A new algorithm for solving the $rSUM$ problem

A determined algorithm is presented for solving the $rSUM$ problem for any natural $r$ with a sub-quadratic assessment of time complexity in some cases. In terms of an amount of memory used the obtained algorithm is the $n \log^3 n$ order.

§ 1. Introduction

In computational complexity theory, the $3SUM$ problem asks if a given set of $n$ integers, each with absolute value bounded by some polynomial in $n$, contains three elements that sum to zero. [1, 2]. The generalized version, $rSUM$, asks the same question for $r$ elements. [1, 2].

The $3SUM$ problem was initially set in [1]. Gajentaan and Overmars collected a large list of geometric problems, which may be solved in an order of quadratic complexity, and nobody knows, how to do it faster [1].

Hereinafter, we understand the order of complexity as asymptotic complexity of the algorithm, namely: the computational complexity (number of operations) of a given algorithm is bounded from above with function $f(n)$ (which is the order of complexity) with accuracy to the constant multiplier and for the sufficiently large input length $n$.

The $3SUM$ problem has a simple and obvious algorithm for solving in the order of $n^2$ operations [1, 2].

There are a probabilistic, sub-quadratic algorithms [3] in the computational model, which implies parallel memory operation.

A determined algorithm of solving the $3SUM$ problem based on the Fast Fourier Transformation was suggested in [4]. However it assumes that absolute values of these $n$ numbers are limited by the number $\frac{n^2}{\log n}$.

There are a algorithms based on sorting with partial information [5].

A solution to the generalized version of the problem, $rSUM$, may be found in [2]. Its known order of complexity is $n^{\frac{3}{2}}$ (the "meet-in-the-middle" algorithm).

The paper suggests a determined algorithm of solving the $rSUM$ problem for any $r \in \mathbb{N}$, which is of the order of $n \log^3 n$ in terms of the amount of memory used, with computational complexity of the sub-quadratic order in some cases.

The idea of the obtained algorithm is based not considering integer numbers, but rather $k \in \mathbb{N}$ successive bits of these numbers in the binary numeration system. It is shown that if a sum of integer numbers is equal to zero, then the sum of numbers presented by any $k$ successive bits of these numbers must be sufficiently "close" (see Lemma 2, 3) to zero. This makes it possible to discard the numbers, which a fortiori, do not establish the solution.
§ 2. Algorithm for solving the rSUM problem

Hereinafter, \(|y|\) designates an absolute value of integer number \(y\), \([y]\) is the smallest integer greater than or equal to \(y\), \([y]\) is the smallest integer smaller than or equal to \(y\). A mapping \(\text{sign}(y)\) returns the sign of integer \(y\) (it returns zero for zero).

Introduce mapping \(P^k_j : \mathbb{Z} \mapsto \mathbb{F}_2^k\) for any \(k \in \mathbb{N}\) and \(j \in \mathbb{N} \cup \{0\}\) as follows:

\[
P^k_j(z) = \text{sign}(z)z_j, \quad \forall z = \sum_{i=0}^{\infty} z_i 2^{ik} \in \mathbb{Z},
\]

i.e. \(j\) digit of integer \(z\) in a numeral system with base \(2^k\).

Given: set \(\Omega\) of \(n\) integer numbers, \(m\) is the degree of a polynomial, which bounds the maximum absolute value of input numbers (\(n^m = 2^m \log_2 n\)).

**Algorithm 1.**
1) From among the numbers in question, find \(\zeta\), which is the maximum in terms of its absolute value. Calculate \(l = \lceil \log_2(\zeta) \rceil\).
2) In a cycle on \(j\) from 0 to \(\lfloor \log_2 r \rfloor\) perform the following:
   2.1) Consider the numbers in \(\Omega\) upon application of \(P^{3\lceil \log_2 r \rceil}_j\) and set them down in array \(\Phi_j\) so that the number of identical elements would not exceed \(r\).
   With each \(\gamma \in \Phi_j\) group such ordinals of elements in \(\Omega\), where numbers with such ordinals in \(\Omega\) and only these numbers would be equal to \(\gamma\) after using of \(P^{3\lceil \log_2 r \rceil}_j\). We associate it with table \(\Pi_j\).
   Brute force to find all \(y_1 \in \Phi_j\), where \(\exists y_2, y_3, \ldots, y_r \in \Phi_j\):

\[
|\sum_{i=1}^r P^{3\lceil \log_2 r \rceil}_j(y_i)| < r \mod 2^{3\lceil \log_2 r \rceil},
\]

for \(j = 0\), strict comparison to zero must be performed.

The gotten \(r\)-tuples, namely, their ordinals in \(\Phi_j\), are to be set down in \(\Upsilon_j\).
3) Return \(\Upsilon = \{ \Upsilon_j \}\) and \(\Pi = \{ \Pi_j \}\).

