WIGNER’S LITTLE GROUP AND BRST COHOMOLOGY FOR ONE-FORM ABELIAN GAUGE THEORY

R. P. MALIK *

S. N. Bose National Centre for Basic Sciences,
Block-JD, Sector-III, Salt Lake, Calcutta-700 098, India

and

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

Abstract: We discuss the (dual-)gauge transformations for the gauge-fixed Lagrangian density and establish their intimate connection with the translation subgroup $T(2)$ of the Wigner’s little group for the free one-form Abelian gauge theory in four $(3 + 1)$-dimensions (4D) of spacetime. Though the relationship between the usual gauge transformation for the Abelian massless gauge field and $T(2)$ subgroup of the little group is quite well-known, such a connection between the dual-gauge transformation and the little group is a new observation. The above connections are further elaborated and demonstrated in the framework of Becchi-Rouet-Stora-Tyutin (BRST) cohomology defined in the quantum Hilbert space of states where the Hodge decomposition theorem (HDT) plays a very decisive role.

Keywords: (Dual-)gauge transformations; (co-)BRST symmetries; BRST cohomology; Wigner’s little group

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*E-mail address: malik@boson.bose.res.in
1 Introduction

In the classification scheme of the elementary particles, the Wigner’s little group [1] plays a very important and decisive role [2]. It was first Weinberg [2-4] and later Han et al. [5-7] who demonstrated a very interesting connection between the transformation generated by the Abelian invariant translation subgroup $T(2)$ of the Wigner’s little group and the $U(1)$ gauge transformation for the one-form $(A = dx^\mu A_\mu)$ Abelian gauge field $A_\mu$ of the Maxwell theory in four $(3 + 1)$-dimensions of spacetime. It is a common folklore that the latter symmetry transformations are generated by the first-class constraints (in the language of the Dirac’s classification scheme [8,9]) of the Abelian gauge theory which forms the internal $U(1)$ symmetry group of transformations. On the contrary, the former transformations are generated by the Wigner’s little group that constitutes the spacetime symmetry group of transformations for a given (gauge) theory. In more precise words, the translation subgroup $T(2)$ of the Wigner’s little group keeps the momentum vector $k_\mu$ of the massless (i.e. $k^2 = 0$) gauge particle invariant but changes the polarization vector $e_\mu$ of the (one-form) gauge field in exactly the same manner as the $U(1)$ gauge transformation generated by the first-class constraints of the gauge theory. Thus, the Abelian one-form gauge theory (i.e. free Maxwell theory) provides a fertile ground for the discussion of the internal symmetry and the spacetime symmetry together in a beautiful setting. Recently, in a set of papers [10-13], the gauge transformations connected with a variety of Abelian gauge theories have been shown to be connected with the translation subgroups of the Wigner’s little group.

For some of the (non-)interacting gauge theories in two $(1 + 1)$-dimensions (2D) and 4D of spacetime, it has been found that there exists a discrete symmetry for the Lagrangian density of the theory which corresponds to the existence of a specific kind of “duality” in the theory. This duality entails upon the theory to possess (i) a local dual-gauge symmetry transformation for the Lagrangian density, and (ii) an analogue of the Hodge duality operation of differential geometry. Such a class of (non-)interacting and duality invariant gauge field theories provide a set of tractable field theoretical models for the Hodge theory where the local, covariant and continuous symmetry transformations (and the corresponding generators) are identified with the de Rham cohomological operators of differential geometry. In this context, mention can be made of many interesting field theoretical models such as (i) the free 2D Abelian gauge theory [14-16], (ii) the interacting 2D Abelian gauge theory where there is an interaction between $U(1)$ gauge field and the Dirac fields [17,18], (iii) the self-interacting 2D non-Abelian gauge theory where there is no interaction between the gauge field and the matter fields [16,19], and (iv) the free Abelian 2-form gauge theory in 4D [20]. In a recent paper [21], an interesting connection between the translation subgroup $T(2)$ of the Wigner’s little group and the BRST cohomology has been established for the free Abelian 2-form gauge theory in 4D. In fact, it is because of the study of the Wigner’s little group that it has been possible to obtain the normal mode expansions for the basic fields of the theory [21] that appear for the consideration of the BRST formalism.
(particularly, BRST cohomology) in the framework of Lagrangian formulation.

The purpose of the present paper is to establish a connection between the Wigner’s little group and the gauge (see, e.g., [22-25]) and the dual-gauge symmetry transformation groups (see, e.g., [26] for details) that exist for the free one-form Abelian gauge theory in four dimensions of spacetime. The salient features of the gauge and the dual-gauge transformations are (i) the latter transformations are continuous, non-local and non-covariant whereas the former are continuous, local and covariant. (ii) It is the gauge-fixing term that remains invariant under the latter transformations. The electric and magnetic fields are left invariant under the former transformations. (iii) The magnetic field remains invariant under both the transformations. We demonstrate that both (the local gauge and the non-local dual-gauge) symmetries owe their origin to the Wigner’s little group as the latter encompasses both the symmetries in its folds in a subtle way. Furthermore, we show that the (dual-)gauge (or (co-)BRST) transformed physical states are found to be the sum of the original physical states and the BRST (co-)exact states. Thus, the increment in the physical state due to the (dual-)gauge (or (co-)BRST) transformations turns out to be a cohomologically trivial state. For this proof, we exploit (i) the HDT in the quantum Hilbert space of states (QHSS), and (ii) choose the physical state to be the harmonic state of the Hodge decomposed state in the QHSS. The choice of the harmonic state to be the physical state is guided by some aesthetic reasons because this state is the most symmetrical nontrivial state which is (anti-)BRST invariant as well as (anti-)co-BRST invariant, simultaneously. One of the most crucial points of our whole discussion is the choice of the momentum vectors $k_\mu = (\omega, 0, 0, -\omega)^T$ and $k^\mu = (\omega, 0, 0, \omega)^T$ for the massless ($k^2 = 0$) photon, propagating along the z-direction of the 4D spacetime manifold with energy $\omega$. This choice enables us to get a simple expression for the non-local and non-covariant dual-gauge (or co-BRST) transformations in the phase space. In fact, the ugly features of non-locality and non-covariance disappear for this choice. Furthermore, this choice of the reference frame allows us to get the same physical inferences from the conserved and nilpotent BRST and co-BRST charges when they apply on the physical harmonic state in the requirement of physicality criteria. This unique feature is not present in our earlier works [14-19] where the BRST and co-BRST charges lead to different physical consequences when they are applied on a single photon state for the 2D gauge theories. To be more precise, the BRST charge implies the transversality condition on the 2D photon but the co-BRST charge leads to the “dual” transversality condition between the momentum vector $k_\mu$ and the polarization vector $e_\mu$. An exact generalization of these results has been obtained for the free 2-form Abelian gauge theory in 4D with conserved and nilpotent (co-)BRST charges [20,21].

