Magnetization of mesoscopic disordered networks

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We study the magnetic response of mesoscopic metallic isolated networks. We calculate the average and typical magnetizations in the diffusive regime for non-interacting electrons or in the first order Hartree-Fock approximation. These quantities are related to the return probability for a diffusive particle on the corresponding network. By solution of the diffusion equation on various types of networks, including a ring with arms or an infinite square network, we deduce the corresponding magnetizations. In the case of an infinite network, the Hartree-Fock average magnetization stays finite in the thermodynamic limit.

The problem of persistent currents in mesoscopic rings has been stimulated by a few key experiments in the recent years. Two types of measurements have been devised, single ring experiments and many rings experiments. In the second case, the measured quantity is an average magnetization $\langle M \rangle$ while the first type of experiment can only give the magnetization corresponding to a given disorder configuration. In the last case, the width $M_{typ}$ of the magnetization distribution is also of interest: $M_{typ}^2 = \langle M^2 \rangle - \langle M \rangle^2$.

Essentially two types of methods have been used: i) analytical methods where the diffusive electronic motion is treated in a perturbative way, leading to the famous Cooperon diagrams; non-interacting electron theory [16,17,18,19,21] or Hartree-Fock approximation [11,12,13,14,15] have been considered; ii) strictly 1D models or numerical methods in which either there is no diffusive motion or the system size is too small to give quantitative results [20]. Up to now, the only results for the diffusive regime are given by the perturbative method. The status of the comparison with experiments is not completely clear yet but its seems to be a reasonable agreement with preliminary recent data [6], the typical current being properly well described by the non-interacting theory [11,12] and the average current being described by the Hartree-Fock approximation [1,2,3,11,12].

We propose that a new way to get insight into the problem is to study other geometries than simple rings. In this letter, we calculate analytically the typical and average magnetizations of various types of networks, following the method (i). To do so, we use a semi-classical picture to relate the quantities of interest to the return probability of a classically diffusive particle. Then, this return probability is calculated on different type of networks, giving access to the magnetization. As examples, we treat the case of an isolated ring connected to one or two arms, and the case of an infinite square lattice. Several experiments are proposed (otherwise specified, $\hbar = 1$ throughout the paper).

In the absence of $e-e$ interactions, a finite contribution to the average magnetization comes from the fact that the number $N$ of particles is fixed in each subsystem of the ensemble [22]. It turns out that this contribution is by far smaller than the experimental results. However we will discuss it mainly for pedagogical purpose and comparison with other contributions. With this constraint on $N$, the "canonical"magnetization is given by

$$\langle M_N(H) \rangle = -\frac{\Delta}{2} \frac{\partial}{\partial H} \langle \delta N^2(\mu) \rangle,$$

where $\Delta$ is the mean level spacing and $\langle \delta N^2(\mu) \rangle$ is the sample to sample fluctuation of the number of single-particle states below the Fermi energy $\mu$. It is an integral of the two-point correlation function of the density of states (DOS) $K(\epsilon_1 - \epsilon_2) = \langle \rho(\epsilon_1)\rho(\epsilon_2) \rangle - \rho_0^2 \cdot \rho_0$ is the average DOS. $K(\epsilon)$ has been calculated by Altshuler and Shklovskii [24] and later in the presence of a magnetic flux [14,15,12]. A very useful semiclassical picture has been presented by Argaman et al., which relates the Fourier transform $\tilde{K}(t)$ of $K(\epsilon)$ to the classical return probability $p(\mathbf{r},t)$ for a diffusive particle [13].

$$\tilde{K}(t) = \frac{tP(t)}{4\pi^2},$$

where $P(t) = \int p(\mathbf{r},t)\,d\mathbf{r}$. This return probability has two components, the purely classical one and the interference term which results from interferences between pairs of time-reversed trajectories. In the diagrammatic picture, they are related to the diffuson and Cooperon diagrams. The interference term is field dependent and is solution of the diffusion equation:
\[
\left[ \frac{\partial}{\partial t} - D(\nabla + \frac{2ieA}{\hbar c})^2 \right] p(r, r', t) = \delta(t) \delta(r - r') \tag{3}
\]

From eqs. (3), the average canonical magnetization can be related to the field dependent part of the return probability:

\[
\langle M_N(H) \rangle = -\frac{\Delta}{4\pi^2} \frac{\partial}{\partial H} \int_0^\infty \frac{P(t, H)}{t} dt , \tag{4}
\]

Note that the field dependent part of this integral converges at small times. At large times, the return probability is exponentially cut-off as \(e^{-\gamma t}\) where \(\gamma = \hbar D/L^2\) is the inelastic scattering rate.

