THE SYMBOLIC GENERIC INITIAL SYSTEM OF POINTS
ON AN IRREDUCIBLE CONIC

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Abstract. In this note we study the limiting behaviour of the symbolic
generic initial system \(\{\text{gin}(I(m))\}\) of an ideal \(I \subseteq K[x, y, z]\) corre-
sonding to an arrangement of \(r\) points of \(\mathbb{P}^2\) lying on an irreducible conic. In
particular, we show that the limiting shape of this system is the subset
of \(\mathbb{R}_+^2\) such consisting of all points above the line through \((\min\{\frac{r}{2}, 2\}, 0)\)
and \((0, \max\{\frac{r}{2}, 2\})\).

The general research trend looking at the asymptotic behaviour of col-
clections of algebraic objects is motivated by the idea that there is often
a structure revealed in the limit that is difficult to see when studying in-
dividual objects (see, for example, [ELS01], [Siu01], [Hun92], and [ES09]).
The asymptotic behaviour of a collection of monomial ideals \(a_i \cdot a_j \subseteq a_{i+j}\) (a graded system of monomial ideals) can be described by its
limiting shape \(P\). If \(P_{a_i}\) denotes the Newton polytope of \(a_i\), then the limiting
shape \(P\) is defined to be the limit \(\lim_{m \to \infty} \frac{1}{m} P_{a_m}\) ([May12d]). In addition
to giving a simple geometric interpretation of the limiting behaviour, \(P\)
completely determines the asymptotic multiplier ideals of \(a_i\) (see [How01]).

Generic initial ideals have a nice combinatorial structure, but are often
difficult to compute and usually have complicated sets of generators (see
[Gre98] for a survey or [Cim06] and [May12a] for examples). This moti-
vates a series of work describing the limiting shape of generic initial sys-
tems, \(\{\text{gin}(I(m))\}\), and of symbolic generic initial systems, \(\{\text{gin}(I^{(m)})\}\)
([May12d], [May12c], [May12a], [May12b]). The goal of this paper is to de-
scribe the limiting shape of the symbolic generic initial system of the ideal
of \(r\) points in \(\mathbb{P}^2\) lying on an irreducible conic.

We will see that when \(I\) is the ideal of points in \(\mathbb{P}^2\), each of the polytopes
\(P_{\text{gin}(I(m))}\), and thus \(P\) itself, can be thought of as a subset of \(\mathbb{R}^2\). The
following theorem describes \(P\) in the case we are interested in.

**Theorem 1.** Let \(I \subseteq R = K[x, y, z]\) be the ideal of \(r > 1\) distinct points
\(p_1, \ldots, p_r\) of \(\mathbb{P}^2\) lying on an irreducible conic and let \(P \subseteq \mathbb{R}_+^2\) be the
limiting shape of the reverse lexicographic symbolic generic initial system
\(\{\text{gin}(I^{(m)})\}\). If \(r \geq 4\), then \(P\) has a boundary defined by the line through
the points \((2, 0)\) and \((0, \frac{r}{2})\) (see Figure 7). If \(r = 2\) or \(r = 3\), then \(P\) has a
boundary defined by the line through the points \((\frac{r}{2}, 0)\) and \((0, 2)\).

The proof of this theorem is an application of ideas that have been de-
scribed elsewhere. Rather than repeating arguments here, we refer the
reader elsewhere for details where necessary.
Figure 1. The limiting shape $P$ of $\{\text{gin}(I^{(m)})\}_m$ where $I$ is the ideal of $r \geq 4$ points lying on an irreducible conic.

The following result describes the structure of the individual ideals $\text{gin}(I^{(m)})$ that make up the generic initial system.

**Theorem 2.** Suppose that $I \subseteq K[x,y,z]$ be the ideal of a set of distinct points of $\mathbb{P}^2$. Then the minimal generators of $\text{gin}(I^{(m)})$ are 
$$\{x^{\alpha(m)}, x^{\alpha(m)-1}y^{\lambda_0(m)-1}, \ldots, xy^{\lambda_1(m)}, y^{\lambda_0(m)}\}$$
for some positive integers $\lambda_0(m), \ldots, \lambda_{\alpha(m)-1}$ such that $\lambda_0(m) > \lambda_1(m) > \cdots > \lambda_{\alpha(m)-1}(m)$. Further, if the minimal free resolution of $I^{(m)}$ is of the form
$$0 \rightarrow F_1 = \bigoplus_{i=1}^{\psi} R(-u_i) \rightarrow F_0 = \bigoplus_{i=1}^{\mu} R(-d_i) \rightarrow I^{(m)} \rightarrow 0$$
with $U(m) = \max\{u_i\}$ and $D(m) = \min\{d_i\}$, then
$$\alpha(m) = D(m)$$
and
$$\lambda_0(m) = U(m) - 1.$$  

