A double-pivot degenerate-tolerable simplex algorithm for linear programming

Yaguang Yang∗ and Fabio Vitor†

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Abstract

A double pivot algorithm that combines features of two recently published papers by these authors is proposed. The proposed algorithm is implemented in MATLAB. The MATLAB code is tested, along with a MATLAB implementation of Dantzig’s algorithm, for several test sets, including a set of cycling LP problems, Klee-Minty’s problems, randomly generated linear programming (LP) problems, and Netlib benchmark problems. The test result shows that the proposed algorithm is (a) degenerate-tolerance as we expected, and (b) more efficient than Dantzig’s algorithm for large size randomly generated LP problems but less efficient for Netlib benchmark problems and small size randomly generated problems in terms of CPU time.

Keywords: Double-pivot algorithm, degenerate-tolerable, simplex method, linear programming, Klee-Minty cube.

MSC classification: 90C05 90C49.

∗US Department of Commerce, 305 Deer Meadow Lane, Rockville, 20850. Email: yaguang.yang@verizon.net.
†Department of Mathematics, University of Nebraska at Omaha. Email: ftorresvitor@unomaha.edu.
1 Introduction

It has been more than 70 years since Dantzig formulated a linear programming (LP) problem and proposed the simplex method [5]. The main idea of this algorithm is to search for an optimizer from a vertex to the next vertex in the polyhedra formed by the linear constraints using Dantzig’s pivoting rule. Since then, linear programming has been one of the mostly studied problems in mathematics. The simplex method remained to be a mainstream technique for LP until Klee and Minty [15] found an example that shows that, in the worst case scenario, Dantzig’s pivot rule needs exponentially many iterations to find an optimal solution of the LP problem. Klee and Minty’s work inspired researchers to find different ways to solve LP problems. Interior-point methods emerged as a computationally feasible alternative technique, which admits algorithms that find an optimal solution in finite iterations bounded by a polynomial of the problem size. In the last three decades, most researches in linear programming focused on finding novel interior-point methods [24, 27]. However, the simplex method never faded away. In 2018, motivated by enhancing the computational performance of the simplex method, Vitor and Easton [21, 22, 23] proposed a double pivot algorithm, which updates two pivot variables in one iteration using Dantzig’s rule. In 2020, motivated by improving the bound of iteration number of the simplex method, Yang [26] independently proposed a different double pivot rule that uses a combined criteria to select two pivot variables.

Realizing the merits of the pivoting rule of [26] and the slope algorithm (SA) proposed in [21, 22, 23], in this paper, we propose and implement a new double pivot algorithm that combines the pivoting rule of [26] and the SA algorithm designed for a two-dimensional linear programming (LP) [21, 22, 23]. This algorithm selects two entering variables based on three criteria. The first entering variable is selected based on Dantzig’s rule, the second entering variable is selected based on the longest step size rule, the combination of the coefficients of these two variables are determined by the criterion that optimizes the cost reduction, which is equivalent to solving a linear programming problem of two variables. The SA algorithm of [21, 22, 23] was developed specifically for this problem.

The intuitions behind the pivoting criteria of [26] are as follows: first, Dantzig’s rule is selected because the computational experience in decades shows that it is probably the most efficient among popular pivoting rules [18]; second, specially designed examples show that none of the popular pivoting rules is better than others [17] in the worst case, hence, using a combination of different rules in a random way can be beneficial and has been proved that the strategy results in some polynomial simplex algorithms in the sense of statistics [8, 13]; third, using the longest step size rule will decrease the chance to enter a degenerate basic feasible solution and increase the chance to get out of a degenerate solution [28]; fourth, the longest step rule may be beneficial to reduce the iteration numbers [26]; finally, using deterministic pivoting rules may give us some hope to find a strongly polynomial algorithm to solve LPs, which is not possible for randomized algorithms such as the ones proposed in [8, 13].

Throughout the paper, we use capitalized bold font for matrices, small case bold font for vectors, and normal font for scalars. To save space, we write stacked vector
\([x^T, y^T]^T\) as \((x, y)\). The remainder of the paper is organized as follows. Section 2 presents the proposed algorithm. Section 3 discusses some implementation details. Numerical test problems and test results are provided in Section 4. Conclusions remarks are summarized in Section 5.

2 The proposed algorithm

We consider the primal linear programming problem in standard form:

\[
\begin{align*}
\min & \quad c^T x, \\
\text{subject to} & \quad Ax = b, \quad x \geq 0,
\end{align*}
\]

where \(A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m, \ c \in \mathbb{R}^n\) are given, and \(x \in \mathbb{R}^n\) is the vector to be optimized. Associated with the primal linear program is the dual problem that is also presented in standard form:

\[
\begin{align*}
\max & \quad b^T y, \\
\text{subject to} & \quad A^T y + s = c, \quad s \geq 0,
\end{align*}
\]

where \(y \in \mathbb{R}^m\) is the dual variable vector, and \(s \in \mathbb{R}^n\) is the dual slack vector. In the discussion below, we make the following assumptions:

1. \(\text{rank}(A) = m\).

2. The primal problem (1) has an optimal solution.

These assumptions are standard. A feasible solution of the linear program satisfies the conditions of \(Ax = b\) and \(x \geq 0\) which exists because of Assumption 2. Among all feasible solutions, we will consider only basic feasible solutions which corresponds to the vertices of the convex polytope described by the constraints of (1). We will denote by \(B \subset \{1, 2, \ldots, n\}\) the index set with cardinality \(|B| = m\) and \(N = \{1, 2, \ldots, n\} \setminus B\) the complement set of \(B\) with cardinality \(|N| = n - m\) such that matrix \(A\) and vector \(x\) can be partitioned as \(A = [A_B, A_N]\) and \(x = (x_B, x_N)\) with \(x_B \geq 0\) and \(x_N = 0\). Moreover, if \(A_B x_B = b\), we call \(x = (x_B, 0)\) as the basic feasible solution. Because some components of \(x_B\) may be zeros, and/or even worse, \(A_B\) may not be full rank, this partition admits the degenerate feasible solution which we have seen many times in Netlib benchmark problems [3]. We denote by \(B\) the set of all bases \(B\) and by \(N\) the set of all non-bases \(N\). Therefore, the linear programming problem (1) can be written as:

\[
\begin{align*}
\min & \quad c_B^T x_B + c_N^T x_N, \\
\text{subject to} & \quad A_B x_B + A_N x_N = b, \quad x_B \geq 0, \quad x_N \geq 0.
\end{align*}
\]

Let superscript \(k\) represent the \(k\)th iteration. Thus, the matrices and vectors in the \(k\)th iteration are then denoted by \(A_{B^k}, A_{N^k}, c_{B^k}, c_{N^k}, x_{B^k}, x_{N^k}\), where \(x^k = (x_{B^k}, x_{N^k})\)
is the basic feasible solution of (1) with \( x_{B^k} \geq 0 \) and \( x_{N^k} = 0 \). Similarly, we denote by \( x^* = (x_{B^*}, x_{N^*}) \) the optimal basic solution of (1) with \( A_{B^*} x_{B^*} = b, \ x_{B^*} \geq 0 \), and \( x_{N^*} = 0 \), by \( z^* = c^T x^* \) the optimal value. It is worthwhile to note that the partition of \( (B^k, N^k) \) keeps updating and it is different from the partition \( (B^*, N^*) \) before an optimizer is found. If \( A_{B^k} \) is full rank, the reduced cost vector can be calculated as

\[
\bar{c}_{N^k}^T = (c_{N^k} - A_{N^k}^T A_{B^k}^T c_{B^k})^T,
\]

and the iterate of \( x^k = A_{B^k}^{-1} b \).

