GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF WEAK SOLUTIONS FOR TIME-SPACE FRACTIONAL KIRCHHOFF-TYPE DIFFUSION EQUATIONS

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Abstract. In this paper, we investigate initial boundary value problems for Kirchhoff-type diffusion equations
\[ \partial_t^\beta u + M(\|u\|^2_{H^s_0(\Omega)})(-\Delta)^su = \gamma|u|^\rho u + g(t, x) \]
with the Caputo time fractional derivatives and fractional Laplacian operators. We establish a new compactness theorem concerning time fractional derivatives. By Galerkin method, let \(0 < \rho < 2^* - 2\) when \(\gamma < 0\), and \(0 < \rho < \min\left\{\frac{4s}{N}, \frac{2s}{N-2s}\right\}\) when \(\gamma > 0\), then we obtain the global existence and uniqueness of weak solutions for Kirchhoff problems. Furthermore, we get the decay properties of weak solutions in \(L^2(\Omega)\) and \(L^{2^*+2}(\Omega)\). Remarkably, the decay rate differs from that in the case \(\beta = 1\).

1. Introduction. Let \(\Omega \subset \mathbb{R}^N\) be a smooth bounded domain. We consider the following time-space fractional Kirchhoff type diffusion problems
\[
\begin{cases}
\partial_t^\beta u + M(\|u\|^2_{H^s_0(\Omega)})(-\Delta)^su = \gamma|u|^\rho u + g(x, t), & \text{in } \Omega \times (0, \infty), \\
u(x, 0) = f(x), & \text{in } \Omega,
\end{cases}
\] (1)
where \(0 < \beta < 1, 0 < s < 1, N > 2s, 0 < \rho < 2^* - 2 = \frac{4s}{N-2s}\), and the diffusion coefficient has the specific form \(M(\sigma) = a + b\sigma\) with parameters \(a, b > 0\). Furthermore, \(\gamma \neq 0\) is a given parameter, and \(g \in L^\infty(0, T; L^2(\Omega))\) is given data. Here the \(\beta\) order Caputo fractional operator \(\partial_t^\beta\) with respect to time is defined by
\[
\partial_t^\beta u = \partial_t(J^{1-\beta}(u - u(0))),
\]
where \(J^{1-\beta}\) denotes \(1 - \beta\) order Riemann-Liouville fractional integral operator, see \([15]\) for details. The fractional Laplacian operator \((-\Delta)^s\) can be represented by
\[
(-\Delta)^s u(x) = C(N, s) \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, x \in \mathbb{R}^N,
\]
where \(C(N, s) = (\int_{\mathbb{R}^N} \frac{1}{|x|^{N+2s}} dx)^{-1}\).

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A strong motivation of study problem (1) comes from its application to model some phenomena in many areas such as physics, mechanics chemistry, population dynamic, biomedical engineering and so on. Time fractional differential equations can model some situations in which there is “memory effect”. Moreover, both time and space fractional differential equations have been exploited for anomalous diffusion or dispersion where particles spread at a rate inconsistent with Brown motion, see [19, 3]. In the case of time fractional derivatives, particles with “memory effect” propagates slowly, which we call anomalous subdiffusion. Different from the former, spatial fractional diffusion equations are used to describe macroscopic transport and usually result in superdiffusion phenomenon. In recent years, the theory of fractional partial differential equations has been developed significantly, see [31, 25, 13, 14, 30, 1, 33, 18, 34).

It is well-known that the parabolic Kirchhoff equation
\[ u_t - M(\|u\|_{H^1_0(\Omega)}^2)\Delta u = g(x,t) \]  
(2)
can be used to describe the motion of a nonstationary fluid or gas in a nonhomogeneous and anisotropic medium, and the nonlocal term $M$ can describe a possible change in the global state of the fluid or gas caused by its motion in the considered medium. In [2], Chipot, Valente and Caffarelli obtain existence, uniqueness and asymptotic behavior of solutions to (2). When $M$ depends on the integral of $u$ on the entire domain, (2) can be used to describe the growth and movement of a particular species (for instance of bacteria), where $u$ could represent the density of a population subject to spreading. In recent years, much interest has been grown on Kirchhoff type diffusion problems, see [21, 29, 20, 22, 28, 12, 5, 11]. In [11], the authors research the following Kirchhoff equation
\[ u_t - M(\|u\|_{H^s_0(\Omega)}^2)(-\Delta)^s u = |u|^\rho u, \text{ in } \Omega \times (0, \infty). \]
where $M : \mathbb{R}_+^+ \to \mathbb{R}_+^0$ is continuous and satisfies $M(\sigma) \geq m_0\sigma^{\theta-1}$ with $m_0 > 0$ and $\theta > 1$. They obtain local existence of nonnegative solutions by Galerkin method, and prove that the local solution blows up in finite time by a differential inequality. Moreover, they give an estimate for lower and upper bounds of the blow-up time. On the basis of [28], in [5], Ding and Zhou consider the following Kirchhoff type equation
\[ u_t + M(\|u\|_{H^s_0(\Omega)}^2)(-\Delta)^s u = |u|^\rho u, \text{ in } \Omega \times (0, \infty), \]
where $L_K$ is a nonlocal integro-differential operator which generalizes the fractional Laplacian operator $(-\Delta)^s$. They obtain the global existence of nontrivial, nonnegative weak solutions for any $\theta \geq \frac{2^*_s}{s}$, and get the conditions of global existence and blow-up for nontrivial and nonnegative weak solutions.

However, to the best of our knowledge, there have been few results concerning the global existence and decay property for problem (1). A difficulty arising from problem (1) is the time fractional derivatives term $\partial_t^\beta u$, since there is no compact theorems concerning time fractional derivatives. Therefore, we establish a new
compactness theorem for time fractional derivatives. With this tool and Galerkin method, we prove that problem (1) admits a unique global weak solution. Moreover, we also consider asymptotic behavior of weak solutions. By a differential inequality derived by Vergara and Zacher [26], we obtain the decay properties of solutions in $L^2(\Omega)$ and $L^{p+2}(\Omega)$. Remarkably, the decay rate differs from that in the case $\beta = 1$.

Now we give the definition of weak solution for problem (1).

**Definition 1.1.** We call $u \in L^\infty(0, T; H^s_0(\Omega)) \cap L^{p+2}(\Omega)$ with $\partial_t^\beta u \in L^2(0, T; L^2(\Omega))$ a weak solution of problem (1) if $u(x, 0) = f(x)$ and

$$
\int_0^T \int_\Omega v(x, t) \partial_t^\beta u(x, t) \, dx \, dt 
+ \frac{C(N, s)}{2} \int_0^T M(\|u\|_{H^s_0(\Omega)}) \int_{\mathbb{R}^{2N}} \frac{(u(x, t) - u(y, t))(v(x, t) - v(y, t))}{|x - y|^{N + 2s}} \, dx \, dy \, dt 
= \gamma \int_0^T \int_\Omega |v(x, t)|^\rho u(x, t)v(x, t) \, dx \, dt + \int_0^T \int_\Omega g(x, t)v(x, t) \, dx \, dt
$$

for any $v \in C^0([0, T]; H^s_0(\Omega))$ with $v(T) = 0$. Moreover, we call $u$ a global weak solution of problem (1) if $T > 0$ can be arbitrarily chosen.

The main results of this paper are the following.

**Theorem 1.2.** Suppose that $f \in H^s_0(\Omega)$. Let

$$
0 < \rho < \frac{4s}{N - 2s}
$$

when $\gamma < 0$, and

$$
0 < \rho < \min\left\{\frac{4s}{N}, \frac{2s}{N - 2s}\right\}
$$

when $\gamma > 0$, then problem (1) admits a unique global weak solution $u$ satisfying

$$
\|u\|_{H^s_0(\Omega)} + \|\partial_t^\beta u\|_{L^2(0, T; L^2(\Omega))} \leq C(\|f\|_{H^s_0(\Omega)} + \|f\|_{L^{p+2}(\Omega)} + \|g\|_{L^\infty(0, T; L^2(\Omega))}).
$$

**Theorem 1.3.** Let $u$ be the weak solution of problem (1) and $g = 0$, then

(i) For $\gamma < 0$ and $0 < \rho < \frac{4s}{N - 2s}$, we have

$$
\|u\|_{L^2(\Omega)} \leq \min\left\{\frac{C_1}{1 + t^\beta}, \frac{C_2}{1 + t^{\frac{2s}{N}}}, \frac{C_3}{1 + t^{\frac{2s}{N}}}\right\}, \quad t \in (0, T),
$$

where $C_i = C_i(f, \beta, s, \gamma, a, b, \rho, \Omega), i = 1, 2, 3$, are different constants.

(ii) For $\gamma > 0$ and $0 < \rho < \frac{4s}{N}$, let $\gamma$ be small enough such that

$$
C(b) - C(a, \gamma) > 0,
$$

then we have

$$
\|u\|_{L^2(\Omega)} \leq \min\left\{\frac{C_1}{1 + t^\beta}, \frac{C_2}{1 + t^{\frac{2s}{N}}}, \frac{C_3}{1 + t^{\frac{2s}{N}}}\right\}, \quad t \in (0, T),
$$

where $C_i = C_i(f, \beta, s, \gamma, a, b, \rho, \Omega), i = 1, 2, 3$, are different constants.

