Generalized intelligent states of the $su(N)$ algebra

M. Daoud

Max Planck Institute for the Physics of Complex Systems
Dresden, Germany

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Abstract

Schrödinger-Robertson uncertainty relation is minimized for the quadrature components of Weyl generators of the algebra $su(N)$. This is done by determining explicit Fock-Bargamann representation of the $su(N)$ coherent states and the differential realizations of the elements of $su(N)$. New classes of coherent and squeezed states are explicitly derived.
1 Introduction

Coherent states [1] and squeezed states [2-4] are usually associated to the mini-
mization of the Heisenberg uncertainty relation. However, it was proven that a
relation more accurate should be used to minimize the fluctuations of two observ-
able when their commutator is not a multiple of unity [5-8]. This relation, known
as Schrödinger-Robertson uncertainty inequality [9-10], gives the so-called general-
ized intelligent states (see the pioneering works [5-7]). Following this new way to
generalize the usual coherent states, there has been much interest for generalized
intelligent states for the quadrature components corresponding to the generators of
$su(2)$ and $su(1,1)$ algebras [8, 11-13]. They were also defined for exactly solvable
quantum systems as eigenstate of complex combination of lowering and raising op-
\-erators [14-17].

In this work, we extend this study to higher symmetries by developing an analytial
approach that provides the generalized intelligent states for $su(N)$ algebra. The
approach is based on the Fock-Bargmann representation of $su(N)$ coherent states.
The analytic representation, presented here, enables us to convert the algebraic
eigenvalue equations, arising from the Schrödinger-Robertson uncertainty relation,
into quasi-linear differential equations. Solving these equations, one to obtain the
explicit forms of the needed intelligent states.

The letter is organised as follows. Keeping in mind the utility of the analytic repren-
station of coherent states in determining the intelligent states, we first give the
explicit expression of $su(N)$ coherent states. This construction is based on the
bosonic realization of the algebra $su(N)$. We introduce the Fock-Bargmann space
of entire analytic functions which gives a framework to simplify and to "minimize"
the problem of the derivation of intelligent states. We give the differential actions
of the $su(N)$ elements on this space. In section 3, the analytic representation, thus
constructed, are used to derive the states minimizing the Schrödinger-Robertson
relation for the quadrature components of Weyl elements of $su(N)$ algebra. The
advantage of the analytic approach is clearly established. New classes of coherent
and squeezed, as it will be explained, emerge. In the last section, remarks and a
number of interesting open problems are enumerated

2 Analytic representations of $su(N)$-coherent states

The algebra $su(N)$ is defined by the generators $e_i$, $f_i$, $h_i$ ($i = 1, 2, \ldots, N - 1$) and
the relations

$$[e_i, f_j] = \delta_{ij} h_j$$
$$[h_i, e_j] = a_{ij} e_j \quad [h_i, f_j] = -a_{ij} f_j$$
$$[e_i, e_j] = 0 \quad \text{for} \quad |i - j| > 1$$
$$e_i^2 e_{i\pm 1} - 2e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 = 0 \quad (4)$$
$$f_i^2 f_{i\pm 1} - 2f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0 \quad (5)$$
where \((a_{ij})_{i,j=1,2,\ldots,N-1}\) is the Cartan matrix of \(su(N)\), i.e. \(a_{ii} = 2, a_{i,i+1} = -1\) and \(a_{ij} = 0\) for \(|i-j| > 1\). Many aspects of Lie algebras are best considered after choosing a special type of the representation basis. Since one would write down the \(su(N)\) coherent states, the most convenient choice is the bosonic realization. Indeed, an adapted basis is given in term of \(N\) bosonic pairs of creation and annihilation operators; They satisfy the commutation relations

\[
[a_k^-, a_l^+] = \delta_{kl}
\]

where \(k, l = 1, 2, \ldots, N\). The occupation numbers are \(a_k^+ a_k^-\). The Fock space is generated by the eigenstates \(|n_1, n_2, \ldots, n_N\rangle\) of number operators, namely,

\[
|n_1, n_2, \ldots, n_N\rangle = \frac{(a_1^+)^{n_1} (a_2^+)^{n_2} \cdots (a_N^+)^{n_N}}{\sqrt{n_1!} \sqrt{n_2!} \cdots \sqrt{n_N!}} |0, 0, \ldots, 0\rangle
\]

