SEMIGLOBAL EXPONENTIAL STABILIZATION OF NONAUTONOMOUS SEMILINEAR PARABOLIC-LIKE SYSTEMS

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Abstract. It is shown that an explicit oblique projection nonlinear feedback controller is able to stabilize semilinear parabolic equations, with time-dependent dynamics and with a polynomial nonlinearity. The actuators are typically modeled by a finite number of indicator functions of small subdomains. No constraint is imposed on the sign of the polynomial nonlinearity. The norm of the initial condition can be arbitrarily large, and the total volume covered by the actuators can be arbitrarily small. The number of actuators depends on the operator norm of the oblique projection, on the polynomial degree of the nonlinearity, on the norm of the initial condition, and on the total volume covered by the actuators. The range of the feedback controller coincides with the range of the oblique projection, which is the linear span of a subspace spanned by a suitable finite number of eigenfunctions of the diffusion operator. For rectangular domains, it is possible to explicitly construct/place the actuators so that the stability of the closed-loop system is guaranteed. Simulations are presented, which show the semiglobal stabilizing performance of the nonlinear feedback.

1. Introduction. Nonlinear parabolic equations appear in many models of real world evolution processes. Therefore, the study of such equations is important for real world applications. In particular, it is of interest to know whether it is possible to drive the evolution to a given desired behavior or whether it is possible to stabilize such evolution process, by means of suitable controls. The simplest model involving parabolic equations is the heat equation, modeling the evolution of the temperature in a room [22, Chapitre II]. Parabolic equations also appear in models for population dynamics [15, 4], traffic dynamics [41], and electrophysiology [42].

Usually, controlled parabolic equations can be written as a nonautonomous evolutionary system in the abstract form

\[ \dot{y} + Ay + A_{rc}(t)y + N(t, y) - \sum_{i=1}^{M} u_i(t)\Psi_i = 0, \quad y(0) = y_0, \]

where \( y \) is the state, \( y_0 \) and \( \Psi_i, i \in \{1, 2, \ldots, M\} \), are given in a Hilbert space \( H \), and \( u = (u_1, \ldots, u_M) \) is a control function at our disposal, \( u(t) \in \mathbb{R}^M, t \geq 0 \). The

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linear operator $A$ is a diffusion-like operator and the linear operator $A_{rc}$ is a time-dependent reaction-convection-like operator. The operator $\mathcal{N}$ is a time-dependent nonlinear operator. The general properties asked for $A$, $A_{rc}$, and $\mathcal{N}$ will be precised later.

In the linear case, $\mathcal{N} = 0$, is has been proven in [31] that the closed-loop system
\[
\dot{y} + Ay + A_{rc}(t)y - K_{U,M}^{F,M}(t, y) = 0, \quad y(0) = y_0 \in H, \tag{2}
\]
is globally exponentially stable, with the feedback control operator
\[
y \mapsto K_{U,M}^{F,M}(t, y) := \Pi_{U,M}^F (Ay + A_{rc}(t)y - \mathcal{F}(y)), \tag{3}
\]
where
\[
\mathcal{F}(y) = \lambda \text{Id} y,
\]
provided the condition
\[
\Pi_M := \alpha_{M+1} - \left(6 + 4 \left| \Pi_{U,M}^F \right|_{L(H)}^2 \right) |A_{rc}|_{L^\infty((0, +\infty), L(H, V^*))}^2 > 0 \tag{5}
\]
holds true. In (3) and (4), $\text{Id}$ is the identity operator, $\lambda > 0$ is an arbitrary constant, and $\Pi_{U,M}^F$ stands for the oblique projection in $H$ onto the closed subspace $U_M$ along the closed subspace $E_M^\perp$. Where $U_M := \text{span}\{\Psi_i \mid i \in \{1, 2, \ldots, M\}\}$ is the linear span of our $M$ linearly independent actuators and $E_M := \text{span}\{e_i \mid i \in \mathcal{M}\}$, with $\mathcal{M} = \{1, 2, \ldots, M\}$, is the linear span of “the” first $M$ linearly independent eigenfunctions of the diffusion operator $A: D(A) \to H$, with domain $D(A) \overset{d,c}{\to} H$.

In (5), $V'$ is a Hilbert space, $H \overset{d,c}{\to} V'$, and $\alpha_{M+1}$ is the $(M+1)$st eigenvalue of $A$. The eigenvalues of $A$, denoted by $\alpha_i$, are supposed to satisfy
\[
Ae_i = \alpha_i e_i, \quad 0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \ldots, \quad \lim_{i \to +\infty} \alpha_i = +\infty.
\]

**Remark 1.1.** Note that $K_{U,M}^{F,M}(t, y) = \sum_{i=1}^{M} u_i(t)\Psi_i$ for suitable $u_i(t) \in \mathbb{R}$.

It is not difficult to see that we can follow the arguments in [31, Thms. 3.5, 3.6, and Rem. 3.8] to conclude that system (2) is still stable if we replace (4) by
\[
\mathcal{F}(y) = Ay + \lambda \text{Id} y.
\]

Observe that (5) concerns a single $M \in \mathbb{N}$ and a single pair $(U_M, E_M)$. The following result, which follows straightforwardly from the sufficiency of (5), concerns a sequence of pairs $(U_M, E_M)_{M \in \mathbb{N}}$.

**Theorem 1.2.** Assume that we can construct a sequence $(U_M, E_M)_{M \in \mathbb{N}}$ such that $\left| \Pi_{U,M}^F \right|_{L(H)} \leq C_P$ remains bounded, with $C_P > 0$ independent of $M$. Then system (2) is globally exponentially stable for large enough $M$, with $\mathcal{F}(y) \in \{\lambda \text{Id} y, Ay + \lambda \text{Id} y\}$.

Our main goal is to prove that an analogous explicit feedback allow us to semiglobally stabilize nonlinear systems as (1), for a suitable class of nonlinearities. We underline that we shall not assume any condition on the sign of the nonlinearity $\mathcal{N}$, which means that the uncontrolled solution may blow up in finite time. For results concerning blow up of solutions, see [7, 36, 34]. In particular, this means that we will have to guarantee that the controlled solution does not blow up, which is a non-trivial task/problem. This is a problem we do not meet when dealing with linear systems, because solutions of linear systems do not blow up in finite time.
In the linear case the number $M$ of actuators that allow us to stabilize the system does not depend on the initial condition, while in the nonlinear case it does. We shall prove that $M$ depends only on a suitable norm of the initial condition, this dependence is what motivates the terminology “semiglobal stability” we use throughout the paper.

For nonlinear systems, previous results in the related literature are concerned with local stabilization, and such results are often derived through a suitable nontrivial fixed point argument. In such situation the feedback operator is linear and is such that it globally stabilizes the linearized system, with $N = 0$. In general, such linearization based feedback will be able to stabilize the nonlinear system only if the initial condition is small enough, in a suitable norm. Here, in order to cover arbitrary large initial conditions, and thus obtain the semiglobal stabilization result for (1), we will use a nonlinear feedback operator. Instead of starting by constructing a feedback stabilizing the linearized system, we deal directly with the nonlinear system.

1.1. The main result. We show that, for a suitable Hilbert space $V \xrightarrow{d.c.} H$, and for an arbitrary given $R > 0$, system (1)

\[ \dot{y} + Ay + A_{rc}(t)y + N(t, y) - K_{U_M}^{F,M,N}(t, y) = 0, \quad y(0) = y_0, \]  

with the feedback

\[ y \mapsto K_{U_M}^{F,M,N}(t, y) := P_{U_M}^{E_M}(Ay + A_{rc}(t)y + N(t, y) - F(y)) \]

is stable, provided the initial condition is in the ball \( \{ v \in V \mid |v|_V < R \} \) and the pair \((U_M, E_M)\) satisfies a suitable “nonlinear version” of (5). The number $M$ of actuators needed to stabilize the system will (or may) increase with $R$. A precise statement of the main stability result concerning a single pair \((U_M, E_M)\), together a “nonlinear version” of the sufficient stability condition (5) is given hereafter, once we have introduced some notation and terminology. A consequence of that result will be the following “nonlinear version” of Theorem 1.2.

**Theorem 1.3.** Assume that we can construct a sequence \((U_M, E_M)_{M \in \mathbb{N}}\) such that \( \|P_{U_M}^{E_M}\|_{\mathcal{L}(H)} \leq C_P \) remains bounded, with $C_P > 0$ independent of $M$. Then, with \( F(y) = Ay + \lambda \text{Id} y \), system (6) is exponentially stable, for large enough $M$ depending on $|y_0|_V$.

The operator choice $F(y) = \lambda \text{Id} y$, used in previous works for linear systems, will not necessarily satisfy the assumptions hereafter (Assumption 3.6, in particular). That is, we cannot conclude/guarantee (from our results) that such choice will semiglobally stabilize the nonlinear system. To better understand the differences between the two choices, we will consider a general operator $F(y) = F_M(P_{E_M}y)$ depending only on the orthogonal projection $P_{E_M}$ of the state $y$ in $H$ onto $E_M$.

Further $F_M \colon E_M \to E_M$ is a continuous operator. Notice that, with $\varsigma \in \{0, 1\}$ we have that $P_{E_M}(\varsigma A + \lambda \text{Id})P_{E_M}$ is continuous, and the feedback in (6b) satisfies $K_{U_M}^{\varsigma A + \lambda \text{Id}, M,N} = K_{U_M}^{\varsigma A + \lambda \text{Id}, M,N}P_{E_M}$, because $P_{E_M}$ commutes with both $A$ and Id and because $P_{U_M}^{E_M}P_{E_M} = P_{U_M}^{E_M}$. Notice also that when $F$ is linear and $N \neq 0$, then $y \mapsto K_{U_M}^{F,M,N}(t, y)$ is linear, while $y \mapsto K_{U_M}^{F,M,N}(t, y)$ is nonlinear.
1.2. Motivation and short comparison to previous works. We find systems in form (1) when, for example, we want to stabilize a system to a trajectory \( \hat{z} \). That is, suppose \( \hat{z} \) solves the nonlinear system

\[
\dot{z} + A \hat{z} + f(\hat{z}) = 0, \quad \hat{z}(0) = \hat{z}_0,
\]

and that \( \hat{z} \) has suitable desired properties (e.g., it is essentially bounded and regular). In many situations, it may happen that the solution issued from a different initial condition \( z_0 \) may present a non-desired behavior (e.g., not remaining bounded, or even blowing up in finite time). In such situation, we would like to find a control \( u(t) = \sum_{i=1}^{M} u_i(t) \), such that the solution of

\[
\dot{z} + Az + f(z) + u = 0, \quad z(0) = z_0,
\]

(7) approaches the desired behavior \( \hat{z} \). More precisely, we would like to have

\[
|z(t) - \hat{z}(t)|_{\mathfrak{H}} \leq Ce^{-\mu t} |z(0) - \hat{z}(0)|_{\mathfrak{H}},
\]

for some normed space \( \mathfrak{H} \) and some \( \mu > 0 \). Observe that the difference \( y := z - \hat{z} \) satisfies a dynamics as (1), because from Taylor expansion (for regular enough \( f \)) we may write \( f(z) - f(\hat{z}) = A_{rc}(t) y + \mathcal{N}(t, y) \), with \( A_{rc}(t) = \frac{d}{dt} f(\hat{z}) \) and with a remainder \( \mathcal{N}(t, y) \). Notice that \( \mathcal{N} \) vanishes if, and only if, \( f \) is affine, otherwise \( \mathcal{N}(t, y) \) is nonlinear. Therefore, stabilizing (7) to the targeted trajectory, is equivalent to stabilizing system (1) (to zero), because (8) reads

\[
|y(t)|_{\mathfrak{H}} \leq Ce^{-\mu t} |y(0)|_{\mathfrak{H}}.
\]

In previous works on internal stabilization of nonautonomous parabolic-like systems including [11, 30, 29, 14, 46], the exact null controllability of the corresponding linearized systems (by means of infinite dimensional controls, see [17, 19, 20, 21, 23, 26, 57]) played a key role in the proof of the existence of a stabilizing control. See also [3] for the weakly damped wave equation. We would like to underline that for the proof of the stability of an oblique projection based closed-loop system, we do not need to assume the above null controllability result.

Our results are also true for the particular case of autonomous systems, which has been extensively studied. However, in such case other tools may be, and have been, used. Among such tools we have the spectral properties of the system operator \( A + A_{rc} \). We refer to the works [43, 49, 6, 8, 10, 12, 9, 24, 40, 16] and references therein. See also the comments in [31, Sect. 6.5]. Finally we refer to the examples in [56], showing that in the nonautonomous case, the spectral properties of \( A + A_{rc}(t) \), at each time \( t \geq 0 \), are not appropriate for studying the stability of the corresponding nonautonomous system.

Though we do not deal here with boundary controls, we refer to [44, 48, 50] for works on the stabilization of the Navier–Stokes equation, evolving in a bounded domain \( \Omega \subset \mathbb{R}^3 \), to a targeted trajectory. In [44, 48] the targeted trajectory is independent of time (autonomous case), while in [50] it is time-dependent (nonautonomous case). In [44] the global stability of the closed-loop is shown to hold in \( L^2 \)-norm for at least one (not necessarily unique) appropriately defined “weak” solution. In [48] the local stability of the closed-loop system has been shown to hold in the Sobolev \( W^{s,2} \)-norm, with \( s \in (\frac{1}{2}, 1) \), and the solutions of the closed-loop system are more regular and unique. In [50] the local stability of the closed-loop system has been shown to hold in the \( W^{1,2} \)-norm and the solutions are unique. Recall that \( L^2 \supset W^{0,2} \supset W^{s,2} \supset W^{s+2,2} \), for \( 0 < s_1 < s_2 \).

Our results can be used to conclude the semiglobal stability of nonautonomous oblique projection based closed-loop parabolic-like systems with internal controls, where semiglobal stability lies between local and global stability. The stability of
the closed-loop system is shown to hold in the $W^{1,2}$-norm, and the solutions are unique. In previous results concerning local stability of parabolic systems, the control domain $\omega$ can be arbitrary and fixed a priori. For our results the volume of the support of the actuators can still be arbitrarily small and fixed a priori, but the support itself is not fixed a priori. See Section 2.2.

Finally, though we consider here the case of parabolic-like systems and are particularly interested in the case where blow up may occur for the free dynamics and on the case our control is finite dimensional, the stabilization problem is still an interesting problem for other types of evolution equations, where blow up does not occur, like those conserving the energy and/or other quantities. For stabilization results (by means of infinite-dimensional control) for nonparabolic-like systems we refer the reader to [53, 5, 33, 52] and references therein.

1.3. **Computational advantage.** We underline that the feedback operators in (3) and (6b) are explicit and the essential step in their practical realization involves the computation of the oblique projection. A classical approach to find a feedback stabilizing control is to compute the solution of the Hamilton–Jacobi–Bellman equation, which is known to be a difficult numerical task, being related with the so-called “curse of dimensionality”, for example see the recent paper [27] (for the autonomous case), where the authors, in order to compute the Hamilton–Jacobi–Bellman feedback, need to approximate a parabolic equation by a 14-dimensional ordinary differential equation (previous works deal with even lower-dimensional approximations). This also means that standard discretization methods as finite elements approximations are not appropriate for computing the Hamilton–Jacobi–Bellman solution, because a 14-dimensional finite elements approximation of a parabolic equation is hardly accurate enough. In the linear case (and with quadratic cost) the Hamilton–Jacobi–Bellman feedback reduces to the (algebraic) Riccati feedback. In this case finite elements approximations can be used, but the computational effort increases considerably as we increase the number of degrees of freedom. For parabolic systems, the computation of the feedback in (3) and in (6b) is considerably cheaper, because the numerical computation of the feedback operators $P_{U_M}^{E_M}$ amounts to the computation of the $M$ eigenfunctions $\{e_i \mid i \in \mathbb{M}\}$, and the computation of the inverse of the matrix $\Theta_M = \{(E_M, U_M)_{H}\} = [(e_i, \Phi_j)_{L^2}] \in \mathbb{R}^{M \times M}$, see [51]. Note that the size of $\Theta_M$ is defined by the number $M$ of actuators, and thus it is independent of the number of degrees of freedom of the space discretization, that is, computing $\Theta_M^{-1}$ does not become a harder task as we refine our discretization.

Even in case we are able to compute an approximation of an Hamilton–Jacobi–Bellman based feedback control, such (approximated) feedback may not guarantee stabilization for arbitrary initial conditions, as reported in [27, Sect. 5.2, Test 2], though we likely obtain a neighborhood of attraction larger than that of the Riccati closed-loop system.

Finally, the main idea behind solving the Riccati or Hamilton–Jacobi–Bellman equations is that of finding a feedback (closed-loop) stabilizing control or an optimal control, under the assumption/knowledge that a stabilizing (open-loop) control does exist. Instead, in this paper, the proof of existence of such a stabilizing control is included in the results.

1.4. **Contents and general notation.** The rest of the paper is organized as follows. In Section 2 we recall suitable properties of oblique projections, present an example of application of our results, and recall previous global and local exponential
stability results, which are related to the problem we address in this manuscript. In Section 3 we introduce the general properties asked for the operators $A$, $A_c$, and $\mathcal{N}$ in (1), and also the properties asked for the triple $(U_M, E_M, \mathcal{F})$ defining the feedback operator. In Section 4 we prove our main result. In Section 5 we show that our results can be applied to the stabilization of semilinear parabolic equations with polynomial nonlinearities. In Section 6 we present the results of numerical simulations showing the performance of the proposed nonlinear feedback. Finally, the appendix gathers proofs of auxiliary results used in the main text.

Concerning the notation, we write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we define $\mathbb{R}_r := (r, +\infty)$ and $\mathbb{R}_r := [r, +\infty)$, for $r \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

For an open interval $I \subseteq \mathbb{R}$ and two Banach spaces $X$, $Y$, we write $W(I, X, Y) := \{y \in L^2(I, X) \mid \dot{y} \in L^2(I, Y)\}$, where $\dot{y} := \frac{\partial}{\partial x}y$ is taken in the sense of distributions. This space is endowed with the natural norm $\|y\|_{W(I, X, Y)} := (|y|^2_{L^2(I, X)} + |\dot{y}|^2_{L^2(I, Y)})^{1/2}$. In the case $X = Y$ we write $H^1(I, X) := W(I, X, X)$.

If the inclusions $X \subseteq Z$ and $Y \subseteq Z$ are continuous, where $Z$ is a Hausdorff topological space, then we can define the Banach spaces $X \times Y$, $X \cap Y$, and $X + Y$, endowed with the norms defined as $|(a, b)|_{X \times Y} := (|a|^2_X + |b|^2_Y)^{1/2}$, $|a|_{X \cap Y} := |(a, a)|_{X \times Y}$, and $|a|_{X + Y} := \inf_{(a, a') \in X \times Y} \{|(a^X, a^Y)|_{X \times Y} \mid a = a^X + a^Y\}$, respectively. In case we know that $X \cap Y = \{0\}$, we say that $X + Y$ is a direct sum and we write $X \oplus Y$ instead.

If the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$. We write $X \overset{d}{\hookrightarrow} Y$, respectively $X \overset{c}{\hookrightarrow} Y$, if the inclusion is also dense, respectively compact.

The space of continuous linear mappings from $X$ into $Y$ is denoted by $\mathcal{L}(X, Y)$. In case $X = Y$ we write $\mathcal{L}(X) := \mathcal{L}(X, X)$. The continuous dual of $X$ is denoted $X' := \mathcal{L}(X, \mathbb{R})$.

The space of continuous functions from $X$ into $Y$ is denoted by $C(X, Y)$, and its subspace of increasing functions, defined in $\overline{E_0}$ and vanishing at 0, by:

$$C_{0,1}(\overline{E_0}, \mathbb{R}) := \{n \mid n \in C(\overline{E_0}, \mathbb{R}), \ n(0) = 0, \ and \ n(x_2) \geq n(x_1) \ if \ x_2 \geq x_1 \geq 0\}.$$  

Next, we denote by $C_{b,1}(X, Y)$ the vector subspace

$$C_{b,1}(X, Y) := \{f \in C(X, Y) \mid \exists n \in C_{0,1}(\overline{E_0}, \mathbb{R}) \forall x \in X : |f(x)|_Y \leq n(|x|_X)\}.$$  

Given a subset $S \subset H$ of a Hilbert space $H$, with scalar product $(\cdot, \cdot)_H$, the orthogonal complement of $S$ is denoted $S^\perp := \{h \in H \mid (h, s)_H = 0 \ for \ all \ s \in S\}$.

