THE MOMENT GRAPH FOR BOTT-SAMELSON VARIETIES
AND APPLICATIONS TO QUANTUM COHOMOLOGY

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Abstract. We give a description of the moment graph for Bott-Samelson varieties in arbitrary Lie type. We use this, along with curve neighborhoods and explicit moduli space computations, to compute a presentation for the small quantum cohomology ring of a particular Bott-Samelson variety in Type A. We also show the conjecture $O$ of Galkin, Golyshev, and Iritani holds for that Bott-Samelson variety.

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1. Introduction

The (small) quantum cohomology of homogeneous varieties has been studied extensively due to its connection with questions in enumerative geometry; see the introduction in [FP97] for a discussion of enumerative results relating to the quantum cohomology of the projective plane. The key to obtaining a presentation for the quantum cohomology ring is to quantize the relations in the ordinary cohomology ring. For flag manifolds $G/B$, this quantization involves the Toda lattice; see [Kim99].

More generally, the quantum cohomology of toric varieties is well understood (originally due to Batyrev [Bat93]; see [AK06] for a modern discussion, Proposition 2.5 in particular.) Vakil in [Vak00] computed Gromov-Witten invariants for Hirzebruch surfaces and (all but two) del Pezzo surfaces in all genera, and shows these invariants are enumerative; see Section 8 in [Vak00] for a discussion of curve counts for Hirzebruch surfaces. Little is known about the quantum cohomology of varieties which are not convex and/or not toric. For example, see [Pec13] and [MS17].
Bott-Samelson varieties are generally not convex in dimension greater than one, and are generally not toric in dimension greater than two, however they are intimately related to homogeneous spaces $G/B$, so it is natural consider their quantum cohomology.

In this article, we describe the moment graph of Bott-Samelson varieties with a view towards describing the (small) quantum cohomology ring of Bott-Samelson varieties. If $X$ is a variety on which an algebraic torus $T$ acts with finitely many fixed points, the moment graph is defined as follows: the vertices are the set of fixed points $X^T$, and two vertices $x, y \in X^T$ are connected by an edge if there is a $T$-stable curve containing both $x, y$.

In order to describe our results more concretely, we recall some constructions and fix some notation. Let $G$ be a simple Lie group over $\mathbb{C}$. Fix a maximal torus contained in a Borel subgroup $T \subset B \subset G$; the Weyl group is denoted $W := N_G(T)/T$. The associated root system is denoted $\Phi = \Phi(G, T)$, the base corresponding to the fixed Borel subgroup is denoted $\Delta \subset \Phi$, and the set of positive roots is denoted $\Phi^+$. The minimal parabolic subgroup corresponding to $\alpha \in \Delta$ is denoted $P_\alpha$.

For a sequence of simple reflections $(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n})$ in $W$, the corresponding Bott-Samelson variety is denoted $Z = Z(\alpha_1, \ldots, \alpha_n)$. A Bott-Samelson variety is a tower of $\mathbb{P}^1$-bundles

$$Z(\alpha_1, \ldots, \alpha_n) \xrightarrow{\pi_{n-1}} Z(\alpha_1, \ldots, \alpha_{n-1}) \xrightarrow{\pi_{n-2}} \cdots \xrightarrow{\pi_2} Z(\alpha_1) \xrightarrow{\pi_1} \{pt\}$$

where each bundle has a natural section $s_k : Z(\alpha_1, \ldots, \alpha_{k-1}) \rightarrow Z(\alpha_1, \ldots, \alpha_k)$, and each Bott-Samelson variety has a morphism $\theta_k : Z(\alpha_1, \ldots, \alpha_k) \rightarrow G/B$.

Bott-Samelson varieties are $T$-varieties (this is described in Section 2). The fixed point set $Z^T$ is easy to describe; the $T$-fixed points correspond to subsequences of $(s_{\alpha_1}, \ldots, s_{\alpha_n})$. The combinatorial object which corresponds to the $T$-fixed points are $\varepsilon \in \{0, 1\}^n$; for $x \in Z^T$, we will denote the binary $n$-tuple corresponding to $x$ by $\varepsilon_x$.

The $n$-tuple $\varepsilon_x$ can be described inductively as follows: $\varepsilon_x$ is obtained from the $(n-1)$-tuple $\varepsilon_{\pi(x)}$ by appending either a zero or one according as $x \in s(Z')$ or $x \notin s(Z')$ respectively. The next definition comes from [Wil04, Section 2]; we have slightly modified the notation.

**Definition 1.1.** For $\varepsilon \in \{0, 1\}^n$, denote by $\pi_+(\varepsilon)$ the set of entries $i$ such that $\varepsilon_i = 1$. Define

$$w_k(\varepsilon) = \prod_{1 \leq i \leq k} s_{\alpha_i}^{\varepsilon_i}$$

($w_k(\varepsilon) = 1$ if $\{1 \leq i \leq k, i \in \pi_+(\varepsilon)\} = \emptyset$, set $w(\varepsilon) = w_n(\varepsilon)$, and define $\varepsilon(\alpha_k) = w_k(\varepsilon)\alpha_k \in \Phi$ for each $1 \leq k \leq n$.

The next definition gives us a notation for the fixed points which lie on the same fiber as a given fixed point.

**Definition 1.2.** For $x \in Z^T$, define $\varepsilon^0_x$ and $\varepsilon^\infty_x$ by adjoining either a 0 or 1 respectively to $\varepsilon_{\pi(x)}$. Note, $\varepsilon^0_x$ corresponds to the $T$-fixed point in $s(Z')$ which is also contained in the fiber of $\pi : Z \rightarrow Z'$ containing $x$, and $\varepsilon^\infty_x$ corresponds to the other $T$-fixed point in that fiber.
Our main theorem is an inductive characterization of the set of $T$-stable curves in $Z$. Since any $T$-stable curve is a union of irreducible $T$-stable curves, we characterize the points $x, y \in Z^T$ which are joined by irreducible $T$-stable curves.

**Theorem 1.** Let $x, y \in Z^T$, and let $k$ be the first index where $\varepsilon_x, \varepsilon_y$ differ. If $k = n$, then $x, y$ are joined by an $T$-stable fiber of $\pi : Z \to Z'$. Otherwise, $k < n$ and we suppose that $\pi(x), \pi(y)$ are joined by an irreducible $T$-stable curve. Let $C$ be one such $T$-stable curve joining $\pi(x), \pi(y)$, and let $h$ be the class of the fiber of $\pi$.

There are four possibilities for the restriction of the moment graph to $\{\varepsilon_0^0, \varepsilon_0^\infty, \varepsilon_y^0, \varepsilon_y^\infty\}$:

**Case I.**

**Case II.**

**Case III.**

**Case IV.**

The cases are characterized as follows:

I. $\varepsilon_0^0(\alpha_k) \neq \varepsilon_0^0(\alpha_n)$ and $\varepsilon_0^\infty(\alpha_k) \neq \varepsilon_0^\infty(\alpha_n)$;

II. $\varepsilon_0^0(\alpha_k) = \varepsilon_0^0(\alpha_n)$ and $\varepsilon_0^\infty(\alpha_k) = \varepsilon_0^\infty(\alpha_n)$;

III. $\varepsilon_0^0(\alpha_k) = \varepsilon_0^0(\alpha_n)$ and $\varepsilon_0^\infty(\alpha_k) \neq \varepsilon_0^\infty(\alpha_n)$;

IV. $\varepsilon_0^0(\alpha_k) \neq \varepsilon_0^0(\alpha_n)$ and $\varepsilon_0^\infty(\alpha_k) = \varepsilon_0^\infty(\alpha_n)$.

In each case, the unlabeled curves have the same homology class, which are as follows:

I.

\[
\begin{cases}
    s_*(C) - (\alpha_k, \alpha_n^0) h, & w(\varepsilon_0^0) \neq w(\varepsilon_0^\infty) \\
    s_*(C), & w(\varepsilon_0^0) = w(\varepsilon_0^\infty)
\end{cases}
\]

II. $s_*(C) - h$

III. $s_*(C) + h$

IV. $s_*(C) + h$
In cases II, III, and IV, the bold line indicates there is a one-dimensional family of $T$-stable curves joining those fixed points.

**Remark 1.1.** Since $\pi : Z \to Z'$ is proper, if $x, y$ are joined by a $T$-stable curve, then $\pi(x), \pi(y)$ are either joined by a $T$-stable curve, or $\pi(x) = \pi(y)$. In particular, the inductive hypothesis in Theorem 1 is a necessary condition for $x, y$ to be joined by a $T$-stable curve.

Moreover, since $\pi : Z \to Z'$ is a $\mathbb{P}^1$-bundle with a section $s$, the push-forward $s_* : H_\ast(Z') \to H_\ast(Z)$ is the inclusion $H_\ast(Z') \subset H_\ast(Z)$. Thus, the homology classes of $T$-stable curves are characterized inductively by our theorem.

**Example 1.1.** In type $A_2$, the Bott-Samelson variety $Z(\alpha_1, \alpha_2)$ is the Hirzebruch surface $\mathbb{F}_1$, and hence the moment graph is

```
01 [C] 11
[F] [F]
00 [E] 10
```

where $F$ is the fiber, $E$ is the exceptional divisor (i.e. the curve with self-intersection $-1$,) and $C$ is related to $E$ and $F$ in Pic $\mathbb{F}_1$ by $C = E + F$.

In the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1)$, the restriction of the moment graph to the fixed points $\{000, 100, 001, 101\}$ is:

```
00 000 101
\[Z_{001}\] \[Z_{001}\] \\
000 100
\[Z_{100}\]
```

where the diagonal curves have homology class $\[Z_{100}\] - \[Z_{001}\]$. We draw the complete moment graph for this Bott-Samelson variety in Example 1.

In the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1, \alpha_2)$, the moment graph restricted to the fixed points $\{0000, 1010, 0001, 1011\}$ is:

```
0000 0001 1001 1011
\[Z_{1000}\] - \[Z_{0001}\] \[Z_{1000}\] + \[Z_{0001}\]
000 [Z_{1000}] - [Z_{0001}]
```

We draw the complete moment graph for this Bott-Samelson variety in Example 1.
All homology classes have been expressed in the Bott-Samelson subvariety basis, which we describe in Section 2.

In order to obtain a presentation for the quantum cohomology of Bott-Samelson varieties, we need to show that certain Gromov-Witten invariants vanish, and we need to compute some nonzero Gromov-Witten invariants. We address the non-zero invariant calculations first.

The Gromov-Witten invariants $I_{\gamma}(\gamma_1, \ldots, \gamma_n)$ are defined by intersection theory in the moduli space. However, in general we cannot control the geometry of the moduli space $M_{0,1}(Z, \beta)$. Under some conditions on the curve class $\beta$ and the Lie group $G$, we are able to prove the moduli space $M_{0,1}(Z, \beta)$ is smooth, and so we are able to perform the intersection theory calculations directly to determine certain Gromov-Witten invariants.

**Theorem 2.** Suppose $\beta$ is indecomposable and effective, and suppose $G$ is of simply laced type. Then $M_{0,1}(Z, \beta)$ is unobstructed; that is, $M_{0,1}(Z, \beta)$ is smooth, irreducible, and has the expected dimension

$$\dim M_{0,1}(Z, \beta) = \dim Z + \int_{\beta} c_1(T_Z) - 2.$$

Theorem 2 has an important corollary which allows us to carry out the necessary calculations.

**Corollary 2.1.** If $h \in H_2(Z)$ is the class of the fiber of $\pi : Z \to Z'$, then $ev : M_{0,1}(Z, h) \to Z$ is an isomorphism.

Corollary 2.1 lets us convert intersection theory calculations in the moduli space, into intersection theory calculations on the Bott-Samelson variety $Z$.

With the explicit calculations described above, and the vanishing of certain Gromov-Witten invariants that will be discussed in a few paragraphs, the final ingredient in obtaining a presentation for the (small) quantum cohomology of $Z(\alpha_1, \alpha_2, \alpha_1)$ is a brute-force calculation. Curve neighborhood techniques and the moduli space results allow us to compute some of the necessary Gromov-Witten invariants to quantize the relations in the ordinary cohomology. The remaining unknown invariants (of which there are 111), save for one, can be computed simply by imposing the relations that the quantum cohomology ring is commutative (this is a system of 192 (generally) nonlinear equations in 111 unknowns). The final Gromov-Witten invariant is computed using a technique of Manolache [Man12, Section 5.4].

