GEOMETRIC MODELS FOR FIBRANT RESOLUTIONS OF MOTIVIC SUSPENSION SPECTRA

ANDREI DRUZHININ

Abstract. We construct geometric models for the \( \mathbb{P}^1 \)-spectrum \( M_{\mathbb{P}^1}(Y) \), which computes in Garkusha-Panin’s theory of framed motives \(^7\) a motivically fibrant \( \Omega_{\mathbb{P}^1} \) replacement of \( \Sigma_{\mathbb{P}^1}^\infty Y \) in positive degrees for a smooth scheme \( Y \in \text{Sm}_k \) over a perfect field \( k \). Namely, we get the \( T \)-spectrum in the category of pairs of smooth ind-schemes that defines \( \mathbb{P}^1 \)-spectrum of pointed sheaves termwise motivically equivalent to \( M_{\mathbb{P}^1}(Y) \).

1. Introduction

1.1. The models given by \( T \)-spectra of smooth ind-pairs. Consider the category \( \text{Sm}^\text{pair}_k \) with objects being the pairs \( (X, U) \) with \( X \in \text{Sm}_k \) and an open subscheme \( U \subset X \) over a field \( k \), and morphisms being morphisms of pairs. Let \( \text{ind-Sm}^\text{pair}_k \) be the category of sequences

\[
(1) \quad (X_1, U_1) \to (X_2, U_2) \to \ldots (X_i, U_i) \to \ldots
\]

of closed embeddings of pairs. We call such sequences by smooth ind-pairs.

Denote by \( T \) the pair \( (\mathbb{A}^1, \mathbb{A}^1 - 0) \), and for \( (X, U) \in \text{Sm}^\text{pair}_k \) denote the pair \( (X, U) \wedge T = (X \times \mathbb{A}^1, U \times \mathbb{A}^1 \cup X \times (\mathbb{A}^1 - 0)) \). The last definition extends in a natural way to ind-pairs as well. Then we can consider \( T \)-spectra of ind-pairs, by which we mean the sequences

\[
(R^0, \ldots R^l, \ldots), f_i: R^i \wedge T \to R^{i+1},
\]

where the terms \( R^i \) are ind-pairs and morphisms \( f_i \) are morphisms of ind-pairs. Denote the category of such sequences by \( \text{Spec}_T \text{ind-Sm}^\text{pair}_k \).

Any smooth pair \( (X, U) \) defines the Nisnevich sheaf \( X/U \) that is a factor sheaf of the sheaves represented by \( X \) and \( U \). Then any ind-pair defines a Nisnevich sheaf, and consequently any \( T \)-spectrum of ind-pairs defines a \( T \)-spectrum of Nisnevich sheaves. Thus since any Nisnevich sheaf can be considered as a motivic space we get the functor

\[
\text{Spec}_T \text{ind-Sm}^\text{pair}_k \to \text{SH}(k)
\]

where by \( \text{SH}(k) \) we mean the model for the stable motivic homotopy category given by \( T \)-spectra of motivic spaces.

Theorem 1. Let \( Y \in \text{Sm}_k \) over a perfect field \( k \). Then there are a \( T \)-spectrum \( M'_T(Y) \) in the category \( \text{ind-Sm}^\text{pair}_k \)

\[
M'_T(Y) = (R^0, \ldots R^l, \ldots),
\]

such that \( M'_T(Y) \simeq \Sigma_{T}^\infty Y \) in \( \text{SH}(k) \), where \( M'_T(Y) \) is considered as object in \( \text{SH}(k) \) via the functor \( \mathcal{G} \), and

\[
R^i(Y) \simeq \mathcal{H}om_{\mathcal{H}^s(k)}(T, R^{i+1}(Y)).
\]

The construction is natural on the class of \( Y \) with an affine neighbourhood for any finite set of points.

1991 Mathematics Subject Classification. 14F42.
The research is supported by “Native towns”, a social investment program of PJSC “Gazprom Neft”.

1
In particular this implies the representability of stable motivic homotopy groups as unstable ones, 
$\pi^B_m(F^r(Y)) = [G_m^{p+1} \wedge S^{q-1}, Y]_{SH(k)}$, $q > 0$, though the claim requires us to represent not only the terms $L^i$, but also to represent the structure morphisms $L^i \wedge T \to L^{i+1}$.

Our result is the application of the theory of framed motives [7], which gives in particular the computations of positively motivically fibrant replacements of infinite suspension spectra. The theory implies in addition that the spectra $C^s(M^I_f(Y))$, where $C^s : F \mapsto F(- \times \Delta^s)$, $\Delta$ is the standard simplicial object in $Sm_k$ given by affine spaces, and $(-)_f$ denotes termwise application of the Nisnevich local replacement on simplicial pointed sheaves, gives the $\Omega$-replacement in positive degrees of $\Sigma^f_p Y$.

We show that the simplicial pointed sheaf $C^s(L^i)$ is represented in the category of simplicial schemes, and we expect that more accurate analyse in our technique will show that these schemes are smooth. The representability of such type we have also for the resolution of the bispectrum $\Sigma^\infty_0 \Sigma^\infty_S Y$.

1.2. Framed motives. As noted above our result is the application of the theory of framed motives.

Studying of framed correspondences, and spectra of (pre)sheaves with framed transfers were suggested in the unpublished notes [12] by Voevodsky as an alternative approach to the stable motivic homotopy theory [13, 10, 11, 12] that would be suitable for computational results. The idea had grow to the theory of framed motives introduced and developed by Garkusha and Panin [7, 9] (based on [8] and in co-authorship Ananievski [11] and Neshitov [6]).

In particular, for a smooth scheme $Y$ over a perfect filed $k$ [7, theorem 4.1] gives a computation of positively motivically fibrant replacement of the infinite $\mathbb{P}^1$-suspension $\Sigma^\infty_p Y$. Namely, it is given by the stably motivically equivalence of the $\mathbb{P}^1$-spectrum of pointed Nisnevich sheaves $\Sigma^\infty_p Y \simeq M_{Fr}(Y)_f = (C^s Fr(Y)_f, C^s Fr(Y \wedge T^1)_f, \ldots C^s Fr(Y \wedge T^d)_f, \ldots)$, where $M_{Fr}(-)_f$ is motivically fibrant $\Omega_{\mathbb{P}^1}$-spectra in positive degrees; $(-)_f$ denotes the Nisnevich local fibrant replacement of simplicial (pointed) sheaves; $C^s : Y \mapsto Hom_{Sh\mathcal{A}_q}(\Delta^s, Y)$, and $Fr$ are the sheaves of framed correspondences.

Let us briefly recall that for $X,Y \in Sm_k$ the elements in $Fr(Y \wedge T^1)(X)$ are given by the equivalence classes of the data $c = (Z \hookrightarrow \Delta^n_X, \psi : V \to \Delta^n_X, \alpha : V \to \Delta^n \times \Delta^1 \times Y)$, where

- $V$ is an etale neighbourhood of a closed subscheme $Z$ in $\Delta^n_X$, and $Z = \Delta^n_X \times (\psi, \alpha) \Delta^n \times \Delta^1$,
- the equivalence relation annihilates the choice of $V$, and identifies $(Z, V, \alpha)$ with $(0 \times Z, \Delta^1 \times V, t_1, \alpha \circ pr)$, where $t_1$ denotes coordinate functions on $\Delta^n_X$, and $pr : \Delta^1 \times V \to V$.

So our question precisely is cloud the spectrum $M_{Fr}(Y)$ for a smooth scheme $Y$ be represented up to termwise motivic equivalences by a spectra of smooth schemes, or pairs of smooth schemes?

Since we ask the question up to motivic equivalences we just need to represent the morphisms $Fr(Y \wedge T^1) \to Fr(Y \wedge T^{1+1})$ in the category of smooth schemes for a smooth $Y$.

Remark 1. Let us note that all mentioned constructions of models does not depend of the properties of the base scheme $S$ (though they depend on the properties of $Y$) at least for an affine $Y$, while the computations given by the theory of framed motives holds for an arbitrary smooth schemes but requires the assumption of a perfect base.

(Representability for $M_{Fr}$.) Firstly, we recall the summery of representability results for the case from [4], where the theory of framed motives is studied form the $\infty$-categorical view point.

Theorem ([3, Theorem 5.1.8]). For a smooth $Y \in Sm_k$ over a perfect filed $k$ such that any finite set in $Y$ has an affine neighbourhood there is a pointed smooth ind-scheme $H^{Fr}(Y)$ such that $C^s(H^{Fr}(Y))$ is equipped with a canonical structure of $\mathcal{E}_{\infty}$-space such that there is a canonical equivalence $\Omega^p T \Sigma^p Y \simeq (C^s(h^{Fr}(Y)))_f^{op}$. In particular $\Omega^p_0 \Sigma^p Y \simeq (C^s(H^{fr}(\mathbb{A}^\infty))_f)^{op}$, where $H^{fr}(\mathbb{A}^\infty)$ parametrises finite subschemes $Z$ in $\mathbb{A}^\infty$ with a trivialisation of a (co)normal sheaf $I(Z)/I^2(Z)$. 

The mentioned above result can be considered as the representability for the $S^1$-spectra $M_{fr}(Y)$ called as a framed motive of $Y$, that gives the computation of a positively motivically fibrant replacement of $\Omega_1^\infty \Sigma_1^\infty S(Y)$ given by the framed motive $M_{fr}(Y)$ [4, def. 5.2, th. 11.7].

Let us note that the results of the theory of framed motives, namely the mentioned computation [4, th. 11.7] recovered in [1] cor. 5.5.15 and additivity theorem [7, th. 6.4], [4, proposition 2.2.11], yields that any model for $Fr(Y)$ natural with respect to morphisms
\[ Fr(A \amalg B) \rightarrow Fr(A), A = Y \amalg \cdots \amalg Y, B = Y \amalg \cdots \amalg Y \]
gives such a representability for $\Omega_1^\infty \Sigma_1^\infty Y$. So up to the mentioned results the theorem is a corollary of the following

**Theorem** ([4], theorem 5.1.5(iii) in combination with corollary 2.2.21). The pointed sheaves $Fr(Y)$, where $Y \in \text{Sm}_S$ is such that any finite set of points has an affine neighbourhood, are motivically equivalent in a natural way to the sheaves represented by pointed smooth ind-schemes $H^{fr}(Y)$.

Precisely, it is proven in [4] theorem 5.1.5(iii]) the representability of the (pre)sheaves $Fr^{nr}(Y)$ that are motivically equivalent to $Fr(Y)$ by [4] corollary 2.2.21. The (pre)sheaves $Fr^{nr}(Y)$ are defined by replacing the etale neighbourhood $V$ in the definition of $Fr(Y)$ by the smallest possible domains for morphisms $(\phi_1 \ldots \phi_n)$ and $g$. Namely the the functions $(\phi_1 \ldots \phi_n)$ are defined on the first order thickening of the support $Z$ in $\mathbb{A}^N_\mathbb{Z}$ and $g$ is defined on $Z$. $Fr^{nr}(Y)$ are so-called normally framed correspondences firstly shared in the specialists community by A. Neshitov and they was independently and deeply studied in [4].

In the present text we recover the mentioned above results of [4] obtaining a model for $Fr(Y)$ with additional properties

**Proposition 1.** The pointed (pre)sheaves $Fr(Y)$, where $Y \in \text{Sm}_S$ is such that any finite set of points has an affine neighbourhood, can be represented up to a motivic equivalences by pointed smooth ind-schemes $\mathcal{F}(Y)$ in a such way that $\mathcal{F}(pt) = \lim_n \mathcal{F}_n$ with $\mathcal{F}_n \subset \mathbb{A}^N_\mathbb{Z}$ being a smooth full intersection of a quadrics such that the projection $\mathcal{F}_n \rightarrow \mathbb{A}^{\dim \mathcal{F}_n}$ is etale.

Our result is obtained using another replacement of $Fr(Y)$ that replaces the domain of the functions $(\phi_1 \ldots \phi_n)$, in distinct to $Fr^{nr}(Y)$, by the maximal possible one (namely $\mathbb{A}^N_\mathbb{Z}$).

(Representability for $M_{fr}^{Gm}$) In view of the representability question for $M_{fr}^{Gm}(Y)_f$, which computes over a perfect filed the motivically fibrant $\Omega$-bi-spectrum replacement of $\Sigma_1^\infty \Sigma_1^\infty Y$ [7, theorem 11.1], models for the sheaves $Fr(Y)$ gives the following

**Theorem 2.** For a smooth scheme $Y$ over a base $S$ such that any finite set of points of $Y$ has an affine neighbourhood there is a functor
\[ L: \Gamma^{op} \rightarrow \text{Spec}_{Gm} \text{ind-Sm}^{cl-pair} \]
such that $C^*(L)$ is a simplicial object in the category of $Gm$-spectra of Segal’s $\Gamma$-spaces in the category $\text{ind-Sch}^{cl-pair}$, and the corresponding $Gm$-$S^1$-bi-spectrum is termwise motivically equivalent to $M_{fr}^{Gm}(Y)$ in positive degrees with respect to $S^1$ direction.

Let us note again that this can be immediately deduced using any natural model for $Fr(Y)$, the only one point that requires extra checking is the representability in the category of ind-schemes for $Fr(- \times \Delta^p, Y)$, but if we don’t care about smoothness then it is not complicated.

One can note that seeking about the representability for the $S^1$-spectra and bi-spectra we actually mean the representability of a $E_\infty$-spaces ($\Gamma$-spaces) and moreover even just about the functors $\Gamma^{op} \rightarrow \text{ind-Sm}$ (or $\text{ind-Sm}^{cl-pair}$). Then the structure morphisms of the corresponding $S^1$-spectra (bi-spectra) are given by morphisms of simplicial smooth schemes (or we can replace them by non-smooth ind-schemes), but the author don’t know the model for $S^0$ is smooth schemes. Actually, the representability
if $E_\infty$-spaces looks being much more natural question. Nevertheless if we want to deduce the data to the pure algebra-geometric objects, then it looks being much more natural to go to the $T$ (or $\mathbb{P}^1$) spectra.

(Representability for $M_{\mathbb{P}^1}$.) The main result of the article is the theorem

**Theorem 3.** Let $Y \in \text{Sm}_S$ over a base scheme $S$ of a finite Krull dimension, and assume that any finite set of points in $Y$ has an affine neighbourhood. Then $M_{\mathbb{P}^1}$ is termwise motivically equivalent to a $\mathbb{P}^1$-spectrum of pointed sheaves defined by a $T$-spectrum of inductive systems of open pairs over $S$.

Precisely, there is a morphism of spectra of simplicial pointed sheaves $f: M'_{\mathbb{P}^1} \to M_{\mathbb{P}^1}$,

$$M'_{\mathbb{P}^1} = (L^0(Y), L^1(Y), \ldots, L^l(Y)) \in \text{SH}_*(S), \ L^i(Y) \simeq \lim_n (\mathcal{E}_n^i(Y)/\mathcal{E}_{n+1}^i(Y)),$$

such that

(-) $f$ is a term-wise $\mathbb{A}^1$-Nis-equivalence;

(-) $\mathcal{F}_n(Y)/\mathcal{E}_n(Y)$ denotes the factor sheaves for the inductive system of a pair of smooth $S$-scheme $\mathcal{F}_n(Y)$ and an open subscheme $\mathcal{E}_n(Y)$;

(-) the structure morphisms $L^i(Y) \wedge (\mathbb{P}^1, \infty) \to L^{i+1}(Y)$ are given by the morphisms of schemes $\varepsilon_1: \mathcal{F}_n(Y) \times \mathbb{A}^1 \to \mathcal{F}_{n+1}(Y)$ such that $\varepsilon_1^{-1}(\mathcal{E}_n^{i+1}(Y)) \supset (\mathcal{E}_n^i(Y) \times \mathbb{A}^1) \cup (\mathcal{F}_n^i(Y) \times (\mathbb{A}^1 - 0))$ in composition with the standard morphism of pointed sheaves $(\mathbb{P}^1, \infty) \to T$.

**Remark 2.** Note that if we don’t care to get the natural model with respect to $Y$ and if $S = \text{Spec} k$ for a regular noetherian ring $k$, then the case of an arbitrary smooth $Y$ can be reduced to the case of a smooth affine $Y$ by Jouanolou-Tomason’s trick.

We give also modifications for the case of open or closed pairs $Y$ and $U \subset Y$. We give two proofs for the theorem. The first one is presented in a sketching way and this proof gives result with more generality, while the second proof is more elementary and precise.

1.3. **Notations and conventions.** For a closed subscheme $Z \subset X$ denote by $I(Z) \subset \mathcal{O}(X)$ the sheaf of ideals of functions vanishing on $Z$. For a sheaf of ideals $I \subset \mathcal{O}(X)$ denote by $Z(I)$ the (nonreduced) vanishing locus of $I$.

For a coherent sheaf $M$ on a scheme $X$ over $S$ we denote by $\Gamma(X, M)$ the coherent sheaf on $X$ that is direct image of $M$ under the canonical projection $X \to S$.

We call the motivic equivalences on the categories of pointed (simplicial) presheaves and sheaves also as $\mathbb{A}^1$-Nis-equivalences.

1.4. **Acknowledgement.** The author thanks Grigory Garkusha for the consultations with reading of [7] and for the consultations on possible criteria of unstable motivic equivalences. The author thanks Marc Hoyois for the consultation with the representability of the Weil restriction functors.

2. **Framed correspondences and positive $\Omega_{\mathbb{P}^1}$ spectra.**

2.1. **Framed correspondences.** Here is the first our list of variations of framed correspondences.

Let $Y \in \text{Sm}_S$ and $U \subset Y$ is open.

**Definition 1.** ((Nisnevich) framed corr. $Fr = Fr^{\text{Nis}}$, [14], or def. 2.1 in [7], or def. 2.1.2 in [4]). $Fr_n(Y/U \wedge T^i)$ is a pointed sheaf of sets with the sections $Fr_n(X,Y/U \wedge T^i)$ for $X \in \text{Sch}_S$ given by the equivalence classes of the data $(Z, V, \alpha)$, where $V \to \mathbb{A}^n_X$ is an etale neighbourhood of a closed subscheme $Z \subset \mathbb{A}^n_X$ finite over $X$, and $\alpha =: V \to \mathbb{A}^n \times \mathbb{A}^i \times Y$ is a morphism of schemes such that $Z = V \times_{\alpha, \mathbb{A}^n+1 \times Y, (0 \times i)} (0 \times (Y \setminus U))$, $i: Y \setminus U \to Y$; all elements $(Z, V, \alpha)$ with $Z = \emptyset$ are pointed; the equivalence is up to a choice of the etale neighbourhood $V$.

