A note on the jump locations of Markov processes

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Abstract

For a continuous-time Markov process, we characterize the law of the first jump location when started from an arbitrary initial distribution, in terms of the invariant distribution of an auxiliary Markov process. This could be of interest in the burgeoning fields of piecewise-deterministic Markov chain Monte Carlo methods and quasi-stationary Monte Carlo methods.

Keywords: Inhomogeneous Poisson process, quasi-stationary distribution, piecewise-deterministic Markov Chain Monte Carlo methods, quasi-stationary Monte Carlo methods.

1 Introduction

Consider a continuous-time càdlàg Markov process \( Y = (Y_t)_{t \geq 0} \) evolving on a Polish space \( \mathcal{X} \) with a distinguished strict measurable subset \( \partial \subset \mathcal{X} \). We write \( E := X \setminus \partial \), and derive the distribution of the process the moment before entering \( \partial \) for the first time. We assume that the process enters \( \partial \) exclusively through jumps, namely we can construct a locally bounded function \( \kappa : E \to [0, \infty) \), where \( \kappa(x) \) dictates the rate of transfer from \( x \in E \) to \( \partial \). In other words, we can express the first hitting time, \( \tau_\partial \), of the process into \( \partial \) as

\[
\tau_\partial = \inf \left\{ t \geq 0 : \int_0^t \kappa(Y_s) \, ds > \xi \right\},
\]

where \( \xi \sim \text{Exp}(1) \) and is independent of \( Y \). Of course, this \( \tau_\partial \) is a stopping time, that is, a Markov random time. \( \tau_\partial \) can be interpreted as the first arrival time of an inhomogeneous Poisson process with (stochastic) rate function \( t \mapsto \kappa(Y_t) \).

Since we are interested only in the behaviour of the process up until \( \tau_\partial \), without loss of generality we can assume that \( \partial \) is a cemetery (absorbing) state, and imagine \( \kappa \) to be a killing rate. In this work we will make connections with the established theory of quasi-stationarity, which precisely concerns the asymptotic behaviour of such killed Markov processes, conditional on survival. For an introduction to this area, see Collet et al. [5].

Recall that a probability distribution \( \pi \) on \( E \) is quasi-stationary if for all \( t \geq 0 \),

\[
\mathbb{P}_\pi(Y_t \in \cdot | \tau_\partial > t) = \pi(\cdot),
\]

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where $P_\tau$ denotes the law of the process $Y$ with initial distribution $Y_0 \sim \pi$.

In this note we characterize the distribution of the location at which the process was killed, that is, the distribution of $Y_{\tau_\partial -}$. We offer an interpretation for the general case in terms of the invariant distribution of an auxiliary Markov process.

The recent developments in computational statistics of piecewise-deterministic Markov chain Monte Carlo (PD-MCMC) methods and quasi-stationary Monte Carlo (QSMC) methods, for instance Vanetti et al. [15], Fearnhead et al. [9] and Pollock et al. [12], have particularly motivated the study of such Markov processes. For these methods, the aim is to simulate a continuous-time stochastic process whose behaviour alters at the arrival times of an inhomogeneous Poisson process precisely as in (1). Practically, the simulation of these arrival times is typically performed using Poisson thinning: points are drawn from some homogeneous dominating Poisson process, and these are accepted or rejected depending on the true killing rate at those times. (Even in cases where the true killing rate is potentially unbounded this can still be done through localization techniques; see Pollock et al. [12].) Implementations of PD-MCMC and QSMC methods will thus typically include the limiting locations $Y_{\tau_\partial -}$ at the times $\tau$ as a free by-product. Given that we have access to these points, we may expect to be able to use them for statistical inference, once we understand how their distribution is related to the the process that generates them. This particular application will be discussed further in Section 4.4.

2 Mathematical background

We assume that we have a non-explosive càdlàg un killed Markov process $\tilde{Y} = (\tilde{Y}_t)_{t \geq 0}$ evolving in continuous time on a Polish space $E$, which will be momentarily augmented with killing. As in Collet et al. [5], we assume that $Y$ is a standard process in the sense of Definition 9.2 of Blumenthal and Getoor [2], so in particular it is strong Markov. We write $P_0^x$ for the law of this un killed process started at $\tilde{Y}_0 = x$ and $L^0$ for its infinitesimal generator.

