We establish both the lim sup and the lim inf law of the iterated logarithm (ilil), for the capacity of the range of a simple random walk in any dimension \(d \geq 3\). While for \(d \geq 4\), the order of growth in \(n\) of such ilil at dimension \(d\) matches that for the volume of the random walk range in dimension \(d - 2\), somewhat surprisingly this correspondence breaks down for the capacity of the range at \(d = 3\). We further establish such ilil for the Brownian capacity of a 3-dimensional Brownian sample path and novel, sharp moderate deviations bounds for the capacity of the range of a 4-dimensional simple random walk.

1. Introduction and Main Results. Let \(\tau_A\) denote the first positive hitting time of a finite set \(A\) by a simple random walk (srw) on \(\mathbb{Z}^d\), denoted hereafter \((S_m)_{m \geq 0}\). Recall that the corresponding (Newtonian) capacity is given for \(d \geq 3\), by

\[
\text{Cap}(A) := \sum_{x \in A} P^x(\tau_A = \infty) = \lim_{|z| \to \infty} \frac{P^0(\tau_A < \infty)}{G(0, z)}
\]

(where \(G(x, y)\) denotes the Green’s function of the walk). The asymptotics of the capacity \(R_n := \text{Cap}(\mathcal{R}_n)\) of the random walk range \(\mathcal{R}_n := \{S_1, \ldots, S_n\}\) is relatively trivial for \(d = 2\) (for then \(R_n = \frac{2\sqrt{n}d(1)}{\pi} \log (\text{diam} \mathcal{R}_n)\), see [26, Lemma 2.3.5]). In contrast, for \(d \geq 3\) such asymptotics is of an on going interest. Indeed, the strong law

\[
\lim_{n \to \infty} \frac{R_n}{n} = \alpha_d \quad \text{a.s., for all } d \geq 3,
\]

is an immediate consequence of the subadditive ergodic theorem, with \(\alpha_d > 0\) iff \(d \geq 5\) (as shown in [22]). Recall Green’s function for the \(d\)-dimensional Brownian motion

\[
G_B(x, y) := \int_0^\infty (2\pi t)^{-d/2} e^{-|x-y|^2/(2t)} dt = \begin{cases} \frac{1}{2\pi} |x-y|^{-1}, & d = 3, \\ \frac{1}{2\pi} |x-y|^{-2}, & d = 4, \end{cases}
\]

and the corresponding Brownian capacity of \(D \subset \mathbb{R}^d\),

\[
\text{Cap}_B(D)^{-1} := \inf \left\{ \int \int G_B(x, y) \mu(dx) \mu(dy) : \mu(D) = 1 \right\}.
\]

More recently, Chang [14] showed that for \(d = 3\),

\[
\frac{R_n}{\sqrt{n}} \overset{D}{\to} \frac{1}{3 \sqrt{3}} \text{Cap}_B(B[0, 1]),
\]

whereas Asselah et al [5] showed that in this case further, for some \(C \) finite,

\[
C^{-1} \sqrt{n} \leq E[R_n] \leq C \sqrt{n}.
\]
We denote throughout by $\overline{X}$ the centering $\overline{X} = X - E[X]$ of a generic random variable $X$. In higher dimensions $d \geq 4$, the centered capacity $\overline{R}_n$, converges after proper scaling to a non-degenerate limit law, which is Gaussian iff $d \geq 5$ (see [7] for $d = 4$ and [5, 36] for $d \geq 5$). For $d \geq 5$, estimates of the corresponding large and moderate deviations are provided in [2] (but they are not sharp enough to imply a LIL), while the central limit theorem (CLT) is further established in [18] for $R_n$ and a class of symmetric $\alpha$-stable walks, provided $d > 5\alpha/2$. We note in passing that similar questions for critical branching random walk on $\mathbb{Z}^d$ conditioned to have total population $n$ have also been studied in [8, 9, 10].

In view of these works, a natural question, which we fully resolve here, is to determine the almost sure fluctuations of $n \mapsto R_n$ for the srw, in the form of some LIL (possibly after centering $R_n$ when $d \geq 4$). Specifically, using hereafter $\log_k a = \log(\log_{k-1} a)$ for $k \geq 2$, with $\log_1 a$ for the usual logarithm, here is our first main result, about the srw in $\mathbb{Z}^3$.

**Theorem 1.1.** For $d = 3$, almost surely,

\begin{equation}
\limsup_{n \to \infty} \frac{R_n}{h_3(n)} = 1, \quad \liminf_{n \to \infty} \frac{R_n}{h_3(n)} = 1,
\end{equation}

where

\begin{equation}
h_3(n) := \frac{\sqrt{6\pi}}{9}(\log_3 n)^{-1} \sqrt{n \log_2 n}, \quad \hat{h}_3(n) := \frac{\sqrt{6\pi^2}}{9} \sqrt{n(\log_2 n)^{-1}}.
\end{equation}

Utilizing (3.21), we also get from Theorem 1.1 the following consequence about the Brownian capacity of the 3-dimensional (Brownian) sample path.

**Corollary 1.2.** For $d = 3$, almost surely,

\begin{equation}
\limsup_{n \to \infty} \frac{\text{Cap}_B(B[0,n])}{3 \sqrt{3}h_3(n)} = 1, \quad \liminf_{n \to \infty} \frac{\text{Cap}_B(B[0,n])}{3 \sqrt{3}h_3(n)} = 1.
\end{equation}

**Remark.** From the variational characterization of $\text{Cap}_B(B[0,n])$, with $\mu(\cdot)$ the pushforward of the uniform law on $[0,n]$ by the Brownian path $t \mapsto B_t$, we get that

$$
\frac{\pi n^2}{\text{Cap}_B(B[0,n])} \leq \int_0^n dt \int_0^t |B_t - B_s|^{-1} ds =: \eta([0,n]_{\infty}).
$$

It thus follows from the lim sup-LIL of [15, Thm. 1.2] for $\eta([0,n]_{\infty})$, that almost surely,

\begin{equation}
\liminf_{n \to \infty} \frac{\text{Cap}_B(B[0,n])}{3 \sqrt{3}h_3(n)} \geq \frac{3}{8 \sqrt{2}\pi \rho},
\end{equation}

where $\rho$ is given by [15, formula (1.15)] for $d = 3$, $\sigma = 1$, $\psi(\lambda) = \lambda^2/2$.

We next provide the LIL for the centered capacity $\overline{R}_n$ of the range, first in case of the srw on $\mathbb{Z}^4$ and then for srw on $\mathbb{Z}^d$, $d \geq 5$.

**Theorem 1.3.** For $d = 4$, almost surely,

\begin{equation}
\limsup_{n \to \infty} \frac{\overline{R}_n}{h_d(n)} = 1, \quad \liminf_{n \to \infty} \frac{\overline{R}_n}{h_d(n)} = -1,
\end{equation}

where for some non-random $0 < c_* < \infty$,

\begin{equation}
h_d(n) := \frac{\pi^2 n \log_3 n}{8 (\log n)^2}, \quad \hat{h}_d(n) := c_* \frac{n \log_2 n}{(\log n)^2},
\end{equation}

\begin{equation}
h_d(n) := \frac{\pi^2 n \log_3 n}{8 (\log n)^2}, \quad \hat{h}_d(n) := c_* \frac{n \log_2 n}{(\log n)^2}.
\end{equation}
Theorem 1.4. For any \( d \geq 5 \), the \( \text{LIL-ss} \) (1.4) hold almost surely, now with

\[
h_d(n) = \hat{h}_d(n) := \sigma_d \sqrt{2n(1 + 1_{[d=5]}) \log n} \log_2 n, \quad d \geq 5,
\]

where the non-random, finite \( \sigma_d^2 > 0 \) are given by the leading asymptotic of \( \text{var}(R_n) \) (c.f. [36, Thm. A] for \( \sigma_5 \) and [5, Thm. 1.1] for \( \sigma_d, \ d \geq 6 \).

Remark 1.5. Our proof of Theorem 1.4 via Skorokhod embedding, also yields Strassen’s \( \text{LIL} \) for the a.s. set of limit points in \( C([0, 1]) \) of the functions \( t \mapsto h_d(n)^{-1} \overline{R}_n \), for any \( d \geq 5 \).

Remark 1.6. The moment generating function of the limit in law of \( -((\log n)^2 / n) \overline{R}_n \) for the \( \text{srw} \) on \( \mathbb{Z}^d \), blows-up at a finite, positive \( \lambda \). The value of \( \lambda \) is identified in [15, Theorem 1.3]. In Lemma 4.6 we establish the uniform in \( n \) boundedness of the moment generating function of \( -((\log n)^2 / n) \overline{R}_n \) for a small enough argument.

We note that \( R_n \approx n \mathbb{E}[\hat{P}^{S^{\infty}}(\hat{\tau}_{\hat{R}_n} = \infty)] \) at any fixed \( n \gg 1 \) and \( d \geq 3 \), where \( \hat{P} \) and \( \hat{\tau}_A \) denote the law and the first hitting time by an i.i.d. copy of the \( \text{srw} \). Similarly, the volume of \( \hat{R}_n \) in any dimension \( (d \geq 1) \), is approximately \( n \mathbb{P}^{\tau_0 > n} \). It has been observed before, see for example [5, Sec. 6], that the typical order of growth of \( \mathbb{E}[\hat{P}^{S^{\infty}}(\hat{\tau}_{\hat{R}_n} = \infty)] \) at any \( d \geq 3 \), matches that of \( \mathbb{P}(\tau_0 > n) \) at \( d' = d - 2 \), yielding the same order of growth in \( n \) for \( R_n \) at \( d \geq 3 \) and for the volume of \( \hat{R}_n \) at \( d' = d - 2 \). In Theorems 1.3 and 1.4, our \( \text{LIL} \) for \( d \geq 4 \) adheres to such a match with the scale for the \( \text{LIL} \) of the volume of \( \hat{R}_n \) at \( d' = d - 2 \) (see [11, 12, 24] for the latter \( \text{LIL} \) at any \( d' \geq 2 \), as well as the limit distribution results for the volume of the Wiener sausage at \( d' \geq 2 \), and the corresponding \( \text{LIL} \) at \( d' \geq 3 \), in [29] and [17, 19], respectively). In contrast, this relation breaks down at the \( \text{lim sup} \) \( \text{LIL} \) for \( d = 3 \), with the appearance of the novel factor \( (\log_3 n)^{-1} \) in Theorem 1.1. Nevertheless, even at \( d = 3 \) the relevant deviations of \( R_n \) are due to those in the diameter of \( \hat{R}_n \), except that the upper tails for these two variables differ in their growth rates. Specifically, our proofs in Sections 3.1 and 3.3 yield the following (sharper) result.

Proposition 1.7. Let \( M_n := \max_{1 \leq i \leq n} |S_i| \). For \( \text{srw} \) of \( \mathbb{Z}^3 \) and any \( \epsilon > 0 \),

\[
P(R_n \geq (1 - \epsilon)h_3(n)) \cap \{M_n \geq (1 - \epsilon)\psi(n)\} \quad \text{i.o.} = 1,
\]

\[
P(R_n \leq (1 + \epsilon)\hat{h}_3(n)) \cap \{M_n \leq (1 + \epsilon)\hat{\psi}(n)\} \quad \text{i.o.} = 1,
\]

where \( \psi(n) := \sqrt{(2/3)n \log_2 n} \) and \( \hat{\psi}(n) := \pi \sqrt{(1/6)n(\log_2 n)^{-1}} \).

We note in passing that Proposition 1.7 is a rotation-invariant result, and in particular it applies also under any (fixed) rotation of the \( \text{srw} \) lattice \( \mathbb{Z}^3 \). Further, Proposition 1.7 implies that almost surely, the \( \text{lim sup} \) (resp. \( \text{lim inf} \)) of \( R_n \) are essentially attained simultaneously with those for \( M_n \), since for \( d = 3 \), almost surely,

\[
\limsup_{n \to \infty} \frac{M_n}{\psi(n)} = 1, \quad \liminf_{n \to \infty} \frac{M_n}{\hat{\psi}(n)} = 1.
\]

Indeed, by the invariance principle it suffices for proving (1.7) to show the equivalent a.s. statement for 3-dimensional Bessel process, and the latter follows by mimicking the proof of Chung’s one-dimensional \( \text{LIL} \) (see [16]), starting with the estimate (3.24). We note in passing that while the scaling \( \psi(n) \) of the upper fluctuation of \( M_n \) is the same as that for a single coordinate of our \( \text{srw} \), this is not true about the scaling \( \hat{\psi}(n) \), which is about the tail probability of confining the walk to stay within an Euclidean ball in \( \mathbb{R}^3 \).

In a followup work, [1, Corollary 1.2] determines the value of \( c_* \) of Theorem 1.3 in terms of the best constant in a generalized Gagliardo-Nirenberg inequality. In contrast, the following analog of Proposition 1.7 in case \( d \geq 4 \), is still open.
Open problem 1.8. Consider the srw $S_i = (S_i^1, \ldots, S_i^d) \in \mathbb{Z}^d$, $d \geq 4$. For $d' = d - 2$, let $\hat{S}_i = (\hat{S}_i^1, \ldots, \hat{S}_i^{d'}, 0, 0)$ and $V_i={(|\hat{S}_i^1|, \ldots, |\hat{S}_i^{d'}|)}$. Pick non-random $\psi_{d'}(n)$ such that $a.s.$

$$\limsup_{n \to \infty} \frac{V_i}{\psi_{d'}(n)} = 1.$$ 

We then conjecture that for $h_d(n)$ of Theorems 1.3-1.4 and any $\epsilon > 0$,

$$P(\{R_n \geq (1 - \epsilon)h_d(n)\} \cap \{V_i \geq (1 - \epsilon)\psi_{d'}(n)\} \ i.o.) = 1.$$ 

While we consider throughout only the discrete time srw whose increments are the $2d$ neighbors of the origin in $\mathbb{Z}^d$, due to sharp concentration of Poisson variables, all our results apply also for the continuous time srw with i.i.d. Exponential(1) clocks and up to the scaling $n \mapsto (1 - \rho)n$, also to the $\rho$-lazy discrete time srw. By definition of $R_n$, our results apply to any random walk on a group with a finite symmetric set of generators, whose words are isomorphic to those of the srw (e.g. an invariance of our results under any non-random, invertible affine transformation of the walk). We note in passing the recent work [32] on the strong law for any symmetric random walk on a group of growth index $d$ and the corresponding cfr in case $d \geq 6$, suggesting the possibility of a future extension of our $\liminf$-s in this context.

Beyond the intrinsic interest in $R_n$, its asymptotic is also relevant for the study of intersections between two independent random walks (e.g., see [26, Ch. 3]). Similarly, [3, 4] utilize bounds on $R_n$ to gain insights about the so called Swiss cheese picture for $d = 3$. Further, to understand Sznitman’s [37] random interlacement model, one may use moment estimates for the capacity of the union of ranges (c.f. [14] and the references therein). Finally, the capacity equals the summation of all entries of the inverse of the (positive definite) Green’s function matrix (see (2.2)), a point of view which [33] uses, for $d = 2$, to estimate the geometry of late points of the walk.

As for the organization of this paper, we prove Theorem 1.1 in Section 3, relying on certain relations between the capacity and Green’s function which we explore in Section 2. Our proof of Theorem 1.1 further indicates that the $\limsup$-$\liminf$ is due to exceptional time where $R_n$ has a cylinder-like shape, with one dimension being about $\psi(n)$ while the other two are $O(\psi(n)/(\log_2 n))$ (see Lemma 2.1 and Section 3.2). In contrast, the $\liminf$-$\liminf$ seems to be due to times where the shape of $R_n$ is close enough to a ball of radius $\hat{\psi}(n)$ to approximately match the capacity of such a ball (see (3.31) and (3.32)).

Sections 4 and 5 are devoted to the proofs of Theorems 1.3 and 1.4, respectively. Our proofs rely on the decomposition (4.2) of $R_n$, as the sum of $k$ independent variables $\{U_j\}$ which are the capacities of the walk restricted to the $k$ parts of a partition of $[1, n_k]$, minus some random $\Delta_{n,k} \geq 0$ (which ties all these parts together). For any $d \geq 5$ the effect of $\Delta_{n,k}$ on the $\liminf$ is negligible, so upon coupling $R_n$ with a one-dimensional Brownian motion, we immediately get the $\liminf$ for the former out of the standard $\liminf$ for the latter. As seen in Section 4, the situation is way more delicate for $d = 4$, where $E\Delta_{n,k} \approx h_4(n)$ dominates for a suitable slowly growing $k = k_n$ the fluctuations of the i.i.d. $\{U_j\}$. The $\limsup$-$\liminf$ is then due to the exceptional (random) sequence $\{n_k\}$ where $\Delta_{n,k} = o(h_4(n_k))$, while the $\liminf$-$\liminf$ is due to the exceptional $\{n_k\}$ for which $\Delta_{n,k} \approx h_4(n_k) \gg E\Delta_{n,k}$. Indeed, whereas Theorem 1.3 is proved via the framework developed in [12, Section 4] for the $\liminf$ for the volume of $R_n$ in the planar case ($d' = 2$), special care is needed here in order to establish tight control on the moderate deviations of $\Delta_{n,k}$ and $U_j$ in case $d = 4$ (c.f. Lemmas 4.1, 4.3, 4.6 and 4.7, which may be of independent interest).

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2. Capacity geometry and Green’s function. The following asymptotics for the 3-dimensional capacity of cylinder-like domains (which we prove at the end of this section), is behind the factor \((\log_2 n)^{-1}\) in the lim sup-LIL of (1.2).

**Lemma 2.1.** For \(m \geq 1\) and \(r \geq k \in \mathbb{N}\), let

\[
C_m(\ell, r) := (\mathbb{Z}^3)^3 \cap \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \leq r^2, 1 \leq x_3 \leq m\}.
\]

Fix \(b < 2/3\), \(r_m = o(m)\), \(r_m \uparrow \infty\). If \(C_m(1, r_m) \supseteq C_m \supseteq C_m(\ell, r_m)\) for some \(\ell \leq r_m^b\), then

\[
\lim_{m \to \infty} \frac{\text{Cap}(C_m)}{m(\log(m/r_m))^{-1}} = \frac{\pi}{3}.
\]

**Remark 2.2.** In the sequel we prove a stronger result, namely that the upper bound in (2.1) holds as soon as \(C_m\) is contained in a union \(\bigcup C_m^*(r_m)\) of at most \(m/r_m^b\) balls \(B(z_i, r_m)\) of radius \(r_m\), of centers such that \(|z_{i+1} - z_i| \leq r_m\) for \(1 \leq i < m/r_m\) (where \(C_m(1, r)\) is merely one possible choice for \(C_m^*(r)\)).

Indeed, in Section 3 we will see that lim sup of \(R_n\) is roughly attained on the event \(\{S_{n^i}^i \geq \psi(n)\} \) for \(\psi(n)\) of Proposition 1.7, with \(R_n\) then having approximately the shape of such \(C_m\) for \(m = \psi(n)\), and \(r_m = cm/\log_2 n\), hence from Lemma 2.1 we find that

\[
R_n \approx \text{Cap}(C_m) \approx \frac{\pi}{3} m \log(m/r_m) \approx \frac{\pi}{3} \psi(n) \log_3 n^{-1},
\]

which is precisely \(h_3(n)\) of Theorem 1.1.

