INTRINSICALLY LIPSCHITZ SECTIONS AND APPLICATIONS TO METRIC GROUPS

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ABSTRACT. We introduce a notion of intrinsically Lipschitz graphs in the context of metric spaces. This is a broad generalization of what in Carnot groups has been considered by Franchi, Serapioni, and Serra Cassano, and later by many others. We proceed by focusing our attention on the graphs as subsets of a metric space given by the image of a section of a quotient map and we require an intrinsically Lipschitz condition. We shall not have any function on a topological product, not we shall consider a metric on the base of the quotient map. Our results are: an Ascoli-Arzelà compactness theorem, an Ahlfors regularity theorem, and some extension theorems for partially defined intrinsically Lipschitz sections. Known results by Franchi, Serapioni, and Serra Cassano, and by Vittone will be our corollaries.

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1. Introduction

Nowadays, in the setting of subRiemannian Carnot groups, there is a rich theory of geometric analysis: from geometric measure theory to partial differential equations [BLU07, CDPT07, Cor20, CM20, Mag13, Pan89, NY18, Rig20, SC16]. Objects that play the role of Lipschitz submanifolds are the intrinsically Lipschitz graphs, originally introduced by Franci, Serapioni, and Serra Cassano, in order to have an adapted notion of rectifiability of (boundaries of) finite-perimeter sets [ASCV06, AK00, AKLD09, ADDDL20, AM21, AS09, BCSC15, BSC10a, BSC10b, CFO19, CM06, DD20a, DD20b, FMS14, FSSC06, FSSC07, FSSC11, JNGV20, MV12, NSC19].

The purpose of the present article is to axiomatize the notion of intrinsically Lipschitz graphs to the setting of metric spaces. We shall prove that some of the geometric results from the last decade are valid in a broad generality. Our setting is the following. We have a metric space $\mathcal{X}$, a topological space $\mathcal{Y}$, and a quotient map $\pi : \mathcal{X} \to \mathcal{Y}$, meaning continuous, open, and surjective. The standard example for us is when $\mathcal{X}$ is a metric group $\mathcal{G}$ (meaning a topological group $\mathcal{G}$ equipped with a left-invariant distance that induces the topology), for example a subRiemannian Carnot group (see [LD17]), and $\mathcal{Y}$ is the space of left cosets $\mathcal{G}/H$, where $H < \mathcal{G}$ is a closed subgroup and $\pi : \mathcal{G} \to \mathcal{G}/H$ is the projection modulo $H$, $g \mapsto gH$. The inexperienced reader may find a more basic example in Example 2.5.

The objects of study in this paper are the following type of maps, which we shall prove generalize the ones in [FSSC01, FSSC03b, FSSC03a], see also [SC16, FS16].

**Definition 1.1.** Given a quotient map $\pi : \mathcal{X} \to \mathcal{Y}$ between a metric space $\mathcal{X}$ and a topological space $\mathcal{Y}$, we say that a map $\varphi : \mathcal{Y} \to \mathcal{X}$ is an intrinsically Lipschitz section of $\pi$ with constant $L$, with $L \in [1, \infty)$, if

1. $\pi \circ \varphi = \text{id}_{\mathcal{Y}}$, and
2. $d(\varphi(y_1), \varphi(y_2)) \leq L d(\varphi(y_1), \pi^{-1}(y_2))$, for all $y_1, y_2 \in \mathcal{Y}$.

Here $d$ denotes both the distance between points on $\mathcal{X}$, and, as usual, for a subset $A \subset \mathcal{X}$ and a point $x \in \mathcal{X}$, we have $d(x, A) := \inf \{d(x, a) : a \in A\}$.

We shall also briefly say that a map as in Definition 1.1 is an intrinsically $L$-Lipschitz section or that it is an intrinsically Lipschitz section if there is no need to specify $\pi$, nor $L$. In Section 2.2, we will give other characterizations of intrinsically Lipschitz sections, also in terms of the fact that the images have trivial intersection with some particular subsets of $\mathcal{X}$, which unlike in the Carnot setting don’t have anymore a structure of cones, because there is no dilation/homogeneous structure assumed on $\mathcal{X}$.

We shall call the image $\varphi(\mathcal{Y})$ of some intrinsically Lipschitz section $\varphi : \mathcal{Y} \to \mathcal{X}$ an intrinsically Lipschitz graph. We stress that, even if the set $\varphi(\mathcal{Y})$ is parametrized by $\mathcal{Y}$, the geometric regularity of $\varphi(\mathcal{Y})$ depends only on the ambient distance in $\mathcal{X}$, and not on the one of $\mathcal{Y}$, which a priori we haven’t metrized. In particular, the set $\varphi(\mathcal{Y})$ may not be biLipschitz equivalent to $\mathcal{Y}$. In some situations we might have a natural way of metrizing $\mathcal{Y}$, but it might not be the case that $\pi$ becomes a Lipschitz quotient, e.g. a submetry (see Section 2.1 for these definitions). In the case $\pi$ is a Lipschitz quotient, the results trivialize, since in this case being intrinsically Lipschitz is equivalent to being a biLipschitz embedding, see Proposition 2.4. In the context of groups, one has a Lipschitz quotient when one takes a normal subgroup $N \triangleleft \mathcal{G}$ of a metric Lie group $\mathcal{G}$ and $\pi : \mathcal{G} \to \mathcal{G}/N$.
In case $G$ is a Carnot group that can be written as product of two complementary homogeneous subgroups $G = H_1 \cdot H_2$ then our notion of intrinsically Lipschitz graph coincides with the one given by Franchi-Serapioni-SerraCassano, see Section 6.6. However, we stress that in the case of an arbitrary decomposition $G = H_1 \cdot H_2$, with $H_1$ and $H_2$ not necessarily homogeneous, one could naturally identify $G/H$ with $H_1$ but, first, if $H_2$ is not normal, then the cosets of $H_2$ are not parallel within $G$, and even if $H_2$ is normal, then the projection onto $H_1$ may not be a Lipschitz quotient. It is crucial to remark that in the case of Franchi, Serapioni, and Serra Cassano the homogeneity assumption will imply such a property, see Section 6.6. The aim of our work is to provide generalizations to the metric setting of results known in the case of homogeneous decomposition of groups.

The first series of our results is about the equicontinuity of intrinsically Lipschitz sections with uniform constant and consequently a compactness property à la Ascoli-Arzelá. Be aware that, in the literature on metric spaces, another term for boundedly compact is properly proper.

**Theorem 1.2** (Equicontinuity and Compactness Theorem). Let $\pi : X \to Y$ be a quotient map between a metric space $X$ and a topological space $Y$.

(i) Every intrinsically Lipschitz section of $\pi$ is continuous.

Next, assume in addition that closed balls in $X$ are compact (we say that $X$ is boundedly compact).

(ii) For all $K' \subset Y$ compact, $L \geq 1$, $K \subset X$ compact, and $y_0 \in Y$ the set

$$\{ \varphi_{|_{K'}} : K' \to X \mid \varphi : Y \to X \text{ intrinsically } L\text{-Lipschitz section of } \pi, \varphi(y_0) \in K \}$$

is equibounded, equicontinuous, and closed in the uniform-convergence topology.

(iii) For all $L \geq 1$, $K \subset X$ compact, and $y_0 \in Y$ the set

$$\{ \varphi : Y \to X \mid \varphi \text{ intrinsically } L\text{-Lipschitz section of } \pi, \varphi(y_0) \in K \}$$

is compact with respect to the topology of uniform convergence on compact sets.

Without the assumption that $\pi$ is an open map, intrinsically Lipschitz sections may not be continuous. See Example 2.10 for some pathological intrinsically Lipschitz sections.

We stress, as similarly done before, that given two intrinsically Lipschitz sections $\varphi_1, \varphi_2 : Y \to X$, the two sets $\varphi_1(Y)$ and $\varphi_2(Y)$ may not be biLipschitz equivalent. However, following the influential paper [FS16], we prove that, in the presence of a nice measure on $Y$, then if $\varphi_1(Y)$ is an Ahlfors regular set then so is $\varphi_2(Y)$. The assumption of the existence of such a measure is necessary, see Example 4.1. More precisely, our next result is the following.

**Theorem 1.3** (Ahlfors regularity, after Franchi-Serapioni-SerraCassano). Let $\pi : X \to Y$ be a quotient map between a metric space $X$ and a topological space $Y$ such that there is a measure $\mu$ on $Y$ such that for every $r_0 > 0$ and every $x, x' \in X$ with $\pi(x) = \pi(x')$ there is $C > 0$ such that

$$\mu(\pi(B(x, r))) \leq C \mu(\pi(B(x', r))), \quad \forall r \in (0, r_0).$$

We also assume that there is an intrinsically Lipschitz section $\varphi : Y \to X$ of $\pi$ such that $\varphi(Y)$ is locally $Q$-Ahlfors regular with respect to the measure $\varphi_* \mu$, with $Q \in (0, \infty)$.

Then, for every intrinsically Lipschitz section $\psi : Y \to X$ of $\pi$, the set $\psi(Y)$ is locally $Q$-Ahlfors regular with respect to the measure $\psi_* \mu$. 

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Namely, in Theorem 1.3 the local Ahlfors $Q$-regularity of $\varphi(Y)$ means that the measure $\varphi_*\mu$ is such that for each point $x \in \varphi(Y)$ there exist $r_0 > 0$ and $C > 0$ so that
\begin{equation}
C^{-1}r^Q \leq \varphi_*\mu(B(x, r) \cap \varphi(Y)) \leq Cr^Q, \quad \text{for all } r \in (0, r_0). 
\end{equation}
The same inequality will hold for $\psi(Y)$ and $\psi_*\mu$ with a possibly different value of $C$. See Section 3.

In some settings one has the necessity of describing “regular” submanifolds as zero sets of distinguished functions, which is useful to possibly extend partially defined objects. Generalizing a result of Vittone [Vit20], we next show that this can also be done with intrinsically Lipschitz graphs, at least when we have a good control on fibers. In next result, we say that a metric-space-valued map $f$ on $X$ is $L$-biLipschitz on fibers (of $\pi$) if on each fiber of $\pi$ it restricts to an $L$-biLipschitz homeomorphism.

**Theorem 1.4** (Extensions as level sets, after Vittone). Let $\pi : X \to Y$ be a quotient map between a metric space $X$ and a topological space $Y$.

(i) If $Z$ is a metric space, $z_0 \in Z$ and $f : X \to Z$ is $L$-Lipschitz and $L$-biLipschitz on fibers, with $L \geq 1$, then there exists an intrinsically $(1 + L^2)$-Lipschitz section $\varphi : Y \to X$ of $\pi$ such that
\begin{equation}
\varphi(Y) = f^{-1}(z_0).
\end{equation}

(ii) Vice versa, assume that $X$ is geodesic and that there exist $k, L \geq 1$, $\rho : X \times X \to \mathbb{R}$ $k$-biLipschitz equivalent to the distance of $X$, and $\tau : X \to \mathbb{R}$ $k$-Lipschitz and $k$-biLipschitz on fibers such that
\begin{enumerate}
\item for all $\tau_0 \in \mathbb{R}$ the set $\tau^{-1}(\tau_0)$ is an intrinsically $k$-Lipschitz graph of a section $\varphi_{\tau_0} : Y \to X$;
\item for all $x_0 \in \tau^{-1}(\tau_0)$ the map $\delta_{\tau_0} : X \to \mathbb{R}, x \mapsto \delta_{\tau_0}(x) := \rho(x_0, \varphi_{\tau_0}(\pi(x)))$ is $k$-Lipschitz on the set $\{|\tau - \tau_0| \leq kL\delta_{\tau_0}\}$.
\end{enumerate}

Let $Y' \subset Y$ be a set. Then for every intrinsically $L$-Lipschitz section $\varphi : Y' \to \pi^{-1}(Y')$ of $\pi|_{\pi^{-1}(Y')} : \pi^{-1}(Y') \to Y'$, there exists a map $f : X \to \mathbb{R}$ that is $K$-Lipschitz and $K$-biLipschitz on fibers, with $K := 2k(Lk + 2)$, such that
\begin{equation}
\varphi(Y') \subseteq f^{-1}(0).
\end{equation}
In particular, each ‘partially defined’ intrinsically Lipschitz graph $\varphi(Y')$ is a subset of a ‘globally defined’ intrinsically Lipschitz graph $f^{-1}(0)$.