Given a sequence $(a_j)_{j \in \{1,2,...,n\}}$ of real constants, $n \in \mathbb{N}_0$, $a_i \geq 0$, we denote $|a| := \max_{1 \leq j \leq n} a_j$. Further, by $\overline{C_{[a_1,...,a_n]}}$ we denote a nonnegative function that increases in each of its nonnegative arguments.

Finally, $C, C_i$, $i = 0, 1, \ldots$, stand for unessential positive constants.

2. Preliminaries. We introduce/recall here specific notation and terminology concerning oblique projections and stability.

2.1. Actuators and eigenfunctions. In the stability condition (5), as we increase $M \in \mathbb{N}_0$ we have sequences of subspaces

$$E_{(1)}, E_{(1,2)}, E_{(1,2,3)}, \ldots \quad \text{and} \quad U_1, U_2, U_3, \ldots \quad (9)$$

where the $M$th term of each sequence is an $M$-dimensional space, $\dim E_M = M = \dim U_M$. 


Motivated by the results in [31, Sect. 4.8] (see also [31, Rem. 3.9]), in order to prove the boundedness of the norm \( \frac{\|F_{U_M}^E\|}{\|E\|_{\mathcal{L}(H)}} \leq C_F \), uniformly on \( M \), it may be convenient to consider different sequences.

To simplify the exposition, we denote by \( \#Z \in \mathbb{N} \) the number of elements of a given finite set \( Z \subseteq Y \). See [25, Sect. 13]. For \( N \in \mathbb{N}_0 \), \( \#Z = N \) simply means that there exists a one-to-one correspondence from \( \{1, 2, \ldots, N\} \) onto \( Z \). Of course \( \#Z = 0 \) means that \( Z = \emptyset \), the empty set. We also denote the collection

\[ \Psi_N(Y) = \{ Z \subseteq Y \mid \#Z = N \}. \]

Now, instead of (9), we consider a more general sequence as follows

\[ E_{\{\sigma_1, \sigma_2, \ldots, \sigma_{M_J}\}}, E_{\{\sigma_1, \sigma_2, \ldots, \sigma_{M_J}\}}, E_{\{\sigma_1, \sigma_2, \ldots, \sigma_{M_J}\}}, \ldots, \text{ and } U_{[\sigma_1]}, U_{[\sigma_2]}, U_{[\sigma_3]}, \ldots \]

that is, denoting \( M_\sigma := \{\sigma_1^M, \sigma_2^M, \ldots, \sigma_{M_J}^M\} \), we have \( \#M_\sigma = |\sigma^M| \) and the sequences

\[ E_{M_\sigma} := \text{span}\{e_i \mid i \in M_\sigma\} \]

where for each \( M \in \mathbb{N}_0 \), the \( M \)-th term of each sequence is a \( \#M_\sigma \)-dimensional space, \( \dim E_{M_\sigma} = \#M_\sigma = \dim U_{\#M_\sigma} \), and the function \( \sigma^M: \{1, 2, \ldots, \#M_\sigma\} \to M_\sigma \in \Psi_{\#M_\sigma}(N_0) \), \( i \mapsto \sigma_i^M \), is a bijection.

For a given \( M \in \mathbb{N}_0 \), we will also need to underline two particular eigenvalues defined as

\[ \alpha_{M_\sigma} := \max\{\alpha_i \mid i \in M_\sigma\}, \quad \alpha_{M_\sigma^+} := \min\{\alpha_i \mid i \notin M_\sigma\}. \quad (11) \]

Notice that, the sequence (9) is the particular case of (10) where \( \sigma_i^M = i, \#M_\sigma = M, M_\sigma = \{1, \ldots, M\}, \alpha_{M_\sigma} = \alpha_M, \) and \( \alpha_{M_\sigma^+} = \alpha_{M+1} \).

Essentially, the results in [31] tell us that the linear closed-loop system

\[ \dot{y} + Ay + A_{rc}(t)y - K_{U_{\#M_\sigma}}(t, y) = 0, \quad y(0) = y_0 \in H, \]

is \textit{globally} exponentially stable, with the feedback control operator

\[ y \mapsto K_{U_{\#M_\sigma}}^F(t, y) := F_{U_{\#M_\sigma}}^E (Ay + A_{rc}(t)y - F_{M_\sigma}(P_{E_{M_\sigma}}y)), \quad (12) \]

with \( F_{M_\sigma} \in \{\lambda \text{Id}, A + \lambda \text{Id}\} \), provided the condition

\[ \overline{m}_M := \alpha_{M_\sigma^+} - \left( 6 + 4 \left\| F_{U_{\#M_\sigma}}^E \right\|_{\mathcal{L}(H)}^2 \right) |A_{rc}|^2_{L^\infty((0, +\infty); \mathcal{L}(H, V'))} > 0 \quad (13) \]

holds true, which is a slightly relaxed version of (5). In case we have that \( \alpha_{M_\sigma^+} \to +\infty \) as \( M \to +\infty \), then we also have Theorem 1.2 with \( \alpha_{M_\sigma^+} \) and \( (U_{\#M_\sigma}, E_{M_\sigma}) \) in the roles of \( \alpha_{M+1} \) and \( (U_M, E_M) \), respectively.

### 2.2. Example of application

We recall here that we can choose \( U_{\#M_\sigma} \) and \( E_{M_\sigma}^E \) so that (13) is satisfied for parabolic equations evolving in rectangular domains. Let \( M = \{1, 2, \ldots, M\} \) and \( r \in (0, 1) \). For 1D parabolic equations, evolving in a nonempty interval \( \Omega^1 = (0, L_1) \subset \mathbb{R} \), we have that the norm of the projection \( P_{E_{M_\sigma}}^E \) remains bounded if we take for the actuators the indicator functions \( 1_{\omega_j}(x_1) \), \( j \in \{1, 2, \ldots, M\} \), defined as follows,

\[ 1_{\omega_j}(x_1) := \begin{cases} 1, & \text{if } x_1 \in \Omega^1 \cap \omega_j, \\ 0, & \text{if } x_1 \not\in \Omega^1 \cap \omega_j, \end{cases} \quad \omega_j := (c_j - \frac{rL_1}{2M}, c_j + \frac{rL_1}{2M}), \quad c_j := \frac{(j-1)L_1}{2M}. \quad (14) \]
This boundedness result holds true for both Dirichlet and Neumann boundary conditions, see [51, Thms. 4.4 and 5.2]. From [31, Sect. 4.8] we also know that, for nonempty rectangular domains $\Omega^\times = \prod_{n=1}^{d} (0, L_n) \subset \mathbb{R}^d$, the operator norm of the projections $P_{U_{\#M_\alpha}}$ remains bounded, if we take $\#M_\alpha = M^d$, and the cartesian product actuators and eigenfunctions as follows

$$U_{\#M_\alpha} = \text{span}\{1_{\omega_j} \mid j \in M^d\} \quad \text{and} \quad E_{M_\alpha} = \text{span}\{e_j^\times \mid j \in M^d\},$$

and $\omega_j^\times := \{(x_1, x_2, \ldots, x_d) \in \Omega^\times \mid x_n \in \omega_j^{\alpha_n}\}$ and $e_j^\times(x_1, x_2, \ldots, x_d) := \prod_{n=1}^{d} e_j^{\alpha_n}(x_n)$.

Notice that we can also write $1_{\omega_j} = \prod_{n=1}^{d} 1_{\omega_j^{\alpha_n}}(x_n)$, and after ordering the eigen-pairs $(\alpha_i, e_i)$ of $-\Delta + \text{Id}$ in $\Omega^\times$, we can find $\sigma^M$ so that $\{e_i \mid i \in M_\alpha\} = \{e_j^\times \mid j \in M^d\}$, roughly speaking $M_\alpha \sim M^d$. Furthermore, the total volume covered by the actuators is given by $r^d \text{vol}(\Omega^\times) = \prod_{n=1}^{d} r L_n$. That is, the total volume covered by the actuators can be taken arbitrarily small. However, for smaller $r$ we may need a larger number $M$ of actuators, because the norm of $P_{U_{\#M_\alpha}}$ will increase as $r$ decreases, see [31, Sect. 4.8.1] and [51, Thms. 4.4 and 5.2].

Observe that we have $\alpha_{M_\alpha} \geq \pi^2(\frac{1}{L^2}(\frac{d-1}{2}+M+1)^2) + 1$ under Dirichlet boundary conditions, and $\alpha_{M_\alpha} \geq \pi^2(\frac{d^2}{L^2}+M^2) + 1$ under Neumann boundary conditions, where $L = \max\{L_j \mid 1 \leq j \leq n\}$, which implies that in either case $\alpha_{M_\alpha} \to +\infty$ as $M \to +\infty$, and so condition (13) will be satisfied for large enough $M$. Recall that $e_i^n(x_n) = (\frac{2}{L_n})^{\frac{1}{2}} \sin(\frac{\pi x_n}{L_n})$, $i \geq 1$, under Dirichlet boundary conditions, and $e_i^\alpha(x_n) = (\frac{1}{L_n})^{\frac{1}{2}}$ and $e_i^n(x_n) = (\frac{2}{L_n})^{\frac{1}{2}} \cos(\frac{(i-1)\pi x_n}{L_n})$, $i \geq 2$, under Neumann boundary conditions.

For nonrectangular domains $\Omega \subset \mathbb{R}^d$, with $d \geq 2$, we do not know whether we can choose the actuators (as indicator functions) so that (13) is satisfied (again, in case the total volume of actuators is fixed a priori and arbitrarily small). This is an interesting open question. Numerical simulations in [31] and [32] show the stabilizing performance of a linear feedback $K^A_{U,M}$ in a nonrectangular domain.

Remark 2.1. For the nonlinear systems, to derive the semiglobal stability result hereafter we will also need that $\frac{\alpha_{M_\alpha}}{\alpha_{M_\alpha}^\times}$ remains bounded. This is again satisfied for the choice above for rectangular domains. Indeed, under Dirichlet boundary conditions we have $\alpha_{M_\alpha} \leq d \pi^2(\frac{M}{L})^2 + 1$, where $L = \min\{L_j \mid 1 \leq j \leq n\}$, which implies $\frac{\alpha_{M_\alpha}}{\alpha_{M_\alpha}^\times} \leq \frac{d \pi^2 \frac{M}{L}^2 + 1}{\pi^2 \frac{M}{L}^2 + 1} = d \frac{\pi^2 \frac{M}{L}^2 + 1}{\pi^2 \frac{M}{L}^2 + 1}$. Analogously, under Neumann boundary conditions

$$\alpha_{M_\alpha} \leq d \pi^2(\frac{M-1}{L}^2)^{\frac{1}{2}} + 1$$

and $\frac{\alpha_{M_\alpha}}{\alpha_{M_\alpha}^\times} \leq \frac{d \pi^2 (M-1)^2 + 1}{\pi^2 \frac{(M-1)^2}{L^2} + 1} = d \frac{\pi^2 (M-1)^2 + 1}{\pi^2 \frac{(M-1)^2}{L^2} + 1}$. That is, for either boundary conditions we have $\lim_{M \to +\infty} \frac{\alpha_{M_\alpha}}{\alpha_{M_\alpha}^\times} \leq d \frac{\pi^2 (M-1)^2 + 1}{\pi^2 \frac{(M-1)^2}{L^2} + 1}$, which implies that $\frac{\alpha_{M_\alpha}}{\alpha_{M_\alpha}^\times} \leq \Lambda$ for a suitable $\Lambda > 0$ independent of $M$.

2.3. Global, local, and semiglobal exponential stability. We recall 3 different exponential stability concepts, in order to better explain the result. Let $\mathfrak{F} \geq 1$, $l > 0$,
and let $\mathcal{F}$ be a normed space. Let us consider the dynamics in (1),
\[ \dot{y} + Ay + A_{rc}(t)y + \mathcal{N}(t, y) - \mathfrak{F}(t, y) = 0, \quad y(0) = y_0, \quad t \geq 0, \] (15)
with a general feedback control operator $\mathfrak{F}$ taken from a suitable class $\mathcal{F}$.

**Definition 2.2.** Let us fix $\mathfrak{F} \in \mathcal{F}$. We say that system (15) is *globally* ($\mathfrak{F}, \mathcal{N}, I, \mathcal{F}$)-exponentially stable if for arbitrary given $y_0 \in \mathfrak{F}$, the corresponding solution $y_{\mathfrak{F}}$ is defined for all $t \geq 0$ and satisfies $|y_{\mathfrak{F}}(t)|^2_{\mathcal{F}} \leq \mathfrak{K} e^{-\mathfrak{L} t} |y_0|^2_{\mathfrak{F}}$.

**Definition 2.3.** Let us fix $\mathfrak{F} \in \mathcal{F}$. We say that system (15) is *locally* ($\mathfrak{F}, \mathcal{N}, I, \mathcal{F}$)-exponentially stable if there exists $\epsilon > 0$, such that for arbitrary given $y_0 \in \mathfrak{F}$ with $|y_0|_{\mathfrak{F}} < \epsilon$, the corresponding solution $y_{\mathfrak{F}}$ is defined for all $t \geq 0$ and satisfies $|y_{\mathfrak{F}}(t)|^2_{\mathcal{F}} \leq \mathfrak{K} e^{-\mathfrak{L} t} |y_0|^2_{\mathfrak{F}}$.

**Definition 2.4.** Let us be given a class of operators $\mathcal{F}$. We say that (15) is *semiglobally* ($\mathcal{F}, \mathcal{N}$)-exponentially stable if for arbitrary given $R > 0$, we can find $\mathfrak{F} \in \mathcal{F}$, $\mathfrak{K} \geq 1$, and $I > 0$, such that: for arbitrary given $y_0 \in \mathfrak{F}$ with $|y_0|_{\mathfrak{F}} < R$, the corresponding solution $y_{\mathfrak{F}}$ is defined for all $t \geq 0$ and satisfies $|y_{\mathfrak{F}}(t)|^2_{\mathcal{F}} \leq \mathfrak{K} e^{-\mathfrak{L} t} |y_0|^2_{\mathfrak{F}}$.

We will consider system (15) evolving in a Hilbert $H$, which will be considered as a pivot space, $H = H'$. Let $D(A) \xrightarrow{d.c} H$ be the domain of the diffusion-like operator, and denote $V := D(A^{\frac{1}{2}}) \xrightarrow{d.c} H$, and its dual by $V'$. From the results in [31] we know that if $\mathcal{N} = 0$ and (13) holds true, then there exist suitable constants $C_1 \geq 1, \mu_1 > 0$, and $M > 0$ so that system (15) is globally $(K_{\mathcal{F}^m_{\mathcal{N}}}, C_1, \mu_1, 1)$-exponentially stable, with $\mathcal{F}^m_{\mathcal{N}} \in \{\lambda \text{Id}, A + \lambda \text{Id}\}$.

In (13) we assume $A_{rc} \in L^\infty((0, +\infty), \mathcal{L}(H, V'))$. If $A_{rc} \in L^\infty((0, +\infty), \mathcal{L}(V, H))$ (also) holds, then we will (also) have strong solutions for system (15) which will lead to the smoothing property
\[ |y(s + 1)|^2_V \leq C_2 |y(s)|^2_H, \quad \text{for all } s \geq 0, \]
for a suitable constant $C_2 > 0$, independent of $s$. Hence, by standard estimates (e.g., following [46, Sect. 3], see also [32, Sect. 4]), we can conclude that there is $C_3 > 0$ such that system (15), again with $\mathcal{N} = 0$, is again globally $(K_{\mathcal{F}^m_{\mathcal{N}}}, C_3, \mu_1, V)$-exponentially stable.

Afterwards, by a rather standard, still nontrivial, fixed point argument, we can derive that for a suitable constant $C_4 > 0$, the perturbed system
\[ \dot{y} + Ay + A_{rc}(t)y + \mathcal{N}(t, y) - K_{\mathcal{F}^m_{\mathcal{N}}}(t, y) = 0, \quad y(0) = y_0 \in V, \] (16)
is locally $(K_{\mathcal{F}^m_{\mathcal{N}}}, C_4, \mu_1, V)$-exponentially stable, for a general class of nonlinearities $\mathcal{N}$.

Let us now consider the nonlinear feedback operator (cf. (6b)),
\[ y \mapsto K_{\mathcal{F}^m_{\mathcal{N}}}(t, y) := P_{\mathcal{F}^m_{\mathcal{N}}}(t, y) + \mathcal{N}(t, y) - \mathcal{F}_{\mathcal{M}_v}(P_{\mathcal{E}_{\mathcal{M}_v}}(y)) \] (17)
and the class
\[ \mathcal{F} := \left\{ K_{\mathcal{F}^m_{\mathcal{N}}} \mid M \in \mathbb{N}, U_{\#M_v} \text{ is a } \#M_v\text{-dimensional subspace of } H, \right\} \] (18)
We will show that the closed-loop system (15) is semiglobally $(\mathcal{F}, V)$-exponentially stable, with $\mathcal{F}$ as in (18) and under general conditions on the state operators $A, A_{rc}$, and $\mathcal{N}$, in (15), under general conditions on $\mathcal{F}_{\mathcal{M}_v}$, and under a particular condition
on the oblique projections $P_{U\oplus M_\sigma}^F$, i.e., under a suitable “nonlinear version” of condition (13) (see condition (26) hereafter). In other words, for arbitrary given $R > 0$ we want to find $M \in \mathbb{N}$, $M_\sigma \in \mathfrak{P}_{\#M_\sigma}(N_0)$, and a set of $\#M_\sigma$ actuators spanning $U_{\#M_\sigma}$ such that the solution of system (15) with $\bar{\sigma} = K_{U\oplus M_\sigma}$ satisfies

$$|y(t)|_V^2 \leq C_5 e^{-\mu_2 t} |y_0|_V^2, \quad \text{for all } t \geq 0 \quad \text{and all } \quad y_0 \in \{v \in V \mid |v|_V < R\},$$

(19)

with $(C_5, \mu_2, M)$ independent of $y_0$. Note that here $(C_5, \mu_2, M)$ may depend on $R$, though.

The assumptions on the state operators, on the “partial feedback” $F_{M_\sigma}$, and on the oblique projection are given in the following sections. Such assumptions will lead to the following relaxed/generalized version of Theorem 1.3, with $F_{M_\sigma} = A + \lambda \text{Id}$, whose proof is given in Section 4.5.

**Theorem 2.5.** Suppose we can construct a sequence $(U_{\#M_\sigma}, E_{M_\sigma})_{M \in \mathbb{N}}$ so that the norm $\|P_{U\oplus M_\sigma}^F\|_{\mathcal{L}(H)} \leq C_P$ and the ratio $\frac{\alpha_{\#M_\sigma}}{\alpha_{\#M_\sigma}+} \leq \Lambda$ remain bounded, with both $C_P$ and $\Lambda > 0$ independent of $M$. Then, for arbitrary given $R > 0$ we can find $M \in \mathbb{N}$ large enough so that the solution of system (15), with $\bar{\sigma} = K_{U\oplus M_\sigma}$, satisfies (19), with $(C_5, \mu_2, M)$ independent of $y_0$. That is, system (15) is semiglobally $(F, V)$-exponentially stable.

3. **Assumptions and mathematical setting.** Here we present the mathematical setting and the sufficient conditions for stability of the closed-loop system.

3.1. **Assumptions on the state operators.** Let $H$ and $V$ be separable Hilbert spaces, with $V \subseteq H$. We will consider $H$ as pivot space, $H' = H$.

**Assumption 3.1.** $A \in \mathcal{L}(V, V')$ is an isomorphism from $V$ onto $V'$, $A$ is symmetric, and $(y, z) \mapsto \langle Ay, z \rangle_{V', V}$ is a complete scalar product on $V$.

From now on we suppose that $V$ is endowed with the scalar product $(y, z)_V := \langle Ay, z \rangle_{V', V}$, which still makes $V$ a Hilbert space. Therefore, $A: V \rightarrow V'$ is an isometry.

