Performance Bounds for Antenna Domain Systems

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Abstract—In this work, we investigate the so-called antenna domain formation problem, as the optimal assignment of users to antenna domains, in cloud radio access networks. We focus on theoretical aspects of the problem, namely, the finding of lower bounds, on the interference leakage in the network. We formulate the antenna domain formation problem as an integer optimization problem, and propose an iterative algorithm for obtaining a solution. Motivated by the lack of optimality guarantees on such solutions, we opt to find lower bounds on the problem, and the resulting interference leakage in the network. We thus derive the corresponding Dantzig-Wolfe decomposition, formulate the dual problem, and show that the former offers a tighter bound than the latter. We highlight the fact that the bounds in question consist of linear problems with an exponential number of variables, and adapt known methods aimed at solving them. In addition to shedding light on the tightness of the bounds in question, our numerical results show sum-rate gains of at least 200% (over a simple benchmark), in the medium SNR region.

Index Terms—Cloud-RAN, antenna domain formation, interference coupling coefficients, Block-Coordinate Descent, L-grange relaxation, Dantzig-Wolfe decomposition,

I. INTRODUCTION

It has been reported that the total volume data is expected to increase by tenfold, between 2013 and 2019 [1], whereby mobile data constitutes the largest fraction. In view of meeting this exponentially increasing demand, 5G systems have to deliver ever-increasing data rates. Such increases are usually met by acquiring new spectrum, higher spectral efficiency, and densification. From a historical perspective, a significant fraction of the gains in data rates are due to densification [2]. Since densification inevitably incurs significant interference, this requires coordination among different base stations.

In the context of cellular networks, an extensive body of work suggests that in general, coordination among (clusters of) base stations is a key to achieving higher sum-rate in the network, namely, the ideas of Coordinated Multi-point [3], [4] and Interference Alignment (IA) [5], [6]. However, in traditional cellular networks, the overhead associated with such techniques has been identified as a (potentially) limiting factor of the sum-rate gains (e.g., [7]–[10]). In contrast, due to its inherent centralized nature Cloud Radio Access Network (C-RAN), enables the tight coordination of antennas in a dense deployment, in a rather economical way. Typically, C-RAN consist of Remote Radio Heads (RRHs) (assumed to have limited baseband/processing capabilities) which are connected to the so-called Aggregation Nodes (ANs) (assumed to have perfect and global CSI), via wired/wireless links. In that sense, aggregation nodes act as centralized compute nodes, that gather all the required CSI from a cluster of connected radio-heads, perform the required optimization (e.g., precoding), and send the resulting parameters to the relevant radio-heads. An Antenna Domain (AD) is the collection of radio-heads connected to a particular aggregation node. It is envisioned that each antenna domain consists of a few (up to tens of) radio-heads, and serves a few dozen (up to a hundred) users. The investigation of such setups, i.e., where base stations are connected via backhauls, was originally done in [11].

While C-RAN promises to effectively manage intra-AD interference, the issue of inter-AD interference remains open (all work thus far assumes the presence of one antenna domain / aggregation node). The authors in [12] investigated dynamic clustering of base stations, where users within each cluster are served in a Joint Transmission (JT)-like manner. The same model was adopted in [13] and [14], where the authors consider the problem of forming BS clusters in the presence of caching and multicast transmission. A similar model for coordination was employed in [15], focusing on energy efficient transmission instead. In [16], (looser) coordination among the radio-heads within the antenna domain was investigated (where Coordinated Beamforming (CB)-type precoding was employed). Obviously, the model adopted by such approaches - where all the network is coordinated by one aggregation node, is not scalable.

Going to the multi-AD/multi-AN setting, the management of inter-AD interference becomes critical (but yet unaddressed). This translates into the so-called Antenna Domain Formation (ADF) problem: the optimal assignment of a) radio-heads to antenna domains, and b) users to antenna domains, to minimize the total interference. a) was addressed in our most recent work [17], where both intra-AD and inter-AD interference were to be balanced. Intra-AD interference was inherently present due to the CB-type of precoding (Weighted-MMSE [18]) that was used within each antenna domain.

In contrast, investigating b) under the assumption that intra-AD interference is nulled by the proposed precoding, will be the main objective of the current work. We focus on theoretical aspects of the so-called antenna domain formation (ADF) problem: Given an initial state (i.e., assignment of users to radio-heads, and radio-heads to aggregation nodes), we study the optimal assignment of users to antenna domains, using the total interference leakage as performance metric.
In contrast to our earlier work [17], we assume tighter coordination within each antenna domain. We formulate the ADF problem as an integer optimization problem, and then employ Block-Coordinated Descent (BCD) - that we earlier developed in [17], to iteratively solve the problem. The lack of theoretical guarantees on the obtained solution, as well as the complicated nature of the problem, motivates us to find useful and meaningful lower bounds on the ADF problem (since it represents the total interference leakage). For that purpose, we derive the corresponding Dantzig-Wolfe (DW) decomposition (a Linear Program (LP) with exponentially many variables), and adapt the Column Generate Method (CGM) to compute the DW lower bound. We also derive the dual problem (a natural lower bound), characterize the duality gap, and show that the DW lower bound is tighter than that of the dual problem (and consequently all related methods such as dual subgradient ascent, and Lagrange relaxation). Finally, we provide some numerical results that highlight the performance of our proposed algorithm.

Sect. II is dedicated to developing the system model/problem formulation, Sect. III to presenting our proposed algorithm, Sect. IV to detailing the proposed relaxations/decompositions, and Sect. V to addressing a special case of the ADF problem. Practical issues and the system-level operation of the algorithm are addressed in Sect. VI, and numerical results are presented and discussed in Sect. VII.

II. System Model

A. Notation

We use bold upper-case letters to denote matrices, bold lower-case to denote vectors, and calligraphic letters to denote sets. Furthermore, for a given matrix \( A \), \( \sigma_{\max}(A) / \sigma_{\min}(A) \) denote the largest/smallest singular value, \( \text{tr}(A) \) denotes its trace, \( |A|_{i,j} \) denotes the matrix formed by taking columns \( i \) to \( j \) of \( A \), \( \|A\|_F^2 \) its Frobenius norm, \( A^T \) its transpose, and \( A^\dagger \) its conjugate transpose. \( A_{i,j} = a_{i,j} \) denotes element \( (i,j) \) in a matrix \( A \), and \( [a_i] \), element \( i \) in a vector \( a \). For any two vectors \( x, y \) (resp. matrices \( X, Y \)), inequalities such as \( x \leq y \) (resp. \( X \preceq Y \)) hold element-wise. While \( I_n \) denotes the \( n \times n \) identity matrix, \( I_n \) denotes the \( n \times 1 \) vector of ones, \( 0_n \) denotes the \( n \times 1 \) vector of zeros, \( e_n \) is the \( n \)th elementary vector of appropriate dimension, \( \mathbb{E} \) denotes the binary mapping, \( \mathbb{Z}_+ \) denotes the set of natural numbers. Given a set \( X \), \( |X| \) denotes its cardinality, and \( \text{conv}(X) \) its convex hull.

B. Model and Assumptions

Consider a large area, comprising of \( A \) aggregation nodes, \( N_T \) remote radio-heads, and \( K_T \) users. The \( k \)th aggregation node is connected to \( N \) radio-heads, via wireless/wired links, where each radio-head is serving a set of users. We refer to the collection of radio-heads connected to each aggregation node, as an antenna domain. Thus, from a system-level perspective, each antenna domain is serving a set of users (thereby abstracting the operation of the radio-heads in the system). A small toy example of the considered system model is illustrated in Fig. [1]. With that in mind, each antenna domain comprises of \( N \) radio-heads and \( K \) users. We denote by \( A \) the set of aggregation nodes, and \( I \) the set of all users, i.e., \( I = \{j_n \mid 1 \leq j \leq A, 1 \leq n \leq K \} \). Since all the quantities defined above are time-varying, the proposed model is for each scheduling time-slot (thus, any time-related indexes are omitted).

Radio-heads are assumed to have limited baseband/processing capabilities, restricted to precoding only. Moreover, aggregation nodes act as "large" centralized processors, that gather channel state information (CSI) from all the users, perform the required processing/optimization, and communicate the optimal precoders to the radio-heads. We assume that each radio head is equipped with \( M \) antennas, while users have a single antenna. We assume that the different radio-heads within each antenna domain are tightly synchronized, e.g. phase-level synchronization, essentially acting as a large virtual antenna array. Each antenna domain consists of a small number of radio-heads (typically \( 2 \sim 4 \)). Moreover, global and perfect CSI is assumed to be a priori available at each of the aggregation nodes. The precoding within each antenna domain is designed in such a way that no intra-antenna domain interference is present (no interference among users within the same antenna domain), i.e., zero intra-AD interference. We underline at this point that such assumptions are quite common for C-RAN related performance studies (e.g., [13]-[14]).

