On Stability of Nodal $L^p_k$-Maps I

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1 Introduction

In this paper and its sequels, we introduce the concept of weak stability for general nodal $L^p_k$-maps as a natural generalization of stability for $J$-holomorphic maps; then give a complete characterization of the weakly stable nodal $L^p_k$-maps in term of their isotropy groups. Among weakly stable $L^p_k$-maps, the stable ones are those whose isotropies are finite. As a consequence, we prove the Hausdorffness, without any stability assumption, of the space of unparametrized nodal $L^p_k$-maps modeled on a fixed tree.

We will only deal with the genus zero case. This is justified since for a nodal $L^p_k$-map, the part of its reparametrization group with positive dimension consists of the reparametrizations of its unstable genus zero components. In this paper, we only consider the nodal $L^p_k$-maps modeled on a fixed tree $T$. The general cases allowing the changes of the topological types of the domains and targets will be treated in the sequels of this paper. We now describe the main results of this paper.

Fix a (minimal) label $L$ of a given tree $T$ such that the labeled tree $\hat{T} = (T, L)$ is stable. Let $\mathcal{M}_{\hat{T}}$ be the moduli space of genus zero stable curves with $n$ marked points modeled on $\hat{T}$ and $\mathcal{U}_{\hat{T}} \rightarrow \mathcal{M}_{\hat{T}}$ be the universal curve over $\mathcal{M}_{\hat{T}}$. Here $n$ is the minimal number of marked points added to a genus zero nodal surface modeled on $T$ to make it stable. Fix a Riemannian manifold $M$. Let $\tilde{\mathcal{B}}_{\hat{T}} = \tilde{\mathcal{B}}_{\hat{T}}^{T,k,p}$ be the set of $L^p_k$-maps $f : \Sigma \rightarrow M$ with $\Sigma$ being
one of the fibers of the universal family $\mathcal{U}_T \to \mathcal{M}_T$. One can show that $\tilde{\mathcal{B}}^{\hat{T}}_{k,p}$ is a Banach manifold of class $[m_0]$ (see Section 2). Here $m_0 = k - 2/p$ is the Sobolev differentiability of $f$. Through this paper, we will always assume that $[m_0] \geq 1$ so that each component of $f$ is at least of class $C^1$. Roughly speaking, the space $\tilde{\mathcal{B}}^{\hat{T}}$ can be thought as the space of parametrized nodal $L^p_k$-maps modeled on $T$. Let $\mathcal{B}^{\hat{T}}$ be the space of equivalence classes of nodal $L^p_k$-maps. It can be obtained from $\tilde{\mathcal{B}}^{\hat{T}}$ as orbit space under the actions of the reparametrization groups $G_f$ of the elements $f$ in $\tilde{\mathcal{B}}^{\hat{T}}$. The normal subgroup of $G_f$ preserving the components of $f$ is independent of $f$ in the sense that they can be identified each other canonically. Denote the resulting group by $G_T$. Note that the quotient group $G_{f/T} = G_f/G_T$ is a finite group that exchanges the components of $f$. Thus $\mathcal{B}^{\hat{T}}$ can be obtained by first forming the global quotient $\tilde{\mathcal{B}}^{\hat{T}}/G_T$, then quotient out a further locally finite equivalence relation by the actions of $G_{f/T}$. Note that the action of $G_T$ is continuous (see Sec. 2).

**Theorem 1.1** The space $\tilde{\mathcal{B}}^{\hat{T}}$ is $G_T$-Hausdorff in the sense that for any two different $G_T$-orbits $G_Tf_1$ and $G_Tf_2$, there exist $G_T$-neighborhoods $G_TU_1$ and $G_TU_2$ such that $G_TU_1 \cap G_TU_2 = \varnothing$. Therefore, the global quotient $\tilde{\mathcal{B}}^{\hat{T}}/G_T$ is Hausdorff.

The proof of the above theorem implies the following

**Proposition 1.1** The space of unparametrized nodal $L^p_k$-maps, $\mathcal{B}^{\hat{T}}$ is always Hausdorff.

Note that in general a nodal map $f$ may have components, such as trivial unstable component, so that the isotropy group $\Gamma_f$ is not compact. In this case, the action of $G_T$ on $\tilde{\mathcal{B}}^{\hat{T}}$ is certainly not proper so that $f$ is not stable in any reasonable sense. Yet, the above results show that when the topological type of the domains is fixed given by $T$, the Hausdorffness of $\tilde{\mathcal{B}}^{\hat{T}}/G_T$ and $\mathcal{B}^{\hat{T}}$ still holds without requiring any stability conditions. This seems to contradict to our experience in Gromov-Witten theory. Indeed, when the topological type of the domains is allowed to change, the corresponding space $\mathcal{B}$ of unparametrized nodal $L^p_k$-maps is not Hausdorff anymore. But this non-Hausdorffness occurs in a rather definite manner mainly caused by the appearance of the extra trivial bubbles obtained by a non-convergence
sequence of reparametrizations. Once such degenerations are prohibited, the Hausdorffness will be restored even allowing the change of the topological types of the domains. In other words, for any subspace of $B$ which do not contain a sequence convergent to a nodal map with extra trivial bubbles, the Hausdorffness still holds. The detail of this will be given in a sequel of this paper.

Next we define the weak stability.

**Definition 1.1** A nodal $L^p_k$-map is said to be weakly stable if none of its unstable component is a trivial map.

Let $\tilde{B}^{ws}_T$ be the subspace of $\tilde{B}^T$ consisting of weakly stable nodal $L^p_k$-maps. As before, $\tilde{B}^{ws}_T$ can be thought as the space of parametrized weakly stable nodal maps modeled on $T$. Let $B^{ws}_T$ be the corresponding space of unparametrized weakly stable nodal maps.

**Note:** Applying the above definition to $J$-holomorphic nodal maps, we get one of the standard definitions for stable $J$-holomorphic maps in Gromov-Witten theory. In this sense the weak stability for $L^p_k$-maps here is a natural generalization of the stability for $J$-holomorphic maps. However, as we will see below, in general the isotropy group $\Gamma_f$ for a weakly stable $L^p_k$-map $f$ is not finite but only compact.

**Theorem 1.2** The action of $G_T$ on $\tilde{B}^{ws}_T$ is proper in the sense that for any $f_1$ and $f_2$ in $\tilde{B}^{ws}_T$, there exist the corresponding open neighborhoods $U_1$ and $U_2$ containing $f_1$ and $f_2$ respectively and compact subsets $K_1$ and $K_2$ in $G_T$ such that for any $h_1$ in $U_1$ ($h_2$ in $U_2$) and $g_1$ in $G_T \setminus K_1$ ($g_2$ in $G_T \setminus K_2$), $g_1 \cdot h_1$ is not in $U_2$ ($g_2 \cdot h_2$ is not in $U_1$).

**Note:** It was explained in [L1] that for the finite dimensional case, the definition of properness above is equivalent to the usual definition.

**Corollary 1.1** For any weakly stable nodal map $f$, the isotropy $\Gamma_f$ of the $G_T$-action or $G_f$-action is always compact.

In fact the same is true for the isotropy groups of the non-trivial components of a nodal $L^p_k$-map.

**Note:** It follows from the continuity of the $G_T$-action that $\Gamma_f$ is closed in $G_T$ so that it is a compact Lie subgroup of $G_T$. 

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Corollary 1.2 A nodal $L^p_k$ map $f$ is weakly stable if and only if the isotropy $\Gamma_f$ is compact.

This gives a complete characterization of weak stability of nodal $L^p_k$-maps in term of their isotropy groups.

Definition 1.2 A weakly stable nodal $L^p_k$-map $f$ is said to be stable if it has no infinitesimal automorphisms. In other words, the dimension of the Lie algebra $\text{Lie}(\Gamma_f)$ is equal to zero.

Since $\Gamma_f$ is compact for a weakly stable nodal map, the above condition is equivalent to the following definition.

Definition 1.3 A weakly stable nodal $L^p_k$-map $f$ is said to be stable if its isotropy group $\Gamma_f$ is a finite group.

Using the above characterization for weakly stable nodal $L^p_k$-map, we get the following corollary.

Corollary 1.3 A nodal $L^p_k$-map $f$ is stable if and only if the isotropy $\Gamma_f$ is finite.

Recall that the corresponding characterization for stable $J$-holomorphic maps is the following well-known proposition in GW theory.

Proposition 1.2 A $J$-holomorphic nodal map $f$ is stable if and only if the isotropy $\Gamma_f$ is finite.

Thus stable $L^p_k$-maps and stable $J$-holomorphic maps have exactly the same characterizations in term of their isotropies. One the other hand, we have seen that in term of the definitions, weakly stable $L^p_k$-maps are the natural generalizations of stable $J$-holomorphic maps.

Let $\tilde{B}_T^s$ be the set of stable nodal $L^p_k$-maps and $B_T^s$ be the corresponding quotient space of unparametrized stable nodal maps. Next theorem follows from the definition.
Theorem 1.3 The actions of $G_T$ and $G_f$ on $\tilde{\mathcal{B}}^s_T$ are proper with finite isotropies so that $\mathcal{B}^s_T$ is a topological Banach orbifold. Moreover, the action of $G_T$ has local slices in $\tilde{\mathcal{B}}^s_T$ that provide local uniformizers for the orbifold structure of $\mathcal{B}^s_T$. In particular, the subspace of $\mathcal{B}^s_T$ consisting of stable nodal maps with the trivial isotropy is a topological Banach manifold.

To get a better understanding of the stability, we divide the unstable components of weakly stable maps into the following two classes. Let $f_v : \Sigma_v \simeq \mathbb{CP}^1 \to M$ be such a component. Then $f_v$ is said to be 2-dimensional if there is a point $x_0$ in $\Sigma_v$ such that the rank of $(df_v)_x$ is equal to two, otherwise it is one-dimensional.

It follows from the definition that

Lemma 1.1 If all unstable components of a weakly stable map are 2-dimensional, it is stable.

Thus we only need to consider a nontrivial unstable component $f_v : \Sigma_v \simeq \mathbb{CP}^1 \to M$ of $f$ that is one-dimensional. The reparametrization group of $f_v$ is the group $G_{\Sigma_v}$ considered as subgroup of $\text{PSL}(2, \mathbb{C})$ that fixes the double points of $\Sigma_v \simeq \mathbb{CP}^1$ (if there are any). Note that the group $G_T$ is the product of these reparametrization groups $G_{\Sigma_v}$ of the unstable components. Denote the isotropy group inside $G_T$ by $\Gamma^T_f$. Then $\Gamma^T_f$ is the product of the corresponding isotropy groups $\Gamma^T_{f_v} \subset G_{\Sigma_v}^T$ of the unstable components. Then if the dimension of $\Gamma_f$ is positive, so is $\Gamma^T_f$. This implies that the same is true for some $\Gamma^T_{f_v}$ with $f_v$ non-trivial.

By our assumption $\Gamma^T_f$ and $\Gamma^T_{f_v}$ are compact. Since $f_v$ is nontrivial, the action of $\Gamma^T_{f_v}$ on $\Sigma_v$ can not be transitive (even for the case that $f : S^2 \to M$) so that the compact subgroup $SU(2)$ of $\text{PSL}(2, \mathbb{C})$ is not contained in $\Gamma^T_{f_v}$. Since $f_v$ is one-dimensional, the compactness of $\Gamma^T_{f_v}$ then implies that its identity component $(\Gamma^T_{f_v})^0$ is the standard $S^1 \simeq SO(2) \subset G_{\Sigma_v}$ up to conjugations. Let $\mu : \mathbb{CP}^1 \to \mathfrak{so}(2)$ be the moment map for the standard action of $S^1$ on $\mathbb{CP}^1$. Then we have proved the following lemma.

Lemma 1.2 If $f_v : \Sigma_v \simeq \mathbb{CP}^1 \to M$ is an one-dimensional unstable component of a weakly stable map of class $C^1$, then upto a conjugation by the action of $G_{\Sigma_v}$, $f_v = \tilde{f}_v \circ \mu$. Here $\tilde{f}_v : I = [-1, 1] \to M$ is a $C^1$-map, and $\mu : \Sigma_v \simeq \mathbb{CP}^1 \to I \subset \mathbb{R}^1 \simeq \mathfrak{so}(2)$ is the “height” function above.
Any one-dimensional map with the form in the above lemma (upto the action of reparametrizations) will be called a standard $S^1$-invariant map.

**Lemma 1.3** If none of the unstable component of a weakly stable map is standard $S^1$-invariant, then it is stable.

A one-dimensional non-trivial map $f_v : \Sigma_v \simeq \mathbb{CP}^1 \to M$ is said to be standard if $f_v = \tilde{f}_v \circ \pi$. Here $\tilde{f}_v : I = [-1, 1] \to M$ and $\pi : \Sigma_v \simeq \mathbb{CP}^1 \to I \subset \mathbb{R}^1$ are two $C^1$ maps. Consider the special case that $\pi : \Sigma_v \simeq \mathbb{CP}^1 \to \mathbb{R}^1$ is a $C^1$ map with only two critical points. Clearly any such $\pi$ is conjugate to the moment map $\mu$ by a diffeomorphism of $\Sigma_v$.

**Lemma 1.4** Even within the space of the standard one-dimensional $C^1$ maps with two critical points, the generic elements are not standard $S^1$-invariant maps.

Thus we have large supplies of stable maps even all the unstable components are standard one-dimensional maps with two critical points.

The next proposition summaries the discussion so far on the continuous part of the isotropy group of a nodal map.

