Stability of \(N\)-soliton molecules in dispersion-managed optical fibers

Abdeláali Boudjemáa\(^1\) and U. Al Khawaja\(^2\)

\(^1\)Department of Physics, Faculty of Sciences, Hassiba Benbouali University of Chlef, P.O. Box 151, 02000, Chlef, Algeria.
\(^2\)Physics Department, United Arab Emirates University, P.O. Box 15551, Al-Ain, United Arab Emirates.

We investigate the stability of \(N\)-soliton molecules in dispersion-managed optical fibers with focus on the recently realized 2- and 3-soliton molecules. We calculate their binding energy using an averaged nonlinear Schrödinger equation. A combination of variational and numerical solutions to this equation shows that it describes well the intensity profiles and relative separations of the experimental molecules. Extending the calculation to larger values of \(N\), the binding energy per soliton is found to saturate at \(N \geq 7\).

PACS numbers: 42.81.Dp, 42.65.Tg, 42.79.Sz

In the last two decades, developments in fiber-optic communications have demonstrated that dispersion management (DM) presents a novel attractive type of a nonlinear carrier of information in optical fiber links. A few years ago, a stable bound state of two DM solitons in optical fibers was realized experimentally [1] and most recently three-soliton molecules in DM optical fibers were also realized by the same group [2]. The main motivation behind creating such molecules is to increase the bit-rate of data transfer in optical fibers. Coding with two or more solitons per clock period increases the alphabet beyond the binary scheme of a single soliton. In this manner, the Shannon limit [3], which soon will be reached, may be exceeded [4].

The main concern in soliton molecules being data carriers is their stability against disintegration. Hence, intensive interest in their stability has emerged [5–10]. The existence of a nonzero binding energy of the soliton molecules is an indication on its stability. The energy of a stable soliton molecule should have a minimum for a finite separation between the solitons. In Ref. [11] it was shown that the energy of a 2-soliton molecule indeed exhibits such a minimum and the potential of interaction was also shown to be of molecular type.

The main aim of the present work is to provide a theoretical framework that explains the stability of 2- and 3-soliton molecules as observed by Mitschke and co-workers [2, 4]. Specifically, we will show variationally that there is indeed a nonzero binding energy for 3-soliton molecules in DM fibers. The calculation provides an estimate for the strength of the bond between the solitons and shows regions in the parameter space where the molecule becomes unstable. Here, we address the problem of calculating the binding energy of the soliton molecule using an averaged nonlinear Schrödinger equation (NLSE). It was shown in Ref. [12] that solitons in DM fiber can be described by an effective nonlinear Schrödinger equation with constant coefficients and a quadratic potential. The averaged equation is more appealing for capturing the main features of the binding mechanism. At first, we show that the averaged equation is not integrable, hence variational and numerical approaches will be followed. For both cases we compare the intensity profiles with the experimental ones and obtain a good agreement. Finally, the calculation is then extended to the larger values of \(N\) up to \(N = 12\).

We first show that the evolution of solitons in dispersion managed dissipative optical fibers obeys a NLSE with a quadratic potential [12] which is integrable [13]. It turns out, however, that integrability restricts the time dependence of the dispersion to the nonrealistic case of exponential form. Therefore, an effective NLSE will be derived by averaging over one period of the dispersion map [12]. The effective equation will then be used to calculate the binding energy of the soliton molecule.

Solitons in dispersion-managed dissipative optical fibers are described by the following NLSE:

\[
i q_2 + \frac{d(z)}{2} q_t + |q|^2 q = -i \Gamma(z) q, \quad (1)
\]

where \(q(t, z)\) is the envelope function of the soliton and the subscripts denote partial derivatives. Here \(z\) and \(t\) are normalized distance and time , \(d(z)\) corresponds to the dispersion management map defined by

\[
d(z) = \begin{cases} 
  d^+, & 0 \leq z \leq L^+,
  d^-, & L^+ < z \leq L^+ + L^-,
\end{cases}
\]

where \(d^+, d^-\) are constant group velocity dispersions of the fiber segments \(L^+\), respectively. The loss(gain) corresponds to positive (negative) \(\Gamma(z)\).

