The Fokas–Lenells equations: Bilinear approach

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Abstract
In this paper, the Fokas–Lenells (FL) equations are investigated via bilinear approach. We bilinearize the unreduced FL system, derive double Wronskian solutions, and then, by means of a reduction technique we obtain variety of solutions of the reduced equations. This enables us to have a full profile of solutions of the classical and nonlocal FL equations. Some obtained solutions are illustrated based on asymptotic analysis. As a notable new result, we obtain solutions to the FL equation, which are related to real discrete eigenvalues and not reported before in the analytic approaches. These solutions behave like (multi)periodic waves or solitary waves with algebraic decay. In addition, we also obtain solutions to the two-dimensional massive Thirring model from those of the FL equation.

KEYWORDS
bilinear, double Wronskian, Fokas–Lenells equation, nonlocal, real eigenvalue

1 | INTRODUCTION

The Fokas–Lenells (FL) equation,

\[ iu_t - νu_{xx} + γ|u|^2(u + ivu_x) = 0, \quad σ = ±1, \]

as a novel generalization of the nonlinear Schrödinger (NLS) equation, was first derived using bi-Hamiltonian structures of the NLS equation,\(^1\) where ν and γ are real parameters. This equation is integrable and belongs to the derivative nonlinear Schrödinger (DNLS) hierarchy\(^2\) that is related
to the Kaup-Newell (KN) spectral problem. It is notable that the FL equation (1) is equivalent to the following one\(^3\):

\[ u_{xt} + u - 2i\delta|u|^2u_x = 0, \quad \delta = \pm 1. \tag{2} \]

The latter is derived for modeling propagation of nonlinear pulses in monomode optical fibers where \( u \) is assumed to describe the slowly varying envelope of the pulse.\(^3\) It is interesting that Equation (2) is a reduced potential form of the first negative member in the KN hierarchy,

\[ u_{xt} + u - 2iuvu_x = 0, \tag{3a} \]

\[ v_{xt} + v + 2iuvv_x = 0, \tag{3b} \]

by imposing reduction \( v = \delta u^* \), where \( i \) is the imaginary unit and \( * \) denotes complex conjugate. Both (1) and (2) can be called the FL equation, and in the following we call (3) the pKN(−1) for convenience. Note that the pKN(−1) system (3) is also known as the Mikhailov model (cf. Refs. 4, 5). It is A.V. Mikhailov in 1976 who first gave a Lax pair for the two-dimensional massive Thirring model in laboratory coordinates. Later it was shown that the Lax pair in light-cone coordinates is gauge equivalent to the KN spectral problem\(^6\) and solutions of the massive Thirring model could be obtained by solving Equation (2), i.e., the reduced pKN(−1).\(^4,6\) We will explain how Equation (2) and the massive Thirring model are related in Appendix A.

Before Fokas and Lenells’ work, Equation (2) has been solved using direct linearization approach.\(^7\) More recently, by virtue of a clear integrable background associated with the well-studied KN spectral problem, solutions of the FL equation (1) or (2) have been derived by means of the Riemann–Hilbert method or inverse scattering transform,\(^2,8,9\) dressing chain,\(^10\) algebro-geometric method,\(^11\) Darboux transformation,\(^12–14\) and a variable separation technique.\(^15\) Note that in Ref. 14 Equation (2) is shown to be related to the Zakharov–Shabat and Ablowitz–Kaup–Newell–Segur (ZS-AKNS) spectral problem.

Without using Lax pairs, the FL equation (2) was bilinearized and determinantal solutions of the bilinear FL equations were constructed in a delicate direct way by introducing determinants of the Cauchy matrix type.\(^16,17\) Another direct approach was presented in Ref. 18, where a chain of Bäcklund transformations of the pKN(−1) (3) was constructed and viewed as semidiscrete equations in the Merola–Ragnisco–Tu hierarchy (cf. Ref. 19) and solutions of the FL equation (2) were given in terms of Cauchy matrix by using the connection between the Merola–Ragnisco–Tu hierarchy and the Ablowitz–Ladik hierarchy. From the bilinear form given in Ref. 16 it is easy to get a three-soliton solution in terms of Hirota’s polynomials of exponential functions.\(^20\) However, it is difficult to give double Wronskian solutions to those FL equations. As a matter of fact, Equation (3) is the potential form KN(−1) (i.e., the first negative member in the KN hierarchy, see Equation (11)). To our understanding, it is difficult to express the integral \( u = \delta x^{-1} q \) in terms of double Wronskians.

In this paper, we aim to construct double Wronskian solutions for the FL equation (2). This will allow us to have more freedom to understand possible distributions of eigenvalues, present different kinds of solutions (e.g., solitons, breathers, and multiple pole solutions), explore their interactions (cf. Refs. 21, 22), and as a result, give a full profile of the FL equations from the bilinear approach and double Wronskian forms. We will start from the pKN(−1) system (3). First, we will bilinearize (3) and prove their double Wronskian solutions. Note that the double Wronskians that we employ in the paper have different structures from those of the AKNS hierarchy (cf. Ref. 23), the
KN equation (cf. Ref. 24), and the Chen–Lee–Liu equation (cf. Ref. 25). After that we will impose reductions on the double Wronskians using the technique developed in Ref. 23. This allows us to have solutions not only for the FL equation (2) but also for its nonlocal partner

\[ u_{xt} + u - 2i\delta uu(-x, -t)u_x = 0, \quad \delta = \pm 1, \quad (4) \]

which is reduced from (3) via a nonlocal reduction \( v(x, t) = \delta u(-x, -t) \). Note that nonlocal integrable systems were first systematically proposed in 2013 in Ref. 26 and have received intensive attention (e.g., Refs. 27–41). The reduction also enables us to see how the distribution of eigenvalues varies with the constraints imposed in the local and nonlocal reductions. It is worthy to mention that, apart from those solutions related to complex discrete eigenvalues (cf. Refs. 2, 8, 9), the FL equation (2) allows solutions related to real discrete eigenvalues. These solutions exhibit (multi)periodic behaviors, and also provide solitary waves with algebraic decay as \( |x| \to \infty \), which are not found in the analytic approaches based on spectral analysis (cf. Refs. 2, 8, 9).

The paper is organized as follows. As preliminary, in Section 2 we recall integrable backgrounds of the pKN(−1) system (3) and give notations of double Wronskians and some determinantal identities. Then, in Section 3 we bilinearize (3), present double Wronskian solutions, and implement the reduction technique to get solutions for (2) and (4). Next, dynamics of some obtained solutions are investigated for the FL equation (2) in Section 4 and for the nonlocal FL equation (4) in Section 5. Finally, concluding remarks are given in Section 6. There are three appendices. The first one introduces known results that how the massive Thirring model, the KN spectral problem, and the FL equation are connected. The second one presents \( N \)-soliton solution formula of the pKN(−1) system (3) via Hirota’s expression, and the third one consists of a detailed proof of double Wronskian solutions.

## 2 | PRELIMINARY

### 2.1 | Integrability of the FL equations

As an integrable background let us recall the relation between the FL equation and the KN hierarchy. The KN spectral problem reads: \cite{6,42}

\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix}_x = \begin{pmatrix}
\frac{i}{2}\lambda^2 & \lambda q \\
\lambda r & -\frac{i}{2}\lambda^2
\end{pmatrix}
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix},
\]

from which one can derivative the well-known KN hierarchy

\[
\begin{pmatrix}
q \\
r
\end{pmatrix}_{tn} = L^n
\begin{pmatrix}
-q \\
r
\end{pmatrix},
\]

where the recursion operator \( L \) is

\[
L = \partial_x
\begin{pmatrix}
-i + 2q\partial_x^{-1}r & 2q\partial_x^{-1}q \\
2r\partial_x^{-1}r & i + 2r\partial_x^{-1}q
\end{pmatrix},
\]
and $\partial_x = \frac{\delta}{\partial x}, \partial_x^{-1} \partial_x = \partial_x \partial_x^{-1} = 1$. Here $\lambda$ is the spectral parameter, and both $q$ and $r$ are functions of $(x, t)$. When $n = 2$, the hierarchy (6) yields the second-order KN system (KN(2) for short)

$$q_t + iq_{xx} - 2(q^2 r)_x = 0, \quad (8a)$$

$$r_t - ir_{xx} - 2(qr^2)_x = 0, \quad (8b)$$

from which the DNLS equation,

$$q_t + 2\delta(|q|^2 q)_x = 0, \quad \delta = \pm 1, \quad (9)$$

is obtained by imposing reduction $r = \delta q^*$. When $n = -1$, we have KN(−1) equations, i.e.,

$$L \begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -q \\ r \end{pmatrix}, \quad (10)$$

which reads (with $t \to -it$)

$$q_t + \partial^{-1} q - 2iq\partial^{-1}(q\partial^{-1} r + r\partial^{-1} q) = 0, \quad (11a)$$

$$r_t + \partial^{-1} r + 2ir\partial^{-1}(q\partial^{-1} r + r\partial^{-1} q) = 0. \quad (11b)$$

Introduce potentials $(u, v)$ by

$$q = u_x, \quad r = v_x, \quad (12)$$

then (11) can easily be written into the local form (3). Thus, the latter is the potential form of the KN(−1) equations (11). As we mentioned before, the pKN(−1) system (3) allows two reductions, $v = \pm u^*$ and $v(x, t) = \pm u(−x, −t)$.

### 2.2 Wronskians and some determinantal identities

Let $\phi$ and $\psi$ be $(N + M)$-th order column vectors

$$\phi = (\phi_1, \phi_2, ..., \phi_{N+M})^T, \quad \psi = (\psi_1, \psi_2, ..., \psi_{N+M})^T, \quad (13)$$

where elements $\phi_j$ and $\psi_j$ are smooth functions of $(x, t)$. A standard double Wronskian is a determinant of the following form:

$$W^{[N,M]}(\phi; \psi) = |\phi, \partial_x \phi, ..., \partial_x^{N-1} \phi; \psi, \partial_x \psi, ..., \partial_x^{M-1} \psi|.$$

We introduce short-hand (cf. Refs. 23, 43, 44)

$$W^{[N,M]}(\phi; \psi) = |\phi^{[N−1]}; \psi^{[M−1]}|,$$
where by $\hat{\phi}[N-1]$ we mean consecutive columns $(\phi, \partial_x \phi, ..., \partial_x^{N-1} \phi)$. Without making any confusion, we also employ the conventional compact notation

$$W^{[N,M]}(\phi; \psi) = |0, 1, ..., N - 1; 0, 1, ..., M - 1| = |\hat{N}; \hat{M} - 1|$$

that was introduced in Ref. 45.

Following the above notations, we introduce four more double Wronskians

$$|\hat{N}; \hat{M} - 1| = |1, 2, ..., N; 0, 1, ..., M - 1|,$$
$$|\hat{N}; \hat{M} - 1| = |0, 1, ..., N; 1, 2, ..., M - 1|,$$
$$|\hat{N}; \hat{M}| = |2, 3, ..., N; 0, 1, ..., M|,$$
$$|\hat{N}; \hat{M}| = |1, 2, ..., N; 1, 2, ..., M|,$$

which will be employed in presenting solutions of the bilinear FL equations in next section.

We also need the following identities, which will be used in verifying solutions of bilinear equations.

**Lemma 1** $^{46}$

$$|M, a, b| |M, c, d| - |M, a, c| |M, b, d| + |M, a, d| |M, b, c| = 0,$$

where $M$ is an arbitrary $N \times (N - 2)$ matrix, and $a, b, c,$ and $d$ are $N$th-order column vectors.

**Lemma 2** $^{21,47}$. Let $\Xi = (a_{js})_{N \times N}$ be an $N \times N$ matrix with column vectors $\{\alpha_j\}$. $\Gamma = (\gamma_{js})_{N \times N}$ is an $N \times N$ operator matrix and each $\gamma_{js}$ is an operator. Then we have

$$\sum_{j=1}^{N} |\Gamma_j * \Xi| = \sum_{j=1}^{N} |(\Gamma^T)_j * \Xi^T|,$$

where

$$|A_j * \Xi| = |\Xi_1, ..., \Xi_{j-1}, A_1 \circ \Xi_j, \Xi_{j+1}, ..., \Xi_N|,$$

and

$$A_j \circ B_j = (A_{1,j}B_{1,j}, A_{2,j}B_{2,j}, ..., A_{N,j}B_{N,j}),$$

in which $A_j = (A_{1,j}, A_{2,j}, ..., A_{N,j})^T$ and $B_j = (B_{1,j}, B_{2,j}, ..., B_{N,j})^T$ are $N$th-order column vectors.

