1. Introduction

We are concerned with the well-posedness of linear elliptic systems of the form

\[-\text{div} \, C : \partial u = f,\]

\[u(x) \sim 0, \quad \text{as} \quad |x| \to \infty,\]  

where \( C \in C(\mathbb{R}^d; \mathbb{R}^{m^2d^2}) \) is bounded and satisfies the Legendre–Hadamard condition,

\[C^\alpha_\partial (x)v_i v_j k_\alpha k_\beta \geq c_0 |v|^2 |k|^2 \quad \forall x, k \in \mathbb{R}^d, \quad v \in \mathbb{R}^m.\]

The functions \( f, u : \mathbb{R}^d \to \mathbb{R}^m \) (we will define the precise function spaces to which \( f \) and \( u \) belong to later on), and \( C : G = (C^\alpha_\partial G^\beta_\partial)_{\alpha\beta} \) denotes the contraction operator.

The concrete problem of interest, for which we require this theory, arises from the linearization of the equations of anisotropic finite elasticity in infinite crystals, however, our results are more generally applicable to translation-invariant problems posed on \( \mathbb{R}^d \).

Some of the main challenges to be overcome in translation-invariant problems on infinite domains are the absence of Poincaré-type inequalities, and the interpretation of boundary conditions.

A common approach to PDEs on infinite domain, as well as for exterior problems, is the formulation in weighted function spaces (see, e.g., [6, 9]). Our aim in this note is to outline a more straightforward existence, uniqueness, and regularity theory in Sobolev spaces of Beppo Levi type (also called homogeneous Sobolev spaces). Such spaces have previously been analyzed in detail in [1] and used for the solution of elliptic PDEs (see, e.g., [7, 8, 3, 5]).

In the present work we describe a version of the homogeneous Sobolev space approach. Variants (and sometimes generalisations) of most of our results can be found in the cited literature; however, the equivalence class viewpoint considered here is not normally taken and the growth characterisation given in Theorem 2.2 appears to be new. This research note is intended as an elementary introduction to and reference for some key ideas.

We wish to define the homogeneous Sobolev space as a closure of smooth functions with compact support. The following cautionary example was discussed by Deny & Lions [1]: let \( u_n : \mathbb{R} \to \mathbb{R} \) be defined by

\[u_n(x) := n \max(0, 1 - |x|/n^3),\]

where \( u_n \) has compact support and \( u'_n(x) = \pm \frac{1}{n^2} \) in \( \pm (0, n^3) \), and hence \( \|\partial u_n\|_{L^2} \to 0 \) as \( n \to \infty \). However, \( u_n \) clearly does not converge in the topology of \( D' \).

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Theorem 2.1. Let a special case of [8, Thm. 1].

Let $\nabla u$ be a linear space of equivalence classes $[u]$ of integrable functions. We have shown that there exists $u \in L^1$ such that $\|\partial u\|_{L^2} = \|\partial v_n\|_{L^2} \to 0$ as $n \to \infty$.

1.1. Notation. $B_R$ denotes the open ball, centre 0, radius $R$ in $\mathbb{R}^d$, $d \in \{1, 2, \ldots\}$; $p \in [1, \infty]$, $p' = p/(p-1)$, and $p^*$ denotes the Sobolev conjugate of $p \in [1, d]$, $d > 1$, defined by $1/p^* = 1/p-1/d$. For Lebesgue and Sobolev spaces of functions defined on the whole of $\mathbb{R}^d$ we shall suppress the symbol $\mathbb{R}^d$ in our notations for these function spaces, and will simply write $L^p$ and $W^{1,p}$, respectively, instead of $L^p(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$. We define the integral average $(u)_A$ of a locally integrable function $u \in L^1_{\text{loc}}(\Omega)$ over a measurable set $A \subset \mathbb{R}^n$, $|A| := \text{meas}(A) < \infty$, by $(u)_A := |A|^{-1} \int_A u(x) \, dx$. Throughout this note $\int$ will signify $\int_{\mathbb{R}^d}$.

Assuming that $\Omega_k$, $k = 1, 2, \ldots$, in an increasing sequence of bounded open sets in $\mathbb{R}^n$, $L^p_{\text{loc}}(\Omega)$ is equipped with the family of seminorms

$$
\|u\|_{L^p(\Omega_k)} := \left(\int_{\Omega_k} |u(x)|^p \, dx\right)^{\frac{1}{p}}.
$$

The linear space $L^p_{\text{loc}}(\Omega)$ is then a Fréchet space (i.e., a metrizable and complete topological vector space).

