Free energy of bipartite Sherrington-Kirkpatrick model

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Abstract. In this paper we study the bipartite version of Sherrington-Kirkpatrick model. We prove that the free energy density is given by an analogue of the Parisi formula, that contains both the usual overlap and an additional new type of overlap. Following Panchenko, we prove the upper bound of the formula by Guerra’s replica symmetry breaking interpolation, then the matching lower bound by Ghirlanda-Guerra identities and the Aizenman-Sims-Starr scheme. Based on this result we study the stability of the replica symmetric solution. A new phase exhibiting partial replica symmetry breaking is observed, where the broken phase is realized in the larger group only.

Keywords: spin glass, Sherrington-Kirkpatrick model, partial replica symmetry breaking

1. Introduction and main results

A multi-species Sherrington-Kirkpatrick (SK) model was recently proposed in [1] where the spins are classified into a number of species such that the interaction between two spins only depends on the species they belong. The free energy of multi-species SK model was obtained together by [1] and [2], where [1] proved the upper bound of free energy under the assumption that the group interaction strength matrix is non-negative definite, and [2] proved the matching lower bound. In particular, at the very end of [2] has been conjectured that in multi species SK models a new mixed phase is possible, where the Replica Symmetry Breaking (RSB) transition is realized by some species and not other.

In this paper we study the SK model with bipartite interactions, that is the simplest case of multi-species SK model where the non-negative definite condition is violated. The model has been studied using replicas in [3]. When the groups of spins have equal size this model is also known as asymmetric Little model [4, 5], and has been studied in [6, 7].
Consider two groups of spins $\sigma \in \Sigma_{N_1} = \{-1, +1\}^{N_1}$ and $\tau \in \Sigma_{N_2} = \{-1, +1\}^{N_2}$, and let $N = N_1 + N_2$. The Hamiltonian of the bipartite SK model is

$$H_{\text{BSK}}(\sigma, \tau) = -\frac{1}{\sqrt{N}} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} g_{ij} \sigma_i \tau_j,$$

where the interactions $(g_{ij})_{1 \leq i \leq N_1, 1 \leq j \leq N_2}$ are i.i.d. standard Gaussian random variables, and there are no interactions within each group. The densities $\lambda_s = N_s/N$ for $s = 1, 2$ are fixed for different $N$. The partition function of bipartite SK is

$$Z_N = \sum_{\sigma \in \Sigma_{N_1}} \sum_{\tau \in \Sigma_{N_2}} \exp(-\beta H_{\text{BSK}}(\sigma, \tau)).$$

Due to the bipartite nature, we can sum over one of the groups of spins in the partition function, say $\tau$,

$$Z_N = \sum_{\sigma \in \Sigma_{N_1}} \exp(-H_N(\sigma)),$$

where $H_N(\sigma)$ is the effective Hamiltonian supported by $\sigma$

$$H_N(\sigma) = -\sum_{j=1}^{N_2} \log \left( \cosh \left( \frac{1}{\sqrt{N}} \beta g_j \cdot \sigma \right) \right).$$

Here $g_j$ denotes the $j$th column of $(g_{ij})$. For notation convenience, hereafter we always assume that the Gibbs averages $\langle \cdot \rangle_N$ are taken respect to an infinitely replicated measure

$$\langle \cdot \rangle_N = \sum_{\sigma^1 \in \Sigma_{N_1}} G_N(\sigma^1) \sum_{\sigma^2 \in \Sigma_{N_1}} G_N(\sigma^2) \ldots \sum_{\sigma^\alpha \in \Sigma_{N_1}} G_N(\sigma^\alpha) \ldots (\cdot)$$

where $G_N(\sigma)$ is the Gibbs measure associated to the considered Hamiltonian

$$G_N(\sigma) = \frac{1}{Z} \exp(-H_N(\sigma)).$$

The effective Hamiltonian itself belongs to two families of spin glass models, that proposed in [8] to approximate the perceptron, and the generalized Hopfield model [9]. The replica symmetric solution and the phase diagrams have been studied for these two models, see Chapter 2 of [8] and [9].

In the following we will prove the analogue Parisi formula [10] [11] of bipartite SK model by working with the effective Hamiltonian (4). The functional order parameter of SK model is the distribution of overlaps. Here we call SK type overlap the quantity

$$R_{12}^{N_1} = \frac{1}{N_1} \sum_{i=1}^{N_1} \sigma_i^1 \sigma_i^2.$$

For bipartite SK, we will show that the Parisi formula need an additional generalized overlap

$$P_{12}^{N_2} = \frac{1}{N_2} \sum_{j=1}^{N_2} \tanh \left( \frac{1}{\sqrt{N}} \beta g_j \cdot \sigma^1 \right) \tanh \left( \frac{1}{\sqrt{N}} \beta g_j \cdot \sigma^2 \right).$$
The meaning of the generalized overlap above will be discussed in Section 2 (as we shall see it appears naturally in the cavity computations of the Aizenman-Sims-Starr scheme in Section 6). Note that \( P_{12}^{N_2} \) is a random function depending on \((g_{ij})\) for finite systems but in the thermodynamic limit it’s nonrandom. To be more specific, define the following

\[
P_{12}(R_{12}) = \mathbb{E} \tanh \left( a \sqrt{\frac{1 + R_{12}}{2}} x_1 + a \sqrt{\frac{1 - R_{12}}{2}} x_2 \right) \times \tanh \left( a \sqrt{\frac{1 + R_{12}}{2}} x_1 - a \sqrt{\frac{1 - R_{12}}{2}} x_2 \right),
\]

where \( x_1 \) and \( x_2 \) are i.i.d. standard Gaussians and \( a = \sqrt{\lambda_1 \beta} \). For notation convenience the dependence of \( P_{12}(R_{12}) \) on \( a \) won’t be written explicitly. By the properties of Gaussian random variables \( P_{12}(R_{12}) \) is a bounded continuous function of \( R_{12} \). Then it’s easy to verify that for any sequence of \( R_{12}^{N_1} \) such that \( \lim_{N \to \infty} R_{12}^{N_1} = R_{12} \),

\[
\lim_{N \to \infty} P_{12}^{N_2} = P_{12}(R_{12})
\]

almost surely along the sequence. Note also that this \( P_{12}^{N_2} \) type overlap has already appeared in the replica symmetric solution of the perceptron model [5] by Talagrand.

To write the analogue Parisi formula we introduce the following functions of the SK overlap \( R_{12} \)

\[
\phi(R_{12}) = \lambda_1 \beta^2 R_{12}, \quad \psi(R_{12}) = \lambda_2 \beta^2 P_{12}(R_{12}), \quad \theta(R_{12}) = \lambda_1 \lambda_2 \beta^2 R_{12} P_{12}(R_{12}).
\]

We remark that \( \theta(R_{12}) \) depends only on \( R_{12} \) since \( P_{12}(R_{12}) \) is determined by \( R_{12} \). Similar to SK model, given an integer parameter \( r \geq 1 \) consider the two sequences of parameters

\[
0 < \zeta_0 < \zeta_1 < \cdots < \zeta_{r-1} < 1, \quad 0 = q_0 < q_1 < \cdots < q_r = 1.
\]

Let \((\eta_p)_{1 \leq p \leq r}, (\omega_p)_{1 \leq p \leq r}\) be i.i.d standard Gaussian random variables and define the following random functions

\[
X_r = \log \cosh \sum_{1 \leq p \leq r} \eta_p (\phi(q_p) - \phi(q_{p-1}))^{1/2},
\]

\[
Y_r = \log \cosh \sum_{1 \leq p \leq r} \omega_p (\psi(q_p) - \psi(q_{p-1}))^{1/2},
\]

where \( \phi(\cdot) \) and \( \psi(\cdot) \) are defined in (11). Then recursively over \( 0 \leq l \leq r - 1 \) define

\[
X_l = \frac{1}{\zeta_l} \log \mathbb{E}_t \exp \zeta_l X_{l+1}, \quad Y_l = \frac{1}{\zeta_l} \log \mathbb{E}_t \exp \zeta_l Y_{l+1}.
\]

Here \( \mathbb{E}_t \) is the expectation to \( \eta_{l+1} \) or \( \omega_{l+1} \) only. The Parisi functional for the bipartite SK model is

\[
\mathcal{P}(\zeta, q) = \log 2 + \lambda_2 X_0 + \lambda_1 Y_0 - \frac{1}{2} \sum_{0 \leq p \leq r-1} \zeta_p \left( \theta(q_{p+1}) - \theta(q_p) \right).
\]

Define the pressure per spin \( p_N = (\log Z_N) / N \), the main technical result of the paper is the following
**Theorem 1.** The pressure of bipartite SK model is given by

$$
\lim_{N \to \infty} p_N = \inf \mathcal{P} (\zeta, q),
$$

where the infimum is taken over $r \geq 1$ and the sequences defined in (12a) and (12b).

