AN ALGORITHM TO CLASSIFY RATIONAL 3-TANGLES

BO-HYUN KWON

Abstract. A 3-tangle $T$ is the disjoint union of 3 properly embedded arcs in the unit 3-ball; it is called rational if there is a homeomorphism of pairs from $(B^3, T)$ to $(D^2 \times I, \{x_1, x_2, x_3\} \times I)$. Two rational 3-tangles $T$ and $T'$ are isotopic if there is an orientation-preserving self-homeomorphism $h : (B^3, T) \rightarrow (B^3, T')$ that is the identity map on the boundary. In this paper, we give an algorithm to check whether or not two rational 3-tangles are isotopic by using a modified version of Dehn’s method for classifying simple closed curves on surfaces.

1. Introduction

Tangles were introduced by J. Conway. In 1970, he proved that every rational 2-tangle defines a rational number and two rational 2-tangles are isotopic if and only if they have the same rational number. However, there is no similar invariant known which classifies rational 3-tangles. In this paper, I describe an algorithm to check whether or not two rational 3-tangles are isotopic.

A $n$-tangle is the disjoint union of $n$ properly embedded arcs in the unit 3-ball; the embedding must send the endpoints of the arcs to $2n$ marked (fixed) points on the ball’s boundary. Without loss of generality, consider the marked points on the 3-ball boundary to lie on a great circle. The tangle can be arranged to be in general position with respect to the projection onto the flat disk in the $xy$-plane bounded by the great circle. The projection then gives us a tangle diagram, where we make note of over and undercrossings as with knot diagrams. A rational $n$-tangle is a $n$-tangle $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$ in a 3-ball $B^3$ such that there exists a homeomorphism of pairs $\overline{H} : (B^3, \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n) \rightarrow (D^2 \times I, \{p_1, p_2, \cdots, p_n\} \times I)$, where $I = [0, 1]$.

We note that there exists a homeomorphism $K : (D^2 \times I, \{p_1, p_2, \cdots, p_n\} \times I) \rightarrow (B^3, \epsilon_1 \cup \epsilon_2 \cup \cdots \cup \epsilon_n)$, where $\epsilon_1 \cup \epsilon_2 \cup \cdots \cup \epsilon_n$ is the $\infty$ tangle as in Figure 1.

Therefore, alternatively, a $n$-tangle $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$ is rational if there exists a homeomorphism of pairs: $\tilde{H} = (\overline{H})^{-1}K^{-1} : (B^3, \epsilon_1 \cup \epsilon_2 \cup \cdots \cup \epsilon_n) \rightarrow (B^3, \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n)$.

Two rational $n$-tangles, $T, T'$, in $B^3$ are isotopic, denoted by $T \approx T'$, if there is an orientation-preserving self-homeomorphism $h : (B^3, T) \rightarrow (B^3, T')$ that is the identity map on the boundary.

Let $\Sigma_{0,6}$ be the six punctured sphere and let $\epsilon = \epsilon_1 \cup \epsilon_2 \cup \epsilon_3$ be the $\infty$ 3-tangle as in Figure 2. Then, for two orientation preserving homeomorphisms $f$ and $g$ from $\Sigma_{0,6}$ to $\Sigma_{0,6}$,
we say they are *isotopic*, denoted by $f \sim g$, if there is a continuous map $H : \Sigma_{0,n} \times I \to \Sigma_{0,n}$ so that $H(x, 1) = f(x)$ and $H(x, 0) = g(x)$ and $h_t(x) = H(x, t)$ is a homeomorphism for all $t$. Also, we say that a subset $C_1$ of $\Sigma_{0,n}$ is isotopic to $C_2$, denoted by $C_1 \sim C_2$, if there is a homeomorphism $h$ of $\Sigma_{0,6}$ with $h(C_1) = C_2$ such that $h \sim id$.

To demonstrate the effectiveness of this algorithm, we will show that the rational 3-tangles in Figure 1 are not isotopic to each other. Note that for every string of $T$, if we consider the other two strings then they are isotopic to a trivial rational 2-tangle in $B^3$. However, we will show that $T$ is not isotopic to the $\infty$ tangle. So, $T$ is similar to the Borromean rings. We will also show that $T$ is not isotopic to the tangle $T'$ which is obtained from $T$ by reversing all the crossings in $T$.

The algorithm is based on the following facts, which are proved in Section 2.

Up to isotopy, orientation preserving homeomorphisms $f$ and $g$ from $\Sigma_{0,6}$ to $\Sigma_{0,6}$ which fix the puncture 1 can be obtained by four half Dehn twists $\sigma_1, \sigma_2, \sigma_3$ and $\sigma_4$ which are the generators of the braid group $\mathbb{B}_5$. (Refer to [2].)

Then we can get extensions $F, G : B^3 \to B^3$ of $f$ and $g$ which fix the set of six points, $\epsilon \cap \partial B^3$, setwise giving two rational 3-tangles $T_F := F(\epsilon)$ and $T_G := G(\epsilon)$.

We will show that a tangle can be “presented” by an element of $\mathbb{B}_5$ and our algorithm will decide whether or not two elements of $\mathbb{B}_5$ present equivalent tangles. We will discuss this in Section 2.

We say that a disk $D$ is essential in $B^3 - T$ for a rational 3-tangle $T$ if $D$ is a properly embedded disk in $B^3 - T$ but it is not boundary parallel in $B^3 - T$.

**Theorem 2.3** For two rational 3-tangles $T_F$ and $T_G$, $T_F \approx T_G$ if and only if $G^{-1}F(\partial E)$ bounds essential disks in $B^3 - \epsilon$, where $E = E_1 \cup E_2 \cup E_3$ is a fixed union of “standard essential disks” in $B^3 - \epsilon$. (Refer to Figure 2.)

In fact, if $G^{-1}F(\partial E_i)$ and $G^{-1}F(\partial E_j)$ bound essential disks in $B^3 - \epsilon$ then $G^{-1}F(\partial E_k)$ also bounds an essential disk in $B^3 - \epsilon$ for $\{i, j, k\} = \{1, 2, 3\}$. Therefore, if two of $G^{-1}F(\partial E)$ bound essential disks then $T_F \approx T_G$. 

![Figure 1. Examples of rational 3-tangles](image-url)
We note that there is another way to check whether $G^{-1}F(\partial E_i)$ bounds an essential disk in $B^3 - \epsilon$ or not by using a fundamental group argument. There is an induced map $i_* : \pi_1(\Sigma_{0,6}) \to \pi_1(B^3 - \epsilon)$ from the inclusion map $i : \Sigma_{0,6} \to B^3 - \epsilon$. Then we know that if $i_*(\left[G^{-1}F(\partial E_i)\right]) = 1$ then $G^{-1}F(\partial E_i)$ bounds an essential disk in $B^3 - \epsilon$ by Dehn’s Lemma. This method is conceptually simple but it appears to be awkward to implement due to having to deal with arbitrarily long words in a free group. However, if one uses the algorithm given below to check whether or not $G^{-1}F(\partial E_i)$ bounds an essential disk in $B^3 - \epsilon$ then we will be dealing with integer vectors of fixed dimension. I will give an example to compare the two algorithms later.

Let $C$ be the set of isotopy classes of closed essential simple closed curves in $\Sigma_{0,6}$. A simple closed curve $\gamma$ is essential in $\Sigma_{0,6}$ if $\gamma$ does not bound a disk in $\Sigma_{0,6}$ and $\gamma$ does not enclose a single puncture of $\Sigma_{0,6}$.

The algorithm is as follows:

**Step 1:** We represent $\Sigma_{0,6}$ as the union of two hexagons $H$ and $H^c$ so that the vertices are the punctures of $\Sigma_{0,6}$. Then we use a variation of normal curve theory to parameterize $C$. Each curve has a “hexagon diagram”. A set of “weights” $w_{ij}$ and $w_{kl}$ for the hexagon diagram parameterizes the set of isotopy classes $[G^{-1}F(\partial E_s)]$. Using certain formulas the weights can be obtained easily from the words in $\sigma_1, \cdots, \sigma_4$ which describe $F$ and $G$, but the weights are hard to use directly to decide whether or not $G^{-1}F(\partial E_s)$ bounds an essential disk.

**Step 2:** We find a simple closed curve $\gamma'$, possibly not isotopic to $\gamma$, which bounds an essential disk in $B^3 - \epsilon$ if and only if the component $\gamma$ of $G^{-1}F(\partial E_i)$ does. We take a decomposition of $\Sigma_{0,6}$ into three 2-punctured disks $E'_i$ and one pair of pants $I$, where each 2-punctured disk contains one component of $H \cap H^c$. We specify the isotopy class $[\gamma']$ by using a modified version of Dehn’s method. (See [5].) We define the Dehn parameters $p_i, q_i$ and $t_i$ ($1 \leq i \leq 3$) of $[\gamma']$ in $E'_i$ and the weights $x_{jk}$ ($1 \leq j, k \leq 3$) of $[\gamma']$ in $I$. The $x_{jk}$ are determined by $p_i, q_i$ and $t_i$. We note that the Dehn parameters $p_i, q_i$ and $t_i$ ($1 \leq i \leq 3$) of $[\gamma']$ are obtained from the weights $w_{ij}$ and $w_{kl}$ for the hexagon diagram, where $i, j, k, l \in \{1, 2, 3, 4, 5, 6\}$.

**Step 3:** We modify $\gamma'$ into $\gamma_0$, possibly not isotopic to $\gamma'$ or $\gamma$, which is in “standard position” and bounds an essential disk in $B^3 - \epsilon$ if and only if $\gamma'$ does. Then we get “standard...
weights" $m_i \geq 0$ ($1 \leq i \leq 11$) of $\gamma_0$ from the Dehn parameters. Standard position is slightly reminiscent of train track theory, but involves fewer diagrams.

Step 4: We define three homeomorphisms $\delta_1, \delta_2$ and $\delta_3$ so that $\gamma_0$ bounds an essential disk in $B^3 - \epsilon$ if and only if both $\delta_1 \delta_2^{-1}(\gamma_0)$ and $\delta_3(\gamma_0)$ bound essential disks in $B^3 - \epsilon$. Then, we repeatedly apply Theorem 9.3 below to check whether $\gamma_0$ bounds an essential disk in $B^3 - \epsilon$ where $I'$ is certain regular neighborhood of $I$.

**Theorem 9.3** Suppose that $\gamma_0$ bounds an essential disk in $B^3 - \epsilon$ and $\gamma_0$ is in standard position in $I'$ and $m_3 > 0$. Then applying one of the homeomorphisms $(\delta_1 \delta_2^{-1})^{\pm 1}$ and $\delta_3^{\pm 1}$ reduces the sum of the $p_i$ for the image of $\gamma_0$.

Suppose that a simple closed curve $\gamma_0$ is in standard position and has $m_3 > 0$. If we can reduce the sum of the standard weights of $\gamma_0$ by using one of the four homeomorphisms then we take the new simple closed curve $\gamma_1$ which is obtained by applying one of the four homeomorphisms. If not, then $\gamma_0$ does not bound an essential disk. If $\gamma_1$ still has $m_3 > 0$, then we will go on. Suppose $m_3 = 0$. Then $\gamma_1$ is isotopic to one of the $\partial E_k$ if $m_i = 0$ for all $i$. It does not bound an essential disk in $B^3 - \epsilon$ if $m_i \neq 0$ for some $i$. Since the sum of the standard weights is finite, the algorithm will end in a finite number of steps.

Recall that $\gamma_0$ bounds an essential disk in $B^3 - \epsilon$ if and only if $G^{-1}F(\partial E_i)$ does. So, the given procedures form an algorithm to classify rational 3-tangles.

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2. **Presentations of rational 3-tangles**

We recall that a rational 3-tangle $T$ can be arranged to be in general position with respect to the projection onto the flat disk $Q$ in the $xy$-plane bounded by the great circle $C$. Then we will have a tangle diagram $TD$ of the rational 3-tangle $T$. Let $p$ be the number of crossings of the tangle $T$ in the diagram.

Now, we say that a tangle diagram $TD$ is standard if for the nested disks $Q_1 \subset Q_2 \subset \cdots \subset Q_{p+1}$, $Q_1$ contains the $\infty$ tangle and each annulus $N_j = Q_{j+1} - Q_j$ contains exactly one crossing of the crossings of $T$ as in Figure 3.

Then we define a rational 3-tangle $T$ to be in standard position if the projection of $T$ onto the flat disk in the $xy$-plane bounded by $C$ is a standard diagram.

Let $\sigma_i$ be the half Dehn twist supported on the twice punctured disk $K_i$ as in Figure 4.

Then we have an extension $\tau_i$ of $\sigma_i$ to $B^3$ as follows.

Take a ball $B_i$ in $B^3$ so that $B_i \cap \partial B^3 = \overline{K_i}$ and $B_i \cap \epsilon$ is two trivial subarcs of $\epsilon$ as in Figure 5.
Figure 3. A standard diagram of a rational 3-tangle expressed by \( w = \tau_5^{-1}\tau_4^{-1}\tau_3\tau_2\tau_1 \).

Figure 4. Generators of the mapping class group of \( \Sigma_{0,6} \)

Figure 5. The extension \( \tau_i \) of a half Dehn twist \( \sigma_i \) to \( B^3 \)

Then, we define \( \tau_i \) so that \( \tau_i|_{B^3-B_i} = id \) and \( \tau_i|_{B_i} \) is an extension of \( \sigma_i \) to \( B_i \) which twists the two trivial simple subarcs in \( B_i \) to have a positive crossing as in Figure 5.

Lemma 2.1. Suppose that \( F \) is an orientation preserving homeomorphism from \( B^3 \) to \( B^3 \) so that \( F(\{1,2,3,4,5,6\}) = \{1,2,3,4,5,6\} \). Then, there exists an orientation preserving homeomorphism \( F_1 : B^3 \to B^3 \) so that \( F_1(1) = 1 \) and \( F_1(\epsilon) = F(\epsilon) \).
Proof. First, we claim that there is a homeomorphism \( R_i : (B^3, \epsilon) \to (B^3, \epsilon) \) so that \( R_i(1) = i \) for \( i \in \{1, 2, 3, 4, 5, 6\} \).

We note that there is a homeomorphism \( \rho_j : (B^3, \epsilon) \to (B^3, \epsilon) \) so that \( \rho_j(1) = j \) for \( j = 1, 3 \) or \( 5 \) by using \( 0^\circ, 120^\circ \) or \( 240^\circ \) counterclockwise rotation in the plane. Then we remark that it preserves \( \epsilon \) setwise. So, \( R_i = \rho_i \) for \( i = 1, 3, 5 \).

We note that \( \tau_k \) switches the two endpoints of \( \epsilon_{1 + k/4} \) for \( k = 0, 2, 4 \).

We let \( R_i : (B^3, \epsilon) \to (B^3, \epsilon) \) be \( R_i = \tau_{i-2} \circ \rho_{i-1} \) for \( i = 2, 4, 6 \). We check that \( R_i(1) = i \) for \( i = 2, 4, 6 \).

Suppose that \( F(p) = 1 \). Then we let \( F_1 = F \circ R_p \). So, we have \( F_1(1) = F(R_p(1)) = F(p) = 1 \). Also, we know that \( F_1(\epsilon) = F(R_p(\epsilon)) = F(\epsilon) \) since \( R_p(\epsilon) = \epsilon \).

Now, by Lemma 2.1 we may assume \( F(1) = 1 \). This will be assumed throughout the rest of the paper.

Let \( f = F|_{\Sigma_{0,6}} \). Then it is an orientation preserving homeomorphism from \( \Sigma_{0,6} \) to \( \Sigma_{0,6} \). The mapping class group of \( \Sigma_{0,6} \) is \( \text{MCG}(\Sigma_{0,6}) = \text{Homeo}^+(\Sigma_{0,6})/\sim \).

Let \( \text{MCG}_1(\Sigma_{0,6}) = \{ [h] \in \text{MCG}(\Sigma_{0,6}) \mid h(1) = 1 \} \), where \([h] \) is the isotopy class of \( h \). Then, \([f] \in \text{MCG}_1(\Sigma_{0,6}) \).

Recall the two rational 3-tangles \( T \) and \( T' \) in Figure 4 which can be arranged as standard diagrams.

In Figure 4 we see that \( T \approx \tau_3 \circ \tau_1 \circ \tau_0^{-1} \circ \tau_3 \circ \tau_1 \circ \tau_5(\epsilon) \) and \( T' \approx \tau_5^{-1} \circ \tau_1^{-1} \circ \tau_5 \circ \tau_3^{-1} \circ \tau_1^{-1} \circ \tau_5^{-1}(\epsilon) \), where \( \tau_i \) is an extension of \( \sigma_i \) to \( B^3 \).

We say that a rational 3-tangle \( F(\epsilon) \) is presented by an element of \( \mathbb{B}_5 \) if \( F(\epsilon) \) is isotopic to \( G(\epsilon) \) so that \( G \) is the composition of a sequence of extensions \( \tau_i \) for \( i \in \{1, 2, 3, 4\} \). We note that the four generators of \( \mathbb{B}_5 \) are associated with the four isotopy classes \([\tau_i] \) \( (1 \leq i \leq 4) \). The later of this section, we will show that every rational 3-tangle can be presented by an element of \( \mathbb{B}_5 \). For example, \( T \approx \tau_5 \circ \tau_1 \circ \tau_0^{-1} \circ \tau_3 \circ \tau_1 \circ \tau_5(\epsilon) \approx \tau_5 \circ \tau_3 \circ \tau_2^{-1} \circ \tau_1 \circ \tau_2 \circ \tau_3(\epsilon) \).

The following Lemma 2.2 and Theorem 2.3 appear as Lemma 4.4.1 and Theorem 4.5 of [3].

**Lemma 2.2** (Alexander [2]). If \( g : D^n \to D^n \) is a homeomorphism from the unit n-ball to itself which fixes the \( (n - 1) \)-sphere \( S^{n-1} = \partial D^n \) pointwise, then \( g \) is isotopic to the identity under an isotopy which fixes \( S^{n-1} \) pointwise. If \( g(0) = 0 \), then the isotopy may be chosen to fix 0.

**Theorem 2.3** (Birman [2]). If \( n \geq 2 \), then \( \text{MCG}(\Sigma_{0,n}) \) admits a presentation with generators \( \sigma_0, \cdots, \sigma_4 \) .
Corollary 2.4. Suppose that $F$ is a homeomorphism of $B^3$ so that $F(\Sigma_{0,6}) = \Sigma_{0,6}$. Then there exists a homeomorphism $G$ of $B^3$ so that $G(\epsilon) \approx F(\epsilon)$ and $G$ is the composition of a sequence of extensions $\tau_i^{\pm 1}$ of $\sigma_i^{\pm 1}$ for $i \in \{0, 1, 2, 3, 4\}$.

Proof. By Theorem 2.3, $F|_{\Sigma_{0,6}}$ is isotopic to $g$ in $\Sigma_{0,6}$ which is the composition of a sequence of $\sigma_i^{\pm 1}$ for $i \in \{0, 1, 2, 3, 4\}$. Then, By Lemma 2.2 the extension $G$ of $g$ which is the composition of the sequence of $\tau_i^{\pm 1}$ is isotopic to $F$. \qed

Lemma 2.5. If two homeomorphisms $f$ and $g$ of $\Sigma_{0,6}$ are isotopic, then for any two extensions $\mathcal{F}$ and $\mathcal{G}$ of $f$ and $g$ to $B^3$, $\mathcal{F}(\epsilon) \approx \mathcal{G}(\epsilon)$.

Proof. First, take a collar $N(\partial B^3) = S^2 \times [0, 1]$ in $B^3$ so that $S^2 \times \{0\} = \partial B^3$, $S^2 \times \{1\}$ is a properly embedded sphere in $B^3$ and $\epsilon \cap N(\partial B^3) = \{1, 2, 3, 4, 5, 6\} \times [0, 1]$.

We note that there exists a homeomorphism $\phi : \Sigma_{0,6} \times [0, 1] \to \Sigma_{0,6} \times [0, 1]$ so that $\phi(x, 0) = (f(x), 0)$ and $\phi(x, 1) = (g(x), 1)$ since $f \sim g$. Then we define $\bar{\phi} : S^2 \times [0, 1] \to S^2 \times [0, 1]$ by filling in the six punctures of $\Sigma_{0,6}$ for each time $t$.

Let $\bar{f}$ and $\bar{g}$ be the extensions of $f$ and $g$ to $S^2$ by filling in the six punctures of $\Sigma_{0,6}$.

Also, we know that there exists a homeomorphism $\psi : \Sigma_{0,6} \times [0, 1] \to \Sigma_{0,6} \times [0, 1]$ so that $\psi(x, t) = (g(x), t)$ for all $t$. Then we define $\bar{\psi} : S^2 \times [0, 1] \to S^2 \times [0, 1]$ by filling in the six punctures of $\Sigma_{0,6}$ for each time $t$.

Now, we define a homeomorphism $F : B^3 \to B^3$ so that $F|_{S^2 \times [0, 1]} = \bar{\phi}$ and $F|_{B^3 - (S^2 \times [0, 1])}$ is a homeomorphism of $B^3 - (S^2 \times [0, 1])$ which extends $\bar{\phi}|_{S^2 \times \{1\}}$.

Also, we define a homeomorphism $G : B^3 \to B^3$ so that $G|_{S^2 \times [0, 1]} = \bar{\psi}$ and $G|_{B^3 - (S^2 \times [0, 1])} = F|_{B^3 - (S^2 \times [0, 1])}$. We remark that $\bar{\phi}(x, 1) = (\bar{g}(x), 1) = \bar{\psi}(x, 1)$.

We see that $F(\epsilon) = G(\epsilon)$.

We remark that for any extension $\mathcal{F}$ of $f$ to $B^3$, $\mathcal{F}(\epsilon) \approx F(\epsilon)$ and any extension $\mathcal{G}$ of $g$ to $B^3$, $\mathcal{G}(\epsilon) \approx G(\epsilon)$ by Lemma 2.2 since $F$ is the extension of $f$ to $B^3$ and $G$ is the extension of $g$ to $B^3$.

This implies that $\mathcal{F}(\epsilon) \approx \mathcal{G}(\epsilon)$ since $F(\epsilon) = G(\epsilon)$. \qed

Lemma 2.6. For a rational 3-tangle $F(\epsilon)$, there exists a rational 3-tangle $G(\epsilon)$ so that $G(\epsilon) \approx F(\epsilon)$ and $G(\epsilon)$ is in standard position.

Proof. By Theorem 2.3, there exists a homeomorphism $g$ of $\Sigma_{0,6}$ which is isotopic to $F|_{\Sigma_{0,6}}$ and $g$ is a composition of a sequence of $\sigma_i^{\pm 1}$ for $0 \leq i \leq 4$.

Now, we construct an extension of $g$ to $B^3$ as follows:

Suppose that $g = \sigma^1_{j_1} \sigma^2_{j_2} \cdots \sigma^m_{j_m}$ for some $\sigma_{j_k} \in \{\sigma_0, \sigma_1, \ldots, \sigma_5\}$ and integers $a_k$. Let $p = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_m|$. Now, consider the projection of $B^3$ onto the flat disk $Q$ in the
$xy$-plane bounded by $C$ and having the infinite tangle diagram in $Q_1 \subset Q$. Then take nested disks $Q_2, \cdots Q_{p+1}$ so that $Q_1 \subset Q_2 \subset \cdots \subset Q_{p+1}$. Let $N_l = Q_{l+1} - Q_l$.

