Hull and geodetic numbers for some classes of oriented graphs

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November 26, 2019

Abstract

An oriented graph $D$ is an orientation of a simple graph, i.e. a directed graph whose underlying graph is simple. A directed path from $u$ to $v$ with minimum number of arcs in $D$ is an $(u,v)$-geodesic, for every $u, v \in V(D)$. A set $S \subseteq V(D)$ is (geodesically) convex if, for every $u, v \in S$, all the vertices in each $(u,v)$-geodesic and in each $(v,u)$-geodesic are in $S$. For every $S \subseteq V(D)$ the (convex) hull of $S$ is the smallest convex set containing $S$ and it is denoted by $[S]$. A hull set of $D$ is a set $S \subseteq V(D)$ whose hull is $V(D)$. The cardinality of a minimum hull set is the hull number of $D$ and it is denoted by $\vec{hn}(D)$. A geodetic set of $D$ is a set $S \subseteq V(D)$ such that each vertex of $D$ lies in an $(u,v)$-geodesic, for some $u, v \in S$. The cardinality of a minimum geodetic set is the geodetic number of $D$ and it is denoted by $\vec{gn}(D)$.

In this work, we first present an upper bound for the hull number of oriented split graphs. Then, we turn our attention to the computational complexity of determining such parameters. We first show that computing $\vec{hn}(D)$ is NP-hard for partial cubes, a subclass of bipartite graphs, and that computing $\vec{gn}(D)$ is also NP-hard for directed acyclic graphs (DAG). Finally, we present a positive result by showing how to compute such parameters in polynomial time when the input graph is an oriented cactus.

Keywords: Convexity, oriented graphs, hull number, geodetic number, computational complexity

1 Introduction

For basic notions on graph theory and computational complexity, the reader is referred to [4, 14]. All graphs in this work are simple and finite, unless explicitly

*This work was partially supported by a CNPq/Funcap project PNE-0112-00061.01.00/16 SPU: 4543945/2016, a CNPq Universal project 401519/2016-3 and CNPq grants 310234/2015-8 and 130467/2018-9.
stated otherwise.

Although the first papers related to convexity in graphs study directed
graphs \[21, 18, 11\], most of the papers we can find in the literature about graph
convexities deal with undirected graphs. For instance, the hull and geodetic
numbers with respect to undirected graphs \[12, 15\] were first studied in the
literature around a decade before their corresponding directed versions \[5, 7\].

An oriented graph \(D\) is an orientation of a simple graph, i.e. a directed
graph whose underlying graph is simple. A directed path from \(u\) to \(v\) with
minimum number of arcs in \(D\) is an \((u, v)\)-geodesic, for every \(u, v \in V(D)\). A
set \(S \subseteq V(D)\) is (geodesically) convex if, for every \(u, v \in S\), all the vertices in
each \((u, v)\)-geodesic and in each \((v, u)\)-geodesic are in \(S\). For every \(S \subseteq V(D)\)
the (convex) hull of \(S\) is the smallest convex set containing \(S\) and it is denoted
by \([S]\). A hull set of \(D\) is a set \(S \subseteq V(D)\) whose hull is \(V(D)\). The cardinality
of a minimum hull set is the hull number of \(D\) and it is denoted by \(\rightarrow \text{hn}(D)\).

It is important to emphasize that as \(D\) is an orientation of a simple graph,
then it cannot have both arcs \((u, v)\) and \((v, u)\), for distinct \(u, v \in V(D)\). Thus,
the parameters \(\rightarrow \text{hn}(D)\) and \(\rightarrow \text{gn}(D)\) are not equivalent to their undirected versions.
For instance, the hull and geodetic numbers of a path \(P\) on \(2k\) edges, for some
positive integer \(k\), are both equal to two in the undirected version, while if \(D\)
is an orientation of \(P\) we can have both \(\rightarrow \text{hn}(D)\) and \(\rightarrow \text{gn}(D)\) ranging from 2 to
\(2k + 1\).

With respect to the directed case, most results in the literature provide
bounds on the maximum and minimum values of \(\rightarrow \text{hn}(D(G))\) and \(\rightarrow \text{gn}(D(G))\)
among all possible orientations \(D(G)\) of a given undirected simple graph \(G\) \[5, 7, 13\]. It is important to emphasize the results on the parameter \(\text{hn}^+(G)\), the
upper orientable hull number of a graph \(G\), since these are the only ones re-
lated to the upper bound we present. Such parameter is defined in \[5\], as the
maximum value of \(\rightarrow \text{hn}(D(G))\) among all possible orientations \(D(G)\) of a simple
graph \(G\). In the same article, the authors prove that for a non-oriented graph
\(G\), \(\text{hn}^+(G) = n(G)\) if and only if there is an orientation \(D(G)\) such that ev-
ey vertex is extreme. They also compare this parameter with others, such as
the lower orientable hull number \((\text{hn}^-(G))\) and the lower and upper orientable
geodetic numbers \((\text{gn}^-\text{ } (G))\) and \(\text{gn}^+\text{ } (G)\) respectively, defined analogously.

There are also few results about some related parameters: the forcing hull
and geodetic numbers \[20, 6\], the pre-hull number \[19\] and the Steiner num-
ber \[16\] are a few examples.

In this work, we first present a general tight upper bound on the hull number
of an oriented split graph, in Section 3. Note that such bound is also an upper
bound to \(\text{hn}^+(G)\), whenever \(G\) is a split graph.

Then, we consider as input an oriented graph \(D\) and we study the computa-
tional complexity of determining \(\rightarrow \text{hn}(D)\) and \(\rightarrow \text{gn}(D)\), when the underlying graph
of \(D\) belongs to some particular graph class. Up to our best knowledge, this is
the first work to consider such questions.

It is known that determining the hull number of an undirected partial cube is NP-hard [1]. In Section 4, we show that such result can be used to prove that determining whether \( \overrightarrow{hn}(D) \leq k \), when \( D \) is an oriented partial cube, is NP-complete. Although the proof requires a careful analysis, the idea is quite simple: by replacing each edge of a partial cube \( G \) with a directed \( C_4 \), we obtain an oriented graph \( D \) whose underlying graph is a partial cube, and whose hull number \( \overrightarrow{hn}(D) \) is the same as \( hn(G) \). It is important to recall that partial cubes are bipartite graphs. In the same section, we also prove that determining if \( \overrightarrow{gn}(D) \leq k \) is also an NP-complete problem, even if \( D \) is a directed acyclic graph (DAG) whose underlying graph is bipartite.

Finally, we prove in Section 5 that \( \overrightarrow{hn}(D) \) and \( \overrightarrow{gn}(D) \) can be computed in polynomial time if \( D \) is a cactus, i.e., a graph whose blocks are either edges or induced cycles.

In Section 6, we present avenues for further research.

2 Preliminaries

A directed graph \( D = (V, A) \) whose underlying graph is simple is an oriented graph. Given an oriented graph \( D \), a (directed) \( (u, v) \)-path \( P \) is a subgraph of \( D \) such that \( V(P) = \{u, u_0, u_1, \ldots, u_k = v \} \) and \( A(P) = \{(u_{i-1}, u_i) \mid i \in \{1, \ldots, k\} \} \). We can also denote it by \( (u, u_1, \ldots, u_{k-1}, v) \); to represent a path in a non-oriented graph \( G \) we remove the parentheses. When \( w \) is a vertex of \( P \) different than \( u \) and \( v \) we say that it is an internal vertex of \( P \). The set of internal vertices of \( P \) we call the interior of \( P \). The length of a path \( P = (u, u_1, \ldots, u_k, v) \) is \( k \). An \( (u, v) \)-path that uses the least number of arcs possible is called an \( (u, v) \)-geodesic. We denote its length by \( d_D(u, v) \) which represents the distance in \( D \) from \( u \) to \( v \). Notice that \( d_D(u, v) \) might not be equal to \( d_D(v, u) \), since we are dealing with directed graphs. In the sequel, whenever \( D \) is clear in the context we only use \( d(u, v) \).

Given a vertex \( v \in V(D) \) we define \( N^-(v) := \{ u \in V(D) \mid (u, v) \in A(D) \} \) and \( N^+(v) := \{ u \in V(D) \mid (v, u) \in A(D) \} \). Moreover we respectively define the indegree and the outdegree of \( v \) by \( d^-(v) := |N^-(v)| \) and \( d^+(v) := |N^+(v)| \).

For two oriented graphs \( D_1, D_2 \) such that \( D_1 \subseteq D_2 \). Given \( D \) an oriented graph and \( C \subseteq D \) such that its underlying graph is a cycle, we say that \( C \) is simply a cycle; the fact that it is oriented is already implied by being a subgraph of \( D \). However, when \( C \) is such that \( V(C) = \{v_1, \ldots, v_n\} \) and \( A(D) = \{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\} \) we say that it is a directed cycle.

The interval function \( I: \mathcal{P}(V(D)) \to \mathcal{P}(V(D)) \) satisfies that, for each vertex set \( S \subseteq V(D) \) with at least two elements, \( I[S] \) is the set of all vertices in an \((u, v)\)-geodesic \((u \text{ and } v \text{ included})\), for every \( u, v \in S \); when \( S \) is unitary we have \( I[S] = S \). For every positive integer \( n \) we recursively define \( I^0[S] = S \) and \( I^n[S] := I[I^{n-1}[S]] \), for every \( n \geq 1 \). A subset \( S \subseteq V(G) \) is convex when
I[S] = S; if this happens, we say that \( S^c = V(D) \setminus S \) is co-convex. The convex hull of \( S \) is the smallest convex set which contains \( S \) and is denoted by \([S]\). There are two interesting properties for this set. One is that it is the intersection of all convex sets containing \( S \). A noteworthy consequence of this fact is that if \( S \) does not intersect a given co-convex set, then its convex hull also does not intersect it. The other is that it is obtained when we iterate the interval function on \( S \) until we reach a convex set \( I^k[S](= [S]) \). Assuming that \( V(D) \) is finite, the convex hull for every subset of \( V(D) \) is well-defined.

