We consider Randall-Sundrum (RS) model in generalized gravities and see that the localization of gravity happens in generic situations though its effectiveness depends on the details of the configuration. It is shown that RS picture is robust against quantum gravity corrections (\(\phi R\)) as long as the correction is reasonably small. We extend our consideration to the model of scalar (dilaton) coupled gravity which leads us to the specific comparison between RS model and inflation models. The exponential and power law hierarchy in RS model are shown to correspond to the exponential and power law inflation respectively.

1 Introduction

Why the gravitational interaction is so weak compared to other gauge interactions is the main question that made people to consider the extension of the Standard Model (SM). For several decades weak scale supersymmetry was believed to be the most popular explanation for the gauge hierarchy. Recently the presence of D-brane opened new way of thinking that gravitational excitations propagate through the full spacetime while gauge interactions and matter fields are confined on the hypersurface, so called branes.

The ‘brane world’ scenario changes many conventional viewpoints toward the problems. First, the weakness of gravity at low energy is understood by the largeness of the volume of the extra dimensions. This model needs a mechanism of radion stabilization at large values for its completion which is not easy without introducing large parameters.

Recently, Randall and Sundrum (RS) proposed a new idea which can explain the gauge hierarchy by localizing gravity on a ‘Planck brane’ and assuming we are living in a tail (‘TeV brane’) of those localized gravity. From now on this will be called RS I (two brane) model to distinguish it with the single brane setup RS II which has been proposed as an alternative to the compactification. The localization of gravity yields the physical mass scale on the TeV brane suppressed by an exponentially small warp factor. This

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scenario can be realized with one extra dimension where $AdS_5$ ends at two boundary 3 branes (Planck and TeV branes). The 4-D Minkowski solution on the TeV brane can be achieved only through two exact fine tunings among bulk cosmological constant and the brane tensions. If the exact relations among the parameters are not satisfied, we obtain unstable configurations generically in which the extra dimension collapses or inflates in addition to the inflation along the 3 spaces parallel to the branes. The solution including black holes has been obtained in $^6$, and the completion of $^5$ has been done in $^7$. The brane inflationary solution with fixed extra dimension $^8$, $^9$ has been obtained first. See also $^{10}$.

In this paper we investigate the properties of RS model that remain unchanged when we modify the original simple setup. Generally quantum corrections alter the simple picture and it is essential to check whether all nice properties are valid even after full consideration of quantum corrections which arise naturally.

2 Framework

In this section, we review the general solution generating technique in the context of scalar coupled gravity mainly following $^1$ though the numerical coefficients appearing in the equations are not the same as in $^1$ due to the difference in the metric sign convention or normalization of the Ricci scalar (or 5-D Planck scale).

Our starting point is the action on $M^4 \times S^1/Z_2$ with the metric convention $g_{MN} = (-1,1,1,1,1)$ and $S = S_{bulk} + S_1 + S_2$, whose bulk action is given by

$$S_{bulk} = \int d^5x \sqrt{-g} \left\{ \frac{1}{2} R - \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - V(\phi) \right\},$$

(1)

where $\phi$ is a scalar field and $V(\phi)$ represents generalized bulk potential including the bulk cosmological constant and other bulk scalar potentials. The brane action is

$$S_i = \int d^4x \sqrt{-g_{wall}} \left\{ -V_i(\phi) + \mathcal{L}_i \right\}.$$

(2)

Here $V_i(\phi)$ denotes the brane tension of the $i$-th brane ($i = 1,2$) and $\mathcal{L}_i$ stands for the other part of the brane Lagrangian including SM matters which is assumed to give negligible effects on the spacetime geometry.

$^a$Single scalar field is enough for our purpose since we are interested in the bulk scalar or dilaton. The generalization including many scalar fields can be done without any difficulty.
The Einstein equation and the equation of motion of $\phi(y)$ are
\[ R_{AB} - \frac{1}{2} g_{AB} R = T_{AB}, \quad \nabla^2 \phi = \frac{\partial V}{\partial \phi}, \]
where the energy momentum tensors are
\[ T_{aa} = -\frac{1}{2} \phi'^2 - V(\phi) - \sum_i V_i(\phi) \delta(y - y_i), \quad (3) \]
\[ T_{55} = \frac{1}{2} \phi'^2 - V(\phi), \quad (4) \]
where $A, B, \cdots$ denote 4+1 dimensional indices, and $a = 1, \cdots, 3$ denotes the spatial index.

