Time-dependent Schrödinger equations having isomorphic symmetry algebras. II. Symmetry algebras, coherent and squeezed states.

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ABSTRACT

Using the transformations from paper I, we show that the Schrödinger equations for: (1) systems described by quadratic Hamiltonians, (2) systems with time-varying mass, and (3) time-dependent oscillators, all have isomorphic Lie space-time symmetry algebras. The generators of the symmetry algebras are obtained explicitly for each case and sets of number-operator states are constructed. The algebras and the states are used to compute displacement-operator coherent and squeezed states. Some properties of the coherent and squeezed states are calculated. The classical motion of these states is demonstrated.

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1 Introduction

In this paper, we continue the investigation that began in paper I\[1\] of three classes of 1-dimensional Schrödinger equations: equations with time-dependent quadratic Hamiltonians ($TQ$), equations with time-dependent masses ($TM$), and equations for time-dependent oscillators ($TO$). They are described thusly:

The $TQ$ class of Schrödinger equations, in units of $m = \hbar = 1$, is

$$S_1 \Phi(x, t) = \{-[1 + k(t)] P^2 - 2T + h(t)D + g(t)P - 2h^{(2)}(t)X^2 - 2h^{(1)}(t)X - 2h^{(0)}(t)I\} \Phi(x, t) = 0,$$

(1)

where $D = \frac{1}{2}(XP + PX)$. The $TM$ class of equations is

$$S_2 \Theta(x, t) = \{-f(t)P^2 - 2T - 2f^{(2)}(t)X^2 - 2f^{(1)}(t)X - 2f^{(0)}(t)I\} \Theta(x, t) = 0.$$

(2)

In these $m = 1$ units, $1/f(t)$ represents a time-dependent mass. Rather than use the most general ($TM$) Eq. (2), we shall work with the more restrictive (see Sec. 4.1 of paper I)

$$\hat{S}_2 \hat{\Theta}(x, t) = \{-e^{-2\nu(t)}P^2 - 2T - 2f^{(2)}(t)X^2 - 2f^{(1)}(t)X - 2f^{(0)}(t)I\} \hat{\Theta}(x, t) = 0,$$

(3)

$$f(t) = \exp[-2\nu(t)],$$

(4)

where the function $\nu(t)$ will be defined below.

Finally, the time-dependent oscillator Schrödinger equations ($TO$) have the form

$$S_3 \Psi(x, t') = \{-P^2 + 2T' - 2g^{(2)}(t')X^2 - 2g^{(1)}(t')X - 2g^{(0)}(t')I\} \Psi(x, t') = 0.$$

(5)

In this paper, we have four main objectives:

First, in Section 2, we compute the relationship between the symmetry algebras of the three classes of Schrödinger equations, $TQ$, $TM$, and $TO$. To do this, we start with the Lie algebra of space-time symmetries associated with $TO$ equations. This Lie symmetry algebra for $TO$ equations is known\[2\]-\[5\] to be the Schrödinger algebra

$$\mathcal{SA}_1^c = su(1, 1) \odot w_1^c.$$

(6)
Then, using the transformation developed in paper I [1], we show that all three classes of Schrödinger equations have symmetry algebras isomorphic to $SA_c^1$.

In Section 3, we start with the symmetry generators associated with $TO$ equations. We then “work backwards” to construct the generators of space-time transformations for first $TM$ and then $TQ$. These calculations constitute the second objective.

In Section 4, working with the oscillator subalgebras of $(SA)^c_1$ for each class of Schrödinger equations, we obtain a set of solutions for each. This has already been done for $TO$ [3, 5, 6]. Here we extend the method to the $TM$ and $TQ$ equations. This completes the third objective.

The fourth objective concerns the calculation of coherent states (Section 5) and squeezed states (Section 6) for the three classes of equations. It is natural, in this context, to take advantage of the Lie symmetry to construct displacement-operator coherent states (DOCS) [8] and displacement-operator squeezed states (DOSS) [5, 9, 10]. We make extensive use of the isomorphism of the symmetry algebras and the results of Section 4 to calculate properties of the coherent (Section 5) and squeezed (Section 6) states.

We close with comments on uncertainty relations and the classical equations of motion.

Elsewhere [11], we shall apply the general results of paper I and this article to the calculation of space-time number-state, coherent-state, and squeezed-state wave functions for $TM$ systems that have been studied by others. We shall give detailed accounts of the solutions.

2 Lie Symmetries

Starting with the $TQ$ Schrödinger equation (1) and using the definitions in Eq. (I-6), we express its Lie symmetries as

$$L_1 = -iA_1(x, t)T + iB_1(x, t)P + C_1(x, t)I.$$  

(7)

For $L_1$ to be a Lie symmetry of Eq. (1), it must satisfy the operator equation

$$[S_1, L_1] = \lambda_1(x, t)S_1,$$  

(8)

where $S_1$ is the $TQ$ Schrödinger operator from Eq. (1). The function $\lambda_1$ depends on the variables $x$ and $t$. As a consequence, if $\Phi(x, t)$ is a solution of the $TQ$ equation (1), then
$L_1 \Phi(x, t)$ is also a solution to this equation.

Next, denote the Lie symmetries of the TM Schrödinger equation (3) by

$$\hat{L}_2 = -i\hat{A}_2(x, t)T + i\hat{B}_2(x, t)P + \hat{C}_2(x, t)I.$$  \(9\)

If $\hat{S}_2$ is the TM Schrödinger operator given in Eq. (3), then for $\hat{L}_2$ to be a symmetry of Eq. (3), it must satisfy the commutator relation $[\hat{\lambda}_2]$ is a function of $x$ and $t$]

$$[\hat{S}_2, \hat{L}_2] = \hat{\lambda}_2(x, t)\hat{S}_2.$$  \(10\)

Finally, for the TO Schrödinger equation (5), its Lie symmetries are

$$L_3 = -iA_3(x, t')T' + iB_3(x, t')P + C_3(x, t')I,$$  \(11\)

where $T' = i\partial_{t'}$. For the operator, $L_3$, to be a symmetry of Eq. (5), it must satisfy

$$[S_3, L_3] = \lambda_3(x, t')S_3,$$  \(12\)

where $S_3$, is the Schrödinger operator given in Eq. (5) and $\lambda_3$ is a function of $x$ and $t'$ (not $t$).

To obtain the coefficients of the operators $T$ or $T'$, $P$, and $I$, in Eqs. (7), (9), and (11), we could substitute these operators into Eqs. (8), (10), and (12), respectively, and solve the three sets of coupled partial differential equations for the coefficients of $T$ or $T'$, $P$, and $I$. This has been done elsewhere for the TO class of Schrödinger equations [2]-[3].

