A DYNAMICAL RESOLUTION OF
THE SIGMA TERM PUZZLE

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Abstract

We propose a resolution of the puzzle posed by the discrepancy between the value of the pion–nucleon sigma term inferred from pion–nucleon scattering, and that deduced from baryon mass splittings using the Zweig rule. We show that there is a significant hypercharge–dependent dynamical contribution to baryon masses, not hitherto included in the analysis, which may be estimated using the scale Ward identity, and computed by solution of the Schwinger–Dyson equation for the quark self–energy. We find that the discrepancy is completely resolved without any need for Zweig rule violation.

Submitted to: \textit{Nuclear Physics B}

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1. The Sigma Term Puzzle

The sigma term puzzle \([1]\) is one of several instances in which experimental data seem to disagree with our understanding of the structure of the nucleon based on naive quark model intuition. The pion–nucleon sigma term is defined as the nucleon matrix element of the light quark mass term in the QCD Hamiltonian:

\[
\sigma = \bar{m} \langle N | (\bar{u}u + \bar{d}d) | N \rangle, \tag{1.1}
\]

where, ignoring isospin breaking, \(\bar{m} = \frac{1}{2}(m_u + m_d)\). It is physically interesting because it relates the pattern of chiral symmetry breaking in QCD to the quark content of the nucleon \([1][2]\).

The value of \(\sigma\) can be determined experimentally by using current algebra to connect the matrix element (1.1) to the value of the isospin even pion–nucleon scattering amplitude at zero momentum transfer \([3]\). Taking several subtleties in the extraction of the matrix element from the scattering data into account\([4]\) leads to the value \(\sigma \approx 45\) MeV, with an overall experimental and theoretical uncertainty of perhaps up to 10 MeV.

On the other hand, the sigma term may also be related to the mass splittings in the baryon octet \([5]\). If we assume that there are no strange quarks in the nucleon (according to the Zweig rule), or more specifically that \(\langle N | \bar{s}s | N \rangle = 0\), then the sigma term (1.1) is equal to the octet sigma term

\[
\sigma_8 = \bar{m} \langle N | (\bar{u}u + \bar{d}d - 2\bar{s}s) | N \rangle, \tag{1.2}
\]

which in turn is proportional to the nucleon matrix element of the octet portion \(H_8\) of the quark mass term in the QCD Hamiltonian:

\[
\sigma_8 = \frac{3}{1 - m_s/\bar{m}} \langle N | H_8 | N \rangle \tag{1.3}
\]

where

\[
H_8 = \frac{1}{3} (\bar{m} - m_s) (\bar{u}u + \bar{d}d - 2\bar{s}s). \tag{1.4}
\]

However, in the quark model the mass splittings are assumed, using first order perturbation theory, to be due to the SU(3) octet component of the (effective) strong Hamiltonian \(O_8\); this leads to the relation

\[
M_B = M_0 + M_1 \langle B | Y | B \rangle + M_2 \langle B | (I(I + 1) - \frac{1}{2}Y^2) | B \rangle, \tag{1.5}
\]

where \(I\) is the isospin operator, and \(Y\) the hypercharge. Eq.(1.3), which expresses the masses of all the octet baryons in terms of two parameters, is in excellent agreement with the data. The matrix element of the symmetry breaking operator \(O_8\) between nucleon states can then be determined by SU(3) algebra;

\[
\langle N | O_8 | N \rangle = M_1 - \frac{1}{2} M_2 \\
= M_\Lambda - M_\Xi = \frac{2}{3} M_N - \frac{1}{3} (M_\Xi + M_\Sigma). \tag{1.6}
\]
Now, if $O_8$ is identified with the octet component of the QCD Hamiltonian, namely $H_8$, using the value of the ratio $m_s/\bar{m} = 25\pm 5$ determined through current algebra,

\[ \sigma_8 = 25 \text{ MeV}, \quad \text{with an uncertainty of about 5 MeV.} \]

The large percentage discrepancy between the values of $\sigma$ and $\sigma_8$ is known as the sigma term puzzle.

It has now become rather fashionable to see this discrepancy as a consequence of the failure of the assumption upon which the identification of $\sigma$ and $\sigma_8$ is based, namely the Zweig rule; one then infers that the strange matrix element $\langle N | \bar{s}s | N \rangle$ is relatively large. More than 300 MeV of the nucleon mass would then be due to the strange quark condensate, and kaons could condense out in nuclear matter at low densities. Alternatively, the discrepancy could be interpreted as a breakdown of first-order perturbation theory, i.e., of the linear dependence of the mass splittings on the symmetry breaking operator which is used to derive eq. (1.5) and relate the mass splittings to the matrix element (1.3).

Both of these alternatives are rather unpalatable, since they are extremely hard to reconcile with the successfulness of the quark model mass formulae which are derived using the same assumptions. For example, the Gell-Mann–Okubo mass formula (1.6) is accurate to a few percent. Also, Weinberg has determined the current quark masses assuming again a linear dependence of baryon masses on quark masses; assuming that the operator $\bar{\psi}\psi$ is proportional to the quark plus antiquark number, the mass splittings are given by the difference in quark content of the various hadrons. This is essentially equivalent to eq. (1.5) with $M_2 = 0$, and is phenomenologically accurate to about 20% in agreement with the fact that fitting eq. (1.5) to the baryon octet spectrum leads to $M_2 \approx \frac{1}{3} M_1$. 

