On actions of connected algebraic groups

Michel Brion

Abstract

We obtain a version of the theorem of the square and a local structure result for actions of connected algebraic groups on seminormal varieties in characteristic 0, and arbitrary varieties in positive characteristics.

1 Introduction and statement of the main results

Consider a connected algebraic $k$-group $G$ acting on a normal $k$-variety $X$, where $k$ is a field. When $G$ is linear, every line bundle $L$ on $X$ is $G$-invariant, and some positive power $L^\otimes n$ is $G$-linearizable. Moreover, $X$ is covered by $G$-stable open subvarieties which admit equivariant embeddings into projectivizations of $G$-modules.

These fundamental theorems, due to Sumihiro (see [Su74, Su75]), do not extend to actions of arbitrary algebraic groups: they fail e.g. for the action of an elliptic curve on itself by translations. But some analogues are known when $k$ is algebraically closed: firstly, the theorem of the square, which asserts in loose words that the polarization map $G(k) \to \text{Pic}(X), \ g \mapsto g^*(L) \otimes L^{-1}$ is a group homomorphism (a more general result is due to Raynaud in the setting of group schemes, see [Ra70, Thm. IV.3.3]). Secondly, $X$ is covered by $G$-stable open subvarieties which admit equivariant embeddings into projectivizations of $G$-homogeneous vector bundles over abelian varieties, quotients of $G$ (see [Br10, Thm. 1, Thm. 2]).

In this article, we extend the theorem of the square and the above local structure of $G$-actions to possibly nonnormal $G$-varieties. We use results and methods from our article [Br14] (based in turn on work of Weibel in [We91]), which extends Sumihiro’s theorems to seminormal varieties in characteristic 0, and to all varieties in positive characteristics, with the Zariski topology replaced by the étale topology. The first results along these lines are due to Alexeev for proper seminormal varieties and their families; he also obtained a version of the theorem of the square for actions of semiabelian schemes (see [Al02, Sec. 4]).

To state our main results in a precise way, we introduce some notation and conventions. We consider schemes and their morphisms over a fixed field $k$. Schemes are assumed to be separated, and of finite type unless otherwise stated. For any group scheme $G$, we denote by $(x, y) \mapsto xy$ the multiplication in $G$, and $e_G \in G(k)$ the neutral element. A $G$-scheme is a scheme $X$ equipped with a $G$-action $\alpha : G \times X \to X, (g, x) \mapsto g \cdot x$. A variety is a reduced scheme. An algebraic group is a smooth group scheme.

Also, we briefly recall the notion of seminormality: a reduced scheme $X$ is seminormal if every integral bijective morphism $f : X' \to X$ which induces an isomorphism on all
residue fields, is an isomorphism. Every reduced scheme has a seminormalization map, which is a universal homeomorphism. We refer to [Ko13, Sec. 10.2] for an overview, and to [T70, GT80, Sw80] for more on this notion.

We may now state our version of the theorem of the square:

**Theorem 1.1.** Let $G$ be a connected group scheme, $X$ a connected $G$-scheme (assumed seminormal if $\text{char}(k) = 0$), and $L$ a line bundle on $X$. Consider the line bundle on $G \times G \times X$ defined by

$$\mathcal{L} := (\mu \times \text{id}_X)^* \alpha^*(L) \otimes p_{13}^* \alpha^*(L^{-1}) \otimes p_{23}^* \alpha^*(L^{-1}) \otimes p_3^*(L),$$

where $\mu : G \times G \to G$ denotes the multiplication, $\alpha : G \times X \to X$ the action, and $p_{13}, p_{23} : G \times G \times X \to G \times X, p_3 : G \times G \times X \to X$ the projections. Then there exists a positive integer $n$ and a line bundle $M$ on $G \times G$ such that $\mathcal{L}^\otimes n = p_{12}^*(M)$.

In particular, by pulling back $\mathcal{L}$ to $\{g\} \times \{h\} \times X$ for $g, h \in G(k)$, we see that the line bundle $(gh)^* (L) \otimes g^*(L^{-1}) \otimes h^*(L^{-1}) \otimes L$ is $n$-torsion; this is closer to the classical formulation of the theorem of the square. More generally, the assertion on $\mathcal{L}$ is equivalent to the assignment $G \to \text{Pic}_X, g \to g^*(L^\otimes n) \otimes L^\otimes -n$ being a homomorphism of group functors. Here $\text{Pic}_X$ denotes the Picard functor, $S \mapsto \text{Pic}(X \times S)/p_2^* \text{Pic}(S)$. In view of [Ra70, Prop. IV.3.1], it follows that the assertion amounts to the (seemingly quite different) formulation of the theorem of the square in [loc. cit., IV.1.4].

As a special case of [loc. cit., Thm. IV.2.6], the assertion of Theorem 1.1 holds with $n = 1$, if $k$ is perfect and $X$ is geometrically normal and geometrically integral. This does not extend to connected schemes over a field of characteristic $p > 0$, as shown by the example of the cuspidal cubic curve $(y^2z - x^3 = 0) \subset \mathbb{P}_k^2$ on which $G = \mathbb{G}_m$ acts via $t \cdot (x, y, z) := (t^2x, t^3y, z)$: then the theorem of the square fails with $n = 1$ (see [Al02, 4.1.5]), but one can check that it holds with $n = p$.

As for earlier versions of the theorem of the square, we deduce Theorem 1.1 from a version of the theorem of the cube, which may have independent interest:

**Theorem 1.2.** Let $G_1, G_2$ be connected group schemes, $X$ a connected scheme (assumed seminormal if $\text{char}(k) = 0$), and $L$ a line bundle on $G_1 \times G_2 \times X$. If the pull-backs of $L$ to $\{e_{G_1}\} \times G_2 \times X$ and $G_1 \times \{e_{G_2}\} \times X$ are trivial, then there exists a positive integer $n$ and a line bundle $M$ on $G_1 \times G_2$ such that $L^\otimes n \cong p_{12}^*(M)$.

This readily implies Theorem 1.1, since (with the notation and assumptions of that theorem) the pull-backs of $\mathcal{L}$ to $\{e_{G_1}\} \times G \times X$ and $G \times \{e_{G_2}\} \times X$ are both trivial. The proof of Theorem 1.2 is presented in Section 2 after two technical lemmas, we proceed via a succession of reductions to the classical theorem of the cube (see [Mum70, Chap. III, §10]). Section 2 concludes with examples showing that the connectedness and seminormality assumptions cannot be omitted in Theorem 1.2 also, that theorem fails for $n = 1$. We do not know if that theorem holds with $n$ being a power (depending on $L$) of the characteristic exponent of $k$.

Next, we turn to the local structure of algebraic group actions. We will need the following:
**Definition 1.3.** Let $G$ be a connected algebraic group and $A$ an abelian variety, quotient of $G$ by a normal subgroup scheme $H$.

(i) We say that a vector bundle $\pi : E \to A$ is $G$-homogeneous if the scheme $E$ is equipped with an action of $G$ which commutes with the natural action of $\mathbb{G}_m$ and lifts the $G$-action on $A = G/H$ by translations. Equivalently, $E$ is the vector bundle $G \times^H V$ associated with some $H$-module $V$.

(ii) We say that a $G$-scheme $X$ is $G$-quasiprojective if it admits an equivariant embedding in the projectivization of some $G$-homogeneous vector bundle.

(iii) We say that a $G$-scheme $X$ is locally $G$-quasiprojective if it admits an étale covering $(f_i : U_i \to X)_{i \in I}$, where each $U_i$ is a $G$-quasiprojective scheme, and each $f_i$ is $G$-equivariant.

When $G$ is linear, the $G$-homogeneous vector bundles are just the $G$-modules, and hence the above notion of (local) $G$-quasiprojectivity gives back that of [Br14, Def. 4.6]; in this setting, $G$-quasiprojectivity is equivalent to the existence of a $G$-linearized ample line bundle. On the other hand, given a abelian variety $A$, the vector bundles on $A$ which are $G$-homogeneous for some extension $G$ of $A$ are exactly the translation-invariant bundles; over an algebraically closed field, they have been classified by Miyanishi and Mukai (see [Miy73, Thm. 2.3], [Muk78, Thm. 4.17]).

**Theorem 1.4.** Let $G$ be a connected group scheme and $X$ a quasiprojective $G$-scheme (assumed seminormal if $\text{char}(k) = 0$). Then $X$ is locally $G$-quasiprojective.

A key ingredient of the proof is the following:

**Theorem 1.5.** Let $G$ be a group scheme, $H \subset G$ a subgroup scheme such that $G/H$ is finite, and $X$ an $H$-scheme. If $X$ admits an $H$-linearized ample line bundle, then it also admits a $G$-linearized ample line bundle.

Theorem 1.5 is proved in Section 3. Note that given $G$, $H$, $X$ as in that theorem, and an $H$-linearized ample line bundle $L$ on $X$, it may well happen that no positive power of $L$ is $G$-linearizable (e.g., when $G$ is the group of order 2 acting on $X = \mathbb{P}^1 \times \mathbb{P}^1$ by exchanging both factors, $H$ is trivial, and $L$ has bidegree $(m, n)$ with $m \neq n$). So we construct the desired line bundle by a process of induction, analogous to that of representation theory.

The proof of Theorem 1.4 is presented in Section 4; it combines reduction steps as in the proof of Theorem 1.2 with arguments from [Br10].

Finally, we construct a projective seminormal surface $X$ equipped with an action of a semiabelian variety $G$ with a unique closed orbit, and such that no finite étale cover of $X$ is $G$-projective. As a consequence, Theorem 1.4 fails for a stronger notion of local $G$-quasiprojectivity, where the morphisms $f_i$ are assumed to be finite. In characteristic 0, the seminormality assumption in that theorem cannot be suppressed, as shown by the example of the cuspidal cubic curve $X$ equipped with its action of $G = \mathbb{G}_a$ (indeed, one checks that every étale $G$-equivariant morphism $f : U \to X$ with image containing the singular point is an isomorphism; also, $X$ is not $G$-projective in view of [Br14 Ex. 2.16]). We do not know if Theorem 1.4 extends to arbitrary $G$-schemes (seminormal in characteristic 0), without the quasiprojectivity assumption.
2 Proof of Theorem 1.2

We will need a preliminary result on Picard groups of torsors, based on results and methods from [Br14]. To formulate this result, we introduce some additional notation, which will be used throughout this article.

