Optimal Control of Diffusion Processes with Terminal Constraint in Law

Samuel Daudin

Received: 19 January 2021 / Accepted: 18 May 2022 / Published online: 25 June 2022
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
Stochastic optimal control problems with constraints on the probability distribution of the final output are considered. Necessary conditions for optimality in the form of a coupled system of partial differential equations involving a forward Fokker–Planck equation and a backward Hamilton–Jacobi–Bellman equation are proved using convex duality techniques.

Keywords Stochastic control · Constraints in law · Hamilton–Jacobi–Bellman equation · Fokker–Planck equation · Mean field games · Minmax · Convex duality

1 Introduction
This paper is devoted to the study of stochastic optimal control problems with constraints on the law $\mathcal{L}(X_T)$ of the controlled process at the terminal time. Our problem takes the following form:

$$\inf_{\alpha_t \in A} \mathbb{E} \left[ \int_0^T (f_1(t, X_t, \alpha_t) + f_2(t, \mathcal{L}(X_t)))dt + g(\mathcal{L}(X_T)) \right]$$

under the constraint $\Psi(\mathcal{L}(X_T)) \leq 0$ for the diffusion:

$$dX_t = b(t, X_t, \alpha_t)dt + \sqrt{2}\sigma(t, X_t, \alpha_t)dB_t$$

with the initial condition given by $\mathcal{L}(X_0) = m_0$ for some $m_0$ in $\mathcal{P}_2(\mathbb{R}^d)$, the space of probability measures over $\mathbb{R}^d$ with finite second-order moment. Here, $f_1 : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$

Communicated by Dylan Possamaï.

Samuel Daudin
daudin@ceremade.dauphine.fr

CEREMADE, PSL Research University, Université Paris-Dauphine, Place de Lattre de Tassigny, 75016 Paris, France
\begin{equation}
\mathbb{R}^d \times A \rightarrow \mathbb{R} \quad \text{and} \quad f_2 : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R} \quad \text{are the instantaneous costs,} \quad g : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R} \quad \text{is the terminal cost,} \quad \Psi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R} \quad \text{is the final constraint,} \quad b : [0, T] \times \mathbb{R}^d \times A \rightarrow \mathbb{R}^d \quad \text{and} \quad \sigma : [0, T] \times \mathbb{R}^d \times A \rightarrow \mathcal{S}_d(\mathbb{R}) \quad \text{are, respectively, the drift and the volatility of the controlled process} X \quad \text{and} \quad \alpha \quad \text{is the control process valued in the control space} A. \quad \text{We look in particular for optimal Markov policies that is control processes} (\alpha_t) \quad \text{which are optimal among all admissible controls and for which there exists some measurable function} \quad \alpha : [0, T] \times \mathbb{R}^d \rightarrow A \quad \text{such that, for all} \quad t \in [0, T], \quad \alpha_t = \alpha(t, X_t).
\end{equation}

We are going to show that optimal Markov policies are related to the solutions of the following system of partial differential equations, where the unknown \((\lambda, u, m)\) belongs to \(\mathbb{R}^+ \times C^{1,2}_b([0, T] \times \mathbb{R}^d) \times C^0([0, T], \mathcal{P}_2(\mathbb{R}^d))\):

\begin{equation}
\begin{cases}
-\partial_t u(t, x) + H(t, x, Du(t, x), D^2u(t, x)) = \frac{\delta f_2}{\delta m}(t, m(t), x) \quad \text{in} \quad [0, T] \times \mathbb{R}^d \quad (1a) \\
\partial_t m - \text{div}(\partial_p H(t, x, Du(t, x), D^2u(t, x))m) + \sum_{i,j} \partial^2_{ij}((\partial_M H(t, x, Du(t, x), D^2u(t, x)))m) = 0 \quad \text{in} \quad [0, T] \times \mathbb{R}^d \quad (1b) \\
u(T, x) = \lambda \frac{\delta \Psi}{\delta m}(m(T), x) + \frac{\delta g}{\delta m}(m(T), x) \quad \text{in} \quad \mathbb{R}^d, m(0) = m_0 \quad (1c) \\
\lambda \Psi(m(T)) = 0, \quad \Psi(m(T)) \leq 0, \quad \lambda \geq 0, \quad (1d)
\end{cases}
\end{equation}

where \(H(t, x, p, M) := \sup_{a \in A} \{-b(t, x, a), p - \sigma^t \sigma(t, x, a), M - f_1(t, x, a)\} \) is the Hamiltonian of the system. The forward equation, Equation 1b, is a Fokker–Planck equation which describes the evolution of the probability distribution \(m\) of the optimally controlled process. The backward equation, Equation 1a, is a Hamilton–Jacobi–Bellman equation satisfied by the adjoint state \(u\). The nonnegative parameter \(\lambda\) is the Lagrange multiplier associated with the terminal constraint. The forward and backward equations are coupled through the source term for the HJB equation, the terminal condition for the HJB equation and the exclusion condition \(\lambda \Psi(m(T)) = 0\).

Our main result see Theorem 2.2 states that, under suitable growth and regularity assumptions, optimal Markov policies \(\alpha \in L^0([0, T] \times \mathbb{R}^d, A) \) exist and satisfy:

\[
\alpha(t, x) \in \arg\max_{a \in A} \left\{-b(t, x, a), Du(t, x) - \sigma^t \sigma(t, x, a), D^2u(t, x) - f_1(t, x, a)\right\}
\]

for some solution \((\lambda, u, m)\) of the above system of PDEs. Notice that we do not a priori require \(\Psi\) to be a convex function. When \(\Psi(m) = \int_{\mathbb{R}^d} h(x)m(dx)\) for some function \(h : \mathbb{R}^d \rightarrow \mathbb{R}\) and for all \(m \in \mathcal{P}_2(\mathbb{R}^d)\), we say that the constraint is linear. When the costs \(f_2\) and \(g\) are linear as well, we recover the problem of stochastic optimal control under expectation constraint (as in [10, 15, 38]).

Such problems arise in economy and finance when an agent tries to minimize a cost (maximize a utility function) under constraints on the probability distribution of the final output. These types of constraints can take into account the risk given by the dispersion of the cost. There has recently been a surge of interest for this kind of problems.
For instance, [23, 24] use similar formulations to study, respectively, the problem of calibration of local-stochastic volatility models and the problem of portfolio allocation with prescribed terminal wealth distribution. Probability constraints of the form \( P[h(X_T) \leq 0] \leq 1 - \epsilon \) also fall into our analysis since they can be written as functions of the law \( L(X_T) \) of \( X_T \). In state constrained problems, the constraint is directly imposed on the process \( X_T \) and must be satisfied almost-surely. Such constraints might be too stringent or even impossible to satisfy, and probability constraints might allow to find controls with a better reward and a controlled probability of failure/success.

Stochastic control problems with terminal constraints have been extensively studied in the literature. Optimal control problems under stochastic target constraints have been studied in Bouchard et al. [9] using the geometric dynamic programming principle proposed in Soner and Touzi [41]. In Föllmer and Leukert [22], the authors introduce the notion of quantile hedging to relax almost-sure constraints into probability constraints. In Yong and Zhou [49] Chapter 3, necessary optimality conditions are proved in the form of a system of forward/backward stochastic differential equations. More recently, the problem with constraints on the law of the process has been studied in Pfeiffer [37] and in Pfeiffer, Tan and Zhou [38]. In these works, the authors prove that the problem can be reduced to a “standard” problem (without terminal constraint) by adding a term involving \( \lambda^* h \)—in the case where the constraint has the form \( \mathbb{E}[h(X_T)] \leq 0 \)—to the final cost for some optimal Lagrange multiplier \( \lambda^* \). A dual problem over the Lagrange multipliers associated with the constraints is exhibited using abstract duality results. In Pfeiffer, Tan and Zhou [38], the authors provide necessary and sufficient optimality conditions for problems with multiple equality and inequality expectation constraints with much less restrictions on the data than we do and in a path-dependent framework. However, [38] needs to assume some controllability condition (Assumption 3.1.ii) and works with a compact control set. In our framework, the corresponding controllability condition would be to assume a priori that there exists some control \( \alpha \) such that \( \mathbb{E}(h(X^\alpha_T)) < 0 \). In our analysis, we are able to prove such controllability condition when \( H \) satisfies suitable assumptions.

The novelty of the present work is to provide a framework in which both controllability and existence of strong regular solutions for the stochastic control problem can be proved. We also believe that our necessary conditions for optimality can lead to efficient numerical methods using techniques already developed for similar kind of coupled PDE systems as in Achdou and Capuzzo Dolcetta [1]. We are also able to handle costs of mean-field type.

Our strategy is to study a relaxed problem which is an optimal control problem for the Fokker–Planck equation and then rely on the regularity of the data to show that optimal controls for the relaxed problem yield optimal controls for the original problem. The relaxed problem is the following:

\[
\inf_{(m, \omega, W)} \int_0^T \int_{\mathbb{R}^d} L(t, x, \frac{d\omega}{dt} \otimes dm(t, x), \frac{dW}{dt} \otimes dm(t, x)) dm(t)(x) dt + \int_0^T f_2(t, m(t)) dt + g(m(T)),
\]
where
\[ L(t, x, q, N) := \sup_{(p, M) \in \mathbb{R}^d \times S_d(\mathbb{R})} \{-p \cdot q - M \cdot N - H(t, x, p, M)\} \]
\[ = H^*(t, x, -q, -N) \]
and the infimum is taken over the triples \((m, \omega, W) \in C^0([0, T], \mathcal{P}_1(\mathbb{R}^d)) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \times \mathcal{M}([0, T] \times \mathbb{R}^d, S_d(\mathbb{R}))\) for which \(\omega\) and \(W\) are absolutely continuous with respect to \(m(t) \otimes dt\) and \((m, \omega, W)\) satisfy in the sense of distributions the Fokker–Planck equation:
\[ \partial_t m + \text{div}\omega - \sum_{i,j} \partial_{ij}^2 W_{ij} = 0 \]

Together with the initial condition \(m(0) = m_0\) and the terminal constraint \(\Psi(m(T)) \leq 0\). Notice that here and in the following, we denote by \(S_d(\mathbb{R})\) the space of symmetric matrices of size \(d\), endowed with the inner product \(M \cdot N := \text{Tr}(MN)\) and by \(\mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)\) (respectively, by \(\mathcal{M}([0, T] \times \mathbb{R}^d, S_d(\mathbb{R}))\)) the space of \(\mathbb{R}^d\)-valued (respectively, \(S_d(\mathbb{R})\)-valued) Borel measures on \([0, T] \times \mathbb{R}^d\) with finite total variation.

In order to study the relaxed problem, we rely on duality techniques that originated in the theory of optimal transport (see [7, 39, 45, 46]) and were further developed in the theory of mean field games. Indeed, when the game has a potential structure—see, for instance, Lasry and Lions [31], Cardaliaguet et al. [13], Briani and Cardaliaguet [12] and Orrieri et al. [36]—the system of partial differential equations which describes the distribution of the players and the value function of a typical infinitesimal player can be obtained as optimality conditions for an optimal control problem for the Fokker–Planck equation. In this framework, the necessary conditions are obtained through convex duality techniques, using generally the Fenchel–Rockafellar theorem as in [12, 13] or the von Neumann theorem as in [36]. We follow this path, and—when the final constraint as well as the costs \(f_2\) and \(g\) is linear—we are able to exhibit a dual problem, which is an optimal control problem for the HJB equation involving the Lagrange multiplier \(\lambda \in \mathbb{R}^+\) associated with the terminal constraint. It takes the following form:

\[ \sup_{(\lambda, \phi)} \int_{\mathbb{R}^d} \phi(0, x) dm_0(x), \]
where the supremum runs over the couples \((\lambda, \phi) \in \mathbb{R}^+ \times \mathcal{C}^{1,2}_b([0, T] \times \mathbb{R}^d)\) satisfying
\[
\begin{cases}
-\partial_t \phi(t, x) + H(t, x, D\phi(t, x), D^2\phi(t, x)) \leq f'_2(t, x) \text{ in } [0, T] \times \mathbb{R}^d \\
\phi(T, x) \leq \lambda h(x) + g'(x) \text{ in } \mathbb{R}^d,
\end{cases}
\]
and where \(f'_2 : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) and \(g' : \mathbb{R}^d \to \mathbb{R}\) are such that \(f_2(t, m) = \int_{\mathbb{R}^d} f'_2(t, x) dm(x)\) and \(g(m) = \int_{\mathbb{R}^d} g'(x) dm(x)\).
The necessary conditions for optimality then follow from the lack of duality gap between the relaxed and the dual problems. We can then address more general constraints $\Psi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and costs $f_2 : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, $g : \mathcal{P}_2(\mathbb{R}^d)$ by “linearizing” the costs and the constraint around solutions of the relaxed problem.

Using convex duality techniques to solve optimal control problems for diffusion processes is of course not new. It can be traced back at least to Fleming and Veremes [21], where the philosophy is very close to ours. In Tan and Touzi [43], the authors extend the usual Monge–Kantorovitch optimal transportation problem to a stochastic framework. The mass is transported along a continuous semimartingale, and the initial and terminal distributions are prescribed. Studying optimal control problems for the Fokker–Planck equation in order to understand the stochastic control problem is less common, and it seems adapted to problems where the constraints only act on the law of the process. We refer to the works of Blaquière [8] and, more recently, Mikami [33] and Mikami and Thieullen [34] where similar approaches are developed in connection with the so-called Schrödinger problem. This approach has been followed recently by Guo et al. [23, 24] for problems with various expectation constraints. In both papers, the authors show that their original problem is in duality with a problem of optimal control of sub-solutions of an HJB equation. This dual problem is solved numerically. Our relaxation is in the spirit of classical works in convex analysis (see [17]), but usually probabilists prefer to study another relaxation of the initial problem through the martingale problem (see Stroock and Varadhan [42]), as in El Karoui et al. [18] or Lacker [29]. These different ways to relax the initial problem are, of course, connected, and the correspondences between the diffusion processes, the martingale problem and the Fokker–Planck equation are now well established starting from the seminal work of [42] and more recently Figalli [19] and Trevisan [44].

Under very general assumptions, as in [21], one is usually able to see that the original problem is in duality with a problem of optimal control of the HJB equation. However, existence of solutions for this dual problem is much harder to come by and requires particular structural conditions. Essentially, the dual problem has a solution if the Hamilton–Jacobi–Bellman equation admits a regular solution. This is of course rather difficult to obtain. Regularity results for the Hamilton–Jacobi–Bellman equation where the control appears in the volatility as in Fleming and Soner [20] Chapter IV.4 usually rely upon three things: the regularity of the coefficients of the diffusion and of the costs functionals, the compactness of the control set and finally the uniform parabolicity of the equation. The last point means that there must be some $\Lambda^- > 0$ such that the volatility coefficient satisfies (uniformly in the time/state/control variables) $\sigma^t \sigma \geq \Lambda^- I_d$.

In studying terminal constraints, compact control sets are not satisfactory since we would not be able to show, in full generality, that the constraint can indeed be reached with a finite cost. Part of the challenge of the paper is to find a framework in which the process is sufficiently “controllable,” but the HJB equation is still solvable. For that we need to impose restrictions on the coefficients.

In particular, we require some growth assumptions on the Hamiltonians and its derivatives. This allows us to use the weak Bernstein method as in Ishii and Lions [26], Barles [6], Lions and Souganidis [32] and Armstrong and Cardaliaguet [5] (among
others) to prove that the viscosity solution of the HJB equation is Lipschitz in time and space.

As it is well known, controllability for such systems is related to the coercivity of the Hamiltonian $H$ in the momentum variable. As we will show, imposing a strictly super-linear polynomial growth (in $p$) for $H(t, x, p, 0) := \sup_{a \in A} -b(t, x, a) . p - f_1(t, x, a)$ allows to show that the agent can take (with a relaxed control) any instantaneous drift without paying too big a cost.

The rest of the paper is organized as follows: In Sect. 2, we present our assumptions and the precise statement of the problem. We also give our main results there. In Sect. 3, we introduce and study the problem of optimal control of the Fokker–Planck equation. Our main results of Theorems 2.2 and 2.3 are then proved in Sect. 4. Finally, we give in Sect. 5 a detailed study of the Hamilton–Jacobi–Bellman equation which is crucial to our analysis.

## 2 Main Results

In this section, we first present our notations and our standing assumptions. Then, we briefly discuss some properties of the Lagrangian $L$ and finally we state our main results.

### 2.1 Notations and Functional Spaces

The $d$-dimensional Euclidean space is denoted by $\mathbb{R}^d$ and the space of real matrices of size $d$ by $M_d(\mathbb{R})$. The space of symmetric matrices of size $d \times d$ is denoted by $S_d(\mathbb{R})$. The subset of $S_d(\mathbb{R})$ consisting of positive symmetric matrices is denoted by $S^+_d(\mathbb{R})$, and $S^{++}_d(\mathbb{R})$ is the subset of $S^+_d(\mathbb{R})$ consisting of definite-positive symmetric matrices.

Recall that $S^+_d(\mathbb{R})$ is endowed with a smooth (analytic) square root: $\sqrt{\cdot} : S^+_d(\mathbb{R}) \to S^{++}_d(\mathbb{R})$ (see, for instance, [42] Lemma 5.2.1). Sometimes we will use $S_p(A)$ to denote the set of eigenvalues of a square matrix $A$. The Euclidean space $\mathbb{R}^d$ is endowed with its canonical scalar product: $\langle x, y \rangle := \sum_{i=1}^d x_i y_i$ and the associated norm $\|x\|^2 := \sum_{i=1}^d x_i^2$.