**Algorithm 2. Algorithm for solving the rSUM problem**
1) Perform Algorithm 1: \(\Upsilon^1, \Pi^1\).
2) Shift the elements of \(\Omega\) cyclically by \(\lceil \log_2 r \rceil\) bits to the right, that the sign bit is retained for all numbers.
3) Perform Algorithm 1 on conditions that for \(j = 0\) inequality must be performed rather than comparison, and assume the last \(\lceil \log_2 r \rceil\) bits of numbers from \(\Omega\) to be zero bits: \(\Upsilon^2, \Pi^2\).
4) Shift the elements of \(\Omega\) cyclically by \(\lceil \log_2 r \rceil\) bits to the right, that the sign bit is retained for all numbers.
5) Perform Algorithm 1 on conditions that for \(j = 0\) inequality must be performed rather than comparison, and assume the last \(2\lceil \log_2 r \rceil\) bits of numbers from \(\Omega\) to be zero bits: \(\Upsilon^3, \Pi^3\).
6) Shift the elements of \(\Omega\) cyclically by \(2\lceil \log_2 r \rceil\) bits to the left, that the sign bit is retained for all numbers.
7) Return \(\bigcap_{i,j} \Upsilon^i_j\) relative to elements of \(\Omega\).
We are now to prove that the presented algorithms are correct.

**Lemma 1.** For any \( y_i \in \mathbb{Z}, i = 1, \ldots, r, \) it is true that:

1) if \( \sum_{i=1}^{r} y_i = 0, \) then \( \sum_{i=1}^{r} y_i \equiv 0 \mod{2^k}, \) where \( k \in \mathbb{N}. \)

2) if \( \sum_{i=1}^{r} y_i \equiv 0 \mod{2^t}, \) then \( \sum_{i=1}^{r} y_i = 0. \)

**Proof.** Obvious. This forms the basis of computer algebra.

The second statement is right because of \( \sum_{i=1}^{r} y_i \equiv 0 \mod{2^t} = r2^t. \)

**Lemma 2.** For any \( y_i \in \mathbb{Z}, i = 1, \ldots, r, \) it is true that:

if \( \sum_{i=1}^{r} y_i = 0, \) then \( |\sum_{i=1}^{r} y_i| < r \mod{2^k}, \)

\( j = 0, \ldots, \lfloor \frac{r}{k} \rfloor, \) \( l = \max(\lfloor \log_2(y_i) \rfloor + \lfloor \log_2 r \rfloor), \) \( k > \lfloor \log_2 r \rfloor \in \mathbb{N}. \)

**Proof.** For \( j = 0 \) the condition of Lemma 2 is met by virtue of Lemma 1.

Assume the opposite meaning that for a value \( j = s, \) for some \( r \) numbers meeting the condition of Lemma 2, the required inequality is wrong. At the same time, by virtue of Lemma 1:

\[ \sum_{i=1}^{r} y_i \equiv 0 \mod{2^{sk}}. \]

Present each \( y_i \mod{2^{sk}} \) as a sum of the value \( P^k_s (\text{the last } k \text{ bits of numbers } \text{sign}(y_i)(|y_i| \mod{2^{sk}})) \) and the residue by module \( 2^{(s-1)k}, \) then

\[ 2^{(s-1)k} \sum_{i=1}^{r} P^k_s (y_i) \equiv -\left( \sum_{i=1}^{r} \text{sign}(y_i)(|y_i| \mod{2^{(s-1)k}}) \right) \equiv \delta 2^{(s-1)k} \mod{2^{sk}}, \]

where \( |\delta| < r, \) as the sum of \( r \) numbers, the absolute value of which is smaller than \( 2^{j} \) for a natural \( j, \) cannot exceed \( r2^{j} - r. \) Besides, we know from Lemma 1 that \( \sum_{i=1}^{r} y_i \equiv 0 \mod{2^{(s-1)k}}. \) From here, we obtain the required.

**Lemma 3.** For any \( y_i \in \mathbb{Z}, i = 1, \ldots, r, \) it is true that:

if \( \sum_{i=1}^{r} y_i = 0, \) then for \( \tilde{y}_i, \) the inequality \( \sum_{i=1}^{r} P^k_j(\tilde{y}_i) < r \mod{2^k} \) is true, where \( \tilde{y}_i \)

is obtained from \( y_i \) by arithmetic shift to the right by \( t \) bits.

\( t, k > \lfloor \log_2 r \rfloor \) are any natural numbers, and \( j \) is any non-negative integer.

**Proof.**

\[ 2^{t+k(j-1)} \sum_{i=1}^{r} P^k_j(\tilde{y}_i) \equiv -\left( \sum_{i=1}^{r} \text{sign}(y_i)(|y_i| \mod{2^{t+k(j-1)}}) \right) \mod{2^{t+kj}}. \]

Further on, the proof totally replicates the proof of Lemma 2.

