Critical points at infinity for analytic combinatorics

Abstract: On complex algebraic varieties, height functions arising in combinatorial applications fail to be proper. This complicates the description and computation via Morse theory of key topological invariants. Here we establish checkable conditions under which the behavior at infinity may be ignored, and the usual theorems of classical and stratified Morse theory may be applied.

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1 Introduction

Recent studies of multivariate generating functions motivate several problems in the topology of complex algebraic varieties. Even when they fall within the scope of well-established theories, these questions are often outside of the realm of existing literature. Here we study one such problem, which lies at the heart of effective methods for the field of analytic combinatorics in several variables (ACSV), and give its solution using tools from deformation theory, Morse theory and algebraic geometry. We begin by describing the purely topological problem; its combinatorial origin is discussed below.

Fix a real Laurent polynomial $Q \in \mathbb{R}[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]$ and a nonzero real vector $r \in \mathbb{R}^d$ with normalization $\hat{r} = r / |r|$, where $|r| = |r_1| + \cdots + |r_d|$. Let $V$ denote the affine algebraic variety defined by $V = \{ Q = 0 \} \subset \mathbb{C}^d$, and $V_* := V \cap (\mathbb{C}^*)^d$ with $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Furthermore, define the function $h : V_* \to \mathbb{R}$ by

$$h(z) := \hat{r} \cdot \{ \log |z| \} = \hat{r}_1 \log |z_1| + \cdots + \hat{r}_d \log |z_d|$$

and assume that $h$ is Morse on $V$ (i.e. the Hessian of $h$ at each critical point has full rank).

**Problem.** Use standard methods of (stratified) Morse theory to give a topological decomposition of $H_d((\mathbb{C}^*)^d \setminus V)$ in terms of relative cycles near critical points of the potentially non-proper height function $h$.

The key word here is ‘non-proper’: most basic results of standard and stratified Morse theory require that the height function $h$ be a proper map (see [Mil63] and [GM88], respectively). Although stratified Morse theory contains results for certain types of non-proper height functions, they apply primarily in the case where the topological space in question fails to be complete because a point was removed [GM88, Chapter 10]. In our situation the function $h$, fixed by the underlying combinatorics, fails to be proper not only near points where one of the coordinates is zero but also due to points at infinity (for certain vectors $\hat{r}$ there might be ‘critical point at infinity’ which stay at finite height). For this reason, books and papers on analytic combinatorics in several variables [PW08, PW13] often use Morse-theoretic heuristics to motivate certain constructions, but cannot use Morse theory outright to prove general results.

One solution to the problem of non-properness is to compactify. Suppose we can embed $(\mathbb{C}^*)^d$ in a compact space $K$ such that $h$ may be extended smoothly to the closure $\overline{V}$ of $V$ in $K$. Then Morse theory gives a decomposition of the topology of $K$ in terms of cycles local to (stratified) critical points. Chains in $V_*$ may be decomposed into sums of relative homology cycles near these critical points, some of which are critical points for $h$ on $V$, and some of which may be **critical points at infinity**. Conjecture 2.11 of [Pem10] posits that a compactification $K$ may always be found on which $h$ extends smoothly. In Appendix A of this paper we include a one-page argument relying on standard results in toric geometry which resolves the conjecture. Unfortunately, this solution works only when $r$ is an integer vector and the structure of the compactification depends on $r$. This is not ideal for the underlying combinatorial applications, as it hinders a complete analysis of multivariate generating functions and hides strong links between the different sequences they encode.

Furthermore, for generic parameter $\hat{r}$, there are no critical points at infinity, and no need to embark on a cumbersome computational exercise. For this reason, we were motivated to find a solution better adapted to use in the combinatorial setting. Instead of trying to construct a compactification of $V$ compatible with
h, we give instead a computationally effective criteria under which the requisite Morse decomposition on \( V \) remains confined to a compact region.

Our main results are the following.

(i) A definition of critical points at infinity for a polynomial \( Q \) with respect to a direction \( r \) (Definition 2.3).
(ii) An algorithm for detecting critical points at infinity (Section 3), including a Maple implementation.
(iii) Theorem 2.4, which states that in the absence of critical points at infinity of the height function \( h \) the usual Morse theoretic decomposition of the topology of \( V \) applies.

In the remainder of this section we discuss the combinatorial origins of the problem.

Section 2 sets the notation for the study of stratified spaces and critical points, formulates the definition of critical points at infinity and states the main result. Section 3 shows how to determine all critical points at infinity using a computer algebra system. Some examples are given in Section 4. Section 5 constructs a Morse deformation in the absence of critical points at infinity, and Section 6 concludes by proving Theorem 2.4.

Motivation from ACSV

Analytic combinatorics in several variables (ACSV) studies coefficients of multivariate generating functions via analytic methods; see, for example, [PW13]. The most developed part of the theory is the asymptotic determination of coefficients of multivariate series \( F(z) = \sum_{\mathbf{r}} a_{\mathbf{r}} z^{\mathbf{r}} \), wherein the coefficients \( a_{\mathbf{r}} \) are defined by the multivariate Cauchy integral

\[
a_{\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{T} z^{-r} F(z) \frac{dz}{z},
\]

with \( T \) an appropriate chain of integration (a certain torus defined by \( |z_k| = \epsilon_k > 0 \) for \( k = 1, \ldots, d \)). In many applications, \( F(z) = P(z)/Q(z) \) is rational function with a power series expansion whose coefficients are indexed by \( \mathbf{r} \), an integer vector. More generally, one often looks for asymptotics in the directions \( \hat{\mathbf{r}} \) that vary over a convex cone \( K \subseteq \mathbb{R}^d \).

Given \( \mathbf{r} \in \mathbb{R}^d \) with \( \hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}| \) we define the functions

\[
\begin{align*}
\tilde{h}_R(z) := &-\Re(\hat{\mathbf{r}} \cdot z) \\
\tilde{h}_C(z) := &-\hat{\mathbf{r}} \cdot z \\
h_R(z) := &-\Re(\hat{\mathbf{r}} \cdot \log z) \\
h_C(z) := &-\hat{\mathbf{r}} \cdot \log z,
\end{align*}
\]

where \( \log \) is taken coordinate-wise and \( \Re(z) \) denotes the real part of complex \( z \).

In order to estimate \( a_{\mathbf{r}} \) for \( |\mathbf{r}| \to \infty \) and \( \mathbf{r}/|\mathbf{r}| \to \hat{\mathbf{r}} \), one isolates the exponential part of the Cauchy integral (1.1) by writing

\[
\left| z^{-r} \frac{F(z)}{z} \right| = \exp(|\mathbf{r}| h(z)) \left| \frac{F(z)}{z_1 \cdots z_d} \right|.
\]

One then deforms \( T \) to a chain \( \mathcal{C} \) on which the maximum value of \( h_R(z) \) is minimized; the maximum value on \( \mathcal{C} \) will occur at some (stratified) critical point of \( h_R \) on \( V \). Let \( \mathcal{M} := (\mathbb{C}^*)^d \setminus V \) denote the domain of holomorphy of the integrand. The value of the Cauchy integral depends on \( T \) only via its homology class \( [T] \in H_d(\mathcal{M}) \). In the absence of the non-properness issue discussed above, and for smooth \( V \), Morse theory
would allow us to find a basis for $H_d(M)$ composed of tubes around unstable manifolds of the critical points of $h$ on $V$ (the critical points are necessarily of of index $(d - 1)$). Resolving the class $[T]$ into this basis results in the decomposition $[T] = \sum_{j=1}^k b_j [C(x_j)]$ where $b_j$ are integers and $C(x_j)$ are tubes around certain cycles $\gamma(x_j)$ corresponding to the critical points $x_j$.

Taking residues results in a sum of integrals

\[ \int_{\gamma(x_j)} z^{-r-1} F(z) dz \]

which are generally well understood asymptotically, computed either as stationary phase integrals or by more difficult singularity theory in the case where $x_j$ is a singular point of the variety $V$.

## 2 Definitions and results

### 2.1 Stratified spaces and critical points

Let $T := \mathbb{R}/2\pi \mathbb{Z}$ and $L$ denote $\mathbb{R}^d \times T^d$, referred to as “log space”. Log space is diffeomorphic to $(\mathbb{C}^*)^d$ via the map $\Phi : L \to (\mathbb{C}^*)^d$ defined by

\[ \Phi : (x, y) \mapsto \exp(x + iy). \]  

We use a tilde to denote the result of pulling back to the log space via $\Phi$. Most of our constructions will take place on the log space, with the analogous constructions on $(\mathbb{C}^*)^d$ defined as the image under $\Phi$. The reason for keeping both $(\mathbb{C}^*)^d$ and $L$ around is that the geometric constructions are more transparent in $L$ but polynomial computations via computer algebra are carried out in $(\mathbb{C}^*)^d$.

