Stress tensor for extreme 2D dilatonic Reissner-Nordström black holes

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We calculate the expectation value of the stress energy tensor for a massless dilaton-coupled 2D scalar field propagating on an extremal Reissner-Nordström black hole formed by the collapse of a timelike shell, showing its regularity on the horizon.

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Extremal black holes are peculiar objects which enter various and important contexts of gravitational physics. Their interest is related to the fact that, being characterized by a zero Hawking temperature, they represent the natural end-state configuration of the evaporation of non extremal charged black holes. It would be rather disappointing if these configurations turned out to be singular. In a previous work [1] we calculated, within the Polyakov theory, the expectation value of the energy-momentum tensor for a 2D massless scalar field propagating on an extreme Reissner-Nordström black hole, showing, unlike previous results [2, 3], regular behaviors on the horizon. The problem was treated considering the process of formation of an extremal black hole through the collapse of a charged null shell. Our purpose here is to generalize the previous work to show that the basic result does not depend on the specific model chosen. To this end we shall consider the more general case of a dilaton coupled scalar field theory, and the formation of the hole through the collapse of a timelike shell. This is a non trivial extension with respect to the pure 2D Polyakov theory, since new divergent terms due to the presence of the dilaton appear; they look quite different from the ones emerging in Polyakov theory and their regularity is not obvious. In quantum field theory in curved spacetimes the mean value of the matter fields energy momentum tensor plays a fundamental role since it determines the behavior of the solutions of the semiclassical evolution equations. The main problem is then to find a covariant expression for this quantum stress tensor. This is a very difficult task for an arbitrary 4D spacetime, where no analytical expression for $T_{\mu\nu}$ exists. Nevertheless, starting from a 4D spherically symmetric theory, one can perform a dimensional reduction leading to a 2D model, where detailed predictions can be made. In the two dimensional theory obtained in this way, the field is coupled not only to the 2D metric, but also to the dilaton $\phi$, which is linked to the radius of the two-sphere and reminds the four-dimensional origin of the theory we started from. Let us consider a four-dimensional massless minimally coupled scalar field with action

$$S^{(4)} = -\frac{1}{8\pi} \int d^4x \sqrt{-g^{(4)}} (\nabla f)^2.$$ (1)

A spherically symmetric four-dimensional metric can be written as

$$ds^{2(4)} = ds^{2(2)} + r_0^2 e^{-2\phi} d\Omega^2,$$ (2)

where we have parameterized the radius of the two-sphere $r^2 = r_0^2 e^{-2\phi}$ through a dilaton field $\phi$. $r_0$ is a scale factor which can be set to 1 without loss of generality. We assume that the field $f$ is a function of the $(t, r)$ coordinates only. Since any field in a spherically symmetric background can be expanded in spherical harmonics, this hypothesis corresponds to picking up only the $s$-wave component. Through the hypothesis of spherical symmetry for both the metric $g_{\alpha\beta}$ and the field $f$, the 4D action (1) can be integrated with respect to the angular coordinates, to obtain the two-dimensional action

$$S^{(2)} = -\frac{1}{2} \int d^2x \sqrt{-g^{(2)}} e^{-2\phi} (\nabla f)^2,$$ (3)

where $g^{(2)}$ represents the metric in the $(t, r)$ plane. This shows that in the 2D theory the scalar field acquires a non trivial coupling with the dilaton field besides the usual one to the metric. Since every two-dimensional spacetime is conformally flat, it is always possible to write the line element in conformal coordinates $\{x^\pm\}$ as:

$$ds^{2(2)} = -e^{2\phi(x)} dx^+ dx^-.$$ (4)