In this bosonic representation, we define the generators of \(su(N)\) as

\[
e_i = a_i^+ a_{i+1}^- \quad f_i = a_i^- a_{i+1}^+ \quad h_i = a_i^+ a_i^- - a_{i+1}^+ a_{i+1}^-
\]

The generators \(e_i, f_i\) are called step, ladder or Weyl operators. The Cartan sub-algebra is generated by the elements \(h_i\). They act on the representation space of dimension \(\frac{N(N+1)}{2}\) that is obtained from the Fock space, generated by eigenvectors \(|j_1, 0, \ldots, 0\rangle\) of \(h_i\). The generators \(su(N)\) having a nontrivial action (non-vanishing and non-diagonal) on the fiducial vector \(|j_1, 0, \ldots, 0\rangle\) are

\[
F_2 \equiv f_1 \quad F_i = [f_{i-1}, F_{i-1}]
\]

for \(i = 3, 4, \ldots, N\) and \(E_i = F_i^\dagger\). At this stage, one can define the coherent state as

\[
|z_1, z_2, \ldots, z_{N-1}\rangle = D(z_1, z_2, \ldots, z_{N-1}) |j_1, 0, \ldots, 0\rangle
\]

where the displacement operator is

\[
D(z_1, z_2, \ldots, z_{N-1}) = \exp\left(\sum_{i=1}^{N-1} (z_{i+1}^- E_i - z_i E_{i+1})\right).
\]

Expanding the operator \(D(z_1, z_2, \ldots, z_{N-1})\) and using the actions of creation and annihilation operators on the restricted Fock space \(\mathcal{F} = \{|n_1, n_2, \ldots, n_{N-1}\}; n_1 + n_2 + \ldots + n_{N-1} = j_1\}\), one get

\[
|z_1, z_2, \ldots, z_{N-1}\rangle = \sum_{j_1=0}^{j_1} \sum_{j_2=0}^{j_2} \cdots \sum_{j_{N-1}=0}^{j_{N-1}} z_1^{j_1} z_2^{j_2} \cdots z_{N-1}^{j_{N-1}} \times \prod_{j_2} I_{j_2}^1(|z_1\rangle) I_{j_2}^2(|z_2\rangle) \cdots I_{j_{N-1}}^{N-1}(|z_{N-1}\rangle) |j_1 - j_2, j_2 - j_3, \ldots, j_N\rangle.
\]

where

\[
I_{j_{s+1}}^s(|z_s\rangle) = \sum_{k=0}^{\infty} \frac{(-)^k (|z_s|)^k}{(j_{s+1} + 2k)!} P(j_{s+1} + 1, k),
\]
for \( s = 1, 2, \ldots, N - 1 \). The quantities \( P \) occurring in (13) are give by

\[
P(j_{s+1} + 1, k) = P(j_{s+1} + 1, 0) \sum_{l_1=1}^{j_{s+1} + 1} p_s(l_1) \sum_{l_2=1}^{l_1+1} p_s(l_2) \cdots \sum_{l_k=1}^{l_{k-1}+1} p_s(l_k)
\]

(14)

with \( P(j_{s+1} + 1, 0) = \frac{j_{s+1}}{(j_s - j_{s-1})!} \) and \( p_s(l) = (j_s - l + 1)l \). They satisfy the following recursion formula

\[
P(j_{s+1} + 1, k) = \sqrt{p_s(j_{s+1})} P(j_{s+1}, k) + \sqrt{p_s(j_{s+1} + 1)} P(j_{s+1} + 2, k - 1).
\]

