On updates of high order cumulant tensors

Krzysztof Domino* and Piotr Gawron†

Institute of Theoretical and Applied Informatics,
Polish Academy of Sciences,
Bałtycka 5, 44-100 Gliwice, Poland

January 24, 2017

Abstract

High order cumulants carry information about statistics of non-normally distributed multivariate data. Such cumulants are utilised in extreme events analysis, small target detection or outliers detection. In this work we present a new algorithm for updating high order cumulant tensors of random multivariate data, if new package of data is recorded. We show algebraically and numerically, that the proposed algorithm is faster than a naïve cumulants recalculation algorithm. For investigated computer generated data our algorithm appears to be faster than a naïve one by $1 - 2$ orders of magnitude. That update algorithm makes the online updates of multivariate data statistics much faster, and can be used for the data streaming analysis.

Further we propose the map reduce algorithm of cumulants calculation, that is based on introduced cumulants updates algorithm. This map reduce algorithm can be used to collect statistics about multivariate confidential data that are held by many agents, without sharing those data.

* kdomino@iitis.pl
† gawron@iitis.pl
1 Introduction

1.1 The motivation

The motivation for this work comes from the fact, that many new applications of high order cumulant tensors have recently appeared. As high order cumulant tensor we understand $n \geq 3$ mode tensor, where $n$ is called the order of the cumulant. High order cumulant tensors have recently been used to analyse non-normally distributed multivariate data such as: financial data \cite{1,2,3}, hyper-spectral data \cite{4} and auto-correlated multivariate data \cite{5}. There are also other potential applications of high order cumulants resulting from non-normality of: weather data \cite{6,7}, medical data \cite{8} and cosmological data \cite{9}.

Many of listed above types of multivariate data can be recorded and examined in real time. Since cumulants are caring information about their frequency distribution \cite{10}, cumulants online update can be used for rapid determination of any change in such data statistics. E.g. in the search for anomalies such as: incoming crash on financial markets \cite{11}, weather anomalies, anomalies recorded in medical data or cosmological anomalies.

The idea of cumulants update leads also to idea of scattered computation used to collect statistics about confidential data. For this purpose, suppose we have a few agents possessing confidential multivariate data with the same number of variables. Examples of such confidential data are: different types of financial data, sales records of priory determined products, industrial data or costs records of some operations. If we want to gather information about the statistics of such data, they have to be preprocessed by each agent separately. To maintain the confidentiality, only results of preprocessing are collected. Henceforth we propose the following map reduce algorithm. First each agent map data onto moment tensors, and further those tensors are reduced into cumulants, by the central agent.

1.2 The cumulant’s definition

We can now move to the formal definition of cumulant’s tensors. Analogically to \cite{12} let us start with $T$ samples of $M$ dimensional random variable represented by a matrix $X \in \mathbb{R}^{T,M}$, or represented by a sequence of $M$ marginal variables $X_j \in \mathbb{R}^T$:

$$X = \begin{bmatrix} x_{1,1} & \ldots & x_{1,M} \\ \vdots & x_{i,j} & \vdots \\ x_{T,1} & \ldots & x_{T,M} \end{bmatrix} = [X_1, \ldots, X_j, \ldots, X_M]. \quad (1)$$
here $x_{t,j}$, the $t^{th}$ record of $j^{th}$ marginal variable $X_j = [x_{1,j}, \ldots, x_{T,j}]^\top$. Using this notation the cumulant generation function $[10, 13]$ is:

$$K(X, \tau) = \log \left( \frac{\sum_{t=1}^T \exp \left( [x_{t,1}, \ldots, x_{t,m}, \ldots, x_{t,M}] \cdot \tau^\top \right)}{T} \right), \quad (2)$$

where $\tau$ is an argument vector of the cumulant generation function and dot represents scalar product.

