SEMICYLASSICAL FOCUSING NLS
WITH STEPLIKE INITIAL DATA

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ABSTRACT

We study the semiclassical behavior of the focusing nonlinear Schrödinger equation in 1+1 dimensions under discontinuous "barrier" initial data and we describe the violent oscillations arising in terms of theta functions. The construction of proofs relies on

(i) the analysis of the associated Riemann-Hilbert factorization problem

(ii) the analysis of the resulting maximin variational problem for a Green’s potential with external field and in particular the proof of existence of a regular solution, which enables the construction of the so-called g-function transformation and hence the asymptotic deformation of the Riemann-Hilbert factorization problem to one that is explicitly solvable in terms of theta functions.

In particular we show that the finite genus ansatz is generically valid for all times.

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0. INTRODUCTION

The semiclassical limit of the 1+1-dimensional, integrable nonlinear Schrödinger equation with cubic focusing nonlinearity has been the subject of recent investigations. Several numerical studies have appeared since 1998 ([MK], [BK], [CT], [C]) and rigorous analyses of the initial value problem under real analytic data have already appeared ([KMM] in 2000 and [TVZ] in 2004) in cases where the initial data is analytic. The present paper makes use of the method and results in [KMM] for the study of a very particular problem with discontinuous (barrier) data. It has been shown by A. Cohen and T. Kappeler [CK] that weak solutions exist for all time under barrier data and, even more, that the inverse scattering technique is still applicable to the integration of the problem. In fact, it is known ([DZ2]) that there exists a unique solution in $C(\mathbb{R}_+, L^2(dx) \cup (L^\infty(dx) \otimes L^4_{\text{loc}}(dt)))$.

Here, we use the inverse scattering method of [CK] to pose an associated Riemann-Hilbert factorization problem, which we then asymptotically (as $h \to 0$) reduce to the Riemann-Hilbert factorization problem that can be explicitly solved in terms of theta functions.

The real aim of this paper is to indicate that a discontinuity in the initial data does not necessarily alter the behavior of the semiclassical focusing NLS problem. Some changes of course have to happen. For example, if the Euler system that appears as a formal limit of the focusing semiclassical NLS does not even admit a solution for small times, it is obvious that the genus zero ansatz cannot hold uniformly for small times.

A natural generalization of the barrier data problem is the problem of general step data. For such data, the eigenvalue density $\rho^0$ has several nonanalyticity points on the "spike" where the eigenvalues accumulate to the complex plane. The analysis of the present paper can be immediately extended to that case. The appropriate contour still exists and the finite genus ansatz still applies. The proof of that fact is not very different from the proof of the simplest case that we present here.
1. FOCUSING NLS WITH BARRIER DATA

We consider the nonlinear Schrödinger equation (1+1-dimensional, integrable, focusing case), on the half-line

\[ iu_t^h(x,t) + \frac{h^2}{2} u_{xx}^h(x,t) + |u^h(x,t)|^2 u^h(x,t) = 0, \]

\[ u^h(x,0) = u_0(x), \]

under barrier-like initial data:

\[ u_0(x) = A, \quad -1/2 < x < 1/2, \]

\[ = 0, \quad otherwise. \]

Here \( A \) is a fixed positive constant and \( h \) is a small positive constant. Eventually we will take \( h \to 0 \). We will assume that \( h \) is staying away from the discrete set \( \{ \frac{2A}{(2k+1)\pi} \}, k = 0, 1, 2, ... \). For simplicity, we will actually require that

\[ h \text{ takes values in the discrete set } \{ \frac{A}{k\pi} \}, \; k = 0, 1, 2, ... \]

Setting

\[ \rho^h = |u^h|^2, \]

\[ \mu^h = -i \frac{h}{2} (\bar{u}^h u_x^h - u^h \bar{u}_x^h), \]

(1) is transformed to

\[ \partial_t \rho^h + \partial_x \mu^h = 0, \]

\[ \partial_t \mu^h + \partial_x \left( \frac{(\mu^h)^2}{\rho^h} \right) - \partial_x (\rho^h)^2 / 2 = \frac{h^2}{4} \partial_x (\rho^h \partial_x^2 \log \rho^h). \]

The formal limit as \( h \to 0 \) is the Euler system

\[ \partial_t \rho + \partial_x \mu = 0, \]

(1a)

\[ \partial_t \mu + \partial_x \left( \frac{\mu^2}{\rho} \right) - \partial_x (\rho)^2 / 2 = 0. \]

The initial data become \( \rho = u_0^2(x), \mu = 0. \)

This initial value problem admits a weak solution for all time, as shown by A.Cohen and T.Kappeler ([CK]). Furthermore, the inverse scattering theory is still applicable.
The associated linear system is

\[
\hbar \psi_x = \begin{pmatrix} -i\lambda & u(x) \\ -u^*(x) & i\lambda \end{pmatrix} \psi,
\]

where * denotes complex conjugation. Jost functions \( \phi, \psi \) are defined as column vector solutions of (3) satisfying the asymptotic conditions

\[
\psi \sim \begin{pmatrix} 0 \\ e^{i\lambda x/\hbar} \end{pmatrix}, \text{ as } x \to +\infty,
\]

\[
\phi \sim \begin{pmatrix} e^{-i\lambda x/\hbar} \\ 0 \end{pmatrix}, \text{ as } x \to -\infty.
\]

One can define the reflection and transmission coefficient as follows.

Let \( \psi^0 \) be the transpose of \((\psi^*_2, -\psi^*_1)\). Then, following [CK] define \( a_+, b_+ \) by

\[
\phi = a_+ \psi^0 + b_+ \psi.
\]

The reflection and transmission coefficients are given by

\[
r_+ = b_+/a_+,
\]

\[
t_+ = 1/a_+.
\]

Similarly, one can define coefficients \( r_-, t_- \) by normalizing the Jost functions at the opposite infinities.

From [CK], we have

\[
\psi(x, \lambda) = \begin{pmatrix} 0 \\ e^{i\lambda x/\hbar} \end{pmatrix}, \quad x < -1/2,
\]

\[
= \begin{pmatrix} Ae^{i\lambda/\hbar}(x-1)/\hbar)^{\sin[(A^2+\lambda^2)^{1/2}((x-1)/\hbar)]} \\ e^{i\lambda/\hbar}\cos[(A^2+\lambda^2)^{1/2}((x-1)/\hbar)] + \lambda e^{i\lambda/\hbar}\sin[(A^2+\lambda^2)^{1/2}((x-1)/\hbar)]/(A^2+\lambda^2)^{1/2}((x-1)/\hbar) \end{pmatrix}, \quad -1/2 < x < 1/2,
\]

\[
= \begin{pmatrix} \beta(\lambda, h)e^{i\lambda x/\hbar} \\ \alpha(\lambda, h)e^{i\lambda x/\hbar} \end{pmatrix}, \quad x > 1/2,
\]

where

\[
\alpha = e^{i\lambda/\hbar}(\cos[(A^2+\lambda^2)^{1/2}/\hbar] - i\lambda \sin[(A^2+\lambda^2)^{1/2}(1/\hbar)]/(A^2+\lambda^2)^{1/2}),
\]

\[
\beta = -A e^{i\lambda/\hbar}\sin[(A^2+\lambda^2)^{1/2}(1/\hbar)]/(A^2+\lambda^2)^{1/2}.
\]
The coefficients \( a_+, b_+, r \) are given by

\[
\begin{align*}
a_+ &= \frac{\alpha}{\sqrt{\alpha^2 + |\beta|^2}}, \\
b_+ &= \frac{\beta}{\sqrt{\alpha^2 + |\beta|^2}}, \\
r(\lambda) &= \frac{\beta}{\alpha} = -\frac{1}{A \cot((A^2 + \lambda^2)^{1/2}(1/h))(A^2 + \lambda^2)^{1/2} - i\lambda}.
\end{align*}
\]

The eigenvalues are the zeros of \( a_+ \). They lie on the imaginary interval \([-iA, iA]\) and are given by

\[
\lambda = i\eta,
\]

\[
\tan\left[\frac{(A^2 - \eta^2)^{1/2}}{h}\right] = \frac{(A^2 - \eta^2)^{1/2}}{\eta}.
\]

In other words,

\[
\frac{(A^2 - \eta^2)^{1/2}}{h} = \arctan\left[\frac{(A^2 - \eta^2)^{1/2}}{\eta}\right] + k\pi, \quad k \in \mathbb{Z}.
\]

As \( h \to 0 \), we get an asymptotic expression for the eigenvalues \( \lambda_k = i\eta_k \).

\[
(A^2 - \eta_k^2)^{1/2} \sim hk\pi, \quad k \in \mathbb{Z}.
\]

The limiting density of eigenvalues is then given by

\[
\rho_0^0(\lambda) = \frac{\lambda}{\pi(A^2 + \lambda^2)^{1/2}}.
\]

or, in terms of \( \lambda = i\eta \),

\[
(9a) \quad \rho_0^0(\lambda) = \frac{\lambda}{\pi(A^2 + \lambda^2)^{1/2}}.
\]

Here the branch is chosen such that \( \rho_0^0(\lambda) \sim \frac{1}{\pi\lambda} \) as \( \lambda \to i\infty \).

We note here that as a consequence of the simplifying condition (2a) we have

\[
\int_0^{iA} \rho_0^0(\lambda)d\lambda = -iA/\pi = ikh, \quad \text{for some integer } k.
\]

Hence

\[
(9b) \quad \exp\left[\frac{\pi}{h} \int_0^{iA} \rho_0^0(\lambda)d\lambda\right] = 1.
\]

This will simplify the analysis of the "local" Riemann-Hilbert problem on a cross centered at the origin; see section 5, paragraph 6. Another consequence of (2a) is that \( r(0) = 0 \).
It is easily seen that the associated norming constants can only take the values 
$-1, 1$ (by symmetry in $x$) and that in fact they have to oscillate between these two 
values (by a Sturm-Liouville oscillation argument; see [KMM]).

REMARK. The density $\rho^0$ blows up at $iA$. This is due to the local "flatness"
and not the discontinuity of the initial data. There are some minor implications of
this fact. The external field of the associated variational problem (see later) is still
continuous at $iA$ and analytic nearby. On the other hand we do not want to allow
that the band endpoints $\lambda_j$ (defined later) are equal to $iA$. Generically (in $x, t$)
this will not happen. If, say, $\lambda_0 = iA$, then the analysis of [KMM] breaks down. In
particular it is not clear how to solve the scalar Riemann-Hilbert problem for the
g-function.

We end this section by defining the three Pauli matrices; we will be using them
later.

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

2. THE RIEMANN-HILBERT PROBLEM

We can now state the following Riemann-Hilbert problem, following [KMM]. Let
$C$ be a piecewise smooth loop encircling all eigenvalues in the upper half-plane
and lying entirely in the upper half-plane except for the point 0. Let $C^*$ be its
conjugate, encircling the eigenvalues in the lower half-plane. Also give the following
orientation:

(i) the real axis is oriented from left to right,

(ii) the loop $C$ is oriented counterclockwise,

(iii) the loop $C^*$ is also oriented counterclockwise.

We use the following convention: the $+\text{-side}$ of an oriented contour is always to
its left, according to the given orientation.

