Heat transfer in a complex medium

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Abstract

The heat equation is considered in the complex medium consisting of many small bodies (particles) embedded in a given material. On the surfaces of the small bodies an impedance boundary condition is imposed. An equation for the limiting field is derived when the characteristic size \( a \) of the small bodies tends to zero, their total number \( N(a) \) tends to infinity at a suitable rate, and the distance \( d = d(a) \) between neighboring small bodies tends to zero: \( a << d \), \( \lim_{a \to 0} \frac{d}{N(a)} = 0 \). No periodicity is assumed about the distribution of the small bodies. These results are basic for a method of creating a medium in which heat signals are transmitted along a given line. The technical part for this method is based on an inverse problem of finding potential with prescribed eigenvalues.

Keywords:
heat transfer; many-body problem; transmission of heat signals; inverse problems; materials science.

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1 Introduction and results

In this paper the problem of heat transfer in a complex medium consisting of many small impedance particles of an arbitrary shape is solved. Equation for the effective limiting temperature is derived when the characteristic size \( a \) of the particles tends to zero while their number tends to infinity at a suitable rate while the distance \( d \) between closest neighboring particles is much larger than \( a \), \( d >> a \).

These results are used for developing a method for creating materials in which heat is transmitted along a line. Thus, the information can be transmitted by a heat signals.

The contents of this paper is based on the earlier papers of the author cited in the bibliography, especially [6], [18] and [19].

Let many small bodies (particles) \( D_m \), \( 1 \leq m \leq M \), of an arbitrary shape be distributed in a bounded domain \( D \subset \mathbb{R}^3 \), \( \text{diam} D_m = 2a \), and the boundary
of $D_m$ is denoted by $S_m$ and is assumed twice continuously differentiable. The small bodies are distributed according to the law

$$N(\Delta) = \frac{1}{a^2 - \kappa} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0. \tag{1}$$

Here $\Delta \subset D$ is an arbitrary open subdomain of $D$, $\kappa \in [0, 1)$ is a constant, $N(x) \geq 0$ is a continuous function, and $N(\Delta)$ is the number of the small bodies $D_m$ in $\Delta$. The heat equation can be stated as follows:

$$u_t = \nabla^2 u + f(x) \text{ in } \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m, := \Omega, \quad u|_{t=0} = 0, \tag{2}$$

$$u_N = \zeta_m u \text{ on } S_m, \quad 1 \leq m \leq M, \quad Re\zeta_m \geq 0. \tag{3}$$

Here $N$ is the outer unit normal to $S$,

$$S := \bigcup_{m=1}^{M} S_m, \quad \zeta_m = \frac{h(x_m)}{a^\kappa}, \quad x_m \in D_m, \quad 1 \leq m \leq M,$$

and $h(x)$ is a continuous function in $D$, $Reh \geq 0$.

Denote

$$U := U(x, \lambda) = \int_0^\infty e^{-\lambda t} u(x, t)dt.$$ 

Then, taking the Laplace transform of equations (2) - (3) one gets:

$$- \nabla^2 U + \lambda U = \lambda^{-1} f(x) \text{ in } \Omega, \tag{4}$$

$$U_N = \zeta_m U \text{ on } S_m, 1 \leq m \leq M. \tag{5}$$

Let

$$g(x, y) := g(x, y, \lambda) := \frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|\sqrt{\lambda}|x-y|}, \tag{6}$$

$$F(x, \lambda) := \frac{1}{\lambda} \int_{\mathbb{R}^3} g(x, y)f(y)dy. \tag{7}$$

Look for the solution to (4) - (5) of the form

$$U(x, \lambda) = F(x, \lambda) + \sum_{m=1}^{M} \int_{S_m} g(x, s)\sigma_m(s)ds, \tag{8}$$

where

$$U(x, \lambda) := U(x), \tag{9}$$

and $U(x)$ depends on $\lambda$.

The functions $\sigma_m$ are unknown and should be found from the boundary conditions (5). Equation (4) is satisfied by $U$ of the form (3) with arbitrary
continuous $\sigma_m$. To satisfy the boundary condition (5) one has to solve the following equation obtained from the boundary condition (5):

$$\frac{\partial U_e(x)}{\partial N} + \frac{A_m \sigma_m - \sigma_m}{2} - \zeta_m U_e - \zeta_m T_m \sigma_m = 0 \text{ on } S_m, \quad 1 \leq m \leq M,$$

where the effective field $U_e(x)$ is defined by the formula:

$$U_e(x) := U_{e,m}(x) := U(x) - \int_{S_m} g(x, s) \sigma_m(s) ds,$$

the operator $T_m$ is defined by the formula:

$$T_m \sigma_m = \int_{S_m} g(s, s') \sigma_m(s') ds',$$

and $A_m$ is:

$$A_m \sigma_m = 2 \int_{S_m} \frac{\partial g(s, s')}{\partial N_s} \sigma_m(s') ds'.$$

In deriving equation (10) we have used the known formula for the outer limiting value on $S_m$ of the normal derivative of a simple layer potential.

