Quantum Integrated Information Theory

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Integrated Information Theory (IIT) has emerged as one of the leading research lines in computational neuroscience to provide a mechanistic and mathematically well-defined description of the neural correlates of consciousness. Integrated Information (Φ) quantifies how much the integrated cause/effect structure of the global neural network fails to be accounted for by any partitioned version of it. The holistic IIT approach is in principle applicable to any information-processing dynamical network regardless of its interpretation in the context of consciousness. In this paper we take the first steps towards a formulation of a general and consistent version of IIT for interacting networks of quantum systems. A variety of different phases, from the dis-integrated (Φ = 0) to the holistic one (extensive log Φ), can be identified and their cross-overs studied.

\textbf{Introduction.} – Over the last decade Integrated Information Theory (IIT), developed by G. Tononi and collaborators, has emerged as one of the leading research lines in computational neuroscience. IIT aims at providing a mechanistic and mathematically well-defined description of the neural correlates of consciousness [1\textsuperscript{4}].

The idea is to quantify the amount of cause/effect power in the neural network that is holistic in the sense that goes beyond and above the sum of its parts. This is done in a bottom-up approach by quantifying how arbitrary parts of the network (“mechanisms”), in a given state, influence the future and constrain the past of other arbitrary parts (“purviews”), in a way that is irreducible to the separate (and independent) actions of parts of the mechanism over parts of the purview.

Iterated at the global network level this process gives rise to a so-called conceptual structure, comprising a family of mechanisms and purviews, where the latter represent the integrated core causes/effects of the former [2]. A suitable measure of the distance between this conceptual structure with the closest one obtainable from a partitioned network (where the connections between the two subsystems are “noised”) quantifies how much of the cause/effect structure of dynamical newtwork fails to be reducible to the sum of its parts. This minimal distance is the Integrated Information (Φ) of the network.

In IIT it is then boldly postulated that the larger Φ, the higher is the degree of consciousness of the network in the given state. The irreducibility of the causal information-processing structure of the network measured by Φ is independent of the specific “wetware” implementing the brain circuitry. It follows that the IIT approach to consciousness seems to unavoidably lead to panpsychist, and a rather controversial \textsuperscript{[5]} view of the world \textsuperscript{[4]}.

Importantly, besides (and irrespective of) the applications to consciousness, the IIT approach is in principle applicable to any information-processing dynamical network. For example, applications of IIT to Elementary Cellular Automata and Adapting Animats have been discussed \textsuperscript{[6]}. Furthermore, potential extensions of IIT to more general systems, including quantum ones, have been investigated in [7\textsuperscript{8}] by M. Tegmark (see also [9]).

In this paper we shall make an attempt at formulating a general and consistent version of IIT for interacting networks of finite-dimensional quantum systems (qudits). Naturally, classical probability distributions are being replaced by (non-commutative) density matrices and probability transition matrices by completely positive (CP) maps [10]. We would like to emphasize that 1) our goal is not to account for potential quantum features of consciousness 2) the quantum extension that we are going to discuss differs in some essential ways from its classical counterpart and it is not claimed to be unique [11].

\textbf{Quantum IIT.} – Let Λ be a set of cardinality |Λ| < ∞. For each \(j \in \Lambda\) there is an associated \(d\)-dimensional quantum system with Hilbert space \(H_j \cong \mathbb{C}^d\). Given any \(\Omega \subset \Lambda\) we define \(\mathcal{H}_\Omega = \bigotimes_{j \in \Omega} H_j\), with dimension \(d^{|\Omega|}\). We will denote by \(\mathcal{L}(\mathcal{H}_\Lambda)\) (\(\mathcal{S}(\mathcal{H}_\Lambda)\)) the associated operator-algebra (state-space). One has that \(\mathcal{H}_\Lambda \cong \mathcal{H}_\Omega \otimes \mathcal{H}_{\Omega'}\), where \(\Omega'\) denotes the complement of \(\Omega\) (in \(\Lambda\)). Let the dynamics be described by a \textit{unital} CP-map \(\mathcal{U}: \mathcal{L}(\mathcal{H}_\Lambda) \rightarrow \mathcal{L}(\mathcal{H}_\Lambda)\) with \(\mathcal{U}(1) = 1\). This map has to be thought of as the one-step evolution of a discrete time process. Adopting the IIT jargon we will refer to subsets \(M \subset \Lambda\) as to \textit{mechanisms}. Given \(\Omega \subset \Lambda\) we define the \textit{noising} CP-map \(\mathcal{N}_\Omega\) by \(\mathcal{N}_\Omega: \mathcal{L}(\mathcal{H}_\Lambda) \rightarrow \mathcal{L}(\mathcal{H}_\Lambda): X \mapsto (\mathcal{Tr}_\Omega X) \otimes \frac{\mathbb{I}_{\Omega'}}{d^{|\Omega'|}}\). The first step of is to define a quantum version of the cause/effect repertoires of classical IIT [3\textsuperscript{4}].

\textbf{Definition 1a: cause/effect Repertoires:} Given the unital \(\mathcal{U}\), the state \(\Psi_\Lambda \in \mathcal{L}(\mathcal{H}_\Lambda), (\Psi_\Lambda \geq 0, \text{Tr} \Psi_\Lambda = 1)\) and the mechanisms \(M, P \subset \Lambda\), we define the effect (e) and cause (c) repertoire of \(M\) over the purview \(P\), by [12]

\[
\rho^{(e)}(P|M) := \text{Tr}_P \mathcal{U}^{(x)} \circ \mathcal{N}_M^*(\Psi_\Lambda), \quad (x = e, c) \tag{1}
\]

\(\mathcal{U}^{(c)} = \mathcal{U}\) and \(\mathcal{U}^{(e)} = \mathcal{U}^*\) (Hilbert-Schmidt dual of \(\mathcal{U}\)). The set of density matrices \(\rho^{(c)}(P|M)\) \((\rho^{(e)}(P|M))\) encode how the dynamics constrain the future (past) of \(P\), given that the system is initialized in \(\Psi_\Lambda\) and noised over \(M'\). Note, \(\rho^{(e/c)}(0|M) = 1 (\forall M)\), by trace normalization.
The second step is to define the cause/effect information by the information-theoretic distance between the conditioned and the un-conditioned repertoire \( \rho(x)(P|\emptyset) \). In classical IIT the distance between repertoires (and related objects) is usually taken to be the Wasserstein distance \([3]\). In this paper, in view of its salient quantum-information theoretic properties and simplicity, we will adopt the trace distance between density matrices \( \rho \) and \( \sigma \) as a natural measure of statistical distinguishability, i.e., \( D(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1 \in [0,1] \).