We also choose $C$ so that it approaches 0 (from left and right) at a non-zero,
non-straight angle with the real axis.
THEOREM 1 (discrete version). Let \( d\mu = \sum_k h(\delta_{\lambda_k^+} - \delta_{\lambda_k^-}) \) be a finite sum of point measures supported at the eigenvalues of the system (3), as given by (8). Let
\[
X(\lambda) = -(A^2 + \lambda^2)^{1/2}.
\]

Letting \( M_+ \) and \( M_- \) denote the limits of \( M \) on \( \Sigma = C \cup C^* \) from left and right respectively, we define the Riemann-Hilbert factorization problem
\[
M_+(\lambda) = M_-(\lambda)J(\lambda),
\]
where
\[
J(\lambda) = \begin{pmatrix} \frac{1}{r^*(\lambda)e^{2i\lambda x + 2i\lambda^2 t}} & r(\lambda)e^{-2i\lambda x - 2i\lambda^2 t} \\ r*(\lambda)e^{2i\lambda x + 2i\lambda^2 t} & 1 + |r(\lambda)|^2 \end{pmatrix}, \lambda \in \mathbb{R},
\]
\[
= v(\lambda), \lambda \in C,
\]
\[
= \sigma_2 v(\lambda^*)^*\sigma_2, \lambda \in C^*,
\]
\[
\lim_{\lambda \to \infty} M(\lambda) = I,
\]
where
\[
v(\lambda) = \begin{pmatrix} 1 & -i \exp\left(\frac{1}{h} \int \log(\lambda - \eta)d\mu(\eta)\right)\exp\left(-\frac{1}{h}(2i\lambda x + 2i\lambda^2 t - X(\lambda))\right) \\ 0 \end{pmatrix},
\]
and \( r \) is the reflection coefficient defined in (7):
\[
r(\lambda) = -A \frac{1}{\cot((A^2 + \lambda^2)^{1/2}(1/h))(A^2 + \lambda^2)^{1/2} - i\lambda}.
\]
Note that \( r(\lambda) \) can have a real singularity at \( \lambda = 0 \) if \( A/h \) is an odd multiple of \( \pi/2 \). By assumption, we have excluded such values of \( h \).

The above Riemann-Hilbert problem admits a solution and the solution of (1) can be recovered from the solution of (11) as follows.
\[
u(x, t) = 2ih \lim_{\lambda \to \infty}(\lambda M^{12}(\lambda)),
\]
where the index 12 here denotes the (12)-entry of a matrix.

PROOF: Standard; see Chapter 2 of [KMM]. The quantity \(-i \exp\left(\frac{1}{h}(X(\lambda_k))\right)\) oscillates between \(-1\) and \(1\). The function \(-i \exp\left(\frac{1}{h}(X(\lambda))\right)\) is thus an extrapolation of the norming constants.
REMARKS. 1. In [KMM] the Riemann-Hilbert problem jump involves some parameters denoted by $K, \sigma, J$. Here we are simply choosing $K = -1, \sigma = 1, J = 1$. This is compatible with the discussion in [KMM] as long as we focus our attention to the case $x \geq 0$. By the obvious symmetry $x \rightarrow -x$ this is acceptable.

2. Obviously, the contour $C$ can be deformed anywhere in the upper half-plane, as long as it passes through 0 (we only require it to be non-tangent to either the real axis or the imaginary axis at 0) and does not touch the linear segment $[0, iA]$. For the asymptotic analysis, the contour $C$ will be eventually fixed by the choice of the ”g-function” in the next section. Similarly for its conjugate $C^*$. In a sense we are seeking a steepest descent contour, on which the Riemann-Hilbert problem will reduce to one that is explicitly solvable.

We note the following factorization of the jump $J(\lambda)$ on the real line.

\[
J(\lambda) = \left( \begin{array}{cc} r^*(\lambda)e^{\frac{2i\lambda x + 2i\lambda^2}{2\lambda}} & r(\lambda)e^{\frac{-2i\lambda x - 2i\lambda^2}{2\lambda}} \\ 1 + |r(\lambda)|^2 & 1 \end{array} \right) = L(\lambda)U(\lambda), \tag{15+}
\]

where

\[
L(\lambda) = \left( \begin{array}{cc} 1 + |r(\lambda)|^2 & 0 \\ r^*(\lambda)e^{\frac{2i\lambda x + 2i\lambda^2}{2\lambda}} & 1 \end{array} \right), \quad U(\lambda) = \left( \begin{array}{cc} 1 & r(\lambda)e^{\frac{-2i\lambda x - 2i\lambda^2}{2\lambda}} \\ 0 & 1 \end{array} \right).
\]

A different factorization is also possible.

\[
J(\lambda) = \left( \begin{array}{cc} r(\lambda)e^{\frac{-2i\lambda x - 2i\lambda^2}{2\lambda}} & 1 + |r(\lambda)|^2 \\ r^*(\lambda)e^{\frac{2i\lambda x + 2i\lambda^2}{2\lambda}} & 1 \end{array} \right) = S(\lambda)D(\lambda)T(\lambda), \tag{15-}
\]

where

\[
S(\lambda) = \left( \begin{array}{cc} 1 & r(\lambda)e^{\frac{-2i\lambda x - 2i\lambda^2}{2\lambda}} \\ 0 & 1 + |r(\lambda)|^2 \end{array} \right), \quad D(\lambda) = \left( \begin{array}{cc} (1 + |r(\lambda)|^2)^{-1} & 0 \\ 0 & 1 + |r(\lambda)|^2 \end{array} \right), \quad T(\lambda) = \left( \begin{array}{cc} r^*(\lambda)e^{\frac{2i\lambda x + 2i\lambda^2}{2\lambda}} & 1 \\ 1 + |r(\lambda)|^2 & 0 \end{array} \right).
\]

Both factorizations will be useful later.

REMARK: From now on we will substitute $d\mu = (\rho^0(\eta) + (\rho^0)^*(\eta^*))d\eta$ in (12), where $\rho^0$ is the asymptotic density of eigenvalues given by (9a), and supported on the linear segment $[0, iA]$. In other words, we will approximate a discrete density of
eigenvalues by a continuous one. This is not a trivial assumption, but it is true. We refer to Chapter 3 of [KMM] for a rigorous justification. Of course the justification of [KMM] is devised for real analytic data, but the proof goes through unaltered since the nice behavior of the asymptotic density of eigenvalues at the crucial point 0 is not altered even when we consider steplike data.

We then restate Theorem 1 as follows.

THEOREM 1 (continuous version). Let
\[ d\mu = \rho_0(\eta) + (\rho_0^*(\eta^*)) \, d\eta, \]
where \( \rho_0 \) is the asymptotic density of eigenvalues given by (9a), and supported on the linear segment \([0, iA]\). Set
\[ X(\lambda) = -(A^2 + \lambda^2)^{1/2}. \]

Letting \( M_+ \) and \( M_- \) denote the limits of \( M \) on \( \Sigma = C \cup C^* \) from left and right respectively, we define the Riemann-Hilbert factorization problem
\[ M_+(\lambda) = M_-(\lambda)J(\lambda), \]
where
\[ J(\lambda) = \begin{pmatrix} 1 & r(\lambda)e^{-2i\lambda x - 2i\lambda^2 t} \\ -r^*(\lambda)e^{2i\lambda x + 2i\lambda^2 t} & 1 + |r(\lambda)|^2 \end{pmatrix}, \lambda \in \mathbb{R}, \]
\[ = v(\lambda), \lambda \in C, \]
\[ = \sigma_2 v(\lambda^*)^* \sigma_2, \lambda \in C^*, \]
\[ \lim_{\lambda \to \infty} M(\lambda) = I, \]
where
\[ v(\lambda) = \begin{pmatrix} 1 & -i \exp\left(\frac{1}{2} \int \log(\lambda - \eta) \, d\mu(\eta)\right) \exp\left(-\frac{1}{h}(2i\lambda x + 2i\lambda^2 t - X(\lambda))\right) \\ 0 \end{pmatrix}, \]
\[ r(\lambda) = -A \frac{1}{\cot[(A^2 + \lambda^2)^{1/2}(1/\hbar)][(A^2 + \lambda^2)^{1/2} - i\lambda}. \]

The above Riemann-Hilbert problem admits a solution and the solution of (1) can be asymptotically recovered from the solution of (11) as follows. As \( h \to 0, \)
\[ u(x,t) \sim 2ih \lim_{\lambda \to \infty} (\lambda M^{12}(\lambda)), \]
where the index 12 here denotes the (12)-entry of a matrix.

**PROOF:** See Chapter 3 of [KMM] in the case where the maximizing contour of the variational problem (see Appendix 2) does not touch the spike [0, iA]. For the general case, follow the discussion in Appendix 2.

## 3. ASYMPTOTIC ANALYSIS OF THE RIEMANN-HILBERT PROBLEM.

The ideas underlying the discussion of this section are the following.

(i) Our given Riemann-Hilbert problem needs to be asymptotically deformed to an explicitly solvable one. To do this, some analyticity properties of the jump matrix are crucial. Proofs of the passage from one Riemann-Hilbert problem to another rely heavily on the equivalence of Riemann-Hilbert problems and special singular integral equations (see [DZ1] for the first rigorous description of the deformation method).

(ii) One of the crucial deformations involves a certain conjugation and the introduction of an appropriate "g-function" (first introduced in [DVZ1]; see also [DVZ2]). See (17) below.

(iii) For the focusing nonlinear Schrödinger problem and because the associated Lax operator is *not self–adjoint* the appropriate introduction of the g-function requires finding a "steepest descent contour"; see (18) below. The term "nonlinear steepest descent method" often applied to the Riemann-Hilbert deformation theory initiated in [DZ1] thus acquires full meaning. For this we follow the ideas of [KMM], [KR]; see also Appendix A.2.

We begin with the following observation. Consider the reflection coefficient given by (13). For \( \lambda \) in the upper half-plane, at least away from the real line and the eigenvalues given by (8), we have

\[
\tag{16}
r(\lambda) \sim \frac{-iA}{(A^2 + \lambda^2)^{1/2} + \lambda}.
\]

while for \( \lambda \) in the lower half-plane, at least away from the real line and the eigen-
values given by (8), we have

\[
(16^*) \quad r(\lambda) \sim \frac{iA}{(A^2 + \lambda^2)^{1/2} - \lambda}.
\]

This means that \(r(\lambda)e^{\frac{2i\lambda t + 2i\lambda^2 x}{\hbar}}\) is exponentially decaying (growing) in the upper half-plane, at least away from the real and imaginary axes, as long as \(\text{Re}(\lambda) < -x/2t\) \((\text{Re}(\lambda) > -x/2t)\), while \(r^*(\lambda^*)e^{\frac{2i\lambda t + 2i\lambda^2 x}{\hbar}}\) is exponentially decaying (growing) in the lower half-plane, at least away from the real and imaginary axes, as long as \(\text{Re}(\lambda) < -x/2t\) \((\text{Re}(\lambda) > -x/2t)\).

The above suggests that using the factorizations of the Riemann-Hilbert problem defined by (15-), (15+) and applying the obviously suggested deformations, the jump across the real line should be reduced to a diagonal matrix independent of \(x,t\).

To do this of course, we must ensure that the solution of the Riemann-Hilbert problem is uniformly bounded \((in \ x,t)\) as \(h \to 0\). This is actually not true as will be seen later. What is true however, is that a judicious conjugation of the jump matrix will ”deform” it to a new Riemann-Hilbert problem whose solution is uniformly bounded \((in \ x,t)\) as \(h \to 0\). At that point we can neglect all terms involving \(r\). Furthermore, the very same conjugation will ensure that the new Riemann-Hilbert is explicitly solvable (asymptotically). The deformed Riemann-Hilbert problem will be defined with respect to a steepest descent contour \(C\).

More precisely, we introduce the change of variables

\[
(17) \quad Q(\lambda) = M(\lambda)e^{\frac{\sigma_3}{\hbar}},
\]

where \(\sigma_3 = \text{diag}(1,-1)\) (a Pauli matrix) and the complex-valued function \(g\) is
constrained by the following conditions:

\[ g(\lambda) \text{ is independent of } h. \]

\[ g(\lambda) \text{ is analytic for } \lambda \in \mathbb{C} \setminus (C \cup C^*). \]

\[ g(\lambda) \to 0 \text{ as } \lambda \to \infty. \]

\[ g(\lambda) \text{ assumes continuous boundary values from both sides of } C \cup C^*, \]

denoted by \( g_+ \) (\( g_- \)) on the left (right) of \( C \cup C^* \).

\[ g(\lambda^*) + g(\lambda)^* = 0 \text{ for all } \lambda \in \mathbb{C} \setminus (C \cup C^*). \]

These conditions of course do not define \( g \) uniquely. They will be augmented by two conditions below, that will also implicitly define an admissible steepest descent contour \( C \).