**Theorem 1.4.** Let $u$ be the weak solution of problem (1) and $g = 0$, then

(i) For $\gamma < 0$ and $0 < \rho < \frac{2s}{N - 2s}$, we have

$$
\|u\|_{L^{p+2}(\Omega)} \leq \min\left\{\frac{C_1}{1 + t^\beta}, \frac{C_2}{1 + t^{\frac{2s}{N}}}, \frac{C_3}{1 + t^{\frac{2s}{N}}}\right\}, \quad t \in (0, T),
$$

where $C_i = C_i(f, \beta, s, \gamma, a, b, \rho, \Omega), i = 1, 2, 3$, are different constants.
(ii) For $\gamma > 0$ and $0 < \rho < \frac{2s}{N-2s}$, let $\gamma$ be small enough such that

$$C(b) - C(a, \gamma) > 0,$$

then we have

$$\|u\|_{L^{p+2}(\Omega)} \leq \min\{\frac{C_1}{1 + \rho^2}, \frac{C_2}{1 + t^2}\}, \ t \in (0, T),$$

where $C_i = C_i(f, \beta, s, \gamma, a, b, \rho, \Omega), i = 1, 2,$ are different constants.

The paper is organized as follows. In Section 2, we provide preliminaries and some useful lemmas including compactness theorem. In Section 3, we establish the existence and uniqueness of Galerkin approximation solutions for problem (1). In Section 4, we obtain the global existence and uniqueness of weak solutions for problem (1). In Section 5, we obtain the decay properties of weak solutions for problem (1).

2. Preliminaries. The fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy < \infty \}$$

endowed with the norm

$$\|u\|^2_{H^s(\mathbb{R}^N)} = \frac{C(N, s)}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy + \|u\|^2_{L^2(\mathbb{R}^N)}.$$

Similar to [7], let

$$H_0^s(\Omega) = \{ u \in H^s(\mathbb{R}^N) : u = 0 \ a.e. \ in \ \mathbb{R}^N \setminus \Omega \}.$$

In the sequel, we take

$$\|u\|^2_{H_0^s(\Omega)} = \frac{C(N, s)}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy$$

as norm on $H_0^s(\Omega)$. It is easily seen that $(H_0^s(\Omega), \| \cdot \|_{H_0^s(\Omega)})$ is a Hilbert space with inner product

$$(u, v)_{H_0^s(\Omega)} = \frac{C(N, s)}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy.$$

Let $H^{-s}(\Omega)$ be the dual space of $H_0^s(\Omega)$. Denote by

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots \leq +\infty$$

the distinct eigenvalue and $w_k$ the eigenfunction corresponding to $\lambda_k$ of eigenvalue problem

$$\begin{cases}
(-\Delta)^s u = \lambda u, & x \in \Omega, \\
u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}$$

By [24], we obtain for $k \in \mathbb{N}$,

$$\lambda_k = \frac{C(N, s)}{2} \min_{w \in \mathcal{P}_k \setminus \{0\}} \frac{\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy}{\int_{\Omega} |u(x)|^2 dx},$$

where $\mathcal{P}_1 = H_0^s(\Omega)$ and

$$\mathcal{P}_k = \{ u \in H_0^s(\Omega) : (u, w)_{H_0^s(\Omega)} = 0, \forall j = 1, 2, \ldots, k - 1 \}, \ k \geq 2.$$
Lemma 2.1 ([4, 3, 7]). (1) If $\Omega$ has a Lipschitz boundary, then the embedding $H_0^s(\Omega) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for any $p \in [1, 2^*_s)$.

(2) The embedding $H_0^s(\Omega) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ is continuous.

The Yosida approximation of the time-fractional derivative operator is an important tool for equations with Caputo fractional derivative operator. For detailed information, we can refer to [31, 32, 30, 13]. Let $1 \leq p < \infty$ and $X$ be a real Banach space. For fractional derivative operator

$$Bu = \frac{d}{dt}(g_{1-\beta} \ast u), \quad D(B) = \{u \in L^p([0, T]; X) : g_{1-\beta} \ast u \in \omega W^{1,p}([0, T]; X)\},$$

its Yosida approximation $B_n$, defined by $B_n = n B_n (n + B)^{-1}$, $n \in \mathbb{N}$, enjoy the property that for any $u \in D(B)$, $B_n u \to Bu$ in $L^p([0, T]; X)$ as $n \to \infty$. For convenience, we only provide some important properties of $g_{1-\beta,n}$ and $B_n$ which are the following:

- The kernel $g_{1-\beta,n}$ is nonnegative and nonincreasing for all $n \in \mathbb{N}$, and $g_{1-\beta,n} \in W^{1,1}([0, T])$.
- $g_{1-\beta,n} \to g_{1-\beta}$ in $L^1([0, T])$ and $B_n u \to Bu$ in $L^p([0, T]; X)$ as $n \to \infty$.

Lemma 2.2 ([31]). Let $H$ be a real Hilbert space and $T > 0$. Then for any $k \in W^{1,1}([0, T])$ and any $v \in L^2(0, T; H)$, there holds

$$\left(\frac{d}{dt}(k \ast u)(t), u(t)\right)_H = \frac{1}{2} \frac{d}{dt} \left(k*|u(\cdot)|^2\right)(t) + \frac{1}{2} k(t)|u(t)|^2_H,$$

$$+ \frac{1}{2} \int_0^t [-k(s)]|u(t) - u(t - s)|^2_H ds, \quad \text{a.e. } t \in (0, T).$$

Lemma 2.3 ([30]). Let $H \in C^1(\mathbb{R})$ and $k \in W^{1,1}([0, T])$, for a sufficiently smooth function $u$, then there holds for a.e. $t \in (0, T)$

$$H'(u(t)) \frac{d}{dt}(k \ast u)(t) = \frac{d}{dt}(k \ast H(u))(t) + [-H(u(t)) + H'(u(t))u(t)]k(t),$$

$$+ \int_0^t [H(u(t) - s) - H(u(t)) - H'(u(t))(u(t) - u(t - s))] [-k(s)] ds.$$

Lemma 2.2 and 2.3 are crucial to establish some priori estimates of solutions for problem (1). Moreover, we also derive some priori estimates for Galerkin approximation solutions, see Lemma 3.2 in Section 2 and Lemma 4.1 in Section 3.

Lemma 2.4 ([9]). Suppose that $u_m = \sum_{j=1}^m a_{mj} w_j$ and $0 < \rho < \frac{4s}{\alpha - 2s}$, then we have

$$\int_\Omega |u_m|^\rho u_m (-\Delta)^s u_m dx \geq \frac{2}{\rho + 2} \|u_m\|_{H^s_\rho(\Omega)}^2.$$ 

From [8], we consider the eigenvalue problem

$$\begin{cases}
\Lambda u = -\frac{d^2u}{dt^2} = \mu u, \quad t \in (0, T), \\
u(0) = \frac{d}{dt} u(T) = 0.
\end{cases}$$

(3)

Let $0 < \mu_1 < \mu_2 < \cdots < \infty$ be the eigenvalues and $e_k, k \in \mathbb{N}$ be the corresponding eigenfunctions of $\Lambda$ which satisfy

$$\mu_k = \frac{(2k-1)!^2(\pi^2)^s}{4^{2s}}, \quad e_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \sin \sqrt{\mu_k} t.$$

Therefore, we conclude that the $\{e_k\}_{k \in \mathbb{N}}$ is a normal orthonormal basis of $L^2(0, T)$. Further, we can define $\alpha$-order fractional operator $\Lambda_\alpha$ and fractional Sobolev spaces.
Definition 2.5. Let $0 \leq \alpha \leq 1$. Define
\[
H^\alpha(0, T) = \{ u \in L^2(0, T) : \| u \|^2_{H^\alpha(0, T)} = \sum_{k=1}^\infty \mu_k^\alpha |(u, e_k)_{L^2(0, T)}|^2 < \infty \}
\]

Next, we show the compactness theorem.

Lemma 2.6 (Compactness Theorem). Let $B_0, B_1$ and $B_i$ be three Hilbert spaces. Assume that $B_0 \subset B \subset B_1$ and $B_i, i = 0, 1,$ are separable. Suppose also that $B_0$ is compactly embedded into $B$ and $T$ is finite. Let
\[
W = \{ u | u \in L^2(0, T; B_0), u \in H^\alpha(0, T; B_1) \}
\]
equipped with the norm
\[
\| u \|_{L^2(0, T; B_0)} + \| u \|_{H^\alpha(0, T; B_1)}.
\]

Then $W$ is compactly embedded into $L^2(0, T; B)$.

Proof. Suppose that a sequence $\{ u_n \}_{n \in \mathbb{N}} \subset W$ is bounded such that
\[
\| u_n \|_W \leq C.
\]

Due to the reflective of $W$, there exists a subsequence (still denoted by $\{ u_n \}$) such that $u_n \rightarrow u$ in $W$. Later, we will prove that there exists a subsequence (still denoted by $\{ u_n \}$) such that $u_n \rightarrow u$ in $L^2(0, T; B)$.

