FEEDBACK SYNCHRONIZATION OF FHN CELLULAR NEURAL NETWORKS

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Abstract. In this work we study the synchronization of ring-structured cellular neural networks modeled by the lattice FitzHugh-Nagumo equations with boundary feedback. Through the uniform estimates of solutions and the analysis of dissipative dynamics, the synchronization of this type neural networks is proved under the condition that the boundary gap signal exceeds the adjustable threshold.

1. Introduction

Cellular neural network (briefly CNN) was invented by Chua and Yang [6, 7]. CNN consists of network-like interacted processors of continuous-time analog or digital signals. In some sense the dynamics of CNN can be analyzed as a lattice dynamical system oftentimes generated by lattice differential equations in time [8, 9, 21].

From the theoretical viewpoint, the theory of cellular neural networks and, more recently, of convolutional neural networks (also called CNN) and variants of complex neural networks is closely linked to discrete nonlinear partial differential equations and delay differential equations as well as the spatially discrete Fourier transform and constrained optimization [1, 4, 8]. In the dramatically broadened application fronts to machine learning, deep learning, and general artificial intelligence, the most prominent area is the image processing especially in medical visualization techniques [7, 9, 20].

Many complicated computational problems can be formulated as multi-layer and parallel tasks for processing signal values on a geometric grid with direct interaction and transmission in a local neighborhood. The cloning template of each cell model and the coupling design mimic biological pattern formation in the brain and nerves are the two features of CNN. The concepts and models of CNN are based on some aspects of neurobiology [21] in terms of sheet-like layers or arrays of massively interconnected excitable neurons and the implementation counterpart by electronic integrated circuit chips on the other hand.

Date: June 18, 2020.

2010 Mathematics Subject Classification. 34A33, 34D06, 37L60, 37N99, 92B20.

Key words and phrases. Cellular neural network, discrete FitzHugh-Nagumo equations, feedback synchronization, dissipative dynamics.
Rapidly expanding applications of CNN and all kinds of complex neural networks \[9, 12, 14, 22\] stimulate frontier and interdisciplinary researches from data analysis to mathematical and statistical modeling to deep learning algorithms, softwares, and robotics.

In this work, we consider a CNN model as the discrete replacement of the biological neural networks described by the partial diffusive FitzHugh-Nagumo equations with different coupling patterns \[11, 19, 20, 23, 24\],

\[
\begin{align*}
\frac{\partial u_i}{\partial t} &= a \Delta u_i + f(u_i) - bw_i, \\
\frac{\partial w_i}{\partial t} &= cu_i - \delta w_i.
\end{align*}
\]

We shall use the discrete Laplacian templates to approximate the Laplacian operator on a 1D domain \[4, 21\]. For the prototype FitzHugh-Nagumo equations, the nonlinear term is

\[f(s) = s(s - \alpha)(1 - s)\]

and the parameters \(0 < \alpha < 1\) and \(0 < c \ll 1\).

Consider a layer of cellular neural network, which consists of 1D cells at the grid points \(\{ih : i = 1, 2, \ldots, n\}\) for \(h > 0\), in a ring-structure. We shall study synchronization dynamics of the FitzHugh-Nagumo lattice equations:

\[
\begin{align*}
\frac{dx_i}{dt} &= a(x_{i-1} - 2x_i + x_{i+1}) + f(x_i) - by_i + pu_i, \quad 1 \leq i \leq n, \\
\frac{dy_i}{dt} &= cx_i - \delta y_i, \quad 1 \leq i \leq n,
\end{align*}
\]

where \(t > 0\), the integer \(n \geq 4\), and the discrete Laplacian operator

\[D_i(x) = a(x_{i+1} - 2x_i + x_{i-1})\]

can be called the synaptic law of cell coupling. In this model of CNN, we impose the periodic boundary condition

\[x_0(t) = x_n(t), \quad x_{n+1}(t) = x_1(t)\] (1.2)

and the boundary feedback control \(\{u_i\}_{i=1}^n\),

\[
\begin{align*}
u_1(t) &= u_{n+1}(t) = x_n(t) - x_1(t), \\
u_i(t) &= 0, \quad 2 \leq i \leq n - 1, \\
u_n(t) &= u_0(t) = x_1(t) - x_n(t).
\end{align*}
\]

(1.3)

for \(t \geq 0\), where \(p > 0\) is the controllable feedback constant and \(x_n(t) - x_1(t)\) measures the boundary gap signal between the two endpoints of the cellular neural network.
The initial conditions for the system (1.1) are denoted by
\[ x_i(0) = x_i^0 \in \mathbb{R} \quad \text{and} \quad y_i(0) = y_i^0 \in \mathbb{R}, \quad 1 \leq i \leq n. \] (1.4)

All the parameters in this system (1.1) are positive constants.

