SINGULARITIES OF AXIALLY SYMMETRIC VOLUME PRESERVING MEAN CURVATURE FLOW

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ABSTRACT. We investigate the formation of singularities under the volume preserving mean curvature flow. We study axially symmetric surfaces with Neumann boundary conditions and prove that under an additional lower height bound on the boundary of a specific region, the first singularity that forms is of type I.

1. INTRODUCTION

Let $M^n$ be an $n$-dimensional manifold and let $x_0: M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of $M^n$ into $\mathbb{R}^{n+1}$. Consider a one-parameter family of smooth immersions $x_t : M^n \to \mathbb{R}^{n+1}$ with $M_t = x_t(M^n)$ and $x_t = x(\cdot, t)$ satisfying

$$\frac{d}{dt} x(l, t) = -H(l, t)\nu(l, t), \quad l \in M^n, t > 0.$$  

By $\nu(l, t)$ we denote a designated outer unit normal of $M_t$ at $x(l, t)$ (outer normal in case of compact surfaces without boundary), and by $H(l, t)$ the mean curvature with respect to this normal. Here the hypersurfaces $M_t$ are evolving by mean curvature.

If the evolving compact surfaces $M_t$ are assumed to enclose a prescribed volume $V$ the evolution equation changes as follows:

$$\frac{d}{dt} x(l, t) = -(H(l, t) - h(t))\nu(l, t), \quad l \in M^n, t > 0,$$

where $h(t)$ is the average of the mean curvature,

$$h(t) = \frac{\int_{M_t} H \, dg_t}{\int_{M_t} \, dg_t},$$

and $g_t$ denotes the metric on $M_t$. This flow is known to decrease the surface area while the enclosed volume remains constant.

Huisken [15] proves that uniformly convex, compact surfaces become asymptotically spherical under mean curvature flow. Grayson [14] proves that mean curvature flow makes smooth embedded curves in the plane shrink to a point, becoming spherical in the limit. Ecker and Huisken [10] prove that entire graphs over $\mathbb{R}^n$ “flatten out” with time. In [1] Altschuler, Angenent and Gigastudy mean curvature flow of surfaces which are axially symmetric and prove that immediately after a singularity the surface becomes smooth and thus continue to study the flow after singularities. In [19] Huisken and Sinestrari avoid the formation of the singularity by performing surgery to the surface before the singularity develops. In this procedure they remove the part of the surface which

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becomes singular, and then patch up the remaining surface by a smooth cap, and afterwards let the surface evolve once again.

If a uniformly convex, compact object without boundary flows by mean curvature while keeping the enclosed volume constant, Huisken [16] proves that it converges to a sphere. The first author [4] proves that an axially symmetric hypersurface between two parallel planes, which encloses a sufficiently large volume, having Neumann boundary data, converges to a cylinder under volume preserving mean curvature flow. In [5] she proves that type I singularities of axially symmetric volume preserving mean curvature flow are self-similar and asymptotically cylindrical. Escher and Simonett [12] investigate the volume flow near spheres. They prove that there exist global solutions to this flow starting from non-convex initial hypersurfaces. Cabezas-Rivas and Miquel [7] study the volume flow in Hyperbolic space.

As the initial surface we choose a compact \( n \)-dimensional hypersurface \( M_0 \), with boundary \( \partial M_0 \neq \emptyset \). We assume \( M_0 \) to be smoothly embedded in the domain between the two parallel planes \( x_1 = a \) and \( x_1 = b, a, b > 0 \). Here, except for the volume constraint, we have a free boundary. We consider an axially symmetric hypersurface contained in the region between the planes \( x_1 = a \) and \( x_1 = b \). Motivated by the fact that the stationary solution to the associated Euler Lagrange equation of an energy minimizing liquid bridge satisfies a Neumann boundary condition, we also assume the surface to meet the planes at right angles along its boundary. We study the first singularity that develops under this flow.

We assume a lower height bound detailed in Assumption 7.4 to prove the following theorem.

**Theorem 1.1.** Under the Assumption 7.4, when the first singularity develops the second fundamental form \( |A|^2 \) satisfies
\[
\max_{M_t} |A|^2 \leq \frac{C}{T - t},
\]
for all \( t < T \), where \( T \) is the blow up time and \( C \) is a constant.

In [17] Huisken obtains type I singularities in the mean curvature flow setting for axially symmetric hypersurfaces with positive mean curvature. In our case, we have no conditions on the curvature, but we have a lower height bound on the boundary of a specific domain.

The paper is organized as follows:
In Section 2 we introduce some notation.
In Section 3 we discuss parabolic maximum principles for non-cylindrical domains. As we use the parabolic maximum principles for specific regions of the hypersurface, our domains change with time, and therefore may not be cylindrical. When applying the maximum principles to a specific region of the hypersurface, it does not help to consider a periodic hypersurface by reflecting it along the planes \( x_1 = a \) and \( x_1 = b \), which alleviates the need to take the boundary data into account. Hence the boundary data of the specific region plays an important role in our case. This version of the parabolic maximum principle is particularly useful as it rules out some specific regions of the
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parabolic boundary from attaining a maximum.

In Section 4 we compute the evolution equations and introduce different regions used in the rest of the paper. These different regions give us a better understanding of the hypersurface and later we prove that the singularity can only develop in certain regions.

In Section 5 we compute the height, gradient and curvature estimates. In particular we prove that a specific region has a lower height bound for all time $t < T$ and also the second fundamental form $|A|$ is bounded in that region. We also prove that the mean curvature is bounded from below on the entire hypersurface.

In Section 6 we discuss the linear and non-linear versions of the Sturmian Theorem and its applications in volume flow. We study the behaviour of zeros of particular functions and discuss the implications of the zeros.

In Section 7 we prove Theorem 1.1 by studying the different cases in which a singularity can develop.

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2. Notation

We follow Huisken’s [17] and Athanassenas’ [4] notation in describing the $n$-dimensional axially symmetric hypersurface. Let $\rho_0 : [a, b] \to \mathbb{R}$ be a smooth, positive function on the bounded interval $[a, b]$ with $\rho_0'(a) = \rho_0'(b) = 0$. Consider the $n$-dimensional hypersurface $M_0$ in $\mathbb{R}^{n+1}$ generated by rotating the graph of $\rho_0$ about the $x_1$-axis. We evolve $M_0$ along its mean curvature vector keeping its enclosed volume constant subject to Neumann boundary conditions at $x_1 = a$ and $x_1 = b$. Equivalently, we could consider the evolution of a periodic surface defined along the whole $x_1$ axis. By definition the evolution preserves axial symmetry. The position vector $x$ of the hypersurface satisfies the evolution equation

$$\frac{d}{dt}x = -(H - h)\nu = H + h\nu,$$

where $H$ is the mean curvature vector. Since $\Delta x = H$, where $\Delta$ denotes the Laplacian on the surface, we obtain,

(2.1) $$\frac{d}{dt}x = \Delta x + h\nu.$$

Let $i_1, \ldots, i_{n+1}$ be the standard basis in $\mathbb{R}^{n+1}$, corresponding to $x_1, \ldots, x_{n+1}$ axes and $\tau_1(t), \ldots, \tau_n(t)$ be a local orthonormal frame on $M_t$ such that

$$\langle \tau_l(t), i_1 \rangle = 0, \text{ for } l = 2, \ldots, n, \text{ and } \langle \tau_1(t), i_1 \rangle > 0.$$

Let $\omega = \hat{x} / |\hat{x}| \in \mathbb{R}^{n+1}$ denote the unit outward normal to the cylinder intersecting $M_t$ at the point $x(l, t)$, where $\hat{x} = x - \langle x, i_1 \rangle i_1$. Let
Here $y$ is the **height function**. We call $v$ the **gradient function**. We note that $v$ is a geometric quantity, related to the inclination angle; in particular $v$ corresponds to $\sqrt{1 + \rho'^2}$ in the axially symmetric setting. The quantity $v$ has facilitated results such as gradient estimates in graphical situations (see for example [8, 10]).

We introduce the quantities (see also [17])

$$
p = \langle \tau_1, i_1 \rangle_{y^{-1}}, \quad q = \langle \nu, i_1 \rangle_{y^{-1}},
$$

so that

$$
p^2 + q^2 = y^{-2}.
$$

The second fundamental form has $n - 1$ eigenvalues equal to $p = \frac{1}{ \rho \sqrt{1 + \rho'^2}}$ and one eigenvalue equal to

$$
k = \langle \nabla_1 \nu, \tau_1 \rangle = \frac{-\rho''}{(1 + \rho'^2)^{3/2}}.
$$

We note that $\rho(x_1, t)$ is the radius function such that $\rho : [a, b] \times [0, T) \to \mathbb{R}$, whereas $y(l, t)$ is the height function and $y : M^n \times [0, T) \to \mathbb{R}$. These are two different interpretations of the same physical object.

There are cases where pinch off develops in the axially symmetric setting (see [5]). We assume that a singularity develops for the first time at $t = T < \infty$.

### 3. Maximum Principles

We are interested in maximum principles in non-cylindrical domains. This section is an extension of Ecker’s ([9] Proposition 3.1) and Lumer’s [21] version of maximum principles to our setting. In [21] the maximum principles are proved in an operator theoretic setting, which has been adapted to the manifold setting here.

Let $\Omega = M^n$. Let $V \subset \Omega \times (0, T)$ be an open non-cylindrical domain. Let $\Omega_t = \Omega \times \{t\}$, and for $t \neq 0$ let $V_t = \Omega_t \cap V$. Let $\overline{V}$ denote the closure of $V$. Let $V_0 = \Omega_0 \cap \overline{V}$. The boundary of $V$ is $\partial V = \overline{V} \setminus V$. The parabolic boundary $\Gamma_V = \partial V \setminus \Omega_T$. To describe the horizontal parts of the boundary of $V$ in the spacetime diagram, we define the following: let $Z_t$ be the largest subset of $\Omega_t \cap \partial V$, which is open in $\partial V$ and that can be reached from “below” in $V$, where $t$ is the vertical axis. Let $Z_V = \bigcup_{0 < t < T} Z_t$. Let $\delta_V = \Gamma_V \setminus Z_V$.

**Proposition 3.1. (Maximum Principle)** Let $(M_t)_{t \in (0, T)}$ be a solution of volume preserving mean curvature flow consisting of hypersurfaces $M_t = x_t(\Omega)$ where $x_t = x(\cdot, t) : \Omega \times [0, T) \to \mathbb{R}^{n+1}$ and $\Omega$ is compact. Suppose $f \in (C^{2,1}(V) \cap C(\overline{V}))$ and $f : V \to \mathbb{R}$ satisfies an inequality of the form

$$
\left( \frac{d}{dt} - \Delta \right) f \leq \langle a, \nabla f \rangle,
$$

where the Laplacian $\Delta$ and the gradient $\nabla$ are computed on the manifold $M_t$ (for the vector field $a : V \to \mathbb{R}^{n+1}$ we only require that it is continuous in a neighbourhood of all maximum points of
f \). Then

$$\sup_{V} f \leq \sup_{\delta V} f,$$

for all \( t \in [0, T) \).

Assuming \( f \) to have a positive supremum in \( V \) then

$$\sup_{V} f \leq \sup_{\delta V} f,$$

for all \( t \in [0, T) \).

**Proof. Part A.** We show that

$$\sup_{V} f(l, t) \leq \sup_{\Gamma V} f(l, t).$$

Let \( \tilde{f} = f - \epsilon_1 t \), where \( \epsilon_1 > 0 \). It holds \( \tilde{f}(l, 0) = f(l, 0) \) and on \( \Gamma V \), \( \tilde{f}(l, t) \leq f(l, t) \). We note that

$$\frac{d\tilde{f}}{dt} = \frac{df}{dt} - \epsilon_1, \quad \Delta \tilde{f} = \Delta f, \quad \nabla \tilde{f} = \nabla f.$$

Therefore

$$\left( \frac{d}{dt} - \Delta - a \cdot \nabla \right) \tilde{f} < 0.$$

At an interior point of \( V \), where for the first time \( \max_{V} \tilde{f} \) reaches a value larger than \( \sup_{\Gamma V} \tilde{f} \), the standard derivative criteria at the local maximum say

$$\frac{d\tilde{f}}{dt} \geq 0, \quad \frac{d^2\tilde{f}}{dx_i dx_j} = 0, \quad \frac{d^2\tilde{f}}{dx_i dx_j dx_k} \leq 0.$$

As

$$\Delta \tilde{f} = g^{ij} \left( \frac{d^2\tilde{f}}{dx_i dx_j} - \Gamma^k_{ij} \frac{d\tilde{f}}{dx_k} \right) \quad \text{and} \quad \nabla \tilde{f} = g^{ij} \frac{df}{dx_j dx_i},$$

by choosing the parametrization such that \( g_{ij} = \delta_{ij} \) at the point that corresponds to the interior maximum we have

$$\left( \frac{d}{dt} - \Delta - a \cdot \nabla \right) \tilde{f} \geq 0.$$

This is a contradiction. Hence \( \tilde{f}(l, t) \) is bounded by the values of \( \sup_{\Gamma V} \tilde{f} \) at all times. Therefore

$$\sup_{V} (f(l, t) - \epsilon_1 t) \leq \sup_{\Gamma V} \tilde{f} \leq \sup_{\Gamma V} f,$$
\[ \sup_{V} f(l, t) - \epsilon_{1} T \leq \sup_{\Gamma_{V}} (f(l, t) - \epsilon_{1} t) \leq \sup_{V} f, \]
\[ \sup_{V} f(l, t) \leq \sup_{\Gamma_{V}} f + \epsilon_{1} T \quad \text{for all} \quad \epsilon_{1} > 0, \]
giving us
\[ \sup_{V} f(l, t) \leq \sup_{\Gamma_{V}} f, \]
which completes Part A.

