A general integral identity

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Abstract

The identity

$$\int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin \phi F(x \sin \phi \sin \theta) = \pi^2 \int_0^1 F(xt) dt,$$

where $F$ is any function, is derived. Several extensions are given and a few examples of physical interest are described.

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1. Introduction

A family of integrals, famous for their ubiquity, difficulty and aesthetic features, are the Watson integrals. These are also known as lattice Green functions and have representations as integrals of the complete elliptic integral $K(k) = \int_0^1 (1 - k^2 \sin^2 \theta)^{-1/2}$. An iconic example is $\int d\vec{k} (1 - \cos k_1 \cos k_2 \cos k_3)^{-1}$ integrated over the parallellepiped $[0, \pi]^3$. In this paper, the identity

$$\int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin \phi F(x \sin \phi \sin \theta) = \frac{\pi^2}{2} \int_0^1 F(xt) dt$$

is derived for arbitrary expressions $F$ which may prove useful in simplifying integrals of this type among others.

2. Derivation

We begin by noting two formulas from the familiar reference [1]. The Bessel transform (6.567(13)) ($x > 0$)

$$\int_0^1 \frac{dt \; t \; I_0(xt)}{\sqrt{1 - t^2}} = \frac{\sinh(x)}{x},$$

(1)

1 Described, e.g., by Eric W Weisstein ‘Watson’s Triple Integrals’ (MathWorld—A Wolfram Web Resource, available at http://mathworld.wolfram.com/WatsonsTripleIntegrals.html).
where \( I_0(z) = J_0(iz) \), and the series form of the modified Struve function (8.551)

\[
L_0(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m+1}}{\Gamma^2(m + 3/2)}. 
\tag{2}
\]

From (2) with \( z = xt \), substituting \( t^2 = u \), integrating term by term and using Euler’s beta integral (8.380), we find

\[
\int_0^1 \frac{dt}{\sqrt{1 - t^2}} L_0(xt) = \frac{\sqrt{\pi}}{2} \sum_{m=0}^{\infty} \frac{(x/2)^{2m+1}}{\Gamma(m + 3/2)\Gamma(m + 2)} = \frac{\cosh(x) - 1}{x}, 
\tag{3}
\]

where the last expression was found by inspection. Consequently,

\[
\int_0^1 \frac{dt}{\sqrt{1 - t^2}} [I_0(xt) - L_0(xt)] = \frac{1 - e^{-x}}{x} = \int_0^1 e^{-xt} \, dt. 
\tag{4}
\]

However [2],

\[
\pi^2 [I_0(z) - L_0(z)] = \int_0^1 \frac{dt}{\sqrt{1 - t^2}} e^{-zt}. 
\tag{5}
\]

Therefore,

\[
\int_0^1 \frac{u \, du}{\sqrt{1 - u^2}} \int_0^1 \frac{dr}{\sqrt{1 - r^2}} e^{-xur} = \frac{\pi}{2} \int_0^1 e^{-xt} \, dt, 
\tag{6}
\]

which can be written as

\[
\int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin \phi e^{-x \sin \phi \sin \theta} = \frac{\pi}{2x} \int_0^x e^{-t} \, dt. 
\tag{7}
\]

Next, let \( F \) belong to the class of functions which are Laplace transforms, i.e. for some real \( f \)

\[
F(s) = \int_0^\infty e^{-st} f(t) \, dt. 
\tag{8}
\]

Now, in (7) replace \( x \) by \( xu \), multiply both sides by such an \( f(u) \) and integrate both sides with respect to \( u \) over \([0, \infty]\), with the further restriction (on \( f \)) that the order of integration can be freely interchanged. The result is

\[
\int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin \phi F(x \sin \phi \sin \theta) = \frac{\pi}{2x} \int_0^x F(t) \, dt = 0. 
\tag{9}
\]

The left-hand side of (9) is a linear functional on an ideal of the class of all real integrable functions. The Hahn–Banach lemma [3] ensures that (9) can be extended at least to the class of all piecewise continuous real-valued functions on the positive real line.