**Assumption 3.2.** The inclusion $V \subseteq H$ is continuous, dense, and compact.

Necessarily, we have that the operator $A$ is densely defined in $H$, with domain $D(A) := \{u \in V \mid Au \in H\}$ endowed with the scalar product $(y, z)_{D(A)} := \langle Ay, Az \rangle_H$, and the inclusions

$$D(A) \xrightarrow{d.c.} V \xleftarrow{d.c.} H \xrightarrow{d.c.} V' \xleftarrow{d.c.} D(A)' .$$

Further, $A$ has compact inverse $A^{-1}: H \rightarrow D(A)$, and we can find a nondecreasing system of (repeated) eigenvalues $(\alpha_i)_{i \in \mathbb{N}_0}$ and a corresponding complete basis of eigenfunctions $(e_i)_{i \in \mathbb{N}_0}$:

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_i \leq \alpha_{i+1} \rightarrow +\infty \quad \text{and} \quad Ae_i = \alpha_i e_i .$$

For every $\beta \in \mathbb{R}$, the power $A^\beta$ of $A$ is defined by

$$A^\beta \sum_{i=1}^{\infty} y_i e_i := \sum_{i=1}^{\infty} \alpha_i^{\beta} y_i e_i ,$$

for every $\beta \in \mathbb{R}$, the power $A^\beta$ of $A$ is defined by

$$A^\beta \sum_{i=1}^{\infty} y_i e_i := \sum_{i=1}^{\infty} \alpha_i^{\beta} y_i e_i .$$
with respective domain $D(A^{1}) := \{ y \in H \mid A^{1}y \in H \}$, and $D(A^{-|\beta|}) := D(A^{\beta})'$. We have $D(A^{\beta}) \xrightarrow{dc} D(A^{\beta_1})$, for all $\beta > \beta_1$, and we can see that $D(A^{0}) = H$, $D(A^{1}) = D(A)$, $D(A^{2}) = V$.

For the time-dependent operators we assume the following:

**Assumption 3.3.** For all $t > 0$ we have $A_{rc}(t) \in \mathcal{L}(V,H)$, and there is a nonnegative constant $C_{rc}$ such that, $|A_{rc}|_{L \rightarrow (L^{2}(\Omega), \mathcal{L}(V,H))} \leq C_{rc}$.

**Assumption 3.4.** We have $\mathcal{N}(t, \cdot) \in C_{b,1}(D(A), H)$ and there exists constants $C_{\mathcal{N}} \geq 0$, $n \in \mathbb{N}_{0}$, $\zeta_{1j} \geq 0$, $\zeta_{2j} \geq 0$, $\delta_{1j} \geq 0$, $\delta_{2j} \geq 0$, with $j \in \{1,2,\ldots,n\}$, such that for all $t > 0$ and all $(y_{1}, y_{2}) \in H \times H$, we have

$$|\mathcal{N}(t,y_{1}) - \mathcal{N}(t,y_{2})|_{H} \leq C_{\mathcal{N}} \sum_{j=1}^{n} \left( |y_{1}|^{\zeta_{1j}}_{V} |y_{1}|^{\zeta_{2j}}_{D(A)} + |y_{2}|^{\zeta_{1j}}_{V} |y_{2}|^{\zeta_{2j}}_{D(A)} \right) d_{1}^{\delta_{1j}} |d_{2}^{\delta_{2j}}_{D(A)} ,$$

with $d := y_{1} - y_{2}$, $\zeta_{2j} + \delta_{2j} < 1$ and $\delta_{1j} + \delta_{2j} \geq 1$.

**Examples.** We can show that our Assumptions 3.1–3.4 on the linear and nonlinear operators will be satisfied for parabolic equations evolving in a bounded smooth, or rectangular, domain $\Omega \subset \mathbb{R}^{d}$, $d \in \{1,2,3\}$, as

$$\frac{\partial}{\partial t} y + (-\nu \Delta + \text{Id})y + (a - 1)y + b \cdot \nabla y - \sum_{i=2}^{n} \hat{a}_{i} y^{i} + (\hat{b}(t) \cdot \nabla y) y = 0, \quad y(0) = y_{0},$$

with $n \leq 4$ if $d = 3$, and $n \in \mathbb{N}$ if $d \in \{1,2\}$. Under either Dirichlet or Neumann homogeneous boundary conditions. For example, here we may take $A = -\nu \Delta + \text{Id}$ as the shifted Laplacian. The same assumptions are also satisfied for the Navier–Stokes equations under homogeneous Dirichlet boundary conditions, where we may take $A = P_{H}(-\nu \Delta + \text{Id})$ as the shifted Stokes operator, where $P_{H}$ is the orthogonal projection in $L^{2}(\Omega, \mathbb{R}^{d})$ onto the space $H$ of divergence free vector fields which are tangent to the boundary $\partial \Omega$ of $\Omega$. More comments and details on these examples are given later in Section 5.

### 3.2. Auxiliary estimates for the nonlinear terms

Besides the assumptions on the state operators, presented in Section 3.1, we will need also assumptions on the triple $(F_{M_{s}}, E_{M_{s}}, U_{\#M_{s}})$, which defines the feedback operator. Before, we need to present suitable estimates resulting from Assumption 3.4. These are the content of the following Proposition, whose proof follows by straightforward computations.

The proof is, however, not trivial and is given in the Appendix, Section A.1.

Recall the notation $|a| \ := \max_{1 \leq j \leq n} \{ a_{j} \}$, for a sequence of constants $a_{j} \geq 0$. We will also denote

$$\|P\|_{\mathcal{L}} := \left| P_{E^{\text{cov}}_{M_{s}}} \right|_{\mathcal{L}(H)},$$

which will not lead to ambiguity, as soon as the pair $(E_{M_{s}}, U_{\#M_{s}})$ is fixed.

**Proposition 3.5.** If Assumptions 3.1, 3.2, and 3.4 hold true, then there are constants $\overline{C}_{\mathcal{N}_{1}} > 0$, and $\overline{C}_{\mathcal{N}_{2}} > 0$ such that: for all $\gamma_{0} > 0$, all $t > 0$, all $(y_{1}, y_{2}) \in H \times H$, and all $(q,Q) \in E_{M_{s}} \times E_{M_{s}}^{\perp}$, we have

$$2 \left( P_{E^{\text{cov}}_{M_{s}}} (\mathcal{N}(t,y_{1}) - \mathcal{N}(t,y_{2})) , A(y_{1} - y_{2}) \right)_{H} \leq \gamma_{0} \left| y_{1} - y_{2} \right|^{2}_{D(A)}$$

$$+ \left( 1 + \gamma_{0}^{-\frac{1+\|\beta\|}{1+2\|\beta\|}} \right) \overline{C}_{\mathcal{N}_{1}} \sum_{j=1}^{n} \left( |y_{1}|^{2\zeta_{1j}}_{V} |y_{1}|^{2\zeta_{2j}}_{D(A)} + |y_{2}|^{2\zeta_{1j}}_{V} |y_{2}|^{2\zeta_{2j}}_{D(A)} \right) d_{1}^{\beta_{1j}} |d_{2}^{\beta_{2j}}_{D(A)} ,$$

where $\gamma_{0}$ is the unique positive solution of $\gamma_{0}^{1+\|\beta\|} = \frac{1+\|\beta\|}{1+2\|\beta\|} \gamma_{0}^{1+\|\beta\|}$.
with $\delta := y_1 - y_2$, and

$$2 \left( P^{E_{\delta}^{\perp} \mathbb{R}}_{E_{\delta}^{\perp}} (N(t,q + Q) - N(t,q), Aq) \right)_H \leq \hat{\gamma}_0 |Q|_{D(A)}^2$$

$$+ \left( 1 + \frac{\hat{\gamma}_0}{1 + |\zeta|^2} \right) \mathcal{C}_N (1 + |q|^2_{V} |\zeta|^2_{D(A)}) \left( 1 + |q|^2_{V} |\zeta|^2_{D(A)} - 2 \right) |Q|_{V}^2.$$  

Further, the constants $\mathcal{C}_{N1}$ and $\mathcal{C}_{N2}$ are of the form $\mathcal{C}_{N1} = \mathcal{C} \left[ n, \frac{1}{1 + |\zeta|^2}, \mathcal{C}, \|P\| \right]$ and $\mathcal{C}_{N2} = \mathcal{C} \left[ n, \|\zeta\|, \mathcal{C}, \frac{1}{1 + |\zeta|^2}, \mathcal{C}, \|P\| \right]$.

Inequality (21) will be used to prove the existence of a solution for the closed-loop system, while (20) will be used to prove the uniqueness of the solution.

### 3.3. Assumptions on the oblique projection based feedback

We present here the assumptions on the triple $(\mathcal{F}_{\#}, E_{\delta}^{\perp}, U_{\#})$. Observe that, from (16) and (17), the orthogonal projection $\hat{q} := P_{E_{\delta}^{\perp}} y$ satisfies

$$\dot{\hat{q}} = -\mathcal{F}_{\#} (q),$$

For the exponential stability of (16) we need $q(t)$ to decrease exponentially to zero. We will also ask for integrability of $\hat{q}$ and $\dot{\hat{q}}$ as follows.

**Assumption 3.6.** We have $\mathcal{F}_{\#} \in C_{0,1}(E_{\delta}^{\perp}, E_{\delta}^{\perp})$ and there are constants $C_{q_0} \geq 0$, $C_{q_1} \geq 0$, $C_{q_2} \geq 0$, $C_{q_3} \geq 0$, $\xi \geq 1$, $\lambda > 0$, $\beta_0 \geq 0$, $\beta_1 \geq 0$, $\beta_2 \geq 0$, $\eta_1 \geq 0$, $\eta_2 \geq 0$, and $1 < r < \frac{1}{\|\zeta\|}$, all independent of $\delta$, such that:

$$|\mathcal{F}_{\#} (\hat{q})|_{H} \leq C_{q_0} |\hat{q}|_{H(A)}^2, \quad \text{for all } \hat{q} \in E_{\delta}^{\perp},$$

$$\beta_1 + \beta_2 \geq 1, \quad r |\zeta_1 + \delta_1 + (\eta_1 + \eta_2)(\zeta_2 + \delta_2)| \geq 1,$$

$$\left( \left\| \frac{\zeta_1 + \delta_1}{1 - \zeta_2 - \delta_2} \right\| - 1 \right) \beta_2 \leq 1 - \left\| \frac{\zeta_2}{1 - \zeta_2 - \delta_2} \right\|, \quad \left( \left\| \frac{\zeta_1 + \delta_1}{1 - \zeta_2 - \delta_2} \right\| - 1 \right) \|\zeta_2 + \delta_2\| r \eta_2 < 1 - \left\| \frac{\zeta_2}{1 - \zeta_2 - \delta_2} \right\|,$

and every solution $q$ of system (22) satisfies, for all $t \geq 0$,

$$|q(t)|_{V} \leq C_{q_1} e^{-\lambda t} |q(0)|_{V}, \quad |q|_{L^2(\mathbb{R}, D(A))} \leq C_{q_2} |q(0)|_{H(A)}^\eta_1 |q(0)|_{D(A)}^{\eta_2}, \quad \text{and}$$

$$|Aq - \mathcal{F}_{\#} (q)|_{L^2(\mathbb{R}, H)} \leq C_{q_3} |q(0)|_{H(A)}^{\beta_1} |q(0)|_{D(A)}^{\beta_2}.$$

As we see, when $\left\| \frac{\zeta_1 + \delta_1}{1 - \zeta_2 - \delta_2} \right\| \neq 1$, we ask for small enough magnitudes of $\beta_2$. Observe that with $\mathcal{F}_{\#} = \lambda I$, we can take $(\beta_1, \beta_2) = (0, 1)$, but we cannot take $\beta_2 < 1$, that is, Assumption 3.6 does not necessarily hold true, being satisfied only if $\|\zeta_2\|_{1 - \zeta_2 - \delta_2} > 2$. Instead, with $\mathcal{F}_{\#} = A + \lambda I$ the Assumption 3.6 is always satisfied, because we can take $(\beta_1, \beta_2) = (\eta_1, \eta_2) = (1, 0)$. This is why in Theorem (2.5) we write only $A + \lambda I$, and exclude $\lambda I$.

Finally, we present the assumptions involving $P_{U_{\#}}^{E_{\delta}^{\perp}}$. Note that both $U_{\#}$ and $E_{\delta}^{\perp}$ are closed subspaces. Thus, the oblique projection $P_{U_{\#}}^{E_{\delta}^{\perp}} : H \mapsto U_{\#}$ is well defined if, and only if, we have the direct sum $H = U_{\#} \oplus E_{\delta}^{\perp}$. In particular, by considering the feedback (3), we are necessarily assuming the following.

**Assumption 3.7.** We have the direct sum $H = U_{\#} \oplus E_{\delta}^{\perp}$. 


Recall that \( \#M_\sigma = \dim(U \# M_\sigma) = \dim(E_{M_\sigma}) \). Recall also that Assumption 3.7 means that for every given \( h \in H \) there exists one, and only one, pair \((h_{U \# M_\sigma}, h_{E_{M_\sigma}})\) satisfying

\[
h = h_{U \# M_\sigma} + h_{E_{M_\sigma}} \quad \text{with} \quad (h_{U \# M_\sigma}, h_{E_{M_\sigma}}) \in U \# M_\sigma \times E_{M_\sigma}^\perp.
\]

Hence we simply take \( P_{U \# M_\sigma}^\perp h \equiv h_{U \# M_\sigma} \). Similarly, the oblique projection in \( H \) onto \( E_{M_\sigma}^\perp \) along \( U \# M_\sigma \) is defined by \( P_{E_{M_\sigma}^\perp}^U h \equiv h_{E_{M_\sigma}} \). Observe that \( P_{U \# M_\sigma}^\perp h \) is the only element in the set \((h + E_{M_\sigma}^\perp) \cap U \# M_\sigma \), and \( P_{E_{M_\sigma}^\perp}^U h \) is the only element in the set \((h + U \# M_\sigma) \cap E_{M_\sigma}^\perp \).

The oblique projection \( P_{E_{M_\sigma}^\perp}^U \) is orthogonal if, and only if, \( U \# M_\sigma = E_{M_\sigma} \). The operator norm of an orthogonal projection onto a closed subspace \( F \subset H \) is always equal to 1, if \( F \neq \{0\} \), that is, \( \left\| P_F \right\|_{\mathcal{L}(H)} = 1 \). If \( F = \{0\} \), then \( \left\| P_F \right\|_{\mathcal{L}(H)} = 0 \).

The operator norm of an oblique nonorthogonal projection is strictly larger than 1. In particular, in case \( U \# M_\sigma \neq E_{M_\sigma} \) we have that \( \left\| P_{U \# M_\sigma} \right\|_{\mathcal{L}(H)} > 1 \).

Orthogonal projections \( P_F^\perp \) will be denoted by \( P_F \), for simplicity. We have the following properties, which are useful in the computations hereafter.

\[
\begin{align*}
P_{E_{M_\sigma}} &= P_{E_{M_\sigma}} P_{E_{M_\sigma}^\perp}, & P_{U \# M_\sigma}^\perp &= P_{E_{M_\sigma}^\perp} P_{E_{M_\sigma}}, \\
P_{E_{M_\sigma}^\perp} = P_{E_{M_\sigma}^\perp} P_{E_{M_\sigma}} P_{E_{M_\sigma}^\perp}, & P_{E_{M_\sigma}}^U = P_{E_{M_\sigma}^\perp} P_{E_{M_\sigma}} P_{E_{M_\sigma}^\perp}. \tag{23a}
\end{align*}
\]

\[
\begin{align*}
P_{E_{M_\sigma}^\perp} = P_{E_{M_\sigma}^\perp} P_{E_{M_\sigma}} P_{E_{M_\sigma}^\perp}, & P_{E_{M_\sigma}}^U = P_{E_{M_\sigma}^\perp} P_{E_{M_\sigma}} P_{E_{M_\sigma}^\perp}. \tag{23b}
\end{align*}
\]

For further comments on oblique projections we refer to [31, Sect. 2.2] and [51, Sect. 3].

The next assumption is less trivial and it is the one that gives us the stability condition. In order to state the assumption we start by recalling the particular eigenvalues \( \alpha_{M_\sigma} \) and \( \alpha_{E_{M_\sigma}} \), defined in (11). Then we define suitable functions as follows. For a given triple \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3 \) with positive coordinates, and a given function \( q \in L^\infty(\mathbb{R}_0, E_{M_\sigma}) \), we define

\[
\begin{align*}
a_0 &:= 2 - \gamma_1 - \gamma_2 - \gamma_3, & a_1 &:= \gamma_1^{-1} \left| P_{E_{M_\sigma}}^U \right|_{\mathcal{L}(H)}^2 C_{\mathcal{R}^3}^2, \\
a_2 &:= \frac{\gamma_3^2}{\gamma_1^3 + \gamma_2^3 + \gamma_3^3} C_{\mathcal{N}^2}, & q &:= \max_{\ell \geq 0} \left( 1 + \left| q \right|_{L^\infty}^2 \right) \left( 1 + \left| q \right|_{D(A)} \right)^2, \\
p &:= \left| \frac{\gamma_1 + \gamma_2}{\gamma_1^3 + \gamma_2^3 + \gamma_3^3} \right| - 1, & b &:= \gamma_2^{-1} \left| \mathcal{M}(q) \right|_{H} \tag{24b},
\end{align*}
\]

with the constants \( C_{\mathcal{R}^3} \) and \( C_{\mathcal{N}} \) as in Assumptions 3.3 and 3.4, respectively, and

\[
\begin{align*}
\mathcal{M}(q) &:= -P_{E_{M_\sigma}}^U \left( Aq + A_{\mathcal{R}}(t)q + \mathcal{N}(t,q) \right) - P_{E_{M_\sigma}} P_{E_{M_\sigma}^\perp}^U \mathcal{F}_{E_{M_\sigma}}(q) \\
&= -P_{E_{M_\sigma}}^U \left( Aq + A_{\mathcal{R}}(t)q + \mathcal{N}(t,q) - \mathcal{F}_{E_{M_\sigma}}(q) \right), \quad q \in E_{M_\sigma}. \tag{25}
\end{align*}
\]
Assumption 3.8. With \( \tau > 1 \) as in Assumption 3.6, we have the strict inequality
\[
\alpha_{M_{\tau+}} > \frac{1}{a_0} \left( a_1 + a_2 q + \tau + (p + 1) a_2 q \left( |Q_0|^2 + (\frac{\tau}{\tau-1}) |h|_{L^*(R_0,R)} \right)^p \right).
\] (26)

Remarks and examples. Note that Assumption 3.6 holds true with, for example, \( F_{\mathcal{M}_c} = A + \lambda \text{Id} \). Of course it would also hold true with \( F_{\mathcal{M}_c} = \lambda \text{Id} \) if we would not ask for the constants in there to be independent of \( \mathcal{M}_c \). Such independence is helpful to prove that, in particular situations as in Corollary 3.9 below, Assumption 3.8 will be satisfied for large enough \( n \). It is also helpful to prove, later on, that the number of actuators depend only in the \( V \)-norm of the initial condition \( y(0) = q(0) + Q(0) \), with \( q(0), Q(0) \in E_{\mathcal{M}_c} \times E_{\mathcal{M}_c}^\perp \) (cf. Thm. 2.5).