Here we will show that there is a third possibility which successfully resolves the puzzle; namely that whereas the mass splittings are indeed given by a linear mass formula according to eq. (1.3), the identification of matrix elements of $O_8$ with those of $H_8$ is incorrect due to the fact that most of the baryon mass, and thus a substantial proportion of the mass splittings, arises dynamically from the trace anomaly. Once this contribution is taken into account, it follows that the octet operator responsible for mass splittings differs from $H_8$ due to the presence of an additional term, generated dynamically, which may be viewed as a nonsinglet gluonic contribution to the mass splittings. Thus the puzzle is resolved, but the quark model results, which follow from the assumption of linear dependence on an octet operator, are preserved.

In section II we show that the conformal anomaly equation leads to an exact relation between the sigma term and mass splittings, which implies that the identification of the nonsinglet matrix elements of $O_8$ and $H_8$ is indeed spoiled by the presence of a dynamical contribution to the masses. We then use a Ward identity to argue that this contribution is nonsinglet and large enough to account for the observed discrepancy. In section III we test this explanation by attempting a computation of the nonsinglet part of the dynamically generated mass, solving the Schwinger–Dyson equation for the quark self-energy in various approximations. We find that this gives a value for the sigma term perfectly consistent with the data, despite large uncertainties in the infrared dynamics responsible for the mass generation. Conclusions are drawn in section IV.
2. The Trace Anomaly and Dynamical Mass Generation

In order to discuss the relationship of the mass splittings to the matrix elements of $H_8$ we exploit an exact relation between the matrix elements of the sigma term and the nucleon masses. Classically the sigma term (1.4) is equal to the divergence of the Noether current for scale transformations, which in turn equals the trace of the energy-momentum tensor. In the quantized theory, this implies a Ward identity that allows us to relate the matrix elements of the trace of the energy-momentum tensor to the masses of physical states.

2.1. The Scale Ward Identity

Consider the dilation current $j_D^\mu = x_\mu T_{\mu
u}$. This is the Noether current for scale transformations; its divergence is equal to the trace of the energy-momentum tensor $T_{\mu
u}$. In the quantized theory, this implies a Ward identity that allows us to relate the matrix elements of the trace of the energy-momentum tensor to the masses of physical states.

On the other hand, the forward matrix element of the trace of $T_{\mu\nu}$ between hadron states is just the mass of the hadron, due to Lorentz invariance and the absence of a massless scalar Goldstone boson [12]. It follows that the baryon matrix element of eq.(2.1) is

$$\langle B | (1 + \gamma_m) \sum_i m_i \bar{\psi}_i \psi_i | B \rangle = M_B,$$

(2.2)

which is the desired Ward identity.

Eq.(2.2) shows that the first-order perturbative expression which equates mass splittings to nonsinglet matrix elements of the mass term is protected by the scale Ward identity, and exact up to quantum corrections. These appear in eq.(2.2) in two distinct instances, namely, the nonzero value of the anomalous dimension $\gamma_m$, which effectively rescales the current masses, and the presence of the conformal anomaly contribution. We can thus separate $M_B$ into a “current” contribution $M^C_B$ and a dynamical contribution $M^D_B$; $M_B = M^C_B + M^D_B$ where

$$M^C_B = \langle B | \sum_i m_i \bar{\psi}_i \psi_i | B \rangle, \quad M^D_B = \langle B | \frac{\beta(\alpha_s)}{4\alpha_s} G_{a\mu\nu} G_{a\mu\nu} + \gamma_m \sum_i m_i \bar{\psi}_i \psi_i | B \rangle.$$

(2.3)

Notice that this is a scale invariant separation, since both $M^C_B$ and $M^D_B$ are renormalization group invariant [13]. Accordingly, we can separate the mass splittings into current and
dynamical contributions by defining $M_1^C$, $M_2^C$ and $M_1^D$, $M_2^D$ in analogy with (1.3), with
$M_1 = M_1^C + M_1^D$ and $M_2 = M_2^C + M_2^D$. The same algebra that led to (1.6) also gives
$\langle N|\mathcal{H}_8|N\rangle = M_1^C - \frac{1}{2}M_2^C$, so that
\[
\Delta M \equiv \langle N|\mathcal{O}_8|N\rangle - \langle N|\mathcal{H}_8|N\rangle = M_1^D - \frac{1}{2}M_2^D.
\] (2.4)

In the conventional argument [3], $M_B^D$ is neglected with the result that $\Delta M$ vanishes, and
the sigma term is given by eqns (1.3) and (1.6). This is presumably done on the
grounds that the term in (2.3) proportional to the mass anomalous dimension $\gamma_m$ is small
enough to be ignored, whereas that containing the isosinglet gluon operator $\frac{\beta(\alpha_s)}{4\alpha_s} G_\mu^a G_\mu^a$ can be assumed to make no contribution to mass splittings. We will now show that this
naive assumption is incorrect; away from the chiral limit $M_1^D$ is actually rather large,
and with opposite sign to $M_1^C$, so that the magnitude of $\langle N|\mathcal{H}_8|N\rangle$, and thus of $\sigma_8$, is
significantly increased.\footnote{Of course a small nonzero value of $M_1^D$ is present due to the term proportional to the
mass anomalous dimension. This is actually as large as $30\%$ of $M_1^C$, since at the nucleon scale
$\gamma_m(1\text{ GeV}) = 0.27$ (to two loop order) [14]. This contribution has (obviously) the same sign
as $M_1^C$ and accordingly if taken into account (while neglecting anything else) would make the
disagreement between the value of $\sigma$ and that of $\sigma_8$ worse, because in eq. (1.3) $\sigma_8$ would be replaced
by its rescaled value $\sigma_8^R = \frac{1}{1+\gamma_m}\sigma_8$. However including such a contribution while neglecting that
of the gluonic operator is a rather dubious procedure, since the separation of $M_B^D$ into a “quark”
and “gluon” piece is necessarily dependent on the renormalization scale.}