**Part B.** For this part we suppose that \( f \) has a positive maximum in \( V \). We will prove by contradiction that

\begin{equation}
\sup_{V} f \leq \sup_{\delta_{V}} f.
\end{equation}

Suppose that (3.3) does not hold. As \( \sup_{V} f(l, t) \leq \sup_{\Gamma_{V}} f \) this can only happen if the maximum of \( f \) is achieved on \( Z_{V} \). In particular the maximum of \( f \) can only be achieved at an interior point of \( Z_{V} \) for (3.3) to be contradicted. We denote by \( Z_{\text{max}} \) the union of \( Z_{t} \)'s on which the maximum is achieved. Let \( t_{*} \) denote the first time that the maximum is achieved on \( Z_{V} \). Let

\[ K = \{(l, t) \in V : f(l, t) = \sup_{V} f\}. \]

(We note that \( K \cap \partial V \) is non-empty as the maximum is achieved in \( Z_{\text{max}} \) and also that \( K \cap \partial V \subset Z_{V} \).) Therefore there exists a \( \beta > 0 \) such that

\[ \sup_{V} f > \beta > \sup_{\Gamma_{V}\setminus Z_{\text{max}}} f. \]

As \( \delta_{V} \subset (\Gamma_{V}\setminus Z_{\text{max}}) \) we have

\[ \sup_{\Gamma_{V}\setminus Z_{\text{max}}} \geq \sup_{\delta_{V}} f. \]

Define

\[ K_{\beta} = \{(l, t) \in V : f(l, t) \geq \beta \} \neq \emptyset. \]

Here \( K_{\beta} \) may not be a connected set. It holds that \( Z_{t_{*}} \subset K_{\beta} \). We work with the connected component of \( K_{\beta} \), which has \( Z_{t_{*}} \) as a part of its boundary. Let \( 0 < \epsilon_{2} < \sup_{V} f - \beta \). As \( V \) is open there exists \( (l_{2}, t_{2}) \in V \) (depending on \( \epsilon_{2} \)) such that \( f(l_{2}, t_{2}) \geq \sup_{V} f - \epsilon_{2} \). We take \( t_{2} \) to be the earliest time such that \( f(l, t) \geq \sup_{V} f - \epsilon_{2} \) is satisfied. Take \( t_{3} \in (t_{2}, t_{*}) \) and choose a smooth
function \( \phi: [0, T] \to \mathbb{R} \), such that \( 0 \leq \phi \leq 1 \), \( \phi' < 0 \) on \((t_2, t_3)\), and \( \phi = 1 \) on \([0, t_2]\), and \( \phi = 0 \) on \([t_3, T]\). The set

\[
K'_\beta = \{(l, t) \in K_\beta : t \leq t_3\} \neq \emptyset ,
\]

is a compact subset in \( V \). The set \( K'_\beta \) is in the interior of \( V \), thus \( K'_\beta \cap Z_V = \emptyset \). As \((\phi f)(l_2, t_2) = f(l_2, t_2)\), and as the maximum of \( f \) is achieved in \( Z_{l_*} \), and as \( \phi \geq 0 \), the supremum of \( \phi f \) must be at least as big as \( f(l_2, t_2) \). Also

\[
\max_{K_\beta} \phi f = \max_{K'_\beta} \phi f ,
\]
as \( \phi = 0 \) on \([t_3, T]\). Hence

\[
\max_{K'_\beta} \phi f \geq f(l_2, t_2) \geq \sup_{V} f - \epsilon_2 > \beta > 0 .
\]

On the other hand, \( V \setminus K'_\beta = (V \setminus K_\beta) \cup (K_\beta \setminus K'_\beta) \). On \( (V \setminus K_\beta) \cap \{(l, t) \in V : f \geq 0\} \) we have \( \phi f \leq f < \beta \), and on \( (V \setminus K_\beta) \cap \{(l, t) \in V : f < 0\} \) we have \( \phi f < \beta \). On \( K_\beta \setminus K'_\beta \), \( \phi = 0 \). As a result

\[
(3.4) \quad \sup_{V \setminus K'_\beta} \phi f < \beta < \max_{K'_\beta} \phi f .
\]

Therefore if we denote by \((l_\phi, t_\phi)\) the point at which \((\phi f)(l_\phi, t_\phi) = \sup_{V} (\phi f)\), we can see that \((l_\phi, t_\phi) \in K'_\beta\). As

\[
\frac{d}{dt}(\phi f) = \phi \frac{df}{dt} + \phi' f , \quad \Delta (\phi f) = \phi \Delta f , \quad \nabla (\phi f) = \phi \nabla f ,
\]

we have

\[
\left( \frac{d}{dt} - \Delta - a \cdot \nabla \right) (\phi f) = \phi \left( \frac{d}{dt} - \Delta - a \cdot \nabla \right) f + \phi' f .
\]

As \( \phi'(l_\phi, t_\phi) \leq 0 \) and \( f(l_\phi, t_\phi) > 0 \) we obtain

\[
\left( \frac{d}{dt} - \Delta - a \cdot \nabla \right) (\phi f) \leq 0 .
\]

By using Part A with \( \phi f \) replacing \( f \) we have

\[
\sup_{V} (\phi f) \leq \sup_{\Gamma_{V}} (\phi f) .
\]

But this is a contradiction as \( \sup_{V} (\phi f) = (\phi f)(l_\phi, t_\phi) \) with \((l_\phi, t_\phi) \notin \Gamma_{V} \), and because \( \Gamma_{V} \subset V \setminus K'_\beta \) and \( (3.4) \) holds. Therefore our original assumption is wrong. Hence a maximum of \( f \) does not occur in \( Z_V \), that means \( K \cap \partial V \not\subset Z_V \). Therefore we conclude that \( (3.3) \) is true.

\[\square\]

4. Evolution Equations and Different Regions of the Hypersurface

4.1. Evolution equations. The computation of the evolution equations is very similar to that of ([17] Lemma 5.1 ) and ([4] Lemma 3 ).

**Lemma 4.1.** We have the following evolution equations:

(i) \( \frac{d}{dt} x_i = \Delta x_i + hq y \; \)

(ii) \( \frac{d}{dt} y = \Delta y - \frac{a-1}{y} + hpy \; \)
we find

\( \frac{d}{dt}q = \Delta q + |A|^2 q + q((n - 1)p^2 + (n - 3)q^2 - 2kp) - h(q) \);

(iv) \( \frac{d}{dt}p = \Delta p + |A|^2 p + 2q^2(k - p) - h(p^2) \);

(v) \( \frac{d}{dt}k = \Delta k + |A|^2 k - 2(n - 1)q^2(k - p) - h(k^2) \);

(vi) \( \frac{d}{dt}H = \Delta H + (H - h)|A|^2 \);

(vii) \( \frac{d}{dt}|A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2h(c) \);

(viii) \( \frac{d}{dt}v = \Delta v - |A|^2 v + (n - 1)\frac{q}{y} - \frac{2}{n}|\nabla v|^2 \);

(ix) \( \frac{d}{dt}\rho = \rho' + \frac{\rho}{1 + \rho^2} - \frac{n - 1}{\rho} + h\sqrt{1 + \rho^2} \);

where \( c = g^{ij}g^{kl}g^{mn}h_{ik}h_{jm}h_{nj} \).

\[ \text{Proof.} \] The first identity is immediate from (2.2) and (2.1).

Since \( |x| = y \), we write \( y = (|x|^2 - |\langle x, i_1 \rangle|^2)^{1/2} \) as. To obtain the second identity we compute

\[
(4.1) \quad \frac{d}{dt}y = y^{-1}(\langle H, x \rangle - \langle x, i_1 \rangle \langle H, i_1 \rangle + h \langle \nu, x \rangle - hqy \langle x, i_1 \rangle), \quad \text{and} \quad \\
\Delta y = y^{-1}(n + \langle x, H \rangle - \langle x, i_1 \rangle \langle H, i_1 \rangle - |\langle \tau_1, i_1 \rangle|^2 - |\nabla y|^2).
\]

Using the relation

\[
|\langle \tau_1, i_1 \rangle|^2 = y^2 p^2 = 1 - y^2 q^2 = 1 - |\nabla y|^2,
\]

we find

\[
\left( \frac{d}{dt} - \Delta \right) y = y^{-1}(h \langle \nu, x \rangle - hqy \langle x, i_1 \rangle - (n - 1)).
\]

Then (ii) follows after considering \( qy = \langle \nu, i_1 \rangle \) and

\[
\langle \nu, x \rangle - \langle \nu, i_1 \rangle \langle x, i_1 \rangle = \langle \nu, \langle x, i_1 \rangle i_1 + \hat{x} \rangle - \langle \nu, i_1 \rangle \langle x, i_1 \rangle, \\
= \langle \nu, \hat{x} \rangle = |\hat{x}| \langle \nu, \omega \rangle = yv^{-1} = py^2.
\]

Also using (4.2) and (4.1) we derive an alternate evolution equation for \( y \) which will be of use later.

\[
\frac{d}{dt}y = \frac{-(H - h)}{y}(\langle \nu, x \rangle - \langle x, i_1 \rangle \langle \nu, i_1 \rangle), \\
\frac{d}{dt}\frac{d}{dt}y = \frac{-(H - h)}{y}py^2 = -(H - h)py.
\]

For (iii) we note that \( \frac{d}{dt}\nu = \nabla H \) (as shown in [15], Lemma 3.3), and using Codazzi’s equation we obtain

\[
\frac{d}{dt} \langle \nu, i_1 \rangle = \Delta \langle \nu, i_1 \rangle + |A|^2 \langle \nu, i_1 \rangle.
\]

We note that

\[
\frac{d}{dt} \langle \nu, i_1 \rangle = \frac{d}{dt}(qy) = q \frac{dy}{dt} + y \frac{dq}{dt} \quad \text{and} \quad \\
\Delta \langle \nu, i_1 \rangle = \Delta(qy) = q\Delta y + y\Delta q + 2(\nabla q, \nabla y).
\]

Combining this with (ii), we find

\[
y\frac{dq}{dt} = (q\Delta y + y\Delta q + 2(\nabla q, \nabla y)) - q \left( \Delta y - (n - 1)y^{-1} + hpy \right) + |A|^2 qy,
\]

which gives
Using this in (4.4) gives the evolution equation for \( q \) from (ii) and (iii) we can now compute

\[
\nabla i y = \delta_{i1} q y, \quad \nabla_1 (\nu, i_1) = k p y, \quad \nabla_i q = \delta_{i1} (q^2 + k p).
\]

Hence we obtain

\[
\frac{2}{y} (\nabla q, \nabla y) + (n - 1) \frac{q}{y^2} = q (-2 k p + (n - 1) p^2 + (n - 3) q^2).
\]

Using this in (4.4) gives the evolution equation for \( q \).

From (ii) and (iii) we can now compute

\[
\frac{d}{dt} \rho = \frac{d}{dt} (y^2 - q^2)^{1/2},
\]

\[
= \frac{1}{2p} \left( \frac{2}{y^2} \frac{dy}{dt} - 2q \frac{dq}{dt} \right),
\]

\[
= \frac{1}{p} \left\{ \frac{-1}{y^2} \left( \Delta y - \frac{n - 1}{y} + hpq \right) - q \left( \Delta q + |A|^2 q + q((n - 1) p^2 + (n - 3) q^2 - 2k p) - hpq \right) \right\},
\]

\[
= \frac{1}{p} \left\{ \frac{1}{y^3} \Delta y + q \Delta q + \frac{n - 1}{y^4} - \frac{hp}{y^2} - |A|^2 q^2 - q^2 ((n - 1) p^2 + (n - 3) q^2 - 2k p) + hpq^2 \right\},
\]

\[
\Delta p = \frac{1}{p} \left( \frac{1}{y^3} \Delta y + q \Delta q - |\nabla q|^2 + \frac{3}{y^4} |\nabla y|^2 - |\nabla p|^2 \right),
\]

\[
\frac{d}{dt} \rho = \frac{n - 1}{py^4} - \frac{h}{y^2} - |A|^2 \frac{q^2}{p} - q^2 ((n - 1) p^2 + (n - 3) q^2 - 2k p) + hpq^2 + |\nabla q|^2 + \frac{3}{py^4} |\nabla y|^2 + |\nabla p|^2.
\]

Equation (iv) follows if we combine (4.5) and the following relations:

\[
\nabla i p = \delta_{i1} q (p - k), \quad |A|^2 = k^2 + (n - 1) p^2, \quad y^{-4} = p^4 + 2 p^2 q^2 + q^4.
\]

The evolution equations for \( H \) and \( |A|^2 \) were derived in [16], while (v) follows from (vi), (iv), and the fact that \( H = k + (n - 1) p \). The evolution equation for \( v \) was derived in [4] and for \( \rho \) in [5].