Our present study is essential primarily on three counts. First and foremost, to the best of our knowledge, the derivation of the connection between the continuous, non-local and non-covariant dual-gauge (or co-BRST) transformations and the Wigner’s little group is a new result which is different from the well-known connection between the usual continuous, local and covariant $U(1)$ gauge transformations for the massless 1-form gauge
field and the $T(2)$ subgroup of the little group. Second, the free Maxwell $U(1)$ Abelian
gauge theory is one of the simplest 1-form gauge theories where the gauge and dual-gauge
symmetry transformations co-exist together for the gauge-fixed Lagrangian density of the
theory. Thus, the study of the deeper reasons for their existence in the framework of the
translation subgroup $T(2)$ of the Wigner’s little group is the first step towards our main goal
of understanding the more complicated (e.g. 2-form, 3-form, etc.) gauge theories which
have relevance in the context of (super)string theories and their close cousins D-branes.
Finally, to corroborate the above assertions, it is worthwhile to state that, in the context
of the 4D free Abelian 2-form gauge theory, it has been shown (see, e.g., [20,21] for details)
that, in some sense, the transformations generated by the little group on the polarization
tensor are more fundamental than the corresponding changes brought about by the first-
class constraints of the gauge and dual-gauge symmetry groups. This is because of the
fact that, only when we demand the consistency of the transformations generated by the
usual (dual-)gauge groups with that of the little group, do we get a set of certain specific
relationships between the parameters of the little group and the (dual-)gauge groups. This
relationship, finally, entails upon the (anti-)ghost fields of the BRST formalism to obey
certain specific restrictions. This, in turn, allows us to obtain the normal mode expansions
for the ghost fields (see, [21] for details) which play very important roles in the proof of
the quasi-topological nature of the free 2-form Abelian gauge theory in the framework of
BRST cohomology [20,21]. Thus, the study of the Wigner’s little group does shed some
light on the formal aspects of the gauge field theories and their discussion in the framework
of BRST formalism. It is well-known that the 2-form Abelian gauge fields are important
in the context of (super)string theories, D-branes and noncommutative geometry.

The material of our present paper is organized as follows. In section 2, we briefly discuss
the (dual-)gauge symmetry transformations for the gauge-fixed Lagrangian density of the
free Abelian (one-form) gauge theory and show that the restrictions on the (dual-)gauge
parameters are similar. These transformations are upgraded to the nilpotent (co-)BRST
transformations in section 3. The central of the present paper are sections 4 and 5 where
we show the connection between the Wigner’s little group and the (dual-)gauge transfor-
mations and comment on such relationship in the language of the BRST cohomology where
the HDT in the QHSS plays a very decisive role. Finally, we make some concluding remarks
in section 6 and point out a few future directions that can be pursued later.

2 (Dual-)gauge transformations

Let us start off with the gauge-fixed Lagrangian density $\mathcal{L}_0$ for the four $(3+1)$-dimensional
free Abelian gauge theory in the Feynman gauge (see, e.g., [22-25])

$$\mathcal{L}_0 = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial \cdot A)^2 \equiv \frac{1}{2} (E^2 - B^2) - \frac{1}{2} (\partial \cdot A)^2,$$

where $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is the anti-symmetric second-rank curvature tensor (with $F_{0i} = E_i \equiv E = \text{electric field}, B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \equiv B = \text{Magnetic field}$) defined through the 2-form
\[ F = dA = \frac{1}{2}(dx^\mu \wedge dx^\nu)(F_{\mu\nu}). \]

As is evident, this 2-form is derived by the application of the exterior derivative
\[ d = dx^\mu \partial_\mu \] (with \( d^2 = 0 \)) on the connection one-form \( A = dx^\mu A_\mu \) which defines \( A_\mu \) as the vector potential \( \dagger \). The gauge-fixing term \( (\partial \cdot A) = (- * d * A) \) is defined through the application of the dual-exterior derivative \( \delta = - * d * (\text{with } \delta^2 = 0) \) on the one-form \( A \). Here the \( * \) operation is the Hodge duality operation on the 4D spacetime manifold. The application of the Laplacian operator \( \Delta = (d + \delta)^2 = d\delta + \delta d \) on the one-form \( A \) leads to \( \Delta A = dx^\mu \Box A_\mu \). In fact, the equation of motion \( (\Box A_\mu = 0) \), emerging from the above gauge-fixed Lagrangian density, is captured by the Laplacian operator in the sense that it (i.e. \( \Box A_\mu = 0 \)) can be derived by the requirement of the validity of the Laplace equation \( \Delta A = 0 \). Together the set of geometrical operators \( (d, \delta, \Delta) \) define the de Rham cohomological properties of the differential forms and obey the algebra: \( d^2 = \delta^2 = 0, \Delta = (d + \delta)^2 = \{d, \delta\}, [\Delta, d] = [\Delta, \delta] = 0 \) (see, e.g., [27-30]). It is unequivocally clear that both the terms of the above Lagrangian density have deep connections with the key cohomological operators of the differential geometry. Their invariances, therefore, will play some prominent roles in our whole discussions about the (dual-)gauge transformations as well as the corresponding (dual-)BRST transformations. In fact, the nomenclature of the gauge (or BRST) and the dual-gauge (or dual(co)-BRST) symmetry transformations owes its origin to the invariances of the above terms.