We shall next apply our study to the case of metric groups and more specifically to the case of Carnot groups. We shall then see how our theorems give known results as immediate consequences. Initially, we shall barely consider the case of a metric group $G$ and a closed subgroup $H$ of $G$. In such a way, we shall rephrase the notion of intrinsically Lipschitz section of the quotient map $\pi : G \to G/H$. To have further geometric properties of intrinsically Lipschitz sections in groups, we shall require a splitting $G = H_1 \cdot H_2$ with $H_1, H_2$ closed subgroups. As commonly done, writing $G = H_1 \cdot H_2$ means that $G = \{h_1h_2 : h_1 \in H_1, h_2 \in H_2\}$ and $H_1 \cap H_2 = \{1_G\}$. For example, if in addition $H_1$ is a normal subgroup, then $G$ has the structure of semidirect product $G = H_1 \rtimes H_2$. In the presence of a splitting $G = H_1 \cdot H_2$, we have the two naturally defined projection maps $\pi_{H_i} : G \to H_i$. We stress that such maps may not be Lipschitz, not even when one of the groups is normal. An important setting, in
which we have several equivalences, is when the map $\pi_{H_1}$ is Lipschitz at $1_G$, i.e.,

$$(7) \quad d(1_G, \pi_{H_1}(g)) \leq Kd(1_G, g), \quad \forall g \in G,$$

where, with $1_G$ we denote the identity element of the group $G$. See Section 6.2 for other equivalent conditions for the Lipschitz property at $1_G$, especially in the case when $H_1$ is normal. For example, for us an important equivalent property is that the inclusion $H_1 \hookrightarrow G$ is an intrinsically Lipschitz section for the projection $G \rightarrow G/H_2$.

When we have a splitting $G = H_1 \cdot H_2$ then the left-coset of $H_1$ are sections of the projection modulo $H_2$. In general, such sections may not be intrinsically Lipschitz (as we just said, they are if and only if $\pi_{H_1}$ is Lipschitz at $1_G$). We introduce a notion of being intrinsically Lipschitz with respect to these sections, see Definition 2.6. The two notions of intrinsically Lipschitz sections coincide under the assumption that the left cosets of $H_1$ are intrinsically Lipschitz (see Corollary 6.14). These types of facts hold in the general setting of metric spaces, as in the next result.

**Proposition 1.5.** Let $X$ be a metric space, $Y$ a topological space, $\pi : X \rightarrow Y$ a quotient map, and $L \geq 1$. Assume that every point $x \in X$ is contained in the image of an intrinsically $L$-Lipschitz section $\psi_x$ for $\pi$. Then for every section $\varphi : Y \rightarrow X$ of $\pi$ the following are equivalent:

1. for some $L_1 \geq 1$ and for all $x \in \varphi(Y)$ the section $\varphi$ is intrinsically $L_1$-Lipschitz with respect to $\psi_x$ at $x$ (see Definition 2.6);
2. the section $\varphi$ is intrinsically $L_2$-Lipschitz.

Next we make the link with the notion of intrinsically Lipschitz maps in the sense of Franchi, Serapioni, and Serra Cassano. Given a splitting $G = H_1 \cdot H_2$, for $\psi : H_1 \rightarrow H_2$ we set

$$\Gamma_\psi := \{n\psi(n) : n \in H_1\}.$$

We say that $\psi$ is an intrinsically Lipschitz map in the FSSC sense if exists $K > 0$ such that

$$(8) \quad d(1_G, \pi_{H_2}(x^{-1}x')) \leq Kd(1_G, \pi_{H_1}(x^{-1}x')), \quad \forall x, x' \in \Gamma_\psi.$$

This last definition has several equivalent expressions when $H_1$ is normal. We point out that if a metric Lie group is a semidirect product $G = N \rtimes H$, then on $G/N$ there is a natural metric that makes the quotient $\varphi : G \rightarrow G/N$ a submetry, see [LDR16, Corollary 2.11]. However, under the identification $G/N \simeq H$, this natural distance is biLipschitz equivalent to the one of $G$ restricted to $H$ exactly when the projection on $H$ is Lipschitz, see Proposition 6.10.

**Proposition 1.6.** Let $G = N \rtimes H$ be a metric group that is a semidirect product.

1. If $G$ is a Carnot group with $N$, $H$ homogeneous subgroups, then
   - (i) $\pi_H : G \rightarrow H$ is a Lipschitz homomorphism and
   - (ii) for all $g \in G$ the set $gN$ is an intrinsically Lipschitz graph.

2. $N$ is normal. If (1.6.i.a) or (1.6.i.b) holds then the following three properties hold:
   - (i) $\pi_{H_1}$ is Lipschitz at $1_G$.

   In general, if (1.6.i.a) or (1.6.i.b) holds then the following three properties hold:

   1. If $\psi : N \rightarrow H$ is intrinsically Lipschitz map in the FSSC sense, then $\varphi : G/H \rightarrow G$ defined as
      $$(9) \quad \varphi(gH) := \pi_N(g)\psi(\pi_N(g)), \quad \forall g \in G$$

      is an intrinsically Lipschitz section of the projection $\pi : G \rightarrow G/H$, with $\varphi(G/H) = \Gamma_\psi$. 5
Vice versa, if $\varphi : G/H \to G$ is an intrinsically Lipschitz section of $\pi : G \to G/H$, then the map $\psi : N \to H$ defined as
\begin{equation}
\psi(n) := n^{-1} \varphi(nH), \quad \forall n \in N
\end{equation}
is an intrinsically Lipschitz map in the FSSC sense, with $\varphi(G/H) = \Gamma_\psi$.

The rest of the paper is organized as follows. In Section 2 we discuss the definition of intrinsically Lipschitz sections, we show some basic properties like their continuity, we prove Proposition 1.5, and finally we show that in the case when the metric space $X$ is geodesic and the fibers of the projection $\pi$ are one-dimensional and continuously oriented, the infima of each family of intrinsically Lipschitz sections is so too (see Proposition 2.11). Section 3 contains the proof of Ascoli-Arzelà compactness theorem, Theorem 1.2. Section 4 is dedicated to Ahlfors regularity, i.e., the proof of Theorem 1.3. Section 5 contains the proof of the Extension Theorem (Theorem 1.4) using the equivalence between intrinsically Lipschitz sections and level sets of Lipschitz maps that are biLipschitz on fibers. Section 6 is specialized to the applications of this theory when the metric space $X$ is a Carnot group or, more generally, a metric group.

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2. INTRINSICALLY LIPSCHITZ SECTIONS

2.1. Preliminaries. In this paper $X$ will denote a metric space, whose distance will be denote arbitrarily by $d$, or $d_X$ if there might be confusion with other distances. Instead, the set $Y$ will sometimes be a topological space, and some other times will be a metric space with topology induced by the distance.

As common in topology, a map $\pi : X \to Y$ is called a quotient map if it is continuous, surjective and open. Distinguished examples of quotient maps are Lipschitz quotients and in particular submetries, whose definition now we recall. Such notions have been introduced in [BJL+99, VN88].

A map $\pi : X \to Y$ between metric spaces is said to be a Lipschitz quotient with constant $k$, with $k \geq 1$ (or briefly a $k$-Lipschitz quotient or Lipschitz quotient, if there is no need to specify $k$) if
\begin{equation}
B_{d_Y}(\pi(x), r/k) \subset \pi(B_{d_X}(x, r)) \subset B_{d_Y}(\pi(x), kr), \quad \forall x \in X, \forall r > 0.
\end{equation}
If $k = 1$, the map $\pi$ is called submetry and \((\text{II})\) simplifies as
\begin{equation}
\pi(B_{d_X}(x, r)) = B_{d_Y}(\pi(x), r), \quad \forall x \in X, \forall r > 0.
\end{equation}

We stress that being a Lipschitz quotient is more restrictive that being a quotient map that is Lipschitz. In fact, \((\text{III})\) also gives a co-Lipschitz condition. Hence, Lipschitz quotients are uniformly open. In next remark we show that every quotient map has some type of uniform openness.
Remark 2.1. Let $\pi : X \to Y$ be an open map, $K \subset X$ be a compact set and $y \in Y$. Then $\pi$ is uniformly open on $K \cap \pi^{-1}(y)$, in the sense that, for every $\varepsilon > 0$ there is a neighborhood $U_\varepsilon$ of $y$ such that

$$U_\varepsilon \subset \pi(B(x,\varepsilon)), \quad \forall x \in K \cap \pi^{-1}(y).$$

Indeed, since $\pi$ is open, for every $x \in \pi^{-1}(y)$ there is a neighborhood $U_{\varepsilon,x}$ of $y$ that is contained in $\pi(B(x,\frac{\varepsilon}{2}))$. Moreover, because $K$ is compact, we know that there is a finite $\frac{\varepsilon}{2}$-net $N \subset K \cap \pi^{-1}(y)$. Finally, if we put $U_{\varepsilon} := \bigcap_{x \in N} U_{\varepsilon,x}$, we have that for all $x \in K \cap \pi^{-1}(y)$ there is a point $\bar{x} \in N$ such that $d(x,\bar{x}) < \frac{\varepsilon}{2}$ and

$$U_{\varepsilon} \subset U_{\varepsilon,x} \subset \pi(B(\bar{x},\varepsilon/2)) \subset \pi(B(x,\varepsilon)),$$

as wished.

2.2. Equivalent definitions for intrinsically Lipschitz sections.

Definition 2.2 (Intrinsic Lipschitz section). Let $X = (X,d)$ be a metric space and let $Y$ be a topological space. We say that a map $\varphi : Y \to X$ is a section of a quotient map $\pi : X \to Y$ if

$$\pi \circ \varphi = \text{id}_Y.$$

Moreover, we say that $\varphi$ is an intrinsically Lipschitz section with constant $L$ if in addition

$$d(\varphi(y_1),\varphi(y_2)) \leq Ld(\varphi(y_1),\pi^{-1}(y_2)), \quad \text{for all } y_1, y_2 \in Y.$$

Necessarily, $L \geq 1$. Equivalently, we are requesting that that

$$d(x_1, x_2) \leq Ld(x_1, \pi^{-1}(x_2)), \quad \text{for all } x_1, x_2 \in \varphi(Y).$$

We further rephrase the definition as saying that $\varphi(Y)$, which we call the graph of $\varphi$, avoids some particular sets (which depend on $L$ and $\varphi$ itself):

Proposition 2.3. Let $\pi : X \to Y$ be a quotient map between a metric space and a topological space, $\varphi : Y \to X$ be a section of $\pi$, and $L \geq 1$. Then $\varphi$ is $L$-intrinsically Lipschitz if and only if

$$\varphi(Y) \cap R_{x,L} = \emptyset, \quad \text{for all } x \in \varphi(Y),$$

where

$$R_{x,L} := \{ x' \in X \mid Ld(x', \pi^{-1}(x)) < d(x', x) \}.$$

Proposition 2.3 is a triviality, still its purpose is to stress the analogy with the FSSC theory. Indeed, the sets $R_{x,L}$ are the intrinsic cones considered in Carnot groups, see Section 6.6.

The case when $\pi : X \to Y$ is a Lipschitz quotient should be considered as the trivial case of our study. Indeed, Condition (11) implies that

$$\frac{1}{k} d(\pi(x_1), \pi(x_2)) \leq d(x_1, \pi^{-1}(\pi(x_2))) \leq kd(\pi(x_1), \pi(x_2)), \quad \forall x_1, x_2 \in X.$$

Hence, being intrinsically Lipschitz is equivalent as being a biLipschitz embedding:

$$\hat{L}^{-1} d(y_1, y_2) \leq d(\varphi(y_1), \varphi(y_2)) \leq \hat{L} d(y_1, y_2), \quad \text{for all } y_1, y_2 \in Y.$$

We formally state this easy proposition for the record:

Proposition 2.4. Let $\pi : X \to Y$ be a quotient map between a metric space and a topological space. If one can metrize $Y$ in such a way that $\pi : X \to Y$ becomes a Lipschitz quotient, then a section $\varphi : Y \to X$ of $\pi$ is intrinsically Lipschitz if and only if it is a biLipschitz embedding.
Example 2.5. The reader could keep in mind the classical fundamental example: For \( n, m \in \mathbb{N} \) one considers the projection map \( \pi : \mathbb{R}^{n+m} \to \mathbb{R}^n \) on the first \( n \) variables, so that every map \( f : \mathbb{R}^n \to \mathbb{R}^m \) has a graphing map \( x \in \mathbb{R}^n \mapsto (x, f(x)) \in \mathbb{R}^{n+m} \) that is a section of \( \pi \). Moreover, such a section is intrinsically Lipschitz (in the sense of Definition\( \ref{def:intrinsic_lipschitzness} \)) if and only if \( f \) is Lipschitz in the classical sense.

2.3. Intrinsic Lipschitz with respect to families of sections. In this section we continue to fix a quotient map \( \pi : X \to Y \) between a metric space \( X \) and a topological space \( Y \).

Definition 2.6 (Intrinsic Lipschitz with respect to a section). Given sections \( \varphi, \psi : Y \to X \) of \( \pi \). We say that \( \varphi \) is intrinsically \( L \)-Lipschitz with respect to \( \psi \) at point \( \hat{x} \), with \( L \geq 1 \) and \( \hat{x} \in X \), if

1. \( \hat{x} \in \psi(Y) \cap \varphi(Y) \);  
2. \( \varphi(Y) \cap C_{\hat{x},L}^\psi = \emptyset \),

where

\[
C_{\hat{x},L}^\psi := \{ x \in X : d(x, \psi(\pi(x))) > Ld(\hat{x}, \psi(\pi(\hat{x}))) \}.
\]

Remark 2.7. Definition\( \ref{def:intrinsic_lipschitzness} \) can be rephrased as follows. A section \( \varphi \) is intrinsically \( L \)-Lipschitz with respect to \( \psi \) at point \( \hat{x} \) if and only if there is \( \hat{y} \in Y \) such that \( \hat{x} = \varphi(\hat{y}) = \psi(\hat{y}) \) and

\[
d(x, \psi(\pi(x))) \leq Ld(\hat{x}, \psi(\pi(\hat{x}))), \quad \forall x \in \varphi(Y),
\]

which equivalently means

\[
d(\varphi(y), \psi(y)) \leq Ld(\psi(\hat{y}), \psi(y)), \quad \forall y \in Y.
\]

Remark 2.8. We stress that Definition\( \ref{def:intrinsic_lipschitzness} \) does not induce an equivalence relation, because of lack of symmetry in the right-hand side of \( \ref{eq:inequality} \). Still, obviously every section is intrinsically Lipschitz with respect to itself.

