Evolutionarily Stable Strategies in Quantum Hawk-Dove Game

Ahmad Nawaz* and A.H. Toor†
National Centre for Physics, Quaid-i-Azam University Campus, Islamabad, Pakistan
† Department of Physics, Quaid-i-Azam University, Islamabad 45320, Pakistan

March 31, 2022

Abstract

We quantized the Hawk-Dove game by using the most general form of a pure initial state to investigate the existence of pure and mixed Evolutionarily Stable Strategies (ESS). An example is considered to draw a comparison between classical and quantum version of the game. Our choice of most general initial quantum state enables us to make the game symmetric or asymmetric. We show that for a particular set of game parameters where there exist only mixed ESS in the classical version of the game, however, quantization allows even a pure strategy to be an ESS for symmetric game in addition to fixed ESS. On the other hand only pure strategy ESS can exist for asymmetric quantum version of the Hawk-Dove game.

1 Introduction

The concept of evolutionarily stable strategies (ESS) was originally introduced in evolutionary biology [1]. Later it has been incorporated as a central concept of stable equilibrium in evolutionarily game theory. Consider a population in which majority is playing a particular strategy and a very small fraction of this population, called mutants, start playing a different strategy. If the mutant strategy remains at disadvantage against the majority strategy then as a consequence mutants strategy gradually disappears. In such a situation the majority strategy is called an ESS. The mathematical theory of ESS was developed for game theoretical analysis of animal conflicts where they are undergoing natural selection. Interestingly the dynamics of evolution resulting from Darwinian idea

*email: ahmad@ele.qau.edu.pk
† ahtoor@qau.edu.pk
of survival of fittest implies that ESS must remain stable against small perturbations caused by mutants. Game theory itself derived much from such analysis of evolutionary mechanism. Our motivation in present paper is to give a thought to evolution, as concretely described in notion of ESS, in circumstances when the game played in population becomes quantum in its setting.

In an interesting development Meyer [2] examined the game theory from quantum mechanical perspective showing that a player having an access to quantum strategies can enhance his/her payoff with respect to an opponent who have access to classical strategies only. Later, Eisert et. al. [3] analyzed a famous game of Prisoner Dilemma in its quantum mechanical version and showed that the dilemma no more exists in quantum version of this game. They also successfully constructed a quantum strategy which always wins over any classical strategy. Marinatto and Weber [8] proposed another interesting quantization scheme for games where players can implement their ‘tactics’ on an initial strategy by probabilistic choice of applying the identity operator $I$ and the flip operator $C$. They applied their scheme to a famous game of Battle of Sexes and showed that both the players can get better payoff in a quantum version of this game.

A consideration of ESS in quantum games is presented by Iqbal and Toor [10, 11]. They analyzed the quantum games of Prisoner’s Dilemma and Battle of Sexes from the point of view of evolutionary stability. Underlying assumption in their approach is that a population is playing a quantum game in two player conflict scenario. They showed that evolutionary stability of Nash equilibria in symmetric as well as asymmetric games can be controlled by changing initial quantum state. Recently Prisoner Dilemma has also been investigated for ESS that is played by using Einstein-Podolsky-Rosen type of setting [12].

In this paper we study an interesting game from evolutionary biology called Hawk-Dove game. In this game Hawk and Dove are the strategies available to the players to get some valuable resource. The most interesting feature of this game in its classical form, which is symmetric, is that neither Hawk nor Dove strategy can be a pure ESS, however, there can exist a mixed ESS. We have used a most general initial quantum state in quantization that enables us to make the game symmetric or asymmetric. Our main result in this paper is that quantization of the game allows even a pure strategy to be an ESS for symmetric game in addition to mixed ESS. On the other hand only pure strategy ESS can exist for asymmetric quantum version of the Hawk-Dove game.

The rest of the paper is organized as follows: in Sect. 2 ESS are discussed from mathematical point of view. Section 3 deals with classical Hawk-Dove game while in Sect. 4 we present our main result.

