Flat manifolds, harmonic spinors, and eta invariants

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Abstract

The aim of this paper is to calculate the eta invariants and the dimensions of the spaces of harmonic spinors of an infinite family of closed flat manifolds $F_{CHD}$. It consists of some flat manifolds $M$ with cyclic holonomy groups. If $M \in F_{CHD}$, then we give explicit formulas for $\eta(M)$ and $h(M)$. The are expressed in terms of solutions of appropriate congruences in $\{-1, 1\}^{\left\lfloor \frac{n-1}{2} \right\rfloor}$. As an application we investigate the integrability of some $\eta$ invariants of $F_{CHD}$-manifolds.

Key words and phrases: Spin structure, harmonic spinor, eta invariant, flat manifold.

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1 Introduction

In this paper we consider Dirac operators on an infinite family $F_{CHD}$ of closed flat manifolds. It consists of flat manifolds $M$ with cyclic holonomy groups of odd order equal to the dimension of $M$. The family $F_{CHD}$ is particularly simple and the investigation of different properties of multidimensional flat manifolds should start with the investigation of them in this particular case. Some $F_{CHD}$ manifolds arises in the classification of flat manifolds whose holonomy groups have prime order (cf. [1]). We describe the eta invariants

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of the Dirac operators arising from different spin structures and we give necessary and sufficient conditions of the existence of nontrivial harmonic spinors. The methods used here extends that used in [12]. We apply them to much wider class of manifolds and we consider related general questions.

To formulate the main results we need some definitions. Let $n = 2k + 1$ be an odd number, and let $a_1, \ldots, a_n$ be a basis of $\mathbb{R}^n$. Consider the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(a_j) = a_{j+1}$ for $j < n-1$, $A(a_{n-1}) = -a_1 - \ldots - a_{n-1}$, and $A(a_n) = a_n$. Let $a = \frac{1}{n} a_n$ and let $g(x) = A(x) + a$. An $n$-dimensional flat manifold $M \in \mathcal{F}_{CHD}$ can be written as $\mathbb{R}^n / \Gamma$, where $\Gamma = \langle a_1, \ldots, a_{n-1}, g \rangle$. The linear part $A$ of $g$ has two lifts $\alpha_+, \alpha_- \in \text{Spin}(n)$ such that $\alpha_n^+ = \text{id}$ and $\alpha_n^- = -\text{id}$ (see Section 2). This defines two spin structures on $M$.

To formulate the result describing $\eta_{M^n}(0)$ for $M^n \in \mathcal{F}_{CHD}$ we need some combinatorial invariants. It is known that $\eta_{M^n}(0) = 0$ if $k$ is even (cf. [1, p. 61]) so we consider the case when $k$ is odd. Let

$$c(k) = \begin{cases} 0 & \text{if } \frac{k(k+1)}{2} \text{ is even} \\ \frac{1}{2} & \text{if } \frac{k(k+1)}{2} \text{ is odd} \end{cases}.$$ 

For every $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k$ consider

$$\mu_\epsilon = \sum_{j=1}^k \epsilon_j j \quad \text{and} \quad \nu(\epsilon) = \epsilon_1 \cdots \epsilon_k.$$ 

Let $D_+ = \{ \epsilon \in \{-1, 1\}^k : \nu(\epsilon) = \epsilon_1 \epsilon_2 \cdots \epsilon_k = 1 \}$, let $r \in \{0, \ldots, n-1\}$, let

$$A^+_r = 2\# \{ \epsilon \in D_+ : \frac{\mu_\epsilon}{2} + c(k)n \equiv r \mod (n) \}$$

in the case of $\alpha_+$, and let

$$A^-_r = 2\# \{ \epsilon \in D_+ : \frac{\mu_\epsilon}{2} + c(k)n + k \equiv r \mod (n) \}$$

in the case of $\alpha_-$. The numbers $A^\pm_r$ are well defined (cf. Remark 1).

**Theorem 1.** Let $k$ be an odd positive integer and let $n = 2k + 1$. If $M^n \in \mathcal{F}_{CHD}$, then

(1) \[ \eta_{M^n, \alpha_+}(0) = \sum_{r=1}^{n-1} A^+_r \left(1 - \frac{2r}{n}\right), \]
\[ \eta_{M^n,\alpha_-}(0) = \sum_{r=0}^{n-1} A_r \left( 1 - \frac{2r + 1}{n} \right). \]

Applying Theorem 1 we prove that some \( \eta \)-invariants of \( \mathcal{F}_{CHD} \)-manifolds are integral (Corollary 1) and that \( \eta_{M,\alpha_+} - \eta_{M,\alpha_-} \in 2\mathbb{Z} \) (Corollary 2). Let \( \mathfrak{h}(V) \) be the dimension of the vector space of harmonic spinors.

Now we state another result of the paper.

**Proposition 1.** Let \( k \) be a positive integer and let \( n = 2k + 1 \). If \( M \in \mathcal{F}_{CHD} \), then

a) \( \mathfrak{h}(M, \alpha_+) > 0 \) if and only if \( n \geq 5 \).

b) \( \mathfrak{h}(M, \alpha_-) = 0 \).

