Short-range Interaction and Nonrelativistic $\phi^4$ Theory in Various Dimensions

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Abstract

We employ the effective field theory method to systematically study the short-range interaction in two-body sector in 2, 3 and 4 spacetime dimensions, respectively. The $\phi^4$ theory is taken as a specific example and matched onto the nonrelativistic effective theory to one loop level. An exact, Lorentz-invariant expression for the S-wave amplitude is presented, from which the nonperturbative information can be easily extracted. We pay particular attention to the renormalization group analysis in the 3 dimensions, and show that relativistic effects qualitatively change the renormalization group flow of higher-dimensional operators. There is one ancient claim that triviality of the 4-dimensional $\phi^4$ theory can be substantiated in the nonrelativistic limit. We illustrate that this assertion arises from treating the interaction between two nonrelativistic particles as literally zero-range, which is incompatible with the Uncertainty Principle. The S-wave effective range in this theory is identified to be approximately $16/3\pi$ times the Compton wavelength.
I. INTRODUCTION

Short-range forces commonly arise in nuclear and condensed-matter system. Because the microscopic dynamics is often not well understood, one usually models the potential with some arbitrary parameters based on empirical assumptions. These parameters are then adjusted by trial and error from numerical solution of the Schrödinger equation.

Because of the \textit{ad hoc} nature in constructing the potential, this traditional method suffers some severe shortcomings. Most notably, it involves the uncontrolled approximations, thus making a reasonable error estimate impossible.

These difficulties can be overcome if the effective field theory (EFT) approach is instead exploited. EFT provides a model-independent means to address problems with separated scales (For a modern and comprehensive review on this topic, see Ref. [1]). When non-relativistic particles interact through short-range forces, their de Broglie wavelengths are much longer than the typical range of interaction. Therefore, they can be treated as point particles, and the low energy dynamics can be described by a local nonrelativistic EFT. The coexistence of two disparate scales, the momentum $k$ and the cutoff $\Lambda$, which is roughly the inverse of the interaction range, validates a systematic expansion in powers of $k/\Lambda$. Contrary to phenomenological potential model, this framework accommodates a transparent power counting, so that the error estimate can be performed systematically. Recently, this method has been fruitfully applied to the few nucleon system [2] and cold dilute Bose gas [3] and Fermion gas [4].

The $\delta$-function potential is usually considered as the prototype of the short-range interaction [5–7]. Simple enough as it may look, this zero-range potential turns out to be too singular when iterated in the Schrödinger equation (except in one spatial dimension), so that one has to regulate and renormalize the resulting ultraviolet divergences.

There is no systematic way to deal with renormalization in nonrelativistic quantum mechanics. In contrast, EFT is the paradigm for the modern understanding of renormalization [1]. Furthermore, the contributions suppressed by inverse powers of $\Lambda$, and relativistic effects proportional to the powers of $k/m$, can be conveniently incorporated in this field-theoretic framework. Neither of these can be easily coped with in the Schrödinger formalism with contact potentials.

Short-range interaction in four dimensions has already been extensively explored in literature, but not much effort has been devoted to the lower dimensional cases. The goal of this paper is to employ the EFT method to systematically investigate the short-range interactions in the two-body sector in various spacetime dimensions. The highlight of this work is to present an exact, Lorentz-invariant expression for the S-wave scattering amplitude. This represents an improvement to the previous results, where the relativistic effects are usually ignored, or only incorporated to the first order.

The capability to arrive at an exact amplitude is rooted in the tremendous simplicity of the nonrelativistic EFT, where one can sum all the four-point Green functions analytically. Some useful nonperturbative insight can be gained from this exact expression. Specifically, this ability is indispensable when discussing the 2-body scattering in 2D and 3D, where the amplitude at fixed order of perturbative expansion is plagued by the zero-momentum singularity. In contrast, taking into account the singular terms to all orders, this exact
amplitude is well behaved in the $k \to 0$ limit.

It is always nice to have some concrete example at hand. The $\phi^4$ theory constitutes one simple, but instructive example. In the nonrelativistic limit, this theory is expected to simulate the $\delta$-function potential. This consideration has motivated some discussions on different aspects of the $\phi^4$ theory in this limit [8–11]. Unfortunately, each work is more or less isolated and the underlying rationale of EFT is not fully realized.

Unlike the nuclear or atomic system, where the microscopic dynamics is too complicated to pinpoint analytically, the full knowledge of this model theory allows us to determine all the EFT parameters to any desired order. Benefiting from the power of EFT, we are able to develop some detailed understanding of this theory in the nonrelativistic limit. On the other hand, the $\phi^4$ theory also provides some useful guidances in deducing the Lorentz-invariant amplitude in the nonrelativistic EFT.

The benchmark feature of the 4D $\phi^4$ theory, as well as other non asymptotically-free theories, is triviality, in the sense that the renormalized coupling $\lambda$ has to vanish, if one insists that this theory be valid all the way down to the arbitrarily short distance (say, Planck length) [12]. It is worth emphasizing, this symptom needs not to be regarded as a serious trouble, since this theory has to merge into a more fundamental theory at some point long before approaching the Planck scale. As long as treated as a low energy effective theory, the predictivity of this theory is not sacrificed.

Nearly two decades ago, Beg and Furlong claimed that triviality of the $\phi^4$ theory can be explicitly corroborated in the nonrelativistic limit [8]. They regarded this as another piece of evidence for the triviality, complimentary to the more rigorous “proof” from formulating this theory nonperturbatively on the spacetime lattice. If their claim were true, it would open an easier way to access this rather formal problem.

This paper is organized as follows: in Sec. II, we describe the most general nonrelativistic effective Lagrangian which is relevant to the two-body S-wave scattering. After the pitfall of the real $\phi^4$ theory in the nonrelativistic limit is pointed out, we match this theory onto the EFT at the tree level. We also sketch how to implement relativistic corrections in the nonrelativistic EFT. In Sec. III, we show that, contrary to what Beg and Furlong claimed, the triviality of the 4D $\phi^4$ theory can not be substantiated in the nonrelativistic limit. The flaw of their argument is traced, and attributed to the incorrect renormalization of the power-law divergences in the cutoff scheme, or equivalently, treating the two-body interaction in this theory literally as contact.

Sec. IV is the main body of this paper, in which a detailed analysis of short-range forces in different space-time dimensions: 2, 3 and 4 is presented. An exact, nonperturbative and Lorentz-invariant expression for the S-wave amplitude is obtained when the dimensional regularization is employed. The $\phi^4$ theory is taken as a concrete example to illustrate the systematics of matching beyond tree level. The peculiarity for each spacetime dimensions is discussed. In particular, we present a detailed renormalization group analysis for the 3D case, and elaborate on the role played by relativity. We also identify the effective range in the 4D $\phi^4$ theory roughly to be the Compton wavelength. We summarize our results in Sec. V.
II. EFFECTIVE LAGRANGIAN AND TREE-LEVEL MATCHING

Before moving on to the nonrelativistic EFT, we first discuss some features of the real \( \phi^4 \) theory. It has the Lagrangian
\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4, \tag{1}
\]
where the natural unit \( \hbar = c = 1 \) is adopted. Vacuum stability requests a positive \( \lambda \), corresponding to a repulsive contact interaction. We don’t specify the spacetime dimension in which (1) is defined.

It should be cautioned, the real \( \phi^4 \) theory is intrinsically hostile to a nonrelativistic description, because it doesn’t respect the particle number conservation. A better candidate to study the nonrelativistic limit is the complex \( \phi^4 \) theory with a \( U(1) \) charge [8]. Provided that one prepares \( N \) nonrelativistic particles of the same charge, excluding those of opposite charge, charge conservation will guarantee that \( N \) is conserved.

Lacking any continuous internal symmetry notwithstanding, the real \( \phi^4 \) theory still possesses a legitimate nonrelativistic limit for 2-body and 3-body interactions, thanks to the energy-momentum conservation which forbids the number of particles to further decrease.

However, it no longer makes sense to talk about the nonrelativistic limit for more than 3 particles in this theory. For example, Fig. 1 shows that even though all the four particles in the initial state are nearly at rest, they can readily annihilate into two highly relativistic particles. There have been discussions about the many-body phenomena of the real \( \phi^4 \) theory in the nonrelativistic limit [11]. For the reason just outlined, the starting point of Ref. [11] looks rather questionable.

In this work, we will confine ourselves in the 2-body nonrelativistic scattering, so we are allowed to stay with the real \( \phi^4 \) theory. For simplicity, we will focus on the S-wave (isotropic) scattering only, which is the most important partial wave in the low energy limit.

Symmetry is the guidepost in constructing a low energy effective theory. For a nonrelativistic system, the most important symmetries are particle number conservation, Galilean invariance \( (\mathbf{k} \rightarrow \mathbf{k} + m\mathbf{v}) \), time reversal and parity. The most general form of the effective Lagrangian compatible with these symmetries is
\[
\mathcal{L}_{\text{NR}} = \Psi^* \left( i \partial_t + \frac{\nabla^2}{2m} \right) \Psi - \frac{C_0}{4} (\Psi^* \Psi)^2 - \frac{C_2}{8} \nabla (\Psi^* \Psi) \cdot \nabla (\Psi^* \Psi) + \cdots. \tag{2}
\]
One can read off the nonrelativistic propagator to be $i/(E - k^2/2m + i\epsilon)$, where $E$ is the kinetic energy, and $k$ is the 3-momentum. In addition to the one-body kinetic operators, we also include the two-body operators which describe the S-wave scattering. Because Bose statistics and parity forbid the P-wave scattering (and all other odd partial waves), the term proportional to $C_2$ is the only possible one with two gradients allowed by Galilean invariance. The next tower of operators enter at 4th order of $\nabla$, which contribute to both the S- and D-wave scattering.

In an arbitrary reference frame, if two initial-state particles have momenta $k_1$ and $k_2$, two final-state particles carry momenta $k'_1$ and $k'_2$, then the tree-level amplitude of (2) reads

$$A_{\text{tree}}^0 = - C_0 - \frac{C_2}{4} [(k_1 - k'_1)^2 + (k_1 - k'_2)^2] - \cdots ,$$

which is clearly Galilean-invariant. In the center-of-momentum (C.M.) frame, the 3-momentum of each particle has the equal magnitude $k$, and the second term in the right-hand side collapses to $-C_2 k^2$. Thus we are reassured that the Lagrangian (2) indeed describes the S-wave scattering.

The Wilson coefficients $C_0, C_2, \ldots$ encode all the short-distance information. For two-nucleon system and Bose gas, experimental input is needed to deduce these coefficients. For the $\phi^4$ theory, they can be determined from the matching procedure, i.e., by requiring that the effective theory (2) reproduces the same physical observable as the full theory (1), up to a prescribed accuracy in powers of $k/\Lambda$, order by order in loop expansion [1]. Since the relativistic and nonrelativistic theories usually adopt different conventions in normalization of states, one should be careful in specifying the matching condition.

To quickly access the nonrelativistic behavior of (1), it is customary to parameterize the relativistic field $\phi$ as [8,9]

$$\phi = \frac{1}{\sqrt{2m}} \left( e^{-imt}\Psi + e^{imt}\Psi^* \right),$$

where the field $\Psi$ only excites the nonrelativistic degree of freedom. Plugging (4) back into (1), and dropping terms containing the rapidly oscillating phases $\exp(\pm 2imt)$, we obtain a new Lagrangian:

$$L' = \Psi^* \left( \frac{i}{2} \partial_t + \frac{\nabla^2}{2m} - \frac{\partial_i^2}{2m} \right) \Psi - \frac{\lambda}{16m^2} (\Psi^*\Psi)^2 .$$

The one-body operator with two time derivatives is not present in the standard nonrelativistic Lagrangian (2). The function of this term is to recover the Lorentz symmetry. To see this lucidly, one can rewrite the relativistic scalar propagator as

$$\frac{i}{k^2 - m^2 + i\epsilon} = \frac{1}{2m} \frac{i}{E - \frac{k^2}{2m} + \frac{E^2}{2m} + i\epsilon} ,$$

where $E \equiv k^0 - m$ is the kinetic energy.

The higher-derivative operators account for the dynamical short-distance effects, and their corresponding Wilson coefficients depend on the specific system under investigation.
In contrast, relativistic corrections represent a purely kinematic effect, thus universal in any system.

In a realistic system, relativistic effects are usually less important than the higher-dimensional operators, because the cutoff \( \Lambda \) is normally much less than the particle mass. For the \( \phi^4 \) theory, we instead have \( \Lambda \sim m \), so the relativistic corrections, as well as those terms proportional to powers of \( k/\Lambda \), should be included simultaneously.