**Definition 1.1.** The element of the $n^{th}$ cumulant at multi-index $I = (i_1, \ldots, i_n) \forall j \; i_j \in [1, \ldots, M]$ is:

$$(C_n)_{i_1,\ldots,i_n}(X) = \frac{\partial^n}{\partial \tau_{i_1} \ldots \partial \tau_{i_n}} \log \left( K(X, \tau) \right) \bigg|_{\tau=[0,\ldots,0]} . \quad (3)$$

A first cumulant is a mean vector, a second one is a covariance matrix, a third is a 3 mode tensor. The wider discussion of cumulant’s features, including their symmetry and meaning is presented in [12].

### 1.3 High order cumulants and frequency distribution

For multivariate normally distributed data, cumulant generation function Eq. (2) is quadratic in parameter $\tau$ and hence its third and higher derivatives are zero $[10, 13]$. Hence, cumulants of order higher or equal 3 are zero. If the multivariate frequency distribution of data is not normal, the characteristic function can be expanded in more terms than quadratic and cumulants of order higher or equal 3 are not zero. It is why, high order cumulants are caring information about frequency distribution of non–normally distributed data, especially about extreme and cross–correlated events. To show the meaning of extreme events, consider financial data, where an extreme event is a crash, a breakdown of stock market or a bankruptcy.

The paper is organized as follows: in Section 2 general procedure of cumulants calculation introduced in [12], is discussed. In Section 3 the algorithm of cumulant’s updates and cumulant’s calculation is introduced and tested against computer generated data. In Section 4 the map reduce algorithm used to collect statistic from confidential data is presented.

## 2 Cumulant’s calculation

### 2.1 Required definitions

In [12] the authors have presented a general recurrence formula for calculation of the cumulant tensor of order $n$, given the $n^{th}$ moment tensor, and cumulant
tensors of lower orders. That formula will be used in Section 3 to construct cumulants update algorithm. To demonstrate the recurrence formula from [12] we need some definitions.

**Definition 2.1.** Let $X \in \mathbb{R}^{T \times M}$ be as in Eq. (1), the $n$\textsuperscript{th} moment tensor $M_n(X)$ has elements:

$$m_{i_1,\ldots,i_n} = \frac{1}{T} \sum_{t=1}^{T} x_{t,i_1} \cdot \ldots \cdot x_{t,i_n}. \quad (4)$$

**Definition 2.2.** Set partition of the multi–index $P_\sigma(I)$, into $\sigma$ sub multi–indices. Let $I = (i_1, \ldots, i_n)$ be a multi–index, and let $1 < \sigma \leq n$. Consider the division of multi–index $I$ into a $\sigma$-tuple of sub multi–indices: $P_\sigma(I) = (I_1, \ldots, I_k, \ldots, I_\sigma)$ where $I_k = (i_{k_1}, \ldots, i_{k_r})$, such that $\bigcup_{k=1,\ldots,\sigma} \{k_s\} = \{1,2,\ldots,n\}$ and $\forall_{k \neq k'} I_k \cap I_{k'} = \emptyset$. In other words the sum of sets of all sub multi–indices gives an original multi index $I$, and each pair of sub multi–indices is disjoint.

**Definition 2.3.** Consider the following set partition of multi–index $I$: $P_\sigma(I) = (I_1, \ldots, I_{\sigma})$ and $P'_\sigma(I) = (I'_1, \ldots, I'_{\sigma})$. Let us introduce the equivalence relation:

$$P_\sigma(I) \sim P'_\sigma(I) \iff (\exists \pi' \forall_{k \in [1,\ldots,\sigma]} \exists_{\pi_k} (I'_1, \ldots, I'_{\sigma}) = \pi' (\pi_1(I_1), \ldots, \pi_{\sigma}(I_{\sigma}))). \quad (5)$$

Where $\pi' \in S(I_1,\ldots,I_{\sigma})$ is a permutation of sub multi–indices and $\pi \in S(i_1,\ldots,i_{\sigma})$ a permutation of indices inside a sub multi–index. This relation defines the abstraction class. Further, following [12], we will take only one representative of each abstraction class and denote it simply by $[P_\sigma(I)]$.