The space $M_d(\mathbb{R})$ is endowed with its canonical scalar product: $\langle M, N \rangle := \text{Tr}(tMM^t)$ and the associated norm $\|M\|^2 := \text{Tr}(tMM^t)$, where $\text{Tr}(M)$ is the trace of $M$ and $tM$ is the transpose of $M$. Sometimes we will use the operator norm on $M_d(\mathbb{R})$:

$$
\|M\| := \sup_{x \in \mathbb{R}^d} \frac{|Mx|}{|x|}.
$$

For two real numbers $r_1$ and $r_2$, $r_1 \wedge r_2$ is the minimum of $r_1$ and $r_2$ and $r_1 \vee r_2$ is the maximum of $r_1$ and $r_2$. If $\eta$ is a $\sigma$-finite positive measure on a measurable space $(\Omega, \mathcal{F})$ and $\mu$ is a $\sigma$-finite vector measure on $(\Omega, \mathcal{F})$, we write $\mu \ll \eta$ if $\mu$ is absolutely continuous with respect to $\eta$ and we write $\frac{d\mu}{d\eta} \in L^1(\eta)$ for the Radon–Nikodym derivative of $\mu$ with respect to $\eta$. If $E$ is a locally compact, complete, separable metric space and $l \geq 1$ is an integer, $C_0(E, \mathbb{R}^l)$ is the space of $\mathbb{R}^l$-valued continuous functions on $X$, vanishing at infinity. It is endowed with the topology of uniform convergence. Its topological dual $(C_0(E, \mathbb{R}^l))^\prime$ can be identified thanks to Riesz theorem as the space $\mathcal{M}(E, \mathbb{R}^l)$ of $\mathbb{R}^l$-valued Borel measures with finite total variation on $E$, normed by total variation. We will often consider the weak-* topology on $\mathcal{M}(E, \mathbb{R}^l)$. When $l = 1$, we simply note $C_0(E)$ and $\mathcal{M}(E)$. $\mathcal{M}^+(E) \subset \mathcal{M}(E)$ is
the cone of finite nonnegative measures. The set of Borel probability measures over \( E \) is denoted by \( \mathcal{P}(E) \). If \( r \geq 1, \mathcal{P}_r(E) \) is the set of Borel probability measures over \( E \) with finite moment of order \( r \). It is endowed with the topology given by the Wasserstein distance \( d_r \) of order \( r \). If \( X \) is a random variable taking values into \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\), its law is denoted by \( \mathcal{L}(X) \in \mathcal{P}(\mathbb{R}^d) \). We say that \( U : \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R} \) is \( C_1 \) if there is a bounded continuous function \( \frac{\delta U}{\delta m} : \mathcal{P}_1(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R} \) such that, for any \( m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^d) \),

\[
U(m_1) - U(m_2) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1 - t)m_2 + tm_1, x)(m_1 - m_2)(dx)dt.
\]

This derivative is defined up to an additive constant, and we use the standard normalization convention: \( \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m, x)m(dx) = 0 \). See [14] for details on the notion(s) of derivatives in the space of measures.

We consider a finite, fixed horizon \( T > 0 \). The set of continuous functions from \([0, T]\) to \( \mathcal{P}(\mathbb{R}^d) \) and from \([0, T]\) to \( \mathcal{P}_r(\mathbb{R}^d) \) for \( r \geq 1 \) are, respectively, denoted by \( \mathcal{C}^0([0, T], \mathcal{P}(\mathbb{R}^d)) \) and by \( \mathcal{C}^0([0, T], \mathcal{P}_r(\mathbb{R}^d)) \). The space of measurable functions defined on \([0, T] \times \mathbb{R}^d \) with values into the measurable space \( Y \) is denoted by \( L^0([0, T] \times \mathbb{R}^d, Y) \). If \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is sufficiently smooth, \( Du : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) and \( D^2u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) denote, respectively, the differential and the Hessian of \( u \) with respect to the space variable \( x \). The space of continuous functions \( u \) on \([0, T] \times \mathbb{R}^d \) for which \( \partial_t u, Du \) and \( D^2u \) exist and are continuous is denoted by \( \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d) \), and \( \mathcal{C}^{1,2}_b([0, T] \times \mathbb{R}^d) \) is the subspace of \( \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d) \) consisting of functions \( u \) for which \( u, \partial_t u, Du \) and \( D^2u \) are bounded. If \( n \in \mathbb{N}^* \) and \( \alpha \in (0, 1) \), \( \mathcal{C}_b^{n,\alpha}(\mathbb{R}^d) \) is the space of bounded continuous real functions on \( \mathbb{R}^d \) for which the first \( n \)-derivatives are continuous and bounded and the \( n \)-th derivative is \( \alpha \)-Hölder continuous. We say that \( \phi : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) is in \( \mathcal{C}_b^{n,\alpha}(\mathbb{R}^d) \) if \( \phi \) is continuous in both variables together with all derivatives \( D_t^rD_x^s\phi \) with \( 2r + s \leq n \). Moreover, \( \|\phi\|_{\alpha, n, \alpha} \) is bounded, where

\[
\|\phi\|_{n, \alpha} := \sum_{2r + s \leq n} \|D_t^rD_x^s\phi\|_\infty + \sum_{2r + s = n} \sup_{t \in [0, T]} \|D_t^rD_x^s\phi(t, .)\|_\alpha + \sum_{0 < n + \alpha - 2r - s < 2} \sup_{x \in \mathbb{R}^d} \|D_t^rD_x^s\phi(., x)\|_{n + \alpha - 2r - s}.
\]

### 2.2 Assumptions

In all the following, \( A \) is a closed subset of an Euclidean space, \( T > 0 \) is a finite horizon and \( r_2 \geq r_1 > 1 \) are two parameters. The conjugate exponents of \( r_1 \) and \( r_2 \) are, respectively, denoted by \( r_1^* \) and \( r_2^* \). The data are:

\[
(b, \alpha, f_1) : [0, T] \times \mathbb{R}^d \times A \to \mathbb{R}^d \times \mathcal{S}_d^1(\mathbb{R}) \times \mathbb{R},
\]

\[
f_2 : [0, T] \times \mathcal{P}_1(\mathbb{R}^d) \to \mathbb{R},
\]
We define the Hamiltonian of the system, for all $(t, x, p, M) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^d(\mathbb{R})$:

$$H(t, x, p, M) = \sup_{a \in A} \left\{ -b(t, x, a) \cdot p - \sigma(t, x, a)' \sigma(t, x, a) \cdot M - f_1(t, x, a) \right\}.$$ 

1. Assumptions on $b, \sigma, f_1, f_2$ and $g$

   (a) For all $R > 0$, $b, \sigma$ and $f_1$ as well as the partial derivatives $\partial_x b, \partial_p b, \partial^2_x b, \partial_x \sigma, \partial_t \sigma, \partial_{xx} \sigma, \partial_x f_1, \partial_t f_1, \partial^2_{xx} f_1$, are continuous and bounded on $[0, T] \times \mathbb{R}^d \times (A \cap \bar{B}(0, R))$; $\partial_x b$ and $\partial_x \sigma$ are globally bounded.

   (b) $b$ has at most a linear growth and $\sigma$ satisfies $\Lambda^- I_d \leq \sigma' \sigma(t, x, a) \leq \Lambda^+ I_d$ for some $\Lambda^+ \leq \Lambda^- > 0$ uniformly in $(t, x, a)$.

   (c) $f_1$ is continuous and coercive with respect to $a$: There is $\delta > 0$ and $C_1, C_2 > 0$ such that, for all $(t, x, a)$, $f_1(t, x, a) \geq C_1 |a|^1 + \delta - C_2$.

   (d) $f_2$ is continuous and bounded and has one linear derivative in $m$. The first-order functional derivative $\frac{\delta f_2}{\delta m} : [0, T] \times \mathcal{P}_1(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ is globally Lipschitz continuous and bounded and $x \to \frac{\delta f_2}{\delta m}(t, m, x)$ belongs to $C_b^{3+\alpha}(\mathbb{R}^d)$ with bounds uniform in $(t, m)$.

   (e) $g$ is continuous and bounded and has one functional derivative in $m$ such that $x \to \frac{\delta g}{\delta m}(m, x)$ belongs to $C_b^{3+\alpha}(\mathbb{R}^d)$ with bounds uniform in $m$.

2. Assumptions on the Hamiltonian

   (a) $H$ is $C^1$ in $(t, x, p, M)$. The partial derivatives $\partial_x H, \partial_p H$ and $\partial_M H$ are Lipschitz in $[0, T] \times \mathbb{R}^d \times B(0, R) \times B(0, R)$ for all $R > 0$.

   (b) There is some $\alpha_1, \alpha_2 > 0$ and $C_H > 0$ such that, for all $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

   $$\alpha_1 |p|^{r_1} - C_H \leq H(t, x, p, 0) \leq \alpha_2 |p|^{r_2} + C_H.$$ 

   (c) $\partial_t H(t, x, p, M)$ is bounded over $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R})$.

   (d) There is some positive constant $C_{D_p H}$ and an exponent $\nu \geq 1$ such that for all $(t, x, p, M) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R})$

   $$|D_p H(t, x, p, M)| \leq C_{D_p H}(1 + |p|^\nu).$$

   (e) $D_x H$ is uniformly in $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ Lipschitz continuous in $M$.

   (f) i. Either $f_2 = 0$ and the limit $\lim_{|p| \to +\infty} \frac{|p|^2 + \partial_x H(t, x, p, 0)}{H^2(t, x, p, 0)} = 0$ holds uniformly in $(t, x) \in [0, T] \times \mathbb{R}^d$,
ii. or $f_2 \neq 0$ and there is some $C_{D_xH} > 0$ such that $|D_xH(t, x, p, 0)| \leq C_{D_xH}(1 + |p|)$.

3. Assumptions on the constraint $\Psi$

(a) $\Psi$ is continuous and admits a functional derivative such that $x \rightarrow \frac{\delta \Psi}{\delta m}(m, x)$

belongs to $C^{3+\alpha}(\mathbb{R}^d)$ with bounds uniform in $m$.

(b) There is at least one $m \in P_1(\mathbb{R}^d)$ such that $\Psi(m) < 0$.

(c) For all $m \in P_1(\mathbb{R}^d)$ such that $\Psi(m) = 0$ there exists $x_0 \in \mathbb{R}^d$ such that

$\frac{\delta \Psi}{\delta m}(m, x_0) < 0$.

Remark 2.1 Assumption 1 is sufficient to uniquely define the controlled process $X^\alpha$

for any control $\alpha \in \mathcal{A}$ (see below for the definitions). If $\mathcal{A}$ were compact with $f_2 = 0$,

we would be in the setting of [20] Chapter IV.4 and these assumptions would guarantee the

existence of a smooth value function (in $C^{1,2}_{b}([0, T] \times \mathbb{R}^d)$).

Remark 2.2 The upper bound in Assumption 2b is a coercivity assumption on the cost

$f_1$ relatively to the drift $b$. Taking the definition of $H$, we see that it is equivalent to

ask that, for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times A$, $f_1(t, x, a) \geq \alpha^2_x |b(t, x, a)|^2 - C_H$,

for some $\alpha^2_x > 0$. It will be a source of compactness throughout the paper. The lower

bound in Assumption 2b is a “weak”-controllability condition, and we will discuss it

further in Lemma 2.2.

Remark 2.3 Using the envelope theorem (see, for instance, [35]), we see that $H$ being

$C^1$—Assumption 2a—in the $p, M$-variables implies that, for any $a(t, x, p, M) \in \mathcal{A}$

such that $H(t, x, p, M) = -b(t, x, a(t, x, p, M)), p - \sigma \sigma(t, x, a(t, x, p, M)) - f_1(t, x, a(t, x, p, M))$ we get

$\partial_p H(t, x, p, M) = -b(t, x, a(t, x, p, M))$ and $\partial M H(t, x, p, M) = -\sigma \sigma(t, x, a(t, x, p, M))$. Consequently, drift and volatility must agree

on potentially different optimal controls with common values $-\partial_p H(t, x, p, M)$ and

$\sqrt{-\partial M H(t, x, p, M)}$, respectively. Notice that the growth conditions on the cost $f_1$

and the drift $b$ ensure that for any $(t, x, p, M) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times S_d(\mathbb{R})$, there

exists at least one such $a(t, x, p, M) \in \mathcal{A}$.

Remark 2.4 Using the envelope theorem and the uniform ellipticity condition in

Assumption 1b, we see that for all $(t, x, p, M), \Lambda^- I_d \leq -\partial M H(t, x, p, M) \leq \Lambda^+ I_d$,

a fact that we will repeatedly use throughout the paper.

Remark 2.5 We use (the restrictive) Assumptions 2c, 2d, 2e, 2f in order to find Lipschitz

estimates for the solution of the Hamilton–Jacobi–Bellman equation and to deduce

that it is well-posed in $C^{1,2}_{b}([0, T] \times \mathbb{R}^d)$. Assumption 2a is then sufficient to show

that the solution is actually in $C^{3+\alpha}_{b}([0, T] \times \mathbb{R}^d)$ when Assumption 2b holds,

Assumption 2(f)ii is stronger than Assumption 2(f)i, but we use it to find Lipschitz

estimates which are independent from the time regularity of the source term of the

HJB equation.

Remark 2.6 Assumption 3c is a transversality condition. When $\Psi$ is convex, this

assumption is equivalent to the existence of some probability measure $m \in P_1(\mathbb{R}^d)$

such that $\Psi(m) < 0$. 
The following observations will be useful in order to translate the properties of the Hamiltonian $H$ into properties of the Lagrangian $L$ defined for all $(t, x, q, N) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R})$ by

$$L(t, x, q, M) := \sup_{(p, M) \in \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R})} \{-p.q - M.N - H(t, x, p, M)\}.$$  

Taking convex conjugates in 2b, we see that this assumption can be reformulated in terms of $L$: for all $(t, x, q) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\alpha_2'| q | r_i^2 - C_H \leq L(t, x, q, 0) \leq \alpha_1' | q | r_i^* + C_H,$$

where, for $i = 1, 2$, $\alpha_i' = \alpha_i^{-\frac{1}{r_i-1}}$ ( $r_i - 1$ ) $r_i^{\frac{-r_i}{r_i-1}}$ and $r_i^* = \frac{r_i}{r_i-1}$ is the conjugate exponent of $r_i$.

Throughout the article, the following dual representation for $L$ will be useful.

**Lemma 2.1** Under Assumption 1, for all $(t, x, p, M) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R})$, $L(t, x, p, M) < +\infty$ if and only if there is $q_A \in \mathcal{P}_1(A)$ such that

$$\int_A b(t, x, a)d\mathbf{q}_A(a) = q \quad \text{and} \quad \int_A \sigma'\sigma(t, x, a)d\mathbf{q}_A(a) = N$$

and in this case

$$L(t, x, q, N) = \min_{\mathbf{q}_A} \int_A f(t, x, a)d\mathbf{q}_A(a),$$

where the minimum is taken over the $\mathbf{q}_A \in \mathcal{P}_1(A)$ such that

$$\int_A b(t, x, a)d\mathbf{q}_A(a) = q$$

and

$$\int_A \sigma'\sigma(t, x, a)d\mathbf{q}_A(a) = N.$$  

**Proof** It is elementary to show that for all $(t, x, p, M) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R})$,

$$H(t, x, p, M) = \sup_{\mathbf{q}_A \in \mathcal{P}_1(A)} \left\{ \int_A (-b(t, x, a)p - \sigma'\sigma(t, x, a) - f_1(t, x, a))d\mathbf{q}_A(a) \right\}$$

and therefore $L$ reads as follows for all $(t, x, q, N) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R})$,

$$L(t, x, q, N) = \sup_{p, M} \left\{ \inf_{\mathbf{q}_A \in \mathcal{P}_1(A)} \left\{ p.q + M.N + \int_A (f(t, x, a) - b(t, x, a)p - \sigma'\sigma(t, x, a)M)d\mathbf{q}_A(a) \right\} \right\}.$$  

The result follows by exchanging the “sup” and the “inf.” To this end, we use von Neumann Theorem A.1 in Appendix. The coercivity of $f_1$ as well as results of [4] (Proposition 7.1.5) about the lower semicontinuity of functions defined on the space of probability measures allows to ensure that the use of the minmax theorem is licit.  

$\square$
From this dual representation, we can see that the lower bound on \( H(t, x, p, 0) \)—or equivalently the upper bound on \( L(t, x, q, 0) \)—is a “weak”-controllability condition. It ensures that the agent can take any drift with a relaxed (i.e., measure-valued) control without paying more than the \( r_1^* \)-power of the drift:

**Lemma 2.2** Fix \((t, x) \in [0, T] \times \mathbb{R}^d\). It holds that \( H(t, x, p, 0) \geq \alpha_1|p|^2 - CH \) for all \( p \in \mathbb{R}^d \) if and only if, for all \( q \in \mathbb{R}^d \), there exists \( q_A \in \mathcal{P}_1(A) \) such that

\[
q = \int_A b(t, x, a) d q_A(a) \quad \text{and} \quad \int_A f_1(t, x, a) d q_A(a) \leq \alpha_1'|q|^2 + CH.
\]

For example, the growth condition on \( H \) is satisfied if \( \text{Conv}(\text{Im}(b(t, x, .))) = \mathbb{R}^d \) for all \((t, x) \in [0, T] \times \mathbb{R}^d \) and for all \((t, x, a) \in [0, T] \times \mathbb{R}^d \times A, \alpha_2^*|b(t, x, a)|^2 - CH \leq f_1(t, x, a) \leq \alpha_1'|b(t, x, a)|^2 + CH.

