Generalizations of Maximal Inequalities to Arbitrary Selection Rules

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Abstract

We present a generalization of the maximal inequalities that upper bound the expectation of the maximum of $n$ jointly distributed random variables. We control the expectation of a randomly selected random variable from $n$ jointly distributed random variables, and present bounds that are at least as tight as the classical maximal inequalities, and much tighter when the distribution of selection index is near deterministic. A new family of information theoretic measures were introduced in the process, which may be of independent interest.

1 Introduction

Throughout this paper, we consider $n$ random variables $Z_i, 1 \leq i \leq n$ such that $E[Z_i] = 0$, where $n$ is a finite positive integer. The zero mean condition can be satisfied via the operation $Z_i' = Z_i - E[Z_i]$ upon assuming that all $Z_i$'s are integrable. The following two maximal inequalities are well known in the literature and serve as the motivational results for this work.

Lemma 1. Let $\psi \geq 0$ be a convex function defined on the interval $[0, b)$ where $0 < b \leq \infty$. Assume that $\psi(0) = 0$. Set, for every $t \geq 0$,

$$\psi^*(t) = \sup_{\lambda \in (0, b)} (\lambda t - \psi(\lambda)).$$

Suppose that $\ln E[e^{\lambda Z_i}] \leq \psi(\lambda)$ for all $\lambda \in [0, b), 1 \leq i \leq n$. Then,

$$E[\max_i Z_i] \leq \psi^{*-1}(\ln n),$$

where $\psi^{*-1}(y)$ is defined as

$$\psi^{*-1}(y) = \inf \{t \geq 0 : \psi^*(t) > y\}.$$

To introduce the second inequality, we say a function $\psi$ is an Orlicz function if $\psi : [0, \infty) \rightarrow [0, \infty]$ is a convex function vanishing at zero and is also not identically 0 or $\infty$ on $(0, \infty)$. We define the Luxemburg $\psi$ norm of a random variable $X$ as

$$\|X\|_{\psi} = \inf \left\{ \sigma > 0 : E \left[ \psi \left( \frac{|X|}{\sigma} \right) \right] \leq 1 \right\}.$$
Lemma 2. Let $\psi$ be an Orlicz function. Suppose $\|Z_i\|_{\psi} \leq \sigma, 1 \leq i \leq n$. Then,

$$E[\max_i Z_i] \leq \sigma \cdot \psi^{-1}(n),$$

where $\psi^{-1}(y)$ is defined as $\psi^{-1}(y) = \inf\{t \geq 0 : \psi(t) > y\}$.

This paper generalizes Lemma 1 and 2 to arbitrary selection rules. Concretely, suppose $T \in \{1, 2, \ldots, n\}$ is a random variable jointly distributed with $Z_1, Z_2, \ldots, Z_n$. We would like to upper bound $E[Z_T]$, which subsumes the maximal inequality $T = \arg \max_i Z_i$ as a special case. Naturally, since

$$E[Z_T] \leq E[\max_i Z_i],$$

we would like to obtain bounds that are at least as strong as Lemma 1 and 2 but dependent on the joint distribution of $T, Z_1, Z_2, \ldots, Z_n$. In particular, the upper bound should be zero if $T$ is deterministic since we have already assumed that $E[Z_i] = 0$ for all $1 \leq i \leq n$.

A generalization of Lemma 1 was achieved in [2] using the Donsker–Varadhan representation of the relative entropy, which is a generalization of the sub-Gaussian case in [3]. Denote the entropy of a discrete random variable $T$ as

$$H(T) = \sum_t P_T(t) \ln \frac{1}{P_T(t)},$$

and the mutual information $I(X; Y)$ between $X$ and $Y$ as

$$I(X; Y) = \begin{cases} \int \frac{dP_{XY}}{d(P_XP_Y)} dP_{XY} & \text{if } P_{XY} \ll P_XP_Y \\ \infty & \text{otherwise} \end{cases}. $$

The following was shown in [2].

