On path factors of (3, 4)-biregular bigraphs

Armen S. Asratian*, Carl Johan Casselgren†

Abstract. A (3, 4)-biregular bigraph $G$ is a bipartite graph where all vertices in one part have degree 3 and all vertices in the other part have degree 4. A path factor of $G$ is a spanning subgraph whose components are nontrivial paths. We prove that a simple (3, 4)-biregular bigraph always has a path factor such that the endpoints of each path have degree three. Moreover we suggest a polynomial algorithm for the construction of such a path factor.

Keywords: path factor, biregular bigraph, interval edge coloring

1 Introduction

We use [9] and [7] for terminology and notation not defined here and consider finite loop-free graphs only. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. A proper edge coloring of a graph $G$ with colors $1, 2, 3, \ldots$ is a mapping $f : E(G) \rightarrow \{1, 2, 3, \ldots\}$ such that $f(e_1) \neq f(e_2)$ for every pair of adjacent edges $e_1$ and $e_2$. A bipartite graph with bipartition $(Y, X)$ is called an $(a, b)$-biregular bigraph if every vertex in $Y$ has degree $a$ and every vertex in $X$ has degree $b$. A path factor of a graph $G$ is a spanning subgraph whose components are nontrivial paths. Some results on different types of path factors can be found in [1, 2, 17, 18, 20, 23]. In particular, Ando et al [2] showed that a claw-free graph with minimum degree $d$ has a path factor whose components are paths of length at least $d$. Kaneko [17] showed that every cubic graph has a path factor such that each component is a path of length 2, 3 or 4. It was shown in [18] that a 2-connected cubic graph has a path factor whose components are paths of length 2 or 3.

In this paper we investigate the existence of path factors of (3, 4)-biregular bigraphs such that the endpoints of each path have degree three. Our investigation is motivated by a problem on interval colorings. A proper edge coloring of a graph $G$ with colors $1, 2, 3, \ldots$ is called an interval (or consecutive) coloring if the colors received by the edges incident with each vertex of $G$ form an interval of integers. The notion of interval colorings was introduced in 1987 by Asratian and Kamalian [5] (available in English as [6]). Generally, it is an NP-complete problem to determine whether a given bipartite graph has an interval coloring [22]. Nevertheless, trees, regular and

*Linköping University, Linköping Sweden, arasr@mai.liu.se.
†Umeå University, Umeå, Sweden, carl-johan.casselgren@math.umu.se.
complete bigraphs [13, 16], doubly convex bigraphs [16], grids [12] and all outerplanar bigraphs [8, 11] have interval colorings. Hansen [13] proved that every \((2, \beta)\)-biregular bigraph admits an interval coloring if \(\beta\) is an even integer. A similar result for \((2, \beta)\)-biregular bigraphs for odd \(\beta\) was given in [14, 19]. Only a little is known about \((3, \beta)\)-biregular bigraphs. It follows from the result of Hanson and Loten [15] that no such a graph has an interval coloring with fewer than \(3 + b - \gcd(3, b)\) colors, where \(\gcd\) denotes the greatest common divisor. We showed in [3] that the problem to determine whether a \((3, \beta)\)-biregular bigraph has an interval coloring is \(\text{NP}\)-complete in the case when 3 divides \(\beta\).

It is unknown whether all \((3, 4)\)-biregular bigraphs have interval colorings. Pyatkin [21] showed that such a graph \(G\) has an interval coloring if \(G\) has a 3-regular subgraph covering the vertices of degree four. Another sufficient condition for the existence of an interval coloring of a \((3, 4)\)-biregular bigraph \(G\) was obtained in [4, 10]: \(G\) admits an interval coloring if it has a path factor where every component is a path of length not exceeding 8 and the endpoints of each path have degree three. It was conjectured in [4] that every simple \((3, 4)\)-biregular bigraph has such a path factor. However this seems difficult to prove.

In this note we prove a little weaker result. We show that a simple \((3, 4)\)-biregular bigraph always has a path factor such that the endpoints of each path have degree three. Moreover, we suggest a polynomial algorithm for the construction of such a path factor.