If the convex hull of \( S \) is \( V(D) \) we say that \( S \) is a hull set of \( D \). When \( S \) is a hull set of minimum cardinality, the hull number of \( D \) is defined as \( \text{hn}(D) = |S| \) [5]. Similarly, if \( I[S] = V(D) \) we say that \( S \) is a geodetic set of \( D \). When \( S \) is a geodetic set of minimum cardinality, the geodetic number of \( D \) is defined as \( \text{gm}(D) = |S| \) [7]. Notice that a geodetic set is also a hull set, therefore every assertion we make for all hull sets of a given oriented graph \( D \) is also valid for the geodetic sets.

Now that we have presented the main parameters of our research, we define a very important type of vertex. It was first introduced in [12] for the undirected case, however we use the definitions given in [5]. A vertex \( v \in V(D) \) is called extreme if it is of one of the three types below:

1. Transmitter (source): \( d^-(v) = 0 \) and \( d^+(v) \geq 0 \);
2. Receiver (sink): \( d^+(v) \geq 0 \) and \( d^-(v) = 0 \);
3. Transitive: \( d^-(v) > 0, d^+(v) > 0 \) and \( (u, w) \in A(D) \) for every \( u \in N^-(v) \) and \( w \in N^+(v) \).

We denote the set of extreme vertices of an oriented graph \( D \) as \( \text{Ext}(D) \). For undirected graphs we have the simplicial vertices, which are the ones with a clique for neighborhood. What is so interesting about the extreme vertices is that they must be in every hull set of the oriented graph, as shown in [5]. There is a similar result for the simplicials in [8].

### 3 Upper bound on the hull number of an oriented split graph

The hull number problem for (undirected) split graphs has already been studied in [8]. Given a split graph \( G = (S \cup C, E) \) with \( S \) a maximal stable set and \( C \) a clique, the authors prove that \( \text{hn}(G) \in \{|S|, |S| + 1, |\text{Ext}(G)|\} \). Moreover, they prove that \( \text{hn}(G) \) can be computed in linear time, when \( G \) is split.

Notice that in a split graph the vertices of \( S \) are simplicials. Thus, they belong to any hull set in the non-oriented case. However, in an orientation of \( G \), maybe none of these vertices is extreme. The extreme vertices of \( S \) must be in every hull set of \( D \) (an orientation of \( G \)), leaving the non-extreme of \( S \) to analyse.
Lemma 3.1. If $D = (S \cup C, A)$ is an oriented split graph, then $S \setminus \text{Ext}(D) \subseteq I[C]$.

Proof. Given a non-extreme vertex $v \in S$ (notice that $d_G(v) \geq 2$), by definition there must be $u, w \in C$ such that $(u, v), (v, w), (w, u) \in A(D)$, which means that $v \in I[\{u, w\}]$. Using that argument for every non-extreme vertex of $S$ gives us the proof of the lemma.

We then focus on the hull number problem for tournaments. From now on, until we state otherwise, $D$ will denote a tournament.

Let us first observe that extreme vertices in a tournament do not need to be considered when applying the interval function, as no $(u, v)$-geodesic with non-empty interior starts or ends in an extreme vertex.

Lemma 3.2. Let $D$ be a tournament and $u, v \in V(D)$ two distinct vertices such that $u \in \text{Ext}(D)$. Then $I[\{u, v\}] = \{u, v\}$.

Proof. First suppose that $u$ is a source. Since $D$ is a tournament the only $(u, v)$-geodesic has exactly one arc: $(u, v)$. Seen as $u$ is a source there cannot be a $(v, u)$-path, which concludes the argument for $I[\{u, v\}] = \{u, v\}$. The case in which $u$ is a sink is analogous.

Now let $u$ be a transitive extreme vertex. Without loss of generality we assume that $(u, v) \in A(D)$, which means that the only $(u, v)$-geodesic has just one arc. Suppose that there is a $(v, u)$-path $P = (v = v_0, v_1, \ldots, v_k = u)$, where $k \geq 2$. If we had an $i \in \{1, \ldots, k-1\}$ such that $(v_i, u), (u, v_{i-1}) \in A(D)$, by the transitivity of $u$ we would also have $(v_i, v_{i-1}) \in A(D)$, contradicting our assumption of $P$. Therefore, since $(v_{k-1}, u) \in A(D)$ we must have $(v_{k-2}, u) \in A(D)$, which in turn leads to $(v_{k-3}, u) \in A(D)$ and so on. Eventually, we conclude that $(v_1, u) \in A(D)$. However, since $(u, v) \in A(D)$ then $(v_1, v) \in A(D)$, which cannot happen because $(v, v_1) \in A(P)$. Thus, we reach the conclusion that there cannot be a $(v, u)$-path, which implies $I[\{u, v\}] = \{u, v\}$.

A consequence of this fact is that the extreme vertices will not exert any influence on the hull sets.

Corollary 3.3. If $D$ is a tournament, then $S$ is a hull set of $D$ if, and only if, $\text{Ext}(D) \subseteq S$ and $|S \setminus \text{Ext}(D)| = |V(D) \setminus \text{Ext}(D)|$.

Proof. Let $X = \text{Ext}(D)$ to simplify. If $S \subseteq V(D)$ is such that $X \subseteq S$ and $|S \setminus X| = |V(D) \setminus X|$ then $|S| \geq |S \setminus X| \cup X = V(D)$, which implies that $S$ is a hull set.

Now assume that $S$ is a hull set of $D$. It is straightforward that $X \subseteq S$. By
Lemma 3.2, we have:

\[ I[S] = I[X \cup (S \setminus X)] = \bigcup_{u,v \in S} I[u, v] \]
\[ = \left( \bigcup_{u,v \in X} I[u, v] \right) \cup \left( \bigcup_{u \in S \setminus X, \ v \in X} I[u, v] \right) \cup \left( \bigcup_{u,v \in S \setminus X} I[u, v] \right) \]
\[ = X \cup (X \cup S) \cup I[S \setminus X] = X \cup I[S \setminus X]. \]

Since \( I[S \setminus X] \) does not contain any extreme vertex, we can iterate the above argument and claim that \( I^n[S] = X \cup I^n[S \setminus X] \), for every \( n \in \mathbb{N} \). We thus conclude that \( V(D) = [S] = X \cup [S \setminus X] ; V(D) \setminus X = [S \setminus X]. \)

Let us then analyse the non-extreme vertices of \( D \). Given a non-extreme vertex \( v \in V(D) \) there must be \( u, w \in V(D) \) such that \((u, v), (v, w), (w, u) \in A(D)\), which implies that \( u, w \) are also non-extreme vertices. This means that each non-extreme vertex \( v \) lies in a directed \( C_3 \subseteq D \), and consequently \( v \) is an internal vertex of an \((u, w)\)-geodesic.

Thus, for each non-extreme vertex \( v \) that does not belong to a minimum hull set \( S \) of \( D \), at most two other vertices must belong to \( S \) to ensure that \( v \in [S] \).

After presenting a few definitions, we will be ready to prove the main result for tournaments.

Since we will only use \( C_3 \)'s in our arguments, assuming \( |S| \geq 2 \), we define \( I_{[S]} \) analogous to \( I \) but using \( I_{[S]} \).

Let \( D \) and \( D' \) be oriented graphs. The lexicographic product of \( D \) by \( D' \), denoted by \( D \circ D' \), is the oriented graph which satisfies \( V(D \circ D') = V(D) \times V(D') \) and \((u_1, v_1), (u_2, v_2) \in A(D \circ D')\) if, and only if, either \((u_1, u_2) \in A(D)\) or \(u_1 = u_2\) and \((v_1, v_2) \in A(D')\). In other words, for each vertex \( v \in V(D) \) we take a copy of \( D' \), namely \( D'_v \), and if \((u, v) \in A(D)\), then we add the arcs \((u', v')\), for each \( u' \in V(D'_v) \) and each \( v' \in V(D'_v) \).

A transitive orientation of a simple graph \( G \) is an oriented graph \( D \) obtained from \( G \) such that every vertex is extreme. In this case, a transitive orientation of a complete graph is a transitive tournament.

**Proposition 3.4.** Let \( D \) be a tournament. Then \( \overrightarrow{hn}(D) \leq |Ext(D)| + \frac{2}{3}|V(D)\setminus Ext(D)| \) and this bound is tight. Moreover, there is a set \( S \subseteq V(D) \) such that \( |S| \leq \frac{2}{3}(n(D) - |Ext(D)|) \) and \( [S]_{C_3} = V(D) \setminus Ext(D) \).

**Proof.** By Corollary 3.3, we have that \( S' = Ext(D) \cup S \) with \( Ext(D) \cap S = \emptyset \) is a hull set of \( D \) if and only if \( |S| = V(D) \setminus Ext(D) \). Thus, we only need to find an \( S \) respecting the conditions presented above.

To make the writing easier, we denote \( X := Ext(D) \) and \( V := V(D) \setminus X \). Since \( v \in V \) is not extreme and \( D \) is a tournament, there are \( u, w \in V \) such
that \((u, v), (v, w), (w, u) \in A(D)\). Since \((u, w) \in A(D)\), we have that a \((w, u)\)-geodesic must have length at least two. Then, as the path \((u, v, w)\) exists in \(D\), it is a geodesic. We thus conclude that \(v \in I_{C_3}^{-1}\{(u, w)\}\).

Next, we iteratively construct the set \(S\). Initially let \(S = \{u_1, u_2\}\), with \(u_1, u_2\) belonging to a common \(C_3\) (notice that \(\|S\|_{C_3} \geq 3\)) and take \(v \in V \setminus S \cap C_3\). As we already said there are \(v_1, v_2 \in V\) such that \((v, v_1), (v_1, v_2), (v_2, v) \in A(D)\). We cannot have \(v_1, v_2 \in S\) because it would result in \(v \in S \cap C_3\), contradicting its choice. If \(v_1, v_2\) are both not in \(S\), then we repeat the process and add them to \(S\). Observe that we will have \(v \in S\) \(\cap C_3\), thus obeying the bound.

