Algebraic Computation of Hierarchical Renormalization Group Fixed Points and their $\epsilon$-Expansions

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Abstract

Nontrivial fixed points of the hierarchical renormalization group are computed by numerically solving a system of quadratic equations for the coupling constants. This approach avoids a fine tuning of relevant parameters. We study the eigenvalues of the renormalization group transformation, linearized around the non-trivial fixed points. The numerical results are compared with $\epsilon$-expansion.
1 Introduction

The renormalization group (RG) is a nonperturbative approach to critical phenomena in statistical mechanics and Euclidean field theory. It offers a satisfactory explanation for the behavior of statistical models in the critical regime, in particular the appearance of universal quantities. It also suggests block spin transformations as an actual scheme for computations of critical quantities. Last not least it has proved to be a powerful tool in rigorous investigations. Although a lot of efforts have been put into the subject since the seminal work of Wilson it is far from being closed.

The main idea is to think of a critical system in terms of the flow of effective actions generated by a block spin transformation. This flow will have fixed points. An instructive example is a system with two possible phases separated by a second order transition in the absence of an external field. Then we typically find two stable fixed points characterizing the different phases and one unstable fixed point signalling a phase transition. Critical indices can be calculated as eigenvalues of a linearized transformation at the unstable fixed point. Given any critical system the most important task is thus the determination of its RG flow fixed points and the flow in their vicinity.

Let us consider a Euclidean scalar field on a \(d\)-dimensional unit lattice \(\Lambda\). The action is

\[
S(\phi) = S_0(\phi) + V(\phi) \quad \text{with} \quad V(-\phi) = V(\phi).
\]

The interaction is called local if the Boltzmann factor \(\exp(-V(\phi))\) factorizes into a product \(\prod_{x \in \Lambda} F(\phi(x))\). Unfortunately, locality is only preserved in an approximate sense by block spin transformations. A main technical difficulty therefore is to find an efficient parametrization of the effective actions generated by the flow. Also it is not a priori clear what the best choice for the block spin is. A guideline is to demand the effective actions to be approximately as local as possible.

The block spin transformation which we will consider here relies on a decomposition of the Gaussian measure \(d\mu_v(\phi)\) determined by the free action \(S_0(\phi) = \frac{1}{2}(\phi, v^{-1}\phi)\) into a background and a fluctuation part. The free covariance (propagator) is \(v = (-\Delta)^{-1}\). The block spin transformation then consists of an integration step, a rescaling step of the background field and the block lattice, and a subsequent subtraction of the field independent term. In this setup the non-locality of the effective interaction is traced back to the fact that the fluctuation covariance despite its exponential decay couples fields further apart than a block distance.

In the hierarchical model the free covariance \((-\Delta)^{-1}\) is replaced by a hierarchical counterpart \((-\Delta)^{-1}_{\text{hier}}\). Both covariances share a similar long distance decay, although the hierarchical covariance is not translation invariant, and the models therefore behave similarly in the critical region. As a gain the hierarchical block spin transformation preserves locality in a strict sense. A hierarchical RG step turns out to be a nonlinear transformation of the local Boltzmann factor \(F(\phi)\), which is a function of a single variable rather than a functional of a field configuration.

One point of view is to think of a hierarchical model as an approximation to its full brother. This is obviously only justified if there exists an interpolation between the two which admits the computation of systematic corrections around the hierarchical situation.
Such a scheme is still lacking although a number of first steps into this direction can be found in the literature, see, e.g., [3]. The main difference between the hierarchical and the full model is the absence of wave function renormalization in the former. Wave function renormalization comes about in the full model because the effective interaction contains a kinetic term as a part of the quadratic piece. This is clearly discarded once the effective action is forced to be ultra-local.

A second point of view is to think of hierarchical models as nontrivial statistical models by themselves which are perfectly suited for a RG approach. Due to their nontrivial interaction they are far from being exactly solvable. Their phase structure and critical behavior is as rich and complicated as in the full case. This point of view seems somewhat academic since we do not know of any real ensembles which belong to the universality class of a hierarchical model. Real systems do not show this peculiar breaking of translation invariance.

The third point of view is to consider hierarchical models as a test ground for both concepts and methods towards a RG treatment of the full model. The problem of non-localities in the effective action is believed to be of purely technical nature. This is convincingly supported by rigorous studies of many full critical models [4] which have been preceded by studies of their hierarchical counterparts. A full RG flow will always contain as a part a flow of local quantities which is accurately modelled by a hierarchical flow.

An exciting possibility within the hierarchical context is to tune the dimension parameter $d$ away from integer values. The hierarchical transformation does not involve any lattice geometry and contains $d$ and a scale parameter $L$ as variables which can be taken real valued. Let us stress that this can be done completely independent of perturbation theory. A beautiful picture emerges that shows a sequence of thresholds between two and four dimensions where new fixed points appear. For instance, four dimensions is the threshold below which an unstable double well fixed point exists which governs three dimensional physics. This picture is believed to be true also in the full model although we do not know of a RG formulation which incorporates lattice geometry only in form of a dimension parameter which can be tuned at will.

In this paper we investigate the question of RG flows in the hierarchical model. Usually small coupling constants are required to do analytically such an investigation. To study infrared fixed points we have no small coupling parameters at hand. Nevertheless, we will show that we can do RG calculations for models which are not asymptotically free by using some algebraic equations.

After splitting off the quadratic fixed point, which corresponds to the unordered phase, we expand the Boltzmann factor in terms of local operators. We will use two kinds of operators, simple powers of fields and normal ordered products. Then the fixed point equation becomes a set of algebraic equations for the coupling constants. We truncate this system and solve it numerically. This turns out to be much better than searching for fixed points by iterating block spin transformations and fine tuning the initial values. It allows us to determine even the higher fixed points with multi-dimensional unstable manifolds to high accuracy. Then we linearize this set of equations and diagonalize it.
numerically around the various fixed points. This directly computes the critical indices of the hierarchical model without having to rely on any kind of expansions.

The algebraic equations can also be written down for the full model in terms of a polymer expansion. It is less clear then how to truncate this system to a reasonable size. But we believe that this is manageable and will admit a similar numerical treatment.

The algebraic equations can be solved for any value of \( d > 2 \). Inserting a power series ansatz in \( \epsilon \) for the coupling constants at \( d = 4 - \epsilon \), we obtain the \( \epsilon \)-expansion for the double well fixed point and analogously at the other critical dimensions. The resulting set of equations can be put into a recursive form which we iterate exactly using computer algebra. We compare this \( \epsilon \)-expansion with the numerical results of the previous computations both for the fixed points and the critical indices.

Finally, we investigate the flow around the fixed points in terms of the eigenoperators of the linearized flow. It turns out that the eigenoperators at the double well fixed point are approximately normal ordered powers in a small field region but show exponential decay at large fields.

Hierarchical models have been investigated by many authors. Closest to our approach is the constructive work of Koch and Wittwer on the nontrivial fixed point \cite{KochWittwer}. The bifurcation picture is also described within the framework of a continuous hierarchical RG (as opposed to the lattice approach) in the article of Felder \cite{Felder}. Another investigation of the \( \epsilon \)-expansion for the double well fixed point was done by Collet and Eckmann \cite{ColletEckmann}.

The present contribution to the subject is an investigation of the flow on the computer in terms of an algebraic formulation both for the fixed points, the critical indices, the eigenoperators, and the \( \epsilon \)-expansion.

