SYMMETRY AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR FRACTIONAL SYSTEMS

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Abstract. We consider the following fractional Hénon system
\[
\begin{cases}
(-\Delta)^{\alpha/2} u = |x|^a v^p, & x \in \mathbb{R}^n, \\
(-\Delta)^{\alpha/2} v = |x|^b u^q, & x \in \mathbb{R}^n,
\end{cases}
\]
for $0 < \alpha < 2$ and $a, b \geq 0$, $n \geq 2$. Under rather weaker assumptions, by using a direct method of moving planes, we prove the nonexistence and symmetry of positive solutions in the subcritical case where $1 < p < \frac{n+\alpha+2a}{n-\alpha}$ and $1 < q < \frac{n+\alpha+2b}{n-\alpha}$.

1. Introduction. The well-known Hénon equation
\[
\begin{cases}
(-\Delta)^{\alpha/2} u = |x|^a u^p, & x \in \mathbb{R}^n, \\
u \geq 0, & x \in \mathbb{R}^n
\end{cases}
\] (1)
has attracted widespread attention. In particular, the problem of existence and nonexistence of solutions (known as the Liouville theorem), but has not yet been fully answered.

For $\alpha = 2$, Gidas and Spruck [20] showed that there is no positive solutions with $0 < p < \frac{n+2}{n-2}$, $a = 0$ and $n > 2$. Phan and Souplet [33] showed the nonexistence of positive bounded solution when $n = 3$, $a > -2$ and $1 < p < 5 + 2a$. Fazly and Ghoussoub [17] proved that there exists no positive solution with finite Morse index to (1) with $n \geq 3$, $a \geq 0$ and $1 < p < \frac{n+2+2a}{n-2}$. For $\alpha = 2m$, $m \in N, 1 < 2m < n$, Fazly [15] proved the nonexistence of positive bounded solution to (1) with $1 < p < \frac{n+2m+2a}{n-2m}$. D’Amborsio and Mitidieri [13] showed that there exist no positive solutions when $1 < p \leq \frac{n}{n-2m}$.

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When $a = 0$, (1) is known as the Lane-Emden equation. It has been the central part in the progression of nonlinear analysis in the past few decades. Such fundamental results as the critical point theory (also known as the Mountain Pass theory), the a priori estimates, the eigenfunction theory and the Liouville-type theorems have been obtained (see [1, 2, 12, 19, 20] and the references therein).

In this paper, we consider the following system involving the fractional Laplacian

$$\begin{cases}
(-\Delta)^{\alpha/2} u = |x|^a v^p, & x \in \mathbb{R}^n, \\
(-\Delta)^{\alpha/2} v = |x|^b u^q, & x \in \mathbb{R}^n, \\
u \geq 0, v \geq 0, & x \in \mathbb{R}^n,
\end{cases}$$

where $0 < \alpha < 2$, $1 \leq p, q < \infty$, $a, b \geq 0$, $n \geq 2$.

The famous Lane-Emden conjecture states that (2) has no classical solutions in the subcritical case

$$\frac{n + a}{p + 1} + \frac{n + b}{q + 1} > n - \alpha.$$
Theorem 1.1. Let \((u, v) \in L_\alpha \cap C^{1,1}_loc\) be a pair of nonnegative solutions to (2) and \(1 < p < \frac{n+a+\alpha}{n-\alpha}, \ 1 < q < \frac{n+a+b}{n-\alpha}\). Then \(u\) and \(v\) are radially symmetric and decreasing about the origin.

Theorem 1.2. Let \((u, v) \in L_\alpha \cap C^{1,1}_loc\) be a pair of nonnegative solutions to (2) with \(a + \alpha > 0, \ b + \alpha > 0\) and \(1 \leq p, q < \infty\). Then \(u\) and \(v\) satisfy
\[
\begin{aligned}
  u(x) &= C_0 \int_{\mathbb{R}^n} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}} \, dy, \\
v(x) &= C_1 \int_{\mathbb{R}^n} \frac{|y|^b u^q(y)}{|x-y|^{n-\alpha}} \, dy,
\end{aligned}
\]
and vice versa.

When \(a = b = 0\) in (3), the integral system is frequently studied for the Hardy-Littlewood-Sobolev (HLS) inequality and best constants. We refer the readers to [9, 10, 18, 22] and the references therein.

Remark 1. Re-writing system (2) as (3) has been a long-standing technique in dealing with PDEs (see [7]). Quite recently, it caught our attention that in [38], Zhuo et al. considered the following fractional equation
\[
\begin{aligned}
  (-\Delta)^{\alpha/2} u(x) &= 0, & x \in \mathbb{R}^n, \\
u(x) &\geq 0, & x \in \mathbb{R}^n.
\end{aligned}
\]
They proved that the only solution for (4) is constant. This extends the classical Liouville theorem from the Laplacian to the fractional Laplacian. As an immediate application, we use it to establish the equivalence between the differential equations (2) and the integral system (3). This proof is totally different from that in [7].

Through a combined use of the above theorems, we arrive at the following result.

Theorem 1.3. Let \(u, v \in L_\alpha \cap C^{1,1}_loc\), assume that \((u, v)\) is a pair of nonnegative solutions to (2) or (3), \(1 < p < \frac{n+a+\alpha}{n-\alpha}\) and \(1 < q < \frac{n+a+b}{n-\alpha}\). Then \((u, v) = (0, 0)\).

As a byproduct of the proof for Theorem 1.3, we come to the conclusion below.

Theorem 1.4. Let \(u, v \in L_\alpha \cap C^{1,1}_loc\) be a pair of nonnegative solutions to (2) or (3). If
\[
\int_{\mathbb{R}^n} \frac{|y|^a v^p(y)}{|x-y|^{n-\alpha}} \, dy < \infty \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|y|^b u^q(y)}{|x-y|^{n-\alpha}} \, dy < \infty
\]
with \(\frac{n+a}{q+1} + \frac{n+b}{p+1} \neq n-\alpha\), then \((u, v) = (0, 0)\).

