Note on the phase space of asymptotically flat gravity in Ashtekar–Barbero variables

Miguel Campiglia

Raman Research Institute Bangalore 560080, India
Instituto de Física, Facultad de Ciencias Montevideo 11400, Uruguay
E-mail: campi@fisica.edu.uy

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Abstract
We describe the canonical phase space of asymptotically flat gravity in Ashtekar–Barbero (AB) variables. We show that the Gauss constraint multiplier must fall off slower than previously considered in order to recover ADM phase space. The generators of the asymptotic Poincare group are derived within the AB phase space without reference to the ADM generators. The resulting expressions are shown to agree, modulo Gauss constraint terms, with those obtained from the ADM generators. A payoff of this procedure is a new expression for the generator of asymptotic rotations, which is polynomial in the triad and hence better suited for quantum theory. Our treatment complements an earlier description by Thiemann in the context of self-dual variables.

Keywords: Ashtekar–Barbero variables, asymptotically flat gravity, canonical phase space

1. Introduction

In [1] Thiemann adapted the description of asymptotically flat canonical gravity in ADM variables [2] given in [3, 4] to Ashtekar variables [5, 6]. Among other things, he obtained the generators of the asymptotic Poincare group, showed their agreement with the ADM generators, and verified their Poisson brackets reproduce the Poincare algebra.

Here we revisit Thiemann’s analysis in the context of real $SU(2)$ variables [7] with an arbitrary Barbero–Immirzi parameter $\gamma$ [8, 9]. We take the same parity conditions and leading fall-off terms as those used in [1, 3]. For subleading terms we use the more general fall-offs given in [4]. We also show that, independently of the choice of subleading fall-offs, the leading term of the Gauss constraint multiplier must fall off slower than what was considered in [1] if one wants to recover ADM phase space.

To obtain the Poincare generators, we follow the strategy of [3] where one seeks for boundary terms to the Hamiltonian and diffeomorphism constraint that yield well defined...
phase space functions when then lapse and shift have asymptotic values that correspond to
Poincare transformations. This strategy is followed within the AB phase space, without
resorting to the ADM expressions.

For spacetime translations, we recover the known expressions given in [1, 6]. For boosts
we obtain a generator, equation (3.46), that is shown to agree on the Gauss constraint surface
with the one obtained in [1]. The situation is the most subtle with rotations. In [1] the
generator of rotations was obtained from the ADM generator. The resulting expression
involves the spin connection and hence is non-polynomial in the triad. On the other hand the
generator obtained here, equation (3.31), is polynomial in the triad. Showing that the two
agree (modulo a Gauss constraint term with phase-space dependent multiplier) requires
careful comparison of the expressions.

The motivation for the present study comes from its application to quantum theory. In
particular, the expression for angular momentum obtained here facilitates the unitary
implementation of asymptotic rotations described in [10]. Beyond its possible quantum
applications, we hope the present work represents a contribution to the existing rich literature
on asymptotics of gravity in first order form [11–15].

The organization of the material is as follows. In section 2 we review the asymptotically
flat ADM phase space as treated in [4]. The section serves to set up notation, display the
ADM Poincare generators for later comparison, and present the guiding principle of refer-
cences [3, 4] that we follow in section 3. In section 3.1 we describe the AB phase space
counterpart of the ADM phase space of section 2. In 3.2 we discuss the Gauss constraint and
corresponding asymptotic behavior of its multiplier. In 3.2.2 we discuss the Hamiltonian and
diffeomorphism constraints, and in 3.3 the Poincare generators. The discussion of rotations
will be more detailed than for the other generators, since it is here that the comparison with
ADM and Thiemann’s expression is more subtle. In section 4 we show the Poincare gen-
erators of section 3 coincide with the ADM and Thiemann ones. We finish with conclusions
in section 5.

2. Review of the ADM case

2.1. Phase space

In the asymptotically flat case, the Cauchy slice $\Sigma$ is such that it admits, outside a compact set,
cartesian coordinates $x^I$, $I = 1, 2, 3$ that extend to infinity. Let $q_{ab}$ be the flat background
metric associated with the cartesian coordinates so that $q_{IJ} = \delta_{IJ}$. Let
\[ r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \]
and $\hat{x}^I := x^I/r$. The phase space is then given by the standard
canonical pair $(q_{ab}, \pi^{ab})$ satisfying the following fall-off conditions in the cartesian coordinate system$^1$:
\begin{align*}
q_{IJ} &= \hat{q}_{IJ} + \frac{h_{IJ}(\hat{x})}{r} + O\left(r^{-1-\epsilon}\right), \quad (2.1) \\
\pi^{IJ} &= \frac{p^{IJ}(\hat{x})}{r^2} + O\left(r^{-2-\epsilon}\right), \quad (2.2)
\end{align*}

$^1$ $q_{ab}$ and $\pi^{ab}$ are taken to be $C^\infty$. A tensor $f$ is $O(r^{-\beta})$ (denoted by $O^\alpha(r^{-\beta})$ in [4]) if for $\alpha = 0, 1, \ldots$, the nth partial derivatives of $f$ in the cartesian chart, $\partial_{x^1} \ldots \partial_{x^n} f$, are bounded by $c_n r^{-n-\beta}$ for constants $c_n$. 

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where \( \epsilon > 0 \) and \( h_{IJ} \) and \( p_{IJ} \) are of even and odd parity respectively:

\[
h_{IJ}(\hat{x}) = h_{IJ}(\hat{x}),
\]

(2.3)

\[
p_{IJ}(\hat{x}) = -p_{IJ}(\hat{x}).
\]

(2.4)

The fall-off conditions ensure the symplectic structure

\[
\Omega(\delta_1, \delta_2) = \frac{1}{16\pi G} \int_\Sigma \left( \delta_1 q_{ab} \delta_2 \pi^{ab} - \delta_2 q_{ab} \delta_1 \pi^{ab} \right),
\]

(2.5)

is well defined, and allows the existence of non-trivial Poincare generators \([3, 4]\).

We will be dealing with phase space functions \( F[q, \pi] \) that are integrals over \( \Sigma \) of local functions of \( q_{ab} \) and \( \pi^{ab} \). The two basic conditions that are required for such expressions are:

(i) \( F \) should be finite, i.e., the integral over \( \Sigma \) should be convergent

(ii) \( F \) should admit a Hamiltonian vector field, i.e.

\[
\delta F = \Omega(\delta_F, \delta) \forall \delta.
\]

Above \( \delta = (\delta q_{ab}, \delta\pi^{ab}) \) is any variation and \( \delta F = (\delta_F q_{ab}, \delta_F \pi^{ab}) \) is the Hamiltonian vector field of \( F \). Both \( \delta \) and \( \delta_F \) are vector fields in phase space and hence respect the fall-off and parity conditions given above. Condition (ii) encompasses the ‘functional differentiability’ requirement that \( \delta F \) contains no surface terms, and the requirement that the action of \( F \) preserves the fall-off and parity conditions. Finally, given two functions \( F \) and \( G \) satisfying (i) and (ii), their Poisson bracket is defined by \( \{F, G\} = \Omega(\delta_F, \delta_G) \).

