On Fock Space Representations of Quantized Enveloping Algebras Related to Non-commutative Differential Geometry.

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**Abstract**

In this paper we construct explicitly natural (from the geometrical point of view) Fock space representations (contragradient Verma modules) of the quantized enveloping algebras. In order to do so, we start from the Gauss decomposition of the quantum group and introduce the differential operators on the corresponding $q$-deformed flag manifold (assumed as a left comodule for the quantum group) by a projection to it of the right action of the quantized enveloping algebra on the quantum group. Finally, we express the representatives of the elements of the quantized enveloping algebra corresponding to the left-invariant vector fields on the quantum group as first-order differential operators on the $q$-deformed flag manifold.

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1 Introduction

Let $G$ denote a simple and simply connected complex Lie group and $K \subset G$ its compact form. The purpose of this paper is to construct an explicit representation of the quantized enveloping algebra $\mathcal{U}_h(\mathfrak{g})$ of the quantum group $K_q$ in terms of (local) holomorphic coordinates and differential operators on the homogeneous space $(K_0 \setminus K)_q = (P_0 \setminus G)_q$. This extends unambiguously to a representation of $\mathcal{U}_h(\mathfrak{k})$. Starting point for this construction is the Gauss decomposition of the canonical element of the quantum double of $G_q$ (which yields the Gauss decomposition of the vector corepresentation) as described in [8]. Using the projection from $G_q$ to $(P_0 \setminus G)_q$ we introduce partial differential operators on the homogeneous space. The representation of the elements of the universal enveloping algebra in terms of these differential operators and holomorphic functions can be viewed as a natural Fock space representation of $\mathcal{U}_h(\mathfrak{g})$ ($\mathcal{U}_h(\mathfrak{k})$) (the contragradient of the Verma module). The explicit formulas are presented in the case $G = SL(N)$ and $P_0 = B_-$ with $B_-$ being the Borel subgroup of lower triangular matrices. However, we hope that from our exposition it is clear that the general case can be treated similarly. We assume a generic value of $q = e^{-h}$, which becomes real, while referring to the compact forms.

Let us mention that there are already many papers devoted to the subject. We mention just a few [1], [20], which seem to be most closely related to our approach. Nevertheless, we wish to stress the geometric origin of our construction, which employs the Borel–Weil-like description of the irreducible representations of the quantized enveloping algebra $\mathcal{U}_h(\mathfrak{g})$ ($\mathcal{U}_h(\mathfrak{k})$) (the contragradient of the Verma module). The explicit formulas are presented in the case $G = SL(N)$ and $P_0 = B_-$ with $B_-$ being the Borel subgroup of lower triangular matrices. However, we hope that from our exposition it is clear that the general case can be treated similarly. We assume a generic value of $q = e^{-h}$, which becomes real, while referring to the compact forms.

2 Preliminaries, notation

In this section we repeat some of the results described in [8]. For the general construction of the quantum double and its relation to quantum groups, we refer to [3], [16], [18].

Let $\mathcal{U}_h(\mathfrak{g})$ be the quantized enveloping algebra related to a simple Lie algebra $\mathfrak{g}$ and $F_q(G)$ the dual Hopf algebra of quantized functions on the corresponding simple Lie group $G$.

Let further $\rho$ be the canonical element

$$
\rho = \sum x_s \otimes a_s \in \mathcal{U}_h(\mathfrak{g})^{op} \otimes F_q(G),
$$
with \( \{ x_\alpha \} \) and \( \{ a_\alpha \} \) being mutually dual bases. Its basic properties are \((S \text{ is the antipode, } \Delta \text{ the comultiplication})\)

\[
\rho^{-1} = (\text{id} \otimes S)\rho, \\
(\Delta \otimes \text{id})\rho = \rho_{23}\rho_{13}, \quad (\text{id} \otimes \Delta)\rho = \rho_{12}\rho_{13}.
\]

(1)

Denote by \( b_+ \subset \mathfrak{g} \) the Borel subalgebras and by \( h = b_+ \cap b_- \) the Cartan subalgebra.