### 2 Evolutionarily Stable Strategies

Consider a large population in which members are randomly paired to contest against each other in a game. In this pair wise contest the average payoff for a group of members playing strategy $A$ against a small fraction, $\epsilon$, of the total
population playing strategy $B$ is $[A, (1 - \epsilon)A + \epsilon B]$. Mathematically a strategy $A$ will be an ESS against a strategy $B$, if

$$[A, (1 - \epsilon)A + \epsilon B] > [B, (1 - \epsilon)A + \epsilon B],$$

there exists a sufficiently small but positive $\epsilon \in [0, \epsilon_0]$ \[4\]. Here $\epsilon_0$ is called the invasion barrier. The strategy played by the smaller group is generally called mutant strategy and in our case if the fraction of members playing strategy $B$ becomes larger than $\epsilon_0$, then the corresponding mutant strategy $B$ would be able to invade and strategy $A$ would no longer be an ESS. In the case of symmetric bi-matrix game between the pairs of members, the above condition for ESS becomes \[4, 5\]:

$$(A, A) > (B, A)$$

and if $(A, A) = (B, A)$ then $(A, B) > (B, B)$. For asymmetric case Nash equilibrium with strict inequality must hold to ensure that strategy $A$ is an ESS \[6\]. For example, a strategy pair $(A^*, B^*)$ is an ESS if $(A^*, B^*) > (A, B)$ for all $A^* \neq A$ and $B(A^*, B^*) > B(A^*, B)$ for all $B \neq B^*$.

## Classical Hawk-Dove Game

Hawk-Dove is a simple two player game where two different behavioral strategies are available to the players to obtain some resources \[7\]. Hawks are very aggressive and always fight to take possession of resource. These fights are very brutal and loser is the one who first sustains the injury. The winner takes the sole possession of the resource. Mathematical description of the game requires that the Hawks fully recover before the next contest. In case both the players opt for Hawk, then the winning probability for both is equal.

On the other hand Doves never fights for a resource. It displays and if attacked immediately withdraws to avoid injury. Thus it will always lose a conflict against Hawk without sustaining any injury. In other words Doves fitness remains unaffected. In case two Doves face each other, there will be period of displaying with some cost (time, energy for display) to both but without any injury. It is assumed that both the Doves are equally good in displaying and waiting for random time. In the Dove-Dove contest, both have equal chance of winning. The winner would be the one with more patience \[7\].

Let $v$ and $i$ are the value of resource and cost of injury, respectively. Cost of losing a resource is 0, while the cost of displaying is $d$. For mathematical description of the game we assume that $v$ is a positive number and both $i$ and $d$ are negative numbers. The payoff matrix for the game takes the form:

$$
\begin{array}{ccc}
\text{Hawk (H)} & \text{Dove (D)} \\
\text{Hawk (H)} & \left(\frac{v}{2} + \frac{i}{2}, \frac{v}{2} + \frac{i}{2}\right) & (v, 0) \\
\text{Dove (D)} & (0, v) & (\frac{v}{2} + d, \frac{v}{2} + d) \\
\end{array}
$$

(2)
It is straightforward to conclude from the above payoff matrix that strategy Hawk is an ESS if either

\[ $(H, H) > (D, H) \] \quad (3) \]

or

\[ $(H, H) = (D, H) \text{ and } (H, D) > (D, D) \] \quad (4) \]

For the above payoff matrix (2), the first condition translates to \((v + i) > 0\) and the second to \(v + i = 0\) and \(\frac{v}{i} > d\). Since \(\frac{v}{i} > d\) always holds, therefore, Hawk is an ESS whenever \(v + i \geq 0\). Moreover, it is important to note that \((H, D) > (D, D)\), hence strategy Dove can never be an ESS [7]. To illustrate our point, let's consider the following example where \((v + i) < 0\) [7];

Value of resource \(v = 50\)
Injury to self \(i = -100\)
Cost of display \(d = -10\)
Resource cost \(c = 0\) \quad (5)

and the payoff matrix (2) takes the form

\[
\begin{array}{c|cc}
 & H & D \\
\hline
H & (−25, −25) & (50, 0) \\
D & (0, 50) & (15, 15) \\
\end{array}
\] \quad (6)

It is clear that there is no ESS for any pure strategy in this game [7]. Next we look at the possibility of ESS in mixed strategies.

