An explicit family of unitaries with exponentially minimal length Pauli geodesics

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Abstract
Recently, Nielsen et al [1, 2, 3, 4] have proposed a geometric approach to quantum computation. They've shown that the size of the minimum quantum circuits implementing a unitary $U$, up to polynomial factors, equals to the length of minimal geodesic from identity $I$ through $U$. They've investigated a large class of solutions to the geodesic equation, called Pauli geodesics. They've raised a natural question whether we can explicitly construct a family of unitaries $U$ that have exponentially long minimal length Pauli geodesics? We give a positive answer to this question.

1 Preliminary

1.1 Pauli basis and Pauli metrics
We define general Pauli matrix $\sigma$ as tensor product of identity matrix or Pauli matrices $X$, $Y$ or $Z$. We define pauli weight of a general Pauli matrix $\sigma$ as the total number of $X$, $Y$ and $Z$ in $\sigma$, noted by $\text{pw}(\sigma)$. Given a control Hamiltonian $H$, we can write $H$ in terms of the Pauli operator expansion $H = \sum_{\sigma} \lambda_{\sigma} \sigma + \sum_{\tau} \lambda_{\tau} \tau$, where in the first sum $\sigma$ ranges over all possible one and two-body interactions with $\text{pw}(\sigma) \leq 2$, while the second sum $\tau$ ranges over all general Pauli matrix $\tau$ with $\text{pw}(\tau) \geq 3$. We then define several Pauli metrics as following:

$$F_q(H) = \sqrt{\sum_{\sigma} \lambda_{\sigma}^2 + q^2 \sum_{\tau} \lambda_{\tau}^2}$$

$$F_2(H) = \sqrt{\sum_{\sigma} \lambda_{\sigma}^2 + \sum_{\tau} \lambda_{\tau}^2}$$

$$F_1(H) = \sqrt{\sum_{\sigma} |\lambda_{\sigma}| + \sum_{\tau} |\lambda_{\tau}|}$$

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1.2 Geodesic Equation

A geodesic is a locally length-minimizing curve. In the plane, the geodesics are straight lines. On the sphere, the geodesics are great circles. The geodesics in a manifold under the Finsler metric, i.e., curves in $SU(2^n)$ which are local extrema of the Finsler length are determined by geodesic equation, which is a second-order differential equation. The length of a geodesic from I to U equals to $\int_{t=0}^{t=1} F_q(H(t))dt$. Here $H(t)$ is a time-dependent Hamiltonian matrix satisfying Schrodinger’s equation $\frac{dV}{dt} = -iH(t)V; V(0) = I ; V(1) = U$.

1.3 Constant Geodesics

The shortest path is a solution of geodesic equation. In general, it is hard to figure out the global minima (geodesics with shortest length) from all possible local minima. However we can study some special geodesics corresponding to time independent Hamiltonian systems.

Definition 1.1. Constant geodesics is a geodesic of the form $V(t) = e^{-iHt}$ which satisfied Schrodinger’s equation $\frac{dV}{dt} = -iHV; V(0) = I ; V(1) = U$.

In this case, H corresponds to a time-independent Hamiltonian system. The length of the constant geodesic equals to $F_q(H)$. If we restrict the H to be a sum of commuting Pauli matrices, then it is a Pauli geodesic. In some condition, the global minimal length geodesic could be a constant geodesic.

Proposition 1.1. Let U be diagonal in the computational basis. Suppose the minimal length geodesic $s$ between I and U is unique, then $s$ must be a Pauli geodesic.

2 Result and Proofs

Theorem 2.1. There exists an explicit family of unitary matrices U which has exponentially long minimal length constant geodesics under $F_q$ metric.

Proof. Set $N = 2^n$. Let $U \in SU(N)$ be a diagonal unitary matrix in the computational basis. Suppose U has N different eigenvalues satisfying $e^{-i\theta_0}$, $e^{-i\theta_1}$, ..., $e^{-i\theta_{N-1}}$ and $0 \leq h_k < 2\pi$ for $k = 0, 1, ..., N-1$. Let $H = diag(h_0, h_1, ..., h_{N-1})$. Let $\mathcal{J}$ be the set of diagonal matrices $J = 2\pi diag(j_0, j_1, ..., j_{N-1})$ where $j_k$ is an arbitrary integer for $k = 0, 1, ..., N-1$. Then the whole set of constant geodesics is $\{e^{-i(H-J)t} | J \in \mathcal{J}\}$. The length is given by $F_q(H - J)$, therefore the length of the minimal constant geodesic from I through U is given by:

$$\min_{J \in \mathcal{J}} F_q(H - J)$$

The set $\mathcal{J}$ forms an integer lattice. The problem of finding shortest constant geodesic is equivalent to finding a closest point in the lattice under the $F_q$ metric on group $SU(N)$.