The proof of Proposition\( \ref{prop:main_result} \) is an immediately consequence of the following result.

Proposition 2.9. Let \( X \) be a metric space, \( Y \) a topological space, and \( \pi : X \to Y \) a quotient map. Let \( L \geq 1 \) and \( y_0 \in Y \). Assume \( \varphi_0 : Y \to X \) is an intrinsically \( L \)-Lipschitz section of \( \pi \). Let \( \varphi : Y \to X \) be a section of \( \pi \) such that \( x_0 := \varphi(y_0) = \varphi_0(y_0) \). Then the following are equivalent:

1. For some \( L_1 \geq 1 \), \( \varphi \) is intrinsically \( L_1 \)-Lipschitz with respect to \( \varphi_0 \) at \( x_0 \);
2. For some \( L_2 \geq 1 \), \( \varphi \) satisfies

\[
d(x_0, \varphi(y)) \leq L_2d(x_0, \pi^{-1}(y)), \quad \forall y \in Y.
\]

Moreover, the constants \( L_1 \) and \( L_2 \) are quantitatively related in terms of \( L \).

Proof. \([\text{(1)} \Rightarrow \text{(2)}] \) For every \( y \in Y \), it follows that

\[
d(\varphi(y), x_0) \leq d(\varphi(y), \varphi_0(y)) + d(\varphi_0(y), x_0) \leq (L_1 + 1)d(\varphi_0(y), x_0) \leq L(L_1 + 1)d(x_0, \pi^{-1}(y))
\]

where in the first inequality we used the triangle inequality, and in the second one the intrinsically Lipschitz property of \( \varphi \) with respect to \( \varphi_0 \) at \( x_0 \). Then, in the third inequality we used the intrinsically Lipschitz property of \( \varphi_0 \).
[(2) ⇒ (1)] For every \( y \in Y \), we have that
\[
d(\varphi(y), \varphi_0(y)) \leq d(\varphi(y), x_0) + d(x_0, \varphi_0(y))
\]
\[
\leq (L_2 + 1)d(\varphi_0(y), x_0),
\]
where in the first equality we used the triangle inequality, and in the second one we used \([LG]\) and that \( \varphi_0(y) \in \pi^{-1}(y) \).

\[\square\]

2.4. **Continuity.** An intrinsically \( L \)-Lipschitz section \( \varphi : Y \to X \) of \( \pi \) is a continuous map. Indeed, fix a point \( y \in Y \) and let \( x := \varphi(y) \in X \). Since \( \pi \) is open at \( x \), for every \( \varepsilon > 0 \) we know that there is an open neighborhood \( U_\varepsilon \) of \( \pi(x) = y \) such that \( U_\varepsilon \subset \pi(B(x, \varepsilon/L)) \).

Hence, if \( y' \in U_\varepsilon \) then there is \( x' \in B(x, \varepsilon/L) \) such that \( \pi(x') = y' \). That means \( x' \in \pi^{-1}(y') \) and, consequently,
\[
d(\varphi(y), \varphi(y')) \leq Ld(\varphi(y), \pi^{-1}(y')) \leq Ld(x, x') \leq \varepsilon,
\]
i.e., \( \varphi(U_\varepsilon) \subset B(x, \varepsilon) \).

**Example 2.10.** We underline that the fact that \( \pi \) is open is a fundamental property in order to obtain the continuity of \( \varphi \). Indeed, if we consider \( X = Y = \mathbb{R} \), \( A \subset \mathbb{R} \) and \( f : A \subset \mathbb{R} \to \mathbb{R} \) be a non-necessarily continuous function with graph \( \Gamma_f \). Then the function \( \pi : \Gamma_f \to A \) defined as
\[
\pi(a, f(a)) = a, \quad a \in A
\]
may not be open but the function \( \varphi : A \to \Gamma_f \) given by \( \varphi(a) = (a, f(a)) \) for \( a \in A \), is an intrinsically Lipschitz section of \( \pi \). On the other hand, it is easy to see that \( \pi \) is open if and only if \( f \) is a continuous map.

2.5. **Infima of intrinsically Lipschitz maps.** In the case when the metric space \( X \) is geodesic and the fibers of the projection \( \pi \) are one-dimensional and are continuously oriented, we could consider infima of a family of sections. Possibly, we need to deal with the possibility of values equal to \(-\infty\). In next result we prove that if we have a family of intrinsically uniformly Lipschitz sections, then the infimum is an intrinsically Lipschitz section, with the possibility of a different value of the intrinsically Lipschitz constant. This latter fact is in accord with Franchi-Serapioni result \([FS16, Proposition 4.0.8]\).

**Proposition 2.11.** Let \( X \) be a metric space, \( Y \) a topological space, and \( \pi : X \to Y \) a quotient map. Assume that \( X \) is a geodesic space, that there exists a continuous map \( \tau : X \to \mathbb{R} \) that is a homeomorphism on the fibers of \( \pi \), and that for each \( y \in Y \) the set \( \tau|_{\pi^{-1}(y)}(\mathbb{R}) \) is boundedly compact. Let \( k \geq 1, J \) a set, and for \( j \in J \) let \( \varphi_j : Y \to X \) be an intrinsically \( k \)-Lipschitz sections. Then either the function
\[
y \in Y \mapsto \inf\{\tau(\varphi_j(y)) : j \in J\} \in \{-\infty\} \cup \mathbb{R}
\]
is constantly equal to \(-\infty\) or the map \( \varphi : Y \to X \) defined as
\[
\varphi(y) := \tau|_{\pi^{-1}(y)}(\inf\{\tau(\varphi_j(y)) : j \in J\})
\]
is well defined on all of \( Y \) and it is an intrinsically \( k \)-Lipschitz section.
Proof. For each \( y \) we define \( h(y) := \inf \{ \tau(\varphi_j(y)) : j \in J \} \in [-\infty, \infty) \). Assume that there exists \( y_0 \in Y \) such that \( h(y_0) \neq -\infty \). We shall prove some bounds that will also imply that \( h(y) \neq -\infty \), for all \( y \in Y \). Let \( y_1, y_2 \in Y \). For the moment, let us assume that \( h(y_1), h(y_2) \neq -\infty \) so that \( \varphi(y_i) = \tau i^{-1}(y_i)(h(y_i)), i = 1, 2, \) is defined as a point in \( X \). By the definition of infimum, for all \( \varepsilon > 0 \) there is \( j_i \in J \) such that \( h(y_i) \leq \tau(\varphi_{j_i}(y_i)) \leq h(y_i) + \varepsilon \), with \( i = 1, 2 \), and since \( \tau i^{-1} \) is continuous, we can also assume that

\[
(17) \quad d(\varphi_{j_i}(y_i), \varphi(y_i)) < \varepsilon.
\]

Fix \( \varepsilon > 0 \) and set \( x_i := \varphi_{j_i}(y_i) \).

We want to prove that

\[
(18) \quad d(x_1, x_2) \leq k'd(x_2, \pi^{-1}(y_1)).
\]

We consider \( x_i \in \pi^{-1}(\pi(x_1)) \) such that \( d(x_2, \pi^{-1}(\pi(x_1))) = d(x_2, x_1) \). Let \( \gamma \) be a geodesic between \( x_2 \) and \( x_1 \). Without loss of generality we assume that

\[
\tau(\varphi_{j_1}(y_1)) \leq \tau(\varphi_{j_2}(y_1)) \quad \text{and} \quad \tau(\varphi_{j_2}(y_2)) \leq \tau(\varphi_{j_1}(y_2)).
\]

Hence, on the curve \( \pi(\gamma) \) there is a point \( y^* \) such that

\[
\tau(\varphi_{j_1}(y^*)) = \tau(\varphi_{j_2}(y^*)) \quad \text{and hence} \quad z^* := \varphi_{j_1}(y^*) = \varphi_{j_2}(y^*).
\]

Be aware that \( z^* \) may not be along \( \gamma \), let \( z \) be a point on \( \gamma \) that is mapped via \( \pi \) to \( y^* \).

Then we use the triangle inequality with \( z^* \), the intrinsically Lipschitz property of \( \varphi_{j_i} \) (since both \( z^* \) and \( x_1 \) are in its graph), the triangle inequality with \( x_2 \), the intrinsically Lipschitz property of \( \varphi_{j_2} \) (since both \( z^* \) and \( x_2 \) are in its graph), that \( z \) is along \( \gamma \), we obtain

\[
d(x_1, x_2) \leq d(x_1, z^*) + d(z^*, x_2) \leq kd(x_1, z^*) + d(z^*, x_2)
\]

\[
\leq k(d(x_1, x_2) + d(x_2, z^*)) + d(z^*, x_2)
\]

\[
\leq kd(x_1, x_2) + k(k + 1)d(z, x_2) \leq (2k + k^2)d(x_1, x_2).
\]

Thus we proved (18).

Finally, putting together (18) and (17) and letting \( \varepsilon \to 0 \) we get that

\[
(19) \quad d(\varphi(y_1), \varphi(y_2)) \leq k d(\varphi(y_2), \pi^{-1}(y_1)).
\]

Now we shall discuss why in the case of the existence of \( y_0 \in Y \) such that \( h(y_0) \neq -\infty \), then we have that the map \( \varphi \) is well posed, i.e. \( h(y) \neq -\infty \), for all \( y \in Y \). The reason is that the same calculation that lead us to (19) with \( y_1 = y_0 \) and \( y_2 = y \) arbitrary will give that the values \( \{ \varphi_j(y) : j \in J \} \) leave in a bounded subset of the fiber \( \pi^{-1}(y) \). Because fibers are assumed to be boundedly compact, we have that \( \{ \tau \varphi_j(y) : j \in J \} \) is a bounded subset of \( \mathbb{R} \), which therefore admits a finite minimum. \( \square \)

2.6. The induced distance.

**Definition 2.12.** Let \( X \) be a metric space, \( Y \) a topological space, and \( \pi : X \to Y \) a quotient map. We define the function \( d_\pi : X \times X \to \mathbb{R}^+ \) as

\[
(20) \quad d_\pi(g_1, g_2) := \frac{1}{2} (d(g_1, F_{g_2}) + d(g_2, F_{g_1})), \quad \text{for all} \ g_1, g_2 \in X,
\]

where \( F_{g_i} := \pi^{-1}(\pi(g_i)) \) for \( i = 1, 2 \) and \( d(g_1, F_{g_2}) := \inf \{d(g_1, p) : p \in F_{g_2}\} \).

In general, the map \( d_\pi \) satisfy the following properties:

(i) \( d_\pi \) is symmetric, by construction;
(ii) \(d_\mathcal{F}(g_1, g_2) = 0\) if and only if \(\mathcal{F}_{g_1} = \mathcal{F}_{g_2}\);

(iii) \(d_\mathcal{F}\) does not necessarily satisfies the triangle inequality (see Proposition 2.13);

(iv) if we restrict \(d_\mathcal{F}\) to a subset of the form \(\varphi(Y)\) with \(\varphi\) a section of \(\pi\) as in Definition 2.2 then every two points of \(\varphi(Y)\) have positive distance.

In (ii), we used that each leave \(\mathcal{F}_g\) is a closed set; indeed, \(d_\mathcal{F}(g_1, g_2) = 0\) if and only if \(d(g_1, \mathcal{F}_{g_2}) = d(g_2, \mathcal{F}_{g_1}) = 0\) which is equivalent to say that \(g_1\) and \(g_2\) belong to the same leaf of \(X\).

Notice that

(1) if \(\pi : X \to Y\) be a k-Lipschitz quotient, then \(d(g_1, \mathcal{F}_{g_2}) \leq kd(\mathcal{F}_{g_1}, \mathcal{F}_{g_2})\);

(2) if \(\pi : X \to Y\) be a submetry, then \(d(g_1, \mathcal{F}_{g_2}) = d(\mathcal{F}_{g_1}, \mathcal{F}_{g_2})\).

