q-Deformed quaternions and su(2) instantons

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Abstract. We have recently introduced the notion of a $q$-quaternion bialgebra and shown its strict link with the $SO_q(4)$-covariant quantum Euclidean space $\mathbb{R}^4_q$. Adopting the available differential geometric tools on the latter and the quaternion language we have formulated and found solutions of the (anti)selfduality equation [instantons and multi-instantons] of a would-be deformed $su(2)$ Yang-Mills theory on this quantum space. The solutions depend on some noncommuting parameters, indicating that the moduli space of a complete theory should be a noncommutative manifold. We summarize these results and add an explicit comparison between the two $SO_q(4)$-covariant differential calculi on $\mathbb{R}^4_q$ and the two 4-dimensional bicovariant differential calculi on the bi- (resp. Hopf) algebras $M_q(2), GL_q(2), SU_q(2)$, showing that they essentially coincide.

1. Introduction

The construction of gauge field theories on noncommutative manifolds has been the subject of quite a lot of work in recent years. A crucial test of it is the search of instantonic solutions, especially after the discovery [29] that deforming $\mathbb{R}^4$ into the Moyal-Weyl noncommutative Euclidean space $\mathbb{R}^4_\theta$ regularizes the zero-size singularities of the instanton moduli space (see also [36]). Various other noncommutative geometries have been considered (see e.g. [9, 4, 10, 25]). They do not always completely fit Connes’ standard framework of noncommutative geometry [7], thus stimulating attempts of generalizations. Among the available deformations of $\mathbb{R}^4$ there is also the Faddeev-Reshetikhin-Takhtadjan noncommutative Euclidean space $\mathbb{R}^4_q$ covariant under $SO_q(4)$ [12]. This, as other quantum group covariant noncommutative spaces (shortly: quantum spaces), is maybe even more problematic for the formulation [23] of a gauge field theory on like $\mathbb{R}^4_q$. One main reason is the lack of a proper (i.e. cyclic) trace to define gauge invariant observables (action, etc). Another one is the $\star$-structure of the differential calculus, which for real $q$ is problematic. Nevertheless, in our main Ref. [19] we have left these two issues aside and investigated about (anti)selfduality equations on it and their solutions. Here we summarize these results adding some detail.

As a first step we recall our notion [19] of a $q$-deformed quaternion as the defining matrix of a copy of $SU_q(2) \times \mathbb{R}_{\geq}^2$ ($\mathbb{R}_{\geq}^2$ denoting the semigroup of nonnegative real numbers), or equivalently of the $2 \times 2$ defining quantum matrix of $M_q(2)$ endowed with the same $\star$-structure of $SU_q(2)$ (more details will be given in [20]), and that its entries can be regarded also as coordinates of $\mathbb{R}^4_q$. As on ordinary $\mathbb{R}^4$, this will much simplify the search and classification of instantons in Yang-Mills theory. We also recall that the quantum sphere $S^4_q$ of [10] can be regarded as a compactification.
of the corresponding $\ast$-algebra. We then show that the two $SO_q(4)$-covariant differential calculi on $\mathbb{R}_q^4$ [5] coincide with the two 4-dimensional bicovariant differential calculi [34, 35] on the bi-
(resp. Hopf) algebras $M_q(2), GL_q(2)$, so that upon imposing the unit $q$-determinant condition one obtains Woronowicz pioneering $4D\pm$ bicovariant differential calculi [43, 33] on $SU_q(2)$ (this had been only announced in [19]). Using the Hodge duality map [16, 17] on $\mathbb{R}_q^4$ in $q$-quaternion language we have formulated (anti)self-duality equations and found [19] solutions $A$, in the form of 1-form valued $2 \times 2$ matrices, that closely resemble their undeformed counterparts (instantons) in $su(2)$ Yang-Mills theory on $\mathbb{R}^4$. [The (still missing) complete gauge theory might be however a deformed $u(2)$ rather than $su(2)$ Yang-Mills theory.]. The projector characterizing the instanton projective module (playing the role of the vector bundle) of [10] in $q$-quaternion language takes exactly the same natural form as in the undeformed theory. The “coordinates of the center” of the instanton are nevertheless noncommuting parameters, differently from the Nekrasov-Schwarz theory. We have also found multi-instantons solutions: they are again parametrized by noncommuting parameters playing the role of “size” and “coordinates of the center” of the (anti)instantons. This indicates that the moduli space of a complete theory should be a noncommutative manifold. This is similar to what was proposed in [22] for $\mathbb{R}_q^4$ for selfdual deformation parameters $\theta_{\mu\nu}$.

2. The $q$-quaternion bialgebra $C(\mathbb{H}_q)$

Any element $X$ in the (undeformed) quaternion $\ast$-algebra $\mathbb{H}$ is given by

$$X = x_1 + x_2i + x_3j + x_4k,$$

with $x \in \mathbb{R}^4$ and imaginary $i, j, k$ fulfilling

$$i^2 = j^2 = k^2 = -1, \quad ijk = -1.$$

Replacing $i, j, k$ by Pauli matrices $\times$ imaginary unit $i$ we get

$$X \leftrightarrow x \equiv \begin{pmatrix} x_1 + x_4 & x_3 + x_2i \\ -x_3 + x_2i & x_1 - x_4i \end{pmatrix} := \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

(where $\alpha, \gamma \in \mathbb{C}$); the quaternionic product becomes represented by matrix multiplication, and the quaternionic conjugation becomes represented by hermitean conjugation of the matrix $x$. Therefore $\mathbb{H}$ essentially consists of all complex $2 \times 2$ matrices of this form.

This can be $q$-deformed as follows. We just pick the pioneering definition of the (Hopf) $\ast$-

algebra $C(SU_q(2))$ [41, 42] without imposing the $\det_q = 1$ condition: for $q \in \mathbb{R}$ consider the unital associative $\ast$-algebra $A \equiv C(\mathbb{H}_q)$ generated by elements $\alpha, \gamma, \alpha^*, \gamma^*$ fulfilling the commutation relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\alpha^* = q\alpha^*\gamma, \quad \gamma^*\alpha^* = q\alpha^*\gamma^*,$$

$$[\alpha, \alpha^*] = (1 - q^2)\beta\gamma^* \quad [\gamma^*, \gamma] = 0.$$  \hfill (1)

Introducing the matrix

$$x \equiv \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} := \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

we can rewrite these commutation relations as

$$\hat{R}x_{12} = x_{12}\hat{R}$$ \hfill (2)

and the conjugation relations as $x^{\alpha\beta\ast} = e^{\beta\gamma}x^{\delta\epsilon}\epsilon_{\delta\alpha}$, i.e.

$$x^\dagger = \bar{x} \quad \text{where } \bar{a} := -1^a T \epsilon \quad \forall a \in M_2. \hfill (3)$$
Here we have used the $\epsilon$-tensor and the braid matrix of $M_q(2), GL_q(2), SU_q(2),\ldots$

$$\epsilon \equiv \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix} = -q\epsilon^{-1}, \quad \hat{R}^{\alpha\beta}_{\gamma\delta} = q\delta^{\alpha\beta}_{\gamma\delta} + \epsilon^{\alpha\beta}\epsilon_{\gamma\delta}. \quad (4)$$

