ON THE TRIPLE JUNCTION PROBLEM ON THE PLANE WITHOUT SYMMETRY HYPOTHESES

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Abstract. We investigate the Allen-Cahn system
\[ \Delta u - W_u(u) = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}^2, \]
where \( W \in C^2(\mathbb{R}^2, [0, +\infty)) \) is a potential with three global minima. We establish the existence of an entire solution \( u \) which possesses a triple junction structure. The main strategy is to study the global minimizer \( u_\varepsilon \) of the variational problem
\[ \min \int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dz, \quad u = g_\varepsilon \text{ on } \partial B_1. \]
The point of departure is an energy lower bound that plays a crucial role in estimating the location and size of the diffuse interface. We do not impose any symmetry hypothesis on the solution.

1. Introduction

This paper is concerned with the existence of an entire, bounded, minimizing solution to the system
\[ \Delta u - W_u(u) = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}^2, \]
where \( W \) is a triple-well potential with three global minima. A key feature of the present work is that we make no a priori hypotheses of symmetry on the solution. Specifically for \( W \) we assume
\[ (H1). \ W \in C^2(\mathbb{R}^2; [0, +\infty)), \{ z : W(z) = 0 \} = \{ a_1, a_2, a_3 \}, \ W_u(u) \cdot u > 0 \text{ if } |u| > M \text{ and } \ c_2 |\xi|^2 \geq \xi^T W_u(u_i) \xi \geq c_1 |\xi|^2, \ i = 1, 2, 3. \]
for some positive constants \( c_1 < c_2 \) depending on \( W \).

(H2). For \( i \neq j, \ i, j \in \{1, 2, 3\} \), let \( U_{ij} \in W^{1,2}(\mathbb{R}, \mathbb{R}^2) \) be an 1D minimizer of the action
\[ \sigma_{ij} := \min_{\mathbb{R}} \left( \frac{1}{2} |U_{ij}|^2 + W(U_{ij}) \right) \, d\eta, \quad \lim_{\eta \to -\infty} U_{ij}(\eta) = a_i, \quad \lim_{\eta \to +\infty} U_{ij}(\eta) = a_j. \]
\( \sigma_{ij} \) satisfies
\[ (1.2) \quad \sigma_{ij} \equiv \sigma > 0 \quad \text{for } i \neq j \in \{1, 2, 3\} \text{ and some constant } \sigma. \]

Note that \[ (1.1) \] is the Euler-Lagrange equation corresponding to
\[ (1.3) \quad J(u, \Omega) := \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \, dz, \quad \forall \text{ bounded open set } \Omega \subset \mathbb{R}^2. \]

It is an easy fact that if \( J(u, \mathbb{R}^2) < \infty \) for a solution \( u \) of \[ (1.1) \], then \( u \) is a constant (cf. [1]). For this reason the construction of the solution cannot be achieved directly. Moreover, the definition of the minimizing solution is as follows.

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**Definition 1.1.** $u$ is a minimizing solution of (1.1) in the sense of De Giorgi if

\begin{equation}
J(u, \Omega) \leq J(u + v, \Omega), \quad \forall \text{ bounded open set } \Omega \subset \mathbb{R}^2, \forall v \in C^1_0(\Omega).
\end{equation}

We note that the hypothesis (H2) on $W$ that all $\sigma_{ij}$ are equal is mainly for convenience and does not imply any symmetry for the potential or the solution. In particular, one can construct examples of non-symmetric potentials (i.e. $W$ is not in the equivariant class of the symmetry group of the equilateral triangle) with three connections that have equal actions $\sigma_{ij} \equiv \sigma$. Take $W(z) = |z^3 - 1|^2$, $z \in \mathbb{C}$. The connections $U_{ij}$ are explicitly known (see [6] Page 79) and they avoid a neighborhood $B_{\rho}$ of $z = 0$. When $\rho$ is sufficiently small one can modify $W$ in $B_{\rho}$ in a non-symmetric way and keep the connections unaffected at the same time.

For the scalar solution $u : \mathbb{R}^n \rightarrow \mathbb{R}$ of the (1.1), there is a relationship (cf. [27, 36, 10]) between minimizing solutions and minimal surfaces. Many deep results have been obtained in the process of understanding this relationship, see [7, 13, 20, 29, 35, 11, 12, 22, 24, 28] and the references therein. In the vector case, minimizing solutions are related to minimal partitions, see Baldo [8], Sternberg [32] and Fonseca & Tartar [18].

The solution we are after can be considered as the diffuse analog of the singular minimal cone. For the scalar solution $u : \mathbb{R}^n \rightarrow \mathbb{R}$ of the (1.1), there is a relationship (cf. [27, 36, 10]) between minimizing solutions and minimal surfaces. Many deep results have been obtained in the process of understanding this relationship, see [7, 13, 20, 29, 35, 11, 12, 22, 24, 28] and the references therein. In the vector case, minimizing solutions are related to minimal partitions, see Baldo [8], Sternberg [32] and Fonseca & Tartar [18].

We now state our main results.

**Theorem 1.2.** Fix $\gamma < \min\{\gamma_0, \min_{i,j \in \{1,2,3\}} \frac{1}{2}|a_i - a_j|, \sqrt{\frac{\gamma}{2C_W}}\}$, where $\gamma_0, C_W$ are constants defined later. There is a constant $C_0$ which depends on $\gamma, W$ such that under the hypotheses (H1), (H2), there exists an entire, bounded minimizing solution of (1.1) with the following triple junction structure.

- a. For every $r > 0$, there exists a point $P(r)$ such that $|u(P(r)) - a_1| \leq \gamma$.
- b. There exists $\{Q_j\}_{j=1}^\infty \cup \{R_j\}_{j=1}^\infty$ such that
  \begin{align*}
  |u(Q_j) - a_2| \leq \gamma, & \quad |u(R_j) - a_3| \leq \gamma, \\
  \text{dist}(Q_j, 0) \leq 32jC_0, & \quad \text{dist}(R_j, 0) \leq 32jC_0.
  \end{align*}

  The pairwise distance of any two points from $\{Q_j\}_{j=1}^\infty \cup \{R_j\}_{j=1}^\infty$ is larger than or equal to $6C_0$.
- c. For each $Q_j$, there exists $P_j$ such that
  \[ \text{dist}(Q_j, P_j) \leq C_0, \quad |u(P_j) - a_1| \leq \gamma. \]

  This property also holds for $\{R_j\}$.
- d. For any sequence $r_k \rightarrow +\infty$ one can extract a subsequence, still denoted by $\{r_k\}$, such that
  \begin{equation}
  u(r_kx) \rightarrow u_0(x) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^2),
  \end{equation}

  where $u_0(x) = \sum_{i=1}^3 a_i \chi_{D_i}$. Here $\chi$ is the characteristic function. $\mathcal{P} = \{D_1, D_2, D_3\}$ provides a minimal partition of $\mathbb{R}^2$ into three sectors of the angle $\frac{2\pi}{3}$ and $\partial \mathcal{P}$ is a triod centered at $0$. The partition $\mathcal{P}$ may depend on $\{r_k\}$. Also there exists a sequence $\{r'_k\}_{k=1}^\infty$, such that
for each $\xi \in D_j \ (j = 1, 2, 3), \ |\xi| = 1,$

\[ \lim_{k \to \infty} \frac{1}{r_k^j} \int_{0}^{r_k^j} u(s\xi) \, ds = a_j, \]

and the convergence is uniform for $\xi$ in compact sets of $S^1 \setminus \partial P$.

In 1996, Bronsard, Gui and Schatzman [9] established the existence of an entire triple junction solution for the triple-well potential $W$ in the equivariant class of the reflection group $G$ of the symmetries of the equilateral triangle. Their solution satisfies the estimate

\[ |u(x) - a_i| \leq Ke^{-kd(x, \partial D_i)}, \ i = 1, 2, 3 \]

and hence in particular it connects the minima of $W$ at infinity along rays emanating from the origin, $u(\lambda \xi) \to a_i$ as $\lambda \to +\infty$, for $x \in D_i$, $i = 1, 2, 3$. And thus it is a stronger result on the asymptotic behavior of the solution than our theorem above. In spite of its great importance, this result suffers from the fact that their solution is obtained as a minimizer in the equivariant class $u(gx) = gu(x), g \in G$, hence is not necessarily even stable under general perturbations.

Since that time, the problem has been understood for an abstract reflection group on $\mathbb{R}^n$ via different and quite general methods rendering also the complete stratification of the solution. The general theory under optimal hypotheses on $W$ covers in particular the diffuse analogs of the two minimal singular cones in $\mathbb{R}^3$: the triod with a spine and the tetrahedral Plateau complex ([34]). The second one was originally obtained in 2008 by Gui and Schatzman [23]. We refer to the book [6] and to the references therein.

In 2021, Fusco [18] succeeded in establishing essentially the result of [9] in the equivariant class of the rotation subgroup of $G$ (by $\frac{2}{3}\pi$), thus eliminating the two reflections. This was a significant achievement since the problem now cannot be reduced to a fundamental domain containing a single minimum of $W$. Nevertheless it suffers from the same shortcoming as before, not allowing general perturbations. Symmetry fixes the center of the junction at the origin, which is a fact that simplifies considerably the analysis.

Schatzman [30] is probably the only previously known entire minimizing solution that does not assume any symmetry. Her result later was revisited in [17, 26, 31]. Triple junction solutions for bounded domains without symmetry assumptions were constructed by Sternberg and Ziemer [33] for clover-shaped domains in $\mathbb{R}^2$ via $\Gamma$-convergence, and for more general domains by Flores, Padilla and Tonegawa [15] by the means of a mountain pass argument. These results do not seem to provide estimates that allow to pass to the limit and establish the existence of a triple junction solution on $\mathbb{R}^2$.

We now list some of the key steps in the proof of our results. We begin by rescaling the problem on the unit disk.

\[ \min_{u=g_\varepsilon \text{ on } \partial B_1} \int_{B_1(0)} \left( \frac{1}{2} \nabla u)^2 + \frac{1}{\varepsilon} W(u) \right) \, dz =: \min_{u=g_\varepsilon \text{ on } \partial B_1} J_\varepsilon(u), \ z = (x, y) \in \mathbb{R}^2, \]

where $g_\varepsilon$ is a given smooth function connecting the phases $(a_1 \to a_2, a_2 \to a_3, a_3 \to a_1)$ in $O(\varepsilon)$-intervals $I_i, i=1,2,3,$ and otherwise equals to constants $(a_1, a_2, a_3)$ in the complement $\partial B_1 \setminus \bigcup_{i=1}^{3} I_i$. A possible choice of $g_\varepsilon$ is given in (2.17).

First a general remark is in order. The main thrust of our work is in obtaining estimates for the minimizer(s) of (1.6), that is working at the $\varepsilon$–level. We make use of the limiting minimal partition problem and $\Gamma$–convergence only in the final stages of the proof of Property (d) in Theorem 1.2 specifically (1.5) and (1.5').
The first step is the derivation of a tight upper/lower bound for the minimizer,

\[
3\sigma - C\varepsilon^{\frac{4}{3}} \leq \int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) \, dz \leq 3\sigma + C\varepsilon.
\]

The usefulness of such an estimate for extracting qualitative information for the minimizer as \(\varepsilon \to 0\) was the major point in the joint paper [5] by G. Fusco and one of the authors. However the geometry of the examples treated in [5] is simple, and the transition along the interface involves only one of the variables, hence only part of the gradient in (1.7) was involved. In the present work the transition is 2D and so the whole gradient is present. Thus the proof of (1.7) utilizes several new ideas, and in particular it introduces the center \((x^*, y^*)\) of the junction. We remark that the particular power \(\frac{4}{3}\) in the lower bound above does not have a particular significance in our treatment.

The second step is the derivation of the estimate

\[
|x^*|, |y^*| \leq C\varepsilon^{\frac{4}{3}},
\]

which makes use of both the lower and the upper bound in (1.7).

The third step is the localization of the diffuse interface

\[
I_{\varepsilon, \gamma} := \{ z \in B_1 : |v_\varepsilon(z) - a_i| > \gamma \text{ for all } i = 1, 2, 3 \}
\]

in an \(O(\varepsilon^{\frac{1}{3}})\) neighborhood of the triod. The proof of this utilizes all the previous estimates and also the variational maximum principle in [1] (see also [6, Theorem 4.1]).

The final tool utilized in the proof is an estimate on the width of the diffuse interface \(I_{\varepsilon, \gamma}\). Let

\[
\Gamma^i_{\varepsilon, \gamma} := \{ z \in \overline{B}_1 : |v_\varepsilon(z) - a_i| = \gamma \}, \quad i = 1, 2, 3.
\]

For an arbitrary \(z_1 \in \Gamma^i_{\varepsilon, \gamma}\) set

\[
r_1 := \max \{ r : B(z_1, r) \cap \left( \bigcup_{j \neq i} \Gamma^j_{\varepsilon, \gamma} \right) = \emptyset \}.
\]

Then we have the estimate

\[
r_1 \leq C(\gamma, W)\varepsilon.
\]

This estimate provides a substitute for the lack of an \(O(\varepsilon)\) localization of the diffuse interface from the triod, which may very well be an effect of the absence of symmetry. The derivation of (1.12) is mainly based on the vector version of the Caffarelli-Córdoba density estimate [5] (see also [6, Theorem 5.2]). Specifically, it does not utilize any of the previous estimates.

Next utilizing these estimates in the proof of the key Lemma [6.3] for any \(k \in \mathbb{N}^+\) and \(\gamma\) small, when \(\varepsilon < \varepsilon(k, \gamma, W)\) we can obtain a set of points \(\{Q^k_1, Q^k_2, ..., Q^k_{2^k}, R^k_1, R^k_2, ..., R^k_{2^k}\}\) in the ball \(B_2(0)\) satisfying

\[
\text{dist}(Q^k_j, \Gamma_{\varepsilon, \gamma}^1) \leq C_0\varepsilon, \quad \text{dist}(R^k_j, \Gamma_{\varepsilon, \gamma}^1) \leq C_0\varepsilon, \quad \forall j = 1, ..., 2^k.
\]

\[
|v_\varepsilon(Q^k_j) - a_2| \leq \gamma, \quad |u_\varepsilon(R^k_j) - a_3| \leq \gamma, \quad \forall j = 1, ..., 2^k.
\]

\[
\text{dist}(A, B) \geq 6C_0\varepsilon, \quad \forall A, B \in \{Q^k_1, ..., Q^k_{2^k}, R^k_1, ..., R^k_{2^k}\}.
\]

Here \(C_0 = C(\gamma, W)\) is the constant in (1.12). This argument has a more discrete nature and is accomplished via an elaborate induction argument. With Lemma [6.3] we can study the blow up map at the scale \(\varepsilon\) and let \(\varepsilon \to 0\) to obtain a minimizing solution \(u(\cdot)\) of (1.1), depending on \(\gamma\), which satisfies Properties (a), (b), (c) in Theorem [1.2].