Definition 1b: cause/effect Information: The cause/effect information of \( M \) over \( P \) is given by

\[
x_i(\rho^e(\emptyset)|\rho^c(\emptyset)) = \frac{1}{dP_M} D[\rho^{(x)}(P|M), \frac{1}{dP_M}], \quad (x = e, c).
\]

Since \( \mathcal{U} \) and \( \mathcal{U}^* \) are unital \( \rho^{(e/c)}(\emptyset) = \frac{1}{dP_M} \); these are the unconditioned repertoires.

Remarks: 1) Since pure states have the maximum distance from the maximally mixed state and by distance monotonicity under partial traces \( xi(P|M) \leq \min \{1 - d^{-|P|}, 1 - d^{-|M|}\} \).

2) From (2) and distance monotonicity under partial traces and unitary invariance follows \( Q \subset P \Rightarrow xi(Q|M) \leq xi(P|M) \).

3) For unitary \( \mathcal{U} \) generated by a local Hamiltonian \( xi(P|M) \), \( (x = e, c) \) fulfill a Lieb-Robinson type inequality \([13]\), see \([12]\).

Definition 2: Integrated information for mechanisms.- Given the mechanism \( M \) and the purview \( P \) we consider all possible bi-partitions of them \( \{M_1, M_2\} \) and \( \{P_1, P_2\} \), where \( X_1 \times X_2 = \emptyset, X_1 \times X_2 = X = (M, P) \). We define the (cause/effected) integrated information (ii) of \( M \) over \( P \) by \( \varphi(x)(P|M) = \min_{\rho^{(x)}(P_1|M_1)} D[\rho^{(x)}(P|M), \rho^{(e)}(P_1|M_1) \otimes \rho^{(x)}(P_2|M_2)] \in [0,1], (x = e, c) \).

In this definition the minimum is taken over all the \( 2^{|P_1|+|M_1|} - 1 \) possible pairings \( (P_i, M_i) \) \((i = 1, 2) \) different from the trivial one \( (\emptyset, \emptyset) \), \( (P, M) \), which would make any repertoire factorizable. Notice that, since the \( \rho^{(x)}(P_1|M_1) \)'s \((i = 1, 2) \) are not the reduced density matrices of \( \rho^{(x)}(P|M) \), the factorizability of the latter is a necessary, but not sufficient condition for the vanishing of \( \varphi(x)(P|M) \). If \( (P_1, P_2) \) is the partition of \( P \), which achieves the minimum, then the entanglement of \( \rho^{(x)}(P|M) \), measured by its distance from the set of separable states over \( \mathcal{H}_{P_1} \otimes \mathcal{H}_{P_2} \), provides a lower-bound to \( \varphi(x)(P|M) \). Moreover, cause/effect information gives an upper bound to the integrated information \([14]\)

\[
\varphi^+(P|M) \leq xi(P|M), \quad (x = e, c).
\]

In particular, Eq. (3) and remark 3) above imply that integrated information obeys a Lieb-Robinson type bound for \( \mathcal{U} \)'s generated by local-Hamiltonians. This shows that \( \varphi \) obeys locality in the usual sense allowed in non-relativistic quantum theory \([13]\).

Definition 3: Core causes, effects.- The purview \( P^e/c_1 \) is a core effect/cause of \( M \), if \( P^e/c_1 = \arg \max_{\rho} \varphi^{(e/c)}(P|M) \). The corresponding value of \( \varphi \) will be denoted by \( \varphi(x)(M) := \max_{\rho} \varphi^{(x)}(P|M) = \varphi(P^e(x)|M), (x = e, c) \). The associated (global) repertoires are given by \( \rho^{(x)}(M) := \rho^{(x)}(P^e(x)|M) \otimes \frac{1}{dQ^{(x)}} \).

where \( Q^{(x)} := (P^e(x)) \) is the complement of the core effect/cause of \( M \) \([15]\). The integrated cause/effect information of \( M \) is given by \( \varphi(M) = \min \{\varphi^e(M), \varphi^c(M)\} \).

If \( \varphi(M) = 0 \), then either \( \varphi^{(e)}(P|M) = 0, \forall P \) or \( \varphi^{(c)}(P|M) = 0, \forall P \). In the first (second) case, the mechanism \( M \) fails to constrain the future (past) on any purview \( P \) in an integrated fashion. Either way, such a mechanism is not regarded as an integrated part of the network and it is dropped out of the picture.

Definition 4: Conceptual Structure operators.- For any mechanism \( M \subset \Lambda \) the triple \( (\rho^e(M), \rho^c(M), \varphi(M)) \) with \( \varphi(M) > 0 \) is called a concept. The totality of concepts forms a conceptual structure (CS) \([3]\). Formally one can encode a CS on a positive semi-definite operator over \( (C^2)^{\otimes |\Lambda|} \mathbb{C}^2 \otimes \mathcal{H}_A \), given by

\[
C(U) := \frac{1}{2} \sum_{M, \alpha} \varphi(U(M)|\alpha M) \otimes \rho^\alpha(M),
\]

where \( M \subset \Lambda, \alpha = e, c \), and we have made explicit the \( \mathcal{U} \)-dependence (but kept implicit the \( \Psi \) one).

Given two CS's, \( C_1 \) and \( C_2 \) associated to \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \), respectively, we define the distance between them as the (trace-norm) distance between the associated CS operators \( D(C_1, C_2) = \frac{1}{2}||C_1 - C_2||_1 \). More explicitly,

\[
D(C_1, C_2) = \frac{1}{4} \sum_{M, \alpha} ||\varphi_1(M)\rho^\alpha_1(M) - \varphi_2(M)\rho^\alpha_2(M)||_1.
\]

In particular, \( D(C(U_1), C(U_2)) = 0 \) iff \( \forall M \subset \Lambda \) one has either \( \varphi_{U_1}(M) = \varphi_{U_2}(M) \neq 0 \) and \( \rho^\alpha_{U_1}(M) = \rho^\alpha_{U_2}(M), (\alpha = e, c) \), or \( \varphi_{U_1}(M) = \varphi_{U_2}(M) = 0 \). In words: two conceptual structures are the same iff all the core effects/causes repertoires and the associated integrated-information coincide for all concepts \([15]\).

