A Magnetic Potential-field Downward Continuation Method Based on Accelerated Landweber Iteration

Ze Wang\textsuperscript{1,a}, Qi Zhang\textsuperscript{1,b}, Mengchun Pan\textsuperscript{1,e}, Dixiang Chen\textsuperscript{1,d\textsuperscript{*}}, Zhongyan Liu\textsuperscript{1,e}, Jiafei Hu\textsuperscript{1,f}, Zhuo Chen\textsuperscript{1,g}, Xintian Ren\textsuperscript{1,h}, Zhe Zeng\textsuperscript{1,i} and Zhenxiong Wang\textsuperscript{1,j}

\textsuperscript{1}College of Intelligence Science and Technology, National University of Defense Technology, Changsha, Hunan, China

\textsuperscript{a}email: renxintian14@nudt.edu.cn, \textsuperscript{b}email: 13141317443@163.com, \textsuperscript{c}email: panmengchun1963@sina.com, \textsuperscript{d}email: garfield_nudt@163.com, \textsuperscript{e}email: liuzhongyan2008@163.com, \textsuperscript{f}email: lansedexin26@126.com, \textsuperscript{g}email: 1876067012@qq.com, \textsuperscript{i}email: zengzhe@gfkd.edu.cn, \textsuperscript{h}email: chendixiang@nudt.edu.cn

Abstract. A high-precision geomagnetic database is the basis of geomagnetic matching navigation. In the construction of a geomagnetic database, it is necessary to use magnetic field extension technology to construct the vertical height relationship of geomagnetic data, in which downward continuation is mathematically ill-posed, and Landweber iterative method is a common solution. However, the convergence rate of the Landweber iterative method for the "non-smooth" solution is very slow, resulting in low delay accuracy in dealing with local magnetic anomalies. Therefore, to solve the problem of slow convergence of the Landweber iterative method, a downward continuation method based on accelerated Landweber iteration is proposed in this paper. In the framework of the iterative method, stable upward continuation is used to connect the observation surface with the magnetic field values on the plane with the same height as upward continuation and downward continuation. Then the residual term is constructed by the vertical first-order derivative relation, and then the residual term is modified by using the low-pass filtering characteristic of the upward continuation operator to suppress the high-frequency noise in the residual term and finally update the iterative value. The simulation results of magnetic field models with different field source depths show that the improved method has faster convergence speed and higher extension accuracy when dealing with shallow local anomalies and deep regional anomalies, and under the condition of shallow source model, the continuation error of the improved algorithm is only 60% of that of Landweber iterative method. At the same time, the measured data also verify that the improved method can achieve fast and high-precision downward continuation of a magnetic field.

1. Introduction

Aerial geophysical exploration provides a means to obtain magnetic anomaly data with high efficiency and high precision and has the advantages of high speed and wide range. The interpretation and analysis of aeromagnetic data are inseparable from the processing and conversion of geomagnetic data, in which downward extension is the core of magnetic field data processing. In magnetic exploration, the downward continuation technique can highlight the local field sources in the shallow layer and suppress large regional anomalies, which is helpful to improve the reliability of geological interpretation. In geomagnetic matching navigation, a high-precision geomagnetic database is the...
premise of geomagnetic matching navigation, and the geomagnetic database can be constructed quickly by using downward continuation technology.

It can be seen that the downward continuation of the potential field plays an important role not only in the geological interpretation of magnetic exploration but also in the establishment of geomagnetic database of a geomagnetic matching navigation.

Mathematically, the upward continuation can obtain a very stable extension result by solving the boundary integral equation, which belongs to the definite solution problem, but the downward continuation calculation is a typical ill-posed problem, in the case of noise in the data, the solution of the problem may not be unique, even if the approximate unique solution is obtained through constraints, the solution will be unstable.

The iterative strategy is usually used to solve the ill-posedness of downward continuation, in which the original downward continuation operator is equivalent to a high-pass filter, and the iterative method is essentially a series of low-pass filters for the original downward continuation operator. However, the low-pass filter is constrained by the number of iterations. when the number of iterations is in a certain range, it can constrain the original downward extension operator and get a stable extension result, but with the continuous increase of the number of iterations, the iterative downward continuation process gradually becomes the original downward extension process, which makes the extension result divergent again.