It is straightforward to check that the above Lagrangian density, under the following local \( U(1) \) gauge [22-25] and dual-gauge transformations (see, e.g., [26])

\[
\begin{align*}
A_\mu(x) & \rightarrow A^{(g)}_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x), \\
A_0(x) & \rightarrow A^{(dg)}_0(x) = A_0(x) + i \beta(x), \\
A_i(x) & \rightarrow A^{(dg)}_i(x) = A_i(x) + i \frac{\partial_\circ \partial_i}{\nabla^2} \beta(x),
\end{align*}
\]

(2.2)

remains invariant if the parameters of the transformations are restricted to obey \( \Box \alpha = 0, \Box \beta = 0 \). Under the infinitesimal version of the above (dual-)gauge transformations \( \delta^{(dg)} \), the following changes occur

\[
\begin{align*}
\delta_g A_\mu & = \partial_\mu \alpha, & \delta_g E_i & = 0, & \delta_g B_i & = 0, & \delta_g (\partial \cdot A) & = \Box \alpha, \\
\delta_{dg} A_0 & = i \beta, & \delta_{dg} A_i & = i \frac{\partial_\circ \partial_i}{\nabla^2} \beta, & \delta_{dg} B_i & = 0, \\
\delta_{dg} (\partial \cdot A) & = 0, & \delta_{dg} E_i & = i \left( \frac{\partial_\circ \partial_i}{\nabla^2} - 1 \right) \partial_i \beta \equiv i \frac{\Box}{\nabla^2} \partial_i \beta.
\end{align*}
\]

(2.3)

Some of the relevant points, at this stage, are as follows. First, it is the kinetic energy term (more precisely the 2-form curvature tensor \( F_{\mu\nu} \) itself) and the gauge-fixing term (more

\textsuperscript{1}We adopt here the conventions and notations in such a way that the flat 4D Minkowski spacetime metric \( \eta_{\mu\nu} = \text{diag} \{+1, -1, -1, -1\} \) and the totally antisymmetric Levi-Civita tensor obeys \( \varepsilon_{\mu\nu\kappa\rho} \varepsilon^{\mu\nu\kappa\rho} = -4! \varepsilon_{\mu\nu\kappa\rho} \varepsilon^{\mu\nu\kappa\rho} = -3! \delta^2_3 \), etc., \( \varepsilon_{0123} = +1 = - \varepsilon^{0123} \), \( \varepsilon_{ijk} = \delta_{ijk} \) and \( \Box = (\partial_0)^2 - (\partial_i)^2 \equiv (\partial_0)^2 - (\nabla)^2 \). Here the Greek indices \( \mu, \nu, \kappa, \ldots = 0, 1, 2, 3 \) correspond to the spacetime directions on the 4D manifold and the Latin indices \( i, j, k, \ldots = 1, 2, 3 \) stand only for the space directions on the 3D submanifold. The 3-vectors on the submanifold are represented by the bold faced letters.
precisely \((\partial \cdot A)\) itself that remain invariant under the gauge and dual-gauge transformations, respectively. Second, exactly the same restrictions (i.e. \(\Box \alpha = \Box \beta = 0\)) are imposed on the (dual-)gauge parameters for the invariance of the Lagrangian density under the (dual-)gauge transformations. Finally, the latter transformations in (2.2) are christened as the dual-gauge transformations because \((\partial \cdot A)\) and \(F_{\mu \nu}\) are ‘Hodge-dual’ to each other from the point of view of their derivation using the operation of the (co-)exterior derivatives \((\delta \) and \(d)\) on the one-form \(A = dx^\mu A_\mu\) defining the vector (gauge) potential.

Using the restriction \(\Box \beta = 0 \rightarrow \partial_0 \partial_0 \beta = \nabla^2 \beta\) as an input, it can be checked that the above dual-gauge transformations on the vector field \(A_\mu\) can be re-expressed as

\[
\tilde{\delta}_{dg} A_\mu(x) = i \partial_\mu \left( \frac{\partial_0}{-\nabla^2} \beta(x) \right),
\]

which imply \(\delta_{dg} A_0 = i \beta, \delta_{dg} A_i = i (\partial_0 \partial_i / \nabla^2) \beta\) as is the case in (2.3) only when \(\partial_0 \partial_0 \beta = \nabla^2 \beta\) is used explicitly. It will be noted, however, that the above form of the dual-gauge transformation \(\tilde{\delta}_{dg}\) does not keep the gauge-fixing term invariant (i.e. \(\tilde{\delta}_{dg}(\partial \cdot A) \neq 0\)). Thus, we shall not use both the forms of non-local dual-gauge transformations (cf. (2.3) and (2.4)) for our later discussions in sections 4 and 5. We shall focus on the transformations (2.3) only for its generalization to the co-BRST symmetry transformations (as the gauge-fixing term remains invariant under it). There are more general discussions [31] on the non-covariance and non-locality of the transformations (2.3) in the framework of BRST formalism. However, the generalized BRST-type symmetries turn out to be nilpotent only for a specific value of the parameter of the theory [31]. Thus, we shall avoid deeper discussions on the symmetry transformations of [31] and concentrate only on the continuous, (non-)local and (non-)covariant symmetry transformations discussed in [26].

3 (Co-)BRST symmetries

The gauge-fixed Lagrangian density \(\mathcal{L}_0\) of equation (2.1) can be generalized to the BRST invariant Lagrangian density \(\mathcal{L}_b\) as (see, e.g., [22-25])

\[
\mathcal{L}_b = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} (\partial \cdot A)^2 - i \partial_\mu \bar{C} \partial^\mu C,
\]

where the anticommuting \((\bar{C}^2 = C^2 = 0, \bar{C} \bar{C} + \bar{C} C = 0)\) (anti-)ghost fields \((\bar{C})C\) are required in the theory to maintain the unitarity and “quantum” gauge (i.e. BRST) invariance together at any arbitrary order of perturbative calculations (see, e.g., [32]). The above Lagrangian density (3.1) respects the following on-shell \((\Box C = \Box \bar{C} = 0)\) nilpotent \((s^2_{(a)b} = 0)\) (anti-)BRST \(s_{(a)b}\) (with \(s_b s_{ab} + s_{ab} s_b = 0\)) symmetry transformations

\[
\begin{align*}
s_b A_\mu & = \partial_\mu C, \\
\bar{s}_{ab} A_\mu & = \partial_\mu \bar{C}, \\
 s_b C & = 0, \\
\bar{s}_{ab} C & = 0, \\
s_b \bar{C} & = -i (\partial \cdot A), \\
\bar{s}_{ab} \bar{C} & = +i (\partial \cdot A).
\end{align*}
\]