Fixing a maximal Weyl element, one orders the set \( \Delta^+ \) of positive roots as \((\beta_1, \ldots, \beta_\Delta), \quad d = |\Delta^+|\). To each root \( \beta_j \) there are related elements \( E(j) \in \mathcal{U}_h(b_+) \) and \( F(j) \in \mathcal{U}_h(b_-) \), so that the elements

\[
E(d)^{n_1} \cdots E(1)^{n_1} H_1^{m_1} \cdots H_1^{m_1}
\]

(2)

\((n_i, \ m_i \in \mathbb{Z}_+)\), form a basis in \( \mathcal{U}_h(b_+) \). The vectors \( H_i \) can be replaced by any elements forming a basis in \( h \). A similar assertion is valid also for \( \mathcal{U}_h(b_-) \). In the limit \( h \downarrow 0 \) the elements \( E(j) \) and \( F(j) \) become the root vectors \( X_{\beta_j} \in \mathfrak{n}_+ \) and \( X_{-\beta_j} \in \mathfrak{n}_- \), respectively. We recall that the universal \( R \)-matrix can be written in the form [3], [4]

\[
R^n = \exp_{q_d}(\mu_d F(d) \otimes E(d)) \cdots \exp_{q_1}(\mu_1 F(1) \otimes E(1)) \exp(\kappa),
\]

(3)

where \( \exp_q \) are the \( q \)-deformed exponential functions, \( \mu_j \) are some coefficients depending on the parameter \( h \), and \( \kappa \) is some element from \( \mathcal{U}_h(h) \otimes \mathcal{U}_h(h) \).

We make use of the fact that \( \mathcal{U}_h(\mathfrak{g})^{op\Delta} \) is a factor algebra of \( \mathcal{U}_h(b_-)^{op\Delta} \otimes_{\text{twist}} \mathcal{U}_h(b_+)^{op\Delta} \) and \( F_q(\mathbb{G}) \simeq \mathcal{U}_h(\mathfrak{g})^* \) is a subalgebra in \( \mathcal{U}_h(b_-)^{op} \otimes \mathcal{U}_h(b_-)^{op\Delta} \). The canonical element \( \tilde{\rho} \) in

\[
\left( \mathcal{U}_h(b_-)^{op\Delta} \otimes_{\text{twist}} \mathcal{U}_h(b_+)^{op\Delta} \right) \otimes \left( \mathcal{U}_h(b_+)^{op} \otimes \mathcal{U}_h(b_-)^{op\Delta} \right)
\]

(4)

can be decomposed as follows [3]

\[
\tilde{\rho} = \sum (e_j \otimes e^k) \otimes (f^j \otimes f_k)
\]

\[
= \sum (e_j \otimes 1 \otimes f^j \otimes 1) \cdot (1 \otimes e^k \otimes 1 \otimes f_k)
\]

\[
= \tilde{R}_{13} \tilde{R}_{24}.
\]

(5)

Here \( \{ e_j \} \), \( \{ e^k \} \), \( \{ f^j \} \) and \( \{ f_k \} \) stand for bases in the corresponding factors, \( \{ e_j \} \) and \( \{ f^j \} \) are dual and the same is assumed of \( \{ e^k \} \) and \( \{ f_k \} \); the dot in the third member of equalities [3] indicates multiplication in the double and \( \tilde{R}' \) is obtained from \( \tilde{R} \) by reversing the order of multiplication. To express \( \rho \) we shall again use bases of the type [2]. In the notation adopted here, the elements \( F(j), \ E(j), \ E'(j) \) and \( \tilde{F}(j) \) belong in this order to the individual factors in [3]. Factorizing off the redundant Cartan elements we have [8].

