, Hokkaido Math. J. 32 (2003), no. 2, 317–333.

[Ul] B. Ulrich, Gorenstein rings and modules with high numbers of generators, Math. Z. 188 (1984), 23–32.

[Wat] K. Watanabe, Certain invariant subrings are Gorenstein. I, Osaka J. Math. 11 (1974), 1–8.

[WY] K. Watanabe and K. Yoshida, Hilbert-Kunz multiplicity and an inequality between multiplicity and colength, J. Algebra 230 (2000), 295–317.

[Wem] M. Wemyss, Reconstruction algebras of type A, Trans. Amer. Math. Soc. 363 (2011), 3101–3132.

[Wun1] J. Wunram, Reflexive modules on cyclic quotient surface singularities, Lecture Notes in Mathematics, Springer-Verlag 1273 (1987), 221–231.

[Wun2] J. Wunram, Reflexive modules on quotient surface singularities, Math. Ann. 279 (1988), no. 4, 583–598.

[Yos] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, London Mathematical Society Lecture Note Series, 146, Cambridge University Press, Cambridge, (1990).

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