Theoretical Limb Darkening for Classical Cepheids:
II. Corrections for the Geometric Baade-Wesselink Method.

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ABSTRACT

The geometric Baade-Wesselink method is one of the most promising techniques for obtaining a better calibration of the Cepheid period-luminosity relation by means of interferometric measurements of accurate diameters. In this paper we present new wavelength- and phase-dependent limb darkening corrections based on our time-dependent hydrodynamic models of the classical Cepheid ζ Gem. We show that a model simulation of a Cepheid atmosphere, taking into account the hydrodynamic effects associated with the pulsation, shows strong departures from the limb darkening otherwise predicted by a static model. For most of its pulsational cycle the hydrodynamic model predicts a larger limb darkening than the equivalent static model. The hydrodynamics affects the limb darkening mainly at UV and optical wavelengths. Most of these effects evolve slowly as the star pulsates, but there are phases, associated with shocks propagating into the photosphere, in which significant changes in the limb darkening take place on time-scales of the order of less than a day. We assess the implication of our model LD corrections fitting the geometric Baade-Wesselink distance of ζ Gem for the available near-IR PTI data. We discuss the effects of our model limb darkening on the best fit result, and analyze the requirements needed to test the time-dependence of the limb darkening with future interferometric measurements.

1We are saddened to report that our colleague, Costas Papaliolios, died 2002 June 6.
1. Introduction

With the recent advances in interferometry, the Baade-Wesselink (BW) method (Baade 1926; Wesselink 1946) has become one of the most promising techniques to obtain independent distance measurements of pulsating stars. A “geometric” variation of this classical method (Sasselov & Karovska 1994) can in principle be used to derive the distances of nearby classical Cepheids from the variations, as they pulsate, of their angular diameter and radial velocity. The potential accuracy of this method can improve the calibration of the Period-Luminosity relation of Classical Cepheids (Leavitt 1906), which is a fundamental step in the cosmological distance ladder.

The geometric BW method has been recently applied on two Cepheids in the northern hemisphere, ζ Gem (Lane et al. 2000) and η Aql (Lane et al. 2002). Both stars were observed at near-IR wavelength (H-band) with the Palomar Testbed Interferometer (PTI). The uniform brightness disk diameter was measured at several epochs for both stars, with enough precision to detect the changes in the stellar radius due to the pulsation. The measured angular diameters were then fitted with an appropriate model derived from the radial velocities of the two stars, obtaining the distance with an estimated accuracy of \( \sim 10\% \).

As explained in detail by Lane et al. (2002), the main source of uncertainty in the measurements is related to (1) the conversion of photospheric line velocities into radial motion (the so-called projection factor, or \( p \)-factor), and (2) the estimate of the limb darkening (LD). These two quantities can be derived by modeling the Cepheid atmosphere, taking into account the spherical geometry of the star and the time-dependent hydrodynamics of the pulsations. The correct way to determine the \( p \)-factor for pulsating Cepheids have been described in detail in Sabbey et al. (1995) and Krockenberger et al. (1997). More recently, we have presented a new method for computing accurate time- and wavelength-dependent center-to-limb brightness distributions for classical Cepheids (Marengo et al. 2002, hereafter paper I).

The model described in paper I provides a significant improvement of the limb darkening coefficients with respect to the tabulated values currently used to analyze interferometric data. These tables (Parsons 1971; Manduca 1979; Kurucz 1993a; Claret et al. 1995) are based on hydrostatic model atmospheres of non-pulsating yellow supergiants, having similar \( T_{\text{eff}} \) and \( \log g \) as classical Cepheids. Our models have the advantage of being specific for each
simulated Cepheid, and provide the appropriate limb darkening for any pulsational phase at arbitrary wavelength. As shown in paper I, the limb darkening predicted by our models is significantly different from the one expected for a static yellow supergiant.

In this paper we analyze how the inclusion of hydrodynamic effects in our Cepheid models can affect the interferometric distance determination with the geometric BW method. We first compute the limb darkening corrections as a function of wavelength and pulsational phase (section 2), starting from the center-to-limb intensity profiles presented in paper I for the classical Cepheid ζ Gem. Then, we derive a new estimate for the geometric BW distance of this star, correcting the PTI data presented by Lane et al. (2002) with our phase and wavelength dependent LD corrections (section 3). The accuracy of the best fit values, and the importance of various error sources, are described in detail. We then conclude by analyzing the extent of the corrections induced by our models in the geometric BW distances, and the level of accuracy that will be required by present and future interferometers to detect the hydrodynamic effects predicted by our models.