We proceed with two lemmas relating the capacity of saw with its Green’s function,

\[
G(x, y) = \sum_{i=0}^{\infty} P^x(S_i = y).
\]

To this end, partition \(\Omega\) by the last time the walk visits \(X = \{x_1 \neq x_2 \cdots \neq x_j\}\), to get that

\[
1 = \sum_{\ell=1}^{j} G(x_i, x_\ell) P^x(\tau_X = \infty), \quad \forall 1 \leq i \leq j.
\]

**Lemma 2.3.** For any set \(X = \{x_1, \ldots, x_j\}\), with \(x_j\) not necessarily distinct,

\[
\max_{1 \leq \ell \leq j} \left\{ \sum_{i=1}^{j} G(x_i, x_\ell) \right\} \leq \text{Cap}(X) \leq \min_{1 \leq \ell \leq j} \left\{ \sum_{i=1}^{j} G(x_i, x_\ell) \right\}.
\]

**Proof.** The set \(X\) of size \(|X| = k \leq j\) consists wlog of distinct points \(\hat{X} = \{\hat{x}_1 \neq \hat{x}_2 \cdots \neq \hat{x}_k\}\), where \(\hat{x}_v\) appears \(m_v \geq 1\) times in \(X\) (and \(\sum_{v=1} m_v = j\)). Though (2.3) follows from the characterization of \(\text{Cap}(\hat{X})\) as in [23, Lemma 2.2(i)], we proceed instead with a direct, short and elementary proof of these bounds. Specifically, setting \(v(\ell)\) for the index of \(x_\ell\) in \(\hat{X}\) and \(q_v := (m_v)^{-1} P^x(\tau_{\hat{X}} = \infty)\), we see that

\[
\text{Cap}(X) = \text{Cap}(\hat{X}) = \sum_{v=1}^{k} P^x(\tau_{\hat{X}} = \infty) = \sum_{\ell=1}^{j} q_{v(\ell)}
\]

and moreover, summing (2.2) over \(i \leq j\), we get that

\[
j = \sum_{i=1}^{j} \sum_{v=1}^{k} G(x_i, \hat{x}_v) P^x(\tau_{\hat{X}} = \infty) = \sum_{\ell=1}^{j} q_{v(\ell)} \sum_{i=1}^{j} G(x_i, x_\ell).
\]

The bounds of (2.3) are an immediate consequence of (2.4) and (2.5).
Lemma 2.4. For $Z_1 = \{x_1, \ldots, x_j\}$, $Z_2 = \{x_{j+1}, \ldots, x_{j+j}\}$ with $x_i$ not necessarily distinct,

$$\text{Cap}(Z_1 \cup Z_2) \leq \text{Cap}(Z_2) + \frac{j_1 + j_2}{\min_{x \in Z_1 \setminus Z_2} \{\sum_{i=1}^{j_1+j_2} G(x_i, x)\}}.$$ \(\Box\)

Proof. Since $\tau_{Z_1 \cup Z_2} \leq \tau_{Z_2}$, it follows that

$$(2.6) \quad \text{Cap}(Z_1 \cup Z_2) \leq \text{Cap}(Z_2) + \sum_{x \in Z_1 \setminus Z_2} P_x(\tau_{Z_1 \cup Z_2} = \infty).$$

For $\mathcal{X}$ enumerating the distinct points in $Z_1 \cup Z_2$, $v(\ell)$, $q_v$ as in Lemma 2.3, we have that

$$\sum_{x \in Z_1 \setminus Z_2} P_x(\tau_{Z_1 \cup Z_2} = \infty) = \sum_{\ell=1}^{j_1+j_2} q_v(\ell) \mathbb{1}_{\{|x_i| \in Z_1 \setminus Z_2\}},$$

$$j_1 + j_2 = \sum_{\ell=1}^{j_1+j_2} q_v(\ell) \sum_{i=1}^{j_1+j_2} G(x_i, x_\ell).$$

Combining these identities with (2.6) yields the stated upper bound. \(\Box\)

Remark 2.5. In particular, applying Lemma 2.4 for

$$Z_1 = \bigcup_{i \in \{j,j-j\}} \hat{Z}_i, \quad Z_2 = \bigcup_{i \in \{1,\ldots,J-j\}} \hat{Z}_i,$$

we have that for any $\hat{Z}_i \subset \mathbb{R}^d$, $2j < J$,

$$\text{Cap}(Z_1 \cup Z_2) \leq \text{Cap}(Z_2) + \frac{\sum_{i=1}^{J} |\hat{Z}_i|}{\min_{x \in Z_1} \{\sum_{i=1}^{J} \sum_{y \in Z_2} G(x, y)\}}.$$ \(\Box\)

Proof of Lemma 2.1. By the monotonicity of $A \rightarrow \text{Cap}(A)$, it suffices to provide a uniform

in $\ell \leq r_m^{b}$ lower bound on $\text{Cap}(C_m(\ell, r_m))$ and a matching upper bound on $\text{Cap}(C_m^*(r_m))$, valid for any union $C_m^*(r_m)$ of at most $m/r_m$ balls $B(z_i, r_m)$ of radius $r_m$ in $\mathbb{Z}^3$ and centers such that $|z_{i+1} - z_i| \leq r_m$ for $1 \leq i < m/r_m$. With $|C_m(\ell, r_m)| = (1 + o(1))\pi mr_m^2 \ell^{-3}$, we get such a lower bound from Lemma 2.3, upon showing that for sw on $\mathbb{Z}^3$,

$$(2.7) \quad \sum_{y \in C_m(\ell, r_m)} G(x, y) \leq 3(1+o(1))r_m^2 \ell^{-3} \log(m/r_m), \quad \forall \ell \leq r_m^b, \forall x \in C_m(\ell, r_m).$$

Fixing $b < 2/3$, since the right-side of (2.7) diverges in $m_r$ uniformly in $\ell \leq r_m^{b}$, we can ignore any bounded contribution to its left-side. In particular, with $m/r_m \uparrow \infty$ and $G(x, y)$ bounded, it suffices to sum in (2.7) only over $y \in C_m(\ell, r_m)$ with $|x - y| \geq r_m^{2/3}$ and use the asymptotics

$$(2.8) \quad G(x, y) = \frac{3 + o(1)}{2\pi} \frac{1}{|x - y|}$$

(for example, see [26, Thm. 1.5.4]). Setting $u_m = m$, we have for any $v_m \uparrow \infty$ and $r \in \{r_m^{2/3}, v_mr_m\}$, at most $Cr^2 \ell^{-3}$ points $y \in C_m(\ell, r_m)$ with $|x - y| \in [r, r+1]$, while for each $r \in \{v_mr_m, u_m\}$ there are at most $(2\pi + o(1))r_m^2 \ell^{-3}$ such points in $C_m(\ell, r_m)$. Thus, taking $v_m^2 \ll \log(m/r_m)$ yields

$$\sum_{y \in C_m(\ell, r_m)} G(x, y) \leq \frac{3 + o(1)}{2\pi \ell^3} \left[C \int_{r_m^{2/3}}^{v_mr_m} r dr + 2\pi r_m^2 \int_{r_m^{2/3}}^{\min(r, v_mr_m)} r^{-1} dr\right]$$

$$= (3 + o(1))r_m^2 \ell^{-3} \log(u_m/(r_m v_m)), \quad (2.9)$$
from which (2.7) immediately follows. Turning to upper bound on $\text{Cap}(C^*_m(r_m))$, take now $u_m := (m/r_m)^{1-\epsilon_w}$ and $v_m := (m/r_m)^{\epsilon_w} \uparrow \infty$ for some $\epsilon_w \to 0$, splitting $C^*_m(r_m)$ to $Q_1 \cup Q_2$, where

$$Q_1 := \bigcup_{i \in (u_m, (m/r_m) - u_m)} \mathbb{B}(z_i, r_m), \quad Q_2 := \bigcup_{i \in (u_m, (m/r_m) - u_m)} \mathbb{B}(z_i, r_m).$$

Note that $C^*_m(r_m)$ has at most $(4\pi/3 + o(1))r^2_m$, possibly overlapping, points. Thus, combining Lemma 2.4 with the upper bound of Lemma 2.3, we get the upper bound of (2.1), once we show that for some $\delta_m \to 0$,

$$\sum_{y \in C^*_m(r_m)} G(x, y) \geq (4 + \delta_m)r^2_m \log(m/r_m), \quad \forall x \in Q_1,$$

$$\sum_{y \in Q_2} G(x, y) \geq \frac{|Q_2|}{m \delta_m} \log(m/r_m), \quad \forall x \in Q_2.$$  

Fixing $x \in \mathbb{B}(z_i, r_m) \subset Q_1$, consider only the contribution to the $\text{lhs}$ of (2.10) from all points $y \in \mathcal{B}(z_j, r_m)$ with $|j - i| \in [v_m, u_m]$. For such a pair $|y - x| \leq (|j - i| + 3)r_m$, hence by (2.8),

$$G(x, y) \geq \frac{3 + o(1)}{2\pi} r^{-1}_m |j - i|^{-1},$$

resulting with

$$\sum_{y \in C^*_m(r_m)} G(x, y) \geq \frac{2}{2\pi} \frac{3 + o(1)}{r_m} \|\mathbb{B}(0, r_m)\| \sum_{j=v_m}^{v_m} j^{-1} \geq \frac{(4 + o(1))}{r^2_m} \log(u_m/v_m),$$

which for our choices of $u_m$ and $v_m$ is as stated in (2.10) (for some $\delta_m \to 0$, uniformly over $x \in Q_1$). Further, $Q_2$ consists of two sets with an equal number of elements, each of diameter at most $(1 + o(1))u_m r_m$. Thus, we get by (2.8) that for some $c > 0$,\n
$$\sum_{y \in Q_2} G(x, y) \geq c |Q_2| (u_m r_m)^{-1}, \quad \forall x \in Q_2,$$

and (2.11) follows upon choosing $\epsilon_m \to 0$ slow enough so that $(m/r_m)^{\epsilon_m} \gg \log(m/r_m)$. \hfill \blacksquare

3. \textbf{Lil for srw on $\mathbb{Z}^3$: Proof of Theorem 1.1.} To ease the presentation we omit hereafter the integer-part symbol $\lceil \cdot \rceil$ and divide the section to four parts, establishing the lower and then upper bounds, first for the lim sup-\text{LIL} of (1.2) and then for the lim inf-\text{LIL} of (1.2). Our proofs for the lim sup-\text{LIL} are \textit{discrete} in nature, relying on Remark 2.2, Lemma 2.3 and the direct evaluation of certain sharp tail probabilities for srw. In contrast, we prove the lim inf-\text{LIL} by first replacing $R_n$ by the corresponding quantity about the Brownian capacity of the range of the 3D Brownian motion.

3.1. \textbf{The lower bound in the limsup-\text{LIL}.} Recall

$$\psi(t) := \sqrt{2/3} t \log t, \quad h_3(t) := \frac{\pi}{3} \psi(t)(\log t)^{-1},$$

of Proposition 1.7 and Theorem 1.1, respectively, and for the srw $(S_m)$ on $\mathbb{Z}^3$ set

$$A_t := \{ S^1_t \geq \psi(t) \}, \quad V_I := 1_{A_t} \sum_{t \in I} G(0, S_t), \quad I \subseteq [0, t] \cap \mathbb{Z}.$$  

The lower bound in our lim sup-\text{LIL} is attained via partial sums on disjoint intervals $I_n$ of suitably growing length $t_n$ and controllable Green function values (utilizing the \text{LHS} of (2.3)). The key for this is our next lemma (whose proof is deferred to the end of this sub-section), showing that $A_{t_n}$ yields the appropriate bound on the sum $V_{[0, t_n]}$ of Green function values.
Lemma 3.1. Fixing $\delta \in (0, 1)$, for $\gamma_t := t(\log_2 t)^{-1}(\log_3 t)^{3/2}$ and some $\zeta_t \to 0$ when $t \to \infty$,

\begin{equation}
P(V_{[0,Y_t]} \geq \delta t) \leq \zeta_t P(A_t),
\end{equation}

\begin{equation}
P(V_{[Y_t,1]} \geq \frac{(1 + 2\delta) t}{2h_3(t)}) \leq \zeta_t P(A_t).
\end{equation}

Indeed, from (3.1)-(3.2) we see that for any $\delta > 0$, there exists $\zeta_t \to 0$ as $t \to \infty$, such that

\begin{equation}
P(A_t^*) \leq 2\zeta_t P(A_t), \quad A_t^* := \{V_{[0,Y_t]} \geq \frac{(1 + 4\delta) t}{2h_3(t)}\}.
\end{equation}

Proceeding to deduce the lower bound in the limsup-l.1 out of (3.3), given $\epsilon > 0$, we choose $q > 1$ large and $\delta > 0$ small so for $t_n := q^n - q^{n-1}$ and all large $n$,

\begin{equation}
\frac{1 - \delta}{1 + 4\delta}h_3(t_n) \geq (1 - \epsilon)h_3(q^n).
\end{equation}

We then partition $\mathbb{Z}_+$ to disjoint intervals $I_n := (q^{n-1}, q^n] \cap \mathbb{Z}$ of length $t_n$ and set the events

\begin{equation}
\hat{A}_n := \{S_{q^n}^1 - S_{q^{n-1}}^1 \geq \psi(t_n)\}, \quad \hat{H}_n(i) := \{V_n(i) \geq (1 + 4\delta)\frac{t_n}{h_3(t_n)}\},
\end{equation}

where

\begin{equation}
\hat{V}_n(i) := \sum_{t \in I_n} G(S_i, S_t).
\end{equation}

Setting

\begin{equation}
H_t(i) := \left\{\sum_{t=i}^{t} G(S_i, S_t) \geq (1 + 4\delta)\frac{t}{h_3(t)}\right\}, \quad i \in [1, t],
\end{equation}

note that by the stationarity of the srw increments, $(\hat{A}_n, \hat{H}_n(q^{n-1} + i)) \overset{d}{=} (A_t, H_t(i))$. Further, the event $A_t$ is invariant to any permutation of the srw increments $\{X_j, j \leq t\}$, whereas given $A_t$ the event $H_t(i)$ depends only on $\{S_{\ell} - S_i, \ell \in [1, t]\}$. Further, the permuted increments $\hat{X}_j = X_i(j + i)_{\mod(t)}$ result with $\hat{S}_\ell = S_{\ell+i} - S_i$ for all $\ell \in [1, t - i]$, whereas $\hat{X}_j = X_i(j-i)_{\mod(t)}$ result with $\hat{S}_\ell = S_i - S_{i+\ell}$ for all $\ell \in [1, i]$. Since $G(x, y) = G(-x, -y) \geq 0$, it thus follows that conditional on $A_t$, the random sum in each of the events $H_t(i)$ is stochastically dominated by twice the random sum in the event $A_t^*$ of (3.3). Consequently, (3.3) yields that

\begin{equation}
\max_{i \in I_n} P(\hat{H}_n(i) | \hat{A}_n) \leq 4\zeta_{t_n} \to 0.
\end{equation}

Next, consider the independent events $G_n := \{|\Lambda_n| \geq (1 - \delta)t_n\}$, where $\Lambda_n$ is the subset of all those $i \in I_n$ for which $\hat{H}_n(i)$ does not hold. From Markov’s inequality and (3.5) it follows that

\begin{equation}
P(G_n^c | \hat{A}_n) = P(|\Lambda_n| \geq \delta t_n | \hat{A}_n) \leq \frac{1}{\delta t_n} \sum_{i \in I_n} P(\hat{H}_n(i) | \hat{A}_n) \leq \frac{4\zeta_{t_n}}{\delta} \to 0.
\end{equation}

Setting $x(t) := \sqrt{2\log_2 t}$, note that $(S^1_{nt})$ is the partial sum of $\{-1, 0, 1\}$-valued, zero-mean, i.i.d. variables of variance $1/3$. By the asymptotic normality of the moderate deviations for such partial sums (see [34, Thm. VIII.2.1]), we have that for some $o(1) \to 0$ as $t \to \infty$, uniformly over $x/t^{1/6}$ small,

\begin{equation}
P((t/3)^{-1/2} S_{nt}^1 \geq x) = (1 + o(1))\tilde{\Phi}(x), \quad \tilde{\Phi}(x) := \int_{x}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} du.
\end{equation}
In particular,
\[
P(\hat{A}_n) = P(A_{n+}) = P\left((t_n/3)^{-1/2} S_{n+}^1 \geq x(t_n)\right)
= (1 + o(1))\Phi(x(t_n)) \geq c_1x(t_n)^{-1}e^{-x(t_n)^2/2} \geq \frac{c_2}{n \sqrt{\log n}},
\]
for some positive \(c_1\) and \(c_2 = c_2(q)\). Hence, by (3.6), for all \(n\) large enough,
\[
P(G_n) \geq P(\hat{A}_n \cap G_n) \geq \frac{1}{2} P(\hat{A}_n) \geq \frac{c_2}{2n \sqrt{\log n}}.
\]
Having \(\{G_n\}\) independent with \(\sum_n P(G_n) = \infty\), we deduce by the second Borel-Cantelli lemma that a.s. the events \(G_n\) hold for infinitely many values of \(n\). Since
\[
\hat{R}_{q^*} := \{S_i\}_{i \in \Lambda_n} \subseteq \mathcal{R}_{q^*},
\]
we have by the monotonicity of \(A \mapsto \text{Cap}(A)\), the non-negativity of \(G(x,y)\), Lemma 2.3 and the definition of \(\Lambda_n \subseteq I_n\), that
\[
R_{q^*} \geq \text{Cap}(\hat{R}_{q^*}) \geq \frac{|\Lambda_n|}{\max_{i \in \Lambda_n} |\hat{V}_n(i)|} \geq \frac{|\Lambda_n| h_3(t_n)}{(1 + 4\delta)t_n}.
\]
Consequently, in view of (3.4) we have on the event \(G_n\) that
\[
R_{q^*} \geq \frac{1 - \delta}{1 + 4\delta} h_3(t_n) \geq (1 - \epsilon)h_3(q^n),
\]
which since \(G_n\) holds infinity often, yields the lower bound in the lim sup-LIL (along the subsequence \(q^n\), and with \(\mathcal{R}_{q^*} \supseteq \hat{R}_{q^*}\) of roughly the shape of \(C_{\phi(t_n)}\) of Lemma 2.1).

Turning to the task of proving Lemma 3.1, we give two sharp tail estimates for the path of the one-dimensional walk \((S_{n+}^1)\), that we will use later for proving (3.1) and (3.2), respectively.

**Lemma 3.2.** Fixing \(\delta \in (0, 1)\), for some \(C < \infty\) and all \(t\) large enough,
\[
\sup_{\ell \leq \gamma_t} P(S_{n+}^1 \geq \psi(t) - \sqrt{\ell}) \leq C P(A_t),
\]
(3.8)
\[
P(\L_t | A_t) \leq 4 \zeta_t \to 0, \quad \L_t := \bigcap_{\ell \in (\gamma_t, t]} \left\{ S_{n+}^1 \geq \frac{\psi(t)\ell}{(1 + \delta)t}\right\},
\]
(3.9)

**Remark 3.3.** In (3.8) we claim that the decay \(t \mapsto P(A_t)\) of our moderate deviations upper-tail event \(A_t\) is within a universal factor of that for such an event with a granted (free) upper fluctuation of \(\sqrt{\ell}\) in the first \(\ell \leq \gamma_t\) steps of \(S_{n+}^1\). The event \(\L_t\) requires \(S_{n+}^1\) to stay above a linear slope which is \(1/(1 + \delta) < 1\) of the slope \(\psi(t)/t\) of \(A_t\). Thus, if \(\ell \mapsto S_{n+}^1\) was a srw on \(\mathbb{Z}\), we could have applied en-route to (3.9) a ballot theorem (after conditioning on \(S_{n+}^1\) and \(S_{n+}^1\)). It is not so here, due to the additional randomness in number of steps of the srw \(S_{n+}^1 \in \mathbb{Z}\) along the other two coordinate axis. We thus resort to proving (3.9) via Gaussian approximations.

**Proof of Lemma 3.2.** Let \(\eta_t := \gamma_t / t = (\log_3 t)^{3/2} / (\log_2 t)\), setting \(x(t, r) := (x(t) - \sqrt{2r})/\sqrt{1 - r}\) for \(r \leq \eta_t\) and \(x(t) = x(t, 0) := \sqrt{2\log_2 t}\). Then, in view of the uniform Gaussian approximation of (3.7), we get (3.8) once we show that uniformly in \(r \leq \eta_t\), the standard Gaussian measure of \([x(t, r), \infty)\), is at most \(C\) times the Gaussian measure of \([x(t), \infty)\). Note that \(\eta_t \to 0\) and \(x(t) \to \infty\) as \(t \to \infty\), hence \(x(t, r)/x(t) \to 1\) uniformly in \(r \leq \eta_t\). It thus remains only to show that for some \(C < \infty\) and all \(t\) large enough,
\[
\inf_{r \leq \eta_t} \{x(t, r)^2 - x(t)^2\} \geq -2 \log C.
\]
(3.10)
Next, our expression for $x(t, r)$ is such
\[(1 - r)[x(t, r)^2 - x(t)^2] = (x(t) \sqrt{r} - \sqrt{3})^2 + 3r - 3 \geq -3,
\]
yielding (3.10) and thereby also (3.8).