**Theorem 1.** Algorithm 2 will issue the solution of the \( rSUM \) problem.

**Proof.** As follows from Lemmas 1, 2, 3, if there exists a solution of the \( rSUM \) problem then, after execution of Algorithm 2, and even more so after execution of Algorithm 1, these numbers will stay within \( \Omega. \)
The cycle on \( j \) in Algorithm 1 finishes at iteration \( \left\lfloor \frac{j+\lceil \log_2 r \rceil}{3\lceil \log_2 r \rceil} \right\rfloor \) by virtue of the second if-clause in Lemma 1.

After step 1), for each \( y_1, y_2, \ldots, y_r \in \Omega \) takes place \( | \sum_{i=1}^{r} P^3[y] \left( y_i \right) | < r \mod 2^{3\lceil \log_2 r \rceil} \)
for any \( j \) under consideration, for \( j = 0 \) comparison to zero is performed.

It is about the numbers as such, not some values of \( P_j^3[y] \) of various numbers at each step on \( j \); this is why we remembered ordinals in \( r \)-tuples for — to coincide at each step of cycle \( j \).

Hence
\[
\sum_{i=1}^{r} y_i = \sum_{i=1}^{\left\lfloor \frac{j+\lceil \log_2 r \rceil}{3\lceil \log_2 r \rceil} \right\rfloor} z_i2^{3i\lceil \log_2 r \rceil}, \text{ where } |z_i| < 2r - 1,
\]
as, considering \( y_i \) after using of \( P_j^3[y] \), we may lose in \( \sum_{i=1}^{r} P_j^3[y_j] \) \( r - 1 \) carry bits by absolute value relative to the sum \( P_j^3[y_j] \left( \sum_{i=1}^{r} y_i \right) \) (see the proof in Lemma 2); besides, the very inequality from Lemma 2 makes it possible to differentiate from zero by absolute value to \( r - 1 \).

Yet, at step 3), the sum \( P_j^3[y_j] \) of \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_r \), where \( \tilde{y}_i \) is \( y_i \) at step 2) cyclically shifted to the right by \( \lceil \log_2 r \rceil \), will not meet the necessary inequality for module \( 2^{3\lceil \log_2 r \rceil} \) (see Lemma 3) for the first \( j : z_j = 0 \), if \( z_j < r \), as in the latter case, this \( z_j \) will not be constituted by the least significant \( \lceil \log_2 r \rceil \) bits of a \( 3\lceil \log_2 r \rceil \)-bit number in the binary numeral system, but by more significant bits, which is determined by the fact that
\[
\sum_{i=1}^{r} \tilde{y}_i = t + \sum_{i=1}^{\left\lfloor \frac{j+\lceil \log_2 r \rceil}{3\lceil \log_2 r \rceil} \right\rfloor} z_i2^{3i\lceil \log_2 r \rceil} - \lceil \log_2 r \rceil, \text{ where } |t| < r.
\]

The correctness of this presentation of the sum \( \tilde{y}_i \) follows from ideas presented in Lemmas 2, 3, as, with a cyclic shift of numbers \( y_i \), we may lose \( r - 1 \) carry bits by absolute value.

At step 5) we will exclude these \( y_1, \ldots, y_r \), if the first \( z_j \neq 0 \) is larger than \( r - 1 \), for the same considerations.

**§ 3. Computational complexity of suggested algorithm**

**Lemma 4.** Algorithm’s 1 order of complexity is \( n \log n \).

**Proof.** Calculating the maximum element by absolute value is \( n \) operations. Applying \( P_j^3[y] \) to elements of \( \Omega \) is no more than \( 2n \) operations (taking in modulus and cyclic shift). Adding the obtained values to \( \Phi_j \) after applying of \( P_j^3[y] \), containing no more \( r \) identical elements, using insertion sort with binary search, is not more than \( n(r2^{3\lceil \log_2 r \rceil} + 4\lceil \log_2 r \rceil) \) operations, where we use \( 4\lceil \log_2 r \rceil \) to assess the complexity of binary search, \( r2^{3\lceil \log_2 r \rceil} \) is the number of shifts of elements in an array for insertion to a proper place.

At step 2.1) we solve the \( rSUM \) problem by modulus \( 2^{3\lceil \log_2 r \rceil} \) for a quantity of different numbers not exceeding \( r2^{3\lceil \log_2 r \rceil} \), though there may be more than
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one solution. The exhaustive enumeration of all the variants requires $r^r 2^{3r \lceil \log_2 r \rceil}$ operations.

All the above-calculated was a single iteration on cycle of $j$.