2. Limb Darkening Corrections

The models described in paper I consist of a series of synthetic atmospheres computed with second-order one-dimensional hydrodynamic calculations, performed assuming a spherical symmetry. Each model simulates the atmosphere of a Cepheid at a certain pulsational phase. In paper I we have described the general procedure to obtain such models, and have shown the specific results for the classical Cepheid ζ Gem.

The pulsational period of ζ Gem (of approximately 10.15 days) was covered by a total of 49 individual models (the time-step in our model grid is determined by the convergence criteria in the hydrodynamic simulation). The time resolution of our model sequence is thus ~ 0.2 days. For each model in the sequence, we have computed realistic spectral intensity distributions. This step was done by approximating the dynamic models with a plane-parallel atmosphere. The radiative transfer problem for the static atmosphere was then solved in LTE conditions using the ATLAS code (Kurucz 1970, 1979, 1993b) and its opacity library. The end result of this procedure is a set of 49 spectral intensity distributions describing the ζ Gem spectrum as it changes while the star pulsates.

In paper I we showed how this procedure allows us to compute accurate limb darkening profiles for ζ Gem, which are (pulsational) phase and wavelength dependent. We proceed here to the next step, which is to describe how these models can be used in interferometry.

Interferometric measurements are usually expressed in terms of uniform intensity disk
(UD) or *limb darkened* diameters. The UD diameters are derived by fitting the normalized fringe visibilities (squared) $V^2$ with a uniform disk model, which assumes that the star is a disk of uniform brightness:

$$V_{UD}^2(\rho) = \left( \frac{2J_1(\pi \theta_{UD} \cdot \rho)}{\pi \theta_{UD} \cdot \rho} \right)^2$$

where $\theta_{UD}$ is the UD angular diameter and $\rho$ the spatial frequency of the measurement. Since a uniform disk model is derived assuming that the source is a disk of uniform brightness, it does not depend on the wavelength. Real stars, however, are limb darkened and have limb-to-center brightness distributions which are a function of wavelength, as shown in paper I.

For any practical purpose, we can consider the physical radius of the stellar photosphere to be a well defined quantity. The contribution functions of observable spectral features in the stellar spectra does peak at different heights in the atmosphere, but the difference is very small compared to the full diameter of the star. For this reason, we can assume that the angular diameter of a Cepheid is the same at all wavelengths, after being corrected for LD. This diameter, which we call “limb darkened diameter”, $\theta_{LD}(\phi)$, since it takes into account the fact that the star has LD, will thus only depend on the pulsational phase of the star, and not on $\lambda$. Fitting fringe visibilities with a UD model, on the other hand, will produce a different $\theta_{UD}$ according to the wavelength, because of the inability of UD models to take into account the spectral properties of the LD. For this reason the UD angular diameter measured from fringe visibilities of a LD star will instead depend on both the pulsational phase and the wavelength: $\theta_{UD}(\lambda, \phi)$.

To obtain the real angular diameter of a star from the fringe visibilities, one can either fit the $V^2$ data with an appropriate limb darkening model, or correct the UD measurement with a specific wavelength dependent limb darkening correction. The first approach is certainly preferable, because preserves the wavelength dependence of the measured visibilities, and thus allow a test the limb darkening models. There are cases, however, in which the original visibilities are not available, as for example when mixing heterogeneous data from different interferometers.

In this section we present accurate LD corrections derived using our models. They are based on hydrodynamic models and therefore they are phase and wavelength dependent. Even though the models are specific for $\zeta$ Gem, we use them here as an example to describe the more general case of LD corrections for the generic classical Cepheids, and their implications for interferometry and the BW method.

The conversion factor between UD and limb darkened diameters is usually defined as
\[ k(\lambda, \phi) = \theta_{UD}(\lambda, \phi)/\theta_{LD}(\phi). \] This factor is in many cases approximated by the model of a star having similar spectral type as the source of interest (in the case of Cepheids, a yellow supergiant, as described in Claret et al. 1995). Such approximations do not take into account the phase dependence for variable sources, and may be incorrect in their dependence on \( \lambda \), since the atmosphere of a Cepheid cannot be properly approximated with a static supergiant (see discussion in paper I).

