Singular Perturbation Analysis for a Coupled KdV–ODE System

Swann Marx ©, Member, IEEE, and Eduardo Cerpa ©, Member, IEEE

Abstract—Asymptotic stability is with no doubts an essential property to be studied for any system. This analysis often becomes very difficult for coupled systems and even harder when different time-scales appear. The singular perturbation method allows to decouple a full system into what are called the reduced-order system and the boundary layer system to get simpler stability conditions for the original system. In the infinite-dimensional setting, we do not have a general result making sure this strategy works. This article is devoted to this analysis for some systems coupling the Korteweg-de Vries equation and an ordinary differential equation with different time scales. More precisely, we obtain stability results and Tikhonov-type theorems.

Index Terms—Automatic control, distributed parameter systems, partial differential equations (PDEs).

I. INTRODUCTION

This article is devoted to the stability analysis of a system composed by a linear Korteweg-de Vries (for short KdV) equation coupled with a scalar ordinary differential equation (ODE) with different time scales. Such a situation may appear when the control (appearing in the ODE) can only be used through a dynamics (given by the ODE), and when one of the equations is faster than the other one. More precisely, we are interested in the system

\[
\begin{align*}
\varepsilon y_t + y_x + y_{xxx} &= 0, \quad (t, x) \in \mathbb{R}_+ \times [0, L] \\
y(t, 0) &= y(t, L) = 0, \quad t \in \mathbb{R}_+ \\
y_x(t, L) &= az(t), \quad t \in \mathbb{R}_+ \\
y(0, x) &= y_0(x), \quad x \in [0, L] \\
z(t) &= bz(t) + cy_x(t, 0), \quad t \in \mathbb{R}_+ \\
z(0) &= z_0
\end{align*}
\]

(1)

and the system

\[
\begin{align*}
y + y_x + y_{xxx} &= 0, \quad (t, x) \in \mathbb{R}_+ \times [0, L] \\
y(t, 0) &= y(t, L) = 0, \quad t \in \mathbb{R}_+ \\
y_x(t, L) &= az(t), \quad t \in \mathbb{R}_+ \\
y(0, x) &= y_0(x), \quad x \in [0, L] \\
\varepsilon z(t) &= bz(t) + cy_x(t, 0), \quad t \in \mathbb{R}_+ \\
z(0) &= z_0
\end{align*}
\]

(2)

where \(a, b, c \in \mathbb{R}\) and \(\varepsilon > 0\). The parameter \(\varepsilon\) is supposed to be small, meaning that in (1) the KdV equation is faster than the ODE, and in (2), the ODE is faster than the KdV equation. Such a situation may appear when considering a control system where the output is given by \(y_x(t, 0)\) and the control is driven by the dynamics, faster, or slower than the \(y\)-dynamics, of \(z\). This may happen with integral actions, which corresponds to the case where \(b = 0\) and \(c = 1\), and where the fast dynamics is the controller one, as explained in [20]. This case has been developed in [3] with different techniques. To analyze these systems from an asymptotic stability viewpoint, we will follow techniques borrowed from the singular perturbation literature (see, e.g., [18], [19], and [20] for the finite-dimensional case, and [11], [34], and [35] for the infinite-dimensional case). Roughly speaking, this technique proposes to decouple the full system into two approximated systems assuming that \(\varepsilon\) is sufficiently small. The approximated slow system is called the reduced-order system while the approximated fast one is called the boundary layer system. It is known that, in the finite-dimensional case, if both systems are asymptotically stable, then the full system is asymptotically stable as well for sufficiently small \(\varepsilon\), and this can be applied for instance in context of output regulation as illustrated in [20]. In general, this is no longer the case in the infinite-dimensional case, as illustrated in [10] and [34] for some hyperbolic equations coupled with an ODE. Therefore, the singular perturbation techniques become very challenging for infinite-dimensional systems, even in the linear case.

It is worth noticing that our analysis could be also applied in the case where \(a, b, c\) would be matrices. Stability conditions on these parameters would hold on their matrix norms. To ease the reading of this article, we choose to avoid such technicalities. Regarding the partial differential part of our systems, we note that even in the case where the KdV equation is not coupled with any ODE, the asymptotic stability analysis is not trivial at all. Indeed, if \(L \in \mathcal{N}\), with

\[
\mathcal{N} := \left\{ 2\pi \sqrt{k^2 + kl + l^2} : k, l \in \mathbb{N} \right\}
\]

(3)
then the equilibrium point 0 of the KdV equation becomes stable, but not attractive, while, if \( L \notin \mathcal{N} \), 0 is exponentially stable. In fact, this is linked to a lack of observability of the output defined as \( y_\varepsilon(t,0) \). With Neumann boundary control (i.e., a control that is acting on \( y_\varepsilon(t,L) \)), the system is not controllable if \( L \in \mathcal{N} \), as shown in [30]. However, when looking at the nonlinear version of the KdV equation, one has better controllability results for any \( L \in \mathcal{N} \) [5, 8, 13, 14] and better stability results for some \( L \in \mathcal{N} \) [12, 16, 26, 32]. In addition to these interesting results, let us mention [2], [7], [15], [22], and [33], which propose to apply the backstepping method on the KdV equation with various boundary control, [9] where a feedback law is designed thanks to a Gramian methodology, [23] that deals with a saturated distributed control, [6] and [31] that propose both a survey about the KdV equation, or [3] where a PI controller is designed to achieve output regulation. This latter article is interesting because it is based on the forwarding method (see, e.g., [24] for the finite-dimensional case, and [21], [36], and [37] for some extensions to the infinite-dimensional case), which requires the existence of an input to state stability (ISS)-Lyapunov functional (see, e.g., [25] for an introduction on ISS in the infinite-dimensional context). In [3], an ISS-Lyapunov functional is built thanks to some strictification technique borrowed from [29] at the price of assuming that \( L \notin \mathcal{N} \). This Lyapunov functional, which was not available before [3], will be crucial to analyze (1) and (2) following the classical procedure of the singular perturbation analysis. Hence, all along this article, we will assume that

\[
L \notin \mathcal{N}. \tag{4}
\]

It is important to note that our technique, in this article, relies not only on an existing Lyapunov functional, but also on the forwarding method, which is known to be a systematic method for the analysis of system of coupled ODEs [24]. We believe, therefore, that this method, being systematic, and for which some extensions are known in the infinite-dimensional setting [21], could be used to deduce more general results for coupled partial differential equations (PDEs) with different time scales. The conditions would be stronger than in the finite-dimensional case, due to the existing counter examples provided in [10] and [34].

Using Lyapunov functionals is not the unique method to analyze asymptotic stability of coupled systems with different time scales, either infinite dimensional or finite dimensional. The linear case, in particular, can be treated with a frequency-domain approach; see, e.g., [19] for systems of coupled ODEs and [1] for hyperbolic PDEs coupled with ODEs. We may also mention articles dealing with output regulation for infinite-dimensional systems, where PDEs coupled with ODEs are treated with a frequency-domain approach, but where the singular perturbation approach is not used; see, e.g., [27]. Optimal results can be deduced from this approach, and in particular, the counter examples, provided in [10] and [34], of the singular perturbation method in the infinite-dimensional setting can be found using the frequency-domain approach. We believe that such a method could be used in our case, but, to obtain precise results, we would have needed a complete description of the spectrum of the linear KdV, which is not available yet, but which is also a topic that would deserve a full analysis. Moreover, since the original KdV equation is nonlinear, having a Lyapunov approach is fruitful, since one can easily deduce a (local) asymptotic stability for the nonlinear system from its linearized version. In this article, we do not treat the nonlinear case, to avoid technicalities that would make this article too complex to follow, but one can follow the analysis done in [3] to deduce a result for the nonlinear version of the KdV equation.

In this article, we have several contributions. First, for each coupled system (1) and (2), we propose some conditions on the parameters \( a, b, \) and \( c \) so that the exponential stability is ensured for any \( \varepsilon > 0 \). For each of the systems, different conditions will be given, because we are going to use different Lyapunov functionals for (1) and (2). Second, for each coupled system (1) and (2), we apply the singular perturbation analysis to find the boundary layer system and the reduced-order system. The stability of these subsystems will imply the stability of the original system as soon as \( \varepsilon \) is small enough. Third, for each coupled system (1) and (2), we provide an analysis of the asymptotic behavior of the solutions with respect to \( \varepsilon \) by obtaining some Tikhonov theorems. To the best of our knowledge, this is the first time that a singular perturbation analysis is applied on a KdV equation, from a control viewpoint. Moreover, we are able in this article to treat the case where the ODE is fast (while, in the case where the wave equation is coupled with a fast ODE, counter-example is given, as illustrated in [10]), but at the price of assuming regular initial conditions.