Concerning Assumption 3.7, it is needed to define the oblique projection \( P_{U^\perp \mathcal{M}_c} \) and it is not difficult to find the actuators such that it holds true. What is not clear is whether we can find the actuators, for example a finite number of indicator functions \( 1_{\omega_i} \) in the setting of parabolic equations, so that Assumption 3.8 also holds true. Indeed, recalling (25) and (24), and using Assumption 3.4, we obtain
\[
|h|_{L^*(R_0,R)} = \gamma_2^{-1} |\mathcal{M}(q)|_{L^{2r}(R_0,H)}^2
= \gamma_2^{-1} | -E_{\mathcal{M}_c}^{U^\perp \mathcal{M}_c} (Aq + A_{rc}(t)q + N(t,q) - F_{\mathcal{M}_c}(q)) |_{L^{2r}(R_0,H)}^2
\leq \mathcal{T}_1[\|P\|_\mathcal{L}] \left( |Aq - F_{\mathcal{M}_c}(q)|_{L^{2r}(R_0,H)}^2 + |A_{rc}(t)q + N(t,q)|_{L^{2r}(R_0,H)}^2 \right)
\leq \mathcal{T}_1[\|P\|_\mathcal{L}] \left( C_{\mathcal{M}} q(0)|_{V}^{2\beta_1} |q(0)|_{D(A)}^{2\beta_2} + C_{\mathcal{M}} C_{\mathcal{M}}^{2} (4\lambda)^{-\frac{\beta_1}{2}} |q(0)|_{V}^2 \right.
+ \left. C_{\mathcal{M}} \sum_{j=1}^{n} \left| q_{|V}^{2\tau(\zeta_j+\delta_{1j})} |q(0)|_{D(A)}^{2\tau(\zeta_j+\delta_{2j})} \right|^2 \right).
\]

Observe that from Assumption 3.6 we have
\[
\int_{R_0} |q|_{V}^{2\tau(\zeta_j+\delta_{1j})} |q|_{D(A)}^{2\tau(\zeta_j+\delta_{2j})} \, ds \leq \left( \int_{R_0} |q|_{V}^{\frac{2\tau(\zeta_j+\delta_{1j})}{1-\tau(\zeta_j+\delta_{2j})}} \, ds \right)^{1-\tau(\zeta_j+\delta_{2j})} \left( \int_{R_0} |q|_{D(A)}^{2\tau(\zeta_j+\delta_{2j})} \, ds \right)
\leq \mathcal{T}_1[C_{\mathcal{M}},C_{\mathcal{M}}^{2},\frac{1}{\lambda}] \left| q(0)|_{V}^{2\tau(\zeta_j+\delta_{1j})} |q(0)|_{D(A)}^{2\tau(\zeta_j+\delta_{2j})} \right| \left| q(0)|_{V}^{2\tau(\zeta_j+\delta_{2j})} \right|,
\]

which allow us to derive that, with \( \mathcal{T}_1 = \mathcal{T}_1[\|P\|_\mathcal{L},C_{\mathcal{M}}^{2},C_{\mathcal{M}}^{2},C_{\mathcal{M}}^{2},C_{\mathcal{M}}^{2},\frac{1}{\lambda}] \),
\[
|h|_{L^*(R_0,R)} \leq \mathcal{T}_1 \left( |q(0)|_{V}^{2\beta_1} + \alpha_{\mathcal{M}_c}^{\delta_2} |q(0)|_{V}^{2\beta_2} \right.
+ \left. \sum_{j=1}^{n} \alpha_{\mathcal{M}_c}^{\tau\eta_2(\zeta_j+\delta_{2j})} |q(0)|_{V}^{2\tau(\zeta_j+\delta_{1j}+(\eta_1+\eta_2)(\zeta_j+\delta_{2j}))} \right)
\leq \mathcal{T}_1[\|P\|_\mathcal{L},C_{\mathcal{M}}^{2},C_{\mathcal{M}}^{2},C_{\mathcal{M}}^{2},C_{\mathcal{M}}^{2},\frac{1}{\lambda}] \left( 1 + \alpha_{\mathcal{M}_c}^{\delta_2} + \alpha_{\mathcal{M}_c}^{\tau\eta_2(\zeta+\delta_2)} \right) |q(0)|_{V}^{2}.
\] (27)
Recall also that $\beta_1 + \beta_2 \geq 1$ and $\varepsilon \|\zeta_1 + \delta_1 + (\eta_1 + \eta_2)(\zeta_2 + \delta_2)\| \geq 1$.

**Corollary 3.9.** Suppose that $\frac{\alpha_{\mathbb{M}_+}}{\alpha_{\mathbb{M}_+}} \leq \Lambda$ and that $\left| P_{E_0^{\mathbb{M}_+}}^{U_{\mathbb{M}_+}} \right|_{\mathcal{L}(H)} \leq C_P$ both remain bounded, with $(\Lambda, C_P)$ independent of $M$. Then Assumption 3.8 is satisfied.

*Proof.* We know that $\lim_{M \to +\infty} \alpha_{\mathbb{M}_+} = +\infty$, then for fixed $\gamma \in \mathbb{R}^3$, such that $a_0 > 0$, and $\varepsilon > 0$, we see that (26) will be satisfied for large enough $M$, because $0 \leq \max\{\beta_2 p, \gamma_2\} + \|\zeta_2 + \delta_2\| \leq 1$, due to Assumption 3.6. Note that the constant $\overline{C}$ in (27) is independent of $\alpha_{\mathbb{M}_+}$.

The boundedness of the ratio $\frac{\alpha_{\mathbb{M}_+}}{\alpha_{\mathbb{M}_+}} \leq \Lambda$ is assumed in Theorem 2.5. This ratio depends only on the choice of $\mathbb{M}_0$ (cf. Rem. 2.1). On the other hand, the boundedness of $\left| P_{E_0^{\mathbb{M}_+}}^{U_{\mathbb{M}_+}} \right|_{\mathcal{L}(H)} \leq C_P$ is generally a nontrivial assumption. However, as we have seen in Section 2.2 for parabolic equations evolving in a bounded rectangular domain we can choose $(E_{\mathbb{M}_0}, U_{\mathbb{M}_0})$ so that the operator norm of the projection remains bounded, furthermore the total volume $\text{vol}(\bigcup_{i=1}^M \omega_i) = \text{vol}(\Omega)$ can be arbitrarily small. On the other hand we have also mentioned in Section 2.2 that for general domains such choice of $(E_{\mathbb{M}_0}, U_{\mathbb{M}_0})$ is an open interesting question.

**4. Stability of the closed-loop system.** Here we prove that system (15) is exponentially stable with the feedback in (17), provided the above assumptions are satisfied by the state operators and the triple $(F_{\mathbb{M}_0}, U_{\mathbb{M}_0}, E_{\mathbb{M}_0})$.

Let $a_0, a_1, a_2, p, q$, and $h$, be as in (24), and $\varepsilon$ be as in (26). Note that, if Assumption 3.8 is satisfied, then

$$\exists \gamma \in \mathbb{R}^3 : a_0 \alpha_{\mathbb{M}_+} > a_1 + a_2 q + \varepsilon + (p + 1) a_2 q \left( |Q_0|_V^2 + \left( \frac{r}{\varepsilon - 1} \right) - \frac{r}{\varepsilon - 1} |h|_{L^r(\mathbb{R}_0, \mathbb{R})} \right)^p.$$  

(28a)

We define the constants

$$\varepsilon := a_0 \alpha_{\mathbb{M}_+} - a_1 - a_2 q - a_2 q |Q_0|_V^2,$$  

(28b)

$$\varepsilon := a_0 \alpha_{\mathbb{M}_+} - a_1 - a_2 q - a_2 q(p + 1) \left( |Q_0|_V^2 + \left( \frac{r}{\varepsilon - 1} \right) - \frac{r}{\varepsilon - 1} |h|_{L^r(\mathbb{R}_0, \mathbb{R})} \right)^p.$$  

(28c)

and observe that, for $\gamma$ as in (28a), we have

$$\varepsilon \geq \varepsilon \geq \varepsilon > 0.$$  

(28d)

**Theorem 4.1.** Let Assumptions 3.1–3.4 and 3.6–3.8 be satisfied. Then system (15) is exponentially stable, with $\tilde{\mathcal{S}} = K_{U_{\mathbb{M}_0}, \mathcal{N}}$ as in (17). Further its solution $y$ satisfies $|y(t)|_V \leq \overline{C} e^{-\mu t} |y(0)|_V$, for all $t \geq 0$, where $0 < \mu < \min\{\varepsilon, 2\lambda\}$, $\varepsilon$ is as in (28), and $\overline{C} = \frac{\alpha_{\mathbb{M}_+}}{\alpha_{\mathbb{M}_+}} \left[ \|P\|_{\mathcal{L}(E_0^{\mathbb{M}_+}, \mathcal{N})}, \|C_{E_0^{\mathbb{M}_+}}(0)|_{\mathcal{L}(E_0^{\mathbb{M}_+}, \mathcal{N})}, \frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}} \right]$.

**4.1. Orthogonal decomposition of the solution.** Observe that we may write system (15), with $\tilde{\mathcal{S}} = K_{U_{\mathbb{M}_0}, \mathcal{N}}$, for time $t \geq 0$, as

$$\dot{y} + \left( \text{Id} - P_{E_0^{\mathbb{M}_+}}^{U_{\mathbb{M}_0}} \right) \left( Ay + A_r c(t) y + \mathcal{N}(t, y) \right) + P_{E_0^{\mathbb{M}_0}}^{U_{\mathbb{M}_0}} F_{\mathbb{M}_0}(P_{E_0^{\mathbb{M}_0}} y) = 0, \quad y(0) = y_0.$$  

(29)
Splitting $y$ as $y = P_{E_{\delta \sigma}} y + P_{E_{\delta \sigma}^\perp} y$, with $(q, Q) := (P_{E_{\delta \sigma}} y, P_{E_{\delta \sigma}^\perp} y)$ we obtain, using the properties in (23),

$$\dot{q} + F_{E_{\delta \sigma}}(q) = 0,$$

$$\dot{Q} + P_{E_{\delta \sigma}^\perp}^{U_{\delta \sigma}} \left( Ay + A_{rc}(t)y + N(t,y) \right) + P_{E_{\delta \sigma}^\perp}^{E_{\delta \sigma}} F_{E_{\delta \sigma}}(q) = 0.$$ 

Now we write

$$N(t, q, Q) := N(t, y) - N(t, q),$$

and

$$\dot{Q} + P_{E_{\delta \sigma}^\perp}^{U_{\delta \sigma}} \left( AQ + A_{rc}(t)Q + N(t, q, Q) \right) = M(q)$$

and satisfies, for all $t$

$$\dot{q} + F_{E_{\delta \sigma}}(q) = 0, \quad q(0) = q_0 \in E_{\delta \sigma}, \quad (30a)$$

$$\dot{Q} + P_{E_{\delta \sigma}^\perp}^{U_{\delta \sigma}} \left( AQ + A_{rc}(t)Q + N(t, q, Q) \right) = M(q), \quad Q(0) = Q_0 \in E_{\delta \sigma}^\perp \cap V. \quad (30b)$$

We will start by studying (30b), for a given function $q$ taking values in $E_{\delta \sigma}$.

**Theorem 4.2.** Let Assumptions 3.1–3.4 and 3.6–3.8 hold true, for a suitable space $U_{\#M_\sigma} \subset H$ and a subset $M_\sigma \subset \Psi_{\#M_\sigma}(\mathbb{N}_0)$. Let also $Q_0 \in E_{\delta \sigma}^\perp \cap V$, and $q \in L^\infty(\mathbb{R}_0, E_{\delta \sigma})$. Then there exists a global strong solution $Q \in W_{loc}(\mathbb{R}_0, D(A), H)$, for the system (30b), taking its values in $E_{\delta \sigma}^\perp \cap V$. Moreover the solution is unique and satisfies, for all $t \geq 0$,

$$|Q(t)|_V^2 \leq e^{-\varepsilon t} |Q_0|_V^2 + \int_0^t e^{-\bar{\varepsilon}(t-s)} |b(s)|_\varepsilon^R \, ds, \quad (31a)$$

$$|Q(t)|_V^2 \leq |Q_0|_V^2 + \left( \frac{\varepsilon}{\bar{\varepsilon}} \right)^{-\frac{1}{1}} |h|_{L^\varepsilon(\mathbb{R}_0, R)}, \quad (31b)$$

with $\varepsilon$, $\bar{\varepsilon}$, and $\varepsilon$ as in (28).

The proof is given hereafter in Section 4.3, where the local stability of (30b) is reduced to the local stability of a suitable scalar ODE system in the form

$$\dot{w} = -\left( \tilde{C}_1 - \tilde{C}_2 |w|_{L^\varepsilon_R}^p \right) w + |h|_\varepsilon, \quad w(0) = w_0 \in \mathbb{R}. \quad (32)$$

where $\tilde{C}_1 > 0$, $\tilde{C}_2 > 0$, and $w$ takes its values in $\mathbb{R}$, say for some given $\tau > 0$ we have $w(t) \in \mathbb{R}$ for $t \in [0, \tau]$.

## 4.2. Auxiliary ODE stability results

Below $\tilde{C}_1 > 0$ and $\tilde{C}_2 > 0$ are positive constants. We will look at (32) as a perturbation of the system

$$\dot{w} = -\left( \bar{C}_1 - \bar{C}_2 |w|_{L^\varepsilon_R}^p \right) w, \quad w(0) = w_0 \in \mathbb{R}. \quad (33)$$

**Proposition 4.3.** Let $p > 0$. If $|w_0|_{L^\varepsilon} < \left( \frac{\bar{C}_1}{\bar{C}_2} \right)^\frac{1}{p}$, then the solution of system (33) satisfies

$$e^{-\bar{C}_1(t-s)} |w(s)|_\varepsilon \leq |w(t)|_\varepsilon \leq e^{-\varepsilon(t-s)} |w(s)|_\varepsilon, \quad \text{for all} \ t \geq s \geq 0, \quad (34)$$

with $\varepsilon := \bar{C}_1 - \bar{C}_2 |w_0|_{L^\varepsilon_R}^p > 0$.

The proof is straightforward. For the sake of completeness we give it in the Appendix, Section A.2.

Next, for the perturbed ODE we have the following.
Lemma 4.4. Let $p > 0$, $r > 1$, and $h \in L^r(\mathbb{R}_0, \mathbb{R})$. If there exists $\bar{\varepsilon} > 0$ such that the inequality
\[
|w_0|_\mathbb{R} + \left(\frac{r}{r-1}\right)^{\frac{r-1}{r}} |h|_{L^r(\mathbb{R}_0, \mathbb{R})} \leq \left(\frac{\bar{C}_1 - \bar{\varepsilon}}{\bar{C}_2(p+1)}\right)^\frac{1}{p}
\] (35)
is satisfied, then the solution $w = w^h$ of system (32) satisfies, for all $t \geq 0$
\[
|w^h(t)|_\mathbb{R} \leq e^{-\varepsilon t} |w_0|_\mathbb{R} + \int_0^t e^{-\tilde{\varepsilon}(t-s)} |h(s)|_\mathbb{R} \, ds,
\] (36a)
\[
|w^h(t)|_\mathbb{R} \leq |w_0|_\mathbb{R} + \left(\frac{r}{r-1}\right)^{\frac{r-1}{r}} |h|_{L^r(\mathbb{R}_0, \mathbb{R})},
\] (36b)
with $\varepsilon := \bar{C}_1 - \bar{C}_2 |w_0|_\mathbb{R}^p$ and $\bar{\varepsilon} := \bar{C}_1 - \bar{C}_2(p+1) |w_0|_\mathbb{R}^p$. Notice that by the assumption (35) the initial condition $w(t) = \bar{w}(0) \in \mathbb{R}$ for all $t \in \mathbb{R}$, reads
\[
\dot{z} = - (\bar{C}_1 - \bar{C}_2(p+1) |\bar{w}|_\mathbb{R}^p) z, \quad z(0) = z_0 \in \mathbb{R},
\] (37)
which is exponentially stable if $\bar{C}_1 > \bar{C}_2(p+1) |\bar{w}|_\mathbb{R}^p$. That is, denoting the solution of (37) by
\[
z(t) = \mathcal{Z}_{\bar{w}}(t;0)z_0 = \mathcal{Z}_{\bar{w}}(t;s)\mathcal{Z}_{\bar{w}}(s;0)z_0 = \mathcal{Z}_{\bar{w}}(t;s)z(s), \quad t \geq s \geq 0, \quad z_0 \in \mathbb{R}, \quad \bar{w} \in \mathbb{R},
\]
we have that, with $z(s) = z_1 \in \mathbb{R}$,
\[
|\mathcal{Z}_{\bar{w}}(t;s)z_1|_\mathbb{R} = e^{-\tilde{\varepsilon}(t-s)} |z_1|_\mathbb{R}, \quad \text{for all} \quad t \geq s \geq 0, \quad \tilde{\varepsilon} := \bar{C}_1 - \bar{C}_2(p+1) |\bar{w}|_\mathbb{R}^p. \tag{38}
\]
Let us also denote the solutions of systems (33) and (32), for $t \geq s \geq 0$, respectively by
\[
w^0(t) =: w^0(t;s,w_1), \quad t \geq s \geq 0, \quad w^0(s) = w_1 \in \mathbb{R},
\]
\[
w^h(t) =: w^h(t;s,w_1), \quad t \geq s \geq 0, \quad w^h(s) = w_1 \in \mathbb{R}.
\]
Notice that by the assumption (35) the initial condition $w_0$ satisfies
\[
0 \leq |w_0|_\mathbb{R} < \left(\frac{\bar{C}_1}{\bar{C}_2}\right)^\frac{1}{p},
\]
which due to Proposition 4.3 implies that $w^0(t)$ is defined for all $t \geq 0$ and satisfies (34). We also know that $w^h(t)$ will be defined for $t \geq 0$ in a maximal time interval, say for $t \in (0,\tau^h)$ with $\tau^h > 0$. We show now that $\tau^h = +\infty$. Indeed if $\tau^h \neq +\infty$ then we would have that
\[
\lim_{t \to \tau^h} |w^h(t)|_\mathbb{R} = +\infty. \tag{39}
\]
Thus we want to show that (39) does not hold with (finite) $\tau^h \in \mathbb{R}_0$. Let us fix an arbitrary $\tau_1 \in (0,\tau^h)$, then both solutions remain bounded in $[0,\tau_1]$. That is, for a suitable large enough $\rho > 0$,
\[
\{ w^0(t), w^h(t) \} \in (w_0 - \rho, w_0 + \rho), \quad \text{for all} \quad t \in [0,\tau_1].
\]
From [13, Lem. 3], since (37) is the linearization of (33), we know that we can write
\[
w^h(t;0,w_0) = w^0(t;0,w_0) + \int_0^t \mathcal{Z}_{w^h(s;0,w_0)}(t;s)h(s) \, ds, \quad t \in [0,\tau_1].
\]
Next we prove that we actually have
\[ |w^h(\tau_1)|_{\mathbb{R}} \leq |w_0|_{\mathbb{R}} + \left( \frac{r-1}{r} \right)^{-\frac{r-1}{r}} |h|_{L^r(\mathbb{R}, \mathbb{R})}. \] (40)
For this purpose, let \( h \neq 0 \) and suppose that there exists \( \tau_2 \in (0, \tau_1) \) such that
\[ |w^h(\tau_2)|_{\mathbb{R}} = |w_0|_{\mathbb{R}} + \left( \frac{r-1}{r} \right)^{-\frac{r-1}{r}} |h|_{L^r(\mathbb{R}, \mathbb{R})}, \] (41a)
\[ |w^h(t)|_{\mathbb{R}} < |w_0|_{\mathbb{R}} + \left( \frac{r-1}{r} \right)^{-\frac{r-1}{r}} |h|_{L^r(\mathbb{R}, \mathbb{R})}, \quad \text{for all } t \in [0, \tau_2). \] (41b)
From (38), we find that
\[
|w^h(\tau_2; 0, w_0)|_{\mathbb{R}} \leq e^{-\varepsilon \tau_2} |w_0|_{\mathbb{R}} + \int_0^{\tau_2} e^{-\varepsilon (\tau_2 - s)} |h(s)|_{\mathbb{R}} \, ds
\]
\[
\leq e^{-\varepsilon \tau_2} |w_0|_{\mathbb{R}} + \left( \frac{r-1}{r} \right)^{-\frac{r-1}{r}} |h|_{L^r((0, \tau_2), \mathbb{R})},
\] (42)
which combined with (41a) and with the fact that \( \varepsilon < \bar{\varepsilon} \), gives us \( \left( \frac{r-1}{r} \right)^{-\frac{r-1}{r}} > \left( \frac{r-1}{r-\bar{\varepsilon}} \right)^{-\frac{r-1}{r}} \).

