MEAN FIELD LIMIT WITH PROLIFERATION

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Abstract. An interacting particle system with long range interaction is considered. Particles, in addition to the interaction, proliferate with a rate depending on the empirical measure. We prove convergence of the empirical measure to the solution of a parabolic equation with non-local nonlinear transport term and proliferation term of logistic type.

1. Introduction. We are concerned with a macroscopic limit result for a system of particles which interact by a mean field potential and may proliferate; we want to prove the convergence of the empirical measure to a parabolic equation with nonlinear non-local transport term, like in classical mean field models, plus a growth term corresponding to the proliferation. Our starting motivation for this research came from Mathematical Oncology, where cells interact by random dynamics and proliferate, and one would like to discover appropriate macroscopic limits (PDE), for instance because a cell-level simulation of a tumor is too demanding, being involved a number of particles of the order of 10^9.

1.1. The microscopic model. The microscopic system, defined on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\), is composed of particles at positions \(X_a^{i,N} \in \mathbb{R}^d\). We label particles by a multi-index \(a = (k, i_1, \ldots, i_n)\) with \(i_1, \ldots, i_n \in \{1, 2\}\) and \(k = 1, \ldots, N\), where \(N \in \mathbb{N}\) is the number of particles at time \(t = 0\). The particles already alive at time \(t = 0\) are those with label \(a = (k)\), \(k = 1, \ldots, N\). Their descendants require the additional labeling; setting \((a, -) := (k, i_1, \ldots, i_n-1)\) if \(a = (k, i_1, \ldots, i_n)\), we may say that \(a\) is a descendant of \((a, -)\). Each particle “lives” only during a random time interval: particle with label \(a\) lives on \(I_{a,N} = [T_{a,N}^0, T_{a,N}^1] \subset [0, \infty)\) where \(T_{a,N}^0, T_{a,N}^1\) are \(\mathcal{F}_t\)-stopping times: it was born at time \(T_{a,N}^0\) (or it exists from \(t = 0\) if \(T_{a,N}^0 = 0\)) and “dies” at time \(T_{a,N}^1\) when it is replaced by two independent particles (this is a proliferation event); the number of alive particles can only increase. Setting \(X_{T_{a,N}^1}^{a,N} := \lim_{t \uparrow T_{a,N}^1} X_t^{a,N}\) we impose \(T_{a,N}^0 = T_{1,(a,-),N}^0\) and \(X_{T_{a,N}^1}^{a,N} = X_{T_{1,(a,-),N}}^{(a,-),N}\). Denote by \(A^N\) the set of all labels \(a\) and

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by \( A_t^N \) the set of particle labels alive at time \( t \), namely the set of \( a \in A^N \) such that \( t \in T_{a,N}^t \); the empirical measure at time \( t \) is defined as

\[
S_t^N = \frac{1}{N} \sum_{a \in A_t^N} \delta_{X_t^a,N}.
\]

In the random time interval \( f^a,N \), particle with label \( a \) interacts with all other living particles through a potential \( V \in C^2_c(\mathbb{R}^d) \); the dynamics of \( X_t^a,N \) is described by the gradient system

\[
dX_t^a,N = \frac{1}{N} \sum_{\tilde{a} \in A_t^N} \nabla V \left(X_t^a,N - X_t^{\tilde{a},N}\right) dt + \sigma dB_t^a
\]

where \( B_t^a \) are independent Brownian motions in \( \mathbb{R}^d \) and \( \sigma > 0 \).

Each particle proliferates at a random rate \( \lambda_t^{a,N} \) which, in most applications, depends on the density of particles it has in the neighborhood. We prescribe a general structure of the form

\[
\lambda_t^{a,N} = F_N \left(S_t^N, X_t^a,N\right)
\]

where the properties of the measurable functionals \( F_N : M_+(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}_+ \) will be specified below (\( M_+(\mathbb{R}^d) \) is the set of Borel finite positive measures on \( \mathbb{R}^d \)). The precise meaning of the previous sentences is that we have standard Poisson processes \( \Lambda_{t,a}^N \), independent between themselves and with respect to the Brownian motions and initial conditions \( X_0^{(k),N} \), and we change their times randomly by setting \( \Lambda_{t,a}^{N,a} := \Lambda_{t,a}^N \) where \( \Lambda_{t,a}^{N,a} = \int_0^t \sum_{s \in I_{a,N}} \lambda_{s,a}^{N,a} ds \); the process \( \Lambda_{t,a}^{N,a} \) is a jump process with rate \( \lambda_{t,a}^{N,a} \) and the stopping time \( T_1^{a,N} \) is the time when \( \Lambda_{t,a}^{N,a} \) jumps from 0 to 1 (and then remains equal to 1).

1.2. Macroscopic limit. Denote by \( W_{1,2}^1(\mathbb{R}^d) \) the subset of the Sobolev space \( W_{1,2}(\mathbb{R}^d) \) made of all non negative functions and by \( C_b^\beta(\mathbb{R}^d) \), \( \beta \in (0,1) \), the space of \( \beta \)-Hölder continuous functions; the \( \beta \)-Hölder seminorm of \( f \) will be denoted by

\[
[|f|]_{\beta} := \sup_{x \neq y} \left( |f(x) - f(y)| / |x - y|^\beta \right).
\]

We say that a map \( F : W_{1,2}^1(\mathbb{R}^d) \to C_b^\beta(\mathbb{R}^d) \) satisfies the mild Lipschitz conditions if it is Lipschitz in the \( C_b^\beta(\mathbb{R}^d) \)-norm and has linear growth in the \( C_b^\beta(\mathbb{R}^d) \)-norm, namely for every \( u, v \in W_{1,2}^1(\mathbb{R}^d) \)

\[
\|F(u) - F(v)\|_{\infty} \leq L_F \|u - v\|_{W_{1,2}} \tag{2}
\]

\[
[F(u)]_{\beta} \leq C \left( \|u\|_{W_{1,2}} + 1 \right). \tag{3}
\]

The macroscopic limit result below requires a number of natural assumptions that we list now, plus the more critical assumption (5) that we emphasize in the statement of the theorem. Let \( \{\theta_N\} \) be a classical family of compact support smooth mollifiers of the form \( \theta_N(x) = \epsilon_N^{-d} \theta(\epsilon_N^{-1} x) \); let us introduce the mollified empirical measure (the theoretical analog of the numerical method of kernel smoothing) \( h_t^N(x) \) defined as

\[
h_t^N = \theta_N \ast S_t^N.
\]

For technical reasons we assume \( \sup_{N \in \mathbb{N}} \epsilon_N^{-d-2} < \infty \); moreover, at least in the case of example (8)-(9) below, the role of \( \theta_N \) is auxiliary, it does not appear in the model, hence the restriction does not have biological relevance. Concerning the interaction potential \( V \), assume that \( V \in C_2^c(\mathbb{R}^d) \). Concerning the initial conditions, assume
that \( u_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) and \( \langle S^N_0, \phi \rangle \) be convergent in probability to \( \langle u_0, \phi \rangle \), as \( N \to \infty \), for every \( \phi \in C_{c,0}^\infty(\mathbb{R}^d) \). Moreover, assume the following technical condition:

\[
\lim_{R \to \infty} \sup_{N \in \mathbb{N}} P \left( \|h^N_0\|^2_{L^2(\mathbb{R}^d)} > R \right) = 0. \tag{4}
\]

A simple and relevant sufficient condition for this property is given by Proposition 1 below. Finally, the definition of weak solution of the PDE (7) is given below in Section 3.1.

**Theorem 1.1.** Assume that, for some \( \beta \in (0,1) \), there is a map \( F : W^{1,2}_+ (\mathbb{R}^d) \to C^1_b (\mathbb{R}^d) \) which satisfies the mild Lipschitz condition (2)-(3) above and there is a sequence of positive real numbers \( \{\alpha_N\} \) converging to zero such that

\[
\|F(\theta_N * \mu) - F_N(\mu, \cdot)\| \leq \alpha_N (1 + \|\mu, 1\|) \tag{5}
\]

Assume moreover that there exists a constant \( C_F > 0 \) such that

\[
|F_N(\mu, x)| \leq C_F \quad \|F(u)\|_\infty \leq C_F \tag{6}
\]

for every \( \mu, x, u \in M_+(\mathbb{R}^d) \times \mathbb{R}^d \times W^{1,2}_+(\mathbb{R}^d) \). Then the process \( h^N_t(x) \) converges in probability, as \( N \to \infty \), to the unique weak solution of the PDE

\[
\partial_t u_t = \text{div} \left( (\nabla V * u_t) u_t \right) + \frac{\sigma^2}{2} \Delta u_t + F(u_t) u_t, \quad u|_{t=0} = u_0. \tag{7}
\]

The topologies of convergence of \( h^N_t(x) \) to \( u_t(x) \) are

- the strong topology of \( L^2_{\text{loc}}([0,T] \times \mathbb{R}^d) \),
- the weak topology of \( L^2([0,T] \times W^{1,2}(\mathbb{R}^d)) \) and
- the weak star topology of \( L^\infty(0,T; L^2(\mathbb{R}^d)) \).