The assumptions above permit us to write \( g \) in terms of a measure \( \rho \) defined on the contour \( C \). Indeed

\[ g(\lambda) = \frac{1}{2} \int_{C \cup C^*} \log(\lambda - \eta) \rho(\eta) d\eta, \]

for an appropriate definition of the logarithm branch (and for \( x > 0 \); if \( x < 0 \) there is a sign change in [KMM] but of course we can restrict ourselves to the case \( x > 0 \) here because of the symmetry of NLS).

For \( \lambda \in C \), define the functions

\[ \theta(\lambda) := i(g_+(\lambda) - g_-(\lambda)), \]

\[ \phi(\lambda) := \int_0^{iA} \log(\lambda - \eta) \rho(\eta) d\eta + \int_{-iA}^0 \log(\lambda - \eta) \rho(\eta)^* d\eta \]

\[ + 2i\lambda x + 2i\lambda^2 t + i\pi \int_{\lambda}^{iA} \rho(\eta) d\eta - g_+(\lambda) - g_-(\lambda). \]

Now we spell out the two important conditions which determine the (steepest descent) contour \( C \) (not uniquely) and the function \( g \).

\[ \rho(\eta) d\eta \text{ is a real measure}, \]

\[ \Re(\phi(\lambda)) \leq 0, \]

i.e. a "measure reality" condition and what can be interpreted (see Chapter 8 of [KMM]) as a "variational inequality" condition. In fact, one can eventually show
that the measure $\rho(\eta)d\eta$ has to be nonpositive: strictly zero in the "gaps" and strictly negative in the "bands"; see section 4 later.

REMARKS. 1. The fact that there exist a contour $C$ and a function $g$ defined implicitly by (17), (17a), (18), (18a), (18b) is highly non-trivial. In [KMM] we had not proved this fact for all times. Instead it was only proved for small (finite) times and it was simply assumed for larger times. A result for larger times appeared only later in [KR] (see Appendix 2 for a discussion). Having the results of [KR] (or more accurately a modification of them as in Appendix 2) we do not need to dwell too much in the details of the construction of $C, g$ any more.

2. The function $g$ and the steepest descent contour $C$ depend only on the density of the eigenvalues, $\rho^0$. They are independent of the reflection coefficient $r$. By (17) the Riemann-Hilbert problem for $Q$ is

$$Q(\lambda) = \begin{pmatrix} 1 & r(\lambda)^* e^{-2i\lambda^2 t} \frac{e^{\lambda x}}{1 + |r(\lambda)|^2} \\ r(\lambda)^* e^{-2i\lambda^2 t} \frac{e^\lambda}{1 + |r(\lambda)|^2} & 0 \end{pmatrix}, \lambda \in \mathbb{R},$$

$$v_Q(\lambda) = \begin{pmatrix} e^{\frac{2\lambda - \psi}{\hbar}} - i \exp\left(\frac{1}{\hbar} \int \log(\lambda - \eta)d\mu(\eta)\right)\exp\left(-\frac{1}{\hbar}(g_+ - g_- + 2i\lambda x + 2i\lambda^2 t - X(\lambda))\right) \\ 0 \end{pmatrix},$$

for $\lambda \in C$,

$$v_Q(\lambda) = \begin{pmatrix} e^{\frac{\lambda - \psi}{\hbar}} \exp\left(\frac{1}{\hbar} \int \log(\lambda - \eta)d\mu(\eta)\right)\exp\left(\frac{1}{\hbar}(g_+ - g_- + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*))\right) \\ e^{-\frac{\psi - \lambda}{\hbar}} \end{pmatrix},$$

for $\lambda \in C^*$. Also $\lim_{\lambda \to \infty} Q(\lambda) = I$.

Now, we will be able to treat the terms involving the reflection coefficient $r$ by arguing as follows. First we note that $g$ is purely imaginary for real $\lambda$; this follows from (18)-(18b). So, write $g = i\psi$, so that $\psi(\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R}$. Let $\zeta(x, t, \lambda) = \psi - \lambda x - \lambda^2 t$. Clearly, $\zeta(\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R}$.

Next, divide the real line into a union of (finitely many) intervals, say $J_k$, such that for $\lambda \in \text{interior}(J_k)$, either $\frac{d\zeta}{d\lambda} > 0$ or $\frac{d\zeta}{d\lambda} < 0$. Denote by $J^+_l$ the intervals in which $\frac{d\zeta}{d\lambda} > 0$ and by $J^-_l$ the intervals in which $\frac{d\zeta}{d\lambda} < 0$. Naturally, the (nonzero) endpoints of the intervals $J_k$ are given by the condition $\frac{d\zeta}{d\lambda} = 0$. Note that $\zeta$ is real analytic in $\mathbb{R} \setminus 0$, and $\zeta \sim -\lambda^2 t$, as $\lambda \to \pm \infty$. The finiteness of the number of
intervals follows from the analyticity of $\zeta$, at least away from $0$. But also near $0$, we will show that $\frac{d\zeta}{d\lambda} < 0$.

We first show that when $\lambda = 0$, then $\frac{d\zeta}{d\lambda} = -\infty$. Recall that

$$g(\lambda) = \frac{1}{2} \int_{C \cup C^*} \log(\lambda - \eta) \rho(\eta) d\eta,$$

where $C \cup C^*$ is oriented counterclockwise and $\rho(\eta) d\eta$ is a non-positive measure on $C$ which is strictly negative in a non-trivial subset of $C$ and is extended to $C^*$ by the condition

$$\rho(\eta^*) = (\rho(\eta))^*.$$

Differentiating, one gets

$$\frac{dg(\lambda)}{d\lambda} = \frac{1}{2} \int_{C \cup C^*} (\lambda - \eta)^{-1} \rho(\eta) d\eta,$$

and at $\lambda = 0$,

$$\frac{dg(\lambda)}{d\lambda} = -\frac{1}{2} \int_{C \cup C^*} \frac{\rho(\eta)}{\eta} d\eta.$$

Making use of the symmetry with respect to complex reflection and remembering that the orientation is counterclockwise for both $C$ and $C^*$ one gets

$$\left[ \frac{dg(\lambda)}{d\lambda} \right](\lambda = 0) = i \int_{C} \frac{\rho(\eta) \text{Im}(\eta)}{|\eta|^2} d\eta.$$

The integrand is strictly negative and the integral diverges because $\rho$ is nonzero at zero. It follows that at $\lambda = 0$, $\frac{d\zeta}{d\lambda} = -i\infty$, so $\frac{d\zeta}{d\lambda} = -\infty$.

Now, if $\lambda$ is real and close but not equal to $0$,

$$\left[ i \frac{dg(\lambda)}{d\lambda} \right] \sim i \int_{C} \frac{\rho(\eta) \text{Im}(\eta)}{|\lambda - \eta|^2} d\eta,$$

so $\frac{d\zeta}{d\lambda} < 0$. Hence the point $\lambda = 0$ belongs to the interior of some interval $J^-_i$.

Consider first the intervals $J^+_i$. By the Cauchy-Riemann relations we have $d(\text{Im}\zeta)/d(\text{Im}\lambda) > 0$, for $\lambda \in J^+_i$, in the positive direction perpendicular to the real line. This means that in an area of the upper half-plane, close to $J^+_i$, the real part of $i\zeta$ is negative, so $\exp(i\zeta/h)$ is exponentially decaying. Similarly, in an area of the upper half-plane, close to $J^-_i$, the real part of $i\zeta$ is positive, so $\exp(-i\zeta/h)$
is exponentially decaying. In the particular case of $\lambda = 0$ it is also easy to check that $\exp(-i\zeta/h)$ is exponentially decaying as $\lambda$ moves upwards along the positive imaginary axis.

Let us now introduce the following lens-like contours. For each interval $J_i^+$ consider small piecewise linear deformations upwards, say $J_i^{+,up}$, keeping the end points fixed, but otherwise lying entirely in the upper half-plane. Similarly, consider small piecewise linear deformations downwards, say $J_i^{-,dn}$, keeping the end points fixed, but otherwise lying entirely in the lower half-plane. For each interval $J_i^-$ consider small piecewise linear deformations upwards, say $J_i^{-,up}$, keeping the end points fixed, but otherwise lying entirely in the upper half-plane. Similarly, consider small piecewise linear deformations downwards, say $J_i^{-,dn}$, keeping the end points fixed, but otherwise lying entirely in the lower half-plane. All orientations are compatible with $\Re(\lambda)$ increasing. We also make sure that $J_i^{+,up}, J_i^{+,dn}, J_i^{-,up}, J_i^{-,dn}$ cut the real line at angles $\neq 0, \pi/2$. See Figure 1.

Note here that one of the lens-like contours $J_i^{-,up}$ will cut the contour $C$ while one of the lens-like contours $J_i^{-,dn}$ will cut the contour $C^*$.

Let the open regions bounded by $J_i^+$ and $J_i^{+,up}$ be denoted by $D_i^{+,up}$ and the open regions bounded by $J_i^+$ and $J_i^{+,dn}$ be denoted by $D_i^{+,dn}$. Similarly, let the open regions bounded by $J_i^-$ and $J_i^{-,up}$ be denoted by $D_i^{-,up}$ and the open regions bounded by $J_i^-$ and $J_i^{-,dn}$ be denoted by $D_i^{-,dn}$. We make sure that $\Re(i\zeta) < 0$, in $D_i^{+,up} \cup J_i^{+,up} \cup D_i^{-,dn} \cup J_i^{-,dn}$. Similarly, we make sure that $\Re(i\zeta) > 0$, in
\[ D^{-,up}_l \cup J^{-,up}_l \cup D^{+,dn}_l \cup J^{+,dn}_l. \]

Next, we can make use of the factorizations given in (15+), (15-). Set
\[
Z(\lambda) = Q(\lambda), \text{ for } \lambda \in \mathbb{C} \setminus \bigcup_l \tilde{D}^{+,up}_l \cup \tilde{D}^{+,dn}_l \cup \tilde{D}^{-,up}_l \cup \tilde{D}^{-,dn}_l,
\]
\[
Z(\lambda) = Q(\lambda)U^{-1}(\lambda), \text{ for } \lambda \in \cup_l D^{+,up}_l,
\]
\[
Z(\lambda) = Q(\lambda)U(\lambda), \text{ for } \lambda \in \cup_l D^{+,dn}_l,
\]
\[
Z(\lambda) = Q(\lambda)T^{-1}(\lambda), \text{ for } \lambda \in \cup_l D^{-,up}_l,
\]
\[
Z(\lambda) = Q(\lambda)S(\lambda), \text{ for } \lambda \in \cup_l D^{-,dn}_l.
\]

The Riemann-Hilbert problem for \( Z \) is described by
\[
v_Z(\lambda) = \begin{pmatrix} 1 & -r(\lambda) e^{\frac{2\pi i}{n} [2g_+ - 2i\lambda x - 2i\lambda^2 t]} \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in \cup_l J^{+,up}_l,
\]
\[
v_Z(\lambda) = \begin{pmatrix} 1 & 0 \\ -r^*(\lambda^*) e^{-\frac{2\pi i}{n} [2g_+ - 2i\lambda x + 2i\lambda^2 t]} & 1 \end{pmatrix}, \text{ for } \lambda \in \cup_l J^{+,dn}_l,
\]
\[
v_Z(\lambda) = \begin{pmatrix} 1 & 0 \\ r(\lambda^*) e^{-\frac{2\pi i}{n} [2g_+ - 2i\lambda x + 2i\lambda^2 t]} & 1 \end{pmatrix}, \text{ for } \lambda \in \cup_l J^{-,up}_l,
\]
\[
v_Z(\lambda) = \begin{pmatrix} 1 & 0 \\ 1 e^{\frac{2\pi i}{n} [2g_+ - 2i\lambda x - 2i\lambda^2 t]} & 1 \end{pmatrix}, \text{ for } \lambda \in \cup_l J^{-,dn}_l,
\]
\[
v_Z(\lambda) = \begin{pmatrix} e^{\frac{2\pi i}{n} [g_+ - g_-]} & -i \exp\left(\frac{i}{n} \int \log(\lambda - \eta) d\mu(\eta) \right) \exp\left(-\frac{i}{n} \left( -g_+ - g_- + 2i\lambda x + 2i\lambda^2 t - X(\lambda) \right) \right) \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in \mathbb{C},
\]
\[
v_Z(\lambda) = \begin{pmatrix} e^{\frac{2\pi i}{n} [g_+ - g_-]} & i \exp\left(\frac{i}{n} \int \log(\lambda - \eta) d\mu(\eta) \right) \exp\left(\frac{i}{n} \left( -g_+ - g_- + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*) \right) \right) e^{\frac{2\pi i}{n} [g_+ - g_-]} \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in \mathbb{C}^*,
\]
\[
v_Z(\lambda) = \begin{pmatrix} (1 + |r(\lambda)|^2)^{-1} & 0 \\ 0 & 1 + |r(\lambda)|^2 \end{pmatrix}, \text{ for } \lambda \in \cup_l J^{-},
\]

with the normalization \( \lim_{\lambda \to \infty} Z(\lambda) = I \). For a part of the extended contour, away from 0, and between the points \( a_i \) and \( a_{i+1} \), including the small lenses, see Figure 1.