Let \( j_n \) denote the index of the \( n \)th user, in the \( j \)th antenna domain, \( j_n \in I \). Then, its received signal is given by (assuming a downlink transmission scenario),

\[
y_{j_n} = \sum_{q=1}^{K} h_{j,j_n} s_{j_n} \sqrt{p} + \sum_{i \neq j}^{A} \sum_{m=1}^{K} h_{i,j_n} s_{i,m} \sqrt{p} + n_{j_n}
\]

(1)

where \( h_{i,j_n} \in \mathbb{C}^{1 \times MN} \) is the (MISO) channel from antenna domain \( i \) to user \( j_n \), \( s_{i,m} \in \mathbb{C}^{MN \times 1} \) is the (MIMO) channel's beamforming vector to user \( i_m \in I \), \( s_{i} \) is the data symbol for user \( i_m \in I \) such that \( E[|s_{i,m}|^2] = 1 \), \( n_{j_n} \) is the AWGN noise for user \( j_n \in I \) such that \( E[n_{j_n}] = 0 \), \( \sigma^2_{j_n} \) is the transmit power for antenna domain \( j_n \). Moreover, the proposed precoding design, i.e., zero intra-AD interference (Sect. II-B), translates to the following.

\[
h_{j,j_n} v_{j_n} = \begin{cases} \beta_j, & \forall n = q \\ 0, & \forall n \neq q \end{cases}, \quad \forall j \in A
\]

(2)

where \( \beta_j > 0 \) is a free parameter that is chosen to satisfy the maximum transmit power constraint (per antenna domain), i.e.,

\[
\sum_{q=1}^{K} ||v_{j_n}||_2^2 \leq K, \quad \forall j \in A
\]

(3)

The resulting received signal and SINR are,

\[
y_{j_n} = \beta_j s_{j_n} \sqrt{p} + \sum_{i \neq j}^{A} \sum_{m=1}^{K} h_{i,j_n} v_{i,m} s_{i,m} \sqrt{p} + n_{j_n}
\]

\[
\gamma_{j_n} = \frac{\beta_j^2}{\sum_{i \neq j}^{A} \sum_{m=1}^{K} \beta_j^2 |h_{i,j_n} v_{i,m}|^2 + \sigma^2_{j_n}}
\]

(4)

where \( \text{SNR}_{j_n} = \frac{p_j \beta_j^2}{\sigma^2_{j_n}} \) is the SNR of user \( j_n \). Assuming optimal encoding/decoding, and treating interference as noise,
the achievable sum-rate of the network is given by,
\[ R_\Sigma = \sum_{j=1}^{A} \sum_{n=1}^{K} \log_2(1 + \gamma_{jn}) \]  
\(5\)

**C. Motivation**

We denote by \(\psi_{im,jn}\) the so-called interference coupling coefficient between users \(i_m\) and \(j_n\),
\[ \psi_{im,jn} = \begin{cases} p_j|h_{ij}|^2, & \forall \ (im,jn) \in \mathcal{I}^2, \ i_m \neq j_n, \\
0, & \forall i = j \end{cases} \]

\(\psi_{im,jn}\) denotes the interference that user \(i_m\) \(\in\mathcal{I}\) causes to user \(j_n\) \(\in\mathcal{I}\). Moreover, we recall that \(\psi_{im,jn} \neq \psi_{jn,i_m}\). Let \(\Psi \in \mathbb{R}_+^{K \times K}\) be the matrix formed by gathering all the coupling coefficients, i.e.,
\[ \Psi_{jn,i_m} = \begin{cases} \psi_{jn,i_m}, & \forall i \neq j \\
0, & \forall i = j \end{cases}, \forall \ (jn,i_m) \in \mathcal{I}^2 \]

(6)

and
\[ x_{k,jn} \in \{0,1\}, \ \forall \ j_n \in \mathcal{I}, \ k \in \mathcal{A} \]  
(7)

be the assignment variable for user \(j_n\) to antenna domain \(k\). With that in mind, \(g_{j_n}\), the total interference leakage seen by user \(j_n\) \(\in\mathcal{I}\), is given by,
\[ g_{j_n}(\{x_{k,jn}\}) = \sum_{k \in \mathcal{A}} \sum_{i \neq j, j \in \mathcal{I}} x_{k,jn} \psi_{im,jn} x_{l,jn} \]  
(8)

where \(\{x_{k,jn}\}\) denotes the set of all assignment variables. The total interference leakage, \(f\), is then
\[ f(\{x_{k,jn}\}) = \sum_{j_n \in \mathcal{I}} g_{j_n}(\{x_{k,jn}\}) \]  
and can be rewritten as,
\[ f(\{x_{k,jn}\}) = \sum_{k \in \mathcal{A}} \sum_{i \neq j} x_{k,i} \psi_{im,jn} x_{l,jn} \]  
(10)

Recall that due to the proposed precoding (i.e., zero intra-AD interference), the inter-AD interference leakage coincides with the total interference leakage, \(f\), in the system.

**Example 1 (Motivating Example).** Consider the following toy example with \(A = 2, K = 2, N = 1\) (Fig. 1). Then the cost in \(10\) reduces to,
\[ f(\{x_{k,jn}\}) = \sum_{im,jn \neq \emptyset} x_{1,im} \psi_{im,jn} x_{2,jn}, \ \mathcal{I} = \{11, \cdots, 22\} \]

Now the intuition behind the above cost becomes clear: the cost of having users \(i_m\) and \(j_n\) in different antenna domains is \(\psi_{im,jn} + \psi_{jn,i_m}\), and zero otherwise. That same criterion is the reason that intra-AD interference is not accounted for, in \(f\). The last equation shows that the total interference leakage in this network (Fig. 1) corresponds to setting all the assignment variables to one, i.e., \(f(\{x_{k,jn} = 1\})\) - a “naive” assignment. Thus, better performance can be reaped-off with a “smarter” assignment. This is the main motivation for using a cost function such as \(10\).

**D. Problem Formulation**

In the last part, we motivated the effect of assigning users to antenna domains. That is the so-called antenna-domain formation (ADF) problem, that is formalized below.

**Definition 1** (Antenna Domain Formation (ADF)). Given an initial state (i.e., assignment of users to radio-heads, and radio-heads to aggregation nodes), the ADF problem is given by the optimal assignment of users to antenna domains, w.r.t. minimizing the total interference leakage in the system. The corresponding optimization problem is the integer programming problem shown below,
\[
\begin{align*}
\min_{\{x_{k,jn}\}} f = \sum_{k=1}^{A} \sum_{i \neq j} x_{k,i} \psi_{im,jn} x_{l,jn} \\
\text{s. t.} & \sum_{i \in \mathcal{I}} x_{k,i} = \rho_k, \ \forall k \in \mathcal{A} \\
& \sum_{k=1}^{A} x_{k,i} \leq 1, \ \forall i \in \mathcal{I} \\
& x_{k,i} \in \{0,1\}, \ \forall (k,i) \in \mathcal{A} \times \mathcal{I}
\end{align*}
\]  
(11)

The first constraint specifies that \(\rho_k \in \mathbb{Z}_+\) users are to be assigned to each antenna domain, i.e., the loading constraint. Such a constraint is needed for the sake of load balancing on the backhaul (i.e., to prevent highly asymmetric cases where all users get assigned to one antenna domain, while the rest are idle). Moreover, the second constraint, i.e., the assignment constraint, ensures that each user is assigned to at most one antenna domain. Another way to interpret (P) is from a user assignment/selection perspective: given an initial state with \(K\) users (where \(K\) large), the goal is to select the optimal subset (of size \(\rho_k < K\)), that minimizes the interference leakage.

We first start by rewriting \(P\) in vector and matrix form - both of which will be used later in the text (keeping in mind that all are equivalent). Let \(\mathbf{x}_k\) be the aggregate assignment vector for antenna domain \(k\) to all other users, and \(\mathbf{X}\) the aggregate assignment matrix for the system,
\[ \mathbf{x}_k = [x_{k,1,1}, \cdots, x_{k,A_k}]^T, \ \mathbf{x}_k \in \mathbb{B}^{K \times A}, \ \forall k \in \mathcal{A} \]
\[ \mathbf{X} = [\mathbf{x}_1, \cdots, \mathbf{x}_A] \in \mathbb{B}^{K \times A} \]
Proposition 1: \((P)\) can be rewritten in equivalent vector form,
\[
(P) \left\{ \begin{array}{l}
\min f(\{x_k\}) = \sum_{k=1}^{A} \sum_{f} x_k^T \Psi x_k, \\
\text{s. t. } \sum_{k=1}^{A} x_k \leq 1_{KT}, \\
1_{KT} x_k = \rho_k, \ x_k \in B_{KT}, \ \forall k \in A
\end{array} \right.
\]
and matrix form
\[
(P) \left\{ \begin{array}{l}
\min f(X) = \text{tr}(X^T \Psi X) \\
\text{s. t. } X 1_A \leq 1_{KT}, \ X \in S_p
\end{array} \right.
\]
where we denote by \(S_p\) the set of all \(KT \times A\) binary matrices, that satisfy the loading constraint, i.e.,
\[
S_p \triangleq \{Q \in B_{KT \times A} | Q^T 1_{KT} = \rho\}
\]
\[
\rho \triangleq [\rho_1, ..., \rho_A]^T, \text{ and } \Omega \triangleq 1_A 1_A^T - I_A.
\]

Proof: The derivations are shown in Appendix A.