2.2. Estimation of the Dynamical Mass Splitting

To this purpose, it is convenient to rewrite the matrix element on the left hand side of
eq.(2.2) in terms of one-particle irreducible physical couplings, analogously to what is done
in the pseudoscalar case in order to derive the isosinglet Goldberger-Treiman relation[13].
This can be done by defining a connected generating functional, $W(S_\mu^D, S_A)$ where $S_\mu^D$
is the source for the dilation current $j_\mu^D$, and $S_A$ are sources for the set of fields $\Phi_A \equiv
(B, \bar{B}, \phi_i \equiv \bar{\psi}_i \psi_i, Q \equiv \frac{\beta(\alpha_s)}{4\alpha_s} G_\mu^a G_\mu^a)$ (which includes the baryons, the scalar quark
condensates, and the gluon condensate respectively). Also, as in ref.[13] we define the
Zumino effective action $\Gamma(S_\mu^D, \Phi_A^{cl})$ by Legendre transformation of $W(S_\mu^D, S_A)$ with respect
to the fields $\Phi_A$ (but not $j_\mu^D$), so that $\Gamma(S_\mu^D, \Phi_A^{cl})$ generates diagrams which are one-particle
irreducible with respect to $\Phi_A^{cl}$.

Rewriting the Ward identity (2.2) in terms of the generating functionals $W$ and $\Gamma$ it can
then be shown [11] that
\[
M_B = (1+\gamma_m) \sum_i m_i \frac{\delta^3 W}{\delta S_i \delta S_B \delta S_B} + \frac{\delta^3 W}{\delta S_Q \delta S_B \delta S_B}
\] (2.5)
\[
= -\Delta_{\bar{B}B} \sum_i \frac{\delta^3 \Gamma}{\delta \phi_i^{cl} \delta B \delta B} \langle \phi_i^{cl}\rangle \Delta_{BB},
\] (2.6)
where $\Delta_{BB}$ ($\Delta_{\overline{B}\overline{B}}$) is the baryon (antibaryon) propagator. It follows that the one particle reducible contributions to the matrix elements of the gluon and quark condensates in eq.(2.5) must cancel against each other in order to yield the one particle irreducible coupling of eq.(2.6), namely
\[
\langle B | \frac{\beta(\alpha_s)}{4\alpha_s} G^\mu\nu G^a_{\mu\nu} | B \rangle^{\text{opr}} + (1 + \gamma_m) \langle B | \sum_i m_i \bar{\psi}_i \psi_i | B \rangle^{\text{opr}} = 0,
\]
(2.7)
or, in the notation of eq.(2.3),
\[
(M_C^B)^{\text{opr}} + (M_D^B)^{\text{opr}} = 0.
\]
(2.8)
This shows immediately that it is quite unreasonable to assume $M_D^B$ to be isosinglet given that (just as in the pseudoscalar sector, where similar results hold [15], [16]) one would expect the one particle reducible contributions to $M_C^B$ to have a significant flavor–nonsinglet component.

More specifically, the one particle reducible matrix elements are presumably dominated by diagrams where the various operators couple directly to a meson state with the appropriate quantum numbers, which then in turn couples irreducibly to the baryon;
\[
(M_C^B)^{\text{opr}} = \langle B | \sum_i m_i \bar{\psi}_i \psi_i | B \rangle^{\text{opr}} \approx \sum_a \frac{1}{m_{\phi_a}} \langle \phi_a | \bar{B} B \rangle,
\]
(2.9)
where $\{\phi_a\}$ are the scalar meson states, and the sum over states implicitly includes an integration over momenta. We can roughly estimate the order of magnitude of this correction by assuming that it is dominated by the exchange of two ideally mixed scalar mesons $\phi_l$ and $\phi_s$, which are respectively an isosinglet pure $u$ and $d$ state, and a pure $s$ state.

Introducing meson-to-vacuum coupling constants $g_{\phi}$ and meson-baryon couplings $g_{\phi B B}$, normalized according to
\[
\langle 0 | \frac{1}{2} \left( \bar{u} \bar{u} + \bar{d} \bar{d} \right) | \phi_l \rangle = g_{\phi_l} m_{\phi_l}^2, \quad \langle 0 | \bar{s} s | \phi_s \rangle = g_{\phi_s} m_{\phi_s}^2,
\]
\[
\langle \phi_a | \bar{B} B \rangle = (2\pi)^4 \delta^4(p_{\phi} - p_B + p_{\overline{B}}) g_{\phi_a B B},
\]
(2.10)
we find
\[
(M_D^B)^{\text{opr}} = -(M_C^B)^{\text{opr}} \approx -2\bar{m} g_{\phi_l} g_{\phi_l B B} - m_s g_{\phi_s} g_{\phi_s B B}
\]
\[
\simeq -(4\bar{m} + m_s)g' + (m_s - 2\bar{m}) \langle B | Y | B \rangle g' ,
\]
(2.11)
where in the second line we further assume the couplings to be given approximately by the additive quark model with SU(3) symmetry, so that $g = g_{\phi_l} \simeq g_{\phi_s}$ while $g_{\phi_l B B} \simeq \langle B | Y + 2 B \rangle g'$ and $g_{\phi_s B B} \simeq \langle B | 1 - Y | B \rangle g'$.