Since \( x \) is a free parameter, let us replace \( x \) by \( x \sin \beta \) and integrate over \( \beta \) to obtain

\[
\int_0^{\pi/2} d\beta \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin \phi F[x \sin \beta \sin \phi \sin \theta] = \frac{\pi}{2x} \int_0^x \frac{d\beta}{\sin \beta} \int_0^{x \sin \beta} \sin \beta \, F(t) \, dt. 
\tag{10}
\]

By integrating the right-hand side of (10) by parts, followed by the substitution \( u = \cos \beta \), we have the intriguing identity

\[
\int_0^1 \int_0^1 \int_0^1 \frac{u \, du \, dv \, dw}{\sqrt{(1 - u^2)(1 - v^2)(1 - w^2)}} F(x uvw) = \frac{\pi}{4} \int_0^1 du \ln \left( \frac{1 + \sqrt{1 - u^2}}{1 - \sqrt{1 - u^2}} \right) F(xu). 
\tag{11}
\]
Clearly, this process can be repeated to obtain a reduction formula for an \( n \)-fold multiple integral, \( n = 4, 5, 6, \ldots \), to at most an \( (n - 2) \)-fold integral.

It seemed unlikely that such a general and useful identity as (9) has no precedents in the classical mathematical literature; a bit of searching showed that such a connection does exist. Let us set \( F(x) = f'(x) \) and note that the second term on the left-hand side of (9) is formally \( [f(x) - f(0)] \) \( \sqrt{x} \). Next write \( h(x) = f(0) + x \int_0^{\sqrt{x}} d\phi \sin \phi f'(x \sin \phi) \), so (9) can be expressed as \( f(x) = (2/\pi) \int_0^{\sqrt{x}} h(x \sin \theta) d\theta \), which is precisely Schlämilch’s integral equation [4]. Thus, it is possible that Schlämilch, or Abel, since his eponymous equation is equivalent to Schlömilch’s, was aware of some form of (9).

It is reasonable, therefore, to seek a more formal derivation of (9) that does not rely on manipulating specific functions. The following argument places it in principle among a family of such identities and displays its elementary character. Consider the double integral

\[
S = \int_0^1 dx \int_0^1 dy \frac{f(x + y)F(xy)}{\sqrt{(1-x^2)(1-y^2)}},
\]

where \( f \) and \( F \) are arbitrary. The change of variables

\[
x = \frac{1}{2}[u + \sqrt{u^2 - 4v}]
\]

\[
y = \frac{1}{2}[u - \sqrt{u^2 - 4v}]
\]

having Jacobian \( (u^2 - 4v)^{-1/2} \) leads to

\[
S = 2 \int_0^{1+\sqrt{v}} du F(v) \int_2 \sqrt{u} \frac{f(u)}{\sqrt{(u^2 - 4v)((1 + v)^2 - u^2)}} \]

Now let \( f(x) = x \). The \( u \)-integral yielding (9) after a trigonometric substitution. Every choice of \( f \) gives a possibly new integral identity. For example, let \( f(x) = 1 \). Since

\[
\int_0^{1+\sqrt{v}} \frac{du}{\sqrt{(u^2 - 4v)((1 + v)^2 - u^2)}} = \frac{1}{v + 1} K \left( \frac{1 - v}{1 + v} \right),
\]

after an elementary change of integration variable, (12) becomes

\[
\int_0^1 \frac{dt}{1+t} K(t) F \left( \frac{1-t}{1+t} \right) = \frac{1}{2} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi F(\sin \theta \sin \phi).
\]

