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Pseudorandomness of the Ostrowski sum-of-digits function

par Lukas SPIEGELHOFER

Résumé. Pour un nombre irrationnel $\alpha \in (0, 1)$, nous étudions la fonction somme des chiffres d’Ostrowski $\sigma_\alpha$. Étant donné un nombre $\alpha$ à quotients partiels bornés et un nombre $\vartheta \in \mathbb{R} \setminus \mathbb{Z}$, nous montrons que la fonction $g : n \mapsto e(\vartheta \sigma_\alpha(n))$, où $e(x) = e^{2\pi i x}$, est pseudo-à-léatoire dans le sens suivant : pour tout $r \in \mathbb{N}$ la limite

$$\gamma_r = \lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq n < N} g(n + r)\overline{g(n)}$$

existe et on a

$$\lim_{R \to \infty} \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r|^2 = 0.$$

Abstract. For an irrational $\alpha \in (0, 1)$, we investigate the Ostrowski sum-of-digits function $\sigma_\alpha$. For $\alpha$ having bounded partial quotients and $\vartheta \in \mathbb{R} \setminus \mathbb{Z}$, we prove that the function $g : n \mapsto e(\vartheta \sigma_\alpha(n))$, where $e(x) = e^{2\pi i x}$, is pseudorandom in the following sense: for all $r \in \mathbb{N}$ the limit

$$\gamma_r = \lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq n < N} g(n + r)\overline{g(n)}$$

exists and we have

$$\lim_{R \to \infty} \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r|^2 = 0.$$

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1. Introduction and main results

Let $g$ be an arithmetical function. The set of $\beta \in [0, 1)$ satisfying

$$\limsup_{N \to \infty} \frac{1}{N} \left| \sum_{0 \leq n < N} g(n) e(-n\beta) \right| > 0$$

is called the Fourier–Bohr spectrum of $g$.

The function $g$ is called pseudorandom in the sense of Bertrandias [4] or simply pseudorandom if the limit

$$\gamma_r = \lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq n < N} g(n + r) \overline{g(n)}$$

exists for all $r \geq 0$ and is zero in quadratic mean, that is,

$$\lim_{R \to \infty} \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r|^2 = 0.$$ 

Pseudorandomness can be understood as a property of the spectral measure associated to $g$: Assume that the correlation $\gamma_r$ of $g$ exists for all $r \geq 0$. By the Bochner representation theorem there exists a unique measure $\mu$ on the Torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ such that

$$\gamma_r = \int_{\mathbb{T}} e(rx) \, d\mu(x)$$

for all $r$. Then $g$ is pseudorandom if and only if the discrete component of $\mu$ vanishes. We refer to [9] for more details.

It is known that pseudorandomness of a bounded arithmetic function $g$ implies that the spectrum of $g$ is empty, which can be proved using van der Corput’s inequality. For the convenience of the reader, we give a proof of this fact in Section 2.

The converse of this statement does not always hold. However, it is true for $q$-multiplicative functions $g : \mathbb{N} \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, which has been proved by Coquet [5, 6, 7]. Here a function $g : \mathbb{N} \to \mathbb{C}$ is called $q$-multiplicative if $f(q^k n + b) = f(q^k n) f(b)$ for all integers $k, n > 0$ and $0 \leq b < q^k$.

The purpose of this paper is to prove an analogous statement for the Ostrowski numeration system, that is, for $\alpha$-multiplicative functions. Assume that $\alpha \in (0, 1)$ is irrational. The Ostrowski numeration system has as its scale of numeration the sequence of denominators of the convergents of the regular continued fraction expansion of $\alpha$. More precisely, let $\alpha = [0; a_1, a_2, \ldots]$ be the continued fraction expansion of $\alpha$ and $p_i/q_i = \ldots$
[0; \alpha_1, \ldots, \alpha_i]$ the $i$-th convergent to $\alpha$, where $i \geq 0$. By the greedy algorithm, every nonnegative integer $n$ has a representation

$$n = \sum_{k \geq 0} \varepsilon_k q_k$$

such that

$$\sum_{0 \leq k < K} \varepsilon_k q_k < q K$$

for all $K \geq 0$. This algorithm yields the unique expansion of the form (1.1) having the properties that $0 \leq \varepsilon_0 < \alpha_1$, $0 \leq \varepsilon_k \leq \alpha_{k+1}$ and $\varepsilon_k = \alpha_{k+1} \Rightarrow \varepsilon_{k-1} = 0$ for $k \geq 1$, the Ostrowski expansion of $n$.

