Invariant Operators vs Heisenberg Operators for Time-Dependent Generalized Oscillators

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We investigate the relation between the invariant operators satisfying the quantum Liouville-von Neumann and the Heisenberg operators satisfying the Heisenberg equation. For time-dependent generalized oscillators we find the invariant operators, known as the Ermakov-Lewis invariants, in terms of a complex classical solution, from which the evolution operator is derived, and obtain the Heisenberg position and momentum operators. Physical quantities such as correlation functions are calculated using both the invariant operators and Heisenberg operators.

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I. INTRODUCTION

In the last several decades there have been many attempts to develop formalisms for time-dependent quantum systems and apply them to physical systems, in particular, oscillators with time-dependent mass and frequency. For a time-dependent oscillator Lewis found an invariant operator, which satisfies the quantum Liouville-von Neumann (LvN) equation, and whose eigenfunctions up to time-dependent phase factors satisfy the time-dependent Schrödinger equation. In classical theory Erniovakov had already found such an invariant long before the advent of quantum theory. The Ermakov-Lewis invariant operators since then have been employed to develop the formalism for time-dependent oscillators and applied to many physical systems such as quantum optics. The original Ermakov-Lewis invariants are quadratic in position and momentum operators and the linear invariants are also introduced.

The invariant operators method is a useful and convenient tool to find various kinds of exact quantum states in terms of the solutions of the auxiliary equation and has been widely applied to physical systems. The relation between the Ermakov-Lewis invariant operators and the Heisenberg operators in terms of the evolution operator was noticed by Dodonov and Man’ko. Also the Heisenberg picture was used applied to time-dependent oscillators. However, the relation between the quantum LvN equation for the invariant operators and the Heisenberg equation for the Heisenberg operators has not been seriously investigated.

The purpose of this paper is to clarify the role of the quantum LvN equation in the invariant operators method and compare it with the Heisenberg picture. The Schrödinger and Heisenberg pictures are the two popular descriptions of quantum theory. In the Schrödinger picture, quantum states evolve in time but operators do not change, whereas in the Heisenberg picture, operators evolve in time but quantum states do not change. The quantum theory of systems are equivalently described in both pictures. It is shown that the quantum LvN equation provides another description of quantum theory independently of the Heisenberg picture. In terms of the evolution operator \( \hat{U}(t) \), the Heisenberg operators evolve according to \( \hat{O}_H(t) = \hat{U}^\dagger(t)\hat{O}\hat{U}(t) \) and satisfy the Heisenberg equation, whereas the invariant operators evolve according to \( \hat{O}_L(t) = \hat{U}(t)\hat{O}\hat{U}^\dagger(t) \) and satisfy the LvN equation. To illustrate this point, we apply the quantum theory based on the LvN equation to the time-dependent generalized oscillators. Further, we use the evolution operator from the invariant operators to find the Heisenberg operators in terms of the classical solution.

The organization of the paper is as follows. In Sec. II we discuss the Schrödinger and Heisenberg pictures. We also discuss the difference and relation between the Heisenberg equation and the LvN equation. In Sec. III, using a time-independent generalized oscillator, we illustrate the difference between the invariant operators and the Heisenberg operators. In Sec. IV we apply the invariant operators and the Heisenberg operators to time-dependent oscillators. The linear invariant operators are found directly from the LvN equation in terms of a classical solution. The evolution operator derived from the invariant operator is used to find the Heisenberg operators.

II. INVARIANT OPERATORS VS HEISENBERG OPERATORS

For stationary (time-independent) quantum systems two pictures have been used: the Schrödinger and Heisenberg pictures. In this paper we consider both time-independent and time-dependent systems simultaneously. The Hamiltonian operator for a time-dependent system has time-dependent coefficients of each
Schrödinger operator $\hat{O}_k$:

$$\hat{H}(t) = \sum_k h_k(t)\hat{O}_k. \tag{1}$$

In this sense the time-dependent Hamiltonian is called a Schrödinger operator, though it depends on time explicitly.