[with $\epsilon \equiv (\epsilon_{\alpha\beta})$ and $\epsilon^{-1} \equiv (\epsilon^{\alpha\beta})$]; note that $\hat{R}^T = \hat{R}$. So $\mathcal{A} := C(\mathbb{H}_q)$ can be endowed also with a bialgebra structure (we are not excluding the possibility that $x \equiv 0_2$), more precisely a real section of the bialgebra $C(M_q(2))$ of $2 \times 2$ quantum matrices [11, 42, 12]. Since the coproduct

$$\Delta(x^{\alpha\gamma}) = (ax)^{\alpha\gamma}$$

is an algebra map, the matrix product $ax$ of any two matrices $a, x$ with mutually commuting entries and fulfilling (2-3) again fulfills the latter. Therefore we shall call any such matrix $x$ a $q$-quaternion, and $\mathcal{A} := C(\mathbb{H}_q)$ the $q$-quaternion bialgebra.

As well-known, the so-called ‘$q$-determinant’ of $x$

$$|x|^2 \equiv \det_q(x) := x^{11}x^{22} - qx^{12}x^{21} = \alpha^*\alpha + \gamma^*\gamma \sim x^{\alpha\alpha'}x^{\beta\beta'}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}, \quad (5)$$

is central, manifestly nonnegative-definite and group-like. Therefore at representation level it will annihilate a state iff $x$ does. Relations (2) can be also equivalently reformulated as

$$x\bar{x} = \bar{x}x = |x|^2 I_2 \quad (6)$$

($I_2$ denotes the unit $2 \times 2$ matrix). If we extend $C(\mathbb{H}_q)$ assuming the existence of a new (central, positive-definite) generator $|x|^{-1}$ (this will imply that $x$ cannot vanish on any state), one finds that $x$ is invertible with inverse

$$x^{-1} = \frac{x}{|x|^2}. \quad (7)$$

$C(\mathbb{H}_q)$ becomes a Hopf $*$-algebra [a real section of $C(GL_q^+(2))$. The matrix elements of $T := \frac{x}{|x|}$ fulfill the relations (2) and

$$T^\dagger = T^{-1} = T, \quad \det_q(T) = 1, \quad (8)$$

namely generate as a quotient algebra $C(SU_q(2))$ [41, 42], therefore in this case the entries of $x$ generate the (Hopf) $*$-algebra of functions on the quantum group $SU_q(2) \times GL^+(1)$, in analogy with the $q = 1$ case.

3. Identification of $\mathbb{H}_q$ with $\mathbb{R}_q^4$, and links with other algebras

As a $*$-algebra, $\mathcal{A} := C(\mathbb{H}_q)$ coincides with the algebra of functions on the $SO_q(4)$-covariant quantum Euclidean Space $\mathbb{R}_q^4$ of [12], identifying their generators as

$$x^1 = qx^{11}, \quad x^2 = x^{12}, \quad x^3 = -qx^{21}, \quad x^4 = x^{22}. \quad (9)$$

We shall denote by $B \equiv (B^a_{\alpha\alpha'})$ this (diagonal and invertible) matrix entering the linear transformation $x^a = B^a_{\alpha\alpha'}x^{\alpha\alpha'}$. We illustrate the relation between the two starting from the braid matrix of $SO_q(4)$, which is obtained as

$$\hat{R} \equiv (\hat{R}^{ab}_{cd}) = q^{-1}B(\hat{R} \otimes_C \hat{R})B^{-1} \quad (10)$$

($\hat{R}$ fulfills the braid equation because $\hat{R}$ does), where $B^{ab}_{\alpha\alpha'\beta\beta'} := B^a_{\alpha\alpha'}B^b_{\beta\beta'}$. Its decomposition

$$\hat{R} = qP_a - q^{-1}P_A + q^{-3}P_8 \quad (11)$$
in orthogonal projectors follows from that of the braid matrix of \( M_q(2), GL_q(2), SU_q(2) \),

\[
\hat{R} = qP_s - q^{-1}P_a,
\]  

(12)

since \( \mathcal{P} := \mathcal{B}(\mathcal{P} \otimes \mathcal{P}^{'}) \mathcal{B}^{-1} \) is a projector whenever \( \mathcal{P}, \mathcal{P}' \) are\(^1\) In fact,

\[
\begin{align*}
P_s & = \mathcal{B}(\mathcal{P}_s \otimes \mathcal{P}_s) \mathcal{B}^{-1}, \\
P_t & = \mathcal{B}(\mathcal{P}_t \otimes \mathcal{P}_t) \mathcal{B}^{-1}, \\
P_a & = \mathcal{B}(\mathcal{P}_a \otimes \mathcal{P}_a) \mathcal{B}^{-1}, \\
P_{a'} & = \mathcal{B}(\mathcal{P}_{a'} \otimes \mathcal{P}_{a'}) \mathcal{B}^{-1}, \\
P_A & = \mathcal{P}_a + \mathcal{P}_{a'}.
\end{align*}
\]  

(14)

\( \mathcal{P}_s, \mathcal{P}_a, \mathcal{P}_t, \mathcal{P}_{a'} \) are respectively \( GL_q(2) \)-covariant deformations of the symmetric and antisymmetric projectors, and have dimension 3.1. They can be expressed in terms of the \( q \)-deformed \( \epsilon \)-tensor by

\[
\begin{align*}
\mathcal{P}_{\alpha \beta} & = - \frac{\epsilon^{\alpha \beta} \epsilon_{\gamma \delta}}{q + q^{-1}}, \\
\mathcal{P}_{\alpha \beta} & = \delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} + \frac{\epsilon^{\alpha \beta} \epsilon_{\gamma \delta}}{q + q^{-1}}.
\end{align*}
\]  

(15)

where the 4 \( \times \) 4 matrix \( g_{ab} \) (denoted as \( C_{ab} \) in [12]) is given by

\[
g_{ab} = B_{a}^{-1} \alpha \alpha' B_{b}^{-1} \beta \beta' \epsilon_{\alpha \beta} \epsilon_{\alpha' \beta'},
\]  

(17)

it is the \( SO_q(4) \)-isotropic 2-tensor, deformation of the ordinary Euclidean metric, and “Killing form” of \( U_{q, so}(4) \).

