Approximation by a Generalized Szász-Bézier Operators

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Abstract. The application of Bézier type operators is very extensive and has attracted people’s attention. In the year 2017, Ren established a generalized Bernstein-Bézier type operators acting on \(C[0, 1]\). Inspired by this, in this paper, a generalized Szász-Bézier type operators, with Gamma function defined on the positive semi-axis, is extended. Then, the equivalent theorem and the Voronovskaja type asymptotic formulas are also obtained.

1. Introduction

The approximation properties of the classical Szász operators \(S_n(f; x)\) were widely investigated in the literature\textsuperscript{[1–6]}. During the past thirty years, the Bézier basis function was extensively used to construct various generalizations of many classical approximation processes\textsuperscript{[7–14]}. In 2017, Ren et al\textsuperscript{[14]} introduced a generalized Bernstein-Bézier type operators acting on \(C[0, 1]\), have been considered in connection with Beta function, and obtained a Jackson type direct theorem. In order to get the Bernstein type inverse theorem, in [15], we introduced a kind of Bernstein-Bézier operators with parameters. This paper is concerned with generalized Szász-Bézier type operators acting on functions defined on the positive semi-axis, with Gamma function. The Szász-Bézier operators are defined as follows:

\[
S_{n, \alpha}(f; x) = \sum_{k=0}^{\infty} s_{n, k}(x) \cdot \left( J_{\alpha}^{k}(x) - J_{\alpha}^{k+1}(x) \right),
\]

where \(\alpha \geq 1\), \(J_{n, k}(x) = \sum_{j=k}^{\infty} S_{n, j}(x), k = 0, 1, ..., s_{n, k}(x) = \frac{(nx)^{k}}{k!} e^{-nx}\), \(J_{n, k}(x)\) is the Szász-Bézier basis function. Obviously, when \(\alpha = 1\), \(S_{n, r}(f; x)\) becomes \(S_{n}(f; x)\), and for \(x \in [0, \infty)\), one has\textsuperscript{[11]} \(J_{n, 0}(x) \geq J_{n, 1}(x) \geq ... J_{n, k}(x) \geq J_{n, k+1}(x) = s_{n, k}(x)\).

In this paper, we are going to study a new kind of Szász type operators for \(f(x) \in C_{B}[0, \infty)\) as follows:

\[
D_{n, \beta}(f; x) = f(0)s_{n, 0}(x) + \sum_{k=1}^{\infty} s_{n, k}(x)T_{n, k}^{(0)}(f),
\]

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where $0 \leq \beta \leq 1$,
\[
T^{(\beta)}_{n,k}(f) = \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1} e^{-nt} f \left( \beta t + (1 - \beta) \frac{k}{n} \right) dt,
\]
$\Gamma(\cdot)$ is the Gamma function. When $\beta = 0$, $D_{n,\beta}(f; x)$ becomes $S_n(f; x)$.

We will also study a generalized Szász-Bézier-type operators for $f(x) \in C_\beta[0, \infty)$ as follows:
\[
D^{(\alpha)}_{n,\beta}(f; x) = f(0) G^{(\alpha)}_{n,\beta}(x) + \sum_{k=1}^{\infty} C^{(\alpha)}_{n,\beta}(x) T^{(\beta)}_{n,k}(f),
\]
where $0 \leq \beta \leq 1, \alpha \geq 1, G^{(\alpha)}_{n,\beta}(x) = I^{\alpha}_{n,k}(x) - J^{\alpha}_{n,k+1}(x), I_{n,k}(x)$ and $T^{(\beta)}_{n,k}(f)$ are defined as above.

The operators $D^{(\alpha)}_{n,\beta}(f; x)$ are bounded and positive on $C_\beta[0, \infty)$. When $\alpha = 1$, $D^{(1)}_{n,\beta}(f; x)$ becomes $D_{n,\beta}(f; x)$.

When $\beta = 0$, $D^{(1)}_{n,\beta}(f; x)$ becomes $S_{n,\beta}(f; x)$.

The goal of the paper is to investigate the rate of convergence. Direct and inverse theorems are proved using Ditzian-Totik modulus of smoothness. The Voronovskaja type asymptotic formulas are also obtained.

**Remark 1** Throughout this paper, $M$ is a positive constant independent of $n$ and $x$, the value of $M$ may be different in different places.

**Remark 2** In this paper, for $f(x) \in C_\beta[0, \infty) = \{ f : f$ is continuous and bounded on $[0, \infty) \}$, the norm of $f(x)$ is defined as $\|f\| = \max \{|f(x)| : x \in [0, \infty]\}$.

**Remark 3**
1. $S_n(1; x) = 1$;
2. $S_n(t; x) = x$;
3. $S_n(t^2; x) = x^2 + \frac{3}{2} x$;
4. $S_n(t^3; x) = x^3 + \frac{3}{2} x + \frac{3}{4} x^2$;
5. $S_n(t^4; x) = x^4 + \frac{5}{2} x^2 + \frac{15}{8} x^3 + \frac{\gamma}{4}$.

2. Estimates of the moments

By the definition of $T^{(\beta)}_{n,k}(f), D_{n,\beta}(f; x)$, Remark 3 and using the integral by parts, we have Lemma 2.1, Lemma 2.2 and Lemma 2.3. Here we omit the details.

**Lemma 2.1** For $T^{(\beta)}_{n,k}(t^i), i = 0, 1, 2, 3, 4, 0 \leq \beta \leq 1$, we have
1. $T^{(\beta)}_{n,k}(1) = 1$;
2. $T^{(\beta)}_{n,k}(t) = \frac{1}{n}$;
3. $T^{(\beta)}_{n,k}(t^2) = \frac{1}{n} + \frac{\beta}{n} t$;
4. $T^{(\beta)}_{n,k}(t^3) = \frac{1}{n} + \frac{3 \beta t}{n^2} + \frac{2 \beta t^2}{n}$;
5. $T^{(\beta)}_{n,k}(t^4) = \frac{1}{n} + \frac{6 \beta t}{n^2} + \frac{(3 \beta^2 + 8 \beta t) t}{n^3} + \frac{6 \beta t^2}{n^4}$.