The integral iterative method and its related iterative method directly use the residual term to update the iterative value without mapping transformation of the residual term, which is easily disturbed by noise and leads to the divergence of the extension results in the later stage of iteration [1]. Taylor series iteration method modifies the residual term through Taylor expansion, which can well retain the useful information of low frequency, but it has no obvious effect on the suppression of high-frequency noise and can not eliminate the noise accumulation in the iterative process [2-3]. The derivative iteration method makes full use of the observed data to improve the convergence speed through the vertical first-order derivative, but it can not guarantee stability and is easily disturbed by high-frequency noise. The Landweber iterative method and its related improved methods are stable downward continuation algorithms, which use the low-pass characteristics of the upward continuation operator to deal with the residual spectrum, which can suppress the high-frequency components [4]. Because the distribution of the magnetic field data in the spatial domain corresponds to the frequency component in the frequency domain, and the filter frequency band of the upward continuation operator is only related to the continuation height and grid resolution, it can not be adjusted according to the frequency component of the data. therefore, the extension effect of the magnetic data of the high-frequency component is poor, that is, the convergence speed of the "non-smooth" solution is very slow [5].

Therefore, because of the slow convergence speed and low extension precision of the Landweber iterative method, this paper first calculates the magnetic field value of the $h$ height above the observation surface as the theoretical value of the plane through stable upward continuation and then extends the preset magnetic field data of the $h$ height below the observation surface upward, respectively, to obtain the continuation value of the $h$ height of the observation surface and the $2h$ height above the observation surface. The difference between the theoretical value and the continuation value of the above two height planes is used as the residual term, and then the upward continuation operator is used to filter the residual term. Finally, the iterative value is updated until the error converges to the desired accuracy. Through the model simulation comparison of different buried depths of field sources, it is shown that the improved method can effectively accelerate the convergence speed, and the extension accuracy is better than that of the Landweber iterative method, and stable extension results are obtained on the measured data.
2. Landweber Iteration Method and Improved Method

2.1. Landweber Iteration Method

In the following paragraphs, all equations are based on space rectangular coordinate system, and the positive direction of the $Z$ axis is set downward. Assuming that the geomagnetic observation plane is $Z = 0$, and its corresponding potential field is:

$$u(x, y, z) |_{z=0} = u(x, y, 0) \quad (1)$$

The plane to be solved is $Z = h$ ($h > 0$), whose potential field is:

$$u(x, y, z) |_{z=h} = u(x, y, h) \quad (2)$$

From the known observation plane $u(x, y, 0)$, the potential field of the plane $u(x, y, h)$, which is the downward continuation of the observation plane for height $h$, can be solved as [6]:

$$u(x, y, 0) = \frac{h}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{u(\xi', \gamma', h)}{[(x - \xi')^2 + (y - \gamma')^2 + h^2]^3} d\xi' d\gamma' \quad (3)$$

Equation (3) belongs to Fredholm integral equations of the first kind, which contains a real symmetric kernel. Therefore, it can be converted into a two-dimensional convolution.

$$u(x, y, 0) = K(x, y) * u(x, y, h) \quad (4)$$

With

$$K(x, y) = \frac{h}{[2\pi(x^2 + y^2 + h^2)^{\frac{3}{2}}]} \quad (5)$$

Such convolution in the space domain can be converted into products in the wavenumber domain, accelerating the operation speed. Furthermore, equation (4) can be transformed into the wavenumber domain by Fourier transform:

$$U(k_x, k_y, 0) = H_{ap}(k_x, k_y) \times U(k_x, k_y, h) \quad (6)$$

In equation (6), $U(k_x, k_y, 0)$ and $U(k_x, k_y, h)$ are the Fourier transforms of $u(x, y, 0)$ and $u(x, y, h)$. They are called the observation plane potential field spectrum and the downward continuation plane potential field spectrum, where $k_x$ and $k_y$ are the wavenumber in $x$ and $y$ axis, respectively.
\[ H_{up}(k_x, k_y) = e^{-h \sqrt{k_x^2 + k_y^2}} \]  \hfill (7)