### 2.3 Main Results

Throughout the article, we consider a fixed filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) with \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions and supporting an adapted, standard \(d\)-dimensional Brownian motion \((B_t)_{t \geq 0}\). We fix a \(\mathcal{F}_0\)-measurable random variable \(X_0\), independent of \((B_t)\) and such that \(X_0\) belongs to \(L_{1}^{\infty}(\Omega \times [0, T])\)-norm. We denote by \(\mathcal{A}\) the set of control processes. From the Cauchy–Lipschitz theorem, we know that for every \(\alpha \in \mathcal{A}\), there exists a unique \(\mathbb{F}\)-adapted process \(X^\alpha\) satisfying:

\[
dX_t = b(t, X_t, \alpha_t)dt + \sqrt{2}\sigma(t, X_t, \alpha_t)dB_t
\]

with the initial condition \(X^\alpha_0 = X_0\). A particular class of controls which is of interest is one of the Markovian controls (or Markov policies). A control process \(\alpha\) is a Markovian control if there is a measurable function \(\alpha : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d\) such that, for all \(t \in [0, T]\), \(\alpha_t = \alpha(t, X^\alpha_t)\). We now introduce the cost functional \(J_{SP} : A \rightarrow \mathbb{R} \cup \{+\infty\}\)

\[
J_{SP}(\alpha) := E\left[\int_0^T (f_1(s, X^\alpha_s, \alpha_s) + f_2(s, L(X^\alpha_T))) ds + g(L(X^\alpha_T))\right].
\]

The optimal control problem we are interested in is to minimize \(J_{SP}(\alpha)\) over \(\alpha \in \mathcal{A}\) under the constraint \(\Psi(L(X_T)) \leq 0\).

If there exists a continuous function \(h : \mathbb{R}^d \rightarrow \mathbb{R}\) such that, for all \(m \in \mathcal{P}_1(\mathbb{R}^d)\), \(\Psi(m) = \int_{\mathbb{R}^d} h(x)dm(x)\), then will say that the final constraint is linear. We define the set of admissible controls \(U_{ad}\)

\[
U_{ad} := \{\alpha \in \mathcal{A} : \Psi(L(X^\alpha_T)) \leq 0 \text{ and } J_{SP}(\alpha) < +\infty\}.
\]

The problem in strong formulation is thus:

\[
\inf_{\alpha \in U_{ad}} J_{SP}(\alpha).
\]
Theorem 2.1 [HJB equation] Take \( g' \in C^{3+\alpha}_b(\mathbb{R}^d) \) and \( f'_2 \in C_b([0, T], \mathbb{R}^{3+\alpha}_b(\mathbb{R}^d)) \) such that \( t \to f'_2(t, x) \in C^\alpha([0, T]) \) for all \( x \in \mathbb{R}^d \) with bounds uniform in \( x \). Assume further that Assumptions 1 and 2 hold with 2(i)i in force if \( f'_2 = 0 \) and 2(f)ii in force if \( f'_2 \neq 0 \). Then, the Hamilton–Jacobi–Bellman equation

\[
\begin{cases}
-\partial_t \phi(t, x) + H(t, x, \Phi(t, x), D_\Phi(t, x), D^2 \Phi(t, x)) = f'_2(t, x) & \text{in } [0, T] \times \mathbb{R}^d \\
\phi(T, x) = g'(x)
\end{cases}
\]

admits a unique strong solution \( \phi \in C^{3+\alpha}_b(\mathbb{R}^d) \). 

Theorem 2.2 [General Constraint] Under Assumptions 1, 2 and 3, there exist optimal Markov policies. Moreover, if \( (\alpha, \sigma, \gamma, m) \in \mathbb{R}^+ \times C^{1,2}_b([0, T] \times \mathbb{R}^d) \times C^0([0, T], \mathcal{P}_2(\mathbb{R}^d)) \) such that for \( m(t) \otimes dt \)-almost all \( (t, x) \) in \( [0, T] \times \mathbb{R}^d \)

\[
H(t, x, \Phi(t, x), D^2 \Phi(t, x)) = -b(t, x, \alpha(t, x)).D\Phi(t, x) - \sigma'(t, x, \alpha(s, x)).D^2 \Phi(t, x) - f_1(t, x, \alpha(t, x))
\]

and \( (\lambda, \phi, m) \) satisfies the system of optimality conditions:

\[
\begin{cases}
-\partial_t \phi(t, x) + H(t, x, \Phi(t, x), D_\Phi(t, x), D^2 \Phi(t, x)) = \frac{\delta f_2}{\delta m}(t, m(t), x) & \text{in } [0, T] \times \mathbb{R}^d \\
\partial_t m - \text{div}(\partial_\phi H(t, x, \Phi(t, x), D^2 \Phi(t, x)))m + \sum_{i,j} \partial^2 \phi_{ij}(t, x, \Phi(t, x), D^2 \Phi(t, x))) \lambda \partial_{ij} m = 0 & \text{in } [0, T] \times \mathbb{R}^d \\
\phi(T, x) = \lambda \frac{\delta \Psi}{\delta m}(m(T), x) + \frac{\delta g}{\delta m}(m(T), x) & \text{in } \mathbb{R}^d, m(0) = m_0, \\
\lambda \Psi(m(T)) = 0 & \text{in } \mathbb{R}^d, m(0) = m_0, \\
\lambda \Psi(m(T)) \leq 0, \lambda \geq 0.
\end{cases}
\]

(OC)

Furthermore, \( m(t) \) is actually the law of the optimally controlled process \( X^\alpha \) and the value of the problem—denoted by \( V_{SP}(X_0) \)—is given by

\[
V_{SP}(X_0) := \inf_{\alpha \in \mathcal{A}} J_{SP}(\alpha) = \int_{\mathbb{R}^d} \phi(0, x)dm_0(x) + \int_0^T f_2(t, m(t))dt + g(m(T)).
\]

When the constraint and the costs \( f_2 \) and \( g \) are convex in the measure variable, we are able to show that the conditions are also sufficient:

Theorem 2.3 [Convex constraint and convex costs] If \( \Psi, f_2 \) and \( g \) are convex in the measure argument and Assumptions 1, 2 and 3 hold, then the conditions of Theorem ...
are also sufficient conditions: If \( \alpha \in L^0([0, T] \times \mathbb{R}^d, A) \) satisfies (3) for some \((\phi, m, \lambda)\) satisfying OC, then the SDE

\[
\mathrm{d}X_t = b(t, X_t, \alpha(t, X_t)) \mathrm{d}t + \sqrt{2} \sigma(t, X_t, \alpha(t, X_t)) \mathrm{d}B_t
\]

starting from \(X_0\) has unique strong solution \(X_t\), it holds that \(m(t) = \mathcal{L}(X_t)\) and \(\alpha_t := \alpha(t, X_t)\) is a Markovian solution to \(\text{SP}\).

**Remark 2.7** Using standard parabolic PDE techniques and the regularity of \(\phi\), we can show that provided \(m_0\) admits a density in \(C^{2,\alpha}_b(\mathbb{R}^d)\), \(m(t)\) in Theorem 2.2 admits a density \(m(t, x)\) with respect to the Lebesgue measure such that \(m(t) \in C^{2,\alpha}_b(\mathbb{R}^d)\).

**Remark 2.8** In Theorems 2.2 and 2.3, the stochastic basis \((\Omega_1, \mathcal{F}, \mathcal{F}, \mathbb{P})\) and the Brownian motion \((B_t)\) introduced at the beginning of this section are a priori fixed. In the terminology of stochastic control, it means that we deal with strong solutions to the stochastic control problem.

**Remark 2.9** In the spirit of the Karush–Kuhn–Tucker theorem, multiple inequality constraints \(\Psi_i(m(T)) \leq 0 \forall i \in [1, n]\) can be considered provided they satisfy some qualification condition. We would say that the constraint is qualified at \(\tilde{m} \in \mathcal{P}_1(\mathbb{R}^d)\) provided there exists some \(m \in \mathcal{P}_1(\mathbb{R}^d)\) such that \(\int_{\mathbb{R}^d} \frac{\delta \Psi_i}{\delta m}(\tilde{m}, x) m(x) \, \mathrm{d}m(x) < 0\) for all \(i \in [1, n]\) such that \(\Psi_i(\tilde{m}) = 0\). If \(n = 2\), a sufficient condition would be \(\frac{\delta \Psi_1}{\delta m}(\tilde{m}, .) \in L^2(\mathbb{R}^d)\) for \(i = 1, 2\) and \(\int_{\mathbb{R}^d} \frac{\delta \Psi_1}{\delta m}(\tilde{m}, x) \frac{\delta \Psi_2}{\delta m}(\tilde{m}, x) \, \mathrm{d}x > 0\). For \(n \geq 2\), the condition would be satisfied everywhere if the constraints \(\Psi_i\) are convex and satisfy Assumption 3 and if there is some \(m \in \mathcal{P}_1(\mathbb{R}^d)\) such that \(\Psi_i(m) < 0\) for all \(i \in [1, n]\).

### 3 A Relaxed Problem: Optimal Control of the Fokker–Planck Equation

**Definition 3.1** The relaxed problem is

\[
\inf_{(m, \omega, W) \in \mathcal{K}} J_{RP}(m, \omega, W), \quad (\text{RP})
\]

where \(\mathcal{K}\) is the set of triples \((m, \omega, W) \in C^0([0, T], \mathcal{P}_1(\mathbb{R}^d)) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathcal{S}_d(\mathbb{R}))\) such that \(\omega\) and \(W\) are absolutely continuous with respect to \(m(t) \otimes \mathrm{d}t\),

\[
\partial_t m + \text{div} \omega - \sum_{i,j} \partial_{ij}^2 W_{ij} = 0
\]
holds in the sense of distributions, \( m(0) = m_0 \) and \( \Psi(m(T)) \leq 0 \). The cost \( J_{RP} \) is defined on \( \mathbb{K} \) by

\[
J_{RP}(m, \omega, W) := \int_0^T \int_{\mathbb{R}^d} L \left( t, x, \frac{d\omega}{dt} \otimes \frac{dm}{dt}, x, \frac{dW}{dt} \otimes \frac{dm}{dt} \right) dm(t)(x)dt + \int_0^T f_2(t, m(t))dt + g(m(T)).
\]

Notice that the first term in the objective function \( J_{RP} \) is convex in the variables \((m, \omega, W)\) and that the Fokker–Planck equation and the initial condition are linear in \((m, \omega, W)\). Therefore, the problem is linear/convex when the final constraint as well as the costs \( f_2 \) and \( g \) is convex.

We say that \((m, \omega, W)\) in \( C^0([0, T], \mathcal{P}_1(\mathbb{R}^d)) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathcal{S}_d(\mathbb{R})) \) satisfies the Fokker–Planck equation (FPE) (4) with initial condition \( m(0) = m_0 \) if and only if, for all \( \varphi \in \mathcal{C}(\mathbb{R}^d) \) with compact support and all \( \phi \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d) \) with compact support, we have

\[
\int_0^T \int_{\mathbb{R}^d} \partial_t \phi(t, x)dm(t)(x)dt + \int_0^T \int_{\mathbb{R}^d} D\phi(t, x).d\omega(t, x) + \int_0^T \int_{\mathbb{R}^d} D^2\phi(t, x).dW(t, x) = 0
\]

and the initial condition

\[
\int_{\mathbb{R}^d} \varphi(x)dm(0)(x) = \int_{\mathbb{R}^d} \varphi(x)dm_0(x).
\]

Moreover, if \( \omega \) and \( W \) are absolutely continuous with respect to \( m(t) \otimes dt \), the above relations hold if \( \varphi \) and \( \phi \) are, respectively, taken in \( \mathcal{C}_b(\mathbb{R}^d) \) and \( \mathcal{C}^{1,2}_b([0, T] \times \mathbb{R}^d) \) (see [44] Remark 2.3). In this case, we have for all \( \phi \in \mathcal{C}^{1,2}_b([0, T] \times \mathbb{R}^d) \) and for all \( t_1, t_2 \in [0, T] \)

\[
\int_{\mathbb{R}^d} \phi(t_2, x)dm(t_2)(x) = \int_{\mathbb{R}^d} \phi(t_1, x)dm(t_1)(x) + \int_{t_1}^{t_2} \left[ \partial_t \phi(t, x) + D\phi(t, x).\frac{d\omega}{dm \otimes dt}(t, x) + \frac{dW}{dm \otimes dt}(t, x).D^2\phi(t, x) \right] dm(t)(x)dt.
\]

Let us recall some known results about the link between solutions of the FPE and solutions to the SDE.

**Proposition 3.1**

1. Suppose that \( m \) is a solution to the Fokker–Planck equation

\[
\begin{cases}
\partial_t m + \text{div}(b(t, x)m) - \sum_{i,j} \partial_{i,j} \left( (\sigma'\sigma(t, x))_{ij} m \right) = 0 \\
m(0) = m_0.
\end{cases}
\]

\[ \square \] Springer
with coefficients $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$, Borel functions satisfying
\[
\int_0^T \int_{\mathbb{R}^d} \left( |b(t, x)| + |\sigma(t, x)|^2 \right) dm(t)(x) dt < +\infty.
\]

Then, there is a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$, an adapted Brownian motion $(B_t)_{t \geq 0}$ and an adapted process $(X_t)_{0 \leq t \leq T}$ such that
\[
\mathcal{L}(X_0) = m_0,
\]
\[
dX_t = b(t, X_t) dt + \sqrt{2} \sigma(t, X_t) dB_t.
\]

Moreover, for all $t \in [0, T]$, $\mathcal{L}(X_t) = m(t)$.

2. Conversely, suppose that $(X_s)_{s \geq 0}$ is a strong solution of the stochastic differential equation
\[
\begin{cases}
  dX_s = b(s, X_s) ds + \sqrt{2} \sigma(s, X_s) dB_s \\
  X|_{t=0} = X_0
\end{cases}
\]
on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ endowed with an adapted Brownian motion $(B_t)$ with $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ Borel-measurable functions such that
\[
\mathbb{P} \left[ \int_0^T \left( |b(s, X_s)| + |\sigma(s, X_s)|^2 \right) ds < +\infty \right] = 1
\]
and let $m(t) := \mathcal{L}(X_t) = X_t \# \mathbb{P}$, then $m$ satisfies the Fokker–Planck equation (5).

**Proof** The second part follows from Itô’s lemma and is standard. For the first part, we need to combine the argument of [27, 44]. From [44] Theorem 2.5 we know that this statement is equivalent to the existence of a solution to the so-called martingale problem and from [27] Chapter 4, we know that existence of a solution to the martingale problem is equivalent to the existence of a weak solution to the SDE.

Let $V_{RP}(m_0)$ be the value of the relaxed problem. The link with the usual compactification/convexification (see [18, 29]) method in stochastic optimal control is the following:

**Proposition 3.2**
\[
V_{RP}(m_0) = \inf_{\mathcal{Q}, \mathcal{M}} \left\{ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{A}} f_1(t, x, a) d\mathcal{Q}(t, x)(a) dm(t)(x) dt \\
+ \int_0^T f_2(t, m(t)) dt + g(m(T)) \right\},
\]

\(\odot\) Springer
where the infimum is taken over the couples \((q_A, m) \in L^0([0, T] \times \mathbb{R}^d, \mathcal{P}_1(A)) \times C^0([0, T], \mathcal{P}_1(\mathbb{R}^d))\) that satisfy in the sense of distributions the Fokker–Planck equation

\[
\partial_t m + \text{div}\left( \int_A b(t, x, a) dq_A(t, x)(a)m \right) - \sum_{i,j} \partial_{ij}^2 \left( \left( \int_A \sigma^i \sigma^j(t, x, a) dq_A(t, x)(a) \right) m \right) = 0
\]

together with the initial condition \(m(0) = m_0\) and the terminal constraint \(\Psi(m(T)) \leq 0\).