Next, setting $s_j := j t / (\log_2 t)$, we partition $(\gamma_i, t)$ into the disjoint intervals $J_j = (s_j, s_{j+1}]$. $j \in [(\log_2 t)^{3/2}, \log_2 t]$. We likewise partition the events $A_i \cap L^*_{t} \cap \gamma_i$ according to whether the stopping time $\tau_i := \inf\{\ell \geq \gamma_i : S_{\ell}^1 < \frac{\psi(\delta \log \eta \ell y)}{1 + \delta}\}$ equals $\gamma_i$ or alternatively which interval $J_j$ contains $\tau_i$.

Note that if $\tau_i > s_j$ then $S_{\tau_i}^1 \geq \frac{\psi(\delta \log \eta \ell y)}{1 + \delta}$ and conditioning on the saw filtration at $\tau_i \in J_j$, we get by the strong Markov property (and i.i.d. increments), of the saw, that
\[P(A_i \cap \{\tau_i \in J_j\}) \leq q_i(s_j, 0) \sup_{s \in J_j} p_i(t - s, 0),\]
where
\[q_i(s, y) \equiv P(S_{\tau_i}^{1, +} < \frac{\psi(t)}{1 + \delta} y), \quad p_i(t, r) \equiv P(S_{\tau_i}^{1, +} \geq \frac{\psi(t)}{1 + \delta} y + r),\]
and $(S_{\tau_i}^{1, +}) := S_{\tau_i}^1 \setminus \emptyset$. The only other way for the event $L^*_{t} \cap \gamma_i$ to occur, is by having
\[S_{\tau_i}^1 \leq \frac{\psi(t)}{1 + \delta} \eta i := \Delta_i \log_2 t.
\]
Partitioning to $\{S_{\tau_i}^1 \in I_i\}$, for $I_i := [(\log_2 t - i)\Delta_i - \Delta_i, (\log_2 t - i)\Delta_i)$ when $0 \leq i < \log_2 t$, and $I_{\log_2 t} := (-\infty, 0)$, yields that
\[P(A_i \cap \{S_{\tau_i}^1 \in I_i\}) \leq q_i(\gamma_i, i\Delta_i + \Delta_i) p_i(t - \gamma_i, i\Delta_i),\]
and consequently,
\[P(A_i \cap L^*_{t} \cap \gamma_i) \leq \sum_i P(A_i \cap \{S_{\tau_i}^1 \in I_i\}) + \sum_j P(A_i \cap \{\tau_i \in J_j\}) \leq (1 + \log_2 t) \left[ \sup_{y \in [0, \Delta_i \log_2 t]} [\sup_{y \in [0, \Delta_i \log_2 t]} \{q_i(\gamma_i, y + \Delta_i)p_i(t - \gamma_i, y)\}] + \sup_{s \in J_j, j \geq (\log_2 t)^{3/2}} q_i(s_j, 0)p_i(t - s, 0)\right].\]

As $P(A_i) = p_i(t, 0)$, we thus get (3.9), if for some $\xi_i \to 0$
\begin{align}
(3.11) \quad q_i(s, 0)p_i(t - s, 0) & \leq \frac{\xi_i}{\log_2 t} p_i(t, 0), \quad \forall j \geq (\log_2 t)^{3/2}, s \in J_j, \\
(3.12) \quad q_i(\gamma_i, y + \Delta_i)p_i(t - \gamma_i, y) & \leq \frac{2\xi_i}{\log_2 t} p_i(t, 0), \quad \forall y \in [0, \Delta_i \log_2 t].
\end{align}

Proceeding to verify (3.11) and (3.12), we rely on (3.7) to replace both $q_i(\cdot)$ and $p_i(\cdot)$ by the Gaussian measure of the corresponding intervals. We claim that when doing so, all partial sums appearing in (3.11) and (3.12) be at time index $t \geq 6\psi(t)/2 \to \infty$. Indeed, note that $p_i(r, y) = 0$ for $y \geq 0$, unless the time index $r$ is at least $6\psi(t)/2$, whereas in all the terms $q_i(\cdot)$ that appear there, such time indices are $s_j \geq \gamma_i \geq 6\psi(t)/2$. Further, the argument $x$ of $\Phi(\cdot)$ in such Gaussian approximations of the probabilities $q_i(\cdot)$ and $p_i(\cdot)$ that appear in (3.11), is $\frac{\psi(t)}{1 + \delta} \sqrt{s_j^1/t}$ and $\frac{\psi(t)}{1 + \delta} \sqrt{s_j^1/t + \xi_i}$ at $\xi_i = 0$, respectively (where $x(t) = \sqrt{3}\psi(t)/\sqrt{t}$). Taking instead
\[\xi_i := (\log_2 t)^{-1/6}\] guarantees that all such space arguments $x$ be uniformly of $o(t^{1/6})$. The Gaussian approximations then hold, as in (3.7), with the same $o(1)$ relative error for all the terms, which we thus ignore hereafter. Note also that $s/t \in [\eta_i, 1]$ with $s_j/t \geq s/t - \xi_i$ for $\xi_i := (\log_2 t)^{-1}$. In conclusion, we have, by the preceding it suffices for (3.11) to show that
\begin{align}
(3.13) \quad \sup_{u \in [\eta_i, 1]} \left[ \Phi\left( \frac{x(t)}{1 + \delta} \sqrt{u - \xi_i} \right) \Phi\left( \frac{x(t)(1 + \delta - u)}{1 + \delta} \sqrt{1/u + \xi_i} \right)\right] & \leq \frac{\xi_i}{\log_2 t} \Phi(x(t)).
\end{align}
The arguments of $Φ(⋅)$ in (3.13) grow to infinity with $t$, uniformly over $u \in [ηt, 1)$. Thus, recalling that $|\log Φ(y) + \log y + y^2/2|$ is bounded at $y \to \infty$, upon taking the logarithm of both sides of (3.13), noting that $ηt \geq 2εt$ and ignoring all the uniformly bounded terms, such as $x(t)^2εt$ and $\log x(t) - 1/2 \log 3t$, it suffices to show that for some $ξt \to 0$ as $t \to \infty$,

$$\sup_{u \in (ηt, 1)} \left\{ \frac{1}{2} x(t)^2 \frac{1}{(1 + \delta)^2} - \frac{1}{2} x(t)^2 \frac{(1 + \delta - u)^2}{(1 + \delta)^2(1 - \eta t)} - \frac{1}{2} \log u \right\} \leq \log ξt - \frac{1}{2} \log 3t.$$  

With $ηt \gg ξt$, it is easy to verify that for any $u \geq ηt$

$$\frac{u}{(1 + \delta)^2} + \frac{(1 + \delta - u)^2}{(1 + \delta)^2(1 - \eta t)} - \frac{1}{(1 - u + ξ t)} \delta^2 =: θ_t(u).$$

Hence, substituting $x(t)^2 = 2 \log 2t$ in the preceding, we arrive (after some algebra) at

(3.14) $$\sup_{u \in (ηt, 1)} \left\{ -(\log 2t) θ_t(u) - \frac{1}{2} \log u \right\} \leq \log ξt - \frac{1}{2} \log 3t.$$  

Since $u \mapsto θ_t(u)$ is non-decreasing, the supremum on the left-side of (3.14) is attained at $u = ηt$, where for large $t$ it is at most $-\frac{δ^2}{10} (\log 3t)^3/2$. This is more than enough for (3.14) to hold, thereby establishing (3.11). We next turn to (3.12), where the Gaussian approximation of (3.7) again applies for all three probabilities, with uniform, hence negligible, $o(1)$ relative errors. Thus, analogously to (3.13), the bound (3.12) is a consequence of having

(3.15) $$\sup_{v \in [0, 1]} \left\{ \frac{\log \Phi(\frac{x(t)}{1 + \delta}) \sqrt{η t} (v - εt)}{(1 + \delta) \sqrt{1 - η t}} \right\} \leq \frac{ξt}{\log 2t} \log \Phi(\frac{x(t)}{1 + \delta})$$  

(temporarily setting $Φ(x) = 1$ whenever $x < 0$). Considering first $v \leq 1/2$, we bound the leftmost term of (3.15) by one, and as before, take the logarithm of both sides, replace $\log Φ(y)$ by $-\log y - y^2/2$ and eliminate all uniformly bounded terms to find that (3.15) holds because

$$\inf_{v \in [0, 1]} \left\{ \frac{(1 + \delta - v η t)^2}{(1 + \delta)^2(1 - η t)} - 1 \right\} \geq \frac{η t \delta}{1 + \delta}$$

and $x(t)^2 η t = 2(\log 3t)^3/2 \gg \log 3t$. To complete the proof of (3.15), it thus suffices (similarly to (3.14)), to have for some $ξt \to 0$, that

(3.16) $$\sup_{v \in [1/2, 1]} \left\{ -(\log 2t) θ_t(v) \right\} \leq \log ξt - \log 3t,$$

where, recalling that $ηt \gg εt$, it is easy to check that for any $v \in [1/2, 1]$,

$$θ_t(v) := \frac{η t (v - εt)^2}{(1 + \delta)^2} + \frac{(1 + \delta - v η t)^2}{(1 + \delta)^2(1 - η t)} - 1 \geq θ_t(1) \geq \frac{η t \delta^2}{(1 + \delta)^2}.$$  

The preceding suffices for (3.16) and thereby for (3.12), thus completing the proof.

**Proof of Lemma 3.1.** Starting with (3.1), note that for some $C_1 < \infty$ and any $t \geq 0$,

$$E[G(0, S) \mathbb{1}_{A_t}] = \sum_{i=1}^{\infty} P^0(S_i = 0) \leq C_1 (1 + \sqrt{t})^{-1}.$$  

Further, recall from (2.8) that $G(0, y) \leq C_2/(1 + |y|)$ for some $C_2 < \infty$ and all $y \in \mathbb{Z}^3$. Hence,

(3.17) $$E[G(0, S) \mathbb{1}_{A_t}] \leq C_2 (1 + \sqrt{t})^{-1} P(A_t) + \sum_{|y| \leq \sqrt{t}} G(0, y) P(|S_t = y) \cap A_t).$$
With \((S_\ell, S_{\ell} - S_\ell) \overset{d}{=} (S_\ell, \hat{S}_{\ell-\ell})\) for skw \((\hat{S}_m)\) which is independent of \((S_m)\), if \(y \in \mathbb{Z}^3\) is such that \(|y| \leq R\), then

\[
P(|S_\ell = y \cap A_\ell| \leq P(S_\ell = y)P(\hat{S}_{\ell-\ell} \geq \psi(t) - \sqrt{t}).
\]

Therefore, thanks to (3.8), for any \(\ell \leq \gamma_i\) the right-most term in (3.17) is at most,

\[
E[G(0, S_\ell)]P(\hat{S}_{\ell-\ell} \geq \psi(t) - \sqrt{t}) \leq C(1 + \sqrt{t})^{-1}CP(A_\ell).
\]

We thus deduce from (3.17), that for some \(C_3, C_4 < \infty\),

\[
E[V_{[0, \gamma_i]}] = \sum_{\ell=0}^{\gamma_i} E[G(0, S_\ell)1_{A_\ell}] \leq C_3P(A_\ell) \sum_{\ell=0}^{\gamma_i} (1 + \sqrt{t})^{-1} \leq C_4P(A_\ell) \sqrt{\gamma_i}.
\]

Consequently, by Markov’s inequality and our choice of \(\gamma_i\),

\[
P\left(V_{[0, \gamma_i]} \geq \frac{\delta t}{h_3(t)}\right) \leq \zeta_iP(A_t),
\]

where our choice of \(\gamma_i\) results with \(\zeta_i := C_4h_3(t)\sqrt{\gamma_i}/(\delta t) \to 0\) as \(t \to \infty\).

Turning to (3.2), recall our choice of \(\gamma_i\) implying that

\[
\sum_{\ell \in \{\gamma_i, \ell\}} \frac{1}{\ell} \leq \log(t/\gamma_i) \leq \log_3 t.
\]

Further, with \(h_3(t) = (\pi/3)\psi(t)(\log_3 t)^{-1}\), it follows that \(t/(2h_3(t)) = \frac{3}{2\pi}(t/\psi(t))(\log_3 t)\). Consequently, for all \(t\) large enough, the event \(L_t\) of (3.9) implies, thanks to (2.8), that

\[
V_{[\gamma_i, \ell]} \overset{\ell}{\leq} \sum_{\ell \in \{\gamma_i, \ell\}} G(0, S_\ell) \leq \frac{3 + o(1)}{2\pi} \sum_{\ell \in \{\gamma_i, \ell\}} \left|S_\ell\right|^{-1} \leq \frac{3 + o(1)}{2\pi} \frac{(1 + \delta)t}{\psi(t)}(\log_3 t) \leq \frac{(1 + 2\delta)t}{2h_3(t)}.
\]

Since the same conclusion applies when \(A_t^c\) holds (in which case \(V_{[\gamma_i, \ell]} = 0\)), we see that (3.2) is an immediate consequence of (3.9).

3.2. The upper bound in the limsup-LIL. Recall that \(R_n\) is non-decreasing. Further, for any \(t_n = q^n, q > 1\) we have that eventually \(h_3(t_n)/h_3(t_{n-1}) \leq q\). It thus suffices to prove the upper bound in our limsup-LIL only along each such sequence \(t_n\) (thereafter taking \(q \downarrow 1\) to complete the proof). To this end, fix \(q > 1\) and \(\delta, \eta > 0\), and set \(m := (1 + \delta)^3\psi(t_n)\), with \(r_m = m/b_m \uparrow \infty\) for \(b_m := 2\eta(1 + \delta)^6(\log_2 t_n)\) (so \(r_m := \psi(t_n)(2\eta(1 + \delta)^3\log_2 t_n))\). We aim to cover \(\mathcal{R}_n\) by the union \(C_m^*(r_m)\) of \(b_m\) balls of radius \(r_m\) each, with the centers of consecutive balls at most \(r_m\) apart. Indeed, as shown in Section 2 (see Remark 2.2), this would yield \(R_n \leq (1 + \delta + o(1))^3h_3(t_n)\), so we then conclude by taking \(\delta \downarrow 0\) and \(q \downarrow 1\).

Specifically, starting at \(T_0 = 0\), set the increasing stopping times

\[
T_i := \inf\{k > T_{i-1} : |S_k - S_{T_{i-1}}| > r_m - 1\}, \quad \forall i \geq 1,
\]

noting that the event \(\{T_{b_n} \geq t_n\}\) implies the aforementioned containment \(\mathcal{R}_n \subseteq C_m^*(r_m)\). Further, with \(exp(-(1 + \delta)\log_2 t_n) \leq Cn^{-(1+\delta)}\) summable, upon employing the first Borel-Cantelli lemma, it remains only to establish the following key lemma.

**Lemma 3.4.** For any \(q > 1\) and small \(\delta > 0\), there exist \(\eta > 0\) and \(C < \infty\) such that

\[
P(T_{b_n} < t_n) \leq C \exp(-(1 + \delta)\log_2 t_n) \quad \forall n.
\]
Proof. By the strong Markov property and the independence of increments of the walk, we see that $T_{b_m}$ is the sum of $b_m$ i.i.d. copies of the first exit time $T_1$ of the (discrete) ball $B(0, r_m - 1)$, by the 3D-srw. As $r_m \uparrow \infty$, Skorokhod’s embedding implies, see [28, Lemma 3.2], that for some $m_0$, $e < \infty$ and $e > 0$ (depending only on $\delta > 0$), all $m \geq m_0$, and $u \geq 0$,

$$P(T_1 < u) \leq ce^{-t_u} + P(3T < u) = ce^{-t_u} + P(3(1 + \delta)^{-1}r_m^2 \hat{T}_1 < u),$$

with $\hat{T}_1 := \inf\{t \geq 0 : |B_t| \geq 1\}$ the Brownian hitting time of the unit sphere $\mathbb{S}^2$ and $T := \inf\{t \geq 0 : |B_t| \geq r_m/\sqrt{1 + \delta}\}$ (so the identity above is merely Brownian scaling). Further, here $b_m e^{-t_u} \ll \exp(-2\log_2 t_n)$ and $t_n = 3r_m^2 \eta b_m$ with $\eta / (2\eta) = 3m^2/(2t_n) = (1 + \delta)^6(\log_2 t_n)$. It thus suffices to show that for some $\eta = \eta(\delta) > 0$ and all $m$,

$$P\left(\frac{1}{b_m} \sum_{i=1}^{b_m} \hat{T}_i < \eta\right) \leq e^{-(1-\delta)b_m/(2\eta)},$$

where $\hat{T}_i$ are i.i.d. copies of $\hat{T}_1$. To this end, covering $\mathbb{S}^2$ by $c_\delta$ balls of radius $\delta$ each, centered at some $\theta_i \in \mathbb{S}^2$, we have by the triangle inequality that

$$\max_i \{(\theta_i, B_{\hat{T}_i})\} \geq 1 - \delta.$$ 

Hence, fixing $\lambda > 0$, we get upon applying Doob’s optional stopping theorem for the martingale $M_t = \sum_i \exp(\lambda(\theta_i, B_t) - \lambda^2 t/2)$ at the stopping time $\hat{T}_1$, that

$$c_\delta = M_0 = E[M_{\hat{T}_1}] \geq e^{\lambda(1-\delta)}E[e^{-\lambda^2 \hat{T}_1/2}].$$

Consequently, by Markov’s inequality, we have for any $\eta, \lambda, \delta > 0$ and integer $b \geq 1$, that

$$P\left(\frac{1}{b} \sum_{i=1}^{b} \hat{T}_i < \eta\right) \leq e^{-b\eta/2}E[e^{-\lambda^2 \hat{T}_1/2}]^b \leq \left(c_\delta e^{\epsilon\eta/2 - \lambda(1-\delta)}\right)^b.$$ 

Taking the optimal $\lambda = (1-\delta)/\eta$ it is easy to check that for $\eta \leq \eta(\delta) = \delta(1-\delta)^2/(2\log c_\delta)$, the left-hand side of (3.19) is at most $\exp(-(1-\delta)^2b/(2\eta))$. We thus got (3.19) for any $\delta > 0$ provided $\eta \leq \eta(\delta)$, thereby completing the proof. 

**Remark 3.5.** One has for any $\delta > 0$ small and all $t$ large enough, the classical bound

$$P(\max_{1 \leq k \leq t} |S_k| \geq (1 + \delta)^3 \psi(t)) \leq Ce^{-(1+\delta)\log_2 t}.$$ 

We need in (3.18) a stronger result, since for any $b_m$ and $r_m \gg 1$,

$$\{ \max_{1 \leq k \leq t} |S_k| \geq b_mr_m \} \subset \{ T_{b_m} < t_n \},$$

and while $b_mr_m = (1 + \delta)^3 \psi(t_n)$, our crude use of $\delta$-cover of $\mathbb{S}^2$ in proving Lemma 3.4 requires us to also have $b_m / (\log_2 t_n) \to 0$ as $\delta \downarrow 0$.