**Remark 3.** The remarkable Voevodsky’s lemma, see [7] prop 3.5, [4], cor. A.1.7] states that

$$Fr_n(Y/U \wedge T^i) = \text{Sh}_{\text{Nis}^*}(\mathbb{P}^n_X/\mathbb{P}^{n-1}_X, (\mathbb{A}^{n+1}/\mathbb{A}^{n+1} - 0) \wedge (Y/U)).$$
**Definition 2** (first order framed corr. \( Fr^{1th} \)). \( Fr^{1th}(Y/U \wedge T^i) \) is a pointed sheaf of sets with the sections on \( X \in \text{Sch}_S \) given by the data \( (Z, \alpha) \), where \( Z \subset \mathbb{A}^n_X \) is closed, and \( \alpha : Z_2 = Z(I^2(Z)) \) is a morphism of schemes such that \( Z = Z_2 \times_{\mathbb{A}^n \times Y} (0 \times (Y \setminus U)) \). The element with \( Z = \emptyset \) is pointed.

**Definition 3** (normally framed corr. \( Fr^{nr} \)). For the case \( U = \emptyset \) and \( l = 0 \) it is agreed with \( [2] \) def 4.1 and \([4] \), def. 2.2.2.). \( Fr^{n}_{nr}(Y/U \wedge T^i) \) is a pointed sheaf with the sections \( Fr^{nr}(X, Y/U \wedge T^i) \) for \( X \in \text{Sch}_S \) given by the data \( (Z, W, \tau, \beta) \), where \( Z \subset W \subset Z(I(Z)^2) \subset \mathbb{A}^n_X \) are closed, \( \tau : I(W)/(I^2(Z)) \simeq \mathcal{O}^n(W), \beta : W \to \mathbb{A}^l \times Y \) such that \( Z = W \times_{\mathbb{A}^l \times Y} (0 \times (Y \setminus U)) \).

**Definition 4** (Zariski framed corr. \( Fr^{Zar} \)). For \( Y \in \text{Sm}_S \), and an open \( U \subset Y, Fr^{Zar}(X, Y/U \wedge T^i) \) is a sheaf with the sections given by the data \( (Z, V, \phi, \beta) \), where \( V \to \mathbb{A}^n_X \) is a Zariski neighbourhood of a closed subscheme \( Z \subset \mathbb{A}^n_X \), \( W \subset V \) is closed, and \( \phi : V \to \mathbb{A}^n, \beta : W \to \mathbb{A}^l \times Y \) are morphisms of schemes such that \( W = V \times_{\phi, \mathbb{A}^n, 0} 0, Z = W \times_{\mathbb{A}^l \times Y} 0 \times (Y \setminus U) \).

**Remark 4.** The Zariski framed corr. does not satisfy Zariski version of the Voevodsky’s lemma since \( g \) is a map \( W \to Y \) but not \( V \to Y \).

**Definition 5** (polynomial framed correspondences). The sections of the presheaf \( Fr^{pol}(X, Y/U \wedge T^i) \) are given by the data \( (\phi, \beta) \) where \( \phi : \mathbb{A}^n_X \to \mathbb{A}^l, \beta : W \to \mathbb{A}^l \times Y, W = \mathbb{A}^n_X \times_{\phi, \mathbb{A}^n, 0} 0 \) are such that \( Z = W \times_{\mathbb{A}^l \times Y_U} (0 \times (Y \setminus U)) \) is finite over \( X \), where \( i : Y \setminus U \to Y \) is the canonical embedding.

**Remark 5.** If \( U = \emptyset \) and \( l = 0 \) the \( W \) in the definitions above is an excessive data, and \( W = Z \). In particular, \( Fr^{nr}(X, Y) \) consists of sets \( (Z, \tau, g) \) where \( Z \subset \mathbb{A}^n_X \) is closed and finite over \( X, \tau : \mathcal{O}(Z)^n \simeq I(Z)/I^2(Z), g : Z \to Y \).

Let us note that \( Fr^{nr}(Y \wedge T^i) \) alternatively can be defined by the following (see \([2] \) def 4.1): a pointed sheaf with the sections \( Fr^{nr}(X, Y/U \wedge T^i) \) for \( X \in \text{Sch}_S \) given by the data \( (Z, \phi, \psi, g) \), where \( Z \subset \mathbb{A}^n_X \) is a closed finite over \( X, (\phi, \psi) : Z(I^2(Z)) \to \mathbb{A}^{n+1}, g : Z \to Y \) such that \( Z = Z(I^2(Z)) \times_{\mathbb{A}^{n+1}} 0 \).

**Remark 6.** Under the above definitions \( Fr^*(T^{n+1}) = Fr^*(\mathbb{A}^1/\mathbb{G}_m) \). In the same time we can replace \( W \) in the above definitions by the smaller subscheme.

For example if we define the sections of \( Fr^{pol}(X, Y \wedge T^i) \) by the data \( (\phi, \psi, g) \), where \( (\phi, \psi) : \mathbb{A}^n_X \to \mathbb{A}^{n+1}, and g : W \to Y, W = \mathbb{A}^n_X \times_{\phi, \mathbb{A}^n, 0} 0 \) are such that \( Z = W \times (Y \setminus U) \) is finite over \( X \). Then new \( Fr^{pol}(Y/U \wedge T^i) \) differs from the above one, but it is \( \mathbb{A}^1 \)-Nis-equivalent.

Denote by \( Fr^{n}_{\ast}(l) \) (and \( Fr^{n}_{\ast} \)) the bi-functor on the product of \( \text{Sch}^{op}_S \) with the category of pairs \((Y, U)\), which is the full subcategory in the category of arrows of \( \text{Sm}_S \), (or on \( \text{Sch}^{op}_S \times \text{Sm}_S \)) given by \( Fr^*(X, Y/U \wedge T^i) \) (or \( Fr^*(X, Y) \)). Then there is a sequence of natural morphisms

\[
\begin{align*}
Fr^{n}_{\ast}(l) & \rightarrow Fr^{Zar}_{n}(l) \rightarrow Fr_{n}(l) \rightarrow Fr^{1th}_{n}(l) \rightarrow Fr^{nr}_{n}(l) \\
(\phi, \beta) & \rightarrow (Z, W, V, \phi, \beta) \rightarrow (Z, V, (\phi, \beta)) \rightarrow (Z, V, (\phi, \beta)|_{Z_2}) \rightarrow (Z, W, \tau, \beta|_{W}) \\
W \triangleq V & \triangleq \mathbb{A}^n_X \\
Z_2 & = Z(I^2(Z)) \\
W & \triangleq Z(\phi), \tau \triangleq d\alpha^*
\end{align*}
\]

**Definition 6.** Define the maps \( \sigma^* : Fr^*(X, Y/U \wedge T^i) \rightarrow Fr^*_{n+1}(X, Y/U \wedge T^i) \)

\[
\begin{align*}
Fr^{Nis}_{n} & (Z, V, \alpha) \rightarrow (0 \times Z, \mathbb{A}^1 \times V, t, \alpha) \\
Fr^{1th}_{n} & (Z, \alpha) \rightarrow (0 \times Z, t, \alpha) \\
Fr^{nr}_{n} & (Z, W, \tau, \beta) \rightarrow (0 \times Z, 0 \times W, (dt, \tau), \beta) \\
Fr^{Zar}_{n} & (Z, W, V, \phi, \psi, g) \rightarrow (0 \times Z, 0 \times W, \mathbb{A}^1 \times V, t, \phi, \psi, g) \\
Fr^{pol}_{n} & (\phi, g) \rightarrow (t, \phi, g)
\end{align*}
\]

Define \( Fr^*(Y \wedge T^i) = \lim \frac{Fr^*}{n} \).
Remark 7. In the list above the bi-functors $Fr^{th}_{\ast}$, and $Fr_{\ast}$ define the graded categories, but others does not. Bi-functors $Fr^{th}_{\ast}$ and $Fr_{\ast}$ define 'categories' with the associativity up to a canonical $\mathbb{A}^1$-homotopy, while all others define 'categories' with a 'composition' up to $\mathbb{A}^1$-homotopy.

Let us note also that $Fr^{th}_{\ast}(Y)$ is represented by a scheme in a similar way as $Fr^{nr}_{\ast}(Y)$ in [4, theorem 5.1.5].

It is proven in [4, corollary 2.2.21] that the morphism $Fr(Y) \to Fr^{nr}(Y)$ is an $\mathbb{A}^1$-Nis-equivalence, and a close statement for the connected components for the correspondences in $T \wedge T^1$ is written in [2] cor. 4.9. We generalize this by the following.

Proposition 2. For an affine $Y \in Sm_S$ and open $U \subset Y$ the morphisms of the sequence $\mathfrak{3}$ induces $\mathbb{A}^1$-equivalences on affines after the $\sigma$-stabilization.

For an arbitrary smooth $Y$ the morphism induces motivic equivalences of sheaves after the $\sigma$-stabilization.

Proof. The claim follows from lemmas 21 and 22 in the Appendix C 9. The second one follows form lemma 19. □

2.2. Positively motivically fibrant $\Omega_{\mathbb{P}^1}$ replacements. Let $k$ be a perfect field.

Theorem 4 (Garkusha-Panin, theorem 4.1 in [4]). Let $Y \in Sm_k$. Then $\Sigma^\infty_{\mathbb{P}^1} \simeq M_{\mathbb{P}^1}(-)_f: Sm_k \to SH(k)$, where

$$M_{\mathbb{P}^1}(Y)_f = (C^* Fr(Y)_f, C^* Fr(Y \wedge T^1)_f, \ldots, C^* Fr(Y \wedge T^d)_f, \ldots)$$

and $M_{\mathbb{P}^1}(-)_f$ lands in motivically fibrant $\Omega_{\mathbb{P}^1}$-spectra in positive degrees, where $(-)_f$ is the Nisnevich local fibrant replacement functor $(-)_f$ on the category of simplicial (pointed) sheaves $SSh_\bullet$, and $C^*$: is the endo-functor $C^*: \mathcal{G} \to \text{Hom}_{Pres}(\Delta^*, \mathcal{G})$ on $SSh_\bullet$.

Corollary 1. Let $Fr^*$ be a bi-functor on $Sm_S \times Sm^\text{pair}_S$ that restriction on the category of affine schemes is $\mathbb{A}^1$-equivalent to $Fr(Y \wedge T^i)$ Then $M^\infty_{\mathbb{P}^1}(Y)_f = (C^* Fr^*(Y)_f, C^* Fr^*(Y \wedge T^1)_f, \ldots, C^* Fr^*(Y \wedge T^d)_f, \ldots)$ satisfies the same properties as $M^\infty_{\mathbb{P}^1}(Y)$ in theorem 4.

Proof. It is enough to consider the case of $\mathbb{A}^1$-equivalences on affines $Fr^*(Y \wedge T^i) \to Fr(Y \wedge T^i)$ or $Fr(Y \wedge T^i) \to Fr^*(Y \wedge T^i)$. Then the morphism $M^\infty_{\mathbb{P}^1}(Y) \to M_{\mathbb{P}^1}(Y)$ (or $M_{\mathbb{P}^1}(Y) \to M^\infty_{\mathbb{P}^1}(Y)$) is a (term-wise) motivic equivalence. Hence $\Sigma^\infty_{\mathbb{P}^1} Y \to M^\infty_{\mathbb{P}^1}(Y)$ is a stable motivic equivalence, and $M^\infty_{\mathbb{P}^1}(Y)$ (and $M_{\mathbb{P}^1}(Y)_f$) is a positively $\Omega_{\mathbb{P}^1}$-spectra of motivic spaces.

So to get the claim we need to check that $M^\infty_{\mathbb{P}^1}(Y)$ is positively motivically fibrant. By assumption the morphism $C^* Fr^*(Y \wedge T^i) \to C^* Fr(Y \wedge T^i)$ (or $C^* Fr(Y \wedge T^i) \to C^* Fr(Y \wedge T^i)$) is a section wise (simplicial homotopy) equivalence on affines for smooth affine $Y$. So it is Nis-equivalence for a smooth $Y$, and hence $C^* Fr^*(Y \wedge T^i)_f \to C^* Fr(Y \wedge T^i)_f$ (or $C^* Fr(Y \wedge T^i)_f \to C^* Fr(Y \wedge T^i)_f$) is (sectionwise simplicial homotopy) equivalence. Thus $C^* Fr^*(Y \wedge T^i)_f$ is positively motivically fibrant. □

3. Smooth model for $Fr(Y)$ and $M_{\mathbb{P}^1}$ (the first approach).

In the section we present our first approach to the construction of the geometric models for the $P^1$-spectra $M_{\mathbb{P}^1}(Y)$ (and $M_{\mathbb{P}^1}(Y/U \wedge T^i)$). The idea is to replace the presheaf $Fr(Y/U \wedge T^i)$ by the factor $Fr(Y \times A^1)/Fr(U \times A^1 \cup Y \times A^1 - 0)$ or $Fr(Y \times P^i)/Fr(U \times P^i \cup Y \times P^i - 1)$ and use an appropriate model for the presheaves $Fr(Y)_f$, $Y \in Sm_S$.

The advantages of our model for $Fr(Y)$ with respect to the model obtained in [4] are that our model for $Fr(pt)$ is equipped with the canonical (globally defined) etale map to an affine space, and the structure morphism in $H(S)$ of presheaves $Fr(Y) \wedge (P^1, \infty) \to Fr(Y \wedge T^1)$ is representable by morphisms of schemes. The representability of the Weil restriction functor is used in the present model for $Fr(Y)$ for of the same reason as in [4], namely to parametrize the regular maps $g: Z \to Y$ in the definition of framed correspondence.
3.1. Standard idempotent framed corr. The replacement $F_{p^{\text{pol}}}$ extends the framing functions $\phi_i$, $i = 1 \ldots n$, defining the map $V \to \mathbb{A}^n$ in the definition of framed correspondences to the maximal possible neighbourhood of the support $Z$, namely $\phi_i \in O(\mathbb{A}_X^n)$. But it leads to that the vanishing locus $W = Z(\phi_1, \ldots \phi_n)$ could not be finite over $X$ itself in general, and even for the case of pairs $(Y, \emptyset)$, $l = 0$, the vanishing locus $Z(s_1, \ldots s_n)$ would intersect infinity $\mathbb{P}^{n-1}_X$, for any $s_i \in \Gamma(\mathbb{P}^{n-1}_X, O(d_i))$, $\phi_i = x_i/x_i^d_i$.

Nevertheless there is a modification of the definition such that $W$ is finite over $X$, and moreover all $W_i = Z(\phi_{i+1}, \ldots \phi_n)$ are finite over $\mathbb{A}_X^1$ under the projection $\mathbb{A}_X^n \to \mathbb{A}_X^1$ with respect to first $i$ coordinates, and furthermore $\phi_i$ are polynomials with leading terms defining empty vanishing locus on $\mathbb{P}^{n-1}_X$. This property is useful with respect to the representability question, since it guarantees that $Z$ is finite over $X$ by purely algebraically condition (and even linearly algebraically).

Since in this section we are concentrated on the case of pairs $(Y, \emptyset)$, $l = 0$, the only one definition we actually need for the rest part of the section is def. [9] in the definition [8] we just write how to apply this approach for the general case.

**Definition 7** (standard idempotent framed corr. $F_{p^{\text{st-id}}}$).

(st. id. corr.) Define $F_{p^{\text{st-id}}}(Y)$ as the sheaf with sections on $X \in \text{Sch}_S$ given by the data $(Z, s_1, \ldots s_n, g)$, $s_i \in \Gamma(\mathbb{P}^{n-1}_X, O(3^n-i)), s_i|_{\mathbb{P}^{n-1}} = t_i^{3^n-i}$, such that $Z(s_1, \ldots s_n) = Z \amalg \tilde{Z}$ for some $\tilde{Z}$, and $g: Z \to Y$.

(hyperbolic corr.) Denote by $F_{p^{\text{hyp}}}(X, Y) \subset F_{p^{\text{st-id}}}(X, Y)$ consisting of such $(Z, s_1, \ldots s_n, g)$ that $Z = Z(s_1, \ldots s_n)$.

(stabilisation) Define the maps

$$
F_{p^{\text{st-hyp}}}(X, Y) \to F_{p^{\text{st-id}}}(X, Y) : (s_1, \ldots s_n) \mapsto (s'_1, \ldots s'_n, t_{n+1})
$$

$$
F_{p^{\text{st-id}}}(X, Y) \to F_{p^{\text{st-id}}}(X, Y) : (Z, s_1, \ldots s_n, g) \mapsto (Z \times 0, s'_1, \ldots s'_n, t_{n+1}, g)
$$

Denote $F_{p^{\text{hyp}}}(X) = \varinjlim_n F_{p^{\text{hyp}}}(X, Y), F_{p^{\text{st-id}}}(X, Y) = \varinjlim_n F_{p^{\text{st-id}}}(X, Y)$.

To explain the place of this definition with more details let us give few more replacements of fr. corr. The dashed arrows in the diagram exists only in the unstable level, and for the case of $U = \emptyset$ all arrow are $\mathbb{A}^1$-Nis equivalences on the unstable level, but in general only un-dashed are so.

**Definition 8** (idempotent framed corr. $F_{p^{\text{id}}}$).

Define $F_{p^{\text{id}}}(Y)$ as the sheaf with sections on $X \in \text{Sch}_S$ given by the data $(Z, \phi_1, \ldots \phi_n, g)$, $Z \subset \mathbb{A}_X^n$ is finite, $\phi_i \in O(\mathbb{A}_X^n)$, $g: Z \to Y$, such that $Z(\phi_1, \ldots \phi_n) = Z \amalg \tilde{Z}$ for some $\tilde{Z}$.

Define $F_{p^{\text{id}}}(Y/U \setminus T^i)$ as the sheaf with sections on $X \in \text{Sch}_S$ given by the data $(Z, \phi_1, \ldots \phi_n, g)$, $Z \subset \mathbb{A}_X^n$ is finite, $\phi_i \in O(\mathbb{A}_X^n)$, $g: Z \to Y$, such that $Z(\phi_1, \ldots \phi_n) = Z \amalg \tilde{Z}$ for some $\tilde{Z}$.

