Mass Insertions vs. Mass Eigenstates calculations in Flavour Physics

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Outline

1 Introduction

2 MI expansion of scalar amplitudes

3 MI expansion of fermionic amplitudes

4 Automatized MI expansion: MassToMI Mathematica package

5 Conclusions

Based on: Dedes, Paraskevas, JR, Suxho, Tamvakis, JHEP 1506 (2015) 151 and JR, Comput.Phys.Commun. 201 (2016) 144-158.
Introduction

QFT models defined by specifying the Lagrangian - choice of “field basis” not unique!

Usual approach to construct QFT model:

- assume some symmetries, local or global
- choose particle content and their quantum numbers
- define interactions - add to Lagrangian all (or subset) of terms allowed by the symmetries of the theory and extra requirements - renormalizability etc.

Result: Lagrangian in the “interaction basis”
Fields in “interaction basis” Lagrangian may not represent the physical degrees of freedom!

Further steps may be required:

■ spontaneous symmetry breaking
■ rediagonalization of mass matrices
■ ...

Result: Lagrangian in the “mass eigenstates” basis

Advantage: redefined fields represent physical degrees of freedom.
Disadvantage: couplings in “mass eigenstates” basis are usually complicated functions of the initial interaction basis parameters.
Toy example: self-energy in model with real scalar field $\eta$ and complex scalar multiplet $\Phi_I$:

$$L_{\text{int}} = (\partial^\mu \Phi_I^\dagger) (\partial_\mu \Phi_I) - M^2_{IJ} \Phi_I^\dagger \Phi_J + \frac{1}{2} (\partial^\mu \eta) (\partial_\mu \eta) - \frac{1}{2} m^2_\eta \eta^2 - Y_{IJ} \eta \Phi_I^\dagger \Phi_J$$

Transition to mass eigenstates basis: $\Phi_I = U_{Ii} \phi_i$

$$U^\dagger M^2 U = m^2 = \text{diag}(m^2_1, \ldots, m^2_N)$$

$$y_{ij} = U^\dagger_{iI}(M^2) Y_{IJ} U_{Jj}(M^2)$$

Mass eigenstates basis: $\Sigma_{ji}(p) = -\frac{1}{(4\pi)^2} y_{j\ell} B_0(p; m^2_\ell, m^2_\eta) y_{\ell i}$

- simple diagram and compact expression
- physical states on external legs
- but: result in terms of $y = y(Y, M), m = m(M)$
Interaction basis: thick dots represent “mass insertions” - off diagonal elements of $M_{IJ}^2$.

- infinite series of diagrams, complicated calculation and expression
- unphysical states on the external legs
- but: result in terms of the initial “interaction basis” parameters $Y, M$

\[
\hat{\Sigma}_{JI}(p) = -\frac{1}{(4\pi^2)} Y_{JK} Y_{LI} \left( \delta_{KL} B_0(p; M_K^2, m_\eta^2) + \hat{M}_{KL}^2 C_0(0, p; M_K^2, M_L^2, m_\eta^2) + \hat{M}_{KN}^2 \hat{M}_{NL}^2 D_0(0, 0, p; M_K^2, M_N^2, M_L^2, m_\eta^2) + \ldots \right),
\]
Transition to physical states $U_{jJ}^\dagger \hat{\Sigma}_{JI}(p) U_{II} = \Sigma_{ji}(p)$:

$$U_{K\ell} \ B_0(p, m_{\ell}^2, m_{\eta}^2) \ U_{\ell L}^\dagger = \delta_{KL} B_0(p; M_K^2, m_{\eta}^2)$$

$$+ \ \hat{M}_{KL}^2 C_0(0, p; M_K^2, M_L^2, m_{\eta}^2)$$

$$+ \ \hat{M}_{KN}^2 \hat{M}_{NL}^2 D_0(0, 0, p; M_K^2, M_N^2, M_L^2, m_{\eta}^2) + \ldots$$

Very particular relation!