The idea for the next transformation is to get rid of the jump across the real line.

We define \( \tilde{d} \) as follows. Let \( \tilde{d} \) be analytic in \( \mathbb{C} \setminus \cup J^{-} \), such that the normal limits from above and below the real line \( d_{\pm} \) exist (apart from the endpoints) and such
that
\[ \tilde{d}_+(\lambda) = \tilde{d}_-(\lambda)(1 + |r(\lambda)|^2), \; \lambda \in \bigcup_{int}J_i^- , \]
\[ \lim_{\lambda \to \infty} \tilde{d}(\lambda) = 1. \]

One can even give an explicit formula for \( \tilde{d} \). Denote
\[ J_i^- = [a_i, b_i]. \]
Then
\[ (21) \quad \tilde{d}(\lambda) = \exp \left[ \sum_l \int_{a_i}^{b_i} \log(1 + |r(\zeta)|^2) \frac{d\zeta}{2 \pi i (\zeta - \lambda)} \right], \; \lambda \in \mathbb{C} \cup \bigcup J_i^- \]
where the principal branch of the logarithm is chosen.

It is clear that the definitions above are not singular and indeed \( \tilde{d}, \tilde{d}^{-1} \) are bounded in the complex Riemann sphere. In fact, it is easy to check that both \( \tilde{d}(\lambda) \) and \( [\tilde{d}(\lambda^*)]^{-1} \) satisfy the conditions of the scalar Riemann-Hilbert problem. Hence they must be equal, from which it easily follows that
\[ |\tilde{d}_-|^2(1 + |r(\lambda)|^2) = 1, \; \text{for} \; \lambda \in \bigcup J_i^- , \]
\[ |\tilde{d}_-| \leq 1, \; \text{for} \; \lambda \in \bigcup J_i^- , \]
\[ |\tilde{d}_+| \leq 1 + \sup |r(\lambda)|^2 < \infty, \; \text{for} \; \lambda \in \bigcup J_i^- , \]
and hence, by the maximum principle, \( |\tilde{d}(\lambda)| \leq 1 + \sup |r(\lambda)|^2 \) \( < \infty \). So \( |\tilde{d}(\lambda)| \) is bounded uniformly in the complex plane. Similarly, \( |\tilde{d}^{-1}(\lambda)| \) is uniformly bounded in the complex plane.

Near the points \( a_i \) the local behavior of \( \tilde{d} \) off the jump contour is
\[ (21b) \quad \tilde{d} \sim d_i (\lambda - a_i)^{i \nu_i}, \]
and near \( b_i , \)
\[ (21c) \quad \tilde{d} \sim f_i (\lambda - b_i)^{i \mu_i}, \]
where \( \nu_i = \frac{1}{2 \pi}(1 + \log |r(a_i)|^2) \), \( \mu_i = \frac{1}{2 \pi}(1 + \log |r(b_i)|^2) \) and \( d_i, f_i \) are independent of \( \lambda \). In fact
\[ (21d) \]
\[ d_j = \exp \left( \int_{-\infty}^{a_j} \log(z - a_j) \frac{d\log(1 + |r(z)|^2)}{2 \pi i} + \sum_{l \neq j} \int_{a_l}^{b_i} \log(1 + |r(\zeta)|^2) \frac{d\zeta}{2 \pi i (\zeta - a_l)} \right), \]
\[ f_j = \exp \left( - \int_{-\infty}^{b_j} \log(z - b_j) \frac{d\log(1 + |r(z)|^2)}{2 \pi i} + \sum_{l \neq j} \int_{a_l}^{b_i} \log(1 + |r(\zeta)|^2) \frac{d\zeta}{2 \pi i (\zeta - b_l)} \right), \]
as can be shown by integration by parts. The next transformation is then
\[
Y(\lambda) = Z(\lambda) \Delta(\lambda),
\]
(22)
where
\[
\Delta(\lambda) = \begin{pmatrix}
\tilde{d}(\lambda) & 0 \\
0 & \tilde{d}^{-1}(\lambda)
\end{pmatrix},
\]
Then \( Y_+ = Y_- v_Y \), where
(23)
\[
v_Y(\lambda) = \begin{pmatrix}
e^{\frac{\lambda - \eta}{h}} & -i \tilde{d}^{-2} \exp\left[\frac{1}{h} \left( \int \log(\lambda - \eta) d\mu(\eta) + g_+ + g_+ - 2i\lambda x - 2i\lambda^2 t + X(\lambda) \right) \right] \\
0 & e^{\frac{\eta - g}{h}}
\end{pmatrix},
\]
for \( \lambda \in C \),
\[
v_Y(\lambda) = \begin{pmatrix}
e^{\frac{\lambda - \eta}{h}} & 0 \\
0 & -i \tilde{d}^{-2} \exp\left( -\int \log(\lambda - \eta) d\mu(\eta) + g_+ - g_+ + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*) \right)
\end{pmatrix},
\]
for \( \lambda \in C^* \),
\[
v_Y(\lambda) = \begin{pmatrix}
1 & -r(\lambda) \tilde{d}^{-2} e^{\frac{2\lambda^2 - 2i\lambda x - 2i\lambda^2 t}{h}} \\
0 & 1
\end{pmatrix},
\]
for \( \lambda \in \cup J_k^{+,up} \),
\[
v_Y(\lambda) = \begin{pmatrix}
r^*(\lambda)^\ast \tilde{d}^{-2} e^{\frac{2\lambda^2 - 2i\lambda x - 2i\lambda^2 t}{h}} & 0 \\
0 & 1
\end{pmatrix},
\]
for \( \lambda \in \cup J_k^{+,dn} \),
\[
v_Y(\lambda) = \begin{pmatrix}
\tilde{d}^{-2} \frac{r(\lambda)}{1 + |r(\lambda)|^2} & 0 \\
0 & 1
\end{pmatrix},
\]
for \( \lambda \in \cup J_k^{-,up} \),
\[
v_Y(\lambda) = \begin{pmatrix}
1 & \tilde{d}^{-2} \frac{r^*(\lambda)}{1 + |r(\lambda)|^2} e^{\frac{2\lambda^2 - 2i\lambda x - 2i\lambda^2 t}{h}} \\
0 & 1
\end{pmatrix},
\]
for \( \lambda \in \cup J_k^{-,dn} \).

Also \( \lim_{\lambda \to \infty} Y(\lambda) = I \).

At this point, we still have a Riemann-Hilbert that is equivalent to the original one (11)-(12), at least accepting the discrete-to-continuous passage in Theorem 1. Now, we can finally start considering the limit \( h \to 0 \). Indeed, all terms involving \( r(\lambda) \) and not supported on the real line can be neglected, not because \( r \) itself is small (it is not, see (16)-(16*)) but only because it always appears multiplied by something exponentially small.

In general jump matrices of the form \( I + \text{exponentially small} \) can be neglected asymptotically as long as it is proved that the solution is uniformly (in \( x, t \)) bounded as \( h \to 0 \). We shall see eventually that this is the case.

Assuming for the moment that this is true, we can simply delete the non-real part of the contour, at least away from the endpoints of the intervals \( J_k \). Eventually (see section 5 later) we can also delete the remaining small crosses centered at such
We end up with a matrix valued function $W$, such that $W \sim Y$ near infinity, and

$$W_+(\lambda) = W_-(\lambda) v_W(\lambda),$$

where

$$v_W(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{\hbar}} & -i \frac{d^2}{d\lambda^2} \exp \left( \frac{1}{\hbar} \int \log(\lambda - \eta) d\mu(\eta) \right) \exp \left( \frac{1}{\hbar} \left( -g_+ - g_- + 2i\lambda x + 2i\lambda^2 t - X(\lambda) \right) \right) \\ 0 & e^{\frac{g_+ - g_-}{\hbar}} \end{pmatrix},$$

for $\lambda \in C$, and

$$v_W(\lambda) = \begin{pmatrix} e^{\frac{g_+ - g_-}{\hbar}} & \frac{d}{d\lambda} \exp \left( \frac{1}{\hbar} \int \log(\lambda - \eta) d\mu(\eta) \right) \exp \left( \frac{1}{\hbar} \left( -g_+ - g_- + 2i\lambda x + 2i\lambda^2 t + X^*(\lambda^*) \right) \right) \\ 0 & e^{\frac{g_+ - g_-}{\hbar}} \end{pmatrix},$$

for $\lambda \in C^*$. Also $\lim_{\lambda \to \infty} W(\lambda) = I$.

We will eventually check that the solution of the Riemann-Hilbert problem for $W$ exists and is uniformly (in $x, t$) bounded as $\hbar \to 0$. This will justify neglecting the exponentially small terms in (23). The passage from (23) to (25) will then be a posteriori justified, again at least except for the remaining small crosses centered at a finite number of real points. Concerning the crosses see section 5 (Remark 5) and Appendix A.1.

After applying the above transformation $Y \to W$, we must use another lens transformation to simplify the jump across the contour $C \cup C^*$. We will not describe this new lens transformation, since the discussion is exactly as in Chapter 4 of [KMM]. We will simply state the end result of these lens transformations and the transformation $Y \to W$. The result is the so-called outer problem. We shall show in the next section how it can be treated along the lines of Chapter 4 of [KMM].

THE OUTER PROBLEM.

We first define the analytic arcs $I_j, I_j^*, j = 1, ..., G/2$ as follows (they come in conjugate pairs). Let the points $\lambda_j, j = 0, ..., G$, in the open upper half-plane be the branch points of the function $g$. (The fact that there are $G + 1$ of them is a consequence of the definition of $g$, according to the "finite genus ansatz"; see
below.) All such points lie on the contour $C$ and we order them as $\lambda_0, \lambda_1, ..., \lambda_G$, according to the direction given to $C$. The points $\lambda_0^*, \lambda_1^*, ..., \lambda_G^*$ are their complex conjugates. Then let $I_0 = [0, \lambda_0]$ be the subarc of $C$ joining points $0$ and $\lambda_0$. Similarly, $I_j = [\lambda_{2j-1}, \lambda_{2j}]$, $j = 1, ..., G/2$. The points $\lambda_j$, $j = 0, ..., G$ lie in the open upper half-plane and they are determined by a set of transcendental equations that follow directly from the definitions of $g$ and $C$ (cf. Remark after Lemma 5.1.5 of [KMM]). The connected components of the set $\mathbb{C} \setminus \bigcup_j (I_j \cup I_j^*)$ are the so-called "gaps", for example the gap $\Gamma_1$ joins $\lambda_0$ to $\lambda_1$, etc. The subarcs $I_j$ are the "bands".

ASSUMPTION (G). For simplicity, we will make the generic (in $x, t$) assumption that $\lambda_j \neq iA$, $j = 0, ..., G$.