In the Schrödinger picture, the quantum states evolve according to the Schrödinger equation with the Hamiltonian $\hat{H}$

$$i\hbar \frac{\partial}{\partial t}|\Psi, t\rangle = \hat{H}(t)|\Psi, t\rangle. \tag{2}$$

All the information of the system is contained in the state $|\Psi, t\rangle$. The physically measurable quantity corresponding to an observable $\hat{O}$ is given by the expectation value

$$\langle \hat{O} \rangle_S(t) = \langle \Psi, t | \hat{O} | \Psi, t \rangle. \tag{3}$$

We introduce the evolution operator satisfying

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t) = \hat{H}(t)\hat{U}(t). \tag{4}$$

The solution is formally written as the time-ordered integral

$$\hat{U}(t) = T \exp\left( -\frac{i}{\hbar} \int_0^t \hat{H}(t')dt' \right). \tag{5}$$

Note that for a time-dependent system, $\hat{H}(t)$ and $\hat{H}(t')$ do not commute in general for $t \neq t'$. For a stationary (time-independent) system with $\hat{H}_0$, the evolution operator is simply given by $\hat{U}(t) = e^{-i\hat{H}_0 t/\hbar}$. The state vector is then written as

$$|\Psi, t\rangle = \hat{U}(t)|\Psi\rangle, \tag{6}$$

where $|\Psi\rangle$ is any state independent of time.

On the other hand, in the Heisenberg picture, quantum states do not change in time but operators evolve as

$$\hat{O}_H(t) = \hat{U}(t)\hat{O}\hat{U}^+(t). \tag{7}$$

The Heisenberg operators $\hat{O}_H(t)$ carrying all the information of the system satisfy the Heisenberg equation

$$i\hbar \frac{\partial}{\partial t} \hat{O}_H(t) + [\hat{H}_H(t), \hat{O}_H(t)] = 0, \tag{8}$$

where $\hat{H}_H(t)$ is the Heisenberg operator

$$\hat{H}_H(t) = \sum_k h_k(t)\hat{O}_{kH}(t). \tag{9}$$

For the stationary system $\hat{U}(t)$ commutes with $\hat{H}_0$, so $\hat{H}_H = \hat{H}_0$. The expectation value of the observable $\hat{O}$ in the Heisenberg picture now takes the form

$$\langle \hat{O} \rangle_H(t) = \langle \Psi | \hat{O}_H(t) | \Psi \rangle = \langle \hat{O} \rangle_S(t). \tag{10}$$

We now introduce another interesting operator, the so-called invariant or the LvN operator,

$$\hat{O}_L(t) = \hat{U}(t)\hat{O}\hat{U}^+(t). \tag{11}$$

The invariant operator $\hat{O}_L(t)$ is the backward evolution of the Schrödinger operator $\hat{O}$, whereas the Heisenberg operator $\hat{O}_H(t)$ is the forward evolution. It follows that the invariant operators satisfy the quantum LvN equation

$$i\hbar \frac{\partial}{\partial t} \hat{O}_L(t) + [\hat{O}_L(t), \hat{H}(t)] = 0. \tag{12}$$

Note that the Hamiltonian $\hat{H}(t)$ appeared in Eq. (12) is a Schrödinger operator, whereas the Heisenberg Hamiltonian operator appears in Eq. (8).

Let us now suppose that the LvN equation (12) be directly solved by some techniques. For time-dependent oscillators, these operators are explicitly given in terms of the solution of an auxiliary equation and widely known as the Ermakov-Lewis invariants. In terms of the eigenstates of an invariant operator $\hat{O}_L(t)$,

$$\hat{O}_L(t)|\lambda, t\rangle = \lambda |\lambda, t\rangle, \tag{13}$$

the exact quantum state of the Schrödinger equation is given by

$$|\Psi, t\rangle = \exp\left( \langle \lambda, t | - \frac{i}{\hbar} \hat{H}(t) + \frac{\partial}{\partial t} |\lambda, t \rangle \right) |\lambda, t\rangle. \tag{14}$$

It follows from Eqs. (11) and (6) that

$$\hat{O}|\lambda\rangle = |\lambda\rangle. \tag{15}$$

Thus invariant operators carry all the information and provide a Hilbert space of quantum states without the direct knowledge of the evolution operator. The expectation value of the observable takes the same value as the Schrödinger picture

$$\langle \hat{O} \rangle_L(t) = \langle \Psi, t | \hat{O} | \Psi, t \rangle = \langle \hat{O} \rangle_S(t). \tag{16}$$

In conclusion, it is shown that the quantum LvN equation (12) self-consistently provides another description of quantum theory because all the invariant operators $\hat{O}_L$ satisfying Eq. (12) carry all information of the system. Therefore, the invariant operators method may be called the LvN picture just as quantum theory based Eq. (8) is called the Heisenberg picture, in both of which time-dependent operators carry all quantum information.