It can be seen from (12) that \(f\) is not jointly convex in all the variables, due to the coupling among them. However, we underline the inherent multi-linear nature of \(f\) (taken separately in each variable, \(f\) is linear), that we exploit for the optimization.

III. PROPOSED ALGORITHM

Our proposed approach consists of two parts, where the first one is concerned with the investigation of the optimal assignment of users to antenna domains, i.e., obtaining a solution to the so called ADF problem in (11). The approach was presented in parts, in our earlier work [17] - albeit for a different system model, but is still summarized here for completeness. In the second part, we develop the preceding mechanism. We first present the following definition.

Definition 2 (Integrality Property for Linear Programming). Consider the following linear binary program (LP),
\[
(P) \ \hat{x}^* = \min \ c^T \hat{x}, \text{ s. t. } \hat{x} \in C, \ \hat{x} \in B^N,
\]
and its continuous relaxation (CR) (also known as LP relaxation),
\[
(CR) \ \tilde{x} = \min \ c^T \tilde{x}, \text{ s. t. } \tilde{x} \in C, \ 0_N \leq \tilde{x} \leq 1_N,
\]
where \(C\) is a convex set. \(C\) is said to satisfy the integrality property if all its vertices correspond to integers: it is well-known for such cases, that the so-called continuous relaxation (CR) is optimal [19], and consequently, \(\tilde{x}\) is integer as well, and \(\tilde{x} = \hat{x}\).

A. Algorithm Description

Due to the coupled nature of the objective function in (12), we leverage the well known Block-Coordinate Descent (BCD) method, that has been applied to several areas of signal processing, namely, transmitter and receiver optimization in cellular networks [18], [20]-[22]. In what follows, \(n\) denotes the iteration number, i.e., \(x_k^{(n)}\) denotes the value of \(x_k\) at the \(n\)th iteration. We denote by \(z_k^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, ..., x_A^{(n)})\) the block of fixed variables, for the \(k\)th update at the \(n\)th iteration.

Our exposition here will be summarized, since the full details of the algorithm are shown in [17]. We let \(f(x_k, z_k^{(n)})\) denote the function \(f(x_k)\), when the variables in block \(z_k^{(n)}\) are fixed, which can be written as
\[
f(x_k, z_k^{(n)}) = x_k^T \Psi \left( \sum_{l=1}^{A} z_l^{(n)} + \sum_{l=0}^{A} x_l^{(n)} \right)
\]
\[
\triangleq x_k^T r_k^{(n)}
\]
where \(r_k^{(n)}\) is referred to as the residual of antenna domain \(k\), at the \(n\)th iteration. Looking at the above equation, \(f(x_k, z_k^{(n)})\) is linear in \(x_k\), implying that \(f\) is linear in each block of variables. The application of BCD yields the following update for \(x_k\) at the \(n\)th iteration.
\[
x_k^{(n+1)} = \arg \min f(x_k, z_k^{(n)}) \text{ s. t. } 1_{KT} x_k = \rho_k, \ x_k \leq \omega_k, \ x_k \in B_{KT}
\]
As seen from (15), when \(\{x_k^{(n)}\}_{k\neq k}\) are fixed, the cost function decouples in \(x_k\) and can thus be solved locally at antenna domain \(k\), in a fully distributed manner: Each aggregation node solves its own subproblem (a linear program), without any loss in optimality. The optimal update for \(x_k\) at antenna domain \(k\), is a function of the assignments at all the other antenna domains (that thus have to be shared): Given assignments from other antenna domains, \((x_1^{(n)}, x_2^{(n)}, ..., x_A^{(n)})\), antenna domain \(k\) forms the residual \(r_k^{(n)}\), and can proceed to solve its optimization problem locally, and update \(x_k^{(n+1)}\). The process is formalized in Algorithm 1.

Algorithm 1 ADF via BCD

Input: \(\Psi, K_T, \rho, A\)
for \(n = 0, 1, \cdots, L - 1\) do
\[\text{// procedure at each aggregation node} \]
\[\text{obtain } \{x_1^{(n+1)}, x_2^{(n+1)}, ..., x_A^{(n+1)}\}\]
at antenna domain \(k\)
\[\text{compute residual } r_k^{(n)} \text{ using (15)}\]
\[\text{compute feasible assignment } \omega_k \text{ using (16)}\]
\[\text{compute } x_k^{(n+1)} \text{ as solution to (17)}\]
end for

Output: \(X^{(L)} = [x_1^{(L)}, x_2^{(L)}, ..., x_A^{(L)}]\)
B. Convergence

Let \( \mathbf{x}_{k}^{(n)} \) be the sequence iterates produced by the BCD in (17), and \( \mathbf{x}_{k}^{(n)} \rightarrow \lim_{n \to \infty} \mathbf{x}_{k}^{(n)} \). The monotonic nature of the BCD iterates was established in our earlier work [17], and is presented below for completeness.

**Lemma 2 (Monotonicity).** With each update \( \mathbf{x}_{k}^{(n)} \rightarrow \mathbf{x}_{k+1}^{(n)} \), \( f \) is non-increasing. Moreover, the sequence of function iterates \( \{f(\mathbf{x}_{1}^{(n)}, \ldots, \mathbf{x}_{A}^{(n)})\}_{n} \) converges to a limit point \( f(\{\mathbf{x}_{k}^{(n)}\}) \).

**Proof:** Refer to Appendix [D]

Although the above result establishes the convergence of the proposed BCD method, it only establishes convergence to a limit. However, showing that this limit is a stationary point of \( f \) is not possible under the BCD framework, due to the coupled nature of the assignment constraint (12). Even the strongest BCD convergence results such as [23] cannot establish convergence to a stationary point.

C. Precoding

This far, we have only focused on the specifics of the ADF problem, while ignoring the precoding. The main idea behind the precoder design is to null all intra-AD interference, as the precoder design is to null all intra-AD interference, as shown in (2) and (3). More intuition could be gained by rewriting the signal model (1) in vector form,

\[
y_j = H_{j,i}V_j s_j \sqrt{\beta_i} + \sum_{i \neq j} H_{i,j} V_i s_i \sqrt{\beta_i} + n_j
\]

(18)

where \( y_j \) is the vector of received signals for users served by antenna domain \( j \). In the above,

\[
H_{i,i} = \begin{bmatrix}
H_{i,j_1} & \cdots & H_{i,j_A}
\end{bmatrix}, \quad V_i = [v_{i_1}, \ldots, v_{i_K}] \text{ and } s_i = \begin{bmatrix}s_{i_1} \\
\vdots \\
s_{i_K}
\end{bmatrix}
\]

(19)

denote the channel between the antennas of antenna domain \( i \) and the users of antenna domain \( j \), the matrix of precoding vectors for antenna domain \( i \), and the vector of transmit symbols for users of antenna domain \( i \), respectively. Then, zero intra-AD interference condition, i.e., (3) and the maximum transmit power constraint, i.e., (5) are equivalently written,

\[
H_{i,i} V_i = \beta_i I_K, \quad \|V_i\|_F^2 \leq K, \quad \forall i \in A
\]

(20)

Note that the total interference leakage \( f \), can be equivalently written as,

\[
f = \sum_{i=1}^{A} \sum_{j \neq i} \|H_{i,j} V_i\|_F^2 = \sum_{i=1}^{A} \sum_{j \neq i} \text{tr}(H_{i,j} V_i V_i^H)
\]

\[
= \sum_{i=1}^{A} \left\| V_i^H H_{i,i} V_i \right\|_F = \sum_{i=1}^{A} \text{tr}(V_i^H R_i V_i) = h(\{V_i\})
\]

(21)

Note that while \( f \) in (19) denotes the interference leakage expressed as a function of the assignment variables, \( h \) denotes interference leakage expressed as function of the precoders. Though the two are equal, we make that distinction for clarity.

