QUANTIZATIONS OF CHARACTER VARIETIES AND QUANTUM KNOT INVARIANTS

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ABSTRACT. Let $G$ be a simple complex algebraic group and $\mathfrak{g}$ its Lie algebra. We show that the $\mathfrak{g}$-Witten-Reshetikhin-Turaev quantum invariants determine a deformation-quantization, $\mathbb{C}_q[X_G(\text{torus})]$, of the coordinate ring of the $G$-character variety of the torus. We prove that this deformation is in the direction of the Goldman’s bracket. Furthermore, we show that every knot $K \subset S^3$ defines an ideal $I_K$ in $\mathbb{C}_q[X_G(\text{torus})]$. We conjecture that the homomorphism $\varepsilon : \mathbb{C}_q[X_G(\text{torus})] \to \mathbb{C}[X_G(\text{torus})]$, $q \to 1$, maps $I_K$ to the ideal whose radical is the kernel of the map $\mathbb{C}[X_G(\text{torus})] \to \mathbb{C}[X_G(S^3 \setminus K)]$. This conjecture is related to AJ-conjecture for $sl(2, \mathbb{C})$. The results of this paper are inspired by the theory of $q$-holonomic relations between quantum invariants of Garoufalidis and Le. Along the way, we disprove Conjecture 2 in [Le2].

1. Statements of theorems and conjectures

1.1. Quantum link invariants. For a simple complex Lie algebra $\mathfrak{g}$, denote by $WRT_{\mathfrak{g}, V}(L)$ the $(\mathfrak{g}, V)$-Witten-Reshetikhin-Turaev invariant of a framed oriented link $L \subset S^3$ whose all components are labeled by a finite dimensional representation $V$, [RT]. It is a polynomial in $q^{\pm \frac{1}{D(\mathfrak{g})}}$, where $D(\mathfrak{g})$ is the smallest positive integer such that the dual Killing form, $(\cdot, \cdot) : \Lambda^*_\mathfrak{g} \times \Lambda^*_\mathfrak{g} \to \mathbb{Q}$, has all its values in $\mathbb{Z}$. (A denotes the weight lattice of $\mathfrak{g}$. The form is normalized so that $(\lambda, \lambda) = 2$ for all short roots $\lambda$.)

$$D(\mathfrak{g}) = \begin{cases} n & \text{for } \mathfrak{g} = sl(n) \\ 1 & \text{for } \mathfrak{g} = sp(2n), so(4n+1), E_6, F_4, G_2 \\ 2 & \text{for } \mathfrak{g} = so(4n), so(4n+3), E_7 \\ 3 & \text{for } \mathfrak{g} = E_6, \\ 4 & \text{for } \mathfrak{g} = so(4n+2). \end{cases}$$

Given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and fixed positive roots of $\mathfrak{g}$, each finite dimensional irreducible representation of $\mathfrak{g}$ is determined by its highest weight $\lambda$. We denote that representation by $V(\lambda)$. Then $\lambda \to WRT_{\mathfrak{g}, V(\lambda)}(L)$ is a function defined on the set of all dominant weights.

1.2. Extension to Verma modules. For the purpose of relating Witten-Reshetikhin-Turaev invariants of links to the topology of their complements, we need to extend the above function to the entire weight lattice of $\mathfrak{g}$. However, if $\lambda$ is not dominant

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$q$-quantum invariants are also called by the compact Lie group corresponding to the real compact form of $\mathfrak{g}$. For example $sl(n, \mathbb{C})$-quantum invariants are also called $SU(n)$-quantum invariants.
then all representations of \( g \) with highest weight \( \lambda \) are infinite dimensional. In fact, each \( \lambda \in \Lambda_g \) defines the Verma module \( M(\lambda) \) which is an infinite-dimensional indecomposable \( g \)-module of highest weight \( \lambda \) with the universal property that each indecomposable \( g \)-module of highest weight \( \lambda \) is a quotient of \( M(\lambda) \). Rozanski (for \( sl(2) \), [Re]) and Le (for all \( g \)) made the following surprising observation: Reshetikhin-Turaev construction of quantum invariants for knots (but not links) extends verbatim for all Verma modules of \( g \). (Details in Sec. [2]) Furthermore,

\[
J_{g,M(\lambda)}(K) = WRT_{g,V(\lambda)}(K),
\]

for all dominant weights \( \lambda \). Let

\[
\rho, \text{ where } \rho \text{ is the half-sum of positive roots of } g.
\]

We call it the \( g \)-Witten-Reshetikhin-Turaev function of \( K \). (The motivation for the shift by \( \rho \) comes from Proposition [3])

**Example 2.** For \( g = sl(2) \), \( \rho = 1 \in \Lambda_g = \mathbb{Z} \). \( J_{g,K}(0) = 0 \), \( J_{g,K}(1) = 1 \), and \( J_{g,K}(2) \) for a zero-framed knot \( K \) is the Jones polynomial of \( K \). More generally, \( J_{g,K}(n) \) is the Jones polynomial of \( K \) colored by the \( n \)-dimensional representation for \( n \geq 1 \) and \( J_{g,K}(n) = -J_{g,K}(-n) \) for negative \( n \).

**Example 2.** It follows from [Le1 1.4.4] that

\[
J_{g,U}(\lambda) = \frac{\sum_{w \in W} \text{sgn}(w)q^{(\lambda,w(\rho))}}{\sum_{w \in W} \text{sgn}(w)q^{(\rho,w(\rho))}},
\]

for the unknot \( U \) and for every \( \lambda \in \Lambda_g \). The sum is over the Weyl algebra of \( g \).

We are going to see that \( J_{g,K} \) has nice algebraic properties and it encodes the \( g \)-quantum invariants of \( K \) in a form which is very convenient for the purpose of relating them to the topology of \( S^3 \setminus K \).

1.3. \( q \)-holonomicity. The next statement follows immediately from the argument of the proof of [GH Thm 6] – see comments in Sec. [2]

**Theorem 3.** \( J_{g,K} \) a \( q \)-holonomic function on \( \Lambda_g \) for all \( g \neq G_2 \).

In order to define \( q \)-holonomicity of \( J_{g,K} \), consider the \( \mathbb{C}[q^{\pm1/D(\mathfrak{g})}] \)-vector space \( F(\Lambda_g, \mathbb{C}[q^{\pm1/D(\mathfrak{g})}]) \) of all \( \mathbb{C}[q^{\pm1/D(\mathfrak{g})}] \)-valued functions on \( \Lambda_g \) and consider two families of operators on it:

\[
E_\alpha f(\beta) = f(\alpha + \beta), \quad Q_\alpha f(\beta) = q^{(\alpha,\beta)} f(\beta),
\]

for all \( \alpha, \beta \in \Lambda_g \). Let \( \mathcal{A}_g \) be the algebra of \( \mathbb{C}[q^{\pm1/D(\mathfrak{g})}] \)-linear endomorphisms of \( F(\Lambda_g, \mathbb{C}[q^{\pm1/D(\mathfrak{g})}]) \) generated by \( E_\alpha \)'s and \( Q_\alpha \)'s for \( \alpha \in \Lambda_g \).