**Theorem 3.** Let $Z = Z(\alpha_1, \alpha_2, \alpha_1)$. The (small) quantum cohomology ring $QH^*(Z)$ is isomorphic to a quotient of $\mathbb{Z}[\sigma_{100}, \sigma_{010}, \sigma_{001}, q_1, q_2, q_3]$, subject to the following relations:

- $\sigma_{100}^2 = q_1 q_3 - q_3 \sigma_{100} + q_3 \sigma_{010}$
- $\sigma_{010}^2 = q_1 q_3 + 2q_1 \sigma_{100} - q_1 \sigma_{010} + q_1 \sigma_{001} + \sigma_{110}$
- $\sigma_{001}^2 = q_1 q_3 + q_2 - q_3 \sigma_{100} + q_3 \sigma_{010} - 2\sigma_{101} + \sigma_{011}$

Under this isomorphism, the generators $\sigma_{100}, \sigma_{010}, \sigma_{001}$ are Poincaré dual to certain Bott-Samelson subvarieties in $Z$. For example, $\sigma_{001}$ is dual to the fiber of $\pi : Z \to Z'$. The other two classes $\sigma_{100}, \sigma_{010}$ arise in a similar way. The quantum
parameters $q_1, q_2, q_3$ correspond curve classes $\beta_1, \beta_2, \beta_3$ which generate the cone of effective 1-cycles.

The connection between Theorem 1 and quantum cohomology is given by curve neighborhoods, which will allow us to show that certain Gromov-Witten invariants vanish; curve neighborhoods were used to study the quantum cohomology and quantum $K$-theory of homogeneous spaces and “almost homogeneous” spaces (see [BM15], [BCMP16], [MS17], [MM18]).

Given an effective curve class $\beta$, and a closed subvariety $\Omega \subset Z$, the curve neighborhood $\Gamma(\beta)(\Omega)$ is the union of all curves of class $\beta$ which intersect $\Omega$; our definition later will be written in terms of the moduli space of stable maps, but is equivalent to the one just given. It will be clear from the alternative definition that curve neighborhoods are closed. Moreover, it is clear if $\Omega$ is $B$-stable. Finding all $B$-stable subvarieties in a Bott-Samelson variety is more difficult than finding all $B$-stable subvarieties in a homogeneous space. However, as we will show in Section 5, the unexpected $B$-stable curves collapse under $\theta : Z \to G/B$. So, with some ad hoc arguments, we can use the moment graph to describe curve neighborhoods for Bott-Samelson varieties.

Acknowledgments. I would like to thank my advisor, Leonardo Mihalcea, for his help and encouragement throughout this project.

2. Preliminaries

Our main reference in this section is [BK05, Chapter 2]. Given a sequence of simple reflections $w = (s_{\beta_1}, s_{\beta_2}, \ldots, s_{\beta_k})$, consider the space $P_w := P_{\beta_1} \times \cdots \times P_{\beta_k}$ equipped with the $B^k$-action

$$(b_1, \ldots, b_k) \circ (p_1, \ldots, p_r) = (p_1 b_1^{-1}, b_1^{-1} p_2 b_2, \ldots, b_{k-1}^{-1} p_k b_k)$$

Definition 2.1. The Bott-Samelson variety $Z_w$ is the coset space

$$Z_w := P_w / B^k$$

The points in $Z_w$ will be denoted by $[p_1, \ldots, p_k]$.

There is a natural $B$-action given by

$$b.[p_1, p_2, \ldots, p_k] = [bp_1, p_2, \ldots, p_k]$$

$Z_w$ contains an affine open cell $Z^o_w$ defined by

$$Z^o_w := B s_{\beta_1} B \times \cdots \times B s_{\beta_k} B / B^k$$

As in the introduction, we index the subwords of $w$ by binary $k$-tuples $\varepsilon \in \{0, 1\}^k$. For example, if $w = (s_1, s_2, s_1)$ and $\varepsilon = (1, 1, 0)$, then $w(\varepsilon) = (s_1, s_2)$. For the same $w$, if $\varepsilon = (1, 0, 1)$, then $w(\varepsilon) = (s_1, s_1)$.

For any subword $w(\varepsilon)$, there is a natural morphism $\pi_{\varepsilon} : Z_w \to Z_{w(\varepsilon)}$; if $w(\varepsilon)$ is the initial subword of length $m$, we will denote $Z_{w(\varepsilon)}$ by $Z_{w[m]}$ and $\pi_{\varepsilon}$ by $\pi_m$.

The length of $\varepsilon$, denoted $\ell(\varepsilon)$, is the number of ones in $\varepsilon$. If $\ell(\varepsilon) = 1$, we will denote $\varepsilon = (i)$ where $i$ is the nonzero entry of $\varepsilon$. When $\varepsilon, \varepsilon'$ have no common components, we say they are transverse, denoted $\varepsilon \perp \varepsilon'$.

For each word $w$, the product of the simple reflections (in order) is an element of the Weyl group which we denote by $w(\varepsilon)$; see Definition 1.1.

Proofs for all the statements in the following proposition (with one small exception) can be found in [BK05 pp. 64-67].
Proposition 2.1. Let $Z_w$ be a Bott-Samelson variety, $X = G/B$ the flag variety, and let $\mathfrak{w}(\varepsilon)$ be a subword of $\mathfrak{w}$.

(a) $Z_w$ is a smooth, projective variety.

(b) The natural morphism $\pi : Z_w \to Z_{w[k-1]}$ defined by

$$\pi([p_1, \ldots, p_k]) = [p_1, \ldots, p_{k-1}]$$

is $B$-equivariant, and realizes $Z_w$ as a $\mathbb{P}^1$-bundle over $Z_{w[k-1]}$.

(c) The map $\theta_w : Z_w \to X$ defined by

$$\theta_w([p_1, \ldots, p_k]) = (p_1 p_2 \cdots p_k) B$$

is a $B$-equivariant morphism. Moreover, if $s_{\beta_1} s_{\beta_2} \cdots s_{\beta_k} = w(\mathfrak{w})$ is a reduced word decomposition, then $\theta_w$ is a birational equivalence: $Z_w^o \cong X(w(\mathfrak{w}))^o$.

(d) The map $j_\varepsilon : Z_w(\varepsilon) \to Z_w$ defined by

$$j_\varepsilon([p_1, \ldots, p_k]) = [1, 1, \ldots, p_1, 1, \ldots, 1]$$

(where ones are placed in the components where $\varepsilon$ is zero) is a $B$-equivariant closed immersion.

For the $(k-1)$-initial subword, the morphism will be denoted by $s_{w : Z_{w[k-1]} \to Z_w}$ and is a section of $\pi_w$.

(e) The natural commutative diagram

$$
\begin{array}{ccc}
Z_w & \xrightarrow{\theta_w} & X \\
\pi_w \downarrow & & \downarrow p_{\beta_k} \\
Z_{w[k-1]} & \xrightarrow{p_{\beta_k} \theta_w} & G/P_{\beta_k}
\end{array}
$$

is Cartesian; that is, $Z_w$ is the fiber product $Z_{w[k-1]} \times_{G/P_{\beta_k}} X$. The $B$-action on the fiber product is diagonal.

Proof. As mentioned before the statement of the proposition, parts (a)-(d) are discussed in [BK05, pp. 64-67]. Part (e) is Exercise 2.2.E.1 in [BK05], with the exception of the $B$-action statement. This follows easily since each of the maps $\pi_w$ and $\theta_w$ are $B$-equivariant.

The cells $Z_\varepsilon^o := j_\varepsilon(Z_w(\mathfrak{w}(\varepsilon))^o)$ form an affine cell decomposition of $Z_w$:

$$Z_w = \bigcup_\varepsilon Z_\varepsilon^o$$

In particular, $\{[Z_\varepsilon] : \varepsilon \in \{0, 1\}^k\}$ is an additive basis for $H^*(Z_w)$; the dual basis (under the Poincare pairing) is denoted $\{\sigma_\varepsilon : \varepsilon \in \{0, 1\}^k\}$.

A presentation for the (ordinary) cohomology of Bott-Samelson varieties was obtained in [Dua05, Lemma 4.5]; we record the result here.

Proposition 2.2. The cohomology of $Z_w$ is generated by $\{\sigma_\varepsilon : \varepsilon \in \{0, 1\}^k\}$ with relations

$$\sigma_\varepsilon \sigma_{\varepsilon'} = \sigma_{\varepsilon + \varepsilon'}, \quad \text{if } \varepsilon \perp \varepsilon'$$

$$\sigma_{(j)}^2 = \sum_{i<j} -((\alpha_i, \alpha_j)) \sigma_{(i)+(j)}$$

where $(\alpha, \beta^\vee)$ is the usual root/coroot pairing.
Remark 2.1. It is known, though the author was unable to find a reference, that the dual classes $\sigma_\varepsilon$ are preserved under pullback along the morphisms $\pi_{w[r]} : Z_w \to Z_{w[r]}$ ($1 \leq r \leq k - 1$). We will write formulas such as: $\pi^{\ast}_{w[r]}(\sigma_\varepsilon) = \sigma_\varepsilon$. One should interpret $\varepsilon \in \{0, 1\}^r$ as a binary $k$-tuple ($k > r$) by appending zeros to the end of $\varepsilon$.

We include a proof that the dual classes are preserved under pull-back for completeness.

Proof. It suffices to show that $\sigma(i)$ is preserved under pullback for all $i$. Let $i < k$ and consider $\sigma(i) \in H^\ast(Z')$. Then

$$\int_{Z} \pi^\ast \sigma(i) \cdot [Z_\varepsilon] = \int_{Z'} \sigma(i) \cdot \pi^\ast [Z_\varepsilon].$$

However, $Z_\varepsilon$ is either the preimage under $\pi$ of a Bott-Samelson subvariety in $Z'$, in which case $\pi^\ast [Z_\varepsilon] = 0$, or $Z_\varepsilon$ is the image of $Z'_\varepsilon$ under the canonical section $s : Z' \to Z$ and $\pi^\ast [Z_\varepsilon] = [Z'_\varepsilon]$. Therefore, $\pi^\ast \sigma(i)$ is Poincaré dual to $Z(i)$, as claimed. \hfill \Box

To conclude this section, we define the cone of effective curves.

Definition 2.2. The cone of effective curves in $H_2(Z)$ is the set of all effective 1-cycles; that is, a positive combination of the fundamental classes of irreducible curves in $Z$.

In [And15, Lemma 2.1], it is stated that for complete, irreducible varieties $X$, the cone of effective $k$-cycles on $X$ is generated by the classes of $B$-invariant $k$-cycles. Thus, the cone of effective curves for a Bott-Samelson variety is generated by the classes of the $B$-stable curves. We characterize those curves in the next section.

First, we give a quick proof of [And15, Lemma 2.1].

Proof. Let $d \in N_k(X)_\mathbb{R}$ be an irreducible, effective $k$-cycle on $X$, and let $C_d^X$ denote the Chow variety for $X$ of degree $d$. The $B$-action on $X$ naturally lifts to $C_d^X$, and by the Borel fixed point theorem, there is a $B$-fixed point in $C_d^X$. This fixed point corresponds to a $B$-invariant $k$-cycle on $X$ with degree $d$.

Therefore, every $k$-cycle on $X$ can be written as a sum of $B$-invariant $k$-cycles. \hfill \Box

3. The moment graph

Bott-Samelson varieties are sympletic varieties with respect to the given torus action, and thus are equipped with a moment map $Z \to t^\ast$; see [Esc14] Section 4.1 for more details on the moment map for Bott-Samelson varieties, along with a description of the images of the $T$-fixed points under this map.

The image of the 1-skeleton of $Z$ under this map (that is, the image of the $T$-fixed points and $T$-stable curves) is called the moment graph for $Z$. The $T$-fixed points in a Bott-Samelson variety were discussed in Section 2, so it remains to describe the $T$-stable curves.

To begin, we characterize the $T$-stable curves on a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Let $X(T)$ the character group of $T$, and suppose $p : \Sigma \to \mathbb{P}^1$ is a $T$-equivariant $\mathbb{P}^1$-bundle. Moreover, we assume the $T$-actions on $\mathbb{P}^1$ and $\Sigma$ are nontrivial. For $x \in \Sigma^T$, the
weights of \( \Sigma \) at \( x \) are \( \chi, \psi \in X(T) \) where

\[
\begin{align*}
t.v &= \chi(t)v, \quad v \in T_{p(x)\mathbb{P}^1} \\
t.w &= \psi(t)w, \quad w \in T_x F
\end{align*}
\]

where \( F \) is the (geometric) fiber \( p^{-1}(x) \).

**Lemma 3.1.** There are infinitely many \( T \)-stable curves passing through \( x \in \Sigma \) if and only if the weights at \( x \), \( \chi \) and \( \psi \), are equal. Otherwise, there are exactly two (irreducible) \( T \)-stable curves passing through \( x \).

**Proof.** Since \( \Sigma \) is smooth and projective, there is a \( T \)-stable affine open neighborhood of \( x \) which is \( T \)-isomorphic to \( T_x \Sigma \) ([ByB73, Theorem 2.5].) Choose local coordinates \( X, Y \) so that \( C[T_x \Sigma] \simeq C[X,Y] \) with \( t.X = \chi(t)X \) and \( t.Y = \psi(t)Y \).