For a nonnegative integer $n$ let $(\varepsilon_k(n))_{k \geq 0}$ be its Ostrowski expansion. An arithmetic function $f$ is $\alpha$-additive resp. $\alpha$-multiplicative if

$$f(n) = \sum_{k \geq 0} f(\varepsilon_k(n) q_k) \quad \text{resp.} \quad f(n) = \prod_{k \geq 0} f(\varepsilon_k(n) q_k)$$

for all $n$. Examples of $\alpha$-additive functions are the functions $n \mapsto \beta n$ (for $\beta \in \mathbb{R}$) and the $\alpha$-sum of digits of $n$ [8]:

$$\sigma_\alpha(n) = \sum_{i \geq 0} \varepsilon_i(n).$$

We refer the reader to [3] for a survey on the Ostrowski numeration system. In particular, we want to note that the Ostrowski numeration system is a useful tool for studying the discrepancy modulo $1$ of $n\alpha$-sequences, see for example the references contained in the aforementioned paper.

Moreover, see [2] for a dynamical viewpoint of the Ostrowski numeration system, see also [1, 12] for more general numeration systems.

Our main theorem establishes a connection between the Fourier–Bohr spectrum and pseudorandomness for $\alpha$-multiplicative functions.

**Theorem 1.1.** Assume that $g$ is a bounded $\alpha$-multiplicative function. The Fourier–Bohr spectrum of $g$ is empty if and only if $g$ is pseudorandom.

Using a theorem by Coquet, Rhin and Toffin [11, Theorem 2], we obtain the following corollary.

**Corollary 1.2.** Assume that $\alpha \in (0, 1)$ is irrational and has bounded partial quotients and $\vartheta \in \mathbb{R} \setminus \mathbb{Z}$. Then $n \mapsto e(\vartheta \sigma_\alpha(n))$ is pseudorandom.

In particular, this holds for the Zeckendorf sum-of-digits function, which corresponds to the case $\alpha = (\sqrt{5} - 1)/2 = [0; 1, 1, \ldots]$. This special case can be found in the author’s thesis [14].

We first present a series of auxiliary results, and proceed to the proof of Theorem 1.1 in Section 3.
2. Lemmas

We begin with the well-known inequality of van der Corput.

Lemma 2.1 (Van der Corput’s inequality). Let \( I \) be a finite interval in \( \mathbb{Z} \) and let \( a_n \in \mathbb{C} \) for \( n \in I \). Then

\[
\left| \sum_{n \in I} a_n \right|^2 \leq \frac{|I| - 1 + R}{R} \sum_{0 \leq |r| < R} \left( 1 - \frac{|r|}{R} \right) \sum_{n \in I} a_{n+r} \overline{a_n}
\]

for all integers \( R \geq 1 \).

In the definition of pseudorandomness for bounded arithmetic functions \( g \), we do not actually need the square.

Lemma 2.2. Let \( g \) be a bounded arithmetic function such that the correlation \( \gamma_r \) of \( g \) exists for all \( r \geq 0 \). The function \( g \) is pseudorandom if and only if

\[
\lim_{R \to \infty} \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r| = 0.
\]

For the proof of sufficiency we note that we may without loss of generality assume that \( |g| \leq 1 \). The other direction is an application of the Cauchy-Schwarz inequality.

As we noted before, pseudorandomness of \( g \) implies that the spectrum of \( g \) is empty.