The main interest of this paper is to look at the scaling behaviors of different theories, and we assume that the radion has been stabilized. From now on, we focus on the solutions which keep the 4-D Poincare invariance ($\Lambda_{\text{eff}} = 0$). This can be achieved by tuning the parameter which corresponds to the cosmological constant problem in 4 dimensional theory. To keep 4-D Poincare invariance, we set $\phi = \phi(y)$. From the above conditions, the metric is expressed in a simple form $ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2$, and we can write the equation of motion in terms of $\phi(y)$ and $A(y)$. The integration over $y$ goes from 0 to $L$ since we fixed the scale factor $g_{55} = 1$.

The Einstein equations can be rewritten in terms of $A(y)$ and $\phi(y)$ as
\[ 3A'' = -\phi'^2 - \sum_i V_i(\phi) \delta(y - y_i), \quad (5) \]
\[ 6A'^2 = \frac{1}{2} \phi'^2 - V(\phi), \quad (6) \]
\[ \phi'' + 4A' \phi' = \frac{\partial V}{\partial \phi} + \sum_i \frac{\partial V_i(\phi)}{\partial \phi} \delta(y - y_i). \quad (7) \]

Though there are three equations, only two of them are independent. The third equation is derived from the first two equations. The second equation reminds us of the relation between the superpotential $W$ and the potential $V$. If we introduce $W$ and set $\phi' = \frac{\partial W}{\partial \phi}$ and $A' = \gamma W$, then from the first equation $\gamma = -\frac{1}{3}$ is determined. Once we obtain $W(\phi)$ for given $V(\phi)$ from the relation
\[ V(\phi) = \frac{1}{2} \left( \frac{\partial W}{\partial \phi} \right)^2 - \frac{2}{3} W^2, \quad (8) \]
the Einstein equations become two first order equations
\[ \phi' = \frac{\partial W}{\partial \phi}, \quad A' = -\frac{1}{3} W \quad (9) \]
provided the boundary jump conditions
\[
A'|_{y_i+\epsilon} = -\frac{1}{3} V_i(\phi(y_i)), \quad \phi'|_{y_i-\epsilon} = \frac{\partial V_i(\phi(y_i))}{\partial \phi}
\]  
are satisfied. Since we are interested in the scaling behavior of the metric in the bulk along the extra dimension \(y\), we choose \(V_i\) such that the previous jump conditions are always satisfied.

This technique can be applied only if we can find \(W(\phi)\) satisfying the relation and is independent of supersymmetry. ‘Superpotential’ like function \(W\) allows us to get an analytic solution even for some nontrivial potentials.

One of the interesting aspects of RS I model (two brane) is the radion phenomenology. The Goldberger-Wise(GW) stabilization mechanism could generate an exponentially small warp factor \(e^{-2kL} \sim M_W/M_{Pl} \sim 10^{-16}\) without introducing any large (or small) numbers in the model.

Csaki et. al., as well as Goldberger and Wise, noted that the radion kinetic term arising from the Ricci scalar \(R\) is exponentially small \((-e^{-4kL})\) as a result of nontrivial cancellations between various terms of order one. Compared to the case of order one kinetic term, this gives dramatically different radion phenomenology. Radion mass is essentially of the order of weak scale and radion couplings to the standard model fields are of the order of weak interaction strength. If the kinetic term were of order one, one would have radion mass \(m^2 \sim (M^2_W/M^2_{Pl})M^2_W\) and radion couplings of gravitational strength.

However, this peculiar phenomenon is due to the choice of unnatural gauge and disappears when we choose more natural gauge. The actual wave function of radion is concentrated on TeV brane, and it is very reasonable to have a weak scale radion mass since the mass is not suppressed by large volume factor.

In the following sections, we see how the radion potential changes when the quantum corrections are added.

### 3 Conformal transformation

Now we have a tool which is very useful for the system with Einstein-Hilbert action and general kinds of scalar potential. However, we need more than this to consider \(\phi R\) correction. It is well known that generalized action of the following form can be transformed to the Einstein frame by conformal transformation

\[
S_{bulk} = \int d^5x \sqrt{\text{det} g} \left\{ \frac{1}{2} f(\phi) R - \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - \mathcal{V}(\phi) \right\}.
\]  

