2-stratifold groups have solvable Word Problem

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To Professor Maria Teresa Lozano on the occasion of her 70th birthday

Abstract

2-stratifolds are a generalization of 2-manifolds in that there are disjoint simple closed curves where several sheets meet. We show that the word problem for fundamental groups of 2-stratifolds is solvable.

1 Introduction

Simple stratified spaces arise in Topological Data Analysis [2], [9]. A related class of 2-complexes, called 2-foams, has been defined and studied by Khovanov [8] and Carter [3]. A special class of stratified spaces, called 2-stratifolds have been introduced and some of their properties have been studied in [4], [5], [6] and similar spaces, called multibranched surfaces, have been investigated in [11]. A 2-stratifold $X$ is a compact space with empty 0-stratum and empty boundary and contains a collection of finitely many disjoint simple closed curves, the components of the 1-stratum $X^1$ of $X$, such that $X - X^1$ is a 2-manifold and a neighborhood of each interval contained in $X^1$ consists

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of $n \geq 3$ sheets (the precise definition is given in section 2). A 2-stratifold is essentially determined by its associated labelled graph. In [4] it is shown that a simply connected 2-stratifold is homotopy equivalent to a wedge of 2-spheres and the simply connected 2-stratifolds whose graph is a linear tree are classified. Furthermore an efficient algorithm (in terms of the associated graph) is developed for deciding whether a trivalent 2-stratifold (where a neighborhood of each component $C$ of $X^1$ consists of 3 sheets) is simply-connected and in [5] an efficient algorithm is given for deciding whether a given 2-stratifold is homotopy equivalent to $S^2$.

Very few 2-stratifolds occur as spines of closed 3-manifolds. For example, fundamental groups of 3-manifolds are residually finite, but there are simple 2-stratifolds with non-residually finite fundamental group. Since 3-manifold groups have solvable word problem ([1]), the question arises whether this is true for 2-stratifold groups. The main goal of this paper is to prove that this is indeed the case.

2 Fundamental group of a graph of groups

In this section we show that the word problem is solvable for fundamental groups of certain graphs of groups. A similar result for graphs of groupoids has been obtained by [7]. Our proof for the graph of groups is more direct, using Serre’s normal form. We first describe the fundamental group of a graph of groups $(G, \Gamma)$ following Serre [14].

A graph of groups $(G, \Gamma)$ consists of a graph $\Gamma$ with vertex set $\text{vert}\Gamma$ and (oriented) edge set $\text{edge}\Gamma$, an associated group $G_v$ to each $v \in \text{vert}\Gamma$ and a group $G_e$ to each $e \in \text{edge}\Gamma$ such that $G_e = G_{\bar{e}}$, where $\bar{e}$ is the inverse edge of $e$. (If $e \in \Gamma$, then $\bar{e} \in \Gamma$, $\bar{\bar{e}} = e$, $e \neq \bar{e}$ and the initial edge $o(e) = t(\bar{e})$, the terminal edge of $\bar{e}$). For each $e \in \text{edge}\Gamma$ with terminal vertex $t(e)$ there is a monomorphism $\delta_{t(e)} : G_e \rightarrow G_{t(e)}$.

The group $F(G, \Gamma)$ is generated by the groups $G_v$ ($v \in \text{vert}\Gamma$) and $\text{edge}\Gamma$, subject to the relations

$\bar{e} = e^{-1}$ and $e\delta_{t(e)}(a)e^{-1} = \delta_{o(e)}(a)$, for each edge $e \in \text{edge}\Gamma$ with initial edge $o(e)$ and terminal edge $t(e)$ and $a \in G_e$. 

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For a fixed vertex \( v_0 \), the fundamental group \( \pi_1(G, \Gamma, v_0) \) of the graph of groups \((G, \Gamma)\) is the subgroup of \( F(G, \Gamma) \) generated by all words

\[
\omega = r_0e_1r_1e_2\ldots e_nr_n
\]

where \( v_0 - v_1 - v_2 - \cdots - v_n \) is an edge path with initial and terminal vertex \( v_0 = v_n \) (i.e. a cycle based at \( v_0 \)) and \( r_i \in G_{v_i} \).

