ON \textit{f}-HARMONIC MORPHISMS BETWEEN RIEMANNIAN MANIFOLDS

YE-LIN OU ∗

Abstract
\textit{f}-Harmonic maps were first introduced and studied by Lichnerowicz in \cite{18} (see also Section 10.20 in Eells-Lemaire’s report \cite{10}). In this paper, we study a subclass of \textit{f}-harmonic maps called \textit{f}-harmonic morphisms which pull back local harmonic functions to local \textit{f}-harmonic functions. We prove that a map between Riemannian manifolds is an \textit{f}-harmonic morphism if and only if it is a horizontally weakly conformal \textit{f}-harmonic map. This generalizes the well-known Fuglede-Ishihara characterization for harmonic morphisms. Some properties and many examples as well as some non-existence of \textit{f}-harmonic morphisms are given. We also study the \textit{f}-harmonicity of conformal immersions.

1. \textit{f}-HARMONIC MAPS VS. \textit{F}-HARMONIC MAPS

1.1 \textit{f}-harmonic maps

Let \( f : (M, g) \rightarrow (0, \infty) \) be a smooth function. An \textit{f}-harmonic map is a map \( \phi : (M^m, g) \rightarrow (N^n, h) \) between Riemannian manifolds such that \( \phi|_\Omega \) is a critical point of the \textit{f}-energy (see \cite{18}, and \cite{10}, Section 10.20)

\[
E_f(\phi) = \frac{1}{2} \int_{\Omega} f |d\phi|^2 dv_g,
\]

for every compact domain \( \Omega \subseteq M \). The Euler-Lagrange equation gives the \textit{f}-harmonic map equation (\cite{5}, \cite{25})

\[
\tau_f(\phi) \equiv f \tau(\phi) + d\phi(\text{grad } f) = 0,
\]

where \( \tau(\phi) = \text{Tr}_g \nabla d\phi \) is the tension field of \( \phi \) vanishing of which means \( \phi \) is a harmonic map.

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Example 1. Let $\varphi, \psi, \phi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined as

\[
\varphi(x, y, z) = (x, y),
\psi(x, y, z) = (3x, xy), \quad \text{and}
\phi(x, y, z) = (x, y + z).
\]

Then, one can easily check that both $\varphi$ and $\psi$ are $f$-harmonic map with $f = e^z$, $\varphi$ is a horizontally conformal submersion whilst $\psi$ is not. Also, $\phi$ is an $f$-harmonic map with $f = e^{y-z}$, which is a submersion but not horizontally weakly conformal.

1.2 $F$-harmonic map

Let $F : [0, +\infty) \rightarrow [0, +\infty)$ be a $C^2$-function, strictly increasing on $(0, +\infty)$, and let $\varphi : (M, g) \rightarrow (N, h)$ be a smooth map between Riemannian manifolds. Then $\varphi$ is said to be an $F$-harmonic map if $\varphi|_\Omega$ is a critical point of the $F$-energy functional

\[
E_F(\varphi) = \int_\Omega F\left(\frac{|d\varphi|^2}{2}\right) v_g,
\]

for every compact domain $\Omega \subseteq M$. The equation of $F$-harmonic maps is given by (2)

\[
\tau_F(\varphi) \equiv F'(\frac{|d\varphi|^2}{2}) \tau(\varphi) + \varphi_* \left( \grad F'(\frac{|d\varphi|^2}{2}) \right) = 0,
\]

where $\tau(\varphi)$ denotes the tension field of $\varphi$.

Harmonic maps, $p$-harmonic maps, and exponential harmonic maps are examples of $F$-harmonic maps with $F(t) = t$, $F(t) = \frac{1}{p} (2t)^{p/2}$ ($p > 4$), and $F(t) = e^t$ respectively (2).

In particular, $p$-harmonic map equation can be written as

\[
\tau_p(\varphi) = |d\varphi|^{p-2} \tau(\varphi) + d\varphi(\grad |d\varphi|^{p-2}) = 0,
\]

1.3 Relationship between $f$-harmonic and $F$-harmonic maps

We can see from Equation (1) that an $f$-harmonic map with $f = \text{constant} > 0$ is nothing but a harmonic map so both $f$-harmonic maps and $F$-harmonic maps are generalizations of harmonic maps. Though we were warned in [5] that $f$-harmonic maps should not be confused with $F$-harmonic maps and $p$-harmonic maps, we observe that, apart from critical points, any $F$-harmonic map is a special $f$-harmonic maps. More precisely we have

**Corollary 1.1.** Any $F$-harmonic map $\varphi : (M, g) \rightarrow (N, h)$ without critical points, i.e., $|d\varphi_x| \neq 0$ for all $x \in M$, is an $f$-harmonic map with $f = F'(\frac{|d\varphi|^2}{2})$. 
In particular, a p-harmonic map without critical points is an f-harmonic map with $f = |d\varphi|^{p-2}$

**Proof.** Since $F$ is a $C^2$-function and strictly increasing on $(0, +\infty)$ we have $F'(t) > 0$ on $(0, +\infty)$. If the $F$-harmonic map $\varphi : (M, g) \to (N, h)$ has no critical points, i.e., $|d\varphi_x| \neq 0$ for all $x \in M$, then the function $f : (M, g) \to (0, +\infty)$ with $f = F'(\frac{|d\varphi|^2}{2})$ is a smooth and we see from Equations (2) and (1) that the $F$-harmonic map $\varphi$ is an $f$-harmonic map with $f = F'(\frac{|d\varphi|^2}{2})$. The second statement follows from the fact that for a $p$-harmonic map, $F(t) = \frac{1}{2p}(2t)^{p/2}$ and hence $f = F'(\frac{|d\varphi|^2}{2}) = |d\varphi|^{p-2}$.