**Proof.** The first part of the theorem is Corollary 2.9 of [May12a] and follows from results in [BS87] and [HS98]. The second statement follows the a result of Hilbert-Burch, which says that, with the notation in the theorem, the minimal free resolution of $\text{gin}(I^{(m)})$ is of the form
$$0 \rightarrow G_1 \rightarrow G_0 \rightarrow \text{gin}(I^{(m)}) \rightarrow 0$$
where $G_1 = \bigoplus_{i=0}^{\alpha(m)} R(-\lambda_i(m) - i - 1)$ and $G_0 = [\bigoplus_{i=0}^{\alpha(m)} R(-\lambda_i(m) - i)] \oplus R(-\alpha(m))$ (Corollary 4.15 of [Gre98]). A **consecutive cancellation** takes a sequence $\{\beta_{i,j}\}$ to a new sequence by replacing $\beta_{i,j}$ by $\beta_{i,j} - 1$ and $\beta_{i+1,j}$ by $\beta_{i+1,j} - 1$. The ‘Cancellation Principle’ says that the graded Betti numbers of $J$ can be obtained by the graded Betti numbers of $\text{gin}(J)$ by making a series of consecutive cancellations (see Corollary 1.21 of [Gre98]). Since $\lambda_0(m) + 1 > \lambda_i(m) + i$ for all $i$, $\beta_{i,\lambda_0+1} \geq 1$ does not change with any such consecutive cancellation; thus, $R(-\lambda_0(m) - 1)$ is the summand of $F_1$ with
the largest shift. Likewise, $\alpha(m) < \lambda_i(m) + i + 1$ for all $i$ so $\beta_{0,\alpha(m)} \geq 1$ does not change with a consecutive cancellation; thus, $R(-\alpha(m))$ is the summand of $F_0$ with the smallest shift.

In the case where $m$ is even and $I$ is the ideal of $r \geq 3$ points lying on an irreducible conic in $\mathbb{P}^2$, we will see in Proposition 5 that we can write down the entire minimal free resolution of $I^{(m)}$. This will give us $D(m)$ and $U(m)$ when $m$ is even so that we can find the powers of $x$ and $y$, $x^{\alpha(m)} = x^{D(m)}$ and $y^{\lambda(m)} = y^{U(m)} - 1$, that appear in a minimal generating set of $\text{gin}(I^{(m)})$. In particular, Proposition 5 implies the following.

**Lemma 3.** Suppose that $I$ is the ideal of $r \geq 3$ points in $\mathbb{P}^2$ lying on an irreducible conic and use the notation of the previous theorem.

(a) If $r \geq 4$ is even, $D(m) = 2m$ and $U(m) = \frac{rm}{2} + 2$.
(b) If $r > 4$ is odd and $m$ is even, then $D(m) = 2m$ and $U(m) = \frac{rm}{2} + 2$.
(c) If $r = 3$ and $m$ is even, then $D(m) = \frac{3m}{2}$ and $U(m) = 2m + 1$.

By Lemma 2, each of the generic initial ideals $\text{gin}(I^{(m)})$ is generated in the variables of $x$ and $y$, so we can think of each Newton polytope $P_{\text{gin}(I^{(m)})}$, and thus the limiting shape $P$ itself, as a subset of $\mathbb{R}^2$.

The following result is the key for proving the main theorem: it describes when the limiting polytope $P$ of the symbolic generic initial system in $\mathbb{P}^2$ is defined by a single boundary line. The proof is contained in [May12a].

**Proposition 4** (Corollary 2.16 of [May12a]). Let $I \subseteq K[x, y, z]$ be the ideal of $r$ distinct points in $\mathbb{P}^2$ and let $P$ be the limiting shape of the symbolic generic initial system $\{\text{gin}(I^{(m)})\}_m$. Suppose that the $x$-intercept $\gamma_1$ and the $y$-intercept $\gamma_2$ of the boundary of $P$ are such that $\gamma_1 \cdot \gamma_2 = r$. Then the limiting polytope $P$ has a boundary defined by the line passing through $(\gamma_1, 0)$ and $(0, \gamma_2)$.