Having in mind Fisher-Kolmogorov-Petrovskii-Piskunov equation and in general the concept of logistic growth, the most natural example of functional \( F(u) \) is \( F(u)(x) = (1 - u(x))^+ \), which means that proliferation rate decreases when we approach the threshold \( u = 1 \) (the value 1 is conventional). More precisely, due to the term \( \text{div} \left( (\nabla V * u_t) u_t \right) \), the solution \( u_t \) may overcome any threshold, so one has to correct the classical logistic term and consider

\[
F(u)(x) = (1 - u(x))^+ \tag{8}
\]

(proliferation is completely inhibited when the threshold is crossed). Our theory covers this case only in dimension \( d = 1 \), where \( W^{1,2}_+(\mathbb{R}^d) \subset C^1_b(\mathbb{R}^d) \) for some \( \beta \in (0,1) \). In this case we take

\[
F_N(\mu, x) = (1 - (\theta_N * \mu)(x))^+ \tag{9}
\]

and assumption (5) is obviously satisfied; the others are elementary. In dimension \( d > 1 \) this example is not covered but we can treat the case

\[
F(u)(x) = (1 - (W * u)(x))^+ \tag{8}
\]

where \( W \) is a Lipschitz continuous compact support probability density. In this case we simply take

\[
F_N(\mu, x) = (1 - (W * \mu)(x))^+. \tag{9}
\]

The validity of assumptions (2)-(3)-(5)-(6) in this case is shown in Lemma 3.10 below.

A variety of macroscopic limit results has been proved in the literature; let us quote, related to the present one, [11], [12], [14], [13], [8], [9], [10] and references
therein, from which we have taken several elements of inspiration. However, a result of mean field type with proliferation, in the sense described here, is not treated in the previous references. Let us also mention a different class of macroscopic limit results, for instance [18], [17], which require very different techniques. In the more applied literature, two examples of works related to our problem are [1], [3].

A feature of our approach, shared by some of the previously quoted references, is that we do not use only results of tightness of measure-valued processes but of processes. A more distinguished feature, probably shared only by [9] (which however is very different), is that we use typical tools of the theory of stochastic Partial Differential Equations, for instance the tightness criterion used for stochastic Navier-Stokes equations in [6], [2], see also [5], [4].

1.3. Motivations from mathematical oncology. Although the aim of this paper is mostly theoretical, we have been inspired by lectures about the emerging field of Mathematical Oncology in the choice of the problem and of some details. In this area, roughly speaking, models are classified as macroscopic, when described by partial differential equations, or microscopic, when described by stochastic ordinary differential equations or, even more often, cellular automata and other discrete stochastic models - in addition there are multiscale models with mixture of the previous two cases. Macroscopic models look at the tumor at the tissue level, microscopic ones at the cellular level. The link between the two descriptions is of interest for various reasons, in particular because a precise justification of the macroscopic models is difficult from general arguments based on fluid dynamics or mechanical, a biological tissue being something different. At the cellular level it is easier to be more realistic and thus results on the macroscopic limit of cellular systems is a way to justify or improve the macroscopic models; and also to have different interpretations of the constants appearing in the models - characterizing these constants is a major problem for the real applications. After these general comments, which clarify why we investigate the macroscopic limit, let us say that most models used nowadays in Mathematical Oncology are more complex than our one, since they involve different cell types - e.g. normoxic and hypoxic tumor cells, or cells with different genetic mutations, or cells of the extracellular matrix - molecular fields like oxygen and growth factors, and possibly objects related to the angiogenic cascade. See for instance [16] as an example of complex model. Our model, chosen as a starting point, captures only a few features of such complexity: i) the interaction between cells which may incorporate for instance a certain degree of repulsion resulting from the fact that cells cannot press each other too much - see the term called “crowding effect” in [16], different from our one but corresponding to a similar mechanism; ii) cancer cell proliferation, always present in each model.

Concerning proliferation in detail, most often in the literature of Mathematical Oncology it is taken of logistic form \( u (1 - u) \) (which corresponds to \( F (u) (x) = (1 - u (x))^+ \) above). Such choice of \( F \) simply corresponds to the fact that cells proliferate better when the neighbor is not so crowded. However, other forms of \( F \) may be interesting as well. A phenomenon observed in vitro is that certain isolated cancer cells, even if embedded in a liquid that is rich of nutrients, do not proliferate: they need to adhere to other cells to proliferate. A functional \( F \) which charge - in the sense that decreases proliferation rate - not only the excessive presence of cells in the neighbor but also the opposite case, an excessive isolation, may be more realistic: cells which separate from the main cloud by random motion will continue
their travel to meet blood vessels and lead to metastasis, but along the trip they do not proliferate so often as the cells close to the main tumor body. This, although vague, could be a motivation for investigating a general proliferation mechanism of the form \( \lambda_t^{a,N} = F_N \left( S_t^N, X_t^{a,N} \right) \) above; more modeling work is necessary and is part of our research program.

2. Preparation. Starting from this section, we drop the suffix \( N \) in \( X_t^{a,N} \), \( I_t^{a,N} \), \( T_t^{a,N} \), \( \lambda_t^{a,N} \), \( X_t^{\alpha,N} \) to simplify notations. Let \( \delta \) denote a point outside \( \mathbb{R}^d \), the so called grave state, where we assume the processes \( X_t^a \) live when \( t \notin I^a \). Hence, whenever a particle proliferates and therefore dies, it stays forever in the grave state \( \delta \). In the sequel, the test functions \( \phi \) are assumed to be defined over \( \mathbb{R}^d \times \{ \delta \} \) and be such that \( \phi(\delta) = 0 \). Using Itô formula over random time intervals, one can show that \( \phi(X_t^a) \), with \( \phi \in C^2(\mathbb{R}^d) \), satisfies

\[
\phi(X_t^a) = \phi(X_{T_0}^a) 1_{t \geq T_0} - \phi(X_{T_1}^a) 1_{t \geq T_1} + \int_0^t 1_{s \in I^a} \nabla \phi(X_s^a) dX_s^a + \frac{\sigma^2}{2} \int_0^t 1_{s \in I^a} \Delta \phi(X_s^a) ds.
\]

With a few computations, one can see that the empirical measure \( S_t^N \) satisfies

\[
d \langle S_t^N, \phi \rangle = - \left( \langle \nabla V * S_t^N \rangle S_t^N, \nabla \phi \right) dt + \frac{\sigma^2}{2} \langle S_t^N, \Delta \phi \rangle dt + \langle F_N \left( S_t^N, \cdot \right) S_t^N, \phi \rangle dt + dM_t^{1,\phi,N} + dM_t^{2,\phi,N} \tag{10}
\]

for every \( \phi \in C^2(\mathbb{R}^d) \) and where

\[
M_t^{1,\phi,N} := \frac{\sigma}{N} \sum_{a \in AN} \int_0^t 1_{s \in I^a} \nabla \phi(X_s^a) dB_s^a
\]

\[
M_t^{2,\phi,N} := \frac{1}{N} \sum_{a \in AN} \phi(X_{T_0}^a) 1_{t \geq T_0} - \frac{1}{N} \sum_{a \in AN} \int_0^t \phi(X_s^a) \lambda_s^a ds.
\]

We deduce that \( h_t^N(x) \) satisfies

\[dh_t^N(x) = \left( \text{div} \left( \theta_N * \left( \nabla V * S_t^N \right) \right) (x) \right) + \frac{\sigma^2}{2} \Delta h_t^N(x) + \left( \theta_N * \left( F_N \left( S_t^N, \cdot \right) \right) \right) (x) dt + dM_t^{1,N}(x) + dM_t^{2,N}(x)
\]

where

\[
M_t^{1,N}(x) := - \frac{\sigma}{N} \sum_{a \in AN} \int_0^t 1_{s \in I^a} \nabla \theta_N(x - X_s^a) \cdot dB_s^a,
\]

\[
M_t^{2,N}(x) := \frac{1}{N} \sum_{a \in AN} \theta_N(x - X_{T_0}^a) 1_{t \geq T_0} - \frac{1}{N} \int_0^t \sum_{a \in AN} \theta_N(x - X_s^a) \lambda_s^a ds.
\]

The total relative mass

\[ [S_t^N] := S_t^N(\mathbb{R}^d) = \langle S_t^N, 1 \rangle = \frac{\text{Card} \left( A_t^N \right)}{N} \]
plays a central role. Since, in our model, the number of particles may only increase, we have the inequality
\[ S_N^t \leq S_T^t \]
for all \( t \in [0, T] \), (11) that we shall use very often. The quantity \( S_T^t \) is, moreover, exponentially integrable, uniformly in \( N \), see Lemma 3.11 below. Sometimes, however, having \( S_T^t \) in the inequalities could spoil properties associated to adaptedness, so we also introduce, for every \( R > 0 \), the stopping time
\[ \tau_R^N := \inf \left\{ r \geq 0 : [S_N^r] > R \text{ or } \| h_N^r \|_{L^2(\mathbb{R}^d)} > R \right\} \]
(12) \((\tau_R^N = +\infty \text{ if the set is empty})\). We also repeatedly use the identity
\[ \int_{\mathbb{R}^d} h_N^t(x) dx = [S_N^t] \]
which follows from Fubini Theorem. Other simple rules of calculus we often use are
\[ |\theta_N^*(fS_N^t)(x)| \leq \| f \|_{\infty} h_N^t(x), \quad |(f * S_N^t)(x)| \leq \| f \|_{\infty} [S_N^t] \]
(14) for every bounded measurable \( f : \mathbb{R}^d \to \mathbb{R} \). Finally, we often use the inequalities
\[ \frac{1}{N} \int_{\mathbb{R}^d} |\nabla \theta_N(x)|^2 dx \leq C, \quad \frac{1}{N} \int_{\mathbb{R}^d} |\theta_N(x)|^2 dx \leq C\varepsilon^2_N \]
(15) which holds with a suitable constant \( C > 0 \). The bound on the first term comes from
\[ \frac{1}{N} \int_{\mathbb{R}^d} |\nabla \theta_N(x)|^2 dx = \frac{e^{-d/2}}{N} \int_{\mathbb{R}^d} \frac{e^{-d/2}}{N} |\nabla \theta(x\varepsilon_N^{-1} x)|^2 dx = \frac{e^{-d/2}}{N} \int_{\mathbb{R}^d} |\nabla \theta(x)|^2 dx \]
the assumption that \( \theta \) has compact support and the assumption \( \sup_N \varepsilon_N^{-d/2}/N < \infty \); the bound on the second term is similar.