Thus, \( h \) vanishes for \( t > \tau_2 \) and we find \( w^h(\tau_1; \tau_2, w^h(\tau_2)) = w^0(\tau_1; \tau_2, w^h(\tau_2)) \), and
\[
|w^h(\tau_1; \tau_2, w^h(\tau_2))|_{\mathbb{R}} \leq e^{-\varepsilon (\tau_1 - \tau_2)} |w^h(\tau_2)|_{\mathbb{R}} \leq |w_0|_{\mathbb{R}} + \left( \frac{r-1}{r} \right)^{-\frac{r-1}{r}} |h|_{L^r(\mathbb{R}, \mathbb{R})}.
\]

Therefore, with \( h \neq 0 \), if there is \( \tau_2 \in (0, \tau_1) \) satisfying (41), then (40) holds true. Of course, if there is no such \( \tau_2 \) then necessarily (40) holds true.

Since (40) holds for arbitrary \( \tau_1 < \tau^h \), and since \( |w_0|_{\mathbb{R}} + \left( \frac{r-1}{r} \right)^{-\frac{r-1}{r}} |h|_{L^r(\mathbb{R}, \mathbb{R})} \)
is independent of \( \tau_1 \), it follows that necessarily (39) cannot hold with \( \tau^h \in \mathbb{R} \).

Therefore \( \tau^b = +\infty \), and (40) and (42) hold true for all \( \tau_1 \geq 0 \) and all \( \tau_2 \geq 0 \). That is, the estimates in (36) are satisfied.

4.3. Proof of Theorem 4.2. We can show the existence of the solution as a weak limit of Galerkin approximations of the system, following a standard argument. By taking the scalar product, in \( H \), with \( 2AQ \) in (30b), we obtain for all \( t > 0 \),
\[
\frac{d}{dt} |Q|^2_V = -2 |Q|^2_{D(A)} - 2 \left( P^U_{E_{\bar{\rho}_N}} \left( A_{rc}(t)Q + N(t, q, Q) \right), AQ \right)_H + 2(M(q), AQ)_H.
\]

Using Assumption 3.8, we fix a quadruple \( \gamma = (\gamma_1, \gamma_2, \gamma_3, \bar{\varepsilon}) \in \mathbb{R}^4 \) satisfying (26).

From Assumption 3.3,
\[
2 \left( I_{E_{\bar{\rho}_N}} A_{rc}(t)Q, AQ \right)_H \leq \gamma_1 |Q|^2_{D(A)} + \gamma_1^{-1} \left| I_{E_{\bar{\rho}_N}} \right|^2_{L(\mathcal{H})} C_{rc}^2 |Q|^2_V,
\] (43a)
\[
2(M(q), AQ)_H \leq \gamma_2 |Q|^2_{D(A)} + \gamma_2^{-1} |M(q)|^2_H
\] (43b)
and, from (21), with \( \gamma_0 = \gamma_3 \), we find
\[
2 \left( P^U_{E_{\bar{\rho}_N}} N(t, q, Q), AQ \right)_H \leq \gamma_3 |Q|^2_{D(A)}
\] (43c)
\[
+ \gamma_3 - \left( \frac{\gamma_3}{1 + \gamma_3} \right) C \left( 1 + |q|^2_{D(A)} \right) \left( 1 + |q|^2_{D(A)} \right) \left( 1 + |Q|^2_{D(A)} \right) |Q|^2_V.
\]
Hence, the estimates in (43) lead us to
\[
\frac{d}{dt} |Q|_{V}^2 \leq -a_0 |Q|_{D(A)}^2 + \left( a_1 + a_2 q \right) |Q|_{V}^2 + \varrho,
\]
with \(a_0, a_1, a_2, q, \varrho, p, \) and \(h\) as in (24).

Recall that for \((q, Q) \in E_{M_{\alpha}} \times E_{M_{\alpha}}^{p}\), we have \(|Q|_{D(A)}^2 \geq \alpha_{M_{\alpha}} |Q|_{V}^2 \) and \(q|_{D(A)}^2 \leq \alpha_{M_{\alpha}} |q|_{V}^2\), see (11). Thus,
\[
\frac{d}{dt} |Q|_{V}^2 \leq -(a_0 \alpha_{M_{\alpha}} - a_1 - a_2 q - a_2 q |Q|_{V}^{2p}) |Q|_{V}^2 + \varrho,
\]
and we conclude that \(|Q(t)|_{V}^2 \leq w(t)\), where \(w\) solves
\[
\dot{w} = -(a_0 \alpha_{M_{\alpha}} - a_1 - a_2 q - a_2 q |w|_{\Re}^{2p}) w + \varrho, \quad w(0) = |Q_0|_{V}^2.
\]

Note that, from Assumption 3.8, we have \(a_0 \alpha_{M_{\alpha}} - a_1 - a_2 q - \varrho > 0\) and
\[
|Q_0|_{V}^2 + \left( \frac{\|M\|}{\varrho} \sigma \right)^{-\frac{1}{p+1}} |h|_{L^p}(\Re) < \left( a_0 \alpha_{M_{\alpha}} - a_1 - a_2 q - \varrho \right) \sigma \wedge \left( \frac{\|M\|}{\varrho} \right)^{\frac{1}{p+1}} \sigma,
\]
which shows that the requirements in Lemma 4.4 are fulfilled with
\[
w_0 = |Q_0|_{V}^2, \quad p = p, \quad h = h, \quad C_1 = a_0 \alpha_{M_{\alpha}} - a_1 - a_2 q, \quad \text{and} \quad C_2 = a_2 q.
\]

Thus, with \(\varepsilon := C_1 - C_2 |w_0|_{\Re}^{2p} \) and \(\tilde{\varepsilon} = C_1 - C_2 (p+1) |w_0|_{\Re}^{2p} \), we arrive at (31).

We have just proven that (31) holds true, for any given strong solution. The existence of a strong solution follows from the fact that the previous estimates hold true for Galerkin approximations \(Q^N\) taking values in the finite-dimensional space \(E_{M_{\alpha}}^{1,N} := E_{M_{\alpha}}^{1} \cap E_{N}^{p}\),
\[
N \in \mathbb{N}, \quad E_N := \text{span}\{e_i \mid i \in \{1, 2, \ldots, N\}\}
\]
and \(P_{E_N} : H \rightarrow E_N\) is the orthogonal projection in \(H\) onto \(E_N\), which solve the finite-dimensional system, with initial condition \(Q_0 := P_{E_N} Q_0\),
\[
Q^N + P_{E_N} P_{E_{M_{\alpha}}^{N}} (A Q^N + A_{rc}(t) Q^N + N(t, q, Q^N)) = P_{E_N} M(q), \quad Q^N(0) = Q_0^N.
\]

We fix an arbitrary \(s > 0\). From (the analogous to) (31) we find \(|Q^N|_{L^{\infty}(0,s),V} \leq C_3\), where \(C_3\) can be taken independent of \(N\) and \(s\). Then, by integrating (44) we obtain that, with \(\mathcal{J}_s := (0, s)\),
\[
|Q(s)|_{V}^2 + a_0 |Q^N|_{L^2(\mathcal{J}_s,D(A))}^2 \leq |Q(0)|_{V}^2 + |h|_{L^{1}(\mathcal{J}_s,R)} + s \left( (a_1 + a_2 q) |Q^N|_{L^{\infty}(\mathcal{J}_s,V)}^2 + a_2 q |Q^N|_{L^{2p+2}(\mathcal{J}_s,V)}^2 \right).
\]
Since \(a_0 > 0\), because \(a_0 > \frac{a_1 + a_2 q}{\alpha_{M_{\alpha}}^2} > 0\), we conclude that \(|Q^N|_{L^2(\mathcal{J}_s,D(A))} \leq C_4\), where \(C_4\) can be taken independent of \(N\). Finally, from Assumption 3.3, (30b), (21), and \(q \in L^{\infty}(\Re, D(A))\), it follows that
\[
\left| \frac{d}{dt} Q^N \right|_{H} \leq C_5 \left( 1 + |Q^N|_{D(A)} + |M(q)|_{H} \right),
\]
from which we have that \(\left| \frac{d}{dt} Q^N \right|_{L^2(\mathcal{J}_s,H)} \leq C_7\), with \(C_7\) independent of \(N\). Thus, we can conclude the existence of the weak limit \(Q^\infty\) of a suitable subsequence of \(Q^N\).
We can assume strong convergence \( Q^N \xrightarrow{L^2(\mathcal{J}, D(A))} Q^\infty \) and consequently to \( \hat{Q}^N \xrightarrow{L^2(\mathcal{J}, H)} \hat{Q}^\infty \). We can assume strong convergence \( Q^N \xrightarrow{L^2(\mathcal{J}, D(A))} Q^\infty \), due to \( W(\mathcal{J}, D(A), H) \xrightarrow{\mathcal{J}} L^2(\mathcal{J}, V) \), see [55, Ch. 3, Sect. 2.2, Thm. 2.1]. Next, we show that

\[
\hat{Q}^N + P_{E_N} \mathcal{P}^f \mathcal{P}_{E_N} \left( AQ^N + A_{rc}(t)Q^N + N(t, q, Q^N) \right) - P_{E_N} \mathcal{M}(q)
\]

from the fact that \( y_1 = q + Q^N \) and from the Hölder inequality, with

\[
|N(t, q, Q^N)| \leq |A_{rc}(t)|^2 Q^N + A_{rc}(t)Q^N
\]

from which we find that \( Q^\infty \) solves (30b). We know that \( \hat{Q}^N \xrightarrow{L^2(\mathcal{J}, H)} \hat{Q}^\infty \), and

\[
AQ^N + A_{rc}(t)Q^N \xrightarrow{L^2(\mathcal{J}, H)} A Q^\infty + A_{rc}(t)Q^\infty
\]

follows straightforwardly. Since \( q \in L^2(\mathcal{J}, D(A)) \) is fixed, we can also derive that \( P_{E_N} \mathcal{M}(q) \xrightarrow{L^2(\mathcal{J}, H)} \mathcal{M}(q) \). Hence, since \( P_{E_N} \mathcal{P}^f \mathcal{P}_{E_N} \in \mathcal{L}(H) \), to show (46) it remains to show

\[
|N(t, q + Q^N) - N(t, q + Q^\infty)| \leq C_N \sum_{j=1}^{n} \left( |y_k|^{\delta_j} \right) \left( \left| D^N \right|^{\delta_j} \right) \xrightarrow{L^2(\mathcal{J}, H)} \mathcal{N}(t, q, Q^\infty).
\]

Actually, we have strong convergence \( \mathcal{N}(t, q, Q^N) \xrightarrow{L^2(\mathcal{J}, H)} \mathcal{N}(t, q, Q^\infty) \). Indeed, from the fact that \( Q^\infty \in W(\mathcal{J}, D(A), H) \rightarrow C([0, s], V) \) and the fact that the sequence \( Q^N \) is uniformly bounded in the space \( W(\mathcal{J}, D(A), H) \), from Assumption 3.4 and from the Hölder inequality, with \( y_1 = q + Q^N \) and \( y_2 = q + Q^\infty \), it follows that, with \( D^N := Q^N - Q^\infty \), and since \( \delta_2j + \zeta_2j < 1 \),

\[
|N(t, q + Q^N) - N(t, q + Q^\infty)| \xrightarrow{L^2(\mathcal{J}, H)} \mathcal{N}(t, q, Q^\infty).
\]

From \( \delta_1j + \delta_2j \geq 1 \), it follows that

\[
\frac{\delta_1j}{2} \xrightarrow{L^2(\mathcal{J}, H)} \mathcal{N}(t, q, Q^\infty),
\]

because \( D^N \) is uniformly bounded in \( L^\infty(\mathcal{J}, V) \). Observe also that by the Young inequality

\[
|y_k| \left. \left| D^N \right| \right|_{D(A)} \leq |y_k|^2 \left( \left| D^N \right| \right)^2_{D(A)},
\]

which leads us to

\[
\left( \left( \left| y_k \right|^2 \right) \left( \left| D^N \right| \right)^2_{D(A)} \right) \leq \left( \left( \left| y_k \right|^2 \right) \left( \left| D^N \right| \right)^2_{D(A)} \right) \leq \left( \left( \left| y_k \right|^2 \right) \left( \left| D^N \right| \right)^2_{D(A)} \right) \leq \left( \left( \left| y_k \right|^2 \right) \left( \left| D^N \right| \right)^2_{D(A)} \right) \leq \left( \left( \left| y_k \right|^2 \right) \left( \left| D^N \right| \right)^2_{D(A)} \right)
\]

and consequently to

\[
\mathcal{N}(t, q + Q^N) \xrightarrow{L^2(\mathcal{J}, H)} \mathcal{N}(t, q + Q^\infty) \xrightarrow{L^2(\mathcal{J}, H)} \mathcal{N}(t, q + Q^\infty).
\]
In order to finish the proof of Theorem 4.2, it remains to prove the uniqueness in $W((0, s), D(A), H)$. For this purpose, observe that given two solutions $Q_1$ and $Q_2$ in $W((0, s), D(A), H)$, we find that $G = Q_2 - Q_1 \in W((0, s), D(A), H)$ solves

$$G + P_{E_{M_0}^{\infty}}^U (AG + A_{rc}(t)G + N(t, q, Q_2) - N(t, q, Q_1)) = 0, \quad G(0) = 0 \in E_{M_0}^{\infty}. $$

Thus, from (20) with $\tilde{\gamma}_0 = 1$, and the Young inequality, with $y_1 = q + Q^N$ and $y_2 = q + Q^\infty$, it follows

$$2\left(D_{E_{M_0}^{\infty}}^U (N(t, y_1) - N(t, y_2)), AG\right)_H$$

$$\leq |G|^2_{D(A)} + C_{N_1} \sum_{j=1}^n \left( |y_1|_{1^{-2\tilde{\gamma}/(2-\tilde{\gamma})}}^2 + |y_2|_{1^{-2\tilde{\gamma}/(2-\tilde{\gamma})}}^2 + |y_1|^2_{D(A)} + |y_2|^2_{D(A)} \right) |G|_V^{2\tilde{\gamma}/(2-\tilde{\gamma})}$$

$$= |G|^2_{D(A)} + \Phi(t) |G|_V^2,$$

with $\Phi(t) := C_{N_1} \sum_{j=1}^n \left( |y_1|_{1^{-2\tilde{\gamma}/(2-\tilde{\gamma})}}^2 + |y_2|_{1^{-2\tilde{\gamma}/(2-\tilde{\gamma})}}^2 + |y_1|^2_{D(A)} + |y_2|^2_{D(A)} \right) |G|_V^{2\tilde{\gamma}/(2-\tilde{\gamma})} - 2$.

Recall that we have $\frac{2\tilde{\gamma}}{2-\tilde{\gamma}} \geq 2$.

By using Assumption 3.3 and (43) with $\gamma_1 = 1$, we find

$$\frac{d}{dt} |G|^2_V \leq -2 |G|^2_{D(A)} + |G|^2_{D(A)} + \Phi(t) |G|^2_V + |G|^2_{D(A)} + \left| P_{E_{M_0}^{\infty}}^U \right|^2_{L(H)} C_{rc} |G|^2_V$$

$$\leq \Phi_2(t) |G|^2_V,$$

with $\Phi_2(t) := \left| P_{E_{M_0}^{\infty}}^U \right|^2_{L(H)} C_{rc} + \Phi(t)$. We see that $\Phi_2$ is integrable on $(0, s)$, due to $q \in L^\infty([0, s], D(A))$ and $\{Q_1, Q_2\} \subset C([0, s], V) \cap L^2((0, s), D(A))$. Hence, by the Gronwall inequality,

$$|G(t)|^2_V \leq e^{\int_0^t \Phi_2(r) dr} |G(0)|^2_V = 0, \quad \text{for all} \ t \in [0, s].$$

That is, $G = 0$ and $Q_2 = Q_1 + G = Q_1$. We have shown that for arbitrary $s > 0$ there exists one, and only one, strong solution $Q \in W((0, s), D(A), H)$ for (30b). In other words, there exists one, and only one, global solution $Q \in W_{loc}(\mathbb{R}_0, D(A), H)$ for (30b). This finishes the proof of Theorem 4.2. \qed

4.4. Proof of Theorem 4.1. Theorem 4.1 follows from Theorem 4.5 below. \qed

Theorem 4.5. If Assumptions 3.1–3.4 and 3.6–3.8 are satisfied, then system (30) is exponentially stable. The solution $y = q + Q$, satisfies $|y(t)|_{V} \leq C e^{-\tilde{\varepsilon} t} |y(0)|_{V}$, for all $t \geq 0$, where $0 < \mu < \min \{ \tilde{\varepsilon}, 2\lambda \}$ and $\tilde{\varepsilon}$ is as in (28). Furthermore, $C = \overline{C} \left[ n, ||P||_{C, r, C_{N, 1/4, C_{1+\varepsilon}}, C_1}, 3^{-1} \tau_0, |Q(0)|_{V}, 1, \frac{1}{1-\mu}, \frac{1}{1-\mu}, \alpha, \xi_0 \right].$

Proof. We have $q \in L^\infty(\mathbb{R}_0, D(A))$ because $q \in L^\infty(\mathbb{R}, H)$ and $E_{M_0}^{\infty}$ is finite dimensional, $E_{M_0}^{\infty} \subset D(A) \subset H$. By Theorem 4.2, we conclude that $Q$ satisfies, for all $t \geq 0$,

$$|Q(t)|^2_V \leq e^{-\varepsilon t} |Q(0)|^2_V + \int_0^t e^{-\tilde{\varepsilon}(t-s)} \gamma_3^{-1} |M(q(s))|^2_H ds.$$
Hence we obtain, using Assumptions 3.4 and 3.6,

\[ |Q(t)|_V^2 - e^{-\varepsilon t} |Q(0)|_V^2 \leq C_{n,P} \frac{\gamma^{-1}}{3} \int_0^t e^{-\tilde{\varepsilon}(t-s)} \left( |Aq(s)|_H^2 + |q(s)|_V^2 \right) \, ds \]

\[ \leq C_{n,P} \frac{\gamma^{-1}}{3} \int_0^t e^{-\tilde{\varepsilon}(t-s)} \left( \alpha_{M_\varepsilon} + 1 + \sum_{j=1}^n \alpha_{M_\varepsilon} |q(s)|_V^{2(\delta_j + \alpha_j)} + \alpha_{M_\varepsilon} |q(s)|_V^{2(\delta_j)} \right) \, ds \]

\[ \leq C_{n,P} \frac{\gamma^{-1}}{3} \int_0^t e^{-\tilde{\varepsilon}(t-s)} e^{-2\lambda s} \, ds. \]

Through straightforward computations we can obtain, with \( \mu < \min\{\tilde{\varepsilon}, 2\lambda\}, \)

\[ \int_0^t e^{-\tilde{\varepsilon}(t-s)} e^{-2\lambda s} \, ds \leq \int_0^t e^{-\tilde{\varepsilon}(t-s)} e^{-\mu s} \, ds \leq \left| \tilde{\varepsilon} - \mu \right|^{-1} e^{-\mu t}, \]

which leads us to

\[ |Q(t)|_V^2 \leq e^{-\varepsilon t} |Q(0)|_V^2 + \tilde{D} e^{-\mu t} |q(0)|_V^2 \]

with \( \tilde{D} = C_{n,P} \frac{\gamma^{-1}}{3} \int_0^t e^{-\tilde{\varepsilon}(t-s)} e^{-2\lambda s} \, ds. \) Hence, \( |y(t)|_V^2 = |Q(t)|_V^2 + |q(t)|_V^2 \) satisfies for all \( t \geq 0, \)

\[ |y(t)|_V^2 \leq e^{-\varepsilon t} |Q(0)|_V^2 + \tilde{D} e^{-\mu t} |q(0)|_V^2 + C_{q_1} e^{-2\lambda t} |q(0)|_V^2 \]

\[ \leq e^{-\varepsilon t} |Q(0)|_V^2 + C_{q_1} + \tilde{D} e^{-\mu t} |y(0)|_V^2 \leq (1 + C_{q_1} + \tilde{D}) e^{-\mu t} |y(0)|_V^2, \quad (47) \]

which finishes the proof. \( \square \)