**Proof** The proof follows from the dual representation of \(L\) in Lemma 2.1 and a measurable selection argument as in [42] Theorem 12.1.10. For every competitor \((m, \omega, W)\) such that \(L(t, x, \frac{d\omega}{dt \otimes dm}(t, x), \frac{dW}{dt \otimes dm}(t, x)) < +\infty\), one can find a measurable function \(q_A : [0, T] \times \mathbb{R}^d \to \mathcal{P}_1(A)\) such that for every \((t, x) \in [0, T] \times \mathbb{R}^d\), one has

\[
L(t, x, \frac{d\omega}{dt \otimes dm}(t, x), \frac{dW}{dt \otimes dm}(t, x)) = \int_A f_1(t, x, a) dq_A(t, x)(a),
\]

and

\[
\left( \frac{d\omega}{dt \otimes dm}(t, x), \frac{dW}{dt \otimes dm}(t, x) \right) = \left( \int_A b(t, x, a) dq_A(t, x)(a), \int_A \sigma^i \sigma^j(t, x, a) dq_A(t, x)(a) \right).
\]

\(\square\)

### 3.1 Analysis of the Relaxed Problem

We will need the following facts:

**Lemma 3.1** There exists \((m, \omega, W) \in \mathbb{K}\) such that \(J_{RP}(m, \omega, W) < +\infty\).

**Proof** We have to check that we can indeed reach the final constraint with a finite cost. By continuity of \(\Psi\), we can find \(x_0, \ldots, x_n \in \mathbb{R}^d\) such that \(\Psi\left( \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \right) < 0\). Fix some \(\delta > 0\). Let \(i\) be in \([0, n]\). For all \(c > 0\), we can find \(q^c \in L^0([0, T] \times \mathbb{R}^d, \mathcal{P}_1(A))\) such that \(\int_A b(t, x, a) dq^c(t, x)(a) = c(x_i - x)\) and \(\int_A f_1(t, x, a) dq^c(t, x)(a) \leq \alpha c^r_i |x_i - x|^r_i + C_H\) for all \((t, x) \in [0, T] \times \mathbb{R}^d\) (see Lemma 2.2). We define the measurable function \(\tilde{\sigma}_c(t, x) := \left( \int_A \sigma^i \sigma^j(t, x, a) dq^c(t, x)(a) \right)^{\frac{1}{2}}\). Notice that \(\Lambda^- I_d \leq \tilde{\sigma}_c(t, x) \leq \Lambda^+ I_d\) for all \((t, x) \in [0, T] \times \mathbb{R}^d\). We can use the result of Krylov in part 2.6 of [28] (existence of weak solutions to stochastic differential equations with bounded measurable coefficients and uniformly non-degenerate volatility)

\(\square\) Springer
and find a filtered probability space \((\Omega^1, \mathcal{F}^1_\cdot, \mathbb{P}^1, \mathbb{F}^1_\cdot)\) satisfying the usual conditions, an adapted Brownian motion \((B_t)\), a \(\mathcal{F}^1_0\) measurable random variable \(X_0\) with law \(m_0\) and a solution \(Y^c_t\) of the stochastic differential equation

\[
dY^c_t = ce^{ct}x_i dt + \sqrt{2}\tilde{\sigma}'(t, Y^c_t)dB_t
\]

starting from \(X_0\) with \(\tilde{\sigma}'(t, y) = e^{ct}\tilde{\sigma}_c(t, e^{-ct}y)\). By Ito’s lemma, \(X^c_t := e^{-ct}Y^c_t\) solves the SDE

\[
dX^c_t = c(x_i - X^c_t)dt + \sqrt{2}\tilde{\sigma}_c(t, X^c_t)dB_t
\]

starting from \(X_0\) and we have for all \(t \in [0, T]\)

\[
X^c_t = x_i + (X_0 - x_i)e^{-ct} + \sqrt{2}e^{-ct}\int_0^t \tilde{\sigma}_c(s, X^c_s)e^{cs}dB_s.
\]

Using the Burkholder–Davis–Gundy inequality and the upper bound on \(\sigma^t\sigma\), we get

\[
\mathbb{E}^1(|X^c_t - x_i|^r_t) \leq 2r^{r_t - 1}e^{-r_tct}\mathbb{E}^1(|X_0 - x_i|^r_t)
+ 2\frac{3r_t^{r_t - 2}}{2}e^{-r_tct}\mathbb{E}^1\left(\left|\int_0^t \tilde{\sigma}_c(s, X^c_s)e^{cs}dB_s\right|^r_t\right)
\]

\[
\leq 2r^{r_t - 1}e^{-r_tct}\mathbb{E}^1(|X_0 - x_i|^r_t)
+ 2\frac{3r_t^{r_t - 2}}{2}e^{-r_tct}\mathbb{E}^1\left(\left|\int_0^t \text{Tr}(\tilde{\sigma}' \tilde{\sigma}_c(s, X^c_s)e^{2cs}ds\right|^r_t\right)
\]

\[
\leq 2r^{r_t - 1}e^{-r_tct}\mathbb{E}^1(|X_0 - x_i|^r_t)
+ 2\frac{3r_t^{r_t - 2}}{2}dA^\frac{r_t}{2}\left(\frac{e^{-2ct} - 1}{2c}\right)^\frac{r_t}{2},
\]

where \(\mathbb{E}^1\) is the expectation under \(\mathbb{P}^1\). In particular, taking \(t = T\) we see that for \(c\) sufficiently large we have \(d_{r_1}(\mathcal{L}(X^c_T), \delta_{x_i}) \leq \delta\). Now, for such a \(c\), we let \(m_i(t) = \mathcal{L}(X^c_t), \omega^i = c(x_i - x)m_i, W^i = \tilde{\sigma}_c^i\tilde{\sigma}_c(t, x)m_i\). Since \(f_2\) and \(g\) are bounded functions, thanks to the upper bound on \(f_1\) we have that

\[
J_{RP}(m^i, \omega^i, W^i) \leq C \left(1 + \int_0^T \mathbb{E}^1(|X^c_t - x_i|^r_t)dt\right) < +\infty.
\]

Now, we do the same for all \(i \in [0, n]\) and we let \((m, \omega, W) := \frac{1}{n} \sum_{i=1}^n (m^i, \omega^i, W^i)\). The triple \((m, \omega, W)\) solves the Fokker–Planck equation starting from \(m_0\). Now by convexity of
\[(m, \omega, W) \rightarrow \int_0^T \int_{\mathbb{R}^d} L \left( t, x, \frac{d\omega}{dm \otimes dt}(t, x), \frac{dW}{dm \otimes dt}(t, x) \right) dm(t)(x)dt \]

and using the fact that \(f_2\) and \(g\) are bounded we get that \(J_{RP}(m, \omega, W) < +\infty\). Finally,

\[d_{r_1}(\frac{1}{n} \sum_{i=1}^n m^i(T), \frac{1}{n} \sum_{i=1}^n \delta_{x_i}) \leq C(n)\delta\]

for some nonnegative constant \(C(n)\). For \(\delta\) small enough, we get that \(J_{RP}(m, \omega, W) < +\infty\) and \(\Psi(m(T)) < 0\) which concludes the proof. \(\square\)

**Lemma 3.2**

1. Any point \((m, \omega, W) \in \mathbb{K}\) with \(J_{RP}(m, \omega, W) < +\infty\) satisfies the following estimate for some constant \(C_{r_2}\) depending only on \(r_2\): for any \(0 < s \leq t < T\),

\[d_{r_2}^*(m(s), m(t)) \leq C_{r_2}(t-s)^{r_2-1} \int_0^T \int_{\mathbb{R}^d} \left| \frac{d\omega}{dt \otimes dm}(u, x) \right|^{r_2} dm(u)(x)du + C_{r_2} \Lambda^+ (t-s)^{r_2}. \tag{6}\]

2. There exists some \(M > 0\) such that

\[\sup_{t \in [0, T]} \int_{\mathbb{R}^d} |x|^{r_2} dm(t)(x) + |\omega|([0, T] \times \mathbb{R}^d) + |W|([0, T] \times \mathbb{R}^d) \leq M \tag{7}\]

whenever \(J_{RP}(m, \omega, W) \leq \inf J_{RP} + 1\).

**Proof** First observe that since \(J_{RP}(m, \omega, W) < +\infty\), by the dual formula for \(L\) of Lemma 2.1, we know that \(m(t) \otimes dt\)-almost everywhere: \(\Lambda^- I_d \leq \frac{dW}{dt \otimes dm} \leq \Lambda^+ I_d\). Let \((\Omega, \mathcal{F}, \mathbb{P}, (X, B))\) be a weak solution to the SDE

\[dX_t = \frac{d\omega}{dt \otimes dm}(t, X_t)dt + \sqrt{2} \frac{dW}{dt \otimes dm}(t, X_t)dB_t\]

with \(\mathcal{L}(X_t) = m(t)\) for all \(t \in [0, T]\). The existence of such a solution is ensured by the fact that \(m\) solves the FPE with coefficients \(\frac{d\omega}{dt \otimes dm}, \frac{dW}{dt \otimes dm}\) (see Proposition 3.1). Now, for all \(0 \leq s < t \leq T\), with \(M_{r_2}\) and \(C_{r_2}\) positive constants depending only on \(r_2\) we have

\(\square\) Springer
Finally, that particular choice of \( \lambda \).

Using the estimate proven in the first part of the lemma, we see that for all $r > 0$ any $\gamma^\alpha_2 $ such that $\gamma^\alpha_2 \leq 1$ for some new $\gamma^\alpha_2$.

For the second part of the lemma, let us take $m, \omega, W \in \mathbb{K}$ such that $J_R (m, \omega, W) \leq \inf J_{R}+1$. From the growth assumptions on $L$, there exists $M_1 > 0$ (which does not depend on the particular $(m, \omega, W)$) such that

$$
\int_{\mathbb{R}^d} \int_{0}^{T} \left| \frac{d \omega}{dt \otimes dm}(u, x) \right| \left| \gamma^\alpha_2 \right| dm(u)(x)du \leq M_1.
$$

Using the estimate proven in the first part of the lemma, we see that for all $t, s \in [0, T]$, $d_{\gamma^\alpha_2}(m(s), m(t)) \leq M'_1$ for some $M'_1 > 0$ which, once again, does not depend on the particular choice of $(m, \omega, W)$. This yields the uniform estimate $\int_{\mathbb{R}^d} |x| \gamma^\alpha_2 dm(t)(x) < M''_1$ for some new $M''_1 > 0$. The uniform estimate on $|\omega|$ follows by Hölder’s inequality

$$
|\omega|([0, T] \times \mathbb{R}^d) \leq \left( \int_{0}^{T} \int_{\mathbb{R}^d} dm(t)(x)dt \right)^{1/r_2} \times \left( \int_{0}^{T} \int_{\mathbb{R}^d} \left| \frac{d \omega}{dt \otimes dm}(t, x) \right| \left| \gamma^\alpha_2 \right| dm(t)(x)dt \right)^{1/r_2} \leq T^{1/r_2} M'_1^{1/r_2}.
$$

Finally, $m(t) \otimes dt$-almost everywhere $S_\mu \left( \frac{d W}{dt \otimes dm} \right) \in [\Lambda^-, \Lambda^+]$ which means that $|W|([0, T] \times \mathbb{R}^d) \leq \sqrt{d} \Lambda^+$. The claim follows taking $M = M''_1 + T^{1/r_2} M'_1^{1/r_2} + \sqrt{d} \Lambda^+$. \qed
From this, we can conclude with:

**Theorem 3.1** $J_{RP}$ achieves its minimum at some point $(\bar{m}, \bar{\omega}, \bar{W})$ in $\mathbb{K}$.

**Proof** This follows from the direct method of calculus of variations. Let $(m_n, \omega_n, W_n)$ be a minimizing sequence such that, for all $n \in \mathbb{N}$, $J_{RP}(m_n, \omega_n, W_n) \leq \inf J_{RP} + 1$. Using Estimate (7) in Lemma 3.2, we can use Arzela–Ascoli theorem on the one hand and Banach–Alaoglu theorem on the other hand to extract a subsequence (still denoted $(m_n, \omega_n, W_n)$) converging to $(\bar{m}, \bar{\omega}, \bar{W}) \in C^0([0, T], \mathcal{P}_{r_+^+}(\mathbb{R}^d)) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \times \mathcal{M}([0, T] \times \mathbb{R}, S_d(\mathbb{R}))$ in $C^0([0, T], \mathcal{P}_0(\mathbb{R}^d)) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \times \mathcal{M}([0, T] \times \mathbb{R}^d, S_d(\mathbb{R}))$ for any $\delta \in (1, r_+^-)$. It remains to show that $(\bar{m}, \bar{\omega}, \bar{W})$ belongs to $\mathbb{K}$ and is indeed a minimum. The Fokker–Planck equation and the initial and final conditions are easily deduced from the weak-* convergence of measures. To conclude we can use Theorem 2.34 of [3] to show that absolute continuity of $\omega_n$ and $W_n$ with respect to $m_n(t) \otimes d\tau$ is preserved when we take limits and that $J_{RP}(\bar{m}, \bar{\omega}, \bar{W}) \leq \liminf_{n} J_{RP}(m_n, \omega_n, W_n)$. So $(\bar{m}, \bar{\omega}, \bar{W})$ is indeed a minimum of $J_{RP}$ in $\mathbb{K}$. \qed

### 3.2 Necessary Conditions for the Linear Case

In this section, we suppose that $\Psi$ is linear: There is a function $h : \mathbb{R}^d \to \mathbb{R}$ such that, for all $m \in \mathcal{P}_1(\mathbb{R}^d)$, $\Psi(m) = \int_{\mathbb{R}^d} h(x)m(dx)$. We also suppose that $h$ belongs to $C^{3+\alpha}_b(\mathbb{R}^d)$ for some $\alpha \in (0, 1)$ and that there exists $x_T \in \mathbb{R}^d$ such that $h(x_T) < 0$. Under these assumptions, $\Psi$ satisfies Assumption 3. We also suppose that $f_2$ and $g$ are linear in $m$ with $f_2(t, m) = \int_{\mathbb{R}^d} f_2(t, x)dm(x)$ and $g(m) = \int_{\mathbb{R}^d} g'(x)dm(x)$ with $g' \in C^{3+\alpha}_b(\mathbb{R}^d)$ and $f_2'$ satisfying the assumptions of Theorem 2.1. Let us introduce a dual problem for $\text{RP}$.

**Definition 3.2** [Dual Problem] The dual problem is:

$$\sup_{(\lambda, \phi) \in \mathbb{A}^+ \times \mathbb{H}^0} \int_{\mathbb{R}^d} \phi(0, x)m_0(dx),$$

where $\mathbb{A} = C^{1, 2}_b([0, T] \times \mathbb{R}^d)$ and, for all $(\lambda, \phi) \in \mathbb{R}^+ \times \mathbb{A}$, $\phi$ belongs to $\mathbb{H}^0(\lambda h + g')$ if and only if:

$$\begin{cases} -\partial_t \phi(t, x) + H(t, x, D\phi(t, x), D^2\phi(t, x)) \leq f_2'(t, x) \text{ in } [0, T] \times \mathbb{R}^d \\ \phi(T, x) \leq \lambda h(x) + g'(x) \text{ in } \mathbb{R}^d \end{cases}$$

The main theorem of this part is a duality result between $\text{RP}$ and $\text{DP}$:

**Theorem 3.2**

$$\min_{(m, \omega, W) \in \mathbb{K}} J_{RP}(m, \omega, W) = \sup_{(\lambda, \phi) \in \mathbb{A}^+ \times \mathbb{H}^0} \int_{\mathbb{R}^d} \phi(0, x)m_0(dx).$$
To prove Theorem 3.2, the idea is to write the relaxed problem $\mathcal{RP}$ as a min/max problem and use the von Neumann theorem to conclude. The statement of the von Neumann theorem is given in Appendix A.

**Proof of Theorem 3.2**

First we need to enlarge the space of test functions $\mathcal{A}$ to allow for functions with linear growth. More precisely, we define $\mathcal{A}'$ as the subset of $C^{1,2}([0, T] \times \mathbb{R}^d)$ consisting of functions $\phi$ such that

$$\| (\partial_t \phi)^- \|_\infty + \| \phi^+ \|_\infty + \| D\phi \|_\infty + \| D^2 \phi \|_\infty + \left\| \frac{|\phi| + |\partial_t \phi|}{1 + |x|} \right\|_\infty < +\infty.$$  

Owing to the estimates of Lemma 3.2 and using an approximation argument similar to [44] Remark 2.3, we see that any minimizer of the relaxed problem $\mathcal{RP}$ satisfies the Fokker–Planck equation against any function $\phi \in \mathcal{A}'$. Now, we define $\mathcal{B}$ to be the set of tuples $(m, \omega, W, n)$ in $\mathcal{M}^+([0, T] \times \mathbb{R}^d) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathbb{R}^d) \times \mathcal{M}([0, T] \times \mathbb{R}^d, \mathcal{S}_d(\mathbb{R})) \times \mathcal{M}^+([0, T] \times \mathbb{R}^d)$ such that $\omega$ and $W$ are absolutely continuous with respect to $m$. $J_{RP}'$ is defined on $\mathcal{B}$ by

$$J_{RP}'(m, \omega, W, n) = \int_0^T \int_{\mathbb{R}^d} \left[ L \left( t, x, \frac{d\omega}{dm}(t, x), \frac{dW}{dm}(t, x) \right) + f_2'(t, x) \right] dm(t, x) + \int_{\mathbb{R}^d} g'(x)dn(x).$$

If $(\tilde{m}, \tilde{\omega}, \tilde{W})$ is a solution of the relaxed problem, we claim that

$$J_{RP}(\tilde{m}, \tilde{\omega}, \tilde{W}) = J_{RP}'(\tilde{m}, \tilde{\omega}, \tilde{W}, \tilde{m}(T, dx)) = \inf_{(m, \omega, W, n)} J_{RP}'(m, \omega, W, n),$$

where the infimum is taken over the $(m, \omega, W, n)$ in $\mathcal{B}$ satisfying,

$$\forall \phi \in \mathcal{A}', \int_0^T \int_{\mathbb{R}^d} (\partial_t \phi m + D\phi \cdot \omega + D^2 \phi \cdot W) + \int_{\mathbb{R}^d} \phi(0, x)dm_0(x)$$

$$- \int_{\mathbb{R}^d} \phi(T, x)dn(x) = 0, \quad (9)$$

$$\int_{\mathbb{R}^d} h(x)dn(x) \leq 0. \quad (10)$$

Indeed, since $(\tilde{m}, \tilde{\omega}, \tilde{W}, \tilde{m}(T))$ belongs to $\mathcal{B}$ and satisfies the Fokker–Planck equation, it is clear that $J_{RP}'(\tilde{m}, \tilde{\omega}, \tilde{W}, \tilde{m}(T)) \geq \inf_{(m, \omega, W, n)} J_{RP}'(m, \omega, W, n).$ Now, let us take $(m, \omega, W, n) \in \mathcal{B}$ satisfying $(9)$ for every $\phi \in \mathcal{A}'$ and such that $J_{RP}'(m', \omega', W', n') < +\infty.$ Testing $(9)$ against space-independent functions, we see that the time marginal of $m$ is the Lebesgue measure on $[0, T]$ and that

$$\int_{\mathbb{R}^d} dm(t)(x) = 1 \, dt$$

almost everywhere in $[0, T]$ for any flow of measures $t \to m(t)$ arising from the disintegration of $m$ with respect to $dt$. Now, we can follow Lemma...
and the discussion below in the proof of Theorem 3.1 to deduce that $m$ admits a continuous representative $m' \in \mathcal{C}([0, T], \mathcal{P}_+^\infty(\mathbb{R}^d))$. We then get $n = m'(T)$ from (9).

