Stable and Efficient Structures
in Multigroup Network Formation
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Abstract—In this work we present a strategic network formation model predicting the emergence of multigroup structures. Individuals decide to form or remove links based on the benefits and costs those connections carry; we focus on bilateral consent for link formation. An exogenous system specifies the frequency of coordination issues arising among the groups. We are interested in structures that arise to resolve coordination issues and, specifically, structures in which groups are linked through bridging, redundant, and co-membership interconnections. We characterize the conditions under which certain structures are stable and study their efficiency as well as the convergence of formation dynamics.

Index Terms—strategic network formation, game theory, multigroup connectivity models

I. INTRODUCTION
A. Motivation and problem description
To study the coordination and control features of a group task, the multiple groups performances must be fitted together. An enduring postulate in organization science is that coordination and control cannot be achieved strictly by the authority structure, but must also entail informal communication and influence networks that link the members of different task-oriented groups; we focus on formation of such network structures. As the size of a connected social network increases, multigroup formations that are distinguishable clusters of individuals become a characteristic and important feature of network topology. The connectivity of multigroup networks may be based on edge bundles connecting multiple individuals in two disjoint groups, bridges connecting two individuals in two disjoint groups, or co-memberships. A large-scale network may include instances of all of these connectivity modalities. We set up populations of multiple groups and propose a dynamic model for formation of these intergroup connectivity structures.

Our economic dynamical model explains and predicts whether a network evolves into different coordination and control structures. Medium and large scale organizations adopt these multigroup structures to tackle complex nested tasks. Among the multitude of possible coordination and control structures, we study formation of multigroup connectivity structures shown in Fig. 1, which are familiar constructs in the field of social network science. For this purpose we apply a game-theoretic framework in which strategic agents take actions based on the rate or importance of coordination problems. In other words, a value is assigned to the coordination problem between any two distinct groups, so that all control and coordination problems among groups are described by a square non-negative matrix, as illustrated in Fig. 2. In our setting, agents are myopic, self-interested, and have thorough knowledge of graph topology and the utility they acquire from any other agent.

B. Related literature
Bridge, edge bundle, and co-membership connectivity models have been studied extensively in [17], where implications of these structures are investigated and generative models are proposed for each. These prototypical structures can mitigate coordination and control loss in an organization. Coordination and control importance of bridge connected structure, in which communication between subgroups are based on single contact edges, is the emphasis of the [9], [22], and [21] models. Coordination and control importance of the redundant ties structure, in which multiple redundant contact edges connect pairs of groups, is the emphasis of [8], Chapter 8, [7], and [23]. Co-membership intersection structures, in which subgroups have common members, is the emphasis of the linking-pin model structures shown in Fig. 1, which are familiar constructs in the field of social network science. For this purpose we apply
by Likert [14], as well as [6] and [2], [25] and [24] propose a community detection algorithm for overlapping networks.

Jackson and Wolinsky introduced a strategic network formation model in their seminal paper [12]. They studied pairwise stability, where bilateral agreement is required for link formation. Homogeneity and common knowledge of current network to all players are two assumptions in this model. Jackson and Watts studied strategic network formation in a dynamic framework in [11]. The network formation model we present in this work is closely related to [12] and [11]. Jackson and Rogers examined an economic model of network formation in [10] where agents benefit from indirect relationships. They showed that small-world features necessarily emerge for a wide set of parameters.

In [1], Bala and Goyal proposed a dynamic model to study Nash and strict Nash stability. In their model, starting from any initial network, each player with some positive probability plays a best response (or randomizes across them when there is more than one); otherwise the player exhibits inertia. A Markov chain on the state space of all networks is defined whose absorbing states are strict Nash networks. The authors proved that starting from any network, the dynamic process converges to a strict Nash network (i.e., the empty network or a center-sponsored star) with probability 1.

In [18], Olaizola and Valenciano extended the model in [1] and studied network formation under linking constraints. An exogenous link-constraining system specifies the admissible links. Players in the same component of the link-constraining network have common knowledge of that component. This model collapses to the unrestricted setting in [1] (when the underling constraining network is complete graph). The set of Nash networks is a subset of Bala and Goyal’s unrestricted Nash network sets.

In the network formation game by Chasparis and Shamma in [5] and [3], agents form and sever unidirectional links with other nodes, and stable networks are characterized through the notion of Nash equilibrium. Pagan and Dörfler [19] studied network formation on directed weighted graphs and considered two notions of stability: Nash equilibrium to model purely selfish actors, and pairwise-Nash stability which combines the selfish attitude with the possibility of coordination among agents. McBride dropped the common knowledge assumption and studied the effects of limited perception (each player perceives the current network only up to a certain distance) in [16]. Song and van der Schaaf [20] studied a dynamic network formation model with incomplete information.

Community networks and their growth into potential socially robust structures is studied in [15]. Bringmann et al. analyzed the evolution of large networks to predict link creation among the nodes in [3]. [13] studied link inference problem in heterogeneous information networks by proposing a knapsack-constrained inference method.

D. Preliminaries

Each undirected graph is identified with the pair \((\mathcal{V}, \mathcal{E})\). The set of graph nodes \(\mathcal{V} \neq \emptyset\) represents individuals or groups of individuals in a social network. \(|\mathcal{V}| = n\) is the size of the network. The pair \((i, j)\) is called an edge and it indicates the interaction between the two individuals \(i\) and \(j\). The set of graph edges \(\mathcal{E}\) represents the social interactions or ties among all individuals. Throughout this paper, since the individuals are unchangeable, we refer to the network \((\mathcal{V}, \mathcal{E})\) simply as \(\mathcal{E}\).

The density of a graph is given by the ratio of the number of its observed to possible edges, \(\frac{2|\mathcal{E}|}{n(n-1)}\). In a complete graph every pair of distinct nodes is connected by an edge. We denote the complete graph of size \(n\) by \(K_n\). A clique is a subset of vertices of a graph in which every two distinct vertices are adjacent. We say two graphs are adjacent if they differ in precisely one edge. A path of length \(k\) is a sequence of nodes \(i_1i_2\ldots i_k\) such that \(\{(i_s, i_{s+1})\} \in \mathcal{E}\). A walk of minimum length between two nodes \(s\) is the shortest path. \(d_{ij}(\mathcal{E})\) denotes
the distance between nodes \(i\) and \(j\), which is defined as the length of the shortest path beginning at \(i\) and ending at \(j\).

II. Multigroup Network Formation Model

Consider a society of \(n\) individuals \(\mathcal{V}\), divided into \(m\) groups. The set of \(m\) groups is denoted by \(\{1, \ldots, m\}\), \(m \leq n\). \(P = \{P_1, \ldots, P_m\}\) represents the partitioning of individuals into the groups and is a set partition of size \(n\), i.e., \(\mathcal{V} = \bigcup_{\gamma=1}^{m} P_{\gamma}\), and \(\bigcap_{\gamma=1}^{m} P_{\gamma} = \emptyset\). We use the shorthand notation \(s_{\gamma} = |P_{\gamma}|\) denoting the size of group \(\gamma\). Throughout this paper, we assume that \(s_{\gamma} \geq 3\) for all \(\gamma \in \{1, \ldots, m\}\).

Group coordination importance matrix (data): is given as \(F \in \mathbb{R}^{m \times m}\), where \(0 \leq F_{\alpha\beta} \leq 1\) for \(\alpha, \beta \in \{1, \ldots, m\}\) represents importance/frequency of coordination problem between groups \(\alpha\) and \(\beta\). We assume \(F\) is a symmetric matrix with diagonal entries equal to 1.