We now apply the ideas and methods for solving many-body scattering problems developed in [6] - [9].

Let us call $U_{e,m}$ the effective (self-consistent) value of $U$, acting on the $m$-th body. As $a \to 0$, the dependence on $m$ disappears, since

$$\int_{S_m} g(x, s) \sigma_m(s) ds \to 0 \text{ as } a \to 0.$$

One has

$$U(x, \lambda) = F(x, \lambda) + \sum_{m=1}^{M} g(x, x_m) Q_m + \mathcal{J}_2, \quad x_m \in D_m,$$

where

$$Q_m := \int_{S_m} \sigma_m(s) ds,$$

$$\mathcal{J}_2 := \sum_{m=1}^{M} \int_{S_m} |g(x, s') - g(x, x_m)| \sigma_m(s') ds'.$$

Define

$$\mathcal{J}_1 := \sum_{m=1}^{M} g(x, x_m) Q_m.$$

We prove in Lemma 3, Section 4 (see also [6] and [18]) that

$$|\mathcal{J}_2| << |\mathcal{J}_1| \text{ as } a \to 0$$

(17)
provided that
\[
\lim_{a \to 0} \frac{a}{d(a)} = 0,
\]  
(18)
where \(d(a) = d\) is the minimal distance between neighboring particles.

If (17) holds, then problem (1) - (5) is solved asymptotically by the formula
\[
U(x, \lambda) = F(x, \lambda) + \sum_{m=1}^{M} g(x, x_m)Q_m, \quad a \to 0,
\]  
(19)
provided that asymptotic formulas for \(Q_m\), as \(a \to 0\), are found.

To find formulas for \(Q_m\), let us integrate (10) over \(S_m\), estimate the order of the terms in the resulting equation as \(a \to 0\), and keep the main terms, that is, neglect the terms of higher order of smallness as \(a \to 0\).

