Superconductivity in a Dilute Array of Sites with Strong Onsite Electron Attraction

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Superconductivity in a conducting material with a dilute array negative-\(U\) sites with strong electron–electron attraction is studied. It is shown that at the attraction exceeding a certain threshold value, Cooper pairs arise at the sites. At low temperatures, a global coherent superconducting state emerges due to the intersite Andreev scattering. The dependences of the superconducting transition temperature and of the second critical magnetic field on the magnitude of attraction are calculated.

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1. INTRODUCTION

IV–VI semiconductors with a low charge carrier density can have relatively high superconducting transition temperatures [1]. A specific feature of such systems is the presence of resonant sites that arise in the main electron band owing to doping and ensure Fermi level pinning. It is found that the highest superconducting transition temperature is attained if the Fermi level becomes pinned for the degenerate states with an even number of electrons. This suggests that a strong attraction between electrons can manifest itself at such resonant sites [2–4]. Recent theoretical and experimental studies of IV–VI compounds and the relation between superconductivity and the features of doping are reported in [5, 6].

The electron gas becomes superconducting at a weak attraction between the particles. The formation of electron pairs and their subsequent condensation occur at the same temperature. With the strong attraction, the binding energy of electron pairs can exceed the condensation temperature [7, 8]. This phenomenon is also favored by the narrow electron bands [9] and by the presence of flat bands [10].

The models with local attraction at impurities near the range of electron degeneracy can be reduced to a system of pseudospins interacting with conduction electrons via the exchange coupling. Superconductivity here corresponds to the long-range order of pseudospins, see [11] and references therein.

We use the Hubbard–Stratonovich transformation for the model with negative-\(U\) sites to study superconductivity in both weak and strong coupling limits. In the latter case, a local order with uncorrelated phases at different sites appears with an increase in attraction. The coherent state arises due to the Andreev scattering between the sites. The upper critical magnetic field is calculated. It is shown that in the strong-coupling limit, the magnetic field does not suppress the local order, but destroys coherence between the sites.

2. MAIN DEFINITIONS

The Hamiltonian of the system with negative-\(U\) sites slightly overlapping with conducting states has the form

\[
H = \sum_{p,\sigma} \epsilon(p) - \mu a^+(p, \sigma)a(p, \sigma) + U_{\text{imp}} + \sum_{i,\sigma} U[a^+(r_i, \sigma)a_i(\sigma) + a_i^+(\sigma)a(r_i, \sigma)] + \sum_i [E_R(n_{i,\downarrow} + n_{i,\uparrow}) - Wn_{i,\downarrow}n_{i,\uparrow}].
\]

Here, \(a^+(p, \sigma)\) and \(a(p, \sigma)\) are the creation and annihilation operators of conduction electrons with spectrum \(\epsilon(p)\), respectively; \(\mu\) is the chemical potential; \(U_{\text{imp}}\) describes the random scattering of conduction electrons; the third term describes a weak hybridization between the resonant and conduction states, where \(a_i^+(\sigma)\) and \(a_i(\sigma)\) are the creation and annihilation operators of electrons in the resonant state at the point \(r_i\), respectively; and the last term is the sum of Hamiltonians describing resonant sites, where \(E_R\) is the resonance energy measured with respect to the chemical potential and \(W > 0\) is the onsite electron–electron attraction energy. We use the units, for which \(\hbar = k_B = 1\), where \(\hbar\) is the reduced Planck constant and \(k_B\) is the Boltzmann constant.
The Green’s function of the resonant site at the Matsubara frequency \( \omega_n = (2n + 1)\pi T \) has the form

\[
g_r(r, r', \omega_n) = \frac{\psi(r - r')\psi(r' - r)}{i\omega_n - E_R + i\gamma \text{sgn} \omega_n}. \tag{2}
\]

It is assumed that the wavefunction obeys the relation \( \psi(r) \sim a^{-3/2} \) in a region having the size of the order of the localization length \( a \). Due to hybridization with band states, the level is broadened by \( \gamma = \pi v_0 |U|^2 \), where \( v_0 \) is the single-spin density of conduction states at the Fermi level.

At a low density of resonant centers, the mean free path \( \tau \) of band electrons is determined mainly by scattering at nonresonant sites. The Green’s function of conduction electrons, averaged over scattering by impurities, is determined by the expression \[ \tag{3} \]

For the Hubbard model, the strong disorder regime was discussed in [13].