A Whitney stratification of a complex algebraic variety $V$ is a decomposition of $V$ into a disjoint union of complex manifolds of dimensions between 0 and $d$ with so-called Whitney conditions on how the tangent planes of the different manifold interact (see, for example, [PW13, Definition 5.4.1]). Such a stratification of $V$ always exists and may be chosen to be compatible with one for $V^*$. If $\{S_\alpha : \alpha \in A\}$ is a Whitney stratification of $V_\alpha$, we let $\{\tilde{S}_\alpha : \alpha \in A\}$ denote the stratification of log $V_\alpha$, where $\tilde{S}_\alpha$ is the inverse image of $S_\alpha$ under $\Phi$.

We recall the stratified definition of an affine critical point of $V$. Note that critical points are only defined relative to stratifications; a stratification $\{S_\alpha\}$ is assumed but does not appear in the notation.

**Definition 2.1 (critical points).** The function $h$ is said to have a critical point (in the stratified sense) at $z \in V_\alpha$ if and only if $dh(z)|_{T_z S} = 0$ for the unique stratum $S$ containing $z$ (the projection of the differential of $h$ onto the tangent space of the stratum is zero). The set of critical points is denoted $\text{crit}(\tilde{r})$; it depends on $\tilde{r}$ because $h_{\mathbb{R}}$ depends on $\tilde{r}$. Later we will call these affine critical points to distinguish them from critical points at infinity.

**Proposition 2.2 (complex versus real).** The following conditions are equivalent:

- The point $z$ is a stratified critical point for $h_{\mathbb{R}}$ on $V_\alpha$.
- The point $z$ is a stratified critical point for $h_{\mathbb{C}}$ on $V_\alpha$. 


• The point \( \Phi^{-1}(z) \) is a stratified critical point for \( \tilde{h}_R \) on \( \tilde{V} \)
• The point \( \Phi^{-1}(z) \) is a stratified critical point for \( \tilde{h}_C \) on \( \tilde{V} \)

We summarize in figure 1.

![Diagram](image)

Figure 1: Critical points are the same in \( \mathcal{L} \) and \((\mathbb{C}^*)^d\), and for the complex valued height or its real part.

**Proof:** By functoriality, critical points of \( h_R \) for the stratification \( \{S_\alpha\} \) pull back to critical points of \( \tilde{h}_R \) for the stratification \( \{\tilde{S}_\alpha\} \) and likewise for \( h_C \) and \( \tilde{h}_C \). It remains to see that \( \tilde{h}_R \) and \( \tilde{h}_C \) have the same critical points. This is equivalent to showing

\[
d\tilde{h}_C|_S = 0 \iff d\tilde{h}_R|_S = 0.
\]

Clearly \( d\tilde{h}_C = 0 \) implies \( d\tilde{h}_R = 0 \) on any subspace. On the other hand, if \( d\tilde{h}_C(\xi) = \zeta \neq 0 \) for some tangent vector \( \xi \), then because we require stratifications to have complex structure, the vector \(-i\xi\) is also a tangent vector to \( S \), and either \( d\tilde{h}_R(\xi) = \Re\{d\tilde{h}_C(\xi)\} \) or \( d\tilde{h}_R(-i\xi) = 3\{d\tilde{h}_C(\xi)\} \) will be nonzero. \( \Box \)

Except in degenerate cases, the set \( \text{crit}(r) \) of critical points in direction \( r \) is a zero-dimensional ideal, easily computed in any computer algebra system; see, for example, [Mel17, Chapter 8].

### 2.2 Critical points at infinity and main deformation result

The logspace \( \mathcal{L} \) is an Abelian group acting on itself, and therefore one can introduce a shift-invariant Hermitian structure on it. In what follows, when we refer to orthogonal projections or angles between tangent vectors in the logspace, it is this shift-invariant structure that we mean.

**Definition 2.3 (critical point at infinity).** Given \( Q, V, \tilde{r} \) and a stratification \( \{S_\alpha : \alpha \in A\} \), let \( \{\tilde{S}_\alpha\} \) be the corresponding stratification in the log-space. A **critical point at infinity** is a sequence \( \{z_n\} \) of points in some stratum \( S \) such that the pullbacks \( \tilde{z}_n \) approach infinity and the length \( \ell_n := |\pi_{T_{\tilde{z}_n}(\tilde{S})}(\tilde{r})| \) of the projection of \( \tilde{r} \) to the tangent space of \( \tilde{S} \) at \( \tilde{z}_n \) goes to zero. The **set of heights** of a critical point at infinity (possibly empty) is the limit set of values \( \{h_R(z_n)\} \). If \( c \in \mathbb{R} \) is the limit of a sequence \( h_R(z_n) \) for a critical point at infinity \( \{z_n\} \), we say \( \{z_n\} \) **witnesses** a critical point at infinity at height \( c \). When there is no critical point at infinity in the direction \( \tilde{r} \) then there is none for directions in some neighborhood of \( \tilde{r} \).

For any space \( S \) with height function \( h \) and any real \( b \), we denote by \( S_{\leq b} \) the set \( \{x \in S : h(x) \leq b\} \). Our main result is the following theorem, together with an algorithm in Section 3 which shows that critical points at infinity are easily computed.

**Theorem 2.4 (no critical point at infinity implies Morse results).**

(i) Suppose there are no critical points at infinity with heights in \([a, b]\), nor any ordinary critical points with heights in \([a, b]\). Then \( V_{\leq b} \) is homotopy equivalent to \( V_{\leq a} \) via the downward gradient flow.
(ii) Suppose there is a single critical point \( x \) with critical value \( c \in [a, b] \), and there is no critical point at infinity with height in \([a, b]\). Then for any compact cycle \( C \) supported on \( M \leq b \setminus M \leq a \) there are \( T, \varepsilon > 0 \) such that the downward gradient flow run for time \( T \) takes \( C \) to a cycle supported on \( B(x, \varepsilon) \cup M \leq c - \varepsilon \), where \( B(x, \varepsilon) \) denotes the ball of radius \( \varepsilon \) around \( x \). In other words, every cycle can be pushed down so it is supported on the union of a neighborhood of \( x \) and the part of \( M \) below height \( c - \varepsilon \).

This immediately implies the following corollary. Because the cycle \( T \) can be pushed down at least until hitting the first critical point corresponding to direction \( \hat{r} \), the magnitude of coefficients in this direction is bounded above by the Cauchy integral over a contour at this height. This is given only as a conjecture in \([PW13]\) because it was not known under what conditions \( T \) could be pushed down to the critical height.

**Corollary 2.5.** Fix a Laurent polynomial \( Q \) and \( \hat{r} \) in the cone \( K \) supporting the Laurent expansion \( P(z)/Q(z) = \sum_{r \in K} a_r z^r \). Let \( c = \max_{x \in \text{crit}(\hat{r})} h_{\hat{r}}(x) \) be the maximal height of an affine critical point. Assume there are no critical points at infinity with height in \([c, \infty)\). Then

\[
\limsup_{r \to \infty} \frac{1}{r^{|r|}} \log |a_r| \leq c.
\]

In the remainder of Section 2 we outline how Theorem 2.4 is used to obtain Morse theoretic representations of integrals. These results, already quoted in several preprints, can be skipped if one is only interested in examples, proofs and computations of critical points at infinity.

### 2.3 Intersection classes

It is useful to be able to transfer between \( H_d(M) \) and \( H_{d-1}(\nu^*_x) \): topologically this is the Thom isomorphism and, when computing integrals, this corresponds to taking a single residue. We outline this construction, which goes back at least to Griffiths \([Gri69]\). Assume \( \nabla Q \) does not vanish on \( \nu \), so that \( \nu \) is smooth. Then the well known Collar Lemma \([MS74, \text{Theorem 11.1}]\) states\(^1\) that an open tubular vicinity of \( \nu \) is diffeomorphic to the space of the normal bundle to \( \nu \). We require a stratified version: if \( \nu \) intersects a manifold \( X \) transversely, then an open tubular neighborhood of the pair \((\nu, \nu \cap X)\) is diffeomorphic to the product \((\nu, \nu \cap X) \times D\) where \( D \) is a two-dimensional disk (because the normal bundle to \( \nu \) is \( \mathbb{C}^* \)). This statement follows from the version of Thom’s first isotopy lemma given in \([GM88, \text{Section 1.5}]\); the proof is only sketched, but the integral curve argument there mirrors what is given explicitly in \([PW13, \text{Section A.4}]\).

In particular, for any \( k \)-chain \( \gamma \) in \( \nu \), one can define a \((k + 1)\)-chain \( \partial \gamma \), obtained by taking the boundary of the union of small disks in the fibers of the normal bundle. The radii of these disk should be small enough to fit into the domain of the collar map, but can (continuously) vary with the point on the base. Different choices of the radii matching over the boundary of the chain lead to homologous tubes. We will be referring to \( \partial \gamma \) informally as the “tube around \( \gamma \). Similarly, the symbol \( \bullet \gamma \) denotes the product with the solid disk. The elementary rules for boundaries of products imply

\[
\partial(\partial \gamma) = \partial(\partial \gamma);
\]

\[
\partial(\bullet \gamma) = \partial(\bullet \gamma).
\]  

\(^1\)See \([Lan02]\) for a full proof.
Because $o$ commutes with $\partial$, cycles map to cycles, boundaries map to boundaries, and the map $o$ on the singular chain complex of $V_*$ induces a map on homology $H_*(\mathbb{C}^* \setminus V)$; we also denote this map on homology by $o$ to simplify notation.