2 The Model

We consider hierarchical RG transformations, defined by

\[
F \mapsto F', \quad F'(\phi) = \int d\mu_\gamma(\zeta) F(\zeta + \beta \phi)^{L^d}, \quad \beta = L^{1-d/2}.
\]

In Appendix 1, we discuss the hierarchical model as an approximation of a field theoretic model and motivate the hierarchical RG transformation.

\( d\mu_\gamma \) denotes the 1-dimensional Gaussian measure with covariance \( \gamma > 0 \), defined through

\[
\int d\mu_\gamma(x)f(x) = (2\pi\gamma)^{-1/2} \int_{-\infty}^{\infty} dx \exp \left( -\frac{x^2}{2\gamma} \right) f(x).
\]

\( L \) denotes the block size of the corresponding blocking transformation. It is not difficult to convince oneself that

\[
F_{HT}(\phi) = L^{1/(L^d-1)} \cdot \exp \left( -\frac{L^2 - 1}{2\gamma L^d} \phi^2 \right).
\]
is a fixed point of the transformation eq. (4). $F_{HT}$ is called high temperature fixed point. Let us write the Boltzmannian $F$ as a product,

$$F(\phi) = F_{HT}(\phi) Z(\phi). \quad (4)$$

The RG transformation of $Z$ is again of the type eq. (4),

$$Z \mapsto Z', \quad Z'(\phi) = \int d\mu(\zeta) Z(\zeta + \beta' \phi)^{L^d}. \quad (5)$$

with the changed parameters $\gamma'$ and $\beta'$ given by

$$\beta' = L^{-1-d/2}, \quad \gamma' = L^{-2} \gamma. \quad (6)$$

In the following, we shall always choose $L^d = 2$. As we shall see below, this choice will lead to quadratic fixed point equations. Note, however, that it is not obvious how to realize a block spin transformation for such a scale factor on a lattice. For $L^d = 2$, we have

$$\beta = 2^{-(d-2)/(2d)}, \quad \beta' = 2^{-(d+2)/(2d)}. \quad (7)$$

Furthermore, we choose $\gamma$ such that $\gamma' = \frac{1}{2}(1 - \beta'^2)$, i.e.,

$$\gamma = \frac{1}{2} \left( 2^{2/d} - \frac{1}{2} \right). \quad (8)$$

This type of conventions agrees with those of ref. [5]. We study in this paper the fixed points of the RG transformation eq. (5) that are even in $\phi$, i.e., the even solutions of the fixed point equation

$$Z(\phi) = \int d\mu(\zeta) Z(\zeta + \beta' \phi)^{L^d}. \quad (9)$$

Let us remark that if $Z$ is a solution of the fixed point equation (5), then $Z_\delta$, where $Z_\delta(\phi) := Z(\delta \phi), \delta > 0$, is a solution of eq. (5) if $\gamma'$ is replaced by $\gamma' \delta^{-2}$. The consequence of dealing with $Z$ instead of $F$ is that $\beta$ is replaced by $\beta' < \beta$. This has important consequences in the constructive approach of Koch and Wittwer [5].

### 3 Computation of Nontrivial RG Fixed Points

A necessary ingredient for a successful study of the transformation (4) is a good choice of coordinates. Good here means that already a reasonable number of coupling constants provide a good approximation of the full problem. Furthermore, good coordinates should be also easy to deal with in practical calculations.

To study the transformation (4), one might consider the expansion of $F(\phi)$ in powers of $\phi$. However, if such an expansion is truncated at a finite order, the resulting approximation will probably not define a reasonable (i.e., positive) Boltzmannian. The problem is that
the coefficients change their sign from order to order, like in the Taylor expansion of $\exp(-x^2)$.

A good choice (though perhaps not the best) is to use the split eq. (4), and consider a Taylor expansion of $Z(\phi)$ in powers of $\phi$. This choice of coordinates has been successfully employed in a rigorous proof that the transformation (11) has a nontrivial fixed point in $d = 3$ dimensions [5]. It has the advantage that the expansion coefficients do not fluctuate in sign. Furthermore, the correct large $\phi$ behavior of the Boltzmannian is implemented automatically.

3.1 Expansion of $Z$ in Powers of $\phi^2$

Let us define a rescaled function $P(\phi)$ through

$$Z(\phi) = P\left(\phi/\sqrt{2\gamma}\right).$$

(10)

The RG transformation for $P$ is

$$P \rightarrow P', \quad P'(\phi) = \int d\mu_{1/2}(\zeta) P(\zeta + \beta'\phi)^2.$$  

(11)

We expand $P(\phi)$ in powers of $\phi^2$,

$$P(\phi) = \sum_{l \geq 0} p_l \frac{\phi^{2l}}{\sqrt{(2l)!}}.$$  

(12)

The specific choice of normalization of the $p_l$ in eq. (12) turned out to be suitable for the numerical fixed point solver to be described below. In terms of the expansion coefficients $p_l$, the RG reads

$$p_l \rightarrow p'_l = \sum_{m,n} S_{mn}^{lm} p_m p_n,$$  

(13)

with

$$S_{lm}^{mn} = \begin{cases} \beta'^2 \frac{(\frac{1}{2})^{l+m+n}}{(m+n-l)! \sqrt{(2l)! (2m)! (2n)!}} & \text{if } 0 \leq l \leq m + n, \\ 0 & \text{else}. \end{cases}$$

(14)

If we look for a RG fixed point, we have to study the infinite set of quadratic equations

$$0 = p_l^* - \sum_{m,n} S_{lm}^{mn} p_m^* p_n^*.$$  

(15)

3.2 Numerical Solution of the Fixed Point Equations

A straightforward numerical treatment of the problem defined by eq. (15) becomes possible if we truncate the sum over $l$ in eq. (12), introducing a highest index $l_{\text{max}}$.

$$P(\phi) = \sum_{l=0}^{l_{\text{max}}} p_l \frac{\phi^{2l}}{2^l \sqrt{(2l)!}}.$$  

(16)
Then, of course, also the sums over $m$ and $n$ run only from 0 to $l_{\text{max}}$, and the fixed point problem consists in the study of $l_{\text{max}} + 1$ quadratic equations,

$$
 f_l = 0, \quad 0 \leq l \leq l_{\text{max}},
 f_l = p_l - \sum_{m,n \leq l_{\text{max}}} S_{lmn} S_{lmn} p_m p_n.
$$

(17)

We shall denote solutions of these equations by $p^*_l$. We used the routine C05NBF from the NAGLIB library for a numerical solution of the fixed point equations in the range $2 < d < 4$. Notice that the dimension dependence enters through the $\beta'$-dependence of the structure coefficients $S_{lmn}$. The program requires an initial guess of the solution. If the program is successful, it returns a solution, together with the values of the $f_l$ for this particular solution. In table 1, we show as an example the $p^*_l$ for $l_{\text{max}} = 10$ and $d = 3$, together with the $f^*_l$ that can be considered as a measure of the error of the solution.

We used truncation parameters $l_{\text{max}}$ in the range 10...50. For the 2-well fixed points, $l_{\text{max}} = 20$ was completely sufficient. To get an impression of the finite $l_{\text{max}}$ effects, the reader is invited to study table 2 where we give the first 11 coefficients of the 2-well fixed point in three dimensions for $l_{\text{max}} = 10, 20, and 30.$

We define an effective potential $V(\phi)$ through

$$
 V(\phi) := -\ln \left( \frac{F(\phi)}{F(\phi = 0)} \right).
$$

(18)

In figure 1, we show our results for the 2-well potential for $d = 2.1$ through $d = 3.8$ in steps of 0.1. In all cases we used $l_{\text{max}} = 20$. The deepest potential corresponds to $d = 2.1$. With increasing dimension, the 2-well gets flatter and flatter until it vanishes in four dimensions.