The spectra of the Dirac operators on flat tori were described in [6]. The spectra of the Dirac operators on closed 3-dimensional flat manifolds and their eta invariants were calculated in [12]. We should also mention about ([11]) where the authors consider spin structure and the Dirac operators on flat manifolds with \( \mathbb{Z}_p \), (p-prime number), and non-cyclic holonomy.

Throughout this paper the following notation will be used. If \( G \) is a group and \( g_1, \ldots, g_l \in G \), then \( \langle g_1, \ldots, g_l \rangle \) is the subgroup of \( G \) generated by \( g_1, \ldots, g_l \). The symbol \( X^G \) stands for the set of the fixed points of an action of \( G \) on \( X \). For every \( g \in G \), \( X^g = \{ x \in X : gx = x \} \). By \( \Gamma \) (or \( \Gamma_n \)) we denote the deck group of a closed flat manifold \( M \), by \( \hat{h} \) the holonomy homomorphism of \( M \), and by \( \hat{h} \) its lift to \( \text{Spin}(n) \). The standard epimorphism from \( \text{Spin}(n) \) to \( \text{SO}(n) \) will be denoted by \( \lambda \) (cf. Section 2). The letter \( \Gamma_0 \) stands for the maximal abelian subgroup of \( \Gamma \) consisting of all translations belonging to \( \Gamma \), (cf. [11] and [15]). By \( a_1, \ldots, a_n \) we usually denote a basis of \( \Gamma_0 \). The subspace of a vector space spanned by vectors \( v_1, \ldots, v_l \) will be denoted by \( \text{Span}[v_1, \ldots, v_l] \). The symbols \( \alpha_+, \alpha_-, \mathfrak{h}(M), c(k), \mu, \nu(\epsilon), \) and \( A_r \) were defined above. The cyclic group \( \langle A \rangle \) will be denoted by \( G \).

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2 Spin structures on $\mathcal{F}_{\text{CHD}}$-manifolds and Dirac operators

Let $k \in \mathbb{N} \cup \{0\}$ and let $n = 2k + 1$. Let $\Gamma$ be as in the introduction, and let $(\cdot, \cdot)^*$ be an $A$-invariant scalar product in $\mathbb{R}^n$. From definition (cf. [4] and [15]) $M = \mathbb{R}^n/\Gamma$ is a closed, orientable, flat manifold. Moreover the eigenvalues of the generator $A$ of the holonomy group of $M$ are equal to $e^{\frac{2\pi i j}{n}}$, $j = 1, \ldots, n$. In fact, for every $j = 2, \ldots, n - 1$, consider the $(j \times j)$-matrix

$$M_j = \begin{bmatrix} 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{bmatrix}.$$ 

Let $M_j(z) = M_j - zI$. Then

$$\det(A - zI) = (1 - z) \det M_{n-1}(z).$$

Applying the Laplace expansion with respect to the first row we have

$$\det M_j(z) = -z \det M_{j-1}(z) + (-1)^j.$$ 

Using this it is easy to check that $\det M_j(z) = (-1)^j \sum_{l=0}^{j} z^l$. Hence

$$\det(A - zI) = (1 - z) \det M_{n-1}(z) = -z^n + 1.$$ 

Let $e_1, \ldots, e_n$ be an orthonormal basis in $(\mathbb{R}^n, (\cdot, \cdot)^*)$. Throughout the rest of the paper we shall always assume (cf. [11] page 61 and [10] Proposition 1.3) that:

(i) $e_1, \ldots, e_{n-1} \in \text{Span } [a_1, \ldots, a_{n-1}]$ and $e_n = a_n$,

(ii) for every for $j \leq n - 1$ : $A(e_{2j-1}) = \cos(2\pi j/n)e_{2j-1} + \sin(2\pi j/n)e_{2j}$, and

$A(e_{2j}) = -\sin(2\pi j/n)e_{2j-1} + \cos(2\pi j/n)e_{2j}$.

Let $\text{Cliff}(n)$ be the Clifford algebra in $\mathbb{R}^n$ and let $\text{Cliff}_C(n)$ be its complexification. The group $\text{Spin}(n)$ is the set of products $x_1 \cdots x_{2r}$, where $r \in \mathbb{N}$, and where $x_1, \ldots, x_{2r}$ are the elements of the unit sphere in $\mathbb{R}^n$. The standard covering map $\lambda : \text{Spin}(n) \to SO(n)$ carries $y \in \text{Spin}(n)$ onto $\mathbb{R}^n \ni x \to yxy^*$,
where \((e_j, \ldots e_j)^* = e_j, \ldots e_j\). A spin structure on an orientable flat manifold \(M = \mathbb{R}^n/\Gamma\) is determined by the lift \(\hat{h}: \Gamma \to \text{Spin}(n)\) of the holonomy homomorphism \(h: \Gamma \to SO(n)\). Recall that \(h\) carries \(\gamma \in \Gamma\) onto its linear part \(h(\gamma)\), (cf. [15, Chapter III]). For \(M \in \mathcal{F}_{CHD}\) we have \(h(\Gamma) = \langle A \rangle \cong \mathbb{Z}_n\) and any lift \(\hat{A}\) of \(A\) to \(\text{Spin}(n)\) defines the lift \(\hat{h}\) of \(h\), given by the formulas \(\hat{h}(a_j) = 1\) for \(j \leq n - 1\), \(\hat{h}(g) = \hat{A}\). In order to construct \(\hat{A}\) consider \(\beta = \pi \frac{n}{r_j} = \cos(j\beta) e_{2j-1} + e_{2j} \sin(j\beta)\), and \(\alpha = \prod_{j=1}^{k} r_j\). Clearly \(r_i r_j = r_j r_i\) for \(i, j \in \{1, \ldots, k\}\). A direct calculation yields

\[
\lambda(r_j)(e_l) = \begin{cases} 
\cos(2j\beta)e_{2j-1} + \sin(2j\beta)e_{2j} & \text{for } l = 2j - 1 \\
- \sin(2j\beta)e_{2j-1} + \cos(2j\beta)e_{2j} & \text{for } l = 2j \\
e_l & \text{for } l \notin \{2j - 1, 2j\} 
\end{cases}
\]