Note that \((3, 4)\)-biregular bigraphs with multiple edges need not have path factors with the required property. For example, consider the graph \(G\) formed from three triple-edges by adding a claw; that is, the pairs \(x_iy_i\) have multiplicity three for \(i \in \{1, 2, 3\}\), and there is an additional vertex \(y_0\) with neighborhood \(\{x_1, x_2, x_3\}\). Clearly, there is no path factor of \(G\) such that the endpoints of each path have degree 3.

### 2 The result

A **pseudo path factor** of a \((3, 4)\)-biregular bigraph \(G\) with bipartition \((Y, X)\) is a subgraph \(F\) of \(G\), such that every component of \(F\) is a path of even length and \(d_F(x) = 2\) for every \(x \in X\). Let \(V_F = \{y \in Y : d_F(y) > 0\}\).

**Theorem 1.** Every simple \((3, 4)\)-biregular bigraph has a pseudo path factor.

**Proof.** Let \(G\) be a simple \((3, 4)\)-biregular bigraph with bipartition \((Y, X)\). The algorithm below constructs a sequence of subgraphs \(F_0, F_1, F_2, \ldots\) of \(G\), where \(V(F_0) = V(G), \emptyset = E(F_0) \subset E(F_1) \subset E(F_2) \subset \cdots\) and each component of \(F_j\) is a path, for every \(j \geq 0\). At each step \(i \geq 1\) the algorithm constructs \(F_i\) by adding to \(F_{i-1}\) one or two edges until the condition \(d_{F_i}(x) = 2\) holds for all \(x \in X\), where \(j \geq 1\). Then \(F = F_j\) is a pseudo path factor of \(G\). Parallelly the algorithm constructs a sequence of subgraphs \(U_0, U_1, U_2, \ldots\) of \(G\), where \(V(U_0) = V(G), \emptyset = E(U_0) \subset E(U_1) \subset E(U_2) \subset \cdots \subset E(U_j)\). The edges of each \(U_i\) will not be in the final pseudo
path factor $F$. The algorithm is based on Properties 1-4. During the algorithm the vertices in the set $Y$ are considered to be unscanned or scanned. Initially all vertices in $Y$ are unscanned. At the beginning of each step $i \geq 1$ we have a current vertex $x_i$. The algorithm selects an unscanned vertex $y_i$, adjacent to $x_i$, and determines which edges incident with $y_i$ will be in $F_i$ and which ones in $U_i$. If $d_{F_i}(v) = 2$ for each $v \in X$, the algorithm stops. Otherwise the algorithm selects a new current vertex and goes to the next step.

Algorithm

Initially $F_0 = (V(G), \emptyset)$, $U_0 = (V(G), \emptyset)$ and all vertices in $Y$ are unscanned.

Step 0. Select a vertex $y_0 \in Y$. Let $x_0, x_1, w$ be the vertices in $X$ adjacent to $y_0$ in $G$. Put $F_1 = F_0 + \{wy_0, y_0x_0\}$ and $U_1 = U_0 + y_0x_1$. Consider the vertex $y_0$ to be scanned. Go to step 1 and consider the vertex $x_1$ as the current vertex for step 1.

Step $i$ ($i \geq 1$). Suppose that a vertex $x_i$ with $d_{F_{i-1}}(x_i) \leq 1$ was selected at step $(i - 1)$ as the current vertex. By Property 4 (see below), $d_{U_{i-1}}(x_i) \leq 2$. Therefore there is an edge $x_iy_i$ with $y_i \in Y$ which neither belongs to $F_{i-1}$, nor to $U_{i-1}$. Then, by Property 3, the vertex $y_i$ is an unscanned vertex and therefore the subgraph $F_{i-1} + x_iy_i$ does not contain a cycle. Since $d_G(y_i) = 3$, the vertex $y_i$, besides $x_i$, is adjacent to two other vertices, $w_1^{(i)}$ and $w_2^{(i)}$.

Case 1. $d_{F_{i-1}}(w_1^{(i)}) = 2 = d_{F_{i-1}}(w_2^{(i)})$.

Put $F_i = F_{i-1} + x_iy_i$ and $U_i = U_{i-1} + \{y_iw_1^{(i)}, y_iw_2^{(i)}\}$. Consider the vertex $y_i$ to be scanned. If $d_{F_i}(v) = 2$ for every vertex $v \in X$ then Stop. Otherwise select an arbitrary vertex $x_{i+1} \in X$ with $d_{F_i}(x_{i+1}) \leq 1$, go to step $(i + 1)$ and consider $x_{i+1}$ as the current vertex for step $(i + 1)$.