- Hold for 1-loop functions only?
- Can it be generalized? How? → **Flavor Expansion Theorem**
Idea: Typical term in QFT mass-eigenstates amplitude: 
\( U_{II} f(m_i^2) U_{JI}^* \) can be expressed as an element of function of the matrix:

\[
U_{II} f(m_i^2) U_{JI}^* = \left( c_0 + c_1 m_i^2 + c_2 m_i^4 + \ldots \right) U_{JI}^* = \left( c_0 + c_1 M^2 + c_2 M^4 + \ldots \right)_{IJ} = f(M^2)_{IJ}
\]

Disadvantage: each power of mass insertion appears in infinite number of terms of Taylor series!

\[
(M^2)^n_{IJ} \supset M^2_{II} M^2_{II} \ldots M^2_{IJ}
\]

Can we derive another series (not Taylor) for \( f(M^2) \) in powers of the off-diagonal elements of \( M^2 \) only?

Answer - yes, on purely algebraic ground!
Flavor Expansion Theorem

**Definition (Divided differences)**

Divided differences are defined recursively as

\[
\begin{align*}
    f^{[0]}(x) & \equiv f(x) \\
    f^{[1]}(x_0, x_1) & \equiv \frac{f(x_0) - f(x_1)}{x_0 - x_1} \\
    \ldots & \\
    f^{[k+1]}(x_0, \ldots, x_k, x_{k+1}) & \equiv \frac{f^{[k]}(x_0, \ldots, x_{k-1}, x_k) - f^{[k]}(x_0, \ldots, x_{k-1}, x_{k+1})}{x_k - x_{k+1}} \\
\end{align*}
\]

**Properties:**

- symmetric functions of all arguments
- smooth degeneracy limit

\[
\lim_{\{x_0, \ldots, x_m\} \to \{\xi, \ldots, \xi\}} f^{[k]}(x_0, \ldots, x_k) = \frac{1}{m!} \frac{\partial^m}{\partial \xi^m} f^{[k-m]}(\xi, x_{m+1} \ldots, x_k),
\]

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Theorem (Flavor Expansion Theorem or “FET”)

Let’s decompose and Hermitian matrix $\mathbf{A}$ as a sum of diagonal and non-diagonal part $\mathbf{A} = \mathbf{A}_0 + \hat{\mathbf{A}}$:

$$A^I_0 \equiv A_{II}, \quad \hat{A}_{IJ} \equiv A_{IJ}, \quad \hat{A}_{II} = 0, \quad (I, J = 1, \ldots, n).$$

Then, matrix element $f(\mathbf{A})_{IJ}$ is given by (no sum over $I, J$):

$$f(\mathbf{A})_{IJ} = \delta_{IJ} f(\mathbf{A}_0^I) + f^{[1]}(\mathbf{A}_0^I, \mathbf{A}_0^J) \hat{A}_{IJ}$$

$$+ \sum_{K_1} f^{[2]}(\mathbf{A}_0^I, \mathbf{A}_0^J, \mathbf{A}_0^{K_1}) \hat{A}_{IK_1} \hat{A}_{K_1J}$$

$$+ \sum_{K_1, K_2} f^{[3]}(\mathbf{A}_0^I, \mathbf{A}_0^J, \mathbf{A}_0^{K_1}, \mathbf{A}_0^{K_2}) \hat{A}_{IK_1} \hat{A}_{K_1K_2} \hat{A}_{K_2J} + \ldots$$

Series coefficients: divided differences of $f(\mathbf{A}_0)$

Expansion parameters: non-diagonal elements of $\hat{\mathbf{A}}$. 

Mass Insertions vs. Mass Eigenstates calculations in Flavour Physics
holds for any analytic function, not just loop functions - as long as the RHS is convergent!

formal proof: rather technical → see arXiv:1504.00960

bonus features:
  ▶ degenerate eigenvalues and/or diagonal matrix elements treated uniformly due to smooth degeneracy limit of the divided differences
  ▶ natural relation to Passarino-Veltman loop functions

Common application of FET: for many processes leading order terms cancel and only higher ones are left - diagrammatic MI calculation tedious and error prone.
FET expansion of PV functions

Any 1-loop amplitude can be expressed in terms of Passarino-Veltman functions of the order $n$:

$$\frac{i}{(4\pi)^2} PV_{n}^{\mu_1 \cdots \mu_l}(p_1, \ldots, p_{n-1}; m_1^2, \ldots, m_n^2) =$$

$$\int \frac{d^4 k}{(2\pi)^4} \frac{k^{\mu_1} \cdots k^{\mu_l}}{(k^2 - m_1^2) \prod_{j=2}^{n} ((k + p_1 + \cdots + p_{j-1})^2 - m_j^2)}$$