We know that the extension $\tau^\pm_{j\pm}$ of $\sigma^\pm_{j\pm}$ generates the crossing which may be in $N_p$. We note that the extension $\tau_i$ of a half Dehn twist $\sigma_i$ in Figure 5 makes a positive crossing as in the last diagram of Figure 6. Then, we isotope the crossing into $N_1$. After this, we generate the next crossing by the extension of the next element either $\sigma^\pm_{j\pm}$ or $\sigma^\pm_{j\pm-1}$. Then we isotope the crossing into $N_2$ while we fix $Q_2$. By reading off the sequence of the composition from the right to the left and doing this procedure repeatedly, we can construct an extension $G$ of $g$ so that $G$ is in standard position.

Finally, by using Lemma 2.2 we complete the proof of this lemma.

We say that a crossing in a standard diagram is expressed by an extension $\tau^\pm_{j\pm}$ of $\sigma^\pm_{j\pm}$ if the crossing is obtained by applying $\tau^\pm_{j\pm}$ as above.

Figure 6. Flippings

Now, we will prove that every rational 3-tangle can be presented by an element of $\mathbb{B}_5$ with generators $\sigma_1, \sigma_2, \sigma_3$ and $\sigma_4$. So, our algorithm will decide whether or not two elements of $\mathbb{B}_5$ present equivalent tangles.

Lemma 2.7. Suppose that $G(\epsilon)$ is in standard position and the crossing in $N_p$ is expressed by $\tau^\pm_0$ as in the first diagram, or $\tau^\pm_5$ as in the third diagram in Figure 6. Then $(\tau_2\tau_3\tau_2\tau_4\tau_3\tau_2)^\pm_1$ or $(\tau_1\tau_2\tau_1\tau_3\tau_2\tau_1)^\pm_1$ can replace $\tau^\pm_0$ or $\tau^\pm_5$, respectively, so that the diagram of the new expression is still standard as in the second or fourth diagram in Figure 6.

Especially, the number of crossings in $Q_p$ is fixed, where $Q_p$ is the disk inside of $C$ as in Figure 6.

Proof. Consider the dotted line which passes through the center of $C$ for each case as in Figure 6.

Flip the disk $Q_p$ about the dotted line to eliminate the crossing associated to $\tau^\pm_0$ or $\tau^\pm_5$. Then, this procedure shows the lemma.

Remark 2.8. In Lemma 2.7, let $\phi^{-1}_0$ be the isotopy move to flip the disk $Q_p$ to eliminate the crossing associated to $\tau_0$ as in Figure 6. Then let $\phi_0$ be the isotopy move to flip the disk $Q_p$.
counter clockwise to eliminate the crossing associated to $\tau_{0}^{-1}$. Similarly, let $\phi_{5}$ be the isotopy move to flip the disk $Q_{p}$ to eliminate the crossing associated to $\tau_{5}^{-1}$ as in Figure 6. Also, let $\phi^{-1}_{5}$ be the isotopy move to flip the disk $Q_{p}$ clockwise to eliminate the crossing associated to $\tau_{5}$. 

Suppose that $G = \tau_{j_{1}}^{\alpha_{1}} \tau_{j_{2}}^{\alpha_{2}} \cdots \tau_{j_{m}}^{\alpha_{m}}$ for some $\tau_{j_{k}} \in \{\sigma_{0}, \sigma_{1}, \cdots, \tau_{5}\}$ and integers $\alpha_{k}$ which expresses the crossings in $Q_{p+1}$.

Then, the crossings in $Q_{p}$ are expressed by $\tau_{j_{1}}^{\alpha_{1}} \tau_{j_{2}}^{\alpha_{2}} \cdots \tau_{j_{m}}^{\alpha_{m} \pm 1}$. 

Then we note that the crossings of $\phi_{0}^{-1}(Q_{p})$ are expressed by $\tau_{j_{1}}^{\alpha_{1}} \tau_{j_{2}}^{\alpha_{2}} \cdots \tau_{j_{m}}^{\alpha_{m} \pm 1}$, where $j_{i} \equiv -j_{i} \pmod{6}$.

Similarly, we note that the crossings of $\phi_{5}^{-1}(Q_{p})$ are expressed by $\tau_{j_{1}}^{\alpha_{1}} \tau_{j_{2}}^{\alpha_{2}} \cdots \tau_{j_{m}}^{\alpha_{m} \pm 1}$, where $j_{i} \equiv 4 - j_{i} \pmod{6}$.

**Theorem 2.9.** A rational 3-tangle can be presented by an element of $B_{5}$.

**Proof.** First, we assume that a rational 3-tangle is in standard position. So, the projection onto the plat disk $Q$ in the $xy$-plane is a standard diagram. Let $p$ be the number of crossings in $Q$.

We remark that the two flippings in Figure 6 will not change the tangle type in $Q_{1}$. i.e., $Q_{1}$ still contains the $\infty$ tangle after flippings.

Also, we know that $\tau_{0}^{\pm 1}$ is replaced by $(\tau_{2} \tau_{3} \tau_{2} \tau_{4} \tau_{3} \tau_{2})^{\pm 1}$ and $\tau_{5}^{\pm 1}$ is replaced by $(\tau_{1} \tau_{2} \tau_{1} \tau_{3} \tau_{2} \tau_{1})^{\pm 1}$ after flipping.

We note that the expression in terms of the crossings in $Q_{p}$ will be changed after flipping as in Remark 2.8, but the number of crossings in $Q_{p}$ is fixed.

If the crossing in $N_{p}$ is not expressed by either $\tau_{0}^{\pm 1}$ or $\tau_{5}^{\pm 1}$, then we consider the next crossing in $N_{p-1}$.

If the crossing in $N_{p}$ is expressed by either $\tau_{0}^{\pm 1}$ or $\tau_{5}^{\pm 1}$, then we flip $Q_{p}$ to eliminate the crossing associated to either $\tau_{0}^{\pm 1}$ or $\tau_{5}^{\pm 1}$.

Then, we also know that the number of crossings in $Q$ of the original diagram is more than the number of crossings in $Q_{p}$.

We remark that $(\tau_{1} \tau_{2} \tau_{1} \tau_{3} \tau_{2})^{\pm 1}$ and $(\tau_{2} \tau_{3} \tau_{2} \tau_{4} \tau_{3} \tau_{2})^{\pm 1}$ do not contain $\tau_{0}^{\pm 1}$ or $\tau_{5}^{\pm 1}$ factors.

By repeating this procedure, we can have another expression of $G(\epsilon)$ which involves only $\sigma_{1}^{\pm 1}, \cdots, \sigma_{4}^{\pm 1}$. □

Now, I give an example about Theorem 2.9.
Figure 7. A procedure to find a presentation which involves only $\sigma_1^{\pm 1}, \ldots, \sigma_4^{\pm 1}$

Consider the rational 3-tangle expressed by $w = \tau_5^{-1}\tau_0^{-1}\tau_4\tau_5^{-1}\tau_1$ as in the first diagram of Figure 7.

Flip the disk $Q_5$ to have a new expression $w_1 = (\tau_1\tau_2\tau_1\tau_3\tau_2\tau_1)^{-1}\tau_4^{-1}\tau_1^{-1}\tau_3$ as in Figure 7.

Then, we note that $w_1 = (\tau_1\tau_2\tau_1\tau_3\tau_2\tau_1)^{-1}\tau_4^{-1}\tau_1^{-1}\tau_3 = (\tau_1\tau_2\tau_1\tau_3\tau_2\tau_1)^{-1}\tau_4^{-1}\tau_0\tau_5^{-1}\tau_3$.

So $w_1 = (\tau_1\tau_2\tau_1\tau_3\tau_2\tau_1)^{-1}\tau_4^{-1}w_1'$, where $w_1' = \tau_0\tau_5^{-1}\tau_3$.

We note that $Q_4$ contains the crossings which are expressed by $\tau_0\tau_5^{-1}\tau_3$ and $Q_3$ contains the crossings which are expressed by $\tau_5^{-1}\tau_3$.

Now, flip the disk $Q_3$ to have a new expression $w_2' = (\tau_2\tau_3\tau_2\tau_4\tau_3\tau_2)^{-1}\tau_5^{-1}\tau_3$ of $w_1'$.

We note that $w_2' = (\tau_2\tau_3\tau_2\tau_4\tau_3\tau_2)^{-1}\tau_5^{-1}\tau_3 = (\tau_2\tau_3\tau_2\tau_4\tau_3\tau_2)^{-1}\tau_1$.

Therefore, we have a new expression $w_3 = (\tau_1\tau_2\tau_1\tau_3\tau_2\tau_1)^{-1}\tau_4^{-1}(\tau_2\tau_3\tau_2\tau_4\tau_3\tau_2)^{-1}\tau_3$ of $w$ which involves only $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \sigma_3^{\pm 1}$ and $\sigma_4^{\pm 1}$. 
3. Equivalence of rational 3-tangles

In this section, we will prove Theorem 3.2 which tells us alternative method to decide whether or not two rational 3-tangles are isotopic.

Let $E_1, E_2$ and $E_3$ be the three disjoint essential disks as in Figure 8. Then $E_1, E_2$ and $E_3$ separate $B^3$ into four components. Let $H_i$ be the component which contains $\epsilon_i$ and $P = cl(B^3 - (H_1 \cup H_2 \cup H_3))$.

Let $E_i'$ be the disk in $\partial B^3$ so that $\partial E_i' = \partial E_i$ and $E_i \cup E_i'$ bounds the ball $H_i$ in $B^3$. Let $E = E_1 \cup E_2 \cup E_3$, $E' = E_1' \cup E_2' \cup E_3'$ and $\partial E = \partial E' = \partial E_1 \cup \partial E_2 \cup \partial E_3$.

We say that a properly embedded simple arc $C$ in $B^3$ is *unknotted* if there is an isotopy $\phi_t : B^3 \to B^3$ that is identity on $\partial B^3$ so that $\phi_1(S) = C$, where $S$ is the straight line arc with the endpoints $\partial C$.

Then we can prove Lemma 3.1 below.

**Lemma 3.1.** If $\alpha$ and $\alpha'$ are properly embedded unknotted simple arcs in $B^3$ with $\partial \alpha = \partial \alpha' \subset S^2$, then $\alpha \approx \alpha'$.

**Proof.** Since $\alpha$ and $\alpha'$ are properly embedded unknotted simple arcs in $B^3$ with $\partial \alpha = \partial \alpha'$, $\alpha \sim \alpha_g \sim \alpha'$ for the straight line arc $\alpha_g$ in $B^3$ from $a$ to $b$. (where $\partial \alpha = \{a, b\}$.) We have a path $f_\alpha : I \to \alpha$ so that $f_\alpha(0) = a$ and $f_\alpha(1) = b$. Similarly, we also have paths $f_{\alpha'}$ and $f_{\alpha_g}$. Let $H$ and $J$ be the isotopies from $B^3 \times I$ to $B^3$ so that $H(x, 0) = f_\alpha(x)$ and $H(x, 1) = f_{\alpha_g}(x)$, and $J(x, 0) = f_{\alpha'}(x)$ and $J(x, 1) = f_{\alpha_g}(x)$. Now, we define the isotopy $K : B^3 \times I \to B^3$ so that $K(x, t) = H(x, 2t)$ for $0 \leq t \leq \frac{1}{2}$ and $K(x, t) = J(x, 2 - 2t)$ for $\frac{1}{2} \leq t \leq 1$. Then $K$ is an isotopy from $\alpha$ to $\alpha'$ in $B^3$.

Now consider orientation preserving homeomorphisms $f$ and $g$ from $\Sigma_{0,6}$ to $\Sigma_{0,6}$. Then we have $F$ and $G$ which are extensions to $B^3$ of $f$ and $g$ respectively.

**Theorem 3.2.** For two rational 3-tangles $T_F$ and $T_G$, $T_F \approx T_G$ if and only if $G^{-1}F(\partial E)$ bounds essential disks in $B^3 - \epsilon$.

**Proof.** ($\Rightarrow$) Suppose that there exists a homeomorphism $H$ from $(B^3, F(\epsilon))$ to $(B^3, G(\epsilon))$ so that $H|_{\partial B^3} = id|_{\partial B^3}$. Then we know that $HF(\partial E) = F(\partial E)$ since $H|_{\partial B^3} = id|_{\partial B^3}$. Also,
$G^{-1}HF(\epsilon) = \epsilon$ since $H(F(\epsilon)) = G(\epsilon)$. Also, $G^{-1}HF(\partial E) = G^{-1}F(\partial E)$. We claim that $G^{-1}HF(E)$ are essential disks in $B^3 - \epsilon$. Since $E$ are essential disks in $B^3 - \epsilon$, $F(E)$ are essential disks in $B^3 - F(\epsilon)$. Then $H(F(E))$ are properly embedded disks in $B^3$ which are disjoint with $H(F(\epsilon)) = G(\epsilon)$. Therefore, $H(F(E))$ are essential disks in $B^3 - G(\epsilon)$. Finally, we know that $G^{-1}(H(F(E)))$ are properly embedded disks in $B^3$ which are disjoint with $G^{-1}(G(\epsilon)) = \epsilon$. So, $G^{-1}HF(E)$ are disks in $B^3 - \epsilon$ and essential since each simple closed curve of $G^{-1}HF(\partial E)$ encloses two punctures in $\Sigma_{0,4}$. This implies that $G^{-1}F(\partial E)$ bound essential disks in $B^3 - \epsilon$.

$(\Leftarrow)$ Since $G^{-1}F(\partial E)$ bounds essential disks in $B^3 - \epsilon$, $F(\partial E)$ bounds essential disks in $B^3 - G(\epsilon)$. Let $D_i$ be the properly embedded disk in $B^3 - G(\epsilon)$ so that $\partial D_i = F(\partial E_i)$. We also know that $F(\partial E_i)$ bounds a disk $F(E_i') (= K_i)$ in $\partial B^3$ which contains two punctures. Then, $F(E_i') \cup K_i$ bounds a ball $M_i$ in $B^3$ and $M_i$ contains $F(\epsilon_i)$. Similarly, $D_i \cup K_i$ bounds a ball $N_i$ in $B^3$ so that $N_i$ contains $G(\epsilon_i)$. Now, we can define a homeomorphism $h_i(1 \leq i \leq 3)$ from $M_i$ to $N_i$ so that $h_i|_{F(E_i')} = id_{F(E_i')}$ and $h_i(F(\epsilon_i)) = G(\epsilon_i)$ by using Lemma 2.2 and the Alexander trick. Also, we can define $h_4$ from $B^3 - (M_1 \cup M_2 \cup M_3) \circ$ to $B^3 - (N_1 \cup N_2 \cup N_3) \circ$ so that $h_4|_{\partial B^3 - (K_1 \cup K_2 \cup K_3)} = id$ and $h_4|_{F(E_i)} = h_i|_{F(E_i)}$. Then we have a homeomorphism $H$ from $B^3$ to $B^3$ so that $H|_{\partial B^3} = id$ and $H(F(\epsilon)) = G(\epsilon)$.

In fact, if two of $G^{-1}F(\partial E)$ bound essential disks in $B^3 - \epsilon$ then $T_F \approx T_G$ by Lemma 3.3 below. So, two disjoint non-parallel simple closed curves which bound essential disks in $B^3 - \epsilon$ determine the infinite tangle.

Lemma 3.3. Suppose that two essential simple closed curves $\alpha, \beta$ ($\not\approx \alpha$) bound disjoint disks in $B^3 - \epsilon$. If $\gamma$ is an essential simple closed curve which encloses two punctures, is disjoint with $\alpha$ and $\beta$ and is non-parallel to $\alpha$ and $\beta$, then $\gamma$ bounds an essential disk in $B^3 - \epsilon$.

Proof. Let $D_1$ and $D_2$ be the two disks in $B^3 - \epsilon$ so that $\partial D_1 = \alpha$ and $\partial D_2 = \beta$. Cut $B^3 - \epsilon$ along the two disks. Then we have three balls $P_i$ which contains $\epsilon_i$. Suppose that $\gamma \subset P_1$ without loss of generality. Then $\gamma$ divides $\partial P_1$ into two regions $Q$ and $R$. Assume that $Q$ contains the two punctures in $P_1$. To have a disk $D_3$ in $P_1$, push $R^c$ from $\partial D_1$ to the interior of $P_1$ a little bit.

□

4. Step 1: Hexagon parameterization of $C$

Recall $C$ which is the set of isotopy classes of essential simple closed curves in $\Sigma_{0,6}$. In this section, we will describe how to parameterize $C$ by using the hexagon diagram. To do this, we define the hexagon as follows.

Let $\partial \epsilon_1 = \{1, 2\}$, $\partial \epsilon_2 = \{3, 4\}$ and $\partial \epsilon_3 = \{5, 6\}$ as in Figure 3. By connecting the punctures in $\Sigma_{0,6}$ as in Figure 3, we can make the hexagon $H$. Let $a_i$ be the dotted open intervals as in Figure 3. Then let $\overline{a_i}$ be the closed interval which is obtained from $a_i$ by adding the two punctures. For example, $\overline{a_1} = a_1 \cup \{1, 2\}$ and $\overline{a_3} = a_3 \cup \{5, 6\}$.

A family $C$ of smooth simple closed curves disjointly embedded in $\Sigma_{0,6}$ so that no component of $C$ is either null-homotopic or homotopic into a puncture is called a multiple curve in $\Sigma_{0,6}$; moreover, we require that two distinct components of $C$ cannot be isotopic to each
other. Define a multicurve in $\Sigma_{0,6}$ to be the isotopy class of a multiple curve in $\Sigma_{0,6}$. Let $\Gamma$ be a graph so that the vertices of $\Gamma$ are the punctures $1, 2, 3, 4, 5, 6$ and the edges of $\Gamma$ are $\overline{\sigma_i}$. Then, we define the pseudo-graph $\Gamma^\circ = \Gamma - \{1, 2, 3, 4, 5, 6\}$. Then a multiple curve $\gamma$ is in general position with respect to $\Gamma^\circ$ in $\Sigma_{0,6}$ if $\gamma$ meets $\Gamma^\circ$ transversely. Also, a multiple curve $\gamma$ is in minimal general position with respect to $\Gamma^\circ$ in $\Sigma_{0,6}$ if $\gamma$ is in general position with respect to $\Gamma^\circ$ and $\gamma$ has a minimal number of intersections with $\Gamma^\circ$ up to isotopy.

Now, consider orientation preserving homeomorphisms $f = \sigma_1^{a_1} \sigma_2^{b_1} \sigma_3^{c_1} \sigma_4^{d_1} \cdots \sigma_1^{a_k} \sigma_2^{b_k} \sigma_3^{c_k} \sigma_4^{d_k}$ and $g = \sigma_1^{a_i} \sigma_2^{b_i} \sigma_3^{c_i} \sigma_4^{d_i} \cdots \sigma_1^{a_m} \sigma_2^{b_m} \sigma_3^{c_m} \sigma_4^{d_m}$ from $\Sigma_{0,6}$ to $\Sigma_{0,6}$ for some $a_i, b_i, c_i, d_i, a_j, b_j, c_j, d_j \in \mathbb{Z}$. Then by Theorem $2.3$, $g^{-1}f(\partial E_{\nu}) = \sigma_4^{-d_m} \sigma_3^{-c_m} \sigma_2^{-b_m} \sigma_1^{-a_m} \cdots \sigma_4^{-d_i} \sigma_3^{-c_i} \sigma_2^{-b_i} \sigma_1^{-a_i} \sigma_1^{a_m} \sigma_2^{b_m} \sigma_3^{c_m} \sigma_4^{d_m}$. Let $\gamma$ be in minimal general position with respect to $\Gamma^\circ$. Then let $\Gamma^\circ = \Gamma^\circ \cap \partial E_{\nu}$ be the isotopy class of a multiple curve in $\Sigma$ which is parallel to a puncture. Also, we define $w^{ijkl}$ to be the number of arcs which are from $a_k$ to $a_l$ in the complement of the hexagon $H^c$. These are called weights. We notice that $w^{ijkl} = w^{ijkl}$ and $w^{ijkl} = w^{ijkl}$. Also, we know if $w^{ijkl} \neq 0$ for $i, j$ such that $i = j \pm 1 \pmod{6}$ then $w^{ijkl} = 0$ and if $w^{ijkl} \neq 0$ for $k, l$ such that $k = l \pm 1 \pmod{6}$ then $w^{ijkl} = 0$. If not, we have a simple closed curve which is parallel to a puncture. We notice that $w^{ijkl} = w^{ijkl} = 0$ for all $i$ since $\gamma$ is in minimal general position with respect to $\Gamma^\circ$.

First, we will show that the weights $w^{ijkl}$ and $w^{ijkl}$ for the isotopy class $[\gamma]$ are well defined.

Let $a_7, a_8, \ldots, a_{12}$ be the open arcs which connect two punctures as in Figure $10$. Let $\Gamma^\circ = \cup_{i=1}^{12} \sigma_i$ and $\Gamma^\circ = \Gamma^\circ - \{1, 2, 3, 4, 5, 6\}$. Then let $\Gamma^\circ$ be a subgraph of $\Gamma^\circ$. Then we define $\Gamma^\circ = \Gamma^\circ - \{1, 2, 3, 4, 5, 6\}$.

For two simple subarcs $\lambda$ and $\nu$ of a union $K$ of finitely many simple arcs in a surface $\Sigma$, $(\Delta, \lambda, \nu)$ is a bigon in the surface $\Sigma$ if $\lambda \cup \nu$ bounds a disk $\Delta$ in $\Sigma$ and $\partial \lambda = \partial \nu = \lambda \cap \nu$ and $(\text{int } \Delta) \cap K = \emptyset$. Then we say that two unions $A$ and $B$ of simple arcs in a surface $\Sigma$ have a bigon if there exist two simple subarcs $\lambda$ and $\nu$ in $A$ and $B$ respectively so that $(\Delta, \lambda, \nu)$ is a bigon.
Let \(|A \cap B|\) be the number of intersections between \(A\) and \(B\).

**Lemma 4.1.** Suppose that \(\delta\) is a simple closed curve in \(\Sigma_{0,6}\) so that \(\delta\) is in general position with respect to \(\Gamma_+^\circ\), but \(\delta \cap \Gamma_+^\circ\) is not minimal. Then \(\delta\) and \(\Gamma_+^\circ\) have a bigon in \(\Sigma_{0,6}\).