The \( k(\lambda, \phi) \) corrections discussed in the following sections are derived with the following procedure. At each phase in the model grid, and for a set of wavelengths in the optical and near-IR range, the center-to-limb profiles computed in paper I are converted into fringe visibilities by applying the Hankel transform (see e.g. Koechlin 1988):

\[ V_{LD}^2(\rho, \phi) = \left[ 2 \int_0^\infty I_\nu(w, \lambda, \phi) J_0(w \rho) w \, dw \right]^2 \]

where \( \rho \) is the spatial frequency of the interferometric visibility, \( w = (\theta_{LD}/2) \cdot \sin(\alpha) \) is the projection of the stellar angular radius on the disk and \( I_\nu(w, \lambda, \phi) \) is the model stellar intensity spectrum at the projected radius, computed for the given pulsational phase \( \phi \).

The simulated \( V_{LD}^2 \) visibilities are then fitted with an UD \( V_{UD}^2 \) model, in order to derive the UD diameter \( \theta_{UD} \) which the interferometer would have measured, at each wavelength, for the model source. The limb darkening correction \( k(\lambda, \phi) \) is the ratio between the best fit \( \theta_{UD} \) and the value of the angular diameter \( \theta_{LD} = 1 \) mas that we set to give a physical dimension to the simulated visibilities in the Hankel transform. The simulated visibilities were computed over a spatial frequency range up to the first minimum, to match the typical conditions encountered when fitting real data with currently available interferometers. We have however tried several other combinations of model \( \theta_{LD} \) and spatial frequency ranges (to simulate different interferometer baselines), confirming that the resulting \( k(\lambda, \phi) \) are not very sensitive to these parameters, as long as the simulated star is at least partially resolved.

### 2.1. Spectral Properties of Limb Darkening

An example of the wavelength dependence of the LD correction \( k(\lambda) \) is shown in Figure 1. The figure shows the LD correction derived from our hydrodynamic simulations (thick line). The thin line is the LD correction obtained from hydrostatic atmospheres (Kurucz models) having the \( T_{\text{eff}}(\phi) \) and \( \log g(\phi) \) measured as a function of the pulsational phase by Krockenberger et al. (1997). Note that the \( T_{\text{eff}} \) of the dynamic and static models, at each pulsational phase, is the same, since the observational effective temperature was used as an
input parameter to compute the dynamic model. The log $g$ is instead different in the two cases, as a consequence of the procedure followed to solve the radiative transfer in the hydrodynamic case, and due to the specific definition of the gravity terms in the hydrodynamic equations.

The $k$ correction is strictly related to the spectral properties of the source. The limb darkening is higher (lower $k$) at UV and visible wavelengths, and converges toward unity (no limb darkening) with increasing wavelengths toward the infrared. Spectral lines are less limb darkened than the continuum: having higher optical depth, they probe upper atmospheric layers, where the temperature gradient is lower. Among the spectral features that are visible in our low spectral resolution computations, is the Balmer jump in the UV, the Ca H&K doublet ($\lambda \simeq 395$ nm), H$\beta$ ($\lambda \simeq 486$ nm) and H$\alpha$ ($\lambda \simeq 656$ nm).

The first panel in Figure 1 shows the limb darkening correction at minimum radius. At this phase, the Cepheid atmosphere is most compressed, giving rise to a steeper temperature gradient. This is responsible for an increase of the limb darkening at all wavelengths. The LD correction for the hydrostatic atmosphere, on the other hand, does not show this same effect, having a flatter limb darkening despite the fact that $T_{\text{eff}}$ is the same in the two models. The difference is as much as $\Delta k \simeq 0.01$ at visible wavelengths, and less in the near-IR. Accurate measurements of the LD with optical interferometers should then be able to verify this effect. The difference between the static and dynamic models is even larger at the wavelengths of the main spectral features, which appear stronger in the more compressed hydrodynamic atmosphere.

In the next section we will show that in most cases the effects of the hydrodynamics on the atmospheric structure result in a larger LD. There are however phases in which the free expansion of the atmosphere results in a quasi-static structure. In these phases (before and after maximum luminosity), the hydrodynamics is less important and the two $k(\lambda)$ are virtually indistinguishable.

