A PROOF OF THE FUNCTIONAL EQUATION CONJECTURE

ADRIANO GARSIA, ANGELA HICKS, AND GUOCE XIN

ABSTRACT. In the early 2000’s the first and second named authors worked for a period of six years in an attempt of proving the Compositional Shuffle Conjecture [1]. Their approach was based on the discovery that all the Combinatorial properties predicted by the Compositional Shuffle Conjecture remain valid for each family of Parking Functions with prescribed diagonal cars. The validity of this property was reduced to the proof of a functional equation satisfied by a Catalan family of univariate polynomials. The main result in this paper is a proof of this functional equation. The Compositional Shuffle Conjecture was proved in 2015 by Eric Carlsson and Anton Mellit [3]. Our proof of the Functional Equation removes one of the main obstacles in the completion of the Garsia-Hicks approach to the proof of the Compositional Shuffle Conjecture. At the end of this writing we formulate a few further conjectures including what remains to be proved to complete this approach.

1. Introduction

Our manipulations rely heavily on the plethystic notation used in [6]. In fact, all the notations used in this paper is introduced in full detail in the first section of [6].

Recall that Dyck paths in the \( n \times n \) lattice square \( R_n \) are paths from \((0,0)\) to \((n,n)\) proceeding by north and east unit steps, always remaining weakly above the main diagonal of \( R_n \). These paths are usually represented by their area sequence \((a_1,a_2,\ldots,a_n)\), where \(a_i\) counts the number of complete cells between the north step in the \( i^{th} \) row and the main diagonal. Notice that the \( x \)-coordinate of the north step in the \( i^{th} \) row is simply the difference \( u_i = i - 1 - a_i \).

A parking function \( PF \) supported by the Dyck path \( D \in R_n \) is obtained by labeling the north steps of \( D \) with 1,2,\ldots,\( n \) (usually referred as “cars”), where the labels increase along the north segments of \( D \). Parking functions can be represented as two line arrays

\[
PF = \begin{pmatrix}
c_1 & c_2 & \cdots & c_n \\
a_1 & a_2 & \cdots & a_n
\end{pmatrix}
\]

with cars \( c_i \) and area numbers \( a_i \) listed from bottom to top. We also set

\[
area(PF) = \sum_{i=1}^{n} a_i, \quad dinv(PF) = \sum_{1 \leq i < j \leq n} \left( \chi(c_i < c_j \ & a_i = a_j) + \chi(c_i > c_j \ & a_i = a_j + 1) \right).
\]

Moreover, the word \( w(PF) \) is the permutation obtained by reading the cars in the two line array by decreasing area numbers and from right to left.

Notice that the diagonals of \( R_n \) can be so ordered that car \( c_i \) lies in diagonal \( a_i \). Where diagonal 0 is the main diagonal of \( R_n \). A given Dyck path \( D \) can hit the main diagonal in at most \( n \) distinct lattice points ((0,0) not counted). We will write \( p(D) = p \) for \( p = (p_1,p_2,\ldots,p_{\ell(p)}) \) a composition of \( n \), if and only if the components of \( p \) give the sizes of the intervals between successive main diagonal hits of \( D \). If a Parking Function \( PF \) is supported by \( D \) and \( p(D) = p \) it will be convenient to write \( p(PF) = p \). This given, we set

\[
\Pi_p[X; q,t] = \sum_{p(PF)=p} t^{area(PF)} q^{dinv(PF)} F_{des(w(PF))}[X], \tag{1.1}
\]

where the last factor in 1.1 is the Gessel Fundamental quasi-symmetric function indexed by the descent set of the inverse of the word \( w(PF) \). It is shown in [17], that the right-hand side of 1.1 defines a symmetric function for any composition \( p \).

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If \( p = (p_1, p_2, \ldots, p_{\ell(p)}) \) then the Compositional Shuffle Conjecture states that the same symmetric function can be obtained by setting

\[
\nabla C_p[X; q, t] = \nabla C_{p_1}C_{p_2} \cdots C_{p_{\ell(p)}} 1
\]

1.2

Where \( \nabla \) is the eigen-operator of the Modified Macdonald polynomial \( \tilde{H}_\mu[X; q, t] \) with eigenvalue \( T_\mu = t^{\alpha(\mu)}q^{\beta(\mu)} \). Here, the family \( \{ \tilde{H}_\mu[X; q, t] \}_\mu \) is defined as the unique symmetric function basis that satisfies the two triangularity conditions, (with partition inequalities indicating dominance)

\[
a) \quad \tilde{H}_\mu[X; q, t] = \sum_{\lambda \leq \mu} s_\lambda \left[ \frac{X}{1-q} \right] c_{\lambda,\mu}(q, t), \quad \quad b) \quad \tilde{H}_\mu[X; q, t] = \sum_{\lambda \geq \mu} s_\lambda \left[ \frac{X}{1-q} \right] d_{\lambda,\mu},
\]

1.3

together with the normalizing condition

\[
(\tilde{H}_\mu[X; q, t], s_n) = 1.
\]

1.3

We can also set for any symmetric function \( F[X] \)

\[
C_aF[X] = (1 - \frac{1}{q})^{-1} F[X - \frac{1-r/q}{z}] \sum_{m \geq 0} z^m h_m[X] \bigg|_{z^*}.
\]

1.4

It is well known and easily verified from 1.4 that for any pair of positive integers \((a, b)\) we have

\[
q(C_bC_a + C_{a-1}C_{b+1}) = (C_aC_b + C_{b+1}C_{a-1})
\]

1.5

Applying this operator to 1 and then applying \( \nabla \) to the resulting symmetric function, the equality of the two polynomials in 1.1 and 1.2, gives the identity

\[
q(\Pi_{b,a}[X; q, t] + \Pi_{a-1,b+1}[X; q, t]) = \Pi_{a,b}[X; q, t] + \Pi_{b+1,a-1}[X; q, t]
\]

1.6

This identity suggests the existence of a bijection from the family of Parking Functions with diagonal compositions \((a, b)\) or \((b+1, a-1)\) onto the family of Parking Functions with diagonal compositions \((b, a)\) or \((a-1, b+1)\) with the following properties

1. preserves area,
2. preserves the Gessel Fundamental index,
3. increases dinv exactly by one unit.

Our proof of the Functional Equation implies the existence of such a bijection even when the family of Parking Functions is restricted by the requirement of having a pre-selected collection of cars in each diagonal. This fact suggests that it may be possible to state and prove the Compositional Shuffle conjecture as a quasi-symmetric function identity. This circumstance has been the main driving force in our effort to prove the functional equation.

A legal schedule is simply a sequence of integers \( W = (w_1, w_2, \ldots, w_{k-1}) \) with the property that \( 1 \leq w_1 \leq 2 \) and \( 1 \leq w_2 \leq w_{k-1} + 1 \). The corresponding alphabet is \( X_k = \{ x_1, x_0, x_1, \ldots, x_{k-1} \} \). This given, we construct a family of multivariate polynomials \( P_W(X_k; q) \) by the following recursion

**Definition 1.1**

For any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and any \( 1 \leq w \leq w_{k-1} + 1 \) we set

\[
P_{W,w}[X_{k+1}; q] = \frac{x_w-x_w^w}{1-q} P_W[X_k; q] + \frac{1-x_w}{1-q} \left( P_W[X_k; q] |_{x_{k-1}, \ldots, q x_{k-1}; 1 \leq i \leq w} \right)
\]

1.7

with base case \( P_0[X_0; q] = qx_1 + x_0 \).

Our next ingredient is the family of univariate polynomials \( Q_W(x; q) \) defined by setting

\[
Q_W(x; q) = P_W[X_k; q] |_{x_i = x; -1 \leq i \leq k-1}
\]

1.8

It is easy to derive from 1.7 that the polynomial \( Q_W(x; q) \) has degree \( k \) in the \( x \) variable and has no constant term, thus it may be written in the form

\[
Q_W(x; q) = \sum_{r=1}^{k} A_r(q)x^r.
\]

1.9

This given, computer data led the second named author in [1] to state the following
Conjecture I

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \), the coefficients \( A_s(q) \) in the expansion 1.9 satisfy the identities

\[
q(A_s+1(q) + A_{k+1-s}(q)) = A_{k-s}(q) + A_s(q) \quad \text{(for all } 0 \leq s \leq k) \tag{1.10}
\]

An equivalent form of these identities may be stated as follows,

Conjecture II

For all legal schedules \( W = (w_1, w_2, \ldots, w_{k-1}) \) the polynomials \( Q_W(x; q) \) satisfy the following functional equation

\[
(1 - \frac{q}{2})Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(A_k - qA_1) = (1 + x^k)(1 - q^2) \prod_{i=1}^{k-1}[w_i]_q \tag{1.11}
\]

Since 1.11 is invariant under the replacement \( x \rightarrow 1/x \), it follows that, to prove this equivalence, we only need to show that the equalities in 1.10 are obtained by equating coefficients of \( x^s \) on both sides of 1.11 for \( 0 \leq s \leq k \).

To this end, notice that the coefficient of \( x^s \) in the left hand side of 1.11 is

\[
A_k - qA_{k+1} + A_{k-s} - qA_{k+1-s}.
\]

and this must vanish for \( 1 \leq s \leq k-1 \) if 1.11 must hold true. Now for \( s = 0 \) this reduces to \(-qA_1 + A_k\), which is precisely what the right hand side of 1.11 gives. Finally if \( s = k \) we obtain \( A_k - A_1 \) which is again what the right hand side of 1.11 gives. The last equality in 1.1 follows by setting \( x = 1 \) in the left hand side of 1.11 and using formula 1.19 proved later.

Remark 1.1

It should be apparent that 1.6 and 1.10 are closely related. In the last section this fact will play a crucial role in conveying the significance of our proof of the functional equation.

Our next task is the introduction of the basic tool that will be used in our proof. These are the so called “Bar Diagrams”. This tool was created in \[1\] precisely for this purpose. In fact, the device has so far been very effective proving special cases of the functional equation. Roughly speaking Bar Diagrams are none other than combinatorial structures that give a visual representation of the terms of the polynomials \( P_W(X_k; q) \). More precisely, each of these polynomials can be written in the form

\[
P_W(X_k; q) = \sum_{S \subseteq X_k} A_S(q) \prod_{x_i \in S} x_i \tag{1.12}
\]

where the sum is over subsets of the alphabet \( X_k = \{x_{-1}, x_0, x_1, \ldots, x_{k-1}\} \) with the following special properties. Given the schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \), define for \( 1 \leq i \leq k-1 \)

\[
act(x_i) = \{x_{i-1}, \ldots, x_{i-w_i}\}. \tag{1.13}
\]

This given, we have, for each \( S \) in 1.12

(i) For each \( x_i \in S \), \(|S \cap act(x_i)| \geq 1\),

(ii) For each \( x_i \in S^c \), \(|S^c \cap act(x_i)| \geq 1\),

(iii) \(|\{x_{-1}, x_0\} \cap S| = 1\).

where \( S^c \) denotes the complement of \( S \) in \( X_k \).

To better understand the mechanism that yields the coefficient \( A_S(q) \) in 1.12 we need to introduce the notion of “Labelled Bar Diagram” corresponding to a given schedule \( W \). By summing over all these diagrams we will obtain a polynomial \( \tilde{P}_W(Y_k; q) \) in the non-commutative alphabet \( Y_k = \{y_{-1}, y_0, y_1, \ldots, y_{k-1}\} \) with expansion

\[
\tilde{P}_W(Y_k; q) = \sum_{LBD \in LBD_W} m_{LBD}(q) \omega_{LBD}(Y_k). \tag{1.14}
\]

Here the sum is over the family of \( LBD's \) constructed from \( W \), \( \omega_{LBD}(Y_k) \) is an injective word in the alphabet \( Y_k \) and \( m_{LBD}(q) \) is a monomial in \( q \) depending only on \( LBD \). Moreover, if we denote by \( \omega_{LBD}(X_k) \) the result of replacing each letter \( y_i \) in \( \omega_{LBD}(Y_k) \) by the corresponding letter \( x_i \) then there is a subset \( S \subseteq X_k \) satisfying properties (i) and (ii) such that \( \omega_{LBD}(X_k) = \prod_{x_i \in S} x_i \).
In particular it will follow that the coefficient $A_S(q)$ in 1.12 can be expressed as

$$A_S(q) = \sum_{\omega_{LBD}(X_k) = S} m_{LBD}(q).$$

1.15

A Labelled Bar Diagram of the polynomial $\tilde{P}_{2,3,4,4,3,4}(Y_k; q)$ is constructed as depicted below

In the left of this display we have a table of powers of $q$, for each bar. This table will be used for labeling each bar, whether or not the bar is placed above or below the ground line. In the middle of the display we illustrate the mechanism that is used to color the cells of each bar. If we rotate counterclockwise by 90° bar $x_0$, we see that it will touch two previous bars. Here we have depicted by the thin rectangle the final position of the rotated bar. Accordingly, the first two cells are colored yellow and the cells above them are colored red. All the other bars above the ground line are colored by the same mechanism. For the bars below the ground line the mechanism is the same, except that in this case we rotate the bar by 90° clockwise. As a result we see that bar $x_4$, after this rotation, will touch two previous bars. Accordingly, the two cells closest to the ground line are colored yellow and the remaining lower cells are colored red. The subset $S \subseteq X_k$ corresponding to the resulting Bar Diagram is obtained by placing $x_1$ in $S$ if and only if its bar is above the ground line. This given, the only additional property we will require is that in all our Bar Diagrams each bar has at least one yellow cell. This is to assure the properties (i) and (ii). The way we draw bar $x_{-1}$ and bar $x_0$ assures that only one of them will be up in all Bar Diagrams. This to assure property (iii).

It remains to describe how the labeling is done. Notice first that in bars $x_1, x_2, x_5$ there is only one yellow cell. In a labelled Bar Diagram, each bar must be given a label. The labels of the yellow cells of these bars, are obtained from the table. Next notice that, for our example, in each of bars $x_3, x_4, x_6$ there are two yellow cells. Since we are only allowed one label per bar we have $2 \times 2 \times 2$ possible choices here. We displayed only one of them. However, for each of the bars, the chosen cell must be labeled according to the table. Finally, notice the $q$ in the top cell of bar $x_0$. This will be always the case whenever bar $x_{-1}$ is up and bar $x_0$ is down and only then.