Lemma 4.2. We have \( \nabla \tau \omega = (1 - \delta_{11}) \frac{\omega}{y} \tau \), where \( \nabla \) is the covariant derivative on \( \mathbb{R}^{n+1} \).

Proof. For any unit vector \( \eta = \eta_k i_k \) we have

\[
\nabla_\eta \dot{x} = \eta_k \nabla_{i_k} \dot{x} = \sum_{k=1}^{n+1} \eta_k \sum_{j=1}^{n+1} \left( \frac{\partial}{\partial x_k} x_j + \Gamma^j_{ik} x_l \right) i_j
\]

\[
= \sum_{k=1}^{n+1} \eta_k \sum_{j=2}^{n+1} \delta_{kj} i_j = \sum_{k=1}^{n+1} \eta_k i_k = \eta - \langle \eta, i_1 \rangle i_1,
\]

as \( \Gamma^j_{ik} \), the induced connection on \( \mathbb{R}^{n+1} \) is zero with orthonormal coordinates. And

\[
\eta( |\dot{x}| ) = \eta_k i_k ( |\dot{x}| ) = \eta_k \frac{\partial}{\partial x_k} |\dot{x}|
\]

\[
= \eta_k \omega_k = \langle \eta, \omega \rangle,
\]

where \( \omega_k \) is the connection on \( \mathbb{R}^{n+1} \).
as \( \omega_1 = 0 \). So we obtain
\[
\nabla_\eta \omega = \nabla_\eta \left( \frac{\dot{x}}{|x|} \right)
= \frac{1}{|x|} \nabla_\eta \dot{x} - \frac{1}{|x|^2} \eta(|x|) \dot{x}
= \frac{1}{y} (\eta - \langle \eta, i_1 \rangle i_1 - \langle \eta, \omega \rangle \omega).
\]

Therefore for \( \eta = \tau_1 \) we have
\[
(4.7) \quad \nabla_\tau \omega = \frac{1}{y} (\tau_1 - \langle \tau_1, i_1 \rangle i_1 - \langle \tau_1, \omega \rangle \omega) = 0.
\]

For \( \eta = \tau_l, l \neq 1 \) we have
\[
(4.8) \quad \nabla_\tau \omega = \frac{1}{y} (\tau_l - \langle \tau_l, i_1 \rangle i_1 - \langle \tau_l, \omega \rangle \omega) = \frac{1}{y} \tau_l.
\]

\[\square\]

**Corollary 4.3.** The gradient function \( v \) satisfies the following alternate evolution equation
\[
\left( \frac{d}{dt} - \Delta \right) v = -|A|^2 v + \frac{p^2 v^3 - 2v k^2 (v^2 - 1)}{p^2}.
\]

**Proof.** First we compute an expression for \(|\nabla v|\) as follows:
\[
\nabla v = \tau_i \langle \nu, \omega \rangle^{-1} \tau_i
= -v^2 \left( \langle \nabla_i \nu, \omega \rangle + \langle \nu, \nabla_i \omega \rangle \right) \tau_i
= -v^2 \left( \langle h_{ij} g^{jk} \tau_k, \omega \rangle + \langle \nu, (1 - \delta_{1l}) \frac{1}{y} \tau_l \rangle \right) \tau_i\]

as \( \nabla_i \nu = h_{ij} g^{jk} \tau_k \) and from Lemma 4.2,
\[
= -v^2 h_{ij} g^{jk} \langle \tau_k, \omega \rangle \tau_i
= -v^2 h_{ij} g^{j1} \langle \tau_1, \omega \rangle \tau_i
= -v^2 h_{11} g^{11} (-qy) \tau_1,
\]

as \( q = \langle \nu, i_1 \rangle y^{-1} \) and \( \langle \nu, i_1 \rangle = -\langle \tau_1, \omega \rangle \)
\[
= v^2 k q y \tau_1 = v k q p \tau_1, \quad \text{as} \quad p = \frac{1}{vy},
\]

\[
(4.9) \quad |\nabla v|^2 = v^2 k^2 \frac{q^2}{p^2}.
\]

From Lemma 4.1 we know
\[
\left( \frac{d}{dt} - \Delta \right) v = -|A|^2 v + \frac{v}{y^2} - \frac{2}{v} |\nabla v|^2.
\]

Substituting \( p = \frac{1}{vy} \), (4.9) and \( \frac{q^2}{p^2} = v^2 - 1 \), we obtain
\[
\left( \frac{d}{dt} - \Delta \right) v = -|A|^2 v + \frac{p^2 v^3 - 2v k^2 (v^2 - 1)}{p^2}.
\]

\[\square\]
4.2. **Bounds on** $h$. We state [5] Proposition 1.4 here.

**Proposition 4.4.** (Athanassenas). Assume $\{M_t\}$ to be a family of smooth, rotationally symmetric surfaces, solving (1.2) for $t \in [0, T)$. Then the mean value $h$ of the mean curvature satisfies

$$0 < c_2 \leq h \leq c_3,$$

with $c_2$ and $c_3$ constants depending on the initial hypersurface $M_0$.

This is an important result that will be used repeatedly in our paper.

4.3. **Different regions of the volume flow surface.** Depending on the situation, we are interested in different parts of the hypersurface. Therefore we subdivide the hypersurface in different regions as follows:

4.3.1. **The regions $\tilde{\Omega}_t$ and $\hat{\Omega}_t$.** Let

$$\tilde{\Omega}_t = \{ x(l, t) \in M_t : H(l, t) \leq c_2 \}$$

and

$$\hat{\Omega}_t = \bigcup_{t < T} \tilde{\Omega}_t,$$

and

$$\tilde{\Omega}_t = \{ x(l, t) \in M_t : H(l, t) > c_2 \}$$

and

$$\hat{\Omega}_t = \bigcup_{t < T} \tilde{\Omega}_t,$$

such that $M_t = \tilde{\Omega}_t \cup \hat{\Omega}_t$. In view of Proposition 4.4 and equation (4.3) we can see that in the region $\tilde{\Omega}$ the height function is bounded from below.

4.3.2. **The regions $\tilde{\Omega}^t_1$, $\tilde{\Omega}^t_2$ and $\hat{\Omega}^t_2$.** Let

$$\tilde{\Omega}^t_1 = \{ x(l, t) \in M_t : H(l, t) \leq \frac{c_2}{2} \}$$

and

$$\tilde{\Omega}^t_1 = \bigcup_{t < T} \tilde{\Omega}^t_1,$$

and

$$\tilde{\Omega}^t_2 = \{ x(l, t) \in M_t : H(l, t) > \frac{c_2}{2} \}$$

and

$$\tilde{\Omega}^t_2 = \bigcup_{t < T} \tilde{\Omega}^t_2,$$

such that $M_t = \tilde{\Omega}^t_1 \cup \tilde{\Omega}^t_2$. We also define

$$\hat{\Omega}^t_2 = \{ x(l, t) \in M_t : H(l, t) \leq c_2 - \delta, \delta > 0 \}$$

and

$$\hat{\Omega}^t_2 = \bigcup_{t < T} \hat{\Omega}^t_2,$$

which will be used occasionally.

4.3.3. **The regions $\Omega^k_{t+}$ and $\Omega^k_{t-}$.** Let

$$\Omega^k_{t+} = \{ x(l, t) \in M_t : k(l, t) \geq 0 \}$$

and

$$\Omega^k_{t-} = \{ x(l, t) \in M_t : k(l, t) < 0 \}$$

and

$$\Omega^k_{t+} = \{ x(l, t) \in M_t : H(l, t) \leq 0 \}$$

and

$$\Omega^k_{t-} = \{ x(l, t) \in M_t : H(l, t) > 0 \}.$$

4.3.4. **The regions $\Omega^\flat_{t}$ and $\Omega^\sharp_{t}$.** For a constant $c_{00} \geq 2$, which we will choose later, let

$$\Omega^\flat_{t} = \{ x(l, t) \in M_t : \frac{|k|}{p} \leq \sqrt{\frac{c_{00}}{2(c_{00} - 1)}} \}$$

and

$$\Omega^\flat_{t} = \bigcup_{t < T} \Omega^\flat_{t},$$

and

$$\Omega^\sharp_{t} = \{ x(l, t) \in M_t : \frac{|k|}{p} > \sqrt{\frac{c_{00}}{2(c_{00} - 1)}} \}$$

and

$$\Omega^\sharp_{t} = \bigcup_{t < T} \Omega^\sharp_{t},$$

such that $M_t = \Omega^\flat_{t} \cup \Omega^\sharp_{t}$. 
5. Height, Gradient and Curvature Estimates

5.1. Height estimates. The first author proves in ([4], 2A Remark (iii)) that the height $y$ satisfies

$$y \leq R,$$

for some $R > 0$ determined by the initial hypersurface $M_0$.

We will show that the height function $y$ has a lower bound in the region $\hat{\Omega}'$.

Lemma 5.1. There exist constants $c, c' > 0$ such that

$$\inf_{\hat{\Omega}'} y = \inf_{\Gamma_{\hat{\Omega}'} \setminus \hat{\Omega}'} y \geq c$$

and

$$\inf_{\hat{\Omega}''} y = \inf_{\Gamma_{\hat{\Omega}''} \setminus \hat{\Omega}''} y \geq c',$$

where $\Gamma_{\hat{\Omega}'}$ and $\Gamma_{\hat{\Omega}''}$ denote the parabolic boundary of $\hat{\Omega}'$ and $\hat{\Omega}''$ (see figure 3) respectively.

Proof. As $\frac{dy}{dt} = -(H - h)y > 0$ in $\hat{\Omega}'$, the height increases in this region. Therefore

$$\inf_{\hat{\Omega}'} y = \inf_{\Gamma_{\hat{\Omega}'} \setminus \hat{\Omega}'} y.$$

We claim that $\inf_{\Gamma_{\hat{\Omega}''}} y \neq 0$. To prove this suppose $\inf_{\Gamma_{\hat{\Omega}''}} y = 0$, at a point $x(l_0, t_0) \in \hat{\Omega}'$, (see figure 3) where $\hat{\Omega}'$ is the closure of $\hat{\Omega}'$ and $t_0$ may equal to $T$. If the height is zero at the point $x(l_0, t_0) \in \hat{\Omega}'$, then the height has to decrease near the point just before $t_0$. That means there exists a neighbourhood $N$ of $x(l_0, t_0)$, such that $N$ is “past” in time, $t < t_0$, and $N \subset \hat{\Omega}$, and $\frac{dy}{dt}|_N < 0$. But this is not possible, since $\frac{dy}{dt}|_{\hat{\Omega}''} > 0$, where $\hat{\Omega}''$ denotes the interior of $\hat{\Omega}$. Therefore there exists a constant $c$ such that, on the parabolic boundary of $\hat{\Omega}'$, $\inf y \geq c > 0$.

When we consider $M_0$ as a periodic hypersurface, we have $\partial \hat{\Omega}' \cap \hat{\Omega}' = \partial \hat{\Omega}' \cap \hat{\Omega}'$, that is $\Gamma_{\hat{\Omega}'} \setminus \hat{\Omega}'' = \Gamma_{\hat{\Omega}'} \setminus \hat{\Omega}''$. As $\inf_{M_0} y \neq 0$ we have the desired result.

Remark 5.2. The lower height bound similar to that in $\hat{\Omega}'$ can be obtained in $\hat{\Omega}''$ for any $\delta > 0$.

5.2. A Gradient estimate.

Lemma 5.3. There exists a constant $c_4$ depending only on the initial hypersurface, such that $vy < c_4$, independent of time.
Proof. We calculate from Lemma 4.1
\[
\frac{d}{dt}(yv - c_3 t) = \Delta(yv) - \frac{2}{v} \langle \nabla v, \nabla (yv) \rangle - yv|A|^2 + h - c_3.
\]
As \( h \leq c_3 \) we get by the parabolic maximum principle
\[
yv - c_3 t \leq \max_{M_0} yv, \quad yv \leq \max_{M_0} yv + c_3 T =: c_4.
\]
\( \Box \)

5.3. Curvature estimates.

**Proposition 5.4.** There is a constant \( c_1 \) depending only on the initial hypersurface, such that \( k_p < c_1 \), independent of time.

**Proof.** Similar to equation (19) of [17] we calculate from Lemma 4.1
\[
\frac{d}{dt} \left( \frac{k}{p} \right) = \Delta \frac{k}{p} + \frac{2}{p} \left( \nabla p, \nabla \left( \frac{k}{p} \right) \right) + \frac{2q^2}{p^2} (p - k) ((n - 1)p + k) + \frac{hk}{p} (p - k).
\]
If \( \frac{k}{p} \geq 1 \) then \( (p - k) < 0 \). By the parabolic maximum principle we obtain
\[
\frac{k}{p} \leq \max \left( 1, \max_{M_0} \frac{k}{p} \right) =: c_1.
\]
\( \Box \)

**Proposition 5.5.** At points \( x(l, t) \) of \( M_t \) where \( H \geq 0 \) we have \( \frac{|k|}{p} \leq \max(c_1, n - 1) \).