Now suppose that for every pair \(v_1, v_2\) as described above we have \(v_1 \in S \cap C_3\) and \(v_2 \notin S \cap C_3\), without loss of generality. Since \(v_1 \in S \cap C_3\) there are \(v_3, v_4 \in S \cap C_3\) such that \((v_1, v_3), (v_3, v_4), (v_4, v_1) \in A(D)\). If we had \((v_2, v_4) \in A(D)\), we would also have the directed \(C_3\) with vertex set \(\{v_1, v_2, v_4\}\), contradicting our assumption. Thus, we must have \((v_4, v_2) \in A(D)\). We can use an analogous argument to show that we must also have \((v_3, v_1) \in A(D)\). This leads us to a contradiction: the arc \((v, v_4)\) would give us the directed \(C_3\) with vertex set \(\{v, v_4, v_2\}\), and the arc \((v_4, v)\) would give us the directed \(C_3\) with vertex set \(\{v, v_3, v_4\}\). Therefore, this situation cannot happen, which means that we always have \(v_1, v_2\) in one of the two cases presented in the previous paragraph.

So, we always add two vertices at a time to \(S\), which in turn increases the cardinality of \(S \cap C_3\) by at least three units. In the end, we obtain a set \(S\) such that \(\|S\|_{C_3} = V\) and \(\|S\| \leq \frac{2}{3}(n(D) - \|X\|)\). With that in mind, we deduce that \(\overline{\text{hn}}(D) \leq \|X\| + \frac{2}{3}(n(D) - \|X\|)\), as \(S \cup X\) is a hull set of \(D\).

For the second part of this proof, we construct a tournament \(D\) such that \(\overline{\text{hn}}(D) = \|X\| + \frac{2}{3}(n(D) - \|X\|)\). See Figure 1 for an illustration of the following construction. To ease our arguments we can take a tournament without extreme vertices. Let \(K_5\) be a transitive orientation of \(K_5\) such that \(V(K_5) = \{1, 2, 3, 4, 5\}\) and \(A(K_5) = \{(i, j) | i, j \in V(K_5)\text{ and } i < j\}\), and let \(C_3\) be a directed \(C_3\) with vertex set \(\{u, v, w\}\). Our tournament will be \(D = K_5 \circ C_3\).

Take \((i, u), (j, v) \in V(D)\) such that \(i < j\), then \((i, u), (j, v) \in A(D)\). Define \(D_i\), for every \(i \in [5]\), as the copy of \(C_3\) in \(D\) corresponding to the vertex \(i\) of \(K_5\). The existence of a \((i, j, v), (i, u))\)-path implies that there are \((i', v'), (j', v') \in V(D)\) such that \(i' < j'\) and \((i, j, v), (i', v') \in A(D)\), thus contradicting the definition of \(K_5\). Moreover, that eliminates the possibility of an \((i, u), (i, v))\)-path that is not contained in \(D_i\). From this we conclude that for every \(S \subseteq V(D)\) we have \(I[S] = \bigcup_{i=1}^{5} I[S \cap V(D_i)]\) and \(I[S \cap V(D_i)] \subseteq V(D_i)\) for every \(i \in [5]\). Moreover, that also eliminates the possibility of \((i, u), (i, v))\)-path that is not contained in \(D_i\). From this we conclude that for every \(S \subseteq V(D)\) we have \(I[S] = \bigcup_{i=1}^{5} I[S \cap V(D_i)]\) and \(I[S \cap V(D_i)] \subseteq V(D_i)\) for every \(i \in [5]\). Therefore, \(\|S\| = V(D)\) if and only if \(\|S \cap V(D_i)\| = V(D_i)\) for every \(i \in [5]\), which means that \(\overline{\text{hn}}(D) = \sum_{i=1}^{5} \overline{\text{hn}}(D_i) = 5\overline{\text{hn}}(C_3)\).

Since \(V(C_3) \geq 2\) we need at least two vertices to form a hull set of \(C_3\). Defining \(A(C_3) = \{(u, v), (v, w), (w, u)\}\) it is easy to see that \(\overline{\text{hn}}(D_2) = 2\). We then have \(\overline{\text{hn}}(D) = 10 = \frac{2}{3}15 = \frac{2}{3}n(D)\).

One may observe that the tight example in Proposition 3.4 can be easily
generalized to have an arbitrarily large number of vertices, by replacing the transitive orientation of a $K_5$ with one of a $K_n$ for any positive integer $n$.

![Figure 1: Tight example to Proposition 3.4.](image)

Now we return to oriented split graphs. Since in Proposition 3.4 we use only paths of length at most two, we can still find a subset of vertices $C'$ in the clique $C$ such that $[C' \cup \text{Ext}(D[C])] \supseteq C$ and $|C'| \leq \frac{2}{3}|C \setminus \text{Ext}(D[C])|$. We have seen before that $S \setminus \text{Ext}(D) \subseteq I[C]$, from where we may deduce the following:

**Corollary 3.5.** Let $D = (S \cup C, A)$ be an oriented split graph such that $S$ is maximal and $|C| \geq 2$. Then, $\overrightarrow{hn}(D) \leq |\text{Ext}(D) \cap S| + |\text{Ext}(D[C])| + \frac{2}{3}|C \setminus \text{Ext}(D[C])|$.

**Proof.** This is a direct consequence of Lemma 3.1 and Proposition 3.4.

## 4 NP-completeness for oriented bipartite graphs

In this section, we prove that, given a directed bipartite graph $G$ and a positive integer $k$, determining whether $\overrightarrow{hn}(G) \leq k$ or whether $\overrightarrow{gn}(G) \leq k$ are both NP-complete problems. Firstly, we study the hull number of a subclass of bipartite graphs called partial cubes.

### 4.1 Hull number

In the undirected case, it was proven that determining the hull number of a graph is an NP-hard problem, even for bipartite graphs [2, 1]. Recall that if arcs in both ways were allowed, then this result would imply the NP-hardness on the oriented case. As we just consider oriented graphs, i.e. orientations of simple graphs, we first prove that replacing each edge with a directed $C_4$ has roughly the same effect in the class of bipartite graphs.

Given a (non-oriented) bipartite graph $G$, let $G_{\overrightarrow{C_4}}$ be the oriented bipartite graph such that: $V(G_{\overrightarrow{C_4}}) = V(G) \cup \{v_{i,j}, v_{j,i} \mid v_iv_j \in E(G)\}$ and $A(G_{\overrightarrow{C_4}}) = \{(v_i, v_{i,j}), (v_{i,j}, v_j), (v_j, v_{j,i}), (v_{j,i}, v_i) \mid v_iv_j \in E(G)\}$.
Thus, we replaced each edge with a directed $C_4$.

The first important detail about this procedure is that it “doubled” the length of each path in $G$. In other words, if we have a path $P = v_{i_0}, v_{i_1}, \ldots, v_{i_k}$ in $G$ we also have the directed paths: $P_1 = (v_{i_0}, v_{i_0,i_1}, v_{i_1}, \ldots, v_{i_{k-1},i_k}, v_{i_k})$ and $P_2 = (v_{i_k}, v_{i_k,i_{k-1}}, v_{i_{k-1},i_{k-2}}, \ldots, v_{i_0})$ in $G_{C_4}$.

Moreover, having $P_1$ (resp. $P_2$) in $G_{C_4}$ implies the existence of the paths $P_2$ (resp. $P_1$) in $G_{C_4}$ and $P$ in $G$. Therefore we say that these three are corresponding paths.

Notice that for each $v_iv_j$-path of length $d$ in $G$ we have corresponding $(v_i, v_j)$-path and $(v_j, v_i)$-path in $G_{C_4}$, both of length $2d$. Which means that $2d_G(v_i, v_j) = d_{G_{C_4}}(v_i, v_j) = d_{G_{C_4}}(v_j, v_i)$ for all $v_i, v_j \in V(G)$. Consequently, the corresponding paths of a geodesic are also geodesics.

The first result about this transformation does not yet require the graph to be bipartite.

**Lemma 4.1.** Given a graph $G$, any of its hull sets is also a hull set of $G_{C_4}$.

Consequently, $\overrightarrow{hn}(G_{C_4}) \leq hn(G)$.

**Proof.** Take a hull set $S$ of $G$. Since $S \subseteq V(G)$, every geodesic considered in order to obtain $I_{G_{C_4}}[S]$ has a corresponding geodesic in $G$, and the same is valid for $I_G[S]$. We then have that $I_G[S] = I_{G_{C_4}}[S] \cap V(G)$. Thus, since there are vertices $v_{i,j}$ in $I_{G_{C_4}}[S]$ we can state that $I_G[S] \subseteq I_{G_{C_4}}[S] \cap V(G)$ for every $n \geq 2$. We know that $[S]_G \subseteq I_{G_{C_4}}[S]$ for some natural $n$ and that every $v_{i,j}$ lies in the $(v_i, v_j)$-geodesic. Therefore, $I_{G_{C_4}}^{n+1}[S] \supseteq V(G_{C_4})$. 

Next we show the “converse” of the above statement: from any hull set of $G_{C_4}$ we can obtain a hull set of $G$ with at most the same cardinality. However, for this to be true the graph needs to be bipartite.

**Proposition 4.2.** Let $G$ be a connected bipartite graph and $S$ be a hull set of $G_{C_4}$. Then, there exists a hull set $S'$ of $G$ such that $|S'| \leq |S|$. Therefore, $hn(G) \leq \overrightarrow{hn}(G_{C_4})$.

**Proof.** Let $S$ be a hull set of $G_{C_4}$. We are going to construct another hull set $S'$ of $G_{C_4}$ such that $|S'| \leq |S|$ and $S' \subseteq V(G)$. Because of the corresponding paths, we deduce that $S'$ is a hull set of $G$ and that the proposition holds.

If $S \subseteq V(G)$, then take $S' = S$ and there is nothing to prove. Otherwise, denote by $S_0 = S$. We inductively construct $S_d$ from $S_{d-1}$, for $d > 0$, by replacing a vertex in $S_{d-1} \setminus V(G)$ by a vertex in $V(G)$.

Let $v_{i,j} \in T_{d-1} \setminus V(G)$. If there is a $w' \in T_{d-1} \setminus \{v_{i,j}\}$ with some $(v_{i,j}, w')$-geodesic $\overrightarrow{P} = (v_{i,j}, v_j, v_{j,i}, v_i, \ldots, w')$, notice that there is a $(v_{j,i}, w')$-geodesic containing $v_i$ in $G$. Defining $S_d = (T_{d-1} \setminus \{v_{i,j}\}) \cup \{v_j\}$ gives us $v_i, v_j \in I[S]$, which implies that $v_{i,j} \in I^2[S]$. Thus $S_{d-1} \subseteq I^2[S]$ and, consequently, $V(G_{C_4}) \subseteq [S_{d-1}] \subseteq [S_d]$. If we have a $(w', v_{i,j})$-geodesic $(w', v_j, v_{j,i}, u_j, u_{j,i}, v_{i,j})$, we analogously take $S_d = (S_{d-1} \setminus \{v_{i,j}\}) \cup \{v_j\}$.