Relativistic effects can be systematically accounted for in the field-theoretic framework, which sharply contrasts to the Schrödinger formalism. There are two different methods to implement relativistic corrections. One popular way is first redefining the field [13]:

\[
\Psi = \left(1 + \frac{\nabla^2}{4m^2} + \frac{5\nabla^4}{32m^4} + \cdots \right) \Psi' = \left(\frac{m}{\sqrt{m^2 + k^2}}\right)^{1/2} \Psi',
\]

and upon using equation of motion, one trades that extra operator with two time derivatives in (5) for an infinite tower of terms with spatial gradients:

\[
L'' = \Psi^{\dagger} \left(i\partial_t + \frac{\nabla^2}{2m} + \frac{\nabla^4}{8m^3} + \frac{\nabla^6}{16m^5} + \cdots \right) \Psi' + \cdots.
\]

From this equation, one can recover the relativistic dispersion relation order by order:

\[
E \equiv \sqrt{m^2 + k^2} - m = \frac{k^2}{2m} - \frac{k^4}{8m^3} + \frac{k^6}{16m^5} - \cdots.
\]

In calculating relativistic corrections, one replaces the kinetic part of the effective Lagrangian (2) by (8). These higher-derivative one-body operators are understood to be perturbatively inserted in the loop, the energy-momentum relation for the external legs should also be readjusted according to (9).

However, as pointed out in Ref. [14], working with (8) is somewhat cumbersome, because every term in the effective Lagrangian (2) is subject to the field redefinition (7). For example, the field redefinition exerting on the lowest dimensional four-boson operator will induce a new operator \(-C_0/(8m^2)[(\nabla^2\Psi^{\dagger}\Psi')\Psi^{\dagger}\Psi' + (\Psi^{\dagger}\nabla^2\Psi')\Psi^{\dagger}\Psi']\), which mixes with the operator proportional to \(C_2\). It adds a new contribution \(C_0/(4m^2)(k_1^2 + k_2^2 + k_3^2 + k_4^2)\) to the tree-level amplitude (3). In the C.M. frame, the new amplitude becomes

\[
\mathcal{A}'_{\text{tree}} = -C_0 \left(1 - \frac{k^2}{m^2} + \cdots \right) - C_2 k^2 - \cdots.
\]

It turns out that directly replacing the kinetic part of (2) by that of (5), without invoking the field redefinition (7), is much more convenient [14]. Firstly, we are free from a plethora of induced new operators. Secondly, we just need insert that single one-body operator with two time derivatives iteratively in the loop, instead of facing an infinite number of higher-derivative one-body operators as indicated in (8).

Taking into account the trivial rescaling from the relativistic field \( \phi \) to the nonrelativistic field \( \Psi \) in (4) (or see Eq. (6)), we obtain the matching equation:

\[
T_0 = 4m^2 \mathcal{A}_0,
\]
where $T_0$ is the S-wave partial amplitude projected out of the $T$-matrix element in the full theory. This matching formula holds to any loop order.

If we had used (8) to implement the relativistic corrections, we should have multiplied the right-hand side of (11) by $(\sqrt{m^2 + k^2/m})^{4/2} = 1 + k^2/m^2$, to compensate for the modification of the residue of the propagator by the field redefinition (7), in compliance with the Lehmann-Symanzik-Zimmermann reduction formula [13]. Therefore, the matching formula in this scheme becomes

$$T_0 = 4 (m^2 + k^2) A'_0. \tag{12}$$

Although the tree level amplitude in the relativistic $\phi^4$ theory, $T_0^{\text{tree}} = -\lambda$, is innocently simple, the matching in this scheme becomes unnecessarily involved.

We will employ (5) to implement the relativistic corrections throughout this work. In this scheme, tree level matching is trivial. From the matching formula (11) and the tree-level EFT amplitude (3), the Wilson coefficients can be simply determined:

$$C_0 = \frac{\lambda}{4m^2} + O(\lambda^2), \tag{13}$$
$$C_2 = 0 + O(\lambda^2). \tag{14}$$

They can also be obtained by comparing the Lagrangian (2) with (5). Since the above derivations don’t assume the specific spacetime dimensions, these results hold for the $\phi^4$ theory living in arbitrary dimensions. Once one goes beyond the tree level, however, the matching results will generally depend upon the dimensions. We will explore the one-loop matching for the $\phi^4$ theory in different dimensions in Sec. IV.

### III. CAN TRIVIALITY BE SEEN IN NONRELATIVISTIC LIMIT?

Triviality of the 4D $\phi^4$ theory is intimately connected with the self-consistency of a non asymptotically-free quantum field theory, and necessarily involves the consideration of very short distance physics. Beg and Furlong’s assertion is based on the nonrelativistic quantum mechanics. Since this framework must cease to work when probing the distance shorter than the Compton wavelength, their claim seems rather counterintuitive—how can a conclusion be trusted when the applicability of the underlying framework is in trouble?

In this Section, we will examine where the flaw of their argument originates. Their strategy is to convert this field-theoretic problem to a quantum mechanical one. They assume that the nonrelativistic limit of $\phi^4$ theory can be described by a $\delta$-function potential. For such a simple potential, Schrödinger equation (or technically more correct, Lippmann-Schwinger equation) can be reduced into an algebraic equation and solved analytically. Nonetheless, the $\delta^3(\mathbf{r})$ potential is too singular that one has to regularize the severe ultraviolet divergences. Upon renormalization, they claim that the two-body scattering in a contact potential leads to a trivial renormalized S-matrix, thus establish the triviality of the $\phi^4$ theory in the nonrelativistic limit.

Instead of repeating their derivation using Schrödinger formalism, we try to reproduce their renormalization formula in the EFT language. The main advantage of the field-theoretic method over quantum mechanics is that renormalization can be dealt with in
a systematic manner. For comparison to their results, we first choose cutoff as the regulator. As will be elucidated, it is the problematic way of removing the ultraviolet power divergences that leads to their incorrect conclusion. After clarifying the pitfall of renormalization in cutoff scheme, we then switch to more convenient dimensional regularization scheme.

A. No triviality if cutoff scheme used properly

Since time can only flow forward in the nonrelativistic theory (Equivalent to say, only one pole is present in the nonrelativistic propagator), a gratifying fact is that loop calculation in Lagrangian (2) becomes enormously simple. A further simplification arises from vanishing of many diagrams. For example, the self-energy diagrams vanish to all orders, so no need for wave-function renormalization. Particle number conservation reduces the two-body amplitude to the bubble diagrams as shown in Fig. 2. This is in sharp contrast with the relativistic $\phi^4$ theory, where the higher order Feynman diagrams can be arbitrarily complicated.

One remarkable feature is that these bubble chain diagrams simply form a geometric series, therefore can be summed analytically. This should be of no much surprise, as it merely echoes the fact that one can solve Schrödinger equation analytically for sufficiently simple potential. For comparison with Beg and Furlong’s results, it suffices for us to consider the bubble chain with only $C_0$ vertex, as shown in Fig. 2. Incorporating the higher dimensional operators proportional to $C_{2n}$ in the bubble sum is straightforward, and will be considered in Section IV. Summing the bubble chain in Fig. 2 renders

$$\mathcal{A}_{0}^{\text{sum}} = -C_0^B - C_0^B I_0 C_0^B - C_0^B I_0 C_0^B I_0 C_0^B - \cdots$$

$$= - \left[ \frac{1}{C_0^B} - I_0 \right]^{-1}$$

(15)

where $C_0^B$ is the bare coupling, and $I_0$ is the one-loop integral

$$I_0 = \frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \frac{i}{E + q_0 - q^2/2m + i\epsilon} \cdot \frac{i}{E - q_0 - q^2/2m + i\epsilon}$$

$$= - \frac{m}{2} \int_{\Lambda} \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 - 2mE - i\epsilon}$$

$$= - \frac{m}{4\pi^2} \Lambda + \frac{m}{8\pi} \sqrt{-2mE - i\epsilon} + O(1/\Lambda).$$

(16)
We have chosen to work in the C.M. frame, in which each external particle carries momentum \( k \), and its corresponding kinetic energy \( E \) is given in (9). We first carry out the \( q^0 \) contour integral with the aid of Cauchy’s theorem. The remaining 3-momentum integral is ultraviolet-divergent and needs regularization. We impose a momentum cutoff \( \Lambda \) to regulate the integral, and the leading term is linearly divergent. Recall the one-loop integral in the relativistic \( \phi^4 \) theory is only logarithmically divergent. This is because EFT is designed to reproduce the correct low energy property of the full theory, at a price in distorting the true UV behavior. It is generic that EFT always produces worse UV divergences than the full theory.

The second term is finite and imaginary, and the residual \( O(1/\Lambda) \) term is small and regulator-dependent, and can be neglected. This imaginary term is ordered by the optical theorem, so has the physical significance and doesn’t depend on the regularization prescription. To respect the positivity of the imaginary part of the amplitude, we have chosen the convention \( \sqrt{-2mE - i\epsilon} \equiv -i\sqrt{2mE} \). If we neglect the relativistic correction to the energy-momentum relation, this term equals \(-ik\).

The linear divergence stems from the ultraviolet part of the loop integral, whose effect cannot be correctly described by the nonrelativistic effective theory, therefore renormalization must be invoked. This divergence can be absorbed into the unknown bare coupling \( C^B_0 \), by introducing the renormalized coupling \( C^R_0 \):

\[
\frac{1}{C^R_0} = \frac{1}{C^B_0} + \frac{m^4}{4\pi^2 \Lambda}, \tag{17}
\]

which is finite and cutoff-independent. Now the resumed amplitude (15) can be expressed in terms of \( C^R_0 \):

\[
A^{\text{sum}}_0 = -\left[ \frac{1}{C^B_0} + \frac{imk}{8\pi} \right]^{-1}, \tag{18}
\]

which is also finite and cutoff-independent.

The renormalization relation (17) is the same as what Beg and Furlong have obtained from solving the regularized Lippmann-Schwinger equation, except an insignificant discrepancy in the coefficient of the \( \Lambda \) term.

Beg and Furlong argue, for any value of the bare coupling \( C^B_0 \), when one takes the limit \( \Lambda \to \infty \), the renormalized \( C^R_0 \) is forced to vanish, so is the amplitude in (18). They then conclude, the renormalized \( \lambda \), related to \( C^R_0 \) through (13), must also vanish. Thus \( \phi^4 \) theory in the nonrelativistic limit is said to be trivial.

The key point is that, are we allowed to send the cutoff to infinity in (17)? The emergence of power-law UV divergence is a warning sign, that this theory cannot hold true at arbitrarily high scale, and must break down somewhere. The cutoff \( \Lambda \) should be taken close to the scale where the theory is expected to fail. For a realistic system, the cutoff scale is set roughly by the inverse of the range of interaction. For the \( \phi^4 \) theory, the cutoff \( \Lambda \) is of order the scalar mass. If we assume the bare coupling \( C^B_0 \sim 1/m^2 \), then the renormalized coupling \( C^R_0 \sim 1/m^2 \), which is finite. Thus the effective theory (2) makes unambiguous and nontrivial predictions. Generally speaking, taking \( \Lambda \to \infty \) is unacceptable for any effective field theory, because it pushes the theory way beyond its range of applicability.
Clearly, Beg and Furlong’s assertion is more general than the $\phi^4$ theory, and applies to any system with the true $\delta$-function potential. They try to provide a physical explanation for their assertion—because two point particles cannot perceive each other in a $\delta^3(\mathbf{r})$ potential, therefore no scattering can occur, so the $S$-matrix is trivial. This viewpoint, that zero-range interaction leads to a noninteracting theory, is further corroborated in Ref. [15], and attributed to the consequence of Friedman’s theorem [16].

We need inspect, to which extent, the $\delta^3(\mathbf{r})$ potential is relevant to the reality? There are undoubtedly various realistic systems with a repulsive, short-range, but nontrivial interaction. These systems can be successfully described by a local nonrelativistic field theory. If one identifies local operators with contact potentials, then one is puzzled by the fact why Beg and Furlong’s assertion doesn’t apply here. Of course, the true reason that local nonrelativistic field theory can correctly describe the reality is not because the interaction is literally contact, but because the range of the interaction is short compared with the de Broglie wavelength of particles.

One may still argue, because the scalar particles in the $\phi^4$ theory can be viewed as point-like, and this theory is supposed to be valid at the distance much shorter than $1/m$, the two-body interaction may be thought of as truly zero-range.

The very notion of short-range interaction in nonrelativistic quantum mechanics deserves some elaboration. As is well known, one cannot probe the distance between two nonrelativistic particles with a resolution better than their Compton wavelengths. Otherwise, according to the Uncertainty Principle, the energy fluctuation becomes of order $m$, the relativistic effects such as pair creation and annihilation will invalidate the nonrelativistic quantum mechanics, and we must resort to the relativistic quantum field theory for a correct description. Therefore, the shortest distance in nonrelativistic quantum mechanics which is still meaningful to talk about is the Compton wavelength. This is another way to say that, in a nonrelativistic problem, the cutoff should never be taken much bigger than $m$.