Lemma 2.3. Let \( g \) be a bounded arithmetic function. If \( g \) is pseudorandom, then the Fourier–Bohr spectrum of \( g \) is empty.

Proof. The proof is an application of van der Corput’s inequality (Lemma 2.1). We have for all \( R \in \{1, \ldots, N\} \)

\[
\left| \frac{1}{N} \sum_{0 \leq n < N} g(n) e(n \beta) \right|^2 \leq \frac{N - 1 + R}{RN^2} \sum_{0 \leq |r| < R} \left( 1 - \frac{|r|}{R} \right) e(r \beta) \sum_{0 \leq n, n+r < N} g(n+r) \overline{g(n)}
\]

\[
\ll \frac{1}{R} \sum_{0 \leq r < R} \left| \frac{1}{N} \sum_{0 \leq n < N} g(n+r) \overline{g(n)} \right| + O \left( \frac{R}{N} \right).
\]

Let \( \varepsilon \in (0,1) \). By hypothesis and Lemma 2.2 we may choose \( R \) so large that

\[
\frac{1}{R} \sum_{0 \leq r < R} |\gamma_r| < \varepsilon^2.
\]
Moreover, we choose \( N_0 \) in such a way that \( R/N_0 < \varepsilon^2 \) and
\[
\left| \frac{1}{N} \sum_{0 \leq n < N} g(n + r) \overline{g(n)} - \gamma_r \right| < \varepsilon^2
\]
for all \( r < R \) and \( N \geq N_0 \). Then for \( N \geq N_0 \) we have
\[
\left| \frac{1}{N} \sum_{0 \leq n < N} g(n) e(n\beta) \right|^2 \\
\leq \frac{1}{R} \sum_{0 \leq r < R} \left| \frac{1}{N} \sum_{0 \leq n < N} g(n + r) \overline{g(n)} - \gamma_r \right| + \frac{1}{R} \sum_{0 \leq r < R} |\gamma_r| + O\left( \frac{R}{N_0} \right)
\leq \varepsilon^2. \quad \Box
\]

The following lemma is a generalization of Dini’s Theorem.

**Lemma 2.4.** Assume that \((f_i)_{i \geq 0}\) is a sequence of nonnegative continuous functions on \([0, 1]\) converging pointwise to the zero function. Assume that \( |f_{i+1}(x)| \leq \max\{|f_i(x)|, |f_{i-1}(x)|\} \) for \( x \in [0, 1] \). Then the convergence is uniform in \( x \).

**Proof.** For \( \varepsilon > 0 \) and nonnegative \( N \) we set
\[
A_N(\varepsilon) = \{x \in [0, 1] : f_N(x) < \varepsilon \text{ and } f_{N+1}(x) < \varepsilon\}.
\]
Note that this is an open set. By induction, using the property \( |f_{i+1}(x)| \leq \max\{|f_i(x)|, |f_{i-1}(x)|\} \), we obtain
\[
A_N(\varepsilon) = \{x \in [0, 1] : f_n(x) < \varepsilon \text{ for all } n \geq N\}.
\]
Trivially, we have \( A_N(\varepsilon) \subseteq A_{N+1}(\varepsilon) \). For each \( x \in [0, 1] \) there is an \( N(x) \) such that \( f_n(x) < \varepsilon \) for all \( n \geq N(x) \). Then \( x \in A_{N(x)}(\varepsilon) \), therefore \( (A_{N(x)}(\varepsilon))_{x \in [0, 1]} \) is an open cover of the compact set \([0, 1]\). Choose \( x_1, \ldots, x_k \) such that \( A_{N(x_1)}(\varepsilon) \cup \cdots \cup A_{N(x_k)}(\varepsilon) \supseteq [0, 1] \) and set \( N = \max\{N(x_1), \ldots, N(x_k)\} \). By monotonicity of the sets \( A_N(\varepsilon) \), we obtain \( A_N(\varepsilon) \supseteq [0, 1] \), in other words, \( f_n(x) < \varepsilon \) for all \( x \in [0, 1] \) and all \( n \geq N \). \( \Box \)