As shown in the bottom panel, however, the full force of a shockwave crossing the photosphere results in high excitation states and local expansion. Even though at this phase $\zeta$ Gem is contracting (this happens one day before minimum radius), the energy deposited by the shock in the region where the visible photons are created generates a lower temperature gradient. As a consequence, the LD of the hydrodynamic simulation is lower than in the static case. Note, however, that this effect is mostly appreciable at visible wavelengths. An optical interferometer should detect at this phase a decrease of the limb darkening of $\Delta k \simeq 0.01$. This phase is very brief and, at least in the case of $\zeta$ Gem, the effects of the shock are already dissipated after less than 20 hours, when the LD is maximum again as the star approaches minimum radius. This timescale is related to the shock propagation speed,
which is very well constrained by observations (see Figure 8 of Sasselov & Lester 1994).

2.2. Limb Darkening and Pulsational Phase

By convolving the $k(\lambda, \phi)$ models with the filter passbands used by interferometers, we can study the variations of the LD correction as a function of the pulsational phase. For a convenient comparison with observations, the best choice is to adopt the phase as defined by the optical lightcurve, where the zero phase, $\phi_L = 0$, coincide with maximum luminosity. As explained in paper I, however, our dynamic simulations are computed as a function of the phase $\phi_V$ based on the pulsational velocity, in which zero phase is at minimum radius. To convert our LD models to the lightcurve phases we have used the phase shift $\Delta \phi = \phi_L - \phi_V$ between maximum luminosity and minimum radius computed by Bersier et al. (1994a,b).

Figure 2 shows the limb darkening corrections plotted as a function of the visible lightcurve phase for two wavelengths. The top panel shows $k(\phi)$ for the near-IR H filter passband used by the PTI interferometer. The bottom panel shows $k(\phi)$ for the bluest available channel ($\lambda \simeq 570$ nm) of the Navy Prototype Interferometer (NPOI) in the optical. The curves have been filtered to remove numerical noise with an adaptive gaussian kernel, and the 1 $\text{rms}$ error bands in our LD corrections associated to the numerical uncertainty are indicated by two thin lines bracketing each LD curve.

The plots show that $k(\phi)$ is roughly constant for most phases, with a slow increase as the star expands, as a consequence of the decreasing temperature gradient in the photosphere. Coincident with the passage of a shockwave through the photosphere (at $\phi_L \simeq 0.6$), $k(\phi)$ shows a sudden rise in the optical, indicative of a sharp decrease in the limb darkening. The energy deposited by the shock is responsible for this effect. It increases the excitation in the photospheric layers, and thus flattens the temperature gradient. This extra energy dissipates in a short timescale, after which the atmosphere resumes its normal state. At $\phi_L \simeq 0.7$ $\zeta$ Gem is close to minimum radius, where the temperature gradient is higher (due to the compression of the atmosphere) and thus the star appears more limb darkened.

The sharp increase in $k(\phi)$ observed in the optical, at the time of the shockwave, is not predicted for the H band. This is because in this wavelength range the effects of the shock are less pronounced. This might be due to the fact that the emergent spectrum in the H-band originates deepest in the atmosphere as a result of the broad minimum of the H$^{-}$ opacity at 1.6 $\mu$m, and the shocks increasingly steepen and disturb the atmosphere as they propagate down the density gradient (up into the atmosphere).

It is important to understand how much of this behavior in the LD curves is due to
hydrodynamic effects, or just to the changes in $T_{\text{eff}}$ and log $g$ during the stellar pulsation. Solid lines in Figure 3 show our hydrodynamic simulations for two wavelengths. The dotted lines show the LD correction derived for hydrostatic atmospheres having at each phase the $T_{\text{eff}}$ and log $g$ measured by Krockenberger et al. (1997), which we used as starting points in our models. The overall value of $k(\phi)$ is similar, at each wavelength, between the hydrodynamic and hydrostatic atmospheres. The phase dependence is instead completely different. Figure 4 reveals that static model LD corrections closely follow the change of $T_{\text{eff}}$ with phase. In a hydrostatic atmosphere a higher $T_{\text{eff}}$ is responsible for a flatter photospheric temperature gradient, and thus for less limb darkening (i.e. higher $k$). In a hydrodynamic atmosphere the temperature gradient is determined predominantly by the pulsation dynamics, and therefore the phase dependency of the LD is different. In a hydrostatic atmosphere, gravity seems to have little or no effect determining the changes in the limb darkening. When computing the spectral energy distribution of our dynamic atmosphere, we constrain the model $T_{\text{eff}}$ to the observed values, leaving log $g$ as a free parameter to be fitted with a grid of static atmospheres (see paper I). The rationale for this choice was that $T_{\text{eff}}$ has a dominant role in determining the spectral properties of the atmosphere. This result confirms the validity of our choice.