With these results in mind, we can now prove the main theorem.

**Proof of Theorem 1.** Suppose first that $r \geq 4$ and that $m$ is even if $r$ is odd. By Theorem 2 and Lemma 3, $x^{D(m)} = x^{2m}$ and $y^{U(m)} - 1 = y^{\frac{rm}{2} - 1}$ are the smallest powers of $x$ and $y$ contained in $\text{gin}(I^{(m)})$. This means that the intercepts of the boundary of $P_{\text{gin}(I^{(m)})}$ are $(2m, 0)$ and $(0, \frac{rm}{2} + 1)$. Thus, the intercepts of the boundary of the limiting polytope $P$ of the entire symbolic generic initial system are $(\lim_{m \to \infty} \frac{2m}{m}, 0) = (2, 0)$ and $(0, \lim_{m \to \infty} \frac{rm/2}{m} + 1) = (0, \frac{r}{2})$.

By Proposition 4, the fact that $\frac{r}{2} \cdot 2 = r$ implies that the limiting polytope $P$ is as claimed.

Now suppose that $r = 3$ and that $m$ is even. By the same argument as above, the intercepts of the boundary of the limiting polytope $P$ are $(\lim_{m \to \infty} \frac{3m/2}{m}, 0) = (\frac{3}{2}, 0)$ and $(0, \lim_{m \to \infty} \frac{2m}{m}) = (0, 2)$. Since $\frac{3}{2} \cdot 2 = 3$, the limiting polytope is as claimed.

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1In the case where $r$ is odd we take the limits over even $m$. 

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The case where \( r = 2 \) follows from the main theorem of [May12d] since \( I \) is a type (1,2) complete intersection in this case.

It remains to prove Lemma 3 which follows immediately from the next proposition. In particular, we will write the minimal free resolutions of the ideals \( I^{(m)} \) when \( I \) is the ideal of \( r \geq 3 \) points on an irreducible conic and \( m \) is even if \( r \) is odd.

**Proposition 5.** Let \( I \) be the ideal of \( r \geq 3 \) points of \( \mathbb{P}^2 \) lying on an irreducible conic and suppose that the minimal free resolution of \( I^{(m)} \) is of the form

\[
0 \rightarrow G_1 \rightarrow G_0 \rightarrow I^{(m)} \rightarrow 0.
\]

(a) If \( r \) is even,

\[
G_0 = \bigoplus_{j=0}^{m} R\left(-2(m-j) - \frac{rj}{2}\right)
\]

and

\[
G_1 = \bigoplus_{j=1}^{m} R\left(-2(m-j) - \frac{rj}{2} - 2\right).
\]

(b) If \( r \geq 5 \) is odd and \( m \) is even,

\[
G_0 = \left[ \bigoplus_{j=0}^{m/2} R(-2m-j(r-4)) \right] \oplus \left[ \bigoplus_{j=0}^{m/2-1} R^2(-2m-j(r-4) - \frac{r-1}{2} + 1) \right]
\]

and

\[
G_1 = \left[ \bigoplus_{j=1}^{m/2} R(-2m-j(r-4) - 2) \right] \oplus \left[ \bigoplus_{j=0}^{m/2-1} R^2(-2m-j(r-4) - \frac{r-1}{2}) \right].
\]

(c) If \( r = 3 \) and \( m \) is even,

\[
G_0 = R\left(-\frac{3m}{2}\right) \oplus \left[ \bigoplus_{j=0}^{m/2-1} R^3(-\frac{3m}{2} - j - 1) \right]
\]

and

\[
G_1 = \bigoplus_{j=0}^{m/2-1} R^3\left(-\frac{3m}{2} - j - 2\right).
\]

To prove this proposition we will follow the results of Catalisano described in [Cat91] that can be used to compute the minimal free resolution of any fat point ideal

\[
I_{(m_1,\ldots,m_r)} = I_{p_1}^{m_1} \cap I_{p_2}^{m_2} \cap \cdots \cap I_{p_r}^{m_r}
\]

as long as the points \( p_1, \ldots, p_r \) lie on an irreducible conic. The following is a specialization of Catalisano’s work to the case where \( r \geq 4 \) and \( m_i = m \) for all \( i \) (that is, when \( I \) is the ideal of a uniform fat point subscheme).
Proposition 6 (Cat91). Let $I$ be the ideal of $r \geq 4$ points of $\mathbb{P}^2$ lying on an irreducible conic. Suppose that the minimal free resolution of $I^{(t-1)}$ is of the form