The precoder optimization problem for antenna domain \( i \), is then formulated as follows.

\[
V_i^* = \arg\min_{V_i} \text{tr}(V_i^H R_i V_i) \quad \text{s. t. } H_{i,i} V_i = \beta_i I_K
\]

(22)

Note that the transmit power constraint can be explicitly enforced, since it can be satisfied by changing the free parameter \( \beta_i \).

**Proposition 2.** Consider the following convex problem.

\[
V_i^* = \arg\min_{V_i \in \mathbb{C}^{M \times N_d}} \text{tr}(V_i^H R_i V_i) \quad \text{s. t. } H_{i,i} V_i = \beta_i I_{d_i}, \quad d_i \in \mathbb{Z}^+
\]

(23)

where \( R_i = \sum_{i \neq j} H_{i,j}^H H_{i,j} \), \( H_{i,j} \in \mathbb{C}^{d_i \times MN} \), \( \beta_i \) is a free parameter chosen to satisfy the transmit power constraint, \( \|V_i\|_F^2 = d_i \). The globally optimal solution is given by

\[
V_i^* = \frac{\sqrt{d_i} R_i^{-1} H_{i,i}^H \left( H_{i,i} R_i^{-1} H_{i,i}^H \right)^{-1}}{\|R_i^{-1} H_{i,i}^H \left( H_{i,i} R_i^{-1} H_{i,i}^H \right)^{-1}\|_F}
\]

(24)

Moreover, for \( d_i \leq MN \), the problem is feasible almost surely.

**Proof:** Refer to Appendix [B]

IV. RELAXATIONS AND PERFORMANCE BOUNDS

It should be clear at this stage that problems such as (P) are quite challenging. This is further highlighted by the findings of the previous section: despite the widespread effectiveness of methods such as BCD, one is not able to show any stationarity of the obtained solution (i.e. no local optimality can be established). Moreover, it is hard to theoretically ascertain how ‘close’ is an obtained solution to optimality. To compensate for those shortcomings, finding meaningful lower bounds on \( (P) \) is of interest: that is particularly relevant for our case, since the cost function, \( f \), represents an actual physical quantity. Moreover, as discussed earlier in Sec. II[C] finding lower bounds on the interference leakage \( f \), result in finding upper bounds on the sum-rate. For the problem at hand we derive the corresponding Dantzig-Wolfe (DW) decomposition, and establish that although the resulting problem is a LP, it has exponentially many variables. We thus adapt the Column Generation Method (CGM), for our particular problem. We also derive the dual problem for \( (P) \), and show that it yields a looser lower bound on \( (P) \). We thus conclude that methods that are based on the dual problem (e.g., Dual Subgradient Ascent, Lagrange Relaxation), offer worse lower bounds, than ones based on the DW decomposition (e.g., CGM).

A. Preliminaries

We here summarize some relevant concepts and definitions that will be applied extensively, later in the work.

**Definition 3 (Inner Representation of Bounded Polyhedron).** Let \( \mathcal{P} \) be a bounded polyhedron (the intersection of finitely many half-spaces), i.e. \( \mathcal{P} = \{x \in \mathbb{R}^n \mid Ax = e\} \). Then, every
point \( \mathbf{x} \in \mathcal{P} \) is expressed as a convex combination of its extreme points,
\[
\mathbf{x} = \sum_{j=1}^{J} \psi_j \mathbf{w}_j, \quad \sum_j \mathbf{w}_j = 1, \; \mathbf{w}_j \geq 0, \; \forall j \in \mathcal{V}
\]
where \( \mathcal{V} = \{ \psi_j \}_{j=1}^{J} \) is the set of extreme points of \( \mathcal{P} \).

**Definition 4** (Special LPs). Consider the following LP,
\[
(LP) \quad \mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathbb{R}^n} c^T \mathbf{x}, \; s.\; t. \; \mathbf{1}^T \mathbf{x} = 1, \; \mathbf{x} \geq 0,
\]
Let \( \mathcal{V} \) be the set of vertexes (extreme points) for \((LP)\). Note that, \( \mathcal{V} \) can be written as \( \mathcal{V} = \{ \mathbf{e}_i \}_{i=1}^{n} \), where \( \mathbf{e}_i \) is the \( i \)th elementary vector in \( \mathbb{R}^n \). Moreover, for LPs, the optimal solution lies within \( \mathcal{V} \) - a fundamental result for LPs.

\[
(LP) \quad \mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathcal{V}} c^T \mathbf{x}
\]
and consequently, \( \mathbf{x}^* = \mathbf{e}_{j^*} \). For such problems, the solution reduces to searching over the cost \( c \). A simple toy example is shown in Fig. 2.

**Fig. 2:** Feasible region of a special LP in \( \mathbb{R}^3 \) (solid lines). All vertexes are elementary vectors, i.e., binary.

In what follows, we define the following notation,
\[
\mathcal{S}_\rho \triangleq \{ \mathbf{Q}_j \}_{j=1}^{S}, \; S = |\mathcal{S}_\rho| \nonumber
\]
\[
\alpha_j \triangleq \text{tr}(\mathbf{Q}_j^T \Psi \Omega_j), \; \forall j = 1, \cdots, \mathcal{S} \nonumber
\]
\[
\mathbf{q}_j \triangleq \mathbf{Q}_j \mathbf{1}_{K_T} - \mathbf{1}_{K_T}, \; \forall \mathbf{Q}_j \in \mathcal{S}_\rho, \; \mathbf{q}_j \in \mathbb{Z}^{K_T} \nonumber
\]
\[
\mathbf{\Gamma} \triangleq [\mathbf{q}_1 \cdots \mathbf{q}_S], \; \mathbf{\Gamma} \in \mathbb{Z}^{K_T \times S}
\]
Moreover, note that \( \mathcal{S}_\rho \) has a decomposable structure, i.e., \( \mathcal{S}_\rho \triangleq \mathcal{W}(|\mathbf{\rho}_i|) \times \cdots \times \mathcal{W}(|\mathbf{\rho}_A|) \) where \( \mathcal{W}(\mathbf{\rho}_i) = \{ \mathbf{x} \in \mathbb{B}^{K_T} | \mathbf{1}_{K_T}^T \mathbf{x} = \rho_i \} \). Thus,
\[
S = |\mathcal{S}_\rho| = \prod_{i=1}^{A} |\mathcal{W}(\mathbf{\rho}_i)| \text{ where } |\mathcal{W}(\mathbf{\rho}_i)| = \left( \frac{K_T}{\rho_i} \right)
\]

**Remark 1.** As it will become clear in this section, some of the decompositions/relaxations in question are computationally demanding. However, we underline the fact that such methods are intended for benchmarking purposes: they are intended to run during an offline training phase, where “enough” computational resources are available. Thus, the computation of quantities such as \( \Psi, \mathcal{S}_\rho, \alpha_j, \mathbf{q}_j \) and \( \mathbf{\Gamma} \) is not a limiting factor.

**B. Dantzig-Wolfe Decomposition**

Initially proposed in their seminal paper [24], the Dantzig-Wolfe decomposition has been widely adopted by the operations research community, for finding bounds on integer programming problems. Based on our above definitions in (26), we can rewrite \( \mathcal{S}_\rho \) and \( (P) \) as,
\[
\mathcal{S}_\rho = \{ \mathbf{X} = \sum_{j=1}^{S} w_j \mathbf{Q}_j | \sum_{j=1}^{S} w_j = 1, \; w_j \in \mathbb{B}, \forall j \} \nonumber
\]
\[
(P) \begin{cases} \min \; f(\mathbf{X}) = \text{tr}(\mathbf{X}^T \Psi \mathbf{X} \Omega) \\ \text{s. t.} \; \mathbf{X} \in \mathcal{S}_\rho, \; \mathbf{1}_A \leq \mathbf{1}_{K_T} \end{cases} \nonumber
\]

The above problem is still difficult to tackle, due to the combinatorial nature of \( \mathbf{X} \in \mathcal{S}_\rho \). The DW decomposition then proceeds by relaxing \( \mathbf{X} \in \mathcal{S}_\rho \), into a convex one, by taking its convex hull, i.e.,
\[
\text{conv}(\mathcal{S}_\rho) = \{ \mathbf{X} = \sum_{j=1}^{S} w_j \mathbf{Q}_j | \sum_{j=1}^{S} w_j = 1, \; \mathbf{0}_S \leq \mathbf{w} \}
\]

As a result, every point in \( \text{conv}(\mathcal{S}_\rho) \) is represented as a convex combination of the extreme points of \( \text{conv}(\mathcal{S}_\rho) \) (detailed in Definition 3). Since \( \mathcal{S}_\rho \subseteq \text{conv}(\mathcal{S}_\rho) \), the resulting problem \( (P_{DW}) \), is a lower bound on \( (P) \).
\[
(P_{DW}) \begin{cases} \min \; f(\mathbf{X}) = \text{tr}(\mathbf{X}^T \Psi \mathbf{X} \Omega) \\ \text{s. t.} \; \mathbf{X} \in \text{conv}(\mathcal{S}_\rho), \; \mathbf{1}_A \leq \mathbf{1}_{K_T} \end{cases}
\]

