RECOGNIZING GALOIS REPRESENTATIONS OF K3 SURFACES

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Abstract. Under the assumption of the Hodge, Tate and Fontaine-Mazur conjectures we give a criterion for a compatible system of $\ell$-adic representations to be isomorphic to the second cohomology of a K3 surface.

1. Introduction

This paper grew out of an attempt to answer a question on the section conjecture for moduli spaces of K3 surfaces, inspired by recent work of Patrikis, Voloch and Zarhin [PVZ16]. In this paper, the authors study the section conjecture for the moduli space of principally polarized abelian varieties. The section conjecture for a (geometrically connected) variety $X$ over a number field $K$ relates the set of rational points $X(K)$ with the sections of the fundamental sequence

$$1 \rightarrow \pi_1(X) \rightarrow \pi_1(X) \rightarrow \Gamma_K \rightarrow 1$$

(we omit base points for the étale fundamental group and write $\Gamma_K = \text{Gal}(\overline{K}/K)$ for the absolute Galois group for a fixed algebraic closure $\overline{K}$, and $X$ the basechange of $X$ to $\overline{K}$). Given a rational point $x: \text{Spec}(K) \rightarrow X$ functoriality of $\pi_1$ gives a section $\Gamma_K \rightarrow \pi_1(X)$ and this defines a map $\sigma_X: X(K) \rightarrow H(K,X)$ where $H(K,X)$ is the set of sections up to conjugation by $\pi_1(X)$. The section conjecture for $X$ states that the map $\sigma_X$ is a bijection. Of course, for general $X$ this map is far from a bijection, so we would want to find a class of varieties suitably determined by their fundamental groups. These are the so called anabelian varieties introduced by Grothendieck in his letter to Faltings [Gro98]. Grothendieck suggested that hyperbolic curves, moduli spaces of curves and (less emphatically) moduli spaces of abelian varieties should all be anabelian.

It is known that moduli spaces $A_g$ of abelian varieties should not be anabelian by results of Ihara and Nakamura [IN97]. However, theorem 1.1 of [PVZ16] shows that under the assumption of well known motivic conjectures, a large subset of sections $S_0(K,A_g) \subset H(K,A_g)$ is contained in the image of $\sigma_{A_g}$, where the sections $S_0(K,A_g)$ are those coming from points locally. The authors are able to prove this by reducing to a question about Galois representations. More specifically there is a short exact sequence

$$1 \rightarrow \pi_1(A_g) \rightarrow \pi_1(A_g) \rightarrow \Gamma_K \rightarrow 1$$

$$
\begin{array}{ccccc}
1 & \rightarrow & \pi_1(A_g) & \rightarrow & \Gamma_K & \rightarrow & 1 \\
\downarrow \cong & & \downarrow & & \downarrow & \\
1 & \rightarrow & \text{Sp}_{2g}(\hat{\mathbb{Z}}) & \rightarrow & \text{GSp}_{2g}(\hat{\mathbb{Z}}) & \rightarrow & \hat{\mathbb{Z}}^\times & \rightarrow & 1
\end{array}
$$

(1)

Given a section $s: \Gamma_K \rightarrow \pi_1(A_g)$ composition with the middle arrow gives a collection of $\ell$-adic representations $\{\rho_{\ell}: \Gamma_K \rightarrow \text{GSp}_{2g}(\mathbb{Z}_\ell)\}_\ell$. The fact that the left arrow is an isomorphism shows that the sections $H(K,A_g)$ are determined by their associated $\ell$-adic representations.
Then the authors use well known conjectures to find conditions on a collection of $\ell$-adic representations $\{\rho_\ell\}$ that ensure they are isomorphic to the $\ell$-adic Tate module of an abelian variety [PVZ16, Thm 3.3]. The proof of theorem 3.3 proceeds by using these conjectures to find a motive underlying the collection of $\ell$-adic representations. Taking Betti realization of this motive gives a Hodge structure that has the Hodge weights of an abelian variety. Using Riemann’s theorem one can show that this Hodge structure is isomorphic to the Hodge structure on the Tate module of an abelian variety.

One might ask whether [PVZ16, Thm 3.3] can be generalized to other classes of varieties. In order for the above method to work, such a class of varieties would require an analogue of Riemann’s theorem, which gives a criterion for an abstract Hodge structure to appear in the (co)homology of a variety. After abelian varieties, the most natural class of varieties with this property is K3 surfaces, where surjectivity of the period map is known. Our main theorem is precisely the analogue for K3 surfaces of [PVZ16, Thm 3.3]. We find a set of conditions on a collection of $\ell$-adic representations to relate them to $H^2(X_{K'}, \mathbb{Z}_\ell)$. To be more precise, let $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 3} \oplus 3\mathbb{E}_8$ be the K3 lattice. Fix a basis $e, f$ of the first copy of the hyperbolic plane $U$ such that $(e^2) = (f^2) = 0$ and $(e, f) = 1$. We prove the following

**Theorem 1.1.** Let $K$ be a number field. Assume the Hodge, Tate and Fontaine-Mazur conjectures. Let $\{\rho_\ell: \Gamma_K \to O(\Lambda \otimes \mathbb{Z}_\ell)\}$ be a weakly compatible system of semisimple representations such that

1. For all but finitely many primes $\ell$, $(e + df) \otimes 1 \in (\Lambda \otimes \mathbb{Z}_\ell(1))^{\Gamma_K}$
2. For some $\ell_0$, $\rho_{\ell_0}$ is de Rham at all $\ell | \ell_0$.
3. For some $\ell_1$, $\text{End}(\rho_{\ell_1} \otimes \mathbb{Q}_{\ell_1}) = \mathbb{Q}_{\ell_1} \oplus \mathbb{Q}_{\ell_1}$.
4. For some $\ell_2$ and some $\ell | \ell_2$, $\rho_{\ell_2}|_{\Gamma_{K'}}$ is de Rham with Hodge-Tate weights 0, 1, 2 of multiplicities 1, 20, 1.

