A Remark on a Theorem of \texttt{math.AG/0511155}

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Abstract

We give another proof of a theorem of H. Kajiura, K. Saito, and A. Takahashi \cite{9} based on the theory of weighted projective lines by Geigle and Lenzing \cite{5, 6} and a theorem of Orlov \cite{13} on triangulated categories of graded B-branes. The content of this paper appears in the appendix to \texttt{math.AG/0511155}.

1 Introduction

The Milnor lattice of a simple singularity is isomorphic to the root lattice of the corresponding simple Lie algebra, and the simple singularity can be constructed inside this simple Lie algebra as the intersection of the nilpotent cone with a transversal slice of a subregular nilpotent orbit \cite{3}. In search of a good class of singularities where this relationship with Lie algebras might generalize, Saito introduced the notion of regular weight system \cite{14} and considered a generalization of root systems coming from the Milnor lattices of the associated singularities \cite{17}. However, Milnor lattices are hard to handle due to the transcendental nature of vanishing cycles, and to circumvent this difficulty, Saito asked for an algebraic or combinatorial construction of the root system starting from a regular weight system \cite{15}.

In \cite{19}, Takahashi proposed an answer to the problem of Saito based on mirror symmetry for Landau-Ginzburg orbifolds and its relation \cite{18} with the duality of weight systems \cite{16}. He introduced the triangulated category $D^b_Z(\mathbb{A}_f)$ of graded matrix factorizations for a weighted homogeneous polynomial $f$ building on earlier papers by Eisenbud \cite{4}, Hori and Walcher \cite{8}, Kapustin and Li \cite{10, 11}, and Orlov \cite{12}, which is equivalent to the triangulated categories of graded B-branes defined by Orlov \cite{13}. He conjectured that this category for a polynomial associated with a regular weight system has a full exceptional collection and its Grothendieck group is isomorphic to the Milnor lattice of the singularity associated with the dual regular weight system. He in particular conjectured that when $f$ is the defining polynomial
of a simple singularity, then $D^b_Z(A_f)$ is equivalent as a triangulated category to the bounded derived category of finite-dimensional representations of a Dynkin quiver. The latter conjecture is solved by Takahashi [19] for $A_n$-singularities and by Kajiura, Saito, and Takahashi [9] for all the simple singularities:

**Theorem 1 ([9, Theorem 3.1]).** The triangulated category of graded B-branes on a simple singularity is equivalent as a triangulated category to the bounded derived category of representations of a Dynkin quiver of the corresponding type.

We give another proof of the above theorem in this paper, which avoids the use of the classification of Cohen–Macaulay modules on simple singularities due to Auslander [1], and is based on the theory of weighted projective lines by Geigle and Lenzing [5, 6] and a theorem of Orlov [13] on triangulated categories of graded B-branes.

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## 2 Weighted projective lines of Geigle and Lenzing

Let $k$ be a field. For a sequence $p = (p_0, p_1, p_2)$ of nonzero natural numbers, let $L(p)$ be the abelian group of rank one generated by four elements $\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{c}$ with relations $p_0 \bar{x}_0 = p_1 \bar{x}_1 = p_2 \bar{x}_2 = \bar{c}$, and consider the $k$-algebra

$$R(p) := k[x_0, x_1, x_2]/(x_0^{p_0} + x_1^{p_1} + x_2^{p_2})$$

graded by $\deg(x_s) = \bar{x}_s \in L(p)$ for $s = 0, 1, 2$. Define the category of coherent sheaves on the weighted projective line of weight $p$ as the quotient category

$$\text{qgr } R(p) := \text{gr } R(p)/\text{tor } R(p)$$

of the abelian category $\text{gr } R(p)$ of finitely-generated $L(p)$-graded $R(p)$-modules by its full subcategory $\text{tor } R(p)$ consisting of torsion modules. This definition is equivalent to the original one by Geigle and Lenzing due to Serre’s theorem in [5, section 1.8].
Let $\pi : \text{gr}-R(\mathbf{p}) \to \text{qgr} R(\mathbf{p})$ be the natural projection. For $M \in \text{gr}-R(\mathbf{p})$ and $\vec{x} \in L(\mathbf{p})$, let $M(\vec{x})$ be the graded $R(\mathbf{p})$-module obtained by shifting the grading by $\vec{x}$; $M(\vec{x}) = M_{\vec{n} + \vec{x}}$ and put $\mathcal{O}(\vec{n}) = \pi R(\mathbf{p})(\vec{n})$. Then the sequence

$$(E_0, \ldots, E_N) = (\mathcal{O}, \mathcal{O}(\vec{x}_0), \mathcal{O}(2\vec{x}_0), \ldots, \mathcal{O}((p_0 - 1)\vec{x}_0),$$

$$(\mathcal{O}(\vec{x}_1), \mathcal{O}(2\vec{x}_1), \ldots, \mathcal{O}((p_2 - 1)\vec{x}_2), \mathcal{O}(\vec{c}))$$

of objects of $\text{qgr} R(\mathbf{p})$, where $N = p_0 + p_1 + p_2 - 2$, is a full strong exceptional collection by [5, Proposition 4.1]. Define the dualizing element $\vec{v} \in L(\mathbf{p})$ by

$$\vec{v} = \vec{c} - \vec{x}_0 - \vec{x}_1 - \vec{x}_2$$

and a $\mathbb{Z}$-graded subalgebra $R'(\mathbf{p})$ of $R(\mathbf{p})$ by

$$R'(\mathbf{p}) = \bigoplus_{n \geq 0} R'(\mathbf{p})_n, \quad R'(\mathbf{p})_n = R(\mathbf{p})_{-n\vec{v}}. \quad (1)$$