2.2. Constraints and Poincare generators

In \([4]\) it is shown that the Hamiltonian and diffeomorphism constraints:

\[
H_0[N] := \frac{1}{16\pi G} \int_\Sigma N \left( q^{-1/2} \left( \pi_{ab} \pi^{ab} - \frac{1}{2} \pi^2 \right) - q^{1/2} R \right),
\]

(2.6)

\[
D_0[N] := -\frac{1}{8\pi G} \int_\Sigma N_\sigma D_\sigma \pi^{ab},
\]

(2.7)

satisfy (i) and (ii) when the lapse and shift have the following \( r \to \infty \) asymptotic behavior:

\[
N = S(\hat{x}) + O(r^{-\epsilon}),
\]

(2.8)

\[
N^I = S^I(\hat{x}) + O(r^{-\epsilon}),
\]

(2.9)

with

\[
S(-\hat{x}) = -S(\hat{x}),
\]

(2.10)

\[
S^I(-\hat{x}) = -S^I(\hat{x}).
\]

(2.11)

\( H_0[N] \) and \( D_0[N] \) with lapse and shift obeying (2.8), (2.9) generate the gauge transformation of the theory\(^2\).

On the other hand, asymptotic Poincare transformations correspond to lapse and shift satisfying the asymptotic conditions:

\[
N \to \alpha + B + \ldots
\]

(2.12)

\(^2\) Transformations associated to nontrivial \( S \) and \( S^I \) are referred in the literature as time and spatial ‘supertranslations’ \([4]\).
\[ N^I \to \alpha^I + R^I + \ldots, \]  
where \( \alpha, \alpha^I \) are constants that represent spacetime translations

\[ B = \beta_I x^I \]  
\[ R^I = \beta_I^J x^J \]

with constant \( \beta \)'s and \( \beta_{IJ} = -\beta_{JI} \) represent boost and rotations, and the dots indicates ‘pure gauge’ terms as in (2.8), (2.9).

In [4] it is shown that the generators of the asymptotic Poincare transformations satisfying (i) and (ii) above are given by:

\[ H[N] := H_0[N] + \frac{1}{8\pi G} \oint_S \mathfrak{d}S_a \hat{q}^{1/2} q^{ab} q^{cd} \left( N \hat{D}_b q_{ca} - \hat{D}_b N \left( q_{ca} - q^{ca} \right) \right), \]  
\[ D[\vec{N}] := D_0[\vec{N}] + \frac{1}{8\pi G} \oint_S \mathfrak{d}S_b \hat{\pi}^{ab} = -\frac{1}{16\pi G} \int_{\Sigma} \hat{\pi}^{ab} \vec{N} q_{ab}, \]

where \( \oint_\infty \equiv \lim_{r \to \infty} \oint_S \) with \( S \) the two-sphere of radius \( r \) with respect to the cartesian system \( x^I \) and \( \hat{D} \) the derivative compatible with \( q_{ab} \).

We emphasize that it is the ‘total’ \( H[N] \) and \( D[\vec{N}] \) that satisfy (i) and (ii). The writing of the generators as ‘volume plus surface term’ is for convenience; each term is not in itself a well defined phase space function. In particular, the surface integrals can be divergent. On the constraint surface however, finiteness of the generators imply finiteness of the surface integrals, which give then the value of the corresponding Poincare charges (e.g. angular momentum in the case of rotations).

3. AB variables

3.1. Phase space

The Cauchy slice \( \Sigma \) and the cartesian coordinates \( \{x^I\} \) are taken as in the previous section. The canonical pair is now given by an \( su(2) \) connection one-form \( A_a = A_a^\tau \tau \) and conjugate electric field \( E^a = E^a_i \tau \), with \( \tau_i \) \( i = 1, 2, 3 \) an \( su(2) \) basis satisfying \( [\tau_i, \tau_j] = \epsilon_{ijk} \tau_k \). As will shortly become clear, in order to have a well defined symplectic structure it is necessary that the electric fields asymptote to a fixed densitized triad \( \vec{E}_i \)'s (whose associated metric is taken to agree with \( \hat{q}_{ab} \) of equation (2.1)). We chose this fixed, zeroth order asymptotic electric field to be given by \( \vec{E}_i = \delta_i^I \). The phase space is then given by pairs \( (A^I_a, E^a_i) \) satisfying the analogue of the ADM asymptotic conditions [1]:

\[ E^I_i = \vec{E}^I_i + \frac{j^I_i(\hat{x})}{r} + O(r^{-1-\epsilon}), \]  
\[ A^I_i = \frac{a^I_i(\hat{x})}{r^2} + O(r^{-2-\epsilon}). \]

\[ \text{(3.1)} \]  
\[ \text{(3.2)} \]

\[ \text{This can happen for asymptotic rotations and boosts, and for phase space points outside the constraint surface } H_0 = D_0 = 0. \]  
\[ \text{This 'off-shell' divergence of surface terms does not occur with the fall-off conditions used in [3], which are schematically of the form } q_{ab} = \delta_{ab} + (\text{even})r^{-1} + O(r^{-2}) + O(r^{-2-\epsilon}) \]  
\[ \text{and } \hat{\pi}^{ab} = (\text{odd})r^{-2} + O(r^{-3}) + O(r^{-3-\epsilon}). \]
with
\[
f_i^I(\hat{x}) = f_i^I(\hat{x})
\]
(3.3)
\[
a_i^I(\hat{x}) = -a_i^I(\hat{x}).
\]
(3.4)

It is not difficult to verify that these fall-offs and parity conditions imply the ADM ones described in the previous section. The fall-off conditions ensure the symplectic structure
\[
\Omega(\delta_1, \delta_2) = \frac{1}{8\pi G} \int_{\Sigma} \left( \delta_1 A^i_\alpha^a \delta_2 E^a_i - \delta_2 A^i_\alpha^a \delta_1 E^a_i \right),
\]
(3.5)
is well defined. We now see the need to keep fixed the zeroth order electric field in (3.1): had we allowed for all possible \(SU(2)\)-rotated \(E^a_i\)'s (so that the asymptotic metric still satisfies (2.1)), we could not have ensured convergence of the integral (3.5).

Below we will be dealing with phase space functions \(F[A, E]\) that are integrals over \(\Sigma\) of local functions of \(A^i_\alpha^a\) and \(E^a_i\). Such functions will be required to satisfy the conditions (i) and (ii) described at the end of section 2.1 (now with respect to the symplectic form (3.5)).