We fix an algebraic closure $\bar{k}$ of $k$. For any scheme $X$ and any field extension $k'/k$, we denote by $X_{k'}$ the scheme obtained from $X$ by base change to $k'$. Also, we denote by $\mathcal{O}(X)$ the $k$-algebra of global sections of the structure sheaf, and by $\mathcal{O}(X)^*$ the group of units of that algebra. If $X$ is a $G$-scheme for some group scheme $G$, we denote by $\mathcal{O}(X)^G \subset \mathcal{O}(X)$ the subalgebra of $G$-invariants, and by $\text{Pic}^G(X)$ the group of isomorphism classes of $G$-linearized line bundles on $X$. When $G$ is an algebraic group, its character group may be viewed as an étale sheaf over $\text{Spec}(k)$, that we denote by $\hat{G}$.

Next, consider a connected algebraic group $G$ and a $G$-torsor $f: X \to Y$ for the étale topology, where $X$ and $Y$ are varieties. If $G$ is linear (and in addition reductive when $k$ is imperfect) and if $Y$ is smooth and geometrically integral, then there is a natural exact sequence

$$0 \to \mathcal{O}(Y)^* \to \mathcal{O}(X)^* \to \hat{G}(k) \to \text{Pic}(Y) \to \text{Pic}(X) \to \text{Pic}(G)$$

and $\text{Pic}(G)$ is finite (see [Sa81, Sec. 6] for these results and further developments). In particular, the cokernel of the pull-back map $\text{Pic}(Y) \to \text{Pic}(X)$ is $n$-torsion for some $n = n(G)$.

The latter result does not extend to an arbitrary variety $Y$. For example, when $Y$ is affine, the pull-back map $\text{Pic}(Y) \to \text{Pic}(Y \times \mathbb{G}_m)$ is an isomorphism if and only if $Y$ is seminormal (see [Tr70, Thm. 3.6]). Under that assumption, we have an isomorphism $\text{Pic}(Y \times \mathbb{G}_m) \cong \text{Pic}(Y) \times H^1_{\text{ét}}(Y, \mathbb{Z})$, as follows from [We91, Thm. 7.5]; in particular, the pull-back map $\text{Pic}(Y) \to \text{Pic}(Y \times \mathbb{G}_m)$ is not an isomorphism when $Y$ is a nodal curve. Yet we have the following partial generalization:

**Lemma 2.1.** Let $G$ a connected algebraic group, and $f: X \to Y$ a $G$-torsor for the étale topology, where $X$ and $Y$ are varieties.

(i) There is an exact sequence of étale sheaves on $Y$

$$0 \to \mathbb{G}_m \to f_* (\mathbb{G}_m) \xrightarrow{\chi} \hat{G} \to 0,$$

where $\chi$ assigns to any étale morphism $V \to Y$ and to any $\varphi \in \mathcal{O}(V \times_Y X)^*$, the character $\chi \in \hat{G}(V \times_Y X)$ such that $\varphi(g \cdot z) = \chi(z) \varphi(z)$ identically.

(ii) There is an exact sequence

$$0 \to \mathcal{O}(Y)^* \xrightarrow{f^*} \mathcal{O}(X)^* \xrightarrow{\gamma} \hat{G}(X) \xrightarrow{\varphi} \text{Pic}(Y) \xrightarrow{f^*} H^1_{\text{ét}}(Y, f_* (\mathbb{G}_m)) \xrightarrow{\chi} H^1_{\text{ét}}(Y, \hat{G}),$$

where $\gamma$ is the characteristic homomorphism that assigns to any $\lambda \in \hat{G}(X)$, the class of the associated line bundle on $Y$.

(iii) The pull-back by $f$ yields isomorphisms

$$\mathcal{O}(Y)^* \cong \mathcal{O}(X)^G, \quad \hat{G}(Y) \cong \hat{G}(X), \quad \text{Pic}(Y) \cong \text{Pic}^G(X), \quad H^1_{\text{ét}}(Y, \hat{G}) \cong H^1_{\text{ét}}(X, \hat{G}).$$
(iv) There is an exact sequence

\[ 0 \to H^1_{\text{ét}}(Y, f_*(\mathbb{G}_m)) \to \text{Pic}(X) \to H^0_{\text{ét}}(Y, R^1 f_*(\mathbb{G}_m)) \]

and an isomorphism for any geometric point \( \overline{y} \) of \( Y \)

\[ R^1 f_*(\mathbb{G}_m)_{\overline{y}} \cong \text{Pic}(G \times \text{Spec}(\mathcal{O}_{Y, \overline{y}})). \]

(v) The exact sequences (3) and (5) are compatible with pull-backs in the following sense: For any homomorphism of connected algebraic groups \( \varphi : G \to G' \) and any commutative square of morphisms of varieties

\[
\begin{align*}
X & \xrightarrow{\psi} X' \\
\downarrow f & \quad \downarrow f' \\
Y & \xrightarrow{\eta} Y'
\end{align*}
\]

such that \( f' \) is a \( G' \)-torsor and \( \psi(g \cdot x) = \varphi(g) \cdot \psi(x) \) identically, the formation of (3) and (5) commutes with pull-backs by \( \varphi, \psi \) and \( \eta \).

(vi) If \( G \) is linear and \( X \) is geometrically seminormal, then \( H^0_{\text{ét}}(Y, R^1 f_*(\mathbb{G}_m)) \) is \( n \)-torsion for some positive integer \( n \) depending only on \( G \).

**Proof.** (i) and (ii) The short exact sequence (2) is a consequence of [Br14, Lem. 2.7], in view of the local triviality of \( f \) for the étale topology. The associated long exact sequence of cohomology yields (3), since \( H^0_{\text{ét}}(Y, \mathbb{G}_m) = \mathcal{O}(Y)^* \) and \( H^1_{\text{ét}}(Y, \mathbb{G}_m) = \text{Pic}(Y) \) (see [Mil80, Prop. III.4.6]).

(iii) The assertions on \( \widehat{G}(Y) = H^0_{\text{ét}}(Y, \widehat{G}) \) and \( H^1_{\text{ét}}(Y, \widehat{G}) \) follow from [Br14, Prop. 3.8]. The assertions on \( \text{Pic}(Y) \) and \( \mathcal{O}(Y)^* \) are obtained by descent: since \( f \) is \( G \)-invariant, the pull-back of any line bundle on \( Y \) is equipped with a \( G \)-linearization. Conversely, a \( G \)-linearization of a line bundle \( L \) on \( X \) is exactly a descent datum for \( L \), since \( f \) is a \( G \)-torsor. So the assertion on Picard groups follows from [SGA1, Exp. VIII, Cor. 1.3, Prop. 1.10]. Likewise, the pull-back by \( f \) induces an isomorphism \( \mathcal{O}(Y) \cong \mathcal{O}(X)^G \), which yields the assertion on units.

(iv) follows from the Leray spectral sequence for \( f \), as in the proof of [Br14, Lem. 3.1].

(v) is checked by using Čech cohomology, as in the proof of [Br14, Lem. 3.1] again.

(vi) The group \( \text{Pic}(G \times \text{Spec}(\mathcal{O}_{Y, \overline{y}})) \) is \( n \)-torsion for some \( n = n(G) \), in view of [Br14, Lem. 3.10]. Together with (iv), this yields the assertion. \( \square \)

**Remark 2.2.** For any \( G \)-variety \( X \), there is an exact sequence

\[ 0 \to \mathcal{O}(X)^* \to \mathcal{O}(X)^* \to \widehat{G}(X) \to \text{Pic}^G(X) \xrightarrow{\varphi} \text{Pic}(X) \xrightarrow{\psi} \text{Pic}(G \times X)/\mu_2^2 \text{Pic}(X), \]

where \( \varphi \) denotes the forgetful map, and \( \psi \) the obstruction map that assigns to \( L \) the image of \( \alpha^*(L) \) in \( \text{Pic}(G \times X)/\mu_2^2 \text{Pic}(X) \) (see [Br14, Prop. 2.10]). When \( X \) is the total space of a \( G \)-torsor, (6) gives back the first 4 terms of the exact sequence (3) in view of the isomorphisms (1). On the other hand, when \( G \) is linear and \( X \) is normal, (3) can also be related to (6) by using the fundamental construction of equivariant intersection theory (see [EG98, To99]): there exists an affine space \( V \) on which \( G \) acts linearly, and a
$G$-stable open subscheme $U \subset V$ having a quotient $U \to U/G$ which is a $G$-torsor, and such that $V \setminus U$ has codimension at least 2 in $V$. Then we have an associated torsor $X \times U \to (X \times U)/G =: X_G$ in view of [EGA III, Prop. 23], and $\text{Pic}(X_G) \cong \text{Pic}^G(X \times U)$. Moreover, the projection $p_1 : X \times U \to X$ induces isomorphisms $\mathcal{O}(X)^* \cong \mathcal{O}(X \times U)^*$ (since $\mathcal{O}(X \times U) \cong \mathcal{O}(X \times V)$ is a polynomial ring on $\mathcal{O}(X)$); $\hat{G}(X) \cong \hat{G}(X \times U)$ (since $U$ is geometrically connected); $\text{Pic}(X) \cong \text{Pic}(X \times U)$ (since $X$ is normal and $U$ is rational); and likewise $\text{Pic}(G \times X) \cong \text{Pic}(G \times X \times U)$. In view of [Br14] Prop. 2.10, it follows that $p_1^* \text{Pic}^G(X) \cong \text{Pic}^G(X \times U)$. This identifies the first 4 terms of (6) with those of the exact sequence (3) for the torsor $X \times U \to X_G$.