**Definition 9** ((standard) equational framed corr. $F_{p^{\text{eq}}}$).

Define $F_{p^{\text{eq}}}(Y)$ as the pointed sheaf with sections on $X \in \text{Sch}_S$ given by the data $(e, \phi_1, \ldots \phi_n, g)$, $\phi_i \in O(\mathbb{A}_X^n)$, $(e^2 - e)|_{Z(\phi_1, \ldots \phi_n)} = 0, g: Z(e, \phi_1, \ldots \phi_n) \to Y$.

Define $F_{p^{\text{eq}}}(Y \setminus T^i)$ as the pointed sheaf with sections on $X \in \text{Sch}_S$ given by the data $(e_1, e_2, \phi_1, \ldots \phi_n, \beta)$, $\phi_i \in O(\mathbb{A}_X^n)$, $(e_1^2 - e_1)|_{Z(\phi_1, \ldots \phi_n)} = 0, g: Z(e, \phi_1, \ldots \phi_n) \to Y; (e_2^2 - e_2)|_{\beta^{-1}(0\times(Y \setminus U))} = 0$. 


Define $Fr_n^{st-en}(Y)$ as the pointed sheaf $s_i \in \Gamma(\mathbb{P}_X^n, O(3^n-i))$, $s_i|_{Fr_{n-1}} = t_i^{3^n-i}$, and $e \in O(\mathbb{A}_X^n)$ such that $(e^2 - e)|_{Z(s_1 \ldots s_n)} = 0$, and $g: Z(e, s_1, \ldots s_n) \to Y$.

(standard equational corr.) Define $Fr_n^{st-en}(Y)$ as the pointed sheaf with sections on $X \in \text{Sch}_S$ given by the data $(e, s_1, \ldots s_n, g)$, $s_i \in \Gamma(\mathbb{P}_X^n, O(3^n-i))$, $s_i|_{Fr_{n-1}} = t_i^{3^n-i}$, and $e \in O(\mathbb{A}_X^n)$ such that $(e^2 - e)|_{Z(s_1 \ldots s_n)} = 0$, and $g: Z(e, s_1, \ldots s_n) \to Y$.

(stabilisation) Define the maps

\[ s'_i = s_i(t_{2i+1} - s_i^2), \quad d_i = 3^n - i, \quad 1 \leq i < n, \]

\[ Fr_n^{st-en}(X, Y) \to Fr_n^{st-id}(X, Y) : (e, s_1, \ldots, s_n, g) \to (e, s'_1, \ldots s'_n, t_{n+1}, g) \]

Denote $Fr_n^{st-hyp}(X) = \lim_{\rightarrow n} Fr_n^{st-hyp}(X)$, $Fr_n^{st-id}(X, Y) = \lim_{\rightarrow n} Fr_n^{st-id}(X, Y)$.

**Proposition 3.** For any affine $Y \in \text{Sm}_S$ and open $U \subset Y$ there are natural $\mathbb{A}^1$-equivalences of sheaves $Fr_n^{st-id}(Y) \to Fr(Y)$, and $Fr_n^{st-en}(Y/U \wedge T) \to Fr(Y/U \wedge T)$.

For any $Y \in \text{Sm}_S$ and open $U \subset Y$ there is natural $\mathbb{A}^1$-Nis-equivalence of motivic spaces $Fr_n^{st-id}(Y) \to Fr(Y)$, and equivalences $Fr_n^{st-en}(Y/U \wedge T) \to Fr(Y/U \wedge T) \to Fr(Y/U \wedge T)$.

**Proof.** Clearly the sheaf $Fr_n^{st-id}(Y)$ satisfy the closed gluing. By lemma 22 and proposition 6 to prove the first claim of the lemma it is enough to prove the lifting property with respect to closed embeddings of affines for the morphism $Fr_n^{st-id} \to Fr_n^{st-en}$.

Let $c = (Z, W, \tau, g) \in Fr_n^{st-en}(X, Y)$, $(Z_0, s'_1, \ldots, s'_n, g') \in Fr_n^{st-id}(X_0, Y)$ define the element in $Fr_n^{st-en}(X, Y) \times Fr_n^{st-id}(X_0, Y)/Fr_n^{st-id}(X_0, Y)$.

Applying lemma 13 we can get for all large enough $b$ an etale neighbourhood $r_i \in \Gamma(\mathbb{P}_X^n, O(d_i))$, $d_{n+i} = 3^{-b-i}$, $i = 1, \ldots, m$, $r_i|_{\mathbb{P}_X^n} = t_{n+i}^d$, $C \subset Z(f_1, \ldots f_m) \subset \mathbb{A}_X^n + b$, where $f_i \in cO(\mathbb{A}_X^n + b)$ is the inverse image of $r_i/t_{n+i}^d$ under the projection, such that $Z(f_1 dots f_m) - C \subset \mathbb{A}_X^n$ is an etale neighbourhood of $W \times_{X_0 \times Z} Z_0$.

Then for all large enough $b$ there are sections $s_i \in \Gamma(\mathbb{P}_X^n, O(d_i))$, $d_{n+i} = 3^{-b-i}$, $s_i|_{\mathbb{P}_X^n} = t_{n+i}^d$, $s_i/t_{n+i} \mid Z_0 = s_i^0$, $s_i/t_{n+i} \mid Z(P(W))$ is agreed with $\tau$. Now $(s_1, \ldots, s_n, s_{n+1}, \ldots, s_{n+m}, t_{n+m+1}(1 - t_{n+m+1}^{b-1})) \in Fr_n^{st-id}(X, Y)$ is the required lift of $\sigma^{m+b}$, where $s_{n+i}$ are sections such that $s_{n+i}/t_{n+i}^d \mid Z_0$ is equal to the inverse image of $f_i$.

In a similar way we get the equivalence $Fr_n^{st-en}(Y/U \wedge T) \to Fr_n^{st-nr}(Y/U \wedge T)$. The equivalence $Fr_n^{eq}(Y/U \wedge T) \to Fr(Y)$ in view of proposition 10 follows immediate from Chinese remainder theorem.

3.2. ind-smooth model for $Fr_n(Y)$. According to the above definition we get the sequence of forgetful functors

\[ Fr_n^{st-id}(Y) \to Fr_n^{st-id}(pt) \to Fr_n^{st-hyp}(pt), \]

where the first one is just the composition with the canonical morphism $Y \to pt$ and the second one forgets the choice of the disjoint component.

**Proposition 4.** For any $Y \in \text{Sm}_S$, such that any finite set of points in $Y$ has an affine neighbourhood, the sheaves $Fr_n^{st-id}(Y)$, and $Fr_n^{st-hyp}(Y)$ are represented in $\text{Sm}_S$ by smooth schemes $Fr_n^{st-id}(Y)$, $Fr_n^{st-hyp}(pt)$, such that there is a sequence of morphisms

\[ Fr_n^{st-id}(Y) \to Fr_n^{st-id}(pt) \to Fr_n^{st-hyp}(pt) \cong \mathbb{A}_N, \]

with the first morphism being smooth and the second one being etale.

The natural isomorphisms $Fr_n^{st-id}(Y) \cong \lim_{\rightarrow n} Fr_n^{st-id}(Y)$, and $Fr_n^{st-hyp}(pt) = \lim_{\rightarrow n} Fr_n^{st-hyp}(pt)$ are compatible with the morphisms in sequence (4).
Proof. Denote $N_n = \sum_{i=1}^{n} (\dim \Gamma(P^n, O(3^{n-i})) - \dim \Gamma(P^{n-1}, O(3^{n-i})))$. Then since by definition the elements of $Fr_n^{\text{st, hyp}}(X, pt)$ are the sets of sections $(s_1, \ldots, s_n)$, $s_i \in \Gamma(P^n, O(3^{n-i}))$, $s_i|_{P^{n-1}} = f_i^{3n-i}$, there is a one-to-one correspondence between $Fr_n^{\text{st, hyp}}(X, pt)$ and the $X$-points of $\mathbb{A}^{N_n}$.

Next, if we are given with the element in $Fr_n^{\text{st, id}}(X, pt)$ all what we need to define an element in $Fr_n^{\text{st, id}}(X, pt)$ is to choose a disjoint component of the schemes $Z(s_1, \ldots, s_n)$. Since $Z(s_1, \ldots, s_n)$ is finite and flat over $X$ and $O(Z(s_1, \ldots, s_n)) \simeq O(X)^d$ for some $d$ the second arrow is represented by an étale morphism by lemma [1].

The first arrow in [4] is the Weil restriction functor and it is represented by smooth morphism by [5, section 7.6, proposition 5(h)].

**Lemma 1.** Let $f : Z \to X$ be a finite locally free morphism. Denote by $V \to X$ the total space of the vector bundle over $X$ defined by the coherent sheaf $A = f_*O(Z)$. Denote by $\text{Id}(Z) \to X$ the closed subscheme in $V$ defined by the equation $e^2 - e = 0$, where $e^2$ is the square with respect to the algebra structure on the sheaf $A$. Then the morphism $\text{Id}(Z) \to X$ is étale.

**Proof.** Firstly let’s note that $\text{Id}(Z)$ is quasi-finite over $X$, since there is only finite set of idempotents in a finite dimensional algebra over a field. Then it follows that the endo-morphism $w : V \to V : e \mapsto e^2 - e$ is quasi-finite over some neighborhood $W$ of the zero section $0_X \subset V$. Since $V$ is smooth over $X$ it follows that $w$ is flat over $W$. Hence $\text{Id}(Z)$ is flat over $X$.

Since the algebra $A$ is commutative the differential $d(e^2 - e)$ is equal to $2e - 1$. Since the vanish locus $Z(e^2 - e, 2e - 1) \subset Z(e^2, 2e - 1)$ is empty, $\Omega_{\text{Id}(Z)/X} = 0$. Now since idempotents satisfy the non-separable descent, it follows that the morphism $V \to X$ is unramified.

(Comment of the non-separable descent: Since the disjoint components of the reduced subscheme are the same as reduced subschemes of the disjoint components, it follows that idempotents in the finite dimensional algebra over any field satisfy the descent with respect to local artin algebras. Since the number of disjoint components of the product is equal to the product of numbers of the disjoint components, it follows that the descent with respect to local artin algebras implies the descent with respect to non-separable extensions.)

3.3. Construction of the Ind-smooth models for $Fr(Y \wedge T^l)$ and $M_{P^1}(Y)$. Here we give the general construction of the smooth model for $M_{P^1}(Y/U)$ and a sketch of the proof.

**Lemma 2.** Assume that the base scheme $S$ is of a finite Krull dimension. There are motivic equivalences of pointed Nisnevich sheaves

$$Fr(Y \wedge T^l) = Fr(Y \times (\mathbb{A}^l/(\mathbb{A}^l - 0))) \simeq Fr(Y \times (\mathbb{A}^l/(\mathbb{A}^l - 0))),$$

$$Fr(Y \wedge T^l) = Fr(Y \wedge (\mathbb{P}^l/(\mathbb{P}^l - 0))) \simeq Fr(Y \times (\mathbb{P}^l/(\mathbb{P}^l - 0))) \simeq Fr(Y \times (\mathbb{P}^l/\mathbb{P}^l - 1)), $$

$$Fr(Y \wedge T^l) = Fr(Y \times (\mathbb{P}^l/(\mathbb{P}^l - 1)) \wedge l) \simeq Fr(Y \times (\mathbb{P}^l/\mathbb{P}^l - 1)).$$

Moreover these morphisms being restricted to affines became $\mathbb{A}^1$-equivalences.

**Proof.** The first isomorphism follows by the precise computation.

The second equivalence $Fr(Y \times (\mathbb{A}^l/(\mathbb{A}^l - 0))) \simeq Fr(Y \times (\mathbb{A}^l/(\mathbb{A}^l - 0)))$ follows from the Nisnevich equivalence $Fr^{\text{st}}(Y \times (\mathbb{A}^l/(\mathbb{A}^l - 0))) \simeq Fr^{\text{st}}(Y \times (\mathbb{A}^l/(\mathbb{A}^l - 0)))$, which is straightforward. And similarly we get the second equivalences in other rows.

The last isomorphism in the second row follows since $Fr(Y \times (\mathbb{P}^l - 0)) \to Fr(Y \times \mathbb{P}^l - 1)$ is an $\mathbb{A}^1$-equivalence (by cirteria [15]).

**Remark 8.** In the above proof we do not use the equivalence $M_{Fr}(Y/U) \simeq M_{P^1}(Y/U)$ in general case.

The second equivalences actually follows from the constructions and arguments form [6] and [5]. Let us note that results in [6] and [5] is formulated for the case of sheaves of abelian groups because of the aim of the computation of the framed motive, but the arguments can be translated to the case of sheaves of sets.
Namely the equivalence given by Nisnevich equivalence $Fr(Y \times (k^l/(k^l-0)) \simeq Fr^{stf}(Y \times (k^l/(k^l-0)))$, and $\mathbb{A}^1$ equivalence $Fr^{stf}(Y \times (k^l/(k^l-0))) \simeq Fr(Y \times (k^l/(k^l-0)))$. where $Fr^{stf}(X,Y/U \land T^l)$ are correspondences such that the subscheme $Z(\phi)$ is quasi-finite over $X$, where $\phi: V \to \mathbb{A}^n$ is framing functions. Actually, the first morphism induces the simplicial equivalences on the section on henselian local schemes. The $\mathbb{A}^1$-homotopy inverse for the second morphism is the morphism of the telescope simplicial set corresponding to the filtrations on $Fr(Y \times (k^l/(k^l-0)))$ defined similarly as the ones in the proof of [5 theorem 6] in view of [5 proposition 3].

Remark 9.

Let $Y \in \text{Sm}_S$ and $U \subset Y$ is an open subscheme. Define the $T$-spectrum of pairs of schemes

$$M_{T,n}(Y/U) = (L^0_n, \ldots, L^l_n, \ldots),\quad L^l_n = Fr^{stf}_n(Y \times \mathbb{A}^l)/(U \times \mathbb{A}^l \cup Y \times (\mathbb{A}^l-0)),$$

where $\phi_i, \psi_j: \mathbb{A}^n_X \to \mathbb{A}^l, Z(\phi, \psi) = Z \bot \hat{Z}, g: Z \to Y, x \in \mathbb{A}^l$. We consider the spectrum $M_{T,n}(Y/U)$ as a $\mathbb{P}^1$-spectrum of pointed sheaves via the canonical morphism $(\mathbb{P}^1, \infty) \to T \in \text{Sh}_S$.

Let $Y \in \text{Sm}_S$ and $U \subset Y$ is a closed smooth subscheme. Define the $\mathbb{P}^1$-spectrum of pairs of schemes

$$M'_{\mathbb{P}^1,n}(Y/U) = (L^0_n, \ldots, L^l_n, \ldots),\quad L^l_n = Fr^{stf}_n((Y, U) \land (\mathbb{P}^1, \infty)^{\vee l}),$$

Let

$$M_T(Y/U) \simeq \lim_{n \to \infty} M_{T,n}(Y/U), \quad M'_{\mathbb{P}^1}(Y/U) \simeq \lim_{n \to \infty} M'_{\mathbb{P}^1,n}(Y/U)$$

be the termwise inductive limit of spectra of pair of schemes. We consider them as a spectra of points Nisnevich sheaves that are factor-sheaves represented by pairs.

Now immediate The first two claims follow directly from lemma\[2\] All the rest follows form corollary \[1\] and \[7\] theorem 4.1.

Theorem 5. For any $Y \in \text{Sm}_S$ over a finite Krull dimensional scheme $S$ such that any finite set of points in $Y$ has an affine neigbourhood, and $U \subset Y$ be an open subscheme, the canonical morphism $g(M_T(Y/U)) \to M'_{\mathbb{P}^1}(Y/U)$ is a termwise motivic equivalence, where $g: \text{Spec}_T \text{Sh}_S \to \text{Spec}_p \text{Sh}_S$ is the standard forgetful functor. If $U \subset Y$ is a smooth closed subscheme then the canonical morphism $M'_{\mathbb{P}^1}(Y/U) \to M'(Y/U)$ is a termwise motivic equivalence.

Assume $S = \text{Spec} k$ for a perfect field $k$. Then the canonical morphism $Y \to Fr^{stf:id}(Y)$ induces a motivic equivalence of $\mathbb{P}^1$-spectra of pointed sheaves $\Sigma^\infty_{\mathbb{P}^1} Y/U \to M'_{\mathbb{P}^1}(Y/U).$ The $\mathbb{P}^1$-spectra $C^*(M^{stf}_{\mathbb{P}^1}(Y/U))$ is a positively motivically fibrant $\Omega_{\mathbb{P}^1}$ spectrum.

Moreover if $U = \emptyset$ or $Y$ is quasi-affine then $g(M_T(Y/U)) \to M'_{\mathbb{P}^1}(Y/U)$ are a termwise motivic equivalences, where $M_T(Y/U)$ is defined similarly to $M'_{\mathbb{P}^1}$ [7 section 4] using $Fr(Y/U \land T^l)$.

Proof. For the case of $U = \emptyset$ the $g(M_T(Y/U)) \to M'_{\mathbb{P}^1}(Y/U)$ and $M'_{\mathbb{P}^1}(Y/U) \to M_{\mathbb{P}^1}(Y/U)$ follow directly from lemma\[2\] and all the rest follows form corollary\[1\] and \[7\] theorem 4.1. The last statement for the case of quasi-projective $Y$ is the consequence of \[5\].