Then there is a K3 surface $X$ over a finite extension $K'$ of $K$ such that $\rho_\ell|_{\Gamma_{K'}} \cong H^2(X_{K'}, \mathbb{Z}_\ell)$ for all $\ell$. Further if we assume

5. For some $n > 2$, the mod $n$ representation $(e + df) \otimes \mathbb{Z}/n$ is trivial.

then $K'$ can be taken to be a quadratic extension.

The primes $\ell_0, \ell_1, \ell_2$ could all be the same. Conditions (1), (2) and (4) are of course necessary conditions to have the collection $\{\rho_\ell\}$ to be isomorphic to $H^2(X_{K'}, \mathbb{Z}_\ell)$ for $X$ a K3 surface with a primitive polarization of degree $2d$. Condition (5) is a technical condition needed in a descent argument and can always be arranged after taking a finite extension of $K$ (whose degree can be easily bounded). Condition (3) is an irreducibility condition (similar to the hypothesis of absolute irreducibility in [PVZ16, Thm 3.3]), and is satisfied by the cohomology of the generic K3 surface, i.e. those of geometric Picard rank 1 with $\text{End}_{\Gamma_K}(H^2(X_{K'}, \mathbb{Q}_\ell)) \cong \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$.

1.1. **Questions.** In the recent preprint [Bal18], Baldi independently proves an analogue of theorem 1.1 above for K3 surfaces whose Picard rank $\rho$ satisfies $12 \leq \rho < 20$. This is theorem 1.2 of [Bal18] where he shows for representations

$$\{\rho_\ell: \Gamma_K \to \text{Gl}_{22-\rho}(\mathbb{Q}_\ell)\}$$
satisfying analogues of conditions (2), (3) and (4) above, there is a finite extension $L$ of $K$ such that $\rho_\ell|_{\Gamma_L}$ is isomorphic to $T(X_L^*)_{\mathbb{Q}_\ell}$. However Baldi does not give a bound on the degree of $L$ over $K$.

In both Baldi's work and this paper, the arguments of [PVZ16] are immediately extended to get a $\mathbb{Q}$-Hodge structure $V$ of K3 type from the collection of $\ell$-adic representations. The next key step, which is unique to the K3 case, is to produce a lattice inside this $\mathbb{Q}$-Hodge structure and to show that this lattice is isomorphic to a (sub-)polarized $\mathbb{Z}$-Hodge structure of $H^2(X, \mathbb{Z})$ for some complex K3 surface $X$. In Baldi's paper, this is done by picking any lattice $T$ and then using an embedding theorem of Nikulin to get an embedding of $T$ into the K3 lattice $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Then one shows that the Hodge structure induced by $T$ on $\Lambda$ is of K3 type and hence, by surjectivity of the period map, isomorphic to $H^2(X, \mathbb{Z})$ for a K3 surface $X$ over $\mathbb{C}$. Baldi's requirement that $12 \leq \rho < 20$ is a consequence of the fact that this is the range for which the embedding theorem holds. For our proof, we show that $V \cong \Lambda \otimes \mathbb{Q}$ as quadratic spaces and thus produce a Hodge structure on the K3 lattice $\Lambda$ from which we can apply surjectivity of the period map. In light of our theorem and Baldi's theorem it is natural to ask whether such a theorem holds for 'Picard rank' $2 \leq \rho \leq 11$, i.e. compatible system of representations $\{\rho_\ell: \Gamma_K \to O(\Lambda \otimes \mathbb{Z}_\ell)\}$ which decompose as $\rho_\ell \otimes \mathbb{Q}_\ell \cong V_\ell \oplus \mathbb{Q}_\ell(-1)^{\oplus \rho}$ with $V_\ell$ irreducible. The author does not believe that the proofs of this paper or those of Baldi can be adapted to prove analogous theorem for $2 \leq \rho \leq 11$.

It is unknown to the author whether the quadratic extension $K'$ of $K$ in theorem 1.1 is necessary. It is the consequence of the fact that for generic K3 surfaces $X$ there are two distinct K3 surfaces in $S(X)$, the isogeny class of $X$ (see lemma 3.3). According to remark 3.4 the other element is $M(v)$, a moduli space of sheaves on $X$ with Mukai vector $v$ for some $v$. It would be interesting to know which Mukai vector $v$ has $M(v)$ isogenous to $X$.

Our original motivation was to apply theorem 1.1 to answer a question about the section conjecture for moduli spaces of K3 surfaces, as was done in [PVZ16, Thm 1.1] for abelian varieties. The moduli space we are interested in is the space $F_{2d}$, using the notation of [Riz05], classifying primitively polarized K3 surfaces of degree $2d$. Using the following diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(\overline{F_{2d}}) & \longrightarrow & \pi_1(F_{2d}) & \longrightarrow & \Gamma_K & \longrightarrow & 1 \\
& & & \alpha \downarrow & & \downarrow & & \\
& & & O(\Lambda_{2d} \otimes \mathbb{Z}) & & & & \\
\end{array}
$$

we can associate to each section $s \in H(F_{2d}, K)$ an $O(\Lambda_{2d} \otimes \mathbb{Z})$ representation. If we knew that this map was a bijection then an analogue of [PVZ16, Thm 1.1] could be proven. However, a computation of the group $\pi_1(\overline{F_{2d}})$ seems difficult. One approach might be to compute the topological fundamental group of the complex analytic space $F_{2d}^{an}$ and then compare the profinite completion to a suitable orthogonal group. The domain $F_{2d}^{an}$ has an explicit description as the quotient by an orthogonal group of the complement of an infinite union of hyperplane sections in a period domain $D_{2d}$, see [Huy16, Remark 6.3.7]. Given this explicit description, one may be able to compute the topological fundamental group. This is expected to be very large, containing an infinitely generated free group generated by loops around the hyperplane sections. These are the sorts of groups that are not residually finite,
and the kernel to the profinite completion can be very large, see [Tol93]. So while $\pi_1^{\text{top}}(\mathcal{F}_{2d})$ is very far from any orthogonal group, it may happen that the ‘non-orthogonal’ part gets killed in the profinite completion.

Finally, the last question that naturally follows from this is paper is whether there are analogues of theorem 1.1 for hyperkahler varieties. There are known results on surjectivity of the period map and global Torelli theorems are known for hyperkahler manifolds, see [Huy11], so it may be reasonable that methods in this paper could work for such varieties.