In case such a vertex $w'$ does not exist in $S_{d-1}$, then for every $w \in S_{d-1} \setminus \{v_{i,j}\}$ the $(v_{i,j}, w)$-geodesics do not contain $v_i$ and the $(w, v_{i,j})$-geodesics do
not contain \( v_j \). Take a vertex \( w \in S_{d-1} \setminus \{v_{i,j}\} \) and a directed cycle \( C = G_{\bar{C}_3}[\{v_k, v_{k,i}, v_{i,j}, v_{k,j}\}] \) such that \( w \in V(C) \). Let \( P_1 \) and \( P_2 \) respectively be a \((v_{i,j}, w)\)-geodesic and a \((w, v_{i,j})\)-geodesic. In case \( V(P_1) \cap V(P_2) \setminus \{v_{i,j}, w\} \neq \emptyset \), then let \( w' \) belong to this set so that the value of \( \min(v_{i,j}, w') := \min\{d(w', v_{i,j}), d(v_{i,j}, w')\} \) is as small as possible. If there are \( r, s \) such that \( w' = v_{r,s} \) then we must also have \( v_r \in V(P_1) \) and \( v_s \in V(P_2) \). Notice that the \((v_{i,j}, v_r)\)-path contained in \( P_1 \) and the \((v_s, v_{i,j})\)-path contained in \( P_2 \) are both geodesics, from where we get \( d(v_{i,j}, v_r) = d(v_{i,j}, w') - 1 \) and \( d(v_s, v_{i,j}) = d(w', v_{i,j}) - 1 \). Thus we have that \( \min(v_{i,j}, v_r) \) or \( \min(v_{i,j}, v_s) \) is smaller than \( \min(v_{i,j}, w') \), contradicting the choice of \( w' \). We must then have \( w' \in V(G) \).

Observe that the \((v_j, w')\)-geodesic \( P_{i,j}' \) contained in \( P_1 \) has an even number of arcs, \( q_1 \), of the form \((v_{r,s}, v_{r,s})\) and \( q_1 \) of the form \((v_{r,s}, v_{s,r})\). The same is true for the \((w', v_i)\)-geodesic \( P_2' \) contained in \( P_2 \), with length \( 2q_2 \). Now, take the \((w', v_i)\)-path \( P \) such that \( V(P) = (V(P_1') \cap V(G)) \cup \{v_{r,s} \mid v_{r,s} \in V(P)\} \) and \( A(P) = \{(v_{s,r}, v), (v, v_{r,s}) \mid (v_{r,s}, v) \in A(P_1')\} \). Notice that we can extend \( P \) to obtain a \((w', v_i)\)-path of length \( 2q_2 + 2 \) containing \( v_j \), which is not a geodesic due to the case we are working on. Thus \( 2q_1 + 2 > 2q_2 \Rightarrow q_1 \geq q_2 \); analogously \( q_2 \geq q_1 \), giving us \( q_1 = q_2 \). We then have that \( P_1', P_2' \) and the arcs \((v_{i,j}, v_j), (v_j, v_{i,j})\) together form a directed cycle of length \( 2q_1 + 2q_2 = 2(2q_1 + 1) \). In \( G \) there is a corresponding undirected cycle with length \( 2q_1 + 1 \), contradicting the fact that \( G \) is bipartite. Since the existence of \( w' \) always results in a contradiction we must have \( V(P_1) \cap V(P_2) = \{v_{i,j}, w\} \).

By the analysis made in the previous paragraph we cannot have \( w \in \{v_k, v_1\} \). Thus assume without loss of generality that \( w = v_{k,l} \). Remember that we are working on the case in which there is no \((w, v_{i,j})\)-geodesic and no \((v_{i,j}, w)\)-geodesic containing both \( v_k, v_l \). Next we show that we can define \( S_d := (S_{d-1} \setminus \{v_{i,j}\}) \cup \{v_k, v_l\} \).

Let us analyse some \((v_j, v_k)\)-geodesic \( P_3 \), which uses neither \( v_j \) nor \( v_l \) and has even length \( 2p \). Let \( P_3' \) be the \((v_j, v_k)\)-geodesic contained in \( P_1 \), which also has even length; notice that \( |A(P_3)| = |A(P_3')| \) must be even. If \( |A(P_3)| = |A(P_3')| \) we would have a directed cycle of length \( 4p + 2 \), which has a corresponding cycle in \( G \) with odd length. If \( |A(P_3)| \leq |A(P_3')| - 2 \) then we could take a path \( P_3' = (v_{i,j}, v_j, v_{j,i}, v_i, \ldots, v_k, v_{k,l}) \) containing \( P_3 \) with length \( |A(P_3')| = |A(P_3)| + 4 \leq |A(P_3')| + 2 = |A(P_1)| \). Consequently it would be a \((v_{i,j}, v_{k,l})\)-geodesic containing both \( v_i \) and \( v_j \), thus contradicting the hypothesis for this case. Therefore \( |A(P_3)| \geq |A(P_3')| + 2 \). The argument is analogous for the \((v_j, v_k)\)-geodesics not containing \( v_i, v_k \). Thus the \((v_{i,j}, v_{k,l})\)-path \( P_1' \) with \( V(P_1') = V(P_1') \cup \{v_{i,j}, v_{k,l}\} \) and \( A(P_1') = A(P_1') \cup \{(v_{i,j}, v_{k,l}), (v_{k,l}, v_{i,j})\} \) is a geodesic. Defining the \((v_k, v_l)\)-path \( P_2' \) analogously we conclude that it is also a geodesic. We then have that denoting \( S_d := (S_{d-1} \setminus \{v_{i,j}, v_{k,l}\}) \cup \{v_k, v_l\} \) gives us \( v_{i,j}, v_{k,l} \in V(P_1') \cup V(P_2') \subset I[S_d] \), which means that \( S_d \subset I[S_d] \) and consequently \( V(D) \subseteq [S_d] \).

Following these steps inductively we obtain a hull set \( S' \subset V(G) \) of \( G_{\bar{C}_3} \). Notice that there might be \( v_{i,j}, v_{k,l} \in S_d \) with \( \{i,j\} \cap \{k,l\} \neq \emptyset \) and that they were both exchanged for the same vertex of \( V(G) \). Now we only have left to prove that \( S' \) is also a hull set of \( G \). We show by induction on \( k \) that
$I^k_C[S'] = I^k_{C_{G,1}}[S'] \cap V(G)$.

Since $T' \subseteq V(G)$, for all pairs of vertices $v_i, v_j \in S'$ every $(v_i, v_j)$-geodesic has its corresponding $v_i v_j$-geodesic in $G$. Thus it is easy to see that for $k = 1$ the result follows. Now assume that it is true for $k-1$. Seen as $I^{k-1}_G[S'] = I^{k-1}_{C_{G,1}}[S'] \cap V(G)$, by the same argument used above it follows that $I^k_G[S'] = I^{k-1}_{C_{G,1}}[S'] \cap V(G) \cap V(G) \subseteq I^k_{C_{G,1}}[S'] \cap V(G)$ Next we show that $I^k_{C_{G,1}}[S'] \cap V(G) \subseteq I^k_G[S']$.

Taking $w \in \left(I^{k-1}_{C_{G,1}}[S'] \cap V(G)\right) \setminus I^{k-1}_{C_{G,1}}[S']$, we know that there are $w_1, w_2 \in I^{k-1}_{C_{G,1}}[S']$ such that $w$ lies in some $(w_1, w_2)$-geodesic. If both $w_1, w_2$ are vertices of $G$ then we clearly have $w \in I^k_G[S']$. Suppose that there are distinct $i, j \in \{1, \ldots, n(G)\}$ such that $w_1 = v_{i,j}$. Since $T' \subseteq V(G)$ let $k' \in [k-1]$ be such that $w_1 \notin I^{k'}_{C_{G,1}}[S']$ and $w_1 \in I^{k}_{C_{G,1}}[S']$. Thus there are $w_3, w_4 \in I^{k-1}_{C_{G,1}}[S']$ such that $w_1$ is interior to some $(w_3, w_4)$-geodesic. We know that both indegree and outdegree of $w_1$ are one, which means that $u_i, u_j \in I^{k}_{C_{G,1}}[S']$. Define $w'_i$ as $w_1$ if this belongs to $V(G)$, and as $u_j$ otherwise. Similarly define $w'_2$. In any event we have a $(w'_1, w'_2)$-geodesic contained in the $(w_1, w_2)$-geodesic mentioned before. It is easy to see now that $w \in I^k_G[S']$. This concludes our demonstration. □

**Corollary 4.3.** If $G$ is a bipartite graph, then $hn(G) = h\overrightarrow{n}(G_{C,1})$.

Thus, one can combine the result in [2] with Corollary 4.3 to deduce that, given an oriented bipartite graph $D$ and a positive integer $k$, deciding whether $\overrightarrow{g}(D) \leq k$ is an NP-complete problem. However, one can observe that such reduction can also be applied to partial cubes, a subclass of bipartite graphs, which we define in the sequel.

The hypercube graph of dimension $n$, $Q_n$, is a (undirected) graph such that its vertex set is $V(Q_n) = \{0, 1\}^n$. We express each vertex as $v = (v^1, \ldots, v^n)$, where $v_i \in \{0, 1\}$ for every $i \in \{1, \ldots, n\}$. The edge set of $Q_n$ is $E(Q_n) = \{uv \mid$ there is exactly one $i \in \{1, \ldots, n\}$ such that $u^i \neq v^i\}$. A partial cube graph $G$ is an isometric subgraph of some $Q_n$, meaning that $d_G(u, v) = d_{Q_n}(u, v)$ for every $u, v \in V(G)$.