The word \( \omega = r_0e_1r_1e_2\ldots e_nr_n \) of length \( n \) is reduced, if

- for \( n = 0 \), \( r_0 \neq 1 \in G_{v_0} \);
- for \( n \geq 2 \), \( r_i \notin \delta_{t(e_i)}(G_{e_i}) \), for each index \( i \) such that \( e_{i+1} = \bar{e}_i \) (backtracking at vertex \( v_i \)).

Serre proves [14] Theorem 11:

If \( \omega \in \pi_1(G, \Gamma, v_0) \) is a reduced word then \( \omega \neq 1 \) in \( \pi_1(G, \Gamma, v_0) \).

**Theorem 1.** Let \((G, \Gamma)\) be a graph of groups with finite graph \( \Gamma \). Suppose that

(i) The word problem for each vertex group \( G_v \) and each edge group \( G_e \) is solvable.

(ii) For each edge \( e \) of \( G \), the membership problem with respect to \( \delta_{t(e)}(G_e) \) is solvable in \( G_{t(e)} \).

Then \( \pi_1(G, \Gamma, v_0) \) has a solvable word problem.

**Proof.** Let \( g \in \pi_1(G, \Gamma, v_0) \) be represented by \( \omega = r_0e_1r_1e_2\ldots e_nr_n \), a word of length \( n \).

If \( n = 0 \) then \( g = 1 \) if and only if \( r_0 = 1 \) in \( G_{v_0} \) and by (i) we can effectively decide whether this is the case.

If \( n = 1 \) then \( \omega \) is reduced and so \( g \neq 1 \).

If \( n \geq 2 \) we check if there is backtracking at \( v_i \). If there is no backtracking at each \( i = 1, \ldots, n-1 \), then \( \omega \) is reduced and \( g \neq 1 \).

If there is backtracking at \( v_i \) then by (ii) we can effectively check whether \( r_i \in \delta_{t(e_i)}(G_{e_i}) \). If this is the case we find \( a \in G_{e_i} \) such that \( \delta_{t(e_i)}(a) = r_i \). Then \( e_ie_{i+1} = \delta_{t(e_i)}(a) \in G_{v_i-1} \) and we replace \( \omega = \ldots r_{i-1}e_ir_{i+1}r_{i+1} \ldots \)
by \( \omega' = \ldots (r_{i-1} \delta_{o(e_i)}(a) r_{i+1}) \ldots \), a word of length \( n - 2 \) which represents the same \( g \in \pi_1(G, \Gamma, v_0) \).

Therefore we can effectively decide whether the word \( \omega \) of length \( n \geq 2 \) representing \( g \) is reduced and, if \( \omega \) is not reduced, effectively find a word of length \( n - 2 \) representing the same \( g \).

\[ \square \]

3 The graph of a 2-stratifold.

We first review the definition of a 2-stratifold \( X \) and its associated graph \( G_X \) given in [4]. A 2-stratifold is a compact, Hausdorff space \( X \) that contains a closed (possibly disconnected) 1-manifold \( X_1 \) as a closed subspace with the following property: Each point \( x \in X_1 \) has a neighborhood homeomorphic to \( \mathbb{R} \times CL \), where \( CL \) is the open cone on \( L \) for some (finite) set \( L \) of cardinality \( >2 \) and \( X - X_1 \) is a (possibly disconnected) open 2-manifold.

A component \( C \approx S^1 \) of \( X_1 \) has a regular neighborhood \( N(C) = N_\pi(C) \) that is homeomorphic to \( (Y \times [0, 1])/(y, 1) \sim (h(y), 0) \), where \( Y \) is the closed cone on the discrete space \( \{1, 2, \ldots, d\} \) and \( h : Y \to Y \) is a homeomorphism whose restriction to \( \{1, 2, \ldots, d\} \) is the permutation \( \pi : \{1, 2, \ldots, d\} \to \{1, 2, \ldots, d\} \). The space \( N_\pi(C) \) depends only on the conjugacy class of \( \pi \in S_d \) and therefore is determined by a partition of \( d \). A component of \( \partial N_\pi(C) \) corresponds then to a summand of the partition determined by \( \pi \). Here the neighborhoods \( N(C) \) are chosen sufficiently small so that for disjoint components \( C \) and \( C' \) of \( X_1 \), \( N(C) \) is disjoint from \( N(C') \). The components of \( N(C) - C \) are called the sheets of \( N(C) \).