Finally we come to the derivation of Property (d). We begin with the minimizing solution \(u : \mathbb{R}^2 \to \mathbb{R}^2\) satisfying Properties (a), (b), (c) in Theorem [1.2]. Modica [25] proved that entire
minimizers \( u : \mathbb{R}^n \to \mathbb{R} \) of the scalar Allen–Cahn equation, with bistable \( W \), converge along subsequences to a minimizing cone \( u_0 \),

\[ u(r_k x) \to u_0(x) \quad \text{in } L^1_{\text{loc}}, \quad r_k \to +\infty. \]

His proof utilizes the monotonicity formula for minimal surfaces (see Giusti [21, Theorem 9.3]). Modica’s argument can be adjusted to the problem at hand, where minimal partitions are the relevant objects. A convenient set-up is that of flat chains introduced in Fleming [14] and further developed by White [37]. In particular minimal partitions can be identified with minimal flat chains of top dimension, which also satisfy the monotonicity formula. Consequently we obtain that \( u_0 \) in (1.5) is a planar minimizing cone, and therefore either a straight line or a triod. However, Properties (a), (b), (c) together imply an energy lower bound that excludes the straight line, and so (1.5) is established. (1.5’) follows from the cone property of \( u_0 \) via [6, Proposition 5.6]. This concludes the sketch of the proof of Theorem 1.2.

Concluding, we remark that all the arguments in the present paper utilize the variational character of \( u \) only, with the exception of the pointwise exponential estimate in (4.37) that makes use of linear elliptic theory.

A previous version of this work was uploaded in the arXiv in December, 2022. In a personal communication, Peter Sternberg has brought to our attention that in a joint work with Etienne Sandier, they have obtained comparable results.

The article is organized as follows. In Section 2 we present some preliminary results from [5] and [6]. In Section 3 we establish the lower bound in (1.7). Then we show in Section 4 the estimate (1.8) and the localization of the diffuse interface within an \( O(\varepsilon^{1/4}) \) neighborhood of the triod. Estimate (1.12) of the width of the diffuse interface is established in Section 5. Finally in Section 6 we give the proof of Theorem 1.2. In the Appendix, we give a proof of the upper bound in (2.18). A similar estimate was derived in Fusco [18].

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2. Preliminaries

Throughout the paper we denote by \( z = (x, y) \) a 2D point and by \( B(z, r) \) the 2D ball centered at the point \( z \) with radius \( r \). In addition, we let \( B_1 \) denote the unit 2D ball centered at the origin. We recall the following basic results which are important in our analysis.

**Lemma 2.1** (Lemma 2.1 in [5]). *The hypotheses on \( W \) imply the existence of \( \delta_W > 0 \), and constants \( c_W, C_W > 0 \) such that*

\[
|u - a_i| = \delta \quad \Rightarrow \quad \frac{1}{2} c_W \delta^2 \leq W(u) \leq \frac{1}{2} C_W \delta^2, \quad \forall \delta < \delta_W, \quad i = 1, 2, 3.
\]

*Moreover if \( \min_{i=1,2,3} |u - a_i| \geq \delta \) for some \( \delta < \delta_W \), then \( W(u) \geq \frac{1}{2} c_W \delta^2 \).*
Lemma 2.2 (Lemma 2.3 in [5]). Take $i \neq j \in \{1, 2, 3\}$, $\delta < \delta_W$ and $s_+ > s_-$ be two real numbers. Let $v : (s_-, s_+) \to \mathbb{R}^2$ be a smooth map that minimizes the energy functional

$$J(v) := \int_{s_-}^{s_+} \left( \frac{1}{2} |\nabla v|^2 + W(v) \right) \, dx$$

subject to the boundary condition

$$|v(s_-) - a_i| = |v(s_+) - a_j| = \delta.$$

Then

$$J(v) \geq \sigma - C_W \delta^2,$$

where $C_W$ is the constant in Lemma 2.1.

For most of the paper, we consider the variational problem

$$\min \int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dz,$$

where $W$ satisfies Hypotheses (H1) and (H2). We denote by $u_\varepsilon \in W^{1,2}(B_1, \mathbb{R}^2)$ a global minimizer of the functional (2.16) with respect to the boundary condition in polar coordinates

$$u_\varepsilon(1, \theta) = g_\varepsilon(\theta) \text{ on } \partial B_1,$$

where $g_\varepsilon : [0, 2\pi)$ is given by

$$g_\varepsilon(\theta) := \begin{cases} a_2 + g_0\left(\frac{\theta}{2c_0\varepsilon} - c_0\varepsilon, -\frac{\theta}{2c_0\varepsilon} + c_0\varepsilon\right), & \theta \in \left[\frac{\pi}{4} - c_0\varepsilon, \frac{\pi}{4} + c_0\varepsilon\right) \\ a_1, & \theta \in \left[\frac{\pi}{2} - c_0\varepsilon, \frac{\pi}{2} + c_0\varepsilon\right), \\ a_3 + g_0\left(\frac{\theta}{2c_0\varepsilon} + c_0\varepsilon\right), & \theta \in \left[\frac{7\pi}{8} - c_0\varepsilon, \frac{7\pi}{8} + c_0\varepsilon\right), \\ a_3, & \theta \in \left[\frac{5\pi}{8} + c_0\varepsilon, \frac{11\pi}{8} - c_0\varepsilon\right), \\ a_2 + g_0\left(\frac{\theta}{2c_0\varepsilon} + c_0\varepsilon\right), & \theta \in \left[\frac{11\pi}{8} - c_0\varepsilon, \frac{11\pi}{8} + c_0\varepsilon\right), \\ a_2, & \theta \in \left[\frac{13\pi}{8} + c_0\varepsilon, 2\pi\right) \cap [0, \pi - c_0\varepsilon), \\ \end{cases}$$

where $g_0 : [0, 1] \to [0, 1]$ is a strictly increasing smooth function that satisfies $g_0(0) = 0$, $g_0(1) = 1$ and $|g_0'(x)| \leq 2$. From the definition there exists a positive constant $M$ such that $|g_\varepsilon| \leq M$.

The energy $J_\varepsilon(u_\varepsilon)$ satisfies the following upper bound.

Lemma 2.3. There is a constant $C = C(W)$ such that

$$\int_{B_1} \left\{ \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right\} \, dz \leq 3\sigma + C\varepsilon.$$

A similar upper bound is derived by Fusco in [18] estimate (3.18)]. For completeness we present the proof of Lemma 2.3 in Appendix A.

In addition, we can control $|u_\varepsilon|$ and $|\nabla u_\varepsilon|$ thanks to the smoothness assumption of $W$ and standard elliptic regularity theory.

Lemma 2.4. Let $u_\varepsilon$ minimize the functional (2.16) with the boundary condition $u_\varepsilon = g_\varepsilon$ on $\partial B_1$. There is a constant $M$ independent of $\varepsilon$, such that

$$|u_\varepsilon(z)| \leq M, \quad |\nabla u_\varepsilon(z)| \leq \frac{M}{\varepsilon}, \quad \forall z \in B_1.$$

We omit the proof.
3. Lower bound for \(J_\varepsilon(u_\varepsilon)\)

**Proposition 3.1.** (weak lower bound) There exist constants \(C_1\) and \(\varepsilon_0\), such that for any \(\varepsilon \leq \varepsilon_0\), it holds

\[
\int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) \, dz \geq 3\sigma - C_1\varepsilon^\frac{1}{4}.
\]

**Proof.** For the sake of convenience we write \(u_\varepsilon = u\) throughout the proof. Firstly we set the family of horizontal line segments \(\gamma_y\) for \(y \in [-\frac{1}{2}, 1]\) as

\[
\gamma_y := \{(x, y) : x \in \mathbb{R}\} \cap B_1.
\]

Then we define functions \(\lambda_1(y), \lambda_2(y), \lambda_3(y)\) for \(y \in [-\frac{1}{2}, 1]\),

\[
\lambda_i(y) := \mathcal{L}^1(\gamma_y \cap \{|u(x, y) - a_i| < \varepsilon^\frac{1}{4}\}), \quad i \in \{1, 2, 3\}.
\]

Here \(\mathcal{L}^1\) denotes the 1-dimensional Lebesgue measure. Then by the boundary condition we know for any \(y \in [-\frac{1}{2} + c_0\varepsilon, 1 - c_0\varepsilon]\), it holds that \(\lambda_1(y) > 0\) and \(\lambda_2(y) > 0\).

Let \(y^*\) be the constant defined by

\[
y^* := \min\{y \in [-\frac{1}{2} + c_0\varepsilon, 1] : \lambda_1(y) + \lambda_2(y) \geq \mathcal{L}^1(\gamma_y - \varepsilon^\frac{1}{4})\}.
\]

Denote the subsets

\[
\Omega_1 := B_1 \cap \{(x, y) : y \geq y^*\},
\]

\[
\Omega_2 := B_1 \cap \{(x, y) : y < y^*\}.
\]

We will calculate the energy in \(\Omega_1\) and \(\Omega_2\) respectively. In \(\Omega_1\), if \(y^* \geq 1 - c_0\varepsilon\), we simply estimate

\[
\int_{\Omega_1} \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \, dx \, dy \geq 0.
\]

Otherwise for any \(y^* < y < 1 - c_0\varepsilon\), (2.17) implies that

\[
 u(-\sqrt{1-y^2}, y) = a_1, \quad u(\sqrt{1-y^2}, y) = a_2.
\]

Integrating the energy density on \(\gamma_y\) gives

\[
\int_{\Omega_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \geq \sigma.
\]

Therefore we obtain

\[
\int_{\Omega_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \geq \int_{\Omega_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \geq \max\{\sigma(1 - c_0\varepsilon - y^*), 0\}.
\]

On domain \(\Omega_2\), we claim that there exists a constant \(C\) such that

\[
\mathcal{L}^1(\{y : -\frac{1}{2} + c_0\varepsilon < y < y^*, \lambda_3(y) = 0\}) < C\varepsilon^\frac{1}{4}.
\]

To prove this claim, first we note that when \(y^* < -\frac{1}{2} + c_0\varepsilon + C\varepsilon^\frac{1}{4}\), the statement is trivial. Therefore we only consider the situation \(y^* \geq -\frac{1}{2} + c_0\varepsilon + C\varepsilon^\frac{1}{4}\).

Set

\[
S := \{y : -\frac{1}{2} + c_0\varepsilon < y < y^*, \lambda_3(y) = 0\}.
\]

For any \(y \in S\), definitions of \(y^*\) and \(S\) imply that \(\lambda_1(y) + \lambda_2(y) + \lambda_3(y) < \mathcal{L}^1(\gamma_y) - \varepsilon^\frac{1}{4}\), i.e.

\[
\mathcal{L}^1(\{x \in [-\sqrt{1-y^2}, \sqrt{1-y^2}] : |u(x, y) - a_i| > \varepsilon^\frac{1}{4}, \forall i\}) > \varepsilon^\frac{1}{4}.
\]
By our assumption on the potential function $W$, 

$$W(u(x, y)) \geq \frac{1}{2} c_W \varepsilon^{\frac{1}{4}}, \quad \text{when } |u(x, y) - a_i| > \varepsilon^{\frac{1}{4}}, \quad \forall i.$$  

From the energy upper bound (2.18) we get 

$$4\sigma \geq \frac{1}{\varepsilon} \int_{\gamma_y} W(u) \, dx \, dy \geq \frac{c_W}{2\varepsilon} \mathcal{L}^1(S) \varepsilon^{\frac{1}{4}} \varepsilon^{\frac{1}{4}} \varepsilon^{\frac{1}{4}} \varepsilon^{\frac{1}{4}}$$ 

$$\Rightarrow \mathcal{L}^1(S) \leq C \varepsilon^{\frac{1}{4}}$$  

for some constant $C$ depending on $W$.

Now we calculate the energy in $\Omega_2$. If $-\frac{1}{2} \leq y^* < -\frac{1}{2} + c_0 \varepsilon + C \varepsilon^{\frac{1}{4}}$, then we define the set 

$$K_0 := \{ x \in [-\frac{\sqrt{3}}{2} + c_0 \varepsilon, \frac{\sqrt{3}}{2} - c_0 \varepsilon] : |u(x, y^*) - a_i| < \varepsilon^{\frac{1}{4}}, \quad i = 1 \text{ or } 2 \}.$$  

According to the definition of $y^*$, $\mathcal{L}^1(K_0) \geq \sqrt{3} - 2c_0 \varepsilon - \varepsilon^{\frac{1}{4}}$. We have 

$$\int_{\Omega_2} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \geq \int_{K_0} \int_{-\sqrt{1-x^2}}^{y^*} \left( \frac{\varepsilon}{2} |\partial_y u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dy \, dx \geq (\sqrt{3} - 2c_0 \varepsilon - \varepsilon^{\frac{1}{4}})(\sigma - C W \varepsilon^{\frac{1}{4}}),$$  

where $C_W$ is a constant only depending on the potential $W$ and the last estimate follows from Lemma 2.2 (3.21) and (3.22) imply that when $-\frac{1}{2} \leq y^* < -\frac{1}{2} + c_0 \varepsilon + C \varepsilon^{\frac{1}{4}}$, 

$$(3.23) \int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \geq (\frac{3}{2} + \sqrt{3} - 3c_0 \varepsilon - (C + 1) \varepsilon^{\frac{1}{4}})(\sigma - C W \varepsilon^{\frac{1}{4}}),$$  

which satisfies (3.20) when $\varepsilon$ is sufficiently small.

Now it suffices to consider the case $y^* \geq -\frac{1}{2} + c_0 \varepsilon + C \varepsilon^{\frac{1}{4}}$. For any $x \in [-\frac{\sqrt{3}}{2} + c_0 \varepsilon, \frac{\sqrt{3}}{2} - c_0 \varepsilon]$, we set 

$$\zeta(x) := \min\{y^*, \sqrt{1-x^2}\}.$$  

We also introduce the sets $K$, $M$,

$$K := \{ x \in [-\frac{\sqrt{3}}{2} + c_0 \varepsilon, \frac{\sqrt{3}}{2} - c_0 \varepsilon] : |u(x, \zeta(x)) - a_i| < \varepsilon^{\frac{1}{4}}, \quad i = 1 \text{ or } 2 \},$$

$$M := \{ y \in [-\frac{1}{2} + c_0 \varepsilon, y^*] : \lambda_3(y) > 0 \}.$$  

Moreover, let $\theta \in (0, \frac{\pi}{2})$ be a parameter that will be determined later. In the next step we will split the potential $W$ into two parts $W = (\sin^2 \theta)W + (\cos^2 \theta)W$ and compute the energy in the vertical direction and the horizontal direction respectively.