**Proof.** In a region or at any given point where \( H \) is positive, if \( k \) is positive as well (in \( \Omega_k \)) we have by Proposition 5.4 \( \frac{|k|}{p} = \frac{k}{p} \leq c_1 \). If \( k \) is negative, then
\[
k + (n - 1)p \geq 0, \quad -|k| + (n - 1)p \geq 0,
\]
\[
\frac{|k|}{p} \leq (n - 1).
\]
\( \Box \)

**Proposition 5.6.** For given \( \epsilon > 0 \), let \( S_t \subset M_t \) and \( S = \bigcup_{t < T} S_t \) be a region such that \( y|S \geq \epsilon > 0 \) and \( H|\partial S_t \geq 0 \) for all \( t < T \). Then the norm of the second fundamental form \( |A| \) is bounded in \( S \).

**Proof.** We proceed as in ([11], proof of Theorem 3.1) and ([4], Proposition 5) and calculate the evolution equation for the product \( g = |A|^2 \varphi(v^2) \), where \( \varphi(r) = \frac{r}{\lambda \varphi'(r^2)} \), with some constants \( \lambda, \mu > 0 \) to be chosen later and \( v = \langle \nu, \omega \rangle^{-1} \). From the evolution equation of \( g \) we find the inequality
\[
\left( \frac{d}{dt} - \Delta \right) g \leq -2\mu g^2 - 2\lambda \varphi v^{-3} \langle \nabla v, \nabla g \rangle - \frac{2\lambda \mu}{(\lambda - \mu u^2)^2} |\nabla v|^2 g - 2hC \varphi(v^2) + \frac{2(n - 1)}{y^2} v^2 \varphi |A|^2.
\]
We estimate the second last term as in [4] using Young’s inequality and obtain
\[-2hC \varphi (v^2) \leq 2h |A|^3 \varphi (v^2) \]
\[ \leq \left( \frac{3}{2} |A|^4 \varphi (v^2) + \frac{1}{2} h^4 \varphi^{-2} (v^2) \right) \]
\[ = \frac{3}{2} g^2 + \frac{1}{2} h^4 \varphi^{-2} (v^2) . \]

Note that by Lemma 5.3 \( v \) is bounded in \( S \), \( v \leq \frac{c_4}{t} \). We choose \( \mu > \frac{3}{4} \) and \( \lambda > \mu \max v^2 \). As \( \varphi' v^2 = \frac{\lambda}{(\lambda - \mu v^2)^2} \varphi \) we have
\[ \frac{2(n - 1)}{y^2} v^2 \varphi' |A|^2 = \frac{2(n - 1)}{y^2(\lambda - \mu v^2)} g \leq c_8 g . \]

Therefore we obtain
\[ \left( \frac{d}{dt} - \Delta \right) g \leq - \left( 2 \mu - \frac{3}{2} \right) g^2 + c_8 g - 2 \lambda \varphi v^{-3} (\nabla v, \nabla g) + \frac{1}{2} h^4 \varphi^{-2} . \]

We note that \( h^4 \varphi^{-2} (v^2) \) and \( 2 \lambda \varphi v^{-3} \) are bounded as \( v \) is bounded in \( S \). By relabeling the constants we obtain
\[ \left( \frac{d}{dt} - \Delta \right) g \leq - c_9 \left( g - \frac{c_8}{2c_9} \right)^2 - c_{10} (\nabla v, \nabla g) + c_{11} . \]

For \( g > \frac{c_8}{2c_9} + \sqrt{\frac{c_{11}}{c_9}} \) the right hand side of the last inequality is negative. From the parabolic maximum principle
\[ g \leq \max \left( \max_{S_0} g , \frac{c_8}{2c_9} + \sqrt{\frac{c_{11}}{c_9}} \right) , \]

where \( \Gamma_S \) denotes the parabolic boundary of \( S \). Therefore
\[ (5.3) \quad |A|^2 \varphi (v^2) \leq \max \left( \max_{S_0} |A|^2 \varphi (v^2) , \max_{t < T} \frac{c_8}{2c_9} + \sqrt{\frac{c_{11}}{c_9}} \right) . \]

As \( H|_{\partial S_t} \geq 0 \) for all \( t < T \), we have \( \frac{\partial}{\partial t} |_{\partial S_t} < \max (c_1, n - 1) =: c_1 \) for all \( t < T \), by Proposition 5.5. Therefore
\[ |A|^2 |_{\partial S_t} = (k^2 + (n - 1)p^2)|_{\partial S_t} \leq (n - 1 + c_1)p^2 |_{\partial S_t} , \]
\[ \leq (n - 1 + c_1)y^{-2} |_{\partial S_t} \leq (n - 1 + c_1)\epsilon^{-2} , \]

for all \( t < T \). Note that \( \varphi (v^2) > 0 \) and is bounded from above as long as \( v \) is bounded, which holds for any points that are at a distance larger than \( \epsilon \) from the axis of rotation (Lemma 5.3). Therefore \( \max_{t < T} |A|^2 \varphi (v^2) \) is bounded and this completes the proof.

\[ \square \]

**Lemma 5.7.** For \( x \) satisfying (1.2), the norm of the second fundamental form \( |A| \) is bounded in the region \( \tilde{\Omega}' \).

**Proof.** From Lemma 5.1 we know that in \( \tilde{\Omega}' \), \( \inf y \geq c > 0 \). On the boundary of \( \tilde{\Omega}' \), \( H = \frac{c_2}{t} > 0 \) for all \( t < T \). Therefore by Proposition 5.6 there exists a constant \( C' \) such that \( |A|^2 |_{\tilde{\Omega}'} \leq C' < \infty \).

\[ \square \]
Lemma 5.8. There exists a constant $C$ independent of time such that $H(l, t) \geq -C^2$ for all $x(l, t) \in M_t$.

**Proof.** By definition $\Omega' = \bigcup_{t < T} \{ x(l, t) \in M_t : H(l, t) \leq \frac{C}{2} \}$, $H$ can only be negative for $x(l, t) \in \tilde{\Omega}'$. But by the above result $|A|^2|_{\tilde{\Omega}'} < C' < \infty$. As $\frac{1}{n} H^2 \leq |A|^2$ we deduce

$$H|_{M_t} \geq -\sqrt{nC'} =: -C^2$$

Now we refine Proposition 5.6 to show that no singularities develop away from the axis of rotation.

**Proposition 5.9.** For given $\epsilon > 0$, let $S_t \subset M_t$ and $S = \bigcup_{t < T} S_t$, such that $y|S \geq \epsilon > 0$, for all $t < T$. Then the norm of the second fundamental form $|A|$ is bounded in the region $S$.

**Proof.** The proof is exactly the same as in Proposition 5.6 up to (5.3). We compute the evolution equation for $|A|^2 \varphi(v^2)$ and end up with

$$|A|^2 \varphi(v^2) \leq \max \left( \max_{S_0} |A|^2 \varphi(v^2), \max_{\partial S^+_{t < T}} |A|^2 \varphi(v^2), \max_{\partial S^-_{t < T}} |A|^2 \varphi(v^2), \frac{c_8}{2c_9} + \frac{c_{11}}{c_9} \right).$$

Let

$$\partial S^+_{t} = \{ x(l, t) \in \partial S_t : H(l, t) \geq 0 \} \quad \text{and} \quad \partial S^-_{t} = \{ x(l, t) \in \partial S_t : H(l, t) < 0 \},$$

so that $\partial S_t = \partial S^+_{t} \cup \partial S^-_{t}$. Continuing from (5.4)

$$|A|^2 \varphi(v^2) \leq \max \left( \max_{S_0} |A|^2 \varphi(v^2), \max_{\partial S^+_{t < T}} |A|^2 \varphi(v^2), \max_{\partial S^-_{t < T}} |A|^2 \varphi(v^2), \frac{c_8}{2c_9} + \frac{c_{11}}{c_9} \right).$$

Here we look at the term $\max_{\partial S^-_{t < T}} |A|^2 \varphi(v^2)$, as the other terms are taken care of in Proposition 5.6. As $H < 0 < \frac{C}{2}$ on $\partial S^-_{t}$, $\partial S^-_{t} \subset \tilde{\Omega}'$. From Lemma 5.7, $|A|^2$ is bounded in $\tilde{\Omega}'$. As $\partial S^+_{t} \subset \tilde{\Omega}'$, $|A|^2$ is bounded on $\partial S^-_{t}$ for all $t < T$. As $\varphi(v^2) > 0$ and is bounded from above at points away from the axis, $\max_{\partial S^-_{t < T}} |A|^2 \varphi(v^2)$ is bounded and thus we get the desired result.

6. Sturmian Theorem and Its Applications

6.1. Introduction. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ to be a solution of

$$(6.1) \quad f_t = a(z, t)f_{zz} + b(z, t)f_z + c(z, t)f,$$

on $Q = \{ (z, t) \in \mathbb{R}^2 : 0 \leq z \leq 1, 0 \leq t \leq \tilde{T} \}$ with Dirichlet or Neumann boundary conditions.

The number of zeros of $f(\cdot, t)$ is defined as the supremum of all $k$ such that there exist $0 < z_1 < z_2 < \cdots < z_k < 1$ with

$$f(z_i, t)f(z_{i+1}, t) < 0, \quad i = 1, 2, \ldots, k - 1.$$

For $t \in (0, \tilde{T})$ let

$$Z_t(f) = \{ x \in \mathbb{R} : f(x, t) = 0 \},$$

and $Z_0(f)$ is defined as the supremum of all $k$ such that there exist $0 < z_1 < \cdots < z_k < 1$ with

$$f(z_i, t)f(z_{i+1}, t) < 0, \quad i = 1, 2, \ldots, k - 1.$$
be the zero set of \( f \). In [2] Angenent proves the following theorem.

**Theorem 6.1. (Angenent [2], Theorem C).** Let \( f : [0, 1] \times [0, \bar{T}] \to \mathbb{R} \) be a bounded solution of (6.1) which satisfies either Dirichlet, Neumann or periodic boundary conditions. Assume the coefficients of (6.1) to satisfy

\[
\begin{align*}
& a > 0, \ a, a^{-1}, a_t, a_z, a_{zz} \in L^\infty, \\
& b, b_t, b_z \in L^\infty, \\
& c \in L^\infty,
\end{align*}
\]

and in addition, in the case of Neumann boundary conditions, assume that \( a \equiv 1 \) and \( b \equiv 0 \). Let \( Z_t \) denote the number of zeros of \( f(\cdot, t) \) in \([0, 1]\). Then

(a) for \( t > 0 \), \( Z_t \) is finite.

(b) if \((x_0, t_0)\) is a multiple zero of \( f \), then for all \( t_1 < t_0 < t_2 \) we have \( Z_{t_1} > Z_{t_2} \).

By change of coordinates it is possible for an equation of type (6.1) to be reduced to an equation in which the coefficients \( a \equiv 1 \) and \( b \equiv 0 \) (see [3, 13]).

The non-linear case has been studied by Galaktionov [13] and Angenent has obtained similar results in [3]. As the non-linear case is important to us we will state it briefly. Consider

\[
f_t = L(x, t, f, f_x, f_{xx}) \text{ in } Q,
\]

where \( L(x, t, p, q, r) \) is nondecreasing relative to the last argument \( r \in \mathbb{R} \) (the parabolicity condition). Let \( \mathcal{F} \) denote the family of all continuous solutions to (6.2) and let \( B \subset \mathcal{F} \). We define the difference \( w(x, t) = f(x, t) - V(x, t) \), where \( f \in \mathcal{F} \) and \( V \in B \) and the number of intersections of \( V \) as the number of sign changes of \( w(x, t) \):

\[
\text{Int}(t, V) = Z_t(w).
\]

If \( L \) is sufficiently smooth , the difference \( w \) satisfies a linear parabolic equation

\[
w_t = aw_{xx} + bw_x + cw,
\]

where the coefficients are given by Hadamard’s formulæ

\[
\begin{align*}
& a = \int_0^1 L_r(x, t, f, f_x, f_{xx}, \theta f_{xx} + (1 - \theta)V_{xx})d\theta \geq 0, \\
& b = \int_0^1 L_q(x, t, f, \theta f_x + (1 - \theta)V_x, V_{xx})d\theta, \\
& c = \int_0^1 L_p(x, t, \theta f + (1 - \theta)V, V_x, V_{xx})d\theta.
\end{align*}
\]

If these coefficients satisfy the conditions of Theorem 6.1 then we can draw the same conclusions about the number of zeros of \( w \).

6.2. **Sturmian Theorem in Mean Curvature Flow and Volume Flow.** Using the Sturmian theorem Altschuler, Angenent and Giga [1] proved the following:
The number of necks of an axially symmetric surface evolving by mean curvature is finite and nonincreasing for \( 0 \leq t < T \) ([1] Lemma 4.7).

The necks converge to points on the axis of rotation as \( t \to T \) ([1] Lemma 5.1).

If the two limit points of nearby necks are distinct, then for any compact interval between the two limit points, there exists a uniform \( \epsilon > 0 \) such that \( \rho(x_1,t) \geq \epsilon \) in that compact interval for all \( t \in (0,T) \) ([1] Lemma 5.2).

The limit \( \lim_{t \to T} \rho(x_1,t) = \rho(x_1,T) \) exists, and \( \rho(x_1,t) \) converges uniformly to \( \rho(x_1,T) \) as \( t \to T \) and \( \rho(x_1,t) \in C^\infty(\mathbb{R} \times (0,T)) \) as long as \( \rho(x_1,t) > 0 \) ([1] Lemma 5.3).

The first author ([5] Corollary 1.2, Lemmata 2.3, 2.4, 2.5) proved the equivalent for volume preserving mean curvature flow.