The propagation of an electron pair among sites at multiple scattering is described by the propagator shown in Fig. 1. It is determined by the equation

\[
(-D\nabla^2 + |\omega_n - \omega_m|)G(r, r', \omega_n, \omega_m) = \Theta(-\omega_n, \omega_m)\delta(r - r'), \tag{4}
\]

where \( D \) is the diffusion constant of conduction electrons scattered by impurities.

In the applied magnetic field, we should make the following substitution in the gradient terms: \( \nabla \rightarrow \nabla - \frac{2e}{c} iA \).

**3. SUPERCONDUCTING INSTABILITY AT A WEAK ATTRACTION**

The system instability with respect to the superconducting transition corresponds to the divergence of the sum of the ladder diagrams [12] shown in Fig. 2. We take into account the electron–electron interaction only at resonant sites and describe the propagation of electron pairs between them by the propagator (4).

The electron loop \( \Pi(i, m) \) in Fig. 2 consists of the local and long-range contributions.

The local contribution \( \Pi(0) \) involves the Green’s functions (2) for the resonant sites.

In the long-range contribution corresponding to the last term in Fig. 2, we can replace the summation over \( m \neq i \) sites by the integration over the site positions \( r_k \), weighted with the density \( n_{r_k} \) of the resonant sites.

The superconducting transition temperature is determined by the equation

\[
1 - W/\Pi(0) = W_{nR}\int d\mathbf{r}\Pi(r). \tag{5}
\]

Dispersion equation (5) determining the transition temperature has the following form after the substitution of Green’s functions (2) and propagator (4):

\[
1 - W/\Pi_c = \frac{2W_{nR}\gamma^2}{\pi v_0} \sum_{\omega_n} \frac{C(\mathbf{q} = 0, \omega_n, -\omega_n)}{((\omega_n + \gamma \text{sgn} \omega_n)^2 + E_R^2)^2}. \tag{6}
\]

Here,

\[
W_{c}^{-1} = T \sum_{\omega_n} \frac{1}{\omega_n + \gamma \text{sgn} \omega_n^2 + E_R^2}. \tag{7}
\]

At \( T \ll \gamma |E_R| \), the critical magnitude of the interaction reads \( W_c = 2|E_R| \left( 1 - \frac{\pi}{2} \arctan \frac{\gamma}{|E_R|} \right)^{-1} \).

The right-hand side of Eq. (6) is

\[
W_{nR} \frac{\gamma^2}{\pi v_0} \ln(E_0/T).
\]
SUPERCONDUCTIVITY IN A DILUTE ARRAY OF SITES

with the following logarithmic cutoff in propagator (4):

\[ E_0 \sim \min(\gamma, |E_R|) \]

The parameter

\[ \Lambda \equiv \frac{1}{(E^2_R + \gamma^2)^{3/2}} \]

(8)
corresponds to the \([g(\omega_n)g(-\omega_n)]^2\) term in the limit \(\omega_n \to 0\). It arises at all definitions of the transition temperature and of the critical magnetic fields.

In the weak-coupling limit, the superconducting transition temperature determined by Eq. (6) is

\[ T_c = E_0 \exp \left\{ -\frac{\pi^2 v_0 (1 - W/W_c)}{\gamma^2 \Lambda W n_R} \right\} \]

(9)

The strong coupling limit corresponds to \(W/W_c < 1\), for which the dispersion equation is valid due to the long-range interaction \(\Pi(r)\), which gives rise to the logarithmic contribution in Eq. (6). At \(W \ll W_c\), expression (9) agrees with the result reported in [14].

Note that for positive \(E_R > \gamma\), the strong-coupling limit \(W \sim W_c = 2E_R\) corresponds to the condition determining the degeneracy of states with the number of particles equal to 0 and 2 [4].

4. STRONG COUPLING \(W/W_c \geq 1\)

In the strong-coupling limit, the dispersion equation (5) is satisfied even with only the local contribution. It is convenient to treat such a limit using the Ginzburg–Landau functional. Let us divide this functional into the sum of local contributions of the individual resonant sites and contributions related to the interaction between these sites.

4.1. Local Part of the Ginzburg–Landau Functional

The local part of the Ginzburg–Landau functional is represented by the sum over the resonant sites. After the decoupling of the interaction at the isolated centers using complex Hubbard–Stratonovich fields \(\Delta_i\), we obtain in the saddle point approximation

\[ F_{\text{loc}} = \sum_i (W^{-1} - W_c^{-1})|\Delta_i|^2 + b|\Delta_i|^4 \]  

(10)

Here, the minimum necessary \(|\Delta_i|^2\) powers are retained; the coefficient \(b > 0\) is given by the expression

\[ b = T \sum_{\omega_n > 0} \left( (\omega_n + \gamma)^2 + E^2_R \right)^2 \]  

(11)

\(W_c\) and \(b\) are the functions of \(T, E_R\), and level broadening \(\gamma\), and \(b^{-1} \sim W_c^3 \sim \max(|E_R|^3, \gamma^3)\).