From naive power counting and the studies of section 5, one expects that $n$-well fixed points occur when the dimension goes below the threshold $d_n = 2n/(n - 1)$. Note that

| $l$ | $p^*_l$ | $f^*_l$ |
|-----|---------|---------|
| 0   | 0.752806717034 · 10$^{-10}$ | 0.423398205312 · 10$^{-16}$ |
| 1   | 0.481272697982 · 10$^{-10}$ | 0.113124959171 · 10$^{-17}$ |
| 2   | 0.313506765870 · 10$^{-10}$ | 0.347242613663 · 10$^{-17}$ |
| 3   | 0.186261032043 · 10$^{-10}$ | 0.179714118083 · 10$^{-17}$ |
| 4   | 0.10696164171 · 10$^{-10}$ | 0.395262886632 · 10$^{-17}$ |
| 5   | 0.49270725225 · 10$^{-10}$ | 0.11750777471 · 10$^{-16}$ |
| 6   | 0.22892987663 · 10$^{-10}$ | 0.57051788570 · 10$^{-17}$ |
| 7   | 0.97756379148 · 10$^{-10}$ | 0.622745240266 · 10$^{-17}$ |
| 8   | 0.390718140134 · 10$^{-10}$ | 0.471372511658 · 10$^{-17}$ |
| 9   | 0.1454689930 · 10$^{-10}$ | 0.307351198867 · 10$^{-17}$ |
| 10  | 0.515497714660 · 10$^{-10}$ | 0.12253903732 · 10$^{-17}$ |

Table 1: The $p^*_l$ and $f^*_l$ for $l_{\text{max}} = 10$ in $d = 3$ dimensions
Table 2: The first 11 coefficients $p_l^* \, $ of the 2-well in $d = 3$ dimensions for three different values of $l_{\text{max}}$

| $l$ | $l_{\text{max}} = 10$           | $l_{\text{max}} = 20$           | $l_{\text{max}} = 30$           |
|-----|--------------------------------|--------------------------------|--------------------------------|
| 0   | $0.752806717034 \cdot 10^{-0}$ | $0.752859732932 \cdot 10^{-0}$ | $0.752859732933 \cdot 10^{-0}$ |
| 1   | $0.481272697982 \cdot 10^{-0}$ | $0.481191004612 \cdot 10^{-0}$ | $0.481191004610 \cdot 10^{-0}$ |
| 2   | $0.313506765870 \cdot 10^{-0}$ | $0.313445974082 \cdot 10^{-0}$ | $0.313445974081 \cdot 10^{-0}$ |
| 3   | $0.186261032043 \cdot 10^{-0}$ | $0.186254920100 \cdot 10^{-0}$ | $0.186254920100 \cdot 10^{-0}$ |
| 4   | $0.100696164171 \cdot 10^{-0}$ | $0.100729195011 \cdot 10^{-0}$ | $0.100729195011 \cdot 10^{-0}$ |
| 5   | $0.499270725225 \cdot 10^{-1}$ | $0.499755196476 \cdot 10^{-1}$ | $0.499755196485 \cdot 10^{-1}$ |
| 6   | $0.228929876623 \cdot 10^{-1}$ | $0.229416956222 \cdot 10^{-1}$ | $0.229416956222 \cdot 10^{-1}$ |
| 7   | $0.97756329148 \cdot 10^{-2}$  | $0.981866078064 \cdot 10^{-2}$ | $0.981866078150 \cdot 10^{-2}$ |
| 8   | $0.390718410134 \cdot 10^{-2}$ | $0.394316100256 \cdot 10^{-2}$ | $0.394316100333 \cdot 10^{-2}$ |
| 9   | $0.146546809430 \cdot 10^{-2}$ | $0.149410716099 \cdot 10^{-2}$ | $0.149410716173 \cdot 10^{-2}$ |
| 10  | $0.515497714660 \cdot 10^{-3}$ | $0.536648340900 \cdot 10^{-3}$ | $0.536648341727 \cdot 10^{-3}$ |

exactly at these thresholds the operators $\phi^{2n}$ become relevant with respect to the Gaussian fixed point. Thus 2-wells are expected to exist for $d < 4$, 3-wells for $d < 3$, 4-wells for $d < 8/3$ and so on.

By starting our program with a suitable initial guess, we were able to find the 3-well potentials. Figure 2 shows the results for $d = 2.1$ through $d = 2.7$ in steps of 0.1. The deepest potential corresponds to $d = 2.1$, the flattest one to $d = 2.7$. In all cases we used again $l_{\text{max}} = 20$.

The search for the 4-wells was a bit more difficult. It turned out that one needed more than 20 couplings for a reasonable parameterization. Figure 3 shows the 4-well potential in $d = 2.1$ and in $d = 2.3$ dimensions. We used $l_{\text{max}} = 30$ in both cases.

In all cases, we made a further check on our results as follows: We represented the Boltzmannian $F$ by its values on a grid of typically 300 sites on the interval $0 \leq \phi \leq \Lambda$, where $\Lambda$ was typically 10 or 20. The RG steps were then done by numerically performing the integral using the NAGLIB routine D01GAF. This routine determines an approximation of an integral if the integrand is given on a finite number of points. All fixed points determined by the algebraic method were converted to functions on the $\phi$-grid and then checked for stability under iterated application of the integration method.

4 The Linearized Renormalization Group

In this section we shall report on a numerical study of the eigenvalues of the hierarchical RG, linearized around the nontrivial fixed points. Note that the eigenvalues related to the full transformation (1) are the same as those related to the transformation (5) for $Z$. The reason is that fixed points $F^*$ and $Z^*$ differ only by the factor $F_{HT}$ that is a fixed point.
Table 3: The 6 leading eigenvalues $a_i$ at $d = 2.1$ for the 2-well, the 3-well, and the 4-well, respectively.

| $i$ | 2-well     | 3-well     | 4-well     |
|-----|------------|------------|------------|
| 0   | 2.1000000  | 2.1000000  | 2.1000000  |
| 1   | 0.4787297  | 1.9715290  | 1.9851950  |
| 2   | -1.1724335 | 0.4611953  | 1.0966180  |
| 3   | -3.3445644 | -0.7546772 | 0.4105210  |
| 4   | -5.8303542 | -2.2270214 | -5.6565547 |
| 5   | -8.4946711 | -3.9790124 | -1.7706071 |

The matrix $R$ is not symmetric. It can, however, be shown that its eigenvalues are real, cf. lemma 5.4 below. We used the NAGLIB procedure F01AKF and F02APF to compute all eigenvalues of $\lambda_i$ of $R$, with $i = 0, \ldots, l_{\text{max}}$. It turns out that all eigenvalues are positive and can therefore be exponentiated. We define

$$\lambda_i = L^{a_i},$$

where $L$ denotes the block size, in our case $2^{1/d}$.