The key idea in IIT is to quantitatively compare the global cause/effect structure, encoded in \( \mathcal{U} \), with the one associated to the factorized maps. In this way one wants to assess how the “whole goes beyond and above the sum of its parts”. The standard way in classical IIT to produce factorized maps is by bi-partitioning the total set \( \Lambda \) and by “cutting the connections between the two halves by injecting them with noise” \([8]\). We adopt here a natural quantum version of this procedure.

Formally, given the (non-trivial) partition \( \mathcal{P} = \{A_1, A_2 = A_1^c\} \), one can define \( U_P = U_1 \otimes U_2 \), where \( U_i \) acts on \( L(H_{A_i}) \) by \( U_i: X \mapsto U_i(X) := \text{Tr}_A(U(X \otimes \frac{1}{dP}(x))) \), \((i = 1, 2) \). Now we define the fundamental global quantity of the paper: the Integrated Information of the whole network. Qualitatively, \( \Phi \) measures how the integrated cause/effect structure of the quantum network fails to be described by any factorized version of it.
Definition 5: Integrated Information.— We define the Integrated Information (II) by
\[ \Phi(\mathcal{U}) := \min \frac{D(C(\mathcal{U}), C(\mathcal{U}_P))}{D(C(\mathcal{U}), \mathcal{C})}, \]  

The minimum here is taken over the set of \(2^{\lvert \mathcal{A} \rvert} - 1\) bi-partitions of \(\mathcal{A}\) [17]. If \(\Phi(\mathcal{U}) = 0\), we say that the network is dis-integrated.

It is important to stress that, in spite of the simplified notation, II depends crucially on \(\Psi_\mathcal{A}\) as well. In this paper we focus on completely factorized pure states \(\Psi_\mathcal{A} = \otimes_{i \in \mathcal{A}} |\psi_i\rangle \langle \psi_i|\). Different state choices, e.g., entangled, may result in dramatically different results [18], and will be considered elsewhere [19]. Notice that \(\Phi\) obeys a sort of time-reversal symmetry \(\Phi(\mathcal{U}^*) = \Phi(\mathcal{U})\) [20].

The bi-partition \(P_{MIP}\), for which the minimum in Eq. (6) occurs, is referred to as the Maximal Irreducible Partition (MIP) in classical IIT, i.e., \(\Phi(\mathcal{U}) = D(C(\mathcal{U}), C(\mathcal{U}_{P_{MIP}}))\). Of course factorizability of \(\mathcal{U}\) is a sufficient (but not necessary [21]) condition for vanishing II. More in general, if the network is dis-integrated \(C(\mathcal{U}) = C(\mathcal{U}_{P_{MIP}})\), namely there exists a “cut and noising” of the network in two halves that does not affect its global (integrated) cause/effect structure. The system does not exist as a whole per se, but fragments into causally decoupled parts.

Permutational networks.— In order to illustrate the above definitions and ideas we will now discuss three specific cases of Permutational networks, in which \(\Psi_\mathcal{A} = \otimes_{i \in \mathcal{A}} |\psi_i\rangle \langle \psi_i|\) and \(\mathcal{U}(X) = U_\sigma X U_\sigma^d\), where \(U_\sigma\) acts as the permutation \(\sigma \in \mathcal{S}_\mathcal{A}\) over \(\mathcal{H}_\mathcal{A} \cong (\mathbb{C}^d)^{\otimes \lvert \mathcal{A} \rvert}\). One can see that
\[ \rho^{(c)}(P|\mathcal{M}) = \Psi_{\sigma^{-1}(P)\sigma(\mathcal{M})} \otimes \rho^{(s)}(\mathcal{M}^c) \otimes \rho^{(c)}(\mathcal{M}^c) = \rho^{(c)}(\sigma(M) \cap P|\mathcal{M}) \otimes \rho^{(s)}(\sigma(M^c) \cap P|\mathcal{M}^c) \otimes \rho^{(c)}(\sigma(M^c) \cap P|\mathcal{M}^c) \],

The same equations hold, with \(\sigma^{-1}\) replacing \(\sigma\), for the cause repertoires. From this totally factorized form one sees that the only irreducible (MIP) in classical IIT, i.e., \(\Phi(\mathcal{U}) = 0\), is \(\mathcal{U}(X) = U_\sigma X U_\sigma^d\), where \(U_\sigma\) acts as the permutation \(\sigma \in \mathcal{S}_\mathcal{A}\) over \(\mathcal{H}_\mathcal{A} \cong (\mathbb{C}^d)^{\otimes \lvert \mathcal{A} \rvert}\). One can then obtain \(\varphi\) for each mechanism \((i = 1, 2, 3)\), for the core effect of \(|\mathcal{A}\rangle\) itself (\(\varphi(2)\)) using the core effect of \(|\mathcal{A}\rangle\) itself (\(\varphi(2)\)). The existence of the intermediate, high \(\Phi\), delocalized phase originates from the commutator term in \(\rho^{(c)}(\mathcal{A}|\mathcal{M})\). It can be regarded as a genuine quantum feature, i.e., it would disappear if the dynamics were a just probabilistic mixture of identity and swap.