Similarly as in [12] we simply divide a multi–index into a $\sigma$-tuple of sub multi–indices in such a way that the ordering of indices inside each sub multi–indices does not matter and the ordering of sub multi–index inside the $\sigma$-tuple does not matter either.

### 2.2 The formula

For the cumulants calculation purpose, we can refer to [12] and Eq. (24) within. An element of $n$\textsuperscript{th} cumulant tensor at a multi–index $I$ of size $n = |I|$, can be computed using the following recurrence relation:

$$c_I(X) = m_I(X) - \sum_{\sigma=2}^{n} \sum_{[P_\sigma(I)]} \left( \prod_{I^* \in P_\sigma(I)} c_{I^*}(X) \right). \quad (6)$$
here \( c_I \) are elements of cumulant tensors of order \(|I^*| < n\). The number of all partitions of a multi–index of size \( n \) is determined by the Bell number \( B_n \) [14], where \( B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15 \) etc. Henceforth, we have \( B_n - 1 \) elements of sums in Eq. (6), the partition which is the identity relation \( I \rightarrow I \) is excluded due to the \( \sigma \geq 2 \) condition. Number of multiplications for each partition is \( \sigma - 1 \), since there are \( \sigma \) elements \( c_I \) that are multiplied. But \( 2 \leq \sigma \leq n \), hence the number of multiplications required to calculate the second element of the RHS of Eq. (6), for each multi–index \( I \), is limited by \((B_n - 1) \cdot (n - 1)\) from above.

For presentation clarity, let us now show some examples of the use of Eq. (6).

**Example 2.1.** The first cumulant is simply a mean vector \( c_i(X) = m_i(X) \).
Consider now, the second cumulant. There is only one way to divide 2 elements multi–index into 2 sub multi–indices according to Def. 2.3, hence we have:

\[
c_{(i_1,i_2)}(X) = m_{(i_1,i_2)}(X) - c_{(i_1)}(X) \cdot c_{(i_2)}(X).
\]

(7)

**Example 2.2.** Consider now the third cumulant. The 3 elements multi–index can be divided into 2 sub multi–indices of size 1 and 2 in 3 ways and into 3 sub multi–indices of size 1 each, in 1 way:

\[
c_{(i_1,i_2,i_3)}(X) = m_{(i_1,i_2,i_3)}(X) - c_{(i_1, i_2)}(X) \cdot c_{(i_3)}(X) - c_{(i_1, i_3)}(X) \cdot c_{(i_2)}(X)
- c_{(i_2,i_3)}(X) \cdot c_{(i_1)}(X) - c_{(i_1)}(X) \cdot c_{(i_2)}(X) \cdot c_{(i_3)}(X).
\]

(8)

3 The idea of cumulants updates

Having the cumulant’s calculation formula, we can move now to the novel idea of cumulant tensors updates. Let \( X \in \mathbb{R}^{T,M} \) be as in Eq. (1). Let us consider the update \( X_{up} \in \mathbb{R}^{T_{up},M} \). The updated matrix \( X' \in \mathbb{R}^{T,M} \) is constructed in such a way, that we remove first \( T_{up} \) rows of \( X \) and add all \( T_{up} \) rows of \( X_{up} \) at the end of reminding. In a formal way, elements of \( X' \) are:

if \( 1 \leq t \leq T - T_{up} \) \( x_{t,i}' = x_{(t+T_{up},i)} \), if \( T - T_{up} < t \leq T \) \( x_{t,i}' = x_{up(T_{up}+t,i)} \).

(9)

Having the updated matrix of data, we can discuss the update of arbitrary order moment’s tensor.

