Comments on $T\bar{T}$ double trace deformations and boundary conditions

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Abstract

We study the UV dynamics of $\mu T\bar{T}$ deformed conformal field theories formulated as a deformation of generating functions. We explore the issue of non-perturbative completion of the $\mu$ expansion by deriving an integral expression using the Fourier/Legendre transform technique, and show that it is more natural to impose Neumann, as opposed to the Dirichlet, boundary condition, for the metric at the cut-off surface recently proposed by McGough, Mezei, and Verlinde. We also comment on interesting connection to boundary conformal field theories.
There is a very interesting proposal [1] that a CFT in 1+1 dimensions which also admits a holographic description is deformable by an irrelevant operator of the form

$$\mu \int dz \, d\bar{z} \, T(z) \bar{T}(\bar{z})$$

and that the resulting system is 1) ultraviolet complete and dynamically well defined as a quantum system, and 2) exhibits a physical cut-off in the number of degrees of freedom in the ultraviolet. This proposal is based mostly on the work of [2], and was subjected to tests in [1] by comparing the group velocity of small fluctuations at finite temperature and the general features of the spectrum of states of the system.

Situations where an irrelevant deformation of a CFT actually makes sense do not occur generically, although they are not strictly forbidden either. It is something for which we don’t have much intuition mostly due to lack of experience. Examples that have been explored can be found in [3]. The fact that the ultraviolet is cut off implies that these are not conformal field theories in the usual sense.

The goal of this note is to explore the ultraviolet dynamics of this system. We will take a closer look at the cut-off from the both the field theory and holographic point of view in the context of AdS/CFT correspondence. Specifically, we will suggest that it is more natural to impose Neumann, rather than Dirichlet, boundary condition for the metric at the holographic boundary in order to match the prescription of [2]. We will also offer several comments which follow as a consequence of this proposal. Interesting discussions relevant to this topic can be found in [4–7]. Also, a slightly different perspective on irrelevant deformation with a different UV completion can be found in [8–10].

We will approach this problem by assuming that a solution to a CFT, in $d$ dimensions, has been provided abstractly. Solving a CFT implies providing a list of all operators and their correlation functions. Information needed to encode all of that can be presented in the form of a generating functional

$$Z_D[g_{ij}^\infty(x), A_i^\infty(x), \phi^\infty(x), \ldots]$$

where the list of fields contained in the argument of $Z_D[\ldots]$ is in one to one correspondence with the list of operators. The functional derivative with respect to these fields will then generate the correlation function. The field $g_{ij}$ is special in that it encodes the rigid geometry on which the CFT lives, and variation with respect to small perturbation of $g_{ij}(x)$ corresponds to the insertion of the stress energy tensor. $Z_D[\ldots]$ can sometimes be computed using bootstrap methods. Another approach is to invoke the AdS/CFT correspondence, in which case, the fields in the argument of $Z_D[\ldots]$ are realized as dynamical fields living in an $AdS_{d+1}$ bulk. $Z_D[\ldots]$ is then computed by computing the quantum gravity partition function.
where the bulk fields are usually subjected to Dirichlet boundary condition\(^1\) at the boundary of \(AdS_{d+1}\). Of course, at the moment, the full quantum gravity partition function is beyond the scope of what we are able to practically compute, but one can extract information about \(Z_D[...]\) reliably in the classical gravity approximation. This amounts to working to leading order in a large

\[
\mathcal{N} = \frac{L^{d-1}}{16\pi G_N}
\]

expansion, where \(L\) is the radius of \(AdS_{d+1}\) and \(G_N\) is the Newton constant in \(d + 1\) dimensions.

In this abstract formalism, it is easy to formulate what one means by (1). One simply considers the deformed generating functional\(^2\)

\[
Z_{\text{def}}[g_{ij}^\infty(x), A_i^\infty(x), \phi^\infty(x), \ldots] = e^{-\frac{\mu}{2} \int d^dx \frac{\delta}{\delta g_{ij}^\infty(x)} \frac{\delta}{\delta g_{kl}^\infty(x)} g_{kl}^\infty} Z_D[g_{ij}^\infty(x), A_i^\infty(x), \phi^\infty(x), \ldots].
\]

This expression is easy to interpret as an expansion in \(\mu\) which gives rise to an expression with natural conformal perturbation theory interpretation. Formally, this expression appears to define a generating function \(Z_{\text{def}}[...]\) for the deformed theory. The formal expression also highlights the one important subtlety, namely whether the expression is well defined non-perturbatively in \(\mu\). In other words, does the expansion in \(\mu\) admits an unambiguous resummation? If so \(Z_{\text{def}}[...]\) is completely well defined and ultraviolet complete.

