A LOWER BOUND FOR THE KÄHLER-EINSTEIN DISTANCE FROM THE DIEDERICH-FORNÆSS INDEX

ANDREW ZIMMER

Abstract. In this note we establish a lower bound for the distance induced by the Kähler-Einstein metric on pseudoconvex domains with positive hyperconvexity index (e.g. positive Diederich-Fornæss index). A key step is proving an analog of the Hopf lemma for Riemannian manifolds with Ricci curvature bounded from below.

1. Introduction

Every bounded pseudoconvex domain $\Omega \subset \mathbb{C}^d$ has a unique complete Kähler-Einstein metric, denoted by $g_{KE}$, with Ricci curvature $-(2d-1)$. This was constructed by Cheng and Yau [CY80] when $\Omega$ has $C^2$ boundary and by Mok and Yau [MY83] in general.

Let $d_{KE}$ be the distance induced by $g_{KE}$. Since $g_{KE}$ is complete, if we fix $z_0 \in \Omega$, then

$$\lim_{z \to \partial \Omega} d_{KE}(z, z_0) = \infty.$$  

(1)

In this note we consider quantitative versions of Equation (1). In particular, it is natural to ask for lower bounds on $d_{KE}(z, z_0)$ in terms of the distance to the boundary function $$\delta_{\Omega}(z) = \min\{\|w - z\| : w \in \partial \Omega\}.$$ Mok and Yau proved for every $z_0 \in \Omega$ there exists $C_1, C_2 \in \mathbb{R}$ such that

$$d_{KE}(z, z_0) \geq C_1 + C_2 \log \log \frac{1}{\delta_{\Omega}(z)}$$

for all $z \in \Omega$, see [MY83, pg. 47]. Further, by considering the case of a punctured disk, this lower bound is the best possible for general pseudoconvex domains.

However, for certain classes of bounded pseudoconvex domains, there are much better lower bounds. For instance, if $\Omega$ is convex, then for any $z_0 \in \Omega$ there exists $C_1, C_2 > 0$ such that

$$d_{KE}(z, z_0) \geq C_1 + C_2 \log \frac{1}{\delta_{\Omega}(z)}$$

for all $z \in \Omega$, see [Fra91]. In this note, we show that Estimate (2) holds for a large class of domains - those with positive hyperconvexity index.
First we recall the well studied Diederich-Fornæss index. Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain. A number $\tau \in (0, 1)$ is called a Diederich-Fornæss exponent of $\Omega$ if there exist a continuous plurisubharmonic function $\psi : \Omega \to (-\infty, 0)$ and a constant $C > 1$ such that

$$\frac{1}{C} \delta_{\Omega}(z)^{\tau} \leq -\psi(z) \leq C \delta_{\Omega}(z)^{\tau}$$

for all $z \in \Omega$. Then the Diederich-Fornæss index of $\Omega$ is defined to be

$$\eta(\Omega) := \sup \{ \tau : \tau \text{ is a Diederich-Fornæss exponent of } \Omega \}.$$ 

It is known that $\eta(\Omega) > 0$ for many domains. For instance, Diederich-Fornæss [DF77] proved that $\eta(\Omega) > 0$ when $\partial \Omega$ is $C^2$. Later, Harrington [Har08] generalized this result and proved that $\eta(\Omega) > 0$ when $\partial \Omega$ is Lipschitz.

The hyperconvexity index, introduced by Chen [Che17], is a similar quantity associated to a bounded pseudoconvex domain $\Omega \subset \mathbb{C}^d$. In particular, a number $\tau \in (0, 1)$ is called a hyperconvexity exponent of $\Omega$ if there exist a continuous plurisubharmonic function $\psi : \Omega \to (-\infty, 0)$ and a constant $C > 1$ such that

$$-\psi(z) \leq C \delta_{\Omega}(z)^{\tau}$$

for all $z \in \Omega$. Then the hyperconvex index of $\Omega$ is defined to be

$$\alpha(\Omega) := \sup \{ \tau : \tau \text{ is a hyperconvexity exponent of } \Omega \}.$$ 

By definition $\alpha(\Omega) \geq \eta(\Omega)$. Further, it is sometimes easier to verify that the hyperconvexity index is positive (see [Che17, Appendix]).

For domains with positive hyperconvexity index we will establish the following lower bound for $d_{KE}$.

**Theorem 1.1.** Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with $\alpha(\Omega) > 0$. If $z_0 \in \Omega$ and $\epsilon > 0$, then there exists some $C = C(z_0, \epsilon) \leq 0$ such that

$$d_{KE}(z, z_0) \geq C + \left( \frac{\alpha(\Omega)}{2d - 1} - \epsilon \right) \log \frac{1}{\delta_{\Omega}(z)}$$

for all $z \in \Omega$.