We will now augment the probability space to include an independent random variable $\xi \sim \text{Exp}(1)$ and use (1) to define the killing time $\tau_\partial$. We consider the space $\mathcal{X} = \{0, \partial\} \times E$. Given $\tilde{Y}$ a realization of the distribution $\sim P_0^x$, we define a pair of processes $(e, Y)$, where $Y_t = \tilde{Y}_{t \wedge \tau}$, and

$$e_t = \begin{cases} 0 & \text{if } t < \tau_\partial, \\ \partial & \text{if } t \geq \tau_\partial. \end{cases}$$

Then the paired process $Z = (Z_t)_{t \geq 0}$, taking values in $\mathcal{X}$, is a Markov process. Let $L_Z$ denote its infinitesimal generator and $P_x$ the projection of the probability measure onto $\mathcal{X}$. This representation augments the un killed process with killing in such a way that it also records the location at which the killing event occurred. This will be crucial for our subsequent analysis. For any probability measure $\mu$ on $E$, we set $P_\mu = \int \mu(dx) P_x$ corresponding to the law where $Y$ has initial distribution $\mu$. Throughout, we will write $\delta_x$ for a point mass at $x \in E$.

We assume that for any starting point killing is certain; that is, for any $x \in E$,

$$P_x(\tau_\partial < \infty) = 1.$$
Until recently Markov processes with soft killing as in (1) have been relatively neglected in the literature on quasi-stationarity. Key contributions in this area for the continuous state-space setting have been Steinsaltz and Evans [14], Kolb and Steinsaltz [10], Champagnat and Villemonais [4, Section 4.4], Vellere et al. [16] and the recent work on QSMC methods, Pollock et al. [12] and Wang et al. [18]. These works all provide results in the diffusion setting, which is one of the main conceived application for this present work, namely application to QSMC methods.

The exit locations of Markov processes has also been relatively understudied in the quasi-stationarity literature. One exception is the following elegant result.

**Theorem 2.1.** (Martínez [11], Proposition 2, repeated in Collet et al. [5], Theorem 2.6.) Let $\pi$ be a quasi-stationary distribution for the process $Z$. Then $\tau_\partial$ and $Y_{\tau_\partial}$ are $P_\pi$-independent random variables.

In Martínez [11] and Collet et al. [5] the authors also give the relation

$$\frac{d}{dt} \mathbb{P}_\pi(Z_t \in \{\partial\} \times A) \bigg|_{t=0} = \theta(\pi) \mathbb{P}_\pi(Z_{\tau_\partial} \in \{\partial\} \times A) = \theta(\pi) \mathbb{P}_\pi(Y_{\tau_\partial} \in A) \tag{3}$$

for any measurable set $A \subset E$, where $\theta(\pi)$ in this case is given by

$$\theta(\pi) = \int_E \kappa(x)\pi(dx) < \infty.$$ 

$\theta(\pi)$ must be finite, otherwise this would contradict $\pi$ being a quasi-stationary distribution (see, for instance, Theorem 2.2 of Collet et al. [5]). $\theta(\pi)$ is the quasi-stationary killing rate: we have

$$\mathbb{P}_\pi(\tau_\partial > t) = e^{-\theta(\pi)t} \quad \forall t \geq 0. \quad \tag{4}$$

In particular, (3) allows us immediately to characterize the distribution of $Y_{\tau_\partial}$ under $P_\pi$.

**Proposition 2.2.** For any measurable $A \subset E$,

$$\mathbb{P}_\pi(Y_{\tau_\partial} \in A) = \frac{\int_A \kappa(x)\pi(dx)}{\theta(\pi)}.$$ 

This states that the law of $Y_{\tau_\partial}$ under $P_\pi$ is (proportional to) $\kappa(x)\pi(dx)$.

**Proof.** Immediate from the relation (3) and the definition of $\tau_\partial$ in (1). □

The goal of this note is to characterise the law of $Y_{\tau_\partial}$ when started in any initial distribution $\mu$; the distribution of $Y_{\tau_\partial}$ under $P_\mu$.