However, here we shall adopt a different approach. Our present objective is to establish a connection between the Lie symmetries of the $TQ$, $TM$, and $TO$ equations; (1), (3), and (5), respectively. We achieve this by starting with the TO symmetries and transforming them into the TM symmetries. Then, we obtain the $TQ$ symmetries from the $TM$ symmetries.

In the first step, we transform from the $(x, t')$ to the $(x, t)$ coordinate system, taking us from TO to TM. Making use of $1 = e^{2\nu} (\partial t'/\partial t) \equiv e^{2\nu} f(t)$, Eq. (12) becomes

$$[\hat{S}_2, \hat{L}] = \hat{\lambda}(x, t)\hat{S}_2,$$  \(13\)

where $\hat{\lambda}(x, t) = (\lambda_3 \circ t')(x, t)$. The generator, $\hat{L}$, takes the form

$$\hat{L} = -i\hat{A}(x, t)T + i\hat{B}(x, t)P + \hat{C}(x, t)I,$$  \(14\)
\[
\hat{A}(x, t) = (A_3 \circ t')(x, t)e^{2\nu}, \quad \hat{B}(x, t) = (B_3 \circ t')(x, t), \quad \hat{C}(x, t) = (C_3 \circ t')(x, t).
\]

From \( \hat{S}_2 = e^{-2\nu}\hat{S}_2 \), we obtain
\[
[e^{2\nu}\hat{S}_2, \hat{L}] = \hat{\lambda}(x, t)e^{2\nu}\hat{S}_2,
\]
which, after rearranging, yields
\[
[\hat{S}_2, \hat{L}] = \left( \hat{\lambda} + e^{-2\nu}[\hat{L}, e^{2\nu}] \right) \hat{S}_2 = \left( \hat{\lambda}(x, t) + 2\hat{A} \frac{d\nu}{dt} \right) \hat{S}_2,
\]
where \( \hat{A} \) is given in Eq. (15). Therefore, \( \hat{L} \) is a symmetry of \( \hat{S}_2 \). Comparing Eqs. (10) and (17), we can identify \( \hat{L}_2 \) with \( \hat{L} \) if
\[
\hat{\lambda}_2 = \hat{\lambda} + 2\hat{A} \frac{d\nu}{dt},
\]
\[
\hat{A}_2(x, t) = \hat{A}(x, t), \quad \hat{B}_2(x, t) = \hat{B}(x, t), \quad \hat{C}_2(x, t) = \hat{C}(x, t).
\]

This means that \( \hat{L} \) is a symmetry of both Eq. (3) and \( \hat{S}_2 \hat{\Theta}(x, t) = 0 \), but with different “lambda” functions; \( \hat{\lambda}_2 \) and \( \hat{\lambda} \), respectively.

In the final step, we transform the commutator bracket (10) with the transformation \( R(\mu, \nu, \kappa) \) of Eq. (I-14), thereby going from \( TM \) to \( TQ \). Inverting the transformation, \( \hat{S}_2 = R(\mu, \nu, \kappa)\hat{S}_1 R^{-1}(\mu, \nu, \kappa) \), we obtain the commutator (8), where
\[
\lambda_1(x, t) = \hat{\lambda}_2 \hat{A} = \hat{\lambda}_2 \hat{S}_2 R(\mu, \nu, \kappa)\hat{S}_1 R^{-1}(\mu, \nu, \kappa), \quad \lambda_1(x, t) = \hat{\lambda}_2 \hat{A} = \hat{\lambda}(x, t) + \left( 8h^{(2)}(t)\kappa - h(t) \right) \hat{A}.
\]

Here, \( d\nu/dt \) is given by Eq. (I-32).

For \( TO \) Schrödinger equations (3), both \( A_3 \) and \( \lambda_3 \) are functions of \( t' \) only [2]-[4]. Therefore, \( \hat{\lambda} \) is a function of \( t \) only and, according to Eq. (18), \( \hat{\lambda}_2 \) is a function of \( t \) only. Since the transformation \( R(\mu, \nu, \kappa) \) involves no time derivatives, we have
\[
\lambda_1(t) = \hat{\lambda}_2(t) = \hat{\lambda}(t) + \left( 8h^{(2)}(t)\kappa - h(t) \right) \hat{A}.
\]
3 Lie Symmetries for Each Class of Schrödinger equation

3.1 TO Symmetries

The six generators of Lie space-time symmetries for the $TO$ Schrödinger equation (5) have been calculated previously [2]-[4]. They form a basis for the Lie algebra $sl(2,\mathbb{R})$ [2, 4]. We prefer to use its complexification [3, 4], which we have called the Schrödinger algebra, denoted by $SA_1^c$ in Eq. (6). We shall work with the generators of $SA_1^c$ only.

First, the three generators which form a basis for the Heisenberg-Weyl subalgebra, $w_1^c$, are

\begin{align*}
J_{3-} &= i\left\{\xi P - X\dot{\xi} + CI\right\}, \\
J_{3+} &= i\left\{-\xi P + X\ddot{\xi} - C\dot{I}\right\}, \\
I &= 1.
\end{align*}

Both $\xi$ and its complex conjugate, $\bar{\xi}$, are functions of $t'$ and are two linearly independent solutions of the second-order, ordinary differential equation [2]-[4]

\begin{equation}
\ddot{\gamma} + 2g^{(2)}(t')\gamma = 0.
\end{equation}

The Wronskian of the two solutions is a constant,

\begin{equation}
W(\xi, \bar{\xi}) = \xi\ddot{\bar{\xi}} - \dot{\xi}\dddot{\bar{\xi}} = -i.
\end{equation}

A dot over the function indicates differentiation by $t'$, i.e. $\dot{\xi} = d\xi/dt'$. The function $C(t')$ is

\begin{align*}
C(t') &= c(t') + C^0, \\
c(t') &= \int_{t_0}^{t'} ds g^{(1)}(s)\xi(s),
\end{align*}

where $C^0$ is an integration constant [4].

