Retarded integral inequalities of Gronwall-Bihari type

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ABSTRACT. We establish two nonlinear retarded integral inequalities. Bounds on the solution of some retarded equations are then obtained.

1. Introduction and Preliminaries

In the recent paper \cite{2} M. Denche and H. Khellaf study, under some conditions on the involved functions, the following two inequalities:

\begin{align*}
(1) \quad u(t) \leq a(t) + \int_a^t f(s)u(s)ds + \int_a^t f(s)W \left( \int_a^s k(s, \tau)\Phi(u(\tau))d\tau \right)ds,
\end{align*}

and

\begin{align*}
(2) \quad u(t) \leq a(t) + \int_a^t f(s)g(u(s))ds + \int_a^t f(s)W \left( \int_a^s k(s, \tau)\Phi(u(\tau))d\tau \right)ds.
\end{align*}

Such inequalities have been then used on general time scales, including discrete-time versions of (1) and (2) (see \cite{3}). In the present note we generalize both inequalities (1) and (2) in a different direction, by considering more general retarded inequalities, i.e., by letting the upper limit of the integrals to be $C^1$ non-decreasing functions less than or equal to $t$ (cf. (4) and (10) below). Moreover, our generalized inequalities (4) and (10) are considered under less restrictive assumptions on the involved functions, e.g., in (2) the function $\Phi(\cdot)$ is assumed to be subadditive and submultiplicative, while here we only assume submultiplicativity.

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We invite the reader to compare Theorems 2.1 and 2.3 of [2] with Theorems 2.2 and 2.6 of this paper, respectively.

2. Main Results

We start by proving a useful lemma. A similar result to Lemma 2.1 was proved in [4, Theorem 1.1] with differentiability assumptions on the function $f(\cdot, \cdot)$.

**Lemma 2.1.** Suppose that $\alpha(\cdot) \in C^1([a, b], \mathbb{R})$ is a nondecreasing function with $a \leq \alpha(t) \leq t$, for all $t \in [a, b]$. Assume that $u(\cdot), a(\cdot), b(\cdot) \in C([a, b], \mathbb{R}_0^+)$ and let $(t, s) \to f(t, s) \in C([a, b] \times [a, \alpha(b)], \mathbb{R}_0^+)$ be nondecreasing in $t$ for every $s$ fixed. If

$$u(t) \leq a(t) + b(t) \int_a^{\alpha(t)} f(t, s)u(s)ds,$$

then

$$u(t) \leq a(t) + b(t) \int_a^{\alpha(t)} \exp \left( \int_s^{\alpha(t)} b(\tau)f(t, \tau)d\tau \right) f(t, s)a(s)ds.$$

**Proof.** The result is obvious for $t = a$. Let $t_0$ be an arbitrary number in $(a, b]$ and define the function $z(\cdot)$ as

$$z(t) = \int_a^{\alpha(t)} f(t, s)u(s)ds, \quad t \in [a, t_0].$$

Then, $u(t) \leq a(t) + b(t)z(t)$ for all $t \in [a, t_0]$, and $z(\cdot)$ is nondecreasing. Hence,

$$z'(t) = f(t_0, \alpha(t))u(\alpha(t))\alpha'(t) \leq f(t_0, \alpha(t))[a(\alpha(t)) + b(\alpha(t))z(\alpha(t))]\alpha'(t) \leq f(t_0, \alpha(t))[a(\alpha(t)) + b(\alpha(t))z(t)]\alpha'(t).$$

The last inequality can be rearranged as

$$z'(t) - f(t_0, \alpha(t))b(\alpha(t))z(t)\alpha'(t) \leq f(t_0, \alpha(t))a(\alpha(t))\alpha'(t).$$

