Theoretical developments during the past several years have shown that large scale properties of the Quantum Hall system can be successfully described by effective field theories which use the Chern-Simons interaction. In this article, we first recall certain salient features of the Quantum Hall Effect and their microscopic explanation. We then review one particular approach to their description based on the Chern-Simons Lagrangian and its variants.

\[\text{ABSTRACT}\]

Theoretical developments during the past several years have shown that large scale properties of the Quantum Hall system can be successfully described by effective field theories which use the Chern-Simons interaction. In this article, we first recall certain salient features of the Quantum Hall Effect and their microscopic explanation. We then review one particular approach to their description based on the Chern-Simons Lagrangian and its variants.

\[1\] To be published in a volume in honour of Professor R. Vijayaraghavan.
1. INTRODUCTION

Since the discovery of the Quantum Hall Effect (QHE) in 1980, there have been significant developments in its theoretical as well as experimental investigations \(^{(1,2)}\).

QHE was observed in effectively two dimensional systems of electrons (experimentally realized in terms of inversion layers formed at the interface between a semiconductor and an insulator or between two semiconductors) subjected to strong magnetic fields. For such a system, the classical Hall conductivity is given by

\[ \sigma_H = \frac{nec}{B} \]  

(1.1)

where \( n \) is the electron concentration, \( e = -|e| \) is the charge of the electron and \( B \) is the magnetic field perpendicular to the plane of the system.

However it was observed that at very low temperatures instead of linearly rising with \( n/B \), the Hall conductivity becomes quantized and develops a series of plateaus given by

\[ \sigma_H = -\frac{\nu e^2}{\hbar} \]  

(1.2)

where \( \nu \) is an integer.

Theoretical understanding of this Integer Quantum Hall Effect (IQHE) was provided in terms of a noninteracting electron system. States of a two dimensional electron system (without any impurities) in a magnetic field normal to the two dimensional surface are discrete Landau levels. We can define the filling factor of Landau levels as \( \nu = \frac{n}{n_B} \) where \( n_B = \frac{1}{2\pi l^2} \) is the number of states per unit area of a Landau level (\( l \) here being the magnetic length, \( l = (hc/eB)^{1/2} \)). Due to gaps in the single particle density of states, diagonal resistance vanishes when a Landau level is full and the Fermi level lies in the gap between occupied levels. Presence of impurities broadens Landau levels and leads to the presence of localized states in the energy gaps. These localized states can not
carry any current. Therefore increase in the occupation of these states does not change Hall conductivity. As long as the extended states in the $\nu^{th}$ Landau level are completely occupied and it is only the localized states (lying in between $\nu^{th}$ and $\nu + 1^{st}$ Landau levels) which are being further occupied as $n/B$ is increased, the Hall conductivity will remain constant at a value given by Eq.(1.1) with $n = \nu n_B$. This gives the result of Eq.(1.2) explaining the plateaus. Longitudinal resistance becomes nonzero and the Hall conductivity makes a transition from one plateau to the next when extended states in the next Landau level start getting filled.

Soon after the discovery of IQHE, it was discovered in 1982 that in systems with extremely low impurity concentrations and at very low temperatures, the value of $\nu$ in Eq.(1.2) can assume certain rational fractional values $f$. This effect is known as the Fractional Quantum Hall Effect (FQHE). It was further observed that if $f = p/q$ where the integers $p$ and $q$ have no common factor, then $q$ was necessarily odd. [We may point out that some later observations also revealed even denominator values of $f$. In terms of the Laughlin theory of FQHE to be described below, these even denominators are understood as arising from an electron system where electrons are bound in pairs and the pairs behave like bosons (cf. ref. 2) or alternatively are “bound” to magnetic vortices in a sense we indicate in Section 3.]

Although FQHE and IQHE seem very similar, the theoretical understanding of FQHE required radically new concepts. While the theory of IQHE was based on a model of a noninteracting system of electrons, the theory of FQHE utilizes strong correlations among electrons. Laughlin proposed that the ground state of the electron system in FQHE is a translationally invariant liquid state and that the lowest energy charge excitations in the system are fractionally charged quasiparticles and quasiholes. These excitations obey fractional statistics and therefore are anyons. Hall plateaus observed in FQHE experiments are then explained as being due to localization of these fractionally charged
quasiparticles. Quantized Hall conductivity is still given by Eq.(1.2), but now the charge carriers are fractionally charged quasiparticles leading to fractional values of the filling factor $\nu$ in Eq.(1.2) as we will explain below.

Let us describe the constant imposed magnetic field $\vec{B} = (0, 0, B)$ in a gauge where the vector potential $\vec{A}$ is $1/2 (\vec{B} \times \vec{x})$, $\vec{x} = (x, y, 0)$ being a point on the two-dimensional surface. The wave function proposed by Laughlin for FQHE corresponding to the $\nu = \frac{1}{m}$ ground state is, in this gauge,

$$\Psi_m = \prod_{j<k} (z_j - z_k)^m \prod_{j=1}^N e^{-|z_j|^2/4l^2} \quad (1.3)$$

where $N$ is number of electrons, $z_j = x_j + iy_j$ is the position of the $j^{th}$ electron in complex coordinates and $l$ is the magnetic length introduced earlier. $m$ is an odd integer so that $\Psi_m$ is antisymmetric in $z_j$ as required by the Fermi statistics of electrons.

The probability density of electrons in state $\Psi_m$ can be written as

$$|\Psi_m|^2 = e^{-H_m}$$

where

$$H_m = -2m \sum_{j<k} \ln|z_j - z_k| + \sum_j |z_j|^2/2l^2 \quad (1.4)$$

We note that $H_m$ can be identified with the potential energy of a two-dimensional, one component plasma where particles of charge $m$ repel one another via logarithmic interaction and are attracted to the origin by a uniform neutralizing background charge density $\rho = \frac{1}{2\pi l^2}$. Charge neutrality of the plasma will then be achieved when the electron density is equal to $1/m$ times the charge density of the equivalent plasma and thus equal to
\[ \rho_m = \frac{1}{2\pi ml^2} \]  

(1.5)

This therefore will give rise to plateaus in the Hall conductivity at filling factors \( \nu = 1/m \). At high densities, \( m = 1 \) (full Landau level) is energetically most favorable and as the neutralizing background density of the original electron system is decreased, first the \( m = 3 \) state becomes stable and then \( m = 5 \) etc.

Starting from a ground state as in Eq.(1.3), one can easily establish that quasiparticle and quasihole excitations of state \( \Psi_m \) are fractionally charged particles obeying fractional statistics and are therefore anyons. The states with \( m = 1, 3, 5, \ldots \) etc. correspond to filling factors \( \nu = 1, 1/3, 1/5, \ldots \) and are called parent states. The whole hierarchy of Fractional Quantum Hall (FQH) states can be generated by essentially repeating the whole process of construction of the state \( \Psi_m \) for the quasiparticles. Thus we consider a system of quasiparticles condensing into a FQH state of the form given in Eq.(1.3) where \( m \) should be replaced by an appropriate real number consistent with the statistical properties of quasiparticles. This process can be iterated for higher levels of the hierarchy.

Though there are important differences between IQHE and FQHE in experimental as well as theoretical aspects, there are strong similarities as well underlying these Quantum Hall (QH) systems. For example Hall conductivity (and Hall conductance) do not depend on microscopic details (such as frequencies etc.) or on the geometrical shape of the sample. These aspects of QH systems are similar to the ones in the theory of critical phenomena in statistical mechanics where certain quantities like critical exponents characterizing continuous phase transitions depend only on the large scale properties of the underlying statistical system and not on the microscopic dynamics. Motivated by this universal behavior of QH systems, many authors\(^{(3,4)}\) have developed effective field theories for the large scale behavior of QH systems. The work of Fröhlich and Kerler, and Fröhlich and Zee\(^{(4)}\) in this regard is of particular interest for the present article. They argued that large
scale properties of QH systems can be described in terms of a pure abelian Chern-Simons theory where the Hall conductivity turns out to be inversely proportional to the coefficient of the Chern-Simons action.