Case 2. $d_{F_{i-1}}(w_1^{(i)}) = 2$ and $d_{F_{i-1}}(w_2^{(i)}) \leq 1$.

Put $F_i = F_{i-1} + x_iy_i$, $U_i = U_{i-1} + \{y_iw_1^{(i)}, y_iw_2^{(i)}\}$ and consider the vertex $y_i$ to be scanned. Furthermore put $x_{i+1} = w_2^{(i)}$, go to step $(i + 1)$ and consider the vertex $x_{i+1}$ as the current vertex for step $(i + 1)$.

Case 3. $d_{F_{i-1}}(w_1^{(i)}) \leq 1$ and $d_{F_{i-1}}(w_2^{(i)}) \leq 1$.

Subcase 3a. $d_{F_{i-1}}(w_1^{(i)}) = 0$ or $d_{F_{i-1}}(w_2^{(i)}) = 0$.
We assume that $d_{F_{i-1}}(w_1^{(i)}) = 0$. Put $F_i = F_{i-1} + \{x_iy_i, y_iw_1^{(i)}\}$, $U_i = U_{i-1} + y_iw_2^{(i)}$ and consider the vertex $y_i$ to be scanned. Furthermore put $x_{i+1} = w_2^{(i)}$, go to step $(i + 1)$ and consider the vertex $x_{i+1}$ as the current vertex for step $(i + 1)$.

Subcase 3b. $d_{F_{i-1}}(w_1^{(i)}) = 1 = d_{F_{i-1}}(w_2^{(i)})$.
Since $y_i$ is an unscanned vertex and $F_{i-1} + x_iy_i$ does not contain a cycle, the vertex $y_i$ is an endvertex of only one path in $F_{i-1} + x_iy_i$. Then at least one of the graphs $F_{i-1} + \{x_iy_i, y_iw_1^{(i)}\}$ and $F_{i-1} + \{x_iy_i, y_iw_2^{(i)}\}$ does not contain a cycle. Assume, for example, that $F_{i-1} + \{x_iy_i, y_iw_1^{(i)}\}$ does not contain a cycle. Then put $F_i = F_{i-1} + \{x_iy_i, y_iw_1^{(i)}\}$, $U_i = U_{i-1} + y_iw_2^{(i)}$ and consider the vertex $y_i$ to be scanned. Furthermore put $x_{i+1} = w_2^{(i)}$, go to step $(i + 1)$ and consider the vertex $x_{i+1}$ as
the current vertex for step \((i + 1)\).

Now we will prove the correctness of the algorithm. At the beginning of step \(i\) we have that \(x_i\) is the current vertex, \(y_i\) is an unscanned vertex adjacent to \(x_i\) and \(w_1^{(i)}, w_2^{(i)}\) are the two other vertices adjacent to \(y_i\). The following two properties are evident.

**Property 1.** The algorithm determines which edges incident with \(y_i\) will be in \(F_i\) and which edges will be in \(U_i\). The vertex \(y_i\) is then considered to be scanned and the algorithm will never consider \(y_i\) again.

**Property 2.** The current vertex \(x_{i+1}\) for step \((i + 1)\) is selected among the vertices \(w_1^{(i)}\) and \(w_2^{(i)}\), except the case \(d_{F_i}(w_1^{(i)}) = d_{F_i}(w_2^{(i)}) = 2\) when an arbitrary vertex \(x_{i+1} \in X\) with \(d_{F_i}(x_{i+1}) \leq 1\) is selected as the current vertex.

Properties 1 and 2 imply the next property:

**Property 3.** If \(x \in X\), \(y \in Y\) and the edge \(xy\) neither belongs to \(F_{i-1}\), nor to \(U_{i-1}\), then the vertex \(y\) is unscanned at the beginning of step \(i\).

**Property 4.** If \(x \in X\) and \(d_{F_{i-1}}(x) \leq 1\) then \(d_{U_{i-1}}(x) \leq 2\).

**Proof.** The statement is evident if \(d_{U_{i-1}}(x) = 0\). Suppose that \(d_{U_{i-1}}(x) \geq 1\) and \(j\) is the minimum number such that \(j < i\) and an edge incident with \(x\) was included in \(U_{j-1}\) at step \((j - 1)\). Then the statement of Property 4 is evident if \(j = i - 1\).