The three generators of the $su(1,1)$ subalgebra have the form

\begin{align*}
M_{3-} &= -\left\{-\phi_1 T' + \frac{1}{2}\phi_1 D + \varepsilon_1 P - \frac{1}{4}\phi_1 X^2 - \dot{\phi}_1 X + (D_1 + g^{(0)}\phi_1)I\right\}, \\
M_{3+} &= -\left\{-\phi_2 T' + \frac{1}{2}\phi_2 D + \varepsilon_2 P - \frac{1}{4}\phi_2 X^2 - \dot{\phi}_2 X + (D_2 + g^{(0)}\phi_2)I\right\}, \\
M_3 &= -\left\{-\phi_3 T' + \frac{1}{2}\phi_3 D + \varepsilon_3 P - \frac{1}{4}\phi_3 X^2 - \dot{\phi}_3 X + (D_3 + g^{(0)}\phi_3)I\right\}.
\end{align*}

The three functions, $\phi_j$, $j = 1, 2, 3$, are defined as (note that $\phi_3$ is a real function of $t'$) [4]

\begin{align*}
\phi_1 &= \xi^2, \\
\phi_2 &= \bar{\xi}^2, \\
\phi_3 &= 2\xi\bar{\xi}.
\end{align*}
The remaining \( t' \)-dependent functions are defined in terms of \( \xi, \bar{\xi}, C, \) and \( \bar{C} \)

\[
\begin{align*}
E_1 &= -\xi C, \quad E_2 = -\bar{\xi} \bar{C}, \quad E_3 = -\xi \bar{C} - \bar{\xi} C, \\
D_1 &= -\frac{1}{2}C^2, \quad D_2 = -\frac{1}{2}\bar{C}^2, \quad D_3 = -C \bar{C}.
\end{align*}
\]

Both \( E_3 \) and \( D_3 \) are real functions of \( t' \).

3.2 TM Symmetries from TO Symmetries

First, we calculate the generators, \( \hat{J}_{2\pm} \). In the initial step, we transform the operators \( J_{3\pm} \) from the \((x, t')\) coordinate system to the \((x, t)\) system, as described in Sections 3 and 4.1 of paper I:

\[
\begin{align*}
\hat{J}_{2-} &= i \left\{ \hat{\xi} P - X \hat{\xi} + \hat{\bar{C}} I \right\}, \\
\hat{J}_{2+} &= i \left\{ -\hat{\bar{\xi}} P + X \hat{\bar{\xi}} - \hat{C} I \right\},
\end{align*}
\]

\[
\hat{\xi}(t) = (\xi \circ t')(t), \quad \hat{\bar{\xi}}(t) = (\hat{\xi} \circ t')(t), \quad \hat{C}(t) = (C \circ t')(t) = (c \circ t')(t) + C^0.
\]

In Eq. (32) we have used definition (26) for \( C \).

It is important to keep in mind that, in general,

\[
\frac{d}{dt} \hat{\xi}(t) \neq \hat{\xi'}(t).
\]

The ‘dot’ over a function will always indicate differentiation by \( t' \). Differentiation by \( t \) will always be written in full notation. Also, an important relationship between \( \hat{\xi} \) and \( \hat{\bar{\xi}} \) is

\[
\hat{\xi} \hat{\bar{\xi}} - \hat{\bar{\xi}} \hat{\xi} = -i,
\]

which follows from the Wronskian (25) and the definitions in Eq (32).

Now, we proceed in the same way as before to obtain the operators spanning the \( su(1,1) \) algebra. First, because of Eq. (I-51) and \( f(t) = e^{-2\nu} \), we note that

\[
T' = e^{2\nu} T.
\]

The three generators of \( su(1,1) \) have the form

\[
\hat{M}_{2-} = - \left\{ -\hat{\phi}_1 e^{2\nu} T + \frac{1}{2}\hat{\bar{\phi}}_1 D + \hat{E}_1 P - \frac{1}{4}\hat{\bar{\phi}}_1 X^2 - \hat{E}_1 X + (\hat{D}_1 + \hat{g}(0) \hat{\phi}_1) I \right\}.
\]
\[ \dot{M}_{2+} = - \left\{ -\dot{\phi}_2 e^{2\nu} T + \frac{1}{2} \dot{\phi}_2 D + \dot{E}_2 P - \frac{1}{4} \dot{\phi}_2 X^2 - \dot{E}_2 X + (\dot{D}_2 + \dot{g}(0) \dot{\phi}_2) I \right\}, \]
\[ \dot{M}_2 = - \left\{ -\dot{\phi}_3 e^{2\nu} T + \frac{1}{2} \dot{\phi}_3 D + \dot{E}_3 P - \frac{1}{4} \dot{\phi}_3 X^2 - \dot{E}_3 X + (\dot{D}_3 + \dot{g}(0) \dot{\phi}_3) I \right\}, \]  
(36)

where, for \( j = 1, 2, 3 \) and again keeping in mind the comment on ‘dots’ following Eq. (33),

\[ \dot{D}_j(t) = (D_j \circ t')(t), \quad \dot{E}_j(t) = (E_j \circ t')(t), \quad \dot{\phi}_j(t) = (\dot{\phi}_j \circ t')(t), \]
\[ \dot{\phi}_j(t) = (\dot{\phi}_j \circ t')(t), \quad \dot{\phi}_j(t) = (\dot{\phi}_j \circ t')(t). \]  
(37)

Also, according to Eq. (I-70), we have

\[ \dot{g}(0)(t) = (g(0) \circ t')(t) = h(0)(t) e^{2\nu} + h(1)(t) e^{3\nu} \mu + h(2)(t) e^{4\nu} \mu^2 = e^{2\nu} f(0)(t). \]  
(38)

### 3.3 TQ Symmetries from TM Symmetries

A basis for the Lie symmetries associated with the TQ Schrödinger equation can be obtained by applying the transformation in Eq. (20) to the generators for the TM class of Lie symmetries obtained in the previous subsection. From Eq. (31), and using (see Ref. [13])

\[ e^A Be^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \cdots, \]  
(39)

we obtain

\[ J_{1-} = R^{-1}(\mu, \nu, \kappa) \dot{J}_{2-} R(\mu, \nu, \kappa) = i \left\{ \Xi_P P - X \Xi_X + \Xi_I I \right\}, \]  
(40)

\[ J_{1+} = R^{-1}(\mu, \nu, \kappa) \dot{J}_{2+} R(\mu, \nu, \kappa) = i \left\{ -\Xi_P P + X \Xi_X - \Xi_I I \right\}, \]  
(41)

\[ \Xi_P(t) = \dot{\xi}(t) e^{\nu} + 2 \dot{\xi}(t) \kappa e^{-\nu}, \quad \Xi_X(t) = \ddot{\xi}(t) e^{-\nu}, \quad \Xi_I(t) = \dot{\chi}(t) + \mu \dot{\chi}(t). \]  
(42)

The analogue of Eqs. (25) and (24) is

\[ \Xi_P \ddot{\Xi}_X - \ddot{\Xi}_P \Xi_X = -i. \]  
(43)