Multiplying both sides of inequality (3) by $\exp \left( -\int_a^{\alpha(t)} b(s)f(t_0, s)ds \right)$, we get

$$\left[ z(t) \exp \left( -\int_a^{\alpha(t)} b(s)f(t_0, s)ds \right) \right]' \leq \exp \left( -\int_a^{\alpha(t)} b(s)f(t_0, s)ds \right) f(t_0, \alpha(t))a(\alpha(t))\alpha'(t).$$
Integrating from $a$ to $t$ and noting that $z(a) = 0$, we obtain successively that

$$z(t) \leq \exp\left(\int_a^t b(s)f(t_0, s)ds\right) \times \int_a^t \exp\left(-\int_a^s b(\tau)f(t_0, \tau)d\tau\right)\times f(t_0, \alpha(s))a(\alpha(s))\alpha'(s)ds \times f(t_0, \alpha(s))a(\alpha(s))\alpha'(s)ds$$

$$= \int_a^t \exp\left(\int_a^s b(\tau)f(t_0, \tau)d\tau\right) f(t_0, \alpha(s))a(\alpha(s))\alpha'(s)ds$$

$$= \int_a^t \exp\left(\int_a^s b(\tau)f(t_0, \tau)d\tau\right) f(t_0, s)a(s)ds.$$

Since $u(t) \leq a(t) + b(t)z(t)$, we have for $t = t_0$ that

$$u(t_0) \leq a(t_0) + b(t_0) \int_a^{\alpha(t_0)} \exp\left(\int_a^{\alpha(t_0)} b(\tau)f(t_0, \tau)d\tau\right) f(t_0, s)a(s)ds.$$

The intended conclusion follows from the arbitrariness of $t_0$. \qed

We are now in conditions to prove the following result:

**Theorem 2.2.** Suppose that $\alpha(\cdot), \beta(\cdot) \in C^1([a, b], \mathbb{R})$ are nondecreasing functions with $\alpha(t), \beta(t) \in [a, t]$ for all $t \in [a, b]$. Assume that $u(\cdot), a(\cdot), b(\cdot) \in C([a, b], \mathbb{R}^+_0)$, $(t, s) \rightarrow f(t, s) \in C([a, b] \times [a, \alpha(b)], \mathbb{R}^+_0)$ is nondecreasing in $t$ for every $s$ fixed, $g(\cdot, \cdot) \in C([a, b] \times [a, \alpha(b)], \mathbb{R}^+_0)$, and $(s, \tau) \rightarrow k(s, \tau) \in C([a, \beta(b)] \times [a, \beta(b)], \mathbb{R}^+_0)$ is nondecreasing in $s$ for every $\tau$ fixed. Let $W(\cdot), \Phi(\cdot) \in C(\mathbb{R}^+_0, \mathbb{R}^+_0)$ be nondecreasing functions, $\Phi(\cdot)$ submultiplicative with $\Phi(x) > 0$ for $x \geq 1$. Define

$$G(x) = \int_0^x ds \frac{1}{\Phi(1 + W(s))}, \quad x \geq 0,$$

$$\eta(\tau) = \max \left\{a(\tau), \int_a^{\beta(\tau)} g(\tau, \theta)d\theta\right\}, \quad \tau \in [a, \max\{\alpha(b), \beta(b)\}],$$

and

$$p(s) = \int_a^s k(s, \tau)\Phi\left(\eta(\tau) + b(\tau)\int_a^{\alpha(\tau)} \exp\left(\int_a^{\alpha(\tau)} b(\theta)f(\tau, \theta)d\theta\right) f(\tau, \xi)\eta(\xi)d\xi\right) d\tau.$$

If for $t \in [a, b]$

$$u(t) \leq a(t) + b(t) \int_a^{\alpha(t)} f(t, s)u(s)ds + \int_a^{\beta(t)} g(t, s)W\left(\int_a^s k(s, \tau)\Phi(u(\tau))d\tau\right) ds,$$
then there exists \( t_* \in (a, \beta(b)) \) such that \( p(t) \in \text{Dom}(G^{-1}) \) for all \( t \in [a, t_*] \), 
\[ G^{-1}(\cdot) \]
the inverse function of \( G(\cdot) \), and

\[
\begin{align*}
 u(t) &\leq q(t) + b(t) \int_a^t \exp \left( \int_s^t b(\tau) f(t, \tau) d\tau \right) f(t, s) q(s) ds, \\
\text{where} \\
q(t) &= a(t) + \int_a^{\beta(t)} g(t, s) W(G^{-1}(p(s))) ds.
\end{align*}
\]