□

Another relationship between $f$-harmonic maps and harmonic maps can be characterized as follows.

**Corollary 1.2.** A map $\phi : (M^m, g) \to (N^n, h)$ is $f$-harmonic if and only if $\phi : (M^m, f^{\frac{2}{m-2}}g) \to (N^n, h)$ is a harmonic map.

**Proof.** The statement that the $f$-harmonicity of $\phi : (M^m, g) \to (N^n, h)$ implies the harmonicity of $\phi : (M^m, f^{\frac{2}{m-2}}g) \to (N^n, h)$ was stated and proved in [18] (see also Section 10.20 in [10]). It is not difficult to see that the converse is also true. In fact, let $\bar{g} = f^{\frac{2}{m-2}}g$, a straightforward computation shows that

$$\tau(\varphi, \bar{g}) = f^{\frac{m}{m-2}}\tau_f(\varphi, g),$$

which completes the proof of the corollary. □

### 1.4 A physical motivation for the study of f-harmonic maps:

In physics, the equation of motion of a continuous system of spins with inhomogeneous neighbor Heisenberg interaction (such a model is called the inhomogeneous Heisenberg ferromagnet) is given by

$$\frac{\partial u}{\partial t} = f(x)(u \times \Delta u) + \nabla f \cdot (u \times \nabla u),$$

where $\Omega \subseteq \mathbb{R}^m$ is a smooth domain in the Euclidean space, $f$ is a real-valued function defined on $\Omega$, $u(x, t) \in S^2$, $\times$ denotes the cross products in $\mathbb{R}^3$ and $\Delta$ is the Laplace operator on $\mathbb{R}^m$. Physically, the function $f$ is called the coupling function, and is the continuum limit of the coupling constants between the neighboring spins. Since $u$ is a map into $S^2$ it is well known that the tension field of $u$ can be written as $\tau(u) = \Delta u + |\nabla u|^2 u$, and one can easily check that the right hand side of the inhomogeneous Heisenberg spin system (4) can be written as $u \times (f\tau(u) + \nabla f \cdot \nabla u)$. It follows that $u$ is a smooth stationary solution of (4) if and only if $f\tau(u) + \nabla f \cdot \nabla u = 0$, i.e., $u$ is an $f$-harmonic map. So
there is a one-to-one correspondence between the set of the stationary solutions of the inhomogeneous Heisenberg spin system (4) on the domain \( \Omega \) and the set of \( f \)-harmonic maps from \( \Omega \) into 2-sphere. The above inhomogeneous Heisenberg spin system (4) is also called inhomogeneous Landau-Lifshitz system (see, e.g., [6], [7], [9], [14], [16], [17], for more details).

Using Corollary 1.2 we have the following example which provides many stationary solutions of the inhomogeneous Heisenberg spin system defined on \( \mathbb{R}^3 \).

Example 2. \( u : (\mathbb{R}^3, ds_0) \rightarrow (N^n, h) \) is an \( f \)-harmonic map if and only if \( u : (\mathbb{R}^3, f^2 ds_0) \rightarrow (N^n, h) \) is a harmonic map. In particular, there is a 1-1 correspondence between harmonic maps from 3-sphere \( S^3 \setminus \{N\} \equiv (\mathbb{R}^3, \frac{4ds_0}{(1+|x|^2)^2}) \rightarrow (N^n, h) \) and \( f \)-harmonic maps with \( f = \frac{2}{1+|x|^2} \) from Euclidean 3-space \( \mathbb{R}^3 \rightarrow (N^n, h) \). When \( (N^n, h) = S^2 \), we have a 1-1 correspondence between the set of harmonic maps \( S^3 \rightarrow S^2 \) and the set of stationary solutions of the inhomogeneous Heisenberg spin system on \( \mathbb{R}^3 \). Similarly, there is a 1-1 correspondence between harmonic maps from hyperbolic 3-space \( H^3 \equiv (D^3, \frac{ds_0}{1-|x|^2}) \rightarrow (N^n, h) \) and \( f \)-harmonic maps \( (D^3, ds_0) \rightarrow (N^n, h) \) with \( f = \frac{2}{1-|x|^2} \) from unit disk in Euclidean 3-space.

1.5 A little more about \( f \)-harmonic maps

Corollary 1.3. If \( \phi : (M^m, g) \rightarrow (N^n, h) \) is an \( f_1 \)-harmonic map and also an \( f_2 \)-harmonic map, then \( \nabla (f_1/f_2) \in \ker d\phi \).

Proof. This follows from

\[
\tau_{f_1}(\phi) \equiv f_1 \tau(\phi) + d\phi(\nabla f_1) = 0, \\
\tau_{f_2}(\phi) \equiv f_2 \tau(\phi) + d\phi(\nabla f_2) = 0,
\]

and hence

\[
d\phi(\nabla \ln (f_1/f_2)) = 0. \]

Proposition 1.4. A conformal immersion \( \phi : (M^m, g) \rightarrow (N^n, h) \) with \( \phi^* h = \lambda^2 g \) is \( f \)-harmonic if and only if it is \( m \)-harmonic and \( f = C \lambda^{m-2} \). In particular, an isometric immersion is \( f \)-harmonic if and only if \( f = \text{const} \) and hence it is harmonic.