**Proposition 2.6** (intersection classes). Suppose $Q$ vanishes on a smooth variety $V$ and let $T$ and $T'$ be two cycles in $M$ that are homologous in $(\mathbb{C}^*)^d$. Then there exists a class $\gamma \in H_{d−1}(V_*)$ such that

$$T - T' = o\gamma \quad \text{in } H_d(M).$$

The class $\gamma$ is represented by the cycle $\partial (H \cap V)$ for any cobordism $H$ between $T$ and $T'$ in $(\mathbb{C}^*)^d$ that intersects $V$ transversely, all such intersections yielding the same class in $H_{d−1}(V_*)$.

Accordingly, we may define the **intersection class** of $T$ and $T'$ by

$$I(T, T') := [\gamma] \in H_{d−1}(V_*)$$

which, by Proposition 2.6, is well defined.

**Proof:** Let $C$ be any $(d+1)$-chain in $(\mathbb{C}^*)^d$. If $C$ intersects $V$ transversely, let $I(C)$ denote the intersection of $C$ with $V$. We claim that $I$ induces a map $I : H_{d+1}((\mathbb{C}^*)^d) \to H_{d−1}(V_*)$. This follows from taking $I$ to be $I$ on any representing cycle $C$ transverse to $V$, provided that $I$ maps cycles to cycles and boundaries to boundaries. To see that this is the case, observe that by transversality and the product formula, $\partial(C \cap V) = 0$ and $C = \partial D$ implies $C \cap V = \partial(D \cap V)$.

The Thom-Gysin long exact sequence implies exactness in the following diagram,

$$0 \cong H_{d+1}(\mathbb{C}^*_d) \overset{\partial}{\longrightarrow} H_{d−1}(V_*) \overset{o}{\longrightarrow} H_d(M) \to H_d(\mathbb{C}^*_d). \quad (2.3)$$

This may be found in [Gor75, page 127], taking $W = (\mathbb{C}^*)_d$, though in the particular situation at hand it goes back to Leray [Ler50].

Consider now two $d$-cycles $T$ and $T'$, homologous in $(\mathbb{C}^*)^d$ and both avoiding $V_*$. We define the intersection class $I(T, T')$ as follows. Let $H$ be any cobordism in $(\mathbb{C}^*)^d$ between $T$ and $T'$, generically perturbed if necessary so as to be transverse to $V$. Let $\gamma = I(H)$, therefore

$$\partial \gamma = \partial (H \cap V) = (T - T') \cap V. \quad (2.4)$$

If $H'$ is another such cobordism then $H - H'$ is a $(d+1)$-cycle in $(\mathbb{C}^*)^d$, hence null-homologous, so $I(H - H') = 0$ and $I(H) = I(H')$ in $H_{d−1}(V_*)$. $\square$

### 2.4 Relative homology

Morse theory decomposes the topology of a manifold into the direct sum of relative homologies of attachment pairs. In the absence of critical points at infinity, we can harness this for the particular Morse functions $h_R$ of interest. Given $\mathbf{r}$, suppose the affine critical values, listed in decreasing order, are $c_1 > c_2 > \cdots > c_m$. Assume that there is precisely one critical point at each affine critical height.

Even without ruling out critical points at infinity, one has the following decomposition which works at the level of filtered spaces. Fix real numbers $b_0, \ldots, b_m$ interleaving the critical values, so that $b_0 > c_1 > b_1 >$
\[ \cdots > b_{m-1} > c_m > b_m, \text{ and for } 0 \leq j \leq m \text{ define } X_j := M_{\leq b_j}. \] Without ambiguity we may let \( \iota \) denote any of the inclusions of \( X_{j+k} \) into \( X_j \) and \( \pi \) denote any of the projections of a pair \((X_{j-\iota}, X_{j+k})\) to a pair \((X_{j-\iota}, X_j)\). A simple diagram chase proves the following result.

**Proposition 2.7** ([PW13, Lemma B.2.1]). Let \( \mathcal{C} \) be any class in \( H_d(M) \). The least \( j \) such that \( \mathcal{C} \) is not in the image of \( \iota_\ast : M \to X_0 \) is equal to the least \( j \) such that \( \pi_\ast \mathcal{C} \neq 0 \) in \( H_d(M, X_j) \). Denoting this by \( n \), there is a class \( \sigma \in H_d(X_{n-1}) \) such that \( \pi_\ast \mathcal{C} = \iota_\ast \sigma \) in \( H_d(M, X_n) \). The class \( \sigma \) is not unique but the projection \( \pi_\ast \sigma \) to \( H_d(X_{n-1}, X_n) \) is unique.

The interpretation is that the relative homology classes \( H_d(X_{j-1}, X_j) \) are the possible obstructions, and that \( \mathcal{C} \) may be pushed down until the first obstruction, which is a well defined relative homology element. Taking \( C_1 := C - \sigma \), we see that \( n(C_1) > n(C) \) so we may iterate, arriving at \( C = \sigma(C) + \sigma(C_1) + \cdots + \sigma(C_k) + \Gamma \) where \( \Gamma \) is a class supported on \( M_{\leq b_m} \).

When there are no critical points at infinity, Theorem 2.4 allows us to upgrade this to the following result. Informally, until you hit a critical value at infinity, you can push \( C \) down to each critical point, subtract the obstruction, and continue downward. The obstructions are the homology groups of the attachment pairs.

**Proposition 2.8** (the Morse filtration). Fix \( k \leq m \) and suppose there are no critical points at infinity with height in \([c_k, \infty)\). Let \( \mathcal{C} \in H_d(M) \) be any class. Then for sufficiently small \( \varepsilon > 0 \) and all \( j \in \{1, \ldots, k\} \), the pair \((X_{j-1}, X_j)\) is naturally homotopy equivalent to the pair \((M_{\leq c_j + \varepsilon}, M_{c_j - \varepsilon})\) which is further homotopic to \((M_{c_j - \varepsilon} \cup B_j, M_{c_j - \varepsilon})\) for any ball \( B_j \) about the critical point that reaches past \( c_j - \varepsilon \). It follows that there are \( \sigma_j \in H_d(M_{c_j - \varepsilon} \cup B_j) \) such that each \( \pi_\ast \sigma_{j+1} \in H_d(M_{c_{j+1} - \varepsilon} \cup B_{j+1}, M_{c_{j+1} - \varepsilon}) \) is well defined given \( \sigma_1, \ldots, \sigma_j \) and \( C - \sum_{i=1}^j \sigma_i \) projects to zero in \( H_d(M, M_{c_j - \varepsilon}) \).

**Proof:** The first homotopy equivalence follows from part (i) of Theorem 2.4 and the second from part (ii). The remainder follows from Proposition 2.7 and the homotopy equivalences.

Combining this representation of \( C \) as \( \sum_{i=1}^j \sigma_i \) with the construction of intersection classes in Proposition 2.6 immediately proves a similar result directly in terms of the homology of \( V_\ast \).

**Theorem 2.9** (filtration on \( V_\ast \)). Fix \( k \leq m \) and suppose there are no critical points at infinity with height in \([c_k, \infty)\). Let \( \mathcal{C} \in H_d(M) \) be any class that is homologous in \((C^\ast)^d\) to some cycle in \( M_{\leq c_k - \varepsilon} \). Then for all \( j \leq k \) there are classes \( \gamma_j \in H_{d-1}(V_{\leq c_j - \varepsilon} \cup (B_j \cap V)) \) such that

\[
C - \sum_{i=1}^j \sigma_i \gamma_j = 0 \text{ in } H_d(V_\ast, V_{\leq c_j - \varepsilon}).
\]

The least \( j \) for which \( \gamma_j \neq 0 \) and the projection \( \pi_\gamma \gamma_j \) of \( \gamma_j \) to \( H_d(V_{c_j + \varepsilon}, V_{c_j - \varepsilon}) \) are uniquely determined. Each \( \gamma_j \) is either zero in \( H_{d-1}(V_{\leq c_j - \varepsilon} \cup (B_j \cap V)) \) or nonvanishing in \( H_{d-1}(V_{\leq c_j - \varepsilon} \cup (B_j \cap V), V_{\leq c_j - \varepsilon}) \). \( \square \)

### 2.5 Integration

Integrals of holomorphic forms on a space \( X \) are well defined on homology classes in \( H_\ast(X) \). Relative homology is useful for us because it defines integrals up to terms of small order. Throughout the remainder of the paper, \( F = F/Q \) denotes a quotient of polynomials except when a more general numerator is explicitly noted. Let \( \text{amoeba}(Q) \) denote the amoeba \( \{ \log |z| : z \in V_\ast \} \) associated to the polynomial \( Q \), where \( \log \) and
| · | are taken coordinatewise. Components $B$ of the complement of $\text{amoeba}(Q)$ are open convex sets and are in correspondence with Laurent expansions $F(z) = \sum_{r \in E} a_r z^r$, each Laurent expansion being convergent when $\log |z| \in B$ and determined by the Cauchy integral (1.1) over the torus $\log |z| = \mathbf{x}$ for any $\mathbf{x} \in B$. The support set $E \subseteq \mathbb{Z}^d$ will be contained in the dual cone to the recession cone of $B$; see [BP11, Section 2.2] for details.