3 | SOLUTIONS OF THE FL EQUATIONS

In this section, we derive bilinear form of the pKN$(-1)$ equation (3), present its double Wronskian solutions, and apply reduction technique to obtain solutions of classical and nonlocal FL equations.
3.1 Bilinear form and double Wronskian solution

With dependent variable transformation

\[ u = \frac{g}{f}, \quad v = \frac{h}{s}, \]

the pKN(−1) equation (3) can be bilinearized as the following:

\[ D_x D_t \ g \cdot f + g f = 0, \]  
\[ D_x D_t \ h \cdot s + h s = 0, \]  
\[ D_x D_t \ f \cdot s + i D_x \ g \cdot h = 0, \]  
\[ D_t \ f \cdot s + i g h = 0, \]

where \( D \) is the Hirota bilinear operator defined as

\[ D^m D^n f \cdot g \equiv (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x, t) g(x', t')|_{x'=x, t'=t}. \]

\[ N \]-soliton solution in Hirota’s form is presented in Appendix B. With regard to double Wronskian solutions, we have the following.

**Theorem 1.** The bilinear equations (19) admit double Wronskian solutions

\[ f = |\tilde{N}; \tilde{M}-1|, \quad g = |\tilde{N}; \tilde{M}-1|, \quad h = -\frac{i}{2}|\tilde{N}; \tilde{M}|, \quad s = |\tilde{N}; \tilde{M}|, \]

where the elementary column vectors \( \phi \) and \( \psi \) satisfy

\[ \phi_x = \frac{i}{2} A^2 \phi, \quad \phi_t = -\frac{1}{4} \partial_x^{-1} \phi, \]  
\[ \psi_x = -\frac{i}{2} A^2 \psi, \quad \psi_t = -\frac{1}{4} \partial_x^{-1} \psi, \]

Here \( A \) is an arbitrary invertible constant matrix in \( \mathbb{C}^{(N+M)\times(N+M)} \). A general form of \( \phi \) and \( \psi \) obeying (22) is

\[ \phi = \exp\left(\frac{i}{2} A^2 x + \frac{i}{2} A^{-2} t\right) C, \]  
\[ \psi = \exp\left(-\frac{i}{2} A^2 x - \frac{i}{2} A^{-2} t\right) B, \]

where \( B \) and \( C \) are \( (N + M) \)-th order constant column vectors.

The proof is given in Appendix C.
Proposition 1. A and its any similar form lead to the same \( u \) and \( v \) through (18) and (21).

Proof. Let \( \Lambda = \Gamma A \Gamma^{-1} \) be a matrix similar to \( A \) and \( \phi' = \Gamma \phi, \quad \psi' = \Gamma \psi \). Then, \( (\phi', \psi') \) also satisfy (22) but with \( A \) replaced by \( \Lambda \), and the double Wronskians composed by \( (\phi, \psi) \) and by \( (\phi', \psi') \) are simply connected by

\[
w(\phi', \psi') = |\Gamma|w(\phi, \psi),
\]

where \( w \) can be \( f, g, h, \) and \( s \). Thus, the double Wronskians composed by \( (\phi', \psi') \) also solve the bilinear equations (19) and lead to the same \( u \) and \( v \) as before. 

This proposition enables us to have a full profile of solutions for the pKN(−1) equations (3) by considering canonical forms of \( A \).

3.2 Reductions and solutions

In this subsection, we impose suitable constraints on \( \phi \) and \( \psi \) given in (23), so that \( u \) and \( v \) defined through (18) and (21) can satisfy the relations \( v(x, t) = \pm u^*(x, t) \) and \( v(x, t) = \pm u(-x, -t) \). We will also look for explicit forms of \( \phi \) and \( \psi \) that obey those constraints. As a result, explicit double Wronskian solutions for the classical FL equation (2) and nonlocal FL equation (4) will be obtained.

3.2.1 Case of the classical FL equation

We note that, compared with the double Wronskian solutions of the AKNS hierarchy (cf. eq. (20) in Ref. 23), the solutions presented in (21) for the pKN(−1) system are more complicated. To implement a reasonable reduction, let us take \( M = N \) and assume

\[
A^2 = \delta SS^*, \quad \delta = \pm 1,
\]

where \( S \) is an undetermined invertible matrix in \( \mathbb{C}_{2N \times 2N} \). Note that relation (24) indicates

\[
A^2 S = SA^{*2}.
\]

Then we immediately have from (23) that

\[
S\Phi^* = S \exp \left[ -\frac{i}{2}(A^{*2}x + (A^*)^{-2}t) \right] C^*,
\]

\[
= \exp \left[ -\frac{i}{2}(A^2 x + A^{-2}t) \right] SC^* = \psi,
\]

where

\[
\Phi^* = \phi^* S^{-1} + \psi S^{-1},
\]
where we have taken $B = SC^*$. With this relation it then follows that we can write the double Wronskians (21) in terms of only $\phi$:

$$f = |\bar{N}; N - 1| = \left(\frac{i}{2}\right)^N [A^2 \phi^{[N-1]}; S\phi^{[N-1]}]^*], \quad (26a)$$

$$g = |\hat{N}; N - 1| = \left(-\frac{i}{2}\right)^{N-1} [\phi^{[N]}; A^2 S\phi^{[N-2]}]^*], \quad (26b)$$

$$h = -\frac{i}{2} |\bar{N}; \hat{N}| = -\left(\frac{i}{2}\right)^{2N-1} [A^4 \phi^{[N-2]}; S\phi^{[N]}]^*, \quad (26c)$$

$$s = |\bar{N}; \bar{N}| = \left(\frac{i}{2}\right)^{2N} (-1)^N |A|^2 [\phi^{[N-1]}; S\phi^{[N-1]}]^*]. \quad (26d)$$

Then, using (24) (i.e., $A^{*2} = \delta S^* S$), we have

$$f^* = \left(-\frac{i}{2}\right)^N [A^*^2 \phi^{[N-1]}; S^* \phi^{[N-1]}],$$

$$= \left(\frac{i}{2}\right)^N [S^* \phi^{[N-1]}; A^2 \phi^{[N-1]}]^*]$$

$$= \left(\frac{i}{2}\right)^N [S^* [A^2 \phi^{[N-1]}]; (S^*)^{-1} A^2 \phi^{[N-1]}]^*]$$

$$= \left(\frac{i}{2}\right)^N [S^* [A^2 \phi^{[N-1]}]; \delta S \phi^{[N-1]}]^*]$$

$$= (2i)^N \delta^N |S|^{-1} s,$$

and

$$g^* = \left(\frac{i}{2}\right)^{N-1} [\phi^{[N]}; A^*^2 S^* \phi^{[N-2]}]$$

$$= \left(-\frac{i}{2}\right)^{N-1} [A^*^2 S^* \phi^{[N-2]}; \phi^{[N]}]^*]$$

$$= \left(-\frac{i}{2}\right)^{N-1} [S^* A^2 \phi^{[N-2]}; \phi^{[N]}]^*]$$

$$= \left(-\frac{i}{2}\right)^{N-1} [S|^{-1} [SS^* A^2 \phi^{[N-2]}]; S\phi^{[N]}]^*]$$

$$= \left(-\frac{i}{2}\right)^{N-1} [S|^{-1} [\delta A^4 \phi^{[N-2]}]; S\phi^{[N]}]^*]$$

$$= (2i)^N \delta^{N-1} |S|^{-1} h.$$
This leads to, by noting that $\delta = \pm 1$ and $|A|^2 = |S||S|^*$, $g^* = \delta \frac{h}{s}$, i.e., $v(x, t) = \delta u^*(x, t)$.

Let us summarize the above results by the following lemma.

**Lemma 3.** For the Wronskians (21) with (23), taking $M = N$ and assuming (24) and $B = SC^*$, where $S$ is some invertible matrix in $\mathbb{C}_{2N \times 2N}$, we have the relation

$$\psi = S\phi^*$$

(27)

and

$$f^* = (2i)^N \delta^N |S|^{-1}s,$$

(28a)

$$g^* = (2i)^{N-1} \delta |S|^{-1}h.$$  

(28b)

These give rise to $v(x, t) = \delta u^*(x, t)$ when $u$ and $v$ are defined by (18).

In practice we replace $S$ by $S = AT$ where $T \in \mathbb{C}_{2N \times 2N}$, and assume

$$AT = TA^*, \quad TT^* = \delta I_{2N},$$

(29)

where $I_{2N}$ is the identity matrix of $2N$ order. Equation (29) is a sufficient condition for (24). In fact,

$$A^2 = \delta A^2 TT^* = \delta A(AT)T^* = \delta ATA^*T^* = \delta SS^*.$$  

Thus, we can write the above lemma in terms of $T$.

**Theorem 2.** The classical FL equation (2) admits the following solution:

$$u(x, t) = \frac{|\hat{N}; N-1|}{|\tilde{N}; N-1|},$$

(30)

where the elementary vector $\phi$ is given by (23a) and

$$\psi = AT \phi^*,$$

(31)

and $A, T \in \mathbb{C}_{2N \times 2N}$ are invertible and satisfy Equation (29). In addition, the double Wronskians (21) composed by the above $\phi$ and $\psi$ satisfy the following bilinear FL equations:

$$D_xD_t \ g \cdot f + gf = 0,$$

(32a)

$$D_xD_t \ f \cdot f^* + i\delta D_x \ g \cdot g^* = 0,$$

(32b)

$$D_t \ f \cdot f^* + i\delta gg^* = 0,$$

(32c)
TABLE 1  $T$ and $A$ for (29)

| Case | $\delta$ | $T$ | $A$ |
|------|----------|-----|-----|
| (1)  | $\pm 1$  | $T_1 = T_4 = 0_N$, $T_2 = \delta T_1 = I_N$ | $K_1 = K_4^* = K_N \in C_{N \times N}$ |
| (2)  | 1        | $T_1 = \pm T_4 = I_N$, $T_2 = T_3 = 0_N$ | $K_1 = K_N \in R_{N \times N}$, $K_4 = H_N \in R_{N \times N}$ |

and the envelope $|u|^2$ can be given by

$$|u|^2 = i\delta \left( \ln \frac{f}{f^*} \right)_t = 2\delta \left( \arctan \frac{\text{Re}[f]}{\text{Im}[f]} \right)_t. \quad (33)$$

Next, we give explicit expression of $\phi$. To achieve that, we assume both $T$ and $A$ are $2 \times 2$ block matrices

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \quad A = \begin{pmatrix} K_1 & 0_N \\ 0_N & K_4 \end{pmatrix}, \quad (34)$$

where $T_i$ and $K_i$ are $N \times N$ matrices. Equation (29) allows the following solutions (cf. Ref. 23) as we list in Table 1, where $|K_N||H_N| \neq 0$.

Let us introduce

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix}, \quad (35)$$

where $\phi^\pm = (\phi_1^\pm, \phi_2^\pm, \ldots, \phi_N^\pm)^T$. When $A$ takes the form in (34), the elementary vector $\phi$ given in (23a) can be written as in (35), where

$$\phi^+ = \exp \left[ \frac{i}{2}(K_1^2 x + K_1^{-2} t) \right] C^+, \quad \phi^- = \exp \left[ \frac{i}{2}(K_4^2 x + K_4^{-2} t) \right] C^-, \quad (36)$$

and $C^\pm = (c_1^\pm, c_2^\pm, \ldots, c_N^\pm)^T$. Vector $\psi$ is defined by (31).