Note that the assignment constraint can be written in an equivalent form,
\[
\mathbf{X}\mathbf{1}_A \leq \mathbf{1}_{K_T} \iff (\sum_j w_j \mathbf{Q}_j) \mathbf{1}_A \leq \mathbf{1}_{K_T} \iff (\sum_j w_j \mathbf{Q}_j) \mathbf{1}_A \leq \mathbf{1}_{K_T}
\]
\[
\sum_j w_j \mathbf{q}_j + (\sum_j w_j) \mathbf{1}_{K_T} \leq \mathbf{1}_{K_T} \iff (\mathbf{w})^T \mathbf{\Gamma} \leq 0_{K_T}
\]
where the last one follows from the fact that \( \sum_j w_j = 1 \) (as defined by the DW decomposition). Moreover, recalling that \( \alpha_j \triangleq \text{tr}(\mathbf{Q}_j^T \Psi \Omega_j), \; \forall j \), and letting \( \mathbf{w} = (w_1, \cdots, w_S)^T \), (31) is equivalent to,
\[
(P_{DW}) \begin{cases} \min \; \mathbf{\alpha}^T \mathbf{w} \\ \text{s. t.} \; \mathbf{\Gamma} \mathbf{w} \leq 0_{K_T}, \; \mathbf{1}_S^T \mathbf{w} = 1, \; \mathbf{w} \geq 0_S \end{cases}
\]
A few remarks are in order at this stage. Note that despite the combinatorial and non-convex nature of \( (P) \), the DW always results in a linear program (provided that \( \mathcal{S}_\rho \) is a bounded polyhedron). However, there is the additional caveat that though (32) is a LP, it has an exponential number of variables, \( S \): it is unfit for conventional LP solvers. We thus adapt the Column Generate Method (CGM), for our particular problem.

**Remark 2.** We note that (30) clearly shows that the DW decomposition is a mapping from \( \mathbf{X} \) in (13), to \( \mathbf{w} \) in (32). However, this mapping is evidently not one-to-one, since \( \mathbf{X} \) uniquely reconstructs from \( \mathbf{w} \), but not vice versa.

1) Solution via Column Generation Method: The Column Generation Method (CGM), attempts to iteratively solve (32), thereby mitigating the need for directly solving it: starting from \( \mathbf{1}_A \) - a matrix consisting of a subset of \( n_a \) columns of \( \mathbf{1} \), one first solves the resulting restricted master problem (RMP), i.e. a reduced version of (32). Then, at the \( l \)th iteration, one selects an additional column that is added to \( \mathbf{1}_A \) (or multiple ones), and solves the resulting RMP. Given a subset \( \mathcal{X} \) of \( \mathcal{S}_\rho \),
we define, $\mathbf{G}(X) \in \mathbb{Z}_+^{K \times |X|}$ as the matrix generated by the $X$ columns of $\mathbf{G}$, and $\mathbf{a}(X) \in \mathbb{R}^{|X|}$ the corresponding sub-vector of $\mathbf{a}$.

The procedure is formalized below. Let $T_0 \subset S_p$ be the initial subset of columns for $\mathbf{G}$, such that $|T_0| = m_o$. At iteration $l \geq 1$, given the previous selected columns $T_{l-1}$, and the corresponding optimal solutions for the RMP, $\mu_{l-1}^*$ and $\pi_{l-1}^*$, the vectors of reduced costs is defined as,

$$d_l \triangleq \mathbf{a}(Z_l) - \Gamma(Z_l)^T \mu_{l-1}^* - \pi_{l-1}^*|Z_{l-1}|,$$

where $Z_l \triangleq S_p/T_{l-1}$ and $\Gamma(Z_{l-1})^T = [-\Gamma(Z_{l-1})^T, \mathbf{I}_{K_T}]$. Then, the index of the column to be updated is defined as,\n
$$i_l^* \triangleq \text{argmin}_{i \in Z_l} |d_l|,$$

and the set of active columns is updated as follows,\n
$$\mathcal{T}_l = \mathcal{T}_{l-1} \cup \{i_l^*\}.$$\n
Essentially, $i_l^*$ is the index of the column in $\mathbf{G}$, that is added to the RMP. Then, the updated RMP at iteration $l$, is denoted by $(R_l)$,

$$(R_l) : \mathbf{w}^*(T_l) \left\{ \begin{array}{ll} \text{argmin} & \mathbf{a}(T_l)^T \mathbf{w}(T_l) \\ \text{s. t.} & \Gamma(T_l)\mathbf{w}(T_l) \leq 0_{K_T}, \\ & 1^T_{m_l} \mathbf{w}(T_l) = 1, \quad \mathbf{w}(T_l) \geq 0_{m_l} \end{array} \right.$$\n
(35)

The above problem is a simple LP, and assuming that it is feasible, strong duality holds. Then, it can be verified that its equivalent dual is written as,

$$\left( \mu_l^*, \pi_l^* \right) \left\{ \begin{array}{ll} \text{argmax} & \pi_l \\ \text{s. t.} & \Gamma(T_l)^T \mu_l + \pi_l |1_{m_l} \leq \mathbf{a}(T_l) \end{array} \right.$$\n
(36)

where $m_l \triangleq |T_l| = m_o + l$, and $\Gamma(T_l)^T = [-\Gamma(T_l)^T, \mathbf{I}_{K_T}]$. The steps are detailed in Table I. Note that, in the worst case, CGM ends up adding all columns in $\mathbf{G}$, i.e., solving the original problem (32). However, most often, the algorithm will terminate much earlier than that.

When all reduced costs are non-negative, the optimal solution has been found, i.e., the solution of the current RMP is the same as the original problem. Let $L$ be that iteration number, and $\mathbf{w}^*(T_L)$, $(\mu_L^*, \pi_L^*)$ be the corresponding optimal primal-dual pair corresponding to $(R_L)$. Then, the optimal solution $\mathbf{w}^*$ of the original problem, (32) is given by,

$$\mathbf{w}^* = \left\{ \begin{array}{ll} \mathbf{w}^*(T_L) & \text{if } i \in T_L, \\ 0, & \text{otherwise} \end{array} \right.$$\n
(37)

Looking at (37), the solution that CGM yields consists only of the component in $\mathbf{w}$ that have a contribution to the solution (32), while setting the rest to zero. Interestingly, in most cases, despite the exponential size of $\mathbf{w}$, it will have only a few non-zero entries. It is a well-known fact that despite its iterative nature, CGM is an exact method, i.e., $\mathbf{w}^*$ in (37) is the globally optimal solution to $(P_{DW})$.

**Remark 3.** Note that the algorithm can be extended to taking $\Delta \geq 1$ columns at each iteration, that correspond to columns with negative reduced cost, thereby speeding up the algorithm. However, for simplicity of exposition, we stick with the above formulation, where one column is added at each iteration.

**Initialization:** $T_0, m_0$

for $l = 1, 2, \ldots, S - m_o$ do

$Z_l \leftarrow S_p/T_l$

if update $\mu_l^*, \pi_l^*$

Generate $\Gamma(T_l), \Gamma(T_l)\mathbf{a}(T_l)$

Compute $\mu_l^*, \pi_l^*$ by solving (36)

if update reduced costs and active columns

$d_l \triangleq \mathbf{a}(Z_l) - \Gamma(Z_l)^T \mu_l^* - \pi_l^* |Z_l|$

$i_l^* \leftarrow \text{argmin}_{i \in Z_l} |d_l|$

end for

**Output:** $\mathbf{w}^*$

**TABLE I: DW solution via CGM**

C. Dual Problem

In addition to being a natural lower bound on $(P)$, the dual problem, $(D)$, is the basis of several techniques for obtaining lower bounds. For instance, it is the “optimal bound” that the Lagrange Relaxation - one of the most widely adopted methods for finding lower bounds, can yield. Moreover, methods such as Dual Subgradient Ascent - the analog of gradient ascent for non-differentiable problems, converge to the optimal solution of $(D)$. With that in mind we derive the dual problem $(D)$, associated with $(P)$, characterizing the resulting duality gap, and show that the DW decomposition offers a tighter bound than the dual problem (and hence all the associated methods described above).