Using this it is easy to check that

\[
\alpha^n = (-1)^{\frac{k(k+1)}{2}}
\]

and \(\lambda(\alpha) = A\). Now we can define

\[
\alpha_+ = (-1)^{\frac{k(k+1)}{2}} \alpha, \quad \alpha_- = -(-1)^{\frac{k(k+1)}{2}} \alpha.
\]

Since \(n\) is odd,

\[
\alpha_+^n = 1 \quad \text{and} \quad \alpha_-^n = -1.
\]

We have.

**Lemma 1.** \(H_1(M, \mathbb{Z}) \cong \mathbb{Z} \oplus H\), where \(H\) is a finite abelian group of odd order and \(H^1(M, \mathbb{Z}_2) \cong \mathbb{Z}_2\).

**Proof:** The group \(\Gamma_0 = \langle a_1, \ldots, a_n \rangle\) is the maximal abelian subgroup of \(\Gamma\) and the following sequence

\[
0 \to \Gamma_0 \to \Gamma \to \langle A \rangle \to 1
\]

is exact (cf. [4, Proposition 4.1], [15, Theorem 3.2.9]). From [8, Corollary 1.3] we have \(\dim_{\mathbb{Q}}(\mathbb{Q} \otimes H_1(\Gamma, \mathbb{Z}) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes \Gamma_0^d)) = 1\). Hence \(H_1(M, \mathbb{Z}) \cong \mathbb{Z} \oplus H\), where \(H\) is a finite group. According to [8, Chapter 3], there are homomorphisms \(\text{res} : H_*(M, \mathbb{Z}) \cong H_*(\Gamma, \mathbb{Z}) \to H_*(\Gamma_0, \mathbb{Z})\) and \(\text{cor} : H_*(\Gamma_0, \mathbb{Z}) \to\)
$H_*(M, \mathbb{Z})$ such that $\text{cor} \circ \text{res}$ is the multiplication by $n$. Since the group $H_*(\Gamma_0, \mathbb{Z}) \cong H_*(T^n, \mathbb{Z})$ is torsion free we have $nH = 0$. In particular, the order of $H$ is odd. For the proof of the last statement we have $H^1(M, \mathbb{Z}_2) \cong \text{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z}_2) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}_2) \cong \mathbb{Z}_2$. □

Since $\alpha_+, \alpha_-$ are different lifts of the holonomy homomorphism $h$ to $\text{Spin}(n)$, the spin structures determined by them are different. It is known that spin structures on $M$ correspond to the elements of $H^1(M, \mathbb{Z}_2)$ ([7, p. 40]).

By [7, Section 1.3], the irreducible complex Cliff$_\mathbb{C}(n)$-module $\Sigma_{2k}$ can be described as follows. Consider

$$g_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Let $\Sigma_{2k} = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$ and let $\alpha(j) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ 2 & \text{if } j \text{ is even} \end{cases}$. Take an element $u = u_1 \otimes \ldots \otimes u_k$ of $\Sigma_{2k}$ and the orthonormal basis $e_1, \ldots, e_n$ considered above. Then

$$e_ju = (I \otimes \ldots \otimes I \otimes g_{\alpha(j)} \otimes T \otimes \ldots \otimes T)(u),$$

for $j \leq n - 1$, and

$$e_nu = i(T \otimes \ldots \otimes T)u.$$

A spin structure on $M$ determines a complex spinor bundle $P\Sigma_{2k}$ with fiber $\Sigma_{2k}$. This bundle is the orbit space of $\mathbb{R}^n \times \Sigma_{2k}$ by the action of $\Gamma$ given by

$$\gamma(x, v) = (\gamma x, \hat{h}(\gamma)v),$$

where $\gamma \in \Gamma$, $x \in \mathbb{R}^n$ and $v \in \Sigma_{2k}$. Clearly

$$\hat{h}(a_j) = 1 \text{ for } j \leq n - 1$$

and

$$\hat{h}(g) = \alpha_{\pm}.$$

Since $\text{Span}[e_1, \ldots, e_{n-1}] = \text{Span}[a_1, \ldots, a_{n-1}]$ and $a_n = e_n$ we conclude that

$$\hat{h}(e_j) = 1 \text{ for } j \leq n - 1.$$
Consider the covering $T^n = \mathbb{R}^n/\Gamma_0 \to M$. We have $\hat{h}(a_n) = \pm 1$. The lift $P_1 \Sigma_{2k}$ of $P\Sigma_{2k}$ to $T^n$ is the orbit space $(\mathbb{R}^n \times \Sigma_{2k})/\Gamma_0$, where the action of $\Gamma_0$ on $\mathbb{R}^n \times \Sigma_{2k}$ is given by the formula (1).