\[\text{We follow here the notations and conventions of Weinberg [22]. Actually, in its full glory, the nilpotent} (\delta^2_{(A)B} = 0) \text{ (anti-)BRST transformations} (\delta_{(A)B}) \text{ are the product} (\delta_{(A)B} = \eta s_{(a)b}) \text{ of an anticommuting} (\bar{C} + C \eta = 0, \bar{C} \bar{C} + \bar{C} \bar{C} \eta = 0) \text{ spacetime independent parameter} \eta \text{ and} s_{(a)b} \text{ with} s^2_{(a)b} = 0.\]
The salient features, at this juncture, are (i) the physical fields $E_i$ and $B_i$ remain invariant ($s_{(a)b}E_i = 0, s_{(a)b}B_i = 0$) under the (anti-)BRST transformations. (ii) The (anti-)BRST transformations are the generalization of the gauge transformations of (2.3) without the restriction like $\Box \alpha = 0$. (iii) The bosonic gauge parameter $\alpha$ has been replaced by $\alpha = \eta C$ and $\alpha = \eta \bar{C}$ for the derivation of the (anti-)BRST transformations.

The non-local and non-covariant dual gauge transformations of (2.3) can be elevated to the on-shell ($\Box C = \Box \bar{C} = 0$) nilpotent ($s_{(a)d}^2 = 0$) symmetry transformations by replacing the dual gauge parameter $\beta = \eta C$ and $\beta = \eta \bar{C}$ corresponding to the (anti-)dual BRST (i.e. (anti-)co-BRST) $s_{(a)d}$ symmetry transformations (with $s_{ad} s_{ad} + s_{ad} s_{ad} = 0$) as

$$
\begin{align*}
    s_{d} A_0 &= i \dot{C}, \quad s_{d} A_i = i \frac{\partial_0 \partial_i}{\nabla^2} C, \quad s_{d} \bar{C} = 0, \quad s_{d} C = -A_0 + \frac{\partial_0 \partial_i}{\nabla^2} A_i, \\
    s_{ad} A_0 &= i \dot{C}, \quad s_{ad} A_i = i \frac{\partial_0 \partial_i}{\nabla^2} C, \quad s_{ad} C = 0, \quad s_{ad} \bar{C} = A_0 - \frac{\partial_0 \partial_i}{\nabla^2} A_i.
\end{align*}
$$

(3.3)

The key and relevant points, at this stage, are (i) it is the gauge-fixing term that remains invariant ($s_{(a)d}(\partial \cdot A) = 0$) under the (anti-)co-BRST transformations $s_{(a)d}$. (ii) A single dual gauge symmetry transformations $\delta_{dg}$ in (2.3) leads to the existence of a couple of nilpotent (anti-)co-BRST symmetry transformations. (iii) The physical magnetic field $B_i$ remains invariant (i.e. $s_{(a)d} B_i = 0$) under $s_{(a)d}$ but the electric field $E_i$ transforms ($s_{d} E_i = i(\Box / \nabla^2) \partial_i C, s_{ad} E_i = i(\Box / \nabla^2) \partial_i C = 0$) such a way as to cancel the contributions coming from the transformation of the ghost term under $s_{(a)d}$. It is obvious that a bosonic symmetry $s_w$ (with $s_w^2 \neq 0$) can be obtained from the anticommutator of the nilpotent symmetries (i.e. $s_w = \{ s_b, s_d \} = \{ s_{ad}, s_{ab} \}$). However, for our discussions, this symmetry is not required in its full glory. Some elementary discussions about it can be found in [26].

The generators for the above (non-)local, continuous, (non-)covariant and nilpotent (co-)BRST transformations can be computed from the Noether conserved current as listed below

$$
\begin{align*}
    Q_d &= \int d^3 x \left[ \frac{\partial_0 (\partial \cdot A)}{\nabla^2} \dot{C} - (\partial \cdot A) \dot{C} \right], \\
    Q_b &= \int d^3 x \left[ \partial_0 (\partial \cdot A) C - (\partial \cdot A) \dot{C} \right].
\end{align*}
$$

(3.4)

From the above expressions, the conserved and nilpotent charges corresponding to the anti-BRST and anti-co-BRST transformations can be computed by the replacement $C \rightarrow \pm i \dot{C}, \bar{C} \rightarrow \pm i \dot{C}$ which turns out to be the discrete symmetry transformation for the ghost part of the Lagrangian density (3.1). For any generic field $\Phi = A_\mu, C, \bar{C}$ of the theory, the (anti-)BRST and (anti-)co-BRST transformations of (3.2) and (3.3) can be succinctly expressed as

$$
\begin{align*}
    s_r \Phi &= -i [\Phi, Q_r]_{\pm}, \quad r = b, ab, d, ad, w, g,
\end{align*}
$$

(3.5)

where the subscripts (+)− on the brackets correspond to the (anti-)commutators for the generic field $\Phi$ being (fermionic)bosonic in nature and $Q_w$ (i.e. $Q_w = \{ Q_b, Q_d \} = \{ Q_{ad}, Q_{ab} \}$) is the generator for the bosonic symmetry transformation and $Q_g$ is the ghost conserved charge that corresponds to the infinitesimal scale symmetry transformations.
\( s_C = -\lambda C, s_{\bar{C}} = +\lambda \bar{C}, s_A = 0 \) under which the action remains invariant. Here the transformation parameter \( \lambda \) is global. Thus, in total, there are six continuous, (non-)local and (non-)covariant symmetries in the theory.