Proposition 2.13. Let \(X\) be a metric space, \(Y\) a topological space, and \(\pi : X \to Y\) a quotient map. If \(\varphi : Y \to X\) is an intrinsically L-Lipschitz section of \(\pi\) with \(L \geq 1\), then

(i) when restricted to \(\varphi(Y)\), the functions d and \(d_\mathcal{F}\) are L-biLipschitz equivalent; more precisely, it holds

\[
d_\mathcal{F}(p_1, p_2) \leq d(p_1, p_2) \leq Ld_\mathcal{F}(p_1, p_2), \quad \forall p_1, p_2 \in \varphi(Y).
\]

(ii) \(d_\mathcal{F}\) when restricted to \(\varphi(Y)\) is a pseudo distance satisfying the weaker triangle inequality up to multiplication by \(L\);

(iii) it holds

\[
\pi\left(B\left(p, \frac{r}{L}\right)\right) \subset \pi(B(p, r) \cap \varphi(Y)) \subset \pi(B(p, r)), \quad \forall p \in \varphi(Y), \forall r > 0.
\]

Proof. (i). The left inequality in (21) follows from the simple fact that \(p_i \in \mathcal{F}_{p_i}\) and so \(d(p_i, \mathcal{F}_{p_2}) \leq d(p_1, p_2)\). Regarding the right one, since \(\varphi\) is intrinsically Lipschitz, we have that

\[d(p_1, p_2) \leq Ld(p_1, \mathcal{F}_{p_2}) \quad \text{and} \quad d(p_1, p_2) \leq Ld(p_2, \mathcal{F}_{p_1}),\]

and, consequently,

\[d(p_1, p_2) \leq \frac{1}{2}L(d(p_1, \mathcal{F}_{p_2}) + d(p_2, \mathcal{F}_{p_1})) = Ld_\mathcal{F}(p_1, p_2).
\]

Hence, (21) holds.

(ii). We observe that \(d_\mathcal{F}\) is symmetric, by construction and \(d_\mathcal{F}(p, p) = 0\) because \(p \in \mathcal{F}_p\).

Moreover, the function \(d_\mathcal{F}\) satisfies the weaker triangle inequality thanks to (i) and to the fact that \(d\) satisfies the triangle inequality; indeed, we get that

\[d_\mathcal{F}(p_1, p_2) \leq d(p_1, p_2) \leq d(p_1, p_3) + d(p_3, p_2) \leq L(d_\mathcal{F}(p_1, p_3) + d_\mathcal{F}(p_3, p_2)),\]

for every \(p_1, p_2, p_3 \in \varphi(Y)\).

(iii). Regarding the first inclusion, fix \(p \in \varphi(Y), r > 0\) and \(q \in B(p, \frac{r}{L})\). We need to show that \(\pi(q) \in \pi(\varphi(Y) \cap B(p, r))\). Actually, it is enough to prove that

\[
\varphi(\pi(q)) \in B(p, r),
\]

because if we take \(g := \varphi(\pi(q))\), then \(g \in \varphi(Y)\) and

\[\pi(g) = \pi(\varphi(\pi(q))) = \pi(q) \in \pi(\varphi(Y) \cap B(p, r)).\]

Hence using the intrinsically Lipschitz property of \(\varphi\) and the fact that \(\mathcal{F}_q = \mathcal{F}_g\) because \(\pi(g) = \pi(q)\), we have that

\[
d(p, g) \leq Ld(p, \mathcal{F}_g) = Ld(p, \mathcal{F}_q) \leq Ld(p, q) < \frac{r}{L} = r,
\]
i.e., (23) holds, as desired.

Finally, the second inclusion in (22) is trivial, since $\varphi(Y) \cap B(p, r) \subset B(p, r)$. \hfill \Box

3. An Ascoli-Arzelà compactness theorem

In this section we finish the proof of Theorem (1.2). We already proved (1.2.ii) in Section 2.1. We next restate the missing part.

Theorem 3.1 (Compactness Theorem). Let $\pi : X \to Y$ be a quotient map between a metric space $X$ for which closed balls are compact and a topological space $Y$. Then:

(i) For all $K' \subset Y$ compact, $L \geq 1$, $K \subset X$ compact, and $y_0 \in Y$ the set

$$A_0 := \{ \varphi|_{K'} : K' \to X \mid \varphi : Y \to X \text{ intrinsically } L\text{-Lipschitz section of } \pi, \varphi(y_0) \in K \}$$

is equibounded, equicontinuous, and closed in the uniform convergence topology.

(ii) For all $L \geq 1$, $K \subset X$ compact, and $y_0 \in Y$ the set

$$\{ \varphi : Y \to X : \varphi \text{ intrinsically } L\text{-Lipschitz section of } \pi, \varphi(y_0) \in K \}$$

is compact with respect to the uniform convergence on compact sets.

Proof. (i). We shall prove that for all $K' \subset Y$ compact, $L \geq 1$, $K \subset X$ compact, and $y_0 \in Y$ the set $A_0$ is

(a): equibounded;

(b): equicontinuous;

(c): closed.

(a). Fix a compact set $K' \subset Y$ such that $y_0 \in K'$. We shall prove that for every $y \in K'$

$$A := \{ \varphi(y) : \varphi \in A_0 \}$$

is relatively compact in $X$. Fix a point $x_0 \in K$ and let $k := \text{diam}_d(K)$ which is finite because $K$ is compact in $X$. Then, for every $\varphi$ that belongs to $A_0$, we have that

$$d(x_0, \varphi(y)) \leq d(x_0, \varphi(y_0)) + d(\varphi(y_0), \varphi(y)) \leq k + LD(\pi^{-1}(y), \varphi(y_0)) \leq k + L \max_{x \in K} d(\pi^{-1}(y), x),$$

where in the first equality we used the triangle inequality, and in the second one we used the fact that $\varphi \in A_0$ and $x_0 \in K$. Finally, in the last inequality we used again $\varphi(y_0) \in K$ and that the map $X \ni x \mapsto d(\pi^{-1}(y), x)$ is a continuous map and so admits maximum on compact sets. Since closed balls on $X$ are compact, we infer that the set $A$ is relatively compact in $X$, as desired.

(b). We shall prove that for every $y \in K'$ and every $\varepsilon > 0$ there is an open neighborhood $U_y \subset K' \subset Y$ such that for any $\varphi \in A$ and any $y' \in U_y$, it follows

$$d(\varphi(y), \varphi(y')) \leq \varepsilon.$$

Because of equiboundedness, we have that for every $\varphi \in A_0$ and $y \in Y$ the set $\varphi(y)$ lies within a compact set $K_y$ and so, by Remark 2.1, $\pi$ is uniformly open on $K_y \cap \pi^{-1}(y)$. Now let $U_\varepsilon$ an neighborhood of $y$ such that $U_\varepsilon \subset \pi(B(x, \varepsilon/L))$ for every $x \in K_y \cap \pi^{-1}(y)$. Then we want to show that such neighborhood $U_\varepsilon$ of $y$ is the set that we are looking for. Take $y' \in U_\varepsilon$ and let $x = \varphi(y)$. Hence, there is $x' \in B(x, \varepsilon/L)$ with $\pi(x') = y'$ and, consequently, $x' \in \pi^{-1}(y')$. Thus we have that for all $\varphi$ belongs to $A_0$

$$d(\varphi(y), \varphi(y')) \leq LD(\varphi(y), \pi^{-1}(y')) \leq LD(x, x') \leq L \frac{\varepsilon}{L} \leq \varepsilon,$$

i.e., (23) holds. Finally, since the bound is independent on $\varphi$, we proved the equicontinuity.
(c). By (a) and (b) we can apply Ascoli-Arzelà Theorem to the set $A_0$. Hence, every sequence in it has a converging subsequence. Moreover, this set is closed since if $\varphi_h$ is a sequence in it converging pointwise to $\varphi$, then $\varphi \in A_0$. Indeed, taking the limit of
\[
d(\varphi_h(y), \varphi_h(y')) \leq L d(\pi^{-1}(y), \varphi_h(y')),
\]
one gets
\[
d(\varphi(y), \varphi(y')) \leq L d(\pi^{-1}(y), \varphi(y')).
\]
Finally, it is trivial that the condition $\varphi_h(y_0) \in K$ passes to the limit since $K$ is compact.

\[\square\]

4. PROOF OF AHLFORS REGULARITY

This section is devoted just to the proof of Theorem 1.3. The proof is elementary and only uses the inclusions (22) from Section 2.6. Still, we shall see in Section 6.7 how this new result implies the theorem for intrinsically Lipschitz maps in the FSSC sense.

Proof of Theorem 1.3. Let $\varphi$ and $\psi$ intrinsically $L$-Lipschitz sections, with $L \geq 1$. Fix $y \in Y$. By Ahlfors regularity of $\varphi(Y)$ with respect to $\varphi, \mu$, we know that there are $c_1, c_2, r_0 > 0$ such that
\[
(26) \quad c_1 r^Q \leq \varphi_* \mu(B(\varphi(y), r) \cap \varphi(Y)) \leq c_2 r^Q,
\]
for all $0 \leq r \leq r_0$. We would like to show that there are $c_3, c_4 > 0$ such that
\[
(27) \quad c_3 r^Q \leq \psi_* \mu(B(\psi(y), r) \cap \psi(Y)) \leq c_4 r^Q,
\]
for every $0 \leq r \leq r_0$. We begin noticing that, by symmetry and (3)
\[
(28) \quad C^{-1} \mu(\pi(B(\psi(y), r))) \leq \mu(\pi(B(\varphi(y), r))) \leq C \mu(\pi(B(\psi(y), r))).
\]
Moreover,
\[
(29) \quad \psi_* \mu(B(\psi(y), r) \cap \psi(Y)) = \mu(\psi^{-1}(B(\psi(y), r) \cap \psi(Y))) = \mu(\pi(B(\psi(y), r) \cap \psi(Y))),
\]
and, consequently,
\[
\psi_* \mu(B(\psi(y), r) \cap \psi(Y)) \geq C^{-1} \mu(\pi(B(\varphi(y), r/L))) \geq C^{-1} \mu(\pi(B(\varphi(y), r/L)))
\]
\[
\geq C^{-1} \mu(\pi(B(\varphi(y), r/L) \cap \varphi(Y))) = C^{-1} \varphi_* \mu(B(\varphi(y), r/L) \cap \varphi(Y))
\]
\[
\geq c_1 C^{-1} L^{-Q} r^Q,
\]
where in the first inequality we used the first inclusion of (22) with $\psi$ in place of $\varphi$, and in the second one we used (28). In the third inequality we used the second inclusion of (22) and in the first equality we used (29) with $\varphi$ in place of $\psi$. Moreover, in a similar way we have that
\[
\psi_* \mu(B(\psi(y), r) \cap \psi(Y)) \leq \mu(\pi(B(\varphi(y), r))) \leq C \mu(\pi(B(\varphi(y), r)))
\]
\[
\leq C \mu(\pi(B(\varphi(y), Lr) \cap \varphi(Y))) = C \varphi_* \mu(B(\varphi(y), Lr) \cap \varphi(Y))
\]
\[
\leq c_2 C L^Q r^Q.
\]
Hence, putting together the last two inequalities we have that (27) holds with $c_3 = c_1 C^{-1} L^{-Q}$ and $c_4 = c_2 C L^Q$.\[\square\]
Example 4.1. Here is an example where some intrinsically Lipschitz sections gives Ahlfors regular graphs and some don’t. One can modify Example 2.10 to obtain more pathological examples. Let \( Y = [0, 1] \) be the unit interval and let \( X := I_0 \cup I_1 \subset \mathbb{R}^2 \) with \( I_i := \{(x, i) : x \in [0, 1]\}, \) for \( i = 0, 1 \). Here, \( X \) is endowed with the following distance: on pair of points in \( I_0 \) we consider the Euclidean distance \( d_{E} \) from the plane \( \mathbb{R}^2 \), on pair of points in \( I_1 \) we consider \( \sqrt{d_{E}} \), and the distance from a point in \( I_0 \) to one in \( I_1 \) is equal to 1, so the triangle inequality is satisfied. Let the projection \( \pi : X \to Y \) be \( \pi(x, y) := x \). Then for \( i = 0, 1 \) we consider the sections \( \varphi_i : Y \to X \) defined as \( \varphi_i(x) := (x, i) \). Both these two sections are intrinsically 1-Lipschitz. However, \( \varphi_0(Y) \) is 1-Ahlfors regular and \( \psi_1(Y) \) is 2-Ahlfors regular. The example could easily be modified to also have a connected space \( X \). And considering instead of \( \sqrt{d_{E}} \) any other distance on \( I_1 \), with diameter 1, one can have that \( \psi_1(Y) = I_1 \) is not Ahlfors regular.

5. Level sets and extensions

In this section we prove Theorem 1.4. We shall both generalize and simplify Vittone’s argument from \cite[Theorem 1.5]{Vit20}. We need to mention that there have been several earlier partial results on extensions of Lipschitz graphs, as for example in \cite[Proposition 4.8]{FSSC06}, \cite[Proposition 3.4]{Vit12}, \cite[Theorem 4.1]{FS16}. Regarding extension theorems in metric spaces, the reader can see \cite{AP20} and its references.