2
Proposition 1. The canonical element for the quantum double $U_h(g)^\alpha \Delta \otimes F_q(G)$ has the form

$$\rho = \exp_{q_d}(\mu_d F(d) \otimes \tilde{E}(d)) \ldots \exp_{q_1}(\mu_1 F(1) \otimes \tilde{E}(1)) \exp(\kappa)$$

$$\times \exp_{q_1}(\mu_1 E(1) \otimes \tilde{F}(1)) \ldots \exp_{q_d}(\mu_d E(d) \otimes \tilde{F}(d)). \quad (6)$$

Let further $\Pi_0$ denote any subset of the set of simple roots $\Pi$ and let us denote by $U_h(g_0)$ the Hopf subalgebra in $U_h(g)$ generated by all Cartan elements $H_i$, and only by those elements $X_i^\pm$ for which $\alpha_i \in \Pi_0$. Similarly we shall denote by $U_h(p_0)$ the Hopf subalgebra in $U_h(g)$ generated by all $H_i$, $X_i^-$ and those $X_i^+$ for which $\alpha_i \in \Pi_0$. The maximal Weyl element can be chosen such that there exists $p \in \mathbb{Z}_+$, $p \leq d$, such that the vectors $X_{-\beta_1}, \ldots, X_{-\beta_d}, H_1, \ldots, H_i, X_{\beta_1}, \ldots, X_{\beta_p}$ form a basis of $p_0$. Then $X_{\beta_{p+1}}, \ldots, X_{\beta_d}$ form a basis of a nilpotent subalgebra $n_0$ and $g = p_0 \oplus n_0$. This means that all elements $F(j)$ belong to $U_h(p_0)$, while $E(j)$ belongs to $U_h(p_0)$ only for $j = 1, \ldots, p$. Notice that in the generic case $\Pi_0 = \emptyset$ and hence $p = 0$, $p_0 = b_-$ and $n_0 = n_+$. As it is easy to see, $U_h(g_0)$ is again a quasitriangular Hopf algebra with the universal $R$-matrix $Q^u$ given by

$$Q^u = \exp_{q_p}(\mu_p F(p) \otimes E(p)) \ldots \exp_{q_1}(\mu_1 F(1) \otimes E(1)) \exp(\kappa). \quad (7)$$

The canonical element $\rho$ can be written as a product

$$\rho = \Lambda Z,$$

where

$$\Lambda = \exp_{q_d}(\mu_d F(d) \otimes \tilde{E}(d)) \ldots \exp_{q_1}(\mu_1 F(1) \otimes \tilde{E}(1)) \exp(\kappa)$$

$$\times \exp_{q_1}(\mu_1 E(1) \otimes \tilde{F}(1)) \ldots \exp_{q_p}(\mu_p E(p) \otimes \tilde{F}(p)),$$

$$Z = \exp_{q_{p+1}}(\mu_{p+1} E(p+1) \otimes \tilde{F}(p+1)) \ldots \exp_{q_d}(\mu_d E(d) \otimes \tilde{F}(d)). \quad (8)$$

Let us also denote

$$A = \exp(\kappa) \exp_{q_1}(\mu_1 E(1) \otimes \tilde{F}(1)) \ldots \exp_{q_p}(\mu_p E(p) \otimes \tilde{F}(p)),$$

so that $\Lambda Z$ can be identified with the canonical element of the quantum double $(U_h(b_+)^{op\Delta} \otimes U_h(b_-)^{op\Delta})$; $A$ itself is the canonical element of the quantum double of $(U_h(b_{0+})^{op\Delta} \otimes U_h(b_{0-})^{op\Delta})$, where $U_h(b_{0-})$ is the Hopf subalgebra in $U_h(b_-)$ generated by all $H_i$ and those $X_i^-$ for which $\alpha_i \in \Pi_0$. The projections $P_{\pm} : U_h(b_{\pm}) \rightarrow U_h(b_{0\pm})$ which send $E(j)$ and $F(j)$ for $j = k+1, \ldots, d$ to 0 are Hopf algebra homomorphisms $[8]$. It also holds that $(id \otimes P_-)R^u = (P_+ \otimes id)R^u = Q^u$. As a simple consequence of the above-mentioned facts we have

Proposition 2. It holds that

$$A_{23}^{-1}Z_{13}A_{23} = (Q_{12}^u)^{-1}Z_{13}Q_{12}^u,$$  

(10)
\[(\Delta \otimes \text{id})Z = A_{23}^{-1}Z_{13}A_{23}Z_{23} = (Q_{12}^u)^{-1}Z_{13}Q_{12}^uZ_{23}, \quad (11)\]

\[R_{12}^u(Q_{12}^u)^{-1}Z_{13}Q_{12}^uZ_{23} = (Q_{21}^u)^{-1}Z_{23}Q_{21}^uZ_{13}R_{12}^u. \quad (12)\]

Here $\Delta$ means the original comultiplication in $U_h(g)$, contrary to (1).