We get
\[
\int_{S_m} \frac{\partial U_e}{\partial N} ds = \int_{D_m} \nabla^2 U_e dx = O(a^3).
\]  
(20)
Here we assumed that \(|\nabla^2 U_e| = O(1), a \to 0\). This assumption is valid since \(U = \lim_{a \to 0} U_e\) is smooth as a solution to an elliptic equation. One has
\[
\int_{S_m} \frac{A_m \sigma_m - \sigma_m}{2} ds = -Q_m[1 + o(1)], a \to 0.
\]  
(21)
This relation is proved in Lemma 2, Section 4, see also [6]. Furthermore,
\[
-\zeta_m \int_{S_m} U_e ds = -\zeta_m |S_m|U_e(x_m) = O(a^{2-\kappa}), \quad a \to 0,
\]  
(22)
where \(|S_m| = O(a^2)\) is the surface area of \(S_m\). Finally,
\[
-\zeta_m \int_{S_m} ds \int_{S_m} g(s, s')\sigma_m(s') ds' = -\zeta_m \int_{S_m} ds' \sigma_m(s') \int_{S_m} ds g(s, s')
\]  
\[
= Q_m O(a^{1-\kappa}), \quad a \to 0.
\]  
(23)
Thus, the main term of the asymptotic of \(Q_m\), as \(a \to 0\), is
\[
Q_m = -\zeta_m |S_m|U_e(x_m).
\]  
(24)
Formulas (24) and (19) yield
\[
U(x, \lambda) = F(x, \lambda) - \sum_{m=1}^{M} g(x, x_m)\zeta_m |S_m|U_e(x_m, \lambda),
\]  
(25)
and
\[
U_e(x_m, \lambda) = F(x_m, \lambda) - \sum_{m' \neq m, m'=1}^{M} g(x_m, x_{m'})\zeta_{m'} |S_{m'}|U_e(x_{m'}, \lambda). \]  
(26)
Denote
\[ U_e(x_m, \lambda) := U_m, \quad F(x_m, \lambda) := F_m, \quad g(x_m, x_{m'}) := g_m(x_m, x_{m'}), \]
and write (26) as a linear algebraic system for \( U_m \):
\[
U_m = F_m - a^{2-\kappa} \sum_{m' \neq m} g_{mm'} h_m c_m U_{m'}, \quad 1 \leq m \leq M, \tag{27}
\]
where \( h_{mm'} = h(x_{mm'}), \quad \zeta_{mm'} = \frac{h_{mm'}}{a^2}, \quad c_{mm'} := |S_{mm'}| a^{-2}. \)
Consider a partition of the bounded domain \( D \), in which the small bodies are distributed, into a union of \( P << M \) small nonintersecting cubes \( \Delta_p, 1 \leq p \leq P \), of side \( b \),
\[
b >> d, \quad b = b(a) \to 0 \quad \text{as} \quad a \to 0 \quad \lim_{a \to 0} \frac{d(a)}{b(a)} = 0.
\]
Let \( x_p \in \Delta_p, |\Delta_p| = \text{volume of} \ \Delta_p \). One has
\[
a^{2-\kappa} \sum_{m'=1, m' \neq m}^M g_{mm'} h_m c_m U_{m'} = a^{2-\kappa} \sum_{p'=1, p' \neq p}^P g_{pp'} h_{p'} c_{p'} U_{p'} \sum_{x_{m'} \in \Delta_{p'}} 1 = \sum_{p' \neq p} g_{pp'} h_{p'} c_{p'} U_{p'} N(x_{p'}) |\Delta_{p'}| [1 + o(1)], \quad a \to 0. \tag{28}
\]
Thus, (27) yields a linear algebraic system (LAS) of order \( P << M \) for the unknowns \( U_p \):
\[
U_p = F_p - \sum_{p' \neq p, p' = 1}^P g_{pp'} h_{p'} c_{p'} U_{p'} |\Delta_{p'}|, \quad 1 \leq p \leq P. \tag{29}
\]
Since \( P << M \), the order of the original LAS (27) is drastically reduced. This is crucial when the number of particles tends to infinity and their size \( a \) tends to zero. We have assumed that
\[
h_{mm'} = h_{pp'}[1 + o(1)], \quad c_{mm'} = c_{pp'}[1 + o(1)], \quad U_{mm'} = U_{pp'}[1 + o(1)], \quad a \to 0, \tag{30}
\]
for \( x_{mm'}, \in \Delta_{pp'} \). This assumption is justified, for example, if the functions \( h(x), \ U(x, \lambda), \)
\[
c(x) = \lim_{x_{mm'} \in \Delta, a \to 0} \frac{|S_{mm'}|}{a^2},
\]
and \( N(x) \) are continuous, but these assumptions can be relaxed.

The continuity of the \( U(x, \lambda) \) is a consequence of the fact that this function satisfies elliptic equation, and the continuity of \( c(x) \) is assumed. If all the small bodies are identical, then \( c(x) = c = \text{const} \), so in this case the function \( c(x) \) is certainly continuous.

The sum in the right-hand side of (29) is the Riemannian sum for the integral
\[ \lim_{\alpha \to 0} \sum_{p' = 1, p' \neq p}^P g_{pp'} h_{pp'} N(x_{p'}) \Delta'_{p'} = \int_D g(x, y) h(y) c(y) N(y) U(y, \lambda) dy. \]

Therefore, linear algebraic system (29) is a collocation method for solving integral equation

\[ U(x, \lambda) = F(x, \lambda) - \int_D g(x, y) c(y) h(y) N(y) U(y, \lambda) dy. \] (31)

Convergence of this method for solving equations with weakly singular kernels is proved in [10], see also [11] and [12].

Applying the operator \(-\nabla^2 + \lambda\) to equation (31) one gets an elliptic differential equation:

\[ (-\Delta + \lambda) U(x, \lambda) = \frac{f(x)}{\lambda} - c(x) h(x) N(x) U(x, \lambda). \] (32)

Taking the inverse Laplace transform of this equation yields

\[ u_t = \Delta u + f(x) - q(x) u, \quad q(x) := c(x) h(x) N(x). \] (33)

Therefore, the limiting equation for the temperature contains the term \(q(x)u\). Thus, the embedding of many small particles creates a distribution of source and sink terms in the medium, the distribution of which is described by the term \(q(x)u\).

If one solves equation (31) for \(U(x, \lambda)\), or linear algebraic system (29) for \(U_p(\lambda)\), then one can Laplace-invert \(U(x, \lambda)\) for \(U(x, t)\). Numerical methods for Laplace inversion from the real axis are discussed in [13] - [14].

If one is interested only in the average temperature, one can use the relation

\[ \lim_{\lambda \to 0} \frac{1}{\lambda} \int_0^\infty u(x, t) dt = \lim_{\lambda \to 0} \lambda U(x, \lambda). \] (34)

Relation (34) is proved in Lemma 1, Section 4. It holds if the limit on one of its sides exists. The limit on the right-hand side of (34) let us denote by \(\psi(x)\).