It is worth noting that this choice is not unique, but infinite sets of conformal coordinates can be obtained from $\{x^\pm\}$ through conformal transformations.
Let us briefly remind the expression for the expectation value of the covariant quantum stress tensor for this 2D dilatonic theory in a generic quantum state $|\Psi\rangle$. Using the anomalous transformation law of the normal ordered stress tensor combined with equivalence principle argument (for details see [1]) one gets

$$\langle \Psi | T_{\pm\pm}(x^+, x^-) | \Psi \rangle = \langle \Psi | T_{\pm\pm}(x^+, x^-) : | \Psi \rangle + \frac{1}{12\pi} \left( \partial_{\pm\pm} \partial_{\pm\pm} - \partial^2_{\pm\pm} \right) \rho \Omega(\partial_{\pm\pm} \phi) + \left( \partial_{\pm\pm} \partial_{\pm\pm} + \rho(\partial_{\pm\pm} \phi)^2 \right).$$

where

$$\Omega(\partial_{\pm\pm} \phi) = (1 - M/r)^2.$$  

Now we want to apply this formalism to the 2D extremal Reissner-Nordström spacetime, whose line element is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} = -f(r)du dv$$

where $f(r) = (1 - M/r)^2$. Extreme black holes are charged Reissner-Nordström black holes whose charge assumes its maximum value, i.e. $|Q| = M$ and possesses a single degenerate horizon at $r = M$ with zero surface gravity.  $u$ and $v$ are respectively the retarded and advanced Eddington-Finkelstein coordinates

$$u = t - r^*, \quad v = t + r^*$$

where

$$r^* = \int \frac{dr}{(1 - \frac{M}{r})^2} = r + 2M \ln \left( \frac{r}{M} - 1 \right) - \frac{M^2}{(r - M)}.$$  

In the literature it has been shown that a static $(T_{\mu\nu})$ calculated in this background is divergent on the horizon [2]. In a recent paper [1] it was shown how such diverging behavior completely disappears in the case of a collapse if one takes into account the time-dependent contribution to $(T_{\mu\nu})$ induced by the collapse itself. Incoming modes, which are asymptotically $1/r e^{-i\omega \tau}$, reflected through the origin, emerge just before the horizon formation transformed into complicated redshifted modes which are positive frequency with respect to a Kruskal retarded coordinate $U$ [3]. The relation between the $u$ and the Kruskal $U$ coordinate is given at late times as

$$u = -4M \left[ \ln \left( \frac{U}{M} \right) + \frac{M}{2U} \right].$$

This was obtained in the case of a collapsing null shell [3]. Now we are going to show how this result is completely independent on the details of the collapse. To this aim we will perform the same calculation for a timelike shell and we will show that indeed we recover the same result. Outside the collapsing shell the line element is

$$ds^2 = -f(r)du dv,$$

$$u = t - r^* + R^*_0,$$

$$v = t + r^* - R^*_0,$$

$$R^*_0 = \text{const},$$

and the dilaton is linked to the radial coordinates through

$$\phi = -\ln r.$$  

Inside the shell instead

$$ds^2 = -dU dV,$$

$$U = \tau - r + R_0,$$

$$V = \tau + r - R_0,$$
where the relation between \( R_0 \) and \( R_0' \) is the same as that between \( r \) and \( r^* \).

Before \( \tau = 0 \) the shell is at rest with its surface at \( r = R_0 \), while for \( \tau > 0 \) the shell will follow the world line \( R(\tau) \). Matching the inner and the outer metrics along the collapsing shell (for details see [9]), in the near horizon limit we finally find

\[
\frac{dU}{du} \sim \frac{(\dot{R} - 1)}{2R} f(R) \tag{22}
\]

where the dot represents differentiation with respect to \( \tau \).

Let us expand \( R(\tau) \) around the value it assumes on the horizon:

\[
R(\tau) = R_H + \dot{R}_H (\tau - \tau_H) + \frac{1}{2} \ddot{R}_H (\tau - \tau_H)^2 + O((\tau - \tau_H)^3)
\]

where the suffix \( H \) means the value the function assumes on the horizon. Note that \( \dot{R}_H < 0 \) since the ball is shrinking.