Equation (7) is the wavenumber response in upward continuation. By inverse operations, the original downward continuation operator can be obtained:

\[ U(k_x, k_y, h) = H_{up}^{-1}(k_x, k_y) \times U(k_x, k_y, 0) \]  \hfill (8)

In the formula, the original downward continuation operator in the wavenumber domain is

\[ H_{down}(k_x, k_y) = H_{up}^{-1}(k_x, k_y) = e^{h \sqrt{k_x^2 + k_y^2}} \]  \hfill (9)

Equation (9) describes \( H_{down}(k_x, k_y) \) as an exponential amplifier, which will result in great response error for minor perturbations in the data. The continuation result is easily disturbed by noise, which leads to the instability of the downward continuation process. Therefore, the Landweber iterative method is used to suppress the high-frequency over-amplification effect of the original lower extension operator, and the iteration calculation process is as follows:

(1) Firstly, the observed plane potential field data \( u(x, y, 0) \) is transformed into wavenumber domain by Fourier transform, and its spectrum \( \hat{u}(k_x, k_y) \) is obtained.

(2) The potential field spectrum \( u_0(k_x, k_y) \) of the observation plane is taken as the initial value \( u^{(0)}_n(k_x, k_y) \) of the potential field of the plane to be extended.

(3) The \( u^{(0)}(k_x, k_y) \) is extended upward to the observation plane to obtain the continuation value \( H_{up}(k_x, k_y) \times u^{(0)}(k_x, k_y) \), and the residual spectrum \( \epsilon^{(0)} = u^{(0)}(k_x, k_y) - H_{up}(k_x, k_y)u^{(0)}(k_x, k_y) \) between the observed potential field spectrum and the upward continuation spectrum is calculated.

(4) Modification of residual Spectrum \( \Delta \epsilon = \epsilon^{(0)} \) by upward continuation operator.

(5) Update the plane spectrum \( u^{(i)}(k_x, k_y) = u^{(i)}_0(k_x, k_y) + \Delta \epsilon \) to be extended;

(6) Repeat steps (3) ~ (5) until the error \( |\epsilon^{(i)}| < \varepsilon \) (\( \varepsilon \) is a very small number) or reach the maximum number of iterations, that is, stop the iteration;

(7) Finally, the potential field data \( u(x, y, h) \) of the downward continuation plane is obtained by inverse Fourier transform of the plane spectrum \( u^{(n)}_n(k_x, k_y) \) to the spatial domain.

2.2. Improvement Iteration Method

Through the analysis of the iterative process of the Landweber iterative method, it is found that the main factors affecting the convergence speed are as follows:

(1) The residual term in the iterative process does not make full use of the high-frequency component of the geomagnetic data;

(2) The filter operator for correcting the residual term is an upward extension operator, which has the function of smoothing, suppressing noise, and filtering some high-frequency useful signals at the same time.

The high-frequency component of geomagnetic data can accelerate the iterative speed, in which the high-frequency component includes high-frequency useful signals and noise. The idea of the improved algorithm is to make full use of high-frequency useful signals and restrain the accumulation of noise in the iterative process. Prevent the extension result from diverging. Therefore, on the premise of ensuring the stability of the algorithm, the upward continuation operator is still used to modify the residual term. Starting from the definition of the vertical first derivative of the potential field, the observation surface is approximately connected with the potential field value on the same height plane of upward continuation and downward continuation, and the residual term is improved [7].
\[ \Delta U(x, y) = \frac{U(k_x, k_y, 0) - U(k_x, k_y, h)}{h} \quad (h \to 0) \quad (10) \]
\[ \Delta U(x, y) = \frac{U(k_x, k_y, 0) - U(k_x, k_y, -h)}{-h} \quad (h \to 0) \quad (11) \]

Among them: \( U(k_x, k_y, 0) \) is the potential field spectrum on the observation surface, which is known; \( U(k_x, k_y, h) \) is the potential field spectrum on the upward continuation the \( h \) height plane \( U(k_x, k_y, 0) \), which can be obtained; \( U(k_x, k_y, -h) \) is the potential field wave spectrum of the downward continuation the \( h \) height plane \( U(k_x, k_y, 0) \).