Useful recursive relation: divided difference of 1-loop functions of the order $n$ is a 1-loop function of the order $n + 1$!

$$PV_{n}^{X}(\ldots p_{j-1} \ldots; \ldots m_j^2 \ldots) - PV_{n}^{X}(\ldots p_{j-1} \ldots; \ldots m_j'^2 \ldots)$$

$$\frac{m_j^2 - m_j'^2}{m_j^2 - m_j'^2}$$

$$= PV_{n+1}^{X}(\ldots p_{j-1}, 0 \ldots; \ldots m_j^2, m_j'^2 \ldots)$$
FET formula for PV functions:

\[
\left[ PV^{(n)}(\ldots, M^2, \ldots) \right]_{IJ} = \delta_{IJ} PV^{(n)}(\ldots, (M^2)_{II}, \ldots) \\
+ PV^{(n+1)}(\ldots, (M^2)_{II}, (M^2)_{JJ}, \ldots) \hat{M}^2_{IJ} \\
+ \sum_K PV^{(n+2)}(\ldots, (M^2)_{II}, (M^2)_{JJ}, (M^2)_{KK}, \ldots) \hat{M}^2_{IK} \hat{M}^2_{KJ} \\
+ \ldots
\]

- immediately reproduces the relation discussed in the “toy model”
- applies to expansion with degenerate diagonal elements - PV functions not singular in this limit, no need to calculate derivatives
- explicit condition for FET series convergence
Convergence of FET expansion of PV functions

Most useful case for flavor physics - vanishing external momenta.

Define matrix of moduli of “dimensionless mass insertions”

\[
\Delta_{IJ} = \frac{|\hat{M}^2_{IJ}|}{\sqrt{(M_0)_{II}^2 (M_0)_{JJ}^2}} \quad \Delta_{II} = 0
\]

FET expansion of PV functions at vanishing external momenta converges if moduli of \( \Delta \) matrix eigenvalues are smaller then 1 (proof in arXiv:1504.00960)

Simpler but weaker sufficient (but not necessary) condition: norm of any row or column of \( \Delta \) must be smaller then 1.
FET expansion for fermionic amplitudes

FET applies to the function of Hermitian matrices - works for bosons (scalar or vector particles).

General mass term for chiral fermions:

$$-\bar{\Psi} \left( M_P L + M^\dagger P_R \right) \Psi$$

where $M$ is **general complex** mass matrix diagonalized by 2 **unitary rotations** $U, V$:

$$\Psi_{LA} = U_{Ai} \psi_{Li}, \quad \Psi_{RA} = V_{Ai} \psi_{Ri}$$

$$V^\dagger M U = m = \text{diag}(m_1, \ldots, m_N)$$

Can FET expansion be used for fermionic amplitudes?
Fermion propagator can be decomposed as

\[
\frac{i}{k - MP_L - M^\dagger P_R} = (M^\dagger P_L + \frac{i}{k} P_L) \frac{i}{k^2 - MM^\dagger} + (MP_R + \frac{i}{k} P_R) \frac{i}{k^2 - M^\dagger M}
\]

- Loop functions depend always on Hermitian matrices $MM^\dagger$ or $M^\dagger M$.
- Only some combinations of mixing matrices can appear in fermionic amplitudes:

\[
U_{Bi} f(m_i^2) U^*_{Ai} = f(M^\dagger M)_{BA}
\]
\[
V_{Bi} f(m_i^2) V^*_{Ai} = f(MM^\dagger)_{BA}
\]
\[
U_{Bi} m_i f(m_i^2) V^*_{Ai} = M^\dagger_{BC} f(MM^\dagger)_{CA} = f(M^\dagger M)_{BC} M^\dagger_{CA}
\]
\[
V_{Bi} m_i f(m_i^2) U^*_{Ai} = M_{BC} f(M^\dagger M)_{CA} = f(MM^\dagger)_{BC} M_{CA}
\]

- All can be expressed using FET formula, works for fermions as well!

Applies also to Majorana fermions – then $M = M^T$ and $U = V^*$

Mass Insertions vs. Mass Eigenstates calculations in Flavour Physics
Great advantage of FET technique - can be easily automatized!