In this sense, $\delta$-function potential should be viewed as an idealized mathematical construct, and there is no any physical system possessing zero-range interaction. Therefore, even for the $\phi^4$ theory, which supposedly accommodates a contact interaction, the interaction range in nonrelativistic scattering is of order $1/m$, instead of zero. We will explicitly compute the $S$-wave effective range in the $\phi^4$ theory in Section IV.

B. Dimensional Regularization

Although cutoff is a very physical and intuitive regulator, it is somewhat awkward for practical use. For instance, when we include the higher-derivative operators or relativistic corrections, severer power-law divergences will be confronted. In the cutoff scheme, the relationship between renormalized couplings and bare couplings in general is very complicated, and a particular drawback is that lower-dimensional operators get renormalized by the higher dimensional ones [17].

Physics certainly shouldn’t rely on which regulator to use, but one judiciously chosen regulator may be more convenient than another. Dimensional regularization (DR) is the preferred one to use practically, especially for the EFT calculation [1]. The particular advantages of this scheme include that power-law divergences are automatically subtracted,
the spacetime symmetry (Galilean or Lorentz) is automatically preserved. In this scheme, the higher dimensional operators never renormalize the lower dimensional ones.

For reader’s convenience, we present the master formula of DR here, which will be heavily used in this work:

$$
\int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{(q^2)^\beta}{(q^2 + \Delta)^\alpha} = \frac{1}{(4\pi)^{\frac{D-1}{2}}} \frac{\Gamma[\beta + \frac{D-1}{2}] \Gamma[\alpha - \beta - \frac{D-1}{2}]}{\Gamma[\frac{D-1}{2}] \Gamma[\alpha]} \Delta^{\beta - \alpha - \frac{D-1}{2}}. \quad (19)
$$

Using this formula, we recalculate the one-loop integral $I_0$ in DR:

$$
I_0 = -\left(\frac{m}{2}\right) \left(\frac{\mu}{2}\right)^{4-D} \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{1}{q^2 - 2mE - i\epsilon} = -\left(\frac{m}{2}\right) \left(\frac{\mu/2}{4\pi}\right)^{4-D} \Gamma \left[\frac{3-D}{2}\right] (-2mE - i\epsilon)^{\frac{D-3}{2}}. \quad (20)
$$

It is standard to use minimal subtraction (MS) in conjunction with DR. Since this integral doesn’t exhibit a $D = 4$ pole, so MS basically does nothing:

$$
I_0 = \frac{m}{8\pi} \sqrt{-2mE - i\epsilon}, \quad (21)
$$

which is automatically finite, and doesn’t depend on the subtraction scale $\mu$. Note only the finite imaginary part in (16) survives in the MS scheme. Plugging (21) back into (15), and replacing $C^B_0$ by $C^R_0$ there, we reproduce the renormalized amplitude (18) which is previously obtained in the cutoff scheme. From now on, unless stated otherwise, we will always assume that MS (or $\overline{\text{MS}}$) is used. To simplify the notation, we will suppress the superscript $R$ which stands for the renormalized coupling.

It is worth emphasizing that, throwing away the power divergence should not be taken for granted. This is permissible only when no delicate cancellation occurs between two terms in the right-hand side of (17), so that the renormalized $C_0$ is small. The $\phi^4$ theory satisfies this criterion.

However, there are a class of interesting systems, where the underlying short-distance physics is both nonperturbative and finely-tuned, so that the two terms in the right side of (17) conspire to nearly cancel each other, and results in an unnaturally large renormalized $C_0$ (or S-wave scattering length). Two nucleons in S-channel constitutes such an example, where the deuteron manifests as a shallow S-wave bound state. In this case, cutoff plays an important role in delineating the fine tuning, and should not be discarded.

If one persists to use DR, the rule of MS must be altered correspondingly to limn the effects of fine tuning. An ingenious scheme, power divergence subtraction (PDS), has been introduced for better describing such a finely-tuned system [18]. It is a generalization to MS, and we will encounter it in the next Section.

**IV. ONE-LOOP MATCHING AND BUBBLE CHAIN SUM**

In this Section, we will present a systematic study of short-range force in various spacetime dimensions: two, three and four. This Section is divided into three parts, each of
which devotes a detailed discussion to each case. Among them, the 3D case is especially interesting, where the renormalization group technique can be fruitfully applied.

In each part, we first take the $\phi^4$ theory as a concrete example of short-range interaction, and match it onto the nonrelativistic EFT at one-loop level. This is a simple, but ideal place to illustrate the generic features of perturbative matching, e.g. cancellation of non-analytic terms and infrared divergences, etc.

We then proceed to explore the nonrelativistic EFT sector. It is shown that the bubble chain diagrams can be summed analytically, with higher-dimensional operators and relativistic effects fully incorporated. The resumed amplitude can be framed in a compact form. From this exact result, we can gain nonperturbative understanding of short-range interactions in the two-body sector.

Let us now sketch how to carry out the one-loop matching. The three one-loop diagrams in the relativistic $\phi^4$ theory are shown in the top row of Fig. 3. The $s$-channel diagram contains an imaginary part, the other two, the $t$- and $u$-channel diagrams are real.

After Feynman parameterization and integrating over loop momentum, the $T$-matrix in the $D$-dimensional $\phi^4$ theory is [19]

$$T^{1\text{-loop}} = -\lambda + \frac{\lambda^2}{2m^4-D} \frac{\Gamma[2-D/2]}{(4\pi)^{D/2}} \int_0^1 dx \left\{ 1 - x(1-x) s/m^2 - i\epsilon \right\}^{D/2-2}$$

$$+ (s \rightarrow t) + (s \rightarrow u) \} - \delta\lambda,$$

where $\delta\lambda$ is the counterterm. The Mandelstam variables $s$, $t$ and $u$ are defined by $s \equiv (k_1 + k_2)^2$, $t \equiv (k_1 - k')^2$, $u \equiv (k_1 - k'_2)^2$. In the C.M. frame, these variables become $s = 4(m^2 + k^2)$, $t = -2k^2(1 - \cos \theta)$ and $u = -2k^2(1 + \cos \theta)$, where $\theta$ is the angle between $k_1$ and $k'_1$.

The amplitude in (22) is the superposition of all the partial waves of even angular momentum, and only the S-wave amplitude $T_0$ needs to be projected out. Bose statistics implies that the occurrences of $t$ and $u$ are symmetric. Because $|t|, |u| \ll m^2$, we can expand the amplitude in terms of $t/m^2$, $u/m^2$ and only keep the linear combination of these two variables:
which doesn’t have angular dependence. This guarantees that correct $T_0$ is projected out up to $O(k^2)$, because the P-wave amplitude is absent and the D-wave contribution starts at order $k^4$.

We also need consider the diagrams in the EFT sector, as shown in the bottom row of Fig. 3. The S-wave amplitude to one-loop order is

$$\mathcal{A}_0^{1-\text{loop}} = -C_0 - C_0 (I_0 + \tilde{I}_0) C_0 - C_2 k^2 - \delta C_0 - \delta C_2 k^2.$$ (24)

Here $I_0$ stands for the one loop integral corresponding to Fig. 3b), and $\tilde{I}_0$, denoted by Fig. 3c) and d), represents the one loop integral with one insertion of the relativistic vertex. The tree level contribution from $C_2$ vertex is shown in Fig. 3e). Counterterms are also implied when the subtraction scheme is specified.

Requesting the S-wave amplitude calculated in the full theory and the EFT to be equal, we can deduce the values of $C_0$, $C_2$, . . . , respectively.

A. Two dimensions

Everyone is familiar with the one-dimensional $\delta$-function potential. The transmission and reflection probability can be easily obtained by solving Schrödinger equation. For the attractive $\delta$-function potential, there is also a bound state solution. However, it is difficult to apply the Schrödinger formalism to more general situations, e.g., inclusion of $\delta''(x)$ potential, and incorporating relativistic corrections, and so on. As we will see, these questions are best addressed by the field-theoretic approach.

1. Matching of 2D $\phi^4$ theory

The two-dimensional $\phi^4$ theory has a coupling $\lambda$ of mass dimension two, so is super-renormalizable. Since no ultraviolet divergences emerge, renormalization is not required.

Substituting $D = 2$ into (22), we work out the following integral associated with the s-channel diagram:

$$\int_0^1 dx \left[1 - x(1-x) s/m^2 - i\epsilon\right]^{-1} = \frac{2m}{\sqrt{s}} \frac{m}{k} \left(\tanh^{-1} \beta + \frac{i\pi}{2}\right)$$

$$\approx 1 - \frac{2k^2}{3m^2} + \frac{i\pi m}{2k} \left(1 - \frac{k^2}{2m^2}\right),$$ (25)

where $\beta = \sqrt{1 - 4m^2/s}$ is the velocity in the C.M. frame. We have expanded both the real and imaginary part to the relative order $k^2$.

The leading imaginary part is singular at small momentum, which can be understood from the optical theorem, due to the too limited phase space. This should be viewed as an artifact of the perturbative expansion, and the sensible answer will be obtained only if the singular terms are summed to all orders.
For the $t$- and $u$-channel diagrams, the corresponding integrals are
\[
\int_0^1 dx \ \left\{ [1 - x(1 - x) t/m^2]^{-1} + (t \to u) \right\} \approx 2 + \frac{t + u}{6m^2} = 2 - \frac{2k^2}{3m^2}.
\]  
(26)

We first expand the integrand in $t/m^2$ and $u/m^2$ to the first order, combine them in compliance with (23), then carry out the integration over $x$. Note the result is convergent at small $k$.

Merging (25) and (26), we obtain the amplitude to one loop order\(^1\):
\[
T^{1-\text{loop}} = -\lambda + \frac{\lambda^2}{8\pi m^2} \left[ 3 + \frac{i\pi m}{2k} \left( 1 - \frac{k^2}{2m^2} \right) - \frac{4k^2}{3m^2} \right].
\]  
(27)

We now move on to the EFT sector. Since (20) doesn’t exhibit $D = 2$ pole, we can directly substitute $D = 2$ into this equation, and find that Fig. 3b) equals
\[
I_0 = -\frac{m}{4\sqrt{-2mE - i\epsilon}} \approx -\frac{im}{4k} \left( 1 + \frac{k^2}{8m^2} \right),
\]  
(28)

where we have retained the first-order relativistic correction in $E$, in conformity with (9).

It is straightforward to evaluate the one-loop diagrams with one relativistic vertex insertion, Fig. 3c) and d):
\[
\tilde{I}_0 = -i \left( \frac{\mu}{2} \right)^{2-D} \frac{d^D q}{(2\pi)^D} \left( \frac{i}{E + q_0 - q^2/2m + i\epsilon} \right)^2 \frac{i(E + q^0)^2}{2m} \cdot \frac{i}{E - q_0 - q^2/2m + i\epsilon}
\approx -\frac{1}{8m} \left( \frac{\mu}{2} \right)^{2-D} \int \frac{d^{D-1} q}{(2\pi)^{D-1}} \frac{3k^4 - 2k^2 q^2}{(q^2 - k^2 - i\epsilon)^2}
= \left( \frac{\mu}{2} \right)^{2-D} \left( \frac{1}{4\pi} \right)^{D-1} \left( 3 \Gamma \left[ \frac{5 - D}{2} \right] + (D - 1) \Gamma \left[ \frac{3 - D}{2} \right] \right) \left( \frac{k^2}{8m} \right) (-k^2 - i\epsilon)^{D-3}.
\]  
(29)

After integrating out the variable $q^0$, we simplify the expression little bit by employing the fact that scaleless integrals vanish in DR. We have also replaced $2mE$ by $k^2$, since the induced error is of relative order $k^4$.

This formula doesn’t display a $D = 2$ pole either, so we simply replace $D$ by 2 everywhere, and the result is
\[
\tilde{I}_0 = \frac{im}{4k} \left( \frac{5k^2}{8m^2} \right).
\]  
(30)

According to (24), we obtain the one-loop EFT amplitude by combining (28) and (30):

---

\(^1\)Since angular momentum cannot be defined in one spatial dimension, so partial wave expansion loses its meaning. We therefore drop the subscript 0, which stands for the S-wave.
\[ \mathcal{A}^{\text{1-loop}} = -C_0 + i C_0^2 \frac{m}{4k} \left( 1 - \frac{k^2}{2m^2} \right) - C_2 k^2. \tag{31} \]

Comparing (27) and (31) through (11), we see that the full theory and the EFT share the same non-analytic (imaginary) terms. Note the structures of relativistic corrections are the same in both sectors. All of these are ensured by the general principles of EFT. Consequently, we obtain the Wilson coefficients:

\[ C_0 = \frac{\lambda}{4m^2} - \frac{3}{2\pi} \left( \frac{\lambda}{4m^2} \right)^2 + O(\lambda^3), \tag{32} \]
\[ C_2 = \frac{2}{3\pi m^2} \left( \frac{\lambda}{4m^2} \right)^2 + O(\lambda^3). \tag{33} \]

Through the one-loop matching, \( C_0 \) receives an \( O(\lambda^2) \) correction, and a nonzero coefficient is generated for \( C_2 \). Tracing back to the relativistic \( \phi^4 \) theory calculation, we can identify how those three different channels contribute in the matching. Because the \( t, u \)-channel processes don’t have counterparts in the EFT sector, their effects are entirely encoded in the Wilson coefficients, and not responsible to the relativistic corrections\(^2\). In contrast, the \( s \)-channel diagram not only contains dynamical effects, but also contains the purely kinematic effects-- the relativistic corrections. At one loop order, the latter only influences its non-analytic (imaginary) part.