3.2. Constraints

3.2.1. Gauss constraint. In connection variables, there appears the additional Gauss law constraint
\[
G[A] = \frac{1}{8\pi G} \int_{\Sigma} A^i D_a E^a_i,
\]
(3.6)
where \(D_a\) is the covariant derivative associated to the connection \(A^i_\alpha^a\), acting as
\[
D_a f^i = \partial_a f^i + \epsilon^{ijk} A^j_\alpha^a f^k.
\]
Since both terms in \(D_a E^a_i\) fall off as \(r^{-2} + O(r^{-2-\epsilon})\), the minimal condition on the multiplier \(A^i\) ensuring convergence of the integral is:
\[
A^i = \hat{\lambda}^i(\hat{x}) + O(r^{-1-\epsilon})
\]
(3.7)
with
\[
\hat{\lambda}^i(\hat{x}) = \lambda^i(\hat{x}).
\]
(3.8)

We now verify that with this fall-off and parity condition, \(G[A]\) also satisfies (ii):
\[
\delta G = \frac{1}{8\pi G} \int_{\Sigma} A^i \left( \partial_a \delta E^a_i + \epsilon^{ijk} A^j_\alpha^a \delta E^a_k + \epsilon^{ijk} \delta A^j_\alpha^a E^a_k \right)
\]
\[
= \frac{1}{8\pi G} \int_{\Sigma} \left( \delta A^i_\alpha^a \delta E^a_i - \delta A^i_\alpha^a \delta E^a_i \right) \equiv \Omega(\delta A, \delta),
\]
(3.9)
\[
\frac{1}{8\pi G} \int_{\Sigma} \left( \delta A^i_\alpha^a \delta E^a_i - \delta A^i_\alpha^a \delta E^a_i \right) \equiv \Omega(\delta A, \delta),
\]
(3.10)
where
\[
\delta A^i_\alpha^a = -\partial_a A^i + \epsilon^{ijk} A^j_\alpha^a A^k = -D_a A^i
\]
(3.11)
\[
\delta A^i_\alpha^a = \epsilon^{ijk} A^j_\alpha^a E^a_k.
\]
(3.12)

In going from (3.9) to (3.10), we performed the integration by parts:
\[
\int_{\Sigma} A^i \partial_a \delta E^a_i = \oint dS \partial_a A^i \delta E^a_i - \int_{\Sigma} \partial_a A^i \delta E^a_i = -\int_{\Sigma} \partial_a A^i \delta E^a_i,
\]
(3.13)
where the surface term being (odd)(even)$r^{-1}$(even)$r^{-1} + O(r^{-2-\epsilon})$ vanishes. It is easy to verify that (3.11) and (3.12) preserve the fall-off and parity conditions, and hence is a well defined phase space variation. Finally, the relation $[G[A], G[A']] = G[[A, A']]$ can be verified thanks to the vanishing of the surface term:

$$\oint_{\gamma} d\sigma_{\nu} e_{\mu\lambda} E_{\nu}^{\lambda} A_{\mu} = 0, \quad (3.14)$$

as implied by the fall-off and parity condition (3.7).

We now show that the leading term in (3.7) is crucial for the recovery of ADM phase space in that it accounts for ‘pure SU(2) gauge’ components of the $r^{-1}$ term of the triad (3.1). The doubly densitized inverse metric is given by $q_{IJ} = E_{i}^{I} E_{j}^{J} = E_{i}^{I} E_{j}^{J} + 2\tilde{E}_{i}^{(IJ)} f_{ij}^{f}$, where $f_{ij}^{f}$ is given by:

$$f_{ij}^{f} = -2\tilde{E}_{i}^{(ij)} f_{ij}^{f} \equiv -2 f_{ij}^{f}.$$  \hspace{1cm} (3.15)

Define $f_{IJ} = -2\tilde{E}_{i}^{(IJ)} f_{ij}^{f}$, which is in agreement with the previous ‘pure gauge’ interpretation of $f_{ij}^{f}$.

In the following sections we will often encounter Gauss constraints (3.6) smeared with phase space dependent multipliers. We now verify properties (i) and (ii) are still satisfied in such cases. For $A' = A'(A, E)$ satisfying (3.7) and (3.8), finiteness follows by the same fall-off/parity argument given before. From (3.10) it follows that the variation of $G[A]$ is now given by

$$\delta G[A] = \Omega(\delta A_{(A,E)} A, \delta) + \frac{1}{8\pi G_{T}} \int_{\Sigma} \delta A_{(A,E)} D_{a} E_{a}.$$  \hspace{1cm} (3.17)

If $A' = A'(A, E)$ does not depend on derivatives of $A_{i}$ or $E_{i}$, then (3.17) is already differentiable. Otherwise one needs to integrate by parts the second term in (3.17) to obtain an integrand that does not depend on derivatives of the variations $\delta A_{i}$ and $\delta E_{i}$. We now argue that the corresponding surface terms will always be (odd)$r^{-2} + O(r^{-2-\epsilon})$ and hence vanish. Consider a term in $\delta A'_{i}$ of the form $\phi_{ai}^{b} [A, E] \delta_{a} \delta E_{i}$. Since $\delta A' = (even) r^{-1}$ and $\delta_{a} \delta E_{i} = (odd) r^{-2}$ it follows that $\phi_{ai}^{b} = (odd) r^{-2}$.

The surface term will then be

$$\oint_{\gamma} d\sigma_{\nu} e_{\mu\lambda} E_{\nu}^{\lambda} A_{\mu} = 0, \quad (3.14)$$

as implied by the fall-off and parity condition (3.7).
Whereas the full $G[A]$ is well defined, the two terms in the rhs (3.18) are not necessarily well defined by themselves. Indeed, it can be easily seen that the fall-offs (3.7) do not ensure convergence of the surface term in (3.18).

3.2.2. Hamiltonian and diffeomorphism constraints. We start with the following form of the Hamiltonian and diffeomorphism constraints [17]:

$$H_0[N] := \frac{1}{16\pi G} \int_\Sigma \left( e_{ijk} F_{ab}^i E_j^a E_k^b - 2(1 + \gamma^2) K_{[a}^l K_{b]}^r E_j^a E_j^b \right),$$

(3.19)

$$D_0 \left( \vec{N} \right) := \frac{1}{8\pi G} \int_\Sigma E_j^a \mathcal{L}_a \vec{A}_j^a,$$

(3.20)

where $F_{ab}^i := \partial_0 A_b^i - \partial_0 A_a^i + e_{ijk} A_j^a A^b_k$, $K_{[a}^l := \gamma^{-1}(A_{[a}^l - T_{[a}^l)$ and $N$ of density weight $-1$. The relation between the constraints $H_0$ and $D_0$ of this section and those of section 2 will be described in section 4.

The minimal conditions for (3.19) and (3.20) to be finite are as in the ADM case:

$$N = S(\vec{x}) + O(r^{-\infty}),$$

(3.21)

$$N^I = S^I(\vec{x}) + O(r^{-\infty}),$$

(3.22)

with $S$ and $S^I$ odd. It is easy to verify that $D_0$ and the first term in $H_0$ satisfy (ii). We now argue the second, ‘KKEE’ term in $H_0$ also satisfies (ii). Under variations of this term, the potentially problematic surface contribution come from derivatives of the triad in $T^i_a$. Schematically:

$$\left( \int \text{NEEK} \delta \mathcal{F} \right)_{\text{bdy}} = \oint \text{NEEK} \delta \mathcal{F} = 0,$$

(3.23)

where the vanishing occurs since the integrand of the surface term falls off as $r^{-3}$. Finally, it is easy to verify that the contribution from the KKEE piece to the Hamiltonian vector field preserves the fall off and parity conditions.