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The conformal transformation 
\[ \hat{g}_{MN} = e^{2\omega} g_{MN} \]  \hspace{1cm} (12)
with \( \omega = \frac{1}{3} \log(f(\phi)) \) brings us to the Einstein-Hilbert action 
\[ S_{\text{bulk}} = \int d^5 x \sqrt{-\hat{g}} \left\{ \frac{1}{2} \hat{R}(\hat{g}) - \frac{1}{2} \hat{g}^{AB} \partial_A \phi \partial_B \Phi - \mathcal{U}(\Phi) \right\} \]  \hspace{1cm} (13)
where the potential in Einstein frame is 
\[ \mathcal{U}(\Phi) = (f(\phi))^{-\frac{3}{2}} \mathcal{V}(\phi). \]  \hspace{1cm} (14)
and the canonically normalized field \( \Phi \) which has a relation with \( \phi \) is
\[ \Phi = \int d\phi \left( \frac{f(\phi) + \frac{2}{3} \left( \frac{df(\phi)}{d\phi} \right)^2}{f^2(\phi)} \right)^{\frac{1}{2}}. \]  \hspace{1cm} (15)
All the cases appearing in the following are analyzed in two steps based on these tools. First, take a conformal transformation to Einstein frame. Second, find \( W \) and solve the first order differential equations.

3.1 Small \( \phi R \) correction

Let’s consider \( \phi R \) correction first. When scalar fields are present, the effective action is expected to contain \( \phi R \) term which is induced by quantum corrections 
\[ S_{\text{bulk}} = \int d^5 x \sqrt{-g} \left\{ \frac{1}{2} (1 + \varepsilon \phi^2) R - \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - \mathcal{V}(\phi) \right\}. \]  \hspace{1cm} (16)
It is reasonable to assume that the coefficient of the induced term is small \( \varepsilon \ll 1 \) (e. g. \( \varepsilon \sim 1/100 \)) since it usually contains a loop suppression factor. We can take the conformal transformations such that 
\[ S_{\text{bulk}} = \int d^5 x \sqrt{-g} \left\{ \frac{1}{2} \hat{R}(\hat{g}) - \frac{1}{2} \hat{g}^{AB} \partial_A \Phi \partial_B \Phi - \mathcal{U}(\Phi) \right\}, \]  \hspace{1cm} (17)
where \( \Phi \) and \( \mathcal{U}(\Phi) \) are determined by (15), (14) as 
\[ \Phi = \phi - \varepsilon \phi^3 + \mathcal{O}(\varepsilon^2), \quad \mathcal{U}(\Phi) = (1 - \frac{5}{3} \varepsilon \phi^2) \mathcal{V}(\phi) + \mathcal{O}(\varepsilon^2). \]  \hspace{1cm} (18)

The action on the brane has not been specified, and we assume that the correction on the brane is such that the relation (11) is satisfied to keep the brane configuration static.

Before considering quantum correction, let’s review the back reaction of GW scalar field. When there was no potential for the scalar \( \phi \) and \( \mathcal{V} = \Lambda = \)
$-24k^2$, we recover the RS model with $W = 6k$ and $A = -2ky$. To check whether $\phi R$ correction destabilize the GW mechanism, we should consider massive bulk scalar at first. Massive bulk scalar changes the potential and now $V = \Lambda + m^2 \phi^2 = -24k^2(1 + \eta \phi^2)$ in which we introduce new parameters, $k$ representing the bulk cosmological constant and $\eta$, the ratio of the scalar mass to the bulk cosmological constant. Now $W = 6k(1 + \frac{1}{2} \eta \phi^2) + O(\eta^2)$ where $\eta \ll 1$ when the mass is small compared to the bulk cosmological constant which is the case for stabilizing the radion at order 10 value ($kL \approx 37$). The bulk geometry is not pure $AdS_5$ due to the back reaction of the bulk scalar, and we obtain the metric $g_{\mu\nu} = e^{-2A} \eta_{\mu\nu}$ from eq. (9) as
\begin{equation}
A = -2k(1 + \frac{1}{2} \eta \phi^2)y + O(\eta^2),
\end{equation}
where $\phi_0$ is the value of $\phi$ at $y = 0$. Though there is a nontrivial(not linear) $y$ dependence due to the back reaction appearing at $O(\eta^2)$, we neglect it in this section since first order approximation is enough if $\eta$ is small enough ($\eta \ll 1$).