Remark 2. In [14], the authors applied the method of moving planes in integral forms to study the symmetry of positive solutions for the fractional system (2), then derived the nonexistence of positive solutions. However, due to some technical restrictions, the authors have to assume that \(u \in L^\beta_{loc}(\mathbb{R}^n)\) and \(v \in L^\gamma_{loc}(\mathbb{R}^n)\) where 
\[
\beta = \frac{n(p-1)}{(n-\alpha)q+b-n}, \quad \gamma = \frac{n(p-1)}{(n-\alpha)p+a-n}.
\]
In this paper, we manage to derive the same nonexistence result without imposing extra integrability conditions, by using a direct method of moving planes to the differential equations.
In recent years, the fractional Hénon-type problem has received a lot of attention. In [28], the authors considered the integral equation

\[ u(x) = \int_{\mathbb{R}^n} \frac{w(y)}{|x-y|^{n-\alpha}|y|^a} dy, \]  

where \( \frac{n-a}{n-\alpha} < p < \alpha^*(a) - 1 \) with \( \alpha^*(a) = \frac{2(n-a)}{n-a} \). They proved the nonexistence of positive solutions using an integral method by establishing the equivalence between the above integral equation and the following partial differential equation

\[ (-\Delta)^{\frac{\alpha}{2}} u(x) = |x|^{-a} u^p. \]  

In [27], the authors studied the following weighted system:

\[
\begin{cases}
(-\Delta)^{\alpha/2} u = |x|^{-a} v^p & \text{in } \mathbb{R}^n, \\
(-\Delta)^{\alpha/2} v = |x|^{-b} u^q & \text{in } \mathbb{R}^n,
\end{cases}
\]  

\[ u \geq 0, v \geq 0. \]

They first established the equivalence between the differential system and an integral system

\[
\begin{cases}
u(x) = \int_{\mathbb{R}^n} \frac{w(y)}{|x-y|^{n-\alpha}|y|^a} dy, \\
v(x) = \int_{\mathbb{R}^n} \frac{w(y)}{|x-y|^{n-\alpha}|y|^b} dy.
\end{cases}
\]

Then, in the critical case \( \frac{n-a}{p+1} + \frac{n-b}{q+1} = n - \alpha \), they showed that every pair of positive solutions \((u(x), v(x))\) is radially symmetric about the origin. While in the subcritical case, they proved the nonexistence of positive solutions. For more similar results, please see [7, 11, 15] and the references therein.

**Remark 3.** The Lane-Emden equations and systems involving negative exponents \(a, b\) are discussed in [27, 28] and the references therein. Here we consider the cases with \(a, b > 0\). Because the proofs in [27, 28] are no longer valid for the case of positive exponents, we turned to the direct method of moving planes. Part of the technical requirement that comes along is the strict monotonicity on the right hand of (2), \(|x|^a u^p\) and \(|x|^b v^q\). Obviously, (2) does not have it. To overcome this difficulty, we consider \(\bar{u}\) and \(\bar{v}\), the Kelvin transforms of \(u\) and \(v\) respectively. Then the right hand side of (2) becomes \(|x|^{-\gamma} u^p\) and \(|x|^{-\beta} v^q\), where \(\gamma = n + \alpha + a - p(n-\alpha)\) and \(\beta = n + \alpha + b - q(n-\alpha)\). Now we can carry out the method of moving planes to \(\bar{u}\) and \(\bar{v}\). We will be more specific in Section 2.

The paper is organized as follows. In Section 2, we use the method of moving planes to prove the symmetric of positive solutions. In Section 3, we establish the equivalence between problem (2) and integral system (3). Finally, in Section 4 we complete the proof of Theorem 1.3 and 1.4.

### 2. The symmetry of positive solutions

Without any decay conditions on \(u\) and \(v\), we are not able to carry the method of moving planes on \(u\) and \(v\) directly. To circumvent this difficulty, we employ the Kelvin transformation of \(u\) and \(v\). Let

\[
\begin{aligned}
\Pi(x) &= \frac{1}{|x|^{n-\alpha}} u\left(\frac{x}{|x|^2}\right), \\
\Pi(x) &= \frac{1}{|x|^{n-\alpha}} v\left(\frac{x}{|x|^2}\right).
\end{aligned}
\]

Without any decay conditions on \(u\) and \(v\), we are not able to carry the method of moving planes on \(u\) and \(v\) directly.
Then we have
\[
(-\triangle)^{\alpha/2}u(x) = \frac{1}{|x|^{n+a}}(-\triangle)^{\alpha/2}u\left(\frac{x}{|x|^2}\right) = \frac{1}{|x|^{n+a}}v^p\left(\frac{x}{|x|^2}\right)
\]
\[
= \frac{1}{|x|^{n+a-a+p(n-a)}}\pi^\gamma(x).
\] (11)

In a similar way, we have
\[
(-\triangle)^{\alpha/2}v(x) = \frac{1}{|x|^{n+a-a+q(n-a)}}\pi^\gamma(x).
\] (12)

Let \(\gamma = n + \alpha + a - p(n-a), \beta = n + \alpha + b - q(n-a)\). Combining (2), (10), (11) and (12), it follows that
\[
\begin{cases}
(-\triangle)^{\alpha/2}\pi = |x|^{-\gamma}v^p,
(-\triangle)^{\alpha/2}\pi = |x|^{-\beta}v^q,
\pi \geq 0, \pi \geq 0.
\end{cases}
\] (13)

In order to apply the method of moving planes, we first give some basic notations and lemmas. Let
\[
T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda, \lambda \in \mathbb{R}\}
\] (14)
be the moving plane,
\[
\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}
\]
be the region to the left of the plane, and
\[
x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)
\]
be the reflection of the point \(x = (x_1, x_2, \ldots, x_n)\) about the plane \(T_\lambda\).

Assume that \((\pi, \pi)\) is a pair of nonnegative solutions to the fractional system (13). To compare the values of \(\pi(x)\) with \(\pi(x^\lambda)\) and \(\pi(x)\) with \(\pi(x^\lambda)\), we denote
\[
\begin{cases}
U_\lambda(x) = \pi(x^\lambda) - \pi(x),
V_\lambda(x) = \pi(x^\lambda) - \pi(x).
\end{cases}
\] (15)

Then system (13) becomes
\[
\begin{cases}
(-\triangle)^{\alpha/2}U_\lambda(x) = \frac{1}{|x^\lambda|^{\gamma}}\pi^\gamma(x^\lambda) - \frac{1}{|x|^\gamma}\pi^\gamma(x),
(-\triangle)^{\alpha/2}V_\lambda(x) = \frac{1}{|x^\lambda|^\beta}\pi^\gamma(x^\lambda) - \frac{1}{|x|^\beta}\pi^\gamma(x).
\end{cases}
\] (16)