Table 3 shows, as an example, the first five eigenvalues $a_i$ of the linearized RG around the 2-well, the 3-well, and the 4-well in $d = 2.1$ dimensions, respectively. Positive eigenvalues $a_i$ are called relevant, negative eigenvalues are called irrelevant, and a zero eigenvalue is called marginal. In accordance with the expectation, the $n$-wells have $n - 1$ (nontrivial) relevant eigenvalues. One always observes an eigenvalue $a_0 = d$ that corresponds to the trivial volume operator. It corresponds to the fact that the fixed point $Z^*$ itself is an eigenvector with eigenvalue $\lambda_0 = 2$. The next eigenvalue $a_1$ is related to the critical exponent $\nu$, via

$$\nu = \frac{1}{a_1},$$

where $L$ denotes the block size, in our case $2^{1/d}$.
Table 4: Results for the critical exponent $\nu$ in the hierarchical model for the 2-well

| $d$ | $\nu$   | $d$ | $\nu$   |
|-----|---------|-----|---------|
| 2.1 | 2.08886 | 3.0 | 0.64957 |
| 2.2 | 1.36234 | 3.1 | 0.62570 |
| 2.3 | 1.09916 | 3.2 | 0.60484 |
| 2.4 | 0.95704 | 3.3 | 0.58640 |
| 2.5 | 0.86534 | 3.4 | 0.56995 |
| 2.6 | 0.79985 | 3.5 | 0.55516 |
| 2.7 | 0.74993 | 3.6 | 0.54182 |
| 2.8 | 0.71011 | 3.7 | 0.52973 |
| 2.9 | 0.67729 | 3.8 | 0.51877 |

In table 4 we show our results for the exponent $\nu$ for $d = 2.1$ through $d = 3.8$ in the case of a 2-well. It is interesting to compare the $d = 3$ result with the exponent $\nu$ for the 3-dimensional Ising model. The best known estimates for the latter are in the range $0.624 \ldots 0.630$ [10]. For $d \to 2$, the deviation of the exponent $\nu$ from the value in the full model increases. We know from the exact solution of the 2-dimensional Ising model that $\nu = 1$ for $d = 2$. In the hierarchical model we observe $\nu > 2$ already at $d = 2.1$.

It is interesting to look also at the eigenvectors of the linearized RG transformation. As an example we consider the eigenvectors of the transformation eq. (1), linearized around the trivial fixed point $F^*(\phi) = 1$ and around the nontrivial 2-well fixed point in $d = 3$ dimensions. The eigenvectors $O_i^{(0)}(\phi)$ for the linearization around $F^* = 1$ can be given exactly:

$$O_i^{(0)}(\phi) = \text{const} \cdot H_{2n} \left( \frac{\phi}{2\tilde{\gamma}} \right), \quad (24)$$

with

$$\tilde{\gamma} \equiv \frac{\gamma}{1 - \beta^2}. \quad (25)$$

The $H_k$ denote the Hermite polynomials. We compare these functions for $i = 1, 2$ with the corresponding eigenvectors of the RG linearized around the 2-well fixed point in $d = 3$, see figure 5. The figure shows that the eigenvectors in $d = 3$ are of similar shape as the corresponding functions for the trivial fixed point. Note, however, that the “full” eigenvectors contain a factor $F_{HT}$ and thus have a completely different large $\phi$ behavior than the Hermite polynomials.

5 $\epsilon$-Expansion

In this section, we want to compare the numerical results obtained in the previous section with expansions in $\epsilon = d^* - d$, where $d^*$ is one of the threshold dimensions $d_n = 2n/(n-1)$. 


This section is organized as follows: In subsection 5.1 a parametrization of the hierarchical Boltzmannian is introduced that is suitable for the \( \epsilon \)-expansion. The recursion relations for the expansion coefficients \( a^{(k)}_l \) shall be derived in subsection 5.2. In subsection 5.3 we will derive the recursion relations for the computation of the \( \epsilon \)-expansion for the eigenvalues and eigenvectors of the linearized \( \epsilon \)-expansion. In subsection 5.4 we shall give some comparisons of the \( \epsilon \)-expansion results with the numerical results.

### 5.1 Expansion of \( F \) in Wick Monomials

We consider the expansion of the Boltzmannian \( F \) in terms of normal ordered powers of \( \phi^2 \). This expansion shall be the basis for the \( \epsilon \)-expansion to be studied below. One expands in terms of normal ordered monomials

\[
F(\phi) = \sum_{l \geq 0} \frac{a_l}{\gamma^l} : \phi^{2l} :_{\bar{\gamma}}.
\]

(26)

The reason to divide the coefficients \( a_l \) by \( \gamma^l \) is that \( a_l \) becomes \( \gamma \)-independent for the fixed point. Under a RG step the coordinates \( a_l \) transform according to

\[
a_l \rightarrow a'_l = \beta^{2l} \sum_{m,n} C_{lmn} a_m a_n.
\]

(27)

The sum in eq. (27) is restricted to \( |m - n| \leq l \leq m + n \), and the ‘structure coefficients’ are given by

\[
C_{lmn} = \frac{(2m)! (2n)!}{(m + n - l)! (l + n - m)! (l + m - n)!}.
\]

(28)

Normal ordering with respect to a covariance \( \bar{\gamma} \) is defined through

\[
: \exp(a \phi) :_{\bar{\gamma}} = \exp \left( -\frac{\bar{\gamma}}{2} a^2 + a \phi \right).
\]

(29)

The normal ordered powers of \( \phi \) can be expressed in terms of Hermite polynomials,

\[
: \phi^n :_{\bar{\gamma}} = \left( \frac{\bar{\gamma}}{2} \right)^{n/2} H_n \left( \frac{\phi}{\sqrt{2\bar{\gamma}}} \right).
\]

(30)

That we use a direct expansion of \( F \) as a basis for the \( \epsilon \)-expansion seems to be in contradiction to our statement above that it should be much better to use the split \( F = F_{HT} Z \), and then expand \( Z \). The \( \epsilon \)-expansion is, however, an expansion about \( F = 1 \) (and \textit{not} about \( Z = 1 \)). Furthermore, as we shall see from the recursion relations to be derived below, the effective Boltzmannians to a given order in \( \epsilon \) live in a finite dimensional space of coupling constants. It is thus irrelevant which coordinates are chosen in this space as long as they span this space.
5.2 $\epsilon$-expansion for the $a_l$

In this subsection we derive the recursive relations for the coefficients $a_l^{(k)}$ defined through

$$ a_l = \sum_{k=0}^{\infty} a_l^{(k)} \epsilon^k. $$

(31)

We determine by $\epsilon$-expansion the coefficients of the infrared fixed point at $d = d^* - \epsilon$ dimensions starting from the trivial fixed point in $d^* := 2l^*/(l^* - 1)$ dimensions, where $l^* \in \{2, 3, \ldots\}$. The following lemma shows how to compute the $a_l^{(k)}$ recursively.