By the interpolation inequality (see Lemma 5.1, [16]), we have for any $\eta > 0$ there exists a $c(\eta)$ such that
\[
\| v \|_B \leq \eta \| v \|_{B_0} + c(\eta) \| v \|_{B_1}, \forall v \in B_0.
\]

Therefore,
\[
\| u_n - u \|_{L^2(0, T; B)} \leq \eta \| u_n - u \|_{L^2(0, T; B_0)} + c(\eta) \| u_n - u \|_{L^2(0, T; B_1)}.
\]

Fix $\eta$, then we need to show that $\| u_n - u \|_{L^2(0, T; B_1)} \rightarrow 0$. $u_n$ and $u$ can be written as
\[
u_n = \sum_{k=1}^\infty u_{n, k} e_k, u = \sum_{k=1}^\infty u_{k} e_k,
\]
where $u_{n, k} = (u_n, e_k)_{L^2(0, T)}$ and $u_{k} = (u, e_k)_{L^2(0, T)}$. Then,
\[
\| u_n - u \|^2_{L^2(0, T; B_1)} \leq \int_0^T \left( \sum_{k=1}^\infty \| u_{n, k} - u_{k} \|^2_{B_1} \right) \left( \sum_{j=1}^\infty \| u_{n, j} - u_{j} \|^2_{B_1} \right) dt
\]
\[
= \sum_{k=1}^{N_0} \| u_{n, k} - u_{k} \|^2_{B_1} + \sum_{k=N_0+1}^{\infty} \frac{1}{\mu_k^\alpha} \| u_{n, k} - u_{k} \|^2_{B_1}
\]
\[
\leq \sum_{k=1}^{N_0} \| u_{n, k} - u_{k} \|^2_{B_1} + \frac{1}{\mu_{N_0}^\alpha} \sum_{k=N_0+1}^{\infty} \| u_{n, k} - u_{k} \|^2_{B_1}
\]
\[
\leq \sum_{k=1}^{N_0} \| u_{n, k} - u_{k} \|^2_{B_1} + \frac{1}{\mu_{N_0}^\alpha} \| u_{n, k} - u_{k} \|^2_{H^\alpha(0, T; B_1)}.
\]
Let $N_0$ be large enough such that $\frac{c^2}{\rho N_0} < \frac{\alpha}{2}$. Next, we will prove
\[
\sum_{k=1}^{N_0} \|u_{n,k} - u_k\|_{B_1}^2 \to 0,
\]
as $n \to \infty$. Hence, we need only to show that there holds $u_{n,k} \to u_k$ in $B_1$ for $k = 1, 2, \ldots, N_0$. As we have $u_n \to u$ in $L^2(0, T; B_0)$, for any $\varphi \in L^2(0, T)$,
\[
\int_0^T (u_n - u) \varphi dt \to 0 \text{ in } B_0,
\]
as $n \to \infty$. Taking $\varphi = e_k$, then
\[
\int_0^T (u_n - u)e_k dt = \sum_{j=1}^\infty \int_0^T (u_{n,j} - u_j)e_k dt = u_{n,k} - u_k \to 0 \text{ in } B_0,
\]
as $n \to \infty$. Because $B_0$ is compactly embedded into $B$, the embedding $B_0 \hookrightarrow B_1$ is compact. Thus there exists a subsequence (still denoted by $\{u_{n,k}\}$) such that $u_{n,k} \to u_k$ in $B_1$ as $n \to \infty$.

**Remark 1.** In [17], Li and Liu obtain some compactness criteria for weak solutions of time fractional PDEs, which gives the existence of weak solution for a special case of time fractional compressible Navier-Stokes equations with constant density and time fractional Keller-Segel equations in $\mathbb{R}^N$, see [17] for details.

**Lemma 2.7 ([23]).** A set $F$ is relatively compact in $L^p(0, T)$ when $1 \leq p < \infty$, or in $C(0, T)$ when $p = \infty$, if and only if:

- \( \exists a_1 < a_2 \) such that $\int_{a_1}^{a_2} f(t) dt$ is bounded uniformly for $f \in F$,
- $\int_{0}^{T-h} |f(t+h) - f(t)|^p dt \to 0$ as $h \to 0$ uniformly for $f \in F$.

**Lemma 2.8 ([26]).** Let $1 < p < \infty$ and $k \in H^1([0, T])$ be nonnegative and nonincreasing. Then for any $u_0 \in L^p(\Omega)$ and $u \in L^p(0, T; L^p(\Omega))$ there holds
\[
|u(t)|_{L^p(\Omega)}^{-1} \partial_t (k * |u|_{L^p(\Omega)}) \leq \int_\Omega |u|^{p-2} u \partial_t (k * |u - u_0|) dx,
\]
for a.e. $t \in (0, T)$.

### 3. The existence and uniqueness of Galerkin approximation solutions.

Our main tool is Galerkin method, see [6, 9, 16, 27, 31] for examples. Assume that $\{w_k\}$ is normal in $L^2(\Omega)$ and
\[
f_m = \sum_{j=1}^m b_{mj}w_j \to f \text{ in } H^s_0(\Omega),
\]
then we can look for Galerkin approximation solutions $u_m = u_m(t)$ of the form
\[
u_m(t) = \sum_{j=1}^m a_{mj}(t)w_j, m = 1, 2, \ldots,
\]
where $a_{mj}$ satisfies that
\[
\begin{cases}
(a^2_t u_m, w_j) + M(\|u_m\|^2_{H^2_0(\Omega)}) \lambda_j a_{mj} = \gamma(|u_m|^p u_m, w_j) + (g, w_j), \\
a_{mj}(0) = b_{mj},
\end{cases}
\]
(4)
for \( j = 1, 2, \ldots, m \). Then, problem (1) becomes a nonlinear fractional ordinary differential systems. Next, we show that problem (4) has a unique solution for every \( m \in \mathbb{N} \).

**Theorem 3.1.** Suppose that \( f \in H^0_0(\Omega) \). Let \( 0 < \rho < \frac{4s}{N-2s} \) when \( \gamma > 0 \) and \( 0 < \rho < \min\{2, \frac{4s}{N-2s}\} \) when \( \gamma < 0 \), for every \( m \in \mathbb{N} \), problem (4) has a unique solution in \( C([0, T]; \mathbb{R}^m) \).

First, by Lemma 2.2, we give a priori estimate for problem (4).

**Lemma 3.2.** Suppose that \( u_m = \sum_{j=1}^m a_{mj}w_j \) solves (4), then we have

\[
\|u_m\|_{L^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^\infty(0,T;L^2(\Omega))}).
\]

**Proof.** Multiply (4) by \( a_{mj} \) and sum over \( j \), then we have

\[
\langle \partial_t^\beta u_m, u_m \rangle + M(\|u_m\|_{H^\rho_0(\Omega)}^2)((-\Delta)^s u_m, u_m) = \gamma(\|u_m\|^2 u_m, u_m) + (g, u_m),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product of \( L^2(\Omega) \). By the Yosida approximation, we have

\[
(\frac{d}{dt}(g_{1-\beta,n} * u_m), u_m) + M(\|u_m\|_{H^\rho_0(\Omega)}^2)((-\Delta)^s u_m, u_m) = \gamma(\|u_m\|^2 u_m, u_m) + (g, u_m) + g_{1-\beta,n}(f_m, u_m) + R_{mn},
\]

where

\[
R_{mn} = \frac{d}{dt}(g_{1-\beta,n} * (u_m - f_m)) - \frac{d}{dt}(g_{1-\beta} * (u_m - f_m)), u_m).
\]

By Lemma 2.2 with \( H = L^2(\Omega) \), we have

\[
\frac{1}{2} \frac{d}{dt}(g_{1-\beta,n} * u_m) \|u_m\|_{L^2(\Omega)}^2 \|u_m\|_{L^2(\Omega)}^2
\]

Therefore, we obtain

\[
\frac{1}{2} \frac{d}{dt}(g_{1-\beta,n} * u_m) \|u_m\|_{L^2(\Omega)}^2 \|u_m\|_{L^2(\Omega)}^2 + M(\|u_m\|_{H^\rho_0(\Omega)}^2)\|u_m\|_{L^2(\Omega)}^2
\]

\[
\leq \gamma(\|u_m\|^2 u_m, u_m) + (g, u_m) + g_{1-\beta,n}(f_m, u_m) + R_{mn}.
\]

On the other hand, by Cauchy-Schwarz inequality and Hölder inequality, we have

\[
g_{1-\beta,n}(f_m, u_m) \leq g_{1-\beta,n}\|f_m\|_{L^2(\Omega)}\|u_m\|_{L^2(\Omega)}
\]

\[
\leq \frac{1}{2} g_{1-\beta,n}\|f_m\|_{L^2(\Omega)}^2 + \frac{1}{2} g_{1-\beta,n}\|u_m\|_{L^2(\Omega)}^2.
\]

Then, we obtain

\[
\frac{1}{2} \frac{d}{dt}(g_{1-\beta,n} * u_m) \|u_m\|_{L^2(\Omega)}^2 + \frac{1}{2} M(\|u_m\|_{H^\rho_0(\Omega)}^2)\|u_m\|_{L^2(\Omega)}^2
\]

\[
\leq \gamma(\|u_m\|^2 u_m, u_m) + \|g\|_{L^2(\Omega)}^2 + \frac{1}{2} g_{1-\beta,n}\|f_m\|_{L^2(\Omega)}^2 + R_{mn}.
\]

If \( \gamma < 0 \), then we drop the second and third terms and convolve (5) with \( g_\beta \). Let \( n \) go to \( \infty \) and select an appropriate subsequence (if necessary), then we arrive at

\[
\|u_m\|_{L^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^\infty(0,T;L^2(\Omega))}).
\]