The simultaneous formula (10) and (11), the potential field spectrum relations of three different height planes \( z = h, z = 0, z = -h \):
\[ U(k_x, k_y, h) = 2U(k_x, k_y, 0) - U(k_x, k_y, -h) \quad (12) \]

Among them, when \( h \to 0 \) is established, by the integral formula of the upward continuation the \( h \) height \( U(k_x, k_y, -h) \) can be obtained:
\[ U(k_x, k_y, -h) = H_{up}(k_x, k_y) \times U(k_x, k_y, 0) \quad (13) \]

By using the upward continuation formula of the potential field, the first order approximate spectrum of the upper field of the known observation plane \((z = 0)\) and the observation plane \((z = -h)\) of the upward continuation the \( h \) height plane can be obtained from \( U^{(0)}(k_x, k_y, h) \), that is:
\[ U^{(1)}(k_x, k_y, 0) = H_{up}(k_x, k_y) \times U^{(0)}(k_x, k_y, h) \quad (14) \]
\[ U^{(1)}(k_x, k_y, -h) = [H_{up}(k_x, k_y)]^2 \times U^{(0)}(k_x, k_y, h) \quad (15) \]

The \( h \) height and \( 2h \) height of the potential field spectrum to be extended downwards are compared with the theoretical potential field spectrum of these two height planes respectively, and the error correction term is obtained by using the formula (12) vertical equal height potential field relation.
\[ \Delta U = 2 \times [U(k_x, k_y, 0) - U^{(n+1)}(k_x, k_y, 0)] - [U(k_x, k_y, -h) - U^{(n+1)}(k_x, k_y, -h)] \quad (16) \]

In the error correction term, the observed data and the potential field data of \( h \) height are regarded as the theoretical "true values" of the \( z = 0 \) and \( z = -h \) planes. Through stable upward extension, the useful information in the observation data is fully utilized to speed up the convergence rate. At the same time, the noise accumulation effect is also accelerated accordingly, so the error correction term is processed by low-pass filtering with the help of the upward continuation operator to suppress the interference of noise. The iterative process is updated mainly through low-frequency useful information:
\[ U^{(n+1)}(k_x, k_y, h) = U^{(n)}(k_x, k_y, h) + H_{up} \times \Delta U \quad (17) \]

Therefore, based on ensuring stability, the improved algorithm introduces the vertical first derivative of the potential field to make full use of the high-frequency useful signals in the geomagnetic data to speed up the convergence speed. The specific algorithm flow is as follows:
(1) First of all, the observed plane potential field data \( u(x, y, 0) \) is transformed into the wavenumber domain by Fourier transform, and the spectral \( U(k_x, k_y, 0) \) of the observed plane potential field data is obtained.

(2) The potential field spectrum \( U(k_x, k_y, 0) \) of the observation plane is taken as the initial value \( U^{(0)}(k_x, k_y, h) \) of the potential field of the plane to be extended downward.

(3) By using the formula (13), the \( U(k_x, k_y, 0) \) is extended upward to the observation plane, and the \( U(k_x, k_y, -h) \) is obtained.

(4) By using the formulas (14) and (15), the \( U^{(0)}(k_x, k_y, h) \) is extended upward to \( h \) height and \( 2h \) height, respectively, and the upward continuation value \( U^{(0)}(k_x, k_y, h), U^{(0)}(k_x, k_y, -h) \) is obtained.

(5) Through the vertical first derivative relation, the error correction term is obtained by using the formula:

\[
\Delta U = 2\times[U(k_x, k_y, 0) - U^{(0)}(k_x, k_y, 0)] - [U(k_x, k_y, -h) - U^{(0)}(k_x, k_y, -h)]
\]

(18)

(6) After the error correction term is processed by the upward continuation operator, the plane spectrum \( U^{(1)}(k_x, k_y, h) = U^{(0)}(k_x, k_y, h) + H_{\text{up}} \times \Delta U \) to be extended downward is updated.

(7) Repeat step (4) \( \sim \) (6) and iterate continuously to correct the continuation value until the error \( \max|U(k_x, k_y, 0) - U^{(0)}(k_x, k_y, 0)| < \varepsilon \) (\( \varepsilon \) gives a smaller number) or when the maximum number of iterations is reached, that is, the iteration is stopped.