3. Tightness. In this section, about \( F_N \), we only use (6).

**Definition 3.1.** We define, for a function \( f : \mathbb{R} \to \mathbb{R} \),
\[ f(t-) := \lim_{s \to t^-} f(s), \quad \text{the left limit of } f \text{ at } t, \]
\[ Jf(t) := f(t) - f(t-) \), \quad \text{the jump size of } f \text{ at } t. \]
Further, let \( (X_t)_{t \geq 0} \) be a stochastic process. We define its quadratic variation, when the limit exists and is independent of the partitions,
\[ [X]_t = \mathbb{P} - \lim_{k \to \infty} \sum_{j=0}^{n-1} \left( X_{t_{j+\lambda}} - X_{t_{j+\lambda}} \right)^2, \]
where the maximal distance of two consequent sites in the partition \( \{0 = t_0^k < t_1^k < \cdots < t_n^k = T\} \) converges to 0 as \( k \to \infty \). We denote the continuous part of the quadratic variation by \([X]^c\), i.e.
\[ [X]^c_t = [X]_t - \sum_{s \leq t} JX^2_s. \]
In the following we need a lemma, also known as generalized Itô formula, see for example [15, page 245].
Lemma 3.2. Let $X$ be a one-dimensional semimartingale such that $X_t, X_{t-}$ take values in an open set $U \subset \mathbb{R}$ and $f : U \rightarrow \mathbb{R}$ twice continuously differentiable. Then $f(X)$ is a semimartingale and

\[
    f(X_t) = f(X_0) + \int_0^t f'(X_s-)dX_s + \frac{1}{2} \int_0^t f''(X_s) d[X]_s + \sum_{s \leq t} (Jf(X_s) - f'(X_{s-})JX_s).
\]

We use the generalized Itô formula to derive the following energy estimate.

Lemma 3.3.

\[
    \frac{1}{2} \left\| h_t^N \right\|^2_{L^2} + \frac{\sigma^2}{2} \int_0^t \left\| \nabla h_t^N \right\|^2_{L^2} dr
    = \frac{1}{2} \left\| h_0^N \right\|^2_{L^2} - \int_0^t \left\langle \theta_N * \left( (\nabla V * S^N_r) S^N_r \right), \nabla h_t^N \right\rangle dr
    + \int_0^t \left\langle \theta_N * \left( F_N \left( S^N_r, \cdot \right) S^N_r \right), h_t^N \right\rangle dr + \frac{1}{2} \left\| [M^1,N]_t \right\|_{L^1}
    + \int_0^t \int_0^t \left( h^N_{r-}(x) \right) d \left( M^1,N(x) + M^2,N(x) \right) dx
    + \frac{1}{2} \int_0^t \frac{1}{N^2} \sum_{\alpha \in A^N} \left( \theta_N(x - X^2_{T_r}) \right)^2 1_{t \geq T_r} dx.
\]

Proof. For every $x \in \mathbb{R}^d$, we apply the generalized Itô formula to $((h(x))^2)_{t \geq 0}$ and integrate over $x \in \mathbb{R}^d$:

\[
    \left\| h_t^N \right\|^2_{L^2} = \left\| h_0^N \right\|^2_{L^2} + 2 \int_0^t \int \left( h^N_{r-}(x) d h^N_r(x) \right) dx + \int \left[ h^N(x) \right]^c_t dx + \int \sum_{\alpha \in A^N} \left( J \left( h^N_r(x) \right) - 2 h^N_{r-}(x) J h^N_r(x) \right) dx
    = \left\| h_0^N \right\|^2_{L^2} + \int \left[ M^1,N(x) \right]_t dx + \sigma^2 \int_0^t \int h^N_r(x) \Delta h^N_r(x) dx dr
    + 2 \int_0^t \int h^N_r(x) \text{div} \left( \theta_N * \left( (\nabla V * S^N_r) S^N_r \right) \right)(x) dx dr
    + 2 \int_0^t \int h^N_r(x) \theta_N * \left( F_N \left( S^N_r, \cdot \right) S^N_r \right)(x) dx dr
    + 2 \int_0^t \int h^N_{r-}(x) d \left( M^1,N(x) + M^2,N(x) \right) dx
    + \int \sum_{\alpha \in A^N} \left( J \left( h^N_r(x) \right) - 2 h^N_{r-}(x) J h^N_r(x) \right) dx
\]

where we have used the fact that $\left[ h^N(x) \right]^c_t = [M^1,N(x)]_t$, and that $h^N_{r-}(x) = h^N_r(x)$ for a.e. $r$ (hence we may replace $h^N_{r-}(x)$ by $h^N_r(x)$ in ordinary Lebesgue integrals).

It remains to understand the last line of the previous formula. At every jump time $r$, we have

\[
    J \left( h^N_r(x) \right) - 2 h^N_{r-}(x) J h^N_r(x)
\]
Lemma 3.4.

Moreover, \( Jh_N^{T_1}(x) = \frac{1}{N} \theta_N(x - X_{T_1}^a) \) hence

\[
\sum_{r \leq t} (Jh_N^r(x))^2 = \sum_{a \in A^N} \left( Jh_N^{T_1}(x) \right)^2 1_{t \geq T_1} = \frac{1}{N^2} \sum_{a \in A^N} \left( \theta_N(x - X_{T_1}^a) \right)^2 1_{t \geq T_1}.
\]

In the next lemmata, we generically denote by \( C > 0 \) any constant depending only on \( \sigma, T, \|\nabla V\|_{L_{\infty}}, C_F, \sup_N \epsilon^{-d-2}/N \), moments of \( [S_N^T] \) (see Lemma 3.11), \( \int_{\mathbb{R}^d} |\theta(x)|^2 \, dx \) and \( \int_{\mathbb{R}^d} |\nabla \theta(x)|^2 \, dx \) (recall that \( \theta \) has compact support).

Lemma 3.4.

\[
\left| \int_0^t \langle \theta_N * ((\nabla V * S_N^t) S_r^t), \nabla h_r^t \rangle \, dr \right| \leq \frac{\sigma^2}{4} \int_0^t \|\nabla h_r^t\|_{L^2}^2 \, dr + C \left[ S_N^t \right]^2 \int_0^t \|h_r^t\|_{L^2}^2 \, dr,
\]

\[
\left| \int_0^t \langle \theta_N * (F_N (S_r^t, \cdot) S_r^t), h_r^t \rangle \, dr \right| \leq C \int_0^t \|h_r^t\|_{L^2}^2 \, dr.
\]

Proof. By Hölder inequality we have

\[
\left| \int_0^t \langle \theta_N * ((\nabla V * S_N^t) S_r^t), \nabla h_r^t \rangle \, dr \right| \leq \frac{\sigma^2}{4} \int_0^t \|\nabla h_r^t\|_{L^2}^2 \, dr + C \frac{\sigma^2}{\sigma^2} \int_0^t \|\theta_N * ((\nabla V * S_N^t) S_r^t)\|_{L^2}^2 \, dr
\]

and then we handle the second term by means of the bound (using also (11))

\[
|\theta_N * ((\nabla V * S_N^t) S_r^t)(x)| \leq \|\nabla V\|_{L_{\infty}} \left[ S_N^t \right] h_r^N(x)
\]

which follows from (14). The left-hand-side of the second inequality of the lemma is bounded above by

\[
\leq \int_0^t \left| \langle \theta_N * (F_N (S_r^t, \cdot) S_r^t), h_r^N \rangle \right| \, dr \leq C_F \int_0^t \|h_r^N\|_{L^2}^2 \, dr
\]

because \( |\theta_N * (F_N (S_r^t, \cdot) S_r^t)(x)| \leq C_F h_r^N(x) \) by (14) and (6).

\[
\square
\]

Lemma 3.5.

\[
E \left[ \sup_{t \in [0,T]} \|M^{1,N}_t\|_{L^1} \right] \leq C
\]

\[
E \left[ \sup_{t \in [0,T]} \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A^N} \left( \theta_N(x - X_{T_1}^a) \right)^2 1_{t \geq T_1} \, dx \right] \leq C.
\]

Proof. Since

\[
[M^{1,N}(x)]_t = \frac{\sigma^2}{N^2} \sum_{a \in A^N} \int_0^t 1_{s \in I^a} |\nabla \theta_N(x - X_s^a)|^2 \, ds
\]
we have (using the change of variable $x \mapsto x - X_s^a$ in the Lebesgue integral over $\mathbb{R}^d$)
\[
\int_{\mathbb{R}^d} \left[ M_{1,N}^a(x) \right] d\mu = \frac{\sigma^2}{N^2} \sum_{a \in A^N} \int_0^t 1_{s \in I^a} \left( \int_{\mathbb{R}^d} |\nabla \theta_N(x - X_s^a)|^2 \, dx \right) \, ds
\]
\[
= \frac{\sigma^2}{N^2} \sum_{a \in A^N} \int_0^t 1_{s \in I^a} \left( \int_{\mathbb{R}^d} |\nabla \theta_N(x)|^2 \, dx \right) \, ds = \frac{\sigma^2}{N} \int_{\mathbb{R}^d} |\nabla \theta_N(x)|^2 \, dx \int_0^t [S_N^N] \, ds.
\]
We conclude the first estimate of the lemma using Lemma 3.11 and (15). Similarly, since
\[
\int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A^N} \left( \theta_N(x - X_s^a) \right)^2 1_{t \geq T^a_s} \, dx \leq \frac{1}{N} \int_{\mathbb{R}^d} \, dx \int_0^t \sup_s \left[ S_N^N \right] \, ds.
\]
we deduce the second estimate of the lemma.