Proof. It is not difficult to check (see also [26]) that for a conformal immersion \( \phi : (M^m, g) \rightarrow (N^n, h) \) with \( \phi^* h = \lambda^2 g \), the tension field is given by

\[
\tau(\phi) = m\lambda^2 \eta + (2 - m) d\phi(\nabla \ln \lambda),
\]
so we can compute the $f$-tension field to have

$$\tau_f(\phi) = f [m\lambda^2 \eta + d\phi (\text{grad} \ln(\lambda^{2-m} f))],$$

where $\eta$ is the mean curvature vector of the submanifold $\phi(M) \subset N$. Noting that $\eta$ is normal part whilst $d\phi (\text{grad} \ln(\lambda^{2-m} f))$ is the tangential part of $\tau_f(\phi)$ we conclude that $\tau_f(\phi) = 0$ if and only if

$$\begin{cases} m\lambda^2 \eta = 0, \\ d\phi (\text{grad} \ln(\lambda^{2-m} f)) = 0. \end{cases}$$

It follows that $\eta = 0$ and $\text{grad} \ln(\lambda^{2-m} f) = 0$ since $\phi$ is an immersion. From these we see that $\phi$ is a minimal conformal immersion which means it is an $m$-harmonic map ([26]) and that $f = C\lambda^{m-2}$. Thus, we obtain the first statement.

The second statement follows from the first one with $\lambda = 1$. □

2. $f$-Harmonic morphisms

A horizontally weakly conformal map is a map $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds such that for each $x \in M$ at which $d\varphi_x \neq 0$, the restriction $d\varphi_x |_{H_x} : H_x \rightarrow T_{\varphi(x)} N$ is conformal and surjective, where the horizontal subspace $H_x$ is the orthogonal complement of $V_x = \ker d\varphi_x$ in $T_x M$. It is not difficult to see that there is a number $\lambda(x) \in (0, \infty)$ such that $h(d\varphi(X), d\varphi(Y)) = \lambda^2(x) g(X, Y)$ for any $X, Y \in H_x$. At the point $x \in M$ where $d\varphi_x = 0$ one can let $\lambda(x) = 0$ and obtain a continuous function $\lambda : M \rightarrow R$ which is called the dilation of a horizontally weakly conformal map $\varphi$. A non-constant horizontally weakly conformal map $\varphi$ is called horizontally homothetic if the gradient of $\lambda^2(x)$ is vertical meaning that $X(\lambda^2) \equiv 0$ for any horizontal vector field $X$ on $M$. Recall that a $C^2$ map $\varphi : (M, g) \rightarrow (N, h)$ is a $p$-harmonic morphism ($p > 1$) if it preserves the solutions of $p$-Laplace equation in the sense that for any $p$-harmonic function $f : U \rightarrow \mathbb{R}$, defined on an open subset $U$ of $N$ with $\varphi^{-1}(U)$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \rightarrow \mathbb{R}$ is a $p$-harmonic function. A $p$-harmonic morphism can be characterized as a horizontally weakly conformal $p$-harmonic map (see [11], [15], [19] for details).

**Definition 2.1.** Let $f : (M, g) \rightarrow (0, \infty)$ be a smooth function. A $C^2$-function $u : U \rightarrow \mathbb{R}$ defined on an open subset $U$ of $M$ is called $f$-harmonic if

$$\Delta_f^M u \equiv f \Delta^M u + g(\text{grad} f, \text{grad} u) = 0. \tag{5}$$

A continuous map $\phi : (M^m, g) \rightarrow (N^n, h)$ is called an $f$-harmonic morphism if for every harmonic function $u$ defined on an open subset $V$ of $N$ such that $\phi^{-1}(V)$ is non-empty, the composition $u \circ \phi$ is $f$-harmonic on $\phi^{-1}(V)$. 
Theorem 2.2. Let \( \phi : (M^m, g) \to (N^n, h) \) be a smooth map. Then, the following are equivalent:

1. \( \phi \) is an \( f \)-harmonic morphism;
2. \( \phi \) is a horizontally weakly conformal \( f \)-harmonic map;
3. There exists a smooth function \( \lambda^2 \) on \( M \) such that
   \[
   \Delta^M_f (u \circ \phi) = f \lambda^2 (\Delta^N u) \circ \phi
   \]
   for any \( C^2 \)-function \( u \) defined on (an open subset of) \( N \).

Proof. We will need the following lemma to prove the theorem.

Lemma 2.3. (15) For any point \( q \in (N^n, h) \) and any constants \( C_\alpha, C_{\alpha \beta} \) with
\[ C_{\alpha \beta} = C_{\beta \alpha} \quad \text{and} \quad \sum_{\alpha=1}^n C_{\alpha \alpha} = 0, \]
there exists a harmonic function \( u \) on a neighborhood of \( q \) such that \( u_\sigma(q) = C_\sigma, u_{\alpha \beta}(q) = C_{\alpha \beta} \).