If \( \gamma > 0 \), by Young inequality, then we have

\[
|\gamma|\|u_m\|_{L^2(\Omega)}^2 \leq C\|u_m\|_{H^\rho_0(\Omega)}^2 \leq \frac{b}{2}\|u_m\|_{H^\rho_0(\Omega)}^2 + C.
\]
From (5), we have
\[
\frac{1}{2} \frac{d}{dt} (g_{1-\beta,n} \ast u_m) + \frac{\alpha}{2} ||u_m||^2_{H^\beta(\Omega)} \\
\leq ||g||^2_{L^2(\Omega)} + \frac{1}{2} g_{1-\beta,n} ||f_m||^2_{L^2(\Omega)} + C + R_{mn}.
\]
Drop the second term and convolute with \( g_{\beta} \). Let \( n \) go to \( \infty \) and select an appropriate subsequence (if necessary), then we obtain
\[
||u_m||_{L^2(\Omega)} \leq C(||f||_{L^2(\Omega)} + ||g||_{L^\infty(0,T;L^2(\Omega))}).
\]
\[\] \[\] \[\]
Next, we prove Theorem 3.1.
\[\]
**Proof.** First, problem (4) is equivalent to the problem
\[
\begin{align*}
\partial_t^\beta \psi(t) + P(\psi(t)) &= R(\psi(t)) + G(t), \\
\psi(0) &= \xi,
\end{align*}
\]
where \( \psi(t) = (a_{mj}(t)) \in \mathbb{R}^m, \xi = (b_{mj}) \in \mathbb{R}^m, (G(t))_j = (g, w_j), \)
\[
(P(\psi(t)))_j = (a + b ||u_m||^2_{H^\beta(\Omega)}) \lambda_j a_{mj}(t),
\]
and
\[
(R(\psi(t)))_j = \gamma (|u_m|^p u_m, w_j).
\]
By Laplace transform or convoluting with \( g_{\beta} \), we transform (6) into Volterra system
\[
\psi(t) = \xi + g_{\beta} * G - g_{\beta} * [P(\psi(t)) - R(\psi(t))].
\]
Therefore, we need only to prove that system (7) admits a unique continuous solution. Notice that
\[
||u_m(t)||^2_{L^2(\Omega)} = \sum_{j=1}^m a_{mj}(t), ||\psi(t)||^2_{\mathbb{R}^m} = \sum_{j=1}^m a_{mj}^2(t).
\]
Then, \( ||u_m(t)||_{L^2(\Omega)} = ||\psi(t)||_{\mathbb{R}^m} \). Let
\[
R_0 = 2C||f||_{H^\beta(\Omega)} + \frac{T^\beta}{\Gamma(1+\beta)} ||g||_{L^\infty(0,T;L^2(\Omega))},
\]
then we obtain a prior estimate \( ||\psi(t)||_{\mathbb{R}^m} \leq R_0 \). Further, let \( \psi^{(1)} \), \( \psi^{(2)} \) correspond to \( u_m^{(1)}, u_m^{(2)} \). By Hölder inequality, we have
\[
||R(\psi^{(1)}) - R(\psi^{(2)})||_{\mathbb{R}^m} \leq \gamma^2 \sum_{j=1}^m (\int_\Omega (\rho + 1) \max(|u_m^{(1)}|^\rho, |u_m^{(2)}|^\rho)|u_m^{(1)} - u_m^{(2)}|w_j dx)^2
\]
\[
\leq C(||u_m^{(1)}||^\rho \ast_L ||u_m^{(2)}||^\rho \ast_L ||u_m^{(1)} - u_m^{(2)}||^2_{L^2(\Omega)})
\]
\[
\leq C C_0^2 \rho^+ ||u_m^{(1)}||^\rho_{H^\beta(\Omega)} + ||u_m^{(2)}||^\rho_{H^\beta(\Omega)} ||u_m^{(1)} - u_m^{(2)}||^2_{H^\beta(\Omega)}
\]
\[
\leq C A_1 \rho^+ (||u_m^{(1)}||^\rho_{L^2(\Omega)} + ||u_m^{(2)}||^\rho_{L^2(\Omega)}) ||u_m^{(1)} - u_m^{(2)}||^2_{L^2(\Omega)}
\]
\[
\leq C (||\psi^{(1)}||^\rho_{\mathbb{R}^m} + ||\psi^{(2)}||^\rho_{\mathbb{R}^m}) ||\psi^{(1)} - \psi^{(2)}||^2_{\mathbb{R}^m}
\]
\[
= A_1 (\psi^{(1)})^2 ||\psi^{(1)} - \psi^{(2)}||^2_{\mathbb{R}^m},
\]
\[\] \[\] \[\]
where $C_0$ is the embedding constant in Lemma 2.1.(2). Moreover, we also have
\[
\|P(\psi^{(1)}) - P(\psi^{(2)})\|_{\mathbb{R}^m} \leq (a + b)\|u^{(1)}\|_{L^2(\Omega)}^2 \lambda_m \|\psi^{(1)} - \psi^{(2)}\|_{\mathbb{R}^m} + b\lambda_m \|\psi^{(2)}\|_{\mathbb{R}^m} \\
\leq (a + b\lambda_m(\|\psi^{(1)}\|_{\mathbb{R}^m} + \|\psi^{(1)}\|_{\mathbb{R}^m}^2)\lambda_m \|\psi^{(1)} - \psi^{(2)}\|_{\mathbb{R}^m} \\
\equiv A_2(\|\psi^{(1)} - \psi^{(2)}\|_{\mathbb{R}^m},) \\
\] 
where
\[
\|\psi\|_{\mathbb{R}^m} = \|u^{(1)}\|_{L^2(\Omega)}^2 - \|u^{(2)}\|_{L^2(\Omega)}^2.
\]
Let $F(\psi) = P(\psi) - R(\psi)$, then we conclude that the function $F : \mathbb{R}^m \to \mathbb{R}^m$ satisfies the Lipschitz condition. Set
\[
K_m = \max_{(t, \psi) \in [0, T] \times B(\xi, 2R_0)} \|F(\psi(t))\|_{\mathbb{R}^m}, \\
T_m = \min\{T, (\frac{R_0\Gamma(1 + \beta)}{M_m})^\frac{1}{\beta}\}.
\]
Define
\[
D(T_m, R_0) = \{(t, \psi)|0 \leq t \leq T_m, \|\psi - \xi - g_\beta \ast G\|_{\mathbb{R}^m} \leq R_0\}.
\]
Then, we acquire from (8) and (9) that
\[
\|F(\psi^{(1)}) - F(\psi^{(2)})\|_{\mathbb{R}^m} \leq A\|\psi^{(1)} - \psi^{(2)}\|_{\mathbb{R}^m}, \forall \psi^{(1)}, \psi^{(2)} \in B(\xi, 2R_0),
\]
where
\[
A = \max_{\psi^{(1)}, \psi^{(2)} \in B(\xi, 2R_0)} [A_1(\psi^{(1)}, \psi^{(2)}) + A_2(\psi^{(1)}, \psi^{(2))}]
\]
Therefore, we say that $F(\psi)$ satisfies Lipschitz condition in $D(T_m, R_0)$. Next, we prove that system (7) has a unique solution for $0 \leq t \leq T_m$.

First, we construct an iterative sequence
\[
\psi_0(t) = \xi + g_\beta \ast G, \psi_n(t) = \xi + g_\beta \ast G - g_\beta \ast F(\psi_{n-1}).
\]

Second, we prove that $\{\psi_n(t)\}$ uniformly converges to a solution. Formally, $\psi_n(t)$ is written as partial sum of the series
\[
\psi_0(t) + \sum_{j=1}^{\infty} [\psi_j(t) - \psi_{j-1}(t)].
\]
Thus, we need only to prove that the above series is uniformly convergent in $[0, T_m]$.

The proof is divided into two steps.

**Step 1:** For $\forall n \geq 1$, we conclude by mathematical induction
\[
\|\psi_n(t) - \xi - g_\beta \ast G\|_{\mathbb{R}^m} \leq g_\beta \ast \|F(\psi_{n-1}(t))\|_{\mathbb{R}^m} \leq K_m T_m^{\beta} \frac{\Gamma(1 + \beta)}{\Gamma(1 + n\beta)} \leq R_0.
\]
Then, there holds $(t, \psi_n(t)) \in D(T_m, R_0)$. Furthermore, we get the following inequality by the same method
\[
\|\psi_n(t) - \psi_{n-1}(t)\|_{\mathbb{R}^m} \leq K_m A^{n-1} T_m^{\beta} \frac{\Gamma(1 + n\beta)}{\Gamma(1 + n\beta)}.
\]