We will need another preliminary result, on iterated Frobenius morphisms. Assume that $k$ has characteristic $p > 0$. Given a scheme $X$ and a positive integer $r$, we denote by

$$F_X^r : X \longrightarrow X^{(r)}$$

the $r$th relative Frobenius morphism considered e.g. in [SGA3, Exp. VIIA] (when $k$ is perfect, this coincides with the $r$th Frobenius morphism of [Ja03, I.9.2, App. F]). Recall that the formation of $F^r$ commutes with products and base extensions. Also, for any group scheme $G$, the $r$th twist $G^{(r)}$ has a natural structure of group scheme such that $F_X^r : G \to G^{(r)}$ is a homomorphism (see [SGA3, Exp. VIIA, 4.1]). Moreover, the quotient group scheme $G/\text{Ker}(F_G^r)$ is smooth for $r \gg 0$, in view of [SGA3, Exp. VIIA, Prop. 8.3].

We may now state the following lemma, most of which follows readily from results and arguments in [SGA3, Exp. VIIA] and [Ja03, App. F]; we will give a proof for completeness.

**Lemma 2.3.** With the above notation and assumptions, consider a group scheme $G$, a $G$-scheme $X$, and a line bundle $L$ on $X$; denote by $F^r(X) \subset X^{(r)}$ the scheme-theoretic image of $F_X^r$.

(i) $F_X^r$ is finite and bijective, and $X^{(r)}$ is of finite type. Moreover, the formation of $F^r(X)$ commutes with products.

(ii) $F^r(X)$ is geometrically reduced for $r \gg 0$.

(iii) $F^r(L)$ is a line bundle on $F^r(X)$, trivial if so is $L$. Moreover, $(F_X^r)^*(F^r(L)) \cong L^\otimes p^r$.

(iv) $L$ is ample if and only if $F^r(L)$ is ample.

(v) $F^r(X)$ is equipped with an action of $G$ via its quotient $F^r(G) \cong G/\text{Ker}(F_G^r)$. If $L$ is $G$-linearized, then $F^r(L)$ is $G/\text{Ker}(F_G^r)$-linearized.

**Proof.** (i) By construction, we have $F_X^r = F_X^r \circ F_X^{r-1}$. Thus, we may assume that $r = 1$. Since $F_X$ is affine, we may also assume that $X$ is affine. Let $X = \text{Spec}(A)$; then the comorphism $F_X^\#$ is the map $k \otimes_F A \to A$, $t \otimes a \mapsto ta^p$, where $k \otimes_F A$ denotes the tensor product of $k$ and $A$ over $k$ acting on $k$ via $t \cdot u = t^p u$, and on $A$ via $t \cdot a = ta$. In particular, the image of $F_X^\#$ is the $k$-subalgebra of $A$ generated by all $p$th powers. Also, if $A = k[X_1, \ldots, X_m]/(f_1, \ldots, f_n)$, then $k \otimes_F A = k[X_1, \ldots, X_m]/(f_1^{(1)}, \ldots, f_n^{(1)})$, where $f_i^{(1)}$ is obtained from $f_i \in k[X_1, \ldots, X_m]$ by raising all coefficients to the $p$th power. This implies readily our assertions.

(ii) As above, it suffices to treat the case where $r = 1$ and $X = \text{Spec}(A)$; then $F(X) = \text{Spec} F_X^\#(k \otimes_F A)$. We may further assume that $k$ is perfect, as the formation of
$F_X$ commutes with field extensions; then $F_X^{\#}(k \otimes_F A)$ is the $k$-subalgebra of $A$ consisting of all $p^r$th powers. Since the $k$-algebra $A$ is finitely generated, there exists a positive integer $n$ such that $a^n = 0$ for any nilpotent $a \in A$. It follows that $F(X)$ is reduced for any $r$ such that $p^r \geq n$.

(iii) We may reduce again to the case where $r = 1$. If $L$ is the trivial line bundle $X \times A^1 \to X$, then so is $L^{(1)}$, since $F$ commutes with products and takes $A^1$ to $A^1$.

If $L$ is an arbitrary line bundle on $X$, then we may choose local trivializations $(U_i, \eta_i)_{i \in I}$, where the $U_i$ form a covering of $X$ by open affine subschemes, and

$$\eta_i : L_{U_i} \xrightarrow{\sim} U_i \times A^1.$$  

Then $L$ is defined by the cocycle $(\omega_{ij} := (\eta_i \eta_j^{-1})_{i,j})$, where $\omega_{ij} \in O(U_i \cap U_j)^\ast$. We may view $X^{(1)}$ as the ringed space $(X, (F_X)_{\ast}(O_X))$, where $(F_X)_{\ast}(O_X)(U_i) = k \otimes_F O(U_i)$. Then one checks that $L^{(1)}$ is the line bundle on $X^{(1)}$ defined by the cocycle $(1 \otimes \omega_{ij})_{i,j}$; thus, $F_X^{\ast}(L^{(1)}) \cong L^{\otimes p}$. As a consequence, $F(L)$ is a line bundle, and $F_X^{\ast}(F(L)) \cong L^{\otimes p}$.

(iv) follows from (iii) by using the following fact: for any finite surjective morphism of schemes $f : X \to Y$ and any line bundle $M$ on $Y$, the ampleness of $M$ is equivalent to that of $f^\ast(M)$. This fact is proved in [Ha70 Prop. I.4.4] when $k$ is algebraically closed; the general case follows, since $M$ is ample if and only if so is $M_k$ (see [SGA1, Exp. VIII, Cor. 5.8]).

(v) The morphism $F^{\ast}_G : G \to G^{(r)}$ factors through an isomorphism of group schemes $G/\ker(F^{\ast}_G) \cong F^{(r)}(G)$ by [SGA3, Exp. VIA, 5.4]. The remaining assertions follow from the fact that the formation of $F^{\ast}(X)$ commutes with products.

\begin{remark}
One can show that $F^{(r)}(X) = X^{(r)}$ for some $r \geq 1$ if and only if $X$ is geometrically reduced. In particular, a group scheme $G$ is smooth if and only if the natural map $G/\ker(F^{\ast}_G) \to G^{(r)}$ is an isomorphism for some $r \geq 1$ (see also [SGA3, Exp. VIIA, Cor. 8.3.1]).
\end{remark}

\begin{proof}[End of the proof of Theorem 1.2]
When $G_1$, $G_2$ are abelian varieties and $k$ is algebraically closed, the assertion follows from the classical theorem of the cube (see [Mum70 Chap. III, §10]). We will reduce to this setting in several steps.

We first show that we may assume $X$ \textit{geometrically connected}. Consider the canonical morphism $f : X \to \pi_0(X)$, where $\pi_0(X)$ denotes the (finite étale) scheme of connected components. Since $X$ is connected and of finite type, there exists a finite separable extension $k'/k$ of fields such that $\pi_0(X) = \text{Spec}(k')$. Denote by $K$ the Galois closure of $k'$ in $\bar{k}$, and by $\Gamma$ the Galois group of $K/k$. Then $\pi_0(X_K) = \pi_0(X)_K = \text{Spec}(k' \otimes_k K) = \text{Spec}(K([k':k]))$. It follows that the connected components of $X_K$ are geometrically connected and permuted transitively by $\Gamma$. Choose such a component $X'$ and denote by $\Gamma' \subset \Gamma$ its stabilizer. Since the natural map $X_K \to X$ is a finite Galois cover with group $\Gamma$, its restriction to $X'$ yields a finite Galois cover $\pi : X' \to X$ with group $\Gamma'$. Moreover, the assertions of the theorem hold for the field $K$ and $(G_1)_K$, $(G_2)_K$, $X'$, and $L' := (\text{id}_{G_1 \times G_2} \times \pi)^\ast(L)$. If $L'^{\otimes n} \cong p_{12}^{\ast}(M')$ for some line bundle $M'$ on $(G_1 \times G_2)_K$, then the norm $N(L')$ is a line bundle on $G \times G \times X$, which satisfies

$$L'^{\otimes n} \otimes \Gamma' \cong N(L'^{\otimes n}) \cong p_{12}^{\ast}(N(M'))$$

\end{proof}
in view of [EGA II.6.5.2.4, II.6.5.4]. This yields our first reduction.

Next, we show that we may further assume \( k \) algebraically closed. Note that
the assumptions of Theorem \( \text{[1.2]} \) (with connexity replaced by geometric connexity) still hold after base change to \( \overline{k} \); moreover, if \( L_k^{\otimes n} \cong p_{12}^*(M) \) for some line bundle \( M \) on \( (G_1 \times G_2)_k \), then there exists a finite extension \( k' \) of \( k \) and a line bundle \( M' \) on \( (G_1 \times G_2)_{k'} \) such that \( M \cong M'_{k'} \) and \( L_{k'}^{\otimes n} \cong p_{12}^*(M') \). Thus, we obtain as above

\[
L^{\otimes [k':k]} \cong N(L_{k'}^{\otimes n}) \cong p_{12}^*(N(M')).
\]

We now show that we may further assume \( G_1, G_2 \) smooth and \( X \) reduced, under the assumption that \( \text{char}(k) > 0 \). By Lemma \( \text{[2.3]} \) the assumptions of Theorem \( \text{[1.2]} \) still hold when \( G_1, G_2, X, L \) are replaced with \( F^r(G_1), F^r(G_2), F^r(X), F^r(L) \) for any positive integer \( r \); also, we may choose \( r \) so that \( F^r(G_1), F^r(G_2) \) are smooth and \( F^r(X) \) is reduced. If \( F^r(L)^{\otimes n} \cong p_{12}^*(M) \) for some integer \( n \geq 1 \) and some line bundle \( M \) on \( F^r(G_1) \times F^r(G_2) \), then we obtain \( L^{\otimes np^r} \cong p_{12}^*(F_{G_1 \times G_2}^{r})^*(M) \) by using Lemma \( \text{[2.3]} \) again.