If \( \chi = \psi \), then it is clear that the lines \( V(X - \alpha Y) \subset T_x \Sigma \) (\( \alpha \in \mathbb{C} \)) are \( T \)-stable curves; in particular, there are infinitely many \( T \)-stable curves in \( \Sigma \) passing through \( x \).

If \( \chi \neq \psi \), the characters are linearly independent. Therefore, the only \( T \)-stable curves passing through \( x \) are (in local coordinates) \( V(X), V(Y) \). \( \square \)

If there is a \( T \)-fixed point \( x \in \Sigma \) with repeated weights, it is also easy to show there are exactly two \( T \)-fixed points with repeated weights. Moreover, there are additional \( T \)-stable curves in \( \Sigma \).

**Lemma 3.2.** Let \( \Sigma = \mathbb{P}^1 \times \mathbb{P}^1 \) (where each \( \mathbb{P}^1 \) is equipped with a nontrivial \( T \)-action). If the \( T \)-fixed points in \( \Sigma \) all have distinct weights, then the \( T \)-stable fibers of the two projections are the only \( T \)-stable curves. Otherwise, the images of the sections \( s_t : \mathbb{P}^1 \to \Sigma \) (\( t \in T \)), defined by \( s_t(z) = (z,t.z) \), are also \( T \)-stable, and these exhaust the set of \( T \)-stable curves in \( \Sigma \).

**Proof.** By the previous lemma, the only case that requires analysis is when there are repeated weights in \( T_x \Sigma \) for some \( x \in \Sigma^T \); fix such a point \( x \in \Sigma \). Since \( T_x \Sigma \) has equal weights, the \( T \)-actions on each factor \( \Sigma = \mathbb{P}^1 \times \mathbb{P}^1 \) are equal. Thus,

\[
s_t(t'.z) = (t'.z,t.(t'.z)) = (t'.z,t'(t.z)) = t'(z,t.z)
\]

that is, \( s_t \) is a \( T \)-equivariant section of \( p_1 : \Sigma \to \mathbb{P}^1 \).

These curves exhaust the set of (irreducible) \( T \)-stable curves in \( \Sigma \) since any other \( T \)-stable curve \( C \), aside from the \( T \)-stable fibers, intersects one of the sections \( s_t \) in a point not fixed by \( T \). Therefore, \( C \) shares a dense open orbit with the section \( s_t \) and so is equal to the image \( s_t(\mathbb{P}^1) \). \( \square \)

**Example 3.1.** In [Mag98, Section 2.1], a configuration variety interpretation of Bott-Samelson varieties is provided in type \( A \). For the Bott-Samelson variety \( Z(\alpha_1, \alpha_2, \alpha_1) \), the points correspond to configuration diagrams
where $V_2, V_3, V_{23}$ are vector subspaces of $\mathbb{C}^3$, $\dim V_2 = \dim V_3 = 1$ and $\dim V_{23} = 2$, with a line between two subspaces meaning inclusion (so, $V_2 \subset V_{23}$ and $V_2 \subset \mathbb{C}^2$ in the standard basis).

The sub Bott-Samelson variety $Z_{101}$ (note this subword is not reduced) is characterized by $V_{23} = \mathbb{C}^2$, so the configuration diagram is

Thus $Z_{101} \simeq \mathbb{P}^1 \times \mathbb{P}^1$; this isomorphism is the natural morphism

and is $T$-equivariant for the diagonal $T$-action on $X(s_1) \times X(s_1)$. The moment graph for $Z_{101}$ is

\[
\begin{array}{ccc}
\text{00} & \text{01} & \text{10} \\
\text{11} & & \\
\end{array}
\]
where the bold line is an infinite family of $T$-stable curves (given by the images of the morphisms $s_t$).

Before proving our main theorem, we prove a critical lemma.

**Lemma 3.3.** Let $C \subset Z$ be an irreducible $T$-stable curve.

(a) If $C$ is not contained in a fiber of $\theta : Z \to G/B$, then $\theta|_C$ is an isomorphism.

(b) If $C$ is not a fiber of $\pi : Z \to Z'$, then $\pi|_C$ is an isomorphism.

In particular, $C \cong \mathbb{P}^1$.

**Proof.** We’ll prove (a); the proof of (b) is similar. First, observe that $\theta|_C$ is an isomorphism if $C$ is a fiber of $\pi$, thus we may assume $C$ is not a fiber of $\pi$; let $C' = \pi(C)$. By induction on $\dim Z$, $\theta|_{C'}$ is an isomorphism. There are two possibilities:

1. $(p_{\beta_k} \theta')(C')$ is an isomorphism. In this case, the preimage $\pi^{-1}(C')$ is isomorphic (under $\theta$) to a surface in $G/B$. In particular, $\theta|_C$ is an isomorphism.

2. $(p_{\beta_k} \theta')(C')$ is a $T$-fixed point in $G/P_{\beta_k}$. In this case, the preimage $\pi^{-1}(C')$ is $T$-equivariantly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with the diagonal $T$-action (coming from $G/B$). Hence, $C$ is a section of $\theta$ over $X(s_{\beta_k})$ by Lemma 3.2. In particular, $\theta|_C$ is an isomorphism.

From (b), any $T$-stable curve $C$ is isomorphic to a fiber, and hence is isomorphic to $\mathbb{P}^1$. $\square$

**Lemma 3.4.** Let $C \subset Z(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a $T$-stable curve, and $k$ the largest integer so that $\dim \pi_k(C) = 0$. Then, if $x \in C^T$, the weight at $x$ is $\varepsilon_x(\alpha_k)$.

**Proof.** If $C$ is a fiber of $\pi$, then the result follows since $\theta|_C$ is an isomorphism. Otherwise, by Lemma 3.3, $\pi|_C$ is an isomorphism and the result follows by induction on $n = \dim Z$. $\square$

We are now ready to prove our main theorem, Theorem 1, stated in the introduction.

**Proof of Theorem 1.** For clarity, we will describe the moment graph pictures as Diagrams I-IV, and the characterizations in terms of $\varepsilon(\alpha_k)$ as Cases I-IV.

We only consider the case where $\pi(x), \pi(y)$ are joined by a $T$-stable curve $C$, since the other case is obvious. Let $\Sigma$ be defined as the pull-back $\Sigma = \pi^{-1}(C)$ as in the fiber diagram

\[
\begin{array}{ccc}
\Sigma & \longrightarrow & Z \\
\downarrow \pi|_\Sigma & & \downarrow \pi \\
C & \longrightarrow & Z'
\end{array}
\]

In particular, $\Sigma$ is the fiber product

\[
\begin{array}{ccc}
\Sigma & \overset{\theta|_\Sigma}{\longrightarrow} & G/B \\
\downarrow \pi|_\Sigma & & \downarrow p_{\alpha_n} \\
C & \overset{p_{\alpha_n}}{\longrightarrow} & G/P_{\alpha_n}
\end{array}
\]

Lemma 3.3 shows $p_{\alpha_n} \theta'|_C$ is either an isomorphism or a point map. If an isomorphism, then $\Sigma$ is isomorphic (under $\theta$) to a surface in $G/B$. In particular, the moment graph restricted to $\Sigma$ is diagram I.
Since the only curve whose homology class is not clear is the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$, we compute the homology class of that curve only. From Lemma 3.4, the weight along $C$ at $\pi(x)$ is $\pi(\varepsilon_x)\alpha_k$, and the weight at $\pi(y)$ is $\varepsilon_y\alpha_k$. Since $\theta_{\pi(C)} = \theta_{\pi(C)}|C$ is an isomorphism, the weights are preserved, and since $\theta(C)$ is a translate of $X(s_{\alpha_k})$, we have $(\theta s)_*|C = [X(s_{\alpha_k})]$. In particular, $w(\varepsilon_x^0) = w(\varepsilon_y^0)s_{\alpha_k}$. Since $\alpha_k \neq \alpha_n$, so case I holds.

We can then compute the relation between $w(\varepsilon_x^0)$, $w(\varepsilon_y^\infty)$:

$$w(\varepsilon_x^\infty) = w(\varepsilon_x^0)s_{\alpha_n} = (w(\varepsilon_y^0)s_{\alpha_k})s_{\alpha_n} = (w(\varepsilon_y^0)s_{\alpha_n})s_{\beta} = w(\varepsilon_y^\infty)s_{\beta}$$

where $\beta = s_{\alpha_n}(\alpha_k) = \alpha_k - (\alpha_k,\alpha_n)\alpha_n$. Thus, the degree of the image of the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$ is $[X(s_{\alpha_k})] - (\alpha_k,\alpha_n)[X(s_{\alpha_n})]$. Note, that since $\alpha_k \neq \alpha_n$, $(\alpha_k,\alpha_n) \leq 0$.

Moreover, the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$ has homology class $s_*|C + ah$ for some constant $a$. Equating the push-forward calculations, we obtain $a = -(\alpha_k,\alpha_n)$. Therefore, the homology class of the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$ is $s_*|C - (\alpha_k,\alpha_n)h$.

We now consider the case where $\alpha_n \theta|C$ is a point map. In this case, $\Sigma$ is the trivial $\mathbb{P}^1$-bundle $\Sigma = C \times X(s_{\alpha_n})$. According to Lemma 3.3, there are cases corresponding to whether all $T$-fixed points in $\Sigma$ have distinct weights or not.

As above, the weights along $C$ at $\varepsilon_x^0, \varepsilon_y^0$ are $\varepsilon_x^0(\alpha_k), \varepsilon_y^0(\alpha_k)$ respectively. Thus, there are repeated weights in cases II, III, and IV. Indeed, there can only be exactly two fixed points with repeated weights, and they cannot be $\varepsilon_x^\infty, \varepsilon_y^\infty$, otherwise there could not be a $T$-stable curve joining $\varepsilon_x^0, \varepsilon_y^0$. By Lemma 3.2, there are infinitely many $T$-stable curves joining the two fixed points with repeated weights, hence we get the moment graph pictures which correspond to diagrams II, III, and IV.

If all fixed points have distinct weights, we get the moment graph in diagram I, and the curve joining $\varepsilon_x^\infty, \varepsilon_y^\infty$ is a fiber of the second projection. Therefore, if $w(\varepsilon_x^0) = w(\varepsilon_y^0)$ and case I holds, the homology class of the unlabeled curve is $s_*|C$.

If $\varepsilon_x^0, \varepsilon_y^0$ are the fixed points with repeated weights, then the moment graph is given by diagram II. Moreover, $w(\varepsilon_x^0) = w(\varepsilon_y^0)s_{\alpha_n}$ so case II holds. From this, $w(\varepsilon_x^0) = w(\varepsilon_y^\infty)$ and vice-versa. Therefore, the homology class of the diagonal curves is $s_*|C - h$.

If $w(\varepsilon_x^0) = w(\varepsilon_y^0)$, and there are two points with repeated weights, they must be diagonally adjacent since both $\varepsilon_x^0, \varepsilon_y^0$ cannot have repeated weights, in particular cases III and IV hold. Diagrams III and IV are the cases where $\varepsilon_x^0$ and $\varepsilon_y^0$ respectively have repeated weights. In both cases, the diagonal family of curves have pushforward $[X(s_{\alpha_n})]$, while $(\theta s)_*|C = 0$. Therefore, the homology class of the diagonal curves is $s_*|C + h$.

\begin{example} \label{3.2} \end{example}

Consider the Bott-Samelson variety $Z = Z(\alpha_1, \alpha_2, \alpha_3)$ in type $A_2$. As stated in Section 2, the cone of effective 1-cycles is generated by the $B$-stable 1-cycles (see [And15] Lemma 2.1.) A basis for the cone of effective 1-cycles is given by $\beta_1 = [Z_{010}], \beta_2 = [Z_{001}], \beta_3 = [Z_{100}] - [Z_{001}]$. The entire moment graph for the Bott-Samelson variety $Z$ is depicted in Figure 1.
4. Moduli space of stable maps

We now turn to our computation of the quantum cohomology of the Bott-Samelson variety $Z = Z(\alpha_1, \alpha_2, \alpha_1)$. In order to compute a presentation $\mathcal{QH}^*(Z)$, we need to compute many Gromov-Witten invariants. Some will vanish using curve neighborhoods, however we will need to compute some nonzero invariants to use brute-force calculations to finish the presentation. This section provides the tools we need to compute these nonzero invariants. We start with a definition.

**Definition 4.1.** We say an effective curve class $\beta \in H_2(Z)$ is **indecomposable** if $\beta$ cannot be expressed: $\beta = \beta_1 + \beta_2$, where $\beta_1, \beta_2 \in H_2(Z)$ are effective.

**Example 4.1.** In the threefold $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ in type $A_2$, the classes $[Z_{010}]$ and $[Z_{001}]$ are indecomposable, but $[Z_{100}] = [Z_{001}] + \beta_3$ where $\beta_3 = [C]$, the fiber of $\theta : Z \to \text{Fl}(3)$ over the identity.