Now we consider the case \(j < i - 1\). Clearly, \(d_{F_{j-1}}(x) \leq 1\) because \(F_{j-1} \subseteq F_{i-1}\) and \(d_{F_{j-1}}(x) \leq d_{F_{i-1}}(x) \leq 1\). Let \(xy_{j-1}\) be the edge included in \(U_{j-1}\) at step \((j - 1)\). Since \(d_{U_{j-1}}(x) = 1\) and \(d_{U_{j-1}}(x) \leq 1\), there is an edge \(xy_j\) with \(y_j \in Y\) which neither belongs to \(F_{j-1}\), nor to \(U_{j-1}\). Then, by Property 3, the vertex \(y_j\) is an unscanned vertex and therefore the subgraph \(F_{j-1} + xy_j\) does not contain a cycle. According to the description of the algorithm, the edge \(xy_j\) will be in any case included in \(F_j\) at step \(j\), that is, \(d_{F_j}(x) \geq 1\). Then \(d_{F_k}(x) = 1\) for every \(k\), \(j \leq k \leq i - 1\), because \(F_j \subseteq F_k \subseteq F_{i-1}\) and \(1 \leq d_{F_j}(x) \leq d_{F_k}(x) \leq d_{F_{i-1}}(x) \leq 1\). Now we will show that \(d_{U_{k-1}}(x) = 1\) for each \(k\), \(j \leq k < i - 1\). Suppose to the contrary that \(d_{U_{k-2}}(x) = 1\) and \(d_{U_{k-1}}(x) = 2\) for some \(k\), \(j < k < i - 1\), that is, another edge incident with \(x\) was included in \(U_{k-1}\) at step \((k - 1)\). Then the conditions \(d_{U_{k-1}}(x) = 2\) and \(d_{F_{k-1}}(x) = 1\) imply that there is an edge \(e \neq y_jx\) incident with \(x\) which neither belongs to \(F_{k-1}\), nor to \(U_{k-1}\). Using a similar argument as above we obtain that the edge \(e\) should be included in \(F_k\) at step \(k\). But then \(d_{F_{i-1}}(x) \geq d_{F_k}(x) = 2\), which contradicts our assumption \(d_{F_{i-1}}(x) \leq 1\). Thus \(d_{U_{k-1}}(x) = 1\) for each \(k\), \(j \leq k < i - 1\). It is possible that an edge incident with \(x\) will be included in \(U_{i-1}\) at step \((i - 1)\). Therefore \(d_{U_{i-1}}(x) \leq 2\).

The description of the algorithm and Properties 1-4 show that the algorithm will stop at step \(i\) only when \(d_{F_i}(x) = 2\) for every \(x \in X\), that is, when \(F_i\) is a pseudo path factor of \(G\). The proof of Theorem \(\text{II}\) is complete.
Now we will prove that every pseudo path factor of a (3,4)-biregular bigraph $G$ can be transformed into a path factor of $G$, such that the endpoints of each path have degree 3.

**Lemma 2.** Let $G$ be a (3,4)-biregular bigraph with bipartition $(Y, X)$. Then $|X| = 3k$ and $|Y| = 4k$, for some positive integer $k$.

This is evident because $|E(G)| = 4|X| = 3|Y|$.

**Lemma 3.** Let $F$ be a pseudo path factor of a (3,4)-biregular bigraph $G$ with bipartition $(Y, X)$.

Then $F$ has a component which is a path of length at least four.

**Proof.** By Lemma 2 we have that $|X| = 3k$ and $|Y| = 4k$ for some integer $k$. We also have that $d_F(x) = 2$ for each vertex $x \in X$. If the length of all paths in $F$ is two, then $|Y| \geq 2|X| = 6k$ which contradicts $|Y| = 4k$. Therefore $F$ has a component which is a path of length at least four.