**Proposition 4.** \( \mathcal{A}_g \) is the \( \mathbb{C}[q^{\pm1/D(\mathfrak{g})}] \)-algebra of polynomials in non-commuting variables \( E_\alpha, Q_\beta, \alpha, \beta \in \Lambda_g \), subject to conditions:

\[
E_\alpha E_\beta = E_{\alpha + \beta}, \quad Q_\alpha Q_\beta = Q_{\alpha + \beta}, \quad E_\alpha Q_\beta = q^{(\alpha,\beta)} Q_\alpha E_\beta, \quad E_0 = Q_0 = 1.
\]

We call \( \mathcal{A}_g \) the quantum Weyl algebra of \( g \). \( \mathcal{A}_{sl(2)} \) is the \( q \)-Weyl algebra of [EQ] and \( q \)-torus algebra of [GE].

\[
\mathcal{A}_{sl(2)} = \mathbb{C}(E_1^{\pm1}, Q_1^{\pm1}) / E_1 Q_1 - q^{1/2} Q_1 E_1.
\]
algebra of $GL$ and from other "quantum Weyl algebras" appearing for example in $[Ga1, Ga2, GL]$. For other $g$, $A_g$ appears to be different from the $q$-torus algebra of $[GL]$ and from other "quantum Weyl algebras" appearing for example in $[DP, Gl] [Ha] [IZ, Ma] [Pa, Ri]$. For any $f : A_g \to \mathbb{C}[q^{\pm 1/D(g)}]$ the set

$$I_f = \{ P \in A_g : Pf = 0 \} \subset A_g$$

is a left-sided ideal in $A_g$ called the recursive ideal of $f$, c.f. $[Ga1]$. This term reflects the fact that each element of $I_f$ represents a recursive relation for $f$. Function $f$ is $q$-holonomic iff $I_f$ is $q$-holonomic, which intuitively means "as large as possible" (and, in particular, non-trivial). More precisely, a left ideal $I \triangleleft A_g$ is $q$-holonomic if its homological codimension is at least the rank of $g$, that is

$$\text{hed}(I) = \min \{ j : \text{Ext}_A^j (A_g/I, A_g) \neq 0 \} \geq \text{rank } g.$$

This definition is a modification of that of $[GL]$ to functions defined on $A_g$ rather than on $\mathbb{N}^n$. We denote the recursive ideal of $J_{g,K}$ by $I_{g,K}$.

1.4. The action of the Weyl group. $J_{g,K}(\lambda)$ is equivariant with respect to the Weyl group $W$ action:

**Proposition 5.** (Proof in Sec. 2)

$$J_{g,K}(w \cdot \lambda) = \text{sgn}(w) \cdot J_{g,K}(\lambda),$$

where $\text{sgn}(w) = \pm 1$ is the sign of $w \in W$.

In particular, $J_{g,K}(\lambda)$ vanishes for weights $\lambda$ in the boundaries of Weyl chambers.

The Weyl group acts on $F(A_g, \mathbb{C}[q^{\pm 1/D(g)}])$ by $w \cdot f(\alpha) = f(w^{-1} \cdot \alpha)$. The inverse is needed to make sure that this is a left action.) Additionally, $W$ acts on $A_g$ via $w \cdot E_\alpha = E_{w^{-1} \cdot \alpha}$, $w \cdot Q_\alpha = Q_{w^{-1} \cdot \alpha}$, and the product $A_g \times F(A_g, \mathbb{C}[q^{\pm 1/D(g)}]) \to F(A_g, \mathbb{C}[q^{\pm 1/D(g)}])$ is $W$-equivariant. Furthermore, by Proposition 5, $w \cdot I_{g,K} = I_{g,K}$, for every $w \in W$.

We call the $W$-invariant part of the recursive ideal, $I_{g,K}^W \triangleleft A_g^W$, the invariant $g$-recursive ideal of $K$.

**Proposition 6.** (Proof in Sec. 2.2) (1) For the unknot $U$,

$$I_{\text{sl}(2,\mathbb{C}), U}^W = \langle E + E - (q^{1/2} + q^{-1/2}) , EQ + E^{-1}Q^{-1} - q(Q + Q^{-1}) \rangle.$$

(2) For the left-sided trefoil, $I_{\text{sl}(2,\mathbb{C}), K}^W$ is generated by elements

$$q^{5/4}(EQ^5 - EQ^{-1}Q^5) - q^{-7/4}(EQ^{-1} - E^{-1}Q) - q^{-3/4}(Q^5 + Q^{-5}) + q^{1/4}(Q + Q^{-1}),$$

$$q^3(EQ^6 + E^{-1}Q^6) + (q^{3/2} + q^{-3/2})(E + E^{-1}) - (q^{1/3} + q^{-5/3})(EQ^{-6} + E^{-1}Q^6) + (Q^6 + Q^{-6}) - 2(q + q^{-1}),$$

$$-q^{-7/2}(E^2Q^{-7} + E^{-2}Q^7) + q^3(EQ^{-7} + E^{-1}Q^7) + (q^2 - q^{-1})(EQ^{-3} + E^{-1}Q^3) - q(EQ^{-1} + E^{-1}Q) - (q^{1/2} - q^{-1/2})(Q^3 + Q^{-3}) + q^{-3/2}(Q + Q^{-1}).$$

(3) The generators of the invariant recursive ideal of the right-handed trefoil are obtained from those above after substitution $Q \to Q^{-1}, \ q \to q^{-1}$.

**Conjecture 7.** For every $g$ and $K$, (1) $J_{g,K}$ is uniquely determined by a finite number of its values together with the recursive relations of $I_{g,K}$.

(2) $J_{g,K}$ is uniquely determined among $W$-equivariant functions (i.e. functions
satisfying the statement of Proposition 8 by a finite number of its values together with the recursive relations of $I_{g,K}$.

1.5. $A_g^W$ is a quantization-deformation of the $G$-character variety of the torus. For a given complex reductive algebraic group $G$, denote the $G$-character variety of a finitely generated (discrete) group $\Gamma$ by $X_G(\Gamma)$, c.f. Sec 3. Additionally, denote the connected component of the trivial character in $X_G(\Gamma)$ by $X_G^0(\Gamma)$. (If $G$ is simply connected, for example if $G = GL(n, \mathbb{C}), SL(n, \mathbb{C}), Sp(n, \mathbb{C})$, then the $G$-character variety is connected, $\text{Rec}$, and hence $X_G^0(\Gamma) = X_G(\Gamma)$.) We often abbreviate $X_G(\pi_1(Y))$ and $X_G^0(\pi_1(Y))$ by $X_G(Y)$ and $X_G^0(Y)$ for a topological space $Y$. Goldman proved that for any closed orientable surface $F$, $X_G(F)$ is a singular holomorphic symplectic manifold, $\text{Go1}$, c.f. Sec. 4. This symplectic structure defines a Poisson bracket on the space of holomorphic functions on $X_G^0(F)$ called the Goldman bracket.

Theorem 8. (Precise statement in Cor. 22 and Thm. 23) (1) For every complex reductive algebraic group $G$ and its Lie algebra $g$, $A_g^W$ is a deformation-quantization of $\mathbb{C}[X_G^0(\text{torus})]$.

(2) For every classical group, $G = GL(n, \mathbb{C}), SL(n, \mathbb{C}), SO(n, \mathbb{C}), Sp(n, \mathbb{C})$, this deformation-quantization is in the direction of the Goldman bracket.

By $\text{FGSa}$, $A_{SL(2)}^W$ is the Kauffman bracket skein algebra of the torus. Therefore, Theorem 8(2) generalizes the result of $\text{BFK}$ for torus to higher rank classical groups. We will discuss the relations between the present work and skein modules of higher rank (in particular those of $\text{S1}$) in an upcoming $\text{S2}$.

1.6. $I_{g,K}^W$ as a quantization of the $G$-representations of $\pi_1(S^3 \setminus K)$. By Theorem 8 we have a $\mathbb{C}$-algebra homomorphism

$$\varepsilon : A_g^W \rightarrow \mathbb{C}[X_G^0(\mathbb{Z}^2)]$$

given by evaluation $q = 1$.

Given a knot $K \subset S^3$, let $M_K$ be the compactification of $S^3 \setminus K$ with boundary torus, $\partial M_K = T$. The embedding $\partial M_K \hookrightarrow M_K$ defines a homomorphism $\phi_K : \mathbb{C}[X_G^0(T)] \rightarrow \mathbb{C}[X_G^0(M_K)]$ whose kernel we denote by $A_{G,K}$. We call it the $A_{G}$-ideal of $K$. The $A_{SL(2, \mathbb{C})}$-ideal of $K$ determines the $A$-polynomial of $K$ of $\text{CCGLS}$. (That is the motivation for the name ”$A_{G}$-ideal”.)

Conjecture 9. The zero set of $\varepsilon(I_{g,K}^W) \subset \mathbb{C}[X_G^0(T)]$ is the closure of the image of $X_G^0(M_K) \rightarrow X_G^0(T)$. Equivalently,

$$\sqrt{\varepsilon(I_{g,K}^W)} = A_{G,K},$$

where $\sqrt{\cdot}$ denotes the nil-radical.

Let $\mathbb{Z}^2 = \langle L, M \rangle$. By Theorem 8 $\mathbb{C}[X_{SL(2, \mathbb{C})}(\mathbb{Z}^2)] = \mathbb{C}[E^{\pm 1}, Q^{\pm 1}, t], \quad \varepsilon(E) = E^{-1}, \quad \varepsilon(Q) = Q^{-1}$.