Proof of Theorem 1.4.i. Let \( f : X \to Z \) and \( z_0 \in Z \) as in the assumptions of part 1.4.i. We begin recalling that by assumption for every \( y \in Y \) the map \( f|_{\pi^{-1}(y)} : \pi^{-1}(y) \to Z \) is a biLipschitz homeomorphism and so it is surjective. Namely, for every \( y \in Y \) there is a unique \( x \in \pi^{-1}(y) \) such that \( f(x) = z_0 \). Hence, it is natural to define \( \varphi(y) := x \) in such a way (29) holds trivially. Moreover, we claim that the just-defined section \( \varphi : Y \to X \) is intrinsically \((1 + L^2)\)-Lipschitz. Indeed, for each \( y_1, y_2 \in Y \) we consider the only points \( x_1 \in \pi^{-1}(y_1) \cap f^{-1}(z_0) \) and \( x_2 \in \pi^{-1}(y_2) \cap f^{-1}(z_0) \), and then we shall prove (2), with constant \( 1 + L^2 \), showing that

\[
(30) \quad d(x_1, x_2) \leq (1 + L^2)d(x_1, \pi^{-1}(y_2)).
\]

For each \( \varepsilon > 0 \), let \( \bar{x}_2 \in \pi^{-1}(y_2) \) such that

\[
(31) \quad d(x_1, \bar{x}_2) \leq d(x_1, \pi^{-1}(y_2)) + \varepsilon.
\]

Then it follows that

\[
d(x_1, x_2) \leq d(x_1, \bar{x}_2) + d(\bar{x}_2, x_2) \\
\leq d(x_1, \bar{x}_2) + Ld(f(\bar{x}_2), f(x_2)) \\
= d(x_1, \bar{x}_2) + Ld(f(\bar{x}_2), f(x_1)) \\
\leq (1 + L^2)d(x_1, \bar{x}_2) \leq (1 + L^2)(d(x_1, \pi^{-1}(y_2)) + \varepsilon),
\]

where in the first inequality we used the triangle inequality and in the second inequality we used the co-Lipschitz property of \( f \) on the fiber \( \pi^{-1}(y_2) \); in the equality we used the fact that \( f(x_1) = f(x_2) = z_0 \) and finally we used the Lipschitz property of \( f \). Consequently, by the arbitrariness of \( \varepsilon \), we deduce that (30) is true and the proof is complete. \( \square \)
Proof of Theorem 1.4.ii. Let \( k, L, \rho, \tau, \text{ and } \{\varphi_{\tau_0}\}_{\tau_0} \) as in the assumptions of part 1.4.ii. Fix \( x_0 \in X \), for the moment; and consider \( \tau_0 := \tau(x_0) \). Recall that we have \( \tau^{-1}(\tau_0) = \varphi_{\tau_0}(Y) \) by assumption. We also consider the function \( \delta_{\tau_0} \), as \( \delta_{\tau_0}(x) := \rho(x, \varphi_{\tau_0}(\tau(x))) \), which is \( k \)-Lipschitz on the set \( \{ |\tau - \tau_0| \leq kL\delta_{\tau_0} \} \) and satisfies \( \delta_{\tau_0}(x_0) = 0 \). Then, for each such a \( x_0 \), and \( \tau_0 \), we consider the function \( f_{x_0} : X \to \mathbb{R} \) defined as

\[
\begin{align*}
 f_{x_0}(x) = \begin{cases}
 2(\tau(x) - \tau(x_0)) - \alpha \delta_{\tau_0}(x) & \text{if } |\tau(x) - \tau(x_0)| \leq kL\delta_{\tau_0}(x) \\
 \tau(x) - \tau(x_0) & \text{if } |\tau(x) - \tau(x_0)| > kL\delta_{\tau_0}(x) \\
 3(\tau(x) - \tau(x_0)) & \text{if } |\tau(x) - \tau(x_0)| < -kL\delta_{\tau_0}(x),
\end{cases}
\end{align*}
\]

where \( \alpha := kL \). We prove that the continuous \( f_{x_0} \) satisfies the following properties:

(i): \( f_{x_0} \) is \( K \)-Lipschitz;

(ii): \( f_{x_0}(x_0) = 0 \);

(iii): \( f_{x_0} \) is \( 3k \)-biLipschitz on fibers, giving the same orientation that \( \tau \) does.

where \( K = \max\{3k, 2k + \alpha k\} = 2k + \alpha k \) because \( \alpha > 1 \). The property (i) follows using that \( \tau, \delta_{\tau_0} \) are both Lipschitz and \( X \) is a geodesic space. On the other hand, (ii) is true since \( \delta_{\tau_0}(x_0) = 0 \) Finally, for every \( y \in Y \) and \( x, x' \in \pi^{-1}(y) \) we have that \( \rho(x_0, \varphi_{\tau_0}(\tau(x))) = \rho(x_0, \varphi_{\tau_0}(\tau(x'))) \), i.e., \( \delta_{\tau_0} \) is constant on fibers. Thus, the function \( f_{x_0} \) is biLipschitz on fibers because \( \tau \) is so too, and actually, the biLipschitz constant is \( 3 \) times the constant for \( \tau \) and \( f_{x_0} \) grows on fibers in the same direction that \( \tau \) does. Hence (iii) holds.

Now that we have the family \( \{f_{x_0}\}_{x_0} \), given \( \varphi : Y' \to X \) intrinsically \( L \)-Lipschitz section, we consider the map \( f : X \to \mathbb{R} \) given by

\[
f(x) := \sup_{x_0 \in \varphi(Y')} f_{x_0}(x), \quad \forall x \in X,
\]

and we want to prove that it is the map we are looking for. The Lipschitz properties are valid since the function \( \delta_{x_0} \) is constant on the fibers, and (iii) holds. Consequently, the only non trivial fact to show is (iii). Fix \( \bar{x}_0 \in \varphi(Y') \). By (ii) we have that \( f_{\bar{x}_0}(\bar{x}_0) = 0 \) and so it is sufficient to prove that \( f_{x_0}(\bar{x}_0) \leq 0 \) for \( x_0 \in \varphi(Y') \). Let \( x_0 \in \varphi(Y') \). Then using in addition that \( \tau \) is \( k \)-Lipschitz, and that \( \varphi \) is intrinsically \( L \)-Lipschitz, we have

\[
|\tau(\bar{x}_0) - \tau(x_0)| \leq kd(\bar{x}_0, x_0) \leq Lkd(x_0, \tau^{-1}(\tau(\bar{x}_0))) \leq Lkd(x_0, \varphi_{\tau_0}(\tau(\bar{x}_0))) = \alpha \delta_{\tau_0}(\bar{x}_0),
\]

and so

\[
f_{x_0}(\bar{x}_0) = 2(\tau(\bar{x}_0) - \tau(x_0) - \alpha \delta_{\tau_0}(\bar{x}_0)) \leq 0,
\]

i.e., (iii) holds. \( \square \)

6. Applications to Groups

In this section shall apply the theory developed in the previous sections to the case of groups. The general setting is a topological group \( G \) together with a closed subgroup \( H \) of \( G \) in such a way that the quotient space \( G/H := \{gH : g \in G\} \) naturally is a topological space for which the map \( \pi : g \mapsto gH \) is continuous, open, and surjective: it is a quotient map.

A section for the map \( \pi : G \to G/H \) is just a map \( \varphi : G/H \to G \) such that \( \varphi(gH) \in gH \), since we point out the trivial identity \( \pi^{-1}(gH) = gH \). To have the notion of intrinsically Lipschitz section we need the group \( G \) to be equipped with a distance which we assume left-invariant. We refer to such a \( G \) as a metric group.
The inequality in definition of intrinsically Lipschitz section (see Definition 1.1) rephrases as
\begin{equation}
 d(\varphi(g_1H), \varphi(g_2H)) \leq Ld(\varphi(g_1H), g_2H), \quad \text{for all } g_1, g_2 \in G. \tag{33}
\end{equation}

The concept of sections and intrinsically Lipschitz sections is preserved by left translation: namely, if \( \Sigma \subset G \) is the graph (i.e., the image) of an intrinsically Lipschitz section and \( \hat{g} \in G \), then \( \hat{g}\Sigma \) is the graph of some (possibly different) intrinsically Lipschitz section (see Proposition 6.1). As a consequence, as done by Franchi, Serapioni, and Serra Cassano, one could see the intrinsically Lipschitz condition as a condition near the identity element \( 1_G \) of \( G \) when the graph is translated at \( 1_G \). In fact, in the special case in which \( \varphi(H) = 1_G \) equation (33), for \( g_1 = 1_G \), becomes
\begin{equation}
 d(1_G, \varphi(gH)) \leq Ld(1_G, gH), \quad \text{for all } g \in G. \tag{34}
\end{equation}

**Proposition 6.1** (Left-invariance of sections). For each \( \hat{g} \in G \) and section \( \varphi : G/H \to G \), the set \( \hat{g}\varphi(G/H) \) is the image of the section \( \varphi_{\hat{g}} : G/H \to G \) defined as
\begin{equation}
 \varphi_{\hat{g}}(gH) := \hat{g}\varphi(\hat{g}^{-1}gH), \quad \forall gH \in G/H. \tag{35}
\end{equation}
Moreover, \( \varphi_{\hat{g}} \) is an intrinsically \( L \)-Lipschitz section, if so is \( \varphi \).

**Proof.** It is clear that \( \varphi_{\hat{g}} \) is a section, since, being \( \varphi \) a section, we have \( \varphi(\hat{g}^{-1}gH) \in \hat{g}^{-1}gH \). It is also evident that the image of \( \varphi_{\hat{g}} \) is \( \hat{g}\varphi(G/H) \). Hence, we are just left to prove that if \( \varphi \) satisfies (33), then so does \( \varphi_{\hat{g}} \). We use the left invariance of the distance and the intrinsically Lipschitz property of \( \varphi \) to obtain
\begin{align*}
 d(\varphi_{\hat{g}}(g_1H), \varphi_{\hat{g}}(g_2H)) &= d(\hat{g}\varphi(\hat{g}^{-1}g_1H), \hat{g}\varphi(\hat{g}^{-1}g_2H)) \\
 &= d(\varphi(\hat{g}^{-1}g_1H), \varphi(\hat{g}^{-1}g_2H)) \\
 &\leq Ld(\varphi(\hat{g}^{-1}g_1H), \hat{g}^{-1}g_2H) \\
 &= Ld(g_1H, g_2H) \\
 &= Ld(\varphi_{\hat{g}}(g_1H), g_2H),
\end{align*}
for every \( g_1, g_2 \in G \), as desired. \( \square \)

6.1. **Splitting of groups and semidirect products.** Next we shall consider setting where the subgroup \( H \) of the metric group \( G \) splits, or even more particularly, it splits with respect to a normal subgroup. In these situations we will have an identification of \( G/H \) with a subgroup of \( G \), which in our opinion it helps in representing points in the quotient space, but confuses the geometric interpretation of intrinsically Lipschitz sections.

In this section \( G \) will be a metric group that admits a splitting \( G = H_1 \cdot H_2 \), as explained in the introduction: \( H_1 \) and \( H_2 \) are two closed subgroups of \( G \) for which every element \( g \in G \) can be written uniquely as \( g = h_1h_2 \) with \( h_1 \in H_1 \) and \( h_2 \in H_2 \). We shall denote \( h_1 \) as \( \pi_{H_1}(g) \) and have a map \( \pi_{H_1} : G \to H_1 \), and similarly with \( \pi_{H_2} : G \to H_2 \).

A special splitting is given by semidirect-product structures: one of the factor is normal. Namely, a group \( G \) is a semidirect product if it admits a splitting \( G = N \cdot H \) with \( N \) normal within \( G \). In other words, the group \( G \) is isomorphic to the structure of semidirect product \( N \rtimes H \) of two groups \( N \) and \( H \) where \( H \) acts on \( N \) by automorphisms. When \( H \) is seen as subgroup, it acts on \( N \) by conjugation\(^1\)
\begin{equation}
 C_h(n) := hnh^{-1} \in N, \quad \text{for all } h \in H \text{ and } n \in N. \tag{36}
\end{equation}

\(^1\) We shall repeatedly use the following identity: for any \( m, n \in N \) and \( \ell \in H \)
\begin{equation}
 \pi_N(m\ell n) = m\pi_N(\ell),
\end{equation}

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For the sake of shortness, we shall write that $G = H \cdot H$ is a splitted metric group if it is a metric group that admits the splitting $G = H \cdot H$. If moreover the splitting is a semidirect product we write that $G = H \cdot H$ is a semidirect metric group.

We stress that if one has a splitting $G = H \cdot H$ then $G$ also admits the splitting $G = H \cdot H$. However, the projection maps may be different. For this reason, in this paper we fix the convention that

we always only consider sections of the quotient with respect to the group on the right

$$h_1 h_2 \in H \cdot H \xrightarrow{\pi} h_1 H_2 \in G/H_2.$$  

Of course, in the case of a splitting, we have an identification of $G/H_2$ with $H_1$ element wise. However, as we will see soon, this identification has very little algebraic or geometric significance.

6.2. Lipschitz property at the identity element. We shall consider the setting of splitted groups $H_1 \cdot H_2$ and consider the various notions of intrinsic Lipschitz graphs. The key property that will make us develop a theory in a way that links the various notions studied in literature with the very general one that we propose is a type of Lipschitz property for the projection map $\pi_{H_1} : h_1 h_2 \in H_1 \cdot H_2 \mapsto h_1$. The condition is like the Lipschitz property but fixes one of the two considered points to be the identity element 1 of the group. We recall that, as defined in (7), we say that $\pi_{H_1}$ is $K$-Lipschitz at 1 if $d(1, \pi_{H_1}(g)) \leq K d(1, g)$, for all $g \in G$. Equivalently, this condition requests that

$$d(1, h_1) \leq K d(1, h_1 H_2), \quad \forall h_1 \in H_1.$$  

The Lipschitz property at the identity element may not hold even in Carnot groups with a semidirect product (see next example), unless the subgroups are homogeneous, see Proposition 1.6.