1.2. Terminology. Throughout the paper, $K$ will be a number field with a fixed algebraic closure $\overline{K}$. We write $\Gamma_K = \text{Gal}(\overline{K}/K)$ for the absolute Galois group.

If $K$ is a field, and $F$ a field of characteristic 0, we denote by $\mathcal{M}_{K,F}$ the category of pure numerical motives over $K$ with coefficients in $F$. If $F = \mathbb{Q}$ we simply write $\mathcal{M}_K$. The functors $H_\ell, H_B, H_{dR}$ are the $\ell$-adic, Betti and algebraic de Rham realization functors. Implicitly whenever we write any of these functors, we are assuming the conjecture that numerical equivalence is equal to homological equivalence for that cohomology theory, and in this way the realization functors may be defined on numerical motives.

See section 2 for notation about quadratic forms and lattices. If $V, W$ are $\mathbb{Z}$ (resp. $\mathbb{Q}$) Hodge structures equipped with pairings (e.g. polarizations) a map $V \to W$ is a $\mathbb{Z}$ (resp. $\mathbb{Q}$) Hodge isometry if it is an isomorphism of Hodge structures that also respects the pairings. All K3 surfaces are smooth and projective. We write $\mathbb{Q}$-HS for the category of $\mathbb{Q}$-Hodge structures.

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2. Quadratic forms and lattices

2.1. Notation. If $F$ is a field of characteristic zero, a quadratic space over $F$ consists of a vector space $V$ with a non degenerate symmetric bilinear pairing $V \otimes V \to F$ (or equivalently a quadratic form on $V$). Of interest to us are quadratic spaces over $\mathbb{Q}, \mathbb{Q}_p$ and $\mathbb{R}$. When we talk about a lattice $T$, we mean a finitely generated free abelian group $T$ with a pairing $(\cdot, \cdot): T \times T \to \mathbb{Z}$ that is non degenerate.

2.2. Lattices associated to K3 surfaces. We write $U$ to denote the hyperbolic plane and $E_8$ the lattice associated to the Dynkin diagram $E_8$.

Example 2.1. Let $\Lambda$ be the lattice $U^\oplus 3 \oplus E_8(-1)^\oplus 2$. The discriminant $d(\Lambda) = -1$ so $\Lambda$ is even, unimodular and has signature $(3, 19)$. This is called the K3 lattice, because if $X$ is a K3 surface over $\mathbb{C}$, then the singular cohomology $H^2(X, \mathbb{Z})$ with the cup product pairing is isomorphic to $\Lambda$. See [Huy16].

We will need the following lemma for the proof of the main theorem.

Lemma 2.2. Let $\Lambda$ be the K3 lattice. Then there exists a lattice $\Gamma$ of signature $(r, s)$ such that $\Gamma \otimes \mathbb{Z}_p \cong \Lambda \otimes \mathbb{Z}_p$ for all finite primes $p$ if and only if $(r, s)$ is one of the following pairs: $(19, 3), (15, 7), (11, 11), (7, 15), (3, 19)$
Proof. By [Cas08, Thm 1.2, pg 129] we reduce to reduce to finding a \( \mathbb{Q} \)-quadratic space \( V \), equivalent over \( \mathbb{Q}_p \) to \( \Lambda \otimes \mathbb{Q}_p \) for all finite primes, with signature \((r, s)\). Now by [Cas08, Thm 1.3, pg 77] and the fact that we know the local data imply that \((-1)^{s}\) (here we use theorem 1.3, page 77; theorem 1.2, page 56 loc. cit.). Again those same two theorems and the fact that we know the local data imply that \((-1)^{s}\) + 1 \equiv 0 \pmod{4}. By the Grunwald Wang theorem, we can assume that \(d(V) = d(\Lambda)\). Therefore by [Cas08, Thm 1.2, pg 56] we have \((-1)^s = -1\). We conclude from theorem 1.3 page 77 loc. cit. that the required \( V \) will exist if and only if \( s \equiv 3 \mod 4 \). Therefore the possible signatures are \((19, 3), (15, 7), (11, 11), (7, 15), (3, 19)\).

\]

\[\prod_{p \neq \infty} c_p(V) = \prod_{p \neq \infty} c_p(\Lambda) = c_\infty(\Lambda) = -1\]

(here we use theorem 1.3, page 77; theorem 1.2, page 56 loc. cit.).

3. K3 surfaces

3.1. Facts about K3 surfaces. For convenience of the reader, we recall facts about K3 surfaces that we will use. All proofs may be found in the book [Huy16] whose terminology we use.

Let \( V \) be a finite free \( \mathbb{Z} \) or \( \mathbb{Q} \) module. A Hodge structure of K3 type on \( V \) is a weight 2 Hodge structure such that \( V^{2,0}, V^{0,2} \) are 1 dimensional and \( V^{i,j} = 0 \) for \( |i - j| > 2 \). Let \( \Lambda \) be the K3 lattice (see example 2.1). The period domain \( D \) is defined as

\[D := \{x \in \mathbb{P}(\Lambda) : (x, \overline{x}) > 0, \quad (x)^2 = 0\}\]

Given an element \( x \in D \) we get a unique Hodge structure of K3 type on \( \Lambda \) satisfying \( \Lambda^{2,0} = x \) and \( x \perp \Lambda^{1,1} \). The key facts that we use about K3 surfaces are the global Torelli theorem and surjectivity of the period map, which we recall here.

**Theorem 3.1** (Global Torelli theorem, [Huy16, Thm 5.3, pg 135]). Two complex K3 surfaces \( X, X' \) are isomorphic if and only if there exists a Hodge isometry \( H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z}) \). Moreover for any Hodge isometry \( \psi : H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z}) \) such that \( \psi(\mathcal{K}_X) \cap \mathcal{K}_{X'} \) is non empty, there is a unique isomorphism \( f : X' \to X \) with \( f^* = \psi \) (where \( \mathcal{K}_X \) is the Kahler cone).

**Theorem 3.2** (Surjectivity of the period map, [Huy16, Rmk 3.3, pg 112]). For any \( x \in D \) there exists a K3 surface \( X \) and a Hodge isometry \( \varphi : H^2(X, \mathbb{Z}) \isom \Lambda \) such that \( \varphi^{-1}(x) \) spans \( H^{2,0}(X) \).