Next, we need to describe how the word $\omega$ is constructed. To begin, the word $\omega$ must be constructed one letter at the time starting with $y_{-1}$ or $y_0$ according as which of bars $x_{-1}$ or bar $x_0$ is up. Next, a letter $y_i$ appears in $\omega$ if and only if bar $x_i$ is up. This given, the power of $q$ used to label bar $x_i$ dictates where in $\omega$ the letter $y_i$ is to be inserted. For instance, if the label is $q^k$ then $y_i$ must be inserted exactly to the left of $k$ from the letters $y_{i-1}, y_{i-2}, \ldots, y_{i-w_i}$. The resulting monomial $m_{\omega}(q)$ is the product of the powers of $q$ used in the labels.

We will see in the final section of this paper that Labelled Bar Diagrams are in bijection with Parking Functions with composition of length 2. Our goal here is to use un-labeled Bar Diagrams constructed for the schedule $W = (w_1, w_2, \ldots, w_{k-1})$ to represent the terms in the expansion of the polynomial $P_W(X_k; q)$. The idea can be communicated by a single example. We simply label all the empty yellow cells with the power of $q$ as dictated by the table. As illustrated in the above display. This given, the term of $P_{2,3,4,4,3,4}(X_k; q)$ produced by this un-labeled Bar Diagram $BD$ is none other than

$$A_{BD}(q) m_{BD}(X_k) = q^4(1 + q)(q^2 + q^3)(1 + q) x_{-1} x_2 x_3 x_6.$$
The following result shows that we can use labelled bar diagrams to geometrically represent our polynomials $P_W(X_k; q)$ and ultimately prove results about the polynomials $Q_W(x; q)$.

**Theorem 1.1**

For all legal schedules $W = (w_1, w_2, \ldots, w_{k-1})$ we have

$$P_W(X_k; q) = \sum_{BD \in BD_W} A_{BD}(q) \ m_{BD}(X_k)$$

1.16

where $BD_W$ denotes the family of all bar diagrams corresponding to the schedule $W$, with $A_{BD}(q)$ and $m_{BD}(X_k)$ represent the polynomial in $q$ obtained from $BD$, and the monomial $\prod_{x_i \in S} x_i$ obtained by letting $S$ be the subset of $X_{k-1}$ of the elements whose bars are up in $BD$.

**Proof**

We need only show that the polynomial on the right hand side of 1.16 can be obtained by the recursive algorithm of definition 1.1. Since the base case 1.16 reduces to

$$P_0[X_0; q] = q \ x_{-1} + x_0,$$

we will proceed by induction on the length of the schedule. Let us assume that 1.16 is valid for schedules of length $k-1 \geq 0$. Now given $W = (w_1, w_2, \ldots, w_{k-1})$ and any integer $1 \leq w \leq w_{k-1} + 1$, we can obtain all the un-labelled bar diagrams in $BD_{W,w}$ by starting with an un-labelled Bar Diagram in $BD \in BD_W$ and appending to it first an $x_k$ up bar of length $w$ and then an $x_k$ down bar of same length. Calling the resulting Bar diagrams $BD^{(1)}$ and $BD^{(2)}$.

To determine the contributions that these two Bar Diagrams yield to the polynomial

$$\sum_{BD \in BD_{W,w}} A_{BD}(q) \ m_{BD}(X_k+1)$$

we only need to know one integer. Namely, if $m_{BD}(X_k) = \prod_{x_i \in S} x_i$ then what we need is the cardinality

$$a = |S \cap \{x_{k-1}, x_{k-2}, \ldots, x_{w-1}\}|$$

1.17

This given, we see that an up bar of length $w$ appended at the end of the Bar Diagram $BD$ will necessarily have exactly $a$ yellow cells. For the same reason a down bar of length $w$ appended at the end of $BD$ will have exactly $w - a$ yellow cells. This gives, for $0 < a < w$

$$A_{BD^{(1)}}(q) \ m_{BD^{(1)}}(X_{k+1}) = A_{BD}(q) \ m_{BD}(X_k) \ x_k (1 + \cdots + q^{a-1}) = A_{BD}(q) \ m_{BD}(X_k) \ \frac{x_k - q^a}{1 - q}$$

$$A_{BD^{(2)}}(q) \ m_{BD^{(2)}}(X_{k+1}) = A_{BD}(q) \ m_{BD}(X_k) (q^a + \cdots + q^{w-1}) = A_{BD}(q) \ m_{BD}(X_k) \ \frac{q^a - q^w}{1 - q}.$$  

In case $a = w$ or $a = 0$ then one of the two terms vanishes, but we can still write

$$A_{BD^{(1)}}(q) \ m_{BD^{(1)}}(X_{k+1}) + A_{BD^{(2)}}(q) \ m_{BD^{(2)}}(X_{k+1})$$

$$= A_{BD}(q) \ m_{BD}(X_k)(\frac{x_k - q^a}{1 - q} + \frac{q^a - q^w}{1 - q}) = A_{BD}(q) \ m_{BD}(X_k)(\frac{x_k - q^w}{1 - q} + q^a \frac{1 - x_k}{1 - q})$$

$$= \frac{x_k - q^w}{1 - q} A_{BD}(q) \ m_{BD}(X_k) + \frac{1 - x_k}{1 - q} A_{BD}(q) \ m_{BD}(X_k)|_{x_{k-1} \rightarrow qx_{k-1} ; 1 \leq i \leq w}$$

It follows from this that by summing over all $BD \in BD_W$ we will necessarily obtain the identity

$$\sum_{BD \in BD_{W,w}} A_{BD}(q) \ m_{BD}(X_{k+1})$$

$$= \frac{x_k - q^w}{1 - q} \sum_{BD \in BD_{W,w}} A_{BD}(q) \ m_{BD}(X_k) + \frac{1 - x_k}{1 - q} \sum_{BD \in BD_{W,w}} A_{BD}(q) \ m_{BD}(X_k)|_{x_{k-1} \rightarrow qx_{k-1} ; 1 \leq i \leq w}$$

By the inductive hypothesis, this is none other than

$$\sum_{BD \in BD_{W,w}} A_{BD}(q) \ m_{BD}(X_{k+1}) = \frac{x_k - q^w}{1 - q} P_W(X_k; q) + \frac{1 - x_k}{1 - q} P_W(X_k; q)|_{x_{k-1} \rightarrow qx_{k-1} ; 1 \leq i \leq w}$$

1.18

and Definition 1.1 gives

$$\sum_{BD \in BD_{W,w}} A_{BD}(q) \ m_{BD}(X_{k+1}) = P_{W,w}[X_{k+1}; q],$$

completing the induction and the proof.
Remark 1.2
An immediate by-product of this proof is the following identity
\[ P_W[X_k; q] \bigg|_{x_i = 1, \forall i} = (1 + q) \prod_{i=1}^{k-1} [w_i]_q \]  
1.19
In fact it follows from 1.18, by setting \( x_k = 1 \), that we have
\[ P_{W, w}[X_k; q] \bigg|_{x_i = 1, \forall i} = [w]_q P_W[X_k; q] \bigg|_{x_i = 1, \forall i} \]
This is precisely the step need to prove 1.19 by induction starting from the base case.

The following result gives an entirely explicit expression for the polynomials \( A_S(q) \) in 1.12.

Theorem 1.2
For a schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) and a given \( BD \in BD_W \) let \( m_{BD} = \prod_{x_i \in S} x_i \) and let \( a_i = a_i(BD) \) denote the number of \( x_i - 1, x_i - 2, \ldots, x_i - w_i \) that are in \( S \) then
\[ A_S(q) = q^{k-1} \prod_{x_i \in S} [a_i]_q = \prod_{x_i \in S} q^{a_i} [w_i - a_i]_q \]
1.20
Proof
This is one of the by-products of the proof of Theorem 1.1. See the equalities after the display in 1.17.

This paper has three more sections. In the next section we gather all the auxiliary results we need for our proof. In the third section we give the proof of the Functional Equation. In the last section we state what remains to be proved and state some further conjectures.

2. Auxiliary results

This section contains the minimal set of facts we need from [1] for the proof of the Functional Equation. We will include their proofs for sake of completeness

The Complementation identity
A close look at a single labelling \( LBD \) of a bar diagram \( BD \) that contributes to the sum in 1.14 reveals that the polynomial \( P_W(X_k; q) \) possesses a degree flipping involution onto itself. This useful fact will play a significant role in establishing various properties of these polynomials. In the display below we have depicted an \( LBD \) for the schedule \( W = (2, 3, 2, 3, 3, 4, 3, 4, 5, 5, 3, 4) \) together with the labelled diagram obtained by flipping \( LBD \) across the ground line

We can immediately see, in the above display, that if for bar \( x_i \) of \( LBD \) the label is \( q^{r_i} \), then the label of bar \( x_i \) in \( flip(LBD) \) is necessarily \( q^{s_i} \) with \( r_i + s_i = w_i - 1 \). This simple example makes it evident that we have a involution of the \( LBD \)’s which complements the monomials \( m_{BD}[X_k] = x_{\epsilon} \prod_{j=1}^{k-1} x_j^{x(bar \ j \ is \ up)} \), as subsets of the alphabet \( X_k \) and complements the power of \( q \) giving the total the weight of \( LBD \). More precisely we get for some \( \epsilon \in \{-1, 0\} \)

\[ \text{weight}_{\text{flip}(BD)}[X_k] = \frac{x_{\epsilon} \prod_{j=1}^{k-1} x_j^{x(bar \ j \ is \ up)}}{x_{\epsilon} \prod_{j=1}^{k-1} x_j^{x(bar \ j \ is \ up)}} \times q^{1 + \sum_{i=1}^{k-1} (w_i - 1)} \prod_{i=1}^{k-1} q^{r_i(LBD)} \]

When these identities are summed over all \( LBD \) we obtain a simple proof of the following basic result
Theorem 2.1 (The complementation identity)
For all legal schedules $W = (w_1, w_2, \ldots, w_{k-1})$ and $X_k = \{x_{-1}, x_0, \ldots, x_{k-1}\}$
$$P_W(X_k; q) = x_{-1} \cdots x_{k-1} q^{1+\sum_{i=1}^{k-1} (w_i-1)} P_W(\frac{1}{x_{-1}}, \frac{1}{x_0}, \ldots, \frac{1}{x_{k-1}}; \frac{1}{q})$$ \text{2.1}

As immediate corollary we have

Theorem 2.2
For all legal schedules $W = (w_1, w_2, \ldots, w_{k-1})$ the coefficients in the expansion

$$Q_W(x; q) = \sum_{b=1}^{k} x^b A_b(q)$$

satisfy the equalities

$$A_b(q) = q^{1+\sum_{i=1}^{k-1} (w_i-1)} A_{k+1-b}(1/q) \quad \text{(for 1 \leq b \leq k)} \quad \text{2.2}$$

Proof
Since by definition

$$Q_W(x; q) = \left. P_W[X_k; q] \right|_{x_i = x_i; q}$$

from 2.1 it follows that

$$Q_W(x; q) = q^{1+\sum_{i=1}^{k-1} (w_i-1)} x^{k+1} Q_W(1/x; 1/q).$$

Equating coefficients of $x_b$ gives 2.2.

The validity of the Functional equation when adding a component 1 to the schedule

The power of bar diagrams can be gauged by the following surprising general fact.

Theorem 2.3
For any legal schedule $W = (1, w_2, \ldots, w_{k-1})$ the polynomial $Q_W(x; q)$ satisfies the functional equation. In particular we have

$$Q_W(x; q) = A_1(q)x + A_k(q)x^k \quad \text{(with } A_1(q) = qA_k(q)) \quad \text{2.3}$$

Proof
In the display below we have depicted 6 diagrams when $w_1 = 1$. On the left we have examples of the case when bar $x_{-1}$ is up and on the right we have the case when bar $x_0$ is up.

Notice that since $x_1$ only acts on $x_0$ we see, in both cases, that if bar $x_1$ is not on the same side as bar $x_0$ we end up with an illegal coloring. But even if bar $x_1$ is on the same side but bar $x_2$ is not then again we end up with an illegal coloring, (remember that $w_i \leq w_{i-1} + 1$). In conclusion we see that in any cases all bars $x_1, \ldots, x_{k-1}$ must be on the same side as bar $x_0$, This proves 2.3.

Let us now substitute 2.3 into the functional equation. This gives

$$A_1x + A_kx^k - qA_1 - qA_kx^{k-1} + A_1x^{k-1} + A_k - qA_1x^k - qA_kx = (1 + x^k)(A_k - qA_1)$$

Thus we see that the functional equation will be satisfied if and only if $A_1 = qA_k$ and $A_k = \prod_{i=1}^{k-1} [w_i]_q$. But one look at the labeling in the above display will clearly show that these two conditions are trivially always satisfied.

Theorem 2.3 shows there is no loss restricting all our legal schedules to start with $w_1 = 2$. This is what we will do for the rest of this paper. The next result, proved in [3], shows that the same conclusion can be drawn even when a later component happens to be equal to 1.
Definition

Let us say that a legal schedule \( W \) is tame if and only if \( Q_W(x; q) \) satisfies the functional equation.

Our next aim here is to show that if two legal schedules \( W' = (w_1, \ldots, w_{k-2} \) and \( W = (w_1, \ldots, w_{k-2}, w_{k-1}) \) are both tame then all the legal schedules \( (w_1, \ldots, w_{k-1}, 1, w_{k+1}, \ldots, w_{\ell}) \) will be tame. It will, be convenient here to introduce the operator \( B_{k,w} \) whose action on a polynomial \( A(X_k, q) \) is defined by setting, for \( 1 \leq w \leq w_{k-1} + 1 \)

\[
B_{k,w}A(X_k, q) = \left. \frac{x^{q^w} - q^w}{1-q} A(X_k, q) + \frac{1-x^q}{1-q} A(X_k, q) \right|_{x_{k-1} = q^{x_{k-1}}; 1 \leq i \leq w}.
\] 2.15

Next, given a legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) we have the natural decomposition

\[
P_W(X_k; q) = A_W(X_{k-1}, q) + x_{k-1} B_W(X_{k-1}, q).
\] 2.16

This induces the decomposition

\[
Q_W(x; q) = A_W(x, q) + x B_W(x; q),
\] 2.17

where by a slight abuse of notation we have set

\[
a) \quad A_W(x, q) = A_W(X_{k-1}) \bigg|_{x_i = x_i^q}, \quad \text{b) } \quad B_W(x, q) = B_W(X_{k-1}) \bigg|_{x_i = x_i^q}.
\] 2.18

It is clear (from 2.17) that both \( A_W(x; q) \) and \( B_W(x; q) \) are of \( x \)-degree at most \( k - 1 \), but we can be more precise.