Intuitively, we have a N dimensional Euclidean space. Dimensions corresponding to some generalized Pauli matrix $\sigma$ which is tensor products of some Is and Zs. A point in this space is a diagonal Hermitian matrix. Its coordinate equals to the coefficient of Pauli operator expansion of the matrix. This
$F_q$ metric is an anisotropic generalization of normal distance on $N$ dimensional Euclidean space. Set $\mathcal{J}$ forms a lattice in this space. Our goal is to find an explicit point so that the distance from this point to the closest point in the lattice is exponential.

The lattice has some nice property which is crucial to our proofs. If we project all lattice points to some dimension $\sigma$, the set of points after projection is discrete and there are interval between two consecutive projected points. Therefore we can pick some point in the middle of this interval, which is far away from any vertex in the lattice.

Let $\text{col}(H) = (h_0, h_1, ..., h_{N-1})^T$, A diagonal matrix can be written as linear combination of general pauli matrices which are tensor products of Is and Zs. We denote $\text{col}(\lambda^H) = (\lambda_0, \lambda_1, ..., \lambda_{N-1})^T$ as $N = 2^n$ coefficients of pauli expansion. Instead of $H$, we use $M$ ($M_{ij} = (-1)^{i\cdot j}$) to denote Hadamard matrix to avoid confusion. It is not hard to prove following lemma:

**Lemma 2.1.**

$$\frac{1}{N} M^{\otimes n} \text{col}(H) = \text{col}(\lambda^H)$$

**Proof.**

$$\sum_i h_i |i><i| = \sum_i h_i \left( \frac{1}{N} \sum_j (-1)^{i\cdot j} \sigma_j \right) = \frac{1}{N} \sum_j \left( \sum_i h_i (-1)^{i\cdot j} \sigma_j \right)$$

Set $H_0 = \pi N \sigma$, $\sigma$ is tensor product of Is and Zs and $pw(\sigma) \geq 3$. Suppose $\sigma$ is the $i$th general pauli matrix. We have

**Lemma 2.2.**

$$\min_{J \in \mathcal{J}} F_q(H_0 - J) \geq \frac{\pi}{N} q$$

**Proof.**

$$\min_{J \in \mathcal{J}} F_q(H_0 - J) \geq q \min_{J \in \mathcal{J}} \lambda^{H_0 - J}$$

$$= q \min_{J \in \mathcal{J}} \left\{ \lambda^{H_0} - \lambda^{J} \right\}$$

$$= q \min_{J \in \mathcal{J}} \left\{ \frac{\pi}{N} \sigma - \left( \frac{1}{N} M^{\otimes n} \text{col}(J) \right)_i \right\}$$

$$\geq q \min_{k \in \mathbb{Z}} \left\{ \frac{\pi}{N} - \frac{2k\pi}{N} \right\}$$

$$= \frac{\pi}{N} q$$

Therefore, if we set $q$ to be exponential large and we perturb $U = e^{-iH_0}$ a little bit to make the eigenvalues differ from each other, we can get a $U'$, whose shortest length constant geodesics is exponential.
3 Discussions

In the previous section, we’ve showed some explicit unitary matrix which has exponentially long minimal constant geodesic. However in the shortest constant geodesic is not necessarily to be shortest among all geodesics. In our example, the Hamiltonian can be simulated by polynomial quantum circuit, therefore the globally minimizing geodesic is only of polynomial length. More interesting question is following:

**Problem 3.1.** Can one show an explicit unitary matrix which has exponentially long minimal geodesic?

We know that if the minimal length geodesic is unique, then it must be a Pauli geodesic. One approach to solving this problem is to find some unitaries with exponential length constant geodesics while the minimal length geodesic is unique. The other direction is to prove similar results for other Pauli metrics.

**Proposition 3.1.** The length of constant geodesics under $F_2$ metric is no more than $2\pi$ for any unitary matrix $U$.

**Proof.**

$$\min\{F_2(H) \mid e^{-iH} = U\} = \min\{\sqrt{\frac{\text{tr}(H^2)}{N}} \mid e^{-iH} = U\} \leq \sqrt{\frac{(2\pi)^2 N}{N}} = 2\pi$$

**Problem 3.2.** Is there an explicit family of unitary matrices $U$ which has exponentially long minimal length constant geodesics under $F_1$ metric?

Finally, Pauli metric seems to be related to the time complexity of simulating Hamiltonian by quantum circuit. If a family of unitaries $U$ has poly length of geodesics under $F_q$ metric where $q$ is exponential, then it can be simulated by quantum circuits in polynomial time.

**Problem 3.3.** Can one show similar relation between Pauli metrics $F_1$ and time complexity of simulating Hamiltonian system?

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References

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