A SHORT PROOF OF KLEE’S THEOREM

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ABSTRACT. In 1959, Klee proved that a convex body $K$ is a polyhedron if and only if all of its projections are polygons. In this paper, a new proof of this theorem is given for convex bodies in $\mathbb{R}^3$.

1. INTRODUCTION

This paper will begin by summarizing the relevant work of Mirkil [2] and Klee [1]. Let $V$ be an $n$-dimensional real vector space and $C \subset V$. The set $C$ is said to be a convex cone if and only if $C$ is stable under both vector addition and multiplication, and polyhedral if and only if $C$ is the intersection of a finite number of closed halfspaces. For a set $K$ embedded in an $n$-dimensional affine space $E$ and a point $p \in K$, define $K$ to be polyhedral at $p$ if and only if some neighborhood of $p$ relative to $K$ is polyhedral. For a set $K \subset E$ and a point $p \in E$, we will denote the smallest cone containing $K$ with vertex $p$ as cone$(p,K)$. A $j$-flat is a $j$-dimensional affine subspace of $E$, and a hyperplane is a $(n-1)$-dimensional affine subspace of $E$.

In Mirkil [2], the following theorem is proven:

**Theorem 1.1.** If $C$ is a closed convex cone, then $C$ is polyhedral if and only if every 2-dimensional projection of $C$ is closed.

**Sketch of Proof.** The forward direction of this statement follows from the fact that every projection of $C$ is polyhedral. The main idea to prove the converse is as follows: If $H$ is a hyperplane, then for all $x \in C \cap H$, there exists a neighborhood $N$ which contains no extreme points except possibly $x$. □

**Example 1.2.** Let our vector space be $\mathbb{R}^3$ with the standard Cartesian coordinate system. Take $C$ to be a circular cone supported by the $(x,y)$ plane so that the infinite half-line of support lies on the $x$-axis, and let $\pi_{(y,z)}(C)$ be the horizontal projection of $C$ into the $(y,z)$ plane. We see that $\pi_{(y,z)}(C)$ may be expressed as

$$\pi_{(y,z)}(C) = \{(0,a,b) : a \in \mathbb{R}, b > 0\} \cup \{(0,0,0)\}$$

Note that $\pi_{(y,z)}(C)$ is not closed, in accordance with Theorem 1.1.
Motivated by Theorem 1.1 comes the extensive work of Klee [1], which includes the following theorem:

**Theorem 1.3.** If $K$ is a $n$-dimensional convex subset of an affine space, $p \in K$, and $2 \leq j \leq n$, then $K$ is polyhedral at $p$ if and only if $\pi(K)$ is polyhedral at $p$ whenever $\pi(K)$ is an affine projection of $K$ into a $j$-flat through $p$.

*Sketch of Proof.* To prove the “only if” portion, the fact that a convex set $K$ is polyhedral at point $p \in K$ if and only if $\text{cone}(p, K)$ is polyhedral is used repeatedly on $K$, $\text{cone}(p, K)$, and their affine projections.

To prove the converse, the fact that all $j$-dimensional projections of $K$ are polyhedral follows directly from the statement. In particular, all 2-dimensional projections of $K$ are polyhedral, thus all intersections with hyperplanes are polyhedral. Furthermore, Klee proves a cone is polyhedral if and only if its intersections with elements of a specific parameterized set of hyperplanes are polyhedral, thus $\text{cone}(p, K)$ is polyhedral. Using again that a convex set $K$ is polyhedral at point $p \in K$ if and only if $\text{cone}(p, K)$ is polyhedral, Klee proves $K$ is polyhedral at $p$. $\square$

From Theorem 1.3, Klee establishes a corollary:

**Corollary 1.4.** With $2 \leq j \leq n$, a $n$-dimensional bounded convex subset is polyhedral if and only if all its projections into $j$-flats are polyhedral.

The following statement follows from the previous corollary:

**Theorem 1.5.** If $K$ is a convex body in $\mathbb{R}^3$ whose orthogonal projection into every plane is a polygon, then $K$ is a polyhedron.

The proof of this theorem is simplified if instead of reformulating the problem in terms of closed projections of convex cones, one shows that all points $x \in K$ are located within a neighborhood which contain no extreme points except possibly $x$. The next sections explain this proof in detail.

2. DUAL REFORMULATION

For a plane $P$ and sets $X, Y$ embedded in $\mathbb{R}^3$, denote the orthogonal projection of $X$ into $P$ by $\pi_P(X)$, the union and intersection of $X$ and $Y$ by $X \cup Y$ and $X \cap Y$ respectively, the convex hull of $X$ by $\text{conv}(X)$, and the boundary of $X$ by $\partial X$. For points $p, q, r$ in $\mathbb{R}^3$, denote the triangle with vertices $p, q, r$ by $\triangle pqr$, and the line segment bounded by $p$ and $q$ by $[pq]$.