Again consider a large population where strategy \(H\) is being played by a fraction \(h\) of the total population and remaining population is playing strategy \(D\). In a pairwise contest the corresponding fitness functions \(W(H), W(D)\) are defined as [7]

\[
W(H) = $(H, H)h + $(H, D)(1 − h), \\
W(D) = $(D, H)h + $(D, D)(1 − h). \quad (7a)
\]

For a mixed strategy to be an ESS, for both the strategies we must have equal fitness functions, which implies

\[
h = \frac{2d - v}{2d + i}. \quad (8)
\]

Let's explore the possibility of mixed strategy ESS for the set of values considered in the above example [see Eq. (5)]. Putting these values in the Eq. (8), we see that there exist an ESS for \(h = .583\).
4 Quantum Hawk-Dove Game

We follow the Marinatto and Weber’s scheme to quantize the strategy space for Hawk and Dove game [8]. Assuming that two players, Alice and Bob, share the following entangled state:

$$|\psi_{in}\rangle = a|HH\rangle + b|DD\rangle + c|HD\rangle + d|DH\rangle \quad (9)$$

where $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$ and the first slot is reserved for Alice’s strategy and the second for Bob’s. Let $C$ be a unitary and Hermitian operator (i.e., $C = C^\dagger = C^{-1}$), such that

$$C|H\rangle = |D\rangle, \quad C|D\rangle = |H\rangle. \quad (10)$$

If Alice uses $I$, the identity operator, with probability $p$ and $C$ with probability $(1-p)$ and Bob uses these operators with probability $q$ and $(1-q)$, respectively. Then the final density matrix of the bipartite system takes the form [7]:

$$\rho_f = pq \left[ (I_A \otimes I_B) \rho_{in} \left( I_A^\dagger \otimes I_B^\dagger \right) \right] + p(1-q) \left[ (I_A \otimes C_B) \rho_{in} \left( I_A^\dagger \otimes C_B^\dagger \right) \right] + q(1-p) \left[ (C_A \otimes I_B) \rho_{in} \left( C_A^\dagger \otimes I_B^\dagger \right) \right] + (1-p)(1-q) \left[ (C_A \otimes C_B) \rho_{in} \left( C_A^\dagger \otimes C_B^\dagger \right) \right]. \quad (11)$$

Here $\rho_{in} = |\psi_{in}\rangle \langle \psi_{in}|$. The payoff operators for Alice and Bob are defined as [7]

$$P_A = \left( \frac{\nu}{2} + \frac{i}{2} \right) |HH\rangle \langle HH| + \nu |HD\rangle \langle HD| + \left( \frac{\nu}{2} + d \right) |DD\rangle \langle DD|, \quad (12)$$

$$P_B = \left( \frac{\nu}{2} + \frac{i}{2} \right) |HH\rangle \langle HH| + \nu |DH\rangle \langle DH| + \left( \frac{\nu}{2} + d \right) |DD\rangle \langle DD|. \quad (13)$$