As $l = m \lceil \log_2 n \rceil + \lceil \log_2 r \rceil$ and $r$, $m$ are fixed numbers, we obtain the required assessment.

Remark 1. It is convenient to assume that each element in the $r$-tuple from $\Upsilon_j$ (where elements of the $r$-tuple are ordinals of elements in $\Phi_j$, as determined by us) is a column of such ordinals of elements in $\Omega$, that the numbers corresponding to these ordinals in $\Omega$ upon application of $P_j^{3\lceil \log_2 r \rceil}$ will be equal to an element with this ordinal. We may assume so, because we have a table of association of the elements in $\Phi_j$ with elements in $\Omega$.

Theorem 2. Algorithm's 2 order of complexity is sub-quadratic for some cases.

Proof. All steps of the Algorithm 2 except step 7) do not exceed the $n \log n$ order (see Lemma 4).

How to compute $\bigcap_{i,j} \Upsilon^i_j$ relative to elements of $\Omega$?

All $r$-tuples from $\Upsilon^i_j$ are tables, see Remark 1.

$\Upsilon^i_j$ contains no more $2^{r}r^{2^{3\lceil \log_2 r \rceil}(r-1)}$ items. Comparing a $r$-tuple with another according to ordinals in $\Omega$ will not make more than $rn \log n$ operations. Consider $\log_2 n$ as elements in $\Omega$ are read successively, and hence, ordinals of elements of $\Omega$, related to an element of $\Phi_j$, are set down in an orderly way, which means that we may use binary search. Every time we create new $r$-tuple with common ordinals of $\Omega$ in columns in one $r$-tuple and the other, if there is at least one common element in each column.

As cycle $j$ ends $\lceil m \lceil \log_2 n \rceil \rceil / 3 \lceil \log_2 r \rceil$ in Algorithm 1 and there are 3 execution of Algorithm 1 in Algorithm 2, we get upper bound of vertices of such comparing $r$-tuples tree:

$$(2r!r^{2^{3(r-1)\lceil \log_2 r \rceil}})^\lceil m \lceil \log_2 n \rceil \rceil / 3 \lceil \log_2 r \rceil \rceil.$$

It’s a lot, that’s why we compute $\Gamma_s = \bigcap_{i,j=sh,...,(s+1)h-1} \Upsilon^i_j$, where $i = 1, 2, 3$, $h = \lceil \log_2 \log_2 n \rceil / 9r \lceil \log_2 r \rceil$, $s = 0, \ldots, \lceil m \lceil \log_2 n \rceil \rceil / 3h \lceil \log_2 r \rceil$.

Cardinality of $\Gamma_s$ is less than

$$(2r!r^{2^{3\lceil \log_2 r \rceil}(r-1)\lceil \log_2 \log_2 n \rceil / 3 \lceil \log_2 r \rceil \rceil} \leq \log_2 n.$$

So, the order of complexity of the computation of all $\Gamma_s$ is less than $n \log_3^2 n$.

Find $\lceil \log_3 \lceil \log_2 n \rceil / 3 \rceil \rceil$ sets $\Gamma_s$ with the smallest number of elements (it is of the order of $n \log n$ operation) and compute confluence of them $\Theta$ (it is of the order of $n \log_3 n$ operations).

To count the quantity of all variants produced by each $r$-tuple from $\Theta$, relative to elements of $\Omega$, takes no more than $2r n \log_2 n$ operations (amount of options generated by fixed $r$-tuple is the product of the number of items in a columns of this $r$-tuple).
If the total number of r-tuples from $\Theta$, relative to elements of $\Omega$, is less than $n^{\frac{3r}{2}}$, we get sub-quadratic time for our algorithm (brute force all of variants).

If the total number of r-tuples from $\Theta$, relative to elements of $\Omega$, is less than $\frac{n^{\frac{3r}{2}}}{\log \frac{1}{r} n}$, brute force still would be faster than using known algorithms.

**Theorem 3.** Algorithm 2 requires an amount of memory of an order $n \log^3 n$ relative to storage of integers.

**Proof.** As will readily be observed, the most memory-consuming step is 7).
Step 7) of Algorithm 2 requires some memory for $\Upsilon^i_j$ (constant quantity) and $\Pi^i_j$ associating elements in $\Upsilon^i_j$ with elements in $\Omega$ (not more than the order of $n$), $i = 1, 2, 3$, $j = 0, \ldots, m \log n + \log r$.
All together $\Gamma_s$ require the order of $n \log^3 n$ memory, see Theorem 2.

**Remark 2.** What is it about the constant in asymptotic complexity?
As follows from Theorem 2 and Lemma 4 the constant would not exceed $3mr^{4r}$.

**Remark 3.** As time and memory complexity of suggested algorithm is of the sub-quadratic order, it seems to be useful to perform it at the beginning of any other known algorithm.

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