$G[A], H_0[N]$ and $D_0[N]$ with multipliers satisfying the conditions above are the constraints/gauge generators of the theory.

3.3. Poincare generators

We now want to extend $H_0$ and $D_0$ in order to obtain well defined generators for lapse and shift corresponding to asymptotic Poincare transformations:

$$N \rightarrow \alpha + B + \ldots$$

(3.24)

$$N^I \rightarrow a^I + R^I + \ldots,$$

(3.25)

with $B = \beta x^I, R^I = \beta x^I$ as in section 2.2 and the dots indicate gauge terms (3.21), (3.22). Following the strategy of [3] we will start by adding surface terms that cancel the unwanted boundary contribution of the variations of $H_0$ and $D_0$.

First, we notice that the ‘KKEE’ term of the Hamiltonian constraint (3.19) is still well defined for the more general lapse (3.24); the leading term in the lapse is now (odd) $r$ so that $NKKEE \sim (\text{odd}) r^{-3} + O(r^{-3-\mathcal{C}})$ and the integral converges; the potentially problematic surface term (3.23) is now (odd) $r^{-2} + O(r^{-2-\mathcal{C}})$ and again vanishes. It is also easy to verify that the corresponding Hamiltonian vector field preserves the fall off and parity conditions.
Thus, the surface terms that cancel the unwanted boundary contributions are the same as in the self-dual formulation [6]:

\[
H_1[N] = H_0[N] - \frac{1}{8\pi G} \oint_{\infty} dS_n \varepsilon_{ijk} A^i_k E^a_j E^b_j ,
\]

(3.26)

\[
D_1[N] = D_0[N] - \frac{1}{8\pi G} \oint_{\infty} dS_{\kappa\lambda} N^\kappa A^\kappa_i E^\lambda_i = -\frac{1}{8\pi G} \int_{\Sigma} A^\kappa_i \mathcal{L}_{N} E^\kappa_i .
\]

(3.27)

For the case of asymptotic spacetime translations (so that \( B = 0 \) and \( R^j = 0 \) in (3.24) and (3.25)), \( H_1 \) and \( D_1 \) yield well defined phase space generators which agree with the ADM ones [1, 6]. This result will be recovered as a particular case of the general Poincare generators discussed below.

At a formal level, even for boosts and rotations the variations of \( H_1 \) and \( D_1 \) have no surface terms However \( H_1 \) and \( D_1 \) are no longer guaranteed to be finite.

As pointed out in [1], the reason these functions are not well defined for nonzero rotations or boosts becomes clear when one realizes their action would change the zeroth order part of the triad and thus map us out of phase space. This suggests one should modify the expression by adding a suitable Gauss piece in such a way that the zeroth order part of the triad is kept fixed. In the following we implement this idea.

### 3.3.1. Rotations

It will be convenient to work with the last expression in (3.27). When the shift has a nonzero rotation at infinity, the integrand has the following asymptotic behavior:

\[
-\mathcal{A}^i_1 \mathcal{L}_{\vec{N}} E^j_i = A^i_1 E^j_i \partial_j \mathcal{N}^j + \ldots \quad \xrightarrow{[r\to\infty]} \quad A^j_1 \mathcal{E}_{\hat{i}j} + (\text{odd})r^{-3} + O(r^{-3-c}).
\]

(3.28)

The first term falls off as \((\text{odd})r^{-2} + O(r^{-2-c})\) and thus we cannot ensure converge of the integral. This is the same term responsible for rotating the zeroth order part of the triad. In order to compensate, we subtract an appropriate Gauss term \( G[A_R] \) with

\[
A^j_R = \mathcal{A}^j_R + \mathcal{N}',
\]

where

\[
\mathcal{A}^j_R := -\frac{1}{2} \varepsilon_{ijk} \mathcal{E}^i_j \mathcal{E}^j_k \beta^l_j = \frac{1}{2} \varepsilon_{ijk} \mathcal{L}_{\vec{N}} E^a_j
\]

(3.30)

is a constant \(( \partial_j \mathcal{A}^j_R = 0 \) zeroth order term, and \( \mathcal{N}' \) a ‘pure gauge’ multiplier as in (3.7), (3.8).

By subtracting \( G[A_R] \) from \( D_1[N] \) we cancel the term responsible for the divergence in (3.28). This also introduces a new divergent and non-differentiable piece which is removed by including an appropriate boundary term. The final expression is:

\[
D_1[N] = D_0[N] - G[A_R] + \frac{1}{8\pi G} \oint_{\infty} dS_{\kappa\lambda} E^\kappa_i \mathcal{A}^i_R .
\]

(3.31)

We now verify (3.31) satisfies (i) and (ii). In section 4 we show (3.31) agrees, modulo pure gauge Gauss constraint terms, with ADM and Thiemann’s expressions.
To show finiteness, we write (3.31) as a volume integral:

\[
D \left[ N \right] = \frac{1}{8\pi G_T} \int \left( -A_i^t \mathcal{L}_{\hat{R}} E_i^a - \epsilon_{ijk} A_k^a A_j^b E_i^a - \left( \Lambda^t_R - \hat{\Lambda}_R^i \right) \delta_{a} E^a \right),
\]

(3.32)

where we used that \( \partial_a (\hat{\Lambda}_R^i E_i^a) = \hat{\Lambda}_R^i \partial_a E_i^a \). By construction the first two terms in (3.32) combine to give a convergent fall-off:

\[
-\Lambda_i^t \mathcal{L}_{\hat{R}} E_i^a - \epsilon_{ijk} A_k^a A_j^b E_i^a = (\text{odd}) r^{-3} + O \left( r^{-3-\epsilon} \right),
\]

(3.33)

where the cancelation of the would-be divergent terms can be explicitly verified by substituting (3.30) in (3.33). The last term in (3.32) is clearly convergent. We thus conclude (3.31) is finite. Let us now verify (ii). As mentioned earlier the first term in (3.32) is functional differentiable. By the same arguments given for the differentiability of the Gauss constraint one finds that the last term in (3.32) is also functional differentiable. The total variation can finally be written as

\[
\delta D \left[ N \right] = \Omega \left( \delta D \left[ \hat{N} \right], \delta \right),
\]

(3.34)

with

\[
\delta D \left[ \hat{N} \right] A^j_a = \mathcal{L}_N A^j_a - \delta \Lambda_a A^j_a
\]

(3.35)

and

\[
\delta D \left[ \hat{N} \right] E_i^a = \mathcal{L}_N E_i^a - \delta \Lambda_a E_i^a,
\]

(3.36)

where \( \delta \Lambda_a \) is given by (3.11), (3.12) with \( \Lambda = \Lambda_R \). It is easy to verify that \( \delta D \left[ \hat{N} \right] \) preserves the falloff and parity conditions\(^4\).

Note that in the above discussion the shift was of the general type (3.25). If \( \hat{R}^t = 0 \) then \( \hat{\Lambda}_R^i = 0 \) and we recover the generator of translations (3.27) (up to a possible 'pure gauge' Gauss term).

