Sums of Polynomials and Clique Roots

Hossein Teimoori Faal
Department of Mathematics and Computer Science,
Allameh Tabatabai University, Tehran, Iran
hossein.teimoori@atu.ac.ir

Abstract
In this paper, pursuing the same line of ideas in the proof of an old longstanding open conjecture of Kadison-Singer, we introduce a key lemma which we call it the interlacing lemma which indicates a necessary condition for having a real root for sums of polynomials with (at least) one real root. Then, as an immediate application of this simple but potentially useful lemma we characterize several class of graphs which have only clique roots. Finally, we conclude our paper with several interesting open problems and conjectures for interested readers.

1 Introduction and Motivation
The property of having only real roots for graphs polynomials of some special classes of graphs is of special interest for algebraic graph theorists in recent years. The reason is that having only real roots for a graph polynomial results in detecting many combinatorial features for the given graph.
One of the most interesting breaking through in mathematical sciences in recent years was the solution of the well-known Kadison-Singer conjecture which was open for more than 50 years. The main idea behind the proof was introducing an interlacing family of polynomials (see [1] and reference therein for more details).
We are not really aware of the origin of interlacing idea, but it seems that one of the early motivations comes from the famous Rolle’s theorem in the calculus of one-variable functions. An immediate corollary of Rolle’s theorem simply states that for a real polynomial $f(x)$, roots of $f'(x)$ interlace those of $f(x)$.
It seems that by using similar line of proofs for the Rolle’s theorem, one can get the following simple but interesting lemma.
Lemma 1.1. Let \( \{f_i(x)\}_i \) be a finite collection of polynomials with all real coefficients, a positive leading coefficient and at least one real root. We also let \( R_i \) and \( r_i \) be the largest and second largest real roots (respectively) of \( \{f_i(x)\}_i \). In the case \( f_i(x) \) has only one real root, by convention we put \( r_i = -\infty \). If \( \max_j r_j \leq \min_j R_j \), then \( f(x) = \sum_j f_j(x) \) has a real root \( R \) which satisfies \( \min_j R_j \leq R \).

The importance of the above key lemma is that it simply gives interesting necessary conditions for having a real root for sums of polynomials. Needless to say, finding necessary conditions for sums of polynomials to having a real roots can be interesting in several occasions where a new polynomial graph invariant can be expressed as a sum of polynomial invariant of its particular subgraphs. For instance, the first derivatives of several graph polynomials like characteristic polynomial, matching polynomial, independence polynomial and the clique polynomials of a given graph can be obtained as a sum of its vertex-deleted subgraphs [3].

In this paper, we will concentrate on the clique polynomial of a graph. By applying the key lemma to the combinatorial interpretations of the first and second derivatives of clique polynomials we obtain several classes of graphs which have only clique roots. We finally conclude the paper with the couple of open questions and conjectures.

2 Basic Definitions and Notations

Throughout this paper, we will assume that our graphs are all finite, simple and undirected. For the definitions which are not appear here, one may refer to [3].

A subset of vertices of a graph \( G \) that are pairwise adjacent is called a complete subgraph or a clique of \( G \). A clique with \( k \) vertices is called a \( k \)-clique. The number of \( k \)-cliques will be denoted by \( c_k(G) \). For a subset \( S \) of vertices, the graph with vertex set \( S \) and edges with end-vertices only on \( S \) is called an induced subgraph of \( G \) and denoted by \( G[S] \). The set vertices adjacent to a vertex \( v \) is called the (open) neighborhood of \( v \) and is denoted by \( N(v) \). The subgraph obtained by deleting the vertex \( v \) from \( G \) will be denoted by \( G - v \). In a similar way, a subgraph obtained by only removing an edge \( e \) from \( G \) is denoted by \( G - e \).

A chord of a cycle is an edge connecting two non-adjacent vertices in the cycle. A chordal graph is a graph that any cycle of length greater than three has a chord.

For a given graph \( G = (V, E) \), the ordinary generating function of the num-
3 Main Results

ber of cliques of $G$ is called the \textit{clique polynomial} of $G$ and is denoted by $C(G, x)$. More precisely, we have

$$C(G, x) = \sum_{k=0}^{\omega(G)} c_k(G)x^k, \quad (2.1)$$

where $\omega(G)$ is the size of the largest clique of $G$. By convention, we may assume $c_0(G) = 1$ for any graph $G$. The real root of the clique polynomial of $G$ is called the \textit{clique root} of $G$.

The clique polynomial of a graph satisfies the following vertex-recurrence relation:

$$C(G, x) = C(G - v, x) + C(G[N(v)], x). \quad (2.2)$$

Similarly, we have the following edge-recurrence relation:

$$C(G, x) = C(G - e, x) + C(G[N(e)], x), \quad (2.3)$$

in which $N(e) = N(u) \cap N(v)$, for $e = \{u, v\} \in E(G)$.