In order to gain a feel for this question, it is useful to explore the analogous issue when the operators being inserted are scalars or vectors. One can, for instance, consider a deformation of the type

\[
Z_{\text{scalar}}[g_{ij}^\infty(x), A_i^\infty(x), \phi^\infty(x), \ldots] = e^{-\mu \int d^dx \frac{\delta}{\delta \phi^\infty(x)} \frac{\delta}{\delta \phi^\infty(x)} Z_D[g_{ij}^\infty(x), A_i^\infty(x), \phi^\infty(x), \ldots]}.
\]

This deformation is interpretable as an insertion of double trace operator built out of the operator sourced by \(\phi^\infty(x)\) in the standard AdS/CFT terminology.

For this case, the issue of resummation in \(\mu\) can be addressed systematically. The idea is to recognize that derivatives are like conjugate variable. One formally inserts a functional delta function

\[
\delta(\phi(x) - \varphi(x)) = \int [DJ] e^{i \int d^dx J(x)(\phi^\infty(x) - \varphi^\infty(x))}
\]

so that we can write

\[
Z_{\text{scalar}}[g_{ij}^\infty(x), A_i^\infty(x), \phi^\infty(x), \ldots]
\]

\(^1\)Which is why there is a subscript \(D\) in \(Z_D\) and a superscript \(\infty\) in the arguments of \(Z_D[...]\)

\(^2\)The sign of \(\mu\) is chosen that the positive \(\mu\) corresponds to negative \(\tilde{\mu}\) in the convention of figure 1 of [1]. We are also following the convention of [1] where \(TT = T_{\mu\nu}T^{\mu\nu}/8 - (T^\mu_{\mu})^2/16.\)
\[ Z_{\text{scalar}}[g_{ij}^{\infty}(x), A_i^{\infty}(x), \phi^{\infty}(x), \ldots] = \int [D\varphi^{\infty}] [D\mu] e^{-\int d^d x N(\alpha \varphi^{\infty}(x)^2 + \beta \varphi^{\infty}(x) + \gamma \varphi^{\infty}(x)^2)} Z_D[g_{ij}^{\infty}(x), A_i^{\infty}(x), \phi^{\infty}(x), \ldots] \quad (8) \]

This is an expression of the form \[ [11] \]

\[ Z_{\text{scalar}}[g_{ij}^{\infty}(x), A_i^{\infty}(x), \phi^{\infty}(x), \ldots] = \int [D\varphi^{\infty}] e^{-\int d^d x \frac{1}{4\pi} \varphi^{\infty}(x) (\partial \varphi^{\infty}(x))^2} Z_D[g_{ij}^{\infty}(x), A_i^{\infty}(x), \phi^{\infty}(x), \ldots] \quad (7) \]

and is interpretable as the relevant deformation of the Neumann theory by a double trace operator constructed out of the operator associated to the bulk field \( \phi(x) \) satisfying the Neumann boundary condition\[3\] \[12\] \[13\]. The expression \[8\] is simply a convolution of \( Z_D[\ldots] \) by a Gaussian, and as such appears to be a well defined expression. Typically, in the standard Dirichlet prescription, the scalar \( \phi \) is assigned a dimension \( 2\Delta_+ \) where

\[ \Delta_\pm = \frac{d \pm \sqrt{d^2 + 4m^2}}{2}, \quad (9) \]

and so the double trace operator built out of it is irrelevant. What we seem to have here is that in attempting to understand the deformation \[5\] at the non-perturbative level, we are naturally led to the full RG flow \[14\] \[15\], which consists of a Neumann theory in the UV being deformed by a relevant operator of dimension \( 2\Delta_- \) and ultimately flowing to the Dirichlet in the IR theory whose leading irrelevant deformation is the dimension the \( 2\Delta_+ \) operator. We can also relate the magnitude of the deformations from the UV and the IR perspectives of this flow as follows:

\[ \mu = -\frac{1}{4N\alpha}. \quad (10) \]