Proof. Let

\[
z(t) = a(t) + \int_a^{\beta(t)} g(t, s) W \left( \int_a^s k(s, \tau) \Phi(u(\tau)) d\tau \right) ds, \quad t \in [a, b].
\]

Then, (4) can be restated as

\[
(5) \quad u(t) \leq z(t) + b(t) \int_a^t f(t, s) u(s) ds.
\]

Applying Lemma 2.1 to (5), we obtain

\[
(6) \quad u(t) \leq z(t) + b(t) \int_a^t \exp \left( \int_s^t b(\tau) f(t, \tau) d\tau \right) f(t, s) z(s) ds.
\]

In order to estimate \( z(t) \), we define the function \( v(\cdot) \) by

\[
v(s) = \int_a^s k(s, \tau) \Phi(u(\tau)) d\tau.
\]

We have that \( z(x) = a(x) + \int_a^{\beta(x)} g(x, \theta) W(v(\theta)) d\theta \) and

\[
v(s) \leq \int_a^s k(s, \tau) \Phi \left[ z(\tau) + b(\tau) \int_a^{\alpha(\tau)} \exp \left( \int_\xi^{\alpha(\tau)} b(\theta) f(\tau, \theta) d\theta \right) f(\tau, \xi) z(\xi) d\xi \right] d\tau
\]

\[
\leq \int_a^s k(s, \tau) \Phi \left[ \eta(\tau)(1 + W(v(\tau))) \\
+ b(\tau) \int_a^{\alpha(\tau)} \exp \left( \int_\xi^{\alpha(\tau)} b(\theta) f(\tau, \theta) d\theta \right) f(\tau, \xi) \eta(\xi) d\xi (1 + W(v(\tau))) \right] d\tau
\]

\[
\leq \int_a^s k(s, \tau) \Phi \left[ \eta(\tau) \\
+ b(\tau) \int_a^{\alpha(\tau)} \exp \left( \int_\xi^{\alpha(\tau)} b(\theta) f(\tau, \theta) d\theta \right) f(\tau, \xi) \eta(\xi) d\xi \right] \Phi(1 + W(v(\tau))) d\tau.
\]
Let $a < t_s \leq \beta(b)$ be a number such that $p(t) \in \text{Dom}(G^{-1})$ for all $t \in [a, t_s]$. Define $r(\cdot)$ on $[a, s_0]$, where $a < s_0 \leq t_s$ is an arbitrary fixed number, by

$$r(s) = \int_a^s k(s_0, \tau) \Phi \left[ \eta(\tau) \right. \left. + b(\tau) \int_\alpha^\alpha(\tau) \exp \left( \int_\xi^\alpha(\tau) b(\theta) f(\tau, \theta) d\theta \right) f(\tau, \xi) \eta(\xi) d\xi \right] \Phi(1 + W(v(\tau))) d\tau.$$ 

Then,

$$r'(s) = k(s_0, s) \Phi \left[ \eta(s) \right. \left. + b(s) \int_a^\alpha(s) \exp \left( \int_\xi^\alpha(s) b(\theta) f(s, \theta) d\theta \right) f(s, \xi) \eta(\xi) d\xi \right] \Phi(1 + W(v(s))) \leq k(s_0, s) \Phi \left[ \eta(s) \right. \left. + b(s) \int_a^\alpha(s) \exp \left( \int_\xi^\alpha(s) b(\theta) f(s, \theta) d\theta \right) f(s, \xi) \eta(\xi) d\xi \right] \Phi(1 + W(r(s))),$$

that is,

$$\frac{r'(s)}{\Phi(1 + W(r(s)))} \leq k(s_0, s) \Phi \left[ \eta(s) \right. \left. + b(s) \int_a^\alpha(s) \exp \left( \int_\xi^\alpha(s) b(\theta) f(s, \theta) d\theta \right) f(s, \xi) \eta(\xi) d\xi \right].$$