The proof of Theorem 1 is in three parts. In the first we prove the existence of the thermodynamic limit, then we prove that the above Parisi formula gives the upper bound of the free energy by Guerra’s replica symmetry breaking interpolation [12]. Finally, the matching lower bound is proved following the Aizenman-Sims-Starr Scheme [13] and by virtue of the Ghirlanda-Guerra identities [14]. These methods have been already successful in solving the SK model and we demand to [15] for a comprehensive introduction.

As start, in Section 2 the physical interpretations and some important properties of the generalized overlaps are discussed. In Section 3 we prove that the thermodynamic limit of the pressure $\lim_{N \to \infty} p_N$ exists, and by the properties of $P_{12}(R_{12})$ that are shown in Section 2 we where able to describe the distributions of $R_{12}$ and $P_{12}(R_{12})$ in terms of Ruelle Probability Cascades simultaneously (Section 4). Then in Section 5 we prove the upper bound and in Section 6 the lower bound. Finally, in Section 7 we use the Parisi formula to compute the phase diagram of the bipartite SK model and locate this new type of partial replica symmetry breaking solution, first conjectured in [2], where the smaller group is in a Replica Symmetric (RS) phase while the larger is in a full Replica Symmetry Breaking (RSB) phase.

### 2. Interpretations and properties of generalized overlap

The Parisi functional defined in (16) contains a term corresponding to the generalized overlap in the group $\sigma$. For comparison, in Parisi formula of multi-species SK model this term is analogously replaced by the SK type overlap of the group $\tau$. So first we will discuss the physical interpretations of the generalized overlap $P_{12}$. Given a fixed configuration of $\sigma$, the conditional probability of a spin $\tau_j = 1$ is

$$
G(\tau_j = 1|\sigma) = \frac{1}{1 + \exp \left( \frac{-2}{\sqrt{N}} \beta g_j \cdot \sigma \right)}.
$$

Then for two fixed configurations $\sigma^1, \sigma^2$, the conditional expectation of $\tau_j^1 \tau_j^2$ under Gibbs measure is

$$
\langle \tau_j^1 \tau_j^2 | \sigma^1, \sigma^2 \rangle = \sum_{\tau_j^1, \tau_j^2} \tau_j^1 \tau_j^2 G(\tau_j = \tau_j^1 | \sigma^1) G(\tau_j = \tau_j^2 | \sigma^2) =
$$

$$
= \tanh \left( \frac{1}{\sqrt{N}} \beta g_j \cdot \sigma^1 \right) \tanh \left( \frac{1}{\sqrt{N}} \beta g_j \cdot \sigma^2 \right).
$$

Average over all the spins of $\tau$, $P_{12}$ can be understood as the conditional expectation of the overlap $\tau^1 \cdot \tau^2 / N_2$, while for the multi-species SK model with non-negative definite interaction strength matrix the Parisi formula is defined in terms of the real overlap over $\tau$. 
The explicit form of the function of $P_{12}(R_{12}) = \lim_{N \to \infty} P_{12}^N$ is not known except for $\beta \to \infty$, in which we have $P_{12}(R_{12}) = \frac{2}{\pi} \arcsin R_{12}$. However some important properties of $P_{12}(R_{12})$ can be obtained for general $\beta$, by which are sufficient to prove the Parisi formula.

It worth notice the following. Let consider a Gaussian process $g(\sigma)$ indexed by $\sigma$ and with variance $\mathbb{E} g(\sigma)^2 = a^2$. If we rescale the variance to $\mathbb{E} g(\sigma)^2 = 1$ the covariance is $\mathbb{E} g(\sigma^1)g(\sigma^2) = R_{12}$, then $P_{12}(R_{12}) = \mathbb{E} \tanh(ag(\sigma^1)) \tanh(ag(\sigma^2))$.

We now prove two general results that will be useful when dealing with Hamiltonians containing an interpolation parameter $t$. The following lemmas will be used many times in next sections.

**Lemma 1.** $P_{12}(R_{12})$ is an increasing function of $R_{12}$ for $a > 0$.

**Proof.** Start from definition (9). By taking derivative respect to $R_{12}$ and using Gaussian integration by parts,

$$
\frac{\partial P_{12}(R_{12})}{\partial R_{12}} = a^2 \mathbb{E} \left( 1 - \tanh^2 \left( a \sqrt{\frac{1 + R_{12}}{2}} x_1 + a \sqrt{\frac{1 - R_{12}}{2}} x_2 \right) \right) \times \\
\times \left( 1 - \tanh^2 \left( a \sqrt{\frac{1 + R_{12}}{2}} x_1 - a \sqrt{\frac{1 - R_{12}}{2}} x_2 \right) \right). \quad (20)
$$

Then it’s clear that $\partial P_{12}(R_{12})/\partial R_{12} > 0$ for $0 < a < \infty$. When $a \to \infty$, $\tanh(ax)$ converges to $2u(x) - 1$ with $u(x)$ the step function, thus

$$
\lim_{a \to \infty} \mathbb{E} \tanh \left( ag(\sigma^1) \right) \tanh \left( ag(\sigma^2) \right) = \\
= 4\mathbb{E}u \left( g(\sigma^1) \right) u \left( g(\sigma^2) \right) + 1 - 2\mathbb{E}u \left( g(\sigma^1) \right) - 2\mathbb{E}u \left( g(\sigma^2) \right). \quad (21)
$$

For the last two terms $2\mathbb{E}u \left( g(\sigma^1) \right) = 2\mathbb{E}u \left( g(\sigma^2) \right) = 1$ by symmetry of Gaussian, then by the known formula

$$
\mathbb{E}u \left( g(\sigma^1) \right) u \left( g(\sigma^2) \right) = P(g(\sigma^1) > 0, g(\sigma^2) > 0) = \frac{1}{2\pi} \arcsin(R_{12}) + \frac{1}{4} \quad (22)
$$

we finally obtain:

$$
\lim_{a \to \infty} \mathbb{E} \tanh \left( ag(\sigma^1) \right) \tanh \left( ag(\sigma^2) \right) = \frac{2}{\pi} \arcsin(R_{12}), \quad (23)
$$

which is an increasing function of $R_{12}$. Combine the proof for for $0 < a < \infty$ and $a \to \infty$ the claim follows.

**Lemma 2.** $R_{12}$ and $P_{12}(R_{12})$ have the same sign.

**Proof.** When $R_{12} = 0$, $g(\sigma^1)$ and $g(\sigma^2)$ are uncorrelated and thus independent. Since $\tanh(\cdot)$ is odd function and by symmetry $P_{12} = 0$. By using Lemma 1 we can see that $R_{12}$ and $P_{12}$ have the same sign.
3. Thermodynamic limit of bipartite SK model

As a preparation for the proof of the Parisi formula, in this section we show that the limit \( \lim_{N \to \infty} p_N \) exists by Guerra-Toninelli interpolation [16]. The proof is based on interpolating between two systems and their union (large system), similar to that of the SK model. The main difference with SK is that now the pressure depends on the generalized overlap \( P_{12}^{N_2} \) which is random for finite systems. In particular, the overlap can be decomposed as \( P_{12}^{N_2} = P_{12}(R_{12}^{N_1}) + \Delta P_{12}^{N_2} \), where the function \( P_{12}(R_{12}^{N_1}) \) is defined in (9) and the correction \( \Delta P_{12}^{N_2} = P_{12}^{N_2} - P_{12}(R_{12}^{N_1}) \) is a finite size random term. With this there is not possible to get superadditivity and apply Fekete’s lemma as usual. However we will show that the contributions from \( \Delta P_{12}^{N_2} \) can be controlled and does not change the thermodynamic limit.

In the proof as well as rest of the paper, Gaussian integration by parts techniques will be frequently used. First we state a Gaussian integration by parts formula that will be used in the proof of thermodynamic limit.