It is straight forward to generalize this analysis to the deformation by double trace operator of the Thirring type

\[ \mu \int d^d x J_\mu(x) J_\mu(x) \quad (11) \]

by considering

\[ Z_{\text{vector}}[g_{ij}^{\infty}(x), A_i^{\infty}(x), \phi^{\infty}(x), \ldots] = e^{-\mu \int d^d x g_{ij}^{\infty}(x) \frac{\delta}{\delta A_i^{\infty}(x)} \frac{\delta}{\delta A_j^{\infty}(x)}} Z_D[g_{ij}^{\infty}(x), A_i^{\infty}(x), \phi^{\infty}(x), \ldots] \quad (12) \]

Repeating the same steps which we took for the scalars, we arrive at

\[ Z_{\text{vector}}[g_{ij}^{\infty}(x), A_i^{\infty}(x), \phi^{\infty}(x), \ldots] \]

\[ ^{3}\text{So } \phi^{\infty}(x) \text{ is the coefficient of the subleading term in the expansion near the boundary.} \]
\[
\int \frac{[Dq_i^\infty][D\sigma]}{\text{Vol}(G)} e^{\int d^4x_N \mathcal{L}(\frac{1}{2}g_{ij}^\infty(x)\partial_i\sigma\partial_j\sigma - \frac{1}{2\mu}(a_i - \partial_i\sigma)A^i + \frac{1}{4\mu}A_iA^i)} \\
\times Z_D[g_{ij}^\infty(x), a_i^\infty(x), \phi^\infty(x), \ldots].
\] (13)

Precisely the same expression in slightly different notation was presented in (4.5) of [16]. The interpretation of (13) is essentially identical to that of the scalars. In particular, the Thirring deformation of the Dirichlet theory is being interpreted as the IR limit of a Neumann theory in the UV with mixed boundary conditions [17]. The UV theory is once again a Gaussian convolution of the IR theory. The relation (10) between \(\alpha\) and \(\mu\) continues to hold. The only subtlety we wish to highlight is that because the Gaussian convolution involves integrating over a spin 1 field, care is needed to ensure that the path integral can be set up in a manifestly gauge invariant form. We have therefore made the quotient by the gauge orbit and the Stueckelburg field manifest in (13).

We are now in a position to generalize these results to stress energy tensor. To the extent that we are exploring Legendre/Fourier transform to convert the functions of functional derivatives to functions of conjugate momenta, we should anticipate the analogues of (8) and (13) to involve functional integral over the metric field \(g_{ij}\). In other words, the resummation requires quantum gravity. The problem of quantum gravity of course is not solved in general. However, quantum gravity in \(d = 2\) is an exceptional case where we do know how to perform the path integral [18]. It appears that the reason the \(TT\) deformation of [2] is UV complete is closely related to the fact that quantum gravity in \(d = 2\) make sense.

Following a similar line of reasoning, we first define the Neumann theory for the tensor theory as follows

\[
Z_N = \int \frac{[Dg_{ij}^\infty]}{\text{Vol}(\text{Diff})} Z_D[g_{ij}^\infty(x), a_i^\infty(x), \phi^\infty(x)].
\] (14)

Note that we have removed all arguments in \(Z_N\). The reason is that in a theory of gravity, we do not except there to be any gauge invariant local operators with which to compute correlation functions [19]. The observables that we are allowed to consider are generally of the non-local type, and include world sheet partition functions in various topologies as well as \(S\)-matrix elements for the fluctuations on the world sheet.

In order to incorporate the relevant deformation of the UV theory to mimic the RG flow to the IR theory, we need the analogue of the \(\alpha\) term but it needs to be presented in a form which is manifestly invariant with respect to diffeomorphism. The natural candidate which behaves as a mass term for small fluctuations yet respect diffeomorphism invariance at the non-linear level is the bare cosmological constant term. We therefore have

\[
Z_{\text{tensor}}[\alpha] = \int \frac{[Dg_{ij}^\infty]}{\text{Vol}(\text{Diff})} e^{-2N\alpha \int d^4x \sqrt{g}} Z_D[g_{ij}^\infty(x), a_i^\infty(x), \phi^\infty(x)].
\] (15)
where we have made the $\alpha$ dependence of the partition function manifest. Note also that
bare cosmological constant is typically included when considering the quantization of generic
matter in $d = 2$ coupled to quantum gravity in a Liouville formalism, reviewed e.g. in [20].