To deal with the spectrum of the Dirac operator $D$ it is convenient to describe it in terms of the spectrum of $D^2$. We state without proofs some related results of [12] that will be used later. Identify the parallel section $E$ of $D$ by describing it in terms of the spectrum of $\Gamma f$ where $\phi$ is induced by the action of $g$ on the set of sections of $\mathbb{R}^n \times \Sigma_{2k}$, corresponding to eigenvectors of $D^2$ on $T^n$, and the elements of $\{f_i v : v \in \Sigma_{2k}, b \in \mathcal{B}\}$ correspond to eigenvectors of $D$ on $M$.

Let $\Gamma_0^*$ be the dual lattice of $\Gamma_0$. Let $\mathcal{B}$ be $\Gamma_0^*$ in the case of $\alpha_+$ and $\Gamma_0^* + \frac{1}{2} e_n$ in the case of $\alpha_-$. The action of $g$ on the set of sections of $\mathbb{R}^n \times \Sigma_{2k}$, induced by the action of $g$ on $\mathbb{R}^n$, is given by the formula

$$g(\phi)(x) = \hat{h}(g)\phi(g^{-1} x),$$

(3)

where $\phi$ is a spinor on $\mathbb{R}^n$.

Consider $f_b(x) = e^{2\pi i(b,x)}$. By immediate calculation or following [12] we have

$$D^2(f_b v) = 4\pi^2 \|b\|^2 f_b v.$$  

(4)

Hence the sections $f_b v, b \in \mathcal{B}, v \in \Sigma_{2k}$, correspond to eigenvectors of $D$ on $T^n$, and the elements of $\{ f_b v : v \in \Sigma_{2k}, b \in \mathcal{B}\}$ correspond to eigenvectors of $D$ on $M$.

For $b \in \mathcal{B}$, let us denote the corresponding $D^2$-eigenspace by $E_b(D^2) = \text{Span}\{ f_b v : v \in \Sigma_{2k}\}$. We have the decomposition $E_b(D^2) = E_{b+}(D) \oplus E_{b-}(D)$, where

$$E_{b\pm}(D) = \{ p \in E_b(D^2) : Dp = \pm 2\pi \|b\| p \}.$$

Since

$$(f_b \circ g^{-1})(x) = e^{-2\pi i(A(b),a)} f_{A(b)}(x)$$

(5)

we have $A E_b \subseteq E_{A(b)}$, (cf. [12, Lemma 4.1]). Denote $< A >$ by $G$.

Let $\mathcal{B}_{sym} = \{ b \in \mathcal{B} : \# G(b) = \# G \}, \mathcal{B}_{pas} = \{ b \in \mathcal{B} : \# G(b) < \# G \}$ and

$$D_S = D|_{\bigoplus_{b \in \mathcal{B}_{sym}} E_b(D^2)}^{\gamma}, \quad D_{Pas} = D|_{\bigoplus_{b \in \mathcal{B}_{pas}} E_b(D^2)}^{\gamma}.$$  

(6)
Clearly \( B \) is the disjoint union of \( B_{\text{Sym}} \) and \( B_{\text{Pas}} \) and the Dirac operator \( D \) on \( M \) can be identified with \( D_S \oplus D_{\text{Pas}} \). If \( b \in B_{\text{Sym}} \) and

\[
V_b^\pm = \bigoplus_{h \in G} E_{h(b\pm)}(D^2)
\]

then \( \dim(V_b^\pm)^g = \dim E_b^\pm(D) = 2^{k-1} \) (cf. [12, Theorem 4.2, Corollary 4.3]).

3 Eta invariants of \( F_{CHD} \)-manifolds

The aim of this section is to prove Theorem 1. Recall that the \( \eta \)-invariant of the Dirac operator on a closed spin manifold \( M \) is defined as follows.

As \( D \) is elliptic formally self adjoint, it has discrete real spectrum and the series \( \sum_{\lambda \neq 0} \text{sgn} \lambda |\lambda|^{-2} \) converges for \( z \in \mathbb{C} \) with \( \text{Re}(z) \) sufficiently large ([1, Theorem 3.10]). Here summation is taken over all nonzero eigenvalues \( \lambda \) of \( D \), each eigenvalue being repeated according to its multiplicity. The function \( z \to \sum_{\lambda \neq 0} \text{sgn} \lambda |\lambda|^{-2} \) can be extended to a meromorphic function \( \eta_M \) in the whole complex plane such that 0 is a regular point of \( \eta_M \) ([1, Theorem 3.10]). The \textit{eta-invariant} of \( M \) is \( \eta_M(0) \).

Define an endomorphism \( \rho_1 \) of \( \mathbb{C}^2 \) by the formula

\[
\rho_1(u) = \cos \beta u + \sin \beta g_1 g_2 u.
\]

The matrix of \( \rho_1 \) is equal to

\[
\cos \beta I + \sin \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}
\]

so that the matrix of \( \rho_1^j \) is equal to

\[
\cos(\beta j) I + \sin(\beta j) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

The following lemma is crucial.