Low-integration and Holistic Phases.— As customary in statistical mechanics one can consider families of increasingly large networks \((\mathcal{U}_\mathcal{A}, 3, \mathbb{F}_\Lambda)\) and study how \(\Phi\)

![FIG. 1. (Color online) Two qubit network. Solid blue curve: \(U = \exp(itS)\). For small \(t\) (near \(\pi/2\)) the network is in the “identity” (“swap”) phase. The discontinuities at \(t = \cos^{-1}(\sqrt{2}/3)\) and \(t = \cos^{-1}(1/\sqrt{3})\) are due to a jump and delocalization of the core cause/effect repertoires. The dashed red curve shows the average of \(\Phi\) where \(U = \exp(itH_{GUE})\) and \(H_{GUE}\) is sampled from the Gaussian unitary ensemble (GUE) with unit variance.](image)
behaves in the “thermodynamical limit” (TDL) $|\Lambda| \to \infty$ [22]. If $O(\mathcal{U}) = \lim_{|\Lambda| \to \infty} \frac{\log_2 \Phi(\mathcal{U}|\Lambda)}{|\Lambda|} > 0$, we say that the network is the holistic phase in the TDL. The quantity $O(\mathcal{U})$ can be referred to as the holistic parameter and it is at most one [23]. In the holistic phase the system shows the maximal level of integration and intrinsic existence as an irreducible causal whole.

To study the different integration phases it is useful to consider the following upper bound to $\Phi$ [21]

$$\Phi(\mathcal{U}) \leq \sum_{M \in \Lambda} \varphi_U(M) \leq N_{c}(\mathcal{U}),$$

where $N_{c}(\mathcal{U}) := \#\{M \subset \Lambda / \varphi_U(M) > 0\}$ is the total number of concepts in $C(\mathcal{U})$. If $N_{c}(\mathcal{U}) = O(|\Lambda|^\kappa)$ with $\kappa > 0$ then, from [5], it follows that the holistic parameter is vanishing: $O(\mathcal{U}) = \lim_{|\Lambda| \to \infty} |\Lambda|^{-1} \log_2 \Phi(\mathcal{U}) \leq \lim_{|\Lambda| \to \infty} \frac{\log_2 |\Lambda|^\kappa}{|\Lambda|} = 0$. On the other hand, in order to be in the holistic phase, the network needs to have a number of concepts asymptotically lower bounded by $2^{|\Lambda|}(1 \geq \alpha > 0)$. Whence, if the concepts are supported only on mechanisms $M$, such that $|M| = \kappa = O(1)$, then the network is necessarily in the non-holistic (low II) phase, e.g., permutational networks.

Now we discuss a sufficient condition for a network to be in the holistic phase and provide a physical example. First, Eqs. [5] and [6] imply the bound:

$$\Phi(\mathcal{U}) \geq \frac{1}{2} \sum_{M \in \partial \mathcal{P}_{MIP}} \varphi_U(M) \geq |\partial \mathcal{P}_{MIP}|^2 \varphi_0,$$

where $\varphi_0 := \min_{M \in \partial \mathcal{P}_{MIP}} \varphi_U(M)$ [23]. Notice that $|\partial \mathcal{P}_{MIP}| \geq 2^{|\Lambda|-1} - 1$, therefore, in view of the lower bound above, one is guaranteed to be in the holistic phase if $\varphi_0$ is lower-bounded by a non-zero constant. This situation can be realized by a $|\Lambda|$-local interaction. Let us consider a qubit network with $U(\mathcal{X}) = e^{iH^Z X - \omega Z}$, $Z := \sum_{t \in \Lambda} \sigma_t^z$. In [12] it is shown that $\varphi_{U}(M|M) \geq 2 \sin^2(2t), \forall M \in \partial \mathcal{P}_{MIP})$. Since, by definition, $\varphi_{U}(M) \geq \varphi_{U}(M|M)$, by setting, e.g., $t = \pi/2$, one finds $\varphi_0 \geq \frac{1}{2}$. This argument also shows that the holistic parameter $O(\mathcal{U})$ is one for all $t \neq 0, \pi/2$, where it is ill-defined as $\Phi = 0$. Turning on the global interaction $Z$ (or mixing it with the identity) results in a direct transition from the dis-integrated phase to the holistic one. In general, one may speculate that $k$-local ($k = O(1)$) interactions will give rise to low-integration networks with sub-extensive $\log_2 \Phi$ [19]. Preliminary numerical results are shown in Fig. 2.

**Conclusions.**—The main goal of classical Integrated Information Theory (IIT) [1][4] is to provide a mathematical and conceptual framework to study the neural correlates of consciousness. In this paper we took the first steps towards a possible quantum version of IIT irrespective of its applications to consciousness.

Our approach is a quantum information-theoretic one, where neural networks are being replaced by networks of qubits, probability distributions by non-commutative density matrices, and markov processes by completely positive maps. The irreducible cause/effect structure of the global network is encoded by a so-called conceptual structure operator. The minimal distance of the latter from those obtained by factorized versions of the network, defines the quantum Integrated Information $\Phi$.

We have studied quantum effects in small qubit networks and provided examples, analytical and numerical, of families of low integration networks. Also, we have demonstrated sufficient conditions for the existence of highly integrated ones and given illustrations.

The scaling of $\Phi$ with the network size defines different phases distinguished by a different level of integration of their global cause/effect structure. The study of those phases and cross-overs, their relation to locality and entanglement, and in general the question whether the quantum IIT discussed in this paper has any direct bearing on standard quantum information processing, are challenging tasks for future investigations.

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[1] G. Tononi, BMC Neuroscience 5, 42 (2004).
[2] M. Oizumi, L. Albantakis, and G. Tononi, PLOS Computational Biology 10, 5 (2014).
If the maps From Eq. (1) one sees that causes and effects are exchanged by replacing $U$ with its dual $U^\ast$, moreover $\varphi(M)$ is symmetric under this exchange $(\forall M)$. Therefore, $C(U^\ast) = X C(U^\ast) X^\dagger$, where $X := 1 \otimes \sigma^x \otimes 1$. Since the distance $D$ in Eq. (3) is unitarily-invariant one gets $\Phi(U^\ast) = \Phi(U)$.

For a given $\Psi_\Lambda$, vanishing $\Phi$ is a weaker property than factorizability of the dynamical map. Take, e.g., any non-factorizable unitary $U$ that is diagonal in a tensor product basis and $\Psi_\Lambda$ to be a basis element. One has that $U \Psi_\Lambda \otimes \frac{1}{d(M)} = \Psi_\Lambda \otimes \frac{1}{d(M)}$, $(\forall M \subset \Lambda)$. The action of $U$, for this $\Psi_\Lambda$, is the same of the identity map and therefore $\Phi(U) = 0$. A different $\Psi_\Lambda$, would generally lead to $\Phi \neq 0$.