Computing the hull number of partial cubes is also an NP-hard problem [1]. Thus, we only have to show that the above procedure applied on a partial cube returns an oriented partial cube.

**Proposition 4.4.** If $G$ is a partial cube, then $G_{C,1}$ is an oriented partial cube.

**Proof.** Let $G$ be a connected partial cube with $V(G) = \{v_1, \ldots, v_l\}$. If $k$ is the smallest natural number such that $G \subseteq Q_k$, each vertex of $G$ can be considered as an element of $\{0, 1\}^k$. Thus let $H \subseteq Q^k$ be a graph initially with $V(H) = \{u_1, \ldots, u_l\} \subseteq \{0, 1\}^{2k}$ and, for each $u_i = (u^1_i, u^2_i, \ldots, u^k_i)$ and $v_i = (v^1_i, v^2_i, \ldots, v^k_i)$, $u^j_i = u^j_{i+1}$ for every $j \in \{1, \ldots, k\}$. Notice that each two vertices $u_i$ and $u_j$ differ in an even number of entries.

Take two distinct vertices $u_i, u_j \in V(H)$ such that there is only one $m \in \{1, \ldots, k\}$ so that $u^m_i = u^m_j$ and $u^{m-1}_i = u^{m-1}_j$ (notice that $v_i, v_j$ are adjacent...
in \( G \). For each such pair add the vertices \( u_{i,j} = (u_{j,1}^1, \ldots, u_{j,2^m-1}^1, u_{j,2^m}^2, u_{j,2^m+1}^2, \ldots, u_{j,2^k}^2) \) and \( u_{j,i} = (u_{i,1}^1, \ldots, u_{i,2^m-1}^1, u_{i,2^m}^2, u_{i,2^m+1}^2, \ldots, u_{i,2^k}^2) \) to \( V(H) \), which differ in the \( 2m-1 \)th and \( 2m \)th entries. Notice also that both \( u_i \) and \( u_j \) differ in exactly one entry from both \( u_{i,j} \) and \( u_{j,i} \). Moreover add the edges \( u_iu_j, u_{i,j}u_j, u_ju_{i,j}, u_p,iu_i \) to \( E(H) \), which if oriented as \((u_i, u_{i,j}), (u_{i,j}, u_j), (u_j, u_{i,j}), (u_{i,j}, u_i)\) result in \( G_{C^2} \).

Next we analyze each kind of pair of vertices of \( H \). If our pair is \( u_p, u_q \) we already know that they have at least two distinct entries. If it is \( u_p, u_{q,r} \) with \( p \in \{q, r\} \) we also know that they have all but one entries in common. Now take \( p, q, r \in \{1, \ldots, l\} \) such that \( u_{p,q} \in V(H) \) and it is not adjacent to \( u_r \). Define \( m: \{1, \ldots, l\}^2 \to \{0, 1, \ldots, k\} \) as \( m(i,j) = 0 \) if either \( i = j \) or if \( u_i, u_j \) are not adjacent, and as the only \( m \in \{1, \ldots, k\} \) such that \( u_{p,q}^m \neq u_{r}^m \) for every other pair \( (i,j) \). Since \( u_r \neq u_p, u_q \), there must be \( m \in \{1, \ldots, k\} \setminus \{m(p,q)\} \) such that \( u_{p,q}^m \neq u_{r}^m \neq u_{q}^m \neq u_{p,q}^m \). We know that the \( 2m-1 \)th and the \( 2m \)th entries of \( u_p \) and \( u_{p,q} \) are equal, thus \( u_r \) and \( u_{p,q} \) have at least two distinct entries.

The last case is a pair \( u_{p,q}, u_{r,s} \). If \( d_G(v_i, v_j) = 1 \) for every \( i \in \{p, q\} \) and every \( j \in \{r, s\} \), since \( v_p \) and \( v_q \) are adjacent we would have triangles in \( G \), which is not possible since it is bipartite. Thus there are \( i \in \{p, q\} \) and \( j \in \{r, s\} \) such that \( d_G(v_i, v_j) \geq 2 \), which means that we have \( m_1, \ldots, m_k \in \{1, \ldots, l\} \) such that \( v_i^m \neq v_j^m \) for every \( m \in \{m_1, \ldots, m_k\} \) and \( k = d_G(v_i, v_j) \). Suppose that there is some \( m \) in the previous set which is not in \( \{m(p,q), m(r,s)\} \), then \( u_{p,q}^m = u_{r,s}^m = u_{r}^m = u_{u,q}^m \neq u_{r}^m \neq u_{q}^m \neq u_{r,q}^m \). Consequently, \( u_{p,q} \) and \( u_{r,s} \) have at least two distinct entries. Else we must have \( k = 2 \), \( m_1 = m(p,q) \) and \( m_2 = m(r,s) \), without loss of generality. Notice that \( u_{p,q}^m \neq u_{r,s}^m = u_{r,s}^m = u_{r}^m = u_{s}^m \implies u_{p,q}^m \neq u_{r,s}^m \), which means that either the \( 2m_1 \)th or the \( 2m_2 \)th positions of \( u_{r,s} \) and \( u_{p,q} \) are different. The same can be said about the \( 2m_2 \)th and \( 2m_1 \)th positions, thus \( u_{p,q} \) and \( u_{r,s} \) have at least two distinct entries. Therefore, if we combine the results of these last two paragraphs with the construction of \( H \), we conclude that two vertices of \( H \) are adjacent if and only if they have only one distinct entry.

That being said, all that is left to prove is that \( d_H(u, v) = d_{Q_{2s}}(u, v) \) for every \( u, v \in V(H) \). First consider \( u_1 \) and \( u_2 \). If \( v_1 \) and \( v_2 \) have \( p \) different entries, then their corresponding vertices in \( H \) differ in \( 2p \) entries implying that \( d_{Q_{2s}}(u_1, u_2) = 2p \). Since \( G \) is a partial cube we know that \( d_G(v_i, v_j) = d_{Q_{2s}}(v_i, v_j) = p \), thus there is a \( v_i, v_j \)-path with length \( p \) in \( G \). Due to the construction of \( H \), we also have in it an \( u_1, u_2 \)-path of length \( 2p \). Now suppose that there is an \( u_1, u_2 \)-path in \( H \) with length \( q < 2p \). Seen as the neighbors in \( H \) have exactly one distinct entry, then the extreme vertices of this path differ in at most \( q \) entries, which is a contradiction. Thus \( d_H(u_1, u_2) = 2p \).

Next we analyze the distance between \( u_{i,j} \) and \( u_{r,s} \). Let \( \{u, u'\} = \{u_i, u_j\} \) and \( \{v, v'\} = \{u_r, u_s\} \) be such that \( u \) and \( v \) are as close as possible. It is straightforward that \( d_H(u_{i,j}, u_{r,s}) = 2 + d_H(u, u') = 2(p + 1), \) where \( 2p \) is the number of distinct entries of \( u \) and \( v \). Since we change one entry from \( u \) to \( u_{i,j} \) and one from \( v \) to \( u_{r,s} \) we must have \( 2p + 2 \) distinct entries between \( u_{i,j} \) and \( u_{r,s} \), meaning that \( d_{Q_{2s}}(u_{i,j}, u_{r,s}) = 2p + 2 \). This also covers the proof that
\[ d_H(u_{i,j}, u_r) = d_{Q_2}(u_{i,j}, u_r). \]

**Corollary 4.5.** Given an oriented partial cube \( D \) and a positive integer \( k \), it is \( \text{NP-complete} \) to decide whether \( \text{hn}(D) \leq k \).

### 4.2 Geodetic number

Our goal in this section is to prove that the following problem is \( \text{NP-complete} \):

**Geodetic number**

*Input:* Oriented graph \( D \) and a positive integer \( k \)

*Output:* \( \text{gn}(D) \leq k \)?

**Theorem 4.6.** Geodetic number is an \( \text{NP-complete} \) problem, even if the input oriented graph \( D \) has no directed cycle and a bipartite underlying graph.

**Proof.** Given a subset of vertices \( S \subseteq V(D) \), one can compute \((u, v)\)-geodesics, for every \( u, v \in V(D) \), and decide whether \( S \) is a geodetic set in polynomial time similarly to the undirected case [3]. Consequently, the problem is in \( \text{NP} \).

We reduce the well-known Set Cover [17] problem to Geodetic number:

**Set Cover**

*Input:* \( U = \{1, 2, \ldots, n\} \), \( F \subseteq \mathcal{P}(U) \) such that \( \bigcup F = U \) and a positive integer \( k \)

*Output:* Does there exist \( F' \subseteq F \) such that \( \bigcup F' = U \) and \( |F'| \leq k \)?

Let \( (U = \{1, 2, \ldots, n\}, F = \{F_1, \ldots, F_m\}, k) \) be an input to Set Cover. We shall construct an oriented graph \( D \) such that \( (U, F, k) \) is an YES-instance if, and only if, \( \text{gn}(D) \leq k + 3 \).

The vertex set of \( D \) is composed by two subsets of vertices \( X \) and \( Y \) union three vertices \( u, v \) and \( w \). In \( X \) there is one vertex \( u_i \) corresponding to \( F_i \in F \), for every \( i \in \{1, \ldots, m\} \). In \( Y \) there is a vertex \( v_j \) corresponding to each element in \( U \), for every \( j \in \{1, \ldots, n\} \).

In the arc set of \( D \) there is the arc \((u_i, v_j) \in A(D) \) whenever \( j \in F_i \), for every \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, n\} \). Moreover, \( A(D) \) has the arcs \((u, u_i), (u_i, w) \) for every \( i \in \{1, \ldots, m\} \), the arcs \((v_j, v) \) for every \( j \in \{1, \ldots, n\} \), and finally the arc \((u, v) \).