2. Sobolev spaces of equivalence classes

For any measurable function $u : \mathbb{R}^d \to \mathbb{R}$, let $[u] := \{u + c \mid c \in \mathbb{R}\}$ denote the equivalence class of all translations of $u$. Let $\mathcal{D}$ denote the space of test functions ($C^\infty$ functions with compact support in $\mathbb{R}^d$), and let $\hat{\mathcal{D}} := \{[u] \mid u \in \mathcal{D}\}$ be the associated linear space of equivalence classes $[u]$ of translations of $u \in \mathcal{D}$.

We denote the linear space of equivalence classes $[u]$ of functions $u \in W^{1,p}_{\text{loc}}$ with $p$-integrable gradient by

$$
\hat{W}^{1,p} := \{[u] \mid u \in W^{1,p}_{\text{loc}}, \partial u \in L^p\},
$$

equipped with the norm $\|[u]\|_{\hat{W}^{1,p}} := |u|_{W^{1,p}} = \|\partial u\|_{L^p}$, $u \in [u]$.

Proposition 2.1. $\hat{W}^{1,p}$ is a Banach space.

Proof. It is clear that $\|\bullet\|_{\hat{W}^{1,p}}$ is a semi-norm on $\hat{W}^{1,p}$. To show that it is a norm, suppose that $\|[u]\|_{\hat{W}^{1,p}} = 0$. Then $\partial u = 0$ and hence $u$ is a constant, that is, $[u] = [0]$.

To prove that $\hat{W}^{1,p}$ is complete, suppose that $([u_j])_{j \in \mathbb{N}}$ is a Cauchy sequence. Let $u_j \in [u_j]$ be defined through the condition that $(u_j)_{B_1} = 0$. Then, it is straightforward to show that there exist $u \in L^p_{\text{loc}}$ and $g \in L^p$ such that $u_j \to u$ in $L^p_{\text{loc}}$ and $\partial u_j \to g$ in $L^p$. By the uniqueness of the distributional limit, it then follows that $g = 0$. Hence we have shown that there exists $u \in W^{1,p}_{\text{loc}}$ such that $\partial u_j \to \partial u$ in $L^p$, that is, $[u_j] \to [u]$ in $\hat{W}^{1,p}$ as $j \to \infty$. \qed

The next result establishes that test functions are dense in $\hat{W}^{1,p}$. This result is a special case of [8, Thm. 1].

Theorem 2.1. Let $p \in (1, \infty)$ or $p = 1$ and $d > 1$; then, $\hat{\mathcal{D}}$ is dense in $\hat{W}^{1,p}$.
**Proof.** Suppose first that \( d > 1 \). Fix \( u \in [u] \subseteq \dot{W}^{1,p} \). Since \( \mathcal{D} \) is dense in \( W^{1,p} \) it is sufficient to show the existence of a sequence \( (u_n) \subseteq W^{1,p} \) such that \([u_n] \to [u]\) in \( \dot{W}^{1,p} \).

Let \( \eta \in C^1([0, \infty)) \) be a cut-off function satisfying

\[
\eta(r) = \begin{cases}
1, & r \leq 1, \\
0, & r \geq 2.
\end{cases}
\]

For each \( n \in \mathbb{N} \), let \( A_n := B_{2n} \setminus B_n \) and define

\[
u_n(x) := \eta(|x|/n) \left(u(x) - (u)_{A_n}\right).
\]

Hence,

\[
\partial u_n(x) = n^{-1} \eta'(|x|/n) \frac{x}{|x|} \left(u - (u)_{A_n}\right) + \eta(|x|/n) \partial u.
\]

Since \( u \in W^{1,p}_{\text{loc}} \) and \( u_n \) has compact support, it is clear that \( u_n \subseteq W^{1,p} \). Further, since \( \eta' \) is uniformly bounded, we can estimate

\[
\|\partial u - \partial u_n\|_{L^p} \leq \|n^{-1} \eta'(u - (u)_{A_n})\|_{L^p} + \|(1 - \eta) \partial u\|_{L^p} \\
\leq C n^{-1} \|u - (u)_{A_n}\|_{L^p(A_n)} + \|\partial u\|_{L^p(R^d \setminus B_n)}.
\]