3.1 Moment’s tensor update

We can show, that the \( n \)th moment of updated data \( M_n(X') \), can be calculated given the original moment’s tensor \( M_n(X) \) the original data \( X \) and the
update $X_{up}$:

$$m_{i_1,...,i_n}(X') = \frac{1}{T} \sum_{t=1}^{T} x'_{t,i_1} \cdot \ldots \cdot x'_{t,i_n}$$
$$= \frac{1}{T} \left( \sum_{t=1}^{T-T_{up}} x_{(t+T_{up},i_1)} \cdot \ldots \cdot x_{(t+T_{up},i_n)} + \sum_{t=T-T_{up}+1}^{T} x_{up(T_{up}-T+t,i_1)} \cdot \ldots \cdot x_{up(T_{up}-T+t,i_n)} \right)$$
$$= \frac{1}{T} \left( \sum_{t=1}^{T} x_{t,i_1} \cdot \ldots \cdot x_{t,i_n} - \sum_{t=1}^{T_{up}} x_{t,i_1} \cdot \ldots \cdot x_{t,i_n} + \sum_{t=1}^{T_{up}} x_{up(t,i_1)} \cdot \ldots \cdot x_{up(t,i_n)} \right)$$
$$= m_{i_1,...,i_n}(X) - \frac{T_{up}}{T} m_{i_1,...,i_n}(X') + \frac{T_{up}}{T} m_{i_1,...,i_n}(X_{up}),$$

(10)

where $X' \in \mathbb{R}^{T_{up} \times M}$ are first $T_{up}$ rows of $X$: $\forall t \leq T_{up}$ $x_{t,i}'' = x_{t,i}$.

Let us now discuss a number of operations required to update a moment’s tensor. We need $T_{up} \cdot (n-1)$ multiplications to compute each element $m_{i_1,...,i_n}(X'')$ and $T_{up} \cdot (n-1)$ multiplications to compute each element $m_{i_1,...,i_n}(X_{up})$. Henceforth update the $n$th moment, we need $2 \cdot T_{up} \cdot (n-1)$ multiplications for each its element at a given multi-index $(i_1, \ldots, i_n)$. On the other hand, the naïve moment recalculation would require the computation of $M_n(X')$ i.e. $T \cdot (n-1)$ multiplications for each its element at a given multi-index. Concluding, the theoretical speed-up factor is $\frac{2T_{up}}{T}$, what is significantly large since usually $T_{up} \ll T$.

### 3.2 The cumulant calculation and update

Using Eq (6), any cumulant can be computed recursively given moment’s tensors. This approach is desirable for data analysis purpose, since there one often requires a series of cumulants up to the given order, see for example [2, 3]. For this purpose, in Algorithm 1 we propose the cumulant’s calculation recursive scheme. Next in Algorithm 2 we propose the cumulant’s update scheme.

In details, we propose the following cumulant’s calculation/update scheme. First we calculate or update moments of order $1, \ldots, n$ i.e. $M_1(X), \ldots, M_n(X)$, further given moments, we calculate cumulants of order $1, \ldots, n$ using Eq. (6). The scheme is presented in Algorithm 1. Number of multiplications, required to compute the $n$th cumulant would be:

$$\# \text{comp}(n) = (T \cdot (n-1) + (B_n - 1) \cdot (n-1)) \cdot M^n.$$  

(11)
Algorithm 1: cumulants calculation using formula Eq. (6)

1: **Input** – moment tensors $\mathcal{M}_1(\mathbf{X}), \ldots, \mathcal{M}_n(\mathbf{X})$, such that $\mathcal{M}_k(\mathbf{X}) \in \mathbb{R}^{M \times k}$.

2: **Output** – cumulant tensors $C_1(\mathbf{X}), \ldots, C_n(\mathbf{X})$, such that $C_k(\mathbf{X}) \in \mathbb{R}^{M \times k}$.