Due to the \(e - e\) interactions, a larger contribution to the average magnetization exists, which has been calculated by Ambegaokar and Eckern \[16\], in the Hartree-Fock approximation. It can be written as \[16,10,18,19\]:

\[
\langle M_{ee}(H) \rangle = -\frac{U}{4} \frac{\partial}{\partial H} \int \langle n(r)^2 \rangle dr \tag{5}
\]

Where \(U\) is an effective screened interaction and \(n(r)\) is the local density. The integrand is related to the fluctuations of the local DOS which in turn can be related to the return probability \[19\]. One gets:

\[
\langle M_{ee}(H) \rangle = -\frac{U\rho_0}{\pi} \frac{\partial}{\partial H} \int_0^\infty \frac{P(t, \phi)}{t^2} dt \tag{6}
\]

In a similar way, the typical magnetization can also be straightforwardly written in terms of \(K(\varepsilon)\) \[13,21\]. By Fourier transform, one has:

\[
M_{typ}^2(H) = \frac{1}{8\pi^2} \int_0^\infty P^{\prime\prime}(t, H) \left| \frac{d}{dt} \right| H dt , \tag{7}
\]

where \(P^{\prime\prime}(t, H) = \partial^2 P/\partial H^2|_0 - \partial^2 P/\partial H^2|_H\).

To be complete, we remind that the weak-localization correction to the conductance of a connected mesoscopic sample can be also be related to the return probability \[25,26\]:

\[
\Delta \sigma(r) = -\frac{2}{\pi\rho_0}\sigma_0 C_\gamma(r, r) \tag{8}
\]

\(\sigma_0\) is the Drude conductivity. The Cooperon \(C_\gamma(r, r, H)\) is the time integrated field-dependent return probability:

\[
C_\gamma(r, r, H) = \int_0^\infty p(r, r, t, H) dt \tag{9}
\]

It appears that all the quantities of interest are obtained as time integrals of the return probability with various power-law weighting functions. Noting that \(P(t)\) has the form \(P_0(t) e^{-\gamma t}\) and that

\[
\int \frac{P_0(t)}{t} e^{-\gamma t} dt = \int_0^\infty d\gamma \int C_\gamma(r, r, H) dr \tag{10}
\]

the different magnetizations can be given in terms of the successive integrals of \(C_\gamma(r, r, H)\):

\[
\langle M_N(H) \rangle = -\frac{\Delta}{4\pi^2} \frac{\partial}{\partial H} \int C_\gamma^{(1)}(r, r, H) dr \tag{11}
\]
\[
\langle M_{ee}(H) \rangle = -\frac{U\rho_0}{\pi} \frac{\partial}{\partial H} \int C_\gamma^{(2)}(r, r, H) dr \tag{12}
\]
\[
M_{typ}^2(H) = \frac{1}{8\pi^2} \frac{\partial^2}{\partial H^2} \int C_\gamma^{(3)}(r, r, H) dr \tag{13}
\]

where \(C^{(n)} = \int_0^\infty d\gamma_1 ... \int_0^\infty d\gamma_n \int_0^\infty d\gamma' C_{\gamma'}.\) These are the key equations of this paper since the different magnetizations can be calculated from the knowledge of the return probability \(C_\gamma(r, r, H)\) on the different lattices considered and can be deduced from each other or related to weak-localization correction by \(H - \gamma - \text{derivatives or integrations.}\)

For the case of weak-localization correction, an extensive study of this quantity on various lattices has been carried out by Douçot and Ramírez \[26\]. Considering networks made of quasi-1D wires so that the diffusion can be considered as one-dimensional, the Cooperon \(C_\gamma(r, r')\) obeys the diffusion equation

\[
[\gamma - \hbar D(\nabla + \frac{2ieA}{\hbar c})^2] C_\gamma(r, r') = \delta(r - r') \tag{14}
\]

with the continuity equations

\[
\sum_{\beta} \left(-i \frac{\partial}{\partial r} - \frac{2eA}{\hbar c}\right) C_{\gamma(\alpha, r')} = \frac{i}{DS} \delta_{\alpha, r'} \tag{15}
\]