For any $x \in K$,

$$|u(x, \zeta(x)) - a_i| < \varepsilon^{\frac{1}{4}}, \quad \text{for } i = 1 \text{ or } 2, \quad u(x, -\sqrt{1-x^2}) = a_3.$$  

We estimate the energy in the vertical direction,

$$\int_{-\sqrt{1-x^2}}^{\zeta(x)} \left( \frac{\varepsilon}{2} |\partial_y u|^2 + \frac{\sin^2 \theta}{\varepsilon} W(u) \right) \, dy$$

$$\geq \sin \theta \int_{-\sqrt{1-x^2}}^{\zeta(x)} \left( \frac{\varepsilon}{2 \sin \theta} |\partial_y u|^2 + \frac{\sin \theta}{\varepsilon} W(u) \right) \, dy$$

$$\geq \sin \theta (\sigma - C W \varepsilon^{\frac{1}{4}}).$$
Similarly, for any $y \in M$, it holds that

$$u(-\sqrt{1-y^2}, y) = a_1, \ u(\sqrt{1-y^2}, y) = a_2, \ \exists (x_0, y) \in \gamma_y \text{ s.t. } |u(x_0, y) - a_3| < \varepsilon^{\frac{1}{3}}. $$

$$\int_{\gamma_y} \left( \frac{\varepsilon}{2} |\partial_x u|^2 + \frac{\cos^2 \theta}{\varepsilon} W(u) \right) \, dx$$

$$= \cos \theta \left\{ \int_{x_0}^{x_0 \sqrt{1-y^2}} + \int_{x_0}^{\sqrt{1-y^2}} \right\} \left( \frac{\varepsilon}{2 \cos \theta} |\partial_x u|^2 + \frac{\cos \theta}{\varepsilon} W(u) \right) \, dx$$

$$\geq 2 \cos \theta (\sigma - C W \varepsilon^{\frac{1}{3}}).$$

Using (3.24) and (3.25) we obtain

$$\int_{\Omega_2} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \geq \left( \sin \theta L^1(K) + 2 \cos \theta L^1(M) \right) (\sigma - C W \varepsilon^{\frac{1}{3}}).$$

Since the estimate above holds for any $\theta \in (0, \frac{\pi}{2})$, we can maximize the lower bound by taking

$$\theta = \arctan \left( \frac{L^1(K)}{2 L^1(M)} \right).$$

Consequently we have

$$\int_{\Omega_2} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \geq \sqrt{L^1(K)^2 + 4 L^1(M)^2} (\sigma - C W \varepsilon^{\frac{1}{3}}).$$

Moreover, by the definition of $K, M$ and the boundary condition (2.17),

$$\mathcal{L}^1(K) \geq \sqrt{3} - 2 c_0 \varepsilon - \varepsilon^{\frac{1}{3}}, \ \mathcal{L}^1(M) \geq y^* + \frac{1}{2} - C \varepsilon^{\frac{1}{3}} - c_0 \varepsilon.$$ 

As a consequence, there exists a constant $C$ such that for small enough $\varepsilon$,

$$\sqrt{L^1(K)^2 + 4 L^1(M)^2} \geq \sqrt{3 + 4(y^* + \frac{1}{2})^2} - C \varepsilon^{\frac{1}{3}}.$$
From this and (3.26) we obtain

\begin{equation}
\int_{\Omega_2} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \geq \sqrt{3 + 4} (y^* + \frac{1}{2})^2 \cdot \sigma - C\varepsilon^{1/3},
\end{equation}

where $C$ is a constant independent of $\varepsilon$. Finally, we combine the estimates (3.21) and (3.27) to get

\begin{align*}
\int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \\
\geq (1 - y^* + \sqrt{3 + 4} (y^* + \frac{1}{2})^2) \sigma - C\varepsilon^{1/3} - c_0 \sigma \varepsilon
\end{align*}

which is just the lower bound (3.20). Note that in the last step we have used the fact that the function $1 - x + \sqrt{3 + 4(x + \frac{1}{2})^2}$ obtains its minimal value 3 at $x = 0$.

We can slightly modify the proof above to get the following refinement of the lower bound estimate.

**Proposition 3.2** (lower bound of order $\varepsilon^{1/2}$). There exist constants $C(W)$ and $\varepsilon_1$, such that for any $\varepsilon \leq \varepsilon_1$, it holds

\begin{equation}
\int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) \, dz \geq 3\sigma - C\varepsilon^{1/2}.
\end{equation}

**Proof.** Define all the analogous functions and subsets corresponding to $\varepsilon^{1/2}$.

\begin{align*}
\lambda_i &:= \mathcal{L}^1(\gamma_y \cap \{|u(x, y) - a_i| < \varepsilon^{1/2}\}), \quad i \in \{1, 2, 3\}, \\
y^* &:= \min\{y \in [-\frac{1}{2}, 1] : \lambda_1(y) + \lambda_2(y) \geq \mathcal{L}^1(\gamma_y) - \alpha \varepsilon^{1/2}\}, \\
\Omega_1 &:= B_1 \cap \{(x, y) : y \geq y^*\}, \quad \Omega_2 := B_1 \cap \{(x, y) : y < y^*\}, \\
\zeta(x) &:= \min\{y^*, \sqrt{1 - x^2}\}, \\
K &:= \{x \in [-\frac{\sqrt{3}}{2} + c_0 \varepsilon, \frac{\sqrt{3}}{2} - c_0 \varepsilon] : |u(x, \zeta(x)) - a_i| < \varepsilon^{1/2}, \ i = 1 \text{ or } 2\}, \\
M &:= \{y \in [-\frac{1}{2} + c_0 \varepsilon, y^*] : \lambda_3(y) > 0\}.
\end{align*}

Here $\alpha$ is a constant only depending on the potential function $W$, whose value will be determined later. Also we emphasize that in the rest of the paper we use (3.29) as the definition of $y^*$.

When $-\frac{1}{2} \leq y^* < -\frac{1}{2} + c_0 \varepsilon$, using the same calculation as in (3.22) and (3.23) we know the energy is strictly larger than $3\sigma$ when $\varepsilon$ is suitably small. Thus it suffices to consider the case $y^* \geq -\frac{1}{2} + c_0 \varepsilon$, for which the set $M$ is well defined.

Note that for $y \in [-\frac{1}{2} + c_0 \varepsilon, y^*] \setminus M$, it holds that

\begin{equation}
\lambda_1(y) + \lambda_2(y) + \lambda_3(y) < \mathcal{L}^1(\gamma_y) - \alpha \varepsilon^{1/2},
\end{equation}

\begin{equation}
\int_{\gamma_y} W(u) \, dx \geq \frac{1}{2} c_W \alpha \varepsilon.
\end{equation}
We split $W = \frac{3}{4}W + \frac{1}{4}W$. Thanks to the analogous estimates as in (3.21) and (3.25), we have

\[
\int_{\Omega_2} \left( \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right) \, dx \, dy \\
\geq \int_K \int_{-\frac{1}{2} - \epsilon} \left( \frac{\epsilon}{2} |\partial_y u|^2 + \frac{3}{4\epsilon} W(u) \right) \, dy \, dx + \int_M \int_{\gamma_y} \left( \frac{\epsilon}{2} |\partial_x u|^2 + \frac{1}{4\epsilon} W(u) \right) \, dx \, dy \\
+ \int_{[-\frac{1}{2} + \alpha_0 \epsilon, \frac{1}{2} + \alpha_0 \epsilon]} \int_{\gamma_y} \frac{1}{4\epsilon} W(u) \, dx \, dy
\]

(3.31)

\[
\geq \left( \frac{\sqrt{3}}{2} \mathcal{L}^1(K) + \mathcal{L}^1(M) \right) \left( \sigma - C W \epsilon \right) + \beta \frac{c W \alpha}{8},
\]

where $\beta := y^* + \frac{1}{2} - \alpha_0 \epsilon - \mathcal{L}^1(M) \geq 0$. We have

\[
\int_{B_1} \left( \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right) \, dx \, dy \\
= \int_{\Omega_2} \left( \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right) \, dx \, dy + \int_{\Omega_1} \left( \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right) \, dx \, dy \\
\geq \left( \frac{\sqrt{3}}{2} \mathcal{L}^1(K) + \mathcal{L}^1(M) \right) \left( \sigma - C W \epsilon \right) + \frac{c W \alpha \beta}{8} + \sigma (1 - c_0 \epsilon - y^*)
\]

(3.32)

\[
= 3 \sigma - (2 + y^*) C W \epsilon + \beta \left( \frac{c W \alpha}{8} - \sigma + C W \epsilon \right) \\
- \frac{\sqrt{3}}{2} \alpha \epsilon \left( \sigma - C W \epsilon \right) - c_0 \epsilon \left[ (\sigma - C W \epsilon) (1 + \sqrt{3}) + \sigma \right]
\]

\[
\geq 3 \sigma - \left( 3 C W + \frac{\sqrt{3}}{2} \alpha \sigma \right) \epsilon \geq \beta \left( \frac{c W \alpha}{8} - \sigma + C W \epsilon \right) - (2 + \sqrt{3}) c_0 \sigma + \frac{\sqrt{3}}{2} \alpha C W \epsilon.
\]

Note that from the second line to the third line we have used the estimates (3.31) and (3.21). Now we choose $\alpha = \alpha(W)$ such that $\frac{c W \alpha}{8} \geq 2 \sigma$. Then it follows easily from (3.32) that there exists a constant $C = C(W)$ such that the lower bound (3.28) holds for sufficiently small $\epsilon$.

4. Localization of the Transition Layer

Let $\lambda_i$ ($i = 1, 2, 3$), $y^*$, $\Omega_j$ ($j = 1, 2$), $\zeta(x)$, $K$, $M$ be defined in the same way as in the proof of Proposition 3.2. Also recall the definition $\beta := y^* + \frac{1}{2} - \alpha_0 \epsilon - \mathcal{L}^1(M)$. We can further derive the following lemma.

**Lemma 4.1.** There exists a constant $C$, which depends on the potential functional $W$, such that

\[
|y^*| \leq C \epsilon^{1/4},
\]

where $y^*$ is defined in (3.29).

**Proof.** First of all, by the energy upper bound (2.18) and $\frac{c W \alpha}{8} \geq 2 \sigma$, we get from (3.32)

\[
\beta \leq C \epsilon^{1/4},
\]
where $C$ is a constant depending only on $W$. Now we take the ratio $\kappa = \frac{2L^1(M)}{L^1(K)}$ and calculate in the same way as in the proof of Proposition 3.1 to get
\[
(4.35) \quad \int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right) \, dz \geq (1 - y^* + \sqrt{3 + 4(y^* + \frac{1}{2})^2})\sigma - C\varepsilon^{\frac{1}{2}}.
\]
Combining (4.35) and (2.18) yields
\[
1 - y^* + \sqrt{3 + 4(y^* + \frac{1}{2})^2} \leq 3 + C\varepsilon^{\frac{1}{2}} \Rightarrow |y^*| \leq C\varepsilon^{\frac{1}{4}}
\] for some constant $C = C(W)$.

\[\square\]

Remark 4.1. By applying the procedure above to each of the “legs” of the triod we obtain an equilateral triangle centered at the origin and of size $O(\varepsilon^{\frac{1}{4}})$. The “center” of the junction $(x^*, y^*)$ is located in this triangle.

Let $T$ denote the union of three line segments (the so-called triod), each two of which form an angle of $\frac{2\pi}{3}$.

\[
T := \{(0, y), \ y \in [0, 1]\} \cap \left\{(x, \frac{\sqrt{3}}{3}x), \ x \in \left[-\frac{\sqrt{3}}{2}, 0\right]\right\} \cap \left\{(x, -\frac{\sqrt{3}}{3}x), \ x \in \left[0, \frac{\sqrt{3}}{2}\right]\right\}.
\]

$T$ divides the closed unit disk $\overline{B_1}$ into three regions:

\[
D_i := \{(r \cos \theta, r \sin \theta) : \ 0 < r < 1, \ \frac{2(i-2)\pi}{3} < \theta < \frac{2(i-1)\pi}{3}\}, \ i = 1, 2, 3.
\]

We mention in passing that according to the $\Gamma$-convergence result in Gazoulis [19], it holds that
\[
\lim_{\varepsilon \to 0} \|u_\varepsilon - u_0\|_{L^1(B_1)} = 0,
\]
where
\[
u_0 = \sum_{i=1}^{3} a_i \chi_{D_i}.
\]
The proposition below provides a quantitative refinement of this convergence.

For $\gamma \geq 0$, we recall the diffuse interface is defined as
\[
I_{\varepsilon, \gamma} := \{z \in B_1 : |u_\varepsilon(z) - a_i| > \gamma \text{ for all } i = 1, 2, 3\}.
\]

The main result of this section is to show that $I_{\varepsilon, \gamma}$ is contained in a $O(\varepsilon^{\frac{1}{4}})$ neighborhood of $T$, for fixed $\gamma$ and suitably small $\varepsilon$. We have the following proposition.

**Proposition 4.2.** There exists a constant $\gamma_0$ such that for any $0 < \gamma \leq \gamma_0$, there exist constants $C = C(\gamma, W)$ and $\varepsilon(\gamma, W)$ satisfying
\[
\forall \varepsilon < \varepsilon(\gamma, W), \ I_{\varepsilon, \gamma} \subset N_{C\varepsilon^{\frac{1}{4}}}(T) := \{z \in B_1 : \text{dist}(z, T) \leq C\varepsilon^{\frac{1}{4}}\}.
\]
Moreover, there are positive constants $K$ and $k$ such that
\[
|u_\varepsilon(z) - a_i| \leq Ke^{-\frac{k}{2}(\text{dist}(z, T) - C\varepsilon^{\frac{1}{4}})^+}, \ z \in D_i, \ i = 1, 2, 3.
\]
Proof. For simplicity we write $u_\varepsilon = u$. We fix $C_0$ as the constant in (4.33), i.e.

\begin{equation}
(4.38)
    y^* \leq C_0 \varepsilon^{\frac{1}{4}}.
\end{equation}

For any $t \in [0, \frac{1}{2}]$, we define the line segments (see Figure 2)

\begin{align}
    l_1^t &:= \{ (-t, y) : y \geq \frac{\sqrt{3}}{3} t \} \cap \bar{D}_1, \\
    l_2^t &:= \{ (x, y) : y - \frac{\sqrt{3}}{3} t \frac{x}{x + t} = \frac{\sqrt{3}}{3}, x \leq -t \} \cap \bar{D}_1.
\end{align}

Thanks to (4.38), we have

\begin{align}
    t \geq \sqrt{3} C_0 \varepsilon^{\frac{1}{4}} \Rightarrow l_1^t \subset \Omega_1.
\end{align}

Let $\gamma_0$ be a positive constant which will be determined later. Now we fix $\gamma \leq \gamma_0$ and set

\begin{equation}
    A := \{ t : t \in [\sqrt{3} C_0 \varepsilon^{\frac{1}{4}}, \frac{1}{2}], \max_{z \in l_1^t} |u(z) - a_1| > \gamma \}.
\end{equation}

If $A = \emptyset$ then we can proceed to (4.43) below. We will show the measure of $A$ is of order $\varepsilon^{\frac{1}{4}}$. The proof will utilize that the part of the lower bound over $\Omega_1$ derived above is based entirely on the horizontal gradient, and thus the vertical gradient can be added to produce an improvement. See [5, Lemma 4.3] for a similar idea.