3: **function** OUTERPRODUCT($C_1(\mathbf{X}), \ldots, C_{k-1}(\mathbf{X})$)

4: for $i_1 \leftarrow 1$ to $M$, ..., $i_k \leftarrow 1$ to $M$

5: for $p \in \text{PARTITIONS}([i_1, \ldots, i_k])$ do

6: $r = \text{map(length, } p\text{)}$, $\sigma = \text{length}(r)$

7: $I_1 = (p[1,1], \ldots, p[1,r_1]), \ldots, I_{\sigma} = (p[\sigma,1], \ldots, p[\sigma,r_{\sigma}])$

8: $c_{i_1,\ldots,i_n} = c_{I_1^r} \cdot \cdots \cdot c_{I_{\sigma}^r}$

9: end for

10: end for

11: return $C_k(\mathbf{X})$.

12: end function

13: **function** CUMULANTS($\mathcal{M}_1(\mathbf{X}), \ldots, \mathcal{M}_n(\mathbf{X})$)

14: $C_1(\mathbf{X}) = \mathcal{M}_1(\mathbf{X})$

15: for $k \leftarrow 2$ to $n$

16: $C_k(\mathbf{X}) = \mathcal{M}_k(\mathbf{X}) - \text{OUTERPRODUCT}(C_1(\mathbf{X}), \ldots, C_{k-1}(\mathbf{X}))$

17: end for

18: return $C_1(\mathbf{X}), \ldots, C_n(\mathbf{X})$.

19: end function

The first term of the sum is the number of multiplications required to compute the $n$th moment’s tensor and the second term of the sum is the number of multiplications required to calculate products of lower order cumulants, see second term of RHS of Eq. (6). Since $T \gg n$, if we compute a series of cumulants of order $1, 2, \ldots, n$ the number of multiplications required to compute the $n$th cumulant would be dominant.

The cumulant’s update scheme that is presented in Algorithm 2 would require approximately

$$
\# \text{up}(n) = (T_{up} \cdot (n - 1) + (B_n - 1) \cdot (n - 1)) \cdot M^n
$$

(12)
multiplications to update the $n$th cumulant. Here also if we compute a series of cumulants of order $1, 2, \ldots, n$, the number of multiplications required to update the $n$th cumulant would be dominant.
Algorithm 2 cumulants update

1: **Input** – data $X \in \mathbb{R}^{T,M}$, $X_{up} \in \mathbb{R}^{T_{up},M}$, moment tensors $\mathcal{M}_1(X), \ldots, \mathcal{M}_n(X) : \mathcal{M}_k(X) \in \mathbb{R}^{M \times k}$.

2: **Output** – cumulant tensors $C_1(X'), \ldots, C_n(X')$, such that $C_k(X') \in \mathbb{R}^{M \times k}$.

3: **function** CUMULANTSUPDATE($X, X_{up}, \mathcal{M}_1(X), \ldots, \mathcal{M}_n(X)$)

4: $X'' = X_{[1:T_{up}]}$

5: for $k \leftarrow 1$ to $n$ do

6: $\mathcal{M}_k(X') = \mathcal{M}_k(X) - \frac{T_{up}}{T} \mathcal{M}_k(X'') + \frac{T_{up}}{T} \mathcal{M}_k(X_{up})$  \triangleright moments update, using Eq. (10)

7: end for

8: return CUMULANTS($\mathcal{M}_1(X'), \ldots, \mathcal{M}_n(X')$)  \triangleright function from Algorithm 1

9: **end function**

3.3 Number of operations

The speed-up of cumulant’s update in compare with the naïve approach, where moments are recalculated, is roughly $\frac{\text{comp}(n)}{\text{up}(n)} \approx \frac{T}{2T_{up} + B_n}$, since in our case $T \gg B_n$.