### 2.1 LU decomposition vs. pseudo inverse solution

Avoiding the calculation of the inverse is not only useful to reduce the operational cost, but also imperative to make sure that the algorithm works when \( A_B \) is singular. In our implementation, we use LU decomposition to achieve this. Let

\[
A_{B^k} = LU,
\]

where \( L \) is a full rank permuted lower triangular matrix and \( U \) is an upper triangular matrix with the same rank of \( A_{B^k} \). Therefore, if \( A_{B^k} \) is not full rank, \( U \) is not full rank, i.e., some diagonal elements of \( U \) are zeros. Given the LU decomposition (5), the reduced cost vector can be calculated as follows: let \( p = A_{B^k}^T c_{B^k} \), i.e., \( A_{B^k}^T p = c_{B^k} \), \( p \) is obtained by sequentially solving two linear systems of equations

\[
U^T p = c_{B^k}, \quad L^T \bar{p} = p,
\]

then the reduced cost vector is obtained by calculating

\[
\bar{c}_{N^k} = c_{N^k} - A_{N^k}^T \bar{p}.
\]

The iterate \( x^k = A_{B^k}^{-1} b \) can be obtained by sequentially solving two linear systems of equations as follows:

\[
Lq = b, \quad Ux^k = q.
\]

When \( A_{B^k} \) is not full rank, and some diagonal elements of \( U \) are zeros, one may think about using the pseudo inverse of \( A_{B^k} \). A simple and careful analysis shows that this would not work in some cases. The pseudo inverse is equivalent to the following process. Let \( \bar{U} \) be the matrix that removes rows with zero diagonal elements from \( U \) and \( \bar{q} \) be the vector that removes the corresponding elements in \( q \); then Equation (8) becomes \( \bar{U}x^k = \bar{q} \) and the solution is given by

\[
x^k = \begin{cases} 
\bar{U}^T (\bar{U}\bar{U}^T)^{-1} \bar{q} & \text{if } \bar{q} \neq 0, \\
x^k & \text{otherwise}.
\end{cases}
\]

\( \bar{U}^\perp \) can be obtained by using QR decomposition for \( \bar{U} \). It is easy to see that the least squared solution [2] of (9) may not meet the equations corresponding to the deleted rows of \( U \). Testing on some Netlib problems verified the analysis. Our strategy is to add a small \( \epsilon \) to the zero diagonal elements of \( U \) when it is singular. This will avoid difficulty in solving (6) and (8).
2.2 A double pivot algorithm

The double pivot algorithm in this paper is based on [26]. First, we briefly describe how the entering variables are selected. Let \( C^k \) be the index set defined as follows:

\[
C^k \in \{ j^k \mid \bar{c}_{j^k} < 0 \},
\]

and the cardinality \( |C^k| = p \). Clearly, if \( \bar{c}_{N^k} \geq 0 \), then \( C^k = \emptyset \), and an optimizer is found. If \( \bar{c}_{j^k} < 0 \) for some \( j^k \in C^k \), then an entering variable \( x_{j^k} \) in the next vertex is chosen from the set of \( \{ j^k \mid \bar{c}_{j^k} < 0 \} \) because by increasing \( x_{j^k} \), the objective function \( c^T x = c_B^T x_B^k + \bar{c}_{j^k} x_{j^k} \) will be reduced. The first entering variable \( x_{j_1^k} \) is selected by using Dantzig’s rule:

\[
J_1^k := \{ j^k \mid \bar{c}_{j^k} = \min_{j^k \in C^k} \bar{c}_{j^k} \}. \tag{11}
\]

If there is a tie, the minimum index of \( j_1^k \) will be used to break the tie. Denote \( \bar{b} = A_B^{-1} b \) and \( \bar{a}_{j} = A_B^{-1} A_{j^k} \), where \( j^k \) is in \( C^k \) and determined by (11). Also denote by \( \bar{b}_i \) the \( i \)th element of \( \bar{b} \) and by \( \bar{a}_{j^k,i} \) the \( i \)th element of \( \bar{a}_{j^k} \). Then, the leaving variable for Dantzig’s rule is to select \( x_{i_1^k} \) that satisfies

\[
x_{i_1^k} = \min_{i \in \{1,\ldots,m\}} \frac{\bar{b}_i}{\bar{a}_{j_1^k,i}}, \quad \text{subject to} \quad \bar{a}_{j_1^k,i} > 0. \tag{12}
\]

If there is a tie, the minimum index of \( i_1^k \) will be used to break the tie. The reduced cost value is given by \( \bar{c}_{i_1^k} x_{i_1^k} \).

If the cardinality \( |C^k| \geq 2 \), the second entering variable whose index is \( j_2^k \) will maximize the step-size, i.e.,

\[
x_{j_2^k} = \max_{\bar{c}_{j^k} < 0} \left\{ \min_{i \in \{1,\ldots,m\}} \frac{\bar{b}_i}{\bar{a}_{j^k,i}}, \quad \text{subject to} \quad \bar{a}_{j^k,i} > 0 \right\}. \tag{13}
\]

The leaving variable \( x_{i_2^k} \) is determined by the index that achieves the max-min value in (13). Given the entering and leaving variables determined by the longest step size rule, we can determine the next iterate \( x^{k+1} \) and the reduced cost \( \bar{c}_{i_2^k} x_{i_2^k} \).