The rest of this article is organized as follows. Section II is devoted to state and prove the well-posedness and stability results for (1) and (2) for any value of the parameter \( \varepsilon \). In Sections III and IV, we provide an asymptotic analysis of (1) and (2), respectively, by applying singular perturbation analysis for small values of the parameter \( \varepsilon \). Finally, Section V concludes this article. Appendix recalls a crucial result borrowed from [3] for the KdV equation subject to disturbances.

II. Analysis for Any Value of \( \varepsilon \)

A. Well-Posedness

This short section deals with the well-posedness of (1) and (2) for any parameter \( a, b, \) and \( c \). We state and prove that there exists a unique solution to both equations. Our proof relies on classical semigroup arguments. Without loss of generality, we assume that \( \varepsilon = 1 \), because, in the well-posedness proof, this parameter does not play any role. Thus, we can deal in a unified way with both systems (1) and (2) studying

\[
\begin{align*}
&y_t + y_x + y_{xxx} = 0, \quad (t, x) \in \mathbb{R}_+ \times [0, L] \\
&y(t, 0) = y(t, L) = 0, \quad t \in \mathbb{R}_+ \\
&y_x(t, L) = a\varepsilon(t), \quad t \in \mathbb{R}_+ \\
&y(0, x) = y_0(x), \quad x \in [0, L] \\
&\dot{z}(t) = bz(t) + cy_x(t, 0), \quad t \in \mathbb{R}_+ \\
&z(0) = z_0.
\end{align*}
\tag{5}
\]

Theorem 1: Let \( a, b, c \in \mathbb{R} \). For any initial condition \((y_0, z_0)\) \(\in H^3(0, L) \times \mathbb{R}\) satisfying the compatibility conditions \(y_0(0) = y_0(L) = y_0(L) = a\varepsilon_0\), there exists a unique strong solution \((y, z) \in C([\mathbb{R}_+; H^3(0, L) \times \mathbb{R}) \cap C^1([\mathbb{R}_+; L^2(0, L) \times \mathbb{R}])

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In addition, for any initial condition \((y_0, z_0) \in L^2(0, L) \times \mathbb{R}\), there exists a unique weak solution \((y, z) \in C([0, T); L^2(0, L) \times \mathbb{R})\) to system \((5)\).

**Proof:** Applying [17, Corollary 2.2.3], we will prove the well-posedness of \((5)\). To do so, we focus on the operator
\[
A : D(A) \subset L^2(0, L) \times \mathbb{R} \to L^2(0, L) \times \mathbb{R}
\]
where
\[
D(A) := \{(y, z) \in H^3(0, L) \times \mathbb{R} \mid y(0) = y(L) = 0, y'(L) = az\}
\]
and
\[
A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y' - y'' \\ bz + cy'(0) \end{pmatrix}.
\] (6)

Our goal is to prove that there exists \(\omega > 0\) such that \(A - \omega I_{L^2(0, L)}\) and its adjoint operator generate a strongly continuous semigroup of contractions, where \(I_{L^2(0, L)}\) denotes the identity operator in \(L^2(0, L)\). As explained in [17, Corollary 2.3.3], and noticing moreover that \(A\) is a closed operator, following the proofs given in [4, Th. A.1.], such a condition is sufficient to prove that \(A\) generates a strongly continuous semigroup. Consider in \(L^2(0, L) \times \mathbb{R}\), the scalar product
\[
\langle y_1, z_1 \rangle, \langle y_2, z_2 \rangle = \int_0^L y_1 y_2 dx + z_1 z_2.
\] (7)

One has for all \((y, z) \in D(A)\)
\[
\langle A \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle = \langle \begin{pmatrix} -y' - y'' \\ bz + cy'(0) \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle
\] (8)
meaning that
\[
\langle A \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle = -\int_0^L y(y'' + y')dy + z(bz + cy'(0)).
\] (9)

Doing some integrations by parts, one obtains for all \((y, z) \in D(A)\)
\[
\int_0^L y'y'
dx = 0, \int_0^L yy''dx = \frac{1}{2} (y'(0)^2 - y(L)^2).
\] (10)

Therefore, using the definition of the domain \(D(A)\), one has for all \((y, z) \in D(A)\)
\[
\langle A \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle = \frac{1}{2} a^2 z^2 - \frac{1}{2} y'(0)^2 + bz^2 + ccy'(0).
\] (11)

Using Young’s Lemma, one obtains, for \((y, z) \in D(A)\)
\[
\langle A \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle \leq \frac{1}{2} a^2 z^2 - \frac{1}{2} y'(0)^2 + bz^2 + \frac{1}{2\alpha} c^2 z^2
\]
\[+ \frac{\alpha}{2} y'(0)^2.\] (12)

If one takes \(\alpha = 1\), one can prove easily that there exists a positive constant \(C\) such that for all \((y, z) \in D(A)\)
\[
\langle A \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle \leq C \left(\|y\|^2_{L^2(0, L)} + z^2\right).
\] (13)

Then, for any \(\omega > C\), one has that \(A - \omega I_{L^2(0, L)}\) is dissipative.

One can prove that the adjoint operator of \(A\), denoted by \(A^*\), is defined as
\[
A^* : D(A^*) \subset L^2(0, L) \times \mathbb{R} \to L^2(0, L) \times \mathbb{R}
\]
where \(D(A^*) := \{(y, z) \in H^3(0, L) \times \mathbb{R} \mid y(0) = y(L), y'(0) = cz\}\) and
\[
A^* \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y' - y'' \\ bz + ay'(L) \end{pmatrix}.
\] (14)

As a sketch of proof, to compute the adjoint operator, we have used the identity
\[
\langle A \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle = \langle \begin{pmatrix} y \\ z \end{pmatrix}, A^* \begin{pmatrix} y \\ z \end{pmatrix} \rangle.
\] (15)

Using the same scalar product than before, and performing some integrations by parts, one has for all \((y, z) \in D(A^*)\) that
\[
\langle A^* \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle = \frac{1}{2} c^2 z^2 - \frac{1}{2} y'(L)^2 + bz^2 + az y'(L).
\] (16)

Again, thanks to the Young’s inequality, one can prove that
\[
\langle A^* \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle \leq c^2 z^2 - y'(L)^2 + bz^2 \frac{1}{2\alpha} a^2 z^2
\]
\[+ \frac{\alpha}{2} y'(L)^2.\] (17)

Setting \(\alpha = 1\), one can prove that there exists a positive constant \(C\) such that for all \((y, z) \in D(A^*)\)
\[
\langle A^* \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \rangle \leq C \left(\|y\|^2_{L^2(0, L)} + z^2\right).
\] (18)

Then, for any \(w > C\), one can prove that \(A^* - \omega I_{L^2(0, L)}\) is dissipative. Then, applying the Lumer-Phillips Theorem [28, Corollary 4.4, Ch. 1], one can deduce the result.

## B. Stability Conditions for System \((1)\) and \((2)\)

Before starting this subsection, we emphasize on the fact that, all along this article, we will need the following assumption.

**Assumption 2:** We suppose that \(L \notin \mathcal{N}\).

Indeed, in order to use Proposition 11, provided in the Appendix, such an assumption is crucial.

In this subsection, we provide two sets of sufficient conditions ensuring exponential stability for \((1)\) and \((2)\) for every \(\varepsilon > 0\). We first prove that, using a change of variable, studying the stability of \((1)\) or \((2)\) is similar for large \(\varepsilon\). Indeed, if one sets \(\tau = \frac{1}{\varepsilon}\), \((1)\) becomes
\[
\begin{cases}
y_{\tau} + y_{\tau \tau} + y_{\tau \tau \tau} = 0, \quad (\tau, x) \in \mathbb{R}_{+} \times [0, L] \\
y(\tau, 0) = y(\tau, L) = 0, \quad \tau \in \mathbb{R}_{+} \\
y_{\tau}(\tau, L) = az(\tau), \quad \tau \in \mathbb{R}_{+} \\
\frac{1}{\varepsilon^2} y_{\tau \tau} = bz(\tau) + cy_{\tau}(\tau, 0), \quad \tau \in \mathbb{R}_{+}
\end{cases}
\]
which corresponds to \((2)\) for large \(\varepsilon\). Therefore, studying the stability of \((1)\) is equivalent to studying the stability of \((2)\) for large \(\varepsilon\). This means that the sets of sufficient conditions that we
will state later one hold also either for (2) or (1) as soon as ε is large enough.