To obtain the basis of the \( su(1, 1) \) Lie subalgebra, we observe that

\[ M_{1-} = R^{-1}(\mu, \nu, \kappa) \dot{M}_{2-} R(\mu, \nu, \kappa), \quad M_{1+} = R^{-1}(\mu, \nu, \kappa) \dot{M}_{2+} R(\mu, \nu, \kappa), \]
\[ M_1 = R^{-1}(\mu, \nu, \kappa) \dot{M}_2 R(\mu, \nu, \kappa). \]  
(44)
Keeping in mind that the functions, $\mu$, $\nu$, and $\kappa$ are $t$-dependent, we find that

$$ M_{1-} = - \left\{ -C_{1,T}T + C_{1,P}T^2 + C_{1,D}D + C_{1,P}P + C_{1,X}X^2 + C_{1,X}X + C_{1,I}I \right\}, $$

$$ M_{1+} = - \left\{ -C_{2,T}T + C_{2,P}T^2 + C_{2,D}D + C_{2,P}P + C_{2,X}X^2 + C_{2,X}X + C_{2,I}I \right\}, $$

$$ M_{1} = - \left\{ -C_{3,T}T + C_{3,P}T^2 + C_{3,D}D + C_{3,P}P + C_{3,X}X^2 + C_{3,X}X + C_{3,I}I \right\}, \quad (45) $$

where, for $j = 1, 2, 3$, the coefficients are

$$ C_{j,T} = \hat{\phi}_j(t)e^{2\nu}, \quad (46) $$

$$ C_{j,P} = \hat{\phi}_j(t) \left( \frac{1}{2} k(t) - 4h^{(2)}(t)\kappa^2 \right) e^{2\nu} - \hat{\phi}_j(t)\kappa - \hat{\phi}_j(t)\kappa^2 e^{-2\nu}, \quad (47) $$

$$ C_{j,D} = \hat{\phi}_j(t) \left( -\frac{1}{2} h(t) + 4h^{(2)}(t)\kappa^2 \right) e^{2\nu} + \frac{1}{2} \hat{\phi}_j(t) + \hat{\phi}_j(t)\kappa e^{-2\nu}, \quad (48) $$

$$ C_{j,X} = \hat{\phi}_j(t) \left( -\frac{1}{2} g(t) + 2h^{(1)}(t)\kappa \right) e^{2\nu} + \hat{\phi}_j(t)\kappa e^{-2\nu}, $$

$$ C_{j,I} = \hat{\phi}_j(t) \left( -\frac{1}{2} g(t)\mu e^{-2\nu} - \hat{\phi}_j(t)\mu e^{-2\nu} + 2\hat{\phi}_j(t)\mu e^{-2\nu}, \quad (49) $$

$$ C_{j,X} = -\frac{1}{2} \hat{\phi}_j(t)e^{-2\nu}, \quad (50) $$

$$ C_{j,X} = -\hat{\phi}_j(t)e^{-\nu} + \hat{\phi}_j(t)\mu e^{-\nu}, \quad (51) $$

$$ C_{j,I} = \hat{\phi}_j(t) \left( h^{(0)}(t) + h^{(1)}(t)\mu e^{-2\nu} + h^{(2)}(t)\mu^2 e^{2\nu} \right) e^{2\nu}. \quad (52) $$

### 3.4 Commutation relations and algebraic structure

The commutation relations for the symmetry operators have been worked out previously \[3\]-\[5\] and the structure of the Lie algebra is known to be $su(1, 1)\Box w_1^c$. The nonzero $TO$ commutators are as follows: For the $w_1^c$ subalgebra:

$$ [J_{3-}, J_{3+}] = I, \quad (53) $$

For the $su(1, 1)$ subalgebra:

$$ [M_{3+}, M_{3-}] = -M_3, \quad [M_3, M_{3-}] = -2M_3, \quad [M_3, M_{3+}] = +2M_3. \quad (54) $$
The nonzero commutators involving operators from each of the two subalgebras are
\[
\begin{align*}
[M_3, J_{3-}] &= -J_{3-}, & [M_3, J_{3+}] &= +J_{3+}, \\
[M_{3-}, J_{3+}] &= -J_{3-}, & [M_{3+}, J_{3-}] &= +J_{3+}.
\end{align*}
\]
(55)

Since commutation relations are preserved by each segment of the transformation \((TO \to TM)\) and \((TM \to TQ)\), the Lie algebras of operators associated with \(TM\) Schrödinger equations and \(TQ\) Schrödinger equations are isomorphic to \(su(1,1) \sqcup \omega_1^c\). We take advantage of this isomorphism to define a set of generic operators, \(\{M, M_{\pm}, J_{\pm}, I\}\), where the subset \(\{M, M_{\pm}\}\) forms a subalgebra with the \(su(1,1)\) structure:
\[
\begin{align*}
[M_+, M_-] &= -M, & [M, M_{\pm}] &= \pm 2M_{\pm},
\end{align*}
\]
and the subset \(\{J_{\pm}, I\}\) forms a subalgebra with the \(\omega_1^c\) structure:
\[
[J_-, J_+] = I.
\]
(57)

The nonzero commutation relations between operators from each of the two subalgebras are
\[
\begin{align*}
[M, J_{\pm}] &= \pm J_{\pm}, & [M_-, J_+] &= -J_-, & [M_+, J_-] &= +J_+.
\end{align*}
\]
(58)

The operators \(M\) and \(M_{\pm}\) are identified with \(M_j\) and \(M_{j\pm}\), respectively, and the \(J_{\pm}\) are identified with \(J_{j\pm}\). This is for \(j = 1, 2, 3\), with or without hats.

4 Eigenstates of the Number Operator

4.1 Casimir Operators

In the following analysis, we do not require the operators \(M_{\pm}\). We shall consider only the subalgebra consisting of the operators \(\{M, J_{\pm}, I\}\), satisfying the nonzero commutation relations
\[
\begin{align*}
[M, J_{\pm}] &= \pm J_{\pm}, & [J_-, J_+] &= I,
\end{align*}
\]
and its representation spaces. Regardless of the system we are working with, we refer to this subalgebra as the oscillator subalgebra, denoted by \(os(1)\), with one Casimir operator
\[
C = J_+ J_- - MI.
\]
(60)
For the $TQ$ class of equations, we have the expressions

$$J_{1+}J_{1-} = -\frac{1}{2}\hat{\phi}_3(t)e^{2\nu S_1} + M_1 - \frac{1}{2},$$  \hspace{1cm} (61)

$$C_1 = J_{1+}J_{1-} - M_1 I = -\frac{1}{2}\hat{\phi}_3(t)e^{2\nu S_1} - \frac{1}{2},$$  \hspace{1cm} (62)

where $S_1$ is the $TQ$ Schrödinger operator in Eq. (1). The operators, $J_{1\pm}$ are given in Eqs. (40) and (41) and $M_1$ is found in Eq. (45).

