Entanglement of electron orbital motion in open two-electron quantum dots

A. V. Chizhov
Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia
E-mail: chizhov@theor.jinr.ru

Abstract. In the present report, the effects of decoherence in a two-electron quantum dot coupled with its environment are investigated. Environmental effects on the quantum dynamics of electrons are modeled by coupling the dot with a thermo-bath at fixed temperatures. In such an approach, the special emphasis is placed on the analysis of the entanglement degradation of electron orbital motion depending on the quantum dot parameters (the strength of electron interaction and quantum dot shapes) and on the temperature of environment.

1. Introduction
Quantum dots (QDs) represent mesoscopic systems of artificially trapped electrons in a localized space of about a few hundreds Angstroms of a linear size. All characteristics of QDs (size, number of electrons, dielectric environment) as well as their external parameters (electric and magnetic fields, temperature) can be kept under control that allows one to use QDs as handy laboratories to study quantum effects at low-dimensional scale. In particular, they are expected to give a natural realization of a quantum bit or “qubit” and can be used in quantum computing and information storage. In systems of electrons the realization of qubits is usually associated with spins of electrons [1, 2], since they have just two projections onto any axe. At the same time, a quantized character of QD energy spectra allows one to represent qubits in terms of pseudospin based on the orbital motion of electrons [3, 4, 5]. Therefore, the study of quantum correlations of the spatial function in the dynamics of electrons, i.e., orbital entanglement of electrons, may be of great importance for the realization of multi-qubit logic gates [6]. Creation of such logic gates is needed for the accomplishment of general algorithms in the quantum information theory [7]. Thus, the question how to efficiently produce and control the orbital entanglement is one of fundamental as well as technological problems.

Recently, an exactly solvable model of an isolated two-electron QD was treated for the study of quantum correlation properties of electrons [8]. A two-electron QD is the simplest pattern with the essential features of more complex systems. In this model, electrons were assumed to move in a two-dimensional parabolic confining potential, whereas the interaction between electrons was approximated by the effective harmonic potential of the Johnson-Payne type. Within the bounds of such a model it was possible to get an analytical description of the electron orbital motion and the electron energy spectrum in accordance with the shell structure. In the presence of a perpendicular magnetic field a detailed study of the degree of the orbital entanglement of
the QD ground state and its first excited states was performed for various values of a QD shape, the strength of the interaction and the magnetic field [9].

Entanglement is known to be very sensitive to the environmental effects which lead to the phenomenon of decoherence. Time of decoherence is one of the crucial problems appeared in quantum systems. The aim of this communication is to evaluate the entanglement degradation phenomenon of decoherence. Time of decoherence is one of the crucial problems appeared in quantum systems. The aim of this communication is to evaluate the entanglement degradation in few-electron QDs is usually assumed, to a good approximation, as parabolic [13] with two-electron Hamiltonian with respect to the mutual permutation of electrons. The confining potential of electron-electron interaction which is taken in the approximation of the Johnson-Payne type [14] the interaction terms in the forms of sums in order to comply with the symmetry demands of the full Hamiltonian with respect to the mutual permutation of electrons. The confining potential in few-electron QDs is usually assumed, to a good approximation, as parabolic [13] with two confining frequencies \( \omega_x \) and \( \omega_y \), which are not equal in a general case, so that

\[
\hat{H}_2e = \sum_{i=1}^{2} \left[ \frac{\hat{p}_{xi}^2 + \hat{p}_{yi}^2}{2m^*} + \frac{m^*}{2} \left( \omega_x^2 x_i^2 + \omega_y^2 y_i^2 \right) \right] + \hat{V}(\mathbf{r}_1, \mathbf{r}_2). \tag{2}
\]

Here \( m^* \) is an effective mass for the conduction electrons, and operator \( \hat{V}(\mathbf{r}_1, \mathbf{r}_2) \) describes the electron-electron interaction which is taken in the approximation of the Johnson-Payne type [14]

\[
\hat{V}(\mathbf{r}_1 - \mathbf{r}_2) = V_0 - \lambda \frac{m^*}{2} (\mathbf{r}_1 - \mathbf{r}_2)^2, \tag{3}
\]

where \( V_0 \) and \( \lambda \) are positive parameters within the model.