Following the definition of [Huy17], two complex K3 surfaces \( X, X' \) are isogenous if there exists a \( \mathbb{Q} \)-Hodge isometry \( H^2(X, \mathbb{Q}) \isom H^2(X', \mathbb{Q}) \). We record the following lemma to be used later.

**Lemma 3.3.** Let \( X \) be a complex K3 surface such that \( \text{End}_{\text{Q-HS}}(H^2(X, \mathbb{Q})) = \mathbb{Q} \oplus \mathbb{Q} \) (in particular \( \rho(X) = 1 \)). Let \( H_X \in \text{NS}(X) \) be a primitive ample class and suppose \( (H_X)^2 = 2d \). Then the set

\[S(X) := \{Y : Y \text{ isogenous to } X \text{ and } Y \text{ has a primitive ample class } H_Y \text{ with } (H_Y)^2 = 2d\}/ \cong \]

has one element if \( d = 1 \) and two elements if \( d > 1 \).
Proof. Suppose $\varphi: H^2(X, \mathbb{Q}) \rightarrow H^2(Y, \mathbb{Q})$ is a Hodge isometry. Write $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ for the K3 lattice and let $e, f$ be a basis of the first copy of the hyperbolic plane with $(e)^2 = (f)^2 = 0, (e, f) = 1$ and $v = e + df$. According to [Huy16, Cor 14.1.10] we can fix an isometry $\psi_X: \Lambda \rightarrow H^2(X, \mathbb{Z})$ with $\psi_X(v) = H_X \in \text{NS}(X)$ and write $\Lambda(X)$ for $\Lambda$ with the Hodge structure making $\psi_X$ an isomorphism of $\mathbb{Z}$-Hodge structures. Since $\varphi$ is an isomorphism of Hodge structures we have that $k(\text{NS}(Y)) = 1$. Let $H_Y$ be an primitive ample class in $\text{NS}(Y)$ with $(H_Y)^2 = 2d$. As above, fix a marking $\psi_Y: \Lambda \rightarrow H^2(Y, \mathbb{Z})$ with $\psi_Y(v) = H_Y$ and write $\Lambda(Y)$ for $\Lambda$ with this Hodge structure. Since $\psi_X, \psi_Y$ are $\mathbb{Z}$-Hodge isometries, the map $g: \Lambda(X) \rightarrow \Lambda(Y)\mathbb{Q}$ given as the composition

$$
\Lambda(X)\mathbb{Q} \rightarrow H^2(X, \mathbb{Q}) \xrightarrow{\varphi^*} H^2(Y, \mathbb{Q}) \rightarrow \Lambda(Y)\mathbb{Q}
$$

is a $\mathbb{Q}$-Hodge isometry. Since the Picard numbers of $X$ and $Y$ are 1 we have an orthogonal direct sum decomposition $\Lambda(X)\mathbb{Q} \cong \mathbb{Q} \cdot v \oplus W$ with $W = (\mathbb{Q} \cdot v)^{\perp}$, and likewise $\Lambda(Y)\mathbb{Q} \cong \mathbb{Q} \cdot v \oplus W$. As $g$ preserves the line $\mathbb{Q} \cdot v$ it must also preserve $W$. On each piece $g$ must be multiplication by a scalar. Since $g$ is an isometry, and $\text{End}(H^2(X, \mathbb{Q})) = \mathbb{Q} \oplus \mathbb{Q}$ the scalar must be $\pm 1$. In particular $g = (\epsilon_1 \text{id}_{\mathbb{Q} \cdot v}, \epsilon_2 \text{id}_W)$ with $\epsilon_i = \pm 1$. If $\epsilon_1 = \epsilon_2$ then $g$ preserves the lattice $\Lambda \subset \Lambda\mathbb{Q}$ and it gives a $\mathbb{Z}$-Hodge isometry

$$H^2(X, \mathbb{Z}) \rightarrow \Lambda(X) \xrightarrow{g} \Lambda(Y) \rightarrow H^2(Y, \mathbb{Z})$$

Thus by the global Torelli theorem, $X$ and $Y$ are isomorphic.

It is now clear that $|S| \leq 2$. To show equality for $d > 1$, consider the isometry given by $g(v) = v$ and $g(w) = -w$ for $w \in W$. It is clear that $g$ preserves the second two copies of the hyperbolic plane (the ones not containing $v$) and it preserves the two copies $E_8(-1)$. When restricted to the first copy of $U$ with basis $e, f$, $g$ sends $v = e + df$ to $e + df$. It sends $e - df$ to $-e + df$. Hence the matrix in the $e, f$ basis is

$${\begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix}}.$$ 

If $d > 1$ this does not preserve $U$, so $g$ cannot preserve $\Lambda$. We claim that $g$ does not preserve the Hodge structure on $\Lambda(X)$ either. If $x \in \Lambda\mathbb{C}$ determines the Hodge structure $\Lambda(X)$ (thinking of $x$ as an element of the period domain) then $g$ preserves the Hodge structure on $\Lambda(X)\mathbb{Q}$ if and only if $g(x) = \lambda x$ for some $\lambda \in \mathbb{C}^{\times}$. Write $x = av + w$ with $a \in \mathbb{C}$ and $w \in W\mathbb{C}$. Then $g(x) = av - w$ so we would get $av - w = \lambda av + \lambda w$ from which we conclude $\lambda = 1$ and $w = 0$. However $(av)^2 = 2da^2$ is 0 only if $a = 0$, i.e. $x = 0$, which is a contradiction. If we write $\Lambda(Y)\mathbb{Q}$ for $\Lambda\mathbb{Q}$ with the Hodge structure determined by $g(x)$, then by surjectivity of the period map, there is a complex K3 surface $Y$ and a Hodge isometry $H^2(Y, \mathbb{Z}) \cong \Lambda \subset \Lambda(Y)\mathbb{Q}$. By the above discussion, there is no Hodge isometry between $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ so $X, Y$ are not isomorphic. \[\Box\]