4.5. **Proof of Theorem 2.5.** We show that Theorem 2.5 follows as a corollary of Theorem 4.1. Indeed, let us suppose we have a sequence \((U_{\#M_\varepsilon}, E_{M_\varepsilon})_{M \in \mathbb{N}}\) so that \( C^M_P := \left| P_{E_{M_\varepsilon}}^0 \right|_{L(H)} \leq C_P \) and \( \alpha_{M_\varepsilon} \leq \Lambda, \) with \( C_P \) and \( \Lambda \) independent of \( M. \)

Let us also fix \( \gamma = \tilde{\pi} \in \mathbb{R}^3_+ \) so that \( a_0 = a_0^\gamma > 0, \) and fix also \( \tau > 0. \)

Recalling (24) and (25), we see that \( \tilde{a}_1^\gamma, \tilde{a}_2^\gamma, \) and \( \tilde{b}_1^\gamma, \) are the only terms in (26) depending on \( C^M_P. \) However, these terms remain bounded if \( C^M_P \) does. Hence, defining

\[ \tilde{a}_1^\gamma := a_1^\gamma(C_P) = \gamma_1^{-1} C_P^2 C_{\#}^2 > a_1^\gamma(C^M_P), \]

\[ \tilde{a}_2^\gamma := a_2^\gamma(C_P) = \gamma_2^{-1} \frac{C_N^2}{C_{\#}^2} C^2(C_P) > a_2^\gamma(C^M_P), \]

\[ \tilde{b}_1^\gamma := b_1^\gamma(C_P) \geq b_1^\gamma(C^M_P), \]
we observe that Assumption 3.8, taking \( r = \frac{1}{\|z + \tau_2\|} \) as in Assumption (3.6), follows from
\[
\alpha_{M_{\tau +}} > \inf_{(\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3) \in \mathbb{R}_+^3, \vec{\tau}_1, \vec{\tau}_2 > 0, \vec{\tau}_3 > 0} \frac{1}{\alpha^{\varphi}_{M_{\tau +}}} \left( \vec{a}_1 + \vec{a}_2 \tilde{q} + \vec{p} + 1 \vec{a}_2 \tilde{q} \left( |Q_0|_{L^r(\mathbb{R}_+, \mathbb{R})} \right)^p \right).
\] (48)

Note that for \( M \) large enough it follows that \( \alpha_0 \alpha_{M_{\tau +}} - \vec{a}_1 - \vec{a}_2 \tilde{q} - \vec{p} > 0 \). Now, with \( y_0 = q_0 + Q_0 \),
\[
\lim_{M \to +\infty} \frac{\vec{a}_1 + \vec{a}_2 \tilde{q} + \vec{p} + 1 \vec{a}_2 \tilde{q} \left( |Q_0|_{L^r(\mathbb{R}_+, \mathbb{R})} \right)^p}{\alpha_{M_{\tau +}}} = 0,
\] (49a)
\[
\lim_{M \to +\infty} \frac{\alpha_{M_{\tau +}}}{\alpha_{M_{\tau +}}} \leq \mathcal{C}_{[C_{\varphi} : \|q_0\|_V]} \lim_{M \to +\infty} \frac{\|h\|^p}{\alpha_{M_{\tau +}}^{\varphi_{M_{\tau +}}}} \leq \mathcal{C}_{[C_{\varphi} : \|q_0\|_V]} \lim_{M \to +\infty} \Lambda \left( \frac{\|h\|^p}{\alpha_{M_{\tau +}}^{\varphi_{M_{\tau +}}}} \right)^{-1} = 0,
\] (49b)

since \( \|\frac{\gamma_2}{1 - \beta_2}\| < 1 \), and
\[
\lim_{M \to +\infty} \left( \frac{\alpha_{M_{\tau +}}}{\alpha_{M_{\tau +}}} \right) \frac{\|h\|^p}{\alpha_{M_{\tau +}}^{\varphi_{M_{\tau +}}}} \leq \mathcal{C}_{[C_{\varphi} : \|q_0\|_V]} \lim_{M \to +\infty} \frac{\|h\|^p}{\alpha_{M_{\tau +}}^{\varphi_{M_{\tau +}}}} = 0
\] (49c)

From (27) we have that
\[
|\tilde{b}|_{L^r(\mathbb{R}_+, \mathbb{R})} \leq \mathcal{C}_{[n, C_{\varphi}, C_{\lambda}, C_{\lambda'}]} \left( 1 + \alpha_{M_{\tau +}}^{\beta_2 + \zeta_2} \right) \left( 1 + \alpha_{M_{\tau +}}^{\beta_2 + \zeta_2} \right) \|q(0)\|_V,
\]
which leads us to
\[
\lim_{M \to +\infty} \frac{\|h\|^p}{\alpha_{M_{\tau +}}^{\varphi_{M_{\tau +}}}} \leq \mathcal{C}_{[n, C_{\varphi}, C_{\lambda}, C_{\lambda'}]} \lim_{M \to +\infty} \left( \frac{\|h\|^p}{\alpha_{M_{\tau +}}^{\varphi_{M_{\tau +}}}} \right)^{-1} \left( 1 + \alpha_{M_{\tau +}}^{\beta_2 + \zeta_2} \right) \left( 1 + \alpha_{M_{\tau +}}^{\beta_2 + \zeta_2} \right) = 0,
\] (49d)

since, by Assumption 3.6, \( \max \left\{ \frac{\gamma_2}{1 - \beta_2}, 1 + \beta_2 \right\} - 1 + \beta_2 \left( \frac{\gamma_2}{1 - \beta_2} \right) - 1 + \gamma_2 \left( \frac{\gamma_2}{1 - \beta_2} \right) < 0 \). Therefore, from the inequalities in (49) we can conclude that necessarily (48) holds true for large enough \( M \), with
\[
M = \mathcal{C}_{[n, C_{\varphi}, C_{\lambda}, C_{\lambda'}]} \left( 1 + \alpha_{M_{\tau +}}^{\beta_2 + \zeta_2} \right) \left( 1 + \alpha_{M_{\tau +}}^{\beta_2 + \zeta_2} \right) |q(0)|_V |Q(0)|_V.
\] (50)

In particular, (50) means that \( M \) increases (or may increase) with the norm \( |q(0)|_V \), of the initial condition \( y(0) = q(0) + Q(0) \), but it also means that, for arbitrary given \( R > 0 \), \( M \) can be taken the same for all initial initial conditions in the ball \( \{ z \in V \mid |z|_V \leq R \} \). \( \square \)

4.6. Boundedness of the control. In applications, besides the existence of a stabilizing feedback, it is important that the total “energy” spent to stabilize the system is finite. We show here that the control given by our nonlinear feedback operator in (17) is indeed bounded, with a bound increasing with the norm of the initial condition. Note that (15) and (29) are the same system.

**Theorem 4.6.** Let \( u(t) := F(t, y(t)) = K_{\varphi}^{\tau_0} N(t, y) \) be the control given by the operator (17) stabilizing system (15), with initial condition \( y_0 \) as in Theorem 4.1.
Then
\[ \sup_{t \geq 0} |K_{\mathcal{T}^\beta_{42}, \mathcal{N}}^{E_{42}, \alpha, \lambda}(t, z)|_H \leq C([|P|]_{L^2(V,V,C_r,C_{N},C_{q_0})} + 1 + |z|_{H^1(D(A))}), \text{ for all } z \in D(A), \text{ and} \]
\[ |u|_{L^{2r}((0,\infty),H)} \leq \frac{C}{|z|_{H^1(D(A))}}, \text{ where} \]
\[ |u|_{L^{2r}((0,\infty),H)} \leq \frac{C}{|z|_{H^1(D(A))}}, \text{ and } \parallel P\parallel_{L^2(U,\mathcal{M},\mathcal{N})} \leq \frac{E_{42}^{E_{42}, \alpha, \lambda}}{L(H)}. \]

**Proof.** Recall (17). The boundedness of \( K_{\mathcal{T}^\beta_{42}, \mathcal{N}}^{E_{42}, \alpha, \lambda} \) follows from \( |A|_{L^2(D(A),H)} = 1 \) and
\[ |A_{ec}(t)|_{L^2(D(A),V)} \leq C[c_{ec}] |Id|_{L^2(D(A),V)}, \]
\[ |N(t,z)|_{H} \leq C[c_{N}] (1+|z|_{H^1(D(A))}), \]
and from \( |Id|_{L^2(D(A),V)} = \alpha_1^{-\frac{1}{2}}, \xi \geq 1, \text{ and } \|\zeta_2 + \delta_2\| < 1. \)

To show the boundedness of the spent “energy” \( |u|_{L^{2r}((0,\infty),H)} \), observe that
\[ |u|_{L^{2r}((0,\infty),H)} \leq \|P\|_{L^2(U,\mathcal{M},\mathcal{N})} \left| P_{E_{42}, \alpha, \lambda} \left( Ay + A_{ec}(\cdot) y + N(\cdot, y) - \mathcal{F}_{M_{+}}(P_{E_{42}, \alpha, \lambda} y) \right) \right|_{L^{2r}((0,\infty),H)} \]
\[ \leq \frac{C}{|z|_{H^1(D(A))}} \left( |y_0|_{V} \|y_0\|_{H^1(D(A))} + |y|_{L^2((0,\infty),V)} + \sum_{j=1}^{n} |y_0|_{V} \|y_0\|_{H^1(D(A))} \right) \]
\[ \leq \frac{C}{|z|_{H^1(D(A))}} \left( |y_0|_{V} \|y_0\|_{H^1(D(A))} + |y|_{L^2((0,\infty),V)} + \sum_{j=1}^{n} |y_0|_{V} \|y_0\|_{H^1(D(A))} \right) \]
with \( q_0 := P_{E_{42}, \alpha, \lambda} y_0 \), and where we have used (47). Observe that \( |y_0|_{V}^{\zeta_1 + \delta_1 + \zeta_2 + \delta_2} \leq |y_0|_{V} |y_0|_{V} + |y_0|_{V} |y_0|_{V}, \) because \( \zeta_1 + \delta_1 + \zeta_2 + \delta_2 \geq \delta_1 + \delta_2 \geq 1. \) Recall also that \( \beta_1 + \beta_2 \geq 1. \)

**Remark on the transient bound.** We have seen that, see (50) and (47), in Theorem 2.5 we may take
\[ M = \frac{C}{|z|_{H^1(D(A))}} \left[ n, c_{ec}, c_{N}, c_{q_0}, c_{q_2}, c_{q_3}, \frac{1}{2}, |y(0)|_{V} \right], \]
\[ 0 < \mu_2 < \min \left( \frac{\xi}{2}, \lambda \right), \text{ and } C_5 = \frac{C}{|z|_{H^1(D(A))}} \left[ n, c_{ec}, c_{N}, c_{q_0}, c_{q_1}, c_{q_3}, \frac{1}{2}, |y(0)|_{V}, \right. \]
Observe that by taking a larger \( M \) we still have a stable closed-loop system, but since the transient bound \( C_5 \) depends on \( \alpha_{M_{+}} \), the transient time \( t_{tr} = \frac{\log C_5}{\mu_2} \) may also depend on \( \alpha_{M_{+}} \). Note also that, from (28), \( \mu_2 \) will depend on \( \alpha_{M_{+}} \) if \( |h|_{L^\infty((0,\infty),H)} \) does. We see that \( C_5 \) gives us an upper bound for the norm of the closed-loop solution, \( \max \{ |y(t)|_{V} \mid t \geq 0 \} \leq C_5 |y(0)|_{V}, \) and for time \( t \geq t_{tr} \) we necessarily have that \( |y(t)|_{V} \leq |y(0)|_{V}. \) Therefore, it could be interesting to understand whether we can make \( C_5 \) and \( t_{tr} \) as small as possible. Though we do not study this possibility in here, we would like to say that a positive answer does not follow from above, due to the dependence on \( \alpha_{M_{+}} \). Finding a positive answer to this question will likely require the derivation of new appropriate estimates.

5. Example of application. Parabolic equations with polynomial nonlinearities. We consider parabolic equations, evolving in a bounded domain \( \Omega \subset \mathbb{R}^d \), \( d \in \{1,2,3\} \) under homogeneous Dirichlet or Neumann boundary conditions and with a general polynomial nonlinearity. We assume that \( \Omega \) is regular enough so that \( V \subset H^1(\Omega), D(A) \subset H^2(\Omega), \) with equivalent norms, say \( C_1 |y|_{V} \leq |y|_{H^2(\Omega)} \leq C_2 |y|_{V} \).
\[ C_2 |y|_{V'} \text{ and } C_3 |y|_{D(A)} \leq |y|_{H^2(\Omega)} \leq C_4 |y|_{D(A)} \] for suitable positive constants \( C_1, C_2, C_3, \) and \( C_4 \).

We check Assumptions 3.1–3.4 and Assumptions 3.6–3.8 for the system
\[ \frac{\partial}{\partial t} y + (-\nu \Delta + \text{Id}) y + (a(t, x) - 1)y + b(t, x) \cdot \nabla y + \mathcal{N}(t, x, y) = 0, \quad y(0) = y_0, \]
with either Dirichlet or Neumann homogeneous boundary conditions.

**Remark 5.1.** Below we use the Agmon embedding \( D(A) \subseteq H^2(\Omega) \to L^\infty(\Omega) \) which holds true for \( d \in \{1, 2, 3\} \). For the case \( d \geq 4 \), which we do not consider here in this section, we would need a different argument because we do not necessarily have \( H^2(\Omega) \to L^\infty(\Omega) \), see [1, Lem. 13.2].

### 5.1. The linear operators
We check Assumptions 3.1–3.3. We take \( A = -\nu \Delta + \text{Id} : D(A) \to H \), with \( H = L^2(\Omega) \) and \( D(A) = \{ u \in H^2(\Omega) \mid Gu|_{\partial \Omega} = 0 \} \). Under Dirichlet boundary conditions we have \( G = \text{Id} \) and \( V = H^1(\Omega) \), and under Neumann boundary conditions we have \( G = \n \cdot \nabla \) and \( V = H^1(\Omega) \), where \( \n \) is the unit outward normal vector to the boundary \( \partial \Omega \) of \( \Omega \). It is straightforward to see that Assumptions 3.1–3.2 are satisfied. Assumption 3.3 is satisfied if \( a \in L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R}) \) and \( b \in L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R}^3) \).

### 5.2. Polynomial reactions and convectons in case \( \Omega \subset \mathbb{R}^3 \)
In case \( d = 3 \) we show now that Assumption 3.4 is satisfied for nonlinearities in the form
\[ \mathcal{N}(t, y) = \mathcal{N}(t, x, y) = \sum_{j=1}^{n} \left( \hat{a}_j(t, x) |y|_{\mathbb{R}}^{r_j} y + \left( \hat{b}_j(t, x) \cdot \nabla y \right) |y|_{\mathbb{R}}^{r_j-1} y \right), \]
with \( r_j \in (1, 5) \), \( s_j \in [1, 2] \), \( \hat{a}_j \in L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R}) \), \( \hat{b}_j \in L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R}^3) \), \( \hat{y} = y(t, x), \) and \( (t, x) \in \mathbb{R}_0 \times \Omega \).

#### 5.2.1. The reaction components
We start by considering the terms \( \mathcal{N}_j(t, y) = \hat{a}_j(t, x) |y|_{\mathbb{R}}^{r_j} y \). We fix \( (t, j, \tau) \) and note that \( \phi_j(x, s) := \hat{a}_j(t, x) |s|_{\mathbb{R}}^{r_j-1} s \), \( (t, s) \in \mathbb{R}^2 \), is differentiable with respect to \( s \), because \( \phi_j(x, \cdot) \in C(\mathbb{R}, \mathbb{R}) \) and \( \frac{\partial}{\partial s} \phi_j(x, \cdot) \in C(\mathbb{R}, \mathbb{R}) \), with \( \frac{\partial}{\partial s} \phi_j(x, \tau) = r_j \hat{a}_j(t, x) |\tau|_{\mathbb{R}}^{r_j-1} \). We also have the growth bounds
\[ \phi_j(x, s) \leq |\hat{a}_j(t, \cdot)|_{L^\infty(\Omega)} |s|_{\mathbb{R}}^{r_j} \quad \text{and} \quad \frac{\partial}{\partial s} \phi_j(x, \tau) \leq r_j |\hat{a}_j(t, \cdot)|_{L^\infty(\Omega)} |\tau|_{\mathbb{R}}^{r_j-1}. \]

With \( r > 1 \) and \( \hat{a}(t, \cdot) \in L^\infty(\Omega) \), the Nemytskij operator \( y \to \mathcal{N}(t, y) := \hat{a} |y|_{\mathbb{R}}^{r_j} y \) and its Fréchet derivative \( d\mathcal{N}|_y \) satisfy:
\[ \mathcal{N}(t, \cdot) \in C(L^{p'}, L^p) \text{ and } d\mathcal{N}|_y = r \hat{a} |y|_{\mathbb{R}}^{r_j-1} \in C(L^{p'}, C(L^{p'}, L^p), \ p \geq 1. \]

Indeed, with \( q > 1 \), we have
\[ \left| r \hat{a} |y|_{\mathbb{R}}^{r_j-1} h \right|_{L^p} \leq r |\hat{a}(t, \cdot)|_{L^\infty} |y|_{L^{p/(r-1)}}^{r_j-1} \left| h \right|_{L^{p/(r-1)}}. \]

Setting \( q = \frac{r_j-1}{r_j-1} \) we obtain
\[ \left| r \hat{a} |y|_{\mathbb{R}}^{r_j-1} h \right|_{L^p} \leq r |\hat{a}(t, \cdot)|_{L^\infty} |y|_{L^{p(r-1)}}^{r_j-1} \left| h \right|_{L^{p(r-1)}}. \quad (52) \]

By the Mean Value Theorem (see, e.g., [2, Thm. 1.8]) we can conclude that
\[ |\mathcal{N}_j(t, y_1) - \mathcal{N}_j(t, y_2)|_{L^2} \leq r_j |\hat{a}_j(t, \cdot)|_{L^\infty} \left( |y_1|_{L^{r_j}}^{r_j-1} + |y_2|_{L^{r_j}}^{r_j-1} \right) |y_1 - y_2|_{L^{2r_j}}. \quad (53) \]
Shortening the notation as $D_{N_j} := N_j(t, y_1) - N_j(t, y_2)$ and $D_y := y_1 - y_2$, we obtain
\[ |D_{N_j}|_{L^2} \leq C_{d_3} r_j |\tilde{a}_j(t, \cdot)|_{L^\infty}(\sum_{k=1}^n |y_k|_{L^\infty}^{2(r_j-5)/(r_j-1)} |y_k|_{L^6}^{3/(r_j-1)} |D_y|_{L^r}^{2r_j-6} |D_{N_j}|_{L^r}^{r_j} |D_y|_{L^6}^{3/r_j}, \quad (r_j \leq 3). \]
\[ |D_{N_j}|_{L^2} \leq C_5 r_j |\tilde{a}_j(t, \cdot)|_{L^\infty}(\sum_{k=1}^n |y_k|_{L^\infty}^{2(r_j-6)/(r_j-1)} |y_k|_{L^6}^{3/(r_j-1)} |D_y|_{L^r}^{2r_j-6} |D_{N_j}|_{L^r}^{r_j} |D_y|_{L^6}^{3/r_j}, \quad (r_j > 3). \]

Where we have used the Sobolev embedding inequality $|z|_{L^6} \leq C |z|_{V^r}$, see [18, Thm. 4.57], and the Agmon inequality $|z|_{L^\infty} \leq C |z|_{V^r}^1 |z|_{(D,A)}^2$, see [1, Lem. 13.2],[54, Sect. 1.4].

We see that $N_j$ satisfies the inequality in Assumption 3.4 when $1 < r_j < 5$, with
\[ \zeta_1 = r_j - 1, \quad \zeta_2 = 0, \quad \delta_1 = 1, \quad \delta_2 = 0, \]
in case $r_j \in (1, 3]$, and with
\[ \zeta_1 = \frac{(2r_j-6)(r_j-1)}{4r_j}, \quad \zeta_2 = \frac{(2r_j-6)(r_j-1)}{4r_j}, \quad \delta_1 = \frac{2r_j+6}{4r_j}, \quad \delta_2 = \frac{2r_j-6}{4r_j}, \]
in case $r_j \in (3, 5)$. In either case $\zeta_2 + \delta_2 < 1$ and $\delta_1 + \delta_2 = 1$. For $r_j \in (3, 5)$ we have $\zeta_2 + \delta_2 = \frac{2r_j-6}{4r_j} = \frac{r_j-3}{2} < 1$.