From equations (7) and (31) it follows that \(\psi\) satisfies the equation

\[ \psi = \varphi - B \varphi, \]

where

\[ \varphi := \int_\Omega g_0(x, y) f(y) dy, \]

\[ g_0(x, y) := \frac{1}{4\pi |x - y|}, \]
\[ B \varphi := \int_{\Omega} g_0(x, y) q(y) \varphi(y) dy, \]

and

\[ q(x) := c(x) h(x) N(x). \]

The function \( \varphi \) can be calculated by the formula

\[ \psi(x) = (I + B)^{-1}\varphi. \] (35)

From the physical point of view the function \( h(x) \) is non-negative because the flux \(-\nabla u\) of the heat flow is proportional to the temperature \( u \) and is directed along the outer normal \( N \): \(-u N = h_1 u\), where \( h_1 = -h < 0 \). Thus, \( q \geq 0 \).

It is proved in [15] - [16] that zero is not an eigenvalue of the operator \(-\nabla^2 + q(x)\) provided that \( q(x) \geq 0 \) and

\[ q = O\left(\frac{1}{|x|^{2+\epsilon}}\right), \quad |x| \to \infty, \]

and \( \epsilon > 0 \).

In our case, \( q(x) = 0 \) outside of the bounded region \( D \), so the operator \((I + B)^{-1}\) exists and is bounded in \( C(D) \).

Let us formulate our basic result.

**Theorem 1.** Assume (1), (18), and \( h \geq 0 \). Then, there exists the limit \( U(x, \lambda) \) of \( U\varphi(x, \lambda) \) as \( \lambda \to 0 \), \( U(x, \lambda) \) solves equation (31), and there exists the limit (34), where \( \psi(x) \) is given by formula (35).

Methods of our proof of Theorem 1 are quite different from the proof of homogenization theory results in [1] and [3].

The author’s plenary talk at Chaos-2015 Conference was published in [20].

## 2 Creating materials which allows one to transmit heat signals along a line

In applications it is of interest to have materials in which heat propagates along a line and decays fast in all the directions orthogonal to this line.

In this Section a construction of such material is given. We follow [19] with some simplifications.

The idea is to create first the medium in which the heat transfer is governed by the equation

\[ u_t = \Delta u - q(x) u \quad \text{in} \quad D, \quad u|_{S} = 0, \quad u|_{t=0} = f(x), \] (36)

where \( D \) is a bounded domain with a piece-wise smooth boundary \( S, D = D_0 \times [0, L], D_0 \subset \mathbb{R}^2 \) is a smooth domain orthogonal to the axis \( x_1, x = (x_1, x_2, x_3), x_2, x_3 \in D_0, 0 \leq x_1 \leq L. \)
Such a medium is created by embedding many small impedance particles into a given domain \( D \) filled with a homogeneous material. A detailed argument, given in Section 1 (see also [6] and [18]), yields the following result.

Assume that in every open subset \( \Delta \) of \( D \) the number of small particles is defined by the formula:

\[
\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x) dx [1 + o(1)], \quad a \to 0, \tag{37}
\]

where \( a > 0 \) is the characteristic size of a small particle, \( \kappa \in [0,1) \) is a given number and \( N(x) \geq 0 \) is a continuous in \( D \) function.

Assume also that on the surface \( S_m \) of the \( m \)-th particle \( D_m \) the impedance boundary condition holds. Here

\[
1 \leq m \leq M = \mathcal{N}(D) = O\left(1 + \frac{1}{a^{2-\kappa}}\right), \quad a \to 0,
\]

and the impedance boundary conditions are:

\[
u_N = \zeta_m u \quad \text{on} \ S_m, \quad \Re \zeta_m \geq 0, \tag{38}\]

where

\[
\zeta_m := \frac{h(x_m)}{a^{\kappa}}
\]

is the boundary impedance, \( x_m \in D_m \) is an arbitrary point (since \( D_m \) is small the position of \( x_m \) in \( D_m \) is not important), \( \kappa \) is the same parameter as in (37) and \( h(x) \) is a continuous in \( D \) function, \( \Re h \geq 0, N \) is the unit normal to \( S_m \) pointing out of \( D_m \). The functions \( h(x), N(x) \) and the number \( \kappa \) can be chosen as the experimenter wishes.