Individual coordination importance matrix: \(\hat{F} \in \mathbb{R}^{n \times n}\), is obtained from \(F\) and the partition \(P\), i.e., \(\hat{F} = f(F, P)\). We construct \(\hat{F}\) as follows:

\[
\hat{F}_{ij} = \begin{cases} 
F_{\alpha\beta}, & i \in P_{\alpha}, j \in P_{\beta}, i \neq j \\
0, & i = j.
\end{cases}
\]

For the setting where groups are all of equal size \(s\), one can write

\[
\hat{F} = F \otimes 1_s 1_s^T - I_n
\]

At edge set \(\mathcal{E}\), the payoff function for individual \(i \in \mathcal{V}\) is

\[
U_i(\mathcal{E}) = \sum_{k=1}^{n} \hat{F}_{ik} d_{ik}(\mathcal{E}) - \sum_{k \in N_i(\mathcal{E})} c,
\]

where \(d_{ik}(\mathcal{E})\) is the number of steps from individual \(i\) to \(k\), \(\delta < 1\) is the one-hop benefit, and \(c\) is the cost of each link. The value of network \(\mathcal{E}\) is defined as the sum of all individuals’ payoffs, i.e., \(v(\mathcal{E}) = \sum_{i=1}^{n} U_i(\mathcal{E})\), and it indicates the social welfare. For a given society \(\mathcal{V}\) and value function \(v\), \(\mathcal{E}^*\) is an efficient structure if its social welfare\(\text{value}\) is maximized over all possible edge sets on \(\mathcal{V}\), i.e., \(\mathcal{E}^* = \arg\max_{\mathcal{E}} v(\mathcal{E})\). Given the pair \((i, j)\) in network \(\mathcal{E}\), we say that individual \(i\) benefits from edge \(\{i, j\}\) if \(U_i(\mathcal{E} \cup \{(i, j)\}) > U_i(\mathcal{E} \setminus \{(i, j)\})\).

Formation Dynamics: Time periods are represented with countable infinite set \(\mathbb{N} = \{1, 2, \ldots, t, \ldots\}\). In each period, a pair \((i, j)\) is uniformly randomly selected and is added to, or removed from, the network \(\mathcal{E}\) according to the following rules:

- if \(\{(i, j)\} \notin \mathcal{E}\), then it is added when its addition is marginally beneficial to the pair of individuals (i.e., either both individuals benefit or one individual is indifferent and the other benefits); the edge \((i, j)\) is not added when its addition causes a drop in the payoff of either or both individuals or both individuals are indifferent towards it; and
- if \(\{(i, j)\} \in \mathcal{E}\), then \((i, j)\) is removed when its removal benefits at least one of the two individuals; no action is taken when both sides are either indifferent or benefit from the existence of the edge.

Definition II.1. (Pairwise Stability) A network \(\mathcal{E}\) is pairwise stable if,

\[
\text{for all } \{(i, j)\} \in \mathcal{E},
U_i(\mathcal{E}) \geq U_i(\mathcal{E} \setminus \{(i, j)\}) \text{ and } U_j(\mathcal{E}) \geq U_j(\mathcal{E} \setminus \{(i, j)\});
\]

and for all \(\{(i, j)\} \notin \mathcal{E},
\text{if } U_i(\mathcal{E}) < U_i(\mathcal{E} \cup \{(i, j)\}) \text{, then } U_j(\mathcal{E}) > U_j(\mathcal{E} \cup \{(i, j)\})\).

Remark II.2. According to Definition II.1, if the edge \((i, j)\) belongs to the pairwise stable network, removing it results in a loss for \(i\) or \(j\); and if the edge \((i, j)\) does not belong to the pairwise stable network, adding it makes no difference or causes loss for \(i\) or \(j\).

Definition II.3. \(\mathcal{E}'\) defeats \(\mathcal{E}\) if either \(\mathcal{E}' = \mathcal{E} \setminus \{(i, j)\}\) and \(U_i(\mathcal{E}') > U_i(\mathcal{E})\), or \(\mathcal{E}' = \mathcal{E} \cup \{(i, j)\}\) and \(U_i(\mathcal{E}') \geq U_i(\mathcal{E})\) and \(U_j(\mathcal{E}') \geq U_j(\mathcal{E})\) with at least one inequality holding strictly.

Lemma II.4. A network is pairwise stable if and only if it does not change under Formation Dynamics.

Proof. To prove necessity, we refer to Remark II.2. According to the definition, if a network is pairwise stable, no network can defeat it, i.e., no links can be added to or severed from it. To show sufficiency, note that a network not being changed by Formation Dynamics, implies that:

(i) adding a link makes no difference or causes loss for at least one individual;
(ii) removing a link results in loss for at least one individual. Therefore, the network is pairwise stable.

According to Lemma II.4, if there exists some time \(t^*\) such that from \(t^*\) on, no additional links are added to or severed from a network by Formation Dynamics, then the network has reached the pairwise stable structure.

We define the following terms that we will frequently use throughout this paper indicating the density of the interconnections among the groups.

Definition II.5. We say that a society of individuals consists of the disjoint union of groups if there exists no interconnection among any pairs of groups. For a connected pair, we say it is

(i) minimally connected if there exists exactly one interconnection among the pair;
(ii) redundantly connected if there exist at least two interconnections among the pair;
(iii) maximally connected if all of the possible interconnections among the pair of groups exist.

Fig. 3 represents a schematic illustration of the terms discussed above.

Remark II.6. A minimally connected pair corresponds to the bridge connection (Fig. 1c), redundantly connected to the ridge connection (Fig. 1b), and maximally connected to a full co-membership connection (Fig. 1a.)

We next define the Price of Anarchy (PoA) as a measure of how the efficiency of a system degrades due to the selfish behavior of its individuals. It is calculated as follows:

\[
\text{PoA} = \frac{\max_{\mathcal{E}} v(\mathcal{E})}{\min_{\mathcal{E}, \text{stable}} v(\mathcal{E})}.
\]
Throughout this paper we use the following threshold functions $y_1(s, \delta)$, $y_2(s, \delta)$, and $y_3(\delta)$ defined by
\[
y_1(s, \delta) = \delta + (s - 1)\delta^2, \\
y_2(s, \delta) = \delta - \delta^2 + (s - 1)\delta^2 - (s - 1)\delta^3 = (1 - \delta)y_1(s), \\
y_3(\delta) = \delta - \delta^2.
\]
In what follows we will often suppress the argument $\delta$ in the interest of simplicity.

Under the conditions $0 < \delta < 1$ and $s \geq 3$, we claim that,
\[
0 < y_3 < y_2(s) < y_1(s).
\]
The proof is as follows: it is easy to see that $y_2(s) < y_1(s)$. To verify $y_3 < y_2(s)$, we rewrite $y_2(s)$ as $(\delta - \delta^2)(1 + \delta(s - 1)) = y_3(1 + \delta(s - 1)) > y_3$. Plots of these three threshold functions for $0 < \delta < 1$, where $s = 3$ are depicted in Fig. 4.

![Fig. 4. Plots of y1, y2, and y3 for s = 3.](image)

III. RESULTS ON FORMATION OF DISJOINT CLIQUES

We first study the inner structure of each group in a pairwise stable network. Throughout this paper, we assume that the dynamics does not start with an initial state containing any interconnection. We define the invariant set of all subgraphs of disjoint cliques as $S = \{ \bigcup_{\gamma=1}^{m} \mathcal{E}_{\gamma} \mid \mathcal{E}_{\gamma} \subset \mathcal{E}_{K_{s_{\gamma}}}, \mathcal{E}_{\gamma} \text{ indicates the inner-network of group } P_{s_{\gamma}} \}.$

**Theorem III.1** (Formation of Cliques: Pairwise Stability, Efficiency, Convergence). Consider $n$ individuals partitioned into groups $P_1, \ldots, P_m$. Then, each one of these $m$ groups is a clique in the pairwise stable and in the efficient structure if and only if $c < y_3$. Moreover, starting from any state in the invariant set $S$, each group $P_\alpha$ will form a $s_\alpha$-size clique along Formation Dynamics (introduced in Section II).