To test the assumption that log $g$ does play a minor role, we have also computed the LD correction for the static models having both $T_{\text{eff}}$ and log $g$ identical to the hydrodynamic simulations. The results are very similar. Despite the significantly different log $g$ with respect to the observed values, $k(\phi)$ still closely follows the variations of $T_{\text{eff}}$. Only just after minimum radius (when $T_{\text{eff}}$ is maximum and the best fit log $g$ is minimum), log $g$ plays some role in lowering the LD correction. This confirms that, except when log $g$ is very small ($\sim 1$, in this case), the effective temperature is the decisive parameter for the LD in a hydrostatic atmosphere.

In a hydrodynamic atmosphere, however, $T_{\text{eff}}$ is not the dominant parameter. As shown in Figure 3, when hydrodynamic effects are taken into account, the relation between $T_{\text{eff}}$ and the LD breaks. Shocks are the source of the most dramatic effects in $k(\phi)$, but even when they are absent, the LD corrections are strikingly different from the ones predicted by hydrostatic atmospheres. This is because they are not determined by $T_{\text{eff}}$, but by the time-dependent structure of the expanding/contracting atmosphere. Given that our hydrodynamic models generally have a steeper temperature gradient than a similar hydrostatic atmosphere, the resulting LD is larger for most phases.

Figure 3 also shows the LD corrections used in Lane et al. (2002) for $\zeta$ Gem, which has the constant value of $k = 0.96 \pm 0.01$. This correction has been derived from tables published by Claret et al. (1995), based on hydrostatic models. Note that the average value of our LD
correction is outside the error bars of the tabulated value. The reason of this discrepancy is that our model closely follows the $T_{\text{eff}}$ and $\log g$ derived for $\zeta$ Gem from spectroscopic observations, while the Claret et al. (1995) values are computed for a generic grid of yellow supergiants matching the Cepheids spectral type.

Note finally that, as expected, the LD correction is much smaller at near-IR wavelengths, and larger toward the blue. At bluer wavelengths the amplitude of the $k$ variations with phase (especially at the time of the shockwave) is also much larger. Therefore, to test observationally the details of the hydrodynamic effects on limb darkening, interferometers operating at short wavelengths are favored.

3. Effects of the LD corrections on the geometric BW method

The LD deviations introduced by the hydrodynamics are relatively small (1% in the optical, and less in the infrared), and are thus usually ignored when treating interferometric measurements. In order to obtain accurate distances with the BW geometric method, however, higher levels of accuracy are needed. Under such stringent requirements, the time and wavelength dependent hydrodynamic effects play an important role, and should be taken into account.

A detailed description of the problems involved with the application of the geometric BW method to pulsating Cepheids is given in Sasselov & Karovska (1994). This method allows the determination of the distance and the average radius of a pulsating star from both its angular diameter and radial velocity at several phases. This is done by means of a $\chi^2$ fit of the following function:

$$\chi^2 = \sum_i \left[ \frac{\Theta_0 + 2\Delta R_i}{\sigma_i} - \theta_i \right]^2$$  \hspace{1cm} (3)

where $\Theta_0 = 2R_0/D$ is the average angular radius of the star, and $\Delta R_i$ its radial displacement at the time of the interferometric measurement $\theta_i$. The variation of the stellar radius $\Delta R(\phi_L)$ is derived, as a function of the lightcurve phase $\phi_L$, by integrating the pulsational velocity over time:

$$\Delta R(\phi_L) = - \int_{\phi_0}^{\phi_L} p [v_r(\phi') - \gamma] \, d\phi'$$  \hspace{1cm} (4)
where \( v_r(\phi) \) is the radial velocity, which is corrected by the systemic velocity \( \gamma \) and the \( p \)-factor to yield the true pulsational velocity. The systemic velocity can be derived by requiring the conservation of the radius over one period (see paper I). Appropriate \( p \)-factors have been computed for pulsating Cepheids such as \( \zeta \) Gem; we use here the value of 1.43 (the same adopted by Lane et al. 2002), derived from the hydrodynamic model we used to compute our LD profiles, which was published in Sabbey et al. (1995).