In the rest of the article, we will elaborate on the theoretical developments centered around the pure Chern-Simons description of QH systems. One of the aspects of these systems which plays a very important role in understanding their large scale behavior is the existence of edge currents\(^{(5)}\). Therefore, in Section 2, we first describe the origin of these edge currents from the microscopic physics of a Quantum Hall system. In Section 3, we will discuss the relation between the theory of the QH system and Chern-Simons gauge theory following the papers of Fröhlich and Kerler, and Fröhlich and Zee\(^{(4)}\) and show how Chern-Simons theory leads to fractional quantization of \(\sigma_H\). In Section 4, we will show how the edge currents of a Quantum Hall system can be obtained from a Chern-Simons theory following the approach of Balachandran et al.\(^{(6)}\). [The existence of edge states in a Chern-Simons theory is first due to Witten\(^{(7)}\).]

2. MASSLESS EDGE CURRENTS IN QUANTUM HALL SYSTEMS

It was pointed out by Halperin\(^{(5)}\) that in a QH system, there are current carrying edge states which extend along the perimeter of the system. Following the discussion in ref. 5, let us consider a system of electrons on a two-dimensional plane with annular geometry as shown in Fig. 1. There is a uniform magnetic field \(B\) through the annulus perpendicular to the plane and in addition there is a flux \(\Phi\) going through the hole \((r < r_1)\). [Here \(r = [(x^1)^2 + (x^2)^2]^{1/2}, (x^1, x^2)\) being the coordinates of the plane. We have slightly changed notations from those in Section 1 for later convenience.] We work in a gauge where \(A_r = 0\) and
\[ A_\theta = \frac{1}{2} Br + \frac{\Phi}{2\pi r}, \] (2.1)

\( A_r \) and \( A_\theta \) being the radial and azimuthal components of the vector potential \( A \).

Due to the azimuthal symmetry, the third component of orbital angular momentum is a good quantum number and the states of electrons in the interior of the annulus (at distances from the edges large compared to the magnetic length \( l \)) are given by the Landau states

\[ \Psi_{m,\nu}(\vec{r}) = \text{const.} \times e^{im\theta} f_\nu(r - r_m). \] (2.2)

Here \( m \) and \( \nu \) are integers, \( m \) being the magnetic quantum number, and \( f_\nu \) is the \( \nu + 1 \)st eigenstate of a one dimensional shifted harmonic oscillator with center \( r_m \) given by

\[ B\pi r_m^2 = m\Phi_0 - \Phi, \]

\[ \Phi_0 = \frac{hc}{e} = \text{Flux quantum.} \] (2.3)

\( f_\nu \) is localized near \( r_m \), decreasing exponentially away from \( r_m \) with typical scale \( l \). The energy of the Landau state \( \Psi_{m,\nu} \) is

\[ E_{m,\nu} = \hbar \omega_c (\nu + \frac{1}{2}) \] (2.4)

where \( \omega_c = |eB|/m^*c \) is the cyclotron frequency, \( m^* \) being the effective mass of the electrons. The current carried by state \( \Psi_{m,\nu} \) is

\[ I_{m,\nu} = \frac{e}{m^*} \int_0^\infty dr |\Psi_{m,\nu}(\vec{r})|^2 \left[ \frac{\hbar^2}{r} - \frac{eA(r)}{c} \right] \]
where the integration over $r$ is performed at a fixed $\theta$. On using Eq.(2.1) and (2.3) and the fact that $|\Psi_{m,\nu}(\vec{r})|^2$ is symmetric about $r_m$ decreasing exponentially away from $r_m$, it becomes

$$I_{m,\nu} \simeq \frac{e^2 B_0}{m^* c} \int_0^{\infty} dr |\Psi_{m,\nu}(\vec{r})|^2 (r_m - r). \quad (2.5)$$

Now we note that since $|\Psi_{m,\nu}|^2$ is symmetric about $r = r_m$ for the interior of the annulus, the integral in Eq. (2.5) effectively vanishes when $r_m$ is far from the edges of the annulus. However, when $r_m$ is close to the edges of the annulus (closer than a few times $l$), the boundary condition that wave functions vanish at the edges makes $|\Psi_{m,\nu}|^2$ asymmetric about $r_m$ in that region and the integral need not vanish. There are thus currents at the edges of the sample. The effect of boundary conditions at the edges is to shift the energy levels as shown in Fig. 2.

The existence of these edge currents can be easily demonstrated using very general arguments as well. In the absence of externally applied potential differences, if the electrons are confined in a two dimensional surface with boundaries, then there must be potential barriers at the boundaries so that electrons do not escape. The gradient of such a potential near an edge will give rise to a force which acts like an electric field $\vec{E}$ directed radially outwards and whose net effect is to confine the electrons. One can think of this electric field as arising from an accumulation of a net positive charge near the edge of the region which the electron is trying to escape. When a magnetic field $\vec{B}$, perpendicular to the plane of the sample, is also present, this $\vec{E}$ gives rise to a Hall current in the direction $\vec{E} \times \vec{B}$ which is tangent to the edge. This current is hence confined to the edge. The potential barrier at the edge will be expected to change $E_{m,\nu}$ as shown in Fig.2. As mentioned above, this is indeed also what happens to the energy levels as a consequence of the boundary conditions on wave functions requiring them to vanish at the edges.

Using Eq.(2.1) and (2.3), we can get the following expression for the edge currents:
\[ I_{m,\nu} = -e \frac{\partial E_{m,\nu}}{\partial \Phi} = \frac{e}{\hbar} \frac{\partial E_{m,\nu}}{\partial m}. \] (2.6)

From this we note that the currents have opposite directions near the outer and inner edges of the annulus. It was further shown by Halperin that a moderate amount of disorder does not destroy these edge currents.

We thus find theoretically that when \( \nu \) Landau levels are filled (the filling factor is \( \nu \)), there are \( \nu \) current carrying states at each of the edges of the sample. Further, the edge currents at the two edges flow in opposite directions. These conclusions have been confirmed in numerical work as well by Rammal et al. (see the article by Prange in the first book of ref. 1), who find that for filling factor \( \nu \), there are \( \nu \) pairs of eigenstates localized at the two edges with opposite momenta.

In this discussion, we have neglected the spin degree of freedom of the electrons, this neglect being justified if the magnetic field is strong. The edge excitations for strong magnetic fields are therefore scalar, chiral fermions propagating in one dimension along the edges of the sample. These fermions will approximately have the dynamics of a “relativistic” massless particle with the one dimensional momentum given by

\[ k = \frac{m - m_{1,2}^F}{r_{1,2}} \] (2.7)

and speed \( k/E_{m,\nu} \), this speed being the substitute for the speed of light for these particles. Here \( m_1^F \) and \( m_2^F \) are the magnetic quantum numbers corresponding to filled Fermi levels at the inner and outer edges of the sample respectively. [“Relativistic” kinematics enters this problem for certain standard reasons: The energy \( E \) of an edge state as measured from the Fermi energy \( E^F \) is, to leading order, proportional to \( k \).]

This concludes the demonstration of the existence of massless edge currents in QH systems using microscopic arguments. In the next two Sections, we explore the universal aspects of QH systems for large scale observations and outline their description using the
Chern-Simons gauge theory.

3. RELATION OF QHE TO CHERN-SIMONS GAUGE THEORY

In this Section we will review certain results due to Fröhlich and Kerler, and Fröhlich and Zee (4) who show that the QH System is related to pure Chern Simons gauge theory and to certain rational conformal field theories. Such relations could have been anticipated if it is recalled that there are chiral edge currents in a QH system and that according to Witten (7), Chern-Simons theory on a two-dimensional space is equivalent to a chiral current algebra on the boundary of that space. One can thus guess the existence of a correspondence between Chern-Simons theory and the QH system.