Let’s back to the quantum correction $\phi R$. The above eq. (18) then become after conformal transformation
\begin{align}
\mathcal{U}(\Phi) &= -24k^2(1 + (\eta - \frac{5}{3} \epsilon) \phi^2) + O(\epsilon^2, \eta^2, \epsilon \eta), \\
\mathcal{W}(\Phi) &= 6k(1 + \frac{1}{2} (\eta - \frac{5}{3} \epsilon) \phi^2) + O(\epsilon^2, \eta^2, \epsilon \eta).
\end{align}
These equations give us very simple interpretations of the $\phi R$ correction as the correction of the bulk scalar masses. The change in the geometry is given as the same way as in eq. (19) with the effective mass by replacing $\eta \to (\eta - \frac{5}{3} \epsilon)$. This opens new possibility of generating bulk scalar masses through quantum gravity corrections. Scalar mass can be obtained from this quantum correction even though we started from the setup with a massless bulk scalar.

All the pictures in RS I (with two branes), e.g., the generation of huge hierarchy between Planck brane and TeV brane, Goldberger-Wise stabilization mechanism, remain the same as long as $\epsilon$ is small enough, for instance, $\epsilon \leq 10^{-2}$. The effects of $\phi R$ is to modify the scalar mass. Though it is possible to imagine that scalar mass is originated from quantum gravity correction, we have kept in mind that the small scalar mass is based on the symmetry which is slightly broken, e.g., bulk supersymmetry. In that case the quantum correction to the scalar mass is proportional to the tree level scalar mass itself and is loop suppressed, $\epsilon \sim \frac{1}{100} \eta \ll \eta$. Therefore $\phi R$ correction is not harmful to the radion potential, and the RS solution to the gauge hierarchy can be kept stable against small $\phi R$ correction.
3.2 Brans-Dicke Theory

In this section, we consider different types of theories which have entirely different properties than RS model. String theory has a dilaton which couples directly with scalar curvature and give Brans-Dicke (BD) type theory as its low energy effective theory. This kind of theory shows very different behavior than the RS model. Already it has been shown that exponential and power law hierarchy are obtained in the usual supergravity and the gauged supergravity respectively in the framework of strongly coupled heterotic string [13].

Now all the features of RS scenario change if we consider BD type interactions between scalar fields and gravity. To see the qualitative features, we consider the following action

$$S_{bulk} = \int d^5x \sqrt{-g}\left\{\frac{1}{2}\varepsilon\phi^2 R - \frac{1}{2}g^{AB}\partial_A\phi\partial_B\phi - \mathcal{V}(\phi)\right\}. \quad (22)$$

Actually typical form of BD theory is

$$S_{bulk} = \int d^5x \sqrt{-g}\left\{\frac{1}{2}\tilde{\phi} R - \frac{1}{2}g^{AB}\omega^{\phi}\partial_A\tilde{\phi}\partial_B\tilde{\phi} - \mathcal{V}(\tilde{\phi})\right\}, \quad (23)$$

but this is equivalent to the previous one with the relation $\varepsilon = \frac{1}{4\omega}$. The equivalence relation can be easily checked by changing the BD kinetic term to the canonical form.

Conformal transformation brings the action into

$$S_{bulk} = \int d^5x \sqrt{-g}\left\{\frac{1}{2}R - \frac{1}{2}g^{AB}\partial_A\Phi\partial_B\Phi - \mathcal{U}(\Phi)\right\}, \quad (24)$$

where $\Phi$ and $\mathcal{U}(\Phi)$ are determined from $\phi$ and $\mathcal{V}(\phi)$ using eq. (15) and (14). Now the situation is entirely different from the previous case. $\Phi$ and $\phi$ are not related linearly,

$$\Phi = \sqrt{\frac{1 + \frac{32}{3}\varepsilon}{\varepsilon}} \log(\phi/\phi_0) = a \log(\phi/\phi_0), \quad (25)$$

and the potential has exponential dependence

$$\mathcal{U}(\Phi) = \varepsilon^{\frac{2}{3}}\phi_0^{\frac{16}{3}}e^{-\frac{10}{3}a\Phi}\mathcal{V}(\phi(\Phi)). \quad (26)$$

When the bulk cosmological constant is dominant,

$$W(\phi) = 6k, \quad \mathcal{V}(\phi) = -24k^2, \quad (27)$$

we get

$$\mathcal{W}(\Phi) = 6k\varepsilon^{-\frac{2}{3}}\phi_0^{-\frac{16}{3}}e^{-\frac{10}{3}\Phi} = ce^{\Phi} \quad (28)$$
and can solve the differential equation

\[
A = \frac{3a^2}{25} \log(1 - \frac{25}{9a^2}cy) = \frac{3 + 32\varepsilon}{25\varepsilon} \log(1 - \frac{25\varepsilon}{9 + 96\varepsilon}cy)
\]