To estimate $(M_D^B)^{\text{opr}}$ we return to Weinberg's determination [9] of the quark masses, which is performed assuming that the baryon matrix elements of $\bar{\psi} \psi$ are proportional to the quark number, and is phenomenologically accurate to about 20%. Because all contributions to $\bar{\psi} \psi$ such that this identification is correct come from one particle irreducible
diagrams, the successfullness of this approach may now be understood as a consequence of the cancellation eq. (2.8), thus implying that \( (M_B^D)^{opi} \) is approximately flavor singlet.

In view of this, we may conclude that the bulk of the flavor nonsinglet component of \( M_B^D \) is provided by \( (M_B^D)^{opr} \), and thus that \( M_1^D \approx (m_s - 2\bar{m})gg' \) while \( M_2^D \approx 0 \). Thus we estimate \( \Delta M \) in eqn. (2.4) to be very roughly of order 200 MeV if the couplings are of order unity, which is of the same order of magnitude, and the same sign, as the matrix element \( \langle N|O_8|N \rangle \) deduced from the mass splittings (1.6). Consequently the theoretical estimate of \( \sigma_8 \) is approximately doubled, to around 50 MeV, which is quite compatible with the identification of \( \sigma_8 \) and \( \sigma \) without the necessity for any violation of the Zweig rule.

Whereas this estimate of the size of \( \Delta M \) relies on the crude pole-dominance approximation (2.9) used to deduce eq. (2.11), and on the neglect of \( (M_B^D)^{opi} \), the failure of the identification of the mass splitting operator \( O_8 \) with \( H_8 \) may be inferred from the exact results (2.2), (2.4) and (2.8). Clearly all of the usual consequences of the assumption that baryon mass splittings are given by matrix elements of an octet operator \( O_8 \), such as the Gell-Mann–Okubo mass formula, are intact; it is only the identification of \( H_8 \) with that operator which is modified. The same applies to results derived assuming that an additive quark model picture applies to the matrix elements of \( \bar{\psi}\psi \), such as Weinberg’s determination of current quark masses, since these only test the identification of baryon matrix elements of \( O_8 \) with \( (M_C^B)^{opi} \) which is correct due to the cancellation eq. (2.8). The sigma term puzzle is thus resolved by the observation that the gluon condensate provides a positive contribution to the parameter \( M_1 \) (which is negative) in eq. (1.3); the quark’s contribution to the splittings is necessarily underestimated if this is not taken into account.

### 3. Computation of Dynamical Quark Masses

How can one test quantitatively the explanation of the sigma term puzzle proposed in the previous section? Because the gluon condensate provides the bulk of the baryons’ masses, the SU(3)-dependence of its matrix elements correspond to a similar SU(3) dependence of the dynamical contribution to constituent quark masses. In particular, the dynamically generated mass should display a strong dependence on the current mass, anticorrelated to it.

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2 A discussion of the dynamical reason for this within the light-cone parton model is given in ref. [10].
3.1. The Quark Self Energy

We can try to find the flavor dependence of the dynamically generated mass by studying the quark self energy, which we may compute by solving the quark Schwinger–Dyson equation [17]–[19]:

\[ S^{-1}(p) = Z_2 S_0^{-1}(p) - Z_1 g^2 \int \frac{d^4k}{(2\pi)^4} \lambda^a \gamma_\mu D_{\mu\nu}^{ab}(p - k) S(k) \Gamma^b_\nu(p, k), \]

where \( S(p) \equiv Z(p^2)/(g' + \Sigma(p^2)) \) is the full quark propagator, \( \Sigma(p^2) \) is the quark self energy, \( S_0(p) \equiv (g' + m_0)^{-1} \) is the bare (or “current”) quark propagator, \( D_{\mu\nu}^{ab}(p - k) \) the full gluon propagator, and \( \Gamma^b_\nu(p, k) \) the full quark-gluon vertex function. \( Z_1, Z_2 \) and \( Z_3 \) are the usual renormalization factors for the vertex, quark and gluon fields, respectively. The generators \( \lambda_a \) of the color group \( SU(3) \) are normalized as \( \text{tr}(\lambda_a \lambda_b) = \frac{1}{2} \delta_{ab} \), and henceforth we will work throughout in Euclidean space.

As it stands eq.(3.1) is not consistent with the renormalization group equations for \( S, D \) and \( \Gamma \), so it must be renormalization group improved [20]; this results in the replacement of the coupling constant with a running coupling \( \bar{g}^2(p, k) \) inside the integral, where \( \bar{g}(k, p) = \bar{g}(p, k) \) and \( \bar{g}(k, p) \sim g(k^2) \) when \( k^2 \gg p^2 \), \( g(k^2) \) being the usual running coupling [21][19].