The transformation \(q(t, z) = \exp(-\int \Gamma(z) dz) u(t, z)\), moves the loss term to the coefficient of the nonlinear term:

\[
i u_2 + \frac{d(z)}{2} u_{tt} + e^{-2 \int \Gamma(z) dz} |u|^2 u = 0. \quad (3)
\]

A quadratic phase chirp develops due to the propagation of the soliton in the fiber which corresponds to the transformation \(u(t, z) = w(t, z) \exp(i C(z) t^2 / 2)\), where \(C(z)\) is a real function. Substituting this transformation in the last equation, the NLSE takes the form
With the scaling transformation $\tau = p(z)t$, $u(t,z) = W(t,z)a(z)$, where $p(z) = \exp(-\int C(z)dz)$ and $a(z)$ is a real function, this equation takes the form

$$i\left(w_z + \frac{d}{2}w_{tt} + e^{-2\int \Gamma dz} |w|^2 w - \frac{1}{2}(\dot{C} + dC^2) x^2 w + \frac{i}{2}C dw = 0. \right)$$

For a numerical purposes, it is useful to reduce Eq. (11) into a dimensionless form. First we introduce the parameters $\zeta = A_0 z$, $\beta' = \beta/A_0$ and $K_0 = K_0/A_0$. Then, Eq. (11) becomes

$$iW_{z'} + \frac{\beta'}{2}W_{\tau'} + |W|^2 W - \frac{1}{2}K_0\tau^2 W = 0. \right)$$

We introduce the dimensionless variables $Z = z'/L'$, $T = \tau/\tau_m$ and $\Psi = W/\sqrt{L'}$ where $\tau_m$ is the characteristic time scale equal to the pulse duration of the laser source and $L' = (L^+ + L^-)A_0$ is the length of the dispersion map period. In terms of these parameters, the dimensionless NLSE takes the form

$$i\Psi_Z + \frac{D}{2}\Psi_{TT} + |\Psi|^2\Psi - \frac{1}{2}BT^2\Psi = 0, \right)$$

where $D = \beta'L'/\tau_m^2$ and $B = K_0L'^2\tau_m^2$. We use the experimental parameters for the DM map corresponding to the setup of [2]. The pulse duration $\tau_m = 0.25$ ps, $d^- = -4.259$ ps/km, $d^+ = 5.159$ ps/km, $A_0 = 1.7 W^{-1}km^{-1}$, $L^+ = 24$ m, $L^- = 22$ m, $L' = 0.078W^{-1}$. Note that $d(z)$ here is the negative of that in Refs. [2, 4]. Using these experimental values in the Nijhof’s method, as described above, we get $\beta = <dp^2> = 0.71$, $K_0 = <\kappa > = -0.0156$. Notice that $A_0 = c_1^2 < p > = 1.7$ is given as an experimental parameter which accounts for a specific selection of the arbitrary constant $c_1$. Thus, the scaled coefficients $D$ and $B$ take the values $D = 0.521$ and $B = -4.5 \times 10^{-5}$.

We use a variational calculation to show that 2- and 3-soliton molecules have indeed nonzero binding energy. This will be evident from the minimum of the energy in terms of a finite separation between solitons. The depth of the minimum will give an estimate to the strength of the bond in the molecule.

We employ the following three solitons trial wavefunction

$$\Psi(Z,T) = A \sum_{j=1}^{3} A_j \exp \left[ -\frac{(T - \eta_j)^2}{q^2} + i\phi_j \right], \right)$$

where $A$ guarantees the normalization of $\Psi$ to the number of solitons in the molecule, namely $N = 3$. The variational parameters $q(Z), \phi(Z)$ and $g(Z)$ correspond respectively to the width, the phase, and the peak position of the soliton. The energy functional corresponding to Eq. (13) reads

$$E = \int_{-\infty}^{\infty} \left[ \frac{D}{2} |\nabla \Psi|^2 + BZ^2 |\Psi|^2 - \frac{1}{2} |\Psi|^4 \right] dT. \right)$$

In real units, this energy can be expressed as $E[J] = (\tau_m/L')E$.