**Proof.** We first provide the proof of sufficiency for pairwise stability: for any individual $i \in P_\alpha$, a direct link with individual $j$, $(j \neq i)$ from the same group provides a profit of $\delta - c$. Without a direct link, this profit is equal to $\delta d_{ij}$, where $d_{ij} > 1$ is the distance between $i$ and $j$ in $\mathcal{E} \setminus \{(i, j)\}$. Since $\delta - \delta^2 > c$, we have $\delta - c > \delta^2 > \ldots > \delta^3$, meaning that all agents prefer direct links to any indirect link. Thus, if agents $i$ and $j$ in group $P_\alpha$ are not directly connected, they will form a link and each will gain at least $(\delta - c) - \delta d_{ij} > 0$, i.e.,

\[
\forall \{(i, j)\} \notin \mathcal{E}, \quad i, j \in P_\alpha, \quad i \neq j \\
U_i(\mathcal{E}) < U_i(\mathcal{E} \cup \{(i, j)\}), \quad \text{and} \quad U_j(\mathcal{E}) < U_j(\mathcal{E} \cup \{(i, j)\}).
\]

Moreover, no node has an incentive to break any link since its payoff strictly decreases if it do so, i.e.,

\[
\forall \{(i, j)\} \in \mathcal{E}, \quad i, j \in P_\alpha, \quad i \neq j, \\
U_i(\mathcal{E}) > U_i(\mathcal{E} \setminus \{(i, j)\}), \quad \text{and} \quad U_j(\mathcal{E}) > U_j(\mathcal{E} \setminus \{(i, j)\}).
\]

Thus, each group forms a clique and no intra-connection is removed after being formed, and according to Lemma II.4, these $m$ groups are cliques in the pairwise stable structure. To prove necessity, assume we have a pairwise stable clique. For $P_\alpha$ to remain a clique, all pairs of nodes belonging to the same group should prefer to keep one-hop links rather than having links with larger lengths, and thus $\delta - c > \delta^2 > \delta^3 > \ldots$. This proves the claim that each group $P_\alpha$ is a clique if and only if $c < \delta - \delta^2$. Convergence of dynamics to cliques can be obtained directly from the same argument.

We now continue by first proving that if $c < \delta - \delta^2$, in the efficient structure each group is a clique. From the analysis above, when $c < \delta - \delta^2$, we have:

\[
v(\mathcal{E} \cup \{(i, j)\}) - v(\mathcal{E} \setminus \{(i, j)\}) \\
\geq U_i(\mathcal{E} \cup \{(i, j)\}) + U_j(\mathcal{E} \cup \{(i, j)\}) \\
- U_i(\mathcal{E} \setminus \{(i, j)\}) - U_j(\mathcal{E} \setminus \{(i, j)\}) \\
\geq 2(\delta - c - \delta^2) > 0
\]

which holds for each pair $(i, j)$ belonging to the same group, meaning that each group is a clique in the efficient structure. We next prove necessity for efficiency: assume $\mathcal{E}$ is the efficient structure and each group is a clique, i.e., $\{(i, j)\} \in \mathcal{E}$ for any two individuals $i, j, (i \neq j)$ from the same group.

Then, we have:

\[
v(\mathcal{E}) - v(\mathcal{E} \setminus \{(i, j)\}) \\
= U_i(\mathcal{E}) + U_j(\mathcal{E}) - U_i(\mathcal{E} \setminus \{(i, j)\}) - U_j(\mathcal{E} \setminus \{(i, j)\}) \\
= 2(\delta - c - \delta^2) > 0,
\]

which results in $c < y_3$. □

**Note:** Theorem III.1 implies that formation of cliques requires $c < 1/4$.

**Theorem III.2** (Pairwise Stable Structures and Convergence: Disjoint Union of Cliques). Consider $n$ individuals partitioned into groups $P_1, \ldots, P_m$ of sizes $s_1, \ldots, s_m$ respectively. Assume that $c < y_3$. Then, the unique pairwise stable structures consists of disjoint union of cliques equal to the groups $P_1, \ldots, P_m$ if and only if $F_{\alpha\beta} = \max_{s \in \{s_{\alpha}, s_{\beta}\}} \frac{c}{y_1(s)}$ for all $\alpha, \beta \in \{1, \ldots, m\}, \alpha \neq \beta$. Moreover, starting from any state...
in the invariant set $S$, Formation Dynamics (introduced in Section II) converges to this pairwise stable structure.

Proof. We first prove sufficiency. Since no interconnection exists in the invariant set $S$, for any network $E \in S$, suppose two individuals $i \in P_a$ and $j \in P_b$ are picked to decide whether to add the corresponding interconnection or not. We know that $U_i(E \cup \{i, j\}) \leq F_{a\beta} y_1(s_{a\beta}) - c$ and $U_j(E \cup \{i, j\}) \leq F_{a\beta} y_1(s_a) - c$, where equalities hold when both groups form cliques. Since $F_{a\beta} \leq \max_{s \in \{s_a, s_{a\beta}\}} y_1(s)$, at least one of $U_i(E \cup \{i, j\}) \leq 0$ and $U_j(E \cup \{i, j\}) \leq 0$ holds. Therefore, the interconnection $\{i, j\}$ does not belong to pairwise stable structure and it does not form. From Theorem III.1, we know that unique stable state consists of the disjoint union of cliques equal to the groups $\{1, \ldots, m\}$. Suppose that all groups form cliques at a time $t^\ast$. From then on, no link will be added or removed. According to Lemma II.4, the pairwise stable structure consists of the disjoint union of cliques equal to the groups $\{1, \ldots, m\}$, and the network converges to this unique stable state.

To prove necessity, take any two individuals $i \in P_a$ and $j \in P_b$. Since $\{i, j\}$ does not belong to pairwise stable structure, we have at least one of $U_i(E \cup \{i, j\}) \leq 0$ and $U_j(E \cup \{i, j\}) \leq 0$ holds, and therefore, $F_{a\beta} \leq \max_{s \in \{s_a, s_{a\beta}\}} y_1(s)$.

IV. TWO GROUP CONNECTIVITY STRUCTURE

In this section we study pairwise stable and efficient structures when individuals are partitioned into two groups.

A. Pairwise Stability

In what follows we give the sufficient and necessary condition for pairwise stable structures.

**Theorem IV.1** (Pairwise Stability and Convergence with Two groups). Consider $n$ individuals partitioned into groups $P_1$ and $P_2$ of sizes $s_1$ and $s_2$ respectively. Then, under the assumption $c < y_3$, the network has

(i) a unique pairwise stable structure consisting of minimally connected cliques if and only if

$$\max_{s \in \{s_1, s_2\}} \frac{c}{y_1(s)} \leq F_{12} < \max_{s \in \{s_1, s_2\}} \frac{c}{y_2(s)};$$

(ii) a unique pairwise stable structure consisting of exact $k$ ($2 \leq k \leq \min\{s_1, s_2\}$) interconnections if and only if

$$\max_{s \in \{s_1, s_2\}} \frac{c}{y_2(s) - (k - 2)\delta y_3} \leq F_{12} < \max_{s \in \{s_1, s_2\}} \frac{c}{y_2(s) - (k - 1)\delta y_3}.$$  \hspace{1cm} (2)

(iii) a unique pairwise stable structure consisting of maximally connected cliques if and only if

$$\frac{c}{y_3} < F_{12} \leq 1.$$  

Moreover, starting from any state in the invariant set $S$, the Formation Dynamics (introduced in Section II) converges to the corresponding pairwise stable structure.

Proof. From Theorem III.2, we know that at least one interconnection will be formed under Formation Dynamics if and only if $F_{12} \geq \max_{s \in \{s_1, s_2\}} \frac{c}{y_1(s)}$. Without loss of generality, suppose that $E$ contain two cliques and at least one interconnection between them. Assume that agents $i$ and $j$, respectively, are picked from $P_1$ and $P_2$, are connected in $E$.