\(r, r'\) are linear coordinates on the network and \(\alpha, \beta\) are nodes. The term \(\gamma\) in eq. (4) describes the inelastic scattering. Integration of the differential equation (14) with the boundary conditions (15) leads to the so-called network equations which relate \(C_\gamma(\alpha, r')\) to the neighboring nodes \(\beta\) \[23\]:

\[
\sum_{\beta} \coth\left(\frac{\hbar c}{L_\varepsilon}\right) C(\alpha, r') - \sum_{\beta} \frac{C(\beta, r') e^{-\gamma_{\alpha\beta}}}{\sinh(\hbar c/L_\varepsilon)} = \frac{L_\varepsilon}{DS} \delta_{\alpha, r'} \tag{16}
\]

\(l_{\alpha\beta}\) is the length of the link \((\alpha, \beta)\) and \(\gamma_{\alpha\beta} = (4\pi/\phi_0) \int_\alpha A d\ell\) is the circulation of the vector potential along this link. Then \(C_\gamma(r', r')\) is calculated in terms of the \(C_\gamma(\alpha, r')\). Finally, spatial integration give access to the magnetizations.

As an example, we have first considered the case of a ring of perimeter \(L\) connected to an arm of length \(b\). Such a geometry has been considered in the strictly 1D case without disorder \[27,28,29\]. It is expected that since the electrons will spend some time in the arm where there are not sensitive to the flux, the persistent current will be decreased. From eqs. (4,5), the function \(C_\gamma(r, r)\) can be straightforwardly calculated on the arm and on the ring. Spatial integration gives:

\[
\langle M_N \rangle = \frac{\Delta S}{\pi \phi_0} \frac{\sin 4\pi \phi}{\frac{b}{L_\varepsilon} \tanh \frac{b}{L_\varepsilon} \sinh \frac{L_\varepsilon}{L_\varepsilon} + \cosh \frac{L_\varepsilon}{L_\varepsilon} - \cos 4\pi \varphi} \tag{17}
\]
where $\varphi = \phi/\phi_0$. $\phi$ is the flux through the ring, $\phi_0 = h/e$ is the flux quantum and $S$ is the area of the ring. Writing this magnetization as $\langle M_N \rangle = \sum_m (M_N)_m \sin 4\pi m \varphi$, the harmonics content is given by

$$\langle M_N \rangle_m = \frac{2\Delta S}{\pi \phi_0} e^{-m \arg \cosh \frac{L}{L_\varphi} + \frac{1}{2} \tanh \frac{L}{L_\varphi} \sinh \frac{L}{L_\varphi}}$$

(16)

Well-known results are recovered when $b = 0$. In the case where $L \approx L_\varphi$, the harmonics content can be simply written as:

$$\langle M_N \rangle_m = \frac{2\Delta S}{\pi \phi_0} \left( \frac{2}{2 + \tanh b/L_\varphi} \right)^m e^{-m L/L_\varphi}$$

(17)

so that in the limit $b \to \infty$, the harmonics are reduced by a ratio $(2/3)^m$, compared to the ring. However the arm has another much more dramatic effect which is to decrease the interlevel spacing: $\Delta(b) = \Delta(0)L/(L + b)$. This reduction does not exist for $\langle M_{ee} \rangle_m$ and $M_{typ}$ which can also be simply calculated by successive integrations on $\gamma$. Here we give the result for an arm of length $b \gg L_\varphi$:

$$\langle M_{ee}(\infty) \rangle_m = \left( \frac{2}{3} \right)^m \langle M_{ee}(0) \rangle_m$$

(18)

$$M_{typ}(\infty)_m = \left( \frac{2}{3} \right)^m M_{typ}(0)_m$$

(19)

where $\langle M_{ee}(0) \rangle_m$ are the magnetizations of the isolated loop. Interestingly, it is seen that the magnetizations $\langle M_{ee} \rangle_m$ and $M_{typ}$ do not decrease to 0 when $b \to \infty$ but saturate to finite values with respective reductions of the first harmonics in the ratios $2/3$ et $\sqrt{2}/3$. In the limit $b \gg L_\varphi$, the magnetization should be unchanged if a reservoir is attached to the arm [27]. The case of a ring connected to two arms of length $b$ shown in figure 1 can be treated in a very similar way. In this case we find that the lowest harmonics of the $e-e$ average current is reduced in a ratio $4/9$ and the typical current is reduced by a factor $2/3$. We propose that single ring experiments with appropriately designed arms could be able to measure these reductions.