We have discussed two special cases of the double-pivot algorithm. For the general case, given the two entering variables \( x_{j_1^k} \) and \( x_{j_2^k} \), we need to determine the two leaving variables which is equivalent to solving a linear programming problem with constraints in a two-dimensional space. Let \( \bar{A}_{(j_1,j_2)} \) be composed of the \( j_1 \) and \( j_2 \) columns of \( \bar{A}_N \), and \( j_1 \) and \( j_2 \) be determined by (11) and (13). Let \( \bar{A}_{(j_1,j_2)} = A_B^{-1} A_{(j_1,j_2)} \) and \( \bar{c}_{(j_1,j_2)} < 0 \) be the two corresponding elements in \( \bar{c}_N \). For the two entering indexes \( (j_1, j_2) \in C^k \) satisfying \( x_{(j_1,j_2)} = (x_{j_1}, x_{j_2}) \geq 0 \), we need \(^1\)

\[
x_{B^{k+1}} = A_B^{-1} b - A_B^{-1} A_{N^k} x_{N^k} = \bar{b} - \bar{A}_{(j_1,j_2)} x_{(j_1,j_2)} \geq 0. \tag{14}
\]

\(^1\)Observe that instead of using matrix inverse, \( \bar{b} \) and \( \bar{A}_{(j_1,j_2)} \) are obtained by using LU decomposition and then solving the linear systems of equations discussed in the previous section.
Therefore, the problem of finding a new vertex is reduced to minimize the following linear programming problem:

\[
\min \bar{c}^T_{(j_1,j_2)} x_{(j_1,j_2)},
\]

subject to \( \bar{A}^T_{(j_1,j_2)} x_{(j_1,j_2)} \leq \bar{b}, \ x_{(j_1,j_2)} \geq 0. \) (15)

Here, the third merit criterion is introduced, which is to determine the values of the two entering variables to minimize the objective function under the constraints of (15).

**Remark 2.1** Using multiple merit criteria to select entering variables has been proved to be a good idea and has been used in randomized pivoting algorithms [8, 13]. Since almost all popular deterministic simplex pivoting rules are proved not polynomial [7, 9, 12, 17], using a combined deterministic merit criteria to select entering variables may give us some fresh ideas to find a strongly polynomial algorithm (defined in [19]) to solve linear programming problems.

**Remark 2.2** Using the longest step size rule to select an entering parameter will increase the chance to have a non-degenerate basic feasible solution because the entering variable is more likely to be greater than zero than any other criteria. Since the double pivot rule will optimize the cost function, the step-sizes for the two pivot variables will not be all zeros if the longest step size is not zero. Therefore, we claim that the double pivot algorithm is degenerate-tolerable. This claim has been observed in numerical experiments to be discussed later.

Although problem (15) can be solved by using standard simplex algorithms, there is a much more efficient algorithm [23] for this special LP problem, which is discussed in the next section.

### 2.3 The slope algorithm to solve two-variable linear programs

The slope algorithm is a fast technique that runs in O(m log m) time and can find both an optimal solution and an optimal basis of two-variable linear programs. The slope algorithm is proposed in [21, 22, 23] and can quickly solve problem (15). Formally, define a two-variable linear program (2VLP) as:

\[
\max c^T x', \quad \text{subject to} \quad A' x' \leq b', \ x' \geq 0,
\]

where \( A' \in \mathbb{R}^{(m+2) \times 2}, \ b' \in \mathbb{R}^{m+2}, \ c' \in \mathbb{R}^2 \) are given, and \( x' \in \mathbb{R}^2 \) is the vector of decision variables. For a 2VLP, \( A'_{m+1,1} = A'_{m+2,2} = -1, \ A'_{m+1,2} = A'_{m+2,1} = 0, \) and \( b'_{m+1} = b'_{m+2} = 0. \) Observe that these two constraints are exactly the nonnegative conditions of (15). For simplicity, the slope algorithm assumes that \( c'_1 > 0, \ c'_2 > 0, \) and

---

2 Again, the vector \( \bar{c}^T_{(j_1,j_2)} \) is obtained by using LU decomposition and solving the linear systems of equations discussed in the previous section.
\( b'_i \geq 0 \) for each \( i \in \{1, 2, ..., m, m + 1, m + 2\} \). One can easily see that problem (15) satisfies these assumptions.

Overall, the slope algorithm contrasts the “slope” formed by the objective function coefficients of the 2VLP, \( c'_1 \) and \( c'_2 \), with the “slope” of every constraint in the problem. The slope of each constraint \( i \in \{1, 2, ..., m, m + 1, m + 2\} \) is defined by \( \alpha_i \) and can be computed as:

\[
\alpha_i = \begin{cases} 
-2M & \text{if } A'_{i,1} = 0 \text{ and } A'_{i,2} < 0 \\
-M + \frac{A'_{i,2}}{A'_{i,1}} & \text{if } A'_{i,1} > 0 \text{ and } A'_{i,2} < 0 \\
-M & \text{if } A'_{i,1} > 0 \text{ and } A'_{i,2} = 0 \\
\frac{A'_{i,2}}{A'_{i,1}} & \text{if } A'_{i,1} > 0 \text{ and } A'_{i,2} > 0 \\
M & \text{if } A'_{i,1} = 0 \text{ and } A'_{i,2} > 0 \\
M - \frac{A'_{i,1}}{A'_{i,2}} & \text{if } A'_{i,1} < 0 \text{ and } A'_{i,2} > 0 \\
2M & \text{if } A'_{i,1} < 0 \text{ and } A'_{i,2} = 0 \\
3M & \text{if } A'_{i,1} = 0 \text{ and } A'_{i,2} = 0 \\
3M & \text{if } A'_{i,1} < 0 \text{ and } A'_{i,2} < 0 
\end{cases} \quad (17)
\]

where

\[ M > \max \left\{ M', M'' , \frac{c'_2}{c'_1} \right\} \quad (18) \]

is a sufficiently large positive number and

\[ M' = \max \left\{ \frac{A'_{i,1}}{A'_{i,2}} : A'_{i,2} \neq 0 \forall i \in \{1, 2, ..., m, m + 1, m + 2\} \right\} \quad (19) \]

\[ M'' = \max \left\{ \frac{A'_{i,2}}{A'_{i,1}} : A'_{i,1} \neq 0 \forall i \in \{1, 2, ..., m, m + 1, m + 2\} \right\}. \quad (20) \]

Notice that \( M \) is used to determine some not well-defined slopes and differentiate the order of constraints. Figure 1 shows eight out of the nine types of constraints in a 2VLP (except constraints where \( A'_{i,1} = 0 \) and \( A'_{i,2} = 0 \) since those define the entire two-dimensional space) and their corresponding \( \alpha \) values. Furthermore, the reader may observe that viewing the constraints in ascending order of the \( \alpha \) values defines a counterclockwise orientation of the constraints.