III. TIME-INDEPENDENT GENERALIZED OSCILLATOR

As a simple and illustrative model, we consider the time-independent generalized oscillator

$$\hat{H}_0 = \frac{X_0}{2} \hat{p}^2 + \frac{Y_0}{2} (\hat{p} \hat{q} + \hat{q} \hat{p}) + \frac{Z_0}{2} \hat{q}^2, \tag{17}$$

where \(X_0, Y_0, Z_0\) are constants. The eigenvalues of the operator \(\hat{H}_0\) are given by

$$\lambda_n = X_0 \left( \frac{n^2 \pi^2}{L^2} \right) + \frac{Y_0}{2} (n \pi)^2 + \frac{Z_0}{2} \left( \frac{n^2 \pi^2}{L^2} \right),$$

for \(n = 0, 1, 2, \ldots\), and the corresponding eigenstates are

$$|\lambda_n\rangle = \left( \frac{L}{n \pi} \right)^{1/4} \sin \left( \frac{n \pi x}{L} \right) e^{-i\Phi} \left[ e^{i \frac{\sqrt{Z_0} \pi^2 n^2}{2L^2}} + e^{-i \frac{\sqrt{Z_0} \pi^2 n^2}{2L^2}} \right]^{-1/2} \left( \frac{L}{n \pi} \right)^{1/4}, \tag{18}$$

where \(\Phi = \sqrt{Y_0} n \pi / \sqrt{X_0} L\). The operators \(\hat{O}_L\) are then

$$\hat{O}_L = \sum_n \lambda_n |\lambda_n\rangle \langle \lambda_n|.$$
where it is assumed
\[ \omega_0^2 = X_0 Z_0 - Y_0^2 \geq 0. \] (18)
We introduce the annihilation (lowering) and creation (raising) operators
\[ \hat{a} = \sqrt{\frac{X_0}{2\hbar\omega_0}} \left[ i\hat{p} + \frac{1}{X_0} (\omega_0 + iY_0)\hat{q} \right], \]
\[ \hat{a}^\dagger = \sqrt{\frac{X_0}{2\hbar\omega_0}} \left[ -i\hat{p} + \frac{1}{X_0} (\omega_0 - iY_0)\hat{q} \right]. \] (19)
The annihilation and creation operators satisfy the usual commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\). Inverting Eq. (19) for the position and momentum operators, we obtain
\[ \hat{q} = \sqrt{\frac{\hbar X_0}{2\omega_0}} (\hat{a} + \hat{a}^\dagger), \]
\[ \hat{p} = \sqrt{\frac{\hbar}{2\omega_0 X_0}} [-i(\omega_0 + Y_0)\hat{a} + (i\omega_0 - Y_0)\hat{a}^\dagger]. \] (20)
Then the Hamiltonian takes the form
\[ \hat{H}_0 = \hbar \omega_0 \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \] (21)
Now, using the evolution operator
\[ \hat{U}(t) = e^{-i\omega_0 t(\hat{a}^\dagger + \hat{a})/2}, \] (22)
we find the Heisenberg operators
\[ \hat{a}_H(t) = e^{-i\omega_0 t} \hat{a}, \quad \hat{a}^\dagger_H(t) = e^{i\omega_0 t} \hat{a}^\dagger, \] (23)
and the invariant operators
\[ \hat{a}_L(t) = e^{i\omega_0 t} \hat{a}, \quad \hat{a}^\dagger_L(t) = e^{-i\omega_0 t} \hat{a}^\dagger. \] (24)
The Heisenberg operators \(\hat{a}_H(t)\) satisfy the Heisenberg equation, whereas the invariant operators \(\hat{a}_L(t)\) satisfy the quantum LvN equation. The invariant operator \(\hat{a}_L(t)\) has the positive frequency in contrast with the negative frequency of the Heisenberg operator \(\hat{a}_H(t)\).

