A method for solving boundary value problems of mathematical theory of thick transversely isotropic plates

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Abstract. A method for solving the boundary value problems of a variant of the mathematical theory of thick transversely isotropic plates has been developed. Systems of high-order equilibrium differential equations are reduced by this method to second-order differential equations. The constructed version of the theory makes it possible to solve three-dimensional boundary value problems with any high accuracy. All components of the stress-strain state and boundary conditions are considered functions of three coordinates. They are represented by infinite mathematical series in transverse coordinate with the help of Legendre polynomials. The three-dimensional boundary-value problem is reduced to two-dimensional using the Reissner variational principle. The transverse normal and tangential stresses are displayed in such a way that the boundary conditions on the horizontal faces of the plate are exactly satisfied. Boundary value problems for semifinite transversely isotropic plates of arbitrary thickness with different boundary conditions are considered. Boundary effects are taken into account. The analysis of the stress-strain state is performed.

1. Introduction and literature review

The classical theories of plates and shells, the Timoshenko-Reissner theory [1, 2], their modifications, theories based on certain models of deformation [3-5], and other theories using various hypotheses, as a rule, cannot describe the state of these elements for the most diverse tasks with the necessary accuracy. The main disadvantage of model theories is the inability to refine the results within the same theories, since the accuracy is determined by a set of accepted hypotheses. In mathematical terms, such theories are reduced to systems of differential equations (DE) of relatively low order. At the same time, obtaining analytical solutions of boundary value problems for plates and shells based on three-dimensional elasticity theory is fraught with great mathematical difficulties [6-9]. A variant of mathematical theory (MT) [10-17] is based on the representation of all components of the stress-strain state (SSS) in the form of an infinite mathematical series with three coordinates. It can be different mathematical series: tensor, power, using Legendre polynomials. In all variants of the MT, three-dimensional problems of the theory of elasticity are reduced to two-dimensional by various methods: projection, variational, and others. Depending on the method of obtaining the basic relations, equilibrium equations, boundary conditions, the accuracy and efficiency of each theory are different. From this point of view, a MT variant, started in [17] for skew-symmetric deformation of plates in the first two approximations, and developed for plates and shells of arbitrary constant thickness in [18-22] for arbitrary approximations and transverse loads, gives high accuracy already at low approximations. As the number of terms in the mathematical series increases for the SSS, the order of the systems of equations increases, but also the accuracy of solving boundary-value problems in determining the internal SSS and boundary effects increases. At present, theories of the Tymoshenko-Reissner type and their refinement variants are usually used in Western studies. These theories are, as a rule, partial cases of a variant of the MT [17-22]. They can be obtained as a first approximation of a variant of the MT with some modifications and refinements. An extensive bibliography of Western works is given in...
The development of the theory of calculation of plates and shells can be found, in particular, in [3, 4, 6, 8, 9, 16].

2. Method of solving boundary value problems for plates

The method of solving the boundary value problems of the constructed variant of the theory is as follows: 1. All components of the SSS develop into mathematical series with the help of Legendre polynomials. Boundary conditions and systems of differential equilibrium equations in the components of displacements are obtained from the Reisner variational equation. 2. The initial system of equations is reduced to two defining subsystems by algebraic, differential and operator transformations. One subsystem (homogeneous) describes the vortex boundary effect and is reduced to Helmholtz differential equations. The other subsystem is heterogeneous and describes the internal SSS and the potential boundary effect, which are further separated. This subsystem depends on the new functions, its left parts are the same, and the right ones are structurally the same and contain Helmholtz operators. By the developed method of order reduction the last system is reduced to inhomogeneous differential equations of the second order. 3. Next, we find the general solutions of the defining system of differential equations, and then find the general solutions of the initial systems of equilibrium equations and the components of the SSS by inverse transformations.

3. Variant of mathematical theory of plates of arbitrary thickness

Here are the main provisions and equations of a variant of the MT of plates of arbitrary thickness [18-20, 22] (hereinafter referred to as a variant of the MT). The variant takes into account all SSS components, which are considered to be functions of three variable coordinates \( x, y, z \). The axes \( x, y \) belong to the middle plane, which coincides with the plane of isotropy. The \( z \) axis is perpendicular and directed upward, \( z \in [-h/2, h/2] \), where \( h \) is the plate thickness. The plate is subjected to the downward distributed loads \( q_1(x, y) \) and \( q_2(x, y) \). The \( q_1(x, y) \) load acts on the upper face of the plate and \( q_2(x, y) \) acts on the lower face.

The boundary conditions (for \( z = \pm h/2 \)) are as follows:

\[
\sigma_x(z = \pm h/2) = (\mp q(x, y) - p(x, y))/2; \quad \sigma_{xx}(z = \pm h/2) = \sigma_{yy}(z = \pm h/2) = 0,
\]

where \( p(x, y) \) and \( q(x, y) \) are symmetrical and obliquely symmetrical loads relative to the median plane.