We write $P_t(x, dy)$ for the sub-Markovian transition kernel of the killed process $Y$. That is, for $x \in E$,

$$P_t(x, f) = \int_E f(y)P_t(x, dy) = \mathbb{E}_x \left[ f(Y_t) \mathbb{1}\{\tau_\partial > t\} \right].$$
Recall that the resolvent operator $\mathcal{R}$, mapping nonnegative measurable functions to nonnegative measurable functions, is defined by

$$\mathcal{R}f(x) = \int_0^\infty P_t(x, f) \, dt.$$  

(5)

(Here we are following the notation of Wang et al. [17].) This is the Green’s function, as presented for instance in Dynkin [7]. Exchanging the order of integration (by Tonelli’s Theorem),

$$\mathcal{R}\kappa(x) = \mathbb{E}_x \left[ \int_0^{\tau_{\partial}} \kappa(Y_s) \, ds \right] = \mathbb{E}_x[\xi] = 1.$$

Given a probability measure $\mu$ on $E$ we define a new process $\text{FR}(\mu)$, the *Fixed Rebirth process*, which we introduced in Wang et al. [17], and is analogous to the immediate-return procedure of Doob referenced in Rogers and Williams [13, III.26]. The invariant measure of this process will give a very useful interpretation of our main result.

The $\text{FR}(\mu)$ process is a càdlàg (unkilled) Markov process $X = (X_t)_{t \geq 0}$ evolving on $E$. This process evolves according to the law of the killed process $Y$, except at killing events, which similarly occur at rate $t \mapsto \kappa(X_t)$, the location is resampled according to the fixed measure $\mu$, and then continues to evolve independently from there. Recall that $L^0$ denotes the infinitesimal generator of the unlanded Markov process $\hat{Y}$, so $L^\kappa = L^0 - \kappa$ is the generator of the sub-Markovian process $Y$ killed at time $\tau = 1$. The generator of the Markovian $\text{FR}(\mu)$ process $X$ is then

$$L_X f(x) = L^0 f(x) + \kappa(x) \int (f(y) - f(x)) \mu(dy)$$

for functions $f$ in the appropriate domain.

In Wang et al. [17] it was shown that an invariant distribution of the $\text{FR}(\mu)$ process is given by $\Pi(\mu)$, where

$$\Pi(\mu)(f) \propto \mu \mathcal{R} f = \int_E \mu(dx) \mathcal{R}(f)(x),$$

provided that this is integrable, namely that

$$\mu \mathcal{R} 1 = \mathbb{E}_\mu[\tau_\partial] < \infty.$$  

(6)

This was proved for diffusions on compact manifolds as Lemma 2.6 of Wang et al. [17], but the proof carries over directly to the more general setting.

3 Main Result

**Theorem 3.1.** For any measurable $A \subset E$,

$$\mathbb{P}_\mu(Y_{\tau_\partial} \in A) = \mu \mathcal{R}(\kappa 1_A).$$

Given the preceding discussion, this immediately implies the following useful interpretation.
Corollary 3.2. When the stationary probability distribution $\Pi(\mu)$ exists, that is, when (6) holds, we have that for any measurable $A \subset E$,

$$P_\mu(Y_{\tau_0} \in A) = \frac{\int_A \kappa(x)\Pi(\mu)(dx)}{\int_E \kappa(x)\Pi(\mu)(dx)}.$$

This states that the law of $Y_{\tau_0}$ under $P_\mu$ is (proportional to) $\kappa(\cdot)\Pi(\mu)$; cf. Proposition 2.2.

Proof of Theorem 3.1. Since $P_\mu = \int \mu(dx)P_x$, it suffices to prove the result for point masses $\mu = \delta_x$, for $x \not\in A$.

Because $\{\partial\} \times A$ is an absorbing set for the process $Z$, the event $\{Y_{\tau_0} \in A\}$ is equivalent to the event $\{\lim_{t \to \infty} 1_{\{\partial\} \times A}(Z_t) = 1\}$. Thus, by the Kolmogorov Forward Equation and the Monotone Convergence Theorem we have

$$P_x(Y_{\tau_0} \in A) = \lim_{t \to \infty} E_x[1_{\{\partial\} \times A}(Z_t)]$$

$$= \lim_{t \to \infty} \int_0^t E_x[\kappa(Y_s)1_{A}(Y_s)1_{\{\tau_0 > s\})] ds$$

$$= \int_0^\infty P_t(x, \kappa 1_A) dt$$

$$= \mathcal{R}(\kappa 1_A)(x). \, \Box$$

4 Examples and remarks

4.1 Martin boundary theory

Related questions concerning exit locations have been considered in the literature on Martin boundary theory.