Let us now assume (neglecting ghost contributions) that \( \Gamma^a_\mu(p, k) = \lambda^a \Gamma_\mu(p, k) \), where \( \Gamma_\mu(p, k) \) satisfies the Ward–Takahashi identity

\[ (p - k)_\mu \Gamma_\mu(p, k) = S^{-1}(p) - S^{-1}(k). \]

Besides ensuring that \( Z_1 = Z_2 \), eq.(3.2) may be used to find a consistent expression for the vertex in terms of the quark propagator. A suitable solution was given long ago by Landau [22]:

\[ \Gamma_\mu(p, k) = \frac{(p - k)_\mu}{(p - k)^2} (S^{-1}(p) - S^{-1}(k)) + T_{\mu\nu}(p - k) \gamma_\nu \bar{Z}(p, k)^{-1}, \]

where \( T_{\mu\nu} \equiv (\delta_{\mu\nu} - p_\mu p_\nu/p^2) \), \( \bar{Z}(p, k) = \bar{Z}(k, p) \) and \( \bar{Z}(p, k) \sim Z(k^2) \) for \( k^2 \gg p^2 \). Even though this ansatz has a kinematic singularity in the infrared (i.e. as \( p \to k \)), it is sufficient for our purposes since it guarantees multiplicative renormalizability, and thus will give a quark self energy which has the correct gauge independent asymptotic behaviour at large \( p^2 \) (see ref.[20] for a more complete discussion). This confines ambiguities to the infrared region over which we have little control anyway. Analogously, the gluon propagator can be taken to have its asymptotic form, up to a momentum-dependent function \( d(p) \) which parametrizes infrared uncertainties;

\[ D_{\mu\nu}^{ab}(p) = \delta^{ab} d(p) \frac{T_{\mu\nu}(p)}{p^2} + \xi p_\mu p_\nu/p^2, \]

where \( d(p) \to 1 \) as \( p^2 \to \infty \).
Using the vertex (3.3) and the propagator (3.4), the Schwinger–Dyson equation (3.1) becomes

\[ S^{-1}(p) = Z_2 S_0^{-1}(p) + 3C_2 Z_2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{g}^2(p,k)d(p-k)}{(p-k)^2} \times \left[ T_{\mu\nu}(p-k)\gamma^\mu S(k)\gamma^\nu \bar{Z}(p,k) - \frac{\xi (\bar{g} - k\gamma^\mu)}{(p-k)^2} \left( S(k)S^{-1}(p) - 1 \right) \right], \]  \hspace{1cm} (3.5)

where the colour factor \( C_2 = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3} \).

Eq. (3.5) can be further simplified by assuming that the a priori unknown factors \( \bar{g}(p,k), d(p-k), \) and \( \bar{Z}(p,k) \) can each be approximated by the asymptotic values they take when one of their arguments is much larger than the other, i.e. \( \bar{g}(p,k)^2 d(p-k) \approx g(\max(p^2, k^2))^2 \) and \( \bar{Z}(p,k) \approx Z(\max(p^2, k^2)) \) \[ 22 \]. We may then perform the angular integrals analytically to give

\[ Z^{-1}(p^2) = Z_2 + \frac{\xi Z_2 C_2}{16\pi^2} \left[ \frac{1}{2} \frac{g^2(p^2)}{p^4} \int_0^{\Lambda^2} dk^2 \frac{k^2 Z(k^2) \Sigma(k^2)}{k^2 + \Sigma^2(k^2)} \right], \]

\[ \Sigma(p^2) = Z(p^2) Z_2 m_0(\Lambda) + \frac{3C_2}{16\pi^2} \left[ (1 + \frac{1}{3} \xi) \frac{g^2(p^2)}{p^2} \int_0^{\Lambda^2} dk^2 \frac{k^2 Z(k^2) \Sigma(k^2)}{k^2 + \Sigma^2(k^2)} \right. \]

\[ + \left. Z(p^2) \int_{p^2}^{\Lambda^2} \frac{d^2 k^2 (k^2) \Sigma(k^2)}{k^2 + \Sigma^2(k^2)} + \frac{1}{3} \xi \Sigma(p^2) \int_{p^2}^{\Lambda^2} \frac{d^2 k^2 [g^2(k^2) Z(k^2)]}{k^2 + \Sigma^2(k^2)} \right], \]  \hspace{1cm} (3.6)

where we have introduced an ultraviolet momentum cutoff \( \Lambda \). In the Landau gauge \( \xi = 0, Z(p^2) = Z_2^{-1} \), and it is not difficult to see that (3.6) reduces to the same equation as obtained in the (renormalization group improved) ladder approximation \[ 13 \].

Eq. (3.6) is sufficiently simple that it may be solved numerically for the quark self energy \( \Sigma(p^2) \). All infrared uncertainties (in the regions \( k^2 \sim M^2, p^2 \sim M^2 \) and \( (p-k)^2 \sim M^2 \), where \( M^2 \) is the “QCD scale”) have been rather crudely absorbed into the infrared uncertainty in the form of the running coupling \( g(p^2) \). By taking suitable parameterizations of this uncertainty we hope to get some feel for the overall uncertainty due to our ignorance of the physics of the infrared region. We may assume for example that

\[ g^2(p^2) = \frac{1}{\beta_0 \ln(\delta + p^2/M^2)}, \]  \hspace{1cm} (3.7)

where \( \beta_0 = \frac{11N_c - 2N_f}{48\pi^2} \), and \( \delta \) may be adjusted at will to give various values of the strong coupling at \( p^2 = M^2 \).
3.2. Current Mass and Dynamical Mass

After performing such a calculation, it is still necessary to adopt some procedure to extract the quark “mass” from its self energy. Since we only have a rather approximate form for the self energy in Euclidean space, there is no question of being able to continue to Minkowski space to search for a pole in the quark propagator, even if one believed that such a pole should exist. We may however estimate the region of \( p^2 \) in which the self energy makes its dominant contribution to the mass of an on–shell baryon. The simplest such estimate is to take the “pole–mass” defined as the solution to \( m_{\Sigma} = \Sigma(m_{\Sigma}^2) \): the self energy is taken to yield the mass when evaluated at the scale set by the mass itself.