**Proof.** Let \(\delta'\) be a simple closed curve in \(\Sigma_{0,6}\) so that \(\delta' \sim \delta\) and \(\delta'\) is in minimal general position with respect to \(\Gamma_+^\circ\). Then by the transversality theorem we can choose an isotopy \(H : S^1 \times [0, 1] \to \Sigma_{0,6}\) so that \(H(S^1 \times \{0\}) = \delta\), \(H(S^1 \times \{1\}) = \delta'\) and \(H^{-1}(\Gamma_+^\circ)\) is a collection of 1-manifolds in \(S^1 \times [0, 1]\). Let \(m = |H^{-1}(\Gamma_+^\circ) \cap (S^1 \times \{0\})|\) and \(n = |H^{-1}(\Gamma_+^\circ) \cap (S^1 \times \{1\})|\). Then we notice that \(m > n\) since \(\delta' \cap \Gamma_+^\circ\) is minimal in \(\Sigma_{0,6}\), but \(\delta \cap \Gamma_+^\circ\) is not minimal. Therefore, there exists a properly embedded arc \(\alpha\) in \(S^1 \times [0, 1]\) so that \(\alpha\) is parallel to an arc \(\beta\) of \(S^1 \times \{0\}\) and \(H(\alpha) \subset \Gamma_+^\circ\). So, \(\alpha \cup \beta\) bounds a disk \(D\) in \(S^1 \times [0, 1]\). Let \(c_1\) and \(c_2\) be the common endpoints of \(\alpha\) and \(\beta\). Let \(d_1 = H(c_1)\) and \(d_2 = H(c_2)\). Now, consider \(H|_D\). Let \([d_1, d_2]\) be the segment between \(d_1\) and \(d_2\) in \(\Gamma_+^\circ\). Now, we choose a homeomorphism \(K : \alpha \to [d_1, d_2]\) with \(K(c_1) = d_1, K(c_2) = d_2\). Then we remark that \(K \simeq H|_\alpha\) rel \(\{c_1, c_2\}\). We define \(\overline{K} : \alpha \cup \beta \to \Sigma_{0,6}\) so that \(\overline{K}(x) = K(x)\) for \(x \in \alpha\) and \(\overline{K}(y) = H(y)\) for \(y \in \beta\). So, \(H|_{\alpha \cup \beta} \simeq \overline{K}|_{\alpha \cup \beta}\) rel \(\{c_1\}\). Let \([d_1, d_2] = \alpha'\) and \(H(\beta) = \beta'\). Let \(p_1\) be a path from \(H(c_1)\) to \(H(c_2)\) along \(\alpha'\) and let \(p_2\) be a path from \(H(c_2)\) to \(H(c_1)\) along \(\beta'\). Then \(p_1 \cdot p_2\) is a loop with base point \(H(c_1)\). Then we notice that \(p_1 \cdot p_2\) is null-homotopic in \(\Sigma_{0,6}\). Therefore, \(\alpha' \cup \beta'\) bounds a disk \(D'\) in \(\Sigma_{0,6}\). This implies that \(\delta\) and \(\Gamma_+^\circ\) have a bigon in \(\Sigma_{0,6}\).

\(\square\)

**Corollary 4.2.** If \(\gamma\) and \(\Gamma_+^\circ\) have no bigons then \(\gamma\) is in minimal general position with respect to \(\Gamma_+^\circ\). Moreover, \(\gamma \cap a_i\) also has a minimal intersection for all \(i \in \{1, 2, ..., 12\}\).

**Proof.** From Lemma 4.1 we know that if \(\gamma\) and \(\Gamma_+^\circ\) have no bigons then \(\gamma\) is in minimal general position with respect to \(\Gamma_+^\circ\). Now, suppose that \(\gamma \cap a_i\) does not have a minimal intersection. Then \(\gamma\) and \(a_i\) have a bigon in \(\Sigma_{0,6}\) by Lemma 4.1. So, we have closed intervals \(\lambda \subset a_i\) and \(\nu \subset \gamma\) so that \(\lambda \cup \nu\) bounds a disk \(\Delta\) in \(\Sigma_{0,6}\). This implies that \(\gamma\) and \(\Gamma_+^\circ\) have a bigon since \(\nu\) is homotopic to \(\lambda\). This contradicts the fact that \(\gamma \cap \Gamma_+^\circ\) has a minimal intersection. Therefore, \(\gamma \cap a_i\) is minimal for all \(i \in \{1, 2, ..., 12\}\). \(\square\)
Using Corollary 4.2 we will show the weights of isotopy classes are well defined.

Recall that $w_{ij}$ is the number of arcs of $\gamma$ which are from $a_i$ to $a_j$ in the hexagon $H$ and $w_{kl}$ is the number of arcs of $\gamma$ which are from $a_k$ to $a_l$ in the complement of the hexagon $H^c$.  

**Lemma 4.3.** The weights $w_{ij}$ and $w_{ij}$ of $[\gamma]$ for $i,j \in \{1,2,\ldots,6\}$ are well defined.

**Proof.** Suppose that $\gamma$ is in minimal general position with respect to $\Gamma_+^\gamma$. Let $m_i$ be the number of intersections between $\gamma$ and $a_i$ for $i \in \{1,2,\ldots,12\}$. Let $A_i$ be the regions as in Figure 10.

For the three sides $a_1, a_6$ and $a_7$ of a region $A_1$, let $s_{16}, s_{17}, s_{67}$ be the numbers of arcs from $a_i$ to $a_j$ in $A_1$, where $i,j \in \{1,6,7\}$. Then we know that $s_{16} + s_{17} = m_1$, $s_{16} + s_{67} = m_6$ and $s_{67} + s_{17} = m_7$. By solving these equations for $s_{ij}$, we have $s_{16} = \frac{m_1 + m_6 - m_7}{2}$, $s_{17} = \frac{m_1 + m_7 - m_6}{2}$ and $s_{67} = \frac{m_6 + m_7 - m_1}{2}$. So, the weights in $A_1$ are determined by $m_1$, $m_6$ and $m_7$. Similarly, the weights $t_{27}, t_{28}, t_{78}$ in $A_2$ are determined by $m_2$, $m_7$ and $m_8$.

Since $\gamma$ is in minimal general position with respect to $\Gamma_+^\gamma$, $\gamma$ and $\Gamma_+^\gamma$ have no bigon. By Corollary 4.2 we know that $m_k$ is unique for $k = 1,2,\ldots,12$. This implies that the weights $s_{ij}$ and $t_{kl}$ in $A_1$ and $A_2$ respectively are also unique since $m_k$ is unique.

Now, consider $A_1 \cup A_2$. Then for the four sides $a_1, a_2, a_6$ and $a_8$ of the rectangle $A_1 \cup A_2$, let $y_{pq}$ be the number of arcs from $a_p$ to $a_q$ in $A_1 \cup A_2$, where $p,q \in \{1,2,6,8\}$. Then the weights $y_{12}, y_{16}, y_{18}, y_{26}, y_{28}$ and $y_{68}$ are determined by $\{s_{16}, s_{17}, s_{67}, t_{27}, t_{28}, t_{78}\}$ as follows. $y_{12} = \min(s_{17}, t_{27}), y_{16} = s_{16}, y_{18} = s_{17} - y_{12} = s_{17} - \min(s_{17}, t_{27}), y_{26} = t_{27} - y_{12} = t_{27} - \min(s_{17}, t_{27}), y_{28} = t_{28} - y_{18} = t_{28} - s_{17} + y_{12} - t_{78} - s_{17} + \min(s_{17}, t_{27})$.

For four sides $a_3, a_4, a_5$ and $a_8$ of $A_3 \cup A_4$, let $z_{uv}$ be the number of arcs from $a_u$ to $a_v$ in $A_3 \cup A_4$, where $u,v \in \{3,4,5,8\}$. Then we note that $z_{uv}$ are determined by the six weights in $A_3$ and $A_4$.

Now, consider $H = A_1 \cup A_2 \cup A_3 \cup A_4$. Then we claim that the weights $w_{ij}$ in $H$ are determined by $y_{pq}$ and $z_{uv}$ as follows. $w_{12} = y_{12}, w_{13} = \max(0, y_{18} + z_{38} - \max(z_{38}, y_{18} + y_{28})), w_{15} = \max(0, y_{18} + z_{58} - \max(z_{58}, y_{18} + y_{68})), w_{14} = y_{18} - w_{13} - w_{15}, w_{16} = y_{16}; w_{23} = \min(y_{28}, z_{38}), w_{24} = \max(0, y_{48} + z_{28} - \max(z_{28}, y_{48} + y_{38})), w_{25} = y_{28} - w_{23} - w_{24}, w_{26} = y_{26}; w_{34} = z_{34}, w_{35} = z_{35}, w_{36} = z_{38} - w_{13} - w_{23}; w_{45} = z_{45}, w_{46} = \max(0, y_{48} + z_{68} - \max(z_{68}, y_{48} + y_{58})); w_{56} = \min(y_{68}, z_{58})$.

In order to get the formula for $w_{13}$, we need to consider the two cases that $z_{38} \geq y_{18} + y_{28}$ and $z_{38} < y_{18} + y_{28}$.

If $z_{38} \geq y_{18} + y_{28}$, then we see that $w_{13} = y_{18}$.

If $z_{38} < y_{18} + y_{28}$, then we see that $w_{13} = \max(y_{18} - \max((y_{18} + y_{28}) - z_{38}), 0) = \max(z_{38} - y_{28}, 0)$.
By combining the two cases, we get \( w_{13} = \max(0, y_{18} + z_{38} - \max(z_{38}, y_{18} + y_{28})) \).

Similarly, we can get the formulas for \( w_{15}, w_{24} \) and \( w_{46} \).

Therefore, \( w_{ij} \) of \( [\gamma] \) for \( i, j \in \{1, 2, 3, 4, 5, 6\} \) are unique if \( \gamma \) is in minimal general position.

By using symmetry, we also know \( w^{kl} \) are determined by the weights in \( A_i \) for \( i = 5, 6, 7, 8 \).

Therefore, \( w_{jk} \) and \( w^{jk} \) for \( j, k \in \{1, 2, ..., 6\} \) are determined by \( m_i \) for \( i = 1, 2, ..., 12 \) and this proves the theorem. \( \square \)

From Theorem 4.2, we define that \( w_{ij} \) and \( w^{ij} \) are the weights for the isotopy class \( [\gamma] \) in the hexagon parameterization if \( w_{ij} \) and \( w^{ij} \) are the weights for a simple closed curve \( \delta \) which is isotopic to \( \gamma \) and has no bigons with the hexagon. Now, we want to calculate the weight changes by a half Dehn twist.

**Figure 11.** The half Dehn twist supported on \( K_P \)

First, we are getting a new curve \( \hat{\gamma} \) as in Figure 11 which is a representative of \( [\sigma_P(\gamma)] \) that may have a bigon, and so the new weight \( w'_{ii} \) may be non-zero. Then \( \hat{\gamma} \) will be isotoped to remove all bigons and get the new weights \( v_{ij} \) for \( [\sigma_P(\gamma)] \).

**Figure 12.** The weight changes by the half Dehn twist \( \sigma_1 \)

Let \( w'_{ij} \) and \( w'^{ij} \) be the weights for \( \sigma_1(\gamma) \) as in the middle diagram of Figure 12.
Theorem 4.4. Let $w_{ij}$ and $w^{ij}$ be the weights for $[\gamma]$. Then the following formulas give the weights $w'_{ij}$ and $w''_{ij}$ for $\sigma_1(\gamma)$.

\[
\begin{align*}
  w'_{12} &= w_{12} + w_{26}, \\
  w'_{13} &= w_{13} + w_{36}, \\
  w'_{14} &= w_{14} + w_{46}, \\
  w'_{15} &= w_{15} + w_{56}, \\
  w'_{16} &= 0; \\
  w'_{23} &= w_{23}, \\
  w'_{24} &= w_{24}, \\
  w'_{25} &= w_{25}, \\
  w'_{26} &= 0; \\
  w'_{34} &= w_{34}, \\
  w'_{35} &= w_{35}, \\
  w'_{36} &= 0; \\
  w'_{45} &= w_{45}, \\
  w'_{46} &= 0; \\
  w'_{56} &= w_{16} + w_{26} + w_{36} + w_{46} + w_{56}; \\
  w'_{11} &= w_{16}; \\
  w'_{52} &= w_{52} + w_{26}, \\
  w'_{53} &= w_{53} + w_{36}, \\
  w'_{54} &= w_{54} + w_{46}, \\
  w'_{51} &= w_{51} + w_{16}, \\
  w'_{56} &= 0; \\
  w'_{23} &= w_{23}, \\
  w'_{24} &= w_{24}, \\
  w'_{21} &= w_{21}, \\
  w'_{26} &= 0; \\
  w'_{34} &= w_{34}, \\
  w'_{31} &= w_{31}, \\
  w'_{36} &= 0; \\
  w'_{41} &= w_{41}, \\
  w'_{46} &= 0; \\
  w'_{16} &= w_{56} + w_{26} + w_{36} + w_{46} + w_{16}; \\
  w'_{55} &= w_{56}.
\end{align*}
\]

Proof. To see weight changes by a half Dehn twist $\sigma_1$, consider Figure 12.

From the two points $x, y$, we have 12 arcs which connect one of the two points and the middle of $a_i$. Then this diagram shows all possibilities of the weights. For example, there are arcs from $a_1$ to $x$ and from $x$ to $a_2$. These two arcs show the possibilities for $w_{12}$. Let $\Theta$ be the graph with the two vertices and the six edges. Now, take a proper two punctured disk $D_1$ which contains $a_6$ so that every component of $D_1 \cap \gamma$ is essential in $D_1$. We note that $D_1$ contains punctures 2 and 3. Now, apply a half Dehn twist $\sigma_1$ supported on $D_1$ counter clockwise to the first diagram to get the second diagram. Let $\Theta'$ be the graph which is obtained from $\Theta$ by $\sigma_1$. Let $w'_{ij}$ and $w''_{kl}$ be the weights for $\sigma_1(\gamma)$. We point out that $\sigma_1(\gamma)$ is not isotoped to have minimal intersection with the hexagon when $w'_{ij}$ and $w''_{kl}$ are
computed. That will happen when $v_{ij}$ and $v^{kl}$ are computed. The formulas above give the weights $w_i'$ and $w_{kl}$.

We remark that if we use the transposition $(1, 5)$ then we can get the formulas for $w_i^{12}$ from the formulas for $w_i'$. i.e., we switch the indices 1 and 5. For example, we get

$$w_i^{16} = w_i^{36} + w_i^{26} + w_i^{36} + w_i^{16} + w_i^{16}$$

from $w_i' = w_i + w_i + w_i + w_i + w_i$.

We notice that if we have a subarc of $\sigma_1(\gamma)$ for $w_i'\text{ or } w^{ii}$ then we can isotope the subarc across $a_i$ so that eventually $w_i' = w^{ii} = 0$. Let $\Phi$ be the graph which is obtained from $\Theta$ by the isotopy to have $w_i' = w^{ii} = 0$. Then we have the following theorem.

**Theorem 4.5.** Let $w_i'$ and $w^{ii}$ be the weights for $\sigma_1(\gamma)$. Then the following formulas give the weights $v_{ij}$ and $v^{ij}$ for $[\sigma_1(\gamma)]$ which has $v_{ii} = v^{ii} = 0$ for all $i \in \{1, 2, 3, 4, 5, 6\}$.

$$
\begin{align*}
v_{12} &= w_{12}', v_{13} = w_{13}', v_{14} = w_{14}', \\
v_{15} &= \max(w_{15}^i - w_{15}^i, 0), \\
v_{16} &= \min(w_{15}^i, w_{15}^i); \\
v_{23} &= w_{23}', v_{24} = w_{24}', \\
v_{25} &= \min(w_{25}^i, \max(w_{15}^i + w_{15}^i + w_{15}^i + w_{15}^i - w_{15}^5, 0)), \\
v_{26} &= \min(w_{25}^i, \max(w_{15}^i - w_{15}^i - w_{15}^i - w_{15}^i, 0)); \\
v_{34} &= w_{34}', \\
v_{35} &= \min(w_{35}^i, \max(w_{15}^i + w_{15}^i + w_{15}^i - w_{15}^5, 0)), \\
v_{36} &= \min(w_{35}^i, \max(w_{15}^i - w_{15}^i - w_{15}^i - w_{15}^i, 0)); \\
v_{45} &= \min(w_{45}^i, \max(w_{15}^i + w_{15}^i - w_{15}^5, 0)), \\
v_{46} &= \min(w_{45}^i, \max(w_{15}^i - w_{15}^i - w_{15}^i, 0)); \\
v_{56} &= w_{56}' - (w_{15}^5).
\end{align*}
$$

$$
\begin{align*}
v_{25} &= w_{25}^i, v_{35} = w_{35}, v_{45} = w_{45}, \\
v_{15} &= \max(w_{15}^i - w_{11}^i, 0), \\
v_{56} &= \min(w_{11}^i, w_{15}^i); \\
v_{23} &= w_{23}, \\
v_{24} &= w_{24}, \\
v_{12} &= \min(w_{12}^i, \max(w_{12}^i + w_{12}^i + w_{12}^i + w_{12}^i - w_{11}^i, 0)), \\
v_{26} &= \min(w_{12}^i, \max(w_{15}^i - w_{13}^i - w_{15}^i - w_{15}^i, 0)); \\
v_{34} &= w_{34}, \\
v_{13} &= \min(w_{13}^i, \max(w_{13}^i + w_{14}^i + w_{15}^i - w_{11}^i, 0)), \\
v_{36} &= \min(w_{13}^i, \max(w_{15}^i - w_{14}^i - w_{15}^i, 0)); \\
v_{14} &= \min(w_{14}^i, \max(w_{14}^i + w_{15}^i - w_{11}^i, 0)), \\
v_{46} &= \min(w_{14}^i, \max(w_{15}^i - w_{15}^i, 0)); \\
v_{16} &= w_{16}^i - (w_{11}^i).
\end{align*}
$$

Let $\Theta''$ be the union of arcs in the hexagon diagram as in Figure 13-(a) which shows the details of $\Theta'$. Similarly, let $\Phi'$ be the union of arcs in the hexagon diagram as in Figure 13-(b) which shows the details of $\Phi$.

I want to remark that the arcs in $\Theta''$ and $\Phi'$ carry $\sigma_1(\gamma)$. 
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Proof.

From the diagram 13-(a), we obtain the diagram 13-(b) by pushing the arcs for $w'_{11}$ across $a_1$ so that we change $w'_{11}$ to 0. I note that each subarc of $\sigma_1(\gamma)$ from $a_i$ to $a_j$ carries a weight. For example, in 13-(a), assume that the two arcs from $a_1$ to $a_6$ carries the weights $w'^{16} - w'_{11}$ and $w'_{11}$ respectively so that the sum of weights is $w'^{16}$. Let $h_1$ be the isotopy move to push the arcs that carring the weight $w'^{11}$ across $a_1$. Also, we know that $h_1 \circ \sigma_1$ is isotopic to $\sigma_1$. Let $h_1(\sigma_1(\gamma)) = \gamma'$ and $w''_{ij}$ and $w'^{nkl}$ be the weights for $\gamma'$. Then we have weight changes by $\sigma_1$ from $\gamma$ to $\gamma'$ which has $w''_{11} = 0$.

We note that $w''_{ij} = w'_{ij}$ for $i \neq j \in \{1, 2, 3, 4, 5, 6\}$ and $w''_{ii} = 0$ for all $i \in \{1, 2, 3, 4, 5, 6\}$.

Now, consider the nine cases to find the formulas for $w'^{nkl}$ as in Figure 14.

I want to emphasize that the bands in the diagrams of Figure 14 now carries the weight of $\gamma'$.

(1) $w'_{11} = 0$
(2) $0 < w'_{11} < w'^{15}$
(3) $w'_{11} = w'^{15}$
(4) $w'^{15} < w'_{11} < w'^{14} + w'^{15}$

(5) $w'_{11} = w'^{14} + w'^{15}$
(6) $w'^{14} + w'^{15} < w'_{11} < w'^{13} + w'^{14} + w'^{15}$

(7) $w'_{11} = w'^{13} + w'^{14} + w'^{15}$
(8) $w'^{13} + w'^{14} + w'^{15} < w'_{11} < w'^{12} + w'^{13} + w'^{15} + w'^{15}$

(9) $w'_{11} = w'^{12} + w'^{13} + w'^{14} + w'^{15}$

Then, we have the following formulas for $w'^{nkl}$.

$w'^{25} = w'^{25}$, $w'^{35} = w'^{35}$, $w'^{45} = w'^{45}$.
Figure 14. Subcases for the weight changes from $w'^{11}$

\[ w'^{23} = w'^{23}, \quad w'^{24} = w'^{24}, \]
\[ w'^{34} = w'^{34}, \]
\[ w'^{16} = w'^{16} - (w'^{11}). \]

Claim (a): $w'^{56} = \min(w'^{11}, w'^{15})$.

Proof. We remind that $w'^{56} = 0$.

(i) If $w'^{11} < w'^{15}$ then by (1) and (2) we have $w'^{56} = w'^{11}$.

(ii) If $w'^{11} \geq w'^{15}$ then by (3) – (9) we have $w'^{56} = w'^{15}$.

Claim (b): $w'^{15} = \max(w'^{15} - w'^{11}, 0)$.

Proof. (i) If $w'^{11} < w'^{15}$ then by (1) and (2) we have $w'^{15} = w'^{15} - w'^{11}$.

(ii) If $w'^{11} \geq w'^{15}$ then by (3) – (9) we have $w'^{15} = 0$. 

\[ w'^{23} = w'^{23}, \quad w'^{24} = w'^{24}, \]
\[ w'^{34} = w'^{34}, \]
\[ w'^{16} = w'^{16} - (w'^{11}). \]
Claim (c): \( w'^{26} = \min(w'^{12}, \max(w'^{11} - w'^{13} - w'^{14} - w'^{15}, 0)) \).

Proof. Recall that \( w'^{26} = 0 \).

(i) If \( w'^{11} \leq w'^{13} + w'^{14} + w'^{15} \) then by (1)–(7) we have \( w'^{26} = 0 \). Since \( w'^{11} - w'^{13} - w'^{14} - w'^{15} \leq 0 \), \( \max(w'^{11} - w'^{13} - w'^{14} - w'^{15}, 0) = 0 \). So, \( w'^{26} = \min(w'^{12}, 0) = 0 \).

(ii) If \( w'^{13} + w'^{14} + w'^{15} < w'^{11} < w'^{12} + w'^{13} + w'^{14} + w'^{15} \) then by (8) we have \( w'^{26} = w'^{11} - w'^{13} - w'^{14} - w'^{15} \). Since \( w'^{11} - w'^{13} - w'^{14} - w'^{15} > 0 \), \( \max(w'^{11} - w'^{13} - w'^{14} - w'^{15}, 0) = w'^{11} - w'^{13} - w'^{14} - w'^{15} \). Since \( w'^{11} - w'^{13} - w'^{14} - w'^{15} < w'^{12} \), we have \( w'^{26} = \min(w'^{12}, w'^{11} - w'^{13} - w'^{14} - w'^{15}) \).

(iii) If \( w'^{11} = w'^{12} + w'^{13} + w'^{14} + w'^{15} \) then by (9) we have \( w'^{26} = w'^{12} \). Since \( w'^{11} - w'^{13} - w'^{14} - w'^{15} = w'^{12} \), \( \max(w'^{11} - w'^{13} - w'^{14} - w'^{15}, 0) = w'^{11} - w'^{13} - w'^{14} - w'^{15} = w'^{12} \). So, \( \min(w'^{12}, w'^{12}) = w'^{12} \).

Claim (d): \( w'^{12} = \min(w'^{12}, \max(w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11}, 0)) \).

Proof. (i) If \( w'^{13} + w'^{14} + w'^{15} > w'^{11} \) then by (1)–(6) we have \( w'^{12} = w'^{12} \). Since \( w'^{13} + w'^{14} + w'^{15} - w'^{11} > 0 \), we have \( \max(w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11}, 0) = w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11} = w'^{12} \). So, \( w'^{12} = \min(w'^{12}, w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11}) = w'^{12} \) since \( w'^{13} + w'^{14} + w'^{15} > w'^{11} \).

