Letter

Mid-range order in trapped quasi-condensates of bosonic atoms

V I Yukalov and E P Yukalova

1 Bogolubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia
2 Instituto de Fisica de São Carlos, Universidade de São Paulo, CP 369, São Carlos 13560-970, São Paulo, Brazil
3 Laboratory of Information Technologies, Joint Institute for Nuclear Research, Dubna 141980, Russia

E-mail: yukalov@theor.jinr.ru and yukalova@theor.jinr.ru

Received 4 March 2019
Accepted for publication 7 March 2019
Published 3 May 2019

Abstract
Finite Bose systems cannot display a genuine Bose–Einstein condensate with infinite long-range order. But, if the number of trapped atoms is sufficiently large, a kind of Bose–Einstein condensation does occur, with the properties of the arising quasi-condensate being very close to the genuine condensate. Although the quasi-condensate does not enjoy long-range order, it has mid-range order. This paper shows that the level of mid-range order in finite Bose systems can be characterized by order indices of density matrices.

Keywords: finite Bose systems, quasi-condensate, mid-range order, order indices

(Some figures may appear in colour only in the online journal)

1. Introduction

Bose–Einstein condensation of trapped atoms has been a hot topic in recent years, intensively investigated both experimentally and theoretically (see, for example, review articles and books [1–19]). Trapped atoms represent finite systems. In finite systems, strictly speaking, there can be no phase transitions with the arising long-range order. Specifically, there can be no genuine phase transition of Bose–Einstein condensation in a finite trap. Nevertheless, when a finite system is sufficiently large, its properties can be so close to a bulk system that, with a good approximation, one can speak of phase transitions there. In that sense, one studies Bose–Einstein condensation in finite traps, and one calls the resulting quasi-condensate simply a Bose condensate.

However, it is interesting as to what extent the quasi-condensate in a finite system differs from the genuine condensate. More precisely, how would it be possible to distinguish infinite off-diagonal order, corresponding to the genuine condensate accompanied by spontaneous breaking of global gauge symmetry, from a quasi-condensate in a finite system? Is it possible to define a measure quantifying the level of a quasi-long-range order?

In the present paper, we show that the Bose quasi-condensate in a finite system is characterized by mid-range order that can be quantified by order indices.

2. Order indices

Order indices were introduced for density matrices in [20] and considered for several macroscopic systems [21–24]. The notion of order indices was generalized for arbitrary matrices and operators in [25].

Let an operator $\hat{A}$ acting on a Hilbert space $\mathcal{H}$ possess a finite norm $\|\hat{A}\|$ and a trace. The order index of the operator is defined as

$$\omega(\hat{A}) = \frac{\log \|\hat{A}\|}{\log |\text{Tr}_{\mathcal{H}}\hat{A}|}.$$  (1)

One says that an operator $\hat{A}_1$ is better ordered than $\hat{A}_2$ if and only if the order index of $\hat{A}_1$ is larger than that of $\hat{A}_2$. If their
order indices are equal, then one says that these operators are equally ordered. The convenient norm in definition (1) is the Hermitian operator norm

$$\| \hat{A} \| = \sup_\varphi \left| \frac{\hat{A} \varphi}{\varphi} \right| = \sup_\varphi \left| \frac{\langle \hat{A} \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \right|^{1/2},$$

where \( \varphi \in \mathcal{H} \) is not zero. If the operator is Hermitian and \( \{ \varphi_k \} \) is an orthonormal basis in \( \mathcal{H} \), then the Hermitian norm reduces to

$$\| \hat{A} \| = \sup_\varphi \left| \frac{\langle \varphi, \hat{A} \varphi \rangle}{\varphi} \right| = \sup_{\varphi_k} (\varphi_k, \hat{A} \varphi_k) \quad (\hat{A} = \hat{A}^\dagger) .$$