Algorithm 2.1 depicts the slope algorithm step by step. The input to Algorithm 2.1 is a 2VLP and the method begins by computing a sufficiently large positive number \( M \), calculating the “slope” \( \alpha_i \) of each constraint, and sorting the constraints in ascending order according to \( \alpha_i \). From there, the slope algorithm finds two constraints, \( \eta_j' \) and \( \eta_k' \), which slope falls in between the slope of the cost coefficients \( c'_1 \) and \( c'_2 \) based on the sorted order. The method then checks whether the given 2VLP is unbounded or not.
This can be done by the checking the presence/absence of some specific constraints that define a ray of unboundedness. That is, 2VLP is unbounded if:

$$\alpha_{\eta_{j'}} = -2M \text{ and } \alpha_{\eta_{k'}} \geq M, \text{ or}$$

$$-2M < \alpha_{\eta_{j'}} < -M \text{ and } \alpha_{\eta_{k'}} = 2M, \text{ or}$$

$$\alpha_{\eta_{j'}} = -M \text{ and } \alpha_{\eta_{k'}} = 2M, \text{ or}$$

$$-2M < \alpha_{\eta_{j'}} < -M \text{ and } M < \alpha_{\eta_{k'}} < 2M, \text{ and } \frac{A'_{\eta_{j'},2}}{A'_{\eta_{j'},1}} \leq \frac{A'_{\eta_{k'},2}}{A'_{\eta_{k'},1}}. \tag{21}$$

If the given 2VLP is bounded, the slope algorithm continues and finds whether the intersection of constraints $\eta_{j'}$ and $\eta_{k'}$ is feasible on every other constraint. If not feasible, the algorithm replaces one of the two constraints with the constraint that violates the feasibility check, and repeat the process until a pair of feasible constraints is found. When this process is completed, the slope algorithm returns the optimal solution and also the two constraints that intersect at the optimal basis. That is, an optimal simplex basic feasible solution from where there does not exist a feasible improving search direction. This is possible because the slope algorithm finds a pair of constraints such that $\alpha_{\eta_{j'}} < \frac{c'_{j}}{x_{i}} \leq \alpha_{\eta_{k'}}$ and $\alpha_{\eta_{k'}} - \alpha_{\eta_{j'}}$ is minimized. The reader is encouraged to see [21, 22, 23] for additional theoretical results.
Algorithm 2.1

1: Data: Matrix $A'$, vectors $b'$ and $c'$
2: Compute a sufficiently large positive number $M$ according to (18).
3: Compute $\alpha_i$ for each constraint $i \in \{1, 2, ..., m, m + 1, m + 2\}$ according to (17).
4: Let $H = (\eta_1, \eta_2, ..., \eta_m, \eta_{m+1}, \eta_{m+2})$ be the list of constraint indices sorted in ascending order according to their $\alpha$ values.
5: Find constraints $j'$ and $k' \in \{1, 2, ..., m, m + 1, m + 2\}$ such that $\alpha_{\eta_{j'}} < \frac{c'_{j'}}{c_1} \leq \alpha_{\eta_{k'}}$.
6: if (21) is satisfied then
7: Report 2VLP is unbounded.
8: else
9: $j \leftarrow j'$.
10: $k \leftarrow k'$.
11: Find the intersection of constraints $\eta_{j'}$ and $\eta_{k'}$ and let its solution be $\bar{x} = (\bar{x}_1, \bar{x}_2)$.
12: while $j > 1$ or $k < m + 2$ do
13: if $j > 1$ then
14: $j \leftarrow j - 1$.
15: end if
16: if $A'_{\eta_1,1}\bar{x}_1 + A'_{\eta_1,2}\bar{x}_2 > b'_{\eta_1}$ then
17: $j' \leftarrow j$.
18: Find the intersection of constraints $\eta_{j'}$ and $\eta_{k'}$ and let its solution be $\bar{x} = (\bar{x}_1, \bar{x}_2)$.
19: end if
20: if $k < m + 2$ then
21: $k \leftarrow k + 1$.
22: end if
23: if $A'_{\eta_k,1}\bar{x}_1 + A'_{\eta_k,2}\bar{x}_2 > b'_{\eta_k}$ then
24: $k' \leftarrow k$.
25: Find the intersection of constraints $\eta_{j'}$ and $\eta_{k'}$ and let its solution be $\bar{x} = (\bar{x}_1, \bar{x}_2)$.
26: end if
27: end while
28: Report the optimal solution $x' = \bar{x}$ along with $c^T \bar{x}'$, $\eta_{j'}$, and $\eta_{k'}$.
29: end if

Observe that finding an optimal basis to 2VLPs is critical when trying to solve problem (15). This is because selecting two constraints that define an optimal solution but not an optimal basis (e.g. a 2VLP with an optimal degenerate solution) may result in an unchanged bases of the double pivot algorithm. This may result in unnecessary extra pivots and potentially, the algorithm may never terminate. The following section presents the proposed double pivot method step by step and shows how the slope algorithm can be used to solve its subproblems.
2.4 The complete double pivot algorithm

The complete double pivot algorithm is provided as follows:

Algorithm 2.2

1: Data: Matrix $A$, vectors $b$ and $c$.
2: Phase 1: To get initial basic feasible solution $x^0$, and its related partitions $x_{B^0}$, $x_{N^0}$, $A_{B^0}$, $A_{N^0}$, $c_{B^0}$, $c_{N^0}$.
3: Using (5), (6), and (7) to calculate the reduced cost $\bar{c}^T_{N^0} = c^T_{N^0} - c^T_{B^0}A_{B^0}^{-1}A_{N^0}$, and determine the reduced cost vector index set $C^k$ using (10).
4: while $\min(\bar{c}_{N^k}) < 0$ do
5: Use Dantzig’s rule to determine the entering variable $x_{i^k_1}$, use (12) to determine the leaving variable $x_{i^k_2}$, then calculate the candidate $x_{B^k+1}$ and the corresponding reduced cost $f_1$.
6: if the cardinality $|C^k| \geq 2$ then
7: Use the longest step size rule for the set $C^k \setminus \{i^k_1\}$ to determine the entering variable $x_{i^k_2}$, use (13) to determine the leaving variable $x_{i^k_2}$, then calculate the candidate $x_{B^k+1}$ and the corresponding reduced cost $f_2$.
8: For the two entering variable $x_{i^k_1}$ and $x_{i^k_2}$, solve the two-dimensional LP problem (15) using Algorithm 2.1 to determine the leaving variable $x_{i^k_2}$ and $x_{i^k_2}$, then calculate the candidate $x_{B^k+1}$ and the corresponding reduced cost $f_3$.
9: end if
10: if the cardinality $|C^k| = 1$ then
11: Dantzig’s pivot rule is used to update $B^k$ and $N^k$
12: LU decomposition is used for $A_{B^k}$, and $\bar{c}_{N^k}$
13: else if the cardinality $|C^k| \geq 2$ then
14: The double pivot rule is used to update $B^k$ and $N^k$
15: LU decomposition is used for $A_{B^k}$, and $\bar{c}_{N^k}$
16: end if
17: $k \leftarrow k + 1$.
18: end while

3 Implementation details

Some implementation details, which are important for improving the efficiency and robustness of the algorithm, are discussed in this section.

3.1 Pre-process

Pre-process or pre-solver is a major factor that can significantly affect the numerical stability and computational efficiency. Many literatures have been focused on this topic, for example, [1, 4, 16]. In this paper, we use the pre-process of [25] which has been
proved to be efficient and effective. Let \( A_{i,j} \) denote for the \( i \)th row of \( A \), \( A_{i,j} \) for the \( j \)th column of \( A \), and \( A_{i,j} \) for the element at \((i, j)\) position of \( A \). While reducing the problem, we express the objective function into two parts, \( c^T x = f_{\text{obj}} + \sum_k c_k x_k \). The first part \( f_{\text{obj}} \) at the beginning is zero and is updated all the time as we reduce the problem (remove some \( c_k \) from \( c \)); the terms in the summation in the second part are continuously reduced and \( c_k \) are updated as necessary when we reduce the problem.

