Modeling the dynamics of tidally-interacting binary neutron stars up to merger

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The data analysis of the gravitational wave signals emitted by coalescing neutron star binaries requires the availability of an accurate analytical representation of the dynamics and waveforms of these systems. We propose an effective-one-body (EOB) model that describes the general relativistic dynamics of neutron star binaries from the early inspiral up to merger. Our EOB model incorporates an enhanced attractive tidal potential motivated by recent analytical advances in the post-Newtonian and gravitational self-force description of relativistic tidal interactions. No fitting parameters are introduced for the description of tidal interaction in the late, strong-field dynamics. We compare the model energetics and the gravitational wave phasing with new high-resolution multi-orbit numerical relativity simulations of equal-mass configurations with different equations of state. We find agreement within the uncertainty of the numerical data for all configurations. Our model is the first semi-analytical model which captures the tidal amplification effects close to merger. It thereby provides the most accurate analytical representation of binary neutron star dynamics and waveforms currently available.

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\textbf{Introduction.} — One of the key aims of the upcoming detections of gravitational wave (GW) signals from coalescing binary neutron stars (BNS) is to inform us on the equation of state (EOS) of matter at supranuclear densities \[5\] via the measurement of the tidal polarizability coefficients (or Love numbers) \[6\] that enter both the interaction potential and the waveform. A necessary requirement for this program is the availability of faithful waveform models that capture the strong-gravity and tidally-dominated regime of the late-inspiral of BNS up to merger. Such models are presently missing; the aim of this work is to close this gap so as to help developing GW astronomy.

The theoretical modeling of BNS waveforms is challenging, and requires synergy between analytical and numerical approaches to the general relativistic two body problem. Traditional post-Newtonian (PN) analytical methods reach their limits during the late BNS inspiral, and are a major limitation for GW data analysis \[5\] \[11\] \[12\]. In recent years numerical relativity (NR) simulations have become fairly robust \[13\] \[18\], though the achievable precision is under debate and exploring the physical parameter space at the necessary accuracy (waveform length and phase errors) is certainly out of reach \[14\] \[16\] \[17\]. The difficulties related to PN and NR modeling carry over in the construction of hybrid PN-NR templates \[19\]. Presently, the effective-one-body (EOB) formalism \[19\] \[22\] offers the most accurate analytical description of the relativistic two body problem. By combining information coming both from analytical results and numerical simulations, the EOB framework succeeds in describing the energetics and the GW signals of coalescing and merging black hole binaries (BBH) \[23\].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The main radial gravitational potential $A(R)$ in various EOB models. Finite-mass ratio effects ($\nu$) make the gravitational interaction less attractive than the Schwarzschild relativistic potential $A_{\text{Schw}} = 1 - 2M/R$, while tides ($\kappa_T^2$) make it more attractive (especially at short separations).}
\end{figure}