**Solutions corresponding to Case (1) in Table 1:**

More explicitly, when $K_N$ is a diagonal matrix

$$K_N = \text{Diag}(k_1, k_2, \ldots, k_N), \quad k_j \in C, \quad (37)$$

we have

$$\phi^+ = (c_1^+ e^{\eta(k_1)}, c_2^+ e^{\eta(k_2)}, \ldots, c_N^+ e^{\eta(k_N)})^T, \quad (38a)$$

$$\phi^- = (c_1^- e^{\eta(k_1)}, c_2^- e^{\eta(k_2)}, \ldots, c_N^- e^{\eta(k_N)})^T, \quad (38b)$$

where

$$\eta(k) = \frac{i}{2}(k^2 x + k^{-2} t). \quad (39)$$
When $K_N$ is a Jordan block matrix $J_N(k)$,

$$J_N(k) = \begin{pmatrix} k & 0 & \cdots & 0 \\ 1 & k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & k \end{pmatrix} \in \mathbb{R}^{N\times N},$$

we have

$$\phi^+ = A_N \left( c^+ e^{\eta(k)}, \partial_k(c^+ e^{\eta(k)}), \frac{1}{2!} \partial_k^2 (c^+ e^{\eta(k)}), \ldots, \frac{1}{(N-1)!} \partial_k^{N-1} (c^+ e^{\eta(k)}) \right)^T,$$

$$\phi^- = B_N \left( c^- e^{\eta(k^*)}, \partial_{k^*}(c^- e^{\eta(k^*)}), \frac{1}{2!} \partial_{k^*}^2 (c^- e^{\eta(k^*)}), \ldots, \frac{1}{(N-1)!} \partial_{k^*}^{N-1} (c^- e^{\eta(k^*)}) \right)^T,$$

where $\partial_k = \frac{\partial}{\partial k}, k, c^\pm \in \mathbb{C}, A_N$ and $B_N$ belong to an Abelian group $G_N$, which is composed by all invertible lower triangular Toeplitz matrices (LTTMs) of the following form:

$$G_N = \begin{pmatrix} g_0 & 0 & 0 & \cdots & 0 \\ g_1 & g_0 & 0 & \cdots & 0 \\ g_2 & g_1 & g_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{N-1} & g_{N-2} & g_{N-3} & \cdots & g_0 \end{pmatrix}, \quad g_i \in \mathbb{C}, \quad g_0 \neq 0.$$

Note that the LTTMs have been widely used in presenting multiple pole solutions (cf. Refs. 21, 22, 47). From (31) one can find that $\psi$ always takes a form

$$\psi = \begin{pmatrix} K_N \phi^- \\ \delta K_N^* \phi^+ \end{pmatrix}.$$

**Solutions corresponding to Case (2) in Table 1**: In this case, both $K_N$ and $H_N$ are real. $\phi^+$ in (35) is governed by $K_N$. When $K_N$ is diagonal, i.e.,

$$K_N = D[k_j]_{j=1}^N = \text{Diag}(k_1, k_2, \ldots, k_N), \quad k_j \in \mathbb{R},$$

one has

$$\phi^+ = (c^+_1 e^{\eta(k_1)}, c^+_2 e^{\eta(k_2)}, \ldots, c^+_N e^{\eta(k_N)})^T,$$

where $\eta$ is defined by (39), and we note that $c^+_j \in \mathbb{C}$. When $K_N$ is a Jordan block matrix $K_N = J_N(k)$ as given in (42) where $k \in \mathbb{R}$, one has

$$\phi^+ = A_N \left( c^+ e^{\eta(k)}, \partial_k(c^+ e^{\eta(k)}), \frac{1}{2!} \partial_k^2 (c^+ e^{\eta(k)}), \ldots, \frac{1}{(N-1)!} \partial_k^{N-1} (c^+ e^{\eta(k)}) \right)^T,$$
where $\mathcal{A}_N$ is a real element in $G_N$ but $c^+$ is complex. $\phi^-$ in (35) is determined by $H_N$. When $H_N$ is a diagonal matrix

$$H_N = D[h_j]_{j=1}^N = \text{Diag}(h_1, h_2, \ldots, h_N), \quad h_j \in \mathbb{R},$$

one has

$$\phi^- = (c_1^+e^{\eta(h_1)}, c_2^+e^{\eta(h_2)}, \ldots, c_N^+e^{\eta(h_N)})^T,$$

where $\eta$ is defined by (39) and $c_j^+ \in \mathbb{C}$. In Jordan block case when $H_N = J_N(h)$ as given in (42) where $h \in \mathbb{R}$, one has

$$\phi^- = B_N \left( c^{-e^{\eta(h)}, \partial h(c^{-e^{\eta(h)}}, \frac{1}{2!}\partial^2 h(c^{-e^{\eta(h)}}, \ldots, \frac{1}{(N-1)!}\partial^{N-1} h(c^{-e^{\eta(h)}})) \right)^T,$$

where $B_N$ is a real element in $G_N$ but $c^-$ is complex. In this case, $\phi$ takes the form (35) where $\phi^+$ can be either (45) or (46) and $\phi^-$ can be either (48) or (49), and consequently $\psi$ takes a form

$$\psi = \begin{pmatrix} K_N \phi^+ \cr -H_N \phi^- \end{pmatrix}.$$  

(50)

Note that, for the above both cases, because (29) is bilinear w.r.t both $A$ and $T$, from Table 1, when $\delta = 1$, one can combine the above cases and get mixed solutions. In detail, when $\delta = 1$, Equation (29) allows a more general solution

$$T = \begin{pmatrix} I_{N_1} & 0_{N_1} \\ 0_{N_1} & -I_{N_1} \\ 0_{N_2} & I_{N_2} \\ I_{N_2} & 0_{N_2} \end{pmatrix}, \quad A = \begin{pmatrix} K'_{N_1} & 0_{N_1} \\ 0_{N_1} & H'_{N_1} \\ K_{N_2} & 0_{N_2} \\ 0_{N_2} & K^*_{N_2} \end{pmatrix},$$  

(51)

where $K'_{N_1}, H'_{N_1} \in \mathbb{R}^{N_1 \times N_1}$, $K_{N_2} \in \mathbb{C}^{N_2 \times N_2}$, $N_1 + N_2 = N$. Obviously, explicit expression for $\phi$ of this case can be easily composed accordingly by referring to the above two cases.

Dynamics of some obtained solutions will be investigated in Section 4.
3.2.2 Case of the nonlocal FL equation

The nonlocal relation

\[ v(-x,-t) = \delta u(x,t), \quad \delta = \pm 1 \]  \( (52) \)

reduces the pKN\((-1)\) system (3) to a one-field equation, the nonlocal FL equation (4). In the following, from (18) and (21), we recover the above nonlocal relation and get solutions to the nonlocal FL equation (4).

Let us consider \( M = N \) and impose constraint on (23) by

\[ \psi(x,t) = S \phi(-x,-t). \]  \( (53) \)

This holds if

\[ A^2 = \delta S^2, \]  \( (54) \)

and \( B = SC \). Note that (54) indicates \( A^2 S = S A^2 \). Next, for convenience we introduce a notation (cf. Refs. 23, 43)

\[ \hat{\phi}^{[N]}(ax, bt)_{\text{ex}} = (\phi(ax, bt), \partial_{cx} \phi(ax, bt), \partial_{cx}^2 \phi(ax, bt), \ldots, \partial_{cx}^N \phi(ax, bt)). \]  \( (55) \)

Thus, the double Wronskians (21) with the constraint (53) are written as

\[ f(x,t) = |\tilde{N}; N - 1| = \left( i \right)^N \left| A^2 \hat{\phi}^{[N-1]}(x,t)_{[x]}; S \hat{\phi}^{[N-1]}(-x,-t)_{[x]} \right|, \]  \( (56a) \)

\[ g(x,t) = |\tilde{N}; N - 1| = \left( -i \right)^{N-1} \left| \hat{\phi}^{[N]}(x,t)_{[x]}; A^2 S \hat{\phi}^{[N-2]}(-x,-t)_{[x]} \right|, \]  \( (56b) \)

\[ h(x,t) = -i \left| \tilde{N}; \tilde{N} \right| = -\left( i \right)^{2N-1} \left| A^4 \hat{\phi}^{[N-2]}(x,t)_{[x]}; S \hat{\phi}^{[N]}(-x,-t)_{[x]} \right|, \]  \( (56c) \)

\[ s(x,t) = |\tilde{N}; \tilde{N}| = \left( i \right)^{2N} (-1)^N \left| A^2 |\hat{\phi}^{[N-1]}(x,t)_{[x]}; S \hat{\phi}^{[N-1]}(-x,-t)_{[x]} \right|. \]  \( (56d) \)

Then we find that

\[ f(-x,-t) = \left( i \right)^N \left| A^2 \hat{\phi}^{[N-1]}(-x,-t)_{[-x]}; S \hat{\phi}^{[N-1]}(x,t)_{[-x]} \right| \]

\[ = \left( i \right)^N \left| A^2 \hat{\phi}^{[N-1]}(-x,-t)_{[x]}; S \hat{\phi}^{[N-1]}(x,t)_{[x]} \right| \]

\[ = \left( i \right)^N \left| S \hat{\phi}^{[N-1]}(x,t)_{[x]}; A^2 \hat{\phi}^{[N-1]}(-x,-t)_{[x]} \right| \]

\[ = \left( i \right)^N \left| S \hat{\phi}^{[N-1]}(x,t)_{[x]}; S^{-1} A^2 \hat{\phi}^{[N-1]}(-x,-t)_{[x]} \right|. \]
\[ \left( \frac{i}{2} \right)^N |S| |\hat{\phi}|^{N-1}(x, t) |\delta S |^{N-1}(-x, -t) |\]
\[ = (-2i)^N |\delta|^N |S|^{-1} s(x, t), \]

and in a similar way,
\[ g(-x, -t) = (-2i)^N |\delta|^N |S|^{-1} h(x, t). \]

These results immediately give rise to
\[ \frac{g(-x, -t)}{f(-x, -t)} = \delta \frac{h(x, t)}{s(x, t)}, \]

i.e., the relation (52).

Introduce \( S = AT \) and to keep (54) we assume that
\[ AT = TA, \quad T^2 = \delta I_{2N}. \]

Then, the solutions of the nonlocal case are summarized as the following.

**Theorem 3.** The nonlocal FL equation (4) admits solutions
\[ u(x, t) = \frac{|\hat{N}; \hat{N} - 1|}{|N; \hat{N} - 1|}, \]

where the elementary vector \( \phi \) is given by (23a) and
\[ \psi(x, t) = AT \phi(-x, -t), \]

and \( A, T \in \mathbb{C}_{2N \times 2N} \) are invertible and satisfy Equation (57). The double Wronskians (21) composed by the above \( \phi \) and \( \psi \) satisfy the following bilinear nonlocal FL equations:
\[ D_x D_t g(x, t) \cdot f(x, t) + g(x, t) f(x, t) = 0, \]
\[ D_x D_t f(x, t) \cdot f(-x, -t) + i \delta D_x g(x, t) \cdot g(-x, -t) = 0, \]
\[ D_t f(x, t) \cdot f(-x, -t) + i \delta g(x, t) g(-x, -t) = 0. \]

A special solution to (57) is given by block matrices form (34) with
\[ T_1 = -T_4 = \sqrt{\delta} I_N, \quad T_2 = T_3 = 0_N, \quad K_1 = K_N \in \mathbb{C}_{N \times N}, \quad K_4 = H_N \in \mathbb{C}_{N \times N}. \]

Explicit expression of \( \phi \) is given through the form (35) where \( \phi^\pm \) are given by those formulas from (44) to (49) but at this stage \( k, k_j, h, k_j \in \mathbb{C} \), and \( \psi \) is
\[ \psi = \begin{pmatrix} \sqrt{\delta} K_N \phi^+(-x, -t) \\ -\sqrt{\delta} H_N \phi^-(x, -t) \end{pmatrix}. \]
4 | DYNAMICS OF THE CLASSICAL FL EQUATION (2)

In this section we analyze dynamics of solutions of the FL equation (2), which we obtained in the previous section. We will investigate solutions related to discrete complex eigenvalues and also discrete real eigenvalues, i.e., Case (1) and Case (2) in Table 1. In the first case, one-soliton feature and two-soliton interactions were already considered in Ref. 16, so we will focus more on breathers and double pole solutions. The second case contributes solutions related to real discrete eigenvalues, which, to our knowledge, were not reported in the past literatures. These solutions allow periodic and double-periodic waves, and quite interestingly, solitary waves with algebraic decays as $|x| \to \infty$.

4.1 | Solutions related to complex eigenvalues

Let us consider Case (1) in Table 1 where we take $\delta = 1$. Note that in this case, when $K_N$ is diagonal one will obtain the usual $N$-soliton solutions, which coincide with those results that have been obtained from the inverse scattering transform (or Riemann–Hilbert method),$^{2,3,8,9}$ dressing method,$^{10}$ Darboux transformation,$^{13,14}$ with zero as a seed solution, and bilinear method by Matsuno,$^{16}$ including the solutions for the FL equation (1) in light of the transformation that converts Equation (1) to Equation (2) (see proposition 1 in Ref. 3). Apart from the diagonal $K_N$, when $K_N$ is a Jordan matrix or contains Jordan blocks, the solution, in principle, can be obtained by a limit procedure from soliton solutions (e.g., Refs. 47, 49).