To have a seamless implementation in the post-process, we store several vectors for every pre-process: \( c_{\text{orig}} = c \) for the original coefficients of the objective function; \( x_{\text{final}} \), which is set to zero at the beginning, will be used to store the optimal solution in the original unreduced coordinate; at the beginning of the pre-process, \( x_{\text{idx}} = (1, 2, \ldots, n) \), which will be reduced to keep a mapping between \( x \) in the reduced coordinate system and \( x_{\text{final}} \) in the unreduced coordinate system. For pre-process 5, we will store a sparse matrix and a few more vectors: \( A^\text{post} \), which is empty at the beginning of the pre-process, is used to store equations which are eliminated in the pre-process 5 but need to be resolved in the final stage to recover the variables in the original coordinate system; \( b^\text{post} \) is a vector associated with \( A^\text{post} \) to recover the variables in the original coordinate system; and \( x_{\text{tobe}} \) is a vector of coordinate information of the variables to be resolved in the final stage. These matrix and vectors need to be updated in the pre-process so that we can recover the solution expressed in the original coordinate system. We describe these updates only for the pre-processes that are chosen to be implemented.

1. **Empty row**
   If \( A_{i,i} = 0 \) and \( b_i = 0 \), this row can be removed. If \( A_{i,i} = 0 \) but \( b_i \neq 0 \), the problem is infeasible.

2. **Empty column**
   If \( A_{i,i} = 0 \) and \( c_i \geq 0 \), \( x_i = 0 \) is the right choice for the minimization. Remove the \( i \)th column \( A_{i,i} \) and \( c_i \). Also remove \( x_{\text{idx}}(i) \). If \( A_{i,i} = 0 \) but \( c_i < 0 \), the problem is unbounded as \( x_i \to \infty \).

3. **Row singleton**
   If \( A_{i,i} \) has exact one nonzero element, i.e., \( A_{i,k} \neq 0 \) for some \( k \), and for \( \forall j \neq k \) \( A_{i,j} = 0 \); then \( x_k = b_i/A_{i,k} \) and \( c^T x = c_k b_i/A_{i,k} + \sum_{j \neq k} c_j x_j \). For \( \ell \neq i \), \( A_{\ell,i} x = b_\ell \) can be rewritten as \( \sum_{j \neq k} A_{\ell,j} x_j = b_\ell - A_{\ell,k} b_i/A_{i,k} \). This suggests the following update: (i) if \( x_k < 0 \), the problem is infeasible, otherwise, continue, (ii) \( f_{\text{opt}} + c_k b_i/A_{i,k} \to f_{\text{opt}} \); (iii) remove \( c_k \) from \( c \), (iv) \( b_\ell - A_{\ell,k} b_i/A_{i,k} \to b_\ell \), (v) insert \( x_k = b_i/A_{i,k} \) into \( x_{\text{final}}(x_{\text{idx}}(k)) \), and (vi) remove \( x_{\text{idx}}(k) \). With these changes, we can remove the \( i \)th row and the \( k \)th column.

4. **Fixed variable defined by a single row**
   If \( b_i < 0 \) and \( A_{i,j} \geq 0 \) with at least one \( j \) such that \( A_{i,j} > 0 \), then, the problem is infeasible. Similarly, If \( b_i > 0 \) and \( A_{i,j} \leq 0 \) with at least one \( j \) such that \( A_{i,j} < 0 \), then, the problem is infeasible. If \( b_i = 0 \), but either \( \max(A_{i,j}) \leq 0 \) or \( \min(A_{i,j}) \geq 0 \), then for any \( j \) such that \( A_{i,j} \neq 0, x_j = 0 \) has to hold. Therefore, we can remove all such rows in \( A \) and \( b \), and such columns in \( A \) and \( c \). Also remove all corresponding elements in \( x_{\text{idx}} \).
5. Positive variable defined by signs of $A_{i, \alpha}$ and $b_i$

Since

$$x_i = \frac{1}{A_{\alpha,i}} \left( b_{\alpha} - \sum_{k \neq i} A_{\alpha,k} x_k \right),$$

if the sign of $A_{\alpha,i}$ is the same as $b_{\alpha}$ and opposite to all $A_{\alpha,k}$ for $k \neq i$, then $x_i \geq 0$ is guaranteed. We can solve $x_i$, and substitute back into $Ax = b$ and $c^T x$. This suggests taking the following actions: (i) if $A_{\beta,i} \neq 0$, $b_{\beta} - \frac{A_{\beta,i} b_{\alpha}}{A_{\alpha,i}} \rightarrow b_{\beta}$, (ii) moreover, if $A_{\alpha,k} \neq 0$, then $A_{\beta,k} - \frac{A_{\beta,i} A_{\alpha,k}}{A_{\alpha,i}} \rightarrow A_{\beta,k}$, (iii) $f_{\text{obj}} + \frac{c_{\alpha} b_{\alpha} A_{\alpha,i}}{A_{\alpha,i}} \rightarrow f_{\text{obj}}$, (iv) $c_k - \frac{c_{\alpha} A_{\alpha,k}}{A_{\alpha,i}} \rightarrow c_k$, (v) add a row to the end of $A_{\text{post}}$, such that $A_{\alpha,i} = 1$ and $A_{\alpha,k} A_{\alpha,i} \rightarrow A_{\alpha,k}$ if $A_{\alpha,k} \neq 0$, (vi) add $b(\alpha)/A_{\alpha,i}$ to the end of $b_{\text{post}}$ and add $x_{\text{idx}}(i)$ to the end of $b_{\text{tobe}}$, (vii) remove the $\alpha$th row and $i$th column from $A$ and remove $x_{\text{idx}}(i)$.

The MATLAB code of the pre-process is available from Netlib (http://www.netlib.org/numeralgo/) as part of the na43 package [25].

### 3.2 Post-process

With the implementation described in the pre-process, the post-process is very simple and is given as the following MATLAB code.

**Algorithm 3.1**

1: for $i = 1 : \text{length}(x_{\text{idx}})$ do
2: \hspace{1em} $x_{\text{final}}(x_{\text{idx}}(i)) = x(i)$;
3: end for
4: for $ii = \text{length}(x_{\text{tobe}}) : 1 : 1$ do
5: \hspace{1em} idxp = find($A_{\text{post}}(ii,:)$);
6: \hspace{1em} $x_{\text{final}}(x_{\text{tobe}}(ii)) = b_{\text{post}}(ii)$;
7: \hspace{1em} for $jj = 1 : \text{length}(idxp)$ do
8: \hspace{2em} if $\text{idxp}(jj) = x_{\text{tobe}}(ii)$ then
9: \hspace{3em} $x_{\text{final}}(x_{\text{tobe}}(ii)) = x_{\text{final}}(x_{\text{tobe}}(ii)) - A_{\text{post}}(ii, \text{idxp}(jj)) \ast x_{\text{final}}(\text{idxp}(jj))$;
10: \hspace{2em} end if
11: \hspace{1em} end for
12: end for

The MATLAB code of the post-process is available from Netlib (http://www.netlib.org/numeralgo/) as part of the na43 package [25].