The associated labelled graph \( G = G_X \) of a given 2-stratifold \( X = X_G \) is a bipartite graph with black vertices and labelled white vertices and edges. The white vertices \( w \) of \( G_X \) are the components \( W \) of \( M := \overline{X} - \bigcup_j N(C_j) \) where \( C_j \) runs over the components of \( X^1 \); the black vertices \( b_j \) are the \( C_j \)’s. An edge \( e \) corresponds to a component \( S \) of \( \partial M \); it joins a white vertex \( w \) corresponding to \( W \) with a black vertex \( b \) corresponding to \( C_j \) if \( S = W \cap N(C_j) \). Note that the number of boundary components of \( W \) is the number of adjacent edges of \( W \).
The label assigned to a white vertex $W$ is its genus $g$; the label of an edge $e$ is an integer $m$, where $|m|$ is the summand of the partition $\pi$ corresponding to the component $S \subset \partial N(\pi(C))$ and the sign of $m$ is determined by an orientation of $C_j$ and $S$. (Here we use Neumann’s [12] convention of assigning negative genus $g$ to nonorientable surfaces; for example the genus $g$ of the projective plane or the Moebius band is $-1$, the genus of the Klein bottle is $-2$). Note that the partition $\pi$ of a black vertex is determined by the labels of the adjacent edges. If all white vertices have labels $g < 0$ or if $G$ is a tree, then the labeled graph determines $X$ uniquely.

4 Natural presentation of $\pi_1(X_G)$

In this section we describe a natural presentation for the fundamental group of a 2-stratifold $X$. First we fix a notation.

$X = X_G$ is a 2-stratifold with associated bipartite graph $G = G_X$.

$N(C_{b_j})$ is a regular neighborhood of $C_{b_j}$, a component of $X^{(1)}$ corresponding to the black vertex $b_j$ of $G_X$.

$W_i$ is a component of $M = \overline{X - \bigcup_j N(C_j)}$ corresponding to the white vertex $w_i$ of $G_X$.

c$_{ijk}$ are the components of $W_i \cap N(C_j)$ corresponding to the edges $e_{ijk}$ of $G_X$.

For a given white vertex $w$, the compact 2-manifold $W$ has conveniently oriented boundary curves $c_1, \ldots, c_p$ such that

\[(\ast) \quad \pi_1(W) = \langle c_1, \ldots, c_p, y_1, \ldots, y_n : c_1 \cdots c_p \cdot q = 1 \rangle\]

where $q = [y_1, y_2] \cdots [y_{2g-1}, y_{2g}]$, if $W$ is orientable of genus $g$ and $n = 2g$, $q = y_1^2 \cdots y_n^2$, if $W$ is non-orientable of genus $-n$.

Let $\mathcal{B}$ be the set of black vertices, $\mathcal{W}$ the set of white vertices and choose a fixed maximal tree $T$ of $G$. 

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We choose orientations of the black vertices and of all boundary components of $M$ such that all labels of edges in $T$ are positive.

Then $\pi_1(X_G)$ has a natural presentation with generators: \{$b\}_{b \in B}$, one set for each $w \in \mathcal{W}$, as in ($\ast$) \{$t_i\}$, one $t_i$ for each edge $c_i \in G - T$ between $w$ and $b$ and relations: $c_1 \cdots c_p \cdot q = 1$, one for each $w \in \mathcal{W}$, as in ($\ast$) $b^m = c_i$, for each edge $c_i \in T$ between $w$ and $b$ with label $m \geq 1$ (corresponding to $W \cap N(C_b)$) $t_i^{-1}c_it_i = b^{m_i}$, for each edge $c_i \in G - T$ between $w$ and $b$ with label $m_i \in \mathbb{Z}$.

\[5 \quad \text{The graph of groups of } X_G\]

Let $X = X_G$ be the 2-stratifold associated to the labeled graph $G$; we assume a maximal tree $T$ of $G$ is given and the labels of edges of $T$ are positive, so the labeling is unique. We first define a graph of CW-complexes as in [13], with underlying graph that of $G$.