Lemma 3. Let $\psi \geq 0$ be a convex function defined on the interval $[0, b)$ where $0 < b \leq \infty$. Assume that $\psi(0) = 0$. Set, for every $t \geq 0$,

$$\psi^*(t) = \sup_{\lambda \in (0, b)} (\lambda t - \psi(\lambda)).$$

Suppose that $\ln E[e^{\lambda Z_i}] \leq \psi(\lambda)$ for all $\lambda \in [0, b), 1 \leq i \leq n$, and $E[Z_i] = 0, 1 \leq i \leq n$. Then,

$$E[Z_T] \leq \psi^*(-1)(I(T; Z))$$

$$\leq \psi^*(-1)(H(T))$$

where $\psi^*(-1)(y)$ is defined as

$$\psi^*(-1)(y) = \inf\{t \geq 0 : \psi^*(t) > y\}.$$
Lemma 3 is clearly stronger than Lemma 1 since \( I(T; Z) \leq H(T) \leq \ln n \). It is also interesting to observe that the soft bound is maximized when \( T \) follows a uniform distribution, and it is zero when \( T \) is deterministic.

Similar attempts were made to generalize Lemma 2 in [2]. However, it was not satisfactory since even in the case of \( \psi(x) = x^p, p \geq 1, x \geq 0 \), the generalization bound obtained in [2] may be infinity when \( 1 \leq p < 2 \), while Lemma 2 shows that it is universally bounded by \( \sigma \cdot n^{1/p} \) for every \( p \geq 1 \).

Our main contribution in this paper is the generalization of Lemma 2 to arbitrary selection rules. Our generalization satisfies the following properties:

1. It is at least as strong as Lemma 2: in other words, it can be shown that the worst case joint distribution of \( T \) and \( Z \) would not incur an upper bound larger than \( \sigma \cdot \psi^{-1}(n) \), which is the upper bound in Lemma 2.

2. It admits a closed form expression for the \( p \)-norm case, i.e., the case where \( \psi(x) = x^p, p \geq 1, x \geq 0 \). In other words, it defines another information theoretic measure paralleling the Shannon entropy \( H(T) \) in Lemma 3. Concretely, for any \( 1 \leq q \leq \infty \), we introduce functional \( H(T; q) \) as

\[
H(T; q) = \begin{cases} \frac{1}{2} \mathbb{1}(H(T) \neq 0) & q = \infty \\ \left( \sum_t \left( P_T(t)^{1/(1-q)} + (1 - P_T(t))^{1/(1-q)} \right)^{1-q} \right)^{1/q} & 1 < q < \infty \\ \sum_t \min \{ P_T(t), 1 - P_T(t) \} & q = 1 \end{cases}
\]

where \( \mathbb{1}(A) = \begin{cases} 1 & A \text{ is true} \\ 0 & \text{otherwise} \end{cases} \), and \( H(T) \) is the Shannon entropy functional. The \( H(T; q) \) functional satisfies the following properties:

(a) \( 0 \leq H(T; q) \leq 1 \);

(b) \( H(T; q) = 0 \iff T \) is deterministic.

The rest of the paper is organized as follows. We present and discuss our main results in Section 3. Auxiliary lemmas and their proofs are provided in Section A, and the proofs of Lemma 1 and 2 are provided in Section B for completeness.

### 2 Preliminaries

The \( \beta \)-norm of a random variable \( X \) for \( \beta \geq 1 \) is defined as

\[
\|X\|_\beta = \begin{cases} \left( \mathbb{E}|X|^{\beta} \right)^{1/\beta} & 1 \leq \beta < \infty \\ \text{ess sup} |X| & \beta = \infty \end{cases}
\]

where the essential supremum is defined as

\[
\text{ess sup} \ X = \inf \{ M : \mathbb{P}(X > M) = 0 \}.
\]
The Fenchel–Young inequality states that for any function \( f \) and its convex conjugate \( f^* \), we have
\[
f(x) + f^*(y) \geq \langle x, y \rangle, \text{ for all } x \in X, y \in X^*,
\]
which follows from the definition of convex conjugate \( f^*(y) = \sup_{x \in X} \{ \langle x, y \rangle - f(x) \} \). It follows from the Fenchel–Moreau theorem that \( f = f^{**} \) if and only if \( f \) is convex and lower semi-continuous.