As an application of (16), consider the family of integrals

\[
K_n = \int_0^1 K(k) \left( \frac{1}{1 + k^n} \right) \frac{dk}{1 + k^n}
\]

of which only the members \( n = 0, 1 \) appear to be well known [1, 7]. First, with \( F(x) = 1 \) we obtain immediately \( K_1 = \pi^2/8 \). Now, by setting \( F(x) = x^n \), writing \( 1 - k = 2 - (1 + k) \) and employing the binomial series, we find the recursion relation

\[
K_{n+1} = \frac{\pi}{2n+3} \left[ \frac{1}{2} - \frac{3}{2} \cdot \frac{n}{2} \right] - \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} 2^{-k} K_{n+1-k}
\]

An interesting formula results from selecting \( F \) in such a way as to cancel the elliptic integral in (16):

\[
\int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \frac{1}{K \left( \frac{1 - \sin \theta \sin \phi}{\sin \theta \sin \phi} \right)} = 2 \ln(2).
\]
2.1. Other examples

In a recent study of Feynman diagrams in two-dimensional quantum field theories [5] and related work [6], attention was drawn to various moments of powers of the complete elliptic integral of the first kind $K(k)$. A number of these were evaluated and the values of several others were conjectured. The form of (9) strongly suggests that it may prove useful in this connection. For example, we have [7]

$$\int_0^{\pi/2} K(a \sin \theta) \, d\theta = K^2 \left( \sqrt{1 - \sqrt{1 - a^2}} \right)$$

(20)

$$\int_0^1 K(ux) \, dx = \frac{\pi}{4} \, _3F_2(1/2, 1/2, 1/2; 1, 3/2; x^2).$$

(21)

Therefore, by setting $F = K$ in (9), we obtain

$$\int_0^1 \frac{u \, du}{\sqrt{1 - u^2}} \, K^2 \left( \sqrt{1 - \sqrt{1 - x^2 u^2}} \right) = \frac{\pi^2}{4} \, _3F_2(1/2, 1/2, 1/2; 1, 3/2; x^2)$$

(22)

and for $x = 1$, after some simplification,

$$\int_0^{1/\sqrt{2}} k K^2(k) \, dk = \frac{1}{4\pi} G,$$

(23)

where $G$ is Catalan’s constant. Both (22) and (23) appear to be new and there is evidence that (23) is the only analytically tractable moment of $K^2$ over a sub-unit interval [8].

In the same vein, let us set $F(t) = \frac{1}{\sqrt{1 - t^2}}$ in (11) with $x = 1$. This results in

$$\int_0^{1/\sqrt{2}} k K^2(k) \, dk = \frac{1}{4\pi} G.$$

(24)

An obvious application of (11) and its higher dimensional generalizations is the so-called Watson integrals [9], which are integrals over a polytope of a ratio of trigonometric polynomials. Thus, (11) gives immediately

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sin \beta \, d\beta \, d\phi \, d\theta}{1 - x \sin \beta \sin \phi \sin \theta} = \frac{\pi}{4x} \left[ \arccos^2(x) - 2\pi \arccos(x) + \frac{3\pi^2}{4} \right].$$

(25)

An alternative form of (16) is

$$\int_0^1 K(u) f(u) \, du = \int_0^{\pi/2} \int_0^{\pi/2} \frac{d\theta \, d\phi}{1 + \sin \theta \sin \phi} \, f \left( \frac{1 - \sin \theta \sin \phi}{1 + \sin \theta \sin \phi} \right).$$

(26)

Reference [7] contains over 100 integrals having the form of the left-hand side of (26) allowing many, apparently new, double trigonometric integrals to be found. For example,

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{d\theta \, d\phi}{\sin \theta \sin \phi (1 + \sin \theta \sin \phi)} = 4k K(k) K'(k),$$

(27)

where $k = \sqrt{2} - 1$. With judicious selection of rational functions $F$, many striking, and potentially useful, results can be worked out in this area, which will be the subject of a future report. The results derived here also follow from the invariance of an integral over the surface of an $n$-sphere under a permutation of the angular hyperspherical co-ordinates [10].
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