**Lemma 2.1** (Narrow region principle [5]). Let \(\Omega\) be a bounded narrow region in \(\Sigma_\lambda\), such that it is contained in \(\{x \mid \lambda - l < x_1 < \lambda\}\) with small \(l\). Suppose that \(\varphi \in L_\alpha \cap C_{loc}^{1,1}(\Omega)\) and is lower semi-continuous on \(\overline{\Omega}\). If \(C(x)\) is bounded from below in \(\Omega\), then
\[
\begin{cases}
(-\triangle)^{\alpha/2}\varphi(x) + C(x)\varphi(x) \geq 0 \quad \text{in} \quad \Omega,
\varphi(x) \geq 0 \quad \text{in} \quad \Sigma_\lambda \setminus \Omega,
\varphi(x^\lambda) = -\varphi(x) \quad \text{in} \quad \Sigma_\lambda,
\end{cases}
\] (17)

then for sufficiently small \(l\), we have
\[
\varphi(x) \geq 0 \text{ in } \Omega.
\]

Furthermore, if \(\varphi = 0\) at some point in \(\Omega\), then
\[
\varphi(x) = 0 \text{ almost everywhere in } \mathbb{R}^n.
\]
These conclusions hold for unbounded region $\Omega$ if we further assume that

$$\lim_{|x| \to \infty} \varphi(x) \geq 0.$$ 

Lemma 2.2 (Decay at infinity [5]). Let $\Omega$ be an unbounded region in $\Sigma_\lambda$. Assume that $\varphi \in L^\alpha \cap C^{1,1}_{\text{loc}}(\Omega)$ is a solution of

$$\begin{cases}
( -\triangle)^{\alpha/2} \varphi(x) + C(x)\varphi(x) \geq 0 \quad \text{in} \quad \Omega,
\varphi(x) \geq 0 \quad \text{in} \quad \Sigma_\lambda \setminus \Omega,
\varphi(x^\lambda) = -\varphi(x) \quad \text{in} \quad \Sigma_\lambda,
\end{cases}$$

with

$$\lim_{|x| \to \infty} |x|^\alpha C(x) \geq 0,$$

then there exists a constant $R_0 > 0$ such that if

$$\varphi(x_0) = \min_{\Omega} \varphi(x) < 0,$$

then

$$|x_0| \leq R_0.$$ 

2.1. The method of moving planes. Now, we show the symmetry of $\bar{u}$ and $\bar{v}$ for system (13) by using the method of moving planes and we provide the proof in a few steps.

**Step 1:** We show that when $\lambda$ sufficiently negative,

$$U_\lambda(x), V_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}. \quad (19)$$

Let $W_\lambda(x) = U_\lambda(x) + V_\lambda(x)$, it follows from (16) that

$$(-\triangle)^{\alpha/2} W_\lambda(x) = (-\triangle)^{\alpha/2} (U_\lambda(x) + V_\lambda(x)),$$

$$= \frac{1}{|x|^\gamma} \varphi(x^\lambda) - \frac{1}{|x|^\gamma} \varphi(x) + \frac{1}{|x|^\beta} \pi_\delta(x^\lambda) - \frac{1}{|x|^\beta} \pi_\delta(x).$$

We claim that

$$W_\lambda(x) \geq 0. \quad (20)$$

If (20) doesn’t hold, then there exists $x_0 \in \Sigma_\lambda \setminus \{0^\lambda\}$, such that

$$W_\lambda(x_0) = \min_{\Sigma_\lambda \setminus \{0^\lambda\}} W_\lambda(x) < 0.$$

This is guaranteed by the fact that, for $\epsilon$ sufficiently small and $\lambda$ sufficiently negative,

$$U_\lambda(x), V_\lambda(x) \geq C > 0, \quad x \in B_\epsilon(0^\lambda) \setminus \{0^\lambda\},$$

we will prove it in the Appendices. Without loss of generality, we suppose that $U_\lambda(x_0) < 0$, then we have $V_\lambda(x_0) < 0$. To see this, we suppose that $V_\lambda(x_0) \geq 0$. It follows from $U_\lambda(x_0) < 0$ that there exists $x_1$, such that $U_\lambda(x_1) = \min_{\Sigma_\lambda \setminus \{0^\lambda\}} U_\lambda(x) < 0$.
0. Thus

\[-\frac{\alpha}{\lambda + 1}U(x_1) - C_{\alpha}PV \int_{\mathbb{R}^n} \frac{U(x_1) - U(y)}{|x_1 - y|^{n+\alpha}} dy\]

\[= C_{\alpha}PV \int_{\mathbb{R}^n} \frac{U(x_1) - U(y)}{|x_1 - y|^{n+\alpha}} dy + C_{\alpha}PV \int_{\mathbb{R}^n} \frac{U(x_1) - U(y)}{|x_1 - y|^{n+\alpha}} dy\]

\[= C_{\alpha}PV \int_{\mathbb{R}^n} \frac{U(x_1) - U(y)}{|x_1 - y|^{n+\alpha}} dy + C_{\alpha}PV \int_{\mathbb{R}^n} \frac{U(x_1) - U(y)}{|x_1 - y|^{n+\alpha}} dy\]

\[\leq C_{\alpha} \int_{\mathbb{R}^n} \frac{U(x_1) - U(y)}{|x_1 - y|^{n+\alpha}} dy + C_{\alpha}PV \int_{\mathbb{R}^n} \frac{U(x_1) - U(y)}{|x_1 - y|^{n+\alpha}} dy\]

\[\leq C_{\alpha} \int_{\mathbb{R}^n} \frac{2U(x_1)}{|x_1 - y|^{n+\alpha}} dy < 0.\] (21)

On the other hand, we have

\[-\frac{\alpha}{\lambda + 1}U(x_1) = \left(\frac{1}{|x_1|^\gamma} - \frac{1}{|x_1|^\gamma}\right)\tilde{p}^p(x_1)\]

\[\leq \frac{1}{|x_1|^\gamma}\tilde{p}^p(x_1) - \frac{1}{|x_1|^\gamma}\tilde{p}^p(x_1) + \frac{1}{|x_1|^\gamma}\tilde{p}^p(x_1) - \frac{1}{|x_1|^\gamma}\tilde{p}^p(x_1)\]

\[= \frac{1}{|x_1|^\gamma}(\tilde{p}^p(x_1) - \tilde{p}^p(x_1)) + \tilde{p}^p(x_1)(\frac{1}{|x_1|^\gamma} - \frac{1}{|x_1|^\gamma})\]

\[= \frac{1}{|x_1|^\gamma}(\tilde{p}^p(x_1) - \tilde{p}^p(x_1)).\] (22)