2. Bubble chain sum in 2D EFT

The \( k \to 0 \) singularity in the one-loop amplitude (31) implies that fixed-order perturbative expansion is not reliable. Since each loop contributes a factor of \( im/4k \), higher order diagrams become more singular. To remedy this nonphysical singularity, it is mandatory to sum the infrared-divergent terms to all orders.

This is an almost intractable task in the relativistic \( \phi^4 \) theory, because any diagram, no matter how complicated, so long as containing one \( s \)-channel subdiagram, needs to be included. However, this problem becomes rather transparent in the nonrelativistic EFT. Let us first consider summing the most singular series-- the bubble chain comprising of \( C_0 \) vertices only. Analogous to (15), this can be easily accomplished:

\[ \mathcal{A}^{\text{sum}} = - \left[ \frac{1}{C_0} + \frac{im}{4k} \right]^{-1}. \tag{34} \]

We immediately see that, the infrared singularity confronted in the fixed-order perturbative expansion is now removed. As \( k \to 0 \), the resumed amplitude vanishes as \( 4ik/m \), not dependent of \( C_0 \) at all. This result can be obtained from solving Schrödinger equation, \(^2\)

Note \( t, u \) are simple polynomials of momentum \( k \). In contrast, the \( s \)-channel integral contains the factor of \( \sqrt{s} \), which is an infinite power series in \( k \).
and can be understood from that a particle carrying very long wavelength cannot penetrate through the one-dimensional infinitely high barrier.

This equation also encodes another important nonperturbative information. The pole of the amplitude located at the positive imaginary momentum, \( k = i\kappa (\kappa > 0) \), signals a bound state with binding energy

\[
E_B \equiv 2 (\sqrt{m^2 - \kappa^2} - m) = -\frac{\kappa^2}{m} - \cdots .
\] (35)

One easily finds the location of the pole in (34), \( \kappa = -mC_0/4 \). In order to have a positive \( \kappa \), we must require \( C_0 < 0 \), which corresponds to an attractive \( \delta \)-function potential. The binding energy is then about \( -mC_0^2/16 \). This is nothing but the familiar result for a particle with the reduced mass \( m/2 \) in the \( -|C_0|^2\delta(x) \) potential. Clearly, the weaker the coupling, the shallower the bound state is.

Including those higher-derivative operators in the bubble diagrams ameliorates the infrared behavior, but not sufficient to justify a fixed-order calculation, since the singularities will emerge when enough bubble diagrams are retained. To obtain physically sensible result, we must also sum bubble chains containing these operators.

The loop integral needed to evaluate these bubble diagrams is

\[
I_n = -\left( \frac{i}{2} \right) \left( \frac{\mu}{2} \right)^{2-D} \int \frac{d^Dq}{(2\pi)^D} \frac{q^{2n}}{E + q_0 - q^2/2m + i\epsilon} \frac{i}{E - q_0 - q^2/2m + i\epsilon} \frac{i}{E + q_0 - q^2/2m + i\epsilon} \frac{i}{E - q_0 - q^2/2m + i\epsilon}.
\]

\[
= -\left( \frac{m}{2} \right) \left( \frac{\mu}{2} \right)^{2-D} \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{q^{2n}}{q^2 - 2mE - i\epsilon} \frac{i}{E + q_0 - q^2/2m + i\epsilon} \frac{i}{E - q_0 - q^2/2m + i\epsilon} \frac{i}{E + q_0 - q^2/2m + i\epsilon} \frac{i}{E - q_0 - q^2/2m + i\epsilon}.
\]

\[
= -\frac{1}{(4\pi)^{D/2}} \Gamma \left[ \frac{3 - D}{2} \right] (-2mE - i\epsilon)^{D-3} (2mE)^n.
\] (36)

Therefore, the following relation holds in any dimensions:

\[
I_n = I_0 (2mE)^n \approx I_0 k^{2n},
\] (37)

where we neglect the relativistic effect in the last equality.

This relation allows that the factors of \( q \) inside the loop get converted into factors of the external momentum \( k \). To appreciate this gratifying feature, let us consider one explicit example. First consider the one loop diagram with \( C_0 \) and \( C_2 \) as its vertices. It contributes to the amplitude with

\[
C_0 C_2 \left( \frac{m}{2} \right) \left( \frac{\mu}{2} \right)^{2-D} \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{q^{2n}}{q^2 - 2mE - i\epsilon} \frac{i}{E + q_0 - q^2/2m + i\epsilon} \frac{i}{E - q_0 - q^2/2m + i\epsilon} \frac{i}{E + q_0 - q^2/2m + i\epsilon} \frac{i}{E - q_0 - q^2/2m + i\epsilon}.
\]

\[
= -C_0 C_2 \left( k^2 I_0 + I_1 \right) \approx -2 C_0 C_2 I_0 k^2.
\] (38)

Similarly, the one loop diagram with two \( C_2 \) vertices contributes

\[
\left( \frac{C_2}{2} \right)^2 \left( \frac{m}{2} \right) \left( \frac{\mu}{2} \right)^{2-D} \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \frac{(k^2 + q^2)^2}{q^2 - 2mE - i\epsilon} \frac{i}{E + q_0 - q^2/2m + i\epsilon} \frac{i}{E - q_0 - q^2/2m + i\epsilon} \frac{i}{E + q_0 - q^2/2m + i\epsilon} \frac{i}{E - q_0 - q^2/2m + i\epsilon}.
\]

\[
= -\left( \frac{C_2}{2} \right)^2 (k^4 I_0 + 2k^2 I_1 + I_2) \approx -C_2^2 I_0 k^4.
\] (39)
Recall the one loop diagram with two $C_0$ vertices contributes to the amplitude with $-C_0^2 I_0$. These three terms can be combined into a simple form, $-(C_0 + C_2 k^2)^2 I_0$.

This suggests that, we can lump all the two-body S-wave operators together, and treat them as a single effective operator. Consequently, we replace each internal $C_0$ vertex in the bubble chain depicted in Fig. 2 by an effective vertex $-\sum C_2^n k^{2n}$ [18]. It can be checked that combinatorics is correctly taken into account. The full bubble chain sum thus gives

$$\mathcal{A}^\text{sum} = -\left[\frac{1}{C_0 + C_2 k^2 + \cdots} + \frac{im}{4k}\right]^{-1}. \quad (40)$$

From this equation, one can infer that a bubble chain containing $n$ $C_0$ vertices and one $C_2$ vertex contributes with $(-1)^{n+1}(n+1)C_0^n C_2 (im/4k)^n k^2$. Taking $n = 1$, one reproduces the one loop result in (38).

When the momentum gets small, the higher-dimensional terms become negligible relative to the $C_0$ term. Ultimately in the $k \to 0$ limit, the amplitude is again governed by the reciprocal of the imaginary factor $I_0$.

Including higher-dimensional terms will shift the bound state pole. From this equation, one finds that the bound state pole moves to

$$\kappa = \frac{2 \left(1 - \sqrt{1 - |C_0| C_2 m^2 / 4}\right)}{m C_2} \approx \frac{m |C_0|}{4} \left[1 + \frac{|C_0| C_2 m^2}{16}\right]. \quad (41)$$

In order for $\kappa$ to have a real solution, one needs impose $C_2 < 4/(m^2 |C_0|)$. For small enough $C_2$, the corresponding binding energy $E_B \approx -\frac{m C_2^2}{16}[1 + |C_0| C_2 m^2 / 8]$.

Thus far, the relativistic corrections have been omitted in the resumed amplitude. Intuitively, relativistic effects ought to be negligible at small $k$. However, because of the rising of infrared singularity, we are not allowed to ignore them a priori. To elucidate this point, let us consider a $n$-loop bubble chain consisting entirely of $C_0$ vertices, but with the first-order relativistic correction included. It contributes with $(im/4k)^n k^2$, the same order as the bubble chain containing $n$ $C_0$ vertices and one $C_2$ vertex. At $n = 3$, infrared divergence arises and keep deteriorating as $n$ increases. Therefore, to have a physically meaningful answer, it is mandatory to sum all the singular terms induced by the relativistic corrections.

At first sight, including relativistic effects in the resumed amplitude is an impossible task, because the pattern of relativistic corrections seems too complicated and random to identify. However, this is just a disguise, and a thorough scrutiny shows that it is in fact feasible to fully incorporate the Lorentz symmetry. In any event, a correct resummation formula must first reproduce the one-loop amplitude (31), where the first-order relativistic correction is included. It is then natural to propose the following formula:

$$\mathcal{A}^\text{sum} = -\left[\frac{1}{C_0 + C_2 k^2 + \cdots} + \frac{im}{4k} \left(1 - \frac{k^2}{2m^2} + \cdots\right)\right]^{-1}. \quad (42)$$

One can deduce from this equation, that the coefficient of $(im/4k)^n k^2$, which arise from the $n$-loop $C_0$ bubble chain implementing the first-order relativistic correction, is $(-1)^n n C_0^{n+1}$. 

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This can be easily confirmed by direct calculation. We also verify this resummation formula by computing the first-order relativistic correction to the n-loop bubble diagram containing n \( C_0 \) vertices and one \( C_2 \) vertex.

To complete this resummation formula, we need know the successive terms in the parenthesis in (42). For instance, to pinpoint the \( O(k^4) \) term, we need expand \( I_0 \) to incorporate the second-order relativistic correction, include the first-order relativistic correction in \( \tilde{I}_0 \), and also calculate the one loop diagram with two insertions of the relativistic vertex.

Fortunately, we actually don’t need go through this computation, thanks to a shortcut provided by the \( \phi^4 \) theory. Relativistic correction, as a solely kinematic effect which only depends on the spacetime symmetry, must be identical in the full theory and the EFT calculation. Therefore, we can directly recognize the pattern of relativistic corrections from the one-loop s-channel integral in the \( \phi^4 \) theory. Inspecting the imaginary part in (25), one finds that the full series in the parenthesis in (42) is nothing but

\[
\gamma^{-1} = \frac{2m}{\sqrt{s}} = 1 - \frac{k^2}{2m^2} + \frac{3k^4}{8m^4} - \cdots.
\]

(43)

where \( \gamma \equiv 1/\sqrt{1 - \beta^2} \) is the familiar dilation factor.

This resummation formula can be confirmed by miscellaneous straightforward calculations. This is an amazing result—though the intermediate stage of computing relativistic corrections looks rather involved and desultory, the final results obey a very simple pattern.

Note when the relativistic corrections are incorporated, this resummed amplitude is still well behaved at small \( k \). Specifically, as \( k \to 0 \), one finds \( A_{\text{sum}} \approx 4i\gamma k/m \). Now it is safe to conclude \textit{a posteriori}, that relativistic effects are indeed unimportant at small momentum.

Finally let us investigate the impact of relativistic effects on the bound state pole. Inspecting (42), we find that the pole shifts from (41) by an additional amount of \( \Delta \kappa \approx mC_0^3/64 \). The corresponding binding energy is then

\[
E_B = -\frac{\kappa^2}{m} - \frac{\kappa^4}{4m^3} - \cdots \approx -m \frac{C_0^2}{16} \left[ 1 + \frac{|C_0| C_2 m^2}{8} - \frac{7 C_0^2}{64} \right],
\]

(44)

where the net relativistic effect is encoded in the third term, which slightly reduces the binding energy. Note its size is comparable to the second term, when \( C_2 \) respects the bound imposed earlier. Apparently, this result can not be easily obtained from solving Schrödinger equation with the potential \(-\frac{|C_0|}{2} \delta(x) - \frac{C_2}{4} \delta''(x)\).

\textbf{B. Three dimensions}

The nonrelativistic planar system displays rich physics. For example, point particles on a plane coupled with the Chern-Simmons gauge field, can be used to formulate the Aharonov-Bohm effect and Fractional Quantum Hall effect in a field-theoretic framework [20].

We will focus on the simplest case, where the external sources are absent and only the short-range interactions among particles themselves are present. Even this case is quite nontrivial. The fullest discussion on this topic by far which utilizes the field-theoretic language, is given by Bergman [9]. In the following, we will expand his results to incorporate
the effects of higher-derivative operators and relativity. As will be seen, the latter plays an important role in influencing the renormalization group flow of the former.

1. Matching of 3D $\phi^4$ theory

In the 3-dimensional $\phi^4$ theory the coupling $\lambda$ has mass dimension one. This theory is super-renormalizable, and ultraviolet finite at one loop.