**Proposition 1.3** The identity component $\Gamma_f^0$ of the isotropy group of the nontrivial components of a nodal $L^p_k$-map $f$ is a torus $T^n = (S^1)^n$. In particular, $\Gamma_f^0 \simeq T^n$ for a weakly stable $L^p_k$-map $f$.

**Note:** In the finite dimensional case, the slice theorem for a Lie group $G$ acting smoothly and properly on a manifold $M$ states that for any $m \in M$, there is a diffeomorphism from the disc bundle $G \times_{\Gamma_m} D$ onto a neighborhood of the orbit $G \cdot m$ in $M$. Here $\Gamma_m$ is the (compact) isotropy group of $m$. The properness and the Sc smoothness of the $G_T$-action on $\tilde{B}_f^{\text{ws}}$ implies that the same statement holds in the Sc setting. The detailed proof of this will be given some where else.

In the formulation of the nodal maps, we have suppressed the information about the homology classes $[f_v] \in H_2(M, \mathbb{Z})$ represented by the components $f_v$. Let $A_T = \{ A_v \in H_2(M, \mathbb{Z}), v \in T \}$ be a collection of second homology classes associated to $T$. Denote the space of nodal $L^p_k$-maps modeled on $\hat{T}$ that represent class $A_T$ by $\tilde{B}_{A_T}^\text{ws}$.
Proposition 1.4 Assume that for all unstable component, the corresponding $A_v \neq 0$. Then all elements of $\widetilde{\mathcal{B}}^T_{A_T}$ are stable. Consequently the actions of $G_T$ and $G_f$ are proper with finite isotropies, and the quotient space $\mathcal{B}^T_{A_T}$ of the unparametrized nodal maps of class $A_T$ is a topological Banach orbifold.

This paper is organized as follows.

Section 2 provides the preliminaries used to define the space $\widetilde{\mathcal{B}}^T_{\hat{T}}$ of $L^p_k$-maps modeled on a labeled tree $\hat{T}$.

Section 3 proves the main theorems here as well as some related results. The results here generalize the corresponding ones in [L1] for $L^p_k$-maps with domain $S^2$.

Section 4 gives an elementary discussion on the classification of the group $S_T$ of the automorphisms of a tree $T$. A comparison of $S_T$ with the discrete part of the reparametrization group of a nodal surface of type $T$ is made in this section, which gives the constraints on the discrete part of the reparametrization group.

2 Preliminaries

2.1 Nodal curves, stable curves modeled on $\hat{T}$

Recall that a tree $T$ is a connected 1-dimensional (abstract) simplicial complex without cycles. We still use $T$ to denote the set of its vertices. For the vertices $v$ and $u$ in $T$, the edge relation will be denoted by $uEv$.

Given two trees $T_1$ and $T_2$, a map $\phi : T_1 \to T_2$ is said to a pre-morphism if the following condition (1) holds: for any vertices $u$ and $v$ in $T_1$ with $uEv$, either $\phi(u) = \phi(v)$ or $\phi(u)E\phi(v)$. A pre-morphism $\phi : T_1 \to T_2$ is said to be a morphism if in addition the following condition (2) holds: for any $u_2$ in $T_2$ the inverse image $\phi^{-1}(u_2)$ is a subtree of $T_1$ (or empty).

To see the meaning of the condition (2), consider the following simple example.

Example: Let $T_1 = \{v_1, v_2, v_3\}$ with $v_1E v_2, v_2E v_3$ be the chain connecting $v_1$ and $v_3$, and $T_2 = \{u_1, u_2\}$ with $u_1E u_2$. Let $\phi : T_1 \to T_2$ sending $\phi(v_1) = \phi(v_3) = u_1$ and $\phi(v_2) = u_2$. Then $\phi$ satisfies (1) (hence is a pre-morphism) but not (2).

Note that the two sides adjacent to $v_2$ is flipped and identified by $\phi$. In general, a pre-morphism $\phi : T_1 \to T_2$ is said to have flipped identifications if
there is a subchain of length two in $T_1$ that is gotten flipped and identified into a single edge at its middle vertex under $\phi$.

**Lemma 2.1** A pre-morphism $\phi : T_1 \to T_2$ is a morphism if and only if it does not contain any flipped identifications.

**Proof:**

Let $C_1$ and $C_2$ be two connected components of $\phi^{-1}(u)$ in $T_1$. Chose a shortest chain $C(v_1, v_2)$ from $v_1 \in C_1$ and $v_2 \in C_2$ connecting the two components. Since $\phi(v_1) = \phi(v_2) = u$, either the image $\phi(C(v_1, v_2))$ is a cycle that is impossible or the map $\phi : C(v_1, v_2) \to T_2$ contains a flipped identifications.

The above condition (2) is justified in Gromov-Witten theory since for the gluing construction at a double, the induced map on corresponding trees does not contain flipped identifications so that it is a morphism.

More specifically, given a edge $[vu]$ at $v \in T$, let $T_{v;u}$ be the tree obtained by contracting the edge into the vertex $v$. The corresponding map denoted by $\phi_{v;u} : T \to T_{v;u}$ will be called a contracting map (of $[vu]$) at $v$. It is a surjective morphism.

Note that such maps generate all surjective morphism from a tree $T$ in the sense that any surjective morphism $\phi : T \to T_1$ can be factorized as $\phi = \rho \circ \phi'$. Here $\rho : T \to T'$ is composition of a sequence of above basic edge-contracting maps, and $\phi' : T \to T$ is an isomorphism.

The full meaning of the condition (2) will become clear in the lemma of this section on the compatibility of the total orders of the two n-labeled stable trees with their underlying trees being related by a morphism defined above.

As usual, the notion of morphism gives rise the notion of isomorphisms between trees. An automorphism of $T$ is a self isomorphism.

A vertex $v$ is said to unstable if its valence $Val(v) \leq 2$. An unstable vertex $v$ is said to be a tip of the tree $T$ if $Val(v) = 1$.

**Lemma 2.2** An isomorphism $\phi : T_1 \to T_2$ is determined by its action on the tips of $T_1$.

**Proof:**

Argue by induction on the numbers of edges. The starting point is the tree with only one edge and two vertices.
Note that in this case, the condition (2) is automatically true. So we start with a one-to-one map \( \phi : T_1 \to T_2 \) between the vertex sets that satisfies condition (1). Let \( v_1 \in T_1 \) be a tip of \( T_1 \) and \( u_1 \in T_1 \) is the vertex next to \( v_1 \). Denote their images in \( T_2 \) under \( \phi \) by \( v_2 \) and \( u_2 \). Let \( T'_1 = T_1 \setminus \{v_1u_1\} \) and consider \( \phi' : T'_1 \to T'_2 = \phi(T'_1) \). By induction \( \phi' \) is determined by its values on the tips of \( T'_1 \).

If \( Val(u_1) > 2 \), tips of \( T'_1 \) are also the tips of \( T_1 \) so that \( \phi \) is determined by its values on the tips.

If \( Val(u_1) = 2 \), the \( u_1 \) becomes a tip in \( T'_1 \), but the value \( \phi'(u_1) \) is already determined by \( \phi : [v_1u_1] \to [v_2u_2] \). Hence in this case by induction \( \phi' \) is determined by the values of \( \phi \) on the tips of \( T_1 \).

□

**Lemma 2.3** Let \( \phi_{v;u} : T \to T_{v;u} \) be the contracting map of edge \([vu]\) at \( v \). Then \( \phi_{v;u} \) is determined by its values on the tips.

**Proof:**

Choosing a tip \( v_1 \neq v \) and edge \([v_1u_1]\) of \( T \), and consider \( T' = T \setminus [v_1u_1] \) and \( T'_{v;u} = T_{v;u} \setminus [v_1u_1] \). Let \( \phi' : T' \to T'_{v;u} \). By induction \( \phi' \) is determined by its values on the tips of \( T' \). The rest of the argument is the same as the one in the previous lemma.

□

**Corollary 2.1** A morphism \( \phi : T_1 \to T_2 \) is determined by its values on the tips of \( T_1 \).

**Example:** Consider again the example with \( T_1 = \{v_1, v_2, v_3\} \) being the chain \([v_1, v_2, v_3]\) of length two and \( T_2 = \{u_1, u_2\} \) with the edge \([u_1u_2]\). Let \( \phi : T_1 \to T_2 \) with \( \phi(v_1) = \phi(v_3) = u_1 \), \( \phi(v_2) = u_2 \) and \( \psi(v_i) = u_1, i = 1, 2, 3 \) be two maps. Then both maps have the same values on the tips of \( T_1 \) and satisfies condition (1) in the definition of morphism. This shows that the condition (2) in the definition of morphism is crucial for the above corollary.

An ordered tree \( T_o \) is a pair \((T, O)\) consisting of a tree \( T \) with an order \( O \) for its tips. The above lemma implies

**Lemma 2.4** Any order preserving automorphism of an ordered tree is the identity map.
Lemma 2.5 An order on the tips of $T$ gives rise a total order to all the vertices and edges described in the proof below. Let $\phi_{v:u} : T \to T_{v:u}$ be a contracting map at the edge $[vu]$. In the case that $v$ is the initial tip of $T$, assume that it is still so in $T_{v:u}$. Then the total order of $T$ obtained above naturally induces an order of the tips of $T_{v:u}$ as well as the total order of $T_{v:u}$ such that $\phi_{v:u} : T \to T_{v:u}$ is ”order preserving” in the obvious sense. Consequently, a surjective morphism $\phi : T \to T_1$ carries the total order on $T$ defined above to the corresponding total order on $T_1$ such that the map $\phi$ is order preserving.

Proof:

We definition of the total order induced by $T_0$ first.

Let \{t_0, t_2, \cdots, t_l\} be the ordered set of tips of $T$. Consider the unique directed chain $[t_0, t_1]$. This gives rise an order for all vertices and edges lying on the directed chain. Assume that the union $T^{k-1}$ of the first $k - 1$ such directed connecting chains has totally ordered. Note that intersection of two connecting chain $[t_0t_m]$ and $[t_0t_k]$ is a chain of the form $[t_0v]$ (including $t_0 =: [t_0t_0]$). Hence for a fixed $k$ the intersection $T^{k-1} \cap [t_0t_k]$ is a subchain of $[t_0t_k]$ of the form $[t_0v_k]$. Hence part of the $k$-th directed connecting chains, $[t_0v_k]$ is already totally ordered from that of $T^{k-1}$. Then rest part of $[t_0t_k]$, $(v_k,t_k]$ (with $v_k$ not included ) is totally ordered in the obvious manner so that $T^k$ is totally ordered.

To prove the last part of the lemma, note that either the connecting chain $[t_0v] = [t_0u][uv]$ or $[t_0u] = [t_0v][vu]$. We may assume that $[t_0v] = [t_0u][uv]$. Then there is a smallest $k \geq 1$ such that $[t_0uv]$ is the initial part of the chain $[t_0t_k]$. Then $T^{k-1} = T_{v:u}^{k-1}$ so that they have the same total order. If $v \neq t_k$, then $t_k \in T_{v:u}^k$, so that the set of tips of $T^k$ and $T_{v:u}^k$ is the same. It is easy to see that in this case there is a compatible total order for $T_{v:u}^k$ which extends to a total order on $T_{v:u}$ compatible with $\phi_{v:u}$.

The remaining case is that $v = t_k$. In this case, $t_k = v$ is not a tip of $T_{v:u}$ anymore. It is replaced by $t_k' = u$ with the connecting chain $[t_0t_k']$ in $T_{v:u}$. Then total order of $T^k$ gives the compatible total order $T_{v:u}^k$ in this case. As before, the extension of the total order from $T^k$ to $T$ gives the induced compatible extension of the total order from $T_{v:u}^k$ to $T_{v:u}$.

Note that in above argument, one can choose any vertex rather than a tip $t_0$ as the initial one.

An $n$-labeled tree $\hat{T} = (T, L)$ consists of a tree $T$ and a label $L$ which is a map $L : n = \{1, 2, \cdots, n\} \to T$. 

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A morphism $\phi : (T_1, L_1) \rightarrow (T_2, L_2)$ between two $n$-labeled trees is defined to be a morphism $\phi : T_1 \rightarrow T_2$ of the underlying trees such that $\phi \circ L_1 = L_2$.

An $n$-labeled tree $\hat{T} = (T, L)$ is said to be stable if for any $v \in T$, $Val(v) + \#(L^{-1}(v)) \geq 3$.

Given an stable $n$-labeled tree $\hat{T} = (T, L)$, on each tip $t$ of $T$, let $i \in n$ be the smallest number in $L^{-1}(t) \subset n$. We label $t$ by $t_i$ so that all the tips of $T$ are ordered. Then above lemma shows that the tree $T$ is totally ordered. This total order can be extended into a total order of $\hat{T}$ including all the subsets $L^{-1}(v), v \in T$ as follows. For each vertex $v$, elements in $L^{-1}(v)$ are considered as ”edges” (loops) connecting $v$ to itself and ordered by the natural order on $L^{-1}(v)$.

Now consider a morphism $\phi : (T_1, L_1) \rightarrow (T_2, L_2)$ between two stable $n$-labeled trees with the underlying morphism $\phi : T_1 \rightarrow T_2$ being surjective. Then the label $L_1$ determines an total order on $T_1$, and hence a compatible total order on $T_2$ as well by above lemma. However this induced total order on $T_2$ map not be the same as the one induced from $L_2$ even $L_2 = \phi \circ L_1$. As an extreme case of this, assume that $T_2$ is a single point. In general induced order on the set $n$ from the total order on $T_1$ determined by $L_1$ may not be the natural order for $n$. On the other hand, the induced order on $n$ by $L_2$ is just the natural order.