Poincaré’s inequality on \( A_1 \) and a standard scaling argument then imply that

\[
C n^{-1} \|u - (u)_{A_n}\|_{L^p(A_n)} \leq (C n^{-1})(C_{\text{p,n}}) \|\partial u\|_{L^p(A_n)} \leq C \|\partial u\|_{L^p(R^d \setminus B_n)};
\]

that is, \( \|\partial u - \partial u_n\|_{L^p} \leq C \|\partial u\|_{L^p(R^d \setminus B_n)} \). Since \( \|\partial u\|_{L^p} \) is finite it follows that this upper bound tends to zero as \( n \to \infty \).

Hence, we have constructed a sequence \( (u_n) \subseteq W^{1,p} \) such that \( \partial u_n \to \partial u \) in \( L^p \), or, equivalently \([u_n] \to [u]\) in \( W^{1,p} \).

If \( d = 1 \), then \( A_n \) is not simply connected and hence the Poincaré inequality does not hold. Instead, we prove that for any \( u \in W^{1,p}_{\text{loc}} \) with \( u' = \chi_{(a,b)} \) (the characteristic function of an interval) we can construct a sequence \([u_n] \subseteq \mathcal{D}\) approaching \([u]\). Density of the span of characteristic functions in \( L^p \) then implies the stated result for \( d = 1 \). Let \( u_n \) be defined by

\[
u'_n(x) = \begin{cases}
1, & x \in (a,b), \\
-1/n, & x \in (b, b + n(b - a)), \\
0, & \text{otherwise},
\end{cases}
\]

then it is a straightforward computation to show that \( u'_n \to u' = \chi_{a,b} \) in \( L^p \) for any \( p > 1 \), but not in \( L^1 \).

**Remark 2.2.** If \( d = 1 \) then \( \mathcal{D} \) is not dense in \( \dot{W}^{1,1} \). If this were the case, then all functions \( u \in \dot{W}^{1,1} \) would satisfy \( \int_{\mathbb{R}} u' \, dx = 0 \). However, it is clear that the equivalence class of the function \( u(x) = \max(0, \min(x, 1)) \) belongs to \( \dot{W}^{1,1} \), but does not satisfy this condition.

Our next result classifies the growth or decay of classes \([u] \subseteq \dot{W}^{1,p} \) at infinity. Case (i) is essentially contained in [5, Prop. 2.4(i)]; cases (ii) and (iii) are new to the best of our knowledge.

**Theorem 2.2.** There exist linear maps \( J_{\infty} : \dot{W}^{1,p} \to C^\infty \) and \( J_0 : \dot{W}^{1,p} \to W^{1,p} \) such that

\[
[u] = [J_{\infty}[u] + J_0[u]], \quad \text{for } [u] \in \dot{W}^{1,p},
\]
and
\[ \| \partial J_\infty[u] \|_{L^p} \leq \| \partial u \|_{L^p}, \quad \| \partial J_\infty[u] \|_{L^\infty} \leq \| \partial u \|_{L^p} \quad \text{and} \quad \| J_0[u] \|_{W^{1,p}} \leq C \| \partial u \|_{L^p}, \]
where \( C = C(d) > 0 \).

Moreover, \( J_\infty \) may be chosen to satisfy the following growth conditions at infinity:

(i) If \( p < d \), then \( W^{1,p} \) is continuously embedded in \( L^{p^*} \), in the sense that, for each \( [u] \in W^{1,p} \) there exists a unique \( u_0 \in [u] \) such that \( u_0 \in L^{p^*} \) and \( \| u_0 \|_{L^{p^*}} \leq C \| \partial u_0 \|_{L^p} \), where \( C \) is a positive constant independent of \( u_0 \). In particular, \( J_\infty[u](x) \to 0 \) as \( |x| \to \infty, \ x \in \mathbb{R}^d \).

(ii) If \( p > d \), then \( |J_\infty[u](x)| \leq C \| [u] \|_{W^{1,p}} |x|^{1/p^*}, \ x \in \mathbb{R}^d \).

(iii) If \( p = d \), then \( |J_\infty[u](x)| \leq C \| [u] \|_{W^{1,p}} \log(2 + |x|), \ x \in \mathbb{R}^d \).