**Step 2:** For $n = 1$, by (12) we have
\[
\|\psi_1(t) - \psi_0(t)\|_{\mathbb{R}^m} = \|\psi_0(t) - \xi - g_\beta \ast G\|_{\mathbb{R}^m} \leq g_\beta \ast K_m = \frac{K_m T_m^{\beta}}{\Gamma(1 + \beta)}.
\]
Assume for \( n = k \), (13) holds. Then, by Lipschitz condition (10), we have
\[
\|\psi_{k+1}(t) - \psi_k(t)\|_{\mathbb{R}^m} = g_\beta \| [F(\psi_k(t)) - F(\psi_{k-1}(t))]\|_{\mathbb{R}^m} \\
\leq g_\beta \| \psi_k(t) - \psi_{k-1}(t)\|_{\mathbb{R}^m} \\
\leq g_\beta * A \| \psi_k(t) - \psi_{k-1}(t)\|_{\mathbb{R}^m} \\
\leq g_\beta * A \frac{K_m A^{k-1} \beta k}{\Gamma(1 + k\beta)} \\
\leq \frac{K_m A^{k} \beta (k+1)}{\Gamma(1 + (k+1)\beta)}.
\]
Therefore, (13) holds. From (13), it follows that
\[
\|\psi_n(t) - \psi_{n-1}(t)\|_{\mathbb{R}^m} \leq \frac{K_m A^{n-1} T_m^{n\beta}}{\Gamma(1 + n\beta)}.
\]
By the Mittag-Leffler function, we have
\[
\sum_{n=1}^{\infty} \frac{K_m A^{n-1} T_m^{n\beta}}{\Gamma(1 + n\beta)} = \frac{K_m}{A} \sum_{n=1}^{\infty} \frac{M(A T_m^{\beta})^n}{\Gamma(1 + n\beta)} = \frac{K_m}{A} (E_{\beta,1}(AT_m^{\beta}) - 1).
\]
By the Weierstrass test, the series is uniformly convergent to a continuous function \( \phi_m \) in \([0, T_m]\), i.e. \( \{\psi_n\} \) uniformly converges to \( \phi_m \) in \([0, T_m]\). Let \( n \to \infty \) in (11), then
\[
\phi_m(t) = \xi + g_\beta * G - g_\beta * [P(\phi_m(t)) - R(\phi_m(t))].
\]
Finally, we prove the uniqueness of the solution. Suppose that \( \tilde{\phi} \) is also a continuous solution of (7), then we have
\[
\|\phi(t) - \tilde{\phi}(t)\|_{\mathbb{R}^m} = \|F(\phi(t)) - F(\tilde{\phi}(t))\|_{\mathbb{R}^m} \leq g_\beta * A \|\phi(t) - \tilde{\phi}(t)\|_{\mathbb{R}^m}.
\]
Since \( \phi(t) - \tilde{\phi}(t) \) is continuous on \([0, T_m]\), there exists a constant \( L \) such that \( \|\phi(t) - \tilde{\phi}(t)\|_{\mathbb{R}^m} \leq L, \forall t \in [0, T_m] \). By (14), we obtain
\[
\|\phi(t) - \tilde{\phi}(t)\|_{\mathbb{R}^m} \leq \frac{ALt^\beta}{\Gamma(1 + \beta)}.
\]
By (14) and (15), we have
\[
\|\phi(t) - \tilde{\phi}(t)\|_{\mathbb{R}^m} \leq \frac{A^2 L t^{2\beta}}{\Gamma(1 + 2\beta)}.
\]
Therefore, by iteration we get
\[
\|\phi(t) - \tilde{\phi}(t)\|_{\mathbb{R}^m} \leq \frac{A^n L t^{n\beta}}{\Gamma(1 + n\beta)},
\]
where the right term is \( n \)-th term of series of the Mittag-Leffler function \( E_{\beta,1}(AT^\beta) \). Let \( n \to \infty \), then we obtain \( \|\phi(t) - \tilde{\phi}(t)\|_{\mathbb{R}^m} \to 0 \). Therefore, we deduce that \( \phi(t) = \tilde{\phi}(t) \). Then, for every \( m \), we obtain a unique continuous solution in \([0, T_m]\).

Further, we may expand \([0, T_m]\) to \([0, T]\). Let \( T_m = t_1 \) and \( \phi(t_1) \) be a initial value, then repeat the above process and get a unique continuous solution in \([t_1, t_1 + T_m]\). We can divide \([0, T]\) into \([\lfloor - 1\rfloor T_m, jT_m], j = 1, 2, ...K\) where \( \frac{T}{K} \leq T_m \). Then, we
obtain a unique continuous solution $\phi_m$ in $[0, T]$ defined by

$$
\phi_m(t) = \begin{cases} 
\phi^1_m(t), & t \in [0, T_m], \\
\phi^2_m(t), & t \in (T_m, 2T_m], \\
\cdots \\
\phi^K_m(t), & t \in ((K - 1)T_m, KT_m].
\end{cases}
$$

Then, we get that for every $m \in \mathbb{N}$, problem (4) has a unique solution in $C([0, T]; \mathbb{R}^n)$. That is, we obtain the approximation solutions $u_m(t) = \sum_{j=1}^n a_{mj}(t)w_j$ for problem (1).

\[ \square \]

**Remark 2.** The proof of Theorem 3.1 is proved by iterative method similar to Picard-Lindelöf Theorem for ordinary differential equations, see Chapter 2 of [10].

4. **Global existence.** By Lemmas 2.2 and 2.3, we obtain the following estimate for (1).

**Lemma 4.1.** Suppose that $u_m = \sum_{j=1}^n a_{mj}w_j$ solves (4). If $\|u_m\|^2_{H^\gamma(\Omega)} \leq \rho$ and $\gamma > 0$, then we have

$$
\|\partial_t^\gamma u_m\|_{L^2(0,T;L^2(\Omega))}^2 \leq C(\|f\|^2_{H^\gamma(\Omega)} + \|f\|_{L^{p+2}(\Omega)} + \|g\|^2_{L^\infty(0,T;L^2(\Omega))}),
$$

where the constant $C = C(a, b, \rho, T, \gamma, \rho)$.

**Proof.** Multiply (4) by $\partial_t^\gamma a_{mj}$ and sum over $j$, then

$$
\|\partial_t^\gamma u_m\|_{L^2(\Omega)}^2 + M(\|u_m\|^2_{H^\gamma(\Omega)})((\partial_t^\gamma u_m, \partial_t^\gamma u_m) = \gamma(\|u_m\|_{L^2}(\Omega), \partial_t^\gamma u_m) + (g, \partial_t^\gamma u_m).
$$

By the Yosida approximation of time Riemann-Liouville fractional derivative, Lemma 2.2 and Hölder inequality, we have

$$
((\Delta) u_m, \partial_t^\gamma u_m) = \left(\frac{d}{dt}g_{1-\beta,n} \ast u_m, (-\Delta)^\gamma u_m\right) - g_{1-\beta,n}(f_m, (-\Delta)^\gamma u_m) - R_{mn}^1,
$$

where

$$
R_{mn}^1 = \left(\frac{d}{dt}(g_{1-\beta,n} \ast (u_m - f_m)) - \frac{d}{dt}(g_{1-\beta,n} \ast (u_m - f_m)), (-\Delta)^\gamma u_m\right).
$$

Similarly, by Lemma 2.3 and Young inequality, we have

$$
(\|u_m\|_{L^p(\Omega)}, \partial_t^\gamma u_m)
$$

$$
= (\frac{d}{dt}g_{1-\beta,n} \ast u_m, |u_m|_{L^p(\Omega)} - g_{1-\beta,n}(f_m, |u_m|_{L^p(\Omega)}) - R_{mn}^2,
$$

where

$$
R_{mn}^2 = \left(\frac{d}{dt}(g_{1-\beta,n} \ast (u_m - f_m)) - \frac{d}{dt}(g_{1-\beta,n} \ast (u_m - f_m)), |u_m|_{L^p(\Omega)}\right).$$
Therefore, we derive
\[
\|\partial_t^\beta u_m\|_{L^2(\Omega)}^2 + \frac{1}{2} M(\|u_m\|_{H^\beta_0(\Omega)}) \frac{d}{dt} g_{1-\beta,n} * \|u_m\|_{H^\beta_0(\Omega)}^2
\]
\[
+ \frac{\gamma}{\rho + 2} \frac{d}{dt} g_{1-\beta,n} * \|u_m\|_{L^{p+2}(\Omega)}^{p+2}
\]
\[
\leq \frac{a + b\varrho}{2} g_{1-\beta,n} \|f_m\|_{H^\beta_0(\Omega)}^2 + (a + b\varrho) R^1_{mn} + \gamma R^2_{mn}
\]
\[
+ \|g\|_{L^2(\Omega)} \|\partial_t^\beta u_m\|_{L^2(\Omega)} + \frac{\gamma}{\rho + 2} g_{1-\beta,n} \|f_m\|_{L^{p+2}(\Omega)}^{p+2}.
\]
Integrate from 0 to T and drop the second and third terms. Let \(n \to \infty\), then we obtain
\[
\|\partial_t^\beta u_m\|_{L^2(0,T;L^2(\Omega))}^2 \leq C(\|f\|_{H^\beta_0(\Omega)}^2 + \|f\|_{L^{p+2}(\Omega)}^{p+2} + \|g\|_{L^\infty(0,T;L^2(\Omega))}^2).
\]

Next, we prove Theorem 1.2.

**Proof.** Suppose that \(u_m\) is the approximation solution of problem (1), then
\[
(\partial_t^\beta u_m) + M(\|u_m\|_{H^\beta_0(\Omega)})(-\Delta)^* u_m = (\gamma|u_m|^p u_m + g, w_j),
\]
for \(j = 1, 2, ..., m\). Multiplying by \(a_{mj} \lambda_j\) and summing over \(j\), we have
\[
(\partial_t^\beta u_m + M(\|u_m\|_{H^\beta_0(\Omega)}))(-\Delta)^* u_m = (\gamma|u_m|^p u_m + g, (-\Delta)^* u_m).
\]
By the Yosida approximation of time Riemann-Liouville fractional derivative, we have
\[
\left(\frac{d}{dt} g_{1-\beta,n} * u_m, (-\Delta)^* u_m\right) + M(\|u_m\|_{H^\beta_0(\Omega)}) \|(-\Delta)^* u_m\|_{L^2(\Omega)}^2
\]
\[
= \gamma(\|u_m|^p u_m, (-\Delta)^* u_m) + g_{1-\beta,n}(f_m, (-\Delta)^* u_m) + (g, (-\Delta)^* u_m) + \tilde{R}_{mn},
\]
where
\[
\tilde{R}_{mn} = \left(\frac{d}{dt} g_{1-\beta,n} * (u_m - f_m)\right) - \frac{d}{dt} (g_{1-\beta} * (u_m - f_m)), (-\Delta)^* u_m).
\]
By Lemma 2.2 with \(H = H^\beta_0(\Omega)\), we have
\[
\left(\frac{d}{dt} g_{1-\beta,n} * u_m, (-\Delta)^* u_m\right)
\]
\[
= \left(\frac{d}{dt} g_{1-\beta,n} * u_m, u_m\right)_{H^\beta_0(\Omega)}
\]
\[
\geq \frac{1}{2} \frac{d}{dt} \|g_{1-\beta,n} * u_m\|_{H^\beta_0(\Omega)}^2 + \frac{1}{2} g_{1-\beta,n} \|u_m\|_{H^\beta_0(\Omega)}^2.
\]

\[
\right)
\]
By Lemma 2.4, we get
\[
(|u_m|^\rho u_m, (-\Delta)^s u_m) \geq \frac{2}{\rho + 2} \|u_m\|_{H^s_0(\Omega)}^{\frac{\rho+2}{2}}.
\]