Substituting $D = 3$ into (22), we carry out the $s$-channel integral and expand it:

$$
\int_0^1 dx \left[ \frac{1 - x(1-x)}{s/m^2 - i\epsilon} \right]^{-\frac{1}{2}} = \frac{m}{\sqrt{s}} \left( \ln \left[ \frac{\sqrt{s} + 2m}{\sqrt{s} - 2m} \right] + i\pi \right)
$$

$$
\approx \left( 1 - \frac{k^2}{2m^2} \right) \left( \ln \left( \frac{2m}{k} \right) + \frac{i\pi}{2} \right) + \frac{k^2}{4m^2}.
$$

This integral is logarithmically divergent as $k \to 0$, though milder than the linear divergence encountered in 2D. As emphasized previously, this infrared singularity is a symptom that the fixed-order perturbation series is untrustworthy at small momentum.

The $t$- and $u$-channel integrals can be performed by exploiting the same trick as in (26):

$$
\int_0^1 dx \left\{ \left[ 1 - x(1-x) \frac{t/m^2}{t/m^2} \right]^{\frac{1}{2}} + (t \to u) \right\} \approx 2 + \frac{t + u}{12m^2} = 2 - \frac{k^2}{3m^2},
$$

which is finite in the $k \to 0$ limit.

Thus, to the one-loop order, the S-wave amplitude in the $\phi^4$ theory is

$$
T_{0}^{1\text{-loop}} = -\lambda + \frac{\lambda^2}{16\pi m} \left[ 2 + \left( 1 - \frac{k^2}{2m^2} \right) \left( \ln \left( \frac{2m}{k} \right) + \frac{i\pi}{2} \right) - \frac{k^2}{12m^2} \right].
$$

This expression has previously been obtained in Ref. [10].

Loop integrals in the 3D EFT exhibit a novel feature, e.g. emergences of logarithmic UV divergences (look at the $D = 3$ pole in (20), (29), (36)). The log divergence implies that different momentum regions are coupled, so the coefficients of the logarithms are regularization-scheme independent.

The one-loop integral corresponding to Fig. 3b) is

$$
I_0 = -\frac{m}{2} \left( \frac{e^\gamma \mu^2}{4\pi} \right)^\epsilon \int \frac{d^{2-2\epsilon} q}{(2\pi)^{2-2\epsilon}} \frac{1}{q^2 - 2mE - i\epsilon}
$$

$$
= -\frac{m}{8\pi} \left[ \frac{1}{\epsilon} + \ln \mu^2 - \ln(-2mE - i\epsilon) \right],
$$

where we have rewritten $D = 3 - 2\epsilon$, and $\gamma = 0.5772 \cdots$ is the Euler’s constant. We adopt $\overline{MS}$, by replacing $\mu^2 \to e^\gamma \mu^2/4\pi$. Expanding $2mE$ to include the first-order relativistic correction, we obtain

$$
I_0 \approx -\frac{m}{8\pi} \left[ \frac{1}{\epsilon} + 2 \ln \left( \frac{\mu}{k} \right) + i\pi + \frac{k^2}{4m^2} \right].
$$
Fig. 3c) and d) can be easily deduced from (29):

\[
\tilde{I}_0 = -\frac{1}{8m} \left( \frac{e^2 \mu^2}{4\pi} \right)^\epsilon \int \frac{d^2 q}{(2\pi)^{2-2\epsilon}} \frac{3k^4 - 2k^2 \mathbf{q}^2}{(\mathbf{q}^2 - k^2 - i\epsilon)^2} \\
= \frac{m}{8\pi} \left( \frac{k^2}{2m^2} \right) \left[ \frac{1}{\epsilon} + 2 \ln \left( \frac{\mu}{k} \right) + i\pi + \frac{1}{2} \right].
\]  

(50)

The appearance of \( k^2/\epsilon \) pole implies that \( C_2 \) is renormalized by \( C_0 \) through the relativistic correction.

Combining (49) and (50), we obtain the S-wave amplitude in the EFT sector:

\[
A^{1\text{-loop}}_0 = -C_0 + C_0^2 \left( \frac{m}{4\pi} \right) \left( 1 - \frac{k^2}{2m^2} \right) \left[ \ln \left( \frac{\mu}{k} \right) + i\pi + \frac{1}{2} \right] - C_2 k^2,
\]  

(51)

where we have introduced the following counterterms to absorb the divergences:

\[
\delta C_0 = \frac{m}{8\pi} \frac{C_0^2}{\epsilon},
\]

(52)

\[
\delta C_2 = -\frac{1}{2(8\pi)m} \frac{C_0^2}{\epsilon}.
\]

(53)

Note when \( I_0 \) and \( \tilde{I}_0 \) are added together, those real analytic \( O(k^2) \) pieces exactly cancel. It is generic that the linear combination \( \ln(\mu/k) + i\pi/2 \) constitutes the only allowed loop factor accompanying powers of \( k \) in the amplitude. Because of this cancellation, whether including relativistic corrections or not will not mess up with the to-be-determined analytic part of the \( C_2 \) coefficient.

Comparing (51) with (47) via (11), it is easy to check that the non-analytic terms of the form \( \ln k \) and \( k^2 \ln k \) are exactly identical in the full theory and the EFT. This is the designed feature of matching [1]. Of course, the structure of the relativistic corrections must be the same.

Consequently, we can determine the Wilson coefficients:

\[
C_0(\mu) = \lambda \frac{m}{4m^2} - \left( \frac{\lambda}{4m^2} \right)^2 \left( \frac{m}{4\pi} \right) \left[ 2 + \ln \left( \frac{2m}{\mu} \right) \right] + O(\lambda^3),
\]

(54)

\[
C_2(\mu) = \left( \frac{\lambda}{4m^2} \right)^2 \left( \frac{1}{8\pi m} \right) \left[ \frac{1}{6} + \ln \left( \frac{2m}{\mu} \right) \right] + O(\lambda^3).
\]

(55)

Both of the coefficients are \( \ln \mu \) dependent.

We notice that, the authors of Ref. [10] don’t realize that the subleading non-analytic term, \( k^2 \ln(2im/k) \) in (47), should be identified with the relativistic correction. They introduce two \( \text{ad hoc} \) four-boson operators at second order of \( \nabla \), and adjust their coefficients to reproduce this term by considering the one loop diagram with these operators and \( C_0 \) as vertices. It should be reminded that, relativistic corrections represent solely kinematic effects, and to reproduce them don’t need involve any unknown parameters. In the one-loop matching considered above, knowing the tree-level value of \( C_0 \) suffices to give the correct answer.
In order not to spoil the perturbative matching, one should choose the matching scale $\mu$ around $2m$, to avoid large logarithms. From (54) and (55), one sees that $C_0$ decreases and $C_2$ increases as $\mu$ decreases. The logarithms grow as $\mu$ declines, and ultimately it is more secure to call for the renormalization group (RG) equation to sum these large logs.

2. 3D EFT and renormalization group

Having considered the 3D $\phi^4$ theory as a specific example to illustrate the matching procedure, we now turn to general discussions on the planar system accommodating short-range interactions. We will derive the exact RG equations for the Wilson coefficients, and also present an exact nonperturbative expression for the S-wave scattering amplitude.

We start with deriving the RG equation for $C_0$. The bare $C_0$ can be expressed as

$$C_0^B = \mu^{2\epsilon} \left[ C_0 + \frac{m}{8\pi} \frac{C_0^2}{\epsilon} + \left( \frac{m}{8\pi} \right)^2 \frac{C_0^3}{\epsilon^2} + \cdots \right]$$

$$= \frac{\mu^{2\epsilon} C_0}{1 - \frac{m}{8\pi} \epsilon}.$$  \hspace{1cm} (56)

The leading-order counterterm is given in (52), and the successive ones simply form a geometric series. The absence of subleading poles at any loop order indicates that an exact RG equation for $C_0$ can be deduced. Acting $\mu \frac{d}{d\mu}$ to the above equation, applying the chain rule, one obtains the $\beta$ function for $C_0$:

$$\beta(C_0, \epsilon) = -\frac{2\epsilon}{C_0^B} \left. \frac{\partial C_0^B}{\partial C_0} \right|_{\epsilon}$$

$$= -2\epsilon C_0 + \frac{m}{4\pi} C_0^2.$$  \hspace{1cm} (57)

Since the right-hand side is positive, the free coupling limit is the infrared fixed point. The solution of this RG equation is [9]

$$C_0(\mu) = \left[ \frac{1}{C_0(\Lambda)} + \frac{m}{4\pi} \ln \left( \frac{\Lambda}{\mu} \right) \right]^{-1},$$  \hspace{1cm} (58)

where $\Lambda$, which characterizes the breakdown scale of the nonrelativistic EFT, together with $C_0(\Lambda)$ comprise the boundary conditions. For the nonrelativistic $\phi^4$ theory, we may choose $\Lambda = 2m$, and $C_0(2m)$ can be read off from (54).

In the $k \to 0$ limit, fixed-order bubble diagrams with $C_0$ vertex suffers from logarithmic singularity. One hopes that after a nonperturbative bubble sum, the infrared behavior will ameliorate. Analogous to (15), the bubble diagrams consisting entirely of $C_0$ vertices can be easily summed:

$$A_0^{\text{sum}} = -\left[ \frac{1}{C_0(\mu)} + \frac{m}{4\pi} \left[ \ln \left( \frac{\mu}{k} \right) + \frac{i\pi}{2} \right] \right]^{-1}.$$  \hspace{1cm} (59)

Requesting it to be $\mu$ independent, one quickly reproduces the RG solution (58). To avoid the large logarithm, the optimal renormalization scale $\mu$ should be chosen around $k$. From
(58), we see that the effective coupling $C_0(\mu)$ vanishes inverse logarithmically as $\mu$ approaches zero. As $k \to 0$, this resumed amplitude can be approximated by $A_0^{\text{sum}} \approx -C_0(k)$, therefore smoothly vanishes. It is encouraging that infrared singularities disappear in this resumed amplitude.

The running coupling $C_0(\mu)$ depends on the boundary conditions $\Lambda$ and $C_0(\Lambda)$, each of which cannot be determined separately. Analogous to introducing $\Lambda_{\text{QCD}}$ in Quantum Chromodynamics (QCD), it is useful to trade them for a new scale $\rho$:

$$\rho = \Lambda \exp \left[ \frac{4\pi}{m C_0(\Lambda)} \right]. \quad (60)$$

One can easily verify that $\rho$ is RG invariant.

Because mass in the nonrelativistic theory is merely a passive parameter, it can be transformed away, and consequently $C_0$ can be tuned dimensionless. In this sense, one observes that the nonrelativistic system with a $\delta^2(\mathbf{r})$ potential is classically scale-invariant [9]. However, renormalization necessarily generates a dynamic scale $\rho$, thus breaks the scale invariance quantum-mechanically. This phenomenon is the manifestation of the so-called dimensional transmutation [21], also referred to as scale anomaly in Ref. [9].

One can invert (60) and express the effective coupling $C_0$ in term of $\rho$,

$$C_0(\mu) = \frac{4\pi}{m} \ln^{-1} \left( \frac{\rho}{\mu} \right). \quad (61)$$

When $\mu$ approaches $\rho$, the effective coupling $C_0$ becomes strong, finally diverges at $\mu = \rho$.

Short-range interaction in two spatial dimensions can be classified into two categories. Firstly, for $C_0(\Lambda) > 0$, which we refer to as repulsive interaction, the scale $\rho$ is larger (and can be much larger) than the cutoff $\Lambda$. The coupling $C_0(\mu)$ monotonically decreases as $\mu$ descends from $\Lambda$.

Another case, $C_0(\Lambda) < 0$, referred to as attractive interaction, is more interesting$^3$. Here we have $\rho < \Lambda$. When $\mu$ descends from $\Lambda$ to $\rho$, the coupling $C_0(\mu)$ drops to $-\infty$. This bears some resemblance with QCD, where the strong coupling $\alpha_s(\mu)$ increases as $\mu$ decreases, finally blows up near $\mu = \Lambda_{\text{QCD}}$. However, very different from QCD, just across $\mu = \rho$ infinitesmally, the effective coupling abruptly jumps to $+\infty$ and then gradually diminishes as $\mu$ further decreases. Note that regardless of the sign of $C_0(\Lambda)$, the effective interaction at sufficiently small momentum is always weakly repulsive.

It is well known that $\Lambda_{\text{QCD}}$, as an integration constant, cannot be pinned down from QCD itself. However, if QCD is indeed embedded in a more fundamental theory, e.g. the Grand Unified Theory, then $\Lambda_{\text{QCD}}$ can be unambiguously determined.