4.1.1 | One-soliton solution (1SS)

When $K_N$ is given in (37) with $N = 1$, we get ISS

$$u = \frac{g}{f}, \quad f = |\phi_x; \psi|, \quad g = |\phi, \phi_x|,$$

which reads

$$u = \frac{c_1 d_1 (k_1^2 - k_1^{*2})}{|k_1|^2 \left[ k_1^* |d_1|^2 e^{-i(k_1^2 x + \frac{i}{k_1})} - k_1 |c_1|^2 e^{-i(k_1^* x + \frac{i}{k_1})} \right]},$$

where $c_1 = c_1^+, \quad d_1 = c_1^-$. The carrier wave is expressed as

$$|u|^2 = \frac{8a_1^2 b_1^2}{(a_1^2 + b_1^2)^3} \cosh \left( 4a_1 b_1 x - \frac{4a_1 b_1 t}{(a_1^2 + b_1^2)^{3/2}} + 2 \ln \frac{|d_1|}{|c_1|} \right) - \frac{a_1^2 - b_1^2}{a_1^2 + b_1^2}.$$
where we have taken $k_j = a_j + ib_j$, $a_j, b_j \in \mathbb{R}$. Equation (64) describes a single direction soliton traveling with amplitude $\frac{2|a_1|}{a_1^2 + b_1^2}$, initial phase $2 \ln \frac{|d_1|}{|c_1|}$, velocity $\frac{1}{(a_1^2 + b_1^2)^2}$, and trajectory (top trace) $x(t) = \frac{1}{(a_1^2 + b_1^2)^2} t - \frac{1}{2a_1 b_1} \ln \frac{|d_1|}{|c_1|}$.

Obviously, $a_1 b_1$ should not be zero, which means $k_1$ cannot be real or pure imaginary. This coincides with the assumption on the distribution of eigenvalues from scattering analysis (cf. Refs. 2, 8). Equation (64) is depicted in Figure 1(A).

### 4.1.2 Two-soliton solution (2SS)

2SS is obtained when $K_N$ is given in (37) with $N = 2$. It can be expressed as

$$u_{2SS} = \frac{g}{f},$$

with

$$f = |\partial_x \phi, \partial^2_x \phi; \psi, \partial_x \psi|, \quad g = |\phi, \partial_x \phi, \partial^2_x \phi; \partial_x \psi|,$$

where

$$\phi = (c_1 e^{\eta(k_1)}, c_2 e^{\eta(k_2)}, d_1 e^{\eta(k_1^*)}, d_2 e^{\eta(k_2^*)})^T, \quad \psi = (k_1 d_1^* e^{-\eta(k_1)}, k_2 d_2^* e^{-\eta(k_2)}, k_1^* c_1^* e^{-\eta(k_1^*)}, k_2^* c_2^* e^{-\eta(k_2^*)})^T,$$

$\eta$ is defined by (39), $k_j, c_j, d_j \in \mathbb{C}$. 2SS has been investigated in Ref. 16 where the solution is expressed in terms of determinants of Cauchy matrix type. We can conduct similar analysis on two-soliton interaction and present same results as in Ref. 16. For completeness of this paper, in the following we skip details but only sketch main results.

To analyze two-soliton interaction, we rewrite ISS (63) in the following form:

$$u_{1SS}[\xi_1; \lambda_1] = \frac{(k_1^2 - k_{1s}^2)y_1}{|k_1|^2(k_1^* - k_1|y_1|^2)},$$

FIGURE 1 Shape and motion of ISS and 2SS of the FL equation (2). (A) ISS given by (64) for $k_1 = 1 + 0.5i$, $c_1 = d_1 = 1$. (B) 2SS $|u|^2$ where $u$ is given in (68) with $k_1 = 1 + 0.5i$, $k_2 = 0.5 + 0.5i$, $c_1 = d_1 = c_2 = d_2 = 1$.
where
\[ y_j = e^{\xi_j + i\chi_j}, \quad \xi_j = -2a_j b_j(x_m t), \quad \chi_j = (a_j^2 - b_j^2)(x + m_j t), \quad m_j = \frac{1}{|k_j|^4}, \]

and we also assume \( a_j b_j > 0, c_j = d_j = 1 \) without loss of generality.

With these notations, the 2SS is written as
\[
u_{2SS} = \frac{g}{f}, \quad (68)
\]
where
\[
g = \frac{1}{|k_1 k_2|^2} \left[ k_1^2 (k_1^2 - k_2^2) (k_1^2 - k_2^2) (k_1^2 - k_2^2) (k_1^2 - k_2^2) y_1 y_2 + k_1^2 (k_1^2 - k_2^2) (k_1^2 - k_2^2) (k_1^2 - k_2^2) (k_1^2 - k_2^2) y_1^* y_2^* \right.
\[
-k_1^2 (k_1^2 - k_2^2) (k_1^2 - k_2^2) (k_1^2 - k_2^2) (k_1^2 - k_2^2) y_1^* y_2^* + k_1^2 (k_1^2 - k_2^2) (k_1^2 - k_2^2) (k_1^2 - k_2^2) (k_1^2 - k_2^2) y_1^* y_2^* \left. \right],
\]
\[
f = |k_1^2 - k_2^2|^2 \left[ k_1^2 y_1 y_2^* + k_1^2 y_2 y_1^* \right] - |k_1^2 - k_2^2|^2 \left[ k_1^2 y_1^* y_2 - k_2^2 y_1 y_2^* \right] + (k_1^2 - k_2^2) (k_1^2 - k_2^2) \left| k_1^2 y_1 y_2^* + k_2^2 y_1 y_2^* \right|
\]

Then we have (cf. Ref. 16), in the coordinate frame \((\xi_j, t)\),
\[
u_{2SS} \sim \nu_{1SS}[\xi_j + \Delta \xi^{(\pm)}_j, \chi_j + \Delta \chi^{(\pm)}_j], \quad t \to \pm \infty, \quad (j = 1, 2),
\]
where \( \nu_{1SS}[\xi_j; \chi_j] \) is given as \((67)\),
\[
\Delta \xi^{(\pm)}_1 = \pm \ln \left| \frac{k_1^2 - k_2^2}{k_1^2 - k_2^2} \right|, \quad \Delta \chi^{(\pm)}_1 = \pm \arg \left( \frac{k_1^2 - k_2^2}{k_1^2 - k_2^2} \right) \pm \arg \left( \frac{k_2^2}{k_1^2 - k_2^2} \right),
\]
\[
\Delta \xi^{(\pm)}_2 = \mp \ln \left| \frac{k_1^2 - k_2^2}{k_1^2 - k_2^2} \right|, \quad \Delta \chi^{(\pm)}_2 = \mp \arg \left( \frac{k_1^2 - k_2^2}{k_1^2 - k_2^2} \right) \mp \arg \left( \frac{k_1^2}{k_1^2 - k_2^2} \right).
\]

This indicates that, after interaction, the soliton described by \( k_j \) gets a phase shift
\[
2(-1)^{j-1} \left( \frac{\Delta \xi^{(\pm)}_j}{-2a_j b_j} + \frac{i \Delta \chi^{(\pm)}_j}{a_j^2 - b_j^2} \right)
\]

Such an interaction is depicted in Figure 1(B).
4.1.3 | Breathers

Note that velocity of a single soliton is governed by $1/|k_j|^4$. This means in 2SS when $|k_1| = |k_2|$ there will be two parallel solitons, while in this case periodic interactions, i.e., breathers, occur.

When $|k_1| = |k_2|$, the envelop of the 2SS (68) is

$$|u|^2 = \frac{G(x,t)}{2(a_1^2 + a_2^2)^2 F(x,t)}, \quad (69)$$

with

$$G(x,t) = 16 \{ a_2 b_2 \left[ Z_1 \left( X^2 - 4A_1 A_2 \right) - 2Z_2 X(A_1 + A_2) \right] \cosh 2Y_1 \cos(Y_3 - Y_4)$$

$$- a_2 b_2 \left[ Z_2 \left( X^2 - 4A_1 A_2 \right) + 2Z_1 X(A_1 + A_2) \right] \sinh 2Y_1 \sin(Y_3 - Y_4)$$

$$+ a_1 b_1 \left[ Z_3 \left( X^2 - 4A_1 A_2 \right) + 2Z_4 X(A_1 - A_2) \right] \cosh 2Y_2 \}^2$$

$$+ 16 \{ a_2 b_2 \left[ Z_2 \left( X^2 - 4A_1 A_2 \right) + 2Z_1 X(A_1 - A_2) \right] \sinh 2Y_1 \cos(Y_3 - Y_4)$$

$$+ a_2 b_2 \left[ Z_1 \left( X^2 - 4A_1 A_2 \right) - 2Z_2 X(A_1 + A_2) \right] \cosh 2Y_1 \sin(Y_3 - Y_4)$$

$$+ a_1 b_1 \left[ Z_4 \left( X^2 - 4A_1 A_2 \right) - 2Z_3 X(A_1 - A_2) \right] \sinh 2Y_2 \}^2,$$

$$F(x,t) = \left[ 16 a_1 a_2 b_1 b_2 |k_1|^2 \cos(Y_3 - Y_4)$$

$$- A_3 \left( X^2 - 4A_1^2 \right) \cosh 2(Y_1 - Y_2) + A_4 \left( X_1^2 + 4A_2^2 \right) \cosh 2(Y_1 + Y_2) \right]^2$$

$$+ \left[ A_5 \left( X^2 + 4A_1^2 \right) \sinh 2(Y_1 - Y_2) + A_6 \left( X^2 + 4A_2^2 \right) \sinh 2(Y_1 + Y_2) \right]^2,$$

where

$$X = a_1^2 - b_1^2 - a_2^2 + b_2^2, \quad \theta_1 = x - \frac{t}{(a_1^2 + b_1^2)^2}, \quad \theta_2 = x + \frac{t}{(a_1^2 + b_1^2)^2},$$

$$Y_1 = a_1 b_1 \theta_1, \quad Y_2 = a_2 b_2 \theta_1, \quad Y_3 = (a_1^2 - b_1^2) \theta_2, \quad Y_4 = (a_2^2 - b_2^2) \theta_2,$$

$$Z_1 = 3a_2^2 b_1 - b_1^3, \quad Z_2 = a_1^3 - 3a_1 b_1^2, \quad Z_3 = 3a_2^2 b_2 - b_2^3, \quad Z_4 = a_1^3 - 3a_2 b_2^2,$$

$$A_1 = a_1 b_1 + a_2 b_2, \quad A_2 = a_1 b_1 - a_2 b_2, \quad A_3 = a_1 a_2 + b_1 b_2, \quad A_4 = a_1 a_2 - b_1 b_2,$$

$$A_5 = a_1 b_2 + a_2 b_1, \quad A_6 = a_1 b_2 - a_2 b_1,$$

and we have taken $c_j = d_j = 1$ without loss of generality. In particular, on the line $x = t/|k_1|^4$, the value of $|u|^2$ is

$$|u|^2_{x=t/|k_1|^4} = \frac{G_1}{2(a_1^2 + a_2^2)^2 F_1}, \quad (71)$$

where

$$G_1 = 16a_1^2 b_2^2 \left[ Z_1 \left( X^2 - 4A_1 A_2 \right) - 2Z_2 X(A_1 + A_2) \right]^2 + 16a_1^2 b_1^2 \left[ Z_3 \left( X^2 + 4A_1 A_2 \right)$$

$$+ 2Z_4 X(A_1 - A_2) \right]^2 + 32a_1 b_1 a_2 b_2 \left[ Z_3 \left( X^2 + 4A_1 A_2 \right) + 2Z_4 X(A_1 - A_2) \right].$$
Breathers of the FL equation (2) \(|u|^2\) give by (69) with
\[ k_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad k_2 = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \quad c_1 = 0.5, \]
\[ d_1 = 1, \quad c_2 = 0.5 \text{ and } d_2 = 1. \]
(B) \(|u|^2\) give by (69) with \(k_1 = 1 + \frac{1}{2}i, \quad k_2 = \frac{1}{2} - i, \quad c_1 = 0.5, \quad d_1 = 1, \quad c_2 = 1, \) and \(d_2 = 1\).

\[
F_1 = \left[ 16a_1a_2b_1b_2|k_1|^2 \cos \frac{2Xt}{(a_1^2 + b_1^2)^2} - A_3 \left( X^2 + 4A_1^2 \right) + A_4 \left( X^2 + 4A_2^2 \right) \right]^2.
\]

Thus, it is obvious to see that the period of interaction is given by

\[
T = \frac{(a_1^2 + b_1^2)^2\pi}{(a_1^2 - b_1^2) - (a_2^2 - b_2^2)}.
\]

To summarize, we have the following,

**Proposition 2.** A breather from a 2SS occurs when \(k_1 \in \mathbb{C}, \ a_1b_1 \neq 0, \ |k_2| = |k_1|, \) but \((a_1^2 - a_2^2)(b_1^2 - b_2^2) \neq 0, \) i.e., \(k_2\) is not any reflection point of \(k_1\) w.r.t. \(x\)-axis, \(y\)-axis, or the origin; the breather travels along the line \(x = t/|k_1|^4\) with period \(T\) given in (72).