**Lemma 2.5.** Assume that \( \lambda \geq 1 \) and let \((w_i)_{i}\) be the increasing enumeration of the integers \( n \) such that \( \varepsilon_0(n) = \cdots = \varepsilon_{\lambda-1}(n) = 0 \). The intervals \([w_i, w_{i+1})\) constitute a partition of the set \( \mathbb{N} \) into intervals of length \( q_\lambda \) and \( q_{\lambda-1} \), where \( w_{i+1} - w_i = q_{\lambda-1} \) if and only if \( \varepsilon_\lambda(w_i) = a_{\lambda+1} \).

**Proof.** Assume first that \( \varepsilon_\lambda(w_i) = a_{\lambda+1} \). We want to show that \( w_{i+1} = w_i + q_{\lambda-1} \). Assume that \( \lambda \geq 2 \) and let \( w_i < n < w_i + q_{\lambda-1} \). Then the Ostrowski expansion of \( n \) is obtained by superposition of the expansions of \( w_i \) and of \( n - w_i \). In particular, for \( w_i < n < w_i + q_{\lambda-1} \) we have \( \varepsilon_j(n) \neq 0 \).
for some \( j < \lambda - 1 \). (Trivially, this also holds for \( \lambda = 1 \).) Moreover, in the addition \( w_i + q_{\lambda-1} \) a carry occurs, producing \( \varepsilon_j(w_i + q_{\lambda-1}) = 0 \) for \( j \leq \lambda \), therefore \( w_{i+1} = w_i + q_{\lambda-1} \). The case \( \varepsilon_{\lambda}(w_i) < a_{\lambda+1} \) is similar, in which case \( w_{i+1} = w_i + q_{\lambda} \).

For an \( \alpha \)-multiplicative function \( g \) and an integer \( \lambda \geq 0 \) we define a function \( g_\lambda \) by truncating the digital expansion: we define \( \psi_\lambda(n) = \sum_{i<\lambda} \varepsilon_i(n)q_i \) and

\[
g_\lambda(n) = g(\psi_\lambda(n)).
\]

We will need the following carry propagation lemma for the Ostrowski numeration system.

**Lemma 2.6.** Let \( \lambda \geq 1 \) be an integer and \( N, r \geq 0 \). Assume that \( \alpha \in (0,1) \) is irrational and let \( g \) be an \( \alpha \)-multiplicative function. Then

\[
(2.1) \quad \left| \left\{ n < N : g(n + r)g(n) \neq g_\lambda(n + r)g_\lambda(n) \right\} \right| \leq N \frac{r}{q_{\lambda-1}}.
\]

**Proof.** The statement we want to prove is trivial for \( r \geq q_{\lambda-1} \), we assume therefore that \( r < q_{\lambda-1} \). Let \( w \) be the family from Lemma 2.5. For \( w_i \leq n < w_{i+1} - r \), we have \( \varepsilon_j(n + r) = \varepsilon_j(n) \) for \( j \geq \lambda \). It follows that

\[
\left| \left\{ n \in \{w_i, \ldots, w_{i+1} - 1\} : g(n + r)g(n) \neq g_\lambda(n + r)g_\lambda(n) \right\} \right| \leq r.
\]

By concatenating blocks, the statement follows therefore for the case that \( N = w_i \) for some \( i \). It remains to treat the case that \( w_i < N < w_{i+1} \) for some \( i \). To this end, we denote by \( L(N) \) resp. \( R(N) \) the left hand side resp. the right hand side of (2.1). For \( w_i \leq N \leq w_{i+1} \) we have

\[
L(N) = \begin{cases} 
L(w_i), & N \leq w_{i+1} - r; \\
L(w_i) + N - (w_{i+1} - r), & N \geq w_{i+1} - r.
\end{cases}
\]

Note that \( L \) is a polygonal line that lies below \( R(N) \) for \( N \in \{w_i, w_{i+1} - r, w_{i+1}\} \) and therefore for all \( N \in [w_i, w_{i+1}] \). By concatenating blocks, the full statement follows.