The equivalences $g(M_T(Y/U)) \to M_{\mathbb{P}^1}(Y/U)$ and $M'_{\mathbb{P}^1}(Y/U) \to M_{\mathbb{P}^1}(Y/U)$ follow from the ones for $Y$ and $U$. The properties of positively motivically $\Omega$-spectra $g(M_T'(Y/U)) \to M'_{\mathbb{P}^1}(Y/U)$ and $M'_{\mathbb{P}^1}(Y/U) \to M_{\mathbb{P}^1}(Y/U)$ follows by the same arguments as in the proof [7 theorem 4.1] (or alternatively they can be deduced from the result of [2 theorem 10.1]). All properties of $M_T'(Y/U)$ and $M'_{\mathbb{P}^1}(Y/U)$ follows by the same arguments as properties of $M_{\mathbb{P}^1}(Y)$ in [7]. Actually some of them are already stated and proven inside the proof of [7 theorem 4.1].
Consider the category $\text{Sm}^*_{k,\text{pair}}$ with objects being the pairs $(X,U)$ with $X \in \text{Sm}_k$ and an open subscheme $U \subset X$ over a filed $k$, and morphisms being morphisms of pairs. Consider the category $\text{Sm}^\text{cl}_{k,\text{pair}}$ with objects being pairs $(X,Z)$ where $Z \subset X$ is a closed subscheme that is the union $Z = Z_1 \cup \ldots \cup Z_n$ for smooth closed subschemes $Z_i$ such that for any $I \subset \{1, \ldots, n\}, \cap_{i \in I} Z_i$ is smooth. We can see that the categories $\text{Sm}^*_{k,\text{pair}}$ are equipped with closed symmetry monoidal structure

$$(X_1,Y_1) \otimes (X_2,Y_2) = (X_1 \times X_2, Y_1 \times X_2 \cup X_1 \times Y_2).$$

Let $\text{ind-Sm}^*_{\text{pair}}$ be the category with objects being sequences

$$(X_1,U_1) \rightarrow (X_2,U_2) \rightarrow \ldots \rightarrow (X_i,U_i) \rightarrow \ldots$$

of closed embeddings in $\text{Sm}^*_{k,\text{pair}}$ and morphisms being morphisms of sequences. We call such sequences (??) by ind-pairs, precisely either open smooth ind-pairs, either closed ind-pairs. The monoidal structure extends in a natural way to ind-pairs as well.

Denote by $T$ the pair $(\mathbb{A}^1_\mathbb{A}, 0) \in \text{ind-Sm}^*_{\text{pair}}$. Then we can consider $T$-spectra of ind-pairs, by which we mean the sequences

$$(R^0, \ldots, R^l, \ldots), f_i: R^l \rightarrow T \rightarrow R^{l+1},$$

where the terms $R^l$ are ind-pairs and morphisms $f_i$ are morphisms of ind-pairs. Denote the category of such sequences by $\text{Spec}_T \text{ind-Sm}^*_{\text{pair}}$.

Similarly consider $\mathbb{P}^1$ as the pair $(\mathbb{P}^1, \infty) \in \text{ind-Sm}^{\text{cl}}_{\text{pair}}$, and define the category $\text{Spec}_{\mathbb{P}^1} \text{ind-Sm}^{\text{cl}}_{\text{pair}}$ of $\mathbb{P}^1$-spectra of closed ind-pairs.

Any smooth pair $(X,U)$ defines the Nisnevich sheaf $X/U$ that is a factor sheaf of the sheaves represented by $X$ and $U$. Then any ind-pair defined a Nisnevich sheaf, and consequently any $T$-spectrum $(\mathbb{P}^1$-spectrum) of ind-pairs defines a $T$-spectrum $(\mathbb{P}^1$-spectrum) of Nisnevich sheaves. Thus sine any Nisnevich sheaves can be considered as a motivic space we get the functor

$$(6) \quad \text{Spec}_T \text{ind-Sm}^*_{\text{pair}} \rightarrow \text{SH}(k), \text{Spec}_{\mathbb{P}^1} \text{ind-Sm}^{\text{cl}}_{\text{pair}} \rightarrow \text{SH}(k)$$

where by $\text{SH}(k)$ we mean the model for the stable motivic homotopy category given by $T$-spectra $(\mathbb{P}^1$-spectra) of motivic spaces.

Now using the representability obtained in proposition[4] we get

**Theorem 6.** Let $Y \in \text{Sm}_k$ over a perfect filed $k$. Then there are a $T$-spectrum $M'_T(Y)$ in the category $\text{ind-Sm}^*_{k,\text{pair}}$ and a $\mathbb{P}^1$-spectrum $M'_{\mathbb{P}^1}(Y)$ in $\text{ind-Sm}^{\text{cl}}_{k,\text{pair}}$

$$(5) \quad M'_T(Y) = (R^0, \ldots, R^l, \ldots), \quad M'_{\mathbb{P}^1}(Y) = (L^0, \ldots, L^l, \ldots)$$

such that $M'_T(Y) \simeq \Sigma_T^\infty Y$ and $M'_{\mathbb{P}^1}(Y) \simeq \Sigma_{\mathbb{P}^1}^\infty Y$ in $\text{SH}(k)$, and

$$(6) \quad R^l(Y) \simeq \text{Hom}_{\mathcal{H}_k}(T, R^{l+1}(Y)), \quad L^l(Y) \simeq \text{Hom}_{\mathcal{H}_k}(((\mathbb{P}^1, \infty), L^{l+1}(Y)), \quad l \geq 0.$$

The construction is natural on the class of $Y$ with an affine neighbourhood for any finite set of points.

Finally, let us outline the construction that gives us the model with all terms being a inductive sequences of pairs $(X,Z)$ $X \in \text{Sm}_k$ and $Z$ is closed smooth subschemes. But we need to replace the notion of $\mathbb{P}^1$-spectra by the notion of twisted $\mathbb{P}^1$-spectra, where the suspension $(X,Z) \wedge \mathbb{P}^1$ is replaced with the nontrivial $(\mathbb{P}^1, \infty)$ bundle $(\tilde{X}, \tilde{Z}) \rightarrow (X,Z)$ and a morphism $\mathbf{X} \rightarrow X \times \mathbb{A}^1$ that induces isomorphism $\mathbf{X} \times (X \setminus Z) \simeq \mathbb{A}^1 \times (X \setminus Z)$, where $\mathbf{X}$ is the corresponding $\mathbb{A}^1$ bundle over $X$.

Alternatively we can say that the twisted spectrum is the sequence

$$(L^0, L^1, \ldots, L^l, \ldots), L^l \leftarrow \tilde{L}^l \rightarrow \tilde{L}^{l+1},$$

where the morphisms in the right formula are morphisms in $\text{ind-Sm}^{\text{cl}}_{\text{pair}}$, and $L^l = \lim_{n \rightarrow \infty} (X^l_n, Z^l_n), Z^l_n$ are smooth, and $L^l \leftarrow \tilde{L}^l$ is a $(\mathbb{P}^1, \infty)$ bundle, with a rational morphism of ind-pairs $\tilde{L}^l \rightarrow L^l \wedge \mathbb{P}^1$ that in an isomorphism of schemes $\tilde{X}^l_n \times (X^l \setminus Z^l_n) \simeq (X^l_n \wedge \mathbb{P}^1) \times (X^l \setminus Z^l_n)$. 

Then the model for $\Sigma_{p^i}(Y/Z)$ in the twisted $\mathbb{P}^1$-spectra is given by the spectrum with terms Define the $\mathbb{P}^1$-spectrum of pairs of schemes

$$M_{\mathbb{P}^1,n}^0(Y/Z) = (L^0_n, \ldots, L^1_n),$$

$$L^l_n = Fr^{st:id}_n((Y, Z) \wedge (\mathbb{P}^l, \mathbb{P}^l-1)), \quad \Gamma_{t+1} = Fr^{st:id}_n((Y, Z) \wedge (Bl_{(1,1,\ldots,1,0)}(\mathbb{P}^{l+1}), W^l))$$

where $W^l$ is the proper preimage of $\mathbb{P}^l \subset \mathbb{P}^{l+1}$ consisting of points $(0, x_1, \ldots, x_n)$. and morphisms given by the corresponding morphism for $\mathbb{P}^{l+1}$, $\mathbb{P}^l$, and $Bl_{(1,1,\ldots,1,0)}(\mathbb{P}^{l+1})$.

4. Quasi-affine model for $Fr(- \times P, Y/U \wedge T^i)$.

In the section we assume the following context notations

**Context 1.** Let $Y$ be an affine scheme over $S$, $N \in \mathbb{Z}$ is even, and $y_i \in \Gamma(P^N, O(d_p))$, $i = 1 \ldots N - r$, such that $Z(y_1, \ldots, y_{N-r}) \cap \mathcal{A}^N = Y \mid \hat{Y}$. Let $P$ be an affine scheme over $S$ and $\pi_i \in \Gamma(P^M, O(d_p))$, $i = 1, \ldots, q$, $z(p_1, \ldots, p_q) = V \subset P^N$.

For any $n > N$ we consider $Y$ as the subscheme $0 \times Y \times 0 \subset \mathcal{A}^N \times \mathcal{A}^N$, define $y_j \in \Gamma(P^{M+n}, O(d_p))$, $N - r < j \leq n - r$, by $y_j = t_j + r + M(t_\infty - t_j + r + M)^{d_y-1}$, and consider $y_j$, $j \leq n - r$ as sections in $\Gamma(P^{M+n}, O(d_p))$ via the rational projection map $\mathbb{P}^{M+n} \rightarrow \mathbb{P}^N$: $(t_\infty, t_1, \ldots, t_n) \rightarrow (t_\infty, t_{M+1}, \ldots, t_{M+N})$.

Consider the sections $\pi_i$ as sections in $\Gamma(P^{M+n}, O(d_p))$ via the rational projection map $\mathbb{P}^{M+n} \rightarrow \mathbb{P}^M$: $(t_\infty, t_1, \ldots, t_n) \rightarrow (t_\infty, t_{M+1}, \ldots, t_M)$.

Precisely this means that the homogeneous polynomial defining $\pi_j$ on $\mathbb{P}^{M+n}$ is given by the same formula as the homogeneous polynomial defining $\pi_j$ on $\mathbb{P}^M$, and the homogeneous polynomial defining $y_j$ on $\mathbb{P}^{M+n}$ is given by the same formula as the homogeneous polynomial defining $y_j$ on $\mathbb{P}^N$ but under the substitution $t_i \rightarrow t_{M+i}$.

Denote by $\overline{Y}$ and $\hat{Y}$ the closure of $\hat{Y}$ and $Y$ in $\mathbb{P}^N$ (or $\mathbb{P}^n$).

In addition we will work in the following context

**Context 2.** Under the assumptions of the context [7] let $U \subset Y$ be an open.

4.0.1. The sheaves $Fr_{n,d}$.

**Definition 10.** Choose some even integers $d, d_\alpha, d_\beta, d_\gamma, d_\delta, d_\varepsilon$ in $\mathbb{Z}$. Under the context [2] for any $n \in \mathbb{Z}$, $n > N$, define the pair of quasi-affine schemes $Fr_{a}^{qaf}(P, Y/U \wedge T^i) = (Fr_{a}^{qaf}(- \times P, Y \times T^i), Fr_{a}^{qaf}(- \times P, U \times \mathcal{A}^t \cup Y \times (\mathcal{A}^t \setminus \{0\})), \alpha = (n, d, d_\alpha, d_\beta, d_\gamma, d_\delta, d_\varepsilon)$ by the following.

Consider the $S$-affine scheme $\mathcal{F}_n(P, Y \wedge T^i)$ parametrising the vectors $a = (e, s, u, b, w, v, f, h, z, b, \overline{c}, \overline{w})$

$$s = (s_1, \ldots, s_n, s_n + 1, \ldots s_{n+i}), \quad u = (u_1, \ldots, u_n), \quad b = (b_1, \ldots, b_{n+i}), \quad w = (w_{1,j})_{j=1,\ldots,n-r}, \quad v = (v_1, \ldots, v_{n-r}), \quad c = (c_{i,k})_{i=1,\ldots,n+l}, \quad f = (f_1, \ldots, f_{n-r}), \quad b = (b_1, \ldots, b_{n+i}), \quad \overline{w} = (\overline{b}_1, \ldots, \overline{b}_q), \quad \overline{c} = (c_{i,k})_{i=1,\ldots,q}, \quad \overline{v} = (v_{1,j})_{j=1,\ldots,n-r},$$

$$d_\varepsilon = d_b, \quad d_\delta = d_w - 2d_e, \quad d_f = d + d_b - d_\gamma^2, \quad d_\varepsilon^2 = d_\varepsilon + d - d_p, \quad d_\gamma = d_w + d - d_p, \quad d_h = d_b + d - 2d_e,$$
such that
\[
\begin{align*}
\left( t^d + d - d_n \right) e &= \sum_{i=1}^{n+i} b_i s_i + \sum_{i=1}^{q} \overline{b}_i p_i + \sum_{i=1}^{n-r} f_j y_j^2, \\
\left( u_k t^d - d - d_n \right) &= \sum_{i=1}^{n+i} c_i s_i + \sum_{i=1}^{q} \overline{c}_i p_i + \sum_{j=1}^{n-r} z_j y_j^2 \\
\left. u_k \right|_{P_n-1} &= t^d_k \left|_{P_n-1} \\
\left( t^d + d - d_n \right) y_j &= \sum_{i=1}^{n+i} w_{i,j} s_i + \sum_{i=1}^{q} \overline{w}_{i,j} p_i + \sum_{k=1}^{n+r} h_{k,j} u_k + y_j e^2,
\end{align*}
\]
where \( P^{n-1} = Z(t_{∞}, t_1, \ldots, t_M) \subset P^{M+n} \).

Define \( \mathcal{F}_{a}^{q,o}(X \times P, Y \times \mathbb{A}^l) \) as an open subscheme of \( \mathcal{F}_a(P, Y \wedge T^l) \)
\[
\mathcal{F}_{a}^{q,o}(X \times P, Y \times \mathbb{A}^l) = \left\{ (e, s, u, w, v, c) \in \mathcal{F}_a | \Gamma(P^{M+n}, \mathcal{O}(d)) \to \Gamma(Z(I^2(Z(e, s, u))) \mathcal{O}(d)), \right. \\
\left. Z(s_1, \ldots, s_{n+1}) \cap \overline{Y} = \emptyset, Z(s_1, \ldots, s_{n+1}) \cap (Y \setminus Y) = \emptyset \}.
\]

Define \( \mathcal{F}_{a}^{q,o}(X \times P, U \times \mathbb{A}^l \cup Y \times (\mathbb{A}^l - 0)) \) as an open subscheme of \( \mathcal{F}_a^{q,o}(\cdot \times P, Y \times \mathbb{A}^l) \)
\[
\mathcal{F}_{a}^{q,o}(X \times P, U \times \mathbb{A}^l \cup Y \times (\mathbb{A}^l - 0)) = \\
\left\{ (e, s, u, w, v, c) \in \mathcal{F}_a^{q,o}(\cdot \times P, Y \times \mathbb{A}^l) | \mathbb{A}_X^{M+n} \times_{Y \times \mathbb{A}^l} (Y \setminus U) \times 0 = \emptyset \}
\]
where the morphism \( \mathbb{A}_X^{M+n} \to Y \times \mathbb{A}^{n+1} \) is given by
\[
\begin{align*}
\text{pr}_Y : \mathbb{A}_X^{M+n} &\to Y \\
(x, t_1, \ldots, t_M) &\mapsto (t_{M+1}, \ldots, t_{M+N}) \\
\text{pr}_Y : \mathbb{A}_X^{M+n} &\to \mathbb{A}^{n+1} \\
(x, t_1, \ldots, t_{n+1}) &\mapsto (s_1/t_{∞}, s_{n+1}/t_{∞}, e/t_{∞})
\end{align*}
\]

**Remark 10.** In the case of the base fielded the condition \( Z(s_1, \ldots, s_{n+1}) \cap \overline{Y} = \emptyset \) can be replaced by the equation \( s_1/t_{∞} = t_{∞} \).

Moreover we can delete the equations \( y_i \) for \( c \), in other words we can work with the subscheme \( Y \times \mathbb{A}^{N-n} \) instead of \( Y \times 0 \) in \( \mathbb{A}^n = \mathbb{A}^N \times \mathbb{A}^{N-n} \). So this model is defined with less number of equations.

**Definition 11.** Denote by \( \mathcal{F}_{a}^{q,o}(\cdot \times P, Y \times \mathbb{A}^l) \) and \( \mathcal{F}_{a}^{q,o}(\cdot \times P, U \times \mathbb{A}^l \cup Y \times (\mathbb{A}^l - 0)) \) the sheaves represented by the corresponding schemes, and denote by \( \mathcal{F}_{a}^{q,o}(\cdot \times P, Y \times T^l) \) the Nisnevich factor-sheaf represented by the pair \( \mathcal{F}_{a}^{q,o}(P, Y \times T^l) \).

**Lemma 3.** Let \( a = (c, e, u, c, b, w, v, f, h, z, \overline{c}, \overline{c}, \overline{c}) \in M(\mathcal{F}_a(P, Y / P \times T^l)). \) Denote \( Z = Z(s, e) \cap (X \times P) \times \mathbb{A}^{N-N} \), and \( Z(s) = (Z(s) \cap \mathbb{A}_X^{M+n}) \times \mathbb{A}^N \). \( \mathcal{F}_a \) then the following hold
\[
\begin{align*}
\{1\} & Z \text{ is finite over } X \times P \text{ and is contained in } X \times P \times \mathbb{A}^{N-N} \times Y; \\
\{2\} & Z(s) = Z \cap Z(e), \ Z \subset Z(s); \\
\end{align*}
\]
If \( a \in \mathcal{F}_{a}^{q,o}(X \times P, Y \times T^l) \) then
\[
\begin{align*}
\{3\} & Z(\overline{s}) \subset X \times P \times \mathbb{A}^n, \text{ and } Z \text{ is disjoint component of } Z(\overline{s}), \text{ and } Z = Z(\overline{s}, \overline{s}, \overline{s}) \\
\end{align*}

\( \overline{s} \) and \( \overline{s} \) denote the inverse images of \( s \) with respect to morphisms \( X \times P \times Y \to I^{M+n} X \times P \times Z(I^2(Y)) \to P^{M+n}, \) and similarly for \( \overline{s}, \overline{s}, \) and \( \overline{s}, \overline{s}, \overline{s}. \)

**Proof.** It follows from the definitions that
\[
Z = Z(\overline{s}, \overline{s}) \cap X \times P \times \mathbb{A}^n, Z(s) = Z(\overline{s}) \cap X \times P \times \mathbb{A}^n.
\]

The second and the third equations of \( \{1\} \) imply that
\[
Z(\overline{s}) = Z(\overline{s}, \overline{s}) \cap X \times P \times P^{n-1}.
\]
Then the first equation of (7) implies that $Z(\overline{x},\overline{y}) = Z(\overline{y},\overline{z},\overline{p}) \cap Z(\overline{z},\overline{p})$. Finally, the last equation of (7) implies that
\begin{equation}
Z(\overline{e},\overline{f}) \subset X \times P \times Y
\end{equation}
(we mean $X \times P \times Y \times 0$, see context [1]). Since by definition $Z(\overline{x},\overline{y}) \subset X \times P \times Y$, it follows now that $Z(\overline{x},\overline{y}) = Z(\overline{y},\overline{z},\overline{p}) \cap Z(\overline{z},\overline{p})$. Thus combining with (11) we see that $Z(\overline{y}) = Z(\overline{x},\overline{y}) \cap H$.