We will set the speed of light $c$ equal to 1 in this Section so that magnetic flux can be measured in units of $\hbar/e$.

Let us begin our discussion by examining a QH system characterized by zero longitudinal resistance. The conductivity tensor $\sigma$ can then be written as

$$\sigma = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix}$$

In QH systems, $\sigma_H$ is quantized and is a rational multiple of $e^2/h$. The idea pursued in ref. 4 is that this fact may have a universal explanation emerging from rational conformal field theories.

As the longitudinal conductivity $\sigma_L$ is zero for a two-dimensional system with $\sigma$ given by Eq. (3.1), the current density $j$ induced by an electric field $E$ is given by

$$j^\alpha(\vec{x}, t) = \sigma_H \epsilon^{\alpha\beta} E_\beta(\vec{x}, t); \quad \alpha, \beta = 1, 2; \quad \epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \epsilon^{12} = 1.$$  

(3.2)

Here $E_\alpha = -F_{0\alpha}, F_{\mu\nu}$ being the electromagnetic field strength tensor.
Now if \( j^0 \) is the charge density, then we have the continuity equation

\[
\frac{\partial j^0}{\partial x^0} + \vec{\nabla} \cdot \vec{j} = 0, \quad x^0 = t.
\]  

(3.3)

Also \( B \) and \( E \) are related by the Maxwell’s equation

\[
\frac{\partial B}{\partial x^0} = -\epsilon^{\alpha\beta} \partial_\alpha E_\beta.
\]  

(3.4)

where \( B = F_{12} \). Equations (3.2), (3.3) and (3.4) give

\[
\sigma_H \frac{\partial B}{\partial x^0} = \frac{\partial}{\partial x^0} j^0.
\]  

(3.5)

We thus obtain

\[
j^0 = \sigma_H (B + B_c),
\]  

(3.6)

Here \( B_c \) is an integration constant representing a time independent background magnetic field.

Let us assume that the three-dimensional manifold \( M \) has the topology of \( \mathbb{R}^1 \times D \) with \( D \) characterizing the two-dimensional space of the sample, and \( \mathbb{R}^1 \) describing time. Furthermore, let \( \eta = (\eta_{\mu\nu}) \) be any metric of Euclidean or Lorentzian signature on \( M \). Then Eqs. (3.2) and (3.6) can be extended to a generally covariant form valid for arbitrary metrics as well as follows.

Let

\[
J_{\alpha\beta}(x) = |\text{Det } \eta(x)|^{-1/2} \epsilon_{\alpha\gamma\beta} j^\gamma(x), \quad x = \vec{x}, t
\]  

(3.7)

and

\[
j^\alpha(x) = \frac{1}{2} |\text{Det } \eta(x)|^{1/2} \sigma_H \epsilon^{\alpha\beta\gamma} F_{\beta\gamma}(x),
\]  

(3.8)

Here \( \epsilon_{\alpha\beta\gamma} \) is the totally antisymmetric symbol with \( \epsilon_{012} = 1 \) and \( t = x^0 \) is time. Then

\[
J_{\alpha\beta}(x) = \sigma_H F_{\alpha\beta}(x)
\]  

(3.9)

(3.9) reduces to (3.2) and (3.6) for a flat metric.
Using the language of differential forms, we can write Eqs. (3.9) and (3.7) as

\[ J = \sigma_H F ; \] (3.10)

\[ J = * j \] (3.11)

where \( J = \frac{1}{2} J_{\alpha\beta} dx^\alpha \wedge dx^\beta \) and \( * \) is the Hodge dual. The one form \( j(x) \) is defined as

\[ j(x) = \sum_{\alpha} \left( \sum_{\beta} \eta_{\alpha\beta}(x) j^\beta(x) \right) dx^{\alpha}. \]

The continuity equation (3.3) can be written as

\[ dJ = 0 \] (3.12)

where \( d \) is the exterior derivative.

We shall assume that \( \sigma_H \) is a constant. Equation (3.10) then gives the Maxwell equations

\[ dF = 0 \] (3.13)

Here, we can write \( F = dA' \), \( A' = A + A_c \) where \( A_c \) is the vector potential corresponding to a constant magnetic field \( B_c \) (see Eq. 3.6), \( A \) represents the vector potential of a fluctuation field due to localized sources and \( A' \) the total vector potential.

Now, Eq. (3.12) implies that

\[ J = da \] (3.14)

where \( a \) is a one form. Equation (3.10) can then be written in terms of one forms \( a \) and \( A' \) as

\[ da = \sigma_H dA' \]

We now note that this last equation can be obtained from an action principle with the action \( S_{CS} \) given by

\[ S_{CS} = \frac{1}{2\sigma_H} \int_M (a - \sigma_H A') \wedge d(a - \sigma_H A') \] (3.15)
or in terms of components,

\[ S_{CS} = \frac{1}{2\sigma_H} \int_M \varepsilon^{\alpha\beta\gamma}(a_\alpha - \sigma_H A'_\alpha)\partial_\beta(a_\gamma - \sigma_H A'_\gamma)d^3x. \]

The overall normalization of \( S_{CS} \) is here fixed by the requirement that the coupling of \( A'_\mu \) to \( j^\mu \) is by the term \(-j^\mu A'_\mu \) in the Lagrangian density.

The action \( S_{CS} \) is the Chern-Simons action for the gauge field \( a - \sigma_H A' \).

It is important to note at this step that the derivation of Eq. (3.15) from the QHE is valid only in the scaling limit when both length and 1/frequency scales are large. This is because although the continuity equation (3.12) is exact, Eq. (3.2) is experimentally observed to be valid only at large distance and time scales.

The action \( S_{CS} \) can be naturally generalized to the case where there are several independently conserved electric current densities \( j^{(i)}, i = 1, \ldots m \). For example, for \( m \) filled Landau levels, if one neglects mixing of levels (which is a good approximation due to the large gaps between Landau levels), each level can be treated as dynamically independent with electric currents in each level being separately conserved. Then one has

\[ J^{(i)} = \sigma^{(i)} F \]
and

\[ J^{(i)} = da^{(i)}. \]  

The action in this case is given by

\[ S_{CS}(\{a^{(i)}\}, A) = \int_M \sum_{i=1}^m \frac{1}{2\sigma_H}(a^{(i)} - \sigma^{(i)} A') \wedge d(a^{(i)} - \sigma^{(i)} A') . \]  

Let us look back at Eq. (3.15). Recall that \( A' = A + A_c \) where \( A_c \) is the vector potential of the background magnetic field \( B_c \). Let us introduce a change of variable by setting \( a = a + \sigma_H A_c \) and call \( a \) again as \( a \). Then the field equations from Eq. (3.15) relate \( a \) and \( A' \):

\[ da = \sigma_H dA. \]
This equation implies that a vortex of magnetic flux

\[ \Phi = \int dA \]

carries a charge

\[ q = \sigma_H \Phi . \quad (3.19) \]

The statistics obeyed by these quasiparticles can be determined in the following manner. Consider two such quasiparticles each carrying magnetic flux \( \Phi \) and electric charge \( q = \sigma_H \Phi \). Under an exchange of two quasiparticles, the wave function picks up a phase factor \( \exp(2\pi i \theta) \) where \( \theta \) characterizes the quasiparticle statistics. \( \theta \) can be calculated by performing two successive exchanges (which amount to taking one quasiparticle a full circle around the other quasiparticle) and calculating the Aharonov-Bohm phase factor. One gets

\[
\exp [2\pi i \theta] = \exp \left[ -i \frac{q}{2\hbar} \Phi \right] \\
= \exp \left[ -i \frac{\sigma_H (\Phi)^2}{2\hbar} \right] \\
= \exp \left[ -i \frac{\pi \Phi^2}{k} \right]
\]

where we have set \( \sigma_H = \frac{e^2}{\hbar k} \),

\( \Phi = \frac{e}{h} \Phi \).