In this note we have normalized the Kähler-Einstein metric to have Ricci curvature equal to $-(2d - 1)$. If we instead normalized so that the Ricci curvature equals $-(2d - 1)\lambda$ we would obtain the lower bound

$$C + \frac{1}{\lambda \sqrt{\lambda}} \left( \frac{\alpha(\Omega)}{2d - 1} - \epsilon \right) \log \frac{1}{\delta_{\Omega}(z)}.$$

In fact, we will show that Estimate (2) holds for any complete Kähler metric with Ricci curvature bounded from below.

**Theorem 1.2.** Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with $\alpha(\Omega) > 0$, $g$ is a complete Kähler metric on $\Omega$ with $\text{Ric}_g \geq -(2d - 1)$, and $d_g$ is the distance associated to $g$. If $z_0 \in \Omega$ and $\epsilon > 0$, then there exists some $C = C(z_0, \epsilon) \leq 0$ such that

$$d_g(z_0, z) \geq C + \left( \frac{\alpha(\Omega)}{2d - 1} - \epsilon \right) \log \frac{1}{\delta_{\Omega}(z)}$$

for all $z \in \Omega$. 
1.1. Lower bounds on the Bergman metric. It is conjectured that the Bergman
distance on a bounded pseudoconvex domain with $C^2$ boundary also satisfies Es-
timate (2). In this direction, the best general result is due Blocki [Blo05] who
extended work of Diederich-Ohsawa [DO95] and established a lower bound of the
form
$$
C_1 + C_2 \frac{1}{\log \log \left(1/\delta_{\Omega}(z)\right)} \log \frac{1}{\delta_{\Omega}(z)}
$$
for the Bergman distance on a bounded pseudoconvex domain with $C^2$ boundary.

Notice that Theorem 1.2 implies the conjectured lower bound for the Bergman
distance under the additional assumption that the Ricci curvature of the Bergman
metric is bounded from below.

Acknowledgements. I would to thank Yuan Yuan and Liyou Zhang for bring-
ing the hyperconvexity index to my attention. This material is based upon work
supported by the National Science Foundation under grant DMS-1904099.

2. A Hopf Lemma for Riemannian manifolds

The standard proof of the Hopf lemma implies the following estimate:

**Proposition 2.1** (Hopf Lemma). If $D \subset \mathbb{R}^d$ is a bounded domain with $C^2$ boundary
and $\varphi : D \to (-\infty, 0)$ is subharmonic, then there exists $C > 0$ such that
$$
\varphi(x) \leq -C \delta_D(x)
$$
for all $x \in D$.

We will prove a variant of (this version of) the Hopf Lemma for Riemannian manifolds with Ricci curvature bounded below.

Given a complete Riemannian manifold $(X, g)$, let $d_g$ denote the distance induced by $g$, let $\nabla_g$ denote the gradient, and let $\Delta_g$ denote the Laplace-Beltrami operator on $X$. A function $\varphi : X \to \mathbb{R}$ is subharmonic if $\Delta_g \varphi \geq 0$ in the sense of distributions.

**Proposition 2.2.** Suppose that $(X, g)$ is a complete Riemannian manifold with
$\text{Ric}(g) \geq -(2d - 1)$. If $x_0 \in X$, $\epsilon > 0$, and $\varphi : X \to (-\infty, 0)$ is subharmonic, then there exists $C > 0$ such that
$$
\varphi(x) \leq -C \exp \left( - (2d - 1 + \epsilon) d_g(x, x_0) \right)
$$
for all $x \in X$.

We require one lemma. Given a complete Riemannian manifold $(X, g)$, $x \in X$, and $r > 0$ define
$$
B_g(x, r) = \{ y \in X : d_g(x, y) < r \}.
$$

**Lemma 2.3.** Suppose that $(X, g)$ is a complete Riemannian manifold with $\text{Ric}(g) \geq -(2d - 1)$. Then for every $x_0 \in X$ and $\epsilon > 0$, there exists $r_0 > 0$ such that the function
$$
\Phi(x) = \exp \left( - (2d - 1 + \epsilon) d_g(x, x_0) \right)
$$
is subharmonic on $X \setminus B_g(x_0, r_0)$. 

When the function \( x \to d_q(x, x_0) \) is smooth on \( X \setminus \{x_0\} \), the lemma is an immediate consequence of the Laplacian comparison theorem. We prove the general case by simply modifying the proof of the Laplacian comparison theorem given in [Pet16].

**Proof.** Let \( r(x) = d_q(x, x_0) \). We will show that
\[
\Delta_g \Phi(x) \geq \Phi(x) \left( (2d - 1 + \epsilon)^2 - (2d - 1)(2d - 1 + \epsilon) \coth r(x) \right)
\]
in the sense of distributions on \( X \setminus \{x_0\} \), which implies the lemma.