Under the isomorphism which will be defined in (7), the regular function $\tau_{a,b} \in \mathbb{C}[X_{SL(2)}(\mathbb{Z}^2)]$, $\tau_{a,b}(\rho) = tr\rho(L^a M^b)$, corresponds to $E^a Q^b + E^{-a} Q^{-b}$, for any $a, b \in \mathbb{Z}$.\footnote{Note that this statement implies in particular that up to an isomorphism $X_G^0(\text{torus})$ depends on the Lie algebra of $G$ only.}
Corollary 11. For the unknot and for the trefoil Conjecture [2] holds. However, $\varepsilon(I^W_{sl(2,\mathbb{C}), U})$ is not divisible by $w^2$ and hence contained in $\varepsilon(I^W_{sl(2,\mathbb{C}), U})$. Therefore, $\sqrt{\varepsilon(I^W_{sl(2,\mathbb{C}), K})} = A_{G, K}$. On the other hand, $Q = 1$ does not belong to the ideal $\langle Q^2 - Q, w, EQ^2 - E^{-1}Q^{-4} \rangle$, since $(E, Q) = (1, -1)$ belongs to the zero set of that ideal. Consequently, $w(Q - Q^{-1}) \notin \varepsilon(I^W_{g, K})$.

Corollary 11 disproves [Le2, Conj. 2].

Theorem 12. For a given $K$ and $g$ implies that the characteristic and deformation varieties of Garoufalidis coincide and, in particular, AJ conjecture of [Ga2] holds and [Ga2] Question 1 has affirmative answer.

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2. WRT knot functions and quantum Weyl algebras

For every $g$-module $V$, $V[[h]] = V \otimes \mathbb{C}[[h]]$ is a module over the quantum group $U_h(g)$. Let $K'$ be a 1-tangle obtained by cutting a knot $K$ open. Reshetikhin-Turaev construction associates with $K'$ colored by a representation $V$ of $g$ (or $V[[h]]$) a morphism of $U_q(g)$-modules $V[[h]] \to V[[h]]$ which is in the center of $End_{U_q(g)}(V[[h]])$. (This is related to the fact that all 1-tangles commute under composition.) Consequently, if $V$ is irreducible then, by Schur’s Lemma, the WRT invariant of $K'$ is a scalar multiple of the identity. We denote that scalar by $WRT_g(V(K'))$. It lies in $\mathbb{C}[q^{\pm 1}] \subset \mathbb{C}[[h]]$, where $q = e^h$, and

$$WRT_g(U) = WRT_g(V(U)) \cdot WRT_g(V(K'))$$

where the WRT-invariant of the unknot is the quantum dimension of $V$,

$$WRT_g(V(U)) = \dim_q(V) = \frac{\sum_{w \in W} \text{sgn}(w) q^{(\lambda + \rho, w(\rho))}}{\sum_{w \in W} \text{sgn}(w) q^{(\rho, w(\rho))}}$$

and $\rho$ is the half-sum of positive roots. Rozansky (for $sl(2)$ in [Ra]), Le for $sl(2)$, [HL], and for all $g$, [GL] Lemma 7.7, proved that Reshetikhin-Turaev definition of $WRT_g(V(K'))$ makes sense verbatim for all Verma modules $V$ despite the fact that they are infinite dimensional. (This is not obvious, since Reshetikhin-Turaev construction involves quantum traces, which in case of infinite dimensional modules involve sums in $\mathbb{C}[q^{\pm 1}]$ which are a priori infinite. One has to prove that all but finitely many summands in all these sums vanish.)
2.1. Example: \( sl(n) \)-quantum invariants of the unknot. Let \( sl(n) \) be the algebra of traceless \( n \times n \) matrices. Consider the standard Cartan subalgebra of \( sl(n) \) composed of diagonal matrices:

\[
\mathfrak{h} = \left\{ \sum a_i E_{ii} : \sum a_i = 0 \right\}.
\]

The weights \( \alpha_i : \mathfrak{h} \to \mathbb{C} \), such that \( \alpha_i(E_{jj}) = \delta_{ij} \), generate the weight lattice of \( sl(n) \) and the Killing form on \( \mathfrak{h}^* \) is given by

\[
(\alpha_i, \alpha_j) = \begin{cases} 
\frac{n-1}{2} & i = j \\
-\frac{1}{n} & i \neq j,
\end{cases}
\]

c.f. [FH] Formula 15.2]. The Weyl group \( W = S_n \) permutes the weights \( \alpha_1, ..., \alpha_n \).

The positive roots are \( \alpha_i - \alpha_j \), for \( i > j \), and

\[
\rho = \frac{n-1}{2} \alpha_1 + \frac{n-3}{2} \alpha_2 + ... - \frac{n-3}{2} \alpha_{n-1} - \frac{n-1}{2} \alpha_n.
\]

Let \( E_i = E_{\alpha_i}, i = 1, ..., n \). By [H],

\[
\sum_{i=1}^{n} E_i J_{sl(n,\mathbb{C}),U} = \frac{1}{S} \sum_{w \in W} \left( sgn(w) q^{(\lambda,w(\rho))} \sum_{i=1}^{n} q^{(\alpha_i,w(\rho))} \right),
\]

where

\[
S = \sum_{w \in W} sgn(w) q^{(\rho,w(\rho))}.
\]

Since \( (\alpha_i, \rho) = \frac{1}{2}(n+1-2i) \), the second sum in (11) is equal to

\[
\sum_{i=1}^{n} q^{(\alpha_i,w(\rho))} = \sum_{i=1}^{n} q^{(w^{-1}(\alpha_i),\rho)} = \sum_{i=1}^{n} q^{(\alpha_i,\rho)} = \sum_{i=1}^{n} q^{\frac{n+1}{2}-2i} = [n],
\]

for every \( w \in S_n \), where \([n]\) is the \( n \)-th quantum integer, \([n]\) = \( \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}} \).

**Corollary 13.** \( \sum_{i=1}^{n} E_i - [n] \) belongs to the \( sl(n) \)-recursive ideal of the unknot.

**Proof of Proposition** [S] (Suggested by T. Le): By [3] and [8],

\[
J_{\mathfrak{g},K}(\alpha) = J_{\mathfrak{g},U}(\alpha) \cdot WRT_{\mathfrak{g},M(\alpha-\rho)}(K').
\]

By [4],

\[
J_{\mathfrak{g},U}(w \cdot \alpha) = sgn(w) \cdot J_{\mathfrak{g},U}(\alpha).
\]

Therefore, it is enough to prove that \( WRT_{\mathfrak{g},M(\alpha-\rho)}(K') \) is invariant under the action of \( W \) on \( \alpha \).
For every positive element \( w \in W \) in Bruhat ordering (i.e. a product of reflections with respect of positive roots) and for every \( \alpha \in \Lambda_g \), \( M(\lambda - \rho) \) is a submodule of \( M(w \cdot \lambda - \rho) \), c.f. [Kn] Ch V.9 Problem 12\(^3\) This implies that

\[
WRT_{\mathfrak{g}, M(\lambda - \rho)}(K') = WRT_{\mathfrak{g}, M(w \cdot \lambda - \rho)}(K').
\]

Since positive elements in \( W \) generate \( W \), the proof is completed. \( \square \)

**Lemma 14.** Operators \( Q_\alpha E_\beta \), for \( \alpha, \beta \in \Lambda_g \), are linearly independent.

**Proof.** Suppose that

\[
\sum_{\alpha, \beta} c_{\alpha, \beta} Q_\alpha E_\beta = 0
\]

and \( c_{\alpha_0, \beta_0} \neq 0 \) for some \( \alpha_0, \beta_0 \in \Lambda_g \) such that

\[
(\beta_0, \beta_0) = \max\{ (\beta, \beta) : c_{\alpha, \beta} \neq 0 \text{ for some } \alpha \}
\]

and

\[
(\alpha_0, \alpha_0) = \max\{ (\alpha, \alpha) : c_{\alpha, \beta} \neq 0 \}.
\]