Proposition 2.1

Setting \( d = 1 + \sum_{i=1}^{k-1} (w_i - 1) \) we have

\[
a) \quad q^d x^k B_W(1/x; 1/q) = A_W(x; q), \quad \text{b) } \quad q^d x^k A_W(1/x; 1/q) = B_W(x; q).
\] 2.19

From this it follows that

\[
(1) \quad B_W(x; q) \text{ is of } x \text{-degree exactly } k - 1, \quad \quad (2) \quad A_W(x; q) \bigg|_x \neq 0
\]

Proof

From the complementation identity it follows that

\[
q^d \prod_{i=1}^{k-1} x_i \left( A_W(\frac{1}{x_{i-1}}, \ldots, \frac{1}{x_{k-2}}, \frac{1}{q}) + \frac{1}{x_{i-1}} B_W(\frac{1}{x_{i-1}}, \ldots, \frac{1}{x_{k-2}}, \frac{1}{q}) \right) = A_W(X_{k-1}; q) + x_{k-1} B_W(X_{k-1}; q).
\]

Expanding both sides and setting \( x_{k-1} = 0 \) gives

\[
q^d x_{i-1} \cdots x_{k-2} B_W(x_{i-1}^{-1}, \ldots, x_{k-2}^{-1}; 1/q) = A_W(X_{k-1}; q)
\] 2.20

while equating coefficients of \( x_{k-1} \) we get

\[
q^d x_{i-1} \cdots x_{k-2} A_W(x_{i-1}^{-1}, \ldots, x_{k-2}^{-1}; 1/q) = B_W(X_{k-1}; q)
\] 2.21

Thus 2.19 a) and b) follow by replacing all the \( x_i \) by \( x \) in 2.20 and 2.21. Note next that since the insertion of \( x_{k-1} \) is what causes the \( x \)-degree of \( Q_W(x; q) \) to reach \( k \), then \( x \)-degree of \( B_W(x; q) \) must be exactly \( k - 1 \). This given, 2.19 a) yields that we must also have \( A_W(x; q) \bigg|_x \neq 0 \).

We are now ready to present a first surprising fact.

Theorem 2.4

Let \( W = (w_1, \ldots, w_{k-1}) \) be a schedule and \( 1 \leq w \leq w_{k-1} + 1 \), then both \( Q_W(x; q) \) and \( Q_{W,w}(x; q) \) will satisfy the functional equation if and only if

\[
(1 - q/x) A_W,w(x; q) + x^k (1-xq) B_{W,w}(1/x; q) = [w]_q (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q
\] 2.22

In particular, by the complementation result, we must also have

\[
(1 - q/x) B_{W,w}(x; q) + x^k (1-xq) A_{W,w}(1/x; q) = x^k [w]_q (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q
\] 2.23

Proof
The idea is to start by writing

\[ P_W(X_k; q) = \sum_{r=0}^{w} A_r(X_k; q) \]

where \( A_r(X_k; q) \) is the sum of all terms in \( P_W(X_k; q) \) that contain exactly \( r \) of the variables acted upon by \( x_k \). Setting all \( x_i = x \), by a slight abuse of notation we will also write

\[ Q_W(x; q) = \sum_{r=0}^{w} A_r(x; q) \]

with

\[ A_r(x; q) = A_r(X_k; q) \bigg|_{x_i = x; \forall i} \]

\[ P_{W,w}(X_k; q) = \sum_{r=0}^{w} A_r(X_k; q) (\lfloor r \rfloor q x_k + q^r [w-r]_q) \]

thus we may write

\[ Q_{W,w}(x; q) = \sum_{r=0}^{w} A_r(x; q) (\lfloor r \rfloor q x + q^r [w-r]_q) \]

Notice, that 2.26 now yields the decomposition

\[ P_{W,w}(X_k; q) = A_{W,w}(X_k; q) + x_k B_{W,w}(X_k; q) \]

with

\[ A_{W,w}(X_k; q) = \sum_{r=1}^{w} A_r(X_k; q) q^r [w-r]_q \]

and

\[ B_{W,w}(X_k; q) = \sum_{r=1}^{w} A_r(X_k; q) [r]_q \]

and

\[ Q_{W,w}(x; q) = A_{W,w}(x; q) + x B_{W,w}(x; q) \]

with

\[ A_{W,w}(x; q) = \sum_{r=1}^{w} A_r(x; q) q^r [w-r]_q \]

and

\[ B_{W,w}(x; q) = \sum_{r=1}^{w} A_r(x; q) [r]_q. \]

Using 2.25, the functional equation for \( Q_W(x, q) \) may be written as

\[ (1-q/x) \sum_{r=0}^{w} A_r(x; q) + x^k (1-xq) \sum_{r=0}^{w} A_r(1/x; q) = (1+x^k)(1-q^2) \prod_{i=1}^{k-1} [w_i]_q \]

While, using 2.27, the functional equation for \( Q_{W,w}(x; q) \) becomes

\[ (1-q/x) \left( \sum_{r=0}^{w} A_r(x; q) (\lfloor r \rfloor q x + q^r [w-r]_q) \right) + \]

\[ + x^{k+1} (1-xq) \left( \sum_{r=0}^{w} A_r(1/x; q) (\lfloor r \rfloor q x^{-1} + q^r [w-r]_q) \right) = (1+x^{k+1})(1-q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q \]

Notice next that 2.32 multiplied by \([w]_q\) may be written in the form

\[ (1-q/x) \left( \sum_{r=0}^{w} A_r(x; q) (\lfloor r \rfloor q + q^r [w-r]_q) \right) + \]

\[ + x^k (1-xq) \left( \sum_{r=0}^{w} A_r(1/x; q) (\lfloor r \rfloor q + q^r [w-r]_q) \right) = (1+x^k)(1-q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q \]

Subtracting 2.34 from 2.33 and dividing by \( x-1 \) gives

\[ (1-q/x) \sum_{r=0}^{w} A_r(x; q)[r]_q + x^k (1-xq) \sum_{r=0}^{w} A_r(1/x; q)q^r [w-r]_q = x^k (1-q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q \]
This in turn, multiplied by $x$ and subtracted from 2.33 gives

$$(1 - q/x)\sum_{r=0}^{w} A_r(x; q)q^{r} [w - r]_q + x^k(1 - qx)\sum_{r=0}^{w} A_r(1/x; q)[r]_q = (1 - q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q$$  \hspace{1cm} 2.36

Using 2.31, 2.36 may also be rewritten as

$$(1 - q/x)A_{W,w}(x; q) + x^k(1 - qx)B_{W,w}(1/x; q) = (1 - q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q$$  \hspace{1cm} 2.37

Likewise 2.35 becomes

$$(1 - q/x)B_{W,w}(x; q) + x^k(1 - qx)A_{W,w}(1/x; q) = x^k(1 - q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q$$  \hspace{1cm} 2.38

This proves that both 2.22 and 2.23 are consequences of the functional equations of $Q_{W}(x; q)$ and $Q_{W,w}(x; q)$.

Conversely suppose that both 2.37 and 2.38 hold true. Then (using 2.25) we see that their sum is simply

$$(1 - q/x)Q_{W}(x; q)[w]_q + x^k(1 - qx)Q_{W}(1/x; q)[w]_q = (1 + x^k)(1 - q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q$$

and the functional equation of $Q_{W}(x; q)$ follows upon division by $[w]_q$. Similarly, multiplying 2.38 by $x$ and adding it to 2.37 gives

$$(1 - q/x)Q_{W,w}(x; q) + x^{k+1}(1 - qx)Q_{W,w}(1/x; q) = (1 + x^{k+1})(1 - q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q$$

which is the functional equation of $Q_{W,w}(x; q)$.

To complete the proof we need to show that 2.23 is a consequence of 2.22. To do this we will use the identities in 2.19 a) and b) for the schedule $W, w$ in the form

\begin{align*}
a) \quad A_{W,w}(x; q) &= q^{d'}x^{k+1}B_{W,w}(1/x; 1/q), \\
b) \quad B_{W,w}(1/x; q) &= q^{d'}x^{-(k+1)}A_{W,w}(x; 1/q),
\end{align*}

with $d' = 1 + w - 1 + \sum_{i=1}^{k-1} (w_i - 1)$, the $q$-degree of $P_{W,w}(X_{k+1}; q)$. Making these substitutions in 2.22 gives

$$(1 - q/x)q^{d'}x^{k+1}B_{W,w}(1/x; 1/q) + (1 - qx)q^{d'}x^{-1}A_{W,w}(x; 1/q) = (1 - q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q$$

Replacing $x$ and $q$ by $1/x$ and $1/q$ we get

$$(1 - x/q)q^{-d'}x^{-k-1}B_{W,w}(x; q) + (1 - 1/xq)q^{-d'}xA_{W,w}(1/x; q) = -q^{-(d'+1)}(1 - q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q$$

which multiplied $q^{d'+1}x^k$ gives

$$(q/x - 1)B_{W,w}(x; q) + x^k(xq - 1)A_{W,w}(1/x; q) = -x^k(1 - q^2)[w]_q \prod_{i=1}^{k-1} [w_i]_q$$

proving 2.23 and completing our proof.

An immediate Corollary of theorem 4.2 may be stated as follows.

**Theorem 2.5**

If both schedules $W' = (w_1, \ldots, w_{k-2})$ and $W = (w_1, \ldots, w_{k-1})$ are tame then the schedule $W, 1 = (w_1, \ldots, w_{k-1}, 1)$ is also tame as is for any legal schedule $W'' = (W, 1, v_2, v_3, \ldots, v_s)$. In particular, it follows that from now on there is no loss in assuming that $w_1 = 2$.

**Proof**
We will start by proving that $W, 1$ is tame. That will be our base case. From the hypotheses and Theorem 2.4 it follows that we have the two identities

\[(1 - q/x)A_W(x; q) + x^{k-1}(1 - qx)B_W(1/x; q) = (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q \quad 2.39\]

\[(1 - q/x)B_W(x; q) + x^{k-1}(1 - qx)A_W(1/x; q) = x^{k-1}(1 - q^2) \prod_{i=1}^{k-1} [w_i]_q \quad 2.40\]

Moreover we also have that

\[P_W(X_k, q) = A_W(X_k, q) + x_{k-1}B_W(X_k, q)\]

since for the schedule $W, 1$ the indeterminate $x_k$ only acts on $x_{k-1}$, it follows that

\[P_W(1; q) = A_W(X_k, q) + x_{k-1}x_kB_W(X_k, q). \quad 2.41\]

In particular we have

\[Q_W(x) = A_W(x, q) + x^2B_W(x, q).\]

This given, the functional equation for $Q_W(x; q)$ must be

\[(1 - q/x) \left( A_W(x, q) + x^2B_W(x, q) \right) + x^{k+1}(1 - qx) \left( A_W(1/x, q) + x^{-2}B_W(1/x, q) \right) =
\[(1 + x^{k+1})(1 - q^2) \prod_{i=1}^{k-1} [w_i]_q.\]

However, this is easily recognized to be none other than the identity in 2.39 plus the identity in 2.40 multiplied by $x^2$.

This given, we will show next that we have

\[P_{W,1,v_2,v_3,...,v_s}(X_{k+s}, q) = \prod_{i=2}^{s}[v_i]_q \left( A_W(X_{k-1}; q) + x_{k-1}x_kx_{k+1} \cdots x_{k-1+s}B_W(X_{k-1}; q) \right) \quad 2.42\]

In particular it follows that

\[Q_{W,1,v_2,v_3,...,v_s}(x, q) = \prod_{i=2}^{s}[v_i]_q \left( A_W(x; q) + x^{s+1}B_W(x; q) \right) \quad 2.43\]

We start by proving the identity in 2.42 by induction on $s$ with 2.41 as the base case $s = 1$. This given, let us assume 2.42 true for $s - 1$, that is

\[P_{W,1,v_2,v_3,...,v_{s-1}}(X_{k+s-1}, q) = \prod_{i=2}^{s-1}[v_i]_q \left( A_W(X_{k-1}; q) + x_{k-1}x_kx_{k+1} \cdots x_{k-1+s-1}B_W(X_{k-1}; q) \right)\]

Since $x_{k-1+s}$ acts on $x_{k-1+s-1} \cdots x_{k-1+s-v_s}$ and $v_s \leq s$ it follows that $x_{k-1}$ is the last variable which could be acted upon by $x_{k-1+s}$. This implies that

\[P_{W,1,v_2,v_3,...,v_s}(X_{k+s}, q) = \prod_{i=2}^{s-1}[v_i]_q \left( A_W(X_{k-1}; q)[v_s]_q + x_{k-1}x_kx_{k+1} \cdots x_{k-1+s-1}x_{k-1+s}[v_s]_qB_W(X_{k-1}; q) \right)\]

Proving 2.42 and completing the induction. Now we may rewrite 2.39 as

\[(1 - q/x)(A_W(x; q) + x^{k+s}(1 - qx)(x^{-s-1}B_W(1/x; q)) = (1 - q^2) \prod_{i=1}^{k-1} [w_i]_q\]

and 2.40 multiplied by $x^{s+1}$ gives

\[(1 - q/x)x^{s+1}B_W(x; q) + x^{k+s}(1 - qx)A_W(1/x; q) = x^{k+s}(1 - q^2) \prod_{i=1}^{k-1} [w_i]_q\]
adding these two equalities, multiplying by \( \prod_{i=2}^{s}[v_i]_q \), using 2.43 and setting \( W' = W, 1, v_2, v_3, \ldots, v_s \) we finally obtain
\[
(1 - q/x)Q_{W''}(x;q) + x^{k+s}(1 - qx)(Q_{W''}(1/x;q)) = (1 - q^2) \prod_{i=1}^{k-1}[w_i]_q \sum_{i=2}^{s}[v_i]_q(1 + x^{k+s})
\]
proving the functional equation for \( Q_{W,1,v_2,v_3,\ldots,v_s}(x;q) \) as asserted.