Next, we show that we may also assume \( X \) seminormal, again under the assumption that \( \text{char}(k) > 0 \). Consider indeed the seminormalization \( \sigma : X^+ \to X \). Since \( G_1 \) and \( G_2 \) are smooth, the morphism

\[
\text{id}_{G_1 \times G_2} \times \sigma : G_1 \times G_2 \times X^+ \to G_1 \times G_2 \times X
\]

is the seminormalization as well, by [GT80] Prop. 5.1]. Moreover, \( X^+ \) is connected (since \( \sigma \) is a universal homeomorphism), and \( (\text{id}_{G_1 \times G_2} \times \sigma)^*(L) \) pulls back to the trivial bundles on \( \{e_{G_1}\} \times G_2 \times X^+ \) and \( G_1 \times \{e_{G_2}\} \times X^+ \). If there exists a line bundle \( M \) on \( G_1 \times G_2 \) and a positive integer \( n \) such that \( (\text{id}_{G_1 \times G_2} \times \sigma)^*(L^{\otimes n}) \cong p_{12}^*(M) \), then \( L^{\otimes np^r} \cong p_{12}^*(M^{\otimes np^r}) \) for some \( m \geq 1 \), since the map

\[
(\text{id}_{G_1 \times G_2} \times \sigma)^* : \text{Pic}(G_1 \times G_2 \times X) \to \text{Pic}(G_1 \times G_2 \times X^+)
\]

is an isomorphism up to \( p^m \)-torsion for \( m \gg 0 \) (see [Br14] Lem. 4.11).

So we assume that \( k \) is algebraically closed (of arbitrary characteristic), \( G_1, G_2 \) are smooth, and \( X \) is connected and seminormal. By Chevalley’s structure theorem (see [Co02] and [Mil13] for modern proofs), each \( G_i \) sits in a unique extension

\[
1 \to H_i \to G_i \xrightarrow{\pi_i} A_i \to 1,
\]

where \( H_i \) is a connected linear algebraic group, and \( A_i \) an abelian variety. The morphism

\[
f := \pi_1 \times \pi_2 \times \text{id}_X : G_1 \times G_2 \times X \to A_1 \times A_2 \times X
\]

is a torsor under the connected linear algebraic group \( H_1 \times H_2 \); also, note that \( G_1 \times G_2 \times X \) is seminormal. In view of Lemma \( \text{[2.1]} \) (iv), (vi), it follows that there exists \( n = n(G) \) such that the class \( L^{\otimes n} \in \text{Pic}(G_1 \times G_2 \times X) \) sits in the subgroup \( H_1^1(A_1 \times A_2 \times X, f_*(\mathbb{G}_m)) \).

We may thus replace \( L \) with \( L^{\otimes n} \), and assume that \( L \in H_1^1(A_1 \times A_2 \times X, f_*(\mathbb{G}_m)) \).

Next, let \( f_2 := \pi_2 \times \text{id}_X : G_2 \times X \to A_2 \times X \), so that \( f_2 \) is an \( H_2 \)-torsor and sits in a commutative square

\[
\begin{array}{ccc}
G_2 \times X & \xrightarrow{\varepsilon_{G_1 \times G_2} \times \text{id}_X} & G_1 \times G_2 \times X \\
\downarrow f_2 & & \downarrow f \\
A_2 \times X & \xrightarrow{\varepsilon_{A_1 \times A_2} \times \text{id}_X} & A_1 \times A_2 \times X.
\end{array}
\]
This square and the homomorphism $e_{H_1} \times \text{id}_{H_2} : H_2 \to H_1 \times H_2$ satisfy the assumptions of Lemma 2.1 (v). By that lemma, we obtain two commutative squares

$$H^1_{\text{et}}(A_1 \times A_2 \times X, f_* (\mathbb{G}_m)) \xrightarrow{\chi_{H_2}} H^1_{\text{et}}(A_1 \times A_2 \times X, H_1 \times H_2)$$

and

$$H^1_{\text{et}}(A_2 \times X, (f_2)_* (\mathbb{G}_m)) \xrightarrow{\chi_{H_2}} H^1_{\text{et}}(A_2 \times X, \hat{H}_2)$$

where the horizontal arrows of the latter diagram are injective. Since $L$ pulls back to the trivial bundle on $\{e_{G_1}\} \times G_2 \times X$, it follows that $(e_{A_1} \times \text{id}_{A_2 \times X})^*(L) = 0$. Thus, the image of $\chi_{H_1 \times H_2}(L)$ in $H^1_{\text{et}}(A_2 \times X, \hat{H}_2)$ is 0 as well.

We claim that $\chi_{H_1 \times H_2}(L) = 0$. Indeed, by [Br14, Prop. 3.8], the projection $p_3 : A_1 \times A_2 \times X \to X$ induces an isomorphism

$$p^*_3 : H^1_{\text{et}}(X, \hat{H}_2) \to H^1_{\text{et}}(A_1 \times A_2 \times X, H_1 \times H_2).$$

Since $e_{A_1 \times A_2} \times \text{id}_X$ is a section of $p_3$, we see that $(e_{A_1 \times A_2} \times \text{id}_X)^*$ is the inverse of $p^*_3$. Likewise, we have an isomorphism

$$p^*_2 : H^1_{\text{et}}(X, \hat{H}_2) \to H^1_{\text{et}}(A_2 \times X, \hat{H}_2)$$

with inverse $(e_{A_2} \times \text{id}_X)^*$. Moreover, these isomorphisms sit in a commutative square

$$H^1_{\text{et}}(X, H_1 \times H_2) \xrightarrow{p^*_3} H^1_{\text{et}}(A_1 \times A_2 \times X, H_1 \times H_2)$$

and

$$H^1_{\text{et}}(X, \hat{H}_2) \xrightarrow{p^*_2} H^1_{\text{et}}(A_2 \times X, H_1 \times H_2).$$

Thus, $\chi_{H_1 \times H_2}(L) = p^*_3(\lambda)$ for a unique $\lambda \in H^1_{\text{et}}(X, H_1 \times H_2)$ such that $(e_{H_1} \times \text{id}_{H_2})^*(\lambda) = 0$. Exchanging $H_1$ and $H_2$, we also obtain $(\text{id}_{H_1} \times e_{H_2})^*(\lambda) = 0$. Also, in view of the natural isomorphism $H_1 \times \hat{H}_2 \cong \hat{H}_1 \times \hat{H}_2$, the product map

$$(e_{H_1} \times \text{id}_{H_2})^* \times (\text{id}_{H_1} \times e_{H_2})^* : H^1_{\text{et}}(X, H_1 \times H_2) \to H^1_{\text{et}}(X, \hat{H}_1) \times H^1_{\text{et}}(X, \hat{H}_2)$$

is an isomorphism. This completes the proof of the claim.

By that claim and Lemma 2.1 (ii), there exists a line bundle $M$ on $A_1 \times A_2 \times X$ such that $L \cong f^*(M)$. Since $(e_{G_1} \times \text{id}_{G_2 \times X})^*(L)$, we obtain $f_2^*(e_{A_1} \times \text{id}_{A_2 \times X})^*(M) = 0$ in $\text{Pic}(A_2 \times X)$. In view of Lemma 2.1 (ii) (again), there exists $\chi_2 \in \hat{H}_2(A_2 \times X)$ such that $(e_{A_1} \times \text{id}_{A_2 \times X})^*(M) = \gamma_{H_2}(\chi_2)$, where $\gamma_{H_2} : \hat{H}_2(A_2 \times X) \to \text{Pic}(A_2 \times X)$ denotes the characteristic homomorphism. Likewise, there exists $\chi_1 \in \hat{H}_1(A_1 \times X)$ such that $(\text{id}_{A_1} \times e_{A_2} \times \text{id}_X)^*(M) = \gamma_{H_1}(\chi_1)$. Replacing $L$ with $L \otimes \gamma_{H_1}(\chi_1)^{-1} \otimes \gamma_{H_2}(\chi_2)^{-1} = \gamma_{H_1 \times H_2}(\chi_1 \times \chi_2)$, we obtain $f_2^*(e_{A_1} \times \text{id}_{A_2 \times X})^*(M) = 0$ in $\text{Pic}(A_2 \times X)$.
$L \otimes \gamma_{H_1 \times H_2}(\chi_1 \times \chi_2)^{-1}$ (which leaves $\chi_{H_1 \times H_2}(L)$ unchanged), we may thus assume that $(e_{A_1} \times \text{id}_{A_2})^*(M) = 0 = (\text{id}_{A_1} \times e_{A_2} \times \text{id}_X)^*(M)$.

Choose $x \in X(k)$. Then $N := M \otimes p_{12}^*(\text{id}_{A_1} \times A_2 \times x)^*(M^{-1})$ pulls back to the trivial line bundle on $\{e_{A_1}\} \times A_2 \times X, A_1 \times \{e_{A_2}\} \times X$ and $A_1 \times A_2 \times \{x\}$. Thus, $N$ is trivial by the classical theorem of the cube; this yields the desired assertion.

**Remark 2.5.** The assumption that $X$ is connected cannot be omitted in Theorem [1.2]
Consider indeed an elliptic curve $E$ and a scheme $X$, disjoint union of two nonempty closed subschemes $X_1, X_2$. Let $L$ be the line bundle on $E \times E \times X$ such that $L$ is trivial on $E \times E \times X_1$ and $L = p_{12}^*(M)$ on $E \times E \times X_2$, where $M$ is the line bundle on $E \times E$ associated with the divisor $\text{diag}(E) - (E \times \{0\}) - (\{0\} \times E)$. Then $L$ does not satisfy the assertion of Theorem [1.2] since $M$ has infinite order in $\text{Pic}(E \times E)$.