Integrating both members of the last inequality from $a$ to $s$, and having in mind that $G(r(a)) = 0$, we get

$$G(r(s)) \leq \int_a^s k(s_0, \tau) \Phi \left[ \eta(\tau) \right. \left. + b(\tau) \int_\alpha^\alpha(\tau) \exp \left( \int_\xi^\alpha(\tau) b(\theta) f(\tau, \theta) d\theta \right) f(\tau, \xi) \eta(\xi) d\xi \right] d\tau.$$

The choice of $t_s$ permits us to write $r(s_0) \leq G^{-1}(p(s_0))$. Since $s_0$ is arbitrary, we conclude that (the case $s = a$ is trivial)

$$(7) \quad r(s) \leq G^{-1}(p(s)), \quad s \in [a, t_s].$$
To complete the proof, we observe that for \( a \leq s \leq t_* \) the inequality \( \beta(\alpha(s)) \leq t_* \) holds. Hence, we can insert inequality (7) into inequality (6).

\[
\square
\]

**Remark 1.** Theorem 2.2 is new even in the particular setting studied in [2] with \( \alpha(t) = \beta(t) = t, b(t) = 1, \) and \( f(t, s) = g(t, s) = f(s) \). Indeed, one may choose in Theorem 2.2 a submultiplicative function \( \Phi(\cdot) \) that is not subadditive, e.g., \( \Phi(x) = x^2 \) for \( x \geq 0 \). This choice of \( \Phi(\cdot) \) is not a possibility in [2, Theorem 2.1].

To prove the forthcoming results we follow F. M. Dannan [1], introducing the following class of functions:

**Definition 2.3.** A function \( g(\cdot) \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) \) is said to belong to the class \( H \) if

1. \( x \rightarrow g(x) \) is nondecreasing for \( x \geq 0 \) and positive for \( x > 0 \);
2. there exists a continuous function \( \Psi(\cdot) \) on \( \mathbb{R}_0^+ \) with \( g(\alpha x) \leq \Psi(\alpha) g(x) \) for \( \alpha > 0, x \geq 0 \).

**Example 2.4.** Every continuous and nondecreasing function \( g(\cdot) \) on \( \mathbb{R}_0^+ \) with \( g(x) > 0 \) for \( x > 0 \) that is submultiplicative, is of class \( H \) with \( \Psi = g \).

To the best of our knowledge, the following lemma is not found in the literature. Therefore, we give a proof here.

**Lemma 2.5.** Suppose that \( \alpha(\cdot) \in C^1([a, b], \mathbb{R}) \) is a nondecreasing function with \( a \leq \alpha(t) \leq t \) for all \( t \in [a, b] \). Assume that \( u(\cdot), a(\cdot) \in C([a, b], \mathbb{R}_0^+) \) with \( a(\cdot) \) a positive and nondecreasing function, and \((t, s) \rightarrow f(t, s) \in C([a, b] \times [a, \alpha(b)], \mathbb{R}_0^+) \) nondecreasing in \( t \) for every \( s \) fixed. If \( g(\cdot) \in H \) and

\[
u(t) \leq a(t) + \int_a^{\alpha(t)} f(t, s) g(u(s)) ds,
\]

then there exists a function \( \Psi(\cdot) \) and a number \( t_* \in (a, b] \) that depends on \( \Psi(\cdot) \) such that

\[
G(1) + \int_a^{\alpha(t)} f(t, s) \frac{\Psi(a(s))}{a(s)} ds \in \text{Dom}(G^{-1}), \quad t \in [a, t_*],
\]

and

\[
u(t) \leq a(t) G^{-1} \left( G(1) + \int_a^{\alpha(t)} f(t, s) \frac{\Psi(a(s))}{a(s)} ds \right), \quad t \in [a, t_*],
\]

where

\[
G(x) = \int_{x_0}^x \frac{ds}{g(s)}, \quad x > 0, \ x_0 > 0,
\]