Remark 3.4. If $X$ as in the lemma has a primitive polarization of degree $2d$ with $d > 1$, then the unique K3 surface $Y$ that is isogenous to $X$ but not isomorphic to $X$ must be a Fourier-Mukai partner of $X$. Namely, the proof above shows that the $\mathbb{Q}$-Hodge isometry between $X$ and $Y$ will give a $\mathbb{Z}$-Hodge isometry of $T(X)$ and $T(Y)$, so a theorem of Orlov [Orl97] shows that $Y$ is a Fourier Mukai partner of $X$. Further, $Y$ is isomorphic to $M(v)$, a fine moduli space of stable sheaves on $X$ with Mukai vector $v$. 
4. Motivic setup

We recall basics facts about motives and refer the reader to [And04] for details. For a field $K$, write $\mathcal{P}_K$ for the category of smooth projective varieties over $K$. If $F$ is a field of characteristic zero we write $\mathcal{M}_{K,F}$ for the category of pure homological motives over $K$ with coefficients in $F$. There is a functor $h: \mathcal{P}_K \to \mathcal{M}_{K,F}$ that functions as a universal cohomology theory, meaning that if $H^*: \mathcal{P}_K \to F$-Alg is a Weil cohomology theory, then $H^*$ extends uniquely through $h$ to a functor $\mathcal{M}_{K,F} \to F$-Alg. Under the conjecture that numerical equivalence is the same as homological equivalence then the category $\mathcal{M}_{K,F}$ is a semisimple rigid abelian tensor category by [Jan92]. A choice of a Weil cohomology theory $H^*$ that extends to $\mathcal{M}_{K,F}$ is a fiber functor, making $\mathcal{M}_{K,F}$ a neutral Tannakian category.

Thus by general theory, we have an equivalence between $\mathcal{M}_{K,F}$ and $\text{Rep}_{G_{K,F}}(F)$, the category of $F$-representations of the pro-reductive algebraic group $G_{K,F} = \text{Aut}\otimes H^*(F)$, the category of $F$ representations of the pro-reductive algebraic group $G_{K,F} = \text{Aut}\otimes H^*$. We recall the most basic examples of fiber functors, and the extra structures they carry.

Example 4.1. Let $K$ be any field, $K^{\text{sep}}$ be a separable closure and $\ell$ a prime. For a smooth projective variety over $K$, the $\ell$-adic cohomology $H_\ell(X) = H^*_\ell(X^{\text{Ksep}}, \mathbb{Q}_\ell)$ is a Weil cohomology theory on $\mathcal{P}_K$. Further, $H_\ell(X)$ has a natural $\Gamma_K = \text{Gal}(K^{\text{sep}}/K)$ action, and we write $H_\ell: \mathcal{M}_{K,Q_\ell} \to \text{Rep}_{Q_\ell}(\Gamma_K)$ for the enriched $\ell$-adic realization functor. The Tate conjecture asserts that $H_\ell$ is fully faithful when $K$ is a number field.

Example 4.2. Let $K = \mathbb{C}$. For a smooth projective variety $X$ over $\mathbb{C}$ we can form the corresponding analytic manifold $X^{\text{an}}$. Singular cohomology $H_B(X) = H^*_\text{sing}(X^{\text{an}}, \mathbb{Q})$ is a Weil cohomology theory on $\mathcal{P}_\mathbb{C}$. Further, $H_B(X)$ has a $\mathbb{Q}$-Hodge structure, and we write $H_B: \mathcal{M}_\mathbb{C} \to \mathbb{Q}$-HS for the enriched Betti realization functor. The Hodge conjecture asserts that $H_B$ is fully faithful.

Example 4.3. Let $K$ be a field of characteristic 0. For a smooth projective variety $X$ over $K$ we have the algebraic de Rham complex $\Omega^\bullet_X/K$. Algebraic de Rham cohomology $H^{dR}_d(X) = H^\bullet(X, \Omega^\bullet_X/K)$ is a Weil cohomology theory on $\mathcal{P}_K$ (with coefficients in $K$). Further, $H^{dR}(X)$ has a filtration, and we write $H^{dR}: \mathcal{M}_{K,K} \to \text{Fil}_K$ (with $\text{Fil}_K$ the category of filtered $K$-vector spaces) for the corresponding enriched de Rham realization functor.

For a given embedding $\iota: \mathbb{Q} \to \overline{\mathbb{Q}}_\ell$ let $H_\iota$ be the composition $\mathcal{M}_{K,\overline{\mathbb{Q}}} \to \mathcal{M}_{K,\overline{\mathbb{Q}}_\ell} \to \text{Rep}_{\overline{\mathbb{Q}}_\ell}(\Gamma_K)$.

The following lemma is taken from [PVZ16], with a slight weakening of the hypothesis due to [Moo17], in which it is shown that the Tate conjecture implies the Grothendieck-Serre semisimplicity conjecture.

Lemma 4.4 (Lemma 3.3, [PVZ16]). Assume the Tate and Fontaine-Mazur conjectures. If $r_\ell: \Gamma_K \to \text{GL}_N(\mathbb{Q}_\ell)$ is an irreducible geometric Galois representation. Then there exists an object $M$ of $\mathcal{M}_{K,\overline{\mathbb{Q}}}$ such that $r_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \cong H_\iota(M)$.