The components of displacements and stresses are represented as infinite mathematical series in transverse coordinate by means of Legendre polynomials.

Components of displacements:

\[
U(x, y, z) = \sum_{k=0}^{\infty} P_k(2z/h)u_k(x, y) \quad (U, V;\ u, v); \quad W(x, y, z) = \sum_{k=1}^{\infty} P_{k-1}(2z/h)w_k(x, y),
\]

where \( P_k \) is Legendre polynomials; \( u_k, v_k, w_k \) are unknown components of displacement.

If in the mathematical series (2) for the components of the tangent displacements we consider only terms with indices \( k = 0, 1, \ldots, N \), where \( N \) is an odd number, then we call this approximation K0-N an approximation; if we consider the terms with the \( k = 1, 3 \) indices, then this is an approximation of K13.

We write down the partial sums of a series of displacements and stresses in the K0-N approximation.

The displacement components according to (2) are as follows:

\[
U^{(N)}(x, y, z) = \sum_{k=0}^{N} P_k(2z/h)u_k(x, y), (U, V; u, v); \quad W^{(N)}(x, y, z) = \sum_{k=1}^{N} P_{k-1}(2z/h)w_k(x, y),
\]

where the index \( (N) \) at the top indicates the approximation number.
The stress components are determined from the Reisner variation equation [23], taking into account (3):

\[ \sigma^{(N)}_{\alpha \beta}(x, y, z) = \sum_{n=0}^{N+1} P_n t_{\alpha \beta n}; \sigma^{(N)}_{\gamma \zeta}(x, y, z) = \sum_{n=0}^{N+1} P_n s_{\gamma \zeta n}; \sigma^{(N)}_{\iota \kappa}(x, y, z) = \sum_{n=0}^{N+1} P_n t_{\iota \kappa n}, \]

(4)

where

\[ t_{\alpha \beta n}(x, y) = \sum_{j=1,3}^{N} (h_{0n}w_{j,\alpha} + l_{0n}u_{\beta}), \quad (n = 0, 2, ..., N + 1); \]

\[ t_{\alpha \beta n}(x, y) = \sum_{j=1,3}^{N} (h_{0n}w_{j,\alpha} + l_{0n}u_{\beta}), \quad (n = 1, 3, ..., N); \]

\[ s_{\gamma \zeta n}(x, y) = \sum_{j=1,3}^{N} (p_{j}w_{j,\gamma} + g_{n\gamma}q_{\gamma}) + g_{n\beta}p_{\beta}, \quad (n = 2, ..., N + 1); \]

\[ s_{\gamma \zeta n}(x, y) = \sum_{j=1,3}^{N} (p_{j}w_{j,\gamma} + g_{n\gamma}q_{\gamma}) + g_{n\beta}p_{\beta}, \quad (n = 1, 3, ..., 2n + 1); \]

(5)

\[ s_{\iota \kappa n}(x, y) = \sum_{j=1,3}^{N} (a_{n\iota}u_{j,\iota} + a_{wj}v_{j,\kappa}) + a_{nwj}w_{j} + a_{nq}p_{q}, \quad (n = 0, 2, ..., N + 1); \]

\[ s_{\iota \kappa n}(x, y) = \sum_{j=1,3}^{N} (a_{n\iota}u_{j,\iota} + a_{wj}v_{j,\kappa}) + a_{nwj}w_{j} + a_{nq}p_{q}, \quad (n = 1, 3, ..., N + 2); \]

\[ t_{\iota \kappa n}(x, y) = G(u_{n,\iota} + v_{n,\kappa}), \quad (n = 0, 1, ..., N). \]

In formulas (5) \( h, l, p, g, a \) with the lower indices depend on the mechanical and geometric parameters (MGP) of the plate. For the approximations K0-3, K0-5, they are given in [19, 20]. The components of the transverse stresses (4), (5) satisfy the boundary conditions (1).

The system of equilibrium equations is obtained on the basis of the Reisner variation principle. In the K0-N approximation, the system looks like:

\[ \sum_{i=0}^{N} (L_{i,j}u_{i} + L_{i,j}v_{j}) + \sum_{j=1}^{N} L_{i,j}w_{j} = L_{pq}(x, y), \quad (i = 1, 2, ..., 3N + 2), \]

(6)

where \( L_{i,j}, L_{i,j}, L_{i,j} \) - differential operators not higher than the 2nd order; \( L_{pq}(x, y) \) - functions that depend on the intensity of the external load \( p(x, y) \) and \( q(x, y) \). The integration constants of the system of DE (6) are determined from the boundary conditions on the lateral surface.

The boundary conditions follow from the Reisner variational equation and are given in [18–20].