A morphism $\phi : (T_1, L_1) \rightarrow (T_2, L_2)$ between two $n$-labeled trees is said to be an equivalence if it is an isomorphism of the underlying trees. An self equivalence of $\hat{T}$ is called an automorphism.

If $\phi$ an automorphism of a stable $n$-labeled tree $\hat{T} = (T, L)$, then for any tip $t, i_t \in L^{-1}(t) \neq \varphi$, and $t = L(i_t) = \phi \circ L(i_t) = \phi(t)$ so that $\phi$ is identity. In other words, there is no nontrivial automorphism for a stable $n$-labeled tree.

On the other hand, there is a weaker notion of morphism between $n$-labeled trees. An unordered morphism $\phi : (T_1, L_1) \rightarrow (T_2, L_2)$ between two $n$-labeled trees is a morphism $\phi : T_1 \rightarrow T_2$ and a permutation $p \in S_n : n_1 \rightarrow n_2$ such that $\phi \circ L_1 = L_2 \circ p$. The unordered equivalence and automorphism can be defined in the same fashion as before.

It is easy to see that the group of unordered automorphisms of $\hat{T}$ is the same as the group of automorphisms of the underlying tree $T$. In other words, for unordered automorphisms of labeled trees, the labeling by $L$ does not impose any further constraints.

Given a tree $T$, a minimal stabilization of $T$ is an $n$-labeled stable tree $\hat{T} = (T, L)$ such that on each unstable vertex $v \in T$, $Val(v) + \#(L^{-1}(v)) = 3$. 

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There are finitely many such minimal stabilizations corresponding to the possible "stable maps" $L$ from $n$ to $T$. Note that for a minimal $n$-stabilization, $n - 3 = \#(E) + \Sigma_{v \in T, \text{stable}}(\#(d_v) - 3)$. Here $\#(E)$ is the number of edges of $T$ which is half of the number of elements in the edge relation $E$ of $T$.

A genus zero nodal curve modeled on $T$ is the pair $\Sigma = (\Sigma, d)$. Here to each $v \in T$ we associate a component $\Sigma_v$ of $\Sigma$, that is a genus zero Riemann surface, and $d = \cup_{v \in T} d_v$ where $d_v = \{d_{uv}, uEv\}$ is the set of double points on $\Sigma_v$. Then $\Sigma$ is the underlying nodal surface obtained by identifying the corresponding double points, $\Sigma = \bigsqcup_{v \in T} \Sigma_v/duv = d_{vu}, uEv$.

- **Note on notations:** The double points $d$ and $\Sigma$ (or $\Sigma =: \Sigma_d$) determine each other. Later on if there is no confusion, we will use any of them to denote the nodal surface. Similar remark is applicable to stable curves defined below.

A genus zero (stable) curve modeled on an $n$-labeled (stable) tree $\hat{T}$ is the tuple $\Sigma = (\Sigma, d, x)$. The double points set $d$ is defined same as above for a nodal curve. The set of marked points, $x = \cup_{v \in T} x_v$, where $x_v = \{x_{vi}, L(i) = v\}$ is the set of marked points on $\Sigma_v$. Let $p_v = d_v \cup x_v$ be set of special points on $\Sigma_v$. The stability condition implies that on each component $\Sigma_v$, $\#(p_v) \geq 3$.

Like the case of the nodal curves, the underlying stable curve, denoted by $\Sigma$ is obtained by identifying the corresponding double points.

A map $\phi : \Sigma_1 \rightarrow \Sigma_2$ is said to be equivalence between two nodal surfaces $\Sigma_1 = (\Sigma_1, d_1)$ and $\Sigma_2 = (\Sigma_2, d_2)$ modeled on $T_1$ and $T_2$ respectively if it is a homeomorphism such that $(1)\phi_v : (\Sigma_1)_v \rightarrow (\Sigma_2)_{\phi(v)}, v \in T_1$ is biholomorphic. It follows that the condition $(2)$ holds: $uE_1v$ if and only if $\phi(u)E_2\phi(v)$ so that the induced map $\phi : T_1 \rightarrow T_2$ is an equivalence, and $\phi((d_1)_{uv}) = (d_2)_{uv}$ for $uE_1v$ and $\phi(u)E_2\phi(v)$.

Such an equivalence can also be defined as a collection of maps $\phi_v, v \in T$ that satisfies condition $(1)$ and $(2)$ above.

Thus by renaming the vertices of $T_2$, we only need to consider equivalence of two nodal surfaces modeled on the same tree $T$.

This leads to the following more restrictive definition for marked nodal surfaces or stable curves.

Given two $n$-marked nodal curves (or stable curves) $\Sigma_1$ and $\Sigma_1$ modeled on $n$-labeled trees $\hat{T}_1$ and $\hat{T}_2$ respectively with the same underlying tree $T$, an equivalence $\phi : \Sigma_1 \rightarrow \Sigma_2$ between the underlying nodal surface is said to be equivalence between the two $n$-marked nodal curves if $\phi(x_1) = x_2$ such that the induced map $\phi : \hat{T}_1 \rightarrow \hat{T}_2$ is an equivalence of the $n$ label trees.
The case relevant to next subsection is that $\hat{T}_1 = \hat{T}_2 = \hat{T}$. In this case, an equivalence between two marked nodal surfaces modeled on $\hat{T}$ induces an automorphism $\phi: \hat{T} \to \hat{T}$. We will show later in this section that for stable curves, any such induced automorphism is trivial.

### 2.2 Moduli space of the genus zero stable curves modeled on $\hat{T}$

The moduli space of the genus zero stable curves modeled on $\hat{T}$, denoted by $\mathcal{M}_{\hat{T}}$, is defined to be the equivalence classes of such curves.

Note that given a genus zero stable curve modeled on an $n$-labeled tree $\hat{T}$, the induced total order by $\hat{T}$ on the special points $p_v$ of $\Sigma$ gives rise a well defined biholomorphic identification $\phi_v: \Sigma_v \to \mathbb{CP}^1_v, v \in T$ by sending the first three special points on $\Sigma_v$ to 0, 1 and $\infty$. Thus $\mathcal{M}_{\hat{T}}$ is the quotient space of $\tilde{\mathcal{M}}_{\hat{T}}$, as an open set of the product of $\mathbb{CP}^1_v$ factors, by the obvious diagonal action of $H_{\hat{T}} = \prod_{v \in T} \text{PSL}(2, \mathbb{C})_v$. Here $|p|$ is the number of spacial points, and $\tilde{\mathcal{M}}_{\hat{T}}$ is the set of special points $p = \{p_{vi}, v \in T\}$ with $p_{vi}$ being special points on $\mathbb{CP}^1_v (= \text{a copy of } \mathbb{CP}^1)$.

It is easy to see that the action of $H_T$ is free and holomorphic. One can show that the action is proper so that $\mathcal{M}_{\hat{T}}$ is a complex manifold. An other way to show that $\mathcal{M}_{\hat{T}}$ is a complex manifold is to use the slice of the $H_T$-action given by a complex submanifold of $\tilde{\mathcal{M}}_{\hat{T}}$ described in next lemma.

In order to describe the universal curve as well as the gluing construction used in the sequels of this paper, we introduce the ”global coordinate” of $\mathcal{M}_{\hat{T}}$ defined by multi-cross ratios selected by the order of $p_{vi}$ given by $\hat{T}$.

As mentioned above, each point in $\tilde{\mathcal{M}}_{\hat{T}}$ can be consider as a tuple of the special points on the components $\mathbb{CP}^1_v, v \in T$. For each $v \in T$, the special points $p_{vi}$ are ordered by $\hat{T}$ described before given by the index $i$. For simplicity, we may assume that $i = 1, 2, \cdots, I_v$. Then the ”coordinate” for $p_{vi}$ with $3 < i \leq I_v$, is given by the cross-ratio $w_{vi} =: w_{v123i} = (p_{v1} : p_{v2} : p_{v3} : p_{vi})$. The tuple of all such coordinates together, denoted by $w_p$, with the order given by $i$ is the coordinate for the tuple $p =: \{p_{vi}, v \in T\}$ of special points. The proof of the next lemma is clear.

**Lemma 2.6** The map $p \to w_p$ is $H_T$-invariant. It gives rise a global ”coordinate chart” for $\mathcal{M}_T$. The complex submanifold of $\tilde{\mathcal{M}}_{\hat{T}}$, $S\tilde{\mathcal{M}}_{\hat{T}} = \{p_{vi}, v \in T \mid p_{v1} = 0, p_{v2} = 1, p_{v3} = \infty\}$ is a slice of the $H_T$-action. The global coordinate map above is naturally defined on the slice.
Given above slice \( S\tilde{M}_T = \prod v S_v \), we define (a model of) the universal curve as a family \( U_T \rightarrow S\tilde{M}_T \) as follows. For each \( v \in T \) we define \( U_v \rightarrow S_v \) to be the trivial family \( U_v = S_v \times \mathbb{C}P^1_v \rightarrow S_v \) first. Note that for fixed \( i \), the tautological map \( p_{vi} \in S_v \rightarrow p_{vi} \in S_v \) is a holomorphic section of the family so that it gives rise a divisor in \( U_v \). Now for each fixed \( v \in T \), the pull-back of \( U_v \rightarrow S_v \) by the projection map \( S\tilde{M}_T \rightarrow S_v \) gives a family over \( S\tilde{M}_T \), still denoted by \( U_v \rightarrow S\tilde{M}_T \). Then the normalization of the universal family \( U_T \rightarrow S\tilde{M}_T \) to be defined is the disjoint union of these \( U_v \rightarrow S\tilde{M}_T, v \in T \).

Now the desired universal curve \( U_T \) is obtained from \( U_v \) by identify those divisors that are corresponding to the tautological sections from double points. This implies that \( U_T \) is an analytic space with only normal crossing singularities.

### 2.3 The space of nodal maps modeled on \( T \)

Fix a Riemannian manifold \((M, g_M)\). Let \( \Sigma \) be a nodal surface modeled on \( T \). A continuous map \( f : \Sigma \rightarrow M \) map is said to be a nodal \( L^p_k \)-map modeled on \( T \) if each its component \( f_v : \Sigma_v \rightarrow M, v \in T \) is of class \( L^p_k \). Here the \( L^p_k \)-norm is measured with respect the metric \( g_M \) on \( M \) and the pull-back of the Fubini-Study metric on \( \Sigma_v \) by a identification \( \phi_v : \Sigma_v \simeq \mathbb{C}P^1 \). Note that \( \phi_v \) depends on the choice of a minimal stabilization of \( \Sigma \) but the induced \( L^p_k \)-norms are equivalent with respect to different choices.

Two such nodal maps \( f_1 : \Sigma_1 \rightarrow M \) and \( f_2 : \Sigma_2 \rightarrow M \) are said to be equivalent if is an equivalence map \( \phi : \Sigma_1 \rightarrow \Sigma_2 \) such that \( f_1 = f_2 \circ \phi \). Thus using the identifications of the components of the domains given by \( \phi_v : \Sigma_v \simeq \mathbb{C}P^1 \), we may assume that each component of a nodal map has the form \( f_v : \Sigma_v = \mathbb{C}P^1_v \rightarrow M \). Again, here \( \phi_v \) depends on a particular choice of a minimal stabilization of \( \Sigma \) which can be obtained by fixing a minimal stable labeling \( \hat{T} \) of \( T \). This leads to the following definition.

**Definition 2.1** Fix a \( n \)-labeled tree \( \hat{T} \) that is a minimal stabilization of the given tree \( T \). Let \( U_{\hat{T}} \rightarrow \mathcal{M}_{\hat{T}} \) be the universal family of genus zero stable curves with \( n \)-marked points. The space of genus zero stable \( L^p_k \)-maps modeled on \( \hat{T} \) is defined to be

\[
\tilde{B}_{\hat{T}} = \{ (\Sigma, f) | f = \{ f_v : \Sigma_v \rightarrow M, v \in T \}, f_v(d_{vu}) = f_u(d_{uv}) \text{ if } uEv, \| f_v \|_{k,p} < \infty \}.
\]

Here the domain \( \Sigma \) of \( f \) with components \( \Sigma_v, v \in T \) is the underlying nodal curve of the stable curve \( \Sigma \), which is a fiber of the universal family \( U_T \rightarrow \mathcal{M}_T \).
The space $\tilde{B}^\hat{T}$ can be thought as a precise version of the intuitive notion of the space of parametrized nodal maps.

**Proposition 2.1** $\tilde{B}^\hat{T}$ is a Banach manifold of class $[m_0]$.  

**Proof:**

Recall that the universal curve can be realized as $U^\hat{T} \rightarrow S\mathcal{M}_T$. The slice $S\mathcal{M}_T$ is the product $\prod_{v \in T} S_v$, where $S_v = \{p_v\}$ is the set of special points on $\mathbb{CP}_v^1$ with the first three of them are $0, 1, \infty$. Note that for an unstable $v$, $S_v$ has only one element $(0, 1, \infty)$.

Let

$$\tilde{B}_v = \{(p_v, f_v) | p_v \in S_v, f_v : \mathbb{CP}_v^1 \rightarrow M, \|f_v\|_{k,p} < \infty\}.$$  

Then $\tilde{B}_v$ is naturally identified with $S_v \times \tilde{B}(\mathbb{CP}_v^1)$. Here $\tilde{B}(\mathbb{CP}_v^1)$ is the Banach manifold of $L^p_k$ maps with the fixed domain $\mathbb{CP}_v^1$. This implies that $\tilde{B}_v$ and hence $\prod_{v \in T} \tilde{B}_v$ is the Banach manifold.