The EOB model is a relativistic generalization of the well-known Newtonian property that the relative motion of a two-body system is equivalent to the motion of a particle of mass $\mu = M_A M_B / (M_A + M_B)$ in the two-body potential $V(R)$. The Newtonian radial dynamics is governed by the effective potential $V_{\text{eff}}(R; P_\varphi) = P_\varphi^2 / (2\mu R^2) + V(R)$, where the first term, which contains the angular momentum $P_\varphi$, is a centrifugal potential. In the EOB formalism there is an analogous effective relativistic radial potential (setting $G = c = 1$), $W_{\text{eff}}(R; P_\varphi) = \sqrt{A(R)} (\mu^2 + (P_\varphi/R)^2)$, where $A(R)$ is the main radial potential. In the Newtonian approxima-
tion, $A(R)^{\text{Newton}} = 1 + 2V(R)/\mu$, so that $W_{\text{eff}}(R) \approx \mu + V_{\text{eff}}(R; P_r)$. In the test-mass limit, $A(R)$ is simply equal to the Schwarzschild potential $A_{\text{Schw}} = 1 - 2M/R$ (where $M \equiv M_A + M_B$). Beyond the test-mass limit, $A(R)$ is a deformation of $A_{\text{Schw}}$ by two different physical effects: (i) finite-mass-ratio effects, parametrized by $\nu \equiv \mu/M$; and (ii) tidal effects (in BNS systems only), parametrized by relativistic tidal polarizability parameters $\kappa^{(i)}_A$, most important of which is the quadrupolar combination $\kappa^{(2)}_2 = \kappa^{(2)}_A + \kappa^{(2)}_B$. Following [2] [30], tidal interactions are incorporated in the EO formalism by a radial potential of the form $A(R; \nu; \kappa^{(i)}_A) = A^0(R; \nu) + A^T(R; \kappa^{(i)}_A)$ where $A^0(R)$ is the EOBB BH radial potential, and $A^T(R)$ is an additional tidal interaction piece whose structure is discussed below. Figure 4 contrasts (for the SLy EOS, see Table 1) the deformations of $A(R; \nu; \kappa^{(i)}_A)$ away from $A_{\text{Schw}} = A(R; 0; 0) = 1 - 2M/R$ induced either by (i) finite-mass-ratio effects, which make $A^0(R)$ less attractive, or by (ii) tidal effects, which make $A^{\text{BNS}}(R)$ more attractive in the strong-field regime where they dominate over the repulsive finite-mass-ratio effects. Figure 1 also compares a resummed tidal EOB model (incorporating recent advances in the relativistic theory of tidal interactions [31] [33]) with another tidal EOB model that incorporates a tidal potential treating tidal interactions in a nonresummed way, up to the next-to-next leading order (NNLO, fractional 2PN, see below) [34] [35]. The resummed tidal EOB model is significantly more attractive than the NNLO one at small separations. We will consider the evolution of the EO dynamics at separations of the order of the contact between the two NSs, i.e., at the point hereafter called “merger”. The marker in the figure indicates the radial location corresponding to that merger for the resummed EOB model ($R_{\text{merg}} = 6.093M$).

The main result of this paper is to show that the resummed EO model is significantly closer (especially at small separations). We will consider equal-mass binaries in which the tidal part of radiation reaction, $A^T(\nu; \kappa^{(i)}_A)$, is the EOB BBH radial potential, and $A^T(\nu; \kappa^{(i)}_A)$ is analytically known [31] [38]. We do not use here the analytical knowledge of $a^{(i)}_0(\nu)$. We used instead the “effective” value $a^{(i)}_0(\nu) = 3097.3u^2 - 1330.6u + 81.38$ deduced from a recent comparison between the EOB model and a sample of NR data [39] [40]. The tidal contribution to $A(\nu; \nu)$ (omitting the negligible gravitomagnetic part [3]) is

$$A^{(\nu)}_T(u; \nu) = -\frac{4}{\ell+2} \left[ \frac{\kappa^{(i)}_A(\nu)}{u^2} \right]^{\ell+2} A^{(\ell+2)}_A + (A \leftrightarrow B),$$

where $\kappa^{(i)}_A = \frac{2a^{(i)}_A(X_A/C_A)2^{(i)+1}M_B/M_A, X_{A,B} \equiv M_{A,B}/M, \kappa^{(i)}_A$ are the dimensionless Love numbers [7–10] and $C_{A,B} \equiv (M/R_A) \Lambda_{AB}$ the compactness of the stars. In the equal-mass case, the EOS information is essentially encoded in the total dimensionless quadrupolar tidal coupling constant $\kappa^{(2)}_2 = \kappa^{(2)}_A + \kappa^{(2)}_B$. The relativistic correction factors $\tilde{A}^{(\nu)}_A$ formally include all the high PN corrections to the leading-order. The choice of $\tilde{A}^{(\nu)}_A$ defines the two tidal EOB models of this paper. The NNLO tidal EOB model, TEOB_{NNLO}, is defined by using the PN-expanded, fractionally 2PN accurate, expression $\tilde{A}^{(\nu)}_A = 1 + \alpha_1 u + \alpha_2 u^2 + \alpha_3 u^3$ known analytically [35]. The resummed tidal EOB model, TEOB_{resum}, is defined by using for the $\ell = 2$ term in Eq. (1) the expression