Therefore, $(m', \omega, W)$ belongs to $\mathbb{K}$,

$$J_{RP}(m, \omega, W, n) = J_{RP}(m', \omega, W) \geq J_{RP}(\tilde{m}, \tilde{\omega}, \tilde{W})$$

and the claim is proved. Now, observe that for any point $(m, \omega, W, n)$ in $\mathbb{B}$

$$\sup_{(\lambda, \phi) \in \mathbb{R}^+ \times A'} \left[ \int_0^T \int_{\mathbb{R}^d} (\partial_t \phi m + D\phi.m + D^2 \phi.W) + \int_{\mathbb{R}^d} \phi(0, x)dm_0(x) + \int_{\mathbb{R}^d} (\lambda h(x) - \phi(T, x)) dn(x) \right] = \begin{cases} 0 & \text{if } (m, \omega, W, n) \text{ satisfies } 9 \text{ and } 10, \\ +\infty & \text{otherwise}. \end{cases}$$

Therefore, we deduce that

$$V_{RP}(m_0) := \min_{(m, \omega, W) \in \mathbb{K}} J_{RP}(m, \omega, W) = \inf_{(m, \omega, W, n) \in \mathbb{B}} \sup_{(\lambda, \phi) \in \mathbb{R}^+ \times A'} \mathcal{L}(\lambda, \phi, (m, \omega, W, n)),$$

where $\mathcal{L} : \mathbb{R}^+ \times A' \times \mathbb{B} \rightarrow \mathbb{R}$ is defined by:

$$\mathcal{L}(\lambda, \phi, (m, \omega, W, n)) = \int_0^T \int_{\mathbb{R}^d} \left( L \left( t, x, \frac{d\omega}{dm}(t, x), \frac{dW}{dm}(t, x) \right) + f_2'(t, x)dm(t, x) + \int_{\mathbb{R}^d} g'(x)dn(x) \\
+ \int_0^T \int_{\mathbb{R}^d} \partial_t \phi(t, x)dm(t, x) + D\phi(t, x).d\omega(t, x) \\
+ D^2 \phi(t, x).dW(t, x) + \int_{\mathbb{R}^d} \phi(0, x)dm_0(x) \\
- \int_{\mathbb{R}^d} \phi(T, x)dn(x) + \lambda \int_{\mathbb{R}^d} h(x)dn(x). \right)$$

**Step 2: Analysis of the Lagrangian**

We immediately check that for all $(\lambda, \phi) \in \mathbb{R}^+ \times A'$, $(m, \omega, W, n) \rightarrow \mathcal{L}(\lambda, \phi, (m, \omega, W, n))$ is convex and for all $(m, \omega, W, n) \in \mathbb{B}$, $(\lambda, \phi) \rightarrow \mathcal{L}(\lambda, \phi, (m, \omega, W, n))$ is concave. Now, $\mathcal{L}$ can be rewritten as the sum of four terms, $\mathcal{L} = L_1 + L_2 + L_3 + \int_{\mathbb{R}^d} \phi(0, x)m_0(dx)$ where

$$L_1(\lambda, \phi, m) := \int_0^T \int_{\mathbb{R}^d} \left[ \partial_t \phi(t, x) - H(t, x, D\phi(t, x), D^2 \phi(t, x)) + f_2'(t, x) \right] dm(t, x),$$

$$L_2(\lambda, \phi, (m, \omega, W)) := \int_0^T \int_{\mathbb{R}^d} f(\lambda, \phi) \left( t, x, \frac{d\omega}{dm}(t, x), \frac{dW}{dm}(t, x) \right) dm(t, x),$$

$$L_3(\lambda, \phi, n) := \int_{\mathbb{R}^d} \left[ \lambda h(x) + g'(x) - \phi(T, x) \right] dn(x).$$
with \( f^{(\lambda, \phi)}: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times S_d(\mathbb{R}) \to \mathbb{R} \) defined by,

\[
f^{(\lambda, \phi)}(t, x, p, N) = L(t, x, q, N) + H(t, x, D\phi(t, x), D^2\phi(t, x)) + p.D\phi(t, x) + N.D^2\phi(t, x).
\]

Now suppose that \((m_k, \omega_k, W_k, n_k)_{k \in \mathbb{N}}\) weakly-* converges to some point \((m, \omega, W, n)\) and satisfies the uniform estimate

\[
\max\left\{ \int_0^T \int_{\mathbb{R}^d} (1 + |x|)dm_k(t, x), \int_0^T \int_{\mathbb{R}^d} (1 + |x|)dn_k(t, x), \int_0^T \int_{\mathbb{R}^d} \left| \frac{d\omega_k}{dm_k}(t, x) \right|^2 dm_k(t, x), \right. \\
\left. \int_0^T \int_{\mathbb{R}^d} \chi_{[0, \lambda^+]}(\frac{dW_k}{dm_k}(t, x)) dm_k(t, x) \right\} \leq M \tag{11}
\]

for some \( M > 0 \) and for all \( k \in \mathbb{N} \). Then, owing to the fact that the integrands in \( L_1 \) and \( L_3 \) are bounded from below, we have that, for every \((\lambda, \phi) \in \mathbb{R}^d \times A'\),

\[
L_1((\lambda, \phi), m) \leq \liminf_{k \to +\infty} L_1((\lambda, \phi), m_k) \quad \text{and} \quad L_3((\lambda, \phi), n) \leq \liminf_{k \to +\infty} L_3((\lambda, \phi), n_k).
\]

Moreover, \( f^{(\lambda, \phi)} \) is nonnegative and lower-semicontinuous and for all \((t, x) \in [0, T] \times \mathbb{R}^d\), \((q, N) \to f^{(\lambda, \phi)}(t, x, q, N)\) is convex, so we can proceed as in [3] Theorem 2.34 and Example 2.36 to prove that \( \omega, W \) are absolutely continuous with respect to \( m \) and \( L_2((\lambda, \phi), (m, \omega, W)) \leq \liminf_{k \to +\infty} L_2((\lambda, \phi), (m_k, \omega_k, W_k)). \) Finally, we have that

\[
L((\lambda, \phi), (m, \omega, W, n)) \leq \limsup_{k \to +\infty} L((\lambda, \phi), (m_k, \omega_k, W_k, n_k)). \tag{12}
\]

**Step 3: Min/Max Argument**

Now, we are going to use the von Neumann Theorem A.1 to show that

\[
\inf_{(m, \omega, W, n) \in \mathbb{B}} \sup_{(\lambda, \phi) \in \mathbb{R}^d \times A'} L((\phi, \lambda), (m, \omega, W, n)) = \sup_{(\lambda, \phi) \in \mathbb{R}^d \times A'} \inf_{(m, \omega, W, n) \in \mathbb{B}} L((\lambda, \phi), (m, \omega, W, n)).
\]

To check that the hypothesis of the theorem is satisfied, we define \( \varphi^*(t, x) := \sqrt{1 + |x|^2}(t - T - 1) \) and \( \phi^*(t, x) := \left( \sqrt{1 + |x|^2} + C_1 \right)(t - T - 1) + C_2 \), where

\[
C_1 = \| H (., ., D\phi^*(., .), D^2\phi^*(., .)) - f'_2(., .) \|_\infty + 1
\]

and \( C_2 = -\| g' \|_\infty - C_1 - 1 \). Then, we let

\[
C^* := \sup_{(\lambda, \phi) \in \mathbb{R}^d \times A'} \inf_{(m, \omega, W, n) \in \mathbb{B}} L((\lambda, \phi), (m, \omega, W, n)) + 1
\]

and we check that

\[
\mathbb{B}^* := \{(m, n, \omega, W) \in \mathbb{B} \text{ such that } L((0, \phi^*), (m, \omega, W, n)) \leq C^* \}
\]
is not empty and that there exists some $M > 0$ such that any $(m, \omega, W, n) \in B^*$ satisfies Estimate (11). We deduce that $B^*$ is (strongly) bounded and using (12) we see that $B^*$ is weakly-* compact. Now, we can use (11) and (12) once again to show that for all $C > 0$ and all $(\lambda, \phi) \in \mathbb{R}^+ \times A'$,

$$B^* \cap \{(m, n, \omega, W) \in B \text{ such that } \mathcal{L}((\lambda, \phi), (m, \omega, W, n)) \leq C\}$$

is (possibly empty and) compact. Therefore, we can apply the von Neumann theorem, Theorem A.1, to show that

$$\inf_{(m, \omega, W, n) \in B} \mathcal{L}((\lambda, \phi), (m, \omega, W, n)) = \sup_{(\lambda, \phi) \in \mathbb{R}^+ \times A'} \inf_{(m, \omega, W, n) \in B} \mathcal{L}((\lambda, \phi), (m, \omega, W, n)).$$

**Step 4: Computation of the Dual Problem**

Let $(\lambda, \phi) \in \mathbb{R}^+ \times A'$ be fixed and consider the problem

$$\inf_{(m, \omega, W, n) \in B} \mathcal{L}((\lambda, \phi), (m, \omega, W, n)).$$

Recall the definitions of $L_1$, $L_2$ and $L_3$ in Step 2 of the proof and observe first that, for fixed $(m, n)$,

$$\inf_{(\omega, W)} L_2((\lambda, \phi), (m, \omega, W)) = 0$$

with the infimum being achieved if and only if,

$$\begin{cases} 
\omega = -\partial p H(t, x, D\phi(t, x), D^2\phi(t, x))m, \\
W = -\partial M H(t, x, D\phi(t, x), D^2\phi(t, x))m.
\end{cases}$$

Therefore, it holds that

$$\inf_{(m, \omega, W, n) \in B} \mathcal{L}((\lambda, \phi), (m, \omega, W, n)) = \inf_{m \in \mathcal{M}^+ ([0, T] \times \mathbb{R}^d)} L_1((\lambda, \phi), m) + \inf_{n \in \mathcal{M}^+ (\mathbb{R}^d)} L_3((\lambda, \phi), n) + \int_{\mathbb{R}^d} \phi(0, x)dm_0(x)$$

but we have

$$\inf_{m \in \mathcal{M}^+ ([0, T] \times \mathbb{R}^d)} L_1((\lambda, \phi), m) = \begin{cases} 0 & \text{if } -\partial_t \phi + H(t, x, D\phi, D^2\phi) \leq f'_2(t, x) \text{ in } [0, T] \times \mathbb{R}^d, \\
-\infty & \text{otherwise},
\end{cases}$$

and

$$\inf_{n \in \mathcal{M}^+ (\mathbb{R}^d)} L_3((\lambda, \phi), n) = \begin{cases} 0 & \text{if } \phi(T, x) \leq \lambda h(x) + g'(x) \text{ in } \times \mathbb{R}^d, \\
-\infty & \text{otherwise},
\end{cases}$$

\(\square\) Springer
so we can conclude that

\[
\inf_{(m, \omega, W, n)} \sup_{(\lambda, \phi) \in \mathbb{R}^+ \times A'} \mathcal{L}(m, \omega, W, n, (\lambda, \phi)) = \inf_{(m, \omega, W, n) \in \mathbb{B}} \sup_{(\lambda, \phi) \in \mathbb{R}^+ \times A'} \mathcal{L}(m, \omega, W, n, (\lambda, \phi)) = \sup_{(\lambda, \phi) \in \mathbb{R}^+ \times A', \phi \in \mathcal{H}^{-}(\lambda h + g')} \int_{\mathbb{R}^d} \phi(0, x) m_0(dx),
\]

where \( \phi \in \mathcal{H}^{-}(\lambda h + g') \) for some \((\lambda, \phi) \in \mathbb{R}^+ \times A' \) if and only if

\[
\left\{
\begin{align*}
-\partial_t \phi(t, x) + H(t, x, D\phi(t, x), D^2 \phi(t, x)) &\leq f_2'(t, x) \quad \text{in } [0, T] \times \mathbb{R}^d, \\
\phi(T, x) &\leq \lambda h(x) + g'(x) \quad \text{in } \mathbb{R}^d.
\end{align*}
\right.
\]

Finally, we get \( \min_{(m, \omega, W) \in \mathbb{K}} J_{RP}(m, \omega, W) = \sup_{(\lambda, \phi) \in \mathbb{R}^+ \times A', \phi \in \mathcal{H}^{-}(\lambda h + g')} \int_{\mathbb{R}^d} \phi(0, x) m_0(dx). \)

Notice that this duality is not surprising and holds under very general conditions (see, for instance, [21]). In particular, the volatility \( \sigma \) can be degenerate. However, the existence of solutions to the dual problem requires stronger assumptions. In particular, we need strong solutions to the HJB equation and that is why we need Theorem 2.1.

**Lemma 3.3** The dual problem has a finite value which is achieved at some point \((\tilde{\lambda}, \tilde{\phi}) \in \mathbb{R}^+ \times C^{1,2}_b([0, T] \times \mathbb{R}^d) \) such that:

\[
\left\{
\begin{align*}
-\partial_t \tilde{\phi} + H(t, x, D\tilde{\phi}(t, x), D^2 \tilde{\phi}(t, x)) &= f'_2(t, x) \quad \text{in } [0, T] \times \mathbb{R}^d, \\
\tilde{\phi}(T, x) &= \tilde{\lambda} h(x) + g'(x) \quad \text{in } \mathbb{R}^d.
\end{align*}
\right.
\]

**Proof** The finiteness follows from the fact that

\[
\sup_{(\lambda, \phi) \in \mathbb{R}^+ \times \mathcal{H}^{-}(\lambda h + g')} \int_{\mathbb{R}^d} \phi(0, x) dm_0(x) = \min_{(m, \omega, W) \in \mathbb{K}} J_{RP}(m, \omega, W) < +\infty.
\]

Now, take \((\tilde{m}, \tilde{\omega}, \tilde{W}) \in \mathbb{K} \) such that \( J_{RP}(\tilde{m}, \tilde{\omega}, \tilde{W}) < +\infty \) and \( \int_{\mathbb{R}^d} h(x) d\tilde{m}(T)(x) < 0 \) and \((\lambda, \phi) \) a candidate for the dual problem. Since \((\tilde{m}, \tilde{\omega}, \tilde{W}) \) satisfies the Fokker–Planck equation, we have, taking \( \tilde{\phi} \) as a test function,

\[
\int_{\mathbb{R}^d} \phi(T, x) d\tilde{m}(T)(x) = \int_{\mathbb{R}^d} \phi(0, x) dm_0(x)
+ \int_0^T \int_{\mathbb{R}^d} \left[ \partial_t \phi(t, x) + \frac{d\tilde{\omega}}{d\tilde{m}} \otimes \frac{d\tilde{m}}{dt} (t, x).D\phi(t, x) \\
+ \frac{d\tilde{W}}{d\tilde{m}} (t, x).D^2 \phi(t, x) \right] d\tilde{m}(t)(x) dt.
\]
Using the inequations satisfied by $\phi$ and the definition of $L$, we get after reorganizing the terms

$$
\lambda \left( - \int_{\mathbb{R}^d} h(x) d\tilde{m}(T)(x) \right) \leq J_{RP}(\tilde{m}, \tilde{\omega}, \tilde{W}) - \int_{\mathbb{R}^d} \phi(0, x) dm_0(x). \quad (13)
$$

Now, if we take $(\phi_n, \lambda_n)$ a maximizing sequence, the above inequality shows that $(\lambda_n)$ is bounded. Taking a subsequence, we can suppose that $(\lambda_n)$ converges to some $\tilde{\lambda} \geq 0$. By comparison, $(\tilde{\phi}, \tilde{\lambda})$ is a solution of the dual problem, where $\tilde{\phi} \in C^1_{b,2}([0, T] \times \mathbb{R}^d)$ is solution to

$$
\begin{align*}
&-\partial_t \phi(t, x) + H(t, x, D\phi(t, x), D^2\phi(t, x)) = f'_2(t, x) \text{ in } [0, T] \times \mathbb{R}^d \\
&\phi(T, x) = \tilde{\lambda} h(x) + g'(x) \text{ in } \mathbb{R}^d.
\end{align*}
\tag{14}
$$

**Remark 3.1** In the proof of the previous lemma, we showed as a by-product that $\lambda$ is bounded independently from $\phi, m$. In particular, using inequality (13) for a maximizing sequence and using the duality result of Theorem 3.2 we get that $\tilde{\lambda}$ satisfies

$$
\tilde{\lambda} \leq \frac{J_{RP}(\tilde{m}, \tilde{\omega}, \tilde{W}) - V_{RP}(m_0)}{-\int_{\mathbb{R}^d} h(x)d\tilde{m}(T)(x)}
$$

for any candidate $(\tilde{m}, \tilde{\omega}, \tilde{W})$ such that $\int_{\mathbb{R}^d} h(x)d\tilde{m}(T)(x) < 0$.