The validity of the Functional equation when all components of the schedule are equal to 2

In \( W \), it was discovered that if \( W \) is tame then for certain pairs of integers \( u, v \) a legal schedule \( uvW \) is also tame. This given, the following special case will be essential in our proof of the functional equation,

**Theorem 2.6**

*For any legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) we have*

\[
Q_{2,2,W}(x;q) = qxQ_{W}(x;q) + Q_{2,2,W}(x;q) \bigg|_x + Q_{2,2,W}(x;q) \bigg|_{x^{k+2}} x^{k+2}  \tag{2.44}
\]

*In particular, if \( W = (w_1, w_2, \cdots, w_{k-1}) \) is tame then \( 22W = (2, 2, w_1, w_2, \cdots, w_{k-1}) \) is also tame*

**Proof**

We will establish 2.44 by a bar diagram argument. In the diagrams below, the yellow “ellipses” are to represent a generic bar diagrams for the schedule \( W \) with omission of the initial bars \( ↑^x \downarrow^0 \) and \( \downarrow^{x-1}↑^0 \).

The diagrams above will be referred to as class[1], class[2], \ldots, class[6] listed as we scan them from left to right and from top to bottom. We will use them to represent a decomposition of the collection of the bar diagrams of the schedule 2, 2, W into “classes” according to the positions of the first four bars. That is can simply represent them by the symbols

\[
↑^{x-1}↓^0 \downarrow^1 \downarrow^2, \quad ↑^{x-1}↓^0 ↑^1 ↑^2, \quad \downarrow^{x-1}↑^0 ↑^1 ↑^2 \quad \downarrow^{x-1}↑^0 ↑^1 ↓^2 \quad \downarrow^{x-1}↑^0 ↓^1 ↑^2 \quad \downarrow^{x-1}↑^0 ↓^1 ↓^2.
\]

In class[1] and class[3] bar \( x_2 \) has no choices, being only of length 2 in each case it has to be on the same side of the ground line as bar \( x_1 \). For class[2] and class[4] we chose to depict bar \( x_2 \) on the same side of bar \( x_1 \). This forces all the remaining bars to be totally yellow. We can clearly see that for class[1], class[2], class[3] and class[4], the polynomial above or below each diagram gives precisely the contribution that the class makes to \( Q_{2,2,W}(x;q) \).

Finally, we see that class[5] and class[6] are obtained by choosing the other alternative for bar \( x_2 \). The second author’s beautiful idea in her thesis, is to view class[5] and class[6], as the result of prepending the pair \( ↑^{x-1}↓^0 \) to the collection of bars diagrams of the schedule \( W \) that start with \( ↑^{x-1}↓^0 \) and respectively prepending the pair \( ↓^{x-1}↑^0 \) to the collection of bars diagrams of the schedule \( W \) that start with \( ↓^{x-1}↑^0 \). A look at the initial required labelings reveals that the contribution of these two classes to the polynomial \( Q_{2,2,W}(x;q) \) is none other than \( xqQ_{W}(x;q) \). This given, since we can easily see that class[1] and class[2] yield the coefficient of \( x \) and class[2] and class[3] yield the coefficient of \( x^{k+1} \) we obtain the equalities

\[
\begin{align*}
& a) \quad Q_{2,2,W}(x;q) \bigg|_x = (2q^2 + q^3) \prod_{i=1}^{k-1}[w_i]_q, \\
& b) \quad Q_{2,2,W}(x;q) \bigg|_{x^{k+2}} = (1 + 2q) \prod_{i=1}^{k-1}[w_i]_q \tag{2.45}
\end{align*}
\]

This completes our proof of 2.44.
We are left with checking that the polynomial $Q_{2,2,W}(x; q)$ satisfies the functional equation. For convenience let us express it as $Q_{2,2,W}(x; q) = b_1(q)x + xqQ_w(x; q) + b_{k+2}(q)x^{k+2}$. This gives
\[
(1 - q/x)Q_{2,2,W}(x; q) + x^{k+2}(1 - qx)Q_{2,2,W}(1/x; q) = 
\]
\[
= b_1x - q b_1 + b_{k+2}x^{k+2} - q b_{k+2}x^{k+1} + b_1 x^{k+1} - q b_1 x^{k+2} + b_{k+2} - q b_{k+2}x 
\]
\[
+ xq(1 - q/x)Q_W(x; q) + x^{k+2}(1 - qx)(q/x)Q_W(1/x; q) 
\]
\[
= (b_{k+2} - q b_1)(1 + x^{k+2}) + (b_1 - q b_{k+2})(x + x^{k+1}) + 
\]
\[
\text{(since } W \text{ is tame)} + qx(1 + x^k)(1 - q^2)\prod_{i=1}^{k-1}[w_i]_q 
\]
But from 2.45 it follows that
\[
(b_1 - q b_{k+2}) + q(1 - q^2)\prod_{i=1}^{k-1}[w_i]_q = (q^3 - q)\prod_{i=1}^{k-1}[w_i]_q + q(1 - q^2)\prod_{i=1}^{k-1}[w_i]_q = 0. 
\]
Finally, we are left with
\[
b_{k+2} - q b_1 = \left((1 + 2q) - q(2q^2 + q^3)\right)\prod_{i=1}^{k-1}[w_i]_q = (1 - q^2)[2/q][2]_q \prod_{i=1}^{k-1}[w_i]_q. 
\]
This completes our argument.

Since the schedules $W = \{\emptyset\}$ and $W = (2)$ are clearly tame we have the following important corollary.

**Theorem 2.7**

The schedules of the form $W = (2, 2, \ldots, 2)$ are tame.

We should mention that the first proof of this result appeared in [8], however the present proof proves much more with considerably less effort.

**3. Proof of the Functional equation**

After so many years of wondering about the validity of this result, the simplicity and modality of its proof is stunning. The proof assumes that all legal schedules $W = (w_1, w_2, \ldots, w_{h-1})$ with $h < k$ are tame. Then proves it for $k$ by recursing over a finite set.

More precisely, let $F[k]$ denote the family of all legal schedules $W = (w_1, w_2, \ldots, w_{k-1})$ whose components are all $\geq 2$, totally ordered by lex order. The minimal element of $F[k]$ is therefore the schedule $W = (2, 2, \ldots, 2)$ with $k - 1$ 2’s. The proof constructs two recursions $\phi(W) = [W'; W'']$ and $\psi(W) = [W'''; W'''''''$ with the following properties

1. $W', W'' <_{lex} W$, for $\phi$ and $W', W'', W''' <_{lex} W$ for $\psi$.
2. For each $W$ in $F[k]$ one and only one of $\phi$ or $\psi$ applies.
3. For $\phi$: $Q_W = (1 + q)Q_{W'} - qQ_{W''}$
4. For $\psi$: $Q_W = \frac{q}{1+q}Q_{W'} + Q_{W''} - \frac{q}{1+q}Q_{W'''}$
5. The only ones of $W', W'' <_{lex} W$, for $\phi$ and $W', W'', W''', W'''', W''''', W'''''$ for $\psi$, that may be not in $F[k]$ will have one of their component equal to 1.

It is easily shown that these recursions reduce the tameness of $W$ to the tameness of their predecessors $W', W''$ or $W', W''$, $W'''$ as the case may be. The base cases turn out to be legal schedules $W = (w_1, w_2, \ldots, w_{k-1})$ with at least one component 1 or the schedule $W = (2, 2, \ldots, 2)$ with $k - 1$ 2’s. Since the base cases have been shown to be tame it follows that all elements of $F[k]$ are tame.

The remainder of this section is dedicated to the construction of the recursions $\phi$ and $\psi$ and proving their stated properties. We will start with helpful definitions and some auxiliary observations.

Recall that $W = (w_1, w_2, \ldots, w_{k-1})$ is legal if $w_i \leq w_{i-1} + 1$ for all $2 \leq i \leq k - 1$. It is clear that all schedules $W \in F[k]$, except the first, have at least one increase. The last increase will be called the *canonical increase* of $W$. If $w_{h-1} = v$ and $w_h = v + 1$ is an increase of $W$ we will say that
bar $x_r$ splits $w_{h-1}$ and $w_h$ if $r - w_r = h$. Recalling that bar $x_r$ acts on bars $x_{r-1}, x_{r-2}, \ldots, x_{r-w_r}$. That means that bar $x_r$ acts on bar $x_h$ but not on bar $x_{h-1}$. If no bar splits the canonical increase of $W$ we will write $W = U, v, v + 1, V$ where $v, v + 1$ is the canonical increasing pair and define

$$\phi(W) = [U, v, v, V; U, v - 1, v, V] = [W'; W'']$$

3.1

If some bar does split the canonical increase then let $x_r = x_{h+j}$ be the first bar that does. Since that means we have $h + j - w_{h+j} = h$ we can write $W = U, v, v + 1, V', j, V''$ with $j$ in position $h + j$. This given, we define

$$\psi(W) = [U, v, v + 1, V', j - 1, V''; U, v, v, V', j + 1, V''; U, v - 1, v, V', j + 1, V''] = [W'; W''; W'''$$

3.2

It is clear from 3.1 and 3.2 that property (1) of $\phi$ and $\psi$ is satisfied. The following two propositions show that both are well defined.

**Proposition 3.1**

*The schedules $W'$ and $W''$ in 3.1 are legal*

**Proof**

The only problem arises if $V$ would start with a component $v + 2$. But if that were the case then we could write $W = U, v, v + 1, v + 2, V'$ and contradict that $v, v + 1$ is the last increase of $W$.

**Proposition 3.2**

*The schedules $W', W''$ and $W'''$ in 3.2 are legal*

**Proof**

Notice first that $W''$ and $W'''$ would not be legal if $V'$ could start with $v + 2$. But then again as we have seen in the previous proof, that would contradict that $v, v + 1$ is the last increase of $W$. Next, the legality of $W'$ would not hold if $V''$ could start with $j + 1$. But if that were the case then $w_{h+j} = j$ and $w_{h+j+1} = j + 1$ would again contradict that the pair $v, v + 1$ is canonical for $W$. The last remaining issue is the $j + 1$ in $W''$ if $V'$ could end with a component less than $j$. However the legality of $W$ would force that component to be $j - 1$ and we could write $W = U, v, v + 1, V', j - 1, j, V''$ again contradicting that the pair $v, v + 1$ is canonical for $W$.

**Remark 3.1**

For future purposes we need to focus on the schedule obtained from $W = U, v, v + 1, V', j, V''$ by setting $W = U, v, v + 1, V', j + 1, V''$. Notice first that since $W$ is legal the last component of $V'$ must be $\geq j - 1$. But equality here cannot hold for two reasons. Firstly, since the pair $j - 1, j$ would contradict that $v, v + 1$ is canonical for $W$. Secondly, since we must recall that $x_r = x_{h+j}$ was chosen to be the first bar that splits the pair $v, v + 1$. This shows that $W$ is legal. Nevertheless the last component in $V'$ could very well be $j$ and the pair $v, v + 1$ would cease to be canonical for $W$. However, what is of crucial importance is that there are no bars in $W$ that split the pair $v, v + 1$. The reason is that $j$ in $W$ was chosen to be the length of first bar that did. Now in $W$ that length is changed to $j + 1$. Could there be in $V''$ a bar $x_{r'}$ that splits $v, v + 1$? The answer in no. Let us recall that the position of $v + 1$ in $W$ is $h$, thus for $x_{r'}$ to act on $v + 1$ and not on $v - 1$ we must have $r' - w_{r'} = h$. This gives $w_{r'} = r' - h$. Since $r' > r = h + j$ then we would also have $w_{r'} > h + j - h = j = w_r$. But that cannot happen in $W$ since the last increase is $v, v + 1$.

Our definitions of $\phi$ and $\psi$ guarantee property (2). Property (5) is assured by 3.1 and 3.2. In fact, if $W$ has any component 1 there is no need to apply $\phi$ or $\psi$ to it. This given, we see from 3.1 that an application of $\phi$ may produce a $W''$ with a component equal to 1 by the replacement $v_\rightarrow v - 1$. Similarly we see from 3.2 that $W'$ by $j_\rightarrow j - 1$ and $W'''$ by $v_\rightarrow v - 1$ may end up with a component 1.

Thus we are left with proving (3) and (4) and deriving from these identities that the tameness of the corresponding $W$'s forces the tameness of $W$. The following result not only proves (3), but also plays a role in the proof of (5).
Proposition 3.3
Let \( W = U, v, v + 1, V \), where \( v, v + 1 \) is not necessarily the last increase, but suppose that there is no bar \( x_r \) that splits the pair \( v, v + 1 \) then we have
\[
Q_{U,v,v+1,V}[x, q] = (1 + q)Q_{U,v,v,V}[x, q] - qQ_{U,v-1,v,V}[x, q].
\]  
3.3

Proof
Let \( v, v + 1 \) be the components \( w_{h-1}, w_h \) of \( W \). As before let us denote by \( W' \) and \( W'' \) the schedules appearing in the right hand side of 3.3. If \( BD', BD'', BD'' \) are corresponding (same up-down) bar diagrams of \( W, W', W'' \) respectively, it will be convenient to use the short notation
\[
\Gamma^{W-[2]W'+qW''}(BD) = \Gamma^W(BD) - [2]\Gamma^{W'}(BD') + q\Gamma^{W''}(BD''),
\]  
3.4
where \( \Gamma^W, \Gamma^{W'}, \Gamma^{W''} \) denote taking the weights contributed by these bar diagrams to the respective polynomials \( P_W, P_{W'}, P_{W''} \). Now let \( BD^U \) and \( BD^V \) be the portions of \( BD \) contributed by \( U \) and \( V \) respectively. Our proof of 3.3 is based on establishing the following identities.
\[
\Gamma^{W-[2]W'+qW''}(BD_U \downarrow \downarrow BD^V) = \Gamma^{W-[2]W'+qW''}(BD_U \uparrow \uparrow BD^V) = 0,
\]  
3.5

where the symbol \( "BD_U \downarrow \downarrow BD^V" \) denotes the bar diagram \( BD \) which starts with \( BD^U \) followed with bars \( x_{h-1}, x_h \) both down and finishing with \( BD^V \). In the same vein as in 3.4, we are also requiring here the Bar Diagrams, \( BD' \) and \( BD'' \) to have the same up or down state of bars \( x_{h-1}, x_h \) as in \( BD \). To prove these identities let us recall that the contribution to the weight of bar \( x_j \) in a diagram \( BD \) of \( W \) is \( [a_j] \) if bar \( x_j \) is up and \( [w_j] - [a_j] = q^{x_j}[w_j - a_j] \) if bar \( x_j \) is down. Where we have denoted by \( a_j \), the number of up-arrrows acted upon by bar \( x_j \) when it is up.