Indeed, since $C$ is a fiber of $\theta$, $\theta_*[C] = 0$. Furthermore, $C$ is a $T$-stable curve which is not contained in the fiber of $\pi : Z(\alpha_1, \alpha_2, \alpha_1) \to Z(\alpha_1, \alpha_2)$. So by Lemma 3.3 $\pi_*\beta_3 = [Z_{10}]$. Since $\pi_*\beta_3 = \pi_*([Z_{100}] - [Z_{001}])$ and $\theta_*\beta_3 = \theta_*([Z_{100}] - [Z_{001}])$, and $Z$ is a fiber product, we have $\beta_3 = [Z_{100}] - [Z_{001}]$.

In fact, the generators $\beta_1, \beta_2, \beta_3$ of the effective cone for $Z(\alpha_1, \alpha_2, \alpha_1)$ are all indecomposable.

Recall, the moduli space of stable maps $\overline{M}_{0,n}(Z, \beta)$ consists of stable maps $f : C \to Z$, where $C$ is decorated with $n$ non-singular marked points $p_1, \ldots, p_n \in C$, and $f_*(C) = \beta$. (The stability condition is equivalent to the automorphism group of the marked curve $C$ being finite.)
Proposition 4.1. If $\beta \in H_2(Z)$ is indecomposable, and the Dynkin diagram of $G$ is simply laced, then the moduli space $\overline{M}_{0,1}(Z, \beta)$ is unobstructed; that is, $\overline{M}_{0,1}(Z, \beta)$ is smooth, irreducible, and has the expected dimension

$$\dim \overline{M}_{0,1}(Z, \beta) = \dim Z + \int_\beta c_1(T_Z) - 2.$$

Proof. The proof is by induction. Let $[g : C \to Z] \in \overline{M}_{0,1}(Z, \beta)$. Since $\beta$ is indecomposable and $C$ has a single marked point, $C \simeq P^1$. There are two cases:

(a) $\beta$ is the class of the fiber of $\pi : Z \to Z'$. Since $g^*\pi^*T_{Z'}$ is a trivial line bundle, $H^1(C, g^*\pi^*T_{Z'}) = 0$. Moreover, $T_\pi = \theta^*T_{P_{s\alpha_n}}$ and $\theta_s\beta = [X(s\alpha_n)]$ which implies $g^*T_\pi \simeq O_{P^1}(2)$ (since $c_1(T_{P_{s\alpha_n}}) \cap [X(s\alpha_n)] = (\alpha_n, \alpha_n^\vee)$). Thus $H^1(C, g^*T_\pi) = 0$, and therefore $H^1(C, g^*T_\pi) = 0$.

(b) Otherwise, $\pi, \beta \neq 0$ (and effective). Moreover, $\pi, \beta$ is indecomposable, otherwise $\pi, \beta = \beta_1 + \beta_2$ which implies $\beta - s_\beta \beta$ is effective. By induction, $\overline{M}_{0,1}(Z', \pi, \beta)$ is unobstructed, thus $H^1(C, g^*\pi^*T_{Z'}) = 0$. To compute $H^1(C, g^*T_\pi)$ there are two cases:

1. If $\theta_s\beta = 0$, then $g^*T_\pi$ is the trivial line bundle on $C$

$$\int_C c_1(g^*T_\pi) \cap [C] = \int_Z c_1(T_\pi) \cap [\beta] = \int_{G/B} c_1(T_{P_{s\alpha_n}}) \cap \theta_s\beta.$$

Thus, $H^1(C, g^*T_\pi) = 0$.

2. If $\theta_s\beta \neq 0$, then since $\beta$ is represented by a $B$-stable curve, the image of that curve under $\theta$ is a Schubert curve. In particular, there is a unique $\alpha'$ for which the integral

$$\int_{G/B} \theta_s\beta \cdot [Y(s\alpha')] = \int_Z \beta \cdot \theta^*[Y(s\alpha')] = \sum \int_Z \beta \cdot \sigma_{\alpha}$$

has value equal to 1, since $\beta$ is indecomposable. Therefore, $g^*T_\pi \simeq O_{P^1}(d)$ where $d = (\alpha, \alpha') \geq -1$ since the Dynkin diagram for $G$ is simply laced, and so $H^1(C, g^*T_\pi) = 0$.

In either case, we have $H^1(C, g^*T_\pi) = 0$ and $H^1(C, g^*\pi^*T_{Z'}) = 0$, therefore $H^1(C, g^*T_Z) = 0$. \hfill \Box

Corollary 4.1. If $\beta$ is indecomposable and $Z$ is the disjoint union of curves of class $\beta$ (i.e. if $Z$ is a $B$-fibration), then $\text{ev} : \overline{M}_{0,1}(Z, \beta) \to Z$ is an isomorphism.

Proof. Since $\beta$ is indecomposable, $\overline{M}_{0,1}(Z, \beta)$ is smooth. Moreover, every point in $Z$ lies on a unique curve of class $\beta$, therefore $\text{ev} : \overline{M}_{0,1}(Z, \beta) \to Z$ is a bijection. Since both are varieties over $\mathbb{C}$, $\text{ev}$ is an isomorphism. \hfill \Box

Example 4.2. For $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ and $\beta = \beta_3$

$$\text{ev} : \overline{M}_{0,1}(Z, \beta) \to Z_{101}$$

is an isomorphism from Corollary 4.1.

If $h$ is the class of the fiber for any Bott-Samelson variety $Z$, $\text{ev} : \overline{M}_{0,1}(Z, h) \to Z$ is an isomorphism since $h$ is indecomposable.
5. Curve neighborhoods

As we stated in the introduction, the connection between our moment graph result and quantum cohomology is given by curve neighborhoods. Background on curve neighborhoods, particularly in the context of homogeneous spaces, can be found in [BML15]. The main difference here is that curve neighborhoods need not “grow.”

**Definition 5.1.** Let \( X \) be a variety, \( Y \) a subset of \( X \), and \( \beta \in A_1(X) \) an effective curve class. The curve neighborhood \( \Gamma_\beta(Y) \) is defined by

\[
\Gamma_\beta(Y) := ev_1(ev_2^{-1}(Y))
\]

where \( ev_i : \overline{M}_{0,2}(X, \beta) \to X \) \((i = 1, 2)\) are the evaluation morphisms; \( \Gamma_\beta(Y) \) is given the reduced scheme structure.

Observe, if \( X \) is a \( G \)-variety for an algebraic group \( G \), and \( Y \) is a \( G \)-stable closed subvariety of \( X \), then \( \Gamma_\beta(Y) \) is a \( G \)-stable closed subvariety of \( X \). Also, note that since the morphism \( ev : \overline{M}_{0,2}(X, \beta) \to X \) is proper, \( \Gamma_\beta(Y) \) can be realized as the closure of the union of all curves \( C \) of class \( \beta \) passing through \( Y \).

**Proposition 5.1.** For any \( B \)-stable subvariety \( \Omega \subset Z(\alpha_1, \alpha_2, \alpha_1) \), the curve neighborhood \( \Gamma_{\beta_3}(\Omega) \subset Z_{101} \). Furthermore, if \( \theta(\Omega) = X(s_1) \), then \( \Gamma_{\beta_3}(\Omega) = Z_{101} \).

**Proof.** Note that any curve of class \( \beta_3 \) collapses under the map \( \theta : Z(\alpha_1, \alpha_2, \alpha_1) \to G/B \). Since \( \theta \) is an isomorphism outside the locally closed set \( Z_{101} \), any curve of class \( \beta_3 \) is contained in the sub Bott-Samelson variety \( Z_{101} \). Since \( Z_{101} \) is a closed, \( B \)-stable variety in \( Z(\alpha_1, \alpha_2, \alpha_1) \), the first claim follows.

For the second claim, since \( Z_{101} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) where \( \beta_3 \) is the class of the fiber of the second projection, every point \( x \in Z_{101} \) has a unique curve of class \( \beta_3 \) passing through it which also intersects \( \Omega \).

**Lemma 5.1.** The following are curve neighborhoods for the Bott-Samelson variety \( Z(\alpha_1, \alpha_2, \alpha_1) \):

\[
\Gamma_{\beta_3}(Z_{100}) = Z_{101}, \quad \Gamma_{\beta_3}(Z_{010}) = \theta^{-1}(x_c), \quad \Gamma_{\beta_3}(Z_{001}) = Z_{101}, \quad \Gamma_{\beta_3}(Z_{100}) = Z_{110}.
\]

**Proof.** We prove each equality in the order specified above.

\( \Gamma_{\beta_3}(Z_{100}) \): Note that \( Z_{100} \subset Z_{101} \). In fact, the image of \( Z_{100} \) under \( \theta : Z(\alpha_1, \alpha_1) \to X(s_{\alpha_1}) \) is \( X(s_{\alpha_1}) \). Therefore, the union of the fibers of \( pr_2 : Z(\alpha_1, \alpha_1) \to X(s_{\alpha_1}) \) is the curve neighborhood \( \Gamma_{\beta_3}(Z_{100}) \). However this union is the whole space. Therefore, \( \Gamma_{\beta_3}(Z_{100}) = Z_{101} \).

\( \Gamma_{\beta_3}(Z_{010}) \): Since \( Z_{010} \cap Z_{101} = x_{010} \), and since there is a unique curve of class \( \beta_3 \) passing through that point (the curve joining \( x_{000} \) to \( x_{101} \)) the curve neighborhood is \( \Gamma_{\beta_3}(Z_{010}) = C \), the fiber of \( \theta \) over \( x_c \in G/B \).

\( \Gamma_{\beta_3}(Z_{001}) \): A similar analysis to \( \Gamma_{\beta_3}(Z_{100}) \) shows that \( \Gamma_{\beta_3}(Z_{001}) = Z_{101} \) (since the image of \( Z_{001} \) under \( \theta : Z(\alpha_1, \alpha_1) \to X(s_{\alpha_1}) \) is \( X(s_{\alpha_1}) \)).

\( \Gamma_{\beta_3}(Z_{100}) \): There is a \( T \)-stable curve of class \( \beta_1 \) joining \( x_{100} \) to \( x_{110} \). Since the \( B \)-action on the cell \( Z_{110} \) is transitive (\( \theta |_{Z_{110}} \) is an isomorphism) and since \( Z_{100} \) is \( B \)-stable, we see that \( \Gamma_{\beta_3}(Z_{100}) = Z_{110} \).
Remark 5.1. In general, it is clear that curve neighborhoods will be connected. However, it would be useful to have a condition for when a particular curve neighborhood is irreducible. For example, for Schubert varieties, it is known that all such curve neighborhoods are irreducible (see [BCMP13]).

6. Quantum cohomology

Let $X$ be a smooth, projective $\mathbb{C}$-variety. Fix a homogeneous basis $\{\gamma_j\}$ for $H^*(X)$, and a basis $\{\beta_k\}$ for the cone of effective curves in $H_2(X)$. As a $\mathbb{Q}$-vector space, the (small) quantum cohomology $QH^*(X) = H^*(X; \mathbb{Q}) \otimes \mathbb{Q}[q^\beta]$, where there is one quantum parameter $q^\beta_k$ for each generator of the effective cone. The quantum product of two classes $x, y \in H^*(X)$ is defined by

$$x \ast y = \sum_{\beta,j} I_\beta(x, y, \gamma_j^\vee)q^\beta \gamma_j$$

where $\gamma_j^\vee$ is Poincaré dual to $\gamma_j$, and the Gromov-Witten invariant $I_\beta(x, y, \gamma_j^\vee)$ is defined by

$$I_\beta(x, y, \gamma_j^\vee) = \int_{[M_{0,3}(X, \beta)]^{\text{virt}}} ev_1^*(x) \cdot ev_2^*(y) \cdot ev_3^*(\gamma_j^\vee)$$

where $[M_{0,3}(X, \beta)]^{\text{virt}}$ is the virtual fundamental class, a generalization of the fundamental class which is necessary since $M_{0,3}(X, \beta)$ is not generally irreducible or even equidimensional. The class $[M_{0,3}(X, \beta)]^{\text{virt}}$ has the “expected dimension” of the moduli space (see Proposition [41]).

In general, for $\sigma_1, \ldots, \sigma_n \in H^*(X)$, Gromov-Witten invariants are defined on $M_{0,n}(X, \beta)$ as

$$I_\beta(\sigma_1, \ldots, \sigma_n) = \int_{[M_{0,n}(X, \beta)]^{\text{virt}}} \prod_j ev_j^*(\sigma_j).$$

For us, there are two important properties of Gromov-Witten invariants:

1. The Gromov-Witten invariant $I_\beta(\sigma_1, \ldots, \sigma_n) = 0$ unless

$$\sum_j \text{codim } \sigma_j = \dim X + \int_\beta c_1(T_X) + n - 3;$$

this property is called the codimension condition.