**Proposition 2.2** The evaluation map $E_v : \tilde{B}_v \rightarrow M^{d_v}$ given by $(p_v, f_v) \rightarrow f_v(d_v)$ is of class $[m_0]$. Here $d_v$ are the double points among the special points $p_v$ and $f_v(d_v)$ is the tuple of the evaluations of $f_v$ at the doubles of $d_v$ considered as an element in $M^{d_v}$. Moreover, the evaluation map is surjective submersion.

This proposition is proved in [L] and [L?].

Let $E = \prod_{v \in T} E_v : \tilde{B}_v \rightarrow M^{[d]}$ be the total evaluation map at double points. Then it is still a submersion so that it is transversal to the diagonal $\Delta_T \subset M^{[d]}$. Here $\Delta_T = \{m_{uv} = m_{vu} \in M, when uEv for u,v \in T\}$ is a submanifold of $M^{[d]} = \{m_{uv} \in M, uEv\}$. This implies that $\tilde{B}^\hat{T} = E^{-1}(\Delta_T)$ is a Banach manifold of class $[m_0]$.  

In Gromov-Witten theory we are mainly interested in the space of equivalent classes of the nodal $L^p_k$-maps, denoted by $\mathcal{B}^T$. It can be obtained by quotient out the action of the reparametrization groups on $\tilde{B}^\hat{T}$. These actions are induced from the corresponding actions on the domains. We now spill out more details on these actions.

Let $\Sigma = \Sigma_f$ be the domain of a nodal map modeled on the tree $T$ with a fixed minimal stabilization modeled on a $n$ labeled tree $\hat{T}$. Recall that the
order on the vertices and edges of $T$ from $\hat{T}$ gives rise the identifications of each component $\Sigma_v, v \in T$ with $\mathbf{CP}_v^1$. For an unstable component $\mathbf{CP}_v^1$ with one or two double points $d_v$, we will assume that $d_v$ are in the standard position $0$ or $0, \infty$. Then the reparametrization group $G_v = G_{\Sigma_v}$ is defined to be the subgroup $G_i, i = 1, 2$ of $G_0 = \text{PSL}(2, \mathbb{C})$ consisting of the biholomorphic maps of $\mathbf{CP}_v^1$ that fix the double point(s) the standard $d_v$. Let $G_T$ be the product $\prod_v G_v$ with $v$ being unstable components. Then (I) Since $d_v$ are in their standard positions in the unstable components, there is no "moduli" coming from such double points. This implies that $G_T$ is independent of $(\Sigma, d, x) \in \mathcal{M}_T$, that justifies our notation. (II) Any element in $G_v$ is determine by its action on the corresponding (minimal )marked point(s) $x_v$.

Note that any element $\phi \in G_T$ can be considered as an automorphism of $\Sigma$ by requiring that $\phi_v = \text{identity on any stable component } \Sigma_v$. Let $G_f =: G_{\Sigma}$ be the group of reparametrization of the nodal curve $\Sigma$ consisting of all self identifications. Clearly $G_{\Sigma}$ depends on $\Sigma$. This dependence comes from the following finite subgroups.

Let $(\Sigma, [x])$ be marked nodal surface with unordered marking $[x]$ obtained from $x$. A map $\phi : (\Sigma_1, [x]) \to (\Sigma_2, [x])$ is said to be an equivalence between the two such surfaces if $\phi$ is an equivalence of the underlying nodal surfaces such that $\phi$ identifies $x_1$ to $x_2$ sets (without any order).

Denote the finite group of automorphisms (=self equivalence) of $(\Sigma, [x])$ by $G_{\Sigma,[x]}$. Note that here $(\Sigma, x)$ is a minimal stabilization of $\Sigma$ so that each top bubble (=unstable component $\Sigma_v$ with $v$ being a tip of $T$) has two marked points with order in $x$. Let $G_{\Sigma,[x]}^0$ be the subgroup of $G_{\Sigma,[x]}$ consisting automorphisms that preserve the order of the two marked points of each top bubble. For each tip $v \in T$, let $Z_{2,v} \simeq Z_2$ and $Z_{2,T} \prod_{v=\text{tip}} Z_{2,v}$ considered as subgroup of $G_{\Sigma,[x]}$. Here $Z_{2,v}$ is the group of involution of $(\Sigma_v, [x])$ that exchanges the two marked points in $x_v$ for a tip $v \in T$. Note that $G_{\Sigma,[x]}^0$ is a normal subgroup of $G_{\Sigma,[x]}$, and $G_{\Sigma,[x]}$ is the semi direct product of $G_{\Sigma,[x]}^0$ and $Z_{2,T}$.

**Lemma 2.7** The group $G_T$ is the normal subgroup of $G_{\Sigma}$ consisting of self identifications that preserve the components of $\Sigma$. The quotient group $G_f/G_T =: G_f/G_T$ is a finite group that switches the components of $\Sigma$. The exact sequence $1 \to G_T \to G_{\Sigma} = G_f \to G_f/G_T \to 1$ splits in the sense that (I) $G_T$ and $G_{\Sigma,[x]}^0$ generate $G_f/G_T$ and (II) The induced homomorphism $G_{\Sigma,[x]}^0 \to G_f/G_T$ is an isomorphism. Therefore $G_{\Sigma}$ is the semi direct product of $G_T$ and $G_{\Sigma,[x]}^0$.

**Proof:**
First note that (i) the $G^0_{\Sigma,[x]} \cap G_T = \varphi$; (ii) $G_T$ and $G^0_{\Sigma,[x]}$ generate $G_\Sigma$.

Next proposition shows that any element $\phi$ in $G_\Sigma$ is determined by its effect on the marked points $x$. Given $\psi \in G^0_{\Sigma,[x]}$, assume that for a top bubble $\psi_v : (\Sigma_v; x_{v1}, x_{v2}) \to (\Sigma_{v'}; x_{v'1}, x_{v'2})$. Let $\gamma = \{\gamma_v, v \in Tip(T)\} \in G_T$. It is easy to check that $\gamma_v \circ \psi_v = \psi_v \circ \gamma_v$ for $v, v' \in Tip(T)$ above, and that the same relation holds for the unstable components with only one marked point. Since $G_T$ acts trivially on the stable components, this implies that $G_T$ is normal.

□

To prove next proposition, we need a lemma on existence of special level one points on a tree. The tips of a tree $T$ will be called level zero points. Given a vertex $v \in T$ that is not a tip, it is said to be a level one point if there is a tip $u \in T$ such that (i) there is a simple chain $[uv]$ connection $u$ and $v$; (ii) $[uv]$ is a maximal simple chain. A chain connecting $u$ and $v$ is said to be simple if all the intermediate vertices have valence two. Note that the condition (ii) above implies that the valence of $v$ is at least three. Clearly a tree $T$ does not have level one points if and only if $T$ is a simple chain.

**Lemma 2.8** Assume that $T$ has level one points. Then there exits a level one point $v \in T$ such that there exist two tips $u_1$ and $u_2$, and the corresponding maximal simple chains $[u_1v]$ and $[u_2v]$ connecting $u_1, u_2$ to $v$.

**Proof:**
Given the tips of $T$ an order so that $T$ is totally ordered. Then the first vertex $v$ on the last chain (connecting to the last tip) is the sought-after level one point.

□

**Proposition 2.3** Any element $\phi \in G_\Sigma$ is determined by its effect on the unstable components.

**Proof:**
Note that in the case that the tree $T$ has no level points, $T$ itself is a simple chain. Then any component of a nodal curve modeled on $T$ is unstable so that the lemma is automatically true in this case.

Therefore we only need to consider the case that $\Sigma$ has level one points. Let $\phi$ be an automorphism of such a nodal surface $\Sigma$.  

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Let $\Sigma_v$ be a component with $v \in T$ be the terminal vertex of a maximal simple chain $[uv]$ for a tip $u \in T$. Thus $v$ is a level one vertex. We may assume that $v$ satisfies the conclusion of the above lemma so that there is another maximal simple chain $[u_1v]$ with tip $u_1 \in T$. Then by the assumption $\phi$ is already defined on all the corresponding components $\Sigma_w$ for any vertices $w \neq v$ lying on any of these two simple chains. Now remove all components other than $\Sigma_v$ along the chain $[uv]$. If $\Sigma_v$ is stable in the new nodal surface $\Sigma'$ then all the unstable components in $\Sigma'$ remain unstable in $\Sigma$ so that $\phi$ is defined on these components. By induction, $\phi$ is defined on $\Sigma'$. This implies the lemma.

Thus we may assume that $\Sigma_v$ is not stable in $\Sigma'$. Since $Val(v) \geq 3$ in $T$, this can only happen if $\Sigma_v$ has one and only one more double point $d_v$ other than the two double points coming from the simple chains $[uv]$ and $[u_1v]$. Note $\phi$ maps the double points to the double points on the corresponding components. By assumption, the values of $\phi$ at two of the double points on $\Sigma_v$ that joins to the two components on the two maximal simple chains are already determined, lying on the corresponding component $\Sigma_{v'}$ as two of its three double points. Then $\phi$ maps the third double point of $\Sigma_v$ to the third one of $\Sigma_v'$ so that the map $\phi$ is determined on $\Sigma_v$. Now we are in the position to apply the induction for this case too.

□

**Corollary 2.2** The automorphism group $G_{\Sigma, x}$ of a stable curve $(\Sigma, x)$ is trivial.

The group $G_{\Sigma, x}$ can be regarded as the isotropy group of the marked nodal surface $(\Sigma, [x]) \in \mathcal{N}_{n,T}$, where $\mathcal{N}_{n,T}$ is the moduli space of genus zero nodal curves with (unordered ) $n$-marked points modeled on $\hat{T}$.

**Proposition 2.4** The action of the isotropy group $G_{\Sigma, x}$ on the ”central fiber” $(\Sigma, x)$ can be extended into an action on the underlying curves of the universal family $U_{\hat{T}} \rightarrow S\mathcal{M}_{\hat{T}}$ preserving the unordered marked point sets. The extended action above is compatible with the action of $G_T$ so that these actions together gives rise a fiber preserving action of $G_{\Sigma}$ on the universal curve $U_{\hat{T}} \rightarrow S\mathcal{M}_{\hat{T}}$ that is an extension of the action on the fiber $(\Sigma, x)$.

Moreover, on the base $S\mathcal{M}_{\hat{T}}$, the action of $G_T$ is trivial while the action of $G_{\Sigma, x}$ on an sufficiently small open neighborhood $\Lambda(\Sigma, x)$ of $(\Sigma, x)$ in $\mathcal{M}_T \simeq S\mathcal{M}_{\hat{T}}$ gives rise a local uniformizer of $\mathcal{N}_{n,T}$. Here $\mathcal{N}_{n,T}$ is the moduli space
of the stable nodal curves with n unordered marked points. In particular, for any \((\Sigma', x') \in \Lambda(\Sigma, x)\), the action of \(G_{\Sigma,[x]}\) induces a surjective group homomorphism \(G_{\Sigma,[x]} \to G_{\Sigma',[x']}\) which is compatible with their actions on the base. Consequently, on the neighborhood \(\Lambda(\Sigma, x)\), any \(G_{\Sigma',[x']}\)-orbit of a given point is covered by its \(G_{\Sigma,[x]}\)-orbit.

We prove the following corollary first.

**Corollary 2.3** The local action of \(G_{\Sigma}\) on the open neighborhood \(\Lambda(\Sigma, x)\) of \((\Sigma, x)\) in \(SM\hat{T}\) defined above is a local model for the ”moduli space” of nodal curves of type \(T\) in the sense that for any two stable curves as two fibers over \(\Lambda(\Sigma, x)\), their underlying nodal curves are equivalent if and only if they are in the same \(G_{\Sigma}\)-orbit.

**Proof:**

For any two stable curves \((\Sigma'_1, x'_1)\) and \((\Sigma'_2, x'_2)\) in the local universal family over \(\Lambda(\Sigma, x)\), their underlying nodal curves are equivalent if and only if one of them, \((\Sigma'_2, x'_2)\) for instance, is in the \(G_{\Sigma'_1}\)-orbit. Since \(G_{\Sigma'_1}\) is generated by \(G_T\) and \(G_{\Sigma'_1,[x'_1]}\), up to a \(G_T\)-action, \((\Sigma'_2, x'_2)\) is in the \(G_{\Sigma'_1,[x'_1]}\)-orbit.

Since \(G_T\) acts trivially on the base \(\Lambda(\Sigma, x)\) and its actions on base are identical when it is regarded either as a subgroup of \(G_{\Sigma'_1}\) or as a subgroup of \(G_{\Sigma}\). Hence we may assume that the induced identification of the two corresponding points on the base is given by an element in \(G_{\Sigma'_1,[x'_1]}\).

Now by last part of the above proposition, on the base the \(G_{\Sigma'_1,[x'_1]}\)-orbit of \((\Sigma'_1, x'_1)\) is covered by its \(G_{\Sigma,[x]}\)-orbit. However, it follow from the definition that the actions of these two finite symmetry groups come from the corresponding ones acting on the universal family preserving the set of unordered marked sections. Hence as curves (fibers) in the universal family \(G_{\Sigma'_1,[x'_1]}\)-orbit of \((\Sigma'_1, x'_1)\) is still covered by its \(G_{\Sigma,[x]}\)-orbit. This proves that the original identification is realized, up to a \(G_T\) action, by an action of an element in \(G_{\Sigma,[x]}\) so that \((\Sigma'_1, x'_1)\) and \((\Sigma'_2, x'_2)\) are in the same \(G_{\Sigma}\)-orbit.