**Definition 2.10** ($c_*$ and the pair $(\mathcal{M}, -\infty)$). Let $c_*$ denote the infimum of heights of critical points, including both affine critical points and critical points at infinity. Denote by $H_d(\mathcal{M}, -\infty)$ the homology of the pair $(\mathcal{M}, \mathcal{M}_{\leq c})$ for any $c < c_*$. By part (i) of Theorem 2.4, these pairs are all naturally homotopy equivalent.

For functions of $r \in (\mathbb{Z}^+)^d$, let $\simeq$ denote the relation of differing by a quantity decaying more rapidly than any exponential function of $|r|$. Homology relative to $-\infty$ and equivalence up to superexponentially decaying functions are related by the following result.

**Theorem 2.11.** Let $F = G/Q$ with $Q$ rational and $G$ holomorphic. Suppose that $c_* > -\infty$. For $d$-cycles $C$ in $\mathcal{M}$, the $\simeq$ equivalence class of the integral

$$\int_C z^{-r} F(z) \, dz$$

depends only on the relative homology class $C$ when projected to $H_d(\mathcal{M}, -\infty)$.

**Proof:** Fix any $c < c_*$. Suppose $C_1 = C_2$ in $H_d(\mathcal{M}, -\infty)$. From the exactness of

$$H_d(\mathcal{M}_{\leq c}) \to H_d(\mathcal{M}) \to H_d(\mathcal{M}, \mathcal{M}_{\leq c}),$$

observing that $C_1 - C_2$ projects to zero in $H_d(\mathcal{M}, \mathcal{M}_{\leq c})$, it follows that $C_1 - C_2$ is homologous in $H_d(\mathcal{M})$ to some cycle $C \in \mathcal{M}_{\leq c}$. Homology in $\mathcal{M}$ determines the integral exactly. Therefore, it suffices to show that $\int_C z^{-r} F(z) \, dz \simeq 0$.

As a consequence of the homotopy equivalence in part (i) of Theorem 2.4, for any $t < c_*$ there is a cycle $C_t$ supported on $\mathcal{M}_{\leq t}$ and homologous to $C$ in $\mathcal{M}$. Fix such a collection of cycles $\{C_t\}$. Let $M_t := \sup \{|F(z)| : z \in C_t\}$ and let $V_t$ denote the volume of $C_t$. Observe that $|z^{-r}| = \exp(|r|h_{\mathbb{R}}(z)) \leq \exp(t|r|)$ on $C_t$. It follows that

$$\int_C z^{-r} F(z) \, dz = \int_{C_t} z^{-r} F(z) \, dz \leq V_t M_t \exp(t|r|)$$

and is thus seen to be smaller than any exponential function of $|r|$.

When $F = P/Q$ is rational we may strengthen determination up to $\simeq$ to exact equality. The **Newton polytope**, denoted $\mathbf{P}$, is defined as the convex hull of degrees $m \in \mathbb{Z}^d$ of monomials in $Q$. It is known (see, e.g. [FPT00]) that the components of $\text{amoeba}(Q)^t$ map injectively into the integer points in $\mathbf{P}$, and that to each extreme point $\mathbf{P}$ corresponds a non-empty component. Moreover, the recession cone of these components (collection of directions of rays contained in the component) correspond to the dual cones of the vertices. Hence the linear function $- (z \cdot r)$ is bounded on a component $B$ if and only if the vector $r$ is within the dual to the tangent cone of the Newton polytope $\mathbf{P}(Q)$ at the corresponding integer point. Fix the component $B$ corresponding to the Laurent expansion $F = a_r z^r$ and integer point $\mathbf{v}$ in the Newton polytope, with $r$ not in the dual to the tangent cone of the Newton polytope at $\mathbf{v}$. Then there exists another...
component \( B' \) of the amoeba complement with \( r \) in the dual to the tangent cone of the Newton polytope at the corresponding integer point (this integer point lying on the opposite side of the Newton polytope from \( v \)).

**Proposition 2.12.** If \( F = P/Q \) is rational and \( x \in B' \), then
\[
\int_{T(x)} z^{-r} F(z) \, dz = 0
\]
for all but finitely many \( r \in (\mathbb{Z}^+)^d \).

**Proof:** Observe that there is a continuous path moving \( x \) to infinity within \( B' \). On the corresponding tori, the (constant) value of \( h_R \) approaches \( -\infty \). Let \( T(x_t) \) denote such a torus supported on \( M \leq t \). Because the tori are all homotopic in \( M \), the value of the integral
\[
\int_{T(x_t)} z^{-r} F(z) \, dz \quad (2.5)
\]
cannot change. On the other hand, with \( M_t \) and \( V_t \) as in the proof of the first part, both \( M_t \) and \( V_t \) are bounded by polynomials in \( |z| \), the common polyradius of points in \( T(x_t) \). Once any coordinate \( r_j \) is great enough so that the product of the volume and the maximum grows more slowly than \( |z_j|^r \), the integral for that fixed \( r \) goes to zero as \( t \to -\infty \), hence is identically zero. \( \square \)

The utility of this is to represent the Cauchy integral precisely as a tube integral. Let \( T = T(x) \) for \( x \in B \), the component of \( \text{amoeba}(Q)^c \) defining the Laurent expansion, and choose \( T' = T(y) \) for \( y \in B' \) as in Proposition 2.12. By Proposition 2.6, if \( \gamma \) denotes the intersection class \( I(T, T') \), we have \( T = o \gamma + T' \) in \( H_d(M) \). The integral over \( T' \) vanishes by Proposition 2.12, yielding

**Corollary 2.13.** If \( F = P/Q \) is rational and there are no critical points at infinity, then
\[
(2\pi i)^d a_r = \int_{o \gamma} z^{-r-1} F(z) \, dz.
\]

**2.6 Residues**

Having transferred homology from \( M \) to \( V_* \), we transfer integration there as well. The point of this is that the minimax height cycles live on \( V_* \), not on \( M \) where the minimax height is never achieved. Thus we reduce to saddle-point integrals on \( V_* \) whose asymptotics can be approximated. In what follows, \( H^*(X) \) denotes the holomorphic de Rham complex, whose \( k \)-cochains are holomorphic \( k \) forms. The following duality between residues and tubes is well known.

**Proposition 2.14 (residue theorem).** There is a functor \( \text{Res} : H^d(M) \to H_{d-1}(V_*) \) such that for any class \( \gamma \in H_d(V) \),
\[
\int_{o \gamma} \omega = 2\pi i \int_{\gamma} \text{Res} (\omega) \quad (2.6)
\]
The residue functor is defined locally and, when \( Q \) is squarefree, it commutes with products by any locally holomorphic scalar function. If, furthermore, \( F = P/Q \) is rational, there is an implicit formula
\[
Q \wedge \text{Res} (F \, dz) = P \, dz.
\]
For higher order poles, the residue can be computed by choosing coordinates: if \( F = P/Q^k \), and locally \( \{ Q = 0 \} \) defines a graph of a function, \( \{ z_1 = S(z_2, \ldots, z_d) \} \), then

\[
\text{Res}_z \left[ z^{-r} F(z) \frac{dz}{z} \right] := \frac{1}{(k-1)! (\partial Q/\partial z_1)^k} \frac{d^{k-1}}{dz_1^{k-1}} \left[ \frac{P z^{-r}}{Q} \right] \bigg|_{z_1 = S(z_2, \ldots, z_d)} dz_2 \wedge \cdots \wedge dz_d.
\] (2.7)

**Proof:** Restrict to a neighborhood of the support of the cycle \( \gamma \) in the smooth variety \( \mathcal{V} \), coordinatized so that the last coordinate is \( Q \). The result follows by applying the (one variable) residue theorem, taking the residue in the last variable. \( \square \)

Applying this to intersection classes and using homology relative to \( -\infty \) to simplify integrals yields the following representation.

**Theorem 2.15.** Let \( F = P/Q \) be the quotient of Laurent polynomials with Laurent series \( \sum_{r \in E} a_r z^r \) converging on \( T(x) \) when \( x \in B \), for some component \( B \) of \( \text{amoeba}(Q)^c \). Assume the minimal critical value \( c_\ast \) is finite. Let \( B' \) denote the low (with respect to \( \mathbf{r} \) component of \( \text{amoeba}(Q) \) as in Proposition 2.12. Then for any \( x \in B \) and \( y \in B' \),

\[
a_r = \frac{1}{(2\pi i)^{d-1}} \int_{I(T, T')} \text{Res} \left( z^{-r} F(z) \frac{dz}{z} \right)
\]

for all but finitely many \( r \). If \( P \) is replaced by any holomorphic function, the same representation of \( a_r \) holds up to a function decreasing super-exponentially in \( |r| \).