From the definition we know that for any $t \in A$, $l_1^t \subset \Omega_1$ and there exists a point $z_t \in l_1^t$ such that

\begin{equation}
    |u(z_t) - a_1| > \gamma.
\end{equation}

Also from the boundary condition (2.17) it follows that

\begin{equation}
    u((-t, \sqrt{1 - t^2})) = a_1, \quad (-t, \sqrt{1 - t^2}) \in l_1^t \cap \partial B_1, \quad t \geq c_0.
\end{equation}

Therefore for any $t \in A$, there exists a constant $C_1 := C_1(\gamma, W)$ such that

\begin{equation}
    \int_{\sqrt{1 - t^2}}^{\sqrt{1 - t^2}} \frac{\varepsilon}{2} \partial_y u |^2 + \frac{1}{\varepsilon} W(u) \ dy \geq C_1.
\end{equation}
Set
\[ \kappa := \frac{\mathcal{H}^1(A)C_1}{\sigma(1 - c_0\varepsilon - y^*)}, \]
we recalculate the energy on \( \Omega_1 \) using (3.21) and (4.40):
\[ \int_{\Omega_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) dxdy \]
\[ = \int_{\Omega_1} \left( \frac{\varepsilon}{2} |\partial_x u|^2 + \frac{1}{(1 + \kappa^2)\varepsilon} W(u) \right) dxdy \]
\[ \geq \frac{1}{\sqrt{1 + \kappa^2}} \int_{\Omega_1} \left( \frac{\varepsilon\sqrt{1 + \kappa^2}}{2} |\partial_x u|^2 + \frac{1}{\sqrt{1 + \kappa^2}\varepsilon} W(u) \right) dxdy \]
\[ \geq \frac{1}{\sqrt{1 + \kappa^2}} \frac{\kappa}{\sqrt{1 + \kappa^2}} \int_{\Omega_1} \left( \frac{\sqrt{1 + \kappa^2}}{\varepsilon} |\partial_y u|^2 + \frac{\kappa}{\sqrt{1 + \kappa^2}\varepsilon} W(u) \right) dxdy \]
\[ \geq \frac{1}{\sqrt{1 + \kappa^2}} \frac{\kappa}{\sqrt{1 + \kappa^2}} \mathcal{H}^1(A)C_1 \]
\[ = \sqrt{\sigma^2(1 - c_0\varepsilon - y^*)^2 + (C_1\mathcal{H}^1(A))^2}. \]
From (4.41) we can update (4.35) to be
\[ \int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \]
\[ \geq \sqrt{\sigma^2(1 - c_0\varepsilon - y^*)^2 + (C_1\mathcal{H}^1(A))^2 + \sqrt{3 + 4(y^* + \frac{1}{2})^2\sigma} - C\varepsilon^{\frac{1}{2}}} \]
\[ \geq (1 - y^* + \sqrt{3 + 4(y^* + \frac{1}{2})^2\sigma} - C\varepsilon^{\frac{1}{2}} + (C_1\mathcal{H}^1(A))^2) \frac{2\sigma(1 - c_0\varepsilon - y^*)}{2\sigma(1 - c_0\varepsilon - y^*)}, \]
which together with Lemma 4.11 and the upper bound (2.18) implies
\[ \frac{(C_1\mathcal{H}^1(A))^2}{2\sigma(1 - c_0\varepsilon - y^*)} \leq C\varepsilon^{\frac{1}{2}} \quad \Rightarrow \quad \mathcal{H}^1(A) \leq C_2(\gamma, W)\varepsilon^{\frac{1}{4}}. \]
We also set
\[ B := \{ t : t \in [\sqrt{3}C_0\varepsilon^{\frac{1}{2}}, \frac{1}{2}], \max_{z \in l^2} |u(z) - a_1| > \gamma \}. \]
An analogous computation implies
\[ \mathcal{H}^1(B) \leq C_2(\gamma, W)\varepsilon^{\frac{1}{4}}. \]
Set
\[ C_3(\gamma, W) := \sqrt{3}C_0 + 3C_2, \quad \varepsilon(\gamma, W) = \frac{1}{16C_3^4}. \]
We fix a small \( \varepsilon < \varepsilon(\gamma, W) \). Then from (4.42) and (4.44) it follows that there exists \( t_0 \in [\sqrt{3}C_0\varepsilon^{\frac{1}{2}}, C_3\varepsilon^{\frac{1}{2}}] \) (note that \( (\sqrt{3}C_0 + 3C_2)\varepsilon^{\frac{1}{2}} < \frac{1}{2} \) by the choice of \( \varepsilon \)) such that
\[ |u(z) - a_1| \leq \gamma, \quad \forall z \in l^1_0 \cup l^0_2. \]
Let $D^1_\gamma$ denote the region enclosed by $l^0_1$, $l^0_2$ and $\partial B_1$. It follows that

$$|u(z) - a_1| \leq \gamma, \quad \forall z \in \partial D^1_\gamma.$$  

By the variational maximum principle [4] (cf. Theorem 4.1 in [6]), there exists $\gamma_0$, which only depends on $W$, such that if $\gamma < \gamma_0$ and $u$ satisfies

$$u \text{ minimizes } \int_{D^1_\gamma} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dz,$$

then

$$|u(z) - a_1| \leq \gamma \text{ on } \partial D^1_\gamma,$$  

(4.45)

This further implies that the diffused interface $I_{\varepsilon,\gamma} \cap D_1$ is contained in a $C_3 \varepsilon^\frac{1}{2}$ neighborhood of $T$. The same argument works for $I_{\varepsilon,\gamma} \cap D_j$ for $j = 2, 3$ and we conclude the proof of the first part (4.36).

Finally from (4.45) and linear elliptic theory the exponential decay estimate (4.37) follows, which completes the proof. 

5. Width of the transition layer

Set

$$\Gamma^i_{\varepsilon,\gamma} := \left\{ z \in \bar{B}_1 : |u^\varepsilon(z) - a_i| = \gamma \right\}.$$  

Then by definition of the diffused interface $I_{\varepsilon,\gamma}$ we have

$$\partial I_{\varepsilon,\gamma} \subset \bigcup_{i=1}^3 \Gamma^i_{\varepsilon,\gamma}.$$  

The following result shows the “width” of $I_{\varepsilon,\gamma}$ is controlled by $O(\varepsilon)$.

**Proposition 5.1.** Fix $\gamma < \min\{\gamma_0, \min_{i\neq j \in \{1,2,3\}} \frac{1}{2}|a_i - a_j|\}$. There exists a constant $C = C(\gamma, W)$ such that for any $i \in \{1,2,3\}$ and sufficiently small $\varepsilon$,

$$\Gamma^i_{\varepsilon,\gamma} \subset NC\varepsilon \left( \bigcup_{j \neq i} \Gamma^j_{\varepsilon,\gamma} \right) = \left\{ z : \text{dist}(z, \bigcup_{j \neq i} \Gamma^j_{\varepsilon,\gamma}) \leq C\varepsilon \right\}.$$  

(5.46)

**Proof.** For simplicity we write $u^\varepsilon = u$. Without loss of generality, we prove (5.46) for $i = 1$. We pick an arbitrary point $z_1 \in \Gamma^1_{\varepsilon,\gamma}$ and set

$$r_1 := \max \left\{ r : B(z_1, r) \cap \left( \Gamma^2_{\varepsilon,\gamma} \cup \Gamma^3_{\varepsilon,\gamma} \right) = \emptyset \right\}.$$  

Since $\gamma < \min_{i\neq j \in \{1,2,3\}} \frac{1}{2}|a_i - a_j|$, we have $|u(z_1) - a_j| > \gamma$ for $j = 2, 3$. Then from the definition of $r_1$ it is not hard to show that

$$|u(z) - a_j| \geq \gamma, \quad j = 2, 3, \quad z \in B(z_1, r_1).$$
We define the blow up map $v_\varepsilon(\zeta) := u_\varepsilon(z_1 + \varepsilon \zeta)$. Then $v_\varepsilon$ satisfies

$$
v_\varepsilon \text{ minimizes } \int_{\Omega/\varepsilon} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) d\zeta,
$$

$$
|v_\varepsilon(0) - a_1| = \gamma,
$$

$$
|v_\varepsilon(\zeta) - a_j| \geq \gamma, \quad j = 2, 3, \quad \zeta \in B(0, \frac{r_1}{\varepsilon}),
$$

$$
|\nabla v_\varepsilon| \leq M, \quad M \text{ is a constant depending on } W.
$$

Here the gradient bound follows from the smoothness assumption of $W$ and standard elliptic regularity theory. This implies

$$
|v_\varepsilon(\zeta) - a_1| \geq \frac{\gamma}{2}, \quad \zeta \in B(0, \frac{\gamma}{2M}).
$$

Then by the vector version of the Caffarelli-Córdoba density estimate [3] (cf. Theorem 5.2 in [6]), there exists a positive constant $C = C(\gamma, W)$ such that

$$
\mathcal{L}^2 \left( \{|v_\varepsilon(\zeta) - a_1| \geq \frac{\gamma}{2} \} \cap B(0, \frac{r_1}{\varepsilon}) \right) \geq C \left( \frac{r_1}{\varepsilon} \right)^2.
$$

It follows that

$$
\mathcal{L}^2 \left( \{|v_\varepsilon(\zeta) - a_i| \geq \frac{\gamma}{2}, \forall i = 1, 2, 3 \} \cap B(0, \frac{r_1}{\varepsilon}) \right) \geq C \left( \frac{r_1}{\varepsilon} \right)^2.
$$

Then we compute

$$
\int_{B(z_1, r_1)} \frac{1}{\varepsilon} W(u) \, dz = \varepsilon \int_{B(0, \frac{r_1}{\varepsilon})} W(v_\varepsilon) \, d\zeta
$$

$$
\geq \varepsilon \frac{cW}{2} \left( \frac{\gamma}{2} \right)^2 \cdot C \left( \frac{r_1}{\varepsilon} \right)^2
$$

$$
= C(\gamma, W) \frac{r_1^2}{\varepsilon},
$$

(5.47)

On the other hand, we can derive an upper bound for the energy of $u$ inside $B(z_1, r_1)$ by constructing an energy competitor $v$ such that $v = u$ in $\Omega \setminus B(z_1, r_1)$. We can assume $r_1 \geq 2\varepsilon$, otherwise there is nothing to prove. Let

$$
v(z) = \begin{cases} 
   u(z), & |z - z_1| \geq r_1, \\
   \frac{r_1 - |z - z_1|}{\varepsilon} a_1 + \frac{|z - z_1| - r_1 + \varepsilon}{\varepsilon} u(z), & r_1 - \varepsilon \leq |z - z_1| < r_1, \\
   a_1, & |z - z_1| < r_1 - \varepsilon
\end{cases}
$$

(5.48)
We compute the energy of \( v \) inside \( B(z_1, r_1) \).
\[
\int_{B(z_1, r_1)} \frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{\varepsilon} W(v) \, dz
\]
\[
= \int_{r_1 - \varepsilon \leq |z - z_1| < r_1} \frac{\varepsilon}{2} |\nabla u(z) + \nabla \left( \frac{|z - z_1| - r_1}{\varepsilon} (u(z) - a_1) \right)|^2 \, dz
\]
\[
+ \int_{r_1 - \varepsilon \leq |z - z_1| < r_1} \frac{1}{\varepsilon} W(v) \, dz
\]
\[
\leq \int_{r_1 - \varepsilon \leq |z - z_1| < r_1} \frac{\varepsilon}{2} \left( 4 |\nabla u(z)|^2 + 2 |u(z) - a_1|^2 \frac{|\nabla z - z_1|^2}{\varepsilon^2} \right) \, dz
\]
\[
+ \int_{r_1 - \varepsilon \leq |z - z_1| < r_1} \frac{1}{\varepsilon} W(v) \, dz
\]
\[
\leq (2\pi r_1 \cdot \varepsilon) \frac{\varepsilon}{2} \left( 4 \left| \frac{M}{\varepsilon} \right|^2 + 2 \left| \frac{M}{\varepsilon} \right|^2 \right) + (2\pi r_1 \cdot \varepsilon) \cdot \frac{c}{\varepsilon}
\]
\[
\leq C(M) r_1.
\]
Here in the calculation we have used that \(|u(z)| \leq M\), \(|\nabla u(z)| \leq \frac{M}{\varepsilon}\) and that \(W(v)\) is also uniformly bounded by some constant \(c\). From (5.47), (5.49) and the minimality of \(u\) we obtain
\[
C(\gamma, W) r_1^2 \varepsilon \leq C(M) r_1 \Rightarrow r_1 \leq C \varepsilon,
\]
where the last constant \(C\) only depends on \(\gamma\) and \(W\). The proof of Proposition 5.1 is complete. 

\[\square\]

6. Proof of Theorem 1.2

Throughout the section, we denote by \(C_0 = C(\gamma, W)\) the constant defined in Proposition 5.1. Fix a small \(\gamma\) and a small constant \(\varepsilon < \varepsilon(\gamma, W)\). We always assume \(\gamma, \varepsilon\) are suitably small constants, and the requirement on the smallness of these values will be explained along the proof. According to Sard’s Theorem and the Implicit Function Theorem, we can without loss of generality assume that each connected component of \(\Gamma_{\varepsilon, \gamma} \cap B_1\) is a \(C^1\) curve.  