To verify this observation, we have implemented the cumulants update algorithm and naïve cumulants recalculation algorithm in Julia Programming Language [16, 17, 18] and executed it on one core of Intel(R) Core(TM) i7 CPU 3.20GHz. To choose the proper size of computer generated data for tests, let us consider the analysis of high frequency financial data such as share’s price’s of companies recorded each second. We have 3600 records each hour, approximately 30 000 record each day and approximately 600 000 records each month. Suppose we have monthly records of 20 companies shares, and want to examine the cumulants update calculated for those data, given the 1 hour, 2 hours, 5 hours and 1 day update. Henceforth original data are $X \in \mathbb{R}^{T,M}$, where $T = 600 000$ and $M = 20$, and update is $X_{up} \in \mathbb{R}^{T_{up},M}$. Next we have calculated a series of cumulants of order 1to 4 Results of computational time using the naïve algorithm and the update algorithm introduced by authors is presented in Figure 1. We can conclude, that the cumulant calculation speed–up is of one or two orders of magnitude.

4 Collecting statistics on confidential data

Being inspired by the performance of cumulant’s update algorithm we are going to use it for the collection of statistics of confidential data. Suppose we
have data in the following form $X_i \in \mathbb{R}^{T_i,M}$, $X_j \in \mathbb{R}^{T_j,M}$, $X_s \in \mathbb{R}^{T_s,M}$. They are parts of data collection $X \in \mathbb{R}^{T,M}$, where $T = \sum_{j=1}^{s} T_j$. Suppose we need to know cummulats of data $X$ for some statistical analysis purpose. E.g. we want to search for extreme events or outliers. However each piece of data is held by different agents that not wish to share them. To deal with this problem we propose the following map reduce scheme, see Algorithm 3.

Data are mapped by each agent onto moments, see function map in Algorithm 3. Such moments carry only statistical information about data, and can be shared without the break ao confidentiality. The $j$ agent produces following output $[\mathcal{M}_1(X_j), \ldots, \mathcal{M}_n(X_j)]$. Suppose we have $s$ agents, their moments can be reduced into single moments:

$$\forall k \in \{1, n\} \quad \mathcal{M}_k(X) = \sum_{j=1}^{s} \frac{T_j}{T} \mathcal{M}_k(X_j).$$

(13)
Algorithm 3 map reduce scheme

1: **Input** data – $X_1 \in \mathbb{R}^{T_1,M}, \ldots X_s \in \mathbb{R}^{T_s,M}$, cumulants order – $n$.
2: **Output** – cumulant tensors $C_1(X_j), \ldots, C_n(X_j) : C_k(X_j) \in \mathbb{R}^{M \times k}$.
3: function MAP($X_j,n$) $\triangleright$ performed by each agent
4: return $M_1(X_j), \ldots, M_n(X_j), T_j$.
5: end function
6: function REDUCE([$M_1(X_1), \ldots, M_n(X_1), T_1], \ldots, [M_1(X_s), \ldots, M_n(X_s), T_s]$).
7: $T = \sum_{j=1}^{s} T_j$
8: for $k \leftarrow 1$ to $n$ do
9: $M_k(X) = \sum_{j=1}^{s} \frac{T_j}{T} M_k(X_j) \quad \triangleright$ uses Eq. [13]
10: end for
11: return CUMULANTS($M_1(X), \ldots, M_n(X)$) $\triangleright$ function from Algorithm [1]
12: end function

Finally given [$M_1(X), \ldots, M_n(X)$] we calculate cumulants using Algorithm [1].

5 Conclusions

In this paper the idea of cumulants tensors updates of non–Gaussian distributed data is discussed. We introduced a new algorithm for cumulants updates and show algebraically and numerically that it can be faster by one or two orders of magnitude, while comparing with naïve cumulants recalculation. This algorithm is based on cumulants moments recurrence relation that is discussed in [12] in details, and the fact that moment tensors are easy to update. The algorithm can be used everywhere, where data are recorded in real time and cumulants fast update is required. It can also be used to analyse updates of large data set, hence it concerns the big data problem. For example it can be used to examine high frequency financial data recorded online.