2. If $\sigma_1 \in H^2(X)$ (i.e. if $\sigma_1$ is a divisor class,) then

$$I_\beta(\sigma_1, \sigma_2, \ldots, \sigma_n) = \left(\int_\beta \sigma_1 \right) I_\beta(\sigma_2, \ldots, \sigma_n);$$

this property is called the divisor axiom.

As a result of the codimension condition, $QH^*(X)$ is equipped with a grading compatible with the grading on $H^*(X)$:

$$\deg q^\beta = \int_\beta c_1(T_X).$$

Lemma 6.1. For $Z = Z(\alpha_1, \alpha_2, \alpha_1)$, the degrees of the quantum parameters are

$$\deg q^{\beta_1} = 1, \quad \deg q^{\beta_2} = 2, \quad \deg q^{\beta_3} = 1.$$
Therefore, we can write
\[ \omega \]
\[
\text{corresponding to a dominant fundamental weight}
\]
\[
\text{generators have Chern classes}
\]
\[ O_{\text{line bundle}} \]
\[
\text{In order to compute a presentation for the small quantum cohomology}
\]
\[
\text{Thus,}
\]
\[ - 
\]
\[ \text{Proposition 11] for more details. Since the relations in}
\]
\[ H^{*}(Z) \]
\[
\text{it suffices to quantize the relations in the ordinary cohomology}
\]
\[
\text{In [LT04], a basis for the ample cone on a Bott-Samelson variety is given; the}
\]
\[ O_{\text{m}}(1) \]
\[
\text{basis. These line bundles are pullbacks of the line bundles}
\]
\[ L_{\omega_{\alpha_{k}}} \]
\[
\text{corresponding to a dominant fundamental weight}
\]
\[ w \]
\[
\text{Since}
\]
\[ c_{1}(L_{\omega_{\alpha_{k}}}) = [Y(s_{\alpha_{k}})] \]
\[
\text{the line bundle}
\]
\[ O_{\text{m}}(1) \]
\[
\text{has Chern class}
\]
\[ \sigma_{100} + \sigma_{001} \]
\[
\text{for}
\]
\[ w = (1, 2, 1) \]
\[
\text{The other two}
\]
\[ \omega_{\alpha_{k}} \]
\[
\text{generators have Chern classes}
\]
\[ c_{1}(O_{1,2}(1)) = \sigma_{010} \]
\[ c_{1}(O_{1}(1)) = \sigma_{100} \]
\[
\text{Therefore, we can write}
\]
\[ c_{1}(T_{Z}) = c_{1}(O_{1}(1)) + c_{1}(O_{1,2}(1)) + 2c_{1}(O_{1,2,1}(1)). \]
\[
\text{Thus,}
\]
\[ -K_{Z} \]
\[
\text{is ample, that is}
\]
\[ Z \]
\[
\text{is Fano.}
\]
\[
\text{From here on,}
\]
\[ Z \]
\[
\text{denotes the Bott-Samelson variety}
\]
\[ Z(\alpha_{1}, \alpha_{2}, \alpha_{1}) \]
\[
\text{in type}
\]
\[ A_{2} \]
\[
\text{In order to compute a presentation for the small quantum cohomology}
\]
\[ QH^{*}(Z) \]
\[
\text{it suffices to quantize the relations in the ordinary cohomology}
\]
\[ H^{*}(Z) \]
\[
\text{see [FP97, Proposition 11] for more details. Since the relations in}
\]
\[ H^{*}(Z) \]
\[
\text{are all in codimension}
\]
\[ \Box \]

**Remark 6.1.** In the course of proving the previous Lemma, we showed \( c_{1}(T_{Z}) = 3\sigma_{100} + \sigma_{010} + 2\sigma_{001} \).

In [LT04], a basis for the ample cone on a Bott-Samelson variety is given; the
\[ O_{\text{m}}(1) \]
\[
\text{basis. These line bundles are pullbacks of the line bundles}
\]
\[ L_{\omega_{\alpha_{k}}} \]
\[
\text{corresponding to a dominant fundamental weight}
\]
\[ w \]
\[
\text{Since}
\]
\[ c_{1}(L_{\omega_{\alpha_{k}}}) = [Y(s_{\alpha_{k}})] \]
\[
\text{the line bundle}
\]
\[ O_{\text{m}}(1) \]
\[
\text{has Chern class}
\]
\[ \sigma_{100} + \sigma_{001} \]
\[
\text{for}
\]
\[ w = (1, 2, 1) \]
\[
\text{The other two}
\]
\[ \omega_{\alpha_{k}} \]
\[
\text{generators have Chern classes}
\]
\[ c_{1}(O_{1,2}(1)) = \sigma_{010} \]
\[ c_{1}(O_{1}(1)) = \sigma_{100} \]
\[
\text{Therefore, we can write}
\]
\[ c_{1}(T_{Z}) = c_{1}(O_{1}(1)) + c_{1}(O_{1,2}(1)) + 2c_{1}(O_{1,2,1}(1)). \]
\[
\text{Thus,}
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\[ -K_{Z} \]
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\[
\text{In order to compute a presentation for the small quantum cohomology}
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\[ QH^{*}(Z) \]
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\[ H^{*}(Z) \]
\[
\text{see [FP97, Proposition 11] for more details. Since the relations in}
\]
\[ H^{*}(Z) \]
\[
\text{are all in codimension} \]
one, and since it is necessary for the approach we use, we will compute all products $\sigma_{(j)} \ast \sigma_x$. We organize this data in what we call Chevalley matrices.

Since one of the terms of our quantum products will always be a divisor class, the divisor axiom will always be used to reduce three-point Gromov-Witten invariants to two-point invariants. For two-point invariants, we have the following lemma which relates curve neighborhoods to the vanishing of Gromov-Witten invariants.

**Lemma 6.2.** Let $X$ be a smooth, projective $\mathbb{C}$-variety, $\Omega \subset X$ a closed subvariety, $\gamma \in H^*(X)$, and $\beta \in H_2(X)$ an effective curve class. Suppose $\dim \gamma + \dim \Omega = \dim X + \int_{\beta} c_1(T_X) - 1$ (that is, the codimension condition is satisfied.) Denote the irreducible components of $\Gamma_\beta(\Omega)$ by $\Gamma_i$, that is:

$$\Gamma_\beta(\Omega) = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k.$$

1. If $\dim \Gamma_\beta(\Omega) < \dim \gamma$, then $I_\beta(\gamma, [\Omega]) = 0$.
2. If $\dim \Gamma_\beta(\Omega) = \dim \gamma$ and $\int_X \gamma \cdot [\Gamma_i] = 0$ for each $1 \leq i \leq k$, then $I_\beta(\gamma, [\Omega]) = 0$.

**Proof.** Using the projection formula, we have

$$I_\beta(\gamma, [\Omega]) = \int_{(\overline{M}_{0,2}(X,\beta))^{\text{virt}}} ev_1^*(\gamma) \cdot ev_2^*[\Omega]$$

$$= \int_X \gamma \cdot ev_1^*(ev_2^*[\Omega] \cdot (\overline{M}_{0,2}(X,\beta))^{\text{virt}})$$

where, by definition of curve neighborhoods,

$$ev_1^*(ev_2^*[\Omega] \cdot (\overline{M}_{0,2}(X,\beta))^{\text{virt}}) = \sum_{i=1}^k m_i [\Gamma_i];$$

the constants $m_i \geq 0$ are zero when $\dim \Gamma_i < \dim ev_2^*[\Omega]$, otherwise they are the degrees of $ev_1$ restricted to each irreducible component.

The desired conclusion follows in both cases since $\int_X \gamma \cdot [\Gamma_i] = 0$ for all $i$. $\square$

Using the moment graph to compute curve neighborhoods for $Z(\alpha_1, \alpha_2, \alpha_3)$, and Lemma 6.2 we are able to show that certain Gromov-Witten invariants vanish. Combining this with the explicit moduli space results obtained in Section 4, we are able to compute many of the Gromov-Witten invariants needed to compute $QH^*(Z)$. However, there are still some unknown invariants that are needed.

Using a computer, and the fact that $QH^*(Z)$ is a commutative ring, we are able to solve for all of the unknowns in the Chevalley matrices. In each of the following subsections, we record the calculations necessary to produce the Chevalley matrices, the matrix $A$ corresponding to quantum multiplication by $\sigma_{100}$, $B$ corresponding to quantum multiplication by $\sigma_{010}$, and $C$ corresponding to quantum multiplication by $\sigma_{001}$.

**6.1. Chevalley matrix $A$.** Given any $\varepsilon \in \{0, 1\}^3$, we have

$$\sigma_{100} \ast \sigma_\varepsilon = \sum_{\beta, \varepsilon'} I_\beta(\sigma_{100}, \sigma_\varepsilon, [Z_{\varepsilon'}]) q^\beta \sigma_{\varepsilon'}$$

We will calculate the quantum product $\sigma_{100} \ast \sigma_\varepsilon$ for each $\varepsilon \in \{0, 1\}^3$. 


• For ε = 000, since 1 = [Z] ∈ H^*(Z) is also the identity in QH^*(Z) we have
  \[
  \sigma_{100} * 1 = \sigma_{100}
  \]

(6.1.1)

• For ε = 100, 010, or 001, we can use the divisor axiom twice to reduce the three-point Gromov-Witten invariants to one-point invariants.

Using the codimension condition, for the curve classes β ≠ 0 which need to be considered, q^d is either degree 1 or 2. Therefore, the curve classes β for which \(I_\beta(\sigma_{100}, \sigma_\varepsilon, [Z_\varepsilon'])\) is possibly nonzero are

\[\beta_3, \beta_1 + \beta_3, 2\beta_3\]

Corollary 4.1 implies \(ev: \overline{M}_{0,1}(Z, \beta_3) \to Z_{101}\) is an isomorphism. Thus

\[
I_{\beta_3}(Z_{100}) = \int_{Z_{101}} [Z_{100}] = -1
\]

\[
I_{\beta_3}(Z_{010}) = \int_{Z_{101}} [Z_{010}] = 1
\]

\[
I_{\beta_3}(Z_{001}) = \int_{Z_{101}} [Z_{001}] = 0
\]

We will assign variables for the remaining Gromov-Witten invariants; we compute the values of these unknowns using brute force after analyzing all three Chevalley matrices.

\[I_{\beta_1 + \beta_3}([pt]) = x_1, \quad I_{2\beta_3}([pt]) = x_2\]

We can now write the quantum products as follows:

(6.1.2) \[
\sigma_{100} * \sigma_{100} = -q_3 \sigma_{100} + q_3 \sigma_{010} + (q_1 q_3 x_1 + 4q_3^2 x_2)
\]

(6.1.3) \[
\sigma_{100} * \sigma_{010} = \sigma_{110} + q_1 q_3 x_1
\]

(6.1.4) \[
\sigma_{100} * \sigma_{001} = \sigma_{101} + q_3 \sigma_{100} - q_3 \sigma_{010} + (-q_1 q_3 x_1 - 4q_3^2 x_2)
\]

• For ε = 110, 101, 011, we can only use the divisor axiom once to reduce the three-point Gromov-Witten invariant to a two-point invariant. Moreover, using the codimension condition and the divisor axiom, we can reduce the curve classes β which need to be considered to the following:

\[\beta_3, \beta_1 + \beta_3, 2\beta_3, 2\beta_1 + \beta_3, \beta_1 + 2\beta_3, \beta_2 + \beta_3, 3\beta_3\]

From the codimension condition, if \(\ell(\varepsilon) = 2, I_{\beta_3}(\sigma_{100}, \sigma_\varepsilon, [Z_\varepsilon']) = 0\) unless \(\ell(\varepsilon') = 2\). Moreover, the curve neighborhoods for class \(\beta_3\) have been considered in Section 5

\[
\Gamma_{\beta_3}(Z_{110}) = Z_{101}, \quad \Gamma_{\beta_3}(Z_{101}) = Z_{101}, \quad \Gamma_{\beta_3}(Z_{011}) = Z_{101}
\]

Using Lemma 6.2, the curve neighborhoods show \(I_{\beta_3}(\sigma_\varepsilon, [Z_\varepsilon']) = 0\) unless \(\varepsilon = 101\). Thus we have shown that six Gromov-Witten invariants vanish, leaving three more (for curve class \(\beta_3\)) which are unknown.
We assign variables for the remaining Gromov-Witten invariants.