**Lemma 3.** Let \( (x_i(\sigma))_{1 \leq i \leq N_2}, (y_i(\sigma))_{1 \leq i \leq N_2} \) be Gaussian processes indexed by \( \sigma \in \Sigma_{N_1} \), with \( N_1 \) and \( N_2 \) finite. The variances of the Gaussian processes are bounded and the elements of the covariance matrix are given by \( \mathbb{E} x_i(\sigma^1) y_j(\sigma^2) = \delta_{i,j} C_{12} \). Define a random measure \( G(\sigma) \) on \( \Sigma \) as

\[
G(\sigma) = \frac{1}{Z} \exp \left( \sum_{i=1}^{N_2} \log (2 \cosh y_i(\sigma)) \right), \quad Z = \sum_{\sigma \in \Sigma_{N_1}} \exp \left( \sum_{i=1}^{N_2} \log (2 \cosh y_i(\sigma)) \right), \quad (24)
\]

and let \( \langle \cdot \rangle \) indicate the infinitely replicated Gibbs average. Then the following holds:

\[
\mathbb{E} \left\langle \frac{1}{N_2} \sum_{i=1}^{N_2} x_i(\sigma) \tanh (y_i(\sigma)) \right\rangle = \mathbb{E} G(\sigma) - \sum_{\sigma^2 \in \Sigma_{N_1}} C_{12} \mathbb{E} \tanh (y_i(\sigma^1)) \tanh (y_i(\sigma^2)) G(\sigma^1) G(\sigma^2). \quad (25)
\]

**Proof.** Write left side of (25) as

\[
\mathbb{E} \left\langle \frac{1}{N_2} \sum_{i=1}^{N_2} x_i(\sigma) \tanh (y_i(\sigma)) \right\rangle = \mathbb{E} \left( \frac{1}{N_2} \sum_{\sigma^1 \in \Sigma_{N_1}} \sum_{i=1}^{N_2} \mathbb{E} x_i(\sigma^1) \tanh (y_i(\sigma^1)) \right) G(\sigma^1), \quad (26)
\]

then by using Gaussian integration by parts

\[
\mathbb{E} x_i(\sigma^1) \tanh (y_i(\sigma^1)) G(\sigma^1) = C_{11} \mathbb{E} G(\sigma^1) - \sum_{\sigma^2 \in \Sigma_{N_1}} C_{12} \mathbb{E} \tanh (y_i(\sigma^1)) \tanh (y_i(\sigma^2)) G(\sigma^1) G(\sigma^2). \quad (27)
\]

Averaging over \( i \) and summing over \( \sigma^1 \) gives the desired equation (25). \( \square \)
Theorem 2. The thermodynamic limit of pressure \( \lim_{N \to \infty} p_N \) exists.

Proof. Consider two systems with \( \sigma \in \Sigma_{N_1} \) and \( \epsilon \in \Sigma_{M_1} \) with fixed densities \( N_1/N = M_1/M = \lambda_1 \) where \( N = N_1 + N_2 \) and \( M = M_1 + M_2 \). Denote \( \rho = (\sigma, \epsilon) \in \Sigma_{N_1+M_1} \) the large system and define the interpolating Hamiltonian

\[
H_{N,M,t}(\sigma) = -\sum_{j=1}^{N_2} \log \left( 2 \cosh \left( \sqrt{\frac{t}{N+M}} \beta g_j \cdot \rho + \sqrt{\frac{1-t}{N}} \beta g_j' \cdot \sigma \right) \right) + \sum_{k=1}^{M_2} \log \left( 2 \cosh \left( \sqrt{\frac{t}{N+M}} \beta x_k \cdot \rho + \sqrt{\frac{1-t}{M}} \beta x_k' \cdot \epsilon \right) \right)
\]

where \( (g_j)_{1 \leq j \leq N_2}, (g'_j)_{1 \leq j \leq N_2}, (x_k)_{1 \leq k \leq M_2}, (x'_k)_{1 \leq k \leq M_2} \) are i.i.d standard Gaussian random vectors. Denote by \( \langle \cdot \rangle_{N,M,t} \) the average under the (infinitely replicated) Gibbs measure associated to the Hamiltonian above, then define the \( t \) dependent pressure

\[
p_{N,M}(t) = \frac{1}{N+M} \mathbb{E} \log \sum_{\rho \in \Sigma_{N_1+M_1}} \exp -H_t(\sigma),
\]

it is easy to verify that

\[
p_{N,M}(0) = \frac{N}{N+M} p_N + \frac{M}{N+M} p_M, \quad p_{N,M}(1) = p_{N+M}.
\]

Also, define the following quantities

\[
U_{12}^{N,M}(t) = \frac{1}{N_2} \sum_{j=1}^{N_2} \tanh \left( \sqrt{\frac{t}{N+M}} \beta g_j \cdot \rho^1 + \sqrt{\frac{1-t}{N}} \beta g_j' \cdot \sigma^1 \right) \times \tanh \left( \sqrt{\frac{t}{N+M}} \beta g_j \cdot \rho^2 + \sqrt{\frac{1-t}{N}} \beta g_j' \cdot \sigma^2 \right),
\]

\[
W_{12}^{N,M}(t) = \frac{1}{M_2} \sum_{j=1}^{M_2} \tanh \left( \sqrt{\frac{t}{N+M}} \beta x_j \cdot \rho^1 + \sqrt{\frac{1-t}{N}} \beta x_j' \cdot \epsilon^1 \right) \times \tanh \left( \sqrt{\frac{t}{N+M}} \beta x_j \cdot \rho^2 + \sqrt{\frac{1-t}{N}} \beta x_j' \cdot \epsilon^2 \right).
\]

Taking the derivative of the pressure respect to \( t \) and applying Lemma 3 we have

\[
p'_{N,M}(t) = \frac{1}{N+M} d_{N,M}(t) + \frac{1}{N+M} r_{N,M}(t),
\]

with a main part

\[
d_{N,M}(t) = \frac{\lambda_2 \beta^2}{2} \mathbb{E} \langle N_1 \left( R_{12}^{N_1} - R_{12}^{N_1+M_1} \right) P_{12} \left( t R_{12}^{N_1+M_1} + (1-t) R_{12}^{N_1} \right) \rangle_{N,M,t} +
\]

\[
\quad + \frac{\lambda_2 \beta^2}{2} \mathbb{E} \langle M_1 \left( R_{12}^{M_1} - R_{12}^{N_1+M_1} \right) P_{12} \left( t R_{12}^{N_1+M_1} + (1-t) R_{12}^{M_1} \right) \rangle_{N,M,t},
\]

(34)
and a correction part
\[ r_{N,M}(t) = \frac{\lambda_2 \beta^2}{2} \mathbb{E} \left\langle N_1 \left( R_{12}^{N_1} - R_{12}^{N_1+M_1} \right) \Delta U_{12}^{N,M}(t) \right\rangle_{N,M,t} + \]
\[ + \frac{\lambda_2 \beta^2}{2} \mathbb{E} \left\langle M_1 \left( R_{12}^{M_1} - R_{12}^{N_1+M_1} \right) \Delta W_{12}^{N,M}(t) \right\rangle_{N,M,t}. \]

In the above equations
\[ R_{12}^{N_1} = \frac{1}{N_1} \sum_{i=1}^{N_1} \sigma_i^1 \sigma_i^2, \quad R_{12}^{M_1} = \frac{1}{M_1} \sum_{i=1}^{M_1} \epsilon_i^1 \epsilon_i^2, \]
\[ R_{12}^{N_1+M_1} = \frac{N_1}{N_1 + M_1} R_{12}^{N_1} + \frac{M_1}{N_1 + M_1} R_{12}^{M_1}, \]
and the random correction terms are defined as
\[ \Delta U_{12}^{N,M}(t) = U_{12}^{N,M}(t) - P_{12} \left( t R_{12}^{N_1+M_1} + (1-t) R_{12}^{N_1} \right) \]
\[ \Delta W_{12}^{N,M}(t) = W_{12}^{N,M}(t) - P_{12} \left( t R_{12}^{N_1+M_1} + (1-t) R_{12}^{M_1} \right). \]

Notice that the main part in (34) is positive. In fact, without losing generality we can assume the ordering \( R_{12}^{N_1} \geq R_{12}^{N_1+M_1} \geq R_{12}^{M_1} \), then by Lemma 1,
\[ P_{12} \left( t R_{12}^{N_1+M_1} + (1-t) R_{12}^{N_1} \right) \geq P_{12} \left( t R_{12}^{N_1+M_1} + (1-t) R_{12}^{M_1} \right), \]
from which follows the inequality
\[ d_{N,M}(t) \geq \frac{\lambda_2 \beta^2}{2} \mathbb{E} \left\langle \left( N_1 R_{12}^{N_1} + M_1 R_{12}^{M_1} - (N_1 + M_1) R_{12}^{N_1+M_1} \right) \times \right. \]
\[ \left. \times P_{12} \left( t R_{12}^{N_1+M_1} + (1-t) R_{12}^{M_1} \right) \right\rangle_{N,M,t} = 0. \]

By integrating back (33) we obtain the following relation
\[ (N+M)p_{N+M} = N p_N + M p_M + \int_0^1 d_{N,M}(t) dt + \int_0^1 r_{N,M}(t) dt. \]

It will be also convenient to introduce
\[ x_N = N p_N, \quad c_{N,M} = \int_0^1 r_{N,M}(t) dt. \]

Then, since \( d_{N,M}(t) \geq 0 \), we arrive to the perturbed superadditive relation
\[ x_{N+M} \geq x_N + x_M + c_{N,M}. \]

Now, for any fixed \( k \) and \( n \geq 1 \) let \( N = nk \). By induction on \( n \)
\[ x_{nk+M} \geq nk x_k + x_M + \sum_{l=1}^{n} c_{k,(l-1)k+M}. \]