Fix an integer \( N > \max \{ (\beta_0, \beta_0), (\alpha_0, \alpha_0) : c_{\alpha, \beta} \neq 0 \} \) and let \( f : \Lambda_g \rightarrow \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}] \),

\[
f(v) = \begin{cases} 1 & \text{if } v = kN\alpha_0 \text{ for some } k \in \mathbb{Z} \\ 0 & \text{otherwise.} \end{cases}
\]

Then the value of \( \sum_{\alpha, \beta} c_{\alpha, \beta} Q_\alpha E_\beta f \) at \(-\beta_0 + kN\alpha_0\) is zero. On the other hand,

\[
\sum_{\alpha, \beta} c_{\alpha, \beta} Q_\alpha E_\beta f (-\beta_0 + kN\alpha_0) = \sum_{\alpha, \beta} c_{\alpha, \beta} q^{(\alpha, -\beta_0 + kN\alpha_0)} f(\beta - \beta_0 + kN\alpha_0).
\]

By the definitions of \( f \) and \( N \), the sum on the right equals

\[
\sum_{\alpha} c_{\alpha, -\beta_0 + kN\alpha_0} q^{(\alpha, -\beta_0 + kN\alpha_0)}.
\]

and by (13), its leading term is

\[
c_{\alpha_0, -\beta_0 + kN\alpha_0} q^{(\alpha_0, -\beta_0 + kN\alpha_0)}.
\]

It grows exponentially with \( k \) – a contradiction. \( \square \)

**Proof of Proposition 4** Since relations (5) are obviously satisfied by the operators \( E_\alpha \) and \( Q_\beta \), \( \alpha, \beta \in \Lambda_g \), it is enough to prove that all other relations between these operators follow from (5). Let \( P \) be a polynomial in \( E_\alpha \)'s and \( Q_\beta \)'s, \( \alpha, \beta \in \Lambda_g \), which equals to the zero operator on \( F(\Lambda_g, \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}]) \). Relations (5) make possible to express \( P \) as a sum

\[
P = \sum_{\alpha, \beta} c_{\alpha, \beta} Q_\alpha E_\beta,
\]

over \( \Lambda_g \times \Lambda_g \), with \( c_{\alpha, \beta} \in \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}] \). By Lemma 14 all \( c_{\alpha, \beta} \)'s in the above sum vanish. Hence the relation \( P = 0 \) is a consequence of relations (5). \( \square \)

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\(^3\)In Knapp’s book the Verma module \( M(\lambda) \) is denoted by \( V(\lambda + \rho) \).
2.2. Proof of Proposition 6: (1) By [FG, Sa], \( A_{sl(2,\mathbb{C})}^W \) is isomorphic to the Kauffman bracket skein module of the torus and, by [Ga1], \( I_{sl(2,\mathbb{C}),K}^W \) corresponds to the orthogonal ideal under that isomorphism. More specifically, the \( p/q \)-torus knot on the torus corresponds to \((-1)^p q^{-ab/4} (EpQ^4 + E^{-p}Q^{-q})\), [Ga1] Fact 4, and the \( t \) of [FG, Ge] is our \( q^{1/4} \). The orthogonal ideal of the unknot was computed in [FG]. It is generated by two elements:

\[
\text{longitude } + (t^2 + t^{-2}) \quad \text{and } \quad (1,1)\text{-curve } + t^3\text{-meridian}.
\]

Therefore, \( I_{sl(2,\mathbb{C}),U}^W \) is generated by \( E + E^{-1} - (q^{1/2} + q^{-1/2}) \) and \( EQ + E^{-1}Q^{-1} - q(Q + Q^{-1}) \).

(2) The orthogonal ideal of the left and right handed trefoil was computed in [Ge]. The result can be summarized as follows: For \( p, q \) coprime, let \( (p, q) \) be the \( p/q \)-curve on the torus \( T \) considered as an element of the Kauffman bracket skein module of \( T \times I \), so that \((1,0)\) is the longitude and \((0,1)\) meridian with respect to the embedding \( T = \partial M_K \subset M_K \). For \( p, q \) such that \( \text{gcd}(p,q) = n \geq 0 \), let \( (p, q) = T_n((p/n, q/n)) \), where \( T_n(x) \) is the \( n \)-th Chebyshev polynomial: \( T_0(x) = 2, T_1(x) = x, T_{n+1}(x) = xT_n(x) - T_{n-1} \). Then the Kauffman bracket peripheral ideal of the left-handed trefoil \( K \) is generated by elements:

\[
(1, -5) - t^{-8}(1, -1) + t^{-3}(0, 5) - t(0, 1),
(2, -6) - (t^6 + t^{-6})(1, 0) + (t^4 + t^{-4})(1, -6) + (0, 6) - 2(t^4 + t^{-4}),
(2, -7) + t^{-5}(1, -7) + (t^{-5} - t^{-1})(1, -3) - (t^5(1, -1) + (t^2 - t^{-2})1, 3) - t^{-6}(0, 1).
\]

Now the statement follows as in (1).

2.3. Proof of Proposition 10: (1) By Proposition 6,

\[
\varepsilon(I_{g,U}^W) = \langle E + E^{-1} - 2, EQ + E^{-1}Q^{-1} - Q - Q^{-1} \rangle.
\]

It remains to prove that this ideal coincides with \( A_{sl(2,\mathbb{C}),U} \subset \mathbb{C}[X_{SL(2,\mathbb{C})}(\mathbb{Z}^2)] \). It is easy to check that \( E + E^{-1} - 2 \) and \( EQ + E^{-1}Q^{-1} - Q - Q^{-1} \) belong to \( A_{sl(2,\mathbb{C}),U} \). We claim that these two elements generate \( A_{sl(2,\mathbb{C}),U} \). Let \( x_k = EQ^k + Q^{-1}Q^{-k} - (Q^k + Q^{-k}) \). Since \( x_0, x_1 \in A_{sl(2,\mathbb{C}),U} \) and \( x_{k+1} = (Q + Q^{-1})x_k - x_{k-1}, x_k \in A_{sl(2,\mathbb{C}),U} \), for all \( k \).

Any element of that ideal can be reduced by \( E + E^{-1} - 2 \) to a polynomial in \( \mathbb{C}[E^{\pm 1}, Q^{\pm 1}]^{\mathbb{Z}/2} \), of span at most 2 in \( E \). It is easy to see that any such element is of the form \( Ep + q + E^{-1}(p) \), where \( p, q \in \mathbb{C}[Q^{\pm 1}] \). Therefore any \( z \in A_{sl(2,\mathbb{C}),U} \) can be presented as

\[
\sum_k c_k x_k + w(Q),
\]

where \( c_k \in \mathbb{C} \) and \( w(Q) \in \mathbb{C}[Q^{\pm 1}] \). Since all \( x_k \)'s are in \( A_{sl(2,\mathbb{C}),U} \), \( w(Q) \in A_{sl(2,\mathbb{C}),U} \).

However, since the meridian of the unknot can be mapped to \( m = \left( \begin{array}{cc} m & 0 \\ 0 & m^{-1} \end{array} \right) \) for every \( m \in \mathbb{C}^* \), \( w(Q) = 0 \). Hence, every element of \( A_{sl(2,\mathbb{C}),U} \) is a linear combination of \( x_k \) modulo \( E + E^{-1} - 2 \).

(2) The generators of the invariant recursive ideal of the left-handed trefoil listed in Proposition 6(2) are equal to \( w(Q^2 - Q^{-2}), w(1 - E^{-1})(EQ^{-3} + Q^3), w(EQ^4 -

\[4\]We have independently verified that these polynomials generate \( P_{sl(2,\mathbb{C}),K} \). Please note the plus sign in the second term of the third generator, which is missing in [Ge].
$E^{-1}Q^{-4}$ for $q = 1$. In order to compute $A_{sl(2,\mathbb{C}),K}$ observe that since $(E - 1)(EQ^{-6} + 1)$ is the $A$-polynomial of the left handed-trefoil,

$$A_{sl(2,\mathbb{C}),K}^W = (w \cdot \mathbb{C}[E^{\pm 1}, Q^{\pm 1}]) \cap \mathbb{C}[E^{\pm 1}, Q^{\pm 1}]^{Z/2}.$$ 

Since $i(w) = -w$, every element of $A_{sl(2,\mathbb{C}),K}^W$ is of the form $w \cdot p$, where $i(p) = -p$. Hence $p$ is a sum of monomials of the form $r_{a,b} = E^aQ^b - E^{-a}Q^{-b}$. Since $r_{a+1,b} = (E + E^{-1})r_{a,b} - r_{a-1,b}$, $r_{a,b+1} = (Q + Q^{-1})r_{a,b} - r_{a,b-1}$, $p$ is a linear combination of $E - E^{-1}, Q - Q^{-1}, EQ - E^{-1}Q^{-1}$ with coefficients in $\mathbb{C}[E^{\pm 1}, Q^{\pm 1}]^{Z/2}$.