Remark 6.2 (Non-example). There are splittings $N \rtimes H$ of subRiemannian Carnot groups for which the projection on $H$ is not Lipschitz, not the projection on $N$ is Lipschitz at 1, not even locally. Here is an example: Let $\mathbb{H}^1$ be the Heisenberg group seen as $\mathbb{R}^3$ with coordinates $x_1, x_2, x_3$, and let $\{X_1 := \partial_{x_1} - \frac{x_3}{2} \partial_{x_2}, X_2 := \partial_{x_2} + \frac{x_3}{2} \partial_{x_1}, X_3 := \partial_{x_3}\}$ be a basis of its Lie algebra so that the only non-vanishing relation is $[X_1, X_2] = X_3$. This identification of $\mathbb{H}^1$ with $\mathbb{R}^3$ is by means of exponential coordinates associated with $(X_1, X_2, X_3)$. The dilations on $\mathbb{H}^1$ become $\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$, the identity element is $0 = (0, 0, 0)$, and the product law is

$$(x_1, x_2, x_3) \cdot (x'_1, x'_2, x'_3) = (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 + \frac{1}{2}(x_1 x'_2 - x_2 x'_1)).$$

We consider the following splitting of $\mathbb{H}^1$: $N := \{(0, x_2, x_3) \in \mathbb{R} \}$ and $H := \{(x_1, 0, x_1) : x_1 \in \mathbb{R}\}$, so that $N \rtimes H = \mathbb{H}^1$. We notice that $N$ is a normal subgroup and $H$ is a non-homogeneous subgroup of $\mathbb{H}^1$. Let $d$ the left invariant metric on $\mathbb{H}^1$ defined as $d((x_1, x_2, x_3), 0) := \max\{|x_1|, |x_2|, \sqrt{|x_3|}\}$, see LDLR17, pp. 352-353] for the proof that this function gives a distance. If $g = (x_1, 0, 0) \in B(0, r)$ for some $r > 0$, then we have that
\( g = (0, 0, -x_1) \cdot (x_1, 0, x_1) \) with \((0, 0, -x_1) \in N\) and \((x_1, 0, x_1) \in H\). Moreover, we have
\[
d(0, \pi_N(g)) = \sqrt{|x_1|}, \quad d(0, g) = |x_1|, \quad d(0, \pi_H(g)) = \max\{|x_1|, \sqrt{|x_1|}\}.
\]
Consequently there is no \( L > 0 \) and \( r > 0 \) such that for all \( g \in B(1, r) \) we would have
\[
d(1, \pi_N(g)) \leq Ld(1, g), \quad \text{nor} \quad d(1, \pi_H(g)) \leq Ld(1, g).
\]
Next we show that if one has the Lipschitz property at the identity element then the standard sections are intrinsically Lipschitz. In case of a splitting \( G = H_1 \cdot H_2 \), the inclusion \( i : H_1 \hookrightarrow G \) can be seen as a section of \( \pi : G \to G/H_2 \) identifying \( G/H_2 \) with \( H_1 \). Also, after Proposition 6.1 it is useful to recall that \( H_1 \) is the graph of an intrinsically \( k \)-Lipschitz section if and only if for all (or, equivalently, for some) \( g \in G \) the set \( gH_1 \) is the graph of an intrinsically \( k \)-Lipschitz section.

**Proposition 6.3.** Let \( G = H_1 \cdot H_2 \) be a splitted metric group and \( K \geq 1 \). Then the following are equivalent:

1. the inclusion map \( i : H_1 \hookrightarrow G \) is an intrinsically \( K \)-Lipschitz section of \( \pi_{H_1} \);
2. \( \pi_{H_1} \) is \( K \)-Lipschitz at 1;
3. one has
\[
d(1, \pi_{H_1}(g)) \leq Kd(1, gH_2), \quad \forall g \in G.
\]

*Proof.* Condition (1), see (33), is equivalent to
\[
d(h_1, h_1') \leq Kd(h_1, h_1'H_2), \quad \forall h_1, h_1' \in H_1,
\]
which by left-invariance is equivalent to (37), which is equivalent to Condition (2).

In addition, since \( \pi_{H_1}(gH_2) = \pi_{H_1}(g) \), Condition (3) and (37) are also equivalent.

In the case \( H_1 \) is normal, which means we have a semidirect product \( G = N \rtimes H \), then the map \( \pi_{H_1} = \pi_N \) is Lipschitz at 1 exactly when the other projection \( \pi_{H_2} = \pi_H \) is Lipschitz. We stress that this latter map is a group homomorphism since \( N \) is normal. In particular, the map \( \pi_H \) is Lipschitz if and only if it is Lipschitz at 1. These equivalences, with few others, are the subject of next proposition.

**Proposition 6.4.** Let \( G = N \rtimes H \) be a semidirect metric group. The following conditions are equivalent:

1. there is \( C_1 > 0 \) such that \( \pi_H : N \rtimes H \to H \) is a \( C_1 \)-Lipschitz map, i.e.,
\[
d(\pi_H(g), \pi_H(p)) \leq C_1d(g, p), \quad \forall g, p \in G;
\]
2. there is \( C_2 > 0 \) such that
\[
d(1, \pi_H(g)) + d(1, \pi_N(g)) \leq C_2d(1, g), \quad \forall g \in G;
\]
3. there is \( C_3 > 0 \) such that \( \pi_N \) is \( C_3 \)-Lipschitz at 1, i.e.,
\[
d(1, \pi_N(g)) \leq C_3d(1, g), \quad \forall g \in G;
\]
4. there is \( C_4 > 0 \) such that
\[
d(1, \pi_H(g)) \leq C_4d(1, g), \quad \forall g \in G;
\]
(5) there is \( C_5 > 0 \) such that
\[
d(1, \pi_N(g)) \leq C_5 d(g^{-1}, H), \quad \forall g \in G;
\]

(6) there is \( C_6 > 0 \) such that
\[
d(1, \pi_H(g)) \leq C_6 d(g, N), \quad \forall g \in G;
\]

(7) there is \( C_7 > 0 \) such that
\[
d(1, C_{\pi_H(g)}^{-1}(\pi_N(g))) \leq C_7 d(1, g), \quad \forall g \in G;
\]

(8) there is \( C_8 > 0 \) such that
\[
d(1, C_{\pi_H(g)}^{-1}(\pi_N(g))) \leq C_8 d(g, H), \quad \forall g \in G.
\]

**Proof.** The equivalences (2) \( \iff \) (3) \( \iff \) (4) easily follow from the bounds:

- \( d(1, \pi_N(g)) \leq d(1, g) + d(g, \pi_N(g)) = d(1, g) + d(1, \pi_H(g)) \),
- \( d(1, \pi_H(g)) \leq d(1, (\pi_N(g))^{-1}) + d((\pi_N(g))^{-1}, \pi_H(g)) = d(1, \pi_N(g)) + d(1, g) \).

[(1) \( \iff \) (4)] The implication (1) \( \Rightarrow \) (4) follows by taking \( p = 1 \). The implication (4) \( \Rightarrow \) (1) follows because \( \pi_H \) is a homomorphism:
\[
d(\pi_H(g), \pi_H(p)) = d(1, \pi_H(g)^{-1}\pi_H(p)) = d(1, \pi_H(g^{-1}p)) \\
\leq C_4 d(1, g^{-1}p) \\
= C_4 d(g, p).
\]

[(3) \( \iff \) (5)] This follows from Proposition 6.3 (2) \( \iff \) (3).

[(4) \( \iff \) (6)] The implication (6) \( \Rightarrow \) (4) follows immediately taking \( 1 \in N \). The implication (4) \( \Rightarrow \) (6) follows observing that \( \pi_H(Ng) = \pi_H(g) \).

For the equivalence of (7), we show that (2) \( \Rightarrow \) (7) and (7) \( \Rightarrow \) (4). Notice that for any \( nh \in N \times H \)
\[
d(1, C_{h^{-1}}(n)) \leq 2d(1, h) + d(1, n) \leq 2C_2 d(1, nh),
\]
we obtain the implication (2) \( \Rightarrow \) (7). Moreover, the implication (7) \( \Rightarrow \) (4) holds because
\[
d(1, h) \leq d(1, nhh^{-1}n^{-1}h) \leq d(1, nh) + d(1, C_{h^{-1}}(n)) \leq (1 + C_7) d(1, nh),
\]
where in the second inequality we used the fact that \( d(1, C_{h^{-1}}(n)) = d(1, C_{h^{-1}}(n)) \).

Finally, in order to prove the equivalence of (8), we show that (8) \( \Rightarrow \) (7) and (3) \( \Rightarrow \) (8). The implication (8) \( \Rightarrow \) (7) follows immediately from \( d(g, H) \leq d(g, 1) \). The implication (3) \( \Rightarrow \) (8) follows by taking \( n \in N, h, \ell H, \) bounding
\[
d(1, C_{h^{-1}}(n)) = d(1, C_{h^{-1}}(n)) = d(1, \pi_N(C_{h^{-1}}(n)h^{-1}\ell)) \\
\leq C_3 d(1, C_{h^{-1}}(n)h^{-1}\ell) \\
= C_3 d(1, h^{-1}n^{-1}\ell) = C_3 d(nh, \ell),
\]
and taking the infimum over \( \ell H \).

Hence, every two points of the proposition are equivalent and the proof is achieved. \( \square \)

**Remark 6.5.** Notice that many implications in the above proposition are valid also when the splitting is not a semidirect product, e.g., (1) \( \Rightarrow \) (4), (5) \( \Rightarrow \) (3), (6) \( \Rightarrow \) (4), (8) \( \Rightarrow \) (7).

**Remark 6.6.** Using the fact that \( N \) is normal, we can rewrite, in the equivalent way, the inequalities of Proposition 6.4.
Remark 6.7. In every metric group \((N \rtimes H, d)\) one has the following inequalities:

\[
d(g, N) \leq d(1, \pi_H(g)),
\]
\[
d(g, H) \leq d(1, C_{\pi_H(g)}^{-1}(\pi_N(g))),
\]
\[
d(1, g) \leq d(1, C_{\pi_H(g)}^{-1}(\pi_N(g))) + d(1, \pi_H(g)), \quad \forall g \in N \rtimes H.
\]

Indeed, considering \(g \in N \rtimes H\) so that \(g = \pi_N(g) \cdot \pi_H(g)\), we have

\[
d(g, N) \leq d(1, (\pi_H(g))^{-1} \cdot (\pi_N(g))^{-1} \cdot \pi_N(g)) = d(1, \pi_H(g)),
\]
\[
d(g, H) \leq d(1, (\pi_H(g))^{-1} \cdot (\pi_N(g))^{-1} \cdot \pi_H(g)) = d(1, C_{\pi_H(g)}^{-1}(\pi_N(g))).
\]

Moreover, to prove the last inequality it is enough to notice that

\[
g = \pi_N(g) \cdot \pi_H(g) = \pi_H(g) \cdot [\pi_H(g)]^{-1} \cdot \pi_N(g) \cdot \pi_H(g) = \pi_H(g) \cdot C_{\pi_H(g)}^{-1}(\pi_N(g)).
\]

6.3. Lipschitz projections for CC distances. In order to understand why in Carnot groups equipped with a homogeneous splitting the various notions of intrinsically Lipschitz sections coincide, we shall get a criterion to determine when the projection on a factor of a splitting of a group is Lipschitz. In this subsection, we shall focus on Carnot-Carathéodory distances on groups, see an introduction in [LD17] for the notion of CC-metric induced by a distribution \(\Delta\).

Proposition 6.8. Let \(G = N \rtimes H\) be the semidirect product of two Lie groups. Let \(\Delta \subseteq \mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{h}\) be a bracket generating distribution on \(G\). Then the following statements are equivalent:

1. \(\pi_h(\Delta) \subseteq \Delta\).
2. \(\pi_H\) is Lipschitz for every CC-metric induced by \(\Delta\).

In the proposition, we denoted by \(\pi_h\) the projection from the Lie algebra \(\mathfrak{g}\) of \(G\) to the Lie algebra \(\mathfrak{h}\) of \(H\) to the modulo the Lie algebra \(\mathfrak{n}\) of \(N\).

Before the proof of Proposition 6.8 we discuss a lemma.