Also, the seminormality assumption in characteristic $0$ cannot be omitted. Indeed, let $X$ be a scheme such that there exists a nontrivial line bundle $M$ on $X \times \mathbb{A}^1$ which pulls back to the trivial bundle on $X \times \{0\}$ (recall from [H70] Thm. 3.6] that such a line bundle exists whenever $X$ is affine and not seminormal). Consider the multiplication map $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ and the line bundle $L := (\mu \times \text{id}_X)^*(M)$ on $\mathbb{A}^1 \times \mathbb{A}^1 \times X$. Then the pull-backs of $L$ to $\{0\} \times \mathbb{A}^1 \times X$ and $\mathbb{A}^1 \times \{0\} \times X$ are both trivial. But $L$ is nontrivial, since its pull-back to $\{1\} \times \mathbb{A}^1 \times X$ is isomorphic to $M$. In characteristic $p > 0$, any such line bundle $M$ is $p^m$-torsion for some positive integer $m$ depending on $M$, as follows from [Sw80 Cor. 8.2]. So $L \otimes p^m$ is trivial, in agreement with the assertion of Theorem [1.2]. But in characteristic $0$, $M$ (and hence $L$) may well be non-torsion; this happens for example when $X$ is a cuspidal curve (see [Br14 Ex. 2.16] for details).

### 3 Proof of Theorem [1.5]

We will need the following preliminary result, probably known but for which we could not locate any reference:

**Lemma 3.1.** Let $G$ be a group scheme, and $X$ a $G$-scheme. Then $X$ admits a $G$-linearized ample line bundle if and only if $X_k$ admits a $G_k$-linearized ample line bundle.

**Proof.** Assume that $X$ has an ample $G$-linearized line bundle $L$. Then $L_k$ is a line bundle on $X_k$, which is easily seen to be $G_k$-linearized; also, $L_k$ is ample by [SGA1] Exp. VIII, Cor. 5.8]. Thus, $X_k$ is $G_k$-quasiprojective.

For the converse, assume that $X_k$ has an ample $G_k$-linearized line bundle $M$. Then there exists a finite extension of fields $k'/k$ and a line bundle $L'$ on $X_{k'}$ such that $M \cong L'_{k'}$; moreover, $L'$ is ample in view of [SGA1] Exp. VIII, Cor. 5.8] again. We may further assume (possibly by enlarging $k'$) that $L'$ is $G_{k'}$-linearized. Consider the norm $L := N(L')$; then $L$ is an ample line bundle on $X$ (see [EGA II.6.6.2]), unique up to isomorphism. We now show that $L$ is equipped with a $G$-linearization. For this, we obtain an interpretation of the norm in terms of Weil restriction; see [CGP10 App. A.5] for the latter notion and its main properties.

Since $X_{k'}$ is a quasiprojective scheme, the Weil restriction $X' := R_{k'/k}(X_{k'})$ is a quasiprojective scheme as well (see [CGP10 Prop. A.5.8]). Also, $G' := R_{k'/k}(G_{k'})$ is a group scheme in view of [CGP10 Prop. A.5.1]. We have closed immersions $j_X : X \to X'$.
Lemma 3.2. Let \( G \) be a homomorphism of group schemes and \( X' \) is equipped with an action of \( G' \) such that \( j_X(g \cdot x) = j_G(g) \cdot j_X(x) \) identically (see [CGP10, Prop. A.5.2, Prop. A.5.7]).

The bundle map \( \pi : L' \to X_{k'} \) yields a morphism \( \pi' : R_{k'/(L')} \to R_{k'(kX')} = X' \). We claim that via this morphism, \( E := R_{k'/k}(L') \) is a \( G' \)-linearized vector bundle on \( X' \) of rank \( n = [k' : k] \); moreover, \( j_X'(\det(E)) \cong L \).

To show this claim, consider the \( \mathbb{G}_m \)-torsor \( \pi^\times : L'^\times \to X_{k'} \) associated to the line bundle \( L' \). Recall that \( L' = (L'^\times \times \mathbb{A}^1_{k'})/\mathbb{G}_m, k' \), where \( \mathbb{G}_m, k' \) acts simultaneously on \( L'^\times \) and on \( \mathbb{A}^1_{k'} \) by multiplication. Using [CGP10, Prop. A.5.2, Prop. A.5.4], it follows that

\[
E = (R_{k'/k}(L'^\times) \times R_{k'/k}(\mathbb{A}^1_{k'}))/R_{k'/k}(\mathbb{G}_m, k').
\]

This is the fiber bundle associated with the \( R_{k'/k}(\mathbb{G}_m, k') \)-torsor \( R_{k'/k}(L'^\times) \to X' \) and the \( R_{k'/k}((\mathbb{G}_m, k'))\)-scheme \( R_{k'/k}(\mathbb{A}^1_{k'}) \). Moreover, \( R_{k'/k}(\mathbb{A}^1_{k'}) \) is the affine space associated with the \( k' \)-vector space \( k' \) on which \( R_{k'/k}(\mathbb{G}_m, k') \) acts linearly (since for any \( k' \)-algebra \( A \), we have \( R_{k'/k}(\mathbb{A}^1_{k'})(A) = A \otimes_k k' = A_{k'} \) on which \( R_{k'/k}(\mathbb{G}_m, k')(A) = A_{k'}^* \) acts by multiplication). Thus, \( E \) is a vector bundle of rank \( n \) on \( X' \); it is equipped with a \( G' \)-linearization, since Weil restriction commutes with fiber products (see [CGP10, Prop. A.5.2]).

The determinant of \( E \) is the line bundle associated to the above \( R_{k'/k}(\mathbb{G}_m, k') \)-torsor and to the \( R_{k'/k(\mathbb{G}_m, k')} \)-module \( \mathbb{A}^1_{k'}(k') \). To describe the pull-back of this line bundle under \( j_X : X \to X' \), choose a Zariski open covering \( (U_i)_{i \in I} \) of \( X' \) such that the \( (U_i)_{k'} \) cover \( X_{k'} \) and the pull-back of \( L' \) to each \( (U_i)_{k'} \) is trivial (such a covering exists by [EGA, IV.21.8]). Also, choose trivializations

\[
\eta_i : L'_{(U_i)_{k'}} \xrightarrow{\sim} (U_i)_{k'} \times_{k'} \mathbb{A}^1_{k'}.
\]

This yields trivializations

\[
R_{k'/k}(\eta_i) : E_{U_i'} \xrightarrow{\sim} U_i' \times_k k',
\]

where \( U_i' := R_{k'/k}((U_i)_{k'}) \), and \( k' \) is again viewed as a \( k \)-vector space. Note that the \( U_i' \) do not necessarily cover \( X' \), but the \( j_X^{-1}(U_i') = U_i \) cover \( X \). Thus, \( j_X(E) \) is equipped with trivializations

\[
R_{k'/k}(\eta_i) : j_X^*(E)_{U_i} \xrightarrow{\sim} U_i \times_k k'.
\]

Consider the 1-cocycle \( (\omega_{ij} := (\eta_i)^{-1}(U_i \cap U_j) \cdot \eta_j)_{i,j} \) with values in \( \mathbb{G}_m, k' \). Then the line bundle \( j_X^*(\det(E)) = \det(j_X^*(E)) \) is defined by the 1-cocycle \( (\det(\omega_{ij})) \) with values in \( \mathbb{G}_m, k' \), where \( \det(\omega_{ij}) \) denotes the determinant of the multiplication by \( \omega_{ij} \) in \( k' \). It follows that \( j_X^*(\det(E)) \cong N(L') \) in view of the definition of the norm (see [EGA] II.6.4, II.6.5).

This completes the proof of the claim, which in turn yields the desired \( G \)-linearization of \( L \).

Next, we prove Theorem [3] under the assumption that both \( G \) and \( H \) are smooth:

**Lemma 3.2.** Let \( G \) be an algebraic group, \( H \subset G \) a closed subgroup such that \( G/H \) is finite, and \( X \) a \( G \)-scheme. If \( X \) admits an ample \( H \)-linearized line bundle, then it also admits an ample \( G \)-linearized line bundle.
Proof. By Lemma 3.1 we may assume that $k$ is algebraically closed. Also, replacing $H$ by the reduced kernel of its action on $G/H$ by left multiplication, we may further assume that $H$ is normal in $G$. We now adapt the argument of Lemma 3.1 with Weil restriction replaced by an induction process defined as follows.

For any $H$-scheme $Y$, consider the functor of morphisms $\operatorname{Hom}(G, Y)$, and the subfunctor $\operatorname{Hom}^H(G, Y)$ consisting of those morphisms $f$ such that $f(gh^{-1}) = h \cdot f(g)$ identically. We claim that $\operatorname{Hom}^H(G, Y)$ is represented by the scheme $Y^n$, where $n$ denotes the order of $G/H$. Indeed, choose $g_1, \ldots, g_n \in G(k)$ such that $G$ is the disjoint union of the cosets $g_iH$ for $i = 1, \ldots, n$. Then $\operatorname{Hom}^H(G, Y) \cong \prod_{i=1}^n \operatorname{Hom}^H(g_iH, Y)$; moreover, each $\operatorname{Hom}^H(g_iH, Y)$ is represented by $Y$, via $f \mapsto f(g_i)$. This proves our claim; for simplicity, we will still denote by $\operatorname{Hom}^H(G, Y)$ the representing scheme.

Note that $G$ acts on $\operatorname{Hom}^H(G, Y)$ via $(g \cdot f)(g') := f(g^{-1}g')$; moreover, every equivariant morphism of $H$-schemes $\varphi : Y \to Z$ yields an equivariant morphism of $G$-schemes $\operatorname{Hom}^H(G, \varphi) : \operatorname{Hom}^H(G, Y) \to \operatorname{Hom}^H(G, Z)$ by composition. For any functorial points $h$ of $H$ and $f$ of $\operatorname{Hom}^H(G, Y)$, we have

$$(h \cdot f)(g_i) = f(h^{-1}g_i) = g_i^{-1}hg_i \cdot f(g_i).$$

Thus, viewed as an $H$-scheme, $\operatorname{Hom}^H(G, Y)$ is isomorphic to $\prod_{i=1}^n Y^{g_i}$, where $Y^{g_i}$ denotes the scheme $Y$ with $H$-action twisted by the conjugation $\operatorname{Int}(g_i)$.