**Proof.** We shall assume throughout that \( 1 \leq p < \infty \); in case (ii) the choice of \( p = \infty \) can be dealt with separately using an analogous argument to the one for \( d < p < \infty \).

Let \( \eta \in \mathcal{D}, \ 0 \leq \eta \leq 1, \int \eta(x) \, dx = 1, \) fix \( u \in [u] \in W^{1,p} \) and define
\[ v := \eta * u \in C^\infty, \quad \text{and} \quad w := u - v. \]

By Young’s inequality for convolutions, \( \| \partial v \|_{L^p} \leq \| \partial u \|_{L^p} \), and, because of the assumption that \( \eta \leq 1 \), it is also straightforward to show that \( \| \partial v \|_{L^\infty} \leq \| \partial u \|_{L^p} \):
\[ |\partial v(x)| = \left| \int \eta(x - z)\partial u(z) \, dz \right| \leq \left( \int \eta(x - z)|\partial u(z)|^p \, dz \right)^{1/p} \leq \| \partial u \|_{L^p}. \]

Next, we show that \( w \in W^{1,p} \). It follows directly from the definition of \( w \) that \( \partial w = \partial u - \partial v \in L^p \). Hence, \( \| \partial w \|_{L^p} \leq \| \partial u \|_{L^p} + \| \partial v \|_{L^p} \leq 2 \| \partial u \|_{L^p} \). To show that \( w \in L^p \), let \( R > 0 \) be such that \( \text{supp} \eta \subset B_R \). For any \( \xi \in \mathbb{R}^d \) we have
\[ \int_{B_R(\xi)} |w(x)|^p \, dx = \int_{B_R(\xi)} \left| \int \eta(x - z)(u(z) - u(x)) \, dz \right|^p \, dx \leq \int_{B_R(\xi)} \int \eta(x - z)|u(z) - u(x)|^p \, dz \, dx \leq \int_{B_{2R}(\xi)} \int_{B_{2R}(\xi)} |u(z) - u(x)|^p \, dz \, dx \leq C(R) \| \partial u \|_{L^p(B_{2R}(\xi))}. \]

where the last inequality is an immediate consequence of Poincaré’s inequality on the ball \( B_{2R}(\xi) \). We can cover \( \mathbb{R}^d \) with countably many balls \( B_R(\xi), \ \xi \in R \mathbb{Z}^d \), such that the balls \( B_{2R}(\xi) \) have finite overlap, that is, any \( x \in \mathbb{R}^d \) belongs to at most \( m \) balls where \( m \) is independent of \( x \). Summing over all balls gives the result that \( \| w \|_{L^p} \leq C \| \partial u \|_{L^p} \), where \( C \) may depend on the support of \( \eta \) and hence on the dimension \( d \), but is independent of the value of \( p \).

We now distinguish between three cases, depending on the values of \( p \) and \( d \).

(i) \( p < d \): Since \( \mathcal{D} \) is dense in \( W^{1,p} \) and by the Gagliardo–Nirenberg–Sobolev Inequality, it follows that \( W^{1,p} \) is embedded in \( L^{p^*} \) in the sense that for each \( [u] \in W^{1,p} \) there exists a unique \( u_0 \in [u] \) such that \( u_0 \in L^{p^*} \) and \( \| u_0 \|_{L^{p^*}} \leq C_{GNS} \| \partial u_0 \|_{L^p} \), where \( C_{GNS} \) is the constant in the Gagliardo–Nirenberg–Sobolev Inequality. We define
\[ J_\infty[u] := v_0 := \eta * u_0. \]
It is an immediate consequence of this definition that \( v_0 \in L^{p_s} \). Since \( \| \partial v_0 \|_{L^\infty} \) is finite, it follows also that the sequence \( \{ \| v_0 \|_{L^\infty(B_1(\xi))} \} \in \mathbb{Z}^d \) belongs to \( \ell^p(\mathbb{Z}^d) \), and this implies that \( \| v_0 \|_{L^\infty(B_1(\xi))} \to 0 \) uniformly as \( |\xi| \to \infty \). We obtain statement (i) as a special case.