\[
I_{\beta_1}(\sigma_{101}, [Z_{110}]) = x_3, \quad I_{\beta_1}(\sigma_{101}, [Z_{101}]) = x_4, \quad I_{\beta_1}(\sigma_{101}, [Z_{011}]) = x_5,
I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{100}]) = x_6, \quad I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{100}]) = x_7, \quad I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{010}]) = x_8,
I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{010}]) = x_9, \quad I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{010}]) = x_{10}, \quad I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{010}]) = x_{11},
I_{\beta_2}(\sigma_{110}, [Z_{010}]) = x_{12}, \quad I_{\beta_2}(\sigma_{101}, [Z_{100}]) = x_{13}, \quad I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{011}]) = x_{14},
I_{\beta_2}(\sigma_{110}, [Z_{100}]) = x_{15}, \quad I_{\beta_2}(\sigma_{101}, [Z_{010}]) = x_{16}, \quad I_{\beta_2}(\sigma_{101}, [Z_{010}]) = x_{17},
I_{\beta_2}(\sigma_{110}, [Z_{100}]) = x_{18}, \quad I_{\beta_2}(\sigma_{101}, [Z_{100}]) = x_{19}, \quad I_{\beta_2}(\sigma_{011}, [Z_{010}]) = x_{20},
I_{\beta_2}(\sigma_{011}, [Z_{100}]) = x_{21}, \quad I_{\beta_2}(\sigma_{011}, [Z_{100}]) = x_{22}, \quad I_{\beta_2}(\sigma_{011}, [Z_{010}]) = x_{23},
I_{\beta_2+\beta_3}(\sigma_{110}, [pt]) = x_{24}, \quad I_{\beta_2+\beta_3}(\sigma_{101}, [pt]) = x_{25}, \quad I_{\beta_2+\beta_3}(\sigma_{011}, [pt]) = x_{26},
I_{\beta_2+\beta_3}(\sigma_{110}, [pt]) = x_{27}, \quad I_{\beta_2+\beta_3}(\sigma_{101}, [pt]) = x_{28}, \quad I_{\beta_2+\beta_3}(\sigma_{011}, [pt]) = x_{29},
I_{\beta_3}(\sigma_{110}, [pt]) = x_{30}, \quad I_{\beta_3}(\sigma_{101}, [pt]) = x_{31}, \quad I_{\beta_3}(\sigma_{011}, [pt]) = x_{32},
I_{\beta_3}(\sigma_{110}, [pt]) = x_{33}, \quad I_{\beta_3}(\sigma_{101}, [pt]) = x_{34}, \quad I_{\beta_3}(\sigma_{011}, [pt]) = x_{35}
\]

We now record these quantum products:

\[
\sigma_{100} * \sigma_{110} = (q_1q_3x_6 + 2q_2^2x_{15})\sigma_{010} + (q_1q_3x_7 + 2q_2^2x_{16})\sigma_{010}
+ (q_1q_3x_8 + 2q_2^2x_{17})\sigma_{001} + (q_1q_3x_{24} + 2q_2^3x_{27} + q_2q_3x_{30} + 3q_3^2x_{33})
\]

\[
\sigma_{100} * \sigma_{101} = q_3x_3\sigma_{110} + q_3x_4\sigma_{101} + q_3x_5\sigma_{011} + (q_1q_3x_9 + 2q_2^2x_{18})\sigma_{100}
+ (q_1q_3x_{10} + 2q_2^2x_{19})\sigma_{101} + (q_1q_3x_{11} + 2q_2^2x_{20})\sigma_{001}
+ q_1^2q_3x_{25} + 2q_1q_3^2x_{28} + q_3q_3x_{31} + 3q_3^3x_{34}
\]

\[
\sigma_{100} * \sigma_{101} = [pt] + (q_1q_3x_{12} + 2q_2^2x_{21})\sigma_{100} + (q_1q_3x_{13} + 2q_2^2x_{22})\sigma_{010} + (q_1q_3x_{14} + 2q_2^2x_{23})\sigma_{001}
+ q_1^2q_3x_{26} + 2q_1q_3^2x_{29} + q_3q_3x_{32} + 3q_3^3x_{35}
\]

- For \( \varepsilon = 111 \), the curve classes which possibly contribute with non-zero Gromov-Witten invariants are

\[
\beta_3, \beta_1 + \beta_3, 2\beta_3, 2\beta_1 + \beta_3, \beta_1 + 2\beta_3, \beta_2 + \beta_3, 3\beta_3,
3\beta_1 + \beta_3, 2\beta_1 + 2\beta_3, \beta_1 + 3\beta_3, \beta_2 + 2\beta_3, 4\beta_3, \beta_1 + \beta_2 + \beta_3
\]

Using the fundamental class axiom for Gromov-Witten invariants,

\[
I_{\beta_3}(\sigma_{100}, [pt], [Z]) = 0.
\]

We assign variables for the remaining Gromov-Witten invariants.

\[
I_{\beta_1+\beta_3}(\sigma_{100}, [pt], [Z_{110}]) = x_{36}, \quad I_{\beta_1+\beta_3}(\sigma_{100}, [pt], [Z_{101}]) = x_{37}, \quad I_{\beta_1+\beta_3}(\sigma_{100}, [pt], [Z_{011}]) = x_{38},
I_{\beta_2}(\sigma_{100}, [pt], [Z_{110}]) = x_{39}, \quad I_{\beta_2}(\sigma_{100}, [pt], [Z_{101}]) = x_{40}, \quad I_{\beta_2}(\sigma_{100}, [pt], [Z_{011}]) = x_{41},
I_{\beta_2+\beta_3}(\sigma_{100}, [pt], [Z_{100}]) = x_{42}, \quad I_{\beta_2+\beta_3}(\sigma_{100}, [pt], [Z_{101}]) = x_{43}, \quad I_{\beta_2+\beta_3}(\sigma_{100}, [pt], [Z_{011}]) = x_{44},
I_{\beta_3}(\sigma_{100}, [pt], [Z_{100}]) = x_{45}, \quad I_{\beta_3}(\sigma_{100}, [pt], [Z_{101}]) = x_{46}, \quad I_{\beta_3}(\sigma_{100}, [pt], [Z_{011}]) = x_{47},
I_{\beta_3}(\sigma_{100}, [pt], [Z_{100}]) = x_{48}, \quad I_{\beta_3}(\sigma_{100}, [pt], [Z_{101}]) = x_{49}, \quad I_{\beta_3}(\sigma_{100}, [pt], [Z_{011}]) = x_{50},
I_{\beta_3+\beta_3}(\sigma_{100}, [pt], [pt]) = x_{51}, \quad I_{\beta_3+\beta_3}(\sigma_{100}, [pt], [pt]) = x_{52}, \quad I_{\beta_3+\beta_3}(\sigma_{100}, [pt], [pt]) = x_{53},
I_{\beta_3+\beta_3}(\sigma_{100}, [pt], [pt]) = x_{54}, \quad I_{\beta_3+\beta_3}(\sigma_{100}, [pt], [pt]) = x_{55}, \quad I_{\beta_3+\beta_3}(\sigma_{100}, [pt], [pt]) = x_{56},
I_{\beta_3+\beta_3}(\sigma_{100}, [pt], [pt]) = x_{57}, \quad I_{\beta_3+\beta_3}(\sigma_{100}, [pt], [pt]) = x_{58}, \quad I_{\beta_3+\beta_3}(\sigma_{100}, [pt], [pt]) = x_{59}
\]

We now record the final quantum product for the Chevalley matrix $A$:

\begin{align}
\sigma_{100} \ast [pt] &= (q_1 q_3 x_{36} + 2q_3^2 x_{39})\sigma_{110} + (q_1 q_3 x_{37} + 2q_3^2 x_{40})\sigma_{101} + (q_1 q_3 x_{38} + 2q_3^2 x_{41})\sigma_{011} \\
&\quad + (q_1^2 q_3 x_{42} + 2q_1 q_3^2 x_{45} + q_2 q_3 x_{48} + 3q_3^3 x_{51})\sigma_{100} \\
&\quad + (q_1^2 q_3 x_{43} + 2q_1 q_3^2 x_{46} + q_2 q_3 x_{49} + 3q_3^3 x_{52})\sigma_{010} \\
&\quad + (q_1^2 q_3 x_{44} + 2q_1 q_3^2 x_{47} + q_2 q_3 x_{50} + 3q_3^3 x_{53})\sigma_{010} \\
&\quad + q_1^3 q_3 x_{54} + 2q_1^2 q_3^2 x_{55} + 3q_1 q_3^3 x_{56} + 2q_2 q_3^2 x_{57} + 4q_3^4 x_{58} + q_1 q_2 q_3 x_{59}
\end{align}

\section{Chevalley matrix $B$}

As in the previous section, we can write the quantum product as

\[ \sigma_{010} \ast \sigma_\epsilon = \sum_{\beta, \epsilon'} I_\beta(\sigma_{010}, \sigma_\epsilon, [Z_{\epsilon'}]) q^\beta \sigma_{\epsilon'} \]

\begin{itemize}
  \item As before, $1 = [Z] \in H^*(Z)$ is also the identity in $QH^*(Z)$. Therefore,
  \[ \sigma_{010} \ast 1 = \sigma_{010} \]
  \[ \sigma_{010} \ast 1 = \sigma_{010} \]

  \item For $\epsilon = 100, 010, 001$, we can use the divisor axiom twice to reduce the three-point Gromov-Witten invariants to one-point invariants. The curve classes which give possibly non-zero Gromov-Witten invariants are
  \[ \beta_1, \beta_1 + \beta_2, 2\beta_1 \]

  We previously defined $I_{\beta_1 + \beta_2}([pt]) = x_1$, so we only need the following new invariants:

  \[ I_{\beta_1}([Z_{100}]) = y_1, \quad I_{\beta_1}([Z_{010}]) = y_2, \quad I_{\beta_1}([Z_{001}]) = y_3, \quad I_{2\beta_1}([pt]) = y_4 \]

  We can then write the quantum products as

  \begin{align}
  \sigma_{010} \ast \sigma_{100} &= \sigma_{110} + q_1 q_3 x_1 \\
  \sigma_{010} \ast \sigma_{010} &= \sigma_{110} + q_1 y_1 \sigma_{100} + q_1 y_2 \sigma_{010} + q_1 y_3 \sigma_{001} + (q_1 q_3 x_1 + 2q_1^2 y_4) \\
  \sigma_{010} \ast \sigma_{001} &= \sigma_{011} - q_1 q_3 x_1
  \end{align}

  \item For $\epsilon = 110, 101, 011$, we use the divisor axiom to reduce the three-point Gromov-Witten invariants to two-point invariants. The curve classes which give possibly non-zero Gromov-Witten invariants are

  \[ \beta_1, \beta_1 + \beta_2, 2\beta_1 + \beta_3, 2\beta_1 + \beta_3, \beta_1 + 2\beta_3, \beta_1 + \beta_2, 3\beta_1 \]

  We assign variables for the remaining Gromov-Witten invariants (note: many of these Gromov-Witten invariants were already considered in the
Chevalley matrix $A$.)

\[
\begin{align*}
I_{\beta_1}(\sigma_{110}, [Z_{110}]) &= y_5, & I_{\beta_1}(\sigma_{110}, [Z_{101}]) &= y_6, & I_{\beta_1}(\sigma_{110}, [Z_{011}]) &= y_7, \\
I_{\beta_1}(\sigma_{101}, [Z_{110}]) &= y_8, & I_{\beta_1}(\sigma_{101}, [Z_{101}]) &= y_9, & I_{\beta_1}(\sigma_{101}, [Z_{011}]) &= y_{10}, \\
I_{\beta_1}(\sigma_{011}, [Z_{110}]) &= y_{11}, & I_{\beta_1}(\sigma_{011}, [Z_{101}]) &= y_{12}, & I_{\beta_1}(\sigma_{011}, [Z_{011}]) &= y_{13}, \\
I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{100}]) &= x_6, & I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{010}]) &= x_7, & I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{001}]) &= x_8, \\
I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{100}]) &= x_9, & I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{101}]) &= x_{10}, & I_{\beta_1+\beta_3}(\sigma_{101}, [Z_{011}]) &= x_{11}, \\
I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{100}]) &= x_{12}, & I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{101}]) &= x_{13}, & I_{\beta_1+\beta_3}(\sigma_{011}, [Z_{011}]) &= x_{14}, \\
I_{2\beta_1}(\sigma_{110}, [Z_{100}]) &= y_{14}, & I_{2\beta_1}(\sigma_{110}, [Z_{101}]) &= y_{15}, & I_{2\beta_1}(\sigma_{110}, [Z_{011}]) &= y_{16}, \\
I_{2\beta_1}(\sigma_{101}, [Z_{100}]) &= y_{17}, & I_{2\beta_1}(\sigma_{101}, [Z_{101}]) &= y_{18}, & I_{2\beta_1}(\sigma_{101}, [Z_{011}]) &= y_{19}, \\
I_{2\beta_1}(\sigma_{011}, [Z_{100}]) &= y_{20}, & I_{2\beta_1}(\sigma_{011}, [Z_{101}]) &= y_{21}, & I_{2\beta_1}(\sigma_{011}, [Z_{011}]) &= y_{22}, \\
I_{2\beta_1+\beta_3}(\sigma_{110}, [pt]) &= x_{24}, & I_{2\beta_1+\beta_3}(\sigma_{101}, [pt]) &= x_{25}, & I_{2\beta_1+\beta_3}(\sigma_{011}, [pt]) &= x_{26}, \\
I_{2\beta_1+\beta_3}(\sigma_{110}, [pt]) &= y_{23}, & I_{2\beta_1+2\beta_3}(\sigma_{101}, [pt]) &= y_{28}, & I_{2\beta_1+2\beta_3}(\sigma_{011}, [pt]) &= y_{29}, \\
I_{3\beta_1}(\sigma_{110}, [pt]) &= y_{26}, & I_{3\beta_1}(\sigma_{101}, [pt]) &= y_{27}, & I_{3\beta_1}(\sigma_{011}, [pt]) &= y_{28}.
\end{align*}
\]