For a black vertex $b$ representing a singular oriented circle $C_b$, let $o(b)$ be the order of $C_b$ in $\pi_1(X_G)$. Note that, if $e$ is an edge joining a black vertex $b$ to a white vertex $w$ and the label of $e$ is $m$, then $e$ represents an oriented circle $c$ of $\partial W$ whose order in $\pi_1(X_G)$ is $k = o(b)/(o(b), m)$. Here $(o(b), m)$ denotes the greatest common divisor of $o(b)$ and $m$. (If $o(b) = 0$, then $(o(b), m) = m$.

Construct a space $\hat{X}$ from $X$ by attaching disks as follows:

If $b$ is a black vertex of order $o(b) \geq 1$, attach a 2-cell $d_b$ to $C_b$ with degree $o(b)$ (i.e. attach a disk under the attaching map $z \rightarrow z^{o(b)}$). If $e$ is an edge joining $b$ to $w$ with label $m$ and $o(b) \geq 1$, attach to $c$ a 2-cell $d_e$ with degree $k = o(b)/(o(b), m)$. If $o(b) = 0$, do not attach 2-cells). Note that

$\pi(\hat{X}) = \pi(X_G)$.

The graph of spaces associated to $\hat{X}$ has the same underlying graph as $G_X$, with vertices $\hat{X}_b$, $\hat{X}_w$, and edges $\hat{X}_e$, defined as follows:
\(\hat{X}_b\): For a black vertex \(b\) of \(G\), \(\hat{X}_b = N(C_b) \cup d_b \cup (\cup e)\), where \(e\) runs over the edges having \(b\) as an endpoint.

\(\hat{X}_e\): For a white vertex \(w\) of \(G\) let \(\hat{X}_w, W \cup (\cup d)\), where \(e\) runs over the edges incident to \(w\). (Recall that there is one such edge for each boundary curve \(c\) of \(W\)).

\(\hat{X}_e\): For an edge \(e\) joining \(b\) to \(w\), \(\hat{X}_e = (\hat{X}_b \cap \hat{X}_w)\). (If \(o(b) \geq 1\) this is a pseudo-projective plane of degree \(k = o(b)/(o(b), m))\).

Since \(\hat{X}_b, \hat{X}_w\) and \(\hat{X}_e\) are path-connected and the inclusion-induced homomorphisms \(\pi_1(\hat{X}_e) \to \pi_1(\hat{X}_b)\) and \(\pi_1(\hat{X}_e) \to \pi_1(\hat{X}_w)\) are injective, this graph of spaces determines a graph of groups \(G = \{G_b, G_e, G_w\}\) (with the same underlying graph as \(G_X\)). The vertex groups are \(G_b = \pi_1(\hat{X}_b)\) and \(G_w = \pi_1(\hat{X}_w)\), the edge groups are \(G_e = \pi_1(\hat{X}_e)\), the monomorphisms \(\delta : G_e \to G_b\) (resp. \(G_e \to G_w\)) are induced by inclusion. Then (see for example [13], [14])

\[\pi_1 G \cong \pi_1(\hat{X})\]

Note that the groups \(G_b\) of the black vertices and the groups \(G_e\) of the edges are cyclic. For a white vertex \(w\) with edges \(e_1, \ldots, e_p\) labelled \(m_1, \ldots, m_p\) with associated edge space \(X_w = W \cup \bigcup_{i=1}^{p} d\), we have

\[G_w = \pi_1(\hat{X}_w) = \langle c_1, \ldots, c_p, y_1, \ldots, y_n : c_1 \cdots c_p \cdot q = 1, c_1^{k_1} = \cdots = c_1^{k_r} = 1 (1 \leq r \leq p) \rangle\).