**Lemma 3.6.**

\[
E \left[ \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} \int_0^{t \wedge T^a_s} h_s^N(x) \mu(1_{s \in I^a} \, M_{1,N}^a(x) + M_{2,N}^a(x)) \, dx \right| \right] \leq 2 + CRE \left[ \int_0^T \left\| h_{s \wedge T^a_s}^N \right\|_{L^2}^2 \, ds \right].
\]

**Proof.** Obviously the expected value on the left-hand-side above is bounded by
\[
\leq 2 + E \left[ \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} \int_0^{t \wedge T^a_s} h_s^N(x) \mu(1_{s \in I^a} \, M_{1,N}^a(x)) \, dx \right| \right] + E \left[ \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} \int_0^{t \wedge T^a_s} h_{s}^N(x) \mu(1_{s \in I^a} \, M_{2,N}^a(x)) \, dx \right| \right]
\]
(we have also replaced $h_{s}^N(x)$ by $h^N_s(x)$ in the integral with respect to the continuous martingale $M_{1,N}^a$). Concerning the $M_{1,N}^a$-term, we have
\[
E \left[ \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} \int_0^{t \wedge T^a_s} h_s^N(x) \mu(1_{s \in I^a} \, M_{1,N}^a(x)) \, dx \right| \right] = \frac{\sigma^2}{N_2} \left[ \sum_{a \in A^N} \int_0^T \left( 1_{s \in I^a} \, g_s^N(X_s^a) \cdot dB_s^a \right)^2 \right]
\]
(we have used stochastic Fubini theorem) where $g_s^N(y) := \int_{\mathbb{R}^d} h_s^N(x) \nabla \theta_N(x - y) \, dy$,
\[
\leq C \frac{\sigma^2}{N_2} \left[ \sum_{a \in A^N} \int_0^T \left( 1_{s \in I^a} \, g_s^N(X_s^a) \cdot dB_s^a \right)^2 \right]
\]
\[
= C \frac{\sigma^2}{N_2} E \left[ \int_0^T \left( 1_{s \in I^a} \, g_s^N(X_s^a) \right)^2 \, ds \right].
\]
But $\frac{1}{N} \left| g_s^N(y) \right|^2 \leq C \left\| h_s^N \right\|_{L^2}^2$, by (15), hence
\[
E \left[ \sup_{t \in [0, T]} \left| \int_{\mathbb{R}^d} \int_0^{t \wedge T^a_s} h_s^N(x) \mu(1_{s \in I^a} \, M_{1,N}^a(x)) \, dx \right| \right] \leq C \sigma^2 E \left[ \int_0^T \left\| S_s^N \right\| \, \left\| h_s^N \right\|_{L^2}^2 \, ds \right].
\]

Concerning the $M_s^{2,N}$-term, we have

$$E \left[ \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \int_0^{t \wedge \tau_R^N} h_s^N(x) dM_s^{2,N}(x) dx \right|^2 \right]$$

$$= \frac{1}{N^2} E \left[ \sup_{t \in [0,T]} \left| \sum_{a \in A^N} \int_{\mathbb{R}^d} \int_0^{t \wedge \tau_R^N} h_s^N(x) \theta_N(x - X_{s_+}^a) d (\mathcal{N}_s^a - \Lambda_s^a) dx \right|^2 \right]$$

$$= \frac{1}{N^2} E \left[ \sup_{t \in [0,T]} \left| \sum_{a \in A^N} \int_0^{t \wedge \tau_R^N} \tilde{g}_s^N(X_{s_+}^a) \cdot d (\mathcal{N}_s^a - \Lambda_s^a) \right|^2 \right]$$

where $\tilde{g}_s^N(y) := \int_{\mathbb{R}^d} h_s^N(x) \theta_N(x - y) dx$. The process $\sum_{a \in A^N} \int_0^{t \wedge \tau_R^N} \tilde{g}_s^N(X_{s_+}^a) \cdot d (\mathcal{N}_s^a - \Lambda_s^a)$ is a martingale with respect to the filtration $\mathcal{G}_t := \mathcal{F}_{\Lambda_t}$, hence the last expression is bounded by

$$\leq C \frac{N^2}{N} E \left[ \sum_{a \in A^N} \int_0^{t \wedge \tau_R^N} \tilde{g}_s^N(X_{s_+}^a) \cdot d (\mathcal{N}_s^a - \Lambda_s^a) \right]^2.$$ 

Since the jumps of $\mathcal{N}_s^a$ and $\mathcal{N}_s^b$, for $a \neq b$, never occur at the same time, we have

$$E \left[ \left( \int_0^{t \wedge \tau_R^N} \tilde{g}_s^N(X_{s_+}^a) d (\mathcal{N}_s^a - \Lambda_s^a) \right) \left( \int_0^{t \wedge \tau_R^N} \tilde{g}_s^N(X_{s_+}^b) d (\mathcal{N}_s^b - \Lambda_s^b) \right) \right] = 0.$$ 

Hence the last expression is equal to

$$= C \frac{1}{N^2} \sum_{a \in A} E \left[ \int_0^{t \wedge \tau_R^N} \tilde{g}_s^N(X_{s_+}^a) \cdot d (\mathcal{N}_s^a - \Lambda_s^a) \right]^2.$$ 

(16)

It is known that

$$E \left[ \left( \int_0^{t \wedge \tau_R^N} \tilde{g}_s^N(X_{s_+}^a) d (\mathcal{N}_s^a - \Lambda_s^a) \right)^2 \right] = E \left[ \left( \int_0^{t \wedge \tau_R^N} |\tilde{g}_s^N(X_{s_+}^a)|^2 d\Lambda_s^a \right)^2 \right]$$

$$= E \left[ \int_0^{t \wedge \tau_R^N} |\tilde{g}_s^N(X_s^a)|^2 1_{s \in I_s} F_N (S_s^N, X_s^a) ds \right]$$

$$\leq C_F E \left[ \int_0^{t \wedge \tau_R^N} |\tilde{g}_s^N(X_s^a)|^2 1_{s \in I_s} ds \right]$$

hence (16) is bounded above by

$$\leq C \frac{1}{N} E \left[ \int_0^{t \wedge \tau_R^N} \frac{1}{N} \sum_{a \in A} |\tilde{g}_s^N(X_s^a)|^2 1_{s \in I_s} ds \right]$$

$$= C \frac{1}{N} E \left[ \int_0^{t \wedge \tau_R^N} \int_{\mathbb{R}^d} |\tilde{g}_s^N(x)|^2 S_s^N (dx) ds \right].$$

As above for $g_s^N(y)$, using (15) we have $\frac{1}{N} |\tilde{g}_s^N(y)|^2 \leq C \|h_s^N\|_{L^2}^2$. Hence we get

$$\leq CE \left[ \int_0^{t \wedge \tau_R^N} \|h_s^N\|_{L^2}^2 [S_s^N] ds \right].$$
The result of the lemma follows by estimating \([S_N^x]\) by \(R\), being the integral in \(s\) only up to \(T \land \tau_R^N\).

\[\text{Corollary 1.}\]

\[\lim_{R \to \infty} \sup_{N \in \mathbb{N}} P \left( \sup_{t \in [0,T]} \left\| h^N_{t \land \tau_R^N} \right\|_{L^2}^2 + \int_0^T \left\| \nabla h^N_r \right\|_{L^2}^2 \, dr > R \right) = 0.\]

\[\text{Proof. Step 1.}\] From lemmata 3.3 and 3.4 we have

\[\frac{1}{2} \left\| h^N_{t \land \tau_R^N} \right\|_{L^2}^2 + \frac{\sigma^2}{4} \int_0^{t \land \tau_R^N} \left\| \nabla h^N_r \right\|_{L^2}^2 \, dr \leq \frac{1}{2} \left\| h^N_0 \right\|_{L^2}^2 + C \int_0^{t \land \tau_R^N} \left\| S_N^x \right\|_{L^2}^2 \, dr + C \int_0^{t \land \tau_R^N} \left\| h^N_r \right\|_{L^2}^2 \, dr + a_N + b_{N,R}\]

where

\[a_N := \frac{1}{2} \sup_{t \in [0,T]} \left\| [M^{1,N}]_t \right\|_{L^1}^2 + \frac{1}{2} \sup_{t \in [0,T]} \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A_N} \left( \theta_N(x - X^a_{T_t}) \right)^2 \, d\mu_{1 \geq T_t} \, dx\]

\[b_{N,R} := \sup_{t \in [0,T]} \left| \int_{\mathbb{R}^d} \int_0^{t \land \tau_R^N} h^N_s (x) d (M^{1,N}_s + M^{2,N}_s (x)) \right| .\]

Hence, writing \(\chi^N_R = 1 - \left\| h^N_0 \right\|_{L^2}^2 \leq R^2\) from Lemmata 3.5 and 3.6 we have

\[\frac{1}{2} E \left[ \chi^N_R \sup_{t \in [0,s]} \left\| h^N_{t \land \tau_R^N} \right\|_{L^2}^2 \right] \leq R^2/2 + C + C \left( R^2 + R + 1 \right) \int_0^s E \left[ \chi^N_R \sup_{t \in [0,s]} \left\| h^N_{t \land \tau_R^N} \right\|_{L^2}^2 \right] \, ds.