By (21) and (22), we deduce that

\[V_\lambda(x_1) < 0.\]

Then,

\[W_\lambda(x_1) < W_\lambda(x_0),\]

which is a contradiction. This proves that \(V_\lambda(x_0) < 0\). Notice that

\[-\frac{\alpha}{\lambda + 1}W_\lambda(x_0) \geq \frac{1}{|x_0|^\gamma}p\tilde{p}^p(x_0) - \tilde{p}^p(x_0) + \frac{1}{|x_0|^\gamma}q\tilde{p}^p(x_0) - \tilde{p}^p(x_0)\]

\[= \frac{1}{|x_0|^\gamma}p\tilde{p}^p(x_0) - \tilde{p}^p(x_0) + \frac{1}{|x_0|^\gamma}q\tilde{p}^p(x_0) - \tilde{p}^p(x_0)\]

\[\geq \frac{1}{|x_0|^\gamma}p\tilde{p}^p(x_0) - \tilde{p}^p(x_0) + \frac{1}{|x_0|^\gamma}q\tilde{p}^p(x_0) - \tilde{p}^p(x_0)\]

\[\geq \frac{1}{|x_0|^\gamma}p\tilde{p}^p(x_0) + \frac{1}{|x_0|^\gamma}q\tilde{p}^p(x_0) - \tilde{p}^p(x_0),\]

where \(\xi \in (\tilde{p}^p(x_0), \tilde{p}(x_0)), \eta \in (\tilde{p}(x_0), \tilde{p}(x_0))\). Let \(C_1(x_0) = \frac{1}{|x_0|^\gamma}p\tilde{p}^p(x_0), C_2(x_0) = \frac{1}{|x_0|^\gamma}q\tilde{p}^p(x_0), C(x_0) = \max\{C_1(x_0), C_2(x_0)\}\), then

\[-\frac{\alpha}{\lambda + 1}W_\lambda(x_0) - C(x_0)W_\lambda(x_0) \geq 0.\] (23)
On the other hand, similar to (21) we have
\[ (-\Delta)^{n/2}W_\lambda(x_0) - C(x_0)W_\lambda(x_0) \leq W_\lambda(x_0)\left(\int_{\Sigma_x} \frac{1}{|x_0 - y^\lambda|^{n+\alpha}} dy - C(x_0)\right), \quad (24) \]
and
\[ \int_{\Sigma_x} \frac{1}{|x_0 - y^\lambda|^{n+\alpha}} dy \geq \int_{\{x_1 \geq 0\}} \frac{1}{|x_0 - y^\lambda|^{n+\alpha}} dy = \frac{1}{2} \int_{\mathbb{R}^n} \frac{1}{(|x_0| + |y^\lambda|)^{n+\alpha}} dy \]
\[ = \frac{1}{2} \int_0^\infty \int_{B^*_{x_0}} \frac{1}{(|x_0| + |r|)^{n+\alpha}} d\sigma dr = \frac{1}{2} \int_0^\infty \frac{w_{n-1} r^{n-1}}{(|x_0| + |r|)^{n+\alpha}} dr, \quad r = t|x_2| \]
\[ \sim \frac{C_3}{|x_0|^\alpha}, \quad (25) \]
where
\[ C_1 = \frac{1}{|x_0|^\gamma} p^p (x_0) \sim \frac{1}{|x_0|^\gamma} \left(\frac{1}{|x_0|^{n-\alpha}}\right)^{p-1} \sim \frac{1}{|x_0|^{\gamma+(p-1)(n-\alpha)}}, \]
\[ C_2 = \frac{1}{|x_0|^\beta} q^q (x_0) \sim \frac{1}{|x_0|^\beta} \left(\frac{1}{|x_0|^{n-\alpha}}\right)^{q-1} \sim \frac{1}{|x_0|^{\beta+(q-1)(n-\alpha)}}. \]
Note that \( \gamma + (p-1)(n-\alpha) = 2a + a > \alpha \) and \( \beta + (q-1)(n-\alpha) = 2a + b > \alpha \), it implies that
\[ \int_{\Sigma_x} \frac{1}{|x_0 - y^\lambda|^{n+\alpha}} dy - C(x_0) > 0. \quad (26) \]
From (26) and (24), we have
\[ (-\Delta)^{n/2}W_\lambda(x_0) - C(x_0)W_\lambda(x_0) < 0, \]
this is a contradiction with (23). This shows that
\[ W_\lambda(x) \geq 0, \]
and
\[ U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0. \]