Using (11) again we see that $Z(\overline{x},\overline{y}) = Z(\overline{x}) \cap X \times P \times \mathbb{A}^n$, and $Z(\overline{y},\overline{z},\overline{p}) = Z(\overline{z}) \cap X \times P \times \mathbb{A}^n$.

Lemma 4. Let $n$ and $l$ be integers. Then for any $d_n \in \mathbb{Z}$ there is $h \in \mathbb{Z}$ such that for all $d > h$, an affine scheme $X$, sections $u = (u_1, \ldots, u_n) \in \Gamma(\mathbb{P}^n_X, \mathcal{O}(d_n))$, $u_i|_{\mathbb{P}^{n-1}} = t_i^d$, and a closed subscheme $Z \subset Z(u)$, the restriction homomorphism $\Gamma(\mathbb{P}^n_X, \mathcal{O}(d)) \to \Gamma(Z, \mathcal{O}(d)) \simeq \Gamma(Z, \mathcal{O})$ is surjective.

Proof. It follows from the conditions on $u$ that $Z(u)$ and consequently $Z$ is finite over an affine scheme $X$. Hence $\Gamma(Z(u), \mathcal{O}(d)) \simeq \mathcal{O}(Z(u)) \to \mathcal{O}(Z) \simeq \Gamma(Z, \mathcal{O}(d))$. So we do not need generality we can assume $Z = Z(u)$.

Consider the $k$-affine space $\Gamma_u$ with closed points being $\{u = (u_i)|u_i \in \Gamma(\mathbb{P}^n), u_i|_{\mathbb{P}^{n-1}} = t_i^d\}$. Let $u \in \Gamma(\mathbb{P}^n_X, \mathcal{O}(d_n))$ denote the universal vector section. Then by lemma [5] for all large enough $d$ the homomorphism $\Gamma(\mathbb{P}^n_X, \mathcal{O}(d)) \to \Gamma(Z(u), \mathcal{O}(d)) \simeq \Gamma(Z, \mathcal{O})$ is surjective.

For any affine $X$ denote by $p_{\mathbb{P}^n} : \mathbb{P}^n_X \to X$ the canonical projection, and denote by $q_u, X : Z(u) \to X$ for a given vector of sections $u$. Then since $X$ is affine $\Gamma(\mathbb{P}^n_X, \mathcal{O}(d)) \to \Gamma(Z(u), \mathcal{O}(d))$ iff $(p_{\mathbb{P}^n})_* (\mathcal{O}(d)) \to (q_u, X)_* (\mathcal{O}(d))$.

In the same we have
\begin{equation}
(p_{\mathbb{P}^n})_* (\mathcal{O}(d)) = \mathcal{O}((p_{\mathbb{P}^n})_* (\mathcal{O}(d))), (q_u, X)_* (\mathcal{O}(d)) = \mathcal{O}((q_u, X)_* (\mathcal{O}(d))).
\end{equation}
Hence $(p_{\mathbb{P}^n})_* (\mathcal{O}(d)) \to (q_u, X)_* (\mathcal{O}(d))$ implies $(p_{\mathbb{P}^n})_* (\mathcal{O}(d)) \to (q_u, X)_* (\mathcal{O}(d))$. □

Lemma 5. Let $X$ and $P$ be affine schemes. Assume $Z$ is a closed subscheme in $\mathbb{A}^n_{X \times P}$ finite over $X \times P$, and $s = (s_i)_{i=1,\ldots,l}$ be a vector of sections $s_i \in \Gamma(\mathbb{P}^n_X, \mathcal{O}(d))$ such that $(Z(s) \cap \mathbb{A}^n_{X \times P}) \cap \mathcal{M} P = Z \cap \mathbb{Y} \cap (\mathbb{Y} \cup (\mathbb{Y} \setminus Y)) = \emptyset$, and $Z \subset X \times P \times \mathbb{A}^n \times Y$.

Then $\exists h_{e}, h_{u} \forall d_{e} > h_{e}, d_{u} > h_{u} \forall h_{b}, h_{w} \forall d_{b} > h_{b}, d_{w} > h_{w} \in Z$ there is a vector of sections
\begin{equation}
a = (e, s, u, c, b, w, v, f, h, z, \overline{b}, \overline{c}, \overline{w})\end{equation}
as in def. (10) such that equalities (11) hold. Moreover the integers $h_{e}, h_{u}, h_{b}, h_{w}$ can be chosen independently on the affine schemes $X, P, Y$.

Proof. Denote by $\overline{Z}$ and $\overline{Z}$ the closures of $Z$ and $\hat{Z}$. Applying Serre’s theorem [5] to the closed subschemes $\mathbb{P}^{n-1}_{X \times P} \cap \overline{Z}$ and $\overline{Z} \cap \mathbb{P}^{n-1}_{X \times P}$ we find $h_{u}$ and $u$, $h_{e}$ and $e$. The choice in given by Serre’s theorem is independent form $Y$ and moreover by lemma [12] one can see that the choice is independent form the affine schemes $X$ and $P$. Then using lemmas [12, 13] we find $h_{b}, h_{w}$, and $c, b, w, v, f, h, z, \overline{b}, \overline{c}, \overline{w}$. □

Definition 12. For any $n$ and $d$ let us chose and fix some $h_{e}(n, d)$ and $h_{u}(n, d)$ and setting $d_{e} = h_{d}, d_{u} = h_{u}$ choose some $h_{b}(n, d)$ and $h_{w}(n, d)$ such that lemma [5] is satisfied, and the homomorphisms
\begin{equation}
\begin{align*}
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_e)) & \to \Gamma(Z(\overline{s}), \mathcal{O}(d_0 + d)) \\
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_b)^{n+1} \oplus \mathcal{O}(d_f)^{n-1}) & \to \Gamma(\mathbb{P}^n_X, \mathcal{O}(d_0 + d)) \\
d_{b} & \to \sum_{i=1}^{n} d_{b} s_{i} + \sum_{j=1}^{n-r} d_{f} y_{j}^2 \\
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_w)^{n+1} \oplus \mathcal{O}(d_v)) & \to \Gamma(\mathbb{P}^n_X, \mathcal{O}(d_0 + d)) \\
d_{w} & \to \sum_{i=1}^{n} d_{w} c_{i} s_{i} + d_{v} e \\
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)) & \to \Gamma(Z(\overline{s})) \cap \mathbb{P}^{n-1}_{X \times P} \\
d_{u} & \to (u_{k})_{Z(\overline{s}) \cap \mathbb{P}^{n-1}_{X \times P}}
\end{align*}
\end{equation}
are surjective, for any affine $\bar{X}, \bar{Z} \subset Z(\bar{s})$, $\bar{s} \in \Gamma(\mathbb{P}_{X}^{n} \times P, \mathcal{O}(d)^{n+1})$. Note $\vec{\mathbf{u}} \in \Gamma(\mathbb{P}_{X}^{n} \times P, \mathcal{O}(d))$.

Denote by $F_{F,n,d} = F_{F,a(n,d),n,d} = F_{F,a(n,d),n,d}^q$, where $a(n,d) = (n, d, u, d_1, u_1, d_2, u_2)$ and similarly defined sheaves $F_{F,n,d}^q$. Denote by $F_{F,n,d} = F_{F,a(n,d),n,d} = F_{F,a(n,d),n,d}^q$. Let $F = F_{n,d}(P, Y \wedge T^l)$ for some $n, d$. Denote $Z = Z(s, e, u, c, w)$.

Consider the universal vector of sections $s = (e, u, c, w, v, h, z, b, \mathbf{c}, \mathbf{w})$ where $\mathcal{F} = F_{n,d}(P, Y \wedge T^l)$.

Consider the sections $s_{i}^l \in \Gamma(\mathbb{P}_{X}^{n+1}, \mathcal{O}(3d)), i = 1, \ldots, l + 1$, $s_{i}^l = s_{i}(t_{i}^{2d} - s_{i}^2)$, for $i \leq l$, and $s_{l+1} = t_{n+1}(t_{n+1}^{2d} - t_{n+1}^{2d-1})$, where $d_1 = h_{n,d}, d_2 = d_1 + 2d + 3d - 1$, and where $s_{i}$ and $e$ are considered as sections on $\mathbb{P}_{X}^{n+1}$ in a standard way via the inclusion $\mathbb{P}_{X}^{n} \rightarrow \mathbb{P}_{X}^{n+1}$. Then one can see $Z(s') = Z \lim_{n} Z'$, $s_{i}^l|_{Z(T^l(z))} = s_{i}t_{i}^{2d}|_{Z(T^l(z))}$, where $Z$ is considered as a closed subscheme in $\mathbb{P}_{X}^{n+1}$ via the inclusion $\mathbb{P}_{F}^{n} \hookrightarrow \mathbb{P}_{F}^{n+1}$.

Then by Lemma 13, there is a vector of sections $a' = (e', s', u', c', w', v', h', z, b, \mathbf{c}, \mathbf{w}) \in F_{n+1,3d}(P, Y \wedge T^l)$. The section $a'$ gives us a regular map $\varphi_{n,d}: F_{n,d}(P, Y \wedge T^l) \rightarrow F_{n+1,3d}(P, Y \wedge T^l)$ that induces the map

$$\varphi_{n,d}: F_{n,d}(P, Y \wedge T^l) \rightarrow F_{n+1,3d}(P, Y \wedge T^l)$$

**Definition 13.** For any $Y$ and $P$ as in the context, define the pairs of ind-schemes $F_{n,d}(P, Y \wedge T^l)$ and the Nisnevich sheaf $F_{n,d}(P, Y \wedge T^l)$

$$F_{n,d}(P, Y \wedge T^l) = \lim_{\rightarrow} F_{n,d}(P, Y \wedge T^l), F_{n,d}(P, Y \wedge T^l) = \lim_{\rightarrow} F_{n,d}(P, Y \wedge T^l)$$

with the morphisms given by $\varphi_{n,d}$. Then the sheaf $F_{n,d}(P, Y \wedge T^l)$ is represented by the pair of ind-schemes $\lim_{\rightarrow} F_{n,d}(P, Y \wedge T^l)$.

5. **Smoothness**

**Context 3.** Assume the context and let $q = 0, M = 0, P = pt$. Moreover assume that $Y$ is affine and $r = \dim Y$.

Denote $F_{n,d}(P, Y \wedge T^l) = F_{n,d}(P, Y \wedge T^l)$, $F_{n,d}(P, Y \wedge T^l) = F_{n,d}(P, Y \wedge T^l)$. The goal of the section is the following

**Proposition 5.** For any $n, d \in Z$, and a smooth affine scheme $Y$ as in context the quasi-affine scheme $F_{n,d}(Y \wedge T^l)$ is smooth.

5.1. Preliminary lemmas.

**Lemma 16.** Let $f = (f_i)_{i=1,\ldots,r}, y_i \in \mathcal{O}(\mathbb{A}_{X}^n), \text{for an affine } k\text{-scheme } X$. Suppose $Z(f) = Y \amalg \bar{Z} \subset \mathbb{A}_{X}^n$, and $Y$ is smooth over $X$. Let $\varphi = (\varphi_i)_{i=1,\ldots,r}, \varphi_i \in \mathcal{O}(\mathbb{A}_{X}^n)$ such that $Z(\varphi) = Z \amalg \bar{Z}, Z \subset Y$, and let

$$\sum_{i=1,\ldots,r} w_{i,j} \cdot (\varphi_i|_{Z(P)}) = f_j|_{Z(P)}$$

where $w_{i,j} \in \mathcal{O}(Z), 1 \leq i \leq l, 1 \leq j \leq r$.

Then the homomorphism $A: \mathcal{O}(Z)^l \rightarrow \mathcal{O}(Z)^r$ given by the matrix $A = (w_{i,j})_{i=1,\ldots,l,j=1,\ldots,r}$ is surjective.

**Proof.** Nakayama’s lemma implies that the homomorphism $A$ is surjective if and only if for each point $x \in Z$ the rank of the matrix $A(x)$ is equal to $r$. Assume that $z \in Z$ is a point such that $\text{rank } A(z) < r$. Hence there is a linear function of $k(z)^{r}$ (a raw) $u = (u_i)_{i=1,\ldots,r}$ such that $u \cdot A(z) = 0$.

Consider the sections $df_j, j = 1,\ldots,r$ of the cotangent vector space in $\mathbb{A}_{z}^n$ at $z$ $T_z = I(z)/I(z)^2$ defined by the functions $f_i$, i.e. $df_j = f_j|_{Z(P)} \in I(z)/I(z)^2, j = 1,\ldots,r$. Since $Y$ is smooth, the sections $df_j$ are linearly independent. Consider then the sections of the fibre of the conormal sheaf of
Lemma 7. Let $y = (y_i)_{i=1,...,r}, y_i \in \Gamma \left( \mathbb{P}_X^n, \mathcal{O}(d_y) \right)$ be a set of global sections on the projective space over an affine $k$-scheme $X$ such that $Z(y|_{\mathbb{A}^n_X}) = Y \sqcup \hat{Y} \subset \mathbb{A}^n_X$, and $Y$ is smooth over $X$. Let $s = (s_i)_{i=1,...,l}, s_i \in \Gamma \left( \mathbb{P}_X^n \mathcal{O}(d_s) \right)$, be a set of global sections. Suppose that $Z(s) = Z \sqcup \hat{Z}, Z \subset Y$ and let
\[
\sum_{i=1,...,l} w_{i,j} \cdot \left( s_i \big|_{Z(\mathbb{P}(Z))} \right) = t_{\infty}^{d_w + d_s - d_y} y_j \big|_{Z(\mathbb{P}(Z))}, 1 \leq j \leq r,
\]
where $w_{i,j} \in \Gamma \left( Z(\mathbb{P}(Z)), \mathcal{O}(d_w) \right)$.

Then the homomorphism $\Gamma \left( Z, \mathcal{O}(d_s)^l \right) \rightarrow \Gamma \left( Z, \mathcal{O}(d_y)^r \right)$ given by the matrix $A = (w_{i,j}|_Z)$ is surjective.

Proof of Lemma 7. Since $Z(s) \subset \mathbb{A}^n$ then $i_Z^* \mathcal{O}(d) \simeq \mathcal{O}(Z)$, where $i_Z : Z \rightarrow \mathbb{P}^n$ is the canonical inclusion. So Lemma 5 implies that the homomorphism of sheaves $i_Z^* \mathcal{O}(d_s)^l \rightarrow i_Z^* \mathcal{O}(d_y)^r$ defined by $A$ is surjective.

Moreover, since $Z(s)$ is closed in $\mathbb{P}_X^n$, it is projective over $X$, and since $Z(s) \subset \mathbb{A}^n$, then it is finite over $X$. Then $Z$ is finite over the affine scheme $X$, so it is affine too. Thus the homomorphism $\Gamma \left( Z, \mathcal{O}(d_s)^l \right) \rightarrow \Gamma \left( Z, \mathcal{O}(d_y)^r \right)$ is surjective.

5.2. Proof of the smoothness.

Proof of the proposition. Denote by $\Gamma_{source}^n$ the affine space that rational points are $\Gamma \left( \mathbb{P}_n, \mathcal{O}(d_e) \oplus \mathcal{O}(d_w)^l \oplus \mathcal{O}(d_s)^l \oplus \mathcal{O}(d_y)^r \oplus \mathcal{O}(d_e)^n \right)$, where $d = 3^n, d_e = h_q^2, d_s = d_e = h_d, d_w = h_d^2, d_v = d_w + d - d_e$.

It follows from the definition that $\mathcal{F} r^q_{\mathbb{P}^2}(Y \land T^l) = A^{-1}_n(0)$, where $A$ is a regular map of the affine spaces over the base
\[
\begin{align*}
A_n : & \quad \Gamma_{source}^n \rightarrow \Gamma_{target}^n \\
(e, s, b, w, v, c) \rightarrow & \quad \left( E_{q_0}, E_{q_1}, E_{q_2,1}, \ldots, E_{q_2,n-r}, E_{q_3,1}, \ldots, E_{q_3,n}, E_{q_4,1}, \ldots, E_{q_4,n} \right) \\
\Gamma_{target}^n = & \quad \Gamma \left( \mathbb{P}_n, \mathcal{O}(d_b + d) \oplus \mathcal{O}(d_w + d)^{n-r} \oplus \mathcal{O}(d_e + d)^n \right) \oplus \Gamma \left( \mathbb{P}^{n-1}, \mathcal{O}(d_e)^n \right) \\
E_{q_1} = & \quad t_{\infty}^{d_w + d_e - d_s} \left( t_{\infty}^{d_e} - e \right) - \sum_{i=1}^{n-r} f_i y_i^2, \\
E_{q_2,i} = & \quad t_{\infty}^{d_w + d_e - d_s} y_j - \sum_{i=1}^{l} w_{i,j} s_i - v_j e^2 - \sum_{k=1}^{n} h_{k,j} u_k, \\
E_{q_3,k} = & \quad u_k t_{\infty}^{d_w + d_e - d_s} - \sum_{i=1}^{l} c_{i,k} s_i - \sum_{j=1}^{n-r} z_{k,j} y_j^2, \\
E_{q_4,k} = & \quad \left( u_k - t_{\infty}^{d_w} \right) \bigg|_{\mathbb{P}^{n-1}}.
\end{align*}
\]

where $k = 1, \ldots, n, j = 1 \ldots n-r$. 
Consider the differential homomorphism $dA_n : T_{p_{\mathrm{source}}} \to A_n^*(T_{p_{\mathrm{target}}})$. The claim is to prove that $dA_n$ is surjective. This is provided by def. [12] and lemma [7]. Let’s write what is $dA_{n,X}$ precisely:

$$dA_n : T_{p_{\mathrm{source}}} \to A_n^*(T_{p_{\mathrm{target}}})$$

$$(de, ds, db, dw, dv, du, dc) \mapsto (dE_{q_0}, dE_{q_1}, \ldots, dE_{q_2}, \ldots, dE_{q_3}, \ldots, dE_{q_4}, \ldots)$$

Then results of lemmas [8–10] which follow in the text, give us the surjections

$$(13) \quad \Gamma(P^n, \mathcal{O}(d_e) \oplus \mathcal{O}(d_b)^{n+1} \oplus \mathcal{O}(d_f)^{n-r}) \to \Gamma(P^n, \mathcal{O}(d + d_b)):\n
(de, db) \mapsto (t^{d+2d} - 2e)de - \sum_{i=1}^{n+l} (db_i s_i + b_i ds_i) - \sum_{i=1}^{n-r} df_i y_j^2$$

$$(14) \quad \Gamma(P^n, \mathcal{O}(d)^{n+l} \oplus \mathcal{O}(d_w)^{(n+l)} \oplus \mathcal{O}(d_v)^{r(n-l)}) \to \Gamma(P^n, \mathcal{O}(d + d_w)^r)$$

$$(de, dw, dv, dc) \mapsto \sum_{i=1}^{n+l} (dw_{i,j} s_i + w_{i,j} ds_i) + \sum_{k=1}^n dh_{k,j} u_{k,j} + dv_j e^2)_{j=1, \ldots, r} \sum_{j=1}^{n-r} dz_{k,j} y_j^2, du_k\big|_{p_{n-1}}$$

for any $a \in F_{r, qaf}^a (Y \wedge T^4)$.