**Proof:** If \( P \) is polynomial, then

\[
(2\pi i)^{d-1} a_r = \frac{1}{2\pi i} \int_{T(x)} z^{-r} F(z) \frac{dz}{z} = \frac{1}{2\pi i} \int_{T(y)} z^{-r} F(z) \frac{dz}{z} + \frac{1}{2\pi i} \int_{T(x)} z^{-r} F(z) \frac{dz}{z} = \int_{I(T, T')} z^{-r} \text{Res} \left( F(z) \frac{dz}{z} \right) + \frac{1}{2\pi i} \int_{T(y)} z^{-r} F(z) \frac{dz}{z} = \int_{I(T, T')} z^{-r} \text{Res} \left( F(z) \frac{dz}{z} \right).
\]

The first line is Cauchy’s integral formula, the second is Proposition 2.6, the third is (2.6) and the last is Proposition 2.12. If \( P \) is not polynomial, use Theorem 2.11 in place of Proposition 2.12 in the last line. \( \square \)

Combining Theorems 2.9 and 2.15 yields the most useful form of the result: a representation of the coefficients \( a_r \) in terms of integrals over relative homology generators produced by the stratified Morse decomposition. Let \( \sigma_1, \ldots, \sigma_m \) enumerate the critical points of \( \mathcal{V} \), in weakly decreasing order of height \( c_1 \geq c_2 \geq \cdots \geq c_m \). For each \( j \), denote the relevant homology pair by

\[
(X_j, Y_j) := (\mathcal{V}_{\leq c_j - \varepsilon} \cup B_j, \mathcal{V}_{\leq c_j - \varepsilon})
\] (2.8)

where \( B_j \) is a sufficiently small ball around \( \sigma_j \) in \( \mathcal{V} \). Let \( k_j := \dim H_{d-1}(X_j, Y_j) \) and let \( \beta_{j,1}, \ldots, \beta_{j,k_j} \) denote cycles in \( H_{d-1}(X_j) \) that project to a basis for \( H_{d-1}(X_j, Y_j) \) with integer coefficients. In the case where \( \sigma_j \) is a smooth point of \( \mathcal{V} \), stratified Morse theory [GM88] implies that \( k_j = 1 \) and \( \beta_{j,1} \) is a cycle agreeing locally with the unstable manifold for the downward \( h_\mathbf{r} \) gradient flow on \( \mathcal{V} \).
Theorem 2.16 (Stratified Morse homology decomposition). Let $F = P/Q$ be rational. Let the critical points for $h_F$ on $\mathcal{V}$ be enumerated as above and assume there are no critical points at infinity. Then there are integers $\{n_{j,i} : 1 \leq j \leq m, 1 \leq i \leq k_j\}$ such that

$$a_\tau = \frac{1}{(2\pi i)^{d-1}} \sum_{j=1}^{m} \sum_{i=1}^{k_j} n_{j,i} \int_{\beta_{j,i}} \text{Res}(z^{-\tau}F(z)\frac{dz}{z}).$$

(2.9)

For each smooth critical point $\sigma_j$, the cycle $\beta_j$ agrees locally with the unstable manifold at $\sigma_j$ for the downward gradient flow on $\mathcal{V}$.

\[\square\]

3 Computation of Critical Points at Infinity

We begin by recalling some background about stratifications and affine critical points.

Computing a stratification

To compute critical points at all requires a stratification. In practice there is nearly always an obvious stratification. Generically, in fact, the variety $\mathcal{V}$ is smooth and the trivial stratification $\{\mathcal{V}\}$ suffices. In non-generic cases, however, one must produce a stratification of $\mathcal{V}$ before proceeding with the search for affine critical points as well as those at infinity.

There are two relevant facts to producing a stratification. One is that there is a coarsest possible Whitney stratification, called the **canonical Whitney stratification** of $\mathcal{V}$. It is shown in [Tei82], Proposition VI.3.2 that there are algebraic sets $\mathcal{V} = F_0 \supset F_1 \supset \cdots \supset F_m = \emptyset$ such that the set of all connected components of $F_i \setminus F_{i+1}$ for all $i$ forms a Whitney stratification of $\mathcal{V}$ and such that every Whitney stratification of $\mathcal{V}$ is a refinement of this stratification. This canonical stratification is effectively computable; that is, an algorithm exists, given $Q$ (in general, given the generators of any radical ideal), to produce $\{P_{i,s} : 1 \leq i < m, 1 \leq s \leq m_i\}$ where $\{P_{1,1}, \ldots, P_{i,m_i}\}$ is a prime decomposition of the radical ideal corresponding to the Zariski closed set $F_i$. A bound on the computation time, doubly exponential in $m$, is given in [MR91, page 282].

Computing the affine critical points

Assume now that a Whitney stratification $\{S_\alpha : \alpha \in A\}$ is given, meaning the index set $A$ is stored, along with, for each $\alpha \in A$, a collection of polynomial generators $f_{\alpha,1}, \ldots, f_{\alpha,m_\alpha}$ for the radical ideal $\mathcal{I}(S_\alpha)$. The set $S_\alpha$ is the algebraic set $V_\alpha := \mathcal{V}(f_{\alpha,1}, \ldots, f_{\alpha,m_\alpha})$ minus the union of varieties $V_\beta$ of higher codimension. Potentially by replacing $\mathcal{I}(S_\alpha)$ with its prime components, we may assume that the tangent space of $V_\alpha$ at any smooth point has constant codimension $k_\alpha$. Note that although the gradients of $k_\alpha$ generators of $\mathcal{I}(S_\alpha)$ will generate the cotangent space of $V_\alpha$ locally at each $z \in S_\alpha$, the number of generators, $m_\alpha$, may be greater than the codimension $k_\alpha$.

\[\footnote{Formally one must join with the stratification generated by the coordinate planes, but no affine critical point can be on a coordinate plane, so in the end one works only with the single stratum $\mathcal{V}$}.

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Critical points for the height function $h_\mathbf{r}$ are determined by orthogonality of $\mathbf{r}$ to $T_\mathbf{r}(S_\alpha)$ in the log space. In particular, Definition 2.1 is satisfied for $S_\alpha$ if and only if the vector $\mathbf{r}$ is orthogonal to $T_\mathbf{r}(S_\alpha)$; equivalently, if $\mathbf{r}$ is in the span of the normal vectors $\{\nabla f_{\alpha,j}, 1 \leq j \leq m_\alpha\}$. Define, for $z \in S_\alpha$,

$$
T_{\log}(z) := T_2(S_\alpha)\\
(\nabla_{\log, f})(z) := \left(z_1 \frac{\partial f}{\partial z_1}, \ldots, z_d \frac{\partial f}{\partial z_d}\right).
$$

Let $J_\alpha$ be the $(m_\alpha + 1) \times d$ matrix whose rows are the vector $\mathbf{r}$ together with the vectors $\nabla_{\log, f_{\alpha,j}}$ for $1 \leq j \leq m_\alpha$. We then have the following computational definition (see, e.g., [Mum76, Sec. 1A]).

**Proposition 3.1.** Fix a stratum $S_\alpha$ and a point $z \in S_\alpha$. Then $z \in \text{crit}(\mathbf{r})$ if and only if all the $(k_\alpha + 1) \times (k_\alpha + 1)$ minors of $J_\alpha$ vanish. Consequently, the critical points are found by taking the union over $\alpha \in \mathbb{A}$ of the solutions to the polynomial equalities and non-equalities saying that: \(i\) these minors all vanish; \(ii\) that $z \in V_\alpha$ (in other words, for all $j$, $f_{\alpha,j}(z) = 0$); and \(iii\) that $z \notin V_\beta$ for any $\beta$ with $k_\beta > k_\alpha$.

Condition \(i\) is polynomial in both $z$ and $r$. For later use, we let $\{g_{\alpha,j}(z, y)\}$ denote the $(k_\alpha + 1) \times (k_\alpha + 1)$ minors of $J_\alpha$ with $\mathbf{r}$ replaced by a variable vector $y$; note that these polynomials will be homogeneous in the $y$ variables. Condition \(ii\) is of course defined by polynomials $f_{\alpha,j}$ in the $z$ variables. Homogenizing the $g_{\alpha,j}$ and $f_{\alpha,j}$ in the $z$ variables gives a set of polynomials $C_\alpha$ that together define the Zariski closure of the graph of the relation $z \in \text{crit}(\mathbf{r})$ in $\mathbb{C}^{pd} \times \mathbb{C}^{pd-1}$. The actual graph of the relation is obtained by removing $(z, r)$ with $z \in S_\beta$ for some $\beta$ where $k_\beta > k_\alpha$ (a lower dimensional stratum) and removing points when the homogenizing coordinate $z_0$ vanishes (points at infinity).