As mentioned earlier, we fix any ε > 0 and give some conditions on a, b, and c such that the origin is globally exponentially stable for the system (1). These conditions are inspired by the singular perturbation method: we assume that the fast system (here, the KdV equation), without coupling (here, assuming that a = 0), is already exponentially stable (here, L ∈ N). To do so, we have to first introduce a Lyapunov functional, inspired by [3]. This Lyapunov functional has been built thanks to the forwarding method, first designed for finite-dimensional systems [24], and later extended to some infinite-dimensional systems [3], [21], [36], [37]. It is defined as

\[ V_1(y, z) = εW(y) + \frac{1}{2} \left( ε \int_0^L M(x) y(t, x) dx - z(t) \right)^2 \tag{19} \]

where W comes from [3, Th. 2.3]. We recall this result in the Appendix. The function M is the solution to the boundary value problem

\[
\begin{align*}
M''(x) + M'(x) &= 0, & x & \in (0, L) \\
M(0) = M(L) &= 0, & M'(0) &= -c
\end{align*}
\tag{20}
\]

for which we know an explicit solution

\[ M(x) = 2c \frac{\sin \left( \frac{x}{2} \right) \sin \left( \frac{L-x}{2} \right)}{\sin \left( \frac{L}{2} \right)} \in C^\infty([0, L]). \tag{21} \]

This function M is defined through a Sylvester equation, as explained in [3]. Roughly speaking, the idea of the Lyapunov functional defined in (19) is to use the fact that the fast system (i.e., the KdV equation) is already exponentially stable without coupling terms and to add a term in the functional such that z converges to the \( L^2 \)-norm of y (modulo the function M, suitably chosen). The idea is to take advantage of the exponential stability of the KdV equation. This corresponds exactly to the forwarding method.

As proved in [3], this Lyapunov functional is equivalent to the usual norm, i.e., one has the following lemma whose proof is given for the sake of completeness.

**Lemma 3:** There exist \( \varphi_1, \psi_1 > 0 \) such that

\[ \varphi_1 \left( \| y \|^2_{L^2(0, L)} + |z|^2 \right) \leq V_1(y, z) \leq \psi_1 \left( \| y \|^2_{L^2(0, L)} + |z|^2 \right) \tag{22} \]

with

\[ \varphi_1 = \max(ε\varphi + ε^2 M^2(y, z)) \quad \text{and} \quad \psi_1 = \min \left( \frac{ε^2}{2} \epsilon \int_0^L z^2 dx \left( \frac{z}{z^2 + ε^2 M^2(y, z)} \right) \right). \]

Moreover, for \( \epsilon \leq 1 \), one has the existence of a constant \( C > 0 \) such that

\[ \epsilon \left( \| y \|^2_{L^2(0, L)} + |z|^2 \right) \leq V_1(y, z) \leq C \left( \| y \|^2_{L^2(0, L)} + |z|^2 \right). \tag{23} \]

This result is proved in the Appendix.

We are now ready to state and prove our stability result.

**Proposition 4:** Suppose that Assumption 2 holds true. For any \( \epsilon > 0 \), there exist positive constants \( a_\epsilon, k_\epsilon, \) and \( k_\epsilon \) such that, if \( a < a_\epsilon \) and \( b, c \) satisfy \( 0 < k_\epsilon < ac - b < k_\epsilon \), then the origin is globally exponentially stable for the system (1).

**Proof:** Using Proposition 11, setting \( d_1 = 0 \) and \( d_2 = z \), the time derivative of \( V \) along the strong solutions to (1) yields

\[ \frac{d}{dt} V_1(y, z) \leq -\lambda \left( \| y \|^2_{L^2(0, L)} + k_\epsilon^2 z(t)^2 \right) \]

\[ + \left( \int_0^L M(y(t, x) + y_{xx}(t, x)) - b z(t) - c y_x(t, 0) \right) \]

\[ \times \left( \epsilon \int_0^L M(y(t, x) dx - z(t) \right). \tag{24} \]

After some integration by parts, and using in particular that \( M'(L) = c \) thanks to (21), one obtains that for all strong solutions to (1), we have

\[ \frac{d}{dt} V_1(y, z) \leq -\lambda \left( \| y \|^2_{L^2(0, L)} + k_\epsilon^2 z(t)^2 \right) \]

\[ - (b - ac) z(t) \left( \epsilon \int_0^L M(y(t, x) dx - z(t) \right) \]

\[ \leq -\lambda \left( \| y \|^2_{L^2(0, L)} + k_\epsilon^2 z(t)^2 + (b - ac) z(t)^2 \right) \]

\[ - \epsilon (b - ac) z(t) \int_0^L M(y(t, x) dx. \tag{25} \]

Using Young’s Lemma, one obtains that for all strong solutions to (1)

\[ \frac{d}{dt} V_1(y, z) \leq - \left( \lambda + \frac{\alpha}{2} \epsilon^2 \| M \|^2_{L^2(0, L)} \right) \| y \|^2_{L^2(0, L)} \]

\[ + \left( \frac{(b - ac)^2}{\alpha} + (b - ac) + k_\epsilon^2 \right) z(t)^2. \tag{26} \]

Let us choose \( \alpha = \frac{1}{\| M \|^2_{L^2(0, L)}}. \) One has, therefore,

\[ \frac{d}{dt} V_1(y, z) \leq - \lambda \| y \|^2_{L^2(0, L)} \]

\[ + \left( \frac{(b - ac)^2}{\alpha} + (b - ac) + k_\epsilon^2 \right) z(t)^2. \tag{27} \]

Let us consider the polynomial \( \frac{a^2}{\alpha} - X + k_\epsilon^2 \). If \( a^2 < \frac{\alpha}{4k_\epsilon^2} \), this polynomial admits two square roots, defined by

\[ X_1 = \frac{\alpha (1 - \sqrt{1 - \frac{4k_\epsilon^2}{\alpha}})}{2}, \quad X_2 = \frac{\alpha (1 + \sqrt{1 - \frac{4k_\epsilon^2}{\alpha}})}{2}. \]

Then, if \( b - ac \) satisfies

\[ X_1 < -(b - ac) < X_2 \]

then, there exists a positive constant \( \mu \) such that for all strong solutions to (1), we have

\[ \frac{d}{dt} V_1(y, z) \leq -\mu V_1(y, z). \tag{28} \]

Using Lemma 3, we conclude the proof.
One might see this result as an extension of the one provided in [3], where one has \( b = 0 \) and \( c = \varepsilon = 1 \), which corresponds to the case where an integrator is added. In [3], it is proved that, for a sufficiently small \( a \), the origin of (1) (with \( b = 0, c = \varepsilon = 1 \)) is exponentially stable. Therefore, Proposition 4 seems to follow the same line, since \( a \) has to be sufficiently small.

Now, we provide a sufficient condition on \( a, b, \) and \( c \) will be found to ensure the stability of (2) for any \( \varepsilon > 0 \) or for (1) for sufficiently large \( \varepsilon > 0 \). As in the case of Proposition 4, this sufficient condition assumes that, without coupling, the ODE in (2) is already exponentially stable, meaning that \( b < 0 \). To do so, we use the Lyapunov functional

\[
V_2(y, z) := -\frac{\varepsilon \kappa_2 a^2}{b} z^2 + W(y)
\]

where \( W \) is the ISS-Lyapunov functional given in Proposition 11 provided in the Appendix. The constant \( \kappa_2 \) is also given in the Appendix. The following Lemma states that this Lyapunov functional is equivalent to the usual norm.

**Lemma 5:** For any \( b < 0 \), defining \( \nu_2 := \max(\varepsilon, -\frac{\varepsilon \kappa_2 a^2}{b}) \) and \( \nu_2 := \min(\varepsilon, -\frac{\varepsilon \kappa_2 a^2}{b}) \), where the constant \( \kappa_2 \) is given in Proposition 11 provided in the Appendix, the Lyapunov functional defined in (29) satisfies

\[
\nu_2(\|y\|_{L^2(0, L)} + t, t, L) V_2(y, z) \leq \nu_2(\|y\|_{L^2(0, L)} + |z|^2).
\]

We postpone the proof of this Lemma in the Appendix. We have now the following result, which states that, for any \( \varepsilon > 0 \), and under suitable conditions on \( a, b, c \), the origin is exponentially stable for the system (2). As explained later on, these conditions differ from the ones collected in Proposition 4.

**Proposition 6:** Suppose that Assumption 2 holds true. Let \( \varepsilon > 0 \). If \( b < 0 \) and \( \frac{c^2 \varepsilon^2}{b^2} < \frac{\kappa_3}{\kappa_2} \), then the origin is exponentially stable for the system (2).