(15)

Setting

\[
J_{j_{s+1}}^s(|z_s|) = |z_s|^{j_s} P(j_{s+1} + 1, 0) P_{j_{s+1}}^s(|z_s|),
\]

(16)

we get the first order differential equation

\[
\frac{dJ_{j_{s+1}}^s(|z_s|)}{d|z_s|} = J_{j_{s+1} - 1}^s(|z_s|) - (p_s(j_{s+1} + 1))^2 J_{j_{s+1} + 1}^s(|z_s|).
\]

(17)

The solution of this equation takes the simple form

\[
J_{j_{s+1}}^s(|z_s|) = \frac{1}{j_{s+1}!} (\cos(|z_s|))^{j_{s+1} - 1} (\sin(|z_s|))^{j_{s+1}}.
\]

(18)

and the \( su(N) \) coherent states rewrite as

\[
|\zeta_1, \zeta_2, \cdots, \zeta_{N-1}| = \mathcal{N} \sum_{j_1=0}^{j_1} \sqrt{\frac{j_1!}{j_2!(j_1 - j_2)!}} \zeta_1^{j_1} \sum_{j_2=0}^{j_2} \sqrt{\frac{j_2!}{j_3!(j_2 - j_3)!}} \zeta_2^{j_2} \cdots \\
\times \sum_{j_{N-1}=0}^{j_{N-1}} \sqrt{\frac{j_{N-1}!}{j_1!(j_{N-1} - j_1)!}} \zeta_{N-1}^{j_{N-1}} |j_1 - j_2, j_2 - j_3, \cdots, j_{N}| \]

(19)

where the normalisation constant is given by

\[
\mathcal{N} = (1 + |\zeta_1|^2 + |\zeta_2|^2 + \cdots + |\zeta_1|^2 |\zeta_2|^2 \cdots |\zeta_{N-1}|^2)^{-\frac{1}{2}}
\]

(20)

and the new variables \( \zeta_s \) are defined by \( \zeta_s = \frac{z_s}{|z_s|} \) \( \cos(|z_{s+1}|) \) for \( s = 1, 2, \cdots, N - 2 \) and \( \zeta_{N-1} = \frac{z_{N-1}}{|z_{N-1}|} \) \( \sin(|z_{N-1}|) \). It is important to note that the coherent states (19) can be obtained also from Eq.(3.31) of the work [18], but it is necessary to follow the procedure, based on the equations (12-18), presented in this section. In other words, the authors of [18] have avoided the explicit computation of the action of the displacement operator on the highest weight state \( |j_1, 0, \cdots, 0| \). It is evident that for \( N = 2 \), one recover the well known \( su(2) \) coherent states. The states (19) have the property of strong continuity in the label space and overcompletness in the sense that there exists a positive measure such that they solve the resolution to identity. The appropriate form of this resolution is

\[
\int d\mu(\zeta_1, \zeta_2, \cdots, \zeta_{N-1}) \langle \zeta_1, \zeta_2, \cdots, \zeta_{N-1}| = \sum_{j_1=0}^{j_1} \sum_{j_2=0}^{j_2} \cdots \sum_{j_{N}=0}^{j_{N}}
\]

\[
|j_1 - j_2, j_2 - j_3, \cdots, j_N| \zeta_1^{j_1} \zeta_2^{j_2} \cdots \zeta_{N-1}^{j_{N-1}}|j_1 - j_2, j_2 - j_3, \cdots, j_N|.
\]