(29)

where \( c = W(y = 0) \) is the integration constant. Now the metric is

\[
g_{\mu\nu} = e^{2A} \eta_{\mu\nu} = \left(1 - \frac{25\varepsilon}{9 + 96\varepsilon}cy\right)^{\frac{6+9\varepsilon}{25\varepsilon}} \eta_{\mu\nu},
\]

(30)

and it shows the power law dependence along \( y \) which becomes singular at finite \( y \) (\( y_c = \frac{9 + 96\varepsilon}{25\varepsilon} \)). Consideration of exponential potential rather than power law potential gives the same result, and this corresponds to the self-tuning brane models having singularity when we cut off the bulk at the singularity \( \varepsilon \rightarrow 0 \). Cutting the bulk before the singularity occurs gives the usual string settings like Horava-Witten \( \mathcal{L} \). The potential generating power law hierarchy has been studied independently.

By taking the limit \( \varepsilon \rightarrow 0 \) with \( \varepsilon \phi^2 \) fixed, we can recover the RS model as \( g_{\mu\nu} = \varepsilon^{2A} \eta_{\mu\nu} = \varepsilon^{-4\varepsilon} \eta_{\mu\nu} \). In this limit \( \phi \) is frozen since the kinetic term becomes huge, and the system recovers Einstein-Hilbert action. Even for a tiny but nonzero \( \varepsilon \) (\( \varepsilon \ll 1 \)), the metric \( g_{\mu\nu} \) shows the presence of singularity at \( y_c \approx \frac{9}{25\varepsilon} \). At any rate, we can generate the hierarchy \( 10^{-32} \) with \( \varepsilon \approx 1/120 \) and putting TeV brane at \( y = 0.9y_c \approx 40c \sim 40 \) for order one \( c \). If we put TeV brane at \( y = 0.99y_c \), we need \( \varepsilon \approx 1/50 \) and \( y \approx 20c \sim 20 \).

4 Randall-Sundrum vs. Inflation

Understanding of RS geometry can be easily done by thinking \( y \) as time of the inflation models. RS model itself corresponds to the usual inflation models in which the expansion is exponential, and scalar(dilaton) coupled theory gives rise to the power law hierarchy as in the case of extended inflation model in which BD theory gives power law inflation. More concrete relations can be found in the recent paper \( \mathcal{L} \) with a table summarizing it. This analogy shows that singularities away from the brane is inevitable since this corresponds to the initial singularity in inflation models. RS II (single brane) model has AdS geometry even far away from the brane with decreasing warp factor. However, an analogy with inflation models shows that asymptotically AdS geometry with decreasing warp factor is not an attractor of the system. This is the opposite case to the inflation in which asymptotically dS is an attractor. The difference is due to the fact that the direction we are considering is opposite with each other. If we consider time reversal direction, asymptotically dS is not an attractor and generally singularity is developed since the connection.
term in the scalar field equation of motion acts like an anti-friction. This
singularity has already been observed in the study of 5-D supergravity inspired
by AdS/CFT correspondence. The paper \cite{22} addresses the question of which
types of singularities are harmless. In other words the criterion on which types
of the singularities are expected to be resolved is suggested.

5 Conclusion

We have studied general features of RS model by considering several quan-
tum corrections and checked that RS model is robust against small quantum
corrections. Though exponentially suppressed potential looks suspicious to
be stable against possible dangerous corrections, it turned out that nothing is
harmful as far as we are concerned on a theory whose systematic expansion
is possible. We considered $\phi R$ term and showed that it acts like changing
the bulk scalar mass. Therefore, once the mass is small, the correction is
further suppressed and the radion potential remains stable. Also we obtained
the power law $y$ dependence of the metric for the BD type generalization of
RS model. Exponential law $y$ dependence of the warp factor which is im-
portant to localize the gravity can be realized by freezing the dilaton such
that there is no scalar that couples directly to $R$. Otherwise, the singularity
appears and we can not have RS II (single brane model) as an alternative
of compactification. Finally we stated several correspondences between RS
model and inflation model and checked that asymptotically $AdS$ do not come
as an attractor of the system which is necessary for the localization of grav-
ity. This is opposite to the slow roll condition of the inflation models where
the condition is guaranteed by appearing as an attractor. The singularity
entering in general setup with scalar fields is inevitable in RS II (single brane
setup). Nonetheless, RS I model (two brane) remains as very robust one at
least within our consideration since the consideration of quantum effects does
not alter its characteristic features.

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