The fit in equation 3 can be solved analytically. The best fit \( R_0 \) and \( D \) can be derived by minimizing the \( \chi^2 \) relation solving for the two unknown parameters. Note that we do not make use of the color curve of the Cepheid, which can be applied to further constrain the geometric BW solution. A detailed description of this procedure will be given in a separate paper.

The LD corrections \( k(\phi) \) computed in the previous sections enter the fit by allowing the conversion of the UD diameters \( \theta_{UD}^{(i)} \) obtained by the interferometer into true LD diameters of the star: \( \theta_i = \theta_{UD}^{(i)}/k(\phi_i) \), where \( k(\phi_i) \) is the LD correction at the phase of the \( i \)-th measurement, computed for the wavelength of the observation.

The terms \( \sigma_i \) in the fit \( \chi^2 \) relation are the errors associated with each data point. They are the geometric sum of the individual error sources (in terms of angular diameters) for each observation:

\[
\sigma_i^2 = \left[ \sigma_i^{(\theta)} \right]^2 + \left[ \sigma_i^{(\Delta R/D)} \right]^2 + \left[ \sigma_i^{(k)} \right]^2
\]

(5)

where \( \sigma_i^{(\theta)} \) is the error of the interferometric measurements, \( \sigma_i^{(\Delta R/D)} \) the error related to the radial displacement and \( \sigma_i^{(k)} \) the error in the LD correction. According to Lane et al. (2002), the errors in the PTI H-band data vary between 0.01 and 0.06 mas. The errors on the radial displacements are due to the uncertainty in the \( p \)-factor (amounting to \( \sim 4\% \), according to Sabbey et al. 1995), and to the estimated measurement errors in the radial velocity, adding a further 2\% according to Bersier et al. (1994b). The errors in our LD corrections, barring systematic errors and based only on the numeric uncertainty in our model, are of the order of \( \pm 0.02\% \) (see Figure 2), which is a negligible contribution with respect to the other error sources, and is thus ignored.

The best fit results are shown in Table 1 (col. [2]) and Figure 5. Column (3) in Table 1 shows, for comparison, the best fit parameters obtained with the same fixed LD correction used by Lane et al. (2002) of \( k \simeq 0.96 \). The difference in the two best fit values of the geometric BW distance is 11 pc, which is less than \( \sim 3\% \). Even though this difference is less than \( 1/3 \) of the error bars, it is significant: Figure 5 shows that the error regions for our
best fit, the UD fit and the fixed \( k \) fit are mutually exclusive. This is again a consequence of the high level of precision that the available interferometric data already allow in the determination of the average angular radius \( \Theta_0 \).

Note that the error regions shown in Figure 5 do not take into account the uncertainty in the LD correction, which we ignored. The uncertainty quoted for the tabulated LD correction used by Lane et al. (2002) is much larger than our numeric error \( \Delta k \simeq 0.02 \), and would have resulted in a much larger error region, including all the narrow ovals in Figure 5. The main point of this paper, however, is to discuss the consequences of using model derived LD corrections, and Figure 5 shows that in the case of \( \zeta \) Gem using generic tabulated LD corrections can lead to an error of a few percent with respect to the best fit distance. This discrepancy is larger than the error regions determined by all the other error sources, and can thus in principle be tested observationally by directly measuring the LD with longer baseline interferometers.

The difference in the results of the geometric BW fit are entirely due to the LD correction, more precisely to its average value \( \bar{k} \) computed over the pulsational cycle. In the near-IR, and especially in the H band, the phase dependency of \( k(\lambda, \phi) \) is relatively small (less than 2\%, compared with the error bars \( \gtrsim 5 - 10\% \) in the measured angular diameters). This means that, with the current error bars in the interferometric data, we are still not sensitive to the variations in the LD corrections induced by the hydrodynamics. An accuracy of the order of 0.2\% in the \( \theta_i \) is required to be sensitive to such effects in the H band. Note that a more favorable situation is met in the visible, where the temporal variations of \( k(\phi) \) are larger. Given the limitation of the available data, however, the main contribution of our hydrodynamic simulations, at least for now, is not in the detailed dependence of \( k(\phi) \), but in setting the right level of the LD correction for the wavelength of the observation. Contrary to the tabulated values of the LD correction, which are computed for generic yellow supergiants having the same spectral type of Cepheid stars, our corrections are specific for the modeled stars, as they follow the \( T_{\text{eff}} \) and \( \log g \) derived from spectroscopic observations for each star. This is not a trivial matter, as the discrepancy with the tabulated value (\( \Delta k \simeq 0.02 \) in the H band) is the largest error source in the determination of the geometric BW distance.