4 Wigner’s little group and gauge transformations

The Wigner’s little group corresponds to the maximal subgroups of the Lorentz group that leave the four momenta of the free relativistic particles invariant. The internal symmetry properties associated with the gauge particle are captured by the little group which turns out to be locally isomorphic to the three dimensional rotation group and the two-dimensional Euclidean group (see, e.g., [33] for details). The most general form of the Wigner’s little group matrix \( \{W_{\mu}^\nu(\theta, u, v)\} \) for a massless (gauge) particle moving along the \( z \)-direction of the 4D spacetime manifold is [2-7]

\[
\{W(\theta, u, v)\} = \begin{pmatrix}
(1 + \frac{u^2 + v^2}{2}) & (ucos\theta - vsin\theta) & (usin\theta + vcos\theta) & -\left(\frac{u^2 + v^2}{2}\right) \\
\frac{u}{2} & cos\theta & sin\theta & -u \\
\frac{v}{2} & -sin\theta & cos\theta & -v \\
\left(\frac{u^2 + v^2}{2}\right) & (ucos\theta - vsin\theta) & (usin\theta + vcos\theta) & \left(1 - \frac{u^2 + v^2}{2}\right)
\end{pmatrix},
\]

(4.1)

where \( \theta \) is the rotational parameter and \( u, v \) are the translational parameters defining \( T(2) \) in the \( xy \) plane. By definition, this matrix preserves the four momentum \( k^\mu = (\omega, 0, 0, \omega)^T \) of a massless \( (k^2 = 0) \) (gauge) particle with energy \( \omega \) and it can be factorized elegantly into 3D rotation group and 2D Euclidean group. Both these statements can be expressed mathematically, in a combined fashion, as

\[
(k^\mu) \rightarrow (k^\mu)' = W_{\mu}^\nu (k^\nu) = (k^\mu), \quad W(\theta, u, v) = R(\theta) W(0, u, v).
\]

(4.2)

The matrix \( R(\theta) \) in the above represents the rotation about the \( z \)-axis

\[
R(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & cos\theta & sin\theta & 0 \\
0 & -sin\theta & cos\theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(4.3)

and the matrix \( \{W(0, u, v)\} \) is found to be isomorphic to the two parameter translation group \( T(2) \) (i.e. \( T(2) \sim W(0, u, v) \)) in the two-dimensional Euclidean plane \( (xy) \) which is a plane perpendicular to the propagation of the light-like (massless) particle along the \( z \)-direction of the 4D spacetime manifold.

It is obvious from the Lagrangian density (3.1) that the equations of motion for the basic fields are: \( \Box A_\mu = \Box C = \Box \bar{C} = 0 \). These imply the masslessness \( (k^2 = 0) \) of the photon as well as the (anti-)ghost fields. Thus, the choice \( k^\mu = (\omega, 0, 0, \omega)^T \) is consistent
with the equations of motion which imply \( k^2 = 0 \). The transversality \( k_\mu e^\mu = 0 \) of photon implies that the polarization vector \( e^\mu(k) \) of the photon can be chosen to be \( e^\mu(k) = (0, e^1(k), e^2(k), 0)^T \). The \( U(1) \) gauge transformation of (2.2) (generated by the first-class constraints of the theory) can be exploited to express itself in terms of the transformation on \( e^\mu(k) \) as (see, e.g., [10])

\[
e^\mu(k) \rightarrow (e^\mu)^{(g)}(k) = e^\mu(k) + i k^\mu \alpha(k).
\]  

(4.4)

It will be noted that, in general, one can choose \( e^\mu(k) = (e^0, e^1, e^2, e^0)^T(k) \) which will be consistent with the transversality condition \( k_\mu e^\mu = 0 \) with our choice of the reference frame where \( k_\mu = (\omega, 0, 0, -\omega)^T \). However, it can be seen, using the gauge transformation (2.2), that the component \( e^0(k) \) of the polarization vector \( e^\mu(k) \) can be gauged away (i.e. \( (e^0)^{(g)} = 0 \)) by the choice \( \alpha(k) = -(e^0)/i\omega \) in the gauge transformation

\[
e^0(k) \rightarrow (e^0)^{(g)}(k) = e^0(k) + i \omega \alpha(k).
\]  

(4.5)

Thus, ultimately, we end up with \( e^\mu(k) = (0, e^1, e^2, 0)^T(k) \). Now, concentrating on the role of the translation subgroup \( T(2) \sim W(0, u, v) \) in generating the \( U(1) \) gauge transformation on \( e^\mu(k) \), it can be readily checked that

\[
e^\mu(k) \rightarrow (e^\mu)'(k) = W^\mu_\nu(0, u, v) e^\nu \equiv e^\mu(k) + \left(\frac{ue^1 + ve^2}{\omega}\right) k^\mu.
\]  

(4.6)

It is unequivocally clear that both the transformations in (4.4) and (4.6) are identical for the \( \omega_\mu = (\omega, 0, 0, -\omega)^T \). Section, it will be seen that this condition (i.e. \( k \cdot e = 0 \)) emerges automatically due to the requirement that the physical states (i.e. \( \vert \text{phys} \rangle \)) of the theory are annihilated (i.e. \( Q_b \vert \text{phys} \rangle = 0 \)) by the conserved \( (Q_b = 0) \) and nilpotent \( (Q_b^2 = 0) \) BRST charge \( Q_b \) due to the physicality criteria.
It is obvious, at this stage, that (i) the transverse components of the polarization vector do not transform at all under the dual-gauge transformation for the choice \( k^\mu = (\omega, 0, 0, \omega)^T \) and \( k'_\mu = (\omega, 0, 0, -\omega)^T \), and (ii) the scalar and longitudinal components transform in exactly the same manner. This observation should be contrasted with the gauge transformations where the scalar and longitudinal components (with \( e^0 = e^3 \)) can be gauged away and the transverse components transform. Finally, the above transformation (4.9) can be concisely written in the four vector notation as

\[
e^\mu(k) \rightarrow (e^\mu)^{(dg)}(k) = e^\mu(k) + \left( \frac{i\beta(k)}{\omega} \right) k^\mu.
\] (4.10)

It is obvious that both the transformations in (4.10) and (4.6) are identical for the following relationship between the infinitesimal dual-gauge parameter \( \beta(k) \) and the parameters \( u \) and \( v \) of the translation subgroup \( T(2) \) of the Wigner’s little group

\[
\beta(k) = \frac{ue^1 + ve^2}{i}.
\] (4.11)

This establishes the fact that (dual-)gauge transformations owe their origin to the transformations generated by the translation subgroup \( T(2) \) of the Wigner’s little group. In this analysis and treatment, the parameters of the (dual-)gauge transformations are chosen in terms of the parameters of the translation subgroup \( T(2) \) as given in (4.11) and (4.7). It is interesting to note that, for the above choice of the reference frame in the context of the discussion on little group, the infinitesimal (dual-)gauge transformation parameters of \( A_\mu \) differ by a factor of the energy \( \omega \) (i.e. \( \beta = \omega \alpha \)) of the photon (cf. (4.7) and (4.11)). It is worthwhile to emphasize at this juncture that, for the free 2-form Abelian gauge theory, the distinction between the gauge and dual-gauge transformations is quite clear and lucid [20,21]. Furthermore, the trivial relationship like \( \beta = \omega \alpha \) does not exist in the case of the 2-form Abelian gauge theory (see, e.g., [21] for details).