**Corollary 3.1** If $(\tilde{m}, \tilde{\omega}, \tilde{W})$ and $(\tilde{\lambda}, \tilde{\phi}, \tilde{\omega})$ are points where, respectively, the primal and the dual problems are achieved, then

$$
\begin{align*}
\tilde{\omega} &= -\partial_p H(t, x, D\tilde{\phi}, D^2\tilde{\phi}(t, x))\tilde{m}(t) \otimes dt, \\
\tilde{W} &= -\partial_M H(t, x, D\tilde{\phi}, D^2\tilde{\phi}(t, x))\tilde{m}(t) \otimes dt
\end{align*}
$$

and $(\tilde{\lambda}, \tilde{\phi}, \tilde{\omega})$ satisfies the optimality conditions

$$
\begin{align*}
&-\partial_t \tilde{\phi}(t, x) + H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x)) = f'_2(t, x) \quad \text{in } [0, T] \times \mathbb{R}^d \\
&\partial_t \tilde{m} - \text{div}(\partial_p H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x))\tilde{m}) \\
&+ \sum_{i,j} \partial^2_{ij} \left( (\partial_M H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x)))_{ij} \tilde{m} \right) = 0 \quad \text{in } [0, T] \times \mathbb{R}^d \\
&\tilde{\phi}(T, x) = \tilde{\lambda} h(x) + g'(x) \text{ in } \mathbb{R}^d, \tilde{m}(0) = m_0 \\
&\tilde{\lambda} \int_{\mathbb{R}^d} h(x)d\tilde{m}(T)(x) = 0, \int_{\mathbb{R}^d} h(x)d\tilde{m}(T)(x) \leq 0, \tilde{\lambda} \geq 0.
\end{align*}
$$

\text{\copyright Springer}
Proof of Corollary 3.1} Let \((\phi, \lambda) \in \mathbb{A}\) and \((m, \omega, W) \in \mathbb{K}\) points where the primal and the dual problems are achieved. One has \(\int_{\mathbb{R}^d} \phi(0, x)dm_0(x) = J_{RP}(m, \omega, W)\). Given the constraint on \(\phi\) and the fact that \(m\) is nonnegative, we get

\[
\begin{align*}
\int_{\mathbb{R}^d} \phi(0, x)dm_0(x) - \int_0^T \int_{\mathbb{R}^d} (-\partial_t \phi(t, x) + H(t, x, D\phi(t, x), D^2\phi(t, x)))[dm(t)(x)] dt \\
\geq \int_0^T \int_{\mathbb{R}^d} \left[ L(t, x, \frac{d\omega}{dt} \otimes dm(t), \frac{dW}{dt} \otimes dm(t), x) \right] dm(t)(x) dt \\
+ \int_{\mathbb{R}^d} g'(x)dm(T)(x).
\end{align*}
\]

Yet, \((m, \omega, W)\) solves the Fokker–Planck equation and \(\phi(T, x) \leq \lambda h(x) + g'(x)\) for all \(x \in \mathbb{R}^d\) so

\[
\lambda \int_{\mathbb{R}^d} h(x)dm(T)(x) - \int_0^T \int_{\mathbb{R}^d} D\phi(t, x).d\omega(t, x) - \int_0^T \int_{\mathbb{R}^d} D^2\phi(t, x).dW(t, x) \\
\geq \int_0^T \int_{\mathbb{R}^d} \left[ L(t, x, \frac{d\omega}{dt} \otimes dm(t), \frac{dW}{dt} \otimes dm(t), x) \right] dm(t)(x) dt \\
+ H(t, x, D\phi(t, x), D^2\phi(t, x))]
\]

Remember that \(\int_{\mathbb{R}^d} h(x)dm(T)(x) \leq 0\) and \(\lambda \geq 0\) so

\[
\int_0^T \int_{\mathbb{R}^d} \left[ L(t, x, \frac{d\omega}{dt} \otimes dm(t), \frac{dW}{dt} \otimes dm(t), x) \right] dm(t)(x) dt \\
- D\phi(t, x). \frac{d\omega}{dt} \otimes dm(t, x) - D^2\phi(t, x). \frac{dW}{dt} \otimes dm(t, x)]dm(t)(x) dt \\
\leq \lambda \int_{\mathbb{R}^d} h(x)dm(T)(x) \leq 0.
\]

But, by definition of \(L\), the integrand is always nonnegative. So, for \(m(t) \otimes dt\text{-ae}\) we have

\[
\begin{align*}
- D\phi(t, x). \frac{d\omega}{dt} \otimes dm(t, x) - D^2\phi(t, x). \frac{dW}{dt} \otimes dm(t, x) \\
= L(t, x, \frac{d\omega}{dt} \otimes dm(t, x), \frac{dW}{dt} \otimes dm(t, x)) + H(t, x, D\phi(t, x), D^2\phi(t, x))
\end{align*}
\]

and since \(H\) is differentiable, for \(m(t) \otimes dt\text{-ae}\) it holds

\[
\begin{align*}
\frac{d\omega}{dt} \otimes dm(t, x) &= -\partial_P H(t, x, D\phi(t, x), D^2\phi(t, x)) \\
\frac{dW}{dt} \otimes dm(t, x) &= -\partial_M H(t, x, D\phi(t, x), D^2\phi(t, x)).
\end{align*}
\]
Finally, since all the inequalities at the beginning of this proof are actually equalities, we get the necessary conditions for optimality.

\[ \Box \]

4 Proof of the Main Results

4.1 Linearization

Let us fix \((\tilde{m}, \tilde{\omega}, \tilde{W})\) a solution of the relaxed problem. The linearized problem is to minimize

\[
J_{RP}^l(m, \omega, W) := \int_0^T \int_{\mathbb{R}^d} L \left( t, x, \frac{d\omega}{dt} \otimes dm(t, x), \frac{dW}{dt} \otimes dm(t, x) \right) dm(t)(x) dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^d} \frac{\delta f_2}{\delta m}(t, \tilde{m}(t), x) dm(t)(x) dt + \int_{\mathbb{R}^d} \frac{\delta g}{\delta m}(\tilde{m}(T), x) dm(T)(x)
\]

among triples \((m, \omega, W)\) that satisfy the Fokker–Planck equation with \(m(0) = m_0\) and with \(m\) satisfying the linearized constraint

\[
\int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x) dm(T)(x) \leq 0.
\]

Notice that we are in the setting of Sect. 3.2 with \(f_2'(t, x) = \frac{\delta f_2}{\delta m}(t, \tilde{m}(t), x), g'(x) = \frac{\delta g}{\delta m}(\tilde{m}(T), x)\) and \(h(x) = \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x)\).

**Proposition 4.1** Let \((\tilde{m}, \tilde{\omega}, \tilde{W})\) be a fixed solution to the relaxed problem. If \(\Psi(\tilde{m}(T)) = 0\), then \((\tilde{m}, \tilde{\omega}, \tilde{W})\) is a solution of the linearized problem (15). If \(\Psi(\tilde{m}(T)) < 0\), then \((\tilde{m}, \tilde{\omega}, \tilde{W})\) is a solution of the linearized problem (15) without the final constraint.

**Proof** Suppose that \(\Psi(\tilde{m}(T)) = 0\). By condition 3c, there is some \(x_0 \in \mathbb{R}^d\) such that \(\frac{\delta \Psi}{\delta m}(\tilde{m}(T), x_0) < 0\) and we can proceed as in Lemma 3.1 (the constraint being then the linear one: \(\tilde{\Psi}(m) = \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x) dm(x)\)) and find \((m', \omega', W')\) such that

\[
\begin{align*}
m'(0) &= m_0 \\
\partial_t m' + \text{div}(\omega') - \sum_{i,j} \partial_{ij} W_{ij}' &= 0 \\
\int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x) dm'(T)(x) &< 0 \\
J_{RP}^l(m', \omega', W') &< +\infty.
\end{align*}
\]

Now, let \((m, \omega, W)\) be any candidate for the linearized problem (in particular \((m, \omega, W)\) satisfies the linearized constraint 16). Let \(\epsilon \in (0, 1)\) and define
(m^ε, ω^ε, W^ε) := (1−ε)(m, ω, W) + ε(m', ω', W'). (We perturb (m, ω, W) a little bit so that it satisfies strictly the linearized constraint.) Let λ ∈ (0, 1) and define (m^λ_κ, ω^λ_κ, W^λ_κ) := (1−λ)(m̄, ̃ω, ̃W) + λ(m^ε, ω^ε, W^ε). We have that

$$\Psi(m^λ_κ(T)) = \Psi(\tilde{m}(T)) + \lambda \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x)dm^ε(T)(x) + o(\lambda) \quad (17)$$

but

$$\int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x)dm^ε(T)(x) = (1−\epsilon) \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x)dm(T)(x)$$

$$+ \epsilon \int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x)dm'(T)(x) < 0$$

and therefore \( \Psi(m^λ_κ(T)) \leq 0 \) for small enough \( \lambda \). Now, by convexity of

\( (m, \omega, W) \rightarrow \Gamma(m, \omega, W) : \)

$$= \int_0^T \int_{\mathbb{R}^d} L\left( t, x, \frac{d\omega}{dt} \otimes dm(t)(x), \frac{dW}{dt} \otimes dm(t)(x) \right) dm(t)(x)dt$$

and optimality of (\( m̄, ̃ω, ̃W \)) for the linearized problem we have

$$\Gamma(\tilde{m}, ̃ω, ̃W) \leq \Gamma(m^λ_κ, ω^κ, W^κ) + \int_0^T \left[ f_2(t, m^κ_τ(t)) - f_2(t, ̃m(t)) \right] dt + g(m^κ_τ(T)) - g(\tilde{m}(T))$$

$$\leq (1−\lambda)\Gamma(\tilde{m}, ̃ω, ̃W) + \lambda \Gamma(m^ε, ω^ε, W^ε) + \int_0^T \left[ f_2(t, m^κ_τ(t)) - f_2(t, ̃m(t)) \right] dt$$

$$+ g(m^κ_τ(T)) - g(\tilde{m}(T))$$

which gives

$$\Gamma(\tilde{m}, ̃ω, ̃W) \leq \Gamma(m^ε, ω^ε, W^ε) + \frac{1}{\lambda} \int_0^T \left[ f_2(t, m^κ_τ(t)) - f_2(t, ̃m(t)) \right] dt$$

$$+ \frac{1}{\lambda} \left[ g(m^κ_τ(T)) - g(\tilde{m}(T)) \right].$$

Now, we let \( \lambda \) go to 0 and use once again the convexity of \( \Gamma \) to get

$$J^l_{RP}(\tilde{m}, ̃ω, ̃W) = \Gamma(\tilde{m}, ̃ω, ̃W)$$

$$\leq \Gamma(m^ε, ω^ε, W^ε) + \int_0^T \int_{\mathbb{R}^d} \frac{\delta f_2}{\delta m}(t, ̃m(t), x)dm^ε(t)(x)dt$$

$$+ \int_{\mathbb{R}^d} \frac{\delta g}{\delta m}(\tilde{m}(T), x)dm^ε(T)(x)$$

$$\leq J^l_{RP}(m, \omega, W) + \epsilon \left( J^l_{RP}(m', ω', W') - J^l_{RP}(m, \omega, W) \right).$$
We get the result letting $\epsilon \rightarrow 0$. When $\Psi(\tilde{m}(T)) < 0$, there is no need to perturb $(m, \omega, W)$ since (17) shows that $\Psi(m_{\lambda}^0(T)) \leq 0$ for small enough $\lambda$ independently from the sign of $\int_{\mathbb{R}^d} \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x)dm(T)(x)$ and we can take $\epsilon = 0$ in the rest of the proof. 

\[ \square \]

4.2 General Constraint

**Proof of Theorem 2.2** Recall that on the one hand we want to prove the existence of optimal Markovian controls for SP and on the other hand we want to prove that optimal controls, if Markovian, satisfy some necessary conditions. Let $(\tilde{m}, \tilde{\omega}, \tilde{W})$ be a solution of the relaxed problem. We can apply Proposition 4.1 and Corollary 3.1 to find some $(\tilde{\lambda}, \tilde{\phi})$ in $\mathbb{R}^+ \times C_b^{1,2}([0, T] \times \mathbb{R}^d)$ such that $(\tilde{m}, \tilde{\lambda}, \tilde{\phi})$ satisfies the system of optimality conditions 14 with $f_2'(t, x) = \frac{\delta f_2}{\delta m}(t, \tilde{m}(t), x)$, $g'(x) = \frac{\delta g}{\delta m}(\tilde{m}(T), x)$ and $h(x) = \frac{\delta \Psi}{\delta m}(\tilde{m}(T), x)$. Notice that when $\Psi(\tilde{m}(T)) < 0$, we can take $\lambda = 0$ since $(\tilde{m}, \tilde{\omega}, \tilde{W})$ is a solution of the linearized problem without constraint in this case. In general, let $\tilde{a}$ be a measurable function such that, for all $(t, x) \in [0, T] \times \mathbb{R}^d$

\[
H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x)) = -b(t, x, \tilde{a}(t, x)).D\tilde{\phi}(t, x)
- \sigma'(t, x, \tilde{a}(t, x)).D^2\tilde{\phi}(t, x)
- f_1(t, x, \tilde{a}(t, x)).
\]

We use the assumption that $H$ is continuously differentiable in $(p, M)$. Indeed, in this case one has, thanks to the Envelope theorem (see [35]),

\[
\begin{align*}
\partial_p H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x)) &= -b(t, x, \tilde{a}(t, x)), \\
\partial_M H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x)) &= -\sigma'(t, x, \tilde{a}(t, x)).
\end{align*}
\]

(18)

Since $\partial_p H$ and $\partial_M H$ are supposed to be locally Lipschitz continuous, respectively, in $p$ and $M$ and uniformly in $x$ and since $|\partial_M H|$ is bounded from below by $\sqrt{d}\Lambda^- > 0$, using the fact (Theorem 2.1) that $\tilde{\phi}$ belongs to $C_b^{3+\alpha, 3+\alpha}([0, T] \times \mathbb{R}^d)$, we see that the coefficients of the functions, $(t, x) \rightarrow \partial_p H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x))$ and $(t, x) \rightarrow \partial_M H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x))$ are Lipschitz in $x$, uniformly in $t$. Thus, there is a unique strong solution of the SDE

\[ d\tilde{X}_t = b(t, X_t, \alpha(t, X_t))dt + \sqrt{2}\sigma(t, X_t, \alpha(t, X_t))dB_t \]

starting from $X_0$. Therefore, $\mathcal{L}(\tilde{X}_t) = \tilde{m}(t)$ for all $t \in [0, T]$ and, in particular, $\Psi(\mathcal{L}(\tilde{X}_T)) \leq 0$. This means that $\tilde{a}_t := \tilde{a}(t, \tilde{X}_t)$ is admissible for the strong problem.

\[ \square \]
Since $H$ is $C^1$, we know that for all $(t, x, p, M) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S}_d(\mathbb{R})$,

\[
H(t, x, p, M) = p \cdot \partial_p H(t, x, p, M) + M \cdot \partial_M H(t, x, p, M)
- L(t, x, -\partial_p H(t, x, p, M), -\partial_M H(t, x, p, M))
\]  

and therefore (18) implies that

\[
L(t, x, -\partial_p H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x)),
-\partial_M H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x))) = f_1(t, x, \alpha(t, x))
\]

and thus $J_{SP}(\tilde{\alpha}) = J_{RP}(\tilde{m}, \tilde{\omega}, \tilde{W}) = V_{SP}(X_0)$ from which it comes that $V_{RP}(m_0) \geq V_{SP}(X_0)$. The reverse inequality being clear, we get $V_{RP}(m_0) = V_{SP}(X_0)$ and $\tilde{\alpha}$ is a solution to the strong problem. This shows in particular that optimal controls for the strong problem $SP$ do exist. Now, take $\alpha$ a Markovian solution to the strong problem. If $X$ is the corresponding process, we take $(m, \omega, W) = (m, b(x, \alpha^1)m, \sigma^1\sigma(x, \alpha^2)m)$. Then, $(m, \omega, W)$ is admissible for the relaxed problem and we have $J_{RP}(m, \omega, W) \leq J_{SP}(\alpha^1, \alpha^2) = V_{SP}(X_0)$. And thus, $J_{RP}(m, \omega, W) = V_{RP}(m_0)$. Finally, $(m, \omega, W)$ is optimal for the relaxed problem and we can apply Proposition 4.1 and Corollary 3.1 to conclude. Now, if we use $\tilde{\phi}$ in $OC$ as a test function for the Fokker–Planck equation, recalling 19 as well as the convention $\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(\mu, x)d\mu(x) = 0$ for the linear functional derivative we get that

\[
\int_{\mathbb{R}^d} \tilde{\phi}(0)dm_0(x) = \int_0^T \int_{\mathbb{R}^d} L(t, x, -\partial_p H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x)),
-\partial_M H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x))) d\tilde{m}(t)(x)dt
\]

and therefore $V_{SP}(X_0) = \int_{\mathbb{R}^d} \tilde{\phi}(0)dm_0(x) + \int_{\mathbb{R}^d} f_2(t, \tilde{m}(t))dt + g(\tilde{m}(T))$.

### 4.3 Convex Constraint and Convex Costs

Now, we show that the conditions are also sufficient when $\Psi$, $f_2$ and $g$ are convex functions in the measure variable. Notice that this case covers in particular the problem with expectation constraint and costs in expectation form when $\Psi$, $f_2$ and $g$ are linear.

**Proof of Theorem 2.3** Let $(\tilde{\alpha}, \tilde{\phi}, \tilde{m})$ be a solution to the system of optimality conditions $OC$, and let $\tilde{X}_t$ be the solution to

\[
\begin{cases}
  d\tilde{X}_t = b(t, X_t, \tilde{\alpha}(t, \tilde{X}_t))dt + \sqrt{2}\sigma(t, \tilde{X}_t, \tilde{\alpha}(t, X_t))dB_t, \\
  \tilde{X}_0 = X_0.
\end{cases}
\]
for some measurable function $\tilde{\alpha} : [0, T] \times \mathbb{R}^d \to A$ such that, for any $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x)) = -b(t, x, \tilde{\alpha}(t, x))D\tilde{\phi}(t, x) - \sigma'(t, x, \tilde{\alpha}(t, x))D^2\tilde{\phi}(t, x) - f_1(t, x, \tilde{\alpha}(t, x)).$$

Since $b(t, x, \tilde{\alpha}(t, x)) = -\partial_p H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x))$, $\sigma'(t, x, \tilde{\alpha}(t, x)) = -\partial_M H(t, x, D\tilde{\phi}(t, x), D^2\tilde{\phi}(t, x))$ and $\phi$ belongs to $C^{3+\alpha,2+\alpha}_b([0, T] \times \mathbb{R}^d)$, the SDE admits a unique strong solution.