\[
(21)
\]
Assuming the isotropy of the measure $d\mu$, we set
\[ d\mu = \pi^{N-1}N^{-1} \prod_{s=1}^{N-1} h(|\zeta_s|^2)|\zeta_s|d|\zeta_s||d\theta_s \]  
(22)
with $\zeta_s = |\zeta_s|e^{i\theta_s}$. Substituting (22) in Eq.(21), we obtain the following sum
\[ \int_0^\infty x^{j_s+1}h(x)dx = \frac{j_s+1!(j_s-j_{s+1})!}{j_s!}. \]  
(23)
which should be satisfied by the function $h(x = |\zeta_s|^2))$. One get
\[ h(x) = \frac{j_s+1}{(1+x^2)^{j_s+2}}. \]  
(24)
This result can be obtained by using the definition of Meijer’s $G$-function and the Mellin inversion theorem [19]. The resolution to identity is necessary to build up the Fock-Bargamann space based on the set of $su(N)$ coherent states.

It is well established that the use of the Fock-Bargmann representation is a powerful method for obtaining closed analytic expressions for various properties of coherent states. Calculation for some quantum exception values and solutions for some eigenvalue equations are simplified by exploiting the theory of analytical entire functions. Here, we give the Fock-Bargamnn representation for a quantum system whose its dynamical symmetry is described by the Lie algebra $su(N)$. We define the Fock-bargamnn space as a space of functions which are holomorphic. The scalar product is written with an integral of the form
\[ \langle f|g \rangle = \int \bar{f}(\zeta_1, \zeta_2, \cdots, \zeta_{N-1})g(\zeta_1, \zeta_2, \cdots, \zeta_{N-1})d\mu \]  
(25)
where the measure is defined above (see Eqs.(22) and (24)). Due to overcompletion of the coherent sates, it is induced by the scalar product in $\mathcal{F}$. Let
\[ |\psi\rangle = \sum_{n_1+n_2+\cdots+n_N=j_1} a_{n_1,n_2,\cdots,n_N}|n_1,n_2,\cdots,n_N\rangle \]  
(26)
an arbitrary quantum state of $\mathcal{F}$, it can be represented as a function of the complex variables $\zeta_1, \zeta_2, \cdots, \zeta_{N-1}$ as
\[ \psi(\zeta_1, \zeta_2, \cdots, \zeta_{N-1}) = \mathcal{N}^{-1}\langle \bar{\zeta}_1, \bar{\zeta}_2, \cdots, \bar{\zeta}_{N-1}|\psi \rangle \]  
(27)
In particular, the analytic functions associated to elements of the basis of $\mathcal{F}$ are defined as
\[ \psi_{j_1,j_2,\cdots,j_N}(\zeta_1, \zeta_2, \cdots, \zeta_{N-1}) = \mathcal{N}^{-1}\langle \bar{\zeta}_1, \bar{\zeta}_2, \cdots, \bar{\zeta}_{N-1}|j_1-j_2,j_2-j_3,\cdots,j_N \rangle \]  
(28)
We now investigate the form of the action of the operators $e_i, f_i$ and $h_i$ on Fock-Bargmann space generated by the functions (28).
Indeed, any operator $O$ of the algebra $su(N)$ is represented in the space of entire analytical functions by some differential operator $\mathcal{O}$, defined by

$$
\langle \bar{\zeta}_1, \bar{\zeta}_2, \cdots, \bar{\zeta}_{N-1} | O | \psi \rangle = \mathcal{O}\psi(\zeta_1, \zeta_2, \cdots, \zeta_{N-1})
$$

(29)

for any state $| \psi \rangle$ of $\mathcal{F}$.