We need to find the explicit form for the transformations between the \( \{u, v\} \) and \( \{U, V\} \) coordinates in the near horizon limit. To solve eq. (22) it is necessary also to expand \( f(R) \) for \( R \rightarrow R_H \): observe that a double pole for \( R = R_H = M \) appears. From \( U = \tau - R(\tau) + R_0 \) and eq. (23) we get

\[
f(U) \approx \frac{\dot{R}_H^2 U^2}{M^2 (1 - \dot{R}_H) + 2M \dot{R}_H (1 - \dot{R}_H)U + a U^2} \tag{24}
\]

with \( a = (\dot{R}_H^2 + M \ddot{R}_H) \). If we insert the above approximation into (22), a straightforward integration gets

\[
u = -4M \left[ \ln \left( -\frac{U}{cM} \right) + \frac{cM}{2U} \right] \quad \text{for} \ U \rightarrow 0 \tag{25}
\]

with \( c = \frac{R_0 - 1}{R} \big|_{R_H} \) and an opportune choice of the integration constant. Note that eq. (25) has the same form of the coordinate transformation given in eq. (13). More precisely the two equations exactly coincide if we just rescale \( U/c \rightarrow U \). This difference however will not affect the calculation we are facing, so, without loss of generality, in the following we will use eq. (13).

Now using eq. (7) we can compute the stress energy tensor expectation value at late time on the \( |U \rangle \) vacuum. First we shall find the expectation value of the stress tensor on the Boulware state \( |B \rangle \). Eq. (14), with \( \{x^\pm\} = \{u, v\} \) the Eddington-Finkelstein coordinates, yields

\[
\langle B|T_{uu}|B \rangle = \langle B|T_{vv}|B \rangle =
\]

\[
= -\frac{1}{24\pi} \frac{M}{r^3} \left( 1 - \frac{M}{r} \right)^3 + \frac{1}{8\pi r^3} \left( 1 - \frac{M}{r} \right)^3 + \frac{1}{16\pi r^2} f^2 \ln f
\]

where the first term in the r.h.s of eq. (26) represents just the expectation value of the stress tensor in the Boulware state for the minimally coupled case (cfr. [1]), while the other two terms are those induced by the dilaton field.

To obtain the Uruh state expectation values we can apply eq. (7) with \( |x^\pm\rangle = |U \rangle \equiv |U, v \rangle \) and \( |x^\pm\rangle = |B \rangle \equiv |u, v \rangle \). We find:

\[
\langle U|T_{uu}|U \rangle = \langle B|T_{uu}|B \rangle - \frac{1}{24\pi} \left[ \frac{M}{r^3} \left( 1 - \frac{M}{r} \right)^3 \right.
\]

\[
+ \frac{1}{8\pi r^3} \left( 1 - \frac{M}{r} \right)^3 + \frac{1}{16\pi r^2} f^2 \ln f \tag{28}\n\]

\[
\langle U|T_{vv}|U \rangle = \langle B|T_{vv}|B \rangle - \frac{1}{24\pi} \left[ \frac{M}{r^3} \left( 1 - \frac{M}{r} \right)^3 \right.
\]

\[
+ \frac{1}{8\pi r^3} \left( 1 - \frac{M}{r} \right)^3 + \frac{1}{16\pi r^2} f^2 \ln f \tag{29}\n\]

with \( du/dU \) calculated from eq. (13).

To check the regularity of the stress tensor on the future horizon \( H^+ \), it is necessary to express it in a frame regular there. The relevant component we need to compute is

\[
\langle U|T_{UU}|U \rangle = \left( \frac{du}{dU} \right)^2 \langle U|T_{uu}|U \rangle =
\]

\[
= 16M^2 \left( 1 - \frac{M}{r} \right)^2 \cdot \frac{f}{r^3} \left( \frac{M}{r} \right)^3 + \frac{1}{24\pi r^3} \left( 1 - \frac{M}{r} \right)^3 + \frac{1}{16\pi r^2} f^2 \ln f + \frac{1}{24\pi M(2U - M)^3} f^2 \ln f
\]

\[
= \frac{1}{24\pi} \left[ \frac{M}{r^3} \left( 1 - \frac{M}{r} \right)^3 \right.
\]

\[
+ \left. \frac{1}{8\pi r^3} \left( 1 - \frac{M}{r} \right)^3 + \frac{1}{16\pi r^2} f^2 \ln f \right] \tag{30}\n\]