We define the Ameniya norm of a random variable \( X \) as
\[
\|X\|_\psi^A = \inf \left\{ \frac{1 + \mathbb{E}\psi(|X|)}{t} : t > 0 \right\}.
\]

3 Main results

We present our main result below.

**Theorem 1.** Let \( \psi \) be an Orlicz function. Suppose \( \|Z_i\|_\psi \leq \sigma, \mathbb{E}[Z_i] = 0, 1 \leq i \leq n \). Then,
\[
|\mathbb{E}[Z_T]| \leq \sigma \cdot \sum_{i=1}^{n} \inf_{a_i} \left\| P_{T|Z}(i|Z) - a_i \right\|_\psi^A
\]
\[
\leq \sigma \cdot \inf_{t>0} \frac{1}{t} \left( n + \sum_{i=1}^{n} P_T(i) \psi^*(t|1-a_i|) + (1-P_T(i)) \psi^*(t|a_i|) \right). \tag{20}
\]
Furthermore, if \( \|Z_i\|_p \leq \sigma, p \geq 1, \frac{1}{p} + \frac{1}{q} = 1 \), then
\[
|\mathbb{E}[Z_T]| \leq \sigma \cdot n^{1/p} \left( \sum_{i=1}^{n} \inf_{a_i \in \mathbb{R}} \mathbb{E}|P_{T|Z}(i|Z) - a_i|^q \right)^{1/q} \tag{21}
\]
\[
\leq \sigma \cdot n^{1/p} H(T; q), \tag{22}
\]
where \( Z = (Z_1, Z_2, \ldots, Z_n) \), and \( H(T; q) \) is defined in \((14)\).

**Proof.** For any \( t > 0 \), we have the following chain of inequalities:
\[
\mathbb{E}\left[ \frac{Z_T}{\sigma} \right] = \sum_{i=1}^{n} P_T(i) \mathbb{E} \left[ \frac{Z_i}{\sigma} \middle| T = i \right] \tag{23}
\]
\[
= \sum_{i=1}^{n} P_T(i) \int \frac{P_{Z_i|T=i}(dx)}{P_{Z_i}(dx)} x \frac{P_{Z_i}(dx)}{\sigma} \tag{24}
\]
\[
= \sum_{i=1}^{n} P_T(i) \int \left( \frac{P_{Z_i|T=i}(dx)}{P_{Z_i}(dx)} - b_i \right) \frac{x}{\sigma} P_{Z_i}(dx) \tag{25}
\]
\[
= \sum_{i=1}^{n} \int \left( \frac{P_{Z_i|T=i}(dx)}{P_{Z_i}(dx)} - b_i P_T(i) \right) \frac{x}{\sigma} P_{Z_i}(dx) \tag{26}
\]
\[
\leq \sum_{i=1}^{n} \int \left| P_{T|Z}(i|x) - a_i \right| \frac{|x|}{\sigma} P_{Z_i}(dx), \tag{27}
\]
where \( a_i = b_iP_T(i) \), and the vectors \((a_1, a_2, \ldots, a_n)^T\) and \((b_1, b_2, \ldots, b_n)^T\) are deterministic vectors in \( \mathbb{R}^n \). The derivations above hold for any arbitrary vector \((a_1, a_2, \ldots, a_n)^T \in \mathbb{R}^n \).