The commutation relations and \( \ast \)-conjugation relations are preserved by matrix multiplication

\[
x \rightarrow a x b.
\]  

(18)

by the defining matrices \( a, b \) of two copies \( SU_q(2), SU_q(2) \) of the special unitary quantum group, or of two copies \( \mathbb{H}_q, \mathbb{H}_q' \) of the quaternion quantum group, respectively, whose entries commute with each other and with the entries of \( x \). This follows from the fact that the twofold coproduct \( \Delta(2)(x) = axb \) is a \( \ast \)-homomorphism. In other words they are covariant under the (mixed left-right) coactions of \( SU_q(2) \times SU_q(2) = Spin_q(4) \) and of \( \mathbb{H}_q \times \mathbb{H}_q' \). The transformation (18) can be reformulated as the (left) \( SO_q(4) \) (resp. \( SO_q(4) \)) coaction

\[
\Delta_L(x) = \mathbf{T}_j^i \otimes x^j, \\
\mathbf{T}_j^i := B_{a}^{-1} \alpha \alpha' B_{b}^{-1} \beta \beta',
\]  

(19)

In fact, the \( \mathbf{T}_j^i \) are manifestly invariant under the \( \mathbb{Z}_2 \) action defined by the change of signs \( (a, b) \rightarrow (-a, -b) \) and fulfill [19] the defining relations [12] for the generators of the quantum

\(^1\) The orthonormality relations for the \( \mathcal{P}_\mu \), with \( \mu = s, a \),

\[
\mathcal{P}_\mu \mathcal{P}_\nu = \mathcal{P}_\mu \delta_{\mu \nu}, \\
\sum_{\mu} \mathcal{P}_\mu = I,
\]  

(13)

trivially imply the orthonormality relations for the \( \mathcal{P}_\mu \), with \( \mu = s, a, a', t \).
groups \( SO_q(4) = SU_q(2) \times SU_q(2) / \mathbb{Z}_2 \) and of the extension \( \widehat{SO_q(4)} := SO_q(4) \times GL^+(1) = \mathbb{H}_q \times \mathbb{H}_q / GL(1) \) (the quantum group of rotations and scale transformations in 4 dimensions), respectively.

A different matrix version (with no interpretation in terms of \( q \)-deformed quaternions) of a \( SU_q(2) \times SU_q(2) \) covariant quantum Euclidean space was proposed in [27].

Define

\[
\alpha' = \sqrt{2} e^{i\alpha} / (1 + 2|q|^2), \quad \alpha'^* = \sqrt{2} e^{-i\alpha} / (1 + 2|q|^2),
\]

\[
\beta' = \sqrt{2} e^{i\beta} / (1 + 2|q|^2), \quad \beta'^* = \sqrt{2} e^{-i\beta} / (1 + 2|q|^2),
\]

\[
z = 1 - 2|x|^2 / (1 + 2|x|^2)
\]

where \( \alpha, \beta, ... \) fulfill (1) and \( e^{i\alpha}, e^{i\beta} \in U(1) \) are possible phase factors. Then \( \alpha', \beta', z \) fulfill the defining relation (1) of the \( C^* \)-algebra considered in Ref. [10] (where these elements are respectively denoted as \( \alpha, \beta, z \)), in particular

\[
\alpha' \alpha'^* + \beta' \beta'^* + z^2 = 1,
\]

which shows that the noncommutative manifold is a deformation \( S_q^4 \) of the 4-sphere. The invertible function \( z(|x|) \) spans \([-1, 1]\), i.e. all the spectrum of \( z \) except the eigenvalue \( z = 1 \), as \( |x| \) spans all its spectrum \([0, \infty]\).

The redefinitions (20) have exactly the form of a stereographic projection of \( \mathbb{R}^4 \) on a sphere \( S^4 \) of unit radius (recall that \( x \cdot x = 2|x|^2 \)): \( S^4 \) is the sphere centered at the origin and \( \mathbb{R}^4 \) the subspace \( z = 0 \) immersing both in a \( \mathbb{R}^5 \) with coordinates defined by \( X \equiv (Re(\alpha'), Im(\alpha'), Re(\beta'), Im(\beta'), z) \). In the commutative theory the point \( X = (0, 0, 0, 0, 1) \) of \( S^4 \) is the point at infinity of \( \mathbb{R}^4 \), therefore going from \( \mathbb{R}^4 \) to \( S^4 \) amounts to compactifying \( \mathbb{R}^4 \) to \( S^4 \). We can thus regard the transition from our algebra to the one considered in Ref. [10] as a compactification of \( \mathbb{R}^4_q \) into their \( S^4_q \).

### 4. Other preliminaries

The \( SO_q(4) \)-covariant differential calculus \((d, \Omega^*)\) on \( \mathbb{R}^4_q \sim \mathbb{H}_q \) [5] is obtained imposing covariant homogeneous bilinear commutation relations (23) between the \( x^a \) and the differentials \( \xi^a := dx^a \). Partial derivatives are introduced through the decomposition \( d = \xi^a \partial_a = \xi^{\alpha\alpha'} \partial_{\alpha\alpha'} \).

All other commutation relations are derived by consistency. The complete list is given by

\[
P_{\alpha\beta\gamma\delta} x^\gamma x^\delta = 0, \quad \Leftrightarrow \quad x^{\alpha\alpha'} x^{\beta\beta'} = \hat{R}^{\alpha\beta}_{\gamma\delta} \hat{R}^{-1} \gamma\gamma' x^{\gamma'} x^{\delta\delta'},
\]

\[
x^{\alpha\alpha'} x^{\beta\beta'} = \hat{R}^{\alpha\beta}_{\gamma\delta} \hat{R}^{-1} \gamma\gamma' x^{\gamma'} x^{\delta\delta'},
\]

\[
(P_s + P_{ij}) x_h x^i x^j = 0 \quad \Leftrightarrow \quad \partial_{\alpha\alpha'} x^{\beta\beta'} = \hat{R}^{\delta\gamma}_{\beta\alpha} \hat{R}^{-1} \gamma\gamma' x^{\gamma'} x^{\delta\delta'},
\]

\[
\partial_{\alpha\alpha'} x^{\beta\beta'} = \hat{R}^{\delta\gamma}_{\beta\alpha} \hat{R}^{-1} \gamma\gamma' x^{\gamma'} x^{\delta\delta'},
\]

\[
\partial x^i = \delta^i_i + q \hat{R}^{ji}_{k\ell} x^j \partial_{\ell}, \quad \partial_{\alpha\alpha'} x^{\beta\beta'} = \hat{R}^{\delta\gamma}_{\beta\alpha} \hat{R}^{-1} \gamma\gamma' x^{\gamma'} x^{\delta\delta'},
\]

\[
\partial x^i = \delta^i_i + q \hat{R}^{ji}_{k\ell} x^j \partial_{\ell}, \quad \partial_{\alpha\alpha'} x^{\beta\beta'} = \hat{R}^{\delta\gamma}_{\beta\alpha} \hat{R}^{-1} \gamma\gamma' x^{\gamma'} x^{\delta\delta'},
\]
The Laplacian $\square \equiv \partial \cdot \partial := \partial_k g^{hk} \partial_h$ is $SO_q(4)$-invariant and commutes the $\partial_i$. In $\mathcal{H}$ there exists a special invertible element $\Lambda$ such that

$$\Lambda x^i = q^{-1} x^i \Lambda, \quad \Lambda \partial^i = q \partial^i \Lambda, \quad \Lambda \xi^i = \xi^i \Lambda.$$  