(ii) If \( w'^{13} + w'^{14} + w'^{15} \leq w'^{11} < w'^{12} + w'^{13} + w'^{14} + w'^{15} \) then by (7) and (8) we have \( w'^{12} = w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11} \). Since \( w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11} > 0 \), \( \max(w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11}, 0) = w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11} = w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11} = w'^{12} \). So, \( w'^{12} = \min(w'^{12}, w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11}) = w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11} \) since \( w'^{13} + w'^{14} + w'^{15} \leq w'^{11} \).

(iii) If \( w'^{12} + w'^{13} + w'^{14} + w'^{15} = w'^{11} \) then by (9) we have \( w'^{12} = 0 \). Since \( w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11} = 0 \), we have \( \max(w'^{12} + w'^{13} + w'^{14} + w'^{15} - w'^{11}, 0) = 0 \). So, \( \min(w'^{12}, 0) = 0 \).

Claim (e): \( w'^{36} = \min(w'^{13}, \max(w'^{11} - w'^{14} - w'^{15}, 0)) \).

Proof. Recall that \( w'^{36} = 0 \).

(i) If \( w'^{11} \leq w'^{15} + w'^{14} \) then by (1)–(5) we have \( w'^{36} = 0 \). Since \( w'^{11} - w'^{14} - w'^{15} \leq 0 \), \( \max(w'^{11} - w'^{14} - w'^{15}, 0) = 0 \). So, \( w'^{36} = \min(w'^{13}, 0) = 0 \).

(ii) If \( w'^{14} + w'^{15} < w'^{11} < w'^{13} + w'^{14} + w'^{15} \) then by (6) we have \( w'^{36} = w'^{11} - w'^{14} - w'^{15} \). Since \( w'^{11} - w'^{14} - w'^{15} > 0 \), we have \( \max(w'^{11} - w'^{14} - w'^{15}, 0) = w'^{11} - w'^{14} - w'^{15} \). So, \( w'^{36} = \min(w'^{13}, w'^{11} - w'^{14} - w'^{15}) = w'^{11} - w'^{14} - w'^{15} \) since \( w'^{11} < w'^{13} + w'^{14} + w'^{15} \).

(iii) If \( w'^{11} \geq w'^{13} + w'^{14} + w'^{15} \) then by (7)–(9) we have \( w'^{36} = w'^{13} \). Since \( w'^{11} - w'^{13} - w'^{14} - w'^{15} \geq 0 \), \( \max(w'^{11} - w'^{14} - w'^{15}, 0) = w'^{11} - w'^{14} - w'^{15} \). So, \( w'^{36} = \min(w'^{13}, w'^{11} - w'^{14} - w'^{15}) = w'^{13} \) since \( w'^{11} \geq w'^{13} + w'^{14} + w'^{15} \).

Claim (f): \( w'^{13} = \min(w'^{13}, \max(w'^{13} + w'^{14} + w'^{15} - w'^{11}, 0)) \).

Proof. (i) If \( w'^{14} + w'^{15} \geq w'^{11} \) then by (1)–(5) we have \( w'^{13} = w'^{13} \). Since \( w'^{14} + w'^{15} - w'^{11} \geq 0 \), \( \max(w'^{13} + w'^{14} + w'^{15} - w'^{11}, 0) = w'^{13} + w'^{14} + w'^{15} - w'^{11} \). So, \( w'^{13} = \min(w'^{13}, w'^{13} + w'^{14} + w'^{15} - w'^{11}) = w'^{13} \) since \( w'^{11} \geq w'^{13} + w'^{14} + w'^{15} \).
(ii) If \( w_{11}' + w_{15}' < w_{11}' \leq w_{13}' + w_{14}' + w_{15}' \) then by (6) and (7) we have \( w_{13}' = w_{13}' + w_{14}' + w_{15}' - w_{11}' > 0 \), we have \( \max(w_{13}' + w_{14}' + w_{15}' - w_{11}', 0) = w_{13}' + w_{14}' + w_{15}' - w_{11}' \). So, \( w_{13}' = \min(w_{13}' + w_{14}' + w_{15}' - w_{11}') = w_{13}' + w_{14}' + w_{15}' - w_{11} \) since \( w_{14}' + w_{15}' < w_{11}' \).

(iii) If \( w_{13}' + w_{14}' + w_{15}' < w_{11}' \) then by (8) and (9) we have \( w_{13}' = 0 \). Since \( w_{13}' + w_{14}' + w_{15}' - w_{11}' < 0 \), \( \max(w_{13}' + w_{14}' + w_{15}' - w_{11}', 0) = 0 \). So, \( w_{13}' = \min(w_{13}' + w_{14}' + w_{15}' - w_{11}', 0) = 0 \).

Claim (g): \( w_{46}' = \min(w_{14}', w_{11}' - w_{15}') \).

Proof. (i) If \( w_{11}' \leq w_{15}' \) then by (1) – (3) we have \( w_{46}' = 0 \). Since \( w_{11}' - w_{15}' \leq 0 \), \( \max(w_{11}' - w_{15}', 0) = 0 \). So, \( w_{46}' = \min(w_{14}', 0) = 0 \).

(ii) If \( w_{15}' < w_{11}' < w_{14}' + w_{15}' \) then by (4) we have \( w_{14}' = w_{11}' - w_{15}' \). Since \( w_{11}' - w_{15}' > 0 \), \( \max(w_{11}' - w_{15}', 0) = w_{11}' - w_{15}' \). So, \( w_{46}' = \min(w_{14}', w_{11}' - w_{15}') = w_{11}' - w_{15}' \) since \( w_{11}' < w_{14}' + w_{15}' \).

(iii) If \( w_{11}' \geq w_{14}' + w_{15}' \) then by (5) – (9) we have \( w_{46}' = w_{14}' \). Since \( w_{11}' - w_{14}' - w_{15}' \geq 0 \), we have \( \max(w_{11}' - w_{15}', 0) = w_{11}' - w_{15}' \). So, \( w_{46}' = \min(w_{14}', w_{11}' - w_{15}') = w_{14}' \) since \( w_{11}' \geq w_{14}' + w_{15}' \).

Claim (h): \( w_{14}' = \min(w_{14}', \max(w_{14}' + w_{15}' - w_{11}', 0)) \).

Proof. (i) If \( w_{15}' \geq w_{11}' \) then by (1) – (3) we have \( w_{14}' = w_{14}' \). Since \( w_{15}' - w_{11}' \geq 0 \), we have \( \max(w_{14}' + w_{15}' - w_{11}', 0) = w_{14}' + w_{15}' - w_{11}' \). So, \( w_{14}' = \min(w_{14}', w_{14}' + w_{15}' - w_{11}') = w_{14}' \) since \( w_{15}' \geq w_{11}' \).

(ii) If \( w_{15}' < w_{11}' \leq w_{14}' + w_{15}' \) then by (4) and (5) we have \( w_{14}' = w_{14}' + w_{15}' - w_{11}' \). Since \( w_{14}' + w_{15}' - w_{11}' \geq 0 \), we have \( \max(w_{14}' + w_{15}' - w_{11}', 0) = w_{14}' + w_{15}' - w_{11}' \). So, \( w_{14}' = \min(w_{14}', w_{14}' + w_{15}' - w_{11}') = w_{14}' + w_{15}' - w_{11}' \) since \( w_{15}' < w_{11}' \).

(iii) If \( w_{14}' + w_{15}' < w_{11}' \) then by (6) – (9) we have \( w_{14}' = 0 \). Since \( w_{14}' + w_{15}' - w_{11}' < 0 \), we have \( \max(w_{14}' + w_{15}' - w_{11}', 0) = 0 \). So, \( w_{14}' = \min(w_{15}', 0) = 0 \).

Now, by pushing the arcs for \( w_{55}' \) across \( a_5 \) we change \( w_{55}' \) to 0. (See Figure 15.) Let \( h_2 \) be the isotopy move to push the arcs for \( w_{55}' \) across \( a_5 \). We know that \( h_2 \circ h_1 \circ \sigma_1 \) is isotopic to \( \sigma_1 \). Let \( h_2(h_1(\sigma_1(\gamma))) = \gamma'' \) and \( v_{ij} \) and \( v^{kl} \) be the weights for \( \gamma'' \). We notice that pushing the arcs for \( w_{55}' \) does not depend on pushing the arcs for \( w_{11}' \) since \( h_1(\sigma_1(\gamma)) \) does not change the weights \( w_{ij} \) if \( i \neq j \). So, by using the diagram which is obtained by switching \( H \) and \( H' \), we have weight changes of \( v^{kl} \) by \( \sigma \) from \( \gamma \) to \( \gamma'' \) which has no bigons. (See Figure 15.) Actually, we have the formulas for \( v_{ij} \) from \( w_{55}' \) by replacing 1 by 5 and 5 by 1.

\[
\begin{align*}
v_{12} &= w_{12}', \quad v_{13} = w_{13}', \quad v_{14} = w_{14}', \\
v_{15} &= \max(w_{15}' - w_{55}', 0), \\
v_{16} &= \min(w_{15}', w_{15}' - w_{11}), \\
v_{23} &= w_{23}', \quad v_{24} = w_{24}', \\
v_{25} &= \min(w_{25}', \max(w_{15}' + w_{25}'+ w_{35}' + w_{45}' - w_{55}', 0)),
\end{align*}
\]
Figure 15. The way to make $v^{ii} = 0$

$v_{26} = \min(w'_{25}, \max(w_{55} - w'_{15} - w'_{45} - w'_{35}, 0))$;
$v_{34} = w'_{34}$;
$v_{35} = \min(w'_{35}, \max(w'_{15} + w'_{35} + w'_{45} - w_{55}, 0))$;
$v_{36} = \min(w'_{35}, \max(w_{55} - w'_{15} - w'_{45}, 0))$;
$v_{45} = \min(w'_{45}, \max(w'_{15} + w'_{45} - w_{55}, 0))$;
$v_{46} = \min(w'_{45}, \max(w_{55} - w'_{15}, 0))$;
$v_{56} = w'_{56} - (w_{55})$.

We note that $v^{kl} = w''_{kl}$.

Now, we have the following corollary by combining Theorem 4.4 and Theorem 4.5.

**Corollary 4.6.** Let $w_{ij}$ and $w^{ij}$ be the weights for $[\gamma]$. Then the following formulas give the weights $v_{ij}$ and $v^{ij}$ for $[\sigma_1(\gamma)]$ which has $v_{ii} = v^{ii} = 0$ for all $i \in \{1, 2, 3, 4, 5, 6\}$.

$v_{12} = w_{12} + w_{26}$,
$v_{13} = w_{13} + w_{36}$,
$v_{14} = w_{14} + w_{46}$,
$v_{15} = \max(w_{15} + w_{56} - w_{56}, 0)$,
$v_{16} = \min(w_{56}, w_{15} + w_{56})$;
$v_{23} = w_{23}$,
$v_{24} = w_{24}$,
$v_{25} = \min(w_{25}, \max(w_{15} + w_{25} + w_{35} + w_{45} + w_{56} - w_{56}, 0))$,
$v_{26} = \min(w_{25}, \max(w_{56} - w_{15} - w_{56} - w_{45} - w_{35}, 0))$;
$v_{34} = w_{34}$;
$v_{35} = \min(w_{35}, \max(w_{15} + w_{35} + w_{45} + w_{56} - w_{56}, 0))$,
$v_{36} = \min(w_{35}, \max(w_{56} - w_{15} - w_{56} - w_{45}, 0))$;
$v_{45} = \min(w_{45}, \max(w_{15} + w_{45} + w_{56} - w_{56}, 0))$,
$v_{46} = \min(w_{45}, \max(w_{56} - w_{15} - w_{56}, 0))$;
$v_{56} = w_{16} + w_{26} + w_{36} + w_{46} + w_{56} - (w_{56})$.

$v_{52} = w_{52} + w_{26}$.
\[
v_{53} = w_{53} + w_{36},
v_{54} = w_{54} + w_{46},
v_{51} = \max(w_{51} + w_{16} - w_{16}, 0),
v_{56} = \min(w_{16}, w_{51} + w_{16});
v_{23} = w_{23},
v_{24} = w_{24},
v_{12} = \min(w_{12}, \max(w_{12} + w_{13} + w_{14} + w_{15} + w_{16} - w_{16}, 0)),
v_{26} = \min(w_{12}, \max(w_{16} - w_{13} - w_{14} - w_{15} - w_{16}, 0));
v_{34} = w_{34},
v_{13} = \min(w_{13}, \max(w_{13} + w_{14} + w_{15} + w_{16} - w_{16}, 0)),
v_{36} = \min(w_{13}, \max(w_{16} - w_{14} - w_{15} - w_{16}, 0));
v_{14} = \min(w_{14}, \max(w_{14} + w_{15} + w_{16} - w_{16}, 0)),
v_{46} = \min(w_{14}, \max(w_{16} - w_{15} - w_{16}, 0));
v_{16} = w_{16} + w_{26} + w_{36} + w_{46} + w_{56} - (w_{16}).
\]

Also, we can calculate weight changes which are affected by \(\sigma_1^{-1}\) by using the symmetry as follows. \(\partial H\) separates \(\Sigma_{0,6}\) into two disks \(H\) and \(H^c\). Now, we interchange \(H\) and \(H^c\) while fixing \(\sigma_1(\gamma)\). Then we can get the formulas for the weights \(u_{ij}\) and \(u^{kl}\) for \([\sigma_1^{-1}(\gamma)]\). In fact, if we switch the upper indices and lower indices, we get the formulas for the weight changes by \(\sigma_1^{-1}\). For example, we get \(u_{12} = w_{12} + w_{26}\) from \(v_{12} = w_{12} + w_{26}\).

Similarly, we get weight changes which are effected by \(\sigma_i^{\pm 1}\) for \(2 \leq i \leq 4\) by using a multiple of \(60^\circ\) rotation. For example, consider \(\sigma_3\). First, rotate the hexagon diagram \(-120^\circ\) (clockwise) about the center of the hexagon. Let \(f\) be the rotation. Now apply \(\sigma_1\) to \(f(\gamma)\) to have the weights \(u_{ij}\) and \(u^{kl}\) for \([\sigma_1(f(\gamma))]\). After this, rotate the resulting diagram \(+120^\circ\) (counterclockwise) about the center of the hexagon. Then we obtain the weight change formulas for \([\sigma_3(\gamma)]\).

We notice that the permutation \((123456)^{(6-2)} = (153)(264)\) gives the index changes for clockwise \(120^\circ\) rotation. From example, \(u_{12} = w_{56}\). Then we can get the weight \(v'_{ij}\) and \(v'^{kl}\) for \([\sigma_1(f(\gamma))]\) from \(u_{ij}\) and \(u^{kl}\). Now, we switch the indices to have the weights \(v_{ij}\) and \(v^{kl}\) for \([\sigma_3(\gamma)]\) by using the permutation \((123456)^{(6-4)} = (135)(246)\) for counterclockwise \(120^\circ\) rotation.

Now, we can calculate the weights of \(G^{-1}F(\partial E_i)\) in the hexagon parameterization by using the formulas given in this section. We need 30 parameters with integer entries to express a simple closed curve \(\gamma\). We notice that it is possible to have a very long length sequence of five generators of \(\pi_1(\Sigma_{0,6})\) to express \(\gamma\) if we use the fundamental group argument. For example, consider a simple closed curve \(\gamma\) which has weights \(w_{16} = w_{45} = w_{14} = w_{56} = 1, w_{15} = w_{15} = 20001\) and all the other weights are zero. However, we need a 40004 length sequence of five generators of \(\pi_1(\Sigma_{0,6})\) to express \(\gamma\).

Despite this benefit, it is difficult to know whether \(\gamma\) bounds an essential disk or not from the hexagon parameterization. So, we will use the Dehn parameterization as the follows.
5. Step 2-1: Dehn parameterization of $C$

Let $\gamma$ be a simple closed curve in $\Sigma_{0,6}$. Consider the pair of pants $I := \partial B^3 - \{E'_1 \cup E'_2 \cup E'_3\}$.

Figure 16 shows standard arcs $l_{ij}$ in the pair of pants $I$. We notice that we can isotope $\gamma$ into $\delta$ in $\Sigma_{0,6}$ so that each component of $\delta \cap I$ is isotopic to one of the standard arcs and $\delta \cap \partial E_i \subset \omega_i$. Then we say that subarc $\alpha$ of $\delta$ is carried by $l_{ij}$ if some component of $\alpha \cap I$ is isotopic to $l_{ij}$. The closed arc $\omega_i \subset \partial E_i$ is called a window.

Let $I_i = |\delta \cap \omega_i|$. Then $\delta$ can have many parallel arcs which are the same type in $I$. Let $x_{ij}$ be the number of parallel arcs of the type $l_{ij}$ which is called the weight of $l_{ij}$.

Now, consider $E'_1$. Let $j_1$ and $k_1$ be the simple arcs as in Figure 17. We assume that $\partial E'_1 \cup \delta \cup j_1 \cup k_1 \cup l_1$ has no bigon in $\Sigma_{0,6}$. We note that $j_1 \cup k_1 \cup l_1$ separates $E'_1$ into two semi-disks $E'_1^+$ and $E'_1^-$ as in Figure 17. Let $u_{ij}^+$ be the number of subarcs of $\delta$ from $l_1$ to $j_1$ in $E'_1^+$. Also, let $v_{ij}^+$ be the number of subarcs of $\delta$ from $l_1$ to $k_1$ in $E'_1^+$ and let $w_{ij}^+$ be the number of subarcs of $\delta$ from $j_1$ to $k_1$ in $E'_1^+$. Let $m_1 = |\delta \cap j_1|$ and $n_1 = |\delta \cap k_1|$ in $E'_1$. For example, in the third diagram of Figure 17 we have $u_1^+ = 0, v_1^+ = 3, w_1^+ = 4, m_1 = 4$ and $n_1 = 7$.

We notice that each component of $\delta \cap E'_1$ meets $l_1$ exactly once. Also, we know that each such component is essential in $E'_1 - \{1, 2\}$.

The components of $\delta \cap E'_1$ are determined by three parameters $p_1, q_1, t_1$ as in Figure 17 where $p_1 = \min\{|\delta' \cap l_1||\delta' \sim \delta \text{ in } \Sigma_{0,6}\}$, $q_1 \in \mathbb{Z}$, $0 \leq q_1 < p_1$. In order to define $q_1$ and $t_1$, consider $m_1$ and $n_1$. Then we know that $u_{ij}^+ + v_{ij}^+ = p_1$. So, $m_1 - n_1 = (u_{ij}^+ + w_{ij}^+) - (v_{ij}^+ + w_{ij}^+) = u_{ij}^- - v_{ij}^-$. Therefore, $-p_1 = -u_{ij}^- - v_{ij}^- \leq u_{ij}^+ - v_{ij}^+ = m_1 - n_1 = u_{ij}^+ - v_{ij}^- \leq u_{ij}^+ + v_{ij}^+ = p_1$. So, we know $-p_1 \leq m_1 - n_1 \leq p_1$. Now, we define $q_1$ and $t_1$ as follows. If $n_1 - m_1 = p_1$ then $q_1 \equiv m_1 \pmod{p_1}$ and $0 \leq q_1 < p_1$, and $t_1 = \frac{m_1 - q_1}{p_1}$ and if $-p_1 \leq n_1 - m_1 < p_1$ then $q_1 \equiv -m_1 \pmod{p_1}$ and $0 \leq q_1 < p_1$, and $t_1 = \frac{-m_1 - q_1}{p_1}$. Then $t_1$ is called the twisting number in $E'_1$. Also, let $(p_1, q_1, t_1)$ be the three parameters to determine the arcs in $E'_1$. Similarly, we have...
the three parameters \((p_i, q_i, t_i)\) for \(E'_i\) \((i = 2, 3)\). Then \(\gamma\) is determined by a sequence of nine parameters \((p_1, q_1, t_1, p_2, q_2, t_2, p_3, q_3, t_3)\) by Lemma 5.1.

**Lemma 5.1.** \(I_i\) \((i = 1, 2, 3)\) determine the weights \(x_{jk}\). \((j, k \in \{1, 2, 3\})\)

**Proof.** We have two subcases for this. First, suppose that \(I_i < I_j + I_k\) for all distinct \(i, j, k \in \{1, 2, 3\}\). We claim that \(x_{11} = x_{22} = x_{33} = 0\). If not, then \(x_{ii} > 0\) for some \(i\). We notice that \(x_{jj} = x_{kk} = 0\). So we have \(I_i = 2x_{ii} + x_{ij} + x_{jk}, I_j = x_{ij} + I_k = x_{ik}\). This shows that \(2x_{ii} + x_{ij} + x_{jk} < x_{ij} + x_{ik}\). This makes a contradiction. So, \(x_{11} = x_{22} = x_{33} = 0\).

Now, we have \(I_i = x_{ij} + x_{ik}\). This implies that \(x_{ij} = \frac{I_i + I_j - I_k}{2}\).

Now, suppose that \(I_i > I_j + I_k\) for some \(i\). Then we notice that \(\gamma\) has \(I_i = 2x_{ii} + x_{ij} + x_{ik}, I_j = x_{ij}\) and \(I_k = x_{ik}\).

This implies that \(x_{ij} = I_j, x_{ik} = I_k\) and \(x_{ii} = \frac{I_i - I_j - I_k}{2}\). \(\Box\)
Recall that $C$ is the set of isotopy classes of simple closed curves in $\Sigma_{0,6}$. For a given simple closed curve $\delta$ in a hexagon diagram, we define $p_i$, $q_i$ and $t_i$ in $E'_i$ as above. Then let $q'_i = p_i t_i + q_i$ for $i = 1, 2, 3$.

**Theorem 5.2** (Special case of Dehn’s Theorem). There is an one-to-one map $\phi : C \to \mathbb{Z}^6$ so that $\phi(\delta) = (p_1, p_2, p_3, q'_1, q'_2, q'_3)$. i.e., it classifies isotopy classes of simple closed curves.

When $p_1 = p_2 = p_3 = 0$ then $t'_i = 1$ if the simple closed curve is isotopic to $\partial E'_i$ and $t'_j = 0$ if $j \neq i$. Refer [5] to see the general Dehn’s theorem.

We will use a sequence of nine parameters instead of six parameters for convenience.

6. Step 2-2: Hexagon diagram and Dehn diagram

Let $(p_1, q_1, t_1, p_2, q_2, t_2, p_3, q_3, t_3)$ be the nine parameters of $\gamma$ in $\Sigma_{0,6}$. Assume that $\gamma$ bounds an essential disk $A$ in $B^3 - \epsilon$. Then we notice that $|\gamma \cap l_i|$ is an even number and $H_i$ cannot contain a bigon component of the closures of $A - E$. If $H_i$ contains a bigon $\Delta$, then $\Delta$ will meet $\epsilon_i$. This makes a contradiction that $A$ is an essential disk in $B^3 - \epsilon$.

To have an essential disk $A$ which is not parallel to one of the $E_i$, at least two components of the closures of $A - E$ are bigon in $P$ since $H_i$ cannot have a bigon. So, $\gamma$ has $l_{ii}$ arcs in $I$ for some $i$. We notice that we cannot have $l_{ii}$ and $l_{jj}(j \neq i)$ at the same time. We assume that $x_{11} > 0$ and $x_{22} = x_{33} = 0$. If not, then we rotate $\gamma$ a multiple of $120^\circ$ about the center of $H$ to have $x_{11} > 0$. Note that the $120^\circ$ rotation preserves $\infty$ tangle. Essential curves obtained by a sequence of nine parameters The following lemma is very useful to simplify the sequence of parameters of $\gamma$.