2.4. **Proof of Theorem 12** The characteristic variety of Garoufalidis is the Zariski closure of the zero set

$$Z(A_{G,K}) \subset Z(\mathbb{C}[A_{\mathfrak{g}}]) = (\mathbb{C}^*)^n \subset \mathbb{C}^n,$$

where $n$ is the rank of $\mathfrak{g}$. Similarly, the deformation variety is the closure of

$$Z(\varepsilon(I_{\mathfrak{g},K})) \subset Z(\mathbb{C}[A_{\mathfrak{g}}]) = (\mathbb{C}^*)^n \subset \mathbb{C}^n.$$ 

Therefore, it is enough to prove that Conjecture 9 implies that for every $K$ and $\mathfrak{g}$

$$\sqrt{\varepsilon(I_{\mathfrak{g},K})} = \sqrt{A_{G,K}}.$$

To show the inclusion "$\subset"$, it is enough to prove that $\varepsilon(I_{\mathfrak{g},K}) \subset \sqrt{A_{G,K}}$. For each $g \in I_{\mathfrak{g},K}$, the element $\varepsilon(g)$ is a root of the polynomial $\prod_{w \in W}(x - w \cdot \varepsilon(g)) = \sum c_kx^k$ with coefficients $c_k \in \varepsilon(I_{\mathfrak{g},K})$ for $k = 0, \ldots, N - 1$, where $N = |W|$ and $c_N = 1$. Assuming that Conjecture 9 holds, $\varepsilon(I_{\mathfrak{g},K}) \subset A_{G,K}$. Consequently, $\varepsilon(g)^N = -\sum_{k=0}^{N-1} c_k \varepsilon(g)^k \in A_{G,K}$ and $\varepsilon(g) \in \sqrt{A_{G,K}}$.

To show the inclusion "$\supset"$, it is enough to prove that $A_{G,K} \subset \sqrt{\varepsilon(I_{\mathfrak{g},K})}$. Each $h \in A_{G,K}$ is a root of the polynomial $\prod_{w \in W}(x - w \cdot h) = \sum c_kx^k$ with coefficients $c_k \in A_{G,K}^W = \sqrt{\varepsilon(I_{\mathfrak{g},K})}$. Consequently, $h^N = -\sum_{k=0}^{N-1} c_k h^k \in \sqrt{\varepsilon(I_{\mathfrak{g},K})}$. Hence, $h \in \sqrt{\varepsilon(I_{\mathfrak{g},K})}$. 

Despite the fact that $A_{G,K}^W$ is equal its nil-radical, $A_{G,K}$ is often not equal to its nil-radical. Indeed, for the unknot, $(l - l^{-1})^2 \in A_{sl(2),U}$ but $(l - l^{-1}) \notin A_{sl(2),U}$!

Note that the conclusion of Theorem 12 is stronger than the AJ-conjecture of Garoufalidis for $\mathfrak{g} = sl(2)$, [Ga2, Conjecture 1], and its version for higher rank Lie algebras, [Ga2 Question 1].

3. Character varieties

3.1. **Introduction.** Let $G$ be a complex reductive algebraic group. If $\Gamma$ is a (discrete) group generated by $\gamma_1, \ldots, \gamma_n$ then the set of homomorphisms $\text{Hom}(\Gamma, G)$ can be identified with the set of points $(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \in G^n$ taken over all representations $\rho : \Gamma \rightarrow G$. It is an algebraic set which up to an isomorphism does not depend on the choice of generators of $\Gamma$. The group $G$ acts on $\text{Hom}(\Gamma, G)$ by conjugating representations and the categorical quotient of that action,

$$X_G(\Gamma) = \text{Hom}(\Gamma, G) / \Gamma$$

is called the $G$-character variety of $\Gamma$. In simple words $X_G(\Gamma)$ is an algebraic set together with a map $\pi : \text{Hom}(\Gamma, G) \rightarrow X_G(\Gamma)$ which is constant on all $G$-orbits.
and has the universal property that every map $\text{Hom}(\Gamma, G) \to Y$ which is constant on all $G$-orbits factors through $\pi$.

If $\Gamma$ is the fundamental group of a topological space $X$, then $X_G(\Gamma)$ is called the $G$-character variety of $X$ and it is abbreviated by $X_G(X)$.

**Proposition 15.** ([S3]) For $G = \text{SL}(n, \mathbb{C}), \text{O}(n, \mathbb{C}), \text{Sp}(2n, \mathbb{C})$, let $\tau_{\gamma} : X_G(\Gamma) \to \mathbb{C}$ be defined as $\tau_{\gamma}([\rho]) = \text{tr}(\rho(\gamma))$ for $\rho : \Gamma \to G \to \text{GL}(V)$, where $V$ is the defining representation of $G$. (The faithful representation of the smallest dimension.) Then the algebra $\mathbb{C}[X_G(\Gamma)]$ is generated by $\tau_{\gamma}$ for all $\gamma \in \Gamma$.

Proposition 15 does not hold for $\text{SO}(n, \mathbb{C})$, [S3].

Goldman proved that for every complex reductive algebraic group $G$ and any closed orientable surface $F$, $X_G(F) = X_G(\pi_1(F))$ is a singular holomorphic symplectic manifold. More specifically, let $X_G^{\text{sym}}(F)$ be the set of conjugacy classes of all representations $\rho : \pi_1(F) \to G$ such that $\rho(\pi_1(F))$ is not contained in a proper connected algebraic subgroup of $G$. $X_G^{\text{sym}}(F)$ is an open subset of $X_G(F)$ and a smooth (complex) manifold. Goldman defines holomorphic symplectic form on $X_G^{\text{sym}}(F)$. His construction utilizes the fact that the tangent space $T_{[\rho]}X_G(F)$ at $[\rho] \in X_G(F)$ represented by a representation $\rho : \pi_1(F) \to G$ is canonically isomorphic to $H^1(F, \text{Ad}_{\rho}\mathfrak{g})$. Let $B$ be a non-degenerate symmetric bilinear form, $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, invariant under the adjoint $G$ action. For classical groups, the standard choice is the trace form, $B(X, Y) = \text{tr}(XY)$, where the trace is defined by the embedding $G \subset \text{GL}(V) = \text{GL}(n, \mathbb{C})$, for the defining representation $V$ of $G$. The induced cup product

$$\omega : H^1(F, \text{Ad}_{\rho}\mathfrak{g}) \times H^1(F, \text{Ad}_{\rho}\mathfrak{g}) \xrightarrow{\cup} H^2(F, \mathbb{C}) \cap [F] \mathbb{C},$$

defines a symplectic form on $T_{[\rho]}X_G^{\text{sym}}(F)$. Goldman proves by an argument from gauge theory that $\omega$ is closed, [Go1]. This is Goldman’s symplectic form.

### 3.2. Character varieties of the torus.

Character varieties are usually very difficult to describe as solution sets of explicit systems of polynomial equations. Even an explicit description of $X_G(\mathbb{Z}^2)$ is difficult in general, since the number of connected and irreducible components of this set is unknown. If $G$ is simply connected, for example $G = \text{SL}(n, \mathbb{C})$ or $\text{Sp}(n, \mathbb{C})$, then $X_G(\mathbb{Z}^2)$ is connected, by [Ric]. However, $X_G(\mathbb{Z}^2)$ may be not connected in general: Points of $X_G(\mathbb{Z}^2)$ classify flat principal $G$-bundles over the torus. If $E_\rho$ is the bundle corresponding to $[\rho] \in X_G(\mathbb{Z}^2)$ then the second obstruction class to the existence of a global section of $E_\rho$ lies in $H^2(\mathbb{Z}^2, \pi_1(G)) = \pi_1(G)$, with the action of $\mathbb{Z}^2$ on $\pi_1(G)$ given by $\rho$, c.f. [Go2]. The obstruction map $F : X_G(\mathbb{Z}^2) \to \pi_1(G)$ is constant on connected components. Goldman conjectures that $F$ maps bijectively connected components of $X_G(\mathbb{Z}^2)$ onto $\pi_1(G)$, for all semi-simple algebraic groups, [Go2].