\textbf{Lemma 2.} Let \( w_{+1} = (1, -i) \), let \( w_{-1} = (1, i) \), let \( \epsilon = (\epsilon_1, ..., \epsilon_k) \in \{-1, 1\}^k \), and let \( v = w_{\epsilon_1} \otimes ... \otimes w_{\epsilon_k} \). Let \( \beta = \frac{\pi}{n} \) and let \( \mu_i \) be as in Theorem 1. Take \( u = u_1 \otimes ... \otimes u_k \in \Sigma_{2k} \). Then

a) \( \alpha u = \rho_1 u_1 \otimes ... \otimes \rho_1^k u_k \),
b) \( ae_n u = e_n \alpha u \),
c) \( \rho_1(w_{\pm 1}) = e^{\pm i\beta} w_{\pm 1} \),
d) \( \alpha_v e = e^{i\beta u} v \) and \( \{ v : \epsilon \in \{-1, 1\} \} \) is a basis of \( \Sigma_{2k} \),
e) \( e_n v_\epsilon = -i\nu(\epsilon) v_\epsilon \).

**Proof.** a) Since \( T^2 = id \),
\[
e_{2j-1} e_{2j} (u_1 \otimes ... \otimes u_j \otimes ... \otimes u_k) = u_1 \otimes ... \otimes g_1 g_2(u_j) \otimes ... \otimes u_k
\]
and consequently
\[
r_j(u_1 \otimes ... \otimes u_j \otimes ... \otimes u_k) = u_1 \otimes ... \otimes \rho_1^j(u_j) \otimes ... \otimes u_k.
\]
Hence
\[
\alpha(u_1 \otimes ... \otimes u_k) = (r_1 \cdots r_k)(u_1 \otimes ... \otimes u_k) = \rho_1(u_1) \otimes ... \otimes \rho_1^k(u_k).
\]
b) For \( j \leq k \) we have \( e_{2j-1} e_{2j} e_n = e_n e_{2j-1} e_{2j} \) so that \( r_j e_n = e_n r_j \).
c) is obvious.
d) By c), \( r_1(w_{\epsilon_j}) = e^{i\beta \epsilon_j} w_{\epsilon_j} \). Hence
\[
\alpha(v_\epsilon) = r_1(w_{\epsilon_1}) \otimes ... \otimes r_1^k(w_{\epsilon_k}) = e^{i\beta \epsilon_k} v_\epsilon.
\]
Since \( \#\{ v_\epsilon : \epsilon \in \{-1, 1\} \} = 2^k = \dim \Sigma_{2k} \) and the vectors \( v_\epsilon \) are linearly independent, they form a basis of \( \Sigma_{2k} \).
e) We have \( T(w_1) = -w_1 \) and \( T(w_{-1}) = w_{-1} \). It follows that
\[
e_n v_\epsilon = iT w_{\epsilon_1} \otimes ... \otimes T w_{\epsilon_k} = i(-1)^\#\{ j \in \{1, ..., k\} : \epsilon_j = 1 \} v_\epsilon = -i\nu(\epsilon) v_\epsilon.
\]
This finishes the proof of Lemma 2. \( \square \)

Let \( E(\lambda, D_{P_{as}}) \) be the eigenspace of \( \lambda \) for \( D_{P_{as}} \) on \( M \). From the definition and [1], [2] it is easy to see that \( \lambda = 2\pi l \) in the case of \( \alpha_+ \) and \( \lambda = 2\pi (l + \frac{1}{2}) \) in the case of \( \alpha_- \), where \( l \in \mathbb{Z} \). In the case \( \alpha_+ \), \( B_{P_{as}} = \{ le_n : l \in \mathbb{Z} \} \) and we have
\[
D(f_{le_n} v_\epsilon) = -\frac{\partial}{\partial x_n} (e^{2\pi i(lle_n,x)}) i\nu(\epsilon) v_\epsilon = \nu(\epsilon) 2\pi l f_{le_n} v_\epsilon.
\]
Hence \( E(2\pi l, D_{P_{as}}) = \text{Span} [f_{\nu(\epsilon) le_n} v_\epsilon : \epsilon \in \{-1, 1\}^k] \). Similar formulas are also true for \( \alpha_- \), where \( B_{P_{as}} = \{ (l + \frac{1}{2}) e_n : l \in \mathbb{Z} \} \).

Now we are able to describe the spectrum of \( D_{P_{as}} \).
Proposition 2. Let $n, k, M, \mu, \nu(\epsilon)$, and $c(k)$ be as in Theorem 1. Let $b^+ = le_n$ and $b^- = (l + \frac{1}{2})e_n$, where $l \in \mathbb{Z}$.

a) If the spin structure is given by $\alpha_+$, then

$$\mathcal{E}(2\pi l, D_{Pas}) = \text{Span}[f_{\nu(\epsilon)b^+} : \frac{\mu}{2} + c(k)n \equiv \nu(\epsilon)l \mod (n)].$$

b) If the spin structure is given by $\alpha_-$, then

$$\mathcal{E}(2\pi(l + \frac{1}{2}), D_{Pas}) = \text{Span}[f_{\nu(\epsilon)b^-} : \frac{\mu}{2} + c(k)n + \frac{n-1}{2} \equiv \nu(\epsilon)l \mod (n)].$$

Remark 1. Since $\epsilon_j - 1$ are even, the difference $\mu - k(k+1)/2 = \sum_{j=1}^{k} e_j - \sum_{j=1}^{k} j$ is divisible by 2. Using this and the definition of $c(k)$ it is easy to see that $\frac{\mu}{2} + c(k)n$ is an integer.