We conclude the section by verifying (3.31) is SU(2) gauge invariant in the sense that it weakly commutes with the Gauss constraint. Using equations (3.11), (3.12), (3.35), (3.36), one finds:

\[
\left\{ D \left[ \hat{N} \right], G \left[ A \right] \right\} = G \left[ \mathcal{L}_{\hat{R}} A - \left[ A_R, A \right] \right] - \frac{1}{8\pi G_T} \oint \mathcal{S}_u E_i^a \left( \mathcal{L}_{\hat{R}} A^j - \left[ A_R, A^j \right] \right).
\]

(3.37)

The multiplier of the Gauss term in the rhs of (3.37) is \((\text{even}) r^{-1} + O \left( r^{r^{-1} - \epsilon} \right)\) and hence satisfies the conditions of section 3.2.1. We now show that the surface term in (3.37) vanishes. The first term can be written as:

\[
\oint \mathcal{S}_u E_i^a \mathcal{L}_{\hat{R}} A^j = -\oint \mathcal{S}_u \mathcal{L}_{\hat{R}} \left( \hat{E}_i^a \right) A^j.
\]

(3.38)

where we used the fact that \( \mathcal{L}_{\hat{R}} A = \left( \text{even} \right) r^{-1} + O \left( r^{r^{-1} - \epsilon} \right) \) and equation (A8). For the second term we have

\[
-\oint \mathcal{S}_u \epsilon_{ijk} \left[ A_R, A^j \right] = -\oint \mathcal{S}_u \epsilon_{ijk} \hat{E}_i^a A_k^a A^j
\]

(3.39)

\(^4\) Each term in (3.35) is a well defined variation. For (3.36) only the total expression is a valid variation, but not each term independently (except when \( \hat{R}^t = 0 \)).
\[
\frac{1}{2} \oint_S dS_a \epsilon_{ijk} \epsilon_{jmnn} \hat{E}_i^a \hat{E}_b^m \mathcal{L}_R \left( \hat{E}_n^b \right)^\Lambda^i \\
= \frac{1}{2} \oint_S dS_a \left( \mathcal{L}_R \left( \hat{E}_i^a \right)^\Lambda^j - \hat{E}_i^a \mathcal{L}_R \left( \hat{E}_j^b \right)^{\Lambda^a} \right) \\
= \oint_{\infty} dS_a \mathcal{L}_R \left( \hat{E}_i^a \right)^\Lambda^j ,
\]
where we used the fact that \([A_R, A] = (\text{even}) r^{-1} + O(r^{-1-\epsilon})\) and \(\mathcal{L}_R (\hat{E}_j^b) \hat{E}_j^a = -\hat{E}_j^b \mathcal{L}_R \hat{E}_j^a\).

The two terms cancel each other and we conclude that \(D[\hat{N}]\) weakly commutes with the Gauss constraint.

### 3.3.2. Boosts

Expressing the boundary term in (3.26) as a volume integral, one can isolate the divergent term:

\[
H_1[N] = -\frac{1}{8\pi G} \int_\Sigma \tilde{D}_a \in \epsilon_{ijk} A_i^b E_j^a E_k^b + \ldots ,
\]
where the dots represents terms whose integral is convergent for \(N \rightarrow \beta_j x^j\). The divergent piece can be removed by subtracting a Gauss term \(G[A_B]\) with

\[
\Lambda_i^b = \hat{A}_B^i + \Lambda^i ,
\]
where

\[
\hat{A}_B^i = \beta_j \hat{E}_B^i = \tilde{D}_a B \hat{E}_i^a
\]
is a constant, zeroth order term and \(\Lambda^i\) a ‘pure gauge’ multiplier as in (3.7), (3.8). As in the case of rotations, the Gauss term introduces a divergent and non-differentiable piece that can be removed by an appropriate surface term. The final expression is:

\[
H[N] = H_1[N] - \gamma G[A_B] + \frac{1}{8\pi G} \oint_{\infty} dS_a E_i^a \hat{A}_B^i.
\]

We now verify the expression satisfies (i) and (ii). In section 4 we show (3.46) agrees, modulo pure gauge Gauss constraint terms, with ADM and Thiemann’s expressions.

As we did for the rotations, let us express (3.46) as a volume integral

\[
H[N] = (8\pi G)^{-1} \int_\Sigma \rho \text{ with:}
\]

\[
\rho = -N\epsilon_{ijk} A_i^b \tilde{D}_a \left( E_j^a E_k^b \right) - \tilde{D}_a N \epsilon_{ijk} A_i^b E_j^a E_k^b \\
\quad - \epsilon_{ijk} A_i^b A_j^a E_k^b - \left( A_B^i - \hat{A}_B^i \right) \partial_a E^a + \ldots
\]

The dots indicate the ‘NAAEE’ term coming from the non-abelian part of \(F_{ab}\) and the ‘Lorentzian’ ‘NNKKEE’ term. The former is manifestly finite and differentiable. The latter is also finite and differentiable by the discussion given in the beginning of 3.3. We then focus on the terms displayed in (3.47). The first and last terms are easily verified to give a finite and differentiable contribution to \(H\). By construction, the potentially divergent contributions from the second and third term cancel out, as can be verified by using (3.24) and (3.45). We conclude that \(H[N]\) is finite and differentiable. One can further verify that all possible contributions to the Hamiltonian vector field are such that they preserve the fall-off and parity conditions so that \(H[N]\) satisfies (ii).
We finally note that if $B = 0$, then $\dot{A}_B = 0$ and we recover the generator of time translations (3.26), up to a possible ‘pure gauge’ Gauss term.

4. Comparison with ADM and Thiemann’s expressions

4.1. Diffeomorphism constraint, asymptotic translations and rotations

We give a quick re-derivation of Thiemann’s expressions based on the ADM ones, and show they coincide (up to pure gauge Gauss terms) with the expressions from last section. We start with the ADM generator (2.17):

$$D_{\text{ADM}}\left[\vec{N}\right] = \frac{1}{16\pi G} \int_{\Sigma} \pi^{ab} \mathcal{L}_{\tilde{N}} q_{ab},$$

with a general shift of the form (2.13), and seek to rewrite it in terms of $(A_i^t, E_i^a)$ variables. A small computation shows the integrand in (4.1) can be rewritten as

$$\pi^{ab} \mathcal{L}_{\tilde{N}} q_{ab} = -2q^{-1/2} K_{ab} \mathcal{L}_{\tilde{N}} \left(q q^{ab}\right),$$

where we used $\pi^{ab} = q^{1/2}(K^{ab} - K q^{ab})$. Performing the substitution

$$q q^{ab} = E^a_i E^b_i,$$

$$q^{-1/2} K_{ab} = E^a_i K^b_i,$$

in (4.2), the ADM generator (4.1) becomes:

$$D_{\text{ADM}}\left[\vec{N}\right] = -\frac{1}{8\pi G} \int_{\Sigma} \left(K^a_i \mathcal{L}_{\tilde{N}} E^a_i + \epsilon_{ijk} A^i_j K^a_j E^a_k\right),$$

where we have defined

$$A^i_{\tilde{N}} = \frac{1}{2}\epsilon_{ijk} E^a_j \mathcal{L}_{\tilde{N}} E^a_k.$$

Finally, substituting $K^a_i = \gamma^{-1}(A^i_j - \Gamma^i_a)$ we recover the expression given in [1]:

$$D_{\text{ADM}}\left[\vec{N}\right] = D_1\left[\vec{N}\right] - G\left(\Lambda_{\vec{N}}\right) + \frac{1}{8\pi G} \int_{\Sigma} \Gamma^a_i \mathcal{L}_{\tilde{N}} E^a_i,$$

with $D_1$ and $G$ given by (3.27) and (3.6) respectively. As we shall see, the last term in (4.7) can be written as a surface term. In appendix A.3 we display this surface term in the form given in [1].