**Step 2.** Step 1 provides a starting point, from which we can now move the plane \( T_\lambda \) to the right as long as (19) holds to its limiting position. Let
\[ \lambda_0 = \sup \{ \lambda \leq 0 | U_\mu \geq 0, V_\mu \geq 0, \forall x \in \Sigma_\mu \setminus \{0^\lambda\}, \mu \leq \lambda \}. \]
In this part, we show that
\[ \lambda_0 = 0. \]
Suppose that
\[ \lambda_0 < 0, \]
we show that the plane \( T_\lambda \) can be moved further right. To be more rigorous, there exists some \( \delta > 0 \), such that for any \( \lambda \in (\lambda_0, \lambda_0 + \delta) \), we have
\[ W_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \setminus \{0^\lambda\}. \quad (27) \]
This is a contradiction with the definition of \( \lambda_0 \). In fact, when \( \lambda_0 < 0 \), we have
\[ W_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0} \setminus \{0^\lambda\}. \quad (28) \]
If not, there exists some $\hat{x}$ such that
\[
W_{\lambda_0}(\hat{x}) = \min_{x \in \Sigma_{\lambda_0} \setminus \{0^\lambda\}} W_{\lambda_0}(x) = 0,
\]
i.e.,
\[
U_{\lambda_0}(\hat{x}) = 0, \quad V_{\lambda_0}(\hat{x}) = 0. \tag{29}
\]
It follows from (29) that
\[
(-\Delta)^{\alpha/2}U_{\lambda_0}(\hat{x}) = C_{n,\alpha} \text{PV} \int_{\mathbb{R}^n} \frac{-U_{\lambda_0}(y)}{|\hat{x} - y|^{n+\alpha}} dy = 0.
\]
On the other hand,
\[
(-\Delta)^{\alpha/2}U_{\lambda_0}(\hat{x}) = \frac{1}{|\hat{x}^\lambda|} \hat{p}(\hat{x}^\lambda) - \frac{1}{|\hat{x}|} \hat{p}(\hat{x}) = \frac{1}{|\hat{x}^\lambda|} \hat{p}(\hat{x}) - \frac{1}{|\hat{x}|} \hat{p}(\hat{x}) > 0, \tag{30}
\]
a contradiction with (30), so does $V_{\lambda_0}$. This proves (28). We claim that for $\lambda_0 < 0$ and $\varepsilon > 0$ sufficiently small,
\[
U_{\lambda_0}(x), V_{\lambda_0}(x) \geq C > 0, \quad x \in B_{\varepsilon}(0^{\lambda_0}) \setminus \{0^{\lambda_0}\},
\]
the proof will be given in the Appendices. It follows from (28) that there exists a constant $C_0 > 0$, such that
\[
W_{\lambda_0}(x) \geq C_0 > 0, \quad x \in \Sigma_{\lambda_0 - \delta} \cap B_R(0) \setminus \{0^{\lambda_0}\}.
\]
Since $W_{\lambda}$ depends on $\lambda$ continuously, there exists $\varepsilon > 0$ and $\varepsilon < \delta$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, we have
\[
W_{\lambda}(x) \geq 0, \quad \forall x \in \Sigma_{\lambda_0 - \delta} \cap B_R(0) \setminus \{0^{\lambda_0}\}. \tag{32}
\]
From Decay at infinity, we have
\[
W_{\lambda}(x) \geq 0, \quad \forall x \in B_R^c. \tag{33}
\]
From Narrow region principle, we have
\[
W_{\lambda}(x) \geq 0, \quad \forall x \in (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}) \setminus \{0^\lambda\}. \tag{34}
\]
To see this, in Lemma 2.1, we let, for any sufficiently small $\eta > 0$, $H = \Sigma_\lambda \setminus B_\eta(0^\lambda)$ and the narrow region $\Omega = (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}) \setminus B_\eta(0^\lambda)$, while the lower bound of $C(x)$ can be seen from (25). Combining (32) (33) and (34), we conclude that
\[
W_{\lambda}(x) \geq 0, \quad x \in \Sigma_\lambda \setminus \{0^\lambda\}.
\]
This contradicts the definition of $\lambda_0$. Therefore, we must have $\lambda_0 = 0$ and $W_{\lambda_0} \geq 0, \forall x \in \Sigma_\lambda$. Similarly, one can move the plane $T_\lambda$ from the $+\infty$ to the left and show that $W_{\lambda_0} \leq 0, \forall x \in \Sigma_\lambda$. Now we have shown that

$$\lambda_0 = 0, \ W_{\lambda_0} \equiv 0, \ \forall \ x \in \Sigma_\lambda.$$ 

2.2. Proof of Theorem 1.1.

Proof. So far, we have proved that $u, v$ is symmetric about the plane $T_0$. Since the $x_1$ direction can be chosen arbitrarily, we have actually shown that $u, v$ are radially symmetric about 0. Let $x_1, x_2$ be any points centered at 0, i.e.,

$$0 = \frac{x_1 + x_2}{2}, \ |x_1| = |x_2|.$$ \hspace{1cm} (35)

Then, it implies that

$$\overline{u}(x_1) = \overline{u}(x_2), \ \overline{v}(x_1) = \overline{v}(x_2).$$

Let

$$y_1 = \frac{x_1}{|x_1|^2}, \ y_2 = \frac{x_2}{|x_2|^2},$$

by (35), we have

$$\frac{y_1 + y_2}{2} = 0.$$ 

Hence, it follows that

$$u(y_1) = u\left(\frac{x_1}{|x_1|^2}\right) = |x_1|^{n-\alpha}u(x_1) = |x_2|^{n-\alpha}u(x_2) = u(y_2),$$

$$v(y_1) = v\left(\frac{x_1}{|x_1|^2}\right) = |x_2|^{n-\alpha}v(x_1) = |x_2|^{n-\alpha}v(x_2) = v(y_2).$$

This completes the proof. \hfill \square

3. The equivalence between problem (2) and the integral form (3). In this section, we prove the equivalence between problem (2) and (3).

Proof of Theorem 1.2 Let $(u, v)$ be a pair of positive solutions to (2), we first show that

$$\begin{cases}
  u(x) = c_1 + \int_{\mathbb{R}^n} \frac{|y|^a v^p(y)}{|x - y|^{n-\alpha}} dy, \\
  v(x) = c_2 + \int_{\mathbb{R}^n} \frac{|y|^b u^q(y)}{|x - y|^{n-\alpha}} dy.
\end{cases}$$ \hspace{1cm} (36)

Let

$$\begin{cases}
  u_R(x) = \int_{B_R(0)} G_R(x, y)|y|^a v^p(y) dy, \\
  v_R(x) = \int_{B_R(0)} G_R(x, y)|y|^b u^q(y) dy,
\end{cases}$$ \hspace{1cm} (37)

where $G_R(x, y)$ is the Green’s function of fractional Laplacian on $B_R(0)$. It is easy to see that

$$\begin{cases}
  (-\Delta)^{\alpha/2} u_R(x) = |x|^a v^p(x) \quad \text{in} \ B_R(0), \\
  (-\Delta)^{\alpha/2} v_R(x) = |x|^b u^q(x) \quad \text{in} \ B_R(0), \\
  u_R(x) = v_R(x) = 0 \quad \text{on} \ B_R^c(0).
\end{cases}$$ \hspace{1cm} (38)
Let $\varphi_R(x) = u(x) - u_R(x)$ and $\psi_R(x) = v(x) - v_R(x)$. From (2) and (38), we have
\[
\begin{cases}
(-\Delta)^{\alpha/2}\varphi_R(x) = 0 & \text{in } B_R(0), \\
(-\Delta)^{\alpha/2}\phi_R(x) = 0 & \text{in } B_R(0), \\
\varphi_R(x), \phi_R(x) \geq 0 & \text{on } B^c_R(0).
\end{cases}
\]
By the Maximum Principle, we derive
\[
\varphi_R(x), \phi_R(x) \geq 0, \quad x \in R^n.
\]
Therefore, when $R \to \infty$, it follows that
\[
\begin{cases}
u_R(x) \to \tilde{u}(x) = \int_{R^n} \frac{|y|^a v^p(y)}{|x - y|^{n-\alpha}} dy, \\
v_R(x) \to \tilde{v}(x) = \int_{R^n} \frac{|y|^b u^q(y)}{|x - y|^{n-\alpha}} dy.
\end{cases}
\]
Moreover,
\[
\begin{cases}
(-\Delta)^{\alpha/2}\tilde{u}(x) = |x|^a v^p(x), \quad x \in R^n, \\
(-\Delta)^{\alpha/2}\tilde{v}(x) = |x|^b u^q(x), \quad x \in R^n.
\end{cases}
\]
Now, let $\Phi(x) = u(x) - \tilde{u}(x)$ and $\Psi(x) = v(x) - \tilde{v}(x)$. From (2) and (41), we have
\[
\begin{cases}
(-\Delta)^{\alpha/2}\Phi(x) = 0, \quad x \in R^n, \\
(-\Delta)^{\alpha/2}\Psi(x) = 0, \quad x \in R^n, \\
\Phi(x), \Psi(x) \geq 0, \quad x \in R^n.
\end{cases}
\]
From Proposition 2 in [38], we have
\[
\Phi(x) = c_1, \quad \Psi(x) = c_2.
\]
Thus we proved (36).