If \( \gamma > 0 \), then we drop the third term in (16). By (17) and Young inequality, we deduce
\[
\frac{1}{2} \frac{d}{dt} (g_{1-\beta,n} \|u_m\|^2_{H^s_0(\Omega)}) + \frac{1}{2} g_{1-\beta,n}\|u_m\|^2_{H^s_0(\Omega)} + a\|(-\Delta)^s u_m\|_{L^2(\Omega)}^2
\leq g_{1-\beta,n}(f_m, (-\Delta)^s u_m) + (g, (-\Delta)^s u_m) + \tilde{R}_{mn}
\leq g_{1-\beta,n}\|f_m\|_{H^s_0(\Omega)}\|u_m\|_{H^s_0(\Omega)} + \|g\|_{L^2(\Omega)}\|(-\Delta)^s u_m\|_{L^2(\Omega)} + \tilde{R}_{mn}
\leq \frac{1}{2} g_{1-\beta,n}(\|f_m\|^2_{H^s_0(\Omega)} + \|u_m\|^2_{H^s_0(\Omega)}^2) + \frac{a}{2}\|(-\Delta)^s u_m\|_{L^2(\Omega)}^2 + C\|g\|_{L^2(\Omega)}^2 + \tilde{R}_{mn}.
\]
Then, we have
\[
\frac{1}{2} \frac{d}{dt} (g_{1-\beta,n} \|u_m\|^2_{H^s_0(\Omega)}) + \frac{a}{2}\|(-\Delta)^s u_m\|_{L^2(\Omega)}^2
\leq \frac{1}{2} g_{1-\beta,n}\|f_m\|^2_{H^s_0(\Omega)} + C\|g\|_{L^2(\Omega)}^2 + \tilde{R}_{mn}.
\] (18)
Drop the second term and convolute (18) with \( g_{\beta} \). Let \( n \to \infty \), and select an appropriate subsequence, then we conclude
\[
\|u_m\|^2_{H^s_0(\Omega)} \leq C\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \|f\|^2_{H^s_0(\Omega)}.
\]
Integrate from 0 to \( T \) and drop the first term in (18). Let \( n \to \infty \), then we obtain
\[
\|(-\Delta)^s u_m\|^2_{L^2(0,T;L^2(\Omega))} \leq C\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \|f\|^2_{H^s_0(\Omega)}.
\]
If \( \gamma < 0 \), by Gagliardo-Nirenberg inequality and Young inequality, then we have
\[
|\gamma|(\|u_m\|^\rho u_m, (-\Delta)^s u_m) \leq \|u_m\|^\rho u_m\|u_m\|_{L^2(\Omega)}\|(-\Delta)^s u_m\|_{L^2(\Omega)}
\leq C\|u_m\|^\rho\|u_m\|_{L^{2+\theta}(\Omega)}\|(-\Delta)^s u_m\|_{L^2(\Omega)}
\leq C\|u_m\|^\rho\|u_m\|_{H^s_0(\Omega)}\|u_m\|_{L^{2+\theta}(\Omega)}\|(-\Delta)^s u_m\|_{L^2(\Omega)}
\leq C\|u_m\|^\rho\|u_m\|_{H^s_0(\Omega)}\|(-\Delta)^s u_m\|_{L^2(\Omega)}
\leq C + \|u_m\|_{H^s_0(\Omega)}^2 + \|u_m\|_{H^s_0(\Omega)}\|(-\Delta)^s u_m\|_{L^2(\Omega)},
\]
where \( \theta = \frac{\rho N}{2(\rho + 1)} \) satisfying
\[
\frac{1}{2\rho + 2} = \frac{\theta(\rho - 2s)}{2N} + \frac{1 - \theta}{2}.
\]
From (16), it follows that
\[
\frac{1}{2} \frac{d}{dt} (g_{1-\beta,n} \|u_m\|^2_{H^s_0(\Omega)}) + \frac{a}{2}\|(-\Delta)^s u_m\|_{L^2(\Omega)}^2
\leq \frac{1}{2} g_{1-\beta,n}\|f_m\|^2_{H^s_0(\Omega)} + C\|g\|_{L^2(\Omega)}^2 + \|u_m\|^2_{H^s_0(\Omega)} + \tilde{R}_{mn}.
\]
Drop the second term and convolute with \( g_{\beta} \). Let \( n \to \infty \), and select an appropriate subsequence, by Gronwall inequality, then we have
\[
\|u_m\|^2_{H^s_0(\Omega)} \leq C\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \|f\|^2_{H^s_0(\Omega)}.
\]
Integrate from 0 to $T$ and drop the first term. Let $n \to \infty$, then we obtain
\[
\|(-\Delta)^s u_m\|_{L^2((0,T);L^2(\Omega))}^2 \leq C\|g\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|f\|_{L^p(0,T;L^2(\Omega))}^2.
\]

Thus, we get
\[
\{u_m\} \text{ is bounded in } L^\infty(0,T;H_0^s(\Omega)) \cap L^{p+2}(\Omega),
\]
\[
\{M(\|u_m\|_{H_0^s(\Omega)}^2)u_m\} \text{ is bounded in } L^\infty(0,T;H_0^s(\Omega)),
\]
\[
\{(-\Delta)^s u_m\} \text{ is bounded in } L^2(0,T;L^2(\Omega)).
\]

Moreover, we also obtain
\[
\|u_m - f_m\|_{L^\infty(0,T;H_0^s(\Omega))} \cap L^{p+2}(\Omega).
\]

Further, if $\gamma > 0$, by Lemma 4.1, then we have
\[
\|\partial_t^\gamma u_m\|_{L^2(0,T;L^2(\Omega))}^2 \leq C(\|f\|_{H_0^s(\Omega)}^2 + \|\partial^2_t f_{H_0^s(\Omega)}\|^2 + \|g\|_{L^\infty(0,T;L^2(\Omega))}^2).
\]

If $\gamma < 0$, for $\varphi \in L^2(0,T;L^2(\Omega))$ satisfying $\varphi_m(t) = \sum_{j=1}^m c_{mj}(t)w_j \to \varphi(t)$, then we have
\[
(\partial_t^\gamma u_m, \varphi_m) + M(\|u_m\|_{H_0^s(\Omega)}^2)((-\Delta)^s u_m, \varphi_m) = \gamma(\|u_m\|_{H_0^s(\Omega)}^2, \varphi_m) + (g, \varphi_m).
\]

Then, we get
\[
(|\partial_t^\gamma u_m, \varphi|) \leq C(\|(-\Delta)^s u_m\|_{L^2(\Omega)} + \|u_m\|_{H_0^s(\Omega)}^2 + \|\partial_t \varphi\|_{L^2(\Omega)} + \|\partial^2_t \varphi\|_{L^2(\Omega)}).
\]

Then, Hölder inequality, we conclude that
\[
\int_0^T (\partial_t^\gamma u_m, \varphi) dt \leq C(\|f\|_{H_0^s(\Omega)} + \|f\|_{H_0^s(\Omega)}^2 + \|g\|_{L^\infty(0,T;L^2(\Omega))})\|\varphi\|_{L^2(0,T;L^2(\Omega))}.
\]

Therefore, we obtain
\[
\|\partial_t^\gamma u_m\|_{L^2(0,T;L^2(\Omega))} \leq C(\|f\|_{H_0^s(\Omega)} + \|f\|_{H_0^s(\Omega)}^2 + \|g\|_{L^\infty(0,T;L^2(\Omega))})
\]

By Theorem 2.1 in [8], we have
\[
\|u_m - f_m\|_{H^{\beta}(0,T;H^{-s}(\Omega))} \leq C.
\]

By (20) and (21), we get
\[
\{u_m - f_m\} \text{ is bounded in } L^2(0,T;H_0^s(\Omega)),
\]
\[
\{u_m - f_m\} \text{ is bounded in } H^{\beta}(0,T;H^{-s}(\Omega)).
\]

Since $H_0^s(\Omega)$ is compactly embedded into $L^2(\Omega)$, by Compactness Theorem, we deduce that there exists a subsequence (still denoted by $\{u_m - f_m\}$) such that
\[
\begin{align*}
& u_m - f_m \to u - f \text{ in } L^2(0,T;L^2(\Omega)), \\
& u_m - f_m \to u - f \text{ a.e. in } (0,T) \times \Omega.
\end{align*}
\]