(ii) \( p > d \): In this case we define \( J_\infty[u](x) := (\eta * u)(x) - (\eta * u)(0) \) for any element \( u \in [u] \), which does not of course change the foregoing results. If we define \( v := \eta * u \), then we obtain

\[
|J_\infty[u](r)| = |v(r) - v(0)| \leq \left| \int_0^{|r|} \partial v(t \frac{r}{|r|}) \frac{r}{|r|} \, dt \right| \\
\leq |r|^{1/p'} \left( \int_0^{|r|} |\partial v(t \frac{r}{|r|})|^p \, dt \right)^{1/p} \\
\leq |r|^{1/p'} \left( \int_0^{|r|} \int \eta(t \frac{r}{|r|} - z) \partial u(z) \, dz \, dt \right)^{1/p} \\
\leq |r|^{1/p'} \left( \int_0^{|r|} \int \eta(t \frac{r}{|r|} - z) \, dt \right)^{1/p} \left( \int_0^{|r|} |\partial u(z)|^p \, dz \right)^{1/p}.
\]

Since the diameter of the support of \( \eta \) is independent of \( |r| \) it follows that

\[
\int_0^{|r|} \eta(t \frac{r}{|r|} - z) \, dt \leq C
\]

for some universal constant \( C \), which implies (ii).

(iii) \( p = d \): In this critical case, we use the fact that \( \dot{W}^{1,p} \) may be embedded in the space \( \text{BMO} \) of functions of bounded mean oscillation (see [4]), though for simplicity we will not refer to \( \text{BMO} \) directly.

If \( Q \) is a cube in \( \mathbb{R}^d \) with arbitrary orientation and \( u \in [u] \in \dot{W}^{1,p} \), then

\[
\int_Q |u - (u)_Q| \, dx \leq |Q|^{1/p'-1} \| u - (u)_Q \|_{L^p(Q)} \leq C \| \partial u \|_{L^p(Q)} \leq C \| [u] \|_{\dot{W}^{1,p}},
\]

where the second inequality follows on noting that \( |Q|^{1/p'-1} = |Q|^{-1/d} \leq (\text{diam}Q)^{-1} \).

The key observation is that if \( u \in W^{1,p}_{\text{loc}} \) with \( \partial u \in L^p \), then for each \( x \in \mathbb{R}^d \) there exist unit cubes \( Q_x \) centred at \( x \) and \( Q_0 \) centred at 0, such that

\[
\left| (u)_{Q_x} - (u)_{Q_0} \right| \leq C \| \partial u \|_{L^p} \log(2 + |x|). \tag{2.3}
\]

The proof of inequality (2.3) will be given below, after the end of the proof of this theorem. With this in hand, we can define \( J_\infty[u](x) := v_0 := (\eta * u)(x) - (\eta * u)(0) \) to obtain

\[
|v_0(x)| \leq |v_0(x) - (v_0)_{Q_x}| + |(v_0)_{Q_x}| + |(v_0)_{Q_x} - (v_0)_{Q_0}| \\
\leq C \| \partial v_0 \|_{L^\infty} + C \| \partial v_0 \|_{L^\infty} + C \| \partial v_0 \|_{L^p} \log(2 + |x|) \\
\leq C \| \partial u \|_{L^p} \log(2 + |x|),
\]

for a generic (dimension-dependent) constants \( C \). This concludes the proof of case (iii). \( \square \)
Figure 1. Visualization of an argument used in the proof of (2.3).

Proof of (2.3). We assume without loss of generality that \(|x| \geq 1\). Let \(Q_0, Q_x\) be unit cubes centred, respectively, at 0 and \(x\) such that one set of edges of each of the cubes \(Q_0\) and \(Q_x\) is aligned with the direction \(\vec{x}\). There exists \(N \leq C(2 + \log |x|)\) and cubes \(Q_2, \ldots, Q_{N-1}\) with the same alignment as \(Q_0, Q_x\) and with disjoint interior such that, for any two neighbouring cubes, their sidelengths differ by at most a factor 2 and one face of the smaller cube is contained within one face of the large cube. See Figure 1 for a visualization of this argument.