**Proof:** Note that, due to the condition \( b < 0 \), the Lyapunov functional defined in (29) is equivalent to the usual norm, invoking Lemma 5. Using Proposition 11 with \( d_2 = a z \), its derivative along (2) yields, for all strong solutions to (2)

\[
\frac{d}{dt} V_2(y, z) = -\lambda \|y\|_{L^2(0, L)}^2 + \kappa_2 a^2 |z(t)|^2 - \kappa_3 |y_x(t, 0)|^2 - 2 \kappa_2 a^2 |z(t)|^2 - 2 \frac{c^2 \varepsilon^2}{b^2} \kappa_2 |y_x(t, 0)|^2.
\]

Using Young’s Lemma, one obtains

\[
\frac{d}{dt} V_2(y, z) \leq -\lambda \|y\|_{L^2(0, L)}^2 - \kappa_2 (1 - \alpha) a^2 z^2 - \left( \frac{\kappa_3}{\kappa_2} - \frac{c^2 \varepsilon^2}{b^2} \right) |y_x(t, 0)|^2.
\]

Setting \( \alpha = \frac{1}{2} \), one obtains

\[
\frac{d}{dt} V_2(y, z) \leq -\lambda \|y\|_{L^2(0, L)}^2 - \frac{\kappa_2}{2} a^2 z^2 - \left( \frac{\kappa_3}{\kappa_2} - \frac{c^2 \varepsilon^2}{b^2} \right) |y_x(t, 0)|^2.
\]

Then, if \( \frac{c^2 \varepsilon^2}{b^2} < \frac{\kappa_3}{2 \kappa_2} \), and using Lemma 5, the desired result holds true, concluding, therefore, the proof.

Note that the conditions given in Proposition 4 are quite different from the ones introduced in Proposition 6. Indeed, in contrast with Proposition 4, Proposition 6 assumes, with the hypothesis \( b < 0 \), that the ODE is already exponentially stable. As it will be illustrated later on, similar conditions will appear when looking at the reduced-order system and the boundary layer system.

**III. FAST KdV EQUATION COUPLED WITH A SLOW ODE**

**A. Stability for Small \( \varepsilon \)**

The singular perturbation method proposes the decoupling of the different time scales appearing in the system in order to get some subsystems that hopefully can be studied separately in order to conclude properties of the full system. Thus, we are going to compute the subsystems, namely the reduced-order system and the boundary layer system, that are approximations of the KdV equation and the ODE when \( \varepsilon \) is closed to 0. We further prove that the stability conditions for those two systems apply for the full system (1) as soon as \( \varepsilon \) is small enough.

1) **Reduced-Order System:** Finding the reduced-order system needs us to suppose that \( \varepsilon = 0 \). One has, therefore, to study this system

\[
\begin{cases}
    h_x(t, x) + h_{xxx}(t, x) = 0, & t \in \mathbb{R}_+, x \in (0, L)\\
h(t, 0) = h(t, L) = 0, & t \in \mathbb{R}_+\\
h_x(t, t) = az(t), & t \in \mathbb{R}_+.
\end{cases}
\]

which corresponds to the KdV equation given in (1) when \( \varepsilon = 0 \). There exists an explicit solution to the latter equation given by for all \((t, x) \in \mathbb{R}_+ \times [0, L)\)

\[
\begin{aligned}
    h(t, x) &= -2az(t) \frac{1}{\sin \left( \frac{L - x}{2} \right)} \sin \left( \frac{x}{2} \right) \sin \left( \frac{L - x}{2} \right).
\end{aligned}
\]

One can easily check that \( h(t, 0) = h(t, L) = 0 \) for all \( t \geq 0 \). Moreover, one has for all \((t, x) \in \mathbb{R}_+ \times [0, L)\)

\[
\begin{aligned}
    h_x(t, x) &= -az(t) \frac{1}{\sin \left( \frac{L - x}{2} \right)} \cos \left( \frac{x}{2} \right) \sin \left( \frac{L - x}{2} \right) + az(t) \frac{1}{\sin \left( \frac{x}{2} \right)} \sin \left( \frac{L - x}{2} \right) \cos \left( \frac{x}{2} \right).
\end{aligned}
\]

One has \( h_x(t, 0) = -az(t) \) and \( h_x(t, t) = az(t) \) for all \( t \geq 0 \). Moreover, by definition of \( h \), one has \( h_x(t, x) + h_{xxx}(t, x) = 0 \) for all \((t, x) \in \mathbb{R}_+ \times [0, L]. In the following, we will use the following notation:

\[
\begin{aligned}
    h(t, x) := -f(x)z(t) \quad \text{where} \quad f(x) := 2a \frac{1}{\sin(\frac{L - x}{2})} \sin(\frac{x}{2}) \sin(\frac{L - x}{2}).
\end{aligned}
\]

Therefore, since \( h_x(t, 0) = -az(t) \), and setting \( \bar{z} = z \), the reduced-order system is given by

\[
\begin{aligned}
    \dot{z}(t) &= (b - ac)\bar{z}(t), & t \in \mathbb{R}_+\\
    \bar{z}(0) &= z_0.
\end{aligned}
\]

In consequence, if \( b - ac < 0 \), then the origin of (35) is exponentially stable.

2) **Boundary Layer System:** Consider \( \tau = \frac{1}{\varepsilon} \) and \( \bar{y}(\tau, x) := y(\tau, x) + z(\tau) f(x) \) for all \((\tau, x) \in \mathbb{R}_+ \times [0, L]. One can...
check that
\[
\dot{y}_r(\tau, x) = y_r(\tau, x) + \varepsilon \frac{d}{dt} f(x) z(t)
\]
for all \((\tau, x) \in \mathbb{R}_+ \times [0, L]\). Setting \(\varepsilon = 0\) yields \(\dot{y}_r(\tau, x) = y_r(\tau, x)\). One can check also easily that \(\dot{y}_r(\tau, x) + \ddot{y}_r(x, \tau) = y_{rr}(\tau, x) + \ddot{y}_{rex}(\tau, x)\) for all \((\tau, x) \in \mathbb{R}_+ \times [0, L]\). One has also \(\dot{y}(\tau, 0) = \dot{y}(\tau, L) = 0\) and \(\ddot{y}_r(\tau, L) = 0\). Finally, the boundary layer is written as
\[
\begin{aligned}
\dot{y}_r(x) &= \dot{y}(x) + f(x) z(t) \quad \forall (t, x) \in \mathbb{R}_+ \times [0, L].
\end{aligned}
\]
Noticing that \(\dot{y}_r(t, 0) = y_r(t, 0) + a z(t)\), its dynamics together with the one of \(z\) is given by
\[
\begin{align*}
\varepsilon \dot{y}_r + \ddot{y}_r + \dddot{y}_{rex} &= -\varepsilon ((b - ac) z(t) + c \dot{y}_r(t, 0)) f(x) \\
\dot{y}_r(x, L) &= 0 \\
\dot{y}_r(0, x) &= \dot{y}(x) \\
\dot{z}(t) &= (b - ac) z(t) + c \dot{y}_r(t, 0) \\
\bar{z}(0) &= \bar{z}_0.
\end{align*}
\] (37)

We can now state and prove the next result.

**Theorem 7:** Suppose that Assumption 2 holds true. For any \(a, b, c \in \mathbb{R}\) such that \((b - ac) < 0\), there exists \(\varepsilon^* > 0\) such that, for every \(\varepsilon \in (0, \varepsilon^*)\), the origin is exponentially stable for the system (1).

**Proof:** We consider the Lyapunov functional (19). Applying Proposition 11 with \(d_1(t, x) = -\varepsilon ((b - ac) z(t) + c \dot{y}_r(t, 0)) f(x)\) and \(d_2(t) = 0\), one obtains that all strong solutions to (37) satisfy
\[
\begin{align*}
\frac{d}{dt} V_1(\tilde{y}, z) &
\leq -\lambda \|\tilde{y}\|^2_{L^2(0, L)} \\
&+ \varepsilon^2 \kappa_1 \|f\|^2_{L^2(0, L)} ((b - ac) z(t) + c \dot{y}_r(t, 0))^2 - \kappa_3 z(t)^2 \\
&+ \left(\int_0^L M(x) (\tilde{y}(x, x) dx - z(t)\right).
\end{align*}
\] (38)

where \(K := \int_0^L M(x) f(x) dx\). Using Young’s Lemma several times one obtains, that for all strong solutions to (37)
\[
\frac{d}{dt} V_1(\tilde{y}, z)
\leq \left(-\lambda + \frac{\alpha_1}{2} \|M\|^2_{L^2(0, L)} + \frac{\alpha_2}{2} \|M\|^2_{L^2(0, L)}\right) \|\tilde{y}\|^2_{L^2(0, L)}
\]

We consider the Lyapunov functional (19). Applying Proposition 11 with \(d_1(t, x) = -\varepsilon ((b - ac) z(t) + c \dot{y}_r(t, 0)) f(x)\) and \(d_2(t) = 0\), one obtains that all strong solutions to (37) satisfy

\[
\begin{align*}
\frac{d}{dt} V_1(\tilde{y}, z) &
\leq -\lambda \|\tilde{y}\|^2_{L^2(0, L)} \\
&+ \varepsilon^2 \kappa_1 \|f\|^2_{L^2(0, L)} ((b - ac) z(t) + c \dot{y}_r(t, 0))^2 - \kappa_3 z(t)^2 \\
&+ \left(\int_0^L M(x) (\tilde{y}(x, x) dx - z(t)\right).
\end{align*}
\] (38)

where \(K := \int_0^L M(x) f(x) dx\). Using Young’s Lemma several times one obtains, that for all strong solutions to (37)
\[
\frac{d}{dt} V_1(\tilde{y}, z)
\leq \left(-\lambda + \frac{\alpha_1}{2} \|M\|^2_{L^2(0, L)} + \frac{\alpha_2}{2} \|M\|^2_{L^2(0, L)}\right) \|\tilde{y}\|^2_{L^2(0, L)}
\]

Using Lemma 3, one deduces the desired result.