We make the following Assumption: The scalar function \( f \in C^1(\mathbb{R}, \mathbb{R}) \) satisfies
\[
\begin{align*}
  f(s)s &\leq -\lambda s^4 + \beta, \quad s \in \mathbb{R}, \\
  f'(s) &\leq \gamma, \quad s \in \mathbb{R},
\end{align*}
\]
where \( \lambda, \beta \) and \( \gamma \) are positive constants. The typical nonlinearity \( f(s) = s(s - \alpha)(1 - s) \) shown above in the FitzHugh-Nagumo model satisfies the Assumption (1.5):
\[
\begin{align*}
  f(s)s &= -\alpha s^2 + (\alpha + 1)s^3 - s^4 \leq -\alpha s^2 + \left( \frac{1}{2}s^4 + 2^3(\alpha + 1)^4 \right) - s^4 \\
  &\leq -\left( \alpha s^2 + \frac{1}{2}s^4 \right) + 8(\alpha + 1)^4 \leq -\frac{1}{2}s^4 + 8(\alpha + 1)^4, \\
  f'(s) &= -\alpha + 2(\alpha + 1)s - 3s^2 \leq -\alpha + (\alpha + 1)^2 - 2s^2 \leq 1 + \alpha + \alpha^2.
\end{align*}
\]

Synchronization plays a significant role for biological neural networks and for the artificial neural networks as well. Fast synchronization may lead to enhanced functionality and performance of complex neural networks.

In recent years, the dynamical behavior and problems of complex and large-scale networks including convolutional neural networks in machine learning and deep learning, Internet networks, epidemic spreading networks, and social networks attract many interdisciplinary research interests, cf. [2, 14, 22]. Synchronization for the CNN modeled by PDE, delay differential equations, or lattice differential equations is one of the essential topics in the theoretical analysis of artificial intelligence.

Synchronization for biological neural networks has been studied by several mathematical models and methods. This topic has been studied for the diffusive FitzHugh-Nagumo networks of neurons coupled by clamped gap junctions [1, 2, 3, 13, 24], the mean field couplings of Hodgkin-Huxley and FitzHugh-Nagumo neuron networks [10, 17], and the chaotic neural networks and stochastic neural networks [10, 18].

Recently we proved results on the exponential synchronization of the boundary coupled Hindmarsh-Rose neural networks in [15, 16] and the boundary coupled partly diffusive FitzHugh-Nagumo neural networks in [19].

The feature of this work is to provide a sufficient condition for realization of the feedback synchronization of the proposed FitzHugh-Nagumo (FHN) cellular neural networks with boundary control. The quantitative threshold condition for synchronization is explicitly expressed in terms of the parameters and can be adjusted by the feedback strength coefficient \( p \) in applications.
2. Uniform Estimates and Dissipative Dynamics

Define the following Hilbert space:

\[ H = l^2(\mathbb{Z}_n, \mathbb{R}^n) = \{(x, y) = \{(x_i, y_i) : 1 \leq i \leq n\}\} \]

where \(\mathbb{Z}_n = \{1, 2, \cdots, n\}\) and \(n \geq 4\). The norm in \(H\) is denoted and define by

\[ \| (x, y) \|^2 = \| x \|^2 + \| y \|^2 = \sum_{i=1}^{n} (|x_i|^2 + |y_i|^2). \]

The inner-product of \(H\) or \(\mathbb{R}^n\) is denoted by \(\langle \cdot, \cdot \rangle\).

Since there exists a unique local solution in time of the initial value problem (1.1)-(1.4) under the Assumption (1.5) that the right-side functions in (1.1) are locally Lipschitz continuous, in this section we shall first prove the global existence in time of the solutions in the space \(H\). By the uniform estimates we show the dissipative dynamics of the solution semiflow.