**Definitions:**

- $\bigwedge \overset{\xi}{=}^p$ graded algebra generated by the $\xi^i$, where grading $\xi \equiv \text{degree in } \xi^i$; any component $\bigwedge \overset{\xi}{=}^p$ with $\xi = p$ carries an irreducible representation of $U_q so(4)$ and has the same dimension as in the $q = 1$ case.
- $\mathcal{D} \overset{\partial}{=}^p \equiv \xi$-graded algebra generated by $x^i, \xi^i, \partial_i$. Elements of $\mathcal{D} \overset{\partial}{=}^p$ are differential-operator-valued $p$-forms.
- $\Omega \overset{\xi}{=}^p \equiv \xi$-graded subalgebra generated by the $\xi^i, x^i$. By definition $\Omega^0 = A$ itself, and both $\Omega^*$ and $\Omega^p$ are $\mathcal{A}$-bimodules. Also, we shall denote $\Omega^*$ enlarged with $\Lambda^\pm 1$ as $\tilde{\Omega}^*$, and the subalgebra generated by $T^{\alpha \alpha'} := x^{\alpha \alpha'}/|x|$, $dT^{\alpha \alpha'}$ as $\tilde{\Omega}_S^*$ (the latter is 4-dim! See below).
- $\mathcal{H} \equiv$ subalgebra generated by the $x^i, \partial_i$. By definition, $\mathcal{D} \overset{\partial}{=}^0 = \mathcal{H}$, and both $\mathcal{D} \overset{\partial}{=}^*$ and $\mathcal{D} \overset{\partial}{=}^p$ are $\mathcal{H}$-bimodules.

The special $SO_q(4)$-invariant 1-form

$$\theta := \frac{1}{1 - q^{-2}} |x|^{-2} d|x|^2 = \frac{q^{-2}}{q^2 - 1} \xi^{\alpha \alpha'} x^{\beta \beta'} \frac{d \xi^{\alpha \alpha'}}{|x|^2} \epsilon_{\alpha \beta} \epsilon_{\alpha' \beta'},$$

plays the role of ”Dirac Operator” [7] of the differential calculus,

$$d \omega = \{-\theta, \omega\} \equiv -\theta \omega + (-)^p \omega_\theta, \quad \omega_\theta \in \Omega^p,$$

$\theta$ is closed:

$$d \theta = 0, \quad \theta^2 = 0. \quad (28)$$

Applying $d$ to (6) we find

$$x\xi + \xi \bar{x} = (q^2 - 1)\theta |x|^2 I_2, \quad \bar{x}\xi + \bar{\xi}x = (q^2 - 1)\theta |x|^2 I_2. \quad (29)$$

Relation (23) implies $|x|^2 \xi^i = q^2 \xi^i |x|^2$, which we generalize as usual to

$$|x|^{\pm 1} \xi^i = q^{\pm 1} \xi^i |x|^\pm 1, \quad \Rightarrow \quad |x|^{\pm 1} \theta = q^{\pm 1} \theta |x|^\pm 1. \quad (30)$$

However, $d(f^*) \neq (df)^*$, and moreover there is no $\ast$-structure $\ast : \Omega^* \to \Omega^*$, but only a $\ast$-structure

$$\ast : \mathcal{D} \overset{\ast}{=}^* \to \mathcal{D} \overset{\ast}{=}^*$$

[31], with a rather nonlinear character (the latter has been recently [18] recast in a much more suggestive form).

The **Hodge map** [16, 17] is a $SO_q(4)$-covariant, $\mathcal{A}$-bilinear map $\ast : \Omega^p \to \tilde{\Omega}^{4-p}$ such that $\ast^2 = \text{id}$, defined by

$$\ast(\xi^{i_1} \ldots \xi^{i_p}) = q^{-4(p-2)} c_{i_1} \xi^{i_{p+1}} \ldots \xi^{i_3} \xi_{i_1} \ldots i_{p+1} i_1 \ldots i_p \Lambda^{2p-4},$$

where $c_{i_1} \ldots i_{p+1} i_1 \ldots i_p$ are suitable coefficients.
where in our normalization the $\epsilon^{hijk} \equiv q$-epsilon tensor is given by

| $\epsilon^{2 \cdot 1 \cdot 12}$ | $q^{-2}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 12}$ | $= q^{-2}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 1}$ | $= q^{-1}$ |
|-----------------|---------|-----------------|----------|-----------------|--------|-----------------|--------|
| $\epsilon^{2 \cdot 2 \cdot 1 \cdot 12}$ | $= 1$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 1}$ | $= 1$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 1}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 1 \cdot 2}$ | $= q^{-1}$ |
| $\epsilon^{2 \cdot 1 \cdot 2}$ | $= 1$ | $\epsilon^{2 \cdot 1 \cdot 2}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 1 \cdot 2}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 1 \cdot 2}$ | $= q^{-1}$ |
| $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= -1$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ |
| $\epsilon^{2 \cdot 1 \cdot 2 \cdot 1}$ | $= 1$ | $\epsilon^{2 \cdot 1 \cdot 2 \cdot 1}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 1 \cdot 2 \cdot 1}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 1 \cdot 2 \cdot 1}$ | $= q^{-1}$ |
| $\epsilon^{2 \cdot 1 \cdot 2 \cdot 1}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ |
| $\epsilon^{2 \cdot 1 \cdot 2 \cdot 1}$ | $= -q$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= 1$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ |
| $\epsilon^{2 \cdot 1 \cdot 2 \cdot 1}$ | $= -q$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= -q$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ |
| $\epsilon^{2 \cdot 1 \cdot 2 \cdot 1}$ | $= -q$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= -q$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ | $\epsilon^{2 \cdot 2 \cdot 1 \cdot 2}$ | $= q^{-1}$ |

and $c_p$ are suitable normalization factors [17]. Actually this extends to a $H$-bilinear map $*: DC^p \to DC^{1-p}$ with the same features. For $p = 2$ the powers of $\Lambda$ disappear and one even gets a map $*: \Omega^2 \to \Omega^2$ defined by

$$\ast \xi^i \xi^j := \frac{1}{[2]_q} \xi^h \xi^k \varepsilon_{hk}^i j = (P_a - P_a')^i j \xi^h \xi^k,$$

(31)

where $P_a, P_a'$ were defined in (14) and $[2]_q = q + q^{-1}$; the second equality can be proved by a direct computation. $\Omega^2$ (resp. $DC^2$) splits into the direct sum of $A$- (resp. $H$-) bimodules

$$\Omega^2 = \Omega^2 \oplus \Omega^{2'} \quad (\text{resp. } DC^2 = DC^2 \oplus DC^{2'})$$

of the eigenspaces of $*$ with eigenvalues $1, -1$ respectively, whose elements are “self-dual and anti-self-dual 2-forms”. $\Omega^2$ (resp. $DC^2$) is generated by the self-dual exterior forms $(\xi \xi)^{\alpha \beta}$, or equivalently by the ones

$$f^{\alpha \beta} := (\xi \xi)^{\alpha \beta}$$

(32)

through (left or right) multiplication by elements of $A$ (resp. $H$). $f^{\alpha \beta}$ span a $(3,1)$ corepresentation space of $SU_q(2) \times SU_q(2)'$.