Since the three schedules \( W, W', W'' \) only differ at the two entries in positions \( h-1 \) and \( h \), every bar diagram \( BD, BD', BD'' \) shares the same \( a_j \) sequence except for \( a_{h-1} \) and \( a_h \). Keeping this in mind we only need to distinguish two cases:

**Case 1:** \( BD \) has bar \( x_{h-v-1} \) down. There are four states of bars \( x_{h-1} \) and \( x_h \) as listed in the following table. (Here, next to each arrow we place the number of up bars it acts upon, when that bar is up)

| \( x_{h-v-1} \) | \( w_{h-1} = v \) | \( w_h = v + 1 \) | \( w'_{h-1} = v \) | \( w'_h = v \) | \( w''_{h-1} = v - 1 \) | \( w''_h = v \) |
|---|---|---|---|---|---|---|
| \( \downarrow b \) | \( \downarrow b \) | \( \downarrow b \) | \( \downarrow b \) | \( \downarrow b \) | \( \uparrow b \) | \( \uparrow b \) |
| \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) |
| \( \downarrow b \) | \( \downarrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) |
| \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b \) |

Thus we have
\[
\Gamma^{W-[2]W'+qW''}(BD_U \downarrow \downarrow BD^V) = \cdots \times (1 + q)(((v) - [b])((v + 1) - [b]) - (1 + q)((v) - [b])^2 + q((v - 1) - [b])((v) - [b])) = 0,
\]
\[
\Gamma^{W-[2]W'+qW''}(BD_U \uparrow \uparrow BD^V) = \cdots \times ([b][b + 1] + (1 + q)[b][b + 1] + q[b][b + 1]) = 0,
\]
\[
\Gamma^{W-[2]W'+qW''}(BD_U \uparrow \uparrow BD^V) = \cdots \times (1 + q)(((v) - [b])((v) - [b]) - (1 + q)((v) - [b])(v - 1 - [b]) = -q^v[b] \cdots ,
\]
\[
\Gamma^{W-[2]W'+qW''}(BD_U \downarrow \downarrow BD^V) = \cdots \times (q[b][v - 1][v - b + 1] - (1 + q)[b][v - 1][v - b + 1] + q[b][v - 1][v - b + 1]) = q^v[b] \cdots .
\]
Case 2: \( BD \) has \( x_{h-1} \uparrow \). There are also four states of bars \( h-1 \) and \( h \) as listed in the following table.

| \( x_{h-1} \) | \( w_{h-1} = v \) | \( w_{h} = v + 1 \) | \( w'_{h-1} = v \) | \( w'_{h} = v \) | \( w''_{h-1} = v - 1 \) | \( w''_{h} = v \) |
|---|---|---|---|---|---|---|
| \( x_{h-1} \uparrow \) | \( \downarrow b + 1 \) | \( \downarrow b + 1 \) | \( \downarrow b + 1 \) | \( \downarrow b \) | \( \downarrow b \) |
| \( x_{h-1} \uparrow \) | \( \uparrow b + 1 \) | \( \uparrow b + 2 \) | \( \uparrow b + 2 \) | \( \uparrow b \) | \( \uparrow b \) | \( \uparrow b + 1 \) |
| \( x_{h-1} \uparrow \) | \( \downarrow b + 1 \) | \( \uparrow b + 1 \) | \( \uparrow b \) | \( \down b \) | \( \up b \) |
| \( x_{h-1} \uparrow \) | \( \up b + 1 \) | \( \up b + 2 \) | \( \up b + 1 \) | \( \up b \) | \( \up b \) | \( \up b + 1 \) |

Thus we have

\[
\Gamma^{W-2}W' + qW''(BDU \downarrow \downarrow BDV) = \cdots \times \times \left( (|v| - [b + 1]([v + 1] - [b + 1]) - (1 + q)q([v] - [b])(q - [v] - [b]) \right) = 0,
\]

\[
\Gamma^{W-2}W' + qW''(BUU \uparrow BDV) = \cdots \times \times \left( (b + 1)[b + 2] - (1 + q)[b + 1][b + 1] + q[b][b + 1] \right) = 0,
\]

\[
\Gamma^{W-2}W' + qW''(BDU \downarrow \uparrow BDV) = \cdots \times \times \left( (|v| - [b + 1])(|v| - [b + 1]) + q([v] - [b])q[b][b] \right) = (|v| - [b + 1]) \cdots,
\]

\[
\Gamma^{W-2}W' + qW''(BDU \downarrow \uparrow BDV) = \cdots \times \times \left( (|v| - [b + 1])(|v| - [b + 1]) + (1 + q)[b + 1][v - [b + 1]] + q[b][v - [b + 1]] \right) = -(|v| - [b + 1]) \cdots.
\]

Summing the weights of bar diagrams considered in these two cases proves 3.5 and completes the proof of the proposition.

Let \( W = U, v, v + 1, V \) where \( v, v + 1 \) is the canonical increasing pair of \( W \) and suppose that there is a bar \( x_r \) that splits the pair \( v, v + 1 \). As we did before, let \( r \) be the smallest one. That is \( r = h + j \) with \( j \) the smallest such that \( h + j - w_{h+j} = h \). Thus we can write \( W = U, v, v + 1, V', j, V'' \) with \( j \) in position \( h + j \).

This given, from Remark 3.1 it follows that the schedule \( \tilde{W} = U, v, v + 1, V', j, V'' \) is not only legal but Proposition 3.3 is also applicable to it. This gives the identity

\[
Q_{\tilde{W}}[x, q] - (1 + q)Q_{U, v, v + 1, V', j, V''} = 0
\]

We claim that we can also prove

**Proposition 3.4**

Let the pair \( v, v + 1 \) be canonical for the schedule \( W = U, v, v + 1, V', j, V'' \) with \( v + 1 \) in position \( h \) and \( j \) in position \( r = h + j \), so that bar \( x_r = x_{h+j} \) splits \( v, v + 1 \), we can still prove the identity

\[
Q_{U, v, v + 1, V', j, V''} = (1 + q)Q_{U, v, v + 1, V', j, V''} + qQ_{U, v, v + 1, V', j, V''} = 0
\]

In particular it follows that we have

\[
Q_{W} = \frac{q}{1+q}Q_{U, v, v + 1, V', j, V''} + \frac{q}{1+q}Q_{U, v, v + 1, V', j, V''} = \frac{q}{1+q}Q_{U, v, v + 1, V', j, V''}.
\]

**Proof** For convenience let us write

\[
\tilde{W} = U, v, v + 1, \tilde{V} \quad W' = U, v, v + 1, \tilde{V}' \quad W'' = U, v, v + 1, \tilde{V}''
\]

with

\[
\tilde{V} = V', j + 1, V'', \quad \tilde{V}' = V', j, V''', \quad \tilde{V}'' = V', j - 1, V'''.
\]

This given, we will prove 3.9 by simultaneously summing the weights

\[
\Gamma^{\tilde{W}}(BDU \downarrow \downarrow BD\tilde{V}), \quad \Gamma^{W}(BDU \downarrow \downarrow BD\tilde{V}'), \quad \Gamma^{W''}(BDU \downarrow \downarrow BD\tilde{V}'''),
\]

then doing the same by the two arrows "\( \downarrow \downarrow \)" respectively replaced by "\( \uparrow \uparrow \)" and "\( \down \uparrow \)".
Notice that the uniqueness of a splitter of the canonical increasing pair assures (see the end of Remark 3.1) that there will be no bar \( x_{h+j} \) with \( i \neq j \) in \( \bar{V} = V', j + 1, V'' \) and \( \bar{V} = V', j - 1, V'' \) that will also split \( v, v + 1 \). Consequently, we only need to focus on the \( a_i \)-sequences in positions \( h - 1, h \) and \( h + j \).

Keeping this in mind, we will verify the following equalities. The identity in 3.9 then follows by summing over all \( BD^U \) and \( BD^\bar{V} \) with the four up, down states of bars \( x_{h-1} \) and \( x_h \).

\[
\begin{align*}
\text{a) } & \Gamma \bar{W} - [2] W' + q W'' (BD^U \downarrow \downarrow BD^\bar{V}) = \Gamma \bar{W} - [2] W' + q W'' (BD^U \uparrow \uparrow BD^\bar{V}) = 0, \\
\text{b) } & \Gamma \bar{W} - [2] W' + q W'' (BD^U \uparrow \downarrow BD^\bar{V}) = -\Gamma \bar{W} - [2] W' + q W'' (BD^U \downarrow \uparrow BD^\bar{V}).
\end{align*}
\]

To prove these three identities, we observe that \( \bar{W}, W', W'' \) only differ at the \( h + j \)-th entry, so does their corresponding weight of any particular bar diagram. We also need to consider the state of bars \( x_{h-1} \) and \( x_h \) since they cause the difference of the weights of bar \( h + j \) with respect to \( \bar{W}, W', W'' \).

We use the following table for the cases when bar \( h + j \) is up or down:

| \( h-1 \) | \( h \) | \( h+j \) | \( w_{h,j} \) |
|---|---|---|---|
| \( \downarrow \) | \( \downarrow \) | \( \downarrow \) | \( v \) |
| \( \downarrow \) | \( \uparrow \) | \( \uparrow \) | \( v + 1 \) |
| \( \uparrow \) | \( \downarrow \) | \( \downarrow \) | \( v + 1 \) |
| \( \uparrow \) | \( \uparrow \) | \( \uparrow \) | \( v + 2 \) |

The case when bar \( x_{h+j} \) is down can be derived from the case when bar \( x_{h+j} \) is up. In fact, since \( [j + 1] - (1 + q)[j] + q(j - 1) = 0 \), the contribution of bar \( x_{h+j} \) down is simply that of bar \( x_{h+j} \) up times \(-1\). Thus we assume that bar \( x_{h+j} \) is up in the computations below.

**Case 1**: Bars \( x_{h-1} \) and \( x_h \) are both down or both up. We only need to concentrate on bar \( x_{h+j} \) since its contribution forces the whole sum to vanish. In fact, have

\[
\Gamma \bar{W} - [2] W' + q W'' (BD^U \downarrow \downarrow BD^\bar{V}) = \cdots ([a]x - (1 + q)[a]x + q[a]x) = 0,
\]

\[
\Gamma \bar{W} - [2] W' + q W'' (BD^U \uparrow \uparrow BD^\bar{V}) = \cdots ([a + 2]x - (1 + q)[a + 1]x + q[a]x) = 0.
\]

**Case 2**: Exactly one of bars \( x_{h-1} \) and \( x_h \) is up. In this case we need to concentrate on all three bars \( x_{h-1}, x_h \) and \( x_{h+j} \). We have

\[
\begin{align*}
\Gamma \bar{W} - [2] W' + q W'' (BD^U \downarrow \downarrow BD^\bar{V}) & = \cdots q^{v-c}[v-c][c]\left((a+1)x - (1+q)[a+1]x + q[a]x\right) \\
& = -q^{a+c+1}[c][v-c]x \cdots , \\
\Gamma \bar{W} - [2] W' + q W'' (BD^U \uparrow \uparrow BD^\bar{V}) & = \cdots q^{a+c+1}
\end{align*}
\]

This completes the proof of the identity in 3.9.

Setting the two identities 3.8 and 3.9 side by side we notice that we can write them in the form

\[
\begin{align*}
Q_{\bar{W}} & = (1 + q)Q_{U,v,v',V',j+1,V''} - qQ_{U,v-1,v,V',j+1,V''} \\
Q_{\bar{W}} & = (1 + q)Q_{W} - qQ_{U,v,v+1,V',j-1,V''}
\end{align*}
\]

Eliminating \( Q_{\bar{W}} \) proves the identity

\[
Q_{W} = \frac{q}{1+q}Q_{U,v,v+1,V',j-1,V''} + Q_{U,v,v,V',j+1,V''} - \frac{q}{1+q}Q_{U,v-1,v,V',j+1,V''},
\]

which we easily recognize as 3.10. This completes our proof of Proposition 3.4.

The next and final step is to prove the following beautiful fact
Proposition 3.5
Let \( W = (w_1, w_2, \ldots, w_{k-1}) \) be a legal schedule
(a) If we use \( \phi(W) = (W', W'') \) and \( W', W'' \) are tame then \( W \) is tame,
(b) If we use \( \psi(W) = (W', W'', W''') \) and \( W', W'', W''' \) are tame then \( W \) is tame.

Proof
Recall that a legal schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \) is tame if an only if
\[
(1 - q/x)Q_W(x; q) + x^k(1 - qx)Q_W(1/x; q) = (1 + x^k)(1 - q^2)\prod_{i=1}^{k-1}[w_i]_q \quad (3.13)
\]
This given, notice first that the \( \mathbb{Q}[q] \)-linearity operator
\[
\mathcal{L}F(x; q) = (1 - q/x)F(x; q) + x^k(1 - qx)F(1/x; q)
\]
sends a Laurent polynomial in \( x \) into another such polynomial. For convenience, for any positive integral vector \( U = (u_1, u_2, \ldots, u_r) \) set \( \Pi[U] = \prod_{i=1}^r[u_i]_q \). With this notation we may write (3.13) as
\[
\mathcal{L}Q_W = (1 + x^k)(1 - q^2)\Pi[W]
\]
If we use \( \phi \) then from Proposition 3.3 and \( W = U, v, v + 1, V \) then it follows that
\[
Q_{U,v,v+1,V}[x, q] = (1 + q)Q_{U,v,v,V}[x, q] - qQ_{U,v-1,v,V}[x, q].
\]
Thus the \( \mathbb{Q}[q] \)-linearity of \( \mathcal{L} \) gives
\[
\mathcal{L}Q_{U,v,v+1,V} = (1 + q)\mathcal{L}Q_{U,v,v,V} - q\mathcal{L}Q_{U,v-1,v,V} \quad (3.15)
\]
If \( W' = U, v, v, V \) and \( W'' = U, v - 1, v, V \) are tame then 3.15 becomes
\[
\mathcal{L}Q_W = (1 + x^k)(1 - q^2)\Pi[U]\left((1 + q)|v_q|v_q - q[v - 1]_q|v_q|\right)\Pi[V]
\]
and the tameness of \( W \) is reduced to showing the trivial identity
\[
|v_q|v_q - q[v - 1]_q|v_q| = (1 + q)|v_q|v_q - q[v - 1]_q|v_q|.
\]
If we use \( \psi \) and \( W = U, v, v + 1, V', j, V'' \) then from Proposition 3.4 it follows that
\[
Q_W = \frac{q}{1+q}Q_{U,v,v+1,V',j-1,V''} + Q_{U,v,v,V',j+1,V''} - \frac{q}{1+q}Q_{U,v-1,v,V',j+1,V''} \quad (3.16)
\]
Thus the tameness of \( W', W'' \) and \( W''' \) gives
\[
\mathcal{L}Q_W = (1 + x^k)(1 - q^2)\Pi[U]|v_q\left(\frac{q}{1+q}|v+1_q|j-1_q + |v_q|j+1_q - \frac{q}{1+q}|v-1_q|j+1_q\right)\Pi[V']\Pi[V'']
\]
and the tameness of \( W \) is reduced to showing the identity
\[
|v+1_q|j_q = \frac{q}{1+q}|v+1_q|j-1_q + |v_q|j+1_q - \frac{q}{1+q}|v-1_q|j+1_q
\]
which is trivially true. This completes the proof of Proposition 3.5 and the proof of the tameness of all legal schedules.