We now turn to the case of an infinite square lattice whose magnetization will be compared with the one of an array of isolated rings. The eigenvalues of the diffusion equation can be calculated for a rational flux per plaquette $\varphi = \phi/\phi_0 = p/2q$. Denoting by $a$ the lattice parameter, $\eta = a/L_\varphi$ and $\phi = Ha^2$ the flux per plaquette, we find that the canonical magnetization per plaquette is

$$\langle M_N \rangle = \frac{\Delta}{4\pi^2} \frac{\partial}{\partial H} \sum_{i=1}^{q} \{ \ln(4 \cosh \eta - \epsilon_i(\theta, \mu)) \}$$

(20)

where $\{ \ldots \} = \int_0^{2\pi} d\theta \int_0^{2\pi} d\mu \ldots \epsilon_i(\theta, \mu)$ are the solutions of the determinental equation

$$det M = det \begin{pmatrix} 1 & 0 & \ldots & 0 & e^{i\mu} \\ 1 & M_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e^{-i\mu} & 0 & \ldots & 1 & M_q \end{pmatrix} = 0$$

(21)

where $M_n = 2 \cos(4n\pi \varphi + \theta/q) - \epsilon$. $M$ is a matrix associated to the Harper equation known to be also relevant for other related problems like tight-binding electrons in a magnetic field [30] or superconducting networks in a field [31].

The magnetization per plaquette can be compared to the magnetization of a square ring of perimeter $L = 4a$:

$$\langle M_N \rangle = \frac{\Delta}{4\pi^2} \frac{\partial}{\partial H} \ln(\cosh 4\eta - \cos 4\pi \varphi)$$

(22)

Since $\Delta \to 0$ for the infinite network, this canonical magnetization density vanishes for an infinite network as it was already noticed for a chain of connected rings [2]. On the other hand, the $e-e$ contribution stays finite in the thermodynamic limit. It is given by:

$$\langle M_{ee} \rangle = U \rho_0 \frac{eD}{\pi^2 q} \frac{\partial}{\partial \varphi} \sum_{i=1}^{q} \int_0^{\infty} \{ \ln(4 \cosh \eta - \epsilon_i(\theta, \mu)) \} \eta d\eta$$

(23)

and can be compared with the ring magnetization which can be cast in the form:

$$\langle M_{ee} \rangle = U \rho_0 \frac{4eD}{\pi} \int_0^{\infty} \frac{\sin 4\pi \varphi}{\cosh 4\eta - \cos 4\pi \varphi} \eta d\eta$$

(24)

(This integral can be calculated explicitly in terms of the Lobatchevsky function and it has the Fourier decomposition found by Ambegaokar and Eckern [13].) Contrary to the canonical magnetization, the $e-e$ magnetization is an extensive quantity. This magnetization density is plotted on figure 2 for the ring and the infinite lattice. It is first seen that the network magnetization is continuous. Although the field dependence of the eigenvalues of the Harper equation has a very complicated discontinuous behavior (the so-called Hofstadter spectrum),

![FIG. 1. Two geometries considered in this letter](image-url)
the sum on the eigenvalues has a smooth behavior \( [36] \). \( \langle M_{ee} \rangle \) can be calculated easily for large \( q \). In this case the dispersion \( \varepsilon_i(\theta, \mu) \) is very small and the density of states can be approximated by a sum of \( \delta \) functions \( [36] \). A very good approximation of the sum \( [29] \) can be obtained by replacing the integrals on \( \theta \) and \( \mu \) by the value of each term in the sum taken at \( \varepsilon_i(\pi, \pi) \).

![Diagram](image_url)

**FIG. 2.** Average magnetization \( \langle M_{ee} \rangle \) of a single ring (full lines) and magnetization density of the infinite network (dashed lines), for \( L_\omega = \infty, 4\pi \) and \( a \).

It is seen on figure 2 that the network magnetization density is about 25 five times smaller than the ring magnetization. Considering that on the array of square rings already considered experimentally \( [6] \), the distance between rings is equal to the size of the squares, the number of squares is four times larger when they are connected. One then expect only a factor of order 6 between these two magnetizations could be drastically changed. Thus we propose that the measurement of the magnetization of a mesoscopic network should give similar results to the one of an array of disconnected rings.

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