It is proved in Section 1 (see also [6], [18]) that, as \( a \to 0 \), the solution of the problem

\[
u_t = \Delta u \quad \text{in} \quad D \setminus \bigcup_{m=1}^M D_m, \quad \mathcal{N} = \zeta_m u \quad \text{on} \ S_m, 1 \leq m \leq M, \tag{39}\]

\[
u|_{S} = 0, \tag{40}\]

and

\[
u|_{t=0} = f(x), \tag{41}\]

has a limit \( u(x,t) \). This limit solves problem (36) with

\[
g(x) = c_S N(x) h(x), \tag{42}\]

where

\[
c_S := \frac{|S_m|}{a^2} = \text{const}, \tag{43}\]
and \( |S_m| \) is the surface area of \( S_m \). By assuming that \( c_S \) is a constant, we assume, for simplicity only, that the small particles are identical in shape, see [6].

Since \( N(x) \geq 0 \) is an arbitrary continuous function and \( h(x) \), \( \Re h \geq 0 \), is an arbitrary continuous function, and both functions can be chosen by experimenter as he/she wishes, it is clear that an arbitrary real-valued potential \( q \) can be obtained by formula (42).

Suppose that
\[
(-\Delta + q(x))\phi(x) = \lambda_n \phi_n, \quad \phi_n|_S = 0, \quad ||\phi_n||_{L^2(D)} = ||\phi_n|| = 1,
\]
where \( \{\phi_n\} \) is an orthonormal basis of \( L^2(D) := H \). Then the unique solution to (36) is
\[
\begin{align*}
\int \frac{\partial^2 u(x,t)}{\partial t^2} + q(x)u(x,t) = f(x,t), \\
\int u(x,t) = 0, \quad \int u(x,t) = 0, \\
\int \frac{\partial^2 u(x,t)}{\partial t^2} + q(x)u(x,t) = f(x,t), \\
\int u(x,t) = 0, \quad \int u(x,t) = 0.
\end{align*}
\]

(44)

If \( q(x) \) is such that \( \lambda_1 = 0, \lambda_2 \gg 1, \) and \( \lambda_2 \leq \lambda_3 \leq \ldots \), then, as \( t \to \infty \), the series (45) is well approximated by its first term
\[
\begin{align*}
\int u(x,t) = (f, \phi_1) \phi_1 + O(e^{-\lambda_1 t}), \\
\int u(x,t) = (f, \phi_1) \phi_1 + O(e^{-\lambda_1 t}),
\end{align*}
\]
(46)

If \( \lambda_1 > 0 \) is very small, then the main term of the solution is
\[
\begin{align*}
\int u(x,t) = (f, \phi_1) \phi_1 e^{-\lambda_1 t} + O(e^{-\lambda_1 t})
\end{align*}
\]
as \( t \to \infty \). The term \( e^{-\lambda_1 t} \sim 1 \) if \( t << \frac{1}{\lambda_1} \).

Thus, our problem is solved if \( q(x) \) has the following property:
\[
|\phi_1(x)| \text{ decays as } \rho \text{ grows, } \quad \rho = (x_2^2 + x_3^3)^{1/2}.
\]
(47)

Since the eigenfunction is normalized, \( ||\phi_1|| = 1 \), this function will not tend to zero in a neighborhood of the line \( \rho = 0 \), so information can be transformed by the heat signals along the line \( \rho = 0 \), that is, along \( s \)-axis. Here we use the cylindrical coordinates:
\[
x = (x_1, x_2, x_3) = (s, \rho, \theta), \quad s = x_1, \quad \rho = (x_2^2 + x_3^3)^{1/2}.
\]

In Section 3 the domain \( D_0 \) is a disc and the potential \( q(x) \) does not depend on \( \theta \).

The technical part of solving our problem consists of the construction of \( q(x) = c_S N(x) h(x) \) such that
\[
\lambda_1 = 0, \quad \lambda_2 \gg 1; \quad |\phi_1(x)| \text{ decays as } \rho \text{ grows}.
\]
(48)

Since the function \( N(x) \geq 0 \) and \( h(x) \), \( \Re h \geq 0 \), are at our disposal, any desirable \( q, \Re q \geq 0 \), can be obtained by embedding many small impedance particles in a given domain \( D \). In Section 3, a potential \( q \) with the desired properties is constructed. This construction allows one to transform information along a straight line using heat signals.
3 Construction of \( q(x) \)

Let 
\[ q(x) = p(\rho) + Q(s), \]
where \( s := x_1, \rho := (x_2^2 + x_3^2)^{1/2} \). Then the solution to problem (44) is 
\[ u = v(\rho)w(s), \]
where 
\[ - v''_m - \rho^{-1} v'_m + p(\rho) v_m = \mu_m v_m, \quad 0 \leq \rho \leq R, \]
\[ |v_m(0)| < \infty, \quad v_m(R) = 0, \quad (49) \]
and 
\[ - w''_l + Q(s) w_l = \nu w_l, \quad 0 \leq s \leq L, \]
\[ w_l(0) = 0, \quad w_l(L) = 0. \quad (50) \]

One has 
\[ \lambda_n = \mu_m + \nu_l, \quad n = n(m, l). \quad (51) \]

Our task is to find a potential \( Q(s) \) such that \( \nu_1 = 0, \nu_2 \gg 1 \) and a potential \( p(\rho) \) such that \( \mu_1 = 0, \mu_2 \gg 1 \) and \( |v_m(\rho)| \) decays as \( \rho \) grows.