One can find 1-form-valued matrices $a$ such that

$$da^{\alpha \beta} = f^{\alpha \beta};$$

(33)

$a$ is uniquely determined to be

$$a^{\alpha \beta} = P^{\alpha \beta}_{\gamma \delta} (\xi \varepsilon x^T)^{\gamma \delta},$$

(34)

if we require $a^{\alpha \beta}$ to transform as $f^{\alpha \beta}$, i.e. in the $(3,1)$ dimensional corepresentation of $SU_q(2) \times SU_q(2)'$, whereas will be defined up to $d$-exact terms of the form

$$\tilde{a} = a + 1_2 dM(|x|^2)$$

if we just require $\tilde{a}^{\alpha \beta}$ to be in the $(3,1) \oplus (1,1)$ reducible representation. In particular, the 1-form valued matrix

$$\tilde{a} := -\xi \varepsilon,$$

(35)

as well as the one $(dT)\overline{T}$ (see section 5), belong to the latter, therefore are invariant under the right coaction of $SU_q(2)$. In the $q = 1$ limit (34) becomes

$$a^{\alpha \beta} = \left(\xi \varepsilon x^T\right)^{(\alpha \beta)} = -\{Im(\xi \varepsilon)\}^{\alpha \beta}.$$

Similarly, antiself-dual $\Omega^{2'}$, $DC^{2'}$ are generated by $(\xi \xi)^{\alpha' \beta'}$, or equivalently by

$$f'^{\alpha' \beta'} := (\xi \xi)^{\alpha' \beta'},$$

(36)
and one can find 1-forms $d^{\alpha\beta'}$ such that $d d^{\alpha\beta'} = f^{\alpha\beta'}$, etc.

**Integration over** $\mathbb{R}^4_q$ [38, 14, 15] can be introduced by the decompositon

$$\int_{\mathbb{R}^4_q} d^4x = \int_0^\infty d|x| \int_{|x| S_q^2} d^3T$$

Integration over the radial coordinate has to fulfill the scaling property $\int_0^\infty d|x| g(|x|) = \int_0^\infty d(q|x|) g(q|x|)$. Integration over the quantum sphere $S_q^3$ is determined up to normalization by the requirement of $SO_q(4)$-invariance. The algebra of functions on the quantum sphere $S_q^3$ is generated by the $T^{\alpha\beta} := x^{\alpha\beta}/|x|$. This integration over $\mathbb{R}^4_q$ fulfills all the main properties of Riemann integration over $\mathbb{R}^4$, including Stokes’ theorem, except the cyclic property, which is $q$-deformed.

5. **Connection with the bicovariant differential calculi on** $GL_q(2)$ and $SU_q(2)$

We start by recalling that an alternative calculus $(\hat{\Omega}, \hat{d})$ on $\mathbb{R}^4_q$ is obtained by replacing $\hat{R} \leftrightarrow \hat{R}^{-1}$, $q \leftrightarrow q^{-1}$ in relations (23),

$$x^{\beta} \xi^{\hat{\beta}} = q^{-1} \hat{R}^{-1} \hat{x}^{\beta} \hat{x}^{\hat{\beta}} \xi^{\hat{\beta}} \iff x^{\alpha\beta} \xi^{\gamma\delta} = \hat{R}^{-1} \hat{x}^{\alpha\beta} \hat{x}^{\gamma\delta},$$

(23)

and in the following ones [(22) is invariant under these replacements]. As just done, we shall add a $\hat{}$ to label these formulae and the corresponding objects after the replacements.

We first show that the two differential calculi on $\mathbb{R}^4_q$ coincide with the two bicovariant differential calculi on $M_q(2), GL_q(2)$ [34, 35]. We recall that a differential calculus is completely determined by the Leibniz rule and nilpotency for the exterior derivative and by the commutation relations between the generators of the algebra and their differentials. For our calculus $(\hat{\Omega}, \hat{d})$ the latter read (23), whereas for the calculus on $M_q(2), GL_q(2)$ they are (13) in [35]. Now it is straightforward to check that indeed relation (23), in the matrix formulation at the right, amounts to relation (13) in [35], provided we identify $x \rightarrow A$ and recall that $\hat{R} := PR (P$ denoting the permutation matrix), $\hat{T} = \hat{R}$. To complete the ‘dictionary’ we add that our $T, \theta, \xi x$ have to be identified with $T, (q^{-1} - q)^{-1} \xi, -\Omega$ of [35].

We now verify that, restricting as in [35] either calculus to the subalgebra generated by the $T^{\alpha\alpha'} = x^{\alpha\alpha'}/|x|^2$, one obtains differential calculi $(\Omega^\ast, \tilde{d})$, $(\hat{\Omega}^\ast, \tilde{d})$ on $SL_q(2)$, which coincide with Woronowicz 4D parallel calculus $[43, 33]$.

Introduce the 1-form valued matrix $\omega := \xi x/|x|^2$. Using (22), (23), (30), $\hat{T} = \hat{R}$ and

$$\epsilon_{\alpha\lambda} \hat{R}^{\pm 1} \lambda^\mu \beta^\gamma = q^{\pm 1} \hat{R}^{\pm 1} \mu^\alpha \lambda^\gamma \epsilon_{\alpha\beta},$$

[which is a consequence of (4)] it is easy to show that

$$T^{\alpha\alpha'} \omega^{\beta\beta'} = q^{-1} \hat{R}^{\alpha\beta} \hat{R}^{\gamma\delta}_{\beta\gamma} \hat{R}^{\mu\lambda}_{\alpha\delta} \omega^{\lambda\mu} T^{\gamma\alpha'},$$

(37)

On the other hand, by a straightforward computation one finds

$$dT^{\alpha\alpha'} = q^{-1} \xi x^{\alpha'}/|x| + (q^{-1} - 1) \theta T^{\alpha\alpha'},$$
This 1-form-valued ‘Maurer-Cartan’ $2 \times 2$ matrix and the one $(d\bar{T})T$ are by (18) manifestly invariant under respectively the right and left coaction of $SU_q(2)$, or equivalently under the $SU_q(2)'$ and the $SU_q(2)$ part of $SO_q(4)$ coaction. Setting $Q := -\epsilon^{-1}c^T$ one finds

$$\text{tr}[Q(dT)\bar{T}] = \text{tr}[Q^{-1}(d\bar{T})T] = (q-1)(q-q^{-2})\theta;$$