**B. Tikhonov Theorem**

The most relevant part of the singular perturbation method is to use the obtained subsystems in order to approximate the dynamics of the full system. This section is devoted to this more precise analysis of the asymptotic behavior of the solutions with respect to the variable \(\varepsilon\). To do so, we will follow the Tikhonov strategy that has been used, for instance, in [11] and [35] for PDEs. We introduce the error solutions
\[
\hat{z}(t) = z(t) - \bar{z}(t)
\] (43)
and
\[
\hat{y}(t, x) = y(t, x) + f(x) \hat{z}(t) - \bar{y}\left(\frac{t}{\varepsilon}, x\right).
\] (44)

Using the solutions of (1), (35), (34), and (36). One can verify that
\[
\hat{z}(t) = b \hat{z}(t) + c \hat{y}_r(t, 0) - (b - ac) \hat{z}(t)
\]
Noticing that \(\hat{y}_r(t, x) = y_r(t, x) + f'(x) \hat{z}(t) - \bar{y}_r\left(\frac{t}{\varepsilon}, x\right)\), one has
\[
\hat{y}_r(t, 0) = y_r(t, 0) - \hat{z}(t) - \bar{y}_r\left(\frac{t}{\varepsilon}, 0\right)
\]
and because \(\hat{z}(t) = b \hat{z}(t) + c \hat{y}_r(t, 0) + \bar{c} \hat{y}_r\left(\frac{t}{\varepsilon}, 0\right)\), we get
\[
\hat{z}(t) = b \hat{z}(t) + c \hat{y}_r(t, 0) + \bar{c} \hat{y}_r\left(\frac{t}{\varepsilon}, 0\right).
\] (45)
Moreover, \[ \varepsilon \dot{y}(t, x) = \varepsilon y(t, x) + \varepsilon f(x)(b - ac)z(t) - \dot{y}(t, x) + \varepsilon f(x)(b - ac)z(t). \] (46)

Using the dynamics of y [given in (1)] and the one of \( \ddot{y} \) [given in (36)], one obtains
\[ \varepsilon \dot{y}(t, x) = -y_x - y_{xxx} + \dot{y}_x \left( \frac{t}{\varepsilon}, x \right) + \dot{y}_x \left( \frac{t}{\varepsilon}, x \right) + \varepsilon f(x)(b - ac)z(t). \] (47)

Note that
\[ \dot{y}_x + \dot{y}_{xxx} = y_x + y_{xxx} - \dot{y}_x \left( \frac{t}{\varepsilon}, x \right) - \dot{y}_x \left( \frac{t}{\varepsilon}, x \right) + \ddot{z}(t)(f'(x) + f''(x)). \]

Recall that \( h(t, x) = -\ddot{z}(t)f(x) \) and that \( h \) solves (34), i.e., \( \ddot{z}(t)(f'(x) + f''(x)) = 0 \). Hence, one has
\[ \varepsilon \dot{y}(t, x) = -\dot{y}_x - y_{xxx} + \varepsilon f(x)(b - ac)z(t). \] (48)

Using the boundary conditions given in (1), (36), and (34), one has
\[ \dot{y}(0, t) = \ddot{y}(t, L) = 0 \quad \forall t \geq 0. \]

Having in mind that \( \dot{y}_x(t, L) = y_x(t, L) + f'(x)\ddot{z}(t) - \dot{y}_x(\frac{t}{\varepsilon}, L) = a \ddot{z}(t) - a \ddot{z}(t), \) one can write the system
\[
\begin{align*}
\varepsilon \dot{y} + \dot{y}_x + y_{xxx} = \varepsilon f(x)(b - ac)z(t) & \\
\dot{y}(0, x) = 0, & t \in \mathbb{R}_+ \quad \text{(50)}
\end{align*}
\]

where \( K := \int_0^L f(x)M(x)dx \). Using Young’s Lemma several times, one obtains for all strong solutions of (49)
\[
\frac{d}{dt} V_1(\dot{y}, \ddot{z}) \leq -\lambda \| \dot{y} \|_{L^2(0, L)}^2 + \frac{\kappa_2 a^2}{2} (b - ac)^2 z(t)^2 + K^2 \varepsilon^2 (b - ac)^2 \left( \frac{1}{2 \alpha_1} + \frac{1}{2 \alpha_2} \right) \ddot{z}(t)^2 + c^2 \left( \frac{\varepsilon^2}{2 \alpha_4} + \frac{1}{2 \alpha_5} \right) \ddot{y}_x(t, 0)^2.
\]

One selects \( \alpha_1, \alpha_3, \) and \( \alpha_4, \) such that
\[ \alpha_2 + \alpha_5 = -(b - ac). \]

Then, setting \( \mu := \lambda - \left( \frac{\alpha_1}{\alpha_3} \varepsilon^2 + \frac{\alpha_2}{\alpha_5} \frac{\varepsilon^2}{2} \right) \| M \|_{L^2(0, L)}^2 < \lambda, \) one has
\[
\frac{d}{dt} V_1(\dot{y}, \ddot{z}) \leq -\mu_1 \| \dot{y} \|_{L^2(0, L)}^2 + \frac{\kappa_2 a^2}{2} (b - ac)^2 \varepsilon^2 z(t)^2 + K^2 \varepsilon^2 (b - ac)^2 \left( \frac{1}{2 \alpha_1} + \frac{1}{2 \alpha_2} \right) \ddot{z}(t)^2 + c^2 \left( \frac{\varepsilon^2}{2 \alpha_4} + \frac{1}{2 \alpha_5} \right) \ddot{y}_x(t, 0)^2.
\]

Let us now consider the polynomial \( P(X) = \kappa_2 a^2 - \frac{1}{2} X + \frac{\varepsilon^2}{\alpha_5} X^2. \) If \( a \) is such that \( a^2 < \frac{\alpha_3}{16 \varepsilon^2 \kappa_2} \), \( P(X) \) admits two square roots
\[
X_1 = \alpha_3 - \frac{1}{4 \varepsilon^2} \frac{1 - 16 \varepsilon^2 \kappa_2 a^2}{\alpha_3}, \quad X_2 = \alpha_3 + \frac{1}{4 \varepsilon^2} \frac{1 - 16 \varepsilon^2 \kappa_2 a^2}{\alpha_3}.
\]

Replacing \( X \) by \( ac - b \), one has \( P(ac - b) < 0 \) if
\[ X_1 < ac - b < X_2. \]

Then, there exists \( \mu_2 > 0 \) such that for all strong solutions to (49)
\[
\frac{d}{dt} V_1(\dot{y}, \ddot{z}) \leq -\mu_2 \| \dot{y} \|_{L^2(0, L)}^2 - \mu_2 \ddot{z}(t)^2 + K^2 \varepsilon^2 (b - ac)^2 \left( \frac{1}{2 \alpha_1} + \frac{1}{2 \alpha_2} \right) \ddot{z}(t)^2 + c^2 \left( \frac{\varepsilon^2}{2 \alpha_4} + \frac{1}{2 \alpha_5} \right) \ddot{y}_x(t, 0)^2.
\]
Using Proposition 11, the Grönwall’s Lemma and setting \( \mu_3 := \min\left(\frac{\mu_1}{\nu_1}, \frac{\mu_1}{\nu_2}\right) \), one obtains for all \( t \geq 0 \) that
\[
V_1(\hat{y}, \hat{z}) \leq e^{-\mu_3 t} V(\hat{y}_0, \hat{z}_0) + O(e^2) \int_0^t e^{-\mu_3 (t-s)} |\tilde{z}(s)|^2 ds + O(1) \int_0^t e^{-\mu_3 (t-s)} \frac{|\tilde{y}_x|}{\nu}\left(\frac{t}{\varepsilon}, 0\right). \tag{53}
\]
Let us estimate the integrals appearing in (53). Using (35), one has
\[
\tilde{z}(t)^2 \leq e^{-(b-ac)t} \frac{|\tilde{y}_x|}{\nu} |\hat{y}_x|^2. \tag{54}
\]
Moreover, using the Lyapunov functional given in Proposition 11, one obtains for all strong solutions to (26) that
\[
\frac{d}{d\tau} W(\hat{y}) \leq -\frac{\lambda}{\varepsilon} W(\hat{y}) - \kappa_3 \hat{y}_x(\tau, 0)^2.
\]
with \( \tau = \frac{\tau}{\varepsilon} \). Notice that
\[
\mu_3 \leq \frac{\mu_1}{\nu_1} \leq \frac{\mu_1}{\varepsilon \tau + \nu^2 \|M\|^2_{L_2(0,L)}} \leq \frac{\mu_1}{\varepsilon \tau} \leq \frac{\lambda}{\varepsilon} \tag{55}
\]
where we have used the definition of \( \nu_1 \) given in Lemma 3 and the definition of \( \mu_1 \) and \( \mu_3 \). Then, one has for all strong solutions to (36) that
\[
\frac{d}{d\tau} W(\hat{y}) \leq -\varepsilon \mu_3 W(\hat{y}) - \kappa_3 \hat{y}_x(\tau, 0)^2.
\]