Introducing new variables for QD electrons

\[
\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \tag{4}
\]

Hamiltonian \( \hat{H}_{2e} \) is divided into the the center-of-mass and relative motion terms

\[
\hat{H}_{2e} = \hat{H}_{CM} + \hat{H}_{rel} + V_0, \tag{5}
\]

\[
\hat{H}_{CM} = \frac{\hat{p}^2}{2M} + \frac{M}{2} \left[ \omega_x^2 \hat{X}^2 + \omega_y^2 \hat{Y}^2 \right],
\]

\[
\hat{H}_{rel} = \frac{\hat{p}^2}{2\mu} + \frac{\mu}{2} \left[ \omega_1^2 \hat{x}^2 + \omega_y^2 \hat{y}^2 \right], \tag{6}
\]
where
\[ \omega_{1,2}^2 = \omega_{x,y}^2 - 2\lambda^2, \quad M = 2m^*, \quad \mu = \frac{m^*}{2}. \] (7)

In these notations the full Hamiltonian (1) takes the form
\[ \hat{\mathcal{H}} = \hat{\mathcal{H}}^{\text{dis}}_{\text{CM}} + \hat{\mathcal{H}}_{\text{rel}} + V_0 \] (8)

with
\[ \hat{\mathcal{H}}^{\text{dis}}_{\text{CM}} = \hat{\mathcal{H}}_{\text{CM}} + \sum_j \left[ \left( \hat{P}_{xj}^2 + \hat{P}_{yj}^2 \right) / 2M_j + M_j \left( \Xi_{xj}^2 \hat{Q}_{xj}^2 + \Xi_{yj}^2 \hat{Q}_{yj}^2 \right) / 2 \right. \]
\[ + 2\kappa_{xj} \hat{X} \hat{Q}_{xj} + 2\kappa_{yj} \hat{Y} \hat{Q}_{yj} + 2 \frac{\kappa_{xj} \hat{P}_x \hat{P}_x + \kappa_{yj} \hat{P}_y \hat{P}_y}{MM_j \omega_{xj} \Xi_{xj}} \left. \right] . \] (9)

Hamiltonians \( \hat{\mathcal{H}}_{\text{rel}} \) and \( \hat{\mathcal{H}}^{\text{dis}}_{\text{CM}} \) can be written in the secondary quantized representation
\[ \hat{\mathcal{H}}_{\text{rel}} = \hbar \omega_1 (\hat{c}_1 \hat{c}_1^\dagger + 1/2) + \hbar \omega_2 (\hat{c}_2 \hat{c}_2^\dagger + 1/2), \] (10)
\[ \hat{\mathcal{H}}^{\text{dis}}_{\text{CM}} = \hbar \omega_x (\hat{C}_1 \hat{C}_1^\dagger + 1/2) + \hbar \omega_y (\hat{C}_2 \hat{C}_2^\dagger + 1/2) + \hbar \sum_j \left[ \Xi_{xj} \left( \hat{f}_{xj} \hat{f}_{xj}^\dagger + 1/2 \right) + \Xi_{yj} \left( \hat{f}_{yj} \hat{f}_{yj}^\dagger + 1/2 \right) \right. \]
\[ + \varsigma_{xj} \left( \hat{C}_1 \hat{f}_{xj} + \hat{C}_1^\dagger \hat{f}_{xj}^\dagger \right) + \varsigma_{yj} \left( \hat{C}_2 \hat{f}_{yj} + \hat{C}_2^\dagger \hat{f}_{yj}^\dagger \right) \left. \right] \] (11)

with annihilation operators \( \hat{c}_{1,2}, \hat{C}_{1,2}, \) and \( \hat{f}_{x,y,j} \) for relative and center-of-mass motion of electrons, and reservoir bosons, respectively, and the transformed coupling constants \( \varsigma_{x,y,j} = 2\kappa_{x,y,j} / \sqrt{MM_j \omega_{x,y} \Xi_{x,y,j}}. \)