The Heisenberg position and momentum operators are obtained from Eq. (20) through the unitary transformation \(U\) as
\[ \hat{q}_H(t) = \sqrt{\frac{\hbar X_0}{2\omega_0}} [\hat{a}_H + \hat{a}^\dagger_H], \]
\[ \hat{p}_H(t) = \sqrt{\frac{\hbar}{2\omega_0 X_0}} [-i(\omega_0 + Y_0)\hat{a}_H + (i\omega_0 - Y_0)\hat{a}^\dagger_H]. \] (25)
On the other hand, the Schrödinger position and momentum operators in terms of the invariant annihilation and creation operators are given by
\[ \hat{q} = \sqrt{\hbar} [u \hat{a}_L + u^* \hat{a}^\dagger_L], \]
\[ \hat{p} = \frac{\sqrt{\hbar}}{X_0} [(\hat{u} - Y_0 u)\hat{a}_L + (\hat{u}^* - Y_0 u^*)\hat{a}^\dagger_L], \] (26)
where \(u(t)\) is the complex solution to the classical equation of motion
\[ u(t) = \sqrt{\frac{X_0}{2\omega_0}} e^{-i\omega_0 t}. \] (27)

**IV. TIME-DEPENDENT GENERALIZED OSCILLATOR**

We now consider a time-dependent generalized oscillator described by the Hamiltonian \([11, 12]\)
\[ \hat{H}(t) = \frac{X(t)}{2} \hat{p}^2 + \frac{Y(t)}{2} (\hat{p} \hat{q} + \hat{q} \hat{p}) + \frac{Z(t)}{2} \hat{q}^2, \] (28)
where \(X, Y\) and \(Z\) explicitly depend on time. The classical equation of motion corresponding to the Hamiltonian \((28)\) is given by
\[ \frac{d}{dt} \left( \frac{\hat{a}}{X} \right) + \left[ XZ - Y^2 + \frac{XY - X\hat{Y}}{X} \right] \left( \frac{u}{X} \right) = 0. \] (29)

In the LvN picture, we look for the operators satisfying the LvN equation \((12)\). Lewis and Riesenfeld found a quartic invariant \([1]\). Following Ref. \([12]\) we introduce a pair of linear invariant operators
\[ \hat{a}_L(t) = \frac{i}{\sqrt{\hbar}} \left[ u^* \hat{p} - \frac{1}{X}(\hat{u}^* - Y u^*) \hat{q} \right], \]
\[ \hat{a}^\dagger_L(t) = -\frac{i}{\sqrt{\hbar}} \left[ u \hat{p} - \frac{1}{X}(\hat{u} - Y u) \hat{q} \right], \] (30)
where \(u\) is a complex solution to the classical equation of motion \((29)\). The Wronskian condition
\[ \text{Wr}\{u^*, u\} = \frac{1}{X} (u\hat{u}^* - u^* \hat{u}) = i, \] (31)
makes the invariant operators satisfy the standard commutation relation at equal time
\[ [\hat{a}_L(t), \hat{a}^\dagger_L(t)] = 1. \] (32)
The Schrödinger position and momentum operators can be expressed in terms of the invariant annihilation and creation operators as
\[ \hat{q} = \sqrt{\hbar} [u \hat{a}_L + u^* \hat{a}^\dagger_L], \]
\[ \hat{p} = \frac{\sqrt{\hbar}}{X} [(\hat{u} - Y_0 u)\hat{a}_L + (\hat{u}^* - Y_0 u^*)\hat{a}^\dagger_L]. \] (33)
Note that Eq. (26) is the constant parameters limit of Eq. (33).