Let \( \phi : (M^m, g) \to (N^n, h) \) be a map and let \( p \in M \). Suppose that \( \phi(x) = (\phi^1(x), \phi^2(x), \ldots, \phi^n(x)) \)
was the local expression of \( \phi \) with respect to the local coordinates \( \{x^i\} \) in the neighborhood \( \phi^{-1}(V) \) of \( p \)
and \( \{y^\alpha\} \) in a neighborhood \( V \) of \( q = \phi(p) \in N \). Let \( u : V \to \mathbb{R} \) defined on an open subset \( V \) of \( N \). Then, a straightforward computation gives
\[
\Delta^M_f (u \circ \phi) = f \Delta^M (u \circ \phi) + d(u \circ \phi)(\text{grad} f)
\]
\[
= f u_{\alpha \beta} \text{grad} \phi^\alpha \cdot \text{grad} \phi^\beta + f u_\alpha \Delta^M \phi^\alpha + d(u \circ \phi)(\text{grad} f)
\]
\[
(6)
= f g(\text{grad} \phi^\alpha, \text{grad} \phi^\beta) u_{\alpha \beta} + [f \Delta^M \phi^\alpha + (\text{grad} f) \phi^\alpha] u_\sigma.
\]

By Lemma 2.3, we can choose a local harmonic function \( u \) on \( V \subset N \) such that
\[ u_\sigma(q) = C_\sigma = 0 \quad \forall \ \sigma = 1, 2, \ldots, n, \quad u_{\alpha \beta}(q) = 1 \quad (\alpha \neq \beta), \]
and all other \( u_{\rho \sigma}(q) = C_{\rho \sigma} = 0 \) and substitute it into (6) to have
\[
(7)
g(\text{grad} \phi^\alpha, \text{grad} \phi^\beta) = 0, \quad \forall \ \alpha \neq \beta = 1, 2, \ldots, n.
\]

Note that the choice of such functions implies
\[
(8)
h^\alpha^\beta(\phi(p)) = 0, \quad \forall \ \alpha \neq \beta = 1, 2, \ldots, n.
\]

Another choice of harmonic function \( u \) with \( C_{11} = 1, C_{\alpha \alpha} = -1 \quad (\alpha \neq 1) \) and all other \( C_\sigma, C_{\alpha \beta} = 0 \) for Equation (6) gives
\[
(9)
g(\text{grad} \phi^1, \text{grad} \phi^1) - g(\text{grad} \phi^\alpha, \text{grad} \phi^\alpha) = 0, \quad \forall \ \alpha \neq \beta = 2, 3, \ldots, n.
\]

Note also that for these choices of harmonic functions \( u \) we have
\[
(10)
h^{11}(\phi(p)) - h^{\alpha \alpha}(\phi(p)) = 0, \quad \forall \ \alpha \neq \beta = 2, 3, \ldots, n.
\]
It follows from (7), (8), (9) and (10) that the $f$-harmonic morphism $\phi$ is a horizontally weakly conformal map

\begin{equation}
 g(\text{grad}\phi^\alpha, \text{grad}\phi^\beta) = \lambda^2 h^{\alpha\beta} \circ \phi.
\end{equation}

Substituting horizontal conformality equation (11) into (6) we have

\begin{align}
 \Delta^M_{\lambda^2}(u \circ \phi) &= f \lambda^2 (h^{\alpha\beta} \circ \phi)u_{\alpha\beta} + \left[f \Delta^M \phi^\sigma + (\text{grad} f)\phi^\sigma\right]u_{\sigma} \\
 &= f \lambda^2 (\Delta^N u) \circ \phi + \left[f \Delta^M \phi^\sigma + f \lambda^2 (h^{\alpha\beta}\Gamma^\sigma) \circ \phi + (\text{grad} f)\phi^\sigma\right]u_{\sigma} \\
 &= f \lambda^2 (\Delta^N u) \circ \phi + \text{du}(\tau_f(\phi)) \quad (12)
\end{align}

for any function $u$ defined (locally) on $N$. By special choice of harmonic function $u$ we conclude that the $f$-harmonic morphism is an $f$-harmonic map. Thus, we obtain the implication “(1) $\implies$ (2)”. Note that the only assumption we used to obtain Equation (12) is the horizontal conformality (11). Therefore, it follows from (12) that “(2) $\implies$ (3)”. Finally, “(3) $\implies$ (1)” is clearly true. Thus, we complete the proof of the theorem. □

Similar to harmonic morphisms we have the following regularity result.

**Corollary 2.4.** For $m \geq 3$, an $f$-harmonic morphism $\phi : (M^m, g) \longrightarrow (N^n, h)$ is smooth.

**Proof.** In fact, by Corollary 1.1 if $m \neq 2$ and $\phi : (M^m, g) \longrightarrow (N^n, h)$ is an $f$-harmonic morphism, then $\phi : (M^m, f^{2/(m-2)}g) \longrightarrow (N^n, h)$ is a harmonic map and hence a harmonic morphism, which is known to be smooth (see, e. g., [1]). □

It is well known that the composition of harmonic morphisms is again a harmonic morphism. The composition law for $f$-harmonic morphisms, however, will need to be modified accordingly. In fact, by the definitions of harmonic morphisms and $f$-harmonic morphisms we have

**Corollary 2.5.** Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be an $f$-harmonic morphism with dilation $\lambda_1$ and $\psi : (N^n, h) \longrightarrow (Q^q, k)$ a harmonic morphism with dilation $\lambda_2$. Then the composition $\psi \circ \phi : (M^m, g) \longrightarrow (Q^q, k)$ is an $f$-harmonic morphism with dilation $\lambda_1(\lambda_2 \circ \phi)$.