By construction, \( D \) is clearly a DAG whose underlying graph is bipartite (with partition \( V(D) = (X \cup \{v\}) \cup (Y \cup \{u, w\}) \)). Notice that \( u \) is a source and that \( v \) and \( w \) are sinks, thus they belong to any geodetic set. Besides, \( (u, w) \notin A(D) \) and for every \( u_i \in X \) we have the path \((u, u_i, w) \). Since \( (u, v) \in A(D) \) we have \( I[u, v, w] = X \cup \{u, v, w\} \).
Let \( F' = \{ F_i \mid i \in I \} \subseteq F \), for some \( I \subseteq \{1, \ldots, m\} \) such that \( \bigcup_{i \in I} F_i = U \) and \( |I| \leq k \). We then take \( X' = \{ u_i \mid i \in I \} \cup \{ u, v, w \} \), which has cardinality at most \( k + 3 \). Thus, for every \( v_j \in Y \) there is \( u_i \in X' \) such that \( (u_i, v_j) \in A(D) \), from where we have the geodesic \( (u_i, v_j, v) \). Therefore, one can observe that \( X' \) is a geodesic set of \( D \).

On the other hand, let \( S \) be a geodesic set of \( D \) with at most \( k + 3 \) vertices. Thus, \( \{ u, v, w \} \subseteq S \) and at most \( k \) vertices of \( S \) belong to \( X \cup Y \). If some \( v_j \in S \), one can observe that by replacing \( v_j \) with a vertex \( u_i \) such that \( (u_i, v_j) \in A(D) \), we obtain another geodesic set \( S' \) such that \( |S'| \leq k + 3 \). Thus, without loss of generality, we assume that \( S \setminus \{ u, v, w \} \subseteq X \). Let \( I = \{ i \in \{1, \ldots, m\} \mid u_i \in S \} \).

One may observe that the previous reduction can also be used to argue that \textsc{Geodetic number} is \( W[2] \)-hard, when parameterized by the value of the solution \( k \), even if the input graph is a DAG having a bipartite underlying graph.

5 Polynomial-time algorithm for cacti

The hull and geodetic numbers of an undirected tree \( T \) are both equal to the number of leaves of \( T \). Notice that the leaves of a tree are the simplicial vertices. Moreover, any node belongs to an \( uv \)-path for some distinct leaves \( u, v \in V(G) \), and that path is a geodesic because it is unique. That means that the set of simplicial vertices of a tree is a minimum hull and geodetic set. A similar statement is true for the oriented case.

**Proposition 5.1.** Let \( D \) be an oriented tree. Then, \( \text{Ext}(D) \) is both a minimum hull set and a minimum geodetic set of \( D \). Consequently, it is unique and \( \text{hh}(D) = \text{gh}(D) = |\text{Ext}(D)| \).

**Proof.** Let \( u \in V(D) \) be a non-extreme vertex and \( P = (v_1, \ldots, u, \ldots, v_2) \) a maximal (directed) path with \( u \) as an internal vertex. Due to the maximality of \( P \), we must have either \( \delta^-(v_1) = 0 \) or \( N^-(v_1) \subset V(P) \). Since the second alternative would imply that \( D \) has a cycle, which cannot happen in a tree, we must have \( \delta^-(v_1) = 0 \). Analogously, \( \delta^+(v_2) = 0 \). Thus, both \( v_1 \) and \( v_2 \) are extreme. Moreover, we know that there is only one \( v_1 v_2 \)-path in the underlying tree of \( D \), which means that \( P \) is a geodesic. Therefore, \( I[\text{Ext}(D)] = V(D) \).

Thus, one can also observe that the hull and geodetic numbers of oriented trees, a subclass of oriented bipartite graphs, can be both computed in linear time.

This result led us to work on the cacti graphs, a superclass of tree graphs. A graph is called a \textit{cactus} if each block is either an edge or a cycle. Consequently, every cycle is an induced cycle and two cycles intersect in at most one vertex.

For cacti, there are algorithms to compute the geodetic and hull numbers of a non-oriented cactus graph proposed in [10, 2]. As in the undirected case,
the extreme vertices may not suffice in order to obtain a hull or a geodetic set of an oriented cactus. It is necessary to include a few non-extreme vertices of some particular cycles. We introduce below a few important notions in order to define such cycles.

In the remainder of this section, let \( D \) be a cactus graph. Let \( C \subseteq D \) be a cycle and \( u \in V(C) \) be a cut-vertex of \( D \). If there is an arc \((u, v) \in A(D) \setminus A(C)\), we say that \( u \) is a transmitter cut-vertex of \( C \) and use the initials TCV. Analogously, if \((v, u) \in A(D) \setminus A(C)\) we say that \( u \) is a receiver cut-vertex and use the initials RCV. Notice that we can have a cut-vertex which is both an RCV and a TCV.

A cycle \( C \subseteq D \) is called a leaf cycle if it has only one cut-vertex. We say that a trap cycle is a cycle \( C \) such that its cut-vertices are either all exclusively transmitters or all exclusively receivers. If \( C \) is a trap cycle with a TCV we say that it is a transmitter trap cycle. Similarly, a trap cycle with an RCV is a receiver trap cycle. At last, we say that a cycle \( C \) is unsatisfactory if one of the following holds:

Type 1: \( C \) is a trap cycle;

Type 2: \( C \) is a directed leaf cycle that is not a trap cycle;

Type 3: there are only two vertices in \( \text{Ext}(C) \), say \( u_1 \) e \( u_2 \), such that the two \((u_1, u_2)\)-paths in \( C \) have different lengths and the longest one does not have internal cut vertices.

When none of these happens, we say that the cycle is satisfactory. To shorten our text we also use the acronym UC\(i\) for "unsatisfactory cycle of type \( i \)\", with \( i \in \{1, 2, 3\} \).

Recall that any hull set must intersect any co-convex set. Next we show that each unsatisfactory cycle contains a co-convex set and we describe these sets for each type of cycle.

**Lemma 5.2.** Let \( D \) be an oriented cactus graph and \( C \) be an unsatisfactory cycle of \( D \). Then, \( V(C) \) contains a co-convex set \( S \) and if \( C \) is of type:

- 1, then \( S = V(C) \);
- 2, then \( S = V(C) \setminus \{w\} \) where \( w \) is the cut-vertex of \( C \);
- 3 with \( \text{Ext}(C) = \{u, v\} \), then \( S \) consists of the internal vertices of the longest \((u, v)\)-path.

Moreover, there is no intersection between any co-convex sets of different cycles.

**Proof.** First let \( C \) be a receiver trap cycle, without loss of generality. Suppose that there are distinct vertices \( u_1, u_2 \in N(V(C)) \) and an \((u_1, u_2)\)-geodesic \( P \) such that all its internal vertices are in \( V(C) \). We then have \( v \in V(C) \) such that \((u_1, v) \in A(P)\), contradicting the choice of \( C \).
Now let $C$ be an UC2 and let $w \in V(C)$ be its cut vertex in $D$. Notice that $N(V(C) \setminus \{w\}) = \{w\}$ since $w$ is the only cut vertex of $C$. Thus, there cannot be a path $P$ such as the one suggested in the previous paragraph.

At last, let $C$ be an UC3 and let $v_1, v_2$ be the extreme vertices in $C$, respectively the source and the sink. Denote by $P^-$ and $P^+$ respectively the shortest and the longest $(v_1, v_2)$-paths in $C$, we prove next that $V(P^+) \setminus \{v_1, v_2\}$ is a co-convex set. Now take $u_1, u_2 \in N(V(P^+) \setminus \{v_1, v_2\})$ and $P$ an $(u_1, u_2)$-path as the one in the first paragraph. by the definition of $C$ we have that $u_1 = v_1$ and $u_2 = v_2$, which implies $P = P^+$. Since that is not a geodesic, the result follows.

To end this demonstration we need only prove that these co-convex sets do not intersect. Two trap cycles cannot intersect because if they did, the common vertex would be an RCV and a TCV for both cycles. If one of the cycle is not a trap cycle than there is no intersection, since its co-convex set does not contain a cut vertex.

We have proven the necessity of having at least one vertex from each unsatisfactory cycle in any hull set. Now we have left to show that these along with Ext($D$) are enough.

**Lemma 5.3.** Let $D$ be an oriented cactus and $u \in V(D)$. For every $v \in N^+(u)$ ($v \in N^-(u)$) there is a maximal path $P = (v_1, \ldots, v_q)$ with $u = v_1$ and $v = v_2$ ($u = v_q$ and $v = v_{q-1}$) such that $v_q$ ($v_1$) either is extreme or belongs to an unsatisfactory cycle $C$ of type either 1 or 2. Moreover, all the vertices of $C$ are in $V(P)$.

**Proof.** Consider all of the maximal paths (starting in $u$) of the form $P = (u, v_1 = v, v_2, \ldots, v_q)$. We have two options: either $N^+(v_q) = \emptyset$ or $N^+(v_q) \subseteq V(P)$. If the first one occurs then $v_q$ is extreme. Else we have $N^+(v_q) \subseteq V(P)$, and since $D$ is a cactus we must also have $d^+(v_q) = 1$. Moreover we have a directed cycle $C$ whose vertices are all in $V(P)$. If $C$ is either an UC1 or an UC2 then we are done.

Assume that every maximal path as described above ends in a satisfactory cycle $C$. Take $P$ as a maximal path which intersects the largest number of cycles in $D$ and let $C_1, \ldots, C_n$ be those cycles such that $C_i$ is the $i^{th}$ cycle that $P$ intersects (with respect to its orientation). Notice that all of these cycles are pairwise different, because if not we would have a block which would neither be a vertex nor a cycle. Since $C_n$ is satisfactory and directed, there is an arc $(w_1, w_2) \in A(D)$ with $w_1 \in V(C_n) \setminus \{v_q\}$ and $w_2 \notin V(C_n)$. Therefore we can take another maximal path $P^* = (u, v, v_2, \ldots, w_1, w_2, \ldots, v_q)$ ending in a satisfactory directed cycle such that $P \cap P^* = (u, v, w_2, \ldots, w_1)$. It is easy to see that $P^*$ intersects at least one more cycle than $P$, thus contradicting the choice of the latest. We then conclude that there always is a maximal path $(u, v_1 = v, v_2, \ldots, v_q)$ such that either $v_q$ is extreme or the last vertices are the ones of either an UC1 or an UC2.

For the maximal paths $(v_0, v_1, \ldots, v, u)$ the argument is analogous. \qed
With that we can obtain a path between two vertices in any hull set having \( v \) in its interior, but it does not guarantee the existence of a geodesic. However, for some vertices if there is a geodesic not containing \( v \) then that would create a block that is not allowed in a cactus graph. Next we present some restrictions for these paths.