For a semi-positive operator \( \hat{A} \geq 0 \) the norm is not larger than the trace, because of which

$$\omega(\hat{A}) \leq 1 \quad (\hat{A} \geq 0) . \quad (2)$$

Order indices can be introduced for generalized density matrices as follows. Let \( x \) be a set of physical variables and \( \mathcal{A} = \{ \hat{A}(x) \} \) be the algebra of local observables acting on a Fock space \( \mathcal{F} \), and let \( \mathcal{A}_\varphi = \{ \hat{\varphi}(x), \hat{\varphi}^\dagger(x) \} \) be the algebra of field operators on \( \mathcal{F} \). The union \( \mathcal{A}_{\text{ext}} \equiv \mathcal{A} \bigoplus \mathcal{A}_\varphi \) is called the extended local algebra.

For any representative \( \hat{A}(x) \) of the extended local algebra \( \mathcal{A}_{\text{ext}} \), it is possible to define the averages

$$D_A(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = \text{Tr}_\mathcal{F} A(x_1)A(x_2) \ldots A(x_n) \hat{\rho} A^\dagger(y_n)A^\dagger(y_{n-1}) \ldots A^\dagger(y_1),$$

(3)

where \( \hat{\rho} \) is a statistical operator. These averages can be treated as matrix elements of the matrix

$$\hat{D}_A^\rho \equiv \left[ D_A(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \right] ,$$

(4)

which can be called a generalized density matrix.

It is also possible to consider the functions \( \varphi(x_1, x_2, \ldots, x_n) \) of a Hilbert space \( \mathcal{H}_n \) as columns

$$\varphi_n \equiv \left[ \varphi(x_1, x_2, \ldots, x_n) \right].$$

(5)

Then the norm of matrix (4) can be defined as

$$\| \hat{D}_A^\rho \| = \sup_{\varphi_n} \left( \frac{\varphi_n, \hat{D}_A^\rho \varphi_n}{\varphi_n, \varphi_n} \right),$$

(6)

where the scalar product is given by the definition

$$\langle \varphi_n, \varphi_m \rangle = \int \varphi^* (x_1, x_2, \ldots, x_n) \varphi_m^\dagger(x_1, x_2, \ldots, x_n) \prod_{i=1}^n dx_i .$$

Specifically, the trace of the matrix is

$$\text{Tr}\hat{D}_A^\rho = \int D_A(x_1, x_2, \ldots, x_n, x_1, x_2, \ldots, x_n) \prod_{i=1}^n dx_i .$$

(7)

The order index of the generalized density matrix is

$$\omega(\hat{D}_A^\rho) \equiv \frac{\log \| \hat{D}_A^\rho \|}{\log \| \text{Tr}\hat{D}_A^\rho \|} .$$

(8)

A particular case of generalized density matrices is the reduced density matrix [26]

$$\hat{\rho}_n = \hat{D}_n^\rho = \left[ \rho(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) \right] ,$$

(9)

with the matrix elements

$$\rho(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) \equiv \text{Tr}_\mathcal{F} \hat{\varphi}(x_1)\hat{\varphi}(x_2) \ldots \hat{\varphi}(x_n) \hat{\varphi}^\dagger(y_n) \hat{\varphi}^\dagger(y_{n-1}) \ldots \hat{\varphi}^\dagger(y_1) .$$

(10)

The related order index of a density matrix (9) is

$$\omega(\hat{\rho}_n) = \frac{\log \| \hat{\rho}_n \|}{\log \| \text{Tr}\hat{\rho}_n \|} .$$

(11)

If \( \varphi_{nk} \) is an eigenfunction, labeled by a multi-index \( k \), of the density matrix (9), then its eigenvalues are

$$N_{nk} \equiv (\varphi_{nk}, \hat{\rho}_n \varphi_{nk})$$

(12)

and its Hermitian norm is

$$\| \hat{\rho}_n \| = \sup_k N_{nk} .$$

(13)

Because of the trace

$$\text{Tr}\hat{\rho}_n = \frac{N!}{(N-n)!} ,$$

the order index (11) can be represented in the form

$$\omega(\hat{\rho}_n) = \frac{\log \sup_k N_{nk}}{\log[N!/(N-n)!]} .$$

(14)