**Lemma 5.1** Suppose that the coefficients $(\beta - 2l)^{(m)}$ are defined by

$$ \beta - 2l = 2l^* \sum_{m=0}^{\infty} (\beta - 2l)^{(m)} \epsilon^m. $$

(32)

For $l^* \in \{2, 3, \ldots\}$ and $a_l^{(0)} = \delta_{l,0}$, we have

$$ a_l^{(1)} = \alpha \delta_{l,l^*}, \quad \alpha := \frac{2(\beta - 2l^*)^{(1)}}{C_{l^*}^{l^*}}; $$

(33)

and, for all $l \neq l^*$, $N \geq 1$,

$$ a_l^{(N)} = \frac{1}{2l^* - 2} \sum_{k=1}^{N-1} \left( -2l^* (\beta - 2l)^{(N-k)} a_l^{(k)} + \sum_{m,n} C_{l^*}^{mn} a_m^{(N-k)} a_n^{(k)} \right). $$

(34)

For all $N \geq 2$,

$$ a_{l^*}^{(N)} = -\frac{1}{2(\beta - 2l^*)^{(1)}} \left[ -2 \left( (\beta - 2l^*)^{(N)} \alpha + \sum_{k=2}^{N-1} (\beta - 2l^*)^{(N+1-k)} a_{l^*}^{(k)} \right) + \sum_{k=2}^{N-1} \sum_{m,n} C_{l^*}^{mn} a_m^{(N+1-k)} a_n^{(k)} + \right. \\
+ \left. 2 \sum_{m: m \neq l^*} C_{l^*}^{ml^*} a_m^{(N)} \alpha \right]. $$

(35)

**Proof:** The $N$th order term of the fixed point equation

$$ \beta - 2l a_l = \sum_{m,n} C_{l^*}^{mn} a_m a_n $$

(36)

is given by

$$ 2l^* \sum_{k=0}^{N} (\beta - 2l)^{(N-k)} a_l^{(k)} = \sum_{k=0}^{N} \sum_{m,n} C_{l^*}^{mn} a_m^{(N-k)} a_n^{(k)}. $$

(37)
Since \( a_l^{(0)} = \delta_{l,0} \) and \((\beta^0)^{(n)} = 0\) for \( n > 0 \), we obtain

\[
2^{l^*} \left( a_l^{(N)} + \sum_{k=1}^{N-1} (\beta - 2l^*) (N-k) a_l^{(k)} \right) = 2a_l^{(N)} + \sum_{k=1}^{N-1} \sum_{m,n} c_l^{mn} a_m^{(N-k)} a_n^{(k)}.
\]

(38)

We have used \( c_l^{m0} = \delta_{m,l} \). Eq. (38) implies for \( l \neq l^* \) eq. (34). Consider eq. (38) for the case \( N = 1 \):

\[
2^{l^*} a_l^{(1)} = 2 a_l^{(1)}.
\]

(39)

This implies \( a_l^{(1)} = \alpha \delta_{l,l^*} \). To determine the constant \( \alpha \), we have to consider eq. (38) for the case \( N = 2, l = l^* \):

\[
2 (\beta - 2l^*) a_l^{(1)} = C_l^{l^* l^*} (a_l^{(1)})^2.
\]

(40)

This implies

\[
\alpha := \frac{2 (\beta - 2l^*) a_l^{(1)}}{C_l^{l^* l^*}}.
\]

(41)

Let us consider eq. (38) for the case \( N \geq 3, l = l^* \):

\[
2 \sum_{k=1}^{N-1} (\beta - 2l^*) (N-k) a_l^{(k)} = \sum_{k=1}^{N-1} \sum_{m,n} c_l^{mn} a_m^{(N-k)} a_n^{(k)}.
\]

(42)

This implies

\[
2 \left[ (\beta - 2l^*) (N-1) a_l^{(N-1)} + (\beta - 2l^*) a_l^{(N-1)} + \sum_{k=2}^{N-2} (\beta - 2l^*) (N-k) a_l^{(k)} \right]
= 2 \sum_m C_l^{l^* l^*} a_m^{(N-1)} a_l^{(N-1)} + \sum_{k=2}^{N-2} \sum_{m,n} C_l^{mn} a_m^{(N-k)} a_n^{(k)}.
\]

(43)

Thus,

\[
a_l^{(N-1)} = \frac{1}{2((\beta - 2l^*) (1) - C_l^{l^* l^*} \alpha)} \left[ -2 \left( (\beta - 2l^*) (1) + \sum_{k=2}^{N-2} (\beta - 2l^*) (N-k) a_l^{(k)} \right) + \sum_{m: m \neq l^*} C_l^{l^* l^*} a_m^{(N-1)} + \sum_{k=2}^{N-2} \sum_{m,n} C_l^{mn} a_m^{(N-k)} a_n^{(k)} \right].
\]

(44)

Using \( C_l^{l^* l^*} \alpha = 2 (\beta - 2l^*) (1) \) and replacing \( N \) by \( N + 1 \), we obtain eq. (35). \( \square \)

With the following lemma we provide explicit expressions for the \( \epsilon \)-expansion of \( \beta - 2l \).
Lemma 5.2 Suppose that the coefficients $(\beta^{-1})^{(m)}$ of the $\epsilon$-expansion for the term $\beta^{-l}$ are defined by
\[
\beta^{-l} = 2^\frac{l}{d^*} \sum_{m=0}^{\infty} (\beta^{-1})^{(m)} \epsilon^m. \tag{45}
\]
Then, we have $(\beta^{-1})^{(0)} = 1$ and
\[
(\beta^{-1})^{(m)} = \sum_{k=0}^{m-1} \frac{1}{(m-k)!} \left( -\frac{l \ln 2}{d^*} \right)^{m-k} \binom{m-1}{k} d^{*m-k}, \tag{46}
\]
for $m \geq 1$.

Proof: We have
\[
\beta^{-l} = 2^\frac{l}{d^*} \exp \left( \frac{l \ln 2}{d^*} \left( 1 - \frac{1}{1 - \frac{\epsilon}{d^*}} \right) \right). \tag{47}
\]
Expansion of the exponential-function on the rhs of eq. (47) gives
\[
\beta^{-l} = 2^\frac{l}{d^*} \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{l \ln 2}{d^*} \right)^n \left( 1 - \frac{1}{1 - \frac{\epsilon}{d^*}} \right)^n \right]. \tag{48}
\]
Furthermore
\[
\left( 1 - \frac{1}{1 - \frac{\epsilon}{d^*}} \right)^n = \left( -\frac{\epsilon}{d^*} \right)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left( \frac{\epsilon}{d^*} \right)^k. \tag{49}
\]
Insertion of eq. (49) into eq. (48) yields
\[
\beta^{-l} = 2^\frac{l}{d^*} \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{l \ln 2}{d^*} \right)^n \binom{n+k-1}{k} \left( \frac{\epsilon}{d^*} \right)^{n+k} \right]. \tag{50}
\]
Introducing a new variable $m = n + k \in \{1, 2, \ldots\}$, we obtain
\[
\beta^{-l} = 2^\frac{l}{d^*} \left\{ 1 + \sum_{m=1}^{\infty} \sum_{k=0}^{m-1} \frac{1}{(m-k)!} \left( -\frac{l \ln 2}{d^*} \right)^{m-k} \binom{m-1}{k} d^{*m-k} \right\} \epsilon^m. \tag{51}
\]
This implies the assertion. \qed

5.3 $\epsilon$-Expansion for the Linearized RG

We shall now consider the eigenvalue problem for the linearized RG equation. The eigenvalues and eigenvectors are computed by using $\epsilon$-expansion.