Figure 2(A) describes a breather coming from two solitons with the same initial phase, while Figure 2(B) describes a breather coming from two solitons with different initial phases.

### 4.1.4 Double-pole solutions

The simplest Jordan block solution is given through (66),

\[
u_2 = \frac{g}{f},
\]

with

\[
f = |\partial_x \phi, \partial_x^2 \phi; \psi, \partial_x \psi|, \quad g = |\phi, \partial_x \phi, \partial_x^2 \phi; \partial_x \psi|,
\]

but here \(\phi\) and \(\psi\) are taken as

\[
\phi = \left( c_1 e^{-\eta(k_1)}, c_1 k_1 e^{-\eta(k_1)}, d_1 e^{-\eta(k_1)}, d_1 k_1 e^{-\eta(k_1)} \right)^T,
\]

\[
\psi = \left( k_1 d_1^* e^{-\eta(k_1)}, d_1^* k_1 e^{-\eta(k_1)}, c_1 k_1^* e^{-\eta(k_1)}, e_1 e^{-\eta(k_1)} \right)^T.
\]
The corresponding envelope is

\[ |u_2|^2 = \frac{16a^2b^2G_2}{F_2}, \quad (75) \]

with

\[ G_2 = \left[ (B_1 + B_2)e^{2Y_1} - (B_1 - B_2)e^{-2Y_1} \right]^2 + \left[ (B_3 + B_4)e^{2Y_1} + (B_3 - B_4)e^{-2Y_1} \right]^2, \]

\[ F_2 = 16 \left[ B_5(e^{4Y_1} + e^{-4Y_1}) - 2(a^2 + b^2)(a^4 + b^4) - 16a^2b^2t^2 - 32a^2b^2(6a^2b^2 - a^4 - b^4)xt - 16a^2b^2(a^2 + b^2)^4x^2 \right]^2 + 16 \left\{ B_6(e^{4Y_1} - e^{-4Y_1}) + 16a^2b^2(a^2 - b^2)[t - (a^2 + b^2)x] \right\}^2, \]

where \( Y_1 \) is defined as in \((70)\),

\[ B_1 = 4a^3(a^2 + b^2), \quad B_2 = 8a^2b[t - (a^2 - 3b^3)(a^2 + b^2)x], \]

\[ B_3 = 4b^3(a^2 + b^2), \quad B_4 = 8ab^2[t + (3a^2 - b^2)(a^2 + b^2)x], \]

\[ B_5 = (a^2 - b^2)(a^2 + b^2)^2, \quad B_6 = 2ab(a^2 + b^2)^3, \]

and we have taken \( k_1 = a + ib, c_1 = d_1 = 1 \).

To understand asymptotic behavior of \(|u|^2\), we consider \(|u|^2\) in a coordinate frame \((z^{(+)}_{\pm}, t)\), where

\[ z^{(+)}_{\pm} = x - \frac{t}{(a^2 + b^2)^2} \pm \frac{2 \ln t + \gamma}{4ab}, \quad \gamma = \frac{\ln H}{2}, \quad (76a) \]

with

\[ H = \frac{2^{16}a^8b^8}{(a^2 + b^2)^8[(a^2 - b^2)^2 + 4a^2b^2(a^2 + b^2)]}. \quad (76b) \]

In this frame when \( t \to +\infty \), we get

\[ |u|^2 \to \frac{2a^2b^2}{(a^2 + b^2)^2 \sqrt{(a^2 - b^2)^2 + 4a^2b^2(a^2 + b^2)} \cosh z^{(+)}_{\pm} - a^4 + b^4}. \quad (77) \]

Similarly, in the coordinate \((z^{(-)}_{\pm}, t)\), where

\[ z^{(-)}_{\pm} = x - \frac{t}{(a^2 + b^2)^2} \pm \frac{2 \ln(-t) + \gamma}{4ab}, \quad (78) \]

when \( t \to -\infty \), we obtain

\[ |u|^2 \to \frac{2a^2b^2}{(a^2 + b^2)^2 \sqrt{(a^2 - b^2)^2 + 4a^2b^2(a^2 + b^2)} \cosh z^{(-)}_{\pm} - a^4 + b^4}. \quad (79) \]
FIGURE 3  Shape and motion of Jordan block solution to the FL equation (2). (A) Jordan block solution given by (75) with \(k = 1 + 0.5i, c = d = 1\). (B) Trajectories of the solution in (A).

The above asymptotic analysis indicates, as depicted in Figure 3, when \(|t|\) is large enough the wave will separate into two single solitons asymptotically traveling along the curves

\[
x(t) = \frac{t}{(a^2 + b^2)^2} \mp \frac{2 \ln |t| + \gamma}{4ab}.
\]  

Note that in Figure 3(B) we give a density plot of (A) as well as the curves given in (80), see the red curves. This also illustrates our asymptotic analysis.

4.2 Solutions related to real eigenvalues

Case (2) in Table 1 contributes solutions that are related to real discrete eigenvalues. Note that so far these type of solutions are not obtained in inverse scattering transform\(^2\) or Riemann–Hilbert approach,\(^8\) as eigenvalues in those two approaches do not locate on axes.

4.2.1 Periodic and double-periodic solutions

Consider \(K_N\) given in (44) and \(H_N\) given in (47), where \(k_j, h_j \in \mathbb{R}\). Note that \(c^\pm \in \mathbb{C}\). When \(N = 1\) we have

\[
u_{1SS} = \frac{c_1 d_1 (k_1^2 - h_1^2)}{k_1 h_1 \left[ c_1 d_1^* k_1 e^{-i(h_1^2 x + \frac{t}{h_1^2})} + c_1^* d_1 h_1 e^{-i(k_1^2 x + \frac{t}{k_1^2})} \right]},
\]  

and the corresponding envelop is

\[
|\nu_{1SS}|^2 = \frac{(k_1^2 - h_1^2)^2}{k_1^2 h_1^2 [k_1^2 + h_1^2 + 2k_1 h_1 \sin(\omega - \vartheta_1)]},
\]  

where

\[
\vartheta_1 = (k_1^2 - h_1^2) \left( x - \frac{t}{k_1^2 h_1^2} \right), \quad \omega = \arctan \frac{\text{Re}[c_1^2 d_1^* e^2]}{\text{Im}[c_1^2 d_1^* e^2]},
\]  

and
and \( c_1 = c_1^+ , d_1 = c_1^- \). We require \( k_1^2 \neq h_1^2 \), otherwise \( |u|^2 = 0 \). Equation (82) is a periodic wave characterized as the following,

\[
\begin{align*}
\text{top trajectories : } & x(t) = \frac{t}{k_1^2 h_1^2} + \frac{1}{k_1^2 - h_1^2} \arctan \left( \frac{\text{Re}[c_1^2 d_1^2]}{\text{Im}[c_1^2 d_1^2]} \right) + \frac{2\pi \kappa}{k_1^2 - h_1^2}, \quad \kappa \in \mathbb{Z}, \\
\text{amplitude : } & \frac{(k_1^2 - h_1^2)^2}{k_1^2 h_1^2 (k_1^2 + h_1^2 - 2|k_1 h_1|)}, \\
\text{velocity : } & \frac{1}{k_1^2 h_1^2}, \\
\text{period in } x, y \text{ direction : } & T_x = \frac{2\pi}{h_1^2 - k_1^2}, \quad T_y = \frac{2\pi k_1^2 h_1^2}{k_1^2 - h_1^2}, \\
\text{distance between two adjacent trajectories : } & T_d = \frac{2\pi k_1^2 h_1^2}{|k_1^2 - h_1^2| \sqrt{k_1^4 h_1^4 + 1}}.
\end{align*}
\]

The wave is depicted in Figure 4.

In 2SS case, i.e., \( N = 2 \) in (44) and (47), 2SS is given by (66) where

\[
\phi = (c_1 e^{\eta(k_1)}, c_2 e^{\eta(k_2)}, d_1 e^{\eta(h_1)}, d_2 e^{\eta(h_2)})^T, \tag{84a}
\]

\[
\psi = (k_1 c_1^n e^{-\eta(k_1)}, k_2 c_2^n e^{-\eta(k_2)}, -h_1 d_1^n e^{-\eta(h_1)}, -h_2 d_2^n e^{-\eta(h_2)})^T, \tag{84b}
\]

\( \eta \) is defined by (39), \( k_j, h_j \in \mathbb{R} \) and \( c_j, d_j \in \mathbb{C} \).

In this case, \( |u_{2SS}|^2 \) exhibit double-periodic interactions, as illustrated in Figure 5. This is not surprised from the periodic feature of 1SS.
FIGURE 5  Double-periodic solution of FL equation (2) in Case (2).
(A) Envelop $|u|^2$ of 2SS given by (66) with (84) where $k_1 = 2, h_1 = -1.8, k_2 = 1, h_2 = 0.5, c_1 = d_1 = c_2 = d_2 = 1$. (B) Density plot of (a)

FIGURE 6  Jordan block solution of the FL equation (2). (A) $|u|^2$ given by (86) with $k_1 = -1.8, h_1 = 0.8$ and $c_1 = d_1 = 1$. (B) Density plot of (a) overlapped by the lines $X_1 = 0$ and $X_2 = 0$. (C) 2D plot of (a) at $t = 3$. (d) An enlarged plot of (a) at $t = 3$

4.2.2  Solitary waves with algebraic decay

Although when $\mathbf{K}_N$ and $\mathbf{H}_N$ are diagonal with distinct diagonal elements, solutions exhibit (multi)periodic interaction behavior, in resonant case, for example, both $\mathbf{K}_N$ and $\mathbf{H}_N$ are Jordan blocks with $N = 2$, the resonance leads to algebraic decay wave asymptotically, without periodic interaction. See Figure 6 as an example.

Let us consider both $\mathbf{K}_N$ and $\mathbf{H}_N$ to be 2 by 2 Jordan blocks

$$
\mathbf{K}_2 = \begin{pmatrix} k_1 & 0 \\ 1 & k_1 \end{pmatrix}, \quad \mathbf{H}_2 = \begin{pmatrix} h_1 & 0 \\ 1 & h_1 \end{pmatrix}, \quad k_1, h_1 \in \mathbb{R}. \quad (85)
$$

Solution is given by (66) but, from (50)

$$
\phi = (c_1 e^{\eta(k_1)}, c_1 \partial_{k_1} e^{\eta(k_1)}, d_1 e^{\eta(h_1)}, d_1 \partial_{h_1} e^{\eta(h_1)})^T,
$$

$$
\psi = (k_1 c_1^* e^{-\eta(k_1)}, c_1^* e^{-\eta(k_1)} + c_1 \partial_{k_1} e^{-\eta(k_1)}, -h_1 d_1^* e^{-\eta(h_1)}, -d_1 e^{-\eta(h_1)} - d_1^* h_1 \partial_{h_1} e^{-\eta(h_1)})^T,
$$
where \( \eta \) is defined by (39), \( k_1, h_1 \in \mathbb{R} \) and \( c_1, d_1 \in \mathbb{C} \). Envelope is

\[
|u_2(x, t)|^2 = \frac{4(h_1^2 - k_1^2)^2 G_2}{F_2},
\]

where

\[
G_2(x, t) = M_1^2 + M_2^2 + M_3^2 + M_4^2 + 2(M_1 M_3 + M_2 M_4) \cos \theta + 2(M_1 M_4 - M_2 M_3) \sin \theta,
\]

\[
F_2(x, t) = N_1^2 + N_2^2 + N_3^2 + N_4^2 + 2N_1 N_2 \cos 2\theta - 2N_3 (N_1 - N_2) \sin \theta + 2N_4 (N_1 + N_2) \cos \theta,
\]

with \( \theta \) defined in (83),

\[
M_1 = -2k_1(k_1^2 - h_1^2)(h_1^4 x - t), \quad M_2 = -k_1 h_1^2(3k_1^2 + h_1^2),
\]

\[
M_3 = 2h_1(k_1^2 - h_1^2)(k_1^4 x - t), \quad M_4 = -k_1^2 h_1(3h_1^2 + k_1^2),
\]

\[
N_1 = -4h_1^2 k_1^2, \quad N_2 = -4h_1^2 k_1^2, \quad N_3 = -2(k_1^2 - h_1^2)(k_1^4 - h_1^4)(h_1^2 k_1^2 x - t),
\]

\[
N_4 = -h_1^2 k_1^2(k_1^4 + 6h_1^2 k_1^2 + k_1^4) + 4(k_1^2 - h_1^2)^2(t - k_1^4 x)(t - h_1^4 x),
\]

and we have taken \( c_1 = d_1 = 1 \).