Dividing by \( nk + M \) and taking the inferior limit for both side leads to the inequality
\[ \liminf_{n \to \infty} \frac{x_{nk+M}}{nk + M} \geq \liminf_{n \to \infty} \frac{nk x_k + x_M}{nk + M} + \liminf_{n \to \infty} \sum_{l=1}^{n} \frac{c_{k,(l-1)k+M}}{nk + M}. \]
Note that the pressure \( p_N = x_N/N \) is bounded, thus the inferior limit exist and is finite. The second term in (46) is the Cesàro mean along some subsequence of \( c_{k,M} \) with fixed \( k \). Recall that the Cesàro mean has the following property
\[
\liminf_{M \to \infty} \frac{1}{M} \sum_{l=1}^{M} c_{N,l} \geq \liminf_{M \to \infty} c_{N,M}. \tag{47}
\]
This can be seen from that for any \( m < M \)
\[
\frac{1}{M} \sum_{l=1}^{M} c_{N,l} \geq \frac{1}{M} \sum_{l=1}^{m} c_{N,l} + \frac{M - m}{M} \inf_{l \geq m} c_{N,l} \tag{48}
\]
and taking inferior limit of \( M \) on both sides gives the inequality
\[
\liminf_{M \to \infty} \frac{1}{M} \sum_{l=1}^{M} c_{N,l} \geq \inf_{l \geq m} c_{N,M} \tag{49}
\]
for any \( m \), then apply the limit of \( m \) to obtain (47). Now we can go back to the second term of (46). Let name the inferior limit of \( c_{N,M} \)
\[
c_N^* = \liminf_{M \to \infty} c_{N,M}, \tag{50}
\]
using Cesàro property (47) along the subsequence \( d_l := c_{k,(l-1)k+M} \)
\[
\liminf_{n \to \infty} \frac{1}{nk + M} \sum_{l=1}^{n} c_{k,(l-1)k+M} = \frac{1}{k} \liminf_{n \to \infty} \sum_{l=1}^{n} \frac{d_l}{n + M/k} \geq \frac{1}{k} \liminf_{n \to \infty} d_n \geq \frac{1}{k} \liminf_{M \to \infty} c_{k,M} = \frac{c_k^*}{k}, \tag{51}
\]
so that (46) becomes
\[
\liminf_{n \to \infty} \frac{nk+M}{nk} = \liminf_{n \to \infty} \frac{x_n}{n} \geq \frac{x_k}{k} + \frac{c_k^*}{k}, \tag{52}
\]
where the equality follows from that any integer can be written as \( nk + M \). Since this holds for any \( k \), then taking the superior limit over \( k \) we get
\[
\liminf_{n \to \infty} \frac{x_n}{n} \geq \limsup_{k \to \infty} \frac{x_k}{k} + \limsup_{k \to \infty} \frac{c_k^*}{k}. \tag{53}
\]
It is clear that if \( \lim_{N \to \infty} c_N^*/N = 0 \) we would get the desired result
\[
\liminf_{n \to \infty} \frac{x_n}{n} = \limsup_{k \to \infty} \frac{x_k}{k}, \tag{54}
\]
Then, start with the explicit expression for \( c_N^* \)
\[
c_N^* = \frac{\lambda_2 \beta^2}{2} \liminf_{M \to \infty} \left( \int_{0}^{1} \mathbb{E} \left[ N_1 \left( R_{12}^{N_1} - R_{12}^{N_1+M_1} \right) \Delta U_{12}^{N,M}(t) \right]_{N,M,t} dt + \right. \]
\[
+ \left. \int_{0}^{1} \mathbb{E} \left[ M_1 \left( R_{12}^{M_1} - R_{12}^{N_1+M_1} \right) \Delta W_{12}^{N,M}(t) \right]_{N,M,t} dt \right) \tag{55}
\]
First we would like to prove that
\[
\lim_{M \to \infty} \frac{\lambda_2 \beta^2}{2} \int_0^1 \mathbb{E} \left\langle M_1 \left( R_{12}^{M_1} - R_{12}^{N_1+M_1} \right) \Delta W_{12}^{N,M}(t) \right\rangle_{N,M,t} dt = 0, \tag{56}
\]
but to do this we have to deal with the possible divergence of \( M_1 \Delta W_{12}^{N,M}(t) \). The crucial observation is that by definition
\[
R_{12}^{M_1} - R_{12}^{N_1+M_1} = \frac{N_1}{N_1 + M_1} R_{12}^{M_1} - \frac{N_1}{N_1 + M_1} R_{12}^{N_1}, \tag{57}
\]
thus \( M_1 \left( R_{12}^{M_1} - R_{12}^{N_1+M_1} \right) = O(1) \) for fixed \( N \). Now since \( \lim_{M \to \infty} \Delta W_{12}^{N,M}(t) = 0 \) a.s. for any \( t \), applying dominated convergence theorem we get (56), then it only remains to prove that
\[
\lim_{N \to \infty} \frac{c_N}{N} = \frac{\lambda_1 \lambda_2 \beta^2}{2} \lim_{N \to \infty} \lim_{M \to \infty} \int_0^1 \mathbb{E} \left\langle (R_{12}^{N_1} - R_{12}^{N_1+M_1}) \Delta U_{12}^{N,M}(t) \right\rangle_{N,M,t} dt = 0. \tag{58}
\]
Start by noticing that the Gibbs average can be bounded as
\[
\left\langle (R_{12}^{N_1} - R_{12}^{N_1+M_1}) \Delta U_{12}^{N,M}(t) \right\rangle_{N,M,t} \leq 2 \sup_{\rho^1, \rho^2} \left| \Delta U_{12}^{N,M}(t) \right|. \tag{59}
\]
Then notice that \( \Delta U_{12}^{N,M}(t) \) is function of a Gaussian process, and for Gaussian processes the convergence of the covariance matrix imply converge in distribution. Since the covariance of the Gaussian process defined in (31) is given by
\[
\lambda^1 \beta^2 \left( t R_{12}^{N_1+M_1} + (1-t) R_{12}^{N_1} \right)
\]
we conclude that the distribution of \( \sup_{\rho^1, \rho^2} \left| \Delta U_{12}^{N,M}(t) \right| \) depends on \( M \) only through the overlap \( R_{12}^{N_1+M_1} \), and then the supremum can be equivalently taken over a finite subset of \([-1, 1]\), which leads to the following
\[
\sup_{\rho^1, \rho^2} \left| \Delta U_{12}^{N,M}(t) \right| \leq \sup_{R_{12}^{N_1+M_1}, R_{12} \in [-1, 1]} \left| \Delta U_{12}^{N,M}(t) \right|. \tag{60}
\]
Note that now the right side does not depend on \( M \) anymore. To emphasize this we relabel it as follows
\[
\sup_{R_{12}^{N_1+M_1}, R_{12} \in [-1, 1]} \left| \Delta U_{12}^{N,M}(t) \right| = \sup_{R_{12}, R_{12}^{N_1} \in [-1, 1]} \left| \Delta U_{12}^{N,-}(t) \right|. \tag{61}
\]
Take the average over the Gaussians, the integral over \( t \) and take the inferior limit of \( M \) on both sides for fixed \( N \)
\[
\liminf_{M \to \infty} \int_0^1 \mathbb{E} \sup_{\rho^1, \rho^2} \left| \Delta U_{12}^{N,M}(t) \right| dt \leq \int_0^1 \mathbb{E} \sup_{R_{12}, R_{12}^{N_1} \in [-1, 1]} \left| \Delta U_{12}^{N,-}(t) \right| dt. \tag{62}
\]
Since \( \lim_{N \to \infty} \Delta U_{12}^{N,-}(t) = 0 \) for any \( R_{12} \) and \( R_{12}^{N_1} \), by applying dominated convergence theorem twice we find
\[
\lim_{N \to \infty} \int_0^1 \mathbb{E} \sup_{R_{12}, R_{12}^{N_1} \in [-1, 1]} \left| \Delta U_{12}^{N,-}(t) \right| dt = 0. \tag{63}
\]
Thus we get \( \lim_{N \to \infty} c_N^* / N = 0 \) and this finishes the proof that the thermodynamic limit of the pressure exists.
4. Representing Parisi functional with Ruelle Probability Cascades

Ruelle Probability Cascades (RPC) \cite{15,17} is a family of random probability measures that, as for the standard SK model, play a central role in our solution of bipartite SK. A direct consequence of Lemma 1 and 2 is that the distribution of generalized overlap $P_{12}$ can be generated by PRC simultaneously with that of $R_{12}$, since if the support of overlap $R_{12}$ is ultrametric then any increasing transform of $R_{12}$ will preserve the ultrametric property. In this section we introduce the construction and also some important properties of RPC that will be used. These results will be stated without proof, since they are basically similar to that for the SK model with only slight modifications. Complete introductions to RPC can be found in \cite{15}.

