Graphical Representation of SUSY and Application to QFT

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Abstract. We present a graphical representation of the supersymmetry and the graphical calculation. Calculation is demonstrated for 4D Wess-Zumino model and for Super QED. The chiral operators are graphically expressed in an illuminating way. The tedious part of SUSY calculation, due to manipulating chiral suffixes, reduces considerably. The application is diverse.

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The supersymmetry is the symmetry between fermions and bosons. It was introduced in the mid 70’s. At present the experiment does not yet confirm the symmetry, but everybody accepts its importance in nature and expects fruitful results in the future development. The requirement of such a high symmetry costs a sophisticated structure which makes its dynamical analysis difficult. In this circumstance, we propose a calculational technique which utilizes the graphical representation of SUSY. The representation was proposed in [1, 2].

Weyl spinors have the SU(2)_L × SU(2)_R structure. The chiral suffix α, appearing in ψ^α or ψ_α, represents (fundamental representation, doublet representation) SU(2)_L and the anti-chiral suffix ˙α, appearing in ˆψ^α or ˆψ_α, represents SU(2)_R. The raising and lowering of suffixes are done by the antisymmetric tensors ε^{αβ} and ε_αβ.

\[(ε^{αβ}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ (ε_αβ) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \ ε^{αβ} ε_βγ = δ^α_γ, \]

\[ψ^α = ε^{αβ} ψ_β, \ ̂ψ_α = ε_αβ ̂ψ^β. \ (1)\]

They are graphically expressed by Fig.1. We encode them as follows. We use 2 dimensional array with the size 2×2. The four chiral spinors are stored in C-program as the array psi[][].

Fig. 1. Weyl fermions.
(σ^m)_{αβ} \quad (σ^m)_{α} \quad (σ^m)_{λ} \quad (σ^m)_{μ} \quad (σ^m)_{ν} \quad (σ^m)_{μν} \quad (σ^m)_{λμ} \quad (σ^m)_{λμν}

Fig. 2. Elements of SL(2,C) σ-matrices. (σ^m)_{αβ} and (σ^m)_{λ} are the standard form.

(2) Sigma Matrix [Symbol: s ; Dimension: M^0 ]
Sigma matrices σ^m, \bar{σ}^m are graphically expressed in Fig.2. They are stored as the 2×2 array si[𝑖][𝑗].

(3) Superspace coordinate [Symbol: t; Dimension: M^{−1/2} ]
The superspace coordinate θ^α is expressed in the same way as the spinor ψ^α.

θ^α \quad θ^α \quad \bar{θ}^α \quad \bar{θ}^α

(4) Gagino [Symbol: l; Dimension: M^{3/2} ]
The photino λ^α is expressed in the same way as the spinor ψ^α. We take the 2×2 array la[𝑖][𝑗].

λ^α \quad \bar{λ}^α

In the process of SUSY calculation, there appear graphs connected by directed lines (chiral suffixes contraction) and by (non-directed) dotted lines (vector suffixes contraction). We can classify them by some graph indices:

(1)vpairno The number of vector-suffix contractions; (2)Nc-pairO The number of chiral-suffix contractions. This is equal to the number of left-directed wedges; (3)NcpairE The number of anti-chiral-suffix contractions. This is equal to the number of the right-directed wedges; (4)closed-chiral-loop-No The closed-chiral-loop is the case that the directed lines, connected by σ or \bar{σ}, make a loop. In this case NcPairO=NcPairE. The number of closed chiral loops is defined to this index; (5)GrNum A group is defined to be a set of σ’s or \bar{σ}’s which are connected by directed lines. The number of groups is defined to be GrNum.

In TABLE 1-2, we list the classification of the product of σ’s using the graph indices defined above. These tables clearly show the σ-matrices play an important role to connect the chiral world and the space-time (Lorentz) world.

Supersymmetry is most manifestly expressed in the superspace (x^m, θ, \bar{θ}). θ^α = ε^αβθ^β, \bar{θ}^α = ε^{αβ}\bar{θ}_β are spinorial coordinates. They satisfy the relations graphically shown in Fig.4. These relations are exploited in the program in order to sort the SUSY quantities with respect to the power of θθ and \bar{θ}θ.

For the totally anti-symmetric tensor ε^{lmns}, we introduce one dimensional array ep[𝑖] with 4 components.

ep[0]=l \quad ep[1]=m \quad ep[2]=n \quad ep[3]=s 

Symbol: e
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This term appears later and produces topologically important terms such as $v_{lm} \tilde{v}_{lm} = \epsilon_{lmns} \epsilon_{lm} v_{ns}$. As for the metric of the chiral suffix, we do not introduce specific arrays. They play a role of raising or lowering suffixes, which can be encoded in the upper (0) and lower (1) code in arrays. For the Lorentz metric $\eta_{mn}$, we do not need to much care for the discrimination between the upper and lower suffixes because of the even-symmetry with respect to the change of the Lorentz suffixes ($\eta_{mn} = \eta_{nm}$).