For fixed small \(\varepsilon, \gamma\), let \(\hat{D}_{\varepsilon, \gamma}^1\) denote the connected component of the set \(\{z : |u_{\varepsilon}(z) - a_1| < \gamma\}\) that connects to the boundary \(\partial B_1\). \(\hat{D}_{\varepsilon, \gamma}^1\) satisfies the following properties

1. \(\{z \in \partial B_1 : u_{\varepsilon}(z) = a_1\} \subset \partial \hat{D}_{\varepsilon, \gamma}^1\),
2. \(\hat{D}_{\varepsilon, \gamma}^1\) is simply connected by the variational maximum principle argument (cf. [6] Theorem 4.1).
3. By Proposition 4.2, \((\partial \hat{D}_{\varepsilon, \gamma}^1 \setminus \partial B_1) \subset N_{\varepsilon^2}(T)\), which is an \(O(\varepsilon^2)\) neighborhood of the triod.  

This property holds only if \(\varepsilon < \varepsilon(\gamma, W)\) where \(\varepsilon(\gamma, W)\) is the constant in (4.36).

We set
\[
\hat{\Gamma}_{\varepsilon, \gamma}^1 := \text{closure of } (\partial \hat{D}_{\varepsilon, \gamma}^1 \setminus \partial B_1).
\]

\[\text{For fixed } \varepsilon, \text{ from Sard’s Theorem and the Implicit Function Theorem we have for a.a. } \gamma, \text{ each connected component of } \Gamma_{\varepsilon, \gamma}^1 \cap B_1 \text{ is a } C^1\text{ curve. Therefore even if the chosen } \{\varepsilon, \gamma\} \text{ doesn’t satisfy this property, we can always take a slightly smaller } \gamma' \text{ so that this property holds for } \{\varepsilon, \gamma'\}. \text{ Then one can proceed with the rest of the proof using } \{\varepsilon, \gamma'\} \text{ and all the conclusions will not be affected.} \]
Then by the definition of $g_\varepsilon$, $\hat{\Gamma}^1_{\varepsilon,\gamma}$ is a continuous curve that intersects with $\partial B_1$ at only two points. We set $A$ as the intersection of $\partial B_1$ and $\hat{\Gamma}^1_{\varepsilon,\gamma}$ that is $O(\varepsilon)$ close to the point $(0,1)$ and $B$ as the other intersection.

We first prove the following lemma that is a direct consequence of the upper bound Lemma 2.3 and Proposition 5.1.

**Lemma 6.1.** Fix $\gamma < \min\{\gamma_0, \min_{i,j} |a_i - a_j|, \sqrt{\frac{\sigma}{2\delta_{B_1}}}\}$. For any $n > 0$, there exists $\varepsilon_0 := \varepsilon_0(n, \gamma, W)$ such that for any $\varepsilon < \varepsilon_0$, there exists $y_0 \in \left[\frac{3}{8}, \frac{3}{4}\right]$ that satisfies

$$\forall \ z = (x, y) \in \hat{\Gamma}^{1}_{\varepsilon,\gamma} \cap \{y \in [y_0 - nC_0\varepsilon, y_0 + nC_0\varepsilon]\}, \text{ it holds } \dist(z, \Gamma^2_{\varepsilon,\gamma}) \leq \dist(z, \Gamma^3_{\varepsilon,\gamma}).$$

**Remark 6.1.** It is not hard to see from the proof below that the constants $\frac{1}{4}, \frac{3}{8}$ in the above lemma can be replaced by any $0 < r_1 < r_2 < 1$. Here we take these values for simplicity of presentation.

**Proof.** First assume $\varepsilon$ is sufficiently small and fix $n$. We argue by contradiction. If for any $h \in \left[\frac{1}{4}, \frac{3}{8}\right]$, there exists $z(h) = (x, y)$ satisfying $|y - h| \leq nC_0\varepsilon$ and $\dist(z(h), \Gamma^2_{\varepsilon,\gamma}) > \dist(z(h), \Gamma^3_{\varepsilon,\gamma})$, then by Proposition 5.1

$$\dist(z(h), \Gamma^3_{\varepsilon,\gamma}) \leq C_0\varepsilon,$$

which implies

$$\exists \ \xi(h) \in \Gamma^3_{\varepsilon,\gamma}, \ \dist(\xi(h), z(h)) \leq C_0\varepsilon.$$

Since $|\nabla u_\varepsilon| \leq \frac{M}{\varepsilon}$, for any $z$ such that $\dist(z, \xi(h)) \leq \frac{\varepsilon}{4N}$ it holds that

$$|u_\varepsilon(z) - a_3| \leq 2\varepsilon.$$

Set $N = \left\lfloor\frac{1}{32(n+1)C_0\varepsilon}\right\rfloor$ as the greatest integer less than or equal to $\frac{1}{32(n+1)C_0\varepsilon}$. Here we require $\varepsilon \leq \frac{1}{128(n+1)C_0\varepsilon}$ such that $N \geq 4$ and $N$ satisfies

$$\frac{1}{64(n+1)C_0\varepsilon} < N \leq \frac{1}{32(n+1)C_0\varepsilon}.$$

Then we split the interval $[\frac{1}{4}, \frac{3}{8}]$ into $N$ small intervals, which are denoted by $I_1, I_2, \ldots, I_N$:

$$I_j = \left[\frac{1}{4} + 4(j - 1)(n+1)C_0\varepsilon, \frac{1}{4} + 4j(n+1)C_0\varepsilon\right], \quad 1 \leq j \leq N - 1,$$

$$I_N := \left[\frac{1}{4} + 4(N - 1)(n+1)C_0\varepsilon, \frac{3}{8}\right].$$

From the definition we know that $L^1(I_j) = 4(n+1)C_0\varepsilon$ for $j = 1, \ldots, N - 1$ and $4(n+1)C_0\varepsilon \leq L^1(I_N) < 8(n+1)C_0\varepsilon$. We define

$$h_j := \frac{1}{4} + (4j - 2)(n+1)C_0\varepsilon, \quad j = 1, \ldots, N.$$

Then $h_j$ is in $I_j$ for $j = 1, \ldots, N$ and $|h_N - \frac{3}{8}| \geq 2(n+1)C_0\varepsilon$.

For any $j = 1, \ldots, N$, by the assumption above there are points $z(h_j) = (z_1, z_2)$, $\xi(h_j) = (\xi_1, \xi_2)$ such that

$$|z_2 - h_j| \leq nC_0\varepsilon, \quad |\xi_2 - z_2| \leq C_0\varepsilon, \quad z(h_j) \in \hat{\Gamma}^{1}_{\varepsilon,\gamma}, \quad \xi(h_j) \in \Gamma^3_{\varepsilon,\gamma}.$$

Hence for any $h \in (\xi_2 - \frac{\varepsilon}{4N}, \xi_2 + \frac{\varepsilon}{4N})$,

$$\exists \hat{z}(h) = (\hat{x}(h), h), \text{ s.t. } |u_\varepsilon(\hat{z}(h)) - a_3| \leq 2\varepsilon.$$

We define the set of points with such property

$$K := \{h \in \left[\frac{1}{4}, \frac{3}{8}\right]: \exists \hat{z}(h) = (\hat{x}(h), h), \text{ s.t. } |u_\varepsilon(\hat{z}(h)) - a_3| \leq 2\varepsilon\}.$$
The deduction above implies that for any \( j = 1, \ldots, N \),
\[
\mathcal{L}^1(K \cap I_j) \geq \min\{\frac{2\gamma \varepsilon}{M}, 2(n+1)C_0 \varepsilon\} = C_1 \varepsilon,
\]
where \( C_1 \) is a constant depending on \( n, \gamma, W \).

For \( h \in K \), \( u_\varepsilon \) equals to \( a_1 \) at \((-\sqrt{1-h^2}, h)\) and \( a_2 \) at \((\sqrt{1-h^2}, h)\) respectively, while it is close to \( a_3 \) at some point \( \hat{z}(h) \) in the middle. We compute (as before we write \( u_\varepsilon = u \) for simplicity)
\[
\int_{\{y=h\}} \left( \frac{\varepsilon}{2} |\partial_x u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \geq \int_{\hat{z}(h)} \sqrt{1-h^2} \left( \frac{\varepsilon}{2} |\partial_x u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx + \int_{\hat{z}(h)} \sqrt{1-h^2} \left( \frac{\varepsilon}{2} |\partial_x u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \\
\geq 2(\sigma - 4C_W \gamma^2) > \frac{3}{2} \sigma,
\]
where in the last inequality we further require that \( \gamma < \sqrt{\frac{\sigma}{20C_W}} \).

Note that when \( \varepsilon \) is suitably small, \( y_* \leq \frac{1}{4} \). We compute the energy in \( \Omega_1 \),
\[
\int_{\Omega_1 \cap \{y \in \left[\frac{1}{4}, \frac{3}{4}\right]\}} \left( \frac{\varepsilon}{2} |\partial_x u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \\
\geq \sum_j \int_{y \in I_j} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left( \frac{\varepsilon}{2} |\partial_x u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \\
\geq \sum_j \left( \int_{y \in I_j \cap K} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left( \frac{\varepsilon}{2} |\partial_x u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \\
+ \int_{y \in I_j \setminus K} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left( \frac{\varepsilon}{2} |\partial_x u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \right) \\
\geq \sum_j \left( \mathcal{L}^1(I_j \cap K) \cdot \frac{3}{2} \sigma + \mathcal{L}^1(I_j \setminus K) \sigma \right) \\
= \frac{\sigma}{8} + \frac{\sigma}{2} \left( \sum_j \mathcal{L}^1(I_j \cap K) \right) \\
\geq \frac{\sigma}{8} + \frac{\sigma}{2} (C_1 \varepsilon) \frac{1}{64(n+1)C_0 \varepsilon} \\
\geq \frac{\sigma}{8} + C_2 \sigma.
\]
Here \( C_2 = C_2(\gamma, W) \) is a positive constant which only depends on \( n, \gamma, W \). Adding the energy in \( \Omega_1 \) and \( \Omega_2 \), we have
\[
\int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \\
= \int_{\Omega_1} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy + \int_{\Omega_2} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx \, dy \\
\geq (3 + C_2(n, \gamma, W)) \sigma - C_2 \varepsilon^{\frac{1}{2}}.
\]
Here the constant $C$ in the last inequality is the same one as in Proposition 3.2. Taking $\varepsilon$ such that $C\varepsilon^2 \leq \frac{C_2}{2\sigma}$ yields a contradiction with the upper bound, which proves the lemma.

□

Lemma 6.2. Fix $\gamma < \min\{\gamma_0, \min_{i,j\in\{1,2,3\}} \frac{1}{2}|a_i - a_j|, \sqrt{\frac{\sigma}{20c_w}}\}$. There exists a small constant $\varepsilon(\gamma, W)$ such that for any $\varepsilon < \varepsilon(\gamma, W)$, there are three points $P_\varepsilon, Q_\varepsilon, R_\varepsilon \in \overline{B}_1$ that satisfy

- $|u_\varepsilon(P_\varepsilon) - a_1| = \gamma$, $|u_\varepsilon(Q_\varepsilon) - a_2| = \gamma$, $|u_\varepsilon(R_\varepsilon) - a_3| = \gamma$,
- $\max\{\text{dist}(P_\varepsilon, Q_\varepsilon), \text{dist}(P_\varepsilon, R_\varepsilon)\} \leq C_0\varepsilon$.

Proof. From Lemma 6.1 (taking $n = 1$) and Proposition 5.1 there is at least one point $\hat{z}_1 = (x_1, y_1) \in \hat{\Gamma}_{\varepsilon,\gamma}^1$ which satisfies the following

1. $y_1 \in \left[\frac{1}{4}, \frac{3}{8}\right],$
2. $\text{dist}(\hat{z}_1, \Gamma_{\varepsilon,\gamma}^2) \leq C_0\varepsilon$, $\text{dist}(\hat{z}_1, \Gamma_{\varepsilon,\gamma}^3) \geq \text{dist}(\hat{z}_1, \Gamma_{\varepsilon,\gamma}^2),$
3. $\hat{\Gamma}_{\varepsilon,\gamma}^1(\hat{z}_1, B) \cap \{x, y\} \in B_1 : y > y_1\} = \emptyset$, here $\hat{\Gamma}_{\varepsilon,\gamma}^1(\hat{z}_1, B)$ denotes the part of $\hat{\Gamma}_{\varepsilon,\gamma}$ that connects $\hat{z}_1$ and $B$.

Here we have the third property because $\hat{z}_1$ is chosen from the intersections of $\hat{\Gamma}_{\varepsilon,\gamma}$ and the horizontal line $\{y = y_1\}$ where $y_1$ satisfies Lemma 6.1. Therefore we can choose the “last” intersection of $\hat{\Gamma}_{\varepsilon,\gamma}$ and $\{y = y_1\}$ such that the third property holds.