\(\square\)

The key ingredient to prove above proposition is the universal property of the universal curve.

For simplicity we assume that \((\Sigma, x)\) is still minimally stable. This is sufficient for the purpose of this paper.

- Universality of \(U_\hat{T} \to M_\hat{T}\).

The above propositions follows from the universality of \(U_\hat{T} \to M_\hat{T}\) in the proper category of family of stable curves. In algebraic geometry, the usual
formulation of a good family uses the algebraic notion of flat family of such curves. In the genus zero case, the tautological family of stable curves with \( n \) marked points, \( n \geq 3, \mathcal{U}_{0,n} \to \overline{\mathcal{M}}_{0,n} \) is indeed an universal flat family. Since flatness is preserved under the base changes, any other flat family of such curves can be obtained by pull-backs.

In this paper and its sequels, we are working on the categories of smooth or complex manifolds. The existence of the universal family above suggests that good families of genus zero \( n \)-stable curves in these categories can be simply defined to be the pull-backs of the universal family.

**Definition 2.2** A smooth (holomorphic) family of nodal curves on a smooth (complex) manifold \( M \) modeled on \( T \) or \( \hat{T} \) is defined to be (obtained by ) a smooth (holomorphic) map \( h : M \to S\overline{\mathcal{M}}_{\hat{T}} =: \{ h_v : M \to S_v, v \in T \} \). Here \( \hat{T} \) is one of the (finitely many) minimal stabilizations of \( T \).

Thus the smooth or holomorphic family of nodal curves \( \mathcal{C} =: \mathcal{C}_h \to M \) (and the corresponding family of \( n \)-stable curve(\( \mathcal{C}_h, x \)) \to M \), modeled on \( \hat{T} \), are obtained by pull back by \( h \) of the underline universal family of curves (together with the \( n \)-marked sections).

Now we specify the meaning of the universality in this setting so that it is not just a tautology.

To this end, let \( (\Sigma, x) \) be ”central fiber” \( \mathcal{U}_{s_0} \) of the family \( \mathcal{U}_{\hat{T}} \to \overline{\mathcal{M}}_{\hat{T}} =: S\overline{\mathcal{M}} \). Here \( s_0 = \{ s_{0,v} \in S_v, v \in T \} \) is the set of special points on the central fiber. Fix \( m_0 \in M \) with \( h(m_0) = s_0 \). The central fiber \( \mathcal{C}(m_0) \) of the family \( \mathcal{C} \to M \) defined by \( h \) is just a copy of the central fiber \( (\Sigma, x) \) of the universal family. Let \( \phi : \mathcal{C}(m_0) \to \mathcal{U}_{s_0} \) be the identification of the two fibers. Note that \( \hat{\phi} \) as identification of \( n \)-labeled stable curves is unique.

Then the universal property in this case is the claim that above identification of the central fibers has an unique extension \( \Phi : (\mathcal{C}, \mathcal{C}(m_0)) \to (\mathcal{U}_{\hat{T}}, \mathcal{U}_{s_0}) \) that preserves the marked sections.

Of course with the respect to the family \( (\mathcal{C}, \mathcal{C}(m_0)) \to (M, m_0) \) of the type given by a fixed label tree \( \hat{T} \), the universality for \( (\mathcal{U}_{\hat{T}}, \mathcal{U}_{s_0}) \to (S\overline{\mathcal{M}}, s_0) \) stated this way is a tautology except the uniqueness of \( \Phi \).

However, if we relabel the marked points by a different minimal stable label tree \( \hat{T}_1 \) and consider the same underlying family \( (\mathcal{C}, \mathcal{C}(m_0)) \to (M, m_0) \) but with the order of the marked points given by \( \hat{T}_1 \), then the universality with respect all possible minimal stable labeling for the underlying tree \( T \) is
not an tautology anymore. What needs to be proved is stated in the following proposition.

**Proposition 2.5** Given a family $(\mathcal{C}, \mathcal{C}(m_0)) \to (M, m_0)$ of type $\hat{T}_1$ and the family $(\mathcal{U}_\hat{T}, \mathcal{U}_{s_0}) \to (S\hat{M}, s_0)$ of type $\hat{T}$ with an equivalence $\phi : \mathcal{C}(m_0) \to \mathcal{U}_{s_0}$ as stable curves with ordered marked points, then there exists an unique extension of $\phi$, $\Phi : (\mathcal{C}, \mathcal{C}(m_0)) \to (\mathcal{U}_\hat{T}, \mathcal{U}_{s_0})$ that preserves the marked sections.

Since the families $\mathcal{C} \to M$ is defined by pull-backs from the universal ones, the above proposition is equivalent to the following.

**Proposition 2.6** Let $(\mathcal{U}_\hat{T}_1, \mathcal{U}_{s_1}) \to (S\hat{M}, s_1)$ be the family obtained from the universal family $(\mathcal{U}_\hat{T}, \mathcal{U}_{s_0}) \to (S\hat{M}, s_0)$ of type $\hat{T}$ by relabel its double point sections given by $\hat{T}_1$. Assume that there is a self equivalence $\phi : \mathcal{U}_{s_1} \to \mathcal{U}_{s_0}$ as stable curves with ordered marked points with respect to $\hat{T}_1$ and $\hat{T}$ respectively, then there exists an unique extension of $\phi$, $\Phi : (\mathcal{U}_\hat{T}_1, \mathcal{U}_{s_1}) \to (\mathcal{U}_\hat{T}, \mathcal{U}_{s_0})$ that preserves the marked sections.

The proof the above proposition is essentially same as the special case that $s_0 = s_1$. We only give the proof for this case, which can be restated as the following lemma.

**Lemma 2.9** The action of the isotropy group $G_{\Sigma, [x]}$ on the "central fiber" $(\Sigma, [x])$ can be extended into an action on the underlying curves of the universal family $\mathcal{U}_\hat{T} \to S\hat{M}_\hat{T}$ preserving the unordered marked point sets.

**Proof:**

First extend the isotropy action of $G_{\Sigma, [x]}^0$ on the central fiber to the local universal family $\mathcal{U}(\Sigma, x) \to \Lambda(\Sigma, x)$ as a fiberwise analytic action. We will use $\beta \in \Lambda(\Sigma, x)$ to denote the local parameter of the base of the universal family, which describes the locations (upto the actions of $\mathrm{PSL}(2, \mathbb{C})$), the "moduli" of double points on stable components with respect to the reference double points on $\Sigma$. Note that when $\hat{T}$ is fixed, using the total ordering induced by $\hat{T}$, there is a coordinate describing the global "moduli" of double points, given by the multi cross-ratios with respect the first three double points on each stable component. The corresponding fiber will be denoted by $(\Sigma_\beta, x_\beta)$ with the central fiber $(\Sigma, x) = (\Sigma_{\beta_0}, x_{\beta_0})$. Note that here we have used the fact that $(\Sigma, x)$ is minimally stable.

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The desired action of $G^0_{\Sigma,[x]}$ on $U(\Sigma, x) \rightarrow \Lambda(\Sigma, x)$ is defined as follows.

For $\phi \in G^0_{\Sigma,[x]}$, assume that the induced map on $T$ given by $v \rightarrow v'$ for any $v \in T$ so that $\phi_v : (\Sigma_v, p_v) \rightarrow (\Sigma_{v'}, p_{v'})$.

Denote the corresponding map to be defined on the fiber $\Sigma_{\beta}$ by $\phi_{\beta}$. First note that for any $v \in T$ and $\beta$ near $\beta_0$, the component $\Sigma_{\beta,v} \simeq \Sigma_{\beta_0,v} = \Sigma_v$ with canonical identifications given by $\hat{T}$. On each component, $\phi_{\beta,v}$ is defined to be $\phi_v$ with domain $\Sigma_{\beta,v} = \Sigma_v$ and target $\Sigma_{\beta',v'} = \Sigma_{v'}$ for some $\beta'$ to be defined. Using these maps $\phi_{\beta,v}$, target $\Sigma_{\beta'}$ is obtained by gluing the components $\Sigma_{v'}$ along the corresponding double points in the obvious manner. More specifically, for each double $d^\beta_{uv} = d^\beta_{vu}$ on $\Sigma_{\beta}$ joining $\Sigma_{\beta,u}$ and $\Sigma_{\beta,v}$, let $d'^\beta_{uv,v'}$ and $d'^\beta_{v',u'}$ be the images $\phi_{\beta,u}(d^\beta_{uv})$ (by the definition above) and $\phi_{\beta,u}(d^\beta_{vu})$, considered as a double point on $\Sigma_{u'}$ and $\Sigma_{v'}$ respectively. Then the surface $\Sigma_{\beta'}$ is defined to be the collection of the components $\Sigma_{w'}$ joining together at double points $d'^\beta_{w,v'} = d'^\beta_{v',w}$ according to the condition $u'Ev' \leftrightarrow uEv$. Clearly, by the construction, these maps $\phi_{\beta,v}$, $v \in T$ together define the desired $\phi_\beta : (\Sigma_{\beta}, [x]) \rightarrow (\Sigma_{\beta'}, [x])$. The parameter $\beta'$ here is defined as follows.

Since $\phi$ preserves the set of double points of $\Sigma$, $d_{u'v'} = \phi_u(d_{uv})$, $uEv$ as sets. If $\beta$ is close to $\beta_0$, the double points $d'^\beta_{u,v'} = \phi_u(d^\beta_{uv}), uEv$ of the surface $\Sigma_{\beta'}$ are contained the poly-discs centered at double points $d_{u',v'} = \phi_u(d_{uv}), uEv$ of $\Sigma$. Thus with respect to these ”centers” $(d_{uv}, uEv)$, the locations of the double points $d'^\beta_{u,v'}$ (upto the obvious $PSL(2,C)$ action on each component) are described by the unique parameter $\beta'$. This defines the $\beta'$. Again, here we have used the fact that $(\Sigma_{\beta'}, x)$ has no extra marked points so that $\beta'$ serves as a local coordinate of the moduli space $\mathcal{M}_T$.

Now both $\beta$ and $\beta'$ can be considered as the multi-cross ratios of the double points $d^\beta_{uv}$ and $d'^\beta_{u'v'}$ with respect to first three of them on each of the stable components using the same ordering induced from the same $\hat{T}$. On the other hand, by the definition of $\phi_\beta$, the parameter $\beta$ is the same as the multi-cross ratios of the $d'^\beta_{u'v'}$ with respect to first three of them on each of the stable components but using the ordering of $T$ induced from the new label $\phi \circ L$. Thus both $\beta$ and $\beta'$ are (part of all ) multi-ratios of the very same double point sets $d'^\beta_{u'v'}$ selected with respect to two different orderings. Sine both of them already describe the full local moduli, the implicit function theorem implies that $\beta'$ is a holomorphic function of $\beta$, and vice versa for $\beta$ sufficiently close to $\beta_0$.

As mentioned before, when $\hat{T}$ and $\phi$ are fixed, as the multi-cross ratios, $\beta$
and $\beta'$ are defined globally on $SM_T$. Hence $\beta'$ is still a holomorphic function of $\beta$ over $SM_T$. This defines the desired global extended action $\Phi = \{\phi_\beta\}$ on the base $SM_T$. Now using this together with the above existence and uniqueness of local extensions, starting from the central fiber, we get the desired global extended action $\Phi = \{\phi_\beta\}$ on the whole universal family $U_T \to SM_T$ by the usual argument of "analytic continuation."

This finishes the proof for $G^0_{\Sigma,[x]}$. The proof for $G_{\Sigma,[x]}$ is obtained as a special case of the argument below.

It is this special form of universality that is the key step of the proof below for the main proposition before on extension of the $G_{\Sigma}$-action.

• Proof of the proposition on extension of the $G_{\Sigma}$-action:

The action of $G_T$ is defined on any $\Sigma_\beta$ automatically. Indeed, the action of $G_T$ on any stable component is defined to be identity, and since any unstable component $\Sigma_{\beta,v}$ is the same as $\Sigma_{\beta',v}$, the action of $G_T$ on each such $\Sigma_v$ automatically gives the corresponding action on $\Sigma_{\beta,v}$.

Thus we already have the extended actions of the two subgroups $G_T$ and $G^0_{\Sigma,[x]}$. Recall that $G_{\Sigma}$ is the semi direct product of $G_T$ and $G^0_{\Sigma,[x]}$. More specifically, it is generated by $G_T$ and $G^0_{\Sigma,[x]}$ with the only relation $\gamma_v \circ \psi_v = \psi_v \circ \gamma_v, v \in T$ for $\gamma_v, \gamma_v' \in G_T$ acting on $\Sigma_v$ and $\Sigma_{v'}$, and $\psi_v : \Sigma_v \to \Sigma_{v'}$ in $G^0_{\Sigma,[x]}$.

By the definition of the extended actions, on a stable component of $\Sigma_\beta$, the above relation is automatically true for the extended actions since $\gamma_v$ and $\gamma_v'$ are the identity map. One can verify that it is still true on any unstable component.

We will assume the next proposition whose proof will be given in a forthcoming paper.

**Proposition 2.7** The action map $G_T \times B^\hat{T} \to B^\hat{T}$ is continuous. Moreover, for a fixed $g$ in $G_T$ or $G_{\Sigma}$, the map $\Phi_g : B^\hat{T} \to B^\hat{T}$ is a smooth map of class $[m_0]$. Similar results hold for the actions of $(\Sigma, x)$.