We plot in Fig. 2 the energy as a function of the soliton width, $q$, where the experimental values of all other
parameters, as in [4], have been used. A minimum is obtained at \( q = 0.32 \) ps which is close to the experimental value for the three-soliton molecule. In terms of the separation between solitons, \( \Delta_1 = \eta_1 - \eta_2 \) and \( \Delta_3 = \eta_2 - \eta_3 \), Fig. 3 shows that the energy has a local minimum for finite values of separations, namely at \( (\Delta_1, \Delta_2) = (-1.0, -1.2) \). This agrees also with the experimental values for equilibrium separations, as will be shown more clearly below in Fig. 4. The fact that there is also a minimum at \( \Delta_1 = \Delta_2 = 0 \) indicates that the stability of the soliton molecule is reduced due to the possibility of tunneling from the off-centered minima to the one at the origin, signifying the merging of the three solitons into one.

The value of the energy at the local minimum represents a measure of the strength of the bond in the molecule. The binding energy of the 3-soliton molecule relative to that of the single soliton energy, denoted by \( E_{ss} \), is taken from Eq. (15) and equals \( E/E_{ss} = 0.9 \).

In Fig. 4 we compare the intensity profile obtained by our variational calculation with the experimental and simulation results of Ref. [2] for both 2-soliton and 3-soliton molecules. The figure shows a good agreement between the variational calculation at one hand and the experimental data and the direct numerical solution of the NLSE at the other hand. Our curves were calculated using the experimental values of all parameters apart from the separations between solitons which were left as variational parameters. The intensity profile was then calculated using the solitons separations obtained by minimizing the energy functional.

To study the equilibrium properties of \( N \)-soliton molecules, we extend the variational formalism of the previous section to larger values of \( N \). Therefore, we consider the generalized Gaussian trial function

\[
\Psi(T, Z) = A \sum_{j=1}^{N_s} A_j \exp \left[ -\left( \frac{T - \eta_j}{q^2} \right)^2 + i\varphi_j \right],
\]

where \( A_j = A_0(1 + C(-1)^j) \), \( \eta_j = -(N_s - 1)/2 + (j - 1)\Delta \), and \( \varphi_j = \pi(-1)^j \) with \( N_s \) being the number of solitons in the molecule. This represents a string of \( N \) solitons each of width \( q \), amplitude \( A_j \), center-of-mass position \( \eta_j \), and phase \( \varphi_j \). For simplicity, the amplitudes were taken to alternate between \( A_0(1 + C) \) and \( A_0(1 - C) \), the separation between adjacent solitons is \( \Delta \), and phases alternating between \( \pi \) and \( -\pi \). In this manner, the parameter space is reduced to two parameters only, \( C \) and \( \Delta \). Using the trial function, Eq. (16), in the energy functional, Eq. (15), the energy of the 2-soliton molecule takes the form

\[
E_2 = \frac{2\sqrt{\pi q^3}DA_1A_3 + BA_1A_2\sqrt{2\pi} q^4 e^{-\frac{2\pi^2}{q^2}} - 2q^3 A_4}{4\sqrt{\pi q^4} A_1^2},
\]

where

\[
\begin{align*}
\Delta_1 &= (C^2 + 1) e^{\frac{\Delta^2}{2q^2}} - (C^2 - 1) \cos(2\pi\delta), \\
\Delta_2 &= (C^2 + 1) e^{\frac{\Delta^2}{2q^2}} (q^2 + \Delta^2) - q^2 (C^2 - 1) \\
&\times \sqrt{e^{\frac{\Delta^2}{2q^2}} \cos(2\pi\delta)}, \\
\Delta_3 &= q^2 (C^2 + 1) e^{\frac{\Delta^2}{2q^2}} - (C^2 - 1) (q^2 - \Delta^2) \cos(2\pi\delta), \\
\Delta_4 &= -4 (C^4 - 1) e^{\frac{\Delta^2}{2q^2}} \cos(2\pi\delta) + (C^4 + 6C^2 + 1) e^{\frac{\Delta^2}{2q^2}} + (C^2 - 1)^2 (\cos(4\pi\delta) + 2). \\
\end{align*}
\]