**Lemma 2.2** For $D_{n,\beta}((t - x)^i; x), i = 2, 4, 0 \leq \beta \leq 1$, we have
1. $D_{n,\beta}((t - x)^2; x) = \frac{1 + \beta^2}{n} x$;
2. $D_{n,\beta}((t - x)^4; x) = \frac{3 + 6 \beta^2 + 3 \beta^4}{n^2} x^2 + \frac{1 + 6 \beta^2 + 8 \beta^3 + 9 \beta^4}{n^3} x$.

**Lemma 2.3**
1. $\frac{1}{n} \sum_{k=1}^{\infty} I_{n,k}(x) = x$;
Hence (2) By Lemma 2.3 (2), one has

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} k j_{n,k}(x) = \frac{x^2}{2} + \frac{x^3}{3}. \]

(3) \[ \frac{1}{n} \sum_{k=1}^{\infty} k^2 j_{n,k}(x) = \frac{x^2}{2} + \frac{2x^3}{3n} + \frac{x^4}{2n^2}. \]

(4) \[ \frac{1}{n} \sum_{k=1}^{\infty} k^3 j_{n,k}(x) = \frac{x^2}{2} + \frac{2x^3}{3n} + \frac{x^4}{2n^2} + \frac{x^5}{3n^3}. \]

Lemma 2.4 Let \( \alpha \geq 1 \), we have

(1) \[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} j_{n,k}(x) = \frac{x^2}{2}; \]

(2) \[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} k^2 j_{n,k}(x) = \frac{x^2}{3} ; \]

(3) \[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{\infty} k^3 j_{n,k}(x) = \frac{x^4}{4}. \]

Proof (1) For \( \varepsilon > 0 \), \( \delta > 0 \), there exists a positive integer \( N = N(\varepsilon, \delta) \), since

\[ \lim_{n \to \infty} j_{n,k}(x) = \begin{cases} 1, & \text{for } k \leq n(x - \varepsilon); \\ 0, & \text{for } k \geq n(x + \varepsilon), \end{cases} \]

we see that

\[ \left\{ \begin{array}{l} 0 \leq 1 - j_{n,k}(x) < \delta, \quad \text{for } k \leq n(x - \varepsilon); \\ 0 \leq j_{n,k}(x) < \delta, \quad \text{for } k \geq n(x + \varepsilon). \end{array} \right. \]

Since \( \sum_{k=1}^{\infty} j_{n,k}(x) [1 - j_{n,k}^{-1}(x)] \) is convergent, then for \( \delta > 0 \), there exists an enough big \( K, \) such that \( \sum_{k=K}^{\infty} j_{n,k}(x) [1 - j_{n,k}^{-1}(x)] < \delta. \)

By Lemma 2.3 (1),

\[ 0 \leq x - \frac{1}{n} \sum_{k=1}^{\infty} j_{n,k}(x) = \frac{1}{n} \sum_{k=1}^{\infty} j_{n,k}(x) [1 - j_{n,k}^{-1}(x)] = \frac{1}{n} \left[ \sum_{k \geq n(x - \varepsilon)} + \sum_{k \geq n(x + \varepsilon)} + \sum_{n(x - \varepsilon) < k < n(x + \varepsilon)} \right]. \]

With the last three terms denoted by \( \Sigma_1, \Sigma_2, \Sigma_3 \) respectively. For enough big \( n \), the following estimates are easily obtained

\[ 0 \leq \Sigma_1 \leq \frac{\delta}{n} \sum_{k \leq n(x - \varepsilon)} j_{n,k}(x) \leq \frac{\delta}{n} \cdot n x = \delta x, \]

\[ 0 \leq \Sigma_2 \leq \frac{\delta}{n} \sum_{k \geq n(x + \varepsilon)} j_{n,k}(x) [1 - j_{n,k}^{-1}(x)] \leq \delta, \]

\[ 0 \leq \Sigma_3 \leq \frac{1}{n} \sum_{n(x - \varepsilon) < k < n(x + \varepsilon)} = \frac{1}{n} \cdot 2n \varepsilon = 2 \varepsilon. \]

Hence \( x - \frac{1}{n} \sum_{k=1}^{\infty} j_{n,k}(x) \to 0, \) we get Lemma 2.4(1).

(2) By Lemma 2.3 (2), one has

\[ \frac{x^2}{2} - \frac{1}{n^2} \sum_{k=1}^{\infty} k^2 j_{n,k}(x) = \frac{1}{n^2} \sum_{k=1}^{\infty} k j_{n,k}(x) - \frac{x}{n} - \frac{1}{n^2} \sum_{k=1}^{\infty} k^3 j_{n,k}(x). \quad (2.1) \]

From Lemma 2.3 (2), for \( \varepsilon > 0 \), there exists \( K' \in N, \) for \( k > K' \), such that

\[ \left| \frac{1}{n^2} \sum_{k=K'}^{\infty} k j_{n,k}(x) \right| < \varepsilon. \quad (2.2) \]
From Lemma 2.4 (1), there exists \( N' \in N \), for \( n > N' \), such that

\[
| x - \frac{1}{n} \sum_{k=1}^{\infty} f_{n,k}^a(x) | < \varepsilon. \tag{2.3}
\]

For a fixed \( x \in [0, \infty) \), choosing \( N = \max\{N', N''\} \), we write

\[
\left| \frac{x^2}{2} - \frac{1}{n^2} \sum_{k=1}^{\infty} kj_{n,k}^a(x) \right| \leq \left| \frac{1}{n} \sum_{k=1}^{N-1} f_{n,k}^a(x) \right| - \frac{1}{n} \sum_{k=1}^{N-1} f_{n,k}^a(x) + \frac{1}{n^2} \sum_{k=1}^{\infty} kj_{n,k}^a(x) + \frac{x}{n}. \tag{2.4}
\]

Combining Lemma 2.3 (1) and (2.1) ~ (2.4), we have get Lemma 2.4(2).