It is known how to construct \( q(s) \) with the desired properties: the Gel'fand-Levitan method allows one to do this, see [4]. Let us recall this construction.

One has \( \nu_{l0} = l^2 \), where we set \( L = \pi \) and denote by \( \nu_{l0} \) the eigenvalues of the problem (50) with \( Q(s) = 0 \). Let the eigenvalues of the operator (50) with \( Q \neq 0 \) be \( \nu_1 = 0, \nu_2 = 11, \nu_3 = 14, \nu_l = \nu_{l0} \) for \( l \geq 4 \).

The kernel \( L(x, y) \) in the Gel'fand-Levitan theory is defined as follows:
\[ L(x, y) = \int_{-\infty}^{\infty} \sin(\sqrt{\lambda}x) \sin(\sqrt{\lambda}y) d(\rho(\lambda) - \rho_0(\lambda)), \]
where \( \rho(\lambda) \) is the spectral function of the operator (50) with the potential \( Q = Q(s) \), and \( \rho_0(\lambda) \) is the spectral function of the operator (50) with the potential \( Q = 0 \) and the same boundary conditions as for the operator with \( Q \neq 0 \).

Due to our choice of \( \nu_l \) and the normalizing constants \( \alpha_j \), namely: \( \alpha_j = \frac{\pi}{2} \) for \( j \geq 2 \) and \( \alpha_1 = \frac{\pi}{3} \), the kernel \( L(x, y) \) is given explicitly by the formula:
\[ L(x, y) = \frac{3xy}{\pi^3} + \frac{2}{\pi} \left( \frac{\sin(\sqrt{\nu_2}x) \sin(\sqrt{\nu_2}y)}{\sqrt{\nu_2}} + \frac{\sin(\sqrt{\nu_3}x) \sin(\sqrt{\nu_3}y)}{\sqrt{\nu_3}} \right) - \frac{2}{\pi} \left( \sin x \sin y + \sin(2x) \sin(2y) + \sin(3x) \sin(3y) \right), \quad (52) \]
where \( \nu_1 = 0, \nu_2 = 11 \) and \( \nu_3 = 14 \). This is a finite rank kernel. The term \( xy \) is the value of the function \( \frac{\sin \nu x \sin \nu y}{\nu} \) at \( \nu = 0 \), and the corresponding normalizing constant is \( \frac{2}{\pi^3} = ||x||^2 = \int_0^\pi x^2 dx \).
Solve the Gel’fand-Levitan equation:

\[ K(s, \tau) + \int_0^s K(s, s') L(s', \tau) ds' = -L(s, \tau), \quad 0 \leq \tau \leq s, \quad (53) \]

which is uniquely solvable (see [4]). Since equation (53) has finite-rank kernel it can be solved analytically being equivalent to a linear algebraic system.

If the function \( K(s, \tau) \) is found, then the potential \( Q(s) \) is computed by the formula (2, [4]):

\[ Q(s) = 2 \frac{dK(s, s)}{ds}, \quad (54) \]

and this \( Q(s) \) has the required properties: \( \nu_1 = 0, \nu_2 \gg 1, \nu_l \leq \nu_{l+1} \).

Consider now the operator (49) for \( v(\rho) \). Our problem is to calculate \( p(\rho) \) which has the required properties:

\[ \mu_1 = 0, \quad \mu_2 \gg 1, \quad \mu_m \leq \mu_{m+1}, \]

and \( |\phi_m(\rho)| \) decays as \( \rho \) grows.

We reduce this problem to the previous one that was solved above. To do this, set \( v = \frac{\psi}{\sqrt{\rho}} \). Then equation

\[ -v'' - \frac{1}{\rho} v' + p(\rho)v = \mu v, \]

is transformed to the equation

\[ -\psi'' - \frac{1}{4\rho^2} \psi + p(\rho)\psi = \mu \psi. \quad (55) \]

Let

\[ p(\rho) = \frac{1}{4\rho^2} + Q(\rho), \quad (56) \]

where \( Q(\rho) \) is constructed above. Then equation (55) becomes

\[ -\psi'' + Q(\rho)\psi = \mu \psi, \quad (57) \]

and the boundary conditions are:

\[ \psi(R) = 0, \quad \psi(0) = 0. \quad (58) \]

The problem (57)–(58) has the desired eigenvalues \( \mu_1 = 0, \mu_2 \gg 1, \mu_m \leq \mu_{m+1} \).