**Lemma 6.1.** Let $N$ be an essential disk in $B^3 - \epsilon$ and $h$ be the clockwise half Dehn twist supported on $N'$ which is the 2-punctured disk in $\Sigma_{0,6}$ so that $\partial N = \partial N'$. Then $\gamma$ bounds an essential disk in $B^3 - \epsilon$ if and only if $h(\gamma)$ bounds an essential disk in $B^3 - \epsilon$.

**Proof.** Let $B_1$ and $B_2$ be the closures of two components of $B^3 - N$. Assume that $B_1$ contains one arc of $\epsilon_i$ for $i = 1, 2, 3$. Consider an extended homeomorphism $H^{-1}$ of $h^{-1}$ from $(B^3, H(\epsilon))$ to $(B^3, \epsilon)$ so that $H(N) = N$ and $H|_{B_2} = id_{B_2}$. Then, $H^{-1}$ interchanges the
endpoints of the properly embedded arc in $B_1$ without changing the tangle type. So, we know that $(B^3, \epsilon) \approx (B^3, H(\epsilon))$. Now, we know that there exists $i$ so that $E_i'$ contains the two punctures of $N'$ since $N$ is an essential disk in $B^3 - \epsilon$. Let $K_1$ and $K_2$ be the closure of two components of $B^3 - E_i$. Actually, $K_1 = H_i$ and $K_2 = B^3 - H_i$. Let $M_1$ and $M_2$ be the closure of two components of $B^3 - N$. We assume that $K_1$ and $M_1$ contains only the same two punctures. Now, we can construct a homeomorphism $J$ from $(B^3, \epsilon)$ to $(B^3, \epsilon)$ so that $J(K_1) = M_1$, $J(K_2) = M_2$ and $J(\epsilon_i) = \epsilon_i$ for $i = 1, 2, 3$. So, we know that $(B^3, \epsilon) \approx (B^3, J(\epsilon)) = (B^3, \epsilon)$. This implies that $(B^3, H^{-1}(\epsilon)) \approx (B^3, J(\epsilon))$. By using Theorem 3.2, we know that $(H^{-1})^{-1} J(\partial E_i) = H(\gamma) = h(\gamma)$ bounds an essential disk in $B^3 - \epsilon$.

To see the other direction, we consider $h^{-1}$ which is the counter-clockwise half Dehn twist supported on $N'$.

By using this lemma, we notice that a simple closed curve $\gamma'$ which is parameterized by $(p_1, q_1, 0, p_2, q_2, 0, p_3, q_3, 0)$ bounds an essential disk in $B^3 - \epsilon$ if only if $\gamma$ does.

Now, we will discuss how to modify $\gamma$ into $\gamma'$ which is parameterized by $(p_1, q_1, 0, p_2, q_2, 0, p_3, q_3, 0)$.

![Figure 19. Pseudo-hexagon diagram](image)

In $\Sigma_{0,6}$, we choose windows $\omega_i$ and three two punctured disks $E'_i$ to have a pseudo-hexagon diagram as in Figure 19. We isotope $\gamma$ into $\delta$ in $\Sigma_{0,6}$ so that all components of $\delta \cap I$ are parallel to one of the standard arcs as in Figure 19. i.e., $l_{12}$ and $l_{13}$ lie in $H$ and $l_{11}$ meets the hexagon at exactly two times.

Now, consider the graph $(\cup_{j=1}^{6} a_j) \cup \partial E_i' \cup \delta$. Then we assume that $\partial E_i' \cup \partial \cup (E_i' \cap \cup_{j=1}^{6} a_j)$ has no bigon for $i = 1, 2, 3$. Then let $v_{ij}$ be the number of arcs for $\delta$ from $a_i$ to $a_j$ in $H$ and let $v^{ij}$ be the number of arcs for $\delta$ from $a_i$ to $a_j$ in $H^c$.

**Lemma 6.2.** Suppose that $\delta$ is a simple closed curve in pseudo-hexagon diagram for which $x_{11} > 0$. Then $v_{11} = v_{33} = v_{44} = v_{55} = v_{35} = 0$ and $v^{ii} = 0$ for all $i = 1, 2, 3, 4, 5, 6$.

**Proof.** Suppose that $v_{11} > 0$. Then there is a component $\alpha$ of $\delta \cap H$ with $\partial \alpha \subset a_1$. We notice that $\alpha$ cannot be carried by $l_{11}$ since $l_{11}$ arc meets $a_6$ and $a_4$. If $\alpha$ is carried by $l_{12}$, then $\alpha$ needs to meet $\omega_2$ at least two times. However, $\alpha$ cannot meet $a_4$, $a_5$ or $a_6$ in $E_i'$. Therefore,
there exists an arc which is parallel in \( E'_2 \) to a subarc of \( \omega_2 \). So, we can isotope the arc out of \( E'_2 \). This makes a bigon in \( E'_2 \). This contradicts the definition of a pseudo-hexagon diagram. Therefore, \( \alpha \) cannot be carried by \( l_{12} \). Similarly, \( \alpha \) cannot be carried by \( l_{13} \). If it cannot be carried by \( l_{11}, l_{12} \) or \( l_{13} \) then \( \alpha \) is parallel in \( E'_1 \) to an arc in \( a_1 \). This contradicts the fact that each component of \( \delta \cap E'_1 \) meets \( a_1 \) exactly once.

Suppose that \( v_{33} > 0 \). Then there is a component \( \alpha \) of \( \delta \cap H \) with \( \partial \alpha \subset a_3 \). We notice that \( a_3 \subset E''_3 \). So, \( \partial \alpha \) is in \( E''_3 \). If \( \alpha \) is carried by \( l_{13} \), \( \alpha \) needs to pass through \( \omega_1 \). However, it cannot come back to \( \omega_1 \) without meet \( a_6, a_1 \) or \( a_2 \) because it is essential in \( E'_1 \setminus \{1, 2\} \). Therefore, \( \alpha \) is not carried by \( l_{13} \). If \( \alpha \) cannot be carried by \( l_{13} \) then \( \alpha \) is parallel in \( E'_3 \) to an arc in \( a_3 \). This contradicts the fact that each component of \( \delta \cap E'_3 \) meets \( a_3 \) exactly once. With a similar argument, we also can show that \( v_{55} = 0 \).

Suppose that \( v_{44} > 0 \). Then there is a component \( \alpha \) of \( \delta \cap H \) with \( \partial \alpha \subset a_4 \). If \( \alpha \) cannot be carried by \( l_{23} \), then \( \alpha \) is parallel in either \( E''_2 \) or \( E'_3 \) to an arc in \( a_5 \) or \( a_3 \) respectively. This contradicts the fact that each component of \( \delta \cap E'_2 \) or \( \delta \cap E'_3 \) meets \( a_5 \) or \( a_3 \) exactly once. So, \( \alpha \) needs to be carried by \( l_{23} \). However, we know that \( x_{23} = 0 \) since \( x_{11} > 0 \). This implies that \( v_{44} \) also should be zero.

Suppose that \( v_{35} > 0 \). Let \( \alpha \) be an arc for \( v_{35} \). Then the endpoints lie in both \( E'_2 \) and \( E'_3 \). Therefore, \( \alpha \) need to be carried by \( l_{23} \). However, \( x_{23} = 0 \) since \( x_{11} > 0 \). This implies that \( v_{35} = 0 \).

Suppose that \( v_{11} > 0 \). Then there is a component \( \alpha \) of \( \delta \cap H^c \) with \( \partial \alpha \subset a_1 \). We notice that \( \alpha \) is parallel in \( E'_1 \) to an arc in \( a_1 \). This contradicts the fact that each component of \( \delta \cap E'_1 \) meets \( a_1 \) exactly once. Therefore, \( v_{11} = 0 \). With a similar argument, we can show that \( v_{33} = v_{55} = 0 \).

Suppose that \( v_{22} > 0 \). Then there is a component \( \alpha \) of \( \delta \cap H^c \) with \( \partial \alpha \subset a_2 \). We notice that both of the endpoints of \( \alpha \) lies in \( E'_1 \) or \( E'_3 \) since there is no subarc of \( \delta \) from \( E'_1 \) to \( E''_2 \), from \( E'_1 \) to \( E'_3 \) or from \( E'_2 \) to \( E'_3 \) in \( H^c \). So, \( \alpha \) is parallel in \( E'_1 \) or \( E'_3 \) to an arc in \( a_2 \). So, we can isotope \( \alpha \) in \( E'_1 \) or \( E'_3 \) to reduce the intersection number of \( \delta \) with \( \cup_{i=1}^{6} a_i \). This contradicts that there is no bigon in \( E'_3 \) in pseudo-hexagon diagram. Therefore, \( v_{22} = 0 \).

Similarly, we can show that \( v_{44} = v_{66} = 0 \).

This lemma shows that \( v_{22} \) and \( v_{66} \) are the only \( v_{ii} \) which might be positive integers.

Now, we take new windows \( \omega'_i \) as in Figure 20.

Then, we can isotope all arcs for \( v_{22} \) and \( v_{66} \) by pushing across \( a_2 \) and \( a_6 \) respectively to have a new simple closed curve \( \eta \) with windows \( \omega'_i \). Let \( u_{ij} \) be the number of arcs of \( \eta \) from \( a_i \) to \( a_j \) in \( H \). Also, let \( u^{ij} \) be the number of arcs of \( \eta \) from \( a_i \) to \( a_j \) in \( H^c \).

**Lemma 6.3.** Suppose that \( \delta \) is a simple closed curve in pseudo-hexagon diagram. Then \( \eta \) has \( u_{ii} = u^{ii} = 0 \) for all \( i = 1, 2, 3, 4, 5, 6 \).
Proof. We notice that all arcs for $v_{ij}$ are essential in $H$ if $i \neq j$. Also, all arc for $v^{ij}$ are essential in $H^c$ if $i \neq j$. By Lemma 6.2, we know that $v^{ii} = 0$ for all $i$ and $v_{22}$ and $v_{66}$ are the only $v_{ii}$ which might be positive integers.

Now, let $\alpha_1$ be an arc for $v_{22}$. Then, we claim that $\alpha_1$ is carried by $l_{13}$. If $\alpha_1$ is not carried by $l_{13}$ then the endpoints lie in either $E'_1$ or $E'_3$. So, $\alpha_1$ is parallel in $E'_1$ or $E'_3$ to an arc in $a_2$. So, we can isotope $\alpha_1$ in $E'_1$ or $E'_3$ to reduce the intersection of $\delta$ with $a_2$. This contradicts the fact that there is no bigon in $E'_1$ in a pseudo-hexagon diagram. Therefore, $\alpha_1$ is carried by $l_{13}$. Let $x$ be the endpoint of $\alpha_1$ in $E'_1$ and $y$ be the endpoint of $\alpha_1$ in $E'_3$. Let $\beta_1$ be the component of $(\delta - \alpha_1) \cap E'_1$ which contains $x$. Then let $\theta_1$ be the component of $\beta_1 \cap H^c$ which contains $x$. Then the other endpoint $z_1$ of $\theta_1$ lies on either $a_3$ or $a_6$. Similarly, Let $\beta_2$ be the component of $(\delta - \alpha_1) \cap E'_3$ which contains $y$ and $\theta_2$ be the component of $\beta_2 \cap H^c$ which contains $y$. Then the other endpoint $z_2$ of $\theta_2$ lies on either $a_3$ or $a_4$.

Therefore, by pushing $\alpha_1$ across $a_2$ we get an arc for either $u^{13}$, $u^{14}$, $u^{63}$ or $u^{64}$. We notice that each arc for $u^{13}$, $u^{14}$, $u^{63}$ or $u^{64}$ is essential in $H^c$.

Now, let $\alpha_2$ be an arc for $v_{66}$. Then, we know that $\alpha_2$ is carried by $l_{11}$ or $l_{12}$. Let $x'$ be the endpoint of $\alpha_2$ in $E'_1$ and $y'$ be the endpoint of $\alpha_2$ in $E'_3$. Then let $\beta'_1$ be the component of $(\delta - \alpha_2) \cap E'_1$ which contains $x'$ and $\theta'_1$ be the component of $\beta'_1 \cap H^c$ which contains $x'$. Then the other endpoint $z'_1$ of $\theta'_1$ lies on either $a_1$ or $a_2$. Similarly, let $\beta'_2$ be the component of $(\delta - \alpha_2) \cap (E'_3)^c$ which contains $y$ and let $\theta'_2$ be the component of $\beta'_2 \cap H^c$ which contains $y$. Then the other endpoint $z'_2$ of $\theta'_2$ lies on either $a_4$ or $a_5$.

Therefore, by pushing $\alpha_2$ across $a_3$ we get an arc for either $u^{14}$, $u^{15}$, $u^{24}$ or $u^{25}$. We notice that each arc for $u^{14}$, $u^{15}$, $u^{24}$ or $u^{25}$ is essential in $H^c$. It is possible to use an arc $\alpha_3$ for $u^{26}$ to have a new arc $\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3$ so that $|\alpha_1 \cap \alpha_3| = |\alpha_2 \cap \alpha_3| = 1$. So, if we push $\alpha_1$ and $\alpha_2$ across $a_2$ and $a_6$ respectively then we have a new arc which might have $\partial \alpha$ in $a_4$. However, we notice that $\alpha \cap E'_1$ is a non-essential arc in $E'_1 - \{1, 2\}$. This contradicts the fact that each component of $\delta \cap E'_1$ is essential in $E'_1 - \{1, 2\}$. Therefore, this case cannot be happen. This implies that $u_{ii} = 0$ and $u^{ii} = 0$ for all $i$. This completes the proof of this lemma. \□
Then the following lemma is also true.

**Lemma 6.4.** $u_{ij}$ and $u^{ij}$ ($i, j \in \{1, 2, 3, 4, 5, 6\}$) are the weights for a hexagon diagram of $\delta$.

**Proof.** By Lemma [4.3](#) and Lemma [6.3](#), we get this lemma. □

Recall that $I_i = 2(\sum_{k=1}^{6} w_{ki}) = 2p_i$ for $i = 1, 2, 3$. Then by Lemma [5.1](#) we can calculate the weights $x_{ij}$ of $l_{ij}$. We remark that $x_{ij}$ only depends on $p_1$, $p_2$ and $p_3$.

It is clear that $p_i = \frac{1}{2}$. But, it is difficult to find $q_i$ and $t_i$ together. So, I want to find $q_i$ by making $t_i$ zero for all $i$. Actually, if $t_i = 0$ for all $i$, then we can find $q_i$ as follows.

**Lemma 6.5.** Let $\delta$ be a simple closed curve in a Dehn diagram. Let $w_{ij}$ and $w^{ij}$ be the weights of $\delta$ in a hexagon diagram.

Suppose that $t_1 = t_2 = t_3 = 0$. Then $q_1 = w^{26}$, $q_2 = w^{46} - x_{11}$ and $q_3 = w^{24}$.

**Proof.** Consider the pseudo-hexagon diagram of $\delta$ with the weights $v_{ij}$ and $v^{ij}$. Suppose that $t_1 = t_2 = t_3 = 0$. We notice that the graph $\partial E_i'$, $(E_i' \cap (a_1 \cup a_2 \cup a_6)) \cup \delta$. Then consider a subarc $C$ of $H$ so that one of the endpoints of $C$ is on $\omega_1$. Then the other endpoint should be on $a_2$ since $t_1 = 0$. This implies that $v_{66} = 0$. Similarly, we see that $v_{22} = 0$ since $t_3 = 0$. So, $v_{ii} = v^{ii} = 0$ for all $i = 1, 2, \ldots, 6$ by Lemma [6.2](#). Therefore, $v_{ij} = w_{ij}$ and $v^{ij} = w^{ij}$ for $i, j \in \{1, 2, 3, 4, 5, 6\}$. By definition of $q_i$, we have $q_1 = m_1 = v^{26} = w^{26}$ since $t_1 = 0$.

Let $m_2 = |\delta \cap j_2|$ and $n_2 = |\delta \cap k_2|$ in $E_2'$. Also, let $m_3 = |\delta \cap j_3|$ and $n_3 = |\delta \cap k_3|$ in $E_3'$. Then, we also know that $v^{46} - l_{11} = m_2$ and $v^{24} = m_3$. Then this implies that $q_2 = m_2 = v^{46} - l_{11}$ and $q_3 = m_3 = v^{24}$ since $t_2 = t_3 = 0$.

We say that $\gamma$ is right-twisted (or left-twisted) in $E_i'$ if $t_i > 0$ (or $t_i < 0$). If we know $\delta$ is right-twisted (or left-twisted) in $E_i'$, then we apply a half Dehn twist supported on $E_i'$ to decrease (or increase) the twisting number $t_i$ until the simple closed curve is not twisted.

**Lemma 6.6.** $\gamma$ is left-twisted in $E_i'$ if and only if $u_1^+ > 0$.

**Proof.** If $\gamma$ is left-twisted in $E_i'$ then $t_i < 0$ by definition. Then $-p_1 \leq m_1 - n_1 < p_1$. We claim that $u_1^+ > 0$. If $u_1^+ = 0$, then we know that $v_1^+ = p_1$. So, $n_1 - m_1 = v_1^+ - u_1^+ = p_1$. This contradicts that $-p_1 \leq n_1 - m_1 < p_1$. Therefore, $u_1^+ > 0$. Now, suppose that $u_1^+ > 0$ to show the other direction. Then we know that $-p_1 \leq n_1 - m_1 < p_1$. This implies that $t_1 < 0$ by the definition of $t_i$. Therefore, $u_1^+ > 0$ if and only if $\gamma$ is left-twisted in $E_i'$.

**Lemma 6.7.** Suppose that $\gamma$ has $x_{11} > 0$. $\gamma$ is left-twisted in $E_i'$ if and only if $\gamma$ has either

1. $w^{15} + w^{16} > 0$
2. $w^{15} = w^{16} = 0$, $w^{14} > 0$ and $w^{45} + w^{46} < w_{45} + w_{46} + w_{14}$, or
3. $w^{15} = w^{16} = 0$, $w^{14} > 0$, $w_{45} + w_{46} + w_{14} \leq w^{45} + w^{46} < w_{45} + w_{46} + w_{14} + w_{24}$ and $w_{26} + w_{25} + (w^{45} + w^{46} - (w_{45} + w_{46} + w_{14})) < w^{12}$.

**Proof.** ($\Rightarrow$) Suppose that $\gamma$ has $x_{11} > 0$ and $\gamma$ is left-twisted in $E_i'$ with $w^{15} = w^{16} = 0$.

Then $\gamma$ has $u_1^+ > 0$ by Lemma [6.6](#) since $\gamma$ is left-twisted. Let $C$ be the innermost arc for $u_1^+$ with the endpoints $x_1$ and $x_2$ which lie on $l_1$ and $j_1$ respectively. Then there is a
component $\alpha_1$ of $(\gamma - C) \cap H^c$ so that one of endpoints of $\alpha_1$ is $x_2$. Then, the other endpoint $x_3$ lies on either $a_6$, $a_5$ or $a_4$. Because $w^{15} = w^{16} = 0$, we notice that $x_3$ lies on $a_4$. This implies that $w^{14} > 0$. We know that $\alpha_1$ is carried by either $l_{11}$ or $l_{12}$. We also know that $w^{26} = 0$ because of $C \cup \alpha$. We notice that $C \cup \alpha$ is also the innermost arc for $w^{14}$ since $C$ is the innermost arc for $u^+_1$.

Because $x_{23} = 0$, we know that $x_3$ goes along a component $\alpha_2$ of $(\gamma - (C \cup \alpha_1)) \cap H$ and the other endpoint $x_4$ of $\alpha_2$ lies on either $a_5$, $a_6$, $a_1$ or $a_2$. This implies that $w^{45} + w^{46} < w_{45} + w_{46} + w_{14} + w_{24}$. Especially, if $x_4$ lies on $a_2$ then we need a condition that $w_{45} + w_{46} + w_{14} \leq w^{45} + w^{46} < w_{45} + w_{46} + w_{14} + w_{24}$. In this case, $x_4$ needs to go along a component $\alpha_3$ of $(\gamma - (C \cup \alpha_1 \cup \alpha_2)) \cap H^c$ and the other endpoint $x_5$ of $\alpha_3$ should be on $a_1$ since $w^{26} = 0$. Therefore, we have $w_{26} + w_{25} + w_{21} + (w^{45} + w^{46} - (w_{45} + w_{46} + w_{14})) < w^{12}$. We notice that $w_{21} = 0$ since $w^{12} > 0$. Finally, $w_{26} + w_{25} + (w^{45} + w^{46} - (w_{45} + w_{46} + w_{14})) < w^{12}$.

$(\Leftarrow)$ Suppose that $\gamma$ is not left-twisted and $w^{16} = w^{15} = 0$, $w^{14} > 0$ and $w_{45} + w_{46} + w_{14} \leq w^{45} + w^{46} < w_{45} + w_{46} + w_{14} + w_{24}$. Since $\gamma$ is not left-twisted, we know that $n_1 - m_1 = v^+_1 = p_1$.

Let $C$ be the outermost arc for $u^+_1$ with the endpoints $x_1$ and $x_2$ as in Figure 22. Let $\alpha_1$ be the component of $(\gamma - C) \cap H^c$ so that $x_2$ is one of the endpoints of $\alpha_1$. We notice that $C \cup \alpha_1$ needs to be the outermost arc for $w^{14}$. Let $x_3$ be the other endpoint of $\alpha_1$. Then $x_3$ should be on $a_4$.

$x_3$ needs to go alone a component $\alpha_2$ of $(\gamma - (C \cup \alpha)) \cap H$. Then the other endpoint $x_4$ of $\alpha_2$ should be on $a_2$ since $w_{45} + w_{46} + w_{14} \leq w^{45} + w^{46} < w_{45} + w_{46} + w_{14} + w_{24}$. After that, $x_4$ continues to go along a component $\alpha_3$ of $(\gamma - (C \cup \alpha_1 \cup \alpha_2)) \cap H^c$ and the other endpoint $x_5$ of $\alpha_3$ lies on either $a_3$ or $a_4$. Then, we need an equality $w_{26} + w_{25} + (w^{45} + w^{46} - (w_{45} + w_{46} + w_{14})) \geq w^{12}$. This completes the proof of lemma 6.7. \hfill \Box

If $\gamma$ is left-twisted in $E'_1$, then we apply the counter clockwise half Dehn twist supported on $E'_1$ to $\gamma$. If $\gamma$ is not left-twisted in $E'_1$, then we notice that $t_1 = 0$ or $t_1 > 0$. So, if we apply the clockwise half Dehn twist supported on $E'_1$ to $\gamma$ then the modified simple closed
curve $\gamma'$ is either left-twisted in $E'_1$ or not. If not, then we continue to apply the clockwise half Dehn twist to $\gamma'$. If $\gamma'$ is left-twisted in $E'_1$ then we notice that $\gamma$ has $t_1 = 0$. Therefore, this lemma is enough to know if a simple closed curve has $t_1 = 0$ or not.

Now, we need the following lemma to make $t_2$ zero.

**Lemma 6.8.** Suppose that $x_{11} > 0$ and $t_1 = 0$. $\gamma$ is left-twisted in $E'_2$ if and only if $w_{45} > 0$.

**Proof.** We notice that $u_{31} = w_{45}$ if $t_1 = 0$. By a similar argument in lemma 6.7 we see that $\gamma$ is left-twisted in $E'_2$ if and only if $w_{45} > 0$.