We define Fourier coefficients for \( g \):

\[
G_\lambda(h) = \frac{1}{q_\lambda} \sum_{0 \leq u < q_\lambda} g(u) e(-huq_\lambda^{-1}).
\]

**Lemma 2.7.** Assume that \( i \) is such that \( w_{i+1} - w_i = q_\lambda \) and let \( r \geq 0 \). We have

\[
(2.2) \quad \sum_{h < q_\lambda} |G_\lambda(h)|^2 e(hrq_\lambda^{-1}) = \frac{1}{q_\lambda} \sum_{w_i \leq v < w_{i+1}} g_\lambda(v + r)g_\lambda(v) + O \left( \frac{r}{q_\lambda} \right).
\]
Proof.

\[
\sum_{0 \leq h < q} |G_{\lambda}(h)|^2 e(h r q^{-1})
\]

\[
= q^{-1} \sum_{0 \leq u, v < q} g_{\lambda}(u)g_{\lambda}(v) q^{-1} \sum_{0 \leq h < q} e\left(\frac{h}{q}(v + r - u)\right)
\]

\[
= \frac{1}{q} \sum_{0 \leq u, v < q} [v + r \equiv u \mod q] g_{\lambda}(u)\overline{g_{\lambda}(v)}
\]

\[
= \frac{1}{q} \sum_{w_1 \leq u, v < w_1 + 1} [v + r \equiv u \mod q] g_{\lambda}(u)\overline{g_{\lambda}(v)}
\]

\[
= \frac{1}{q} \sum_{w_1 \leq u < w_1 + 1 - r} g_{\lambda}(v + r)\overline{g_{\lambda}(v)} + O\left(\frac{r}{q}\right).
\]

Lemma 2.8. Let \( H \geq 1 \) be an integer and \( R \) a real number. For all real numbers \( t \) we have

\[
\sum_{h < H} \left| \frac{1}{R} \sum_{r < R} e(r(t + h/H)) \right|^2 \leq \frac{H + R - 1}{R}.
\]

This lemma is an immediate consequence of the analytic form of the large sieve, see [13, Theorem 3]. This form of the theorem, featuring the optimal constant \( N - 1 + \delta^{-1} \), is due to Selberg.

Lemma 2.9 (Selberg). Let \( N \geq 1, R \geq 1, M \) be integers, \( \alpha_1, \ldots, \alpha_R \in \mathbb{R} \) and \( a_{M+1}, \ldots, a_{M+N} \in \mathbb{C} \). Assume that \( \|\alpha_r - \alpha_s\| \geq \delta \) for \( r \neq s \). Then

\[
\sum_{r=1}^{R} \left| \sum_{n=M+1}^{M+N} a_n e(n \alpha_r) \right|^2 \leq (N - 1 + \delta^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2.
\]

As an important first step in the proof of Theorem 1.1, we show that for the functions in question we have the following uniformity property.

Proposition 2.10. Let \( g \) be a bounded \( \alpha \)-multiplicative function. Assume that the Fourier–Bohr spectrum of \( g \) is empty, that is,

\[
\left| \sum_{0 \leq n < N} g(n) e(-n \beta) \right| = o(N)
\]

