Cohomology of finite groups without homological algebra.

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Introduction.

This note is devoted to a proof of explicit formulas for the homology and cohomology of a finite group that uses only simple operations such as quotient, tensor product and $G$-invariants. These formulas are interesting because they do not use the notion of a complex. In other words, they do not use homological algebra at all. At the end of the note we prove a few well-known statements in order to show how these formulas can be used.

Let $G$ be a finite group, $\mathbb{Z}G$ be the group ring of $G$, $\Delta$ be the augmentation ideal and $M$ be a $\mathbb{Z}G$-module. We denote by $M^G$ and $M_\Delta$ the abelian group of $G$-invariants and the group of $G$-coinvariants of $M$. The tensor product of $\mathbb{Z}G$-modules is always taken over $\mathbb{Z}$ and is equipped with the diagonal action. We set $\mathcal{N} = \sum_{g \in G} g \in \mathbb{Z}G$. Since $g\mathcal{N} = \mathcal{N}$, the abelian group $\mathcal{N} \cdot \mathbb{Z}$ is a $\mathbb{Z}G$-submodule of $\mathbb{Z}G$. Denote by $\mathbb{Z}G/\mathcal{N}$ the corresponding quotient module. The main goal of this note is to prove the following “elementary” formulas for homology and cohomology of a finite group

\[
H^n(G, M) \cong \frac{((\mathbb{Z}G/\mathcal{N})^n \otimes M)^G}{\mathcal{N} \cdot ((\mathbb{Z}G/\mathcal{N})^n \otimes M)}, \quad H_n(G, M) \cong \frac{((\Delta^\otimes n+1 \otimes M)^G}{\mathcal{N} \cdot ((\Delta^\otimes n+1 \otimes M)},
\]

where $n \geq 1$. Moreover, we prove similar formulas which do not look so elementary but they still do not use notion of a complex. We set $\text{Ann}_\mathcal{N}(M) = \{m \in M \mid \mathcal{N} \cdot m = 0\}$. Then there are isomorphisms

\[
H_n(G, M) \cong \frac{\text{Ann}_\mathcal{N}(\Delta^\otimes n \otimes M)}{\Delta \cdot (\Delta^\otimes n \otimes M)}, \quad H^n(G, M) \cong \frac{\text{Ann}_\mathcal{N}((\mathbb{Z}G/\mathcal{N})^n \otimes M)}{\Delta \cdot ((\mathbb{Z}G/\mathcal{N})^n \otimes M)},
\]

where $n \geq 1$. The last formula for homology is a generalisation of the well-known isomorphism $G_{ab} \cong \Delta/\Delta^2$, where $G_{ab}$ is the abelianization of the group $G$. The proof of these formulas is essentially based on the Tate cohomology theory of finite groups.

1 Tate cohomology.

In this section we remind several facts about Tate cohomology of a finite group (see [1 VI]). Throughout the paper, $G$ will denote a finite group.

A complete resolution of the finite group $G$ is an acyclic complex of projective $\mathbb{Z}G$-modules $T_\bullet$ such that $Z_1(T_\bullet) = \text{Ker}(d_1^T) \cong \mathbb{Z}$, where $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}G$-module. The complete resolution exists and is unique up to a homotopy equivalence. Tate cohomology of the group $G$ with coefficients in a module $M$ is defined by $\hat{H}^n(G, M) = H^n(\text{Hom}_{\mathbb{Z}G}(T_\bullet, M))$ for $n \in \mathbb{Z}$. A short exact sequence of $\mathbb{Z}G$-modules $M_1 \rightarrowtail M_2 \twoheadrightarrow M_3$ induces the long exact sequence of Tate cohomologies

\[
\cdots \rightarrow \hat{H}^n(G, M_1) \rightarrow \hat{H}^n(G, M_2) \rightarrow \hat{H}^n(G, M_3) \rightarrow \hat{H}^{n+1}(G, M_1) \rightarrow \cdots
\]
One of the main properties of Tate cohomology theory is the fact that it “glues” together usual homologies and cohomologies. More precisely, there are isomorphisms

\[ H^n(G, M) \cong \hat{H}^{-n}(G, M), \quad H_n(G, M) \cong \hat{H}^{1-n}(G, M), \]  

for \( n \geq 1 \). These formulas give a description of all Tate cohomologies \( \hat{H}^n(G, M) \), except the indices \( n = -1, 0 \). Let us describe the rest. The multiplication by \( N \) induces a homomorphism \( N : M_G \to M_G \). So \( \hat{H}^{-1}(G, M) \cong \text{Ker}(N), \hat{H}^0(G, M) \cong \text{Coker}(N) \). Hence, we obtain the following isomorphisms:

\[ \hat{H}^{-1}(G, M) \cong \frac{\text{Ann}_N(M)}{\Delta \cdot M}, \quad \hat{H}^0(G, M) \cong \frac{M_G}{N \cdot M}. \]  

If \( A \) is an abelian group (with the trivial action of \( G \)), then \( \hat{H}^n(G, ZG \otimes A) = 0 \) (see [1 VI, §5.3]). On the other hand, the module \( ZG \otimes M \) with the diagonal action is isomorphic to \( ZG \otimes \overline{M} \), where \( \overline{M} \) is the underlying abelian group of the module \( M \) (see [1 III, §5.7]). Therefore, for any module \( M \), we get

\[ \hat{H}^n(G, ZG \otimes M) = 0. \]  

## 2 Main result.

Consider the monomorphism \( N' : Z \to ZG \) given by the formula \((N')(n) = N \cdot n \). Since \( gN = N \), we obtain that it is a \( ZG \)-homomorphism. Its cokernel is equal to \((ZG)/N\). Thus, we get the following short exact sequence of \( ZG \)-modules

\[ 0 \to Z \xrightarrow{N'} ZG \to (ZG)/N \to 0. \]  

We state that \( N' : Z \to ZG \) is a split monomorphism of abelian groups since the composition with the homomorphism \( \sum n_0 g \mapsto n_1 \) is the identity map. Hence, the short exact sequence [2.1] splits as a short exact sequence of abelian groups.