We can then write the quantum products:

\[
\begin{align*}
\sigma_{010} * \sigma_{110} &= q_1y_5\sigma_{110} + q_1y_6\sigma_{101} + q_1y_7\sigma_{011} + (q_1q_3x_6 + 2q_1^2y_{14})\sigma_{100} \\
&+ (q_1q_3x_7 + 2q_1^2y_{15})\sigma_{010} + (q_1q_3x_8 + 2q_1^2y_{16})\sigma_{001} \\
&+ 2q_1^2q_3x_{24} + q_1q_3^2x_{27} + q_1q_2y_{28} + 3q_1^3y_{26}
\end{align*}
\]

\[
\begin{align*}
\sigma_{010} * \sigma_{101} &= [pt] + q_1y_8\sigma_{110} + q_1y_9\sigma_{101} + q_1y_{10}\sigma_{011} + (q_1q_3x_9 + 2q_1^2y_{17})\sigma_{100} \\
&+ (q_1q_3x_{10} + 2q_1^2y_{18})\sigma_{100} + (q_1q_3x_{11} + 2q_1^2y_{19})\sigma_{001} \\
&+ 2q_1^2q_3x_{25} + q_1q_3^2x_{28} + q_1q_2y_{29} + 3q_1^3y_{27}
\end{align*}
\]

\[
\begin{align*}
\sigma_{010} * \sigma_{011} &= [pt] + q_1y_{11}\sigma_{110} + q_1y_{12}\sigma_{101} + q_1y_{13}\sigma_{011} + (q_1q_3x_{12} + 2q_1^2y_{20})\sigma_{100} \\
&+ (q_1q_3x_{13} + 2q_1^2y_{21})\sigma_{100} + (q_1q_3x_{14} + 2q_1^2y_{22})\sigma_{001} \\
&+ 2q_1^2q_3x_{26} + q_1q_3^2x_{29} + q_1q_2y_{25} + 3q_1^3y_{28}
\end{align*}
\]

- For $\varepsilon = 111$, the curve classes which contribute with possibly non-zero Gromov-Witten invariants are

\[
\beta_1, \beta_1 + \beta_3, 2\beta_1, 2\beta_1 + \beta_3, \beta_1 + 2\beta_3, \beta_1 + 2\beta_3, 3\beta_1 + \beta_3, \beta_1 + 2\beta_3, 4\beta_1, \beta_1 + \beta_2 + \beta_3
\]

Using the fundamental class axiom, $I_{\beta_1}(\sigma_{101}, [pt], [Z]) = 0$. We record the remaining Gromov-Witten invariants.

\[
\begin{align*}
I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{110}]) &= x_{36}, & I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{101}]) &= x_{37}, & I_{\beta_1+\beta_3}(\sigma_{110}, [Z_{011}]) &= x_{38}, \\
I_{2\beta_1}(\sigma_{110}, [Z_{110}]) &= y_{29}, & I_{2\beta_1}(\sigma_{110}, [Z_{101}]) &= y_{30}, & I_{2\beta_1}(\sigma_{110}, [Z_{011}]) &= y_{31}, \\
I_{2\beta_1+\beta_3}(\sigma_{110}, [Z_{100}]) &= x_{42}, & I_{2\beta_1+\beta_3}(\sigma_{110}, [Z_{010}]) &= x_{43}, & I_{2\beta_1+\beta_3}(\sigma_{110}, [Z_{001}]) &= x_{44}, \\
I_{\beta_1+2\beta_3}(\sigma_{110}, [Z_{100}]) &= x_{45}, & I_{\beta_1+2\beta_3}(\sigma_{110}, [Z_{101}]) &= x_{46}, & I_{\beta_1+2\beta_3}(\sigma_{110}, [Z_{011}]) &= x_{47}, \\
I_{3\beta_1}(\sigma_{110}, [Z_{100}]) &= y_{35}, & I_{3\beta_1}(\sigma_{110}, [Z_{101}]) &= y_{36}, & I_{3\beta_1}(\sigma_{110}, [Z_{011}]) &= y_{37}, \\
I_{\beta_1+\beta_3}(\sigma_{101}, [pt], [pt]) &= x_{54}, & I_{2\beta_1+2\beta_3}(\sigma_{101}, [pt], [pt]) &= x_{55}, & I_{\beta_1+3\beta_3}(\sigma_{101}, [pt], [pt]) &= x_{56}, \\
I_{2\beta_1+\beta_3}(\sigma_{101}, [pt], [pt]) &= y_{38}, & I_{4\beta_1}(\sigma_{101}, [pt], [pt]) &= y_{39}, & I_{\beta_1+2\beta_2+\beta_3}(\sigma_{101}, [pt], [pt]) &= x_{59}.
\end{align*}
\]
We record the final quantum product for the matrix $B$:

\begin{equation}
\sigma_{010} * [pt] = (q_1 q_3 x_{36} + 2 q_1^2 y_{29}) \sigma_{110} + (q_1 q_3 x_{37} + 2 q_1^2 y_{30}) \sigma_{101} + (q_1 q_3 x_{38} + 2 q_1^2 y_{31}) \sigma_{011} \\
+ (2 q_1^2 q_3 x_{42} + q_1 q_3^2 x_{45} + q_1 q_2 y_{32} + 3 q_1^3 y_{35}) \sigma_{100} \\
+ (2 q_1^2 q_3 x_{43} + q_1 q_3^2 x_{46} + q_1 q_2 y_{33} + 3 q_1^3 y_{36}) \sigma_{010} \\
+ (2 q_1^2 q_3 x_{44} + q_1 q_3^2 x_{47} + q_1 q_2 y_{34} + 3 q_1^3 y_{37}) \sigma_{001} \\
+ 3 q_1^3 q_3 x_{54} + 2 q_1^2 q_3^2 x_{55} + q_1 q_3 x_{56} + 2 q_1^2 q_2 y_{38} + 4 q_1^4 y_{39} + q_1 q_2 q_3 x_{59}
\end{equation}

6.3. Chevalley matrix $C$. As in the previous two sections, we can express the quantum product as follows

$$
\sigma_{001} * \sigma_{\varepsilon} = \sum_{\beta, \varepsilon'} I_\beta(\sigma_{001}, \sigma_{\varepsilon}, [Z_{\varepsilon'}]) q^\beta \sigma_{\varepsilon'}
$$

- For $\varepsilon = 000$, since $1 = [Z] \in H^*(Z)$ is also the identity in $QH^*(Z)$, we have

\begin{equation}
\sigma_{001} * 1 = \sigma_{001}
\end{equation}

- For $\varepsilon = 100, 010, 001$, we use the divisor axiom twice to reduce three-point Gromov-Witten invariants to one-point invariants. The applicable curve classes are 

$$
\beta_3, \beta_1 + \beta_3, \beta_2, 2\beta_3
$$

In the first subsection, we observed that the one-point invariants for curve class $\beta_3$ are explicitly computable:

$$
I_{\beta_3}([Z_{100}]) = \int_{Z_{100}} [Z_{100}] = -1
$$

$$
I_{\beta_3}([Z_{010}]) = \int_{Z_{010}} [Z_{010}] = 1
$$

$$
I_{\beta_3}([Z_{001}]) = \int_{Z_{001}} [Z_{001}] = 0
$$

In fact, all the relevant Gromov-Witten invariants (except $I_{\beta_3}([pt])$) for these curve classes were already considered in the subsection for matrix $A$. Using Corollary 4.1, we compute $I_{\beta_3}([pt]) = 1$. We record the quantum products

\begin{align}
(6.3.2) \quad & \sigma_{001} * \sigma_{100} = \sigma_{101} + q_3 \sigma_{100} - q_3 \sigma_{010} + (-q_1 q_3 x_1 - 4 q_3^2 x_2) \\
(6.3.3) \quad & \sigma_{001} * \sigma_{010} = \sigma_{011} - q_1 q_3 x_1 \\
(6.3.4) \quad & \sigma_{001} * \sigma_{001} = \sigma_{011} - 2 \sigma_{101} - q_3 \sigma_{100} + q_3 \sigma_{010} + (q_1 q_3 x_1 + 4 q_3^2 x_2 + q_2)
\end{align}

- For $\varepsilon = 110, 101, 011$, we use the divisor axiom to reduce three-point Gromov-Witten invariants to two-point invariants. The applicable curve classes are very similar to those in matrix $A$, however $\beta_2 + \beta_3 = [Z_{100}]$. So, by the divisor axiom $I_{\beta_2 + \beta_3}(\sigma_{001}, \sigma_{\varepsilon}, [Z_{\varepsilon'}]) = 0$. The applicable curve classes are

$$
\beta_3, \beta_1 + \beta_3, \beta_2, 2\beta_3, 2\beta_1 + \beta_3, \beta_1 + 2\beta_3, \beta_1 + \beta_2, 3\beta_3
$$
Recall, curve neighborhoods are used to show $I_{\beta_i}(\sigma \varepsilon, [Z_i]) = 0$ unless $\varepsilon = 101$. We record the remaining Gromov-Witten invariants here:

$$I_{\beta_3}(\sigma_{101}, [Z_{110}]) = x_3, \quad I_{\beta_3}(\sigma_{101}, [Z_{101}]) = x_4, \quad I_{\beta_3}(\sigma_{101}, [Z_{111}]) = x_5,$$

$$I_{\beta_3}(\sigma_{110}, [Z_{101}]) = x_6, \quad I_{\beta_3}(\sigma_{110}, [Z_{110}]) = x_7, \quad I_{\beta_3}(\sigma_{110}, [Z_{101}]) = x_8,$$

$$I_{\beta_3}(\sigma_{101}, [Z_{101}]) = x_9, \quad I_{\beta_3}(\sigma_{101}, [Z_{110}]) = x_10, \quad I_{\beta_3}(\sigma_{101}, [Z_{111}]) = x_11,$$

$$I_{\beta_3}(\sigma_{011}, [Z_{101}]) = x_{12}, \quad I_{\beta_3}(\sigma_{011}, [Z_{110}]) = x_{13}, \quad I_{\beta_3}(\sigma_{011}, [Z_{111}]) = x_{14},$$

$$I_{\beta_3}(\sigma_{110}, [Z_{101}]) = z_1, \quad I_{\beta_3}(\sigma_{110}, [Z_{110}]) = z_2, \quad I_{\beta_3}(\sigma_{110}, [Z_{111}]) = z_3,$$

$$I_{\beta_3}(\sigma_{101}, [Z_{101}]) = z_4, \quad I_{\beta_3}(\sigma_{101}, [Z_{110}]) = z_5, \quad I_{\beta_3}(\sigma_{101}, [Z_{111}]) = z_6,$$

$$I_{\beta_3}(\sigma_{011}, [Z_{101}]) = z_7, \quad I_{\beta_3}(\sigma_{011}, [Z_{110}]) = z_8, \quad I_{\beta_3}(\sigma_{011}, [Z_{111}]) = z_9,$$

$$I_{\beta_3}(\sigma_{110}, [Z_{101}]) = x_{15}, \quad I_{\beta_3}(\sigma_{110}, [Z_{110}]) = x_{16}, \quad I_{\beta_3}(\sigma_{110}, [Z_{111}]) = x_{17},$$

$$I_{\beta_3}(\sigma_{101}, [Z_{101}]) = x_{18}, \quad I_{\beta_3}(\sigma_{101}, [Z_{110}]) = x_{19}, \quad I_{\beta_3}(\sigma_{101}, [Z_{111}]) = x_{20},$$

$$I_{\beta_3}(\sigma_{011}, [Z_{101}]) = x_{21}, \quad I_{\beta_3}(\sigma_{011}, [Z_{110}]) = x_{22}, \quad I_{\beta_3}(\sigma_{011}, [Z_{111}]) = x_{23},$$

$$I_{\beta_3}(\beta_{110}, [pt]) = x_{24}, \quad I_{\beta_3}(\beta_{101}, [pt]) = x_{25}, \quad I_{\beta_3}(\beta_{011}, [pt]) = x_{26},$$

$$I_{\beta_3}(\beta_{110}, [pt]) = x_{27}, \quad I_{\beta_3}(\beta_{101}, [pt]) = x_{28}, \quad I_{\beta_3}(\beta_{011}, [pt]) = x_{29},$$

$$I_{\beta_3}(\beta_{110}, [pt]) = x_{30}, \quad I_{\beta_3}(\beta_{101}, [pt]) = x_{31}, \quad I_{\beta_3}(\beta_{011}, [pt]) = x_{32},$$