For larger \( N \) values, the energy expressions become even more lengthy and hence will not be shown here for convenience. One can see from Fig. 5 that the binding energy of three solitons is the largest (in magnitude) among all other molecules. The binding energy starts to stabilize when the number of solitons reaches 7. Another important remark is that the odd number of soliton molecules is more stable than the even ones. A similar behavior is obtained for the separation where it decreases with increasing number of solitons to saturate at \( N_s \geq 7 \). Also, a careful observation to the same figure shows that the separation between the second and the third solitons converges at \( N_s \geq 4 \) to zero, which means that the two solitons merge.

In conclusion, we have considered \( N \)-soliton molecules propagating in dispersion-managed optical fibers. The well-known effective nonlinear Schrödinger equation was first rederived and then shown to be not integrable for the realistic situation. Using a variational calculation, the binding energy of the soliton molecules was calculated with emphasis on the 2- and 3-soliton molecules. From the locations and depths of the local minima in the equilibrium energy of the molecule, the bond length and strength were calculated. The calculated sizes of the 2- and 3-soliton molecules agreed favorably with the experimental values and the numerical simulation as shown in Fig. 3. It should be noted that Raman scattering and higher order dispersion terms aught to affect the equilibrium properties of the molecule. These effects will be investigated in a future work. The calculation was also extended for larger molecules to show a non-monotonic dependence of the molecule’s binding energy and size in terms of the number of its constituent solitons.

We are indebted to Fedor Mitschke and his co-workers for useful discussions and for providing us with the experimental data. We acknowledge support of King Fahd University of Petroleum and Minerals under research group projects RG1217-1 & RG1217-2. U.K. Acknowledges the support of the UAEU-NRF 2011 research grant.
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![FIG. 1. A closed orbit in the $p - C$ plane, which solves Eqs. (6)-(8), with the experimental parameters $d^- = -4.259 \text{ ps}^2/\text{km}$, $d^+ = 5.159 \text{ ps}^2/\text{km}$, $L^+ = 22 \text{ m}$, $L^- = 24 \text{ m}$ of Ref. [4] and the assignments $\kappa^+ = 0.009$ and $\kappa^- = -0.01$.](image)
FIG. 2. Binding energy of three solitons molecule relative to that of the single soliton energy $E_{ss}$ as function of the width $q$ with same experimental parameters of Refs. [2, 4] and for $D = 0.521$ and $B = -4.5 \times 10^{-5}$.

FIG. 3. Binding energy of three solitons relative to that of the single soliton energy $E_{ss}$ as function of the separation $\Delta_1$ and $\Delta_2$ with same experimental parameters of Refs. [2, 4] and for $D = 0.521$ and $B = -4.5 \times 10^{-5}$.

FIG. 4. (Color online) Field amplitude envelopes along dispersion managed fiber of two-soliton molecule (left), three-soliton molecule (right). Green line: our variational calculation, Blue line: experimental data of [2] and Red line: simulation of [2].
FIG. 5. (Color online) Left panel: Energy minimum versus the number of solitons for \( A_0 = 1, C = 0.5, \delta = 0.5, D = 0.521 \) and \( B = -4.5 \times 10^{-5} \). Red line: \( E_{\min 1} \), Blue dotted line: \( E_{\min 2} \) and Green dashed line: \( E_{\min 3} \). Right panel: Solitons separations versus the number of solitons for same parameters. Red line: \( \Delta_{\min 1} \), Blue dotted line: \( \Delta_{\min 2} \) and Green dashed line: \( \Delta_{\min 3} \). \( E_{\min 1,2,3} \) and \( \Delta_{\min 1,2,3} \) correspond respectively to the energy and separation at the first, second and the third minima of the energy.