The linearized RG transformation is given by the matrix $U(a)$, with matrix elements
\[
U_{mn}(a) := 2 \beta^l \sum_{m} C^{(m)}_{nm} a_m. \tag{52}
\]
Lemma 5.3 Consider the eigenvalue equation for the linearized RG equation

\[ U(a)b = \lambda b. \]  

Suppose that the vector \( a \) in \( d = d^* - \epsilon \) dimensions, \( d^* = \frac{2l^*}{l^* - 1}, l^* \in \{2, 3, \ldots\} \) is given by the \( \epsilon \)-expansion

\[ a_m = \sum_{k: k \geq 0} a^{(k)}_m \epsilon^k, \]  

and the \( \epsilon \)-expansion of \( U \) is

\[ U = \sum_k U^{(k)} \epsilon^k. \]  

Suppose that the \( \epsilon \)-expansions of \( b \) and \( \lambda \) are given by

\[ b = \sum_{m: m \geq 0} b^{(m)} \epsilon^m, \quad \lambda = \sum_{n: n \geq 0} \lambda^{(n)} \epsilon^n. \]  

Suppose that \( b^{(0)}_m = \delta_{n,m} \) and \( \lambda^{(0)} = 2^{1-\frac{m}{l}} \), for \( m \in \{0, 1, 2, \ldots\} \). Then, we have

\[ \lambda^{(N)} = \sum_{n=0}^{N-1} \sum_m U^{(N-n)}_{\overline{m}m} b^{(n)}_m, \]  

and, for \( l \neq \overline{m} \),

\[ b^{(N)}_l = \frac{1}{2(2^{1-\frac{m}{l}} - 2^{-\frac{m}{l}})} \sum_{n=0}^{N-1} \sum_m \left( \lambda^{(N-n)} \delta_{l,m} - U^{(N-n)}_{lm} \right) b^{(n)}_m. \]  

Furthermore,

\[ b^{(N)}_{\overline{m}} = 0. \]  

The proof of this lemma will be given later.

Lemma 5.4 \( U(a) \) is symmetric with respect to the canonical scalar product, defined by

\[ (a, b) := \sum_n a_n b_n. \]  

Proof. We have

\[ (u, U(a)v) = \sum_{n,l} u_n T_{nl} v_l, \]  

where

\[ T_{nl} := \frac{\beta^{2n}}{(2n)!} U_{nl}. \]  

Since \( T_{nl} = T_{ln} \), we have

\[ (u, U(a)v) = (U(a)u, v). \square \]  

Lemma 5.4 shows that the eigenvalues of \( U(a) \) are real. The next lemma shows how to compute the \( \epsilon \)-expansion for the eigenvectors and eigenvalues.
Lemma 5.5 Suppose that the linearized RG group equation is given by the following series expansion

$$U(a) = \sum_{n: n \geq 0} U^{(n)}(a) \epsilon^n$$

(64)

and that the 0th order term $U^{(0)}$ is symmetric. Let $b$ be an eigenvector with eigenvalue $\lambda$ of $U(a)$, i.e.

$$U(a)b = \lambda b.$$  

(65)

Let $b^{(0)}$ be a normalized eigenvector with eigenvalue $\lambda^{(0)}$ of $U^{(0)}(a)$, i.e.

$$U^{(0)}(a)b^{(0)} = \lambda^{(0)}b^{(0)}, \quad (b^{(0)}, b^{(0)}) = 1,$$

(66)

where the scalar product $(\ , \ )$ is defined by eq. (60). Suppose that the eigenvalue $\lambda^{(0)}$ is not degenerate. Then, there exists an $\epsilon$-expansion for the eigenvector $b$ with eigenvalue $\lambda$

$$b = \sum_m b^{(m)} \epsilon^m, \quad \lambda = \sum_n \lambda^{(n)} \epsilon^n$$

(67)

such that

$$(b^{(0)}, b^{(N)}) = 0,$$

(68)

for all $N > 0$. The coefficients $b^{(N)}$ and $\lambda^{(N)}$ are recursively determined by

$$\lambda^{(N)} = \sum_{n=0}^{N-1} (b^{(0)}, U^{(N-n)}b^{(n)})$$

(69)

and

$$b^{(N)} = (U^{(0)} - \lambda^{(0)})^{-1} \left[ \sum_{n=0}^{N-1} (U^{(N-n)} - \lambda^{(N-n)})b^{(n)} \right] \perp,$$

(70)

where $u \perp$ is the component of $u$ perpendicular to $b^{(0)}$. $\square$

Proof: The eigenvalue equation implies, for all $N$,

$$\sum_{m, n: m+n=N} (U^{(m)}b^{(n)} - \lambda^{(m)}b^{(n)}) = 0.$$  

(71)

Thus,

$$(U^{(0)} - \lambda^{(0)})b^{(N)} = \sum_{n=0}^{N-1} (U^{(N-n)} - \lambda^{(N-n)})b^{(n)}.$$  

(72)

Since $(U^{(0)} - \lambda^{(0)})b^{(0)} = 0$, we may add to $b^{(N)}$, $N > 0$ on the lhs of eq. (72) a multiple of vector $b^{(0)}$ such that eq. (68) holds. Scalar multiplication of eq. (72) with $b^{(0)}$ gives

$$(b^{(0)}, (U^{(0)} - \lambda^{(0)})b^{(N)}) = \sum_{n=0}^{N-1} (b^{(0)}, (U^{(N-n)} - \lambda^{(N-n)})b^{(n)}).$$  

(73)
Since $U^{(0)}$ is symmetric, the lhs of eq. (73) is zero. Therefore eq. (69) is valid. Since the rhs of eq. (72) is perpendicular to $b^{(0)}$, eq. (70) can be computed by eq. (72). $\blacksquare$

The next lemma presents the $\epsilon$-expansion of the linearized RG transformation $U(a)$.

**Lemma 5.6** Let the $\epsilon$-expansion of $\beta^{2l}$ be

$$\beta^{2l} = 2^{-\frac{l}{l^*}} \sum_{m=0}^{\infty} (\beta^{2l})^{(m)} \epsilon^m, \quad (\beta^{2l})^{(0)} = 1,$$

for $l^* \in \{2, 3, \ldots\}$. Then the $\epsilon$-expansion of $U$

$$U = \sum_k U^{(k)} \epsilon^k$$

is explicitly given by

$$U^{(k)}_{nl} = 2^{1-\frac{k}{l^*}} \sum_{r=0}^{k} \sum_{m} C^{mn}_{l}(\beta^{2l})^{(k-r)} a^{(r)}_{m}.$$

The proof of the foregoing lemma follows immediately from the definitions.

For the recursive computation of the coefficients for the eigenvalues and eigenvectors we need the start values of the recursion relations which are eigenvalues and eigenvectors of $U^{(0)}$.

**Lemma 5.7** Suppose that $a^{(0)}_m = \delta_{m,0}$. The normalized eigenvectors $b^{(0)}$ with eigenvalues $\lambda^{(0)}$ of $U^{(0)}$ are

$$b^{(0)}_m = \delta_{m,\overline{m}}, \quad \lambda^{(0)} = 2^{1-\frac{1}{l^*}},$$

for all $\overline{m} \in \{0, 1, 2, \ldots\}$.

**Proof:** We have, using $a^{(0)}_m = \delta_{m,0}$, $C^{0m}_l = \delta_{l,n}$

$$U^{(0)}_{nl} = 2^{1-\frac{1}{l^*}} \delta_{n,l}.$$

Thus

$$(U^{(0)} b^{(0)})_n = 2^{1-\frac{1}{l^*}} \delta_{n,\overline{m}} = 2^{1-\frac{1}{l^*}} b^{(0)}_n. \quad \Box$$

We finish this subsection with the proof of [83].