The expression analogous to Eq. (61) for the $TM$ class of equations is

$$\hat{J}_{2+}\hat{J}_{2-} = -\frac{1}{2}\hat{\phi}_2(t)e^{2\nu \hat{S}_2} + \hat{M}_2 - \frac{1}{2},$$  \hspace{1cm} (63)

where $\hat{S}_2$ is the $TM$ Schrödinger operator from Eq. (3). The operators $\hat{J}_{2\pm}$ and $\hat{M}_2$, defined in Eqs. (31) and (36), are members of the $TM \, os(1)$ algebra. Its Casimir operator is

$$\hat{C}_2 = \hat{J}_{2+}\hat{J}_{2-} - \hat{M}_2 I = -\frac{1}{2}\hat{\phi}_3(t)e^{2\nu \hat{S}_2} - \frac{1}{2}.$$  \hspace{1cm} (64)

Similarly, as shown in Refs. [3, 5], the Casimir operator for the $TO \, os(1)$ Lie algebra is

$$C_3 = J_{3+}J_{3-} - M_3 I = -\frac{1}{2}\phi_{3}(t')S_3 - \frac{1}{2}.$$  \hspace{1cm} (65)

The second equality follows from the relationship

$$J_{3+}J_{3-} = -\frac{1}{2}\phi_{3}(t')S_3 + M_3 - \frac{1}{2},$$  \hspace{1cm} (66)

where $S_3$ is the $TO$ Schrödinger operator from Eq. (5). The operators $J_{3\pm}$ and $M_3$, defined in Eqs. (23) and (27), are members of the $TO \, os(1)$ algebra.

### 4.2 Number States

Previously, we showed [3] that certain states of the time-dependent oscillator equation (5) form a representation space for the oscillator algebra, $os(1)$. Since the representation spaces depend primarily upon the algebraic structure of the algebra, the same will hold true for the $TM$ and $TQ$ equations. In this representation, the operators $M$ and $C$ are diagonal. If $\mathbb{Z}_0^+$ denotes the set of nonnegative integers and if we denote the spectrum of the operator $M$ by $\text{Sp}(M)$, then

$$\text{Sp}(M) = \left\{ n + \frac{1}{2}, \, n \in \mathbb{Z}_0^+ \right\},$$  \hspace{1cm} (67)
and the spectrum is bounded below. Let \( \{ \Omega_n, n \in \mathbb{Z}_0^+ \} \) be a basis for this representation space. Each vector \( \Omega_n \) in this set is an eigenvector of the operator \( M \) with eigenvalue \( n + 1/2 \).

The extremal state, \( \Omega_0 \), is annihilated by the operator \( J_- \), that is

\[
J_- \Omega_0 = 0. \tag{68}
\]

Furthermore, by requiring that each \( \Omega_n \) be a solution to an appropriate Schrödinger equation in each class, Eq. (65) implies that, for all \( n \in \mathbb{Z}_0^+ \),

\[
C \Omega_n = -\frac{1}{2} \Omega_n. \tag{69}
\]

The action of the basis of the \( TO \) Lie algebra on the vectors in the representation space is

\[
M \Omega_n = \left( n + \frac{1}{2} \right) \Omega_n, \tag{70}
\]

\[
J_+ \Omega_n = \sqrt{n + 1} \Omega_{n+1}, \quad J_- \Omega_n = \sqrt{n} \Omega_{n-1}, \tag{71}
\]

for \( n \in \mathbb{Z}_0^+ \). We indicate this irreducible representation by the symbol \( \uparrow_{-1/2} \), where the subscript is the eigenvalue of the Casimir operator \( C \). From the extremal state, \( \Omega_0 \), we can obtain all higher-order states by repeated application of the raising operator, \( J_+ \):

\[
\Omega_n = \sqrt{\frac{1}{n!}} (J_+)^n \Omega_0. \tag{72}
\]

The states \( \Omega_n \) are also eigenstates of the operator \( J_+ J_- \) with eigenvalue \( n \). We shall refer to the eigenfunctions \( \Omega_n \) as number states. We emphasize that the number states are generally not eigenfunctions of a Hamiltonian. Therefore, we do not refer to the extremal state, \( \Omega_0 \), as a ground state nor to the higher-order states as excited states. We reserve the terms ‘ground state’ and ‘excited state’ for states that are energy eigenstates of the system.

### Table 1. Generic symbols and their values according to class.

|   |   |   |   |   |
|---|---|---|---|---|
|   |   |   |   |   |
| \( TQ \) | \( M_1 \) | \( C_1 \) | \( J_{1+} \) | \( J_{1-} \) | \( \Phi_n \) |
| \( TM \) | \( M_2 \) | \( C_2 \) | \( J_{2+} \) | \( J_{2-} \) | \( \Theta_n \) |
| \( TO \) | \( M_3 \) | \( C_3 \) | \( J_{3+} \) | \( J_{3-} \) | \( \Psi_n \) |
For convenience, in Table 1, we present the connection between the generic symbols, the operators, and states for each class of Schrödinger equation. Recall that the eigenvectors \( \Phi_n \) and \( \hat{\Theta}_n \) are related by \( \hat{\Theta}(x, t) = R(\mu, \nu, \kappa)\Phi(x, t) \) [see Eqs. (I-36) and (I-65)] while \( \hat{\Theta}_n \) and \( \Psi_n \) are connected through \( \Psi(x, t') = (\hat{\Theta} \circ t)(t') \) [see Eqs. (I-50) and (I-65)].

5 Coherent States

5.1 The Displacement Operator

In this and the following section, we shall continue to use generic symbols where convenient. In addition, we shall write the operators \( J_- \) as

\[
J_- = i \{ G_P P - G_X X + G_I I \}, \quad J_+ = i \{ -G_P P + G_X X - G_I I \},
\]

(73)

where the functions \( G_P, G_X, \) and \( G_I \) are given in Table 2. We have used Eq. (42) for the definitions of the TQ functions. For convenience, we have dropped the prime on the variable \( t \), since we do not make explicit use of the relationship between \( t' \) and \( t \) in this and the following section. The coefficients, \( G_P \) and \( G_X \), satisfy the relationship

\[
G_P G_X - G_P G_X = -i.
\]

(74)
In essence, this contains expressions (25), (34), and (43).