**Theorem 2.1.** Under the boundary feedback control (1.3), for any given initial state \((x^0, y^0) = ((x_0^0, y_0^0), \cdots, (x_n^0, y_n^0)) \in H\), there exists a unique solution, \((x(t), y(t)) = ((x_1(t), x_0^1), y_1(t), y_0^1), \cdots, (x_n(t), x_0^n), y_n(t), y_0^n)), t \in [0, \infty), of the initial value problem (1.1)-(1.4) for this cellular neural network.

**Proof.** Multiply the \(x_i\)-equation in (1.1) by \(C_1 x_i(t)\) for \(1 \leq i \leq n\), where the constant \(C_1 > 0\) is to be chosen, then sum them up and by the Assumption (1.5) to get

\[
\frac{C_1}{2} \frac{d}{dt} \sum_{i=1}^{n} |x_i|^2 = C_1 \sum_{i=1}^{n} \left[ a(x_{i-1} - 2x_i + x_{i+1})x_i + f(x_i)x_i - bx_i y_i + pu_i x_i \right]
\]

\[
\leq C_1 \sum_{i=1}^{n} a(x_{i-1} - 2x_i + x_{i+1})x_i
\]

\[
+ C_1 \sum_{i=1}^{n} \left[ -\lambda|x_i|^4 + \beta + \frac{b}{2} |x_i|^2 + \frac{b}{2} |y_i|^2 \right] - C_1 \rho(x_1 - x_n)^2, \quad t \in I_{\text{max}},
\]

where \(I_{\text{max}} = [0, T_{\text{max}}]\) is the maximal existence interval of the solution. According to the discrete ”divergence” formula and \(x_0(t) = x_n(t), x_{n+1}(t) = x_1(t)\) due to the periodic boundary condition (1.2), we have

\[
\sum_{i=1}^{n} (x_{i-1} - 2x_i + x_{i+1})x_i = \sum_{i=1}^{n} (x_{i+1} - x_i)x_i - \sum_{i=1}^{n} (x_i - x_{i-1})x_i
\]

\[
= \left[ \sum_{i=1}^{n-1} (x_{i+1} - x_i)x_i - \sum_{i=2}^{n} (x_i - x_{i-1})x_i \right] + (x_{n+1} - x_n)x_n - (x_1 - x_0)x_1 \quad (2.2)
\]

\[
= -\sum_{i=2}^{n} (x_i - x_{i-1})^2 - (x_1 - x_0)^2 = -\sum_{i=1}^{n} (x_i - x_{i-1})^2 \leq 0.
\]
Then (2.1) with (2.2) yields the differential inequality

\[ C_1 \frac{d}{dt} \sum_{i=1}^{n} |x_i(t)|^2 + 2C_1 \left[ \sum_{i=2}^{n} a(x_i - x_{i-1})^2 + p(x_1 - x_n)^2 \right] \leq C_1 \sum_{i=1}^{N} \left[ -2\lambda|x_i(t)|^4 + 2\beta + b|x_i(t)|^2 + b|y_i(t)|^2 \right], \quad t \in I_{max}. \]  

(2.3)

Next multiply the \( y_i \)-equation in (1.1) by \( y_i(t) \) for \( 1 \leq i \leq n \) and then sum them up. By using Young’s inequality, we obtain

\[ \frac{1}{2} \frac{d}{dt} \sum_{i=1}^{n} |y_i(t)|^2 = \sum_{i=1}^{n} (c x_i y_i - \delta y_i^2) \leq \sum_{i=1}^{n} \left[ \left( \frac{c^2}{\delta} x_i^2 + \frac{1}{4}\delta y_i^2 \right) - \delta y_i^2 \right] \]

\[ = \sum_{i=1}^{n} \left[ \frac{c^2}{\delta} |x_i(t)|^2 - \frac{3}{4}\delta |y_i(t)|^2 \right], \quad \text{for } t \in I_{max}. \]  

(2.4)

Now add the above two inequalities (2.3) and doubled (2.4). We obtain

\[ \frac{d}{dt} \sum_{i=1}^{n} \left( C_1|x_i(t)|^2 + |y_i(t)|^2 \right) + 2C_1 \left[ \sum_{i=1}^{n} a(x_i - x_{i-1})^2 + p(x_1 - x_n)^2 \right] \leq \sum_{i=1}^{n} \left[ \left( C_1 b + \frac{2c^2}{\delta} \right) |x_i(t)|^2 - 2C_1\lambda|x_i(t)|^4 + 2C_1\beta \right] \]

\[ + \sum_{i=1}^{n} \left[ \left( C_1 b - \frac{3\delta}{2} \right) |y_i(t)|^2 \right], \quad t \in I_{mac} = [0, T_{max}). \]  

(2.5)

We choose constant \( \delta \) so that \( C_1 b - \frac{3\delta}{2} = -\delta \).