3. Quantum dynamics of electron subsystem
The operator dynamics of electrons can be found in the Heisenberg-Langevin approximation [15]. In view of the fact that the relative-motion operators are not coupled to the reservoir (see Eq. (10)), they follow the free evolution in time
\[ \hat{c}_1(t) = \hat{c}_1(0) \exp(-i\omega_1t), \quad \hat{c}_2(t) = \hat{c}_2(0) \exp(-i\omega_2t), \] (12)

whereas the center-of-mass motion operators are subject to damping
\[ \hat{C}_1(t) = u_1(t) \hat{C}_1(0) + \sum_j w_{xj}(t) \hat{f}_{xj}(0), \quad \hat{C}_2(t) = u_2(t) \hat{C}_2(0) + \sum_j w_{yj}(t) \hat{f}_{yj}(0), \] (13)

with
\[ u_1(t) = \exp(-i\omega_1t - \gamma_1t/2), \quad u_2(t) = \exp(-i\omega_2t - \gamma_2t/2), \] (14)
\( \gamma_{1,2} = 2\pi |\varsigma_{x,y,j}|^2 \rho(\omega_{x,y}) \) are the damping constants (\( \rho(\omega) \) being the density distribution of the reservoir oscillators), and functions \( w_{x,y,j}(t) \) obey the equalities
\[ \sum_j |w_{xj}(t)|^2 = 1 - \exp(-\gamma_1t), \quad \sum_j |w_{yj}(t)|^2 = 1 - \exp(-\gamma_2t) \] (15)

in order to fulfill the commutation relations for operators \( \hat{C}_{1,2} \). Then the time evolution of two-mode annihilation operators of the two QD electrons confined in the \( xy \) plane
\[ \hat{d}_{x,1,2} = m^* \omega_x \hat{x}_{1,2} + i\hat{p}_{x,1,2}, \quad \hat{d}_{y,1,2} = m^* \omega_y \hat{y}_{1,2} + i\hat{p}_{y,1,2}, \] (16)
can be obtained in accordance with Eqs. (4)

\[\hat{d}_{x,2}(t) = \frac{1}{2\sqrt{2}} \left[ 2\hat{C}_1(t) \pm \left( \sqrt{\frac{\omega_x}{\omega_y}} + \sqrt{\frac{\omega_y}{\omega_x}} \right) \hat{c}_1(t) \mp \left( \sqrt{\frac{\omega_y}{\omega_z}} - \sqrt{\frac{\omega_z}{\omega_y}} \right) \hat{c}_1^\dagger(t) \right], \quad (17)\]

\[\hat{d}_{y,2}(t) = \frac{1}{2\sqrt{2}} \left[ 2\hat{C}_2(t) \pm \left( \sqrt{\frac{\omega_y}{\omega_z}} + \sqrt{\frac{\omega_z}{\omega_y}} \right) \hat{c}_2(t) \mp \left( \sqrt{\frac{\omega_z}{\omega_x}} - \sqrt{\frac{\omega_x}{\omega_z}} \right) \hat{c}_2^\dagger(t) \right], \quad (18)\]

by using Eqs. (12) and (13).