Hence, using the Grönwall’s Lemma again, one obtains for all \( \tau \geq 0 \) that
\[
W(\hat{y}) \leq e^{-\varepsilon \mu_3 \tau} W(\hat{y}_0) - \kappa_3 \int_0^\tau e^{-\varepsilon \mu_3 (\tau-s)} \hat{y}_x(\tau, 0)^2 d\tau.
\]
One can conclude that
\[
\int_0^\tau e^{-\varepsilon \mu_3 (\tau-s)} \hat{y}_x(s, 0)^2 ds \leq \frac{\varepsilon e^{-\varepsilon \mu_3 \tau} \|\hat{y}_0\|^2_{L_2(0,L)}}{\nu \tau}.
\]
Setting \( \varepsilon = \varepsilon \) and using the definition of \( \tau \), one obtains
\[
\int_0^\tau e^{-\varepsilon \mu_3 (\tau-s)} \hat{y}_x(s, 0)^2 ds \leq \frac{e^{-\varepsilon \mu_3 \tau} \|\hat{y}_0\|^2_{L_2(0,L)}}{\nu \tau}. \tag{56}
\]

Then, using (54) and (57), and noticing that, since \( \varepsilon \leq 1 \), one has \( \varepsilon \geq t \), one obtains finally that for all \( t \geq 0 \)
\[
V_1(\hat{y}, \hat{z}) \leq O(1)e^{-\mu_3 t} V_1(\hat{y}_0, \hat{z}_0) + O(e^2) e^{-(b-ac)t} |\hat{z}_0|^2 + O(1)e^{-\mu_4 t} \|\hat{y}_0\|^2_{L_2(0,L)} \tag{58}
\]
Taking \( \mu_4 = \min((b-ac, \mu_3) \), and using the smallness condition on the initial conditions, one obtains that
\[
V_1(\hat{y}, \hat{z}) \leq e^{-\mu_4 t} O(e^3). \tag{59}
\]
Using Lemma 3, one has \( V_1(\hat{y}, \hat{z}) > O(\varepsilon)(\|\hat{y}\|_{L_2(0,L)} + |\hat{z}|)^2 \), concluding thus the proof.

IV. FAST ODE COUPLED WITH A SLOW KdV ALIGN

A. Stability for Small \( \varepsilon \)

Following the steps in the singular perturbation method, we are going to compute the reduced-order system and the boundary layer system for (2). The exponential stability conditions will be drastically different, which explains why we used a different Lyapunov functional. In addition to this different Lyapunov functional, these conditions will hold at the price of considering strong solutions to (2).

1) Reduced-Order System: Setting \( \varepsilon = 0 \), one obtains that \( z(t) = -\frac{a}{b} y_x(t, 0) \). The reduced-order system, whose state is denoted by \( \hat{y} \), satisfies
\[
\begin{align*}
\hat{y}_t + \hat{y}_x + \hat{y}_{xxx} &= 0, (t, x) \in \mathbb{R}_+ \times [0, L] \\
\hat{y}(t, 0) &= \hat{y}(t, L) = 0, t \in \mathbb{R}_+ \\
\hat{y}_x(t, L) &= -\frac{a}{b} \hat{y}_x(t, 0), t \in \mathbb{R}_+ \\
\hat{y}(0, x) &= \hat{y}_0(x), x \in [0, L].
\end{align*}
\tag{60}
\]
Using the Lyapunov functional given in Proposition 11 with \( d_2(t) = -\frac{a}{b} \hat{y}_x(t, 0) \), one has
\[
\frac{d}{d\tau} W(\hat{y}) \leq -\lambda W + \left(\kappa_2 \frac{a^2 c^2}{b^2} - \kappa_3\right) \hat{y}_x(t, 0)^2. \tag{61}
\]
Hence, if \( \frac{a^2 c^2}{b^2} < \kappa_3 \), then the origin of (60) is ensured to be exponentially stable. Note that this conditions looks like the one given in [38]. However, in this latter article, one requires that \( |\frac{a}{b}| < 1 \). This is surely associated to the fact that the Lyapunov approach is more conservative than the one followed in [38].

2) Boundary Layer System: Consider \( \tau = \frac{t}{\varepsilon} \) and \( \tilde{z}(\tau) = z(\tau) + \frac{a}{b} y_x(t, 0) \). One has \( \dot{\tilde{z}}(\tau) = \dot{z}(\tau) + \varepsilon \frac{a}{b} y_{xx}(t, 0) \). With \( \varepsilon = 0 \), one obtains that
\[
\dot{\tilde{z}}(\tau) = \dot{z}(\tau) + b z(\tau) + c y_x(\tau, 0) = b(\tilde{z}(\tau) + \frac{c}{b} y_x(\tau, 0)) = b\tilde{z}(\tau).
\]

Then, the boundary layer system is defined by
\[
\frac{d}{d\tau} \tilde{z}(\tau) = b \tilde{z}(\tau) \tag{62}
\]
which means that its origin is exponentially stable if \( b < 0 \).

3) Full System: Consider \( \tilde{z}(t) = z(t) + \frac{a}{b} y_x(t, 0) \). Then, \( y \) solves the following align:
\[
\begin{align*}
y_t + y_x + y_{xxx} &= 0, (t, x) \in \mathbb{R}_+ \times [0, L] \\
y(t, 0) &= y(t, L) = 0, t \in \mathbb{R}_+ \\
y_x(t, L) &= -\frac{a}{b} y_x(t, 0), t \in \mathbb{R}_+ \\
y(0, x) &= y_0(x), x \in [0, L].
\end{align*}
\tag{63}
\]
One has \( \varepsilon \hat{z} = \varepsilon \dot{z} + \varepsilon \frac{a}{b} y_x(t, 0) = b \dot{z}(t) + \frac{a}{b} y_x(t, 0) + \varepsilon \frac{a}{b} y_{xx}(t, 0) \), i.e.,
\[
\varepsilon \dot{\tilde{z}}(t) = b \tilde{z}(t) + \varepsilon \frac{a}{b} y_x(t, 0). \tag{64}
\]

Since the dynamics of \( \tilde{z} \) introduces the time derivative of \( y_x(t, 0) \), one needs more regularity on \( y \). We consider, therefore, \( \varepsilon = \varepsilon \hat{y} \). The use of the parameter \( \varepsilon \) in the variable \( v \) is due to
the fact that, in (63), it appears the state \( \tilde{z} \). The dynamics of \( v \) is given by
\[
\begin{aligned}
v_t + v_x + v_{xxx} &= 0, \quad (t, x) \in \mathbb{R}_+ \times [0, L] \\
v(0, x) &= -\varepsilon y_0 - \varepsilon y_0', \quad x \in [0, L] \\
v_x(t, L) &= ab\tilde{z}(t), \quad t \in \mathbb{R}_+ \\
\tilde{z}(t) &= b\tilde{z}(t) + \varepsilon(t) v_x(t, 0), \quad t \in \mathbb{R}_+ \\
\tilde{z}(0) &= 0 + 5\varepsilon y_0.
\end{aligned}
\]

Well-posedness of (65) is given by our results in Section II. Because we work in \( L^2 \)-regularity for \( v \) and \( H^3 \)-regularity for \( y \), some compatibility conditions appear on the initial data that are given in an explicit way in the result given below. We are in position to state the following result.