6.3. Zeros of \( \rho'' \) and \( \left( \frac{H}{p} \right) \). We use the Sturmian theorem to prove that the zeros of \( \rho'' \) and of \( \frac{H}{p} \) are discrete and nonincreasing. Furthermore, we show that the limits of the zeros of \( \rho'' \) and \( \frac{H}{p} \) exist as \( t \to T \).

**Corollary 6.2.** Assume \( \{M_i\} \) to be a family of smooth hypersurfaces evolving by (1.2). Assume in addition that \( \rho(x_1,t) \geq \epsilon, \epsilon > 0 \), for \( a \leq x_1 \leq b, t \in (0,T'), T' < T \). Then the set \( Z_t(\rho'') = \{x_1 \in \mathbb{R} : \rho''(x_1,t) = \frac{\partial^2 \rho}{\partial x_1^2} = 0\} \) is a discrete set in \([a,b]\), for all \( t \in (0,T') \). Moreover, the number of zeros of \( \rho'' \) is a nonincreasing function of time.

**Proof.** We know from Lemma 4.1

\[
\frac{d}{dt} \rho = \frac{\rho''}{1 + \rho'^2} - \frac{n-1}{\rho} + h \sqrt{1 + \rho'^2}.
\]

Differentiating this equation with respect to \( x_1 \) twice we find that \( \eta \equiv \rho'' \) satisfies

\[
\frac{d\eta}{dt} = \frac{1}{1 + \rho'^2} \eta'' + \left( \frac{-6 \rho' \eta}{(1 + \rho'^2)^2} + \frac{h}{\sqrt{1 + \rho'^2}} \right) \eta' + \left( \frac{n-1}{\rho^2} - \frac{2 \eta^2}{(1 + \rho'^2)^2} + \frac{h \eta}{\sqrt{1 + \rho'^2}} + \frac{8 \eta \rho'^2}{(1 + \rho'^2)^3} - \frac{h \rho'^2 \eta}{(1 + \rho'^2)^2} \right) \eta - \frac{2(n-1) \rho'^2}{\rho^3}.
\]

As this is a nonlinear equation, we resort to the intersection comparison method discussed above. Suppose we have two axially symmetric surfaces evolving by mean curvature while preserving their respective volumes, then we can define the difference \( w(x_1,t) = \rho''_1(x_1,t) - \rho''_2(x_1,t) \), where \( \rho_1, \rho_2 \) are the respective radius functions of the first and the second surfaces. For \( \rho_i \geq \epsilon > 0, i = 1, 2 \), we have bounds on the second fundamental form and all its covariant derivatives, thus we have smoothness. The difference \( w(x_1,t) \) satisfies (6.3), where the coefficients are computed as in (6.4), (6.5), (6.6). As \( \rho_i \geq \epsilon \) we have bounds for all the derivatives of \( \rho_i \) from ([5] Lemma 2.5). Thus our integrands are bounded. The coefficients satisfy the conditions of the Sturmian theorem as the integrands are bounded, and thus we can apply Theorem 6.1 and conclude that the intersections are discrete and their number nonincreasing in time. In particular if the function \( w \) has a multiple zero at \((x_0,t_0)\) then for all \( 0 < t_1 < t_0 < t_2 < T' \), the strict inequality \( Z_{t_1}(w) > Z_{t_2}(w) \) holds, so the number of intersections of the two surfaces, \( Z_t(w) \) is strictly decreasing at \( t = t_0 \). Finally as
\( \rho_2 = c \) is a solution of (6.7) where \( c \) is a constant, we can conclude that the zeros of \( \rho_1'' := \rho'' \) are discrete and nonincreasing in time.

**Corollary 6.3.** Assume \( \{ M_t \} \) to be a family of smooth hypersurfaces evolving by (1.2). Assume in addition that \( \rho(x_1,t) \geq \epsilon, \epsilon > 0 \), for \( a \leq x_1 \leq b, t \in (0,T'), T' < T \). Then the set
\[
Z_t \left( \left( \frac{H}{p} \right)^2 \right) = \left\{ x_1 \in \mathbb{R} : \left( \frac{H}{p} \right)^2 (x_1,t) = 0 \right\}
\]
is discrete in \([a,b], \) for all \( t \in (0,T') \). Moreover, the number of zeros of \( \left( \frac{H}{p} \right)^2 \) is a nonincreasing function of time.

**Proof.** From Lemma 4.1 we obtain
\[
\left( \frac{d}{dt} - \Delta \right) \left( \frac{H}{p} \right)^2 = \frac{2}{p} \left\langle \nabla p, \nabla \left( \frac{H}{p} \right)^2 \right\rangle + \frac{2(p-k)H}{p} \left( 2 \frac{Hq^2}{p^2} + \frac{k}{p} \right) - 2 \left| \nabla \left( \frac{H}{p} \right) \right|^2 .
\]
As the Laplacian and the gradient in the above equation are computed on the hypersurface, we need the equivalent equation on \( \mathbb{R} \):
\[
(6.8)
\]
\[
\frac{d}{dt} \left( \left( \frac{H}{p} \right)^2 \right) = \frac{1}{(1+\rho'^2)} \frac{d^2}{d^2x_1} \left( \left( \frac{H}{p} \right)^2 \right) - \left( \frac{\rho'\rho''}{(1+\rho'^2)} - \frac{(n-1)\rho'}{\rho(1+\rho'^2)} \right) \frac{d}{dx_1} \left( \left( \frac{H}{p} \right)^2 \right) \\
+ \frac{2}{p(1+\rho'^2)} \frac{dp}{dx_1} \frac{d}{dx_1} \left( \left( \frac{H}{p} \right)^2 \right) + 4\frac{q^2}{p}(p-k) \left( \frac{H}{p} \right)^2 + \frac{2hk(p-k)}{p^2} \\
- \frac{2}{1+\rho'^2} \left( \frac{d}{dx_1} \left( \frac{H}{p} \right) \right)^2 .
\]
As this is a nonlinear equation, we resort to the intersection comparison method discussed above, and as in Corollary 6.2 by comparing \( M_t \) to a cylinder, where \( \left( \frac{H}{p} \right)^2 \) is a constant and \( k = 0 \), we obtain the desired result.

After discarding an initial section of the solution, without loss of generality we may assume that the hypersurface has \( l \) local minima of \( \rho' \), and \( l+1 \) local maxima of \( \rho' \). Let the local minima of \( \rho' \) be located at \( \{ t_j(t) \}_{1 \leq j \leq l} \), and the local maxima of \( \rho' \) be located at \( \{ \chi_j(t) \}_{1 \leq j \leq l+1} \), so that
\[
a < \xi_1(t) < \xi_2(t) < \cdots < \xi_l(t) < b ,
\]
\[
a < \chi_1(t) < \chi_2(t) < \cdots < \chi_l(t) < b.
\]
Similarly we may assume that the hypersurface has \( m \) minima of \( \left( \frac{H}{p} \right)^2 \). Let the minima of \( \left( \frac{H}{p} \right)^2 \) be located at \( \{ \xi_j(t) \}_{1 \leq j \leq m} \), so that
\[
a < \xi_1(t) < \xi_2(t) < \cdots < \xi_m(t) < b .
\]

**Lemma 6.4.** (Convergence of zeros of \( \rho'' \)). The limits
\[
\lim_{t \to T} t_j(t) = \tau_j(T) \quad \text{and} \quad \lim_{t \to T} \chi_j(t) = \tau_j(T)
\]
establish.

**Proof.** The proof is exactly the same as in ([1] Lemma 5.1). For the sake of completeness and as we use it in Lemma 6.5 we include it here.
Assume that $\iota_j(t)$ does not converge as $t \to T$. Then $\liminf_{t \to T} \iota_j(t) < \limsup_{t \to T} \iota_j(t)$, and we can choose an $x_0 \in (\liminf_{t \to T} \iota_j(t), \limsup_{t \to T} \iota_j(t))$. Since $\iota_j(t)$ is continuous, there is an infinite sequence of times $t_k \to T$ at which $\iota_j(t_k) = x_0$, and at which therefore $\rho''(x_0, t_k) = 0$ holds.

Consider the family of curves $\rho'(t)$ on $[a, b]$ and $\rho'(t)$ obtained by reflecting $\rho'(t)$ about the hyperplane $x_1 = x_0$. Here $\tilde{\rho}'(x_1, t) = \rho'(2x_0 - x_1, t)$, and $\rho'(t)$ is defined on $[2x_0 - b, 2x_0 - a]$. The curve $\rho'$ corresponds to $\frac{d}{dt} \rho(2x_0 - x_1, t)$. From ([5] Corollary 1.2) we know that the zeros of $\rho'$ for $x_1 \in [a, b]$ and of $\rho'$ for $x_1 \in [2x_0 - b, 2x_0 - a]$ are finite and nonincreasing. Therefore the zeros of $w = \rho - \rho'$, where $x_1 \in [\max(a, 2x_0 - b), \min(b, 2x_0 - a)]$, are finite and nonincreasing as well. Moreover, the number of zeros of $w$ drops whenever $\rho' - \rho'$ has a multiple zero. If $\rho' - \rho'$ has a multiple zero at $(x_0, t_*)$, $t_* < T$, that means if $\rho'(x_0, t_*) = \rho'(x_0, t_*)$ and $\rho''(x_0, t_*) = \rho''(x_0, t_*)$, then $\rho'(x_0, t_* + \delta) - \rho'(x_0, t_* + \delta)$ has at most one zero in the interval $[x_0 - \epsilon, x_0 + \epsilon]$ (see Theorem 6.1). As $\rho''(x_0, t_k) = -\rho''(x_0, t_k)$ due to the one being the reflection of the other and as $\rho''(x_0, t_k) = 0$, we have $\rho''(x_0, t_k) = 0$ making $\rho''(x_0, t_k) = \rho''(x_0, t_k)$. As $\rho'(x_0, t_k) = \rho'(x_0, t_k)$, this would mean that multiple zeros of $w$ exist at $(x_0, t_k)$ for $t_k > t_*$. This is a contradiction. We must therefore conclude that the $\iota_j(t)$ converges after all.

The same argument also shows that $\chi_j(t)'s$ converge.

\begin{proof}

Lemma 6.5. (Convergence of zeros of $\frac{H}{\rho}$). The limits

$$
\lim_{t \to T} \xi_j(t) = \xi_j(T)
$$

exist. Moreover, the zeros of $\frac{H}{\rho}$ converge as $t \to T$.

Proof. The proof is exactly the same as in Lemma 6.4 after the following changes. Instead of the curves $\rho'$ and $\rho'$ we work with the curves $\left(\frac{H}{\rho}\right)^2$ defined on $[a, b]$ and $\left(\frac{H}{\rho}\right)^2(x_1, t)$:

$$
\left(\frac{H}{\rho}\right)^2(2x_0 - x_1, t) \text{ defined on } [2x_0 - b, 2x_0 - a].
$$

Instead of ([5] Corollary 1.2) we use Corollary 6.3 above to say that the zeros of $w = \left(\frac{H}{\rho}\right)^2 - \left(\frac{H}{\rho}\right)^2$ are finite and nonincreasing. The remainder of the proof is the same as in Lemma 6.4. Thus $\xi_j(t)$ converges as $t \to T$. As the zeros of $\frac{H}{\rho}$ are a subset of the minima of $\left(\frac{H}{\rho}\right)^2$ we conclude that the zeros of $\frac{H}{\rho}$ converge as $t \to T$.

\end{proof}

6.4. Implication of the behaviour of the zeros.

Lemma 6.6. Let $H$ denote the mean curvature of the evolving hypersurface $M_t$. If $|\nabla H| \leq c$ in a closed region $S \subset \Omega \times [0, T]$, then $|\frac{dH}{dt}| \leq c$ in $S$ as well.

Proof. For the magnitude of $\nabla H$ we obtain

$$
|\nabla H|^2 = |\nabla_{\tau_1} H|^2 + \cdots + |\nabla_{\tau_n} H|^2.
$$

As $M_t$ is an axially symmetric surface, the mean curvature $H$ is constant on the $n - 1$ dimensional sphere for a fixed $x_1$ coordinate. Here we let $\mathbf{x}(l, t) = (x_1, \theta_1, \cdots, \theta_{n-1}, t)$, and for $2 \leq i \leq n$

$$
\nabla_{\tau_i} H = \frac{\partial}{\partial \theta_{i-1}} H = 0,
$$

so that

$$|\nabla H|^2 = |\nabla H|^2 = \left|\frac{\partial H}{\partial x_1}\right|^2.$$  

As $|\nabla H|$ is bounded in $S$, we have the same bound for $|\frac{\partial H}{\partial x_1}|$ as well. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{The paths of $H = C_1$ and $H = C_2$ in $\tilde{\Omega}''$}
\end{figure}

We recall that $\tilde{\Omega}'' = \bigcup_{t < T} \tilde{\Omega}''_t = \bigcup_{t < T} \{x(l, t) : H(l, t) \leq c_2 - \delta : \delta > 0\}$. Let us define the following:

**Definition 6.7.** In a connected component of $\tilde{\Omega}''$, consider any two paths where $H = C_1$ and $H = C_2$ such that $0 \leq C_1 < C_2 \leq c_2 - \delta, \delta > 0$. (Recall that $0 < c_2 \leq h(t) \leq c_3$.) Let

$$l_1(t) = \{l \in M^n : H(l, t) = C_1\}, \quad \text{and} \quad x_1(l_1(t), t) = \langle x(l_1(t), t), i_1 \rangle,$$

$$l_2(t) = \{l \in M^n : H(l, t) = C_2\}, \quad \text{and} \quad x_1(l_2(t), t) = \langle x(l_2(t), t), i_1 \rangle,$$

$$\alpha(t) = \min \{x_1(l_1(t), t), x_1(l_2(t), t)\}, \quad \text{and} \quad \beta(t) = \max \{x_1(l_1(t), t), x_1(l_2(t), t)\}.$$  

Here $l_i(t), i = 1, 2$ is the curve in $M^n \times [0, T]$ that parametrizes $H = C_i$ and $x_1(l_i(t), t)$ the corresponding $x_1$ coordinate.