(39)

only in the $q \to 1$ limit these traces vanish. That’s why for generic $q \neq 1$ the four matrix elements of either $(dT)\bar{T}$ or $(d\bar{T})T$ are independent (4-dimensional calculus) and make up alternative bases for $\Omega^*$. Moreover, we see that for $q \neq 1$ the ‘Dirac operator’ $\theta$ can be expressed purely in terms of the matrix elements of $dT$ and $T$, in other words the restriction $(d, \Omega^*)$ of the above calculus to $C(SU_q(2))$ is well defined and 4-dimensional. From (38) one sees that the matrix elements $\omega^{\beta\gamma}$ make up an alternative basis of $\Omega^*_M$; their commutation relations (37) with the $T^\alpha\beta$ completely specify the first calculus. Similarly, setting for the other calculus $\bar{\omega} := \xi|x|^2$ we find

$$T^\alpha\beta, \omega^{\gamma\delta} = q\hat{R}^{-1\alpha\beta}_{\lambda\delta} \hat{R}^{-1\mu\beta}_{\gamma\delta} \omega^{\lambda\mu} T^{\gamma\alpha},$$

(37)

$$\text{tr}[Q(dT)\bar{T}] = \text{tr}[Q^{-1}(d\bar{T})T] = (q-1)(q-q^{-2})\hat{\theta};$$

(39)

and

Let us compare now our results with Woronowicz 4D+ bicovariant differential calculus on $C(SU_q(2))$ [43, 33]. We describe the latter in the $R$-matrix formalism, as done in Ref. [6], where the matrix $T$ was denoted as $M$. Comparing formula (5.8) of the latter with our (3) leads to identify our bi-invariant 1-form ‘Dirac operator’ $\hat{\theta}$ with their $-X/N$. This is consistent as we then find that our (39) coincides with their (5.26) (with $N = 2$). Formula (5.23) of [6] [$\kappa$ denotes the antipode, so $\kappa(M)$ is our $T^{-1} = \bar{T}$] leads to identify our right invariant 1-form valued matrix $(dT)\bar{T}$ with our $T^\alpha\beta$; further comparison of formula (5.25) of [6] with our (38) leads to identify our $\omega^{\alpha\beta}$ with their $\theta^{\alpha\beta} (1 - q^2)/N q^3$ (but they use latin letters instead of greek ones to label matrix rows and columns). This is consistent because the commutation relations (37) coincide with the commutation relations for the $\theta^{\alpha\beta}$, which one obtains after little work from their formulae (4.14), (3.16), (3.20) and the $\ast$-conjugates of the latter. Therefore the differential calculus $(\Omega^*_M, d)$ coincides with Woronowicz 4D+ bicovariant one. Similarly one shows that the differential calculus $(\Omega^*_M, \bar{d})$ coincides with Woronowicz 4D- bicovariant one.

We end by noting that the above identifications and our results about the Hodge map give as a bonus a well-defined Hodge operator $\ast$ and (anti)seldual 2-forms on $M_q(2), GL_q(2), SL_q(2), SU_q(2)$. On $M_q(2), GL_q(2)$ (anti)seldual 2-forms are respectively the $(\xi \xi)_{\alpha\beta}$, $(\xi \xi)^{\alpha\beta}$, whereas on $SL_q(2), SU_q(2)$ are respectively obtained dividing $(\xi \xi)_{\alpha\beta}$, $(\xi \xi)^{\alpha\beta}$ by $|x|^2$ and expressing the results in term of $T, dT$ only:

$$v^{\alpha\beta} := (\xi \xi)_{\alpha\beta} q^{-1} \frac{1}{|x|^2} = [q^2 T\theta \bar{T} \theta + \theta T\bar{T} \theta]^{\alpha\beta} = [q^2 (dT)(d\bar{T}) + (q^2 - 1)(dT)\theta \bar{T}]^{\alpha\beta}$$

(40)

$$v^{\alpha\beta} := (\xi \xi)^{\alpha\beta} q^{-1} \frac{1}{|x|^2} = [q^2 T\theta \bar{T} \theta + \theta T\bar{T} \theta]^{\alpha\beta} = [q^2 (dT)(d\bar{T}) + (q^2 - 1)(dT)\theta \bar{T}]^{\alpha\beta}$$

(41)

fulfill

$$\ast v^{\alpha\beta} = v^{\alpha\beta}, \quad \ast v^{\alpha\beta} = -v^{\alpha\beta'}$$

(42)

[and similarly for the other calculus $(\Omega^*_M, \bar{d})$].
6. Formulations of noncommutative gauge theories

We recall some minimal common elements in the formulations of \( U(n) \) gauge theories on commutative as well as noncommutative spaces \([7, 24, 13, 26]\). In \( U(n) \) gauge theory the gauge transformations \( U \) are unitary \( \mathcal{A} \)-valued (\( \mathcal{A} \) being the algebra of functions on the noncommutative manifold) \( n \times n \) matrices, \( U \in M_n(\mathcal{A}) \equiv M_n(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{A} \). The gauge potential \( A \equiv (A^\alpha_\beta) \) is a \( 1 \)-form-valued \( n \times n \) matrix, \( A \in M_n(\Omega^1(\mathcal{A})) \). The definition of the field strength \( F \in M_n(\Omega^2(\mathcal{A})) \) associated to \( A \) is as usual \( F := dA + AA \). At the right-hand side the product \( AA \) has to be understood both as a (row by column) matrix product and as a wedge product. Even for \( n = 1 \), \( AA \neq 0 \), contrary to the commutative case. The Bianchi identity \( DF := dF + [A,F] = 0 \) is automatically satisfied and the Yang-Mills equation reads as usual \( D^*F = 0 \). Because of the Bianchi identity, the latter is automatically satisfied by any solution of the (anti)self-duality equations

\[ *F = \pm F. \tag{43} \]

The Bianchi identity, the Yang-Mills equation, the (anti)self-duality equations, the flatness condition \( F = 0 \) are preserved by gauge transformations

\[ A^U = U^{-1}(AU + dU), \quad \Rightarrow \quad F^U = U^{-1}FU. \]

As usual, \( A = U^{-1}dU \) implies \( F = 0 \). Up to normalization factors, the gauge invariant ‘action’ \( S \) and ‘Pontryagin index’ (or ‘second Chern number’) \( Q \) are defined by

\[ S = \text{Tr}(F^*F), \quad Q = \text{Tr}(FF) \tag{44} \]

where \( \text{Tr} \) stands for a positive-definite trace combining the \( n \times n \)-matrix trace with the integral over the noncommutative manifold (as such, \( \text{Tr} \) has to fulfill the cyclic property). If integration \( \int \) fulfills itself the cyclic property then this is obtained by simply choosing \( \text{Tr} = \int \text{tr} \), where \( \text{tr} \) stands for the ordinary matrix trace. \( S \) is automatically nonnegative.