$$\tilde{A}^{(2)}_A(u) = 1 + \frac{3u^2}{1 - r_{\text{LR}} u} + X_A \tilde{A}^{(2)+1}\text{SF}_A \left(1 - r_{\text{LR}} u \right)\left(1 - r_{\text{LR}} u \right)^{\gamma/2} + X_B \tilde{A}^{(2)+2}\text{SF} _B \left(1 - r_{\text{LR}} u \right)^{\gamma/2},$$

where the functions $\tilde{A}^{(2)+1}\text{SF}_A(\nu)$ and $\tilde{A}^{(2)+2}\text{SF}_B(\nu)$ are defined as in [33], and where we choose $p = 4$ for the exponent. The $\ell = 3, 4$ contributions of the resummed model are taken as in the NNLO model. A key prescription here is to use as pole location in Eq. (4) the light ring $r_{\text{LR}}(\nu; \kappa^{(i)}_A)$ (i.e., the location of the maximum of $A^{\text{NNLO}}(\nu; \nu; \kappa^{(i)}_A)$), a tidal contribution $\tilde{A}^{(\nu)}_A = 0$, is always set to zero [29] [39]; the tidal part of radiation reaction is completed with the next-to-leading-order tidal contribution [2] [30] [34].

**NR simulations.**—Simulations are performed with the BAM code [41] [42], which solves the Z4c formulation of Einstein’s equations [13] [44] and more general relativistic hydrodynamics. The setup used here is similar to that of [35] [45], numerical details will be discussed elsewhere. We consider equal-mass binaries in which the fluid is described either by a $\Gamma = 2$ polytropic EOS enforcing isentropic evolutions [44] [15], or by a piecewise polytropic representation of cold EOS [40] adding a $\Gamma_{\text{th}} = 1.75$ thermal pressure component [16]. All configurations (Table 1) are simulated at multiple resolutions.
TABLE I. BNS configurations and phasing results. From left to right: name, EOS, $\kappa_T^j$, TEOB$_{\mathrm{NNLO}}$ light-ring location, star compactnesses $C_{A,B}$ and gravitational masses in isolation, initial Arnowitt-Deser-Misner (ADM) mass and angular momentum, $(M^0_{\mathrm{ADM}}, J^0_{\mathrm{ADM}})$. The phase differences $\Delta \phi^X \equiv \phi^X - \phi^\text{NR}$, where $X \equiv (\text{TT4}, \text{TEOB$_{\mathrm{NNLO}}$}, \text{TEOB$_{\text{Resum}}$})$ labels various analytical models, are reported at the moment of NR merger $(0.11 \lesssim M^\text{mer} \lesssim 0.19)$. The phase differences, in radians, are obtained by aligning all waveforms on the frequency interval $I_\omega = (0.04, 0.06)$ (except $\Gamma_{164}$ for which $I_\omega = (0.0428, 0.06)$). The NR uncertainty $\delta \phi^\text{NR}_{\text{Resum}}$ at NR merger is estimated as the difference between the highest and second highest resolution data. The resummed TEOB$_{\text{Resum}}$ model displays the best agreement with NR data.