Lemma 6.9. Let \(N \rtimes H\) be the semidirect product of two Lie groups. Let \(k \in \mathbb{N}\) and \(\Delta \subseteq \mathfrak{n} \rtimes \mathfrak{h}\) be a \(k\)-dimensional linear subspace of the Lie algebra. If \(m := \dim(\pi_h(\Delta))\) then there are \(X_1^b, \ldots, X_m^b \in \mathfrak{h}\) and \(X_1^n, \ldots, X_k^n \in \mathfrak{n}\) such that

\[
X_1^b + X_1^n, \ldots, X_m^b + X_m^n, X_{m+1}^n, \ldots, X_k^n \quad \text{is a basis for} \ \Delta.
\]

Moreover, if

\[
\pi_h(\Delta) \subseteq \Delta,
\]

we may choose \(X_1^n = \ldots = X_m^n = 0\), so that

\[
X_1^b, \ldots, X_m^b, X_{m+1}^n, \ldots, X_k^n \quad \text{is a basis for} \ \Delta.
\]
Proof. Recall that $\pi := \pi_{\mathfrak{h}} : \mathfrak{n} \times \mathfrak{h} \to \mathfrak{h}$ is the projection onto $\mathfrak{h}$ modulo $\mathfrak{n}$. We shall consider the restriction of it to $\Delta$, that is $\pi|\Delta : \Delta \to \pi(\Delta)$. Recall that
\begin{equation}
(42) \quad k = \dim(\Delta) = \dim(\pi(\Delta)) + \dim(\ker(\pi|_{\Delta})) = m + \dim(\ker(\pi|_{\Delta})).
\end{equation}
Thus $\dim(\ker(\pi_{\Delta})) = k - m$. Hence, let $X^n_{m+1}, \ldots, X^n_k \in \mathfrak{n}$ be a basis of $\ker(\pi_{\Delta})$. Also, let $X^1_h, \ldots, X^h_m$ be a basis of $\pi(\Delta)$. In particular, notice that
$X^n_{m+1}, \ldots, X^n_k \in \ker(\pi|_{\Delta}) \subseteq \mathfrak{n} \cap \Delta$
and
$X^1_h, \ldots, X^h_m \in \pi(\Delta) \subseteq \mathfrak{h}$.
For each $i = 1, \ldots, m$, since $X^h_i \in \pi(\Delta)$ and since $\ker(\pi|_{\Delta}) \subseteq \mathfrak{n}$ there exists $X^n_i \in \mathfrak{n}$ such that $X^h_i + X^n_i \in \Delta$. Therefore, from (42) we have that (39) holds true.

If, in addition, we have (40) then we can choose $X^n_i = 0$, for all $i = 1, \ldots, m$. And we conclude (41). \qed

Proof of Proposition 6.8. [(1) $\Rightarrow$ (2)] Let $\Delta' := \Delta \cap \mathfrak{h}$. Fix a left-invariant scalar product on $\mathfrak{g}$. Let $d'$ be the CC-distance on $H$ determined by $\Delta'$. Notice that since for the definition of $d'$ one only considers $\Delta$-horizontal curves within $H$, we have that
\begin{equation}
(43) \quad d' \geq d_H,
\end{equation}
where $d_H$ denotes the CC distance $d$ determined by $\Delta$ on $G$ restricted to $H$.

We notice that $\pi_{\mathfrak{h}}(\Delta) \subseteq \Delta'$ if and only if the (smooth homomorphic) map $\pi_H : (G, d) \to (H, d')$ is Lipschitz on compact sets. Since the map is a group morphism and the distance is geodesic, then there is no difference between Lipschitz and locally Lipschitz. Hence by (43), these last conditions imply that $\pi_H : (G, d) \to (H, d_H)$ is Lipschitz.

[(2) $\Rightarrow$ (1)] By contradiction, we assume that $\pi_{\mathfrak{h}}(\Delta) \nsubseteq \Delta$, i.e., there is $w \in \Delta$ such that $\pi_{\mathfrak{h}}(w) = w_1 \in \pi_{\mathfrak{h}}(\Delta) \setminus \Delta$. Hence, by $\pi_{\mathfrak{h}}(\Delta) \subseteq \mathfrak{h}$, we have that $w_1 \in \mathfrak{h} \setminus \Delta$.

Now if $w = w_1 + w_2$ with $w_2 \in \mathfrak{n}$, then for some $t > 0$ we have that $tw = tw_1 + tw_2 \in B(1, r)$ for some $r > 0$ and so, using the facts $tw \in \Delta$ and $tw_1 \notin \Delta$.

\begin{equation}
(d(1, tw) \sim t, \quad \text{and} \quad d(1, tw_1) \gg t).
\end{equation}

Now, since $\pi_H : (G, d) \to (H, d_H)$ is assumed $L$-Lipschitz, it follows that
\begin{equation}
t \ll d(1, tw_1) \leq Ld(1, tw) \sim t
\end{equation}
and so the contradiction. \qed

6.4. Sections in semidirect products and FSSC conditions. Next we make some links between our notion of intrinsically Lipschitz section and the various notions of intrinsically Lipschitz maps in the sense of Franchi, Serapioni, and Serra Cassano. The setting we are considering is the case of a splitting $G = H_1 \cdot H_2$. There will be a double view point in the objects of study: On the one hand, we might consider sections $\varphi : G/H_2 \to G$. On the other hand, we might consider maps $\psi : H_1 \to H_2$. There is an obvious link between the two objects: A map $\psi$ induces a section $\varphi$ as
$\varphi(gH_2) := \pi_{H_1}(g)\psi(\pi_{H_1}(g)), \quad \forall g \in G$.
A section $\varphi$ induces a map $\psi$ as
$\psi(n) := n^{-1}\varphi(nH_2), \quad \forall n \in H_1$.  

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For a map \( \psi : H_1 \to H_2 \) we set \( \Gamma_\psi := \{ n\psi(n) : n \in H_1 \} \), which is exactly the image of the associated section.

We say that \( \psi \) is an \textit{intrinsically Lipschitz map in the FSSC sense} if exists \( K > 0 \) such that
\[
(44) \quad d(1, \pi_{H_2}(x^{-1}x')) \leq K d(1, \pi_{H_1}(x^{-1}x')) , \quad \forall x, x' \in \Gamma_\psi .
\]
In Lemma 6.12 we shall soon see that this condition is equivalent to require that
\[
(45) \quad d(1, x^{-1}x') \leq K d(1, \pi_{H_1}(x^{-1}x')) , \quad \forall x, x' \in \Gamma_\psi .
\]

The last property that we consider for a section is the following: Given a splitting \( G = H_1 \cdot H_2 \), we say that a section for \( \varphi : G/H_2 \to G \) is \textit{intrinsically Lipschitz with respect to the standard sections} if it is intrinsically Lipschitz with respect to every section of the form \( gH_1 \) for all \( g \in \varphi(G/H_2) \), see Definition 2.6

Explicitly, a set \( \Sigma \subset G \) is the graph of an intrinsically Lipschitz map with respect to the standard sections if and only if
\[
(46) \quad d(g, gH_2 \cap \hat{g}H_1) \leq L d(\hat{g}, gH_2 \cap \hat{g}H_1) , \quad \forall g, \hat{g} \in \Sigma .
\]
When \( gH_2 \cap \hat{g}H_1 \) is a singleton for all \( g, \hat{g} \in \Sigma \) (this happen for instance when \( H_1 \) is normal), this condition is more general than those mentioned above; indeed, for any \( x, x' \in \Sigma = \Gamma_\psi \) we have that
\[
(47) \quad d(x, x') \leq d(x, xH_2 \cap x'H_1) + d(xH_2 \cap x'H_1, x') \leq (L + 1)d(x', xH_2 \cap x'H_1)
\]
\[
\leq (L + 1)d(x', x'H_1) \leq (L + 1)d(1, \pi_{H_1}(x^{-1}x')).
\]

Yet, when \( \pi_{H_1} \) is \( k \)-Lipschitz at 1 and \( H_1 \) is normal, the condition (46) is equivalent to (44) and (45) (see Proposition 6.13).

6.5. **The trivial case when the quotient map is a Lipschitz quotient.** We shall spend some words remarking that in the case we are in a group on which we are taking the quotient modulo a normal subgroup, then the quotient map is a Lipschitz quotient with respect to a distance on the quotient space. Hence, by Proposition 2.4 the theory of intrinsically Lipschitz sections coincides with the one of biLipschitz embeddings. If moreover, the group has a splitting, then our intrinsically Lipschitz sections coincide with the Lipschitz maps between the factors.

**Proposition 6.10.** Let \( G = N \rtimes H \) be a semidirect metric group. Assume \( N \) is boundedly compact so to have a quotient metric \( d_{G/N} \) on \( G/N \) (see [LDR16, Corollary 2.11]). Via the projection on \( H \) given by the semidirect product we see \( d_{G/N} \) as a distance on \( H \). Then, the following facts are equivalent:

1. the projection \( \pi_H : G \to H \) is \( L \)-Lipschitz map;
2. it holds
\[
(48) \quad d_{\mid_H}(h, \ell) \leq L d_{G/N}(h, \ell) , \quad \forall h, \ell \in H .
\]
Moreover, if one of these conditions are true then \( d_{\mid_H} \) and \( d_{G/N} \) are biLipschitz equivalent:
\[
\frac{1}{L} d_{\mid_H}(h, \ell) \leq d_{G/N}(h, \ell) \leq d_{\mid_H}(h, \ell) , \quad \forall h, \ell \in H .
\]

**Proof.** [(1) \( \Rightarrow \) (2)] Fix \( h, \ell \in H \). Recall that there are \( p, q \in G \) such that \( \pi_H(p) = h, \pi_H(q) = \ell \) and \( d_{G/N}(h, \ell) = d(p, q) \), we get that
\[
d_{\mid_H}(h, \ell) = d(\pi_H(p), \pi_H(q)) \leq Ld(p, q) = Ld_{G/N}(h, \ell),
\]
where in the first inequality we used the Lipschitz property of $\pi_H$.

$(2) \Rightarrow (1)$ We notice that for every $p, q \in G$ with $\pi_H(p) = h$ and $\pi_H(q) = \ell$

$$d(\pi_H(p), \pi_H(q)) = d(h, \ell) \leq Ld_{G/N}(h, \ell) = Ld(\pi_H^{-1}(h), \pi_H^{-1}(\ell)) \leq Ld(p, q).$$

The last statement follows from the simple fact that $d_{G/N}(h, \ell) = d(\pi_H^{-1}(h), \pi_H^{-1}(\ell)) = d(Nh, N\ell) \leq d(h, \ell)$.

From Proposition 2.4 we have the following consequence.

**Corollary 6.11.** Let $G = N \rtimes H$ semidirect metric group with $N$ boundedly compact. If the projection $\pi_H : G \to H$ is Lipschitz, then every intrinsically Lipschitz section $\psi : N \to G$ for $\pi_N$ is a Lipschitz embedding.

6.6. **Link between the various notions.** Recall that in a group that admits a splitting $G = H_1 \cdot H_2$, to every map $\psi : H_1 \to H_2$ we associate its graph $\Gamma_\psi := \{n\psi(n) : n \in H_1\} \subset G$.

**Lemma 6.12.** Let $G = H_1 \cdot H_2$ be a splitted metric group. For every $\psi : H_1 \to H_2$, the following are equivalent:

1. $\psi$ is an intrinsically $K$-Lipschitz map in the FSSC sense, as in (14);
2. it holds

$$d(x, x') \leq \tilde{K}d(1, \pi_{H_1}(x^{-1}x')) , \quad \forall x, x' \in \Gamma_\psi.$$

**Proof.** $(1) \Rightarrow (2)$ Using the triangle inequality we have that for any $x, x' \in \Gamma_\psi$

$$d(x, x') = d(1, x^{-1}x') \leq d(1, \pi_{H_1}(x^{-1}x')) + d(1, \pi_{H_2}(x^{-1}x')) \leq (K + 1)d(1, \pi_{H_1}(x^{-1}x')).$$

$(2) \Rightarrow (1)$ Using the left invariant property of $d$ and the triangle inequality we obtain that for any $x, x' \in \Gamma_\psi$

$$d(1, \pi_{H_2}(x^{-1}x')) \leq d(1, x^{-1}x') + d(1, \pi_{H_1}(x^{-1}x')) \leq (\tilde{K} + 1)d(1, \pi_{H_1}(x^{-1}x')).$$

**Proposition 6.13.** Let $G = N \rtimes H$ be a semidirect metric group such that $\pi_N$ is $k$-Lipschitz at 1. For every $\psi : N \to H$, the following are equivalent:

1. $\psi$ is an intrinsically $K$-Lipschitz map in the FSSC sense as in (14);
2. $\Gamma_\psi \subset G$ is the graph of an intrinsically Lipschitz map with respect to the standard sections, i.e., (16) holds for every $g, \hat{g} \in \Gamma_\psi$.

**Proof.** $(2) \Rightarrow (1)$ This follows from (17) noticing that the set $xH \cap x'N$ is a singleton for every $x = n\psi(n), x' = m\psi(m) \in \Gamma_\psi$, with $n, m \in N$. Indeed, using the fact that $N$ is normal, if $nh = m\psi(m)n' \in xH \cap x'N$, for some $h \in H$ and $n \in N$, then

$$nh = m\psi(m)(n')\psi(m),$$

and so by uniqueness of the projection on $N$ and on $H$ we get that $h = \psi(m)$ and $n' = C_{\psi(m)}^{-1}(m^{-1}n)$.