The same reasoning applies to our case too. Since nonrelativistic theory is necessarily only an effective theory, $\rho$ can be determined once the more fundamental theory is known. For example, the $\phi^4$ theory specifies such a microscopic theory. Perturbative expansion in

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$^3$The word “attractive” is somewhat inaccurate, because at small $k$, the effective interaction in this case also becomes repulsive.
this theory is governed by the factor $\lambda/8\pi m$. For definiteness, let us take the coupling $\lambda = 4\pi m$, which lies in the perturbative regime. We then find $\rho \approx 2e^8 m \approx 10^4 m$, which is several order-of-magnitude larger than the scalar mass. At a cursory glance, such a gigantic scale can hardly be associated with any reasonable nonrelativistic observables.

The resumed amplitude (59) can also be expressed in term of $\rho$:

$$A_0^{\text{sum}} = -\frac{4\pi}{m} \left[ \ln \left( \frac{\rho}{k} \right) + \frac{i\pi}{2} \right]^{-1}. \quad (62)$$

For the attractive case, it is possible to tune the momentum equal to $\rho$. At this specific momentum, the amplitude becomes purely absorptive and exceedingly simple, $8i/m$. This scale can be viewed as the transition point between the attractive and repulsive interaction.

One can easily infer the imaginary pole $k = i\kappa$ of the amplitude. It is nothing but $\kappa = \rho$. Since $\rho$ is positive definite, one may naively expect that regardless of the sign of $C_0(\Lambda)$, there always exists a bound state. Nevertheless, recall for repulsive interaction, $\rho > \Lambda$, which is (far) beyond the applicable range of the nonrelativistic effective theory. Therefore, this bound state pole is fictitious and should not be endowed with any physical significance.

On the contrary, for attractive interaction, $\rho$ is exponentially suppressed relative to $\Lambda$, therefore there does exist a true bound state with binding energy

$$E_B \approx -\frac{\Lambda^2}{m} \exp \left[ -\frac{8\pi}{m|C_0(\Lambda)|} \right]. \quad (63)$$

Therefore, the smaller $|C_0(\Lambda)|$ is, the much shallower the bound state becomes. To be specific, let us again take the $\phi^4$ theory as an example. If we allow $\lambda$ to be negative$^4$, and take $\lambda = -4\pi m$ for instance, we find the binding energy $E_B \approx -4e^{-16} m \approx -10^{-7} m$, corresponding to a rather shallow bound state.

One bonus comes from the RG equation. The amplitude in (59), being RG invariant, enables us to organize the logarithms of $\lambda^n \ln^n(2m/k)$ ($m > n$) in the $\phi^4$ theory in a most efficient way. After the tree-level matching (13), we can ascertain the leading logarithms $\lambda^{n+1} \ln^n k$ in $T_0$:

$$(-1)^{n+1} \left[ \frac{\lambda \ln(2m/k)}{16\pi m} \right]^n. \quad (64)$$

They can also be directly inferred from the full theory, which simply follow from the s-channel bubble chain.

Similarly, once the one-loop matching is done, we are able to determine all the next-to-leading logs of the form $\lambda^{n+2} \ln^n(2m/k)$. Setting $\mu = 2m$ in (59), plugging $C_0(2m) = \lambda/4m^2 - (\lambda/4m^2)^2 (m/2\pi)$ in, and expanding a first few terms, for example, one can determine the coefficients of $\lambda^3 \ln k$ and $\lambda^4 \ln^2 k$:

$^4$Let us don’t worry about the vacuum stability problem for a moment.
\[-\frac{\lambda^3}{(8\pi m)^2} \ln\left(\frac{2m}{k}\right) \left(1 + \frac{i\pi}{4}\right), \]  
(65)

\[-\frac{\lambda^4}{2(8\pi m)^3} \ln^2\left(\frac{2m}{k}\right) \left(1 + \frac{3i\pi}{8}\right). \]  
(66)

These results cannot be easily obtained in the relativistic $\phi^4$ theory, because one has to extract these logarithms from rather complicated two-loop and three-loop diagrams.

It is straightforward to generalize (59), to include those higher-derivative operators in the bubble chain sum. Because the relation $I_n \approx I_0 k^{2n}$ holds regardless of the spacetime dimensions, the same argument leading to the resummation formula (40) in 2D also applies here. Therefore, the full bubble chain sum renders

$$A_0^{\text{sum}} = - \left[ \frac{1}{C_0 + C_2 k^2 + \cdots} + \frac{m}{4\pi} \left[ \ln\left(\frac{\mu}{k}\right) + \frac{i\pi}{2}\right] \right]^{-1}. \tag{67}$$

We can infer the RG equation for $C_2$ from this equation through a shortcut. The first term in the bracket can be expanded, 

$$1/(C_0 + C_2 k^2 + \cdots) \approx 1/C_0 - C_2/C_0 k^2 + O(k^4).$$

Notice that $1/C_0(\mu)$ together with the $\ln\mu$ term forms a RG invariant. The residual terms must be $\mu$ independent at any order of $k^2$ individually. Consequently, one can read off the RG equation of $C_2$:

$$\mu \frac{d}{d\mu} \left( \frac{C_2}{C_0^2} \right) = 0. \tag{68}$$

Therefore $C_2(\rho)$ diverges as $C_0^2(\rho)$. One can verify that in general, $C_{2n}(\rho) \propto C_0^{n+1}(\rho)$.

This RG equation can be confirmed by directly working out the counterterms of $C_2$, which arise from the bubble chain containing all $C_0$ vertices but one $C_2$ vertex:

$$C_2^B = \mu^{2\kappa} \left[ C_2 + 2 \left( \frac{m}{8\pi} \right) \frac{C_2 C_0}{\epsilon} + 3 \left( \frac{m}{8\pi} \right)^2 \frac{C_2 C_0^2}{\epsilon^2} + \cdots \right]$$

$$= \frac{\mu^{2\kappa} C_2}{\left( 1 - \frac{m}{8\pi \epsilon} C_0 \right)^2}. \tag{69}$$

Dividing this equation by the square of (56), one immediately recovers (68). With (57) as the input, one readily deduces the $\beta$ function for $C_2$:

$$\beta(C_2, \epsilon) = -2\epsilon C_2 + \frac{m}{2\pi} C_0 C_2. \tag{70}$$

Because the resumed amplitude (67) is RG invariant, we have the freedom to choose any $\mu$ which we prefer. At first sight, setting $\mu = \rho$ leads to great simplification, since all the $C_{2n}(\rho)$ diverge, so the first term in the bracket may be dropped. Consequently, the amplitude still remains the very simple form of (62), and the pole still remains at $\kappa = \rho$.

This is puzzling, for it implies that effects of those higher-derivative operators can be totally discarded. A closer examination discloses that, choosing $\mu \approx \rho$ will instead lead to a highly unstable answer. Recall $C_0$ diverges to either $+\infty$ or $-\infty$, depending on toward
which direction $\mu$ approaches $\rho$. Similar pattern occurs for $C_4$, $C_8$, and so on. Therefore,
for any finite $k$, if one chooses $\mu$ very close to $\rho$, there is a possibility of large cancellations
among different terms in the series $\sum C_{2n} k^{2n}$. As a result, $1/\sum C_{2n}(\rho) k^{2n}$ may not vanish
as one naively expects.

A judicious analysis indicates that the higher-derivative operators do affect the location
of the pole. From (67), one finds that the pole $\kappa$ no longer coincides with $\rho$, but shifts by
an amount of

$$\Delta \kappa = \frac{4\pi}{m} \frac{C_2(\mu)}{C_0^2(\mu)} \rho^3 + O(\rho^5).$$  \hspace{1cm} (71)

Note this shift is RG invariant, as it must be.

The RG equation of $C_2$, (68), indicates that $C_2(\mu) \sim \ln^{-2}(\rho/\mu)$ as $\mu \to 0$, vanishing in
a more rapid speed than $C_0$. However, so far we have neglected the renormalization of $C_2$
by $C_0$ through relativistic corrections. Recall in (55), the relativistic effect tends to enhance
$C_2(\mu)$ as $\mu$ decreases, which counteracts the effect represented by (68). The true RG flow of
$C_2$ will depend upon the competition between them.

Now let us rederive the RG equation for $C_2$, this time including the effects of relativistic
corrections. The leading relativity-induced counterterm can be extracted from $\tilde{I}_0$, and has
been given in (53). Higher-order counterterms can be worked out analogously by computing
the bubble diagrams which contribute at $O(k^2)$. These diagrams can have $C_0$, $\delta C_0$ or lower-
order $\delta C_2$ induced by the relativistic corrections, as their vertices, and may also need one
relativistic vertex insertion in the loops. Adding these new counterterms to (69), we have

$$C_2^B = \mu^{2\epsilon} \left[ \frac{C_2}{(1 - \frac{m C_0}{8\pi \epsilon})^2} - \frac{1}{2(8\pi m)} \frac{C_0^2}{\epsilon} - \frac{1}{(8\pi)^2} \frac{C_3}{\epsilon^2} - \frac{3m}{2(8\pi)^3} \frac{C_0^4}{\epsilon^3} - \cdots \right]$$

$$= \mu^{2\epsilon} \frac{C_2 - \frac{1}{2(8\pi m)} \frac{C_0^2}{\epsilon}}{\left(1 - \frac{m C_0}{8\pi \epsilon}\right)^2}.$$  \hspace{1cm} (72)

Note these new counterterms can also be cast into a geometric series. Acting $\mu \partial/\partial \mu$ on this
equation, applying the chain rule, we obtain the full $\beta$ function for $C_2$:

$$\beta(C_2, \epsilon) = -2\epsilon C_2 + \frac{m}{2\pi} C_0 C_2 - \frac{C_0^2}{8\pi m}.$$  \hspace{1cm} (73)

In deriving this, the knowledge of $\beta(C_0, \epsilon)$ is needed. Two competing forces driving the RG
flow of $C_2$ are manifest in this equation. We pause to point out one subtlety concerning these
two contributions. Whereas the RG equation for $C_2$ obtained alone from the bubble chain
consisting of all $C_0$ vertices except one $C_2$ vertex, (70), is self-consistent, the converse is not true.
It can be easily checked, if one keeps only the counterterms induced by the relativistic
corrections in (72), no sensible $\beta$ function will be obtained$^5$. Therefore, the relativistic effects
cannot be isolated from the higher-derivative operators.

$^5$This means $\beta$ function will contain the uncancelled poles.
Dividing both sides of this equation by $C_2$, one can arrange it into the form:

$$\mu \frac{d}{d\mu} \left( \frac{C_2}{C_0^2} \right) = -\frac{1}{8\pi m}, \quad (74)$$

which can be easily solved:

$$\frac{C_2(\mu)}{C_0^2(\mu)} = \frac{C_2(\rho)}{C_0^2(\rho)} + \frac{1}{8\pi m} \ln \left( \frac{\rho}{\mu} \right). \quad (75)$$

We now see that, contrary to (68), $C_2/C_0^2$ is no longer a constant, but increases logarithmically as the renormalization scale gets lower. In the $\mu \to 0$ limit, the RG flow of $C_2(\mu)$ is dominated by the relativistic corrections. Keeping only the second term in the right-hand side, one finds

$$C_2(\mu) \approx \frac{2\pi}{m^3} \ln^{-1} \left( \frac{\rho}{\mu} \right). \quad (76)$$

Therefore, $C_2$ approaches zero in the same speed as $C_0$. Comparing (61) and (76), one finds a universal relation irregardless of any specific planar system: $C_2(\mu)/C_0(\mu) \approx 1/2m^2$ at sufficiently small $\mu$.

The concise form of (72) suggests that the relativistic effects can also be incorporated in the resumed amplitude. Similar to the consideration leading to (42) in 2D, such a resummation formula must first reproduce the one-loop amplitude (51), which includes the first-order relativistic correction. We thus generalize (67) to

$$A_{0 \text{sum}} = -\left[ \frac{1}{C_0 + C_2 k^2 + \cdots} + \frac{m}{4\pi} \left( 1 - \frac{k^2}{2m^2} + \cdots \right) \left[ \ln \left( \frac{\mu}{k} \right) + \frac{i\pi}{2} \right] \right]^{-1}. \quad (77)$$

Expanding the first term in the bracket to $O(k^2)$, combining it with the $k^2 \ln \mu$ term, and demanding them to be $\mu$ independent, we recover the RG equation for $C_2$, (74). This provides a cogent support for this formula.

We need to know the higher-order relativistic corrections. The $\phi^4$ theory again provides the useful guidance in helping to recognize the pattern. Examining (45), one finds that, interestingly enough, this series is again represented by the dilation factor $\gamma^{-1}$. One can verify this resummation formula by all kinds of straightforward computations.