According this definition, we obtain

$$
E_{i+1} = \partial_i \quad F_{i+1} = j_1 \zeta_i - \zeta_i^2 \partial_i - \zeta_i \sum_{i \neq k} \zeta_k \partial_k
$$

(30)

for $i = 1, 2, \cdots, N-1$ and where $\partial_i$ stands for the derivative in respect to the variable $\zeta_i$. To obtain the above differential realization:

(i) we remark that the coherent states (19) can be also written as

$$
| \zeta_1, \zeta_2, \cdots, \zeta_{N-1} \rangle = N D(\zeta_1, \zeta_2, \cdots, \zeta_{N-1}) | j_1, 0, \cdots, 0 \rangle
$$

(31)

where $D(\zeta_1, \zeta_2, \cdots, \zeta_{N-1}) = \exp(\sum_{i=1}^{N-1} \zeta_i F_{i+1})$,

(ii) we observe that

$$
\partial_i D(\zeta_1, \zeta_2, \cdots, \zeta_{N-1}) = F_{i+1} D(\zeta_1, \zeta_2, \cdots, \zeta_{N-1})
$$

(32)

(iii) we use the Hausdorff formula

$$
e^{-B} A e^B = \sum_{n \geq 0} \frac{1}{n!} (-adB)^n A
$$

(33)

where $(adB)A = [B, A]$,

(iv) we use also the actions of the elements of $su(N)$ on the basis of Fock space $\mathcal{F}$, in particular the fiducial vector $| j_1, 0, \cdots, 0 \rangle$, and the structure relations (1-5) of the algebra $su(N)$.

It follows that the elements $e_i$, $f_i$ and $h_i$ of the algebra $su(N)$ are realized as

$$
f_1 = j_1 \zeta_1 - \zeta_1^2 \partial_1 - \zeta_1 \sum_{k=2}^{N-1} \zeta_k \partial_k, \quad e_1 = \partial_1, \quad h_1 = j_1 - 2 \zeta_1 \partial_1 - \sum_{k=2}^{N-1} \zeta_k \partial_k
$$

(34)

$$
e_i = \zeta_{i+1} \partial_i, \quad f_i = \zeta_i \partial_{i+1}, \quad h_i = \zeta_i \partial_i - \zeta_{i+1} \partial_{i+1},
$$

(35)

for $i = 2, 3, \cdots, N-1$. Hence, as it is clear from the previous considerations, the $su(N)$ generators act as first-order holomorphic differential operators on the Fock-Bargmann space generated by the elements (28). One can verify that the commutation relations (1-5) are preserved. This result combined with eigenvalue equations ensuring the minimization of Schrödinger-Robertson inequality provides the intelligent states as that will be explained in the next section.
3 Robertson states for $su(N)$ Weyl generators

In this section, we will study the fluctuations of quadrature components of Weyl generators which represent creation and annihilation of states for a quantum mechanical system of $su(N)$ symmetry. In this order, to construct the intelligent states of any pair of ladder operators $e_i, f_i$ ($i = 1, 2, \ldots, N-1$), it is natural to introduce the quantum observables $\sqrt{2}p_i = e_i + f_i$ and $i\sqrt{2}q_i = e_i - f_i$ where $i^2 = -1$. These observables obey

$$[p_i, q_i] = i\hbar_i \quad (36)$$

We known that $p_i$ and $q_i$ satisfy, in a given state, the Robertson-Shrödinger uncertainty relation

$$(\Delta p_i)^2(\Delta q_i)^2 \geq \frac{1}{4}(\langle h_i \rangle^2 + \langle c_i \rangle^2) \quad (37)$$

where $\Delta p_i$ and $\Delta q_i$ are the dispersions and the hermitian operator $c_i = \{p_i - \langle p_i \rangle, q_i - \langle q_i \rangle\}$ gives the covariance (correlation) of the observables $p_i$ and $q_i$. The symbol $\{,\}$ stands for the standard definition of the anticommutator. A state $|\Phi\rangle$ providing the equality in (37) is the so-called generalized intelligent state. It was proven that such state satisfy the following eigenvalue equation

$$((1 + \alpha)e_i + (1 - \alpha)f_i)|\Phi\rangle = \lambda|\Phi\rangle \quad (38)$$

where $\alpha \neq 0$ and $\lambda = (1+\alpha)|e_i\rangle + (1-\alpha)|f_i\rangle$ are complex parameters. Furthermore, the variance and covariance, in the intelligent state $|\Phi\rangle$, are given by