The norms of reduced density matrices satisfy [26] the inequalities

$$\| \hat{\rho}_n \| \leq (b_n N)^n$$

(15)

for Bose particles and

$$\| \hat{\rho}_{2n} \| \leq (c_{2n} N)^n, \quad \| \hat{\rho}_{2n+1} \| \leq (c_{2n+1} N)^n$$

(16)

for Fermi particles, where \( b_n \) and \( c_n \) are finite numbers. Therefore, for the order indices of density matrices, under large \( N \gg 1 \), we have

$$\omega(\hat{\rho}_n) \leq 1$$

(17)

for bosons and

$$\omega(\hat{\rho}_{2n}) \leq \frac{1}{2}, \quad \omega(\hat{\rho}_{2n+1}) \leq \frac{n}{2n+1}$$

(18)

for fermions.

3. Dilute gas

As an example of a concrete finite system, let us consider a dilute Bose gas with local interactions

$$\Phi(r) = \Phi_0 \delta(r), \quad \Phi_0 \equiv \frac{4\pi a_s}{m} ,$$

(19)

in which \( m \) is atomic mass and \( a_s \), scattering length. Here and in what follows, the Planck and Boltzmann constants are set to one, \( \hbar = 1 \) and \( k_B = 1 \). The energy Hamiltonian reads as
\[ \hat{H} = \int \hat{\psi}^\dagger(r) \left( -\frac{\nabla^2}{2m} \right) \hat{\psi}(r) \, dr + \frac{1}{2} \phi_0 \int \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(\vec{r}) \hat{\psi}(\vec{r}) \hat{\psi}(r) \, d\vec{r}, \]  

(18)

where \( \hat{\psi}(r) \) is a Bose field operator, generally depending on time \( t \), which is not shown for simplicity of notation.

Below we employ the self-consistent approach, reviewed in [16–18, 27], guaranteeing the correct description of Bose-condensed systems. This approach possesses several unique features: (i) it satisfies all conservation laws; (ii) it gives a gapless spectrum; (iii) it describes Bose–Einstein condensation as a second order phase transition; (iv) it provides for the condensate fraction, as a function of interaction strength, good numerical agreement with Monte Carlo simulations, both for uniform as well as for trapped systems; (v) it leads to the behavior of the ground state energy, under varying interaction strength, which, at weak interactions, yields exactly the Lee–Huang–Yang formula [28–30] and, at strong interactions, it is close to the results of Monte Carlo calculations; (vi) it explains the effect of local condensate depletion at the trap center under strong interactions [31], agreeing well with Monte Carlo simulations.

We start with the Bogolubov shift [32–34]

\[ \hat{\psi}(r) = \eta(r) + \psi_1(r) \]  

(19)

separating the condensate (quasi-condensate) function \( \eta \) from the field operator of uncondensed atoms \( \psi_1 \), with these quantities being mutually orthogonal,

\[ \int \eta^*(r) \psi_1(r) \, dr = 0. \]  

(20)

The condensate function plays the role of an order parameter, so that

\[ \eta(r) = \langle \hat{\psi}(r) \rangle, \quad \langle \psi_1(r) \rangle = 0. \]  

(21)

The total number of atoms \( N = N_0 + N_1 \) is formed by the number of condensed atoms

\[ N_0 = \int |\eta(r)|^2 \, dr \]  

(22)

and the number of uncondensed atoms

\[ N_1 = \langle \hat{N}_1 \rangle, \quad \hat{N}_1 = \int \psi_1^\dagger(r) \psi_1(r) \, dr. \]  

(23)

The grand Hamiltonian, taking into account conditions (21), (22), and (23), reads as

\[ H = \hat{H} - \mu_0 N_0 - \mu_1 \hat{N}_1 - \hat{\Lambda} , \]  

(24)

where

\[ \hat{\Lambda} = \int \left[ \lambda(r) \psi_1^\dagger(r) + \lambda^*(r) \psi_1(r) \right] \, dr. \]

The quantities \( \mu_0, \mu_1, \) and \( \lambda(r) \) are the Lagrange multipliers guaranteeing the validity of these conditions.