### Computing critical points at infinity

To determine whether there exist critical points at infinity we use ideal quotients, corresponding to the difference of algebraic varieties. Recall that the variety $\mathbb{V}(I : J^\infty)$ defined by the saturation $I : J^\infty$ of two ideals $I$ and $J$ is the Zariski closure of the set difference $\mathbb{V}(I) \setminus \mathbb{V}(J)$ (see [CLO92, Section 4.4]), and can be determined through Gröbner basis computations. Let $D_\alpha$ denote the ideal generated by $z_0$ and the homogenizations of all polynomials $f_{\beta,j}$ with $k_\beta > k_\alpha$.

**Definition 3.2** (saturated ideals). Fix $\alpha \in \mathbb{A}$. Let $\mathfrak{C}_\alpha$ denote the result of saturating $C_\alpha$ by the ideal $D_\alpha$. Let $\mathfrak{C}'_\alpha$ denote the result of substituting $z_0 = 0$ and $y = \mathbf{r}$ in $\mathfrak{C}_\alpha$.

Geometrically, the variety $\mathbb{V}(\mathfrak{C}_\alpha)$ is the Zariski closure of $\mathbb{V}(\mathfrak{C}'_\alpha) \setminus \mathbb{V}(z_0) \setminus \mathbb{V}(\mathfrak{T}_{j+1})$, that is, the closure of that part of the graph of the relation $z \in \text{crit}(\mathbf{y})$ in $\mathbb{C}^{pd} \times \mathbb{C}^{pd-1}$ corresponding to points $(z, y)$ whose $z$ component is not on a substratum and not at infinity. Note that in this setting, the Zariski closure equals the classical topological closure [Mum76, Theorem 2.33]. The variety $\mathbb{V}(\mathfrak{C}'_\alpha)$ picks points at infinity in $\text{crit}(\mathbf{r})$ that are limits of affine points in $\text{crit}(\mathbf{y})$ with $y \to \mathbf{r}$.

**Proposition 3.3** (computability of critical points at infinity). The rational function $F(z) = P(z)/Q(z)$ has critical points at infinity in direction $\mathbf{r}$ if and only if there exists $\alpha$ such that $\mathfrak{C}'_\alpha$ has a projective solution, that is, a solution other than $(0, \ldots, 0)$.

**Proof:** First, assume there is a critical point at infinity. Then there exists a sequence $(z^{(n)})$ contained in some $S_\alpha$ and going to infinity, such that the projection of $\mathbf{r}$ onto $T_{\log}(z^{(n)})$ goes to zero. If $\pi_n$ is the projection
computes a superset of what we need, namely all critical points in a fixed direction regardless of height. This includes critical points at infinity with infinite heights, which can be discarded. The examples below show one way of proceeding to determine the relevant heights; when \( \alpha \) is rational, the height(s) may be computed from the start along with the critical points themselves. By the same argument as in Proposition 3.3, we obtain the following corollary.

**Corollary 3.4.** Let \( \mathbf{r} \) be an integer vector. Let \( H_\alpha \) denote the ideal generated by \( C_\alpha \) along with \( H z_0^d - \mathbf{z}^* \). Let \( \mathcal{H}_\alpha \) be the result of saturating by \( D_\alpha \) and let \( \mathcal{H}_\alpha' \) be the result of substituting \( z_0 = 0 \) and \( \mathbf{y} = \mathbf{r} \) in \( \mathcal{H}_\alpha \). Then there exists a critical point at infinity of height \( \log |c| \) if and only if there is a critical point at infinity in direction \( \mathbf{r} \) with \( H \) coordinate equal to \( c \).

Determining a method—uniform in \( \mathbf{r} \) which does not require \( \mathbf{r} \) to have rational coordinates—for computing the heights of the critical points at infinity, or for computing an ideal encoding all critical points at some fixed height, is a good direction for future work.

### 4 Examples

When \( Q \) is square-free and \( \mathcal{V} \) is smooth, \( \mathcal{V} \) itself forms a stratification. The pseudocode in Algorithm 1 computes critical points at infinity in this case. We have implemented this algorithm, and more general variants, in Maple. A Maple worksheet with our code and examples is available from the authors’ webpages.

We give three examples in the smooth bivariate case, always computing in the main diagonal direction \( \mathbf{r} = (1, 1) \) and examining the coefficients \( a_{n,n} \) of \( 1/Q \). In the first example there is no affine critical point and
Algorithm 1: Smooth critical points at infinity

**Input:** Polynomial $Q$ and direction $r \in \mathbb{Z}^d$

**Output:** Ideal $\mathcal{C}'$ in the variables $z$ of $Q$ and a homogenizing variable $z_0$ such that there is a critical point at infinity iff the generators of $I$ have a non-zero solution in the variables of $Q$.

Let $\tilde{Q} = z_0^{\deg Q} Q(z_1/z_0, \ldots, z_d/z_0)$; 
Let $C$ be the ideal generated by $\tilde{Q}$ and 

$$y_j z_1 (\partial \tilde{Q}/\partial z_1) - y_1 z_j (\partial \tilde{Q}/\partial z_j), \quad (2 \leq j \leq d);$$ 

Saturate $C$ by $z_0$ to obtain the ideal $C$; 
Substitute $y_j = r_j$ for $1 \leq j \leq d$ and return the resulting ideal with the generator $z_0$ added.

the critical point at infinity determines the exponential growth rate. In the second there are both critical points at infinity and affine ones, and the one at infinity has too low height to matter. In the third, the critical point at infinity is higher than all the affine ones and controls the exponential growth rate of diagonal coefficients.

**Example 1** (smooth case). Let $Q(x, y) = 2 - xy^2 - 2xy - x + y$, so that $V$ is smooth and we can use the above code for the diagonal direction $r = (1, 1)$. First, an examination of the polynomial system $Q = \partial Q/\partial x - \partial Q/\partial y = 0$ shows there are no affine critical points. Thus, if there were no critical points at infinity, then the diagonal coefficients of $Q(x, y)^{-1}$ would decay super-exponentially. It is easy to see that this does not happen, for example by extracting the diagonal. This may be done via the Hautus-Klarner-Furstenberg method [HK71], which is completely effective [BDS17], giving $\Delta Q(x, y)^{-1} = (1 - z)^{-1/2}/2$.

We compute the existence of critical points at infinity in the diagonal direction using our Maple implementation via the command

```maple
CPatInfty(2 - x*y^2 - 2*x*y - x + y , [1,1]);
```

This returns the ideal $[Z,x]$, where $Z$ is the homogenizing variable, meaning the projective point $(Z : x : y) = (0 : 0 : 1)$ is the sought after critical point at infinity, which must have finite height as the diagonal coefficients of $Q(x, y)^{-1}$ do not decay super-exponentially. Using Corollary 3.4 to find the height(s) of critical points in the diagonal direction produces the ideal

$$[(H - 1)^2 , Z, y(H - 1), x]$$

meaning there is a critical point at infinity with height $\log |1| = 0$. The critical point at infinity is a topological obstruction to the gradient flow across height 0, pulling trajectories to infinity; it implies that on the diagonal coefficients of $1/Q$ do not grow exponentially nor decay exponentially (in fact they decay like a constant times $1/\sqrt{n}$). Further geometric analysis of this example is found in [DeV11, pages 120–121].

**Example 2.** Let $Q(x, y) = 1 - x - y - xy^2$. To look for critical points at infinity in the diagonal direction, execute $CPatInfty(Q, [1,1])$ to obtain the ideal $[Z, y]$, showing that $(x : y : Z) = (0 : 1 : 0)$ is a critical point at infinity. To find the height again we reduce $h - xy$ modulo $Q$ to get $h^2 + h + x^2 - x$, and setting $x = 0$ gives $h = 0$ or $-1$. Therefore if $\log |xy|$ goes to a finite value it goes to $\log |1| = 0$, so there is a critical point in the diagonal direction of height zero. This time, there is an affine critical point $(1/2, \sqrt{2} - 1)$ of greater height. This affine point is easily seen to be a topological obstruction: in the terminology of Proposition 2.7,
n = 1, and therefore controls the exponential growth. Theorem 2.15 allows us to write the resulting integral as a saddle point integral in $V$, over a class local to $(1/2, \sqrt{2} - 1)$, thereby producing an asymptotic expansion with leading term $a_{n,n} \sim cn^{-1/2}(2 + \sqrt{2})^n$.