In order to check $\gamma$ is left-twisted in $E'_3$, we need the following lemma.

**Lemma 6.9.** Suppose that $x_{11} > 0$ and $t_1 = t_2 = 0$. $\gamma$ is left-twisted in $E'_3$ if and only if $w_{13} + w_{14} + w_{63} + w_{64} > 0$.

**Proof.** By using a similar argument in lemma 6.7 $\gamma$ is left-twisted in $E'_3$ if and only if $u_{31} > 0$. Let $C$ be an arc for $u_{31}$.

$(\Rightarrow)$ Suppose that $w_{13} = w_{14} = w_{63} = w_{64} = 0$, then $C$ cannot be a subarc for any $w_{ij}$. Therefore, $\gamma$ is not left-twisted.

$(\Leftarrow)$ Suppose that $\gamma$ is not left-twisted in $E'_3$ then we know that $w_{13} = w_{14} = w_{63} = w_{64} = 0$.

With the three previous lemmas, we can get $\gamma'$ by applying appropriate half Dehn twists to $\gamma$ so that $\gamma'$ has the same $p_i$ and $q_i$, but $t_i = 0$ for all $i$. Also, we know that $\gamma'$ bounds an essential disk in $B^3 - \epsilon$ if and only if $\gamma$ does by Lemma 6.1.

7. **Step 3-1: Pattern diagram of $\gamma'$ in $I'$**

Now, we modify $\gamma'$ into $\gamma_0$ which is in standard position to access the main theorem.
Recall that $E''_i$ is a concentric two punctured disk in the interior of $E'_i$ so that all the components of $\partial A \cap E''_i$ are parallel simple arcs as in Figure 17. Let $I' = S^2 - \{E''_1 \cup E''_2 \cup E''_3\}$. Now, take equators $e_i$ for each $E''_i$ as in Figure 23. Then we can divide $E''_i$ into $E''_i^+$ and $E''_i^-$ by $e_i$.

Recall the fact that we need to have $x_{ii} > 0$ for some $i$ to have an essential disk in $B^3 - \epsilon$ which is not parallel to one of the $E_i$, where $x_{ij}$ is the weight of $l_{ij}$ and $l_{ij}$ is the standard arc from the window $\omega_i$ to the window $\omega_j$. So, we assume that $x_{11} > 0$ and $x_{22} = x_{33} = 0$ without loss of generality.

Let $\gamma$ be a simple closed curve which bounds an essential disk in $B^3 - \epsilon$ and has $x_{11} > 0$. Now, we define a pattern diagram of $\gamma$ in $I'$ which has 11 types of essential arcs in $I'$ as in Figure 23.

The given number shows each type of arc. For example, type 1 is for an arc from $E''_1^+$ to $E''_1^-$. These are patterns of connectivity, not isotopy classes of arcs.

We will discuss the relation between the hexagon diagram and the pattern diagram in Section 8.

8. Step 3-2: Standard diagram of $\gamma_0$ in $I'$

Figure 23. Pattern diagram

Figure 24.
From a simple closed curve $\gamma$ which bounds an essential disk in $B^3 - \epsilon$, we want to construct a new simple closed curve $\gamma_0$ which may be in a different isotopy class in $\Sigma_{0,6}$, but $\gamma_0$ bounds an essential disk in $B^3 - \epsilon$ if and only if $\gamma$ does. Especially, every component of $\gamma_0 \cap I'$ is isotopic to one of the given arc types in one of the diagrams in Figure 24. The two diagrams in Figure 24 are called standard diagrams and a simple closed curve $\gamma_0$ is in standard position if $\gamma_0$ is obtained from one of the standard diagrams by putting weights on these arcs.

In this section, we show how $\gamma_0$ is obtained from $\gamma$ by having a certain properly chosen $t_i$ and the same $p_i$ and $q_i$.

![Figure 25. Standard diagram](image)

First, we modify the hexagon diagram into the new diagram as in Figure 25.

We will let $m_k$ be the number of parallel arcs of $\gamma_0 \cap I'$ that are isotopic to the arc type $k$ in the standard diagram. Then we say that $m_k$ is the weight of the arc type $k$ in the standard diagram. We note that it is possible to have two non-isotopic arcs in $I'$, but they are the same arc type $k$. So, we define $m_{k_1}$ and $m_{k_2}$ of the weights for the two different isotopic arcs with the same arc type $k$ so that $m_{k_1} + m_{k_2} = m_k$ if they exist. Then we have a sequence of weights $m_i$ for $\gamma_0$. It is called a standard parameterization. I want to mention the fact that $m_k$ is going to be computed from the Dehn parameters as described in the following pages.

Now, we will show the following lemmas.

**Lemma 8.1.** Suppose that $\gamma_0$ is a simple closed curve which bounds an essential disk $A$ in $B^3 - \epsilon$ and it is in standard position with $x_{11} > 0$. Then $m_1 + m_3 > 0$.

Consider the standard diagram with $m_2 > 0$, $m_1 = m_3 = 0$ as in Figure 26. By referring to (1) in Figure 24 we can have two possible diagrams to have $m_2 > 0$. We note that the other diagram can be obtained by reflecting the given diagram about the horizontal axis which is passing through the middles of $E_i''$. Then we have a similar argument to check the
Let $r_i$ be the simple arcs as in Figure 26. Then we note that the $r_i \cup \varepsilon_i$ bound three disjoint disks in $B^3$. So, by considering the intersections between $\partial A$ and $r_i$ we can calculate the element $[\partial A]$ of $\pi_1(B^3 - N^\circ(\varepsilon), x_0)$ as in Figure 26.

Now, we consider all the possible cases for the element $[\partial A]$ with respect to the three generators $a, b, c$ of $\pi_1(B^3 - N^\circ(\varepsilon), x_0)$.

Consider the sequence of arc types in the standard diagram which carries $\partial A$. Then, we note that a subarc of $\partial A$ which is carried by one of the arcs in the standard diagram meets at most two times with $r_1 \cup r_2 \cup r_3$. The type 2 and 11 meet only $r_1$, the type 7 meets only $r_2$ and the type 8 meets only $r_3$, but the type 10 meets $r_1$ and $r_3$ to have $(ac^{-1})^{\pm 1}$ with respect to the generators.

We note that the arc type 2 carries $\partial A$ since $m_2 > 0$. So, we take the base point $x_0$ in $\partial E''_1$ as in Figure 26 and start with a subarc of $\partial A$ which is carried by the arc type 2. Then we have $a^{\pm 1}$ with respect to the generators. Let $i_1$ be the upper arc type $i$ and $i_2$ be the lower arc type $i$ as in Figure 26. We note that if we show that $[\partial A]$ with the given orientation is not trivial then $[\partial A]$ with the other orientation is also not trivial.

Now, we give an orientation to each arc type $i$ as in Figure 26. We say an arc type $7$ if the arc type $i$ has the opposite orientation of $i$.

We define a path $p$ of $\partial A$ so that $p \subset \partial A - E''_1$ and $|p \cap E''_1| = 2$. We note that a path $p$ can break $E''_2$ or $E''_3$. A path $p$ is carried by some arc types. So, a path $p$ is represented by an arc type or a sequence of two types. For example, 2 and 9 $8_1$ stand for paths. We also note that each path generates non-trivial element with respect to the generators.

Figure 26.
Now, we consider all the cases for consecutive paths of length 2 to check whether or not there is a cancellation between two paths as follows, where the first element of each pair means consecutive paths of length 2 and the second element of each pair means the element for the given paths of length 2 with respect to the generators.

1. \((2|2, a^{-1} a^{-1}), (2|8_i f, a^{-1} c), (2|9 f, a^{-1} c^{-1}), (2|9 10, a^{-1} c^{-1} a^{-1})\) for \(i = 1, 2\)

2. \((2|2, aa), (2|\overline{t}_i 6, ab^{-1}), (2|\overline{g}_i 7, ab)\) for \(i = 1, 2\)

3. \((6 \overline{t}_i 2, ba^{-1}), (6 \overline{t}_i 8_j f, bc), (6 \overline{t}_i 9 f, 8_j b^{-1}), (6 \overline{t}_i 9 10, bc^{-1} a^{-1}), (6 \overline{t}_i 8_2 11, bca^{-1})\) for \(i, j = 1, 2\)

4. \((\overline{t}_i 6|2, b^{-1} a^{-1}), (\overline{t}_i 6|8_j f, b^{-1} c), (\overline{t}_i 6|9 f, 8_j b^{-1} c^{-1}), (\overline{t}_i 6|9 10, b^{-1} c^{-1} a^{-1}), (\overline{t}_i 6|8_2 11, b^{-1} ca^{-1})\) for \(i, j = 1, 2\)

5. \((8_1 f, 2, ca), (8_1 f, 6 \overline{t}_i, cb), (8_1 f, 6 8_j f, 6, cb^{-1}), (8_1 f, 10 8_j f, cac)\) for \(i = 1, 2\)

6. \((8_2 f, 2, ca), (8_2 f, 6 \overline{t}_i, cb), (8_2 f, 6 8_j f, 6, cb^{-1}), (8_2 f, 10 8_j f, cac), (8_2 f, 10 11, cac a^{-1}), (8_2 f, 10 8_2, cac c^{-1})\) for \(i = 1, 2\)

7. \((8_2 11|8_2 11, cac a^{-1} c^{-1}), (8_2 11|9 8_2, cac a^{-1} c^{-1})\)

8. \((9 8_i f, 2, c^{-1} a), (9 8_i f, 6 \overline{t}_i, c^{-1} b), (9 8_i f, 6, c^{-1} b^{-1}), (9 8_i f, 10 8_j f, c^{-1} ac)\) for \(i = 1, 2\)

9. \((9 8_2 f, 2, c^{-1} a), (9 8_2 f, 6 \overline{t}_i, c^{-1} b), (9 8_2 f, 6 8_j f, c^{-1} b^{-1}), (9 8_2 f, 10 8_j f, c^{-1} ac), (9 8_2 f, 10 11, c^{-1} ac a^{-1}), (9 8_2 f, 10 11, c^{-1} ac c^{-1})\) for \(i = 1, 2\)

10. \((9 10|8_1 f, 9, c^{-1} a^{-1} c), (9 10|9 8_i f, c^{-1} a^{-1} c^{-1}), (9 10|9 8_1 f, c^{-1} a^{-1} c^{-1})\) for \(i = 1, 2\)

11. \((\overline{t}_i 10|9 8_i f, ac a^{-1} c), (\overline{t}_i 10|9 8_i f, ac b^{-1} c), (\overline{t}_i 10|9 8_i f, ac a^{-1} c), (\overline{t}_i 10|9 10 f, ac a^{-1} c^-1)\),\((\overline{t}_i 10|9 10 f, ac a^{-1} c^-1)\) for \(i = 1, 2\)

12. \((\overline{t}_i 11|8_2 f, ac a^{-1} c), (\overline{t}_i 11|9 8_2, ac a^{-1} c^-1), (\overline{t}_i 11|9 10 f, ac a^{-1} c^-1)\)

13. \((\overline{t}_i 10|8_2 f, ac a^{-1} c), (\overline{t}_i 10|9 8_2, ac a^{-1} c^-1), (\overline{t}_i 10|9 10 f, ac a^{-1} c^-1)\)

We note that the path 2 cannot be the next path of paths 9 10, \(\overline{t}_i 10, 8_2 11\) and \(\overline{t}_i 11\). Otherwise, \(\partial A\) has an infinite spiral. Similarly, the paths \(\overline{t}_i 9, \overline{t}_i 11, \overline{t}_i 8_2\) and \(\overline{t}_i 10\) cannot be the next path of the path \(\overline{t}_i\).

By considering all the cases, we note that there is no cancellation between two consecutive paths with respect to the generators \(a, b, c\). This implies that \([\partial A] \neq e\) since \(\pi_1(B^3 - N^\circ(\epsilon), x_0)\) is a free group. However, \([\partial A] = e\) since \(A\) is an essential disk in \(B^3 - \epsilon\). This contradicts the assumption that \(m_1 = m_3 = 0, m_2 > 0\) and this completes the proof. \(\Box\)

**Lemma 8.2.** Suppose that \(\gamma'\) is a simple closed curve which is parameterized by \((p_1, q_1, 0, p_2, q_2, 0, p_3, q_3, 0)\). If \(x_{11} > 0\), then we can construct a simple closed curve \(\gamma_0\) which is parameterized by \((p_1, q_1, t_1, p_2, q_2, t_2, p_3, q_3, t_3)\) for \(t_i \in \mathbb{Z}\) as in the table below, and it bounds an essential
disk in \( B^3 - \epsilon \) if \( \gamma' \) does. Moreover, each component of \( \gamma_0 \cap I' \) is carried by one of the given arc types in one of the standard diagrams.

1. \( q_1 + p_1 < x_{11} + x_{13} : (t_1, t_2, t_3) = (0, -1, 0) \) if \( p_2 \neq 0 \), \( (t_1, t_2, t_3) = (0, 0, 0) \) if \( p_2 = 0 \).

2. \( q_1 + p_1 \geq x_{11} + x_{13} : (t_1, t_2, t_3) = (-1, -1, 0) \) if \( p_2 \neq 0 \), \( (t_1, t_2, t_3) = (-1, 0, 0) \) if \( p_2 = 0 \). Moreover, if we have the following condition then \( \gamma' \) does not bound an essential disk in \( B^3 - \epsilon \).

3. \( x_{11} + x_{13} \leq q_1 + p_1 < x_{11} + x_{12} + x_{13} \) and \( x_{13} \geq q_1 \).

**Figure 27. The TWP diagram**

**Proof.** Consider a diagram which is called the train tracks-window-pattern diagram or the TWP diagram as in Figure 27. We note that \( \gamma' \) meets \( \partial E_i' \) only at the windows \( \omega_i \) for \( i = 1, 2, 3 \) in the TWP diagram. Let \( a, b, \cdots, l \) be the weights for the train tracks as in Figure 27. We also consider \( a, b, \cdots, l \) as the types of arcs. Then we can get the 11 types of essential arcs in \( I' \) as in the pattern diagram. (See Figure 23.) Then we can realize each connectivity pattern in the pattern diagram by an arc carried by this train track. For instance, the arcs for type 1 are carried by only \( g - e - g \) and the arcs for type 3 are carried by \( f - e - f, h - e - f \) or \( h - e - h \).

We note that \( |\gamma' \cap \omega_1| = 2p_1 = 2e + d + i, q_1 = h, x_{11} = e \) and \( x_{13} = i \).

We will now define the standard diagram by modifying the arcs carried by the train track so that they may lie outside the windows.

We set \( t_i = 0 \) if \( p_i = 0 \).

In order to consider all the possible cases, we modify the diagram of Figure 27 into the diagram (a) of Figure 28 Figure 29 and Figure 30 by the following three subcases.

Case 1: \( q_1 + p_1 < x_{11} + x_{13} \). (See the diagram (a) of Figure 28) We note that \( q_1 + p_1 = h + g < e + i = x_{11} + x_{13} \). Because of the given inequality \( h + g < e + i \), we can have the black boxes \( BB_1 \) and \( BB_3 \) for the connectivities. Then we can consider the two more black boxes \( BB_2 \) and \( BB_4 \) for the rest of connectivities of subarcs. We say that the left incoming arc of \( BB_i \) is input of \( BB_i \) and the right incoming arc arc of \( BB_i \) is output
of \( BB_i \). We note that \( f_2 > 0 \) since \( h + g < e + i \). So, we have only one possible output of \( BB_1 \) which is called \( f_1 \). We note that \( f_1 + f_2 = f \). Then the following are all possible connectivities of arcs having arc types from the diagram \( (a) \) of Figure 28 where \( i_1 \) is the upper type \( i \) and \( i_2 \) is the lower type \( i \). Also, we define \( m_k \) as the weight of the types \( k_1 \) and \( k_2 \) if \( \gamma \) has two different isotopy types for type \( k \). Clearly, we have \( m_k = m_{k_1} + m_{k_2} \).

We claim that the diagram \( (b) \) contains all the possible arc types which are obtained from the diagram \( (a) \).

1. We start from \( E_1'' \) with \( f_1 \). Then \( f_1 - e - (h + g) - h \) gives \( x \), \( f_1 - e - (h + g) - g \) gives \( 2 \), \( f_1 - e - f_2 \) gives \( 3 \), \( f_1 - c \) gives \( 6 \), \( f_1 - b \) gives \( 7 \) and \( f_1 - a \) gives \( 6 \).

2. We start from \( E_1'' \) with \( f_2 \). Then \( f_2 - i - j \) gives \( y \), \( f_2 - i - k \) gives \( 11 \) and \( f_2 - i - l \) gives \( 10 \).

3. We start from \( E_1'' \) with \( g \). Then \( g - (h + g) - i - j \) gives \( 8 \), \( g - (h + g) - i - k \) gives \( 9 \) and \( g - (h + g) - i - l \) gives \( 8 \).

4. We start from \( E_1'' \) with \( h \). Then \( h - (h + g) - i - j \) gives \( 10 \), \( h - (h + g) - i - k \) gives \( 11 \) and \( h - i - l \) gives \( z \).
We note that there is no subarc in $I'$ which connects $E''_2$ and $E''_3$. Now, we exclude some cases as follows.

Claim 1: $x$ cannot be realized.

Proof. If $m_x > 0$ then $m_8 = m_9 = m_{10} = m_{11} = 0$. Then we note that $|\gamma' \cap (\partial E''_1 \cap E''_2)| = m_2$. However, $|\gamma' \cap (\partial E''_1 \cap E''_3)| \geq m_2 + 2m_x$. Because of the connectivity in $E''_1$, we have an inequality $m_2 \geq m_2 + 2m_x$. This violates the assumption that $m_x > 0$. Therefore $m_x = 0$. 

Claim 2: $y$ and $z$ cannot be realized.

Proof. If there is an arc for $y$ then the arc for type $8_1$ is the only possible arc which connects $E''_1$ and $E''_3$ and there is no arc to connect $E''_1$ and $E''_3$. We note that $m_{8_1} < j$ if there is an arc for $y$. We have the equality $j + l = k$ for the connectivity in $E''_3$. We point out that $m_y \neq 0$ implies $m_2 = 0$, so the only arc entering $E''_1$ is $8_1$. Now, we have the inequality $m_{8_1} > l + k$ for the connectivity in $E''_3$. So, $m_{8_1} > l + k = l + (j + l) = j + 2l$. This implies that $m_{8_1} > j$. This contradicts that $m_{8_1} < j$.

If there is an arc for $z$ then we note that the two arcs $2$ and $8_2$ are the only arcs that can enter $E''_1$. We note that $m_{8_2} < l$ if there is an arc for $z$. We still have the equality $j + l = k$. Then, we have the inequality that $m_{8_2} > k + j = (j + l) + j = 2j + l$ for the connectivity in $E''_3$. So, $m_{8_2} > l$. This contradicts that $m_{8_2} < l$. 

Now, we set $t_1 = 0$ and $t_2 = -1$ if $p_2 \neq 0$ by applying a half Dehn twist supported on $E''_2$ clockwise to have $\gamma_0$ in the diagram (c) of Figure 28. We set $t_2 = 0$ if $p_2 = 0$. Also, we set $t_3 = 0$. We note that the numbers on the diagram (c) does not match with the numbers of the diagram (b) in Figure 28 since the numbers came from the pattern diagram. Then, we can check that every component of $\gamma_0 \cap I'$ is isotopic to one of the arcs in the standard diagram (1).

We note that $\gamma'$ bounds a disk in $B^3 - \epsilon$ if and only if $\gamma_0$ bounds a disk in $B^3 - \epsilon$ by Lemma 6.1.

Case 2: $x_{11} + x_{13} \leq q_1 + p_1 < x_{11} + x_{12} + x_{13}$. (See the diagram (a) of Figure 29.) First, in the diagram (a) we note that $m_1 = 0$.

Because of the given inequalities $x_{11} + x_{13} \leq q_1 + p_1 < x_{11} + x_{12} + x_{13}$, we need to have the three black boxes $BB_1, BB_2$ and $BB_3$, where $n_1 \geq 0$ and $f - e > 0$ since $x_{12} > 0$. We note that the sum of the inputs of $BB_1$ and $BB_2$ is $x_{11} + x_{12} + x_{13}$. All the output of $BB_2$ should be the input of $BB_3$ since $x_{11} + x_{13} \leq q_1 + p_1$.

Now, we get the diagram (b) of Figure 29 by setting $t_1 = t_2 = -1$ if $p_2 \neq 0$. If $p_2 = 0$ then set $t_2 = 0$. Set $t_3 = 0$. 

We claim that the diagram (c) of Figure 29 contains all the possible arc types which can be obtained from the diagram (b).

The following are all possible connectivities of arcs having arc types from the diagram (b) of Figure 29.

(1) We start from $E_1''$ with $e$. Then $e - n_2 - g$ gives 2 and $e - n_2 - h$ gives 1.

(2) We start from $E_1''$ with $f - e$. Then $(f - e) - c$ gives $5_1$, $(f - e) - b$ gives $4_1$ and $(f - e) - a$ gives $x$.

(3) We start from $E_1''$ with $g$. Then $g - n_1 - c$ gives $7_1$, $g - n_2 - b$ gives 6 and $g - n_1 - a$ gives $7_2$. We note that $g - n_2$ cannot take $i$ since $x_{13} < q_1$.

(4) We start from $E_1''$ with $h$. Then $h - n_1 - c$ gives $y$, $h - n_1 - b$ gives $4_2$, $h - n_1 - a$ gives $5_2$, $h - n_2 - i - j$ gives $8_1$, $h - n_2 - i - k$ gives $9$ and $h - n_2 - i - l$ gives $8_2$.

Claim: $x$ and $y$ cannot be realized.

Proof. Assume that there is a subarc to realize $x$. Then, there is no arc to connect $E_1''$ and $E_2''$. So, $m_{4_1} > m_5 + m_{7_2}$ for the connectivity in $E_2''$ since $m_x > 0$. Also, we have an inequality $m_{7_1} + m_2 \geq m_2 + m_{4_1} + m_5$ for the connectivity in $E_1''$. By combining these two inequalities, we have $m_{4_1} > m_5 + m_{7_2} \geq m_5 + (m_{4_1} + m_5) = m_{4_1} + 2m_5$. However, this is impossible to satisfy. Therefore, the assumption fails.

Now, assume that there is a subarc to realize $y$. Then, there is no arc to connect $E_1''$ and $E_2''$. So, $m_{4_2} > m_5 + m_{7_1}$ for the connectivity in $E_1''$ since $m_y > 0$. Also, we note that the arc $7_1$ is the only arc that can enter $E_1''$. This makes an inequality $m_{7_1} > m_{4_2} + m_5$ for the connectivity in $E_1''$. By combining these two inequalities, we have $m_{4_2} > m_5 + m_{7_1} \geq$
\[ m_5 + (m_{42} + m_5) = m_{42} + 2m_5. \] However, this is impossible to satisfy. Therefore, the assumption fails.

\[ \square \]

We note that if \( x_{13} \geq q_1 \), then \( m_1 = 0 \) because in the diagram (b) every subarc of \( \gamma' \) carried by \( e \) starts from \( E_{11}'' + \) and ends at \( E_{11}'' - \) in the diagram (b) in Figure 29. Also, \( m_3 = 0 \) in the diagram (b) since \( m_1 = 0 \) in the diagram (a). Therefore, \( m_1 + m_3 = 0 \). In this case, \( \gamma' \) cannot bound an essential disk in \( B^3 - \epsilon \) by Lemma 8.1.