Obviously, when both \(|x|\) and \(|t|\) go to infinity, \(|u|^2\) is dominated by

\[
|u|^2 \sim \frac{M_1^2 + M_3^2}{N_3^2 + N_4^2}.
\]

Thus, we consider the above \(|u|^2\) in the coordinate frame \((X_1, t)\) and \((X_2, t)\), respectively, where

\[
X_1 = x - \frac{t}{k_1^4}, \quad X_2 = x - \frac{t}{h_1^4}.
\]

After taking \( t \to \pm \infty \), we arrive at the following.

**Proposition 3.** Asymptotically, \(|u|^2\) given in (86) obeys

\[
|u|^2 \sim \frac{4}{k_1^2(1 + 4k_1^4 X_1^2)}
\]

in \((X_1, t)\), and

\[
|u|^2 \sim \frac{4}{h_1^2(1 + 4h_1^4 X_2^2)}
\]

in \((X_2, t)\), where \( X_j \) are given in (87).

This indicates that, when \( t \) is large enough, \(|u|^2\) are two algebraic decayed waves, with amplitudes \(4/k_1^2\) and \(4/h_1^2\), respectively, as depicted in Figure 6(A). When \( t \) is not large, periodic effect
FIGURE 7  The mixed solution of the FL equation (2) in Case (2). (A) $|u|^2$ given from (66) with (90) and $k_1 = -1$, $k_2 = 1.5$, $h_1 = 0.5$, $c_1 = d_1 = c_2 = d_2 = 1$. (B) Density plot of (a)

can still be observed, see Figure 6(D). It is worthy to mention that the above asymptotic analysis indicates that, asymptotically, there is no phase shift after interaction, which is different from normal soliton interactions. This is illustrated in Figure 6(B) where the density plot of (a) is overlapped by the lines $X_1 = 0$ and $X_2 = 0$.

One may also consider mixed solutions resulted from diagonal $K_N$ and Jordan block $H_N$. When $N = 2$, $u$ is given by (66), where

$$
\phi = \begin{pmatrix} c_1 e^{\eta(k_1)}, c_2 e^{\eta(k_2)}, d_1 e^{\eta(h_1)}, d_1 \delta h_1 e^{\eta(h_1)} \end{pmatrix}^T, \quad \psi = \begin{pmatrix} c_1 k_1 e^{\eta(k_1)}, c_2 k_2 e^{\eta(k_2)}, -d_1^* h_1 e^{\eta(h_1)}, -d_1^* e^{\eta(h_1)} - d_1^* \delta h_1 e^{\eta(h_1)} \end{pmatrix}^T,
$$

and $k_j, h_j \in \mathbb{R}$, $c_j, d_j \in \mathbb{C}$. $|u|^2$ is illustrated in Figure 7, from which we can see one solitary wave is interacting with a periodic wave.

5  |  DYNAMICS OF THE NONLOCAL FL EQUATION (4)

In nonlocal case, $K_N$ and $H_N$ are complex matrices, $\phi$ is given through (35) as described in Section 3.2.2 and $\psi$ takes the form (61). In the following we will mainly investigate 1SS with details, while for 2SS we only list our formulas with figures as illustrations. We only consider the case $\delta = 1$. Besides, note that when $K_N, H_N \in \mathbb{R}_N$, it is possible for $\psi$ (61) to take the same form as the $\psi$ (50), and then the FL equation (2) and nonlocal FL equation (4) share the corresponding solutions.

5.1  |  1SS

1SS of the nonlocal FL equation (4) is given by (62) where

$$
\phi = \begin{pmatrix} c_1 e^{\eta(k_1)}, d_1 e^{\eta(h_1)} \end{pmatrix}^T,
$$

$$
\psi = \begin{pmatrix} c_1 k_1 e^{\eta(k_1)}, -d_1 h_1 e^{\eta(h_1)} \end{pmatrix}^T,
$$
where \( \eta(k) \) is given in (39), \( k_1, h_1, c_1, d_1 \in \mathbb{C} \). The explicit formula is

\[
\begin{align*}
    u_{1,SS} &= \frac{k_1^2 - h_1^2}{k_1 h_1} \left[ k_1 e^{-i\left(h_1^2 x + \frac{t}{h_1}\right)} + h_1 e^{-i\left(k_1^2 x + \frac{t}{k_1}\right)} \right].
\end{align*}
\]

(91)

The corresponding envelop is

\[
|u_{1,SS}|^2 = \frac{(a_1^2 - b_1^2 - m_1^2 + s_1^2)^2 + 4(a_1 b_1 - m_1 s_1)^2}{2|k_1|^3|h_1|^3 e^{2W_1}} \left[ \cosh \left( 2W_2 + \ln \frac{|h_1|}{|k_1|} \right) + \sin(W_3 + \omega_1) \right],
\]

(92)

where we have taken \( k_j = a_j + ib_j, h_j = m_j + is_j, c_1 = d_1 = 1, \) and

\[
\begin{align*}
    W_1 &= (a_1 b_1 + m_1 s_1) x - \left( \frac{a_1 b_1}{|k_1|^4} + \frac{m_1 s_1}{|h_1|^4} \right) t, \\
    W_2 &= (a_1 b_1 - m_1 s_1) x - \left( \frac{a_1 b_1}{|k_1|^4} - \frac{m_1 s_1}{|h_1|^4} \right) t, \\
    W_3 &= (a_1^2 - b_1^2 - m_1^2 + s_1^2) x + \left( \frac{a_1^2 - b_1^2}{|k_1|^4} - \frac{m_1^2 - s_1^2}{|h_1|^4} \right) t, \\
    \omega_1 &= \arctan \frac{a_1 m_1 + b_1 s_1}{a_1 s_1 - b_1 m_1}.
\end{align*}
\]

To understand dynamics of (92) in an analytic way, let us first investigate when \( W_j \) vanish for all \((x, t)\). It can be found that \( W_1 \equiv 0 \) if

\[
(a_1, b_1) = (\pm m_1, \mp s_1), \quad \text{or} \quad (a_1, b_1) = (\pm s_1, \mp m_1),
\]

(93a)

or

\[
a_1 b_1 = m_1 s_1 = 0, \quad \text{but} \quad |k_1||h_1| \neq 0;
\]

(93b)

\( W_2 \equiv 0 \) if (93b) holds, or

\[
(a_1, b_1) = (m_1, s_1), \quad \text{or} \quad (a_1, b_1) = (s_1, m_1);
\]

(94)

\( W_3 \equiv 0 \) if

\[
(a_1^2, b_1^2) = (m_1^2, s_1^2).
\]

(95)

With these in hand, we may obtain desirable solutions by arranging real parameters \( a, b, m, s \). For example, if we take \( a_1 = s_1 = 0 \) but \( |k_1| \neq |h_1| \neq 0 \) so that (93b) holds, we get \( W_1 = W_2 \equiv 0 \).
FIGURE 8  Shape and motion of 1SS of the nonlocal FL equation (4). (A) Periodic wave given by (96) with $k_1 = i$ and $h_1 = 1.5$. (B) Soliton given by (97) with $k_1 = 0.8 + 0.8i$ and $h_1 = 0.8 - 0.8i$

\[
|u|^2 = \frac{(b_1^2 + m_1^2)^2}{b_1^2 m_1 \left[ b_1^2 + m_1^2 - 2|b_1 m_1| \sin \left( \frac{(b_1^2 + m_1^2)(x + \frac{t}{b_1^2 m_1^2})}{2} \right) \right]},
\]

which is a nonsingular periodic wave (by virtue of $|k_1| \neq |h_1|$, i.e., $|k_1| \neq |h_1|$). This wave is depicted in Figure 8(A).

When $W_1 \equiv W_3 \equiv 0$ but $W_2 \neq 0$, which can hold by taking, e.g., $(a_1, b_1) = (m_1, -s_1)$, we get 1SS

\[
|u|^2 = \frac{8a_1^2 b_1^2}{(a_1^2 + b_1^2)^2 (\cosh 4a_1 b_1 W_2' + \sin \omega_1^')},
\]

where $W_2' = x - \frac{t}{(a_1^2 + b_1^2)^2}$ and $\omega_1' = \arctan \frac{b_1^2 - a_1^2}{2a_1 b_1}$, which is depicted in Figure 8(B).

There can have kink-type waves but always with singularities. Considering the case that there is only one number being zero among $(a_1, b_1, m_1, s_1)$, e.g., only $m_1 = 0$, i.e.,

\[
(a_1, b_1, m_1, s_1) = (a_1, b_1, 0, s_1), \quad \text{and} \quad a_1 b_1 s_1 \neq 0,
\]

we have

\[
W_1 = W_2 = a_1 b_1 \left( x - \frac{t}{|k_1|^4} \right),
\]

\[
W_3 = (a_1^2 - b_1^2 + s_1^2) x + \left( \frac{a_1^2 - b_1^2}{|k_1|^4} + \frac{1}{s_1^2} \right) t, \quad \omega_1 = \arctan \frac{b_1}{a_1}.
\]

It is easy to check that the slopes of lines $W_1 = 0$ and $W_3 = 0$ can never be same in light of (98). In this case, (92) turns out to be

\[
|u_{1SS}|^2 = \frac{(a_1^2 - b_1^2 + s_1^2)^2 + 4(a_1 b_1)^2}{|k_1|^2 |h_1|^2 (|h_1|^2 y^2 + |k_1|^2 + 2y |k_1| |h_1| \sin z)},
\]

where

\[
y = e^{2W_1}, \quad z = W_3 + \omega_1,
\]
FIGURE 9  Shape and motion of a kink-type wave of the nonlocal FL equation (4), given by (100) with $k_1 = 1 + i$ and $h_1 = i$

and $W_1, \omega_1$ take the forms in (99). This is a kink-type wave for any given $t$: when $x \to \pm \infty$, $|u|^2$ goes to zero on one side and $\frac{(a_1^2 - b_1^2 + s_1^2)^2 + 4(a_1 h_1)^2}{|k_1|^4 |h_1|^2}$ on the other side, or the other way around, depending on $\text{sgn}[a_1 b_1]$. However, there are infinitely many poles appearing at the intersections

$$\begin{cases} W_1 = \frac{1}{2} \ln \left| \frac{k_1}{h_1} \right|, \\ W_3 + \omega_1 = 2j\pi - \frac{\pi}{2}, \quad j \in \mathbb{Z}, \end{cases}$$

and all poles are located at the line $W_1 = \frac{1}{2} \ln \left| \frac{k_1}{h_1} \right|$. Such a solution is illustrated in Figure 9.

Note that some ISS of the nonlocal FL equation (4) have been explored in Ref. 50 using Darboux transformation.

5.2  2SS

The above analysis we have made for ISS is helpful to understand two-soliton interactions. In the following we only list out illustrations.

2SS is given via (66) where in nonlocal case, when both $K_N$ and $H_N$ are diagonals, we have

$$\phi = (c_1 e^{\eta(k_1)}, c_2 e^{\eta(k_2)}, d_1 e^{\eta(h_1)}, d_2 e^{\eta(h_2)})^T; \quad \psi = (k_1 c_1 e^{-\eta(k_1)}, k_2 c_2 e^{-\eta(k_2)}, -h_1 d_1 e^{-\eta(h_1)}, -h_2 d_2 e^{-\eta(h_2)})^T; \quad \text{(101a)}$$

when both $K_N$ and $H_N$ are Jordan blocks, we have

$$\phi = (c_1 e^{\eta(k_1)}, c_1 \partial_k e^{\eta(k_1)}, d_1 e^{\eta(h_1)}, d_1 \partial_h e^{\eta(h_1)})^T; \quad \psi = (k_1 c_1 e^{-\eta(k_1)}, c_1 e^{-\eta(k_1)} + c_1 k_1 \partial_k e^{-\eta(k_1)}, -h_1 d_1 e^{-\eta(h_1)}, -h_1 d_1 \partial_h e^{-\eta(h_1)})^T; \quad \text{(102a)}$$
and when $K_N$ is diagonal and $H_N$ is a Jordan block, we have

$$\phi = (c_1 e^{\eta(k_1)}, c_2 e^{\eta(k_2)}, d_1 e^{\eta(h_1)}, d_1 h_1 e^{\eta(h_1)})^T,$$

$$(103a)$$

$$\psi = (k_1 c_1 e^{-\eta(k_1)}, k_2 c_2 e^{-\eta(k_2)}, -h_1 d_1 e^{-\eta(h_1)}, -d_1 e^{-\eta(h_1)} - h_1 d_1 h_1 e^{-\eta(h_1)})^T.$$  

$$(103b)$$

Here $k_j, h_j, c_j, d_j \in \mathbb{C}$, and we have taken the LTTMs $A_N = B_N = I_N$.