If the maps $U_\Lambda$ are associated with unitaries $U_\Lambda = e^{-i t_\Lambda H_\Lambda}$ one has to choose how to scale with the system size $|\Lambda|$ both the times $t_\Lambda$ as well as the Hamiltonians $H_\Lambda$. For the former three natural options are given by: a) $t = O(1)$; b) “constant action” $t_\Lambda \|H_\Lambda\| = O(1)$; c) $t_\Lambda := \text{argmax}_t \Phi(U(t))$, where the maximum overtime is taken at fixed $|\Lambda|$ (see [12]).

From Def. 3, Eq. (3) and $\varpi(P|M) \leq \min\{1 - d^{-|P|}, 1 - d^{-|M|}\}$ it follows that $\varphi(M) \leq 1 - d^{-|M|}$. Now $D(C_1,C_2) \leq \frac{1}{2} \sum_{M \subset \Lambda} (\varphi_1(M) + \varphi_2(M)) \leq \sum_{|M| = 0}^{|\Lambda|} \binom{|\Lambda|}{|M|} (1 - d^{-|M|})$.

From [6] one has $\Phi(U) \leq \left(2^{|\Lambda|} - (1 + d^{-1})^{|\Lambda|}\right) \Rightarrow O(U) \leq \frac{1}{\sqrt{d}} \lim_{|\Lambda| \to \infty} \log_2 \left[2^{|\Lambda|} (1 - (\frac{1}{d-1})^{|\Lambda|})\right] = 1$.

By definition $\Phi(U) \leq D[C(U),C(U_{FP})]$, for all partitions $P$. From Eq. (3) one has $D[C(U),C(U_{FP})] = \frac{1}{2} \|C(U) - C(U_{FP})\|_1 \leq \frac{1}{2} (\|C(U)\|_1 + \|C(U_{FP})\|_1) \leq \frac{1}{2} \sum_{M \subset \Lambda} (\varphi_U(M) + \varphi_{U_{FP}}(M)) \leq \frac{1}{2} (N_U(U) + N_{U_{FP}}(U)) \leq N_U(U)$. Here we have used the fact that the number of concepts $N(U_{FP})$ of the partitioned network is always not larger than the one $N_U(U)$ of the un-partitioned one $\forall P$. In the same way one proves $\Phi(U) \leq \sum_{M \subset \Lambda} \varphi_U(M)$.

Given a (non-trivial) bi-partition $P := \{A_1, A_2 = \Lambda\}$ we define $\partial P := \{S \subset \Lambda / S \cap A_1 \neq \emptyset \wedge S \cap A_2 \neq \emptyset\}$. This is the “boundary of the partition” and it contains $|\partial P| = 2^{|A|} - 2^{|A_1|} - 2^{|A_2|} + 1$ elements.

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Supplemental Material for the paper
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QUANTUM CAUSE/EFFECT REPETOIRES

Effects. — We will denote by \( p \) (\( p' \)) the degrees of freedom (DOFs) associated with the purview \( P \) at time \( t + 1 \) (its complement \( P' \)) and by \( m \) (\( m' \)) the DOFs associated to the mechanism \( M \) at time \( t \) (its complement \( M' \)). On purely classical probabilistic grounds one can write

\[
\Pr(p|m) = \frac{\Pr(q,m)}{\Pr(m)} = \sum_{p',m'} \frac{\Pr(q',m,m')}{\Pr(m')} \Pr(m') = \sum_{p',m'} \Pr(q',m,m') \Pr(m'). \tag{S1}
\]

Here we have assumed that the prior of \( m \) and \( m' \) factorizes, i.e., \( \Pr(m,m') = \Pr(m)\Pr(m') \). Now quantum mechanics enters in defining the transition probability \( \Pr(q,q'|m,m') = \langle p,p' | U(|m,m') \rangle |p,p' \rangle \), where \( U \) is the unitary CP-map describing the (one-step) dynamics of the network. Inserting this in the equation above one gets

\[
\Pr(p|m) = \sum_{p'} |p| \langle p' | U(|m,m') \rangle \otimes \sum_{m'} \Pr(m') |m'| \langle p'|p \rangle = \langle p | \text{Tr}_P U(|m,m' \rangle \otimes \frac{1}{d|M'|} |p \rangle). \tag{S2}
\]

Here we have assumed that the prior for \( m' \) is the uniform unconstrained one, i.e., \( \Pr(m) = d^{-|M'|} \). The last equation shows that the probability for the purview being in the state \( p \) at time \( t + 1 \) (conditioned on the mechanism being in the state \( m \) at time \( t \)) is the diagonal element \( |p| \) of the reduced density matrix \( \rho_U(|m,m'|) = \text{Tr}_P U(|m,m' \rangle \otimes \frac{1}{d|M'|} |m,m' \rangle \rangle \). The definition adopted in Eq. (1) in the main text for the quantum effect repertoire amounts to consider this full density matrix, as opposed to just its diagonal entries, like in classical IIT.

Causes. — We now consider cause repertoires and denote by \( p \) (\( p' \)) the degrees of freedom (DOFs) associated with the purview \( P \) at time \( t - 1 \) (its complement \( P' \)) and by \( m \) (\( m' \)) the DOFs associated to the mechanism \( M \) at time \( t \) (its complement \( M' \)). Using Bayes rule and Eq. (S2) (with \( p \) and \( m \) interchanged) one can write

\[
\Pr(p|m) = \frac{\Pr(m,P)\Pr(p)}{\Pr(m)} = \langle m | \text{Tr}_M U(|p \rangle \otimes \frac{1}{d|M'|} |m \rangle \rangle \frac{\Pr(p)}{\Pr(m)} = \text{Tr} \left[ \langle m | \langle m \rangle \otimes \frac{1}{d|M'|} U |p \rangle \otimes \frac{1}{d|M'|} \right]. \tag{S3}
\]

Here we used \( \frac{\Pr(p)}{\Pr(m)} = d^{|M'|-|P'|}|P'| = d^{-|M|} \), Using the Hilbert-Schmidt dual and the properties of reduced density matrices, the equation above becomes

\[
\Pr(p|m) = \text{Tr} \left[ U^* |m \rangle \langle m \rangle \otimes \frac{1}{d|M'|} \right] |p \rangle = \langle p | \text{Tr}_P U^*(|m \rangle \otimes \frac{1}{d|M'|} |p \rangle \rangle. \tag{S4}
\]

Again, the last equation shows that the probability for the purview being in the state \( p \) at time \( t - 1 \) (conditioned on the mechanism being in the state \( m \) at time \( t \)) is the diagonal element \( |p| \) of the reduced density matrix \( \rho_U(|M'|) \). This justifies the definition for cause repertoires given in Eq. (1), we consider this full density matrix as opposed to just its diagonal entries like in classical IIT.