Let $\tau$ designate the irreducible representation of $U_h(g)$ corresponding to the vector corepresentation $T$ of $F_q(G)$, $T = (\tau \otimes \text{id})\rho$. As in [8] we use the entries of the matrix $Z = (\tau \otimes \text{id})Z$ as (local) non-commutative coordinates on the $q$-deformed homogeneous space (generalized flag manifold) $(P_0 \backslash G)_q$. We shall denote the algebra generated by these by $C$. Applying $\tau \otimes \tau$ to eq. (12) we get the commutation relations

\[R_{12}Q_{12}^{-1}Z_{13}Q_{12}Z_{23} = Q_{21}^{-1}Z_{23}Q_{21}Z_{13}R_{12}, \quad (13)\]

where $R$ and $Q$ are used to denote the universal $R$-matrices $R^u$ and $Q^u$ in the vector representation $\tau$. Let us note that (13) are formally of the same form as the defining relations of a quantum braided group [12], [5].

Using the universal element $Z$ we now introduce a mapping (which is an algebra homomorphism in the classical case) from the algebra of quantized functions on $G$ to the functions on the $q$-deformed homogeneous space $C$:

\[\Gamma : F_q(G) \to C : a \mapsto (\langle a, \cdot \rangle \otimes \text{id})Z. \quad (14)\]

Unlike the classical case, the mapping $\Gamma$ is not an algebra homomorphism on $F_q(G)$ (this is obvious from (13)) but the following properties are sufficient for our construction. The mapping $\Gamma$ satisfies:

**Proposition 3.** It holds that

\[(id \otimes \Gamma)\Delta(\Gamma(a)) = (\Gamma \otimes \Gamma)\Delta(a), \quad \Gamma(ab) = \Gamma(a)b, \quad a \in F_q(G), b \in C. \quad (15)\]

**Proof.** The proof of the second equation is rather straightforward using the decomposition of the universal element $\rho$. The proof of the first equation goes as follows. The starting point is

\[(id \otimes id \otimes \Gamma)(id \otimes \Delta)Z = (id \otimes id \otimes \Gamma)(Z_1 \otimes \langle Z_2, \rho_1 \tilde{\rho}_1 \rangle \rho_2 \otimes \tilde{\rho}_2), \quad (16)\]

where we used

\[Z = Z_1 \otimes Z_2, \quad \rho = \rho_1 \otimes \rho_2, \quad Z = Z_1 \otimes \langle Z_2, \rho_1 \rangle \rho_2. \quad (17)\]

Applying the projection $\Gamma$ and using the decomposition of $\rho$ yields

\[(id \otimes id \otimes \Gamma)(id \otimes \Delta)Z = Z_{12}Z_{13}. \quad (18)\]
Inserting $a$ in the first tensor factor gives the result.

For later purposes we also introduce the dual mapping

$$
\tilde{\Gamma} : U_h(g) \to U_h(g) : x \mapsto (id \otimes \langle x, \cdot \rangle)Z.
$$

(19)

3 Differential operators on $C$

The aim of the following paragraph is to introduce the partial derivatives $\frac{\partial}{\partial Z_j}$ with respect to the coordinates $Z_j$ with the help of the projection to $C$ of the right action of $U_h(g)$ on $F_q(G)$, and to express the action of the left-invariant vector fields on $C$ in terms of these.