5 Wigner’s little group and BRST cohomology

Here we shall recapitulate some of the key and pertinent points of the discussion connected with the BRST cohomology by Weinberg [22] for the gauge transformations. To this end in mind, we first express the normal mode expansion for the basic fields \( (A_\mu, C, \bar{C}) \) of the Lagrangian density (3.1) in the (momentum) phase space as

\[
A_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}} \left[ a_\mu^\dagger(k) e^{ik\cdot x} + a_\mu(k) e^{-ik\cdot x} \right],
\]
\[
C(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}} \left[ c^\dagger(k) e^{ik\cdot x} + c(k) e^{-ik\cdot x} \right],
\] (5.1)
\[
\bar{C}(x) = \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}} \left[ \bar{c}^\dagger(k) e^{ik\cdot x} + \bar{c}(k) e^{-ik\cdot x} \right].
\]

The above expansions correspond to the equations of motion \( \Box A_\mu = \Box C = \Box \bar{C} = 0 \) obeyed by the basic fields of the theory. Here \( k_\mu \) are the 4D momenta and \( d^3k = dk_1dk_2dk_3 \) is the
volume in the momentum space. All the dagger operators are the creation operators and the non-dagger operators correspond to the annihilation operators for the basic quanta of the fields. The on-shell nilpotent version of the BRST symmetries (3.2) can be expressed, due to (3.5), in terms of the (anti-)commutators with $Q_b$ as (see, e.g., [22,16] for details)

$$[Q_b, a^\dagger_\mu(k)] = k_\mu \ c^\dagger(k), \quad [Q_b, a_\mu(k)] = -k_\mu c(k),$$
$$\{Q_b, c^\dagger(k)\} = 0, \quad \{Q_b, c(k)\} = 0,$$
$$\{Q_b, c^\dagger(k)\} = i \ k^\mu \ a^\dagger_\mu(k), \quad \{Q_b, c(k)\} = -i \ k^\mu \ a_\mu(k).$$

(5.2)

Similar kinds of (anti-)commutation relations can be obtained with the anti-BRST generators but we do not require them for our present discussions. For aesthetic reasons, we can define the most symmetric physical vacuum ($\left|\text{vac}\right>$) of the present theory as

$$Q_{(a)b} \left|\text{vac}\right> = 0, \quad Q_{(a)d} \left|\text{vac}\right> = 0, \quad Q_w \left|\text{vac}\right> = 0, \quad a_\mu(k) \left|\text{vac}\right> = 0, \quad c(k) \left|\text{vac}\right> = 0, \quad \bar{c}(k) \left|\text{vac}\right> = 0,$$

(5.3)

where $Q_w = \{Q_{(a)b}, Q_{(a)d}\}$ is the generator for the bosonic symmetry transformation that we commented on earlier. In the above, it is clear that the physical vacuum is (anti-)BRST and (anti-)co-BRST invariant which imply the invariance w.r.t. $Q_w$ as well. It is lucid and clear now that a single photon state with polarization $e_\mu(k)$ and momenta $k_\mu$ can be created from the physical vacuum by the application of a creation operator $a^\dagger_\mu(k)$ as [22,16]

$$e^\mu a^\dagger_\mu(k) \left|\text{vac}\right> = \left|e, \text{vac}\right>, \quad k^\mu a^\dagger_\mu(k) \left|\text{vac}\right> = \left|k, \text{vac}\right> = -i \{Q_b, \bar{c}(k)\} \left|\text{vac}\right>,$$

(5.4)

where the latter state $|k, \text{vac}\rangle$ with momenta $k_\mu$ has been expressed by exploiting the anti-commutator $\{Q_b, \bar{c}(k)\} = ik^\mu \ a^\dagger_\mu(k)$ from (5.2). Exploiting the $U(1)$ gauge transformation (4.4) on the polarization vector $e_\mu(k) \rightarrow (e_\mu)^{(g)}(k) = e_\mu(k) + iA k_\mu$, where $A$ is a complex number (which can be expressed taking the help of (4.4) and (4.7) as $A = (ue^1 + ve^2)/(i\omega)$), it is straightforward to check that

$$|e + i \ A \ k, \text{vac}\rangle = |e, \text{vac}\rangle + Q_b \ (A \ \bar{c}(k)) |\text{vac}\rangle, \quad Q_b \ |\text{vac}\rangle = 0.$$

(5.5)

It should be noted that the arbitrariness of $A$ in the gauge transformation, for the discussion of the BRST cohomology, is being traded with the arbitrariness of the parameters of the Wigner’s little group and the energy $\omega$ of the massless field. This key observation is due to the consistency between the transformations generated by the Wigner’s little group and the gauge group. We conclude from (5.5) that a gauge transformed state corresponding to an original single photon state (i.e. $e^\mu(k) a^\dagger_\mu(k) |\text{vac}\rangle \equiv |e, \text{vac}\rangle$ with the polarization vector $e_\mu(k)$) is equal to the sum of the original state $|e, \text{vac}\rangle$ plus a BRST exact state. In more sophisticated language, the gauge transformed state and the original state belong to the same cohomology class w.r.t. the conserved and nilpotent BRST charge $Q_b$. In other words, the increment in the original physical state due to the gauge (or BRST) transformation is a cohomologically trivial state. Thus, the truly physical state remains invariant under the gauge (or BRST) transformations (modulo a cohomologically trivial state which is a BRST exact state).
Now let us focus on the dual-BRST transformations (3.3). These transformations can be expressed in terms of the conserved and nilpotent dual BRST charge $Q_d$ and creation and annihilation operators of (3.1), by exploiting the general expression for the transformation (3.5), as

$$
[Q_d, a_0^\dagger(k)] = e^\dagger(k), \\
[Q_d, a_0(k)] = e(k), \\
[Q_d, a_1^\dagger(k)] = \frac{k_0k_1}{k^2} e^\dagger(k), \\
[Q_d, a_1(k)] = \frac{k_0k_1}{k^2} e(k), \\
\{Q_d, c^\dagger(k)\} = i(a_0^\dagger(k) - \frac{k_0k_1}{k^2} a_1^\dagger(k)), \\
\{Q_d, c(k)\} = i(a_0(k) - \frac{k_0k_1}{k^2} a_1(k)).
$$