Proof of Proposition 2. a) From the definitions of $\alpha_+$ and $c(k)$ it follows that $\alpha_+ = (-1)^{2c(k)}\alpha$. We have

$$g(f_{le_n}(x)v_\epsilon) = f_{le_n}(g^{-1}x)\alpha_+ v_\epsilon = e^{-\frac{2\pi ik}{n}f_{le_n}(x)}(-1)^{2c(k)}\alpha v_\epsilon = e^{-\frac{2\pi ik}{n}(l - \frac{\mu}{2} - c(k)n)}f_{le_n}(x)v_\epsilon.$$

By the above one gets the required conditions.

b) In the case of $\alpha_-$ the eigenvectors of $D_{Pas}$ on $T^n$ can be written as $f_{(l + \frac{1}{2})e_n} v$ for $l \in \mathbb{Z}, v \in \Sigma_{2k}$. We have

$$g f_{(l + \frac{1}{2})e_n}(x)v_\epsilon = e^{-\frac{2\pi ik}{n}(l - c(k)n - \frac{n-1}{2} - \frac{\mu}{2})}f_{(l + \frac{1}{2})e_n}(x)v_\epsilon.$$

Hence $f_{(b^-)}v_\epsilon$ is $g$-equivariant if and only if $l \in n\mathbb{Z} + c(k)n + \frac{n-1}{2} + \frac{\mu}{2}$. The rest of the argument is the same as in a). This finishes the proof of Proposition 2.

Lemma 3. Let $M \in \mathcal{F}_{CHD}$ be an $n$-dimensional with $k = \lfloor \frac{n-1}{2} \rfloor$ odd and with a fixed spin structure. Let $m$ be a natural number such that $m \equiv r \mod (n)$. Assume that $f_{b^+}v_\epsilon \in \mathcal{E}(\lambda, D_{Pas})$. Then

a) $f_{b^-}v_\epsilon \in \mathcal{E}(\lambda, D_{Pas})$.

b) $\dim \mathcal{E}(2\pi(m), D_{Pas}) = A^+_r$ and $\dim \mathcal{E}(2\pi(m + \frac{1}{2}), D_{Pas}) = A^-_r$. 

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Proof. a) If the spin structure on $M$ is $\alpha_+$, then $b = le_n$ for some $l \in \mathbb{Z}$, and, from the equivariance of $f_b v_\epsilon$, it follows that

$$l \equiv \frac{\mu_\epsilon}{2} + c(k)n \mod (n).$$

Hence

$$-l \equiv \frac{\mu_{-\epsilon}}{2} + c(k)n \mod (n)$$

and $f_{-b} v_{-\epsilon}$ is $g$-equivariant. By the assumption that $k$ is odd, $\nu(-\epsilon) = -\nu(\epsilon)$. According to Lemma, $f_{-b} v_{-\epsilon} \in \mathcal{E}(\lambda, D_{P_{\alpha_+}})$.

If the spin structure is $\alpha_-$, then we use the congruence

$$l \equiv \frac{\mu_\epsilon}{2} + c(k)n + k \mod (n).$$

Since $\mu_{-\epsilon} = -\mu_\epsilon$, $c(k)n \equiv -c(k) \mod (n)$, and $-k - 1 \equiv k \mod (n)$ we have

$$-l - 1 \equiv \frac{\mu_{-\epsilon}}{2} + c(k)n + k \mod (n)$$

and consequently $f_{-b} v_{-\epsilon}$ is $g$-equivariant.

b) If the spin structure is $\alpha_+$, then using Proposition and a), we get

$$\dim \mathcal{E}(2\pi(m), D_{P_{\alpha_+}}) = \dim \mathcal{E}(2\pi(r), D_{P_{\alpha_+}})$$

$$= 2\# \{ \epsilon \in \mathcal{D}_+: \frac{\mu_\epsilon}{2} + c(k)n \equiv r \mod (n) \} = A_r^+. $$

In the case of $\alpha_-$ we get

$$\dim \mathcal{E}(2\pi(m + \frac{1}{2}), D_{P_{\alpha_-}}) = \dim \mathcal{E}(2\pi(r + \frac{1}{2}), D_{P_{\alpha_-}})$$

$$= 2\# \{ \epsilon \in \mathcal{D}_+: \frac{\mu_\epsilon}{2} + c(k)n + k \equiv r \mod (n) \} = A_r^-.$$
for \( \text{Re}(z) \) sufficiently large. By Lemma 3 b), \( \dim \mathcal{E}(\lambda, D_{Pa}) = A_0^+ \) for \( \lambda \in \mathcal{S}_r \). If \( A_0^+ \neq 0 \), then the eigenvalue \( \lambda \in \mathcal{S}_0 \) occur together with \(-\lambda\) with the same multiplicity \( A_0^+ \) so that \( \sum_{\lambda \in \mathcal{S}_0 - \{0\}} \frac{A_0^+ \text{sgn}(\lambda)}{\lambda^n} = 0 \) and, for \( \text{Re}(z) \) sufficiently big,