### 3.3 Update $x_{B}^k$

Once we have the updated base and $A_{B^{k+1}}$, we can update the feasible basic solution as $x_{B^{k+1}} := A_{B^{k+1}}^{-1} b$. However, for many Netlib problems, we have singular or nearly
singular $A_{B_{k+1}}$. Thus, the computation of $x_{B_{k+1}}$ using this formula may result in some negative components due to numerical errors. A better implementation is as follows. From (3), we have

$$x_{B_{k+1}}^{k+1} = A_{B_{k}}^{-1}b - A_{B_{k}}^{-1}A_{N_{k}}x_{N_{k}} = \tilde{b} - \tilde{A}_{(j_1,j_2)}x_{(j_1,j_2)} \geq 0. \quad (22)$$

If there is only one negative element in $\tilde{c}_{N_{k}}$, then

$$x_{B_{k}}^{k+1} = A_{B_{k}}^{-1}b - A_{B_{k}}^{-1}A_{N_{k}}x_{N_{k}} = \tilde{b} - \tilde{A}_{j_1,j_1}x_{j_1} \geq 0. \quad (23)$$

It is worthwhile to note that $\tilde{b}$, $\tilde{A}_{j_1}$, and $\tilde{A}_{(j_1,j_2)}$ are calculated by using LU decomposition for efficiency and robustness and the details are discussed in Section 2.1. To obtain $x_{B_{k+1}}^{k+1}$ from $x_{B_{k}}^{k+1}$, we need to remove the component of the leaving variable from $x_{B_{k}}^{k+1}$ and insert the entering variable into $x_{B_{k}}^{k+1}$.

### 3.4 Degenerated solutions

Even though the double pivot rule solves all cycling problems of small size listed in Section 4.3, there are some Netlib benchmark problems which stays in degenerate solutions for a long time. This is a sign that the double pivot rule may still have difficult to handle all cycling problems. During the process of our code development, if a degenerate basic feasible solution moves to another degenerate basic feasible solution and the objective function is not improved after an iteration, then Bland’s rule is applied. Still, this implementation has difficult to solve the Netlib benchmark problem QAP8 due to numerical errors. Therefore, for zero component in $x_{B_{k}}$, we introduce a small positive perturbation to replace the zero components. This strategy clearly improves the stability of the code even when Bland’s rule is NOT applied.

### 4 Numerical test

Computational experiments were performed on an Intel® Xeon® E5-2670 2.60GHz 2 CPU/16 cores per node processor with 62.5GB of RAM per node. The version of MATLAB used was R2020a. The code that implements Algorithm 2.2 has been extensively tested for many problems in some extreme cases, randomly generated problems, and benchmark problems. This section summarizes the test results. These test results have demonstrated the computational merit of the double pivot algorithm.

#### 4.1 Test on the Klee-Minty cube problems

The Klee-Minty cube and its variants have been used to show a serious drawback of popular simplex algorithms, i.e., they need exponential number of iterations, in the worst case, to find an optimal solution. This section provides the test results of the double pivot algorithm for three variants of the Klee-Minty cube [10, 11, 14].
The first variant of the Klee-Minty cube is given in [10]:

\[
\begin{align*}
\text{min} & \quad -\sum_{i=1}^{m} 2^{m-i} x_i \\
\text{subject to} & \quad \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
2 & 1 & 0 & \ldots & 0 & 0 \\
2^3 & 2^2 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2^{m-1} & 2^{m-2} & 2^{m-3} & \ldots & 1 & 0 \\
2^m & 2^{m-1} & 2^{m-2} & \ldots & 2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{m-1} \\
x_m \\
\end{bmatrix} \
\leq 
\begin{bmatrix}
5 \\
25 \\
\vdots \\
5^{m-1} \\
5^m \\
\end{bmatrix} \\
x_i \geq 0 & \quad i = 1, \ldots, m.
\end{align*}
\] (24)

The optimizer is \([0, \ldots, 0, 5^m]\) with optimal objective function \(-5^m\).

The second variant of the Klee-Minty cube is given in [11]:

\[
\begin{align*}
\text{min} & \quad -\sum_{i=1}^{m} 10^{m-i} x_i \\
\text{subject to} & \quad 2 \sum_{j=1}^{i-1} 10^{i-j} x_j + x_i \leq 100^{i-1} & \quad i = 1, \ldots, m,
\end{align*}
\] (25)

The optimizer is \([0, \ldots, 0, 10^{2(m-1)}]\) with optimal objective function \(-10^{2(m-1)}\).

The third variant of the Klee-Minty cube is given in [14]:

\[
\begin{align*}
\text{min} & \quad -\sum_{i=1}^{m} x_i \\
\text{subject to} & \quad x_1 \leq 1, \\
& \quad 2 \sum_{i=1}^{k-1} x_i + x_k \leq 2^k - 1 & \quad k = 2, \ldots, m,
\end{align*}
\] (26)

The optimizer is \([0, \ldots, 0, 2^m - 1]\) with optimal objective function \(-(2^m - 1)\).

It is known that Dantzig’s pivoting rule needs \(2^m - 1\) iterations to find the optimizer for these problems. But the proposed double pivot algorithm finds the optimizer in just one iteration [26]. Therefore, the double pivot algorithm is able to solve these problems with size as large as \(m = 200\), which is impossible for Dantzig’s algorithm because it needs about \(2^{200} \geq 10^{60}\) iterations!

### 4.2 Test on randomly generated problems

We also tested and compared Algorithm 2.2\(^3\) and Dantzig’s pivot algorithm using randomly generated problems. Numerical test is carried out for randomly generated problems which are obtained as follows: first, given the problem size \(m\), a matrix \(M\) with uniformly distributed random entries between \([-0.5, 0.5]\) of dimension \(m \times m\) and an identity matrix of dimension \(m\) are generated. Therefore, \(A = [M \ I]\) is determined and the initial basic solution is composed of the last \(m\) columns. Then a positive vector \(b\)

\(^3\)Notice that Algorithm 2.2 is an improved version of the algorithm in [26] because the slope algorithm is used for the two-dimensional LP problem. Therefore, the numerical results here are better than the ones reported in [26].
with uniformly distributed random entries between \([10, 11]\) of dimension \(m\) and a vector \(c = (c_1, 0)\) with \(m\ c_1\)'s entries uniformly distributed between \([-0.5, 0.5]\) are generated. Two MATLAB codes that implement Dantzig’s pivot algorithm and Algorithm 2.2 are used to solve these randomly generated LP problems. Given the problem size \(m\), 100 random problems are generated and solved by using these two MATLAB codes. The average iteration number and average computational time in seconds are obtained. The test results are presented in Table 1.