In commutative geometry the so-called Serre-Swan theorem \([37, 8]\) states that vector bundles over a compact manifold coincide with finitely generated projective modules \( \mathcal{E} \) over \( \mathcal{A} \). The gauge connection \( A \) of a gauge group (fiber bundle) acting on a vector bundle is expressed in terms of the projector \( \mathcal{P} \) characterizing the projective module. Therefore these projectors can be used to completely determine the connections. In Connes’ standard approach \([7]\) to noncommutative geometry the finitely generated projective modules are the primary objects to define and develop the gauge theory. The topological properties of the connections can be classified in terms of topological invariants (Chern numbers), and the latter can be computed directly in terms of characters of \( \mathcal{P} \) (Chern-Connes characters), in particular \( Q \) can be computed in terms of the second Chern-Connes character, when Connes’ formulation of noncommutative geometry applies.

In the present \( \mathcal{A} \equiv C(\mathbb{R}^4_q) = C(\mathbb{H}_q) \) case there are 2 main problems preventing the application of this formulation of gauge theories:

(i) Integration over \( \mathbb{R}^4_q \) fulfills a deformed cyclic property \([38]\).
(ii) \( d(f^*) \neq (df)^* \), and there is no \(*\)-structure \(* : \Omega^* \to \Omega^* \), but only a \(*\)-structure \(* : \mathcal{D}C^* \to \mathcal{D}C^* \) \([31]\), with a nonlinear character.

A solution to both problems might be obtained

(i) allowing for \( \mathcal{D}C^1\)-valued \( A \Rightarrow \mathcal{D}C^2\)-valued \( F \)'s), and/or
(ii) realizing \( \text{Tr}(\cdot) \) by in the form \( \text{Tr}(\cdot) := \int \text{tr}(W \cdot) \), with \( W \) some suitable positive definite \( \mathcal{H} \)-valued (i.e. pseudo-differential-operator-valued) \( n \times n \) matrix (this implies a change in the hermitean conjugation of differential operators), or even a more general form.

This hope is based on our results \([18]\).
7. The (anti)instanton solution

We first recall the commutative \((q = 1)\) solution of the self-duality eq. \(*F = F\): the instanton solution of [3] in t’Hooft [39] and in ADHM [2] quaternion notation (see [1] for an introduction) reads:

\[
A = dx^i a^a \eta^a_{ij} x^j \left( \frac{1}{\rho^2 + r^2/2} \right)^{A^a_i},
\]

\[
= -Im \left\{ \frac{\xi}{|x|^2} \right\} \left( \frac{1}{1 + \rho^2 |x|^2} \right),
\]

\[
F = \frac{\xi \rho^2}{(\rho^2 + |x|^2)^2},
\]

where \(r^2 := x \cdot x = 2|x|^2\), \(\eta^a_{ij}\) are the so-called ’t Hooft \(\eta\)-symbols and \(\rho\) is the size of the instanton (here centered at the origin). The third equality is based on the identity

\[
\xi \frac{\bar{x}}{|x|^2} = (dT)T + I_2 \frac{d|x|^2}{2|x|^2}
\]

and the observation that the first and second term at the rhs are respectively antihermitean and hermitean, i.e. the imaginary and the real part of the quaternion at the lhs.

Noncommutative (i.e. \(q \neq 1\)) solutions of \(*F = F\). Looking for \(A\) directly in the form

\[
A = \xi \bar{x} l/|x|^2 + \theta I_2 n,
\]

where \(l, n\) are functions of \(x\) only through \(|x|\), one finds a family of solutions parametrized by \(\rho^2\) (a nonnegative constant, or more generally a further generator of the algebra) and by the function \(l\) itself. The freedom in the choice of \(l\) should disappear upon imposing the proper (and still missing!) antihermiticity condition on \(A\), as it occurs in the \(q = 1\) case. For the moment, out of this large family we just pick one which has the right \(q \to 1\) limit and closely resembles the undeformed solutions (45-46):

\[
A = -(dT)T \frac{1}{1 + \rho^2 |x|^2},
\]

\[
F = q^{-1} \xi \bar{x} l/|x|^2 + \rho^2 \frac{1}{(\rho^2 + |x|^2)^2}. \tag{47}
\]

Of course we have to extend the algebras so that they contain the rational functions at the rhs. The matrix elements \(A^{ij}_{ab}\) span a \((3, 1) \oplus (1, 1)\) dimensional corepresentation of \(SU_q(2) \times SU_q(2)'\), suggesting as the ‘fiber’ of the gauge group in the complete theory a (possibly deformed) \(U(2)\) [instead of a \(SU(2)\)].

By the scaling and translation invariance of integration over \(\mathbb{R}^4_q\), if we could find a ‘good’ pseudodifferential operator \(W\) to define gauge invariant “action” and “topological charge” by

\[
Q := \int_{\mathbb{R}^4_q} \text{tr}(WF) = \int_{\mathbb{R}^4_q} \text{tr}(WF^*F) = S
\]

the latter would, as in the commutative case, equal a constant independent of \(\rho, y\) (which by the choice of the normalization of the integral we can make 1).

In the \(q = 1\) case multi-instanton solution are explicitly written down in the so-called ‘singular gauge’. Note that as in the \(q = 1\) case \(T = x/|x|\) is unitary and singular at \(x = 0\). So it can play
the role of a ‘singular gauge transformation’. In fact \( A \) can be obtained through the singular gauge transformation

\[
\hat{A} = T dT + \frac{1}{1 + |x|^2} \frac{1 - \rho^2}{q^2} (dT)T
\]

from the singular gauge potential

\[
\hat{A} = \sum_{\alpha \alpha'} q^{-1} \frac{\bar{\xi}^\alpha}{|x|^2} - \frac{q^{-3} I_2}{1 + q} \left( \xi^{\alpha \beta} x^{\beta \gamma} \frac{\epsilon_{\alpha \beta \gamma} \epsilon_{\alpha' \gamma' \beta'}}{|x|^2} \right).
\]

\( \hat{A} \) can be expressed also in the form

\[
\hat{A} = \phi^{-1} \hat{D} \phi, \quad \phi := 1 + q^2 \rho^2 \frac{1}{|x|^2},
\]

where \( \hat{D} \) is the first-order-differential-operator-valued \( 2 \times 2 \) matrix obtained from the square bracket in (49) by the replacement \( x^{\alpha \alpha'}/|x|^2 \rightarrow q^2 \partial^{\alpha \alpha'} \):

\[
\hat{D} := q \bar{\xi} \partial - \frac{q^{-1} I_2}{q + 1} d
\]

(for simplicity we are here assuming that \( \rho^2 \) commutes with \( \xi^{\alpha \beta} \partial^{\beta \gamma} \)). \( \phi \) is harmonic:

\[
\Box \phi = 0.
\]

This is the analog of the \( q = 1 \) case, and is useful for the construction of multi-instanton solutions.