The payoff functions for Alice and Bob are the mean values of the above operators, i.e.,

$$S_A(p, q) = \text{Tr}(P_A \rho_f),$$

$$S_B(p, q) = \text{Tr}(P_B \rho_f). \quad (14)$$
The expected payoff functions for both the players are obtained using Eqs. \ref{eq:payoff}. \hspace{1cm} (13)

$$
\begin{align*}
\mathbb{S}_A(p, q) &= \left(\frac{\nu}{2} + \frac{i}{2}\right)[pq |a|^2 + p(1 - q) |c|^2 + q(1 - p) |d|^2 + (1 - p)(1 - q) |b|^2] \\
&+ v[pq |c|^2 + p(1 - q) |a|^2 + q(1 - p) |b|^2 + (1 - p)(1 - q) |d|^2] \\
&+ (\frac{\nu}{2} + d)[pq |b|^2 + p(1 - q) |a|^2 + q(1 - p) |c|^2 + (1 - p)(1 - q) |a|^2], \\
\mathbb{S}_B(p, q) &= \left(\frac{\nu}{2} + \frac{i}{2}\right)[pq |a|^2 + p(1 - q) |c|^2 + q(1 - p) |d|^2 + (1 - p)(1 - q) |b|^2] \\
&+ v[pq |d|^2 + p(1 - q) |b|^2 + q(1 - p) |a|^2 + (1 - p)(1 - q) |c|^2] \\
&+ (\frac{\nu}{2} + d)[pq |b|^2 + p(1 - q) |d|^2 + q(1 - p) |c|^2 + (1 - p)(1 - q) |a|^2].
\end{align*}
$$

(15)

Corresponding to the set of values we considered earlier, i.e., Eq. \ref{eq:psi}, the above set of payoff functions becomes:

$$
\begin{align*}
\mathbb{S}_A(p, q) &= p[q(-60 |a|^2 - 60 |b|^2 + 60 |c|^2 + 60 |d|^2) \\
&- 25 |c|^2 + 25 |b|^2 + 35 |a|^2 - 35 |d|^2] \\
&+ q[75 |b|^2 - 75 |d|^2 - 15 |a|^2 + 15 |c|^2] \\
&- 25 |b|^2 + 50 |d|^2 + 15 |a|^2, \\
\mathbb{S}_B(p, q) &= q[p(-60 |a|^2 - 60 |b|^2 + 60 |c|^2 + 60 |d|^2) \\
&- 25 |d|^2 + 25 |b|^2 + 35 |a|^2 - 35 |c|^2] \\
&+ p[75 |b|^2 - 75 |c|^2 + 15 |d|^2 - 15 |a|^2] \\
&- 25 |b|^2 + 50 |c|^2 + 15 |a|^2.
\end{align*}
$$

(16)

As the classical version of the Hawk and Dove game is symmetric, one would expect in the quantum version of the game interchanging $p$ and $q$ would change $\mathbb{S}_A(p, q)$ into $\mathbb{S}_B(p, q)$. However, it is interesting to note that in quantum version of the game would be symmetric, if $c = d$ in initial quantum state $\ket{\psi_{in}}$ \ref{eq:psi}. This observation is consistent with earlier work on ESS where quantum version of the game is shown to be symmetric \cite{9}. \cite{10} \cite{11}. In our case, the possibility of asymmetric game is due to the choice of general initial quantum state instead of one of the Bell states. Next we discuss both symmetric and asymmetric case in our game separately.

### 4.1 Symmetric case

Our generalized treatment of Hawk and Dove game become symmetric for game for $c = d$ in initial state. The corresponding payoff functions, i.e., Eq. \ref{eq:payoff}, become:
\[ A(p, q) = pq \left\{ -60|a|^2 - 60|b|^2 + 120|c|^2 \right\} + 35|a|^2 + 25|b|^2 - 60|c|^2 + q(-15|a|^2 + 75|b|^2 - 60|c|^2) + 15|a|^2 - 25|b|^2 + 50|c|^2, \]
\[ B(p, q) = qp \left\{ -60|a|^2 - 60|b|^2 + 120|c|^2 \right\} + 35|a|^2 + 25|b|^2 - 60|c|^2 + p(-15|a|^2 + 75|b|^2 - 60|c|^2) + 15|a|^2 - 25|b|^2 + 50|c|^2. \] (17)

Being a symmetric game the players are anonymous, therefore the subscripts \( A \) and \( B \) are not necessary, i.e., \( A(p, q) = B(p, q) = (p, q) \). Corresponding NE inequality becomes
\[ (p^*, q^*) - (p, q^*) \geq 0 \] (18)