To better appreciate the difference between this and the non dilatonic theory, we write down also the \( \langle T_{UU}^P \rangle \) obtained in the context of the Polyakov theory:

\[
\langle U|T_{UU}^P|U \rangle = \left( \frac{du}{dU} \right)^2 \langle U|T_{uu}^P|U \rangle =
\]

\[
= 16M^2 \left( 1 - \frac{M}{r} \right)^2 \cdot \frac{f}{r^3} \left( \frac{M}{r} \right)^3 + \frac{1}{24\pi r^3} \left( 1 - \frac{M}{r} \right)^3 + \frac{1}{16\pi r^2} f^2 \ln f \tag{31}\n\]

So eq. (30) can be rewritten as

\[
\langle U|T_{UU}|U \rangle = \langle U|T_{UU}^P|U \rangle +
\]

\[
\text{(other terms)}
\]
In [1] we have already shown how the behavior of \( \langle T^\mu_\nu \rangle \) is regular on the future horizon. Eq. [29] shows clearly that new divergent terms (both polynomial and logarithmic) in the near horizon limit \( U \sim -(r-M) \) appear. We can see how these divergences compensate exactly each other to give a finite net result. For the polynomial terms we have

\[
\frac{16M^2}{U^2} \left(1 - \frac{M}{2U}\right)^2 \left\{ -\frac{1}{8\pi r^3} \left(1 - \frac{M}{r}\right)^3 + \frac{1}{16\pi r^2} f^2 \ln f + \frac{1}{4\pi M(2U-M)^2} f \right\} \sim
\]

\[
\frac{4M^2(M-2r)^2}{(r-M)^4} \left( -\frac{1}{8\pi} \frac{M(r-M)}{r^6} + \frac{1}{8\pi Mr^2(M-2r)^2} \right) = 0.
\]

Analogously the logarithmic terms sum up to give a non diverging result:

\[
\frac{16M^2}{U^2} \left(1 - \frac{M}{2U}\right)^2 \left(\frac{1}{16\pi r^2} f^2 \ln f + \frac{1}{4\pi M(2U-M)^2} \right).
\]

Near the future horizon the limit finally is

\[
\langle U|T_{UU}|U \rangle \sim \frac{1}{4\pi M^2} \ln 2 \sim \infty
\]

that is indeed regular. Had we used eq. (25) instead of eq. (13), we would have obtained a different final value for the limit:

\[
\langle U|T_{UU}|U \rangle \sim \frac{1}{4\pi M^2} \ln 2 \sim \infty
\]

which is finite too. Note that the finite part depends, through the constant \( c \), on the details of the collapse without affecting the validity of our arguments.

So the potentially diverging terms induced by the dilaton field in the stress energy tensor compensate each other, leading to a regular result. No physically unacceptable behavior arises.

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[10] It is possible to face the problem from a different point of view, choosing as effective action for the quantum fields the one obtained by functional integration of the trace anomaly (the so-called "anomaly induced effective action" \( S_{\text{ind}} \)). However this approximation for the effective action is in general not reliable since it predicts an unphysical negative energy outflux for the Schwarzschild black hole. In any case also in these models one can construct a regular \( T_{\mu\nu} \) on the horizon of extremal RN background.
[11] It is also possible to find a new coordinate \( V \) regular on the past horizon \( H^- \), \( v = 4M \ln \left( \frac{V}{2\pi} \right) - \frac{M}{2V} \). Using this \( V \) as a Kruskal coordinate one can show that the expectation value of \( T_{VV} \) on \( H^- \) is regular. This result holds for any transformation \( v \to V \) which reduces to the previous one in the limit \( V \to 0 \).