**Proof of Lemma 5.2.** By Lemma 5.5, eq. (83), follows eq. (77). Since

$$U^{(0)}_{nl} = 2^{1-\frac{1}{l^*}} \delta_{n,l}, \quad \lambda^{(0)} = 2^{1-\frac{1}{l^*}},$$

(80)
we have
\[(U^{(0)} - \lambda^{(0)})_{nl} = 2(2^{-l} - 2^{-l^*})\delta_{n,l} .\]  
Therefore
\[(U^{(0)} - \lambda^{(0)})_{nl}^{1} = \frac{1}{2(2^{-l} - 2^{-l^*})}\delta_{n,l} .\]  
Thus, eq. (70) of Lemma 5.5 implies eq. (58). \(\square\)

### 5.4 Comparison of \(\epsilon\)-Expansion with Numerical Results

We evaluated the recursion relations presented in the preceding subsection using the computer algebra program Maple V Release 2. This allowed us to go to relatively high order. We always used programs that computed everything exactly (in the form of analytical expressions) and programs that solved the recursion relations numerically. Note however, that Maple allows for arbitrary high precision in the numerical computations. It was no problem to compute the coefficients \(a^{(n)}_l\) for \(d^* = 4\) exactly to sixth order in \(\epsilon\). However, the expressions become quite nasty then. As an example we present the coefficients \(a^{(n)}_l\) up to \(n = 3\) in Appendix 2. The general structure for the expansion around \(d^* = 4\) is that at a given order \(n\) the only nonvanishing coefficients are those with \(l \leq 2n\). The corresponding relation at \(d^* = 3\) is \(l \leq 3n\). The \(\epsilon\)-expansion is expected to have zero convergence radius. However, the series are believed to be Borel-summable. For small \(\epsilon\) even the naively summed low order series can be a quite good approximation. In figure 4 we show the comparison of the ‘true’ 2-well potential at \(d = 3.8\) with the 1st and 4th order \(\epsilon\)-expansion. The full line gives the result obtained numerically, and the dashed lines give the 1st and 4th order approximations, respectively.

With the numerical version of the program it was no problem to go to orders like 16. In table 5 we show the intimidating growth of the expansion coefficients when the order becomes large.

| \(n\) | \(a^{(n)}_1\) | \(a^{(n)}_2\) |
|------|----------------|----------------|
| 10   | \(4.59314 \cdot 10^4\) | \(-2.89444 \cdot 10^2\) |
| 11   | \(-3.41664 \cdot 10^2\) | \(2.26283 \cdot 10^3\) |
| 12   | \(2.70284 \cdot 10^3\) | \(-1.87364 \cdot 10^4\) |
| 13   | \(-2.26138 \cdot 10^4\) | \(1.63588 \cdot 10^5\) |
| 14   | \(1.99286 \cdot 10^5\) | \(-1.50104 \cdot 10^6\) |
| 15   | \(-1.84404 \cdot 10^6\) | \(1.44371 \cdot 10^7\) |
| 16   | \(1.78725 \cdot 10^7\) | \(-1.45243 \cdot 10^8\) |

Table 5: Numerical results of \(a_1\) and \(a_2\) for \(n\) from 10 to 16.
Table 6: Comparison of the $\epsilon$-expansion coefficients of $\nu$ in the hierarchical model $\nu_i$ with the ones in the full model $\nu_{i,f}$

| $i$ | $\nu_i$ | $\nu_{i,f}$ |
|-----|---------|-------------|
| 0   | 0.5000  | 0.5000      |
| 1   | 0.0833  | 0.0833      |
| 2   | 0.0556  | 0.0445      |
| 3   | -0.0324 | -0.0190     |
| 4   | 0.1468  | 0.0888      |
| 5   | -0.5743 | -0.2015     |

With the help of the recursion relations of the preceding subsections we also determined the $\epsilon$-expansion for the exponent $\nu$. We again used an exact version of the program that was practicable up to order 6, and a numerical version that could be used to higher order.

In table 6 we show our results for the expansion coefficients of $\nu$, compared with those of the full model [11]. The first two orders are exactly the same (the coefficient $\nu_1$ is $1/12$ in the hierarchical and in the full model). This might be due to the fact that the $\epsilon$-expansion of the exponent $\eta$ starts at order $\epsilon^2$.

In table 6 we give the results of resummed series for $\nu$ up to order $\epsilon^k$, for $k \leq 5$, and $d = 3.0 \ldots 3.8$. For the larger values of $\epsilon$, the signal of the divergence of the series is obvious. For comparison we also quote our numerical result (‘true’) and the result of a Borel-Padé summation of the 6th order $\epsilon$ expansion (BP). The latter was obtained as follows: The $\nu_i$ were divided by $i!$, and the diagonal Padé approximation of the resulting Taylor series was determined. From the resulting rational function $Q(\epsilon)$ the estimate for $\nu$ was then obtained by numerically computing

$$\nu(\epsilon) = \int_0^\infty dt \exp(-t) Q(t\epsilon).$$  \hspace{1cm} (83)

There is a quite good agreement with the ‘true’ results. (For $d = 3$, the diagonal Padé of the Borel transform had a nonintegrable singularity on the positive real axis.)

6 Conclusions

In this paper, we have demonstrated that at least for hierarchical models, an algebraic computation of fixed points and exponents is feasible. An extension to $N$-component models and general values of $L$ could be easily done. Of course, many new ideas are necessary to do the same thing in full models. The crucial question here is the proper choice of parametrization of the Boltzmannian. An interesting question which certainly deserves study is whether the $\epsilon$-expansion for the RG flow (in the Wilson sense) in full
models could be performed to an order that is competitive with what has been done in the conventional framework.

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Appendix 1: Field Theory and Hierarchical Models

This appendix discusses the hierarchical model as an approximation of a field theoretic model and motivates the hierarchical RG equation.

The generating functional for Green's functions of a scalar field theory on a $d$-dimensional continuum $\mathbf{R}^d$ is given by the following (formal) infinite-dimensional integral

$$Z[J] := \mathcal{N} \int \prod_{z \in \mathbf{R}^d} d\phi(z) \exp \left[ -\frac{1}{2}(\phi, -\Delta \phi) \right] \exp \left[ -V(\phi) + (J, \phi) \right].$$

(84)

$\phi$ is a scalar field, and $J$ is an external source on $\mathbf{R}^d$. $V(\phi) = \int_{z \in \mathbf{R}^d} V(\phi(z))$ is the local interaction term, and $\frac{1}{2}(\phi, -\Delta \phi)$ is the free part of the action. $\Delta$ is the Laplacian, and $\mathcal{N}$ is a normalization factor chosen such that

$$d\mu_\nu(\phi) := \mathcal{N} \prod_{z \in \mathbf{R}^d} d\phi(z) \exp \left[ -\frac{1}{2}(\phi, -\Delta \phi) \right]$$

(85)
is a normalized Gaussian measure. \( v \) is called propagator. The canonical scalar product is defined by

\[ (\phi, \psi) := \int_{z \in \mathbb{R}^d} \phi(z)\psi(z) , \tag{86} \]

for fields \( \phi, \psi \) on \( \mathbb{R}^d \).