Example 3. Let $Q(x,y) = -x^2y - 10xy^2 - x^2 - 20xy - 9x + 10y + 20$. This time $CPatInfty(Q, [1,1])$ produces

$$[(2 H^4 - 11 H^3 + 171 H^2 - 1382 H + 3220) (H-1)^2, Z, y (H-1), x]$$

Again, because $x$ and $y$ cannot both vanish, the only height of a critical point at infinity is $\log |1| = 0$. There are four of affine critical points: a Gröbner basis computation produces one conjugate pair with $|xy| \approx 9.486$ and another conjugate pair with $|xy| \approx 4.230$. Both of these lead to exponentially decreasing contributions, meaning the point at infinity could give a topological obstruction for establishing asymptotics. If there is an obstruction, it would increase the exponential growth rate of diagonal coefficients from $4.23^{-n}$ to no exponential growth or decrease. To settle this, we can compute a D-finite equation for the diagonal. This reveals that the diagonal asymptotics are of order $a_{n,n} \sim n^{-1/2}$, meaning the exponential growth rate on the diagonal is in fact zero.

Naively, one might imagine searching for critical points at infinity simply by homogenizing and solving the critical point equations (without performing the ideal saturation) or by examining and saturating the critical point equations with the direction $y = r$ already fixed. Our next example shows this gives spurious critical points at infinity.

Example 4 (naive computation gives spurious points). Let $Q(x,y,z) = 1 - x - y - z - xy$. Homogenizing $Q$ and the two critical point equations with a variable $t$ and computing a Gröbner basis gives $[t, xy]$, meaning there might be critical points at infinity in the diagonal directions at $(x:y) = (1:0)$ and $(x:y) = (0:1)$. On the other hand, running $CPatInfty(Q, [1,1,1])$ gives only the solution $(0,0,0)$ so there is no critical point at infinity.

The two affine critical points may be computed easily:

$$\sigma_1 = \left( -\frac{3 + \sqrt{17}}{4}, -\frac{3 + \sqrt{17}}{4}, \frac{7 + \sqrt{17}}{8} \right)$$

$$\sigma_2 = \left( -\frac{3 - \sqrt{17}}{4}, -\frac{3 - \sqrt{17}}{4}, \frac{7 - \sqrt{17}}{8} \right).$$

Theorem 2.16 then implies

$$a_{n,n,n} = \frac{1}{(2\pi i)^2} \sum_{j=1}^{2} \int_{\beta_j} z^{-r} \text{Res} \left( F(z) \frac{dz}{z} \right)$$

where $\beta_1$ and $\beta_2$ are respectively the downward gradient flow arcs on $V$ at $\sigma_1$ and $\sigma_2$. Saddle point integration gives an asymptotic series for each, the series for $\sigma_1$ dominating the series for $\sigma_2$, yielding an asymptotic expansion for $a_{n,n}$ beginning

$$a_{n,n,n} = \left( \frac{3 + \sqrt{17}}{2} \right)^{2n} \left( \frac{7 + \sqrt{17}}{4} \right)^n \cdot \frac{2}{n\pi \sqrt{26\sqrt{17} - 102}} \left( 1 + O \left( \frac{1}{n} \right) \right).$$
The following application concerns an analysis in a case where \( V \) is not smooth. There is an interesting singularity at \((1/3,1/3,1/3,1/3)\) and a geometric analysis involving a lacuna \([\text{BMP19}]\), which depends on there being no critical points at infinity at finite height. This example illustrates the use of the results in Sections 2.3 – 2.6.

**Example 5** (no critical points at infinity). In \([\text{BMPS18}]\), asymptotics are derived for the diagonal coefficients of several classes of symmetric generating functions, including the family of 4-variable functions \( \{1 - x - y - z - w + Cxyzw : C > 0\} \). The most interesting case is when the parameter \( C \) passes through the critical value 217: diagonal extraction and univariate analysis show the exponential growth rate of the main diagonal to have a discontinuous jump downward at the critical value \([\text{BMPS18}, \text{Section 1.4}]\). This follows from Proposition 6.2 below if we show there are in fact no critical points at infinity above this height. Here there is a single point where the zero set of \( Q \) is non-smooth, the point \((1/3,1/3,1/3,1/3)\). Running a slightly more general version Algorithm 1 yields the ideal \([Z, z^4, -z+y, -z+z, -z+w] \) which has only the trivial solution \( Z = w = x = y = z = 0 \). Thus, there are no critical points at infinity for the diagonal direction.

There are three affine critical points, one at \( (1/3,1/3,1/3,1/3) \), one at \((\zeta, \zeta, \zeta, \zeta)\) and one at \((\zeta, \zeta, \zeta, \zeta)\), where \( \zeta = (-1-i\sqrt{3})/3 \). Call these points \( x(1), x(2), \) and \( x(3) \). The one with the greatest value of \( h_B \) (the diagonal direction is \((1/3,1/3,1/3,1/3)\), which has height \log 81. The other two have height \log 9. By a somewhat involved topological process, it is checked in \([\text{BMP19}]\) that \( \pi T = 0 \) in \( H_4(B \cup M_{log 81 - \varepsilon}, M_{log 81 - \varepsilon}) \), where \( B \) is a small ball centered at \( x(1) \). Crucially for the analysis after that, it follows from Proposition 2.8 that \( T \) is homologous to a chain supported in \( M_{\mathbb{C}^2 + \varepsilon} \). In other words, \( T \) can be pushed down until hitting obstructions at \((\zeta, \zeta, \zeta, \zeta)\) and \((\zeta, \zeta, \zeta, \zeta)\). In fact an alternative analysis using a differential equation satisfied by the diagonal verifies a growth rate of \( 9^n \), not \( 81^n \). By means of Theorem 2.15, the coefficients \( a_{n,n,n,n} \) may be represented as a residue integrals and put in standard saddle point form. When this is done, one obtains the more precise asymptotic \( Cn^{-3/2}9^n \). The value of \( C \) depends on geometric invariants (curvature) and topological invariants (intersection numbers) and can be deduced by rigorous numeric methods. Details are given in \([\text{BMP19}, \text{Section 8}]\).

Finally, we include a computation with three variables.

**Example 6** (trivariate computation). Consider the diagonal direction and
\[
Q(x, y, z) = 1 - x + y - z - 2xy^2.
\]

There are two affine critical points, \( x_+ = \left(\frac{1}{2}, \frac{9 + \sqrt{108}}{10}, \frac{1}{2}\right) \), and running our Maple code shows there is also a critical point at infinity of height \(-\log 1/2 = \log 2\). Since the height of \( x_- \) is \(-\log \left|\frac{9 - \sqrt{108}}{36}\right| > \log 2\), the critical point at infinity does not affect dominant asymptotics of \( 1/Q \). One can get a mental picture of the situation by examining the Newton polytope of \( Q \). Due to the monomials \( x, y, \) and \( z \), the dual cone of the Newton polytope at the origin is the negative orthant. Thus, the component \( B \) of \( \text{amoeba}(Q) \) corresponding to the power series expansion of \( 1/Q \) admits the negative orthant as its recession cone. This implies one cannot move along a direction perpendicular to \((1, 1, 1)\) and stay in \( B \), so if there exists a critical point of \( Q \) in \( \overline{B} \) it must have larger height than any critical point at infinity. The critical point at infinity lies on the closure of the complements of \( \text{amoeba}(Q) \) corresponding to the vertices \((0, 1, 0)\) and \((1, 2, 1)\) of the Newton polytope of \( Q \). One can directly verify that both of these components have a recession cone containing a vector normal to the diagonal direction. Ultimately, the lack of a critical point at infinity of highest height.

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implies an asymptotic expansion of the diagonal of $1/Q$ beginning

$$a_{n,n,n} = \left(\frac{27 + 3\sqrt{105}}{2}\right)^n \cdot \frac{\sqrt{3}}{2n\pi} \left(1 + O\left(\frac{1}{n}\right)\right).$$

## 5 Deformations

In this section we build a vector field in the log space, which will create a flow that we use to prove Theorem 2.4.

**Lemma 5.1.** Fix a unit vector $\hat{r} \in \mathbb{R}^d$ and a stratification $\{S_\alpha : \alpha \in A\}$. Suppose that there are no affine critical points nor critical points at infinity in direction $\hat{r}$ whose heights lie in the interval $[a, b]$. Then there is some $\varepsilon > 0$ such that for all strata $\tilde{S}$ and all $x \in \tilde{V}$ with $h_{\tilde{r}}(x) \in [a, b]$, the projection of $\hat{r}$ onto $T_x(\tilde{S})$ has length at least $\varepsilon$.

**Proof:** There are finitely many strata, therefore if the conclusion fails then for some stratum $\tilde{S}$ there is a sequence $x_n$ in $\tilde{S}$ with $\pi_{T_x(\tilde{S})}(\hat{r}) \to 0$ and $h_{\tilde{r}}(x_n) \in [a, b]$. This contradicts the assumption of no finite or infinite critical points in direction $\hat{r}$ with heights in $[a, b]$.

**Lemma 5.2.** Given a stratification, suppose that for every $\tilde{S}$ and every $x \in \tilde{S}$ with height in $[a, b]$ the projection of $\hat{r}$ onto $T_x(\tilde{S})$ has length at least $\varepsilon > 0$. Then there is a continuous unit vector field $v$ on $h_{\tilde{r}}^{-1}[a, b]$ such that

(i) $x \in \tilde{S}$ implies $v(x) \in T_x(\tilde{S})$ and

(ii) $\hat{r} \cdot v(x) \geq \varepsilon$.