In next section, we will consider the induced action of $G_f =: G_{\Sigma_f}$ on the $G_f$-orbit of a prescribed neighborhoods $U_f \subset B^ws_T$ of $f$ for each $f \in B^ws_T$. 

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3 The properness of $G_T$-action and $G_T$-Hausdorffness of $\hat{\mathcal{B}}$

First note that the following slightly general theorem can be derived from the corresponding main theorem on properness of $G_T$-action in section 1.

**Theorem 3.1** The partially defined action of $G_f =: G_{\Sigma_f}$ on $\hat{\mathcal{B}}_T^{ws}$ is proper in the sense that for any $f_1$ and $f_2$ $\hat{\mathcal{B}}_T^{ws}$ there exist the corresponding open neighborhoods $U_1$ and $U_2$ containing $f_1$ and $f_2$ at which $G_{f_1}$ and $G_{f_2}$ act on respectively and compact subsets $K_1$ and $K_2$ in $G_{f_1}$ and $G_{f_2}$ accordingly such that for any $h_1$ in $U_1$ ($h_2$ in $U_2$) and $g_1$ in $G_{f_1} \setminus K_1$ ($g_2$ in $G_{f_2} \setminus K_2$), $g_1 \cdot h_1$ is not in $U_2$ ($g_2 \cdot h_2$ is not in $U_1$).

Indeed, if the theorem is not true, there are sequences of $h_i \in U_1$ with $\lim_{i \to \infty} h_i = f_1$ and $g_i \in G_{f_1}$ with $g_i$ not staying in any compact set as $i$ goes to infinity such that $g_i \cdot h_i$ is lying in $U_2$. Now $g_i = t_i \circ p_i$ with $t_i$ in $G_T$ and $p_i$ in the finite group $G_{\Sigma_{f_1},[x]}$. Hence after taking a subsequence, we may assume that $p_i$ is a fixed element $p$ in $G_{\Sigma_{f_1},[x]}$ and $t_i$ is not staying in any compact set of $G_T$ as $i$ goes to infinity. Since $G_T$ acts trivially on the base of the universal curve, the assumption that $g_i \cdot h_i$ is lying in $U_2$ implies that the group $G_{\Sigma_{h_1},[x]}$ is "covered" by $G_{\Sigma_{f_2},[x]}$ so that the action of $p$ on $(\Sigma_{h_1},[x])$ comes from a corresponding element $p'$ in $G_{\Sigma_{f_2},[x]}$. Hence after conjugating the actions of $p^{-1}$ and $(p')^{-1}$ accordingly we may assume that there are sequences of $h_i \in U_1$ with $\lim_{i \to \infty} h_i = f_1$ and $g_i \in G_T$ with $g_i$ not staying in any compact set as $i$ goes to infinity such that $g_i \cdot h_i$ is lying in $U_2$. This contradicts to the main theorem on the properness of $G_T$-action.

Clearly to prove the main theorem for $G_T$, we only need to look its action on its unstable components. Since $G_T$-action on the domains does not move the double points, we only need to prove the corresponding statement for a fixed domain $\Sigma \simeq \mathbb{C}P^1$ with $G_T = G_i, i = 0, 1, 2$ of the subgroups of $\text{PSL}(2,\mathbb{C})$ preserving $i$ marked points. The proofs of the three cases are similar. We only prove the hardest case that $G_T = G_0 = \text{PSL}(2,\mathbb{C}) = : G$.

**Proposition 3.1** Let $\hat{\mathcal{B}}^*$ be the Banach manifold of non-trivial $L^p_k$-maps from $\Sigma = \mathbb{C}P^1$ to $M$. Then the action of $G = \text{PSL}(2,\mathbb{C})$ on $\hat{\mathcal{B}}^*$ is proper.

**Proof:**
We start with some elementary linear algebra.

For any \( g \in SL(2, \mathbb{C}) \), let \( g = h \cdot u \) be the decomposition in \( SL(2, \mathbb{C}) \) with \( u \in SU(2) \) and \( h \) being self-adjoint. Here \( h = (g \cdot g^*)^{1/2} \in SL(2, \mathbb{C}) \) and \( u = (g \cdot g^*)^{-1/2} \cdot g \in SU(2) \).

Note that for \( g \notin SU(2) \) the decomposition \( g \cdot g^* = w^* \cdot diag(\lambda_1, \lambda_2) \cdot w \) with \( w \in SU(2) \) and \( \lambda_1 < \lambda_2 \) is unique. So is \( (g \cdot g^*)^{1/2} = w^* \cdot diag(r_1, r_2) \cdot w \). Here \( r_i = (\lambda_i)^{1/2} > 0 \), \( i = 1, 2 \) and \( r_1 < r_2 \). Rename \( w^* \) as \( u \) and \( wu \) as \( v \). Denote \( diag(r_1, r_2) \) by \( D(r) \) for short. Then we have the unique decomposition \( g = u \cdot D(r) \cdot v \) in \( SL(2) \) with \( u \) and \( v \) in \( SU(2) \) for any \( g \notin SU(2) \).

Assume that the proposition is not true. Then for any small neighbourhoods \( U_{\epsilon_i}(f_i), i = 1, 2 \) and any nested sequences of compact sets \( K_1 \subset K_2 \subset \cdots \subset K_i \cdots \) in \( G \), there are sequences \( \{g_n\}_{n=1}^{\infty} \) in \( G \) and \( \{h_n\}_{n=1}^{\infty} \) in \( U_{\epsilon_i}(f_i) \) such that (a) \( g_n \) is not in \( K_n \); (b) \( h_n \circ g_n \) is in \( U_{\epsilon_2}(f_2) \).

Here \( \epsilon_i, i = 1, 2 \) and \( K_n, n = 1, \cdots, \) will be decided below in the proof.

After taking a subsequence we may assume that \( g_n \notin SU(2) \) so that it has the unique decomposition in \( SL(2, \mathbb{C}) \), \( g_n = u_n \cdot D(r_n) \cdot v_n \) with \( u_n \) and \( v_n \) in \( SU(2) \) and \( D(r_n) = diag(r_{n,1}, r_{n,2}) \) with \( 0 < r_{n,1} < r_{n,2} \). We may assume that \( r_{n,2} = 1 \) by considering \( D(r_n) \) as an element in \( PSL(2, \mathbb{C}) \). Denote \( r_{n,1} \) by \( a_n \) and \( D(r_n) \) by \( D(a_n) \) or \( g_n \). Let \( D_2 \) be the collection of all \( 2 \times 2 \) non-singular diagonal matrices with positive entries.

Let \( K_n \subset SU(2) \times D_2 \times SU(2) \) be the set of tuples \((u, D(r), v)\) with \( \frac{1}{n} \leq r_i \leq 1, \, i = 1, 2 \) where \( D(r) = diag(r_1, r_2) \). Denote the corresponding compact set in \( PSL(2, \mathbb{C}) \) by \( K_n \).

In these notations, the condition (a) above implies that for \( g_n = u_n \cdot D(a_n) \cdot v_n, \lim_{n \to \infty} a_n = 0 \).

After taking subsequence, we may assume that \( \lim_{n \to \infty} u_n = u \) and \( \lim_{n \to \infty} v_n = v \) in \( SU(2) \).

Let \( CP^1 = \mathbb{C} \cup \infty \) with complex coordinate of \( z \in \mathbb{C} \). Denote the closed disc of radius \( R \) in \( \mathbb{C} \) centered at origin by \( B(R) \).

Since \( f_2 \) is nontrivial, its energy \( E(f_2) = \delta_2 > 0 \). Then there is a point \( x_0 \in CP^1 \) such that \( e(df_2)(x_0) > 0 \). We may assume that (i) \( x_0 \neq \infty \) so that for \( R \) large enough, \( x_0 \) is in the interior of \( B(R) \); (ii) there are positive constants \( \gamma \) and \( \rho \) small enough such that the disc \( B(x_0; \rho) \) of radius \( \rho \) centered at \( x_0 \) is in \( B(R) \) and that for any \( x \) in \( B(x_0; \rho) \), \( e(df_2)(x) > \gamma \). This implies that for \( N_2 \) sufficient large, we have \( E(f_2|_{B(R)}) \geq E(f_2|_{B(x_0; \rho)}) \geq \delta_2/N_2 \). Clearly when \( \tilde{\epsilon}_2 > 0 \) is small enough, for any \( h \) with \( \|h - f_2\|_{C^1} < \tilde{\epsilon}_2 \), we still have that \( E(h|_{B(R)}) \geq \delta_2/N_2 \). Now \( \|h - f_2\|_{C^1} \leq C_2 \cdot \|h - f_2\|_{k,p} \) by our assumption.
Hence if we choose \( \epsilon_2 < \bar{\epsilon}_2/C_2 \), then for any \( h \) in \( U_{c_2}(f_2) \), \( E(h|_{B(R)}) \geq \delta_2/N_2 \). In particular, since \( h_n \circ g_n \) is in \( U_{c_2}(f_2) \), we conclude that \( E(h_n \circ g_n|_{B(R)}) \geq \delta_2/N_2 \). Fix such \( R, \delta_2 \) and \( N_2 \).

Now under the automorphism \( v \) of \( \mathbb{C}P^1 = C \cup \infty \), the point \( \infty \) maps to \( \infty_v := v(\infty) \), and \( C \) to \( C_v := v(C) \), etc. In particular, \( B(R)_v = v(B(R)) \).

Under the identification \( C_v \simeq C \) the standard coordinate \( z \) becomes the coordinate for \( C_v \).

Then in term of this coordinate of \( C_v \), the action of \( D(a_i) \) restricted to \( C_v \) is given by \( Da_i(z) = a_i \cdot z \) with all \( a_i \leq 1 \). For any fixed \( R > 0 \) and any given \( \epsilon > 0 \), by our assumption, when \( i \) is large enough, we have \( D(a_i)(B(R)_v) \subset (B(\epsilon)_v) \subset C_v \). Hence the area \( A(D(a_i)(B(R)_v)) \leq C_3 \epsilon^2 \).

Here the area is computed with respect to the Fubini-Study metric which is uniformly equivalent to the flat metric on \( B(R)_v \) for fixed \( R \). Note that it follows from the definition of \( B(R)_v \), in above inequality, the constant \( C_3 \) is independent of \( v \in SU(2) \) and \( i \) as long as \( i > i_0 = i_0(\epsilon) \). In particular, for \( v = v_i \) with \( i >> i_0 \), the inequality still holds.

Applying this to \( g_i = u_i \circ D(a_i) \circ v_i \), since \( u_i \) preserves the Fubini-Study metric, we conclude that for \( i \) large enough, \( A(g_i(B(R)) = A(D(a_i)(B(R)_v))) \leq C_3 \epsilon^2 \).

Now

\[
E((h_i \circ g_i)|_{B(R)}) = E(h_i|_{g_i(B(R))}) \\
\leq ||h_i||_{C^1}^2 A(g_i(M(R)) \leq C_3 \cdot ||h_i||_{C^1}^2 \epsilon^2 \\
\leq C_3 ||h_i||_{k,p}^2 \epsilon^2 \leq C_3(||f_1||_{k,p} + ||h_i - f_1||_{k,p})^2 \epsilon^2 \\
\leq C_3(||f_1||_{k,p} + \epsilon_1)^2 \epsilon^2.
\]

This implies that when \( \delta_2 > 0 \), \( N_2 > 0 \) (depending on \( ||f_2||_{k,p} \) and \( \epsilon_2 \)) and \( \epsilon_1 \) are fixed, for any choice of \( \epsilon \), when \( i > i_0 = i_0(\epsilon) \) is large enough, \( \delta_2/N_2 < E((h_i \circ g_i)|_{B(R)}) < C_3(||f_1||_{k,p} + \epsilon_1)^2 \epsilon^2 \). This is impossible.

QED

Applying the main theorem on properness to the case that \( f_1 = f_2 \), we get the first part of the following corollary.

**Corollary 3.1** The isotropy group \( \Gamma_f \) of any weakly stable \( L_k^p \)-map \( f \) is always compact. Moreover the \( G_T \)-orbit and hence \( G_f \)-orbit of \( f \) in \( \mathcal{B}^T \) is closed.

**Proof:**
We only need to prove the last statement. We only prove this for the essential case that \( f \in \hat{\mathcal{B}}_{T}^{ws} \). Rename \( f \) as \( f_1 \). If the corollary is not true, there exist \( g_i \in G \) and \( f_2 \in \hat{\mathcal{B}}_{T}^{ws} \) such that \( f_2 = \lim_{i \to \infty} f_1 \circ g_i \), but \( f_2 \) is not in \( G \cdot f_1 \). Therefore for any \( U_{\varepsilon_2}(f_2) \), when \( i \) is large enough, \( f_1 \circ g_i \) is in \( U_{\varepsilon_2}(f_2) \).

On the other hand, the main theorem with the same notation implies that for all such \( i \), \( g_i \) is in the compact set \( K_1 \). Therefore, we may assume that \( \lim_{i \to \infty} g_i = g \) in \( K_1 \). Consequently, \( f_2 = \lim_{i \to \infty} f_1 \circ g_i = f_1 \circ g \). That is \( f_2 \in G \cdot f_1 \) which is a contradiction.

Essentially the same argument proves the following stronger result.