Similarly, we can obtain Lemma 2.4(3), (4) by some computations.

Noting \( f_{n,0}^a(x) = 1, \sum_{k=0}^{\infty} q_{n,k}^a(x) = 1 \) and Lemma 2.4, by simple calculation, one can get the following Lemma 2.5.

**Lemma 2.5** Let \( \alpha \geq 1, 0 \leq \beta \leq 1 \), we have

(1) \( D_{n,\phi}^0 (1; x) = 1 \);

(2) \( \lim_{n \to \infty} D_{n,\phi}^0 (t; x) = x \);

(3) \( \lim_{n \to \infty} D_{n,\phi}^0 (t^2; x) = x^2 \);

(4) \( \lim_{n \to \infty} D_{n,\phi}^0 (t^3; x) = x^3 \);

(5) \( \lim_{n \to \infty} D_{n,\phi}^0 (t^4; x) = x^4 \).

**Lemma 2.6** Let \( \alpha \geq 1, 0 \leq \beta \leq 1, \varphi^2(x) = x \), we have

\[
(1) D_{n,\phi}^0 ((t-x)^2; x) \leq \frac{\alpha}{n} (1 + \beta^2) \varphi^2(x);
\]

\[
(2) D_{n,\phi}^0 ((t-x)^4; x) \leq \frac{\alpha}{n^2} \left( 3 + 6\beta^2 + 3\beta^4 \right) \varphi^4(x) + \frac{1 + 6\beta^2 + 8\beta^3 + 9\beta^4}{n} \varphi^2(x).
\]

**Proof** Using the mean value theorem for differential calculus, for \( x \in [0, \infty) \), \( \alpha \geq 1, k = 0, 1, \ldots \), we have \( 0 \leq C_{n,k}^a(x) \leq \alpha s_{n,k}(x) \). Since

\[
D_{n,\phi}^0((t-x)^2; x) = x^2 G_{n,\phi}^0(x) + \sum_{k=1}^{\infty} C_{n,k}^a(x) T_{n,k}^\phi((t-x)^2)
\]

\[
\leq \alpha \cdot \left[ x^2 s_{n,0}(x) + \sum_{k=1}^{\infty} s_{n,k}(x) T_{n,k}^\phi((t-x)^2) \right]
\]

\[
= \alpha \cdot D_{n,\phi}^0((t-x)^2; x) = \frac{\alpha}{n} (1 + \beta^2) \varphi^2(x),
\]

we get Lemma 2.6(1). Similarly, we can obtain Lemma 2.6(2), we omit the details.

**Remark 4** By the Korovkin theorem\(^{[1,3]} \) and Lemma 2.5, the following result follows immediately:

For \( f(x) \in C_B[0, \infty) \), the functions \( D_{n,\phi}^a(f; x) \) converge to \( f(x) \) on \( [0, \infty) \).

**3. Direct Theorems**

For \( f(x) \in C_B[0, \infty) \), \( \varphi(x) = \sqrt{x}, 0 \leq \lambda \leq 1 \), let\(^{[1,3]} \)

\[
\omega_{\psi^1}(f; t) = \sup_{0 \leq t \leq \lambda} \sup_{x \in [0, \infty)} \left| f \left( x + \frac{\lambda \psi^1(x)}{2} \right) - f \left( x - \frac{\lambda \psi^1(x)}{2} \right) \right|,
\]

\[
\omega_{\psi^3}(f; t) = \sup_{0 \leq t \leq \lambda} \sup_{x \in [0, \infty)} \left| f \left( x + \frac{\lambda \psi^3(x)}{2} \right) - f \left( x - \frac{\lambda \psi^3(x)}{2} \right) \right|,
\]
be the Ditzian-Totik modulus, and
\[ K_{\psi^t}(f; t) = \inf_{g \in W_1(0, \infty)} \{ \|f - g\| + t\|\psi^t g'\| \}, \]
be the corresponding K-functional, here \( W_1 = \{ g \in A_{\infty \infty} [0, \infty), \|\psi^1 g'\| < \infty \}. \) It is well known that\(^{[1,3]}\)
\[ K_{\psi^t}(f; t) \sim \omega_{\psi^t}(f; t). \]

**Theorem 3.1** For \( f \in C[0, \infty), \alpha \geq 1, \psi(x) = \sqrt{x} \) and \( 0 \leq \beta \leq 1, 0 \leq \lambda \leq 1, \) then we have
\[ |D_{\alpha, \beta}^{(\psi^t)}(f; x) - f(x)| \leq M\omega_{\psi^t}(f; \sqrt{n}). \]

**Proof** Let \( g \in W_1, \) then
\[ |D_{\alpha, \beta}^{(\psi^t)}(f; x) - f(x)| \leq |D_{\alpha, \beta}^{(\psi^t)}(f - g; x)| + |f(x) - g(x)| + |D_{\alpha, \beta}^{(\psi^t)}(g; x) - g(x)|. \]

Since \( g(t) = \int_0^t g'(u)du + g(x), \) \( D_{\alpha, \beta}^{(\psi^t)}(1; x) = 1, \) we know
\[ |D_{\alpha, \beta}^{(\psi^t)}(g; x) - g(x)| \leq \|\psi^1 g'\| \cdot \|D_{\alpha, \beta}^{(\psi^t)}(\int_0^t g'(u)du; x)\|, \]
and by the Hölder inequality, we get
\[ \left| \int_0^t g'(u)du \right| \leq 2\sqrt{\psi^{-1}(x)} \cdot |t - x|. \]
Thus,
\[ |D_{\alpha, \beta}^{(\psi^t)}(g; x) - g(x)| \leq 2\sqrt{\psi^1 g'\| \cdot \psi^{-1}(x) \cdot D_{\alpha, \beta}^{(\psi^t)}(t - x; x)}. \]