$$0 \to F'_1 \to F'_0 \to I^{(t-1)} \to 0$$

where $F'_1 = \bigoplus_{i=1}^{\mu} R(-u_i) + \bigoplus_{i=1}^{\mu} R(-d_i)$ and $F'_0 = \bigoplus_{i=1}^{\mu} R(-u_i - 2) + \bigoplus_{i=1}^{\mu} R(-d_i - 2)$ and that the minimal free resolution of $I^{(t)}$ is of the form

$$0 \to F_1 \to F_0 \to I^{(t)} \to 0.$$

If $rt$ is even

1. $F_1 = \bigoplus_{i=1}^{\mu} R(-u_i - 2) \oplus R\left(-\frac{rt}{2} - 2\right)$ and $F_0 = \bigoplus_{i=1}^{\mu} R(-d_i - 2) \oplus R\left(-\frac{rt}{2}\right)$

while if $rt$ is odd

2. $F_1 = \bigoplus_{i=1}^{\mu} R(-u_i - 2) \oplus R^2\left(-\frac{rt+1}{2} - 1\right)$ and $F_0 = \bigoplus_{i=1}^{\mu} R(-d_i - 2) \oplus R^2\left(-\frac{rt+1}{2}\right)$.

Therefore, one can apply this result $m$ times to find the minimal free resolution of $I^{(m)}$. That is, first find the minimal free resolution of $I^{(1)} = I$ from $0 \to 0 \to R(0) \to I^{(0)} \to 0$, then find the minimal free resolution of $I^{(2)}$ from that of $I$, and so on.

Sketch of Proof of Proposition 6. We will give an idea of how to find the minimal free resolutions of the ideals $I^{(m)}$ using the algorithm in Proposition 6.

If we are in case (a) where $r$ is even, $rt$ is even for all $t$, so to find the resolution of $I^{(t)}$ from the minimal free resolution of $I^{(t-1)}$ we follow the first case of Proposition 6. In particular, to find the resolution of $I^{(m)}$ from the resolution of $I^{(0)}$, we apply part (1) exactly $m$ times for $t = 1, \ldots, m$.

If we are in case (b) where $r$ is odd, $rt$ is odd for odd $t$ and $rt$ is even for even $t$. Thus, we need to apply both cases of Proposition 6 to find the resolution of $I^{(m)}$. To obtain the resolution

$$0 \to F_1 \to F_0 \to I^{(t)} \to 0$$

of $I^{(t)}$ from the resolution

$$0 \to F''_1 \to F''_0 \to I^{(t-2)} \to 0$$

of $I^{(t-2)}$ when $t$ is even, one needs to:

- shift each summand of $F_0$ and $F_1$ by $-4$;
- add $R^2\left(-\frac{rt(t-1)+1}{2} - 3\right)$ and $R\left(-\frac{rt}{2} - 2\right)$ to $F''_1$, and
- add $R^2\left(-\frac{rt(t-1)+1}{2} - 3\right)$ and $R\left(-\frac{rt}{2}\right)$ to $F''_0$.

If $m$ is even, we can follow this procedure $m\frac{t}{2}$ times with $t = 2, 4, 6, \ldots, m$ to find the resolution of $I^{(m)}$ from that of $I^{(0)}$.

For case (c) when $r = 3$ and $m$ is even, one needs to use other results beyond those stated in Proposition 6. The general idea is the same as above: use a sequence of fat point schemes $Z_0 = m(p_1 + p_2 + p_3), Z_1, Z_2, \ldots, Z_H = 0(p_1 + p_2 + p_3)$ and find the minimal free resolution
of $I_{Z_{H-1}}$ from that of $I_{Z_H}$, then find the minimal free resolution of $I_{Z_{H-2}}$ from that of $I_{Z_{H-1}}$, and so on, until we can find the minimal free resolution of $I^{(m)} = I_0$ from that of $I_1$. However, when $r = 3$ not all of the $Z_i$ will be uniform fat point subschemes. In particular, subsequences of the form $Z_l = t(p_1 + p_2 + p_3)$, $Z_{l+1} = (t-1)p_1 + (t-1)p_2 + tp_3$, $Z_{l+2} = (t-1)p_1 + (t-2)p_2 + (t-1)p_3$, $Z_{l+3} = (t-2)(p_1 + p_2 + p_3)$ come together to form the sequence $Z_0, \ldots, Z_H$. See [Cat91] for further details. □

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