With this background, we define generic displacement-operator coherent states (DOCS), 
\( \Omega_\alpha \), for \( \text{os}(1) \)-type systems in the usual way \([6]-[8]\):

\[
\Omega_\alpha = D(\alpha)\Omega_0. \tag{75}
\]

\( \alpha \) is a complex parameter, \( \Omega_0 \) from Table 1 is a generic extremal state, and 
\( D(\alpha) \) is a generic displacement operator

\[
D(\alpha) = \exp [\alpha J_- - \bar{\alpha} J_+]. \tag{76}
\]

\( D(\alpha) \) is unitary since \( J_- \) and \( J_+ \) are Hermitian conjugates and \( \alpha \) is a complex parameter.

By expressing the operators \( x = X \) and \(-i\partial_x = P \) in terms of \( J_- \) and \( J_+ \), we can compute the expectation values in the usual way. Using Eqs. (73) to (74), we find that

\[
X = \bar{G}_P J_- + G_P J_+ + iF_P I, \tag{77}
\]

\[
P = \bar{G}_X J_- + G_X J_+ + iF_X I. \tag{78}
\]

The purely imaginary functions \( F_P \) and \( F_X \) of Table 2 are defined as

\[
F_P = G_P \bar{G}_I - \bar{G}_P G_I, \quad F_X = G_X \bar{G}_I - \bar{G}_X G_I, \tag{79}
\]

and specific values of these two functions for the three classes of systems are given in Table 2.

### 5.2 Position and Momentum Expectation Values

To calculate expectation values for position and momentum we have

\[
\langle x(t) \rangle = \langle \Omega_\alpha | X | \Omega_\alpha \rangle = \langle \Omega_0 | D^{-1}(\alpha) XD(\alpha) | \Omega_0 \rangle = \alpha \bar{G}_P + \bar{\alpha} G_P + iF_P, \tag{80}
\]

\[
\langle p(t) \rangle = \langle \Omega_\alpha | P | \Omega_\alpha \rangle = \langle \Omega_0 | D^{-1}(\alpha) PD(\alpha) | \Omega_0 \rangle = \alpha \bar{G}_X + \bar{\alpha} G_X + iF_X. \tag{81}
\]

To evaluate \( D^{-1}(\alpha) XD(\alpha) \) and \( D^{-1}(\alpha) PD(\alpha) \) in Eqs. (80) and (81) we used the Eqs. (74) and (78), the unitarity of \( D(\alpha) \), Eq. (89), and the commutation relations (I-8) through (I-10).

Let \( x_o \) and \( p_o \) be initial position and momentum:

\[
x_o = \langle x(t_o) \rangle, \quad p_o = \langle p(t_o) \rangle. \tag{82}
\]
Placing a superscript ‘o’ on $G_P$, $G_X$, $F_P$, and $F_X$ (and their corresponding values in Table 2) to indicate $t = t_o$, and using Eq. (74), we find that

$$\alpha = i (G_P^o p_o - G_X^o x_o) + G_P^o F_X^o - G_X^o F_P^o. \quad (83)$$

Substituting Eq. (83) for $\alpha$ into Eqs. (84) and (85), we obtain expressions for the expectation values of $X$ and $P$ in terms of $x_o$ and $p_o$:

$$\langle x(t) \rangle = i (\bar{G}_P G_P^o - G_P \bar{G}_P^o) p_o + i (G_P \bar{G}_X^o - \bar{G}_P G_X^o) x_o$$
\hspace{1cm} + i G_P (\bar{G}_I - \bar{G}_I^o) - i G_P (G_I - G_I^o), \quad (84)$$

$$\langle p(t) \rangle = i (\bar{G}_X G_P^o - G_X \bar{G}_P^o) p_o + i (G_X \bar{G}_X^o - \bar{G}_X G_X^o) x_o$$
\hspace{1cm} + i G_X (\bar{G}_I - \bar{G}_I^o) - i G_X (G_I - G_I^o), \quad (85)$$

where we have used the definitions in Eqs. (74) and (75).

For each of the three classes, we can combine Eqs. (84), and (85) with the functions in Table 2 to obtain explicit expectation values:

**TQ** : \( \langle x(t) \rangle = i \left[ \left( \xi_+ \xi_- \right) + \xi_0 \xi_+ \right] p_o + i \left[ \xi_0 (\bar{\xi}_+ \bar{\xi}_-) p_o + i \left( \bar{\xi}_0 (\bar{\xi}_+ \bar{\xi}_-) \right) x_o + i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) \right] \) \hspace{1cm} (86)

\( \langle p(t) \rangle = i \left[ \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right] p_o + i \left( \bar{\xi}_0 \xi_+ \right) x_o + i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) \) \hspace{1cm} (86)

**TM** : \( \langle x(t) \rangle = i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) p_o + i \left( \bar{\xi}_0 (\bar{\xi}_+ \bar{\xi}_-) \right) x_o + i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) \) \hspace{1cm} (87)

\( \langle p(t) \rangle = i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) p_o + i \left( \bar{\xi}_0 (\bar{\xi}_+ \bar{\xi}_-) \right) x_o + i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) \) \hspace{1cm} (87)

**TO** : \( \langle x(t) \rangle = i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) p_o + i \left( \bar{\xi}_0 (\bar{\xi}_+ \bar{\xi}_-) \right) x_o + i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) \) \hspace{1cm} (88)

\( \langle p(t) \rangle = i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) p_o + i \left( \bar{\xi}_0 (\bar{\xi}_+ \bar{\xi}_-) \right) x_o + i \left( \xi_0 (\bar{\xi}_+ \bar{\xi}_-) \right) \) \hspace{1cm} (88)
5.3 Uncertainties

Now, we compute the uncertainty product for the general case. If we take the uncertainty of an operator $A$ as

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2,$$

then we find that

$$(\Delta x)^2 = G_P G_P, \quad (\Delta p)^2 = G_X G_X,$$

which are both real and positive quantities. Therefore, the uncertainty product has the form

$$(\Delta x)^2 (\Delta p)^2 = G_P G_P G_X G_X = \frac{1}{4} \left[ 1 + (G_P G_X + \bar{G}_P \bar{G}_X)^2 \right].$$

This is both real and always greater than or equal to $1/4$. (We used Eq. (74) in the calculation of the second equality.)

We delay presentations of the particular uncertainties and uncertainty products for each of the classes of systems until the end of the corresponding section for squeezed states.