It is clear that the double pivot algorithm always uses few iterations to find the optimizer than Dantzig’s algorithm does. However, for small size problems, Dantzig’s method uses less CPU time to find the optimizer than the double pivot algorithm does. As the problem size increases, the double pivot algorithm becomes more attractive. For the problem size of \(m = 1000\), the double pivot algorithm improves performance by reducing about 80% of the CPU time to find the optimizer. For the problem size of \(m = 10000\), the double pivot algorithm improves performance by reducing about 23% of the CPU time to find the optimizer. For this paper, percentage improvement is computed as

\[
\left(\frac{t_{\text{Dantzig}} - t_{\text{Double}}}{t_{\text{Dantzig}}}\right) \times 100\%
\]

where \(t_{\text{Dantzig}}\) and \(t_{\text{Double}}\) represents the number of pivots/CPU time required by Dantzig’s implementation and the double pivot algorithm, respectively.

| Problem Size \(m\) | Dantzig’s Implementation | Double Pivot | % Improvement |
|---------------------|--------------------------|--------------|--------------|
|                     | Avg. Iter | Avg. Time (s) | Avg. Iter | Avg. Time (s) | Iter | Time |
| 10                  | 7         | 0.00052       | 5         | 0.00159       | 34%  | -205% |
| 100                 | 165       | 0.08615       | 99        | 0.08227       | 40%  | 4%   |
| 1,000               | 17,334    | 4.57952E+03   | 3,399     | 9.10605E+02   | 80%  | 80%  |
| 10,000              | 160,741   | 6.97135E+04   | 103,967   | 5.38842E+04   | 35%  | 23%  |

Table 1: Comparison test for Dantzig’s pivot rule and double pivot rule for randomly generated problems.

### 4.3 Test on small size cycling problems

Notice that many Netlib problems have cycling issues. Before we tested the large size benchmark Netlib problems, we tested a collection of small size cycling problems listed in [28]. This strategy helped us to develop a robust MATLAB code that appropriately implements the proposed algorithm to make it capable to avoid the cycling issue. As expected in Remark 2.2, the code for Algorithm 2.2 performs well when degenerate solutions are encountered in these problems because the longest step size rule selects the entering variable with the largest value, which increases the chance to avoid the cycling problem. As a result, Algorithm 2.2 solves all problems in this set.
4.4 Test on Netlib problems