1) Suboptimality of Dual Problem Bound :

**Proposition 3.** The dual problem, $(D)$, is defined as,

$$(D) \max \alpha \in \mathbb{R}_{\geq 0} \{ \min \mathbf{w}^T (\mathbf{X}^T \mathbf{X} \mathbf{w}) + \lambda \mathbf{w}^T (\mathbf{X}_1 \mathbf{w} - 1_{K_T}) \}$$

(38)

can be written as follows,

$$(D) \max \mathbf{c}^T \mu \s.t. \mathbf{G}^T \mu \leq \alpha, \quad \mu \geq 0_{K_T}$$

(39)

where $\mathbf{c} = [0_N, 1]^T$, and $\mathbf{G}^T = [-\mathbf{G}^T, 1_S]$.\n
**Proof:** Refer to Appendix B.

$(D)$ in (39) is a LP, and since strong duality holds, we work with its (equivalent) dual form. Moreover, plugging in the values of $\mathbf{G}$ and $\mathbf{c}$, $(D)$ in (39) is equivalent to,

$$(D) \min \mathbf{w}^T \mathbf{w} \s.t. \mathbf{G}^T \mathbf{w} \leq [0_S, 1_S]^T \mathbf{w} \geq 1, \quad \mathbf{w} \geq 0_S$$

(40)

Comparing $(D)$ in (40) to $(P_{DW})$ in (32) quickly reveals that $(D)$ is a relaxation of $(P_{DW})$. Consequently, the bound provided by the DW decomposition is tighter than that of the dual. Thus, methods such as Lagrange Relaxation and Dual Subgradient Ascent (that yield a solution to $(D)$) will result in looser bounds on $(P)$, when compared to methods based on the $(P_{DW})$.\n
2) Characterization of Duality Gap : Due to the non-convex nature of $(P)$, it is quite likely that strong duality does not hold. Since the dual problem is the object of several investigations in this work, it is natural to inquire about the wideness of the duality gap: the difference between the optimal solution of $(P)$, and that of $(D)$. Indeed, such a gap could be large (or potentially unbounded). We note at this point that an exact characterization of the duality gap is clearly infeasible (since one needs optimal solutions for both $(P)$, and $(D)$). We thus provide a bound on the gap, in the result below.

Lemma 3 (Bound on Duality Gap). Let $X^*$ and $\lambda^*$ be optimal solution for the primal problem $(P)$ in (43) and the dual $(D)$ in (38), respectively. Then the duality gap satisfies,

$$\begin{align}
0 \leq f(X^*) - d(\lambda^*) & \leq \eta(\sigma_{\text{max}}[\Psi] - \sigma_{\text{min}}[\Psi]) \\
& + \sum_k \rho_k \min_i |\lambda^*_i|, 
\end{align}
$$

(41)

where $\eta \triangleq \sum_k \sum_{i \neq k} \rho_i \rho_k$.

**Proof:** Refer to Appendix F

V. THE TWO ANTENNA DOMAIN CASE

We focus in this section on the case of two antenna domains, since the problem takes a rather simple form. Moreover, we propose an equivalent reformulation of the ADF problem, that enables a straightforward and simple solution. Firstly, the cost function is given by $f(x_1, x_2) = x_1^T (\Psi + \Psi^T) x_2$. Moreover, note that in this case, the assignment constraint is always satisfied and thus no longer needed. We assume full-load conditions with equal loading for the antenna domains (i.e., $\rho_1 = \rho_2 = K_T/2$). For this special case, $x_2 = 1_{K_T} - x_1$. Thus, the optimization problem can be expressed in terms of $x_1$ only (and one can drop all subscripts). With that in mind, the loading constraint is expressed as, $1_{K_T} x_1 = \rho$. Letting $\Psi = \Psi + \Psi^T$, when $A = 2$, $(P)$ takes the following simple form:

$$(P) : f(x^*) = \min_{x \in S_\rho} f(x) = (x^T \Psi 1_{K_T} - x^T \Psi x)$$

(42)

where $S_\rho = \{ x \in \mathbb{R}^{K_T} \mid 1_{K_T} x = \rho \}$.

A. Equivalent formulation

We use a “DW-like” transformation to reformulate problems such as $(P)$, into an equivalent form. The result below is given for the generic case.

Lemma 4. Let $p(Z)$ be any arbitrary (possibly non-convex) function, and consider the following integer programming problem

$$(Q) \quad Z^* = \arg\min p(Z) \text{ s.t. } Z \in S$$

where $S = \{ W_j \mid j = 1, \ldots, n \}$ is a finite discrete set. Then, the problem is equivalent to

$$(Q) \quad t^* = \begin{cases} \arg\min p_d(t) = t^* \theta \\ \text{s.t. } t^* 1_n = 1, \quad t \geq 0_n, \end{cases}$$

(44)

where $[\theta]_{j} \triangleq p(W_j), \ j = 1, \ldots, n$.

**Proof:** Refer to Appendix C

VI. PRACTICAL ASPECTS

A. System-Level Operation

We next detail the overall operation of the algorithm. Starting from a given deployment of aggregation nodes, radioheads and users, each radio-head is first assigned to an aggregation node (based on some rule, e.g., minimal distance), and then synchronized within each antenna domain. Moreover, users are initially assigned to antenna domains, based on strongest channels. After the CSI acquisition phase (where each aggregation node acquires global CSI), the precoders are computed at each aggregation node, and the matrix of coupling coefficients (consisting of channels and precoders) is computed at each aggregation node. Algorithm 1 is then run across all the aggregation nodes to compute an ADF solution, that is in turn used to (re-)assign users to antenna domains. Finally, the precoders are recomputed based on the latter assignment. The overall system-wide operation of the proposed method is summarized in Algorithm 2.

![Fig. 3: System-level Operation](image-url)

Lemma 4 can be directly applied to rewrite (42) in an equivalent form,

$$(P) \quad w^* = \begin{cases} \arg\min w^T \alpha \\ \text{s.t. } w^T 1_{K_T} = 1, \ w \geq 0_{K_T} \end{cases}$$

where $\alpha = [\alpha_1, \ldots, \alpha_S]^T$, $\alpha_j = u_j^T \Psi_1 1_{K_T} - u_j^T \Psi u_j, \ \forall j = 1, \ldots, S$, and $S_\rho = \{ u_j \}_{j=1}^S$. Note that this last problem falls under the category of special LPs, and following the discussion in Definition 4 its solution is an elementary vector. Thus, the optimal solution to $(P)$ is given by,

$x^* = u_{j^*}, \ j^* = \arg\min_{1 \leq j \leq S} \alpha_j$ (45)

Consequently, for the two antenna domain case, solving for $x^*$ reduces to just finding the minimum of the $S$-dimensional vector, $\alpha$. Although this is similar in complexity to exhaustively searching for $(P)$, it does provide a systematic means of doing that. Moreover, as argued in Remark 1, computing $\alpha$ is not a limiting factor.
Algorithm 2 Precoding and Antenna Domain Formation

// Start with a given users-to-antenna domain assignment
1. Compute precoders using (22)
2. Compute \( \Psi \) (based on CSI and precoders)
3. Compute ADF solution (\( \mathbf{X}^{(L)} \)) in Algorithm 1
4. Assign users to antenna domain based on \( \mathbf{X}^{(L)} \)
5. Recompute precoders using (22)

B. Choice of loading factors

We highlight the existence of an interesting result, regarding the choice of loading factors: when \( \sum \rho_i \leq MN \), then one can show that the leakage can be completely nulled.

Corollary 1. Consider a special case of Proposition 2 by setting \( d_i = \rho_i \), where \( \sum \rho_i \leq MN \).

\[
V_i^* = \arginf_{V_i \in \mathbb{C}^{M \times N \times \rho_i}} h(V_i) = \text{tr}(V_i^* R_i V_i^*)
\]

s. t. \( H_i V_i^* = \beta_i I_{\rho_i} \),

(46)

Then, \( h(V_i^*) = 0 \), almost surely.

Proof: Refer to Appendix [6] ■

Note that the same result of nulling all interference can be achieved by the so-called global zero-forcing (ZF), wherein ZF is performed across all antenna domains thereby suppressing all interference: this turns the whole system into a noise-limited one. While global ZF would require synchronizing all radio-heads in the system, this requirement is absent in our case, and yet it still achieves the same performance. More light will be shed on this matter, in the numerical results section.

C. Communication Overhead and Complexity

In this section - included for completeness, we (roughly) estimate the cost associated with deploying the proposed scheme (Algorithm 2), in terms of total communication overhead. This overhead chiefly consists of ADF overhead (Algorithm 1), the CSI acquisition overhead, the data sharing overhead, and the radio-head synchronization overhead. We use the coarse measure of counting the total number of required training symbols, for each of the previous parts. We assume that the aggregation nodes form a fully connected network. We underline the fact that we are not advocating any specific algorithms for, say, channel estimation or radio-head synchronization. We are rather estimating the number of training symbols that one needs, using well-known methods.