By Gronwall’s lemma, we get, with \(C(R) := (R^2 + 2C) \exp \left( 2C (R^2 + R + 1) \right)\),

\[E \left[ \chi^N_R \sup_{t \in [0,T]} \left\| h^N_{t \land \tau_R^N} \right\|_{L^2}^2 \right] \leq C(R).\]

Moreover, for the same reasons,

\[\frac{\sigma^2}{4} E \left[ \chi^N_R \int_0^{t \land \tau_R^N} \left\| \nabla h^N_r \right\|_{L^2}^2 \, dr \right] \leq R^2/2 + C + C \left( R^2 + R + 1 \right) \int_0^s E \left[ \chi^N_R \sup_{t \in [0,s]} \left\| h^N_{t \land \tau_R^N} \right\|_{L^2}^2 \right] \, ds \leq C_1(R)\]

where \(C_1(R) = R^2/2 + C + C \left( R^2 + R + 1 \right) TC(R)\) and we have used the previous bound in the last term.

\[\text{Step 2.}\] For every \(R_1 > 0\), the probability \(P \left( \sup_{t \in [0,T]} \left\| h^N_t \right\|_{L^2}^2 \geq R_1 \right)\) is bounded above by

\[\leq P \left( \sup_{t \in [0,T]} \left\| h^N_t \right\|_{L^2}^2 \geq R_1, \chi^N_R = 1 \right) + P \left( \left\| h^N_0 \right\|_{L^2}^2 > R \right)\]
We may now find functions $R_{true}$ such that the second half of the claim of the corollary holds. For every $\lim R_{true} \rightarrow \infty$, $C(R) (R(R_1)) / R_1 = 0$, where the function $C(R)$ has been defined in step 1. We deduce, from assumption (4), that

$$\lim_{R_{true} \rightarrow \infty} \sup_{N \in \mathbb{N}} P \left( \sup_{t \in [0,T]} \|h_t^N\|_{L^2}^2 \geq R_1 \right) = 0.$$

This proves one half of the claim of the corollary.

**Step 3.** For every $R_1 > 0$, by similar arguments $P \left( \int_0^T \|\nabla h_t^N\|_{L^2}^2 \, dr \geq R_1 \right)$ is bounded above by

$$\leq P \left( \chi_R^N \sup_{t \in [0,T]} \|h_t^N\|_{L^2}^2 \geq R_1, \tau_R \geq T \right) + P (\tau_R < T) + P \left( \|h_0^N\|_{L^2}^2 \geq R \right)$$

$$\leq P \left( \chi_R^N \sup_{t \in [0,T]} \|h_t^N \|_{L^2}^2 \geq R_1 \right) + P \left( \left[ S_T^N \right] \geq R \right) + P \left( \|h_0^N\|_{L^2}^2 \geq R \right)$$

$$\leq \frac{1}{R_1} \mathbb{E} \left[ \chi_R^N \sup_{t \in [0,T]} \|h_t^N \|_{L^2}^2 \right] + \frac{1}{R} \mathbb{E} \left[ \left[ S_T^N \right] \right] + P \left( \|h_0^N\|_{L^2}^2 \geq R \right)$$

$$\leq \frac{C(R)}{R_1} + C + P \left( \|h_0^N\|_{L^2}^2 \geq R \right).$$

As above, we conclude that the second half of the claim of the corollary holds true.

In order to show tightness of the family of the functions $\{h^N\}_N$, in addition to the previous bound which shows a regularity in space, we also need a regularity in time. See the compactness criteria below.

**Lemma 3.7.** Given any $\alpha \in (0, 1/2)$,

$$\lim_{R \rightarrow \infty} \sup_{N \in \mathbb{N}} P \left( \int_0^T \int_0^T \left\| h_t^N - h_s^N \right\|_{W_{-1,2}}^2 \frac{dsdt}{|t-s|^{1+2\alpha}} > R \right) = 0.$$ 

**Proof.** **Step 1.** We need to estimate $||h_t^N - h_s^N||_{W_{-1,2}}^2$ in such a way that it cancels with the singularity in the denominator at $t = s$. First, we have

$$||h_t^N - h_s^N||_{W_{-1,2}}^2 \leq C \left( \int_s^t \text{div} \left( \theta_N \ast ((\nabla V \ast S^N_r) \ast S^N_r) \right) \right)^2$$
\[\begin{align*}
&+ C \left\| \int_s^t \frac{\sigma^2}{2} \Delta h_r^N \, dr \right\|_{W^{-1,2}}^2 \\
&+ C \left\| \int_s^t \theta_N * (F_N \left( S_r^N, S_r^N \right) \right\|_{W^{-1,2}}^2 \\
&+ C \left\| M_t^{1,N} - M_s^{1,N} \right\|_{W^{-1,2}}^2 + C \left\| M_t^{2,N} - M_s^{2,N} \right\|_{W^{-1,2}}^2
\end{align*}\]

and thus by the Hölder inequality

\[
\begin{align*}
&\leq C (t-s) \int_s^t \| \text{div} \left( \theta_N * \left( (\nabla V * S_r^N) \right) S_r^N \right) \|_{W^{-1,2}}^2 \, dr \\
&+ C (t-s) \int_s^t \| \theta_N * (F_N \left( S_r^N, S_r^N \right) \|_{W^{-1,2}}^2 \, dr \\
&+ C (t-s) \int_s^t \left\| \frac{\sigma^2}{2} \Delta h_r^N \right\|_{W^{-1,2}}^2 \\
&+ C \left\| M_t^{1,N} - M_s^{1,N} \right\|_{W^{-1,2}}^2 + C \left\| M_t^{2,N} - M_s^{2,N} \right\|_{W^{-1,2}}^2.
\end{align*}\]

Notice that \( L^2 \subseteq W^{-1,2} \) with continuous embedding, namely there exists a constant \( C > 0 \) such that \( \| f \|_{W^{-1,2}} \leq C \| f \|_{L^2} \) for all \( f \in L^2 \). Moreover, the linear operator \( \text{div} \) is bounded from \( L^2 \) to \( W^{-1,2} \), namely \( \| \text{div} f \|_{W^{-1,2}} \leq C \| f \|_{L^2} \) and the operator \( \Delta \) is bounded from \( W^{1,2} \) to \( W^{-1,2} \), namely \( \| \Delta f \|_{W^{-1,2}} \leq C \| f \|_{W^{1,2}} \). Therefore (we denote by \( C > 0 \) any constant independent of \( N, h_r^N, t, s \))

\[
\begin{align*}
&\leq C (t-s) \int_s^t \| \theta_N * \left( (\nabla V * S_r^N) \right) S_r^N \|_{L^2}^2 \, dr + C (t-s) \int_s^t \| h_r^N \|_{W^{1,2}}^2 \, dr \\
&+ C (t-s) \int_s^t \| \theta_N * (F_N \left( S_r^N, S_r^N \right) \|_{L^2}^2 \, dr \\
&+ C \left\| M_t^{1,N} - M_s^{1,N} \right\|_{L^2}^2 + C \left\| M_t^{2,N} - M_s^{2,N} \right\|_{L^2}^2
\end{align*}\]

and now using (14), assumption (6) and \( \int_s^t \leq \int_0^T \) in all terms,

\[
\| h_t^N - h_s^N \|_{W^{-1,2}}^2 \leq C (t-s) \int_0^T \left( \left[ S_r^N \right]^2 + 1 \right) \| h_r^N \|_{L^2}^2 \, dr \\
+ C (t-s) \int_0^T \| h_r^N \|_{W^{1,2}}^2 \, dr \\
+ C \left\| M_t^{1,N} - M_s^{1,N} \right\|_{L^2}^2 + C \left\| M_t^{2,N} - M_s^{2,N} \right\|_{L^2}^2
\leq C (t-s) \left( \left[ S_r^N \right]^2 + 1 \right) \sup_{r \in [0,T]} \| h_r^N \|_{L^2}^2 \\
+ C (t-s) \int_0^T \| h_r^N \|_{W^{1,2}}^2 \, dr \\
+ C \sum_{i=1,2} \left\| M_t^{i,N} - M_s^{i,N} \right\|_{L^2}^2.
Accordingly, we split the estimate of $P \left( \int_0^T \int_0^T \frac{\|h^{t-N}_r - h^{s-N}_r\|^2}{|t-s|^{1+2a}} \, ds \, dt > R \right)$ in four more elementary estimates, that now we handle separately; the final result will be a consequence of them.

The number $C_\alpha = \int_0^T \int_0^T \frac{1}{|t-s|^{1+2a}} \, ds \, dt$ is finite, hence the first term is bounded by (renaming the constant $C$)

$$P \left( \int_0^T \int_0^T C (t-s) \left( [S_{r,T}^N] + 1 \right) \sup_{r \in [0,T]} \|h_r^N\|_{L^2}^2 \rangle ds \, dt > R \right)$$

$$= P \left( [S_{r,T}^N] + 1 > \sqrt{R/C} \right) + P \left( \sup_{r \in [0,T]} \|h_r^N\|_{L^2}^2 > \sqrt{R/C} \right)$$

and both these terms are, uniformly in $N$, small for large $R$, due to Lemma 3.11 and the estimates proved in the tightness part. The second addend, the one with $C (t-s) \int_0^T \|h_r^N\|^2_{W^{1,2}} \, dr$, is similar.