Our claim is that this “theory” can be interpreted as the UV complete description of (1) when $d = 2$. One can define correlation functions of local operators order by order as an
expansion in small $\mu$ and compare against predictions from conformal perturbation theory. However, our claim based on the structure of (15) is that a concept of local observables do not
exist microscopically, and as such these observables are not well defined non-perturbatively
in $\mu$.

Let us comment on the holographic interpretation of this picture when the CFT admits
a good gravity description. In [1], it was suggested that the UV cut-off wall be subjected
to Dirichlet boundary condition so that a notion of quasi-local observables can be defined. We are suggesting instead that it is more natural to assign a Neumann boundary condition. A physical setup where the AdS geometry is subjected to an “artificial” boundary subject to Neumann boundary condition was considered in a very different context by Karch and Randall in [21]. The setup of Karch and Randall also involves the bare cosmological constant, which they parameterize using the variable

$$\lambda = 2\alpha L.$$  \hspace{1cm} (16)

One of the main points of [21] is the fact that in order for the geometry of the boundary
to be flat (or equivalently, in order to tune the effective cosmological constant to zero), the
bare cosmological constant needed to be tuned to a finite value. Their analysis for the case
of $d = 4$ can be found in (3) of [21]. In our context, this implies that tuning the RG flow to
go into a CFT on flat space, the $\alpha$ must be non-vanishing. Consequently,

$$\frac{\mu}{2} = -\frac{1}{4N}\alpha$$  \hspace{1cm} (17)

also needs to be non-vanishing. From the perspective of [21], we can also relate the value
of $\lambda = (d - 1)r_c^2/L^3$ to the position of the cut-off surface in Poincare coordinate, and
$\mathcal{N} = c/24\pi$ [22]. In this way, we essentially recover the same relation

$$\mu = -\frac{24\pi L^4}{cr_c^2}$$  \hspace{1cm} (18)

as (1.3) of [1], although we are stressing that the boundary condition on the cut-off surface
that we are imposing is Neumann. This has important consequences on the set of physical
observables.

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4 This also suggests that limits $\mu \to 0_+$ and $\mu \to 0_-$ may be disconnected.
5 The value of $\lambda$ reported by [21] was for $d = 3$ and in a conformal frame where $r_c = L$. One can easily
generalize via a conformal transformation.
6 Recall that our conventions for $\mu$ are the opposite of [1].
We also note that taking $\alpha$ to be smaller than the critical value necessary for flat space, then, according to [21], we would generate a negative effective cosmological constant. This appears then to lead naturally to the setup considered in [6].

The appearance of Neumann boundary also offers an interesting connection to the construction of boundary conformal field theory [23]. In order to make this connection a bit more tangible, it is useful to tune the effective cosmological constant so that the cut-off boundary has the geometry of $AdS_2$. In that case, the cut-off boundary is precisely identifiable as the $Q$ component of the boundary using the notation of [23]. We refer the reader to figure 1 of [23] where the $Q$ and $M$ components of the boundary of a bulk $AdS$ geometry are defined. The same figure is reproduced in figure 1 below. The observables of BCFT are local operators inserted in the $M$ component. These are interpretable as “boundary observables” living on the boundary of the “boundary” $AdS_2$. We however do not insert operators on the Neumann surface $Q$. One can then imagine taking the limit of vanishing effective cosmological constant on $AdS_2$. This amounts to taking the flat space limit where the boundary observables on $AdS_2$ becomes the $S$-matrix on flat space, along the lines of [24]. From this perspective, it is clear that there are non-trivial observables such as the $S$-matrix elements one can compute even when the notion of local correlation functions are absent. At the same time, we also see that identifying correct set of observables are more subtle when quantum gravity is involved.

Finally, let us offer few pieces of evidence in support of our claim that the UV complete description of $T\bar{T}$ deformed CFT in $d = 2$ must be of Neumann type.

One argument is the fact that all observables computed in the literature (to the best of our knowledge) are consistent with the Neumann interpretation. The partition function on a cylinder or a torus, and the $S$-matrix elements, are typically reported to establish the theory being well defined at the quantum level [2,3]. Local correlation functions have been computed in [25] but only to few orders as an expansion in $\mu$. The issue of whether this series is summable at the non-perturbative level appears to be left open in most of these discussions. Here, we have provided an argument based on the structure of Fourier/Legendre transform and gauge invariance. Related arguments have also recently been provided by [7].