The parameters of RPC include a number $r \geq 1$ and the sequences defined in (12a) and (12b). The construction of RPC is as follows. First generate a Poisson process on $(0, \infty)$ with mean measure 

$$\mu(dx) = \zeta_0 x^{-1-\omega_0} dx.$$  

Label the of points of the Poisson process in decreasing order by $u_{01} > u_{02} > \cdots > u_{0n} > \cdots$. Then for each point of the Poisson process, generate another Poisson process with the same form of mean measure in (64) but with parameter $\zeta_1$. Repeat this procedure and an infinitary tree with $r$ layers is generated, and for each vertex in layer $l$ its children is generated by Possion process with parameter $\zeta_l$. For each vertex in the tree, define the following quantity:

$$w_\alpha = \prod_{\beta \in p(\alpha)} u_\beta,$$  

where $p(\alpha)$ denote the set of vertices on the path from $\alpha$ to the root. Then for the leaves $\alpha \in \mathbb{N}^r$ define the weights

$$v_\alpha = \frac{w_\alpha}{\sum_{\beta \in \mathbb{N}^r} w_\beta}.$$  

These weights define a probability measure in $\mathbb{N}^r$ and the measure is called RPC. For $\alpha, \beta \in \mathbb{N}^r$, denote

$$\alpha \wedge \beta = \min\{1 \leq l \leq r | \alpha_1 = \beta_1, \cdots, \alpha_l = \beta_l, \alpha_{l+1} \neq \beta_{l+1}\}.$$  

In other words $\alpha \wedge \beta$ is the layer where $p(\alpha)$ and $p(\beta)$ meet in the tree. Let $(\eta_\beta)$ be a sequence of i.i.d. standard Gaussian random variables definite on all vertices of the tree of RPC. For a nondecreasing function $g : [0, \infty) \to [0, \infty)$, define Gaussian process:

$$C_g(\alpha) = \sum_{\beta \in p(\alpha)} \eta_\beta \left(g(q_{|\beta|}) - g(q_{|\beta|-1})\right)^{1/2}.$$  

Note that $P_{12}$ satisfy the condition of $f$ to be nondecreasing according to Lemma 11 and this makes RPC applicable for bipartite SK model. It’s easy to verify the covariance of the Gaussian process is

$$\mathbb{E}C_g(\alpha)C_g(\beta) = g(q_{\alpha \wedge \beta}).$$  

(70)
This finishes the basic constructions of RPC. Next we state two important results that will be used.

**Lemma 4.** The Parisi functional can be represented by Ruelle Probability Cascades as

\[
\mathcal{P} (\zeta, q) = \lambda_2 \mathbb{E} \log \sum_{a \in \mathbb{N}^r} v_a \cosh C_\phi (\alpha) + \lambda_1 \mathbb{E} \log \sum_{a \in \mathbb{N}^r} v_a \cosh C_\psi (\alpha) + \\
- \mathbb{E} \log \sum_{a \in \mathbb{N}^r} v_a \exp C_\phi (\alpha) + \log 2.
\]

**Proof.** The proof can be found in Lemma 3.1 in [15]. \(\square\)

**Lemma 5.** Let \(C_\phi^i (\alpha)\) with \(i \in \{1, \cdots, m\}\) be i.i.d. copies of \(C_\phi (\alpha)\) defined in (69) and \(C_\psi^j (\alpha)\) with \(j \in \{1, \cdots, n\}\) be i.i.d. copies of \(C_\psi (\alpha)\). The following relation holds

\[
\mathbb{E} \log \sum_{a \in \mathbb{N}^r} v_a \prod_{i=1}^{m} \cosh C_\phi^i (\alpha) \prod_{j=1}^{n} \cosh C_\psi^j (\alpha) = \\
m \mathbb{E} \log \sum_{a \in \mathbb{N}^r} v_a \cosh C_\phi (\alpha) + n \mathbb{E} \log \sum_{a \in \mathbb{N}^r} v_a \cosh C_\psi (\alpha).
\]

Also, for an integer \(l > 0\)

\[
\mathbb{E} \sum_{\alpha} v_\alpha \exp \sqrt{l} (C_\theta (h_\alpha)) = l \mathbb{E} \sum_{\alpha} v_\alpha \exp C_\theta (h_\alpha).
\]

**Proof.** These are standard properties of RPC, see section 2.3 of [15] for the proof. \(\square\)

### 5. Upper bound via Guerra’s Interpolation

With these preparations we can start the proof of the Parisi formula for bipartite SK model. In this section we prove that the Parisi formula gives an upper bound of the free energy by Guerra interpolation.

**Lemma 6.** Let \((x_i (\sigma, \alpha))_{1 \leq i \leq N_2}, (y_i (\sigma, \alpha))_{1 \leq i \leq N_2}\) be Gaussian processes indexed by \(\sigma \in \Sigma_{N_1}\) and \(\alpha \in \mathcal{A}\), where \(N_1, N_2\) are some finite numbers and \(\mathcal{A}\) a countable infinite set. The variances of the Gaussian processes again bounded and the elements of the covariance matrix are given by \(\mathbb{E} x_i (\sigma^1, \alpha^1) y_i (\sigma^2, \alpha^2) = \delta_{i,j} C_{12}\). Note that the cardinality of \(\mathcal{A}\) is independent of \(N_1, N_2\). Define a random measure \(G (\sigma, \alpha)\) on \(\Sigma \times \mathcal{A}\) as

\[
G (\sigma, \alpha) = \frac{1}{Z} \exp \left( \sum_{i=1}^{N_2} \log \left( 2 \cosh y_i (\sigma, \alpha) \right) \right), \quad Z = \sum_{\sigma \in \Sigma_{N_1}, \alpha \in \mathcal{A}} \exp \left( \sum_{i=1}^{N_2} \log \left( 2 \cosh y_i (\sigma, \alpha) \right) \right),
\]

and let \(\langle \cdot \rangle\) be the average over the measure, then the following equality holds:

\[
\mathbb{E} \left\langle \frac{1}{N_2} \sum_{i=1}^{N_2} x_i (\sigma, \alpha) \tanh (y_i (\sigma, \alpha)) \right\rangle = \\
= \mathbb{E} \left\langle C_{11} - \frac{C_{12}}{N_2} \sum_{i=1}^{N_2} \tanh (y_i (\sigma^1, \alpha^1)) \tanh (y_i (\sigma^2, \alpha^2)) \right\rangle.
\]
Proof. First suppose $\mathcal{A}$ is a finite set. Then by identical computations in the proof of Lemma 3 we get (74). Following Lemma 1.2. in [15] the equality can be shown to hold when $\mathcal{A}$ is countable infinite.

As we did for the thermodynamic limit, let us define an Hamiltonian, this time supported by $\Sigma_{N_1} \times \mathbb{N}^r$, interpolating between the actual system and the RPC

$$-H_{N,t}(\sigma, \alpha) = \sum_{j=1}^{N_2} \log \left( 2 \cosh \left( \sqrt{\frac{t}{N}} \beta g_j \cdot \sigma + \sqrt{1-t} C_{\phi,j}(\alpha) \right) \right) +$$

$$+ \sqrt{1-t} \sum_{i=1}^{N_1} C_{\psi,i}(\alpha) \sigma_i + \sqrt{t} N C_{\theta}(\alpha), \quad (75)$$

where $C_{\phi,j}(\alpha)$ are i.i.d copies of $C_{\phi}(\alpha)$ and $C_{\psi,i}(\alpha)$ i.i.d copies of $C_{\psi}(\alpha)$. Denote the infinitely replicated Gibbs average by $\langle \cdot \rangle_{N,t}$. Then, define the $t$ dependent pressure

$$p_N(t) = \frac{1}{N} \mathbb{E} \log \sum_{\sigma, \alpha} v^\alpha \exp (-H_{N,t}(\sigma, \alpha)). \quad (76)$$

For $t = 0$ we can verify that

$$p_N(0) = \frac{1}{N} \mathbb{E} \log \sum_{\alpha \in \mathbb{N}^r} v^\alpha \prod_{j=1}^{N_2} 2 \cosh \left( \sqrt{\frac{t}{N}} \beta g_j \cdot \sigma^1 + \sqrt{1-t} C_{\phi,j}(\alpha) \right) \prod_{i=1}^{N_1} 2 \cosh \left( \sqrt{\frac{t}{N}} \beta g_i \cdot \sigma^2 + \sqrt{1-t} C_{\theta}(\alpha) \right) +$$

$$= \lambda_2 \mathbb{E} X_0 + \lambda_1 \mathbb{E} Y_0 + \log 2, \quad (77)$$

where the second line is by applying Lemma 5. For $t = 1$

$$p_N(1) = p_N + \frac{1}{N} \mathbb{E} \sum_{\alpha} v^\alpha \exp \left( -\sqrt{N} \left( C_{\theta}(h^\alpha) \right) \right) =$$

$$= p_N + \frac{1}{2} \sum_{0 \leq p \leq r-1} \zeta_p \left( \theta(q_{p+1}) - \theta(q_p) \right), \quad (78)$$

and the second line is again by Lemma 5. Taking the difference we have $p_N(1) - p_N(0) = p_N - \mathcal{P}(\zeta, q)$. Now we prove the following bound of pressure in the limit.