**Lemma 6.8.** With the above notation, there exists a constant $c$ such that $|x_1(l_1(t), t) - x_1(l_2(t), t)| \geq c > 0$ for all $t \leq T$.

**Proof.** From Remark 5.2 we know that $y_{\tilde{\Omega}''} \geq y_{\tilde{\Omega}''} \geq \epsilon > 0$. As the height is always positive in $\tilde{\Omega}''$ we have $\rho(x_1, t) \in C^\infty(\mathbb{R} \times [0, T])$ in that region by ([5] Lemma 2.5). We note that this holds in $\tilde{\Omega}''$ even at $t = T$, as the height is strictly positive. Thus, there exists a constant $C$ such that $|\nabla A|_{\tilde{\Omega}''} < C$ for all $t \in [0, T]$. As $|\nabla H|^2 \leq n|\nabla A|^2$ we have bounds for $|\nabla H|$. From Lemma 6.6 we know that there exists a constant $c'$ such that $|\frac{\partial H}{\partial x_1}| \leq c'$ in $\tilde{\Omega}''$, for $t \in [0, T]$. Therefore

$$\left|\int_{\alpha(t)}^{\beta(t)} \frac{\partial H}{\partial x_1} \, dx_1\right| \leq \int_{\alpha(t)}^{\beta(t)} \left|\frac{\partial H}{\partial x_1}\right| \, dx_1 \leq c' \int_{\alpha(t)}^{\beta(t)} \, dx_1,$$

$$\left|H(\beta(t), t) - H(\alpha(t), t)\right| \leq c'(\beta(t) - \alpha(t)),$$

$$\left|\frac{C_2 - C_1}{c'}\right| \leq |x_1(l_1(t), t) - x_1(l_2(t), t)|, \quad \text{for all} \quad t \in [0, T].$$  

\hfill \Box

We recall that the zeros of $\frac{\partial H}{\partial x_1}$ are located at $\{\xi_j(t)\}_{1 \leq j \leq m}$, so that $a < \xi_1(t) < \xi_2(t) < \cdots < \xi_m(t) < b$, and that $\lim_{t \to T} \xi_j(t) = \xi_j(T)$ exist from Lemma 6.5.
Lemma 6.9. For any fixed time \( t \) the one-dimensional Hausdorff measure of \( \xi_i(t) \) on the \( x_1 \) axis satisfies \( \mathcal{H}^1(\xi_i(t)) = 0 \), for all \( i \), if \( \mathcal{H}^1(\xi_i(0)) = 0 \) (i.e. \( \xi_i(t) \) cannot develop any sudden thickness or horizontal length on the spacetime diagram \([a, b] \times (0, T)\)).

Proof. From Lemma 5.7, \( |A|^2 \) is bounded in \( \hat{\Omega}' \) and, as \( p^2 \leq |A|^2 \), we have bounds for \( p \) in \( \hat{\Omega}' \). From ([5] Lemma 2.5), the radius function \( \rho(x_1, t) \) is smooth in the region \( \hat{\Omega}' \) because of the lower height bound. \( \mathcal{H}^1(\xi_i(t)) > 0 \) would imply a horizontal line on the spacetime diagram for a given time \( t \). This would mean multiple zeros of \( \frac{H}{p} \) for the corresponding values of \( x_1 \) on the axis of rotation. From Corollary 6.3 we know that the number of zeros of \( \frac{H}{p} \) is decreasing, therefore this cannot happen. □

In particular we rule out the scenario in figure 5. As the zeros of \( \frac{H}{p} \) can only develop in \( \hat{\Omega}' \) and as the height is always positive in this region, we have \( \rho(x_1, t) \in C^\infty(\mathbb{R} \times [0, T]) \) in \( \hat{\Omega}' \). Thus even at \( t = T \) the zeros of \( \frac{H}{p} \) are well behaved.

![Figure 5. \( H^1(\xi_i(T)) > 0 \)](image)

For the next Lemma we recall that

\[
\hat{\Omega}' = \left\{ x(l, t) : H(l, t) > \frac{C_2}{2} \right\}, \quad \text{and} \quad \Omega' = \bigcup_{t<T} \hat{\Omega}'.
\]

Lemma 6.10. There exists a constant \( c > 0 \), such that the one-dimensional Hausdorff measure of the projection of \( \hat{\Omega}' \) onto the \( x_1 \) axis satisfies \( \mathcal{H}^1(I(\hat{\Omega}')) > c \) for all \( t \leq T \).

Proof. To invoke Lemma 6.8, without loss of generality we choose \( \delta = \frac{C_2}{20} \), \( C_1 = \frac{6C_2}{10} \) and \( C_2 = \frac{9C_2}{10} \). Hence \( \hat{\Omega}_i'' = \left\{ x(l, t) : H(l, t) \leq \frac{19}{20}C_2 \right\} \). From Lemma 6.8 we have \( |x_1(l_i(t), t) - x_1(l_2(t), t)| \geq c > 0 \), for all \( t \leq T \). By the definition of \( \hat{\Omega}' \) and choosing \( l_i(t), i = 1, 2 \), such that \( H(l_i(t), t) = C_i \) we know that \( x(l_i(t), t) \in \hat{\Omega}_i'', i = 1, 2 \). As \( \mathcal{H}^1(I(\hat{\Omega}')) \geq |x_1(l_i(t), t) - x_1(l_2(t), t)| \) for any \( t \), we have the desired result. □

Lemma 6.11. In any compact set \( S \subset \hat{\Omega}' \) where \( y \geq \varepsilon > 0 \) there exists a constant \( c(\varepsilon) \) such that \( \frac{H}{p} \geq c(\varepsilon) > 0 \).

Proof. By definition we know that \( H > \frac{C_2}{2} \) in \( \hat{\Omega}' \) for all \( t < T \). In any compact set \( S \) where \( y \geq \varepsilon > 0 \), by Proposition 5.9 we know that \( |A|^2 \) is bounded and hence we get an upper bound for \( p \). Therefore we have a lower bound for \( \frac{H}{p} \) in \( S \). □

Therefore in \( \hat{\Omega}' \), \( \frac{H}{p} \) can only tend to zero when \( y \) tends zero. We will investigate this in the next section.
7. The Singularity

We break up the investigation of the singularity into different cases, depending on the value of $|k|$. With the aid of an additional assumption which involves a lower height bound, we will prove that the singularity is of type I. From now on all our calculations are done in $\mathbb{R}^3$ for two dimensional surfaces.

First we will state and prove two useful results in a restricted setting. We consider a two dimensional surface in $\mathbb{R}^3$ and follow a method as used by Huisken in [17].

7.1. The region $S$. Let $S_t \subset \Omega_t^i$ and $S = \bigcup_{t < T} S_t$. For this region $S$ we assume that there exist constants $c_{12}, c_{13} > 0$ such that

$$\frac{|A|^2}{H^2} \leq c_{12} \quad \text{and} \quad y|_{\Gamma_t} \geq c_{13}.$$

The following Lemma corresponds to Lemma 5.2 in [17].

**Lemma 7.1.** Under the above assumptions, there exists a constant $c_{15}$ such that $\frac{|q|}{p} \leq c_{15}$ in $S_t$ for all $t < T$.

**Proof.** From Lemma 4.1 we compute the evolution equation for $\frac{q}{H}$

$$\frac{d}{dt} \left( \frac{q}{H} \right) = \Delta \left( \frac{q}{H} \right) + \frac{2}{H} \left( \nabla H \cdot \nabla \left( \frac{q}{H} \right) \right) + \frac{q}{H} \left( (p^2 - q^2 - 2kp) + \frac{h}{H}(k^2 - kp) \right).$$

We know in $S_t$

$$|A| \leq c_{12} H^2, \quad -2kp \leq k^2 + p^2 = |A|^2 \leq c_{12} H^2, \quad 0 < c_2 \leq h \leq c_3, \quad H > \frac{c_2}{2}, \quad \text{such that} \quad \frac{h}{H} \leq \frac{2c_3}{c_2}.$$

When $\frac{q}{H} > 2 \sqrt{\frac{2c_3c_{12}}{c_2}}$, we obtain

$$p^2 - q^2 - 2kp + \frac{h}{H}(k^2 - kp) \leq c_{12}H^2 - 8\frac{c_3}{c_2}c_{12}H^2 + c_{12}H^2 + 2\frac{c_3}{c_2} \left( c_{12}H^2 + \frac{1}{2}c_{12}H^2 \right)$$

$$= c_{12}H^2 \left( 2 - \frac{5c_3}{c_2} \right) \leq 0,$$

as $\frac{c_3}{c_2} \geq 1$. Therefore, when $\frac{q}{H} > 2 \sqrt{\frac{2c_3c_{12}}{c_2}}$, from the parabolic maximum principle and viewing the surface as periodic

$$\frac{q}{H} \leq \max \left( \max_{S_0} \frac{q}{H}, \max_{\partial S_t} \frac{q}{H}, 2 \sqrt{\frac{2c_3c_{12}}{c_2}} \right).$$

We recall that $q = \langle \nu, i_1 \rangle y^{-1}$. As $\max_{\partial S_t} \frac{q}{H} \leq \max_{\partial S_t} \frac{2y^{-1}}{c_2}$ and as $y^{-1}|_{\partial S_t} \leq \frac{1}{c_{13}}$, the right hand side in the above estimate is bounded.

Similarly when $\frac{q}{H} < -2 \sqrt{\frac{2c_3c_{12}}{c_2}}$, we have

$$\frac{d}{dt} \left( \frac{q}{H} \right) \geq \Delta \left( \frac{q}{H} \right) + \frac{2}{H} \left( \nabla H \cdot \nabla \left( \frac{q}{H} \right) \right).$$
Thus from the maximum principle we have \( \frac{|q|}{p} \leq c_{14} \). As \( \frac{k}{p} \leq c_1 \) we obtain
\[
|q| \leq c_{14}H = c_{14}(p + k) \leq c_{15}p,
\]
as desired. \( \square \)

**Remark 7.2.** (i) Note that \( \frac{|q|}{p} \) is a geometric quantity that corresponds to the slope \( |\rho'| \) of the generating curve \( \rho \). Therefore, Lemma 7.1 gives us a gradient bound in the region \( S \).

(ii) Assuming that the singularity develops in \( S \), there is a point on the generating curve that approaches the axis of rotation as \( t \to T \). By definition, as \( y|_\Gamma_S \geq c_{13} \), we have \( y|_{\partial S_t} \geq c_{13} \) for all \( t < T \). As the boundary of the domain has a lower height bound, and as the above gradient bound holds in that domain, \( S_t \) cannot collapse to a point as \( t \) goes to \( T \).

The following Proposition corresponds to ([17], Proposition 5.3).

**Proposition 7.3.** If the singularity develops in the region \( S \), then there exists a constant \( C > 0 \) such that the second fundamental form satisfies
\[
\max_{S_t} |A|^2 \leq C \frac{1}{T - t},
\]
for all \( t < T \).

**Proof.** From (4.3) we know
\[
\frac{d}{dt} y^{-1} = (H - h)py^{-1} \geq (H - c_3)py^{-1}.
\]
As \( p^2 \leq |A|^2 \leq c_{12}H^2 \), and using Lemma 7.1
\[
y^{-2} = p^2 + q^2 \leq (1 + c_{15}^2)p^2,
\]
\[
\frac{d}{dt} y^{-1} \geq \left( \frac{1}{\sqrt{c_{12}}p - c_3} \right) \frac{1}{\sqrt{(1 + c_{15}^2)}} y^{-2}.
\]
As \( S \) contains the singularity, for \( t \) near \( T \), we have \( p \to \infty \). For \( p \geq 2\sqrt{c_{12}c_3} \)
\[
\frac{1}{\sqrt{c_{12}}p - c_3} \geq \frac{1}{2\sqrt{c_{12}}p}.
\]
Therefore
\[
\frac{d}{dt} y^{-1} \geq \frac{1}{2\sqrt{c_{12}}} \frac{1}{\sqrt{(1 + c_{15}^2)}} y^{-2} \geq \frac{1}{2\sqrt{c_{12}(1 + c_{15}^2)}} y^{-3}.
\]
Let \( U(t) = \max_{S_t} y^{-1} \). By renaming the constant \( \frac{1}{2\sqrt{c_{12}(1 + c_{15}^2)}} = \epsilon \) we obtain
\[
\frac{d}{dt} U(t) \geq \epsilon U^3(t) \iff \frac{d}{dt} U^{-2}(t) \leq -2\epsilon.
\]
Since \( U^{-2}(t) \) tends to zero as \( t \to T \), we integrate from \( t \) to \( T \) and obtain
\[
U(t) = \max_{S_t} y^{-1} \leq \frac{1}{\sqrt{2\epsilon(T - t)}}.
\]
As \( |A|^2 \leq c_{12}H^2 \) and \( H = k + p \leq c_1p + p \leq (c_1 + 1)y^{-1} \) we get the result. \( \square \)

These results will be useful when we consider the different cases outlined below.
7.2. **Different cases.** We consider different scenarios, depending on the value of $|k|$. For this we divide the hypersurface into different regions. We now state an additional assumption, which will only be used when $|k|$ is unbounded; it presumes a height bound on the boundary of $\Omega^p_t$ independent of time.