Summing this surjections together we prove surjectivity of $dA_n$.

Consider the tangent sheaf (module) $T_{p_{\mathrm{source}}} = \text{Ker}(dA_n)$. By the above we get that $(T_{p_{\mathrm{source}}})^*_{\mathcal{O}}$ is a free coherent sheaf on $F_{r, qaf}^a (Y \wedge T^4)$ of the rank $\dim F_{r, qaf}^a (Y \wedge T^4)$, which yields that $F_{r, qaf}^a (Y \wedge T^4)$ is smooth.

So all what we need is to prove surjectivity of the homomorphisms [13]. Denote $X = F_{r, qaf}^a (Y \wedge T^4)$, $p_{n,X} : P^n \to X$. Let $a \in F_{r, qaf}^a (X, Y \wedge T^4)$ be the universal section, and denote $Z = Z(e, s, u)$. Denote $\overline{s} = s|_{X \times Z(s^2)}, \overline{e} = e|_{X \times Z(s^2)}, \overline{f} = u|_{X \times Z(s^2)}$, where $Z = Z(y_1^2, \ldots, y_{n-r})$.

**Lemma 8.** For any $a \in F_{r, qaf}^a (X, Y \wedge T^4)$, the homomorphism

$$(p_{n,X})_* (\mathcal{O}(d_e) \oplus \mathcal{O}(d_b)^{n+1} \oplus \mathcal{O}(d_f)^{n-r}) \to (p_{n,X})_* (\mathcal{O}(d + d_b))$$

$$(de, db) \mapsto (t^{d+2d} - 2e)de - \sum_{i=1}^{n+l} (db_i s_i) - \sum_{i=1}^{n-r} df_i y_j^2$$

is surjective.
Proof. It follows from lemma \[3\] that \( Z(\mathcal{F}) \subset X \times \mathbb{A}^n \), and \( (e^2 - e)|_{Z(\mathcal{F})} = 0 \). Hence \( td_{d_0} - 2d_0 \) is invertible on \( Z(\mathcal{F}) \). Thus by definition \[12\] the homomorphisms

\[
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_c)) \to \Gamma(Z(\mathcal{F}), \mathcal{O}(d_0 + d)): \quad dc \mapsto (t^d_{d_0} - 2d_0|_{Z(\mathcal{F})})
\]

are surjective. Now by lemma \[7\] the homomorphism \( (15) \) is surjective. \( \square \)

Lemma 9. The homomorphism

\[
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_w)^{(n+l)} \oplus \mathcal{O}(d_u)^r) \to \Gamma(\mathbb{P}^n_X, \mathcal{I}(Z(\mathcal{F}))(d_0 + d)): \quad (ds, dw, dv) \mapsto \left( \sum_{i=1}^{n+l} dw_{i,j}s_i + w_{i,j}ds_i + \sum_{k=1}^r dh_{k,j}u_k + dv_je^2 \right)_{j=1,...,r}
\]

is surjective.

Proof. By def. \[12\] we see that the homomorphism

\[
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)^n) \to \Gamma(\mathbb{P}^n_X, \mathcal{I}(Z(\mathcal{F}))(d_0 + d)): \quad (du, dv) \mapsto \sum_{i=1}^{n+l} dw_{i,j}s_i + dv_je
\]

is surjective. Now by lemma \[4\] the homomorphism

\[
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)^{n+l}) \to \Gamma(\mathbb{Z}, \mathcal{O}(d + d_u)): \quad (ds) \mapsto \sum_{i=1}^{n+l} w_{i,j}ds_i
\]

is surjective. Whence the exact sequence

\[
0 \to \Gamma(\mathbb{P}^n_X, \mathcal{I}(Z(\mathcal{F}))(d_0 + d)) \to \Gamma(\mathbb{P}^n_X, \mathcal{O}(d + d_u)) \to \Gamma(\mathbb{Z}, \mathcal{O}(d + d_u))
\]

yields that the homomorphism \( (16) \) is surjective. \( \square \)

Lemma 10. The homomorphism

\[
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)^n) \oplus \Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)^{n+1}) \to \Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)^n) \oplus \Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)^{n+1}) : \quad (du, dc) \mapsto (du - \sum_{i=1}^{n+l} dc_{i,k}s_i - \sum_{j=1}^{n-r} dz_{k,j}y_j^2, du_{[p_{n-1}]})_{k=1...n}
\]

is surjective.

Proof. By definition we have \( Z(\mathcal{F}) = Z(s, y_1^2, \ldots, y_{r-1}) \), and it follows from lemma \[3\] that \( Z(\mathcal{F}) \subset X \times \mathbb{A}^n \). According to def. \[12\] we have surjections,

\[
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)^{n+1} \oplus \mathcal{O}(d_u)^{n-r}) \to \Gamma(\mathbb{P}^n_X, \mathcal{I}(Z(\mathcal{F}))(d_0 + d)): \quad (dc, dz) \mapsto \sum_{i=1}^l dc_i + \sum_{j=1}^{n-r} dz_{j}y_j^2,
\]

\[
\Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)) \to \Gamma(Z(\mathcal{F}), \mathbb{P}^n_X, \mathcal{O}(d_u)): \quad (du) \mapsto (u_k|_{Z(\mathcal{F})}, u_k|_{\mathbb{P}^n_X})
\]

where \( k = 1, \ldots, n \). Hence homomorphism \( (16) \) is surjective because of the exact sequence

\[
0 \to \Gamma(\mathbb{P}^n_X, \mathcal{I}(Z(\mathcal{F}))(d_u)) \to \Gamma(\mathbb{P}^n_X, \mathcal{O}(d_u)) \to \Gamma(Z(\mathcal{F}), \mathcal{O}(d_u)).
\]

\( \square \)
6. Models for $M_{p^1}$ and $M_{F_{fr}}^{G_m}$ via $F_r^{qagf}$.

Lemma 3 yields that there is a natural map

$$F_r^{qagf}(- \times P, Y \cup U \cap T^l) \rightarrow Fr_n(- \times P, Y \cup U \cap T^l)$$

(17)

\((e, s, u, w, v, c) \mapsto (Z(e, s), V, s_1/\ell_1, \ldots, s_n/\ell_1, \ldots, s_n/\ell_n, \pi Y, V = \Delta_n \cap -Z(e, s))\)

see (4) for $pr_Y$. Hence there is a natural map $F_r^{qagf}(- \times P, Y \cup U \cap T^l) \rightarrow Fr_n(- \times P, Y \cup U \cap T^l)$. The map (17) is not agreed with the stabilisation according to the definition (13) but the map $F_r^{qagf}(- \times P, Y \cup U \cap T^l) \rightarrow Fr_n(- \times P, Y \cup U \cap T^l)$ is agreed. So we get the natural map

$$F_r^{qagf}(- \times P, Y \cup U \cap T^l) \rightarrow Fr_n(- \times P, Y \cup U \cap T^l).$$

We are going to prove that this is an $\mathbb{A}^1$-Nis-equivalence. The morphism is the composition of the following

(19) $F_r^{qagf}(- \times P, Y \cup U \cap T^l) \rightarrow Fr^{agc}(- \times P, Y \cup U \cap T^l) \rightarrow Fr^{agc}(- \times P, Y \cup U \cap T^l) \rightarrow Fr^{agc}(- \times P, Y \cup U \cap T^l)$

according to the following definitions

**Definition 14.** $Fr^{agc}(- \times P, Y \cup U \cap T^l) = \varinjlim_n Fr_r^{agc}(- \times P, Y \cup U \cap T^l)$ are pointed presheaves with sections $Fr_r^{agc}(X \times P, Y \cup U \cap T^l)$ given by the sets $(e, s)$ with $e$ and $s$ like as in (17) and $Y, U$ are under context (11)

$$Fr^{agc}(- \times P, Y \cup U \cap T^l) \subset Fr^{agc}(- \times P, Y \cup U \cap T^l), (e, s) \in Fr^{agc}(X \times P, Y \cup U \cap T^l) \iff Z(s) \cap (X \times Y) = Z(s) \cap (Y \setminus Y) = \emptyset.$$

**Definition 15.** $Fr^{agc}(- \times P, Y \cup U \cap T^l) = \lim_n Fr_n^{agc}(- \times P, Y \cup U \cap T^l), Fr_n^{agc}(- \times P, Y \cup U \cap T^l) \subset Fr_n^{agc}(- \times P, Y \cup U \cap T^l), (Z, \tau, g) \in Fr_n^{agc}(- \times P, Y \cup U \cap T^l) \iff Z \subset X \times Y \times 0 \times X \times \Delta X \times \Delta^{n-N}, g = pr_N,$

is given by the projection to the first $N$ coordinates, see context (11) for $N$.

**Proposition 6.** Under the context (2) the natural map of presheaves (19) is $\mathbb{A}^1$-Nis-equivalence.

**Proof.** 1) The last morphism restricted to the category of affine schemes is an $\mathbb{A}^1$-equivalence by lemma (15).

Actually there are morphisms of presheaves

$$r_n(X) : Fr_n^{agc}(Y \cap T^l)(X) \rightarrow Fr_n^{agc}(Y \cap T^l)(X) : (Z, \phi, \psi) \mapsto (Z, \phi, \psi, pr_N^*)$$

$$l(X) : Fr_n^{agc}(Y \cap T^l)(X) \rightarrow Fr_n^{agc}(Y \cap T^l)(X) : (Z, \phi, \psi, g) \mapsto (\Gamma, \gamma, \gamma_1, \ldots, \gamma_N, \psi),$$

where $\Gamma \in \mathbb{A}^1$, $Z \rightarrow Y$ considered as a subset in $\mathbb{A}^N \times \mathbb{A}^N$ via inclusions $Y \rightarrow \mathbb{A}^N, Z \rightarrow \mathbb{A}^N$, see context (11) and $\gamma_i = t_i - w_i \circ \tilde{g}$ where $w_i$ denotes coordinates on $\mathbb{A}^N$ and $\tilde{g} : \mathbb{A}^N \rightarrow \mathbb{A}^N$ is a lift of $g$.

To get the claim we need to construct $\mathbb{A}^1$-homotopies

$$r \circ l \sim \sigma_{Fr_n(Y \cap T^l)} : Fr_n^{agc}(X) \rightarrow Fr_n^{agc}(\mathbb{A}^1 \times X), \quad l \circ r \sim \sigma_{Fr_n(Y \cap T^l)} : Fr_n^{agc}(X) \rightarrow Fr_n^{agc}(\mathbb{A}^1 \times X)$$

To get $h_l$ consider the homotopy $h_l(X) : Fr_n^{agc}(X) \rightarrow Fr_n^{agc}(\mathbb{A}^1 \times X) : (Z, \phi, \psi, g) \mapsto (\tilde{\gamma}, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_N, \tilde{\gamma}_i = t_i - \lambda(w_i \circ \tilde{g}) \in O(\mathbb{A}^1 \times \mathbb{A}^{N+n})$.

Then $h_l(X)$ connects $r \circ l \sim P \circ \sigma_{Fr_n(Y \cap T^l)}$, where $P$ is an endomorphism on $Fr_n^{agc}(Y \cap T^l)$ defined by the automorphism of $\mathbb{A}^{n+N}$ given by permutation of coordinates $(t_1, \ldots, t_{n+N}) \mapsto (t_{n+N}, \ldots, t_{n+N})$. Since $N$ is even according to context (11) $P$ is $\mathbb{A}^1$-homotopy equivalent to the identity. The second homotopy $h_r$ is given in a similar way as a composition of the homotopy $h_r : (Z, \gamma, \phi, \psi) \mapsto (\Gamma, \gamma, \phi, \psi)$, where $g = pr_N^*|Z$, and the permutation $P$.

2) The second last morphism is an $\mathbb{A}^1$-Nis-equivalence by proposition (24) because of the isomorphism on presheaves $Fr^{agc} = Fr^{eq}$ on affines. The second morphism in (19) is an $\mathbb{A}^1$-Nis equivalence, since the condition $Z(s) \cap (X \times Y) = \emptyset$ is a condition of infinite codimension.
3) The first morphism in (19) is $\mathbb{A}^1$-Nis-equivalence by proposition 9. Let us skip the checking of the closed glueing that is straightforward. The lifting property for the section \( u \) follows directly from lemma 14 presented in Appendix A: The lifting property for the rest data follows from def. 12 lemma 12 corollary 6 and Lm-remark 1 in Appendix A.

**Corollary 2.** Under context 2 the pointed sheave $Fr(− × P, Y/U \wedge T^i)$ are $\mathbb{A}^1$-Nis-equivalent to the factor sheave represented by the pair of ind-schemes $\mathcal{F}^{qaf}(P, Y/U \wedge T^i)$.

Define the $T$-spectrum

\[
M^f_r(Y) = (Fr^{qaf}(Y), \ldots Fr^{qaf}(Y \wedge T^i))
\]

with

\[
Fr^{qaf}(Y \wedge T^i) \wedge T, (e, s, u, c, b, w, v, f, h, z, \bar{t}, \bar{m}, x) \mapsto (e, s, t_\infty x, u, c, b, w, v, f, h, z, \bar{t}, \bar{m})
\]

Let $M^f_{p_1}(Y)$ denotes the $\mathbb{P}^1$-spectrum obtained from $M^f_r(Y)$ using the standard morphism of pointed sheaves $(\mathbb{P}^1, \mathbb{P}^1 - 0) \simeq T$. Now proposition 6 implies

**Corollary 3.** For any $Y$ under context 2 the canonical morphism of spectra $M^f_{p_1}(Y) \to M^f_1(Y)$ is an equivalence.

**Theorem 7.** Let $Y \in Sm_S$ over a base $S$ and let $Y$ be affine over $S$ or $S = \text{Spec} k$ for a regular noetherian ring $k$. Then there is a $T$-spectrum $M^f_r(Y)$ in the category ind-$\text{Sm}^{\text{exato}}$ with a section wise motivic equivalences of the $\mathbb{P}^1$-spectra of motivic spaces $M^f_{p_1}(Y) \to M^f_1(Y)$, where $M^f_{p_1}(Y)$ is the $\mathbb{P}^1$-spectrum defined by $M^f_r(Y)$.

**Proof.** The case of an arbitrary $Y \in Sm_S$ can be reduced to the case of affine smooth $Y$ by Jouanolou-Tomason’s trick [3, Theorem 1.1]. Next the case of affine smooth $Y$ can be reduced to the case of an affine scheme with the trivial normal bundle 20. Thus since any smooth affine scheme with trivial normal bundle fits into context 1 the claim follows from theorem 4 and corollary 3.

In a similar way to the spectrum $M^f_r(Y)$ in [7, section 11] we can define the spectrum $M^f_{Gr}(Y/U)$ using the sheaves $Fr^{qaf}(Y \wedge T^i)/Fr^{qaf}(U \wedge T^i)$. Now for the case of quasi-projective smooth $Y$ and an open $U \subset Y$ and for an arbitrary smooth $Y$ and closed smooth $U \subset Y$ we have the section-wise equivalence $M^f_{Gr}(Y/U) \simeq M^f_{Gr}(Y/U)$. In the same time the following proposition follows straightforward from the definition.

**Proposition 7.** $M^f_{Gr}(Y/U)_f$ is termwise equivalent to $M^f_{Gr}(Y/U)$ in positive degrees in $S^1$-direction, where $M^f_{Gr}(Y/U)$ is defined like as in [7, section 11] using $Fr(Y \wedge T^i)$.

Finally we have the following representability result

**Theorem 8.** The bi-spectrum $M^u_{Gr}(Y/U)$ is stably motivically equivalent to $\Sigma^\infty_{Gr} \Sigma^\infty_{Gr}(Y/U)$, the termwise fibrant replacement with respect to the injective (Nisnevich) local model structure $M^u_{Gr}(Y/U)$ is motivically fibrant $\Omega$-bi-spectrum in positive degrees with respect to $S^1$-direction.

**Proof.** The case of $U = \emptyset$ follows form the above proposition and that fact that $M^u_{Gr}(Y) = M^u_{Gr}(Y)$ in general the equivalence $M^u_{Gr}(Y/U) \simeq \Sigma^\infty_{Gr} \Sigma^\infty_{Gr}(Y/U)$ follows form the ones for $M^u_{Gr}(Y)$ and $M^u_{Gr}(Y/U)$.

That fact that $M^u_{Gr}(Y/U)$ is motivically fibrant in positive degrees follows by the similar arguments that are used for $M^f_{Gr}$ in [7, section 11].

7. Appendix A: Lifting properties for sections of coherent sheaves.

In the appendix we summarise some results on coherent sheaves used in the article.