**Corollary 3.2** Given any non-constant map \( f \) in \( \hat{\mathcal{B}}^{T} \), there is a small closed \( \delta \)-neighbourhood \( B_\delta(f) \) such that the \( G_T \)-orbit \( G \cdot B_\delta(f) \) is closed in \( \hat{\mathcal{B}}^{T} \). In other words, \( \hat{\mathcal{B}}^{T} \) is \( G_T \)-regular in the sense that for any \( G \)-closed subset \( C \) in \( \hat{\mathcal{B}}^{T} \) and \( f \not\in C \), there are \( G_T \)-open neighbourhoods \( U_1 \) and \( U_2 \) of \( C \) and \( G \cdot f \) respectively such that \( U_1 \) and \( U_2 \) do not intersect.

In fact a similar argument together with the fact that any closed and bounded subset of a Banach space is weakly compact implies the following stronger result.

**Corollary 3.3** Let \( C \) be a closed and "bounded " subset in \( \hat{\mathcal{B}}^{T} \), then its \( G_T \)-orbit \( C^{G_T} \) is closed.

**Proof:**

Again we only give the proof for the essential case that \( f \in \hat{\mathcal{B}}_{T}^{ws} \). This can be reduced further by assuming the domain of the elements in \( f \in \hat{\mathcal{B}}_{T}^{ws} \) is the fixed \( S^2 \). We will assume that the target \( M \) is \( C^\infty \) embedded in some \( \mathbb{R}^n \). The compactness of \( M \) implies that the induced embedded \( \hat{\mathcal{B}}_{T} \to Map_{k,p}(S^2, \mathbb{R}^n) \) is closed. Moreover the topology given by the Banach structure on \( \hat{\mathcal{B}} \) is the same as the one induced from the Banach space \( Map_{k,p}(S^2, \mathbb{R}^n) \). Then the assumption on \( C \) becomes the condition that there is a closed and bounded subset \( \hat{C} \) in \( Map_{k,p}(S^2, \mathbb{R}^n) \) such that \( C = \hat{C} \cap Map_{k,p}(S^2, \mathbb{R}^n) \). Clearly the \( G_T \)-orbit \( C^{G_T} = \hat{C}^{G_T} \cap Map_{k,p}(S^2, \mathbb{R}^n) \). Hence we only need to prove the statement for \( Map_{k,p}(S^2, \mathbb{R}^n) \).

With this understanding, let \( f_i \in C \) and \( g_i \in G \) be two sequences such that \( h = \lim_{i \to \infty} f_i \circ g_i \) in \( \hat{\mathcal{B}}_{T}^{ws} \) with respect to the \( L^p_{k} \)-topology. The convergence
is also with respect to $L^p_{k-1}$-topology. Then for $i$ large enough, $f_i \circ g_i$ is in $U_{\epsilon_2}(h)$ in $L^p_{k-1}$-topology. On the other hand, the assumption on $C$ implies that there exists a $f$ in $\mathcal{B}^{w,s}_{\tilde{T}}$ such that $f = \lim_{i \to \infty} f_i$ in $L^p_{k-1}$-topology so that for $i$ large enough, $f_i$ is in a $L^p_{k-1}$ neighborhood $U_{\epsilon_1}(f)$. Thus the argument before implies that after taking a subsequence, there exists a $g \in G$ such that $g = \lim_{i \to \infty} g_i$.

Denote $f_i \circ g_i$ by $h_i$. Then $h = \lim_{i \to \infty} h_i$ in $L^p_k$-topology. This implies that $f_i = h_i \circ g_i^{-1}$ is convergent to $h \circ g^{-1}$ in $L^p_k$-topology as well. Clearly the $L^p_k$-limit here is the same as the $L^p_{k-1}$-limit $f$ before. Hence $h = \lim_{i \to \infty} f_i \circ g_i = f \circ g$ is in $C^G_T$.

On the other hand, in the case $f_1 \neq f_2$, the conclusion of the main theorem is weaker than the corresponding statement on the $G$-Hausdorffness stated in Theorem 1.1.

Next theorem is a slightly stronger version of the Theorem 1.1

Given $f$ in the space $\mathcal{B}_{\tilde{T}}$ of nodal $L^p_k$-maps modeled on $\tilde{T}$, let $G_f$ be the group of its reparametrizations. Recall that there is sufficiently small neighborhood $U = U_f$ such that the action $G_f$ extends to $U$ such that for any $h \in U$ the action of $G_f$ covers the action of $G_h$ on the corresponding $U_h \subset U_f$. Therefore the local model for the quotient space $\mathcal{B}_{\tilde{T}}$ of nodal $L^p_k$-maps modeled on $T$ is given by above neighborhood $U_f$ quotient by $G_f$.

**Theorem 3.2** The space $\mathcal{B}_{\tilde{T}}$ of nodal $L^p_k$-maps modeled on $\tilde{T}$ is $G_f$-Hausdorff in the sense that for any two different $G_f$-orbits $G_{f_1} f_1$ and $G_{f_2} f_2$, there exist $G_f$-neighborhoods $G_{f_1} U_1$ and $G_{f_2} U_2$ such that $G_{f_1} U_1 \cap G_{f_2} U_2 = \varnothing$. Therefore, the quotient space $\mathcal{B}_{\tilde{T}}$ of unparametrized nodal $L^p_k$-maps is Hausdorff.

**Proof:**

As remarked before, the essential case is the one for $G_T$-action. We only prove this case. In this situation, the proof reduces to consider the two corresponding components defined on the same domain $\Sigma_v \simeq S^2$ with $v \in T$, still denoted by $f_1$ and $f_2$ and the mapping spaces that they lie on.

The proof consists of three parts.

- **Part I:** the case that both $f_1$ and $f_2$ are non-trivial.

By Theorem 1.1, for any $g$ not in the compact set $K_1$ and $h \in U_{\epsilon_1}(f_1)$, $h \circ g$ is not in $U_{\epsilon_2}(f_2)$. By our assumption, we may assume that $U_{\epsilon_1}(f_1)$ and $U_{\epsilon_2}(f_2)$ have no intersection.
• Claim: when $\epsilon_i, i = 1, 2$ are small enough, $(G_T \cdot U_{\epsilon_1}(f_1)) \cap U_{\epsilon_2}(f_2)$ is empty.

Proof:

If this is not true, there are $h_i \in U_{\delta_i}(f_1)$ and $g_i \in K_1$ such that $h_i \circ g_i$ is in $U_{\delta_i}(f_2)$ with $\delta_i \rightarrow 0$. The compactness of $K_1$ implies that after taking a subsequence, we have that $\lim_{i \rightarrow \infty} g_i = g \in K_1$. Since $\delta_i \rightarrow 0$, we have that $f_1 = \lim_{i \rightarrow \infty} h_i$ and $f_2 = \lim_{i \rightarrow \infty} h_i \circ g_i = f_1 \circ g$. Hence, $f_1$ and $f_2$ are in the same orbit which contradicts to our assumption. Note that in the last identity above, we have used the fact that the action map $\Psi : G T \times \tilde{B}\check{T} \rightarrow \hat{B}\check{T}$ is continuous.

Of course the same proof also implies that $(G_T \cdot U_{\epsilon_2}(f_2)) \cap U_{\epsilon_1}(f_1)$ is also empty for sufficiently small $\epsilon_i, i = 1, 2$.

If $h \in (G_T \cdot U_{\epsilon_1}(f_1)) \cap (G_T \cdot U_{\epsilon_2}(f_2))$, then there are $h_i \in U_{\epsilon_i}(f_i)$ and $g_i \in G, i = 1, 2$ such that $h = h_1 \circ g_1 = h_2 \circ g_2$. Hence $h_2 = h_1 \circ g_1 \circ g_2^{-1}$ and $(G_T U_{\epsilon_1}(f_1)) \cap U_{\epsilon_2}(f_2)$ is not empty. This contradicts to the above claim.

• Part II: the case that one of $f_1$ and $f_2$ is trivial but the other is not. Then the desired results follows from the following stronger statement.

**Lemma 3.1** Given any two $L_k^p$-maps $f_1$ and $f_2$ with $E(f_1) \neq E(f_2)$, there exit $G$-neighbourhoods $W(f_1)$ of $f_1$ and $W(f_2)$ of $f_2$ which do not intersect.

In particular if $f_1$ is a constant map and $f_2$ is not, then $E(f_1) = 0 \neq E(f_2)$ and the above conclusion holds.

Proof:

Note that the condition $E(f_1) \neq E(f_2)$ implies that $f_1$ and $f_2$ are not in the same $G$-orbit. We may assume that $E(f_1) < E(f_2)$. For any $E(f_1) < c < E(f_2)$, since the energy function $E : M \rightarrow R$ is continuous and $G$-invariant, the inverse images $E^{-1}((-\infty, c))$ and $E^{-1}((c, \infty))$, denoted by $W(f_1)$ and $W(f_2)$, are two open $G$-sets in $M$ containing $f_1$ and $f_2$ respectively. Clearly $W(f_1)$ and $W(f_2)$ do not intersect.

• Part III: the case that both $f_1$ and $f_2$ are trivial.

Let $B_{c_1}(c_1)$ and $B_{c_2}(c_2)$ be two open balls in $M$, which do not intersect. Here $c_1$ and $c_2$ are the values of the two constant maps $f_1$ and $f_2$ respectively.
Clearly if $||h_i - f_i||_{C^0} = \max_{x \in \Sigma} |h_i(x) - c_i| < \epsilon'_i, i = 1, 2$, then the image of $h_i$ is contained in $B_{\epsilon'_i}(c_i)$. Moreover, since for any $h_i$ and $g_i \in G$, the image of $h_i \circ g_i$ is the image of $h_i$, for any $h_1$ and $h_2$ as above, their $G$-orbits $G \cdot h_1$ and $G \cdot h_2$ do not intersect. Clearly by our assumption for $\epsilon_i << \epsilon'_i$, any $h_i, i = 1, 2$ in $U_{\epsilon_i}(f_i)$ satisfies the condition $||h_i - f_i||_{C^0} < \epsilon'_i$, hence $G \cdot U_{\epsilon_1}(f_1)$ and $G \cdot U_{\epsilon_2}(f_2)$ do not intersect.

\[\square\]

4 Comparison of finite part of the reparametrization group $G_{\Sigma}$ with $S_T$

Recall that the reparametrization group $G_{\Sigma} =: G_{\Sigma,f}$ is generated by its continuous part $G_T$ and the finite part $G^0_{\Sigma,[x]}$.

As mentioned in the introduction, the important subgroups, such as isotropy groups in $\Gamma_f$ in $G_T$ are rather special. For instance, it follows from the discussion above, for a weakly stable nodal $L^p_k$-map $f$, the identity component of $\Gamma_f$ in $G_T$ is a tours $T^n$.

We now show the finite group $G^0_{\Sigma,[x]}$ itself is also quite restrictive.

The finite group $G^0_{\Sigma,[x]}$ is isomorphic to $G_f/G_T$ that exchanges the components of $f$, hence induces an injective homomorphism $\Psi : G^0_{\Sigma,[x]} \rightarrow S_T$, where $S_T$ is the group of automorphisms of $T$. The above homomorphism is also defined on $G^*_{\Sigma,[x]}$. We want to show that the image of $\Psi$ is small so that "most" of symmetries of $T$ can not be realized as the automorphisms of $\Sigma$.

To this end, we start with an elementary discussion on symmetries of $T$. Let $\phi \in S_T$ be an non-trivial symmetry of $T$. Consider the corresponding self map, denoted by $|\phi| : |T| \rightarrow |T|$ of the underlying space of $T$. It follows from Lefschetz fixed point theorem that $\phi$ has at least one fixed point $x_0 \in |T|$. Let $F_{|\phi|}$ be the set of fixed points of $|\phi|$.

The following lemma summaries important properties of the automorphism $\phi$.

**Lemma 4.1** (I) If dimension of $F_{|\phi|}$ is one, then $F_{|\phi|}$ is an subtree of $T$. For any tip $v$ of $F_{|\phi|}$, let $T_v$ be the subtree consists of vertices that are reachable by chains in $T \setminus F_{|\phi|}$ starting from $v$. Then the action of $\phi$ induces an automorphism of $T_v$ with the only one fixed vertex $v$.
(II) If the dimension of $F_{|\phi|}$ is zero, then $|\phi|$ has only one fixed point that is either a vertex $v$ of $T$ or a midpoint $m_0$ of an edge $[v_0v_1]$. In the latter case, $\phi$ is an involution with respect to the midpoint $m_0$.

(III) The underlying space $|T|$ can have at most one such midpoint with respect to which there is an involution $\phi \in S_T$.

Proof:

The proof is elementary. We only give the proof for (III).

Let $T_0$ ($T_1$ respectively) be the subtree consisting of all vertices that are reachable by a chain from $v_0$ ($v_1$) without passing $v_1$ ($v_0$). Let $m'_0 \neq m_0$ be the midpoint of $[v'_0v'_1]$ for another involution $\phi' \in S_T$. Let $T'_0$ and $T'_1$ be the corresponding connecting components. We may assume that $[v'_0v'_1]$ is contained in $T_1$, and the distance from $v'_0$ to $v_1$ is one less than the distance from $v'_1$ to $v_1$. Then the chain $[v_1,v_0']$, the edge $[v_0,v_1]$ and the connecting component $T_0$ is reachable from $v'_0$ without passing $v'_1$. This implies that $T_0 \subset T'_0$, which is impossible unless $T'_i = T_i, i = 0, 1$ and $\phi = \phi'$.

It follows from this lemma that essentially we only need to consider the case that $|\phi|$ only has one fixed point.