4. Problem statement

A semi-infinite transversal isotropic plate of arbitrary constant thickness \((x \in [0; \infty), y \in (-\infty; \infty))\) is considered. The plate has a transverse load, which disappears at \( x \to \infty \):

\[ q(x, y) = q_{0} \exp(-\alpha x) \cos \beta y, \quad (\alpha > 0; \beta > 0; \alpha \neq \beta), \quad q_{0} \text{ - const.}. \]

(7)

On the basis of the MT variant, we investigate the analytically SSS in this plate under skew-symmetric deformation and different boundary conditions at the edge \( x = 0 \). Consider the approximation of K13. In this approximation, a theory variant MT describes with high precision the internal SSS and boundary effects under smooth loads. System (6) is reduced by algebraic and differential transformations to two independent systems of DE: the first – inhomogeneous system of
DE describing the internal SSS and potential boundary effect, the second – homogeneous system of equations of vortex boundary effect [19, 20].

5. Transformation of equations. General solutions

The system of equations of the internal SSS and the potential limit effect is reduced to two defining inhomogeneous differential equations of the eighth order with respect to new functions \( \Phi_1(x, y), \Phi_3(x, y) \) by the operator method.

\[
D_0 D_x D_y \Phi_k(x, y) = a_{ko} D_{ko} q(x, y), \quad (k = 1, 3),
\]

where \( D_0 = \nabla^2, D_i = (\nabla^2 - s_i) \quad (i=1,2), \quad D_{ko} = (\nabla^2 - s_{ko}); \quad a \) and \( s \) with indexes – MGP.

General solution for components of transverse displacements [20]:

\[
w_i(x, y) = \Pi_{3i} \Phi_1(x, y) - \Pi_{1i} \Phi_3(x, y);
\]

\[
w_j(x, y) = -\Pi_{1i} \Phi_1(x, y) + \Pi_{3i} \Phi_3(x, y),
\]

where \( \Pi_{3i}(x, y),..., \Pi_{1i}(x, y) \) – differential operators of the fourth order, \( \Phi_1(x, y) \) and \( \Phi_3(x, y) \) are the general solutions of equations (8), and

\[
\Phi_1 = \Phi_{1s} + \Phi_{1t}; \quad \Phi_3 = \Phi_{3t}.
\]

In Equations (10) \( \Phi_{1s}(x, y) \) and \( \Phi_{1t}(x, y) \) is the general solution of the equation

\[
\nabla^2 \Phi_{1s}(x, y) = 0; \quad (\nabla^2 - s_1)(\nabla^2 - s_2)\Phi_{1t}(x, y) = 0,
\]

\( \Phi_{1s}(x, y), \Phi_{3s}(x, y) \) - partial solutions of inhomogeneous equations (8).

The total solution of \( \Phi_{1t}(x, y) \) equation (11) is equal to the sum of \( \Phi_{11}(x, y) \) and \( \Phi_{12}(x, y) \):

\[
\Phi_{1t}(x, y) = \Phi_{11}(x, y) + \Phi_{12}(x, y),
\]

where \( \Phi_{1i}(x, y) \quad (i = 1, 2) \) is the general solutions of the Helmholtz equations:

\[
(\nabla^2 - s_i)\Phi_{1i}(x, y) = 0.
\]

The second equation (11) with respect to the function \( \Phi_{1t}(x, y) \) describes the potential boundary effect, and the biharmonic equation with respect to the function \( \Phi_{1s}(x, y) \) together with the partial solutions \( \Phi_{1s}(x, y) \) and \( \Phi_{3s}(x, y) \) of equations (8) describe the internal SSS.

The method of reducing a non-homogeneous high-order DE to a low-order inhomogeneous equations [22] has proved that partial solutions \( \Phi_{ki}(x, y) \quad (k=1,3) \) of a non-uniforms DE (8) are represented in the following form:

\[
\Phi_{ki}(x, y) = \frac{a_{ko}}{s_1 s_2} \left( \frac{s_2}{s_1} \left( f_{1s} - f_{0s} \right) + \frac{s_1}{s_2} \left( f_{2s} - f_{0s} \right) + \frac{s_1}{s_2} \left( f_{00s} \right) \right).
\]

In dependencies (14): \( f_{0s}(x, y), f_{is}(x, y), f_{2s}(x, y), f_{00s}(x, y) \) – partial solutions of the following equations, respectively:

\[
D_0 f_{0s}(x, y) = q(x, y) \quad ; \quad D_i f_{is}(x, y) = q(x, y), (i = 1, 2) \quad ; \quad D_0 D_0 f_{00s}(x, y) = q(x, y).
\]
The vortex functions $\psi_1(x, y), \psi_3(x, y)$ are expressed by the operator method through the new function $\psi(x, y)$:

$$ \psi_1(x, y) = (\beta_{333} + \beta_{332} \nabla^2) \psi(x, y); \quad \psi_3(x, y) = -\beta_{13} \psi(x, y). \quad (16) $$

Function $\psi(x, y)$ satisfies DE

$$ (\nabla^2 - r_1)(\nabla^2 - r_2)\psi(x, y) = 0, \quad (17) $$

where $r_1$ is the plate MGP $(r_1 > 0)$.