Knowing the structure of the exact amplitude, we can pin down those logarithms accompanying $k^2$ in the relativistic $\phi^4$ theory, analogous to what we have done in (64)–(66). After the tree-level matching, one can infer the leading logarithms $\lambda^{n+1} k^2 \ln^n k$ in $T_0$ to all orders:

$$(-1)^n \frac{n k^2}{2m^2} \lambda \left[ \frac{\lambda \ln(2m/k)}{16\pi m} \right]^n. \quad (78)$$

In the full theory, these leading logarithms come from the $s$-channel bubble chain with the first-order relativistic correction retained.
Once $C_0$ and $C_2$ are determined through the one-loop matching, we are able to know all the next-to-leading logs of the form $\lambda^{n+2} k^2 \ln^k n$. Taking $\mu = 2m$ in (77), substituting $C_0(2m)$ and $C_2(2m) = (\lambda/4m)^2/(48\pi m^3)$ in, and expanding a first few terms, for example, one can determine the next-to-leading logs at $O(\lambda^3)$ and $O(\lambda^4)$:

\[
\frac{k^2 \lambda^3 \ln(2m/k)}{m^2 (16\pi m)^2} \left( \frac{13}{6} + i\pi \right),
\]

\[
- \frac{k^2 \lambda^4 \ln^2(2m/k)}{4m^2 (16\pi m)^3} (25 + 9i\pi).
\]

Needless to say, these results are difficult to derive in the full theory.

When relativistic effects are included, the pole shifts from $\rho$ by an amount of

\[
\Delta \kappa = \frac{4\pi}{m} \left[ \frac{C_2(\mu)}{C_0^2(\mu)} - \frac{1}{8\pi m} \ln \left( \frac{\rho}{\mu} \right) \right] \rho^3 + O(\rho^5),
\]

\[
= \frac{4\pi}{m} \frac{C_2(\rho)}{C_0^2(\rho)} \rho^3 + O(\rho^5),
\]

where we resort to (75) in the second line. Evidently, this shift is also RG invariant, and much resembles its counterpart without incorporating relativistic corrections, (71). The corresponding binding energy then becomes

\[
E_B = -\frac{\rho^2}{m} \left[ 1 + \frac{8\pi}{m} \frac{C_2(\rho)}{C_0^2(\rho)} \rho^2 + \frac{\rho^2}{4m^2} + O(\rho^4) \right].
\]

Requiring the resumed amplitude (77) to be RG invariant, we can infer the RG equations for all remaining Wilson coefficients. Let us take $C_4$ as a specific example. Expanding $1/\sum C_{2n} k^{2n}$ and $\gamma^{-1} \ln \mu$ to the 4th order of $k$, piecing their $O(k^4)$ coefficients together and demanding it to be $\mu$ independent, we obtain the following coupled RG equation:

\[
\mu \frac{d}{d\mu} \left( \frac{C_4}{C_0^2} - \frac{C_2^2}{C_0^3} \right) = \frac{3}{32\pi m^3}.
\]

Consequently, the $\beta$ function for $C_4$ can be readily identified:

\[
\beta(C_4) = \frac{m}{2\pi} C_0 C_4 + \frac{m}{4\pi} C_2^2 - \frac{C_0 C_2}{4\pi m} + \frac{3 C_0^2}{32\pi m^3}.
\]

The first two terms in the right-hand side are as expected from (67), when the relativistic effects are turned off. The third term arises from implementing the first-order relativistic correction in bubble diagrams containing all $C_0$ vertices but one $C_2$ vertex, whereas the last term stems from the second-order relativistic correction in the bubble diagrams comprising entirely of $C_0$ vertices.

From (83) and the asymptotic behaviour of $C_0$ and $C_2$, one finds that $C_4(\mu)$ in the $\mu \to 0$ limit is approximately

\[
C_4(\mu) \approx -\frac{\pi}{2m^5} \ln^{-1} \left( \frac{\rho}{\mu} \right).
\]
We are now at a stage to understand the general pattern of the asymptotic behaviour of $C_{2n}$ coherently. If we take $\mu = k$ in (77), the terms explicitly depending upon $\ln \mu$ vanish. In the $k \to 0$ limit, the amplitude is well approximated by $- \sum C_{2n}(k) k^{2n}$.

Because the amplitude is RG invariant, we can freely switch to a different $\mu$. Recall we have warned that taking $\mu = \rho$ doesn’t produce a stable result. However, at sufficient small $k$, it is permissible to choose $\mu$ around $\rho$ in (77). In the $k \to 0$ limit, one can approximate $1/ \sum C_{2n}(\rho) k^{2n}$ by $1/C_0(\rho)$, which hence vanishes. The resumed amplitude therefore reduces to

$$A_0^{\text{sum}} \approx - \frac{4\pi}{m} \gamma \ln^{-1} \left( \frac{\rho}{\mu} \right). \quad (86)$$

Recall $\gamma = 1 + k^2/2m^2 - k^4/8m^4 + \cdots$, so we immediately identify the asymptotic forms of $C_{2n}(k)$, and readily reproduce the earlier results in (61), (76) and (85). This signals, regardless of the boundary conditions $C_{2n}(\Lambda)$, all the Wilson coefficients at sufficiently small renormalization scale $\mu$ are effectively generated by the relativistic effects— all of them are inversely proportional to $\ln(\rho/\mu)$, with the coefficients fixed by the dilation factor.

C. Four dimensions

Because of its close connection with the reality, short-range force in three spatial dimensions has been extensively discussed in the literature. Our main new result is to fully incorporate the relativistic effects in the resumed amplitude. We also determine the effective range in the $\phi^4$ theory, which roughly equals the Compton wavelength.

1. Matching of 4D $\phi^4$ theory

In the 4D $\phi^4$ theory, the coupling $\lambda$ is dimensionless and this theory becomes renormalizable. In addition, because the spatial dimension is now big enough, the fixed-order perturbation series no longer suffers the zero-momentum singularity as encountered in 2D and 3D.

It is necessary to specify the renormalization prescription. It is standard to use MS when studying the high energy process, where the running coupling and running mass are valuable notions. However, for the nonrelativistic problem at hand, it is most convenient to choose the on-shell renormalization scheme, in which the renormalized coupling $\lambda$ and the mass $m$ are physical observables. In this scheme, the counterterm $\delta \lambda$ is chosen such that the 2-body amplitude in the zero-momentum limit remains fixed at $-\lambda$ to all orders.

The $T$-matrix to one loop order in this scheme is [19]

$$T = -\lambda - \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left\{ \ln \frac{1 - x(1 - x)s/m^2 - i\epsilon}{1 - 4x(1 - x)} + \ln[1 - x(1 - x)t/m^2] \right\}.$$  \quad (87)

The $s$-channel integral can be worked out and expanded.
\[
\int_0^1 dx \ln[1 - x(1 - x) s/m^2 - i\epsilon] = -2 + \frac{4k}{\sqrt{s}} \left( \tanh^{-1} \beta - \frac{i\pi}{2} \right)
\approx -2 + \frac{2k^2}{m^2} - \frac{i\pi k}{m} \left( 1 - \frac{k^2}{2m^2} \right). \tag{88}
\]

One can carry out the \(t\)-, \(u\)-channel integrals, similar to as in 2D and 3D:
\[
\int_0^1 dx \left\{ \ln[1 - x(1 - x) t/m^2] + (t \to u) \right\} \approx -\frac{t + u}{6m^2} = \frac{2k^2}{3m^2}. \tag{89}
\]

When calculating the \(t\)- and \(u\)-channel diagrams, the authors of Ref. [11] neglect mass of the scalar particle in the loop integral. After Fourier-transforming the amplitude to the coordinate space, they then find that two particles effectively experience a \(-1/r^3\) long-range potential. We should stress, nevertheless, the approximation \(m = 0\) inside the loop is not legitimate. Since the typical virtuality of the internal momenta is \(O(m^2)\), the Uncertainty Principle implies that these virtual particles cannot propagate much farther than the Compton wavelength. Thus the effects of these diagrams should be mimicked by the local operators, instead of by an instantaneous, nonlocal potential.

Piecing (88) and (89) together, we obtain the S-wave amplitude:
\[
T_0^{1\text{-loop}} = -\lambda + \frac{\lambda^2}{32\pi^2} \left[ \frac{i\pi k}{m} \left( 1 - \frac{k^2}{2m^2} \right) - \frac{8k^2}{3m^2} \right]. \tag{90}
\]

Absence of constant term at \(O(\lambda^2)\) is specific to the on-shell renormalization scheme.

Next we consider the one loop calculation in the EFT sector. Fig. 3b) is already known in (21), and we need simply to include the first-order relativistic correction:
\[
I_0 = \frac{m}{8\pi} \sqrt{-2mE - i\epsilon} \approx -\frac{imk}{8\pi} \left( 1 - \frac{k^2}{8m^2} \right). \tag{91}
\]

Fig. 3c) and d), the one loop diagram with one insertion of the relativistic vertex, can be obtained by substituting \(D = 4\) in (29):
\[
\tilde{I}_0 = \frac{imk}{8\pi} \left( \frac{3k^2}{8m^2} \right). \tag{92}
\]

Merging these together, we obtain the S-wave amplitude in the EFT sector:
\[
A_0^{1\text{-loop}} = -C_0 + iC_0^2 \frac{mk}{8\pi} \left( 1 - \frac{k^2}{2m^2} \right) - C_2 k^2. \tag{93}
\]

By construction, \(C_0\) doesn’t receive any modification with respect to (13). Needless to repeat, both the full theory and the EFT share the same non-analytic (imaginary) terms. The relativistic corrections only influence the imaginary parts of the amplitude. Matching (90) onto (93) via (11), we then read off \(C_2\):
\[
C_2 = \frac{1}{3m^4} \left( \frac{\lambda}{4\pi} \right)^2 + O(\lambda^3). \tag{94}
\]
2. Effective range expansion

If the scattering length in a nonrelativistic system is of natural size, we can simply stick to MS in the EFT sector. Contrary to the lower dimensional cases, it is not compulsive here to sum the bubble diagrams to all orders. Retaining first few terms in the perturbation series suffice for practical purpose. Nevertheless, it is still instructive to perform the full bubble sum. We can routinely generalize (18) to incorporate all the higher-derivative terms, by replacing $1/C_0$ with $1/\sum C_{2n} k^{2n}$.

Unlike in 2D and 3D, there is no strong motivation to include relativistic corrections. Nevertheless, for completeness and clarification, we proceed to give a resummation formula which fully implements the relativistic effects.

Analogous to the previous analyses, inspecting the imaginary part of the one-loop $s$-channel integral in the $\phi^4$ theory, (88), we find that the relativistic factor is again represented by $\gamma^{-1}$, exactly the same as in 2D and 3D. Therefore, the Lorentz-invariant resumed amplitude is

$$A_0^{\text{sum}} = -\left[\frac{1}{C_0 + C_2 k^2 + \cdots} + \frac{im}{8\pi} \gamma^{-1} k\right]^{-1}. \hspace{1cm} (95)$$

At small $k$, this amplitude is dominated by $-C_0$. Evidently, it is not essential to include the relativistic corrections.

A by-product of this resumed amplitude is that it conveniently embodies the optical theorem. One can quickly read off the uncalculated higher order imaginary part from the lower order results. For example, at $O(\lambda^3)$, the leading imaginary term in $T_0$ is

$$2i C_0 C_2 \frac{m k^3}{8\pi} (4m^2) = \frac{i\lambda^3}{3(4\pi)^3} \frac{k^3}{m^3}. \hspace{1cm} (96)$$

It will be little bit laborious to infer from the relativistic $\phi^4$ theory.

According to the partial wave expansion, the S-wave partial amplitude can be parameterized as

$$A_0 = \frac{8\pi}{m} \frac{\gamma}{k} e^{i\delta_0} \sin \delta_0, \hspace{1cm} (97)$$

where $\delta_0$ is the S-wave phase shift. It is convenient to rewrite $\exp(i\delta_0) \sin \delta_0 = 1/(\cot \delta_0 - i)$. It is well known that in the low energy scattering, each partial amplitude is insensitive to the fine structure of the short-range potential, instead can be characterized rather accurately by a few parameters only. This idea, akin to the spirit of EFT, is referred to as effective range expansion. According to this ansatz, we parameterize the S-wave phase shift as

---

6The kinematic factor is chosen such to reproduce the cross section formula in the relativistic field theory: $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |T_0|^2$. The phase shift is defined according to $\frac{d\sigma}{d\Omega} = \frac{4\sin^2 \delta_0}{k^2}$, where the effect due to identical bosons is accounted by the factor 4.