5. Proof of main theorem

The first part of the proof follows closely that of [PVZ16, Thm 3.1], the main difference being that we have to worry about carrying the bilinear form through the motivic yoga.
Proof of theorem 1.1. Let \( \{ \rho_\ell : \Gamma_K \to O(\Lambda \otimes \mathbb{Z}_\ell) \} \) be as in the theorem. Fix an embedding \( \iota_0 : \mathbb{Q} \hookrightarrow \mathbb{Q}_\ell \). Then from lemma 4.4 (which obviously extends to semisimple geometric representations) we have a motive \( M \in \mathcal{M}_{K,E} \) such that \( H_{i_0}(M) \cong \rho_{i_0} \otimes \mathbb{Q}_\ell \). In fact, \( M \) has coefficients in some finite extension \( E \) of \( \mathbb{Q} \). Let \( \rho : \mathcal{G}_{K,E} \to GL_{22,E} \) be the associated motivic Galois representation. Now \( \iota_0 \) induces some place \( \lambda_0 \) of \( E \). If \( \lambda | \ell \) is another prime, and we let \( \rho_\lambda \) be the \( \lambda \)-adic realization of \( \rho \) then
\[
\text{tr}(\rho_\lambda(Fr_v)) = \text{tr}(\rho_{i_0}(Fr_v)) = \text{tr}(\iota_0(Fr_v)) = \text{tr}(\rho_\ell(Fr_v))
\]
Our representations are semisimple, so by Brauer-Nesbitt and Chebotarev
\[
\rho_\lambda \cong \rho_\ell \otimes \mathbb{Z}_\ell E_\lambda
\]
for all \( \lambda \). Conditions (1) and (3) of our assumption say for a place \( \lambda_1 | \ell_1 \) that \( \rho_{\ell_1} \otimes E_{\lambda_1}(1) \) splits as a sum of the trivial representation and an absolutely irreducible representation. By the Tate conjecture, we conclude that \( \rho = 1(-1) \oplus \rho' \) (where \( 1(-1) \) is the Tate twist of \( 1 \)) with \( \rho' \) absolutely irreducible. It follows that each \( \rho_\ell \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell \) is isomorphic to \( \mathbb{Q}_\ell(-1) \oplus V_\ell \) with \( V_\ell \) an absolutely irreducible representation of \( \Gamma_K \).

Lemma 3.4 of [PVZ16] shows that \( \rho' \) descends to \( \mathbb{Q} \), and clearly \( 1(-1) \) does, hence \( \rho \) descends to \( \mathbb{Q} \). We have \( \Gamma_K \)-equivariant pairings \( \rho_\ell \otimes \rho_\ell \to \mathbb{Z}_\ell \), and thus there are pairings \( \rho_\lambda \otimes E \rho_\lambda \to E_\lambda \). By the Tate conjecture, there is an isomorphism
\[
\left( \text{Sym}^2 \rho_{\ell}^\vee \right)^{\mathcal{G}_{K,E}} \otimes E_\lambda \cong \left( \text{Sym}^2 \rho^{\vee}_\lambda \right)^{\Gamma_K}
\]
Hence we get a non degenerate \( \mathcal{G}_{K,E} \)-equivariant pairing \( \rho_E \otimes E \rho_E \to E \). However each local pairing descends to \( \mathbb{Q}_\ell \). By Galois descent the map
\[
\text{Hom}_{\mathcal{G}_K}(\rho \otimes \mathbb{Q}, \mathbb{Q}) \to \text{Hom}_{\mathcal{G}_{K,E}}(\rho_E \otimes E, E)^{\Gamma_Q}
\]
is an isomorphism, and therefore the pairing on \( \rho_E \) descends to a pairing \( \rho \otimes \mathbb{Q} \rho \to \mathbb{Q} \).

Now that we have a motivic Galois representation \( \rho : \mathcal{G}_K \to GL_{22,\mathbb{Q}} \) whose \( \ell \)-adic realizations are \( \rho_\ell \otimes \mathbb{Z}_\ell \mathbb{Q}_\ell \), we can do some comparisons. Let \( M \in \mathcal{M}_K \) be the corresponding rank 22 motive. Objects of \( \mathcal{M}_K \) enjoy the de Rham comparison theory of \( p \)-adic Hodge theory. In particular, for \( v, \ell_2 \) as in condition (4) there are isomorphisms
\[
H_{dR}(M) \otimes K_v B_{dR,K_v} \cong H_{\ell_2}(M) \otimes \mathbb{Q}_{\ell_2} B_{dR,K_v}
\]
Hence
\[
H_{dR}(M) \otimes K_{v,\ell_2}(H_{\ell_2}(M)) = \left( H_{\ell_2}(M) \otimes \mathbb{Q}_{\ell_2} B_{dR,K_{v,\ell_2}} \right)^{\Gamma_{K,v}}
\]
By assumption (4) and the comparison isomorphism, the Hodge filtration on \( H_{dR}(M) \) satisfies
\[
\dim_K \text{gr}^i H_{dR}(M) = \begin{cases} 1 & \text{if } i = 0, 2 \\ 20 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}
\]
The Betti de-Rham comparison theorem states that
\[
H_{dR}(M|_C) \cong H_B(M) \otimes \mathbb{Q},
\]
so \( H_B(M) \) is a Hodge structure of K3 type. It is also a \( \mathbb{Q} \)-quadratic space, coming from the motivic pairing. We will show that there is an isomorphism of \( \mathbb{Q} \)-quadratic spaces \( H_B(M) \cong \Lambda \otimes \mathbb{Q} \) with \( \Lambda \) the K3 lattice \( U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \). First, note that we have comparison isomorphisms \( H_B(M) \otimes \mathbb{Q}_\ell \cong H_\ell(M) \) that respect the pairings on both spaces. By assumption
$H_k(M) \cong \Lambda \otimes \mathbb{Q}_\ell$ as quadratic spaces, so to show that $H_B(M) \cong \Lambda \otimes \mathbb{Q}$ it is enough to show that $H_B(M)$ has the same signature as $\Lambda$, which is $(3,19)$. Now $M = 1(-1) \oplus M'$ with $M'$ absolutely irreducible. Thus we have an orthogonal decomposition of Hodge structures $H_B(M) = \mathbb{Q}(-1) \oplus H_B(M')$ with $H_B(M')$ irreducible.