Let us emphasize that in the strong coupling limit at \(W > W_c\), a nonzero value of \(\Delta\) does not imply the presence of superconductivity. For the formation of a coherent state, the following correlation function should be nonzero

\[ \langle |\Delta_i| \Delta_k | \exp i(\phi_i - \phi_k) \rangle \neq 0 \]  

(12)
at large \(|r_i - r_k|\) for distant sites. This condition can be satisfied taking into account nonlocal contributions to the Ginzburg–Landau functional.

4.2. Nonlocal Part of the Ginzburg–Landau Functional

The nonlocal part of the Ginzburg–Landau functional is determined by the Andreev reflections in the system with the fixed \(\Delta_i\) distribution. The relation between \(\Delta_i\) and \(\Delta_k\) is given by the diagram shown in Fig. 3. Its contribution to the Ginzburg–Landau functional is

\[ F(i, k) = -B(i, k)|\Delta_i \Delta_k| \cos \phi_{ik} \]  

(13)

where

\[ B(i, k) = \frac{8\gamma^2}{\pi v_0} \sum_{\omega_n > 0} \frac{\exp(iqr_{ik})}{(2\pi)^2 2\omega_n + Dq^2} \]  

(14)

\[ \times \frac{1}{[(\omega_n + \gamma)^2 + E^2_R]^2} \]

The first term in Eq. (14) determines the power-law dependence on the distance. It is related to propagator (4).

We disregard the quadruple nonlocal terms. The phase-independent terms are reduced to the corrections to the nonlinear part in Eq. (10) since \(|\Delta_i| = |\Delta_m|\).

The terms \(\Delta_i \Delta_k^* \Delta_j \Delta^*_m\) lead to the decay of the propagator (4) in Eq. (13). At low densities \(n_R\), they can be neglected.

The terms related to the Andreev reflections between different sites, \(\Delta_i \Delta_k^* \Delta_m \Delta^*_n\), are neglected. At high temperatures, the corresponding terms are small owing to random phase factors. Within the mean field approach, they do not affect the superconducting transition temperature.
4.3. Ginzburg–Landau Functional at $W \geq W_c$

Expressions (10) and (13) determine the Ginzburg–Landau functional

$$F = \sum_i \left[ \frac{1}{W} - \frac{W}{W_\xi} |\Delta_i|^2 + b|\Delta_i|^4 \right] - \frac{1}{2} \sum_{i \neq k} B(i, k) |\Delta_i| |\Delta_k| \cos(\phi_{ik}).$$  \hspace{1cm} (15)

5. MEAN FIELD

Now, we calculate the superconducting transition temperature using the mean field approach with averaging. For $i = 0$, the mean field equation for the complex function $\theta_1 + i\theta_2$ has the form

$$\Theta \equiv \langle \theta_1 + i\theta_2 \rangle = \frac{\int d\theta_1 d\theta_2 \Delta \exp(-F_{\text{MF}}(\Theta)/T)}{\int d\theta_1 d\theta_2 \exp(-F_{\text{MF}}(\Theta)/T)},$$  \hspace{1cm} (16)

where

$$F_{\text{MF}}(\Theta) = \frac{1}{W} - \frac{W}{W_\xi} |\Delta|^2 + b|\Delta|^4 - \Theta|\delta| \sum_{k \neq 0} B(0, k).$$  \hspace{1cm} (17)

In the limit $\Theta \rightarrow 0$, the equation gives the transition temperature to the coherent state

$$1 = \frac{\langle |\Delta|^2 \rangle}{2T} \sum_{k \neq 0} B(0, k),$$  \hspace{1cm} (18)

where $\langle |\Delta|^2 \rangle$ is the average taken with the local functional

$$\langle |\Delta|^2 \rangle = \frac{\int_0^\infty d\rho \rho \exp\left[ -\left( \frac{1}{W} - \frac{W}{W_\xi} \rho + b\rho^2 \right)/T \right]}{\int_0^\infty d\rho \exp\left[ -\left( \frac{1}{W} - \frac{W}{W_\xi} \rho + b\rho^2 \right)/T \right]}.$$  \hspace{1cm} (19)

Again, similarly to (5), we can replace the summation over the sites by the integration

$$\sum_{k \neq 0} B(0, k) = \frac{2\gamma^2 \lambda n_R}{\pi v_0} \ln\left( \frac{|E_0/T_c|}{n_R} \right).$$  \hspace{1cm} (20)

Expressions (18)–(20) allow us to find the transition temperature both in the weak- and strong-coupling limits.