\[ (3.20) \]

We thus find,

\[ \theta = - \frac{\Phi^2}{2k} \mod N, \quad N \in \mathbb{Z} . \quad (3.21) \]

\( \theta = 0 \) and \( -\frac{1}{2} (\mod N) \) respectively correspond to Bose and Fermi statistics whereas the quasiparticles are anyons when \( \exp(2\pi i \theta) \neq \pm 1 \).
Clearly, the electron must be among the charged excitations of the theory. Since the magnetic field $B_c$ is strong for us, the electron spin is frozen in the direction of $B_c$. The symmetry of the many electron spin wave function implies that the values $\theta_e$ and $\tilde{\Phi}_e$ of $\theta$ and $\tilde{\Phi}$ for electrons are related by

$$\theta_e = \frac{1}{2} \mod N = \frac{\tilde{\Phi}_e^2}{2k} \mod N, \quad N \in \mathbb{Z}$$

which gives

$$\frac{\tilde{\Phi}_e^2}{k} = \pm(2\ell + 1), \quad \ell = 0, 1, 2, \ldots .$$

Equation (3.22) shows that the charge of this particle is given by

$$q_e = e \frac{\tilde{\Phi}_e}{k}.$$  \hspace{1cm} (3.23)

Since we want $q_e = e$, Eqs. (3.22) and (3.23) show that

$$k = \pm(2\ell + 1).$$

With this value of $k$, one can obtain $\sigma_H$ from Eq. (3.20):

$$\sigma_H = \pm \frac{1}{2\ell + 1} \frac{e^2}{\hbar}, \ell = 0, 1, 2, \ldots .$$  \hspace{1cm} (3.24)

Equation (3.24) gives the fractionally quantized Hall conductivities corresponding to the parent states in Laughlin's theory.

Let us recall from Section 1 that there are experiments where even integer values of $k$ have been obtained (cf. ref. 2). These even denominators in $\nu$ arise naturally in the framework of the present model in the following manner. Suppose that the external magnetic field $B_c$ is not very large. Then as long as the temperature of the system is not extremely low, the spins of the electrons will not be completely frozen in the direction of $B_c$ and the N-electron spin wave function need not be completely symmetric. For example, it could be in a singlet state. The electron may then appear as a compound
state of a magnetic vortex (with flux $-\frac{2\hbar}{e}$ and charge $-e$) and of a neutral fermion with spin $\frac{1}{2}$. Allowed statistics $\theta$ for this compound state will then be, from Eq. (3.21), $\theta = 0 \mod N$, $N \in \mathbb{Z}$. From Eq. (3.23) and with the above value of flux for this picture of the electron, we get $k = 2$ and $\sigma_H = \frac{e^2}{2\hbar}$, thereby obtaining an even denominator plateau.

We will now generalize the theory in order to get the higher levels of hierarchies for $\sigma_H$. Let us go back to the case of $m$ filled Landau levels with the Lagrangian given in Eq. (3.17). From the example of one filled Landau level, with Eq. (3.20) defining $k$, we know that $k$ characterizes the level of the hierarchy and is hence related to the filling factor, $k$ being 1 for one completely filled Landau level. Suppose now that we consider $m$ filled Landau levels, with $k = 1$ for each level. The Lagrangian $\mathcal{L}$ will be given by Eq. (3.17) with $\sigma_H^{(i)}$ set equal to $\frac{e^2}{\hbar}$ for all $i$ (see Eq. (3.20)) and $a$ redefined as indicated earlier in order to get rid of $A_c$.

In obtaining (3.17), we had neglected interactions between the electrons in the different Landau levels. Such interactions will lead to mixing between different levels. However, by general arguments we expect that in the scaling limit, the dominant contributions from such interactions can come only from dimension three (Chern-Simons like) terms. If we now assume that the “interaction” to be added to $\mathcal{L}$ should only involve the total electromagnetic current

$$J = \sum_i J^{(i)} = \sum da^{(i)}$$

and not say, just one $J^{(i)}$, then the electron-electron interaction changes the part of the Lagrangian in Eq. (3.17) not involving $A_\mu$ to the form

$$\mathcal{L}^{(1)} = \frac{\hbar}{2e^2}(\sum a^{(i)} \wedge da^{(i)} + p(\sum a^{(i)})(\sum da^{(i)}))$$

(3.25)

where $p$ is some real constant. Although the physical basis of this assumption is not clear to us, we shall accept it and proceed to study (3.25).
The Lagrangian (3.25) can be written in the more compact form

\[ \mathcal{L}^{(1)} = \frac{\hbar}{2e^2} a^T K da \] (3.26)

by introducing a matrix \( K = I + pC \) where \( C \) is the \( m \times m \) matrix with each entry equal to 1. \( a \) here is the column with entries \( a^{(i)} \).

The field equations following from (3.26) and (3.17) are

\[ K da^{(i)} = \frac{e^2}{\hbar} dA. \] (3.27)

For an applied field \( dA \) (independent of \( i \)), it shows that

\[ \sum_i da^{(i)} = e^2 \sum_{i,j} K^{-1}_{i,j} dA \]

and hence that the Hall conductivity is

\[ \sigma_H = \frac{e^2}{\hbar} \sum_{i,j} K^{-1}_{i,j}. \] (3.28)

If we characterize a quasiparticle by a vorticity vector

\[ \Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_m \end{pmatrix} \]

where \( \Phi_i \) is its flux associated with the \( i^{th} \) Landau level, then (3.27) shows that the electric charge is described by the charge vector

\[ q = \begin{pmatrix} q_1 \\ \vdots \\ q_m \end{pmatrix} \]

where

\[ q_i = \frac{e^2}{\hbar} \sum_j (K^{-1})_{ij} \Phi_j. \] (3.29)
We can repeat the steps used in getting (3.21) for this case. Thus the statistics phase \( \theta \) for this case is
\[
\theta = -\frac{1}{2\hbar} \left( \sum_i q_i \Phi_i \right) \mod N, \quad N \in \mathbb{Z} \tag{3.30}
\]
where again \( \theta \) is defined by the phase factor \( e^{2\pi i\theta} \) which the wave function picks up under an exchange of two quasiparticles.

Now since \( C^2 = mC \), we have,
\[
K^{-1} = I - \left( \frac{p}{1+mp} C \right).
\]
Using this in Eqs. (3.28), (3.29) and (3.30) we get
\[
\sigma_H = \frac{e^2}{\hbar} \frac{m}{1+mp}, \quad q_i = \frac{e^2}{\hbar} \left( \Phi_i - \frac{p}{1+mp} \sum_{j=1}^{m} \Phi_j \right)
\]
and
\[
\theta = -\frac{1}{2} \left( \frac{e}{\hbar} \right)^2 \left[ \sum_i \Phi_i^2 - \frac{p}{1+mp} \left( \sum_i \Phi_i \right)^2 \right] \mod N, \quad N \in \mathbb{Z}. \tag{3.31}
\]
For \( \frac{e\Phi_i}{\hbar} = 1, i = 1, \ldots, m \), we therefore find,
\[
\sigma_H = \frac{e^2}{\hbar} \frac{m}{1+mp}, \quad q_i = e \frac{1}{1+mp} \quad \text{and} \quad \theta = -\frac{1}{2} \left( \frac{m}{1+mp} \right) \mod N, \quad N \in \mathbb{Z}. \tag{3.32}
\]
For the simplest case of a single Landau level, \( m = 1 \) and (3.32) becomes
\[
\sigma_H = \frac{e^2}{\hbar} \frac{1}{p+1}, \quad q = e \frac{1}{p+1} \quad \text{and} \quad \theta = -\frac{1}{2(p+1)} \mod N, \quad N \in \mathbb{Z}. \tag{3.33}
\]
The expressions for \( \sigma_H, q_i \) and \( \theta \) in Eq. (3.31) with \( m = 1 \) agree with those obtained earlier in this Section for the single filled Landau level case (Eqs. (3.19), (3.20) and (3.21)). One can therefore repeat the type of arguments used for that case and conclude that \( p+1 \) must be an odd integer giving us the odd denominator fractional Hall conductivity. Note that Eq. (3.32) gives more general values of fractionally quantized Hall conductivities than those found for \( m = 1 \). Fröhlich and Zee\(^{(4)}\) discuss further generalizations of (3.25) leading to even more general possibilities for the fractional Hall conductivity.
We will continue in the next Section with the exploration of the relationship between the Quantum Hall system and the Chern-Simons theory. We will demonstrate how the edge currents (see Section 2) in a Quantum Hall system arise naturally from the Chern-Simons theory. This result is first due to Witten\(^{(7)}\). We follow the approach of Balachandran et al\(^{(6)}\) who derive further results in Chern-Simons theory using this approach.