We may assess the uncertainty in this estimate by observing that the baryon mass could be determined, if we knew its Bethe–Salpeter amplitude, by calculating an integral over the relative momenta of its constituent quarks; the effect of these integrals would be to give the contribution of each quark as some weighted mean of its self energy. We may attempt to simulate this by smearing the quark’s self-energy with a momentum-dependent “form factor” \( \rho(p^2) \), and then defining the mass \( m_{\Sigma}^\rho \) as the solution of
\[
\rho(p^2) \equiv \int_0^\infty dp^2 \rho(p^2) \Sigma(p^2),
\]
where \( \rho \) satisfies the normalization condition \( \langle 1 \rangle_{\rho} = 1 \) and the additional condition \( \langle p^2 \rangle_{\rho} = \langle m_{\Sigma}^2 \rangle_{\rho}^2 \), which ensures that the pole–mass definition is reproduced when \( \rho \) tends to a Dirac delta. A suitable form of \( \rho \) could be \( \rho(p^2) = N p^2 (p^2 + a)^{-n} \), for \( n = 4, 5, 6 \), say, with \( N \) and \( a \) determined by the two conditions.

The masses \( m_{\Sigma}^\rho \) determined in this way may be thought of as “constituent” masses; before we can use them to estimate \( \Delta M \) we must still separate out the renormalized “current” mass \( m_C \) from the dynamical mass \( m_D \) which arises nonperturbatively to break chiral symmetry even when \( m_C \) is small. This can be done by observing that the amount of explicit symmetry breaking may be extracted from the asymptotic behaviour at large \( p^2 \) of the self energy \[17\]
\[
\Sigma(p^2) \sim m \ln^{-\lambda}(p^2/M^2)(1 + O(g(p^2)^2)),
\]
where \( \lambda \equiv \gamma_m^0/\beta_0 = 9(N_c^2 - 1)/(2N_c(11N_c - 2N_f)) = 4/9 \) when \( N_c = N_f = 3 \), and \( \gamma_m^0 \) is the leading coefficient in the perturbative expansion of the mass anomalous dimension. The renormalization group invariant mass parameter \( m \) is related to the running current mass in the usual way:
\[
m_C(m, \mu) \sim m \ln^{-\lambda}(\mu^2/M^2),
\]
where \( \mu \) is the renormalization scale.

By varying the bare mass \( m_0(\Lambda) \) we obtain thus a family of solutions to (3.6), each of which corresponds to a different (unique) value of \( m \). Calculating the constituent mass

\[3\] Both the quark propagators and the Bethe–Salpeter amplitude fall as powers when \( p^2 \) becomes large.
$m_\Sigma$ for each of these solutions gives us $m_\Sigma$ as a function of $m$. The asymptotic behaviour (3.9) in combination with the definition of $m_\Sigma$ tells us immediately that

$$m_\Sigma(m) \sim m \ln^{-\lambda}(m^2/M^2)(1 + O(m^2')). \quad (3.11)$$

To obtain the dynamical mass $m_D(m)$ we should now subtract from $m_\Sigma(m)$ the current mass $m_C$ eq.(3.10), evaluated at the scale of the constituent mass:

$$m_D(m) = m_\Sigma(m) - m_C(m, m_\Sigma(m)). \quad (3.12)$$

Due to the asymptotic behaviors of $m_\Sigma$, eq.(3.11), and $m_C$, eq.(3.10), the dynamical mass defined by eq.(3.12) vanishes for large $m$, as it ought to (the current mass and the constituent mass coincide for large values of $m$).

It is still necessary however to further specify the form of $m_C$ in the infrared. Clearly $m_\Sigma(0) = m_D(0)$, so $m_C(0, m_D(0)) = 0$. Furthermore we may use Weinberg’s definition [9] of the constituent quark mass, i.e., assume that in the chiral limit the dependence of $m_D$ on $m$ is dominated by the linearized dependence of $M_B^C$ on $m$ (which, recalling eq.(2.3), is essentially the dependence of $M_B^C$ on the quark content of the given state). In other words we assume that for small $m$

$$m_\Sigma(m) \simeq m_D(0) + m_C(m, m_\Sigma(m)), \quad (3.13)$$

and, identifying the linear terms in a Taylor expansion in $m$,

$$\frac{dm_C}{dm} \bigg|_{m=0} = \frac{dm_\Sigma}{dm} \bigg|_{m=0}. \quad (3.14)$$

Note that this condition may be enforced only at one particular $m$ since $m_\Sigma(m)$ is nonlinear. In practice we choose a smooth interpolating function to parameterize $m_C$ for all $m$, such as, for example (compare eq.(3.7))

$$m_C(m, m_\Sigma(m)) = m \ln^{-\lambda}(\epsilon + m^2/M^2), \quad (3.15)$$

where the parameter $\epsilon$ is chosen to ensure the satisfaction of (3.14).

The dynamical mass $m_D(m)$ is then a smooth function, flat at the origin, tending gradually to zero on a scale of $m_D(0) \approx 300$ MeV. It is this mass which should be identified with the dynamical contribution $M_B^D$ (see (2.3)) at the constituent quark level, since the “current” quark contribution $M_B^C$ is clearly already accounted for by $m_C$. Thus, referring back to (1.5) and (2.4) we see that assuming a linear mass formula

$$\Delta M \simeq M_1^D \simeq m_D(0) - m_D(m_s), \quad (3.16)$$

where $m_s$ is the value of $m$ appropriate for the strange quark mass; taking the running mass $m_C(m_s, 1 \text{ GeV}) = 175 \pm 50$ MeV [8] we find $m_s = 200 \pm 50$ MeV [8], which is of the same order of magnitude as $m_D(0)$. From the qualitative behaviour of $m_D(m)$ we thus expect $\Delta M$ to be loosely of order $150$ MeV, in rough agreement with the estimate of section 2.2.