For any two neighbouring cubes \(Q_j, Q_{j+1}\) we have
\[
|(u)_{Q_j} - (u)_{Q_{j+1}}| \leq C\|\partial u\|_{L^p},
\]
which is a special case of [4, Lemma 2], but can also be verified directly by enclosing \(Q_j, Q_{j+1}\) in a larger cube of approximately the same size. Hence, defining \(Q_x = Q_N\), we obtain
\[
|(u)_{Q_x} - (u)_{Q_0}| \leq \sum_{j=0}^{N-1} |(u)_{Q_{j+1}} - (u)_{Q_j}| \leq N\|\partial u\|_{L^p}. \tag*{□}
\]

Remark 2.4. The map \(J' := J_\infty + J_0\) defines an embedding of \(\dot{W}^{1,p}\) into \(\mathcal{D}'\). Since \(J_0\) is continuous to \(W^{1,p}\) and \(J_\infty\) is continuous to \(W^{1,\infty}_{\text{loc}}\) it follows that the embedding \(J'\) of \(W^{1,p}\) into \(\mathcal{D}\) is in fact continuous. However, it is not particularly useful for our purposes since we are explicitly interested in operations that are translation invariant, that is, independent of the representative \(u \in [u]\), whenever \([u]\) lies in \(\dot{W}^{1,p}\).

3. Well-posedness and regularity

From now on we restrict our presentation to the case \(p = 2\) and hence define \(\dot{H}^1 := W^{1,2}\). Since we will take particular care that all operators, linear functionals, and bilinear forms we consider are translation invariant, we will drop the brackets in \([u] \in H^1\) and instead write simply \(u \in H^1\) instead, by which we mean an arbitrary representative from the class \([u]\). (For convenience one may take \(u = J_\infty[u] + J_0[u]\).)

Since we consider elliptic systems, we will from now on identify all function spaces with spaces of vector-valued functions, that is, \(L^p = (L^p)^m, \dot{H}^1 = (\dot{H}^1)^m\), and so forth, for some fixed \(m \in \mathbb{N}\).

Before we embark on the analysis of the elliptic system (1.1) we briefly discuss admissible right-hand sides \(f\) for (1.1) as well as the far-field boundary condition (1.2).

3.1. The dual of \(\dot{H}^1\). We denote the topological dual of \(\dot{H}^1\) by \(\dot{H}^{-1}\). Since \(\dot{H}^1\) is a Hilbert space with inner product \((\partial \cdot, \partial \cdot)_{L^2}\) it follows that, for each \(\ell \in \dot{H}^{-1}\), there exists \(F \in L^2\) such that \(\ell = -\text{div} F\) in the distributional sense. (For a generalisation of this result to \(W^{-1,p}\), \(p \in (1, \infty)\) see [5, Lemma 2.2].)
If we wish to define $\ell$ via an $L^2$-pairing, then the following two examples give concrete conditions:

1. Let $f \in L^1_{\text{loc}}$; then we can define $\ell : \mathcal{D} \to \mathbb{R}$ by
   \[ \ell([u]) := \int_{\mathbb{R}^d} f \cdot u^* \, dx, \quad \text{where } u^* \in [u], u^* \in \mathcal{D}. \]  
   (3.1)

   If, moreover, $f = \text{div} g$, where $g \in L^2$, then (3.1) can be extended to a bounded linear functional on $H^1$.

2. More concretely, if $f \in L^1$ with $\int_{\mathbb{R}^d} f \, dx = 0$ and $\text{div} g = f$, $g \in L^2$, then we may define
   \[ \ell([u]) := \int_{\mathbb{R}^d} f \cdot u \, dx, \quad \text{for any } u \in [u] \in \mathcal{D}. \]  
   (3.2)

   Again, (3.2) can be extended to a bounded linear functional on $\dot{H}^1$.

3. Even more concretely, if $f \in L^1 \cap L^\infty$ with $\int_{\mathbb{R}^d} f \, dx = 0$, and $\text{div} g = f$, $g \in L^2$, then this is sufficient to ensure that $\ell$ defined through (3.2) can be extended to a bounded linear functional on $\dot{H}^1$ (see Lemma 3.3 below). We note, however, that right-hand sides with such strong decay assumptions may be more naturally treated within the framework of weighted Sobolev spaces [6].