For RG calculations it is more convenient to use generating functionals with external fields \( \psi \),

\[ Z(\psi) := \int d\mu_v(\phi) \exp \left[ -V(\phi + \psi) \right] . \tag{87} \]

The two generating functionals are related by

\[ Z(\psi) = Z[J] \exp \left[ -\frac{1}{2} (J, vJ) \right] \bigg|_{J = v^{-1}\psi} . \tag{88} \]

For the definition of the hierarchical model, we introduce the notion of a hierarchical lattice (or multigrid). For \( L \in \{2, 3, \ldots\} \) and \( j \in \mathbb{Z} \) divide \( \mathbb{R}^d \) into hypercubes of side length \( a_j := L^{-j}a \), where \( a \) is a unit length. Denote the set of all these hypercubes by \( \Lambda_j \). \( \Lambda_j \) can be considered as a lattice with lattice spacing \( a_j \) by identifying the centers of the hypercubes with the lattice sites. The location of the hypercubes can be chosen in the following way. For a hypercube \( y \in \Lambda_j \) let \( \tilde{y} \) be the open hypercube of \( y \). For \( y \in \Lambda_j \) and \( x \in \Lambda_k \) we suppose that \( \tilde{y} \cap x = \emptyset \) or \( \tilde{y} \subseteq x \). In the latter case we write \( y \ll x \). The hierarchical approximation is given by the following replacement

\[ v \rightarrow \sum_{j \in \mathbb{Z}} v^j , \tag{89} \]

where, for \( z, z' \in \mathbb{R}^d \),

\[ v^j(z, z') := a_j^{2-d} \gamma \delta_{x, x'} . \tag{90} \]

Here, \( x \) and \( x' \) are the uniquely determined hypercubes of \( \Lambda_j \) such that \( z \in x \), \( z' \in x' \). \( v^j \) is called fluctuation propagator. The above replacement represents the fact that in general \( v \) can be decomposed into a sum of fluctuation propagators which are exponentially decaying with decay length \( a_j \). This exponential decay is simulated by the Kronecker delta on the rhs of eq. (90). There are other ways to define the hierarchical approximation. But all hierarchical approximations share the property that the fluctuation propagators \( v^j \) have compact support. Insertion of replacement (89) into eq. (87) yields, using the convolution formula of Gaussian measures,

\[ Z(\psi) = \int \prod_{j \in \mathbb{Z}} d\mu_{v^j}(\phi^j) \exp \left[ -V(\sum_{j \in \mathbb{Z}} \phi^j + \psi) \right] . \tag{91} \]

We define an ultraviolet cutoff by setting the propagator \( v^j = 0 \) if \( j > n \) and an infrared cutoff by setting \( v^k = 0 \) if \( k \leq j \):

\[ Z_j^{(n)}(\psi) = \int \prod_{i=j+1}^{n} d\mu_{v^i}(\phi^i) \exp \left[ -V(\sum_{i=j+1}^{n} \phi^i + \psi) \right] . \tag{92} \]
The effective generating functions $Z^{(n)}_{j}$ and $Z^{(n)}_{j-1}$ are related by

$$Z^{(n)}_{j-1}(\psi) = \int d\mu_{\psi}(\phi^{j})Z^{(n)}_{j}(\phi^{j} + \psi).$$  \hspace{1cm} (93)$$

Since we started with a local interaction $V$, the effective generating functions obey the following factorization property

$$Z^{(n)}_{j}(\psi) = \prod_{y: y \in \Lambda_{j}} Z^{(n)}_{j}(y|\psi).$$  \hspace{1cm} (94)$$

Since the kernel of $v^{j}$ is constant on hypercubes of $\Lambda_{j}$, we can assume that the fields $\psi$ are constant arguments of $Z^{(n)}_{j}$, i.e., do not depend on $z \in \mathbb{R}^{d}$. Therefore, eq. (93) is equivalent to

$$Z^{(n)}_{j-1}(x|\psi) = \prod_{y: y \in x} \left[ \int d\mu_{\psi}(\phi)Z^{(n)}_{j}(y|\phi + \psi) \right],$$  \hspace{1cm} (95)$$

for $x \in \Lambda_{j-1}$. For translation invariant models $Z^{(n)}_{j}(y|\phi)$ does not depend on $y \in \Lambda_{j}$. Let us define

$$Z^{(n)}_{j}(\phi) := Z^{(n)}_{j}(y|a_{j-1}^{-\frac{d}{2}} \phi).$$  \hspace{1cm} (96)$$

From eq. (95) and definition eq. (96) follows, using $\psi \rightarrow a_{j-1}^{-\frac{d}{2}} \psi$,

$$Z^{(n)}_{j-1}(\psi) = \left[ \int d\mu_{\gamma}(\phi)Z^{(n)}_{j}(\phi + L^{1-\frac{d}{2}} \psi) \right]^{L^{d}}.$$  \hspace{1cm} (97)$$

For $F' := (Z^{(n)}_{j-1})^{L^{-d}}$ and $F := (Z^{(n)}_{j})^{L^{-d}}$, we obtain the hierarchical RG transformation eq. (1).

**Appendix 2: Some Results of $\epsilon$-Expansion for the $a^{(n)}_{i}$**

$$R := \ln 2$$
$$T := \sqrt{2}$$

$$a_{0}^{(0)} = 1$$
$$a_{0}^{(2)} = \frac{R^{2}}{864}$$
$$a_{0}^{(3)} = \frac{12 R^{2} (3 R - 2) - R^{2} (19 R - 18) T}{-31104 T + 41472}$$
$$a_{1}^{(2)} = \frac{R^{2}}{216 T - 432}$$
\begin{align*}
  a_1^{(3)} &= \frac{-2 R^2 (7 R - 8) + R^2 (7 R - 12) T}{17280 T - 24192} \\
  a_2^{(1)} &= -\frac{R}{144} \\
  a_2^{(2)} &= \frac{12 R (3 R - 2) - R (19 R - 18) T}{-10368 T + 13824} \\
  a_2^{(3)} &= \frac{R (154 R^2 + 1161 R - 459) - 12 R (8 R^2 + 69 R - 27) T}{-746496 T + 1057536} \\
  a_3^{(2)} &= \frac{R^2}{2592 T - 2592} \\
  a_3^{(3)} &= \frac{-8 R^2 (R - 1) + R^2 (5 R - 6) T}{72576 T - 103680} \\
  a_4^{(2)} &= \frac{R^2}{41472} \\
  a_4^{(3)} &= \frac{-12 R^2 (3 R - 2) + R^2 (35 R - 18) T}{-1492992 T + 1990656} \\
  a_5^{(3)} &= -\frac{R^3}{373248 T - 373248} \\
  a_6^{(3)} &= -\frac{R^3}{17915904}
\end{align*}

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Figure Captions

**Fig. 1:** Our results for the 2-well potential for $d = 2.1$ through $d = 3.8$ in steps of 0.1. The deepest potential corresponds to $d = 2.1$. With increasing dimension, the 2-well gets flatter and flatter until it vanishes in 4 dimension. In all cases we used $l_{\text{max}} = 20$.

**Fig. 2:** The fixed point 3-wells for $d = 2.1$ through $d = 2.7$ in steps of 0.1. The deepest potential corresponds to $d = 2.1$, the flattest one to $d = 2.7$. In all cases we used $l_{\text{max}} = 20$.

**Fig. 3:** The fixed point 4-well potentials for $d = 2.1$ and $d = 2.3$. The potential with the deeper wells corresponds to $d = 2.1$. In both cases we used $l_{\text{max}} = 30$.

**Fig. 4:** Comparison of the ‘true’ 2-well potential at $d = 3.8$ with the 1st and 4th order $\epsilon$-expansion. The full line gives the result obtained numerically, and the dashed lines give the 1st and 4th order approximations, respectively.

**Fig. 5:** Comparison of the two leading eigenvectors of the RG transformation linearized around the trivial fixed point and the 2-well fixed point in $d = 3$. The single well shaped functions correspond to the relevant eigenvalue. The double well shaped functions correspond to the first irrelevant eigenvalue. The full lines belong to the non-trivial fixed point.
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