(2.6)

Then from (2.5) with the fact \( 2C_1[\cdots] \geq 0 \) on the left-hand side and from the choice (2.6), we have

\[ \frac{d}{dt} \sum_{i=1}^{n} (C_1|x_i|^2 + |y_i|^2) \leq \sum_{i=1}^{n} \left[ \left( C_1 b + \frac{2c^2}{\delta} \right) |x_i(t)|^2 - 2C_1(\lambda|x_i(t)|^4 + \beta) - \delta|y_i(t)|^2 \right] \]

and consequently,
\[
\frac{d}{dt} \sum_{i=1}^{n} (C_1|x_i(t)|^2 + |y_i(t)|^2) + \delta \sum_{i=1}^{n} (C_1|x_i(t)|^2 + |y_i(t)|^2) 
\]
\[
\leq \sum_{i=1}^{n} \left[ \left( C_1 b + C_1 \delta + \frac{2c^2}{\delta} \right) |x_i(t)|^2 - 2C_1(\lambda|x_i(t)|^4 + \beta) \right] 
\]
\[
= \sum_{i=1}^{n} \left[ \left( \frac{\delta^2}{2b} + \frac{\delta}{2} + \frac{2c^2}{\delta} \right) |x_i(t)|^2 - \frac{\delta \lambda}{b} |x_i(t)|^4 \right], \quad t \in I_{\text{max}}.
\]

Completing square shows that
\[
\left( \frac{\delta^2}{2b} + \frac{\delta}{2} + \frac{2c^2}{\delta} \right) |x_i(t)|^2 - \frac{\delta \lambda}{b} |x_i(t)|^4
\]
\[
= - \frac{\delta \lambda}{b} \left[ |x_i(t)|^2 - \frac{b}{2\delta \lambda} \left( \frac{\delta^2}{2b} + \frac{\delta}{2} + \frac{2c^2}{\delta} \right) \right]^2 + C_2
\]
and
\[
C_2 = \frac{b}{4\delta \lambda} \left( \frac{\delta^2}{2b} + \frac{\delta}{2} + \frac{2c^2}{\delta} \right)^2.
\]

Therefore, (2.7) yields
\[
\frac{d}{dt} \sum_{i=1}^{n} (C_1|x_i|^2 + |y_i|^2) + \delta \sum_{i=1}^{n} (C_1|x_i|^2 + |y_i|^2) \leq n \left( C_2 + \frac{\delta \beta}{b} \right), \quad t \in I_{\text{max}}. \tag{2.9}
\]

Apply the Gronwall inequality to (2.9). Then we get the following bounded estimate for all the solutions of the system of equations (1.1)-(1.4),
\[
\sum_{i=1}^{n} \left( |x_i(t, x_0^i)|^2 + |y_i(t, y_0^i)|^2 \right)
\]
\[
\leq \frac{1}{\min\{C_1, 1\}} \left[ e^{-\delta t} \sum_{i=1}^{n} (C_1|x_i^0|^2 + |y_i^0|^2) + \frac{n}{\delta} \left( C_2 + \frac{\delta \beta}{b} \right) \right], \quad t \in [0, \infty). \tag{2.10}
\]

Here it is shown that \( I_{\text{max}} = [0, \infty) \) for all the solutions because they will never blow up at any finite time. Thus it is proved that for any given initial state there exists a unique global solution \((x_1(t, x_0^1), y_1(t, y_0^1)), \ldots, (x_n(t, x_0^n), y_n(t, y_0^n))\) in \( H \).

The global existence and uniqueness of the solutions to the initial value problem (1.1)-(1.4) and their continuous dependence on the initial data enable us to define the solution semiflow \( \{S(t) : H \to H\}_{t \geq 0} \) of this system of the FitzHugh-Nagumo cellular neural network:
\[
S(t) : ((x_1^0, y_1^0), \ldots, (x_n^0, y_n^0)) \mapsto ((x_1(t, x_1^0), y_1(t, y_1^0)), \ldots, (x_n(t, x_n^0), y_n(t, y_n^0))).
\]
We call \( \{S(t)\}_{t \geq 0} \) the semiflow of the FitzHugh-Nagumo CNN.