Combining Lemma 2.6 (2), using the Cauchy-Schwarz inequality, we have
\[ |D_{\alpha, \beta}^{(\psi^t)}(g; x) - g(x)| \leq 2\sqrt{(1 + \beta^2)\alpha \cdot \|\psi^1 g'\| \cdot \frac{\psi^{-1}(x)}{\sqrt{n}}}. \]

By the definition of \( D_{\alpha, \beta}^{(\psi^t)}(f; x) \) and Lemma 2.5 (1), we have \( |D_{\alpha, \beta}^{(\psi^t)}(f; x)| \leq \|f\|, \) so
\[ |D_{\alpha, \beta}^{(\psi^t)}(f; x) - f(x)| \leq 2\|f - g\| + 2\sqrt{(1 + \beta^2)\alpha \cdot \|\psi^1 g'\| \cdot \frac{\psi^{-1}(x)}{\sqrt{n}}}. \]

Taking infimum on the right hand side over all \( g \in W_1, \) one can obtain
\[ |D_{\alpha, \beta}^{(\psi^t)}(f; x) - f(x)| \leq MK_{\psi^t}(f; \frac{\psi^{-1}(x)}{\sqrt{n}}) \leq M\omega_{\psi^t}(f; \sqrt{n}). \]

**Theorem 3.2** For \( f(x) \) is continuous and bounded on \([0, \infty), \alpha \geq 1, 0 \leq \beta \leq 1, \psi(x) = \sqrt{x}, \) then
\[ |D_{\alpha, \beta}^{(\psi^t)}(f; x) - f(x)| \leq \sqrt{(1 + \beta^2)\alpha \cdot \{ \|f'\| + \omega(f'; \frac{1}{\sqrt{n}}) \cdot (1 + \sqrt{1 + \beta^2}) \cdot \psi(x) \}} \cdot \psi(x). \]

**Proof** For \( \delta > 0, t, x \in [0, \infty), |t - x| < \delta, \) by the Taylor’s expansion, we get
\[ |f(t) - f(x) - f'(x)(t - x)| \leq \int_x^t |f'(u) - f'(x)| du \leq \omega(f'; \delta) \cdot (|t - x| + \delta\cdot (t - x)^2), \]
applying the Cauchy-Schwarz inequality, we have
\[
\left| D_{n,\delta}^{(a)}(f(t) - f(x) - f'(x)(t - x); x) \right| \\
\leq \omega(f'; \delta) \cdot \left| D_{n,\delta}^{(a)}([t-x]; x) \right| + \delta^{-1} D_{n,\delta}^{(a)}((t-x)^2); x \\
\leq \omega(f'; \delta) \cdot \sqrt{D_{n,\delta}^{(a)}(1; x) + \delta^{-1} \sqrt{D_{n,\delta}^{(a)}((t-x)^2); x}} \cdot \sqrt{D_{n,\delta}^{(a)}(t-x^2); x}.
\]
Thus,
\[
\left| D_{n,\delta}^{(a)}(f; x) - f(x) \right| \\
\leq \|f''\| \cdot D_{n,\delta}^{(a)}([t-x]; x) + \omega(f'; \delta) \cdot \left[ 1 + \delta^{-1} \sqrt{D_{n,\delta}^{(a)}((t-x)^2); x} \right] \cdot \sqrt{D_{n,\delta}^{(a)}(t-x^2); x}.
\]
Taking \( \delta = \frac{1}{N} \), by Lemma 2.6 (1), we can obtain the desired results.

4. Inverse Theorem

Lemma 4.1 Let \( f \in C_0[0, \infty), \varphi(x) = \sqrt{x}, \alpha \geq 1, 0 \leq \beta \leq 1, 0 \leq \lambda \leq 1, \) we have
\[
\left| \varphi^3(x) \cdot \left( D_{n,\delta}^{(a)}(f; x) \right) \right| \leq 9 \alpha \varphi^{3-1}(x) \sqrt{\lambda} \|f\|.
\]
Proof We write
\[
\left( D_{n,\delta}^{(a)}(f; x) \right) = f(0) \left( C_{n,\delta}^{(a)}(x) \right) + \left( \sum_{k=1}^{\infty} C_{n,k}(x) T_{n,k}^{(a)}(f) \right) = R_1 + R_2,
\]
and will estimate \( R_1 \) and \( R_2 \), respectively. Noting that \( T_{n,k}^a(x) = 0 \), we have
\[
|R_1| = |n a f(0) \cdot (1 - e^{-\alpha n \lambda})^{a-1} \cdot e^{-\alpha n \lambda}| \leq \frac{n a |f(0)|}{\alpha^{a n \lambda}}.
\]
For a fixed \( x \in [0, +\infty) \), \( \lim_{n \to \infty} \frac{\sqrt{\lambda}}{e^{\alpha n \lambda}} = 0 \), one may say \( \frac{\sqrt{\lambda}}{e^{\alpha n \lambda}} \leq 1 \), then
\[
\varphi^3(x) \cdot |R_1| \leq \frac{\sqrt{\lambda}}{e^{\alpha n \lambda}} \cdot \alpha \sqrt{\lambda} \|f\| \leq \alpha \varphi^{3-1}(x) \sqrt{\lambda} \|f\|.
\]
\[
R_2 = \alpha \sum_{k=1}^{\infty} T_{n,k}^{(a)}(f) \left( \left[ j_{n,k}^{a-1}(x) - j_{n,k+1}^{a-1}(x) \right] j_{n,k+1}'(x) + j_{n,k+1}'(x) \cdot \psi_{n,k}'(x) \right).
\]
For \( k = 0, 1, 2, 3, \ldots, 1 = j_{n,0}(x) \geq j_{n,1}(x) \geq \ldots \geq j_{n,k}(x) \geq j_{n,k+1}(x) \geq \ldots \geq 0, \) and \( j_{n,0}'(x) = 0, j_{n,k}'(x) = n s_{n,k-1}(x) \geq 0, \)
\[
\left| T_{n,k}^{(a)}(f) \right| = \left[ \frac{n^k}{\Gamma(k)} \int_0^\infty e^{-nt} \cdot f \left( \beta t + (1 - \beta) \frac{k}{n} \right) \right] dt \leq \|f\|,
\]
we have
\[
|R_2| \leq \alpha \|f\| \left( \sum_{k=1}^{\infty} \left| j_{n,k}^{a-1}(x) - j_{n,k+1}^{a-1}(x) \right| j_{n,k+1}'(x) + \sum_{k=1}^{\infty} j_{n,k}^{a-1}(x) \cdot \psi_{n,k}'(x) \right) = \alpha \|f\| (V_1 + V_2).
\]
Noting that \( f_{n,1}'(x) > 0, f_{n,0}'(x) = 0 \), we have