A weight sequence $\mathbf{p} = (p_0, p_1, p_2)$ is called of Dynkin type if

$$\frac{1}{p_0} + \frac{1}{p_1} + \frac{1}{p_2} > 1.$$ 

A weight sequence of Dynkin type yields the rational double point of the corresponding type:

**Proposition 2 ([3, Proposition 8.4.]).** For a weight sequence $\mathbf{p}$ of Dynkin type, the $\mathbb{Z}$-graded algebra $R'(\mathbf{p})$ has the form

$$k[x, y, z]/f_\mathbf{p}(x, y, z)$$

where the homogeneous generators $(x, y, z)$ and the relation $f_\mathbf{p}(x, y, z)$ are displayed in the following table:

| weight | Generators $(x, y, z)$ | $\mathbb{Z}$-degrees | Relations $f_\mathbf{p}$ |
|--------|------------------------|----------------------|-------------------------|
| $A_{p+q}$ | $(1, p, q)$ | $(x_1x_2, x_2^{p+q}, x_1^{p+q})$ | $(1, p, q)$ | $x^{p+q} - yz$ |
| $D_{2l-2}$ | $(2, 2, 2l)$ | $(x_2^2, x_0^2, x_0x_1x_2)$ | $(2, 2l, 2l + 1)$ | $z^2 + x(y^2 + yx')$ |
| $D_{2l-1}$ | $(2, 2, 2l + 1)$ | $(x_2^3, x_0x_1^2, x_0^2x_2)$ | $(2, 2l + 1, 2l + 2)$ | $z^2 + x(y^2 + xz')$ |
| $E_6$ | $(2, 3, 3)$ | $(x_0, x_1x_2, x_3^3)$ | $(3, 4, 6)$ | $z^2 + y^3 + x^2z$ |
| $E_7$ | $(2, 3, 4)$ | $(x_1, x_2^2, x_0x_2)$ | $(4, 6, 9)$ | $z^2 + y^3 + x^2y$ |
| $E_8$ | $(2, 3, 5)$ | $(x_2, x_1, x_0)$ | $(6, 10, 15)$ | $z^2 + y^3 + x^5$ |

Moreover, we have the following:

**Proposition 3 ([3, Proposition 8.5]).** For a weight sequence $\mathbf{p}$ of Dynkin type, there exists a natural equivalence

$$\text{qgr} R(\mathbf{p}) \cong \text{qgr} R'(\mathbf{p}),$$

where $\text{qgr} R'(\mathbf{p})$ is the quotient category of the abelian category $\text{gr}-R'(\mathbf{p})$ of finitely-generated $\mathbb{Z}$-graded $R'(\mathbf{p})$-modules by its full subcategory $\text{tor}-R'(\mathbf{p})$ consisting of torsion modules.
3 Graded B-branes on simple singularities

For the $\mathbb{Z}$-graded algebra $R'(p)$ given above, define the triangulated category of graded B-branes as the quotient category

$$D_{Sg}^{gr}(R'(p)) := D^b(gr-R'(p))/D^b(grproj-R'(p))$$

of the bounded derived category $D^b(gr-R'(p))$ by its full triangulated subcategory $D^b(grproj-R'(p))$ consisting of perfect complexes, i.e., bounded complexes of projective $\mathbb{Z}$-graded modules \[13\].

$D_{Sg}^{gr}(R'(p))$ is equivalent to $HMF_{k[x,y,z]}^{gr}(f_p)$ in the notation of \[9\]. Note that $R'(p)$ is Gorenstein with Gorenstein parameter one, i.e.,

$$\text{Ext}^i_A(k, A) = \begin{cases} k(1) & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

which follows from, e.g., \[12\] Corollary 2.2.8 and Proposition 2.2.10]. Thus there exists a fully faithful functor $\Phi_0 : D_{Sg}^{gr}(R'(p)) \rightarrow D^b(qgr R'(p))$ and a semiorthogonal decomposition

$$D^b(qgr R'(p)) = \langle E_0, \Phi_0 D_{Sg}^{gr}(R'(p)) \rangle$$

by \[13\] Theorem 2.5.(i). Therefore, $D_{Sg}^{gr}(R'(p))$ is equivalent to the full triangulated subcategory of $D^b(qgr R'(p))$ generated by the strong exceptional collection $(E_1, \ldots, E_N)$. Its total morphism algebra $\text{End}(\bigoplus_{i=1}^n E_i)$ is isomorphic to the path algebra of the Dynkin quiver

$$\Delta(p) : \bar{x}_0 \xrightarrow{x_0} 2\bar{x}_0 \xrightarrow{x_0} \cdots \xrightarrow{x_0} (p_0 - 1)\bar{x}_0 \xrightarrow{x_0} \bar{x}_0$$

$$\bar{x}_1 \xrightarrow{x_1} 2\bar{x}_1 \xrightarrow{x_1} \cdots \xrightarrow{x_1} (p_1 - 1)\bar{x}_1 \xrightarrow{x_1} \bar{c}$$

$$\bar{x}_2 \xrightarrow{x_2} 2\bar{x}_2 \xrightarrow{x_2} \cdots \xrightarrow{x_2} (p_2 - 1)\bar{x}_2 \xrightarrow{x_2} \bar{c}$$

of the corresponding type, obtained from the quiver appearing in \[5\] section 4] by removing the leftmost vertex. Since $D_{Sg}^{gr}(R'(p))$ is an enhanced triangulated category, the equivalence $D_{Sg}^{gr}(R'(p)) \simeq D^b(\text{mod-} \mathbb{C}\Delta(p))$ follows from Bondal and Kapranov \[2\] Theorem 1].

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