The **anti-instanton solution** is obtained just by converting unbarred into barred matrices, and conversely, as in the \( q = 1 \) case. For instance, from (47) we obtain the anti-instanton solution in the regular gauge

\[
A' = -(dT)T \frac{1}{1 + \rho^2 |x|^2},
\]

\[
F' = q^{-1} \xi \frac{1}{|x|^2 + \rho^2} \frac{1}{q^2 |x|^2}.
\]

**Recovering the instanton projective module of Ref. [10]**

In commutative geometry the instanton projective module \( \mathcal{E} \) over \( A \) and the associated gauge connection can be most easily obtained using the quaternion formalism, in the way described e.g. in Ref. [1]. \( \mathbb{H} \cong \mathbb{R}^4 \) can be compactified as \( P^1(\mathbb{H}) \sim S^4 \). Let \( (w, x) \in \mathbb{H}^2 \) be homogenous coordinates of the latter, and choose \( w = I_2 \) on the chart \( \mathbb{H} \sim \mathbb{R}^4 \). The element \( u \in \mathbb{H}^2 \) defined by

\[
u \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} I_2 \\ \rho x / |x|^2 \end{pmatrix} \left( 1 + \frac{\rho^2}{|x|^2} \right)^{-1/2}
\]

fulfills \( u^\dagger u = I_2 \mathbf{1} \), and the \( 4 \times 2 \) \( A \)-valued matrix \( u \) has only three independent components. Therefore the \( 4 \times 4 \) \( A \)-valued matrix

\[
\mathcal{P} := uu^\dagger = \begin{pmatrix} I_2 \\ \rho x / |x|^2 \\ \rho^2 / |x|^2 I_2 \end{pmatrix} \frac{1}{1 + \rho^2 / |x|^2}
\]

is a self-adjoint three-dimensional projector. It is the projector associated in the Serre-Swan theorem correspondence to the gauge connection (48), by the formula \( \hat{A} = u^\dagger du \). The associated
projective module \( \mathcal{E} \) is embedded in the free module \( \mathcal{A}^4 \) seen as \( M_4(\mathcal{A}) \), and is obtained from the latter as \( \mathcal{E} = \mathcal{P}M_4(\mathcal{A}) \).

In the present \( q \)-deformed setting we immediately check that the element \( u \in \mathbb{H}^4 \) defined by (52) fulfills \( u^\dagger u = I_2^1 \) again, so that the \( 4 \times 2 \) \( \mathcal{A} \)-valued matrix \( \mathcal{P} \) defined by (53) is again hermitean and idempotent, and has only 3 independent components. Therefore, it defines the ‘instanton projective module’ \( \mathcal{E} = \mathcal{P}M_4(\mathcal{A}) \) also in the \( q \)-deformed case. One can easily verify that \( \mathcal{P} \) reduces to the hermitean idempotent \( e \) of [10] if one chooses the instanton size as \( \rho = 1/\sqrt{2} \) and performs the change of generators (20). Therefore, interpreting the model [10] as a compactification to \( S^4_\theta \) of ours, we can use all the results [10] about the Chern-Connes classes of \( e \).

Unfortunately in the \( q \)-deformed case it is no more true that \( \hat{A} = u^\dagger du \), essentially because the \( |x| \)-dependent global factor multiplying the matrix at the rhs(53) does not commute with the 1-forms of the present calculus (\( |x|\xi^i = q\xi^i|x| \)).

Shifting the ‘center of the instanton’ away from the origin
This can be done by the replacement (or ‘braided coaddition’ [28])
\[
x \rightarrow x - y,
\]
where the ‘coordinates of the center’ \( y^i \) generate a new copy of \( \mathcal{A} \), ‘braided’ with the original one (see below); in fact this replacement maps a solution of the (anti)selfduality equation into a new solution. Therefore the instanton moduli space must be a noncommutative manifold, with coordinates \( \rho, y^i \)!

Performing a dilatation or a rotation of the solution
The transformation (18) maps a solution of the (anti)selfduality equation into a new solution. This is gauge equivalent to the original one if \( |a|, |b| = 1 \), the gauge transformation being \( U = a^{-1} \) (resp. \( U = \bar{b}^{-1} \)), i.e. depending on the additional noncommutative variables in \( a, b \). If either \( |a|, |b| \) is different from 1, the new solution is gauge equivalent to the one with instanton size \( \rho' := \rho|a|^{-1}|b|^{-1} \).

8. Multi-instanton solutions
We have found solutions of the self-duality equation corresponding to \( n \) instantons in the “singular gauge” [39, 40] in the form
\[
\hat{A} = \phi^{-1}\hat{D}\phi,
\]
where \( \phi \) is the harmonic scalar function
\[
\phi = 1 + \rho_1^2 \frac{1}{(x-y_1)^2} + \rho_2^2 \frac{1}{(x-y_1-y_2)^2} + \ldots + \rho_n^2 \frac{1}{(x-y_1-\ldots-y_n)^2}
\]
as in the commutative case. In the commutative limit
\[
\rho_\mu \equiv \text{size of the } \mu \text{-th instanton},
\]
\[
u_i^\mu := \sum_{\nu=1}^{\mu} y_{i\nu}^\mu \equiv i \text{-th coordinate of the } \mu \text{-th instanton}.
\]
are constants ($\mu = 1, 2, \ldots, \nu$). In the noncommutative setting the new generators $\rho_{\mu}^2$, $y_{\nu}^i$ have to fulfill the following nontrivial commutation relations:

$$
\begin{align*}
\rho_{\mu}^2 \rho_{\nu}^2 &= q^2 \rho_{\mu}^2 \rho_{\nu}^2, & \nu < \mu \\
\rho_{\mu}^2 y_{\nu}^i &= y_{\mu}^i \rho_{\nu}^2, & q^{-2} \nu < \mu \\
\rho_{\mu}^2 \xi_i &= \xi_i \rho_{\mu}^2, & 1, \nu \geq \mu \\
y_{\mu}^i y_{\nu}^j &= q R^{ij}_{hk} y_{\mu}^h y_{\mu}^k, & \nu < \mu, \\
\mathcal{P}_{\mu}^{ij} y_{\mu}^i y_{\mu}^j &= 0.
\end{align*}
$$

($\mu, \nu = 0, 1, \ldots, \nu$, and we have set $x^i \equiv y_{\nu}^i$).

The last relation states that for any fixed $\nu$ the 4 coordinates $y_{\nu}^i$ generate a copy of $\mathcal{A}$. The last but one relation states that the various copies of $\mathcal{A}$ are braided [28] w.r.t. each other (this is necessary for the $SO_q(4)$ covariance of the overall algebra).

The obvious consequence of the nontrivial commutation relations (56) is that in a complete theory the instanton moduli space must be a noncommutative manifold.

Not only for $n = 1$, but also for $n = 2$ we have been able to go to a gauge potential $A$ `regular’ in $z_\mu^i := x^i - v_{\mu}^i$ by a ‘singular gauge transformation’, which also depends on $y_{\nu}^i$ (as in the $q = 1$ case [21, 32, 40]):

$$
A_2 = U_2^{-1} \left( \hat{A} U_2 + dU_2 \right), \quad U_2 \equiv U_2(z_1, z_2) := \begin{vmatrix} z_1 & y_2 \\ z_1 & z_2 \end{vmatrix}
$$

(57)
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