The general solution of equation (17) is shown in the form

$$ \psi(x, y) = \psi_{10}(x, y) + \psi_{20}(x, y), \quad (18) $$

where $\psi_{10}(x, y)$ is the general solutions of the Helmholtz equations $(\nabla^2 - r)\psi_{10}(x, y) = 0$.

After determining the transverse displacements of (9) and the vortex functions of (16), we find the tangential displacements of $u_k(x, y), v_k(x, y)$:

$$ u_k(x, y) = \lambda_{k1} \phi_{k1} + \lambda_{k2} \phi_{k3} + \lambda_{k3} \phi_{k3,3} + \lambda_{k4} w_1 + \lambda_{k5} w_3 + \lambda_{k6} q_3; $$

$$ v_k(x, y) = \lambda_{k1} \phi_{k1} + \lambda_{k2} \phi_{k3} - \lambda_{k3} \phi_{k3,3} + \lambda_{k4} w_1 + \lambda_{k5} w_3 + \lambda_{k6} q_3, \quad (19) $$

where $\phi_k = \lambda_{k1} \nabla^2 w_1 + \lambda_{k2} w_3 + \lambda_{k3} \nabla^2 w_3 + \lambda_{k4} q_3 \quad (k = 1, 3)$; $\lambda$ with indexes – MGP.

### 6. Solving boundary value problems. General solutions for the components of transverse displacements and vortex functions

Consider the boundary value problem formulated in Section 4 for different boundary conditions on the side surface.

Given (7), (9)–(19) and the disappearance of all functions at infinity, we obtain:

$$ \Phi_{10}(x, y) = (C_1 + C_2 x) \exp(-\beta x) \cos \beta y; $$

$$ \Phi_{11}(x, y) = C_3 \exp(-\beta \omega_1 x) \cos \beta y, (\omega_1 = \sqrt{1 + s_1 / \beta^2}) 1); $$

$$ \Phi_{12}(x, y) = C_4 \exp(-\beta \omega_2 x) \cos \beta y, (\omega_2 = \sqrt{1 + s_2 / \beta^2}) 1); $$

$$ \Phi_{13}(x, y) = (C_5 \exp(-\beta \omega_2 x) + C_6 \exp(-\beta \omega_2 x)) \cos \beta y; $$

$$ \Phi_{10}(x, y) = (C_1 + C_2 x) \exp(-\beta x) + C_5 \exp(-\beta \omega_1 x) + C_2 \exp(-\beta \omega_2 x) \cos \beta y; $$

$$ \Phi_{30}(x, y) = 0; \quad f_{10}(x, y) = A_1 \exp(-\alpha x) \cos \beta y; \quad A_1 = q_0 / (\alpha^2 - \beta^2); $$

$$ f_{11}(x, y) = A_2 \exp(-\alpha x) \cos \beta y; \quad A_2 = g_0 / (\alpha^2 - \beta^2 - s_1); $$

$$ f_{12}(x, y) = A_3 \exp(-\alpha x) \cos \beta y; \quad A_3 = g_0 / (\alpha^2 - \beta^2 - s_2) \quad (20); $$

$$ \Phi_{1k}(x, y) = (C_1 + C_2 x) \exp(-\beta x) + A_k \exp(-\alpha x) + C_5 \exp(-\beta \omega_1 x) + C_2 \exp(-\beta \omega_2 x) \cos \beta y; $$

$$ \Phi_{1k}(x, y) = A_{k1} \exp(-\alpha x) \cos \beta y, \quad (k = 1, 3); $$

$$ w_{1k}(x, y) = (C_1 b_1 + C_2 b_2 x) \exp(-\beta x) + c_1 \exp(-\alpha x) \cos \beta y + C_5 \exp(-\beta \omega_1 x) + C_2 \exp(-\beta \omega_2 x) \cos \beta y; $$