6 Squeezed States

6.1 The Squeeze Operator

Define the operators

$$K_- = \frac{1}{2} J_-^2, \quad K_+ = \frac{1}{2} J_+^2, \quad K_3 = J_+ J_- + \frac{1}{2},$$

which satisfy the commutation relations

$$[K_+, K_-] = -K_3, \quad [K_3, K_\pm] = \pm 2 K_\pm.$$

Calculating their commutation relations with $J_\pm$, and using Eq. (54), we find that

$$[K_-, J_-] = 0, \quad [K_+, J_-] = -J_+, \quad [K_3, J_-] = -J_-, \quad [K_-, J_+] = J_-, \quad [K_+, J_+] = 0, \quad [K_3, J_+] = +J_+.$$

The algebra of operators, $\{K_\pm, K_3, J_\pm, I\}$,
has the $su(1,1) \Box w_1^1$ structure.

We define a generalized displacement-operator squeezed state [5, 10, 15], $\Omega_{\alpha,z}$, as follows:

$$\Omega_{\alpha,z} = D(\alpha)S(z)\Omega_0,$$  \hspace{1cm} (95)

where $D(\alpha)$ is given in Eq. (76) and $S(z)$ is the squeeze operator

$$S(z) = \exp (zK_+ - \bar{z}K_-).$$  \hspace{1cm} (96)

The parameter $z$ is complex. For computational purposes, it is more convenient to write the squeeze operator in the form of “canonical coordinates of the second kind” [5]. A Baker-Campbell-Hausdorff relation [5, 9, 16] gives this form as

$$S(z) = \exp (\gamma_+ K_+) \exp (\gamma_3 K_3) \exp (\gamma_- K_-),$$  \hspace{1cm} (97)

where

$$\gamma_- = -\frac{\bar{z}}{|z|} \tanh |z|, \quad \gamma_+ = \frac{z}{|z|} \tanh |z|, \quad \gamma_3 = -\ln (\cosh |z|),$$  \hspace{1cm} (98)

$$z = re^{i\theta}, \quad r = |z|.$$  \hspace{1cm} (99)

6.2 Position and Momentum Expectation Values

To compute expectation values of position and momentum, we again employ the operators (77) and (78) and follow the same method of calculation as in Ref. [5].

$$\langle x(t) \rangle = \langle \alpha, z|X|\alpha, z \rangle = \langle 0|S^{-1}(z)D^{-1}(\alpha)|X|D(\alpha)S(z)|0 \rangle,$$  \hspace{1cm} (100)

$$\langle p(t) \rangle = \langle \alpha, z|P|\alpha, z \rangle = \langle 0|S^{-1}(z)D^{-1}(\alpha)|P|D(\alpha)S(z)|0 \rangle.$$  \hspace{1cm} (101)

Making use of Eq. (93) we obtain the adjoint action of the group operators $S(z)$ and $D(\alpha)$ on $X$ and $P$ respectively,

$$S^{-1}(z)D^{-1}(\alpha)XD(\alpha)S(z) = X_{X,-}J_- + X_{X,+}J_+ + X_{X,I}I,$$  \hspace{1cm} (102)

$$S^{-1}(z)D^{-1}(\alpha)PD(\alpha)S(z) = X_{P,-}J_- + X_{P,+}J_+ + X_{P,I}I.$$  \hspace{1cm} (103)
The coefficients of the operators in these two expressions are

\[ X_{-} = \bar{G}_P (e^{\gamma} - \gamma^+ e^{-\gamma}) - G_P e^{-\gamma} = \bar{G}_P \cosh r + G_P e^{-i\theta} \sinh r, \]
\[ X_{+} = \bar{G}_P \gamma^+ e^{-\gamma} + G_P e^{-\gamma} = \bar{G}_P e^{i\theta} \sinh r + G_P \cosh r, \]
\[ X_{I} = \alpha \bar{G}_P + \bar{\alpha} G_P + iF_p, \quad (104) \]
\[ X_{-} = \bar{G}_X (e^{\gamma} - \gamma^+ e^{-\gamma}) - G_X e^{-\gamma} = \bar{G}_X \cosh r + G_X e^{-i\theta} \sinh r, \]
\[ X_{+} = \bar{G}_X \gamma^+ e^{-\gamma} + G_X e^{-\gamma} = \bar{G}_X e^{i\theta} \sinh r + G_X \cosh r, \]
\[ X_{I} = \alpha \bar{G}_X + \bar{\alpha} G_X + iF_X, \quad (105) \]

where \( G_X \) and \( G_P \) are given in Table 2.

Combining Eqs. (100) - (105), we obtain the equations (80) and (81) for \( \langle x(t) \rangle \) and \( \langle p(t) \rangle \), respectively. Identifying an initial position and momentum, as in Eq. (82), we end up with Eqs. (84) and (85) for \( \langle x(t) \rangle \) and \( \langle p(t) \rangle \) in terms of \( x_o \) and \( p_o \). The expectation values for position and momentum for each of the three classes of equations are given in Eqs. (86) through (88).

### 6.3 Uncertainties

To obtain the squeezed-state uncertainty products, we proceed in the same way as we did for the coherent states, but with the operators (102) and (103). The uncertainties in position and momentum are

\[ (\Delta x)^2 = \frac{1}{2} \left( \bar{G}_P^2 e^{-i\theta} + G_P^2 e^{i\theta} \right) \sinh 2r + G_P \bar{G}_X \cosh 2r, \quad (106) \]
\[ (\Delta p)^2 = \frac{1}{2} \left( \bar{G}_X^2 e^{i\theta} + G_X^2 e^{-i\theta} \right) \sinh 2r + G_X \bar{G}_X \cosh 2r. \quad (107) \]

These are both real and positive since \( G_P e^{-i\theta}/2 + \bar{G}_P e^{i\theta}/2 \) and \( G_X e^{-i\theta}/2 + \bar{G}_X e^{i\theta}/2 \) are both real. Particular expressions \( (\Delta x)^2 \) and \( (\Delta p)^2 \), for the \( TQ \), \( TM \), and \( TM \) systems, can be obtained by using the values of \( G_P \) and \( G_X \) in Table 2.

After some manipulation, we obtain the following expression for the uncertainty product:

\[ (\Delta x)^2 (\Delta p)^2 = \frac{1}{4} \left\{ 1 + [(G_P \bar{G}_X + \bar{G}_P G_X) \cosh 2r \right. \]
\[ + \left. (G_P G_X e^{-i\theta} + \bar{G}_P \bar{G}_X e^{i\theta}) \sinh 2r \right]^2 \} \quad (108) \]
As for the coherent states, we have used the identity (74) to aid in obtaining this result.

Note that coefficients of \( \cosh 2r \) in Eqs. (106) and (107) and of \( \cosh^2 2r \) in Eq. (108), are identical to their respective coherent-state expressions (90) and (91). Also, as expected, the uncertainties and uncertainty products for the coherent states can be reclaimed by setting the squeezing parameters \( r = \theta = 0 \) in the above.