**Theorem 2.2.** The semiflow \( \{S(t)\}_{t \geq 0} \) of the FitzHugh-Nagumo CNN in the space \( H \) is dissipative in the sense that there exists a bounded ball

\[
B^* = \{g \in H : \|g\|^2 \leq Q\}
\]

where the constant

\[
Q = \frac{1}{\min\{C_1, 1\}} \left[ 1 + \frac{n}{\delta} \left( C_2 + \frac{\delta \beta}{b} \right) \right]
\]

such that for any given bounded set \( B \subset H \), there is a finite time \( T_B > 0 \) and all the solutions with the initial state inside the set \( B \) will permanently enter the ball \( B^* \) for \( t \geq T_B \).

**Proof.** The uniform estimate (2.10) implies that

\[
\limsup_{t \to \infty} \sum_{i=1}^{n} (|x_i(t, x_i^0)|^2 + |y_i(t, y_i^0)|^2) < Q
\]

for all solutions of (1.1) with any initial data \(((x_1^0, y_1^0), \cdots, (x_n^0, y_n^0)) \in H \). Indeed for any given bounded set \( B = \{g \in H : \|g\|^2 \leq \rho\} \) in \( H \), there is a finite time

\[
T_B = \frac{1}{\delta} \log^+(\rho \max\{C_1, 1\})
\]

such that

\[
e^{-\delta t} \sum_{i=1}^{n} (C_1|x_i^0|^2 + |y_i^0|^2) < 1, \quad \text{for } t \geq T_B,
\]

which means all the solution trajectories started from the set \( B \) will permanently enter the bounded ball \( B^* \) shown in (2.11) for \( t \geq T_B \). Therefore, this semiflow is dissipative. \( \square \)

### 3. Synchronization of the FitzHugh-Nagumo CNN

Define the differences of solutions for two adjacent indexed cells of the FitzHugh-Nagumo CNN (1.1) to be

\[
V_i(t) = x_i(t) - x_{i-1}(t), \quad W_i(t) = y_i(t) - y_{i-1}(t), \quad \text{for } i = 1, \cdots, n.
\]

Consider the system of the **differencing** equations for this CNN. For \( i = 1, \cdots, n, \)

\[
\frac{\partial V_i}{\partial t} = a(V_{i-1} - 2V_i + V_{i+1}) + f(x_i) - f(x_{i-1}) - bW_i + p(u_i - u_{i-1}),
\]

\[
\frac{\partial W_i}{\partial t} = c V_i - \delta W_i.
\]

The periodic boundary condition \( V_0(t) = V_n(t), V_{n+1}(t) = V_1(t) \) holds due to (1.2).
Here is the main result on the feedback synchronization of the proposed FitzHugh-Nagumo cellular neural networks.

**Theorem 3.1.** If the following threshold condition for the boundary gap signal of the FitzHugh-Nagumo cellular neural network \((1.1)-(1.3)\) is satisfied,

\[
\liminf_{t \to \infty} (x_n(t) - x_1(t))^2 > \left(1 + \frac{1}{p} (\delta + \gamma + |c - b|)\right) Q,
\]

where the constant \(Q > 0\) is given in \((2.12)\), then this cellular neural network is asymptotically synchronized in the space \(H\) at a uniform exponential rate.

**Proof.** Multiply the first equation in \((3.1)\) by \(V_i(t)\) and the second equation in \((3.1)\) by \(W_i(t)\). Then sum them up for all \(1 \leq i \leq n\) and use the Assumption \((1.5)\) to get

\[
\frac{1}{2} \frac{d}{dt} \sum_{i=1}^{n} (|V_i|^2 + |W_i|^2) - \sum_{i=1}^{n} a(V_{i-1} - 2V_i + V_{i+1})V_i
\]

\[
= \sum_{i=1}^{n} [(f(x_i) - f(x_{i-1}))V_i + (c - b)V_i W_i - \delta |W_i|^2 + p(u_i - u_{i-1})V_i]
\]

\[
\leq \sum_{i=1}^{n} \left[ f'(\xi x_i + (1 - \xi)x_{i-1})V_i^2 + (c - b)V_i W_i - \delta |W_i|^2 + p(u_i - u_{i-1})V_i \right]
\]