### 3.3. The upper bound in the liminf-UL

For any $A \subset \mathbb{R}^3$ and $r > 0$, let

$$\text{Nbd}(A, r) := \bigcup_{x \in A} B(x, r)$$

denote the $r$-blowup of $A$. Utilizing [14], we first relate $R_n$ with a suitable Brownian capacity, as stated next.
Lemma 3.6. We can couple the srw with a 3D Brownian motion \((B_t, t \geq 0)\), such that

\[
\lim_{n \to \infty} \frac{R_n}{\text{Cap}_B(B[0, n/3])} = \frac{1}{3} \quad \text{a.s.}
\]

and for any \(\delta \in (0, 1/2)\),

\[
\lim_{n \to \infty} \frac{R_n}{\text{Cap}_B(B[0, n/3], n^{1/2-\delta})} = \frac{1}{3} \quad \text{a.s.}
\]

Proof: The results were essentially shown in [14]. Indeed, [14, (4.15)] shows that (3.21) holds when each ratio is restricted to the events \(E_n\), while it is also shown that a.s. \(E_n\) holds for all sufficiently large \(n\) (combine [14, (4.2)] with Borel-Cantelli). Hence (3.21) also holds without such a restriction. Turning to show (3.22), let \(\tilde{P}\) denote the probability of an independent Brownian motion \((\tilde{B}_t)\). Fixing \(\delta \in (0, 1/2)\) and some \(y_n \in \mathbb{R}^3\) such that \([y_n] = n^{1/2+\delta}\), we similarly obtain (after dispensing of events \(E_n\), that by the same argument as in [14, (4.4)], a.s. one has for all large \(n\),

\[
\text{Cap}_B(B[0, n/3], n^{1/2-\delta}) = (2\pi + o(1))n^{1/2-\delta} \tilde{P}(\text{Nbd}(B[0, n/3], n^{1/2-\delta}) \cap (y_n + B(0, \infty)) \neq \emptyset | B[0, n/3]).
\]

By [14, (4.13) & (4.4)], a.s., the latter expression is for all \(n\) large \((1 + o(1))\text{Cap}_B(B[0, n/3])\). Thus,

\[
\lim_{n \to \infty} \frac{\text{Cap}_B(B[0, n/3])}{\text{Cap}_B(B[0, n/3], n^{1/2-\delta})} = 1 \quad \text{a.s.},
\]

which in view of (3.21) completes the proof of (3.22).

Proceeding to show the upper bound in the lim inf-LIL, let \(r(n) := \pi \sqrt{n/(2 \log_2 n)}\), that is \(r(n) = \sqrt{\frac{3}{2}} \hat{r}(n)\) for \(\hat{r}(\cdot)\) of Proposition 1.7. Recall that by [20, Lemma 1.1] or [25], and Brownian scaling, for some \(c > 0\) and all \(t > 0\),

\[
P(\sup_{x \in [0,t]} |B_x| \leq r) \geq 2ce^{-2r^2}.
\]

We have used in (3.23) also that the largest eigenvalue of the Dirichlet Laplacian in the unit ball in \(\mathbb{R}^d\) is \(-j^2\), where \(j = j_{(d-2)/2,1}\) denotes the first positive zero of the Bessel function of the first kind with index \((d-2)/2\), and in particular that \(j_{1/2,1} = \pi\) (see [38, Page 490]). Considering (3.23) for \(r = r(s_n)\), \(s_n = n^c\) and the Brownian increments in the disjoint intervals \([s_{n-1}, s_n)\) of length \(s_n - s_{n-1}\), result with

\[
P\left(\sup_{t \in [s_{n-1}, s_n]} |B_t - B_{s_{n-1}}| \leq r(s_n)\right) \geq c \exp(-\log_3 s_n) = \frac{c}{n \log n}.
\]

Thus, thanks to the independence of Brownian increments on these disjoint intervals, we get from the second Borel-Cantelli lemma that

\[
P\left(\liminf_{n \to \infty} r(s_n)^{-1} \sup_{t \in [s_{n-1}, s_n]} |B_t - B_{s_{n-1}}| \leq 1\right) = 1.
\]

Further, as \(\sqrt{s_{n-1}(\log_2 s_{n-1})} = o(r(s_n))\), by Kinchin’s LIL for the Brownian motion,

\[
P\left(\limsup_{n \to \infty} r(s_n)^{-1} \sup_{t \leq s_n} |B_t| = 0\right) = 1.
\]
Combining (3.25) and (3.26), we deduce that
\[
P\left( \liminf_{n \to \infty} (r(s_n))^{-1} \sup_{t < s_n} |B_t| \leq 1 \right) = 1.
\]
This of course implies that also
\[
(3.27) \quad P\left( \liminf_{n \to \infty} (r(n))^{-1} \sup_{t < n} |B_t| \leq 1 \right) = 1.
\]
Recall that for any \( r > 0 \) one has that \( r^{-1} \text{Cap}_B(B(0, r)) = \text{Cap}_B(\mathbb{B}(0, 1)) = 2\pi (= k_1 \text{ on [13, Page 356]}). \) By (3.27), for any \( \epsilon > 0, \text{ a.s. } B(0, n) \subset \mathbb{B}(0, (1 + \epsilon) r(n)) \) for infinitely many values of \( n, \) in which case also \( \text{Cap}_B(B(0, n)) \leq 2\pi(1 + \epsilon) r(n). \) That is,
\[
(3.28) \quad P\left( \text{Cap}_B(B(0, n)) \leq 2\pi(1 + \epsilon) r(n) \quad \text{i.o.} \right) = 1.
\]
By Brownian scaling, the sequence \( \{ \sqrt{3} \text{Cap}_B(B(0, n/3)) \} \) has the same law as the sequence \( \{ \text{Cap}_B(B(0, n/n)) \}. \) Thus, in view of (3.21), we can also construct a coupling so that for any \( \epsilon > 0, \) we have that a.s.
\[
R_n \leq \frac{(1 + \epsilon)}{3 \sqrt{3}} \text{Cap}_B(B(0, n)),
\]
for all \( n \) large enough. With \( r(n) = \frac{3\sqrt{3}}{2\pi} \hat{h}_3(n) \), it thus follows from (3.28) that
\[
P\left( R_n \leq (1 + \epsilon)^2 \hat{h}_3(n) \quad \text{i.o.} \right) = 1,
\]
and taking \( \epsilon \downarrow 0 \) establishes the stated upper bound in our lim inf-LIL.

3.4. The lower bound of the liminf-LIL. Fixing \( a > 0, \) we have by [13, (1.4)], that for any \( f(t) \uparrow \infty \) such that \( f(t) = o(t^{2/3}), \)
\[
(3.29) \quad \lim_{t \to \infty} \frac{f(t)}{t} \log P\left( |\text{Nbd}(B(0, t], a)| \leq \left( \frac{\pi}{\sqrt{2}} \right)^3 f(t)^{3/2} \omega_3 \right) = -1,
\]
where \( \omega_3 \) denotes the volume of the unit ball (using here that the largest eigenvalue of the Dirichlet Laplacian in the unit volume ball in \( \mathbb{R}^3 \) is \( -\omega_3^2 \pi^2 \)). Fixing \( \delta \in (0, 1/2) \) as in (3.22), Brownian scaling by time factor \( 3n^{2\delta-1} \) yields equality in distribution between the sequences
\[
|\text{Nbd}(B(0, n/3], n^{1/2-\delta})| \overset{d}{=} 3^{-3/2} n^{3/2-3\delta} |\text{Nbd}(B(0, n^{2\delta}], 3^{-1/2})|.
\]
Thus, considering (3.29) for \( a = 3^{-1/2}, t = n^{2\delta} \) and \( f(t) = (1 - \epsilon)^2 n^{2\delta} (\log_2 n)^{-1}, \) we arrive at
\[
P\left( |\text{Nbd}(B(0, n/3], n^{1/2-\delta})| \leq (1 - \epsilon)^2 \hat{\psi}(n)^3 \omega_3 \right)
\]
\[
= P\left( |\text{Nbd}(B(0, n^{2\delta}], 3^{-1/2})| \leq (1 - \epsilon)^2 \left( \frac{\pi}{\sqrt{2}} \right)^3 (n^{2\delta} (\log_2 n)^{-1})^{3/2} \omega_3 \right)
\]
\[
\leq C \exp(-(1 - \epsilon)^{-2}(\log_2 n)).
\]
Considering \( n_k = q^k \), we get by the first Borel-Cantelli lemma, that for fixed \( q > 1 \) and \( \epsilon > 0, \)
\[
(3.30) \quad \liminf_{k \to \infty} |\text{Nbd}(B(0, n_k/3], n_k^{1/2-\delta})| \omega_3 \hat{\psi}(n_k)^3 \geq (1 - \epsilon)^2 \quad \text{a.s.}
\]
With \( n \mapsto |\text{Nbd}(B[0,n/3],n^{1/2-\delta})| \) monotone increasing and \( \hat{\psi}(q^k)/\hat{\psi}(q^{k-1}) \to 1 \) as \( k \to \infty \) followed by \( q \downarrow 1 \), we deduce from (3.30) that
\[
\liminf_{n \to \infty} \frac{|\text{Nbd}(B[0,n/3],n^{1/2-\delta})|}{\omega_3 \hat{\psi}(n)^3} \geq 1 \quad \text{a.s.}
\]
Next, recall the Poincaré-Carleman-Szegö theorem [35], that for any \( r > 0 \),
\[
\inf_{|A| = \omega_3 r^3} |\text{Cap}_A(A)| = \text{Cap}_B(\mathbb{B}(0,r)) = r \text{Cap}_B(\mathbb{B}(0,1)) = 2\pi r.
\]
Recall that \( \hat{h}_3(n) = 2\pi \hat{\psi}(n) \). Hence, by (3.32) for \( A = \text{Nbd}(B[0,n/3],n^{1/2-\delta}) \), we have in view of (3.30) that
\[
\liminf_{n \to \infty} \frac{\text{Cap}_B(\text{Nbd}(B[0,n/3],n^{1/2-\delta}))}{3\hat{h}_3(n)} \geq 1, \quad \text{a.s.,}
\]
which together with (3.22) yields the stated lower bound for the \( \text{liminf-LIL} \) of \( R_n \) in \( \mathbb{Z}^3 \).

4. LIL for \( \text{SRW on } \mathbb{Z}^4 \): Proof of Theorem 1.3. Hereafter we consider for integers \( 0 \leq a \leq b \leq c \), the random variables
\[
R_{a,b} := \text{Cap}(\mathcal{R}(a,b)), \quad V_{a,b,c} := R_{a,b} + R_{b,c} - R_{a,c} \geq 0.
\]
Note that by shift invariance \( R_{a,b} = R_{0,b-a} = R_{b-a} \) and \( R_{a,b} \) is independent of \( R_{b,c} \) (due to the independence of increments). In particular, for any increasing \( \{n_k\} \) starting at \( n_0 = 0 \), one has the decomposition
\[
R_{n_k} := \sum_{j=1}^k U_j - \Delta_{n_k,k},
\]
in terms of the independent variables \( U_j := R_{n_{j-1},n_j} \) and the sum of non-negative variables
\[
\Delta_{n_k,k} = \sum_{j=1}^{k-1} V_{n_{j-1},n_j,n_k} = \sum_{j=1}^{k-1} V_{0,n_j,n_{j+1}}.
\]
We use different non-random sub-sequences \( \{n_k\} \) for \( d = 4 \), for \( d = 5 \) and for \( d \geq 6 \). Also, the subsequences used for the \( \text{limsup-LIL} \) and for the \( \text{liminf-LIL} \) be different. As we shall see, for such suitable \( n_k \), the fluctuation in \( \sum U_j \) is negligible for the \( \text{limsup-LIL} \) of Theorem 1.3 where \( E\Delta_{n_k,k} \approx h_k(n_k) \), the \( \text{limsup-LIL} \) being due to the exceptional times with \( \Delta_{n_k,k} = o(E\Delta_{n_k,k}) \), while the \( \text{liminf-LIL} \) is due to the exceptional times where \( \Delta_{n_k,k} \approx \hat{h}_k(n_k) \Rightarrow E\Delta_{n_k,k} \). In contrast, we show in Section 5 that \( \Delta_{n_k,k} \) has a negligible effect when \( d \geq 5 \), where the \( \text{LIL} \) follows the usual pattern for sums of independent variables (namely, that of the \( \text{LIL} \) for a Brownian motion). We take a relatively small \( k = O(\log^2 n_k) \) for the \( \text{limsup-LIL} \) and \( d = 4 \), with larger \( k = O((\log n_k)^p) \) for the \( \text{liminf-LIL} \) and for any \( d \geq 5 \).

4.1. The lower bound in the \( \text{limsup-LIL} \). We start with the statement of a key lemma about the \( \text{SRW on } \mathbb{Z}^4 \) (which is related to [15, Thm. 2.2] in the 4D Brownian motion case).

**Lemma 4.1.** Suppose \( (S_m) \) and \( (\hat{S}_m) \) are two independent \( \text{SRW on } \mathbb{Z}^4 \). Let
\[
X_n := \frac{1}{n} \sum_{i,j \in [1,n]} G(S_i, \hat{S}_j).
\]
Then, for some \( C < \infty \) and any \( p, n \in \mathbb{N} \),
\[
E[X_n^p] \leq C^p p!.
\]
One immediate consequence of (4.4) is that for any $c < 1/C$, 
\begin{equation}
(4.5) \quad \sup_n E[e^{cx}] < \infty.
\end{equation}

The proof of Lemma 4.1 follows the same scheme as that of [30, Lemma 2]. However, [30] crucially relies on an explicit representation of the moments of the Brownian self-intersection local time via variances of linear combinations of Brownian increments. Lacking any such tool here, our more involved proof of Lemma 4.1 relies instead on the following elementary bounds, where (4.6)-(4.8) implies also that (4.4)-(4.5) hold for \( \int_0^1 \int_0^1 |\beta_s - \beta_t|^{-2} ds dt \) and the independent, standard 4-dimensional Brownian motions \( (\beta_s, s \geq 0), (\beta_t, t \geq 0) \) (which is an improvement over the upper bound of [7, Prop. 4.1]).

**Lemma 4.2.** There exists $C < \infty$ such that for any $t > 0$ and $x, y \in \mathbb{R}^4$,
\begin{align}
(4.6) \quad & E[|\beta_t - x|^{-2}] \leq C \min\{|t^{-1}, |x|^{-2}\} \leq C t^{-1/2}|x|^{-1}, \\
(4.7) \quad & E[|\beta_t - x|^{-1}|\beta_t - y|^{-1}] \leq C t^{-1/2}(|x| \vee |y|)^{-1} \leq 2C t^{-1/2}|y - x|^{-1}, \\
(4.8) \quad & E[|\beta_t - x|^{-2}|\beta_t - y|^{-1}] \leq 2C t^{-1/2}|x|^{-1}|y - x|^{-1}.
\end{align}

Similarly, for $|\cdot|_+ = |\cdot| \vee 1$, any $i \geq 0$ and $x, y \in \mathbb{Z}^4$,
\begin{align}
(4.9) \quad & E[|S_i - x|^{-2}] \leq C \min\{|i|_+^{-1}, |x|_+^{-2}\} \leq C i_+^{-1/2}|x|_+^{-1}, \\
(4.10) \quad & E[|S_i - x|^{-1}|S_i - y|_+^{-1}] \leq C |i|_+^{-1/2}(|x|_+ \vee |y|_+)^{-1} \leq 2C |i|_+^{-1/2}|y - x|_+^{-1}, \\
(4.11) \quad & E[|S_i - x|^{-2}|S_i - y|_+^{-1}] \leq 2C |i|_+^{-1/2}|x|_+^{-1}|y - x|_+^{-1}.
\end{align}

**Proof.** Denoting by $\phi,(x) := (2\pi s)^{-1/2} e^{-\frac{|x|^2}{2s}}$ the density at $x$ of the Gaussian law of $\beta_s$, we get from (1.1) after change of variables that
\begin{equation}
E[|\beta_t - x|^{-2}] = 2\pi^2 E[G_0(\beta_t, x)] = 2\pi^2 \int_0^\infty \phi,(x) dx = t^{-1} \varphi_1(|x|^2/t) = |x|^{-2} \varphi_2(t/|x|^2),
\end{equation}
for the finite decreasing functions $\varphi_1(r) := \frac{1}{2} \int_0^\infty u^{-2} e^{-r(2\pi/u)} du$, $\varphi_2(r) := \frac{1}{2} \int_0^\infty u^{-2} e^{-1/(2\pi/u)} du$. Thus, (4.6) holds for any $C \geq \varphi_1(0) \vee \varphi_2(0)$. Next, by Cauchy-Schwarz and (4.6),
\begin{equation}
E(|\beta_t - x|^{-1}|\beta_t - y|^{-1}) \leq (E[|\beta_t - x|^{-2}])^{1/2} (E[|\beta_t - y|^{-2}])^{1/2} \leq C t^{-1/2}|y|^{-1}.
\end{equation}
Exchanging $x$ with $y$ yields the first inequality in (4.7), whereby the second inequality follows (as $|x - y| \leq |x| + |y| \leq 2|x| \vee |y|$). Now, by the triangle inequality, for $\beta_t \neq x \neq y$,
\begin{equation}
|y - x| |\beta_t - x|^{-2}|\beta_t - y|^{-1} \leq |\beta_t - x|^{-1} (|\beta_t - x|^{-1} \geq |\beta_t - y|^{-1}),
\end{equation}
so taking the expectation using (4.6) and (4.7) to bound the rhs, results with (4.8).

With $S_0 = 0$, clearly (4.9) holds at $i = 0$, while for $i \geq 1$, recall [26, Thm. 1.2.1 and 1.5.4] that for some $C \geq 0$ finite and any $x \in \mathbb{Z}^4$,
\begin{align}
(4.12) \quad & P(S_i = x) \leq C t^{-2} \left[ e^{-2|x|^2/i} + (|x| \vee i)^{-1} \right], \\
(4.13) \quad & C^{-1}|x|_+^{-2} \leq G(0, x) \leq C|x|_+^{-2}.
\end{align}
By (4.13),
\begin{equation}
E[|S_i - x|_+^{-2}] \leq CE[G(S_i, x)] = C \sum_{\ell=i}^\infty P(S_\ell = x).
\end{equation}
Further, for some $C$ finite and all $i \geq 1$, $x \in \mathbb{Z}^4$,

$$\sum_{i=1}^{\infty} E^{-2}(|x|^2 \vee \ell)^{-1} \leq C p^{-1} \min\{1, |x|^2\}.$$ 

Thus, in view of (4.12), the same computation as for (4.6) yields the first inequality of (4.9). The second inequality of (4.9) follows (as $a \wedge b \leq \sqrt{ab}$ for any $a, b > 0$), and since the strictly positive $| \cdot |$, satisfies the triangle inequality on $\mathbb{Z}^4$, we get first (4.10), and then (4.11), by the same reasoning that led to (4.7) and (4.8), respectively.