Next, we will show that $c_1 = c_2 = 0$. Without loss of generality, we may assume that $c_2 > 0$, then from (36), we have
\[
u(x) = c_1 + \int_{R^n} \frac{|y|^a v^p(y)}{|x - y|^{n-\alpha}} dy \geq c_1 + \int_{R^n} \frac{c_2 |y|^a}{|x - y|^{n-\alpha}} dy = \infty.
\]
But it is impossible, hence $c_1 = c_2 = 0$. Therefore, we obtain
\[
\begin{cases}
u(x) = \int_{R^n} \frac{|y|^a v^p(y)}{|x - y|^{n-\alpha}} dy, \\
v(x) = \int_{R^n} \frac{|y|^b u^q(y)}{|x - y|^{n-\alpha}} dy.
\end{cases}
\]
This completes the proof.

4. The nonexistence of positive solutions. In this section, we prove Theorem 1.3 and Theorem 1.4. First we need some Lemmas.

**Lemma 4.1.** Assume that $(u, v)$ is a pair of positive radial solutions for system (2). Then for $r = |x| > 0$, it holds that
\[
u(r) \leq Cr^{-\frac{(b+\alpha)p+n}{pq-1}}, \quad (42)
\]
\[
u(r) \leq Cr^{-\frac{(a+\alpha)q+n}{pq-1}}. \quad (43)
\]
Proof. From (3) and the decreasing property of radial solutions, we have
\[
u(r) = \int_{R^n} \frac{|y|^a u^p(y)}{|x-y|^{n-\alpha}} dy \geq \int_{B_r(0)} \frac{|y|^a u^p(y)}{|x-y|^{n-\alpha}} dy \geq \int_{B_r(0)} \frac{|y|^a u^p(y)}{|x-y|^{n-\alpha}} dy \geq |x|^a u^p(r) \int_{B_r(0)} \frac{|y|^a}{|x-y|^{n-\alpha}} dy = C u^p(r)^{a+\alpha}.
\]
Similarly, we have
\[
v(r) = C u^q(r)^{b+\alpha}.
\]
Combining (44) and (45), it gives
\[
u(r) \geq C u^q(r)^{b+\alpha}.
\]
and
\[
v(r) \geq C u^p(r)^{a+\alpha}.
\]
(42) and (43) follow immediately from the above inequalities.

Lemma 4.2. Assume that \((u, v)\) is a pair of positive radial solutions for system (2), then it holds that
\[
\int_{R^n} |x|^b u^{q+1} dx < \infty \quad \text{and} \quad \int_{R^n} |x|^b v^{q+1} dx < \infty, \quad \text{(46)}
\]
\[
\int_{R^n} |x|^a u^{p+1} dx < \infty \quad \text{and} \quad \int_{R^n} |x|^a v^{p+1} dx < \infty. \quad \text{(47)}
\]
Proof. For any \(R > 0\), we need to show that
\[
\int_{B_R(0)} |x|^b u^{q+1} dx < \infty \quad \text{as} \quad R \to \infty,
\]
\[
\int_{B_R(0)} |x|^a u^{p+1} dx < \infty \quad \text{as} \quad R \to \infty.
\]
Here we only show \(\int_{B_R(0)} |x|^b u^{q+1} dx < \infty, R \to \infty\), the other can be proved in a similar way. The integral will converge only when
\[
|x|^b u^{q+1} \sim o\left(\frac{1}{|x|^n}\right) \quad \text{for} \quad |x| \text{ large}.
\]
Since
\[
|x|^b u^{q+1} \leq |x|^b \left(|x|^{-\frac{(b+\alpha)}{n+\alpha}}\right)^{q+1},
\]
it is sufficient to show that
\[
\frac{(b+\alpha)p + a + \alpha}{pq - 1} (q + 1) - b > n, \quad \text{(48)}
\]
or
\[
p(n - \alpha)(q - \frac{b + \alpha}{n - \alpha}) \leq (n + b) + (a + \alpha)(q + 1). \quad \text{(49)}
\]
Case i: if \(q \leq \frac{b+\alpha}{n-\alpha}\), (49) is automatically true.
Case ii: if \(q \geq \frac{b+\alpha}{n-\alpha}\), then (49) becomes
\[
\frac{p(n - \alpha)}{a + \alpha} < \frac{q + \frac{n+b}{a+\alpha} + 1}{q - \frac{b+\alpha}{n-\alpha}} = 1 + \frac{\frac{n+b}{a+\alpha} + \frac{b+\alpha}{n-\alpha} + 1}{q - \frac{b+\alpha}{n-\alpha}}. \quad \text{(50)}
\]
Notice that \(1 < p < \frac{n + a + a}{n - \alpha}\) and \(1 < q < \frac{n + a + b}{n - \alpha}\), it implies that
\[
\text{LHS of (50)} \leq \frac{n + \alpha + a}{n - \alpha} \cdot \frac{n - \alpha}{a + \alpha} = 1 + \frac{n}{a + \alpha},
\]
and
\[
\text{RHS of (50)} \geq 1 + \frac{n + b}{a + \alpha} + \frac{b + \alpha}{n} = 1 + \frac{n + b}{a + \alpha} + \frac{b + \alpha}{n}
\]

\[
= 1 + \frac{n + b}{a + \alpha} + \frac{n + b}{n} + \frac{b + \alpha}{n}
\]
\[
= 1 + \frac{n + b}{n} \cdot \frac{n + a}{a + \alpha}.
\]
(50) is true as long as
\[
\frac{n}{a + \alpha} < \frac{n + b}{n} \cdot \frac{n + a}{a + \alpha}, \quad (51)
\]
since \(a, b > 0\) and \(a + \alpha > 0\). This completes the proof. \(\Box\)