More generally, we can prove that $f$-harmonic morphisms pull back harmonic maps to $f$-harmonic maps.

**Proposition 2.6.** Let $\phi : (M^m, g) \longrightarrow (N^n, h)$ be an $f$-harmonic morphism with dilation $\lambda$ and $\psi : (N^n, h) \longrightarrow (Q^q, k)$ a harmonic map. Then the composition $\psi \circ \phi : (M^m, g) \longrightarrow (Q^q, k)$ is an $f$-harmonic map.
Proof. It is well known (see e.g., [4], Proposition 3.3.12) that the tension field of the composition map is given by
\[ \tau(\psi \circ \phi) = d\psi(\tau(\phi)) + \text{Tr}_g \nabla d\psi(d\phi, d\phi), \]
from which we have the \( f \)-tension of the composition \( \psi \circ \phi \) given by
\[ \tau_f(\psi \circ \phi) = d\psi(\tau_f(\phi)) + f\text{Tr}_g \nabla d\psi(d\phi, d\phi). \] (13)
Since \( \phi \) is an \( f \)-harmonic morphism and hence a horizontally weakly conformal \( f \)-harmonic map with dilation \( \lambda \), we can choose a local orthonormal frame \( \{ e_1, \ldots, e_n, e_{n+1}, \ldots, e_m \} \) around \( p \in M \) and \( \{ e_1, \ldots, e_n \} \) around \( \phi(p) \in N \) so that
\[ \begin{cases} d\phi(e_i) = \lambda e_i, & i = 1, \ldots, n, \\ d\phi(e_\alpha) = 0, & \alpha = n + 1, \ldots, m. \end{cases} \]
Using these local frames we compute
\[ \text{Tr}_g \nabla d\psi(d\phi, d\phi) = \sum_{i=1}^n \nabla d\psi(d\phi e_i, d\phi e_i) = \lambda^2 \left( \sum_{i=1}^n \nabla d\psi(e_i, e_i) \right) \circ \phi \]
\[ = \lambda^2 \tau(\psi) \circ \phi. \]
Substituting this into (13) we have
\[ \tau_f(\psi \circ \phi) = f d\psi(\tau(\phi)) + f\lambda^2 \tau(\psi) \circ \phi + d(\psi \circ \phi)(\text{grad} f) \]
\[ = d\psi(\tau_f(\phi)) + f\lambda^2 \tau(\psi) \circ \phi, \]
from which the proposition follows. \( \square \)

Theorem 2.7. Let \( \phi : (M^m, g) \rightarrow (N^n, h) \) be a horizontally weakly conformal map with \( \phi^* h = \lambda^2 g|_H \). Then, any two of the following conditions imply the other one.

(1) \( \phi \) is an \( f \)-harmonic map and hence an \( f \)-harmonic morphism;
(2) \( \text{grad}( f\lambda^{2-n} ) \) is vertical;
(3) \( \phi \) has minimal fibers.

Proof. It can be check (see e.g., [4]) that the tension field of a horizontally weakly conformal map \( \phi : (M^m, g) \rightarrow (N^n, h) \) is given by
\[ \tau(\phi) = -(m-n)d\phi(\mu) + (2-n)d\phi(\text{grad} \ln \lambda), \]
where \( \lambda \) is the dilation of the horizontally weakly conformal map \( \phi \) and \( \mu \) is the mean curvature vector field of the fibers. It follows that the \( f \)-tension field of \( \phi \) can be written as
\[ \tau_f(\phi) = -(m-n)f d\phi(\mu) + f d\phi(\text{grad} \ln \lambda^{2-n} ) + d\phi(\text{grad} f), \]
or, equivalently,
\[ \tau_f(\phi) = f[-(m - n)d\phi(\mu) + d\phi(\text{grad} \ln(f^{\lambda^2-n}))] = 0. \]

From this we obtain the theorem. \qed

An immediate consequence is the following

**Corollary 2.8.** (a) A horizontally homothetic map (in particular, a Riemannian submersion) \( \phi : (M^m, g) \rightarrow (N^n, h) \) is an \( f \)-harmonic morphism if and only if \( -(m - n)\mu + \text{grad} \ln f \) is vertical;

(b) A weakly conformal map \( \phi : (M^m, g) \rightarrow (N^m, h) \) with conformal factor \( \lambda \) of same dimension spaces is \( f \)-harmonic and hence an \( f \)-harmonic morphism if and only if \( f = C\lambda^{m-2} \) for some constant \( C > 0 \);

(c) An horizontally weakly conformal map \( \phi : (M^m, g) \rightarrow (N^2, h) \) is an \( f \)-harmonic map and hence an \( f \)-harmonic morphism if and only if \( f = C\lambda^{m-2} \) for some constant \( C > 0 \);

Using the characterizations of \( f \)-harmonic morphisms and \( p \)-harmonic morphisms and Corollary 1.1 we have the following corollary which provides many examples of \( f \)-harmonic morphisms.

**Corollary 2.9.** A map \( \phi : (M^m, g) \rightarrow (N^n, h) \) between Riemannian manifolds is a \( p \)-harmonic morphism without critical points if and only if it is an \( f \)-harmonic morphism with \( f = |d\phi|^{p-2} \).