Definition 1.

$$a \triangleleft X := \Gamma(a \ast X) = \Gamma(a_{(2)})\langle X, a_{(1)} \rangle, \quad a \in F_q(G), X \in U_h(g).$$

(20)

Using the explicit form of the mapping $\Gamma$, it is easy to see that we can write:

$$a \triangleleft X = (\langle a, X \cdot \rangle \otimes id)Z.$$

(21)

The “action” $\triangleleft$ has the following properties:

Proposition 4.

$$
\begin{align*}
    a \triangleleft XY &= \epsilon(X)a \triangleleft Y \quad \text{if } X = (id \otimes \langle X', \cdot \rangle)\Lambda, a \in C, \\
    a \triangleleft XY &= (a \triangleleft X) \triangleleft Y \quad \text{if } Y \in Im\tilde{\Gamma}.
\end{align*}
$$

(22)

The above-defined action (it is really an action of $Im\tilde{\Gamma}$ on $C$ according to (22)) now serves to introduce a complete set of partial differential operators on the space $C$. In order to do so we start from the following observation:

$$Z^a_b \triangleleft X = \tau^a_c(X)Z^c_b, \quad \text{where } \quad X \in Im\tilde{\Gamma}.$$

(23)

To introduce an appropriate set of differential operators on $C$ we choose the following elements of $Im\tilde{\Gamma}$:

$$\beta := (q - q^{-1})^{-1}(\tau \otimes id)(R^aQ^{-1}u - 1).$$

(24)

From now on, we shall restrict ourselves to the case $G = SL(N)$ and $P_0 = B_-$, where $B_-$ is the Borel subgroup of lower triangular matrices. In this case the matrix $\tilde{Z}$ is an upper triangular matrix with units on the diagonal. The $R$-matrices $R$ and $Q$ are then of the form

$$
\begin{align*}
    q^{1/N}R_{st}^{jk} &= \delta^j_s\delta^k_t + (q - q^{\text{sgn}(k-j)})\delta^j_s\delta^k_t, \\
    q^{1/N}Q_{st}^{jk} &= q^{\delta^j_s\delta^k_t}.
\end{align*}
$$
and the relation (13) can be rewritten, for the individual matrix entries, as
\[ q^\delta_k Z^k_i Z^j_t - q^\delta_j Z^k_i Z^j_t = (q^{\text{sgn}(k-j)} - q^{\text{sgn}(s-t)})q^\delta_k Z^k_i Z^j_t. \] (7.17)

Using (23) in the definition of \( \beta \) yields in this case the following result for the right action of the functionals (24) on the matrix of the holomorphic coordinates on the homogeneous space \((P_0\setminus G)_q\),

\[ Z^a_b \triangleright \delta^i_j Z^i_b \quad \text{for} \quad i > j. \] (25)

This identity implies the following ansatz for the \( \beta \)'s in terms of derivatives in the variables \( Z \)
\[ a \triangleright \beta^i_j = Z^i_r \frac{\partial a}{\partial Z^r_j}. \] (26)

Therefore we define the partial derivatives on the space \( C \) through the action of the functionals \( \beta \). In order to obtain a complete description of the partial derivatives \( \frac{\partial}{\partial Z^j} \) we have to specify the deformed Leibnitz rule. This is done by starting from the comultiplication of the \( \beta \)'s. Using
\[ (\Delta \otimes \text{id})Q^u = Q^u_{13}Q^u_{23}, \]
\[ Q^u = Q^u_{21}, \] (27)

one obtains the following comultiplication of the \( \beta \)'s:
\[ \Delta(\beta^i_j) = S(L^{-k}_m)Q^{-1}_r^j \otimes S(L^{-i}_k)Q^{-1}_m^r - \delta^i_j 1 \otimes 1. \] (28)

In order to derive the deformed Leibnitz rule for the derivatives \( \beta \), we make use of the following observation:
\[ \Gamma(a) \triangleright X = a \triangleright \tilde{\Gamma}(X). \] (29)

So in the case where \( X \) is already an element of \( \text{Im} \tilde{\Gamma} \) the right action on any element of \( F_q(G) \) is identical to the right action on its projection. Therefore using the first equation in (13) and the definition of the matrix \( Z \) one obtains for \( f \in C \)
\[ (Z^a_b f) \triangleright \beta^i_j = (T^a_b f) \triangleright \beta^i_j. \] (30)