In the display below, our recursion is applied to the 14 schedules of length 4 in lex order, (omitting \([2, 2, 2, 2]\)). As we can plainly see, by reading the columns from top to bottom and from left to right, every schedule is followed by its image by \( \phi \) or \( \psi \) as dictated by the algorithm.
4. FROM BAR DIAGRAMS TO PARKING FUNCTIONS: OPEN PROBLEMS AND CONJECTURES

We will start by making clearer the connection between Parking Functions and Labeled Bar Diagrams. This may require the repetition of some material covered in the first section. However it is very important that this connection is clearly understood for future applications of the results of this paper.

To carry this out we need first a detailed description of an algorithm used in [1] to construct all the Parking Functions with prescribed diagonal cars. This algorithm extends to the case of Frobenius series what Haglund and Loehr did in [5] for Hilbert series of Diagonal Harmonics.

This is better understood by means of a specific example. Given the permutation

\[ \sigma = [5, 6, 2, 4, 7, 8, 1, 3] \]

The first step is to break it into increasing runs

\[ \text{runs}(\sigma) = 5, 6 | 2, 4, 7, 8 | 1, 3 \]

Our task is to construct all the Parking Functions with sets of cars \{1, 3\}, \{2, 4, 7, 8\}, \{5, 6\}, in diagonals 0, 1, 2 respectively. The display below exhibits such a Parking Function

The idea is to consider a car \( c \) to be placed in diagonal \( d \) as a vertical domino labeled by \( c \) over \( d \). In particular the runs of \( \sigma \) are thus converted into the three runs of dominoes displayed below

\[ \text{dominoes}(\sigma) \]

The algorithm constructs all rearrangements of these dominoes that yield a Parking Function precisely as the array of dominoes on the right of 4.3 may by viewed as yielding the corresponding Parking Function. For such an arrangement to yield a Parking Function we need to obey the following rules, in placing a domino \( [c_2 \ d_2] \) immediately to the right of a domino \( [c_1 \ d_1] \)

- We must have \( d_2 \leq d_1 + 1 \)
- If \( d_2 = d_1 + 1 \) then we must also have \( c_2 > c_1 \).

The algorithm applied to the domino sequence in 4.4, starts with one of the two arrangements

\[ \begin{array}{c|c|c|c|c|c|c}
  1 & 3 & & & & \\
  0 & 0 & 3 & 1 & & \\
\end{array} \]

and then inserts the rest of the dominoes in 4.4, in the right to left order, according to 4.5.

Recall that two cars cause a “primary dinv” if they are in the same diagonal and the one on the left is smaller than the one on the right. Alternatively, two cars cause a “secondary dinv” if the car on the left is in the immediate higher diagonal than the car on the right and it is greater than the car on the right. To obtain a precise description of the algorithm let us say that car \( a \) “acts” on car \( b \) if car \( b \) is on the right of car \( a \) and in the same run, or car \( b \) is in the run immediately to the right of car \( a \) and \( b \) is smaller than \( a \). We can easily see that as soon as the domino of car \( a \) is placed on the left of a domino of a car \( b \) upon which \( a \) acts a new unit of dinv is created.
In the display on the right we have placed for each car the list of the cars it acts upon. This given, we proceed as follows, starting from one of the two pairs in 4.6 we construct a tree $T(\sigma)$, whose nodes are indexed by sequences of dominos. Recursively after the dominos of cars $a_1, a_2, \ldots, a_{i-1}$ have been inserted to obtain a node of $T(\sigma)$, the labels of the children of that node are obtained by successively inserting in the label of the parent, the domino of car $a_i$ immediately to the right of the domino of one of the cars upon which $a_i$ acts. If this is done proceeding from right to left the additional dinvs that are created are $0, 1, \ldots, w_i - 1$ where $w_i$ is the number of cars upon which $a_i$ acts. For simplicity we will omit the fact that in this case car 1 acts on 3. In fact, given that we are going to consider only permutations with last run of length 2, car $\sigma_{n-1}$ will always act only on $\sigma_n$. This given, here and in the following we will denote by $w_i$ the number of cars acted upon by $\sigma_{n-i-1}$. To be consistent with the notation we introduced in the first section, the resulting vector will be denoted $W = (w_1, w_2, \ldots, w_{k-1})$, with $k = n - 1$, and will be called a “schedule” as before. In the display below we have depicted an instance of this construction. We have also added on the label of the parent node, a green line to indicate each of the positions the domino of car 2 can be placed. Finally each new branch of the tree has been labelled by the power of $q$ corresponding to the resulting dinv increase.

The tree obtained when we start from the pair on the left in 4.6 will be called the left subtree. Likewise the tree obtained from the pair on the right of 4.6 will be called the right subtree. The leaves of $T(\sigma)$ yield the domino sequence of the desired Parking Functions. In the following display we illustrate the choices that yield the leaf that gives the Parking Function in 4.3.

The actual path on the tree of $\sigma$ is obtained by appending each step in the above display right under the red disk of the previous step. The final power of $q$ that gives the dinv of the Parking Function corresponding to the leaf at the end of the path is simply obtained by multiplying the labels of the branches encountered along the path. To see that the leaves of the resulting tree give all the domino sequences of the Parking Functions that have the desired diagonal car distribution it suffices to see that the algorithm can be reversed by starting from any of the domino sequence of a desired Parking Functions and removing dominos of cars $\sigma_1, \sigma_2, \ldots, \sigma_n$, as indicated in the following display.

It can be easily seen from this that every time we remove a domino its car is necessarily adjacent either to a smaller car in a following run, or a larger car in the same run. It is also easily noticed that all the parking functions produced by the tree have the same area statistic. This is due to the fact that every car in diagonal $i$ contributes $i$ area units. Using this information it follows that the common area value is none other than the major index of the permutation $\sigma$ that yielded the tree. This given, for a given $\sigma \in S_n$ whose last run is of size 2, the weight of each $PF \in T(\sigma)$ reduces to $q$ to its dinv, the corresponding Gessel fundamental and a power of $x$ to record the diagonal hits of its Dyck path. In summary, we are led to define

$$P_{\sigma}(x; q, F) = \sum_{PF \in T(\sigma)} x^{r(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)},$$

where $r(PF)$ gives the number of cars weakly to the right of the second diagonal car. In particular the diagonal composition of $PF$ is simply given by the pair $(n - r(PF), r(PF))$. 

It should be quite apparent now how Parking Functions can be visually represented by our Labelled Bar Diagrams. For instance the Parking Function in 4.3 is represented by the labelled bar diagram in the display below (the last one). The bijection between Parking Functions and labelled bar diagrams can be easily understood by working on this example.

To begin the length of the bar under car \(a\) is simply given by the number of cars acted upon by \(a\). The cars are ordered as they appear from right to left in the initial permutation. The cars in the last run are not included in the first diagram in the above display. The cells are assigned weight as indicated in the table on the left. The second diagram in the display is constructed by placing a bar above or below the ground line according as the corresponding car occurs after or before the second car on the main diagonal. The position of the two cars in the main diagonal is represented by prepending the diagram with \(↑3\) or \(↓1\) according to whether the smaller car precedes or not the larger car.

The reader by now must have at least a glimpse of the understanding how Labelled Bar Diagrams were created. In particular we should at least understand that the powers of \(q\) are simply to indicate the amount of \(\text{dinv}\) that was created when the corresponding dominos were inserted in the tree \(T(\sigma)\). More precisely, if the insertion of the domino of car \(a\) created \(i\) \(\text{dinv}\) units then we placed \(q^i\) in the cell of that weight in bar \(a\). The last diagram is simply obtained by lowering the cars to replace the powers of \(q\). The latter is the labelled bar diagram that represents the Parking Function in 4.3.

We terminate this section with a list of important properties that easily follow from our construction.

1. The cars whose bars are below the ground line, in a given bar diagram, precede the cars whose bars are above the ground line.
2. A labelled bar diagram occurs in the construction of all the Parking Functions with prescribed diagonal cars if and only if none of its bars are totally red.
3. Given a fixed colored unlabeled diagram, its contribution to the polynomial in 4.7, when we set \(F_{\text{ides}}(PF) = 1\), is the polynomial in \(q\) obtained by summing all monomials obtained by the \(q\) labelling process described above
4. Setting \(F_{\text{ides}}(PF) = 1\) in 4.7 we obtain

\[
P_0(x; q, 1) = \sum_{PF \in T(\sigma)} x^{r(PF)}q^{\text{dinv}(PF)} = \sum_{b=1}^{n-1} A_b(q) x^b = Q_W(x; q).
\]

Where \(n\) gives the number of cars, and \(A_b(q)\) is the sum of the \(\text{dinv}\) contributions of the Labelled Bar Diagrams with \(b\) bars above the ground line. Here the components of \(W\) are given by the successive numbers of cars acted upon by 8, 7, 2, 6, 5.

Recall that the operators \(C_a\) are defined by setting for any symmetric function \(F[X]\)

\[
C_a F[X] = (-\frac{1}{q})^{a-1} F[X - \frac{1-1/q}{z}] \sum_{k \geq 0} z^k h_k[X] |_{z^n} \frac{1}{1}. \]

It is shown in [4] that these operators satisfy the relations

\[
q(C_b C_a + C_{a-1} C_{b+1}) = C_a C_b + C_{b+1} C_{a-1} \quad \text{(for all } b \leq a - 1) \]

\[
4.8
\]
The compositional Shuffle conjecture by J. Haglund, J. Morse, M. Zabrocki \cite{4} (now a theorem) states that for any composition \( p = (p_1, p_2, \ldots, p_k) \) we have

\[
\nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 = \sum_{PF(p)} t^{area(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)} \quad 4.11
\]

Where the sum is over all Parking Functions whose Dyck path hits the 0-diagonal according to \( p \).

The point of departure of the approach in \cite{1} was to split the proof of 4.11 by first reducing to the case when \( p \) is a partition by means of the identity in 4.10. Then solve the partition case using the fact that in the partition case both sides of 4.11 are symmetric function bases. Leaving the latter case aside for a while let us first see how this reduction can be accomplished.

To begin notice that using 4.10 we derive the equality

\[
\nabla C_\alpha C_\beta 1 = \frac{1}{q} \nabla C_\alpha C_\beta 1 + \frac{1}{q} \nabla C_{\alpha-1}C_\beta 1 - \nabla C_{\alpha-1}C_{\beta+1} 1 \quad 4.12
\]

for all \( b \leq a-1 \) and all pairs of compositions \( \alpha \) and \( \beta \). Now notice that every one of the compositions \( \alpha ab\beta \), \( \alpha b+1 a-1 \beta \) and \( \alpha a-1 b+1 \beta \), is lexicographically greater than \( \alpha ba\beta \) for \( b < a-1 \). When \( b = a-1 \) the resulting identity can still be used to reverse the pair \( b=a \). This shows that by a sequence of such uses of the identities in 4.10 we can express any one of the polynomials \( \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 \) as a linear combination of the family of polynomials \( \{\nabla C_{\lambda_1} C_{\lambda_2} \cdots C_{\lambda_k} 1\}_{\lambda \vdash |p|} \) and since this family is independent, the coefficients are uniquely determined by the composition \( p = (p_1, p_2, \ldots, p_k) \). This given, denoting by \( \Pi[p_1, p_2, \ldots, p_k] \) the right hand side of 4.11, we can reduce the proof of 4.11 to the partition case by showing the identities

\[
\Pi[\alpha ab\beta] = \frac{1}{q} \Pi[\alpha ba\beta] + \frac{1}{q} \Pi[\alpha b+1 a-1 \beta] - \Pi[\alpha a-1 b+1 \beta] \quad 4.13
\]

for all \( b \leq a-1 \) and all pairs of compositions \( \alpha \) and \( \beta \).

Now it is not difficult to see that 2.5 would be a consequence of the simpler identity

\[
q(\Pi[ab] + \Pi[a-1 b+1]) = \Pi[a b] + \Pi[b+1 a-1] \quad 4.14
\]

if the latter could be proved by a bijection of the Parking Functions contributing to the right hand side onto those contributing to the left hand side that moves cars only within their diagonals. The reason is very simple, such a bijection would be transferable to a proof of 4.13 since the transfer could not cause area changes nor cause changes in dinvs created by pairs of cars when at least one of the cars is located in the segments covered by \( \alpha \) or \( \beta \). For the same reason the \( \text{ides} \) of a \( PF \) could not be affected under the transfer again for any pair of cars \( i, i+1 \) when at least one of them is in the segments covered by \( \alpha \) or \( \beta \).