$$ w_{1k}(x, y) = (C_1 d_1 \exp(-\beta x) + c_1 \exp(-\alpha x)) \cos \beta y. $$
\text{where here and further } C \text{ with indices – constant integrations,}
\begin{align*}
A_{k\alpha} &= \frac{a_{k\alpha}}{s_1 s_2} (A_1 - A_0) + \frac{s_1}{s_2 (s_1 - s_2)} (A_2 - A_0) + A_3 (\alpha^2 - \beta^2 - s_{\alpha \theta}) ; \\
b_1 &= \mu(s_0) ; b_2(x) = \mu(s_3) ; c_{1r} = A_{1r} b_{1r} + A_{3r} b_{3r} ; b_5 = \mu(s_4) a_5^2 + \mu(s_2) a_5 + \mu(s_3) ; \\
b_2 &= \mu(s_4) a_7^2 + \mu(s_3) ; d_2 = -\mu(s_2) a_7^2 ; c_{3r} = A_{4r} d_{3r} + A_{6r} d_{6r} ; d_5 = -\mu(s_4) a_5^2 + \mu(s_2) a_5 ; \\
d_7 &= -\mu(s_4) a_7^2 + \mu(s_3) ; d_{7r} = -\mu(s_2) a_7^2 ; d_{3r} = \mu(s_3) a_7^2 ; \\
b_h &= \mu(s_4) a_h^2 + \mu(s_3) ; b_{hr} = -\mu(s_2) a_h^2 + \mu(s_2) a_h ; a_2 = -2\beta ; a_h = \beta^2 (\alpha^2 - 1) ; \\
a_r = \beta^2 (\alpha^2 - 1) ; a_{hr} = \alpha^2 - \beta^2 .
\end{align*}

Vortex functions \( \psi_1(x, y) \) and \( \psi_3(x, y) \) are determined from the following dependences:
\begin{align*}
\psi_1(x, y) &= (C_0 k_1 \exp(-\beta \omega_3 x) + C_1 k_2 \exp(-\beta \omega_4 x)) \cos \beta y ; \\
\psi_3(x, y) &= (C_0 k_3 \exp(-\beta \omega_3 x) + C_1 k_4 \exp(-\beta \omega_4 x)) \sin \beta y ,
\end{align*}
where \( k_1 = \beta_32 \beta + \beta_33 ; k_2 = \beta_32 \beta + \beta_33 ; k_3 = -\beta_33 .
\section{Stresses and tangential displacements}
Given (4), (5), (19)–(21), we find the stresses and tangential displacements:
\begin{align*}
u_k(x, y) &= ((C_0 a_{1k\beta} \exp(-\beta \omega_3 x) + C_1 a_{2k\beta} \exp(-\beta \omega_4 x)) \cos \beta y + (C_0 a_{3k\beta} (-\beta \omega_3 x) + C_1 a_{4k\beta} (-\beta \omega_4 x)) \sin \beta y \\
&+ (C_0 a_{5k\beta} (-\beta \omega_3 x) + C_1 a_{6k\beta} (-\beta \omega_4 x)) \cos \beta y) ,
\end{align*}
where
\begin{align*}
a_{1k\beta} &= -b_{k1} \lambda_{k1} ; a_{2k\beta} = \lambda_{k2} ; (\mu(s_3) - \beta b_2(x)) - \beta (\lambda_{k3} t_2 + \lambda_{k4} s_2) - d_2 \lambda_{k5} ; \\
a_{4k\beta} &= \lambda_{k4} ; (t_0 \lambda_{k5} + \lambda_{k6} q_0) + \lambda_{k7} (t_3 + \lambda_{k8} q_3) + (\lambda_{k9} c_r + \lambda_{k10} c_3) + \lambda_{k11} q_0 ; \\
a_{3k\beta} &= \lambda_{k4} ; t_1 + \lambda_{k5} t_2 + \lambda_{k6} t_4 + \lambda_{k7} d_5 ; a_{1k\beta} = \lambda_{k1} \lambda_{k1} k_1 + \lambda_{k2} k_2 + \lambda_{k3} k_3 ; b_{1k\beta} = a_{2k\beta} - \lambda_{k1} \mu(s_3) ; \\
b_{2k\beta} &= \lambda_{k4} ; (t_0 \lambda_{k5} + \lambda_{k6} q_0) + \lambda_{k7} (t_3 + \lambda_{k8} q_3) + (\lambda_{k9} c_r + \lambda_{k10} c_3) + \lambda_{k11} q_0 ; \\
t_{0k} &= \lambda_{k1} a_1 c_i + \lambda_{k2} c_{2r} + \lambda_{k3} \alpha^2 - \beta^2 ; t_{1k} = \lambda_{k2} d_2 - 2\lambda_{k1} \mu(s_3) ; \\
t_{5k} &= \lambda_{k1} d_3 + \lambda_{k2} d_5 + \lambda_{k3} \alpha \beta \alpha - \beta^2 ; t_{1k} = \lambda_{k1} d_1 + \lambda_{k2} d_4 + \lambda_{k3} a_4 .
\end{align*}
Stress components;
\[ \sigma_{x}^{(3)}(x, y, z) = \sum_{n=0,2}^{4} P_{n} t_{x+n} + \sigma_{y}^{(3)}(x, y, z) = \sum_{n=0,2}^{4} P_{n} t_{y+n} + \sigma_{z}^{(3)}(x, y, z) = \sum_{n=0,2}^{4} P_{n} t_{z+n}, \]

where \( n \) is an integer.