---

4 Even if we could compute $m_C(m, \mu)$ reliably for small $\mu$, we would still require extra conditions to fix the (probably rather large) scheme dependence.
3.3. Numerical Results

We may now compute $\Delta M$ eq. (3.16) by solving the Schwinger–Dyson equation in the approximate form (3.6), with $N_c = N_f = 3$ and, for example, the ansatz (3.7), and extracting $m_D(m)$ according to (3.8), (3.12) and (3.15). The results of such calculations with $\alpha_s(0) \equiv (4\pi \beta_0 \ln \delta)^{-1} = 2.5$ and the gauge parameter $\xi = 0$ are displayed in fig.2, for both the “pole” definition and the various averaging definitions (3.8) of the constituent mass; it can be seen that $m_D(m)$ depends very little on which definition we adopt, so all remaining calculations are done with the simpler “pole” definition.

The variation with gauge parameter $\xi$ is explored in fig.3 — again we find that for $|\xi| \lesssim 1$ $m_D$ is approximately independent of the choice of gauge, as expected since the Landau vertex (3.3) satisfies the Ward–Takahashi identity (3.2). Of course for larger values of $\xi$ there is still some gauge dependence, resulting from the infrared uncertainties. This could presumably only be tamed by using a vertex which satisfied the full Slavnov–Taylor identity, and thus including ghost contributions which are significant in the infrared, but this is not possible with the present level of expertise.

However the dominant uncertainty comes not from gauge dependence but from our general ignorance of the infrared. If we vary the strong coupling in the infrared by choosing different values for $\delta$ in (3.7), we can obtain wide variations in $m_D(m)$; displayed in fig.4 are curves for $\delta = 2, 1.5, 1.2, 1.01$ corresponding to $\alpha_s(0) = 0.5, 1, 2.5, 5.0$. Clearly as $\alpha_s(0)$ is increased the dynamical mass becomes firmer, and consequently falls off less rapidly as $m$ is increased; $\Delta M$ is thus reduced. To check this observation we also computed curves for couplings containing various infrared singularities, for example $\delta^4(p^2)$ or $1/(p^2)^a$ with $0 \leq a < 1$ (for $a = 1$ the Schwinger–Dyson equation is infrared divergent) which confirm this expectations: the stronger the singularity, the firmer the dynamical mass.

We conclude that it is not possible to compute $\Delta M$ without further (highly nontrivial) information on the infrared structure of QCD — all we can do is estimate $\Delta M$ to lie in the range $50 \text{ MeV} \lesssim \Delta M \lesssim 250 \text{ MeV}$ which translates (using (2.4)) into the estimate

$$30 \text{ MeV} \lesssim \sigma_8 \lesssim 60 \text{ MeV}. \quad (3.17)$$

in perfectly satisfactory agreement with the value of $\sigma$ inferred from experiment (and with the estimate presented in section 2.2).

Amusingly, we also have sufficient information to compute $\sigma$ using the so-called Feynman–Hellman formula at the constituent quark level, which amounts to the (rather reasonable) assumption that $m_C$ is approximately linear in the small light flavor average mass $\bar{m}$:

$$\sigma \simeq 3\bar{m} \frac{\partial m_C}{\partial \bar{m}} = 3m_C(\bar{m}, m_D(0)) \quad (3.18)$$

where in the last step we used eqns. (3.13), (3.14), and we may compute $m_C(\bar{m}, m_D(0))$ through eq. (3.12) from the known value of the running current mass at the nucleon scale. Taking $m_C(\bar{m}, 1 \text{ GeV}) = 5 \pm 3 \text{ MeV}$, we find $\sigma_8 = 40 \pm 20 \pm 15 \text{ MeV}$, the first error

\footnote{It is however possible to do computations with more complicated vertices [23] which satisfy the Ward–Takahashi identity (3.2) but are however free of the infrared kinematic singularity which plagues (3.3). The results are however little different to those with the simpler Landau vertex.}
being due to the uncertainty in $m_C(\bar{m}, 1 \text{ GeV})$, and the second due to the uncertainty in $\epsilon$, which we estimate by varying $\alpha_s(0)$ from 1 to 50. Since the former is rather large, and the latter probably underestimated, this is clearly a less reliable result than (1.17), but nonetheless it provides a useful independent check on the consistency of the calculation.

The calculations presented here are necessarily extremely uncertain, not only due to our implicit use of a constituent quark picture of the baryon (instead of a linear mass formula, we could have used a more sophisticated one (see for example [24]) in which $M_D^2$ would also be nonzero), but also to uncertainties in the separation of the dynamical contribution from the “current” quark contribution, in the form of the quark–gluon vertex function and, most importantly, our complete ignorance of the behaviour of the gluon propagator in the infrared. Note that the possible effects of quark loops, both light and heavy, are only a part of this uncertainty, in that while they certainly contribute to the gluon propagator in the infrared, there is no reason to believe that they dominate it.

A more accurate computation of the sigma term than that presented here would thus require much more powerful techniques (for example lattice studies), because of its sensitivity to the infrared dynamics of chiral symmetry breakdown. Certainly nothing more can be learnt from models such as that of Nambu–Jona-Lasinio, or the chiral quark model, since these necessarily trivialize this dynamical structure by assuming a constant quark self energy.