The miracle here is the further discovery that we can prove that such a bijection already exits within the collections of Parking Functions obtained from any permutation \( \sigma \) with a last run of size 2. More precisely, given \( \sigma \in S_n \) with a last run of length 2, if we set

\[
\Phi_\sigma[a, b] = \sum_{PF \in T(\sigma)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)} \chi(p(PF) = (a, b)),
\]

we can prove the validity of the identity

\[
q(\Phi_\sigma[ab] + \Phi_\sigma[a-1 b+1]) = \Phi_\sigma[ab] + \Phi_\sigma[b+1 a-1] \quad (\text{for } b \geq 1 \text{ and } a \geq 1) \quad 4.15
\]

This given, in principle, the desired reduction could be achieved by constructing, for any given such \( \sigma \in S_n \), a bijection of the family Parking Functions contributing to the left hand side of 4.15 onto the family contributing to the right hand side which preserves \( \text{ides} \) and increases \( \text{dinv} \) exactly by one. Now it is easily seen that for such \( \sigma \) the polynomial in 4.7 has the expansion

\[
P_\sigma(x; q, F) = \sum_{PF \in T(\sigma)} x^{\text{dinv}(PF)} q^{\text{dinv}(PF)} F_{\text{ides}(PF)} = \sum_{k=1}^{n-1} x^k A_k(q, F) \quad 4.16
\]

Thus, making the replacements \( a = s + 1 \), \( b = n - a \) and \( n = k + 1 \), 4.15, can also rewritten as

\[
q(A_{s+1}(q, F) + A_{k-s+1}(q, F)) = A_{k-s}(q, F) + A_s(q, F) \quad (\text{for } k \geq s \geq 1) \quad 4.17
\]

At this point the existence of the desired bijection seems to require a proof that the coefficients of all the polynomials \( P_\sigma(x; q, F) \) satisfy the identities in 4.17.
That may appear as a tall order. Fortunately, another rather surprising miracle comes to our help. Guided by computer data we discovered that is relatively easy to establish in full generality a factorization of the form

\[ P_\sigma(x; q, F) = P_\sigma^{(1)}(x; q) \times P_\sigma^{(2)}(q; F) \]  

To understand what causes such a factorization, we need some definitions. A string of consecutive integers \( i, i+1, \ldots, i+\ell-1 \) in a given run with both \( i-1 \) and \( i+\ell \) not in this run, is called a maxicon. An index \( i \) that is in the \( \text{ides}(PF) \) of every leaf of \( T(\sigma) \) is called a forced ides. The following observations are immediate consequences of our construction of the polynomials \( P_\sigma(x; q, F) \):

1. The elements of a maxicon \( i, i+1, \ldots, i+\ell-1 \) will appear in the leaves of \( T(\sigma) \) in each of their \( ! \) orders.
2. If a given car \( i \) belongs to a run that follows the run that contains \( i+1 \), then \( i \) will always be in the \( \text{ides}(PF) \) of every leaf of \( T(\sigma) \).

This given, the factorization in 4.18, proved in [1], may be stated as follows.

**Theorem 4.1**

Calling “Ycons” the Young subgroup of \( S_n \) generated by the maxicons of a \( \sigma \in S_n \) with a last run of length 2, we have the factorization

\[ P_\sigma(x; q, F) = \left( \sum_{PF \in T(\sigma)} x^r(PF) q^{\text{dinv}(PF)} \right) \left( \sum_{\alpha \in \text{Ycons}} q^{\text{inv}(\alpha)} F_{\text{ides}(\alpha) \cup \text{fides}(\sigma)} \right) \]  

where for convenience we have let \( \text{fides}(\sigma) \) denote the set of forced ides of \( T(\sigma) \).

This is all beautifully confirmed by the following display that constructs the tree \( T(\sigma) \) for \( \sigma = 45312 \). The runs of \( \sigma \) are 45, 3 and 12. Thus we have 2 maxicons 1, 2 and 4, 5. Since 2 is in the main diagonal, 3 is in diagonal 1 and 4 is in diagonal 2, we deduce that in the word of each leaf of \( T(\sigma) \) we will have 4 to the left of 3 and 3 to the left of 2. Thus 3 and 2 will be both forced ides. In this case the tree yields

\[ P_\sigma(x; q, F) = x^4(q^3 + q^2)F_{1,2,3,4} + x^4(q^2 + q)F_{1,2,3} + x(q^2 + q)F_{2,3,4} + xF_{2,3} \]

whose factorization is at the end of the display.

\[
\begin{array}{cccc}
13 & 12 & q & 45312 \\
1352 & 1352 & 1235 & 1354 \\
13542 & 12354 & 2351 & 2354 \\
23541 & 23541 & 2351 & 2354 \\
45312 & 45312 & 2351 & 2354 \\
54320 & 54320 & 45320 & 54320 \\
q^2F_{1,3,3,4}x & q^2F_{1,3,3}x & q^2F_{1,3,3,4}x^4 & q^2F_{1,3,3}x^4 \\
\end{array}
\]

\[
(x^4 + qx^3F_{1,3,3,4} + q^2F_{1,3,3} + qF_{2,3,4} + F_{2,3})
\]

In particular the validity of the factorization in 4.18 shows that 4.17 may be rewritten as

\[ q \left( P_\sigma^{(1)}(x; q) \right)_{x^{k+1}} + P_\sigma^{(1)}(x; q) \big|_{x^{k+1}}} P_\sigma^{(2)}(q; F) - \left( P_\sigma^{(1)}(x; q) \big|_{x^{k-\ell}} + P_\sigma^{(1)}(x; q) \big|_{x^\ell} \right) P_\sigma^{(2)}(q; F). \]  

Rather than canceling the common factor we make the replacement \( F \rightarrow 1 \) and, using 4.18, rewrite 4.19 as

\[ q \left( P_\sigma(x; q, 1) \big|_{x^{k+1}} + P_\sigma(x; q, 1) \big|_{x^{k-1}}} \right) = P_\sigma(x; q, 1) \big|_{x^{k-\ell}} + P_\sigma(x; q, 1) \big|_{x^\ell} \]
Now it is not difficult to see that, in full generality, we have the equality
\[ P_\sigma(x; q, 1) = Q_W(x, q), \]  
4.21
where for \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \) a permutation with last run of length 2 the components of \( W \) are obtained by the now quite familiar algorithm. That is we first break up \( \sigma \) into runs set \( x_i = \sigma_{n-i-1} \), and, if \( i \geq 1 \), set \( w_i \) to be the number of \( x_j > x_i \) that are either in the same run as \( x_i \) or \( x_j < x_i \) and in the following run. Notice further that, with this convention, \( x_0 = \sigma_{n-1} < \sigma_n = x_{-1} \). Thus all the Parking Functions with car \( x_{-1} \) to the left of car \( x_0 \), will necessarily have and extra primary dinv caused by these two cars. That explains the extra factor \( q \) we assigned to all the monomials produced by the Labelled Bar Diagrams on the left subtree of \( T(\sigma) \). It is easy to see that this creates schedules which can increase by at most one, and that all legal schedules can be constructed this way.

In summary, the bijection between Parking Functions yielded by the leaves of \( T(\sigma) \) and Labelled Bar Diagrams should by itself make evident the equality in 4.21. Moreover, our proof of Theorem 4.21 with the proof of the functional equation we derive the following fundamental fact.

**Theorem 4.2**

The equality in 4.11, that is
\[ \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 = \sum_{PF(p)} \tarea(F) q^{\dinv(F)} F_{\text{ides}(F)} \]  
4.24
holds for all compositions if and only if it holds for all partitions

In other words, to complete the proof of the Compositional Shuffle Theorem we need to show the equality
\[ \nabla C_\lambda 1[X; q, t] = \Pi_\lambda[X; q, t] \]  
4.25
for all partitions \( \lambda \).

It is easy to derive that the family of polynomials on the left hand side of 4.25 are a symmetric function basis. The same can be proved also for the family on the right hand side. This given, our present plan is to derive the equality in 4.25 as an application of the following elementary linear algebra result. In fact, this theorem turned out to be successful in the proof of the Haglund combinatorial formula for the modified Macdonald Polynomials \([9]\) as well as for the balanced path result in \([16]\).

**Theorem 4.3**

Suppose an \( n \) dimensional vector space \( V \) has two bases
\[ \langle \phi \rangle = \langle \phi_1, \phi_2, \ldots, \phi_n \rangle, \quad \langle \psi \rangle = \langle \psi_1, \psi_2, \ldots, \psi_n \rangle. \]
This given, any basis \( \langle H \rangle = \langle H_1, H_2, \ldots, H_n \rangle \) which is Upper triangularly related to the \( \langle \phi \rangle \) basis and Lower triangularly related to the \( \langle \psi \rangle \) basis is uniquely determined by the normalizing condition
\[ F[H_i] = c_i \neq 0 \quad ( \text{for } 1 \leq i \leq n ) \]  
4.26
for a suitable linear functional \( F \).

It turns out that that in \([6]\) it is shown that for all \( \lambda \vdash n \) we have
\[ \langle \nabla C_\lambda 1[X; q, t], e_n \rangle = \langle \Pi_\lambda[X; q, t], e_n \rangle. \]  
4.27
Thus we can use this result to provide the desired functional \( F \) in 4.26. It is not difficult to prove the additional triangularity results for the symmetric function side with

a) \( \langle \phi \rangle = \langle s_\lambda [X^2] \rangle_{\lambda \vdash n} \) and b) \( \langle \psi \rangle = \langle \nabla s_\lambda[X] \rangle_{\lambda \vdash n} \)  
4.28
However, so far we have only been able to prove that the combinatorial side is upper triangularly related to the basis in 4.28 a). But the lower triangularity of the combinatorial side to the basis in 4.28 b) is unlikely to lead to a proof since the still conjectural \[10\] combinatorial interpretation of $\nabla s_\lambda$ has $\nabla e_\alpha$ as a special case.

The upper triangularity result, for both bases, which is highly non trivial for the combinatorial side, will be presented in a forthcoming publication.

In the next few pages we will first derive some further consequences of the functional equation and formulate an interesting open problem. We will terminate the paper with some conjectures that sharpen the results of this paper.

Notice first that the factorization in 4.19 enables us to recover the second factor in 4.18 and thus also the whole polynomial $P_\sigma(x; q, F)$. But it does not stop here. Indeed, there is no reason to limit our Labelled Bar Diagrams to representing leaves of $T(\sigma)$ for $\sigma$ a permutation with last run of length 2. In fact, we need only make a few changes when we place no restriction on the length of the last run. More precisely, for a $\sigma \in S_n$, a permutation with last run of length $r$ the corresponding, polynomials can be written in the form

$$Q_\sigma(Y_r; q, F) = \sum_{p=(p_1, p_2, \ldots, p_r)=n} y_1^{p_1} y_2^{p_2} \cdots y_r^{p_r} \Pi_\sigma(p; q, F)$$

with $Y_r = (y_1, y_2, \ldots, y_r)$, where the sum in 4.24 is over $r$-part compositions of $n$.

For such $\sigma$ the only difference in the construction of the tree $T(\sigma)$, is that the root of this tree will have $r!$ children, according to the permutation of the cars in the last run. It goes without saying that the $r!$ branches of $T(\sigma)$ emanating from its root are labelled with powers of $q$ summing to the polynomial $[r]_q!$. Of course, the Parking Functions yielded by the leaves of $T(\sigma)$ will all have a diagonal composition of length $r$. In particular, we have the expansion

$$\Pi_\sigma(p; q, F) = \sum_{PF \in \mathcal{PF}_r(p)} q^{dinv(PF)} F_{ides(PF)}$$

where the sum is over all Parking Function with diagonal cars given by the runs of $\sigma$ and whose Dyck path hits the main diagonal according to the composition $p$.

Now suppose that the composition $p$ may be decomposed in the form $p = \alpha b a \beta$ with $\alpha$ and $\beta$ such that $p = n$ but otherwise arbitrary, and $1 \leq b \leq a - 1$. Then it follows from the functional equation that we must have the identity

$$q(\Pi_\sigma(\alpha b a \beta ; q, F) + \Pi_\sigma(\alpha (a - 1) (b + 1) \beta ; q, F)) = \Pi_\sigma(\alpha b a \beta ; q, F) + \Pi_\sigma(\alpha (a - 1) (b + 1) \beta ; q, F)$$

and that must holds true for every $\sigma \in S_n$ and for all such $\alpha, \beta$ and $1 \leq b \leq a - 1$.

The presence of all these identities, even when the Parking Functions are restricted to have prescribed diagonal cars, strongly suggests that all the ingredients occurring in the Symmetric Function world must have corresponding specializations to the Quasisymmetric Functions world.

To be precise we are led to conjecture that there must exist Quasisymmetric analogs of Modified Macdonald polynomials, nabla as well as the $C_\alpha$ operators satisfying the fundamental identity in 4.10. Recall that in the Symmetric Function world all the composition identities follow from 4.10. We find it difficult to believe that the phenomenon resulting from the functional equation has no Quasisymmetric function counterpart. Several attempts have been made in the past to carry out such extensions. The first one that seems to be worth investigating is the one given by N. Bergeron and M. Zabrocki in [11].

But there are even more amazing facts that strengthen the validity of these conjectures. Our point of departure is here is the following identity:
Proposition 4.1

For all \( a > b + 1 \) we have
\[
(-\frac{1}{q})^{a+b-3}s_{a-1,b+1}[X] = C_aC_b1 - qC_{a-1}C_{b+1}1 \tag{4.27}
\]

Proof

From 4.9 it follows that
\[
C_b1 = (-\frac{1}{q})^{b-1}h_b[X]
\]
Using 4.9 again we get
\[
(-q)^{a+b-2}C_aC_b1 = h_b[X + \frac{1-q}{q}] \sum_{k \geq 0} z^k h_k[X] \bigg|_{z^*} = h_b h_a + (1-q) \sum_{r \geq 1} h_{b-r} h_{a+r}/q^r
\]
Likewise we obtain
\[
(-q)^{a+b-2}qC_{a-1}C_{b+1}1 = qh_{b+1} h_{a-1} + (1-q) \sum_{r \geq 1} h_{b+1-r} h_{a-1+r}/q^{r-1}
\]
Thus
\[
(-q)^{a+b-2}(C_aC_b1 - qC_{a-1}C_{b+1}1) = h_b h_a - qh_{b+1} h_{a-1} - (1-q)h_b h_a = q(h_b h_a - h_{b+1} h_{a-1})
\]
and the Jacobi-Trudi identity gives 4.27.

Now combining 4.27 with the conjectures in [12] for Nabla of a Schur function we derive the following

Conjecture III

The symmetric polynomial
\[
\nabla C_aC_b1 - q\nabla C_{a-1}C_{b+1}1 = q\nabla C_bC_a1 - \nabla C_{b+1}C_{a-1}1
\]

is Schur positive.