6. TRANSITION TEMPERATURE TO THE COHERENT STATE

In the weak-coupling limit, $W < W_c$, $b|\Delta|^4$ can be neglected in the calculations of the correlation function in (18). In this case, Eqs. (18)–(20) give us the transition temperature in the weak-coupling limit (9).

At the strong attraction, $W > W_c$, the thermal average $|\Delta|$ is nonzero. At $(W - W_c)^2/W_c > T_c$, the contribution to (19) from fluctuations can be neglected, hence the transition temperature is given by the expression

$$T_c(W) = \frac{(W/W_c - 1) n_R \gamma^2 \lambda}{2\pi v_0} \ln\left( |E_0/T_c| \right).$$  \hspace{1cm} (21)

Note that Eq. (21) corresponds to the limit discussed in [11] and in the references therein.

In a small neighborhood of the critical magnitude of attraction, such as $T_c(W) - (W - W_c)^2/W_c$, the term proportional to $\rho$ in the argument of exponentials in (19) can be neglected. In this case, we obtain the transition temperature using (18)

$$T_c(W \sim W_c) = \left[ \frac{n_R \gamma^2 \lambda}{\pi v_0 \sqrt{\pi \rho}} \ln\left( |E_0/T_c| \right) \right]^2.$$  \hspace{1cm} (22)

This expression is valid both at $W < W_c$ and at $W > W_c$. The dependence of the transition temperature on the magnitude of attraction normalized with respect to the critical point value is shown in Fig. 4.

7. UPPER CRITICAL MAGNETIC FIELD

In the applied magnetic field, the mean-field order parameter is independent of the position, $\langle \Delta \rangle = \Theta_i$. Instead of (18), the linear self-consistency equation has the form

$$\Theta_i = \frac{\beta \langle |\Delta|^2 \rangle}{2T} \sum_{k \neq i} B(i, k) \Theta_k.$$  \hspace{1cm} (23)
After changing the summation by integration and the expansion of the order parameter in the magnetic field with $A$ near $r$, in Eq. (23)

$$\Theta_k = \Theta(r) + \frac{\mu_k^2}{2} \left( \frac{d}{dr} - \frac{2e_i A}{c} \right)^2 \Theta(r),$$

(24)

the following factor appears at the gradient term:

$$\int d r \nu^2 \frac{B(r)}{\nu_0 T_c}.$$ (25)

To determine the upper critical magnetic field $H_{c2}$, the eigenstate $\Theta(r)$ of the operator (24) corresponding to the largest eigenvalue should be chosen.

In the weak-coupling limit, $W < W_c$, and at temperatures close to $T_c$, the upper critical magnetic field is given by the expression typical of disordered superconductors.

At a strong attraction, when $W > W_c$ and $\langle |\Delta|^2 \rangle \approx \langle \Delta \rangle^2 = \frac{1 - W/W_c}{2W}$, the critical field is enhanced by a factor of $\ln |E_0/T_c|$.

$$\frac{D\epsilon H_{c2}}{c} = \frac{4}{\pi} (T_c - T) \ln |E_0/T_c|.$$ (26)

It is remarkable that the magnetic field does not affect the local parameter $\Lambda$. Most likely, it reduces the effective Josephson coupling between different resonant sites.

8. CONCLUSIONS

In this work, the superconducting transition in a dilute array of negative-U sites with a strong electron—electron attraction has been studied.

It has been shown that an incoherent superconducting state arises in the system when the attraction exceeds a certain critical value $W_c$, which depends on the energy and width of the resonance levels. Global superconductivity is established at a lower temperature by the Andreev reflection involving the resonant level. This regime resembles the transition from the BCS state to a liquid consisting of composite Bose particles.

The results do not change with change in the sign of the resonance energy $E_R$. Therefore, they are valid not only for the sites with degeneracy with respect to the number of electrons at small $|2E_R - W| \ll E_R$, but also for the array of sites with negative $E_R$ at $|2E_R + W| \ll |E_R|$.  

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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