LIEB-ROBINSON BOUNDS FOR CAUSE/EFFECT INFORMATION

We now assume that the CP map \( U \) is a unitary generated by a local Hamiltonian \( H_M \), i.e., \( U(X) = e^{iH_M X} e^{-iH_M} \). In this case one can show that a Lieb-Robinson type bound holds for the cei (2). Indeed, \( 2 e^{iP} M = \| \rho c(P|M) - \frac{1}{d |P|} \|_1 = \text{Tr}_P \left( O_P (\rho^{(c)}(P|M) - \frac{1}{d |P|}) \right) = \text{Tr} \left( O_P \otimes \frac{1}{d |P|} U(|\Psi_M \rangle \otimes \frac{1}{d |M'|} - \frac{1}{d |P|}) \right) = \text{Tr} \left( \tilde{O}_P \left( \Psi_M \otimes \frac{1}{d |M'|} \right) \right) \), where \( \tilde{O}_P := U^* (O_P \otimes \frac{1}{d |P|} \rangle \). Now one can write \( \Psi_M \otimes \frac{1}{d |M'|} = T_M \left( \frac{1}{d |M'|} \Psi_M \right) \), where \( T_M \) is a “preparation” CP Map, local to the \( M \) mechanism whose Kraus operators can be given by \( A_k = |k \rangle \langle k | \otimes \frac{1}{d |M'|} \). Therefore, \( 2 e^{iP} M = \text{Tr} \left( (T_M - 1)(\tilde{O}_P \frac{1}{d |P|}) \right) \leq \| (T_M - 1)(\tilde{O}_P) \| = \| T_M (\tilde{O}_P) - \tilde{O}_P \| = \| \sum_k (A_k^\dagger \tilde{O}_P A_k - A_k^\dagger A_k \tilde{O}_P) \| \leq \sum_k \| A_k \| \| \tilde{O}_P \| \leq d_M \text{max}_k \| \tilde{O}_P, A_k \| \). Now, since \( \tilde{O}_P \) is the evolution of an operator local to \( P \) and the \( A_k \)'s are local to \( M \), the Lieb-Robinson holds in the form

\[
e^{iP} M \leq c \exp \left( -a (\text{dist}(P,M) - v(t)) \right) =: \epsilon. \tag{S5}
\]
Here $a, c > 0$ are constants depending on $d_M, |M|, |P|$, and $||O_P||$. Moreover, $v > 0$ is the Lieb-Robinson velocity, which depends on $H_A$. Since $U^*$ is also generated by a local Hamiltonian ($-H_A$), an identical proof holds for $ci(P|M)$. Finally, in the light of Eqs. (3) and (5), one has that integrated-information fulfills a Lieb-Robinson bound as well, i.e., $\varphi^{(c)}(P|M) \leq c \exp(-a(\text{dist}(P,M) - v|t|))$ (and a similar one for $\varphi^{(c)}$).

From the Lieb-Robinson bound, by a standard argument, it also follows that $\|\rho^{(c)}(M|M) - \rho^{(c)}(M'|\emptyset) \| \leq \epsilon$, where $M > M$ is a suitable Lieb-Robinson "fattening" of $M$. From this inequality by taking traces with respect $P'$, it follows (again) that the reperatures $\rho^{(c)}(P|M)$ with purviews $P \subset M$, are exponentially close to the unconditioned one $\rho^{(c)}$.

**HOLISTIC PHASE EXAMPLE**

Let us consider a qubit network of size $|\Lambda|$ with

$$U(X) = e^{itZ}Xe^{-itZ}, \quad Z := \otimes_{i \in \Lambda} \sigma_i^z, \quad \Psi_A = \otimes_{i \in A}|+\rangle =: \Pi_A^+.$$ (S6)

For $M \neq \Lambda$, one directly finds $\rho^{(c)}(P|M) = (c_i^2 \Pi_{M\setminus P}^+ + s_i^2 \Pi_{M\setminus P}^-) \otimes \frac{\rho^{(c)}}{2^{d_M}}$, where $\Pi^\pm_X := \otimes_{i \in X}|\pm\rangle\langle\pm|$, and $c_i := \cos t, s_i := \sin t$. We now focus on the $P = M$ case with $|M| > 1$ and look for factorizations of the form $\rho^{(c)}(M^1|M_1) \otimes \rho^{(c)}(M_2|M_2)$, where $(M_1, M_2)$ and $(\bar{M_1}, \bar{M_2})$ are (non-trivial) pairings of $M$. Using the above expression for the reperatures, one has that $\rho^{(c)}(M|M) = \rho^{(c)}(M|M_1) \otimes \rho^{(c)}(M|M_2)$ is equal to

$$X := \Pi_+ \otimes (c_i^2 \Pi_{BC}^+ - c_i^4 \Pi_{BC}^- \Pi_{D}^+ + \Pi_A^+ \otimes (s_i^2 \Pi_{BC}^+ - s_i^4 \Pi_{BC}^- \Pi_{D}^+ - s_i^2 \Pi_{BC}^- \Pi_{D}^+ \Pi_{D}^+ + \Pi_A \otimes \Pi_{D}^+ \otimes \Pi_{D}^+),$$