| Name   | EOS   | $\kappa_T^j$ | $\tau_\text{LR}$ | $C_{A,B}$ | $M_{A,B}[M_\odot]$ | $M^0_{\text{ADM}}[M_\odot]$ | $J^0_{\text{ADM}}[M_\odot^2]$ | $\Delta \phi^{\text{TT4}}_{\text{PN},\text{Resum}}$ | $\Delta \phi^{\text{TEOB$_{\text{NNLO}}$}}_{\text{NR},\text{Resum}}$ | $\Delta \phi^{\text{TEOB$_{\text{Resum}}$}}_{\text{NR},\text{Resum}}$ | $\delta \phi^\text{NR}_{\text{Resum}}$ |
|--------|-------|--------------|-----------------|----------|------------------|-------------------------------|-------------------------------|---------------------------------|-------------------------------|---------------------------------|---------------------|
| 2B135  | 2B    | 23.9121      | 3.253           | 0.2049   | 1.34997          | 2.67762                      | 7.66256                      | $-1.25$                         | $-0.19$                        | $+0.57$                         | $\pm 4.20$          |
| SLy135 | SLy   | 73.5450      | 3.701           | 0.17381  | 1.35000          | 2.67760                      | 7.65780                      | $-2.75$                         | $-1.79$                        | $-0.75$                         | $\pm 0.40$          |
| $\Gamma_{164}$ | $\Gamma = 2$ | 75.0671      | 3.728           | 0.15999  | 1.64388          | 3.25902                      | 11.13113                     | $-2.29$                         | $-1.36$                        | $-0.31$                         | $\pm 0.90$          |
| $\Gamma_{151}$ | $\Gamma = 2$ | 183.3911     | 4.160           | 0.13999  | 1.51484          | 3.00497                      | 9.71561                      | $-2.60$                         | $-1.92$                        | $-1.27$                         | $\pm 1.20$          |
| H4135  | H4    | 210.5866     | 4.211           | 0.14710  | 1.35003          | 2.67768                      | 7.66315                      | $-3.02$                         | $-2.43$                        | $-1.88$                         | $\pm 1.04$          |
| MS1b135 | MS1b  | 289.8034     | 4.381           | 0.14218  | 1.35001          | 2.67769                      | 7.66517                      | $-3.25$                         | $-2.84$                        | $-2.45$                         | $\pm 3.01$          |

* This value is the dephasing at the moment of TEOB$_{\text{Resum}}$ merger, which occurs $\approx 30M$ before NR merger.

FIG. 2. Energetics: comparison between NR data, TEOB$_{\text{Resum}}$, TEOB$_{\text{NNLO}}$ and TPN. Each bottom panel shows the two EOB-NR differences. The filled circles locate the merger points (top) and the corresponding differences (bottom). The shaded area indicates the NR uncertainty. The TEOB$_{\text{Resum}}$ model displays, globally, the smallest discrepancy with NR data (notably for merger quantities), supporting the theoretical, light-ring driven, amplification of the relativistic tidal factor.

The simulations of (SLy, $\Gamma_{151}$, H4) use three resolutions with $(L, M, H) = (64^4, 96^4, 128^3)$ grid points resolving the star diameter, while for $(2B, \Gamma_{164}, \text{MS1b})$ only the $(L, M)$ resolutions are available. Numerical uncertainties are conservatively estimated as the difference between the highest and the second highest available resolutions, in an attempt at including possible systematic errors [15]. Overall, these BNS data are among the longest and most accurate available to date.

EOB-NR comparison: energetics.— First, we compare EOB to NR regarding the energetics of the binary system. This is done through the gauge-invariant re-
lation between the binding energy and the orbital angular momentum \( I^2 \). We work with corresponding dimensionless quantities defined respectively as \( E_b \equiv (|M_{\text{ADM}} - \Delta \mathcal{E}_r)|/M - 1/\nu \) and \( j \equiv (J_0^R - \Delta \mathcal{J}_r)/(M^2 \nu) \), where \( \Delta \mathcal{E}_r (\Delta \mathcal{J}_r) \) is the radiated GW energy (angular momentum). Since the relation \( E_b(j) \) essentially captures the conservative dynamics \( I^2 \), this analysis directly probes the performance of the EOB Hamiltonian, and notably the definition of \( A^2(u; \nu) \).