7Note the relativistic factor $\gamma^{-1}$ is absent in the standard definition.
\[
\gamma^{-1} k \cot \delta_0 = -\frac{1}{a_0} + \frac{r_0}{2} k^2 + \cdots ,
\]
(98)
where \(a_0\) is the S-wave scattering length, and \(r_0\) is the effective range. The S-wave amplitude can thereby be written
\[
A_0 = \frac{8\pi}{m} \left[ -\frac{1}{a_0} + \frac{r_0}{2} k^2 + \cdots - i\gamma^{-1} k \right]^{-1} .
\]
(99)

Comparing (95) and (99), one sees that EFT and effective range expansion are completely equivalent. Note they share the same structure of relativistic corrections. Recalling the values of \(C_0\) and \(C_2\) from the one-loop matching, we can identify the scattering length and the effective range in the \(\phi^4\) theory:
\[
a_0 = \frac{m}{8\pi} C_0 = \frac{\lambda}{32\pi m} , \\
r_0 = \frac{16\pi}{m} \frac{C_2}{C_0^2} = \frac{16}{3\pi m} [1 + O(\lambda)] .
\]
(100)

Clearly, the on-shell renormalization scheme in the \(\phi^4\) theory perfectly matches with the effective range expansion. From this scheme, we acquire an exact scattering length, and an approximate effective range, which yet can be expanded order by order in \(\lambda\). For the coupling lying in the perturbative region \((\lambda < 16\pi^2)\), we always have \(a_0\) smaller than \(r_0\).

It is interesting to note, the effective range in the \(\phi^4\) theory at leading order doesn’t depend on \(\lambda\). It approximately equals the Compton wavelength, consistent with what we have expected in Section III. Because \(k r_0 \sim k/m \ll 1\), so practically it is unnecessary to include any higher partial waves.

One subtlety deserves being pointed out. If the factor \(\gamma^{-1}\) were not absorbed in the definition of the effective range expansion in (98), \(r_0\) would receive an additional correction, \(-32\pi/(\lambda m)\). This is an unacceptable situation, since a very weak coupling would correspond to a very large negative effective range!

We have adopted (5) to implement relativistic corrections in this work. It is worth commenting on what if the alternative scheme, (8), is instead used. It should be of some interest, since this scheme seems to be favored by many authors. To convert the resumed formula into this scheme, one needs divide (95) and (97) by \(\gamma^2\), in compliance with the reduction formula (One can understand this factor by comparing (11) and (12)). The corresponding amplitude in this scheme then reads:
\[
A_0^{\text{sum}} = - \left[ \frac{\gamma^2}{C_0 + C_2 k^2 + \cdots} + \frac{i m}{8\pi} \gamma k \right]^{-1} .
\]
(101)

Keeping the lowest order terms, one readily recovers \(A_0^{\text{tree}}\) in (10). We notice that Ref. [17] presents a similar resummation formula which contains the correct imaginary term in the bracket, but misses the factor \(\gamma^2\) in the first term. Note those two-body operators induced by the field redefinition (7) have not been included in Ref. [17].

Although triviality of the \(\phi^4\) theory cannot be substantiated in the nonrelativistic limit, it is possible to establish a much weaker assertion— that the strong interacting \(\phi^4\) theory may not exist.
It is well known that an unusually large positive (negative) $a_0$ corresponds to a threshold (virtual) bound state. This can be understood from (99), the pole $\kappa$ is located roughly at $1/a_0$ for $a_0 \gg r_0$. For the $\phi^4$ theory to be well defined, we must request a positive $\lambda$, and the repulsive interaction forbids it to host any bound state. It is equivalent to say that this theory cannot possess a large positive scattering length. As a result, the self-coupling $\lambda$ cannot be too strong.

Complimentary evidence comes from (100). If $\lambda$ can be very large, $r_0$ may significantly depart from the Compton wavelength, driven by the higher-order corrections. This seems to conflict with the natural expectation that the effective range in this theory should always be of order $1/m$.

Let us quantify this discussion little bit. The location of bound state pole for general $a_0$, $r_0$ can be solved from (99):

$$\kappa = \frac{1}{a_0} \frac{2}{1 + \sqrt{1 - 2r_0/a_0}},$$

where we neglect the insignificant relativistic correction. To nullify such a pole, one requires $a_0 < 2r_0$, so that $\kappa$ doesn’t admit a real solution. This imposes a bound $\lambda < 32^2/3 \approx 341$. Unfortunately, this bound is too loose to be useful, because when $\lambda$ exceeds $16\pi^2 \approx 158$, the perturbative matching can no longer be trusted, neither can the effective range determined from it.

3. Incorporating Relativity in PDS

The PDS scheme is tailor-made for describing the finely-tuned system in which the scattering length becomes unnaturally large [18]. In this scheme, one not only subtracts the $D = 4$ pole as in MS, but also the $1/(D-3)$ pole which corresponds to the linear divergence at $D = 4$. As a result, the subtracted integral depends on the subtraction scale $\mu$ linearly, which mimics the effects of $\Lambda$ in the cutoff scheme.

The integral $I_n$ in (36) exhibits a $D = 3$ pole, which can be removed by adding to $I_n$ the counterterm

$$\delta I_n = -\left(\frac{m}{8\pi}\right)\frac{(2Em)^n\mu}{D-3},$$

so that the subtracted integral in $D = 4$ is

$$I_n^{\text{PDS}} = I_n + \delta I_n \approx -\left(\frac{m}{8\pi}\right) k^{2n} (\mu + ik),$$

where the relativistic correction has been neglected.

Since the relation $I_n \approx I_0 k^{2n}$ still holds in PDS, according to the preceding discussions, the bubble chain diagrams incorporating all the higher-derivative terms can still be summed analytically, and the result is

$$A_{0}^{\text{sum}} = -\left[\frac{1}{C_0 + C_2k^2 + \ldots} + \frac{m}{8\pi}(\mu + ik)\right]^{-1}.$$  

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At $\mu = 0$, PDS coincides with MS.

Requiring the amplitude to be $\mu$ independent, one can infer the RG equations for the Wilson coefficients. For instance, $C_0$ and $C_2$ satisfy the RG equations:

$$\frac{d C_0}{d\mu} = \frac{m}{8\pi} C_0^2 ,$$

$$\frac{d}{d\mu} \left( \frac{C_2}{C_0^2} \right) = 0 .$$

Note they are identical to their counterparts in 3D, (57) and (68), except one needs replace $\mu$ by $\ln \mu$ and double the right side of both equations. This is not unexpected, because the same $1/(D - 3)$ poles get subtracted in both cases.

The solutions of these RG equations are

$$C_0(\mu) = \frac{8\pi}{m} \left( \frac{1}{a_0} - \mu \right)^{-1} ,$$

$$C_2(\mu) = \frac{4\pi r_0}{m} \left( \frac{1}{a_0} - \mu \right)^{-2} .$$

The scattering length and effective range enter as the boundary condition, which specify the effective coupling at $\mu = 0$ through (100). Note $1/a_0$ here plays the similar role as $\rho$ in 3D. First, the bound state pole is approximately located at $\kappa \approx 1/a_0$; second, all the couplings $C_{2n}$ diverge at $\mu = 1/a_0$. Nevertheless, contrary to the logarithmic running in 3D, the effective couplings in PDS depend linearly on $\mu$, so run quite fast. When the momentum is larger than $1/a_0$, one usually chooses $\mu \sim k$, so that all the Wilson coefficients have definite scaling in momentum, $C_{2n}(\mu) \sim 1/\mu^{n+1} \sim 1/k^{n+1}$ [18].

Let us now take relativistic effects into account. First consider the one loop integral with one relativistic vertex insertion, $\tilde{I}_0$, whose D-dimensional expression is given in (29). To get rid of its $D = 3$ pole, we add the counterterm

$$\delta \tilde{I}_0 = \left( \frac{m}{8\pi} \right) \frac{k^2}{2m^2} \frac{\mu}{D - 3} .$$

Therefore, the one-loop integral $I_0 + \tilde{I}_0$ is

$$(I_0 + \tilde{I}_0)^{\text{PDS}} = - \frac{m}{8\pi} \left( 1 - \frac{k^2}{2m^2} \right) (\mu + ik) .$$

It is identical to its MS counterpart, (93), except $ik$ there should be promoted to $\mu + ik$.

One can check that this pattern is completely general. Therefore, the resummation formula (105) can be generalized into a Lorentz-invariant form:

$$\mathcal{A}_0^{\text{sum}} = - \left[ C_0 + C_2 \frac{k^2}{2m^2} + \cdots + \frac{m}{8\pi} \gamma^{-1}(\mu + ik) \right]^{-1} .$$

This accomplishment should not bring much surprise. At any rate, PDS, like MS, is based on the DR, and it is well known that DR preserves the spacetime symmetries by default.
The RG equation of $C_0$ is not affected by relativistic corrections. After incorporating the relativistic effects, $C_2$ satisfies the following RG equation:

$$\frac{d}{d\mu} \left( \frac{C_2}{C_0^2} \right) = -\frac{1}{16\pi m},$$

which is again very similar to its 3D counterpart, (74). It can be easily solved:

$$C_2(\mu) = \frac{4\pi}{m} \left( r_0 - \frac{\mu}{m^2} \right) \left( \frac{1}{a_0} - \mu \right)^{-2}. \quad (114)$$

For any reasonable subtraction scale $\mu$, the relativistic correction, which is represented by $\mu/m$, is always much smaller that $r_0$.

Let us take the two-nucleon S-wave scattering as an concrete example, to estimate the importance of the relativistic effects. The effective range $r_0$ is roughly about $1/m\pi$. In fitting low-energy scattering data, one usually chooses $\mu = m\pi$ [18]. One finds the relativistic effect reduces $C_2(m\pi)$ given in (109) by $m^2\pi/m^2\Lambda \approx 2\%$. As expected, this is a quite small effect.

V. SUMMARY

In this work, we have presented a rather detailed study of the short-range interaction in the 2-body sector. This study is expedited by employing the EFT approach, which is in many aspects superior to the quantum mechanical formalism.

Considerable effort is devoted to clarifying some confusion concerning the short-range interaction. Based on the Uncertainty Principle, we argue that, any distance scale which can be confidently referred in nonrelativistic quantum mechanics must exceed the Compton wavelength. The effective range should obey this criterion. Therefore, contact interaction, such as the $\delta^3(r)$ potential, should be viewed as an idealized, but unrealistic and physically-irrelevant notion. The tenet can be also stated in another way– the cutoff $\Lambda$ in all the EFTs should be kept finite. Evidently, triviality of the 4D $\phi^4$ theory cannot be substantiated in the nonrelativistic limit. Nevertheless, from other considerations elaborated at the end of Sec. IV, we argue that a much weaker assertion may hold true– there is no strongly interacting 4D $\phi^4$ theory.

There are very few exactly soluble models in quantum field theory. Notably, the expression we obtained for the S-wave amplitude doesn’t involve any approximation. This knowledge allows us to unequivocally extract important nonperturbative information such as the bound state pole, which will never show up at fixed order perturbation series. Although the short-range force is basically a quantum mechanical problem, it is crucial to employ the field-theoretic method to accomplish this. Therefore, this complete solution adds some wealth to the treasury of quantum field theory.

We have considered the short-range interactions in various spacetime dimensions. Among them, the attractive interaction in 3D is especially interesting. It shares some similar features with QCD, e.g., most notably, dimensional transmutation. In this problem, a dynamical scale $\rho$, which could be much smaller compared to the cutoff scale, is generated. Analogous to the low-energy dynamics of QCD, which is mainly governed by $\Lambda_{QCD}$,
the dynamics in this case is largely controlled by $\rho$. In addition, this theory also displays asymptotic freedom in the region $\rho < \mu < \Lambda$.

Relativistic effects are usually thought unimportant in the nonrelativistic limit. However, as we have seen in 3D, relativity plays an important role in governing the RG flows of the higher-dimensional operators at infrared scale.

The EFT method sheds useful light on the behaviour of the $\phi^4$ theory in the nonrelativistic limit. At any fixed-order perturbative expansion, the 2D and 3D $\phi^4$ theories are plagued by infrared singularities. It is only with recourse to the EFT that one can achieve a sensible result at small momentum. Furthermore, the power of EFT is vividly exemplified by the renormalization group. With the aid of the RG equations in 3D EFT, one can easily deduce the next-to-leading logarithms in the 3D $\phi^4$ theory. It is much more efficient than directly extracting them from multi-loop diagrams in the full theory.

The equivalence between the effective range expansion and the resummation formula in 4D EFT is carefully verified, with the relativistic effects fully accounted for. We then pinpoint the effective range in the 4D $\phi^4$ theory to be approximately $16/3\pi m$. This nonzero result provides the compelling evidence to our earlier statement, that no any physical system accommodates a zero-range interaction.

It is interesting, but challenging to infer the $O(\lambda)$ correction to the effective range. To accomplish this, the two-loop matching, hence the two-loop calculations in the $\phi^4$ theory are requested. The two-loop integrals in the full theory are enormously complicated, unlikely to be worked out in a closed form. Fortunately, knowing their approximate expressions, which are expressed in power expansion of $k$, will be sufficient for the purpose of matching. The threshold expansion method [22] can be called for to fulfill this goal.

We have only considered the two-body scattering in this work. An interesting application of the EFT method is to study the many-body phenomena. For instance, the ground-state energy density for a dilute homogeneous gas have been calculated in the EFT framework [3,4]. One can exploit similar techniques to attack the collective phenomena in one and two spatial dimensions.

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