**Assumption 7.4.** We assume that there exists a $c_{00} > 2$ such that $y|_{\partial\Omega^p_t} \geq c$ for some $c > 0$ for all $t < T$.

The different cases we look at are shown in the figure below.

Due to the lower height bound in $\hat{\Omega}'$ (Lemma 5.1), and Proposition 5.6, we know that $|A|^2$ is bounded in $\hat{\Omega}'$. Therefore the singularity can only develop in $\hat{\Omega}'$. As $M_t = \Omega^p_t \cup \Omega^\sharp_t$, we have $\hat{\Omega}' = \left(\hat{\Omega}' \cap \Omega^p\right) \cup \left(\hat{\Omega}' \cap \Omega^\sharp\right)$. Therefore the cases shown in the diagram below present all the possible conditions under which a singularity can develop. The case where $|k|$ is bounded on $M_t$ for all $t < T$ is considered in section 7.3. The case where $|k|$ is unbounded and the singularity develops in $\hat{\Omega}' \cap \Omega^p$ is considered in section 7.4 and the case where $|k|$ is unbounded and the singularity develops in $\hat{\Omega}' \cap \Omega^\sharp$ is considered in section 7.5. The latter two cases are investigated under Assumption 7.4.

7.3. **Case I : $|k|$ is bounded on $M_t$ for all $t < T$.** In this case it can be shown that $\frac{|A|^2}{H^2}$ is bounded in $\hat{\Omega}'$ as follows. As $|k| \leq c_{16}$, we get

$$p^2 - 2c_{16}p \leq k^2 + p^2 - 2c_{16}p \leq k^2 + p^2 + 2kp.$$  

Hence

$$\frac{|A|^2}{H^2} = \frac{k^2 + p^2}{k^2 + p^2 + 2kp} \leq \frac{c_{16}^2 + p^2}{p^2 - 2c_{16}p}.$$  

In the case of a singularity developing, we can assume without loss of generality that $p \geq 4c_{16}$, such that $p - 2c_{16} \geq \frac{p}{2}$. Therefore, for $p \geq 4c_{16}$,

$$\frac{|A|^2}{H^2} = \frac{k^2 + p^2}{k^2 + p^2 + 2kp} \leq \frac{c_{16}^2 + p^2}{p(p - 2c_{16})} \leq \frac{p^2 + p^2}{p^2} = \frac{17}{8}.$$
From Lemma 5.1 we know that \( \inf_{\Gamma(t)} y \geq c' \). By letting \( S_t := \tilde{\Omega}_t' \), from Lemma 7.1 and Proposition 7.3 we conclude that the singularity is of type I.

### 7.4. Case II: \( |k| \) is unbounded and the singularity develops in \( \tilde{\Omega}' \cap \Omega^2 \). Cases II and III are studied under Assumption 7.4. Let

\[
\left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_+ = \left\{ x(l, t) \in \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right) : k(l, t) \geq 0 \right\} \quad \text{and} \quad \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_- = \bigcup_{t<T} \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_+ .
\]

Let

\[
\left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_- = \left\{ x(l, t) \in \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right) : k(l, t) < 0 \right\} \quad \text{and} \quad \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_- = \bigcup_{t<T} \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_- .
\]

In \( \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_+ \), we have \( |A|^2 = \frac{k^2 + p^2}{k^2 + p^2 + 2kp} \leq 1 \) as \( k \geq 0 \).

As \( c_0 > 2 \) we have \( \frac{c_0}{2(c_0 - 1)} < 1 \). Let \( \delta := 1 - \sqrt{\frac{c_0}{2(c_0 - 1)}} > 0 \). In \( \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_- \) we have \( |k| = -k \) as \( k < 0 \). Therefore in \( \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_- \), by definition of the domain (see 4.3.4),

\[
-k \leq \frac{k}{p} \leq 1 - \delta , \quad \delta p \leq H .
\]

This estimate combined with

\[
|A|^2 = k^2 + p^2 \leq \left( \frac{c_0}{2(c_0 - 1)} + 1 \right) p^2 = \frac{3c_0 - 2}{2(c_0 - 1)} p^2 ,
\]

gives us a bound for \( \frac{|A|^2}{H^2} \) in \( \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_- \). As \( \frac{|A|^2}{H^2} \leq 1 \) in \( \left( \tilde{\Omega}_t' \cap \Omega_t^2 \right)_+ \) we have an upper bound for \( \frac{|A|^2}{H^2} \) in \( \tilde{\Omega}_t' \cap \Omega_t^2 \). From Lemma 5.1 we know that \( \inf_{\Gamma(t)} y \geq c' > 0 \) and from the Assumption 7.4 we know that \( y|_{\tilde{\Omega}_t'} \geq c > 0 \) for all \( t < T \). By letting \( S_t := \tilde{\Omega}_t' \cap \Omega_t^2 \), from Lemma 7.1 we get a gradient bound in \( \tilde{\Omega}_t' \cap \Omega_t^2 \). By Proposition 7.3 we conclude the singularity is of type I in this case as well.

### 7.5. Case III: \( |k| \) is unbounded and the singularity develops in \( \tilde{\Omega}_t' \cap \Omega_t^2 \). First we prove a couple of useful results in the domain \( \tilde{\Omega}_t' \cap \Omega_t^2 \). We show that the gradient function is bounded in \( \Omega_t^2 \) in the following Lemma.

**Lemma 7.5.** There exists a constant \( c_{17} \) such that \( v \leq c_{17} \) in \( \Omega_t^2 \) for all \( t < T \).

**Proof.** From Corollary 4.3 we know

\[
\left( \frac{d}{dt} - \Delta \right) v = -|A|^2 v + p^2 v^3 - 2vk^2(v^2 - 1) .
\]
For \( v \geq \sqrt{c_{00}} \) we have \( v^2 - 1 \geq \left( \frac{c_{00} - 1}{c_{00}} \right) v^2 \), giving us

\[
\left( \frac{d}{dt} - \Delta \right) v \leq -|A|^2 v + p^2 v^3 - 2 \left( \frac{c_{00} - 1}{c_{00}} \right) k^2 v^3,
\]

\[
= -|A|^2 v + 2 \left( \frac{c_{00} - 1}{c_{00}} \right) v^3 p^2 \left( \frac{c_{00}}{2(c_{00} - 1)} - \frac{k^2}{p^2} \right) \leq 0,
\]
as \( \frac{|k|}{p} > \sqrt{\frac{c_{00}}{2(c_{00}-1)}} \) in \( \Omega_t^\sharp \). By the parabolic maximum principle

\[
v \leq \max \left( \max_{\Omega_0^t} v, \max_{\partial \Omega_t^\sharp} v, \sqrt{c_{00}} \right).
\]

From Assumption 7.4 we know that \( y \geq c > 0 \) on \( \partial \Omega_t^\sharp = \partial \Omega_t^\sharp \). From Lemma 5.3 we know that \( v y \leq c_4 \) on \( M_t \) for \( t < T \). Therefore \( v \leq \frac{c_4}{c} \) on \( \partial \Omega_t^\sharp \) for all \( t < T \). So we have

\[
v \leq \max \left( \max_{\Omega_0^t} v, \frac{c_4}{c}, \sqrt{c_{00}} \right) =: c_{17}.
\]

\( \square \)

As \( \inf_{\Omega} y \geq c' > 0 \) and \( y|_{\partial \Omega_t^\sharp} > c > 0 \) for all \( t < T \) (Assumption 7.4), and as the gradient is bounded in the domain \( \Omega_t^\sharp \) for all \( t < T \), by an argument similar to that in Remark 7.2(ii) we see that the domain \( \Omega_t \cap \Omega_t^\sharp \) does not collapse as \( t \) tends to \( T \).

Note that for this case (singularity develops in \( \Omega_t^\sharp \cap \Omega_t^\sharp \)) we do not have an upper bound for \( \frac{|A|^2}{H} \) and thus we cannot use Proposition 7.3. Our approach here is to investigate the behaviour of the curvature along paths going to the singularity, i.e. smooth curves in the domain \( \Omega \times [0, T) \) approaching the singularity. First we consider paths along which \( \frac{H}{p} \) stays bounded away from zero.

**Proposition 7.6.** Let \( \Lambda_t \) be a path going to the singularity in \( \Omega_t^\sharp \cap \Omega_t^\sharp \). If \( \frac{H}{p} \geq c_{18} > 0 \) on \( \Lambda_t \) for some constant \( c_{18} > 0 \), then there exists a constant \( C \) such that

\[
|A|^2|_{\Lambda_t} \leq \frac{C}{T - t}.
\]

**Proof.** The proof is along the same lines as that of Proposition 7.3 with only a few changes. From (4.3) we know

\[
\frac{d}{dt} y^{-1} = (H - h)py^{-1} \geq (c_{18}p - c_3)py^{-1}.
\]

As \( p \to \infty \) on \( \Lambda_t \), for \( p \geq 2\frac{c_3}{c_{18}} \) we have \( c_{18}p - c_3 \geq \frac{1}{2} c_{18}p \).

Therefore

\[
\frac{d}{dt} y^{-1} \geq \frac{c_{18}}{2} p^2 y^{-1} = \frac{c_{18} y^{-3}}{2} v^2, \quad \text{as} \quad p = \frac{1}{v y}.
\]

As \( v \leq c_{17} \) in \( \Omega_t^\sharp \)

\[
\frac{d}{dt} y^{-1} \geq \frac{c_{18}}{2 c_{17}^2} y^{-3}.
\]
Let \( U(t) = y^{-1}|_{\Lambda_t} \). By renaming the constant \( \frac{c_{18}}{2\pi^2} = \epsilon \) we obtain
\[
\frac{d}{dt}U(t) \geq \epsilon U^3(t) \iff \frac{d}{dt}U^{-2}(t) \leq -2\epsilon .
\]
Since \( U^{-2}(t) \) tends to zero as \( t \to T \) on \( \Lambda_t \), we integrate from \( t \) to \( T \) and obtain
\[
U(t) = y^{-1}|_{\Lambda_t} \leq \frac{1}{\sqrt{2\epsilon(T-t)}} .
\]
From Proposition 5.5 we know that \( |c| \leq \max(c_1, n-1) \) in \( \hat{\Omega}' \cap \Omega^2 \). As we are considering a two dimensional hypersurface, we have \( |A|^2 = k^2 + p^2 \leq c_1 p^2 + p^2 \leq (c_1 + 1)y^{-2} \), along any path \( \Lambda_t \) in \( \hat{\Omega}' \cap \Omega^2 \) where \( \frac{H}{p} \geq c_{18} > 0 \), and thus we get the result. \( \square \)

In what follows we prove that a singularity that develops in \( \hat{\Omega}' \cap \Omega^2 \) is of type I by way of contradiction. We start by considering a statement which is the negation of a singularity of type I. Then we prove that this statement is false by using a rescaling procedure similar to that used in [18].

**Statement 7.7.** For all \( c > 0 \) there exists \( t(c) \in [0, T) \), such that
\[
\max_{l \in M^n} |A|^2(l, t(c)) > \frac{c}{T - t(c)} .
\]

**Proposition 7.8.** Statement 7.7 is false in \( \hat{\Omega}' \cap \Omega^2 \).

**Proof.** Suppose Statement 7.7 were true in \( \hat{\Omega}' \cap \Omega^2 \). Let us choose a sequence \( \{c_i\}_{i \in \mathbb{N}} \) such that \( \lim_{i \to \infty} c_i = \infty \). If there exists more than one \( t(c_i) \) for a particular \( c_i \), we choose the earliest \( t(c_i) =: t_i \) such that the above inequality is satisfied. Then
\[
\max_{l \in M^n} |A|^2(l, t_i) (T - t_i) > c_i .
\]

The sequence \( \{c_i\} \) going to infinity forces the left hand side of the above inequality to also go to infinity. Therefore \( \max_{l \in M^n} |A|^2(l, t_i) \) has to go to infinity as \( c_i \) goes to infinity. That means that \( t_i \) approach \( T \) as \( i \) goes to infinity.