The following combinatorial interpretation of the first derivative of clique polynomial is given in \cite{2}.

$$\frac{d}{dx}C(G, x) = \sum_{v \in V(G)} C(G[N(v)], x). \quad (2.4)$$

In a similar way, one can obtain the following edge-version of the graph-theoretical interpretation of the second derivative of a clique polynomial.

$$\frac{1}{2!} \frac{d^2}{dx^2}C(G, x) = \sum_{e \in V(G)} C(G[N(e)], x). \quad (2.5)$$

3 Main Results

We first note that based on the recursive definition of chordal graphs using the idea of pasting two complete graphs along a clique, one can show that any $k$-connected chordal graphs has a clique root $-1$ of multiplicity $k$ \cite{2}.

Next, we need another key result.

**Proposition 3.1.** Let $T$ be a tree on $n$ vertices. Then, the graph $T$ has only clique roots. Moreover, the largest and the second largest clique roots are $R = -\frac{1}{n-1}$ and $r = -1$, respectively.
An immediate corollary of Lemma 1.1 and Proposition 3.1 is the following.

**Corollary 3.2.** Let $F$ be a forest with non-trivial components. Then the greatest clique root $R_F$ of $F$ satisfies the inequality $R_F \geq -\frac{1}{n-1}$, where $n$ is the size the smallest component of $F$.

From now on, for simplicity of arguments, we will assume that our graphs are connected.

**Proposition 3.3.** Let $G$ be a triangle-free graph. Then $G$ has only clique roots.

*Proof.* Since $G$ is triangle-free, the neighborhood of each vertex is an independent set. That is $C(G[N(v)], x) = 1 + r_v x$ where $r_v$ is the size of corresponding independent set of $v$. If $r$ denotes the smallest size of all those independent sets, then we clearly have

$$-\infty = \max_j r_j < -1 \leq -\frac{1}{r} = \min_j R_j. \quad (3.1)$$

Hence the conditions of Lemma 1.1 are true and the polynomial

$$\sum_{v \in V(G)} C(G[N(v)], x)$$

has a real root $-1 \leq R < 0$. Now considering formula (2.4), we immediately conclude that $dC(G, x)$ has a real root. Thus the graphs $G$ has only clique roots. \hfill \square

**Proposition 3.4.** Let $G$ be a $K_4$-free connected chordal graph. Then $G$ has only clique roots.

*Sketch of Proof.* Considering the fact that $C(G, x)$ is a quartic polynomial with at least two clique roots $r = -1$, we just need to prove $\frac{d^2}{dx^2} C(G, x)$ has a real root. Now, since for each $e \in E(G)$ the graph $G[N(e)]$ is triangle-free, based on combinatorial formula (2.5) and Lemma 1.1 the proof is complete. \hfill \square

**Proposition 3.5.** Let $G$ be a bi-connected $K_5$-free chordal graph. Then $G$ has only clique roots.

*Sketch of Proof.* considering the fact that $C(G, x)$ is a quartic polynomial with at least two clique roots $r = -1$, we just need to prove $\frac{d^2}{dx^2} C(G, x)$ has a real root. Now, since for each $e \in E(G)$ the graph $G[N(e)]$ is triangle-free, based on combinatorial formula (2.5) and Lemma 1.1 the proof is complete. \hfill \square
4 Open Questions and Conjectures

Considering Lemma 1.1, we come up with the following conjecture.

**Conjecture 4.1.** Let $G$ be a connected $K_4$-free graphs, then $G$ has only clique roots.

Based on Proposition 3.5, the following open question naturally arises.

**Question 4.1.** Is there any 2-connected non-chordal $K_5$-free graph which has only clique roots?

Consider the wheel graph $W_5$ on five vertices with one extra edge connecting two non-adjacent vertices on the outer ring. Now it is clear that $C(G, x) = (1 + x)(1 + 5x + 6x^2 + x^3)$ has only real roots.

**Conjecture 4.2.** The class of all 2-connected $K_5$-free graphs in which each edge belongs to at most two triangles has only clique roots.

**Question 4.2.** Which classes of $K_r$-free chordal graphs has only real roots.

We finally came up with following stronger conjecture.

**Conjecture 4.3.** The class of $l$-connected chordal graphs which are $K_{l+3}$-free has only clique roots.

We believe that the last conjecture is a starting point to find another algebraic graph-theoretic proof of the well-known Turán-type graph theorems.

References

[1] H. Hajiabolhassan and M. L. Mehrabadi, *On clique polynomials*, Australasian Journal of Combinatorics., 18 (1998), 313-316.

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