Prescription:

- calculate amplitude in the mass eigenstates basis - less diagrams, more compact expressions, better suited for numerical computations
- expand result using FET implemented in MassToMI package - recover direct analytic dependence on “interaction basis” parameters (better suited for understanding of various effects)

Avoids tedious and error-prone direct calculations of diagrams with mass insertions.
Example: $H^- \rightarrow d^I \bar{c}^J$ decay in the MSSM, the triangle diagram with chargino (Dirac) $C_n$, neutralino (Majorana) $N_j$ and down squark $D_i$ circulating in the loop.

![Triangle Diagram]

Typical term ($Z, O, U, V$ are scalar, Majorana and Dirac fermion mixing matrices):

$$A \supset Z_D^I Z_D^{J*} O_{N}^{K_j} O_{N}^{L_j*} V_{C}^{M_n} U_{C}^{N_n} m_{C_n} c_0(p, q, m_{C_n}^2, m_{D_i}^2, m_{N_j}^2)$$

(summation convention assumed for the repeating indices)
MassToMI input syntax rules:

\[ Z^I_D \rightarrow \text{SMIX}[D,I,i] \quad \text{scalar} \]
\[ O^K_N \rightarrow \text{NMIX}[N,K,j] \quad \text{Majorana fermion} \]
\[ V^M_{C*} \rightarrow \text{Conjugate}[\text{FMIXL}[C,M,n]] \quad \text{left Dirac fermion} \]
\[ U^M_C \rightarrow \text{FMIXR}[C,M,n] \quad \text{right Dirac fermion} \]
\[ m_{C,n} \rightarrow \text{MASS}[C,n] \quad \text{physical (fermion) mass} \]

Loop integral:

\[ c_0(p,q,m^2_{C,n},m^2_{D,i},m^2_{N,j}) \rightarrow \text{LOOP}[c0,\{\{C,n\},\{D,i\},\{N,j\}\},\{p,q\}] \]
MassToMI expression for amplitude:

\[ A = Z_D^{I_i} Z_D^{J_j} O_N^{K_L} O_N^{L_J} V_C^{M_n} U_C^{N_m} m_{C_n} c_0(p,q,m_{C_n}^2, m_{D_i}^2, m_{N_J}^2) \rightarrow \]

\[ A = \text{SMIX}[D,I,i] \text{Conjugate}[\text{SMIX}[D,J,i]] \]
\[ \text{NMIX}[N,K,j] \text{Conjugate}[\text{NMIX}[N,L,j]] \]
\[ \text{Conjugate}[	ext{FMIXL}[C,M,n]] \text{FMIXR}[C,N,n] \text{MASS}[C,n] \]
\[ \text{LOOP}[c_0,\{\{C,n\},\{D,i\},\{N,j\}\},\{p,q\}] ; \]

Control variables:

- **FetScalarList**=\{\{D,2\}\}. Mixing matrices for scalar \( D \) are expanded up to 2nd order in mass insertions.

- **FetFermionList**=\{\{C,1,MHM\},\{O,1,MMH\}\}. Mixing matrices for fermions \( C, O \) are expanded to 1st order in MI. Parameters \( \text{MMH,MMH} \) - final result is expressed in terms of \( M^\dagger M \) or \( MM^\dagger \).

- **FetMaxOrder**=2. Only mass insertion products of the total order \( \text{FetMaxOrder} \) or lower are kept in the final result.
Function \texttt{FetExpand[ A ]} automatically performs the MI expansion to required order!

MassToMI tested on realistic case set of LFV processes in the MSSM (to be published):

- Initial mass eigenstates expression for amplitude: \textit{few lines}.
- Mathematica code/execution time: \textit{300 lines/up to few hours} on standard PC.
- \textbf{Intermediate expressions:} \textit{\sim 50000} MI terms! Equivalent to tens/hundreds of diagrams in the interaction basis.
- Final expanded interaction basis expressions after simplifications: \textit{few lines} for leading terms.
Conclusions

1. Mass Insertion expansion can be done starting from mass eigenstates amplitude, without direct diagrammatic calculations.

2. “Flavor Expansion Theorem” applies to all types of amplitudes: scalar, vector and fermionic.

3. Natural relation of FET to recursive properties of the 1-loop functions (including explicit convergence criterion for the MI series).

4. FET technique automatized and implemented in MassToMI Mathematica package.

MassToMI can be downloaded from www.fuw.edu.pl/masstomi