Equations of motion, equivalent to the Heisenberg equations [17], are the equation for the condensate fraction

\[ i \frac{\partial}{\partial t} \eta(r,t) = \left\langle \frac{\delta H}{\delta \eta^*(r,t)} \right\rangle \]  

(25)

and the equation for the field operator of uncondensed atoms

\[ i \frac{\partial}{\partial t} \psi_1(r,t) = \frac{\delta H}{\delta \psi_1^\dagger(r,t)}. \]  

(26)

The first-order density matrix is

\[ \hat{\rho}_1 = \left\{ \rho(r,r') \right\}, \quad \rho(r,r') = \langle \hat{\psi}_1^\dagger(r') \hat{\psi}_1(r) \rangle. \]  

(27)

With the Bogolubov shift (19), we have

\[ \rho(r,r') = \eta^*(r') \eta(r) + \langle \hat{\psi}_1^\dagger(r') \hat{\psi}_1(r) \rangle. \]  

(28)

The eigenvalues of the density matrix are

\[ N_{1k} = \int \varphi_k^\dagger(r) \rho(r,r') \varphi_k(r') \, dr \, dr', \]  

(29)

provided that \( \varphi_k \) are the eigenfunctions. Using (28), the eigenvalues can be written as the sum

\[ N_{1k} = N_k + n_k , \]  

(30)

in which

\[ N_k \equiv \left| \int \eta^*(r) \varphi_k(r) \, dr \right|^2 \]  

(31)

and

\[ n_k = \langle a_k^\dagger a_k \rangle, \quad a_k \equiv \int \varphi_k^\dagger(r) \psi_1(r) \, dr. \]  

(32)

In what follows, we shall study the order index of the single-particle density matrix

\[ \omega(\hat{\rho}_1) = \frac{\log \left| \hat{\rho}_1 \right|}{\log N} , \]  

(33)

with the norm of matrix (27)

\[ \left| \left| \hat{\rho}_1 \right| \right| = \sup_k N_{1k} = \sup_k (N_k + n_k) . \]  

(34)

### 4. Box-shaped trap

We consider a trap having the shape of a box of volume \( V = L^3 = Na^3 \). A Bose–Einstein condensate (quasi-condensate) in such a box trap has recently been observed [35]. As usual, the box is assumed to be periodically continued.

The eigenfunctions of the first-order density matrix are plane waves

\[ \varphi_k(r) = \frac{1}{\sqrt{V}} \, e^{ik \cdot r}. \]  

(36)

The condensate function becomes a constant

\[ \eta(r) = \sqrt{\frac{N_0}{V}}. \]  

(37)

The density-matrix eigenvalues are

\[ N_{1k} = N_0 \delta_{k0} + n_k , \]  

(38)

which defines the norm

\[ \left| \left| \hat{\rho}_1 \right| \right| = \sup \left\{ N_0, \sup_k n_k \right\} . \]  

(39)
The average atomic density

$$\rho \equiv \frac{N}{V} = \rho_0 + \rho_1$$  \hspace{1cm} (38)

is the sum of the condensate density $\rho_0$ and the density of uncondensed atoms $\rho_1$,

$$\rho_0 \equiv \frac{N_0}{V}, \quad \rho_1 \equiv \frac{N_1}{V} = \frac{1}{V} \sum_k n_k.$$  \hspace{1cm} (39)