The eigenfunction

\[ \phi_1(x) = v_1(\rho)w_1(s), \]

where \( v_1(\rho) = \frac{\psi_1(\rho)}{\sqrt{\rho}} \), decays as \( \rho \) grows, and the eigenvalues \( \lambda_n \) can be calculated by the formula:

\[ \lambda_n = \mu_m + \nu_l, \quad m, l \geq 1, \quad n = n(m, l). \]
Since $\mu_1 = \nu_1 = 0$ one has $\lambda_1 = 0$. Since $\nu_2 = 11$ and $\mu_2 = 11$, one has $\lambda_2 = 11 \gg 1$.

Thus, the desired potential is constructed:

$$q(x) = Q(s) + \left(\frac{1}{4\rho^2} + Q(\rho)\right),$$

where $Q(s)$ is given by formula (54).

This concludes the description of our procedure for the construction of $q$.

**Remark 1.** It is known (see, for example, [2]) that the normalizing constants

$$\alpha_j := \int_0^\pi \varphi_j^2(s) ds$$

and the eigenvalues $\lambda_j$, defined by the differential equation

$$-\frac{d^2 \varphi_j}{ds^2} + Q(s) \varphi_j = \lambda_j \varphi_j,$$

the boundary conditions

$$\varphi'_j(0) = 0, \quad \varphi_j'(\pi) = 0,$$

and the normalizing condition $\varphi_j(0) = 1$, have the following asymptotic:

$$\alpha_j = \frac{\pi}{2} + O\left(\frac{1}{j^2}\right) \text{ as } j \to \infty,$$

and

$$\sqrt{\lambda_j} = j + O\left(\frac{1}{j}\right) \text{ as } j \to \infty.$$

The differential equation

$$-\Psi''_j + Q(s) \Psi_j = \nu_j \Psi_j,$$

the boundary condition

$$\Psi_j(0) = 0, \quad \Psi_j'(\pi) = 0,$$

and the normalizing condition $\Psi'_j(0) = 1$ imply

$$\sqrt{\lambda_j} = j + O\left(\frac{1}{j}\right) \text{ as } j \to \infty,$$

$$\Psi_j(s) \sim \frac{\sin(js)}{j} \text{ as } j \to \infty.$$

The main term of the normalized eigenfunction is:

$$\frac{\Psi_j}{||\Psi_j||} \sim \sqrt{2/\pi} \frac{\sin(js)}{j} \text{ as } j \to \infty,$$

and the main term of the normalizing constant is:

$$\alpha_j \sim \frac{\pi}{2j^2} \text{ as } j \to \infty.$$
4 Auxiliary results

Lemma 1 If one of the limits $\lim_{t \to \infty} \frac{1}{t} \int_0^t u(s)ds$ or $\lim_{\lambda \to 0} \lambda U(\lambda)$ exists, then the other also exists and they are equal to each other:

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t u(s)ds = \lim_{\lambda \to 0} \lambda U(\lambda),$$

where

$$U(\lambda) := \int_0^\infty e^{-\lambda t}u(t)dt := \bar{u}(\lambda).$$

Proof. Denote

$$\frac{1}{t} \int_0^t u(t)dt := v(t), \quad \bar{u}(\sigma) := \int_0^\infty e^{-\sigma t}u(t)dt.$$ 

Then

$$\bar{v}(\lambda) = \int_\lambda^\infty \bar{u}(\sigma)\frac{d\sigma}{\sigma}$$

by the properties of the Laplace transform.

Assume that the limit $v(\infty) := v_\infty$ exists:

$$\lim_{t \to \infty} v(t) = v_\infty. \quad (59)$$

Then,

$$v_\infty = \lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t}v(t)dt = \lim_{\lambda \to 0} \lambda \bar{v}(\lambda).$$

Indeed $\lambda \int_0^\infty e^{-\lambda t}dt = 1$, so

$$\lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t}(v(t) - v_\infty)dt = 0,$$

and (59) is verified.