**Proof of Lemma 4.1.** From (4.11), for any $p \geq 2$, $1 \leq s_1 \leq \cdots \leq s_p$ and $(y_1, \ldots, y_p) \in \mathbb{Z}^4$,

$$E\left[ \prod_{i=1}^{p-1} |S_{s_i} - y_i|_{+}^{-2} |S_{s_{p-1}} - y_p|_{+}^{-1} \right]$$

$$= E\left[ \prod_{i=1}^{p-2} |S_{s_i} - y_i|_{+}^{-2} |S_{s_{p-2}} - (y_{p-1} - S_{s_{p-2}})|_{+}^{-2} |S_{s_{p-1}} - S_{s_{p-2}} - (y_p - S_{s_{p-2}})|_{+}^{-1} \right]$$

$$\leq C s_{p-1} - s_{p-2}|^{-1/2} |y_p - y_{p-1}|_{+}^{-1} E\left[ \prod_{i=1}^{p-1} |S_{s_i} - y_i|_{+}^{-2} |S_{s_{p-2}} - y_{p-1}|_{+}^{-1} \right],$$

where we set throughout $s_0 = 0$ and $y_0 = 0$. Thus, by induction on $p \geq 1$,

$$E\left[ \prod_{i=1}^{p} |S_{s_i} - y_i|_{+}^{-2} \right] \leq C p^{-1} \prod_{i=1}^{p-1} |s_i - s_{i-1}|_{+}^{-1/2} \prod_{i=1}^{p} |y_i - y_{i-1}|_{+}^{-1}. \tag{4.14}$$

Next, note that by (4.9), for any $p \geq 1$,

$$E\left[ \prod_{i=1}^{p} |S_{s_i} - \tilde{S}_i|_{+}^{-2} \right] \leq E\left[ \prod_{i=1}^{p-1} |S_{s_i} - y_i|_{+}^{-2} |S_{s_{p-1}} - (y_p - S_{s_{p-1}})|_{+}^{-2} \right]$$

$$\leq C |s_p - s_{p-1}||^{-1/2} E\left[ \prod_{i=1}^{p-1} |S_{s_i} - y_i|_{+}^{-2} |S_{s_{p-1}} - y_p|_{+}^{-1} \right].$$

Hence, setting $t_0 = 0$, in view of (4.14) and the independence of $(S_i)$ and $(\tilde{S}_i)$,

$$E\left[ \prod_{i=1}^{p} |S_{s_i} - \tilde{S}_i|_{+}^{-2} \right] \leq C p \prod_{i=1}^{p} |s_i - s_{i-1}|_{+}^{-1/2} E\left[ \prod_{i=1}^{p} |\tilde{S}_i - \tilde{S}_{i-1}|_{+}^{-1} \right]. \tag{4.15}$$

Suppose that $t_{\sigma(1)} \leq \cdots \leq t_{\sigma(p)}$ for some permutation $\sigma$ of $\{1, \ldots, p\}$. Then, conditioning on $(\tilde{S}_i, i \leq t_{\sigma(p-1)})$, we get by (4.10) and the independence of increments, that when $\sigma(p) = \ell$,

$$E\left[ \prod_{i=1}^{p} |\tilde{S}_h - \tilde{S}_{t_{\ell-1}+1}|_{+}^{-1} \right] \leq C |t_{\sigma(p)} - t_{\sigma(p-1)}|_{+}^{-1/2} E\left[ \prod_{i=t_{\ell-1}+1}^{\ell-1} |\tilde{S}_h - \tilde{S}_{t_{\ell-1}+1}|_{+}^{-1} |\tilde{S}_{t_{\ell-1}+1} - \tilde{S}_{t_{\ell-1}+1}|_{+}^{-1} \prod_{i=\ell+2}^{p} |\tilde{S}_h - \tilde{S}_{t_{\ell-1}+1}|_{+}^{-1} \right].$$

Any permutation $\sigma$ of $\{1, \ldots, p\}$ with $\sigma(p) = \ell$ must be a bijection from $\{1, \ldots, p-1\}$ to $\{1, \ldots, \ell-1, \ell+1, \ldots, p\}$. We can thus further bound the right-side of the preceding inequality, inductively according to the values of $\sigma(j)$, for $j = p-1, \ldots, 1$, and thereby arrive at

$$E\left[ \prod_{i=1}^{p} |\tilde{S}_h - \tilde{S}_{t_{j-1}+1}|_{+}^{-1} \right] \leq C \prod_{j=1}^{p} |t_{\sigma(j)} - t_{\sigma(j-1)}|_{+}^{-1/2}, \quad \sigma(0) := 0. \tag{4.16}$$
Combining (4.15) and (4.16), we conclude that for any non-decreasing \((s_i)\) and \((t_{r(j)})\),

\[
E\left[\prod_{i=1}^{p} |S_{s_i} - \tilde{S}_{s_i}|_\ast^2\right] \leq C^{2p} \prod_{i=1}^{p} |s_i - s_{i-1}|_\ast^{-1/2} \prod_{j=1}^{p} |t_{r(j)} - t_{r(j-1)}|_\ast^{-1/2}.
\]

We next bound \(E[X_n^p]\) by (4.13) and enumerate over all words \((s_i)\) and \((t_i)\) of length \(p\) with symbols from \([1, n]\), according to the numbers \(k\) and \(k'\) of distinct symbols in each word. Recalling that \(|1|_\ast = 1\) and having at most \(k^p\) words of length \(p\) composed of given (fixed) \(k\) distinguished symbols, we thus deduce from (4.17) that for any \(n, p \in \mathbb{N}\),

\[
E[X_n^p] \leq \frac{1}{n^p} C^p E\left[\sum_{s_i : i \in [1, n]} \prod_{i=1}^{p} |S_{s_i} - \tilde{S}_{s_i}|_\ast^{-2}\right] \leq C^{2p} \sum_{k=1}^{k^p} n^{-(p-k)/2} J_{k,n}^2,
\]

\[
J_{k,n} := \frac{1}{n^k} \sum_{1 \leq s_1 < \cdots < s_k \leq n} \prod_{i=1}^{k} \left(\frac{s_i}{n} - \frac{s_i-1}{n}\right)^{-1/2}.
\]

Considering \(\theta_i = s_i/n\), we see that \(\{J_{k,n}\}\) are for each \(k \in \mathbb{N}\), the Riemann sums of

\[
J_k := \int_{0=0,\ldots,=1} \prod_{i=1}^{k} (\theta_i - \theta_{i-1})^{-1/2} d\theta_1 \cdots d\theta_k \leq C^k (k!)^{-1/2}
\]

(for the inequality, see e.g. [30, proof of Lemma 2]). Note that \(J_k = (Q^k 1)(0)\) for the positive linear operator \((Q f)(x) = \int_0^1 y^{-1/2} f(x + y) dy\) on \(C([0, 1])\). Setting \((y)_k = [yn]/n\), we have that \(J_{k,n} = (Q^k 1)(0)\) for the positive linear operators \((Q_n f)(x) = \int_0^1 (y)^{-1/2} f((x)_n + (y)_n) dy\). It is easy to see that \((Q_n f) \leq (Q f)\) are both non-increasing whenever \(f(\cdot)\) is non-increasing. By induction on \(k \geq 0\), we thus have that \(Q^k 1 \leq Q^1\), pointwise, so in particular \(J_{k,n} \leq J_1\) for any \(k, n \in \mathbb{N}\). Further, \(k! \geq (k/e)^k\) and \(J_{k,n} = 0\) unless \(k \leq n\). Hence, in view of (4.18) and (4.19) we find that

\[
E[X_n^p] \leq C^{2p} \left[\sum_{k=1}^{k^p} k^{p-1/2} J_k^2\right] \leq C^{2p} \left[\sum_{k=1}^{k^p} k^{(p+k)/2} C^{k} (k/e)^{-k/2}\right] \leq p^2 C^4 p^p p^p,
\]

and the uniform moment bounds (4.4) on \(X_n\) follow.

For any interval \(I\), consider the range \(R_I = \{s_i\}_{i \in I}\) and \(\tilde{R}_I = \{\tilde{s}_i\}_{i \in I}\) of independent swr-s. Fixing \(\alpha > 0\), let \(n_\alpha := n (\log n)^{-\alpha}\) and denoting by \(\hat{P}\) and \(\hat{\tau}_A\) the probability and the hitting time to a set \(A\) by another independent swr \((\tilde{S}_i)\), set for \(i \in [n_{2\alpha}, n - n_{2\alpha}]\),

\[
g_{n,\alpha}(i) := 1_{[s_i, \tau_R(i+n_{2\alpha})]} \tilde{B}^S(\tau_{\tilde{R}_I(i+n_{2\alpha})} = \infty),
\]

with \(g_{n,\alpha}(i) = 1\) for \(i \in [0, n_{2\alpha}) \cup (n - n_{2\alpha}, n]\), and \(\tilde{g}_{n,\alpha}(i)\) defined analogously for the swr \((\tilde{S}_i)\). In particular, for any \(i \in [n_{2\alpha}, n - n_{2\alpha}]\),

\[
\tilde{g}_{n,\alpha} := E[1_{[s_{2\alpha}, n_{2\alpha}]} \tau_{\tilde{R}_I(2\alpha, n_{2\alpha})} = \infty]] = E[g_{n,\alpha}(i)].
\]

The next lemma, whose proof is deferred to the end of this section, allows us to complete the

proof of the limsup-lil lower bound for swr on \(\mathbb{Z}^4\), by comparing

\[
Y_{n,m} := \sum_{i \in [1,n], \ell \in [1,m]} g_{n,\alpha}(i) G(S_i, \tilde{S}_\ell) \tilde{g}_{m,\alpha}(\ell),
\]

with the simpler to analyze

\[
\tilde{Y}_{n,m} := \tilde{g}_{n,\alpha} \tilde{g}_{m,\alpha} \sum_{i \in [1,n], \ell \in [1,m]} G(S_i, \tilde{S}_\ell),
\]

whose moments we bound in Lemma 4.1 (for \(m = n\), see also (4.37) for the decay of \(\tilde{g}_{n,\alpha}\)).
Lemma 4.3. We have that $E Y_{n,m} \leq C \sqrt{n m}$ for some $C$ finite and all $m, n \in \mathbb{N}$, and if $n^s \leq m \leq n$, then for some $C = C(\epsilon, \alpha) < \infty$ and any $\epsilon > 0$, $\alpha > 4$,

\[(4.22)\quad E Y_{n,m} \leq C \sqrt{n m} (\log n)^{-2},\]

\[(4.23)\quad E \left[ (Y_{n,m} - \bar{Y}_{n,m})^2 \right] \leq C n m (\log n)^{-\alpha/2}.
\]

Proof of the limsup-lil lower bound. Equipped with Lemma 4.1 and Lemma 4.3, we derive the limsup-lil lower bound for srw on $\mathbb{Z}^d$, and with $n \mapsto R_n$ having a similar structure as $|R_n|$ for the srw on $\mathbb{Z}^2$, we adapt the proof in [12, Prop. 4.4] of the limsup-lil lower-bound for the latter sequence. Specifically, set $p = [(\kappa + \log_2 n)/(\log 2)]$ with $\kappa < \infty$ large and $k = 2^p$ (so $k = \gamma \log_2 n$ for small $\gamma \leq e^{-\epsilon}$). Centering both sides of (4.2) for $n_j = jm$, $m = n/k$ (assumed for simplicity to be integer), we have that for i.i.d. $U_j := R_{(j-1)m,jm}$ and the non-random $\varphi_n := ER_n$,

\[(4.24)\quad \bar{R}_n = k \varphi_{n/k} - \varphi_n + \sum_{j=1}^k U_j - \sum_{j=1}^{k-1} V_{0,jm,(j+1)m}.
\]

Further, denoting by $\theta$ the time shift $S_i \mapsto S_{i+1}$, we set

$$
\chi(A, B) := \sum_{y \in A} \sum_{z \in B} P^y(\tau_{A \cup B} = \infty) G(y, z) P^z(\tau_B = \infty),
$$

and for all $i \in (0, n]$,

\[(4.25)\quad h_n(i) := 1_{[S_i \not\in R(i,n)]} \hat{P}^i(\hat{\tau}_{R_n} = \infty) \leq g_{n, \alpha}(i)
\]

(see (4.20)), recalling from (4.1) and [7, Prop. 1.6] that

$$
0 \leq V_{0,jm,(j+1)m} \leq \chi(\mathcal{R}_{jm}, \mathcal{R}(jm, (j + 1)m)) + \chi(\mathcal{R}(jm, (j + 1)m), \mathcal{R}_{jm})
$$

\[
\leq 2 \sum_{y \in \mathcal{R}_{jm}} \sum_{z \in \mathcal{R}(jm, (j+1)m)} P^y(\tau_{\mathcal{R}_{jm}} = \infty) G(y, z) P^z(\tau_{\mathcal{R}(jm, (j+1)m)} = \infty)
\]

\[
= 2 \sum_{i=1}^{jm} \sum_{t=1}^m h_{jm}(i) G(S_i, S_{jm+\ell}) h_m(\ell) \circ \theta_{jm}
\]

\[(4.26)\quad \leq 2 \sum_{i=1}^{jm} \sum_{t=1}^m g_{jm, \alpha}(i) G(S_i, S_{jm+\ell}) g_m(\ell) \circ \theta_{jm} := 2W_j.
\]

Setting in addition

\[(4.27)\quad W_j := \hat{g}_{jm, \alpha} \hat{g}_m, \alpha \sum_{i=0}^{m-1} \sum_{\ell=1}^m G(S_{jm-i}, S_{jm+\ell}),
\]

\[(4.28)\quad \hat{W}_j := \hat{g}_{jm, \alpha} \hat{g}_m, \alpha \sum_{i=1}^{jm-1} \sum_{\ell=1}^m G(S_{jm-i}, S_{jm+\ell}),
\]

we see that for any fixed $j$,

\[(4.29)\quad W_j - \hat{W}_j \overset{d}{=} Y_{jm,m} - Y_{jm,m}.
\]

Next, following [7], we let

$$
\chi_n(i, j) := \chi(\mathcal{R}_{n}^{(i,j-1)}, \mathcal{R}_{n}^{(i,j)}), \chi(\mathcal{R}_{n}^{(i,j)}, \mathcal{R}_{n}^{(i,j-1)}), \text{ for } \mathcal{R}_{n}^{(i,j)} := \mathcal{R}[(j-1)2^{-i}n, j2^{-i}n],
$$
and take the expected value in [7, Prop. 2.3], to arrive at

\[
(4.30) \quad k\varphi_{n/k} - \varphi_n = \sum_{i=1}^{p} \sum_{j=1}^{2^{i-1}} E[X_n(i, j)] - \sum_{i=1}^{p} \sum_{j=1}^{2^{i-1}} E[\epsilon_n(i, j)].
\]

In [7, Prop. 2.3] it is shown that for \( p \) fixed, the non-negative right-most sum is at most \( C(\log n)^2 \). The same applies for our choice of growing \( p = p(n) \). Indeed, as each \( \epsilon_n(i, j) \) is bounded by the intersection of the ranges of two independent srw-s of length \( n/2^i \), we have that \( \max_{i,j} E[\epsilon_n(i, j)] \leq \log n \) (see [26, Section 3.4]), and with at most \( 2^p \leq C \log n \) such terms, we conclude that

\[
(4.31) \quad \sum_{i=1}^{p} \sum_{j=1}^{2^{i-1}} E[X_n(i, j)] - C(\log n)^2 \leq k\varphi_{n/k} - \varphi_n \leq \sum_{i=1}^{p} \sum_{j=1}^{2^{i-1}} E[X_n(i, j)].
\]

Recall [7, Prop. 6.1], that

\[
\lim_{n\to\infty} \frac{2(\log n)^2}{\pi^2 n} E[X_n(1, 1)] = \int_{A_1^1} E[G_{\beta}(\beta, \beta)]d\pi s dt, \quad \text{for} \quad A_1^1 = [0, 2^{-1}) \times (2^{-1}, 1].
\]

Now, using (1.1) (at \( d = 4 \), with \( \int_{A_1} |t-s|^{-1}d\pi s dt = \log 2 \) and \( E[\beta_1|\beta|^2] = \frac{1}{2} \), we see that

\[
\int_{A_1} E[G_{\beta}(\beta, \beta)]d\pi s dt = \frac{1}{2\pi^2} E[|\beta|^{-2}] \int_{A_1} |t-s|^{-1}d\pi s dt = \frac{\log 2}{4\pi^2}.
\]

By definition it follows that \( E[X_n(i, j)] = E[X_n(1, 1)] \) for \( n' := n2^{1-i} \) and all \( i, j \). So, in view of the preceding, we deduce that for any \( p = o(\log n) \), as \( n \to \infty \),

\[
\max_{i\leq p} \left\{ \frac{2(\log n')^2}{\pi^2 n'} E[X_n'(1, 1)] - \frac{\log 2}{4\pi^2} \right\} \to 0.
\]

Recall also (see (1.5)), that

\[
p = (1 + o(1)) \frac{\log_3 n}{\log 2}, \quad h_4(n) = \frac{\pi^2 n \log_3 n}{8 (\log n)^2}.
\]

It thus follows that

\[
(4.32) \quad \sum_{i=1}^{p} \sum_{j=1}^{2^{i-1}} E[X_n(i, j)] = (1 + o(1)) p \frac{\log 2}{4\pi^2} \frac{\pi^4 n}{2(\log n)^2} = (1 + o(1)) h_4(n),
\]

and combining (4.31) and (4.32) we arrive at

\[
(4.33) \quad k\varphi_{n/k} - \varphi_n = (1 + o(1)) h_4(n).
\]

In view of (4.24), (4.26) and (4.33), we get our limsup-lim lower bound, precisely as in [12, Proof of Prop. 4.4], once we find for any \( \varepsilon > 0 \), constants \( c_1 < \infty, c_2 > 0 \) and for any \( k = 2^p, m = n/k, p \) as above, some events \( G_k \) such that \( P(G_k) \geq \frac{1}{4} c_2^k \) and

\[
(4.34) \quad G_k \subseteq \bigcap_{j=1}^{k} \{ U_j \geq -\frac{c_1 m}{(\log m)^2} \} := \bigcap_{j=1}^{k} B_j,
\]

\[
(4.35) \quad G_k \subseteq \{ \sum_{j=1}^{k-1} W_j \leq 3\varepsilon n \log_3 n \}.
\]
To this end, it suffices to construct events $F_k$ such that for some $c_3 < \infty$,

$$P(F_k) \geq \frac{c_1}{k^2}, \quad F_k \subseteq \left\{ \max_{j < k} \left( \frac{\hat{W}_j}{m} \right) \leq \frac{c_3 m}{(\log m)^2} \right\}_{j=1}^{k} B_j. \tag{4.36}$$

Indeed, we shall see that $P(C_i) \leq \frac{1}{4} c_2$ for $k \leq \gamma \log_2 n$, $\gamma > 0$ small and $n \to \infty$, where

$$C_1 := \left\{ \sum_{j \text{ odd}} W_j > \epsilon n \log_3 n \left( \frac{m}{\log m} \right)^2 \right\}, \quad C_2 := \left\{ \sum_{j \text{ even}} W_j > \epsilon n \log_3 n \left( \frac{m}{\log m} \right)^2 \right\}, \quad C_3 := \left\{ \max_{j < k} (W_j - \hat{W}_j) > \frac{m}{(\log m)^2} \right\}.$$

Taking $G_k := F_k \cap_{i=1}^{3} C_i^c$, this would imply that $P(G_k) \geq \frac{1}{4} c_2^k$ for large $k$, and it is easy to check, as stated, that both (4.34) and (4.35) then hold for such $G_k$.

Next, utilizing the union bound, (4.29), Markov’s inequality and (4.23), we get that

$$P(C_3) \leq \sum_{j=1}^{k-1} P(W_j - \hat{W}_j \geq \frac{m}{(\log m)^2}) = \sum_{j=1}^{k-1} P(Y_{jm,m} - Y_{jm,m} = \frac{m}{(\log m)^2}) \leq C(\log m)^{4-a/2} \sum_{j=1}^{k-1} j \leq Ck^2(\log m)^{4-a/2}. \tag{4.37}$$

In particular, for $\alpha > 8 + 2\gamma \log(1/c_2)$, $k$ as above and $n = mk \geq n_0$, the preceding bound implies that $P(C_3) \leq \frac{1}{4} c_2^k$. Turning to deal with $\hat{W}_j$ and $\hat{W}_j$, upon expressing (4.21) via the independent sww-s $\hat{S}_i$, $\hat{S}_i^+: = S_{n+i} - S_n$ and $\hat{S}_i^- := S_{n-i} - S_n$, at $n' = n_{2\alpha}$, it follows from [5, (1.4)] that

$$\hat{g}_{n,a} = E \{ 1_{[0,\gamma_3]} \hat{g}^\beta (\hat{\tau}_{R_1^+} \cup \hat{\tau}_{(0,n'-1)}) = \infty \} \tag{4.38}$$

(note that $\log n' = (1 + o(1)) \log n$). In view of (4.27), we note that $\{ m^{-1}(\hat{g}_{m,a})^2 W_j \}$ are, for odd $j$, independent copies of $X_m$ of Lemma 4.1 (except for now including also $\ell = 0$ in $X_m$). It thus follows from (4.5) and (4.37) that for some $c > 0$, and all $k, m$,

$$E \left[ \exp \left( c m^{-1}(\log m)^2 \sum_{j \text{ odd}} \hat{W}_j \right) \right] \leq \exp(k/c).$$

Hence, for $n = mk$, one has by Markov’s inequality that

$$P(C_1) \leq \exp(-ck \log_3 n) \exp(k/c)$$

decays as $n \to \infty$, faster than $\frac{1}{4} c_2^k$. By the same reasoning, this applies also for $C_2$.