And the quantum products are:

(6.3.5) $\sigma_{001} * \sigma_{110} = [pt] + (q_1 x_6 + q_2 z_1 - q_3 x_15)\sigma_{100} + (-q_1 x_7 + q_2 z_2 - q_3 x_16)\sigma_{100} + (-q_1 x_8 + q_2 z_3 - q_3 x_17)\sigma_{011} + (-q_1 x_9 + q_2 z_4 - q_3 x_2)\sigma_{011}$

(6.3.6) $\sigma_{001} * \sigma_{101} = [pt] - q_3 x_11 \sigma_{100} - q_4 x_10 \sigma_{100} - q_3 x_2 \sigma_{101}$

(6.3.7) $\sigma_{001} * \sigma_{111} = [pt] + (-q_1 x_12 + q_2 z_7 - q_3 x_21)\sigma_{100} + (-q_1 x_13 + q_2 z_8 - q_3 x_22)\sigma_{011}$

- For $\varepsilon = 111$, the curve classes which contribute with possibly nonzero Gromov-Witten invariants are:

$$\beta_3, \beta_1 + \beta_3, \beta_2 + \beta_3, \beta_1 + \beta_3, \beta_1 + \beta_2, 3\beta_3$$

$$3\beta_1 + \beta_3, 2\beta_1 + 2\beta_3, \beta_1 + 3\beta_3, 2\beta_2 + \beta_2, 2\beta_3, 4\beta_3$$

Using the fundamental class axiom, $I_{\beta_3}(\sigma_{001}, [pt], [Z]) = 0$. We record the remaining unknown Gromov-Witten invariants:

$$I_{\beta_3}(\beta_{110}, [pt], [Z_{110}]) = x_{36}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{101}]) = x_{37}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{111}]) = x_{38},$$

$$I_{\beta_3}(\beta_{110}, [pt], [Z_{101}]) = z_{10}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{110}]) = z_{11}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{111}]) = z_{12},$$

$$I_{\beta_3}(\beta_{110}, [pt], [Z_{101}]) = x_{39}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{110}]) = x_{40}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{111}]) = x_{41},$$

$$I_{\beta_3}(\beta_{110}, [pt], [Z_{101}]) = x_{42}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{110}]) = x_{43}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{111}]) = x_{44},$$

$$I_{\beta_3}(\beta_{110}, [pt], [Z_{101}]) = x_{45}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{110}]) = x_{46}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{111}]) = x_{47},$$

$$I_{\beta_3}(\beta_{110}, [pt], [Z_{101}]) = x_{48}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{110}]) = x_{49}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{111}]) = x_{50},$$

$$I_{\beta_3}(\beta_{110}, [pt], [Z_{101}]) = x_{51}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{110}]) = x_{52}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{111}]) = x_{53},$$

$$I_{\beta_3}(\beta_{110}, [pt], [Z_{101}]) = x_{54}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{110}]) = x_{55}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{111}]) = x_{56},$$

$$I_{\beta_3}(\beta_{110}, [pt], [Z_{101}]) = x_{57}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{110}]) = x_{58}, \quad I_{\beta_3}(\beta_{110}, [pt], [Z_{111}]) = x_{59}.$$
The final quantum product is

\[ (6.3.8) \]

\[
\sigma_{001} * [pt] = (-q_1 q_3 x_{36} + q_2 z_{10} - 2q_3^2 x_{36})\sigma_{110} + (-q_1 q_3 x_{37} + q_2 z_{11} - 2q_3^2 x_{36})\sigma_{101} + (-q_1 q_3 x_{38} + q_2 z_{12} - 2q_3^2 x_{41})\sigma_{011} + (-q_1^2 q_3 x_{42} - 2q_1 q_2 y_{32} - 3q_3^3 x_{51})\sigma_{100} + (-q_1^2 q_3 x_{43} - 2q_1 q_2 y_{33} - 3q_3^3 x_{52})\sigma_{010} + (-q_1^2 q_3 x_{44} - 2q_1 q_2 y_{34} - 3q_3^3 x_{53})\sigma_{001} + (-q_1^3 q_3 x_{54} - 2q_1^2 q_3^2 x_{55} - 3q_1 q_2 y_{3} - 2q_2^2 z_{13} - 2q_3 q_3^2 x_{57} - 4q_3^4 x_{58}
\]

6.4. **Brute force.** The unknown Gromov-Witten invariants can be computed by imposing the relations

\[
[A, B] = 0, \quad [A, C] = 0, \quad [B, C] = 0;
\]

these are the relations that quantum multiplication commutes. This gives relations among the remaining unknown invariants which can then be solved using brute force. This gives values for all but a single Gromov-Witten invariant

\[
y_3 = I_{p_1}(Z_{001})
\]

We record the matrices here: \((A\) is the matrix obtained from multiplication by \(\sigma_{100}, B\) from multiplication by \(\sigma_{010},\) and \(C\) from multiplication by \(\sigma_{001}\))

\[
A = \begin{pmatrix}
0 & q_1 q_3 y_3 & q_1 q_3 y_3 & -q_1 q_3 y_3 & 0 & 0 & 0 & q_1 q_2 y_3 y_3 \\
1 & -q_3 & 0 & q_3 & q_1 q_3 y_3 & q_1 q_3 y_3 & 0 & 0 \\
0 & q_3 & 0 & -q_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q_1 q_3 y_3 & q_1 q_3 y_3 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & q_3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -q_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q_3 & 0 & q_1 q_3 y_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & q_1 q_3 y_3 & q_1 q_3 y_3 & -q_1 q_3 y_3 & 0 & 0 & 0 & q_1 q_2 y_3 & q_1 q_2 y_3 & q_1 q_2 y_3 \\
0 & 0 & 2q_1 y_3 & 0 & q_1 q_3 y_3 & q_1 q_3 y_3 & 0 & q_1 q_2 y_3 & q_1 q_2 y_3 & q_1 q_2 y_3 \\
1 & 0 & -q_1 y_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_1 y_3 & 0 & q_1 q_3 y_3 & q_1 q_3 y_3 & 0 & 0 \\
0 & 1 & 1 & 0 & -q_1 y_3 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & q_1 y_3 & 0 & 0 \\
0 & 0 & 0 & 0 & q_1 y_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & q_1 q_3 y_3 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
0 & -q_3 q_3 y_3 & -q_3 q_3 y_3 & q_1 q_3 y_3 + q_2 & 0 & 0 & 0 & q_1 q_2 y_3 & 0 \\
0 & q_3 & 0 & -q_3 & q_1 q_3 y_3 & -q_1 q_3 y_3 & q_2 & 0 & q_1 q_2 y_3 \\
0 & -q_3 & 0 & q_3 & 0 & 0 & q_2 & 0 \\
1 & 0 & 0 & 0 & -q_1 q_3 y_3 & -q_1 q_3 y_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -q_3 & 0 & q_2 \\
0 & 1 & 0 & -2 & 0 & q_3 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & -q_3 & 0 & -q_1 q_3 y_3 \\
0 & 0 & 0 & 1 & 1 & 1 & -1 & 0
\end{pmatrix}
\]

In order to compute \(y_1\), we use [Man12, Remark 5.7]. In particular, this remark immediately implies the following result.
Proposition 6.1. Let $Z$ denote the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1)$, and $Z' = Z(\alpha_1, \alpha_2)$. The following commutative diagram is Cartesian:

$$\begin{array}{c}
\overline{M}_{0,1}(Z, \beta_1) \\ \downarrow \varphi \\
\overline{M}_{0,1}(G/B, [X(s_{\alpha_2})])
\end{array} \qquad \begin{array}{c}
\overline{M}_{0,1}(Z', [Z'_0]) \\ \downarrow \varphi_{\alpha_1} \\
\overline{M}_{0,1}(G/P_{\alpha_1}, [X(s_{\alpha_2})])
\end{array}$$

In particular, since $\varphi_{\alpha_1}$ is an isomorphism, $\varphi : \overline{M}_{0,1}(Z, \beta_1) \to \overline{M}_{0,1}(Z', [Z'_0])$ is also an isomorphism.

Combined with Corollary 4.1, we can compute $y_3$:

$$y_3 = I_{\beta_1}([Z_{001}]) = \int ev^*(\sigma_{110}) \cdot [\overline{M}_{0,1}(Z, \beta_1)]$$

$$= \int ev^*([pt]) \cdot [\overline{M}_{0,1}(Z', [Z'_0])] = \int_{Z'} [pt] = 1.$$ 

Proposition 6.2. The Gromov-Witten invariant $y_3 = I_{\beta_1}([Z_{001}]) = 1$.

One can read the entire Chevalley formula for $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ from the Chevalley matrices now that $y_3$ has been computed; we record the presentation for $QH^*(Z)$ so obtained here:

Theorem 6.1. $QH^*(Z)$ is generated by $\sigma_{100}, \sigma_{010}, \sigma_{001}, q_1, q_2, q_3$ subject to the following relations:

$$\begin{align*}
\sigma_{100}^2 &= q_1 q_3 - q_3 \sigma_{100} + q_3 \sigma_{010} \\
\sigma_{010}^2 &= q_1 q_3 + 2q_1 \sigma_{100} - q_1 \sigma_{010} + q_1 \sigma_{001} + \sigma_{110} \\
\sigma_{001}^2 &= q_1 q_3 + q_2 - q_3 \sigma_{100} + q_3 \sigma_{010} - 2\sigma_{101} + \sigma_{011}
\end{align*}$$

We can also record the “Giambelli formula,” the representation of the vector space generators $\sigma_{000}, \sigma_{100}, \ldots, \sigma_{011}, \sigma_{111}$ as polynomials in the algebra generators $\sigma_{100}, \sigma_{010}, \sigma_{001}$.

Corollary 6.1. In $QH^*(Z)$, the Giambelli formulae for the classes $\sigma_{110}, \sigma_{101}, \sigma_{011}, \sigma_{111}$ are as follows:

$$\begin{align*}
\sigma_{110} &= \sigma_{100} \sigma_{010} - q_1 q_3 \\
\sigma_{101} &= \sigma_{100} \sigma_{001} - q_3 \sigma_{100} + q_3 \sigma_{010} + q_1 q_3 \\
\sigma_{011} &= \sigma_{010} \sigma_{001} + q_1 q_3 \\
\sigma_{111} &= \sigma_{100} \sigma_{010} \sigma_{001} + q_1 q_3 \sigma_{100}
\end{align*}$$

Using these formulas, we are able to verify that the ring $QH^*(Z)$ is indeed associative.

7. Conjecture $O$

Consider the operator $\tilde{c}_1 : H^*(Z) \to H^*(Z)$ defined as follows: let $c_1 : QH^*(Z) \to QH^*(Z)$ be the operator defined by multiplication by $c_1(-K_Z)$, and let $\tilde{c}_1$ denote the specialization of $c_1$ with all quantum parameters set equal to one.
Conjecture $O$, which is related to the Gamma conjectures of Galkin, Golyshev, and Iritani (see [CL17]), is a statement concerning the eigenvalues of $\hat{c}_1$. Namely, Conjecture $O$ states that if $Z$ is Fano, then the following properties hold:

1. Let $\delta_0$ denote the maximum modulus among the eigenvalues of $\hat{c}_1$. Then $\delta_0$ is one of the eigenvalues of $\hat{c}_1$, and occurs with multiplicity one;
2. If $\delta$ is any eigenvalue of $\hat{c}_1$ with $|\delta| = \delta_0$, then there is an $r$th root of unity $\zeta$ such that $\delta = \delta_0 \zeta$, where $r$ is the Fano index of $Z$.

Since the Bott-Samelson variety $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ is Fano (see Remark 6.1), and using the Chevalley matrices from the last section, we can write the matrix for $\hat{c}_1$:

$$\hat{c}_1 = \begin{pmatrix}
0 & 2 & 2 & -2 & 0 & 0 & 3 & 4 \\
3 & -1 & 2 & 1 & 2 & 4 & 0 & 3 \\
1 & 1 & -1 & -1 & 0 & 0 & 2 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 1 & 4 & 0 & -1 & 1 & 0 & 2 \\
0 & 2 & 0 & -1 & 1 & -1 & 0 & 0 \\
0 & 0 & 2 & 3 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 3 & 2 & 0
\end{pmatrix}$$

This matrix has eight distinct eigenvalues which are approximated numerically, the eigenvalue with the largest modulus is real, and all other eigenvalues have strictly smaller modulus. In particular, Conjecture $O$ holds for the Bott-Samelson variety $Z(\alpha_1, \alpha_2, \alpha_1)$.

**Theorem 7.1.** The Conjecture $O$ holds for the Bott-Samelson variety $Z = Z(\alpha_1, \alpha_2, \alpha_1)$ (in Type $A_2$).

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