Consider first the case that there is an involution $\phi \in S_T$ with fixed midpoint $m_0$. Let $Z_{2,\phi}$ be the $Z_2$-subgroup of $S_T$ generated by $\phi$. The uniqueness of $m_0$ and $\phi$, implies that for any $\phi_1 \in S_T$, $\phi_1^{-1} \circ \phi \circ \phi_1 = \phi$ and $\phi_1(m_0) = m_0$. Hence in this case, if $\phi_1 \neq \phi$, the dimension of $F_{|\phi_1|}$ is one with $m_0 \in [v_0,v_1] \subset F_{|\phi_1|}$. In other words, once there exists an involution $\phi \in S_T$ with midpoint $m_0$, all the other $\phi_1$ in $S_T$ are in case I above fixing $[v_0,v_1]$ and commutes with $\phi$. It follows that in this case, $S_T$ is isomorphic to the direct product of $Z_{2,\phi}$ and $S_{T(\phi)}$. Here $T(\phi)$ is the quotient tree obtained from $T \setminus (v_0,v_1)$ by the identification of $\phi$.

Thus with the reductions above, we only need to consider the essential case that $\phi$ has only one fixed point that is a vertex $v$.

Example: The simplest case for this is the tree $T(l)$ of $l + 1$ vertices, $v_0, v_1, \cdots, v_l$, with one root $v_0$ and $l$ edges $[v_0v_1], \cdots, [v_0v_l]$. Then $S_T$ is the group $S_l$ of permutations on $l$ letters $v_1, \cdots, v_l$ fixing $v_0$.

For a general $\phi \in S_T$ with only one fixed vertex $v_0$, let $v_1, v_2, \cdots, v_l$ be the $l$ vertices that are adjacent to $v_0$. By the assumption, $\phi$ acts on $\{v_1,v_2,\cdots,v_l\}$ without fixed points. Let $T_1, \cdots, T_l$ be the corresponding subtree with root $v_k$. More precisely $T_k, 1 \leq k \leq l$, is defined to be the subtree of all vertices (and edges) that is reachable from $v_0$ through a chain passing through $v_k$ with
$v_k$ as the root (the initial tip). Then $\phi$ permutes these subtrees. The action of $\phi$ on $v_1, v_2, \ldots, v_l$ is decomposed as a product of independent cyclic cycles. First assume that the action of $\phi$ itself is a cyclic cycle. This implies that $\phi^k, k = 1, \ldots, l - 1$ identifies $T_1$ with $T_k$. By definition $T_k$ is isolated in the sense that each of its vertices has no edge relation with any vertices that is not in $T_k$ except the relation $v_0Ev_k$. Let $T'_k$ be the tree obtained from $T_k$ by adding the edge $[v_0v_k]$. Then $T$ is decomposed as an union of these identical trees with a common root $v_0$. Back to the general case, we may assume that the cyclic cycle decomposition of $\phi$, considered as an element of $S_l$ by the induced action on $\{v_1, \ldots, v_l\}$, is maximal comparing to the decompositions of all other elements of $\Gamma_{v_0}$. Here $\Gamma_{v_0} = \Gamma_{v_k}^T$ be the subgroup of $S_T$ that fixes $v_0$. Indeed, first note that there is a $\phi_1 \in \Gamma_{v_0}$ that brings $v_i'$ in one cycle $C_i$ to $v_j'$ in the other $C_j$ for some $1 \leq i \neq j \leq l$, if and only if there is a identification of subtrees $T_i$ and $T_j$ with $i' \in C_i$ and $j' \in C_j$. Here we have used the fact that all $T_k$ are isolated. Then in this case, all the subtrees $T_k$ with roots in $C_i \cup C_j$ are isomorphic. Again, since all $T_k$ are isolated, this implies that $\Gamma_{v_0}$ contains an element, still denoted by $\phi$, whose cyclic cycle decomposition contains $C_i \cup C_j$.

This proves both existence and uniqueness of such maximal $\phi$. Moreover, each cycle $C_i$ associated to $\phi$ is the collection of all isomorphic subtrees $T_{i'}$ with $i' \in C_i$. Furthermore, the induced action of $\Gamma_{v_0}$ on $\{v_1, \ldots, v_l\}$ preserves each cycle $C_i$. It follows from this that (1) $\Gamma_{v_0}$ is decomposed as a direct product with each factor associated to a cycle $C_k$ above and (2) each factor is mapping surjectively on to the symmetry group permutations of the letters in the cycle $C_k$.

The discussion above implies the following lemma on the structure of each factor.

**Lemma 4.2** Assume that $\phi \in S_T$ has only one fixed vertex $v_0$, which has $l$ adjacent vertices that form a cyclic cycle under the permutation by $\phi$. Then $\phi$ ”generates” a subgroup $S_l^\phi$ of $\Gamma_{v_0}$ that is isomorphic to $S_l$. Let $\Gamma_{v_k}^{T_k}$ be the subgroup of $S_{T_k}$ that fixes the root $v_k$, $k = 1, \ldots, l$. Then $\Gamma_{v_0}$ contains $S_l^\phi$ as its subgroup such that $\Gamma_{v_0}$ maps surjectively on $S_l$ so that the exact sequence $1 \to \prod_{k=1}^l \Gamma_{v_k}^{T_k} \to \Gamma_{v_0} \to S_l \to 1$ splits. Consequently $\Gamma_{v_0}$ is a semi-direct product of $\prod_{k=1}^l \Gamma_{v_k}^{T_k}$ and $S_l^\phi$.

In general, for a splitting exact sequence $1 \to G_1 \to G_0 \to G_2 \to 1$, the group $G_0$ as semi direct product is defined using the conjugation action
of \( G'_2 \) on the normal subgroup \( G_1 \), where \( G'_2 \) is a subgroup of \( G_0 \) mapping isomorphically to \( G_2 \) in the above sequence. For the case at our hands, the action of \( S_l = S_l^\phi \) on \( \prod_{k=1}^l \Gamma_{v_k}^{T_k} \) is the one induced by the conjugation of \( \phi \) that permutes the \( T_k \).

Thus existence of such maximal \( \phi \in S_T \) with only one fixed vertex \( v \) leads to that \( \Gamma_v \) is a direct product of the groups, each still denoted by \( \Gamma_v \) has the form \( \Gamma_v = (\prod_{k=1}^l \Gamma_{v_k}^{T_k}) \times S_l^\phi S_l \), a semi direct product induced by the action of \( S_l^\phi \) on the normal subgroup \( \prod_{k=1}^l \Gamma_{v_k}^{T_k} \).

Finally note that in Gromov-Witten theory and Floer homology, there is a principal component in bubble tree so that the corresponding tree has a preferred vertex \( v_0 \). In this case, \( S_T = \Gamma_{v_0} \) by definition.

It turns out that the existence of \((\phi, v_0)\) as above already implies that \( S_T = \Gamma_{v_0} \). Indeed, for any \( \phi_1 \in S_T \), if \( \hat{v}_0 = \phi_1(v_0) \neq v_0 \), then we may assume that \( \hat{v}_0 \in T_1 \subset T_1' \). Denote the image \( \phi_1(T_k) \) by \( \hat{T}_k \) with the root \( \hat{v}_k = \phi_1(v_k) \). Then \( \hat{v}_1 = \phi_1(v_1) \) is contained in \( T_k \) for some \( k \). We claim that \( k = 1 \). Indeed if \( k \neq 1 \), since \( v_0 E v_1 \) implies that \( \hat{v}_0 E \hat{v}_1 \), this can happen only when \( \hat{v}_0 = v_1 \) and \( \hat{v}_1 = v_0 \). Then \( \phi_1 \) is an involution on \([v_0, v_1]\). The argument before implies that this is impossible. Thus both \( \hat{v}_0 \neq v_0 \) and \( \hat{v}_1 \) are contained in \( T_1 \) with \( \hat{v}_1 \neq v_0 \). Hence both of them are contained in \( T_k \). Now consider any vertex \( v_{1,i} \) lying on \( T_1 \) ”next” to \( v_1 \) with respect to the direction on edges of \( T_k' \) induced by the paths from the root \( v_0 \) to the tips of \( T_1 \). The similar argument implies that \( \hat{v}_{1,i} = \phi_1(v_{1,i}) \) is still contained in \( T_1 \). Inductively, this implies that \( \phi_1 \) maps any vertex of \( T_1 \) into \( T_1 \). This is impossible unless \( \hat{v}_0 = v_0 \).

This proves the following lemma.

**Lemma 4.3** Assume that exist a \( \phi \in S_T \) with only one fixed vertex \( v_0 \). Then \( S_T = \Gamma_{v_0} \). Consequently, \( v_0 \) is a fixed point of \( S_T \).

Thus the above discussion on \( \Gamma_{v_0} \) also gives a inductive ”classification” of \( S_T \) in above essential case. Moreover, the last statement of the above lemma suggests that we can reformulate above classification in term of the fixed point set \( F_{\Gamma_{v_0}} \) of the whole group \( \Gamma_{v_0} \) rather than the fixed point set \( F_{|\phi|} \) of a single element.

Assume that \( \Gamma_{v_0} \subset S_T \) is non-trivial. Then \( v_0 \in F_{\Gamma_{v_0}} \cap_{\phi \in \Gamma_{v_0}} F_{|\phi|} \). This implies that \( F_{\Gamma_{v_0}} \) is a subtree containing \( v_0 \). Here we use the result before that if a vertex \( v_0 \) is in the fixed point set \( F_{|\phi|} \), then \( F_{|\phi|} \) is a subtree. Then there are two cases: (A) dimension of \( F_{\Gamma_{v_0}} \) is zero; (B) dimension of \( F_{\Gamma_{v_0}} \) is one. For the case (B), for any tip \( v \) of \( F_{\Gamma_{v_0}} \), let \( T_v \) be the subtree consists of
vertices that are reachable by chains in $T \setminus F_{\Gamma v_0}$ starting from $v$. Then the action of $\Gamma v_0$ induces automorphisms of $T_v$ with the only one fixed vertex $v$ so that each $T_v$ is in the case (A).

Thus we only need to consider the case (A). In this case, $F_{\Gamma v_0} = v_0$. Then there are two subcases:

(A1) There exists a $\phi \in \Gamma v_0$ such that $v_0$ is the only fixed points. This case has been discussed thoroughly above with an inductive classification of $\Gamma v_0$ which is equal to $S_T$.

(A2) For all $\phi \in \Gamma v_0$, dimension of $F|_\phi$ is always one. We claim that this can not occur. Indeed, if this does happen, consider the vertices $v_1, \ldots, v_l$ adjacent to $v_0$. Since $v_0$ is the only common fixed point, for any $v_k, 1 \leq k \leq l$, there exists a $\phi_{jk}$ such that $\phi_{jk}(v_k) = v_j, 1 \leq j \leq l$ and $j \neq k$. Otherwise $[v_0v_k]$ is a common fixed edge for all $\phi \in \Gamma v_0$. Hence the subtree $T_k$ and $T_j$ are identical. Continue this fashion, we conclude that the $l$ subtrees $T_1, \cdots T_l$ are divided into independent cyclic cycles of length at least 2 such that for each cycle of length $l_i$ there is a $S_{l_i}$ subgroup contained in $\Gamma v_0$ and permutes the corresponding trees over the cycle. Clearly there exists a $\phi \in \Gamma v_0$ whose induced action on $\{v_1, \ldots, v_l\}$ is the product of these cycles. Then $v_0$ is the only fixed point for $\phi$.

Combining with the results before, this proves the following lemma.

**Lemma 4.4** The common fixed point set $F_{\Gamma v_0}$ is the single point $v_0$ if and only if there is a $\phi \in \Gamma v_0$ whose fixed point set $F|_\phi = v_0$. Moreover, in this case, $S_T = \Gamma v_0$.

Thus case (A) here is equivalent to the essential case before.

Now assume that $\phi \in S_T$ with the property in the above lemma is obtained as an induced automorphism on $T$ from an automorphism $\hat{\phi} \in G_{\Sigma,[x]}$. Then $\Sigma$ is decomposed as $l$ identical bubble trees $\Sigma_{T_1'}, \cdots, \Sigma_{T_l'}$ with the common root component $\mathbf{CP}^1_{v_0}$.

The subgroup $\hat{\Gamma}_{v_0}$ corresponding to $\Gamma v_0$ above consists of automorphisms in $G_{\Sigma,[x]}$ that preserve the component $\mathbf{CP}^1_{v_0}$, which induces a homomorphism from $\hat{\Gamma}_{v_0}$ into $\text{PSL}(2, \mathbb{C})$. Of course the image of the homomorphism here contains the cyclic group $C_l$ generated by the image of $\phi$.

The next well-known proposition shows that instead of having $S_l$ as its image, when $l > 6$, the largest possibility for the image of $\hat{\Gamma}_{v_0}$ in $\text{PSL}(2, \mathbb{C})$ is the dihedral group $D_l$. In other words, most of the permutations in $S_l$ can not be realized as the symmetries of $\Sigma$ despite of the fact that $\Sigma$ is decomposed as
identical bubble trees with a common root component. Roughly speaking, inductively, this shows that in each basic factor $S_l$ in the "structure theorem" for $S_T$, only $C_l$ or $D_l$ is realizable as the corresponding symmetries of $\Sigma$.

**Proposition 4.1** Any finite subgroup of $\text{PSL}(2, \mathbb{C})$ is isomorphic to (and conjugate to) one of the following groups: (1) the standard cyclic group $C_l$; (2) the standard dihedral group $D_l$; (3) the rotational symmetry group of a regular polyhedron in $\mathbb{R}^3$.

Note that in the exceptional case (3), the possible finite groups are $A_4$, $S_4$ or $A_5$. In particular, none of these groups contains an element of order greater than six.

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