5.2.2. The convection components. Now for the terms $(\hat{b}_j(t, \cdot) \cdot \nabla y_1)|y_1|_{L^\infty}^{s_j-1} y_1$, we observe that
\[ \left| (\hat{b}_j(t, \cdot) \cdot \nabla y_1)|y_1|_{L^\infty}^{s_j-1} y_1 - (\hat{b}_j(t, \cdot) \cdot \nabla y_2)|y_2|_{L^\infty}^{s_j-1} y_2 \right|_{L^2} \leq \mathcal{G}_1 + \mathcal{G}_2, \quad (54) \]
with
\[ \mathcal{G}_1 := \left| (\hat{b}_j(t, \cdot) \cdot \nabla y_1)|y_1|_{L^\infty}^{s_j-1} y_1 - |y_2|_{L^\infty}^{s_j-1} y_2 \right|_{L^2}, \]
\[ \mathcal{G}_2 := \left| (\hat{b}_j(t, \cdot) \cdot \nabla(y_1 - y_2))|y_2|_{L^\infty}^{s_j-1} y_2 \right|_{L^2}. \]

from which, we obtain
\[ \mathcal{G}_1 \leq C \left( \hat{b}_j(t, \cdot) \right)_{L^\infty} \left| \nabla y_1 \right|_{L^\frac{6}{s_j}} \left| y_1 \right|_{L^\frac{6}{s_j}}^{s_j-1} y_1 - \left| y_2 \right|_{L^\frac{6}{s_j}}^{s_j-1} y_2 \right|_{L^\frac{6}{s_j}}^{\frac{6}{s_j}}, \quad (55a) \]
\[ \mathcal{G}_2 \leq C \left( \hat{b}_j(t, \cdot) \right)_{L^\infty} \left| \nabla(y_1 - y_2) \right|_{L^\frac{6}{s_j}} \left| y_2 \right|_{L^\frac{6}{s_j}}^{s_j-1} y_2 \right|_{L^\frac{6}{s_j}}^{\frac{6}{s_j}}. \quad (55b) \]

Let us consider first the case $s_j \in (1, 2)$. From (52) we find that
\[ \left| y \right|_{L^\frac{6}{s_j}}^{s_j-1} h \left| L^\frac{6}{s_j} \right| \leq \left| y \right|_{L^\frac{6}{s_j}}^{s_j-1} \left| \nabla y \right|_{L^\frac{s_j}{6}}^{\frac{s_j}{6}}, \quad (55a) \]
and, from (55) and the Mean Value Theorem, it follows that
\[ \mathcal{G}_1 \leq C_1 s_j \left| \nabla y_1 \right|_{L^\frac{6}{s_j}}^{\frac{6}{s_j}} \left( \left| y_1 \right|_{L^\frac{6}{s_j}}^{s_j-1} + \left| y_2 \right|_{L^\frac{6}{s_j}}^{s_j-1} \right) \left| y_1 \right|_{L^\frac{6}{s_j}}^{s_j-1}, \quad (56a) \]
\[ \mathcal{G}_2 \leq C_1 \left| \nabla(y_1 - y_2) \right|_{L^\frac{6}{s_j}}^{\frac{6}{s_j}} \left| y_2 \right|_{L^\frac{6}{s_j}}^{s_j-1}. \quad (56b) \]

Since $d = 3$ and $1 < s_j < 2$, we have the Sobolev embeddings, see [18, Thm. 4.57],
\[ H^{\frac{3}{2}} \rightarrow L^{\frac{6}{3}}, \quad \text{and} \quad H^{\frac{3}{2}} \rightarrow L^{\frac{6}{3}}, \]
because \( \chi := \frac{s_j - 1}{2} \) and \( \xi := \frac{2s_j + 1}{2s_j} \), solve
\[
\frac{6}{s_j} = \frac{3d}{a-2\chi} = \frac{6}{3-2\chi}, \quad 2\chi < 3, \quad \frac{6s_j}{s_j - 1} = \frac{6}{3-2\xi}, \quad 2\xi < 3.
\]
Hence, from (56) and \( |\nabla_{x}^\frac{s_j - 1}{H} \leq | \), we find
\[
\mathcal{G}_1 \leq C_2 |y_1|_{V^\frac{s_j + 1}{H}} \left( |y_1|_{V^\frac{s_j - 1}{2s_j + 1}} + |y_2|_{V^\frac{s_j - 1}{2s_j + 1}} \right) |y_1 - y_2|_{L^\frac{2s_j + 1}{2s_j}} \quad \text{(57a)}
\]
\[
\mathcal{G}_2 \leq C_2 |y_1 - y_2|_{H^\frac{s_j + 1}{2s_j}} |y_2|_{V^\frac{s_j + 1}{2s_j}}. \quad \text{(57b)}
\]
Next, we use an interpolation argument, see [35, Ch. 1, Sect. 2.5, Prop. 2.3]. From
\[
\frac{s_j + 1}{2} = 1(1 - \tau_1) + 2\tau_1 \quad \iff \quad \frac{s_j - 1}{2} = \tau_1, \quad \tau_1 \in (0, 1), \quad 1 - \tau_1 = \frac{3-s_j}{2},
\]
\[
\frac{2s_j + 1}{2s_j} = 1(1 - \tau_2) + 2\tau_2 \quad \iff \quad \frac{1}{2s_j} = \tau_2, \quad \tau_2 \in (0, 1), \quad 1 - \tau_2 = \frac{2s_j - 1}{2s_j},
\]
and (56), with \( D_y := y_1 - y_2 \), we arrive at
\[
\mathcal{G}_1 \leq C_3 |y_1|_{V^\frac{3-s_j}{2}} |y_1|_{D(A)} \left( \sum_{k=1}^2 |y_k|_{V^\frac{s_j - 1}{2s_j + 1}} |y_k|_{D(A)} \right) |y_1 - y_2|_{V^\frac{2s_j - 1}{2s_j}} |y_2|_{D(A)}, \quad \text{(58a)}
\]
\[
\mathcal{G}_2 \leq C_1 |\hat{y}_j(t, \cdot)|_{L^\infty} |y_1|_{V^\frac{3-s_j}{2}} |y_2|_{D(A)} \left( \sum_{k=1}^2 |y_k|_{V^\frac{s_j - 1}{2s_j + 1}} |y_k|_{D(A)} \right) |y_1 - y_2|_{V^\frac{2s_j - 1}{2s_j}} |y_2|_{D(A)}. \quad \text{(58b)}
\]
Observe that since \( s_j \in (1, 2) \) we can set \( \kappa \in (\frac{2s_j - 1}{s_j}, \frac{2s_j - 1}{s_j(s_j - 1)}) \subset (1, +\infty) \). Next, since \( \kappa > 1 \), we derive
\[
|y_1|_{V^\frac{s_j - 1}{2s_j}} |y_1|_{D(A)} |y_2|_{V^\frac{s_j - 1}{2s_j}} |y_2|_{D(A)} = |y_1|_{V^\frac{s_j - 1}{2s_j}} |y_1|_{D(A)} + |y_2|_{V^\frac{s_j - 1}{2s_j}} |y_2|_{D(A)}.
\]
We can see that the component (58a) can be bounded as in Assumption 3.4, with
\[
(\zeta_{11}, \zeta_{21}) \in \left\{ \left( \frac{3-s_j}{2} + \frac{(s_j - 1)(2s_j - 1)}{2s_j}, \frac{s_j - 1}{2s_j} \right), \left( \kappa \frac{3-s_j}{2}, \kappa \frac{s_j - 1}{2s_j} \right), \left( \frac{\kappa (s_j - 1)(2s_j - 1)}{2s_j}, \frac{\kappa s_j - 1}{2s_j} \right) \right\}
\]
and with \( (\delta_{11}, \delta_{21}) = (\frac{2s_j - 1}{2s_j}, \frac{1}{2s_j}) \). Note that \( \delta_{11} + \delta_{21} = 1, \frac{s_j - 1}{2s_j} + \frac{s_j - 1}{2s_j} = \frac{s_j - 1}{2s_j} \), and
\[
\zeta_{21} + \delta_{21} \leq \frac{1}{2s_j} + \max \left\{ \frac{s_j - 1}{2s_j}, \kappa \frac{s_j - 1}{2s_j} \right\},
\]
and from \( \frac{1}{2s_j} + \frac{s_j - 1}{2s_j} \), \( \frac{s_j - 1}{2s_j} < \frac{1}{2s_j} + \frac{s_j - 1}{2s_j} = 1, \) and \( \frac{1}{2s_j} + \frac{s_j - 1}{2s_j} = \frac{\kappa s_j - 1}{\kappa - 1} \), together with
\[
\frac{\kappa s_j - 1}{\kappa - 1} < 1 \quad \iff \quad \kappa s_j - 1 < 2\kappa s_j - 2s_j \quad \iff \quad 2s_j - 1 < \kappa s_j,
\]
we can conclude that \( \zeta_{31} + \delta_{21} < 1 \).
We see that the component (58b) can be bounded as in Assumption 3.4, with
\[
(\zeta_{11}, \zeta_{21}, \delta_{11}, \delta_{21}) = (\frac{3-s_j}{2}, \frac{1}{2}, \frac{s_j - 1}{2}, \frac{s_j - 1}{2}) \). Note that \( \delta_{11} + \delta_{21} = 1, \zeta_{21} + \delta_{21} = \frac{1}{2} < 1 \).
Finally, in case $s_j = 1$, (54) implies
\[
\left| \left( \tilde{b}_j(t, \cdot) \cdot \nabla y_1 \right) y_1^{s_j - 1} y_1 - \left( \tilde{b}_j(t, \cdot) \cdot \nabla y_2 \right) y_2^{s_j - 1} y_2 \right|_{L^2} 
\leq \left| \tilde{b}_j(t, \cdot) \right|_{L^\infty} \left( |\nabla y_1|_{L^2} |y_1 - y_2|_{L^\infty} + |\nabla (y_1 - y_2)|_{L^2} |y_2|_{L^\infty} \right)
\leq \left| \tilde{b}_j(t, \cdot) \right|_{L^\infty} \left( |y_1|_V |y_1 - y_2|^1_1 + |y_2|^\frac{1}{2}_V |y_1 - y_2|^\frac{1}{2}_V D(A) + |y_2|^\frac{1}{2}_V |y_2|^{\frac{1}{2}}_{D(A)} |y_1 - y_2|_V \right).
\]
We have a sum as in Assumption 3.4 with $(\zeta_1, \zeta_2, \delta_1, \delta_2) \in \{(1, 0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 1, 0)\}$. Note that in either case $\delta_1 + \delta_2 = 1$ and $\zeta_2 + \delta_2 = \frac{1}{2}$.

**Remark 5.2.** Above in (51), we may replace $|y|_{\tilde{R}}^{\tau_j - 1}$ by $y^{r_j - 1}$ in case $r_j \in \{2, 3, 4\}$ is an integer. The reason the absolute values are taken in (51) is because we want $N(t, x, y(t, x)) \in \mathbb{R}$ in order to have real valued solutions $y(t, x) \in \mathbb{R}$.

### 5.3. The cases $\Omega \subset \mathbb{R}^1$ and $\Omega \subset \mathbb{R}^2$

In the cases $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2\}$ we have the Sobolev embedding inequality $|z|_{L^q} \leq C |z|_V$ for all $q < \infty$, see [18, Thm. 4.57]. In particular from (53) we can obtain that the reaction terms satisfy an inequality as in Assumption 3.4 for all $r_j > 1$, with $(\zeta_1, \zeta_2, \delta_1, \delta_2) = (r_j - 1, 0, 1, 0)$. From (54) and (52), the convection terms satisfy, for $s_j > 1$, and $p > 1$,
\[
\left| \tilde{b}_j(t, \cdot) \cdot \nabla y_1 \right| |y_1|^{s_j - 1}_{\tilde{R}} y_1 - \left( \tilde{b}_j(t, \cdot) \cdot \nabla y_2 \right) y_2^{s_j - 1} y_2 \right|_{L^2} \leq \mathcal{G}_1 + \mathcal{G}_2
\leq C_{2s_j} |\nabla y_1|_{L^2^{\infty}} \left( |y_1|^{s_j - 1}_{\tilde{R}} + |y_2|^{s_j - 1}_{\tilde{R}} \right) |D y|_{V}^{2} + C_{1} |D y|_{H^{\infty}} |y_2|_{V}^{1}.
\]
Now we claim that we can find $\xi_1$ so that
\[
0 < 2\xi_1 < 1, \quad \text{and} \quad 2p \leq \frac{2}{1 - 2\xi_1} \iff \varpi_1(\xi_1) := -4\xi_1 p + 2p - 2 \leq 0.
\]
Indeed, it is enough to observe that $\varpi_1(\frac{1}{2}) = -2 < 0$. This implies that for $d = 1$, we have $H^{\xi_1} \hookrightarrow L^{2p}$. Analogously, we can find $\xi_2$ so that
\[
0 < 2\xi_2 < 2, \quad \text{and} \quad 2p \leq \frac{2}{1 - 2\xi_2} \iff \varpi_2(\xi_2) := -4\xi_2 p + 4p - 4 \leq 0,
\]
due to $\varpi_2(2\xi_2) = -4 < 0$, which implies that for $d = 2$, we have $H^{\xi_2} \hookrightarrow L^{2p}$. Further, note that $0 < \xi_i < 1$, $i \in \{1, 2\}$. Thus taking $\xi = \xi_d$ for $d \in \{1, 2\}$,
\[
\left| \tilde{b}_j(t, \cdot) \cdot \nabla y_1 \right| |y_1|^{s_j - 1}_{\tilde{R}} y_1 - \left( \tilde{b}_j(t, \cdot) \cdot \nabla y_2 \right) y_2^{s_j - 1} y_2 \right|_{L^2} \leq C_{2} y_1|H^{\frac{1}{2} + \varepsilon}| \left( |y_1|^{s_j - 1}_{\tilde{R}} + |y_2|^{s_j - 1}_{\tilde{R}} \right) |D y|_{V} + C_{1} |D y|_{H^{\frac{1}{2} + \varepsilon}} |y_2|_{V} \leq C_{2} y_1|^{\xi - \varepsilon}_{\tilde{R}} |y_1|^{\xi - \varepsilon}_{D(A)} \left( |y_1|^{s_j - 1}_{\tilde{R}} + |y_2|^{s_j - 1}_{\tilde{R}} \right) |D y|_{V} + C_{1} |y_2|_{V} |D y|^{-\xi}_{V} |D y|_{D(A)}^{\xi}.
\]
Finally, setting $1 < \kappa < \frac{1}{4}$, from
\[
|y_1|^{\frac{1}{4} - \xi}_{L^2} \leq |y_1|^{\frac{1}{2}}_{D(A)} |y_2|^{\frac{1}{4} - \xi}_{V} \leq |y_1|^{\frac{1}{2}}_{V} |y_1|^{\frac{1}{2}}_{D(A)} + |y_2|^{\frac{1}{4} - \xi}_{V} (s_j - 1)
\]
once more we arrive to an inequality as in Assumption 3.4, with the quadruples
$(\zeta_1, \zeta_2, \delta_1, \delta_2) \in \{(\kappa(1 - \xi), \kappa\xi, 1, 0), (\frac{\kappa}{4} - (s_j - 1), 0, 1, 0), (1, 0, 1 - \xi, \xi)\}$. For either quadruple we have $\delta_1 + \delta_2 = 1$ and $\zeta_2 + \delta_2 < 1$. Thus, in the cases $\Omega \subset \mathbb{R}^d$, with $d \in \{1, 2\}$, we can take nonlinearities as
\[
N(t, y) = N(t, x, y) = \sum_{j=1}^{n} \left( \hat{a}_j(t, x) |y|^{r_j - 1}_{L^\infty} y + \left( \hat{b}_j(t, x) \cdot \nabla y \right) |y|^{s_j - 1}_{L^2} y \right),
\]
with $r_j > 1$, $s_j \geq 1$, $\hat{a}_j \in L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R})$, $\hat{b}_j(t) \in L^\infty(\mathbb{R}_0 \times \Omega, \mathbb{R}^d)$,
Remark 5.3. Note that, when \( \zeta_2 = \delta_2 = 0 \), by taking \( \mathcal{F}_{M_\sigma} = \lambda \text{Id} \) we find that Assumption 3.6 will follow with \( \eta_2 = \beta_2 = 1 \) and \( \|\zeta_1 + \delta_1\| < 2 \). In particular, Assumption 3.6 is satisfied, in case \( d \in \{1, 2\} \), if we have no nonlinear convection terms and the nonlinear reaction terms satisfy \( 1 < r_j < 2 \). On the other hand we underline that Assumption 3.6 is part of the sufficient conditions for stability of the closed system, we do not claim that Assumption 3.6 is necessary. That is, our results do not show neither that the feedback obtained by taking \( \mathcal{F}_{M_\sigma} = \lambda \text{Id} \) is able to stabilize the system nor that it is not.

6. Numerical results. We present here numerical results in the one dimensional case, showing the stabilizing performance of the controller. Our parabolic equation evolving in the unit interval \((0, L)\) reads

\[
\frac{\partial}{\partial t} y + (-\nu \Delta + \text{Id}) y + (a-1)y + b \nabla y - c_N |y|^{p-1} = \mathbf{K}(y),\quad y(0) = c_{ic} y_0,\quad y|_{\partial(0,L)} = 0,
\]

where Dirichlet boundary conditions, \( y|_{\partial(0,L)} = 0 \), that is \( y(t,0) = y(t,L) = 0 \), are imposed and where we have taken

\[
\nu = 0.1,\quad L = 1,\quad y_0 = \sin(2\pi x),\quad p = \frac{13}{7},\quad c_{ic} \in \{2, 4\},\quad c_N \in \{0, 1\},
\]

\[
a(t, x) := -35\nu x^2 - 10 \cos(4 t) x \cos(x)|_\mathbb{R},\quad b(t, x) := -4 \cos(3 t) - 5(1 - x)^2 + 2,
\]

\[
\mathbf{K} \in \left\{ \mathbf{K}_{U_{\# M_\sigma}^{F_{M_\sigma}}}, \mathbf{K}_{U_{\# M_\sigma}^{F_{M_\sigma} \cdot N}} \right\},\quad \mathcal{F}_{M_\sigma} \in \{ \lambda \text{Id}, -\nu \Delta + \lambda \text{Id} \},\quad \lambda = 1,\quad M_\sigma = \{1, \ldots, M\}.
\]

Above \((t, x) \in (0, +\infty) \times (0, 1)\). Recall that \( \mathbf{K}_{U_{\# M_\sigma}^{F_{M_\sigma}}} \) and \( \mathbf{K}_{U_{\# M_\sigma}^{F_{M_\sigma} \cdot N}} \) are defined in (12) and (17), respectively.

For a given \( M \in \mathbb{N}_0 \), the actuators were taken as in (14), \( U_{\# M_\sigma} = U_M = \text{span}\{1_{\omega_i} \mid i \in \mathbb{M}\} \), with \( r = 0.1 \) and \( L_1 = L = 1 \), that is, the actuators cover 10% of the domain \((0, L)\).

The simulations have been performed for a spatial finite element approximation of the equation, based on piecewise linear elements (hat functions). The interval domain \([0, 1]\) has been discretized by \( N + 1 \) equidistant nodes \([0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1]\), with \( N = 10^4 \). To solve the associated ODEs we followed a Crank–Nicolson scheme with the time interval \((0, +\infty)\) discretized with timestep \( k = 10^{-4} \), \([0, k, 2k, 3k, \ldots]\). For further details, see [51]. In the figures below we are going to plot the behaviour of either \( |y|_H \) or \( |y|_V \). Note that since \( V \hookrightarrow H \), if \( |y|_H \) goes to \( +\infty \) then also \( |y|_V \) does. Analogously, if \( |y|_V \) goes to 0 then also \( |y|_H \) does. These norms have been computed as \( |y(t_j)|_H^2 = \overline{y}(t_j)^\top \mathbf{M}_j \overline{y}(t_j) \) and \( |y(t_j)|_V^2 = \overline{y}(t_j)^\top (\nu \mathbf{S} + \mathbf{M}) \overline{y}(t_j) \). Here \( \mathbf{M} \) and \( \mathbf{S} \) are, respectively, the Mass and Stiffness matrices, and \( \overline{y}(t_j) \) is the discrete solution at a given discrete time \( t_j = jk \). The simulations have been run for time \( t \in [0, 5] \), and have been performed in MATLAB.