\[
V_1 \leq \sum_{k=0}^{\infty} j_{n,k}'(x) f_{n,k+1}'(x) - \left[ \sum_{k=0}^{\infty} j_{n,k+1}'(x) f_{n,k}'(x) - f_{n,1}'(x) f_{n,1}'(x) \right]
\]

\[
= \sum_{k=0}^{\infty} j_{n,k}'(x) f_{n,k}'(x) - \sum_{k=0}^{\infty} j_{n,k}'(x) s_{n,k}'(x) - \sum_{k=0}^{\infty} j_{n,k+1}'(x) f_{n,k+1}'(x)
\]

\[+ (1 - e^{-nx})^{a-1} \cdot ne^{-nx}, \]

thus,

\[
V_1 \leq V_2 + \frac{2n}{e^{\pi x}}. \tag{4.4}
\]

For \( x > 0, s_{n,k}'(x) = \frac{n}{\varphi(x)} \left[ \frac{k}{n} - x \right] \cdot s_{n,k}(x) \), combining the fact that \( f_{n,1}'(x) = (1 - e^{-nx})^{a-1}, s_{n,1}' = ne^{-nx}(1 - nx) \), we write that

\[
\varphi^3(x) V_2 \leq ne^{-nx}(1 + nx) \cdot \varphi^3(x) + \sum_{k=2}^{\infty} f_{n,k}'^{-1}(x) \cdot \left| s_{n,k}'(x) \right| \cdot \varphi^3(x)
\]

\[\leq \frac{\sqrt{n} \cdot \sqrt{nx} \cdot x^{1-\lambda} \cdot 1}{e^{nx}} + \frac{\sqrt{n} \cdot (nx)^{2} \cdot x^{1-\lambda}}{e^{nx}} + n \sum_{k=2}^{\infty} \left| \frac{k}{n} - x \right| \cdot s_{n,k}(x) \cdot \varphi^3(x)
\]

\[\leq 2 \sqrt{n} \cdot \varphi^{a-1}(x) + nq \varphi^{-2}(x) \cdot \left( S_n((t - x)^2, x) \right)^{1/2}, \]

then,

\[
\varphi^3(x) V_2 \leq 3\varphi^{a-1}(x) \sqrt{n}, \tag{4.5}
\]

and \( \varphi^2(x) V_1 \leq 3 \varphi^{a-1}(x) \sqrt{n} + 2 \sqrt{n} \cdot \varphi^{a-1}(x) \cdot \frac{\sqrt{n}}{e^{nx}} \leq 5\varphi^{a-1}(x) \sqrt{n}. \)

So

\[
\varphi^3(x)|R_2| \leq a||f|| \cdot \varphi^3(x) (V_1 + V_2) \leq 8a \sqrt{n}||f|| \cdot \varphi^{a-1}(x). \tag{4.6}
\]

From (4.1)-(4.6), the desired result follows.

**Lemma 4.2** Let \( f \in W_{4,4} \), \( \varphi(x) = \sqrt{x}, \alpha \geq 1, 0 \leq \beta \leq 1, 0 \leq \lambda \leq 1 \), we have

\[
\left| \varphi^3(x) \cdot \left( \left[ D_{n,0}^{(a)}(f; x) \right] \right) \right| \leq 38a||\varphi^3 f'||.
\]

**Proof** Since \( f(x) \left( D_{n,0}^{(a)}(f; 1; x) \right)' = 0 \), we get

\[
\left( D_{n,0}^{(a)}(f; x) \right)' = \left[ f(0) - f(x) \right] \left( C_{n,0}^{(a)}(x) \right)' + \sum_{k=1}^{\infty} \left[ T_{n,k}^{(a)}(f) - f(x) \right] \left[ f_{n,k}'(x) - f_{n,k+1}'(x) \right],
\]

we write

\[
\left( D_{n,0}^{(a)}(f; x) \right)' = H_1 + H_2, \tag{4.7}
\]

and will estimate \( H_1 \) and \( H_2 \) respectively. First, from (3.1), we have

\[
\varphi^3(x)|H_1| \leq \varphi^3(x) \cdot \varphi^3 f' \cdot 2^4 \cdot x^{1-\lambda} \cdot \frac{\alpha n}{e^{\alpha n}} \leq 2a\varphi^3 f'. \tag{4.8}
\]
Next,
\[ H_2 = \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-nt} \cdot \left[ f\left( \beta t + \frac{1 - \beta}{n} \right) - f(x) \right] dt \cdot \left[ \Gamma_{n_k}(x) - \Gamma_{n_{k+1}}(x) \right] \]
\[ = \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-nt} \cdot \int_x^\infty f'(u) du \cdot \left[ \Gamma_{n_k}(x) - \Gamma_{n_{k+1}}(x) \right]. \]