We are going to show that $\tilde{\alpha}_t := \tilde{\alpha}(t, \tilde{X}_t)$ is a solution to the optimal control problem. The law of $\tilde{X}_t$ is $\tilde{m}(t)$, and we deduce that $\Psi(\mathcal{L}(\tilde{X}_T)) \leq 0$ and $\tilde{\alpha}_t$ is admissible. Now, we show that $\tilde{\alpha}_t$ is indeed optimal among the admissible strategies. Let $\alpha_t$ be an admissible control, $X_t$ the corresponding process and $m(t) := \mathcal{L}(X_t)$. Let also $J'_{SP}$ be defined on $U_{ad}$ as follows

$$J'_{SP}(\alpha_t) := \mathbb{E} \left( \int_0^T \left( f_1(t, X_t, \alpha_t) + \frac{\delta f_2}{\delta m}(t, \tilde{m}(t), X_t) \right) dt + \frac{\delta g}{\delta m}(\tilde{m}(t), X_T) + \tilde{\lambda} \frac{\delta \Psi}{\delta m}(\tilde{m}(T), X_T) \right).$$

Using a classical verification argument and the fact that $\tilde{\phi}$ solves the HJB equation, we get that $J'_{SP}(\tilde{\alpha}_t) \leq J'_{SP}(\alpha_t)$. Now by convexity of $\Psi$, $f_2$ and $g$, we get

$$\mathbb{E} \left[ \int_0^T f_2(t, \tilde{m}(t)) dt - \int_0^T f_2(t, m(t)) dt + \int_0^T \frac{\delta f_2}{\delta m}(t, \tilde{m}(t), X_t) dt \right] \leq 0,$$

$$\mathbb{E} \left[ g(\tilde{m}(T)) - g(m(T)) + \frac{\delta g}{\delta m}(\tilde{m}(T), X_T) \right] \leq 0$$

and

$$\tilde{\lambda} \mathbb{E} \left[ \frac{\delta \Psi}{\delta m}(\tilde{m}(T), X_T) \right] = \tilde{\lambda} \left( \Psi(\tilde{m}(T)) + \mathbb{E} \left[ \frac{\delta \Psi}{\delta m}(\tilde{m}(T), X_T) \right] \right) \leq \tilde{\lambda} \Psi(m(T)) \leq 0.$$

Therefore, we get that $J_{SP}(\tilde{\alpha}) \leq J_{SP}(\alpha)$ and $\tilde{\alpha}$ is optimal for the strong problem.

### 5 The HJB Equation

The aim of this section is to show that the HJB equation

$$\begin{align*}
-\partial_t u(t, x) + H(t, x, Du(t, x), D^2u(t, x)) &= f'_2(t, x) \text{ in } [0, T] \times \mathbb{R}^d \\
\|u(T, x) &= g'(x) \text{ in } \mathbb{R}^d
\end{align*}$$

(20)
admits a unique strong solution \( u \in C^{\frac{1+\alpha}{2},3+\alpha}_b([0,T] \times \mathbb{R}^d) \). We first recall that, by classical arguments (see [20] Chapter V), the equation (20) satisfies a comparison principle between bounded uniformly continuous sub- and super-solutions and admits therefore a unique bounded uniformly continuous viscosity solution. We denote by \( u \) this solution and now observe that it is enough to prove that \( u \) is Lipschitz continuous in \((t,x)\) to deduce that it is actually in \( C^{\frac{1+\alpha}{2},3+\alpha}_b([0,T] \times \mathbb{R}^d) \). Indeed, if \( u \) is Lipschitz continuous in space, we can use Theorem VII.3 in [26] to deduce that \( u \) is semiconcave with a modulus of semiconcavity uniform in \((t,x)\). Now using the strict parabolicity of the equation, the fact that \( u \) is Lipschitz and semiconcave, we can prove that \( u \) is also semiconvex (see [25] Theorem 4 with the help of [2]) and therefore a unique bounded uniformly continuous viscosity solution. We denote by \( u \) this solution and now observe that it is enough to prove that \( u \) is Lipschitz continuous in \((t,x)\) to deduce that it is actually in \( C^{\frac{1+\alpha}{2},3+\alpha}_b([0,T] \times \mathbb{R}^d) \). Indeed, if \( u \) is Lipschitz continuous in space, we can use Theorem VII.3 in [26] to deduce that \( u \) is semiconcave with a modulus of semiconcavity uniform in \((t,x)\). Now using the strict parabolicity of the equation, the fact that \( u \) is Lipschitz and semiconcave, we can prove that \( u \) is also semiconvex (see [25] Theorem 4 with the help of [2]) and therefore \( Du \) is continuous and Lipschitz in space. At this point, using the Hölder regularity in time of \( f'_2 \), we can use the results of [47, 48] (see also the last section of [11]) to deduce that \( u \) belongs to \( C^{1,2}_b([0,T] \times \mathbb{R}^d) \). Finally, by differentiating the equation we can use results on uniformly parabolic linear PDEs (Theorem IV.5.1 of [30]) to conclude that \( u \) belongs to \( C^{\frac{1+\alpha}{2},3+\alpha}_b([0,T] \times \mathbb{R}^d) \).

Now, we proceed to show that \( u \) is indeed globally Lipschitz continuous. We first show this when \( f'_2 \) is also globally Lipschitz continuous (and not just Hölder continuous in time) and then we use an approximation argument.

**Lemma 5.1** Suppose that Assumptions 1 and 2 hold. Take \( g' \in C^{3+\alpha}_b(\mathbb{R}^d) \) and suppose that \( f'_2 \in C_b([0,T] \times \mathbb{R}^d) \) is globally Lipschitz continuous in \((t,x)\) and \( C^1 \) in \( x \). Let \( u \) be the unique viscosity solution to (20). Then, \( u \) is also globally Lipschitz-continuous.

**Proof** We first show the regularity in time. We observe that since \( g' \in C^{3+\alpha}_b(\mathbb{R}^d) \), for

\[
C_{g'} \geq \sup_{(t,x)} |H(t,x,Dg'(t,x),D^2g'(t,x)) - f'_2(t,x)|,
\]

\( g'(x) - C_{g'}(T-t) \) and \( g'(x) + C_{g'}(T-t) \) are, respectively, viscosity sub-solution and super-solution to (20). By comparison, we have that \( |u(T-t,x) - g'(x)| \leq C_{g'}t \) for all \( t \in [0,T] \). If we fix \( s \in [0,T] \) and define for all \((t,x) \in [s,T] \times \mathbb{R}^d \),

\[
v(t,x) = u(t-s,x),
\]

it is plain to check that \( v^+(t,x) := v(t,x) - C's \) and \( v^-(t,x) := v(t,x) + C's \) are, respectively, sub- and super-solutions, where \( C' \) is such that

\[
|H(t-s,x,p,M) - H(t,x,p,M)| + |f'_2(t-s,x) - f'_2(t,x)| \leq C's
\]

for all \( s \in [0,T] \), all \( t \in [s,T] \) and all \((x,p,M) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}_d(\mathbb{R}) \). By comparison, we find that for all \( s \in [0,T] \), all \( t \in [s,T] \) and all \( x \in \mathbb{R}^d \),

\[
u(t,x) - v^+(t,x) \leq \sup_{x \in \mathbb{R}^d} u(T,x) - v^+(T,x) \leq \sup_{x \in \mathbb{R}^d} g'(x) - u(T-s,x) + C'Ts \leq C_{g'}s + C'Ts.
\]

Doing the same with \( v^- \), we get that \( |u(t,x) - u(t-s,x)| \leq (C_{g'} + C'T)s \) for all \( s \in [0,T] \), all \( t \in [s,T] \) and all \( x \in \mathbb{R}^d \).
Now, we show the space regularity. Let $K > \|Dg\|_\infty$ such that $H(t, x, p, 0) > L_T + \|f_2\|_\infty$ for all $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ such that $|p| \geq K - 1$ and where $L_T$ is an upper bound for the time-Lipschitz constant of $u$. We are going to show that for all $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, $u(t, x) - u(t, y) \leq K|x - y|$ when $K$ is large enough. Suppose on the contrary that $\delta := \sup_{(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d} |u(t, x) - u(t, y) - K|x - y||$ is positive. Let $\beta$ be a small positive parameter and define

$$\phi_\beta(t, x, y) := u(t, x) - u(t, y) - K|x - y| - \beta|y|^2 - \beta \frac{1}{t}.$$ 

The function $\phi_\beta$ reaches its maximum at some point $(\bar{t}, \bar{x}, \bar{y}) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, and there is $\beta_0 > 0$ such that for $0 < \beta \leq \beta_0$

$$\phi_\beta(\bar{t}, \bar{x}, \bar{y}) \geq \delta \frac{2}{2}. \quad (21)$$

Suppose that $\beta \leq \beta_0$ and $\bar{t} = T$, then

$$\frac{\delta}{2} \leq \phi_\beta(T, \bar{x}, \bar{y}) = u(T, \bar{x}) - u(T, \bar{y}) - K|\bar{x} - \bar{y}| - \beta|\bar{y}|^2 - \frac{\beta}{T} \leq (\|Dg\|_\infty - K)|\bar{x} - \bar{y}|.$$ 

But this is impossible since $K > \|Dg\|_\infty$ and $\delta > 0$. Thus for all $\beta \leq \beta_0$, $\bar{t} \neq T$. From (21), we deduce that $\beta|\bar{y}|^2 \leq 2u|\infty|$ and thus that $\beta|\bar{y}| \rightarrow 0$ as $\beta \rightarrow 0$ but we also deduce that

$$\frac{\delta}{2} \leq u(\bar{t}, \bar{x}) - u(\bar{t}, \bar{y}) - K|\bar{x} - \bar{y}|,$$

in particular, $\bar{x} \neq \bar{y}$. Since $\bar{t} \neq T$ for $\beta \leq \beta_0$, we can apply the maximum principle for semicontinuous functions from [16]. Let $\varphi_\beta(t, x, y) = K|x - y| + \beta \frac{1}{t}$. Computing the various derivatives for $|x - y| > 0$ gives

$$\begin{align*}
\partial_t \varphi_\beta(t, x, y) &= -\beta \\
\partial_x \varphi_\beta(t, x, y) &= \frac{K}{|x - y|} (x - y) \\
\partial_y \varphi_\beta(t, x, y) &= \frac{K}{|x - y|} (y - x) \\
D^2 \varphi_\beta(t, x, y) &= \frac{K}{|x - y|} \begin{pmatrix}
I_d & -I_d & -I_d & -I_d \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1
\end{pmatrix}
\begin{pmatrix}
(x - y) \otimes (x - y) \\
-(x - y) \otimes (x - y) \\
-(x - y) \otimes (x - y) \\
-(x - y) \otimes (x - y)
\end{pmatrix}.
\end{align*}$$

In particular, if $N := (x - y) \otimes (x - y)$, then $N \geq 0$ (rank one symmetric matrices with positive trace) and thus it is elementary to show that $\begin{pmatrix}
N & -N \\
-N & N
\end{pmatrix} \geq 0$. And thus,

$$D^2 \varphi_\beta(t, x, y) \leq \frac{K}{|x - y|} \begin{pmatrix}
I_d & -I_d \\
-I_d & I_d
\end{pmatrix}.$$
Now, from the maximum principle, we get \( v \in \mathbb{R}, X, Y \in \mathbb{S}_d(\mathbb{R}) \) such that
\[
\begin{cases}
(v, K \frac{x-y}{|x-y|}, X) \in \mathcal{P}^2_u(\bar{\bar{u}}),
(v + \frac{\beta}{2}, K \frac{x-y}{|x-y|} - 2\beta \bar{y}, Y) \in \mathcal{P}^2_u(\bar{\bar{u}})
\end{cases}
\]
and
\[
\begin{pmatrix}
X & 0 \\
0 & -(Y + 2\beta I_d)
\end{pmatrix}
\leq 3 \frac{K}{|x-y|} \begin{pmatrix}
I_d & -I_d \\
-I_d & I_d
\end{pmatrix}.
\]

Observe that \(|v|\) is bounded by \(L_T\) the time-Lipschitz constant of \(u\) and thus \(|v|\) is bounded independently of \(K\). Now, we use the equation satisfied by \(K\) to get
\[
H(\bar{\bar{t}}, \bar{\bar{x}}, K \frac{x-y}{|x-y|}, X) - f_2'(\bar{\bar{t}}, \bar{\bar{x}}) \leq v \leq H(\bar{\bar{t}}, \bar{\bar{y}}, K \frac{x-y}{|x-y|} - 2\beta \bar{y}, Y) - f_2'(\bar{\bar{t}}, \bar{\bar{y}}).
\]

From now on, we let \( \bar{\bar{\xi}} := K \frac{x-y}{|x-y|} \) and \( \gamma := \frac{3K}{|x-y|} \). We are going to show that the information
\[
\begin{cases}
|v| \leq L_T \\
|\bar{\bar{\xi}}| = K \\
\begin{pmatrix}
X & 0 \\
0 & -(Y + 2\beta I_d)
\end{pmatrix}
\leq \gamma \begin{pmatrix}
I_d & -I_d \\
-I_d & I_d
\end{pmatrix} \tag{22}
\end{cases}
\]
is inconsistent whenever \( K \) is sufficiently large. Let \( \bar{\bar{\eta}} := \bar{\bar{\xi}} - 2\beta \bar{\bar{y}} \) and for any \( \lambda \in [0, 1], x_\lambda := (1-\lambda)\bar{x} + \lambda \bar{\bar{y}} \) and \( \bar{\bar{\xi}}_\lambda := (1-\lambda)\bar{\bar{\xi}} + \lambda \bar{\bar{\eta}} = \bar{\bar{\xi}} - 2\lambda \beta \bar{\bar{y}} \). From [5], Lemma A.2, there exists a \( C^1 \) map, \( \lambda \rightarrow Z_\lambda \) from \( [0, 1] \rightarrow \mathbb{S}_d(\mathbb{R}) \) such that
\[
\begin{cases}
\frac{d}{d\lambda} Z_\lambda = \gamma^{-1} Z^2_\lambda, \\
Z_0 = X \\
\forall \lambda \in [0, 1], X \leq Z_\lambda \leq Y + 2\beta I_d.
\end{cases}
\]

Let us define \( l : [0, 1] \rightarrow \mathbb{R} \) by \( l(\lambda) = H(\bar{\bar{t}}, x_\lambda, \bar{\bar{\xi}}_\lambda, Z_\lambda) - f_2'(\bar{\bar{t}}, x_\lambda) \), so that \( l(0) = H(\bar{\bar{t}}, \bar{\bar{x}}, \bar{\bar{\xi}}, X) - f_2'(\bar{\bar{t}}, \bar{\bar{x}}) \leq v \) and (using \( Z_1 \leq Y + 2\beta I_d \) and the boundness of \( \partial_M H \))
\[
l(1) = H(\bar{\bar{t}}, \bar{\bar{y}}, \bar{\bar{\eta}}, Z_1) \geq H(\bar{\bar{t}}, \bar{\bar{y}}, \bar{\bar{\eta}}, Y + 2\beta I_d) - f_2'(\bar{\bar{t}}, \bar{\bar{y}}) \geq H(\bar{\bar{t}}, \bar{\bar{y}}, \bar{\bar{\eta}}, Y) - C\beta - f_2'(\bar{\bar{t}}, \bar{\bar{y}}) \geq v - C\beta,
\]
where \( C = 2\Lambda^+ \sqrt{d} \). Thus, \( l \) being \( C^1 \), there exists \( \lambda \in [0, 1] \) such that
\[
l(\lambda) = v - C\lambda \beta, \tag{23}
\]
\[
l'(\lambda) \geq -C\beta. \tag{24}
\]
From inequality (23), we are going to obtain a lower bound on $|Z_{\lambda}| = \sqrt{\text{Tr}(Z_{\lambda}^2)}$, and from inequality (24), we are going to obtain an upper bound on $|Z_{\lambda}|$. Combining the two bounds will get a contradiction for $K$ large enough. First, we exploit (23). It gives us

$$H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, 0) - f'_{\lambda}(\tilde{t}, x_{\lambda}) - v + C\lambda\beta = H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, 0) - H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, Z_{\lambda}) \leq -\partial_{\lambda}H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, Z_{\lambda}).Z_{\lambda} \leq \sqrt{d}\Lambda^+|Z_{\lambda}|,$$

where we used Cauchy–Schwarz inequality at the last step. Therefore, we have

$$|Z_{\lambda}| \geq \frac{H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, 0) - f'_{\lambda}(\tilde{t}, x_{\lambda}) - v + C\lambda\beta}{\sqrt{d}\Lambda^+}. \quad (25)$$

Now, we use (24). Computing the derivative of $l$ gives

$$l'(\lambda) = \partial_{\lambda}H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, Z_{\lambda}).(\tilde{y} - \check{x}) - \partial_{\lambda}f'_{\lambda}(\tilde{t}, x_{\lambda}).(\tilde{y} - \check{x}) - 2\beta\partial_{p}H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, Z_{\lambda}).\tilde{y} + \gamma^{-1}\partial_{\lambda}H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, Z_{\lambda}).Z_{\lambda}^2 \geq -C\beta$$

and since $-\partial_{\lambda}H \geq \Lambda^{-1}I_{d}$, we get

$$|Z_{\lambda}|^2 \leq \frac{1}{\Lambda^{-}} \left[ \gamma C\beta + \gamma \partial_{\lambda}H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, Z_{\lambda}).(\tilde{y} - \check{x}) - \gamma \partial_{\lambda}f'_{\lambda}(\tilde{t}, x_{\lambda}).(\tilde{y} - \check{x}) - 2\beta \gamma \partial_{p}H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, Z_{\lambda})\tilde{y} \right]. \quad (26)$$

Recalling that $\gamma (\check{x} - \tilde{y}) = 3\check{\xi} = 3\xi_{\lambda} + 6\lambda\beta\tilde{y}$, combining 25 and 26 and using Assumption 2e, we get that for some new positive constant $C$ independent from $(K, \delta, \beta, \lambda)$,

$$H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, 0)^2 \leq C \left( 1 + v^2 + |\xi_{\lambda}|^2 + \xi_{\lambda}.\partial_{\lambda}H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, 0) + \gamma \beta \tilde{y}.\partial_{p}H(\tilde{t}, x_{\lambda}, \xi_{\lambda}, Z_{\lambda}) \right).$$

We get a contradiction letting $\beta \to 0$ as soon as $K$ is big enough since $|\xi| = K$, $H(t, x, p, 0) \geq \alpha_1|p|^{r_1} - C_H$ with $r_1 > 1$ for all $(t, x, p)$ and $\partial_{\lambda}H$ satisfies Assumption 2f. \hfill \Box

To conclude with the proof, we need to show Lipschitz estimates which are independent from the time regularity of $f'_{\lambda}$. This is what we do in the following proof of Theorem 2.1.