Further we propose the map reduce algorithm that can be used to compute cumulants from non-Gaussian distributed confidential data, that are held by different agents. The algorithm preserves data confidentiality, and can be used for example in extreme events determination. Examples of non–Gaussian distributed confidential data may be: different types of financial data, sales records of priory determined products, industrial data or costs records of some operations.
Acknowledgments

The research was partially financed by the National Science Centre, Poland – project number 2014/15/B/ST6/05204.

References

[1] J. C. Arismendi and H. Kimura, “Monte Carlo Approximate Tensor Moment Simulations,” Available at SSRN 2491639, 2014.

[2] E. Jondeau, E. Jurczenko, and M. Rockinger, “Moment component analysis: An illustration with international stock markets,” Swiss Finance Institute Research Paper, no. 10-43, 2015.

[3] K. Domino, “The use of the multi-cumulant tensor analysis for the algorithmic optimisation of investment portfolios,” Physica A: Statistical Mechanics and its Applications, vol. 467, pp. 267–276, 2017.

[4] X. Geng, K. Sun, L. Ji, H. Tang, and Y. Zhao, “Joint Skewness and Its Application in Unsupervised Band Selection for Small Target Detection,” Scientific reports, vol. 5, 2015.

[5] E. S. Manolakos and H. M. Stellakis, “Systematic synthesis of parallel architectures for the computation of higher order cumulants,” Parallel Computing, vol. 26, no. 5, pp. 655–676, 2000.

[6] R.-G. Cong and M. Brady, “The interdependence between rainfall and temperature: copula analyses,” The Scientific World Journal, vol. 2012, 2012.

[7] D. Krzysztof, B. Tomasz, and C. Maurycy, “The use of copula functions for predictive analysis of correlations between extreme storm tides,” Physica A: Statistical Mechanics and its Applications, vol. 413, p. 489–497, 2014.

[8] D.-B. Pougaaz, A. Mohammad-Djafari, and J.-F. Bercher, “Using the Notion of Copula in Tomography,” arXiv preprint arXiv:0812.1316, 2008.

[9] R. J. Scherrer, A. A. Berlind, Q. Mao, and C. K. McBride, “From finance to cosmology: The copula of large-scale structure,” The Astrophysical Journal Letters, vol. 708, no. 1, p. L9, 2009.
[10] M. G. Kendall et al., “The advanced theory of statistics,” The advanced theory of statistics., no. 2nd Ed, 1946.

[11] G. L. Vasconcelos, “A guided walk down Wall Street: an introduction to econophysics,” Brazilian Journal of Physics, vol. 34, no. 3B, pp. 1039–1065, 2004.

[12] K. Domino, P. Gawron, and Ł. Pawela, “The tensor network representation of high order cumulant and algorithm for their calculation,” arXiv preprint arXiv:1701.05420, 2017.

[13] E. Lukacs, “Characteristics functions,” Griffin, London, 1970.

[14] L. Comtet, “Advanced combinatorics, reidel pub,” Co., Boston, 1974.

[15] D. E. Knuth, The art of computer programming: sorting and searching, vol. 3B. Pearson Education, 1998.

[16] J. Bezanson, S. Karpinski, V. B. Shah, and A. Edelman, “Julia: A fast dynamic language for technical computing,” arXiv preprint arXiv:1209.5145, 2012.

[17] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, “Julia: A fresh approach to numerical computing,” arXiv preprint arXiv:1411.1607, 2014.

[18] J. Bezanson, J. Chen, S. Karpinski, V. Shah, and A. Edelman, “Array operators using multiple dispatch: A design methodology for array implementations in dynamic languages,” in Proceedings of ACM SIGPLAN International Workshop on Libraries, Languages, and Compilers for Array Programming, p. 56, ACM, 2014.