and, as usual, \( G^{-1}(\cdot) \) represents the inverse function of \( G(\cdot) \).
Proof. Since \( a(\cdot) \) is positive and nondecreasing and \( g(\cdot) \in H \), we obtain from (8) that
\[
\frac{u(t)}{a(t)} \leq 1 + \int_a^{\alpha(t)} f(t,s)g(u(s))ds \leq 1 + \int_a^{\alpha(t)} f(t,s)\frac{\Psi(a(s))}{a(s)}g\left(\frac{u(s)}{a(s)}\right)ds
\]
for some function \( \Psi(\cdot) \) as in the Definition 2.3. Let us now choose a number \( a < t_* \leq b \) such that (9) holds, and define function \( z(\cdot) \) by
\[
z(t) = 1 + \int_a^{\alpha(t)} f(t_0, s)\frac{\Psi(a(s))}{a(s)}g\left(\frac{u(s)}{a(s)}\right)ds, \quad t \in [a, t_0],
\]
where \( t_0 \in (a, t_*] \) is an arbitrary fixed number. Then, with \( x(t) = u(t)/a(t) \), we have
\[
z'(t) = f(t_0, \alpha(t))\frac{\Psi(a(\alpha(t)))}{a(\alpha(t))}g(x(\alpha(t)))\alpha'(t)
\]
\[
\leq f(t_0, \alpha(t))\frac{\Psi(a(\alpha(t)))}{a(\alpha(t))}\alpha'(t)g(z(t)),
\]
because \( x(t) \leq z(t) \) and \( z(t) \) is nondecreasing. Since \( z(t) \) is positive, we can divide both sides of the last inequality by \( g(z(t)) \) and, after integrating both sides on \( [a, t] \), we get
\[
G(z(t)) \leq G(1) + \int_a^{\alpha(t)} f(t_0, s)\frac{\Psi(a(s))}{a(s)}ds.
\]
Hence,
\[
z(t_0) \leq G^{-1}\left(G(1) + \int_a^{\alpha(t_0)} f(t_0, s)\frac{\Psi(a(s))}{a(s)}ds\right).
\]
Since \( x(t_0) = u(t_0)/a(t_0) \leq z(t_0) \) and \( t_0 \) is arbitrary, the result follows for all \( t \in [a, t_*] \). The case when \( t = a \) is obvious. \( \square \)

Theorem 2.6. Let functions \( u(\cdot) \), \( f(\cdot) \), \( g(\cdot) \), \( W(\cdot) \), \( \Phi(\cdot) \), \( \alpha(\cdot) \), \( \beta(\cdot) \), \( p(\cdot) \), and \( G(\cdot) \) be as in Theorem 2.2, and \( a(\cdot) \) be as in Lemma 2.5. If \( h(\cdot) \in H \),
\[
\mathcal{H}(x) = \int_{x_0}^x \frac{ds}{h(s)}, \quad x > 0, \ x_0 > 0,
\]
and
\[
u(t) \leq a(t) + \int_a^{\alpha(t)} f(t, s)h(u(s))ds + \int_a^{\beta(t)} g(t, s)W\left(\int_a^s k(s, \tau)\Phi(u(\tau))d\tau\right)ds,
\]
then there exists a function \( \Psi(\cdot) \) and a number \( t'_* \in (a, \beta(b)] \) depending on \( \Psi(\cdot) \) such that, for all \( t \in [a, t_*'] \),
\[
\mathcal{H}(1) + \int_a^{\alpha(t)} f(t, s)\frac{\Psi(a(s))}{a(s)}ds \in \text{Dom}(\mathcal{H}^{-1}),
\]
\[ p(t) \in \text{Dom}(G^{-1}), \]

and

\[ u(t) \leq \left[ a(t) + \int_a^t g(t, s) W \left( G^{-1}(p(s)) \right) ds \right] q(t), \]

where

\[ q(t) = H^{-1} \left( H(1) + \int_a^t f(t, s) \frac{\Psi(a(s))}{a(s)} ds \right). \]

**Proof.** Define function \( z(\cdot) \) by

\[ z(t) = a(t) + \int_a^t g(t, s) W \left( \int_a^s k(s, \tau) \Phi(u(\tau)) d\tau \right) ds, \quad t \in [a, b]. \]