4. CONFORMAL EDGE CURRENTS

In this final Section, we develop elementary canonical methods for the quantization of the abelian Chern-Simons action (considered earlier) on a disc and show that it predicts the edge currents. They are in fact described by the edge states of Witten carrying a representation of the Kac-Moody\(^{(8)}\) algebra. The canonical expression for the generators of diffeomorphisms (diffeos) on the boundary of the disk are also found and it is established that they are the Chern-Simons version of the Sugawara construction.

The Lagrangians considered here follow from (3.15) by setting

\[
\overline{\alpha} = (a - \sigma_H A')[2\pi / |k\sigma_H|]^{1/2}
\]

and calling \(\overline{\alpha}\) again as \(a\), \(k\) being \(|k|\)\(|\frac{\sigma_H}{\sigma_H}|\). We do so in order to be consistent with the form of the Chern-Simons Lagrangian most frequently encountered in the literature.

In this Section, we will use natural units where \(\hbar = c = 1\).

4.1 THE CANONICAL FORMALISM

Let us start with a U(1) Chern-Simons (CS) theory on the solid cylinder \(D \times R^1\) with
action given by

\[ S = \frac{k}{4\pi} \int_{D \times \mathbb{R}^3} ada, \quad a = a_\mu dx^\mu, \quad ada \equiv a \wedge da \quad (4.1) \]

where \( a_\mu \) is a real field.

The action \( S \) is invariant under diffeos of the solid cylinder and does not permit a natural choice of a time function. As time is all the same indispensable in the canonical approach, we arbitrarily choose a time function denoted henceforth by \( x^0 \). Any constant \( x^0 \) slice of the solid cylinder is then the disc \( D \) with coordinates \( x^1, x^2 \).

It is well known that the phase space of the action \( S \) is described by the equal time Poisson brackets (PB’s)

\[ \{a_i(x), a_j(y)\} = \epsilon_{ij} \frac{2\pi}{k} \delta^2(x - y) \quad \text{for} \ i, j = 1, 2, \quad \epsilon_{12} = -\epsilon_{21} = 1 \quad (4.2) \]

and the constraint

\[ \partial_i a_j(x) - \partial_j a_i(x) \equiv f_{ij}(x) \approx 0 \quad (4.3) \]

where \( \approx \) denotes weak equality in the sense of Dirac\(^{(9)}\). All fields are evaluated at the same time \( x^0 \) in these equations, and this will continue to be the case when dealing with the canonical formalism or quantum operators in the remainder of the paper. The connection \( a_0 \) does not occur as a coordinate of this phase space. This is because, just as in electrodynamics, its conjugate momentum is weakly zero and first class and hence eliminates \( a_0 \) as an observable.

The constraint (4.3) is somewhat loosely stated. It is important to formulate it more accurately by first smearing it with a suitable class of “test” functions \( \Lambda^{(0)} \). Thus we write, instead of (4.3),

\[ g(\Lambda^{(0)}) : = \frac{k}{2\pi} \int_D \Lambda^{(0)}(x) da(x) \approx 0. \quad (4.4) \]

It remains to state the space \( \mathcal{T}^{(0)} \) of test functions \( \Lambda^{(0)} \). For this purpose, we recall that a functional on phase space can be relied on to generate well defined canonical
transformations only if it is differentiable. The meaning and implications of this remark can be illustrated here by varying $g(\Lambda(0))$ with respect to $a_\mu$:

$$
\delta g(\Lambda(0)) = \frac{k}{2\pi} \left( \int_{\partial D} \Lambda(0) \delta a - \int_D d\Lambda(0) \delta a \right). \tag{4.5}
$$

By definition, $g(\Lambda(0))$ is differentiable in $a$ only if the boundary term – the first term – in (4.5) is zero. We do not wish to constrain the phase space by legislating $\delta a$ itself to be zero on $\partial D$ to achieve this goal. This is because we have a vital interest in regarding fluctuations of $a$ on $\partial D$ as dynamical and hence allowing canonical transformations which change boundary values of $a$. We are thus led to the following condition on functions $\Lambda(0)$ in $\mathcal{T}^{(0)}$:

$$
\Lambda(0) |_{\partial D} = 0. \tag{4.6}
$$

It is useful to illustrate the sort of troubles we will encounter if (4.6) is dropped. Consider

$$
q(\Lambda) = \frac{k}{2\pi} \int_D d\Lambda a \tag{4.7}
$$

It is perfectly differentiable in $a$ even if the function $\Lambda$ is nonzero on $\partial D$. It creates fluctuations

$$
\delta a |_{\partial D} = d\Lambda |_{\partial D}
$$

of $a$ on $\partial D$ by canonical transformations. It is a function we wish to admit in our canonical approach. Now consider its PB with $g(\Lambda(0))$:

$$
\{g(\Lambda^0), q(\Lambda)\} = \frac{k}{2\pi} \int d^2 x d^2 y \Lambda(0)(x) \epsilon^{ij} \left[ \partial_j \Lambda(y) \right] \left[ \frac{\partial}{\partial x^i} \delta^2(x - y) \right] \tag{4.8}
$$

where $\epsilon^{ij} = \epsilon_{ij}$. This expression is quite ill defined if

$$
\Lambda(0) |_{\partial D} \neq 0.
$$
Thus integration on \( y \) first gives zero for (4.8). But if we integrate on \( x \) first, treating derivatives of distributions by usual rules, one finds instead,

\[
- \int_D d\Lambda^0 d\Lambda = - \int_{\partial D} \Lambda^0 d\Lambda .
\]  

(4.9)

Thus consistency requires the condition (4.6).