The Fock space of exact quantum states for the Schrödinger equation consists of the number state \(|n, t\) of the number operator \(\hat{N}_L(t) = \hat{a}^\dagger_L(t) \hat{a}_L(t)\), another invariant operator:
\[ \hat{N}_L(t)|n, t\rangle = n|n, t\rangle. \] (34)
The quadratic Ermakov-Lewis invariants \([1, 3]\) are, up to overall constants, the number operator \(\hat{N}(t)\), whose auxiliary field \(\rho\) is the amplitude of the complex solution \(u\) \([12]\). In contrast with the Schrödinger and Heisener
pictures, both the operators and states of the LvN picture depend on time but their eigenvalues (quantum numbers) do not change. For instance, the correlation functions with respect to the number state are given by

\[
\langle n, t | \hat{q}^2 | n, t \rangle = \hbar u^* u (2n + 1), \\
\langle n, t | \hat{p}^2 | n, t \rangle = \frac{\hbar}{X^2} (\hat{u}^* - Y u^*)(\hat{u} - Y u)(2n + 1), \\
\langle n, t | \frac{1}{2} (\hat{q}^4 + \hat{p}^4) | n, t \rangle = \frac{\hbar}{2X} \left[ (\hat{u}^* - Y u^*) u + (\hat{u} - Y u) u^* \right] \\
\times (2n + 1). \quad (35)
\]

The invariant operators \([\hat{a}_L, \hat{a}_L^\dagger] = 0\) are related with the annihilation and creation operators \([\hat{a}, \hat{a}^\dagger] = 1\) through the Bogoliubov transformation

\[
\hat{a}_L(t) = \alpha \hat{a} + \beta \hat{a}^\dagger, \\
\hat{a}_L^\dagger(t) = \alpha^* \hat{a}^\dagger + \beta^* \hat{a}, \quad (36)
\]

where

\[
\alpha = -i \sqrt{\frac{X_0}{2\omega_0}} \left[ \frac{1}{X_0} (i\omega_0 + Y_0) u^* + \frac{1}{X} (\hat{u}^* - Y u^*) \right], \\
\beta = -i \sqrt{\frac{X_0}{2\omega_0}} \left[ \frac{1}{X_0} (-i\omega_0 + Y_0) u^* + \frac{1}{X} (\hat{u}^* - Y u^*) \right]. \quad (37)
\]

The evolution operator \([\hat{U}(t)] = e^{-i\hat{q}t} \hat{a}^\dagger \hat{S}(t) \hat{S}(t) \hat{a}^\dagger \hat{U}^\dagger(t)\]

\[
\hat{U}(t) = e^{-i \frac{1}{2} (\hat{q}^2 + \hat{p}^2)} e^{i \frac{1}{2} \hat{w}_+ \hat{q}^2 / \hbar} e^{i \frac{1}{2} \hat{w}_- \hat{p}^2 / \hbar} e^{i \frac{1}{2} \hat{w}_\pm \hat{q}^2 / \hbar} e^{i \frac{1}{2} \hat{w}_\mp \hat{p}^2 / \hbar}, \quad (43)
\]

has been used in Ref. [17], where \(w\) and \(w_\pm\) are found by directly solving Eq. [14]. It was found there that \(e^{-w}\) and \(e^{-w}\) satisfy the classical equation \([29]\).

Now, using the evolution operator \([33]\), the annihilation and creation operators in the Heisenberg picture are found to be

\[
\hat{a}_H(t) = \hat{U}(t) \hat{a} \hat{U}^\dagger(t) = \alpha \hat{a} - \beta \hat{a}^\dagger, \\
\hat{a}_H^\dagger(t) = \hat{U}(t) \hat{a}^\dagger \hat{U}^\dagger(t) = \alpha^* \hat{a}^\dagger - \beta^* \hat{a}. \quad (44)
\]

Similarly, from Eq. [20] follows the Heisenberg position operator

\[
\hat{q}_H(t) = \hat{U}(t) \hat{q} \hat{U}^\dagger(t) = \sqrt{\frac{\hbar}{X}} [(\hat{u}^* - Y u^*) \hat{a} + (\hat{u} - Y u) \hat{a}^\dagger]. \quad (45)
\]

The Heisenberg momentum operator takes the form

\[
\hat{p}_H(t) = \frac{1}{X} (\hat{q}_H - Y \hat{q}_H) = \sqrt{\frac{\hbar}{X}} [(\hat{u}^* - Y u^*) \hat{a} + (\hat{u} - Y u) \hat{a}^\dagger]. \quad (46)
\]