Applying the generalized Holder’s inequality, we obtain that

\[
\mathbb{E} \left[ \frac{Z_T}{\sigma} \right] \leq \sum_{i=1}^n \inf_{a_i \in \mathbb{R}} \left\| P_{T|Z_i}(i|Z_i) - a_i \right\|^A_{\psi^*}.
\]  

(28)

We further upper bound each term in the summation as follows. For each \( i, 1 \leq i \leq n \),

\[
\left\| P_{T|Z_i}(i|Z_i) - a_i \right\|^A_{\psi^*} = \inf_{t > 0} \frac{1 + \mathbb{E}[\psi^*(t|P_{T|Z_i}(i|Z_i) - a_i|)]}{t} \leq \inf_{t > 0} \frac{1 + \mathbb{E}[\psi^*(t|P_{T|Z}(i|Z) - a_i|)]}{t} \leq \left\| P_{T|Z}(i|Z) - a_i \right\|^A_{\psi^*}.
\]

(29)

(30)

(31)

Here in the second step we have used the fact that \( \psi^*(t|x) \) is a convex function of \( x \), and the fact that \( \mathbb{E}[P_{T|Z}(i|Z)|Z_i] = P_{T|Z_i}(i|Z_i) \).

Hence, we have proved that

\[
\mathbb{E}[Z_T] \leq \sigma \cdot \sum_{i=1}^n \inf_{a_i} \left\| P_{T|Z}(i|Z) - a_i \right\|^A_{\psi^*}.
\]

(32)

It is clear that the inequality above also holds for \(-\mathbb{E}[Z_T]\). Hence, one has

\[
|\mathbb{E}[Z_T]| \leq \sigma \cdot \sum_{i=1}^n \inf_{a_i} \left\| P_{T|Z}(i|Z) - a_i \right\|^A_{\psi^*}.
\]

(33)

We now further upper bound the RHS of (32) to obtain a bound that only depends on the marginal distribution of \( T \) but not the joint distribution of \( T \) and \( Z \). For any \( t > 0, a_1, a_2, \ldots, a_n \in \mathbb{R} \), we have

\[
\mathbb{E}[Z_T] \leq \sigma \cdot \frac{1}{t} \left( n + \mathbb{E} \left[ \sum_{i=1}^n \psi^*(t|P_{T|Z}(i|Z) - a_i|) \right] \right).
\]

(34)

Since \( \psi^*(t|x - a|) \) is a convex function of \( x \) when \( t > 0 \), for any \( x \in [0, 1] \),

\[
\psi^*(t|x - a|) = \psi^*(t|x \cdot 1 + (1 - x) \cdot 0 - a|) \leq x\psi^*(t|1 - a|) + (1 - x)\psi^*(t|a|).
\]

(35)

(36)

Applying the inequality above, we have

\[
\mathbb{E}[Z_T] \leq \sigma \cdot \inf_{t > 0} \frac{1}{t} \left( n + \sum_{i=1}^n P_T(i)\psi^*(t|1 - a_i|) + (1 - P_T(i))\psi^*(t|a_i|) \right).
\]

(37)

Now, we present the results pertaining to the \( p \)-norm, which corresponds to \( \psi(x) = x^p, p \geq 1, x \geq 0 \).
When \( p = 1 \), \( \psi^*(y) = \begin{cases} 0 & y \in [0, 1] \\ \infty & y > 1 \end{cases} \). Hence, if \( T \) is not deterministic, it follows from (37) that

\[
\mathbb{E}[Z_T] \leq \inf_{a_1, a_2, \ldots, a_n \in \mathbb{R}} n \max \{|a_i|, |1 - a_i|\} 
\leq \sigma \cdot \frac{n}{2}.
\]

(38) (39)

When \( T \) is deterministic, we have \( |\mathbb{E}[Z_T]| = 0 \).