If all \(k_i \geq 2\) and \(r = p\) then \(G_w\) is an \(F\)-group ([10] p. 126-127), otherwise it is a free product of cyclic groups.

6 The Word Problem for Fundamental groups of 2-stratifolds

It is well-known that free groups have solvable membership problem with respect to cyclic subgroups. More generally it follows from the Proposition below, which is Corollary 4.16 of [1], that free products of cyclic groups have solvable membership problems.
Proposition 1. Solvability of the membership problem is preserved under taking free products.

We are interested in the membership problem of free products with amalgamation with respect to cyclic groups and give an elementary proof of the following

Lemma 1. Let \( G = A \ast_C B \). Assume that \( C \) has solvable membership problem with respect to cyclic subgroups and \( A \) and \( B \) have solvable membership problem with respect to the subgroup \( C \). Then \( G \) has solvable membership problem with respect to cyclic subgroups.

Proof. For \( g, g' \in A \) or \( B \) we can decide whether \( g(g')^{-1} \) is in \( C \). Therefore, given \( g \in G \) and a fixed choice of right coset representatives of \( C \) in \( A \) (resp. in \( B \)) we can effectively find the (unique) reduced normal form \( w = g_1 \ldots g_n c \) of \( g \), where \( g_i \in A \) or \( B \) are the chosen representatives of the right cosets \( g_i C, c \in C \), and \( g_i, g_{i+1} \) are in different subgroups \( A, B \), for \( i = 1, \ldots, n - 1 \). The length of \( g \) is \( l(g) = l(w) = n \). In particular, \( l(w) = 0 \) iff \( g \in C \). Also, if \( w \) is not cyclically reduced (i.e. \( g_1 \) and \( g_n \) are in the same subgroup \( A \) or \( B \)), then we can effectively reduce \( w \) to a cyclically reduced word.

Let \( t \in G \) of length \( l(t) \geq 0 \) generate an infinite cyclic subgroup \( \langle t \rangle \subset G \) and let \( g \in G \). Now \( w \in \langle t \rangle \) if and only if \( w = t^k \) for some \( |k| \geq 1 \). Since \( w \in \langle t \rangle \) iff \( w^{-1} \in \langle t \rangle \) we may assume \( k \geq 1 \). If \( l(t) = 0 \) then \( l(w) = 0 \) and the result follows since \( C \) has solvable membership problem with respect to cyclic groups. Thus assume \( l(t) \geq 1 \).

If the word \( t \) is cyclically reduced then \( l(t^k) = kl(t) \). Thus there is a unique \( k \) such that \( l(w) = kl(t) \) and we can effectively check whether the reduced words \( w \) and \( t^k \) agree.

If \( t \) is not cyclically reduced then \( t = uru^{-1} \) for some reduced word \( r \) and cyclically reduced word \( u \). Then \( w \in \langle t \rangle \) iff \( u^{-1}wu = r^k \) for some \( k \). We effectively find the reduced word \( w' \) representing \( u^{-1}wu \) and (by the above argument) effectively determine whether \( w' = r^k \).

Corollary 1. Let \( G \) be a free product of cyclic groups or a free product of two such groups amalgamated over a cyclic group. Then the membership problem with respect to cyclic subgroups is solvable.

Proof. This follows from Proposition 1 and Lemma 1.

Corollary 2. \( F \)-groups have solvable membership problem with respect to cyclic subgroups.
Proof. Let $G = \langle c_1, \ldots, c_p, y_1, \ldots, y_n : c_1 \cdots c_p \cdot q = 1, c_1^{k_1} = \cdots = c_1^{k_p} = 1 \rangle$ be an $F$-group.

If $p \geq 1$, let $A = \langle c_1, \ldots, c_p : c_1^{k_1} = \cdots = c_1^{k_p} = 1 \rangle$, $B = \langle y_1, \ldots, y_n : \rangle$, $C$ the infinite cyclic group generated by $c_1 \cdots c_p$ in $A$ resp. by $q$ in $B$. Then $G = A *_C B$.