**Lemma 11.** Let $f: V \to F$ be a homomorphism of coherent sheaves on a scheme $X$, and $V$ be locally free of a finite rank. Then the set of points $x \in X$ such that $\text{Coker}(t_x^*(f)) = 0$ is closed, where $t_x: x \mapsto X$. 
\textbf{Proof.} Since the question is local we can assume that $X$ is affine. Then any coherent sheaf $F$ on $X$ can be represents as a cokernel of a locally free coherent sheaves of a finite rank, and we can assume that $F$ is locally free of a finite rank without lose of generality. Next since $\text{Coker}(i^*_s(f)) = 0$ iff $\text{Coker}(\Lambda^r f) = 0$, $r = \text{rank } F$, we can assume that $F$ is an invertible sheaf. Consider the dual morphism $f^*: D(V) \to D(V), D(F) = \text{Hom}(F, O(X)), D(V) = \text{Hom}(V, O(X))$. Then $\{x \in X : \text{Coker}(i^*_s(f)) = 0\} = \text{Supp}(\text{Ker } f^*)$.

The rest part of the Appendix is about consequences of Serre’s theorem on ample bundles and cohomologies of coherent sheaves. Let us recall the theorem.

**Theorem 9** (Serre’s theorem). Let $F$ be a coherent sheaf on a scheme $X$ and $O(1)$ be an ample bundle. Then for some $N \in \mathbb{Z}$, for all $d > N$ the cohomologies presheaves of $F(d) = F \otimes O(d)$ are trivial.

We use this theorem in the following form:

**Corollary 4.** Let $F \to G$ be a surjective morphism of coherent sheaves on a scheme $X$ with an ample bundle $O(1)$; then for all large enough $d \in \mathbb{Z}$ the homomorphism of global sections $\Gamma(X, F(d)) \to \Gamma(X, G(d))$ is surjective.

Let us also formulate the following particular case

**Corollary 5.** Let $X' \to X$ be a closed embedding, and let $O(1)$ be an ample bundle on $X$; let $F$ be a coherent sheaf of $F$. Then for all large enough $d$ the restriction $\Gamma(X, F(d)) \to \Gamma(X', F(d))$ is surjective, where $F(d) = F \otimes O(d)$.

**Lemma 12.** Let $Y$ be a projective scheme over some base $X$. Let $s_i \in \Gamma(Y, O(d_i)), i = 1, \ldots, l$, Denote $I(Z)(d) = \Gamma(\mathbb{P}^n_X, I(Z)(d)) = \{s \in \Gamma(Y, O(d)) | s|_Z = 0\}$. Then $\exists N \in \mathbb{Z}$ such that for all $d > N$ the map

$$\bigoplus \Gamma(Y, O(d - d_i)) \to I(Z)(d): (\alpha_1, \ldots, \alpha_l) \mapsto \sum_{i=1}^{l} s_i \alpha_i$$

is surjective.

**Proof.** Consider the homomorphism of coherent sheaves $e: \bigoplus_{i=1}^{l} O(-d_i) \to I(Z)$ given by the vector $s = (s_i)_{i=1,\ldots,l}$, where $I(Z)$ denotes the sheaf of ideals corresponding to the closed subscheme $Z = Z(s)$. Then $e$ is surjective, and $e(d): \sum_{i=1}^{l} O(d - d_i) \to I(Z)(d)$ is surjective for an large enough $d$.

**Lemma 13.** Let $e: X' \to X$ be a closed embedding of affine schemes.

Then for all $d \in \mathbb{Z}$ for any sections $s_i \in \Gamma(\mathbb{P}^n_X, O(d_i)), w'_i \in \Gamma(\mathbb{P}^n_X, O(d - d_i), i = 1, \ldots, l$, such that

$$\sum_{i=1}^{l} w'_i e^*(s_i) = 0,$$

there is a vector of sections $w = (w_i), w_i \in \Gamma(\mathbb{P}^n_X, O(d - d_i)), i = 1, \ldots, l$, such that

$$\sum_{i=1}^{l} w_i s_i = 0, e^*(w_i) = w'_i = 0.$$

**Proof.** Consider the morphism of coherent sheaves $h: \bigoplus O(d - d_i) \to O(d): (w_i) \mapsto \sum w_i s_i$, and denote $\mathcal{E} = \text{Ker } h(d)$. Then $w' \in \Gamma(\mathbb{P}^n_X, \mathcal{E}) = \Gamma(X, p_*(\mathcal{E}))$, where $p: \mathbb{P}^n_X$ is the canonical projection. Now since the direct image $p_*(\mathcal{E})$ is a coherent sheaf on the affine scheme, it follows that $\exists w \in \Gamma(X, p_*(\mathcal{E}))$, $e^*(w) = w'$. This finishes the proof.\qed
Lemma 14. Let \( e : X' \to X \) be a closed embedding of affine schemes. Let \( s_i \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d_i)), i = 1 \ldots l \), and assume that \( d \in \mathbb{Z} \) is such that the homomorphism \( \bigoplus_{i=1}^l \mathcal{O}(d - d_i) \to \mathcal{I}(Z(s))(d) : (w_1, \ldots w_l) \mapsto \sum_{i=1}^l w_is_i \) is surjective.

Then for any sections \( a \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d)), w'_i \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d - d_i)), i = 1 \ldots l \), such that
\[
e^*(a) = \sum_{i=1}^l w'_i e^*(s_i), a|_{Z(s_1, \ldots s_l)} = 0,
\]
there is a vector of sections \( w = (w_i), w_i \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d - d_i)), i = 1 \ldots l \), such that
\[
a = \sum_{i=1}^l w_is_i, e^*(w_i) = w'_i.
\]

Proof. By assumption on \( d \) there is some section \( \bar{w}_i \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d - d_i)), i = 1 \ldots l \), such that \( a = \sum_{i=1}^l \bar{w}_is_i \). Now the claim follows form lemma 13 applied to \( \bar{w} \to w' \), where \( \bar{w} = (\bar{w}_i)_{i=1 \ldots l} \), \( w' = (w'_i)_{i=1 \ldots l} \).

Remark 11. Lemma 14 and lemma 12 implies the following result:
For any closed embedding of affine schemes \( e : X' \to X, \exists D \in \mathbb{Z}, \forall d > D \), for any sections \( a \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d)), s_i \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d_i)), i = 1 \ldots l \), such that
\[
e^*(a) = \sum_{i=1}^l w'_i e^*(s_i), a|_{Z(s_1, \ldots s_l)} = 0,
\]
there is a vector of sections \( w = (w_i), w_i \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d - d_i)), i = 1 \ldots l \), such that \( a = \sum_{i=1}^l w_is_i, e^*(w_i) = w'_i \).

Lemma-remark 1. For any affine \( X \) and \( n, d \in \mathbb{Z} \) elements of \( \Gamma(\mathbb{P}^n, \mathcal{O}(d)) \) are homogeneous polynomials of degree \( d \) with coefficients in \( \mathcal{O}(X) \). Since any polynomial of \( n \) variables can be considered as a polynomial of larger amount of variables we get the following:

Proof. For any affine \( X \) and integers \( 0 < n_1 < n_2 \), and \( d \in \mathbb{Z} \) the restriction homomorphism \( \Gamma(\mathbb{P}^{n_2}, \mathcal{O}(d)) \to \Gamma(\mathbb{P}^{n_1}, \mathcal{O}(d)) \) is surjective. Moreover there is a canonical homomorphism \( \Gamma(\mathbb{P}^{n_1}, \mathcal{O}(d)) \to \Gamma(\mathbb{P}^{n_2}, \mathcal{O}(d)) \) that is left inverse to the restriction \( \Gamma(\mathbb{P}^{n_2}, \mathcal{O}(d)) \to \Gamma(\mathbb{P}^{n_1}, \mathcal{O}(d)) \).

Lemma-remark 2. Let \( X \) be an affine scheme, \( d \in \mathbb{Z} \), and \( \mathbb{A}^n \) be a closed subscheme finite over \( X \). Let \( t_\infty \in \Gamma(\mathbb{P}^n, \mathcal{O}(1)), Z(t_\infty) = \mathbb{P}^{n-1} = \mathbb{P}^n \setminus \mathbb{A}^n \). Consider the restriction homomorphisms \( f_{d+1} \in \Gamma(\mathbb{P}_X^n, \mathcal{O}(d + 1)) \to \Gamma(Z, \mathcal{O}(d + 1)), f_d : \Gamma(\mathbb{P}_X^n, \mathcal{O}(d)) \to \Gamma(Z, \mathcal{O}(d)) \). Then \( \text{Image}(f_d) \subset \text{Image}(f_{d+1}) \).

Proof. Actually, \( \Gamma(Z, \mathcal{O}(d + 1)) \to \Gamma(Z, \mathcal{O}(d)) \) is the isomorphism that is defined by the multiplication by \( t_\infty \) and \( t_\infty \). So the homomorphism \( t_\infty : \Gamma(\mathbb{P}_X^n, \mathcal{O}(d)) \to \Gamma(\mathbb{P}_X^n, \mathcal{O}(d)) \) induced by the multiplication by \( t_\infty \) induces the homomorphism \( \text{Image}(f_d) \to \text{Image}(f_{d+1}) \).

8. Appendix B: \( \mathbb{A}^1 \)-Nis-equivalences

To prove \( \mathbb{A}^1 \)-Nis equivalences we use three following criteria. where the first one is straightforward, and the second one is contained inside the proofs form [3].

Lemma 15. Let \( f : F \to G \) and \( g : G \to F \) be a pair of morphisms of presheaves on \( Sm_k \), and let \( h_F : F \to F^{\mathbb{A}^1}, F^o \circ h_F = g \circ f, h_F^* = id_F, h_G : G \to G^{\mathbb{A}^1}, G^o \circ h_G = f \circ g, h_G^* = id_G \), where \( F^{\mathbb{A}^1}(-) = F(- \times \mathbb{A}^1), F^o(-) = i_0^*(-), i_0: 0 \to \mathbb{A}^1 \), and similarly for \( G \) and \( i_1 \).

Proof.
Proposition 8. Let $e: F \to G$ be a morphism of presheaves on $Sm_k$. Suppose that $e$ satisfies the lifting property with respect to closed embeddings of affines, and both presheaves $F$ and $G$ satisfy closed glueing; then $e$ is a $\mathcal{A}^1$-Nisnevich equivalence.

Proof. The proof is contained in the proof of [2] proposition 2.2.21. Let us briefly recall it. The lifting property with respect to closed embedding of affines imply that the morphism is surjective on affines. Hence [2] lemma A.2.6. implies that the morphism $e$ being valued on affines is a trivial Kan fibration. Whence, since any scheme admits an affine Zariski covering, $e$ is an $\mathcal{A}^1$-Nisnevich equivalence.

Definition 16. Denote by $\Pi_n$ the union of the $\Delta^n \times 1 \subset \mathcal{A}^{n+1} \times \mathcal{A}^1$ and $\delta \Delta_n \times \mathcal{A}^1 \subset \mathcal{A}^{n+1} \times \mathcal{A}^1$.

Proposition 9. Let $e: F \to G$ be a morphism of presheaves on $Sm_k$. Suppose that both presheaves $F$ and $G$ satisfy closed glueing, and suppose that for any simplicial model $V \to \Delta$ for the embedding $\delta \Delta_n \to \Delta_n$, a morphism $v: \Delta \to G$, and a lift $v: \delta \to F$, of the morphism $\delta \to G$, there is a lift of $v$ to a morphism $\Delta \to F$.

Then $e$ is a Nisnevich $\mathcal{A}^1$ equivalence.

Proof. Denote by $I$ the model structure on the category of pointed simplicial presheaves on $Sm$ cofibrantly generated by $\mathcal{A}$-geometric realisations of cofibrations in the injective model structure on pointed simplicial sets, and closed embeddings and coverings. Then any trivial fibration in $I$ is a Nisnevich $\mathcal{A}^1$ equivalence (and even $\mathcal{A}^1$ equivalence) in the category of pointed simplicial presheaves.

Consider the fibrant replacement $\tilde{f}: \tilde{F} \to \tilde{G}$ of $f$ with respect to a model structure $I$. Then $G \to \tilde{G}$ and $F \to \tilde{F}$ are Nisnevich $\mathcal{A}^1$ equivalences. The claim now is to prove that $\tilde{f}$ is a Nisnevich $\mathcal{A}^1$ equivalence.

Actually check that $\tilde{f}$ satisfies the condition of the criteria [15]. Let $\tilde{r}: \delta \Delta_n \to \tilde{F}$ $\tilde{g}: \Delta_n \to \tilde{G}$. Then there is $r: \delta \to F$ and $g: \Delta \to G$. Then there is $\Delta \to F$ and hence $\Delta_n \to F$ since $F$ is fibrant with respect to $I$.

Lemma 16. For any Nisnevich neighbourhood $(U, Z) \to (\mathcal{A}^n_X, Z)$, $X \in Sm$, $Z \subset \mathcal{A}^n_X$ is closed, $Z$ is finite over $X$, for all large enough $d_i \in \mathbb{Z}$, there is a refinement $(U', Z) \to (U, Z) \to (\mathcal{A}^n_X, Z)$, and open immersion $U \hookrightarrow Z(s_1, \ldots, s_n) \subset \mathcal{A}^n \times X$ for some $s_i \in \Gamma(\mathcal{A}^n_X, \mathcal{O}(d_i))$, $s_i |_{\mathcal{A}^{n-1} \times X} = x_i^n$. Moreover we can assume that $Z(\mathcal{I}^2(Z)) \times 0 \subset Z(s_1, \ldots, s_n)$.

Proof. Since any Nisnevich neighbourhood $U \hookrightarrow \mathcal{A}^n_X$ is quasi-finite by Zariski main theorem it follows that there is an open embedding $U \hookrightarrow \overline{U}$ and finite morphism $\overline{U} \to X$. Since any finite morphism is affine there is an embedding $\overline{U} \to \mathcal{A}^1 \times \mathcal{A}^n_X$. Moreover since there is the closed embedding $Z \to \mathcal{U}$, which is a lift of $Z \to \mathcal{A}^n_X$, we can choose it in such a way that the image of $Z$ in $\mathcal{A}^1 \times \mathcal{A}^n_X$ is equal to $0 \times Z$.

Denote $Z_2 = Z(\mathcal{I}^2(0 \times Z)) \times \mathcal{A}^1 \times \mathcal{A}^n_X \subset \mathcal{A}^1 \times \mathcal{A}^n_X$. Choose a sections $v_i \in \Gamma(\mathcal{I}^1 \times \mathcal{A}^n_X, \mathcal{O}(d_i))$ for some $d_i \in \mathbb{Z}$, $v_i |_{\mathcal{F}^2(0 \times Z)} = 0$, $v_i |_{\mathcal{I}^2 \times \mathcal{A}^n_X} = x_i^d$, $i = 1 \ldots l$, where $x_i$ are coordinates on $\mathcal{F}^1$. And denote by $d_1$ the minimal integer such that there are $s_i \in \Gamma(\mathcal{A}^n_X, \mathcal{O}(d_1))$ such that $s_i | t_{\infty}^i = v_i | t_{\infty}^i$ for $i = 1 \ldots l$, where $t_{\infty} \in \Gamma(\mathcal{I}^{l+n}, \mathcal{O}(1))$, $Z(t_{\infty}) = \mathcal{I}^{l+n}$.

Let $e \in \mathbb{Z}$ be any integer such that $ed > d_1$ and let $f \in \mathcal{I}[t]$ be a polynomial of degree $e$ with unit leading term and such that $f |_{\mathcal{I}^{l+2}} = t$. Then consider the morphism $p: \mathcal{A}^1 \times \mathcal{A}^n_X \to \mathcal{A}^1 \times \mathcal{A}^n_X; (x_1, \ldots, x_l, x_{l+1} \ldots x_{l+n}, x) \to (f(x_1), \ldots, f(x_l), x_{l+1} \ldots x_{l+n}, x)$. Let $\mathcal{U} = F^{-1}(\mathcal{U})$ and let $U' = F^{-1}(U)$. Then if follows by assumptions that the intersection of the closure of $\mathcal{U}$ in $\mathcal{I}^{l+n}$ with $\mathcal{I}^{l+n-1}$ is contained in the subspace $\mathcal{I}^{l+n-1} = Z(t_1, \ldots, t_l)$, where $t_i$ denotes coordinates on $\mathcal{I}^{l+n}$. Hence for all large enough $d_i \in \mathcal{Z}$ there are sections $s'_i \in \Gamma(\mathcal{A}^n_X, \mathcal{O}(d_i))$, $s'_i |_{\mathcal{A}^{n-1} \times X} = x_i^d$, $s'_i |_{\mathcal{I}^{l+n}} = 0$. Then $\mathcal{U}'$ is a union of some of the irreducible components of $Z(s'_1, \ldots, s'_l)$, and whence $U'$ is an open subset in $Z(s'_1, \ldots, s'_l)$. 


Thus sections $s_i'$ satisfies all conditions except the last one. Finally, since there is a lift of $Z$ along the morphism $p: Z(s_1, \ldots, s_i) \to \mathbb{A}^n_X$ and $p$ is etale on the image of $Z$ under this lift, changing coordinates on $\mathbb{A}^1 \times \mathbb{A}^n_X$ (relatively over $\mathbb{A}^n_X$) we can get that $Z(I^2(Z)) \times 0 \subset Z(s_1, \ldots, s_i)$. □

**Lemma 17.** For any Nisnevich neighbourhood $(V, Z) \to (X, Z)$, $X \in Sm$, $Z \subset X$ is closed, there are $f_i \in \mathcal{O}(\mathbb{A}^n_X)$, $i = 1, \ldots, m$, such that $Z(f_1, \ldots, f_i) \to X$ is a refinement of $V \to X$, and $Z(\phi_1, \ldots, \phi_m) \subset Z(I^2(Z)) \times 0$.

**Proof.** Since the morphism $V \to X$ is quasi-finite by Zariski main theorem it can be passed throw $V \to V' \to X$ with $V' \to X$ being finite. So $V' \to X$ is affine, let $V' \to \mathbb{A}^n_X$ be an embedding. Since there is a lift $Z \to V'$ and $V' \to X$ is etale on $Z$ changing coordinates on $\mathbb{A}^n_X$ we can get that $V' \subset Z(I^2(Z)) \times 0$. Choose a function $r \in \mathcal{O}(\mathbb{A}^n_X)$ such that $r|_{Z(I^2(Z))} = 1$, $r|_{V'} = 0$. Then $(f_1, \ldots, f_m, (1 + t_m)r - 1)$ is the required set of functions. □

**Lemma 18.** Let $f: F \to G$ be a morphism of simplicial presheaves on $Sm_S$, such that $r^*(f)$ is $\mathbb{A}^1$-Nis-equivalence for any $r \times S$ with affine $X$. Then $f$ is $\mathbb{A}^1$-Nis-equivalence.