At each iteration, aggregation node \( k \) updates its assignment vector, and broadcasts the updated vector to all \( A - 1 \) other nodes. To estimate the total overhead, we assume that a given assignment vector (of size \( K_T \)) can be encoded 8-bits at a time (into a symbol), and then broadcast, thereby requiring \( K_T/8 \) symbols. Then the total overhead is given by,

\[
\mathcal{H}_{ADF} = AL(K_T/8) \quad \text{symbols} ,
\]

where \( L \) is the number of iterations of Algorithm 1. We assume a TDD uplink pilot-based channel estimation done in an orthogonal fashion: each of the \( K \) users sends out orthogonal pilot sequences that enables each of the antenna domains to estimate the \( MN \) channel gains. Moreover, each antenna domain has to broadcast its CSI to the other \( A - 1 \), for a total of

\[
\mathcal{H}_{CSI} = KT NT M \quad \text{symbols} .
\]

The precoding implicitly assumes that radio-heads within an antenna domain act as virtual array (Sect. II-B). Thus, the \( K \) data symbols for each antenna domain have to be broadcast to all other ones, for a total of

\[
\mathcal{H}_{DS} = AK = KT \quad \text{symbols} .
\]

Finally, the overhead required to perform phase-level synchronization of the radio-heads within each antenna domain, was studied in our earlier work [16]. Using the latter results, we see that \( K \) training symbols are required to synchronize radio-heads within each antenna domain (if carried out in the uplink phase), thereby resulting in a total of,

\[
\mathcal{H}_{SYNC} = AK = KT \quad \text{symbols} .
\]

Note that each of the aforementioned quantities can occur at the backhaul between aggregation nodes, the backhaul between the radio-heads, and/or over-the-air.

At each aggregation node, the computational complexity of the proposed approach (Algorithm 2) is dominated by the matrix inversion \((MN \times MN)\) step to compute the precoder (Proposition 2), as well as solving a \( K_T \)-dimensional linear program (Algorithm 1). The resulting complexity is approximated as \( C = O(M^3 N^3) + O(K_T^3) \).

VII. NUMERICAL RESULTS

A. Simulation Setup

Recall that \( A \) is the total number of antenna domains, \( N \) and \( K \) the number of radio-heads and users per antenna domain, respectively, and \( M \) the number of antennas at each radio-head. Aggregation nodes/radio-heads/users are dropped uniformly within the area of interest, of size \( AL^2, \Delta = 100 \text{m} \). The position for aggregation nodes/radio-heads/users are kept fixed throughout the simulation, and no mobility is considered. Then, for each simulation run, channels are generated randomly, and averaging is done over 100 different channel realizations. To emulate a realistic setting, channels between radio-heads and users are assumed to be spatially correlated Rician (Kronecker model), with pathloss and shadow fading. The parametrization is discussed at length in our earlier work [16][Sect. VII-A]. The system bandwidth is 200 MHz, and noise level is set to \( \sigma_{j_n} = -91 \text{ dBm} \) (for all users). Moreover, we assume that the loading factors are identical, \( \rho_i \overset{\Delta}{=} \rho \) (i.e., the user load is split equally among the antenna domains). The performance metric under consideration is the sum-rate in (5), as well as the total interference leakage in the system, \( f \).

For the assignment of radio heads to aggregation nodes, we benchmark our proposed ADF algorithm (Algorithm 2) against a simple distance-based assignment heuristic:

- each aggregation node picks the \( N \) nearest radio-heads (to form an antenna domain)
- users are associated to radio-heads (and consequently antenna domains) based on strongest channels (\( \rho_i \) users are associated to antenna domain \( i \))
- each antenna domain performs ZF to its users
Moreover, we use the following upper bound:

- **Global ZF**: whereby an equivalent system is used, with all interference set to zero, i.e., global ZF across all antenna domains (requires synchronization of all radio-heads in the system)

### B. Sum-rate results

We first aim to investigate the sum-rate performance of a relatively small deployment with $A = 2, M = 4, N = 2$ radio-heads per antenna domain, and $K = 8$ users per antenna domain, while varying the loading factors $\rho$. Fig. 4 shows the resulting sum-rate, and one can clearly see an increase in the performance of both schemes, as $\rho$ is decreased: this result is expected since interference decreases as less users are served. More importantly, we see a very significant performance gap between our proposed methods, and the benchmark, for all values of $\rho$. Note that sum-rate values are plotted in log scale, for clarity. Moreover the aforementioned gap is increasing with decreasing $\rho$, becoming massive for $\rho = 4$.

Similar trends are observed by moving on to a larger setup where $A = 4, M = 2, N = 6$ radio-heads per antenna domain, and $K = 6$ users per antenna domain, as evidenced in Fig. 5. However, we clearly see that in that case (Fig. 5), the performance gap is indeed more pronounced than the previous case (Fig. 4): while the performance of the benchmark increases with smaller $\rho$, this increase is significantly more pronounced for our algorithm. In particular, for the case where $\rho = 3$, the gap is over 20 times. As detailed earlier, this is due to the fact that by an appropriate choice of $\rho$, the proposed scheme can totally suppress all interference in the system.

To shed further light on the latter effect, we investigated further deployments with $A = 2, N = 2, K = MN$, and where the loading factor is appropriately chosen as $\rho = K/2$. Fig. 6 shows the sum-rate for such a system, for various values of $M$. Most importantly, in this regime, our proposed algorithm coincides exactly with that of the global ZF upper bound. This is due to the fact that all the latter schemes are able to totally suppress all interference in the network.

### C. Performance bounds

We compare in this section, the performance of the proposed BCD algorithm (Algorithm 1), against the globally optimal solution (found via exhaustive search), as well as the DW lower bound. We first look at the tightness of the DW decomposition, with respect to the globally optimal solution of $(P)$. We consider a small scenario ($A = 2$), assuming no fading, and looking at the (average) total interference leakage $f$, as metric. As seen in Table 1, the error form approximating the globally optimal solution of $(P)$, by the DW lower bound (solved using CGM in Table 1), is quite tolerable (for $\rho = 3, 4$). We note that the case where $\rho = 1$ is too small, and not practically relevant. We also compare in Table 1 the performance of the proposed BCD algorithm (Algorithm 1) against that of the globally optimal solution. With that in mind, we observe a similar trend here, where the proposed BCD algorithm has a similar performance as the globally optimal solution, for relevant cases.

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**Fig. 4:** Average sum-rate performance for $A = 2, M = 4, N = 2, K = 8$

**Fig. 5:** Average sum-rate performance for $A = 4, M = 2, N = 6, K = 6$

**Fig. 6:** Average sum-rate performance for $A = 2, K = MN, \rho = K/2$ (curves for Proposed ADF and Global ZF coincide)
D. Discussions

A clear observation that follows from the above results (Fig. 9), is that massive performance gains can be achieved when the loading factor are appropriate chosen - an expected result. Though the performance of our proposed scheme is extremely close to that of global ZF (Sect. VI-B), it circumvents the corresponding need for synchronizing all radio-heads in the system. Not surprisingly, we observe that the dependence on $M, N$, the total number of transmit antennas in each antenna domain, rather than on $M$ and $N$, individually. Finally, our results also suggest that both the proposed BCD-based algorithm (Algorithm 1), and the DW lower bound approximate well the globally optimal solution to the ADF problem, for practical cases.

VIII. Conclusion

We formulated the ADF problem as an integer optimization problem (using the interference leakage as metric), and showed that it can be tackled using BCD. Motivated by the complicated nature of the problem, we argued the need for "good" lower bounds on the problem (as well as the interference leakage). We investigated several "classical" lower bounds, such as the DW decomposition, the dual problem, and showed that the DW lower bound is tighter. Due to the exponential number of variables present in the DW lower bound, we adapted the Column Generation Method to (globally) solve it. Finally, sum-rate results clearly indicate a large performance gap between our proposed ADF algorithm, and the relevant benchmark. Moreover, in practical setups, the proposed ADF algorithm, and the advocated lower bound, seem approximate the optimal solution to the ADF problem, with acceptable error.