This condition in our example translates to:
\[ (p^* - p)q \left\{ -60|a|^2 - 60|b|^2 + 120|c|^2 \right\} + 35|a|^2 + 25|b|^2 - 60|c|^2 \geq 0 \] (18)

Upon inspection it can be seen that the above inequality holds if both the factors have same sign. Let's consider following three cases:

4.1.1 Case 1:
Let's consider a case of a pure strategy \((p^* = 0, q^* = 0)\) and examine the possibility of it being a NE. The inequality \[(18)\] for this strategy requires \(35|a|^2 + 25|b|^2 - 60|c|^2 < 0\). This holds, for example, when \(|a|^2 = \frac{1}{16}, |b|^2 = \frac{1}{4}, |c|^2 = \frac{11}{32}\). The corresponding payoff functions from Eqs. \[(17)\] are
\[ $(0, 0) = \frac{95}{8}, \]
\[ $(p, 0) = \frac{95}{8} - \frac{195}{16}p. \] (19)

Which means $(0, 0) > $(p, 0) \forall 0 < p < 1$, therefore, the strategy \((p^* = 0, q^* = 0)\) is an ESS. In the context of our evolutionary game if both the players are playing strategy \( C \) no mutant strategy can invade for an initial state, \(|\psi_{in}\rangle = \frac{1}{\sqrt{2}} |HH\rangle + \frac{1}{\sqrt{2}} |DD\rangle + \sqrt{\frac{11}{32}} |HD\rangle + \sqrt{\frac{11}{32}} |DH\rangle\). Therefore, in contrast to the classical version of the game pure strategies can also be a ESS in the quantum version of the game under certain conditions.

4.1.2 Case 2:
Let's examine another case of pure strategy \((p^* = 1, q^* = 1)\) to be a NE. For this strategy the inequality given by Eq. \[(18)\] demands \(-25|a|^2 - 35|b|^2 + 60|c|^2 > 0\). This holds, for example, for \(|a|^2 = \frac{1}{16}, |b|^2 = \frac{1}{4}, |c|^2 = \frac{11}{32}\). The corresponding
payoff functions from Eqs. (17) are

$$\$(1, 1) = \frac{165}{8}$$

$$\$(p, 1) = \frac{35}{16} + \frac{295}{16} p. \quad (20)$$

Since $$\$(1, 1) - \$(p, 1) = \frac{295}{16} (1 - p) > 0 \ \forall \ 0 < p < 1$$, therefore, $$(p^* = 1, q^* = 1)$$ is an ESS. Thus a population engaged in the pure strategy $$(p^* = 1, q^* = 1)$$ cannot be invaded by any mutant strategy if for the initial quantum state the inequality $$-25 |a|^2 - 35 |b|^2 + 60 |c|^2 > 0$$ holds, which correspond to the initial state $$|\psi_{in}\rangle = \frac{1}{4} |HH\rangle + \sqrt{\frac{1}{8}} |DD\rangle + \sqrt{\frac{13}{32}} |HD\rangle + \sqrt{\frac{13}{32}} |DH\rangle$$. 

4.1.3 Case 3:

Let's explore the possibility of mixed NE in the quantum version of the game when the players apply their operators with probability $$0 < p < 1$$. From the inequality given by Eq. (18), the mixed NE is

$$p^* = q^* = \frac{-7 |a|^2 - 5 |b|^2 + 12 |c|^2}{12(-|a|^2 - |b|^2 + 2 |c|^2)}.$$ \quad (21)

Corresponding to the classical version of the game, we get $$(p^* = \frac{7}{12}, q^* = \frac{7}{12})$$. In quantum version of the game we can obtain this value, for example, for $$|a|^2 = \frac{1}{2}, |b|^2 = |c|^2 = \frac{1}{4}$$. The initial state, then, takes the form