As before, let $X_G^0(\Gamma)$ be the connected component of the trivial character. The proof of the following statement appears in [Th]. For the convenience of the reader we include the proof below.

**Theorem 16.** ([Th]) For any complex reductive algebraic group $G$ and its Cartan subgroup (a maximal complex torus) $\mathbb{T}$, the map

$$\mathbb{T}^2 = \text{Hom}(\mathbb{Z}^2, \mathbb{T}) \to \text{Hom}(\mathbb{Z}^2, G) \to \text{Hom}(\mathbb{Z}^2, G)/G = X_G(\mathbb{Z}^2)$$

factors through an isomorphism $\chi : \mathbb{T}^2/W \to X_G^0(\mathbb{Z}^2)$, where the Weyl group $W$ acts diagonally on $\mathbb{T} \times \mathbb{T}$. 
Proof. Let us first give an elementary proof that \( \chi \) is onto for \( G = GL(n, \mathbb{C}) \) and \( SL(n, \mathbb{C}) \). For any commuting matrices \( g_1, g_2 \in G \) there is \( h \in G \) such that \( h g_1 h^{-1}, h g_2 h^{-1} \) are upper triangular. Furthermore, \( h \) can be chosen so that the entries of \( h g_1 h^{-1}, h g_2 h^{-1} \) above diagonal are uniformly arbitrarily small. Therefore, every representation \( \rho : \mathbb{Z}^2 \to G \) has an infinite sequence of conjugates approaching some \( \phi : \mathbb{Z}^2 \to \mathbb{T} \subset G \) in the classical topology and, hence, in the Zariski topology as well. Since all points are closed in \( X_G(\mathbb{Z}^2) \), every point in it is represented by some \( \phi : \mathbb{Z}^2 \to \mathbb{T} \).

In the proof of \( \chi \) being onto for every reductive \( G \) we follow an argument of [Th]. Denote the equivalence class of \( \rho \in \mathbb{Z}^2 \to G \) in \( X_G(\mathbb{Z}^2) \) by \([g_1, g_2]\), where \( g_1 = \rho(1, 0) \), \( g_2 = \rho(0, 1) \). Let \( h \) be a regular semisimple element of \( G \) and let \( T \) be its centralizer. Then every element in some open neighborhood of \( h \) in \( G \) is conjugated to a regular element in \( T \). Hence \([h, e]\) has an open neighborhood \( U \subset X_G(\mathbb{Z}^2) \) in complex topology whose every element is represented by \([g h' g^{-1}, k] = [h', g^{-1} k g]\), where \( h' \in T \) and \( h' \) is regular. Since \( T \) is the centralizer of \( h' \), \( g^{-1} k g \in T \). Hence we proved that the image of \( \chi \) contains the open set \( U \) in complex topology. Consequently, the image of \( \chi \) is dense in Zariski topology. However, by [Ric], the connected component \( \text{Hom}^0(\mathbb{Z}^2) \) of the trivial homomorphism in \( \text{Hom}(\mathbb{Z}^2, G) \) is irreducible for every connected reductive group \( G \). Therefore, \( \chi \) is onto.

To prove that \( \chi \) is 1-1, we need to show that if \( g_1, ..., g_N, g'_1, ..., g'_N \in T \) and \((g'_1, ..., g'_N) = g(g_1, ..., g_N) g^{-1}\) for some \( g \in G \) then \((g'_1, ..., g'_N) = w(g_1, ..., g_N)\), for some \( w \in W \) acting on \( T \). We follow an argument of Borel, [Bo], and Thaddeus, [Th]: The centralizer of \( g_1, ..., g_N, Z(g_1, ..., g_n) \subset G \) is a reductive group by [Hum] 26.2A since the proof there is valid not only for a subtorus but for any subset. \( T \) and \( g^{-1} T g \) are maximal tori in \( Z(g_1, ..., g_n) \) and, therefore, \( T \) is conjugate to \( g^{-1} T g \) by some \( h \in Z(g_1, ..., g_n) \). Then \( h g \in N(T) \) represents \( w \in W \) which sends \((g_1, ..., g_N)\) to \((g'_1, ..., g'_N)\).

More generally, one can prove that \( \chi : T^n / W \to X^0_G(\mathbb{Z}^n) \) is an embedding for every \( n \).

Let \( \Lambda_g \) be the weight lattice of the Lie algebra \( g \) of \( G \). Since every weight \( \alpha \in \Lambda_g \) is a homomorphism \( \alpha : T \to \mathbb{C}^* \) and a regular function on \( T \), there is an natural map \( L : \Lambda_g \to C[T] \). Extending it additively to the group ring of \( \Lambda_g \) we get a \( C \)-algebra homomorphism \( L : C[\Lambda_g] \to C[T] \).

**Lemma 17.** \( L : C[\Lambda_g] \to C[T] \) is an isomorphism of \( C \)-algebras.

**Proof.** Let \( \alpha_1, ..., \alpha_n : T \to \mathbb{C}^* \) be weights of a faithful representation \( V \) of \( G \). Since the intersection of the kernels of these group homomorphisms is trivial, \((\alpha_1, ..., \alpha_n)\) embeds \( T \) into \((\mathbb{C}^*)^n \). Consequently, \( \alpha_i \)’s generate \( C[T] \) and \( L \) is onto. To show that \( L \) is 1-1, note that \( L \) embeds the group \( \Lambda_g \) into the multiplicative group \((C[T])^*\) of the ring \( C[T] \). Since \( C[T] \simeq \mathbb{C}[x_1^\pm 1, ..., x_n^\pm 1] \), the elements of \((C[T])^* = \langle x_1^\pm 1, ..., x_n^\pm 1 \rangle \) are linearly independent in \( C[T] \). Hence \( L \) is 1-1.

Consequently, \( L \otimes L \) is an isomorphism between \( C[\Lambda_g^2] = C[\Lambda_g] \otimes C[\Lambda_g] \) and \( C[T^2] \) restricting to an isomorphism \( C[\Lambda_g^2]^W \to C[T^2]^W = C[T^2 / W] \).

**Corollary 18.** \( C[X^0_G(\mathbb{Z}^2)] \simeq C[\Lambda_g^2]^W \). Consequently, the algebraic variety \( X^0_G(\mathbb{Z}^2) \) depends on the Lie algebra of \( G \) only.
Example 19. If $G = SL(n, \mathbb{C})$ then $\mathbb{T} = \{(x_1, \ldots, x_n) \in (\mathbb{C}^\ast)^n : x_1 \cdot \ldots \cdot x_n = 1\}$ and $X_{SL(n, \mathbb{C})}(T) = \mathbb{T}^2/S_n$ where $\sigma(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)})$, for $\sigma \in S_n$.

Corollary 20. For $G = SL(n, \mathbb{C}), SO(n, \mathbb{C}), O(n, \mathbb{C}), Sp(2n, \mathbb{C})$, the algebra $\mathbb{C}[X^n_G(\Gamma)]$ is generated by $\tau_\gamma$ for all $\gamma \in \Gamma$.

Proof. Since the embedding $X^n_G(\Gamma) \hookrightarrow X_G(\Gamma)$ induces an epimorphism $\mathbb{C}[X_G(\Gamma)] \twoheadrightarrow \mathbb{C}[X^n_G(\Gamma)]$, the statement follows immediately from Proposition 19 for $G = SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$. For $G = SO(n, \mathbb{C})$ the statement follows Corollary 18. \hfill $\square$

4. DEFORMATION-QUANTIZATIONS

If $(M, \omega)$ is a holomorphic symplectic manifold then the space $\mathcal{H}(M)$ of holomorphic functions on it is a Poisson algebra, i.e. a commutative algebra together with a Poisson bracket: if $f, g \in \mathcal{H}(M)$ then
\begin{equation}
\{f, g\} = -V_f(g),
\end{equation}
where $V_f$ is the Hamiltonian vector field of $f$ defined by condition $\omega(V_f, W) = W(f)$ for every vector field $W$.