Thus, we can check that every component of \( \gamma_0 \cap I' \) is isotopic to one of the arcs in the standard diagram 24 (2).

**Figure 30.**

Case 3: \( x_{11} + x_{12} + x_{13} \leq q_1 + p_1 \). (See the diagram (a) of Figure 30) We have the black box \( BB_2 \) for the given inequality. We note the sum of the inputs of \( BB_2 \) is \( x_{11} + x_{12} + x_{13} \) and the sum of the outputs of \( BB_3 \) is \( q_1 + p_1 \). We note that \( n_2 = x_{11} + x_{12} + x_{13} \).

Then, we set \( t_1 = -1 \), and we set \( t_2 = -1 \) if \( p_2 > 0 \) and \( t_2 = 0 \) if \( p_2 = 0 \). Also we set \( t_3 = 0 \) to have the diagram (b) of Figure 30.

Then we have two subcases as follows. We refer to the diagram (b).

Subcase 1: If \( x_{13} \geq q_1 \) then we note that \( m_1 = 0 \) since every arc which starts from \( E_{1}'' + \) with \( f \) should end at \( E_{1}'' - \).

Now, we claim that the diagram (c) of Figure 30 contains all the possible arc types which are obtained from the diagram (b).
The following are the connectivities of arcs having arc types from the diagram (b) of Figure 30.

1. We start from $E_1''$ with $f$. Then $f - e - n_2 - g$ gives 2. We note that $m_1 = 0$ implies $f - e - n_2 - h$ does not occur.

2. We start from $E_1''$ with $g$. Then $g - n_1 - e - n_2 - g$ gives 3, $g - n_2 - c$ gives 7, $g - n_2 - b$ gives 6, $g - n_2 - a$ gives $7_2$, $g - n_2 - i - j$ gives $w$, $g - n_2 - i - k$ gives 11 and $g - n_2 - i - l$ gives 10. We note that $g - n_1 - e - n_2 - h$ does not exist since every arc starting from $E_1''$ with $g - n_1 - e - n_2$ should connect to $g$ since $x_{13} = i \geq h = q_1$.

3. We start from $E_1''$ with $h$. Then $h - n_2 - i - j$ gives $8_1$, $h - n_2 - i - k$ gives 9 and $h - n_2 - i - l$ gives $8_2$.

Claim: $w$ cannot be realized.

Proof. Suppose that there is a subarc to realize $m_w > 0$. Then, we note that the arcs 2 and $8_1$ are the only two arcs that can enter $E_1''$. This makes an inequality $m_{8_1} + m_2 > m_2 + m_{10} + m_{11} + m_w$ for the connectivity in $E_1''$ since $m_3 > 0$ by Lemma 8.1. This implies that $m_{8_1} > m_{10} + m_{11} + m_w$. We also have an equality $m_w + m_{8_1} + m_{10} = m_{11}$ for the connectivity in $E_3''$. By combining this inequality and equality, we have $m_{11} = m_w + m_{8_1} + m_{10} > m_w + m_{10} + m_{11} + m_w + m_{10} > m_{11}$ since $m_w > 0$. However, this is impossible to satisfy. Therefore, the assumption fails.

Thus, we can check that every component of $\gamma_0 \cap I'$ is isotopic to one of the arcs in the standard diagram 24 (1).

![Figure 31](image_url)

Subcase 2: If $x_{13} < q_1$ then $m_1 > 0$ since $e > 0$ because $x_{11} > 0$.

The following are the connectivities of arcs having arc types from the diagram (b) of Figure 30. We claim that the diagram (b) of Figure 31 contains all the possible arc types.

1. We start from $E_1''$ with $f$. Then $f - e - n_2 - h$ gives 1 and $f - e - n_2 - g$ gives 2.
(2) We start from $E_1''$ with $g$. Then $g - n_1 - e - n_2 - h$ gives $2_1$, $g - n_1 - e - n_2 - g$ gives $3$, $g - n_2 - c$ gives $7_1$, $g - n_2 - b$ gives $6$ and $g - n_2 - a$ gives $7_2$.

We note that $g - n_2 - i - j$, $g - n_2 - i - k$ and $g - n_2 - i - l$ do not exist since no arc starting from $E_1''$ with $g - n_2$ can go to $i$ since $i < h$.

(3) We start from $E_1''$ with $h$. Then $h - n_1 - e - n_2 - h$ gives $x$, $h - n_2 - c$ gives $y$, $h - n_2 - b$ gives $x_2$, $h - n_2 - a$ gives $x_5$, $h - n_2 - i - j$ gives $x_1$, $h - n_2 - i - k$ gives $x_9$ and $h - n_2 - i - l$ gives $x_8$.

Claim: $x$ and $y$ cannot be realized.

Proof. First, assume that there is a.subarc to realize $m_x > 0$. Then we note that $2_1$ is the only possible arc which can enter to the $E_1''$. Because of the connectivity in $E_1''$, we have an inequality $m_{21} \geq m_{21} + m_x$. This implies that $m_{21} > m_{21}$ since $x > 0$. However, this is impossible to satisfy. Therefore, $m_x = 0$.

Now, assume that there is a subarc to realize $m_y > 0$. Then we note that $4_2$ is the only possible arc which can enter to the $E_2''$. We have an equality $m_{42} = m_y + m_{71} + m_{52}$ for the connectivity in $E_2''$. Also, we have an inequality $m_{71} \geq m_{9} + m_{42} + m_{52}$ for the connectivity in $E_2''$. Therefore, we have $m_{71} \geq m_{9} + m_{42} + m_{52} = m_y + (m_y + m_{71} + m_{52}) + m_{52} > m_{71}$ since $m_y > 0$. This is impossible to satisfy. Therefore, $m_y = 0$.

Thus, we can check that every component of $\gamma_0 \cap I'$ is isotopic to one of the arcs in the standard diagram $24(2)$.

By combining cases 1, 2 and 3, we note that $\gamma'$ can be modified as a simple closed curve $\gamma_0$ so that all components of $\gamma_0 \cap I'$ are isotopic to arcs of the two diagrams in Figure $24$ but $\gamma_0$ bounds an essential disk if $\gamma'$ does by Lemma 6.1.

We represent $\gamma_0$ by putting weights on these arcs.

Now, we want to calculate $m_i$ of $\gamma_0$ for $i = 1, 2, ..., 11$ by using the parameters $p_j, q_j$ of $\gamma'$ for $j = 1, 2, 3$.

The following is the table of $m_i$ for each case.

Lemma 8.3. (1) $x_{11} + x_{13} > q_1 + p_1$:

\[
\begin{align*}
m_3 &= p_3 - p_2 - q_1, \\
m_2 &= q_1 + p_1 - 2p_3, \\
m_7 &= q_2, \\
m_2 &= p_2 - q_2, \\
m_6 &= p_2, \\
m_{10} &= \min(q_1, p_3 - q_3), \\
m_{10} &= \min(0, \max(2p_3 - p_1 - q_1, 0), q_3), \\
m_{11} &= \max(0, q_1 - p_3 + q_3), \\
m_{12} &= \max(0, \max(2p_3 - p_1 - q_1, 0) - q_3), \\
m_{81} &= \max(0, p_3 - q_3 - p_1), \\
m_{82} &= \max(0, q_3 - \max(2p_3 - p_1 - q_1, 0)), \\
m_9 &= m_{10} + m_8 - m_{11}.
\end{align*}
\]
(2) $x_{11} + x_{13} \leq q_1 + p_1$ and $x_{13} < q_1$:

(a) $q_1 > x_{13} + x_{11}$: $m_1 = p_1 - p_2 - p_3$, $m_{81} = p_3 - q_3$, $m_{82} = q_3$, $m_9 = p_3$, $m_{51} = \min(q_2, p_2 + p_3 - q_1)$, $m_{71} = q_2 - m_{51}$, $m_{41} = p_2 + p_3 - q_1 - m_{51}$, $m_{52} = \min(p_2 - q_2, q_1 + p_2 - p_1 - p_3)$, $m_{72} = p_2 - q_2 - m_{52}$, $m_{42} = q_1 + p_2 - p_1 - p_3 - m_{52}$, $m_6 = m_{5} + m_{7} - m_{4}$.

(b) $x_{13} < q_1 \leq x_{13} + x_{11}$: $m_1 = q_1 - 2p_3$, $m_{81} = p_3 - q_3$, $m_{82} = q_3$, $m_9 = p_3$, $m_2 = p_1 + p_3 - p_2 - q_1$, $m_{51} = \min(q_2, p_2 + p_3 - q_1)$, $m_{71} = q_2 - m_{51}$, $m_{41} = p_2 + p_3 - q_1 - m_{51}$, $m_{72} = p_2 - q_2 - m_{52}$, $m_6 = m_{5} + m_{7} - m_{4}$.

(3) $x_{11} + x_{12} + x_{13} \leq q_1 + p_1$:

(a) $x_{13} \geq q_1$:

\[
m_3 = q_1 - p_2 - p_3, \quad m_2 = p_1 - q_1, \quad m_{71} = q_2, \quad m_{72} = p_2 - q_2, \quad m_6 = p_2, \quad m_{81} = \min(q_1, p_3 - q_3), \quad m_{82} = \max(q_3 - (2p_3 - q_1), 0), \quad m_{10} = \min(2p_3 - q_1, q_3), \quad m_{11} = \max((2p_3 - q_1) - q_3, 0).
\]

(b1) $x_{13} < q_1, \quad p_1 \geq 2q_1 - 2p_3$: $m_2 = p_1 + 2p_3 - 2q_1$, $m_1 = q_1 - 2p_3$, $m_3 = q_1 - p_2 - p_3$, $m_{71} = q_2, m_{72} = p_2 - q_2, m_6 = p_2, m_{81} = p_3 - q_3, m_{82} = q_3$ and $m_9 = p_3$.

(b2) $x_{13} < q_1, \quad p_1 < 2q_1 - x_{13}$:

(i) $q_1 > x_{13} + x_{11}$: $m_{21} = q_1 - p_2 - p_3$, $m_1 = p_1 - q_1, m_{71} = q_2, m_{52} = \min(p_2 - q_2, q_1 - x_{11} - x_{13}), m_{72} = p_2 - q_2 - m_{52} = p_2 - q_2 - \min(p_2 - q_2, q_1 + p_2 - p_1 - p_3)$, $m_{42} = q_1 + p_2 - p_1 - p_3 - m_{52} = q_1 + p_2 - p_1 - p_3 - \min(p_2 - q_2, q_1 + p_2 - p_1 - p_3)$, $m_6 = p_2 - m_{42} = p_1 + p_3 - q_1 + \min(p_2 - q_2, q_1 + p_2 - p_1 - p_3)$, $m_{81} = p_3 - q_3, m_{82} = q_3$ and $m_9 = p_3$.

(ii) $q_1 \leq x_{13} + x_{11}$: $m_{21} = 2q_1 - p_1 - 2p_3$, $m_1 = p_1 - q_1, m_3 = p_1 + p_3 - p_2 - q_1, m_{71} = q_2, m_{72} = p_2 - q_2, m_6 = p_2, m_{81} = p_3 - q_3, m_{82} = q_3$ and $m_9 = p_3$.

Proof. Consider $\gamma_0$ which is in standard position.

First, we recall that $x_{ij} = I_j$, $x_{ik} = I_k$ and $x_{ii} = \frac{I_i - I_j - I_k}{2}$ if $x_{11} > 0$ by the equations in Lemma 5.1. Also, we note that $I_1 = 2p_1, I_2 = 2p_2$ and $I_3 = 2p_3$. So, we can have $x_{11} = p_1 - p_2 - p_3$, $x_{12} = 2p_2$ and $x_{13} = 2p_3$ from the equations.

Case 1: $x_{11} + x_{13} > q_1 + p_1$. Refer to the diagrams of Figure 28 and Figure 32 to get the weights $m_i$ as follows.

First, we note the weight of the bottom arc starting from $E_1^{d-}$ is $\max(x_{13} - (p_1 + q_1), 0)$ as in Figure 32 since if $m_2 > 0$ then the weight is zero and $x_{13} - (p_1 + q_2) < 0$, and if $m_2 = 0$ then the weight is $x_{13} - (p_1 + q_1)$ because the weight starting from $E_1^{d+}$ is $p_1$ in this case.
Then we have \( m_3 = f_2 = (e + i) - (h + g) = (x_{11} + x_{13}) - (q_1 + p_1), \) \( m_2 = x_{11} - m_3 = (q_1 + p_1) - x_{13}, m_7 = q_2, m_7 = p_2 - q_2, m_6 = p_2. \)

We note that \( m_{10_1} = \min(q_1, p_3 - q_3), m_{10_2} = \min(\max(x_{13} - (p_1 + q_1), 0), q_3). \) Then we have \( m_{11_1} = \max(0, q_1 - (p_3 - q_3)) \) since if \( q_1 \geq p_3 - q_3 \) then \( m_{11_1} = q_1 - (p_3 - q_3) \) and if \( q_1 < p_3 - q_3 \) then \( m_{11_1} = 0. \) Similarly, we have \( m_{11_2} = \max(0, \max(x_{13} - (p_1 + q_1), 0) - q_3). \) Also, we have \( m_{8_1} = \max(0, (p_3 - q_3) - q_1) \) since if \( p_3 - q_3 \geq q_1 \) then \( m_{8_1} = (p_3 - q_3) - q_1 \) and if \( p_3 - q_3 < q_1 \) then \( m_{8_1} = 0. \) Similarly, \( m_{8_2} = \max(0, q_3 - \max(x_{13} - (p_1 + q_1), 0)). \) We note that \( m_9 = m_{10} + m_8 - m_{11}. \)

In order to have better formulas as in Lemma 8.3 use \( x_{11} = p_1 - p_2 - p_3, x_{12} = 2p_2 \) and \( x_{13} = 2p_3. \)

**Case 2:** \( x_{11} + x_{13} \leq q_1 + p_1 < x_{11} + x_{12} + x_{13} \) and \( x_{13} < q_1. \)

Then we need to consider the following two cases in Figure 33 which are obtained from the diagram (b) of Figure 29. We note that the first case has \( m_2 = 0 \) and the second case has \( m_2 > 0. \) By referring to the two diagrams of Figure 33 we can get the weights \( m_i \) as follows. We will use a similar argument as in the first case to find \( m_{5_1} \) and \( m_{5_2}. \)

\( (a) \) \( q_1 > x_{13} + x_{11}: m_1 = x_{11}, m_{8_1} = p_3 - q_3, m_{8_2} = q_3, m_9 = p_3, m_5 = \min(q_2, x_{11} + x_{12} + x_{13} - (q_1 + p_1)), m_{7_1} = q_2 - m_{5_1}, m_{4_1} = x_{11} + x_{12} + x_{13} - (q_1 + p_1) - m_{5_1}, m_{5_2} = \min(p_2 - q_2, q_1 - (x_{11} + x_{13})), m_{7_2} = p_2 - q_2 - m_{5_2}, m_{4_2} = q_1 - (x_{11} + x_{13}) - m_{5_2}, m_6 = m_5 + m_7 - m_4. \)

\( (b) \) \( x_{13} < q_1 \leq x_{13} + x_{11}: m_1 = q_1 - x_{13}, m_{8_1} = p_3 - q_3, m_{8_2} = q_3, m_9 = p_3, m_2 = x_{11} + x_{13} - q_1; m_{5_1} = \min(q_2, x_{11} + x_{12} + x_{13} - (q_1 + p_1)), m_{7_1} = q_2 - m_{5_1}, m_{4_1} = x_{11} + x_{12} + x_{13} - (q_1 + p_1) - m_{5_1}, m_{7_2} = p_2 - q_2, m_6 = m_5 + m_7 - m_4. \)

In order to have better formulas as in Lemma 8.3 use \( x_{11} = p_1 - p_2 - p_3, x_{12} = 2p_2 \) and \( x_{13} = 2p_3. \)
We recall that if $q_1 \leq x_{13}$ then $\gamma'$ does not bound an essential disk in $B^3 - \epsilon$.

**Case 3:** $x_{11} + x_{12} + x_{13} \leq q_1 + p_1$. We note that $q_1 + p_1 < 2p_1$ since $0 \leq q_1 < p_1$. Then we have two cases for this.

\[(a)\] $x_{13} \geq q_1$: Refer to the diagram (c) of Figure 30 and the diagram of Figure 34.

Then we have the following formulas for $m_j$. We will use a similar argument as in the first case to find $m_{81}, m_{82}, m_{10}$ and $m_{11}$.

\[
m_3 = (q_1 + p_1) - (x_{11} + x_{12} + x_{13}), \quad m_2 = x_{11} - m_3 = (2x_{11} + x_{12} + x_{13}) - (q_1 + p_1), \quad m_{71} = q_2, \quad w_{72} = p_2 - q_2, \quad w_6 = p_2, \quad m_{81} = \min(q_1, p_3 - q_3), \quad m_{82} = \max(q_3 - (2p_3 - q_1), 0), \quad m_{10} = \min(2p_3 - q_1, q_3), \quad m_{11} = \max((2p_3 - q_1) - q_3, 0).
\]

\[(b)\] $x_{13} < q_1$:

We have the diagram (a) of Figure 35 by referring Figure 30 and Figure 31.
Then we consider the two subcases \( p_1 - q_1 \geq q_1 - x_{13} \) and \( p_1 - q_1 < q_1 - x_{13} \).

(b1) : If \( p_1 - q_1 \geq q_1 - x_{13} \) then we can directly get the formulas for \( m_j \) as follows. (Refer to the diagram (b) of Figure 35)

\[
\begin{align*}
    m_{2_1} &= p_1 + x_{13} - 2q_1, \\
    m_1 &= q_1 - x_{13}, \\
    m_3 &= x_{11} + q_1 - p_1, \\
    m_7 &= q_2, \\
    m_{7_2} &= p_2 - q_2, \\
    m_6 &= p_2, \\
    m_8 &= q_3, \\
    m_9 &= p_3.
\end{align*}
\]

(b2) : If \( p_1 - q_1 < q_1 - x_{13} \) then we have the diagram (c) of Figure 35.

Now, we consider the two subcases for this which are (i) \( q_1 > x_{13} + x_{11} \) or (ii) \( q_1 \leq x_{13} + x_{11} \).

Then we have the two diagrams of Figure 36

(i) \( q_1 > x_{13} + x_{11} \): (Refer to the diagram (i) of Figure 36)

Then we get the formulas for \( m_j \) as follows.

\[
\begin{align*}
    m_{2_1} &= x_{11} + q_1 - p_1, \\
    m_1 &= p_1 - q_1, \\
    m_7_1 &= q_2, \\
    m_{5_2} &= \min(p_2 - q_2, q_1 - x_{11} - x_{13}), \\
    m_7_2 &= p_2 - q_2 - m_{5_2} = p_2 - q_2 - \min(p_2 - q_2, q_1 - x_{11} - x_{13}), \\
    m_4_2 &= q_1 - x_{11} - x_{13} - m_{5_2} = q_1 - x_{11} - x_{13} - \min(p_2 - q_2, q_1 - x_{11} - x_{13}).
\end{align*}
\]
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\[ m_6 = p_2 - m_{42} = p_2 - (q_1 - x_{11} - x_{13} - \min(p_2 - q_2, q_1 - x_{11} - x_{13})) = p_2 - q_1 + x_{11} + x_{13} + \min(p_2 - q_2, q_1 - x_{11} - x_{13}), \]

\[ m_{81} = p_3 - q_3, \quad m_{82} = q_3 \text{ and } m_9 = p_3. \]

(ii) \( q_1 \leq x_{13} + x_{11} \): (Refer to the diagram (ii) of Figure 36)

Then we get the formulas for \( m_j \) as follows.

\[ m_{21} = 2q_1 - p_1 - x_{13}, \quad m_1 = p_1 - q_1, \quad m_3 = x_{11} + x_{13} - q_1, \quad m_{71} = q_2, \quad m_{72} = p_2 - q_2, \]

\[ m_6 = p_2, \quad m_{81} = p_3 - q_3, \quad m_{82} = q_3 \text{ and } m_9 = p_3. \]

In order to have better formulas as in Lemma 8.3 use \( x_{11} = p_1 - p_2 - p_3, \quad x_{12} = 2p_2 \) and \( x_{13} = 2p_3 \).

\[ \square \]

9. Step 4: Main Theorem

Now, we want to discuss the main theorem that can complete my algorithm. In order to do this, we define four homeomorphisms \((\delta_1 \delta_2^{-1})^\pm \) and \( \delta_3^\pm \) as follows.

Let \( \delta_1 \) and \( \delta_2 \) be the clockwise half Dehn twists supported on two punctured disks \( C_1 \) and \( C_2 \) respectively as in Figure 37. Also, let \( \delta_3 \) be clockwise half Dehn twists supported on two punctured disk \( E'_4 \) as in Figure 38. Then we have the two following lemmas.

**Lemma 9.1.** \( \gamma_0 \) bounds an essential disk in \( B^3 - \epsilon \) if and only if \((\delta_1 \delta_2^{-1})^\pm(\gamma_0) \) bounds an essential disk in \( B^3 - \epsilon \).

**Proof.** We notice that an extension \( K \) to \( B^3 \) of \((\delta_1 \delta_2^{-1})^\pm \) changes the position of two strings. So, it preserves the \( \infty \) tangle. Let \( A \) be the essential disk in \( B^3 - \epsilon \) so that \( \partial A = \gamma_0 \). Then, we know that \( K(A) \) bounds a disk in \( B^3 - \epsilon \) and \( K(\gamma_0) \) is essential in \( \Sigma_0,6 \). Therefore, \((\delta_1 \delta_2^{-1})^\pm(\gamma_0) \) bounds an essential disk in \( B^3 - \epsilon \).

\[ \square \]

**Lemma 9.2.** \( \gamma_0 \) bounds an essential disk in \( B^3 - \epsilon \) if and only if \( \delta_3(\gamma_0) \) bounds an essential disk in \( B^3 - \epsilon \).

**Proof.** By lemma 6.1, it is trivial.

\[ \square \]
Now, let \((p_{11}, q_{11}, t_{11}, p_{21}, q_{21}, t_{21}, p_{31}, q_{31}, t_{31})\) be the parameters for \(h(\gamma_0)\) for \(h \in \{(\delta_1 \delta_2^{-1})^{\pm 1}, \delta_3^{\pm 1}\}\). Also, let \((p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31})\) be the Dehn’s parameters for \(h(\gamma_0)\), where \(q'_{11} = p_{11}t_{11} + q_{11}\) for \(i = 1, 2, 3\). We note that the nine parameters can be obtained from the Dehn’s parameters. Also, we know that \(|\gamma_0 \cap \partial E| = 2(p_1 + p_2 + p_3)\) and \(|\delta_1^{-1}\delta_2(\gamma_0) \cap \partial E| = 2(p_{11} + p_{21} + p_{31})\).

Now, here is the main theorem.

**Theorem 9.3.** Suppose that \(\gamma_0\) bounds an essential disk in \(B^3 - \epsilon\) and \(\gamma_0\) is in standard position in \(I'\) and \(m_3 > 0\). Then applying one of the homeomorphisms \((\delta_1 \delta_2^{-1})^{\pm 1}\) and \(\delta_3^{\pm 1}\) reduces the sum of the \(p_i\) for the image of \(\gamma_0\). Especially, the following are the formulas for the Dehn’s parameter changes for each case.