as \( N \to \infty \) for all \( \beta \in \mathbb{R} \). Then

\[
\sup_{\beta \in \mathbb{R}} \left| \sum_{0 \leq n < N} g(n) e(-n \beta) \right| = o(N).
\]
Proof of Proposition 2.10. Without loss of generality we may assume that \(|g| \leq 1\), since the full statement follows by scaling. We first prove the special case
\[
\lim_{i \to \infty} \sup_{\beta \in \mathbb{R}} \frac{1}{q_i} \left| \sum_{0 \leq n < q_i} g(n) e(-n\beta) \right| = 0.
\]
We set \(h(n) = g(n) e(-n\beta)\) and
\[
S_i = S_i(\beta) = \frac{1}{q_i} \sum_{0 \leq n < q_i} h(n).
\]
For all \(i \geq 1\) we have
\[
S_{i+1} = \frac{1}{q_{i+1}} \sum_{0 \leq b < a_{i+1}} \sum_{0 \leq u < q_i} h(u + bq_i) + \frac{1}{q_{i+1}} \sum_{0 \leq u < q_{i-1}} h(u + a_{i+1}q_i)
\]
\[
= \frac{q_i}{q_{i+1}} \left( \sum_{0 \leq b < a_{i+1}} h(bq_i) \right) \cdot S_i + \frac{q_i-1}{q_{i+1}} h(a_{i+1}q_i) S_{i-1}.
\]
Using the recurrence for \(q_i\), it follows that \(|S_{i+1}| \leq \max\{|S_i|, |S_{i-1}|\}\). By Lemma 2.4 we obtain the statement.

We proceed to the general case. We consider partial sums of \(g(n) e(n\beta)\) up to \(N\). Assume that \(w_i \leq N < w_{i+1}\). We have
\[
\left| \sum_{0 \leq n < w_i} g(n) e(n\beta) \right|^2 \leq \sum_{0 \leq n < w_i} g(n) e(n\beta) \leq q_\lambda^2 + 2Nq_\lambda.
\]
We apply the inequality of van der Corput (Lemma 2.1) to obtain
\[
\left| \sum_{0 \leq n < w_i} g(n) e(n\beta) \right|^2 \leq \frac{N + R - 1}{R} \sum_{|r| < R} \left( 1 - \frac{|r|}{R} \right) e(r\beta) \sum_{0 \leq n, n+r < w_i} g(n+r) g(n).
\]
We adjust the summation range by omitting the condition \(0 \leq n + r < w_i\). This introduces an error term \(O(NR)\). Moreover, \(\alpha\)-multiplicative functions satisfy Lemma 2.6, therefore we may replace \(g\) by \(g_\lambda\) for the price of another error term, \(O(N^2 R q_{\lambda-1}^{-1})\). Using (2.2) we get
\[
\left| \sum_{0 \leq n < w_i} g(n) e(n\beta) \right|^2 \ll \frac{N}{R} \sum_{|r| < R} \left( 1 - \frac{|r|}{R} \right) e(r\beta) \left( \sum_{0 \leq n < w_i} g_\lambda(n+r) g_\lambda(n) + O\left( R + \frac{NR}{q_{\lambda-1}} \right) \right)
\]
\[
\ll NR + \frac{N^2 R}{q_{\lambda-1}} + \frac{N}{R} w_i \sum_{h < q_\lambda} |G_\lambda(h)|^2 \sum_{|r| < R} \left( 1 - \frac{|r|}{R} \right) e\left( r\left( \beta + \frac{h}{q_\lambda} \right) \right).
\]
Note that the sum over $r$ is a nonnegative real number. This follows from
the identity
$$
\sum_{|r|<R} (R - |r|) e(rx) = \left| \sum_{0 \leq r < R} e(rx) \right|^2,
$$
which can be proved by an elementary combinatorial argument. We use
this equation and collect the error terms to get

$$
\left( 1 - \frac{1}{N} \sum_{0 \leq n < N} g(n) e(n\beta) \right)^2 \leq \frac{q_\lambda^2}{N^2} + \frac{q_\lambda}{N} + \frac{R}{N} + \frac{R}{q_{\lambda-1}}
$$

which can be proved by an elementary combinatorial argument. We use
the special case proved before and choosing $R$ and $\lambda$ appropriately,
we obtain the statement.

In order to establish the existence of the correlation $\gamma_r$ of $g$ for all $r \geq 0$, we use the following theorem [10, Théorème 4]. (Note that we defined
$\psi_\lambda(n) = \sum_{0 \leq i < \lambda} \varepsilon_i(n)q_i$.)