Two-soliton interactions are illustrated in Figure 10 and Figure 11, from which we can see that, compared with classical case, the interactions of 2SS are more complicated in nonlocal case (see also the nonlocal Gross–Pitaevskii equation\(^5\)).

We also remark that, in principle, the $N$-soliton solutions obtained in this paper for the nonlocal FL equation coincide with those obtained from Darboux transformation\(^5\) with zero as a seed solution. However, as we have shown in this subsection, the independency of $K_N$ and $H_N$ allows more variety in multisoliton and multiple-pole solutions.

6 | CONCLUDING REMARKS

We have derived solutions for the classical FL equation (2) and nonlocal FL equation (4) from bilinear approach. We introduced new double Wronskian expressions (21) that are different from those of the AKNS hierarchy, the KN equation, and the Chen–Lee–Liu equation (cf. Refs. 23–25). The assumption (22) with a general $A$ for Wronskian entries and the reduction technique enable
us to have a full profile for the solutions of the FL equations. 1SS and 2SS were illustrated based on analysis in detail. It is notable that the FL equation (2) also allows solutions related to real discrete eigenvalues. They exhibit (multi)periodic behavior for distinct eigenvalues and algebraic decayed solitary waves for those eigenvalues with multiplicity two. The later case was not found before in the analytic approaches (e.g., Refs. 2, 8, 9) that are based on analyzing analytic domains of wave functions.

Before Fokas and Lenells, the FL equation (2) was already explored around 40 years ago (cf. Refs. 4, 7), as it could generate solutions to the massive Thirring model arose in relativistic quantum field theory. In Appendix A we will recall the links between the FL equation (2) and the massive Thirring model. As a result, all the solutions we obtained for the FL equation (2) can generate solutions to the massive Thirring model (see Theorem 4 in Appendix A).

In this paper solutions are presented in terms of (double) Wronskians. Solutions in this form and similar forms are usually obtained via Darboux transformations or bilinear method. Compared with other popular forms of \( N \)-soliton solutions (e.g., Hirota’s form using polynomials of exponential functions given in Appendix B and dressed Cauchy matrix form obtained in Refs. 10, 14, 16), by virtue of their special structure, Wronskian solutions have advantage in presenting limit solutions, i.e., multiple-pole solutions. In a bilinear approach, such solutions are alternatively obtained by taking, for example, \( A \) in (22) to be composed of Jordan blocks. One may refer to Ref. 49 (see p. 22) for the limit procedure and to Ref. 47 for the connections between the LTTMs and limit solutions. For the multiple-pole solutions in terms of dressed Cauchy matrix form, one may refer to Ref. 52 for the Riemann–Hilbert method and to Ref. 53 for the Cauchy matrix approach. In addition, by employing double Wronskians, coefficient matrix \( A \), and bilinear approach, we have also illustrated an effective reduction technique to obtain solutions for the reduced equations. In this technique, looking for vectors \( \phi \) and \( \psi \) such that \( u \) and \( v \) satisfy desired constraints when they are expressed in terms of double Wronskians \( f, g, h, \) and \( s \), is boiled down to solving the algebraic equation (29) and (57) for nonlocal case. This enables us to approach to new solutions that might be missed before, for example, Case (2) in Table 1 that corresponds to real eigenvalues and the mixed case related to (51).

Finally, as remarks we list several possible interesting questions arising from the current paper. The first is to reinvestigate the coupled KN equation (8) and the DNLS equation (9) using the double Wronskian structure given in (21). Because the couple system (3) is the potential form of the KN(-1) system (11), it is possible to get solutions for the coupled KN equation (8) from \( q = (g/f)_x, r = (h/s)_x \) after redefining \( \phi, \psi \) with the dispersion relation of (8). Similar treatment was done in Ref. 10. The second is to investigate the FL equations with nonzero backgrounds from bilinear approach. Note that the assumption (22) corresponds to \( q = r = 0 \) in the Lax pair. In addition, it would be interesting to reinvestigate possible analytic domains of wave functions for the Cauchy problem where \( |u(x, t = 0)|^2 \) is algebraic decayed as \( |x| \rightarrow \infty \). This was not touched in Refs. 2, 8, 9. Finally, solving the nonlocal FL equation from an analytic approach is also an interesting problem. Note that the analysis in Section 5.1 implies soliton solutions arise from the eigenvalue distribution \( H_N = K_N^* \) or \( H_N = -K_N^* \). However, interactions of 2SS exhibit more varieties in nonlocal case.

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APPENDIX A: THE MASSIVE THIRRING MODEL, KN SPECTRAL PROBLEM, AND THE FL EQUATION

The $pKN(-1)$ (3) is called the Mikhailov model by Gerdjikov and his collaborators.\textsuperscript{4,5} Its reduction gives rise to the FL equation (2). The latter provides solutions to the massive Thirring model, which describes the theory of a massive fermion field coupled to a two-component vector field interacting with itself via a Fermi interaction.\textsuperscript{54,55}

The two-dimensional massive Thirring model is\textsuperscript{54–56}

\[ (-i\partial_\mu \gamma^\mu + m)\mathcal{X} + g\gamma^\mu \mathcal{X}(\overline{\mathcal{X}} \gamma_\mu \mathcal{X}) = 0, \tag{A1} \]

where $m$ stands for mass, $g$ is a parameter, $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2)^T$, $\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\overline{\mathcal{X}} = \mathcal{X}^\dagger \gamma^0 = (\mathcal{X}_1^*, \mathcal{X}_2^*)\gamma^0$, $\gamma_\mu = (\gamma^\mu)^{-1}$, and the Einstein summation convention is used. Denoting $\partial_\mu = \delta_{x_\mu}$, the above equation is written as (with $m = 2, g = 1$)

\begin{align*}
- i(\partial_{x_0} + \partial_{x_1})\mathcal{X}_1 + 2\mathcal{X}_2 + 2|\mathcal{X}_2|^2\mathcal{X}_1 &= 0, \tag{A2a} \\
- i(\partial_{x_0} - \partial_{x_1})\mathcal{X}_2 + 2\mathcal{X}_1 + 2|\mathcal{X}_1|^2\mathcal{X}_2 &= 0. \tag{A2b}
\end{align*}

It is Mikhailov\textsuperscript{56} who first gave a Lax pair of the massive Thirring model, for (A2), which reads\textsuperscript{56,57}

\begin{align*}
\Phi_{x_0} &= M_0 \Phi, \quad M_0 = i \frac{1}{2} \begin{pmatrix} -|\mathcal{X}_1|^2 + |\mathcal{X}_2|^2 - \lambda^2 + \lambda^{-2} & 2\lambda \mathcal{X}_2^* - 2\lambda^{-1} \mathcal{X}_1^* \\ 2\lambda \mathcal{X}_1 - 2\lambda^{-1} \mathcal{X}_2 & |\mathcal{X}_1|^2 - |\mathcal{X}_2|^2 + \lambda^2 - \lambda^{-2} \end{pmatrix}, \tag{A3a} \\
\Phi_{x_1} &= M_1 \Phi, \quad M_1 = i \frac{1}{2} \begin{pmatrix} |\mathcal{X}_1|^2 + |\mathcal{X}_2|^2 - \lambda^2 - \lambda^{-2} & 2\lambda \mathcal{X}_2^* + 2\lambda^{-1} \mathcal{X}_1^* \\ 2\lambda \mathcal{X}_1 + 2\lambda^{-1} \mathcal{X}_2 & -|\mathcal{X}_1|^2 - |\mathcal{X}_2|^2 + \lambda^2 + \lambda^{-2} \end{pmatrix}. \tag{A3b}
\end{align*}

where $\lambda$ is a spectral parameter. In light-cone coordinates $(x, t) = (x_0 + x_1, x_0 - x_1)$, Equation (A2) and its Lax pair are written as

\begin{align*}
\mathcal{X}_{1,x} + i\mathcal{X}_2 + i|\mathcal{X}_2|^2\mathcal{X}_1 &= 0, \tag{A4a} \\
\mathcal{X}_{2,t} + i\mathcal{X}_1 + i|\mathcal{X}_1|^2\mathcal{X}_2 &= 0. \tag{A4b}
\end{align*}
\[ \Phi_x = M' \Phi, \quad M' = \frac{i}{2} \begin{pmatrix} -\lambda^2 + |\lambda_2|^2 & 2\lambda \lambda_2^n \\ 2\lambda \lambda_2 & \lambda^2 - |\lambda_2|^2 \end{pmatrix}, \quad (A5a) \]

\[ \Phi_t = N' \Phi, \quad N' = \frac{i}{2} \begin{pmatrix} |\lambda_1|^2 - \lambda^{-2} & 2\lambda^{-1} \lambda_1^n \\ 2\lambda^{-1} \lambda_1 & -|\lambda_1|^2 + \lambda^{-2} \end{pmatrix}, \quad (A5b) \]

where (A5a) is known as the spectral problem of the derivative Schrödinger equation of Chen–Lee–Liu’s version.\(^{38}\)

Introducing

\[ \Phi = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} \Phi, \quad q = \lambda_2^n e^{-i\beta}, \quad \beta = \int_{-\infty}^{x} |\lambda_2(y)|^2 dy, \]

and noting that \( \partial_x |\lambda_1|^2 = -\partial_t |\lambda_2|^2 \), one can prove that the Lax pair (A5) is gauge equivalent to\(^{6}\)

\[ \Phi_x = M \Phi, \quad M = \begin{pmatrix} -\frac{i}{2} \lambda^2 & i\lambda q \\ i\lambda q^* & \frac{i}{2} \lambda^2 \end{pmatrix}, \quad (A6a) \]

\[ \Phi_t = N \Phi, \quad N = \begin{pmatrix} i(|\lambda_1|^2 - \frac{1}{2} \lambda^{-2}) & i\lambda^{-1} \lambda_1^n e^{-i\beta} \\ i\lambda^{-1} \lambda_1 e^{i\beta} & -i(|\lambda_1|^2 - \frac{1}{2} \lambda^{-2}) \end{pmatrix}, \quad (A6b) \]

where Equation (A6a) coincides with the KN spectral problem (5) with \( r = q^* \) and \( \lambda \to i\lambda \).

The FL equation (2) (with \( \delta = 1 \)) is alternatively written as (also see eq. (4.17) in Ref. 4)

\[ q_t + u - 2i|u|^2 q = 0, \quad u_x = q. \quad (A7) \]

By the transformation (cf. Refs. 4, 6)

\[ \lambda_2 = q^* e^{-i\beta}, \quad \lambda_1 = -iu^* e^{-i\beta}, \quad \beta = \int_{-\infty}^{x} |q|^2 dy, \quad q = u_x, \quad (A8) \]

and noticing that \( \partial_x |q|^2 = -\partial_t |u|^2 \), the complex conjugate of Equation (A7) gives rise to Equation (A4a), and Equation (A4b) holds automatically in light of (A8).

With regard to the solutions between the FL equation and the massive Thirring model, we have the following.