Finally, in view of (4.13), (4.28) and (4.37), for some $C < \infty$ and any $m, j$,

$$\hat{W}_j \leq \frac{Cm^2}{(\log m)^2} \sum_{j=1}^{j-1} \text{dist}(R((s-1)m, sm), R((jm, (j+1)m)))^{-2}. \tag{4.38}$$

As in [12, Proof of Prop. 4.4], fixing a unit vector $u$, we let $F_j := \cap_{i=1}^{j} (A_i \cap B_i)$, while taking here $B_j$ of (4.34), and

$$A_j := \left\{ S_{im} \subset B(i \sqrt{m} u, \sqrt{m}/8), R((i-1)m, im) \subset B \left( (i - \frac{1}{2}) \sqrt{m} u, \frac{3}{4} \sqrt{m} \right) \right\}. \tag{4.39}$$
The event $F_k$ guarantees that for any $s < j$, the distance of $R((s - 1)m, sm]$ from $R((j \cdot m, (j + 1)m))$ be at least $(j - s - 1/2) \sqrt{m}$, so \ref{eq:3.4.1} results with the rhs of \ref{eq:3.4.0} (for $c_1 = C \sum (r - 1/2)^{-2}$ finite). As for the lhs of \ref{eq:3.4.0}, recall \cite[Thm. 1.2]{7} that \((\frac{\log{m}}{m}) \mathbb{E}_m\) converges in law, hence is a uniformly tight sequence. In particular, for any $\delta > 0$ there exists $c_1 = c_1(\delta)$ finite, such that $P(B_1^c) \leq \delta$ for $B_1$ of \ref{eq:3.4.1}, uniformly in $m$. Further, $\{B_j, j \geq 1\}$ are i.i.d. and by the invariance principle, there exists $c_2 > 0$ such that,

$$
\lim_{m \to \infty} \inf_{S_0 \in [0, \sqrt{m}/8]} \{P^S(A_1) = \inf_{\beta_\ell \in [0, 1/4]} \{P(\beta_\ell < 1/4, \sup \{\beta_i - u < 3/2\}) \geq 2c_2\}.
$$

As $F_j$ is measurable on $S(S, i \leq jm)$, by the Markov property of the srw and its independent, stationary increments, for any $j \geq 1$,

$$
P(A_j \cap B_j \cap F_{j-1}) \geq \inf_{S_0 \in [0, \sqrt{m}/8]} \{P^S(A_1) - P(B_1^c) \geq c_2\},
$$

provided $\delta > 0$ is small enough and $m \geq m_0$ finite, thereby establishing the rhs of \ref{eq:3.4.0}. ■

We conclude this sub-section by proving Lemma 4.3. To this end, note that

\begin{equation}
E[(Y_{n,m} - Y_{n,m})^2] = \frac{1}{2} \sum_{(i_1, i_2) \in [1, n]^2} \sum_{(i_1, i_2) \in [1, m]^2} E[g \cdot \mathbb{G}],
\end{equation}

where the sum is over the two permutations $\pi$ of $\{1, 2\}$ and

\begin{equation}
\begin{split}
g & := \left( g_{n,a}(i_1) \tilde{g}_{m,a}(e_{\ell_1}) - \tilde{g}_{n,a} \tilde{g}_{m,a}(i_1) \right) \left( g_{n,a}(i_2) \tilde{g}_{m,a}(e_{\ell_2}) - \tilde{g}_{n,a} \tilde{g}_{m,a}(i_2) \right), \\
\mathbb{G} & := G(S_{i_1}, \tilde{S}_{e_{\ell_1}})G(S_{i_2}, \tilde{S}_{e_{\ell_2}}).
\end{split}
\end{equation}

Setting for $\alpha > 0$,

$$
I_{\alpha}(n) := \left\{ n_{\alpha}, n - n_{\alpha} \right\} \cap \left\{ (i, j) : j - i \geq n_{\alpha} \right\},
$$

the key to \ref{eq:3.2.0} is to bound $|E[g \cdot \mathbb{G}]|/E[\mathbb{G}]$ uniformly over $(i_1, i_2) \in I_{\alpha}(n)$ and $(\ell_1, \ell_2) \in I_{\alpha}(m)$. For $(i_1, i_2) \in I_{\alpha}(n)$, $n' = n_{2\alpha}$, we will show that the contribution from the complement of

$$
H_{i_1}^{(n)} := \left\{ |S_{i_1} - S_{i_1 - n'}| \leq \sqrt{n'} \log n' \right\},
$$

$$
H_{i_1, i_2}^{(n)} := H_{i_1}^{(n)} \cap \left\{ |S_{i_2} - S_{i_2 - n'} + S_{i_1 + n'} - S_{i_1}| \leq \sqrt{n'} \log n' \right\},
$$

is negligible and the same applies for the analogous events $\tilde{H}_{i_1}^{(m)}$, $\tilde{H}_{i_1, i_2}^{(m)}$ defined in terms of the srw $(\tilde{S}_{i})$, $(\tilde{\ell}_1, \tilde{\ell}_2) \in I_{\alpha}(m)$ and $m' = m_{2\alpha}$. Further, from \ref{eq:3.4.2} it follows that $E[g] = 0$ for such $(i_1, i_2)$ and $(\ell_1, \ell_2)$, allowing us to instead bound (in terms of $E[\mathbb{G}]$), the value of

$$
\left| E[g \cdot \mathbb{G} 1_{H_{i_1, i_2}^{(n)} \cap \tilde{H}_{i_1, i_2}^{(m)}}] - E[g 1_{H_{i_1, i_2}^{(n)} \cap \tilde{H}_{i_1, i_2}^{(m)}}] E[\mathbb{G}] \right|.
$$

Decomposing the events $H_{i_1}^{(n)}$ and $H_{i_1, i_2}^{(n)}$ as

$$
H_{i_1}^{(n)} = \bigcup_{\ell \leq n' \log n'} H_{i_1}^{(n')(\ell)}, \quad H_{i_1, i_2}^{(n)} = \bigcup_{\ell, \ell' \leq n' \log n'} H_{i_1, i_2}^{(n')(\ell, \ell')},
$$

$$
H_{i_1}^{(n')(\ell)} := \{|S_{i_1} - S_{i_1 - \ell} = u\}, \quad H_{i_1, i_2}^{(n')(\ell, \ell')} := H_{i_1}^{(n')(\ell)} \cap \{|S_{i_2} - S_{i_2 - \ell} + S_{i_1 + \ell} - S_{i_1} = v\},
$$

and such decomposition for $\tilde{H}_{i_1, i_2}^{(m)}$, we show in the sequel that given $H_{i_1}^{(n')(\ell)(u, v)} \cap \tilde{H}_{i_1, i_2}^{(m')(\tilde{u}, \tilde{v})}$, makes $\mathbb{G}$ independent of $g$, whereby the following estimates shall be utilized.
Lemma 4.4. Fix $\alpha > 2$, $\epsilon > 0$ and a permutation $\pi$ of $\{1, 2\}$. Then, for $n^\epsilon \leq m \leq n$,

\begin{equation}
F_1(u, \tilde{u}) := E[G(S_{i_1}, \tilde{S}_{\ell_1})|H^{(n)}_{i_1}(u) \cap \tilde{H}^{(m)}_{\ell_1}(\tilde{u})] = (1 + O((\log n)^{2-\alpha}))E[G(S_{i_1}, \tilde{S}_{\ell_1})],
\end{equation}

\begin{equation}
F_2(u, v, \tilde{u}, \tilde{v}) := E[G(S_{i_1}, \tilde{S}_{\ell_1})G(S_{i_2}, \tilde{S}_{\ell_2})|H^{(n)}_{i_2}(u, v) \cap \tilde{H}^{(m)}_{\ell_2}((\tilde{u}, \tilde{v}))
\end{equation}

\begin{equation}
= (1 + O((\log n)^{2-\alpha}))E[G(S_{i_1}, \tilde{S}_{\ell_1})G(S_{i_2}, \tilde{S}_{\ell_2})],
\end{equation}

uniformly over $(i_1, i_2) \in I_n(n)$, $(\ell_1, \ell_2) \in I_n(m)$, $|u|, |v| \leq \sqrt{n^\epsilon} \log n^\epsilon$ and $|\tilde{u}|, |\tilde{v}| \leq \sqrt{m^\epsilon} \log m^\epsilon$.

Proof. For $(i_1, i_2) \in I_n(n)$, the law of $(S_{i_1}, S_{i_2})$ given $H^{(n)}_{i_1}(u, v)$ is as $(u + S^{(1)}_{i_1-n^\epsilon}, u + v + S^{(1)}_{i_2-3n^\epsilon})$ for an independent saw $S^{(1)}_i$. Similarly, when $(\ell_1, \ell_2) \in I_n(m)$, the law of $(\tilde{S}_{\ell_1}, \tilde{S}_{\ell_2})$ given $\tilde{H}^{(m)}_{\ell_2}((\tilde{u}, \tilde{v}))$ is as $(\tilde{u} + \tilde{S}^{(1)}_{i_1-n^\epsilon}, \tilde{u} + \tilde{v} + \tilde{S}^{(1)}_{i_2-3n^\epsilon})$. Consequently,

\begin{equation}
F_1(u, \tilde{u}) = E[G(u + S_{i_1-n^\epsilon}, \tilde{u} + \tilde{S}^{(1)}_{\ell_1-n^\epsilon})],
\end{equation}

\begin{equation}
F_2(u, v, \tilde{u}, \tilde{v}) = \begin{cases} E[G(u + S_{i_1-n^\epsilon}, \tilde{u} + \tilde{S}^{(1)}_{\ell_1-n^\epsilon})G(u + v + S_{i_2-3n^\epsilon}, \tilde{u} + \tilde{v} + \tilde{S}^{(1)}_{\ell_2-3n^\epsilon})], & \text{if } \pi_1 = 1, \\
E[G(u + S_{i_1-n^\epsilon}, \tilde{u} + \tilde{v} + \tilde{S}^{(1)}_{\ell_2-3n^\epsilon})G(u + v + S_{i_2-3n^\epsilon}, \tilde{u} + \tilde{S}^{(1)}_{\ell_1-n^\epsilon})], & \text{if } \pi_1 = 2.
\end{cases}
\end{equation}

Note that for some $C = C(\epsilon)$ finite, $c = c(\epsilon) > 0$, any $m \geq n^\epsilon$ and $(\ell_1, \ell_2)$,

\begin{equation}
P(H^{(n)}_{\ell_1}c) \leq P(H^{(m)}_{\ell_2}c) \leq 2P(|S_{2m^\epsilon}| > \sqrt{m^\epsilon} \log m^\epsilon) \leq C e^{-c(\log n)^2},
\end{equation}

with the same bound applying also for $P(H^{(n)}_{i_1, i_2}c)$. Now, by (4.13) and (4.17) (at $p = 1, 2$),

\begin{equation}
E[G(S_{i_1}, \tilde{S}_{\ell_1})] \leq C^{-1/2} \ell^{-1/2},
\end{equation}

\begin{equation}
E[G(S_{i_1}, \tilde{S}_{\ell_1})] \leq C^{-1/2} \ell^{-1/2} \ell_1^{-1/2} (\ell_1 - 1)^{-1/2},
\end{equation}

with the RHS of (4.45) and (4.46) being the expected values of $F_1(\cdot)$ and $F_2(\cdot)$ according to the joint law of the corresponding saw increments (for independent saw $S_i$ and $\tilde{S}_\ell$). In view of (4.44), it thus suffices to bound the maximum fluctuation of $F_1(\cdot)$, $s = 1, 2$ over $|u|, |v| \leq \sqrt{n^\epsilon} \log n^\epsilon$ and $|\tilde{u}|, |\tilde{v}| \leq \sqrt{m^\epsilon} \log m^\epsilon$, by $C(\log n)^{2-\alpha}$ times the RHS of (4.45) and (4.46), respectively. To this end, since $F_1(\cdot)$ depends only on $u - \tilde{u}$ and $F_2(\cdot)$ depends only on $u - \tilde{u}$ and $v - \tilde{v}$ if $\pi_1 = 1$, or on $u + v - \tilde{u}$ and $v + \tilde{v}$ if $\pi_1 = 2$, we may wlog fix $u = \tilde{u} = 0$ and consider the maximum fluctuation of

\begin{equation}
F_1(u) = E[G(u + S_{i_1-n^\epsilon}, \tilde{S}^{(1)}_{\ell_1-n^\epsilon})],
\end{equation}

\begin{equation}
F_2(u, v) = \begin{cases} E[G(u + S_{i_1-n^\epsilon}, \tilde{S}^{(1)}_{\ell_1-n^\epsilon})G(v + S_{i_2-3n^\epsilon}, \tilde{S}^{(1)}_{\ell_2-3n^\epsilon})], & \text{if } \pi_1 = 1, \\
E[G(u + S_{i_1-n^\epsilon}, \tilde{S}^{(1)}_{\ell_1-n^\epsilon})G(v + S_{i_2-3n^\epsilon}, \tilde{S}^{(1)}_{\ell_1-n^\epsilon})], & \text{if } \pi_1 = 2.
\end{cases}
\end{equation}

over $|u|, |v| \leq 3 \sqrt{n^\epsilon} \log n^\epsilon$. Further, with both $n_{i_1} / n^\epsilon$ and $m_{\ell_1} / m^\epsilon$ diverging (as $(\log n)^\alpha$), it follows that uniformly over $(i_1, i_2) \in I_n(n)$, $(\ell_1, \ell_2) \in I_n(m)$ and $m \geq n^\epsilon$, the RHS of (4.45) and (4.46) also bound $F_1(0)$ and $F_2(0, 0)$, respectively. Consequently, it suffices to show that for some $C$ finite and all $|u|, |v| \leq 3 \sqrt{n^\epsilon} \log n^\epsilon$,

\begin{equation}
|F_1(u) - F_1(0)| \leq C(\log n)^{2-\alpha} F_1(0),
\end{equation}

\begin{equation}
|F_2(u, v) - F_2(0, 0)| \leq C(\log n)^{2-\alpha} F_2(0, 0).
\end{equation}

Turning to this task, since $t_2 := \ell_2 - \ell_1 - 2m^\epsilon \geq 0$, $t_3 := i_2 - i_1 - 2n^\epsilon \geq 0$, $G(x, y) = G(y - x, 0)$ and $S_{i_1}^{(d)} = -S_i$, we can further simplify the functions $F_1(\cdot)$ to be

\begin{equation}
F_1(u) = E[G(S_{i_1}, u)],
\end{equation}

\begin{equation}
F_2(u, v) = \begin{cases} E[G(S_{i_1}, u)G(S_{i_1+t_2+t_3}, v)], & \text{if } \pi_1 = 1, \\
E[G(S_{i_1+t_2}, u)G(S_{i_1+t_3}, v)], & \text{if } \pi_1 = 2.
\end{cases}
\end{equation}
where \( t_1 = i_1 + \ell_1 - n' - m' \). Denoting by \( p_j(u) := P(S_j = u) \), it is easy to check that

\[
F_1(u) = \sum_{j_0 > t_1} p_{j_0}(u), \quad F_2(u, v) = \sum_{j_1, j_2} p_{j_1}(u)p_{j_2}(v),
\]

where the sum is over \( j_1 > t_1 \) and \( j_2 > t_1 + t_2 + t_3 \) in case \( \pi = 1 \), and over \( j_1 > t_1 + t_2, j_2 > t_1 + t_3 \) when \( \pi = 2 \). By the local \( \mathbb{C} \)fir for the \( \mathbb{S} \)-walk on \( \mathbb{Z}^d \), we have for some \( C < \infty \) that

\[
\left| \frac{p_j(u) + p_{j+1}(u)}{p_j(0) + p_{j+1}(0)} - 1 \right| \leq \frac{\mathbb{C}_{\text{clt}} n^2}{9j} \leq Cn'(\log n')^2 t_1^{-1} \leq 2C(\log n)^{2-\alpha}
\]

throughout the range of parameters considered here (utilizing the fact that \( j_0, j_1, j_2 \geq t_1 \geq n_a/2 \)). The same bound applies with \( v \) instead of \( u \), and plugging these bounds in (4.49) results with (4.47)-(4.48), thereby completing the proof of the lemma.

**Proof of Lemma 4.3.** First observe that \( g \) of (4.40) can be written also as

\[
g = g_0 - g_1 - g_2 + \hat{g}_{n,a} \hat{g}_{m,a},
\]

\[
g_0 := \hat{g}_{n,a}(i_1) \hat{g}_{m,a}(\ell_1) \hat{g}_{n,a}(i_2) \hat{g}_{m,a}(\ell_2),
\]

\[
g_1 := \hat{g}_{n,a}(i_1) \hat{g}_{m,a}(\ell_1) \hat{g}_{n,a}(i_2)
\]

proceeding to show (4.23), note that \( g_{n,a}(\ell) \) and \( \hat{g}_{m,a}(\ell) \) are measurable on \( \mathcal{F}_i := (S_{i+j} - S_j, j \in (-n', n')) \), and \( \mathcal{F}_\ell := (S_{\ell_1+j} - S_{\ell}, j \in (-m', m')) \), respectively. Further, when \((i_1, i_2) \in I_a(n) \) and \((\ell_1, \ell_2) \in I_a(m) \), the law of \( \mathcal{G} \) of (4.41) given \( \mathcal{F}_i, \mathcal{F}_\ell, \mathcal{F}_1, \) and \( \mathcal{F}_2 \) is determined by \( H_{i_1, i_2}^{(a)}(u,v) \cap \hat{H}_{\ell_1, \ell_2}^{(m)}(u,v) \). Thus, for \( s = 0, 1, 2, \) and any such \((i_1, i_2), (\ell_1, \ell_2)\),

\[
E[g, \mathcal{G} 1_{H_{i_1, i_2}^{(a)}(u,v) \cap \hat{H}_{\ell_1, \ell_2}^{(m)}(u,v)}] = E[g \cdot \mathcal{G} 1_{H_{i_1, i_2}^{(a)}(u,v) \cap \hat{H}_{\ell_1, \ell_2}^{(m)}(u,v)}].
\]

With \( g_s \geq 0 \) and \( E[g_s] = \mathbb{E}_{n,a} \hat{g}_{m,a} \) whenever \((i_1, i_2) \in I_a(n) \) and \((\ell_1, \ell_2) \in I_a(m) \), we get from (4.43) (of Lemma 4.4), that for some universal \( C < \infty \),

\[
\left| E[g, \mathcal{G} 1_{H_{i_1, i_2}^{(a)}(u,v) \cap \hat{H}_{\ell_1, \ell_2}^{(m)}}] - E[g, 1_{H_{i_1, i_2}^{(a)}(u,v) \cap \hat{H}_{\ell_1, \ell_2}^{(m)}}] \right| \leq 2Ce^{-\varepsilon(\log n)^2}.
\]

(4.51)

In addition, with \( g_s \in [0, 1], \) \( \mathcal{G} \) uniformly bounded and \((\log m) / (\log n) \geq \varepsilon \), we have from (4.44) that

\[
E[g, \mathcal{G} 1_{H_{i_1, i_2}^{(a)}(u,v) \cap \hat{H}_{\ell_1, \ell_2}^{(m)}}] + E[g, 1_{H_{i_1, i_2}^{(a)}(u,v) \cap \hat{H}_{\ell_1, \ell_2}^{(m)}}] \leq C_1(P(H_{i_1, i_2}^{(a)}) + P(H_{i_1, i_2}^{(a)})) \leq 2Ce^{-\varepsilon(\log n)^2}.
\]

Combining (4.50) and (4.51) for \( s = 0, 1, 2 \), we thus find for the zero-mean

\[
g = g_0 - g_1 - g_2 + \hat{g}_{n,a} \hat{g}_{m,a},
\]

that uniformly over \( \pi, (i_1, i_2) \in I_a(n) \) and \((\ell_1, \ell_2) \in I_a(m) \),

\[
E[\mathcal{G} 1_{H_{i_1, i_2}^{(a)}(u,v) \cap \hat{H}_{\ell_1, \ell_2}^{(m)}}] \leq 3C(\log n)^{2-\alpha} \mathbb{E}_{n,a} \hat{g}_{m,a} \mathbb{E}[\mathcal{G}] + 6Ce^{-\varepsilon(\log n)^2}.
\]

(4.52)
Further, as $|g| \leq 1$, we have from (4.13) and (4.17) at $p = 2$, that for some $C_2 < \infty$ and uniformly over all $(i_1, i_2) \in [1, n]^2$, $(\ell_1, \ell_2) \in [1, m]^2$,

$$
|E[g \cdot G]| \leq E[G] \leq C_2(i_1 \wedge i_2)^{-1/2}|i_2 - i_1|_{+}^{1/2}(\ell_1 \wedge \ell_2)^{-1/2}|\ell_2 - \ell_1|_{+}^{1/2}.
$$

Next note that from (4.37) we have for some $C_3, C < \infty$,

$$
\sum_{i \in [1, n]^2} E[G] \leq \frac{C_3}{(\log n)^2(\log m)^2} \sum_{i \in [1, n]^2} i_1^{1/2}|i_2 - i_1|_{+}^{1/2}\ell_1^{-1/2}|\ell_2 - \ell_1|_{+}^{1/2}
$$

$$
\leq Cnm(\log n)^2(\log m)^2.
$$

With the right most term of (4.52) being $o(n^{-5})$, it follows that the overall contribution to the right-side of (4.39) from $i_1, i_2 \in [n, n - n_a]$ with $|i_2 - i_1| \geq n_a$ and $\ell_1, \ell_2 \in [m, m - m_a]$ with $|\ell_2 - \ell_1| \geq m_a$ is at most $O(nm(\log n)^{-2})$, as specified in (4.23). Further, the sum over the rhs of (4.53) under any of the following three restrictions

$$
|i_2 - i_1| < n_a, \quad i_1 \wedge i_2 < n_a, \quad i_1 \vee i_2 > n - n_a,
$$

is at most $O(nm(n^a/n)) = O(nm(\log n)^{-a})$. With $(\log m)/(\log n) \geq \epsilon$, this applies also when summing the rhs of (4.53) under each of the analogous restrictions $|i_2 - i_1| < m_a, \ell_1 \wedge \ell_2 < m_a$, or $\ell_1 \vee \ell_2 > m - m_a$. As $\alpha/2 < 2 + \alpha$, we have thus established (4.23).