**Lemma 7.** The thermodynamic limit of pressure is bounded by $\lim_{N \to \infty} p_N \leq \mathcal{P}(\zeta, q)$.

Proof. The proof will again by decomposing the generalized overlap into a deterministic part and a controllable random part. Define

$$V_{12}^{N_2}(t) = \frac{1}{N_2} \sum_{j=1}^{N_2} \tanh \left( \sqrt{\frac{t}{N}} \beta g_j \cdot \sigma^1 + \sqrt{1-t} C_{\phi,j}(\alpha^1) \right) \times$$

$$\times \tanh \left( \sqrt{\frac{t}{N}} \beta g_j \cdot \sigma^2 + \sqrt{1-t} C_{\phi,j}(\alpha^2) \right). \quad (79)$$

Since the Gaussian processes

$$g(\sigma, \alpha, t) = \sqrt{\frac{t}{N}} \beta g_j \cdot \sigma + \sqrt{1-t} C_{\phi,j}(\alpha) \quad (80)$$
has a covariance matrix $E g (\sigma^1, \alpha^1, t) g (\sigma^2, \alpha^2, t) = \lambda_1 \beta^2 \left( t R_{12}^{N_1} + (1 - t) q_{1, \alpha^2} \right)$, and a variance $E g (\sigma, \alpha, t)^2 = \lambda_1 \beta^2$ we can decompose as
\[ V_{12}^{N_2} (t) = P_{12} (t R_{12}^{N_1} + (1 - t) q_{1, \alpha^2}) + \Delta V_{12}^{N_2} (t). \]  
Now take the derivative of $p_N (t)$ with respect to $t$, apply Lemma 6 and Lemma 1.1 in [15], then we obtain
\[ p'_N (t) = d_N (t) + r_N (t) \]  
where there is again a main part
\[ d_N (t) = - \frac{\lambda_1 \lambda_2 \beta^2}{2} E \langle (R_{12}^{N_1} - q_{1, \alpha^2}) P_{12} (t R_{12}^{N_1} + (1 - t) q_{1, \alpha^2}) \rangle_{N,t} + \frac{\lambda_1 \lambda_2 \beta^2}{2} E \langle (R_{12}^{N_1} - q_{1, \alpha^2}) P_{12} (q_{1, \alpha^2}) \rangle_{N,t} \]  
and correction term
\[ r_N (t) = - \frac{\lambda_1 \lambda_2 \beta^2}{2} E \langle (R_{12}^{N_1} - q_{1, \alpha^2}) \Delta V_{12}^{N_2} (t) \rangle_{N,t}. \]  
As before, we first consider the main part in (83). If $R_{12}^{N_1} \geq q_{1, \alpha^2}$, then by Lemma 11 follows $P_{12} (t R_{12}^{N_1} + (1 - t) q_{1, \alpha^2}) \geq P (q_{1, \alpha^2})$, and therefore
\[ (R_{12}^{N_1} - q_{1, \alpha^2}) \left( P_{12} (t R_{12}^{N_1} + (1 - t) q_{1, \alpha^2}) - P_{12} (q_{1, \alpha^2}) \right) \geq 0, \]  
on the other hand if $R_{12}^{N_1} < q_{1, \alpha^2}$ then $P_{12} (t R_{12}^{N_1} + (1 - t) q_{1, \alpha^2}) \leq P (q_{1, \alpha^2})$, and again we find (85). Then we conclude that $d_N (t) \leq 0$ for any $t$. Write (82) as
\[ p_N - \mathcal{P} (\zeta, q) = \int_0^1 d_N (t) dt + \int_0^t r_N (t) dt \]  
and take the limit in $N$ on both sides. Since $\int_0^1 d_N (t) dt \leq 0$ holds for any $N$, then $\lim_{N \to \infty} \int_0^1 d_N (t) dt \leq 0$. Meanwhile since $\lim_{N \to \infty} r_N (t) = 0$ for any $t$, by dominated convergence theorem $\lim_{N \to \infty} \int_0^1 r_N (t) dt = 0$ and we finally get
\[ \lim_{N \to \infty} p_N \leq \mathcal{P} (\zeta, q), \]  
which is the claim. \qed

6. The matching Lower Bound

6.1. Ghirlanda-Guerra identities

As for the standard SK model, the proof of the lower bound of bipartite SK is obtained by applying Ghirlanda-Guerra identities and Aizenman-Sims-Starr representation. The proof that Ghirlanda-Guerra identities holds for bipartite SK model has already been given in [2], see Section 3 of [2]. Briefly, a perturbation term is introduced into the Hamiltonian
\[ H_{\text{BSK}}^\text{pert} (\sigma, \tau) = H_{\text{BSK}} (\sigma, \tau) + sN h_N (\sigma, \tau). \]
The full formula of \( h_N(\sigma, \tau) \) can be found in (24) and (27) in \[2\] by setting \( \omega_s = 0 \) for \( s > 2 \). Let \( \langle \cdot \rangle_{\text{BSK}}^{\text{pert}} \) be the Gibbs average defined by \( H_{\text{BSK}}^{\text{pert}}(\sigma, \tau) \), then the following Ghirlanda-Guerra identities

\[
\mathbb{E} \langle f_1(R^n) f_2(R_{1,n+1}) \rangle_{\text{BSK}}^{\text{pert}} = \frac{1}{n} \mathbb{E} \langle f_1(R^n) \rangle_{\text{BSK}}^{\text{pert}} \mathbb{E} \langle f_2(R_{1,2}) \rangle_{\text{BSK}}^{\text{pert}} + \frac{1}{n} \sum_{l=2}^{n} \mathbb{E} \langle f_1(R^n) f_2(R_{1,l}) \rangle_{\text{BSK}}^{\text{pert}},
\]

hold for any bounded measurable functions \( f_1 = f_1(R^n) \) and \( f_2 = f_2(R_{12}) \).

On an intuitive level, Ghirlanda-Guerra identities describe how to generate quantities depending on \( R_{l,n+1} \) for \( l \leq n \) basing on \( R_n = (R_{l,l'})_{l,l' \leq n} \). It has been shown in \[15\] that if a measure satisfy Ghirlanda-Guerra identities, then the measure can be approximated by RPC in the sense that the overlap array generated is close in distribution to that of RPC. By choosing parameters in (88), the perturbation can be small enough not to change the free energy, but large enough to force the Gibbs measure satisfy Ghirlanda-Guerra identities. Ghirlanda-Guerra identities further imply that the overlap distribution can be approximated by RPC. For complete introduction see \[2, 15\].

A problem is that with the perturbation term \( h_N(\sigma, \tau) \) in (88) one cannot sum over \( \tau \) to get the equivalent system defined in terms of \( \sigma \) only, which make the computation of required Aizenman-Sims-Starr representation difficult. However, since \( P_{12}(R_{12}) \) is a bounded continuous of \( R_{12} \), the full Ghirlanda-Guerra identities in (89) are not actually needed, since they include overlaps both among \( \sigma \) and among \( \tau \). Ghirlanda-Guerra identities defined partially with overlaps of \( \sigma \) will be sufficient for the task.

To prove the partial Ghirlanda-Guerra identities for \( R_{12} \) it will be enough to set \( \omega_2 = 0 \) in (24) of \[2\] and consider only the \( \sigma \) dependent part \( h_N(\sigma) \) of the original perturbation term \( h_N(\sigma, \tau) \). This follows from the proof of Theorem 3 in \[2\]. Infact, from (36) in \[2\] one can directly get the partial Ghirlanda-Guerra identities by approximating \( f_2 \) with polynomials.

This observation makes convenient to prove the Aizenman-Sims-Starr representation with a perturbation term \( h_N(\sigma) \) that does not depend on \( \tau \), since the pressure and Gibbs averages with respect to the Hamiltonian

\[
H_{\text{BSK}}^{\text{pert}}(\sigma, \tau) = H_{\text{BSK}}(\sigma, \tau) + s_N h_N(\sigma)
\]

will be identical to that with

\[
H_N^{\text{pert}}(\sigma) = H_N(\sigma) + s_N h_N(\sigma)
\]

by summing over the \( \tau \) side.