Theorem 24 in Sec. 6 implies:

Remark 21. For every classical group $G$ and for every closed orientable surface $F$, the Poisson bracket on $\mathcal{H}(X^n_G(F))$ restricts to a Poisson bracket on $\mathbb{C}[X^n_G(F)]$.

For any $\mathbb{C}$-subalgebra $R \subset \mathbb{C}[[h]]$, let $\mathbb{C}_0$ be $\mathbb{C}$ considered as an $R$-module via the homomorphism $\varepsilon : R \to \mathbb{C}$, $\varepsilon(h) = 0$. Let $B$ be an associative $R$-algebra, such that the $R$-submodule of $B$ generated by $\ker \varepsilon$ is an ideal. In this case, $B/(\ker \varepsilon) = B \otimes_R \mathbb{C}_0$. Let us assume that this ring is commutative.

Since $(B \otimes_R \mathbb{C}[[h]])/(h) = B \otimes_R \mathbb{C}_0$, for any $x, y \in B$, $x \cdot y - y \cdot x$ is divisible by $h$ in $B \otimes_R \mathbb{C}[[h]]$ and, consequently,
\begin{equation}
\{x, y\} = \frac{1}{h}(x' \cdot y' - y' \cdot x') + hB \otimes_R \mathbb{C}[[h]]
\end{equation}
defines a unique element in $B \otimes_R \mathbb{C}_0$. It is easy to check that $\{x, y\}$ depends on the coset values of $x$ and $y$ in $B \otimes_R \mathbb{C}_0$ only and, therefore, $\{\cdot, \cdot\}$ descends to a bracket on $B \otimes_R \mathbb{C}_0$. Furthermore, it is a Poisson bracket.

Let $A$ be a commutative algebra with a Poisson bracket $\{\cdot, \cdot\} : A \times A \to A$. We say that $B$ as above is a deformation quantization of $A$ in the direction of the $\{\cdot, \cdot\}$ if there is an isomorphism of Poisson algebras $\Psi : B \otimes_R \mathbb{C}_0 \to A$.

(Often, deformation-quantization is defined more restrictively with the conditions: $R = \mathbb{C}[[h]]$ and $B$ is topologically $R$-free.)

Given the embedding $R = \mathbb{C}[q^{\pm 1/D(\mathfrak{g})}] \subset \mathbb{C}[[h]]$, $q = e^h$, $\mathbb{C}_0$ is the $\mathbb{C}[q^{\pm 1/D(\mathfrak{g})}]$-module $\mathbb{C}$ via the homomorphism $q \to 1$. There is an isomorphism
\[\eta : \mathbb{A}_g^W \otimes \mathbb{C}_0 \to \mathbb{C}[\mathbb{A}_g^2]\]
\[\eta(E_\alpha) = (\alpha, 0), \eta(Q_\alpha) = (0, \alpha),\]
for $\alpha \in \mathbb{A}_g$, which restricts to an isomorphism
\[\eta : \mathbb{A}_g^W \otimes \mathbb{C}_0 \to \mathbb{C}[\mathbb{A}_g^2]^W\].

Therefore,
\begin{equation}
\Theta : \mathbb{A}_g^W \otimes \mathbb{C}_0 \xrightarrow{\eta} \mathbb{C}[\mathbb{A}_g^2]^W \to \mathbb{C}[T^2/W] \simeq \mathbb{C}[X^n_G(\mathbb{Z}^2)]
\end{equation}
is an isomorphism as well.
Corollary 22. \( \mathcal{A}_g^W \) together with (17) is a deformation-quantization of \( C[X_G^0(\mathbb{Z}^2)] \).

Now Theorem 8 can be stated more precisely as

Theorem 23. (Proof in Sec. 6) For classical Lie algebras, \( \mathfrak{g} = \mathfrak{sl}(n), \mathfrak{sp}(2n, \mathbb{C}), \) and \( \mathfrak{so}(n, \mathbb{C}) \), the above deformation-quantization is in the direction of the Goldman bracket\(^5\).

We conjecture that the above statement holds for the exceptional Lie algebras \( \mathfrak{g} \) as well.

5. Goldman bracket

Proposition 24. (1) For \( G = \text{SL}(n, \mathbb{C}) \) the Goldman bracket is given by

\[
\{\tau_\alpha, \tau_\beta\} = \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta) \left( \tau_{\alpha_p \beta_p} - \frac{\tau_{\alpha \beta_p}}{n} \right),
\]

where \( \alpha, \beta \) are any smooth closed oriented loops in \( F \) in general position. (We identify closed oriented loops in \( F \) with conjugacy classes in \( \pi_1(F) \).) \( \alpha \cap \beta \) is the set of the intersection points and \( \alpha_p \beta_p \) is the product of \( \alpha \) and \( \beta \) in \( \pi_1(F, p) \), and \( \varepsilon(p, \alpha, \beta) \) is the sign of the intersection:

\[
\begin{array}{cccc}
\alpha & + & \beta & - \\
\downarrow & & \downarrow & \\
\beta & & \alpha & \\
\end{array}
\]

(2) For \( G = \text{SO}(n, \mathbb{C}), \text{Sp}(n, \mathbb{C}) \),

\[
\{\tau_\alpha, \tau_\beta\} = \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta) \left( \tau_{\alpha_p \beta_p} - \tau_{\alpha_p \beta_p^{-1}} \right).
\]

Proof. Let \( f_\alpha = \text{Re} \tau_\alpha \) and \( \tilde{f}_\alpha = \text{Im} \tau_\alpha \). The formulas for Goldman bracket between \( f_\alpha \)'s and \( \tilde{f}_\alpha \)'s appear in [Go2]. Since Goldman’s \( B \) and \( f_\alpha \)'s are twice ours, the formulas for \( \text{SL}(n, \mathbb{C}) \) are

\[
\{f_\alpha, f_\beta\} = \frac{1}{2} \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta) \left( f_{\alpha_p \beta_p} - f_{\alpha_p \beta_p^{-1}} \right),
\]

\[
\{f_\alpha, \tilde{f}_\beta\} = \{\tilde{f}_\alpha, f_\beta\} = \frac{1}{2} \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta) \left( \tilde{f}_{\alpha_p \beta_p} - \tilde{f}_{\alpha_p \beta_p^{-1}} \right),
\]

Now (19) follows from

\[
\{\tau_\alpha, \tau_\beta\} = \{f_\alpha + i\tilde{f}_\alpha, f_\beta + i\tilde{f}_\beta\} = 2\{f_\alpha, f_\beta\} + 2i\{f_\alpha, \tilde{f}_\beta\}.
\]

Let \( G = \text{SO}(n, \mathbb{C}) \) or \( \text{Sp}(2n, \mathbb{C}) \) now. By [Go2],

\[\{f_\alpha, f_\beta\} = \left( f_{\alpha_p \beta_p} - f_{\alpha_p \beta_p^{-1}} \right) \]

5 By Remark 21, the Poisson bracket on \( C[X_G(T)] \) is well defined.
(Since \(so(n, \mathbb{C}) = o(n, \mathbb{C})\), the above formula holds for the Goldman bracket for \(SO(n, \mathbb{C})\) as well.) Furthermore, from [Go2, Lemma 1.11] and Goldman’s product formula, [Go2], we obtain additional formulas
\[
\{ \hat{f}_{\alpha}, \hat{f}_{\beta} \} = \{ f_{\alpha}, f_{\beta} \} = \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta) \left( \hat{f}_{\alpha_p} - \hat{f}_{\alpha_p^{-1}} \right).
\]
\[
\{ \hat{f}_{\alpha}, \hat{f}_{\beta} \} = -\{ f_{\alpha}, f_{\beta} \}.
\]
Now (19) follows from (22).