The components in stresses \( \sigma_{x}(x, y, z), \sigma_{y}(x, y, z) \) have the form:

\[ s_{x} = s_{x}(x, y) = \sum_{k=1}^{3} ((C_{1}c_{1k} + C_{2}s_{2k}) \exp(-\beta x) + c_{aikn} \exp(-\alpha x) \]

\[ + c_{bikn} \exp(-\beta \omega_{x} x) + C_{n} c_{7kn} \exp(-\beta \omega_{x} x)) \]

\[ + c_{y} y_{kn} \exp(-\beta \omega_{y} x) = C_{y} y_{kn} \exp(-\beta \omega_{y} x) + C_{n} y_{7kn} \exp(-\beta \omega_{y} x) \cos \beta y, \quad (n = 1, 3, 5), \]

where

\[ c_{1k} = a_{1k} \beta^2 (a_{n_k} - a_{m_k}) ; c_{2k} = a_{n_k} (a_{1k} \beta - \lambda_{k} \nu_{k} \mu_{k}) \]

\[ c_{3k} = a_{aik} \alpha^2 - a_{aik} \beta a_{k} \beta^2 + a_{aik} \beta a_{k} \beta^2 ; \]

\[ c_{5k} = a_{k} \beta^2 (a_{n_k} \alpha^2 - a_{m_k}) + a_{n_k} d_{k} \beta^2 \]

\[ c_{7k} = a_{k} \beta^2 (a_{n_k} \beta^2 - a_{m_k}) + a_{n_k} d_{k} \beta^2 \]

\[ c_{y} = a_{k} \beta^2 (a_{n_k} \beta^2 - a_{m_k}) + a_{n_k} d_{k} \beta^2 \]

In order to obtain \( S_{y}(x, y) \), it is necessary to replace \( S_{x}(x, y) \) with \( S_{y}(x, y) \) in formulas (24) and \( a_{n_k} \rightarrow a_{n_k}; a_{m_k} \rightarrow a_{m_k} \) in formulas (25).

Formulas for components in other stresses:

\[ t_{x} = t_{x}(x, y) = ((C_{1}d_{1k} + C_{2}d_{2k}) \exp(-\beta x) + d_{aikn} \exp(-\alpha x) \]

\[ + C_{n} d_{7kn} \exp(-\beta \omega_{x} x) + C_{y} d_{kn} \exp(-\beta \omega_{x} x)) \cos \beta y, \quad (n = 0, 2, 4), \]

\[ (t_{x} \rightarrow t_{x} ; d_{x} \rightarrow d_{y}, d_{x} \rightarrow d_{y}, \cos \beta y \rightarrow \sin \beta y) ; \]

\[ s_{y} = s_{y}(x, y) = ((C_{1}d_{1k} + C_{2}d_{2k}) \exp(-\beta x) + d_{aikn} \exp(-\alpha x) \]

\[ + C_{n} d_{7kn} \exp(-\beta \omega_{x} x) + C_{y} d_{kn} \exp(-\beta \omega_{x} x) + C_{y} d_{kn} \exp(-\beta \omega_{x} x) \cos \beta y, \quad (n = 1, 3, 5), \]

where \( c \) and \( d \) with the indices depend on \( \alpha, \beta, n \) and the plate MGP.

Thus, the components in displacements and stresses are as follows:

\[ w_{1}(x, y) = f_{w_{1}}(x, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}) \cos \beta y; \]

\[ w_{2}(x, y) = f_{w_{2}}(x, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}) \cos \beta y; \]

\[ u_{n}(x, y) = f_{u_{n}}(x, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}, C_{11}, t_{x+n}) \sin \beta y; \quad (n = 1, 3); \]

\[ v_{n}(x, y) = f_{v_{n}}(x, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}, C_{11}, t_{x+n}) \sin \beta y; \quad (n = 1, 3); \]

\[ s_{x}(x, y) = f_{s_{x}}(x, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}, C_{11}, d_{x+n}) \cos \beta y; \]

\[ s_{y}(x, y) = f_{s_{y}}(x, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}, C_{11}, d_{x+n}) \cos \beta y; \quad (n = 1, 3, 5); \]

\[ t_{y}(x, y) = f_{t_{y}}(x, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}, C_{11}, d_{x+n}) \sin \beta y; \quad (n = 1, 3); \]

\[ t_{y}(x, y) = f_{t_{y}}(x, C_{1}, C_{2}, C_{3}, C_{4}, C_{5}, C_{11}, d_{x+n}) \sin \beta y; \quad (n = 0, 2, 4); \]
\[ s_{zn}(x, y) = f_{zn}(x, C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11}, d_{zn}) \cos \beta y, \quad (n = 1, 3, 5). \]

The coefficients \( c, a, b, d \) with the indexes depend on \( \alpha, \beta, n \) and the MGP of the plate.