Finally, we discuss the issue of the validity of the best fit \( \chi^2 \) as guide to which LD better represent the data. Is the best fit \( \chi^2 \) an indicator of the best value for the LD correction? If this is the case, then we should conclude that \( \zeta \) Gem is not limb darkened, since the best fit \( \chi^2 \) computed with this same procedure with the UD diameters would be \( \chi^2 \simeq 27.5 \). However, this is not the case.

To explain this, one should first remember that the time dependent hydrodynamic effects are too small to be observed with the available PTI data. While doing the BW fit, we could
thus consider the LD correction as a constant value $\bar{k}$ equal to its average over the pulsational period. In the three fits shown in Figure 5, the value of $\bar{k}$ would be equal to 1 for the UD data, $\sim 0.979$ for our hydrodynamic calculations and 0.96 for Claret et al. 1995 (used by Lane et al. 2002). This coefficient enters the $\chi^2$ equation by dividing the UD $\theta_{UD}^{(i)}$:

$$\chi^2 = \sum_i \left[ \left( \Theta_0 + \frac{2\Delta R_i}{D} \right) - \frac{\theta_{UD}^{(i)}}{\bar{k}} \right]^2$$  (6)

A smaller value of $\bar{k}$ (larger LD), will result in a larger best fit mean angular diameter $\Theta_0$, which will thus scale as $1/\bar{k}$. The net effect of this scaling is that the best fit $\chi^2$ will scale as $1/\bar{k}$. This means that a fit made with larger LD (as long as the phase dependency of $k$ is irrelevant) will have a larger best fit $\chi^2$. The UD fit will thus appear always to be the best.

The conclusion is that, until the accuracy of the measured angular diameters becomes good enough to appreciate the changes in LD due to the pulsation, the geometric BW fit is not a good tool to test the LD models. On the other hand, a good model for the LD is absolutely necessary to obtain a reliable value of the geometric BW distance, since the best fit angular diameter scales linearly as $1/\bar{k}$.

Note, finally, that a different fitting strategy may allow a test our LD models even with present day interferometric data. Our insensitivity to the LD is due to the fact that we are fitting derived data (the angular diameters $\theta_i$) which have been previously obtained from the fringe visibilities assuming a uniform disk model. This initial fit has destroyed the sensitivity of the original data to the stellar LD. The correct way to test the LD is to perform $\chi^2$ fits on the original visibilities. Any test for the absolute value of LD predicted by our models should thus be made using the visibility data taken at different projected baselines and wavelengths.

4. Conclusions

The geometric BW method is a powerful tool to derive the distances of pulsating stars. The detailed analysis of the method, and our discussion of its main current uncertainties described in the previous sections, show that a special care should be taken when assessing the accuracy of the results.

Despite the increasing quality of the interferometric data, the error bars in the individual measurements are the largest contributors to the final errors in the geometric BW distances.
and average radii. After the instrumental errors, however, follows the uncertainty in the limb darkening correction. The importance of LD will grow in the near future, given the fast pace at which the available interferometers are improving their accuracy, and with the new long baseline, large aperture interferometers which are becoming operational. To address this issue, we have presented in this paper a procedure to compute accurate LD corrections for pulsating Cepheids, which are based on time- and wavelength-dependent hydrodynamic models.

We show that our \( k(\lambda, \phi) \) strongly differ from the equivalent corrections computed from hydrostatic atmospheres, even in absence of “strong” hydrodynamic effects like shocks. The main consequence of the hydrodynamic terms in our stellar atmospheric models is an increase in the limb darkening, because of a generally higher temperature gradient in the photosphere. The situation briefly reverses in presence of shockwaves, due to the energy deposited by the shocks.