Employing the Hartree–Fock–Bogolubov decoupling, we find [16–18] the distribution of uncondensed atoms

$$n_k = \frac{\omega_k}{2\varepsilon_k} \coth \left( \frac{\varepsilon_k}{2T} \right) - \frac{1}{2},$$  \hspace{1cm} (40)

where we use the notation

$$\omega_k \equiv mc^2 + \frac{k^2}{2m},$$  \hspace{1cm} (41)

where $\varepsilon_k$ is the spectrum of collective excitations

$$\varepsilon_k = \sqrt{(ck)^2 + \left( \frac{k^2}{2m} \right)^2},$$  \hspace{1cm} (42)

and the sound velocity $c$ satisfies the equation

$$mc^2 = \Phi_0(\rho_0 + \sigma_1).$$  \hspace{1cm} (43)

Here, the anomalous average is

$$\sigma_1 = \frac{1}{V} \sum_k \sigma_k, \quad \sigma_k = -mc^2 2\varepsilon_k \coth \left( \frac{\varepsilon_k}{2T} \right).$$  \hspace{1cm} (44)

Considering zero temperature, we get

$$n_k = \frac{\omega_k - \varepsilon_k}{2\varepsilon_k}, \quad \sigma_k = -mc^2 \varepsilon_k \left( T = 0 \right),$$  \hspace{1cm} (45)

the density of uncondensed atoms

$$\rho_1 = \frac{(mc)^3}{3\pi^2} \left( T = 0 \right),$$  \hspace{1cm} (46)

and the anomalous average

$$\sigma_1 = -mc^2 \int \frac{1}{2\varepsilon_k} \frac{dk}{(2\pi)^3} \left( T = 0 \right).$$  \hspace{1cm} (47)

The anomalous average (47) diverges and requires a regularization. This can be done by resorting to the dimensional regularization that provides asymptotically exact results at low density and weak interactions, and then accomplishing analytic continuation extending the results to finite density and interaction strength [36].

We notice that at asymptotically weak interaction, equation (43) leads to the Bogolubov sound velocity

$$c \to c_B \equiv \sqrt{\frac{\rho}{m} \Phi_0} \quad (\rho \Phi_0 \to 0).$$  \hspace{1cm} (48)

Because of this, at small values of $\rho \Phi_0$, the anomalous average (44) can be represented as

$$\sigma_1 \simeq -mc^2 \int \frac{1}{2\varepsilon_k} \frac{dk}{(2\pi)^3} \quad (\rho \Phi_0 \to 0).$$  \hspace{1cm} (49)

Using the dimensional regularization [2, 18, 37] yields

$$\int \frac{1}{2\varepsilon_k} \frac{dk}{(2\pi)^3} = -\frac{m^2 c^{(n)}}{\pi^2} \quad (\rho \Phi_0 \to 0),$$  \hspace{1cm} (50)

where $c^{(n)}$ is a weak-interaction approximation of sound velocity. The latter, together with (49), gives the corresponding approximation for the anomalous average

$$\sigma_1^{(n)} = -\frac{m^2 c_B}{\pi^2} c^{(n)}.$$  \hspace{1cm} (51)

To analytically continue the sound velocity $c^{(n)}$ to finite values of $\rho \Phi_0$, we employ equation (43) in the form

$$c^{(n+1)} = \left[ \frac{\Phi_0}{m} \left( \rho_0 + \sigma_1^{(n)} \right) \right]^{1/2}.$$  \hspace{1cm} (52)

Combining (51) and (52) gives the iterative equation

$$\sigma_1^{(n+1)} = \frac{(mc)^3}{\pi^2 \sqrt{\rho}} \sqrt{\rho_0 + \sigma_1^{(n)}}.$$  \hspace{1cm} (53)

Defining dimensionless fractions

$$n_0 \equiv \frac{\rho_0}{\rho}, \quad n_1 \equiv \frac{\rho_1}{\rho}, \quad \sigma \equiv \frac{\sigma_1}{\rho}$$  \hspace{1cm} (54)

and dimensionless sound velocities

$$s_0 \equiv \frac{mc}{\rho^{1/3}}, \quad s_B \equiv \frac{mc_B}{\rho^{1/3}}$$  \hspace{1cm} (55)

reduces (53) to the dimensionless equation

$$\sigma^{(n+1)} = \frac{s_B}{\pi^2} \sqrt{n_0 + \sigma^{(n)}}.$$  \hspace{1cm} (56)

It is reasonable to start the iterative procedure from the Bogolubov approximation, that is asymptotically exact at low $\rho \Phi_0 \to 0$, where $\sigma^{(0)} = 0$. In the second order, we obtain

$$\sigma = \frac{s_B}{\pi^2} \left( n_0 + \frac{s_B}{\pi^2} \sqrt{n_0} \right)^{1/2}.$$  \hspace{1cm} (57)

This anomalous average will be used below.