$$(\Delta p_i)^2 = |\alpha|\Delta_i \quad (\Delta q_i)^2 = \frac{1}{|\alpha|}\Delta_i \quad (39)$$

where $\Delta_i = \frac{1}{2}\sqrt{\langle h_i \rangle^2 + \langle c_i \rangle^2}$. Remark that they can be also expressed as

$$(\Delta p_i)^2 = \frac{|\alpha|^2}{u}\langle h_i \rangle \quad (\Delta q_i)^2 = \frac{1}{u}\langle h_i \rangle \quad \langle c_i \rangle = \frac{v}{u}\langle h_i \rangle \quad (40)$$

where the real parameters $u$ and $v$ are such that $u^2 + v^2 = 4|\alpha|^2$ (As example, one can take $u = 2Re\alpha$ and $v = 2Im\alpha$). It is clear that the dispersions and the correlation can be obtained from the mean value of the observable $h_i$. The state $|\Phi\rangle$ satisfying (38) with $|\alpha| = 1$ are coherent because they satisfy $(\Delta p_i)^2 = (\Delta q_i)^2 = \Delta_i$. The fluctuations are equals and minimized in the sense of Schrödinger-Robertson uncertainty relation. The state satisfying (38) with $|\alpha| \neq 1$ are squeezed because if $|\alpha| < 1$, we have $(\Delta p_i)^2 < \Delta_i < (\Delta q_i)^2$ and for $|\alpha| > 1$, we have $(\Delta q_i)^2 < \Delta_i < (\Delta p_i)^2$.

To solve the eigenvalues equation (38), we will use the analytic representations of coherent states as well as the differential realizations of the the generators $e_i$ and $f_i$. So, let us start by deriving the eigenfunctions of Eq.(38) for the first pair $e_1, f_1$. By introducing the analytic function

$$\Phi_1 \equiv \Phi_1(\zeta_1, \zeta_2, \ldots, \zeta_{N-1}, \alpha, \lambda, f_1) = \mathcal{N}^{-1}\langle \bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_{N-1}|\Phi_1\rangle, \quad (41)$$
it can be easily checked that the eigenvalue equation (38) can be converted in the following first order differential equation

\[(j_1 \eta_1 - \lambda') \Phi_1 + (1 - \eta_1^2) \frac{\partial \Phi_1}{\partial \eta_1} - \eta_1 \sum_{i=2}^{N-1} \eta_i \frac{\partial \Phi_1}{\partial \eta_i} = 0, \]  \hspace{1cm} (42)\]

where \(\eta_1 = \sqrt{\frac{1 - \alpha}{1 + \alpha}} \zeta_1, \eta_i \neq 1 = \zeta_i\) and \(\lambda' = \frac{\lambda}{\sqrt{1 - \alpha^2}}\) for \(\alpha \neq \pm 1\). The function \(\Phi_1(\zeta_1, \zeta_2, \ldots, \zeta_{N-1} \alpha, \lambda, j_1)\) and can be expanded as

\[
\Phi_1 = \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} \cdots \sum_{j_N=0}^{j_{N-1}} a_{j_1,j_2,\ldots,j_N} \eta_1^{j_2} \eta_2^{j_3} \cdots \eta_N^{j_N} \]  \hspace{1cm} (43)\]