Now we consider the case of \( p > 1 \). We have that

\[
\mathbb{E}[Z_T] \leq \sigma \sum_{i=1}^{n} \inf_{a_i} \| P_{T|Z}(i|Z) - a_i \|_q, 
\]

where \( X \|_\psi = \| X \|_q \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). Concretely,

\[
\sum_{i=1}^{n} \left\| P_{T|Z}(i|Z) - a_i \right\|_q = \sum_{i=1}^{n} \left( \mathbb{E}[P_{T|Z}(i|Z) - a_i]^q \right)^{1/q} \leq \frac{n}{q} \left( \sum_{i=1}^{n} \mathbb{E}[P_{T|Z}(i|Z) - a_i]^q \right)^{1/q} \leq n \left( \sum_{i=1}^{n} \mathbb{E}[P_{T|Z}(i|Z) - a_i]^q \right)^{1/q} = n^{1/p} \left( \sum_{i=1}^{n} \mathbb{E}[P_{T|Z}(i|Z) - a_i]^q \right)^{1/q},
\]

(40) (41) (42) (43) (44)

where we have used the fact that \( x^{1/q}, x \geq 0 \) is a concave function.

It follows from Jensen’s inequality that for any \( x \in [0, 1], q \geq 1, a \in \mathbb{R} \), we have

\[
|x - a|^q = |x \cdot 1 + (1 - x) \cdot 0 - a|^q \leq x|1 - a|^q + (1 - x)|a|^q,
\]

and the inequality is tight when \( x = 1 \) or \( x = 0 \). Applying the inequality above, we have

\[
\mathbb{E}[P_{T|Z}(i|Z) - a_i]^q \leq \mathbb{E} \left[ P_{T|Z}(i|Z) |1 - a_i|^q + (1 - P_{T|Z}(i|Z)) |a_i|^q \right]
= P_T(i) |1 - a_i|^q + (1 - P_T(i)) |a_i|^q.
\]

(45) (46) (47) (48)

Hence, we have that

\[
\mathbb{E}[Z_T] \leq \sigma \cdot n^{1/p} \inf_{a_1, a_2, \ldots, a_n \in \mathbb{R}} \left( \sum_{i=1}^{n} P_T(i) |1 - a_i|^q + (1 - P_T(i)) |a_i|^q \right)^{1/q} \leq \sigma \cdot n^{1/p} \left( \sum_{i=1}^{n} \left( P_T(i)^{1/(1-q)} + (1 - P_T(i))^{1/(1-q)} \right)^{1-q} \right)^{1/q}.
\]

(49) (50)

It follows from Lemma 5 that

\[
\mathbb{E}[Z_T] \leq \sigma \cdot n^{1/p} \left( \sum_{i=1}^{n} \left( P_T(i)^{1/(1-q)} + (1 - P_T(i))^{1/(1-q)} \right)^{1-q} \right)^{1/q}.
\]

\(\square\)
3.1 Discussions

We now show that the upper bound is at most $\sigma \cdot \psi^{-1}(n)$. Choosing $a_i = 0, 1 \leq i \leq n$, then for any $t > 0$,

$$
\mathbb{E}[Z_T] \leq \sigma \cdot \sum_{i=1}^{n} \frac{1 + \mathbb{E}[\psi^*(t|P_T|Z(i|Z))]}{t}
$$

(51)

$$
= \sigma \frac{1}{t} \left( n + \mathbb{E} \left[ \sum_{i=1}^{n} \psi^*(t|P_T|Z(i|Z)) \right] \right).
$$

(52)

Since $\psi^*(x), x \geq 0$ is a convex function, and $\sum_{i=1}^{n} t|P_T|Z(i|Z) \leq t$, we know that it holds pointwise that

$$
\sum_{i=1}^{n} \psi^*(t|P_T|Z(i|Z)) \leq \psi^*(t).
$$

(53)

Hence, we have

$$
\mathbb{E}[Z_T] \leq \sigma \inf_{t>0} \frac{n + \psi^*(t)}{t}
$$

(54)

$$
= \sigma \cdot \psi^{-1}(n).
$$

(55)