In the figures below \( \mathcal{F}_M = \mathcal{F}_{M_\sigma} \), and "Ktype = Klinz" means that we have taken the linearization based feedback \( \mathbf{K} = \mathbf{K}_{U_{\# M_\sigma}^{F_{M_\sigma}}} \), while "Ktype = Knonl" means that we have taken the nonlinear feedback \( \mathbf{K} = \mathbf{K}_{U_{\# M_\sigma}^{F_{M_\sigma} \cdot N}} \). Note that with \( c_N = 0 \) the system is linear, while with \( c_N = 1 \) the system is nonlinear. Furthermore, FeedOn stands for the time interval on which the control is switched on. For example in Figure 1 the control is switched off on the entire time interval \([0, 5]\), while in Figure 2 it is switched on the entire time interval \([0, 5]\).
In Figure 1, we observe that both the linear and the nonlinear systems are unstable. The linear system is exponentially unstable and the nonlinear system blows up in finite time.

Figure 1. Uncontrolled solutions. Linear and nonlinear systems.

In Figure 2 we see that, with 6 actuators, the linear feedback is able to stabilize the linear system, for both choices of $F_M$. In this example, the choice of $F_M = -\nu \Delta + \lambda \text{Id}$ leads to faster exponential decay rate of the $V$-norm.

Figure 2. Linear systems and linear feedback.

In Figure 3 we see that the same linear feedback, is not able to stabilize the nonlinear system. This is because the initial condition is too big. Recall that it is known that we can expect such linearization based feedback to be able to stabilize the nonlinear system only if the norm of the initial condition is small enough (local stability).

Figure 3. Nonlinear systems and linear feedback.
In Figure 4 we observe that the full nonlinear feedback with 6 actuators and with \( F_M = -\nu \Delta + \lambda \text{Id} \) succeeds to stabilize the solution, while with the choice \( F_M = \lambda \text{Id} \) it fails to. The latter choice \( F_M = \lambda \text{Id} \) succeeds by taking 7 actuators. Figure 5 shows that, for a bigger initial condition, the same nonlinear feedback, with 7 actuators is not anymore able to stabilize the system for both choices \( F_M = -\nu \Delta + \lambda \text{Id} \) and \( F_M = \lambda \text{Id} \). Finally, in Figure 6 we observe that by increasing the number \( M \) of actuators the nonlinear feedback is again able to stabilize the system. This could give raise to the question on whether by increasng \( M \) would also lead to the stability of the linearization based closed-loop system, Figure 7 shows that this is not the case.

**Remark 6.1.** Note that when \( q(0) := P_{E_M} y(0) = 0 \), the feedbacks corresponding to \( F_M = \lambda \text{Id} \) and to \( F_M = -\nu \Delta + \lambda \text{Id} \) do coincide, because necessarily the solution \( q = F_M q \) vanishes in both cases. In particular, the corresponding closed-loop solutions must coincide. This is observed in Figure 8, where we have taken the initial condition \( y(0) = \sin(8\pi x) \). Note that, with \( M = 7 \), \( E_{M_1} = E_{M_2} = \text{span}\{\sin(i\pi x) \mid i \in \{1, 2, \ldots, 7\}\} \) and thus \( q(0) = P_{E_M} \sin(8\pi x) = 0 \).

**Remark 6.2.** We have taken our \( M \) actuators with centers location \( c \in [0,1]^M \) as in (14), which guarantee that the norm \( \|P_{E_M} u_M\|_{L^2(H)} \leq C_P \) remains bounded as \( M \) increases, with \( C_P \) independent of \( M \). Since the number \( M \) of actuators needed to stabilize the system increase with \( C_P \), see (50), it would be interesting to know the/
optimal location $c = c^{opt} \in [0, 1]^M$ for the $M$ actuators minimizing $\| P_{E_M} (c) \|_{\mathcal{L}(H)}$.

We would like to refer to [37, 38, 28, 47, 39] for works related to finding a/the placement (and/or shape) of actuators, though the functional to be minimized in those works is not $\| P_{E_M} (c) \|_{\mathcal{L}(H)}$. 

Figure 6. Nonlinear systems and nonlinear feedback. Increasing the number of actuators.

Figure 7. Nonlinear systems and linear feedback. Increasing the number of actuators.
implies that for all \( (s,t) \leq (0,\gamma_0) \).

Note that \( r := \frac{1}{\gamma_0} \) satisfies \( \frac{1}{r} + \frac{1}{\gamma_0} = 1 \). In particular, \( ab \leq a^r + b^{\gamma_0} \). Assumption 3.4 implies that

\[
2\left( P_{E_{0,\sigma}}^{U_{\delta,\alpha}} (\mathcal{N}(t,y_1) - \mathcal{N}(t,y_2)), A(y_1 - y_2) \right)_H \leq 2 \left| P_{E_{0,\sigma}}^{U_{\delta,\alpha}} \right|_{\mathcal{L}(H)} C_N \sum_{j=1}^{n} \left( \left| y_1 y_j^{(1)} \parallel y_1 \parallel_{D(A)} + \left| y_2 y_j^{(1)} \parallel y_2 \parallel_{D(A)} \right| \right) |y_1 - y_2|_{V} \delta_j^{(1)} y_1 - y_2|_{D(A)}
\]

and the Young inequality gives us for all \( \gamma_0 > 0 \), writing for simplicity \( \|P\|_{\mathcal{L}} := \left| P_{E_{0,\sigma}}^{U_{\delta,\alpha}} \right|_{\mathcal{L}(H)} \),

\[
2 \left( P_{E_{0,\sigma}}^{U_{\delta,\alpha}} (\mathcal{N}(t,y_1) - \mathcal{N}(t,y_2)), A(y_1 - y_2) \right)_H \leq \sum_{j=1}^{n} \left( 1 + \delta_j \right) \gamma_0 \left| y_1 - y_2 \right|_{D(A)}^2
\]

From [45, Prop. 2.6] we have that \( (a_1 + a_2)^s \leq 2^{s-1} (a_1^s + a_2^s) \) for \( (a_1, a_2, s) \in [0, +\infty) \), which implies

\[
2 \left( P_{E_{0,\sigma}}^{U_{\delta,\alpha}} (\mathcal{N}(t,y_1) - \mathcal{N}(t,y_2)), A(y_1 - y_2) \right)_H \leq \sum_{j=1}^{n} \left( 1 + \delta_j \right) \gamma_0 \left| y_1 - y_2 \right|_{D(A)}^2
\]

Observe that if we fix an arbitrary \( \hat{\gamma}_0 > 0 \) and set, in (A.2),

\[
\gamma_0 = \gamma_0 := \left( \frac{2}{1 + \delta_j} \frac{\gamma_0}{n} \right) = \frac{2}{1 + \delta_j} \frac{\gamma_0}{n} \iff \frac{\gamma_0}{n} = \frac{1 + \delta_j \gamma_0}{2} \frac{\gamma_0}{n},
\]

Figure 8. Nonlinear systems and nonlinear feedback. \( y(0) = c_{ic} \sin(8 \pi x) \in E^{1}_{\bar{H}} \).

Appendix.

A.1. Proof of Proposition 3.5. We recall the Young inequality [58] as follows: for all \( (a,b,\gamma_0) \in \mathbb{R}_0^3 \) and all \( s > 1 \), we have

\[
ab = (\gamma_0 a)(\gamma_0^{-1} b) \leq \frac{1}{\gamma_0^s} a^s + \frac{1}{\gamma_0} \gamma_0 \left( \frac{1}{\gamma_0} \right)^s b^s.
\]
then, since $\delta_{2j} < 1$, we obtain

$$
\left( \frac{1}{n} \right)^{\frac{2}{1-2j}} = \left( \frac{n(1+\delta_{2j})}{2} \right)^{\frac{1+\delta_{2j}}{1-2j}} < \left( \frac{n}{70} \right)^{\frac{1+\delta_{2j}}{1-2j}} < 1 + \left( \frac{n}{70} \right)^{\frac{1+\|\delta_2\|}{1-2j}}.
$$

with $\|\delta_2\| := \max_{1 \leq j \leq n} |\delta_{2j}|$. Hence, we arrive at

$$
2\left( P_{E_{M_0}E_{M_0}}^{l_{u_{M_0}}} \left( N(t, y_1) - N(t, y_2) \right), A(y_1 - y_2) \right)_{L} \leq \gamma_0 |y_1 - y_2|^2_{(D(A)}
$$

$$
+ \left( 1 + \tilde{\gamma}_0 \right) \frac{1}{n} \|\delta_2\| \left( \|\|\sigma\|\| \right) \frac{1}{n} \|\delta_2\| \left( \|\|\sigma\|\| \right) \frac{1}{n} \|\delta_2\| \left( \|\|\sigma\|\| \right)
$$

$$
\sum_{j=1}^{n} \left( |y_j| \left| y_j \right| \left| y_j \right| \left| y_j \right| \right) \frac{1}{n} \|\delta_2\| \left( \|\|\sigma\|\| \right) \frac{1}{n} \|\delta_2\| \left( \|\|\sigma\|\| \right) \frac{1}{n} \|\delta_2\| \left( \|\|\sigma\|\| \right)
$$

In the particular case $y_1 = q + Q$ and $y_2 = q$ with $(q, Q) \in E_{M_0} \times E_{M_0}^1$, estimate (A.1) also gives us

$$
2\left( P_{E_{M_0}E_{M_0}}^{l_{u_{M_0}}} \left( N(t, q + Q) - N(t, q) \right), AQ \right)_{L} \leq 2 \|P\|_{C_N} \sum_{j=1}^{n} \left( |q + Q|_{V}^{|c_{ij}|} |q + Q|_{V}^{|c_{ij}|} + |q|_{V}^{|c_{ij}|} |q|_{V}^{|c_{ij}|} \right) \left| Q_{1+\delta_{2j}} \right| \left| Q_{1+\delta_{2j}} \right|
$$

$$
\leq 2(1 + 2\|\zeta - 1\|) \|P\|_{C_N} \sum_{j=1}^{n} \left( |q|_{V}^{|c_{ij}|} \left| Q_{1+\delta_{2j}} \right| Q_{1+\delta_{2j}} + |Q|_{V}^{|c_{ij}|} \right) \left| Q_{1+\delta_{2j}} \right| \left| Q_{1+\delta_{2j}} \right|
$$

$$
\leq 2(1 + 2\|\zeta - 1\|) \|P\|_{C_N} \sum_{j=1}^{n} \left( |q|_{V}^{|c_{ij}|} \left| Q_{1+\delta_{2j}} \right| Q_{1+\delta_{2j}} + |Q|_{V}^{|c_{ij}|} \right) \left| Q_{1+\delta_{2j}} \right| \left| Q_{1+\delta_{2j}} \right|
$$

$$
+ 2(1 + 2\|\zeta - 1\|) \|P\|_{C_N} \sum_{j=1}^{n} \left( |q|_{V}^{|c_{ij}|} \left| Q_{1+\delta_{2j}} \right| Q_{1+\delta_{2j}} + |Q|_{V}^{|c_{ij}|} \right) \left| Q_{1+\delta_{2j}} \right| \left| Q_{1+\delta_{2j}} \right|
$$

$$
\leq 2^2 + \|\zeta - 1\| \|P\|_{C_N} \sum_{j=1}^{n} \left( |q|_{V}^{|c_{ij}|} \left| Q_{1+\delta_{2j}} \right| Q_{1+\delta_{2j}} + |Q|_{V}^{|c_{ij}|} \right) \left| Q_{1+\delta_{2j}} \right| \left| Q_{1+\delta_{2j}} \right|
$$

$$
+ 2^2 + \|\zeta - 1\| \|P\|_{C_N} \sum_{j=1}^{n} \left( |q|_{V}^{|c_{ij}|} \left| Q_{1+\delta_{2j}} \right| Q_{1+\delta_{2j}} + |Q|_{V}^{|c_{ij}|} \right) \left| Q_{1+\delta_{2j}} \right| \left| Q_{1+\delta_{2j}} \right|
$$

with $\|\zeta - 1\| := \max \{|c_{k,j} - 1| : 1 \leq j \leq n, 1 \leq k \leq 2\}$. By the Young inequality, with $\gamma_0 > 0$ and $\tilde{\gamma}_0 > 0$,

$$
2\left( P_{E_{M_0}E_{M_0}}^{l_{u_{M_0}}} \left( N(t, q + Q) - N(t, q) \right), AQ \right)_{L} \leq \sum_{j=1}^{n} \left( \frac{1+\delta_{2j}+c_{ij}}{2} \gamma_0 \frac{2}{1-\delta_{2j}} + \frac{1+\delta_{2j}}{2} \gamma_0 \frac{2}{1+\delta_{2j}} \right) |Q|^2_{(D(A)}
$$

$$
+ \sum_{j=1}^{n} \left( \frac{2+\|\zeta - 1\| \|P\|_{C_N} \gamma_0}{70} \right)^{\frac{1}{1+\delta_{2j}+c_{ij}}} \left( |q|_{V}^{|c_{ij}|} \left| Q_{1+\delta_{2j}} \right| Q_{1+\delta_{2j}} + |Q|_{V}^{|c_{ij}|} \right) \left( \frac{2}{1-\delta_{2j}} \right)^{\frac{1}{1+\delta_{2j}+c_{ij}}}
$$

$$
+ \sum_{j=1}^{n} \left( \frac{2+\|\zeta - 1\| \|P\|_{C_N} \gamma_0}{70} \right)^{\frac{1}{1+\delta_{2j}+c_{ij}}} \left( |q|_{V}^{|c_{ij}|} \left| Q_{1+\delta_{2j}} \right| Q_{1+\delta_{2j}} + |Q|_{V}^{|c_{ij}|} \right) \left( \frac{2}{1-\delta_{2j}} \right)^{\frac{1}{1+\delta_{2j}+c_{ij}}}
$$

Fixing an arbitrary $\tilde{\gamma}_0 > 0$ and setting, in (A.3),

$$
\gamma_0 = \gamma_0 := \left( \frac{2}{1+\delta_{2j}+c_{ij}} \right)^{\frac{1}{1+\delta_{2j}+c_{ij}}} \quad \iff \quad \gamma_0 \frac{2}{1+\delta_{2j}+c_{ij}} = \frac{1+\delta_{2j}}{2} \gamma_0 \frac{2}{1+\delta_{2j}+c_{ij}}.
$$
then, since $\delta_{2j} < 1$, we obtain
\[
\left(\frac{1}{\gamma_0}\right) \frac{|\zeta_2 + \delta_2|}{\gamma_2} = \left(\frac{2(n+1+\delta_{2j})}{\gamma_0}\right) \frac{1+\delta_{2j}}{\gamma_{2j}} < \left(\frac{2n}{\gamma_0}\right) \frac{1+\delta_{2j}}{\gamma_{2j}} < 1 + \left(\frac{2n}{\gamma_0}\right) \frac{1+|\zeta_2 + \delta_2|}{\gamma_{2j}}
\]
with $|\zeta_2 + \delta_2| := \max_{1 \leq j \leq n} |\delta_{2j} + \zeta_{2j}|$.

With $\tilde{\gamma}_0 = \gamma_0 := \frac{2n}{\gamma_0} \tilde{\gamma}_0 = \frac{1+\delta_{2j} + \zeta_{2j}}{\tilde{\gamma}_0}$, we also find that
\[
\left(\frac{1}{\gamma_0}\right) \frac{|\zeta_2 + \delta_2|}{\gamma_2} < 1 + \left(\frac{2n}{\gamma_0}\right) \frac{1+|\zeta_2 + \delta_2|}{\gamma_{2j}}
\]
Now we show that, from (A.3), we can obtain
\[
2\left(P^{\eta_E_{\gamma_0}}_{E_{10}} (N(t, q + Q) - N(t, q)), AQ\right)_H \leq \tilde{\gamma}_0 |Q|_{D(A)}^2
\]
with $|\zeta_2 + \delta_2| := \max_{1 \leq j \leq n} |\delta_{2j} + \zeta_{2j}|$, and with $|\tilde{\zeta}_2| := \max_{1 \leq j \leq n} |\tilde{\zeta}_{2j}|$, and with the constant $\overline{C}$ of the form $\overline{C} = \overline{C}[\eta, \|\zeta_1\|, \|\zeta_2\|, \frac{1}{1+\zeta_2}, \frac{1}{1+\delta_2}, \|P\|, \|Q\|]$.

Indeed, we can use the inequalities
\[
\left(\frac{1}{\gamma_0}\right) \frac{|\zeta_2 + \delta_2|}{\gamma_2} \leq C_0 \left(\frac{|q|_{V}^{2(1+\delta_{2j})}}{|q|_{V}^{2(1+\delta_{2j})}}, |Q|_{V}^{2(1+\delta_{2j})}\right),
\]
\[
\left(\frac{1}{\gamma_0}\right) \frac{|\zeta_2 + \delta_2|}{\gamma_2} \leq C_1 \left(\frac{|q|_{V}^{2(1+\delta_{2j})}}{|q|_{V}^{2(1+\delta_{2j})}}, |Q|_{V}^{2(1+\delta_{2j})}\right),
\]
\[
\left(\frac{1}{\gamma_0}\right) \frac{|\zeta_2 + \delta_2|}{\gamma_2} \leq C_2 \left(\frac{|q|_{V}^{2(1+\delta_{2j})}}{|q|_{V}^{2(1+\delta_{2j})}}, |Q|_{V}^{2(1+\delta_{2j})}\right),
\]
with the constants $C_k, k \in \{1, 2, 3\}$ of the form $C_k = \overline{C}[\|\zeta_1 + \delta_1\|, \frac{1}{1+\zeta_2}, \frac{1}{1+\delta_2}, \|C\|]$. 

A.2. Proof of Proposition 4.3. Observe that, since $p \geq 0$, the function $w \mapsto |w|^p$ is locally Lipschitz. Therefore, the solutions of (45), do exist and are unique, in a small time interval, say for time $t \in [0, \tau)$ with $\tau$ small. When $w_0 = 0$ the solution is the trivial one, that is, $w = 0$. Note that the equilibria of (33), that is, the solutions of $\dot{w} = 0$, are given by $\bar{w}_1 = 0$ and $\bar{w}_2^\pm = \pm (\frac{\zeta_1}{C_2})^{\frac{1}{p}}$. Furthermore, we observe that $\dot{w} < 0$ if $w \in (0, \bar{w}_2^+)$. Indeed, the solution issued from $w(s) \in (0, \bar{w}_2^+)$ at time $t = s$, is globally defined, for all time $t \geq s$, is decreasing, and thus remains in $(0, \bar{w}_2^+)$. Note that
\[
-\bar{C}_1 w \leq \dot{w} \leq -\left(\bar{C}_1 - \bar{C}_2 |w_0|^p\right) w, \text{ for } w \in (0, \bar{w}_2^+).
\]
Therefore we can conclude that (34) holds for $w_0 \in (0, \bar{w}_2^+)$.
with $w^+(t) := S(t, s)(-w_s)$, we have

$$
\frac{d}{d\tau} (-w^+) = -\dot{w} = -\left(-\left(\bar{C}_1 - \bar{C}_2 |w^+|_R^p\right)w^+\right) = -\left(\bar{C}_1 - \bar{C}_2 |w^+|_R^p\right)(-w^+),
$$

$$
-w^+(s) = w_s.
$$

The uniqueness of the solution, implies that $S(t, s)(w_s) = -w^+$. Since $-w_0 \in (0, \pi_2^+)$, it follows, from above, that $|w^+|_R$ satisfies (34) and, from $|S(t, 0)(w_0)|_R = |w^+(t)|_R$, we obtain that (34) holds for $w_0 \in (-\pi_2^+, 0)$.

\[\square\]

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