**Lemma 5.4.** Let \( D \) be an oriented cactus graph with \( C \subseteq D \) a cycle, \( u, v \in V(D) \) and \( P \) an \((u,v)\)-path.

1. If there is a \( w \in V(P) \) that does not belong to any cycle of \( D \) then every \((u,v)\)-path contains \( w \).

2. Let \((w_1,\ldots,w_q) = P' \subseteq P \cap C\) be maximum with \( q \geq 2 \). Then:
   (a) every \((u,v)\)-path goes through \( C \);
   (b) if \( u \notin V(C) \) then \( w_1 \) is the same vertex for every path \( P \); and
   (c) if \( v \notin V(C) \) then \( w_q \) is the same vertex for every path \( P \).

**Proof.** In the first case, by contradiction suppose that there exists an \((u,v)\)-path \( P^* \) that does not contain \( w \). Let \( v_1,v_2,\ldots,v_r \) be the vertices which are in both \( P \) and \( P^* \), ordered according to the orientation of these two. Take \( i \in \{1,\ldots,r-1\} \) such that \( w \) lies in the \((v_i,v_{i+1})\)-path contained in \( P \). The two \((v_i,v_{i+1})\)-paths (the ones contained in \( P \) and in \( P^* \)) are internally disjoint, thus together they form a cycle. This contradicts the choice of \( w \).

For the second, also by contradiction assume that there is another \((u,v)\)-path \( P^* \subset D \) that does not intersect \( C \). Using arguments analogous to the ones above we conclude that there is a block containing \( C \) which is not a cycle.

Now suppose that there is an \((u,v)\)-path \( P^* \subset D \) intersecting \( C \), with \( P'' := P \cap C \) such that its first vertex is \( w'_1 \neq w_1 \). Since \( u \in V(P) \cap V(P^*) \) let \( u' \) be the last vertex (with respect to the orientation of \( P \)) in that intersection before the cycle \( C \). Therefore \( C \), the \((u',w_1)\)-path contained in \( P \) and the \((u',w'_1)\)-path contained in \( P^* \) are in the same block, which again is a contradiction to the fact that \( D \) is a cactus. The argument for \( w_q \) in the statement of this lemma is analogous.

Combining the arguments provided by Lemmas 5.2, 5.3 and 5.4, one may deduce how to obtain a minimum hull set of an oriented cactus.

**Theorem 5.5.** Let \( D \) be an oriented cactus graph. Then there exists a minimum hull set \( S \) of \( D \) composed by the extreme vertices of \( D \) and by exactly one non-extreme vertex of each unsatisfactory cycle. Moreover, \( I[S] \) contains all the vertices that are not in a satisfactory cycle and \( I^2[S] = V(D) \).

**Proof.** Besides the extreme vertices, by Lemma 5.2 a hull set of \( D \) must also contain at least one non-extreme from each unsatisfactory cycle. Next we show how to obtain a hull set \( S \) with only these vertices, which consequentially will be minimum.
Let $C \subset D$ be a receiver trap cycle (the argument for transmitter trap cycles is analogous). Take $u, u' \in V(C)$ such that $(u, u') \in A(C)$ and $u'$ is a cut vertex, which means that there is $w \in N^-(u') \setminus V(C)$. By Proposition 5.3, there is a path $P = (v, \ldots, w, u')$ such that $V(C) \cap V(P) = u'$ and $v$ is either extreme or lies in a cycle $C'$, which is either an UC1 or an UC2, with $V(C') \subset V(P)$. If $v$ is extreme then call it $v'$ from now on. If the second case happens, by Lemma 5.2 we must have a vertex of $C'$ in every hull set; let $v'$ be that vertex and let $P' \subseteq P$ be the $(v', u')$-path. In both events, we can extend $P'$ to get a $(v', u)$-path which contains every vertex of $C$ since it is directed and $(u, u') \in A(D)$.

By Lemma 5.4 every $(v', u)$-path arrives at $C$ by the vertex $u'$, including the geodesics. Therefore, by adding $u$ to $S$ we have $V(C) \subseteq I[S]$.

Now let $C$ be an UC2 and let $v \in V(C)$ be its cut vertex. Since $C$ is not a trap cycle there are $v_1, v_2 \in V(D) \setminus V(C)$ such that $(v_1, v), (v, v_2) \in A(D)$. Thus taking an arbitrary $u \in V(C) \setminus \{v\}$ by Proposition 5.3 there are $w_1, w_2$ in the hull set $S$ such that we have a $(w_1, u)$-path and an $(u, w_2)$-path. Notice that each vertex of $C$ is in at least one of these paths. Moreover by Lemma 5.4 we deduce that each vertex of $C$ other than $u$ and $v$ lies either in a $(w_1, u)$-geodesic or in an $(u, w_2)$-geodesic. Therefore adding $u$ to $S$ results in $V(C) \subseteq I[S]$.

Next we analyse the case where $C$ is an UC3, such that $u_1 \in V(C)$ is a source in $C$ and $u_2 \in V(C)$ is a sink in $C$. Take $v$ an arbitrary interior vertex of the longest $(u_1, u_2)$-path in $C$. If $v_1$ is also an extreme vertex in $D$ we take $w_1 = u_1$. If not then we have $u'_1 \in N^-(u_1) \setminus V(C)$. By Proposition 5.3 there are a vertex $w_1$ either extreme or lying in an UC1 or in an UC2 and a path $w_1, \ldots, u'_1, u_1$. Define $w_2$ analogously. Thus we have an $(w_1, w_2)$-path intersecting $C$ (that intersection being an $(u_1, u_2)$-path), a $(w_1, u)$-path and an $(u, w_2)$-path. By Lemma 5.4 every $(w_1, w_2)$-geodesic contains the $(u_1, u_2)$-geodesic and we also have a $(w_1, u)$-geodesic and an $(u, w_2)$-geodesic. Therefore $V(C) \subseteq I[S]$.

Take a vertex $v$ which is not in any cycle and neither is extreme. Thus its indegree and outdegree are positive. By Proposition 5.3 and Lemma 5.4 there is an $(u, w)$-geodesic containing $v$ such that $u$ and $w$ are each either extreme or in an UC1 or in an UC2, meaning that $u, w \in S$. We then have $v \in I[S]$.

This leaves the satisfactory cycles to be analyzed. First let $C$ be a cycle with at least four extreme vertices in $C$. Notice that each non-extreme vertex of $C$ lies in a path between two extreme vertices of $C$. We thus analyse these paths in $C$. Take $u_1, u_2 \in \text{Ext}(C)$ respectively a source and a sink such that there is an $(u_1, u_2)$-path. Define $w_1$ and $w_2$ like in the case where $C$ is an UC3. Then by Proposition 5.3 and Lemma 5.4 we have a $(u_1, u_2)$-geodesic containing the $(u_1, u_2)$-path in $C$. Applying this argument for every path in $C$ like the $(u_1, u_2)$-path we obtain $V(C) \subseteq I[S]$.

If $C$ has only two extreme vertices (in $C$), $u_1$ the source and $u_2$ the sink, take $w_1, w_2$ as above. Thus for each $(u_1, u_2)$-path there is a $(w_1, w_2)$-path containing it. If both paths are geodesics we can find a $(w_1, w_2)$-geodesic for each one, which gives us $V(C) \subseteq I[S]$. Else let $v$ be an interior cut vertex of the longest $(u_1, u_2)$-path. Without loss of generality let $(v_1, v) \in A(D) \setminus A(C)$. By Proposition 5.3 and Lemma 5.4 there are $w_3 \in S$ and a $(w_1, w_3)$-geodesic containing the $(u_1, v_1)$-geodesic. We then have that all the vertices in this geodesic and the
ones in the \((u_1, u_2)\)-geodesic are in \(I[S]\). Since \(u_2, v_1 \in I[S]\) the vertices in the \((v_1, u_2)\)-path are in \(I^2[S]\).

For the last case let \(C\) be a directed cycle which is neither trap nor leaf. Thus there are distinct cut vertices \(v_1, v_2 \in V(C)\) and \((v'_1, v_1), (v_2, v'_2) \in A(D) \setminus A(C)\). By Proposition 5.3 and Lemma 5.4 there are \(w_1, w_2 \in S\) such that every \((w_1, w_2)\)-path contains the \((v_1, v_2)\)-path \(P\), which implies that \(V(P) \subseteq I[S]\). Therefore, since \(v_1, v_2 \in I[S]\) all the vertices in the \((v_2, v_1)\)-path are in \(I_2[S]\).

The proof argues which vertex must be chosen in each unsatisfactory cycle. All such vertices can be found in linear-time. Thus, \(\text{hn}(D)\) too can be found in linear-time, for every oriented cactus \(D\).

The above theorem motivated us to also work on the geodetic number for these oriented graphs. Since a geodetic set is also a hull set, every geodetic set must have a non-extreme vertex of each unsatisfactory cycle. Besides, as only some satisfactory cycles may have vertices that are not obtained in the first iteration of the interval function, we studied such cycles in order to obtain a minimum geodetic set. As a result, we define the \emph{falsely satisfactory} cycles, FSC for short. These can be of two types:

Type 1: \(\Ext(C) = \{u_1, u_2\}\), \(u_1\) is a source and \(u_2\) is a sink. The \((u_1, u_2)\)-paths have distinct lengths, \(P\) being the longest. \(P\) has length at least three and one of its internal vertices is a cut-vertex in \(D\). Besides, all the following statements hold:

1. If there is an RCV \(v_1\) internal to \(P\), the \((u_1, v_1)\)-path must have length at least two;
2. If there is a TCV \(v_2\) internal to \(P\), the \((v_2, u_2)\)-path must have length at least two;
3. If there are both an RCV \(v_1\) and a TCV \(v_2\) internal to \(P\), we must have \(P = (u_1, \ldots, v_2, \ldots, v_1, \ldots, u_2)\). Moreover, the \((v_2, v_1)\)-path must also have length at least two.

Type 2: The cycle \(C\) is directed and there are distinct RCV \(v'_1\) and TCV \(v'_2\) in \(C\) such that:

1. \(d_C(v'_2, v'_1) \geq 2\); and
2. all the other cut-vertices are internal to \(P = (w_0, w_1, \ldots, w_{k-1}, w_k)\), where \(w_0 = v'_1\) and \(w_k = v'_2\). Besides, if \(w_i\) is an RCV and \(w_j\) is a TCV then \(i \leq j\) for every \(i, j \in \{0, 1, \ldots, k\}\).