Turning to (4.22), we similarly have from (4.42) of Lemma 4.4 that for some $C < \infty$ and $c > 0$, uniformly over $m \in [n^\epsilon, n], \ell \in [n, n - n_a]$ and $\ell \in [m, m - m_a]$,

$$
E[g_{n,\alpha}(i)g_{m,\alpha}(\ell)G(S_i, \bar{S}_\ell)] \leq [1 + C(\log n)^{2\alpha}] g_{n,\alpha}g_{m,\alpha}E[G(S_i, \bar{S}_\ell)] + 2Ce^{-c(\log m)^2}
$$

(4.54)

(\text{using in the latter inequality also (4.37), (4.13) and (4.17) at } p = 1). \text{ As } \log m \geq \epsilon \log n, \text{ the sum of the rhs of (4.54) over } i \leq n \text{ and } \ell \leq m \text{ is at most as specified (ie. } O(\sqrt{nm}(\log n)^{-2}). \text{ Further, even when } i < n_a \text{ or } \ell < m_a \text{ or } i > n - n_a \text{ or } \ell > m - m_a, \text{ we still get the bound } C(i - i_1/\ell - 1/2 \text{ on the rhs of (4.54). The sum of } i^{-1/2} \ell^{-1/2} \text{ subject to any one of the latter four restrictions is at most } O(\sqrt{nm}n) = O(\sqrt{nm}(\log m)^{-2/3}), \text{ which is as required (for } \alpha > 4).}

Finally, recall that for some $C, C_3 \text{ finite and all } m, n \in \mathbb{N},$

$$
EY_{m,n} \leq \sum_{i=1}^n \sum_{\ell=1}^m E[G(S_i, \bar{S}_\ell)] \leq C_3 \sum_{i=1}^n \sum_{\ell=1}^m i^{-1/2} \ell^{-1/2} \leq C \sqrt{nm},
$$

as claimed. 

\[4.2\]. \text{ The upper bound in the limsup-\lil.} \text{ As in case of the capacity limsup-\lil lower bound, we adapt here the relevant element from the proof of the limsup-\lil of } |\mathcal{R}_n| \text{ an sw } \mathbb{Z}^2, \text{ namely } [12, \text{ Prop. 4.1}.]. \text{ To this end, we first establish a key approximate additivity for } \varphi_n := ER_n.$$

\text{Lemma 4.5. There exists } c' \text{ finite, such that for any } a, b \in \mathbb{N},

$$
0 \leq \varphi_a + \varphi_b - \varphi_{a+b} \leq c' \frac{a^{1/4}(a+b)}{(\log(a+b))^2},
$$

(4.55)

\text{where } \varphi_n := ER_n \text{ and } \alpha := \min(a, b)/(a + b).
Proof. Starting at the expected value of (4.1), we get by the same reasoning we have used in deriving (4.26), that

\[ 0 \leq E[V_{a,a+a+b}] = \varphi_a + \varphi_b - \varphi_{a+b} \leq 2E \sum_{i=1}^a \sum_{j=1}^b g_{a,a}(i)G(S_i, \tilde{S}_j)\tilde{g}_{b,a}(\ell) = 2EY_{a,b}. \]

Assuming wlog that \( a \leq b \), it thus suffices to verify that \( EY_{a,b} \leq Ca^{1/4}b^{3/4}/(\log b)^2 \) (yielding (4.55) for some \( c'(C) < \infty \)). Indeed, for \( a \geq \sqrt{b} \) this follows from (4.22), whereas if \( a < \sqrt{b} \) then even the bound \( EY_{a,b} \leq C\sqrt{ab} \) which we have from Lemma 4.3, suffices.

Recall (4.1) that \( R_{a+b} - R_a - R_b \circ \theta_a = -V_{0,a,a+b} \leq 0 \) for any \( a, b \geq 0 \). This implies a non-random bound on the difference of such centered variables, yielding in terms of the non-random \( c' \) of Lemma 4.5 the upper bound

\[ (4.56) \quad \overline{R}_{a+b} - \overline{R}_a - \overline{R}_b \circ \theta_a \leq c' \left( \frac{\min(a, b)}{a+b} \right)^{1/4} \frac{(a+b)}{(\log(a+b))^2}. \]

Utilizing (4.56), we next establish sharp tail estimates for \( \max_{j \leq n} [\overline{R}_j] \) (in particular, improving upon [21, Lemma 2.5]).

**Lemma 4.6.** For some \( c > 0, C < \infty \) and all \( n \),

\[ (4.57) \quad E[e^{cD(n)}] \leq C, \quad D(n) := \frac{(\log n)^2}{n} \max_{0 \leq j \leq n} [\overline{R}_j]. \]

**Proof.** Note that (4.57) matches the statement of [12, (4.4)] for \( G_j := G_j^0 := \frac{(\log n)^2}{n} \overline{R}_j \), \( j \leq n \). It is easy to check that the proof of [12, (4.3) & (4.4)] applies verbatim for any variables \( \{G_j\} \) that satisfy [12, (4.5) & (4.6)], and furthermore, that their argument applies even if the power \( a^{1/2} \) on the right-most term in [12, (4.5)] is replaced by \( a^{1/4} \). Indeed, this is what we have here, with (4.56) yielding that for some non-random \( c_1 < \infty \) and all \( a \leq j \leq n \),

\[ G_j^0 - G_a^0 \leq G_{j-a}^0 \circ \theta_a + c_1 \left( \frac{a}{j} \right)^{1/4}. \]

To finish the proof note that for some \( c_2 < \infty \) and all \( a, b \geq 0 \), we get from [7, Cor. 1.5] that

\[ (4.58) \quad E[(\overline{R}_b \circ \theta_b)^2] \leq \frac{c_2b^2}{(\log b)^4}, \quad \text{hence} \quad E[(G_j^0 \circ \theta_a)^2] \leq c_2 \frac{2^4}{n^2(\log j)^4}, \]

which is precisely [12, (4.6)].

For the limsup-i.i.d. upper bound, by Borel-Cantelli it suffices to show that for any \( q > 1, \gamma > 0 \) and \( \varepsilon > 0 \),

\[ (4.59) \quad \sum_i P \left( \frac{(\log m)^2}{m} \max_{r_{i+1} \leq \ell \leq r_i} \overline{R}_\ell \geq \left( \frac{\pi^2}{8} + 2\varepsilon \right) k \log k \right) < \infty, \]

where \( r_i = q^i, k = 2^p \) for \( p = [(\log \gamma + \log_3 r_i)/\log 2] \) and \( m = [r_i/k] \). Now, considering (4.2) for \( n_j = jm, j < k' \) and \( n_{k'} = \ell \) it follows from (4.55) that

\[ (4.60) \quad \frac{(\log m)^2}{m} \max_{(k-1)m \leq \ell \leq k'm} [\overline{R}_\ell] \leq \sum_{j=1}^{k'} D_j^m + \frac{(\log m)^2}{m} \left( k'\varphi_m - \varphi_{k'm} \right) + c'. \]
where $D_j^{(m)}$ are i.i.d. copies of $D^{(m)}$ of Lemma 4.6. With $k' \mapsto (k' \varphi_m - \varphi_{k'm})$ non-decreasing and $D^{(m)} \geq 0$, the maximum over $k' \leq k$ of the rhs of (4.60), is attained at $k' = k$. Further, en route to (4.33), we showed that as $p = o(\log m) \rightarrow \infty$,

\begin{equation}
\frac{1}{k \log k} \frac{(\log m)^2}{m} (k \varphi_m - \varphi_{km}) \rightarrow \frac{\pi^2}{8}.
\end{equation}

Thus, noting that (4.57) results with

$$P\left( \sum_{j=1}^{k} D_j^{(m)} \geq ek \log k \right) \leq C^k e^{-ek(\log k)},$$

which is summable over $i$ for our choice of $k = \gamma \log i$, we have established (4.59) and thereby completed the proof of the limsup-lil.

4.3. Non-random and positive liminf-lil. Setting hereafter $\tilde{h}_d(n) := n(\log_2 n)/(\log n)^2$, we first show that the $[-\infty, \infty]$-valued,

$$c_* := \liminf_{r \rightarrow \infty} \left\{ \tilde{R}_r \right\},$$

is non-random. Indeed, recall (2.6) that $\text{Cap}(\mathcal{R}_r) - \text{Cap}(\mathcal{R}[k, r]) \in [0, k]$. Thus, for any $k$ finite, changing $\mathcal{R}_k$ without altering $S_k$ yields at most a difference of $k$ in the value of $\text{Cap}(\mathcal{R}_r)$, implying by the Hewitt-Savage zero-one law that $c_*$ is non-random.

Turning next to show that $c_* > 0$, it suffices to establish this for the sub-sequence $r_{j+1} = r_j + 2n_j$, where $n_j := 2^{j^2}$, $r_0 := 0$ and we proceed to show that infinitely many $-\tilde{R}_{r_{j+1}}$ are at least of $O(\tilde{h}_d(n))$, due to heavy tails of the non-negative variables $V_{0,n,2n}/\tilde{h}_d(n)$. Specifically, setting $Q_n := |\mathcal{R}(0, n) \cap \mathcal{R}(n, 2n)|$ we have from [7, Prop. 1.6] that

\begin{equation}
V_{0,n,2n} \geq 2 \inf_{j, \ell \in [1, 2n]} \{ G(S_j, S_\ell) \} R_n R_{n,2n} - Q_n.
\end{equation}

Recall from [26, Section 3.4] that $E Q_n \leq C_0 \log n$ for some $C_0 < \infty$ and all $n$. Hence, by Markov’s inequality

\begin{equation}
P(\tilde{A}_n^\circ) := P(Q_n \geq (\log n)^3) \leq C_0 (\log n)^{-2}.
\end{equation}

Further, recall [7, (1.4) and Cor. 1.5], that $E R_n \geq n/(\log n)$ and $\text{Var}(R_n) \leq C_1 n^2/(\log n)^4$ for some $C_1 < \infty$ and all $n$ large, in which case by Markov’s inequality

$$P(R_n \leq \frac{n}{2 \log n}) \leq \left( \frac{2 \log n}{n} \right)^2 \text{Var}(R_n) \leq \frac{4 C_1}{(\log n)^2}.$$ 

Consequently, by the union bound,

\begin{equation}
P(\tilde{A}_n^\circ) := P\left( \min(R_n, R_{n,2n}) \leq \frac{n}{2 \log n} \right) \leq 8C_1 (\log n)^{-2}.
\end{equation}

Next, from (4.13) we have that

$$F_{k,m} := \left\lfloor \max_{j \leq 2km} |S_j| \leq \sqrt{m} \right\rfloor \implies \inf_{j, \ell \in [1, 2km]} \{ G(S_j, S_\ell) \} \geq (4 C m)^{-1}.$$ 

Setting $c = 1/(10C)$ it thus follows from (4.62) that for all $n = km \geq n'(C)$, on the event $G_n := F_{k,m} \cap \tilde{A}_n \cap A_n$,

$$V_{0,n,2n} \geq 2 \frac{n}{2 \log n} - (\log n)^3 \geq \frac{cnk}{(\log n)^2}.$$
Similarly to our derivation of the l.h.s of (4.36), it follows from the invariance principle that 
\( P(F_{k,m}) \geq c_k^k \) for some \( c_k > 0 \) and any \( k, m \geq 1 \). By (4.63)-(4.64) this implies in turn that 
\( P(G_n) \geq \frac{1}{2} c_n^k \) for \( k = 2^p = [\gamma \log_2 n] \), provided \( \gamma := \gamma \log(1/c_2) < 2 \) and \( n \geq n' \). To summarize, if \( c' = c \gamma > 0 \) and all \( n \geq n' \),
\[
P\left(V_{0,n,2n} \geq c'\overline{h}_4(n)\right) \geq (\log n)^{-\gamma}.
\]
The same applies for the mutually independent \( \{V_{r,r+n,r+r_1}\} \), hence upon fixing \( \gamma' < 1/2 \) we get by the second Borel-Cantelli lemma, that a.s.
\[(4.65) \limsup_{j \to \infty} \{\overline{h}_4(n_j)^{-1}V_{r,r+n,r+r_1}\} \geq c' > 0.\]

Now, as in (4.24), for any \( r \geq 0, n \geq 1 \),
\[(4.66) \overline{R}_{r,r+2n} = 2\overline{\varphi}_n - \overline{\varphi}_{2n} + \overline{R}_{r,r+n} + \overline{R}_{r+n,r+2n} - V_{r,r+n,r+2n}.
\]
Considering (4.31) for \( p = 1 \) (that is, \( k = 2 \)), we see that as \( n \to \infty \),
\[(4.67) \frac{\overline{h}_4(n)^2}{n} \leq \frac{\overline{h}_4(n)^2}{n} E[\chi_{2n}(1,1)] \to \frac{\pi^2 \log 2}{4}.
\]
Further, recall (4.59) that for any \( \delta > 0 \),
\[(4.68) \sum_j P(\overline{R}_{n_j} \geq (1 + \delta)\overline{h}_4(n_j)) < \infty.
\]
The same applies of course also for \( \overline{R}_{n_j} \circ \theta^j \) and \( \overline{R}_{n_j} \circ \theta^{j+n} \), so with \( \overline{h}_4(n)/\overline{h}_4(n) \to 0 \), we deduce from (4.65)-(4.68) (at \( n = n_j \) and \( r = r_j \)), that a.s.
\[
\liminf_{j \to \infty} \left\{ \frac{\overline{R}_{r,j+1}}{\overline{h}_4(n_j)} \right\} \leq \limsup_{j \to \infty} \left\{ \overline{h}_4(n_j)^{-1}V_{r,r,n,r,r_1+1} \right\} \leq -c'.
\]
Now from (4.56) (at \( a = r_j, b = 2n_j \),
\[
\overline{R}_{r,j+1} \leq \overline{R}_{r} + \overline{R}_{r,j+1} + \frac{c'r_{j+1}}{(\log r_{j+1})^2},
\]
and since \( r_{j+1} \leq 3n_j \), dividing by \( \overline{h}_4(n_j) \) and taking limits, yields that
\[
-3c* \leq \liminf_{j \to \infty} \left\{ \overline{h}_4(n_j)^{-1}\overline{R}_{r,j+1} \right\} \leq -c' + \limsup_{j \to \infty} \left\{ \overline{h}_4(n_j)^{-1}\overline{R}_r \right\}.
\]
The last term is a.s. zero (as (4.68) applies also for \( \{r_j\} \) instead of \( \{n_j\} \) and \( \overline{h}_4(n)/\overline{h}_4(n) \to 0 \),
so we conclude that \( c* \geq c'/3 > 0 \).

### 4.4. Finiteness of the \( \liminf \)-lil.  
We show that \( c* \leq c_o < \infty \) by following the proof of the upper bound of [11, Thm. 1.7] (on the \( \liminf \) \( \text{lil} \) of \( |R_n| \) in \( \mathbb{Z}^2 \)), while replacing [11, Thm. 1.5] and [11, Lemma 10.3], respectively, by
\[(4.69) \sup_n \left\{ (\log n)(\log_2 n)^2 P\left( -\overline{R}_n > c_o\overline{h}_4(n) \right) \right\} < \infty,
\]
\[(4.70) \sup_n \left\{ (\log n)^2 P\left( \max_{n/2k \leq k \leq n} (\overline{R}_n - \overline{R}_k) > \epsilon\overline{h}_4(n) \right) \right\} < \infty,
\]
holding for some \( c_o < \infty \), any \( \epsilon > 0 \) and some \( q_0(\epsilon) > 1 \).

Similarly to \( |R_n| \), the capacity is sub-additive (see (4.1)), and upon centering satisfies (4.56), which is the analog of [11, (10.2)]. Thus, the bound (4.70) follows as in [11, Proof.
of Lemma 10.3, now using (4.57) to arrive at [11, (10.14)] and to bound the rhs of [11, (10.15)]).

Since $|R_n - R_{n'}| \leq |n - n'|$, it suffices to prove (4.69) only for some $\{n_i\}$ such that $n_i+1 - n_i = o(h_4(n_i))$. We take here all integers of the form $n = mk$, $k = 2^p$, $p = |(\log_2 n)/ \log 2|$ (thus with gaps of size $k = O(\log n) = o(h_4(n_i))$). Setting such values and $n'_u := 2^{-u}n$ for $1 \leq u \leq p$, we have as in (4.24), now using an alternative expression for $\Delta_{n,k}$ of (4.2), that

$$
(4.71) \quad -R_n = \Delta_{n,k} - \sum_{j=1}^k \bar{U}_j, \quad \Delta_{n,k} = \sum_{u=1}^p \sum_{j=1}^{2^{u-1}} V_{(2j-2)n',(2j-1)n',2n'},
$$

with $k$ i.i.d. copies $\{\bar{U}_j\}$ of $\bar{R}_m$ and the i.i.d. variables $\{V_{(2j-2)n',(2j-1)n',2n'}\}$ per fixed $u \geq 1$. Since $\Var(\bar{R}_m) \leq C_1 m^2/(\log m)^4$ it follows by Markov’s inequality, that

$$
(4.72) \quad P\left(- \sum_{j=1}^k \bar{U}_j \geq \epsilon \bar{c}_o \bar{h}_4(n)\right) \leq (\epsilon \bar{c}_o \bar{h}_4(n))^{-2} \frac{C_1 km^2}{(\log m)^4} \leq \frac{2C_1}{(\epsilon \bar{c}_o)^2 k (\log n)^2}.
$$

Setting $\bar{c}_o > (1 - \epsilon)^{-1} \epsilon_1^{-1}$, we arrive at (4.69) out of (4.71), (4.72) and the following lemma.