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A Auxiliary lemmas

Lemma 4 (Generalized Hölder’s Inequality). Denote an Orlicz function by $\psi$ and its convex conjugate by $\psi^* = \sup \{uv - \psi(v) : v \geq 0\}$. Then,

$$
\mathbb{E}[XY] \leq \|X\|_\psi \|Y\|_{\psi^*}^4.
$$

(56)

Lemma 5. For fixed $a \in [0, 1], q \geq 1, q \in \mathbb{R}$, we have

$$
\min_{x \in [0,1]} a(1-x)^q + (1-a)x^q = \begin{cases} 
(a^{1/(1-q)} + (1-a)^{1/(1-q)})^{1-q} & q > 1 \\
\min\{a, 1-a\} & q = 1
\end{cases}
$$

(57)

Proof. Introduce

$$
f(x) = a(1-x)^q + (1-a)x^q.
$$

(58)

Taking derivative on both sides with respect to $x$, we have

$$
f'(x) = -aq(1-x)^{q-1} + (1-a)qx^{q-1}.
$$

(59)
From now on we only consider $q > 1$, since it is clear that
\[
\lim_{q \to 1^+} \left( a^{1/(1-q)} + (1 - a)^{1/(1-q)} \right)^{1-q} = \min\{a, 1 - a\}
\] (60)
\[
= \min_{x \in [0,1]} a(1 - x) + (1 - a)x.
\] (61)

For any $q > 1$, $f(x)$ is monotonically decreasing for any $x \leq x^*$, and then it is monotonically increasing for $x \geq x^*$. It attains the minimum when $x = x^*$, where $f'(x^*) = 0$. Solving $f'(x^*) = 0$, we obtain that
\[
\frac{x^*}{1 - x^*} = \left( \frac{a}{1 - a} \right)^{1/(q-1)},
\] (62)
which implies that
\[
f(x^*) = \left( a^{1/(1-q)} + (1 - a)^{1/(1-q)} \right)^{1-q}.
\] (63)

\[\square\]

B Proofs of classical maximal inequalities

B.1 Proof of Lemma 1

We have the following chain of inequalities. For any $\lambda \in [0, b)$,
\[
e^{\lambda \mathbb{E}[\max_i Z_i]} \leq \mathbb{E}[e^{\lambda \max_i Z_i}]
\] (64)
\[
= \mathbb{E}[\max_i e^{\lambda Z_i}]
\] (65)
\[
\leq \sum_{i=1}^{n} \mathbb{E}[e^{\lambda Z_i}]
\] (66)
\[
\leq n \cdot e^{\psi(\lambda)}.
\] (67)

Taking logarithm on both sides, we have
\[
\mathbb{E}[\max_i Z_i] \leq \inf_{\lambda \in (0,b)} \left( \frac{\ln n + \psi(\lambda)}{\lambda} \right)
\] (68)
\[
= \psi^{*^{-1}}(\ln n),
\] (69)

where in the last step we have used the fact that
\[
\psi^{*^{-1}}(y) = \inf_{\lambda \in (0,b)} \left( \frac{y + \psi(\lambda)}{\lambda} \right)
\] (70)
as shown in [5] Lemma 2.4, Pg 32].
B.2 Proof of Lemma 2

We have the following chain of inequalities:

\[
\psi \left( \mathbb{E} \left[ \max_i \frac{|Z_i|}{\sigma} \right] \right) \leq \mathbb{E} \left[ \psi \left( \max_i \frac{|Z_i|}{\sigma} \right) \right] \leq \sum_{i=1}^{n} \mathbb{E} \left[ \psi \left( \frac{|Z_i|}{\sigma} \right) \right] \leq n. \tag{71-73}
\]

Hence,

\[
\mathbb{E} \left[ \max_i Z_i \right] \leq \mathbb{E} \left[ \max_i |Z_i| \right] \leq \sigma \cdot \psi^{-1}(n), \tag{74-75}
\]

where in the last step we used the fact that an Orlicz function is nondecreasing.

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