1. \(m_2, m_3 > 0, m_1 = 0\) and \(m_3 > m_2 + 1\): \((p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - 2m_2, q_1' + m_2 - (m_{10} + m_{11}), p_2, q_2', p_3, q_3 + 2(m_{10} + m_{11}))\) by \(\delta_3\).

2. \(m_2, m_3 > 0, m_1 = 0\) and \(m_3 > m_2 + 1\): \((p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - 2m_2, q_1' - m_2 + (m_{10} + m_{11}), p_2, q_2', p_3, q_3 - 2(m_{10} + m_{11}))\) by \(\delta_3^{-1}\).

3. \(m_3 > m_2 > 0\): \((p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - 2m_2, q_1' + m_2, p_2, q_2', p_3, q_3)\) by \(\delta_3\).
(4) $m_1 > m_2 \geq m_3 > 0$: $(p_{11}, q_{11}', p_{21}, q_{21}', p_{31}, q_{31}) = (p_1 - 2m_2, q_1' + (m_3 - m_2), p_2, q_2', p_3, q_3')$

by $\delta_3$.

(5) $m_1 = m_3 = 1$ and $m_i = 0$ for all $i \neq 1, 3$: It bounds an essential disk in $B^3 - \epsilon$.

(6) $m_1 = m_2 = 0$, $m_3 \geq 2$.

(a) $m_{11} = 0$: $(p_{11}, q_{11}', p_{21}, q_{21}', p_{31}, q_{31}) = (p_1 - m_8, q_1' - m_8, p_2, q_2', p_3 - m_8, q_3' + m_8)$

by $\delta_1^{-1} \delta_2$.

(b) $m_8 = 0$: $(p_{11}, q_{11}', p_{21}, q_{21}', p_{31}, q_{31}) = (p_1 - m_{11}, q_1' - m_{11}, p_2, q_2', p_3 - m_{11}, q_3' + m_{11})$

by $\delta_1^{-1} \delta_2$.

(c) $m_8, m_{11} > 0$:

(i) $m_8, m_{11} > 0$: $(p_{11}, q_{11}'', p_{21}, q_{21}'', p_{31}, q_{31}''') = (p_1 - (m_{81} - m_{112}), q_1'' - m_8, p_2, q_2', p_3 - (m_8, m_{112}), q_3' + m_{112})$

by $\delta_1^{-1} \delta_2$.

(ii) $m_8, m_{11} > 0$: $(p_{11}, q_{11}'', p_{21}, q_{21}'', p_{31}, q_{31}''') = (p_1 - (m_8, m_{112}), q_1'' + m_{11}, p_2, q_2', p_3 - (m_8, m_{112}), q_3' - m_8)$

by $\delta_1^{-1} \delta_2$.

Also, if $\gamma_0$ satisfies any of the following conditions then $\gamma_0$ does not bound an essential disk in $B^3 - \epsilon$.

(7) $m_1 + m_3 < 2$ and $m_i > 0$ for some $i$.

(8) $m_2, m_3 > 0$, $m_1 = 0$ and $m_3 \leq m_2 + 1$.

(9) $m_2 \geq m_1$, $m_3 > 0$.

Proof. Suppose that $\gamma_0$ is parameterized by $(p_1, q_1, t_1, p_2, q_2, t_2, p_3, q_3, 0)$ to have $\gamma_0$ which is in standard position in $I'$. We note that $t_2 = 0$ if $p_2 = 0$ and $t_2 = -1$ if $p_2 \neq 0$.

First, we assume that $m_2, m_3 > 0$ and $m_1 = 0$. We note that there are two type 2 in a standard diagram and they cannot coexist. Without loss of generality, we choose the diagram (a) as in Figure 39.

Then we apply $\delta_3$ to $\partial A$ to reduce the minimal intersection number of $\gamma_0$ with $\partial E$. The diagram (b) of Figure 39 shows there is no possibility to have $x_{22} > 0$ or $x_{33} > 0$ for $\delta_3(\gamma_0)$.

We note that if $m_3 \leq m_2 + 1$ then $x_{11}$ of $\delta_3(\gamma_0)$ is less than 2. Therefore, $\delta_3(\gamma_0)$ does not bound an essential disk in $B^3 - \epsilon$ since $x_{11} \geq 2$ if $\delta_3(\gamma_0)$ bounds an essential disk in $B^3 - \epsilon$. This implies that $\gamma_0$ also does not bound an essential disk in $B^3 - \epsilon$. This makes a contradiction. Therefore, $m_3 > m_2 + 1$. So, we have the diagram (c) of Figure 39.

Now, we note that $|\delta_3(\gamma_0) \cap \partial E| = |\gamma_0 \cap \partial E| - 4m_2$ as the diagram (c) of Figure 39.

First, we note that $(p_{21}, q_{21}') = (p_2, q_2')$. Also, we know that $p_{31} = p_3$. We note that the rightmost arc type coming to $E''_3$ is $s_2$ before taking $\delta_3$, but the arc type $s_2$ moved around
counterclockwise by $2(m_{10} + m_{11})$ after taking $\delta_3$. Therefore, $q'_{31} = q_3 + 2(m_{10} + m_{11})$.

We note that $p_{11} = p_1 - 2m_2$. We also note that $q'_{11} = q'_1 + m_2 - (m_{10} + m_{11})$ by considering the incoming of the arc type for $x_{11}$ to the $E''_4$. In the diagram (a), the arc type 2 for $x_{11}$ is coming to the $E''_4$ after the types 10 and 11. However, after applying $\delta_3$ to $\gamma_0$ the arc type 3 for $x_{11}$ is the right of some arcs with the weight $m_2$ which are not for $x_{11}$ in the $E''_4$. Therefore, we have the following formula for the parameter changes.

$$(p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - 2m_2, q'_1 + m_2 - (m_{10} + m_{11}), p_2, q'_2, p_3, q'_3 + 2(m_{10} + m_{11})).$$

Also, we check that $p_{11} + p_{21} + p_{31} = p_1 - 2m_2 + p_2 + p_3 < p_1 + p_2 + p_3$ since $m_2 > 0$.

We note that if $\gamma_0$ has another type 2 then we need to apply $\delta_3^{-1}$ to reduce the sum of $p_i$ for $\gamma_0$.

Actually, the following is the formula for the parameter changes by $\delta_3^{-1}$.
\[(p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - 2m_2, q'_1 - m_2 + (m_{10} + m_{11}), p_2, q'_2, p_3, q'_3 - 2(m_{10} + m_{11})).\]

The case that \(m_1, m_2 > 0\) and \(m_3 = 0\) is analogous to the previous case.

Now, we assume that \(m_1, m_2, m_3 > 0\).

\[\begin{array}{c}
\text{Figure 40.}
\end{array}\]

Then we have the diagram (a) of Figure 39. After applying \(\delta_3\) we can get the diagram (b) of Figure 39.

First of all, we note that if \(m_2 \geq m_1, m_3\) then \(\gamma_0\) does not bound an essential disk in \(B^3 - \epsilon\) since \(\delta_3(\gamma_0)\) has \(x_{11} = 0\) and in the diagram (b) there is no arc can occur \(x_{22}\) and \(x_{33}\). So, we consider the two cases \(m_2 < m_3\) and \(m_3 \leq m_2 < m_1\).

(1) First, we assume that \(m_2 < m_3\). Then we can get the diagram (c) in Figure 39.

We note that \((p_{21}, q'_{21}) = (p_2, q'_2)\) and \((p_{31}, q'_{31}) = (p_3, q'_3)\). Also, \(p_{11} = p_1 - 2m_2\). In the diagram (a), the arc type 2 for \(x_{11}\) is coming to the \(E''_1\) leftmost. However, after applying \(\delta_3\) to \(\gamma_0\) the arc with the weight \(m_2\) which is not for \(x_{11}\) is the left of the arc type 2 in the \(E''_1\). This implies that \(q'_{11} = q'_1 + m_2\). So, we have the following formula in this case.

\[(p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - 2m_2, q'_1 + m_2, p_2, q'_2, p_3, q'_3).\]

Also, we can check that \(p_{11} + p_{21} + p_{31} = p_1 - 2m_2 + p_2 + p_3 < p_1 + p_2 + p_3\) since \(m_2 > 0\).
(2) Now, we assume that \( m_3 \leq m_2 < m_1 \). Then we can get the diagram (d) of Figure 39.

We note that \((p_{21}, q'_{21}) = (p_2, q'_2)\) and \((p_{31}, q'_{31}) = (p_3, q'_3)\). We also can check that \( p_{11} = p_1 - 2m_2 \). We note that the arc with the weight \( m_2 - m_3 \) in \( E_1'' \) cannot connect to the arc with the weight \( m_3 \) since \( m_1 > m_3 \). In the diagram (a), the arc type 2 for \( x_{11} \) is coming to the \( E_1'' \) leftmost. However, after applying \( \delta_3 \) to \( \gamma_0 \) the arc with the weight \( m_3 \) which is for \( x_{11} \) is replacing the position of the arc type 2 in the \( E_1'' \). This implies that \( q'_{11} = q'_{1} + (m_3 - m_2) \). So, we have the following formula in this case.

\[
(p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - 2m_2, q'_1 + (m_3 - m_2), p_2, q'_2, p_3, q'_3).
\]

Also, we can check that \( p_{11} + p_{21} + p_{31} = p_1 - 2m_2 + p_2 + p_3 < p_1 + p_2 + p_3 \) since \( m_2 > 0 \).

\[\text{Figure 41.}\]

Now, we assume that \( m_2 = 0 \).

We note that if \( m_1, m_3 > 0 \) then \( m_1 = m_3 = 1 \) and \( m_i = 0 \) for all \( i \neq 1, 3 \). Otherwise, \( \gamma_0 \) is not a simple closed curve. Moreover, if \( m_1 = m_3 = 1 \) then \( \gamma_0 \) bounds an essential disk in \( B_3 - \epsilon \) and the algorithm stops. So, we may assume that \( m_3 > 0 \), \( m_1 = 0 \) for the rest of this algorithm.

Then, we have consider the following three subcases.

(1) \( m_{11} = 0 \): We have the left diagram of Figure 41. Now, apply \( \delta_1^{-1}\delta_2 \) to \( \gamma_0 \) to get the right diagram of Figure 41. We note that \((p_{21}, q'_{21}) = (p_2, q'_2)\). Also, \( p_{31} = p_3 - m_8 \) and \( q'_{31} = q'_3 + m_8 \). Moreover, \( p_{11} = p_1 - m_8 \) and \( q'_{11} = q'_1 - m_{81} \).

Therefore, we have the following formula for the Dehn’s parameter changes.

\[
(p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - m_8, q'_1 - m_{81}, p_2, q'_2, p_3 - m_8, q'_3 + m_8).
\]

We can check that \( p_{11} + p_{21} + p_{31} = p_1 + p_2 + p_3 - 2m_8 \).

(2) \( m_8 = 0 \): We have the left diagram of Figure 42. Now, apply \( \delta_1 \delta_2^{-1} \) to \( \gamma_0 \) to get the right diagram of Figure 42. We know that \((p_{21}, q'_{21}) = (p_2, q'_2)\). Also, \( p_{31} = p_3 - m_{11} \) and \( q'_{31} = q'_3 + m_{11} \). Moreover, \( p_{11} = p_1 - m_{11} \) and \( q'_{11} = q'_1 - m_{111} \). So, we have the
following formula for the Dehn’s parameter changes.

\[(p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - m_{11}, q'_1 - m_{11}, p_2, q'_2, p_3 - m_{11}, q'_3 + m_{11}).\]

We also can check that \(p_{11} + p_{21} + p_{31} = p_1 + p_2 + p_3 - 2m_{11}.\)

(3) \(m_8, m_{11} > 0: \) Since \(m_8,\) and \(m_{11},\) cannot coexist, we assume that \(m_8 > 0\) without loss of generality. Then \(m_{11} > 0\) and \(m_8 = m_{11} = 0.\) Figure 43 shows this case.

Figure 43.

Now, apply \(\delta_1^{-1}\delta_2\) to the left diagram of Figure 43. For the connectivities in \(E''_1\) and \(E''_3,\) we have \((m_3 + m_7 + m_6) + m_{10} + m_{11} = m_{8_1} + m_9\) and \(m_9 + m_{11} = m_{10} + m_{8_1}.\) So, we have \((m_3 + m_7 + m_6) + 2m_{11} = 2m_{8_1}.\) This implies that \(m_{8_1} > m_{11} \) since \(m_3 > 0.\)

We also note that \(m_3 + m_{11} \geq m_{8_1} + 2\) to have at least two type 3 for \(\delta_1^{-1}\delta_2(\gamma_0).\) So, we know that \((p_{21}, q'_{21}) = (p_2, q'_2).\) Also, we know that \(p_{31} = p_3 - (m_{8_1} - m_{11})\) and \(q'_{31} = q'_3 + m_{8_1}.\) Moreover, \(p_{11} = p_1 - (m_{8_1} - m_{11})\) and \(q'_{11} = q'_1 - m_{8_1}.\) So, we have the following formula for the Dehn’s parameter changes.

\[(p_{11}, q'_{11}, p_{21}, q'_{21}, p_{31}, q'_{31}) = (p_1 - (m_{8_1} - m_{11}), q'_1 - m_{8_1}, p_2, q'_2, p_3 - (m_{8_1} - m_{11}), q'_3 + m_{11}).\] Also, we can check that \(p_{11} + p_{21} + p_{31} = p_1 + p_2 + p_3 - 2(m_{8_1} - m_{11}).\)

For the case that \(m_{8_2} > 0,\) \(m_{11} > 0\) and \(m_{8_1} = m_{11} = 0,\) we still need to apply \(\delta^{-1}\delta_2\) to reduce the minimal intersection number of \(\gamma_0\) with \(\partial E.\) Then we can have the following formula for the Dehn’s parameter changes.
If $\gamma_0$ is in standard position in $I'$ with $m_1 > 0$, then we rotate $\gamma_0$ with 180° about the center of $E'_1$ to have a new simple closed curve $\eta$ which is in standard position in $I'$ with $m_3 > 0$. We note that $\eta$ also bounds an essential disk in $B^3 - \epsilon$ since the rotation preserves $\infty$ tangle.

If the set of weights $m_i$ for $\gamma_0$ satisfies one of the conditions (7) – (9) of Theorem 9.3, then we stop the algorithm to say that $\gamma_0$ does not bound an essential disk in $B^3 - \epsilon$. If the set of weights $m_i$ for $\gamma_0$ satisfies one of the conditions (1) – (4), (6) of Theorem 9.3, then we reduce the sum of $m_i$ by using the formulas for the Dehn’s parameter changes after applying one of four homeomorphism as in Theorem 9.3. Then with the new Dehn’s parameters we can continue to follow this algorithm until either the data in each step fails to bound an essential disk in $B^3 - \epsilon$ or $m_i = 0$ for all $i = 1, 2, ..., 11$. We note that if $m_i = 0$ for all $i$ for $\gamma_0$ then it bounds an essential disk in $B^3 - \epsilon$.

10. Examples of the use of the algorithm

Example 1: $\infty$ tangle and $T$.

Consider $\infty$ tangle as in Figure 4[4]. Then the extension of $\sigma_3 \sigma_1\sigma_2^{-1}\sigma_3\sigma_1$ to $B^3$ makes a rational 3-tangle $T$. For every strings of $T$, if we choose the other two strings then they are isotopic to a trivial rational 2-tangle in $B^3$. However, $T$ is not isotopic to $\infty$ tangle.

In order to show this, we consider $\partial E_2$ which bounds an essential disk in $B^3 - \epsilon$. We notice that $w_{46} = w^{16} = 1$ and all the other weights are zero for $[\partial E_2]$.

Now, consider the simple closed curve $\alpha = \sigma_3 \sigma_1\sigma_2^{-1}\sigma_3\sigma_1(\partial E_2)$. We will show that $\alpha$ does not bound an essential disk in $B^3 - \epsilon$. Also, let $w_{ij}(f)$ and $w^{ij}(f)$ be the weights of $[f(\partial E_2)]$.

Let $f_1 = \sigma_1$, $f_2 = \sigma_3\sigma_1$, $f_3 = \sigma_2^{-2}\sigma_3\sigma_1$, $f_4 = \sigma_1\sigma_2^{-2}\sigma_3\sigma_1$, $f_5 = \sigma_3\sigma_1\sigma_2^{-2}\sigma_3\sigma_1$, and $f_6 = \sigma_5\sigma_3\sigma_1\sigma_2^{-2}\sigma_3\sigma_1$.

From the weight change formulas, we can get $w_{14}(f_1) = w_{56}(f_1) = w^{16}(f_1) = w^{45}(f_1) = 1$ and all the other weights are zero.

From $w_{ij}(f_1)$ and $w^{ij}(f_1)$, we get $w_{15}(f_2) = w_{56}(f_2) = w_{34}(f_2) = w^{35}(f_2) = w^{45}(f_2) = w^{16}(f_2) = 1$ and all the other weights are zero.

From $w_{ij}(f_2)$ and $w^{ij}(f_2)$, we get $w_{14}(f_3) = w_{46}(f_3) = w_{34}(f_3) = w^{36}(f_3) = w^{46}(f_3) = w^{16}(f_3) = 1$ and all the other weights are zero.
From $w_{ij}(f_3)$ and $w_{ij}(f_3)$, we get $w_{14}(f_4) = 2, w_{34}(f_4) = 1, w_{50}(f_4) = 3$ and $w_{15}^{15}(f_4) = w^{35}(f_4) = 1, w^{16}(f_4) = w^{45}(f_4) = 3$ and all the other weights are zero.

From $w_{ij}(f_4)$ and $w_{ij}(f_4)$, we get $w_{15}(f_5) = 4, w_{34}(f_5) = w_{56}(f_5) = 3, w_{35}(f_5) = 1$ and $w_{15}^{15}(f_5) = 1, w^{16}(f_5) = w^{45}(f_5) = 3, w^{35}(f_5) = 4$ and all the other weights are zero. We notice that $w_{ij}(f_5) = w_{ij}(f_6)$ and $w_{kl}(f_5) = w_{kl}(f_6)$.

Finally, we get $w_{15}(f_6) = 4, w_{34}(f_6) = w_{56}(f_6) = 3, w_{35}(f_6) = 1$ and $w_{15}^{15}(f_6) = 1, w^{16}(f_6) = w^{45}(f_6) = 3, w^{35}(f_6) = 4$ and all the other weights are zero. We notice that $w_{ij}(f_5) = w_{ij}(f_6)$ and $w_{kl}(f_5) = w_{kl}(f_6)$.

So, $\alpha$ has $p_1 = w_{15}(f_6) = 4, p_2 = w_{15}(f_6) + w_{56}(f_6) + w_{35}(f_6) = 8$ and $p_3 = w_{34}(f_6) + w_{35}(f_6) = 4$. This implies that $x_{12} = 8, x_{23} = 8$ and all other $x_{ij} = 0$.

Especially, $x_{11} + x_{22} + x_{33} = 0$. Therefore, $\alpha$ does not bound an essential disk in $B^3 - \epsilon$. This implies that $T$ is not isotopic to $\infty$ tangle. We can find the nine parameters for $\alpha$ by using the algorithm to check if $\alpha$ is left-twisted in $E_i'$. We remark that $\alpha$ is parameterized by $(4, 0, -1, 8, 1, -1, 4, 0, 0)$.

Example 2: $T$ and $T'$

Now, consider $T'$ which is obtained by reversing all the crossings in $T$. Then we have Figure 45. We want to check whether $T'$ is isotopic to $T$ or not. Let $f_7 = \sigma_5 f_6, f_8 = \sigma_3 f_7, ..., f_{12} = \sigma_1 f_{11} = \sigma_1 \sigma_3 \sigma_2^{-1} \sigma_1 \sigma_3 \sigma_5 f_6 = (\sigma_5^{-1} \sigma_3^{-1} \sigma_1^{-1} \sigma_2 \sigma_3^{-1} \sigma_1^{-1})^{-1} f_6$. 
We notice that if $T$ is isotopic to $T'$ then $f_{12}(\partial E)$ bound essential disks in $B^3 - \epsilon$.

Consider $\partial E_2$. Then we already got $w_{ij}(f_6)$ and $w^{ij}(f_6)$ in the previous argument.

From $w_{ij}(f_6)$ and $w^{ij}(f_6)$, we can get $w_{ij}(f_7)$ and $w^{ij}(f_7)$. Actually, $w_{ij}(f_7) = w_{ij}(f_6)$ and $w^{ij}(f_7) = w^{ij}(f_6)$.

From $w_{ij}(f_7)$ and $w^{ij}(f_7)$, we get $w_{34}(f_8) = w_{56}(f_8) = 3$, $w_{15}(f_8) = w_{35}(f_8) = 4$ and $w_{15}(f_8) = 1$, $w^{16}(f_8) = w^{45}(f_8) = 3$, $w^{35}(f_8) = 7$.

From $w_{ij}(f_8)$ and $w^{ij}(f_8)$, we get $w_{34}(f_9) = w_{56}(f_9) = 3$, $w_{35}(f_9) = 4$, $w_{15}(f_9) = 7$ and $w^{16}(f_9) = w^{45}(f_9) = 3$, $w^{35}(f_9) = 4$, $w^{35}(f_9) = 7$.

From $w_{ij}(f_9)$ and $w^{ij}(f_9)$, we get $w_{34}(f_{10}) = w_{14}(f_{10}) = 7$, $w_{46}(f_{10}) = 3$, $w_{56}(f_{10}) = 14$ and $w^{16}(f_{10}) = w^{36}(f_{10}) = 7$, $w^{46}(f_{10}) = 3$, $w^{45}(f_{10}) = 14$.

From $w_{ij}(f_{10})$ and $w^{ij}(f_{10})$, we get $w_{15}(f_{11}) = w_{35}(f_{11}) = 7$, $w_{34}(f_{11}) = w_{56}(f_{11}) = 17$ and $w^{16}(f_{11}) = 7$, $w^{36}(f_{11}) = 10$, $w^{35}(f_{11}) = 14$, $w^{45}(f_{11}) = 17$.

Then finally, we get $w_{35}(f_{12}) = 7$, $w_{34}(f_{12}) = w_{56}(f_{12}) = 17$, $w_{15}(f_{12}) = 24$ and $w^{15}(f_{12}) = 7$, $w^{16}(f_{12}) = w^{45}(f_{12}) = 17$, $w^{35}(f_{12}) = 24$ from $w_{ij}(f_{11})$ and $w^{ij}(f_{12})$.

Let $\beta = f_{12}(\partial E_2)$. Then $\beta$ has $p_1 = w_{15}(f_{12}) = 24$, $p_2 = w_{35}(f_{12}) + w_{56}(f_{12}) + w_{15}(f_{12}) = 48$ and $p_3 = w_{35}(f_{12}) + w_{34}(f_{12}) = 24$. Therefore, $x_{12} = 48$, $x_{23} = 48$ and all other $x_{ij} = 0$. 
Especially, $x_{11} + x_{22} + x_{33} = 0$. This implies that $f_{12}(\partial E_2)$ does not bound an essential disk in $B^3 - \epsilon$. Therefore, $T$ is not isotopic to $T'$.

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