\[
\leq \sum_{i=1}^{n} \left[ |V_i|^2 + |c - b||(|V_i|^2 + |W_i|^2) - \delta |W_i|^2 + p(u_i - u_{i-1})V_i \right],
\]

where \(0 \leq \xi \leq 1\). By the periodic boundary condition for the differencing equations \((3.1)\) we have

\[
- \sum_{i=1}^{n} a(V_{i-1} - 2V_i + V_{i+1})V_i = - \sum_{i=1}^{n} a(V_{i+1} - V_i)V_i + \sum_{i=1}^{n} a(V_i - V_{i-1})V_i
\]

\[
= - \left[ \sum_{i=1}^{n} a(V_{i+1} - V_i)V_i - \sum_{i=2}^{n} a(V_i - V_{i-1})V_i \right] = - a(V_{n+1} - V_n) V_n + a(V_1 - V_0) V_1
\]

\[
= \sum_{i=2}^{n} a(V_{i} - V_{i-1})^2 + a(V_1^2 + V_n^2) - a(V_{n+1} V_n + V_0 V_1)
\]

\[
= \sum_{i=2}^{n} a(V_{i} - V_{i-1})^2 + a(V_1^2 + V_0^2) - 2aV_1 V_0
\]

\[
= \sum_{i=2}^{n} a(V_{i} - V_{i-1})^2 + a(V_1^2 - V_0^2) = \sum_{i=1}^{n} a(V_{i} - V_{i-1})^2 \geq 0.
\]
From the above two inequalities, we obtain
\[
\frac{1}{2} \frac{d}{dt} \sum_{i=1}^{n} (|V_i(t)|^2 + |W_i(t)|^2) \leq \sum_{i=1}^{n} \left[ \gamma |V_i|^2 + |c - b|(|V_i|^2 + |W_i|^2) - \delta |W_i|^2 + p(u_i - u_{i-1})V_i \right].
\] (3.4)

The boundary feedback (1.3) and the periodic boundary condition (1.2) infer that
\[
\sum_{i=1}^{n} p(u_i - u_{i-1})V_i = \sum_{i=1}^{n} p(u_i - u_{i-1})(x_i - x_{i-1})
\]
\[
= p \left[ (u_1 - u_0)(x_1 - x_0) + (u_2 - u_1)(x_2 - x_1) + (u_n - u_{n-1})(x_n - x_{n-1}) \right]
\]
\[
= p \left[ (u_1 - u_n)(x_1 - x_n) - u_1(x_2 - x_1) + u_n(x_n - x_{n-1}) \right]
\]
\[
= p \left[ 2(x_n - x_1)(x_1 - x_n) - (x_n - x_1)(x_2 - x_1) + (x_1 - x_n)(x_n - x_{n-1}) \right]
\]
\[
= p \left[ -2(x_n - x_1)^2 + (x_n - x_1)(x_1 - x_2 + x_{n-1} - x_n) \right]
\]
\[
= p \left[ -3(x_n - x_1)^2 + (x_n - x_1)(x_{n-1} - x_2) \right]
\]
\[
\leq p \left[ -2(x_n - x_1)^2 + (x_{n-1} - x_2)^2 \right].
\] (3.5)

Substitute (3.5) into (3.4). Then we get the following differential inequality
\[
\frac{d}{dt} \sum_{i=1}^{n} (|V_i(t)|^2 + |W_i(t)|^2) + 4p(x_n(t) - x_1(t))^2 \leq \sum_{i=1}^{n} \left[ 2\gamma |V_i|^2 + |c - b|(|V_i|^2 + |W_i|^2) - \delta |W_i|^2 \right] + 2p(x_{n-1}(t) - x_2(t))^2.
\]