6.2. Aizenman-Sims-Starr representation

Consider adding \( m_1 \) and \( m_2 \) cavity spins into the \( \sigma \) and \( \tau \) group respectively. The ratios \( m_1/m = \lambda_1 \) and \( m_2/m = \lambda_2 \) are fixed. Denote \( m = m_1 + m_2 \) and assume \( m \ll N \). The lower bound is obtained from the relation

\[
\liminf_{N \to \infty} p_N \geq \frac{1}{m} \liminf_{N \to \infty} ((N + m)p_{N+m} - Np_N).
\]

(92)
Let $\epsilon \in \Sigma_{m_1}$ be the cavity spins for the $\sigma$ group and let $\rho \in \Sigma_{m_2}$ that for the $\tau$ group. Let $(\tilde{g}_{ij})_{1 \leq i \leq N_1, 1 \leq j \leq m_2}$ be the interactions between $\sigma$ and $\rho$, $(\bar{g}_{ij})_{1 \leq i \leq m_1, 1 \leq j \leq N_2}$ between $\epsilon$ and $\tau$ and $(\bar{g}_{ij})_{1 \leq i \leq m_1, 1 \leq j \leq m_2}$ between $\epsilon$ and $\rho$. Define a Hamiltonian of $N$ spins

$$-H'_N(\sigma) = \sum_{j=1}^{N_2} \log \left( 2 \cosh \left( \frac{1}{\sqrt{N+m}} \beta \tilde{g} \cdot \sigma \right) \right).$$

(93)

The Hamiltonian with $m$ cavity spins can be expanded using Taylor series:

$$-H_{N+m}(\sigma, \epsilon) = -H'_N(\sigma) + \sum_{j=1}^{m_2} \log \left( 2 \cosh x^j_N(\sigma) \right) + \sum_{i=1}^{m_1} \epsilon_i y^i_N(\sigma) + R(\sigma, \tau),$$

(94)

where

$$x^j_N(\sigma) = \frac{1}{\sqrt{N+m}} \beta \sum_{i=1}^{N_1} \tilde{g}_{ij} \sigma_i,$$

(95)

$$y^i_N(\sigma) = \frac{1}{\sqrt{N+m}} \beta \sum_{j=1}^{N_2} \sqrt{\tilde{g}_{ij}} \tanh \left( \frac{1}{\sqrt{N+m}} \beta \bar{g}_j \cdot \sigma \right),$$

(96)

and the remainder terms is

$$R(\sigma, \epsilon) = \frac{1}{\sqrt{N+m}} \beta \sum_{j=1}^{m_2} \sum_{i=1}^{N_1} \epsilon_i \sqrt{\tilde{g}_{ij}} \tanh \left( \frac{1}{\sqrt{N+m}} \beta \bar{g}_j \cdot \sigma \right) + \frac{1}{2(N+m)} \beta^2 \sum_{j=1}^{N_2} (\bar{g}_j \cdot \epsilon)^2 \left( 1 - \tanh^2 \left( \frac{1}{\sqrt{N+m}} \beta \bar{g}_j \cdot \sigma \right) \right) + ...$$

(97)

The first term in the remainder correspond to interactions among cavity spins and is thus of a smaller order. For the second term, according to the law of large numbers, in the thermodynamic limit it must converge to a constant and cannot change the free energy in the limit. Higher derivatives of $\tanh(\cdot)$ are bounded thus all higher order terms are of smaller order. All combined, the remainder terms $R(\sigma, \epsilon)$ can be ignored, and the pressure of the $N + m$ system can be represented as

$$(N+m)p_{N+m} = \mathbb{E} \log \left( \prod_{j=1}^{m_2} 2 \cosh(x^j_N(\sigma)) \prod_{i=1}^{m_1} 2 \cosh(y^i_N(\sigma)) \right)' ,$$

(98)

where the average of Gibbs measure $\langle \cdot \rangle'$ is defined with the Hamiltonian $H'_N$. Similarly, for a system of size $N$,

$$-H_N(\sigma) = \sum_{j=1}^{N_2} \log \cosh \left( \frac{1}{\sqrt{N+m}} \beta \tilde{g}_j \cdot \sigma + \sqrt{\frac{m}{N(N+m)}} \beta g'_j \cdot \sigma \right)$$

$$= -H'_N(\sigma) + z_N(\sigma),$$

(99)

where $g'_j$ is another fresh set of i.i.d. Gaussians and

$$z_N(\sigma) = \sum_{j=1}^{N_2} \sqrt{\frac{m}{N(N+m)}} \beta g'_j \cdot \sigma \tanh \left( \frac{1}{\sqrt{N+m}} \beta \bar{g}_j \cdot \sigma \right).$$

(100)
Then the free energy of $N$ spin system is
\[ Np_N = \mathbb{E} \log \langle \exp (z_N (\sigma)) \rangle'. \] (101)

The main difference of this Aizenman-Sims-Starr representation from that of SK is in that $y_N (\sigma)$ and $z_N (\sigma)$ also depend on $(g_{ij})$. Conditioned on $(g_{ij})$, $x_N (\sigma)$, $y_N (\sigma)$ and $z_N (\sigma)$ are Gaussian processes with covariance
\[
\begin{align*}
\mathbb{E}_{\beta} x_N (\sigma^1) x_N (\sigma^2) &= \lambda_1 \beta^2 R^N_{12} + O(N^{-1}), \\
\mathbb{E}_{\beta} y_N (\sigma^1) y_N (\sigma^2) &= \lambda_2 \beta^2 P^N_{12} + O(N^{-1}), \\
\mathbb{E}_{\sigma} z_N (\sigma^1) z_N (\sigma^2) &= m \lambda_1 \lambda_2 \beta^2 R^N_{12} P^N_{12} + O(N^{-1}).
\end{align*}
\] (102)

Note that the expectations here only average on the new randomness introduced in the cavity computations, but not on $(g_{ij})$. Consider any $q \in (1, 2)$, it’s easy to verify that by Marcinkiewicz-Zygmund theorem
\[ P^N_{12} = P_{12} \left( R^N_{12} \right) + o(N^{1/q-1}) \] (103)
amost surely. Then we can rewrite the last two covariances in (102) as
\[
\begin{align*}
\mathbb{E}_{\beta} y_N (\sigma^1) y_N (\sigma^2) &= \lambda_2 \beta^2 P_{12} \left( R^N_{12} \right) + o(N^{1/q-1}), \\
\mathbb{E}_{\sigma} z_N (\sigma^1) z_N (\sigma^2) &= m \lambda_1 \lambda_2 \beta^2 R^N_{12} P_{12} \left( R^N_{12} \right) + o(N^{1/q-1}).
\end{align*}
\] (104)

Now the covariances in (104) are deterministic functions of $R^N_{12}$ and the lower order terms $o(N^{1/q-1})$ are ignored. The following quantity
\[ A_N := \frac{1}{m} \mathbb{E} \log \left( \prod_{i=1}^{m_2} 2 \cosh x^i_N (\sigma) \prod_{j=1}^{m_1} 2 \cosh y^j_N (\sigma) \right)' - \frac{1}{m} \mathbb{E}_{\sigma'} \log \langle \exp z_N (\sigma) \rangle' \] (105)
and the pressure increment can be shown to be related by
\[ (N + m) p_{N+m} - Np_N = mA_N + o(1). \] (106)

In other words it’s safe to replace $P^N_{12}$ by $P_{12} \left( R^N_{12} \right)$ in the Aizenman-Sims-Starr representation. The proof is similar to Theorem 3.6 in [15] and will not be repeated here.

The remaining of the proof is basically the same of SK and multi-species SK model thus we only briefly sketch the procedure, see [2] and [15] for formal statement. When the Hamiltonian contains a perturbation term as in (91), the Aizenman-Sims-Starr representation can be computed similarly. It has been shown in [2] that the difference from replacing $s_{N+m} h_{N+m}(\sigma)$ by $s_N h_N(\sigma)$ can be ignored. With the perturbation term, the Gibbs measure with Hamiltonian $H' (\sigma)$ satisfies Ghirlanda-Guerra identities. Ghirlanda-Guerra identities imply that the overlap distribution can be approximated by Ruelle Probability Cascades. Meanwhile (104) can be proved to be a continuous functional of the overlap distribution as in Theorem 1.3 in [15] and [2]. Thus it can be approximated by RPC as
\[
\begin{align*}
\frac{1}{m} \left( \mathbb{E} \log \sum_{\alpha \in \Lambda^r} v_\alpha \prod_{j=1}^{m_2} 2 \cosh C_{\phi,j}(\alpha) \prod_{i=1}^{m_1} 2 \cosh C_{\psi,i}(\alpha) \right) + \\
-\frac{1}{m} \mathbb{E} \log \sum_{\alpha \in \Lambda^r} v_\alpha \exp \sqrt{m} C_{\phi}(h_\alpha),
\end{align*}
\] (107)
It can be checked that the form of the covariances of $x_N(\sigma)$, $y_N(\sigma)$, $z_N(\sigma)$ and $C_\phi(\alpha)$, $C_\psi(\alpha)$, $\sqrt{m} C_\theta(\alpha)$ are correspondingly the same. By Lemma 5, (107) is exactly the Parisi functional. This finishes the proof of lower bound.

Then, combining with the upper bound in Section 5 finishes the proof of the analogue Parisi formula for the bipartite SK model.

7. Critical lines and partial replica symmetry breaking

Based on the Parisi formula proved above we can compute the critical point equations, the corresponding critical lines, and finally discuss the phase diagram of the bipartite SK model. As proposed in [2], we found that there are regions in the parameter space where the smaller group is in RS phase while other in RSB, namely, partial replica symmetry breaking solutions.