**Theorem 9:** Suppose that Assumption 2 holds true. For any \( a, b, c \in \mathbb{R} \) such that \( \frac{a^2\varepsilon^2 c}{b^2} < \frac{\varepsilon}{\kappa_2} \), where \( \kappa_2 \) and \( \kappa_3 \) are defined in Proposition 11, there exists \( \varepsilon > 0 \) such that for any \( \varepsilon \in (0, \varepsilon^*) \), the origin of (2) is exponentially stable in the \( H^3(0, L) \times \mathbb{R} \) topology for any initial conditions \( y_0, z_0 \in H^3(0, L) \times \mathbb{R} \) such that
\[
y_0(0) = y_0(L) = 0, \quad y_0'(L) = ab \left( z_0 + \frac{c}{b} y_0'(0) \right).
\]

To be more precise, there exist positive constants \( C \) and \( \mu \) such that the strong solution to (2) satisfies the following inequality for all \( t \geq 0 \):
\[
\|y(t, \cdot)\|_{H^3(0, L)} + |z(t)| \leq C e^{-\mu t} \left( \|y_0\|_{H^3(0, L)} + |z_0| \right).
\] (66)

**Proof:** To prove this result, we consider the following Lyapunov functional:
\[
V_3(v, \tilde{z}) = W(v) - \varepsilon \kappa_2 a^2 b^2 \tilde{z}^2
\] (67)
where \( W \) is the Lyapunov functional given in Proposition 11. Using the same proof as in Lemma 5, one has that for any \( b < 0 \), we can define \( \varepsilon_3 = \max(\varepsilon, -\varepsilon \kappa_2 a^2 b) \) and \( \varepsilon_3 = \min(\varepsilon, -\varepsilon \kappa_2 a^2 b) \), where \( \kappa_2 > 0 \) comes from Proposition 11 such that the Lyapunov functional (67) satisfies
\[
L_3(\|v\|_{L^2(0, L)} + |\tilde{z}|^2) \leq V_3(v, \tilde{z}) \leq L_3(\|v\|_{L^2(0, L)} + |\tilde{z}|^2).
\] (68)

Time derivative of (67) along the strong solutions to (65) yields
\[
\frac{d}{dt} V_3(v, \tilde{z}) \leq -\lambda \|v\|_{L^2(0, L)}^2 - \varepsilon \kappa_2 a^2 b^2 \tilde{z}^2 - \varepsilon \kappa_3 |v_x(0)|^2
\]
\[
- 2\varepsilon \kappa_2 a^2 b^2 \tilde{z}^2 + 2\kappa_2 a^2 \left( b \varepsilon \frac{c}{b} v_x(t, 0) \right).
\] (69)

Using Young’s Lemma, one gets
\[
\frac{d}{dt} V_3(v, \tilde{z}) \leq -\lambda \|v\|_{L^2(0, L)}^2 - \varepsilon \kappa_2 a^2 b^2 \tilde{z}^2 - \varepsilon \kappa_3 |v_x(0)|^2
\]
\[
+ \alpha \varepsilon \kappa_2 a^2 b^2 \tilde{z}^2 + \varepsilon \kappa_2 a^2 \kappa_2 b^2 v_x(t, 0)^2 \] (70)
and setting \( \alpha = \frac{1}{2} \), one obtains
\[
\frac{d}{dt} V_3(v, \tilde{z}) \leq -\lambda \|v\|_{L^2(0, L)}^2 - \kappa_2 \varepsilon \kappa_2 b^2 \tilde{z}^2
\]
\[
+ \left( 2\varepsilon \kappa_2 \frac{a^2 b^2 \tilde{z}^2}{b^2} - \kappa_3 \right) v_x(t, 0)^2.
\] (71)

If one takes \( \varepsilon^2 < \frac{1}{2} b, b < 0 \), and \( \frac{a^2 b^2 \varepsilon}{b^2} < \frac{\varepsilon}{\kappa_2} \), then one obtains (68).

It is worth saying that the condition given in Proposition 6 is stronger than the one given in Theorem 9, because of the factor \( \frac{1}{2} \) of Proposition 6. However, such a weaker condition holds at the price of assuming more regular solutions.

**B. Tikhonov Theorem**

This subsection is devoted to the asymptotic analysis of (2) with respect to \( \varepsilon \). As before, such an analysis requires to consider strong solutions to (2). Let us introduce the following two variables:
\[
\hat{z}(t) = z(t) + \frac{c}{b} y_x(t, 0) - \varepsilon \left( \frac{t}{\varepsilon} \right), \quad \hat{y}(t) = y(t) - \hat{y}(t).
\] (72)
One can check that
\[
\varepsilon \hat{z}(t) = b \hat{z}(t) + \frac{c}{b} \left( y_x(t, 0) + \hat{y}_x(t, 0) \right)
\] (73)
and
\[
\begin{aligned}
\hat{y}_t + \hat{y}_x + \hat{y}_{xxx} &= 0, \quad (t, x) \in \mathbb{R}_+ \times [0, L] \\
\hat{y}(t, 0) &= \hat{y}(t, L) = 0, \quad t \in \mathbb{R}_+ \\
\hat{y}_x(t, L) &= \alpha \left( \hat{z}(t) + \hat{z} \left( \frac{t}{\varepsilon} \right) \right), \quad t \in \mathbb{R}_+ \\
\hat{y}(0, x) &= \hat{y}_0(x), \quad x \in [0, L].
\end{aligned}
\] (74)
Since (73) introduces the time-derivative of \( y_x(t, 0) \), let us consider \( \hat{v} = \varepsilon \hat{y} \), where \( \varepsilon \) is introduced because the boundary condition makes appear \( z \) and \( \tilde{z} \). Its dynamics satisfies the following system:
\[
\begin{aligned}
\hat{v}_t + \hat{v}_x + \hat{v}_{xxx} &= 0, \quad (t, x) \in \mathbb{R}_+ \times [0, L] \\
\hat{v}(t, 0) &= \hat{v}(t, L) = 0, \quad t \in \mathbb{R}_+ \\
\hat{v}_x(t, L) &= ab\hat{z}(t) \\
+ \varepsilon \frac{a c}{b} v_x(t, 0) + \hat{y}_x(t, 0) \right) + ab \hat{z} \left( \frac{t}{\varepsilon} \right), \quad t \in \mathbb{R}_+ \\
\hat{v}(0, x) &= \hat{v}_0(x), \quad x \in [0, L].
\end{aligned}
\] (75)
Let us consider the Lyapunov functional
\[
V_4(\hat{v}, \hat{z}) = W(\hat{v}) - 3\varepsilon \kappa_2 a^2 b^2 \hat{z}^2
\] (76)
We can prove, as before, the following. For any \( b < 0 \), defining \( \pi_4 := \max(\pi, -3\varepsilon \kappa_2 a^2 b) \) and \( \varepsilon_4 := \min(\varepsilon, -3\varepsilon \kappa_2 a^2 b) \), where \( \kappa_2 \) comes from Proposition 11, the Lyapunov functional (76) satisfies
\[
L_4(\|\hat{v}\|_{L^2(0, L)} + |\hat{z}|^2) \leq V_4(\hat{v}, \hat{z}) \leq L_4(\|\hat{v}\|_{L^2(0, L)} + |\hat{z}|^2).
\] (77)
Moreover, if \( \varepsilon \leq 1 \), then there exists \( C > 0 \) such that
\[
\varepsilon(\|\hat{v}\|_{L^2(0, L)} + |\hat{z}|^2) \leq V_4(\hat{v}, \hat{z}) \leq C(\|\hat{v}\|_{L^2(0, L)} + |\hat{z}|^2).
\] (78)
We are now in position to state and prove our Tikhonov theorem for (2).

**Theorem 10:** Suppose that Assumption 2 holds true. There exist \( \varepsilon^* \) and \( \mu > 0 \) such that for any \( \varepsilon \in (0, \varepsilon^*) \), for any \( b < 0 \), for any \( a, c \in \mathbb{R} \) such that \( \frac{a^2 b^2 \varepsilon}{b^2} < \frac{\varepsilon}{\kappa_2} \), where \( \kappa_2 \) and \( \kappa_3 \) come.
Choosing \( \frac{a^2c^2}{b^2} \leq \frac{\kappa_1}{17\kappa_2\epsilon^2} \) as in the statement of the theorem, one obtains for all solutions to (73)–(75)

\[
\frac{d}{dt} V_4(v, \hat{z}) \leq -\lambda \| \hat{z} \|_{L^2(0,L)}^2 - \frac{1}{17\kappa_2a^2b^2} \epsilon^2 + 4\kappa_2a^2b^2 \hat{z} \left( \frac{t}{\epsilon} \right)^2 + \epsilon \kappa_2 a^2 b^2 \frac{17\epsilon^2}{b^2} \| y_x(t,0) \|_{H^2}^2 (t,0).
\]

(85)

Denoting by \( \mu := \min \left( \frac{1}{\epsilon}, \frac{1}{17\kappa_2\epsilon^3} \right) \), where \( \epsilon \) comes from Proposition 11, and using the Grönwall’s Lemma, one obtains for all \( t \geq 0 \)

\[
V_4(v, \hat{z}) \leq e^{-\mu t} V_4(v_0, \hat{z}_0) + \int_0^t e^{-\mu (t-s)} \left( O(1) \| \hat{z} \|_{L^2(0,L)}^2 + O(\epsilon^2) \| \hat{z}_x \|_{L^2(0,L)}^2 \right) ds.
\]