**Example 3.** Möbius transformation \( \phi : \mathbb{R}^m \setminus \{0\} \rightarrow \mathbb{R}^m \setminus \{0\} \) defined by
\[ \phi(x) = a + \frac{r^2}{|x-a|^2}(x-a) \]
is an \( f \)-harmonic morphism with \( f(x) = C\left(\frac{r}{|x-a|}\right)^{2(m-2)} \). In fact, it is well known that the Möbius transformation is a conformal map between the same dimensional spaces with the dilation \( \lambda = \frac{r^2}{|x-a|^2} \). It follows from [20] that \( \phi \) is an \( m \)-harmonic morphism, and hence by Corollary 2.3 the inversion is an \( f \)-harmonic morphism with \( f = |d\phi|^{m-2} = (\sqrt{m\lambda})^{m-2} = C\left(\frac{r}{|x-a|}\right)^{2(m-2)} \).

The next example is an \( f \)-harmonic morphism that does not come from a \( p \)-harmonic morphism.

**Example 4.** The map from Euclidean 3-space into hyperbolic plane \( \phi : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+, ds_0^2) \rightarrow H^2 \equiv (\mathbb{R} \times \{0\} \times \mathbb{R}^+, \frac{1}{z^2}ds_0^2) \) with \( \phi(x,y,z) = (x,0,\sqrt{y^2+z^2}) \) is an \( f \)-harmonic morphism with \( f = 1/z \). Similarly, we know from [12] that the map \( \phi : H^3 \equiv (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+, \frac{1}{z^2}ds_0^2) \rightarrow H^2 \equiv (\mathbb{R} \times \{0\} \times \mathbb{R}^+, \frac{1}{z^2}ds_0^2) \) with \( \phi(x,y,z) = (x,0,\sqrt{y^2+z^2}) \) is a harmonic morphism. It follows from Example 2...
that the map from Euclidean space into hyperbolic plane $\phi : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+, ds_0^2) \rightarrow H^2 \equiv (\mathbb{R} \times \{0\} \times \mathbb{R}^+, \frac{1}{2} ds_0^2)$ with $\phi(x, y, z) = (x, 0, \sqrt{y^2 + z^2})$ is an $f$-harmonic map with $f = 1/z$. Since this map is also horizontally weakly conformal it is an $f$-harmonic morphism by Theorem 2.2.

**Example 5.** Any harmonic morphism $\phi : (M^n, g) \rightarrow (N^n, h)$ is an $f$-harmonic morphism for a positive function $f$ on $M$ with vertical gradient, i.e., $d\phi(\text{grad} f) = 0$. In particular, the radial projection $\phi : \mathbb{R}^{m+1} \setminus \{0\} \rightarrow S^m$, $\phi(x) = \frac{x}{|x|}$ is an $f$-harmonic morphism for $f = \alpha(|x|)$, where $\alpha : (0, \infty) \rightarrow (0, \infty)$ is any smooth function. In fact, we know from [4] that the radial projection is a harmonic morphisms and on the other hand, one can check that the function $f = \alpha(|x|)$ is positive and has vertical gradient.

Using the property of $f$-harmonic morphisms and Sacks-Uhlenberg’s well-known result on the existence of harmonic 2-spheres we have the following proposition which gives many examples of $f$-harmonic maps from Euclidean 3-space into a manifold whose universal covering space is not contractible.

**Proposition 2.10.** For any Riemannian manifold whose universal covering space is not contractible, there exists an $f$-harmonic map $\phi : (\mathbb{R}^3, ds_0^2) \rightarrow (N^n, h)$ from Euclidean 3-space with $f(x) = \frac{2}{1+|x|^2}$.

**Proof.** Let $ds_0^2$ denote the Euclidean metric on $\mathbb{R}^3$. It is well known that we can use the inverse of the stereographic projection to identify $(\mathbb{R}^3, \frac{4ds_0^2}{(1+|x|^2)^2})$ with $S^3 \setminus \{N\} = \{(u_1, u_2, u_3, u_4) | \sum_{i=1}^{4} u_i^2 = 1, u_4 \neq 1\}$, the Euclidean 3-sphere minus the north pole. In fact, the identification is given by the isometry

$$\sigma : \left(\mathbb{R}^3, \frac{4ds_0^2}{(1+|x|^2)^2}\right) \rightarrow S^3 \setminus \{N\} \subseteq \mathbb{R}^4$$

with $\sigma(x_1, x_2, x_3) = (\frac{2x_1}{1+|x|^2}, \frac{2x_2}{1+|x|^2}, \frac{2x_3}{1+|x|^2}, \frac{|x|^2-1}{1+|x|^2})$. One can check that under this identification, the Hopf fibration $\phi : \left(\mathbb{R}^3, \frac{4ds_0^2}{(1+|x|^2)^2}\right) \cong S^3 \setminus \{N\} \rightarrow S^2$ can be written as