Eqs. (17) and (18) allow one to study the state evolution of orbital motion of QD electrons. In case the whole system of electrons and reservoir bosons at the initial moment is described by the total density matrix \(\hat{\rho}(0)\), a four-mode quantum state of the electron subsystem in time can be represented in the phase space by the Wigner function

\[W(\alpha_{x1}, \alpha_{y1}, \alpha_{x2}, \alpha_{y2}|t) = \frac{1}{\pi} \int \chi(\beta_{x1}, \beta_{y1}, \beta_{x2}, \beta_{y2}|t) \exp\{\sum_{k=1}^{2} \left[ \alpha_{xk}\beta_{xk}^* + \alpha_{yk}\beta_{yk}^* - c.c. \right]\} d^8\beta, \quad (19)\]

based on a symmetric characteristic function

\[\chi(\beta_{x1}, \beta_{y1}, \beta_{x2}, \beta_{y2}|t) = \text{Tr}\{\hat{\rho}(0)\hat{D}(\beta_{x1}, \beta_{y1}, \beta_{x2}, \beta_{y2}|t)\} \quad (20)\]

with the displacement operator

\[\hat{D}(\beta_{x1}, \beta_{y1}, \beta_{x2}, \beta_{y2}|t) = \exp\{\sum_{k=1}^{2} [\beta_{xk}\hat{d}_k^\dagger(t) + \beta_{yk}\hat{d}_k(t) - \text{h.c.}]\}.\]

4. Degradation of QD charge entanglement

In order to take account of temperature effects of the QD environment on the orbital entanglement of electrons (i.e., the charge entanglement), the state of reservoir bosons might be chosen as chaotic, so that their mean boson numbers

\[\langle n_1\rangle = \langle \hat{F}^\dagger_{yj}(0)\hat{F}_{xj}(0)\rangle = \left\{e^{\hbar\omega_x/k_BT - 1}\right\}^{-1}, \quad \langle n_2\rangle = \langle \hat{F}^\dagger_{yj}(0)\hat{F}_{yj}(0)\rangle = \left\{e^{\hbar\omega_y/k_BT - 1}\right\}^{-1} \quad (21)\]

are appeared to be dependent on the temperature \(T\) (\(k_B\) is the Boltzman constant).

Regarding the initial state of electrons, which is the eigenstate of the Hamiltonian \(\hat{H}_{2e}\), in this case it should be one of the entangled states. In this connection, it is convenient to regard electrons at the initial moment to be in the QD vacuum state \(|0\rangle_c\) (see [9]), that means \(\hat{c}_{1,2}|0\rangle_c = \hat{c}_{1,2}(0)|0\rangle_c = 0\).

Under these conditions, the time evolution of the state of the electron subsystem is described by the Wigner function of a Gaussian type

\[W(\alpha_{x1}, \alpha_{y1}, \alpha_{x2}, \alpha_{y2}|t) = \frac{16}{\pi^4[2\langle n_1\rangle(1 - e^{-\gamma_1t}) + 1][2\langle n_2\rangle(1 - e^{-\gamma_2t}) + 1]} \times \exp\left\{-2 \left[ |\sigma_1|^2 + |\sigma_2|^2 - \frac{|\Sigma|^2}{2\langle n_1\rangle(1 - e^{-\gamma_1t}) + 1} - \frac{|\Sigma|^2}{2\langle n_2\rangle(1 - e^{-\gamma_2t}) + 1} \right]\right\}, \quad (22)\]

where \(\sigma_1 = \sqrt{\omega_{1,2}/\omega_x}\text{Re}(\alpha_{x,y} - \alpha_{x,y}) + i\sqrt{\omega_{x,y}/\omega_{1,2}}\text{Im}(\alpha_{x,y} - \alpha_{x,y}) / \sqrt{2}\) and \(\Sigma = (\alpha_{x,y} + \alpha_{x,y})/\sqrt{2}\).