An analogue of Theorem 3.1 is given in Dynkin [3, Section 6] for the discrete-time, discrete state space setting. Dynkin considered a Markov chain $(X_n)_{n \in \mathbb{N}}$, with transition probabilities $p(n, x, y) := P_x(X_n = y)$, exiting a distinguished subset $D$ for the final time. Setting

$$\tau := \sup \{ n : x_n \in D \},$$

he noted that

$$P_x(X_\tau = y) = \sum_{m=0}^\infty p(m, x, y)P_y(\tau = 0). \quad (7)$$

This is analogous to our operator $\mathcal{R}$ in (5), but no interpretation of this expression is given.

In another piece of work, Dynkin [8] considers the very general continuous-time Markov process setting. Dynkin explores an intimate connection between the distribution of exit locations of Markov processes and excessive functions, which are generalisations of nonnegative superharmonic functions. In the paper [8], Dynkin rigorously derives the existence of exit locations in terms of the
Martin compactum and establishes their fundamental properties, for instance characterising the space of ‘admissible’ exit locations in terms of the Martin function. While he does explore connections with the Green’s kernel – what we have called the resolvent, (5) – [8] is a foundational mathematical paper, and thus is less concerned with the practical question of identifying specific exit distributions.

Our contribution here can be seen as a simple derivation of an analogous continuous-time continuous-state-space result to (7), with the practically useful connections with the theory of quasi-stationarity and the interpretation in terms of the invariant distribution of the FR(μ) process.

4.2 General observations

The quantity \( \mu \mathcal{R}(\kappa_{1A}) \) appearing in Theorem 3.1 has a very natural interpretation, namely since

\[
\mu \mathcal{R}(\kappa_{1A}) = \mathbb{E}_{\mu} \left[ \int_0^{\tau_0} \kappa(Y_t)1\{Y_t \in A\} \, dt \right],
\]

we can think of this as the ‘average amount of killing picked up by the process in set \( A \) when started from \( \mu \)’. Since the average total killing picked up (that is, when \( A = E \)) is 1, this indeed will correspond to the probability of being killed in \( A \).

Theorem 3.1 is valid even in situations where there is no quasi-stationary distribution. For example, consider the case of a continuous-time simple symmetric random walk on \( \mathbb{Z} \), where at position \( i \in \mathbb{Z} \), transitions to states \( i - 1 \) and \( i + 1 \) occur at rate 1, and there are no other transitions. In addition, we have a uniform killing rate of \( \kappa(i) = 1 \) for all \( i \in \mathbb{Z} \). Clearly there can be no quasi-stationary distribution, since conditioning on survival reverts us to the simple symmetric random walk on \( \mathbb{Z} \), which has no stationary distribution. Let us take the initial distribution to be \( \mu = \delta_0 \). Theorem 3.1 is still valid, and tells us that the distribution of the location at which the particle is killed is the same as the invariant distribution of the simple symmetric random walk on \( \mathbb{Z} \) which also has additional jumps to 0 at a uniform rate 1. This process has a unique invariant distribution since, in particular, it is uniformly ergodic, in the sense of Down et al. [6]. The invariant distribution can be computed exactly through routine calculations involving the \( Q \)-matrix.

An example where Theorem 3.1 is still applicable while \( \Pi(\mu) \) is not well-defined (so Corollary 3.2 does not apply) is a continuous-time simple symmetric random walk on \( \mathbb{Z} \) as before, except that killing occurs only at a finite collection of states \( \{1, 2, \ldots, k\} \), with at least some of the killing rates \( \kappa_i, i = 1, \ldots, k \) nonzero. We take \( \mu = \delta_0 \). Since the un killed process is recurrent, we know that killing will occur almost surely at a finite time, from any initial position. However, since the process is null recurrent the expectation of the return times to a given state are infinite. In particular this implies that \( \mathbb{E}_{\mu}[\tau_0] = \infty \). In this setting Theorem 3.1 still holds, and we may infer that the distribution of the point of killing is equivalent to the stationary distribution reweighted by \( \kappa \) of the Markov process on \( \{1, \ldots, k\} \), with transition rates \( q_{ij} = 1 \) if \( |i - j| = 1 \), except for \( q_{21} = 1 + \kappa_2 \), and \( q_{1i} = \kappa_i \) for \( i \geq 3 \).