\[
\eta_M(z) = \sum_{r=1}^{n-1} \sum_{m=-\infty}^{\infty} \frac{A_r^+ \text{sgn}(2\pi(mn + r))}{|2\pi(mn + r)|^z} = \sum_{r=1}^{n-1} \sum_{m=-\infty}^{\infty} \frac{A_r^+ \text{sgn}(2\pi n (m + \frac{z}{n}))}{|2\pi n (m + \frac{z}{n})|^z}
\]

\[
= \sum_{r=1}^{n-1} \frac{A_r^+}{|2\pi n|^z} \left( \sum_{m=0}^{\infty} \frac{1}{(m + \frac{z}{n})^z} - \sum_{m=0}^{\infty} \frac{1}{(m + 1 - \frac{z}{n})^z} \right).
\]

The last two series are known as generalized zeta functions (cf. [13]). They have meromorphic extensions on \( \mathbb{C} \) without poles in \( z = 0 \). Let \( \zeta(z, a) \) denote the function defined by \( \sum_{m=0}^{\infty} \frac{1}{(m + \frac{z}{n})^z} \) for \( \text{Re}(z) \) sufficiently big. One gets for the extension: \( \zeta(0, a) = \frac{1}{2} - \frac{z}{n} \). Hence

\[
\eta_M(0) = \sum_{r=1}^{n-1} A_r^+ \left( 1 - \frac{2r}{n} \right).
\]

b) We use similar arguments as those given in the proof of a). Now the component \( \mathcal{S}_0 \) is not symmetric so that we do not remove \( r = 0 \) from the formula describing \( \eta_M(z) \). The equality

\[
2\pi \left( mn + r + \frac{1}{2} \right) = 2\pi n \left( m + \frac{2r + 1}{2n} \right)
\]

and the above considerations implies that

\[
\eta_M(0) = \sum_{r=0}^{n-1} A_r^- \left( 1 - \frac{2r + 1}{n} \right).
\]

This finishes the proof of Theorem 1.

We have.

**Corollary 1** Let \( n \) be a prime number greater than 3 such that \( n + 1 \) is divisible by 4 and let \( M^n \in \mathcal{F}_{CHD} \) be a flat manifold with a fixed spin structure. Then \( \eta_{M^n} \in \mathbb{Z} \).
Proof: Let \( l = \frac{n+1}{4} \). It is known (cf. [13, chapter 9]) that \( 2^s \) copies of \( M^n \) is a boundary of a spin manifold \( W^{n+1} \) for some \( s \in \mathbb{N} \). By [1, Theorem 4.2] 
\[
\int_{W^{n+1}} \hat{A}_l(p) - \frac{2^s \eta_{M^n}}{2} \in \mathbb{Z},
\]
where \( \hat{A}_l \) is the \( l \)-th \( \hat{A} \)-polynomial on Pontriagin classes. By [9]\[
\int_{W^{n+1}} \hat{A}_l(p) \text{ can be written as } C_{W^{n+1}} \prod_{q} q^{r},
\]
where \( C_{W^{n+1}} \in \mathbb{Z} \) and where \( q_1, \ldots, q_r \in \{2, 3, \ldots, n-1\} \) are prime numbers. From Theorem 1 \( \eta_{M^n} = C_{M^n} \), for some \( C_{M^n} \in \mathbb{Z} \). Since \( \frac{C_{W^{n+1}}}{q_1 \cdots q_r} - 2^s \eta_{M^n} \in \mathbb{Z} \) we have \( 2^s \eta_{M^n} \in \mathbb{Z} \). Hence \( \eta_{M^n} \in \mathbb{Z} \). \( \square \)

Corollary 2 Let \( n \) be an odd number. If \( M^n \in \mathcal{F}_{CHD} \), then \( d = \eta_{M^n, \alpha} - \eta_{M^n, \alpha^-} \in 2\mathbb{Z} \).

Proof: By [3, Theorem 1.1], \( d \in \frac{1}{2}\mathbb{Z} \). From the definition (cf. page 2) all \( A^+_r \) belongs to \( 2\mathbb{Z} \). Hence \( d = \frac{2C}{n} \) for some \( C \in \mathbb{Z} \). Summing up \( \frac{d^n}{2} \in \mathbb{Z} \) and \( d \in 2\mathbb{Z} \). \( \square \)

Example 1. We calculate \( \eta_{M^n, \alpha} \), where \( M^7 \in \mathcal{F}_{CHD} \). Since \( \frac{k(k+1)}{2} = 6 \) is even our equation is
\[
r \equiv \frac{\mu_{\epsilon}}{2} \text{ mod (7)}.
\]
The values of \( \frac{\mu_{\epsilon}}{2} \) and \( r \) for \( \epsilon \in \mathcal{D}_+ \) are given in the following table.

| \( \epsilon \)     | \( \frac{\mu_{\epsilon}}{2} \) | \( r \) |
|------------------|-------------------------------|-----|
| \( (1, 1, 1) \)  | 3                             | 3   |
| \( (1, -1, -1) \)| -2                            | 5   |
| \( (-1, 1, -1) \)| -1                            | 6   |
| \( (-1, -1, 1) \)| 0                             | 0   |

It follows that \( A^+_j = 2 \), for \( j = 0, 3, 5, 6 \) and \( A^+_j = 0 \) for other values of \( j \). By Theorem 1
\[
\eta_{M^7}(0) = 2 \left[ \left( 1 - \frac{6}{7} \right) + \left( 1 - \frac{10}{7} \right) + \left( 1 - \frac{12}{7} \right) \right] = -2.
\]
4 Harmonic spinors on $\mathcal{F}_{CHD}$-manifolds

A harmonic spinor on a closed spin manifold $M$ is an element of the kernel of the Dirac operator on $M$.