**Step 2.** Concerning the martingale terms, we now prove that

$$E \left\| M_t^{i,N} - M_s^{i,N} \right\|_{L^2}^2 \leq C |t-s|$$

for some constant $C > 0$, $i = 1, 2$. By Chebyshev inequality it follows that

$$\lim_{R \to \infty} \sup_{N \in \mathbb{N}} P \left( \int_0^T \int_0^T \frac{\|M_t^{i,N} - M_s^{i,N}\|^2_{L^2}}{|t-s|^{1+2a}} \, ds \, dt > R \right) = 0$$

and the proof will be complete. For notational convenience, we abbreviate, for $i = 1, 2$,

$$M_t^{i,N}(x) = \frac{1}{N} \sum_{a \in A_N} M_t^{i,a}(x).$$

Note, that for every $x \in \mathbb{R}^d$ the processes $M_t^{1,a}(x)$ and $M_t^{2,a}(x)$ are martingales. It follows, with computations similar to those of Lemma 3.6, for $t \geq s$

$$E \left\| M_t^{1,N} - M_s^{1,N} \right\|_{L^2}^2 = \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A_N} E \left[ \left( M_t^{1,a}(x) - M_s^{1,a}(x) \right)^2 \right] \, dx$$

$$= \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A_N} E \left[ \int_s^t 1_{r \in I^a} \nabla \theta_N (x - X_r^a)^2 \, dr \right] \, dx$$

$$= \frac{1}{N} \|\nabla \theta_N\|_{L^2}^2 \int_s^t \frac{1}{N} \sum_{a \in A_N} 1_{r \in I^a} \, dr \leq C (t-s)$$

where in the last inequality we have used (15) and Lemma 3.11. Similarly, for the second martingale,

$$E \left\| M_t^{2,N} - M_s^{2,N} \right\|_{L^2}^2 = \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A_N} E \left[ M_t^{2,a}(x)^2 - M_s^{2,a}(x)^2 \right] \, dx$$
follows that the family probability of the family $u$ converges to a probability measure $u$ converge in probability to the whole sequence $Q$.

3.1. Auxiliary results.

Definition 3.8. Given $u_0 \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, $u_0 \geq 0$, by weak solution of equation (7) we mean a function $u \geq 0$ of class $L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; W^{1,2}(\mathbb{R}^d))$. We shall prove that every such limit...
such that
\[
\langle u_t, \phi \rangle = \langle u_0, \phi \rangle - \int_0^t \langle (\nabla V * u_r) \nabla \phi \rangle \, dr - \frac{\sigma^2}{2} \int_0^t \langle \nabla u_r, \nabla \phi \rangle \, dr + \int_0^t \langle F(u_r) u_r, \phi \rangle \, dr
\]
(18)
for almost every \( t \in [0, T] \) and for all \( \phi \in W^{1,2}(\mathbb{R}^d) \).

Notice that, due to \( u \in L^\infty(0, T; L^2(\mathbb{R}^d)) \), and the assumption that \( \nabla V \) is bounded and compact supported, we have \( (\nabla V * u_r) \) bounded, hence the first integral in the weak equation is well defined. Moreover, since \( F \) is uniformly bounded (see (6)), the last term is also well defined.

**Remark 1.** If \( u \) is a solution in the sense of the definition, then there exists a (unique) re-presentative of \( u \) of class \( C_w([0, T]; L^2(\mathbb{R}^d)) \). Indeed, given \( \phi \in W^{1,2}(\mathbb{R}^d) \), from identity (18) we deduce that the a.e. defined function \( t \mapsto \langle u_t, \phi \rangle \) is a.s. equal to a continuous function \( g_\phi \). Let \( \overline{\pi} : [0, T] \to L^2(\mathbb{R}^d) \) be a bounded measurable representative (bounded in \( L^2(\mathbb{R}^d) \) by \( C \)). Let \( \{\phi_n\} \subset W^{1,2}(\mathbb{R}^d) \) be dense in \( L^2(\mathbb{R}^d) \). Let \( \Upsilon \subset [0, T] \) be a set of measure \( T \) such that \( \langle \overline{\pi}(t), \phi \rangle = g_\phi(t) \) for all \( t \in \Upsilon \). Then, for \( \phi \in L^2(\mathbb{R}^d) \) and \( t, s \in \Upsilon \),
\[
|\langle \overline{u}(t) - \overline{u}(s), \phi \rangle| \leq |g_{\phi_n}(t) - g_{\phi_n}(s)| + 2C\|\phi_n - \phi\|_{L^2}.
\]
From this it follows that \( t \mapsto \langle \overline{\pi}(t), \phi \rangle \) is uniformly continuous on \( \Upsilon \) hence uniquely extendible to a continuous function \( t \mapsto L_t(\phi) \) on \([0, T] \). From this it is easy to extract a re-definition of \( \overline{\pi}(t) \) for \( t \notin \Upsilon \) so that \( t \mapsto \langle \overline{\pi}(t), \phi \rangle \) is continuous on \([0, T] \).

Finally, it is not difficult to show that identity (18) holds for all \( t \in [0, T] \) for the representative of class \( C_w([0, T]; L^2(\mathbb{R}^d)) \).

**Theorem 3.9.** There is at most one weak solution of equation (7).

**Proof. Step 1.** If \( u_t^{(i)}, i = 1, 2 \) are two solutions and \( v_t = u_t^{(1)} - u_t^{(2)} \), from the equation (in weak form)
\[
\partial_t v_t = \frac{\sigma^2}{2} \Delta v_t + \text{div} \left( (\nabla V * v_t) u_t^{(1)} \right) + \text{div} \left( (\nabla V * u_t^{(2)}) v_t \right) + F(u_t^{(1)}) u_t^{(1)} - F(u_t^{(2)}) u_t^{(2)}
\]
and the property \( \partial_t v \in L^2(0, T; W^{-1,2}(\mathbb{R}^d)) \) (see Step 2 below), we have
\[
\frac{1}{2} \left\| v_t \right\|_{L^2}^2 + \frac{\sigma^2}{2} \int_0^t \left\| \nabla v_s \right\|_{L^2}^2 \, ds = \frac{1}{2} \left\| v_0 \right\|_{L^2}^2
\]
\[
- \int_0^t \left\langle (\nabla V * v_s) u_s^{(1)}, \nabla v_s \right\rangle \, ds - \int_0^t \left\langle (\nabla V * u_s^{(2)}) v_s, \nabla v_s \right\rangle \, ds
\]
\[
+ \int_0^t \left\langle F(u_s^{(1)}) u_s^{(1)} - F(u_s^{(2)}) u_s^{(2)}, v_s \right\rangle \, ds.
\]
Since
$$\|\nabla V \ast v_s\|_\infty = \sup_x \left| \int \nabla V(x - y) v_s(y) dy \right| \leq \|\nabla V\|_{L^2} \|v_s\|_{L^2},$$
$$\|\nabla V \ast u_s^{(2)}\|_\infty \leq \|\nabla V\|_{L^2} \|u_s^{(2)}\|_{L^2},$$
the terms on the second line can be bounded by
$$\leq \frac{\sigma^2}{4} \int_0^t \|\nabla v_s\|_{L^2}^2 ds + C \int_0^t \left( \|\nabla V \ast u_s\|_{L^2}^2 \right) ds + C \int_0^t \left( \|\nabla V \ast u_s^{(2)}\|_{L^2}^2 \right) ds$$
$$\leq \frac{\sigma^2}{4} \int_0^t \|\nabla v_s\|_{L^2}^2 ds + C \|\nabla V\|_{L^2}^2 \int_0^t \|v_s\|_{L^2}^2 \left( \|u_s^{(1)}\|_{L^2}^2 + \|u_s^{(2)}\|_{L^2}^2 \right) ds.$$

The terms on the last line, using assumptions (2)-(6), can be bounded by
$$\int_0^t \left| \left\langle F \left( u_s^{(1)} \right) u_s^{(1)} - F \left( u_s^{(2)} \right) u_s^{(2)}, v_s \right\rangle \right| ds$$
$$\leq \int_0^t \left| \left\langle F \left( u_s^{(1)} \right) - F \left( u_s^{(2)} \right) \right| \left| v_s \right| \right| \right| + \int_0^t \left| \left\langle F \left( u_s^{(1)} \right) \right| \left| v_s \right| \right| \right| ds$$
$$\leq \int_0^t L_F \|u - w\|_{W^{1,2}} \|u_s^{(2)}\|_{L^2} \|v_s\|_{L^2} ds + \int_0^t C_F \|v_s\|_{L^2}^2 ds$$
$$\leq \frac{\sigma^2}{8} \int_0^t \|\nabla v_s\|_{L^2}^2 ds + C \int_0^t \|v_s\|_{L^2}^2 \left( 1 + \|u_s^{(2)}\|_{L^2}^2 \right) ds.$$
from \( W^{1,2}_+ (\mathbb{R}^d) \) (in fact \( L^2_+ (\mathbb{R}^d) \)) to \( C_0^\beta (\mathbb{R}^d) \), hence (2)-(3) are true for \( F \) by composition with the Lipschitz bounded function \( r \mapsto (1 - r)^+ \) (we also use the fact that \( W * u \geq 0 \)). Finally,

\[
\left| (1 - (W * \theta_N * \mu) (x))^+ - (1 - (W * \mu) (x))^+ \right| \leq |(W * \theta_N * \mu) (x) - (W * \mu) (x)|
\]

\[
\leq \int \int \theta_N (z - y) |W (x - z) - W (x - y)| \mu (dy) dz
\]

\[
\leq \|\nabla W\|_\infty \int \int \theta_N (z - y) |y - z| \mu (dy) dz
\]

\[
\leq \|\nabla W\|_\infty \epsilon_N \int \int \theta_N (z - y) \mu (dy) dz = \|\nabla W\| \epsilon_N \|\mu\|
\]

which implies (5). \(\square\)