Starting from this equation we finally end up with the following Leibnitz rule on \( C \):
\[ (Z^a_b f) \triangleright \beta^i_j = (Z^a_b \triangleright \beta^i_j) f + q^{-(\delta^a_b + \delta^i_j - \delta^s_k)} Z^a_b (f \triangleright \beta^i_j) \]
\[ + \delta^a_q \delta^s_c (q - q^{\text{sgn}(a-c)}) Z^a_c (f \triangleright \beta^i_c). \] (31)

So (23) together with the Leibnitz rule completely define the derivatives on \( C \). Therefore it is now possible to express the left action on \( C \) of the vector fields on the quantum group through functions of the holomorphic coordinates \( Z \) and the differential
operators $\beta$ (or $\frac{\partial}{\partial Z_j}$).

The vector fields are defined in the standard way [7], [2], [17]

$$\kappa := (q - q^{-1})^{-1}(\tau \otimes id)(1 - R_{21}^u R_u) .$$ (32)

**Proposition 5.** On the space $C$ we obtain the following representation of the vector fields

$$\kappa_j^i * a = -Z_k^{-1} Z_l^i q^{-q} q^k Z_l^i \frac{\partial a}{\partial Z_k}, \quad k > l .$$ (33)

**Proof.** Using (13) it is straightforward to derive the following commutation relation

$$Z_{23} R_{21}^u R_{12}^u = Q_{12}^{-1} Z_{13}^{-1} Q_{12}^u R_{21}^u R_{12}^u Q_{12}^{-1} Z_{13} Q_{12}^u Z_{23} .$$ (34)

Now using

$$(id \otimes X*) \rho = \rho (X \otimes id) ,$$ (35)

and the commutation relation (34) it is possible to derive the following identity on $a \in C$ by applying $(\tau_j^i \otimes a \otimes id)$ on both sides of (34)

$$\kappa_j^i * a = -Z_k^{-1} Z_m^i \beta_j^i Q_{km}^i .$$ (36)

To bring this equation to the desired form, we have to commute $Q$ and $\beta$ and use (22). Starting from (27) and the fact that $U_h(g)$ is quasitriangular, one obtains the following commutation relation (on $a \in C$)

$$a \beta^i_m Q_{jk}^m = Q_{jk}^m Q^k_{jr} \beta^m_j = 0 .$$ (37)

Therefore inserting (37) in (36) yields the expression in Proposition 5 for the left action of the vector fields on $C$ in terms of holomorphic coordinates and derivatives.

**Remark:** It is easy to see that, for $a \in C$:

$$\kappa_j^i * a = (\kappa_j^i * Z_m^i) \frac{\partial a}{\partial Z_m^i} .$$

So if we assume the differential calculus on $SL_q(N)$ with the conventions of, for instance [8], we have for the differential of $a \in C$

$$da = dZ_n^m \frac{\partial a}{\partial Z_n^m} .$$

This observation justifies our ansatz (26) for partial derivatives.
4 Representations

Using the proof of Proposition 5, one can now easily generalize the above formulas to the case of the action of $\kappa^j_i$ in an arbitrary irreducible finite-dimensional representation $\mathcal{T}^\lambda$ of $\mathcal{U}_h(\mathfrak{sl}(N))$ corresponding to a maximal weight $\lambda = (m_1, m_2, ..., m_{N-1})$ [14], [11], [13]. First let us embed the representation space $\mathcal{H}_\lambda$ into $\mathcal{C}$ by $|\Psi\rangle \mapsto \Psi_\lambda := \langle \lambda | \mathcal{Z} | \Psi \rangle$. So it is now clear that instead of applying the second component of the triple tensor product in (34) to an element $a \in \mathcal{C}$ in going from (34) to (36) we have now to compute the matrix element $\langle \lambda | | \Psi \rangle$ of the second component in order to obtain the action of $\kappa^j_i$ on $\Psi_\lambda$. We use the notation $m_i = \tilde{m}_i - \tilde{m}_{i+1}$, $\sum \tilde{m}_i = 0$.