In the following we discuss asymptotic rotations and translations separately. We will write the last term in (4.7) in a way that will facilitate the comparison with the expressions of section 3.

4.1.1. Rotations. Consider the case where $a^t = 0$ so that

$$N^i = R^i + S^i + O\left(r^{-\epsilon}\right),$$

with $R^i$ as in (2.15) and $S^i$ as in (2.11). The last term in the rhs of (4.7) can be shown to be a pure boundary term as follows. First integrate by parts

$$\int \Gamma^a_i \mathcal{L}_{\tilde{N}} E^a_i = \oint \mathcal{S}_a N^a \Gamma^b_i E^b_i - \int E^a_i \mathcal{L}_{\tilde{N}} \Gamma^a_i.$$

(4.9)
The surface integral vanishes since 
$dS_a N^a = (\text{even}) + O(r^{-\epsilon})$, $\Gamma^a_b = (\text{odd}) r^{-2}$ and $E^b_i$ as in (3.1). For the second integral we use the identity (see appendix A.1),

$$E^a_i \delta \Gamma^i_a = -\frac{1}{2} \epsilon_{ijk} \partial_a \left( E^a_i E^a_j \partial_a \right),$$

(4.10)
to write it as a boundary term:

$$-\int E^a_i \mathcal{L} \Gamma^i_a = \frac{1}{2} \int_\infty dS_a \epsilon_{ijk} E^a_i E^a_j \mathcal{L} \tilde{N} E^b_k$$

(4.11)

$$= \int_\infty dS_a E^a_i \Lambda^i_b,$$

(4.12)

where in the second equality we used the definition of $\Lambda_S$ (4.6). We thus obtain:

$$D_{\text{ADM}} \left[ \tilde{N} \right] = D_i \left[ \tilde{N} \right] - G(\tilde{N}) + \frac{1}{8\pi G_f} \int_\infty dS_a E^a_i \Lambda^i_b.$$ (4.13)

Expression (4.13) resembles that of the generator (3.31) given in the previous section. Choosing for simplicity $\Lambda_R = \tilde{\Lambda}_R$ in (3.31), the difference between the two is:

$$D \left[ \tilde{N} \right] - D_{\text{ADM}} \left[ \tilde{N} \right] = G(\tilde{\Lambda}_R) - \frac{1}{8\pi G_f} \int_\infty dS_a E^a_i \lambda^i_R$$ (4.14)

with

$$\lambda^i_R = \Lambda^i_R - \tilde{\Lambda}^i_R = (\text{even}) r^{-1} + O\left(r^{-1-\epsilon}\right).$$ (4.15)

In appendix B we show that

$$\int_\infty dS_a E^a_i \lambda^i_R = 0$$ (4.16)

and hence the difference (4.14) is a ‘pure gauge’ Gauss term $G(\lambda_R)$ (with phase space dependent multiplier).

4.1.2. Translations. For completeness we re-derive the result that for asymptotic translations the generator (3.27) coincides, modulo a Gauss term, with the ADM generator (4.7). Since the considerations from the previous section already account for the $S^i$ and $O(r^{-\epsilon})$ terms in the shift (see appendix B), we now restrict attention to shifts of the form

$$N^i = a^i + O\left(r^{-1-\epsilon}\right).$$ (4.17)

The surface term that was dropped in (4.9) no longer vanishes and so the last term in the rhs of (4.7) now becomes

$$\int_\Sigma \Gamma^i_a \mathcal{L} \tilde{N} E^a_i = \int \mathcal{S}_a \left( N^a \Gamma^i_a E^b_i + E^a_i \Lambda^i_R \right).$$ (4.18)

For the first term in (4.18) we write:

$$\Gamma^i_a E^b_i = -\frac{1}{2} \epsilon_{ijk} E^a_i E^a_j \tilde{D}_b E^b_k,$$

(4.19)

(this follows from equation (A.11) by noting that the additional $(D_b - \tilde{D}_b)$ contribution vanishes). For the second term we write

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where we used that \( \hat{D}_b N^b = O(r^{-2}) \) for the shift (4.17). Since \( \hat{D}_k E^i_k = (\text{odd}) r^{-2} \), the triads in (4.19) and (4.20) that are not being derivated can be set to their zeroth order value.

Defining

\[
X^a := \frac{1}{2} \epsilon_{ijk} \dot{E}^a_i \dot{E}^b_j \dot{E}^c_k,
\]

(4.21)

the surface integral (4.18) becomes

\[
\oint dS_a \left( -N^a \hat{D}_b X^b + N^b \hat{D}_b X^a \right) = \oint dS_a \left( -N^a \hat{D}_b X^b + \mathcal{L}_\mathcal{H} X^a \right) = 0,
\]

(4.22)

where we used equation (A.8). Thus, the surface term vanishes and the generators (4.7) and (3.27) differ by a pure gauge Gauss term.

\[4.2. \text{Hamiltonian constraint, asymptotic time translations and boosts}\]

We start with a quick re-derivation of the Hamiltonian in AB variables in order to ensure no further subtleties arise from the ‘KKKEE’ term. Let

\[
H_{\text{ADM}}[N_{\text{ADM}}] = H_{\text{ADM}}^0[N_{\text{ADM}}] + S[N_{\text{ADM}}]
\]

(4.23)

be the full ADM Hamiltonian (2.16) with \( H_{\text{ADM}}^0[N_{\text{ADM}}] \) given by equation (2.6) and \( S[N_{\text{ADM}}] \) the surface term in (2.16). The lapse is taken to be \( N_{\text{ADM}} = q^{1/2} N \) with \( N \) the density weight \(-1\) lapse satisfying (3.24).