Then, we obtain
\[
\begin{align*}
& u_m \to u \text{ in } L^2(0,T;L^2(\Omega)), \\
& u_m \to u \text{ a.e. in } (0,T) \times \Omega.
\end{align*}
\]
Thus, we conclude that
\[
\partial_t^2 u_m \rightharpoonup \partial_t^2 u \text{ in } L^2(0, T; L^2(\Omega)),
\]
\[
u_m \rightharpoonup u \text{ in } L^\infty(0, T; H^1_0(\Omega) \cap L^{p+2}(\Omega)),
\]
\[
M(\|u_m\|_{H^1_0(\Omega)})u_m \rightharpoonup \eta \text{ in } L^\infty(0, T; H^1_0(\Omega)),
\]
\[
|u_m|^p u_m \rightharpoonup |u|^p u \text{ in } L^\infty(0, T; L^{\frac{2p+2}{p+2}}(\Omega)).
\]
Therefore, we obtain
\[
\int_0^T (\partial_t^2 u, v) dt + \int_0^T (\eta, (-\Delta)^s u) dt = \gamma \int_0^T (|u|^p u, v) dt + \int_0^T (g, v) dt,
\]
for any \(v \in C^1([0, T]; H^1_0(\Omega))\) with \(v(T) = 0\). In particular, taking \(v = u_m\) and letting \(m \to \infty\), we have
\[
\int_0^T (\partial_t^2 u, u) dt + \int_0^T (\eta, (-\Delta)^s u) dt = \gamma \int_0^T (|u|^p u, u) dt + \int_0^T (g, u) dt.
\]
Next, we show that \(\eta = M(\|u\|_{H^1_0(\Omega)})u\). Set
\[
Y_m = \int_0^T M(\|u_m\|_{H^1_0(\Omega)})((-\Delta)^s (u_m - u), u_m - u) dt
\]
\[- \gamma \int_0^T (|u_m|^p u_m - |u|^p u, u_m - u) dt.
\]
Then, we have
\[
Y_m = \int_0^T (g, u_m) dt - \int_0^T (\partial_t^2 u_m, u_m) dt - \int_0^T M(\|u_m\|_{H^1_0(\Omega)})((-\Delta)^s u_m, u) dt + \gamma \int_0^T (|u_m|^p u_m, u_m - u) dt - \int_0^T M(\|u_m\|_{H^1_0(\Omega)})((-\Delta)^s u, u_m - u) dt + \gamma \int_0^T (|u|^p u, u_m - u) dt.
\]
From (19) and (22), we conclude that \(\{M(\|u_m\|_{H^1_0(\Omega)})\}\) is equi-integrable and uniformly bounded in \(L^1(0, T)\). By Lemma 2.7, there exists a subsequence (still denoted by \(\{M(\|u_m\|_{H^1_0(\Omega)})\}\)) such that
\[
M(\|u_m\|_{H^1_0(\Omega)}) \rightharpoonup \varsigma \text{ in } L^1(0, T).
\]
Since \((-\Delta)^s u_m - u \rightharpoonup 0 \text{ in } L^\infty(0, T),\) we have
\[
\int_0^T M(\|u_m\|_{H^1_0(\Omega)})((-\Delta)^s u, u_m - u) \to 0.
\]
Therefore, we have
\[
Y_m \to \int_0^T (g, u) dt - \int_0^T (\partial_t^2 u, u) dt + \int_0^T (\eta, (-\Delta)^s u) dt - \gamma \int_0^T (|u|^p u, u) dt = 0.
\]
Further, if \(\gamma < 0\), then we have \(Y_m \geq 0\). By Simon inequality, we derive that
\[
u_m \rightharpoonup u \text{ in } L^2(0, T; H^1_0(\Omega)),
\]
\[
u_m \rightharpoonup u \text{ in } L^{p+2}(0, T; L^{p+2}(\Omega)).
\]
If $\gamma > 0$, by Cauchy-Schwarz inequality, then we have
\[
0 \leq \int_0^T [a + b\|u_m\|^2_{H_0^s(\Omega)}]((-\Delta)^s(u_m - u), u_m - u) dt
\leq Y_m + C\|u_m - u\|_{L^2(0,T;L^2(\Omega))}.
\]

Let $m \to \infty$, we conclude that
\[
u_m \to u \text{ in } L^2(0,T;H_0^s(\Omega)),
\]
\[
u_m \to u \text{ in } L^{p+2}(0,T;L^{p+2}(\Omega)).
\]

Therefore, there exists a subsequence (still denoted by $\{u_m\}$) such that
\[
u_m \to u \text{ in } H^s(\Omega) \to \tilde{\nu} \text{ a.e. in } (0,T).
\]

Then,
\[
a + b\|u_m\|^2_{H_0^s(\Omega)} \to a + b\|u\|^2_{H_0^s(\Omega)} \text{ a.e. in } (0,T).
\]

Further, from (22) we have
\[
\int_0^T (\partial_t^\beta u, v) dt + \int_0^T M(\|u\|^2_{H_0^s(\Omega)})(u, (-\Delta)^s v) dt = \int_0^T (|u|^p u, v) dt + \int_0^T (g, v) dt,
\]
for any $v \in C^1([0,T]; H_0^s(\Omega))$ with $v(T) = 0$. It follows that $u$ is a weak solution of problem (1). Next, we show that the weak solution is unique. Suppose $u_1$ and $u_2$ are weak solutions of problem (1). Let $u = u_1 - u_2$, then we have
\[
(\partial_t^\beta u, v) + (\Psi u, v) = \gamma(|u_1|^p u_1 - |u_2|^p u_2, v), \text{ a.e. } t \in (0,T),
\]
for any $v \in H_0^s(\Omega)$ where
\[
\Psi u = M(\|u_1\|^2_{H_0^s(\Omega)})(-\Delta)^s u_1 - M(\|u_2\|^2_{H_0^s(\Omega)})(-\Delta)^s u_2.
\]

Taking $v = u$, we get
\[
(\partial_t^\beta u, u) + (\Psi u, u) = \gamma(|u_1|^p u_1 - |u_2|^p u_2, u), \text{ a.e. } t \in (0,T).
\]

By the Yosida approximation, we have
\[
\left(\frac{d}{dt} g_{1-\beta,n} \ast u, u\right) + (\Psi u, u) = \gamma(|u_1|^p u_1 - |u_2|^p u_2, u) + \tilde{R}_n, \text{ a.e. } t \in (0,T),
\]
where
\[
\tilde{R}_n = \left(\frac{d}{dt} (g_{1-\beta,n} \ast u) - \frac{d}{dt} (g_{1-\beta} \ast u), u\right).
\]

By Lemma 2.2, we get
\[
\left(\frac{d}{dt} g_{1-\beta,n} \ast u, u\right) \geq \frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2(\Omega)}. \tag{23}
\]
Furthermore, we have
\[
(\Psi u, u) = a\|u\|_{H^s_0(\Omega)}^2 + (b\|u_1\|_{H^s_0(\Omega)}^2 (-\Delta)^s u_1 - b\|u_2\|_{H^s_0(\Omega)}^2 (-\Delta)^s u_2, u)
\]
\[
= a\|u\|_{H^s_0(\Omega)}^2 + b\|u_1\|_{H^s_0(\Omega)}^2 + b\|u_2\|_{H^s_0(\Omega)}^2
\]
\[
- b(\|u_1\|_{H^s_0(\Omega)}^2 + \|u_2\|_{H^s_0(\Omega)}^2)((-\Delta)^s u_1, u_2)
\]
\[
\geq a\|u\|_{H^s_0(\Omega)}^2 + \Phi,
\]
where
\[
\Phi = b(\|u_1\|_{H^s_0(\Omega)} - \|u_2\|_{H^s_0(\Omega)})^2(\|u_1\|_{H^s_0(\Omega)}^2 + \|u_2\|_{H^s_0(\Omega)}^2 + \|u_2\|_{H^s_0(\Omega)}^2).
\]
If \(\gamma > 0\), by Simon inequality, then we have
\[
(\|u_1\|^\rho u_1 - \|u_2\|^\rho u_2, u) = \int_\Omega (\|u_1\|^\rho u_1 - \|u_2\|^\rho u_2)(u_1 - u_2)dx
\]
\[
\geq \int_\Omega C(\rho)|u_1 - u_2|^\rho + 2 dx \geq 0. \quad (24)
\]
From (23) and (24), we get
\[
\frac{1}{2} \frac{d}{dt}(g_{1-\beta,n} * \|u\|_{L^2(\Omega)}^2) + \|u\|_{H^s_0(\Omega)}^2 \leq \tilde{R}_n, \text{ a.e. } t \in (0, T).
\]
Drop the second term and convolute with \(g_\beta\). Let \(n \to \infty\), select an appropriate subsequence, then we obtain
\[
\|u(t)\|_{L^2(\Omega)}^2 \leq 0, \text{ a.e. } t \in (0, T).
\]
If \(\gamma < 0\), by Young inequality, then we have
\[
|\gamma||\|u_1\|^\rho u_1 - \|u_2\|^\rho u_2, u| \leq C(\max(|\|u_1\|^\rho, |\|u_2\|^\rho|)|u_1 - u_2|, |u|)
\]
\[
\leq C\|\|u_1\|^\rho + \|u_2\|^\rho\|_{L^{2+2/(\rho-2)}(\Omega)}\|u\|_{L^2(\Omega)}
\]
\[
\leq C\|u\|_{H^s_0(\Omega)}\|u\|_{L^2(\Omega)}
\]
\[
\leq \frac{a}{2}\|u\|_{H^s_0(\Omega)}^2 + C\|u\|_{L^2(\Omega)}^2.
\]
Then, we conclude that
\[
\frac{1}{2} \frac{d}{dt}(g_{1-\beta,n} * \|u\|_{L^2(\Omega)}^2) + \frac{a}{2}\|u\|_{H^s_0(\Omega)}^2 \leq C\|u\|_{L^2(\Omega)}^2 + \tilde{R}_n, \text{ a.e. } t \in (0, T).
\]
Drop the second term and convolute with \(g_\beta\). Let \(n \to \infty\), select an appropriate subsequence, then we obtain
\[
\|u(t)\|_{L^2(\Omega)}^2 \leq C g_\beta * \|u\|_{L^2(\Omega)}^2, \text{ a.e. } t \in (0, T).
\]
By Gronwall inequality, we have
\[
\|u(t)\|_{L^2(\Omega)}^2 \leq 0, \text{ a.e. } t \in (0, T).
\]
Therefore, we obtain \(u = 0\) a.e. in \(L^2(\Omega)\), i.e. \(u_1 = u_2\). \(\square\)
5. **Decay properties.** In this section, we assume \( g = 0 \). We consider the decay properties of weak solutions. The following differential inequality is an important tool to obtain the decay properties of weak solutions, which is derived by Vergara and Zacher [26].