$(1) \Rightarrow (2)$ Using Lemma 6.12 and recall that the set $xH \cap x'N$ is a singleton, for any $x, x' \in \Gamma_\psi$, we have that

$$d(x, xH \cap x'N) \leq d(x, x') + d(x', xH \cap x'N)$$

$$\leq (K + 1)d(1, \pi_{H_1}(x')^{-1}x) + d(x', xH \cap x'N)$$

$$\leq C(K + 1)d(1, (x')^{-1}xH) + d(x', xH \cap x'N),$$

where $C$ is a positive constant.
where in the last inequality we used Proposition \((6.3)\) (3). Now we consider \(h_2 \in H\) such that \(d(x', xH \cap x'N) = d(x', xh_2)\) and, consequently,
\[
d(x, xH \cap x'N) \leq C(K + 1)d(x', xh_2) + d(x', xH \cap x'N) = (C(K + 1) + 1)d(x', xH \cap x'N).
\]

In the context of metric groups, Proposition \((1.5)\) is as follows. Regarding Carnot groups, the reader can see [FS16, SC16] and their references.

**Corollary 6.14.** Let \(G = H_1 \cdot H_2\) be a split metric group such that \(\pi_{H_1}\) is \(k\)-Lipschitz at \(1\). Let \(\psi : H_1 \to H_2, h_1 \in H_1\) and \(p = h_1\psi(h_1)\). Then the following statements are equivalent:

1. \(\psi\) is intrinsically \(L\)-Lipschitz in the FSSC sense at \(h_1 \in H_1\);
2. for all \(\hat{L} \geq (L + 1)k\), it holds
\[
p \cdot X_{H_2}(1/\hat{L}) \cap \Gamma_{\psi} = \emptyset,
\]
where \(p \cdot X_{H_2}(\alpha)\) is the cone with axis \(H_2\), vertex \(p\), opening \(\alpha\) defined as the translation of
\[
X_{H_2}(\alpha) := \{ g \in G : d(g^{-1}, H_2) < \alpha d(1, g)\}.
\]

**Proof.** It is enough to combine Lemma \((6.12)\) and Proposition \((6.3)\).

We conclude this section proving Proposition \((1.6)\).

**Proof of Proposition \((1.6)\).** (1.6.i.a) and (1.6.i.b). The statements can be either found in [FS16, Proposition 2.2.9], or, more generally, they follow from Proposition \((6.8)\) and Proposition \((6.4)\).

(1.6.ii) Because \(N\) is a subgroup and because of left-invariance of intrinsically Lipschitz sections (see Proposition \((6.1)\)), it is enough to prove that \(N \simeq G/H \hookrightarrow G\) is an intrinsically Lipschitz section of \(\pi_N\). From Proposition \((6.3)\) we conclude if we have \((1.6.i.b)\) (or, equivalently from Proposition \((6.4)\) if we have \((1.6.i.a)\)).

(1.6.iii) We want to prove \((33)\) for \(\varphi\). Notice that from the definition \((9)\) of \(\varphi\) and the fact that \(\psi\) is ranged into \(H\), we have
\[
\varphi(g_2H)H = \pi_N(g_2)\psi(\pi_N(g_2))H = g_2H.
\]
Since \(\psi : N \to H\) is intrinsically Lipschitz map in the FSSC sense, by Lemma \((6.12)\) we have \((49)\), once we observe that \(\varphi\) is ranged into \(\Gamma_{\psi}\). Hence, we use Proposition \((6.3)\) (3) to get the desired equation \((33)\):
\[
d(\varphi(g_1H), \varphi(g_2H)) \leq \tilde{K}d(1, \pi_N(\varphi(g_1H)^{-1}\varphi(g_2H))) \leq \tilde{K}d(1, \varphi(g_1H)^{-1}\varphi(g_2H)H) \leq \tilde{K}d(\varphi(g_1H), g_2H).
\]

(1.6.iv) Vice versa, we want to prove \((49)\) for the map \(\psi\) defined as in \((10)\), assuming \((33)\). First, for all \(n, m \in N\) observe that, since \(\psi\) is ranged into \(H\) we have that
\[
\pi_N((n\psi(n))^{-1}m\psi(m)) = \pi_N((n\psi(n))^{-1}m\psi(n)) = (n\psi(n))^{-1}m\psi(n),
\]

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where in the last equality we used that $N$ is normal. Then, for all $n, m \in N$ we have
\[ d(n\psi(n), m\psi(m)) \leq Ld(\varphi(nH), mH) \]
\[ = Ld(1, (n\psi(n))^{-1}mH) \]
\[ \leq Ld(1, (n\psi(n))^{-1}m\psi(n)) = Ld(1, \pi_N((n\psi(n))^{-1}m\psi(m))). \]

where in the inequality we used that $\psi$ is ranged into $H$. \hfill \Box

6.7. Ahlfors regularity in groups. A corollary of Theorem 6.13 is Theorem 6.16 which roughly states that an intrinsically Lipschitz graph on a normal Ahlfors-regular subgroup $N$ is Ahlfors-regular.

Remark 6.15. Given a left-Haar measure $\mu_N$ on a closed normal subgroup $N \triangleleft G$, then the measure $\mu_N$ may not be preserved by conjugation by elements in $G$. However, assuming in addition that $G$ is a Lie group, we claim that for all compact sets $K \subseteq G$ there exists $C > 0$ such that for all $g \in K$ we have that on $K$ the Jacobian of $C_g$ with respect to $\mu_N$ is bounded by $C$. Indeed, this last statement follows from the fact that on Lie groups every Haar measure is given by a smooth volume form and each map $C_g : N \to N$ is smooth.

At this point we have an easy rephrasing of Theorem 1.3 in the case of groups. Still, we provide the short proof next.

Theorem 6.16. Let $G = N \rtimes H$ be a semidirect metric Lie group with boundedly compact distance. Assume that $\pi_H : G \to H$ is Lipschitz and that $N$ is locally $Q$-Ahlfors regular. If $\varphi : G/H \to G$ is an intrinsically $L$-Lipschitz section, then $\varphi(G/H)$ is locally $Q$-Ahlfors regular.

Recall that requiring that $N$ is locally $Q$-Ahlfors regular means, first that the $Q$-Hausdorff measure $\mu_N$ of $N$ is locally finite and nonzero, hence, being left-invariant, it is a left-Haar measure; second, we have that for each point $p \in N$ there are $c, r_0 > 0$ so that
\[ c^{-1}r^Q \leq \mu_N(B_N(p, r)) \leq cr^Q, \quad \forall r \in (0, r_0). \]

Proof. We plan to use Theorem 1.3. Let $X = G$, $Y = G/H$, and $\pi : X \to Y$ the projection. We identify $G/H$ with $N$ and $\pi$ with $\pi_N$. We shall show that the $Q$-Hausdorff measure $\mu := \mu_N$ on $N$ is such that for every $r_0 > 0$ and every $x, x' \in G$ with $\pi(x) = \pi(x')$ there is $C > 0$ such that

\[ \mu(\pi(B(x, r))) \leq C\mu(\pi(B(x', r))), \quad \forall r \in (0, r_0). \]

Fix $r_0 > 0$ and $x, x' \in G$ such that $\pi(x) = \pi(x')$, i.e., there is $n \in N$ and $h, h' \in H$ such that $x = nh$, $x' = nh'$. Hence \[ \pi(B(x', r)) = \pi(\{nh'g : g \in G, d(1, g) \leq r\}) \]
\[ = \{nC_{h'}(\pi(g)) : g \in G, d(1, g) \leq r\} \]
\[ = L_nC_{h'}(\pi(B(1, r))). \]

Moreover, using a similar argument, it easy to see that $\pi(B(1, r)) = C_{h^{-1}}L_{n^{-1}}(\pi(B(x, r)))$ and, consequently,
\[ \pi(B(x', r)) = L_nC_{h'}C_{h^{-1}}L_{n^{-1}}(\pi(B(x, r))), \quad \forall r \in (0, r_0). \]
Since \( \pi(\bar{B}(x,r_0)) \) is contained in a compact set \( K \subset N \), we have that on the set \( K \), the map \( N \ni m \mapsto L_nC_hC_{h^{-1}}L_{n^{-1}}(m) \) is smooth and hence has bounded Jacobian with respect to the (smooth) measure \( \mu \), say by \( C > 0 \). Hence, (53) holds and we apply Theorem 1.3 in order to obtain the thesis.

As a consequence, as done by Franchi and Serapioni [FSI16], one could see this result in the context of Carnot groups:

**Corollary 6.17** (FSSC). Let \( G = N \rtimes H \) be a Carnot group that is the semidirect product of two homogeneous subgroups, with \( N \) normal. For every \( \varphi : N \to H \) intrinsically Lipschitz map in the FSSC sense, the set \( \Gamma_{\varphi} \) is locally Ahlfors regular.

**Proof.** In order to justify the application of Theorem 6.16 we stress that the distance \( d \) on each Carnot group is boundedly compact and, since \( N \) is homogeneous, the distance \( d \) restricted on \( N \) is homogenous and hence \( N \) is \( Q \)-Ahlfors regular. Because on Carnot groups intrinsically Lipschitz maps in the FSSC sense are in correspondence (with same graphs) to intrinsically \( L \)-Lipschitz section (see Proposition 1.6), Theorem 6.16 gives the corollary.

### 6.8. Level sets and extensions in groups

In this section we present Theorem 1.4 in Carnot groups which is already proved in [Vit20, Theorem 1.4]. We underline that Vittone shows the result in \( \mathbb{R}^s \) and not only in \( \mathbb{R} \) and he uses the coercivity condition, which corresponds to asking a biLipschitz property of \( f \) on the fibers. However, it is possible to obtain the following result:

**Theorem 6.18** (Vittone). Let \( G = N \rtimes H \) be a Carnot group that is the semidirect product of a normal subgroup \( N \) and a one-dimensional horizontal subgroup \( H \), i.e., \( H = \{ \exp(tX) : t \in \mathbb{R} \} \) for some \( X \) in the first layer of \( G \). If \( S \subset G \) is not empty, then the following statements are equivalent:

1. there exists a map \( \psi : U \subset N \to H \) that is intrinsically Lipschitz in the FSSC sense, here \( U \) is a subset of \( N \), with \( S = \Gamma_{\psi} \);
2. there exists a Lipschitz map \( f : G \to \mathbb{R} \) that is biLipschitz on fibers such that \( S \subset f^{-1}(0) \).

**Proof.** Recall from Theorem 1.6 that there is a dual viewpoint between maps \( \psi : U \subset N \to H \) that are intrinsically Lipschitz in the FSSC sense and maps \( \varphi : U \subset G/H \to G \) that are intrinsically Lipschitz sections of the projection \( \pi : G \to G/H \). We shall use this identification.

The proof of the theorem will be just an application of our Theorem 1.4. We apply the theorem with the following notation: \( X = G, Y = G/H \simeq N, \pi : G \to G/H, Z = \mathbb{R} \).

\( (6.18.2) \Rightarrow (6.18.1) \) From Theorem 1.4i there is an intrinsically Lipschitz section \( \varphi : G/H \to G \) (and equivalently a map \( \psi : N \to H \) that is intrinsically Lipschitz in the FSSC sense) such that \( \Gamma_{\varphi} = \varphi(G/H) = f^{-1}(0) \). Then, it is enough to take \( U := \{ n \in N : n\psi(n) \in S \} \) and restrict the \( \psi \) to \( U \).

\( (6.18.1) \Rightarrow (6.18.2) \) Next we use Theorem 1.4ii. We have that \( X = G \) is geodesic and that admits equivalent homogeneous distances \( \rho \) with the property that the distance from the origin \( 1_G \) is smooth away from \( \exp(V_1) \). We also take \( \tau := \pi_H : G \to \mathbb{R} \), where we identify \( \mathbb{R} \) with \( H \) via the map \( t \mapsto \exp(tX) \). Since \( \tau \) can be seen as the projection modulo the normal subgroup \( N \), then it is Lipschitz. Moreover, the level sets \( \tau^{-1}(\tau_0) \) are left-translations of \( N \), which are intrinsically \( k \)-Lipschitz graph of sections \( \varphi_{g_0}(gH) := g_0\pi_N(g) \), see Proposition
together with \([1.0]\). Next, we check the assumption (2) of \([1.4]\). Because of left invariance, we can just consider the function \(x \mapsto \delta_0(x) := \rho(1_G, \pi_N(x))\) on the set \(|\pi_H(x)| \leq \delta_0(x)\). Notice that, denoting by \(M\), the intrinsic multiplication in the Carnot group, we have \(\pi_H(M(x)) = \varepsilon \pi_N(x)\) and \(\delta_0(M(x)) = \varepsilon \delta_0(x)\). Hence the set \(|\pi_H(x)| < \delta_0(x)\) is dilation invariant and its intersection avoids \(\exp(V_i)\). Consequently, on it the function \(\delta_0\) is the composition of smooth functions, which are therefore Lipschitz on compact sets. Again, by homogeneity, the function is Lipschitz. (This last part of the argument is not very different from Vittone’s original proof.) Applying Theorem \([1.4]\) concludes the existence of the requested function \(f : G \to \mathbb{R}\). \(\square\)

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