First we derive the critical point equations. The replica symmetric pressure is by setting $\zeta_0 = 0$, $\zeta_1 = 1$ and $q_2 = 1$ in the Parisi formula. Then the replica symmetric Parisi functional is

$$P_{RS}(q_1) = \log 2 + \lambda_2 E_{\eta_1} \log \cosh(\eta_1 \sqrt{\phi(q_1)}) + \lambda_1 E_{\omega_1} \log \cosh(\omega_1 \sqrt{\psi(q_1)}) +$$

$$+ \frac{\lambda_2}{2} (\phi(1) - \phi(q_1)) + \frac{\lambda_1}{2} (\psi(1) - \psi(q_1)) - \frac{1}{2} (\theta(1) - \theta(q_1)), \quad (108)$$

which depends only on one parameter $q_1$. The critical point equation comes from taking the derivative to $q_1$ and set it to 0.

**Lemma 8.** The critical point equation of bipartite SK model is

$$\lambda_2 \phi'(q_1) \left( \frac{\psi(q_1)}{\lambda_2 \beta^2} - E_{\eta_1} \tanh^2(\eta_1 \sqrt{\phi(q_1)}) \right) +$$

$$+ \lambda_1 \psi'(q_1) \left( \frac{\phi(q_1)}{\lambda_1 \beta^2} - E_{\omega_1} \tanh^2(\omega_1 \sqrt{\psi(q_1)}) \right) = 0. \quad (109)$$

Next we derive the critical lines of the bipartite SK model. The critical lines are defined as the lines beyond which RS solution becomes unstable. Thus it is an analogue of the de Almeida-Thouless (A-T) line of SK model, but in the parameter space of $\beta$ and $\lambda_1$. Here we do not include the external field. Let $P_{1RSB}$ be the one step replica symmetry breaking Parisi functional. This is obtained by setting $\zeta_0 = 0$, $\zeta_2 = 1$ and $q_3 = 0$. Define Gaussian process

$$A = \eta_1 \sqrt{\phi(q_1)} + \eta_2 \sqrt{\phi(q_2)} - \phi(q_1), \quad (110)$$

$$B = \omega_1 \sqrt{\psi(q_1)} + \omega_2 \sqrt{\psi(q_2)} - \phi(q_1), \quad (111)$$

and take from the definitions in (13) and (14),

$$X_1 = \frac{1}{\zeta_1} \log E_{\eta_2} \exp (\zeta_1 \log \cosh A), \quad Y_1 = \frac{1}{\zeta_1} \log E_{\omega_2} \exp (\zeta_1 \log \cosh B). \quad (112)$$

The 1-RSB Parisi functional is

$$P_{1RSB}(q_1, q_2, \zeta_1) = \log 2 + \lambda_2 E_{\eta_1} X_1 + \lambda_1 E_{\omega_1} Y_1 +$$
Free energy of bipartite Sherrington-Kirkpatrick model

\[ + \frac{\lambda_2}{2} (\phi(1) - \phi(q_2)) + \frac{\lambda_1}{2} (\psi(1) - \psi(q_2)) + \\
- \frac{1}{2} (\theta(1) - \theta(q_2)) - \frac{\zeta_1}{2} (\theta(q_2) - \theta(q_1)). \]  

(113)

Let \( p_{\text{1RSB}} \) and \( p_{\text{RS}} \) the 1-RSB and RS pressure in the thermodynamic limit respectively, we will prove that

**Theorem 3.** For any \( q_1 \) which is a solution of the critical point equation, if it is beyond the critical line, i.e., if

\[ \lambda_2 \beta^2 (\phi'(q_1))^2 \mathbb{E}_{n_1} \cosh^{-4} \left( \eta_1 \sqrt{\phi(q_1)} \right) + \lambda_1 \beta^2 (\psi'(q_1))^2 \mathbb{E}_{\omega_1} \cosh^{-4} \left( \omega_1 \sqrt{\psi(q_1)} \right) + \\
+ \lambda_1^2 \beta^2 \psi''(q_1) \mathbb{E}_{\omega_1} \tanh^2 \left( \omega_1 \sqrt{\psi(q_1)} \right) > 2 \phi'(q_1)\psi'(q_1) + \phi(q_1)\psi''(q_1), \]

then \( p_{\text{1RSB}} < p_{\text{RS}}. \)

**Proof.** The proof is similar to that of the A-T line [8][18] for SK model. Define a function

\[ f(q_2) = \frac{\partial \mathcal{P}_{\text{1RSB}}}{\partial q_1} \bigg|_{q_1 = 1} = \\
- \lambda_2 \mathbb{E}_{n_1} \log \cosh \left( \eta_1 \sqrt{\phi(q_1)} \right) - \lambda_1 \mathbb{E}_{\omega_1} \log \cosh \left( \omega_1 \sqrt{\psi(q_1)} \right) + \\
+ \lambda_2 \mathbb{E}_{n_2} \log \left( \cosh (A) \cosh (A) \right) + \lambda_1 \mathbb{E}_{\omega_2} \log \left( \cosh (B) \cosh (B) \right) + \\
- \frac{\lambda_2}{2} (\phi(q_2) - \phi(q_1)) - \frac{\lambda_1}{2} (\psi(q_2) - \psi(q_1)) - \frac{1}{2} (\theta(q_2) - \theta(q_1)), \]  

(114)

where the dependence on \( q_1 \) is not explicitly shown. It’s easy to verify that \( f(q_1) = 0. \)

The derivative of \( f(q_2) \) to \( q_2 \) is

\[ f'(q_2) = \frac{\lambda_2 \phi'(q_2)}{2} \mathbb{E}_{n_1} \frac{\mathbb{E}_{n_2} \tanh(A) \sinh(A)}{\mathbb{E}_{n_2} \cosh(A)} + \\
+ \frac{\lambda_1 \psi'(q_2)}{2} \mathbb{E}_{\omega_1} \frac{\mathbb{E}_{\omega_2} \tanh(B) \sinh(B)}{\mathbb{E}_{\omega_2} \cosh(B)} - \frac{1}{2} \theta'(q_2). \]  

(115)

Also \( f'(q_1) = 0 \) follows from the critical point equation. The critical line comes out from the second derivative of \( f(q_2) \) at \( q_1 \), which is

\[ f''(q_1) = - \frac{1}{2} \theta''(q_1) + \frac{\lambda_1}{2} \psi''(q_1) \mathbb{E}_{\omega_1} \tanh^2 \left( \omega_1 \sqrt{\psi(q_1)} \right) + \\
+ \frac{\lambda_2}{2} (\phi'(q_1))^2 \mathbb{E}_{n_1} \cosh^{-4} \left( \eta_1 \sqrt{\phi(q_1)} \right) + \\
+ \frac{\lambda_1}{2} (\psi'(q_1))^2 \mathbb{E}_{\omega_1} \cosh^{-4} \left( \omega_1 \sqrt{\psi(q_1)} \right) \]  

(116)

If \( f''(q_1) > 0 \), since \( f'(q_1) = f(q_1) = 0 \), there must be a \( q' > q_1 \) with \( f(q') > 0. \) Then there must be a \( 0 < \zeta_1 < 1 \) such that

\[ \mathcal{P}_{\text{1RSB}}(q_1, q', \zeta_1) < \mathcal{P}_{\text{1RSB}}(q_1, q', 1) = \mathcal{P}_{\text{RS}}(q_1). \]  

(117)

It’s easy to verify that the equality \( \mathcal{P}_{\text{1RSB}}(q_1, q_2, 1) = \mathcal{P}_{\text{RS}}(q_1) \) holds by taking \( \zeta_1 = 1 \) into (113). Then the proof finishes by observing that \( f''(q_1) > 0 \) gives the analogue A-T criteria. ☐
This possibility that one group is in a RS phase and the other in RSB is due to the definition of the effective Hamiltonian, in fact in each of the groups can be summed over the other, this leads to two different critical point equations and also two distinct critical lines. We solved numerically the critical point equations and checked the stability criteria; the resulting phase diagram is shown in Figure 1. It can be seen that the partial replica symmetry breaking solution does exist in a certain region of parameters. By increasing $\beta$ from RS phase, the group with larger size, say group $\sigma$, first become RSB, while the smaller group $\tau$ still in RS. Notice these partial RSB solution only appear if the two populations are different in size, and that the smaller group is always the one in the RS phase.

We conclude with the following remark. By computing the analogue AT lines before we found that in the limit in which the smaller group shares only an infinitesimal fraction of the total spin mass (i.e., $\lambda_1 \to 1$ or $\lambda_1 \to 0$) still there is a nontrivial critical temperature for the larger group. From the phase diagram we find that in this regime the smaller group is in a RS phase at any temperature, while the larger group can enter in a full-RSB phase below the critical temperature. Would be worth investigating what kind of situations would be possibly described by this regime, and if detailed information on how the groups are related could be obtained at least in this special case.

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