**Lemma 1.** For any \( n \in Z \) there are isomorphisms

\[ \hat{H}^n(G, M) \cong \hat{H}^{-n}(G, (ZG/N) \otimes M), \quad \hat{H}^n(G, M) \cong \hat{H}^{n+1}(G, \Delta \otimes M). \]

**Proof.** The short exact sequence \( \Delta \to ZG \to Z \) splits as a short exact sequence of abelian groups because \( Z \) is a free abelian group. Thus, tensoring by \( M \), we obtain the following short exact sequence of \( ZG \)-modules \( \Delta \otimes M \to ZG \otimes M \to M \). Applying the long exact sequence [1.1] to this short exact sequence and using [1.4], we get the isomorphism \( \hat{H}^n(G, M) \cong \hat{H}^{n+1}(G, \Delta \otimes M) \). The other isomorphism can be established similarly using the short exact sequence [2.1]. \( \square \)

**Theorem 1.** Let \( G \) be a finite group and \( M \) be a \( ZG \)-module. Then there are isomorphisms

\[ H^n(G, M) \cong \frac{\left((ZG/N)^{\otimes n} \otimes M\right)^G}{N' \cdot \left((ZG/N)^{\otimes n} \otimes M\right)}, \quad H_n(G, M) \cong \frac{\left(\Delta^{\otimes n+1} \otimes M\right)^G}{N' \cdot \left(\Delta^{\otimes n+1} \otimes M\right)}, \]

\[ H_n(G, M) \cong \frac{\text{Ann}_N(\Delta^{\otimes n} \otimes M)}{\Delta \cdot \left(\Delta^{\otimes n} \otimes M\right)}, \quad H^n(G, M) \cong \frac{\text{Ann}_N((ZG/N)^{\otimes n+1} \otimes M)}{\Delta \cdot \left((ZG/N)^{\otimes n+1} \otimes M\right)}, \]

where \( n \geq 1 \).

**Proof.** Using induction on \( n \) and lemma [1] we obtain isomorphisms \( \hat{H}^n(G, M) \cong \hat{H}^{-n}(G, (ZG/N)^{\otimes n} \otimes M) \) and \( \hat{H}^n(G, M) \cong \hat{H}^{1-n}(G, \Delta^{\otimes n} \otimes M) \). Then, combining them with [1.2] and [1.3], we obtain the required isomorphisms. \( \square \)
3 Corollaries.

In this section we prove three well-known statements using theorem [1] in order to show how these formulas can be used.

**Corollary 1.** If $|G| = k$, then $k \cdot H^n(G, M) = 0$ and $k \cdot H_n(G, M) = 0$ for $n \geq 1$.

*Proof.* It is sufficient to prove that $k \cdot \left( M^G / (N \cdot M) \right) = 0$ for any $\mathbb{Z}G$-module $M$. It follows from the fact that for any $m \in M^G$ the equality $km = Nm$ holds. 

**Corollary 2.** There is an isomorphism $G_{ab} \cong \Delta/\Delta^2$.

*Proof.* Theorem [1] implies the isomorphism $G_{ab} \cong H_1(G) \cong \text{Ann}_N(\Delta)/\Delta^2$, and it is easy to see that $\text{Ann}_N(\Delta) = \Delta$.

Remind that a derivation is a map $\partial : G \to M$ such that $\partial(gh) = \partial(g) + g\partial(h)$ for all $g, h \in G$, and the inner derivation corresponding to $m \in M$ is the derivation $\partial_m : G \to M$ given by $\partial_m(g) = gm - m$. The abelian group of derivations (resp. inner derivations) $G \to M$ is denoted by $\text{Der}(G, M)$ (resp. $\text{Inn}(G, M)$).

**Lemma 2.** There is an isomorphism

$$(\mathbb{Z}G/N \otimes M)^G \cong \text{Der}(G, M)$$

that induces an isomorphism $N \cdot (\mathbb{Z}G/N \otimes M) \cong \text{Inn}(G, M)$.

*Proof.* Put $\bar{g} = g + N\mathbb{Z}$. It is easy to see that $\{ \bar{g} \mid g \in G \setminus \{1\} \}$ is a basis of the abelian group $\mathbb{Z}G/N$. Hence, for any element $x \in \mathbb{Z}G/N \otimes M$ there is a unique collection $(\partial(g))_{g \in G}$ such that $x = \sum g \otimes \partial(g)$ and $\partial(1) = 0$. A straightforward computation shows that $x \in (\mathbb{Z}G/N \otimes M)^G$ if and only if $\partial \in \text{Der}(G, M)$. Moreover, the equalities $\sum \bar{g} \otimes \partial_m(g) = N \cdot (1 \otimes m)$ and $N \cdot (\sum \bar{g} \otimes m_g) = \sum \bar{g} \otimes \partial_{N \cdot m_{g'}}(g)$ show that inner derivations correspond to elements of $N \cdot (\mathbb{Z}G/N \otimes M)$. 

**Corollary 3.** There is an isomorphism $H^1(G, M) \cong \text{Der}(G, M)/\text{Inn}(G, M)$.

References

[1] Brown K.S. *Cohomology of Groups.* (Grad. Texts Math. 87) Berlin Heidelberg New York: Springer 1982.