We compute the possible signatures on $H_B(M')$. First, there is a smooth projective $X$ such that $M' \hookrightarrow h^k(X)(j)$ for some integers $k,j$ where $h^k(X)(j)$ is the motive whose realization is $H^k(X) \otimes \mathbb{Q}(j)$. The pairing on $M'$ is up to a scalar multiple the same as the intersection pairing coming from a polarization $L$ on $X$ because $\dim_{\mathbb{Q}}(\text{Sym}^2 \rho^\vee)^{\otimes k} = 1$. Since $M'$ has weight 2, we know that $k$ is even and $2 = k - 2j$. There is a decomposition of motives [And04, Prop 5.2.5.1]

$$h^k(X) = \bigoplus_{r \leq k} L^r h^{k-2r} \text{prim}(X)(-r)$$

so $M \subset L^r h^{k-2r} \text{prim}(X)$ for some $r$ and thus $H^{1,1}(M') \subset L^r H^{j+1-r,j+1-r} \text{prim}(X)$. The intersection pairing on this subspace is definite by the Hodge index theorem [Vo02, Thm 6.32]. The same reasoning shows that the pairing on $H^{2,0}(M'), H^{0,2}(M')$ is also definite. By lemma 2.2 the only possible signatures of $H_B(M)$ are $(3,19), (7,15), (11,11), (15,7)$ and $(19,3)$. Since we have decomposed $H_B(M) \otimes \mathbb{C}$ into an orthogonal direct sum of three 1-dimensional spaces $(H^{1,1}(1(-1)), H^{2,0}(M'), H^{0,2}(M'))$ and a 19 dimensional subspace on which the form is definite, the only possible signatures are $(3,19)$ or $(19,3)$. The quadratic form restricted to $H_B(1(-1))$ is determined by an element $\alpha$ of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$. The comparison isomorphism

$$H_B(1(-1)) \otimes \mathbb{Q}_\ell \cong H_\ell(1(-1)) = \mathbb{Q}_\ell(e + df) \otimes 1$$

and the fact that $(e + df)^2 = 2d$ shows that the image of $\alpha$ in $\mathbb{Q}_\ell^\times/(\mathbb{Q}_\ell^\times)^2$ is $2d$ for almost all $\ell$. Thus by the Grunwald-Wang theorem, we have that $\alpha = 2d$. Consequently, there is an isomorphism of quadratic spaces $H_B(1(-1)) \cong \mathbb{Q}(2d)$ where the bilinear form on $\mathbb{Q}(2d)$ is given by $(a,b) = 2dab$. In particular, the pairing on $H_B(1(-1))$ is positive definite. We conclude that the signature on $H_B(M)$ is $(3,19)$ which completes the proof that $H_B(M) \cong \Lambda \otimes \mathbb{Q}$ as quadratic spaces. Since the line spanned by $(e + df)$ in $\Lambda \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q}(2d)$ isomorphic to $H_B(1(-1)) \subset H_B(M)$, we can choose by a theorem of Witt [Cas08, Ch. 2 Thm 4.1] an embedding $i: \Lambda \hookrightarrow H_B(M)$ such that $i(e + df)$ spans $H_B(1(-1))$. Write $\Lambda(M)$ for the image of $\Lambda$ under this embedding.

Now $\Lambda(M)$ has an induced Hodge structure from that on $H_B(M)$. By surjectivity of the period map (theorem 3.2) we know that there is a K3 surface $X$ over $\mathbb{C}$ with a Hodge isometry $H^2(X,\mathbb{Z}) \cong \Lambda \subset H_B(M)$. The Hodge conjecture implies that $M|_\mathbb{C} \cong h^2(X)$ and further that this isomorphism respects the pairings on each motive. For each $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ we have

$$h^2(X)^\sigma \cong M|_\mathbb{C}^\sigma = M|_\mathbb{C} \cong h^2(X),$$

where $M|_\mathbb{C}$ is the image of $M$ under base change $\mathcal{M}_K \rightarrow \mathcal{M}_\mathbb{C}$ and $h^2(X)^\sigma, M|_\mathbb{C}^\sigma$ are the $\sigma$ conjugates of $h^2(X)$ and $M|_\mathbb{C}$. Since these isomorphisms respect the pairing, upon applying Betti realization we see that $X$ is isogenous to each conjugate $X^\sigma$. We also have that

$$\text{End}_{\mathbb{Q}_{\text{HS}}}(H^2(X,\mathbb{Q})) = \text{End}_{\mathbb{Q}_{\text{HS}}}(H_B(M)) = \mathbb{Q} \oplus \mathbb{Q}$$

Hence by corollary 6.2, $X$ admits a model, which we denote also by $X$, over a finite extension $L$ of $K$. Note that $X$ has Picard rank 1, and carries a primitive polarization of degree 2d since $e + df$ generates $\text{NS}(X)$ (under the identification $\Lambda \cong H^2(X_\mathbb{C},\mathbb{Z})$).
We have an isomorphism $M|_L \to h^2(X)$ in $\mathcal{M}_L$. For each $\sigma \in \text{Gal}(L/K)$ write $X^\sigma$ for the conjugate variety. Since the motive $h^2(X)$ descends to $K$, $X$ and $X^\sigma$ are isogenous over $\mathbb{C}$. Further, $X^\sigma$ also carries a primitive polarization of degree $2d$. Thus we get an action of $\text{Gal}(L/K)$ on the set $\mathcal{S}(X_\mathbb{C})$ (introduced in lemma 3.3) which has size $\leq 2$ by the same lemma. Consequently $\text{Gal}(L/K')$ stabilizes the isomorphism class of $X_\mathbb{C}$ for some quadratic extension $K' \supset K$. For each $\sigma \in \text{Gal}(L/K')$ we have that $X^\sigma$ is a twist of $X$, i.e. corresponds to a cohomology class in

$$H^1(\Gamma_L, \text{Aut}(X_\mathbb{C})) = \text{Hom}(\Gamma_L, \text{Aut}(X_\mathbb{C}))$$

By the global Torelli theorem $\text{Aut}(X_\mathbb{C}) \to \text{O}_{\text{Hodge}}(H^2(X_\mathbb{C}, \mathbb{Z})) \cong \{\pm 1\}^2$. The last isomorphism is because the endomorphism algebra of $H^2(X_\mathbb{C}, \mathbb{Q})$ is isomorphic to $\mathbb{Q} \oplus \mathbb{Q}$. Therefore, there exists a biquadratic extension $L(\sigma)$ of $L$ over which $X$ and $X^\sigma$ become isomorphic. If we enlarge $L$ to contain all of the $L(\sigma)$, then we have $L$ isomorphisms $f_\sigma: X^\sigma \to X$ for all $\sigma \in \text{Gal}(L/K')$. In fact $f_\sigma$ gives an isomorphism of degree $2d$ primitively polarized K3 surfaces $(X^\sigma, \lambda^\sigma) \to (X, \lambda)$. This is because $f_\sigma^*(\lambda)$ must be in the Kahler cone $K_{X^\sigma}$, and since both varieties have Picard rank 1, it therefore must map to the generator of the Kahler cone which is $\lambda^\sigma$.