**Lemma 2.11** (Coquet–Rhin–Toffin). Let $\lambda \geq 1$ and $a < q_\lambda$. The set
$E(\lambda, a) = \{ n \in \mathbb{N} : \psi_\lambda(n) = a \}$ possesses an asymptotic density given
by
$$
\delta = (q_\lambda + q_{\lambda-1}[0; a_{\lambda+1}, \ldots])^{-1} \quad \text{if } a \geq q_{\lambda-1};
$$
$$
\delta' = \delta(1 + [0; a_{\lambda+1}, \ldots]) \quad \text{if } a < q_{\lambda-1}.
$$

**Lemma 2.12.** Let $g$ be a bounded $\alpha$-multiplicative function. Then for every
$r \geq 0$ the limit
$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n < N} g(n + r)g(n)
$$
exists.

We note that the existence of the correlation was established in [10] for
the special case that $g(n) = e(y\sigma_\alpha(n))$, where $e(x) = e^{2\pi ix}$ and $y \in \mathbb{R}$. 
**Proof.** Let $\lambda, N \geq 0$ and $r \geq 1$ and set $k = \max\{j : w_j \leq N\}$. Moreover, let $a = a(N)$ be the number of indices $j < k$ such that $w_{j+1} - w_j = q\lambda$ and $b = b(N)$ be the number of indices $j < k$ such that $w_{j+1} - w_j = q\lambda - 1$. By Lemma 2.11 $a(N)/N$ and $b(N)/N$ converge, say to $A$ and $B$ respectively. Let $\lambda$ be so large that $r/q\lambda - 1 < \varepsilon$. Moreover, choose $N_0$ so large that 

$$
\left| A - \frac{a(N)}{N} \right| < \varepsilon q^{-1}\lambda - 1 \quad \text{and} \quad \left| B - \frac{b(N)}{N} \right| < \varepsilon q^{-1}\lambda - 1 \quad \text{and} \quad q\lambda/N < \varepsilon$$

for all $N \geq N_0$.

Then by Lemma 2.6 we get

$$\sum_{0 \leq n < N} g(n + r\lambda)g(n) = \sum_{0 \leq n < N} g(n + r)g(n) + O(Nrq^{-1})$$

therefore

$$\left| \frac{1}{N} \sum_{0 \leq n < N} g(n + r\lambda)g(n) - A \sum_{0 \leq n < q\lambda} g(n + r)g(n) - B \sum_{0 \leq n < q\lambda - 1} g(n + r)g(n) \right|$$

$$\ll \left| \frac{1}{N} \sum_{0 \leq n < N} g(n + r\lambda)g(n) - \frac{a}{N} \sum_{0 \leq n < q\lambda} g(n + r)g(n) \right| + 2\varepsilon$$

$$= \left| \frac{1}{N} \sum_{0 \leq n < N} g(n + r\lambda)g(n) - \frac{1}{N} \sum_{0 \leq n < w_k} g(n + r)g(n) \right| + 2\varepsilon$$

$$\ll \frac{q\lambda}{N} + \frac{r}{q\lambda - 1} + 2\varepsilon.$$

By the triangle inequality it follows that the values $\frac{1}{N} \sum_{n < N} g(n + r\lambda)g(n)$ form a Cauchy sequence and therefore a convergent sequence, which proves the existence of the correlation of $g$. □

### 3. Proof of the theorem

Now we are prepared to prove Theorem 1.1. If $g$ is pseudorandom, then by Lemma 2.3 its spectrum is empty. We are therefore concerned with the converse. Let $\ell \geq 0$. We denote by $a$ the number of $i < \ell$ such that $w_{i+1} - w_i = q\lambda$ and by $b$ the number of $i < \ell$ such that $w_{i+1} - w_i = q\lambda - 1$.