Notice that applying Nabla to both sides of 4.27 we can rewrite it in the form
\[
\nabla s_{a,b} = (-q)^{a+b-3}\left(\nabla C_{a+1}C_{b-1}1 - q\nabla C_aC_b1\right)
\]
This given, we should mention that Yeon Kim in [2], in both cases \((a, b) = (n-3,3)\) and \((a, b) = (n-4,4)\), succeeded to construct an injection of the Parking Functions with diagonal composition \((a, b)\) into Parking Functions with diagonal composition \((a+1, b-1)\) preserving area and Gessel Fundamental and increasing the dinv by one unit. Since her work \(n\) was only required to be greater than 4 and 7 respectively in the above two cases she succeeded in giving a Parking Function interpretation to the resulting infinite variety of Nabla Schurs \(\nabla s_{n-3,3}\) and \(\nabla s_{n-4,4}\).

In this connection, computer experimentation revealed that Kim’s type results should be obtainable even with prescribed diagonal cars,

For a more precise description of these findings we must recall the polynomial
\[
\Phi_\sigma[a,b] = \sum_{P \in \mathcal{T}(\sigma)} q^{\text{des}(PF)} F_{\text{ides}(PF)} \chi(p(PF) = (a,b)).
\]
It turns out that computer data strongly suggests that the identity in 4.15 can be sharpened to
\[
\Phi_\sigma[a,b] - q \Phi_\sigma[a-1,b+1] = q \Phi_\sigma[b,a] - \Phi_\sigma[b+1,a-1] \in \mathbb{N}[q] \tag{4.29}
\]
for all \(\sigma \in S_n, a + b = n\) and \(a, b \geq 1\).

From 4.29 it follows that for any composition \(\gamma \vdash d\) we have
\[
\left(\Phi_\sigma[a,b] - q \Phi_\sigma[a-1,b+1]\right)|_{\gamma} = \left(q \Phi_\sigma[b,a] - \Phi_\sigma[b+1,a-1]\right)|_{\gamma} \in \mathbb{N}[q] \tag{4.30}
\]
This is a remarkable parallel to Conjecture III at the Quasi-symmetric function level. It also suggests for instance that there is an injection of Parking Functions with diagonal cars prescribed by \(\sigma\) and diagonal composition \(a - 1, b + 1\) into Parking Functions with diagonal composition \(a, b\) which preserves the Gessel fundamental and increases the dinv by one unit. Likewise there must be a way of injecting Parking Functions with diagonal cars prescribed by \(\sigma\) and diagonal
composition $b + 1, a - 1$ into Parking Functions with diagonal composition $b, a$ which preserves the Gessel fundamental and decreases the dinv by one unit.

But there is one more surprising sharpening of the functional equation by passing to our schedule polynomials $Q_W(x; q)$. To make easier to compare our statements with 4.30, let us write this polynomial, for the schedule $W = (w_1, w_2, \ldots, w_{k-1})$, in the form

$$Q_W(x, q) = \sum_{b \in \{1, k\}} \sum_{a+b=k+1} A_{a,b}^W(q) x^b$$

This given, passing from 4.16 to 4.31 using the factorization result, 4.30 becomes

$$A_{a,b}^W(q) - q A_{a-1,b+1}^W(q) = q A_{a,b}^W(q) - A_{b+1,a-1}^W(q) \in \mathbb{N}[q] \quad (\text{for all } 1 \leq b \leq k)$$

To see what this implies let us explore the case $n = 7$. This gives

\begin{align*}
a) & \quad A_{6,1}^W(q) - q A_{5,2}^W(q) = q A_{4,6}^W(q) - A_{2,5}^W(q) = q A(q) \in \mathbb{N}[q], \\
b) & \quad A_{5,2}^W(q) - A_{4,3}^W(q) = A_{2,5}^W(q) - A_{3,4}^W(q) \in \mathbb{N}[q], \\
c) & \quad A_{4,4}^W(q) = A_{4,4}^W(q) \in \mathbb{N}[q].
\end{align*}

Now a) implies that the polynomial $A_{5,5}^W(q)$ must be divisible by $q$. Likewise b) implies that $A_{3,4}^W(q)$ must be divisible by $q^2$ and finally c) forces $A_{4,4}^W(q)$ to be divisible by $q^3$. Thus writing

\begin{align*}
a) & \quad A_{2,5}^W(q) = q A_{2,5}^W(q) \quad b) \quad A_{3,4}^W(q) = q^2 C(q) \quad c) \quad A_{4,4}^W(q) = q^3 C(q)
\end{align*}

the identities in 4.33 can be rewritten as

\begin{align*}
a) & \quad q A_{1,6}^W - q A_{2,5}^W = q A(q) \quad b) \quad q^2 A_{2,5}^W - q^2 C(q) = q^2 B(q)
\end{align*}

In summary we have

\begin{align*}
c) & \quad A_{3,4}^W = q^2 C \quad b) \quad A_{2,5}^W = q B + q C \quad a) \quad A_{1,6}^W = A + B + C.
\end{align*}

Thus from 4.33 it follows that

\begin{align*}
a) & \quad A_{6,1}^W - q A_{5,2}^W = q (A + B + C) - (q B + q C) \quad b) \quad A_{5,2}^W - q A_{4,3}^W = q^2 B + q^2 C - q^2 C
\end{align*}

and using 4.33 c) we finally obtain

\begin{align*}
a) & \quad A_{5,1}^W = q A + q^3 B + q^5 C \quad b) \quad A_{5,2}^W = q^4 C + q^2 B \quad c) \quad A_{4,3}^W = q^3 C.
\end{align*}

This is typically what happens when $n$ (the number of cars) is odd. Our data not only confirms all of these identities but reveals that the polynomials $A, B, C$ are also unimodal. The implications of these identities are best understood by a visual display. In the figure on the right we have depicted the polynomials $A(q), B(q)$ and $C(q)$ respectively in Green, Red and Blue. The Red down arrow represents a conjectural bijection of a sub polynomial of $A_{1,6}^W$ onto $A_{2,5}^W$ which increases the dinv by one unit. The Red up arrow on the right represents a conjectural bijection of $A_{5,2}^W$ onto a sub polynomial of $A_{5,1}^W$ which increases the dinv by one unit. The top Brown arrow represents a highly non trivial bijection constructed by our second author during her thesis work, and played a fundamental role in [7]. It may not be unlikely that all the intricacies encountered in the proof of this bijection could be explained by means of this display. The middle Brown arrow represents a bijection of $A_{2,5}^W$ onto $A_{5,2}^W$ whose proof in full generality appears difficult. The bottom Brown arrow should be easy to prove.
The following basic result was proved in lecture notes on schedule polynomials:

**Theorem 4.4**

The vector space of polynomials

\[ Q(x, q) = \sum_{i=1}^{k} A_i(q)x^i \quad (4.36) \]

which satisfy the functional equation

\[ (1 - q/x)Q(x) + x^k(1 - qx)Q(1/x) = (1 + x^k)(A_k(q) - qA_1(q)) \]

is \( \lfloor (k + 1)/2 \rfloor \) dimensional with basis the following polynomials

a) When \( k = 2a \)

\[ E_{j,k}(x; q) = \sum_{i=1}^{j} (q^{2j-i}x^i + q^{-1}x^{2a+1-i}) \quad \text{for } 1 \leq j \leq a \]

b) When \( k = 2a - 1 \)

\[ O_{j,k}(x; q) = \sum_{i=1}^{j} (q^{2j-i}x^i + q^{-1}x^{2a-i}) \quad \text{for } 1 \leq j \leq a - 1 \]

\[ \text{and } O_{a,k}(x; q) = \sum_{i=1}^{a-1} q^{2a-i-1}x^i + \sum_{i=1}^{a} q^{i-1}x^{2a-i}. \]

Notice, for instance that from a) it follows that

\[ E_{1,6} = x^6 + qx, \quad E_{2,6} = x^6 + qx^5 + q^2x^2 + q^3x, \]

\[ E_{3,6} = x^6 + qx^5 + q^2x^4 + q^3x^3 + q^4x^2 + q^5x \quad (4.37) \]

The independence of the basis in Theorem 4.4 is due to a very simple reason, easily seen in this example. Namely the fact, that both families \( \{E_{j,2a}(x; q)\}_{1 \leq j \leq a} \) and \( \{O_{j,2a-1}(x; q)\}_{1 \leq j \leq a} \) are triangularly related to the monomials \( x, x^2, \ldots, x^a \).

For any legal schedule \( W = (w_1, w_2, \ldots, w_6) \) the coefficients \( A_i^W(q) \) of the polynomial

\[ Q_W(x; q) = \sum_{i=1}^{6} A_i^W(q)x^i \quad (4.38) \]

must satisfy the identities in 4.33. Thus it follows that it may be rewritten in the form

\[ Q_W(x; q) = (A + B + C)x^6 + (qB + qC)x^5 + q^2Cx^4 + q^3Cx^3 + (q^2B + q^4C)x^2 + (qA + q^3B + Cq^3)x. \]

This fact is an immediate consequence of 4.34 and 4.35. Regrouping terms according to \( A, B, C \) gives

\[ Q_W(x; q) = A(q)x^6 + qx + B(q)x^6 + qx^5 + q^2x^2 + q^3x + \]

\[ + C(q)x^6 + qx^5 + q^2x^4 + q^3x^3 + q^4x^2 + q^5x \]

Comparing the coefficients of \( A, B, C \) with the basis in 4.37 gives the expansion

\[ Q_W(x; q) = A(q)E_{1,6}(x; q) + B(q)E_{2,6}(x; q) + C(q)E_{3,6}(x; q) \]

The computations we carried out in this particular case should clearly indicate how the bases introduced in a) and b) of Theorem 4.4 were discovered.

The algorithm that constructs the coefficients of the expansion of each schedule polynomials in terms of the appropriate basis proceeds very much as was illustrated in the special case \( k = 6 \). More precisely, given a schedule \( W = (w_1, w_2, \ldots, w_{k-1}) \), let

\[ Q_W(x; q) = \sum_{s=1}^{k} A_{k+1-s,s}(q)x^s \]

For \( k = 2a = n - 1 \) set

\[ \alpha_r(q) = q^{-r}(qA_{r,n-r}(q) - A_{r+1,k-r}(q)) \quad \text{for } 1 \leq r \leq a - 1 \quad \text{and} \quad \alpha_a(q) = q^{-a+1}A_{a,a+1}(q) \quad (4.39) \]
For $k = 2a - 1$ set
\[ \alpha_r(q) = q^{-r}(qA_{r,n-r}(q) - A_{r+1,k-r}(q)) \quad \text{for } 1 \leq r \leq a - 1 \quad \& \quad \alpha_a(q) = q^{-a+1}A_{a,a}(q) \quad 4.40 \]
The factor $q^{-r}$ can be shown to be the minimum power of $q$ occurring in the polynomial $qQ_r(q)$.

This given, we have

**Theorem 4.5**

*Using 4.39 or 4.40, as the case may be, the polynomial $Q_W(x, q)$ has the expansion*

\[
Q_W(x; q) = \begin{cases} 
\sum_{r=1}^{a} \alpha_r(q) E_{r,k}(x; q) & \text{if } k = 2a \\
\sum_{r=1}^{a} \alpha_r(q) O_{r,k}(x; q) & \text{if } k = 2a - 1
\end{cases} \quad 4.41
\]

**Proof**

In the first case we need only show that
\[ a) \quad A_{s,n-s}^W = q^{s-1}(\alpha_s + \alpha_{s+1} + \cdots + \alpha_a) \quad \text{and} \quad b) \quad A_{n-s,s}^W = \sum_{s \leq r \leq a} q^{2r-s} \alpha_r. \]

While in the second case we need only show
\[ a) \quad A_{s,n-s}^W = q^{s-1}(\alpha_s + \alpha_{s+1} + \cdots + \alpha_a) \quad \text{and} \quad b) \quad A_{n-s,s}^W = \sum_{s \leq r \leq a-1} q^{2r-s} \alpha_r + q^{2a-s-1} \alpha_a. \]

This can be carried out in a straightforward manner. Since we already illustrated the $k$ even case, we will limit our efforts to verifying the odd case $k = 7$. Using 4.40 for $a = 4$ we obtain
\[
\begin{align*}
\alpha_1(q) &= q^{-1}(qA_{1,7}(q) - A_{2,6}(q)) = q^{-1}(A_{7,1}(q) - qA_{6,2}(q)) \\
\alpha_2(q) &= q^{-2}(qA_{2,6}(q) - A_{3,5}(q)) = q^{-2}(A_{6,2}(q) - qA_{5,3}(q)) \\
\alpha_3(q) &= q^{-3}(qA_{3,5}(q) - A_{4,4}(q)) = q^{-3}(A_{5,3}(q) - qA_{4,4}(q))
\end{align*}
\]

From which we derive that
\[
\begin{align*}
\alpha_1(q) &= A_{1,7}(q) - A_{2,6}(q) = A_{1,7}(1) - A_{6,2}(q) \\
\alpha_2(q) &= A_{2,6}(q) - A_{3,5}(q) = A_{2,6}(1) - A_{5,3}(1) \\
\alpha_3(q) &= A_{3,5}(q) - A_{4,4}(q) = A_{3,5}(1) - A_{4,4}(1)
\end{align*}
\]

This gives
\[
\alpha_4(q) = A_{4,4}(q) = q^3 \alpha_4, \quad A_{3,5} = q^2(\alpha_3 + \alpha_4), \quad A_{2,6} = q(\alpha_2 + \alpha_3 + \alpha_4), \quad A_{1,7} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,
\]
and
\[
A_{5,3} = q^3 \alpha_3 + q^4 \alpha_4, \quad A_{6,2} = q^5 \alpha_4 + q^4 \alpha_3 + q^3 \alpha_2, \quad A_{7,1} = q^6 \alpha_4 + q^5 \alpha_3 + q^3 \alpha_2 + qa_1.
\]

Multiplying $A_{8-r,r}$ by $x^r$ and summing for $1 \leq r \leq 7$ proves the odd case for $a = 4$.

We will terminate by a statement with potentially significant consequences.

**Conjecture IV**

*The Polynomials $\alpha_r(q)$ in 4.41 are unimodal and in $\mathbb{N}[q]$.***
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1DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CA, USA
E-mail address: garsia@math.ucsd.edu

2DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, 14 E. PRKWR AVE, BETHLEHEM, PA 18015
E-mail address: anh316@lehigh.edu

3SCHOOL OF MATHEMATICAL SCIENCES, CAPITAL NORMAL UNIVERSITY, BEIJING 100048, PR CHINA
E-mail address: guoce.xin@163.com