$$\phi(x_1, x_2, x_3) = (|z|^2 - |w|^2, 2zw),$$

where $z = \frac{2x_1}{1+|x|^2} + i\frac{2x_2}{1+|x|^2}, \ w = \frac{2x_3}{1+|x|^2} + i\frac{|x|^2-1}{1+|x|^2}$. It is well known (see, e.g., [4]) that the Hopf fibration $\phi$ is a harmonic morphism with dilation $\lambda = 2$. So, by Corollary [1.1] $\phi : (\mathbb{R}^3, ds_0^2) \rightarrow S^2$ is an $f$-harmonic map with $f = \frac{2}{(1+|x|^2)^2}$. It is easy to see that this map is also horizontally conformal submersion and hence, by Theorem 2.2 it is an $f$-harmonic morphism. On the other hand, by a well-known result of Sacks-Uhlenbeck’s, we know that there is a harmonic map $\rho : S^2 \rightarrow$
(N^n, h) from 2-sphere into a manifold whose covering space is not contractible. It follows from Proposition 2.6 that the composition \( \rho \circ \phi : (\mathbb{R}^3, ds_0^2) \rightarrow (N^n, h) \) is an \( f \)-harmonic map with \( f = \frac{2}{1 + |x|^2} \). □

Remark 1. We notice that the authors in [8] and [14] used heat flow method to study the existence of \( f \)-harmonic maps from closed unit disk \( D^2 \rightarrow S^2 \) sending boundary to a single point. The \( f \)-harmonic morphism \( \phi : (\mathbb{R}^3, ds_0^2) \rightarrow S^2 \) in Proposition 2.10 clearly restrict to an \( f \)-harmonic map \( \phi : (D^3, ds_0^2) \rightarrow S^2 \) from 3-dimensional open disk into \( S^2 \). It would be interesting to know if there is any \( f \)-harmonic map from higher dimensional closed disk into two sphere. Though we know that \( \phi : (M^m, g) \rightarrow (N^n, h) \) being \( f \)-harmonic implies \( \phi : (M^m, f^m=2-g) \rightarrow (N^n, h) \) being harmonic we need to be careful trying to use results from harmonic maps theory since a conformal change of metric may change the curvature and the completeness of the original manifold \( (M^m, g) \).

As we remark in Example 5 that any harmonic morphism is an \( f \)-harmonic morphism provided \( f \) is positive with vertical gradient, however, such a function need not always exist as the following proposition shows.

Proposition 2.11. A Riemannian submersion \( \phi : (M^m, g) \rightarrow (N^n, h) \) from non-negatively curved compact manifold with minimal fibers is an \( f \)-harmonic morphism if and only if \( f = C > 0 \). In particular, there exists no nonconstant positive function on \( S^{2n+1} \) so that the Hopf fibration \( \phi : S^{2n+1} \rightarrow (N^n, h) \) is an \( f \)-harmonic morphism.

Proof. By Corollary 2.8 a Riemannian submersion \( \phi : (M^m, g) \rightarrow (N^n, h) \) with minimal fibers is an \( f \)-harmonic morphism if and only if \( \text{grad} \ln f \) is vertical, i.e., \( d\phi(\text{grad} \ln f) = 0 \). This, together with the following lemma will complete the proof of the proposition.

**Lemma:** Let \( \phi : (M^m, g) \rightarrow (N^n, h) \) be any Riemannian submersion of a compact positively curved manifold \( M \). Then, there exists no (nonconstant) function \( f : M \rightarrow \mathbb{R} \) such that \( d\phi(\text{grad} \ln f) = 0 \).

**Proof of the Lemma:** Suppose \( f : (M^m, g) \rightarrow \mathbb{R} \) has vertical gradient. Consider

\[ (M, e^{\varepsilon f} g) \]

where \( \varepsilon > 0 \) is a sufficiently small constant.

If \( \varepsilon \) is small enough, then \( e^{2\varepsilon f} g \) is positively curved. One can check that

\[ \phi : (M, e^{2\varepsilon f} g) \rightarrow (N, h) \]
is a horizontally homothetic submersion with dilation $\lambda^2 = e^{-2f}$ since $f$ has vertical gradient. By the main theorem in [24] we conclude that the map $\phi$ defined in (14) is a Riemannian submersion, which implies that the dilation and hence the function $f$ has to be a constant. \[\Box\]

Remark 2. It would be very interesting to know if there exists any $f$-harmonic morphism (or $f$-harmonic map) $\phi : S^{2n+1} \to (N^n, h)$ with non-constant $f$. Note that for the case of $n = 2$, the problem of classifying all $f$-harmonic morphisms $\phi : (S^3, g_0) \to (N^2, h)$ (where $g_0$ denotes the standard Euclidean metric on the 3-sphere) amounts to classifying all harmonic morphisms $\phi : (S^3, f^2g_0) \to (N^2, h)$ from conformally flat 3-spheres. A partial result on the latter problem was given in [13] in which the author proves that a submersive harmonic morphism $\phi : (S^3, f^2g_0) \to (N^2, h)$ with non-vanishing horizontal curvature is the Hopf fibration up to an isometry of $(S^3, g_0)$. This implies that there exists no submersive $f$-harmonic morphism $\phi : (S^3, g_0) \to (N^2, h)$ with non-constant $f$ and the horizontal curvature $K_H(f^2g_0) \neq 0$.

Proposition 2.12. For $m > n \geq 2$, a polynomial map (i.e. a map whose component functions are polynomials) $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is an $f$-harmonic morphism if and only if $\phi$ is a harmonic morphism and $f$ has vertical gradient.