(5.6)

Exploiting our choice of the contravariant and covariant momentum vectors $k^\mu = (\omega, 0, 0, \omega)^T$ and $k_\mu = (\omega, 0, 0, -\omega)^T$, the above (anti-)commutators can be expressed in explicit components of $a_\mu(k)$ as

$$
[Q_d, a_0^\dagger(k)] = \bar{e^\dagger}(k), \\
[Q_d, a_0(k)] = \bar{e}(k), \\
[Q_d, a_1^\dagger(k)] = -\bar{e^\dagger}(k), \\
[Q_d, a_1(k)] = -\bar{e}(k), \\
[Q_d, a_2^\dagger(k)] = 0, \\
[Q_d, a_2(k)] = 0, \\
\{Q_d, c^\dagger(k)\} = i(a_0^\dagger(k) + a_1^\dagger(k)), \\
\{Q_d, c(k)\} = i(a_0(k) + a_1(k)).
$$

(5.7)

The above (anti-)commutators between $Q_d$ and the creation and annihilation operators can be re-expressed in the covariant form by exploiting the choice of $k_\mu = (\omega, 0, 0, -\omega)^T$ and $k^\mu = (\omega, 0, 0, \omega)^T$ as

$$
[Q_d, a_\mu^\dagger(k)] = \frac{1}{\omega} k^\mu \bar{e^\dagger}(k), \\
[Q_d, a_\mu(k)] = \frac{1}{\omega} k^\mu \bar{e}(k), \\
\{Q_d, c^\dagger(k)\} = 0, \\
\{Q_d, c(k)\} = 0, \\
\{Q_d, c^\dagger(k)\} = \frac{i}{\omega} k^\mu a_\mu^\dagger(k), \\
\{Q_d, c(k)\} = \frac{i}{\omega} k^\mu a_\mu(k).
$$

(5.8)

The dual gauge transformation on the polarization vector (i.e. $e_\mu(k) \rightarrow (e_\mu)^{(dg)}(k) = e_\mu(k) + iBk_\mu$ where $B$ is a complex number which can be expressed in view of the equations (4.10) and (4.11) as $B = (ue^\dagger + ve^2)/(i)$) will correspond to the following expression in the language of the co-BRST charge $Q_d$

$$
|e + i Bk, vac >= |e, vac > + Q_d (B \omega c^\dagger(k)) |vac >, \\
Q_d |vac >= 0.
$$

(5.9)

Here we have used the anti-commutator $\{Q_d, c^\dagger(k)\} = (i/\omega)k^\mu a_\mu^\dagger(k)$ from (5.8) and invoked the condition $Q_d |vac >= 0$ (cf equation (5.3)). It is worthwhile to mention that the derivation of (5.9) is not dependent on the specific choice of the reference frame. A general proof of (5.9) can be given by exploiting the general (anti-)commutators of (5.6). It is clear, from (5.6), that the following anti-commutation relation is correct, namely;

$$
\{Q_d, k_\mu c^\dagger(k)\} |vac >= i(k_0a_0^\dagger - k_1a_1^\dagger) |vac >= ik^\mu a_\mu^\dagger(k)|vac >= i |k, vac >.
$$

(5.10)
where we have used the masslessness \((k^2 = 0)\) condition which implies \(k_0^2 = k^2\). Thus, in the general case, the analogue of (5.9) is

\[
|e + i B k, \text{vac} > = |e, \text{vac} > + Q_d (B k_0 c^\dagger(k)) |\text{vac} >, \quad Q_d |\text{vac} > = 0. \tag{5.11}
\]

The above equations (5.9) and (5.11) imply that the dual-gauge transformed state in the quantum Hilbert space is equal to the sum of the original state and a BRST co-exact state. With the four nilpotent and conserved charges \(Q_{(a)b}, Q_{(a)d} \) and a bosonic conserved charge \(Q_w\) in the theory, the most symmetric physical state \(|\text{phy} >\) can be defined as \(Q_{(a)b}|\text{phy} > = 0, Q_{(a)d}|\text{phy} > = 0, Q_w|\text{phy} > = 0\). Applying this physicality condition on the single photon state, we obtain the following relationships by exploiting the commutators \([Q_b, a^\dagger_\mu(k)] = k_\mu c^\dagger(k), [Q_d, a^\dagger_\mu(k)] = (k_\mu/\omega) c^\dagger(k)\) of equations (5.2) and (5.8), namely;

\[
\begin{align*}
Q_b |e + i A k, \text{vac} > & = Q_b |e, \text{vac} > + \equiv (k \cdot e) c^\dagger(k) |\text{vac} > = 0, \quad (Q_b^2 = 0), \\
Q_d |e + i B k, \text{vac} > & = Q_d |e, \text{vac} > + \equiv (k \cdot e/\omega) c^\dagger(k) |\text{vac} > = 0, \quad (Q_d^2 = 0),
\end{align*}
\tag{5.12}
\]

which imply the transversality (i.e. \(k \cdot e = 0\)) of the photon because of the fact that \(c^\dagger(k)|\text{vac} > \neq 0, \bar{c}^\dagger(k)|\text{vac} > \neq 0\). It is also obvious from the above discussion that for a single photon state, not satisfying the above transversality condition, the single particle (anti-)ghost state(s) \((\bar{c}^\dagger(k)|\text{vac} >, c^\dagger(k)|\text{vac} >)\) created by the operators \(\bar{c}^\dagger(k)\) and \(c^\dagger(k)\) would turn out to be the BRST (co-)exact states. This explains the no-(anti-)ghost theorem in the context of the BRST cohomology. Physically, it amounts to the well-known fact (see, e.g., [32]) that the contributions coming from the longitudinal and scalar degrees of freedom of the photons are cancelled by the presence of (anti-)ghost fields. This statement is valid at any arbitrary order of perturbative calculations. Ultimately, the physicality criteria \(Q_b|e, \text{vac} > = 0, Q_d|e, \text{vac} > = 0, Q_w|e, \text{vac} > = 0\) on a single photon state implies the transversality and masslessness of the photon.