Fix \( q \in X \) and let \( \sigma : [0, T] \to X \) be a unit speed geodesic joining \( x_0 \) to \( q \). Then for \( \delta \in (0, T) \) consider the function \( r_{q, \delta}(x) = d_q(x, \sigma(\delta)) + \delta \). By the proof of [Pet16] Lemma 7.1.9, \( q \) is not in the cut locus of \( \sigma(\delta) \). In particular, there exists a neighborhood \( O_q \) of \( q \) such that \( r_{q, \delta} \) is \( C^\infty \) and
\[
\| \nabla_g r_{q, \delta} \| = 1
\]
on \( O_q \), see [Sak96] Proposition III.4.8. Further, by the Laplacian comparison theorem
\[
\Delta_g r_{q, \delta}(x) \leq (2d - 1) \coth (r_{q, \delta}(x) - \delta)
\]
on \( O_q \), see [Pet16] Lemma 7.1.9. Next consider the function \( \Phi_{q, \delta} : O_q \to [0, \infty) \) defined by
\[
\Phi_{q, \delta}(x) = \exp \left( - (2d - 1 + \epsilon) r_{q, \delta}(x) \right).
\]

Then
\[
\Delta_g \Phi_{q, \delta}(x) = \Phi_{q, \delta}(x) \left( (2d - 1 + \epsilon)^2 \| \nabla_g r_{q, \delta} \|^2 - (2d - 1 + \epsilon) \Delta_g r_{q, \delta}(x) \right)
\]
\[
\geq \Phi_{q, \delta}(x) \left( (2d - 1 + \epsilon)^2 - (2d - 1 + \epsilon)(2d - 1) \coth (r_{q, \delta}(x) - \delta) \right).
\]

(3)

Fix a partition of unity \( 1 = \sum_{j=1}^{\infty} \chi_j \) subordinate to the open cover \( X = \cup_{q \in X} O_q \). For each \( j \in \mathbb{N}, \) fix \( q_j \in X \) such that \( \text{supp}(\chi_j) \subset O_{q_j} \).

Now suppose that \( \psi : X \setminus \{x_0\} \to [0, \infty) \) is a compactly supported smooth function. Then by the dominated convergence theorem (notice that the sum is finite)
\[
\int_X \Phi(x) \Delta_g \psi(x) dx = \lim_{\delta \to 0^+} \sum_{j=1}^{\infty} \int_{O_{q_j}} \Phi_{q_j, \delta}(x) \Delta_g (\chi_j(x) \psi(x)) dx.
\]

By integration by parts and Equation (3)
\[
\int_{O_{q_j}} \Phi_{q_j, \delta}(x) \Delta_g (\chi_j(x) \psi(x)) dx = \int_{O_{q_j}} \chi_j(x) \psi(x) \Delta_g \Phi_{q_j, \delta}(x) dx
\]
\[
\geq \int_{O_{q_j}} \chi_j(x) \psi(x) \Phi_{q_j, \delta}(x) \left( (2d - 1 + \epsilon)^2 - (2d - 1 + \epsilon)(2d - 1) \coth (r_{q, \delta}(x) - \delta) \right) dx.
\]

So by applying the dominated convergence theorem again
\[
\int_X \Phi(x) \Delta_g \psi(x) dx \geq \int_X \Phi(x) \left( (2d - 1 + \epsilon)^2 - (2d - 1 + \epsilon)(2d - 1) \coth r(x) \right) \psi(x) dx.
\]

Hence
\[
\Delta_g \Phi(x) \geq \Phi(x) \left( (2d - 1 + \epsilon)^2 - (2d - 1 + \epsilon)(2d - 1) \coth r(x) \right)
\]
in the sense of distributions on \( X \setminus \{x_0\} \).
\[ \square \]
Proof of Proposition 2.2. Fix $r_0 > 0$ such that
\[ x \to \exp \left( - (2d - 1 + \epsilon) d_g(x, x_0) \right) \]
is subharmonic on $X \setminus B_g(x_0, r_0)$. Since $\varphi < 0$, there exists $C > 0$ such that
\[ \varphi(x) \leq -C \exp \left( - (2d - 1 + \epsilon) d_g(x, x_0) \right) \]
for all $x \in B_g(x_0, r_0)$. Then consider
\[ f(x) = \varphi(x) + C \exp \left( - (2d - 1 + \epsilon) d_g(x, x_0) \right). \]
Then $f$ is subharmonic on $X \setminus B_g(x_0, r_0)$. Fix $R > r_0$ and let
\[ A_R = B_g(x_0, R) \setminus B_g(x_0, r_0) \]
Then $f(x) \leq 0$ on $\partial B_g(x_0, r_0)$ and
\[ f(x) \leq C \exp \left( - (2d - 1 + \epsilon) R \right) \]
on $\partial B_g(x_0, R)$. So by the maximum principle
\[ f(x) \leq C \exp \left( - (2d - 1 + \epsilon) R \right) \]
on $A_R$. Then sending $R \to 0$ shows that
\[ f(x) \leq 0 \]
on $X \setminus B_g(x_0, r_0)$. So
\[ \varphi(x) \leq -C \exp \left( - (2d - 1 + \epsilon) d_g(x, x_0) \right) \]
for all $x \in X$. \qed