We recall that a similar situation occurs in QED. There, if \( E_j \) is the electric field, which is the momentum conjugate to the potential \( a_j \), and \( j_0 \) is the charge density, the Gauss law can be written as

\[
\varphi(\Lambda^{(0)}) = \int d^3 x \Lambda^{(0)}(x) \left[ \partial_i E_i(x) - j_0(x) \right] \approx 0 .
\]

(4.10)

Since

\[
\delta \varphi(\Lambda^{(0)}) = \int_{r=\infty} r^2 d\Omega \Lambda^{(0)}(x) \hat{x}_i \delta E_i(x) - \int d^3 x \partial_i \Lambda^{(0)}(x) \delta E_i(x), \quad r = | \vec{x} |, \quad \hat{x} = \frac{\vec{x}}{r}
\]

(4.11)

for the variation \( \delta E_i \) of \( E_i \), differentiability requires

\[
\Lambda^{(0)}(x) \big|_{r=\infty} = 0 .
\]

(4.12)

[\( d\Omega \) in (4.11) is the usual volume form of the two sphere]. The charge, or equivalently the generator of the global U(1) transformations, incidentally is the analogue of \( q(\Lambda) \). It is got by partial integration on the first term. Thus let

\[
\overline{q}(\Lambda) = - \int d^3 x \partial_i \Lambda(x) E_i(x) - \int d^3 x \Lambda(x) j_0(x) .
\]

(4.13)

This is differentiable in \( E_i \) even if \( \Lambda \big|_{r=\infty} \neq 0 \) and generates the gauge transformation for the gauge group element \( e^{i\Lambda} \). It need not to vanish on quantum states if \( \Lambda \big|_{r=\infty} \neq 0 \), unlike \( \varphi(\Lambda^{(0)}) \) which is associated with the Gauss law \( \overline{q}(\Lambda^{(0)}) \approx 0 \). But if \( \Lambda \big|_{r=\infty} = 0 \), it becomes the Gauss law on partial integration and annihilates all physical states. It follows that if \( (\Lambda_1 - \Lambda_2) \big|_{r=\infty} = 0 \), then \( \overline{q}(\Lambda_1) = \overline{q}(\Lambda_2) \) on physical states which are thus sensitive only to the boundary values of test functions. The nature of their response determines their
charge. The conventional electric charge of QED is $q(\Omega)$ where $\Omega$ is the constant function with value 1.

The constraints $g(\Lambda^{(0)})$ are first class since

$$\{g(\Lambda_1^{(0)}), g(\Lambda_2^{(0)})\} = \frac{k}{2\pi} \int \! d\Lambda_1^{(0)} d\Lambda_2^{(0)} = \frac{k}{2\pi} \int_{\partial D} \! \Lambda_1^{(0)} d\Lambda_2^{(0)} = 0 \quad \text{for } \Lambda_1^{(0)}, \Lambda_2^{(0)} \in \mathcal{T}^{(0)}. \quad (4.14)$$

$g(\Lambda^{(0)})$ generates the gauge transformation $a \rightarrow a + d\Lambda^{(0)}$ of $a$.

Next consider $q(\Lambda)$ where $\Lambda \big|_{\partial D}$ is not necessarily zero. Since

$$\{q(\Lambda), g(\Lambda^{(0)})\} = -\frac{k}{2\pi} \int \! d\Lambda d\Lambda^{(0)} = \frac{k}{2\pi} \int_{\partial D} \! \Lambda^{(0)} d\Lambda = 0 \quad \text{for } \Lambda^{(0)} \in \mathcal{T}^{(0)}, \quad (4.15)$$

they are first class or the observables of the theory. More precisely observables are obtained after identifying $q(\Lambda_1)$ with $q(\Lambda_2)$ if $(\Lambda_1 - \Lambda_2) \in \mathcal{T}^{(0)}$. For then,

$$q(\Lambda_1) - q(\Lambda_2) = -g(\Lambda_1 - \Lambda_2) \approx 0.$$ 

The functions $q(\Lambda)$ generate gauge transformations $a \rightarrow a + d\Lambda$ which do not necessarily vanish on $\partial D$.

It may be remarked that the expression for $q(\Lambda)$ is obtained from $g(\Lambda^{(0)})$ after a partial integration and a subsequent substitution of $\Lambda$ for $\Lambda^{(0)}$. It too generates gauge transformations like $g(\Lambda^{(0)})$, but the test function space for the two are different. The pair $q(\Lambda), g(\Lambda^{(0)})$ thus resemble the pair $q(\bar{\Lambda}), \bar{g}(\bar{\Lambda}^{(0)})$ in QED. The resemblance suggests that we think of $q(\Lambda)$ as akin to the generator of a global symmetry transformation. It is natural to do so for another reason as well: the Hamiltonian is a constraint for a first order Lagrangian such as the one we have here, and for this Hamiltonian, $q(\Lambda)$ is a constant of motion.

22
In quantum gravity, for asymptotically flat spatial slices, it is often the practice to include a surface term in the Hamiltonian which would otherwise have been a constraint and led to trivial evolution. However, we know of no natural choice of such a surface term, except zero, for the CS theory.

The PB’s of $q(Λ)$’s are easy to compute:

$$\{q(Λ_1), q(Λ_2)\} = \frac{k}{2\pi} \int_D dΛ_1 dΛ_2 = \frac{k}{2\pi} \int_{\partial D} Λ_1 dΛ_2 .$$

(4.16)

Remembering that the observables are characterized by boundary values of test functions, (4.16) shows that the observables generate a U(1) Kac-Moody algebra localized on $\partial D$. It is a Kac-Moody algebra for “zero momentum” or “charge”. For if $Λ |_{\partial D}$ is a constant, it can be extended as a constant function to all of $D$ and then $q(Λ) = 0$. The central charges and hence the representation of (4.16) are different for $k > 0$ and $k < 0$, a fact which reflects parity violation by the action $S$.

Let $θ \pmod{2\pi}$ be the coordinate on $\partial D$ and $φ$ a free massless scalar field moving with speed $v$ on $\partial D$ and obeying the equal time PB’s

$$\{φ(θ), ˙φ(θ')\} = δ(θ - θ') .$$

(4.17)

If $μ_i$ are test functions on $\partial D$ and $\partial_± = ∂_ν ± v∂_θ$, then

$$\left\{ \frac{1}{v} \int μ_1(θ) ∂_± φ(θ), \frac{1}{v} \int μ_2(θ) ∂_± φ(θ) \right\} = ±2 \int μ_1(θ) dμ_2(θ),$$

(4.18)

the remaining PB’s being zero. Also $∂_+ ∂_± φ = 0$. Thus the algebra of observables is isomorphic to that generated by the left moving $∂_+ φ$ or the right moving $∂_- φ$.

The CS interaction is invariant under diffeos of $D$. An infinitesimal generator of a diffeo with vector field $V^{(0)}$ is

$$δ(V^{(0)}) = -\frac{k}{2\pi} \int_D V^{(0)} i a_i da .$$

(4.19)

The differentiability of $δ(V^{(0)})$ imposes the constraint

$$V^{(0)} |_{\partial D} = 0 .$$

(4.20)
Hence, in view of (4.4) as well, we have the result

\[ \delta(V^{(0)}) = -\frac{k}{4\pi} \int_D a\mathcal{L}_{V^{(0)}}a \approx 0, \]  

(4.21)

where \( \mathcal{L}_{V^{(0)}}a \) denotes the Lie derivative of the one form \( a \) with respect to the vector field \( V^{(0)} \) and is given by

\[ (\mathcal{L}_{V^{(0)}}a)_i = \partial_j a_i V^{(0)j} + a_j \partial_i V^{(0)j}. \]

Next, suppose that \( V \) is a vector field on \( D \) which on \( \partial D \) is tangent to \( \partial D \),

\[ V^i \mid_{\partial D} (\theta) = \epsilon(\theta) \left( \frac{\partial x^i}{\partial \theta} \right) \mid_{\partial D}, \]  

(4.22)

\( \epsilon \) being any function on \( \partial D \) and \( x^i \mid_{\partial D} \) the restriction of \( x^i \) to \( \partial D \). \( V \) thus generates a diffeo mapping \( \partial D \) to \( \partial D \). Consider next

\[ l(V) = \frac{k}{2\pi} \left( \frac{1}{2} \int_D d(V^i a_i a) - \int_D V^i a_i da \right) = -\frac{k}{4\pi} \int_D a\mathcal{L}_{V}a . \]  

(4.23)

Simple calculations show that \( l(V) \) is differentiable in \( a \) even if \( \epsilon(\theta) \neq 0 \) and generates the infinitesimal diffeo associated with the vector field \( V \). We show in subsection 4.2 that it is, in fact, related to \( q(\Lambda) \)'s by the Sugawara construction.