If $p = 0$ then $G$ is the fundamental group of a closed surface of genus $g$. If $g \neq 1, -2$, then $G$ can be similarly written as a free product of two free groups with amalgamation over a cyclic group. If $g = 1, -2$ the result is trivial (in this case every element of $G$ has a normal form of length $\leq 2$).

Theorem 2. The fundamental group of a 2-stratifold has solvable word problem.

Proof. From section 5 we know that $\pi(X_G) \cong \pi_1 G$ for a graph of groups where the edge groups and black vertex groups are cyclic and the white vertex groups are $F$-groups or free products of finitely many cyclic groups. By Lemma 1 and Corollary 2 all these groups have solvable membership problem with respect to finite cyclic subgroups. Now the Theorem follows from Theorem 1.

7 Some Consequences

Corollary 3. There is an algorithm to decide whether or not $\pi(X_G)$ is abelian.

Proof. $\pi(X_G)$ is abelian if and only if $[x_i, x_j] = 1$ for $1 \leq i < j \leq n$, where $x_1, \ldots, x_n$ generate $\pi(X_G)$. Since the word problem is solvable, we can decide whether this is true.

A 0-terminal edge of $X_G$ is an edge $b - w$, where $w$ is a terminal white vertex of genus 0. The following deals with a special case of the order problem.

Corollary 4. Let $b - w$ be a 0-terminal edge of $G_X$. One can calculate the (finite) order $o$ of $b$ in $\pi(X_G)$.

Proof. $o$ is one of the (finitely many) divisors of the finite nonzero labels of $b - w$. The Corollary follows since $\pi(X_G)$ has solvable word problem.
In [4] and [6] we obtained for certain classes of 2-stratifolds $X_G$ (namely those with a linear graph $G_X$ or those that are trivalent) an efficient algorithm to decide if $X_G$ is simply connected. These algorithms can be read off from the labelled graph $G_X$. For the general case we do not yet have an efficient algorithm, but we now see that there is an algorithm:

**Corollary 5.** There is an algorithm to decide whether or not $X_G$ is simply-connected.

*Proof.* If $S$ is a finite set of generators of $\pi = \pi_1(X_G)$, the $\pi = 1$ if and only if $s = 1$ in $\pi$ for every $s \in S$. Since $\pi$ has solvable word problem one can decide if every $s$ in $S$ is 1. \hfill \square

In [4] it was shown that a necessary condition for a 2-stratifold $X_G$ to be simply-connected is that $G_X$ is a tree, all white vertices are of genus 0, and all terminal edges are white. If there is an efficient algorithm for the order problem in Corollary [4] then this result may be used in obtaining an efficient algorithm in Corollary [5] as follows:

If $G_X$ is not a tree or if some white vertex has nonzero genus, or if there is a black terminal vertex, then $\pi(X_G) \neq 1$. Otherwise apply repeatedly the following “pruning”:

Calculate the order $o$ in $\pi(X_G)$ of $b$ where $b - w$ is a 0-terminal edge; if $o$ is not 1 then $X_G$ is not simply-connected; if $o = 1$ delete $b$, $w$ and all edges incident to $b$ from $X_G$. Each component $G_i$ of the resulting graph (the “pruned” graph) corresponds to a 2-stratifold $X_{G_i}$, and since $o(b) = 1$ in $\pi(X_G)$ it follows that $X_G$ is simply-connected if and only if each $X_{G_i}$ is simply-connected. Then $X_G$ is 1-connected if and only if we eventually obtain a graph with no edges.

**Corollary 6.** One can decide whether or not $X_G$ is homotopically equivalent to a wedge of $n$ 2-spheres and, if so, calculate $n$.

*Proof.* In [5] it was shown that a simply-connected 2-stratifold $X_G$ is homotopy equivalent to a wedge of 2-spheres and moreover if $n_b$ (resp. $n_w$) denotes the number of black (resp. white) vertices of $G_X$, then $X_G$ is homotopy equivalent to a wedge of $n_w - n_b$ 2-spheres. Now the Corollary follows from Corollary [5] and, if $\pi(X_G) = 1$, then $n = n_w - n_b$. \hfill \square
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