From (3.1), we have
\[ \phi^3(x)\left| \int_x^{\phi+(1-\beta)\frac{1}{2}} f'(u) du \right| \leq \phi^3(x)\| \phi^3 f'\| \cdot \left| \int_x^{\phi+(1-\beta)\frac{1}{2}} \frac{1}{\phi^3(u)} du \right| \]
\[ \leq 2\| \phi^3 f'\| \cdot \left| \beta t + \frac{1 - \beta}{n} - x \right|, \]
then,
\[ \phi^3(x)|H_2| \leq 2\| \phi^3 f'\| \cdot \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-nt} \cdot \left| \beta t + \frac{1 - \beta}{n} - x \right| dt \cdot \left[ \Gamma_{n_k}(x) - \Gamma_{n_{k+1}}(x) \right] \]
\[ = 2\alpha\| \phi^3 f'\| \cdot \left\{ \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-nt} \cdot \left| \beta t + \frac{1 - \beta}{n} - x \right| dt \cdot \left[ \Gamma_{n_k}(x) - \Gamma_{n_{k+1}}(x) \right] \right. \]
\[ \times \left[ \Gamma_{n_{k+1}}(x) \right] + \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-nt} \cdot \left| \beta t + \frac{1 - \beta}{n} - x \right| dt \cdot \left[ \Gamma_{n_{k+1}}(x) \right] \right\}. \]

Write
\[ \phi^3(x)|H_2| \leq 2\alpha\| \phi^3 f'\| \cdot (A + B). \]

We will estimate A and B on \( E_n^C \) and \( E_n \) respectively.

(1). For \( x \in E_n^C = [0, \frac{1}{n}) \):
\[ B \leq \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-nt} \cdot \left| \beta t + \frac{1 - \beta}{n} - x \right| dt \cdot \left[ \Gamma_{n_k}(x) - \Gamma_{n_{k+1}}(x) \right] \]
\[ \leq \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-nt} \cdot \left| \beta t + \frac{1 - \beta}{n} - x \right| dt \cdot n(s_{n,k-1}(x) - s_{n,k}(x)) \]
\[ + \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-nt} \cdot \left| \beta t + \frac{1 - \beta}{n} - x \right| dt \cdot n s_{n,k}(x), \]
we write
\[ B \leq L_1 + L_2. \]

Since \( D_{n\beta}((t - x)^2) = \frac{(1+\beta^2)\gamma}{n} \), by the Cauchy-Schwarz inequality, we get
\[ L_2 \leq n \left( D_{n\beta}((t - x)^2) \right)^{\frac{1}{2}} \leq \sqrt{1 + \beta^2}. \]

Using the fact that \( \Gamma(j+1) = j\Gamma(j) \), and for \( j \geq 1 \)
\[ \int_0^\infty t^{j-1} e^{-nt} \cdot \left| \beta t + \frac{1 - \beta}{n} - x \right| dt \]
\[ \leq \frac{j}{n} \int_0^\infty e^{-nt} t^{j-1} \cdot \left| \beta t + \frac{1 - \beta}{n} - x \right| dt + \frac{\beta}{n} \int_0^\infty e^{-nt} \cdot t^j dt, \]
then

\[
L_1 \leq \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_0^{\infty} t^{j} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{j}{n} - x \right| dt \cdot n s_n (x) \\
+ \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_0^{\infty} t^{j} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{j}{n} - x \right| dt \cdot n s_n (x) \\
= \frac{n}{\Gamma(1)} \int_0^{\infty} t^{0} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{j}{n} - x \right| dt \cdot n s_n (x) \\
+ \sum_{j=1}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_0^{\infty} t^{j-1} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{j}{n} - x \right| dt \cdot n s_n (x) \\
+ \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_0^{\infty} t^{j} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{j}{n} - x \right| dt \cdot n s_n (x)
\]

\[
\leq \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \cdot \frac{n}{\Gamma(1)} \int_0^{\infty} t^{j-1} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{j}{n} - x \right| dt \cdot n s_n (x) \\
+ \sum_{j=1}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \cdot \frac{\beta}{n} \int_0^{\infty} t^{j-1} e^{-nt} \cdot n s_n (x) + \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \cdot \frac{1 - \beta}{n} \int_0^{\infty} t^{j} e^{-nt} \cdot n s_n (x)
\]

\[
\leq \frac{n}{\Gamma(1)} + \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \cdot \frac{1}{n} \int_0^{\infty} t^{j-1} e^{-nt} \cdot dt \cdot n s_n (x) \\
+ \sum_{j=1}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \cdot \frac{\beta}{n} \int_0^{\infty} t^{j-1} e^{-nt} \cdot dt \cdot n s_n (x) + \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \cdot \frac{1 - \beta}{n} \int_0^{\infty} t^{j} e^{-nt} \cdot dt \cdot n s_n (x)
\]

we get

\[
L_1 \leq 3 + L_2 + 1 \leq 4 + \sqrt{1 + \beta^2}, \quad (4.12)
\]

from (4.10)-(4.12), we know \( B \leq 4 + 2 \sqrt{1 + \beta^2} \).

Noting that \( J_{n,0}^x (x) = 0 \), for \( x \in [0, \frac{1}{n}] \), one has

\[
A = \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{k}{n} - x \right| dt \cdot J_{n,k}^{x-1} (x) J_{n,k}^x (x) \\
- \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{k}{n} - x \right| dt \cdot J_{n,k+1}^{x-1} (x) J_{n,k+1}^x (x) \\
- \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{k}{n} - x \right| dt \cdot J_{n,k+1}^{x-1} (x) \cdot J_{n,k+1}^x (x)
\]

\[
\leq L_3 + B - \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1} e^{-nt} \cdot \left| \beta t + (1 - \beta) \frac{k}{n} - x \right| dt \cdot J_{n,k+1}^{x-1} (x) \cdot J_{n,k+1}^x (x),
\]
and

$$L_3 = \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_{0}^{\infty} t e^{-nt} \cdot |\beta t + (1 - \beta) \frac{j + 1}{n} - x| \ x \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x)

\leq \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_{0}^{\infty} t e^{-nt} \cdot |\beta t + (1 - \beta) \frac{j}{n} - x| \ x \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x)

= L_4 + \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_{0}^{\infty} t e^{-nt} \cdot \frac{1 - \beta}{n} \ x \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x)

L_4 \leq \sum_{j=0}^{\infty} \frac{n^j}{\Gamma(j)} \int_{0}^{\infty} e^{-nt} \cdot t^{j-1} \cdot \left|\beta t - x\right| dt \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x)