**Theorem 4.** Let $F$ be a pseudo path factor of a simple (3,4)-biregular bigraph $G$ with bipartition $(Y, X)$. If $V_F \neq Y$ and $y_0$ is a vertex with $d_F(y_0) = 0$, then there is a pseudo path factor $F'$ with $V_{F'} = V_F \cup \{y_0\}$, such that no path in $F'$ is longer than the longest path in $F$.

**Proof.** Let $y_0 \in Y$ and $d_F(y_0) = 0$. We will describe an algorithm which will construct a special trail $T$ with origin $y_0$.

**Step 1.** Select an edge $y_0x_1 \notin E(F)$. Since $d_F(x_1) = 2$, there are two edges of $F$, $x_1y_1$ and $x_1u_1$, which are incident with $x_1$.

**Case 1.** $d_F(y_1) = 2$ or $d_F(u_1) = 2$.

Suppose, for example, that $d_F(y_1) = 2$. Then put $T = y_0 \rightarrow x_1 \rightarrow y_1$ and Stop.

**Case 2.** $d_F(y_1) = 1 = d_F(u_1)$.

Put $T = y_0 \rightarrow x_1 \rightarrow y_1$ and go to Step 2.

**Step $i$ ($i \geq 1$).** Suppose that we have already constructed a trail $T = y_0 \rightarrow x_1 \rightarrow y_1 \rightarrow \cdots \rightarrow x_i \rightarrow y_i$ which satisfies the following conditions:

(a) All edges in $T$ are distinct and $y_{j-1}x_j \notin E(F)$, $x_jy_j \in E(F)$ for $j = 1,\ldots, i$.

(b) The vertices $y_1,\ldots,y_i$ are distinct.

(c) A component of $F$ containing the vertex $x_j$ is a path of length 2, for $j = 1,\ldots,i$.

Select an edge $e \in E(G) \setminus E(F)$ which is incident with $y_i$. The existence of such an edge follows from the conditions (a), (b) and (c). Moreover, the condition (b) implies that $e \notin T$. Let $e = y_ix_{i+1}$. Then $d_F(x_{i+1}) = 2$ because $F$ is a pseudo path factor of $G$. Since $e \notin E(T)$, the conditions (a), (b) and (c) imply that at least one of the edges of $F$ incident with $x_{i+1}$, does not belong to $T$.

**Case 1.** $x_{i+1}$ lies on a component of $F$ which is a path of length two.

Select a vertex $y_{i+1}$ such that $x_{i+1}y_{i+1} \in E(F) \setminus E(T)$, add the edge $x_{i+1}y_{i+1}$ and the vertex $y_{i+1}$ to $T$ and go to step $(i+1)$. Now $T = y_0 \rightarrow x_1 \rightarrow y_1 \rightarrow \cdots \rightarrow x_{i+1} \rightarrow y_{i+1}$. 
Case 2. $x_{i+1}$ lies on a component of $F$ which is a path of length at least four.

There is a vertex $y_{i+1}$ such that $x_{i+1}y_{i+1} \in E(F) \setminus E(T)$ and $d_F(y_{i+1}) = 2$. Add the edge $x_{i+1}y_{i+1}$ and the vertex $y_{i+1}$ to $T$ and Stop. We have now that $T = y_0 \to x_1 \to y_1 \to \cdots \to x_{i+1} \to y_{i+1}$.

By Lemma 3, $F$ has a component which is a path of length at least four. Therefore the algorithm will stop after a finite number of steps. Let the trail $T = y_0 \to x_1 \to y_1 \to \cdots \to x_{i+1} \to y_{i+1}$ be the result of the algorithm, where $i \geq 0$, the vertex $x_j$ lies on a component of $F$ which is a path of length two for each $j \leq i$, the vertex $x_{i+1}$ lies on a component of $F$ which is a path of length at least 4, and $d_F(y_{i+1}) = 2$. We define a new pseudo path factor $F'$ by setting $V(F') = V(F)$ and $E(F') = (E(F) \setminus \{x_jy_j : j = 1, \ldots, i, i+1\}) \cup \{y_{j-1}x_j : j = 1, \ldots, i, i+1\}$.

Clearly, $V_{F'} = V_F \cup \{y_0\}$ and the proof of Theorem 4 is complete.

Theorems 1 and 4 imply the following theorem:

Theorem 5. Every simple $(3, 4)$-biregular bigraph has a path factor such that the endpoints of each path have degree 3.

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