Similarly we can apply the same argument to the transition layer between $a_1$ and $a_3$ and find one point $\hat{z}_2 = (x_2, y_2) \in \hat{\Gamma}_{\varepsilon,\gamma}^3$ satisfying

1. $\sqrt{3}x_2 + y_2 \in \left[-\frac{3}{4}, -\frac{1}{2}\right],$
2. $\text{dist}(\hat{z}_2, \Gamma_{\varepsilon,\gamma}^3) \leq C_0\varepsilon$, $\text{dist}(\hat{z}_2, \Gamma_{\varepsilon,\gamma}^2) \geq \text{dist}(\hat{z}_2, \Gamma_{\varepsilon,\gamma}^3),$
3. $\hat{\Gamma}_{\varepsilon,\gamma}^1(\hat{z}_2, B) \cap \{x, y\} \in B_1 : y < y_1\} = \emptyset$.
(3) \( \tilde{\gamma}_{\varepsilon, \gamma}(A, \tilde{z}_2) \cap \{(x, y) \in B_1 : \sqrt{3}x + y < \sqrt{3}x_2 + y_2\} = \emptyset. \)

From the path-connectedness of \( \tilde{\gamma}_{\varepsilon, \gamma} \) it follows that there is a continuous map \( \hat{f} : [0, 1] \to \tilde{\gamma}_{\varepsilon, \gamma} \) that satisfies \( \hat{f}(0) = \tilde{z}_1, \hat{f}(1) = \tilde{z}_2. \) By the property (3) of \( \tilde{z}_1 \) and \( \tilde{z}_2 \) and simple geometry arguments, we know that

\[
\Gamma_{\varepsilon, \gamma}(\tilde{z}_1, \tilde{z}_2) \subset B_{\frac{1}{2}} \cap N_{C\varepsilon^\frac{1}{2}}(T), \quad \text{when } C\varepsilon^\frac{1}{2} < \frac{1}{8}.
\]

Define the function

\[
\alpha : [0, 1] \to \mathbb{R}, \quad \alpha(t) = \text{dist}(\hat{f}(t), \Gamma_{\varepsilon, \gamma}^2) - \text{dist}(\hat{f}(t), \Gamma_{\varepsilon, \gamma}^3).
\]

Due to the continuity of \( \hat{f} \) and the distance function to a closed set, \( \alpha(t) \) is also continuous. The properties of \( \tilde{z}_1 \) and \( \tilde{z}_2 \) mean that \( \alpha(0) \leq 0 \) and \( \alpha(1) \geq 0. \) Then the intermediate value theorem immediately implies there exists \( t_0 \in [0, 1] \) such that \( \alpha(t_0) = 0. \) Set \( P_\varepsilon = \hat{f}(t_0). \) From Proposition 5.1 it follows that

\[
\text{dist}(P_\varepsilon, \Gamma_{\varepsilon, \gamma}^2) = \text{dist}(P_\varepsilon, \Gamma_{\varepsilon, \gamma}^3) \leq C_0\varepsilon.
\]

Take \( Q_\varepsilon \in \Gamma_{\varepsilon, \gamma}^2 \) and \( R_\varepsilon \in \Gamma_{\varepsilon, \gamma}^3 \) such that

\[
\text{dist}(Q_\varepsilon, P_\varepsilon) = \text{dist}(R_\varepsilon, P_\varepsilon) = \text{dist}(P_\varepsilon, \Gamma_{\varepsilon, \gamma}^2) = \text{dist}(P_\varepsilon, \Gamma_{\varepsilon, \gamma}^3) \leq C_0\varepsilon.
\]

Then it is straightforward to verify that \( P_\varepsilon, Q_\varepsilon \) and \( R_\varepsilon \) satisfy all the properties in the statement of Lemma 6.2. The proof is complete. \( \square \)

Next we present the key lemma for the main Theorem 1.2, which has a more discrete nature.

**Lemma 6.3.** Fix \( \gamma < \min\{\gamma_0, \min_{i,j} |a_i - a_j|, \sqrt{\frac{\sigma}{20C_W}}\}. \) For any \( k \in \mathbb{N}^+, \) there exists \( \varepsilon_0(k, \gamma, W) \) such that for any \( \varepsilon < \varepsilon_0, \) there is a point \( P_\varepsilon \in \tilde{\gamma}_{\varepsilon, \gamma} \cap B_{\frac{1}{2}} \) that satisfies the following,

\[
\exists \{Q^j_k\}_{j=1}^{2^k} \subset \Gamma_{\varepsilon, \gamma}, \{R^j_k\}_{j=1}^{2^k} \subset \Gamma_{\varepsilon, \gamma} \text{ such that} \quad \text{dist}(Q^j_k, P_\varepsilon) \leq (32j + 1)C_0\varepsilon, \quad \text{dist}(R^j_k, P_\varepsilon) \leq (32j + 1)C_0\varepsilon,
\]

For any two points \( A_1, A_2 \in \{Q^j_k\}_{j=1}^{2^k} \cup \{R^j_k\}_{j=1}^{2^k}, \) \( \text{dist}(A_1, A_2) \geq 6C_0\varepsilon, \)

For any point \( A \in \{Q^j_k\}_{j=1}^{2^k} \cup \{R^j_k\}_{j=1}^{2^k}, \) \( \text{dist}(A, \tilde{\gamma}_{\varepsilon, \gamma}) \leq C_0\varepsilon. \)

**Proof.** Assume \( \gamma \) and \( \varepsilon \) are suitably small. The specific conditions on the smallness of \( \gamma, \varepsilon \) will be provided along the proof. We focus on the curve \( \hat{\gamma}_{\varepsilon, \gamma} \subset \Gamma_{\varepsilon, \gamma} \) which is a \( C^1 \) curve that intersects with the boundary \( \partial B_1 \) at two points \( A, B. \)

First of all we parameterize \( \hat{\gamma}_{\varepsilon, \gamma} \) by arclength:

\[
\tilde{\gamma}_{\varepsilon, \gamma} = \{\eta(t) : t \in [0, L], \eta(0) = A, \eta(L) = B, |\eta'(t)| = 1 \text{ a.e. }\},
\]

where \( L = \mathcal{H}^1(\tilde{\gamma}_{\varepsilon, \gamma}) \) is the length of the curve \( \tilde{\gamma}_{\varepsilon, \gamma}. \)

**Step 1.** “Discretize the curve \( \tilde{\gamma}_{\varepsilon, \gamma} \)” We can find a sequence of points \( z_1, \ldots, z_N \) by the following rule:

1. \( z_1 = A = \eta(0), t_1 = 0; \)
2. If we already define \( z_i = \eta(t_i), \) then take

\[
t_{i+1} = \sup_{t \geq t_i} \{t : \text{dist}(\eta(t), z_i) \leq 8C_0\varepsilon\}, \quad z_{i+1} = \eta(t_{i+1});
\]
3. When \( \text{dist}(B, z_N) \leq 8C_0\varepsilon \) for some \( N, \) the process stops.
By the continuity of $\eta(t)$, we have immediately
\begin{equation}
\text{(6.57)} \quad \text{dist}(z_{i+1}, z_i) = 8C_0\varepsilon.
\end{equation}

Since $|\eta'(t)| = 1$ a.e., it holds
\[ t_{i+1} - t_i \geq 8C_0\varepsilon. \]

Hence the process will stop in finite steps and
\[ N \leq \frac{L}{8C_0\varepsilon} + 1. \]

Moreover for any $1 \leq i \neq j \leq N$,
\begin{equation}
\text{(6.58)} \quad \text{dist}(z_i, z_j) \geq 8C_0\varepsilon.
\end{equation}

Otherwise, if there exists $i > j$ such that dist$(z_i, z_j) < 8C_0\varepsilon$, by (6.57) we know $i \geq j + 2$. Then dist$(\eta(t_i), z_j) < 8C_0\varepsilon$ and $t_j > t_{j+1}$ together yield a contradiction with the choice of $t_{j+1}$.

Next we classify $\{z_i\}_{i=1}^N$ into two subsets according to their relative distances to $\Gamma_{\varepsilon,\gamma}^2$ and $\Gamma_{\varepsilon,\gamma}^3$.

\[ Z_2 := \{z_i : \text{dist}(z_i, \Gamma_{\varepsilon,\gamma}^2) \leq \text{dist}(z_i, \Gamma_{\varepsilon,\gamma}^3)\}, \]
\[ Z_3 := \{z_i\}_{i=1}^N \setminus Z_2. \]

\textit{Step 2.} Fix $k \in \mathbb{N}^+$. Taking $n = 2^{k+5}$ in Lemma 6.1, we have that when $\varepsilon < \varepsilon(n, \gamma, W)$, there exists a $\tilde{y} \in [\frac{1}{4}, \frac{3}{8}]$ such that for any $z_i = (x_i, y_i)$ satisfying $|y_i - \tilde{y}| \leq 2^{k+5}C_0\varepsilon$, $z_i \in Z_2$. Define
\[ t_{i_0} := \sup\{t_i : z_i = \eta(t_i) = (x_i, y_i), y_i \geq \tilde{y}\}. \]

This definition of $t_{i_0}$ and (6.57) implies that
\[ |y_{i_0} - \tilde{y}| \leq 8C_0\varepsilon, \]
\[ y_i < \tilde{y}, \forall i > i_0. \]

In particular we have for all $i_0 \leq i \leq i_0 + 2^{k+1} - 1$ it holds that
\[ |y_i - \tilde{y}| \leq 2^{k+4}C_0\varepsilon, \quad y_i \in [\frac{1}{8}, \frac{1}{2}], \quad z_i \in Z_2. \]

Here we only require $2^{k+4}C_0\varepsilon \leq \frac{1}{8}$.

Following the same argument, we can also get a $j_0$ that satisfies the following,
\[ j_0 > i_0; \]
\begin{equation}
\text{(6.59)} \quad \forall i_0 < j \leq j_0, \quad \sqrt{3}x_j + y_j > \frac{3}{4}; \quad y_j < \frac{3}{8};
\end{equation}
\[ \forall j_0 - 2^{k+1} + 1 \leq j \leq j_0, \quad z_j \in Z_3, \quad \sqrt{3}x_j + y_j \in (-1, -\frac{1}{4}). \]

The locations of $\{z_i\}_{i=i_0}^{i_0+2^{k+1}-1}$ and $\{z_j\}_{j=j_0-2^{k+1}+1}^{j_0}$ imply that they are disjoint subsets. Since $j_0 > i_0$, we further infer that
\[ j_0 - 2^{k+1} + 1 > i_0 + 2^{k+1} - 1. \]

Furthermore, (6.59) implies
\begin{equation}
\text{(6.60)} \quad z_i \in B_{\frac{1}{2}}, \quad \forall i_0 \leq j \leq j_0.
\end{equation}
Step 3. For \( i_0 \leq l \leq j_0 - 2^{k+1} + 1 \), define
\[
N(l) := |\{z_l, z_{l+1}, \ldots, z_{l+2^{k+1} - 1}\} \cap \mathbb{Z}_2|
\]
as the number of points in \( \{z_i\}_{i=l}^{l+2^{k+1}-1} \cap \mathbb{Z}_2 \). Then
\[
N(i_0) = 2^{k+1}, \quad N(j_0 - 2^{k+1} + 1) = 0.
\]
Also each time \( l \) is shifted by 1, \( N(l) \) is changed at most by 1, i.e.
\[
|N(l + 1) - N(l)| \leq 1.
\]
Hence by continuity there exists a \( l_0 \) such that
\[
N(l_0) = 2^k, \quad i_0 < l_0 < j_0 - 2^{k+1} + 1.
\]
Next we can find \( l_1 \in \{l_0, \ldots, l_0^2\} \) such that \( |\{z_i\}_{i=l_1}^{l_1+2^k-1} \cap \mathbb{Z}_2| = 2^{k-1} \). Indeed, if \( |\{z_i\}_{i=l_0}^{l_0+2^k-1} \cap \mathbb{Z}_2| = 2^{k-1} \), then we simply take \( l_1 = l_0 \). Otherwise without loss of generality we assume \( |\{z_i\}_{i=l_0+2^k} \cap \mathbb{Z}_2| > 2^{k-1} \), then \( |\{z_i\}_{i=l_0+2^k}^{n+1} \cap \mathbb{Z}_2| < 2^{k-1} \). By continuity we can find \( l_1 \in \{l_0 + 1, \ldots, l_0 + 2^k - 1\} \) such that
\[
|\{z_i\}_{i=l_1}^{l_1+2^{k-1}} \cap \mathbb{Z}_2| = 2^{k-1}.
\]
Proceeding in the same way we finally get \( l_0, l_1, \ldots, l_k \) that satisfy
\begin{enumerate}
\item \( l_j \leq l_{j+1} \leq l_j + 2^{k-j} - 1 \), for \( j = 0, \ldots, k - 1 \).
\item \( |\{z_i\}_{i=l_j}^{l_j+2^{k-j}-1} \cap \mathbb{Z}_2| = |\{z_i\}_{i=l_j}^{l_j+2^{k-j}-1} \cap \mathbb{Z}_3| = 2^{k-j} \) for \( j = 0, \ldots, k \).
\item \( \{z_i\}_{i=l_j}^{l_j+2^{k-j}-1} \subset B(z_k, 2^{k-j} \cdot 8C_0\varepsilon) \) for \( j = 0, \ldots, k \).
\end{enumerate}
Take \( P_\varepsilon = z_k \), then from the above construction we have that
\[
|B(P_\varepsilon, 2j+3C_0\varepsilon) \cap \mathbb{Z}_2| \geq 2^{j-1},
\]
\[
|B(P_\varepsilon, 2j+3C_0\varepsilon) \cap \mathbb{Z}_3| \geq 2^{j-1}.
\]
Recall the definition of \( \mathbb{Z}_2 \), if \( z_i \in \mathbb{Z}_2 \), then there exists a \( Q_i \in \mathbb{Z}_2 \) such that \( \operatorname{dist}(z_i, Q_i) \leq C_0\varepsilon \).
For \( z_i \neq z_j \in \mathbb{Z}_2 \), let \( Q_i, Q_j \) be the corresponding points on \( \Gamma_{\varepsilon, \gamma}^2 \). Then \( Q_i, Q_j \) satisfy \( \operatorname{dist}(Q_i, Q_j) \geq 6C_0\varepsilon \).
Therefore, we have obtained \( \{Q_1, \ldots, Q_{2^k}\} \subset \mathbb{Z}_{\varepsilon, \gamma}, \{R_1, \ldots, R_{2^k}\} \subset \mathbb{Z}_{\varepsilon, \gamma}^3 \) satisfying
\[
\{Q_1, \ldots, Q_{2^k}, R_1, \ldots, R_{2^k}\} \subset B(P_\varepsilon, (2j+4)C_0\varepsilon), \quad \forall \ 0 \leq j \leq k,
\]
\[
A_1, A_2 \subset \{Q_1, \ldots, Q_{2^k}, R_1, \ldots, R_{2^k}\}, \operatorname{dist}(A_1, A_2) \geq 6C_0\varepsilon.
\]
Let \( Q_k^j = Q_j, R_k^j = R_j \). The proof of the lemma is complete.

\[ \square \]

Conclusion of the proof of Theorem 1.2 By Lemma 6.3 we can find a sequence \( \varepsilon_k \to 0 \) and a sequence of points \( P_{\varepsilon_k} = (x_k, y_k) \in \Gamma_{\varepsilon_k, \gamma}^1 \cap B_{\frac{2}{3}} \) such that there exist \( \{Q_{\varepsilon_k}^j\}_{j=1}^{2^k} \) satisfying that the pairwise distance is larger than \( 6C_0\varepsilon_k \) and
\[
Q_{\varepsilon_k}^j \in \Gamma_{\varepsilon_k, \gamma}^2, \quad R_{\varepsilon_k}^j \in \Gamma_{\varepsilon_k, \gamma}^3,
\]
\[
\operatorname{dist}(Q_{\varepsilon_k}^j, \Gamma_{\varepsilon_k, \gamma}^1) \leq C_0\varepsilon_k, \quad \operatorname{dist}(R_{\varepsilon_k}^j, \Gamma_{\varepsilon_k, \gamma}^1) \leq C_0\varepsilon_k,
\]
\[
\operatorname{dist}(Q_{\varepsilon_k}^j, P_{\varepsilon_k}) \leq (32j + 1)C_0\varepsilon_k, \quad \operatorname{dist}(R_{\varepsilon_k}^j, P_{\varepsilon_k}) \leq (32j + 1)C_0\varepsilon_k.
\]
Moreover, since $\Gamma^{1}_{\varepsilon_k, \gamma}$ is connected, for any $r \in (0, \text{dist}(P_{\varepsilon_k}, \partial B_1))$ we have

$$\Gamma^{1}_{\varepsilon_k, \gamma} \cap \partial B(P_{\varepsilon_k}, r) \neq \emptyset.$$ 

We rescale $u_{\varepsilon_k}$ at the point $P_{\varepsilon_k}$:

$$U_k(X,Y) := u_{\varepsilon_k}(x_k + \varepsilon_k X, y_k + \varepsilon_k Y), \quad (X,Y) \in B_{r_{\varepsilon_k}}(0),$$

where $r_{\varepsilon_k} := \frac{\text{dist}(P_{\varepsilon_k}, \partial B_1)}{\varepsilon_k} \geq \frac{1}{2\varepsilon_k}$.