Choose $\varepsilon_r$ such that $|\varepsilon_r| = 1$ and

$$\varepsilon_r \sum_{0 \leq h < q\lambda} |G(\lambda(h))|^2 e(hrq^{-1})$$
is a nonnegative real number. Similarly choose $\varepsilon'_r$ for $\lambda - 1$. We have

$$
\frac{1}{R} \sum_{0 \leq r < R} \frac{1}{w_\ell} \sum_{0 \leq n < w_\ell} g_\lambda(n + r)g_\lambda(n) = \left| \frac{a q_\lambda}{w_\ell} \frac{1}{R} \sum_{0 \leq r < R} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 e\left(\frac{hr}{q_\lambda}\right) + \frac{b q_{\lambda-1}}{w_\ell} \sum_{0 \leq r < R} \sum_{0 \leq h < q_{\lambda-1}} |G_{\lambda-1}(h)|^2 e\left(\frac{hr}{q_{\lambda-1}}\right) \right| + O\left(\frac{a r + b r}{w_\ell} + \frac{b r}{w_\ell}\right)
$$

$$
= \frac{1}{R} \left| \frac{a q_\lambda}{w_\ell} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 \sum_{0 \leq r < R} \varepsilon_r e\left(\frac{hr}{q_\lambda}\right) + \frac{b q_{\lambda-1}}{w_\ell} \sum_{0 \leq h < q_{\lambda-1}} |G_{\lambda-1}(h)|^2 \sum_{0 \leq r < R} \varepsilon'_r e\left(\frac{hr}{q_{\lambda-1}}\right) \right| + O\left(\frac{r}{q_{\lambda-1}}\right)
$$

By Cauchy-Schwarz we obtain

$$
\frac{1}{R^2} \left| \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^2 \sum_{0 \leq r < R} \varepsilon_r e\left(\frac{hr}{q_\lambda}\right) \right|^2 \leq \frac{1}{R^2} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4 \sum_{0 \leq r < R} \sum_{0 \leq h < q_\lambda} \varepsilon_r e\left(\frac{hr}{q_\lambda}\right) \leq \frac{1}{R^2} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4 \sum_{0 \leq r_1, r_2 < R} \sum_{0 \leq h < q_{\lambda-1}} \varepsilon_{r_1} \varepsilon_{r_2} e\left(\frac{hr_1 - r_2}{q_{\lambda-1}}\right)
$$

$$
= \frac{q_\lambda}{R^2} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4 \sum_{0 \leq r_1, r_2 < R} \varepsilon_{r_1} \varepsilon_{r_2} \delta_{r_1, r_2}
$$

$$
= \frac{q_\lambda}{R} \sum_{0 \leq h < q_\lambda} |G_\lambda(h)|^4,
$$
similarly for $\lambda - 1$. Using Lemma 2.6, we get

$$
\frac{1}{R} \sum_{0 \leq r < R} |\gamma_r|
= \lim_{\ell \to \infty} \frac{1}{R} \sum_{0 \leq r < R} \frac{1}{w_\ell} \sum_{0 \leq n < w_\ell} g(n + r)\overline{g(n)}
= \lim_{k \to \infty} \frac{1}{R} \sum_{0 \leq r < R} \frac{1}{w_\ell} \sum_{0 \leq n < w_\ell} g_\lambda(n + r)\overline{g_\lambda(n)} + O\left(\frac{R}{q_{\lambda - 1}}\right)
\leq \left[ \left( \sum_{0 \leq h < q_{\lambda - 1}} |G_{\lambda - 1}(h)|^4 \right)^{1/2} + \left( \sum_{0 \leq h < q_{\lambda}} |G_\lambda(h)|^4 \right)^{1/2} \right] \left( \frac{q_\lambda}{R} \right)^{1/2} + O\left(\frac{R}{q_{\lambda - 1}}\right).
$$

Using the hypothesis and Proposition 2.10, we get $\sup_{h \in \mathbb{Z}} |G_\lambda(h)| = o(1)$ as $\lambda \to \infty$. By Parseval’s identity this implies

$$
\sum_{h < q_\lambda} |G_\lambda(h)|^4 = o(1).
$$

By a straightforward argument we conclude that

$$
\frac{1}{R} \sum_{0 \leq r < R} |\gamma_r| = o(1)
$$

as $R \to \infty$. Since $g$ is bounded, an application of Lemma 2.2 completes the proof of Theorem 1.1.

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