Recall that a *convex body* is a closed bounded convex set with nonempty interior. Fix a convex body $K$ in $\mathbb{R}^3$ so that the origin of $\mathbb{R}^3$ belongs to the interior of $K$. The *polar dual* of $K$ will be denoted as $K^*$; i.e.,

$$K^* = \{ y \mid x \cdot y \leq 1 \text{ for every } x \in K \}$$

Clearly $K^*$ is a convex body and the origin is an interior point of $K^*$. Moreover $K^*$ is a convex polyhedron if and only if so is $K$.

The following statement follows directly from the definition of polar dual.

**Proposition 2.1.** If $P$ is a plane passing through the origin, then

$$K^* \cap P = \pi_P(K)^* \cap P.$$

Note that $\pi_P(K)^* \cap P$ is a polygon if and only if so is $\pi_P(K)$. Using the above proposition, Theorem 1.5 can be reformulated the following way:
Theorem 2.2. Suppose $K^*$ is a convex body in $\mathbb{R}^3$ containing the origin in its interior. If for every plane $P$ passing through the origin, the intersection $P \cap K^*$ is a polygon, then $K^*$ is a polyhedron.

3. Proof of Theorem 2.2

Lemma 3.1. Let $K^*$ be a convex body in $\mathbb{R}^3$ and $p, q, x, y \in K^*$. If $x$ lies between $p$ and $q$, and the line segment $[xy]$ lies completely in $\partial K^*$, then the triangle $\triangle pqy$ lies completely in $\partial K^*$.

Proof. Suppose to the contrary that the point $r \in \triangle pqy$ belongs to the interior of $K^*$. This implies the existence of a line segment $L \subset K^*$ containing $r$ such that the convex hull $\text{conv}(L \cup \triangle pqy)$ is a bipyramid in $\mathbb{R}^3$, and the interior of $\triangle pqy$ lies in the interior of $\text{conv}(L \cup \triangle pqy)$.

Therefore, all the interior points of $\triangle pqy$ belong to the interior of $K^*$. Because the midpoint of $[xy]$ lies in the interior of $\triangle pqy$, the result follows. $\square$

Proposition 3.2. Suppose $K^*$ satisfies the conditions of Theorem 2.2. For all points $p, q \in K^*$, there exists an $\varepsilon > 0$ such that for $r \in K^*$, if $0 < |p - r| < \varepsilon$ and $\angle rpq < \varepsilon$, then $r$ is not extreme.

Proof. The statement is evident if the line segment $[pq]$ passes through the interior of $K^*$, so we can assume that $[pq] \subset \partial K^*$.

Let $x$ denote the midpoint of $[pq]$. Choose a plane $P$ through the origin which intersects $[pq]$ transversely at $x$. The intersection $K^* \cap P$ is a polygon, where the sides extending from $x$ are denoted by the line segments $[xy]$ and $[xz]$. We refer to Figure 1 for clarity.

By Lemma 3.1, the triangles $\triangle pyq$ and $\triangle pzq$ lie completely in $\partial K^*$. Choose a point $s$ in the interior of $K^*$. Clearly there exists an $\varepsilon > 0$ such that for any point $r \in K^*$, if $|p - r| < \varepsilon$ and $\angle rpq < \varepsilon$, then $r$ lies in the convex hull $\text{conv}(\triangle pyq, \triangle pzq, s)$. Hence the result follows. $\square$

Proof of Theorem 2.2. Suppose to the contrary that $\{q_n\}$ is an infinite set of distinct extreme points contained within $K^*$. Pass $\{q_n\}$ to a convergent subsequence $\{q_{n_k}\}$, and let $p \in \partial K^*$ be the point such that $q_{n_k} \rightarrow p$ as $n_k \rightarrow \infty$. Choose the convergent subsequence $\{q_{n_k}\}$ so that the unit vectors $v_{n_k} = \frac{q_{n_k} - p}{|q_{n_k} - p|}$ also converge, say $v_{n_k} \rightarrow u$.

Consider the plane $P$ which passes through $p$, $u$ and the origin. Since the intersection $P \cap K^*$ is a polygon, there is a line segment $[pq] \subset \partial K^*$ pointing from $p$ in the direction of $u$.

Applying Proposition 3.2 we arrive at a contradiction. $\square$

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REFERENCES

[1] V. Klee. Some characterizations of convex polyhedra. *Acta Mathematica*, 102 (1959), 79-107.
[2] H. Mirkil. New characterizations of polyhedral cones. *Canadian Journal of Mathematics*, 9 (1957), 1-4.