4. Conclusion

In this paper we have shown that the traditional argument relating the sigma term to baryon mass splittings [3] is incorrect, due to the neglect of the dynamical contribution to such mass splittings from the trace anomaly. This contribution may be viewed as a nonsinglet gluonic contribution to constituent masses, and as such it does not spoil all the usual quark model results, although it does contradict the naive assumption that mass splittings at the constituent and current level are the same.

We have further attempted a computation of this contribution through the solution of the Schwinger-Dyson equation for the quark self energy, and find that it not only has the right sign, but is also quite large enough to agree with the experimental determination. However since it is rather sensitive to the infrared dynamics, it seems to us to be rather difficult to obtain a more precise determination, and thus a firm bound on the amount of Zweig rule violation, without a better understanding of the details of the mechanism responsible for the dynamical breaking of chiral symmetry. What we can say, however, is that without such an understanding the experimental value of the sigma term cannot be used as evidence of strange quarks in the nucleon.

Rather than providing evidence for the breakdown of the naive quark model, the sigma term appears thus to provide nontrivial information on the dynamics of QCD in the infrared. In view of this, both a better experimental determination of the sigma term, and a better theoretical understanding of nonsinglet contributions to quark masses would be highly desirable.

Acknowledgement: RDB would like to thank the Royal Society for their financial support; JT thanks the SERC. SF thanks M. Bos, V. de Alfaro and E. Predazzi for discussions.
References

[1] See e.g. R. L. Jaffe and C. L. Korpa, Comm. Nucl. Part. Phys., 17 (1987) 163.
[2] J. F. Donoghue and C. R. Nappi, Phys Lett. 168B (1986) 105.
[3] T.-P. Cheng and R. Dashen, Phys. Rev. Lett. 26 (1971) 594.
[4] J. Gasser, H. Leutwyler and M. E. Sainio, Phys. Lett. B253 (1991) 252.
[5] T.-P. Cheng, Phys. Rev. D13 (1976) 2161; see also T.-P. Cheng and L.-F. Li, “Gauge theory of elementary particle physics” (Oxford U.P., Oxford, 1984).
[6] J. Gasser and H. Leutwyler, Phys. Rep., 87 (1982) 77.
[7] J. F. Donoghue, B. R. Holstein and D. Wyler, Phys. Rev. Lett., 69 (1992) 3444.
[8] D. B. Kaplan and A. Nelson, Phys. Lett. 175B (1986) 57.
[9] S. Weinberg, in “A Festschrift for I.I. Rabi”, ed. L. Motz (New York Acad. Sci., New York, 1977).
[10] M. Anselmino and S. Forte, Torino preprint DFTT 92/6 (1992).
[11] S. Forte, Phys. Rev. D47 (1993) 1842.
[12] See e.g. V. De Alfaro, S. Fubini, G. Furlan and C. Rossetti, “Currents in Hadron Physics” (North–Holland, Amsterdam, 1973).
[13] J. C. Collins, A. Duncan and S. Joglekar, Phys. Rev. D16 (1977) 438; N. K. Nielsen, Nucl. Phys. B120 (1977) 212.
[14] R. Tarrach, Nucl. Phys. B196 (1982) 45.
[15] G. M. Shore and G. Veneziano, Phys. Lett. B244 (1990) 75; Nucl. Phys. B381 (1992) 3.
[16] D. G. Gross, S. Treiman and F. Wilczek, Phys. Rev. D19 (1977) 2188.
[17] K. Lane, Phys. Rev. D10 (1974) 2605; T. Appelquist and E. Poggio, Phys. Rev. D10 (1974) 3280.
[18] H. Pagels, Phys. Rev. D19 (1979) 3080.
[19] V. A. Miransky, Sov. Jour. Nucl. Phys. 38 (1983) 280; K. Higashijima, Phys.Rev. D29 (1984) 1228.
[20] R.D. Ball and J. Tigg, Oxford preprint OUTP-92-34P.
[21] C. H. Llewellyn Smith, Acta Phys. Austr. Suppl XIX (1978) 331; Yu. L. Dokshitzer, D. I. Diakonov and S. I. Troyan, Phys. Rep. 58 (1980) 269.
[22] L. D. Landau,A. Abrikosov and I. Khalatnikov, Nuovo Cim. Suppl. 3 (1956) 80.
[23] J. S. Ball and T.-W. Chiu, Phys. Rev. D22 (1980) 2542 ; D. C. Curtis and M. R. Pennington, Phys. Rev. D42 (1990) 4165.
[24] A. De Rujula, H. Georgi and S.L. Glashow, Phys. Rev. D12 (1975) 147; N. Isgur and G. Karl, Phys. Rev. D18 (1978) 4187; D19 (1979) 2653.
Figure Captions

[Fig. 1] The Schwinger–Dyson equation.

[Fig. 2] The dynamical mass $m_D(m)$ vs. the current mass $m$, both in units of $m_D(0)$ (solid); also $m^n_D(m)$ (defined in eqn.(3.8)) for n=4,5,6 (long dashes, short dashes, dots respectively).

[Fig. 3] As fig. 2, but for the gauge parameter $\xi = 0, 1, 2, -1, -2$ (solid, long dashes, short dashes, dots, dots and dashes respectively).

[Fig. 4] As fig. 2, but for $\alpha_s(0) = 0.5, 1, 2.5, 50$ (dots, short dashes, solid, long dashes respectively).