Clearly \( z(\cdot) \) is a positive and nondecreasing function. Hence, we can apply Lemma 2.5 to the inequality

\[ u(t) \leq z(t) + \int_a^t f(t, s) h(s) ds, \]

to obtain

\[ u(t) \leq z(t) H^{-1} \left( H(1) + \int_a^t f(t, s) \frac{\Psi(a(s))}{a(s)} ds \right), \quad t \in [a, t_*], \]

for some function \( \Psi(\cdot) \) and some number \( t_* \in (a, b] \). An estimation of \( z(t) \) can be obtained following the same procedure as in the proof of Theorem 2.2. After that, we obtain

\[ z(t) \leq a(t) + \int_a^t g(t, s) W \left( G^{-1}(p(s)) \right) ds, \quad t \in [a, t_*], \]

where \( G(\cdot) \) and \( p(\cdot) \) are defined as in Theorem 2.2. \( \Box \)

### 3. An Application

Let us consider the following retarded equation:

\[ u(t) = k + \int_0^{\alpha(t)} F \left( s, u(s), \int_0^s K(\tau, u(\tau)) d\tau \right) ds, \quad t \in [a, b], \quad (11) \]

where \( k \geq 0, b > 0, \alpha(\cdot) \in C^1([a, b], \mathbb{R}) \) is a nondecreasing function with \( 0 \leq \alpha(t) \leq t, u(\cdot) \in C([0, b], \mathbb{R}), F(\cdot) \in C([0, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and \( K(\cdot) \in C([0, b] \times \mathbb{R}, \mathbb{R}) \). The following theorem gives a bound on the solution of equation (11).
Theorem 3.7. Assume that functions $F(\cdot, \cdot, \cdot)$ and $K(\cdot, \cdot)$ in (11) satisfy
\begin{align}
|K(t, u)| & \leq k(t)\Phi(|u|), \\
|F(t, u, v)| & \leq t|u| + |v|,
\end{align}
with $k(\cdot)$ and $\Phi(\cdot)$ defined as in Theorem 2.2. If $u(\cdot)$ is a solution of (11), then
\begin{equation}
|u(t)| \leq q(t) + t \int_{0}^{\alpha(t)} \exp (t(\alpha(t) - s)) q(s) ds, \
t \in [a, t_*],
\end{equation}
for some $t_* \in (a, \alpha(b)]$ such that
\begin{equation}
p(t) \in \text{Dom}(G^{-1}), \
t \in [a, t_*].
\end{equation}
Here,
\begin{align}
q(t) & = k + \int_{0}^{\alpha(t)} G^{-1}(p(s)) ds, \\
G(x) & = \int_{0}^{x} \frac{ds}{\Phi(1 + s)}, \quad x \geq 0, \\
p(s) & = \int_{a}^{s} k(\tau) \Phi \left[ \eta(\tau) + \tau \int_{0}^{\alpha(\tau)} \exp (\tau(\alpha(\tau) - \xi)) \eta(\xi) d\xi \right] d\tau, \\
\eta(\tau) & = \max \{ k, \alpha(\tau) \}, \quad \tau \in [0, \alpha(b)],
\end{align}
with $G^{-1}(\cdot)$ representing the inverse function of $G(\cdot)$.

Proof. Let $u(\cdot)$ be a solution of equation (11). In view of (12) and (13), we get
\begin{equation}
|u(t)| \leq k + \int_{0}^{\alpha(t)} \left( t|u(s)| + \int_{0}^{s} k(\tau) \Phi(|u(\tau)|) d\tau \right) ds.
\end{equation}
An application of Theorem 2.2 with $a(t) = k$, $\alpha(t) = \beta(t)$, $f(t, s) = t$, $b(t) = g(t, s) = 1$, and $W(u) = u$, gives the desired conclusion:
\begin{equation}
|u(t)| \leq q(t) + t \int_{0}^{\alpha(t)} \exp (t(\alpha(t) - s)) q(s) ds.
\end{equation}

\[\square\]

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