Substitution of (43) in (42) yields the recursion formula

\[(j_1 + 1 - \sum_{i=2}^{N} j_i) a_{j_1,j_2-1,\ldots,j_N} \lambda' a_{j_1,j_2,\ldots,j_N} + (j_2 + 1) a_{j_1,j_2+1,\ldots,j_N} = 0 \]  \hspace{1cm} (44)\]

which can be solved by the Laplace method. Indeed, we set

\[a_{j_1,j_2,\ldots,j_N} = \int_{-1}^{+1} x^{j_2} f(x) dx \]  \hspace{1cm} (45)\]

that we introduce in (44) to obtain, after partial integration, the simple first order differential equation satisfied by the function \(f(x)\)

\[(x - x^3) \frac{df}{dx} + (2j + 1 - \lambda' x - x^2) = 0. \]  \hspace{1cm} (46)\]

where \(2j = j_1 - \sum_{i=3}^{N} j_i\). The last equation is easily solvable. Replacing in (45), one get

\[a_{j_1,j_2,\ldots,j_N} = \int_{-1}^{+1} x^{j_2-2j-1}(1-x)^{\lambda'/2} (1 + x)^{\lambda'/2 + j} dx, \]  \hspace{1cm} (47)\]

or

\[a_{j_1,j_2,\ldots,j_N} = (-)^{j_2} \frac{\Gamma(\frac{\lambda'}{2} + j + 1) \Gamma(- \frac{\lambda'}{2} + j + 1)}{\Gamma(2j + 2)} \times \frac{\Gamma(2j + 2)}{2F_1(2j - j_2 + 1, \frac{\lambda'}{2} + j + 1, 2j + 2)} \]  \hspace{1cm} (48)\]

using the integral representation for the hypergeometric function \(2F_1\) [19] with the condition \(- (j + 1) < \text{Re}(\lambda' / 2) < (j + 1)\). Comparing the expansion (43) with the general formula (41), we have the decomposition of the intelligent states over the basis of Fock space \(\mathcal{F}\)

\[
|\Phi_1\rangle = \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} \cdots \sum_{j_N=0}^{j_{N-1}} a_{j_1,j_2,\ldots,j_N} \left(1 - \frac{\alpha}{1 + \alpha}\right)^{\frac{j_2}{j_1}} \sqrt{\frac{j_N!}{j_1!}} \sqrt{(j_1 - j_2)!(j_2 - j_3)! \cdots (j_{N-1} - j_N)!} |j_1 - j_2 - j_3, \ldots, j_N\rangle \]  \hspace{1cm} (49)\]
where the coefficients \( a_{j_1,j_2,...,j_N} \) are given by Eq.(48).

Now we consider the construction of intelligent states for the pairs \( e_i, f_j \) with \( i = 2, 3, \cdots, N - 1 \). The eigenvalues equation (38) gives, in this case, the following quasi linear differential equation

\[
\xi_i \frac{\partial \Phi_i}{\partial \xi_i} + \xi_{i+1} \frac{\partial \Phi_i}{\partial \xi_{i+1}} - \lambda' \Phi_i = 0
\]

(50)

where \( \xi_i = \sqrt{\frac{1+i}{1-i}} \xi_i, \xi_i+1 = \xi_i+1 \) and \( \lambda' = \frac{\lambda}{\sqrt{1-\alpha^2}} \).

Here also, we expand the eigenfunction \( \Phi_i \equiv \Phi_i(\xi_1, \xi_2, \cdots, \xi_{N-1}, \alpha, \lambda, j_1) \) as

\[
\Phi_i = \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} \cdots \sum_{j_N=0}^{j_{N-1}} b_{j_1,j_2,...,j_N} \xi_1^{j_2} \xi_2^{j_3} \cdots \xi_{N-1}^{j_N}
\]

(51)

that we insert in the equation (50) to obtain the recursion relation linking the coefficients \( b \)'s

\[
(j_i + 2 + 1)b_{j_1,...,j_{i+1}-1,j_{i+2}+1,...,j_N} - \lambda' b_{j_1,...,j_{i+1},j_{i+2},...j_N} + (j_i + 1 + 1)b_{j_1,...,j_{i+1}+1,j_{i+2}-1,...j_N} = 0
\]

(52)