Finally, from Table 2, the uncertainty products for the three classes of equations are:

\[
TQ : \quad (\Delta x)^2 (\Delta p)^2 = \frac{1}{4} \left\{ 1 + \frac{1}{4} \left[ \left( \hat{\phi}_3 + 8 \xi \kappa e^{-2\nu} \right) \cosh 2r \right. \right. \\
\left. \left. + \left( [\hat{\phi}_1 + 4 \xi^2 \kappa e^{-2\nu}] e^{-i\theta} + [\hat{\phi}_2 + 4 \xi^2 \kappa e^{-2\nu}] e^{i\theta} \right) \sinh 2r \right]^2 \right\}. \tag{109}
\]

\[
TM : \quad (\Delta x)^2 (\Delta p)^2 = \frac{1}{4} \left\{ 1 + \frac{1}{4} \left[ \hat{\phi}_3 \cosh 2r + \left( \hat{\phi}_1 e^{-i\theta} + \hat{\phi}_2 e^{i\theta} \right) \sinh 2r \right]^2 \right\}. \tag{110}
\]

The functions \( \hat{\phi}_j(t), j = 1, 2, 3 \), are in Eq. (34).

\[
TO : \quad (\Delta x)^2 (\Delta p)^2 = \frac{1}{4} \left\{ 1 + \frac{1}{4} \left[ \hat{\phi}_3 \cosh 2r + \left( \hat{\phi}_1 e^{-i\theta} + \hat{\phi}_2 e^{i\theta} \right) \sinh 2r \right]^2 \right\}. \tag{111}
\]

The functions \( \hat{\phi}_j(t'), j = 1, 2, 3 \), are in Eq. (28).

For the harmonic oscillator [5], \( \hat{\phi}_3 = 0, \hat{\phi}_1 = i \exp [2i\omega(t - t_o)], \hat{\phi}_2 = -i \exp [-2i\omega(t - t_o)]. \) (Recall we ignore primes for time in this section.) The uncertainty product becomes

\[
(\Delta x)^2 (\Delta p)^2 = \frac{1}{4} \left[ 1 + \frac{1}{4} \left( s^2 - \frac{1}{s^2} \right)^2 \sin^2 [2\omega(t - t_o) - \theta] \right], \quad s = \exp r. \tag{112}
\]

This expression is well known. See, e.g., Eq. (86) in Ref. [7].

7 Discussion

7.1 Uncertainty Relations

We have been considering Heisenberg-Weyl algebras with \( J_-, J_+ \), and \( I \) satisfying the appropriate commutation relations. Now define

\[
\mathcal{X} = \frac{J_- + J_+}{\sqrt{2}}, \quad \mathcal{P} = \frac{J_ - J_+}{i\sqrt{2}}. \tag{113}
\]
Then, the coherent states we have defined [see Eqs. (75) and (76)],

\[ |\alpha \rangle = D(\alpha)|0\rangle, \quad D(\alpha) = \exp[\alpha J_+ - \bar{\alpha} J_-], \quad (114) \]

minimize the Heisenberg uncertainty relation

\[ (\Delta \mathcal{X})^2(\Delta \mathcal{P})^2 \geq 1/4. \quad (115) \]

In addition, in Eq. (112), we have have defined the $su(1,1)$ algebra with $\mathcal{K}_-, \mathcal{K}_+$, and $\mathcal{K}_3$ satisfying the appropriate commutation relations among themselves and with the HW algebra.

Then the squeezed states [see Eq. (95)],

\[ |\alpha, z \rangle = D(\alpha)S(z)|0\rangle, \quad S(z) = \exp[z\mathcal{K}_- - \bar{z}\mathcal{K}_+], \quad (116) \]

minimize the Schrödinger-Robertson uncertainty relation

\[ (\Delta \mathcal{X})^2(\Delta \mathcal{P})^2 \geq \frac{1}{4} + \frac{1}{4} |\langle \{ \mathcal{X} - \langle \mathcal{X} \rangle, \mathcal{P} - \langle \mathcal{P} \rangle \} \rangle|^2, \quad (117) \]

where $\{ , \}$ is the anticommutator [18].

Note that $x$ and $p$ are not $\mathcal{X}$ and $\mathcal{P}$, but linear combinations of them and $I$, with multiplicative coefficients. Therefore, although the uncertainty products of $x$ and $p$ are correct and physically relevant, they do not necessarily satisfy the equalities in Eqs. (115) and (117). In fact, they tend not to, except possibly for particular times (such as $t = t_0$). They do, however, often tend to be close to minimum uncertainties.

### 7.2 The Classical Motion

For coherent and squeezed states, $\langle x(t) \rangle$ and $\langle p(t) \rangle$ should obey the classical Hamiltonian equations of motion:

\[ \dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}, \quad (118) \]

The ‘dot’ indicates differentiation by $t'$ for $TO$ systems and differentiation by $t$ for $TM$ and $TQ$ systems, and $t$ and $t'$ are related through Eq. (I-67).
The classical Hamiltonians associated with each class of Schrödinger equations are

\[ \text{TO : } H = \frac{p^2}{2} + g(2)(t')x^2 + g(1)(t')x + g(0)(t'), \]  

\[ \text{TM : } \dot{H} = e^{-2\nu p^2} + f(2)(t)x^2 + f(1)(t)x + f(0)(t), \]  

\[ \text{TQ : } H = [1 + k(t)]\frac{p^2}{2} - h(t)x + \frac{g(t)}{2}p + h(2)(t)x^2 + h(1)(t)x + h(0)(t). \]  

Putting these Hamiltonians into Eqs. (118) one finds

\[ \text{TO : } \dot{x} = p, \quad \dot{p} = -2g(2)(t')x - g(1)(t'), \]  

\[ \text{TM : } \dot{x} = e^{-2\nu x}, \quad \dot{p} = -2f(2)(t)x - f(1)(t), \]  

\[ \text{TQ : } \dot{x} = [1 + k(t)]p - \frac{h(t)}{2}x - \frac{g(t)}{2}, \quad \dot{p} = -\frac{h(t)}{2}p - 2h(2)(t)x - h(1)(t). \]  

Now consider the expectation values \( \langle x \rangle \) and \( \langle p \rangle \) in Eqs. (86) through (88) and their time derivatives. Making extensive use of Eqs. (I-32), (I-33), (I-59)-(I-61), (I-66), (25), (34), and (35), one can demonstrate that these quantities satisfy Eqs. (122) to (124) with \( x \rightarrow \langle x \rangle \) and \( p \rightarrow \langle p \rangle \). Thus, the classical motion is satisfied.

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