Introducing the quadrature-component variables defined as \(q_{x,y,k} = \sqrt{2}\text{Re}\alpha_{x,y,k}\) and \(p_{x,y,k} = \sqrt{2}\text{Im}\alpha_{x,y,k}\), the Wigner function (22) can be represented in terms of the covariance matrix \(\mathbf{V}(t)\)
The logarithmic negativity $E_N$ as a function of time and the strength of the electron-electron interaction (left), the QD deformation (right), when $\langle n_1 \rangle = \langle n_2 \rangle = 0.5$ and the damping constants $\gamma_1/\omega_x = \gamma_2/\omega_x = 1$.

$$W(\xi|t) = (16\pi^4 \sqrt{\det V(t)})^{-1} \exp \left\{ -\frac{1}{2} \zeta^T V^{-1}(t) \zeta \right\}.$$  \hspace{1cm} (23)

Here $\zeta^T = (q_{x1}, p_{x1}, q_{y1}, p_{y1}, q_{x2}, p_{x2}, q_{y2}, p_{y2})$ is a transposed eight-vector whose elements are the quadrature-component variables.

For the entanglement measure of the two QD electrons it is appropriate to use the logarithmic negativity based on the trace norm of the partial transpose of the density matrix of a bipartite state [16]. In the case of Gaussian states, the logarithmic negativity can be calculated by the symplectic spectrum of the partial transpose of covariance matrix $V^T_1(t)$, that is $V(t)$ in which all matrix elements connecting momenta $p_1$ of the first electron are multiplied by $-1$. The spectrum $\{\eta_k\} (k = 1, \ldots, 4)$ can be found by the following expression

$$-J V^T_1(t) J V^T_1(t) = \text{diag}(\eta_1, \eta_1, \eta_2, \eta_2, \eta_3, \eta_3, \eta_4, \eta_4),$$ \hspace{1cm} (24)

where the (antisymmetric) symplectic $8 \times 8$ matrix

$$J = \bigoplus_{k=1}^{4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the logarithmic negativity defined as

$$E_N = -\sum_{j=1}^{4} \log_2(\min(1, 2\eta_j))$$ \hspace{1cm} (25)

determines the degree of entanglement of QD electrons for $E_N > 0$.

In the report, it is given an analysis of the behavior of the logarithmic negativity for the orbital motion of QD electrons as a function of time and QD parameters. All model parameters are given with respect to the confining frequency $\omega_x$, which serves as a dimensional parameter. In QDs the Coulomb interaction is substantially screened, so that the effective interaction (3) should be considered as a perturbation. It is why in the numerical analysis the strength of
interaction is assumed to obey the condition $\lambda/\omega_x \leq 0.5$. Regarding the thermal noises that enter into the model in the form of the mean boson numbers (21), they are held as $\langle n \rangle \leq 0.1$. Taking into account the typical mean spacing of QD energy levels $\hbar \omega_x \sim 3$ meV, it corresponds to the temperatures $T \leq 10$ K according to relations (21).

It is seen in Fig. 1 that entangled states of electrons in time are noticeably subjected to the degradation caused by environmental noises. Moreover, the degradation in nearly circular QDs occurs much faster than that in strongly deformed QDs (left plot). It means that deformed QDs are more firm to temperature effects than the circular ones, although the latter can create more strong entanglement.

The dependence of the entanglement degree of QD electrons at the moment $\omega_x t = 1$ on the damping constants (QD openness) and environment thermal excitations is plotted in Fig. 2. One observes the pronounced decrease of the logarithmic negativity for $\gamma > 0$ with the increase of $\langle n \rangle$. This decrease becomes more sudden as the damping constant gains.

5. Summary
A model for a two-electron two-dimensional open QD with the anisotropic parabolic confinement is considered. The effects of the QD openness are taken into account by coupling a QD with a boson reservoir at fixed temperatures. An analysis of the dynamics of the electron orbital motion shows the appreciable degradation of the entanglement of QD electrons caused by quantum noises of reservoir excitations. The degradation is turn out to be most pronounced for circular QDs with a non-negligible degree of openness. The obtained results are of particular interest from the view point of estimation of the coherence time in quantum logic gates based on pseudospin of QD electrons.

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