**Proof.** Let $f^l: F^l \to G^l$ is fibrant replacement with respect to motivic model structure on the category of simplicial presheaves. The claim is to show that for any affine $X$, $r \times X \to S$, the morphism of simplicial sets $f^l(X)$ is an equivalence (i.e. an isomorphism). We know that $r^*(f)$ and consequently $r^*(f)^l$ are an equivalences. Next we know that $r^*(F^l)$ and $r^*(G^l)$ are fibrant. Then $r^*(f^l)$ is an equivalence of simplicial sets. But $r^*(F^l)(X) = F(X)$ and $r^*(G^l)(X) = G(X)$. □

Next lemma we use without a full proof which requires an accurate analyse left for other works.

**Lemma 19.** For any Nisnevich covering $w: W \to Y$ and open subscheme $U \subset Y$ the morphism of simplicial presheaves $Fr^*_n(W \wedge T^i) \to Fr^*_n(Y/U \wedge T^i), \ast \in \{\text{ft}, e, s, t, n, \text{id}, \text{fr}, \text{ft-id}, \text{fr-id}, \text{Zar}, \text{r}, \text{Nis}, \text{1st}, g, \text{n-r}, \text{nr}, \text{fr-id}, \text{fr-id}, \text{Zar}, \text{r}, \text{Nis}, \text{1st}, g, \text{n-r}, \text{nr}\}$, where $W = \{W/W \cap \bigcup p_i^{-1}(w^{-1}(U)) \cdots W_i^{-1}(W \times \{w^{-1}(U) \cup w^{-1}(U) \times Y \xrightarrow{W/\{W \times \{W \times \{Y\}} \xrightarrow{W/U}\}} W/U\}$, $p_i: W \to W, i = 1 \ldots n$, is the Chech simplicial object in the category of pairs in $Sm_S$ for the morphism $W/\{W \times \{W \times \{Y\}} \xrightarrow{W/U}\} \times Y/w^{-1}(U) \to Y/U$.

**Proof.** The proof is similar to the case of framed correspondences $Fr(Y)$. Namely, it is enough to show that the morphism is equivalence on henselian local pairs, which follows for the lifting property of henselian local pairs with respect to Nisnevich coverings. □

**Lemma 20.** For any smooth affine scheme $Y$ there are an integer $d$, and sections $y_i \in \Gamma(\mathbb{P}^N, \mathcal{O}(d)), i = 1 \ldots r$, such $Z(y_1, \ldots, y_r) = Y' \times Y' \subset \mathbb{P}^N$, and $Y'$ is isomorphic to the normal vector bundle under some closed inclusion $Y \to \mathbb{A}^N$. □

**Proof.** The target bundle of the scheme $N_{Y/\mathbb{A}^N}$ is trivial. Hence its normal bundle is stable trivial. So $N_{Y/\mathbb{A}^N}$ is a disjoint component of a complete intersection in some affine space.

9. **Appendix C:** equivalences of framed corr.

**Lemma 21.** For any $n \in \mathbb{Z}_{\geq 0}$, an affine smooth $Y$, and an open $U \subset Y$, there are $\mathbb{A}^1$-equivalences $Fr^*_n(Y/U \wedge T^i) \to Fr^*_n(Y/U \wedge T^i), \ast \in \{\text{Nis}, 1st\}$. □

**Proof.** Both equivalences follows from the criteria given in proposition 9 (used in the proof of 4 corollary 2.2.11). It is not difficult to see that all of the presheaves satisfy the closed glueing, so to get the claim we need to prove the lifting property with respect to closed embeddings of affines.

Let $c = (Z, W, \tau, r, g) \in Fr^*_n(X, Y/U \wedge T^i)$, and let $(Z_0, V_0, \phi_0, g_0) \in Fr_n(X_0, Y/U \wedge T^i)$ is the lift of the restriction of $c$ to an element in $Fr^*_n(X_0, Y/U \wedge T^i)$. It follows from lemma 18 that there is an etale neighbourhood $V \to \mathbb{A}^n$ of $W \times X \times X_0 V_0$ with the lift $g': V \to \mathbb{A}^1 \times Y$ of $g|_{W \times X X_0 V_0} \to \mathbb{A}^1 \times Y$. Let $\phi$ be a regular map $V \to \mathbb{A}^n$ such that $\phi|_{W} = 0$, the restriction $\phi|_{Z(f^2(W))}$ is defined by the trivialisation $\tau$, and $\phi|_{V_0} = \phi_0$. Now $(Z, V, \phi, g) \in Fr(X, Y/U \wedge T^i)$ is the required lift of $c$.
Let $c = (Z, W, \tau, g) \in F_{n}^{nr}(X, Y / U \times T^{l})$. Let $(Z_{0}, \phi_{0}, g_{0}) \in F_{n}^{th}(X_{0}, Y / U \times T^{l})$ is the lift of the restriction of $c$ to an element in $F_{n}^{nr}(X_{0}, Y / U \times T^{l})$. Similar as above by lemma 23 there is an etale neighbourhood $V \rightarrow \mathbb{A}^{n}_{X}$ of $Z$ with the lift $g': V \rightarrow \mathbb{A}^{n} \times \mathbb{A}^{1} \times Y$ of $g \cup_{g_{0}}: W \coprod_{X \times \mathbb{A}^{n}} (Z(I^{2}(Z))) \times X_{0} \rightarrow \mathbb{A}^{1} \times Y$, and a regular function $\phi$ on $V$ that is a lift of $\phi_{0}$ and is defined by $\tau$ on $Z(I^{2}(W))$. Now the image of $(Z, V, \phi, g)$ under the map $F_{n}(X, Y / U \times T^{l}) \rightarrow F_{n}^{th}(X, Y / U \times T^{l})$ gives the required lift.

The arguments in the proof above gives are enough itself for all equivalences of proposition 2 in the case of $U = \emptyset$, $l = 0$. To get the proof in the general case we need two extra definitions.

**Definition 17** (modified framed corr. $F^{n}$). For $Y \in Sm_{S}$, and an open $U \subset Y$, $F_{n}^{r}(Y / U \times T^{l})$ is a pointed sheaf of sets with the sections $F_{n}^{r}(X, Y / U \times T^{l})$ for $X \in Sch_{S}$ given by the equivalence classes of the data $(Z, W, \phi, \psi, g)$, where $V \rightarrow \mathbb{A}^{n}$ is an etale neighbourhood of a closed subscheme $W \subset \mathbb{A}^{n}$ over $X$, and $\alpha = (\phi, \psi, g): V \rightarrow \mathbb{A}^{n} \times \mathbb{A}^{1} \times Y$ is a morphism of schemes such that $W = V \times_{\mathbb{A}^{n}, \mathbb{A}^{1}, Y} 0$, and $W \times_{Y, \mathbb{A}^{1}, Y} (Y \times U) = Z \cup_{Z} i: Y \times U \rightarrow Y$; all elements $(Z, V, \phi, \psi, g)$ with $Z = \emptyset$ are pointed; the equivalence is up to a choice of the etale neighbourhood $V$.

**Definition 18** (globally normally framed corr. $F^{g-nr}$). For $Y \in Sm_{S}$, and an open subscheme $U \subset Y$, $F_{n}^{g-nr}(Y / U \times T^{l})$ is a sheaf with the sections $F_{n}^{g-nr}(X, Y / U \times T^{l})$ for $X \in Sch_{S}$ given by the data $(Z, W, \tau, \beta)$, where $Z \subset W \subset \mathbb{A}^{n}$ are closed, $\tau: I(W) / (I^{2}(W)) \simeq O^{n}(W)$, $\beta: W \rightarrow \mathbb{A}^{1} \times Y$ such that $Z = W \times_{\mathbb{A}^{1}, Y} (0 \times (Y \times U))$; the elements with $Z = \emptyset$ are pointed.

**Lemma 22.** For any $Y \in Sm_{S}$ and open $U \subset Y$ there are natural $\mathbb{A}^{1}$-Nis equivalences of motivic spaces

$$
\begin{align*}
F_{n}^{r}(Y / U) & \rightarrow F^{g-nr}(Y / U), \quad \ast \in \{\text{pol}, \text{Zar}, \tau\}, n \in \mathbb{Z}_{\geq 0}, \\
F_{n}^{g-nr}(Y / U) & \rightarrow F^{n}(Y / U).
\end{align*}
$$

**Proof.** All arrows in the lemma are equivalences by the criteria given by proposition 9. Let us skip the verification of the closed glueing, which is straightforward. So if we check the lifting property on affines for the morphism $F^{g-nr}(Y / U) \rightarrow F^{n}(Y / U)$, this would implies the equivalence. Using lemma 15 we can reduce the question to the case of affine $Z$.

Consider an element of $F_{n}^{g-nr}(X_{0}, Y / U \times T^{l}) \times_{F_{n}^{r}(X, Y / U \times T^{l})} F_{n}^{r}(X, Y / U \times T^{l})$. It is a set $(Z, W, \tau, g)$ such that $Z, W \subset \mathbb{A}^{n}$ are closed,

$$
W \times (X - X_{0}) \subset Z(I^{2}(Z)) \times X (X - X_{0}), \quad \tau: I(W) / (I(W_{2})) \simeq O^{n}(W),
$$

g: W \rightarrow \mathbb{A}^{1} \times Y, Z = W \times_{g, \mathbb{A}^{1}, Y, 0 \times Y} 0 \times (Y \times U),
$$

where $W_{2} = W \times_{X} X_{0} \cup Z(I^{2}(Z))$. The trivialisation $\tau$ defines the regular map $\phi': Z(I^{2}(W)) \rightarrow \mathbb{A}^{n}$ such that $Z(I^{2}(W)) \times_{\mathbb{A}^{n} \times X_{0}} 0 = W$. By lemma 13 there is an etale neighbourhood $V$ of $W$ in $\mathbb{A}^{n}$ and a map $(\hat{\phi}, \hat{g}): V \rightarrow \mathbb{A}^{1} \times Y$ that is a lift of $\phi'$ and $g$. By lemma 17 there are regular functions $f_{i}$ on $\mathbb{A}^{n} \times \mathbb{A}^{m}$, $i = 1 \ldots m$, such that $Z(I^{2}(\hat{Z})) \times 0 \subset Z(f_{1}, \ldots, f_{m}) \subset \mathbb{A}^{n} \times \mathbb{A}^{m}$, $f_{i}|_{Z(I^{2}(Z) \times 0)} = t_{n+i}$, where $t_{i}$ denotes coordinates on $\mathbb{A}^{n} \times \mathbb{A}^{m}$, and $(Z(f_{1}, \ldots, f_{m}), 0 \times Z) \rightarrow (\mathbb{A}^{n}, Z)$ is a refinement of the trivialisation $(V, Z) \rightarrow (\mathbb{A}^{n}, Z)$. Let $W' = Z(\phi', f_{1}, \ldots, f_{m}) \subset \mathbb{A}^{n+m}$, and $\tau': I(W') / I^{2}(W') \simeq O^{n+m}(W')$, and $f_{i}|_{W'} \in F_{n}^{g-nr}(X, Y)$ is the required lift of $\sigma^{m}(Z, W, \tau, \psi, g) \in F_{n}^{g-nr}(X, Y / U \times T^{l})$.

The lifting property with respect to closed embeddings of affines for the arrows $F_{n}^{r}(Y / U) \rightarrow F_{n}^{g-nr}(Y / U)$ is follows immediately from the Chinese remainder theorem.

Finally, we need to check the lifting property for $F_{n}^{pol}(Y / U) \rightarrow F_{n}^{g-nr}(Y / U)$. Let $X_{0} \rightarrow X$ be a closed embedding of affine schemes. Consider an element of $F_{n}^{pol}(X_{0}, Y / U \times T^{l}) \times_{F_{n}^{pol}(X, Y / U \times T^{l})} F_{n}^{g-nr}(X, Y / U \times T^{l})$. It is given by a pair $c = (Z, W, \tau, g) \in F_{n}^{g-nr}(X, Y / U)$, $c_{0} = (\phi^{0}, \beta) \in F_{n}^{pol}(X_{0}, Y / U)$.
Firstly using Jouanolou’s trick (Tomason’s theorem) we reduce the question about the general proof.

Let $\phi = (\phi_i)_{i=1...n}$, on $A^n_X$ such that $\phi_i|_{A^n \times X} = \phi_0$, and $\phi_i|_{Z(I^2(W))}$ are agreed with $\tau$. Then $Z(\phi) = W \setminus W$. Let $\phi_n+1 \in O(A^{n+1}_X)$ be such that $\phi_n+1|_{Z(I^2(W))} = 1$, $\phi_n+1|_{\bar{W}} = 0$. Now the correspondence $(\phi, (t_n+1)\phi_n+1 - 1) \in F^p_{n+1}(X,Y)$ is the required lift of $c$. □

10. Appendix D: Lifting properties of smooth morphisms

Lemma 23. Let $Y$ be a smooth over a base $S$. Let $Z \to X$ be an affine henselian pair with $X$ being regular, and $g: Z \to Y$ a morphism. Then there is a lift of $g$ to a morphism $g': X \to Y$. If in addition $Y$ is affine (and more generally quasi-projective) then $Y \to S$ satisfies the lifting property with respect to any affine henselian pair $Z \to X$.

Proof. Firstly using Jouanolou’s trick (Tomason’s theorem) we reduce the question about the general $Y$ to the case of affine $Y$. Namely, for an arbitrary $Y$ we consider the morphism $T = Y \times X \to X$ as a smooth scheme over an affine scheme $X$. Then by Jouanolou’s trick [3] Proposition 4.3, corollary 4.6] there is an affine bundle $T' \to T$ with affine $T'$. Now since $Z$ is affine $T' \times_Z Z \to Z$ is vector bundle, and there is a lift $Z \to T'$ of the morphism $Z \to T$ defined by the morphism $Z \to Y$. Finally applying the lemma to the affine smooth $X$-scheme $T'$ we get the lift $X \to T'$ of the morphism $Z \to T'$, and thus the composition $X \to T' \to Y$ gives the required lift of the morphism $Z \to Y$.

Assume $Y$ is affine and smooth over $S$. Let $N_Y \to Y$ be a vector bundle such that $N_Y \oplus N_Y$ is trivial, where $N_Y$ is relative normal bundle over $S$. Then it is enough to prove the claim for $N_Y$ instead of $Y$. So we can assume that normal (and tangent) bundle $N_Y$ of $Y$ is trivial, let $r = (r_1m...r_d): O(Y) \cong N_Y$, $d = \dim Y$.

By assumptions the morphism $Y \times_S X \to X$ is affine and admits a section $v: Z \to Y \times_S X$ over $Z$. We need to find a lift to a section $X \to Y \times_S X$. Consider regular functions $f_i$ on $Y \times_S X$ such that $f_i|_v(Z) = 0$, $f_i|_Z = r_i$, where $Z_2 = Z(I^2(v(Z))) \times_X Z$. Let $p: \nabla = Z(f_1,...f_d) \to X$ be the canonical projection. Let $C \subset \nabla$ be the maximal closed subset such that fibres of $\nabla_X$ over $C$ are of dimension at least one. Let $V' = V - (C \cup (p^{-1}(Z) - v(Z)))$. Then $V' \to X$ is quasi-finite, and $V' \times_X Z = v(Z)$. It follows form the condition on $f_i$ that $V'$ is smooth at $v(Z)$ over $S$. Let $V \subset V'$ be an open neighbourhood of $v(Z)$ that is smooth over $S$. Since $V \times_X Z = v(Z)$ it follows that $e: V'' \to X$ is unramified. Hence $e$ etale, and so there is a lift $X \to V$.

Combining this with the notion of formally smoothness we get the following

Corollary 6. For a locally finite type $S$-scheme $Y$ the following are equivalent:
1) $Y$ is smooth;
2) $Y$ is formally smooth, i.e., it satisfies the lifting property with respect to a closed embeddings of affine schemes $Z \to X$ with $I^2(Z) = 0$;
3) $Y$ satisfies the lifting property with respect to a henselian affine pairs $Z \to X$ with $X$ being regular.

If $Y$ is affine (or quasi-projective) in addition then the above conditions are equivalent to
4) $Y$ satisfies the lifting property with respect to a henselian affine pairs.

References

[1] Ananyevskiy, Garkusha, Panin, Cancellation theorem for framed motives of algebraic varieties, arXiv:1601.06642
[2] A. Ananyevskiy and A. Neshitov, Framed and MW-transfers for homotopy modules, arXiv:1710.07412v2.
[3] A. Asok, The Jouanolou-Thomason homotopy lemma, http://www-bcf.usc.edu/~asok/notes/Jouanolou.pdf
[4] E. Elmanto, M. Hoyois, A. A. Khan, V. Sosnilo, M. Yakerson, Motivic infinite loop spaces, arXiv:1711.05248
[5] D., Motives of smooth affine pairs, arXiv:1803.11388 (March 2018).
[6] Garkusha, Neshitov, Panin, Framed motives of relative motive spheres, arXiv:1604.02732
[7] Garkusha, Panin, Framed motives of algebraic varieties (after V. Voevodsky), arXiv:1409.1372
[8] Garkusha, Panin, Homotopy invariant presheaves with framed transfers, arXiv:1504.00849v2
[9] G. Garkusha, I. Panin, *The triangulated categories of framed bispectra and framed motives*, https://arxiv.org/pdf/1809.08006.pdf.

[10] J. F. Jardine, *Motivic symmetric spectra*, Doc. Math., 5 (2000), 445–552

[11] F. Morel and V. Voevodsky, *A¹-homotopy theory of schemes*. Inst. Hautes Etudes Sci. Publ. Math., (90):45-143 (2001), 1999.

[12] F. Morel, *An introduction to A¹-homotopy theory*, Contemporary developments in algebraic K-theory, 357-441, ICTP Lect. Notes, 2004/

[13] V. Voevodsky, *A¹-homotopy theory*, Doc. Math., Extra Vol. I (1998), pp. 579-604.

[14] V. Voevodsky, *Notes on framed correspondences*, unpublished, 2001. Also available at math.ias.edu/vladimir/files/framed.pdf