The expression (4.23) for the diffeo generators of observables given in the first paper of ref. 6 seems to be new.

As final points of this subsection, note that

\[ \{l(V), g(\Lambda^{(0)})\} = g(V^i \partial_i \Lambda^{(0)}) = g(\mathcal{L}_V \Lambda^{(0)}) \approx 0 , \]  

(4.24)

\[ \{l(V), q(\Lambda)\} = q(V^i \partial_i \Lambda) = q(\mathcal{L}_V \Lambda), \]  

(4.25)

\[ \{l(V), l(W)\} = l(\mathcal{L}_V W) \]  

(4.26)

where \( \mathcal{L}_V W \) denotes the Lie derivative of the vector field \( W \) with respect to the vector field \( V \) and is given by

\[ (\mathcal{L}_V W)^i = V^j \partial_j W^i - W^j \partial_j V^i. \]
$l(V)$ are first class in view of (4.24). Further, after the imposition of constraints, they are entirely characterized by $\epsilon(\theta)$, the equivalence class of $l(V)$ with the same $\epsilon(\theta)$ defining an observable.

**4.2 QUANTIZATION**

Our strategy for quantization relies on the observation that if

$$\Lambda |_{\partial D} (\theta) = e^{iN\theta},$$

then the PB's (4.16) become those of creation and annihilation operators. These latter can be identified with the similar operators of the chiral fields $\partial_{\pm}\phi$.

Thus let $\Lambda_N$ be any function on $D$ with boundary value $e^{iN\theta}$:

$$\Lambda_N |_{\partial D} (\theta) = e^{iN\theta}, \quad N \in \mathbb{Z}.$$  \hfill (4.27)

These $\Lambda_N$’s exist. All $q(\Lambda_N)$ with the same $\Lambda_N |_{\partial D}$ are weakly equal and define the same observable. Let $\langle q(\Lambda_N) \rangle$ be this equivalence class and $q_N$ any member thereof. [$q_N$ can also be regarded as the equivalence class itself.] Their PB’s follow from (4.16):

$$\{q_N, q_M\} = -iNk\delta_{N+M,0}.$$  \hfill (4.28)

The $q_N$’s are the CS constructions of the Fourier modes of a massless chiral scalar field on the circle $S^1$.

The CS construction of the diffeo generators $l_N$ on $\partial D$ (the classical analogues of the Virasoro generators) are similar. Thus let

$$< l(V_N) >$$

be the equivalence class of $l(V_N)$ defined by the constraint

$$V_N^i |_{\partial D} = e^{iN\theta} \left( \frac{\partial x^i}{\partial \theta} \right) |_{\partial D}, \quad N \in \mathbb{Z}.$$  \hfill (4.29)
\((x^1, x^2) \mid_{\partial D} (\theta)\) being chosen to be \(R(\cos \theta, \sin \theta)\) where \(R\) is the radius of \(D\). Let \(l_N\) be any member of

\[ < l(V_N) > . \]

It can be verified that

\[ \{l_N, q_M\} = iM q_{N+M} , \quad (4.30) \]

\[ \{l_N, l_M\} = -i(N - M) l_{N+M} . \quad (4.31) \]

These PB’s are independent of the choice of the representatives from their respective equivalence classes. Equations (4.28), (4.30) and (4.31) define the semidirect product of the Kac-Moody algebra and the Witt algebra (Virasoro algebra without the central term) in its classical version.

We next show that

\[ l_N \approx \frac{1}{2k} \sum_M q_M q_{N-M} \quad (4.32) \]

which is the classical version of the Sugawara construction\(^8\).

For convenience, let us introduce polar coordinates \(r, \theta\) on \(D\) (with \(r = R\) on \(\partial D\)) and write the fields and test functions as functions of polar coordinates. It is then clear that

\[ l_N \equiv l(V_N) = \frac{k}{4\pi} \int_{\partial D} d\theta e^{iN\theta} a_\theta^2(R, \theta) - \frac{k}{2\pi} \int_D V_N^l(r, \theta) a_l(r, \theta) da(r, \theta) \quad (4.33) \]

where \(a = a_r dr + a_\theta d\theta\).

Let us next make the choice

\[ e^{iM\theta} \lambda(r), \quad \lambda(0) = 0 , \quad \lambda(R) = 1 \quad (4.34) \]

for \(\Lambda_M\). Then

\[ q_M = q(e^{iM\theta} \lambda(r)) . \quad (4.35) \]

Integrating (4.35) by parts we get

\[ q_M = \frac{k}{2\pi} \left( \int_{\partial D} d\theta e^{iM\theta} a_\theta(R, \theta) - \int_D dr d\theta \lambda(r) e^{iM\theta} f_r\theta(r, \theta) \right) \quad (4.36) \]
where \( f_{r\theta} \) is defined by \( da = f_{r\theta} dr \wedge d\theta \). Therefore

\[
\frac{1}{2k} \sum_M q_{M} q_{N-M} \approx \frac{k}{4\pi} \int_{\partial D} d\theta e^{iN\theta} a_\theta^2(R, \theta) - \frac{k}{2\pi} \int_{D} dr d\theta e^{iN\theta} \lambda(r)a_\theta(R, \theta) f_{r\theta}(r, \theta)
\]

where the completeness relation

\[
\sum_N e^{iN(\theta - \theta')} = 2\pi \delta(\theta - \theta')
\]

has been used.

The test functions for the Gauss law in the last term in (4.37) involves \( f_{r\theta} \) itself. We therefore interpret it to be zero and get

\[
\frac{1}{2k} \sum_M q_{M} q_{N-M} \approx \frac{k}{4\pi} \int_{\partial D} d\theta e^{iN\theta} a_\theta^2(R, \theta) - \frac{k}{2\pi} \int_{D} dr d\theta e^{iN\theta} \lambda(r)a_\theta(R, \theta) f_{r\theta}(r, \theta). \tag{4.38}
\]

Now in view of (4.29) and (4.34), it is clear that

\[
V^l_N(r, \theta) a_l(r, \theta) - e^{iN\theta} \lambda(r)a_\theta(R, \theta) = 0 \quad \text{on} \quad \partial D. \tag{4.39}
\]

Therefore

\[
l_N \approx \frac{1}{2k} \sum_M q_{M} q_{N-M}
\]

which proves (4.32).

We can now proceed to quantum field theory. Let \( G(\Lambda^{(0)}), Q(\Lambda_N), Q_N \) and \( L_N \) denote the quantum operators for \( g(\Lambda^{(0)}), q(\Lambda_N), q_N \) and \( l_N \). We then impose the constraints

\[
G(\Lambda^{(0)})|\cdot\rangle = 0 \tag{4.40}
\]

on all quantum states. It is an expression of their gauge invariance. Because of this equation, \( Q(\Lambda_N) \) and \( Q(\Lambda'_N) \) have the same action on the states if \( \Lambda_N \) and \( \Lambda'_N \) have the same boundary values. We can hence write

\[
Q_N|\cdot\rangle = Q(\Lambda_N)|\cdot\rangle. \tag{4.41}
\]
Here, in view of (4.28), the commutator brackets of $Q_N$ are

$$[Q_N, Q_M] = Nk\delta_{N+M,0} .$$  \hspace{1cm} (4.42)

Thus if $k > 0$ ($k < 0$), $Q_N$ for $N > 0$ ($N < 0$) are annihilation operators (upto a normalization) and $Q_{-N}$ creation operators. The “vacuum” $|0>$ can therefore be defined by

$$Q_N |0>= 0 \text{ if } Nk > 0 .$$  \hspace{1cm} (4.43)

The excitations are got by applying $Q_{-N}$ to the vacuum.