8. Boundary problems and their analysis

8.1. Boundary problems. Let us consider some boundary value problems for a half-infinite plate under the action of transverse loading (7).

**Boundary problem 1.** Consider the following boundary conditions for \( x = 0 \).

\[ U(x = 0, y, z) = 0, \quad V(x = 0, y, z) = 0, \quad W(x = 0, y, z) = 0. \]

These conditions, given (3), are as follows:

\[ u_n(x = 0, y) = v_n(x = 0, y) = w_n(x = 0, y) = 0, \quad (n = 1, 3). \quad (28) \]

Based on (20), (22), (27), (28) we have six linear algebraic equations with respect to constant integrations \( C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11} \):

\[ u_n(x = 0, y) = f_{un}(x = 0, C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11}, a_{u}) = 0; \]
\[ v_n(x = 0, y) = f_{vn}(x = 0, C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11}, b_{v}) = 0; \]
\[ w_n(x = 0, y) = f_{wn}(x = 0, C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11}, c_{w}) = 0, \quad (n = 1, 3). \]

The right-hands sides of these equations depend on \( c_{1}, c_{3}, a_{u}, b_{v}, b_{w} \) respectively.

After determining \( C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11} \), all displacements and stresses according to (3), (20), (22–26) are found.

**Boundary problem 2.** Let us now consider the following boundary conditions:

\[ U(x = 0, y, z) = (A_{1}^{0} P_{1}(2z / h) + A_{3}^{0} P_{3}(2z / h)) \cos \beta y; \]
\[ V(x = 0, y, z) = (B_{1}^{0} P_{1}(2z / h) + B_{3}^{0} P_{3}(2z / h)) \sin \beta y; \]
\[ W(x = 0, y, z) = (C_{1}^{0} P_{1}(2z / h) + C_{3}^{0} P_{3}(2z / h)) \cos \beta y, \quad \text{for} \ A_{1}^{0}, \ldots, C_{3}^{0} \text{—const}. \]

We also obtain a system of six linear algebraic equations with respect to \( C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11} \):

\[ u_n(x = 0, y) = f_{un}(x = 0, C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11}, a_{u}) = A_{u}^{0}; \]
\[ v_n(x = 0, y) = f_{vn}(x = 0, C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11}, b_{v}) = B_{v}^{0}; \]
\[ w_n(x = 0, y) = f_{wn}(x = 0, C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11}, c_{w}) = C_{w}^{0}, \quad (n = 1, 3). \]

**Boundary problem 3.** Consider the following boundary conditions:

\[ X_{x}(x = 0, y, z) = (A_{1}^{x} P_{1}(2z / h) + A_{3}^{x} P_{3}(2z / h)) \cos \beta y, \quad \text{for} \ A_{1}^{x}, A_{3}^{x} \text{—const}; \]
\[ V(x = 0, y, z) = 0, \quad W(x = 0, y, z) = 0. \]

Continuous integrations will be determined from the following equations:

\[ s_{xn}(x = 0, y) = f_{xn}(x = 0, C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11}, a_{xn}) = x_{xn} = A_{xn}; \]
\[ v_{n}(x = 0, y) = f_{vn}(x = 0, C_{1}, C_{2}, C_{5}, C_{7}, C_{9}, C_{11}, b_{vn}) = 0; \]
Boundary problem 4.

\[ X_n(x, y, z) = (A_1 P_1(2\pi / h) + A_3 P_3(2\pi / h)) \cos \beta y; \]
\[ Y_n(x, y, z) = (A_1 P_1(2\pi / h) + A_3 P_3(2\pi / h)) \sin \beta y; \]
\[ W(x, y, z) = 0, (A_1, A_3 - \text{const}). \]

The system of linear algebraic equations for determining the integration constants will look like:

\[ s_{x_n}(x, y) = f_{x_n}(x, C_1, C_2, C_5, C_9, C_{11}, a_{m_0}) = s_{x_n} \equiv A_{m_1}; \]
\[ t_{y_n}(x, y) = f_{y_n}(x, C_1, C_2, C_5, C_9, C_{11}, c_{m_0}) = t_{y_n} \equiv A_{m_2}; \]
\[ w_{n}(x, y) = f_{w_n}(x, C_1, C_2, C_5, C_9, c_{m_n}) = 0, (n = 1, 3). \]

The sequence of determining stresses and displacements in boundary value problems 2, 3 and 4 is similar to boundary value problem 1.