\leq \frac{1}{\Gamma(0)} \int_{0}^{\infty} e^{-nt} \cdot t^{j-1} \cdot \left|\beta t - x\right| dt \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x)

\leq \sum_{j=0}^{\infty} \frac{n^j}{\Gamma(j)} \int_{0}^{\infty} e^{-nt} \cdot t^{j-1} \cdot \left|\beta t + (1 - \beta) \frac{j}{n} - x\right| dt \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x)

\leq \sum_{j=0}^{\infty} \frac{n^j}{\Gamma(j)} \int_{0}^{\infty} e^{-nt} \cdot t^{j-1} \cdot \left|\beta t + (1 - \beta) \frac{j}{n} - x\right| dt \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x),$$

classic, we have that

$$L_3 \leq \left( f_{n_{j+1}}^{(j)}(t; x) + x \right) \cdot e^{\int_{j}^{j+1} e^{-nt} \cdot \left|\beta t + (1 - \beta) \frac{j + 1}{n} - x\right| dt \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x)

\leq \sum_{j=1}^{\infty} \frac{n^j}{\Gamma(j)} \int_{0}^{\infty} t^{j-1} e^{-nt} \cdot \left|\beta t + (1 - \beta) \frac{j}{n} - x\right| dt \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x)

\leq 1 + \sum_{j=1}^{\infty} \frac{n^j}{\Gamma(j)} \cdot t \int_{0}^{\infty} t^{j-1} e^{-nt} \cdot \left|\beta t + (1 - \beta) \frac{j}{n} - x\right| dt \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x)

\leq 2 + \sum_{j=1}^{\infty} \frac{n^j}{\Gamma(j)} \cdot t \int_{0}^{\infty} t^{j-1} e^{-nt} \cdot \left|\beta t + (1 - \beta) \frac{j}{n} - x\right| dt \cdot e^{\int_{n_{j+1}}^{n_j} (x)'} (x),$$

hence $A \leq 2 + |B| \leq 6 + 2 \sqrt{1 + \beta^2}.$

Combing A and B, we have,

$$|\varphi^\lambda(x) H_2| \leq 2a\|\varphi^\lambda f^\prime\| \cdot (10 + 4 \sqrt{1 + \beta^2}) \leq 36a\|\varphi^\lambda f^\prime\||. \quad (4.13)$$
From (4.7), (4.8), (4.13), for \( x \in E_n^c \), we have \( |\varphi^3(x) \cdot (D_{n,0}^{(\alpha)}(f; t))'| \leq 38\alpha||\varphi^3 f'|| \).

(II) \( x \in E_n = [\frac{1}{n}, +\infty) \): Noting \( s_{n,\lambda}(x) = \frac{n}{\varphi^3(x)} \int_0^\infty \frac{k}{1 - x} \cdot s_{n,\lambda}(x) \), using the Cauchy-Schwarz inequality,

\[
B \leq \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^\infty t^{-1} e^{-t} \cdot \left( \beta t + (1 - \beta) \frac{k}{n} - x \right) dt \cdot \left| s_{n,\lambda}(x) \right|
\]

Taking \( \lambda = 1 \) and using Theorem 3.4 and Theorem 4.1, we get the equivalent theorem.

Hence, for \( x \in D \), we have

\[
|\varphi^3(x) H_2| \leq 12\alpha||\varphi^3 f'||.
\]

Hence, for \( x \in E_n \), we have \( |\varphi^3(x) \cdot (D_{n,0}^{(\alpha)}(f; t))'| \leq |\varphi^3(x) H_1| + |\varphi^3(x) H_2| \leq 14\alpha||\varphi^3 f'||.

**Theorem 4.1** Let \( f(x) \in C[0, \infty) \), \( \varphi(x) = \sqrt{x} \), \( 0 \leq \beta \leq 1 \), \( 0 \leq \gamma \leq 1 \), \( 0 \leq \lambda \leq 1 \), if

\[
|D_{n,0}^{(\alpha)}(f; t) - f(x)| = O(n^{-\gamma}), \quad \text{one has} \quad \omega_{\varphi^3}(f; t) = O(t^{\gamma}).
\]

**Proof** By the definition of the \( K \)-functional, for \( g \in W_\lambda \),

\[
K_{\varphi^3}(f; t) \leq \left\| f - D_{n,0}^{(\alpha)}(f; t) \right\| + t \left\| \varphi^3(x) \left(D_{n,0}^{(\alpha)}(f; t)\right)' \right\|
\]

\[
\leq M n^{-\gamma} + t \left( \left\| \varphi^3(x) \left(D_{n,0}^{(\alpha)}(f - g; x)\right)' \right\| \right) + \left\| \varphi^3(x) \left(D_{n,0}^{(\alpha)}(g; x)\right)' \right\|
\]

\[
\leq M n^{-\gamma} + t \sqrt{n} \left( \left\| f - g \right\| + \frac{1}{n} \left\| \varphi^3 g' \right\| \right)
\]

\[
\leq M \left( n^{-\gamma} + t \sqrt{n} \cdot K_{\varphi^3}(f; n^{-\gamma}) \right),
\]

applying to the Berens-Lorentz Lemma[1], then \( K_{\varphi^3}(f; t) = O(t^{\gamma}) \).

We know[1]: \( \omega_{\varphi^3}(f; t) \sim K_{\varphi^3}(f; t) \), then one get Theorem 4.1.

From Theorem 3.4 and Theorem 4.1, we get the equivalent theorem.