**Proof:** Given $x \in \tilde{S}$, let $N_x$ be a neighborhood of $x$ intersecting no strata other than those containing $\tilde{S}$. Let $v_x$ be a unit vector in $T_x(\tilde{S})$ such that $\hat{r} \cdot v_x \geq \varepsilon$, the existence of which is guaranteed by the hypotheses of the lemma and by Proposition 2.2. Let $B_R$ denote the ball of radius $R$ centered at the origin. The neighborhoods $\{N_x\}$ form an open cover of each set $T^d \times B_R$. Intersecting $T^d \times B_R$ with $h_{\tilde{r}}^{-1}[a, b]$ gives a compact set. Let $\{N_x : x \in A_R\}$ be a finite subcover of this compact set. Let $\{\alpha_{R,x} : x \in A_R\}$ be a partition of unity subordinate to this cover. Then

$$v_R(z) := \sum_{x \in A_R} \alpha_x(z)v_x$$

defines a vector field satisfying the conclusions of the lemma for all $z \in (T^d \times B_R) \cap h^{-1}[a, b]$. Any sequence of such vector fields $v_R$ has a subsequential pointwise limit, uniformly on compact sets. Taking any such limit finishes the proof.

## 6 Proof of Theorem 2.4

We prove the homotopy equivalences by exhibiting deformations for any chain $\Gamma$. To prove part (i) we show the following.
Proposition 6.1. Let $[a, b]$ be a real interval and suppose that $\mathcal{V}_*$ has no affine critical points or critical points at infinity with height in $[a, b]$. Then for any $\varepsilon > 0$, any chain $\Gamma$ of maximum height at most $b$ can be homotopically deformed into a chain whose maximum height is at most $a + \varepsilon$.

Proof: Fix a Whitney stratification of $\tilde{\mathcal{V}} \cap \tilde{h}_R^{-1}[a+\varepsilon, b]$. The hypotheses of the theorem along with Lemma 5.1 imply the hypotheses of Lemma 5.2. Choose a vector field $v$ as in the conclusion of Lemma 5.2. Let $H$ denote flow by the vector field $v$, defined by the differential equation

$$ H_t(x, t) = v(x); \quad H(x, 0) = x $$

when $\tilde{h}_R(H(x, t)) > a + \varepsilon$, and stopped when height $a + \varepsilon$ is hit. By construction, $v(x)$ is in the tangent space to $\tilde{\mathcal{V}}$ when $x$ is in a neighborhood of $\tilde{\mathcal{V}}$, therefore for $x \notin \tilde{\mathcal{V}}$, $H(x, \cdot)$ never hits $\tilde{\mathcal{V}}$. By conclusion (ii) of Lemma 5.2, $(d/dt)\tilde{h}_R(H(x, t)) \leq -\varepsilon$ until the flow stops. Therefore, by time $\tau := (b - a - \varepsilon)/\varepsilon$, the flow has stopped and $\Gamma$ (in the log space) is transported to a chain of maximum height at most $a + \varepsilon$. Mapping the whole homotopy by $\Phi$ proves the proposition.

Part (ii) of Theorem 2.4 is proved via a similar construction.

Proposition 6.2. Let $[a, b]$ be a real interval. Suppose that there is precisely one critical point $x \in \mathcal{V}_*$ for which $h_B(x) \in (a, b)$ and no other finite or infinite critical points with heights in $[a, b]$. Then the homotopy type of the pair $(\mathcal{M}_{\leq b}, \mathcal{M}_{\leq a})$ is the same as the homotopy type of $((\mathcal{M} \cap B)_{\leq b}, (\mathcal{M} \cap B)_{\leq a})$ for an arbitrarily small ball $B$ about $x$ and the inclusion $((\mathcal{M} \cap B)_{\leq b}, (\mathcal{M} \cap B)_{\leq a}) \hookrightarrow (\mathcal{M}_{\leq b}, \mathcal{M}_{\leq a})$ induces a homotopy equivalence.

Proof: Map to the log space. Let $X := \Phi^{-1}[B \cup \mathcal{M}_{\leq a}]$ and $Y := \Phi^{-1}[\mathcal{M}_{\leq a}]$. Define a flow $H$ on $\tilde{\mathcal{M}}$ by $H_t(x, t) = v(x)$ for $x \notin X$ and $H_t(x, t) = 0$ for $x \in X$. On $\mathcal{M}_{\leq b} \setminus B$, the rate $d/dt(\tilde{h}_R(H(x, t)))$ is bounded above by $-\delta$ for some positive $\delta$, therefore taking $T := (b-a)/\delta$, the flow started on $\mathcal{M}_{\leq b}$ always stops by time $T$:

$$ H(x, T) \in X \text{ for all } x \in \tilde{\mathcal{M}}_{\leq b}. $$

Let $g(x) := H(x, T)$ and let $\hookrightarrow$ denote inclusions. The proposition, hence part (ii) of Theorem 2.4, is implied by the following homotopy equivalences:

$$ (\Phi^{-1}[\mathcal{M} \cap B]_{\leq b}, \Phi^{-1}[\mathcal{M} \cap B]_{\leq a}) \hookrightarrow (X, Y) \hookrightarrow (\Phi^{-1}[\mathcal{M}_{\leq b}], Y) \overset{\Phi}{\rightarrow} (X, Y). $$

The first map is a homotopy equivalence by excision of the set $\Phi^{-1}[\mathcal{M}_{\leq b} \setminus B]$. The latter two maps are homotopy equivalences because their composition one way is the identity on $(X, Y)$ while the composition the other way is homotopic to the identity via the homotopy $H$.

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A Appendix

We now give an abstract answer to [Pem10, Conjecture 2.11], in a manner suggested to us by Justin Hilburn and Roberta Guadagni.

Let \( H(z) := z^m \) be the monomial function on \((\mathbb{C}^*)^d\), and \( G \subset (\mathbb{C}^*)^d \times \mathbb{C}^* \), its graph. An easy case of toric resolution of singularities (see, e.g., [Kho78]) implies the following result.

**Theorem A.1.** There exists a compact toric manifold \( K \) such that \((\mathbb{C}^*)^d\) embeds into it as an open dense stratum and the function \( H \) extends from this stratum to a smooth \( \mathbb{P}^1 \)-valued function on \( K \).

**Proof:** The graph \( G \) is the zero set of the polynomial \( P_m := h - z^m \) on \((\mathbb{C}^*)^d \times \mathbb{C}^*\), where \( h \) is the coordinate on the second factor. Theorem 2 in [Kho78] implies that a compactification of \((\mathbb{C}^*)^d \times \mathbb{C}^*\) in which the closure of \( G \) is smooth exists if the restrictions of the polynomial \( P_m \) to any facet of the Newton polyhedron of \( P_m \) is nondegenerate (defines a nonsingular manifold in the corresponding subtorus). In our case, the Newton polytope is a segment, connecting the points \((m, 0)\) and \((0, 1)\), and this condition follows immediately. Hence, the closure of \( G \) in the compactification of \((\mathbb{C}^*)^d \times \mathbb{C}^*\) is a compact manifold \( K \). We notice that the projection to \((\mathbb{C}^*)^d\) is an isomorphism on \( G \), and therefore \( K \) compactifies \((\mathbb{C}^*)^d\) in such a way that \( H \) lifts to a smooth function on \( K \).

Lifting the variety \( V \) to \( G \subset K \) and taking the closure produces the desired result: a compactification of \( V \) in a compact manifold \( K \) on which \( H \) is smooth. \( \square \)

A practical realization of the embedding requires construction of a simple fan (partition of \( \mathbb{R}^{d+1} \) into simplicial cones with unimodular generators) which subdivides the fan dual to the Newton polytope of \( h - z^m \). While this is algorithmically doable (and implementations exist, for example in macaulay2), the resulting fans depend strongly on \( m \), and the resulting compactifications \( K \) are hard to work with.

**Definition A.2 (compactified critical point).** Define a compactified critical point of \( H \), with respect to a compactification of \((\mathbb{C}^*)^d\) to which \( H \) extends smoothly, as a point \( x \) in the closure of \( V \) such that \( \partial H \) vanishes at \( x \) on the stratum \( \overline{S(x)} \), and \( H(x) \) is not zero or infinite.

Applying basic results of stratified Morse theory [GM88] to \( K \) directly yields the following consequence.

**Corollary A.3** (no compactified critical point implies Morse results).

(i) If there are no critical points or compactified critical points with heights in \([a, b]\), then \( V_{\leq b} \) is homotopy equivalent to \( V_{\leq a} \) via the downward gradient flow.

(ii) If there is a single critical point \( x \) with critical value in \([a, b]\), and there is no compactified critical point with height in \([a, b]\), then the homotopy type of the pair \((M_{\leq b}, M_{< b})\) is determined by a neighborhood of \( x \), with an explicit description following from results in [GM88]. \( \square \)

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