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}} |HH\rangle + \frac{1}{\sqrt{6}} |DD\rangle + \frac{1}{\sqrt{6}} |HD\rangle + \frac{1}{\sqrt{6}} |DH\rangle \quad (22)$$

Now from Eq. (17)

$$\$(p^*, q^*) = \$(p, q^*) = 8.75,$$

$$\$(q, q) = \frac{-60 q^2 + 20 q + 35}{3},$$

$$\$(p^*, q) = \frac{-600 q + 665}{36}.$$ 

It can be seen that $$\$(p^*, q) - \$(q, q) > 0, \ \forall \ 0 < q < 1$$. This implies that $$\$(p^*, q) > \$(q, q)$$. Therefore $$(p^*, q^*)$$ given by Eq. (21) is a mixed ESS for the above initial quantum state.

4.2 Asymmetric case

Our initial quantum state correspond to asymmetric game for $$c \neq d$$ for which an ESS is defined with strict Nash inequality [6]. In this case a strategy pair $$(A^*, B^*)$$ is an ESS if NE conditions with strict inequalities hold, i.e., $$\$(A^*, B^*) > \$(A, B^*)$$ for all $$A \neq A^*$$ and $$\$(A^*, B) > \$(A^*, B)$$ for all $$B \neq B^*$$. Nash inequalities [10] then yield
\$A(p^*, q^*) - \$A(p, q^*) \geq 0 \\
\Rightarrow (p^* - p)[60q^*(-|a|^2 - |b|^2 + |c|^2 + |d|^2) + 35|a|^2 + 25|b|^2 - 25|c|^2 - 35|d|^2] \geq 0 \tag{23}
\]

and

\$B(p^*, q^*) - \$A(p^*, q) \geq 0 \\
\Rightarrow (q^* - q)[60p^*(-|a|^2 - |b|^2 + |c|^2 + |d|^2) + 35|a|^2 + 25|b|^2 - 35|c|^2 - 25|d|^2] \geq 0 \tag{24}
\]

From these inequalities three Nash equilibria arise

4.2.1 Case 1:

From inequalities (23), (24) with \((p^* = 0, q^* = 0)\), we get

\[35|a|^2 + 25|b|^2 - 25|c|^2 - 35|d|^2 < 0 \tag{25}\]
\[35|a|^2 + 25|b|^2 - 35|c|^2 - 25|d|^2 < 0 \tag{26}\]

respectively. Both these inequalities (25) and (26) are satisfied, for example, for \(|a|^2 = \frac{1}{16}, |b|^2 = \frac{1}{4}, |c|^2 = \frac{9}{16}, |d|^2 = \frac{1}{8}\). Therefore, from Eq. (16)

\[\$A(0,0) = \frac{15}{16} \]
\[\$A(p,0) = \frac{15}{16} - 10p \tag{27}\]
\[\$B(0,0) = \frac{365}{16} \]
\[\$B(0,q) = \frac{365}{16} - \frac{230}{16}q \tag{28}\]

Since \$A(0,0) > \$A(p,0) \forall 0 < p < 1 and \$B(0,0) > \$B(0,q) \forall 0 < q < 1. Therefore, strict inequality holds and strategy \((p^* = 0, q^* = 0)\) is an ESS.

4.2.2 Case 2:

Similarly from inequalities (23), (24) with \((p^* = 1, q^* = 1)\), we get

\[-25|a|^2 - 35|b|^2 + 35|c|^2 + 25|d|^2 \quad > \quad 0 \tag{29}\]
\[-25|a|^2 - 35|b|^2 + 25|c|^2 + 35|d|^2 \quad > \quad 0 \tag{30}\]
respectively. These inequalities are satisfied, for example, for $|a|^2 = \frac{1}{16}$, $|b|^2 = \frac{1}{8}$, $|c|^2 = \frac{9}{16}$, $|d|^2 = \frac{1}{4}$. Hence from eq. (16)

\[
A(1, 1) = \frac{455}{16}, \quad A(p, 1) = \frac{135}{16} + 20p
\] (31)

\[
B(1, 1) = \frac{205}{16}, \quad B(1, q) = \frac{270}{16}q - \frac{65}{16}
\] (32)

Again it shows that $A(1, 1) > A(p, 1)$ ∀ $0 < p < 1$ and $B(1, 1) > B(1, q)$ ∀ $0 < q < 1$. As strict inequality holds in this case, therefore, $(p^* = q^* = 1)$ is an ESS.