Otherwise, we say that the cycle is \emph{truly satisfactory} and use TSC to simplify.

**Lemma 5.6.** Let \(D\) be an oriented cactus graph, \(C \subseteq D\) a satisfactory cycle and \(u_1, u_2, v_1, v_2, v'_1, v'_2\) as in the above definition. Then:

- if \(C\) is truly satisfactory and \(S\) is a minimum hull set of \(D\) then \(I[S] \supseteq V(C)\);
• if C is falsely satisfactory and \( S = N^+(V(C)) \cup N^-(V(C)) \) then \( I[S] \not\subseteq V(C) \). Moreover, the vertices not in \( I[S] \) are the following ones:

  * if C is of type 1, the internal vertices of the \((w_1, w_2)\)-path where \( w_1 \in \{u_1, v_2\} \) and \( w_2 \in \{u_2, v_1\} \) are as close as possible;

  * if C is of type 2, the internal vertices of the \((v'_2, v'_1)\)-path.

And also none of these vertices is a cut-vertex.

**Proof.** First we analyse the truly satisfactory cycles. Let C be a cycle with at least four extreme vertices in C. Take an \((u_1, u_2)\)-path in C such that \( u_1, u_2 \) are extreme in C. If \( u_1 \) is extreme also in D define \( u_1' := u_1 \), else \( d^-(u_1) > 0 \) and by Proposition 5.3 there is \( w_1 \in S \) with a path form it to \( u_1 \). Define \( w_2 \in S \) analogously. We then have a \((u_1, w_2)\)-path containing the \((u_1, u_2)\)-path in C. By Lemma 5.4 the \((u_1, w_2)\)-path is contained in every \((w_1, w_2)\)-path including the geodesics, implying that all its vertices are in \( I[S] \). Using this argument for every \((u_1, u_2)\)-path as described above we have that \( V(C) \subseteq I[S] \).

If \( \text{Ext}(C) = \{u_1, u_2\} \), with \( u_1 \) source and \( u_2 \) sink, take \( w_1, w_2 \) as in the previous case. By Proposition 5.3 and Lemma 5.4 every \((w_1, w_2)\)-path contains one of the two \((u_1, u_2)\)-paths in C. We know that the vertices of an \((u_1, u_2)\)-geodesic are in \( I[S] \). If the two paths are geodesics we are satisfied, else let \( v_1 \) be an RCV and \( v_2 \) be a TCV, both interior to the longest \((u_1, u_2)\)-path. Suppose that there is no TCV in C, then we can take \( v_1 \) such that \( (u_1, v_1) \in A(C) \).

By Proposition 5.3 and Lemma 5.4 there is a \( v'_1 \in S \) and a \((w'_1, w_2)\)-geodesic containing the \((v_1, u_2)\)-path. Since this path uses every interior vertex of the longest \((u_1, u_2)\)-path we have \( V(C) \subseteq I[S] \). Analogously, if there is no RCV we also have \( V(C) \subseteq I[S] \). If there are both RCV and TCV in C we can take \( v_1 \) and \( v_2 \) such that all interior vertices of the longest \((u_1, u_2)\)-path are in the \((v_1, u_2)\)-path or in the \((v_1, u_2)\)-path. Thus we still have \( V(C) \subseteq I[S] \).

Now let C be a directed cycle, neither trap nor leaf. Since it is truly satisfactory there must be \( v_1 \neq v_2 \in V(C) \) such that \( v_1 \) is an RCV and \( v_2 \) is a TCV. If there are such \( v_1 \) and \( v_2 \) with \((v_2, v_1) \in A(C)\), we can take \( w_1, w_2 \in S \) such that there is a \((w_1, w_2)\)-geodesic containing the \((v_1, v_2)\)-path in C, which in turn contains all the vertices of C. If not then take \( v_1, v_2 \) such that every other cut vertex of C lies in the \((v_1, v_2)\)-path. Again by the fact that C is truly satisfactory, we must have an RCV \( v_3 \) and a TCV \( v_4 \), different from one another, such that the \((v_1, v_2)\)-path contains the \((v_4, v_3)\)-path in C. Once more by Proposition 5.3 and Lemma 5.4 there are \( w_3, w_4 \in S \) and a \((w_3, w_4)\)-geodesic containing the \((v_3, v_4)\)-path in C. Knowing that the \((v_4, v_3)\)-path is contained in the \((w_3, w_2)\)-path, we thus conclude that all the vertices of C either in the \((v_3, v_4)\)-path or in the \((v_4, v_3)\)-path will be in \( I[S] \), which implies \( V(C) \subseteq I[S] \).

We next treat the cases where C is an FSC. Take C of type 1 and let \( u_1 \) and \( u_2 \) respectively be the source and the sink in C. Due to previous arguments, it is straightforward that the vertices internal to the \((u_1, u_2)\)-geodesic are in \( I[S] \). If there are only TCV’s in the other \((u_1, u_2)\)-path, with \( v_1 \) being the one such that \( d_C(u_1, v_1) \) is as small as possible, there is at least one vertex in the \((u_1, v_1)\)-path which will not be in \( I[S] \). The case in which there are only RCV’s
is analogous. We thus suppose that there are both RCV’s and TCV’s and let $v_1$ be an RCV and $v_2$ be a TCV such that $d_C(v_2, v_1)$ is the smallest value possible. By repetitive arguments we state that the vertices in the $(u_1, v_2)$-path and in the $(v_1, u_2)$-path are in $I[S]$. However, knowing that there is at least one vertex in the $(v_2, v_1)$-path we have that this will not be in $I[S]$.

Assume now $C$ of type 2, thus there are $v_1$ RCV and $v_2$ TCV such that the other cut vertices are in the $(v_1, v_2)$-path $P$. Moreover if $P = (v_1 = u_0, u_1, \ldots, u_k = v_2)$ and $u_i$ and $u_j$ are respectively an RCV and a TCV we have $i \leq j$. Using again Proposition 5.3 and Lemma 5.4 we have that all the vertices in any $(u_i, u_j)$-path are in $I[S]$, and these are the only ones. Since the $(v_1, v_2)$-path contains every one of these vertices and there is at least one vertex internal to the $(v_2, v_1)$-path in $C$ we have $V(C) \not\subseteq I[S]$.

To close this demonstration let $v$ be a vertex in an FSC not in $I[S]$. If it had any neighbor outside $C$ he would be a cut vertex, thus contradicting the definition of FSC. Since, for each FSC, the vertices not in $I[S]$ are not connected to the rest of the graph, we conclude that every geodetic set must have at least one non-extreme vertex of each FSC.

Theorem 5.7. Let $D$ be an oriented cactus graph. There is a geodetic set composed by all the extreme vertices and one non-extreme of each unsatisfactory and falsely satisfactory cycle. Moreover, this geodetic set is minimum.

Proof. By Theorem 5.5, we can build a minimum hull set $S$ only with the extreme vertices and one non-extreme form each unsatisfactory cycle. Besides, the only vertices not in $I[S]$ are in satisfactory cycles. Moreover, by Lemma 5.6 we can be even more specific and say which vertices are these. So we only have left to show that adding the remaining vertices in the statement to $S$ will give us a minimum geodetic set.

If $C$ is a falsely satisfactory cycle of type 1, let $u_1, u_2, v_1, v_2$ be as in the definition. If there is not any TCV in $C$, take $w \in N^+(u_1)$ and add it to $S$. Once more by Proposition 5.3 and Lemma 5.4 we can take $w_2 \in S$ such that there is a $(w, v_2)$-geodesic containing the $(w, v_2)$-path, which gives us the desired result. The case where there are no RCV’s is analogous. If we have both RCV and TCV, take $v_1$ and $v_2$ as close as possible and take $w$ an internal vertex of the $(v_2, v_1)$-path, which exists by definition. We can take $w_1, w_2 \in S$ such that there are a $(w_1, w)$-geodesic and a $(w, w_2)$-geodesic respectively containing the $(v_1, w)$-path and the $(w, v_2)$-path in $C$.

Now let $C$ be of type 2 and $v_1, v_2$ be as in the definition. Take $v \in N^+(v_2) \cap V(C)$. Notice that there are $w \in S$ and a $(v, w)$-geodesic containing the $(v, v_2)$-path in $C$. Since the last path contains all vertices of $C$ we have $V(C) \subseteq I[S]$.

Once more, such minimum geodetic set can be found in linear time by just analyzing the cycles and determining which ones are (truly/falsely) satisfactory and unsatisfactory.
6 Further research

We first proved that the hull number of an oriented split graph $D = (S \cup C, A)$ is roughly $\frac{3}{4} |C|$ plus the number of its extreme vertices. A natural question is whether a similar bound holds for the geodetic number of an oriented split graph or, at least, a tournament.

Here we also proved that, given an oriented graph $D$, determining $\overrightarrow{hn}(D)$ and $\overrightarrow{gn}(D)$ are NP-hard problems even if the underlying graph of $D$ is bipartite. Equivalent results were known in the literature [1, 9] for the undirected case. We believe that the same is true concerning the class of chordal graphs. Determining $\overrightarrow{hn}(G)$ and $\overrightarrow{gn}(G)$ are NP-hard problems even if $G$ is chordal [3, 9]. A first open problem would be:

**Problem 6.1.** Is it true that, given an oriented graph $D$ and a positive integer $k$, then determining whether $\overrightarrow{hn}(D) \leq k$ or whether $\overrightarrow{gn}(D) \leq k$ are NP-complete problems, even if the underlying graph of $D$ is chordal?

In fact, even determining such parameters for tournaments seems a hard task.

Another natural problem is to find some graph class $G$ for which determining $\overrightarrow{hn}(D)$ is an NP-hard problem, while determining $\overrightarrow{hn}(G)$ can be solved in polynomial time, for some simple graph $G \in G$ and some orientation $D$ of $G$. The same should also be studied for the geodetic number.

Finally, bounds and complexity results for other graph classes (e.g. planar graphs, graphs with bounded treewidth, graphs with few $P_4$’s, etc.) are also widely open.

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