Take agents $i$ from $P_1$ and $j$ from $P_2$. For $i \neq j$, we have $U_i(E \cup \{i, j\}) = (s_1 - 1)\delta + F_{12}\delta + (s_2 - 1)F_{12}\delta^2 - s_1c$ and $U_j(E \cup \{i, j\}) = (s_1 - 1)(\delta - c) + F_{12}\delta^2 + (s_2 - 1)F_{12}\delta^3$, implying $U_i(E \cup \{i, j\}) \leq U_j(E \cup \{i, j\})$ $\iff$ $F_{12} \geq \frac{c}{y_2(s)}$. From $U_j(E \cup \{i, j\}) = (s_2 - 1)\delta + 2F_{12}\delta^2 + (s_2 - 2)F_{12}\delta^2 + (s_2 + 1)c$ and $U_j(E \cup \{i, j\}) = (s_2 - 1)\delta + F_{12}\delta^2 + (s_2 - 1)F_{12}\delta^3 + s_2c$, we obtain; $U_j(E \cup \{i, j\}) \leq U_j(E \cup \{i, j\})$ $\iff$ $F_{12} > \frac{c}{y_3}$. Then, from $y_2(s) > y_3 > 0$, we conclude that an additional interconnection $\{i, j\}$ is added and maintained if and only if $F_{12} \geq \frac{c}{y_3}$. Similarly, an additional interconnection $\{i, j\}$ is added and maintained if and only if $F_{12} > \frac{c}{y_3}$. For $i \neq j, j \neq j$, using a similar argument, we obtain that $U_i(E \cup \{i, j\}) > U_j(E \cup \{i, j\})$ $\iff$ $F_{12} > \frac{c}{y_2(s)}$, which means that an additional interconnection $\{i, j\}$ is added and maintained if and only if $F_{12} > \max_{s \in \{s_1, s_2\}} \frac{c}{y_2(s)}$ (strictly holds when $s_1 = s_2$). Thus, we conclude that at least two interconnections are added and maintained if and only if $F_{12} > \max_{s \in \{s_1, s_2\}} \frac{c}{y_2(s)}$ (strictly holds when $s_1 = s_2$). Therefore, the network contains precisely one interconnection if and only if $\max_{s \in \{s_1, s_2\}} \frac{c}{y_2(s)} < F_{12} < \max_{s \in \{s_1, s_2\}} \frac{c}{y_1(s)}$. Moreover, from the moment when two group form cliques and this unique interconnection builds, the network will not change. According to Lemma II.4, this concludes the proof of statement (i).

To prove (iii), assume that $F_{12} > \max_{s \in \{s_1, s_2\}} \frac{c}{y_2(s)}$. We have shown above that $E$ contains at least two interconnections between two cliques. As a result, for any agent $i$ from $P_1$ and $j$ from $P_2$, the distance between $i$ and $j$ in $E \cup \{i, j\}$ is equal to either 2 or 3. If it is equal to 2, then $U_j(E \cup \{i, j\}) - U_j(E \cup \{i, j\}) = U_j(E \cup \{i, j\}) - U_j(E \cup \{i, j\}) = F_{12}\delta - \delta^2 - c$; and if it is equal to 3, then $U_j(E \cup \{i, j\}) - U_j(E \cup \{i, j\}) = U_j(E \cup \{i, j\}) - U_j(E \cup \{i, j\}) = F_{12}\delta - \delta^3 - c$. Inter-
connection \{({i}, {j})\} is added and maintained if and only if
\(U_i(E \cup \{({i}, {j})\}) - U_i(E \setminus \{({i}, {j})\}) > 0\) and \(U_j(E \cup \{({i}, {j})\}) - U_j(E \setminus \{({i}, {j})\}) > 0\). Thus, we conclude that \{({i}, {j})\} is added
and maintained if and only if \(F_{12} > \max \left\{ \frac{c}{\delta - \delta^2}, \frac{c}{\delta - \delta^3} \right\} \).

Since \(\max \left\{ \frac{c}{\delta - \delta^2}, \frac{c}{\delta - \delta^3} \right\} = \frac{c}{\delta - \delta^2}\) for \(0 < \delta < 1\), \{({i}, {j})\} is added and maintained if and only if
\(F_{12} > \frac{c}{\delta - \delta^2} = \frac{c}{y_3}\). Therefore, the network will not be changed when all agents link with each other. By Lemma II.4,
this concludes the proof of (iii).

From statements (i) and (iii), we know that the pairwise stable structure contains at least 2 but not fully numbers of
interconnections if \(\max_{s \in \{s_1, s_2\}} y_2(s) < F_{12} < \frac{c}{y_3}\). Suppose that
\((i_1, j_1), \ldots, (i_{k-1}, j_{k-1})\) are \(k - 1\) interconnections between
\(P_1\) and \(P_2\). Take agents \(i\) from \(P_1\) and \(j\) from \(P_2\). Similar to the analysis in the proof of statement (i), we have the following two cases:

(a) For \(i \notin \{i_1, \ldots, i_{k-1}\}, j \notin \{j_1, \ldots, j_{k-1}\}\), we have
\[
U_1(E \cup \{({i}, {j})\}) - U_1(E \setminus \{({i}, {j})\}) = F_{12}(y_2(s_2) - (k - 2)\delta y_3),
\]
\[
U_2(E \cup \{({i}, {j})\}) - U_2(E \setminus \{({i}, {j})\}) = F_{12}(y_2(s_1) - (k - 2)\delta y_3),
\]

implying
\[
U_1(E \cup \{({i}, {j})\}) > U_1(E \setminus \{({i}, {j})\}), \quad \text{and} \quad U_2(E \cup \{({i}, {j})\}) > U_2(E \setminus \{({i}, {j})\}) \iff F_{12} > \frac{c}{y_3}.
\]

(b) For \(i \in \{i_1, \ldots, i_{k-1}\}, j \notin \{j_1, \ldots, j_{k-1}\}\) or \(i \notin \{i_1, \ldots, i_{k-1}\}, j \in \{j_1, \ldots, j_{k-1}\}\), we have
\[
U_1(E \cup \{({i}, {j})\}) > U_1(E \setminus \{({i}, {j})\}) \quad \text{and} \quad U_2(E \cup \{({i}, {j})\}) > U_2(E \setminus \{({i}, {j})\}) \iff F_{12} > \frac{c}{y_3}.
\]

Therefore, we conclude then \(k - th\) interconnection is added and maintained if and only if \(F_{12} > \max_{s \in \{s_1, s_2\}} y_2(s) - (k - 2)\delta y_3\). Likewise, the \((k + 1) - th\) interconnection is added and maintained if and only if
\(F_{12} > \max_{s \in \{s_1, s_2\}} y_2(s) - (k - 1)\delta y_3\). It follows that the unique
pair-wise stable structure has exactly \(k\) \((2 \leq k \leq \min\{s_1, s_2\})\) interconnections if and only if
\(F_{12} > \max_{s \in \{s_1, s_2\}} y_2(s) - (k - 1)\delta y_3 < \max_{s \in \{s_1, s_2\}} y_2(s) - (k - \delta) y_3\). This concludes the proof
of (ii).

Finally, we complete the proof of Theorem IV.1 by proving the convergence statement. Since \(c < y_3\), from Theorem III.1
we know that the network structure, regardless of the density of intra-group connections, consists of cliques of sizes \(s_1\) and
\(s_2\). From above analysis, starting from invariable set \(S\), intra-
connections will increase one by one until one more intra-
connection can not bring increasing of benefit for two players.

Then, it follows from Lemma II.4 that the network can not be changed from then on, i.e., the network will converge to the
Corresponding pairwise stable structure under the formation dynamics in Section II.

Theorem IV.1 implies that if \(F_{12}\) equals the boundaries, i.e.,
\[
\max_{s \in \{s_1, s_2\}} y_2(s) - \frac{c}{y_3}, \quad \max_{s \in \{s_1, s_2\}} y_2(s) - \frac{c}{y_3}, \quad \max_{s \in \{s_1, s_2\}} y_2(s) - \frac{c}{y_3}, \quad \ldots,
\]
the pairwise stable structure is not unique.

To show this, we provide the following example for a society consisting of 8 individuals.