where $A := M_1 \cap \bar{M_1}, B := \bar{M_2} \cap M_2, C := \bar{M_2} \cap M_1, D := M_2 \cap M_2$, and $d_{BC} := 2^{\frac{|B|+|C|}}{2^{d_M}}$. The X operator above can be easily diagonalized giving $\sigma(X) = \{0, c_i^2 - c_i^4, s_i^2 - s_i^4, c_i^2 s_i^2, -c_i^4 s_i^4, -s_i^2 s_i^4, -c_i^4, -s_i^2 s_i^4\}$, with degeneracies $d = 4d_{BC}, 1, 1, 2d_{BC}, d_{BC} - 1, d_{BC} - 1$. From this it follows $\|X\| = 2(1 - \frac{c_i^4}{d_{BC}} - \frac{s_i^4}{d_{BC}}) \geq 4s_i^2c_i^2$. The lower bound is achieved when $d_{BC} = 1$, i.e., $M_1 = \bar{M_1}$ and $M_2 = \bar{M_2}$. Finally, $\varphi(M|M) = \min_{(M_1, M_2)} \frac{1}{2} \|\rho^{(c)}(P|M) - \rho^{(c)}(M_1|M_1) \otimes \rho^{(c)}(M_2|M_2)\| = 2s_i^2c_i^2$. The case $M = \Lambda$ gives, with a similar calculation, $\varphi(\Lambda|\Lambda) = |s_i c_i|/1 + |s_i c_i| \geq 2s_i^2c_i^2$.

In Fig. (a) we plot $\Phi(t)$, for different system sizes $|\Lambda| = 3, 4, 5$, given the initial state $\Psi_A = \otimes_{i \in A}|+\rangle$. As predicted by the above analytical calculations, we find $\Phi > 2s_i^2c_i^2$, the red dashed line represents $2s_i^2c_i^2$, and for $t = 0$ and $t = \pi/2$ we have $\Phi = 0$. Fig. (b) shows the holistic behaviour of the network, i.e., the exponential scaling of $\Phi$ with the system size $|\Lambda|$. In particular, we observe an exponential scaling of $\Phi$ for all the three natural prescriptions for fixing the timescale: a) for $t = 0.5$ (solid blue line) we get $\log_2 \Phi = 1.05|\Lambda| - 2.64, b)$ for $t = \pi/2$ (dashed orange line), $\log_2 \Phi = 2.04|\Lambda| - 2.35$ and c) for $t = \arg \max \Phi(t)$ (dotted green line), $\log_2 \Phi = 1.04|\Lambda| - 2.29$. The fits obtained using all the three different prescriptions are consistent with a scaling of the form $\Phi \sim 2^{|\Lambda|}$. The fit of case b) (dashed orange line), which corresponds to the fixed action case $t = \pi/2$, is also shown in the inset of Fig. 2 from the main text.

**LOW-INTEGRATION EXAMPLES**

In this section we numerically calculate $\Phi$ for four different qubit networks of sizes up to $|\Lambda| = 7$. We use $U(X) = e^{-itA}H e^{itA}H$, but now the dynamics are described by a two-body Hamiltonian $H$. Namely, the dynamics are governed by i) the XX Hamiltonian on a ring $H_{XX} = \sum_{i=1}^{|A|} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y)$, ii) the XX Hamiltonian on a fully connected graph $H_{XX} = \sum_{i=1}^{|\Lambda|} \sum_{j=1}^{|\Lambda|} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y)$, iii) by the XXX Hamiltonian on a ring $H_{XXX} = \sum_{i=1}^{|\Lambda|} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z)$, and iv) by the XXZ Hamiltonian on a fully connected graph $H_{XXZ} = \sum_{i=1}^{|\Lambda|} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z)$. The initial state was chosen to be $\Psi_A = \otimes_{i \in A}|+\rangle\langle+|$, whereas in the main text and for the holistic example with $Z = \otimes_i \sigma_i^z$, we used the state $\Psi_A = \otimes_{i \in A}|+\rangle\langle+|$, where $\langle+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |\rangle)$ with $|\rangle = (1, 0)^T$ and $|\rangle = (0, 1)^T$.

First, in Fig. (a) we plot the scaling of the timescale $t_A$, used in the evolution operator of the network $U(X) = e^{-it_A}H e^{it_A}H$. The timescale $t_A$ for the different system sizes $|\Lambda|$ is determined by fixing the "action" to a constant value, i.e., $t_A||H||_{\infty} = 2.5$. Figure (a) shows the scaling of the resulting $t_A$ with the system size on a log$_2$ scale for the XX Hamiltonian on a ring $H_{XX}$. The blue dots show it for an odd number of qubits ($|\Lambda|$ odd) and the orange squares for an even number of qubits ($|\Lambda|$ even) in the network. The same is shown in Fig. (b), (c) and (d) for the
FIG. S3. (Color online) (a) Numerical computation of the Integrated Information $\Phi$ as a function of the parameter $t$, for different system sizes $|\Lambda| = 3, 4, 5$, with $\Psi_A = \otimes_{i \in \Lambda}|+\rangle\langle +|$. The red dashed line shows $2s^2c^2$. (b) Scaling of $\Phi$ with the system size $|\Lambda|$, for the three options to fix the time scale $a) t = 0.5$ (solid blue line) $\log_2 \Phi = 1.04|\Lambda| - 2.60$, $b) t||Z||_\infty = 2.5$ (dashed orange line) $\log_2 \Phi = 1.04|\Lambda| - 2.35$ and $c) t = \arg\max \Phi_A(t)$ (dotted green line) $\log_2 \Phi = 1.04|\Lambda| - 2.29$.

XX Hamiltonian on a fully connected graph, the XXX Hamiltonian on a ring and the XXX Hamiltonian on a fully connected graph. The dashed blue line and the dotted orange line show the fits for $|\Lambda|$ odd and even, respectively.

We observe that for the the XX and the XXX Hamiltonians on a ring, there is a marked even/odd effect in the scaling of $t_\Lambda$ with $|\Lambda|$. This is due to the fact that for the ring geometry the (sup) norm of the Hamiltonian is given by $|E_0|$, where $E_0$ is the ground state of the corresponding antiferromagnetic Hamiltonian. As such especially $||H_{\text{XXX}}||_\infty$ shows a marked even-odd effect, since for odd chains one cannot have a singlet ground state.