**Lemma 3.3.** Suppose that $f \in L^1 \cap L^2$ with $\int_{\mathbb{R}^d} f \, dx = 0$, $f \otimes x \in L^1$, and let $\ell : \hat{\mathcal{D}} \to \mathbb{R}$ be defined through (3.2); then
\[ \ell(u) \leq C \| \partial u \|_{L^2} \quad \forall u \in \hat{\mathcal{D}}. \]

**Proof.** Consider the Fourier transform of $f$, which is defined in a pointwise sense since $f \in L^1$:
\[ \hat{f}(k) = \int f(x) \exp(-ik \cdot x) \, dk. \]

Taking the formal derivative with respect to $k$ we obtain
\[ \partial \hat{f}(k) = -i \int f(x) \otimes x \exp(-ik \cdot x) \, dk. \]

If $f \otimes x \in L^1$ then Lebesgue’s differentiation theorem can be used to make this rigorous. Hence we deduce that $\hat{f} \in W^{1,\infty}$. Therefore, since $\hat{f}(0) = 0$, it follows that $\hat{f}(k)/|k|$ is bounded as $k \to 0$. Because $f \in L^2$, it follows that $\hat{f} \in L^2 \cap L^\infty$, from which is follows easily that $\hat{f}(k)/|k| \in L^2$.

**3.2. The far-field boundary condition.** In this section we interpret the far-field boundary condition (1.2) by showing that the space $\hat{H}^1$ is a natural ansatz space to make this condition rigorous. A simple motivation for selecting $\hat{H}^1$ as space of functions in which a solution to (1.1), (1.2) is sought, is that this space can be understood as the closure of $\mathcal{D}$ in an “energy-norm”. However, we can give a finer interpretation of (1.2) by employing Theorem 2.2.

Let $m = d$. Suppose that an elastic body occupies the reference domain $\mathbb{R}^d$. Deformations of $\mathbb{R}^d$ are sufficiently smooth invertible maps $y : \mathbb{R}^d \to \mathbb{R}^d$. Suppose we apply a far-field boundary condition
\[ y(x) \sim Ax \quad \text{as } |x| \to \infty \]  
(3.4)
for some non-singular matrix \( A \in \mathbb{R}^{d \times d} \), which is usually understood to mean
\[
y(x) = Ax + o(|x|), \quad \text{or, equivalently} \quad \frac{|y(x) - Ax|}{|x|} \to 0 \quad \text{as} \quad |x| \to \infty.
\]

Suppose now that we decompose \( y(x) = Ax + u(x) \); then, the far-field boundary condition (3.4) for the deformation, written in terms of the displacement \( u \), becomes
\[
u(x) = o(|x|), \quad \text{or, equivalently,} \quad \frac{|u(x)|}{|x|} \to 0 \quad \text{as} \quad |x| \to \infty. \tag{3.5}
\]

While the pointwise condition (3.5) cannot be satisfied for classes of Sobolev functions, Theorem 2.2 indicates that (3.5) is satisfied “on average” for the representative \( J_3 \).

**Weak form and well-posedness.** Let \( \mathcal{C} = (\mathcal{C}^{ij\beta}_{\alpha\iota\iota})_{\iota=1,\ldots,m} \in (L^\infty)^{m^2d} ; \) we then define the symmetric bilinear form \( a : \dot{H}^1 \times \dot{H}^1 \to \mathbb{R} \),
\[
a(u, v) := \int \mathcal{C}^{ij\beta}_{\alpha\iota\iota} \partial_{\alpha} u_i \partial_{\beta} v_j \, dx;
\]
where, here and throughout, we employ the summation convention. Clearly, \( a \) is bounded,
\[
a(u, v) \leq c_1 \| \partial u \|_{L^2} \| \partial v \|_{L^2} \quad \forall u, v \in \dot{H}^1,
\]
where \( c_1 = \| \mathcal{C} \|_{L^\infty} \), hence we can pose (1.1), (1.2) in weak form:
\[
a(u, v) = \ell(v) \quad \forall v \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^m), \tag{3.6}
\]
where \( \ell \) is of the form of Example 1 discussed in §3.1. An application of the Lax–Milgram theorem gives the following result.

**Theorem 3.1.** Suppose that \( a \) is also coercive:
\[
a(u, u) \geq c_0 \| \partial u \|_{L^2}^2 \quad \forall u \in \dot{H}^1(\mathbb{R}^d; \mathbb{R}^m), \tag{3.7}
\]
for some constant \( c_0 > 0 \); then, (3.6) possesses a unique solution.