The finite genus ansatz implies that for each $x, t$ there is a finite positive integer $G$ such that the contour $C$ can be divided into bands and gaps as above. In fact, it follows from the conditions defining $\rho, C$ that the measure reality condition $(\rho(\eta)d\eta \text{ real};$ see (18b)) splits into a measure strict negativity condition in the bands and a measure zero condition in the gaps. Furthermore, the function $\theta(\lambda)$ of (18a) defined on $C$ is constant on each of the gaps $\Gamma_j$, taking a value which we will denote by $\theta_j$, while the function $\phi$ of (18a) is constant on each of the bands, taking the value denoted by $\alpha_j$ on the band $I_j$. For the justification of the finite ansatz under the barrier data see Appendix 2.

We are seeking a matrix $O$, which is analytic everywhere except across the contour $C \setminus \Gamma_{G/2+1}$ and its conjugate, with limits that are $L^2(C \setminus \Gamma_{G/2+1})$, converging to the identity at infinity and such that

\begin{equation}
O_+(\lambda) = O_-(\lambda) \begin{pmatrix} 0 & i\tilde{d}^{-2} \exp(-i\alpha_k/h) \\ id^2 \exp(i\alpha_k/h) & 0 \end{pmatrix}, \quad \lambda \in I_k \cup I_k^*, \quad k = 0, 1, ..., G/2,
\end{equation}

\begin{equation}
O_+(\lambda) = O_-(\lambda) \begin{pmatrix} \exp(i\theta_k/h) & 0 \\ 0 & \exp(-i\theta_k/h) \end{pmatrix}, \quad \lambda \in \Gamma_k \cup \Gamma_k^*, \quad k = 1, ..., G/2.
\end{equation}

We here recapitulate the sequence of matrix deformations introduced so far:

\[ M(\text{discrete}) \rightarrow M(\text{continuous}) \rightarrow Q \rightarrow Z \rightarrow Y \rightarrow W \rightarrow O. \]
The first problem in the sequence, for $M$ (in its discrete version), is equivalent to the inverse scattering problem for NLS. The last Riemann-Hilbert problem, for $O$, (26), will be solved explicitly via theta functions (see (33) of section 5).

4. THE RESULT

As explained in Appendix 2, the finite genus ansatz holds for the semiclassical asymptotics under barrier initial data, at least generically.

Assuming the finite genus ansatz, the $x,t$-plane can be divided into (possibly empty) open regions $R_G, G = 0, 2, 4, ....$ (here $G/2 + 1$ is the number of components of the support of the equilibrium measure of $C$, see Appendix A.2), together with their boundaries, such that within each region the asymptotics of the solution of (1) with barrier data as in (2) can be given as follows.

**THEOREM 2.** Let $x_0, t_0$ lie in region $R_G$. The solution $u^h(x, t)$ is asymptotically described (locally) as a slowly modulated $G+1$ phase wavetrain. Setting $x = x_0 + h \hat{x}$ and $t = t_0 + h \hat{t}$, so that $x_0, t_0$ are "slow" variables while $\hat{x}, \hat{t}$ are "fast" variables, there exist parameters

$$a, U = (U_0, U_1, ...., U_G)^T, k = (k_0, k_1, ..., k_G)^T, w = (w_0, w_1, ...., w_G)^T, Y = (Y_0, Y_1, ......., Y_G)^T, Z = (Z_0, Z_1, ......., Z_G)^T$$

depending on the slow variables $x_0$ and $t_0$ and possibly $h$ (but not $\hat{x}, \hat{t}$) such that

$$u^h(x, t) = u^h(x_0 + h \hat{x}, t_0 + h \hat{t}) \sim a(x_0, t_0)e^{iU_0(x_0, t_0)/h}e^{i(k_0(x_0, t_0) \hat{x} - w_0(x_0, t_0) \hat{t})}$$

$$\cdot d^2(\lambda_0) \frac{\Theta(Y(x_0, t_0) + iU(x_0, t_0)/h + i(k(x_0, t_0) \hat{x} - w(x_0, t_0) \hat{t}))}{\Theta(Z(x_0, t_0) + iU(x_0, t_0)/h + i(k(x_0, t_0) \hat{x} - w(x_0, t_0) \hat{t}))}.$$  

Unlike the analogous formula in [KMM], we allow here a dependence of the parameters on $h$. But of course, we can always rearrange terms to arrive at a formula like (27) where the parameters are independent of $h$.

All parameters can be defined in terms of an underlying Riemann surface $X$. The moduli of $X$ are given by $\lambda_j, j = 0, ..., G$ and their complex conjugates $\lambda_j^*, j = 0, ..., G$. The genus of $X$ is $G$. The moduli of $X$ vary slowly with $x, t$, i.e. they depend on $x_0, t_0$ but not $\hat{x}, \hat{t}$. For the exact formulae for the parameters as well as the definition of the theta functions we present the following construction.
The Riemann surface $X$ is constructed by cutting two copies of the complex sphere along the slits $I_0 \cup I_0^*, I_j, I_j^*, j = 1, \ldots, G$, and pasting the "top" copy to the "bottom" copy along these very slits.

In exact analogy with the discussion of [KMM] (as exemplified in Figure 4.5 on p.62 of [KMM]) we define the homology cycles $a_j, b_j$, $j = 1, \ldots, G$ as follows. Cycle $a_1$ goes around the slit $I_0 \cup I_0^*$ joining $\lambda_0$ to $\lambda_0^*$, remaining on the top sheet, oriented counterclockwise, $a_2$ goes through the slits $I_{-1}$ and $I_1$ starting from the top sheet, also oriented counterclockwise, $a_3$ goes around the slits $I_{-1}, I_0 \cup I_0^*, I_1$ remaining on the top sheet, oriented counterclockwise, etc. Cycle $b_1$ goes through $I_0$ and $I_1$ oriented counterclockwise, cycle $b_2$ goes through $I_{-1}$ and $I_1$, also oriented counterclockwise, cycle $b_3$ goes through $I_{-1}$ and $I_2$, and around the slits $I_{-1}, I_0 \cup I_0^*, I_1$, oriented counterclockwise, etc.

Let

$$R(\lambda)^2 = \prod_{k=0}^{G} (\lambda - \lambda_k)(\lambda - \lambda_k^*),$$

choosing the particular branch that is cut along the bands $I_k$ and $I_k^*$ and such that

$$\lim_{\lambda \to \infty} \frac{R(\lambda)}{\lambda^{G+1}} = -1.$$

On $X$ there is a complex $G$-dimensional linear space of holomorphic differentials, with basis elements $\nu_k(P)$ for $k = 1, \ldots, G$ that can be written in the form

$$\nu_k(P) = \sum_{j=0}^{G-1} \frac{c_{kj} \lambda(P)^j}{R_X(P)} \, d\lambda(P),$$

where $R_X(P)$ is a "lifting" of the function $R(\lambda)$ from the cut plane to $X$: if $P$ is on the first sheet of $X$ then $R_X(P) = R(\lambda(P))$ and if $P$ is on the second sheet of $X$ then $R_X(P) = -R(\lambda(P))$. The coefficients $c_{kj}$ are uniquely determined by the constraint that the differentials satisfy the normalization conditions:

$$\oint_{a_j} \nu_k(P) = 2\pi i \delta_{jk}.$$
From the normalized differentials, one defines a $G \times G$ matrix $H$ (the period matrix) by the formula:

$$H_{jk} = \oint_{b_j} \nu_k(P).$$

It is a consequence of the standard theory of Riemann surfaces that $H$ is a symmetric matrix whose real part is negative definite.

In particular, we can define the theta function

$$\Theta(w) := \sum_{n \in \mathbb{Z}^G} \exp\left(\frac{1}{2} n^T H n + n^T w\right),$$

where $H$ is the period matrix associated to $X$. Since the real part of $H$ is negative definite, the series converges.

We arbitrarily fix a base point $P_0$ on $X$. The Abel map $A : X \to Jac(X)$ is then defined componentwise as follows:

$$A_k(P; P_0) := \int_{P_0}^{P} \nu_k(P'), \quad k = 1, \ldots, G,$$

where $P'$ is an integration variable.

A particularly important element of the Jacobian is the Riemann constant vector $K$ which is defined, modulo the lattice $\Lambda$, componentwise by

$$K_k := \pi i + \frac{H_{kk}}{2} - \frac{1}{2\pi i} \sum_{j=1 \atop j \neq k}^{G} \oint_{a_j} \nu_j(P) \int_{P_0}^{P} \nu_k(P')
\left(\frac{d\lambda(P)}{\lambda(P)^2}\right),$$

where the index $k$ varies between 1 and $G$.

Next, we will need to define a certain meromorphic differential on $X$. Let $\Omega(P)$ be holomorphic away from the points $\infty_1$ and $\infty_2$, where it has the behavior

$$\Omega(P) = d\lambda(P) + \left(\frac{d\lambda(P)}{\lambda(P)^2}\right), \quad P \to \infty_1,$$

$$\Omega(P) = -d\lambda(P) + O\left(\frac{d\lambda(P)}{\lambda(P)^2}\right), \quad P \to \infty_2,$$

and made unique by the normalization conditions

$$\oint_{a_j} \Omega(P) = 0, \quad j = 1, \ldots, G.$$
Here \( p \) is some polynomial. In the present context see section 5, equation (31) for its definition.

Let the vector \( U \in \mathbb{C}^G \) be defined componentwise by

\[
U_j := \oint_{b_j} \Omega(P).
\]

Note that \( \Omega(P) \) has no residues.

Let the vectors \( V_1, V_2 \) be defined componentwise by

\[
V_{1,k} = (A_k(\lambda_1^+) + A_k(\lambda_2^+) + A_k(\lambda_3^+) + \cdots + A_k(\lambda_G^+)) + A_k(\infty) + \pi i + \frac{H_{kk}}{2},
\]

\[
V_{2,k} = (A_k(\lambda_1^+) + A_k(\lambda_2^+) + A_k(\lambda_3^+) + \cdots + A_k(\lambda_G^+)) - A_k(\infty) + \pi i + \frac{H_{kk}}{2},
\]

where \( k = 1, \ldots, G \), and the + index means that the integral for \( A \) is to be taken on the first sheet of \( X \), with base point \( \lambda_0 \).

Finally, let

\[
a = \frac{\Theta(Z)}{\Theta(Y)} \sum_{k=0}^{G} (-1)^k \Im(\lambda_k),
\]

\[
k_n = \partial_x U_n, \ w_n = -\partial_t U_n, \ n = 0, \ldots, G,
\]

where

\[
Y = -A(\infty) - V_1, \ Z = A(\infty) - V_1,
\]

and \( U_0 = -(\theta_1 + \alpha_0) \) where \( \theta_1 \) is the (constant in \( \lambda \)) value of the function \( \theta \) in the gap \( \Gamma_1 \) and \( \alpha_0 \) is the (constant) value of the function \( \phi \) in the band \( I_0 \). The fact that these values are actually constants in \( \lambda \) follows from the conditions defining \( g \) and \( C \).

Now, the parameters appearing in formula (27) are completely described.

We simply note here that the \( U_i \) and hence the \( k_i \) and \( w_i \) are real modulo \( O(h) \).

We also note that the denominator in (27) never vanishes (for any \( x_0, t_0, \dot{x}, \dot{t} \)).

**REMARK:** Because \( C \) depends only on the eigenvalue density \( \rho^0 \) and not on the reflection coefficient \( r \), the constructions of \( C \), the Riemann surface \( X \), the holomorphic differential \( \nu_k(P) \), the Abel map \( A \) and the theta functions are all independent of \( r \). The only contribution of \( r \) comes through the factors \( \tilde{a}^2, \tilde{a}^{-2} \).
5. REMARKS AND PROOF OF THEOREM 2.

1. Formula (27) is locally a so-called finite gap expression. It describes violent oscillations of bounded amplitude but high frequency $O(1/h)$.

2. As in [KMM] weak limits of densities exist: $\rho = \lim_{h \to 0} |u_h|^2$ and $\mu = \lim_{h \to 0} -i\hbar (\bar{u}_h u_h - u_h \bar{u}_h)$. These limits are actually strong in the genus zero region.