APPENDIX

A. Proof of Proposition 1

The fact that (P) can be rewritten in vector form, i.e., (12), is straightforward and can be skipped. As for rewriting (P) in matrix form (13), we first recall that for any $Q \in \mathbb{R}^{m \times m}$, $1_m^T Q 1_m = \sum_{i=1}^m \sum_{j=1}^m Q_{i,j}$, and rewrite the cost function in (P) as

$$f = \text{tr}[A^T (X^T \Psi X) 1_A] - \text{tr}(X^T \Psi X)$$

$$= \text{tr}(X^T \Psi X 1_A 1_A^T) - \text{tr}(X^T \Psi X) = \text{tr}(X^T \Psi X \Omega)$$  \hspace{1cm} (51)

where we used the fact that $\text{tr}(AB) = \text{tr}(BA)$, and let $\Omega \triangleq I_A 1_A^T - I_A$. Moreover, the loading constraint can be rewritten as

$$\sum_{k=1}^K x_{k,i,m} = \rho_k, \forall k \iff 1_K^T X = [\rho_1, ..., \rho_K] \iff X^T 1_N = \rho$$

Similarly, the assignment constraint can be reformulated as

$$\sum_{k=1}^A x_{k,i,m} \leq 1, \forall i,m \in I \iff X 1_A \leq 1_K$$

B. Proof of Proposition 2

Note that (22) is a convex problem (quadratic cost and linear constraint), it can be solved using standard Lagrangian techniques. The associated Lagrangian is,

$$L(V_i, M_i) = \text{tr}(V_i^T R_i V_i) + \text{tr}(M_i (H_{i,i} V_i - \beta_i I_{A}))$$

where $M_i \in \mathbb{C}^{d_i \times d_i}$ is the matrix of Lagrange multipliers. Differentiating the latter w.r.t. $V_i$ and setting to zero yields,

$$\nabla_{V_i} L = 0 \iff V_i^* = -\beta_i R_i^{-1} H_{i,i}^* M_i^*$$

where $M_i^*$ is chosen to satisfy the linear constraint, i.e.,

$$H_{i,i} (H_{i,i}^* M_i^*) = \beta_i M_i^* \iff M_i^* = -\beta_i (H_{i,i}^{-1} R_i^{-1} H_{i,i}^*)^{-1}$$

Combining the last two equations yields the optimal solution,

$$V_i^* = \beta_i R_i^{-1} H_{i,i}^* \left( H_{i,i} R_i^{-1} H_{i,i}^* \right)^{-1}$$

The resulting transmit power is

$$\|V_i^*\|_F^2 = \beta_i^2 \|R_i^{-1} H_{i,i}^* \left( H_{i,i} R_i^{-1} H_{i,i}^* \right)^{-1}\|_F^2$$

Thus transmit power constraint is satisfied with equality for

$$\beta_i = \sqrt{d_i}/\|R_i^{-1} H_{i,i}^* \left( H_{i,i} R_i^{-1} H_{i,i}^* \right)^{-1}\|_F$$

Note that when $d_i \leq M_i$, then there always exists at least one $V_i$, such that $H_{i,i} V_i = \beta_i I_{A}$; the Moore-Penrose inverse of $H_{i,i}/\beta_i$. Due to the generic nature of the channels, $H_{i,i}/\beta_i$ is full-rank almost surely, its Moore-Penrose inverse exists almost surely, and the problem is feasible almost surely.

C. Proof of Lemma 2

The proof follows from considering the following “DW-like” mapping,

$$S = \{ Z = \sum_j t_j W_j \mid \sum_j t_j = 1, t_j \in \mathbb{B}, \forall j = 1, ..., n \}$$

$$= \{ Z = \sum_j t_j W_j \mid t^T 1_n = 1, t \in \mathbb{B}^n \} (g.1)$$

Then, the cost in (Q) is written as $p(Z) = \sum_j t_j p(W_j)$. Letting $t = [t_1, ..., t_n]^T$ , and $\theta_j = p(W_j)$, (Q) is equivalent to,

$$\left\{ \begin{array}{l} \text{argmin } p_d(t) = t^T \theta \\ \text{s. t. } t^T 1_n = 1, t \in \mathbb{B}^n \end{array} \right\} (52)$$

It can be verified that the mapping in (g.1) is one-to-one from $Z$ to $t$: every $t$ yields a unique $Z$, and every $Z$ decomposes into a unique $t$. The equivalence between the two problems follows from that.
D. Proof of Lemma [2]

Note that the following is a direct consequence of [17]
\[ f(x^{(n)}_{A}) \geq f(x^{(n-1)}_{A}, z^{(n-1)}_{A}) \geq \cdots \geq f(x^{(n)}_{A}) = f(x^{(n)}_{A}) \]
where the last equality follows from the fact that \( f(x^{(n)}_{A}, z^{(n)}_{A}) \) corresponds to the case where all variables \( x_1, \ldots, x_A \), are updated. It follows that the sequence \( \{f(x^{(n)}_{A}, z^{(n)}_{A})\}_n \) converges to a limit point \( f_0 \).

E. Proof of Proposition [3]

We rewrite (38) in a series of equivalent problems.
\[
\begin{align*}
(D) \max_{\lambda \geq 0, \kappa_T} & \lambda \\
(\text{max} \ d(\lambda)) & = \min_{\alpha, \lambda, \kappa_T} \left\{ \alpha \lambda + \lambda \geq \zeta, \forall j = 1, \ldots, S \right\}
\end{align*}
\]

The result in (39) follows by letting \( \mu = [\lambda, \zeta]^T, c = [0, 1]^T, \) and \( T = [-\Gamma, 1] \).

F. Proof of Lemma [4]

Let \( \eta = \sum k \sum \rho_k \rho_k \). The left inequality, stating that the dual solution is always a lower bound on the primal one, follows immediately from weak duality. Moreover, the right one is obtained from upper bounding \( f(X^*) \) and lower bounding \( d(\lambda^*) \),
\[
f(X^*) = \sum \sum_{k \neq k} x_k^T \sum_{j \neq j} \sigma_{\max}[\Psi]\|x_k\|_2\|x_j\|_2
\]
where (e.1) follows from the fact that \( x_k^* \) must be feasible: thus, \( \|x_k^*\|_2^2 \) is the sum of all non-zero elements, and equal to \( \rho_k \). Using (38), we formulate the optimal dual solution (and its lower bound as)
\[
\begin{align*}
d(\lambda^*) & \leq \min_{X \in S_p} \left\{ \min_{\lambda} Tr(X^T \Psi X \Omega) + \lambda^T (X 1 - 1_{K_T}) \right\} \\
& = \min_{z_k \in S_p} \left\{ \min_{\lambda} \sum_{k \neq k} x_k^T \sum_{j \neq j} \sigma_{\max}[\Psi]\|x_k\|_2\|x_j\|_2 \right\}
\end{align*}
\]
\[
\begin{align*}
(\text{e.2}) & \leq \sigma_{\min}[\Psi]\|x_1\|_2 \leq 1_{K_T} \lambda^* + \sum_{k \neq k} \min_{\lambda} \|x_k\|_2\|x_j\|_2 \lambda^* \\
(\text{e.3}) & \leq \sigma_{\min}[\Psi]\|x_1\|_2 \leq 1_{K_T} \lambda^* + \sum_{k \neq k} \min_{\lambda} \|x_k\|_2\|x_j\|_2 \lambda^* \\
(\text{e.4}) & \leq \sigma_{\min}[\Psi]\|x_1\|_2 \leq 1_{K_T} \lambda^* + \sum_{k \neq k} \min_{\lambda} \|x_k\|_2\|x_j\|_2 \lambda^* \\
(\text{e.5}) & \leq \sigma_{\min}[\Psi]\|x_1\|_2 \leq 1_{K_T} \lambda^* + \sum_{k \neq k} \min_{\lambda} ||\lambda^*||
\end{align*}
\]
Note that (e.2) follows from the fact that \( \|x_k\|_2 = \rho_k \) for any feasible \( x_k \). (e.3) is due to the fact that the problem is a MILP. Furthermore, we show that it satisfied the integrality property (as per Definition 2), then, relaxing the binary constraint into a continuous one, yields the optimal solution. Finally, (e.4) is obtained by letting \( z_k = x_k/\rho_k \), and (e.5) from the fact that the problem is a Special LP whose solution is detailed in Definition 4. The final result follows by combining the above result with (e.1).

G. Proof of Corollary [5]

Let \( Z_i = \bigcup \{H_{i,j} \} \), and \( N_i = null(Z_i) \). Due to the generic random nature of the channels, then one can verify that \( \dim(N_i) = MN - \rho_i \), almost surely. In the case where \( \sum \rho_i \leq MN \), then \( \dim(N_i) \geq \rho_i \Rightarrow \exists T_i \in C^{MN \times \rho_i}, T_i \in N_i \)
\[
\Rightarrow \exists T_i \in C^{MN \times \rho_i}, T_i \big(H_{i,j} = 0_{\rho_i \times \rho_i}, \forall j \neq i, \big)
\]
Then, \( h(V_i) \geq 0 \), since it is a quadratic form. Moreover, since the problem is convex and has a unique optimal solution, then, any solution that makes \( h \) zero, is globally optimal then. Then, consider solutions of the form, \( V_i = T_i \Theta_i \), where \( \Theta_i \) is an arbitrary unitary matrix. Then,
\[
h(V_i) = tr(\Theta_i T_i^T R_i T_i \Theta_i) = tr(T_i^T R_i T_i) = 0
\]

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