We now handle the descent from $L$ to $K'$. Our approach is to exhibit $X$ over $L$ as a Galois invariant point of a moduli space represented by a separated scheme to show $X$ arises from a $K'$ point. The moduli space we will use is $\mathcal{N} = \mathcal{F}_{2d,\mathbb{K}_n,\mathbb{Q}}$ for any $n > 2$ so that $\mathcal{N}$ is a separated scheme, see [Riz05]. An $L$-point of $\mathcal{N}$ consists of a triple $(Y, \lambda, \alpha)$ where $Y$ is a K3 surface over $L$, $\lambda$ is a primitive polarization of degree $2d$ and $\alpha: \Lambda_{2d} \otimes \mathbb{Z}/n \xrightarrow{\sim} P^2(X_\mathbb{T}, \mathbb{Z}/n)$ is a $\Gamma_L$-invariant isomorphism. Here $\Lambda_{2d} = (e - df) \oplus U_{\mathbb{Z}/2} \oplus E_{8}(-1)_{\mathbb{Z}/2}$ is the orthogonal complement of $e + df$ in $\Lambda$ and $\Lambda_{2d} \otimes \mathbb{Z}/n$ has trivial $\Gamma_L$-action.

For our given $X$, we already have a primitive polarization $\lambda$ of degree $2d$. Further from $H^2(X_\mathbb{T}, \mathbb{Q}_\ell) \cong \rho_\ell \otimes \mathbb{Q}_\ell$ we get $P^2(X_\mathbb{T}, \mathbb{Q}_\ell) \xrightarrow{\sim} \rho' \otimes \mathbb{Q}_\ell$, with $\rho' = (e + df)^\perp$. Since both are absolutely irreducible, after possibly scaling by a power of $\ell$, we get an integral isomorphism $P^2(X_\mathbb{T}, \mathbb{Z}_\ell) \xrightarrow{\sim} \rho'$, with $P^2(X_\mathbb{T}, \mathbb{Z}_\ell) = P^2(X_\mathbb{T}, \mathbb{Q}_\ell) \cap H^2(X_\mathbb{T}, \mathbb{Z}_\ell)$. Thus by assumption (5) we get a trivialization with $\alpha: \Lambda_{2d} \otimes \mathbb{Z}/n \xrightarrow{\sim} (e + df)^\perp \otimes \mathbb{Z}/n \xrightarrow{\sim} P^2(X_\mathbb{T}, \mathbb{Z}/n)$

For each $\sigma \in \text{Gal}(L/K')$ there are isomorphisms of polarized varieties $f_\sigma: (X^\sigma, \lambda^\sigma) \to (X, \lambda)$ and hence $f_\sigma^*: P^2(X_\mathbb{T}, \mathbb{Z}/n) \xrightarrow{\sim} P^2(X_\mathbb{T}, \mathbb{Z}/n)$. Thus $\alpha^\sigma = f_\sigma^* \circ \alpha$ is a trivialization of $P^2(X_\mathbb{T}, \mathbb{Z}/n)$ and $f_\sigma: (X^\sigma, \lambda^\sigma, \alpha^\sigma) \to (X, \lambda, \alpha)$ is an isomorphism of primitively polarized K3 surfaces with full level $n$ structure. Consequently $(X, \lambda, \alpha) \in \mathcal{N}(L)^{\text{Gal}(L/K')}$ which is equal to $\mathcal{N}(K')$ as $\mathcal{N}$ is a separated scheme, see [Riz10, Corollary 2.4.3]. In particular, $X$ can be defined over $K'$ as required. \hfill \Box

6. Descent

Lemma 6.1. Let $X$ be a complex variety whose conjugates $X^\sigma$ for $\sigma \in \text{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ fall into finitely many isomorphism classes. Then $X$ admits a model over a number field.

Proof. We can choose $\sigma_1, \ldots, \sigma_n \in \text{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ so that every conjugate of $X$ is isomorphic to $X^{\sigma_i}$ for some $i$. For each $i$, $X^{\sigma_i}$ is defined over a finitely generated field extension $K_i$ over $\overline{\mathbb{Q}}$. Let $K \subset \mathbb{C}$ be the composite in $\mathbb{C}$ of $K_1, \ldots, K_n$ so that $X^{\sigma_1}, \ldots, X^{\sigma_n}$ (hence any conjugate
of $X$) admit models over $K$. Note $K$ finitely generated over $\mathbb{Q}$. Let $\tau \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ be any automorphism with $\tau(K)$, $K$ linearly disjoint over $K$ (see [Mil96, Ch 16, Lem 16.30] for the existence of such a $\tau$). Suppose $X_0$ is a model of $X$ over $K$. As $X^\tau$ is isomorphic (over $\mathbb{C}$) to $X^\sigma_i$ for some $i$, $X_0^\tau$ is a model of $X^\sigma_i$ over $\tau(K)$. Thus $X^\sigma_i$ admits a model over $K$ and $\tau(K)$, and as these are linearly disjoint over $\mathbb{Q}$, $X^\sigma_i$ admits a model over $\mathbb{Q}$. Hence $X$ admits a model over $\mathbb{Q}$. Since $X$ is finite type, the model over $\mathbb{Q}$ will be defined over some finite extension $L$ of $\mathbb{Q}$.

□

**Corollary 6.2.** If $X$ is a complex $K3$ surface with $\text{End}_{\mathbb{Q}-\text{HS}}(H^2(X, \mathbb{Q})) \cong \mathbb{Q} \oplus \mathbb{Q}$ and $X$ is isogenous to $X^\sigma$ for all $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ then $X$ admits a model over a number field.

**Proof.** We have $\text{Pic}(X) \cong \text{Pic}(X^\sigma)$ for all $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ and hence $X^\sigma \in S(X)$ where $S(X)$ is the isogeny class of $X$. From lemma 3.3 we have $|S(X)| \leq 2$ and by the previous lemma it follows that $X$ has a model over a number field. □

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