8.2. Analysis of boundary value problems.

In the considered boundary value problems for a semi-infinite plate subjected to loading (7), we have six conditions for determining six constant integrations of systems of equations.

The internal SSS is described by terms that do not contain integration constants, together with terms containing \( C_1, C_2 \) constants. Terms with \( C_1, C_5 \) constants determine the potential boundary effect, and terms with constants \( C_9, C_{11} \)-vortex boundary effect.

In [18], boundary effects were investigated based on a variant MT. The vortex boundary effect was characterized by decaying exponential solutions containing functions \( \exp(-\lambda x / h) \) and \( \exp(-\lambda y / h) \), (\( \lambda = \sqrt{\rho_1}, \lambda_2 = \sqrt{\rho_2} \)). Parameters \( \lambda_1 \) and \( \lambda_2 \) are indicators of the variability of the eddy effect of the first and second boundary layers. The potential boundary effect was characterized by the disappearing solutions of the exponential form (\( \lambda_3 = \sqrt{s_1}, \lambda_4 = \sqrt{s_2} \) variability indicators) and the disappearing solutions of the oscillating form (\( \lambda_0 \) variability indicator).

The table 1 shows the variability indicators for isotropic (\( V=0.3 \)) and transtropic plates (\( \nu = \nu' = 0.3; \ E' = E \)); \( \lambda_0 \) is variability indicator of the edge effect according to Reissner theory. The depth of penetration of eddy boundary layers is greater than the potential boundary layers for \( G'/G < 1 \). At \( G'/G = 0.7708 \) for the potential boundary effect \( \lambda_3 = \lambda_4 = \lambda_0 = 7.9478 \). At \( G'/G < 0.7708 \), the attenuation is exponential and at \( G'/G > 0.7708 \) it is oscillatory. According to the exact theory of isotropic Lurie plates [24] \( \lambda_1 = 3.142; \lambda_2 = 9.423; \lambda_3 = 7.498; \lambda_4 = 13.90 \).

**Table 1.** Variability indicators of the vortex and potential boundary layers in the K13 approximation.

| \( G'/G \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( \lambda_0 \) |
|-----------|----------------|----------------|----------------|----------------|----------------|
| 0.01      | 0.3142         | 1.011          | 0.3347         | 0.2876         | 0.5622         | 112.4          |
| 0.1       | 0.9935         | 3.196          | 1.058          | 0.9094         | 1.805          | 35.00          |
| 1.0       | 3.142          | 10.11          | 3.347          | 2.876          | -              | -              |
| 10.0      | 9.935          | 31.96          | 10.58          | 9.094          | -              | -              |
| 100.0     | 31.42          | 101.1          | 33.47          | 28.76          | -              | -              |

9
A variant of the theory describes with high accuracy the vortex edge effects in the K13 approximation.

The potential boundary effect needs to be clarified. Other theories describe boundary effects with less accuracy.

A variant of mathematical theory and the developed method can be generalized to solve boundary value problems of dynamics [26].

9. Conclusions

A method for solving the boundary value problems of a variant of the mathematical theory of thick transversely isotropic plates under arbitrary transverse loads has been developed. The proposed method makes it possible to significantly simplify the solution of systems of differential equations of high-order equilibrium by reducing them to second-order differential equations.

The method is used to solve the boundary value problem for a semi-infinite transverse-isotropic plate of arbitrary thickness at a transverse load in the K13 approximation. The general solutions of the problem for the components of displacements and stresses are deduced and their research is carried out.

Four different classes of boundary conditions are considered, covering the boundary-value problem in displacements and the mixed problem. Boundary conditions and algebraic equations for determination of constant integrations are obtained. The terms that describe the internal SSS and boundary effects are highlighted. All components of the state in the variable $x$, which is perpendicular to the edge of the plate, have a decaying exponential character.

The transverse displacements depend on the internal SSS and the potential boundary effect. Other components depend on internal SSS, potential and vortex boundary effects. The internal SSS on the variable $x$ depends exponentially on $\alpha$ and $\beta$. Potential boundary effect is characterized by $\beta \omega_1$ and $\beta \omega_2$, and vortex by $\beta \omega_3$ and $\beta \omega_4$.

As parameter $\beta$ increases, the permeation of the vortex and potential boundary layers for $G' / G < 1$ decreases, with the decrease of $\beta$, the permeability of the vortex and potential boundary layers increases, with $\beta \to 0$ the variability indicators go to the values given in the table.

A high-precision theory variant describes the internal SSS (at smooth loads) and eddy boundary effects in the K13 approximation [18-20, 22]. The potential marginal effect needs clarification. Similarly to the K13 approximation, analytic solutions can be found in the K0-3, K135, K0-5, and others approximations, using the basic equations previously obtained [19] for these approximations. Quantitative results are determined on the basis of analytical solutions.

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