Example IV.2. Suppose that individuals are partitioned into
groups \(P_1 = \{1, 2, 3\}\) and \(P_2 = \{4, 5, 6, 7, 8\}\). Let \(c = 0.2\), \(\delta = 0.5\) and \(F_{12} = \max_{s \in \{s_1, s_2\}} y_2(s) = 0.4\). Assume at time
0, each of the two groups have a line structure, illustrated in
Fig 6. Now consider the following two processes:

Process A. At first, individuals 2 and 6 are chosen to play the
game and interconnection \((2, 6)\) is formed. Then, individuals 3 and 7 are chosen to play the game, and
interconnection \((3, 7)\) is also formed. After that, all possible intra-connections are considered and formed. Finally, the network contains two
interconnections and reaches pairwise stability.

Process B. At first, all possible intra-connections are consid-
ered and formed. Then individuals 2 and 6 are chosen to play the game and interconnection \((2, 6)\) is formed. As a result, the network contains only
1 interconnection and reaches pairwise stability.

Fig. 6. An illustration of two processes in Example IV.2. At each
step, the darker dots are chosen to play the game.

Fig. 5 illustrates the scenarios specified in Theorems III.2
and IV.1 for \(F_{12}\).

Remark IV.3. As the difference between group sizes \(|s_2 - s_1|\)
increases, the bound for formation of bridges increases, and
thus communication between groups becomes harder. To see
why this is true, for a society of \(n = s_1 + s_2\) individuals,
without loss of generality, we assume that \(s_1 \leq s_2\). We know
\[
\max_{s \in \{s_1, s_2\}} y_1(s) = \min_{s \in \{s_1, s_2\}} y_1(s) = y_1(\min\{s_1, s_2\}), \quad \text{and}
\]
since \(\max_{s \in \{s_1, s_2\}} y_1(s) = \frac{c}{y_3} \) is a monotonically decreasing function
of \(s\), as \(s_1\) increases, \(|s_2 - s_1|\), and therefore, \(\max_{s \in \{s_1, s_2\}} y_1(s)\)
decreases, and communication is facilitated. As illustrated in
Fig. 7, for a society of fixed size, as the sizes of the two groups
becomes closer to each other, the number of interconnections
increases.
It follows that

\[ s \text{ of any intra-connection causes loss for both individuals and involved in that link, as well as the social welfare, and removal} \]

Proof.

From Theorem III.1, we know that addition of an intra-connection results in increasing the payoff of both individuals involved in that link, as well as the social welfare, and removal of any intra-connection causes loss for both individuals and decreases social welfare. As a result, the efficient structure consists of two cliques. Suppose \( i \in P_1 \) and \( j \in P_2 \). Let \( E_0 = E_{K_{s_1}} \cup E_{K_{s_2}} \) be the union of two cliques of sizes \( s_1 \) and \( s_2 \). We have

\[
U_k(E_0 \cup \{(i, j)\}) - U_k(E_0) =
\begin{cases}
F_{12}(\delta + (s_2 - 1)\delta^2) - c, & k = i; \\
F_{12}(\delta + (s_1 - 1)\delta^2) - c, & k = j; \\
F_{12}(\delta^2 + (s_2 - 1)\delta^3), & k \in P_1, k \neq i; \\
F_{12}(\delta^2 + (s_1 - 1)\delta^3), & k \in P_2, k \neq j.
\end{cases}
\]

It follows that

\[ v(E_0 \cup \{(i, j)\}) - v(E_0) = 2F_{12}[\delta + (s_1 - 1)\delta^2 + (s_2 - 1)\delta^2 + (s_1 - 1)(s_2 - 1)\delta^3] - 2c. \]

\( E_0 \) is the efficient structure if and only if \( v(E_0 \cup \{(i, j)\}) - v(E_0) \leq 0 \), which is equivalent to:

\[
F_{12} \leq \frac{c}{\delta + (s_1 - 1)\delta^2 + (s_2 - 1)\delta^2 + (s_1 - 1)(s_2 - 1)\delta^3} = \frac{\delta c}{y_1(s_1)y_1(s_2)}. 
\]

This concludes the proof of (i).

Now let \( E_1 = E_0 \cup \{(i, j)\} \). As elaborated above, \( v(E_1) - v(E_0) \geq 0 \) if and only if

\[
F_{12} \geq \frac{c}{\delta + (s_1 - 1)\delta^2 + (s_2 - 1)\delta^2 + (s_1 - 1)(s_2 - 1)\delta^3} = \frac{\delta c}{y_1(s_1)y_1(s_2)}. 
\]

Suppose \( \hat{i} \in P_1 \) and \( \hat{j} \in P_2 \). For the case of \( \hat{i} \neq i \) and \( \hat{j} \neq j \), we have

\[
U_k(E_1 \cup \{(\hat{i}, \hat{j})\}) - U_k(E_1) =
\begin{cases}
F_{12}[y_2(s_2) - c], & k = \hat{i} \\
F_{12}[y_2(s_1) - c], & k = \hat{j} \\
0, & k \in P_2 \text{ and } k \neq \hat{j}, \\
F_{12}(\delta^2 - \delta^3), & k \in P_1 \text{ and } k \neq i.
\end{cases}
\]

For the case of \( \hat{i} = i \) and \( \hat{j} \neq j \), we have

\[
U_k(E_1 \cup \{(\hat{i}, \hat{j})\}) - U_k(E_1) =
\begin{cases}
F_{12}(\delta - \delta^3) - c, & k = \hat{i} \\
F_{12}y_2(s_1) - c, & k = \hat{j} \\
0, & k \in P_2 \text{ and } k \neq \hat{j}, \\
F_{12}(\delta^2 - \delta^3), & k \in P_1 \text{ and } k \neq \hat{i}.
\end{cases}
\]

For the case of \( \hat{i} \neq i \) and \( \hat{j} = j \), we have

\[
U_k(E_1 \cup \{(\hat{i}, \hat{j})\}) - U_k(E_1) =
\begin{cases}
F_{12}(\delta - \delta^3) - c, & k = j \\
F_{12}y_2(s_2) - c, & k = \hat{i} \\
0, & k \in P_1 \text{ and } k \neq \hat{i}, \\
F_{12}(\delta^2 - \delta^3), & k \in P_2 \text{ and } k \neq j.
\end{cases}
\]

It follows that

\[ v(E_1 \cup \{(\hat{i}, \hat{j})\}) - v(E_1) =
\begin{cases}
F_{12}[y_2(s_2) + y_2(s_1) + (s_1 + s_2 - 4)(\delta^2 - \delta^3)] - 2c, & \hat{j} \neq \hat{i}, \hat{i} \neq i, \\
2F_{12}y_2(s_1) - 2c, & \hat{i} = i, \hat{j} \neq j, \\
2F_{12}y_2(s_2) - 2c, & \hat{i} \neq i, \hat{j} = j.
\end{cases}
\]

Since \( y_2(s) \) is a monotonically increasing function of \( s \) and

\[
2c \frac{c}{y_2(s_2) + y_2(s_1) + (s_1 + s_2 - 4)(\delta^2 - \delta^3)} = \frac{c}{y_2(s_1) + (s_2 - 2)(\delta^2 - \delta^3)} = \frac{c}{y_2(s_2) + (s_1 - 2)(\delta^2 - \delta^3)},
\]

(3)
we have
\[
\frac{2c}{y_2(s_2) + y_2(s_1) + (s_1 + s_2 - 4)(\delta^2 - \delta^3)} < \min_{s \in \{s_1, s_2\}} \frac{c}{y_2(s)}.
\]
We then conclude that \(v(E_2 \cup \{(i, j)\}) - v(E_2) \leq 0\) if and only if \(F_{12} \leq \frac{2c}{y_2(s_2) + y_2(s_1) + (s_1 + s_2 - 4)\delta y_3}\). Statement (ii) follows accordingly.