The estimates $|U_k| \leq M, |\nabla U_k| \leq M$ give the following convergence (up to some subsequence)

$$U_k \rightarrow u \text{ uniformly on any compact set } K \subset \mathbb{R}^2.$$ 

$u$ solves the equation (1.1) and inherits the minimality from $U_k$. Finally by a diagonal argument we can get a further subsequence, which is still denoted by $\varepsilon_k$, such that for all $j \in \mathbb{N}^+$ the following limits hold true:

$$Q_j = \lim_{k \rightarrow \infty} Q_{j_{\varepsilon_k}}, \quad R_j = \lim_{k \rightarrow \infty} R_{j_{\varepsilon_k}}.$$

It is easy to verify that $u$ and $\{Q_j, R_j\}_{j=1}^{\infty}$ satisfy Property (a), (b), (c) in Theorem 1.2. The rest of this section is devoted to prove Property (d).

Let $u_t(z) := u(tz), \quad z \in \mathbb{R}^2, \ t > 0$.

Note that this is the blow-down scaling for large $t$.

**Claim.** For any $t_k \rightarrow \infty$, there is a subsequence (still denoted by $t_k$) such that

$$u_{t_k} \overset{L^1_{\text{loc}}(\mathbb{R}^2)}{\rightarrow} u_0 = \sum_{j=1}^{\bar{N}} \bar{a}_j \chi_{D_j},$$

where $\bar{a}_j \in \{W = 0\} = \{a_1, a_2, a_3\}, \ 1 \leq \bar{N} \leq 3, \ \mathcal{P} = \{D_j\}_{j=1}^{\bar{N}}$ is a minimal partition of $\mathbb{R}^2$, $\partial \mathcal{P}$ is a minimal cone, i.e. $\partial \mathcal{P}$ is scaling invariant.

**Proof of the Claim.** We follow the proof of Modica [25, Theorem 3] with the necessary modifications to fit our setting. First we establish the compactness in $L^1_{\text{loc}}(\mathbb{R}^2)$.

Since

$$E_t(u_t, B_r) := \int_{B_r} \left( \frac{1}{2t} |\nabla u_t|^2 + tW(u_t) \right) \, dz$$

$$= \frac{1}{t} \int_{B_{tr}} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \, dz \leq C r,$$

where $C$ is independent of $r$, by a well-known estimate for minimizers (cf. [6, Lemma 5.1]). By Baldo [8, Proposition 2.1], $\|u_t\|_{L^\infty(\mathbb{R}^2)} < C$, and (6.63) we obtain compactness in $L^1(B_r)$, and further in $L^1_{\text{loc}}(\mathbb{R}^2)$ by a diagonal argument. Utilizing [8 Proposition 2.2 & 2.4], we conclude that for $u_0$ as in (6.62), $\mathcal{P}$ is a minimal partition. To obtain that $\partial \mathcal{P}$ is a cone, following Modica, we repeat the argument by considering $u_0(tz)$ and the corresponding $\partial \mathcal{P}_t$. For this purpose we need the analog of Giusti [21, Theorem 9.3] for minimal partitions, which are defined as flat chains of top dimension (see Fleming [14] and White [37]). The key points are

1. the flat norm coincides with the mass norm which in turn equals the $L^1$ norm,
2. the compactness theorem for flat chains,
3. the monotonicity formula,
which all can be checked (see e.g. the expository paper \[2\]).

Thus the claim is established.

Since the only minimal cones of dimension one in \(\mathbb{R}^2\) are the straight line and the triod, it suffices to show that \(\partial P\) is a triod instead of a straight line.

We argue by contradiction. Suppose \(\partial P\) is a straight line, then after a possible rotation we assume
\[
\begin{align*}
& L^1_{\text{loc}}(\mathbb{R}^2) \\
& \Rightarrow u_k \\
& \{a_1, a_2\} \subset \{a_1, a_2, a_3\}, \quad D_1 = \{(x, y) : y \geq 0\}, \quad D_2 = \{(x, y) : y < 0\}.
\end{align*}
\]

By Caffarelli-Córdoba density argument, the \(L^1_{\text{loc}}\) convergence can be strengthened to uniform convergence away from \(\{y = 0\}\). Thus we have
\[
\forall \varepsilon, \text{ there exists a } R = R(\varepsilon) \text{ such that}
\]
\[
\text{dist}(u(z), a_1) \leq \varepsilon, \quad \forall z \in B_R \cap \{y \geq \varepsilon R\},
\]
\[
\text{dist}(u(z), a_2) \leq \varepsilon, \quad \forall z \in B_R \cap \{y \leq -\varepsilon R\},
\]
\[
\int_{B_R} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \, dy \, dx \leq R \sigma (2 + \varepsilon).
\]

Here the last estimate \(6.65\) follows from the estimate \(\sigma \mathcal{H}^1(\partial P \cap B_1) = \lim_{k \to \infty} E_k(u_k, B_1)\), thanks to the \(\Gamma\)-convergence result in Baldo\[8\] that holds also without the mass constraint (see Gazoulis\[10\]).

**Case 1.** When \(a_1 \in \{a_1, a_2\}\), without loss of generality we assume \(a_1 = a_1, a_2 = a_2\). Take \(\varepsilon\) to be a small constant whose value will be determined in the proof process. We will focus on the stripe \(\{z = (x, y) : x \in [-R, R], y \in [-\varepsilon R, \varepsilon R]\}\) (written as \([-R, R] \times [-\varepsilon R, \varepsilon R]\)) where \(R = R(\varepsilon)\).

By Property (b) in Theorem 1.2 there are points \(R_1, \ldots, R_N \in [-R, R] \times [-\varepsilon R, \varepsilon R]\) \((N \geq \frac{R}{2C_0})\) that satisfy
\[
|u(R_i) - a_3| \leq \gamma, \quad \forall 1 \leq i \leq N,
\]
\[
\text{dist}(R_i, R_j) \geq 6C_0, \quad \forall 1 \leq i, j \leq N,
\]
\[
|u(z) - a_3| \leq 2\gamma, \quad \forall z \in B(R_i, \frac{\gamma}{M}).
\]

Define
\[
X = \bigcup_{i=1}^{N} [x_i - \frac{\gamma}{M}, x_i + \frac{\gamma}{M}].
\]

Here \(x_i\) is the abscissa of \(R_i\).

If \(\mathcal{H}^1(X) \geq \delta R\), for some small \(\delta\) which will be determined later, then we can estimate
\[
E(u, B_R) = \int_{-R\sqrt{1-\varepsilon^2}}^{R\sqrt{1-\varepsilon^2}} \int_{-\varepsilon R}^{\varepsilon R} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \, dy \, dx
\]
\[
= \left( \int_X + \int_{[-R\sqrt{1-\varepsilon^2}, R\sqrt{1-\varepsilon^2}] \setminus X} \right) \, dx \left( \int_{-\varepsilon R}^{\varepsilon R} \left( \frac{1}{2} |\partial_y u|^2 + W(u) \right) \, dy \right)
\]
\[
\geq \delta R \cdot (2(2\sigma(1 - C\gamma^2)) + (2\sqrt{1 - \varepsilon^2}R - \delta R)\sigma(1 - C\varepsilon^2))
\]
\[
\geq R \sigma (2 + \delta - 2C\delta \gamma^2 - 2C\varepsilon^2)
\]
\[
\geq R \sigma (2 + \frac{\delta}{2}),
\]
where to get last inequality we have required
\[
2C\gamma^2 \leq \frac{1}{4}, \quad 2C\varepsilon^2 \leq \frac{\delta}{4}.
\]

Note that the first inequality in the estimate above is derived from the fact that when \(x \in X\), on the vertical line \(\{x\} \times [-\varepsilon R, \varepsilon R]\) \(u\) is \(\varepsilon\)-close to \(a_1, a_2\) at \((x, \varepsilon R), (x, -\varepsilon R)\) respectively, and is \(\gamma\)-close to \(a_3\) at some middle point \((x, y)\).

As a result, (6.66) contradicts with (6.65) if we take \(\varepsilon < \frac{\delta}{4}\).

Otherwise \(H^1(X) < \delta R\). We define
\[
\tilde{X} = \bigcup_{i=1}^{N} [x_i - 2C_0, x_i + 2C_0].
\]

Note that we can assume \(\gamma\) satisfies \(\frac{\gamma}{M} \leq C_0\). Then we have
\[
H^1(\tilde{X}) < \frac{2C_0 M}{\gamma} \cdot \delta R.
\]

For every \(R_i\), by Property (c) in Theorem 1.2 there exists a point \(P_i \in B(R_i, C_0)\) such that \(|u(P_i) - a_1| \leq \gamma\). It follows that for any \(z \in B(P_i, \frac{\gamma}{M})\), \(|u(z) - a_1| \leq 2\gamma\). This allows us to estimate the energy from below inside \(B(R_i, 2C_0)\). Actually after suitable translation and rotation we can assume
\[
R_i = (0, 0), \quad P_i = (x_i, 0), \quad \text{for some } x_i \in (0, C_0].
\]

For any \(y_0 \in [-\frac{\gamma}{M}, \frac{\gamma}{M}]\), the line segment \(\{(x, y_0) : 0 \leq x \leq x_i\}\) satisfies
\[
|u((0, y_0)) - a_3| \leq 2\gamma, \quad |u((x_i, y_0)) - a_1| \leq 2\gamma.
\]

Thus by calculating the energy along these line segments that “connect” \(B(P_i, \frac{\gamma}{M})\) and \(B(R_i, \frac{\gamma}{M})\), one has
\[
\int_{B(R_i, 2C_0)} \left( \frac{1}{2|\nabla u|^2 + W(u)} \right) dz
\geq \int_{-\frac{\gamma}{M}}^{\frac{\gamma}{M}} \int_{x_i}^{x} \left( \frac{1}{2|\partial_x u|^2 + W(u)} \right) dx dy
\geq \frac{2\gamma}{M} (\sigma - C\gamma^2) \geq \frac{\gamma}{M} \sigma.
\]
Note that since \( \text{dist}(R_i, R_j) \geq 6C_0 \) for any \( i \neq j \), all \( B(R_i, 2C_0) \) are pairwisely disjoint. We have

\[
\int_{B_R} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \, dz \\
\geq \int_{[-\sqrt{1-\varepsilon^2} R, \sqrt{1-\varepsilon^2} R]} \int_{-\varepsilon R}^{\varepsilon R} \left( \frac{1}{2} |\partial_y u|^2 + W(u) \right) \, dy \, dx \\
+ \sum_{i=1}^{N} \int_{B(R_i, 2C_0)} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) \, dz
\]

(6.68)

where in the last inequality we require that \( (2C + 1)\varepsilon^2 < \frac{\gamma^2}{128C_0 M} \) and \( \delta < \frac{\gamma^2}{256C_0 M} \). If we further require that \( \varepsilon < \frac{\sqrt{2}}{128C_0 M} \), then (6.68) contradicts with the upper bound (6.65).

**Case 2.** If \( \{a_2, a_3\} = \{a_1, \bar{a}_2\} \), then for any \( r \in (0, R) \), there exists \( P(r) \) such that \( |u(P(r)) - a_1| \leq \gamma \).

And \( P(r) \in [-R, R] \times [-\varepsilon R, \varepsilon R] \). Let

\[
\{P_i\}_{i=1}^{N} \quad \text{and} \quad \text{dist}(P_i, P_j) \geq 6C_0,
\]

\( \forall P_i, \exists z_i \in \overline{B}(P_i, C_0) \) such that \( \min\{|u(z_i) - a_2|, |u(z_i) - a_3|\} \leq \gamma \).

Then one can replace \( \{R_i\} \) by \( \{P_i\} \) in the proof of Case 1 and get a contradiction using the same arguments.

Therefore, we conclude that \( \bar{N} = 3 \) and that \( \partial P \) is a triod.

Finally (1.5) follows from [6, Proposition 5.6], and from \( \partial P \) being the triod. The proof of Theorem 1.2 is complete. \( \square \)

**Appendix A. Proof of Lemma 2.3**

For sufficiently small \( \varepsilon \), we want to construct an energy competitor \( u_{test} \in W^{1,2}(B_1, \mathbb{R}^2) \) with the boundary condition \( u_{test}|_{\partial B_1} = g_\varepsilon \) and show that

\[
(A.69) \quad \int_{B_1} \left( \frac{\varepsilon}{2} |\nabla u_{test}|^2 + \frac{1}{\varepsilon} W(u_{test}) \right) \, dz \leq 3\sigma + C\varepsilon,
\]

for a positive constant \( C \) independent of \( \varepsilon \).

Let \( c_0 \) be as in (2.17). Set

\[
r_1 := c_0 \varepsilon, \quad r_2 := 1 - c_0 \varepsilon.
\]
We define
\[ S(s, t; \theta_1, \theta_2) := \{(x, y) = (r \cos \theta, r \sin \theta) : s \leq r \leq t, \theta_1 \leq \theta \leq \theta_2\}, \]
\[ 0 \leq s < t \leq 1, \quad 0 \leq \theta_1 < \theta_2 \leq 2\pi. \]

Since \( W \) satisfies (H2), we only need to construct \( u_{test} \) and estimate its energy in the sector \( S(0, 1; \frac{\pi}{3}, \pi) \). Then one can do the same construction and estimate on the other two sectors \( S(0, 1; \frac{\pi}{3}, \frac{5\pi}{3}) \) and \( S(0, 1; 0, \frac{\pi}{3}) \cup S(0, 1; \frac{5\pi}{3}, 2\pi) \).