5. Order-index behaviour

Now we shall calculate the order index (33) for a finite box, where we take the natural logarithms. For a periodically continued finite system, there exists the minimal wave vector

$$k_{\text{min}} = \frac{2\pi}{L} = 2\pi \left( \frac{\rho}{N} \right)^{1/3}.$$  \hspace{1cm} (58)

The maximal value of $n_k$ occurs at the minimal wave vector,

$$\sup_k n_k = \sup_k \omega_k - \varepsilon_k - \frac{mc}{2k_{\text{min}}},$$

which yields

$$\sup_k n_k = \frac{s}{4\pi} N^{1/3}.$$  \hspace{1cm} (59)

Thus the norm of the first-order density matrix is
Figure 1. Condensate (quasi-condensate) fraction \( n_0 \) (dash-dotted line), anomalous average \( \sigma \) (dashed line), and sound velocity \( s \) (solid line) as functions of the gas parameter \( \gamma \).

\[
\| \hat{n} \| = \sup \left\{ n_0 N, \frac{s}{4\pi N^{1/3}} \right\}. \tag{60}
\]

In that way, we need to study the behavior of the order index

\[
\omega(\hat{n}) = \sup \left\{ 1 + \frac{\ln n_0}{\ln N} \cdot \frac{1}{3} + \frac{\ln(s/4\pi)}{\ln N} \right\}. \tag{61}
\]

It is convenient to introduce the dimensionless gas parameter

\[
\gamma \equiv \alpha \rho^{1/3} \tag{62}
\]

characterizing the interaction strength. Then, the Bogolubov sound velocity takes the form \( \omega = \sqrt{4\pi \gamma} \), and equation (43) becomes

\[
s^2 = 4\pi\gamma(n_0 + \sigma). \tag{63}
\]

The condensate fraction is

\[
n_0 = 1 - \frac{s^3}{3\pi^2}, \tag{64}
\]

and the anomalous average (57) reduces to

\[
\sigma = \frac{8}{\sqrt{\pi}} \gamma^{3/2} \left[ n_0 + \frac{8}{\sqrt{\pi}} \gamma^{3/2} \sqrt{n_0} \right]^{1/2}. \tag{65}
\]

At small gas parameter \( \gamma \to 0 \), the condensate fraction can be expanded as

\[
n_0 \approx 1 - \frac{8}{3\sqrt{\pi}} \gamma^{3/2} - \frac{64}{3\pi} \gamma^3 - \frac{256}{9\pi^{3/2}} \gamma^{9/2} + \frac{61952}{81\pi^2} \gamma^6. \tag{66}
\]

The anomalous average has the expansion

\[
\sigma \approx \frac{8}{\sqrt{\pi}} \gamma^{3/2} + \frac{64}{3\pi} \gamma^3 - \frac{1408}{9\pi^{3/2}} \gamma^{9/2} - \frac{1792}{27\pi^2} \gamma^6, \tag{67}
\]

while the sound velocity behaves as

\[
s \approx \frac{16}{3} \gamma^2 - \frac{64}{9\sqrt{\pi}} \gamma^{7/2} - \frac{4480}{27\pi} \gamma^5. \tag{68}
\]