The result is

**Proposition 6.** The action of the $\kappa^j_i$ in the representation $\mathcal{T}^\lambda$ in terms of holomorphic coordinates and derivatives is given by

$$
(\kappa^j_i * \Psi_\lambda)(Z) = (q - q^{-1})^{-1} q^{-\tilde{m}_i} q^{\tilde{m}_k} q^{\tilde{m}_j} (q^{-\tilde{m}_k} - q^{\tilde{m}_k}) Z^{-1}_{-1} Z^k_{j} \Psi_\lambda(Z)
- q^{\tilde{m}_k} q^{\delta_k} q^{-\tilde{m}_k} q^{\delta_k} Z^{-1}_{-1} Z^k_{j} (\Psi_\lambda(Z) \langle \beta^k \rangle). \quad (38)
$$

**Proof.** Inserting $\tau^j_i$ in the first component of the tensor product (34) we obtain the following expression

$$
\mathcal{Z}_{12}((\tau^j_i \otimes \text{id}) R_{21} R_{12})_1 = (Q^{-1}_{-1} Q^l_m ((\tau^j_i \otimes \text{id}) R_{21} R_{12}) Q^{-1}_{-1} Q^l_p)_{12}(Z^{-1}_{-1} Z^o_p)_{212}. \quad (39)
$$

The restriction to the specific state $\Psi$ in the representation $\mathcal{T}^\lambda$ is done by taking the matrix element between $\langle \lambda |$ and $| \Psi \rangle$ in the first tensor factor (all matrix elements in the following are in the first tensor factor), where $\langle \lambda |$ is defined by the following equations:

$$
Q^l_j = \delta^l_j q^H_i,
\bar{H}_i - \bar{H}_{i+1} = H_i,
\sum_i \bar{H}_i = 0,
\bar{H}_i |\lambda\rangle = \tilde{m}_i |\lambda\rangle,
\langle \lambda | L^l_j = \langle \lambda | \delta^l_j q^{\tilde{m}_i}. \quad (40)
$$

This yields

$$
\langle \lambda | \mathcal{Z}_{12}((\kappa^j_i)_{12}) | \Psi \rangle = q^{-\tilde{m}_i} q^{\tilde{m}_k} Z^{-1}_{-1} Z^l_{j} [(q - q^{-1})^{-1} (q^{-\tilde{m}_k} - q^{\tilde{m}_k}) \delta^k_l \langle \lambda | Q^l_j \mathcal{Z}_{12} | \Psi \rangle
- q^{\tilde{m}_k} \langle \lambda | \beta^k_j \mathcal{Z}_{12} | \Psi \rangle], \quad (41)
$$

where the $\kappa$ are the vector fields defined in (32).

Using the evaluation of $Q$ on $\langle \lambda |$ as defined in (10) and the commutation relations between $\beta$ and $Q$ defined in (34) one obtains

$$
\langle \lambda | \mathcal{Z}_{12}(X^j_j)_{12} | \Psi \rangle = -q^{\delta_k} q^l_j q^{-\tilde{m}_k} q^{\tilde{m}_k} Z^{-1} Z^l_{j} (\langle \lambda | Q^l_{j} \mathcal{Z}_{12} | \Psi \rangle
+ (q - q^{-1})^{-1} q^{-\tilde{m}_i} q^{\tilde{m}_k} q^{\tilde{m}_j} (q^{-\tilde{m}_k} - q^{\tilde{m}_k}) Z^{-1}_{-1} Z^k_{j} \langle \lambda | \mathcal{Z}_{12} | \Psi \rangle. \quad (42)
$$
Identifying
\[ \langle X, \Psi_\lambda \rangle = \langle \lambda | X | \Psi \rangle , \quad X \in \text{Im} \tilde{\Gamma} , \] (43)
and using the definition of the left and right action \( \triangleleft \) (in the form given in (21)) we obtain the desired result.

If we now assume that the generators \( \kappa \) are acting on the whole \( \mathcal{C} \), we can interpret our results as a natural Fock space representation of the universal enveloping algebra of \( \mathcal{U}_q(\mathfrak{g}) \) \( \mathcal{U}_q(\mathfrak{t}) \) as the action is expressed in terms of holomorphic coordinates \( Z_j^i \) and derivatives \( \frac{\partial}{\partial Z_j^i} \). These representations are nothing but the contragradient Verma modules.