Of course, we can also apply Theorem 3.1 to the FR(\( \mu \)) process itself. This
tells us that the distribution of the death locations are independently propor-
tional to $\kappa(x)\Pi(\mu)(dx)$, provided $\Pi(\mu)$ is well-defined.

### 4.3 Quasi-stationary case

Consider the situation where there is a quasi-stationary distribution $\pi$ as in (2).

In this situation, Theorem 3.1 implies Proposition 2.2 since $\pi$ being quasi-
stationary implies that there exists $\theta(\pi)$ such that for all non-negative measur-
able $f$,

$$\int \pi(dx) P_t(x, f) = e^{-\theta(\pi)t} \pi(f).$$

This implies that that $\Pi(\pi) = \pi$. Indeed, in many cases $\Pi(\mu) = \mu$ is necessary
and sufficient for $\mu$ to be quasi-stationary (see Proposition 2.9 of Wang et al.
[17]).

Furthermore, if in addition $\kappa$ is uniformly positively lower bounded, so there
exists some $\epsilon > 0$ such that $\kappa(x) \geq \epsilon$ for all $x \in E$, our result has an elegant
interpretation. Our result states that the distribution of $Y_{\tau_{\partial}}$ under $P_\pi$ can be
written as

$$\frac{\theta'(\pi)}{\theta'(\pi) + \epsilon} (\kappa(x) - \epsilon)\pi(dx) + \frac{\epsilon}{\theta'(\pi) + \epsilon} \pi,$$

where $\theta'(\pi) := \int_E (\kappa(x) - \epsilon)\pi(dx) = \theta(\pi) - \epsilon$. The expression (8) is saying
that the distribution of the exit location is a mixture of the quasi-stationary
distribution $\pi$ and the modified exit location for the process killed at rate $\kappa - \epsilon$.

Since the killing time under $P_\pi$ is exponential with rate $\theta(\pi)$, see (4), this can
be exactly seen as a mixture corresponding to competing independent exponen-
tial clocks, at rates $\theta'(\pi)$ and $\epsilon$. This particular remark is a useful practical
observation, since when implementing QSMC methods, it is often convenient to
choose $\kappa$ with such a uniform positive lower bound.

### 4.4 Application to Monte Carlo methods

As mentioned in the introduction, the practical motivation for this note is the
development of Monte Carlo methods relying on the simulation of continuous-
time Markov processes exhibiting jumps at random times as in (1). For example,
for both the Bouncy Particle Sampler (BPS) of Bouchard-Côté et al. [3] and the
Zig-Zag process of Bierkens et al. [1], the Markov processes in question evolve
deterministically, except at event times as in (1) where the velocity jumps. For
the ScaLE algorithm of Pollock et al. [12], the interarrival dynamics are diffusive,
and arrival times particles are re-weighted.

It is carefully noted in Bierkens et al. [1, Section 6.1] that the distribution
of the Zig-Zag process at the switching times (that is, precisely the distribution
of $Y_{\tau_{\partial}}$) is not equal to the invariant distribution, but rather is ‘biased towards
the tails of the target distribution’. Our work here clarifies exactly what this
bias is.

Practically speaking, the difficulty in simulating such processes is precisely
the simulation of the locations at which the Markov process jumps. This is typically
done by simulating the jump time $\tau_{\partial}$ as in (1) via Poisson thinning, and
then given the time $\tau_{\partial}$, simulating the corresponding location $Y_{\tau_{\partial}}$. Our result
provides a direct characterisation of the distribution of $Y_{\tau_{\partial}}$. Unfortunately,
Theorem 3.1 and Corollary 3.2 are not immediately applicable, since sampling from the object $\Pi(\mu)$ would be as difficult as the original problem. Nevertheless, we hope that our work here provides another perspective on this practical problem.

Finally, since our result characterises the exact distribution of the exit locations, from a methodological perspective, this could motivate the development of further Monte Carlo methods designed to exploit this.

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