In a torus \(T\), the signed number of the intersection points of any two curves \((a, b), (c, d)\) in \(T\) is
\[
\begin{vmatrix}
a & b \\
c & d
\end{vmatrix}
\]
and, therefore, formulas (18) and (19) for Goldman bracket on \(\mathbb{C}[X_G(\mathbb{Z}^2)]\) simplify to

\[
\tau_{a,b}, \tau_{c,d} = \begin{vmatrix}
a & b \\
c & d
\end{vmatrix}
\left( \tau_{a+b+c+d} - \frac{\tau_{a,b} \tau_{c,d}}{n} \right), \text{ for } G = SL(n, \mathbb{C}),
\]
and

\[
\tau_{a,b}, \tau_{c,d} = \begin{vmatrix}
a & b \\
c & d
\end{vmatrix}
\left( \tau_{a+b+c+d} - \tau_{a-c,b-d} \right) \text{ for } G = SO(n, \mathbb{C}), Sp(2n, \mathbb{C}).
\]

6. PROOF OF THEOREM 23

In light of isomorphism (17), it is enough to prove that Poisson bracket (16) on \(A^{W}_p \otimes \mathbb{C}_0\), which we will denote here by \(\{ \cdot, \cdot \}_A\), coincides with Goldman bracket, (23) and (24), on \(\mathbb{C}[X_G(\mathbb{Z}^2)]\) which we denote here by \(\{ \cdot, \cdot \}_G\).

Furthermore, since Poisson brackets are skew-commutative and they satisfy Leibniz’ law:
\[
\{f, gh\} = \{f, g\} h + g\{f, h\}
\]
it is enough to prove that the above brackets coincide for algebra generators only. Hence Theorem 23 follows from Proposition 17 and Propositions 25 and 26.

Proposition 25. For \(G = SL(n, \mathbb{C})\)
\[
\Theta(\{ \tau_{a,b}, \tau_{c,d} \}_A) = \{ \Theta(\tau_{a,b}), \Theta(\tau_{c,d}) \}_G,
\]
for any \(a, b, c, d \in \mathbb{Z}\), where \(\Theta\) is defined in (17).

Proof. Let \(\alpha_i : h \to \mathbb{C}, i = 1, \ldots, n\) be the weights defined in Section 2.1. Denote \(E_{\alpha_i}\) and \(Q_{\alpha_i}\) by \(E_i\) and \(Q_i\) respectively. Then
\[
\Theta \left( \sum_{i=1}^{n} E_i^a Q_i^b \right) = \tau_{a,b}.
\]
Hence, by (10), \(\{ \Theta^{-1} \tau_{a,b}, \Theta^{-1} \tau_{c,d} \}_A\) is the image of
\[
\frac{1}{h} \left( \sum_{i=1}^{n} E_i^a Q_i^b \cdot \sum_{i=1}^{n} E_i^c Q_i^d - \sum_{i=1}^{n} E_i^c Q_i^d \cdot \sum_{i=1}^{n} E_i^a Q_i^b \right)
\]
in \(A^{W}_{SL(n, \mathbb{C})} \otimes \mathbb{C}_0\). By (5) and (9),
\[
\sum_{i=1}^{n} E_i^a Q_i^b \cdot \sum_{i=1}^{n} E_i^c Q_i^d = \sum_{i,j} E_i^a E_j^c Q_i^b Q_j^d \cdot \left\{ \begin{array}{ll}
\frac{1}{n} hbc & \text{for } i \neq j \\
\frac{1}{n-1} hbc & \text{for } i = j
\end{array} \right\} \mod h^2
\]
for \( q = e^h \). Hence (20) equals
\[
\sum_i \left( \frac{1}{n} - 1 \right) (bc - ad) E_i^{a+c} Q_i^{b+d} + \sum_{i \neq j} \frac{1}{n} (bc - ad) E_i^a E_j^c Q_i^b Q_j^d =
\]
\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \sum_i E_i^{a+c} Q_i^{b+d} - \frac{1}{n} \sum_{i,j} E_i^a E_j^c Q_i^b Q_j^d
\]
and, by (20), it equals to
\[
\begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \left( \Theta^{-1} \tau_{a+b} - \frac{\Theta^{-1} \tau_{a,b} \cdot \Theta^{-1} \tau_{a,b}}{n} \right).
\]
Now the statement follows from (20). \( \square \)

**Proposition 26.** For \( G = SO(n, \mathbb{C}), Sp(2n, \mathbb{C}) \),
\[
\Theta \{ \tau_{a,b}, \tau_{c,d} \} = \{ \Theta(a,b), \Theta(c,d) \} G,
\]
for any \( a, b, c, d \in \mathbb{Z} \).

**Proof.** Let \( s_n \) be the \( n \times n \) matrix
\[
s_n = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & \ldots & \vdots & \vdots \\ 1 & \ldots & 0 & 0 \end{pmatrix}.
\]
Define matrices
\[
B_m = \begin{pmatrix} 0 & s_n \\ s_n & 0 \end{pmatrix} y, \text{ for } m = 2n, \quad B_m(x, y) = \begin{pmatrix} 0 & s_n & 0 \\ s_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ for } m = 2n + 1,
\]
and \( \Omega_{2n} = \begin{pmatrix} 0 & -s_n \\ s_n & 0 \end{pmatrix} \).

Then \( Sp(2n) \subset SL(m, \mathbb{C}) \) for \( m = 2n \) and \( SO(m, \mathbb{C}) \subset SL(m, \mathbb{C}) \) are the groups of isomorphisms of \( \mathbb{C}^n \) preserving bilinear forms \( (x, y) \to x^T \Omega_{m} y \) and \( (x, y) \to x^T B_m y \), respectively. The advantage of this definition of \( SO(n, \mathbb{C}) \) over the “standard” one, \( SO(n, \mathbb{C}) = \{ A : A \cdot A^T = I_n, \ det(A) = 1 \} \), is that the intersection of the group of diagonal matrices in \( GL(n, \mathbb{C}) \) with \( SO(n, \mathbb{C}) \) (defined above) is a maximal torus in \( SO(n, \mathbb{C}) \). The Lie algebras \( so(m) \) and \( sp(m) \) for \( m \) even are spaces of matrices \( X \) such that \( B_m X + X B_m = 0 \) and \( \Omega_{m} X + X \Omega_{m} = 0 \) respectively. If \( g = sp(2n), so(2n), so(2n + 1) \), then \( H_i = E_{i_i} - E_{n+i,n+i} \), for \( i = 1, \ldots, n \), form a basis of its Cartan subalgebra \( \mathfrak{h} \). Furthermore, the vectors \( H_1, \ldots, H_n \) are of equal length and are mutually orthogonal with respect of the Killing form.

The weight lattice is generated by weights \( \alpha_1, \ldots, \alpha_n \) dual to \( H_1, \ldots, H_n \), \( \alpha_i(H_j) = \delta_{ij} \), and the dual Killing form on \( \mathfrak{h}^* \) is given by
\[(27) \quad (\alpha_i, \alpha_j) = \delta_{ij}. \]

The Weyl group is composed all signed permutations \( W = (\mathbb{Z}/2)^n \rtimes S_n \) for \( sp(2n) \) and \( so(2n + 1) \) and it is the subgroup of \( (\mathbb{Z}/2)^n \rtimes S_n \) composed of signed permutations with an even number of sign changes, \( (\mathbb{Z}/2)^{n-1} \rtimes S_n \), for \( so(2n) \).
Let $E_i = E_{\alpha_i}$ and $Q_i = Q_{\alpha_i}$ for $i = 1, \ldots, n$. We have

$$\Theta^{-1}(\tau_{a,b}) = \sum_{i=1}^{n} (E_i^a Q_i^b + E_i^{-a} Q_i^{-b}).$$

By (5) and (27),

$$E_i^a Q_i^b \cdot E_i^c Q_i^d = -hbc E_i^{a+c} Q_i^{b+d} \mod h^2,$$

for $q = e^h$. Therefore, $\Theta^{-1}(\tau_{a,b}) \cdot \Theta^{-1}(\tau_{c,d}) = -hbc \sum_{i=1}^{n} E_i^{a+c} Q_i^{b+d} + E_i^{-a-c} Q_i^{-b-d} - E_i^{-a-c} Q_i^{-b-d} - E_i^{a+c} Q_i^{b+d} \mod h^2$.

By (16),

$$\{\tau_{a,b}, \tau_{c,d}\}_A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot (\tau_{a+c,b+d} - \tau_{a-c,b-d}).$$

Now the statement follows from (21).

\begin{flushright} \Box \end{flushright}

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