**Lemma 4.7.** For some $c_1 > 0$ and any $\lambda > 0$,

$$
(4.73) \quad \limsup_{n \to \infty} \frac{1}{\log_2 n} \log P(\Delta_{n,k} \geq \lambda \bar{h}_4(n)) \leq -c_1 \lambda,
$$

where $\Delta_{n,k}$ are as in (4.3) for $k = 2^p$ and $p = |(\log_2 n)/ \log 2|$.

We note in passing [21, Lemma 2.6], which is somewhat related to Lemma 4.7. The proof of Lemma 4.7 relies in turn on our next result.

**Lemma 4.8.** Set $p = |(\log_2 n)/ \log 2|$ and for any $r \in \mathbb{N}$, the partition $I_r^{(r)} := ((i-1)r, ir]$ of $\mathbb{N}$. Consider for $n'_u := 2^{-u}n$ and each $0 \leq u \leq p$, the i.i.d. variables

$$
\alpha_j^{(n'_u)} := \frac{1}{n} \sum_{i \in I_{2j-1}^{(n'_u)}} \sum_{\ell \in I_{2j}^{(n'_u)}} G(S_i, S_\ell), \quad 1 \leq j \leq 2^{u-1}.
$$

Then, for some $c_2 > 0$,

$$
(4.74) \quad \sup_{n \in \mathbb{N}} E[e^{2\Theta_n}] < \infty, \quad \Theta_n := \sum_{u=1}^p \sum_{j=1}^{2^{u-1}} \alpha_j^{(n'_u)}.
$$

**Proof.** With the swrf having independent and symmetric increments, one easily verifies that $\alpha_j^{(n'_u)} \overset{(d)}{=} 2^{-u}X_u$, for $X_n$ of Lemma 4.1. Consequently, from (4.4) and (4.5) we know that

$$
(4.75) \quad \varphi(\lambda) := \sup_{u, n \in \mathbb{N}} E[\exp(\lambda 2^u \alpha_j^{(n'_u)})] \leq 1 + c_2 \lambda^2 < \infty,
$$

for some $c < \infty$ and all $\lambda > 0$ small enough. The uniform mgf bound of (4.74) then follows as in [30, Page 177, Proof of Thm. 1], upon setting $a_n = \alpha_j^{(n'_u)}$, $c_2 = b_{\infty} > 0$ of [30], and noting that (4.75) suffices in lieu of both [30, Lemma 2] and the scale invariance of [30, property (ii)].
Proof of Lemma 4.7. For \( u \leq p \), consider the i.i.d. variables \( W_{u,j} := W^{(u')} \circ \theta_{(2j-2)\nu'} \), with

\[
W^{(u)} := \sum_{i,\ell \in [1, m]} g_{m,\alpha}(i)G(S_{i}, S_{\ell+m})g_{m,\alpha}(\ell) \circ \theta_{m}
\]

having the law of \( W_1 \) of (4.26) and their (i.i.d.) approximations \( \tilde{W}_{u,j} := n^{2}\tilde{g}_{m,\alpha}^{2}(u) \) (for \( g_{m,\alpha}(\cdot) \) and \( \tilde{g}_{m,\alpha} \) of Lemma 4.3). Setting

\[
Z_u := \sum_{j=1}^{2^{u-1}} W_{u,j}, \quad \tilde{Z}_u := \sum_{j=1}^{2^{u-1}} \tilde{W}_{u,j},
\]

it follows by Cauchy-Schwarz and (4.23) that for some \( C < \infty \), any \( u \leq p \) and all \( n \),

\[
E[(Z_u - \tilde{Z}_u)^2] \leq 2^{(u-1)} E[(W_{u,1} - \tilde{W}_{u,1})^2] \leq Cn^2(\log n)^{-\alpha/2}.
\]

In particular, taking \( \alpha > 8 + 2c_1\lambda \), yields, by the union bound and Markov’s inequality that for any \( \epsilon > 0 \),

\[
P(\sum_{u=1}^{p} (Z_u - \tilde{Z}_u) \geq \epsilon \tilde{h}_4(n)) \leq \sum_{u=1}^{p} P(Z_u - \tilde{Z}_u \geq \epsilon p^{-1} \tilde{h}_4(n))
\]

\[
\leq C' p^3 (\epsilon \tilde{h}_4(n))^{-2} n(\log n)^{-\alpha/2} \leq C'' \epsilon^{-2} (\log n)^{-c_1\lambda}.
\]

We also find for our choice of \( k \), see (4.61), that

\[
E \Delta_{n,k} = \frac{\pi^2}{8} \tilde{h}_4(n)(1 + o(1)).
\]

Similarly to (4.26), we have that \( V_{(2j-2)\nu',(2j-1)\nu',2ju'} \leq 2W_{u,j} \) for any \( u, j \). In view of (4.71) and (4.77) it thus suffices to establish (4.73) with \( \tilde{\Delta}_{n,k} \) replaced by

\[
2 \sum_{u=1}^{p} Z_u - \frac{\pi^2}{8} \tilde{h}_4(n),
\]

which in view of (4.37) and the definition of \( \Theta_n \), can be further replaced by

\[
\frac{\pi^2}{8} \frac{n}{(\log n)^2} (\frac{\pi^2}{4} + o(1)) \Theta_n - \log_2 n.
\]

Moreover, by (4.76),

\[
\left| E \sum_{u=1}^{p} Z_u - E \sum_{u=1}^{p} \tilde{Z}_u \right| \leq O\left( \frac{pn}{(\log n)^{\alpha/4}} \right) = o(\tilde{h}_4(n)),
\]

and combining [7, Lemma 6.5] with the considerations as in (4.44) and after (4.54), we deduce that for large \( m \),

\[
2E[W^{(u)}] = (1 + o(1))E[V_{0,m,2m}].
\]

It then follows that

\[
2 \sum_{u=1}^{p} EZ_u = (1 + o(1)) \frac{\pi^2}{8} \tilde{h}_4(n)
\]

and consequently,

\[
\frac{\pi^2}{4} E \Theta_n = (1 + o(1)) \log_2 n.
\]
Thus it suffices to establish (4.73) with $\overline{\alpha}_{a,k}$ replaced by
\[
\frac{n^4}{32 (\log n)^2} \Theta_n,
\]
which in turns follows from (4.74), upon setting $c_1 = 32\pi^{-d}c_2 > 0$.

5.11 for srw on $\mathbb{Z}^d$, $d \geq 5$: Proof of Theorem 1.4. Since $R_n$ for the srw on $\mathbb{Z}^d$, $d \geq 5$, has similar structural properties to the size of the range of the srw on $\mathbb{Z}^{d-2}$, we establish Theorem 1.4 by adapting the proof in [12, Section 3] for the latter of the latter sequence. Specifically, setting $\rho_n := \sqrt{n} \log n$ when $d = 5$ and otherwise $\rho_n := \sqrt{n}$, we have from [36, Thm. A] in case $d = 5$, and from [5, Lemma 3.3] when $d \geq 6$, that
\[(5.1) \lim_{n \to \infty} \frac{1}{\rho_n} \| R_n \|_2 = \sigma_d.\]
Next, recalling for integers $0 \leq a \leq b \leq c$, the notations of (4.1),
\[(5.2) R_{a,b} := \text{Cap}(\mathcal{R}(a,b)), \quad V_{a,b,c} := R_{a,b} + R_{b,c} - R_{a,c} \geq 0,\]
we proceed with the following variant of [5, Lemma 3.2].

**Lemma 5.1.** For any $0 \leq a < b$, set
\[
\overline{V}_{a,b} := \sup_{t \geq b} \{ V_{a,b,t} \}, \quad \hat{V}_{a,b} := \sup_{s \leq a} \{ V_{s,a,b} \}.
\]
Then, for some $C_{d,f}$ finite, any $\ell \geq 1$ and all $a < b$,
\[(5.3) E[\overline{V}_{a,b}] \leq C_{d,\ell} f_d(b - a)^\ell, \quad E[\hat{V}_{a,b}] \leq C_{d,\ell} f_d(b - a)^\ell,
\]
where
\[
f_3(n) = \sqrt{n}, \quad f_6(n) = \log n, \quad f_7(n) = 1 \quad \forall d \geq 7.
\]

**Proof.** By the shift invariance of the srw, we may wlog set $a = 0$. Further, in view of [7, (2.9) & (2.11)], for a fixed set $A$, the function
\[
B \mapsto \text{Cap}(A) + \text{Cap}(B) - \text{Cap}(A \cup B)
\]
is non-decreasing (and bounded above by $\text{Cap}(A)$). In particular, the value of $\overline{V}_{0,n}$ is attained for $t \to \infty$. Thus, from [5, Prop. 1.2] we arrive at

\[(5.4) \overline{V}_{0,n} \leq 2 \sum_{x \in \hat{R}_{d,0}} \sum_{y \in \mathcal{R}(n,\infty)} G(x,y), \quad \overline{V}_{0,n} \leq 2 \sum_{x \in \hat{R}_{d,0}} \sum_{y \in \mathcal{R}_{d,0}} G(x,y),\]

where $\hat{R}_{d,0}$ denotes the range of an independent srw. Similarly, the value of $\overline{V}_{0,n}$ is attained at $s \to -\infty$, with the right-side of (5.4) also bounding $\overline{V}_{0,n}$ (we then have $\mathcal{R}(\infty,0]$ instead of $\mathcal{R}(n,\infty)$ in (5.4)). Thereafter, adapting [5, Section 3.1] yields (5.3). Indeed, with $p_{2\ell}(x,y) := P^{\ell}(S_{2\ell} = y)$ the square of a transition probability, we have as in the proof of [5, Lemma 3.1], that for even $k \geq 0$ and any $a \in \mathbb{Z}^d$,
\[
s_a := \sum_{x,y \in \mathbb{Z}^d} G(0,x)G(0,y)p_k(x,y + a) \leq s_0.
\]
In case of a lazy srw, this applies for any $k \geq 0$, so summing over $k \leq n$ yields that
\[(5.5) \max_{a \in \mathbb{Z}^d} \left\{ \sum_{x,y \in \mathbb{Z}^d} G(0,x)G(0,y)G_n(x,y + a) \right\} = \sum_{x,y \in \mathbb{Z}^d} G(0,x)G(0,y)G_n(x,y) \leq C_d f_d(n),\]
which is the desired result (5.3).
where we have utilized [5, (3.4)] for the latter inequality. Further, as in [5], up to an increase of \(C_d\) value, (5.5) extends to the original srw. Now, similarly to [5, (3.5)], it follows from (5.4) that

\[
E[\hat{V}_{0,n}^\ell] \leq 2^\ell \sum_{x} E \left[ \prod_{i=1}^{\ell} L_n(x_i) \right] E \left[ \prod_{i=1}^{\ell} L_\infty(y_i) \right] \prod_{i=1}^{\ell} G(x_i, y_i),
\]

where \(L_n(x)\) denotes the total srw local time at \(x \in \mathbb{Z}^d\), during time interval \([1, n]\) (and the same bound applies for \(E[\hat{V}_{0,n}^\ell]\)). For \(\ell = 1\) we thus get (5.3) out of (5.5) (as \(E[L_n(x)] = G_n(0, x)\) and \(E[L_\infty(y)] = G(0, y)\)). The general case then follows by an inductive argument, as in the proof of [5, Lemma 3.2], utilizing also that \(a = 0\) is optimal in (5.5).

Utilizing Lemma 5.1, we next establish the analog of [12, Lemma 3.3] for \(R_n\).

**Lemma 5.2.** For any \(d \geq 5\), \(m \geq 3\), there exists \(c_m\) finite, such that for all \(b > a \geq 0\)

\[
\|R_{b} - R_{a}\|_m \leq c_m \rho_{b-a}.
\]

Further, for some \(\tilde{c}_m\) finite and any \(\lambda > 0\), \(b > a \geq 0\),

\[
P(\max_{m(a,b)} |R_{n} - R_{a}| > \lambda \rho_{b-a}) \leq \tilde{c}_m \lambda^{-m}.
\]

**Proof.** From (5.2) we see that \(V_{0,a,b} = R_{a} - R_{a,b} - R_{b} \in [0, \hat{V}_{a,b}]\),

for \(\hat{V}_{a,b}\) of Lemma 5.1. In particular, for any \(m \geq 3\), \(b > a\),

\[
0 \leq E[V_{0,a,b}]^m \leq E[V_{a,b}^m] \leq E[\hat{V}_{a,b}] \leq C_{d,m} f_d(b - a)^m = o(\rho_{b-a}^m).
\]

We can thus replace \(R_{b} - R_{a}\) in (5.6) by \(R_{a,b}\) and thereby due to the shift invariance of the increments, set \(wlog\ \rho_{a} = 0\) (whereupon \(R_{a} = 0\)). Hence, analogously to [12, (3.34)], it suffices for (5.6) to show inductively over \(\ell \geq 1\), that \(\sup_{n}[L_{n,2\ell}]\) is finite for

\[
L_{n,\ell} := \frac{1}{\rho_{n}^\ell} \|\hat{R}_{n}\|_\ell.
\]

The induction basis \(\ell = 1\) is merely (5.1). Further, with

\[
\lim_{m \to \infty} \rho_{2n}^{-1} \sup_{a < 2n} \|\hat{V}_{0,a,2n}\|_\ell = 0,
\]

by the preceding decomposition, we can and shall replace \(R_{2n}\) in the induction step, by

\[
R_{n} + R_{n,2n} \overset{d}{=} R_{n} + \hat{R}_{n},
\]

where \(\hat{R}_{n}\) denotes the capacity of the range of an independent second srw. For any \(\ell \geq 2\), by the induction hypothesis \(\sup_{n}[L_{n,k}]\) are finite for all \(k \leq 2(\ell - 1)\), hence

\[
\sup_{n,k} \left( \sum_{k=2}^{2\ell-2} \binom{2\ell}{k} t^k L_{n,k}^{2(\ell-k)} \right) \leq c_\ell < \infty.
\]

Recalling that \(\rho_{n} \leq 2^{-1/2} \rho_{2n}\), we thus get similarly to [12, (3.37) & (3.38)] that

\[
L_{2n,2\ell} \leq o_{n}(1) + \left( 2^{-\ell(1/2)} L_{n,2\ell} + 2^{-\ell/2} c_\ell \right)^{1/(2\ell)},
\]
Thus, sup \( R_{2j} \) is finite. Finally, for any \( n \in \{2^{j-1}, 2^j\}, j \geq 2 \), we have as in the preceding that
\[
R_{2j} \overset{d}{=} R_n + \hat{R}_{2j-n} - V_{0,n,2j}.
\]

Upon centering, taking the \( 2\ell \)-th power and isolating the \( 2\ell \)-th power of \( R_n \), the preceding identity results with
\[
[L_{2\ell/2\ell} + o_{j}(1)]^{2\ell} + c_{\ell} \geq (\rho_{n}/\rho_{2j})^{2\ell} L_{n,2\ell} \geq 4^{-\ell} L_{n,2\ell}.
\]

Thus, \( \sup_{n} L_{n,2\ell} \) is finite as well, completing the induction step and thereby establishing (5.6). Finally, we get (5.7) out of (5.6) precisely as in deriving [12, (3.39)] out of [12, (3.40)]

Recall the decomposition (4.2)-(4.3), for the independent variables \( U_j := R_{n_{j-1},a_j} \) and any increasing \( \{n_k\} \) starting at \( n_0 = 0 \),
\[
R_{n_k} = \sum_{j=1}^{k} U_j - \Delta_{n_{j,k}}, \quad \Delta_{n_{j,k}} := \sum_{j=1}^{k-1} V_{n_{j-1},a_j,a_k}.
\]

Centering the random variables of the preceding identity we arrive at
\[
\bar{R}_{n_k} = \sum_{j=1}^{k} \bar{U}_j - \bar{\Delta}_{n_{j,k}},
\]

with zero-mean, independent variables \( \bar{U}_j \). Proceeding to show that \( \bar{\Delta}_{n_{j,k}} \) has a negligible effect on \( \bar{R}_{n_k} \), first recall from (5.1) that
\[
\lim_{j \to \infty} \frac{E[|\bar{U}_j|^2]}{\sigma_d^2} = 0,
\]
whereas (5.6) at \( a = n_{j-1}, b = n_j \), amounts to
\[
E[|\bar{U}_j|^m] \leq (c_m \rho_{n_{j-1},a_j})^m.
\]

In case \( d = 5 \) we take the same values of \( \alpha, \beta < 1/2 \) and \( \{n_k\} \) as in the proof of [12, Thm. 2.1]. Lemma 5.1 at \( \ell = 4 \) is then the analog of [12, (3.2)], and utilizing it at \( a = n_{j-1}, b = n_j, j < k \), we find by following verbatim, the derivation of [12, (3.9)], that for some \( c \) finite
\[
\lim_{k \to \infty} \sup \frac{\max_{n_{j,k}} |\Delta_{n_{j,k}}|}{\sqrt{\alpha_k(\log n_k)^\beta}} \leq c \quad \text{a.s.}
\]

Thereafter, substituting (5.12) for [12, (3.9)] and (5.11) to get [12, (3.16)], by the same reasoning as in the proof of [12, Thm. 2.1], we find that a.s.
\[
\lim_{k \to \infty} h_{\ell}(n_k)^{-1} \left[ \bar{R}_{n_k} - \sigma_d B_{\ell, n_k}^2 \right] = 0,
\]

for some one-dimensional standard Brownian motion \( (B_t, t \geq 0) \). As shown after [12, (3.17)] (apart from replacing [12, Lemma 3.3(b)] by (5.7)), the stated i.i.d. is then a direct consequence of Kinchin’s i.i.d. for the latter Brownian motion.

In case \( d \geq 6 \), we take \( \{n_k\} \) again as in the proof of [12, Thm. 2.1], except that now this is done for the choice of \( \alpha = 1 \). Then, by Lemma 5.1, for \( C = C_{d,2} \) and any \( 1 \leq j < k \),
\[
\text{Var}(V_{n_{j-1},n_j,a_k}) \leq C(\log(n_j - n_{j-1}))^2 \leq C(\log n_k)^2.
\]
Thus, for any $\beta > 0$ and all $k$, by Markov’s inequality and the definition of $\Delta_{n,k}$ (see (5.8)),

$$ P\left(\frac{\Delta_{n,k}}{n_k^{\beta}} \leq n_k^{-\frac{2}{\beta}} \text{Var}\left(\Delta_{n,k}\right) \leq C n_k^{-2\beta} k^2 (\log n_k)^2\right). $$

Since $|n_k| \cap [2^\ell, 2^{\ell+1}] | \leq \ell$ for any $\ell \geq 1$, eventually $k \leq (\log n_k)^2$. Hence, by the first Borel-Cantelli lemma, we have that for any $\beta > 0$,

$$ \limsup_{k \to \infty} n_k^{\beta} |\Delta_{n,k}| \leq 1 \quad \text{a.s.} \tag{5.14} $$

We then get (5.13) by following, as for $d=5$, the proof of [12, Thm. 2.1], utilizing again (5.10)-(5.11), while having now, via (5.14) at $\beta < 1/2$, a negligible contribution at scale $\rho_n = \sqrt{n}$ (instead of (5.12) and the scale $\sqrt{n \log n}$ throughout [12, (3.13)-(3.17)]). Finally, recall that $n_{k+1} - n_k \leq n_k / \ell$ whenever $n_k \in [2^\ell, 2^{\ell+1})$. Hence, in view of (5.7) at $m=6$ and $\lambda = \varepsilon h_d(n_k) / \sqrt{n_{k+1} - n_k}$, we have that for some $c_\varepsilon$ finite, any $\varepsilon > 0$ and $n_k \in [2^\ell, 2^{\ell+1})$,

$$ P\left(\max_{n \in (n_k, n_{k+1})} \frac{\left|\bar{R}_n - \bar{R}_{n_k}\right|}{\varepsilon h_d(n_k)} > c_\varepsilon \ell^{-3}\right) \leq c_\varepsilon \ell^{-3}. $$

With at most $\ell$ values of such $n_k$, by the first Borel-Cantelli lemma, the events on the LHS a.s. occur only for finitely many values of $k$ and the stated LIL thus follows, as before, from (5.13).

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