Recall that \( U_{12} \in \mathcal{W}_{1,2}^{1,2}(\mathbb{R}, \mathbb{R}^2) \) is the 1D minimizer of the minimization problem
\[
\min \int_{-\infty}^{\infty} \left( \frac{1}{2} |v'|^2 + W(v) \right) \, d\eta, \quad \lim_{x \to -\infty} v(\eta) = a_1, \quad \lim_{\eta \to \infty} v(\eta) = a_2, \quad v(\mathbb{R}) \subset \mathbb{R}^2 \setminus \{a_1, a_2\}.
\]

The properties of this 1D minimizer play an important role in our construction of \( u_{test} \).

Now we are ready to construct \( u_{test} \). On \( S(r_1, r_2; \frac{\pi}{3}, \frac{2\pi}{3}) \), we set
\[
u_{test} = U_{12}(\frac{r \sin(\frac{\pi}{2} - \theta)}{\varepsilon})
\]

By the exponential decay estimate for the minimizing connection \( U_{12} \) (see [4 Proposition 2.4]), we have the following estimates for \( u_{test} \) on \( \partial S(r_1, r_2; \frac{\pi}{3}, \frac{2\pi}{3}) \):

(A.70) \[ |u_{test}(r_1, \frac{\pi}{3}) - a_2| \leq Ke^{-k\frac{r}{\varepsilon}}, \quad \text{on } \{r_1 \leq r \leq r_2, \theta = \frac{\pi}{3}\} \]

(A.71) \[ |u_{test}(r_2, \frac{2\pi}{3}) - a_1| \leq Ke^{-k\frac{r}{\varepsilon}}, \quad \text{on } \{r_1 \leq r \leq r_2, \theta = \frac{2\pi}{3}\} \]

(A.72) \[ |u_{test}(r_1, \theta) - a_1| \leq Ke^{-k\varepsilon_0 \sin(\theta - \frac{\pi}{2})}, \quad \text{on } \{r = r_1, \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\}, \]

(A.73) \[ |u_{test}(r_1, \theta) - a_2| \leq Ke^{-k\varepsilon_0 \sin(\frac{\pi}{2} - \theta)}, \quad \text{on } \{r = r_1, \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}\}, \]

(A.74) \[ |u_{test}(r_2, \theta) - a_1| \leq Ke^{-r_2\varepsilon_0 \sin(\theta - \frac{\pi}{2})}, \quad \text{on } \{r = r_2, \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\}, \]

(A.75) \[ |u_{test}(r_2, \theta) - a_2| \leq Ke^{-r_2\varepsilon_0 \sin(\frac{\pi}{2} - \theta)}, \quad \text{on } \{r = r_2, \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}\}. \]

Here \( K, k \) are constants that are independent of \( \varepsilon \).

Note that in \( S(r_1, r_2; \frac{\pi}{3}, \frac{2\pi}{3}) \), \( u_{test} \) only depends on the \( x \) variable, thus \( |\partial_y u_{test}| = 0 \). We compute the energy
\[
\int_{S(r_1, r_2; \frac{\pi}{3}, \frac{2\pi}{3})} \left( \frac{\varepsilon}{2} |\nabla u_{test}|^2 + \frac{1}{\varepsilon} W(u_{test}) \right) \, dz
\]
\[
= \int_{S(r_1, r_2; \frac{\pi}{3}, \frac{2\pi}{3})} \left( \frac{\varepsilon}{2} |\partial_x u_{test}|^2 + \frac{1}{\varepsilon} W(u_{test}) \right) \, dz
\]
\[
\leq \int_{r_1}^{r_2} dy \int_{-\frac{\sqrt{3}y}{3}}^{\frac{\sqrt{3}y}{3}} \left( \frac{\varepsilon}{2} |\partial_x u_{test}|^2 + \frac{1}{\varepsilon} W(u_{test}) \right) \, dx
\]
\[
\leq (r_2 - r_1) \sigma
\]
\[
\leq (1 - c_0 \varepsilon) \sigma.
\]
Similarly we set respectively

\[ u_{\text{test}} = U_{31}\left(\frac{r \sin \left(\frac{2\pi}{3} - \theta\right)}{\varepsilon}\right), \quad \text{on } S(r_1, r_2; \pi, \frac{4\pi}{3}), \]

\[ u_{\text{test}} = U_{23}\left(\frac{r \sin \left(\frac{11\pi}{6} - \theta\right)}{\varepsilon}\right), \quad \text{on } S(r_1, r_2; \frac{5\pi}{3}, 2\pi). \]

In particular, when \( \theta = \pi \) it holds that

\[ |u_{\text{test}}(r, \pi) - a_1| \leq Ke^{-k_2 \frac{r}{\varepsilon}}, \quad \text{on } \{ r_1 \leq r \leq r_2, \theta = \pi \}. \]

With (A.71) and (A.77) we are able to define \( u_{\text{test}} \) on \( S(r_1, r_2; \frac{2\pi}{3}, \pi) \) by

\[ u_{\text{test}}(r, \theta) := u_{\text{test}}(r, \frac{2\pi}{3}) \frac{\pi - \theta}{\frac{2\pi}{3} - \theta} + u_{\text{test}}(r, \pi) \frac{\theta - \frac{2\pi}{3}}{\frac{2\pi}{3} - \theta}. \]

And this construction can be extended to \( S(r_1, r_2; \frac{2\pi}{3}, \pi) \) and \( S(r_1, r_2; 0, \frac{2\pi}{3}) \) in the same way. The energy in \( S(r_1, r_2; \frac{2\pi}{3}, \pi) \) will be estimated in polar coordinates.

\[ \int_{S(r_1, r_2; \frac{2\pi}{3}, \pi)} \left( \frac{\varepsilon}{2} \left| \nabla u_{\text{test}} \right|^2 + \frac{1}{\varepsilon} W(u_{\text{test}}) \right) \, dz \]

\[ = \int_{\frac{2\pi}{3}}^\pi \int_{r_1}^{r_2} \left( \frac{\varepsilon}{2} \left( \frac{\partial u_{\text{test}}}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial u_{\text{test}}}{\partial \theta} \right|^2 \right) + \frac{1}{\varepsilon} W(u_{\text{test}}) \right) r \, dr \, d\theta. \]

Note that

\[ \left| \frac{\partial u_{\text{test}}}{\partial r} \right|^2 \leq 2 \left( \frac{\partial u_{\text{test}}(r, \frac{2\pi}{3})}{\partial r} \right)^2 + \left| \frac{\partial u_{\text{test}}(r, \pi)}{\partial r} \right|^2 \leq \frac{1}{2\varepsilon^2} (|U_{12}'(r^2/2\varepsilon)|^2 + \left| U_{12}'(r^2/2\varepsilon) \right|^2) \leq \frac{Ce^{-k_2 \frac{r}{\varepsilon}}}{\varepsilon^2}, \]

\[ \left| \frac{\partial u_{\text{test}}}{\partial \theta} \right|^2 \leq Ce^{-k_2 \frac{r}{\varepsilon}}, \quad W(u_{\text{test}}(r, \theta)) \leq Ce^{-k_2 \frac{r}{\varepsilon}}, \]

where \( C \) is a universal constant.

Substituting these into (A.78) yields

\[ \int_{S(r_1, r_2; \frac{2\pi}{3}, \pi)} \left( \frac{\varepsilon}{2} \left| \nabla u_{\text{test}} \right|^2 + \frac{1}{\varepsilon} W(u_{\text{test}}) \right) \, dz \]

\[ \leq \int_{\frac{2\pi}{3}}^\pi \int_{c_0 \varepsilon}^{1-c_0 \varepsilon} \left( \frac{\varepsilon}{2} \left( \frac{Ce^{-k_2 \frac{r}{\varepsilon}}}{\varepsilon^2} + \frac{Ce^{-k_2 \frac{r}{\varepsilon}}}{r^2} \right) + \frac{1}{\varepsilon} Ce^{-k_2 \frac{r}{\varepsilon}} \right) r \, dr \, d\theta \]

\[ \leq C(W, c_0) \varepsilon, \]

for some constant \( C(W, c_0) \) that does not depend on \( \varepsilon \).

Now that \( u_{\text{test}} \) has been already defined on the annulus \( S(r_1, r_2; 0, 2\pi) \), we proceed to define \( u_{\text{test}} \) in the inner ball \( B_{r_1} \) and the outer layer \( S(r_2, 1; 0, 2\pi) \). First of all we take \( u_{\text{test}} \) to be the harmonic extension in \( B_{r_1} \), with respect to its boundary data.

\[ \Delta u_{\text{test}} = 0 \quad \text{in } B_{r_1}, \]

\[ u_{\text{test}}|_{\partial B_{r_1}} \] is given by the construction on \( S(r_1, r_2; 0, 2\pi) \).

It is not hard to verify that

\[ |u_{\text{test}}| \leq C, \quad |\nabla u_{\text{test}}| \leq \frac{C}{\varepsilon} \quad \text{on } \partial B_{r_1}. \]
where $\nabla_T$ denotes the tangential derivative, $C = C(W, c_0)$ is a constant independent of $\varepsilon$. By elliptic regularity, $|u_{test}|$ and $\varepsilon |\nabla u_{test}|$ are also bounded by some universal constant $C$ inside $B_{r_1}$. Then we have

$$\int_{B_{r_1}} \left( \frac{\varepsilon}{2} |\nabla u_{test}|^2 + \frac{1}{\varepsilon} W(u_{test}) \right) \, dz \leq \pi(c_0 \varepsilon)^2 \left( \frac{\varepsilon}{2} \frac{C}{\varepsilon} \right)^2 + \frac{1}{\varepsilon} C \, dx \leq C(W, c_0) \varepsilon. \tag{A.80}$$

It remains to construct $u_{test}$ on the annulus $S(r_2, 1; 0, 2\pi)$. Set

$$u_{test}(r, \theta) = \frac{1 - r}{c_0 \varepsilon} u_{test}(r_2, \theta) + \frac{r - r_2}{c_0 \varepsilon} g_\varepsilon(\theta), \quad r_2 \leq r \leq 1, \quad \theta \in [0, 2\pi). \tag{A.81}$$

Here $g_\varepsilon$ is the boundary data on $\partial B_1$ defined by (2.17) and $u_{test}(r_2, \theta)$ is given by the construction of $u_{test}$ on $\partial S(r_1, r_2; 0, 2\pi)$. We have

$$\int_{S(r_2, 1; \frac{\pi}{3}, \pi)} \left( \frac{\varepsilon}{2} |\nabla u_{test}|^2 + \frac{1}{\varepsilon} W(u_{test}) \right) \, dz \tag{A.82}$$

$$= \left( \int_{\frac{\pi}{3} + c_0 \varepsilon}^{\frac{2\pi}{3} + c_0 \varepsilon} + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} + \int_{\frac{2\pi}{3} + c_0 \varepsilon}^\pi \right) \int_{r_2}^1 \left( \frac{\varepsilon}{2} \left( \frac{\partial u_{test}}{\partial r}^2 + \frac{1}{r^2} \frac{\partial u_{test}}{\partial \theta}^2 \right) + \frac{1}{\varepsilon} W(u_{test}) \right) r \, dr \, d\theta.$$

We estimate each part separately. In $S(r_2, 1; \frac{\pi}{3} - c_0 \varepsilon, \frac{\pi}{3} + c_0 \varepsilon)$, it holds that

$$\int_{\frac{\pi}{3} - c_0 \varepsilon}^{\frac{2\pi}{3} + c_0 \varepsilon} \int_{r_2}^1 \left( \frac{\varepsilon}{2} \left( \frac{\partial u_{test}}{\partial r}^2 + \frac{1}{r^2} \frac{\partial u_{test}}{\partial \theta}^2 \right) + \frac{1}{\varepsilon} W(u_{test}) \right) r \, dr \, d\theta \leq 2|c_0 \varepsilon|^2 \cdot \left( \frac{\varepsilon}{2} \left( \frac{C}{\varepsilon} \right)^2 + \frac{1}{\varepsilon} C \right) \leq C \varepsilon;$$

In $S(r_2, 1; \frac{\pi}{3}, \frac{2\pi}{3} - c_0 \varepsilon) \cup S(r_2, 1; \frac{\pi}{3} + c_0 \varepsilon, \frac{2\pi}{3})$, by (A.74) and (A.75) we have

$$\int_{\frac{\pi}{3} - c_0 \varepsilon}^{\frac{2\pi}{3} + c_0 \varepsilon} \int_{r_2}^1 \left( \frac{\varepsilon}{2} \left( \frac{\partial u_{test}}{\partial r}^2 + \frac{1}{r^2} \frac{\partial u_{test}}{\partial \theta}^2 \right) + \frac{1}{\varepsilon} W(u_{test}) \right) r \, dr \, d\theta \leq C \int_{r_2}^1 \frac{1}{\varepsilon} e^{-k \sin \theta} \cdot \frac{1}{\varepsilon} r \, dr \, d\theta \leq C \varepsilon;$$

In $S(r_2, 1; \frac{2\pi}{3}, \pi)$, we note that

$$u_{test}(r, \theta) - a_1 = \frac{r - r_2}{c_0 \varepsilon} \left( a_1 \left( u_{test}(r_2, \frac{2\pi}{3}) - a_1 \frac{\pi}{\frac{2\pi}{3}} \right) + (u_{test}(r_2, \pi) - a_1 \theta \frac{\pi}{\frac{2\pi}{3}}) \right),$$

which implies that

$$\int_{r_2}^\pi \int_{r_2}^1 \left( \frac{\varepsilon}{2} \left( \frac{\partial u_{test}}{\partial r}^2 + \frac{1}{r^2} \frac{\partial u_{test}}{\partial \theta}^2 \right) + \frac{1}{\varepsilon} W(u_{test}) \right) r \, dr \, d\theta \leq C e^{-\frac{k}{\varepsilon}} \leq C \varepsilon.$$
Therefore, (A.82) becomes

\[(A.83) \quad \int_{S(r_{2};1,\frac{\pi}{3},\pi)} \left( \frac{\varepsilon}{2} |\nabla u_{test}|^2 + \frac{1}{\varepsilon} W(u_{test}) \right) \, dz \leq C\varepsilon.\]

Finally, using (A.78), (A.79), (A.80), (A.83) we conclude that

\[\int_{S(0,1,\frac{\pi}{3},\pi)} \left( \frac{\varepsilon}{2} |\nabla u_{test}|^2 + \frac{1}{\varepsilon} W(u_{test}) \right) \, dz \leq \sigma + C(W,c_0)\varepsilon.\]

The energies on \(S(0,1,\pi,\frac{5\pi}{3})\) and \(S(0,1;0,\frac{\pi}{3}) \cup S(0,1;\frac{5\pi}{3},2\pi)\) satisfy the same estimate. Adding them up leads to (2.18), which completes the proof.

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