In numerical form, the expansions are:

\[
n_0 \approx 1 - 1.50451\gamma^{3/2} - 6.79061\gamma^3 - 5.10826\gamma^{9/2} + 77.4944\gamma^6,
\]

\[
\sigma \approx 4.51352\gamma^{3/2} + 6.79061\gamma^3 - 28.0954\gamma^{9/2} - 6.72472\gamma^6,
\]

\[
s \approx 3.54491\gamma^{1/2} + 5.33333\gamma^2 - 4.01201\gamma^{7/2} - 52.8159\gamma^5.
\]

For the order index, we find

\[
\omega(\hat{n}) \approx 1 - \frac{8}{3\sqrt{\pi} \ln N} \gamma^{3/2} - \frac{224}{9\pi \ln N} \gamma^3 \quad (\gamma \to 0). \tag{69}
\]

At strong interaction, when \( \gamma \to \infty \), the condensate fraction reads as

\[
n_0 \approx 0.397978 \frac{10^{-4}}{\gamma^{13}} - 0.111251 \frac{10^{-6}}{\gamma^{21}}. \tag{70}
\]

the anomalous average is

\[
\sigma \approx 0.761618 \frac{1}{\gamma} - 0.397978 \frac{10^{-4}}{\gamma^{13}} - 0.202072 \frac{10^{-4}}{\gamma^{14}} + 0.111251 \frac{10^{-6}}{\gamma^{21}}, \tag{71}
\]

and the sound velocity has the expansion

\[
s \approx 3.09367 - 0.410404 \frac{10^{-4}}{\gamma^{13}} + 0.114725 \frac{10^{-6}}{\gamma^{21}}. \tag{72}
\]

The order index behaves as

\[
\omega(\hat{n}) \approx 1 - \frac{1.40167}{\ln N} - \frac{0.132659}{\ln N} \frac{10^{-4}}{\gamma^{13}} \quad (\gamma \to \infty). \tag{73}
\]

Figure 2. Order index \( \omega(\hat{n}) \) as a function of the gas parameter \( \gamma \) for different numbers of trapped atoms: \( N = 10 \) (solid line), \( N = 10^3 \) (dashed line), and \( N = 10^6 \) (dash-dotted line).
In this way, the limits of large $N$ and large $\gamma$ are not com-
mutative. The limit of large $N$, for any finite $\gamma$, gives
\[
\lim_{N \to \infty} \omega(\hat{\rho}_1) = 1 \quad (\gamma < \infty),
\]
which defines the genuine Bose condensate in the thermody-
namic limit. While, if we first take the limit of large $\gamma$, under
finite $N$, and after this, the limit of large $N$, we get
\[
\lim_{N \to \infty} \lim_{\gamma \to \infty} \omega(\hat{\rho}_1) = \frac{1}{3}.
\]
For finite $N$ and $\gamma$, the order index is smaller than one, which
implies that we do not have a genuine condensate, with a long-
range off-diagonal order, but a quasi-condensate possessing
only mid-range order.

The influence of varying the interaction strength $\gamma$ and
the number of trapped atoms $N$ on the system characteristics is
illustrated in figures 1–3. Figure 1 describes the dependence
of the quasi-condensate fraction $n_0$, anomalous average $\sigma$, and
sound velocity $s$ on the strength of the gas parameter $\gamma$.
Figure 2 shows that increasing $\gamma$ diminishes the order index,
which is quite natural, since strong interactions are known to
deplete the condensate, and figure 3 demonstrates that increas-
ing the number of trapped atoms leads to larger values of the
order index.

In conclusion, we have shown that in finite quantum sys-
tems, where there is no long-range order, there can exist
mid-range order, which can be quantified by order indices of
density matrices. Finite systems of trapped Bose atoms can
exhibit a quasi-condensate possessing mid-range order, with
an order index smaller than one. The order index of the first-
order density matrix shows the relation between the norm of
the matrix and the number of atoms in the system,
\[
\| \hat{\rho}_1 \| = N^{\omega(\hat{\rho}_1)}.
\]

For a large number of atoms in a trap, the order index can be
so close to unity that the quasi-condensate becomes almost
indistinguishable from the genuine condensate. But for a not
so large number of atoms, or very strong interactions, the
order index can essentially deviate from unity.