(86)

On the other hand, one can prove that for all \( t \geq 0 \)

\[
\int_0^t e^{-\mu (t-s)} \left( \frac{\epsilon}{\epsilon^3} \right)^2 ds \leq O(1) e^{b^2} |z_0|^2.
\]

On the other hand, consider the variable \( \bar{v}(t, x) = \epsilon \hat{y}_t(t, x) \). It satisfies the following KdV align:

\[
\begin{align*}
\frac{d}{dt} \bar{v}(t, x) & = -\lambda \| \bar{z} \|_{L^2(0,L)}^2, \\
\bar{v}(t,0) & = \bar{v}_x(t,0), \quad \bar{v}_x(t,L) = 0, \quad \bar{v}_t(t,0) = \bar{v}_x(t,L) = 0.
\end{align*}
\]

(87)

Using the ISS-Lyapunov functional given in Proposition 11 with \( d_2(t) = -\frac{\kappa_2}{b^2} \bar{v}_x(t,0) \) along the solutions to (87), one obtains

\[
\frac{d}{dt} W(\bar{v}) \leq -\lambda \| \bar{z} \|_{L^2(0,L)}^2 + \frac{a^2c^2}{b^2} \kappa_2 - \kappa_3 \| v_x(t,0) \|_{L^2}^2.
\]

(88)

Under the condition on \( a, b, \) and \( c \), there exists \( \kappa_4 > 0 \) such that

\[
\frac{d}{dt} W(\bar{v}) \leq -\lambda \| \bar{z} \|_{L^2(0,L)}^2 - \kappa_4 \| v_x(t,0) \|_{L^2}^2.
\]

(89)

Using Proposition 11 and since \( \mu \leq \frac{1}{\epsilon} \), one has for all strong solution to (87)

\[
\frac{d}{dt} W(\bar{v}) \leq -\lambda W(\bar{v}) - \kappa_4 v_x(t,0)^2.
\]

(90)

Thanks to Grönwall’s Lemma, one gets

\[
\int_0^t e^{-\mu (t-s)} |v_x(s,0)|^2 ds \leq \frac{1}{\kappa_4} e^{-\mu t} W(v_0).
\]

(91)

Therefore, using the smallness conditions given in the statement of the theorem, one has for all \( t \geq 0 \)

\[
V_4(v, \hat{z}) \leq e^{-\mu t} V_4(v_0, \hat{z}_0) + O(1) e^{bt} |z_0|^2 + O(\epsilon^2) e^{-\mu t} \| \hat{y}_0 \|_{H^3}^2 \leq e^{-\mu t} O(\epsilon^3)
\]

(92)
Due to inequality (77), one has \( V_4(\hat{v}, \hat{z}) \geq O(\varepsilon) \left( \| \hat{v} \|_{L^2(0,L)}^2 + |\hat{z}|^2 \right) \). Moreover, recall that \( \hat{v} = \varepsilon y_t = -\varepsilon (y_x + y_{xxx}) \). Hence, \( V_4(\hat{v}, \hat{z}) \geq O(\varepsilon^3) \left( \| \hat{y} \|_{L^2(0,L)}^2 + |\hat{z}|^2 \right) \). Then, one can deduce the desired result concluding the proof.

\section{Conclusion}
In this article, we have provided a singular perturbation analysis for two coupled systems composed by a KdV equation and an ODE. In particular, we have proved that, the conditions for the reduced-order system and the boundary layer system to be exponentially stable also work for the full system for \( \varepsilon \) small enough. Different Lyapunov functionals have been introduced for the cases where the KdV equation is faster or the ODE is faster. For both cases, the ISS Lyapunov functional built in [3] has been instrumental. It is also worth mentioning that, when the ODE is faster than the KdV equation, the perturbation analysis can be performed only for sufficiently smooth solutions.

\section*{Appendix: ISS-Lyapunov Functional and Other Technical Results}
This appendix recalls a crucial result provided in [3], which proposes the construction of an ISS-Lyapunov functional for the KdV equation. This result will be instrumental all along this article. To introduce it, let us focus on the following disturbed KdV equation:

\[
\begin{align*}
  y_t + y_{xxx} &= d_1(t, x), & (t, x) &\in \mathbb{R}_+ \times (0, L) \\
  y(t, 0) &= y(t, L) = 0, & t &\in \mathbb{R}_+ \\
  y_x(t, L) &= d_2(t), & t &\in \mathbb{R}_+ \\
  y(0, x) &= y_o(x), & x &\in [0, L]
\end{align*}
\]  
(93)

where \( d_1 \in L^2(0, T; L^2(0, L)) \) and \( d_2 \in L^2(0, T) \), for any \( T \geq 0 \). The well-posedness of (93) can be obtained using the semigroup theory in standard way for strong or mild solutions, depending on the regularity of the data, as explained in [6, Proposition 2]. According to [3, Th. 2.3], one has the following result.

\textbf{Proposition 11:} There exists an ISS-Lyapunov functional for (93), i.e., there exists a function \( W : L^2(0, L) \rightarrow \mathbb{R} \) and positive constants \( \lambda, \kappa_1, \kappa_2, \kappa_3, \tau, \zeta \) such that

\[ \| y \|_{L^2(0, L)}^2 \leq W(y) \leq \zeta \| y \|_{L^2(0, L)}^2 \]  
(94)

and the derivative of \( W \) along the solutions to (93) satisfies

\[ \frac{d}{dt} W(y) \leq -\lambda \| y \|_{L^2(0, L)}^2 + \kappa_1 \| d_1(t, \cdot) \|_{L^2(0, L)}^2 + \kappa_2 \| d_2(t) \|_{L^2(0, L)}^2 - \kappa_3 \| y_x(t, 0) \|_{L^2(0, L)}^2. \]  
(95)

Note that the term \(-\kappa_3 \| y_x(t, 0) \|_{L^2(0, L)}^2\) does not appear in [3, Th. 2.3], but following the proof in that article, one can prove that such a term exists. It will be useful in our context.

Now, we prove Lemma 3.

\textbf{Proof:} First, using Proposition 11 in Appendix and Young’s Lemma, we have

\[ V_1(y, z) \leq (\varepsilon \sigma + \varepsilon^2 \| M \|_{L^2(0, L)}^2) \| y \|_{L^2(0, L)}^2 + |z|^2. \]  
(96)

Second, using again Proposition 11 and Young’s Lemma, we get

\[
V_1(y, z) \geq \varepsilon \sigma \| y \|_{L^2(0, L)}^2 + \frac{1}{2} \left( 1 - \frac{1}{\alpha} \right) \varepsilon \sigma^2 \int_0^L M(x)^2 y(t, x)^2 + \frac{1}{2} (1 - \alpha) z(t)^2.
\]  
(97)

Choose \( \alpha = \frac{\varepsilon^2 \| M \|_{L^2(0, L)}^2}{\varepsilon \sigma^2 \| y \|_{L^2(0, L)}^2 + \varepsilon^2} \). Then, \( 1 - \frac{1}{\alpha} < 0 \), and one has

\[
V_1(y, z) \geq \varepsilon \sigma \| y \|_{L^2(0, L)}^2 - \frac{1}{2} \left( \varepsilon \sigma \| y \|_{L^2(0, L)}^2 \right) \varepsilon^2 \| M \|_{L^2(0, L)}^2 \| y \|_{L^2(0, L)}^2 + \frac{1}{2} \varepsilon^2 \| y \|_{L^2(0, L)}^2 \| z \|_{L^2(0, L)}^2 + \varepsilon \sigma \| y \|_{L^2(0, L)}^2 |z|^2.
\]  
(98)

This concludes the proof.

We prove now the Lemma 5

\textbf{Proof:} Using Proposition 11, one first has

\[ V_2(y, z) \leq \tau \| y \|_{L^2(0, L)}^2 - \frac{\varepsilon \kappa_2 a^2}{b} \leq \tau_2(\| y \|_{L^2(0, L)}^2 + |z|^2) \]  
(99)

where \( \tau_2 = \max \left( \hat{c}, -\varepsilon \kappa_2 a^2 \right) \). Using again Proposition 11, one obtain

\[ V_2(y, z) \geq \tau \| y \|_{L^2(0, L)}^2 - \frac{\varepsilon \kappa_2 a^2}{b} \geq \tau_2(\| y \|_{L^2(0, L)}^2 + |z|^2) \]  
(100)

where \( \tau_2 = \min \left( \hat{c}, -\varepsilon \kappa_2 a^2 \right) \). This concludes the proof.

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