3. Naturally, since the initial data is discontinuous and the limiting Euler system (1a) is elliptic (see [KMM]) one expects the break-time of the limiting system to be zero. This is indeed the case, as numerical experiments by H.Ceniceros and F.Tian have shown [T], or as can be shown analytically by considering the limiting Euler system directly: from the second equation of (1a) it is obvious that $\mu_t$ is infinite at $x = \pm 1/2, t = 0$. Of course a genus zero region still exists but the first caustic (the boundary between the genus zero region and higher genus regions) touches the $t = 0$ axis of the $x,t$-plane at $x = \pm 1/2$.

4. The proof of the results in [KMM] makes use of the assumption that the eigenvalue density can be analytically extended in the upper half-plane with the spike where eigenvalues accumulate deleted. This is the case here, see formula (9a). The branch root singularity at $iA$ is integrable. Integrals like $\int \rho^0 d\eta$ or $\int \log(\lambda - \eta) \rho^0(\eta) d\eta$ can still be deformed and the Cauchy theorem holds. On the other hand, we assume that $x,t$ are such that $\lambda_j \neq iA$, $j = 0, 1, ..., g$, a generic case.

5. A priori, neglecting the exponentially terms in (23) only allows us to “delete” arcs $J_{i,+}^{up}, J_{i,+}^{dn}, J_{i,-}^{up}, J_{i,-}^{dn}$ only away from the real endpoints of $J_i^+, J_i^-$. For entirely rigorous justifications of the deletion of the small crosses centered at each such point remaining after the deletion of the bulk of the arcs $J_{i,+}^{up}, J_{i,+}^{dn}, J_{i,-}^{up}, J_{i,-}^{dn}$, one must construct local parametrices of the Riemann-Hilbert problem and make sure they match with the solution of (25) away from the endpoints. This is a procedure that is by now standard in the literature; more details are given in the
Appendix. In fact, essentially the same situation has appeared in [DZ1]. The "local" Riemann-Hilbert problem can be solved via parabolic cylinder functions. (In [DZ1], of course, the contribution was not negligible, since one was trying to evaluate the term of order $O(t^{-1/2})$ of the asymptotics. Also in [DZ1] the reflection coefficient is independent of the parameter $1/t$ going to zero, but this is irrelevant since the local Riemann-Hilbert problem is solved exactly via parabolic cylinder functions.)

6. As in [KMM], one needs to provide a local parametrix near the origin and then match it with the solution of the "outer problem". This can still be done, using the Fredholm theory described in [KMM]. The "cyclic" relation, that the product of limits of jumps at 0 is the identity, still holds. In view of (9b) the cyclic relation follows easily from the analogous relation in the reflectionless case. The discussion in sections (4.4.3), (4.5.1) and (4.5.2) of [KMM] can be then followed verbatim.

7. The Bohr-Sommerfeld condition (8) is not quite the same as the condition postulated in [KMM] defining the so-called soliton ensemble, which translates as

$$ (A^2 + \lambda_k^2)^{1/2} = \hbar k \pi $$

for our present problem (1)-(2). Now, one can notice that the difference between the two conditions gives rise to a uniform error of higher order $O(\hbar^2)$. The analysis in [KMM] (Chapter 3) of estimates needed for the passage from a "discrete" Riemann-Hilbert problem to a "continuum" Riemann-Hilbert problem is not altered by this innocent modification.

8. From formulae (6)-(7) it is obvious that $\alpha(0) = 0$ if $Ah$ is an odd multiple of $\pi/2$. So, for a particular sequence of $h$ going to zero we have a spectral singularity at the real point 0. The Riemann-Hilbert problem jump becomes singular at 0. By assumption we have excluded such values of $h$. We plan to study the effect of a real spectral singularity on the semiclassical behavior of the focusing NLS in a later publication.

9. The proof of Theorem 2 now follows the discussion of Chapter 4 of [KMM]. One minor change is the extra factor $\tilde{d}^{1/2}$ appearing in the off-diagonal terms of (26).
This factor can be taken care of by the auxiliary scalar Riemann-Hilbert problem (4.38) in section 4.3.1 of [KMM]. The proof goes through with only a minor change: $\alpha_k$ has to be substituted by $\alpha_k + 2i h \log \tilde{d}$. Of course $\tilde{d}$ has to appear in the final formula (27). Granted that the term $h \tilde{d}$ is no more constant on bands, but then the auxiliary scalar Riemann-Hilbert problem is still explicitly solvable.

More precisely, we introduce the scalar problem:

\begin{align}
H_+ (\lambda) - H_- (\lambda) &= -\theta_k, \quad \lambda \in \Gamma_k \cup \Gamma_k^*, \quad k = 1, \ldots, G/2, \\
H_+ (\lambda) + H_- (\lambda) &= -(\alpha_k + 2i h \log \tilde{d}), \quad \lambda \in I_k \cup I_k^*, \quad k = 0, \ldots, G/2,
\end{align}

not specifying any condition at infinity yet. Consider the matrix defined by

$$P(\lambda) = O(\lambda) \exp(i H(\lambda) \sigma_3/h),$$

where $O$ is the solution of the outer Riemann-Hilbert problem (26). It is straightforward to verify that the matrix $P(\lambda)$ has the identity matrix as the jump matrix in all gaps $\Gamma_k$ and $\Gamma_k^*$. Since the boundary values of $O(\lambda)$ and $H(\lambda)$ are continuous, it follows that $P(\lambda)$ is in fact analytic in the gaps. In the bands, the jump relation becomes simply

$$P_+ (\lambda) = iP_- (\lambda) \sigma_1,$$

so the jump relation is the same in all bands. Next, suppose that $\beta(\lambda)$ is a scalar function analytic in the $\lambda$-plane except at the bands, where it satisfies $\beta_+ (\lambda) = -i \beta_- (\lambda)$. Suppose further for the sake of concreteness that $\beta(\lambda) \to 1$ as $\lambda \to \infty$. Then, setting

$$V(\lambda) = \beta(\lambda) P(\lambda),$$

we see that the jump relations for $V(\lambda)$ take on the elementary form:

$$V_+ (\lambda) = V_- (\lambda) \sigma_1, \quad \lambda \in \cup_k (I_k \cup I_k^*).$$

Our purpose in reducing the jump relations to this universal constant form is that it can be explicitly solved in terms of theta functions.
But let us describe the scalar functions \( H(\lambda) \) and \( \beta(\lambda) \). We get

\[
\beta(\lambda) = \frac{\lambda - \lambda_0^*}{\lambda - \lambda_0} \prod_{k=1}^{G/2} \frac{\lambda - \lambda_{2k-1}^*}{\lambda - \lambda_{2k-1}} \cdot \frac{\lambda - \lambda_{2k}^*}{\lambda - \lambda_{2k}},
\]

and for \( \beta(\lambda) \) we select the branch that tends to unity for large \( \lambda \) and that is cut along the bands \( I_k \) and \( I_k^* \). It is easily checked that \( \beta(\lambda) \) as defined here is the only function satisfying the required jump condition and normalization at infinity that has continuous boundary values (except at half of the endpoints). To find \( H(\lambda) \), we introduce the function \( R(\lambda) \) defined by

\[
R(\lambda) = \frac{G}{2} \prod_{k=0}^{G} (\lambda - \lambda_k)(\lambda - \lambda_k^*),
\]

choosing the particular branch that is cut along the bands \( I_k \) and \( I_k^* \) and satisfies

\[
\lim_{\lambda \to \infty} R(\lambda) = -1,
\]

This defines a real function, i.e. one that satisfies \( R(\lambda^*) = R(\lambda)^* \). At the bands, we have \( R_+(\lambda) = -R_-(\lambda) \), while \( R(\lambda) \) is analytic in the gaps. Set

\[
H(\lambda) = k(\lambda)R(\lambda),
\]

where

\[
k(\lambda) = \frac{1}{2\pi i} \sum_{n=1}^{G/2} \theta_n \int_{\Gamma_n \cup \Gamma_n^*} \frac{d\eta}{(\lambda - \eta)R(\eta)} + \frac{1}{2\pi i} \sum_{n=0}^{G/2} \int_{I_n \cup I_n^*} \frac{(\alpha_n + 2iHlog\tilde{d}(\eta))}{(\lambda - \eta)R_+(\eta)} d\eta.
\]

We see that \( k(\lambda) \) satisfies the jump relations:

\[
k_+(\lambda) - k_-(\lambda) = \frac{-\theta_n}{R(\lambda)}, \quad \lambda \in \Gamma_n \cup \Gamma_n^*
\]

\[
k_+(\lambda) - k_-(\lambda) = \frac{-\alpha_n + 2iHlog\tilde{d}(\lambda)}{R_+(\lambda)}, \quad \lambda \in I_n \cup I_n^*,
\]

and is otherwise analytic. So \( H \) satisfies (28).

The function \( k \) blows up like \((\lambda - \lambda_n)^{-1/2}\) near each endpoint, has continuous boundary values in between the endpoints, and vanishes like \(1/\lambda\) for large \( \lambda \). It is the only such solution of the jump relations. The factor of \( R(\lambda) \) renormalizes the
singalities at the endpoints, so that, as desired, the boundary values of $H(\lambda)$ are bounded continuous functions. Near infinity, there is the asymptotic expansion:

$$H(\lambda) = H_G \lambda^G + H_{G-1} \lambda^{G-1} + \cdots + H_1 \lambda + H_0 + O(\lambda^{-1})$$

(31)

$$= p(\lambda) + O(\lambda^{-1}),$$

where all coefficients $H_j$ of the polynomial $p(\lambda)$ can be found explicitly by expanding $R(\lambda)$ and the Cauchy integral $k(\lambda)$ for large $\lambda$. It is easy to see from the reality of $\theta_j$ and $\alpha_j$ that $p(\lambda)$ is a polynomial with coefficients which are real modulo $O(h)$.

So the matrix function $V$ defined in (29) has the following asymptotics at infinity:

$$V(\lambda) \exp[-ip(\lambda)\sigma_3/h] = I + O(\lambda^{-1}).$$

(32)

This together with the jump relations (30) defines a Riemann-Hilbert problem for $V$ that can be explicitly solved via theta functions. Equivalently $P$ and $O$ can be explicitly expressed in terms of theta functions. For example, it is now elementary to check that the solution of the outer Riemann-Hilbert problem (26) for $O$ is given by the formulae:

$$O_{11}(\lambda) = \frac{b^-(\lambda)}{\beta(\lambda)} \frac{\Theta(A(\infty) - V_1)}{\Theta(A(\infty) - V_1 + iU/h)} \frac{\Theta(A(\lambda) - V_1 + iU/h)}{\Theta(A(\lambda) - V_1)} \frac{\Theta(A(\lambda) - V_1)}{\Theta(-A(\lambda) - V_1)} \frac{\Theta(-A(\lambda) - V_1 + iU/h)}{\Theta(-A(\lambda) - V_1 + iU/h)},$$

$$O_{12}(\lambda) = \frac{b^+(\lambda)}{\beta(\lambda)} e^{2iR(\lambda)k_{+}(\lambda_0)/h} \frac{\Theta(A(\infty) - V_1)}{\Theta(A(\infty) - V_1 + iU/h)} \frac{\Theta(-A(\lambda) - V_1 + iU/h)}{\Theta(-A(\lambda) - V_1 + iU/h)} \frac{\Theta(-A(\lambda) - V_1)}{\Theta(-A(\lambda) - V_1 + iU/h)} \frac{\Theta(-A(\lambda) - V_1 + iU/h)}{\Theta(-A(\lambda) - V_1 + iU/h)},$$

(33)

$$O_{21}(\lambda) = \frac{b^+(\lambda)}{\beta(\lambda)} e^{-2iR(\lambda)k_{+}(\lambda_0)/h} \frac{\Theta(-A(\infty) - V_2)}{\Theta(-A(\infty) - V_2 + iU/h)} \frac{\Theta(A(\lambda) - V_2)}{\Theta(-A(\lambda) - V_2 + iU/h)} \frac{\Theta(-A(\lambda) - V_2 + iU/h)}{\Theta(-A(\lambda) - V_2 + iU/h)},$$

$$O_{22}(\lambda) = \frac{b^-(\lambda)}{\beta(\lambda)} \frac{\Theta(-A(\infty) - V_2)}{\Theta(-A(\infty) - V_2 + iU/h)} \frac{\Theta(A(\lambda) - V_2)}{\Theta(-A(\lambda) - V_2 + iU/h)} \frac{\Theta(-A(\lambda) - V_2 + iU/h)}{\Theta(-A(\lambda) - V_2 + iU/h)},$$

where

$$b^\pm(\lambda) = \frac{R(\lambda) \pm (\lambda - \lambda_0^+)(\lambda - \lambda_1)\cdots(\lambda - \lambda_0^+)}{2R(\lambda)}.$$

From the explicit solution, using formula (14) with the obvious substitution of $O$ for $M$ one derives formula (27). This completes the proof of Theorem 2.