The magnitude of these effects depends on the wavelength. The effect is more significant in the visible spectrum (\( \Delta k \sim 0.015 \)) and it is smaller in the infrared (\( \Delta k \sim 0.002 \)). Current interferometers still cannot test these time-dependent variations in the limb darkening, which will however become important in the near future, especially when interferometric observations in spectral lines will become feasible. The phases of rapid variation in the LD associated to the propagation of the shockwave in the photosphere, when significant changes in the LD occur in timescales of hours, appears particularly appealing for an observational test of our models.

Even though the time-dependent variations in the LD are not currently measurable because of the required visible accuracy, our models can already provide an average value of the wavelength-dependent LD correction \( k(\lambda) \) which is significantly different from the tabulated values currently used. In the case of \( \zeta \) Gem, our limb darkening corrections induce a \( \sim 3\% \) change in the best fit value of the BW distance derived from H-band PTI data. Even more important is that, as shown in section 3, the \( 1\sigma \) error regions around the best fit distances derived with our and other LD corrections are mutually exclusive, opening the possibility of an independent test of our models when longer baselines reaching the first minimum in the visibilities will allow a direct test. A preliminary test of our models will also be possible by directly fitting the observed visibilities with our model visibilities, instead of using LD corrections of UD best fit diameters.

As the data for other Cepheids will become available, it will be important to have a large library of models specifically computed for each source. As the models are very dependent on the pulsational characteristics of each star, upon which the hydrodynamic model is built, a “generic” parametric limb darkening correction for all stars is not possible, as any individual
model cannot be extended to be used for a Cepheid with a different pulsational engine. This is the reason why tables of LD corrections as the ones produced by Claret et al. (1995) cannot reproduce the detailed changes in the LD which are required to apply the geometric BW method to classical Cepheids.

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## TABLE 1
BW BEST FIT OF ζ GEM PTI H-BAND LD DATA

| Fit Parameter | $k(\phi)$ (model) | $k = 0.96$ |
|---------------|------------------|------------|
| $R_0 [R_\odot]$ | $67.0^{+5.7}_{-6.9}$ | $66.2^{+8.3}_{-6.6}$ |
| $D [pc]$      | $372^{+49}_{-39}$  | $361^{+46}_{-36}$  |
| $\Theta_0 [mas]$ | $1.665\pm0.007$ | $1.698\pm0.007$ |
| $\chi^2$      | 28.1             | 29.5       |
Fig. 1.— Wavelength dependence of the LD correction $k(\lambda)$ for our $\zeta$ Gem model. From top to bottom, the models are shown at minimum radius, at a quasi-static phase after maximum luminosity, and at the time in which a shockwave is crossing the atmosphere. Solid thick line is $k(\lambda)$ from our hydrodynamic model, while the thin lines are derived from a hydrostatic model having the $T_{\text{eff}}$ and $\log g$ determined from observations (Krockenberger et al. 1997).

Fig. 2.— Pulsational phase dependence of the LD correction $k(\phi)$ for our $\zeta$ Gem model. The LD correction is shown for the near-IR H band (top), and at visible wavelength (570 nm, bottom). The thin lines are the 1 $rms$ numerical uncertainties in our simulations.

Fig. 3.— Phase dependent $k(\phi_L)$ for $\zeta$ Gem in the PTI H band, and in the bluest 570 nm channel of NPOI. Solid lines are the LD corrections computed with our hydrodynamic model; dotted lines are the equivalent corrections for an hydrostatic atmosphere having $T_{\text{eff}}$ and $\log g$ from Krockenberger et al. (1997). The dashed line is the value computed by Claret et al. (1995) and used in Lane et al. (2002).

Fig. 4.— Effective temperature, as a function of lightcurve phase, used to compute the hydrodynamic model of $\zeta$ Gem. The dependence of $T_{\text{eff}}$ from the pulsational phase is derived from measurements by Krockenberger et al. (1997).

Fig. 5.— Best fit parameters and error regions for $\zeta$ Gem PTI H-band data. Top curve is the UD data, middle curve is our best fit for model LD data and the bottom curve is the result obtained using a fixed LD correction of $k \approx 0.96$ as in Lane et al. (2002). The inner error region is the 68% confidence level of the fit (1$\sigma$), while the outer is the 90% confidence level.