Proof. Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial map (i.e. a map whose component functions are polynomials). If $\phi$ is an $f$-harmonic morphism, then, by Theorem 2.2, it is a horizontally weakly conformal $f$-harmonic map. It was proved in [1] that any horizontally weakly conformal polynomial map between Euclidean spaces has to be harmonic. This implies that $\phi$ is also a harmonic morphism, and in this case we have $d\phi(\text{grad } f) = 0$ from (1). \[\Box\]

Example 6. $\phi : \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ with $\phi(t, z) = p(z)$, where $p(z)$ is any polynomial function in $z$, is an $f$-harmonic morphism with $f(t, z) = \alpha(t)$ for any positive smooth function $\alpha$.

Finally, we would like to point out that our notion of $f$-harmonic morphisms should not be confused with $h$-harmonic morphisms studied in [11] and [3], where an $h$-harmonic function is defined to be a solution of $\Delta u + 2g(\text{grad } \ln h), \text{grad}(u)) = 0$ (or equivalently, $h\Delta u + 2du(\text{grad } h) = 0$), and an $h$-harmonic morphism is a continuous map between Riemannian manifolds which pulls back local harmonic functions to $h$-harmonic functions.

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References

[1] R. Ababou, P. Baird, and J. Brossard, *Polynômes semi-conformes et morphismes harmoniques* Math. Z. 231 (1999), no. 3, 589–604.
[2] M. Ara, *Geometry of F-harmonic maps* Kodai Math. J. 22 (1999), no. 2, 243–263.
[3] P. Baird and S. Gudmundsson, *p-Harmonic maps and minimal submanifolds*, Math. Ann., 294 (1992), 611-624.
[4] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monogr. (N.S.) No. 29, Oxford Univ. Press (2003).
[5] N. Course, *f-harmonic maps*, Thesis, University of Warwick, Coventry, CV4 7AL, UK, 2004.
[6] J. Cieśliński, A. Sym, and W. Wessels, *On the geometry of the inhomogeneous Heisenberg ferromagnet: non-integrable case*, J. Phys. A. Math. Gen. 26 (1993) 1353-1364.
[7] J. Cieśliński, P. Goldstein, and A. Sym, *On integrability of the inhomogeneous Heisenberg ferromagnet model: Examination of a new test*, J. Phys. A: Math. Gen., 1994, 27: 1645-1664.
[8] N. Course, *f-harmonic maps which map the boundary of the domain to one point in the target*, New York J. Math. 13 (2007), 423-435.
[9] M. Daniel, K. Porsezian, and M. Lakshmanan, *On the integrability of the inhomogeneous spherically symmetric Heisenberg ferromagnet in arbitrary dimension*, J. Math Phys, 1994, 35(10): 64986510.
[10] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 10 (1978), 1-68.
[11] B. Fuglede, *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier (Grenoble), vol 28 (1978), 107-144.
[12] S. Gudmundsson, *The geometry of harmonic morphisms*, Ph. D. Thesis, University of Leeds, UK, 1992.
[13] S. Heller, *Harmonic morphisms on conformally flat 3-spheres*, Bull. London Math. Soc., doi:10.1112/blms/bdq089, to appear.
[14] P. Huang and H. Tang, *On the heat flow of f-harmonic maps from D^2 into S^2*, Nonlinear Anal. 67 (2007), no. 7, 21492156.
[15] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ., 19(2) (1979), 215-229.
[16] M. Lakshmanan and R. K. Bullough, *Geometry of generalised nonlinear Schrödinger and Heisenberg ferromagnetic spin equations with x-dependent coefficients*, Phys. Lett. A, 1980, 80(4): 287292.
[17] Y. X. Li and Y. D. Wang, *Bubbling location for f-harmonic maps and inhomogeneous Landau-Lifshitz equations*, Comment. Math. Helv. 81 (2) (2006) 433448.
[18] A. Lichnerowicz, *Applications harmoniques et variétés kählériennes*, Symposia Mathematica III, Academic Press, London, 1970, pp. 341–402.
[19] E. Loubeau, *On p-harmonic morphisms*, Diff. Geom. and its appl., 12(2000), 219-229.
[20] J. Manfredi and V. Vespri, *n-harmonic morphisms in space are Möbius transformations*, Michigan Math. J. 41 (1994), 135142.
[21] Y.-L. Ou, *p-Harmonic morphisms, minimal foliations, and rigidity of metrics*, J. Geom. Phys. 52 (4) (2004) 365381.
[22] Y.-L. Ou, *On p-harmonic morphisms and conformally flat Spaces*, Math. Proc. Camb. Phil. Soc. (2005), 139-317.

[23] Y.-L. Ou, *p-Harmonic morphisms, biharmonic morphisms, and nonharmonic biharmonic maps*, J. of Geo. Phy, Vol. 56, No. 3, 2006, 358-374.

[24] Y.-L. Ou and F. Wilhelm, *Horizontally homothetic submersions and nonnegative curvature*. Indiana Univ. Math. J. 56 (2007), no. 1, 243261.

[25] S. Ouakkas, R. Nasri, and M. Djaa, *On the f-harmonic and f-biharmonic maps*, JP J. Geom. Topol. 10 (1), (2010), 11-27.

[26] H. Takeuchi, *Some conformal properties of p-harmonic maps and regularity for sphere-valued p-harmonic maps*, J. Math. Soc. Japan, 46(1994), 217-234.

School of Mathematics and Computer Science, Guangxi University for Nationalities, 188 East Daxue Road, Nanning, Guangxi 530006, P. R. China E-mail: yelinou@hotmail.com