We end this section by once more recapitulating the sequence of matrix deformations used:

$$M(\text{discrete}) \rightarrow M(\text{continuous}) \rightarrow Q \rightarrow Z \rightarrow Y \rightarrow W \rightarrow O.$$
The first problem in the sequence, for $M$ (in its discrete version), is equivalent to the inverse scattering problem for NLS. The last Riemann-Hilbert problem, for $O$, (26), was solved explicitly via theta functions. Since, as is seen, the solution $O$ of the outer problem is uniformly bounded in $x, t$, with $L^2$ limits $O_+, O_-$, as $h \to 0$, so are $W$ and $Y$ and the (asymptotically valid) transformations from $Y$ to $O$ and back are a posteriori justified.
APPENDIX 1. THE CROSS PROBLEM.

The transformation from the Riemann-Hilbert problem (23) to the problem (24)-(25) requires two steps. First, the deletion of the lens contours away from the points $a_l, b_l$. This is immediate because the jump matrices are uniformly exponentially small perturbations of the identity.

Second, one needs to consider the small remaining crosses centered at the points $a_l, b_l$ (see the remark of section 5). Since the jump matrices are not uniformly small there, one needs to find "local" parametrices. In other words, one needs to solve the local Riemann-Hilbert problems.

For example, after translation all problems centered at $a_l$ look as follows: to find a matrix $L$ which is analytic in $\mathbb{C} \setminus \Gamma$ where $\Gamma$ is the cross shown in Figure A.1, centered at 0. The actual angles between the four half-lines emanating from 0 are not important as long as every half-line is in a different quadrant.

![Fig.A.1. The Riemann–Hilbert problem on a cross centered at zero.](image-url)
The jumps for $L$ are

\[ v_L(\lambda) = \begin{pmatrix} 1 & \frac{r_i}{1+|r_i|^2} \lambda^{2\nu_i} e^{-2\nu_i (2\lambda+|\lambda|)^2} \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in J_1, \]

\[ v_L(\lambda) = \begin{pmatrix} \frac{1}{d_i^2} \frac{r_i}{1+|r_i|^2} \lambda^{2\nu_i} e^{-2\nu_i (2\lambda+|\lambda|)^2} & 0 \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in J_2, \]

\[ v_L(\lambda) = \begin{pmatrix} 1 & -d_i^2 r_i \lambda^{2\nu_i} e^{-2\nu_i (2\lambda+|\lambda|)^2} \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in J_3, \]

\[ v_L(\lambda) = \begin{pmatrix} d_i^{-2} r_i^* \lambda^{-2\nu_i} e^{-2\nu_i (2\lambda+|\lambda|)^2} & 0 \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in J_4. \]

Also $\lim_{\lambda \to \infty} L(\lambda) = I$.

Here

\[ \nu_i = \frac{1}{2\pi} \log(1 + |r(a)|^2), \]

\[ g_i = g(\lambda = a_i), r_i = r(a_i) \]

and $d_i$ is defined by the local behavior of $\tilde{d}$ near $a_i$ (see (21b)).

After a conjugation of the jump matrix by $\text{diag}(d_i e^{\frac{g_i+i\pi^2}{\lambda}}, d_i^{-1} e^{-\frac{g_i-i\pi^2}{\lambda}})$, a further translation $\lambda \to \lambda - \frac{x}{2}$ (to complete the square) and a rescaling $\lambda \to \lambda (\frac{t}{\lambda})^{1/2}$, we end up with a problem with jumps on a rescaled cross which is no more small. Extending the cross to infinity (by setting the jump equal to the identity on the extension) we have a new Riemann-Hilbert problem which can be approximated by the following:

\[ \Psi_+ = \Psi_- \nu_\Psi, \text{where } \nu_\Psi(\lambda) = \begin{pmatrix} 1 & \frac{r_i}{1+|r_i|^2} \lambda^{2\nu_i} e^{-2\nu_i (2\lambda+|\lambda|)^2} \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in \tilde{J}_1, \]

\[ \begin{pmatrix} \frac{1}{d_i^2} \frac{r_i}{1+|r_i|^2} \lambda^{2\nu_i} e^{-2\nu_i (2\lambda+|\lambda|)^2} & 0 \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in \tilde{J}_2, \]

\[ \begin{pmatrix} 1 & -d_i^2 r_i \lambda^{2\nu_i} e^{-2\nu_i (2\lambda+|\lambda|)^2} \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in \tilde{J}_3, \]

\[ \begin{pmatrix} d_i^{-2} r_i^* \lambda^{-2\nu_i} e^{-2\nu_i (2\lambda+|\lambda|)^2} & 0 \\ 0 & 1 \end{pmatrix}, \text{ for } \lambda \in \tilde{J}_4, \]

\[ \lim_{\lambda \to \infty} S(\lambda) = I, \]

where $\tilde{J}_i$ is the extension of $J_i$, $i = 1, 2, 3, 4$. 

The last Riemann-Hilbert problem can be explicitly solved via parabolic cylinder functions (see e.g. [DZ1]). Indeed, let $D_a(\lambda)$ denote the standard parabolic cylinder function. Then $D_a(e^{-3i\pi/4}\lambda)$ and $D_a(-e^{-3i\pi/4}\lambda)$ solve the ODE

$$\frac{d^2 D}{d\lambda^2} + \left(\frac{1}{2} - \frac{\lambda^2}{4} + a\right)D = 0,$$

where $a = i\nu_l$. Let

(A.3) \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$

where

\begin{align*}
S_{11} &= e^{-3\pi\nu_l/4}D_a(e^{-3i\pi/4}\lambda), \\
S_{12} &= \frac{r_l^\ast \Gamma(a)}{(2\pi)^{1/2}e^{-i\pi/4}}e^{3\pi\nu_l/4} \frac{d}{d\lambda} D_{-a}(e^{-i\pi/4}\lambda) - \frac{i\lambda}{2} D_{-a}(e^{-i\pi/4}\lambda)], \\
S_{21} &= \frac{r_l \Gamma(-a)}{(2\pi)^{1/2}e^{i\pi/4}}e^{-\pi\nu_l/4} \frac{d}{d\lambda} D_a(e^{-3i\pi/4}\lambda) + \frac{i\lambda}{2} D_a(e^{-3i\pi/4}\lambda)], \\
S_{22} &= e^{\pi\nu_l/4}D_a(e^{-i\pi/4}\lambda),
\end{align*}

for $\text{Im}\lambda > 0$,

and

\begin{align*}
S_{11} &= e^\pi\nu_l/4D_a(e^{i\pi/4}\lambda), \\
S_{12} &= \frac{r_l^\ast \Gamma(a)}{(2\pi)^{1/2}e^{-i\pi/4}}e^{-3\pi\nu_l/4} \frac{d}{d\lambda} D_{-a}(e^{3i\pi/4}\lambda) - \frac{i\lambda}{2} D_{-a}(e^{3i\pi/4}\lambda)], \\
S_{21} &= \frac{r_l \Gamma(-a)}{(2\pi)^{1/2}e^{i\pi/4}}e^{-\pi\nu_l/4} \frac{d}{d\lambda} D_a(e^{3i\pi/4}\lambda) + \frac{i\lambda}{2} D_a(e^{3i\pi/4}\lambda)], \\
S_{22} &= e^{-3\pi\nu_l/4}D_a(e^{3i\pi/4}\lambda),
\end{align*}

for $\text{Im}\lambda < 0$.

Then it is possible to check that

$$S_+ = S_-\nu_S, \text{ where } S(\lambda) =$$

\begin{align*}
&\begin{pmatrix} 1 & r_l \\
0 & 1 + |r_l|^2 \\
\end{pmatrix}, & \text{for } \lambda \in \tilde{J}_1, \\
&\begin{pmatrix} 1 & 0 \\
-\frac{r_l}{1 + |r_l|^2} & 1 \\
\end{pmatrix}, & \text{for } \lambda \in \tilde{J}_2, \\
&\begin{pmatrix} 1 & -r_l \\
0 & 1 \\
\end{pmatrix}, & \text{for } \lambda \in \tilde{J}_3, \\
&\begin{pmatrix} 1 & 0 \\
r_l^\ast & 1 \\
\end{pmatrix}, & \text{for } \lambda \in \tilde{J}_4,
\end{align*}

\text{i}m\lambda \to \infty S(\lambda) = I.
Now, setting

\[(A.7) \quad S = \Psi \lambda^{i\nu_3} e^{-i\lambda^2/4} \sigma_3.\]

and using the well known asymptotics for the parabolic cylinder function at infinity

\[(A.8) \quad D_a(\lambda) = \lambda^a e^{-\lambda^2/4} (1 + O(\lambda^{-2})), \quad |\arg \lambda| < \frac{3\pi}{4},\]

\[(A.8) \quad D_a(\lambda) = \lambda^a e^{-\lambda^2/4} (1 + O(\lambda^{-2})) - \frac{(2\pi)^{1/2}}{\Gamma(-a)} e^{i\pi a} \lambda^{-a-1} e^{\lambda^2/4} (1 + O(\lambda^{-2})), \quad \frac{\pi}{4} < |\arg \lambda| < \frac{5\pi}{4},\]

\[(A.8) \quad D_a(\lambda) = \lambda^a e^{-\lambda^2/4} (1 + O(\lambda^{-2})) - \frac{(2\pi)^{1/2}}{\Gamma(-a)} e^{-i\pi a} \lambda^{-a-1} e^{\lambda^2/4} (1 + O(\lambda^{-2})), \quad -\frac{5\pi}{4} < |\arg \lambda| < -\frac{\pi}{4},\]

it is immediate to check that \(\Psi\) solves \((A.2)\).

It can be easily verified that the back-rescaled local version near \(a_l, b_l\) matches with the solution of the outer problem (as \(h \to 0\)). Thus the issue of the small crosses is settled.

**APPENDIX 2. THE VARIATIONAL PROBLEM AND THE FINITE GENUS ANSATZ.**

The function \(g\) defined by \((18)\) and the conditions stated before \((18)\) is crucial for the asymptotic analysis of the Riemann-Hilbert problem \((11)\). As stated in \([KMM]\) and \([KR]\) the existence of such a function follows from the existence and regularity of a solution to a variational problem. In this section we pose the variational problem and we state the results of \([KR]\) on existence. We also show that a variation of the proofs of \([KR]\) guarantees the validity of the finite genus ansatz for the barrier data problem.

Let \(\mathbb{H} = \{ z : \text{Im} z > 0 \}\), be the complex upper-half plane and \(\overline{\mathbb{H}} = \{ z : \text{Im} z \geq 0 \} \cup \{ \infty \}\) be the closure of \(\mathbb{H}\). Let also \(\mathbb{K} = \{ z : \text{Im} z > 0 \} \setminus \{ z : \text{Re} z = 0, 0 < \text{Im} z \leq A \}\), where \(A\) is a positive constant. In the closure of this space, \(\overline{\mathbb{K}}\), we consider the points \(ix_+\) and \(ix_-\), where \(0 \leq x < A\) as distinct. In other words, we cut a slit in the upper half-plane along the segment \((0, iA)\) and distinguish between the two sides of the slit. The point infinity belongs to \(\overline{\mathbb{K}}\), but not \(\mathbb{K}\). Define \(G(z; \eta)\)
to be the Green’s function for the upper half-plane

\[ G(z; \eta) = \log \frac{|z - \eta^*|}{|z - \eta|} \]

and let \( d\mu^0(\eta) \) be the nonnegative measure \(-\rho^0 d\eta\) on the segment \([0, iA]\) oriented from 0 to \(iA\), where \(\rho^0\) is the density of eigenvalues given by

\[ \rho^0(\lambda) = \frac{\lambda}{\pi(A^2 + \lambda^2)^{1/2}}. \] (9a)

The star denotes complex conjugation. Let the "external field" \( \phi \) be defined by

\[ \phi(z) = -\int G(z; \eta) d\mu^0(\eta) - \text{Re}(i\pi \int_0^A \rho^0 d\eta + 2i(zx + z^2 t)), \] (A.9)

where, without loss of generality \(x > 0\).