(8) Finally, the potential field data \( u^{(n)}(x, y, h) \) of the downward continuation plane is obtained by inverse Fourier transform of the plane spectrum \( U^{(n)}(k_x, k_y, h) \) to the spatial domain.

It can be seen from the iterative process that two upward continuation integral equations with different heights need to be realized in each iteration. Compared with the Landweber iterative method, the observed data information can be more fully utilized, and the high-frequency interference can be eliminated by low-pass filtering of the upward continuation operator.

3. Results & Discussion

3.1. The simulation

The combined sphere model was used in the simulation test, and parameters of five spherical magnetic sources in the model are shown in Table 1 (z-axis direction upward was set as the positive direction).

| X0 (m) | Y0 (m) | Geomagnetic declination | Magnetization deflection angle | Radius(m) | Magnetic susceptibility (A/m) |
|--------|--------|-------------------------|-------------------------------|-----------|-------------------------------|
| 0      | 0      | 45°                     | 45°                           | 100       | 100                           |
| 3000   | 3000   | 45°                     | 45°                           | 100       | 100                           |
| 3000   | 7000   | 45°                     | 45°                           | 100       | 100                           |
| -4000  | 8000   | 45°                     | 45°                           | 100       | 100                           |
| -5000  | -9000  | 45°                     | 45°                           | 100       | 100                           |

Magnetic anomalies are generally superimposed anomalies, in which regional anomalies are mainly caused by middle and deep geological factors which are widely distributed. This anomaly is characterized by large amplitude and a large range in the spatial domain, but small horizontal gradient, corresponding to the low-frequency components of geomagnetic data. Local anomalies mainly refer to
the anomalies caused by shallow materials with small spatial distribution and amplitude, and the horizontal gradient of the anomaly is relatively large, corresponding to the high-frequency components of geomagnetic data.

Then, by setting two groups of simulation experiments with different field source depths to simulate regional anomalies and local anomalies, the extension effects of the Landweber iterative method and improved algorithm in dealing with data of different frequency components are compared.

The extension effects of the two iterative methods are analyzed quantitatively by the root mean square error.

\[
\Delta g_{\text{err}} = \sqrt{\frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} (g_m(n) - g_i(m,n))^2}
\]  

(19)

3.1.1. Shallow source model
The buried depth of the shallow source model was set as 200m underground, and the geomagnetic maps for height planes at Z = 300m (Fig. 2 (a)) and Z = 800m (Fig. 2 (b)) were obtained by simulation as the theoretical planes. The Landweber iterative method (Fig. 3 (a)) and the proposed improved method (Fig. 3 (b)) were used to continue Fig. 2 (b) downward for 500 m, and the number of iterations was set to 750. Results showed that the improved method can lead to perfect continuation in the main region of the magnetic source at extremely high-frequencies, as the results from it almost fit the theoretical plane completely. However, despite that, the Landweber method would weaken the Gibbs effect on the boundary, but the high frequency components of the main body were smoothed, resulting in a poor downward extension and unwanted oscillation.

Additionally, the root means square error curves (Fig. 4) showed that improved method not only accelerated convergence but also exhibited high continuation precision and excellent stability. By contrast, the Landweber iterative method demonstrated poor continuation in high frequencies because of its slow convergence.

![Fig. 2. Simulation magnetogram of two height planes (shallow source model)](image-url)
3.1.2. Deep source model

Nevertheless, when the continuation is performed to process geomagnetic data at other burial depths, the outcomes could be very different. Therefore, the burial depth of the model was reset as 1000m underground, and the geomagnetic maps at two different heights (Fig. 5(a), Fig. 5(b)) under the deep source model were obtained by simulation. The two methods (Fig. 6 (a), Fig. 6 (b)) were still used to continue Fig. 5(b) downward 500m, and the number of iterations was set to 150. It can be seen that both methods can produce better continuation at regions with small variation gradients.