The top panels of Fig. 2 compare for all EOS four energetics \( E_b(j) \): NR, TEOB\(_{\text{resum}}\), TEOB\(_{\text{NNLO}}\), and the PN-expanded tidal energetics TPN, i.e. the (2PN accurate) expansion of the function \( E_b(j) \) in powers of \( 1/c^2 \). The markers on the first three curves identify the corresponding merger points. Following \( \text{[45]} \), we define the moments of merger, intrinsically for each model, as the peak of the modulus of the corresponding \( t \sim m = 2 \) waveform. The two differences \( \Delta E_b(\text{EOBNR}) = E_b(\text{EOB}) - E_b(\text{NR}) \) for TEOB\(_{\text{resum}}\) and TEOB\(_{\text{NNLO}}\) are shown in the bottom panels. The shaded area indicates the NR uncertainty. The main findings of this EOB/NR energetics comparisons are: (i) like in the BBH energetics comparison \( \text{[17]} \), TPN is always above the NR curve with a difference which becomes unacceptably large towards merger; (ii) the location of the TEOB\(_{\text{resum}}\) merger point in the \((E_b,j)\) plane is, in all cases, very significantly away from the corresponding NR merger point; (iii) by contrast, the TEOB\(_{\text{resum}}\) merger point is, in all but one case (2B), rather close to NR (especially when \( \kappa_2^T \) is large); (iv) in all cases, the TEOB\(_{\text{resum}}\) NR differences (bottom panels) closely oscillate around zero during most of the simulated \( \sim \) ten orbits; (v) moreover, such differences keep staying within the NR uncertainty essentially up to (slightly before for H4 and MS1b) the TEOB\(_{\text{resum}}\) merger.

**EOB-NR comparison: phasing.**— The TEOB\(_{\text{resum}}\) resummed tidal waveform is obtained following \( \text{[42, 45]} \). We compare the EOB and NR quadrupole waveforms \( \mathcal{R}h_{\text{22}} \), with \( \mathcal{R}(h_{+} - ih_{\times}) = \sum_{\ell,m} \mathcal{R}_{\ell m}e^{-2\ell Y_{\ell m}} \), by using a standard (time and phase) alignment procedure in the time domain. Relative time and phase shifts are determined by minimizing the \( L^2 \) distance between the EOB and NR phases integrated on a time interval corresponding to the dimensionless frequency interval \( \omega_{L} = M(\omega_{L}, \omega_{R}) = (0.04, 0.06) \) for all EOS, except \( \Gamma_2164 \), for which \( \omega_{L} = (0.0428, 0.06) \) as the simulation starts at higher GW frequency. Such choice for \( \omega_{L} \) allows one to average out the phase oscillations linked to the residual eccentricity \( \sim 0.01 \) of the NR simulations.

A sample of time-domain comparisons for three representative \( \kappa_2^T \)'s is shown in Fig. 3. Top panels: (i) phase and relative amplitude differences between TEOB\(_{\text{resum}}\) and NR; (ii) phase difference between the tidal Taylor T4 with NLO tides and 3PN waveform (TT4) and NR; and (iii) NR phase uncertainty (shaded region). The two vertical lines indicate the alignment interval; as in Fig. 2, the markers indicate the EOB (red) and NR (blue) mergers. The crossing of the radius of the TEOB\(_{\text{resum}}\) last stable orbit (LSO) is also indicated by a green marker. Bottom panels compare the TEOB\(_{\text{resum}}\) and NR waveforms real part and modulus. From this time-domain comparison one sees that for all \( \kappa_2^T \) the TEOB\(_{\text{resum}}\) model is compatible with NR data up to merger within NR uncertainties (at the 2\( \sigma \) level or better, both in phase and amplitude). Note that the TT4 phasing performs systematically worse than TEOB\(_{\text{resum}}\).

Figure 3 is quantitatively completed by Table 1, which compares the phase differences \( \Delta \phi^X \equiv \phi^X - \phi^\text{NR} \) evaluated (after time-alignment) at the moment of NR merger for: the tidal T4 \((X = \text{TT4})\), the NNLO EOB model \((X = \text{TEOB}_{\text{NNLO}})\), the resummed EOB model \((X = \text{TEOB}_{\text{resum}})\). The NR uncertainty at merger \( \delta_{\phi^N} \) is also listed in the table. These numbers indicate how the disagreement with NR systematically decreases when successively considering the analytical models TT4, TEOB\(_{\text{NNLO}}\) and TEOB\(_{\text{resum}}\). Such hierarchy of qualities among analytical models is confirmed by the gauge-invariant phasing diagnostic \( Q_c(\omega) \equiv \omega^2/\dot{\omega} \) \( \text{[15, 16]} \). To clean up the eccentricity-driven oscillations in the NR phase, we based our computation of \( Q_c^\text{NR} \) by starting from a simple, PN-inspired, six-parameter fit of the NR frequency as a rational function of \( z = (\nu(\nu - 1)/5 + d^2)^{-1/8} \) (similarly to \( \text{[49]} \)). For each \( \kappa_2^T \) we find: \( Q_c^\text{NR} \approx Q_c^\text{TEOB}_{\text{resum}} < Q_c^\text{TEOB}_{\text{NNLO}} < Q_c^\text{TT4} < Q_c^\text{BBH} \) (see Fig. 4 for SLy EOS).