Note that another Heisenberg operator \(\hat{U}^\dagger \hat{p} \hat{U}\) is not conjugate to \(\hat{q}_H\) except for the time-independent case. In the Heisenberg picture physical quantities are obtained by taking the expectations of Heisenberg operators with respect to time-independent state relevant to the physical system under study. The correlation functions with respect to the number state \(\langle n \rangle\) of \(\hat{a} \hat{a}^\dagger\) in Sec. II are given by

\[
\langle n | \hat{q}_H^2 | n \rangle = \hbar u^* u (2n + 1), \\
\langle n | \hat{p}_H^2 | n \rangle = \frac{\hbar}{X^2} (\hat{u}^* - Y u^*)(\hat{u} - Y u)(2n + 1), \\
\langle n | \frac{1}{2} (\hat{p}_H \hat{q}_H + \hat{q}_H \hat{p}_H) | n \rangle = \frac{\hbar}{2X} \left[ (\hat{u}^* - Y u^*) u + (\hat{u} - Y u) u^* \right] \\
\times (2n + 1). \quad (47)
\]

A few comments are in order. First, as \(u\) and \(u^*\) satisfy Eq. [29], the Heisenberg position operator \([30]\) satisfies the Heisenberg equation \([31]\) for the time-dependent oscillator \([28]\):

\[
\frac{d}{dt} \left( \frac{\hat{q}_H}{X} \right) + \left[ X Z - Y^2 + \frac{XY - X Y}{X} \right] \left( \frac{\hat{q}_H}{X} \right) = 0. \quad (48)
\]

In fact, the general solution to Eq. [43] is

\[
\hat{q}_H(t) = v_1(t) \hat{q}_1 + v_2(t) \hat{q}_2, \quad (49)
\]

where \(v_1\) and \(v_2\) are the solutions to Eq. [29] and \(\hat{q}_1\) and \(\hat{q}_2\) are two independent operators whose commutator does not vanish. The operator \([44]\) is the special case with \(v_1 = \sqrt{\hbar} \hat{a}, v_2 = \sqrt{\hbar} \hat{a}\) and \(\hat{q}_1 = \hat{a}, \hat{q}_2 = \hat{a}^\dagger\). This choice makes the standard commutation relation satisfied at equal time

\[
[\hat{q}_H(t), \hat{p}_H(t)] = i\hbar. \quad (50)
\]

Second, by comparing Eqs. [33] and Eqs. [14] and [16], we see that the invariant operators have the negative (positive) frequency for the creation (annihilation) operator, which is a consequence of the backward evolution of the invariant operator \([20]\). Third, the correlation functions \([33]\) in the LvN picture are the same as those \([14]\) in the Heisenberg picture.
V. CONCLUSION

In this paper we show that the invariant operators method based on the quantum Liouville-von Neumann equation provides another description of quantum theory, a counterpart of the Heisenberg picture. The invariant operators method, which we prefer to call the invariant picture, shares not only the property of the Schrödinger picture that quantum states are time-dependent, but also the property of the Heisenberg picture that operators are time-dependent, backwardly evolving in time. In terms of the evolution operator $\hat{U}(t)$, the Heisenberg operators evolve according to $\hat{O}_H(t) = \hat{U}(t)\hat{O}(t)\hat{U}^\dagger(t)$ and satisfy the Heisenberg equation, whereas the invariant operators to $\hat{O}_I(t) = \hat{U}(t)\hat{O}U^\dagger(t)$ and satisfy the Schrödinger equation. The invariant picture provides another picture in addition to the Schrödinger and the Heisenberg pictures.

To illustrate the invariant picture we apply it to time-independent and time-dependent generalized oscillators.

The time-independent oscillator shows manifestly the difference between the Heisenberg and the invariant pictures. The invariant operators have the opposite signs of frequencies to the Heisenberg operators. The invariant picture is particularly useful to handle time-dependent quantum systems. The Hilbert space of exact quantum states of the time-dependent Schrödinger equation consists of the eigenstates of an invariant operator. For time-dependent generalized oscillators we find two linear invariant operators and use them to derive the evolution operator. The evolution operator is then used to find the Heisenberg position and momentum operators. It is shown that the invariant picture and the Heisenberg picture provide the same physical results.

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