6 Conclusions

In the present investigation, we have been able to demonstrate a deep connection between the transformations on the polarization vector \(e_\mu(k)\) generated by (i) the translation subgroup \(T(2)\) of the Wigner’s little group (ii) the \(U(1)\) group of gauge symmetry, and (iii) the dual version of the \(U(1)\) gauge symmetry. It turns out that the (dual-)gauge symmetries owe their origin to the Wigner’s little group. The connection between the dual-gauge transformations and the Wigner’s little group is a new observation which has also been found for the 2-form free Abelian gauge theory in 4D [21]. It is worthwhile to point out that the non-local and non-covariant dual gauge symmetries look quite trivial when we choose the reference frame where the momentum vector \(k^\mu\), for a propagating massless photon with energy \(\omega\) along the \(z\)-direction of the 4D Minkowskian flat manifold, takes the form \(k^\mu = (\omega, 0, 0, \omega)^T\). In fact, the virtues of this choice of the reference frame are three fold. First, the non-locality and non-covariance vanish from the co-BRST (or dual gauge)
symmetry transformations. Second, it enables us to tackle the subtleties and intricacies associated with the discussions on Wigner’s little group in a much nicer and better way. Finally, all the (anti-)commutators between the co-BRST charge $Q_d$ and the creation and annihilation operators become very simple. As a consequence of this simplicity, the physicality criteria with the (co-)BRST charges (i.e. $Q_{(d)b}|\text{phys} >= 0$) on the physical harmonic state lead to the same physical inferences on the 4D photon. That is to say, the 4D photon is found to be massless ($k^2 = 0$) and transverse ($k \cdot e = 0$) due to both the conserved and nilpotent (co-)BRST charges $Q_{(d)b}$. However, a closer look at the physicality conditions (5.12) sheds some light on an important difference between the physical inferences drawn from the (co-)BRST charges. Whereas the BRST charge leads to only the transversality ($k \cdot e = 0$) condition, the co-BRST charge implies (i) the transversality ($k \cdot e = 0$) condition for a photon with finite energy $\omega$, and (ii) the transversality condition might not be absolutely essential condition for a highly energetic ($\omega \rightarrow \infty$) photon. In contrast, it would be worthwhile to mention that, the physicality criteria with the (co-)BRST charges for the one-form gauge theory in 2D [16,17] (and 2-form gauge theory in 4D [21]), lead to mathematically two different kinds of relationships between the momentum vector $k_\mu$ and the polarization vector $e_\mu$ of the one-form gauge theory (and the polarization anti-symmetric tensor $e_{\mu\nu}$ of the 2-form gauge theory). As far as the discussion on the BRST cohomology is concerned, we have been able to show that the (dual-)gauge (or (co-)BRST) transformed states are the sum of the original states and the (co-)BRST exact states. Thus, the genuine physical states of the theory remain invariant under the (dual-)gauge (or (co-)BRST) transformations because the increment in the physical states, due to the above transformations, turns out to be cohomologically trivial state which does not change the physical contents in any way.

It would be interesting endeavour to capture these 4D (dual-)gauge (or (co-)BRST) symmetries in the framework of superfield formulation where the geometrical origin for the nilpotent charges can be found out. In fact, for the 2D free Abelian and self-interacting non-Abelian gauge theories, such studies have already been performed [34-39] where the super de Rham cohomological operators ($\tilde{d}, \tilde{\delta}, \tilde{\Delta}$) have been exploited in the (dual-)horizontality conditions. Moreover, the topological properties of these theories have also been encompassed in the framework of the superfield formalism developed on the four $(2+2)$ dimensional manifold where the Lagrangian density and the symmetric energy momentum tensor of the theory have been shown to correspond to the translation of some composite superfields along the Grassmannian directions. Thus, the superfield formulation of the (co-)BRST symmetries with (non-)local transformations, is a key direction that should be pursued in our further investigations. It is interesting to point out that, for the 4D Abelian gauge theory, these (non-)local and (non-)covariant transformations corresponding to the (co-)BRST symmetries have been recently captured in the superfield formulation where the geometrical origin and interpretation for these symmetries have been provided (see, e.g., [40] for details). We have not discussed, in the present work, the non-local and non-covariant
symmetry transformations for the \textit{interacting} theory where there is a coupling between the $U(1)$ gauge field and the Noether conserved current constructed by the matter (Dirac) fields. These transformations for the gauge as well as matter fields have been discussed in detail (see, e.g., [26,31,40] and references therein).

It is an open problem to capture the symmetry transformations on the Dirac (matter) fields in the framework of the Wigner’s little group and/or the superfield formulation. Some thoughts are being given to this problem at the moment. In this connection, it is gratifying to state that the local, covariant, continuous and nilpotent (anti-)BRST transformations for the matter fields have recently been derived in the framework of superfield formalism [41,42] for the 2D and 4D interacting gauge theories. The local, covariant, continuous and nilpotent (anti-)co-BRST symmetries for the Dirac fields in the case of 2D QED have also been derived (see, e.g., [41] for details). However, there is no clue, so far, on such derivations (for the matter fields) in the framework of the Wigner’s little group. The derivation of the gauge symmetry transformations for the non-Abelian gauge field is an open problem in the context of the Wigner’s little group. It is obvious from our earlier works [20,21] that the Wigner’s little group does shed some new light on the structure of the higher-spin (e.g. 2-form) gauge field theories in the framework of BRST formalism. Thus, we hope that such studies for still higher-spin (e.g. 3-form, 4-form, etc.) gauge fields will be relevant to the (super)string theories and the related areas of the extended objects (e.g. D-branes, membranes, etc.) which are of interest at the frontier level of research devoted to the discussions of physics at Planck energy. The above directions are some of the key issues that are under investigation at the moment and our results will be reported elsewhere [43].

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