Also, when $Y$ is a $G$-scheme, the assignment

$$\operatorname{Hom}^H(G, Y) \to \operatorname{Hom}(G/H, Y), \quad f \mapsto (g \mapsto g \cdot f(g))$$

is easily seen to be an isomorphism, equivariant for the action of $G$ on $\operatorname{Hom}(G/H, Y)$ given by $(g \cdot f)(\gamma) := g \cdot f(g^{-1}\gamma)$. Via this isomorphism, the subscheme of $\operatorname{Hom}(G/H, Y)$ consisting of constant morphisms is identified with the image of $Y$ under the closed $G$-equivariant immersion

$$\iota : Y \to \operatorname{Hom}^H(G, Y), \quad y \mapsto (g \mapsto g^{-1}y).$$

Next, let $L$ be an ample $H$-linearized line bundle on $X$. Then the bundle map $\pi : L \to X$ yields a $G$-equivariant morphism $\operatorname{Hom}^H(G, \pi) : \operatorname{Hom}^H(G, L) \to \operatorname{Hom}^H(G, X)$, which may be identified with the product of the natural maps $\pi^{g_i} : L^{g_i} \to X^{g_i}$ for $i = 1, \ldots, n$. Thus, $E := \operatorname{Hom}^H(G, L)$ is a vector bundle of rank $n$ on $\operatorname{Hom}^H(G, X)$. The corresponding action of $\mathbb{G}_m$ on $E$ (by multiplication on fibers) satisfies $(t \cdot f)(g) = t \cdot f(g)$ identically, where $\mathbb{G}_m$ acts on the right-hand side by multiplication on the fibers of $L$. It follows that the actions of $\mathbb{G}_m$ and $G$ on $E$ commute, and hence $E$ is $G$-linearized. Moreover, we have an isomorphism of $H$-linearized vector bundles on $X$

$$\iota^*(E) \cong \bigoplus_{i=1}^n L^{g_i}.$$ 

Taking the determinant, we see that the line bundle $\bigotimes_{i=1}^n L^{g_i}$ on $X$ is equipped with a $G$-linearization. Moreover, this line bundle is ample, since so is each $L^{g_i}$. \qed
Remark 3.3. Given a group scheme $G$, a subgroup scheme $H \subset G$ and an $H$-scheme $Y$, the functor $\text{Hom}^H(G,Y)$ is represented by a scheme under less restrictive assumptions than those of Lemma 3.2. For example, this holds when $G/H$ is projective and $Y$ is $H$-quasiprojective.

Indeed, under these assumptions, the associated fiber bundle $f : G \times^H Y \to G/H$ exists and $G \times^H Y$ is quasiprojective, by [MPK94, Prop. 7.1] applied to the projection $G \times Y \to Y$ and to the $H$-torsor $G \to G/H$. Also, $\text{Hom}^H(G,Y)$ is a closed subfunctor of $\text{Hom}^H(G,G \times^H Y)$, since $Y$ is the fiber of $f$ at the base point of $G/H$. Finally, we have an isomorphism of functors $\text{Hom}^H(G,G \times^H Y) \cong \text{Hom}(G/H,G \times^H Y)$ as seen in the proof of Lemma 3.2 moreover, $\text{Hom}(G/H,G \times^H Y)$ is represented by a disjoint union of schemes of finite type, in view of [Gr60, p. 269].

Also, if $Y$ is the affine space associated with an $H$-module $M$, then $\text{Hom}^H(G,Y)$ is represented by the affine space (possibly of infinite dimension) associated with the induced module $\text{ind}^H_M := (O(G) \otimes M)^H$ (see e.g. [Ja03, I.3.3]). For an arbitrary $H$-scheme $Y$ and a $G$-scheme $X$, we have a natural isomorphism

$$\text{Hom}^H(X,Y) \cong \text{Hom}^G(X,\text{Hom}^H(G,Y)),$$

the analogue of Frobenius reciprocity in representation theory.

**End of the proof of Theorem 1.5.** If $\text{char}(k) = 0$, then every group scheme is smooth, and hence the assertion follows from Lemma 3.2. Thus, we may assume that $\text{char}(k) = p > 0$. By the assumption and Lemma 2.3, $F^r(X)$ admits an $F^r(H)$-linearized ample line bundle for any positive integer $r$. It suffices to show that $F^r(X)$ admits a $F^r(G)$-linearized ample line bundle for some $r$; indeed, such a line bundle $M$ is $G$-linearized via the homomorphism $F^r_\chi : G \to F^r(G)$. Thus, $(F^r_\chi)^*(M)$ is a $G$-linearized line bundle on $X$, which is ample since $F^r_\chi$ is finite.

So we may replace $G$, $H$, $X$ with $F^r(G)$, $F^r(H)$, $F^r(X)$ for $r \gg 0$. In view of Lemma 2.3 we may thus assume that $G$ and $H$ are smooth. Then the assertion follows from Lemma 3.2 again.

## 4 Proof of Theorem 1.4

We first show that we may assume $G$ smooth and $X$ geometrically reduced. In characteristic 0, the smoothness of $G$ is automatic, and the (geometric) reducedness of $X$ follows from the seminormality assumption. In characteristic $p > 0$, the $r$th twist $F^r(G)$ is smooth and $F^r(X)$ is geometrically reduced for $r \gg 0$ (Lemma 2.3). Assume that there exists an étale covering $(U_i \to F^r(X))_{i \in I}$, where each $U_i$ is a $F^r(G)$-quasiprojective scheme, and each $f_i$ is $F^r(G)$-equivariant. Then the $V_i := X \times_{F^r(X)} U_i$ form an étale covering of $X$ by $G$-schemes, and the morphisms $V_i \to X$ are $G$-equivariant. It remains to show that each $V_i$ is $G$-quasiprojective. By assumption, $U_i$ is equivariantly isomorphic to the associated fiber bundle $F^r(G) \times^H_i Y_i$, where $H_i \subset F^r(G)$ is a normal subgroup scheme such that $F^r(G)/H_i$ is an abelian variety, and $Y_i \subset U_i$ is a closed $H_i$-stable subscheme admitting an $H_i$-linearized ample line bundle. Denote by $K_i \subset G$ the pull-back of $H_i$ under $F^r_\chi$; then $V_i \cong G \times^{K_i} Z_i$ for some closed $K_i$-stable subscheme $Z_i \subset V_i$, finite over $Y_i$. As a consequence, $Z_i$ admits a $K_i$-linearized ample line bundle; so $Z_i$ is $K_i$-equivariantly isomorphic.
to a subscheme of the projectivization $\mathbb{P}(V_i)$, where $V_i$ is a $K_i$-module. Thus, $V_i$ admits a $G$-equivariant embedding into $G \times K_i \mathbb{P}(V_i)$, and the latter is the projectivization of a $G$-homogeneous vector bundle (since $G/K_i \cong F^r(G)/H_i$).

Next, we obtain a criterion for $G$-quasiprojectivity. To formulate it, recall a generalization of Chevalley’s structure theorem due to Raynaud (see [Ra70 Lem. IX 2.7] and also [BLR90] 9.2 Thm. 1)): any connected group scheme $G$ sits in an extension

$$1 \rightarrow H \xrightarrow{\iota} G \xrightarrow{\pi} A \rightarrow 1,$$

where $H$ is a connected affine group scheme, and $A$ an abelian variety. Moreover, there is a smallest subgroup scheme $H \subset G$ satisfying the above properties. If $G$ is smooth and $k$ is perfect, then $H$ is smooth as well.

**Lemma 4.1.** With the above notation, the following are equivalent for a geometrically reduced scheme $X$ equipped with a faithful action of a connected algebraic group $G$:

(i) $X$ is $G$-quasiprojective.

(ii) $X$ admits an ample $H$-linearized line bundle.

**Proof.** (i)$\Rightarrow$(ii) By Lemma 3.1, we may assume that $k$ is algebraically closed; then $H$ is a connected linear algebraic group. Then the assertion follows from a theorem of Sumihiro (see [Su75 Thm. 1.6]), since the projectivization of any $G$-homogeneous vector bundle is a smooth quasiprojective $H$-variety.

(ii)$\Rightarrow$(i) We adapt arguments from the proof of [Br10 Thm. 2]. Let $L$ be an ample $H$-linearized line bundle on $X$. Consider the action $\alpha : G \times X \rightarrow X$ and the projection $p_2 : G \times X \rightarrow X$. Then

$$M := \alpha^*(L) \otimes p_2^*(L)^{-1}$$

is a line bundle on $G \times X$, equipped with an $H$-linearization for the $H$-action on $G \times X$ via left multiplication on $G$ (since $\alpha$ is $H$-equivariant and $p_2$ is $H$-invariant). Since $\pi \times id_X : G \times X \rightarrow A \times X$ is an $H$-torsor for the fppf topology, there exists a line bundle $N$ on $A \times X$, unique up to isomorphism, such that

$$M = (\pi \times id_X)^*(N)$$

(as follows from a standard descent argument, already used in the proof of Lemma 2.1 (iii)). The $H$-linearization of $L$ yields an isomorphism of the pull-backs of $\alpha^*(L)$ and $p_2^*(L)$ to $H \times X$, i.e., a trivialization of $(\iota \times id_X)^*(M)$. Since $(\iota \times id_X)^*(M) \cong p_2^*(e_A \times id_X)^*(N)$, and $e_H \times id_X$ is a section of $p_2 : H \times X \rightarrow X$, this yields in turn a rigidification of $N$ along $\{e_A\} \times X$. So we obtain a morphism

$$\varphi : X \rightarrow \text{Pic}_A, \quad x \mapsto (id_A \times x)^*(N)$$

(see e.g. [BLR90 Sec. 8.1]). Recall that Pic$_A$ is an algebraic group, locally of finite type, with neutral component the dual abelian variety of $A$; also, $A$ acts on Pic$_A$ via its action on $A$ by translations, and the $A$-action on Pic$_A$ commutes with that of Pic$_A^0$ by translations.