**Lemma 5.1 ([26]).** Let \( u(t) \geq 0 \) be a solution of the differential inequality
\[
\partial_t^\alpha u + \nu u^l \leq 0, \quad u(0) = u_0,
\]
for \( t \geq 0 \) where \( \nu > 0, l > 0 \). Then there exist constants \( C_1, C_2 > 0 \) such that
\[
\frac{C_1}{1 + t^\beta} \leq u(t) \leq \frac{C_2}{1 + t^\gamma}, \quad t \geq 0,
\]
where \( C_1, C_2 \) are constants depending on \( u_0, \beta \) and \( l \).

First, we prove Theorem 1.3.

**Proof.** Let \( u \) be the weak solution of problem (1), taking \( v = u(t) \), then we have
\[
\int_\Omega \partial_t^\alpha u dx + (a + b\|u\|_{H^s(\Omega)}^2)\|u\|_{H^s(\Omega)}^2 = \gamma\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2}, \text{ a.e. } t \in (0, T).
\]
By Lemma 2.8, we have
\[
\|u\|_{L^2(\Omega)} \partial_t^\alpha \|u\|_{L^2(\Omega)} + (a + b\|u\|_{H^s(\Omega)}^2)\|u\|_{H^s(\Omega)}^2 \leq \gamma\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2}.
\]
If \( \gamma < 0 \), by Lemma 2.1, then we have
\[
\|u\|_{L^2(\Omega)} \partial_t^\alpha \|u\|_{L^2(\Omega)} + C(a)\|u\|_{L^2(\Omega)}^4 + C(b)\|u\|_{L^2(\Omega)}^4 + |\gamma|\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2} \leq 0.
\]
Therefore, by Lemma 5.1, we obtain
\[
\|u\|_{L^2(\Omega)} \leq \min\{ \frac{C_1}{1 + t^\beta}, \frac{C_2}{1 + t^\gamma} \}, \quad t \in (0, T),
\]
where \( C_1, C_2, C_3 \) are different positive constants depending on \( f, \beta, s, \gamma, a, b, \rho, \Omega \). If \( \gamma > 0 \), by Gagliardo-Nirenberg inequality and Young inequality, then we have
\[
|\gamma|\|u\|_{L^{\rho+2}(\Omega)}^{\rho+2} \leq |\gamma|\|C\|\|u\|_{L^2(\Omega)}^{(\rho+2)(1-\theta)}\|u\|_{H^s(\Omega)}^{(\rho+2)\theta}
\leq |\gamma|C(\varepsilon)\|u\|_{L^2(\Omega)}^4 + \varepsilon\|u\|_{H^s(\Omega)}^2
\leq |\gamma|C(a)\|u\|_{L^2(\Omega)}^4 + \frac{a}{2}\|u\|_{H^s(\Omega)}^2,
\]
where
\[
\theta = \frac{N\rho}{2s(\rho + 2)} < \frac{2}{(\rho + 2)},
\]
and
\[
\kappa = \frac{4s(\rho + 2) - 2N\rho}{4s - N\rho}.
\]
Therefore, by Lemma 2.1, we have
\[
\|u\|_{L^2(\Omega)} \partial_t^\alpha \|u\|_{L^2(\Omega)} + C(a)\|u\|_{L^2(\Omega)}^4 + (C(b) - C(a, \gamma))\|u\|_{L^2(\Omega)}^4 \leq 0.
\]
Taking small enough \( |\gamma| \) such that
\[
C(b) - C(a, \gamma) > 0,
\]
by Lemma 5.1, we obtain
\[
\|u\|_{L^2(\Omega)} \leq \min\{ \frac{C_1}{1 + t^\beta}, \frac{C_2}{1 + t^\gamma} \}, \quad t \in (0, T),
\]
where \( C_1, C_2 \) are different positive constants depending on \( f, \beta, s, \gamma, a, b, \rho, \Omega \).
Next, we prove Theorem 1.4.

Proof. Taking \( v(t) = |u|^p u \), we have

\[
\int_\Omega |u|^p u \partial_t^\gamma u \, dx + M(||u||^2_{H^2_0(\Omega)})((-\Delta)^{\gamma} u, |u|^p u) = \gamma ||u||^{2p+2}_{L^{2p+2}(\Omega)}.
\]

If \( \gamma < 0 \), we have

\[
M(||u||^2_{H^2_0(\Omega)})((-\Delta)^{\gamma} u, |u|^p u) \geq C(a + b ||u||^2_{H^2_0(\Omega)})||u||^{2p+2}_{L^{2p+2}(\Omega)}
\]

\[
\geq C(a + b ||u||^2_{H^2_0(\Omega)})||u||^\frac{4p+4}{2p+2}_{L^{2p+2}(\Omega)}
\]

\[
\geq C(a)||u||^{p+2}_{L^{p+2}(\Omega)} + C(b)||u||^{p+4}_{L^{p+4}(\Omega)}.
\]

By Lemma 2.8 and Hölder inequality, we have

\[
||u||^{p+1}_{L^{p+2}(\Omega)} \partial_t^\gamma ||u||_{L^{p+2}(\Omega)} + C(a)||u||^{p+2}_{L^{p+2}(\Omega)} + C(b)||u||^{p+4}_{L^{p+4}(\Omega)} + C(\gamma)||u||^{2p+2}_{L^{2p+2}(\Omega)} \leq 0.
\]

Then, we obtain

\[
\partial_t^\gamma ||u||_{L^{p+2}(\Omega)} + C(a)||u||_{L^{p+2}(\Omega)} + C(b)||u||^{3}_{L^{p+2}(\Omega)} + C(\gamma)||u||^{p+1}_{L^{p+2}(\Omega)} \leq 0.
\]

Therefore, by Lemma 5.1, we obtain

\[
||u||_{L^{p+2}(\Omega)} \leq \min\left\{ \frac{C_1}{1 + t^{\frac{\theta}{2}}, \quad \frac{C_2}{1 + t^{\frac{\theta}{2}}, \quad \frac{C_3}{1 + t^{\frac{\theta}{2}}} \right\}, \quad t \in (0, T),
\]

where \( C_1, C_2, C_3 \) are different positive constants depending on \( f, \beta, s, \gamma, a, b, \rho, \Omega \). If \( \gamma > 0 \), by Gagliardo-Nirenberg inequality and Young inequality, we have

\[
|\gamma||u||^{2p+2}_{L^{2p+2}(\Omega)} \leq |\gamma||C||u||^{(2p+2)(1-\theta)}_{L^{2(1-\theta)}(\Omega)} ||u||^{2p+2}_{L^{2p+2}(\Omega)}
\]

\[
\leq |\gamma||C||u||^{(2p+2)(1-\theta)}_{L^{2(1-\theta)}(\Omega)} ||u||^{2p+2}_{L^{2p+2}(\Omega)}
\]

\[
\leq |\gamma||C||u||^{(2p+2)(1-\theta)}_{L^{2p+2}(\Omega)} ||u||^{2p+2}_{L^{2p+2}(\Omega)}
\]

\[
\leq |\gamma||C(a)||u||^{p+2}_{L^{p+2}(\Omega)} + \frac{a}{4} ||u||^{p+2}_{L^{p+2}(\Omega)}
\]

\[
\leq C(a, \gamma)||u||^{p+4}_{L^{p+2}(\Omega)} + \frac{a}{4} ||u||^{p+2}_{L^{p+2}(\Omega)}
\]

where \( \theta = \frac{N\rho}{2\rho + 2} \) satisfying

\[
\frac{1}{2p + 2} = \frac{1 - \theta}{2} + \frac{\theta}{2},
\]

and

\[
\kappa = (1 + \frac{s\rho}{s\rho + 2s - \rho N})(\rho + 2).
\]

Therefore, by Lemma 2.1, we conclude that

\[
||u||^{p-1}_{L^{p+2}(\Omega)} \partial_t^\gamma ||u||_{L^{p+2}(\Omega)} + C(a)||u||^{p+2}_{L^{p+2}(\Omega)} + (C(b) - C(a, \gamma)||u||^{p+4}_{L^{p+2}(\Omega)} \leq 0.
\]

Then,

\[
\partial_t^\gamma ||u||_{L^{p+2}(\Omega)} + C(a)||u||_{L^{p+2}(\Omega)} + (C(b) - C(a, \gamma)||u||^{p+1}_{L^{p+2}(\Omega)} \leq 0.
\]

Taking small enough \(|\gamma|\) such that

\[
C(b) - C(a, \gamma) > 0,
\]
by Lemma 5.1, we obtain
\[ \|u\|_{L^{\rho+2}(\Omega)} \leq \min\{\frac{C_1}{1+t^\beta}, \frac{C_2}{1+t^\beta}\}, \quad t \in (0, T), \]
where \(C_1, C_2\) are different positive constants depending on \(f, \beta, s, \gamma, a, b, \rho, \Omega\).

**Remark 3.** Theorem 1.3 and Theorem 1.4 show that the decay properties in the case \(0 < \beta < 1\) differ from that in the case \(\beta = 1\), where we have exponential decay as \(u(t) \sim e^{-t}\).

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