Assume \(F_{12} \geq \max_{s \in \{s_1, s_2\}} \frac{c}{y_2(s)}\). From statement (ii), we know that the efficient structure has at least two interconnections. For any agent \(i\) from \(P_1\) and \(j\) from \(P_2\), as elaborated in the proof of Theorem IV.4, one has
\[
v(E \cup \{(i, j)\}) - v(E \setminus \{(i, j)\}) \geq 2[F_{12}(\delta - \delta^2) - c],
\]
which concludes statements (iii) and (iv).

\[\square\]

From the properties of \(y_1(s), y_2(s)\), and \(y_3\), we have:

(i) if \(\delta \geq \max_{s \in \{s_1, s_2\}} \frac{s - 3}{n - 3}\),

\[
\frac{c\delta}{y_1(s_1)y_1(s_2)} < \frac{c}{y_1(s)} \leq \frac{2c}{y_2(s_2) + y_2(s_1) + (s_1 + s_2 - 4)\delta y_3} < \frac{c}{y_2(s)};
\]

(ii) if \(\delta < \max_{s \in \{s_1, s_2\}} \frac{s - 3}{n - 3}\),

\[
\frac{c\delta}{y_1(s_1)y_1(s_2)} < \frac{2c}{y_2(s_2) + y_2(s_1) + (s_1 + s_2 - 4)\delta y_3} < \frac{c}{y_2(s)}.
\]

From Theorems IV.1 and IV.4, we can directly obtain the conditions for the equivalence between pairwise stability and efficiency.

**Corollary IV.5.** Consider \(n\) individuals partitioned into groups, \(P_1\) and \(P_2\) of sizes \(s_1\) and \(s_2\) respectively. Under the assumption \(c < y_3\), the efficient structure has equal or more interconnections than the pairwise stable structure. Moreover, the efficient structure is the pairwise stable structure if

(i) for \(\delta \geq \max_{s \in \{s_1, s_2\}} \frac{s - 3}{n - 3}\),

\[
F_{12} \in \left[0, \frac{c\delta}{y_1(s_1)y_1(s_2)}\right] \cup \left[\frac{c}{y_3}, 1\right],
\]

(ii) for \(\delta < \max_{s \in \{s_1, s_2\}} \frac{s - 3}{n - 3}\),

\[
F_{12} \in \left[0, \frac{c\delta}{y_1(s_1)y_1(s_2)}\right] \cup \left[\frac{c}{y_3}, 1\right]
\]

We illustrate the compatibility (and incompatibility) between the pairwise stable and efficient structures in Fig. 8. The intersections of efficient and pairwise stable structures are highlighted.

**Corollary IV.6.** Consider \(n\) individuals partitioned into groups \(P_1\) and \(P_2\) of sizes \(s_1\) and \(s_2\), respectively. Under the assumption \(c < y_3\), if the efficient structure is not pairwise stable, then the efficient structure has more interconnections than the pairwise stable structure.

**Proof.** If the efficient structure is not pairwise stable, it implies that it can be changed by Formation Dynamics. Suppose that it has less interconnections than pairwise stable structure. Then, adding interconnections can make it pairwise stable, which means that both players involved in the interconnection benefit from this interaction. Then the social welfare increases which is conflict with the fact that the structure is efficient. Thus, the efficient structure has more interconnections than the pairwise stable structure.

\[\square\]

**Remark IV.7.** For the two intervals \(\frac{c\delta}{y_1(s_1)y_1(s_2)} < F_{12} < \frac{\max_{s \in \{s_1, s_2\}} c}{y_1(s)}\) and \(\frac{c}{y_2(s_2) + y_2(s_1) + (s_1 + s_2 - 4)\delta y_3} < F_{12} < \frac{\max_{s \in \{s_1, s_2\}} c}{y_2(s)}\), the pairwise stable structure is not efficient. For \(\frac{\max_{s \in \{s_1, s_2\}} c}{y_2(s)} \geq F_{12} < \frac{c}{y_1(s)}\), the pairwise stable structure might be not efficient. This is because individuals are rational and selfish (i.e., they only care about their own payoffs.) Therefore, total social utility might drop when an individual tries to maximize its own payoff. Specifically,

(i) when \(\frac{c\delta}{y_1(s_1)y_1(s_2)} < F_{12} < \frac{\max_{s \in \{s_1, s_2\}} c}{y_1(s)}\) an individual experiences loss by having an interconnection, although that interconnection could bring profit for other individuals and, therefore, result in an increase of total utility. The individual refuses to add or maintain this interconnection.

(ii) when \(\frac{c}{y_2(s_2) + y_2(s_1) + (s_1 + s_2 - 4)\delta y_3} < F_{12} < \frac{c}{y_3}\), as mentioned in the proof of Theorem IV.4, there exists at least one interconnection \(\{(i, j)\}, i \in P_1, j \in P_2\). For the two individuals \(i \in P_1, j \in P_2 (i \neq i, j \neq j)\), an interconnection causes loss for the player from the larger group, but brings profit for all other individuals, which results in removal of the interconnection or rejection in adding it and, therefore, in total utility loss.

Fig. 9 shows the number of interconnections and social welfare for both efficient and stable structures for two cliques of sizes 3 and 5, where the value of \(\delta = 0.5\) and \(c = 0.2\). Note that:

(i) As shown in Fig. 9a, when \(F_{12} = \frac{c}{y_3}\) which evaluates to 0.8 for the choice of our parameters, there exist non unique pairwise stable and efficient structures, and the number of interconnections for those structures vary between 3 and 15.
V. MULTIGROUP CONNECTIVITY STRUCTURE

In what follows we analyze the more general case of having more than two groups. Consider $n$ individuals partitioned into groups $P_1, \ldots, P_m$, and the payoff function defined above with 1-benefit $\delta < 1$ and edge cost $c$, and functions $y_1$, $y_2$, and $y_3$. From Theorem III.1 we know that each group forms a clique when $y_3 > c$. We introduce the undirected graph $T = (\mathcal{V}_P, \mathcal{E}_T)$, whose nodes represent groups and $(\alpha, \beta) \in \mathcal{E}_T$ if there exists at least one connection between $P_\alpha$ and $P_\beta$.

Theorem V.1 (Sufficiency Condition for Minimally Connected Cliques). Consider $n$ individuals partitioned into groups $P_1, \ldots, P_m$ of sizes $s_1, \ldots, s_m$ respectively. Assume that $c < y_3$ and $T$ is connected. Then, there exists a pairwise stable structure consisting of minimally connected cliques along $T$, if

(i) for all $(\alpha, \beta) \in \mathcal{E}_T$, $\alpha, \beta \in \{1, \ldots, m\}$,

$$\sum_{\lambda \neq \alpha, \lambda = 1}^m F_{\alpha\lambda}(\delta^{d_{\alpha\lambda}} - \delta^{d_{\alpha\lambda}})(1 + (s_\lambda - 1)\delta) > c,$$

(ii) as shown in Fig. 9a and Fig. 9b, for $F_{12} \in (0.67, 0.2) \cup (0.23, 0.53)$, the pairwise stable structure is not efficient and thus the social welfare of efficient structures is always greater than that of pairwise stable structures. Fig. 10 shows plot of PoA as a function of $F_{12}$. We observe that the slope for $F_{12} \in (0.67, 0.2)$ is the highest. This is because in this interval, the pairwise stable structure dose not have any interconnection, whereas the efficient structure has one interconnection. This link brings large value for the overall network and makes a large difference in social welfare of two kinds of structures.

We next discuss the evolution of intra vs. interconnections according to Formation Dynamics, when starting from an empty graph.

**Remark IV.8** (Formation of intra- vs. interconnections). Starting from an empty (or sparse) graph, initially the speed of formation of intra-connections is generally higher than that of interconnections. Fig. 11 illustrates the fact that at the beginning of the formation dynamics, a certain amount of intra-connections is required to produce enough incentives for both groups to communicate through an interconnection.
Fig. 9. Plots of no. of interconnections and social welfare as a function of $F_{12}$ for two groups of sizes 3 and 5, $\delta = 0.5$, $c = 0.2$.

Fig. 10. Plot of price of anarchy as a function of $F_{12}$ for two groups of sizes 3 and 5, $\delta = 0.5$, $c = 0.2$.