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References

[1] Awata, H., Noumi, M., Odake, S.: Heisenberg realization for \( \mathcal{U}_q(\mathfrak{sl}_n) \) on the flag manifold. Lett. Math. Phys. 30, 35 (1994)

[2] Carow-Watamura, U., Schlieker, M., Watamura, S., Weich, W.: Bicovariant differential calculus on quantum groups \( \mathcal{U}_q(\mathfrak{su}(n)) \) and \( \mathcal{U}_q(\mathfrak{so}(n)) \). Commun. Math. Phys. 142, 605 (1991)

[3] Drinfeld, V. G.: Quantum groups. In Proc. ICM Berkley 1986, AMS 1987, p.798

[4] Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Quantum groups. In: Yang, C.N., Ge, M.L. (eds.) Braid groups, knot theory and statistical mechanics. Singapore: World Scientific 1989

[5] Hlavatý, L.: Quantum braided groups. J. Math. Phys. 35, 2560 (1994)

[6] Jimbo, M.: A q–difference analogue of U(\( \mathfrak{g} \)) and the Yang–Baxter equation. Lett. Math. Phys. 10, 63 (1985)

[7] Jurčo, B.: Differential calculus on quantized simple Lie groups, Lett. Math. Phys. 22, 177 (1991)

[8] Jurčo, B., Šťovíček, P.: Coherent States for Quantum Compact Groups. Preprint CERN-TH.7201/94.

[9] Kirillov, A.N., Reshetikhin, N.Yu.: \( q \)–Weyl group and a multiplicative formula for universal \( R \)–matrices. Commun. Math. Phys. 134, 421 (1990)
[10] Levendorskii, S.Z., Soibelman, Ya.S.: Some application of quantum Weyl groups. The multiplicative formula for universal $R$–matrix for simple Lie algebra. J. Geom. Phys. 7(4), 1 (1990)

[11] Lusztig, G.: Quantum deformations of certain simple modules over enveloping algebras. Adv. Math. 70, 237 (1988)

[12] Majid, S.: Proc. Cambridge Philos. Soc. 113, 45 (1993) Majid, S.: J. Pure Appl. Algebra 86 (1993)

[13] Parshall, B., Wang, J.: Quantum linear groups. Rhode Island: AMS 1991

[14] Rosso, M.: Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple Lie algebra. Commun. Math. Phys. 117, 581 (1988)

[15] Reshetikhin, N. Yu., Semenov-Tian-Shansky, M. A.: Quantum $R$–matrices and factorization problems. J. Geom. Phys. 5, 533 (1988)

[16] Reshetikhin, N. Yu., Takhtajan, L. A., Faddeev, L. D.: Quantization of Lie groups and Lie algebras. Algebra i analiz 1, 178 (1989) (in Russian)

[17] Schupp, P., Watts, P., Zumino, B.: Cartan calculus for Hopf algebras and quantum groups. preprint LBL–34215

[18] Woronowicz, S. L.: Compact matrix pseudogroups. Commun. Math. Phys. 111, 613 (1987)

[19] Woronowicz, S.L.: Differential calculus on quantum matrix pseudogroups (quantum groups). Commun. Math. Phys. 122, 125 (1989)

[20] Šťovíček, P.: Quantum Grassmann manifolds. Commumun. Math. Phys. 158, 135 (1993)

Morozov, A.: Bosonization of the coordinate ring of $U_h(sl(N)$. The cases of $N = 2$ and $N = 3$. hep-th/9311412

Shafiekhani, A.: $U_q(sl(n))$ difference operator realizations. hep-th/9408173

Jurčo, B.: On coherent states for the simplest quantum groups. Lett. Math. Phys. 21, 51 (1991)

Dobrev, V.: $q$-difference intertwining operators for $U_q(sl(n))$: General setting and the case $n = 3$. J. Phys. A27 4841 (1994)

Dabrowski, L., Parashar, P.: Left regular representation of $SL_q(3)$: Reduction and intertwiners. hep-th/9405131