Let \( \mathcal{M} \) be the set of all positive Borel measures on \( \bar{\mathbb{K}} \), such that both the free energy

\[ E(\mu) = \int \int G(x, y) d\mu(x) d\mu(y), \quad \mu \in \mathcal{M} \]

and \( \int \phi d\mu \) are finite. Also, let

\[ V^\mu(z) = \int G(z, x) d\mu(x), \quad \mu \in \mathcal{M}. \]

be the Green’s potential of the measure \( \mu \).

The weighted energy of the field \( \phi \) is

\[ E_\phi(\mu) = E(\mu) + 2\int \phi d\mu, \quad \mu \in \mathcal{M}. \]

Now, given any curve \( F \) in \( \bar{\mathbb{K}} \), the equilibrium measure \( \lambda^F \) supported in \( F \) is defined by

\[ E_\phi(\lambda^F) = \min_{\mu \in M(F)} E_\phi(\mu), \]

where \( M(F) \) is the set of measures in \( \mathcal{M} \) which are supported in \( F \), provided such a measure exists.

The finite gap ansatz is equivalent to the existence of a so-called S-curve joining the points \(0_+\) and \(0_-\) and lying entirely in \( \bar{\mathbb{K}} \). By S-curve we mean an oriented
curve $F$ such that the equilibrium measure $\lambda^F$ exists, its support consists of a finite union of analytic arcs and at any interior point of $supp \mu$ the so called S-property is satisfied

(A.10)  \[ \frac{d}{dn_+}(\phi + V\lambda^F) = \frac{d}{dn_-}(\phi + V\lambda^F). \]

In [KR] we show that there is a $C$ such that

(A.11)  \[ E_\phi(\lambda^C) = max_{contours F} E_\phi(\lambda^F) = max_{contours F} min_{\mu \in M(F)} E_\phi(\mu), \]

and that the existence of an S-curve follows from the existence of a contour $C$ maximizing the equilibrium measure.

EXISTENCE THEOREM [KR]. For the external field given by (A.9), there exists a continuum $F \in F$ such that the equilibrium measure $\lambda^F$ exists and

(A.12)  \[ E_\phi[F](= E_\phi(\lambda^F)) = max_{F \in F} min_{\mu \in M(F)} E_\phi(\mu). \]

PROOF: See Theorem 4 in [KR]. Even though the density $\rho$ does not satisfy all of conditions (1) of Theorem 4, the proof goes through unaltered.

For our particular problem we also have

REGULARITY THEOREM. Under the extra assumption that the continuum $F$ does not touch the spike $\{z : Re z = 0, 0 < Im z \leq A\}$ except at a finite number of points, the continuum $F$ is at worst a union of an S-curve and a finite union of real intervals.

PROOF: The proof follows as in Theorem 8 of [KR]. But there are some changes here. The density of eigenvalues given by (9a) does not satisfy all conditions (1) set in [KR]. In particular it is not true that $Im[\rho^0(z)] > 0$, for $z \in (0, iA] \cup \mathbb{R}^+$. Rather $Im[\rho^0(z)] = 0$ on the real line. So a small amendment of the regularity proof is needed.

The point of the assumption that $Im[\rho^0(z)] > 0$, for $z \in (0, iA] \cup \mathbb{R}^+$ is to ensure that the continuum $F$ does not touch the negative real line, except of course at $0_-$ and possibly infinity. In our case, we can argue as follows.
Because of the analyticity properties of the field, the real line can be divided into a finite number of intervals $J_k$ such that in the interior of each $J_k$ either
\[
\frac{d\phi}{dImz} > 0,
\]
or
\[
\frac{d\phi}{dImz} \leq 0.
\]
In the first case, one can see that the continuum $F$ cannot touch the real line except of course at the endpoints of $J_k$. This is because for any configuration that involves a continuum including other points on the real line, we can find a configuration with no other points on the real line, by pushing measures up away from the real axis, which has greater (unweighted and weighted) energy. It is crucial here that if $u \in \mathbb{R}$ then $G(u, v) = 0$, while if both $u, v$ are off the real line $G(u, v) > 0$.

In the second case, we get a finite union of real intervals.

So $F$ consists of a finite union of arcs: some of them are real intervals (at worst) and the rest do not touch the real line except at their endpoints.

To pursue the proof of regularity one needs the following identity.

**THEOREM [KR].** Let $F$ be the maximizing continuum of and $\lambda^F$ be the equilibrium measure. Let $\mu$ be the extension of $\lambda^F$ to the lower complex plane via $\mu(z^*) = -\mu(z)$. Then
\[
\text{Re}(\int \frac{d\mu(u)}{u-z} + V'(z))^2 = \text{Re}(V'(z))^2 - 2\text{Re} \int \frac{V'(z) - V'(u)}{z-u} d\mu(u)
\]
\[
+ \text{Re} [\frac{1}{z^2} \int 2(u + z)V'(u) d\mu(u)].
\]

Here $V$ is the complexification of the real field $\phi$.

**PROOF:** By taking variations with respect to the equilibrium measure; see Theorem 5 of [KR]. Of course, variations cannot be taken near the real intervals, but since the field $\phi = 0$ and the Green’s function is also zero, these real intervals are not part of the support of the equilibrium measure.

From (A.13) it is easy to see that the support of the equilibrium measure of the maximizing continuum is characterized by
\[
\text{Re} \int^z (R_\mu)^{1/2} dz = 0,
\]
where
\[ R_\mu(z) = (V'(z))^2 - 2 \int_{\text{supp}\mu} \frac{V'(z) - V'(u)}{z-u} d\mu(u) \]
\[ + \frac{1}{z^2} \left( \int_{\text{supp}\mu} 2(u+z)V'(u) d\mu(u) \right). \]

Since \( R_\mu(z) \) is a function analytic in \( K \), the locus defined by (A.14) is a union of arcs with endpoints at zeros of \( R_\mu \). But it is readily seen that \( R_\mu \) has finitely many zeros.

(A.10) also follows easily from (A.13); see [KR]. Alternatively, see Chapter 8 of [KMM].

If \( F \) touches the spike \([0, iA]\) at more than a finite number of points, regularity cannot be proved as above because variations cannot be taken. In [KR] we have included a rough idea on how to extend the above proof. Here is a more detailed argument forthcoming (see also [K]).

One wishes to somehow allow the contour \( F \) go through the spike \([0, iA]\). One problem arising is that (the complexification of) the external field is not analytic across the segment \([-iA, iA]\). What is true, however, is that \( V \) is analytic in a Riemann surface consisting of infinitely many sheets, cut along the line segment \([-iA, iA]\). So, the appropriate underlying space for the (doubled up) variational problem should now be a non-compact Riemann surface, say \( L \). Now, compactness is the crucial element in the proof of a maximizing continuum. But we can indeed compactify the Riemann surface \( L \) by mapping it to a subset of the complex plane and compactifying the complex plane. The other problem, of course, is whether the amended variational problem (with the modified field defined on the Riemann surface and with the possibility of \( F \) not enclosing all the original eigenvalues) is still appropriate for the semiclassical NLS. The argument goes roughly as follows:

(i) Proof of the existence of an S-curve \( F \) in \( L \) along the lines of [KR].

(ii) Deformation of the original discrete Riemann-Hilbert problem to the set \( \hat{F} \) consisting of the projection of \( F \) to the complex plane. At first sight, it is clear that \( \hat{F} \) may not encircle the spike \([0, iA]\). It is however possible to append S-loops (not necessarily with respect to the same branch of the external field) and end up with
a sum of S-loops, such that the amended $\hat{F}$ does encircle the spike $[0, iA]$. To see this, suppose there is an open interval, say $(i\alpha, i\alpha_1)$, which lies in the exterior of $\hat{F}$, while $i\alpha, i\alpha_1 \in \hat{F}$. Let us assume, that $\hat{F}$ crosses $[0, iA]$ along bands at $i\alpha, i\alpha_1$ (if not the situation is similar and simpler); call these bands $S, S_1$. Let $\beta^-, \beta^+$ be points (considered in $\mathbb{C}$) lying on $S$ to the left and right of $i\alpha$ respectively, and at a small distance from $i\alpha$. Similarly, let $\beta^-_1, \beta^+_1$ be points lying on $S_1$ to the left and right of $i\alpha_1$ respectively, and at a small distance from $i\alpha_1$. We will show that there exists a "gap" region including the preimages of $\beta^-, \beta^-_1$ lying in the $N$th sheet for $-N$ large enough, and similarly there exists a "gap" region including the preimages of $\beta^+, \beta^+_1$ lying in the $M$th sheet for $M$ large enough, both being regions for which the gap inequalities hold a priori, irrespectively of the actual S-curve, depending only on the external field!

Indeed, note that the quantity $Re(\tilde{\phi}^\sigma(z))$ (which defines the variational inequalities) is a priori bounded above by $-\phi(z)$. For this, see (8.8) in Chapter 8 of [KMM]; there is actually a sign error: the right formula is

$$Re(\tilde{\phi}^\sigma(z)) = -\phi(z) + \int G(z, \eta) \rho^\sigma(\eta) d\eta.$$ 

Next note that the difference of the values of the function $Re(\tilde{\phi}^\sigma(z))$ in consecutive sheets is $\delta Re(\tilde{\phi}^\sigma) = \pm 2\pi Rez$, and hence the difference of the values at points on consecutive sheets whose image under the projection to the complex plane is $i\eta + \epsilon$, where $\eta$ is real and $\epsilon$ is a small (negative or positive) real, is $\delta(Re\tilde{\phi}^\sigma) = \pm 2\pi \epsilon$. This means that on the left (respectively right) side of the imaginary semiaxis, the inequality $Re(\tilde{\phi}^\sigma(z)) < 0$ will be eventually (depending on the sheet) be valid at any given small distance to it.

Applying the theory of [KR] we join the preimages of $\beta^-$ and $\beta^-_1$ (under the projection pf $L$ to $\mathbb{C}$) lying in the $N$th sheet and the preimages of $\beta^+$ and $\beta^+_1$ lying in the $M$th sheet. Finally we connect the preimage of $\beta^+$ lying in the $N$th sheet to the preimage of $\beta^-$ lying in the $M$th sheet, and so on. We thus end up with an S-loop whose projection is covering the "lacuna" $(i\alpha, i\alpha_1)$.

The original discrete Riemann-Hilbert problem can be trivially deformed to a
discrete Riemann-Hilbert on the resulting (projection of the) union of S-loops. All this is possible even in the case where $\hat{F}$ self-intersects.

(iv) Deform the discrete Riemann-Hilbert problem to the continuous one with the right band/gap structure (on $\hat{F}$; according to the projection of the equilibrium measure on $F$), which is then explicitly solvable via theta functions. Both the discrete-to-continuous approximation and the opening of the lenses needed for this deformation are justified as in [KMM] (see also the article [LM] for the delicate study of the Riemann-Hilbert problem near the points where $\hat{F}$ crosses the spike). The g-function is defined by the same Thouless-type formula with respect to the equilibrium measure (cf. section 2(iii)). It satisfies the same conditions as in [KMM] (measure reality and variational inequality) on bands (where the branch of the field turns out to be irrelevant) and on gaps (where the inequalities are satisfied according to the branch of the external field).

We arrive at the following conclusion.

FINITE GAP ANSATZ. The finite genus ansatz is valid generically in $x,t$ for the steplike data given by (2).

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