3. Proof of Theorem 1.2

Suppose $\Omega \subset \mathbb{C}^d$ is a bounded pseudoconvex domain with $\alpha(\Omega) > 0$, $g$ is a complete Kähler metric on $\Omega$ with $\text{Ric}_g \geq -(2d - 1)$, $z_0 \in \Omega$, and $\epsilon > 0$.

Fix $\epsilon_1 > 0$ and a hyperconvexity exponent $\tau \in (0, 1)$ such that
\[ \frac{\tau}{2d - 1 + \epsilon_1} \geq \frac{\alpha(\Omega)}{2d - 1} - \epsilon. \]
Then there exists a continuous plurisubharmonic function $\psi : \Omega \to (-\infty, 0)$ and $a > 1$ such that
\[ -\psi(z) \leq a \delta_\Omega(z)^\tau \]
for all $z \in \Omega$.

Since $\psi$ is plurisubharmonic and $g$ is Kähler, $\psi$ is subharmonic on $(\Omega, g)$. So by Proposition 2.2 there exists $C_0 > 0$ such that
\[ \psi(z) \leq -C_0 \exp \left( - (2d - 1 + \epsilon_1) d_g(x, x_0) \right) \]
for all $z \in \Omega$. Then
\[ -a \delta_\Omega(z)^\tau \leq -C_0 \exp \left( - (2d - 1 + \epsilon_1) d_g(x, x_0) \right) \]
and so there exists $C_1 \in \mathbb{R}$ such that
\[
C_1 + \left( \frac{\tau}{2d - 1 + \epsilon_1} \right) \log \frac{1}{\delta_{\Omega}(z)} \leq d_g(z, z_0)
\]
for all $z \in \Omega$. Since the set \{ $z \in \Omega : \delta_{\Omega}(z) \geq 1$ \} is compact and
\[
\frac{\tau}{2d - 1 + \epsilon_1} \geq \frac{\alpha(\Omega)}{2d - 1} - \epsilon,
\]
there exists $C \in \mathbb{R}$ such that
\[
C + \left( \frac{\alpha(\Omega)}{2d - 1} - \epsilon \right) \log \frac{1}{\delta_{\Omega}(z)} \leq d_g(z, z_0)
\]
for all $z \in \Omega$.

**References**

[Blo05] Zbigniew Błocki. The Bergman metric and the pluricomplex Green function. *Trans. Amer. Math. Soc.*, 357(7):2613–2625, 2005.

[Che17] Bo-Yong Chen. Bergman kernel and hyperconvexity index. *Anal. PDE*, 10(6):1429–1454, 2017.

[CY80] Shiu Yuen Cheng and Shing Tung Yau. On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman’s equation. *Comm. Pure Appl. Math.*, 33(4):507–544, 1980.

[DF77] Klas Diederich and John Erik Fornaess. Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions. *Invent. Math.*, 39(2):129–141, 1977.

[DO95] Klas Diederich and Takeo Ohsawa. An estimate for the Bergman distance on pseudoconvex domains. *Ann. of Math. (2)*, 141(1):181–190, 1995.

[Fra91] Sidney Frankel. Applications of affine geometry to geometric function theory in several complex variables. I. Convergent rescalings and intrinsic quasi-isometric structure. In *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, volume 52 of *Proc. Sympos. Pure Math.*, pages 183–208. Amer. Math. Soc., Providence, RI, 1991.

[Har08] Phillip S. Harrington. The order of plurisubharmonicity on pseudoconvex domains with Lipschitz boundaries. *Math. Res. Lett.*, 15(3):485–490, 2008.

[MY83] Ngaiming Mok and Shing-Tung Yau. Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions. In *The mathematical heritage of Henri Poincaré, Part 1 (Bloomington, Ind., 1980)*, volume 39 of *Proc. Sympos. Pure Math.*, pages 41–59. Amer. Math. Soc., Providence, RI, 1983.

[Pet16] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, Cham, third edition, 2016.

[Sak96] Takashi Sakai. *Riemannian geometry*, volume 149 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. Translated from the 1992 Japanese original by the author.