(ii) for all $(\alpha, \beta) \notin \mathcal{E}_T$, $\alpha, \beta \in \{1, \ldots, m\}$,

\[
\sum_{\lambda \neq \alpha, \lambda = 1}^{m} F_{\alpha \lambda}(\delta_{d_{\alpha}^{\lambda} - \delta_{d_{\alpha}^{\lambda}}})(1 + (s_{\lambda} - 1)\delta) < c, \\
\text{or} \sum_{\lambda \neq \beta, \lambda = 1}^{m} F_{\beta \lambda}(\delta_{d_{\beta}^{\lambda} - \delta_{d_{\beta}^{\lambda}}})(1 + (s_{\lambda} - 1)\delta) < c,
\]

where $d_{\rho}^{\mu_{\lambda}} = d_{\mu_{\lambda}}(\mathcal{E}_T \cup \{(\alpha, \beta)\})$ and $d_{\mu_{\lambda}} = d_{\mu_{\lambda}}(\mathcal{E}_T \setminus \{(\alpha, \beta)\})$.

Proof. Suppose the network $\mathcal{E}_0$ (consisting of disjoint cliques) be connected along $\mathcal{T}$ and satisfy

(i) if $(\alpha, \beta) \in \mathcal{E}_T$, there is only one inter-link between $P_\alpha$ and $P_\beta$;

(ii) for every group $P_\alpha$, only one agent $i_\alpha$ has inter-links.

For any pair of $(\alpha, \beta)$, let $\mathcal{E}'' = \mathcal{E}_0 \cup \{(i_\alpha, i_\beta)\}$ and $\mathcal{E}' = \mathcal{E}_0 \setminus \{(i_\alpha, i_\beta)\}$, it is easy to find that

\[
U_i(\mathcal{E}'') - U_i(\mathcal{E}') = \sum_{k=1, \alpha \neq \alpha}^{m} \sum_{l \in P_\alpha} F_{k\alpha} \sum_{l \in P_\alpha} (\delta_{d_{k}^{\alpha} - \delta_{d_{k}^{\alpha}}})(1 + (s_{\lambda} - 1)\delta) - c.
\]

Therefore, we have

\[
U_{i_\alpha}(\mathcal{E}'') - U_{i_\alpha}(\mathcal{E}') = \sum_{k=1, \lambda \neq \alpha, \lambda = 1}^{m} F_{\alpha \lambda}(\delta_{d_{\alpha}^{\lambda} - \delta_{d_{\alpha}^{\lambda}}})(1 + (s_{\lambda} - 1)\delta) - c
\]

and

\[
U_{i_\beta}(\mathcal{E}'') - U_{i_\beta}(\mathcal{E}') = \sum_{k=1, \lambda \neq \beta, \lambda = 1}^{m} F_{\beta \lambda}(\delta_{d_{\beta}^{\lambda} - \delta_{d_{\beta}^{\lambda}}})(1 + (s_{\lambda} - 1)\delta) - c.
\]

Therefore, for $(\alpha, \beta) \in \mathcal{E}_T$, it follows from (4) that $\mathcal{E}''$ defeats $\mathcal{E}'$. Similar to the proof of Theorem IV.1, since $F_{\alpha \beta} < \max_{s \in (s_{\alpha}, s_{\beta})} \frac{c}{y_{2}(s)}$, there only exists one inter-link between $P_\alpha$ and $P_\beta$. For $(\alpha, \beta) \notin \mathcal{E}_T$, (5) implies that there exist no inter-link between $P_\alpha$ and $P_\beta$.

By Theorem II.4, since network $\mathcal{E}_0$ can not be changed under dynamics, we can conclude that network $\mathcal{E}_0$ is stable. □

Remark V.2. Theorem V.1 answers the question: given a certain matrix $F$ and graph structure $\mathcal{E}$ is $\mathcal{E}$ pairwise stable or not?

Corollary V.3. For the special case of interconnection structure being a star, with $P_\gamma$ as the central group, the sufficient condition of Theorem V.1 can be simplified as follows:

(i) for all $\alpha \in \{1, \ldots, m\}$, $(\alpha \neq \gamma)$

\[
F_{\alpha \gamma} > \max_{s \in (s_{\alpha}, s_{\gamma})} \frac{c}{y_{1}(s)};
\]

(ii) for all $(\alpha, \beta) \in \{1, \ldots, m\}$, $(\alpha, \beta \neq \gamma)$,

\[
F_{\alpha \beta} < \max_{s \in (s_{\alpha}, s_{\beta})} \frac{c}{y_{2}(s)}.
\]

In the following example, we illustrate that due to randomness in choosing the pair of players, Formation Dynamic does
not always converge to a unique stable structure even for the same initial network structure and matrix \( F \).

**Example V.4.** Consider the case where we have \( \frac{c}{y_1(s)} < F_{\alpha\beta} < \frac{c}{y_2(s)} \) for all \( \alpha, \beta \), and that \( \delta < \frac{\sqrt{5} - 1}{2} \). We have five equal size groups named \( \{1, 2, \ldots, 5\} \). Two different processes are shown in Fig. 12.

**Process A.** For the process shown in Fig. 12a the order of pair selection is as follows: \( (1, 2) \rightarrow (1, 3) \rightarrow (1, 4) \rightarrow (1, 5) \). Note that by \( (1, 2) \) we mean an individual selected from group 1 paired with an individual from group 2, which results in the star graph being the convergent pairwise stable structure.

**Process B.** Now, consider the process shown in Fig. 12b for which the order of pairs of groups selected is as follows: \( (2, 3) \rightarrow (1, 2) \rightarrow (1, 3) \rightarrow (1, 4) \rightarrow (3, 5) \rightarrow (2, 4) \rightarrow (4, 5) \). At the very last step, we have:

\[
U_1(\mathcal{E} \cup \{4, 5\}) - U_1(\mathcal{E} \setminus \{4, 5\}) = F_{45}(y_1(s)) - F_{45}(\delta^2 y_1(s)) + F_{35}(\delta y_1(s)) - c.
\]

Since \( F_{35}, F_{45} < \frac{c}{y_1(s)} \) and \( \delta < \frac{\sqrt{5} - 1}{2} \), we conclude that

\[
F_{45}(1 - \delta^3) + F_{35}(\delta - \delta^2) > \frac{c}{y_1(s)}.
\]

Consequently, we obtain

\[
U_1(\mathcal{E} \cup \{4, 5\}) - U_1(\mathcal{E} \setminus \{4, 5\}) > 0
\]

which means that the connection \( (4, 5) \) is formed. Now since we have \( F_{\alpha\beta} < \frac{c}{y_2(s)} \), no connected triad and thereby, no additional links will be formed. Also no link will be removed. Therefore, the final structure in Fig. 12b, which is a ring, is stable.

Example V.4 shows that, based on the order of the sequence of selected pairs, we can have two or possibly more convergent stable structures, and therefore, the convergence results cannot be generalized and the convergent structure is not always unique.

From Theorem III.1 we know that each group forms a clique. We now analyze the interconnections among those cliques. Theorem V.5 addresses the redundancy of interconnections.

**Theorem V.5 (Formation of Redundancies).** Consider \( n \) individuals partitioned into groups \( P_1, \ldots, P_m \) of sizes \( s_1, s_2, \ldots, s_m \). Suppose that \( c < y_3 \). Then, under Formation Dynamics,

(i) redundant interconnections between \( P_\alpha \) and \( P_\beta \) will be formed and never removed, if \( F_{\alpha\beta} > \max_{s \in \{\alpha, \beta\}} \frac{c}{y_2(s)} \),

and

(ii) maximal interconnections between \( P_\alpha \) and \( P_\beta \) will be formed and never removed, if \( \frac{c}{y_3} < F_{\alpha\beta} \leq 1 \).

Fig. 13 illustrates the four scenarios for \( F_{\alpha\beta} \)’s. The horizontal axis corresponds to the values of \( F_{\alpha\beta} \) where \( \{\alpha, \beta\} \) belongs to the edge-set of the spanning tree, and the vertical axis corresponds to all other \( F_{\alpha\beta} \)’s.