**Proof of Theorem 1.3** Let \(u, v \in L_\alpha \cap C^{1,1}_{loc}\), assume that \((u, v)\) is a pair of positive solutions to (2) or (3). By (3), we have
\[
\begin{aligned}
    u(x) &= \frac{1}{|x|^a v^p(x)} \int_{\partial B^0_R} |x|^a v^p(y) \frac{\partial}{\partial n} u(y) \, d\sigma(y), \\
    v(x) &= \frac{1}{|x|^b u^q(x)} \int_{\partial B^0_R} |x|^b u^q(y) \frac{\partial}{\partial n} v(y) \, d\sigma(y).
\end{aligned}
\]
(52)
We differentiate the first equation of (52) with respect to \(k\),
\[
x \cdot \nabla u(x) = (\alpha - n) \int_{\mathbb{R}^n} \frac{x \cdot (x - y) |y|^a v^p(y)}{|x|^n - |y|^{n-\alpha+2}} \, dy, \quad x \neq 0.
\]
Let \(k = 1\), then
\[
x \cdot \nabla u(x) = (\alpha - n) \int_{\mathbb{R}^n} \frac{x \cdot (x - y) |y|^a v^p(y)}{|x|^n - |y|^{n-\alpha+2}} \, dy, \quad x \neq 0.
\]
(53)
Multiply both sides of (53) by \(|x|^b u^q(x)\) and integrate on \(\mathbb{R}^n\), we have
\[
\int_{\mathbb{R}^n} |x|^b u^q(x) (x \cdot \nabla u(x)) \, dx = (\alpha - n) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{x \cdot (x - y) |x|^b u^q(x) |y|^a v^p(y)}{|x|^n - |y|^{n-\alpha+2}} \, dy \, dx.
\]
On the other hand, from the integration by parts formula, it follows that
\[
\int_{B_R(0)} |x|^b u^q(x) (x \cdot \nabla u(x)) \, dx = \frac{1}{q + 1} \int_{B_R(0)} |x|^b (x \cdot \nabla u^{q+1}(x)) \, dx.
\]
\[
= -\frac{n + b}{q + 1} \int_{B_R(0)} |x|^b u^{q+1}(x) \, dx + \frac{1}{p + 1} \int_{\partial B_R(0)} R^{b+1} u^{q+1} \, d\sigma.
\]
From Lemma 4.1, we have
\[
\frac{1}{p + 1} \int_{\partial B_R(0)} R^{b+1} u^{q+1} \, d\sigma \to 0, \quad R \to \infty.
\]
Hence when \(R \to \infty\),
\[
\int_{\mathbb{R}^n} |x|^b u^q(x) (x \cdot \nabla u(x)) \, dx = -\frac{n + b}{q + 1} \int_{\mathbb{R}^n} |x|^b u^{q+1}(x) \, dx.
\]
Therefore,
\[-\frac{n + b}{q + 1} \int_{R^d} |x|^b u^{q+1}(x) \, dx = (\alpha - n) \int_{R^d} \int_{R^d} \frac{x \cdot (x - y)}{|x - y|^{n-\alpha+2}} \, dy \, dx.\]  
(54)

Using the similar argument on the second equation of (52), we also have
\[\int_{R^d} |x|^a v^p(x) (x \cdot \nabla v(x)) \, dx = (\alpha - n) \int_{R^d} \int_{R^d} \frac{x \cdot (x - y)}{|x - y|^{n-\alpha+2}} \, dy \, dx,\]
and
\[\int_{R^d} |x|^a v^p(x) (x \cdot \nabla v(x)) \, dx = -\frac{n + a}{p + 1} \int_{R^d} |x|^a u^{p+1}(x) \, dx.\]
Consequently, it implies that
\[-\frac{n + a}{p + 1} \int_{R^d} |x|^a u^{p+1}(x) \, dx = (\alpha - n) \int_{R^d} \int_{R^d} \frac{x \cdot (x - y)}{|x - y|^{n-\alpha+2}} \, dy \, dx.\]  
(55)

Adding (54) and (55) together, and using the fact $|x - y|^2 = x \cdot (x - y) + y \cdot (y - x)$, we have
\[-\frac{n + a}{p + 1} \int_{R^d} |x|^a u^{p+1}(x) \, dx - \frac{n + b}{q + 1} \int_{R^d} |x|^b u^{q+1}(x) \, dx = \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{x \cdot (x - y)}{|x - y|^{n-\alpha+2}} \, dy \, dx \]
\[+ \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{y \cdot (y - x)}{|x - y|^{n-\alpha+2}} |x|^a v^p(x) \, dx \, dy\]
\[+ \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{x \cdot (x - y)}{|x - y|^{n-\alpha+2}} |y|^b u^q(y) \, dy \, dx\]
\[+ \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{y \cdot (y - x)}{|x - y|^{n-\alpha+2}} |y|^a v^p(y) \, dx \, dy\]
\[= \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{x \cdot (x - y)}{|x - y|^{n-\alpha+2}} \, dy \, dx \]
\[+ \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{y \cdot (y - x)}{|x - y|^{n-\alpha+2}} |x|^a v^p(x) \, dx \, dy\]
\[+ \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{x \cdot (x - y)}{|x - y|^{n-\alpha+2}} |y|^b u^q(y) \, dy \, dx\]
\[+ \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{y \cdot (y - x)}{|x - y|^{n-\alpha+2}} |y|^a v^p(y) \, dx \, dy\]
\[= \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{|x|^a v^p(x) |y|^b u^q(y)}{|x - y|^{n-\alpha}} \, dy \, dx + \frac{\alpha - n}{2} \int_{R^d} \int_{R^d} \frac{|y|^a v^p(y) |x|^b u^q(x)}{|x - y|^{n-\alpha}} \, dy \, dx\]
\[= \frac{\alpha - n}{2} \int_{R^d} |x|^a v^{p+1}(x) \, dx + \frac{\alpha - n}{2} \int_{R^d} |x|^b u^{q+1}(x) \, dx, \]  
(56)
It is trivial for

since

\[
\int_{\mathbb{R}^n} |x|^a v^{p+1}(x)dx = \int_{\mathbb{R}^n} |x|^a |v(x)|^p |y|^b u^q(y) dy dx = \int_{\mathbb{R}^n} \frac{|y|^a v^p(y) |x|^b u^q(x)}{|x-y|^{n-\alpha}} dxdy
\]