Setting \( b_{j_1,...,j_{i+1},j_{i+2},...j_N} \equiv b_{j_{i+1}+1,j_{i+2}} \equiv b_{j_{i+1}+1,l} \) where \( 2l = j_{i+1} + j_{i+2} \), the previous relation can be transformed to

\[
(j_i + 2 + 1)b_{j_{i+1}+1,l-1,l} - \lambda' b_{j_{i+1}+1,l-1,l} + (j_i + 1 + 1)b_{j_{i+1}+1,l+1,l} = 0,
\]

(53)

solvable in a similar manner that one given the solution of recursion formula (44), and one has

\[
b_{j_1,j_2,...,j_N} = (-1)^{j_i+1} \frac{\Gamma\left(\frac{\lambda'}{2} + l + 1\right)\Gamma\left(-\frac{\lambda'}{2} + l + 1\right)}{\Gamma(2l + 2)}
\]

\[
\times _2F_1(2l - j_{i+1} + 1, \frac{\lambda'}{2} + l + 1, 2l + 2, 2)
\]

(54)

where \(-l + 1 < Re(\lambda'/2) < (l + 1)\). Finally, one obtain

\[
|\Phi_i\rangle = \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} \cdots \sum_{j_N=0}^{j_{N-1}} b_{j_1,j_2,...,j_N} \left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{j_2}{2}} \sqrt{\frac{j_N!}{j_1!}}
\]

\[
\sqrt{(j_1-j_2)!(j_2-j_3)!(j_{N-1}-j_N)!}|j_1-j_2-j_3,...,j_N\rangle
\]

(55)

From equations (38) and (39), it is clear that the intelligent states \(|\Phi_i\rangle, (i = 1, 2, \cdots, N - 1)\) given by (49) and (55) are coherent for \( \alpha = e^{i\theta} (\theta \text{ real}) \) and squeezed for \( |\alpha| \neq 1 \) in the sense of Schrödinger-Robertson uncertainty relation. To close this section, let us also note that the Fock-Bargmann representation of the coherent states plays a helpful role in the problem of finding intelligent states of the quadrature of Weyl generators. The procedure described here can be relevant in the determination of intelligent states for quadrature components of type \( e_i, e_j \) and \( f_i, f_j \) \((i \neq j)\).
4 Summary and outlook

We constructed explicitly the coherent states associated with the Lie algebra of type $su(N)$. We have proceeded, in a second stage, in the study of Fock-Bargmann representation based on the obtained states. We gave the differential actions of $su(N)$ generators on this space. We have shown that they act as first order differential operators. As byproducts, simple quasi-linear equations, satisfied by the minimum Schrödinger-Robertson uncertainty states, are solved. Thus, new classes of coherent ($|\alpha| = 1$) and squeezed ($|\alpha| \neq 1$) states are obtained for the quadrature components of $su(N)$ Weyl generators. It is clear that the analytic approach used through this work can be applied for finding Robertson intelligent states associated to the other quadratures of the $su(N)$ generators. They can be also obtained by considering the eigenvalue problem for an operator which is a complex linear combination of all elements of $su(N)$

$$\sum_{i=1}^{N-1} (\alpha_i^+ e_i + \alpha_i^- f_i + \alpha_i^0 h_i) |\Psi\rangle = \lambda |\Psi\rangle.$$  

The solutions of such general problem give the so-called in the literature algebra eigenstates or algebraic coherent states ([11-13] and references therein). Taking specific constraints on the complex parameters occurring in this general eigenvalue equation, one can get various kind of coherent and squeezed states, in particular ones not discussed in this letter. This constitutes the first possible prolongation of this work. The results of this note give the necessary ingredients to write down a complete classification of $A_N$ coherent and squeezed states. Indeed, although we have performed the analysis for the compact algebra $su(N)$, our results also describe the non-compact algebra $su(p,q)$ ($p + q = N$). This can be done following the general procedure for associating a non-compact algebra with a compact one. Also, as continuation, it would be interesting to apply the approach given here to the Lie algebras $B_N$, $C_N$ and $D_N$.

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