Root mean square error curves (Fig. 7) showed that the Landweber iterative method and the improved method exhibited great continuation accuracy in the deep source model, but the improved method demonstrated even higher convergence speed.
Fig. 5. Simulated magnetic map of two height planes (deep source model)

Fig. 6. Extension results of two iterative methods

Fig. 7. Mean square error curves of two iterative methods

Through the model simulation experiments of two groups of different field source depth, the geomagnetic data obtained by the magnetic field model with the depths of 200m account for a large proportion of high-frequency components, and the depth of 1000m is mainly composed of low-frequency components. It is verified that the improved algorithm has a fast convergence speed when dealing with geomagnetic data with different frequency components. The minimum root means square
error in Table 2 shows that the improved algorithm has higher extension accuracy under both regional anomaly and local anomaly magnetic field models. Because the residual term of the Landweber iterative method does not make full use of the high-frequency component of observed magnetic data, the results are different when dealing with different frequency components, and the extension effect of low-frequency data is good, but the extension error is large in high-frequency data.

| Model          | Method          | Landweber method | Improvement method |
|----------------|-----------------|------------------|--------------------|
| Shallow source | 4.9749 nT       | 3.5305 nT        |
| Deep source    | 0.8737 nT       | 0.7556 nT        |

3.2. Simulation with Real Data
The aeromagnetic survey data on the Kluane area was used to examine the practicability of the proposed method, which is the same as that explored by Pilkington and Boulanger [8-9]. The data (Fig. 8(a)) were taken from the Canadian Geological Survey Data Service. Firstly, the aeromagnetic data was continued up to 800m as the observation plane potential field (Fig. 8(b)), followed by downward continuation for 500m by the Landweber method (Fig. 8(c)) and the Improvement method (Fig. 8(d)). It can be seen that the two iterative methods of interest revealed better continuation outcomes in areas with gentle variations in the field value. However, in Fig. 8(d), the morphological features in the high-frequency region in the potential were stably and accurately characterized despite sharp spatial gradients, and the downward continuation anomalies were consistent with the aeromagnetic ones without loss of abnormal information.

Fig. 8. Continuation effect of each iterative method for measured data
4. Conclusions
In this paper, a downward continuation method for accelerating Landweber iteration is proposed, which aims at the problems of poor effect and slow convergence of the Landweber iteration method in local anomaly time delay. The vertical first derivative is introduced, and the difference between the theoretical value and the continuation value of the two height planes is used as the residual term to make full use of the high-frequency component of the observed data and the suppression of high-frequency noise by the low-pass filtering characteristic of the upward continuation operator. The cumulative effect of noise in the iterative process is solved, which can accelerate the convergence speed and obtain higher extension accuracy while improving the anti-noise interference ability of the algorithm. And in the practical application of potential field transformation, the problem of local amplification of conversion factor has been greatly alleviated, which is a more practical method for a downward continuation of the potential field.

References
[1] Zhikui Guo, Chunhui Tao. (2020) Potential field continuation in spatial domain: A new kernel function and its numerical scheme. Computers and Geosciences, 136.
[2] J. Wang, X.H. Meng and Z.W. Zhou. (2018) A Constrained Scheme for High Precision Downward Continuation of Potential Field Data. Pure and Applied Geophysics, 175: 3511-3523.
[3] M. Pilkington and O. Boulanger. (2017) Potential field continuation between arbitrary surfaces — comparing methods. Geophysics, 82: 9-25.
[4] S.Z. Xu. (2006) The integral-iteration method for continuation of potential fields. Chin. J. Geophys, 49: 1054-1060.
[5] G. Cooper. (2004) The stable downward continuation of potential field data. Explor. Geophys. 35: 260-265.
[6] X.N. Zeng, X.H. Li, S.Q. Han et al. (2011) A comparison of three iteration methods for downward continuation of potential fields. Progress in Geophys (in Chinese). 26: 908-915.
[7] L. Landweber, (1951) An iterative formula for Fredholm integral equations of the first kind. Am. J. Math. 73: 615-624.
[8] M. Abedi, A. Gholami and G.H. Norouzi. (2013) A stabled downward continuation of airborne magnetic data: a case study for mineral prospectivity mapping in Central Iran. Comput. Geosci. 53: 269-280.
[9] M. Fedi and G. Florio. (2011) Normalized downward continuation of potential fields within the quasi-harmonic region. Geophys Prospect. 59: 1087-1100.