FIG. S4. (Color online) Scaling of the $t_\Lambda$ with the system size $|\Lambda|$ given by fixing action $t_\Lambda||H||_\infty = 2.5$, for the XX Hamiltonian on a ring (a), the XX Hamiltonian on a fully connected graph (b), the XXX Hamiltonian on a ring (c), and the XXX Hamiltonian on a fully connected graph. Note the more pronounced even-odd effect for $H_{\text{XXX}}$ as discussed in the text.

The scaling of the Integrated Information $\Phi$ with the system size and different prescriptions for fixing the timescale...
$t_\Lambda$ is shown in Fig. S5. Panels (a) and (d) show holistic (exponential) behavior consistent with $\Phi \sim 2^{|\Lambda|}$, whereas in panel (b) and (c) the best fit is obtained with $\Phi \sim |\Lambda|^\alpha$ for some positive $\alpha$ (see figure).

**FIG. S5.** (Color online) Scaling of Integrated Information $\Phi$ as a function of the system size $|\Lambda|$ for the three different scenarios fixing the timescale $t_\Lambda$, i.e., (a) $t_\Lambda = 0.5$, (b) and (c) fixed “action” $t_\Lambda ||H||_\infty = 2.5$ and (d) $t_\Lambda = \text{argmax}_t \Phi_\Lambda(t)$, for the four different two-body Hamiltonians $H_{XX}$, $H_{XXc}$, $H_{XXX}$, $H_{XXXc}$. The initial state was chosen to be $\Psi_\Lambda = \bigotimes_{i \in \Lambda} |\uparrow\rangle \langle \uparrow|$.  

**“ALGORITHM” TO COMPUTE $\Phi$:**

i) Compute $\rho^{(e/c)}(P|M)$ for all non-empty $M, P \subset \Lambda$ (# of repertoires: $2(2^{|\Lambda|} - 2^{|\Lambda|+1} + 1)$)

ii) For each non-trivial $\rho^{(e/c)}(P|M)$ compute $\varphi$ (# of pairings of $(P,M)$: $(2^{|M|+|P|-1} - 1)$)

iii) For each $M \neq \emptyset$ find its core effect/cause $P^{(x)}_\star$ and associated integrated-information $\varphi^{(x)}(M) = \max_P \varphi^{(x)}(P|M) = \varphi^{(x)}(P^{(x)}_\star|M)$, ($x = e,c$). Now the CS $\{1\}$ is defined for the given $U$.

iv) For each partition $P$ of $\Lambda$ compute $\Phi$ by using $[6]$ (# partitions of $\Lambda$: $(2^{|\Lambda|}-1) - 1$).

The total number of steps (for a fixed partition) is

$$\sum_{M, P \subset \Lambda} (2^{|M|+|P|-1} - 1) = \frac{1}{2} \sum_{|M|, |P| = 0}^{|\Lambda|} \left( \frac{|\Lambda|}{|M|} \right) \left( \frac{|\Lambda|}{|P|} \right) (2^{|M|}2^{|P|}-2) = O(3^{|\Lambda|}/2).$$

As in the classical IIT case, the actual computation of $\Phi$ is exponentially costly (in $|\Lambda|$) and therefore provides a challenging task even for networks of moderate size.
PROOF OF A FACTORIZATION PROPERTY

The next result shows how the CS is affected by partitioning the system and simplifies the computation of II [12].

**Proposition** The cause/effect repertoires of the factorized map $U_P$ are themselves factorized:

$$\rho^{(e/c)}(P|M) = \rho^{(e/c)}(P_1|M_1) \otimes \rho^{(e/c)}(P_2|M_2),$$  \eqno{(S7)}

where $P_i := P \cap \Lambda_i$, $M_i := M \cap \Lambda_i$, $(i = 1, 2)$.

From the factorized (S7) form it follows that:

a) if both $M$ and $P$ are on the same side of the partition the $\rho^{(e)}(P|M)$ is unaffected

b) if they are on opposite sides there is zero cause/effect information ($\rho^{(e)}(P|M) = \rho^{(e)}(P|\emptyset) \otimes \rho^{(e)}(\emptyset|M)$) c) if either one the two is straddling between the partition there is zero $\rho^{(e)}(P|M)$.

From a)–c) above one sees that all the concepts such that $M \cup P \in \partial \Lambda$ are dis-integrated whereas all those that $M \cup P \notin \partial \Lambda$ are left invariant. From Eqs. (5) and (6) we see that, for a given $P$ just the former contributes to $\Phi$ (as the latter cancel being identical for $U$ and $U_P$). The MIP is then the partition that dis-integrates the least number of concepts in the CS of the undivided system.

**Proof.** We define $\Psi_{\Omega} = \otimes_{i \in \Omega} \Psi_i$, ($\forall \Omega \subset \Lambda$). From Eqs. (11) and the definition of the factorized map one finds

$$\rho^{(e/c)}(P|M) = \text{Tr}_{P'} U_P \circ N_M'(\Psi_\Lambda),$$

where

$$U_P \circ N_M'(\Psi_\Lambda) = \otimes_{i=1}^2 \text{Tr}_{\Lambda'} U_i \left( \Psi_{M_i} \otimes \frac{1_{M_i' \cap \Lambda_i'}}{d(M_i' \cap \Lambda_i')} \right).$$

First notice that $\frac{1_{M_i' \cap \Lambda_i'}}{d(M_i' \cap \Lambda_i')} \otimes \frac{1_{\Lambda_i'}}{d(\Lambda_i')} = \frac{1_{M_i' \cap \Lambda_i'} \otimes 1_{\Lambda_i'}}{d(M_i' \cap \Lambda_i')}$. The result follows now from $\text{Tr}_{P'} = \text{Tr}_{P' \cap \Lambda_1} \otimes \text{Tr}_{P' \cap \Lambda_2}$ and $\text{Tr}_{P' \cap \Lambda_1} \circ \text{Tr}_{\Lambda'_1} = \text{Tr}_{(P' \cap \Lambda_1) \cup \Lambda'_1} = \text{Tr}_{(P \cap \Lambda_1)' \cup \Lambda'_1}$. Where we used $(P' \cap \Lambda_1) \cup \Lambda'_1 = (P' \cup \Lambda'_1) \cap (\Lambda_1 \cup \Lambda'_1) = P' \cup \Lambda'_1 = (P \cap \Lambda_1)' = P'_1$. 