By arguing as in the proof of [Br10 Lem. 2.2] (iii)$\Rightarrow$(i), we obtain that $\varphi_k : X_k \rightarrow \text{Pic}_{A_k}$ is $G_k$-equivariant (where $G_k$ acts on Pic$_{A_k}$ via $\pi_k : G_k \rightarrow A_k$ and the action of $A_k$ by
transitions), and its image consists of finitely many orbits of $A_k$; moreover, the stabilizer of each of these orbits is finite as well. Thus, $\varphi$ is equivariant.

We now consider the case where $X$ is geometrically connected; then the scheme-theoretic image $Y$ of $\varphi$ is a torsor under the quotient of $A$ by a finite subgroup scheme. In view of [Ro56, Thm. 14] (see [Mi13, Lem. 2.9] for a modern proof, and [Ra70, Chap. XIII] for generalizations), there exists a positive integer $n$ and an $A$-equivariant morphism $\psi : Y \to A/A_n$, where $A_n \subset A$ denotes the kernel of the multiplication by $n$. Composing $\psi$ with $\varphi$ yields a $G$-equivariant morphism $X \to G/H_n$, where $H_n$ is a subgroup scheme of $G$ containing $H$, and $H_n/H$ is finite. Thus, $X$ is equivariantly isomorphic to the associated fiber bundle $G \times H_n Z$ for some closed $H_n$-stable subscheme $Z \subset X$. By our assumption, $Z$ admits an ample $H$-linearized line bundle, and hence an ample $H_n$-linearized line bundle in view of Theorem 1.5. It follows that $X$ is $G$-quasiprojective.

Finally, we handle the general case, where $X$ is no longer assumed geometrically connected. Clearly, we may yet assume $X$ connected. Consider again the image $Y$ of $\varphi$. Then there exists a finite Galois extension $K/k$ of fields such that $Y_K$ is the disjoint union of $A_K$-orbits, each of them having a $K$-rational point. The Galois group $\Gamma$ acts on the set of these orbits; this action is transitive, since $X$ is connected. Let $Y' \subset Y_K$ be an orbit, and $\Gamma' \subset \Gamma$ its stabilizer. Choose $y' \in Y'(K)$ and denote by $y_1', \ldots, y_n'$ its distinct $\Gamma'$-conjugates. For any $y' \in Y'$, there exists a unique $a_i \in A_K$ such that $y' = a_i \cdot y_i'$. The morphism $Y' \to A_K$, $y' \mapsto a_1 + \cdots + a_n$, is $\Gamma'$-invariant, and hence yields a morphism $\psi : Y'/\Gamma' \to A_K/\Gamma = A$. Moreover, $Y'/\Gamma' \cong (\Gamma \times \Gamma')/\Gamma \cong Y_K/\Gamma = Y$; also, $\psi$ satisfies $\psi(a \cdot y) = na \cdot \psi(y)$ identically, and hence may be viewed as an $A$-equivariant morphism $Y \to A/A_n$. So we obtain as above that $X$ is $G$-quasiprojective.  

**END OF THE PROOF OF THEOREM 1.4** We may assume that the $G$-action on $X$ is faithful, by replacing $G$ with its quotient by the kernel of that action.

Consider first the case where $H$ is smooth; in other words, $H$ is a connected linear algebraic group. Choose an ample line bundle $L$ on $X$. By [Br14] (Theorem 4.4 and Section 4.3), there exists an $\hat{H}$-torsor $f : Y \to X$ and a positive integer $n$ such that $f^*(L^\otimes n)$ is $H$-linearizable; note that the $G$-action on $X$ lifts to a unique action on $Y$ which commutes with the $\hat{H}$-action, in view of [Br14] Prop. 4.3 (i)]. Also, $Y$ is locally of finite type; in fact, each $y \in Y$ admits a closed $G$-stable neighborhood $Y_y$ which is finite over $X$, as follows from [Br14] Prop. 4.3 (ii)]. As a consequence, $Y_y$ admits an ample $H$-linearized line bundle and is ´etale over $X$ at $y$; this yields the desired assertion in view of Lemma 4.3.

In the general case, there exists a finite, purely inseparable field extension $k'/k$ and an exact sequence 

$$1 \to H' \to G_{k'} \to A' \to 1,$$

where $H'$ is a connected linear algebraic $k'$-group, and $A'$ an abelian $k'$-variety (as follows from Chevalley’s structure theorem again). Then $H' \subset H_{k'}$, since the image of $H'$ in the abelian variety $A_{k'}$ must be trivial. Moreover, the quotient $H_{k'}/H'$ is finite, since it is an affine subgroup scheme of the abelian variety $A'$. Arguing as in the first step, we obtain an $\hat{H'}$-torsor $f' : Y' \to X_{k'}$, where $Y'$ is a $G_{k'}$-scheme, locally of finite type in the above sense. Since the natural morphism $\varphi : X_{k'} \to X$ is finite and radicial, the map $\varphi^* : H^\delta_k(X, \hat{H'}) \to H^\delta_{k'}(X_{k'}, \hat{H'})$ is an isomorphism (see [Mi80, Rem. II.3.17,
Rem. III.1.6]). So there exists a cartesian square

$$
\begin{array}{ccc}
Y' & \xrightarrow{\psi} & Y \\
\downarrow f' & & \downarrow f \\
X_{k'} & \xrightarrow{\varphi} & X,
\end{array}
$$

where $f$ is an $\widehat{H}'$-torsor; as a consequence, $Y' \cong Y_{k'}$. Also, the $G$-action on $X$ lifts uniquely to an action on $Y$ commuting with the $\widehat{H}'$-action, and every $y \in Y$ admits a closed $G$-stable neighborhood $Y_y$, finite over $X$, by [Br14, Prop. 4.3] again.

We check that $\psi$ is equivariant for the $G_{k'}$-action $\beta'$ on $Y'$, the $G$-action $\beta$ on $Y$ and the natural map $\eta : G_{k'} \to G$, i.e., $\psi(\beta'(g', y')) = \beta(\eta(g'), \psi(y'))$ identically. Since $\varphi$ satisfies a similar equivariance property, we obtain

$$\psi(\beta'(g', y')) = \lambda(g', y) \cdot \beta(\eta(g'), \psi(y'))$$

for a unique $\lambda \in \widehat{H}'(G_{k'} \times Y')$. As $G$ is geometrically connected, $\lambda$ factors through $\mu \in \widehat{H}'(Y')$; evaluating at $e_G$ yields $\psi(y') = \mu(y') \cdot \psi(y')$, and hence $\mu = 0$. This gives the desired equivariance property.

By that property and the first step, $(Y_y)_{k'}$ admits an ample $H'$-linearized line bundle, and hence an ample $H_{k'}$-linearized line bundle by Theorem 1.5. So $Y_y$ admits an ample $H$-linearized line bundle in view of Lemma 3.1. We conclude that $Y_y$ is $G$-quasiprojective, by using Lemma 4.1.

**Example 4.2.** Assume that the ground field $k$ is not locally finite (that is, $k$ is not a finite field or its algebraic closure). We construct a projective surface $X$, geometrically irreducible and seminormal, which is equipped with an action of a connected algebraic group $G$ such that $X$ consists of an open and a closed $G$-orbit (in particular, every $G$-stable neighborhood of the closed orbit is the whole $X$), and $X$ admits no finite étale $G$-projective cover.

By our assumption on $k$, there exists an elliptic curve $E$ having a $k$-rational point $x_0$ of infinite order (see e.g. [ST67]). Denote by $\pi : L \to E$ the line bundle associated with the divisor $(x_0) - (0)$, where $0$ is the origin of $E$. Since $L$ has degree $0$, the complement of the zero section in $L$ has a structure of a commutative connected algebraic group $G$ such that $\pi$ sits in an exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow G \xrightarrow{\pi} E \longrightarrow 0.$$

Consider the projective completion of $L$,

$$\tilde{\pi} : \mathbb{P}(L \oplus O_E) =: \tilde{X} \longrightarrow E.$$

Then $\tilde{X}$ is a smooth projective surface, ruled over $E$ via $\tilde{\pi}$ and equipped with two sections,

$$E_0 := \mathbb{P}(0 \oplus O_E) \quad \text{and} \quad E_\infty := \mathbb{P}(1 \oplus 0).$$

Moreover, $G$ acts on $\tilde{X}$ with three orbits: $E_0 \cong E = G/\mathbb{G}_m$, $E_\infty \cong E$, and $G$ itself.
Let $X$ be the scheme obtained from $\tilde{X}$ by identifying any $x \in E_0$ to $\tau_{x_0}(x) \in E_∞$, where $\tau_{x_0} : E \to E$ denotes the translation (the existence of the pinched scheme $X$ follows from [Fe03 Thm. 5.4]). Denote by $η : \tilde{X} \to X$ the pinching map. Since $\tau_{x_0}$ commutes with the $G$-action on $E \cong G/\mathbb{G}_m$, there is a unique $G$-action on $X$ such that $η$ is equivariant. Moreover, $X$ has two $G$-orbits: a closed orbit, isomorphic to $E$, and an open one, isomorphic to $G$. As a consequence, every open $G$-stable subscheme of $X$ which contains the closed orbit is the whole $X$.

By construction, $X$ is a proper surface, geometrically irreducible and seminormal. We now show that $X$ is projective. For this, we use the description of line bundles on pinched schemes obtained in [Fe03 2.2, 7.4]: via the assignement $M \mapsto η^*(M)$, the line bundles on $X$ correspond bijectively to the line bundles $\tilde{M}$ on $\tilde{X}$ equipped with an isomorphism

$$\tilde{M}_0 \cong \tau_{x_0}^*(\tilde{M}_∞),$$

where $\tilde{M}_0$ (resp. $\tilde{M}_∞$) denotes the pull-back of $\tilde{M}$ to $E_0$ (resp. $E_∞$). Consider the line bundle

$$\tilde{M} := O_\tilde{X}(E_0 + F),$$

where $F$ denotes the fiber of $\tilde{π}$ at 0. Then $\tilde{M}_∞ \cong O_{E_∞}(F) \cong O_E(0)$, since $E_∞ \cap F$ consists of the reduced point 0. Also, $\tilde{M}_0 = O_{E_0}(E_0 + F) \cong L \otimes O_E(0)$, since the normal bundle to $E_0$ in $\tilde{X}$ is isomorphic to $L$, and $E_0 \cap F = \{0\}$ as schemes. As $L \cong τ_{x_0}^*(O_E(0)) \otimes O_E(0)^{-1}$, it follows that $τ_{x_0}^*(\tilde{M}_∞) \cong \tilde{M}_0$. Thus, $\tilde{M} = η^*(M)$ for some line bundle $M$ on $X$.