For the integrand of \( H_{\text{ADM}}^0 \) we use the identities (see for instance [16]):

\[
-q R = \epsilon_{ijk} F_{ab} E^a_i E^b_j + 2D_a \left( E^a_i \hat{D}_b E^b_i \right) - 2q^2 K^i_{[ab} K^j_{b]} E^a_i E^b_j
\]

(4.24)

\[
\pi_{ab} \pi^{ab} - \frac{1}{2} \pi^2 = -2K^i_{[ab} K^j_{b]} E^a_i E^b_j + \frac{1}{2q^2} D_a E^a_i \hat{D}_b E^b_i
\]

(4.25)

where \( D_a \) is the derivative compatible with \( q_{ab} \) (4.3). For the surface term \( S[N_{\text{ADM}}] \) we use the result derived in [1]:

\[
S[q^{1/2} N] = \frac{1}{8\pi G} \oint_{\infty} dS_a \left( -NE^a_i \partial_a E^b_j + D_b NE^a_i \left( E^a_i - \dot{E}^a_i \right) \right).
\]

(4.26)

We now use (4.24)–(4.26) to express (4.23) in \((A^a_i, \dot{E}^a_i)\) variables. Subtracting for convenience the ‘pure gauge’ Gauss piece arising from the second term in (4.25) and integrating by parts the middle term in (4.24), one obtains:

\[
H_{\text{ADM}}^0[q^{1/2} N] = H_{\text{ADM}}[q^{1/2} N] - (4\pi)^{-1} G \left[ N \partial_a E^a \right]
\]

(4.27)

\[
= H[N] - \gamma G \left[ A_N \right] + \frac{1}{8\pi G} \oint_{\infty} dS_a A^a_i \left( E^a_i - \dot{E}^a_i \right),
\]

(4.28)

with \( H_1 \) given by (3.26), \( G \) as in (3.6) and

\[
A^a_k := D_a NE^a_i.
\]

(4.29)

Expression (4.28) corresponds to the one given in [1], written in a way that will facilitate comparison with \( H[N] \) (3.46). Subtracting \( 0 = \oint_{\infty} dS_a \dot{A}^a_k \dot{E}^a_k \) in \( H[N] \) so that the surface term in (3.46) involves the difference \((E^a_i - \dot{E}^a_i)\) as in (4.28), we find:
The difference $A^i_N - A^i_B$ can be seen to be $(even)\, r^{-1}$ as follows. First write

$$A^i_N = \tilde{D}_a N E^a_i + q^{-1/2}\tilde{D}_a q^{1/2} N E^a_i = \tilde{D}_a N E^a_i + (even)\, r^{-1}$$

(4.31)

since $\tilde{D}_a q^{1/2} = (odd)\, r^{-2}$ and $N = (odd)\, r$. Finally

$$\tilde{D}_a N E^a_i - \tilde{A}^a_i = \tilde{D}_a N E^a_i - \tilde{D}_a B\tilde{E}^a_i$$

(4.32)

$$= \tilde{D}_a N \left( E^a_i - \tilde{E}^a_i \right) + \tilde{D}_a (N - B)\tilde{E}^a_i$$

(4.33)

$$= (even)\, r^{-1}.$$  (4.34)

It then follows that the surface term in (4.30) vanishes and the difference (4.30) is given by the ‘pure gauge’ Gauss term $-\gamma G [A^i_N - A^i_B]$. Finally, we note that even though we were here mostly interested in the case of boosts, the lapse $N$ above is of the general form (3.24). If $B = 0$ the expressions reduce (modulo a pure gauge Gauss term) to the generator of asymptotic time translations $H_1[N]$ (3.26).

5. Summary and outlook

Field theories on non-compact spaces must be supplemented with boundary conditions for them to be well-defined. In the case of gravity with zero cosmological constant there exists various approaches depending on the applications one is interested in. Our interest for the analysis presented in this paper came from needs imposed on us when investigating how asymptotic flatness may be imposed in canonical quantum gravity [10]. The work [10] intends to be a step towards a description of asymptotically flat spacetimes in loop quantum gravity (LQG) [16–18].

Since current LQG is based upon real variables, the original treatment by Thiemann of asymptotically flat conditions in self-dual variables had to be reconsidered. At first this seemed like a very minor exercise, but in the process we realized there were improvements we could make and that a detailed analysis was warranted. These improvements have to do with asymptotic conditions and surface terms, and are for the most part independent of the issue of real vs. self-dual variables. The improvements may be summarized as follows.

The first one is regarding the compatibility between AB and ADM descriptions: if one desires to recover ADM phase space as the ‘reduced phase space’ associated to the Gauss constraint, one must allow for a leading $1/r$ term in the $su(2)$ gauge multiplier $A_i$. This is seen in equation (3.16), where the coefficient of this $1/r$ term (referred to by $\lambda'$) accounts for the $su(2)$ gauge redundancy that is present in the $1/r$ term of the triad (referred to by $f_i^a$). The second and perhaps most relevant improvement is the finding of a generator of asymptotic rotations that is polynomial in the variables. This is achieved through a new surface term for the diffeomorphism constraint that is polynomial. The surface term considered for the Hamiltonian constraint is also slightly simpler than the one considered in [1]. A nice feature of the present description is that now both diffeomorphism (3.31) and Hamiltonian (3.46) surface terms have a remarkably similar form.

The present analysis represents a key ingredient for the work in [10]. We hope it will also be of use for future studies of LQG in asymptotically flat spacetimes.
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Appendix A. Assorted results

A.1. Equation (4.10)

We starting from an identity given in [1, 16]:

\[ E_a^i \delta \Gamma^i_a = \frac{1}{2} \partial_a \left( \eta^{abc} e^j_b \delta e^j_c \right) \]  

(A.1)

and rewrite the expression inside the derivative as:

\[ \eta^{abc} e^j_b \delta e^j_c = e^j_b \delta \left( \eta^{abc} e^j_c \right) \]  

(A.2)

\[ = q^{-1/2} e^j_b \delta \left( q^{1/2} \eta^{abc} e^j_c \right) \]  

(A.3)

\[ = E^j_b \delta \left( -e^j_b e^i_d \right) \]  

(A.4)

using the last expression back in (A.1) we obtain (4.10).

A.2. Integration by parts formulas

Let \( \rho \) be a density one scalar and \( X^a \) a density one vector field. \( \rho \) is dual to the three-form \( \rho_{abc} \) and \( X^a \) dual to the two-form \( \omega_{ab} := \eta_{abc} X^c \) so that \( X^a = \frac{1}{2} \eta^{abc} \omega_{bc} \).

Stokes theorem for the integral of \( X^a \) over a volume \( V \) reads

\[ \int_V \partial_a X^a = \int_{\partial V} dS_a X^a. \]  

(A.5)

In particular, if \( \tilde{N} \) is a vector field then \( \mathcal{L}_{\tilde{N}} \rho = \partial_a (\rho \tilde{N}^a) \) and one has:

\[ \int_V \mathcal{L}_{\tilde{N}} \rho = \int_{\partial V} dS_a \tilde{N}^a \rho. \]  

(A.6)

The Lie derivative of \( X^a \) along a vector field \( \tilde{N} \) can be written as

\[ \mathcal{L}_{\tilde{N}} X^a = \tilde{N}^a \partial_b X^b + 2 \partial_b \left( X^a \tilde{N}^b \right), \]  

(A.7)

where the second term is a total derivative: \( 2 \partial_b (X^a \tilde{N}^b) = \eta^{abc} \partial_b Y_c \) with \( Y_c = \eta_{abc} X^d \tilde{N}^c \).