We now rescale \( \hat{\Omega}' \cap \Omega^2 \) as follows: Let
\[
\alpha_i = \max_{l \leq t, t \in \mathbb{N}} |A|(l, t)
\]
and the points \( l_i \) be defined by the equation
\[
|A|(l_i, t_i) = \max_{l \in M^n} |A|(l, t_i) .
\]
Then we have
\[
(7.1) \quad \alpha_i \geq |A|(l_i, t_i) \geq \sqrt{\frac{c_i}{T - t_i}} .
\]
We consider the family of rescaled surfaces \( \mathcal{M}_{i, \tau} \) defined by the following immersions:
\[
(7.2) \quad \tilde{\mathcal{X}}_i(\cdot, \tau) = \alpha_i \left( \mathcal{X}(\cdot, \alpha_i^{-2} \tau + t_i) - \langle \mathcal{X}(l_i, t_i), i_1 \rangle i_1 \right) ,
\]
where \( \tau \in [-\alpha_i^2 t_i, 0] \). For this rescaling we have
\[
\tilde{H}_i(\cdot, \tau) = \alpha_i^{-1} H(\cdot, \alpha_i^{-2} \tau + t_i) \quad \text{and} \quad |\tilde{A}_i|(\cdot, \tau) = \alpha_i^{-1} |A|(\cdot, \alpha_i^{-2} \tau + t_i) .
\]
This rescaling guarantees that $|\tilde{A}_i| \leq 1$. From Proposition 5.5 we know that $\frac{|k|}{p} \leq \max(c_1, 1)$ in $\bar{\tilde{\Omega}}' \cap \tilde{\Omega}^2$. Therefore

$$|A| = \sqrt{k^2 + p^2} \leq c_5 p \leq c_5 y^{-1},$$

where $c_5 = \sqrt{1 + (\max(c_1, 1))^2}$. Hence we obtain

(7.3) \quad |\tilde{A}_i| = \alpha_i^{-1}|A| \leq \alpha_i^{-1} c_5 y^{-1} = c_5 (\alpha_i y)^{-1} = c_5 \hat{y}^{-1}.

Thus the rescaled surfaces do not float away to infinity. As $|\tilde{A}_i| \leq 1$ for all $i$, by employing a diagonal sequence argument, we find that the rescaled surfaces $\mathcal{M}_{i,0}$ converge to a smooth hypersurface, which we denote by $\mathcal{M}_0$. We explain this process in the next paragraph.

Without loss of generality let us assume that the singularity develops at the origin. We translate each generating curve to have $\langle x(l_i, t_i), i_1 \rangle = 0$. We rename them, using the notation for the radius function, $\rho$, and work with them for the rest of the section. As $\tilde{A}_i(l_i, 0) = 1$, from (7.3) we have $\hat{y}(l_i, 0) \leq c_5$ for all $i$. Thus we find a converging subsequence of points. We look at the unit tangent vectors at $x_1 = 0$ of the corresponding subsequence of curves. By identifying these unit vectors with points on the sphere, we observe that a subsequence of these points converges and so obtain a subsequence of tangent vectors that converges. As $\tilde{A}_i \leq 1$ on the hypersurface, we can roll a unit radius ball anywhere on the hypersurface. Thus, after rotating each curve of the earlier subsequence such that all the unit tangent vectors at $x_1 = 0$ are horizontal, we have a uniform gradient bound within a small radius $|x_1| \leq \delta < 1$. Therefore by Arzela-Ascoli we find a subsequence that converges uniformly within the interval $|x_1| \leq \delta$, in which the uniform gradient bound holds. Repeating this process along the curve we can find a uniformly converging solution within any given compact set. This process is explained in detail in [20] Chapter 5.

Now we consider paths that lie in $\hat{\tilde{\Omega}}' \cap \tilde{\Omega}^2$ in the spacetime diagram $\Omega \times [0, T)$ . We look at the value of $\frac{H}{p}$ on these paths. We note that as $\frac{\rho}{p} \leq c_1$ on $M_t$ for all $t < T$ (Proposition 5.4), $\frac{H}{p} \leq 1 + c_1$ on $M_t$ as well. Thus $\frac{H}{p}$ is bounded on $M_t$ for all $t < T$.

From Corollary 6.3 we know that the zeros of $\frac{H}{p}$ are discrete and non-increasing with time. And from Lemma 6.5 we also know that the zeros of $\frac{H}{p}$ converge as $t$ tends to $T$. Thus the zeros of $\frac{H}{p}$ are well behaved.

The quantity $\frac{H}{p}$ can be zero either when $H = 0$ or when $p = \infty$. However, the zeros of $\frac{H}{p}$ that correspond to the zeros of $H$ lie outside $\hat{\tilde{\Omega}}'$. In conclusion the only point where $\frac{H}{p}$ can tend to zero in the domain $\hat{\tilde{\Omega}}' \cap \tilde{\Omega}^2$ is the singular point. That means, if we follow any path in the domain $\hat{\tilde{\Omega}}' \cap \tilde{\Omega}^2$, the value of $\frac{H}{p}$ cannot be zero as long as we are not at the singularity.

We consider at different paths in $\hat{\tilde{\Omega}}' \cap \tilde{\Omega}^2$ for $t \in [0, T)$. We categorize these paths by the behaviour of $\frac{H}{p}$, and show that the sequence $\hat{H}_i$ converges to zero as $i$ goes to infinity, along each of these paths.

(1) **Paths going to the singularity** – paths where $\frac{H}{p} \geq c > 0$ for some positive constant $c$

From Proposition 7.6 we know that on these paths $|A|^2 \leq \frac{C}{t}$ for some positive constant $C$. 


Therefore on these paths
\[ H^2 \leq n|A|^2 \leq \frac{nC}{T-t}. \]
From (7.1)
\[ \tilde{H}_i^2 = \alpha_i^{-2}H^2 \leq \left( \frac{T-t_i}{c_i} \right) \left( \frac{nC}{T-t} \right). \]
As \( \frac{T-t}{T-t_i} < 1 \) for \( t < t_i \),
\[ \tilde{H}_i^2 \leq \frac{nC}{c_i}, \quad \text{for} \quad t < t_i. \]
As \( c_i \) goes to infinity, \( \tilde{H}_i \) converges to zero on these paths.

(2) **Paths going to the singularity – paths where \( \frac{H}{p} \to 0 \) as \( t \to T \)**

As \( \alpha_i \geq p \) for \( t \leq t_i \), on these paths we have
\[ \tilde{H}_i = \alpha_i^{-1}H \leq \frac{H}{p} \to 0. \]
On these paths \( \tilde{H}_i \) converges to zero as well.

(3) **Paths not going to the singularity**

On these paths \( |A|^2 \leq c \) for some constant \( c \), that depends on the end point of the path. As \( \alpha_i \) goes to infinity, we have
\[ \tilde{H}_i = \alpha_i^{-1}H \leq \alpha_i^{-1}\sqrt{n|A|} \leq \alpha_i^{-1}\sqrt{nc} \to 0. \]
Therefore on all paths in \( \hat{\Omega}' \cap \Omega^\beta \), \( \tilde{H}_i \) converges to zero as \( i \) goes to infinity. The limiting solution \( \mathcal{M}_0 \) is a catenoid, as it is the only axially symmetric minimal surface with zero mean curvature.

We are now in a position to show that we have a contradiction: In order to get a better understanding of the original surface we rescale back \( \mathcal{M}_{i,0} \) for large \( i \), and show that the estimate \( vy \leq c_4 \) would not hold on that (the original) surface.

We denote the quantities associated to the catenoid \( \mathcal{M}_0 \) by a hat \( \hat{\cdot} \). We obtain the catenoid \( \mathcal{M}_0 \) by rotating \( \hat{y} = c_5 \cosh(c_5^{-1}\hat{x}_1) \) around the \( x_1 \) axis, where \( \hat{x}_1 \) is the \( x_1 \) coordinate of the limiting surface \( \mathcal{M}_0 \). For any \( \epsilon_1 > 0 \) and for any \( l_0 \in M^n \) we have
\[ |\hat{v}(l_0)\hat{y}(l_0) - \tilde{v}_i(l_0,0)\tilde{y}_i(l_0,0)| \leq \epsilon_1 \quad \text{for large} \quad i. \]
For the catenoid \( \hat{v} = \sqrt{1 + \hat{y}^2} = \sqrt{1 + \sinh^2(c_5^{-1}\hat{x}_1)} = \cosh(c_5^{-1}\hat{x}_1) \). As \( \tilde{y}_i = \alpha_i y \) and \( \tilde{v}_i = v \) we have
\[ c_5 \cosh^2(c_5^{-1}\hat{x}_1(l_0)) - \epsilon_1 \leq \alpha_i v(l_0, \alpha_i^{-2}\tau + t_i)y(l_0, \alpha_i^{-2}\tau + t_i), \]
\[ \frac{c_5}{2\alpha_i} \left( \cosh(2c_5^{-1}\hat{x}_1) + 1 \right) - \frac{\epsilon_1}{\alpha_i} \leq vy \quad \text{for} \quad i > I_0. \]
As
\[ \hat{x}_1(l_0) = \langle \hat{x}(l_0), i_1 \rangle = \lim_{j \to \infty} \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right), \]
where \( x_{1j} := \langle x_j, i_1 \rangle \), for any \( \epsilon_2 > 0 \) we have
\[ \left| \hat{x}_1(l_0) - \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right) \right| \leq \epsilon_2 \quad \text{for large } j, \]

(7.5) \[ \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right) - \epsilon_2 < \hat{x}_1(l_0) \leq \epsilon_2 + \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right), \]
for \( j > J_0 \). For given \( \epsilon_1, \epsilon_2 \) and for a fixed \( i > I_0 \), we pick large \( j > J_0 \), \( j >> i \), such that
\[ \frac{1}{\alpha_j} \left( \epsilon_2 + 2c_5 \log \left( \frac{4\alpha_i}{c_5} \left( c_4 + \frac{\epsilon_1}{\alpha_i} \right) \right) - 1 \right) \ll \epsilon, \]
with \( \epsilon \) sufficiently small and with \( c_4 \) as in Lemma 5.3. Then we choose \( l_0 \in M^\infty \) such that
(7.6) \[ x_1 \left( l_0, \alpha_j^{-2} \tau_0 + t_j \right) > x_{1j} + \epsilon \]
holds. Considering \( M_t \) as a periodic surface, we can find points on the hypersurface that lie an \( \epsilon \) distance away from \( x_{1j} \). Therefore
(7.7) \[ x_1 \left( l_0, \alpha_j^{-2} \tau_0 + t_j \right) > x_{1j} + \frac{1}{\alpha_j} \left( \epsilon_2 + 2c_5 \log \left( \frac{4\alpha_i}{c_5} \left( c_4 + \frac{\epsilon_1}{\alpha_i} \right) \right) - 1 \right) . \]
We will use the inequality (7.7) to obtain a contradiction. By (7.5) and (7.7)
\[ \hat{x}_1(l_0) > \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right) - \epsilon_2 > 2c_5 \log \left( \frac{4\alpha_i}{c_5} \left( c_4 + \frac{\epsilon_1}{\alpha_i} \right) - 1 \right) > 0. \]
Therefore
\[ \frac{c_5}{2\alpha_i} \cosh \left( \frac{1}{2c_5} \left( \left( \alpha_j - x_{1j} \right) - \epsilon_2 \right) \right) < \frac{c_5}{2\alpha_i} \cosh \left( \frac{1}{2c_5} \hat{x}_1(l_0) \right) \quad \text{for } j > J_0. \]
By (7.4)
\[ \frac{c_5}{2\alpha_i} \left( \cosh \left( \frac{1}{2c_5} \left( \alpha_j(x_1 - x_{1j}) - \epsilon_2 \right) \right) + 1 \right) - \frac{\epsilon_1}{\alpha_i} < \frac{c_5}{2\alpha_i} \left( \cosh \left( \frac{1}{2c_5} \hat{x}_1(l_0) \right) + 1 \right) - \frac{\epsilon_1}{\alpha_i} < vy, \]
\[ \frac{c_5}{4\alpha_i} \left( e^{\frac{1}{2c_5}(\alpha_j(x_1 - x_{1j}) - \epsilon_2)} - e^{-\frac{1}{2c_5}(\alpha_j(x_1 - x_{1j}) - \epsilon_2)} + 2 \right) - \frac{\epsilon_1}{\alpha_i} < vy, \]
for \( \epsilon_1, \epsilon_2, i, j \) and \( l_0 \) as previously chosen. As \( \alpha_j \left( x_1(l_0, \alpha_j^{-2} \tau + t_j) - x_{1j} \right) - \epsilon_2 > 0 \), by the choice of \( l_0 \)
(7.8) \[ \frac{c_5}{4\alpha_i} \left( e^{\frac{1}{2c_5}(\alpha_j(x_1 - x_{1j}) - \epsilon_2)} + 1 \right) - \frac{\epsilon_1}{\alpha_i} \leq vy. \]
But from (7.7) we have
(7.9) \[ \frac{c_5}{4\alpha_i} \left( e^{\frac{1}{2c_5}(\alpha_j(x_1 - x_{1j}) - \epsilon_2)} + 1 \right) - \frac{\epsilon_1}{\alpha_i} > c_4. \]
From Lemma 5.3 we know that \( vy \leq c_4 \). Therefore (7.8) and (7.9) contradict Lemma 5.3: by examining the rescaled surfaces we find that the estimate \( vy \leq c_4 \) does not hold on the corresponding, non-rescaled, hypersurfaces near the singular time \( T \).
Therefore we have a contradiction and Statement 7.7 is false. \( \square \)
Hence there exists a constant $c > 0$ such that for all $t \in [0, T)$,
\[
\max_{l \in M^n} |A|^2(l, t) \leq \frac{c}{T - t}.
\]
We have proved that the singularity is of type I if it develops in $\hat{\Omega}' \cap \Omega^2$. The combination of cases I, II and III, gives the proof of Theorem 1.1.

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