The quantum Virasoro generators are the normal ordered forms of their classical expression\(^{(8)}\):

$$L_N = \frac{1}{2k} : \sum_M Q_M Q_{N-M} :$$  \hspace{1cm} (4.44)

They generate the Virasoro algebra for central charge $c = 1$:

$$[L_N, L_M] = (N-M)L_{N+M} + \frac{c}{12}(N^3 - N)\delta_{N+M,0} , c = 1 .$$  \hspace{1cm} (4.45)

When the spatial slice is a disc, the observables are all given by $Q_N$ and our quantization is complete. When it is not simply connected, however, there are further observables associated with the holonomies of the connection $a$ and they affect quantization. We will not examine quantization for nonsimply connected spatial slices here.

The CS interaction does not fix the speed $v$ of the scalar field in (4.18) and so its Hamiltonian, a point previously emphasized by Fröhlich and Kerler, and Fröhlich and Zee\(^{(4)}\). This is but reasonable. For if we could fix $v$, the Hamiltonian $H$ for $\phi$ could naturally be taken to be the one for a free massless chiral scalar field moving with speed $v$. It could then be used to evolve the CS observables using the correspondence of this field and the former. But we have seen that no natural nonzero Hamiltonian exists for the CS system. It is thus satisfying that we can not fix $v$ and hence a nonzero $H$. \hspace{1cm} 28
In the context of Fractional Quantum Hall Effect, as we have seen in Section 3, the following generalization of the CS action has become of interest:

\[ S' = \frac{k}{4\pi} \mathcal{K}_{IJ} \int_{D \times \mathbb{R}^1} a^{(I)} da^{(J)}. \]  

(4.46)

Here the sum on \( I, J \) is from 1 to \( m \), \( a^{(I)} \) is associated with the current \( j^{(I)} \) in the \( I^{th} \) Landau level and \( \mathcal{K} \) is a certain invertible symmetric real \( F \times F \) matrix. By way of further illustration of our approach to quantization, we now outline the quantization of (4.46) on \( D \times \mathbb{R}^1 \).

The phase space of (4.46) is described by the PB’s

\[ \{a^{(I)}(x), a^{(J)}(y)\} = \epsilon_{IJ} \frac{2\pi}{k} \mathcal{K}_{IJ}^{-1} \delta^2(x - y), \quad x^0 = y^0 \]  

(4.47)

and the first class constraints

\[ g^{(I)}(\Lambda^{(0)}) = \frac{k}{2\pi} \int_D \Lambda^{(0)} da^{(I)} \approx 0, \quad \Lambda^{(0)} \in \mathcal{T}^{(0)}. \]  

(4.48)

with zero PB’s.

The observables are obtained from the first class variables

\[ q^{(I)}(\Lambda) = \frac{k}{2\pi} \int_D d\Lambda a^{(I)} \]  

(4.49)

after identifying \( q^{(I)}(\Lambda) \) with \( q^{(I)}(\Lambda') \) if \( (\Lambda - \Lambda')|_{\partial D} = 0 \). The PB’s of \( q^{(I)}'s \) are

\[ \{q^{(I)}(\Lambda_1^{(I)}), q^{(J)}(\Lambda_2^{(J)})\} = \frac{k}{2\pi} \mathcal{K}_{IJ}^{-1} \int_{\partial D} \Lambda_1^{(I)} d\Lambda_2^{(J)}. \]  

(4.50)

Choose a \( \Lambda_N^{(I)} \) by the requirement \( \Lambda_N^{(I)}|_{\partial D}(\theta) = e^{iN\theta} \) and let \( q_N^{(I)} \) be any member of the equivalence class \( < q^{(I)}(\Lambda_N^{(I)}) > \) characterized by such \( \Lambda_N^{(I)} \). Then

\[ \{q_N^{(I)}, q_M^{(J)}\} = -i \mathcal{K}_{IJ}^{-1} N k \delta_{N+M,0}. \]  

(4.51)

As \( \mathcal{K}_{IJ}^{-1} \) is real symmetric, it can be diagonalized by a real orthogonal transformation \( M \) and has real eigenvalues \( \lambda_\rho (\rho = 1, 2, \ldots, m) \). As \( \mathcal{K}_{IJ}^{-1} \) is invertible, \( \lambda_\rho \neq 0 \). Setting

\[ q_N(\rho) = M_\rho q_N^{(I)} \]  

(4.52)
we have
\[
\{q_N(\rho), q_M(\sigma)\} = -i\lambda_\rho N k \delta_{\rho\sigma} \delta_{N+M,0} .
\] (4.53)

(4.53) is readily quantized. If \( Q_N(\rho) \) is the quantum operator for \( q_N(\rho) \),
\[
[Q_N(\rho), Q_M(\sigma)] = \lambda_\rho N k \delta_{\rho\sigma} \delta_{N+M,0} .
\] (4.54)

(4.54) describes \( m \) harmonic oscillators or edge currents. Their chirality, or the chirality of the corresponding massless scalar fields, is governed by the sign of \( \lambda_\rho \).

The classical diffeo generators for the independent oscillators \( q_N(\rho) \) and their quantum versions can be written down using the foregoing discussion. The latter form \( m \) commuting Virasoro algebras, all for central charge 1.

**ACKNOWLEDGEMENTS**

A.P.B. would like to thank Diptiman Sen for useful discussions, especially regarding the edge currents in a Quantum Hall system. The work of A.P.B. was supported by the U.S. Department of Energy under contract number DE-FG02-85ER40231. The work of A.M.S. was supported by the Theoretical Physics Institute at the University of Minnesota and by the U.S. Department of Energy under contract number DE-AC02-83ER40105.
REFERENCES

No attempt will be made here to compile an adequate bibliography. The following books and papers may be consulted for further guide to literature.

1. For reviews of QHE, see, “The Quantum Hall Effect”, edited by R.E. Prange and S. Girvin (Springer-Verlag, New York, 1987); G. Morandi, “Quantum Hall Effect” (Bibliopolis, Napoli, 1988); A.P. Balachandran, E. Ercolessi, G. Morandi and A.M. Srivastava, “The Hubbard Model and Anyon Superconductivity”, Int. J. Mod. Phys. B4 (1990) 2057 and Lecture Notes in Physics, Volume 38 (World Scientific, Singapore, 1990); F. Wilczek, “Fractional Statistics and Anyon Superconductivity” (World Scientific, Singapore, 1990) and articles therein. See also ref. 2.

2. T. Chakraborty and P. Pietiläinen, “The Fractional Quantum Hall Effect”, (Springer-Verlag, Berlin, 1988).

3. S.C. Jhang, T.H. Hansson and S. Kivelson, Phys. Rev. Lett. 62 (1989) 82; B. Blok and X.G. Wen, Institute for Advanced Study preprint IASSNS -HEP - 90/23 (1990).

4. J. Fröhlich and T. Kerler, Nucl. Phys. B 354 (1991) 369; J. Fröhlich and A. Zee, Institute for Theoretical Physics, Santa Barbara preprint NSF-ITP-91-31 (1991).

5. B.I. Halperin, Phys. Rev. B25 (1982) 2185.

6. A.P. Balachandran, G. Bimonte, K.S. Gupta and A. Stern, Syracuse University preprints SU-4228-477 and 487 (1991).

7. E. Witten, Commun. Math. Phys. 121 (1989) 351.

8. For a review, see P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303.
9. Cf. A.P. Balachandran, G. Marmo, B.-S. Skagerstam and A. Stern, “Classical Topology and Quantum States” (World Scientific, Singapore, 1991), Part I.

10. E. Witten, Nucl. Phys. B311 (1988) 46.
FIGURE CAPTIONS

Figure 1 : Electrons are confined in the annular region $r_1 < r < r_2$ in the 1-2 plane with a uniform magnetic field $B_0$ directed along the third axis in that region. Additional magnetic flux $\Phi$ is confined to the region $r < r_1$.

Figure 2 : Shift of the energy levels at the edges $r = r_1$ and $r = r_2$ due to the boundary effects.