The "reduction" formulae (from the cubic $\sigma'$s to the linear one) are expressed as in Fig.5. From Fig.5, we notice any chain of $\sigma'$s can always be expressed by less than three $\sigma'$s. The appearance of the 4th rank anti-symmetric tensor $\epsilon_{lmns}$ is quite illuminating. The following closed-chiral-loop graph reduces to an interesting quantity.

$$= 2(\eta^{lm} \eta^{ns} - \eta^{ln} \eta^{ms} + \eta^{ls} \eta^{mn} - i\epsilon_{lmns}) \quad (2)$$

The transformation between the superfield expression and the fields-components expression is an important subject of SUSY theories. For the purpose, we do the calculation of $\Phi^{\dagger} \Phi$. In this case, the input data is taken from the content of the superfield.

$$\Phi^{\dagger} = -i \sqrt{\sigma^{a}} \partial_{m} A^{a} + \sqrt{\sigma^{a}} \nabla^{a} + \frac{1}{4} \partial^{2} A^{a} + 2 \sqrt{\nabla^{a}} + i \sqrt{\sigma^{a}} \sqrt{\nabla^{a}} + F^{a} + A^{a} \quad (3)$$

where the directed dotted line is the superspace coordinate $\theta^{a}$. This data of $\Phi^{\dagger}$ is stored as
The calculation of $\Phi|\Phi$ leads to the Wess-Zumino Lagrangian.

$$\Phi|\Phi|_{\theta^a\bar{\theta}^b} = -\frac{1}{2} \partial_m A^a \partial^m A^a + \frac{1}{4} \partial^2 A^a \cdot A^a + \frac{1}{4} A^a \partial^2 A^a$$

$$-i \times (-1) \sqrt{\bar{\theta}^a \theta^b} + i \times (-1) \sqrt{\bar{\theta}^a \bar{\theta}^b} + F^a F^a(4)$$

We donot ignore the total divergence here.

We also do the calculation of $W_\alpha W^\alpha$ where $W_\alpha$ is the field strength superfield. They are expressed as follows.

$$W_\alpha = -i \sqrt{\bar{\theta}^a \theta^b} + \theta^a \bar{\theta}^b D$$

$$-i \sqrt{\bar{\theta}^a \theta^b} \sqrt{\bar{\theta}^c \theta^d} \frac{1}{2} \theta^m_{\alpha} + \sqrt{\bar{\theta}^a \theta^b} \bar{\theta}^c \theta^d \sqrt{\bar{\theta}^e \theta^f}$$

This data of $W_\alpha$ is stored as

weight[sf=0,t=0]=0+i(-1)

type[sf=0,t=0,c=0]= t

weight[sf=0,t=1]=1+i(0)

type[sf=0,t=1,c=0]= t

th[sf=0,t=0,c=0,0,0]=1

th[sf=0,t=1,c=0,0,0]=1

type[sf=0,t=0,c=0,1]= s

type[sf=0,t=1,c=1]= t

s[sf=0,t=0,c=0,1]=1

th[sf=0,t=1,c=1,0,1]=1

th[sf=0,t=0,c=1,1,1]=2

type[sf=0,t=1,c=2]= t

th[sf=0,t=0,c=1,2,1,1]=2

type[sf=0,t=1,c=3]= t

th[sf=0,t=0,c=2,1,0]=2

type[sf=0,t=1,c=3,1,0]=2

type[sf=0,t=0,c=3]= B

type[sf=0,t=1,c=4]= C

B[sf=0,t=0,c=3,1]=51

C[sf=0,t=1,c=4,1]=1

\ldots

The kinetic term of the photon and the photino, in the SuperQED, is given by

$$\mathcal{L} = \frac{1}{4} (-W_\alpha W^\alpha_{\bar{\theta}^2} + \bar{W}_\alpha \bar{W}^\alpha _{\bar{\theta}^2})$$

$$= \frac{1}{4} \theta^m_{\alpha} \left( \sqrt{\bar{\theta}^a \theta^b} \sqrt{\bar{\theta}^c \theta^d} - \sqrt{\bar{\theta}^a \theta^b \bar{\theta}^c \theta^d} \right) + \frac{1}{2} \bar{D}^2$$

where we do not ignore the total divergence.

In the history of the quantum field theory, new techniques have produced physically important results. The regularization techniques are such examples. The dimensional regularization by 't Hooft and Veltman\cite{5} produced important results on the renormalization group property of Yang-Mills theory and many scattering amplitude calculations. The lattice regularization in the gauge theory revealed non-perturbative features of hadron physics. In this case, the computor technique of numerical calculation is essential. As for the computer algebraic one, we recall the calculation of 2-loop on-shell counterterms of pure Einstein gravity\cite{6,7}. A new technique is equally important as a new idea.

The SUSY theory is beautifully constructed respecting the symmetry between bosons and fermions, but the attractiveness is practically much reduced by its complicated structure: many fields, chiral properties, Grassmannian algebra, etc. The present approach intends to improve the situation by a computer program which makes use of the graphical technique. (This approach is taken in Ref.\cite{8} for the calculation of product of SO(N) tensors. It was applied to various anomaly calculations.)

The present program should be much more improved. Here we cite the prospective final goal.

1. It can do the transformation between the superfield expression and the component expression.
2. It can do the SUSY trasformation of various quantities. In particular it can confirm the SUSY-invariance of the Lagrangian in the graphical way and give the final total divergence.
3. It can do algebraic SUSY calculation involving $D_\alpha, \bar{D}^\alpha, Q_\alpha$ and $\bar{Q}^\alpha$.

The item 1 above has been demonstrated in the present paper for the simple cases of Wess-Zumino model and the Super QED.

It is impossible to deal with all SUSY calculations. This is simply because which fields appear and which dimensional quantities are calculated depend on each problem. If we obtain a list of (graph) indices which classify all physical quantities (operators) appearing in the output, then the present program works (by adding new lines for the new problem).

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