**Theorem 4.2** Let \( f(x) \in C[0, \infty) \), \( \varphi(x) = \sqrt{x} \), \( 0 \leq \beta \leq 1 \), \( 0 \leq \gamma \leq 1 \), \( 0 \leq \lambda \leq 1 \), we have

\[
\left| D_{n,0}^{(\alpha)}(f; x) - f(x) \right| = O(n^{-\gamma}) \iff \omega_{\varphi^3}(f; t) = O(t^{\gamma}).
\]

5. Voronovskaja type theorem

In this section, we will first prove Voronovskaja type theorems for the operators \( D_{n,0}^{(\alpha)}(f; x) \) by means of the Ditzian-Totik modulus of smoothness \( \omega_{\varphi^3}(f; t) \).
The quantity \(\int_{t}^{\infty} |t - y| \cdot |f''(y) - f'''(x)| dy\) was estimated as follows,
\[
\left|\int_{t}^{\infty} |t - y| \cdot |f''(y) - f'''(x)| dy\right| \leq 2\|f'' - h\| ||(t - x)^2 + 2\|\varphi h\|\varphi^{-1}(x)||t - x||^3,
\]
where \(h \in W_1[0, \infty)\).

Using Lemma 2.6 it follows that there exists a constant \(M > 0\) such that, for \(n\) sufficiently large,
\[
D_{n,\varphi}^{(a)}((t - x)^2; x) \leq \frac{M}{2H^2} \varphi^2(x)\quad \text{and} \quad D_{n,\varphi}^{(a)}((t - x)^4; x) \leq \frac{M}{2H^2} \varphi^4(x),
\]

applying the Cauchy-Schwarz inequality, we get
\[
\left|D_{n,\varphi}^{(a)}(f; x) - f(x) - E_n(x)f'(x) - \frac{1}{2} F_n(x)f''(x)\right| \leq 2\|f'' - h\| ||D_{n,\varphi}^{(a)}((t - x)^2; x) + 2\|\varphi h\|\varphi^{-1}(x)D_{n,\varphi}^{(a)}((t - x)^2; x)\|
\]
\[
\leq \frac{M}{n} \varphi^2(x) ||f'' - h\| || + 2\|\varphi h\|\varphi^{-1}(x) \cdot \left(D_{n,\varphi}^{(a)}((t - x)^2; x)\right)^{1/2} \cdot \left(D_{n,\varphi}^{(a)}((t - x)^4; x)\right)^{1/4}
\]
\[
\leq \frac{M}{n} \varphi^2(x) \left[||f'' - h\| + n^{-1/2}\|\varphi h\||\right] \leq \frac{M}{n} \varphi^2(x)\omega_{\varphi}(f'', n^{-1/2}),
\]
in the last inequality, we have used the relation of K-functional and the modulus of smoothness.

Corollary 5.1 If \(f''(x)\) is continuous and bounded on \([0, \infty)\), \(a \geq 1\), then
\[
\lim_{n \to \infty} n \left\{D_{n,\varphi}^{(a)}(f; x) - f(x) - E_n(x)f'(x) - \frac{1}{2} F_n(x)f''(x)\right\} = 0,
\]
where \(E_n(x)\) and \(F_n(x)\) are defined in Theorem 5.1.

The Grüss type approximation problem has been studied by many authors \([17-20]\). Next, we will provide a Grüss-Voronovskaja type theorem for the operators \(D_{n,\varphi}(f; x)\).

Theorem 5.2 If \(f''(x), g''(x)\) are continuous and bounded on \([0, \infty)\), for each \(x \in [0, \infty)\), we have
\[
\lim_{n \to \infty} n \left\{D_{n,\varphi}(fg; x) - D_{n,\varphi}(f; x)D_{n,\varphi}(g; x)\right\} = (1 + \beta^2)x f''(x)g''(x).
\]
Proof We write that
\[
D_{n,\beta}((fg); x) - D_{n,\beta}(f; x)D_{n,\beta}(g; x) = D_{n,\beta}((fg); x) - f(x)g(x) - E_n(x)(fg)'(x) - \frac{1}{2}F_n(x)(fg)'' \\
- D_{n,\beta}(f; x) \cdot [D_{n,\beta}(g; x) - g(x)] - E_n(x)g'(x) - \frac{1}{2}F_n(x) \cdot g''(x) \\
- g(x) \cdot [D_{n,\beta}(f; x) - f(x) - E_n(x)f'(x) - \frac{1}{2}F_n(x)f''(x)] \\
+ \frac{1}{2}F_n(x) \cdot [(fg)''(x) - g''(x)]D_{n,\beta}(f; x) - g(x)f''(x) \\
+ E_n(x) \cdot [(fg)'(x) - g'(x)]D_{n,\beta}(f; x) - g(x)f'(x)].
\]

From the definition of $E_n(x) = D_{n,\beta}(t - x; x), F_n(x) = D_{n,\beta}((t - x)^2; x)$, and the relation $(fg)''' = (f'g + fg')' = f''g + 2f'g' + fg'', (fg)'' = f'g + fg'$, we can express that
\[
E_n(x) \cdot [(fg)''(x) - g''(x)]D_{n,\beta}(f; x) - g(x)f''(x) = E_n(x) \cdot [(f'g')'(x) - g'(x)]D_{n,\beta}(f; x);
\]

and $E_n(x) \cdot [(fg)'(x) - g'(x)]D_{n,\beta}(f; x) - g(x)f'(x) = E_n(x) \cdot [(f'g')(x) - g'(x)]D_{n,\beta}(f; x)$.

Because $E_n(x) = D_{n,\beta}(t - x; x) = 0$ and Lemma 2.2, Lemma 2.5, combining the Korovkin Theorem[3] and Corollary 5.1, we get
\[
\lim_{n \to \infty} n \{D_{n,\beta}((fg); x) - D_{n,\beta}(f; x)D_{n,\beta}(g; x)\} = 0 \quad \text{with} \quad \lim_{n \to \infty} n f'(x)g'(x)F_n(x) = \frac{1}{2} \lim_{n \to \infty} n g''(x) \cdot [f(x) - D_{n,\beta}(f; x)]F_n(x) \\
+ \lim_{n \to \infty} n g''(x) \cdot [f(x) - D_{n,\beta}(f; x)] \cdot E_n(x) \\
= \lim_{n \to \infty} n f''(x)g'(x)F_n(x) = f''(x)g'(x)q^2(x)(1 + \beta^2) = (1 + \beta^2)x f'(x)g'(x).
\]

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