Fig. 13. An illustration of ranges of parameter space in Theorem V.5

**Theorem V.6. (Efficiency)** For \( n \) individuals partitioned into groups according to \( P \), the efficient structure requires the same node to be chosen from each clique to provide bridges to other cliques, i.e., for a fixed interconnection structure and number of nodes, choosing the same representative from each clique increases the social welfare.

**Proof.** Suppose that the density and the structure of interconnections are fixed. It is easy to see that by choosing the same representative from each clique, the distance between individu-
als from different cliques would be shortened, resulting in the term $\delta d_{ij}$ in equations (1) being larger. Therefore, the social welfare will increase.

However, this structure is unlikely to be pairwise stable since the representative would bear a high cost for maintaining these interconnections. Figure 15 illustrates two networks with the same interconnection structures where one network has higher social welfare due to each group having only one representative.

![Fig. 14. Three groups of sizes 3, 4, and 5, $\delta = 0.5$, $c = 0.2$](image)

VI. CONCLUSION

This paper proposes the first strategic network formation model that, given a matrix specifying the frequency of a coordination problem among groups, identifies the conditions that result in multigroup formation. The model deviates from the seminal papers on strategic network formation in that it accounts for heterogeneous frequency of control problems arising among the individuals and investigates pairwise stability and efficiency of multigroup connectivity structures, as well as convergence of the formation dynamics.

In our model link formations occur bilaterally and thus many of the classical game-theoretic concepts do not apply to our framework. In particular, to study equilibrium structures, we utilize the concept of pairwise stability. A key challenge in our problem stems from the fact that not many tools are available for rigorous analysis or that they cannot be applied to the case of heterogeneous coordination problems among groups.

We identified the ranges of parameters where pairwise stable and efficient structures do and do not coincide and concluded that, for two-group structures, the efficient structures always has the same or a larger number of links than the pairwise stable ones. We also considered the price of anarchy and observed that the highest value occurs for the case when pairwise stable structures consist of disjoint union of cliques and the efficient structure has one link. Similar to the classical models, at the two ends of the spectrum of link values there is an overlap between efficient and stable structures.

We presented the conditions that result in the formation dynamics starting from an invariant set converge to cliques, and provided rigorous results for the number of interconnections in two-group structures. However, exact identification of the boundaries that result in certain number of interconnections among arbitrary number of groups with arbitrary size and interconnection structure is out of scope of this paper.

We note that by providing a full characterization of pairwise stability and efficiency for a two-group model, we focus on local topologies versus global topologies, as the individual interconnections can capture valuable information about the whole network and that all interconnections have subsets of two groups. This can be interpreted into taking the distance only for the people in one’s group or in the next immediate group in the utility function.

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REFERENCES

[1] V. Bala and S. Goyal. A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229, 2000. doi:10.1111/1468-0262.00155.

![Fig. 15. In network a) each cliques has only one representative, whereas in figure b) some cliques have multiple representatives.](image)
[2] S. P. Borgatti and D. S. Halgin. Analyzing affiliation networks. *The Sage Handbook of Social Network Analysis*, pages 417–433, 2011. doi: 10.4135/9781446294413.n28.

[3] B. Bringmann, M. Berlingerio, F. Bonchi, and A. Gionis. Learning and predicting the evolution of social networks. *IEEE Intelligent Systems*, 25(4):26–35, 2010. doi: 10.1109/MIS.2010.91.

[4] G. C. Chasparis and J. S. Shamma. Efficient network formation by distributed reinforcement. In *IEEE Conf. on Decision and Control*, pages 1690–1695, Cancun, Mexico, 2008. doi: 10.1109/CDC.2008.4739163.

[5] G. C. Chasparis and J. S. Shamma. Network formation: Neighborhood structures, establishment costs, and distributed learning. *IEEE Transactions on Cybernetics*, 43(6):1690–1695, 2013. doi: 10.1109/TSMCB.2012.2236553.

[6] B. Cornwell and J. A. Harrison. Union members and voluntary associations: Membership overlap as a case of organizational embeddedness. *American Sociological Review*, 69(6):862–881, 2004. doi: 10.1177/000312240406900606.

[7] N. E. Friedkin. Horizons of observability and limits of informal control in organizations. *Social Forces*, 62(1):54–77, 1983. doi: 10.1093/sf/62.1.54.

[8] N. E. Friedkin. *A Structural Theory of Social Influence*. Cambridge University Press, 1998, ISBN 9780521454827.

[9] M. S. Granovetter. The strength of weak ties. *American Journal of Sociology*, 78(6):1360–1380, 1973. doi: 10.1086/225469.

[10] M. O. Jackson and B. W. Rogers. The economics of small worlds. *Journal of the European Economic Association*, 3:617–627, 2005. doi: 10.1162/jeea.2005.3.2-3.617.

[11] M. O. Jackson and A. Watts. The evolution of social and economic networks. *Journal of Economic Theory*, 106:265–295, 2002. doi: 10.1006/jeth.2001.2903.

[12] M. O. Jackson and A. Wolinsky. A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74, 1996. doi: 10.1006/jeth.1996.0108.

[13] Y. Jia, Y. Wang, X. Jin, Z. Zhao, and X. Cheng. Link inference in dynamic heterogeneous information network: A knapsack-based approach. *IEEE Transactions on Computational Social Systems*, 4(3):80–92, 2017. doi: 10.1109/TCSS.2017.2715069.

[14] R. Likert. *The Human Organization: Its Management and Values*. McGraw-Hill, 1967, ISBN 0070378517.

[15] L. Maccari. Detecting and mitigating points of failure in community networks: A graph-based approach. *IEEE Transactions on Computational Social Systems*, 6(1):103–116, 2019. doi: 10.1109/TCSS.2018.2890483.

[16] M. McBride. Imperfect monitoring in communication networks. *Economic Theory*, 126:97–119, 2006. doi: 10.1016/j.jet.2004.10.003.

[17] S. Mehaghheghi, P. Agharkar, F. Bullo, and N. E. Friedkin. Multigroup connectivity structures and their implications. *Network Science*, pages 1–17, 2019. doi: 10.1017/nws.2019.22.

[18] N. Olazola and F. Valenciano. Network formation under linking constraints. *Physica A: Statistical Mechanics and its Applications*, 392:5194–5205, 2013. doi: 10.1016/j.physa.2013.06.013.

[19] N. Pagan and F. Dörfler. Game theoretical inference of human behavior in social networks. *Nature Communications*, 10(1):5507, 2019. doi: 10.1038/s41467-019-13148-8.

[20] Y. Song and M. van der Schaar. Dynamic network formation with incomplete information. *Economic Theory*, 59:301–311, 2015. doi: 10.1007/s00199-015-0858-y.

[21] W. Stam and T. Elfring. Entrepreneurial orientation and new venture performance: The moderating role of intra-and extraindustry social capital. *Academy of Management Journal*, 51(1):97–111, 2008. doi: 10.5465/amj.2008.30744031.

[22] M. Tortoriello and D. Krackhardt. Activating cross-boundary knowledge: The role of simmelian ties in the generation of innovations. *Academy of Management Journal*, 53(1):167–181, 2010. doi: 10.5465/amj.2010.48037420.

[23] H. C. White, S. A. Boorman, and R. L. Breiger. Social structure from multiple networks. I. Blockmodels of roles and positions. *American Journal of Sociology*, 81(4):730–780, 1976. doi: 10.1086/226141.

[24] J. Yang and J. Leskovec. Community-affiliation graph model for overlapping network community detection. In *IEEE International Conference on Data Mining*, pages 1170–1175, Brussels, Belgium, 2012. doi: 10.1109/ICDM.2012.139.

[25] X. Zhang, C. Wang, Y. Su, L. Pan, and H. Zhang. A fast overlapping community detection algorithm based on weak cliques for large-scale networks. *IEEE Transactions on Computational Social Systems*, 4(4):218–230, 2017. doi: 10.1109/TCSS.2017.2749282.