**Merger characteristics.**— The TEOB\(_{\text{resum}}\) model, in addition to giving good energetics, \( E_b(j) \), and phasing \( \phi(t) \) up to NR merger, has the remarkable feature of intrinsically predicting the frequency location and physical characteristics of merger in good quantitative agreement with NR results. This can have important consequences for building predictive analytical GW templates. More precisely, the two quasianuniversal functional relations \( \text{[45, 49]} E_{b}^\text{min}((\kappa_2^T)^2) \) and \( M\omega_{\text{res}}((\kappa_2^T)^2) \) (as well as \( j_{\text{res}}((\kappa_2^T)^2) \) and the waveform amplitude at merger \( A_{\text{res}}((\kappa_2^T)^2) \)) evaluated (after time-alignment) at the moment of NR merger points. Following \( \text{[49]} \), we define the moments of merger for the six models. Such agreements are remarkable as no NR-accurate model (X = TEOB\(_{\text{resum}}\)). The NR uncertainty at merger \( \delta_{\phi^N} \) is also listed in the table. These numbers indicate how the disagreement with NR systematically decreases when successively considering the analytical models TT4, TEOB\(_{\text{NNLO}}\) and TEOB\(_{\text{resum}}\). Such hierarchy of qualities among analytical models is confirmed by the gauge-invariant phasing diagnostic \( Q_c(\omega) \equiv \omega^2/\dot{\omega} \) \( \text{[15, 16]} \). To clean up the eccentricity-driven oscillations in the NR phase, we based our computation of \( Q_c^\text{NR} \) by starting from a simple, PN-inspired, six-parameter fit of the NR frequency as a rational function of \( z = (\nu(\nu - 1)/5 + d^2)^{-1/8} \) (similarly to \( \text{[49]} \)). For each \( \kappa_2^T \) we find: \( Q_c^\text{NR} \approx Q_c^\text{TEOB}_{\text{resum}} < Q_c^\text{TEOB}_{\text{NNLO}} < Q_c^\text{TT4} < Q_c^\text{BBH} \) (see Fig. 4 for SLy EOS).

Conclusions. — We introduced the first tidal EOB model able to describe the energetics and waveforms of coalescing BNS from the early inspiral up to the moment of merger. The EOB prediction for the binary dynamics as measured by the \( E_b(j) \) curve agrees with NR data.
within their uncertainties for a sample of EOS spanning a significant range of tidal parameters, Fig. 2. The EOB and NR waveform phasing essentially agree within the NR uncertainties up to the moment of merger. This result is a significant improvement with respect to previous work [15, 16], notably because no parameters were tuned. Given the NR intrinsic uncertainties, and the possible residual eccentricity influence, we refrain from further calibrating the model at this stage. Once improved NR data will be available, we expect to be able to NR-inform the model, e.g., by including next-to-quasi-circular corrections to the waveform.

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FIG. 3. Phasing and amplitude comparison (versus NR retarded time $\nu$) between $\text{TEOB}_\text{Resum}$, NR and the phasing of TT4 for three representative models. Waves are aligned on a time window (vertical dot-dashed lines) corresponding to $I_\nu = (0.04, 0.06)$ for SLy and $\Gamma_2 = 151$ and $I_\nu = (0.0428, 0.06)$ for $\Gamma_2 = 164$. The markers in the top panels indicate: the crossing of the $\text{TEOB}_\text{Resum}$ LSO radius; NR and EOB merger moments. The NR merger is also indicated with a dashed vertical line.

FIG. 4. Hierarchy of qualities among analytical models against NR data using the $Q_\omega \equiv \omega^2/\omega$ phasing diagnostics.

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