**Lemma 3.11.** There exists a \( \gamma > 0 \) such that \( \sup_N E \left[ e^{\gamma (S_T^N)} \right] < \infty \).

**Proof.** On the same probability space \((\Omega, \mathcal{F}, P)\) one can construct two processes, \(\{X^t_\alpha; \alpha \in A_N\}\), which is equal in law to our particle system, and \(\{Y^t_\alpha; \alpha \in A_N\}\), such that \(\left< S \right>_t^N, 1 \leq \left< S \right>_t^N, 1 \) a.s. (we distinguish the objects of the different particle systems by adding (“\(X\)” or “\(Y\)”), where the branching rate of \(\{Y^t_\alpha; \alpha \in A_N\}\) is constant and dominates the branching rate of the “\(X\)” system, i.e. \(\lambda(Y)^a) \equiv C_F\) for all \(\alpha \in A(Y)_t^N\) and \(t \in I(Y)^{a,N}\). Further, one has the representation

\[
\left< S \right>_t^N, 1 = \frac{1}{N} \sum_{k=1,...,N} N_T^{(k)},
\]

where \(N^{(k)}\) is a Yule process with birth rate \(C_F\) and start in 1 that describes the number of descendant of \(a(Y) = (k)\) which are alive at time \(T\). Furthermore, \(N_T^{(k)}\) are identically distributed and independent random variables for \(k = 1, ..., N\). Together with Jensen’s inequality it follows

\[
\sup_{n \in N} E \left[ e^{\gamma \left< S \right>_t^N, 1} \right] = \sup_{n \in N} \left( E \left[ e^{\gamma N_T^{(1)}} \right] \right) \leq E \left[ e^{\gamma N_T^{(1)}} \right].
\]

Because \(N_T^{(1)}\) is geometrically distributed with parameter \(C_F\), the right-hand side is finite if \(\gamma < \frac{1}{1 - e^{-1/C_F T}}\). \(\square\)

**Proposition 1.** Assume that \(X^i_0, i = 1, ..., N\), are independent identically distributed r.v. with common probability density \(u_0 \in L^2 (\mathbb{R}^d)\). Let \(\theta_N\) be mollifiers of the form \(\theta_N (x) = \epsilon_N^{-d} \theta (\epsilon_N^{-1} x)\), with \(\theta \in L^2 (\mathbb{R}^d)\) and \(\epsilon_N \geq CN^{-1/d}\). Then

\[
\sup_{N} E \left[ \left\| \theta_N * S_0^N \right\|_{L^2 (\mathbb{R}^d)}^2 \right] < \infty.
\]

The proof is an easy exercise, similar to a classical computation about the weak law of large numbers.

4. **Passage to the limit.** Given \(\chi : [0, T] \rightarrow \mathbb{R}\) of class \(C^1\), with \(\chi_T = 0\) and given \(\psi \in W^{1,2} (\mathbb{R}^d)\), defined \(\phi_t := \chi_t \psi\), we have
We shall prove that this \( \lim \inf \) is zero. It will follow that \( Q \) is continuous on \( Y \) using the identity satisfied by \( Q \) and the set \( \{ \Psi \} \). The lemma follows from the fact that \( Q \) is open since \( \Psi \) is continuous. Hence, for every \( \epsilon > 0 \),

\[
Q \left( u : |\Psi_\phi(u)| > \epsilon \right) \leq \liminf_{k \to \infty} Q \left( u : |\Psi_\phi(u)| > \epsilon \right) = \liminf_{k \to \infty} P \left( |\Psi_\phi(h_N^k)| > \epsilon \right).
\]

We shall prove that this \( \lim \inf \) is zero. It will follow \( Q \left( u : |\Psi_\phi(u)| > \epsilon \right) = 0 \). Since this holds for every \( \epsilon > 0 \), we will deduce \( Q \left( u : \Psi_\phi(u) = 0 \right) = 1 \). From this fact we shall prove the following result.

**Lemma 4.1.** \( Q \) is supported on the set of weak solutions of equation (7).

**Proof.** It remains to prove that \( \liminf_{k \to \infty} P \left( \left| \Psi_\phi \left( h_N^k \right) \right| > \epsilon \right) = 0 \), that the assertion of the lemma follows from the fact that \( Q \left( u \in Y : \Psi_\phi(u) = 0 \right) = 1 \) for every \( \phi \) of the form \( \phi_t = \chi_t \Psi \) as above, and that \( Q \) is concentrated on non negative functions. We prove these claims in three different steps.

**Step 1.** We show that \( \liminf_{k \to \infty} P \left( \left| \Psi_\phi \left( h_N^k \right) \right| > \epsilon \right) = 0 \). Let us write \( N \) in place of \( N_k \) for simplicity of notation. We have

\[
\Psi_\phi \left( h_N^k \right) = \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} h_N^k - \frac{\sigma^2}{2} \Delta \phi_t - \nabla \phi_t \cdot \nabla h_N^k \right) + F \left( h_N^k \phi_t \right) h_N^k dxdt + \langle u_0, \phi_0 \rangle.
\]

Using the identity satisfied by \( h_N^k \) against the test function \( \phi_t \), we have

\[
\Psi_\phi \left( h_N^k \right) = \int_0^T \int_{\mathbb{R}^d} \left[ \theta_N * \left( \nabla h_N^k \cdot \Phi_{\phi} \right) h_N^k \right] dxdt + \langle u_0 - h_0^N, \phi_0 \rangle + \int_{\mathbb{R}^d} \int_0^T \phi_t dM_{t,1,N} dx + \int_{\mathbb{R}^d} \int_0^T \phi_t dM_{t,2,N} dx.
\]
\[ \langle u_0 - h_0^N, \phi_0 \rangle = \langle u_0 - S_0^N, \phi_0 \rangle + \langle S_0^N, \phi_0 - \theta_N (-\cdot) \phi_0 \rangle \]

From Lemmata 4.2 and 4.3 below we have
\[ |\Psi_\phi (h^N) | \leq C (\epsilon_N + \alpha_N) [S_T^N]^2 + C [S_T^N] \epsilon_N^3 \left( \int_0^T \| h_t^N \|_{W^1,2}^2 dt + 1 \right) \]
\[ + |\langle u_0 - S_0^N, \phi_0 \rangle | + \| \theta_N^* \phi_0 - \phi_0 \| [S_T^N] \]
\[ + \int_{\mathbb{R}^d} \int_0^T \phi_t dM_t^{1,N} dx + \int_{\mathbb{R}^d} \int_0^T \phi_t dM_t^{2,N} dx \]

where the constant \( C \) depends only on the quantities described below, before Lemma 4.2. In order to prove \( \lim_{N \to \infty} P (|\Psi_\phi (h^N) | > \epsilon) = 0 \), it is sufficient to prove the same result for each one of the previous terms.

We have \( \lim_{N \to \infty} P \left( C (\epsilon_N + \alpha_N) [S_T^N]^2 > \epsilon \right) = 0 \) from Chebyshev inequality and Lemma 3.11. The same applies to the terms \( C [S_T^N] \epsilon_N^3 \), and \( \| \theta_N^* \phi_0 - \phi_0 \| [S_T^N] \). The term \( |\langle u_0 - S_0^N, \phi_0 \rangle | \) is obvious by the assumption of convergence in probability on \( S_0^N \). The two martingale terms can be treated again by Chebyshev inequality and Lemma 4.4 below. Finally
\[ P \left( C [S_T^N] \epsilon_N^3 \int_0^T \| h_t^N \|_{W^1,2}^2 dt > \epsilon \right) = P \left( [S_T^N] \int_0^T \| h_t^N \|_{W^1,2}^2 dt > \epsilon \epsilon_N^{-\beta} / C \right) \]
\[ \leq P \left( [S_T^N] > \sqrt{\epsilon \epsilon_N^{-\beta} / C} \right) + P \left( \int_0^T \| h_t^N \|_{W^1,2}^2 dt > \sqrt{\epsilon \epsilon_N^{-\beta} / C} \right). \]

The first term goes to zero by Chebyshev inequality and Lemma 3.11. The second one goes to zero, as \( N \to \infty \), by Corollary 1, by a simple argument based on the definition of limit.

**Step 2.** We prove that the assertion of the lemma follows from the fact that \( Q (u \in \mathcal{Y} : \Psi_\phi (u) = 0) = 1 \) for every \( \phi \) of the form \( \phi_t = \chi_t \psi_\phi \) as above. Let \( \{ \chi^n \} \) be a sequence of functions \( \chi^n : [0, T] \to \mathbb{R} \) of class \( C^1 \), with \( \chi^n_t = 0 \), which is dense in \( L^2 (0, T) \). Let \( \{ \psi_m \} \) be a dense sequence in \( W^{1,2} (\mathbb{R}^d) \). Set \( \phi_t^{n,m} := \chi_t^n \psi_t^m \); we have \( Q (u \in \mathcal{Y} : \Psi_{\phi_t^{n,m}} (u) = 0, \forall n, m) = 1 \). Let us prove that the set \( \mathcal{A} := \{ u \in \mathcal{Y} : \Psi_{\phi_t^{n,m}} (u) = 0, \forall n, m \} \) is contained in the set of weak solutions.

If \( u \in \mathcal{A} \), then \( \Psi_{\chi_t^n \psi_t^m} (u) = 0 \) for every \( m \in \mathbb{N} \) and every Lipschitz continuous \( \chi : [0, T] \to \mathbb{R} \) with \( \chi_T = 0 \). The proof of this claim by approximation with the sequence \( \chi^n_t \) is not difficult and we omit the details.

Given \( u \in \mathcal{Y} \), there exists a set \( \mathcal{Y} \subset [0, T) \) of full measure such that
\[ \lim_{n \to \infty} n \int_{t_0}^{t_0 + \frac{1}{n}} \int_{\mathbb{R}^d} \psi_t^m u_t dx dt - \int_{\mathbb{R}^d} \psi_t^m u_{t_0} dx \]
for every \( t_0 \in \mathcal{Y} \) and every \( m \in \mathbb{N} \). Given \( t_0 \in \mathcal{Y} \), take the new sequence \( \chi^n_t \) defined (at least for \( n \) large enough) as \( \chi^n_t = 0 \) for \( t > t_0 + \frac{1}{n} \), \( \chi^n_t = -1 \) for \( t < t_0 \), \( \chi^n_t = -1 + n(t - t_0) \) for \( t \in [t_0, t_0 + \frac{1}{n}] \). We have
\[ \int_0^T \int_{\mathbb{R}^d} \partial_t \phi_t^{m,n} u_t dx dt = n \int_{t_0}^{t_0 + \frac{1}{n}} \int_{\mathbb{R}^d} \psi_t^m u_t dx dt \]
We deduce, from \( \Psi_{\chi_t^n \psi_t^m} (u) = 0, \forall n, m \in \mathbb{N} \), that identity (18) holds at time \( t_0 \), for all \( \phi = \psi_t^m, m \in \mathbb{N} \). Therefore it is true, for each \( \phi = \psi_t^m \), a.s. in time. By density
of \( \{\psi^n\} \) and the regularity properties of \( u \) it is easy to deduce that (18) holds for every \( \phi \in W^{1,2}(\mathbb{R}^d) \).