We present three elementary examples of coercivity (3.7):

1. If \( m = 1 \) and \( \mathcal{C} := (\mathcal{C}^{i\beta}_{\alpha\iota\iota})_{\alpha,\beta=1,\ldots,d} \) is uniformly positive definite, i.e., \( k^T \mathcal{C}(x) k \geq c_0 |k|^2 \) for a.e. \( x \in \mathbb{R}^d \) and for all \( k \in \mathbb{R}^d \), then (3.7) holds.
2. If \( m \in \mathbb{N} \), \( \mathcal{C} \) is a constant tensor and satisfies the Legendre–Hadamard condition
\[
\mathcal{C}^{ij\beta}_{\alpha\iota\iota} v_iv_j k_\alpha k_\beta \geq c_0 |v|^2 |k|^2 \quad \forall v \in \mathbb{R}^m, \quad k \in \mathbb{R}^d,
\]
then (3.7) holds.

This result is classical if \( u \in \mathcal{D} \). Since \( a \) is translation invariant it follows that it also holds for all \( u \in \mathcal{D} \). Since \( a \) is bounded and \( \mathcal{D} \) is dense in \( \dot{H}^1 \) coercivity holds also in the full space \( \dot{H}^1 \).

3. Let \( \mathcal{C} \in \mathbb{R}^{d^2m^2} \) be a constant tensor satisfying \( \mathcal{C}^{ij\beta}_{\alpha\iota\iota} \) with \( c_0 = \bar{c}_0 \); then, (3.7) holds with \( c_0 = \bar{c}_0 - c_1 \| \mathcal{C} \|_{L^\infty} \).

**Remark 3.9.** One may give more general conditions for coercivity of \( a \) (or inf-sup conditions) based on Gårding’s inequality and conditions on the \( L^2 \)-spectrum of \( a \).
3.4. Regularity. Higher regularity of the right-hand side leads to higher regularity of the solution to (3.6). For \( s \in \{2,3,\ldots\} \) we define

\[
H^s := \{ u \in \dot{H}^1 \mid \partial u \in H^{s-1} \}.
\]

In the following theorem we present conditions for \( \dot{H}^2 \) and \( \dot{H}^3 \) regularity. Regularity in \( H^s \) for \( s \geq 4 \) can be established similarly.

**Theorem 3.3.** Let all conditions of Theorem 3.1 be satisfied, and let \( u \) denote the unique solution to (3.6).

(i) Suppose, in addition, that \( C \in C^1 \) and \( f \in L^2 \); then, \( u \in \dot{H}^2 \) and

\[
\| \partial^2 u \|_{L^2} \leq C \left( \| f \|_{L^2} + c_2 \| \partial C \|_{L^\infty} \right).
\]

(ii) Suppose, in addition, that \( C \in C^2 \), \( \partial C \in L^2 \), and \( f \in H^1 \); then, \( u \in \dot{H}^3 \) and

\[
\| \partial^3 u \|_{L^2} \leq C \left( \| f \|_{L^2} + \| \partial C \|_{L^2} \| \partial^2 u \|_{L^2} + c_2 \| \partial^2 C \|_{L^\infty} \right).
\]

**Proof.** Let \( u_* := J_{\infty}[u] + J_0[u] \) denote a concrete representative of the solution \([u] \in \dot{H}^1 \) of the weak form (3.6). Then clearly \( u_* \) satisfies (3.6). The finite difference technique [2, Section 6.3.1] ensures that \( u_* \in H^{s+2}_{\text{loc}} \), and in particular \( \partial u_* \in H^{s+1}_{\text{loc}} \). The latter property is independent of the representative; hence we may say that \( \partial u \in H^{s+1}_{\text{loc}} \).

To obtain the global bound (i), we test (3.6) with \( v' = \partial_\gamma v \) for some \( v \in \mathcal{D} \), \( \gamma \in \{1,\ldots,d\} \). Then,

\[
\int f \cdot \partial_\gamma v \, dx = \int C_{\alpha}^{i\beta} \partial_\alpha u_i \partial_\beta \gamma_j v_j \, dx
\]

\[
= - \int \left( \partial_\gamma C_{\alpha}^{i\beta} \partial_\alpha u_i + C_{\alpha}^{i\beta} \partial_\gamma \partial_\beta u_i \right) \partial_\beta v_j \, dx,
\]

which implies that

\[
c_0 \| \partial(\partial_\gamma u) \|_{L^2} \leq \| f \|_{L^2} + \| \partial C \|_{L^\infty} \| \partial u \|_{L^2} \leq \| f \|_{L^2} + c_2 \| \partial C \|_{L^\infty}.
\]

To prove (ii) we test with \( v' = \partial_\gamma \partial_\delta v \) and perform a similar calculation. \( \square \)

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