Combining (56) and (57), we have

\[
\int_{\mathbb{R}^n} |y|^a v^p(y) |x|^b u^q(x) dy dx = \int_{\mathbb{R}^n} |x|^b u^{q+1}(x)dx. \tag{57}
\]

Combining (56) and (57), we have

\[
\left(\frac{\alpha - n}{2} + \frac{n + b}{q + 1}\right) \int_{\mathbb{R}^n} |x|^b u^{q+1}(x)dx + \left(\frac{\alpha - n}{2} + \frac{n + a}{p + 1}\right) \int_{\mathbb{R}^n} |x|^a v^{p+1}(x)dx = 0.
\]

That is,

\[
(\alpha - n + \frac{n + b}{q + 1} + \frac{n + a}{p + 1}) \int_{\mathbb{R}^n} |x|^b u^{q+1}(x)dx = 0.
\]

Because 1 < p < \frac{n+a+\alpha}{\alpha+n} and 1 < q < \frac{n+a+b}{\alpha+n}, thus \frac{n+b}{q+1} + \frac{n+a}{p+1} \neq n - \alpha and problem (2) admits no positive solutions. This completes the proof.

**Proof of Theorem 1.4** Theorem 1.4 is a direct consequence of Theorem 1.3.

5. Appendices.

**Lemma 5.1.** Let \( U_\lambda(x), V_\lambda(x) \) be given in (15). Then for \( \lambda \) negative large, there exists a constant \( C > 0 \), such that

\[
U_\lambda(x), V_\lambda(x) \geq C > 0, \ x \in B_\varepsilon(0^\lambda) \setminus \{0^\lambda\}. \tag{58}
\]

**Proof.** For \( x \in \Sigma_\lambda \), as \( \lambda \to -\infty \), it is easy to see that

\[
\pi(x) \to 0. \tag{59}
\]

To prove (58), it is sufficient to show that

\[
\pi_\lambda(x) \geq C > 0, \ x \in B_\varepsilon(0^\lambda) \setminus \{0^\lambda\}. \]

Or equivalently,

\[
\pi(x) \geq C > 0, \ x \in B_\varepsilon(0) \setminus \{0\}. \]

Let \( \eta \) be a smooth cut-off function such that \( \eta \in [0,1] \) in \( \mathbb{R}^n \), supp \( \eta \subset B_2 \) and \( \eta \equiv 1 \) in \( B_1 \). Let

\[
(-\Delta)^{\alpha/2} \phi(x) = \eta(x)|x|^a v^p(x).
\]

Then,

\[
\phi(x) = C_{n,-\alpha} \int_{\mathbb{R}^n} \frac{\eta(y)|y|^a v^p(y)}{|x-y|^{n-\alpha}} dy = C_{n,-\alpha} \int_{B_2(0)} \frac{\eta(y)|y|^a v^p(y)}{|x-y|^{n-\alpha}} dy.
\]

It is trivial for \( |x| \) sufficiently large,

\[
\phi(x) \sim \frac{1}{|x|^{n-\alpha}}. \tag{60}
\]

Since

\[
\left\{
\begin{array}{ll}
(-\Delta)^{\alpha/2}(u - \phi) \geq 0, & x \in B_R, \\
(u - \phi)(x) \geq 0, & x \in B^R_R,
\end{array}
\right.
\]

by the Maximum Principle, we have

\[
(u - \phi)(x) \geq 0, \ x \in B_R,
\]
thus, it implies that

$$(u - \phi)(x) \geq 0, \ x \in R^n.$$ 

For $|x|$ sufficiently large, from (60), one can see that for some constant $C > 0$,

$$u(x) \geq \frac{C}{|x|^{n-\alpha}}. \quad (62)$$

Hence for $|x|$ small,

$$u(\frac{x}{|x|^2}) \geq C|x|^{n-\alpha},$$

and

$$\overline{u}(x) = \frac{1}{|x|^{n-\alpha}}u(\frac{x}{|x|^2}) \geq C.$$ 

Together with (59), it yields that

$$U_{\lambda}(x) \geq \frac{C}{2} > 0, \ x \in B_{\varepsilon}(0^{\lambda}) \setminus \{0^{\lambda}\}. \quad (63)$$

Through an identical argument, one can show that (63) holds for $V_{\lambda}(x)$ as well. \hfill \Box

**Lemma 5.2.** For $\lambda_0 < 0$, let $U_{\lambda_0}(x), V_{\lambda_0}(x)$ be given in (15). If either of $U_{\lambda_0}, V_{\lambda_0}$ is not identically 0, then there exist some constant $C$ and $\varepsilon > 0$ small such that

$$U_{\lambda_0}(x), V_{\lambda_0}(x) \geq C > 0, \ x \in B_{\varepsilon}(0^{\lambda_0}) \setminus \{0^{\lambda_0}\}.$$ 

**Proof.** From Lemma 2.2 in [12], we have the integral equation

$$U_{\lambda_0}(x) = \overline{u}_{\lambda_0}(x) - \overline{u}(x)$$

$$= C_{n,\alpha} \int_{\Sigma_{\lambda_0}} \left( |y|^{-\gamma} \overline{p}_{\lambda_0}(y) - |y|^{-\gamma} \overline{p}(y) \right) \left( \frac{1}{|x - y|^{n+\alpha}} - \frac{1}{|x - y^{\lambda_0}|^{n+\alpha}} \right) dy$$

$$\geq C_{n,\alpha} \int_{\Sigma_{\lambda_0}} \left( |y|^{-\gamma} \overline{p}_{\lambda_0}(y) - |y|^{-\gamma} \overline{p}(y) \right) \left( \frac{1}{|x - y|^{n+\alpha}} - \frac{1}{|x - y^{\lambda_0}|^{n+\alpha}} \right) dy.$$

Since

$$V_{\lambda_0}(x) \neq 0, \ x \in \Sigma_{\lambda_0},$$

there exists some $x_0$ such that $V_{\lambda_0}(x_0) > 0$. Thus, for some $\delta > 0$ small, it holds that

$$\overline{u}_{\lambda_0}(y) - \overline{p}(y) \geq C > 0, \ y \in B_{\delta}(x_0).$$

Therefore, we have

$$U_{\lambda_0}(x) \geq \int_{B_{\delta}(x_0)} C \ dy \geq C > 0. \quad (64)$$

In a same way, one can show that $V_{\lambda_0}(x)$ also satisfies (64). \hfill \Box

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