Integrating (A.7) over a two-surface \( S \) without boundary we obtain the relation:

\[ \oint_S dS_a \mathcal{L}_{\tilde{N}} X^a = \oint_S dS_a \tilde{N}^a \partial_b X^b. \]  

(A.8)

A.3. Surface term of equation (4.7) in terms of \( \Gamma^i_a \)’s

For a general asymptotic shift, \( N^i = \alpha^i + R^i + S^i + O(r^{-c}) \), Equations (4.9) and (4.12) expresses the last term in (4.7) as a pure surface term:
\[ \int_\Sigma \Gamma_a^i L^b_i E^a_i = \oint dS_n \left( N^a \Gamma_a^b E^b_i + E^a_i \Lambda^b_i \right). \quad (A.9) \]

If we write \( L^b_i E^a_i \) in (4.6) in terms of the derivative \( D_a \) compatible with \( q_{ab} \) (4.3), the second term on the rhs of (A.9) is given by:

\[ \oint dS_n E^a_i \Lambda^b_i = \frac{1}{2} \oint dS_n \varepsilon_{ijk} E^a_i E^b_j N^b D_b E^c_k - \frac{1}{2} \oint dS_n \varepsilon_{ijk} E^a_i E^b_j E^b_k D_b N^c. \quad (A.10) \]

The integrand of the second term in (A.10) can be written as the total derivative \( \eta^{abc} D_b N_c \) with \( N_i := q_{ij} N^j \) and hence the integral vanishes. The first term can be cast in terms of the spin connection by use of the formula

\[ \Gamma^b_i = -\frac{1}{2} \varepsilon_{ijk} E^b_j D_b E^c_k. \quad (A.11) \]

Doing so one obtains:

\[ \oint dS_n E^a_i \Lambda^b_i = -\oint dS_n D_a N^b E^a_i, \quad (A.12) \]

which together with the first term in the rhs of (A.9) correspond to form of the surface term given in [1].

**Appendix B. Equation (4.16)**

Let us denote the ‘pure gauge’ part of the shift (4.8) by \( \vec{\nu} \) so that:

\[ N^i = R^i + \nu^i, \quad \nu^i = S^i + O(r^{-c}). \quad (B.1) \]

Let

\[ \lambda^b_i := \frac{1}{2} \varepsilon_{ijk} \left( E^b_j L^k_i E^b_j - E^b_j L^k_i E^b_j \right), \quad (B.2) \]

\[ \Lambda^b_i := \frac{1}{2} \varepsilon_{ijk} E^b_j L^c_i E^b_k, \quad (B.3) \]

so that

\[ N^i = R^i + \lambda^b_i + \Lambda^b_i. \quad (B.4) \]

We now show that the contribution to the surface integral (4.16) from each term in (B.4) vanishes. The contribution coming from \( \Lambda^b_i \) can be written as in (A.10) with \( N = \vec{\nu} \). The second term on the rhs of (A.10) is again a total derivative whose integral vanishes, whereas the integrand of the first term is now \( (odd) \) \( r^{-2} + O(r^{-2-c}) \) and hence also vanishes.

To study the contribution from \( \lambda^b_i \), let us write the triad as

\[ E^a_i = \tilde{E}^a_i + g^a_i + h^a_i, \quad (B.5) \]

where \( g^a_i \) represents the (even) \( r^{-1} \) term in (3.1) and \( h^a_i \) the remaining \( O(r^{-1-c}) \) piece. Let

\[ E^a_i = \tilde{E}^a_i + g^a_i + h^a_i \quad (B.6) \]

denote the analogous expansion of the inverse triad so that

\[ g^a_i = -\tilde{E}^b_j E^a_i g^b_j \quad (B.7) \]

The integrand of the second term in (B.10) can be written as the total derivative \( \eta^{abc} D_b N_c \) with \( N_i := q_{ij} N^j \) and hence the integral vanishes. The first term can be cast in terms of the spin connection by use of the formula

\[ \Gamma^b_i = -\frac{1}{2} \varepsilon_{ijk} E^b_j D_b E^c_k. \quad (A.11) \]

Doing so one obtains:

\[ \oint dS_n E^a_i \Lambda^b_i = -\oint dS_n D_a N^b E^a_i, \quad (A.12) \]

which together with the first term in the rhs of (A.9) correspond to form of the surface term given in [1].
The corresponding expansion for $\lambda_R$ is:

$$
\lambda_R^i = \frac{1}{2} \epsilon_{ijk} \left( \hat{E}_i^a \hat{E}_j^b \hat{E}_k^c + \hat{E}_i^a \hat{E}_j^c \hat{E}_k^b + \hat{E}_i^b \hat{E}_j^c \hat{E}_k^a + O\left(r^{-2-\epsilon}\right) \right).
$$

(B.8)

Parity conditions imply that the nontrivial contributions to the integral come from the last two terms in (B.9):

$$
\oint dS_a E^a_{ij} \lambda_R^i = -\frac{1}{2} \oint dS_a \epsilon_{ijk} \left( \hat{E}_i^a \hat{E}_j^b \hat{E}_k^c + \hat{E}_i^a \hat{E}_j^c \hat{E}_k^b + \hat{E}_i^b \hat{E}_j^c \hat{E}_k^a + O\left(r^{-2-\epsilon}\right) \right).
$$

(B.10)

We now integrate by parts the first term (using equation (A.8) and $dS_a R^a = 0$) and use (B.8) for $h_b^k$ in the second term (only the first term in (B.8) contributes, since the ‘gg’ piece is even) to get:

$$
\oint dS_a E^a_{ij} \lambda_R^i = -\frac{1}{2} \oint dS_a \epsilon_{ijk} \left( \mathcal{L}_R \left( \hat{E}_i^a \hat{E}_j^b \right) h_k^b + \hat{E}_i^a \hat{E}_j^b h_k^c \mathcal{L}_R \hat{E}_k^c \right).
$$

(B.11)

Using

$$
\mathcal{L}_R \left( \hat{E}_i^a \hat{E}_j^b \right) = -E_i^c \hat{E}_b^a \hat{D}_c R^a + \hat{E}_i^a \hat{E}_j^b \hat{D}_c R^c
$$

(B.12)

and

$$
\mathcal{L}_R \hat{E}_k^b = -\hat{E}_k^d \hat{D}_d R^b,
$$

we get (B.11) can be written as (after renaming some indices and using $\hat{q}_{ab}$ to raise and lower some indices):

$$
\oint dS_a E^a_{ij} \lambda_R^i = \frac{1}{2} \oint dS_a B^{abk} \hat{q}_{ac} h_k^c
$$

(B.14)

with

$$
B^{abk} = \epsilon_{ijk} \hat{E}_i^a \hat{E}_j^b \hat{D}_c R^a - \epsilon_{ijk} \hat{E}_i^a \hat{E}_j^c \hat{D}_b R^c + \epsilon_{ijk} \hat{E}_i^b \hat{E}_j^c \hat{D}_a R^c.
$$

(B.15)

By writing (B.15) in cartesian coordinates so that $\hat{E}_i^a = \delta_i^a$, etc and using that $\hat{D}_a R^b = \delta_abc \hat{q}_c$ for some constant $\hat{q}_c$, one finds that (B.15) identically vanishes:

$$
B^{abk} = \epsilon_{abc} \epsilon_{cde} \hat{q}_d - \epsilon_{abc} \epsilon_{cde} \hat{q}_d + \epsilon_{abc} \epsilon_{cde} \hat{q}_d
$$

(B.16)

This concludes the proof of equation (4.16).

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