**Step 3.** By the Portmanteau Theorem, the weak convergence of \( Q_{N_k} \) to \( Q \) implies that \( Q (A) \geq \limsup_{k \to \infty} Q_{N_k} (A) \) if \( A \) is a closed set. Note, that \( \{u: u \geq 0\} \) is closed in \( Y \), hence

\[
Q (u: u \geq 0) \geq \limsup_{k \to \infty} Q_{N_k} (u: u \geq 0) = \limsup_{k \to \infty} P (h_{N_k} \geq 0) = 1.
\]

Thus, \( Q \) is concentrated on non-negative functions, which completes the proof. \( \square \)

In the next three lemmata we denote by \( C \) any constant depending only on \( \sigma^2, T, C_F, \|D^2V\|_\infty, \|\theta\|_{L^2}, \|\nabla\theta\|_{L^2}, \sup_N \epsilon_N^{d-2} / N, \|\phi\|_\infty, \|\nabla\phi\|_\infty, E \left[ \left[ S_T^N \right] \right], \) and the diameter of the support of \( \theta \). Lemma 4.2 below treats the convergence of the divergence terms.

**Lemma 4.2.**

\[
\left| \int_0^T \int_{\mathbb{R}^d} \left[ \theta_N \ast \left( (\nabla V \ast S_t^N) S_t^N \right) - (\nabla V \ast h_t^N) h_t^N \right] \cdot \nabla \phi dx dt \right| \leq C \epsilon_N \left[ S_T^N \right]^2.
\]

**Proof.** The left-hand-side is bounded above by

\[
\leq \|\nabla\phi\|_\infty \int_0^T \int_{\mathbb{R}^d} \left| \theta_N \ast \left( (\nabla V \ast S_t^N) S_t^N \right) - (\nabla V \ast S_t^N) h_t^N \right| dx dt
+ \|\nabla\phi\|_\infty \int_0^T \int_{\mathbb{R}^d} \left| (\nabla V \ast S_t^N) h_t^N - (\nabla V \ast h_t^N) h_t^N \right| dx dt.
\]

Let us separately bound the two terms. The first one is a sort of commutation lemma estimate. We have

\[
\left| \theta_N \ast \left( (\nabla V \ast S_t^N) S_t^N \right) (x) - (\nabla V \ast S_t^N) \left( \theta_N \ast S_t^N \right) (x) \right|
\leq \int \theta_N (x - y) \left| (\nabla V \ast S_t^N) (y) - (\nabla V \ast S_t^N) (x) \right| S_t^N (dy)
\leq \|D^2V\|_\infty \left[ S_t^N \right] \int \theta_N (x - y) |x - y| S_t^N (dy).
\]

Since \( \theta_N (x) = \epsilon_N^{-d} \theta (\epsilon_N^{-1} x) \) with \( \epsilon_N \to 0 \), \( \theta \) smooth non-negative compact support with diameter \( K \), we have

\[
\int \theta_N (x - y) |x - y| S_t^N (dy) \leq K \epsilon_N \int \theta_N (x - y) S_t^N (dy) = K \epsilon_N h_t^N (x).
\]

Summarizing, the first term above is bounded by

\[
\leq \|\nabla\phi\|_\infty \|D^2V\|_\infty K \epsilon_N \int_0^T \left[ S_t^N \right] \int_{\mathbb{R}^d} h_t^N (x) dx dr
\leq \|D^2V\|_\infty \|\nabla\phi\|_\infty K \epsilon_N T \left[ S_T^N \right]^2
\]

where in the last inequality we have used \( \int h_t^N (x) dx = \left[ S_t^N \right] \). The second term above is bounded above by

\[
\leq \|\nabla\phi\|_\infty \int_0^T \int_{\mathbb{R}^d} \left| (\nabla V \ast S_t^N) - (\nabla V \ast h_t^N) \right| h_t^N |dx dr
\leq \|D^2V\|_\infty \|\nabla\phi\|_\infty K \epsilon_N \int_0^T \left[ S_t^N \right] \int h_t^N |dx dr \leq \|D^2V\|_\infty \|\nabla\phi\|_\infty K \epsilon_N T \left[ S_T^N \right]^2
\]
hence

\[ |(\nabla V \ast S^N_t) (x) - (\nabla V \ast h_t^N) (x)| = \left| \int \int [\nabla V (x-y) - \nabla V (x-z)] \theta_N (z-y) d\varepsilon N (dy) \right| \leq \|D^2V\|_\infty \int \int |y-z| \theta_N (z-y) d\varepsilon N (dy) \leq \|D^2V\|_\infty K \epsilon_N [S^N_t]. \]

Lemma 4.3 proves the convergence of the proliferation terms.

Lemma 4.3.

\[ \int_0^T \int_{\mathbb{R}^d} [F (h_t^N) h_t^N - \theta_N \ast (F_N (S^N_t, \cdot) S^N_t)] \phi_t dxdt \leq C \left[ \frac{S^N_T}{\epsilon_N} \right] \epsilon_N ^\beta \left( \int_0^T \|h_t^N\|^2_{W^{1,2}} dt + 1 \right) + C \alpha_N [S^N_T]. \]

Proof. We have, using assumptions (3)-(5),

\[ |F (\theta_N \ast \mu) (x) - F_N (\mu, y)| \leq |F (\theta_N \ast \mu) (x) - F (\theta_N \ast \mu) (y)| + |F (\theta_N \ast \mu) (y) - F_N (\mu, y)| \leq C (\|\theta_N \ast \mu\|_{W^{1,2}} + 1) |x-y| + \alpha_N [\mu] \]

hence

\[ ||F (\theta_N \ast \mu) \theta_N \ast \mu - \theta_N \ast (F_N (\mu, \cdot) \mu)|| (x)| \leq \int \theta_N (x-y) |F (\theta_N \ast \mu) (x) - F_N (\mu, y)| \mu (dy) \leq \int \theta_N (x-y) (C (\|\theta_N \ast \mu\|_{W^{1,2}} + 1) |x-y| + \alpha_N [\mu]) \mu (dy) \leq \left( K^\beta (\|\theta_N \ast \mu\|_{W^{1,2}} + 1) \epsilon_N ^\beta + \alpha_N [\mu] \right) (\theta_N \ast \mu) (x) \]

therefore the left-hand-side in the statement of the lemma is bounded above by

\[ \leq \|\phi\|_\infty \int_0^T \left( K^\beta (\|h_t^N\|_{W^{1,2}} + 1) \epsilon_N ^\beta + \alpha_N [S^N_t] \right) \int_{\mathbb{R}^d} h_t^N (x) dxdt \leq \|\phi\|_\infty \int_0^T \left( K^\beta (\|h_t^N\|_{W^{1,2}} + 1) \epsilon_N ^\beta + \alpha_N [S^N_t] \right) [S^N_t] dt. \]

Finally, Lemma 4.4 provides the needed control over the martingale terms.

Lemma 4.4. For \( i = 1, 2 \)

\[ E \left[ \left( \int_{\mathbb{R}^d} \int_0^T \phi_t (x) dM_t^{i,N} (x) dx \right)^2 \right] \leq C \epsilon_N ^2. \]

Proof. The proof is very similar to the one of Lemma 3.6, but taking advantage of the smoothness of \( \phi \) which did not appear there. We only sketch the computations. Denoting
\[ g_t^N(y) := -\int_{\mathbb{R}^d} \phi_t(x) \nabla \theta_N(x-y) \, dx = \int_{\mathbb{R}^d} \nabla \phi_t(x) \theta_N(x-y) \, dx \]

for the first martingale term we have
\[
E \left[ \left( \int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM_{1,N}^t(x) \, dx \right)^2 \right] = \frac{\sigma^2}{N^2} \sum_{a \in A^N} E \left[ \int_0^T 1_{t \in I^a} |g_t^N(X_t^a)|^2 \, dt \right] 
\leq \frac{\sigma^2}{N} \|\theta_N\|_{L^2}^2 \|\nabla \phi\|_{L^\infty}^2 E \int_0^T [S^N_t] \, dt
\]

and then we use (15) and Lemma 3.11. Set \( \tilde{g}_t^N(y) := -\int_{\mathbb{R}^d} \phi_t(x) \theta_N(x-y) \, dx \), then for the second martingale term we have
\[
E \left[ \left( \int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM_{2,N}^t(x) \, dx \right)^2 \right] = \frac{1}{N^2} \sum_{a \in A^N} E \left[ \int_0^T 1_{t \in I^a} |\tilde{g}_t^N(X_t^a)|^2 \lambda_t^a \, dt \right] 
\leq \frac{C \nu}{N} \|\theta_N\|_{L^2}^2 \|\phi\|_{L^\infty}^2 E \int_0^T [S_t^N] \, dt
\]
and we conclude by the same argument. This completes the proof. \( \Box \)

With the preceding three lemmata, which are used in the proof of Lemma 4.1, the proof of Theorem 1.1 is complete.

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