**Theorem 4.** If \( u(x, t) \) is a solution to the FL equation (2) with \( \delta = 1 \), then Equation (A8) provides solutions to the massive Thirring model (A4) in light-cone coordinates.
APPENDIX B: \( N \)-SOLITON SOLUTION IN HIROTA’S FORM

Employing the standard procedure of Hirota’s method, one can derive 1-, 2-, 3-soliton solutions for (19), which obey the following general form:

\[
\begin{align*}
  g_N(x, t) &= \sum_{\mu=0,1} A_2(\mu) \exp \left\{ \sum_{j=1}^{2N} \mu_j \zeta_j + \sum_{1 \leq j < s} \mu_j \mu_s \theta_{j,s} \right\}, \\
  f_N(x, t) &= \sum_{\mu=0,1} A_1(\mu) \exp \left\{ \sum_{j=1}^{2N} \mu_j \zeta_j'' + \sum_{1 \leq j < s} \mu_j \mu_s \theta_{j,s} \right\}, \\
  h_N(x, t) &= \sum_{\mu=0,1} A_3(\mu) \exp \left\{ \sum_{j=1}^{2N} \mu_j \eta_j' + \sum_{1 \leq j < s} \mu_j \mu_s \theta_{j,s} \right\}, \\
  s_N(x, t) &= \sum_{\mu=0,1} A_1(\mu) \exp \left\{ \sum_{j=1}^{2N} \mu_j \eta_j'' + \sum_{1 \leq j < s} \mu_j \mu_s \theta_{j,s} \right\},
\end{align*}
\]

where for \( j, s = 1, 2, \ldots, N \),

\[
\begin{align*}
  \zeta_j &= k_j x - \frac{1}{k_j} t + \zeta_j^{(0)}, \quad w_j = \frac{1}{k_j}, \quad \eta_j = -l_j x + \frac{1}{l_j} t + \eta_j^{(0)}, \quad m_j = -\frac{1}{l_j}, \\
  \zeta_j' &= \zeta_j, \quad \zeta_j^{N+j} = \eta_j + \ln l_j + \frac{\pi}{2} i, \quad \zeta_j'' = \zeta_j + \ln k_j + \frac{\pi}{2} i, \quad \zeta_j^{N+j} = \eta_j, \\
  \eta_j' &= \zeta_j + \ln k_j + \frac{\pi}{2} i, \quad \eta_j^{N+j} = \eta_j, \quad \eta_j'' = \eta_j + \ln l_j + \frac{\pi}{2} i, \quad \eta_j^{N+j} = \zeta_j, \\
  e^{\theta_{j,N+s}} &= \frac{1}{(k_j - l_s)(w_j + m_s)}, \quad (j, s = 1, 2, \ldots, N), \\
  e^{\theta_{j,s}} &= (k_j - k_s)(w_j - w_s), \quad (j < s = 2, 3, \ldots, N), \\
  e^{\theta_{N+j,N+s}} &= -(l_j - l_s)(m_j - m_s), \quad (j < s = 2, 3, \ldots, N),
\end{align*}
\]

\( k_j, l_j, \zeta_j^{(0)}, \eta_j^{(0)} \in \mathbb{C} \), and \( A_1(\mu), A_2(\mu), \) and \( A_3(\mu) \) take over all possible combinations of \( \mu_j = 0, 1 \) \( (j = 1, 2, \ldots, 2N) \) and meanwhile satisfy the constraints \( \sum_{j=1}^{N} \mu_j = \sum_{j=1}^{N} \mu_{N+j}, \sum_{j=1}^{N} \mu_j = 1 + \sum_{j=1}^{N} \mu_{N+j}, \) and \( 1 + \sum_{j=1}^{N} \mu_j = \sum_{j=1}^{N} \mu_{N+j} \), respectively.

Consider reduction

\[
\begin{align*}
  l_j &= -k_j^*, \quad m_j = w_j^*, \quad \eta_j^{(0)} = \zeta_j^{(0)*}, \\
  \zeta_j &= \eta_j^*, \quad e^{\theta_{j,N+s}} = e^{\theta_{j,N,j+s}^*}, \quad \text{and} \quad e^{\theta_{j,s}} = e^{\theta_{j,s}^*} \quad \text{and further} \quad s = f^*, h = g^*.
\end{align*}
\]

Thus, (B8) reduces the bilinear pKN(−1) (19) to a bilinear equation for the bilinear FL equation (32) with \( \delta = 1 \), and its solution is given by (B1a) and (B1b) with \( (j, s = 1, 2, \ldots, N) \),

\[
\begin{align*}
  \zeta_j &= k_j x - \frac{1}{k_j} t + \zeta_j^{(0)}, \quad w_j = \frac{1}{k_j}, \\
  \zeta_j' &= \zeta_j, \quad \zeta_j^{N+j} = \zeta_j + \ln (-k_j^*) + \frac{\pi}{2} i, \quad \zeta_j'' = \zeta_j + \ln k_j + \frac{\pi}{2} i, \quad \zeta_j^{N+j} = \zeta_j^*, \quad (B10)
\end{align*}
\]
\[
e^{\vartheta_{j,N+s}} = \frac{1}{(k_j + k_s^*)(w_j + w_s^*)}, \quad (j, s = 1, 2, \ldots, N), \tag{B11}
\]
\[
e^{\vartheta_{j,s}} = (k_j - k_s)(w_j - w_s), \quad (j < s = 2, 3, \ldots, N). \tag{B12}
\]

**APPENDIX C: PROOF OF THEOREM 1**

From the condition (22) one can calculate derivations of \(f, g, h,\) and \(s:\)

\[
f_x = |\overline{N - 1, N + 1; \overline{M - 1}| + |\overline{N}; \overline{M - 2, M}|, \]
\[
f_t = -\frac{1}{4}(|0, \overline{N}; \overline{M - 1}| + |\overline{N}; -1, \overline{M - 1}|), \]
\[
f_{xt} = -\frac{1}{4}(2|\overline{N}; \overline{M - 1}| + |0, \overline{N - 1, N + 1; \overline{M - 1}|} + |0, \overline{N}; \overline{M - 2, M}| \]
\[
+ |\overline{N - 1, N + 1; \overline{M - 1}|} + |\overline{N}; -1, \overline{M - 2, M}|), \]
\[
g_x = |\overline{N - 1, N + 1; \overline{M - 1}| + |\overline{N}; \overline{M - 2, M}|, \]
\[
g_t = -\frac{1}{4}(|-1, \overline{N}; \overline{M - 1}| + |\overline{N}; 0, \overline{M - 1}|), \]
\[
g_{xt} = -\frac{1}{4}(2|\overline{N}; \overline{M - 1}| + |-1, \overline{N - 1, N + 1; \overline{M - 1}|} + |-1, \overline{N}; \overline{M - 2, M}| \]
\[
+ |\overline{N - 1, N + 1; 0, \overline{M - 1}|} + |\overline{N}; 0, \overline{M - 2, M}|). \]

Substituting them into Equation (19a), the left-hand side gives rise to

\[
D_x D_t g \cdot f + gf = g_{xt} f - g_x f_t - g_t f_x + g f_{xt} + gf \]
\[
= -\frac{1}{4}|\overline{N}; \overline{M - 1}|(2|\overline{N}; \overline{M - 1}| + |-1, \overline{N - 1, N + 1; \overline{M - 1}|} + |-1, \overline{N}; \overline{M - 2, M}| \]
\[
+ |\overline{N - 1, N + 1; 0, \overline{M - 1}|} + |\overline{N}; 0, \overline{M - 2, M}|) + \frac{1}{4}(|0, \overline{N}; \overline{M - 1}| + |\overline{N}; -1, \overline{M - 1}|) \]
\[
(|\overline{N - 1, N + 1; \overline{M - 1}|} + |\overline{N}; \overline{M - 2, M}|) + \frac{1}{4}(|\overline{N - 1, N + 1; \overline{M - 1}|} + |\overline{N}; \overline{M - 2, M}|) \]
\[
+ |0, \overline{N}; \overline{M - 2, M}| + |\overline{N - 1, N + 1; -1, \overline{M - 1}|} + |\overline{N}; -1, \overline{M - 2, M}| + |\overline{N}; \overline{M - 1}|\overline{N - 1, \overline{M - 1}|}. \tag{C1}
\]

To simplify the right-hand side, we make use of Lemma 2. Consider \(\Xi = |\overline{N}; \overline{M - 1}|\) and \(\gamma_{ij} = \partial_{x^{-1}}^{-1}\) for \(j = 1, 2, \ldots, N\) and \(\gamma_{ij} = -\partial_{x^{-1}}^{-1}\) for \(j = N + 1, N + 2, \ldots, N + M.\) Using Lemma 2 and relation (22) we have

\[ -2i \text{Tr}(A^{-2})|\overline{N}; \overline{M - 1}| = |0, \overline{N}; \overline{M - 1}| - |\overline{N}; -1, \overline{M - 1}|, \]

where Tr(A) stands for the trace of matrix A. In a similar way, we have
\begin{align*}
-2i \text{Tr}(A^{-2})|\tilde{N};\tilde{M} - 1| &= | -1, \tilde{N};\tilde{M} - 1| - |\tilde{N}; 0,\tilde{M} - 1|, \\
-2i \text{Tr}(A^{-2})|\tilde{N} - 1, N + 1;\tilde{M} - 1| &= |0, \tilde{N} - 1, N + 1;\tilde{M} - 1| + |\tilde{N};\tilde{M} - 1| \\
-|\tilde{N} - 1, N + 1; -1,\tilde{M} - 1|, \\
-2i \text{Tr}(A^{-2})|\tilde{N};\tilde{M} - 2, M| &= |0, \tilde{N};\tilde{M} - 2, M| - |\tilde{N};\tilde{M} - 1| - |\tilde{N}; -1,\tilde{M} - 2, M|, \\
-2i \text{Tr}(A^{-2})|\tilde{N} - 1, N + 1; M - 1| &= | -1, \tilde{N} - 1, N + 1; M - 1| + |\tilde{N}; M - 1| \\
-|\tilde{N} - 1, N + 1; 0,\tilde{M} - 1|, \\
-2i \text{Tr}(A^{-2})|\tilde{N};\tilde{M} - 2, M| &= | -1, \tilde{N};\tilde{M} - 2, M| - |\tilde{N};\tilde{M} - 1| - |\tilde{N}; 0,\tilde{M} - 2, M|.
\end{align*}

From these relations we have

\begin{align*}
|\tilde{N};\tilde{M} - 1|(| -1, \tilde{N} - 1, N + 1;\tilde{M} - 1| + | -1, \tilde{N};\tilde{M} - 2, M| - |\tilde{N}; 0,\tilde{M} - 2, M| \\
-|\tilde{N} - 1, N + 1; 0,\tilde{M} - 1|) \\
= ([|\tilde{N} - 1, N + 1; M - 1| + |\tilde{N};\tilde{M} - 2, M|][|0, \tilde{N} - 1, N + 1;\tilde{M} - 1| - |\tilde{N}; -1,\tilde{M} - 1|), \\
|\tilde{N};\tilde{M} - 1|(|0, \tilde{N} - 1, N + 1;\tilde{M} - 1| + |0, \tilde{N};\tilde{M} - 2, M| - |\tilde{N} - 1, N + 1; -1,\tilde{M} - 1| \\
-|\tilde{N}; -1,\tilde{M} - 2, M|) \\
= ([|\tilde{N} - 1, N + 1;\tilde{M} - 1| + |\tilde{N};\tilde{M} - 2, M|][| -1, \tilde{N};\tilde{M} - 1| + |\tilde{N}; 0,\tilde{M} - 1|),
\end{align*}

by which we can reduce (C1) to

\begin{align*}
g_{xt}f - g_{x}f_{t} - g_{t}f_{x} + g f_{xt} + g f \\
= -\frac{1}{2} |\tilde{N};\tilde{M} - 1|( |\tilde{N} - 1, N + 1; 0,\tilde{M} - 1| + |\tilde{N}; 0,\tilde{M} - 2, M|) \\
+\frac{1}{2} |\tilde{N}; -1,\tilde{M} - 1|( |\tilde{N} - 1, N + 1;\tilde{M} - 1| + |\tilde{N};\tilde{M} - 2, M|) \\
+\frac{1}{2} |\tilde{N}; 0,\tilde{M} - 1|( |\tilde{N} - 1, N + 1;\tilde{M} - 1| + |\tilde{N};\tilde{M} - 2, M|) \\
-\frac{1}{2} |\tilde{N};\tilde{M} - 1|( |\tilde{N} - 1, N + 1; -1,\tilde{M} - 1| + |\tilde{N}; -1,\tilde{M} - 2, M|),
\end{align*}

in which some terms can vanish by using Lemma 1 and we then come to

\begin{align*}
g_{xt}f - g_{x}f_{t} - g_{t}f_{x} + g f_{xt} + g f \\
= \frac{1}{2} (-|\tilde{N} - 1; M - 1| |\tilde{N} + 1; 0,\tilde{M} - 1| + |\tilde{N};\tilde{M} - 2||\tilde{N}; 0,\tilde{M} | \\
+ |\tilde{N} + 1; M - 1| |\tilde{N} - 1; -1,\tilde{M} - 1| - |\tilde{N};\tilde{M}||\tilde{N}; -1,\tilde{M} - 2|).
\end{align*}
To show the right-hand side being zero, let us employ relation (22) to rewrite some double Wronskians as

\[ |\hat{N} - 1; \hat{M} - 1| = (-1)^M(-2i)^N|M|^{-2} |\hat{N}; \hat{M}|, \]
\[ |\hat{N}; \hat{M} - 2| = (-1)^{-M-1}(-2i)^{N+M}|A|^{-2}|\hat{N} + 1; \hat{M} - 1|, \]
\[ |\hat{N} - 1; -1, \hat{M} - 1| = (-1)^M(-2i)^{N+M}|A|^{-2}|\hat{N}; 0, \hat{M}|, \]
\[ |\hat{N}; -1, \hat{M} - 2| = (-1)^{-M-1}(-2i)^{N+M}|A|^{-2}|\hat{N} + 1; 0, \hat{M} - 1|. \]

Substituting them into (C4) we immediately find the right-hand side vanished. Thus, we have completed the proof for Equation (19a). Equations (19b), (19c), and (19d) can be proved similarly.