4.2.3 Case 3:  

For asymmetric case from inequalities (23), (24), we get

\[
p^* = \frac{-7|a|^2 - 5|b|^2 + 7|c|^2 + 5|d|^2}{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}, \tag{33}
\]

\[
q^* = \frac{-7|a|^2 - 5|b|^2 + 5|c|^2 + 7|d|^2}{12(-|a|^2 - |b|^2 + |c|^2 + |d|^2)}. \tag{34}
\]

Strict inequality does not hold for these values, therefore, it is not an ESS.

5 Summary

Evolutionary game theory with ESS as central idea is an interesting branch of game theory. It was developed by mathematical biologists to model evolutionary dynamics. Introduction of quantum mechanics in evolutionary game theory transpired very interesting situations, e.g., in a quantum version of Rock Scissors Paper (RSP) mixed NE becomes stable contrary to classical version of the game where no stable mixed NE exist [9]. Similarly quantization of the Prisoner Dilemma and the Battle of Sexes showed that evolutionary stability of NE in symmetric as well in asymmetric games can be changed by maneuvering initial quantum state [10, 11].

We quantized the Hawk-Dove game using a pure initial quantum state of two-qubit system in its most general form. We showed that for quantization of this symmetric classical game, initial quantum state plays a crucial role in keeping it symmetric or asymmetric. In other words there is a restriction on initial quantum state for which a classical game remains symmetric in its quantum form. To elaborate our point we considered an example with set of parameters
for which there is no pure ESS for the classical Hawk-Dove game though there exits mixed ESS. However in quantum version of the game, even a pure strategy can be an ESS for certain initial quantum state. We analyzed both symmetric and asymmetric situations and showed that pure ESS can exist in both symmetric and asymmetric whereas mixed ESS exists only in the symmetric form of the quantum version of the game.

6 Acknowledgment

One of us (A. N) is grateful to A. Iqbal and Khalid Loan for their useful help.

References

[1] J. Maynard Smith, G.R. Price, Nature 15, 246 (1973).
[2] D. Meyer, Phy. Rev. Lett. 82, 1052 (1999).
[3] J. Eisert, M. Wilkens, M. Lewenstein, Phys. Rev. Lett. 83, 3077 (1999).
[4] M. Broom, Evolution in knockout conflicts, Center for Statistics and Stochastic Modeling, School of Mathematical Sciences, University of Sussex, UK, 1997.
[5] Game theory Department of Biology, College of Holly cross, Worcester, MA, 1999.
[6] G. Van der Laan, X. Tieman, Evolutionary game theory and the modeling of economic behavior, Research program”Competition and cooperation”of Faculty of Economics and Econometrics, Free University, Amsterdam, 1996.
[7] Game theory, Kenneth Prestwich, Department of Biology college of Holly Cross, [http://science.holycross.edu/departments/biology/kprestwi/behavior/ESS](http://science.holycross.edu/departments/biology/kprestwi/behavior/ESS)
[8] L. Marinatto and T. Weber, Phys. Lett. A 272, 291 (2000).
[9] A. Iqbal and A. H. Toor, Phys. Rev. A 65, 022306 (2002).
[10] A. Iqbal and A.H. Toor, Physics Letters A, 280/5-6, 249 (2001).
[11] A. Iqbal and A.H. Toor, Physics Letters A, 286/4, 245 (2001).
[12] Azhar Iqbal and Derek Abbott, Physics Letters A 373, 2537–2541 (2009).