One has

$$\lim_{\lambda \to 0} \lambda \bar{v}(\lambda) = \lim_{\lambda \to 0} \lambda \int_\lambda^\infty \frac{\lambda}{\sigma} \bar{u}(\sigma)d\sigma = \lim_{\lambda \to 0} \lambda \bar{u}(\lambda), \quad (60)$$

as follows from a simple calculation:

$$\lim_{\lambda \to 0} \int_\lambda^\infty \frac{\lambda}{\sigma} \bar{u}(\sigma)d\sigma = \lim_{\lambda \to 0} \int_0^\infty \frac{\lambda}{\sigma^2} \sigma \bar{u}(\sigma)d\sigma = \lim_{\sigma \to 0} \sigma \bar{u}(\sigma), \quad (61)$$

where we have used the relation $\int_\lambda^\infty \frac{\lambda}{\sigma^2}d\sigma = 1$.

Alternatively, let $\sigma^{-1} = \gamma$. Then,

$$\int_\lambda^\infty \frac{\lambda}{\sigma^2} \sigma \bar{u}(\sigma)d\sigma = \frac{1}{\lambda} \int_0^{1/\lambda} \frac{1}{\gamma} \bar{u}(\frac{1}{\gamma})d\gamma = \frac{1}{\omega} \int_0^\omega \frac{1}{\gamma} \bar{u}(\frac{1}{\gamma})d\gamma. \quad (62)$$
If \( \lambda \to 0 \), then \( \omega = \lambda^{-1} \to \infty \), and if
\[
\psi := \gamma^{-1} \bar{u}(\gamma^{-1}),
\]
then
\[
\lim_{\omega \to \infty} \int_{0}^{\omega} \psi \, d\gamma = \psi(\infty) = \lim_{\gamma \to 0} \gamma^{-1} \bar{u}(\gamma^{-1}) = \lim_{\sigma \to 0} \sigma \bar{u}(\sigma).
\]
Lemma 1 is proved.

\textbf{Lemma 2} Equation (21) holds.
\textit{Proof.} As \( a \to 0 \), one has
\[
\frac{\partial}{\partial N_s} e^{-\sqrt{\lambda} |s-s'|} = \frac{\partial}{\partial N_s} \frac{1}{4\pi |s-s'|} + \frac{\partial}{\partial N_s} \frac{e^{-\sqrt{\lambda} |s-s'|} - 1}{4\pi |s-s'|}.
\]
It is known (see [5]) that
\[
\int_{S_m} ds \int_{S_m} \frac{\partial}{\partial N_s} \frac{1}{4\pi |s-s'|} \sigma_m(s') ds' = -\frac{1}{2} \int_{S_m} \sigma_m(s') ds' = -\frac{1}{2} \bar{Q}_m.
\]
On the other hand, as \( a \to 0 \), one has
\[
\left| \int_{S_m} ds \int_{S_m} \frac{e^{-\sqrt{\lambda} |s-s'|} - 1}{4\pi |s-s'|} \sigma_m(s') ds' \right| \leq |Q_m| \int_{S_m} ds \frac{1 - e^{-\sqrt{\lambda} |s-s'|}}{4\pi |s-s'|} = o(Q_m).
\]
The relations (65) and (66) justify (21).
Lemma 2 is proved.

\textbf{Lemma 3} If assumption (18) holds, then inequality (17) holds.
\textit{Proof.} One has
\[
\mathcal{J}_{1,m} := |g(x, x_m)| Q = \frac{|Q_m| e^{-\sqrt{\lambda} |x-x_m|}}{4\pi |x-x_m|},
\]
and
\[
\mathcal{J}_{2,m} \leq e^{-\sqrt{\lambda} |x-x_m|} \max \left( \sqrt{\lambda} a, \frac{a}{|x-x_m|} \right) \int_{S_m} |\sigma_m(s')| ds'
\]
where \( |x-x_m| \geq d \), and \( d > 0 \) is the smallest distance between two neighboring particles. One may consider only those values of \( \lambda \) for which \( \lambda^{1/4} a < \frac{a}{d} \), because for the large values of \( \lambda \), such that \( \lambda^{1/4} \geq \frac{1}{d} \) the value of \( e^{-\sqrt{\lambda} |x-x_m|} \) is negligibly small. The average temperature depends on the behavior of \( \mathcal{U} \) for small \( \lambda \), see Lemma 1.
One has $|Q_m| = \int_{S_m} |\sigma_m(s')|ds' > 0$ because $\sigma_m$ keeps sign on $S_m$, as follows from equation (24) as $a \to 0$.

It follows from (67) - (68) that

$$\left| \frac{J_2,m}{J_1,m} \right| \leq O\left( \frac{a}{x-x_m} \right) \leq O\left( \frac{a}{d} \right) << 1. \quad (69)$$

From (69) by the arguments similar to the given in [17] one obtains (17). Lemma 3 is proved. \qed
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