Another argument that quantum gravity must be involved in the full story can be made by building on the observation that the $T\bar{T}$ deformation can formally be resummed for the case of free scalar fields to bring it into the form of Nambu-Goto action in static gauge [3]. The question is whether this theory should be thought of as a gauge fixed version of a gauge invariant theory. If so, the full quantization of the Nambu-Goto action must essentially consist of performing the Polyakov path integral [18]. It is difficult to imagine how to quantize the Nambu-Goto action without invoking Polyakov’s technique. One of the salient features
Figure 1: Illustration of locally $AdS_{d+1}$ bulk spacetime $N$ bounded by boundary components $M$ and $Q$. (a) is a reproduction of figure 1 of [23]. (b) is the embedding of (a) inside global anti de Sitter geometry as was illustrated in figure 6 of [21]. The spacetime of boundary field theory from the $T\bar{T}$ deformed CFT point of view is $Q$. Here, we are taking the effective cosmological constant to be negative so that the geometry on $Q$ is $AdS_d$. From the boundary field theory point of view, $M$ is an auxiliary structure which emerges as a dual representation of large number of degrees of freedom that the original CFT contained. Operators are inserted on $M$ and are to be interpreted as boundary observables for the gravity theory on $AdS_d$, or as $S$-matrix elements in the limit where $AdS_d$ approaches flat $d$-dimensional space-time.
Figure 2: An example of non-single valued field configuration which are allowed in Nambu-Goto theory. Similar issue arises in the “tree stump” configuration of Bions illustrated in figure 2 of [26].

of Nambu-Goto and DBI action [26] is the fact that a configuration of the type illustrated in figure 2 where the embedding is not single valued are included in the space of allowed configurations, and in fact has finite action. In other words, singularities corresponding to the derivative of the embedding field reaching infinity is coordinate, as opposed to physical singularity. The issue of how to precisely identify the singular configurations which should, or should not, be included in the path integral, is in fact the central challenge in properly understanding the gauge field formulation of quantum gravity in $d = 3$ [27,28]. In $d = 2$, by invoking diffeomorphism invariance and the uniformization theorem, one can iron out the non-singlevaluedness of configurations like the one illustrated in figure 2, e.g. by going to lightcone gauge [29–32].

Polyakov’s path integral [18] is an elegant solution which overcomes all of these subtleties. It is certainly possible, but it is difficult for us to imagine an alternative, consistent, quantization of a long Nambu-Goto string aside from the Polyakov’s prescription. At the end of section 4.1 of [1], the authors indeed state that the $T\bar{T}$ system “behaves like a causal theory similar to 2D quantum gravity.” We are making a stronger statement that is a theory of 2D quantum gravity, in the sense that it is a quantum theory with diffeomorphism invariance.

That the boundary condition for the metric is Neumann instead of Dirichlet will only affect the UV of the full system. Most of the effective physics in the IR are unaffected. Nonetheless, since the $T\bar{T}$ deformation of [2] is supposed to be UV complete, the distinction between Neumann and Dirichlet is physically meaningful.

It should also be re-iterated that the functional Fourier/Legendre transform made sense mathematically without additional input only for $d = 2$. This suggests that the generaliza-
tion of \cite{2} will likely not work so nicely in dimensions other than \(d = 2\). It would be very interesting to explore if the special features of quantum gravity in \(d = 3\) can be exploited to push the agenda in that direction \cite{27, 28}. Perhaps some version of the story in higher dimension can be constructed following the prescription of \cite{33}.

Finally, let us comment in closing that while we have provided arguments based on the structure of Fourier/Legendre transform and some circumstantial observations, we have not provided a rigorous argument that conformal perturbation theory in \(\mu\) in fact fails to converge. It is likely that correlation functions suffer from breakdown of locality at short distances along the lines found in \cite{34}. It would be very interesting to understand the nature of the asymptotic expansion in \(\mu\) more thoroughly.

Acknowledgements

This work is supported in part by the DOE grant [de-sc0017647]. We would also like to thank Victor Gorbenko and Mehrdad Mirbabayi for interesting conversations.

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