**Proof of Theorem 2.1** When $f'_{\lambda} = 0$, the previous lemma and the discussion at the beginning of this section are enough to conclude. In the general case, take a smooth kernel $\rho$ with support in $[-1, 1]$ and define for all $n \in \mathbb{N}^*$, $\rho_n(r) := n\rho(nr)$ and $f_{\lambda}^{(n)}(t, x) := \int_{-1}^{1} f'_{\lambda}(s, x)\rho_n(t-s)ds$, where we extended $f'_{\lambda}$ to $[-1, T + 1] \times \mathbb{R}^d$. \hfill \textcircled{ Springer}
by \( f_2'(t, x) = f_2'(0, x) \) for \( t \in [-1, 0] \) and \( f_2'(t, x) = f_2'(T, x) \) for \( t \in [T, T + 1] \).

We also define \( u_n \) to be the viscosity solution to

\[
\begin{cases}
-\partial_t u_n(t, x) + H(t, x, Du_n(t, x), D^2 u_n(t, x)) = f_2^{(n)}(t, x) & \text{in } [0, T] \times \mathbb{R}^d \\
u(T, x) = g(x)
\end{cases}
\]

Thanks to the previous lemma and the discussion at the beginning of this section, we know that \( u_n \) actually belongs to \( C^{1+\alpha, 3+\alpha}_b([0, T] \times \mathbb{R}^d) \). If we define \( w_n := \frac{1}{2} e^{\mu t} |Du_n|^2 \) for some \( \mu > 0 \), we get, after differentiating the HJB equation and taking scalar product with \( e^{\mu t} Du_n \):

\[
-\partial_t w_n(t, x) + D_p H(t, x, Du_n(t, x), D^2 u_n(t, x)) \cdot Dw_n(t, x) + D_M H(t, x, Du_n(t, x), D^2 u_n(t, x)) \cdot D^2 w_n(t, x)
= Df_n(t, x) \cdot Du_n(t, x) e^{\mu t} - D_h H(t, x, Du_n(t, x), D^2 u_n(t, x)) \cdot Du_n(t, x) e^{\mu t}
+ e^{\mu t} D_M H(t, x, Du_n(t, x), D^2 u_n(t, x)) \cdot (D^2 u_n(t, x))^2 - \frac{1}{2} \mu e^{\mu t} |Du_n(t, x)|^2
\leq Df_2^{(n)}(t, x) \cdot Du_n(t, x) e^{\mu t} + C_{D_h} (1 + |Du_n(t, x)| + |D^2 u_n(t, x)|) e^{\mu t} |Du_n(t, x)|
- e^{\mu t} \Lambda^- |D^2 u_n(t, x)|^2 - \frac{1}{2} \mu e^{\mu t} |Du_n(t, x)|^2,
\]

where we used the growth assumption 2(f)ii on \( D_h H \), Assumption 2e and the uniform ellipticity of \( H \). Now, we can choose \( \mu = \mu(\|Df\|_\infty, C_{D_h}, \Lambda^-) > 0 \) such that the right-hand side of the above expression is bounded by above and by the maximum principle for parabolic equations we get that \( \|Du_n\|_\infty \leq C \) for some

\( C = C(\|Dg'\|_\infty, \|Df_2^{(n)}\|_\infty, C_{D_h}, \Lambda^-) > 0 \).

Now, let \( v_n := \partial_t u_n \). By differentiating the HJB equation with respect to time, we get that \( v_n \) solves

\[
\begin{cases}
-\partial_t v_n(t, x) + D_p H(t, x, Du_n(t, x), D^2 u_n(t, x)) \cdot Dv_n(t, x) + D_M H(t, x, Du_n(t, x), D^2 u_n(t, x)) \cdot D^2 v_n(t, x) = \partial_t f_2^{(n)}(t, x)
\quad v_n(T, x) = H(T, x, Dg'(x), D^2 g'(x)) - f_2^{(n)}(T, x).
\end{cases}
\]

Fix \((t_0, x_0) \in [0, T] \times \mathbb{R}^d\) and consider a weak solution \( m_n \in C([t_0, T], P_2(\mathbb{R}^d)) \) to the adjoint equation

\[
\begin{cases}
\partial_t m_n - \text{div}(D_p H(t, x, Du_n(t, x), D^2 u_n(t, x)) m_n) + \sum_{i,j=1}^d \partial_{i,j} D_M H(t, x, Du_n(t, x), D^2 u_n(t, x)) m_n = 0 \text{ in } [t_0, T] \times \mathbb{R}^d \\
m_n(t_0) = \delta_{x_0}
\end{cases}
\]

\( \odot \) Springer
Integrating $v_n$ against $m_n$ gives, after integration by part and reorganizing the terms:

$$v_n(t_0, x_0) = \int_{\mathbb{R}^d} \left[ H(T, x, D\gamma'(x), D^2\gamma'(x)) - f_2^{(n)}(T, x) \right] dm^n(T)(x)$$

$$- \int_{t_0}^{T} \int_{\mathbb{R}^d} \partial_t H(t, x, Du_n(t, x), D^2u_n(t, x)) dm^n(t)(x)dt$$

$$+ \int_{t_0}^{T} \int_{\mathbb{R}^d} \partial_t f_2^{(n)}(t, x) dm^n(t)(x)dt$$

But, again by integration by part, we have

$$\int_{t_0}^{T} \int_{\mathbb{R}^d} \partial_t f_2^{(n)}(t, x) dm^n(t)(x)dt = \int_{\mathbb{R}^d} f_2^{(n)}(T, x) dm(T)(x) - f_2^{(n)}(t_0, x_0)$$

$$+ \int_{t_0}^{T} \int_{\mathbb{R}^d} D_p H(t, x, Du_n(t, x), D^2u_n(t, x)).Df_2^{(n)}(t, x) dm^n(t)(x)dt$$

$$+ \int_{t_0}^{T} \int_{\mathbb{R}^d} D_M H(t, x, Du_n(t, x), D^2u_n(t, x)).D^2f_2^{(n)}(t, x) dm^n(t)(x)dt$$

and we can conclude, using the growth assumption on $D_p H$, Assumption 2d, and the boundness of $\partial_M H$ and $\partial_t H$, that $\|\partial_t u_n\|_\infty \leq C$ for some $C > 0$ depending only on $\|Du_n\|_\infty, \|f_2^{(n)}\|_\infty, \|Df_2^{(n)}\|_\infty, \|D^2f_2^{(n)}\|_\infty, \|D^2g'\|_\infty, \|D^2g'\|_\infty$ but not on $\|\partial_t f_2^{(n)}\|_\infty$.

Combining the two above estimates, we can use the stability of viscosity solutions to show that $u_n$ converges locally uniformly to $u$ and that $u$ is therefore a globally Lipschitz function. Following the discussion at the beginning of this section, this is enough to conclude that $u$ belongs to $C^{\frac{3+\alpha}{2}}_{X}([0, T] \times \mathbb{R}^d)$.

### 6 Conclusion

In this paper, we investigated a stochastic control problem with constraints on the probability distribution of the output. By reformulating the problem as a control problem for the PDE satisfied by the time marginals of the process, we were able to prove the existence of solutions and characterize them. The optimal trajectories and associated controls are given by a mean field game system of PDEs associated with an exclusion condition. We proved the sufficiency of these conditions under suitable convexity assumptions.

**Acknowledgements** The author thanks the anonymous referees for their comments and careful proofreading of the paper. The author was partially supported by the ANR (Agence Nationale de la Recherche) project ANR-16-CE40-0015-01 on Mean Field Games. Part of this research was performed, while the author was visiting the Institute for Mathematical and Statistical Innovation (IMSI), which is supported by the National Science Foundation (Grant No. DMS-1929348). The author wishes to thank Professor Pierre Cardaliaguet (Paris Dauphine) for fruitful discussions all along this work.
A Appendix

Since it appears twice in our article and in particular in the proof of Theorem 3.2, we recall the statement of the von Neumann theorem we are using. The statement and proof can be found in Appendix of [36] and in a slightly different setting, in [40].

**Theorem A.1** (Von Neumann) Let $\mathbb{A}$ and $\mathbb{B}$ be convex sets of some vector spaces and suppose that $\mathbb{B}$ is endowed with some Hausdorff topology. Let $\mathcal{L}$ be a function satisfying:

\[
\begin{align*}
  a \to \mathcal{L}(a, b) & \text{ is concave in } \mathbb{A} \text{ for every } b \in \mathbb{B}, \\
  b \to \mathcal{L}(a, b) & \text{ is convex in } \mathbb{B} \text{ for every } a \in \mathbb{A}.
\end{align*}
\]

Suppose also that there exist $a_\ast \in \mathbb{A}$ and $C_\ast > \sup_{a \in \mathbb{A}} \inf_{b \in \mathbb{B}} \mathcal{L}(a, b)$ such that:

\[
\mathbb{B}_\ast := \{ b \in \mathbb{B}, \mathcal{L}(a_\ast, b) \leq C_\ast \} \text{ is not empty and compact in } \mathbb{B},
\]

\[
b \to \mathcal{L}(a, b) \text{ is lower-semicontinuous in } \mathbb{B}_\ast \text{ for every } a \in \mathbb{A}.
\]

Then,

\[
\min_{b \in \mathbb{B}} \max_{a \in \mathbb{A}} \mathcal{L}(a, b) = \sup_{a \in \mathbb{A}} \inf_{b \in \mathbb{B}} \mathcal{L}(a, b).
\]

**Remark A.1** The fact that the infimum in the “inf sup” problem is in fact a minimum is part of the theorem.

**References**

1. Achdou, Y., Capuzzo Dolcetta, I.: Mean field games: numerical methods. SIAM J. Numer. Anal. **48**, 1136–1162 (2010)
2. Alvarez, O., Lasry, J.-M., Lions, P.-L.: Convex viscosity solutions and state constraints. J. Math. Pures Appl. **76**, 265–288 (1997)
3. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs. Oxford University Press, Oxford (2000)
4. Ambrosio, L., Gigli, N., Savare, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures. Birkhäuser, Basel (2005)
5. Armstrong, S., Cardaliaguet, P.: Stochastic homogenization of quasilinear Hamilton–Jacobi equations and geometric motions. J. Eur. Math. Soc. **20**, 797–864 (2018)
6. Barles, G.: A weak Bernstein method for fully nonlinear elliptic equations. Differ. Integral Equ. **4**, 241–262 (1991)
7. Benamou, J.-D., Brenier, Y.: A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem. Numer. Math. **84**, 375–393 (2000)
8. Blaquière, A.: Controllability of a Fokker–Planck equation, the Schrödinger system, and a related stochastic optimal control. Dyn. Control **2**, 235–253 (1992)
9. Bouchard, B., Elie, R., Imbert, C.: Optimal control under stochastic target constraints. SIAM J. Control Optim. **48**, 3501–3531 (2009)
10. Bouchard, B., Elie, R., Touzi, N.: Stochastic target problems with controlled loss. SIAM J. Control Optim. **48**, 3123–3150 (2010)
11. Bourgoing, M.: C1, $\beta$ regularity of viscosity solutions via a continuous-dependence result. Adv. Differ. Equ. **9**, 447–480 (2004)
12. Briani, A., Cardaliaguet, P.: Stable solutions in potential mean field game systems. Nonlinear Differ. Equ. Appl. **25**, 1–26 (2018)
13. Cardaliaguet, P., Graber, P.J., Porretta, A., Tonon, D.: Second order mean field games with degenerate diffusion and local coupling. Nonlinear Differ. Equ. Appl. **22**, 1287–1317 (2015)
14. Carmona, R., Delarue, F.: Probabilistic Theory of Mean Field Games with Applications. I. Springer, Berlin (2018)
15. Chow, Y.L., Yu, X., Zhou, C.: On dynamic programming principle for stochastic control under expectation constraints. J. Optim. Theory Appl. **185**, 803–818 (2020)
16. Crandall, M.G., Ishii, H., Lions, P.-L.: User’s guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. **27**, 1–67 (1992)
17. Ekeland, I., Témin, R.: Convex Analysis and Variational Problems. SIAM, Philadelphia (1999)
18. El Karoui, N., Nguyen, D., Jeanblanc-Picqué, M.: Compactification methods in the control of degenerate diffusions: existence of an optimal control. Stochastics **20**, 169–219 (1987)
19. Figalli, A.: Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. J. Funct. Anal. **254**, 109–153 (2008)
20. Fleming, W.H., Soner, H.M.: Controlled Markov Processes and Viscosity Solutions. Springer, New York (2006)
21. Fleming, W.H., Vermes, D.: Convex duality approach to the optimal control of diffusions. SIAM J. Control Optim. **27**, 1136–1155 (1989)
22. Föllmer, H., Leukert, P.: Quantile hedging. Finance Stoch. **3**, 251–273 (1999)
23. Guo, I., Langrené, N., Loeper, G., Ning, W.: Portfolio optimization with a prescribed terminal wealth distribution. Quant. Finance **22**, 333–347 (2022)
24. Guo, I., Loeper, G., Wang, S.: Calibration of local-stochastic volatility models by optimal transport. Math Financ. **32**(1), 46–77 (2022)
25. Imbert, C.: Convexity of solutions and C1, 1 estimates for fully nonlinear elliptic equations. J. Math. Pures Appl. **85**, 791–807 (2006)
26. Ishii, H., Lions, P.-L.: Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. J. Differ. Equ. **83**, 26–78 (1990)
27. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. Springer, New York (1991)
28. Krylov, N.: Controlled Diffusion Processes. Springer, Berlin (1980)
29. Lacker, D.: Mean field games via controlled martingale problems: existence of Markovian equilibria. Stoch. Process. Appl. **125**, 2856–2894 (2015)
30. Ladyženskaja, O., Solonnikov, V., Ural’ceva, N.: Linear and quasi-linear equations of parabolic type. In: Translations of Mathematical Monographs, vol. 23. American Mathematical Society, Providence, RI (1998)
31. Lasry, J.-M., Lions, P.-L.: Mean field games. Jpn. J. Math. **2**, 229–260 (2007)
32. Lions, P.-L., Souganidis, P.: Homogenization of degenerate second-order PDE in periodic and almost periodic environments and applications. Annales De L Institut Henri Poincare-analyse Non Lineaire **22**, 667–677 (2005)
33. Mikami, T.: Two end points marginal problem by stochastic optimal transportation. SIAM J. Control Optim. **53**, 2449–2461 (2015)
34. Mikami, T., Thieullen, M.: Duality theorem for the stochastic optimal control problem. Stoch. Process. Appl. **116**, 1815–1835 (2006)
35. Milgrom, B.P., Segal, I.: Envelope theorems for arbitrary choice sets. Econometrica **70**, 583–601 (2002)
36. Oellig, C., Porretta, A., Savaré, G.: A variational approach to the mean field planning problem. J. Funct. Anal. **277**, 1868–1957 (2019)
37. Pfeiffer, L.: Optimality conditions in variational form for non-linear constrained stochastic control problems. Math. Control Rel. Fields **10**, 493–526 (2020)
38. Rachev, S.T., Rüschendorf, L.: Mass Transportation Problems—Volume 1: Theory. Springer, New York (1998)
39. Simons, S.: Minimax and Monotonicity. Springer, Berlin (1998)
40. Soner, H.M., Touzi, N.: Dynamic programming for stochastic target problems and geometric flows. J. Eur. Math. Soc. **4**, 201–236 (2002)
41. Stroock, D.W., Varadhan, S.R.S.: Multidimensional Diffusion Processes. Springer, Berlin (1997)
43. Tan, X., Touzi, N.: Optimal transportation under controlled stochastic dynamics. Ann. Probab. 41, 3201–3240 (2013)
44. Trevisan, D.: Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients. Electron. J. Probab. 21, 1–42 (2016)
45. Villani, C.: Topics in optimal transportation. In: Graduate Studies in Mathematics, vol. 58. American Mathematical Society, Providence, RI (2003)
46. Villani, C.: Optimal Transport Old and New. Springer, Berlin (2009)
47. Wang, L.: On the regularity theory of fully nonlinear parabolic equations: I. Commun. Pure Appl. Math. 45, 27–76 (1992)
48. Wang, L.: On the regularity theory of fully nonlinear parabolic equations: II. Commun. Pure Appl. Math. 45, 141–178 (1992)
49. Yong, J., Zhou, X.Y.: Stochastic Controls—Hamiltonian Systems and HJB Equations. Springer, New York (1999)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.