We now show that $M$ is ample. Since $η$ is finite, it suffices to check that $\tilde{M}$ is ample, in view of [Ha70 Prop. I.4.4]. For this, we use the Nakai-Moishezon criterion (see [Ha70 Thm. I.5.1]). Every irreducible curve on $\tilde{X}$ is rationally equivalent to a nonnegative integer combination of $\mathbb{G}_m$-stable irreducible curves. Thus, the cone of curves on $\tilde{X}$ is generated by the classes of $E_0$, $E_∞$ and $F$. Moreover, we have

$$(\tilde{M} \cdot E_0) = ((E_0 + F) \cdot E_0) = 1, \quad (\tilde{M} \cdot E_∞) = 1, \quad (\tilde{M} \cdot F) = 1, \quad (\tilde{M} \cdot \tilde{M}) = 2(E_0 \cdot F) = 2.$$

Next, we show that every morphism $f : X \to A$ is constant, where $A$ is an abelian variety. The map $η \circ f : \tilde{X} \to A$ is constant on the fibers of $\tilde{π}$, since $A$ contains no rational curve. It follows that $η \circ f = g \circ \tilde{π}$ for a unique morphism $g : E \to A$. Since $η$ identifies $E_0$ and $E_∞$ via $τ_{x_0}$, we see that $g$ is invariant under $τ_{x_0}$, and hence is constant as $x_0$ has infinite order. Thus, $f$ is constant as well. As a consequence, $X$ is not $G$-quasiprojective.

Finally, we consider a finite étale cover $φ : X' \to X$, where $X'$ is connected and equipped with an action of $G$ which lifts the action on $\tilde{X}$; we show that $X'$ is not $G$-quasiprojective as well. Form the cartesian square

$$\begin{array}{ccc}
\tilde{X}' & \xrightarrow{\psi} & \tilde{X} \\
\downarrow η' & & \downarrow η \\
X' & \xrightarrow{φ} & X.
\end{array}$$

Then $ψ$ is finite and étale, and $G$ acts on $\tilde{X}'$ so that $ψ$ and $η'$ are equivariant. Consider the connected components $X'_1, \ldots, X'_{n'}$ of $\tilde{X}'$. Each $X'_i$ is stable under the $G$-action, and
hence contains an open $G$-orbit. Since the open orbit in $\tilde{X}$ has trivial stabilizer, the same holds for that in $\tilde{X}_i'$. Thus, $\psi$ restricts to a finite birational map $\psi_i : \tilde{X}_i' \to X'$. By Zariski’s Main Theorem, it follows that $\psi_i$ is an isomorphism. Also, considering the pull-back of the above cartesian square to $E \subset X$, we obtain that $\varphi^{-1}(E)$ is the disjoint union of copies $E_1, \ldots, E_n$ of $E$. We may index the connected components of $\tilde{X}'$ so that each $E_i$ is the image under $\eta'$ of the zero section $E_{i,0} \subset \tilde{X}_i'$. Then we have (with an obvious notation)

$$\eta'^{-1}(E_i) = E_{i,0} \sqcup E_{j,\infty}$$

for a unique $j$, and the assignment $\sigma : i \mapsto j$ is bijective (since $X'$ has nodal singularities along each $E_i$). The unions of the $\eta'(X_{i,\sigma(i)})$ for a fixed $i$ and all $r \geq 0$ are closed and pairwise disjoint. Since $X'$ is connected, it follows that $X'$ is a cycle of copies $X_i'$ of $\tilde{X}$, where $E_{i,0} \subset X_i'$ is identified to $E_{i,1} \subset X_{i+1}'$ via $\tau_{x_0}$. Arguing as in the above step, we obtain that every morphism from $X'$ to an abelian variety is constant.

**Remark 4.3.** The above construction makes sense, more generally, for any algebraically trivial line bundle $L$ on an abelian variety having a $k$-rational point of infinite order. This yields a proper, geometrically irreducible seminormal scheme, which is generally not projective (e.g., when $L$ is trivial).

**Acknowledgements.** Many thanks to Stéphane Druel and Philippe Gille for very helpful discussions.

**References**

[Al02] V. Alexeev, *Complete moduli in the presence of semi-abelian group action*, Ann. of Math. (2) **155** (2002), no. 3, 611–708.

[BLR90] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Ergeb. Math. Grenzgeb. (3) **21**, Springer-Verlag, Berlin, 1990.

[Br10] M. Brion, *Some basic results on actions of nonaffine algebraic groups*, Symmetry and spaces, 1–20, Progr. Math. **278**, Birkhäuser Boston, Inc., Boston, MA, 2010.

[Br14] M. Brion, *On linearization of line bundles*, arXiv:1312.6267v3, to appear at the Kodaira Centennial Issue of J. Math. Sci. Univ. Tokyo.

[Co02] B. Conrad, *A modern proof of Chevalley’s structure theorem for algebraic groups*, J. Ramanujan Math. Soc. **17** (2002), no. 1, 1–18.

[CGP10] B. Conrad, O. Gabber, G. Prasad, *Pseudo-reductive groups*, New Math. Monogr. **17**, Cambridge Univ. Press, Cambridge, 2010.

[EG98] D. Edidin, W. Graham, *Equivariant intersection theory*, Invent. math. **131** (1998), no. 3, 595–634.
[EGA] A. Grothendieck, *Éléments de géométrie algébrique* (rédigés avec la collaboration de J. Dieudonné) : II. Étude globale élémentaire de quelques classes de morphismes, Publ. Math. IHÉS **8** (1961), 5–222; III. Étude cohomologique des faisceaux cohérents, Seconde partie, ibid., **17** (1963), 5–91; IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie, ibid., **32** (1967), 5–361.

[Gr60] A. Grothendieck, *Techniques de construction et théorèmes d’existence en géométrie algébrique IV : les schémas de Hilbert*, Séminaire Bourbaki **6** (1960-1961), Exp. No. 221, 28 p.

[Fe03] D. Ferrand, *Conducteur, descente et pincement*, Bull. Soc. math. France **131** (2003), no. 4, 553–585.

[GT80] S. Greco, C. Traverso: *On seminormal schemes*, Compositio Math. **40** (1980), no. 3, 325–365.

[Ja03] J. C. Jantzen, *Representations of algebraic groups. Second edition*, Math. Surveys Monogr. **107**, American Mathematical Society, Providence, RI, 2003.

[Ha70] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes Math. **156**, Springer-Verlag, New York, 1970.

[Ko13] J. Kollár, *Singularities of the minimal model program. With a collaboration of Sándor Kovács*, Cambridge Tracts in Math. **200**, Cambridge University Press, Cambridge, 2013.

[Mil80] J. Milne, *Étale cohomology*, Princeton Math. Ser. **33**, Princeton Univ. Press, Princeton, N.J., 1980.

[Mil13] J. Milne, *A proof of the Barsotti-Chevalley theorem on algebraic groups*, preprint, [arXiv:1311.6060v2](http://arxiv.org/abs/1311.6060v2).

[Miy73] M. Miyanishi, *Some remarks on algebraic homogeneous vector bundles*, in: Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki, 71–93. Kinokuniya, Tokyo, 1973.

[Muk78] S. Mukai, *Semi-homogeneous vector bundles on Abelian varieties*, J. Math. Kyoto Univ. **18** (1978), 239–272.

[Mum70] D. Mumford, *Abelian varieties*, Oxford University Press, Oxford, 1970.

[MFK94] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory*, Ergeb. Math. Grenzgeb. (2) **34**, Springer-Verlag, Berlin, 1994.

[Ra70] M. Raynaud, *Faisceaux amples sur les schémas en groupes et les espaces homogènes*, Lecture Notes Math. **119**, Springer-Verlag, New York, 1970.

[Ro56] M. Rosenlicht, *Some basic results on algebraic groups*, Amer. J. Math. **78** (1956), no. 2, 401–443.
[SGA1] Revêtements étals et groupe fondamental (SGA 1), Séminaire de Géométrie Algébrique du Bois Marie 1960–1961, Documents Mathématiques 3, Soc. Math. France, Paris, 2003.

[SGA3] Schémas en groupes (SGA3), Séminaire de Géométrie Algébrique du Bois Marie 1960–1964, Lecture Notes Math. 151, 152, 153, Springer-Verlag, New York, 1970.

[Sa81] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, J. Reine Angew. Math. 327 (1981), 12–80.

[ST67] I. R. Shafarevich, J. Tate, The rank of elliptic curves, Dokl. Akad. Nauk SSSR 175 (1967), 770–773.

[Su74] H. Sumihiro, Equivariant completion, J. Math. Kyoto Univ. 14 (1974), no. 1, 1–28.

[Su75] H. Sumihiro, Equivariant completion. II, J. Math. Kyoto Univ. 15 (1975), no. 3, 573–605.

[Sw80] R. Swan, On seminormality, J. Algebra 67 (1980), no. 1, 210–229.

[To99] B. Totaro, The Chow ring of a classifying space, in: Algebraic K-theory (Seattle, WA, 1997), 249–281, Proc. Sympos. Pure Math. 67, Amer. Math. Soc., Providence, RI, 1999.

[Tr70] C. Traverso, Seminormality and Picard group, Ann. Scuola Norm. Sup. Pisa (3) 24 (1970), 585–595.

[We91] C. Weibel, Pic is a contracted functor, Invent. math. 103 (1991), no. 2, 351–377.