Hence it holds that
\[
\frac{d}{dt} \sum_{i=1}^{n} (|V_i(t)|^2 + |W_i(t)|^2) + 2\delta \sum_{i=1}^{n} (|V_i(t)|^2 + |W_i(t)|^2) + 4p(x_n(t) - x_1(t))^2 \leq \sum_{i=1}^{n} \left[ (\delta + \gamma)|V_i(t)|^2 + |c - b|(|V_i|^2 + |W_i|^2) \right] + 2p(x_{n-1}(t) - x_2(t))^2,
\] (3.6)
for $t > 0$. Note that (2.13) in Theorem 2.2 confirms that for all solutions of (1.1),
\[
\limsup_{t \to \infty} \sum_{i=1}^{n} (|x_i(t, x_i^0)|^2 + |y_i(t, y_i^0)|^2) < Q.
\]
Thus for any given bounded set \( B \subset H \) and any initial data \(((x_1^0, y_1^0), \ldots, (x_n^0, y_n^0)) \in B\), there is a finite time \( T_B \geq 0 \) such that

\[
\sum_{i=1}^{n} 2 \left[ (\delta + \gamma)|V_i(t)|^2 + |c - b|(|V_i|^2 + |W_i|^2) \right] + 2p(x_{n-1}(t) - x_2(t))^2 < 4(\delta + \gamma + |c - b|)Q + 4pQ = 4(\delta + \gamma + |c - b| + p)Q, \quad \text{for } t \geq T_B.
\] (3.7)

Combining (3.6) and (3.7), we have shown that

\[
\frac{d}{dt} \sum_{i=1}^{n} (|V_i(t)|^2 + |W_i(t)|^2) + 2\delta \sum_{i=1}^{n} (|V_i(t)|^2 + |W_i(t)|^2) + 4p(x_n(t) - x_1(t))^2 < 4(\delta + \gamma + |c - b| + p)Q, \quad \text{for } t \geq T_B.
\] (3.8)

Under the threshold condition (3.2) of this theorem, for any given initial state \((x^0, y^0) = ((x_1^0, y_1^0), \ldots, (x_n^0, y_n^0)) \in H\) as a set \( B \) of single point, there exists a finite time \( T_{(x^0, y^0)} > 0 \) such that the differential inequality (3.8) holds for \( t > T_{(x^0, y^0)} \) and

\[
\sum_{i=1}^{n} (|x_i(t, x^0_i)|^2 + |y_i(t, y^0_i)|^2) < Q, \quad \text{for } t \geq T_{(x^0, y^0)}.
\]

Moreover,

\[
(x_n(t) - x_1(t))^2 > \left(1 + \frac{1}{p}(\delta + \gamma + |c - b|)\right)Q, \quad \text{for } t \geq T_{(x^0, y^0)},
\]

so that

\[
p(x_n(t) - x_1(t))^2 > (\delta + \gamma + |c - b| + p)Q, \quad \text{for } t \geq T_{(x^0, y^0)}.
\] (3.9)

It follows from (3.8) and (3.9) that

\[
\frac{d}{dt} \sum_{i=1}^{n} (|V_i(t)|^2 + |W_i(t)|^2) + 2\delta \sum_{i=1}^{n} (|V_i(t)|^2 + |W_i(t)|^2) < 0, \quad \text{for } t \geq T_{(x^0, y^0)}.
\] (3.10)

Finally, the Gronwall inequality applied to (3.10) shows that

\[
\sum_{i=1}^{n} (|V_i(t)|^2 + |W_i(t)|^2) \leq e^{-\delta(t-T_{(x^0, y^0)})} \sum_{i=1}^{n} (|V_i(T_{(x^0, y^0)})|^2 + |W_i(T_{(x^0, y^0)})|^2)
\]

\[
\leq 2e^{-\delta(t-T_{(x^0, y^0)})} Q \rightarrow 0, \quad \text{as } t \to \infty.
\] (3.11)

Then it is proved that for all solutions of the problem (1.1)-(1.3) for this FitzHugh-Nagumo CNN with the boundary feedback,

\[
\lim_{t \to \infty} \sum_{i=1}^{n} (|(x_i(t, x^0_i) - x_{i-1}(t, x^0_{i-1}))|^2 + |(y_i(t, y^0_i) - y_{i-1}(t, y^0_{i-1}))|^2) = 0.
\] (3.12)
This FHN cellular neural network with boundary feedback is asymptotically synchronized in the space $H$ at a uniform exponential rate. □

This result provides a sufficient condition for feedback synchronization of the FitzHugh-Nagumo complex neural networks with boundary control. The threshold condition 3.2 needs to be satisfied by the boundary gap signal $\liminf_{t \to \infty} (x_n(t) - x_1(t))^2$ between the two boundary cells. And the threshold in 3.2 is adjustable by the designed feedback coefficient $p$ in applications.

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