Proof of Proposition 1. a) We have

$$g\nu_\epsilon = (-1)^{\frac{k(k+1)}{2}}\alpha \nu_\epsilon = (-1)^{\frac{k(k+1)}{2}} e^{\frac{2\pi i}{2n} \mu_\epsilon} \nu_\epsilon.$$

First consider the case when $k(k+1)/2$ is even. Then $g\nu_\epsilon = \nu_\epsilon$ if and only if

$$\mu_\epsilon \equiv 0 \mod (2n)$$

and $k = 4k_0 + 3$ or $k = 4k_0$. Let $\delta_4$ denote the sequence $1, -1, -1, 1$. If $k = 4k_0 + 3$ and

$$\epsilon = (-1, -1, 1, \delta_4, \ldots, \delta_4),$$

then $\epsilon$ belongs to $\{-1, 1\}^k$ and $\mu_\epsilon = 0$. If $k = 4k_0$ and

$$\epsilon = (\delta_4, \ldots, \delta_4),$$

then $\epsilon$ belongs to $\{-1, 1\}^k$ and $\mu_\epsilon = 0$. In particular, $b(M) > 0$.

Now assume that $k > 2$ and $k(k+1)/2$ is odd. Then $g\nu_\epsilon = \nu_\epsilon$ if and only if

$$\mu_\epsilon \equiv n \mod (2n)$$

and $k = 4k_0 + 1$ or $k = 4k_0 + 2$. If $k = 4k_0 + 1$ consider

$$\epsilon = (1, -1, 1, \delta_4, \ldots, \delta_4, 1, 1)$$

and if $k = 4k_0 + 2$ consider

$$\epsilon = (-1, 1 - 1, 1, \delta_4, \ldots, \delta_4, 1, 1).$$

In both cases $\epsilon \in \{-1, 1\}^k$ and $\mu_\epsilon = n$. It is easily seen that the equation $\mu_\epsilon \equiv 0 \mod (2n)$ have no solutions for $k = 1$ or 2.

b) Since $g^n = -id$, the equation $g\nu = \nu$ have only one solution $\nu = 0$. \qed

It is easy to see that the equality $\alpha_+ \nu_\epsilon = \nu_\epsilon$ implies $\alpha_+ \nu_{-\epsilon} = \nu_{-\epsilon}$. Using this and the arguments given in the proof of Proposition 1 we have.
Corollary 3  If $M \in \mathcal{F}_{CHD}$ and $\dim M = 2k + 1$, then

$$h(M, \alpha) = 2\# \{ \epsilon \in \mathcal{D}_+ : \frac{H_\epsilon}{2} + c(k)n \equiv 0 \mod (2k + 1) \}.$$ 

References

[1] ATIYAH, M.F.; PATODI, V.K.; SINGER I.M.; Spectral asymmetry and Riemannian geometry I. *Math. Proc. Cambridge Philos. Soc.* 77 (1975), 43-69;

[2] BÄR CH.; Dependence of the Dirac spectrum on the spin structure. *Sém. Congr.* 4 (2000), 17-33;

[3] BROWN K.S.: *Cohomology of groups.* Springer, Berlin 1982.

[4] CHARLAP L.S.: *Bieberbach Groups and Flat Manifolds.* Springer-Verlag, 1986.

[5] DAHL M.; Dependence on the spin structure of the eta and Rokhlin invariants. *Topology Appl.* 118 (2002), 345-355;

[6] FRIEDRICH T.; Zur Abhängigkeit des Dirac-Operators von der Spin-Struktur. *Colloq. Math.* 47 (1984), 57-62.

[7] FRIEDRICH T.: *Dirac Operators in Riemannian Geometry.* AMS, Graduate Studies in Math., Vol. 25 Providence, Rhode Island, 2000.

[8] HILLER H., SAH C.H.; Holonomy of flat manifolds with $b_1 = 0$. *Q. J. Math.* 37 (1986), 177-187;

[9] HIRZEBRUCH F., *Topological methods in algebraic geometry,* Springer, Berlin 1966

[10] HITCHIN N.; Harmonic spinors. *Adv. Math.* 14 (1974), 1-55;

[11] MIATELLO R.J.; PODESTA R.A. The spectrum of twisted Dirac operators on compact flat manifolds - preprint 2003, arXive:math.DG/0312004

[12] PFÄFFLE F.; The Dirac spectrum of Bieberbach manifolds. *J. Geom. Phys.* 35 (4) 2000, 367-385;
[13] STONG R.E.: *Notes on cobordism theory*, Princeton University Press, Princeton 1968.

[14] WHITTAKER E.T., WATSON G.N.: *A Course in Modern Analysis* fourth edition, Cambridge University Press, London, 1963

[15] WOLF J.A.: *Spaces of constant curvature*. McGraw-Hill, 1967.

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