References

[1] Courtelle P W, Bagnato V S and Yukalov V I 2001 Laser
Phys. Lett. 11 659
[2] Andersen J O 2004 Rev. Mod. Phys. 76 599
[3] Yukalov V I 2004 Laser Phys. Lett. 1 435
[4] Bongs K and Sengstock K 2004 Rep. Prog. Phys. 67 907
[5] Yukalov V I and Girardeau M D 2005 Laser Phys. Lett. 2 375
[6] Lieb E H, Seiringer R, Solovej J P and Yngvason J 2005 The
Mathematics of the Bose Gas and its Condensation
(Basel: Birkhauser)
[7] Posazhennikova A 2006 Rev. Mod. Phys. 78 1111
[8] Morsch O and Oberthaler M 2006 Rev. Mod. Phys. 78 179
[9] Yukalov V I 2007 Laser Phys. Lett. 4 632
[10] Letokhov V 2007 Laser Control of Atoms and Molecules
(Oxford: Oxford University Press)
[11] Moseley C, Fialko O and Ziegler K 2008 Ann. Physik 17 561
[12] Bloch I, Dalibard J and Zwerger W 2008 Rev. Mod. Phys.
80 885
[13] Proukakis N P and Jackson B 2008 J. Phys. B: At. Mol. Opt.
Phys. 41 203002
[14] Yurovsky V A, Olshanii M and Weiss D S 2008 Adv. At. Mol.
Opt. Phys. 55 6
[15] Pethick C J and Smith H 2008 Bose–Einstein Condensation in
Dilute Gases (Cambridge: Cambridge University Press)
[16] Yukalov V I 2009 Laser Phys. 19 1
[17] Yukalov V I 2011 Phys. Part. Nucl. 42 460
[18] Yukalov V I 2016 Laser Phys. 26 062001
[19] Yukalov V I and Yukalova E P 2017 Laser Phys. Lett.
14 073001
[20] Coleman A J and Yukalov V I 1991 Mod. Phys. Lett. B 5 1679
[21] Coleman A J and Yukalov V I 1992 Nuovo Cimento B 107 535
[22] Coleman A J and Yukalov V I 1993 Nuovo Cimento B
108 1377
[23] Coleman A J, Yukalova E P and Yukalov V I 1995 Physica C
243 76
[24] Coleman A J and Yukalov V I 1996 Int. J. Mod. Phys. B
10 3505
[25] Yukalov V I 2002 Physica A 310 413
[26] Coleman A J and Yukalov V I 2000 Reduced Density Matrices
(Berlin: Springer)
[27] Yukalov V I 2018 Laser Phys. 28 053001
[28] Lee T D and Yang C N 1957 Phys. Rev. 105 1119
[29] Lee T D, Huang K and Yang C N 1957 Phys. Rev. 106 1135
[30] Lee T D and Yang C N 1958 Phys. Rev. 112 1419
[31] Yukalov V I and Yukalova E P 2018 J. Phys. B: At. Mol. Opt.
Phys. 51 085301
[32] Bogolubov N N 1967 Lectures on Quantum Statistics vol 1
(New York: Gordon and Breach)
[33] Bogolubov N N 1970 Lectures on Quantum Statistics vol 2
(New York: Gordon and Breach)
[34] Bogolubov N N 2015 Quantum Statistical Mechanics
(Singapore: World Scientific)
[35] Navon N, Gaunt A L, Smith R P and Hadzibabic Z 2016
Nature 532 7
[36] Yukalov V I and Yukalova E P 2014 Phys. Rev. A 90 013627
[37] Kleinert H 2004 Path Integrals (Singapore: World Scientific)

Figure 3. Order index $\omega(\hat{\rho}_1)$ as a function of $\ln N$ for different
values of the gas parameter: $\gamma = 0.1$ (solid line), $\gamma = 0.5$ (dashed
line), $\gamma = 1$ (dash-dotted line), and $\gamma = 2$ (dotted line).