Our last test set is the Netlib benchmark library provided in [3]. Both Dantzig’s pivoting rule and the double pivot algorithm are tested for these problems. The size of these problems is normally large, and finding the pivot that has the longest step size becomes extremely expensive. To reduce the cost, we therefore select the pivot with longest step size among candidates that satisfies the condition $\bar{c}_i < 0.99 \times \min\{\bar{c}\}$. The test results are provided in Table 2. It is clear that Dantzig’s method uses less CPU time for almost all tested Netlib problems than the double pivot method even though the latter uses few iterations to find the optimal solution. However, the double pivot method still shows some merit on solving cycling problems. For problem DEGEN2, even though the code for Dantzig’s method implemented some safeguard tricks learned from the experience of the small size cycling problems, after more than one hundred million iterations, Dantzig’s pivot rule still cannot find the solution, which is a sign that cycling has happened in this problem, but the double pivot rule finds the solution in 2,350 iterations.
| Problem  | Dantzig’s Implementation | Double Pivot | % Improvement |
|----------|--------------------------|--------------|---------------|
|          | Obj                      | Iter         | Time (s)      | Infea    | Obj                      | Iter         | Time (s) | Infea    | Iter | Time |
| ADLITTLE | 2.25495E+05              | 154          | .09308        | 3.14578E-13 | 2.25495E+05              | 122          | .26763   | 2.89899E-13 | 21%  | -188% |
| AFIRO    | -4.64753E+02             | 10           | .04628        | 7.25485E-14 | -4.64753E+02             | 8            | .05018   | 3.82639E-14 | 20%  | -8%  |
| AGG      | -3.59918E+07             | 469          | .55888        | 1.44443E-10 | -3.59918E+07             | 272          | .84588   | 1.33442E-10 | 42%  | -51% |
| AGG2     | -2.02393E+07             | 560          | .76756        | 1.28699E-10 | -2.02393E+07             | 314          | .88015   | 1.14947E-10 | 44%  | -15% |
| AGG3     | 1.03121E+07              | 577          | .79314        | 8.89876E-11 | 1.03121E+07              | 319          | .91423   | 8.90887E-11 | 45%  | -15% |
| BANDM    | -3.01616E+02             | 581          | .77758        | 2.11801E-13 | -3.01616E+02             | 444          | 2.0567   | 4.01359E-13 | 24%  | -165%|
| BEACONFD | 3.20902E+04              | 79           | .84003        | 2.38277E-13 | 3.20902E+04              | 44           | .92386   | 1.39430E-13 | 44%  | -10% |
| BLEND    | -3.08121E+01             | 103          | .07257        | 1.77005E-14 | -3.08121E+01             | 97           | .17077   | 1.49333E-14 | 6%   | -135%|
| BNL1     | 1.97763E+03              | 3,577        | 9.21727       | 9.23917E-11 | 1.97763E+03              | 2,515        | 21.37082 | 9.15327E-11 | 30%  | -132%|
| BNL2     | 1.77526E+03              | 13,220       | 84.74392      | 2.29648E-10 | 1.77526E+03              | 9,084        | 194.82839| 2.58843E-10 | 31%  | -130%|
| BRANDY   | 1.51851E+03              | 496          | .28671        | 4.65831E-13 | 1.51851E+03              | 399          | .95905   | 7.20008E-13 | 20%  | -235%|
| DEGEN2   | -                        | -            | -             | -          | -                        | -            | -        | -         | -    | -    |
| DEGEN3   | -9.87287E+02             | 27,253       | 1236.393      | 1.71899E-09 | -9.87290E+02             | 24,897       | 1753.8486| 1.53891E-09 | 9%   | -42% |
| FFFFF800 | 5.55680E+05              | 1,464        | 2.02042       | 6.69504E-10 | 5.55680E+05              | 1,101        | 6.59932  | 9.90309E-10 | 25%  | -227%|
| ISRAEL   | -8.96645E+05             | 875          | .58372        | 2.52827E-10 | -8.96645E+05             | 503          | 2.23322  | 3.73152E-10 | 32%  | -283%|
| LOTFI    | -2.52647E+01             | 286          | .43982        | 7.76751E-10 | -2.52647E+01             | 239          | .66401   | 5.30878E-12 | 16%  | -51% |
| MAROS_R7 | 6.93586E+05              | 3,314        | 933.66064     | 3.16666E-10 | 6.93586E+05              | 1,991        | 821.8122 | 3.10953E-10 | 40%  | 12%  |
| OSA_07   | 5.37523E+05              | 6,378        | 303.95223     | 7.38573E-12 | 5.37523E+05              | 3,098        | 378.8818 | 5.69731E-12 | 51%  | -25% |
| OSA_14   | 1.10646E+06              | 13,111       | 1381.86663    | 3.22600E-12 | 1.10646E+06              | 6,787        | 2261.1212| 3.63473E-10 | 48%  | -64% |
| OSA_30   | 2.14214E+06              | 27,623       | 4805.0524     | 3.39458E-10 | 2.14214E+06              | 14,618       | 8331.4974| 3.63473E-10 | 47%  | -73% |
| QAP8     | 2.03500E+02              | 44,232       | 2661.87395    | 3.07977E-09 | 2.03500E+02              | 45,125       | 3860.2692| 3.47153E-09 | -2%  | -45% |
| SC50A    | -6.45751E+01             | 22           | .02811        | 3.57485E-14 | -6.45751E+01             | 24           | .04507   | 2.94039E-14 | -9%  | -60% |
| SC50B    | -7.00000E+01             | 26           | .02467        | 3.17764E-14 | -7.00000E+01             | 22           | .03733   | 8.32427E-14 | 15%  | -51% |
| SCI05    | -5.22021E+01             | 52           | .07303        | 1.03273E-13 | -5.22021E+01             | 53           | .12555   | 1.58714E-13 | -2%  | -68% |
| SC205    | -5.22021E+01             | 91           | .2418         | 2.36857E-13 | -5.22021E+01             | 144          | .49622   | 3.91708E-13 | -58% | -105% |
| SCAGR7   | -2.08339E+06             | 201          | .11251        | 2.83939E-12 | -2.08339E+06             | 164          | .33007   | 3.31881E-12 | 18%  | -193%|
| SCAGR25  | -1.45054E+07             | 1,043        | 2.39327       | 1.58450E-11 | -1.45054E+07             | 1,016        | 7.77863  | 1.55783E-11 | 3%   | -225%|
| SCFXM1   | 1.85315E+04              | 598          | 1.33481       | 1.04481E-09 | 1.85315E+04              | 462          | 1.96483  | 1.04478E-09 | 23%  | -47% |
|       |       |       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| SCFXM2 | 3.68898E+04 | 1,341 | 5.16887 | 3.65015E-12 | 3.68898E+04 | 958 | 7.57217 | 6.05719E-12 29%  -46% |
| SCFXM3 | 5.52456E+04 | 2,047 | 11.03826 | 5.74796E-12 | 5.52456E+04 | 1,457 | 16.92685 | 4.73435E-12 24%  -53% |
| SCRS8  | 9.04297E+02 | 616 | 3.07467 | 1.20544E-13 | 9.04297E+02 | 518 | 4.17995 | 2.89076E-14 16%  -36% |
| SCSD1  | 8.66667E+00 | 217 | .1574 | 1.59068E-11 | 8.66667E+00 | 218 | .37656 | 1.46231E-11 0%  -556% |
| SCSD6  | 5.05000E+01 | 528 | .2355 | 3.13451E-11 | 5.05000E+01 | 401 | 1.05024 | 1.74808E-11 24%  -346% |
| SCSD8  | 9.05000E+02 | 1,242 | 1.43172 | 1.67105E-11 | 9.05000E+02 | 965 | 6.6592 | 1.48737E-11 22%  -365% |
| SCTAP1 | 1.41225E+03 | 691 | .71354 | 3.09856E-10 | 1.41225E+03 | 534 | 2.06893 | 3.01522E-10 23%  -190% |
| SCTAP2 | 1.72481E+03 | 2,366 | 9.34237 | 5.88075E-14 | 1.72481E+03 | 2,228 | 33.37118 | 3.43825E-14 6%  -257% |
| SCTAP3 | 1.42400E+03 | 2,994 | 16.78474 | 6.60119E-10 | 1.42400E+03 | 2,537 | 46.74904 | 3.59637E-14 15%  -179% |
| SHARE1B | -7.01632E+04 | 513 | .2243 | 1.78766E-10 | -7.01632E+04 | 330 | .52793 | 1.96484E-10 36%  -135% |
| SHARE2B | -3.58732E+02 | 166 | .07706 | 5.41047E-11 | -3.58732E+02 | 118 | .16453 | 5.41023E-11 29%  -113% |
| SHIP04L | 1.79315E+06 | 555 | 2.5496 | 3.34155E-14 | 1.79315E+06 | 258 | 12.89924 | 1.96751E-14 54%  -406% |
| SHIP04S | 1.78245E+06 | 399 | 1.3644 | 1.37957E-14 | 1.78245E+06 | 190 | 5.48947 | 3.41211E-14 52%  -302% |
| SHIP08L | 1.90734E+06 | 970 | 6.59985 | 2.41521E-14 | 1.90734E+06 | 650 | 25.11337 | 2.47146E-14 33%  -281% |
| SHIP08S | 1.88314E+06 | 541 | 2.554 | 2.29440E-14 | 1.88314E+06 | 324 | 5.16255 | 2.29741E-14 40%  -102% |
| SHIP12L | 1.46555E+06 | 1,211 | 12.38967 | 3.89742E-14 | 1.46555E+06 | 791 | 28.34739 | 3.11866E-14 35%  -129% |
| SHIP12S | 1.46060E+06 | 600 | 4.28729 | 3.34777E-14 | 1.46060E+06 | 352 | 10.30063 | 4.01002E-14 41%  -140% |
| STOCFOR1 | -4.11320E+04 | 57 | .12437 | 1.60489E-12 | -4.11320E+04 | 39 | .14181 | 1.60489E-12 32%  -14% |
| STOCFOR2 | -3.90244E+04 | 1,733 | 39.92682 | 1.46442E-11 | -3.90244E+04 | 1,303 | 51.91037 | 1.56429E-11 25%  -30% |
| STOCFOR3 | -3.99768E+04 | 14,453 | 2571.60678 | 4.58717E-11 | -3.99768E+04 | 14,590 | 4985.93169 | 4.44197E-11 -1%  -94% |
| TRUSS  | 4.58816E+05 | 12,656 | 99.91333 | 4.58816E+05 | 7,315 | 181.77245 | 2.37071E-11 42%  -82% |

Table 2: Comparison test for Dantzig’s rule and double pivot rule for Netlib benchmark problems.
5 Conclusion

In this paper, we proposed a double pivot algorithm that combines the pivot rule of [26] and the slope algorithm of [21, 22, 23]. We implemented the algorithm as a MATLAB code. We tested the MATLAB code using a small set of cycling problems, three variant of the Klee-Minty problems, randomly generated LP problems, and Netlib benchmark test problems. We compared the performances of the double pivot code and a code that implements Dantzig’s pivoting algorithm. The test results show that the double pivot algorithm performs better than Dantzig’s pivot algorithm for large size randomly generated problems, while the latter performs better for small size randomly generated problems and Netlib benchmark test problems.

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